VERDIER HYPERCOVERING THEOREM FOR MOTIVIC SPECTRA

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Abstract. We prove a Verdier Hypercovering Theorem for cohomology theories arising from motivic spectra. This allows us to construct for smooth quasi-projective complex varieties a natural morphism from étale algebraic to Hodge filtered complex cobordism, which extends the map from étale motivic to Deligne-Beilinson cohomology.

1. Introduction

Let $X$ be a smooth quasi-projective complex variety, and let $H^m_M(X; \mathbb{Z}(n))$ be the motivic cohomology groups, defined as the hypercohomology groups of Bloch’s cycle complex, viewed as a complex of Zariski sheaves (or equivalently, as the hypercohomology groups of Voevodsky’s complex $\mathbb{Z}(n)$). Since these complexes are also complexes of étale sheaves, we have the analogously defined étale motivic cohomology groups $H^m_L(X; \mathbb{Z}(n))$, together with an evident map $H^m_M(X; \mathbb{Z}(n)) \to H^m_L(X; \mathbb{Z}(n))$. It is known that with rational coefficients this comparison map is an isomorphism; however, with integral coefficients these groups are different in general. For example, there is a map $c^n_{L,B} : CH^n_L(X) = H^{2n}_M(X; \mathbb{Z}(n)) \to H^{2n}_B(X; \mathbb{Z}(n))$ from the étale Chow groups to singular cohomology, which is surjective on torsion [16, Theorem 1.1]. Because of the counterexamples to the integral Hodge conjecture given by Atiyah-Hirzebruch [2], this implies that $CH^n_L(X)$ contains more elements than the usual Chow group $CH^n(X)$, and that $c^n_{L,B}$ cannot arise in the usual fashion as a cycle map coming from a cycle on $X$. To give a geometric interpretation of the étale motivic cohomology groups and to define more general maps from étale motivic cohomology to other cohomology theories, it has been shown in [16, Theorem 4.2] that the elements of $H^m_L(X, \mathbb{Z}(n))$ have an interpretation in terms of cycles on étale covers of $X$; more precisely, there is an isomorphism

$$\text{colim} H^m_M(U, \mathbb{Z}(n)) \xrightarrow{\cong} H^m_L(X; \mathbb{Z}(n))$$

where the colimit runs over all étale hypercovers of $X$. The proof of this result in [16, §4] uses rather sophisticated techniques and relies on the proof of the Beilinson-Lichtenbaum conjecture by Voevodsky [19] and Rost-Voevodsky [20].

In this note, we first use homotopy-theoretic methods to prove the above type of Verdier Hypercovering Theorem in a far more general context for cohomology theories arising from motivic spectra:

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Theorem 1.1. Let \( X \) be a smooth quasi-projective scheme over a Noetherian scheme \( S \). If \( E \) is a motivic spectrum over \( S \) and \( U \to X \) is an étale hypercover, let \( E^{m,n}(U) \) and \( E_{\text{ét}}^{m,n}(X) \) be the motivic and the étale motivic \( E \)-cohomology groups of \( U \) and \( X \) respectively. Then there is a natural isomorphism

\[
\colim_{U \to X} E^{m,n}(U) \xrightarrow{\cong} E_{\text{ét}}^{m,n}(X),
\]

where the colimit runs over all étale hypercovers of \( X \).

Taking \( E = \mathbb{H} \) and \( S = \text{Spec}(k) \) for a field \( k \), it follows that the isomorphism (1) holds for a smooth quasi-projective variety over a field, independent of further assumption such as, for example, finite cohomological dimension.

The isomorphism (1) has been used in [16] to construct a map from étale motivic to Deligne-Beilinson cohomology \( c_{L,D}^{m,n} : H^m_L(X; \mathbb{Z}(n)) \to H^m_D(X; \mathbb{Z}(n)) \), where Deligne-Beilinson cohomology is defined as the hypercohomology of a complex of Zariski sheaves [9]. If \( X \) is projective, there is an isomorphism

\[
H^m_D(X; \mathbb{Z}(n)) \cong H^m(X; \mathbb{Z}_D(n)),
\]

where the group on the right is the cohomology of the analytic Deligne complex \( \mathbb{Z}_D(n) \), which is quasi-isomorphic to the homotopy pullback of the diagram of complexes of sheaves arising from the inclusions \( \Omega_X^n \to \Omega_X^\bullet \leftarrow \mathbb{Z} \). In [10] variants of Deligne cohomology theories have been constructed by replacing the complex \( \mathbb{Z} \) (which represents singular cohomology) with a spectrum representing a more general cohomology theory. In particular, this construction applied to the Thom spectrum \( MU \) yields the Hodge filtered cobordism groups \( MU_{\log}^m(n)(X) \) with the property that the map \( MU \to \mathbb{H} \) induces natural homomorphisms \( MU_{\log}^m(n)(X) \to H^m(X; \mathbb{Z}_D(n)) \). Since filtered Hodge cobordism is an oriented motivic cohomology theory, the universal property of algebraic cobordism represented by the motivic spectrum \( MGL \) yields maps

\[
MGL^{m,n}(X) \to MU_{\log}^m(n)(X).
\]

We use Theorem 1.1 to show the following.

Theorem 1.2. Let \( X \) be a smooth quasi-projective complex variety and let \( m, n \) be integers. Then there are natural homomorphisms

\[
MGL_{\text{ét}}^{m,n}(X) \to MU_{\log}^m(n)(X)
\]

such that \( MGL^{m,n}(X) \to MGL_{\text{ét}}^{m,n}(X) \to MU_{\log}^m(n)(X) \) coincides with (3). If \( X \) is projective, the map \( MGL \to \mathbb{H} \) induces a natural commutative diagram

\[
\begin{array}{ccc}
MGL_{\text{ét}}^{m,n}(X) & \xrightarrow{\cong} & MU_{\log}^m(n)(X)
\end{array}
\]

\[
\begin{array}{ccc}
| & & |
\end{array}
\]

\[
\begin{array}{ccc}
H^n_D(X; \mathbb{Z}(n)) & \xrightarrow{\cong} & H^m(X; \mathbb{Z}_D(n))
\end{array}
\]

We remark that for a smooth projective complex variety the restriction of the map in the bottom row [3] to torsion subgroups in an isomorphism, provided \( m \neq 2n \) [16 Theorem 1.2]. It is tempting to ask whether the restriction of the
top row to torsion is an isomorphism as well, allowing to determine the torsion in étale cobordism groups via filtered Hodge cobordism.

2. Verdier’s hypercovering theorem for motivic spectra

2.1. Preliminaries. Let $\text{Sm}_S$ be the category of smooth schemes over a Noetherian scheme $S$, and let $\text{Spc}(S)$ be the category of simplicial presheaves on $\text{Sm}_S$. Thus objects of $\text{Spc}(S)$ are contravariant functors from $\text{Sm}_S$ to the category $\text{sS}$ of simplicial sets, which we refer to as spaces (over $S$). Let $f: \mathcal{X} \to \mathcal{Y}$ be a morphism of spaces. Then $f$ is called

- a projective weak equivalence, if it induces a weak equivalence of simplicial sets $\mathcal{X}(U) \to \mathcal{Y}(U)$ for every object $U$ of $\text{Sm}_S$;
- a projective fibration, if it induces a Kan fibration of simplicial sets $\mathcal{X}(U) \to \mathcal{Y}(U)$ for every object $U$ of $\text{Sm}/S$;
- a projective cofibration, if it has the right lifting property with respect to any acyclic projective fibration.

These classes of morphisms define a closed model structure on $\text{Spc}(S)$, called the projective model structure (see [6]).

We will consider $\text{Sm}_S$ as a site with respect to a Grothendieck topology $\tau$. To obtain a model structure which is sensitive to the topology $\tau$, one needs to modify the above structure. We will consider the cases when $\tau = \text{Nis}$ is the Nisnevich topology or $\tau = \text{ét}$ is the étale topology. Then $f: \mathcal{X} \to \mathcal{Y}$ is called

- a $\tau$-weak equivalence (or just weak equivalence), if it induces a weak equivalence of simplicial sets $\mathcal{X}_x \to \mathcal{Y}_x$ at every $\tau$-point $x$ of the site $\text{Sm}_S$;
- a $\tau$-cofibration (or cofibration), if it is a projective cofibration;
- a $\tau$-local projective fibration (or local projective fibration), if it has the right lifting property with respect to any projective cofibration which is also a weak equivalence.

These classes of morphisms define a closed proper cellular simplicial model structure on $\text{Spc}(S)$, called the local projective model structure (see [6] and [11, Theorem 2.3] for the corresponding injective structure which is Quillen equivalent to the projective one). Since we will only use this projective structure, we will often omit the word ‘projective’. Let $\mathcal{H}_{s,\tau}(S)$ be the homotopy category of $\text{Spc}(S)$, considered as a site with respect to $\tau$. The category $\text{Spc}_s(S)$ of pointed spaces over $S$ has a model structure via the forgetful functor $\text{Spc}_s(S) \to \text{Spc}(S)$ and we write $\mathcal{H}_{s,\tau}(S)$ for the corresponding homotopy category.

Dugger, Hollander and Isaksen [7] have shown that one way to obtain the local projective model structure is to form the localization of the projective model structure with respect to the special class of morphisms called hypercovers. Since these hypercovers will play an essential role in this paper, we will recall their definition following the conventions used in [7]: Given a topology $\tau$ on $\text{Sm}_S$, a map $f$ of simplicial presheaves is called a stalkwise fibration (resp. acyclic stalkwise fibration), if the map of stalks $f_x$ is a Kan fibration (resp. Kan fibration and weak equivalence) of simplicial sets for every $\tau$-point $x$. Let $X$ be an object of $\text{Sm}_S$ and let $U_\bullet$ be a simplicial presheaf, together with
an augmentation map $\mathbf{U}_\bullet \rightarrow X$ in $\text{Spc}(S)$. This map is called a $\tau$-hypercover of $X$ if it is an acyclic stalkwise fibration and each $U_n$ is a coproduct of representables. Note that the projective model structure on $\text{Spc}(S)$ has the property that every hypercover is a morphism of cofibrant objects [6]. Moreover, by [7] the fibrations in the local projective model structure on $\text{Spc}(S)$ admit a characterization in terms of such hypercovers. Following [7], we say that a simplicial presheaf $\mathbf{Y}$ satisfies descent for a hypercover $\mathbf{U}_\bullet \rightarrow X$, if there is a projective fibrant replacement $\mathbf{U}_\bullet \rightarrow \mathbf{Y}'$ with the property that the natural map

\begin{equation}
\text{Map}(X, \mathbf{Y}') \rightarrow \text{Map}(\mathbf{U}_\bullet, \mathbf{Y}')
\end{equation}

is a weak equivalence of simplicial sets, where Map denotes the mapping space in the simplicial structure on spaces. It is easy to see that if $\mathbf{Y}$ satisfies descent for a hypercover $\mathbf{U}_\bullet \rightarrow X$, then the map (6) is a weak equivalence for every objectwise fibrant replacement $\mathbf{Y}'$. Moreover, the local projective fibrant objects in $\text{Spc}(S)$ are exactly those spaces which are projective fibrant and satisfy descent with respect to all hypercovers [7, Corollary 7.1].

\section{2.2. The classical case.}

Let $\tau$ be either the Nisnevich or the étale topology on $\text{Sm}_S$. For stalkwise fibrant spaces $\mathbf{X}$ and $\mathbf{Y}$, simplicial homotopy of maps $\mathbf{X} \rightarrow \mathbf{Y}$ is an equivalence relation. The set $\pi(\mathbf{X}, \mathbf{Y})$ of simplicial homotopy classes of morphisms from $\mathbf{X}$ to $\mathbf{Y}$ is the quotient of $\text{Hom}_{\text{Spc}(S)}(\mathbf{X}, \mathbf{Y})$ with respect to the equivalence relation generated by simplicial homotopies. For $X \in \text{Sm}_S$, we write $\pi HC_\tau/X$ for the category whose objects are the $\tau$-hypercovers of $X$ and whose morphisms are simplicial homotopy classes of morphisms which fit in the obvious commutative triangle over $X$. The category $\pi HC/X$ is filtered (see [7, Proposition 8.5], for instance). A crucial observation, made first by Brown [3, Proof of Theorem 2], is that one can use $\pi HC/X$ to approximate the homotopy category $\mathcal{H}_{s, \tau}(S)$ in the following sense, yielding a generalization of the Verdier Hypercovering Theorem [1, exposé V, 7.4.1(4)] (see also [7, Theorem 8.6] and [12]):

\begin{theorem}
Let $\mathbf{Y}$ be a stalkwise fibrant simplicial presheaf and let $X$ be an object in $\text{Sm}_S$. Then the canonical map induces a bijection

\begin{equation}
\text{colim}_{U_\bullet \rightarrow X \in \pi HC_\tau/X} \pi(\mathbf{U}_\bullet, \mathbf{Y}) \xrightarrow{\cong} \text{Hom}_{\mathcal{H}_{s, \tau}(S)}(X, \mathbf{Y}).
\end{equation}

We apply Theorem 2.1 to obtain a description of $\text{Hom}_{\mathcal{H}_{s, \text{ét}}(S)}(X, \mathbf{Y})$, i.e. the set of maps between a smooth scheme $X$ over $S$ and a projective fibrant space $\mathbf{Y}$ in the étale homotopy category $\mathcal{H}_{s, \text{ét}}(S)$. Since $\mathbf{Y}$ is also stalkwise fibrant for the Nisnevich and the étale topology on $\text{Sm}_S$, we have from Theorem 2.1

\begin{equation}
\text{colim}_{U_\bullet \rightarrow X \in \pi HC_{\text{ét}}/X} \pi(\mathbf{U}_\bullet, \mathbf{Y}) \xrightarrow{\cong} \text{Hom}_{\mathcal{H}_{s, \text{ét}}(S)}(X, \mathbf{Y}).
\end{equation}

Let $\mathbf{Y}$ be a fibrant object in the local Nisnevich model structure. Then $\mathbf{U}_\bullet$ is a cofibrant object, and the set $\pi(\mathbf{U}_\bullet, \mathbf{Y})$ of simplicial homotopy classes of maps is in bijection with the set of morphisms from $\mathbf{U}_\bullet$ to $\mathbf{Y}$ in the homotopy category associated with local Nisnevich model structure on $\text{Spc}(S)_{\text{Nis}}$. In

\begin{equation}
particular, we obtain from (7) the following bijection
\[
\colim_{U \in \pi HC_{\text{ét}}/X} \text{Hom}_{\mathcal{H}_{s, \text{Nis}}(S)}(U \cdot, \mathcal{Y}) \cong \text{Hom}_{\mathcal{H}_{s, \text{ét}}(S)}(X, \mathcal{Y}).
\]

2.3. A motivic variant. We prove a motivic analogue of (8). Let \( \mathcal{Y} \) be a simplicial presheaf on \( \mathbf{Sm}_S \). Recall that \( \mathcal{Y} \) is \( \mathbb{A}^1 \)-local, if for every object for every \( X \in \mathbf{Sm}_S \) the projection \( X \times_S \mathbb{A}^1_S \to X \) induces a weak equivalence
\[
\text{Map}(X, \mathcal{Y}) \to \text{Map}(X \times_S \mathbb{A}^1_S, \mathcal{Y}).
\]
If \( \mathcal{Y} \) is \( \mathbb{A}^1 \)-local, then \( \mathcal{Y} \) is Nisnevich \( \mathbb{A}^1 \)-local (resp. étale \( \mathbb{A}^1 \)-local), if \( \mathcal{Y} \) is Nisnevich local fibrant (resp. étale local fibrant).

Since the motivic model structure is given by a left Bousfield localization with respect to the maps \( X \times_S \mathbb{A}^1_S \to X \) for all \( X \in \mathbf{Sm}_S \), it follows that the Nisnevich \( \mathbb{A}^1 \)-local objects (resp. étale \( \mathbb{A}^1 \)-local objects) are exactly the fibrant objects in the Nisnevich motivic structure (resp. étale motivic model structure) in \( \mathbf{Spc}(S) \). Let \( \mathcal{H}_{\text{Nis}}(S) \) (resp. \( \mathcal{H}_{\text{ét}}(S) \)) be the motivic homotopy category of spaces with respect to the Nisnevich topology (resp. étale topology).

**Lemma 2.2.** Let \( \mathcal{Y} \) be a simplicial presheaf on \( \mathbf{Sm}_S \) which is Nisnevich \( \mathbb{A}^1 \)-local. Then a fibrant replacement of \( \mathcal{Y} \) in the étale local model structure is an étale-\( \mathbb{A}^1 \)-local simplicial presheaf. In particular, for \( X \in \mathbf{Sm}_S \) we have
\[
\text{Hom}_{\mathcal{H}_{s, \text{ét}}(S)}(X, \mathcal{Y}) \cong \text{Hom}_{\mathcal{H}_{\text{ét}}(S)}(X, \mathcal{Y}).
\]

**Proof.** Let \( X \in \mathbf{Sm}_S \) and let \( q: \mathcal{Y} \to R_{\text{ét}} \mathcal{Y} \) be an acyclic cofibration in the étale local model structure with the property that \( R_{\text{ét}} \mathcal{Y} \) is étale local fibrant. Then \( q \) induces the following commutative diagram
\[
\begin{array}{ccc}
\text{Map}(X, \mathcal{Y}) & \to & \text{Map}(X \times_S \mathbb{A}^1_S, \mathcal{Y}) \\
\downarrow & & \downarrow \\
\text{Map}(X, R_{\text{ét}} \mathcal{Y}) & \to & \text{Map}(X \times_S \mathbb{A}^1_S, R_{\text{ét}} \mathcal{Y}).
\end{array}
\]
By assumption \( \mathcal{Y} \) is Nisnevich \( \mathbb{A}^1 \)-local, hence the top horizontal map is a weak equivalence. Since \( q \) is an acyclic cofibration and all objects are cofibrant, we also know that the two vertical maps are weak equivalences. Hence the lower horizontal map is a weak equivalence as well, and \( R_{\text{ét}} \mathcal{Y} \) is étale-\( \mathbb{A}^1 \)-local. For the second assertion note that since \( R_{\text{ét}} \mathcal{Y} \) is étale \( \mathbb{A}^1 \)-local, the diagonal maps in the commutative diagram
\[
\begin{array}{ccc}
\pi(X, R_{\text{ét}} \mathcal{Y}) & \cong & \pi(X, R_{\text{ét}} \mathcal{Y}) \\
& \nearrow & \nwarrow \\
\text{Hom}_{\mathcal{H}_{s, \text{ét}}(S)}(X, \mathcal{Y}) & \cong & \text{Hom}_{\mathcal{H}_{\text{ét}}(S)}(X, \mathcal{Y}).
\end{array}
\]
are bijections. Thus the bottom row is a bijection, which proves (10). \( \square \)

The next Proposition gives the motivic analogue of (8):
Proposition 2.3. Let \( X \in \text{Sm}_S \) and let \( \mathcal{Y} \) be a simplicial presheaf which is Nisnevich-\( \mathbb{A}^1 \)-local. Then the natural map induces a bijection
\[
\text{colim}_{U \cdot \to X \in \pi HC_{/X}} \text{Hom}_{\text{Nis}}(U \cdot, \mathcal{Y}) \cong \text{Hom}_{\text{H}^{\text{et}}}(S)(X, \mathcal{Y}).
\]

Proof. Because \( \mathcal{Y} \) is an objectwise fibrant simplicial presheaf, by Theorem 2.1
\[
\text{colim}_{U \cdot \to X} \pi(U \cdot, \mathcal{Y}) \cong \text{Hom}_{\text{H}^{\text{et}}}(S)(X, \mathcal{Y}).
\]
Since \( \mathcal{Y} \) is \( \mathbb{A}^1 \)-local and Nisnevich fibrant, it is a fibrant object in the Nisnevich motivic local model structure on \( \text{Spc}(S) \). Since all objects in this structure are cofibrant, the set of simplicial homotopy classes \( \pi(U \cdot, \mathcal{Y}) \) computes the set of morphisms in the motivic Nisnevich homotopy category, i.e.
\[
\pi(U \cdot, \mathcal{Y}) \cong \text{Hom}_{\text{Nis}}(S)(U \cdot, \mathcal{Y}).
\]
Again, since \( \mathcal{Y} \) is Nisnevich-\( \mathbb{A}^1 \)-local, we have from Lemma 2.2 a bijection
\[
\text{Hom}_{\text{H}^{\text{et}}}(S)(X, \mathcal{Y}) \cong \text{Hom}_{\text{H}^{\text{et}}}(S)(X, \mathcal{Y});
\]
this proves the assertion. \( \square \)

2.4. Motivic \( E \)-cohomology groups. We use Proposition 2.3 to prove the Verdier Hypercovering Theorem 1.1 for cohomology theories arising from motivic spectra. Let \( \mathbb{P}^1 \) be the projective line over \( S \) pointed at \( \infty \). Recall that a motivic or \( \mathbb{P}^1 \)-spectrum over \( S \) is a sequence \( E = (E_0, E_1, \ldots) \) of pointed spaces \( E_n \in \text{Spc}_*(S) \), together with bonding maps \( \sigma_n : E_n \wedge \mathbb{P}^1 \to E_{n+1} \) in \( \text{Spc}_*(S) \). A morphism \( f : E \to F \) of \( \mathbb{P}^1 \)-spectra is a sequence of maps \( f_n : E_n \to F_n \) in \( \text{Spc}_*(S) \) which commute with the bonding maps. We write \( \text{Spt}(S) \) for the category of motivic spectra.

Given an object \( \mathcal{X} \in \text{Spc}_*(S) \), one can associate to \( \mathcal{X} \) its motivic suspension spectrum, which is given by the sequence of pointed spaces
\[
\Sigma^\infty_{\mathbb{P}^1}(\mathcal{X}) := (\mathcal{X}, \mathcal{X} \wedge \mathbb{P}^1, \ldots, \mathcal{X} \wedge (\mathbb{P}^1)^\wedge n, \ldots)
\]
together with the identity maps as bonding maps. This suspension yields a functor \( \Sigma^\infty_{\mathbb{P}^1} : \text{Spc}_*(S) \to \text{Spt}(S) \), which has a right adjoint \( \Omega^\infty_{\mathbb{P}^1} : \text{Spt}(S) \to \text{Spc}_*(S) \). Starting with a model structure on spaces, one obtains via a formal process a stable model structure on \( \text{Spt}(S) \) such that suspension with \( \mathbb{P}^1 \) induces an equivalences of categories. If we equip \( \text{Spc}_*(S) \) with the Nisnevich (resp. étale) motivic model structure, we obtain the stable Nisnevich (resp. étale) model structure on \( \text{Spt}(S) \). Let \( \mathcal{SH}_{\text{Nis}}(S) \) (resp. \( \mathcal{SH}_{\text{ét}}(S) \)) be the corresponding Nisnevich (resp. étale) stable motivic homotopy category. Then the above pair of adjoint functors becomes a Quillen pair of adjoint functors
\[
\Sigma^\infty_{\mathbb{P}^1} : \text{Spc}_*(S) \leftrightarrows \text{Spt}(S) : \Omega^\infty_{\mathbb{P}^1}.
\]

Recall that there are other suspension operators which play a role in the definition of generalized motivic cohomology groups. For example, if \( K \) is a simplicial set, considered as a constant presheaf, then \( K \) defines a space in
\textbf{Spc}(S) (also denoted by \(K\)). For \(K = S^1\) the simplicial circle, one defines a simplicial suspension operator \(\Sigma_\cdot: \text{sSpc}(S) \to \text{sSpc}(S)\) by the formula

\[ \mathcal{X} \mapsto S^1 \wedge \mathcal{X}. \]

Also, for \(G_m = \mathbb{A}^1 - \{0\}\) over \(S\) pointed at 1, one sets \(\Sigma_{G_m} \mathcal{X} = G_m \wedge \mathcal{X}\), and given integers \(m \geq n\) one defines the motivic sphere \(S^{m,n} \in \text{sSpc}(S)\) by

\[ S^{m,n} = \Sigma_s \Sigma_{G_m} (S^0). \]

These three suspension operators are related in \(\mathcal{H}_*(S)\) by the isomorphisms

\[ \mathbb{P}^1 \cong S^1 \wedge G_m = S^{2,1}, \]

which show that the suspensions \(\Sigma_s\) and \(\Sigma_{G_m}\) become invertible in the stable motivic homotopy category. Thus it makes sense to define \(S^{m,n}\) for all integers \(m, n\); we write \(E \mapsto \Sigma^{m,n} E\) for the induced operator on spectra.

If \(\mathcal{X}\) is a space, let \(\mathcal{X}_+\) be the pointed space obtained by attaching a disjoint base point. Given a \(\mathbb{P}^1\)-spectrum \(E\), the motivic (or Nisnevich motivic) \(E\)-cohomology groups of the space \(\mathcal{X}\) with respect to \(E\) are defined as the groups

\[ E^{m,n}(\mathcal{X}) = \text{Hom}_{\text{SH}_{\text{Nis}}(S)}(\Sigma^{m,n}_{\mathbb{P}^1}(\mathcal{X}_+), \Sigma^{m,n} E), \]

Analogously, the étale motivic \(E\)-cohomology groups of \(\mathcal{X}\) are given by

\[ E^{m,n}_{\text{ét}}(\mathcal{X}) = \text{Hom}_{\text{SH}_{\text{ét}}(S)}(\Sigma^{m,n}_{\mathbb{P}^1}(\mathcal{X}_+), \Sigma^{m,n} E). \]

We now prove Theorem \ref{thm:1.1}.

\textit{Proof.} (of Theorem \ref{thm:1.1}) Let \(E\) be a \(\mathbb{P}^1\)-spectrum which is fibrant in the Nisnevich stable motivic model structure on \(\text{Spt}(S)\). Being fibrant in the Nisnevich (resp. étale) stable model structure means that \(E\) consists of Nisnevich (resp. étale) \(\mathbb{A}^1\)-local spaces \(E_n\) such that if \(\text{Hom}\) is the internal function object, the bonding maps induce weak equivalences

\[ E_n \to \text{Hom}(\mathbb{P}^1, E_{n+1}). \]

By Lemma \ref{lem:2.2} finding a fibrant replacement of \(E\) in the étale stable motivic model structure on \(\text{Spc}_*(S)\) only requires to take functorial fibrant replacements of the spaces \(E_n\) in the étale local model structure on \(\text{sSpc}_*(S)\). Let \(\mathcal{E}(n)[m]\) be the Nisnevich \(\mathbb{A}^1\)-local space \(\Omega_{\mathbb{P}^1} (\Sigma^{m,n} E)\). Then \(\mathcal{E}(n)[m]\) represents \(E\)-cohomology in \(\text{H}_{\text{Nis}}(S)\), i.e. for every \(\mathcal{X}\) we have for the \(E\)-cohomology groups

\[ E^{m,n}(\mathcal{X}) = \text{Hom}_{\text{H}_{\text{Nis}}(S)}(\mathcal{X}, \mathcal{E}(n)[m]). \]

By the previous remark on fibrant spectra, we know that taking a functorial fibrant replacement of the spaces \(E_n\) in the étale local model structure on \(\text{Spc}_*(S)\) yields a fibrant replacement in in the stable étale motivic model structure of \(E\), and hence also a fibrant replacement of \(\mathcal{E}(n)[m]\). This implies that \(\mathcal{E}(n)[m]\) also represents the étale \(E\)-cohomology groups in \(\text{H}_{\text{ét}}(S)\), i.e.

\[ E^{m,n}_{\text{ét}}(\mathcal{X}) = \text{Hom}_{\text{H}_{\text{ét}}(S)}(\mathcal{X}, \mathcal{E}(n)[m]). \]

The Theorem follows now from Proposition \ref{prop:2.3} applied with \(\mathcal{Y} = \mathcal{E}(n)[m]\). \(\square\)
Remark 2.4. We remark that Theorem 1.1 does not state that motivic $E$-cohomology satisfies étale descent in the sense of Thomason [17]. One can formulate such an étale descent statement for a motivic spectrum $E$ as follows: Let $\alpha$ be the change of topology morphism from the étale to the Nisnevich site. There is a pair of adjoint functors
\[
\alpha^*: \mathcal{SH}_{\text{Nis}}(S) \rightleftarrows \mathcal{SH}_{\text{ét}}(S): R_{\text{ét}}\alpha_*,
\]
and a motivic spectrum $E \in \mathcal{SH}_{\text{Nis}}(S)$ satisfies étale descent, if the adjunction
\[
(12) \quad E \rightarrow R_{\text{ét}}\alpha_*\alpha^*E
\]
is an equivalence. Note that (12) is not an equivalence in general; for example, algebraic $K$-theory with finite coefficients satisfies étale descent only after inverting a Bott element, see [17] and [13].

3. Étale algebraic and Hodge filtered cobordism

In this section, we let $S = \text{Spec}(\mathbb{C})$ be the spectrum of the field $\mathbb{C}$ of complex numbers. We use Theorem 1.1 to construct maps from étale algebraic cobordism (represented by Voevodsky’s motivic Thom spectrum $MGL$ [18]) to Hodge filtered cobordism (represented by the spectrum $MU_{\log}$ [10]). Since the construction of $MU_{\log}$ is rather technical, we will only briefly introduce the properties needed for the proof below; for details we refer the reader to [10].

Let $S^1$ be the simplicial circle, viewed as a constant presheaf, and let $\text{Spt}_s(\mathbb{C})$ be the category of $S^1$-spectra in $\text{Sm}_C$. Thus objects of $\text{Spt}_s(\mathbb{C})$ are sequences $F = (F_0, F_1, \ldots)$ of pointed spaces $F_n$, together with bonding maps $F_n \wedge S^1 \rightarrow F_{n+1}$ in $\text{Spc}_s(\mathbb{C})$. We consider $\text{Spc}(\mathbb{C})$ with the Nisnevich local model structure and denote by $\mathcal{SH}_{s,\text{Nis}}(\text{Sm}_C)$ the homotopy category of the induced stable model structure.

Given an integer $n$, we have in the category $\text{Spt}_s(\text{Sm}_C)$ morphisms
\[
(13) \quad H(A^{n+*}_\log(\pi_2, MU \otimes \mathbb{C})) \rightarrow Rf_*H(A^*(\pi_2, MU \otimes \mathbb{C})) \leftarrow Rf_*MU
\]
and the $S^1$-spectrum $MU_{\log}(n)$ is defined as the homotopy pullback resulting from these data. By construction, suitable suspensions of the objects $Rf_*MU$, $Rf_*H(A^*(\pi_2, MU \otimes \mathbb{C}))$ and $H(A^{n+*}_\log(\pi_2, MU \otimes \mathbb{C}))$ represent in $\mathcal{SH}_{s,\text{Nis}}(\mathbb{C})$ complex cobordism, singular cohomology and certain levels of the Hodge filtration respectively. The wedge of the spectra $MU_{\log}(n)$ for all integers $n$ defines a spectrum $MU_{\log}$ in $\text{Spt}_s(\mathbb{C})$ which represents (logarithmic) Hodge filtered cobordism in $\mathcal{SH}_{s,\text{Nis}}(\mathbb{C})$. By [10] Theorem 7.6 and Proposition 7.9, Hodge filtered cobordism is an oriented motivic cohomology theory on $\text{Sm}_C$ and is represented by a $\mathbb{P}^1$-spectrum in the stable Nisnevich motivic homotopy category, which we also denote by $MU_{\log}$. The motivic Hodge filtered cobordism groups of a space $X \in \text{Spc}(\mathbb{C})$ are the groups represented by this spectrum
\[
MU_{\log}^m(n)(X) = \text{Hom}_{\mathcal{SH}_{\text{Nis}}(\mathbb{C})}(\Sigma^\infty_{\mathbb{P}^1}(X_+), \Sigma^m,nMU_{\log}).
\]
We prove Theorem 1.2.
Proof. (of Theorem 1.2) Since $MU_{\log}$ is an oriented motivic cohomology theory [10, Proposition 7.9], it follows from the universal property of algebraic cobordism [15, Theorem 1.1] that there is a canonical map in the motivic stable category

$$MGL \to MU_{\log}.$$ 

In particular, given an étale hypercover $U_\bullet \to X$, we have natural maps

$$(14) \quad MGL^{m,n}(U_\bullet) \to MU_{\log}^m(n)(U_\bullet).$$

Taking the colimit over all such hypercovers, Theorem 1.1 yields the map

$$(15) \quad MGL_{\text{ét}}^m(n)(X) \cong \colim_{U_\bullet \to X} MGL^{m,n}(U_\bullet) \to \colim_{U_\bullet \to X} MU_{\log}^m(n)(U_\bullet).$$

Thus we get maps as in (13), provided Hodge filtered cobordism satisfies étale descent, i.e. for every étale hypercover $U_\bullet \to X$ we have an isomorphism

$$MU_{\log}^m(n)(U_\bullet) \cong MU_{\log}^m(n)(X).$$

In order to show this, it suffices to show that each of the objects appearing in (13) satisfies étale descent. Note that the topological realization functor sends an étale hypercover $U_\bullet \to X$ to a topological hypercover $f^{-1}(U_\bullet) \to f^{-1}(X)$. By [8, Proposition 4.10 and Theorem 5.2], this map induces a weak equivalence of topological spaces

$$\text{hocolim } f^{-1}(U_\bullet) \to f^{-1}(X),$$

which shows that the two objects representing complex cobordism and complex cohomology satisfy étale descent. It remains to check the Hodge filtered part of cohomology. For each component $U_n$ of $U_\bullet$, let $U_n \to Y_n$ be a smooth compactification such that $Y_n = X_n \setminus U_n$ is a normal crossing divisor. The resulting simplicial scheme $X_\bullet$ is a smooth proper hypercover of $X$, and as described in [5] (8.1.19), (8.1.20), and (8.3.3)], the Hodge filtration on the cohomology of the simplicial scheme $U_\bullet$ induces the Hodge filtration on the cohomology of $X$. Moreover, the spectral sequence which relates the cohomology of the components $U_n$ with the cohomology of $U_\bullet$ is compatible with the Hodge filtration. Since cohomology with complex coefficients satisfies étale descent, this spectral sequence abuts to the complex cohomology of $X$ and degenerates at the $E_2$-term. Hence $\text{hocolim } f^{-1}(U_\bullet) \to f^{-1}(X)$ also induces an isomorphism on Hodge filtered cohomology groups, which completes the construction of the maps in (13). It is clear that these maps extend the maps from (13).

The diagram (15) is induced by the map of motivic spectra $MGL \to H \mathbb{Z}$ and the fact that the complex realization $f^{-1}$ of this map in the topological stable homotopy category is equal to $MU \to H \mathbb{Z}$. Moreover, it has been shown in [10] that the map $MU \to H \mathbb{Z}$ induces the indicated map from Hodge filtered cobordism to Deligne cohomology. The commutativity of diagram (15) follows from the universality of $MGL$ in the stable motivic homotopy category and the fact that the horizontal maps in (15) are defined via the colimit of $MGL^{m,n}(U_\bullet)$ for all étale hypercovers $U_\bullet \to X$, together with the isomorphism (12). □
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