THE SPHERICAL TRANSFORM OF A SCHWARTZ FUNCTION ON THE FREE TWO STEP NILPOTENT LIE GROUP

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Abstract. Let $F(n)$ be a connected and simply connected free 2-step nilpotent Lie group and $K$ be a compact subgroup of $\text{Aut}(F(n))$. We say that $(K, F(n))$ is a Gelfand pair when the set of integrable $K$-invariant functions on $F(n)$ forms an abelian algebra under convolution. In this paper, we consider the case when $K = O(n)$. In this case, the Gelfand space $(O(n), F(n))$ is equipped with the Godement-Plancherel measure, and the spherical transform $\hat{\cdot}: L^2_{O(n)}(F(n)) \to L^2(\Delta(O(n), F(n)))$ is an isometry. I will prove the Gelfand space $\Delta(O(n), F(n))$ is equipped with the Godement-Plancherel measure and the inversion formula. Both of which have something related to its corresponding Heisenberg group. The main result in this paper provides a complete characterization of the set $\varphi_{O(n)}(F(n))^\wedge = \{ \hat{f} \ | \ f \in \varphi_{O(n)}(F(n))\}$ of spherical transforms of $O(n)$-invariant Schwartz functions on $F(n)$. I show that a function $F$ on $\Delta(O(n), F(n))$ belongs to $\varphi_{O(n)}(F(n))^\wedge$ if and only if the functions obtained from $F$ via application of certain derivatives and difference operators satisfy decay conditions.

1. Introduction

What can one say about the spherical transform of a $O(n)$-invariant Schwartz function on $F(n)$? More precisely, letting $\varphi_{O(n)}(F(n))$ denote the space of $O(n)$-invariant Schwartz functions on $F(n)$ we seek to characterize the subspace $\varphi_{O(n)}(F(n))^\wedge = \{ \hat{f} \ | \ f \in \varphi_{O(n)}(F(n))\}$ of $C_0(\Delta(O(n), F(n)))$, where $O(n)$-spherical transform $\hat{f} \to C$ for a function $f \in L^1_{O(n)}(F(n))$ is defined by $\hat{f}(\psi) = \int_{F(n)} f(x) \overline{\psi(x)} \, dx$. The main result in this paper is section 4 below, which provides a complete solution to this problem. Before describing the contents of this problem I wish to provide some background and motivation for the study of $\varphi_{O(n)}(F(n))$ via the spherical transform.

Schwartz functions have played an important role in harmonic analysis with nilpotent groups since the work of Kirilov [1]. Let $N$ be connected and simply connected nilpotent Lie group with Lie algebra $n$. The exponential map: $n \to N$ is a polynomial diffeomorphism and one defines the (Frechet) space $\varphi(N)$ of Schwartz functions on $N$ via identification with the usual

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space $\varphi(n)$ of Schwartz function on the vector space $n$: $\varphi(N) := \{ f : N \rightarrow C \mid f \circ exp \in \varphi(n) \}$. $\varphi(N)$ is dense in $L^p(N)$ for each $p$ and carries an algebra structure given by the convolution product. Moreover, it is known that the primitive ideal space for $\varphi(N)$ is isomorphic to that of both $L^1(N)$ and $C^*(N)[2]$. The Heisenberg groups $H_n$ are the simplest groups for which $\varphi(N)$ is non-abelian. Recall that the group Fourier transform for a function $f \in L^1(N)$ associates to $\pi \in \hat{N}$, an irreducible unitary representation of $N$, the bounded operator $\pi(f) = \int_N f(x)\pi(x)dx$ in the representation space of $\pi$. This generates the usual Euclidean Fourier transform for the case $N = \mathbb{R}^n$. The importance of Schwartz functions in Euclidean harmonic analysis arises from the fact that $\varphi(\mathbb{R}^n)$ is preserved by the Fourier transform. It is thus very natural to seek a characterization of $\varphi(N)$ via the group Fourier transform; a problem solved by Roger Howe in [3].

One can sometimes obtain subalgebras of $\varphi(N)$ by considering "radial" functions. This is of interest even when $N = \mathbb{R}^n$. Indeed, the algebra $\varphi_{O(n)}(\mathbb{R}^n)$ of radial Schwartz function on $\mathbb{R}^n$ can be identified with $\varphi(\mathbb{R}^n)$ and the Fourier transform becomes a Hankel transform on $\varphi(\mathbb{R}^+)$. This is the spherical transform for the Gelfand pair obtained from the action of the orthogonal group $O(n)$ on $\mathbb{R}^n[4]$. Theorem 4.4 provides conditions that are both necessary and sufficient for a function $F$ on $\Delta(O(n), F(n))$ to belong to the space $\varphi_{O(n)}(F(n))\wedge$:

1. $F$ is continuous on $\Delta(O(n), F(n))$.
2. The function $F_0$ on $R$ defined by $F_0(r) = F(\phi^r)$ belongs to $\phi(R)$.
3. The map $\lambda \mapsto F(\phi^{r,\alpha,\lambda})$ is smooth on $R^\times$ and the functions $\partial_\lambda^m F(\phi^{r,\alpha,\lambda})$ satisfy certain decay conditions. In particular, $\partial_\lambda^m F(\phi^{r,\alpha,\lambda})$ is a rapidly decreasing sequence in $\alpha$ for each fixed $r \in R$ and $\lambda \in R^\times$.
4. Certain "derivatives" of $F$ also satisfy the three conditions above. These are defined on $\Delta_1(O(n), F(n))$ as specific combinations of $\partial_\lambda$ and "difference operators" which play the role of differentiation in the discrete parameter $\alpha \in \wedge$.

The precise formulation of these conditions can be found in Definition 3.2.1. The "derivatives" of functions in $\varphi_{O(n)}(F(n))\wedge$ referred to above are operators corresponding to multiplication of functions in $\varphi_{O(n)}(F(n))$ by certain polynomials. The difference operators in the discrete parameter $\alpha \in \wedge$ are linear operators whose coefficients are "generalized binomial coefficients". These coefficients were introduced by Z. Yan in [5]. A summary of their properties is given below in Section 3.

I will prove the inversion formula for $(O(n), F(n))$ is of the form:

$$f(x) = \frac{c}{(2\pi)^{n-1}} \int_R \int_{R^\times} \sum_{\alpha \in \wedge} (\text{dim} \mathcal{P}_\alpha) \int (\phi^{r,\alpha,\lambda}(x) |\lambda|^n d\lambda dr, \quad x = exp(X + A) \in F(n), \quad c \text{ is a fixed constant, } \phi^{r,\alpha,\lambda}(x) \text{ is the "type 1" } O(n)\text{-bounded spherical functions. Also, I will show the Godement-Plancherel measure } d\mu \text{ on } \Delta(O(n), F(n)) \text{ is given by:}$$
\[
\int_{\Delta(O(n), F(n))} F(\psi) d\mu(\psi) = \left(\frac{2\pi}{\dim O(n)}\right)^n \int_R \int_{R^+} \sum_{\alpha \in \Lambda} (\dim P_\alpha) \hat{F}(\phi^{r,\alpha,\lambda}) |\lambda|^n d\lambda dr.
\]
Both of them are related to the corresponding formula of the Heisenberg group.

One consequence of the estimates involved in our characterization of \(\varphi_{O(n)}(F(n))\) is that \(f \in \varphi_{O(n)}(F(n))\) can be recovered from \(F = \hat{f}\) via the inversion of the spherical transform.

2. Notation and Preliminaries

Let \(G\) be a connected Lie group, \(K\) a compact subgroup. Let \(\pi\) denote the natural mapping of \(G\) onto \(X = G/K\) and as usual we put \(o = \pi(e)\) and \(\tilde{f} = f \circ \pi\) if \(f\) is any function on \(X\). Let \(D(G)\) denote the set of all left invariant differential operators on \(G\), \(D_K(G)\) the subspace of those which are also right invariant under \(K\) and \(D(G/K)\) the algebra of differential operators on \(G/K\) invariant under all the translations \(\tau(g) : xK \to gxK\) of \(G/K\).

**Definition 2.1.** Let \(\phi\) be a complex-valued function on \(G/K\) of class \(C^\infty\) which satisfies \(\phi(\pi(e)) = 1; \phi\) is called a spherical function if

1. \(\phi^*(k) = \phi\) for all \(k \in K\),
2. \(D\phi = \lambda_D\phi\) for each \(D \in D(G/K)\),

where \(\lambda_D\) is a complex number.

It is sometimes convenient to consider the function \(\tilde{\phi} = \phi \circ \pi\) on \(G\) instead of \(\phi\). We say that \(\tilde{\phi}\) is a spherical function on \(G\) if and only if \(\phi\) is a spherical function on \(G/K\). Then a spherical function \(\phi\) on \(G\) is characterized by being an eigenfunction of each operator in \(D_K(G)\) and in addition satisfying the relations \(\tilde{\phi}(e) = 1, \tilde{\phi}(kgk') = \tilde{\phi}(g)\) for all \(g \in G\) and all \(k, k' \in K\). The last condition will be called bi-invariance under \(K\).

**Theorem 2.2.** Let \(f\) be a complex-valued continuous function on \(G\), not identically 0. Then \(f\) is a spherical function if and only if
\[
\int_K f(xky) dk = f(x)f(y) \quad \text{for all} \quad x, y \in G.
\]

Next, we consider the Heisenberg groups. There are many ways to define Heisenberg group. I introduce two of them here. [6] The first one is as follows:

Let \(X\) be an arbitrary real vector space. Denote \(X^*\) the dual space of \(X\). Let \(\varphi(X)\) be the Schwartz space of \(X\), that is, the space of smooth, rapidly decreasing functions on \(X\). (Unless specified otherwise, functions are complex-valued.) Let \(T \subseteq C\) be the unit circle. Write \(e(t) = e^{2\pi it}\) for the usual exponential map from \(R\) to \(T\). Define operators on \(\varphi(X)\) as follows.

1. \(\rho(x') f(x) = f(x - x')\) for \(x, x' \in X\) and \(f \in \varphi(X)\)
2. \(\rho(\xi) f(x) = e(\xi(x)) f(x)\) for \(\xi \in X^*, \) and \(x, f\) as in (a)
3. \(\rho(z) f(x) = z f(x)\) for \(z \in T, \) and \(x, f\) as in (a)

These operators fit together to form a nice group of operators. Precisely, put...
Define a law of composition on \( H \) by
\[
(x_1, \xi_1, z_1)(x_2, \xi_2, z_2) = (x_1 + x_2, \xi_1 + \xi_2, z_1 z_2 e(\xi_1(x_2))).
\]
It is easily verified that the law in above makes \( H \) into a two-step nilpotent Lie group with center \( T \). We call \( H \) the Heisenberg group. A straightforward computation shows that
\[
\rho : (x, \xi, z) \rightarrow \rho(x)\rho(\xi)\rho(z)
\]
defines an isomorphism of \( H \) to a group of operators on \( \varphi(X) \), or in other words, is a representation of \( H \) on \( \varphi(X) \).

As in usual, we let \( L^2(X) \) be the Hilbert space of functions on \( X \) which are square integrable with respect to Lebesgue measure. We know \( \varphi(X) \) is a dense subspace of \( L^2(X) \), and in particular inherits the inner product from \( L^2(X) \). One checks easily that the operators \( \rho(h) \) for \( h \in H \) are isometries with respect to this inner product, so that in fact \( \rho \) is the restriction to \( \varphi(X) \) of a unitary representation, also denoted \( \rho \), of \( H \) on \( L^2(X) \). As we will see, \( \varphi(X) \) is intrinsically defined in terms if \( \rho \).

For the second definition, we identify \( H_a \) with \( C^a \times R \) with multiplication given by
\[
(z,t)(z',t') = (z + z', t + t' + \frac{1}{2} \omega(z, z')) \quad \text{where} \quad \omega(z, z') := -\text{Im} \langle z, z' \rangle = -\text{Im}(z \cdot z').
\]
It will occasionally be more congenial to have a "coordinate free" model for \( H_a \). In this cases, we assume that \( V \) is an \( a \)-dimensional vector space over \( C \) equipped with a Hermitian inner product \( < \cdot, \cdot > \). \( H_a \) is then identified with \( V \times R \), and the multiplication is given by
\[
(v, t)(v', t') = (v + v', t + t' + \frac{1}{2} \omega(v, v')) \quad \text{where} \quad \omega(v, v') := -\text{Im} \langle v, v' \rangle.
\]
We will write \( H_V \) for the Heisenberg group given by \( (V, < \cdot, \cdot >) \).

The left-invariant vector fields generated by the one-parameter subgroups through \(((0, \cdots, 0), \cdots, (0,0))\) are written explicitly as
\[
Z_j = 2 \frac{\partial}{\partial Z_j} + i \frac{\partial}{\partial \overline{Z}_j}, \quad \overline{Z}_j = 2 \frac{\partial}{\partial \overline{Z}_j} - i \frac{\partial}{\partial Z_j}.
\]
In addition, let \( T := \frac{\partial}{\partial t} \), so that \( \{Z_1, \cdots, Z_a, \overline{Z}_1, \cdots, \overline{Z}_a, T\} \) is a basis for the Lie algebra \( n_a \) of \( H_a \). With these notations one has \([Z_j, \overline{Z}_j] = -2iT\).

Consider a unimodular group \( G \) with \( K \subseteq G \) a compact subgroup. We denote the \( L^1 \)-functions that are invariant under both the left and right actions of \( K \) on \( G \) by \( L^1(G//K) \). These form a subalgebra of the group algebra \( L^1(G) \) with respect to the convolution product \( f*g(x) = \int_G f(y)g(y^{-1}x)dy = \int_G f(xy^{-1})g(y)dy \). According to the traditional definition, one says that \( K \subseteq G \) is a Gelfand pair if \( L^1(G//K) \) is commutative.

Suppose now \( K \) is a compact group acting on \( N \), where \( N \) is a connect and simply connected solvable Lie group. By automorphism via some homomorphism \( \phi : K \to Aut(N) \), one can form the semidirect product \( K \rtimes N \), with group law
\[
(k_1, x_1)(k_2, x_2) = (k_1 k_2, x_1 k_1 x_2), \quad \text{where we write} \quad k.x \text{ for} \quad \phi(k)(x).
\]
Right \( K \)-invariance of a function \( f : K \rtimes N \to C \) means that \( f(k, x) \) depends only on \( x \). Accordingly, if one defines \( f_N : N \to C \) by \( f_N(x) = f(e, x) \),
then one obtains a bijection \( L^1(K \ltimes N/K) \cong L^1_{k_0}(N) \) given by \( f \mapsto f_N \).

Here \( L^1_{k_0}(N) \) denotes the \( K \)-invariant functions on \( N \), i.e. those \( f \in L^1_{k_0}(N) \) such that \( f(k.x) = f(x) \) for all \( x \in N \) and \( k \in K \). One verifies easily that this map respects the conclusion product and we see that \( K \subseteq K \ltimes N \) is a Gelfand pair if and only if, the convolution algebra \( L^1_{k_0}(N) \) is commutative. Thus, the definition given here agrees with the more standard one.

Note that if \((K_1,N)\) is a Gelfand pair and \( K_1 \subseteq K_2 \), \((K_2,N)\) is also a Gelfand pair. Also note that we can assume that \( K \) acts faithfully on \( N \) since we can always replace \( K \) by \( K/\ker \phi \). In this way, we can regard \( K \) as a compact subgroup of \( Aut(N) \).

**Lemma 2.3.** [7] Let \( K,L \) be compact groups acting on \( G \) which are conjugate inside \( Aut(G) \). Then \((K,G)\) is a Gelfand pair if, and only if, \((L,G)\) is a Gelfand pair.

For the \((2a+1)\)-dimensional Heisenberg group \( H_a \), the natural action of the group of \( a \times a \) unitary matrices on \( C^a \) (which we denote by \( k.z \) for \( k \in U(a) \) and \( Z \in C^a \) gives rise to a compact subgroup of \( Aut(H_a) \) via \( k.(z,t) = (kz, t) \). This subgroup, again denoted by \( U_a \), is a maximal connected, compact subgroup of \( Aut(H_a) \) and thus any connected, compact subgroup of \( Aut(H_a) \) is the conjugate of a subgroup \( K \) of \( U_a \). Since conjugates of \( K \) form Gelfand pairs with \( H_a \) if and only if, \( K \) does, and produce the same spherical functions, \( \tilde{\pi} \), I will always assume that I am dealing with a compact subgroup of \( U(a) \).

**Theorem 2.4.** [7] Let \( N \) be a connected, simply connected, nilpotent Lie group. If \( N \) is an \( n \)-step group with \( n \geq 3 \) then there are no Gelfand pairs \((K,N)\).

Suppose the \((2a+1)\)-dimensional Heisenberg group \( H_a \) has Lie algebra \( h_a \) with basis \( X_1, \ldots, X_a, Y_1, \ldots, Y_a, Z \) and structure equations given by \([X_i,Y_j] = Z\). We call the representation theory of \( H_a \). A generic set of coadjoint orbits in \( h_a^* \) is parametrized by nonzero \( \lambda \in \mathbb{R} \), where the orbit \( O_{\lambda} \) is the hyperplane in \( h_a^* \) of all functions taking the value \( \lambda \) at \( Z \). The action of \( U(a) \) on \( h_a^* \) preserves each \( O_{\lambda} \). Hence, if \( \pi(\lambda) \) is the element of \( \hat{H}_a \) corresponding to \( O_{\lambda} \), the \( U(a) \) also preserves the equivalence class of \( \pi(\lambda) \). One can realize \( \pi(\lambda) \) in the Fock space.

The Fock model, for real \( \lambda > 0 \), is defined on the space \( F_\lambda \) of holomorphic functions on \( C^a \) which are square integrable with respect to the measure \( d\omega = (\frac{\lambda}{2\pi})^a e^{-\frac{1}{2}|\omega|^2} d\omega d\overline{\omega} [8] \). The space \( P(C) \) of holomorphic polynomials is dense in \( F_\lambda \).

The representation \( \pi_\lambda \) of \( H_a \) of \( F_\lambda \) is given by

\[
\pi_\lambda(z,t)u(\omega) = e^{i\lambda \cdot <\omega,z> - \frac{1}{4}|z|^2} u(\omega + z)
\]

For \( \lambda < 0 \), \( F_\lambda \) consists of antiholomorphic functions which are square with respect to \( d\overline{\omega} |\lambda| \), and the representation is given by

\[
\pi_\lambda(z,t)u(\overline{\omega}) = e^{i\lambda \cdot <\omega,z> + \frac{1}{4}|z|^2} u(\overline{\omega} + z)
\]
Since the irreducible unitary representations of $H_a$ which are non-trivial on the center $R$, are determined up to equivalence by their centre character. For $k \in U(a)$, the representation $\pi^k_\lambda(z,t) = \pi_\lambda(k.z,t)$ has the same centre character as $\pi_\lambda$ and hence is equivalent to $\pi_\lambda$. For $\lambda > 0$, the operator that intertwines these two representations comes from the standard action of $U(a)$ on $C^a$. More precisely,

$$[\pi_\lambda(k.z,t)u](k.\omega) = [\pi_\lambda(z,t)(k^{-1}.u)](\omega). \text{ where } k \times u(\omega) = u(k^{-1} \times \omega).$$

One has a similar formula for $\lambda < 0$, except that the action of $U(a)$ on antiholomorphic functions is given by $k \times u(\omega) = u(k \times \omega)$.

If we denote $W_\lambda(k)u(\omega) = u(k^{-1}.\omega)$ for $\lambda > 0$ and $W_\lambda(k)u(\omega) = u(k.\omega)$ for $\lambda < 0$ respectively. We have $W_\lambda(k)\pi_\lambda(z,t)W_\lambda(k)^{-1} = \pi_\lambda(k.z,t)$. That is, $U(a)$ is the stabilizer of the equivalence class of $\pi_\lambda \in \tilde{H}_a$ under the action of $U(a)$ and $W_\lambda$. I remark that up to a factor of $\det(k)^{\frac{1}{2}}$, $W_\lambda$ lifts to the oscillator representation on the double cover $MU(a)$ of $U(a)$ (cf[9]) Now I give another way to introduce the oscillator representation. It is as follows.

**Theorem 2.5.** (Moore-Wolf). An irreducible representation $\sigma$ of a nilpotent Lie group $N$ is square-integrable (modulo the centre of $N$) if and only if it is the unique irreducible representation of $N$ with its central character.

Let $\alpha$ be an automorphism of $H$ which acts trivially on $T$. By the uniqueness of $\rho$ we know from general considerations that there is a unitary operator $\omega(\alpha)$ defined up to a scalar multiples, such that $\omega(\alpha)\rho(h)\omega(\alpha)^{-1} = \rho(\alpha(h))$.

**Theorem 2.6.** (Shale-Weil). Let $\tilde{S}p$ be the 2-fold cover of $Sp$. Let $\tilde{g} \rightarrow g$ be the projection map. Then there is a unitary representation $\omega$ of $\tilde{S}p$ on $L^2$ such that

$$\omega(\tilde{g})\rho(h)\omega(\tilde{g})^{-1} = \rho(g(h)).$$

I call $\omega$ the oscillator representation.

Given a compact, connected subgroup $K \subseteq U(n)$, we denote its complexification by $K_C$. The action of $K$ on $C^n$ yields a representation of $K_C$ on $C^n$, and one can view $K_C$ as a subgroup of $Gl(n, C)$.

A finite dimensional representation $\rho : G \rightarrow Gl(V)$ in a complex vector space $V$ is said to be multiplicity free if each irreducible $G$-modules occurs at most once in the associated representation on the polynomial ring $C[V].(\text{given by } (x.p)(z) = p(\rho(x^{-1})z)).$

**Theorem 2.7.** Let $K$ be a compact, connected subgroup of $U(a)$ acting irreducibly on $C^a$. The following are equivalent: (i) $(K,H_a)$ is a Gelfand pair. (ii) The representation of $K_C$ on $C^a$ is multiplicity free. (iii) The representation of $K_C$ on $C_a$ is equivalent to one of the representations in a table[7].

**Theorem 2.8.** [7] $(SO(n), F(n))$ and $(O(n), F(n))$ are Gelfand pairs for all $n \geq 2$.

**Theorem 2.9.** [7] If $K$ is a proper, closed (not necessarily connected) subgroup of $SO(n)$ then $(K,F(n))$ is not a Gelfand pair.
A result due to Howe and Umeda (cf. [10]) shows that $\mathbb{C}[v_R]^K$ is freely generated as an algebra. So there are polynomials $\gamma_1, \ldots, \gamma_d \in \mathbb{C}[v_R]^K$ so that $\mathbb{C}[v_R]^K = \mathbb{C}[\gamma_1, \ldots, \gamma_d]$.

We call $\gamma_1, \ldots, \gamma_d$ the fundamental invariants and we can suppose $\gamma_1(z) = \frac{|z|^2}{2}$.

Invariant differential operators. The algebra $\mathcal{D}(H_a)$ of left-invariant differential operators on $H_a$ is generated by $\{Z_1, \ldots, Z_a, \overline{Z_1}, \ldots, \overline{Z_a}, T\}$. We denote the subalgebra of $K$-invariant differential operators by

$$\mathcal{D}_K(H_a) := \{D \in \mathcal{D}(H_a) \mid D(f \circ k) = D(f) \circ k \text{ for } k \in K, f \in C^\infty(H_a)\}$$

From now on, we always suppose $(K, H_a)$ is a Gelfand pair, and if this is true, $\mathcal{D}_K(H_a)$ is an abelian algebra.

One differential operator will play a key role in the Heisenberg group. This is the Heisenberg sub-Laplacian defined by

$$U = \frac{1}{2} \sum_{j=1}^n (Z_j \overline{Z}_j + \overline{Z}_j Z_j).$$

$U$ is $U(a)$-invariant and hence belongs to $\mathcal{D}_K(H_a)$ for all Gelfand pairs $(K, H_a)$. Note that $U$ is essentially self-adjoint on $L^2(H_a)$.

The eigenvalues of the Heisenberg sub-Laplacian $U$ on the type 1 $K$-spherical functions are given by

$$U(\phi_{a,\lambda}) = -|\lambda| (2|\alpha| + a)\phi_{a,\lambda}.$$

Define $\phi_{a,\lambda}$ for $\alpha \in \wedge$ and $\lambda \in \mathbb{R}^\times$ by

$$\phi_{a,\lambda}(z, t) = \phi_{a,\lambda}(\sqrt{|\lambda|} z, \lambda t),$$

so that $\phi_a = \phi_{a,1}$. The $\phi_{a,\lambda}$’s are distinct bounded $K$-spherical functions. We refer to these elements of $\Delta(K, H_a)$ as the spherical function of type 1. One can show that $\phi_a$ has the general form

$$\phi_a(z, t) = e^{ia}\eta_{a}(z)e^{-\frac{|z|^2}{4}},$$

where $\eta_{a}$ is a $K$-invariant polynomial on $V_R$ with homogeneous component of highest degree given by $(-1)^{[\alpha]}|p_{a}/\dim(P_{a}).$

In addition to the $K$-spherical functions of type 1, there are $K$-spherical functions which arise from the one-dimensional representations of $H_a$. For $\omega \in V$, let

$$\eta_{\omega}(z, t) = \int_{K} e^{i\text{Re}<\omega, k z>} dk = \int_{K} e^{i\text{Re}<z, k \omega>} dk$$

where ”dk” denotes normalized Haar measure on $K$. The $\eta_{\omega}$ are the bounded $K$-spherical functions of type 2. Note that $\eta_0$ is the constant function 1 and $\eta_{\omega} = \eta_{\omega'}$ if and only if $K \omega = K \omega'$. It is shown in [16] that every bounded $K$-spherical function is of type 1 or type 2. Thus we have:

**Theorem 2.10.** The bounded $K$-spherical functions on $H_a$ are parametrized by the set $(\mathbb{R}^\times \times \wedge) \cup (V/K)$ via

$$\Delta(K, H_a) = \{\phi_{a,\lambda} \mid \lambda \in \mathbb{R}^\times, \alpha \in \wedge\} \cup \{\eta_{K \omega} \mid \omega \in V\}$$

Note that, for $\psi \in \Delta(K, H_a)$, one has $\psi(z, t) = e^{i\lambda t}\psi(z, 0),$

where $\lambda = -i\bar{T}(\psi) \in \mathbb{R}.$

Finally, we consider the two-step free nilpotent Lie groups. First Definition. Let $N$ be the (unique up to isomorphism) free two-step nilpotent Lie
algebra with \( n \) generators. The definition using the universal property of the free nilpotent Lie algebra can be found in [11, Chapter V 5]. Roughly speaking, \( \mathcal{N} \) is a (nilpotent) Lie algebra with \( n \) generators \( X_1, \ldots, X_n \), such that the vectors \( X_1, \ldots, X_n \) and \( X_{i,j} = [X_i, X_j], i < j \) form a basis; we call this basis the canonical basis of \( \mathcal{N} \).

We denote by \( V \) and \( Z \), the vector spaces generated by the families of vectors \( X_1, \ldots, X_n \) and \( X_{i,j} = [X_i, X_j], 1 \leq i < j \leq n \) respectively; these families become the canonical base of \( V \) and \( Z \). Thus \( \mathcal{N} = V \oplus Z \), and \( Z \) is the center of \( \mathcal{N} \). With the canonical basis, the vector space \( Z \) can be identified with the vector space of antisymmetric \( n \times n \)-matrices \( A_n \). Let \( z = \dim Z = n(n-1)/2 \).

The connected simply connected nilpotent Lie group which corresponds to \( \mathcal{N} \) is called the free two-step nilpotent Lie group and is denoted \( F_n \). We denote by \( \exp: \mathcal{N} \to F(n) \) the exponential map.

In the following, we use the notations \( X + A \in \mathcal{N}, \exp(X + A) \in F(n) \) when \( X \in V, A \in Z \). We write \( n = 2p' \) or \( 2p' + 1 \).

A Realization of \( \mathcal{N} \). We now present here a realization of \( \mathcal{N} \), which will be helpful to define more naturally the action of the orthogonal group and representations of \( F(n) \).

Let (\( V, \langle, \rangle \)) be an Euclidean space with dimension \( n \). Let \( O(V) \) be the group of orthogonal transformations of \( V \), and \( SO(V) \) its special subgroup. Their common Lie algebra denoted by \( Z \), is identified with the vector space of antisymmetric transformations of \( V \). Let \( \mathcal{N} = V \oplus Z \) be the exterior direct sum of the vector spaces \( V \) and \( Z \).

Let \( [\cdot, \cdot]: V \times V \to Z \) be the bilinear application given by:
\[
[X,Y](V) = \langle X, V \rangle Y - \langle Y, V \rangle X
\]
where \( X,Y,V \in V \).

We also denote by \([\cdot,\cdot]\) the bilinear application extended to \( \mathcal{N} \times \mathcal{N} \to \mathcal{N} \) by:
\[
[\cdot,\cdot]_{\mathcal{N} \times \mathcal{N}} = [\cdot,\cdot]_{\mathcal{N} \times V} = 0
\]
This application is a Lie bracket. It endows \( V \) with the structure of a two-step nilpotent Lie algebra.

As the elements \( X, Y \in V \) generate the vector space \( Z \), we also define a scalar product \( \langle \cdot, \cdot \rangle \) on \( Z \) by:
\[
\langle [X,Y], [X',Y'] \rangle = \langle X, X' \rangle \langle Y,Y' \rangle - \langle X, Y' \rangle \langle X', Y \rangle
\]
where \( X,Y,X',Y' \in V \).

It is easy to see \( V \) as a realization of \( \mathcal{N} \) when an orthonormal basis \( X_1, \ldots, X_n \) of \( (V, \langle, \rangle) \) is fixed.

We remark that \( \langle [X,Y], [X',Y'] \rangle = \langle [X,Y]X', Y' \rangle \), and so we have for an antisymmetric transformation \( A \in Z \), and for \( X,Y \in V \):
\[
\langle A, [X,Y] \rangle = \langle A.X, Y \rangle
\]
This equality can also be proved directly using the canonical basis of \( \mathcal{N} \).

Actions of Orthogonal Groups. We denote by \( O(V) \) the group of orthogonal linear maps of \( (V, \langle, \rangle) \), and by \( O_n \) the group of orthogonal \( n \times n \)-matrices.
On $\mathcal{N}$ and $F(n)$. The group $O(\mathcal{V})$ acts on the one hand by automorphism on $\mathcal{V}$, on the other hand by the adjoint representation $Ad_\mathcal{Z}$ on $\mathcal{Z}$. We obtain an action of $O(\mathcal{V})$ on $\mathcal{N} = \mathcal{V} \oplus \mathcal{Z}$. Let us prove that this action respects the Lie bracket of $\mathcal{N}$. It suffices to show for $X, Y, Z \in \mathcal{V}$ and $k \in O(\mathcal{V})$:

\[
[k.X, k.Y](V) = k.X, V > k.Y - k.Y, V > k.X
\]

(2.11) \[k.(<X, k^t.V > Y - <Y, k^t.V > X = k.[X, Y](k^{-1}.V) = Ad_\mathcal{Z}k.[X, Y].\]

We then obtain that the group $O(\mathcal{V})$ and also its special subgroup $SO(\mathcal{V})$ acts by automorphism on the Lie algebra $\mathcal{N}$, and finally on the Lie group $F(n)$.

Suppose an orthonormal basis $X_1, \ldots, X_n$ of $(\mathcal{V}, <, >)$ is fixed; then the vectors $X_{i,j} = [X_i, X_j], 1 \leq i < j \leq n$, form an orthonormal basis of $\mathcal{V}$ and we can identify:

- the vector space $\mathcal{Z}$ and $\mathcal{A}_n$.
- the group $O(\mathcal{V})$ with $O_n$.
- the adjoint representation $Ad_\mathcal{Z}$ with the conjugate action of $O_n$ and $\mathcal{A}_n$:

\[k.A = kA_k^{-1}, \quad \text{where } k \in O_n, A \in \mathcal{A}_n.\]

Thus the group $O_n \sim O(\mathcal{V})$ acts on $\mathcal{V} \sim \mathbb{R}^n$ and $\mathcal{Z} \sim \mathcal{A}_n$, and consequently on $\mathcal{N}$. Those actions can be directly defined; and the equality $[k.X, k.Y] = k.[X, Y], k \in O_n, X, Y \in \mathcal{V}$, can then be computed.

On $\mathcal{A}_n$. Now we describe the orbits of the conjugate actions of $O_n$ and $SO_n$ on $\mathcal{A}_n$. An arbitrary antisymmetric matrix $A \in \mathcal{A}_n$ is $O_n$-conjugated to an antisymmetric matrix $D_2(\wedge)$ where $\wedge = (\delta_1, \ldots, \delta_{p'}) \in \mathbb{R}^r$ and:

\[
D_2(\wedge) = \begin{bmatrix}
\delta_1 J & 0 & 0 & 0 \\
0 & \ddots & 0 & 0 \\
0 & 0 & \delta_{p'} J & 0 \\
0 & 0 & 0 & (0)
\end{bmatrix}
\]

where $J := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

\((0)\) means that a zero appears only in the case $n = 2p' + 1$ Further-
more, we can assume that $\wedge$ is in $\mathcal{Z}$, where we denote by $\mathcal{L}$ the set of $\wedge = (\delta_1, \ldots, \delta_{p'}) \in \mathbb{R}^r$ such that $\delta_1 \geq \cdots \geq \delta_{p'} \geq 0$.

Parameters. To each $\wedge \in \mathcal{Z}$, we associate $p_0$ the number of $\delta_i \neq 0$, $p_1$ the number of distinct $\delta_i \neq 0$, and $\mu_1, \ldots, \mu_{p_0}$ such that:

\[\{\mu_1 > \mu_2 > \cdots > \mu_{p_1} > 0\} = \{\delta_1 \geq \delta_2 \geq \cdots \geq \delta_{p_0} \geq 0\}\]

We denote by $m_j$ the number of $\delta_i$ such that $\delta_i = \mu_j$, and we put

$m_0 := m'_0 := 0$ and for $j = 1, \cdots p_1$ $m'_j := m_1 + \cdots + m_j$.

For $j = 1, \cdots p_1$, let $pr_j$ be the orthogonal projection of $\mathcal{V}$ onto the space generated by the vectors $X_{2j-1}, X_{2j}$, for $i = m'_j + 1, \cdots m'_j$.

Let $\mathcal{M}$ be the set of $(r, \wedge)$ where $\wedge \in \mathcal{L}$, and $r \geq 0$, such that $r = 0$ if $2p_0 = n$. 

Expression of the bounded spherical functions. The bounded spherical functions of \((K, F(n))\) for \(K = O_n\), are parameterized by

\((r, \wedge) \in \mathcal{M}\) (with the previous notations \(p_0, p_1, \mu, pr_j\) associated to \(\wedge\)),

\(l \in \mathbb{N}^{p_1}\) if \(\wedge \neq 0\), otherwise \(\emptyset\).

Let \((r, \wedge), l\) be such parameters. Then we have the following two types of bounded \(O(n)\)-spherical functions:

For \(n = \exp(X + A) \in N\).

Type 1: \(\phi_{n}^{1, \alpha, \lambda}(n) = \int_{\mathbb{R}} e^{ir <X^*, k.X>} \omega_{\alpha, \lambda}(\Psi_2^{-1}(\mathbb{M}(k.n)))dk\).

Type 2: \(\phi_{n}^{2}(n) = \int_{\mathbb{R}} e^{ir <X^*, k.X>} dk\). Here \(X^*_p\) is the unit \(K_p\)-fixed invariant vector. For a Gelfand pair \((H_{p_0}, K(m; p_1; p_0))\), we have \(\omega_{\alpha, \lambda}\) is the "type 1" bounded \(K(m; p_1; p_0)\)-spherical functions for the Heisenberg group \(H_{p_0}\). We will introduce it next. \(\Psi_2\) is an isomorphism between \(H_{p_0}\) with a group with respect to \(F(n)\), which will be introduced later.

We call \(H_{p_0}\) the Heisenberg group with respect to \(F(n)\).

We use the following law of the Heisenberg group \(H_{p_0}\):

\(\forall h = (z_1, \ldots, z_{p_0}, t), h' = (z'_1, \ldots, z'_{p_0}, t') \in \mathbb{H}^{p_0} = \mathbb{C}^{p_0} \times \mathbb{R}\)

\(h, h' = (z_1 + z'_1, \ldots, z_{p_0} + z'_{p_0}, t + t' + \frac{1}{2} \sum_{i=1}^{p_0} z_i z'_i)\)

The unitary \(p_0 \times p_0\) matrix group \(U_{p_0}\) acts by automorphisms on \(\mathbb{H}^{p_0}\). Let us describe some subgroups of \(U_{p_0}\). Let \(p_0, p_1 \in \mathbb{N}\), and \(m = (m_1, \ldots, m_{p_1}) \in \mathbb{N}^{p_1}\) be fixed such that \(\sum_{j=1}^{p_1} m_j = p_0\). Let \(K(m; p_1; p_0)\) be the subgroup of \(U_{p_0}\) given by:

\(K(m; p_1; p_0) = U_{m_1} \times \ldots \times U_{m_{p_1}}\).

The expression of spherical functions of \((H_{p_0}, K(m; p_1; p_0))\) can be found in the same way as in the case \(m = (p_0), p_1 = 1\) i.e. \(K = U_{p_0}\) (cf.[12]).

Stability group \(K_\rho = \{ k \in K : k.\rho = \rho \} = \{ k \in K \subset G : k.f \in F(n).f \}\).

The aim of this paragraph is to describe the stability group \(K_\rho\) of \(\rho \in T_{X^*_p} + D_2(\wedge)\).

Before this, let us recall that the orthogonal \(2n \times 2n\) matrices which commute with \(D_2(1, \ldots, 1)\) have determinant one and form the group \(Sp_n \cap O_{2n}\).

This group is isomorphism to \(U_n\); the isomorphism is denoted \(\psi_1^{(n)}\), and satisfies:

\(\forall k, X: \psi_1^{(n)}(k.X) = \psi_1^{(n)}(K)\psi_1^{(n)}(X),\)

where \(\psi_1^{(n)}\) is the complexification:

\(\psi_1^{(n)}(x_1, y_1; \ldots; x_n, y_n) = (x_1 + iy_1, \ldots, x_n + iy_n)\).

Now, we can describe \(K_\rho\).

**Proposition 2.12.** Let \((r, \Lambda) \in \mathcal{M}\). Let \(p_0\) be the number of \(\lambda_i \neq 0\), where \(\wedge = (\lambda_1, \ldots, \lambda_{p_1})\), and \(p_1\) the number of distinct \(\lambda_i \neq 0\). We set \(\widetilde{\Lambda} = (\lambda_1, \ldots, \lambda_{p_0}) \in \mathbb{R}^{p_0}\). Let \(\rho \in T_f\) where \(f = rX^*_p + D_2(\Lambda)\).

If \(\wedge = 0\), then \(K_\rho\) is the subgroup of \(K\) such that \(k.rX^*_p = rX^*_p\) for all \(k \in K_\rho\).

If \(\wedge \neq 0\), then \(K_\rho\) is the direct product \(K_1 \times K_2\), where:

\(K_1 = \{ k_1 \begin{bmatrix} \tilde{k}_1 & 0 \\ 0 & Id \end{bmatrix} | \tilde{k}_1 \in SO(2p_0) \ D_2(\wedge) \tilde{k}_1 = \tilde{k}_1 D_2(\wedge) \} \)
$K_2 = \{ k_2 = \begin{bmatrix} \text{Id} & 0 \\ 0 & \tilde{k}_1 \end{bmatrix} \mid \tilde{k}_2.X^*_p = rX^*_p \}.$

Furthermore, $K_1$ is isomorphism to the group $K(m; p_0; p_1)$.

Proof. We keep the notations of this proposition, and we set $A^* = D_2(\wedge)$ and $X^* = rX^*_p$. It is easy to prove:

$K_\rho = \{ k \in K : kA^* = A^*k$ and $kX^* = X^*k \}.$

If $\wedge = 0$, since $K_\rho$ is the stability group in $K$ of $X^* \in Y^* \sim \mathbb{R}^n$. So the first part of Proposition 2.12 is proved.

Let us consider the second part. $\wedge \neq 0$ so we have:

$$A^* = \begin{bmatrix} D_2(\tilde{\wedge}) & 0 \\ 0 & 0 \end{bmatrix} \quad \text{with} \quad D_2(\tilde{\wedge}) = \begin{bmatrix} \mu_1 J_{m_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \mu_{p_1} J_{m_{p_1}} \end{bmatrix}$$

Let $k \in K_\rho$. From above computation, the matrices $k$ and $A^*$ commute and we have:

$$k = \begin{bmatrix} \tilde{k}_1 & 0 \\ 0 & \tilde{k}_2 \end{bmatrix} \quad \text{with} \quad \tilde{k}_1 \in O(2p_0) \quad \text{and} \quad \tilde{k}_2 \in O(n - 2p_0)$$

furthermore, $\tilde{k}_2.X^* = X^*$, and the matrices $\tilde{k}_1$ and $D_2(\tilde{\wedge}^*)$ commute. So $\tilde{k}_1$ is the diagonal block matrix, with block $[\tilde{k}_1]_j \in O(m_j)$ for $i = 1, \ldots, p_1$. Each block $[\tilde{k}_1]_j \in O(m_j)$ commutes with $J_{m_j}$. So on one hand, we have:

$$\det[k]_j = 1, \det[k] = 1, \text{ and one the other hand, } [\tilde{k}_1]_j \in O(m_j)$$

corresponds to a unitary matrix $\psi_1^{(m_j)}([\tilde{k}_1]_j)$. Now we set for $k_1 \in K_1$:

$$\Psi_1(k_1) = (\psi_1^{(m_j)}([\tilde{k}_1]_1), \ldots, \psi_1^{(m_j)}([\tilde{k}_1]_{p_1}))$$

$\Psi_1 : K_1 \rightarrow K(m; p_0; p_1)$ is a group isomorphism. $\square$

Quotient group $\overline{F(n)} = F(n)/ker \rho$. In this paragraph, we describe the quotient groups $F(n)/ker \rho$ and $G/ker \rho$, for some $\rho \in \overline{F(n)}$. This will permit in the next paragraph to reduce the construction of the bounded spherical functions on $F(n)$ to known questions on Euclidean and Heisenberg groups.

For a representation $\rho \in \overline{F(n)}$, we will denote by:

$ker \rho$, the kernel of $\rho$.

$\overline{F(n)} = F(n)/ker \rho$ its quotient group and $\overline{N}$ its lie algebra.

$(\mathcal{H}, \overline{\rho})$ the induced representation on $\overline{F(n)}$.

$\pi \in \overline{F(n)}$ and $\overline{Y} \in \overline{N}$ the image of $n \in F(n)$ and $Y \in N$ respectively by the canonical projections $F(n) \rightarrow \overline{F(n)}$ and $N \rightarrow \overline{N}$.

Now, with the help of the canonical basis, we choose $E_1 = \mathbb{R}X_1 \oplus \cdots \oplus \mathbb{R}X_{2p_0 - 1}$

as the maximal totally isotropic space for $\omega_{D_2(\wedge), r}$. The quotient lie algebra $\overline{N}$ has the natural basis: . You can refer to [12].

Here, we have denoted $|\wedge| = (\sum_{j=1}^{p} \lambda_j^2)^{\frac{1}{2}} = |D_2(\wedge)|$ (for the Euclidean norm on $Z$).
Let $\overline{N}_1$ be the Lie sub-algebra of $\overline{N}$, with basis $X_1, \ldots, X_{2p_0}, B$, and $\overline{N}_1$ be its corresponding connected simply connected nilpotent lie group. We define the mapping: $\Psi_2 : \mathbb{H}^{p_0} \rightarrow \overline{N}_1$ for $h = (x_1 + iy_1, \ldots, x_{p_0} + iy_{p_0}, t) \in H_{p_0}$ by:

$$\Psi_2(h) = \exp\left(\sum_{j=1}^{p_0} \frac{v_0}{x_j} (x_j X_{2j-1} + y_j X_{2j}) + tB\right)$$

We compute that each lie bracket of two vectors of this basis equals zeros, except:

$$[X_{2i-1}, X_{2i}] = \lambda_{X,B}, \ i = 1, \ldots, p_0.$$  

From this, it is easy to see:

**Theorem 2.13.** $\Psi_2$ is a group isomorphism between $\overline{N}_1$ and $H_{p_0}$.

Finally, we note that $\overline{N}_1 \triangleright F(n) \rightarrow \overline{N}_1$ is the canonical projection.

Remark that the (Kohn) sub-Laplacian is $L := -\sum_{i=1}^{p} X_i^2$ (cf. section 6 in [12]). Then we can deduce that $L.\phi^{r,\alpha,\lambda} = (\sum_{j=1}^{p_0} \delta_j (2\alpha_j + m_j) + r^2)\phi^{r,\alpha,\lambda}$.

### 3. Some Knowledge for the Generalized Binomial Coefficients

Suppose $(K, H_a)$ is a Gelfand pair.

Decomposition of $C[V]$.

We decompose $C[V]$ into $K$-irreducible subspaces $P_\alpha$, $C[V] = \sum_{\alpha \in \Lambda} P_\alpha$ where $\Lambda$ is some countably infinite index set. Since the representation of $K$ on $C[V]$ preserves the space $P_m(V)$ of homogeneous polynomials of degree $m$, each $P_\alpha$ is a subspace of some $P_m(V)$. We write $|\alpha|$ for the degree of homogeneity of the polynomials in $P_\alpha$, so that $P_\alpha \subset P_{|\alpha|}(V)$.

We will write $d_\alpha$ for the dimension of $P_\alpha$ and denote by $0 \in \Lambda$ the index for the scalar polynomials $P_0 = P_0(V) = C$.

For $f : H_a \rightarrow C$ we define $f^\circ : V \rightarrow C$ by $f^\circ(z) := f(z, 0)$. We denote $\phi_\alpha^\circ(z) := \phi_{\alpha,1}(z, 0)$.

**Theorem 3.1.** $\{\phi_{\alpha}^\circ \mid \alpha \in \Lambda\}$ is a complete orthogonal system in $L^2_K(V)$ with $\|\phi_\alpha^\circ\|^2_2 = \frac{2\pi^{p_0}}{d_\alpha}$.

For $\alpha, \beta \in \Lambda$, we have a well defined number $\left[ \begin{array}{c} \alpha \\ \beta \end{array} \right]$.

We call the value generalized binomial coefficients for the action of $K$ on $V$ (cf. [24]). We have two important results, they are as follows:

$$\sum_{|\beta| = |\alpha| - 1} \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right] = |\alpha|$$

$$\sum_{|\beta| = |\alpha| + 1} \frac{d_\beta}{d_\alpha} \left[ \begin{array}{c} \beta \\ \alpha \end{array} \right] = |\alpha| + a.$$  

**Definition 3.2.** Given a function $g$ on $\Lambda$, $D^+g$ and $D^-g$ are the functions on $\Lambda$ defined by

$$D^+g(\alpha) = \sum_{|\beta| = |\alpha| + 1} \frac{d_\beta}{d_\alpha} \left[ \begin{array}{c} \beta \\ \alpha \end{array} \right] g(\beta) - (|\alpha| + a)g(\alpha)$$
\[\sum_{|\beta|=|\alpha|+1} \frac{d_{\beta}}{d_{\alpha}} \left[ \alpha \right] (g(\beta) - g(\alpha))\]

\[D^- g(\alpha) = |\alpha| g(\alpha) - \sum_{|\beta|=|\alpha|-1} \left[ \alpha \beta \right] g(\beta)\]

\[\sum_{|\beta|=|\alpha|-1} \left[ \alpha \beta \right] (g(\alpha) - g(\beta))\]

for \(|\alpha| > 0\), and \(D^- g(0) = 0\).

Also we can compute \(\gamma \phi^\alpha = -(D^+ - D^-) \phi^\alpha\), \(\gamma \phi^\alpha,\lambda = -\frac{1}{|\lambda|}(D^+ - D^-) \phi^\alpha,\lambda\).

Thus we can compute

\[(3.3) \quad \partial_\lambda \phi^\alpha,\lambda = \begin{cases} \left(\frac{\gamma}{2}\right) D^- \phi^\alpha,\lambda - \left(\frac{\gamma}{2}\right) \phi^\alpha,\lambda + it \phi^\alpha,\lambda \\ \left(\frac{1}{\lambda}\right) D^+ \phi^\alpha,\lambda + \left(\frac{\gamma}{2}\right) \phi^\alpha,\lambda + it \phi^\alpha,\lambda \end{cases}\]

for \(\lambda > 0\).

and similarly

\[(3.4) \quad \partial_\lambda \phi^\alpha,\lambda = \begin{cases} \left(\frac{\gamma}{2}\right) D^- \phi^\alpha,\lambda + \left(\frac{\gamma}{2}\right) \phi^\alpha,\lambda + it \phi^\alpha,\lambda \\ \left(\frac{1}{\lambda}\right) D^+ \phi^\alpha,\lambda - \left(\frac{\gamma}{2}\right) \phi^\alpha,\lambda + it \phi^\alpha,\lambda \end{cases}\]

for \(\lambda < 0\).

Equivalently

\[(3.5) \quad \left(\frac{\gamma}{2} + it\right) \phi^\alpha,\lambda = \begin{cases} \left(\partial_\lambda - \frac{1}{\lambda} D^+\right) \phi^\alpha,\lambda, & \text{for} \ \lambda \ \text{is greater than} \ 0 \\ \left(\partial_\lambda - \frac{1}{\lambda} D^-\right) \phi^\alpha,\lambda, & \text{for} \ \lambda \ \text{is smaller than} \ 0 \end{cases}\]

and

\[(3.6) \quad \left(\frac{\gamma}{2} - it\right) \phi^\alpha,\lambda = \begin{cases} -(\partial_\lambda - \frac{1}{\lambda} D^-) \phi^\alpha,\lambda, & \text{for} \ \lambda \ \text{is greater than} \ 0 \\ -(\partial_\lambda - \frac{1}{\lambda} D^+) \phi^\alpha,\lambda, & \text{for} \ \lambda \ \text{is smaller than} \ 0 \end{cases}\]

**Definition 3.7.** We say that a function \(F : \land \to C\) is rapidly decreasing if for each \(N \in \mathbb{Z}^+\), there is a constant \(C_N\) for which

\[|F(\alpha)| \leq \frac{C_N}{\left(2|\alpha|+a\right)^N}\]

**Theorem 3.8.** [13] If \(f \in \phi_K(V)\) then \(\hat{f}\) is rapidly decreasing on \(\land\). Conversely, if \(F\) is rapidly decreasing on \(\land\) then \(F = \hat{f}\) for some \(f \in \phi_K(V)\). Moreover, the map

\(\land : \phi_K(V) \to \{F \mid F \text{ is rapidly decreasing on } \land\}\)

is a bijection.

**Lemma 3.9** (13). Let \(F\) be a rapidly decreasing function on \(\land\) and \(G\) be a bounded function on \(\land\). Then

\[
\sum_{\alpha \in \land} d_{\alpha} F(\alpha) D^+ G(\alpha) = -\sum_{\alpha \in \land} d_{\alpha} (D^- + a) F(\alpha) G(\alpha),
\]

\[
\sum_{\alpha \in \land} d_{\alpha} F(\alpha) D^- G(\alpha) = -\sum_{\alpha \in \land} d_{\alpha} (D^+ + a) F(\alpha) G(\alpha),
\]
4. The proof of the third main theorem

In this section, we identity $H_a$ with $H_{p_0}$ and $\phi_{a,\lambda}$ with $\omega_{a,\lambda}$.

**Definition 4.1.** Let $G$ be a function on $\Delta(O(n), F(n))$. We say that $G$ is rapidly decreasing on $\Delta(O(n), F(n))$ if

1. $G$ is continuous on $\Delta(O(n), F(n))$.
2. The function $G_0$ on $R$ defined by $G_0(r) = G(\phi^r)$ belongs to $\phi(R)$.
3. The map $\lambda \to G(\phi^{r,\alpha,\lambda})$ is smooth on $R^\times = (-\infty, 0) \cup (0, \infty)$ for each fixed $\alpha \in \Lambda$ and $r \in R$.
4. For each $m, N \geq 0$, there exists a constant $C_{m,N}$ for which

$$\left| \partial^m \lambda G(\phi^{r,\alpha,\lambda}) \right| \leq \frac{C_{m,N}}{r^{\alpha}(\sum_{i=1}^{\infty} j_i (2\alpha + m_j) + r^2)},$$

for all $(r, \alpha, \lambda) \in R \times \Lambda \times R^\times$.

We say that a continuous function on $\Delta(O(n), F(n))$ is rapidly decreasing if it extends to a rapidly decreasing function on $\Delta(O(n), F(n)) = \Delta_1(O(n), F(n)) \cup \Delta_2(O(n), F(n))$. Since $\Delta_1(O(n), F(n))$ is dense in $\Delta(O(n), F(n))$, such an extension is necessarily unique.

Note that if $G$ is rapidly decreasing on $\Delta(O(n), F(n))$, then $\alpha \to G(\phi^{r,\alpha,\lambda})$ is rapidly decreasing on $\Lambda$, in the sense of Definition 4.1 for each $\lambda \neq 0$. We see that $F$ is bounded by letting $m = N = 0$ and one can show, moreover, that $G$ vanishes at infinity by letting $m = 0$ and $N = 1$. We remark that the functions $\partial^m \lambda G(\phi^{r,\alpha,\lambda})$ defined on $\Delta_1(O(n), F(n))$ need not extend continuously across $\Delta_2(O(n), F(n))$.

**Definition 4.2.** Let $G$ be a function on $\Delta_1(O(n), F(n))$ which is smooth in $\lambda$. $M^+G$ and $M^-G$ are the functions on $\Delta_1(O(n), F(n))$ defined by

$$M^+ G(\phi^{r,\alpha,\lambda}) = \begin{cases} (\partial^\lambda - \frac{1}{\alpha} D^+) G(\phi^{r,\alpha,\lambda}), & \text{for } \lambda \text{ is greater than } 0 \\ (\partial^\lambda - \frac{1}{\alpha} D^-) G(\phi^{r,\alpha,\lambda}), & \text{for } \lambda \text{ is smaller than } 0 \end{cases}$$

and

$$M^- G(\phi^{r,\alpha,\lambda}) = \begin{cases} (\partial^\lambda - \frac{1}{\alpha} D^-) G(\phi^{r,\alpha,\lambda}), & \text{for } \lambda \text{ is greater than } 0 \\ (\partial^\lambda - \frac{1}{\alpha} D^+) G(\phi^{r,\alpha,\lambda}), & \text{for } \lambda \text{ is smaller than } 0 \end{cases}$$

We reminded the reader that the difference operators $D^\pm$ are defined by

$$D^+ G(\phi^{r,\alpha,\lambda}) = \sum_{|\beta| = |\alpha| + 1} \frac{d^{|\alpha|}}{d\alpha} \left[ \beta \right] G(\phi^{r,\beta,\lambda}) - (|\alpha| + 1) G(\phi^{r,\alpha,\lambda})$$

$$D^- G(\phi^{r,\alpha,\lambda}) = |\alpha| G(\phi^{r,\alpha,\lambda}) - \sum_{|\beta| = |\alpha| - 1} \left[ \alpha \beta \right] G(\phi^{r,\beta,\lambda}).$$

**Definition 4.3.** $\tilde{\varphi}(O(n), F(n))$ is the set of all functions $G : \Delta(O(n), F(n)) \to C$ for which $(M^+)^l(M^-)^mG$ is rapidly decreasing for all $l, m \geq 0$.

If $G$ is rapidly decreasing on $\Delta(O(n), F(n))$ then $\lambda \to G(\phi^{r,\alpha,\lambda})$ is smooth on $R^\times$ and we have well defined functions $(M^+)^l(M^-)^mG$ on $\Delta_1(O(n), F(n))$. $G$ belongs to $\tilde{\varphi}(O(n), F(n))$ if and only if these functions extend continuously to rapidly decreasing functions on $\Delta(O(n), F(n))$.

Note that $f(x) = \int_{\Delta(O(n), F(n))} \tilde{\varphi}(\psi(x)) d\mu(\psi)$ Here $x = exp(X + A)$. 

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Theorem 4.4. (The Main Theorem) If \( f \in \varphi_{O(n)}(F(n)) \) then \( \hat{f} \in \hat{\varphi}(O(n), F(n)) \). Conversely, if \( G \in \hat{\varphi}(O(n), F(n)) \), then \( G = \hat{f} \) for some \( f \in \varphi_{O(n)}(F(n)) \). Moreover, the map \( \wedge : \varphi_{O(n)}(F(n)) \to \hat{\varphi}(O(n), F(n)) \) is a bijection.

If \( f \in \varphi_{O(n)}(F(n)) \) and \( \hat{f} = 0 \) then the inverse formula for the spherical transform yields that \( f = 0 \). Thus, the spherical transform is injective on \( \varphi_{O(n)}(F(n)) \). To prove above theorem, it remains to show that \( \varphi_{O(n)}(F(n))^\wedge \subset \hat{\varphi}(O(n), F(n)) \), and that \( \hat{\varphi}(O(n), F(n)) \subset \varphi_{O(n)}(F(n))^\wedge \).

Proof of \( \varphi_{O(n)}(F(n))^\wedge \subset \hat{\varphi}(O(n), F(n)) \). Suppose that \( f \in \varphi_{O(n)}(F(n)) \) and let \( G := \hat{f} \). We begin by showing that \( G \) is rapidly decreasing. \( G \) is continuous on \( \Delta(O(n), F(n)) \), as is the spherical transform of any integrable \( O(n) \)-invariant function. Moreover, \( G_0(r) = (\phi^r) = \int_{F(n)} f(x) \hat{\phi}(r) dx = \int_{F(n)} f(x) \int_{R} e^{-irX_p^rX} dx = \int_{F(n)} f(x) e^{-irX_p} dx \). Since \( f \) is a Schwartz function, so is \( G_0(r) \). Thus \( G \) satisfies the first two conditions in Definition 3.2.1.

Next, I will show that \( G \) satisfies the estimates in Definition 4.1 for \( m = 0 \). Recall that the (Kohn) Sub-Laplacian \( L \) is a self-adjoint operator on \( L^2(F(n)) \) with

\[
L(\phi^r, \alpha, \lambda) = (\sum_{j=1}^{p_1} \delta_j (2\alpha_j + m_j) + r^2) \phi^r, \alpha, \lambda. \quad \text{(cf. [12])}
\]

WLOG, I denote \( G(\phi^r, \alpha, \lambda) = G(r, \alpha, \lambda) \).

Thus we have

\[
(\sum_{j=1}^{p_1} \delta_j (2\alpha_j + m_j) + r^2)^N |G(r, \alpha, \lambda)| = \left| \frac{\partial L^N \phi^r, \alpha, \lambda > 2}{\partial L^N f, \phi^r, \alpha, \lambda > 2} \right| \leq \left| L^N f \right|_1.
\]

Since \( \phi^r, \alpha, \lambda(x) \leq 1 \) for all \( x \in F(n) \). Letting \( C_{0,N} := \|L^N f\|_1 \), we see that the equality in Definition 4.1 hold for \( m = 0 \).

Since \( \phi_{\alpha, \lambda}(z, t) \) is smooth in \( R^\times \) for fixed \( z, t \), \( \phi^r, \alpha, \lambda(x) \) is smooth in \( R^\times \) for fixed \( x \in F(n) \). And \( \hat{\phi}^r, \alpha, \lambda \) is a Schwartz function, \( G(r, \alpha, \lambda) = \hat{f}(r, \alpha, \lambda) \) is smooth in \( \lambda \in R^\times \) with

\[
\partial \lambda G(r, \alpha, \lambda) = \int_{F(n)} f(x) \partial \lambda \phi^r, \alpha, \lambda(x) dx
\]

Equation 3.3 and 3.4 provide formula for \( \partial \lambda \phi_{\alpha, \lambda} = \partial \lambda \phi_{\alpha, -\lambda} \) but I require a different approach here. Note that \( \omega_{\alpha, \lambda}(z, t) \) is a special case for \( \phi_{\alpha, \lambda}(z, t) \). I write \( \omega_{\alpha, \lambda}(z, t) = \omega^0_{\alpha, \lambda}(z, t) e^{it} \) as \( \omega^0_{\alpha, \lambda}(z, t) e^{it} \) so that \( \omega_{\alpha, \lambda}(z, t) = \omega^0_{\alpha, \lambda}(\lambda z, \overline{\lambda t}) e^{-i\lambda t} \).

We see that

\[
\partial \lambda \phi_{\alpha, \lambda}(z, t) = \frac{1}{\lambda} \left( \sum_{j=1}^{n} \Xi_j \frac{\partial \omega^0_{\alpha, \lambda}}{\partial z_j} \right) \omega^0_{\alpha, \lambda}(\lambda z, \overline{\lambda t}) e^{i|\lambda| t} - i t \omega_{\alpha, \lambda}(z, t).
\]
Substituting this expansion onto equation 4.5 and integrating by parts gives
\[ \partial_{\lambda} G(r, \alpha, \lambda) = \int F(n) f(x) e^{-ir \cdot \phi_r} 1_{\{r > 0\}} \left[ (\sum_{j=1}^a Z_j \frac{\partial}{\partial z_j}) \omega_{a, \lambda}(|\lambda| z, \bar{z}) e^{i|\lambda| t} dx \right] \]
\[ - \frac{1}{\lambda} (Df)^{\wedge}(r, \alpha, \lambda) - i \left( \left( \frac{1}{\lambda} \int F(n) f(x) e^{-ir \cdot \phi_r} \phi_r^{\alpha} \frac{1}{(q_1(x^{-1}))} \omega_{a, \lambda}(\Psi_1^{-1}(q_1(x^{-1}))) dx \right)^{\wedge}(r, \alpha, \lambda) \right) \]
where \( Df = -\sum_{j=1}^a \frac{\partial}{\partial z_j} Z_j f \) such that after integration by parts, above equality holds.

Since \( \int F(n) f(x) e^{-ir \cdot \phi_r} \phi_r^{\alpha} \frac{1}{(q_1(x^{-1}))} dx \) and \( Df \) are both Schwartz functions on \( F(n) \), they satisfy estimates above. Thus, given \( N \geq 0 \), one can find constants \( A \) and \( B \) with
\[ |\partial_{\lambda} G(r, \alpha, \lambda)| \leq \frac{A}{|\lambda|^{(\sum_{j=1}^a \delta_j(2\alpha_j + m_j) + r^2)^N}} + \frac{B}{(\sum_{j=1}^a \delta_j(2\alpha_j + m_j) + r^2)^N} \]
where \( C_{1,N} = A + |\lambda| \times B \). By induction on \( m \), we see that \( |\partial_{\lambda}^m G(r, \alpha, \lambda)| \) satisfies an estimate as in Definition 4.1. This completes the proof that \( F \) is rapidly decreasing. Equations 3.5 and 3.6 shows that
\[ M^+ G = ((\frac{2}{\pi} + if)^{\wedge} | \Delta_1(O(n), F(n)) \]
and \( M^- G = -((\frac{2}{\pi} - if)^{\wedge} | \Delta_1(O(n), F(n)) \]
where \( ((\frac{2}{\pi} + if)^{\wedge} = (\phi_r^{\alpha, \lambda} \phi_r^{\alpha, \lambda} \frac{1}{(q_1(x^{-1}))}) \phi_r^{\alpha, \lambda}(\Psi_1^{-1}(q_1(x^{-1}))) dx \]
\[ \phi_r^{\alpha, \lambda}(\Psi_1^{-1}(q_1(x^{-1}))) dx \]
Similarly, \( ((\frac{2}{\pi} - if)^{\wedge} = (\phi_r^{\alpha, \lambda} \phi_r^{\alpha, \lambda} \frac{1}{(q_1(x^{-1}))}) \phi_r^{\alpha, \lambda}(\Psi_1^{-1}(q_1(x^{-1}))) dx \]
Thus \( M^+ G \mid (M^-)^m G \) is the restriction of \( g \) to \( \Delta_1(O(n), F(n)) \)
where \( g = (-1)^m ((\frac{2}{\pi} + if)^{\wedge}((\frac{2}{\pi} - if)^{\wedge})^m f \).

Since \( g \in \varphi(O(n), F(n)) \), it now follows that \( (M^+)^m (M^-)^m G \) is rapidly decreasing. Thus \( G \in \varphi(O(n), F(n)) \) as desired. The following is required to complete the Main theorem.

**Theorem 4.6.** (Inversion formula) If \( f \in L^1_{O(n)}(F(n)) \cap L^2_{O(n)}(F(n)) \) is continuous, one has the Inversion Formula:
\[ f(x) = \frac{c}{(2\pi)^{n+2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sum_{\alpha \in \Lambda} (\text{dim} P_{\alpha}) \hat{f}(\phi_r^{\alpha, \lambda}) \phi_r^{\alpha, \lambda}(x) |\lambda|^a dr d\lambda \]
\[ = \frac{c}{(2\pi)^{n+2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sum_{\alpha \in \Lambda} (\text{dim} P_{\alpha}) \hat{f}(\phi_r^{\alpha, \lambda}) \phi_r^{\alpha, \lambda}(\Psi_1^{-1}(q_1(k,x))) |\lambda|^a dk dr d\lambda \]
where \( c \) is a fixed constant. \( x = \exp(X + A) \in F(n) \).

**Proof.** Note that \( \phi_r^{\alpha, \lambda} \phi_r^{\alpha, \lambda} = \frac{(2\pi)^{n+2}}{d_{\alpha, \beta}} \delta_{\alpha, \beta} \)
\[ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \phi_r^{\alpha, \lambda}(z,t) \phi_r^{\alpha, \lambda}(z,t) dz dt |\lambda|^a d\lambda \]
\[ = \int_{\mathbb{R}^n} \phi_r^{\alpha}(\sqrt{\lambda} |z|) \phi_r^{\alpha}(\sqrt{\lambda} |z|) dz \int_{\mathbb{R}^n} e^{i(\lambda - \lambda')} dt \int_{\mathbb{R}^n} |\lambda|^a d\lambda \]
\[ = \int_{\mathbb{R}^n} \phi_r^{\alpha}(z) \phi_r^{\alpha}(z) dz \times 2\pi \]
Theorem 4.8. \[ \text{Since existence and uniqueness of the Godement-Plancherel measure} \]

\[ \text{Let} \]

\[ \text{Proof.} \]

\[ \text{Suppose} \]

\[ g(x) = \frac{c}{(2\pi)^{n+c}} \int_{O(n)} \int_{R^x} \sum_{\alpha \in \Lambda} (\dim P_\alpha) \hat{f}(\phi^{r,\alpha,\lambda}) |\lambda|^a \, dk d\lambda, \]

\[ \text{then we can deduce} \]

\[ \hat{g}(\phi^{r,\alpha,\lambda}) = \int_{F(n)} g(x) e^{-ir <x_p, x>} \int_{R^x} \sum_{\alpha \in \Lambda} (\dim P_\alpha) \hat{f}(\phi^{r,\alpha,\lambda}) |\lambda|^a \, dk d\lambda, \]

\[ \text{Note that} \]

\[ \int_{R^x} \int_{F(n)} \phi_{\alpha,\lambda}(\Psi_2^{-1}(q_1(k,x))) \phi^{r,\alpha,\lambda}(\Psi_2^{-1}(q_1(x))) |\lambda|^a \, d\lambda \, dk = \]

\[ \frac{1}{c} \int_{R^x} \int_{H_\mu} \phi_{\alpha,\lambda}(z,t) \phi^{r,\alpha,\lambda'}(z,t) \, dz dt |\lambda|^a \, d\lambda, \quad \text{When } k \text{ is not equal to } e \]

\[ = \begin{cases} \frac{1}{c} \frac{(2\pi)^{n+c}}{d_a} \delta_{\alpha,\alpha'} \delta_{\lambda,\lambda'}, & \text{When } k \text{ is e} \\ 0, & \text{When } k \text{ is not equal to } e \end{cases} \]

\[ \therefore \]

\[ g(x) = f(x) \text{ for all } x \in F(n). \]

Lemma 4.7. \[ \text{There exists constants } M_1, M_2 \text{ such that} \]

\[ \frac{M_1}{2^|a|+a} \leq \frac{1}{\sum_{j=1}^{p_1} \delta_j (2\alpha_j + m_j) + r^2} \leq \frac{M_2}{2^|a|+a} \]

\[ \text{Proof. Let} \]

\[ \delta = \min_{1 \leq j \leq p_1} \delta_j. \]

\[ \text{On one hand, we have} \]

\[ \sum_{j=1}^{p_1} \delta_j (2\alpha_j + m_j) + r^2 \geq (2\delta \sum_{j=1}^{p_1} \alpha_j + \delta \sum_{j=1}^{p_1} m_j + r^2) \]

\[ \geq (2\delta |a| + \delta \sum_{j=1}^{p_1} m_j + r^2) = \frac{1}{M_2} (2\delta |a| + a) \]

\[ \text{where } M_2 = a + \delta \sum_{j=1}^{p_1} m_j + r^2. \]

\[ \text{One the other hand, let} \]

\[ \delta' = \max_{1 \leq j \leq p_1} \delta_j \]

\[ \sum_{j=1}^{p_1} \delta_j (2\alpha_j + m_j) + r^2 \leq (2\delta' \sum_{j=1}^{p_1} \alpha_j + \delta' \sum_{j=1}^{p_1} m_j + r^2) \]

\[ \leq (2\sqrt{\sum_{j=1}^{p_1} \delta' + \delta' \sum_{j=1}^{p_1} m_j + r^2)} \]

\[ \leq (\sqrt{\sum_{j=1}^{p_1} \delta' + \delta' \sum_{j=1}^{p_1} m_j + r^2}) (2\delta |a| + a) \]

\[ \text{Therefore, we can take } M_1 = \frac{1}{\sqrt{\sum_{j=1}^{p_1} \delta' + \delta' \sum_{j=1}^{p_1} m_j + r^2}} \text{ and the inequality holds.} \]

Theorem 4.8. \[ \text{The Godement-Plancherel measure } d\mu \text{ on } \Delta(O(n), F(n)) \text{ is given by} \]

\[ \int_{\Delta(O(n), F(n))} G(\psi) d\mu(\psi) = \frac{c}{(2\pi)^{n+c}} \int_{R^x} \sum_{\alpha \in \Lambda} (\dim P_\alpha) \hat{f}(\phi^{r,\alpha,\lambda}) |\lambda|^a \, dr d\lambda. \]

\[ \text{Proof. Since existence and uniqueness of the Godement-Plancherel measure is guaranteed, we need only verify that the equation} \]

\[ \boxed{\int_{\Delta(O(n), F(n))} G(\psi) d\mu(\psi) = \frac{c}{(2\pi)^{n+c}} \int_{R^x} \sum_{\alpha \in \Lambda} (\dim P_\alpha) \hat{f}(\phi^{r,\alpha,\lambda}) |\lambda|^a \, dr d\lambda.} \]
Indeed, $N$ from Theorem 4.9. Let correspondence to $A$ via $f$ and replace by $F$. Therefore, the theorem holds.

\[
\int_{F(n)} |f(x)|^2 \, dx = \frac{1}{(2\pi)^{a+2}} \int_{\Delta(O(n), F(n))} |\hat{f}(\psi)|^2 \, d\mu(\psi).
\]

for all continuous functions $f \in L^1_{O(n)}(F(n)) \cap L^2_{O(n)}(F(n))$.

Since $f(x) = \frac{1}{(2\pi)^{a+2}} \int_R \sum_{\alpha \in \Lambda} (dim P_\alpha) \hat{f}(\phi^{r,\alpha}) \phi^{r,\alpha}(x) |\lambda|^a \, drd\lambda$.

\[
\int_{F(n)} |f(x)|^2 \, dx = \frac{1}{(2\pi)^{a+2}} \int_R \int_{\stackrel{\sim}{\Delta}} \int_{\stackrel{\sim}{\Delta}} \sum_{\alpha \in \Lambda} (dim P_\alpha) \hat{f}(\phi^{r,\alpha}) \phi^{r,\alpha}(x) |\lambda|^a \, drd\lambda dr' d\lambda dx.
\]

Note that $\int_R \int_{\Delta} \int_{\Delta} \phi^{r,\alpha}(z, t) \phi^{r,\alpha'}(z, t) |\lambda|^a \, drd\lambda dx$

\[
= \int_R \int_{\Delta} \int_{\Delta} e^{i\varphi < x_k^r, X>} \phi_{\alpha, \lambda}(\Psi_2^{-1}(q_1(k,x))) e^{i\varphi' < x_k^r, X>}
\]

\[
\phi_{\alpha', \lambda}(\Psi_2^{-1}(q_1(k,x))) |\lambda|^a \, dk dik dr d\lambda dx.
\]

Therefore,

\[
\int_{F(n)} |f(x)|^2 \, dx = \frac{1}{(2\pi)^{a+2}} \int_R \int_{\Delta} \sum_{\alpha \in \Lambda} (dim P_\alpha) \hat{f}(\phi^{r,\alpha}) f(\phi^{r,\alpha}) |\lambda|^a \, drd\lambda.
\]

Therefore, the theorem holds.

\[\square\]

**Theorem 4.9.** Let $G$ be a bounded measurable function on $(\Delta O(n), F(n))$ correspondence to $H_a$ with $|G(r, \alpha, \lambda)| \leq \frac{c_0}{(2\pi)^{a+1} \delta (r_\alpha + r_\lambda)^N}$

for some $N \geq a + 3$ and some constant $c_0$. Then

1. $G \in L^p(O(n), F(n))$ for all $p \geq 1$, and

2. $G = \hat{f}$ for some bounded continuous function $f \in L^2_{O(n)}(F(n))$.

Suppose, for example, that $f$ is a continuous function in $L^1_{O(n)}(F(n))$ with $G = \hat{f}$ rapidly decreasing. Theorem 4.9 shows that $f$ is square integrable and that $G$ is integrable. Thus the inversion formula applies and we can cover $f$ from $G$ via

\[ f(x) = \frac{c}{(2\pi)^{a+2}} \int_{F(n)} e^{i\varphi < x_k^r, X>} \int_R \int_{\Delta} \sum_{\alpha \in \Lambda} (dim P_\alpha) \hat{f}(\phi^{r,\alpha}) \phi_{\alpha, \lambda}(\Psi_2^{-1}(q_1(k,x))) |\lambda|^a \, drd\lambda dx.
\]

In particular, we see that this formula certainly holds for any function $f \in \varphi_{O(n)}(F(n))$.

**Proof.** To establish the first assertion, it suffices to prove that $G \in L^1(\delta(O(n), F(n)))$.

Indeed, $|G(r, \alpha, \lambda)|^p$ satisfies an inequality as in the statement of the theorem with $N$ replace by $pN$. Fixed a number $K > 0$, let

\[
A_1 = \{ \phi^{r,\alpha, \lambda} | 0 \leq r \leq K, |\lambda| (2|\alpha| + a) \leq 1 \},
\]

\[
A_2 = \{ \phi^{r,\alpha, \lambda} | 0 \leq r \leq K, |\lambda| (2|\alpha| + a) > 1 \},
\]

\[
A_3 = \{ \phi^{r,\alpha, \lambda} | r > K, |\lambda| (2|\alpha| + a) \leq 1 \}
\]

and

\[
A_4 = \{ \phi^{r,\alpha, \lambda} | r > K, |\lambda| (2|\alpha| + a) > 1 \},
\]

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Let $M$ be a constant for which $|F(\psi)| \leq M$ for all $\psi \in \Delta(O(n), F(n))$ and let

$$d_m = \dim(P_m(V)) = \binom{m+n-1}{m}$$

Note that $\sum_{\alpha=m} d_\alpha = d_m \leq (m+n-1)^n$.

$$\int A_1 \, G(\psi)d\mu(\psi) = \frac{2c}{(2\pi)^{a+a}} \int_0^{O(n)} \alpha \in \Delta(dimP_\alpha) \int_0^{\frac{1}{2|m+a|}} |F(r,\alpha,\lambda)| |\lambda|^a \, dr \, d\lambda.$$

$$\leq \frac{4cMK}{(2\pi)^{a+a}} \sum_{m=0}^\infty \int_0^{2m+a} \lambda^a \, d\lambda$$

$$= \frac{4cMK}{(2\pi)^{a+a}(a+1)} \sum_{m=0}^\infty \int_0^{2m+a} \lambda^a \, d\lambda$$

Since $d_m = O(m^{a-1})$, we see that the series converges. Hence $|G|$ is integrable over $A_1$. Next we compute

$$\int A_2 \, G(\psi)d\mu(\psi) = \frac{4cK}{(2\pi)^{a+a}} \int_0^{O(n)} \alpha \in \Delta(dimP_\alpha) \int_0^{\frac{1}{2|m+a|}} |G(r,\alpha,\lambda)| |\lambda|^a \, dr \, d\lambda.$$

$$\leq \frac{4KcM}{(2\pi)^{a+a}} \sum_{m=0}^\infty \int_0^{2m+a} \lambda^a \, d\lambda$$

$$= \frac{4KcM}{(2\pi)^{a+a}(N-a-1)} \sum_{m=0}^\infty \int_0^{2m+a} \lambda^a \, d\lambda$$

The hypothesis that $N-n \geq 2$ was used above to evaluate the integral of $\frac{1}{\lambda^{N-a}}$ over $\frac{1}{2m+a} \leq \lambda \leq \infty$. Since $d_m = O(m^{a-1})$, we see that the series in the expansion converges. Hence $|F|$ is integrable over $A_2$. Similarly,

$$\int A_3 \, G(\psi)d\mu(\psi) = \frac{4cK}{(2\pi)^{a+a}} \int_0^{O(n)} \alpha \in \Delta(dimP_\alpha) \int_0^{\frac{1}{2|m+a|}} |G(r,\alpha,\lambda)| |\lambda|^a \, dr \, d\lambda.$$

$$\leq \frac{2c^2}{(2\pi)^{a+a}} \int_r^\infty \frac{1}{\pi} \, dr \sum_{m=0}^\infty \int_0^{2m+a} \lambda^a \, d\lambda$$

$$\leq \frac{1}{(2\pi)^{a+a}} \sum_{m=0}^\infty \int_0^{2m+a} \lambda^a \, d\lambda$$

$$= \frac{1}{(2\pi)^{a+a}(a+1)} \sum_{m=0}^\infty \int_0^{2m+a} \lambda^a \, d\lambda$$

Since $d_m = O(m^{a-1})$, we see that the last series converges. Hence $|F|$ is integrable over $A_3$. Finally we compute

$$\int A_4 \, G(\psi)d\mu(\psi) = \frac{4cK}{(2\pi)^{a+a}} \int_0^{O(n)} \alpha \in \Delta(dimP_\alpha) \int_0^{\frac{1}{\lambda^{N-1-a}}} |G(r,\alpha,\lambda)| |\lambda|^a \, dr \, d\lambda.$$

$$\leq \frac{8c^2M^{-1}}{(2\pi)^{a+a}} \int_0^{\frac{1}{\lambda^{N-1-a}}} \frac{1}{\lambda^N} \lambda^a \, d\lambda$$

$$\leq \frac{8c^2M^{-1}}{(2\pi)^{a+a}} \int_0^{1} \frac{1}{\lambda^{N-1-a}} \lambda^a \, d\lambda$$

The hypothesis $N-a \geq 3$ was used to evaluate the integral of $\frac{1}{\lambda^{N-1-a}}$ over $\frac{1}{2m+a} \leq \lambda \leq \infty$. Since $d_m = O(m^{a-1})$, we see that the series in the last expression converges. Hence $|F|$ is integrable over $A_4$. As $A_1 \cup A_2 \cup A_3 \cup A_4 = (\Delta(O(n), F(n)))$ is a set of full measure in $(\Delta(O(n), F(n)))$, it follows that $F \in L^1(\Delta(O(n), F(n)))$.

Next let $f$ be the function on $F(n)$ defined by

$$f(x) = \int_\Delta(O(n), F(n)) G(\psi) \psi(x) \, d\mu(\psi).$$

Since $G \in L^1(\Delta(O(n), F(n)))$ and the bounded spherical functions are continuous and bounded by 1, we see that $f$ is well defined, continuous and bounded by $\|G\|L^1(\Delta(O(n), F(n)))$. Moreover, since $G \in L^2(\Delta(O(n), F(n)))$ and $: L^2(\Delta(O(n), F(n))) \rightarrow L^2(O(n), F(n))$
is an isometry, we have that \( f \in L^2_{\Delta(O(n))}(F(n)) \) with \( \| f \|_2^2 = \int_{\Delta(O(n), F(n))} |G(\psi)|^2 \, d\mu(\psi) \)
and \( \tilde{f} = G \). This establish the second assertion in this Theorem. \( \square \)

Remark. One can show that the set \( A_1 \) used in the proof of the above theorem is compact in \( \Delta(O(n), F(n)) \). Thus any bounded measurable function is integrable over \( A_1 \). This observation motivates the decomposition used in the proof.

Proof that \( \hat{\varphi}(O(n), F(n)) \subseteq \varphi_{O(n)}(F(n)) \). Suppose now that \( G \in \hat{\varphi}(O(n), F(n)) \).

Above theorem shows that \( \hat{G} = \hat{f} \) where
\[
\hat{f}(x) = \frac{c}{(2\pi)^{n+2}} \int_R \int_R^n \sum_{\alpha \in \Lambda} (\text{dim} P_\alpha) \hat{f}(\hat{\phi}_\alpha^\alpha, \lambda) \phi^{\alpha, \lambda}(x) |\lambda|^\alpha \, d\lambda.
\]

is \( O(n) \)-invariant, continuous, bounded and square integrable. To show that \( f \in \varphi_{O(n)}^F \), I will show that \( f \) is smooth and that
\[
(\frac{1}{4} + t^2)^a (\frac{\partial}{\partial x})^b \tilde{\Delta} f \in L^2_{\Delta(O(n))}(F(n))
\]

where \( \tilde{\Delta} e^{t} + t^2, \frac{\partial}{\partial x} \tilde{\Delta} \) will be defined below.

for all \( a, b, c \geq 0 \). This will follows from the facts

1. \( \tilde{\Delta} f \in L^2_{\Delta(O(n))}(F(n)) \) with \( (\tilde{\Delta} f)^{\wedge} \in \hat{\varphi}(O(n), F(n)) \).

2. \( \frac{\partial}{\partial x} f \in L^2_{\Delta(O(n))}(F(n)) \) with \( \hat{\left( \frac{\partial}{\partial x} f \right)}^{\wedge} \in \hat{\varphi}(O(n), F(n)) \).

3. \( \left( \frac{\partial}{\partial x} \right)^{\wedge} f \in L^2_{\Delta(O(n))}(F(n)) \) with \( \hat{\left( \frac{\partial}{\partial x} f \right)}^{\wedge} \in \hat{\varphi}(O(n), F(n)) \).

which I will prove below. Here \( \Delta = \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_j} \) is in formal.

Using equation \( \hat{U}(\phi_{\alpha, \lambda}) = -|\lambda| (2|\alpha| + a) \) for the eigenvalues of the Heisenberg sub-Laplacian, one obtains

\[
-|\lambda| (2|\alpha| + a) \phi_{\alpha, \lambda} = U \phi_{\alpha, \lambda} = [4\Delta - \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} \right)^2] \phi_{\alpha, \lambda}
\]

\[
= 4\lambda \phi_{\alpha, \lambda} - \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x} \right) \phi_{\alpha, \lambda}.
\]

Since \( \gamma \phi_{\alpha, \lambda} = -\frac{1}{|\alpha|} (D^+ - D^-) \phi_{\alpha, \lambda} \). Therefore,

\[
4\Delta \phi_{\alpha, \lambda} = -|\lambda| (2|\alpha| + a) \phi_{\alpha, \lambda} - |\lambda| (2|\alpha| + a) \phi_{\alpha, \lambda}
\]

\[
= -\frac{|\lambda|}{2} \sum_{|\beta| = |\alpha| + 1} \frac{d}{d\alpha} \left[ \int_0^1 \alpha^{\beta} \phi_{\beta, \lambda} \right] (2|\alpha| + a) \phi_{\alpha, \lambda}
\]

\[
= \sum_{|\beta| = |\alpha| + 1} \frac{d}{d\alpha} \left[ \int_0^1 \alpha^{\beta} \phi_{\beta, \lambda} \right] (2|\alpha| + a) \phi_{\alpha, \lambda}
\]

Define a function \( G_\Delta \) on \( \Delta_1(O(n), F(n)) \) by

\[
G_\Delta (r, \alpha, \lambda) = -\frac{|\lambda|}{2} (D^+ - D^-) G(r, \alpha, \lambda) - |\lambda| (2\alpha + a) G(r, \alpha, \lambda)
\]

\[
= -\frac{|\lambda|}{2} \sum_{|\beta| = |\alpha| + 1} \frac{d}{d\alpha} \left[ \int_0^1 \alpha^{\beta} \phi_{\beta, \lambda} \right] G(r, \beta, \lambda) + (2|\alpha| + a) F(r, \alpha, \lambda)
\]

\[
+ \sum_{|\beta| = |\alpha| + 1} \frac{d}{d\alpha} \left[ \int_0^1 \alpha^{\beta} G(r, \beta, \lambda) \right]
\]

It is not hard to show that \( G_\Delta \in \hat{\varphi}(O(n), F(n)) \).

In particular, note the equations

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\[
\sum_{|\beta|=|\alpha|-1} \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right] = |\alpha|
\]
\[
\sum_{|\beta|=|\alpha|+1} \frac{d_{\alpha}}{d_{\alpha}} \left[ \begin{array}{c} \beta \\ \alpha \end{array} \right] = |\alpha| + a.
\]

\[
\frac{M_1}{2|\alpha|+a} \leq \sum_{j=1}^{\infty} \delta_j (2\alpha_j + m_j + r^2) \leq \frac{M_2}{2|\alpha|+a} \quad \text{for some fixed } M_1 > 0 \text{ and } M_2 > 0.
\]

\[
|G_\Delta(r, \alpha, \lambda)| \leq \frac{\lambda}{2} (|\alpha| + a) \sum_{|\beta|=|\alpha|+1} |G(r, \beta, \lambda)| + (2|\alpha| + a) |G(r, \alpha, \lambda)| \quad \text{for some fixed } M_1 > 0 \text{ and } M_2 > 0.
\]

One uses this to show that \( G_\Delta \) satisfies estimates as in Definition 4.1. Moreover, Lemma 3.9 shows that for each \( \lambda \neq 0 \),
\[
\sum_{\alpha \in \Lambda} d_\alpha G(r, \alpha, \lambda)(D^+-D^-)\phi_{\alpha,\lambda} = \sum_{\alpha \in \Lambda} d_\alpha (D^+-D^-)G(r, \alpha, \lambda)\phi_{\alpha,\lambda}.
\]

and hence also
\[
\frac{1}{(2\pi)^{n}} \int (O(n)) e^{ir \cdot X_\alpha} \int_{R^\times} \sum_{\alpha \in \Lambda} (\dim P_\alpha) G_\Delta(\phi^{r,\alpha,\lambda}) \phi_{\alpha,\lambda}(\Psi^-_2(1(k,x))) |\lambda|^a \, dkdrd\lambda
\]
\[
= \frac{4}{(2\pi)^{n+2}} \int (O(n)) e^{ir \cdot X_\alpha} \int_{R^\times} \sum_{\alpha \in \Lambda} (\dim P_\alpha) G(\phi^{r,\alpha,\lambda}) \Delta \phi_{\alpha,\lambda}(\Psi^-_2(1(k,x))) |\lambda|^a \, dkdrd\lambda
\]
\[
= 4\Delta f.
\]

We conclude \( \hat{\Delta} f \in L^2(O(n), F(n)) \) with \( 4(\Delta f)^\wedge = G_\Delta \in \hat{\varphi}(O(n), F(n)). \)

This proves item 1 above.

Next note that the function defined on \( \Delta_1(O(n), F(n)) \) by \( \lambda G(r, \alpha, \lambda) \)

belongs to \( \hat{\varphi}(O(n), F(n)). \) Since \( \frac{\partial \phi_{\alpha,\lambda}^*}{\partial \alpha} = i\lambda \phi_{\alpha,\lambda}, \)

we see that
\[
\frac{\partial f}{\partial \alpha}(X) = \frac{1}{(2\pi)^n} \int (O(n)) e^{ir \cdot X_\alpha} \int_{R^\times} \sum_{\alpha \in \Lambda} (\dim P_\alpha) \hat{f}(\phi^{r,\alpha,\lambda}) \frac{\partial}{\partial \alpha} \phi_{\alpha,\lambda}(\Psi^-_2(1(k,x))) |\lambda|^a \, dkdrd\lambda
\]
\[
= i\lambda f(x).
\]

Therefore, \( \frac{\partial f}{\partial \alpha} \in L^2(O(n), F(n)) \) with \( (\frac{\partial f}{\partial \alpha})^\wedge = i\lambda G \in \hat{\varphi}(O(n), F(n)). \)

This establishes 2 above.

We begin the proof of item 3 by setting
\[
G(r, \alpha, \lambda, k) = \sum_{\alpha \in \Lambda} (\dim P_\alpha) G(\phi^{r,\alpha,\lambda}) \phi_{\alpha,\lambda}^0 (Pr_V \Psi^-_2(1(k,x))) |\lambda|^a
\]

for each \( \lambda \neq 0 \), so that
\[
f(x) = \frac{1}{(2\pi)^n} \int (O(n)) e^{ir \cdot X_\alpha} \int_{R^\times} \tilde{G}(r, \alpha, \lambda, k) e^{ir Pr_V \Psi^-_2(1(k,x))} \, dkdrd\lambda.
\]

We denote \( Pr_V \Psi^-_2(1(k,x)) = Z_{k,n} \in V. \)

Note that we can compute \( \partial^2 \hat{G} \) by taking derivatives term-wise in above equation. For \( \lambda > 0 \), we have
\[
\partial_\lambda \hat{G}(r, \alpha, \lambda, k) = \sum_{\alpha \in \Lambda} (\dim P_\alpha) d_\alpha a \lambda^a G(\phi^{r,\alpha,\lambda}) \phi_{\alpha,\lambda}^0 (Z_{k,n})
\]
\[
+ \sum_{\alpha \in \Lambda} (\dim P_\alpha) d_\alpha \lambda^a \partial_\lambda G(\phi^{r,\alpha,\lambda}) \phi_{\alpha,\lambda}^0 (Z_{k,n})
\]
\[
+ \sum_{\alpha \in \Lambda} (\dim P_\alpha) d_\alpha \lambda^a G(\phi^{r,\alpha,\lambda}) \partial_\lambda \phi_{\alpha,\lambda}^0 (Z_{k,n})
\]

Since \( G \in \hat{\varphi}(O(n), F(n)) \), the estimates in Definition 4.1 can be applied to show that the first two sum converges absolutely for \( \lambda > 0 \). For the third sum, I use Equation 3.3 for \( \partial_\lambda \phi_{\alpha,\lambda}^0 (Z) = \partial_\lambda \phi_{\alpha,\lambda}(z,0) \), together with the lemma 3.9 to derive two identities.
\[
\sum_{\alpha \in \Lambda} (\dim P_\alpha) G(\phi^{r,\alpha,\lambda}) \phi_{\alpha,\lambda}^0 (Z_{k,n}) \lambda^a
\]
Substituting these identities in the expansion for (4.10)

\[
\partial \tilde{\lambda} \tilde{G}(r, a, \lambda, k) = \begin{cases} 
\gamma(Z_{k,n}) \sum_{\alpha \in \Lambda} (dim P_{\alpha}) \lambda^{\alpha} G(\phi_{r,\alpha,\lambda}) \phi_{0,\alpha,\lambda}(Z_{k,n}) & \\
\frac{1}{2} \gamma(Z_{k,n}) \sum_{\alpha \in \Lambda} (dim P_{\alpha}) \lambda^{\alpha} G(\phi_{r,\alpha,\lambda}) \phi_{0,\alpha,\lambda}(Z_{k,n}) & \\
-\gamma(Z_{k,n}) G(r, a, \lambda, k) - \sum_{\alpha \in \Lambda} (dim P_{\alpha}) \lambda^{\alpha-1} (D^{+} + a) G(\phi_{r,\alpha,\lambda}) \phi_{0,\alpha,\lambda}(Z_{k,n}) & \\
\frac{1}{2} \gamma(Z_{k,n}) \tilde{G}(r, a, \lambda, k) - \sum_{\alpha \in \Lambda} (dim P_{\alpha}) \lambda^{\alpha-1} (D^{+} + a) G(\phi_{r,\alpha,\lambda}) \phi_{0,\alpha,\lambda}(Z_{k,n})
\end{cases}
\]

both valid for \( \lambda > 0 \). We have similar identities for \( \lambda < 0 \):

(4.11)

\[
\partial \bar{\lambda} \bar{G}(r, a, \lambda, k) = \begin{cases} 
\gamma(Z_{k,n}) \sum_{\alpha \in \Lambda} (dim P_{\alpha}) \lambda^{\alpha} G(\phi_{r,\alpha,\lambda}) \phi_{0,\alpha,\lambda}(Z_{k,n}) & \\
\frac{1}{2} \gamma(Z_{k,n}) \sum_{\alpha \in \Lambda} (dim P_{\alpha}) \lambda^{\alpha} G(\phi_{r,\alpha,\lambda}) \phi_{0,\alpha,\lambda}(Z_{k,n}) & \\
-\gamma(Z_{k,n}) G(r, a, \lambda, k) - \sum_{\alpha \in \Lambda} (dim P_{\alpha}) \lambda^{\alpha-1} (D^{+} + a) G(\phi_{r,\alpha,\lambda}) \phi_{0,\alpha,\lambda}(Z_{k,n}) & \\
\frac{1}{2} \gamma(Z_{k,n}) \tilde{G}(r, a, \lambda, k) - \sum_{\alpha \in \Lambda} (dim P_{\alpha}) \lambda^{\alpha-1} (D^{+} + a) G(\phi_{r,\alpha,\lambda}) \phi_{0,\alpha,\lambda}(Z_{k,n})
\end{cases}
\]

Note that \( (\partial \lambda - \frac{1}{\lambda} D^{+}) G \) is the restriction of \( M^{\pm} G \) to \( \Delta_{+}^{\pm} (O(n), F(n)) = \{ \phi_{r,\alpha,\lambda} \mid r \in R, \alpha \in \Lambda, \lambda > 0 \} \) and also of \( M^{\mp} G \) to \( \Delta_{-}^{\pm} (O(n), F(n)) = \{ \phi_{r,\alpha,\lambda} \mid r \in R, \alpha \in \Lambda, \lambda > 0 \} \). Since \( M^{\pm} G \in \tilde{\varphi}(O(n), F(n)) \), \( M^{\pm} G \) is integrable on \( \Delta(O(n), F(n)) \) and equation 4.10, 4.11 show that \( \partial \tilde{G}(r, a, \lambda, k) \) is integrable on \( \Gamma_{k}^{\pm} = \{ \lambda \mid \lambda \neq 0 \} \). We have

[\begin{equation}
\frac{(2\pi)^{a+2}}{c} (\pm \frac{\pi}{2} \pm it) f(x) = \int_{O(n)} e^{irX_{P}X} \int_{R^{+}} G(r, a, \lambda, k) \partial \lambda \left( e^{irP_{r}P_{r}^{-1}(q_{i}(k,x))} \right) dk dr \lambda.
\end{equation}]

It can be shown that the limits \( \lim_{\lambda \to 0^{\pm}} \tilde{G}(r, a, \lambda, k) \) exists and are equal. Here one need to use the hypothesis that \( F \) is continuous across \( \Delta \). We obtained

[\begin{equation}
\frac{(2\pi)^{a+2}}{c} (\pm \frac{\pi}{2} \pm it) f(x) = \int_{O(n)} e^{irX_{P}X} \int_{R^{+}} \partial \lambda \tilde{G}(r, a, \lambda, k) e^{irP_{r}P_{r}^{-1}(q_{i}(k,x))} \ partial \lambda dr dk.
\end{equation}]

We conclude that \( (\pm \frac{\pi}{2} \pm it) f \in L_{0}(F(n)) \) with \( \frac{(2\pi)^{a+2}}{c} (\pm \frac{\pi}{2} \pm it) f \) is an integrable function on \( \Gamma_{k}^{\pm} \). This completes the proof of item 3.
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