GOLAY COMPLEMENTARY SETS WITH LARGE ZERO ODD-PERIODIC CORRELATION ZONES

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(Communicated by Sihem Mesnager)

Abstract. Golay complementary sets (GCSs) are widely used in different communication systems, i.e., GCSs could be used in OFDM systems to control peak-to-mean envelope power ratio (PMEPR). In this paper, inspired by the work on GCSs with large zero correlation zone given by Chen et al in 2018, we investigate the relationship between GCSs and zero odd-periodic correlation zone (ZOCZ) sequence sets, and present GCSs with flexible sequence set sizes, sequence lengths, large ZOCZ and low PMEPR. Those proposed sequences could be applied in OFDM system for synchronization.

1. Introduction

Sequences with good correlation properties are desired for synchronization and channel estimation in different communication systems [9]. Quasi-synchronous code division multiple access (QS-CDMA) system allows signals from different users to have a small delay when they arrive at the receiving end which provides more flexibility. Besides, zero correlation zone (ZCZ) sequences [7] were employed to eliminate both multiple access interference and multipath interference in QS-CDMA systems [18]. In addition to this application, ZCZ sequences also have good performance in multiple input multiple output (MIMO) systems [27], ranging systems [3], and orthogonal frequency division multiplexing (OFDM) systems [14]. Constructions of new ZCZ sequences have received much attention in the literature. Much progress has been made in recent years (see [5, 12, 13, 24, 26, 31, 32] and the references therein).

Golay first proposed aperiodic complementary pairs [8], of which the sum of out-of-phase aperiodic autocorrelation equals to zero. Davis and Jedwab formulated Golay’s aperiodic complementary pairs by using generalized boolean functions [4]. Due to the aperiodic correlation property, Golay sequences have been used to construct
Hadamard matrices for direct sequence code division multiple access (DS-CDMA) system [25], and to control the peak-to-mean envelope power ratio (PMEPR) in several IEEE 802.11 standards [15, 16, 17]. For example, Golay sequences of length 32, 64, 128 have been used in the preamble, single carrier guard interval, beam refinement receive/transmit training and automatic gain control fields [15]. In [16], several resource units defined for multiple users downlink and uplink transmission are given by Golay sequences. In [17], the legacy short training fields (L-STF), the legacy long training fields (L-LTF) and the training units of presentation protocol data unit (PPDU) were generated using Golay sequences/pairs. The autocorrelation properties of the Golay sequences/pairs used in the training field allow for the estimation of the impulse response of the channel between transmitter and receiver.

In [29, 30], ZCZ sequence sets were proposed as training sequences and pilots in OFDM systems, while their PMEPR were not discussed. In [15], Golay sequences were employed as preambles for channel estimation due to their low PMEPR, while their crosscorrelation properties were not utilized. Gong et al discovered some Golay sequences with large zero autocorrelation zones [10], and Golay pairs with large ZCZ property [11]. Recently Chen proposed GCSs with large zero correlation zone widths [2].

Except aperiodic correlation and (even) periodic correlation, odd-periodic correlation is also important [21, 22, 23]. Specifically, Massey and Uhran [19] pointed out that both the even correlation property and the odd-periodic correlation property of sequences should be considered in DS-CDMA systems. In [28], Yang et al investigated the zero odd-periodic autocorrelation zone of Golay sequences, while the odd-periodic crosscorrelation properties of GCPs are not considered. In this paper, inspired by the work in [2] and [28], we will investigate a connection between GCSs and zero odd-periodic correlation zone sets, and propose Golay complementary pairs/sets with large zero odd-periodic correlation zone (ZOCZ) properties, which are called Golay-ZOCZ sequence sets. In particular, the proposed Golay-ZOCZ sequence sets have various constellation sizes, set size, sequence lengths, and ZOCZ widths.

This paper is organized as follows. In Section II, some useful notations and preliminaries are given. In Section III, the ZOCZ properties of Golay sequence pairs/sets are discussed. Finally, we summarize this paper in Section IV. We end this section by fixing the following notations which will be used throughout this paper:

- \( H \geq 2 \) is an even integer, and \( m \geq 2 \) is an integer;
- \( \pi \) is a permutation of \( \{1, 2, \cdots, m\} \);
- \( \xi = \exp(2\pi \sqrt{-1}/H) \) is an \( H \)th root of unity;
- \( \mathbb{Z}_H = \{0, 1, \cdots, H-1\} \) is the integer residue ring of module \( H \);
- \((i_1, i_2, \cdots, i_m)\) and \((j_1, j_2, \cdots, j_m)\) are binary representations of the integers \( i, j \) respectively, \( 0 \leq i, j \leq 2^m - 1 \), i.e., \( i = \sum_{k=1}^{m} i_k \cdot 2^{k-1} \) and \( j = \sum_{k=1}^{m} j_k \cdot 2^{k-1} \).

2. Notation and preliminaries

The aperiodic crosscorrelation function of \( c = (c_t) \) and \( d = (d_t) \) of length \( L \) over \( \mathbb{Z}_H \) at shift \( \tau \) is defined as

\[
C_{c,d}(\tau) = \sum_{t=0}^{L-1-\tau} \xi^{c_t-d_{t+\tau}}, \quad 0 \leq \tau \leq L-1.
\]
When $c = d$, $C_{c,d}(\tau)$ is called the aperiodic autocorrelation function, denoted by $C_c(\tau)$ for short. The (periodic) crosscorrelation and odd-periodic crosscorrelation function of $c = (c_t)$ and $d = (d_t)$ of length $L$ at shift $\tau$, $0 \leq \tau \leq L - 1$ are respectively defined as

$$R_{c,d}(\tau) = C_{c,d}(\tau) + (C_{d,c}(L - \tau))^\ast$$
$$\hat{R}_{c,d}(\tau) = C_{c,d}(\tau) - (C_{d,c}(L - \tau))^\ast,$$

where $x^\ast$ denotes the complex conjugate of the complex number $x$. When $c = d$, $R_{c,d}(\tau)$ (resp. $\hat{R}_{c,d}(\tau)$) is called periodic autocorrelation (resp. odd-periodic autocorrelation) function, denoted by $R_c(\tau)$ (resp. $\hat{R}_c(\tau)$) for short.

A set of $M$ sequences $\{c_0, c_1, \cdots, c_{M-1}\}$ over $\mathbb{Z}_H$ is called a Golay complementary set (GCS) of size $M$ if

$$\sum_{k=0}^{M-1} C_{c_k}(\tau) = \left\{ \begin{array}{ll}
ML, & \tau = 0, \\
0, & \tau \neq 0,
\end{array} \right.$$ (1)

where $L$ is the sequence length. In particular, when $M = 2$, $\{c_0, c_1\}$ is called a Golay complementary pair (GCP), each of which is called a Golay sequence.

A set $C = \{c_0, c_1, \cdots, c_{M-1}\}$ consisting of $M$ sequences of length $L$ is said to be an $(M, L, Z)$-ZOCZ sequence sets with zero odd-periodic correlation zone width $Z$ if

$$\hat{R}_{c_k,c_l}(\tau) = 0, \quad 1 \leq \tau \leq Z, 0 \leq k \leq M - 1,$n$$
$$\hat{R}_{c_k,c_l}(\tau) = 0, \quad 0 \leq \tau \leq Z, 0 \leq k \neq l \leq M - 1.$$

An $(M, L, Z)$-ZOCZ set $C = \{c_0, c_1, \cdots, c_{M-1}\}$ is called a Golay-ZOCZ set if $C$ additionally satisfies (1). $C$ is denoted as an $(M, L, Z)$-Golay-ZOCZ set.

A generalized Boolean function $f(x_1, x_2, \cdots, x_m)$ from $\mathbb{Z}_2^m$ to $\mathbb{Z}_H$ is represented as a linear combination of the $2^m$ monomials:

$$f(x_1, x_2, \cdots, x_m) = \sum_{i=0}^{2^m-1} a_i \prod_{k=1}^{m} x_{ik}, \quad a_i \in \mathbb{Z}_H,$$

where $(i_1, i_2, \cdots, i_m)$ is the binary representation of the integer $i = \sum_{k=1}^{m} 2^{k-1}i_k$. Then a sequence over $\mathbb{Z}_H$ of period $2^m$ can be generated from the truth table of a boolean function $f(x_1, x_2, \cdots, x_m)$ from $\mathbb{Z}_2^m$ to $\mathbb{Z}_H$: the $i$-th element of the sequence is $f(i_1, i_2, \cdots, i_m)$. For example, $m = 3$, $H = 2$, and $f(x_1, x_2, x_3) = x_1x_2 + x_3$, the corresponding sequence $f$ of $f$ is equal to

$$f = (f(0, 0, 0), f(1, 0, 0), f(0, 1, 0), f(1, 1, 0), f(0, 0, 1), f(1, 0, 1), f(0, 1, 1), f(1, 1, 1)),$$

i.e., $f = (0, 0, 0, 1, 1, 1, 1, 0)$. In the following, if the context is clear, then a generalized boolean function $f$ from $\mathbb{Z}_2^m$ to $\mathbb{Z}_H$ will be regarded as a sequence over $\mathbb{Z}_H$ and is denoted by $f$.

For a sequence $a = (a_0, a_1, \cdots, a_{L-1})$ over $\mathbb{Z}_H$, the transmitted time-domain OFDM signal is written as

$$S_a(t) = \sum_{i=0}^{L-1} \xi^i e^{2\pi i f_c t \Delta f / \sqrt{-1}}, \quad 0 \leq t < T,$$

where $f_c$ denotes the carrier frequency and $\Delta f = \frac{1}{T}$ denotes the subcarrier spacing, with $T$ being the OFDM symbol duration. The instantaneous envelope power of...
the signal is the real-valued function \( P_a(t) = |S_a(t)|^2 \). The peak-to-mean envelope power ratio (PMEPR) of \( a \) is given by

\[
PMEPR(a) = \max_{0 \leq i < 1} \frac{P_a(t)}{L}.
\]

It is well-known that for any GCS \( \{c_0, c_1, \ldots, c_{2^m-1}\} \), we have \( PMEPR(c_i) \leq M \) [20].

3. Golay-ZOCZ sequence sets

In this section, we will introduce the Golay-ZOCZ sequence sets. In the following, for any given integers \( \tau \) and \( u \), where \( 1 \leq \tau \leq 2^m - 1 \), \( 0 \leq i \leq 2^m - 1 - \tau \) and \( 0 \leq u \leq \tau - 1 \), setting \( v = u + 2^m - \tau \), and let \((u_1, u_2, \ldots, u_m)\) and \((v_1, v_2, \ldots, v_m)\) be the binary representations of \( u \) and \( v \), respectively.

3.1. GCPs with large ZOCZ. We firstly review the well-known Golay-Davis-Jedwab (GDJ) pair.

**Lemma 3.1** ([4], [20]). Let \( f = (f_i) \) and \( g = (g_i) \) be two sequences over \( \mathbb{Z}_H \) of length \( 2^m \), where

\[
\begin{align*}
    f_i &= \frac{H}{2} \sum_{k=1}^{m-1} i_{\pi(k)} i_{\pi(k+1)} + \sum_{k=1}^{m} c_k i_k + c_0, \\
    g_i &= f_i + \frac{H}{2} i_{\pi(m)},
\end{align*}
\]

\( c_k \in \mathbb{Z}_H \) for \( 0 \leq k \leq m \). Then \( \{f, g\} \) is an \( H \)-ary GCP.

The GCP given by (2) is called the Golay-Davis-Jedwab (GDJ) pair. It turns out that GDJ sequences have PMEPR upper bounded by 2 [4]. It will be seen later that the sequences with zero odd correlation zone proposed in this paper is a special case of GDJ sequences. This means that the proposed sequences have PMEPR at most 2.

Note that the odd-periodic correlation function of two sequences is different from aperiodic correlation function. Before showing some GDJ pairs with large ZOCZ, we will firstly consider the properties of the aperiodic autocorrelation and cross-correlation of \( f \) and \( g \):

\[
\begin{align*}
    C_f(\tau) &= \sum_{i \in J_1(\tau)} \xi^{f_i-f_j} + \sum_{i \in J_2(\tau)} \xi^{f_i-f_j}, \\
    C_g(\tau) &= \sum_{i \in J_1(\tau)} \xi^{g_i-g_j} + \sum_{i \in J_2(\tau)} \xi^{g_i-g_j}, \\
    C_{fg}(\tau) &= \sum_{i \in J_1(\tau)} \xi^{f_i-g_j} + \sum_{i \in J_2(\tau)} \xi^{f_i-g_j},
\end{align*}
\]

where \( J_1(\tau) \) and \( J_2(\tau) \) form a partition of \( \{i : 0 \leq i < 2^m - \tau\} \), and

\[
\begin{align*}
    J_1(\tau) &= \{0 \leq i < 2^m - \tau : i_{\pi(1)} = j_{\pi(1)}\}, \\
    J_2(\tau) &= \{0 \leq i < 2^m - \tau : i_{\pi(1)} \neq j_{\pi(1)}\}.
\end{align*}
\]

**Lemma 3.2.** For any given GDJ pair \( \{f, g\} \), we have

\[
\sum_{i \in J_1(\tau)} \xi^{f_i-f_j} = 0,
\]

\[
\sum_{i \in J_1(\tau)} \xi^{g_i-g_j} = 0,
\]

\[
\begin{align*}
    C_{fg}(\tau) &= \sum_{i \in J_1(\tau)} \xi^{f_i-g_j} + \sum_{i \in J_2(\tau)} \xi^{f_i-g_j},
\end{align*}
\]

where \( J_1(\tau) \) and \( J_2(\tau) \) form a partition of \( \{i : 0 \leq i < 2^m - \tau\} \), and

\[
\begin{align*}
    J_1(\tau) &= \{0 \leq i < 2^m - \tau : i_{\pi(1)} = j_{\pi(1)}\}, \\
    J_2(\tau) &= \{0 \leq i < 2^m - \tau : i_{\pi(1)} \neq j_{\pi(1)}\}.
\end{align*}
\]

**Lemma 3.2.** For any given GDJ pair \( \{f, g\} \), we have

\[
\sum_{i \in J_1(\tau)} \xi^{f_i-f_j} = 0,
\]

\[
\sum_{i \in J_1(\tau)} \xi^{g_i-g_j} = 0,
\]
Lemma 3.3.

If $i \neq j$, we can define $z$ as follows:

$$z = \min\{1 \leq k \leq m : i_{\pi(k)} \neq j_{\pi(k)}\}.$$  

Obviously, $z \geq 2$. For any $i \in J_1(\tau)$, let $i'$ and $j'$ be the two integers whose bits in the binary representation are defined by

$$i'_{\pi(k)} = \begin{cases} i_{\pi(k)}, & k \neq z - 1, \\ 1 - i_{\pi(k)}, & k = z - 1, \end{cases}$$

and

$$j'_{\pi(k)} = \begin{cases} j_{\pi(k)}, & k \neq z - 1, \\ 1 - j_{\pi(k)}, & k = z - 1. \end{cases}$$

In other words, $i'$ and $j'$ are obtained from $i$ and $j$ by “flipping” the $(z-1)$-th bit in $(i_{\pi(1)}, \ldots, i_{\pi(m)})$ and $(j_{\pi(1)}, \ldots, j_{\pi(m)})$. Note that the following facts hold when $\tau$ is given: 1) $i \rightarrow i'$ is a one-to-one mapping; 2) $j' - i' = j - i = \tau$; 3) $i'_{\pi(1)} = j'_{\pi(1)}$.

Therefore, $i'$ enumerates all the elements in $J_1(\tau)$ as $i$ ranges over $J_1(\tau)$. Given $i \in J_1(\tau)$, by the definition of $z$ in (6) we then have $2 \leq z \leq m$ and

$$f_{i} - f_{j} - f_{i'} + f_{j'} = \begin{cases} H/2(i_{\pi(2)} + j_{\pi(2)}), & z = 2, \\ H/2(i_{\pi(z-2)} + j_{\pi(z-2)} + i_{\pi(z)} + j_{\pi(z)}), & z > 2, \end{cases}$$

which indicates that $\xi^{f_{i} - f_{j}}/\xi^{f_{i'} - f_{j'}} = -1$, i.e., $\xi^{f_{i} - f_{j}} + \xi^{f_{i'} - f_{j'}} = 0$. Hence, we have

$$2 \sum_{i \in J_1(\tau)} \xi^{f_{i} - f_{j}} = \sum_{i \in J_1(\tau)} \xi^{f_{i} - f_{j}} + \sum_{i' \in J_1(\tau)} \xi^{f_{i'} - f_{j'}} = \sum_{i \in J_1(\tau)} \left(\xi^{f_{i} - f_{j}} + \xi^{f_{i'} - f_{j'}}\right) = 0.$$ 

Similarly, we can prove the other three equalities. \hfill \Box

Lemma 3.3. If $H \equiv 0 \ (\text{mod} \ 4)$, $\pi(1) = m$ and $c_m \in \{\frac{H}{4}, \frac{3H}{4}\}$, then we have $i_{\pi(2)} \neq j_{\pi(2)}$ for any $i \in J_2(\tau)$, $1 \leq \tau \leq 2^{\pi(2) - 1}$,

$$\sum_{i \in J_2(\tau)} \xi^{f_{i} - f_{j}} - \sum_{u \in J_2(2^{m-\tau})} \xi^{f_{u} - f_{u}} = 0,$$

$$\sum_{i \in J_2(\tau)} \xi^{g_{i} - g_{j}} - \sum_{u \in J_2(2^{m-\tau})} \xi^{g_{u} - g_{u}} = 0,$$

$$\sum_{i \in J_2(\tau)} \xi^{f_{i} - g_{j}} - \sum_{u \in J_2(2^{m-\tau})} \xi^{f_{u} - g_{u}} = 0,$$

$$\sum_{i \in J_2(\tau)} \xi^{g_{i} - f_{j}} - \sum_{u \in J_2(2^{m-\tau})} \xi^{g_{u} - f_{u}} = 0.$$
Proof. We will prove the first equality, and the other three equalities can be similarly derived.

For any given \( i \in J_2(\tau) \), \( i_{\pi(1)} \neq j_{\pi(1)} \) and \( j = i + \tau \) imply that \( i_{\pi(1)} = 0 \) and \( j_{\pi(1)} = 1 \).

First we will show that \( i_{\pi(2)} \neq j_{\pi(2)} \) for any \( i \in J_2(\tau) \); otherwise \( i_{\pi(2)} = j_{\pi(2)} \), and then we have
\[
i + \tau < 2^{n-1} + 2^{\pi(2)-1} = j_{\pi(1)} 2^{\pi(1)-1} + j_{\pi(2)} 2^{\pi(2)-1} \leq j
\]
which indicates that \( i + \tau < j \) since \( 1 \leq \tau \leq 2^{\pi(2)-1} \). This is a contradiction with \( j = i + \tau \). Hence, we has \( i_{\pi(2)} \neq j_{\pi(2)} \).

For any given \( i \in J_2(\tau) \), let \( u \) and \( v \) be the two integers whose bits in the binary representation are defined by
\[
u_{\pi(k)} = \begin{cases} 0, & k = 1, \\ j_{\pi(k)}, & k \neq 1, \end{cases}
\]
and
\[
u_{\pi(k)} = \begin{cases} 1, & k = 1, \\ i_{\pi(k)}, & k \neq 1. \end{cases}
\]

It is obvious to see that \( 0 \leq u, v \leq 2^m - 1 \). Note that the following facts hold: 1) \( i \to j \to u \) is a one-to-one mapping; 2) \( v - u = 2^m - (j - i) = 2^m - \tau \). Therefore, \( u \) enumerates all the elements in \( J_2(2^m - \tau) \) as \( i \) ranges over \( J_2(\tau) \). We also have
\[
(f_u - f_j) - (f_v - f_u) = \left[H/2(i_{\pi(2)} + j_{\pi(2)}) + 2c_{\pi(1)}(i_{\pi(1)} - j_{\pi(1)})\right] = 0,
\]
where the last equal sign is due to \( i_{\pi(2)} \neq j_{\pi(2)} \) and \( 2c_{\pi(1)} = H/2 \). The equality above indicates \( \xi^{f_u - f_j} / \xi^{f_v - f_u} = 1 \), or
\[
\xi^{f_u - f_j} - \xi^{f_v - f_u} = 0.
\]
In this way, the two terms will cancel each other. Computing
\[
\sum_{i \in J_2(\tau)} \xi^{f_i - f_j} - \sum_{u \in J_2(2^m - \tau)} \xi^{f_u - f_u} = \sum_{i \in J_2(\tau)} \left(\xi^{f_i - f_j} - \xi^{f_j - f_u}\right) = 0.
\]

Now we give the ZOCZ property of a GDJ pair.

**Theorem 3.4.** If \( H \equiv 0 \pmod{4} \), \( \pi(1) = m \) and \( c_m \in \{ m, \frac{3m}{4}, \frac{\Delta}{4} \} \), then the GDJ pair \( \{ f, g \} \) given by (2) is a \( (2, 2^m, 2^{\pi(2)-1}) \)-Golay-ZOCZ.

**Proof.** Firstly, we will show that \( \hat{R}_f(\tau) = \hat{R}_g(\tau) = 0 \) for \( 1 \leq \tau \leq 2^{\pi(2)-1} \). By Lemmas 3.2 and 3.3, we have
\[
\hat{R}_f(\tau) = C_f(\tau) - (C_f(2^m - \tau))^* = \sum_{i \in J_1(\tau)} \xi^{f_i - f_j} + \sum_{i \in J_2(\tau)} \xi^{f_i - f_j} - \sum_{u \in J_1(2^m - \tau)} \xi^{f_u - f_u} - \sum_{u \in J_2(2^m - \tau)} \xi^{f_u - f_u} = 0.
\]
and
\[
\hat{R}_{f,g}(\tau) = C_{g}(\tau) - (C_{g}(2^{m} - \tau))^{*} \\
= \sum_{i \in J_{1}(\tau)} \xi^{g_{i}-g_{i}} + \sum_{i \in J_{2}(\tau)} \xi^{g_{i}-g_{i}} - \sum_{u \in J_{1}(2^{m} - \tau)} \xi^{g_{u}-g_{u}} - \sum_{u \in J_{2}(2^{m} - \tau)} \xi^{g_{u}-g_{u}} \\
= 0
\]
for any $\tau$, $1 \leq \tau \leq 2^{2(2)} - 1$.

Secondly, we will show that $\hat{R}_{f,g}(\tau) = \hat{R}_{g,f}(\tau) = 0$ for $0 \leq \tau \leq 2^{2(2)} - 1$.

For any $\tau$, $1 \leq \tau \leq 2^{2(2)} - 1$, by Lemmas 3.2 and 3.3, we have
\[
\hat{R}_{f,g}(\tau) = C_{g}(\tau) - (C_{g}(2^{m} - \tau))^{*} \\
= \sum_{i \in J_{1}(\tau)} \xi^{f_{i}-g_{i}} + \sum_{i \in J_{2}(\tau)} \xi^{f_{i}-g_{i}} - \sum_{u \in J_{1}(2^{m} - \tau)} \xi^{f_{u}-g_{u}} - \sum_{u \in J_{2}(2^{m} - \tau)} \xi^{f_{u}-g_{u}} \\
= 0,
\]
and
\[
\hat{R}_{g,f}(\tau) = C_{g,f}(\tau) - (C_{g,f}(2^{m} - \tau))^{*} \\
= \sum_{i \in J_{1}(\tau)} \xi^{g_{i}-f_{i}} + \sum_{i \in J_{2}(\tau)} \xi^{g_{i}-f_{i}} - \sum_{u \in J_{1}(2^{m} - \tau)} \xi^{g_{u}-f_{u}} - \sum_{u \in J_{2}(2^{m} - \tau)} \xi^{g_{u}-f_{u}} \\
= 0.
\]

For $\tau = 0$, we have
\[
\hat{R}_{f,g}(0) = \sum_{i=0}^{2^{m}-1} \xi^{f_{i}-g_{i}} \\
= \sum_{i=0}^{2^{m}-1} \xi^{-2^{i}t_{2}(m)} \\
= \sum_{i=0}^{2^{m}-1} (-1)^{t_{2}(m)} \\
= 0,
\]
where the last equal sign is due to there are $2^{m-1}$ integers $i$ such that $i_{2}(m) = 1$, and the other $2^{m-1}$ integers $i$ such that $i_{2}(m) = 0$.

Since $\hat{R}_{g,f}(0) = (\hat{R}_{f,g}(0))^{*} = 0$, we have proved that $\hat{R}_{f,g}(\tau) = \hat{R}_{g,f}(\tau) = 0$ for any $\tau$, $0 \leq \tau \leq 2^{2(2)} - 1$.

According to the discussion above, we finish the proof. \hfill \Box

**Remark 1.** In [2], the authors derived the conditions ($H \equiv 0(\text{mod} \ 2)$, and $g_{m} \in \{0, H/2\}$ for a GDJ pair over $Z_{H}$ having a ZCZ. Inspired the work in [2], we get the the conditions ($H \equiv 0(\text{mod} \ 4)$, and $g_{m} \in \{H/4, 3H/4\}$ such that a GDJ pair over $Z_{H}$ have a ZOCZ. Note that for any even $H$, the conditions mentioned above cannot hold at the same time. This means that the proposed sequences with ZOCZ in this paper are not identical to those in [2].

**Example 1.** For $H = 4$ and $m = 3$, let $\pi$ be a permutation of $\{1, 2, 3\}$ with $\pi(1) = 3$, $\pi(2) = 2$, $\pi(3) = 1$, and $(c_{0}, c_{1}, c_{2}, c_{3}) = (0, 1, 0, 0)$. Then by (2),

\[
\text{Advances in Mathematics of Communications} \quad \text{Volume X, No. X (200X), X–XX}
\]
we have \( f = (0, 0, 0, 2, 1, 1, 3, 1) \) and \( g = (0, 2, 0, 0, 1, 3, 3, 3) \). The odd-periodic autocorrelation and crosscorrelation of \( f \) and \( g \) are given as follows:
\[
\{ \hat{R}_f(r) \}_{r=0}^7 = (8, 0, 0, -4i, 0, -4i, 0, 0),
\{ \hat{R}_g(r) \}_{r=0}^7 = (8, 0, 0, 4i, 0, 4i, 0, 0),
\{ \hat{R}_{f,g}(r) \}_{r=0}^7 = (0, 0, 0, -4i, 0, 4i, 0, 0).
\]

Hence \( \{ f, g \} \) is a \((2, 8, 2)\)-Golay-ZOCZ sequence set. In addition, the actual PMEPR of the sequences of \( \{ f, g \} \) are 2.

### 3.2. GCSs with Large ZOCZ

In this subsection, we will recall the Golay complementary sequence sets [1], some of which have a large ZOCZ property.

**Lemma 3.5.** For a positive integer \( m \geq 2 \), and \( 1 \leq k \leq m - 1 \), let \( \{ 1, 2, \cdots, m \} \) be divided into a partition \( I_1, I_2, \cdots, I_k \) and let \( \pi_\alpha \) be a bijection mapping from \( \{ 1, 2, \cdots, m \} \) to \( I_\alpha \) where \( m_\alpha = |I_\alpha| \geq 1 \) for \( \alpha = 2, 3, \cdots, k \) and \( m_1 = |I_1| \geq 2 \). Given the generalized Boolean function

\[
f(x_1, \cdots, x_m) = \frac{H}{2} \sum_{\alpha=1}^{k} \sum_{\beta=1}^{m_\alpha-1} x_{\pi_\alpha(\beta)} x_{\pi_\alpha(\beta+1)} + \sum_{s=1}^{m} g_s x_s + g_0,
\]

where \( g_s \in \mathbb{Z}_H \) for \( s = 0, 1, \cdots, m - 1 \), then

\[
G = \left\{ f(x_1, \cdots, x_m) + \frac{H}{2} \sum_{\alpha=1}^{k} d_\alpha x_{\pi_\alpha(m_\alpha)} : (d_1, \cdots, d_k) \in \mathbb{Z}_2^k \right\}
\]

is a GCS.

In fact, some Golay complementary sets given by Eq. (7) have the large ZOCZ property.

**Theorem 3.6.** Let \( G \) be a GCS given by Eq. (7). If \( g_m \in \{ \frac{H}{4}, \frac{3H}{4} \} \), and \( H \equiv 0 \pmod{4} \), and \( \pi_\alpha(1) = m - \alpha + 1 \) for \( \alpha = 1, 2, \cdots, k \), then \( G \) is a \((2^k, 2^m, 2^{\pi_1(2)-1})\)-Golay-ZOCZ sequence set.

**Proof.** The proof is the same to the Theorem 3.4, and then we omit it here. \( \square \)

**Remark 2.** A GCS given by [1] can not have ZCZ property and ZOCZ property too; otherwise, such GCS has zero aperiodic autocorrelation zone and zero aperiodic crosscorrelation zone, which is impossible. In [1], the authors proposed some GCSs with large ZCZ property when \( H \equiv 0 \pmod{2} \), and \( g_m \in \{ 0, H/2 \} \). Inspired by their work of [2], we derive new GCSs when \( H \equiv 0 \pmod{4} \), and \( g_m \in \{ H/4, 3H/4 \} \). So our proposed sequences are not the sequences of [2].

**Remark 3.** Theorem 2 gives a construction of \((2^k, 2^m, 2^{\pi_1(2)-1})\)-Golay-ZOCZ sequence sets with flexible ZOCZ width which is determined by the value of \( \pi_1(2) \), where \( \pi_1(2) \leq m - k \). Since a larger ZOCZ width is desirable, we take \( \pi_1(2) = m - k \) to propose a family of \((2^k, 2^m, 2^{m-k-1})\)-Golay-ZOCZ sequence sets with large ZOCZ width.

**Example 2.** Let \( H = 4, k = 2, m = 5, I_1 = \{ 5, 3, 2 \} \) and \( I_2 = \{ 4, 1 \} \). Let \( \pi_1(1) = 5, \pi_1(2) = 3, \pi_1(3) = 2, \pi_2(1) = 4, \pi_2(2) = 1 \), and \( f(x_1, x_2, x_3, x_4, x_5) = \cdots \)
\[2x_5x_3 + 2x_3x_2 + 2x_2x_1 + x_5.\] Then by Eq. (7), we can obtain a Golay complementary set \(G\) consisting of four sequences:

\[(0000022020220111331133113),\]
\[(0220000220201133331331313),\]
\[(020200000022131313111113311),\]
\[(022002020000133131131133333).\]

It is easy to check that \(G\) is a \((4, 32, 4)\) Golay-ZOCZ sequence set. In addition, the actual PMEPR of the sequences of GCS \(G\) are upper bounded by 3.4298.

4. CONCLUDING REMARKS

In this paper, we studied the relationship between GCSs and the ZOCZ properties, and introduced two classes of Golay-ZOCZ sequence sets. In particular, a family of \(H\)-ary \((2^k, 2^m, 2^{m-k-1})\)-Golay-ZOCZ sets with flexible set sizes, various sequence lengths, and large ZOCZ widths is provided where \(H \equiv 0\ (\text{mod}\ 4)\) is a positive integer. We summarize known GCPs/GCSs with large ZCZ/ZOCZ property in Table 1. The proposed Golay-ZOCZ sequence sets are good candidates for pilots or synchronization sequences in OFDM systems. For example, they can be employed as synchronization sequences in the frequency domain similar to synchronization signals in 3GPP long term evolution (LTE) system and narrowband internet-of-things (NB-IoT) technology [6]. The synchronization sequences are assigned in the frequency domain, so the PMEPRs of these signals are highly concerned. Note that the PMEPRs of the proposed Golay-ZOCZ sequences are theoretically upper bounded. Moreover, those proposed sequences are easy to be implemented in practical communication systems. One of our future work is to construct Golay-ZOCZ sequence sets with new sequence lengths and ZOCZ widths.

Table 1. Comparison of GCSs/GCPs with Large ZCZ/ZOCZ Property

| Parameters | Constraints | Ref. |
|------------|-------------|-----|
| \((2, 2^m, 2^{m-2})\) Golay-ZCZ | \(g_m \in \{ \frac{H}{4}, 0 \}, H \equiv 0 \ (\text{mod}\ 2)\) | [10, 11] |
| \((2, 2^m, 2^{\pi(2)-1})\) Golay-ZCZ | \(g_m \in \{ 0, \frac{H}{2} \}, H \equiv 0 \ (\text{mod}\ 2)\), \(\pi(1) = m\) | [2] |
| \((2^k, 2^m, 2^{\pi_1(2)-1})\) Golay-ZCZ | \(g_m \in \{ 0, \frac{H}{2} \}, H \equiv 0 \ (\text{mod}\ 2), \pi_{\alpha}(1) = m - \alpha + 1,\) for \(1 \leq \alpha \leq k\) | [2] |
| \((2, 2^m, 2^{\pi(2)-1})\) Golay-ZOCZ | \(c_m \in \{ \frac{H}{4}, \frac{3H}{4} \}, H \equiv 0 \ (\text{mod}\ 4), \pi(1) = m\) | Thm. 1 |
| \((2^k, 2^m, 2^{\pi(2)-1})\) Golay-ZOCZ | \(g_m \in \{ \frac{H}{4}, \frac{3H}{4} \}, H \equiv 0 \ (\text{mod}\ 4), \pi_{\alpha}(1) = m - \alpha + 1,\) for \(1 \leq \alpha \leq k\) | Thm. 2 |

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Received February 2019; revised June 2019.

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