Acoustic waves in monodisperse and polydisperse gas-dust mixtures with intense momentum transfer between phases

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Abstract. The paper presents one-dimensional problems of dusty gas dynamics on plane acoustic wave propagation. In the first problem the dusty gas is considered as polydisperse medium with a zero-volume dust fraction. In the second problem the dusty gas is a monodisperse medium with particles that occupy finite volume. For both problems, dispersion relations are obtained. Then a limiting case when a velocity relaxation time between particles and gas tends to zero is considered. This case corresponds to mixtures with fine dust when intense interaction between phases takes place. For this limiting case expressions for the phase velocity are derived. Moreover, we present formulas for generating particular solutions of corresponding linearized problems. These solutions are intended to be used as a standard in the study of asymptotic, dispersive and dissipative properties of numerical methods. Program codes for generating such solutions are published and available for download.

1. Introduction
Currently, computer simulation of dynamics and heat and mass transfer in mixtures of gas and dust is actively developing. Much of the simulations are based on equations of continuum mechanics. There are several problems in numerical simulations of dynamics of multiphase media. In particular, when modeling mixtures with finely dispersed inclusions, we have to describe short-term processes associated with intense exchange of momentum and energy between gas and particles that take place against the background of long-term processes. It means that we have to numerically solve the problem with one or more small parameters. Such problems are stiff and require asymptotic preserving computational methods. In some cases asymptotic preserving methods need to be developed, for example, when using Lagrangian variables to simulate the dynamics of a polydisperse continuous medium. Thus it is necessary to have a system of demonstrative test problems with known standard solutions for study properties of new computing methods and determine a required numerical resolution for solving practically important problems.

The paper presents problems of dust-gas media dynamics: the propagation of acoustic waves in polydisperse media with a zero-volume dust fraction (section 2), the propagation of acoustic waves in monodisperse media taking into account the volume of dispersed phase (section 3).
both problems, dispersion relations are derived and formulas for generating particular solutions of linearized problems are given. These solutions of linearized problems are intended to be used as standards or reference solutions for studying the asymptotic, dissipative and dispersive properties of numerical methods. To make the practice of getting reference solution easier we developed codes that generates solutions for particular parameters. These codes are available for download at https://github.com/MultiGrainSPH/1D_Dust_DS/tree/master/DustyWave.

Moreover, we considered the limiting case when the time of phase velocity relaxation tends to zero. For this limiting case expressions for the phase velocity are obtained.

2. Polydisperse two-fluid medium with a zero-volume dust fraction

The first problem is a problem of propagating one-dimensional plane waves in isothermal dusty gas medium. We consider two-phase polydisperse medium in which the carrier phase is a compressible non-viscous gas of density $\rho$ and velocity $v$. The gas contains $N$ fractions of dispersed inclusions differing in particle size. For each fraction, the condition of applicability of hydrodynamic medium description is satisfied and the following space-averaged values are defined:

- $u_j$ is a velocity,
- $\rho_j$ is a mass density,
- $t_j$ is the stopping time, where $j$ is a number of the fraction.

The particles of all fractions consist of the same substance of material density $\rho_s$, have a spherical shape and occupy a zero volume in space. The gas-dust medium has a common pressure, which does not act on the particles due to a zero-volume fraction of particles. Mass and heat exchange between the gas and the particles is absent. All dust fractions exchange momentum with the gas but not with each other. Let us write the system of equations.

$$\frac{\partial \rho_g}{\partial t} + \rho_g \frac{\partial v}{\partial x} = 0,$$

$$\frac{\partial \rho_j}{\partial t} + \rho_j \frac{\partial u_j}{\partial x} = 0,$$

$$\frac{\partial v}{\partial t} + \sum_{j=1}^{N} \frac{\varepsilon_j}{t_j} (v - u_j) + \frac{c_s^2}{\rho_g} \frac{\partial \rho_g}{\partial x} = 0,$$

$$\frac{\partial u_j}{\partial t} + \frac{1}{t_j} (u_j - v) = 0,$$

where dust particles index is $j = 1, \cdots, N$ and $\varepsilon_j = \rho_j^0 / \rho_g^0$, $c_s$ is sound speed in the gas.

A stationary solution

$$\rho_g(x,t) = \rho_g^0, \quad \rho_j(x,t) = \rho_j^0, \quad v(x,t) = 0, \quad u_j(x,t) = 0$$

satisfies this system (1), (2). Therefore, in a small neighborhood, the solutions of initial systems and the system linearized on solution (3) are expected to be close.

Let us find the solution of the linearized problem by the Fourier method. We suppose that the solution (1), (2) has the form

$$\rho_g(x,t) = \rho_g^0 + \delta \rho_g(x,t), \quad \rho_j(x,t) = \rho_j^0 + \delta \rho_j(x,t)$$

and

$$v(x,t) = \delta v(x,t), \quad u_j(x,t) = \delta u_j(x,t),$$

where $\delta f(x,t)$ is small amplitude perturbations from the stationary solution. Substituting it into the system (1), (2) and neglecting the terms of the second order of smallness, we obtain the linear system with $2N + 2$ differential equations

$$\frac{\partial \delta \rho_g}{\partial t} + \rho_g^0 \frac{\partial \delta v}{\partial x} = 0,$$

$$\frac{\partial \delta \rho_j}{\partial t} + \rho_j^0 \frac{\partial \delta u_j}{\partial x} = 0,$$

$$\frac{\partial \delta v_g}{\partial t} + \sum_{j=1}^{N} \frac{\varepsilon_j}{t_j} (\delta v - \delta u_j) + \frac{c_s^2}{\rho_g^0} \frac{\partial \delta \rho_g}{\partial x} = 0,$$

$$\frac{\partial \delta u_j}{\partial t} + \frac{1}{t_j} (\delta u_j - \delta v) = 0.$$

To solve this system, we represent each of sought functions (4), (5) as

$$\delta f = \delta f e^{ikx-\omega t},$$

where $k$ and $\omega$ are wave number and frequency, respectively.
where $k$ is a real wave number, $\omega$ is a complex frequency. The substitution of (6) into (4), (5) yields the system of linear equations for perturbations $\delta f$

$$-\omega \delta \hat{f} + \sum_{j=1}^{N} \frac{\varepsilon_j}{t_j} (\delta \hat{u}_j - \delta \hat{u}_j) + i k \frac{\varepsilon_j}{\rho^\delta_k} \delta \hat{\rho}_k = 0, \quad -\omega \delta \hat{u}_j + \frac{1}{t_j} (\delta \hat{u}_j - \delta \hat{v}) = 0, \quad (7)$$

$$-\omega \delta \hat{\rho}_k + i k \rho^0_\delta \delta \hat{v} = 0, \quad -\omega \delta \hat{u}_j + i k \rho^0_\delta \delta \hat{u}_j = 0. \quad (8)$$

The system (7), (8) has nontrivial solutions when and only when its determinant is equal to 0. Setting to zero the determinant of the system we obtain the dispersion relation which relates frequency and wave number

$$P(\omega) = \frac{(-1)^N \omega^N}{\prod_{j=1}^{N} \rho^0_j} \left( \omega^2 \prod_{j=1}^{N} (1 - \omega t_j) + \sum_{p \neq j}^{N} \frac{\varepsilon_j}{w^0_p} \prod_{q \neq p}^{N} (1 - \omega t_p) + \omega^2 \prod_{j=1}^{N} (1 - \omega t_j) \right) = 0, \quad (9)$$

where $\omega_k = kc_s$. By virtue of the fact that $P$ is the polynomial of degree $2N + 2$, equation (9) has $2N + 2$ roots, from which the root $\omega = 0$ has multiplicity $N$. Due to (6), zero perturbation corresponds to root $\omega = 0$, damping waves correspond to real and positive values $\omega$, waves with increasing amplitude correspond to real negative values, and the periodic solutions will correspond to purely imaginary values.

We solved equation (9) using the program we created DustiWaveMulti.sce (available for download at the link presented in Introduction). The solution method that is implemented in DustiWaveMulti.sce is presented in [1]. Among the obtained nonzero roots of the equation, $N$ roots are real roots, and $2$ are complex conjugate roots. We take a wave that corresponds to the complex value of $\omega_{N+1}$ with the negative imaginary part for generating the solution. Through the perturbation of gas density $\delta \hat{\rho}_k$ we define all other perturbations by using the system (7), (8). Let us assume that $\delta \hat{\rho}_k = A$, then the solution in the complex plane corresponds to the wave with $\omega_{N+1}$

$$\delta \hat{\rho}_k(x, t) = Ae^{ikx-\omega_{N+1}t}, \quad \delta \hat{\rho}_j(x, t) = A \frac{\rho^0_j}{\rho^0_k} \frac{1}{(1 - \omega_{N+1} t_j)} e^{ikx-\omega_{N+1}t}, \quad (10)$$

$$\delta \hat{v}(x, t) = -A \frac{\omega_{N+1}}{\omega_k} \frac{c_s}{\rho^0_k} e^{ikx-\omega_{N+1}t}, \quad \delta \hat{u}_j(x, t) = -A \frac{\omega_{N+1}}{\omega_k} \frac{c_s}{\rho^0_k} \frac{1}{(1 - \omega_{N+1} t_j)} e^{ikx-\omega_{N+1}t}, \quad (11)$$

$$j = 1, \cdots, N.$$

The real part of the functions (10), (11) is a solution of the system (4), (5). Then for $t = 0$ this solution has the form

$$\delta f(x) = A[Re(\delta f)cos(kx) - Im(\delta f)sin(kx)].$$

Solution also was generated using the program DustiWaveMulti.sci. At the input, this code takes the parameters of gas-dust medium and size of domain $x \in [0, L], t \in [0, T]$, in which we seek the solution. At the output, we obtain the analytical representation for $\rho_k(x), \rho_j(x), v(x), u_j(x)$ at time moment $t = 0$ and numerical values $\rho_k(x), \rho_j(x), v(x), u_j(x)$ at time moment $t = T$ and numerical values $\rho_k(t), \rho_j(t), v(t), u_j(t)$ at the point $x = x_0$.

Let us study the asymptotic behavior of the system’s solution (4), (5) for small $t_j$. The phase velocity of a wave is defined as $v_{ph} = \omega/k$. For sound waves in the gas $\omega = c_s k$ and $v_{ph} = c_s$. In our case of the medium with polydisperse dust, the frequency is a complex function of wave vector. We need to analyze the expression (12) for $\omega$ which is equivalent to the relation (9) to find the sound speed. Relation (12) is an implicit dependence of frequency on the wave number and parameters of polydisperse medium with $N$ dust components:
\[
\omega^2 \left(1 + \sum_{j=1}^{N} \varepsilon_j^0 \left(1 - \omega t_j \right) \right) + k^2 c_s^2 = 0. \tag{12}
\]

We remind that relation (12) is obtained for the solutions of linearized system with perturbations of the form \( \delta f = \delta f e^{ikx - \omega t} \), i.e., the solution for sound wave corresponds to imaginary values \( \omega \). The expression (12) is simplified to the following form for low-frequency waves \( \omega t_j \ll 1 \)

\[
\omega^2 \left(1 + \sum_{j=1}^{N} \varepsilon_j^0 \right) + k^2 c_s^2 = 0, \tag{13}
\]

From (13) it follows that \( \omega = \pm ikc_s/\sqrt{1 + \sum_{j=1}^{N} \varepsilon_j^0} \). Hence, we obtain that the velocity of sound in polydisperse medium within low-frequency limit

\[
c_s^* = v_{ph} = c_s/\sqrt{1 + \sum_{j=1}^{N} \rho_j^0/\rho_g^0}. \tag{14}
\]

3. Monodisperse medium taking into account the volume of dispersed phase

The second type of standard solutions is considered for the isothermal two-phase medium (gas and particles). We assume that the dust has finite volume and take into account the intrinsic pressure of particles. The dynamics of such system is described by the equations of mass and momentum conservation for each phase and completed by state equations [2]:

\[
\begin{align*}
\frac{\partial \rho_i}{\partial t} + \frac{\partial \rho_i u_i}{\partial x} &= 0, \\
\frac{\partial \rho_g u_g}{\partial t} + \frac{\partial (\rho_g u_g^2 + m_g p_g)}{\partial x} &= p_g \frac{\partial m_g}{\partial x} + f_g, \\
\frac{\partial \rho_d u_d}{\partial t} + \frac{\partial (\rho_d u_d^2 + m_d p_d)}{\partial x} &= \frac{\partial p_d}{\partial x} = p_g \frac{\partial m_d}{\partial x} + f_d, \\
p_g &= \frac{a_d^2 \rho_g}{1 - \frac{a_d}{r}}, \quad p_d = a_d^2 \rho_d, \quad m_g + m_d = 1,
\end{align*}
\]

where \( i = g,d \) and \( r = \rho_{dd} \) is intrinsic or material density of dust grains; \( a_i \) is sound speed; \( \rho_i \) are volume densities; \( u_i \) are velocities of dust and gas; \( m_i \) is volume fraction; \( \rho_t \) is pressure; \( f_g = \rho_d \frac{u_d - u_g}{t_{stop}} \) is force acting on the particle by gas; \( f_d = -f_g \) is force acting on the gas by the particle. System (15) is closed. Substituting the dependent unknown variables into the equations from (15) and expressing \( m_d \) as \( \frac{\rho_d}{\rho_{dd}} \), we obtain system with 4 equations and 4 unknown variables:

\[
\begin{align*}
\frac{\partial \rho_i}{\partial t} + \frac{\partial \rho_i u_i}{\partial x} &= 0, \\
u_g \frac{\partial p_g}{\partial t} + \rho_g \frac{\partial u_g}{\partial x} + (u_g^2 + a_g^2) \frac{\partial p_g}{\partial x} + 2 u_g p_g \frac{\partial u_g}{\partial x} + \frac{a_g^2 \rho_g}{r(1 - \frac{a_d}{r})} \frac{\partial \rho_d}{\partial x} &= f_g, \\
u_d \frac{\partial \rho_d}{\partial t} + \rho_d \frac{\partial u_d}{\partial x} + (u_d^2 + a_d^2) \frac{\partial \rho_d}{\partial x} + 2 u_d p_d \frac{\partial u_d}{\partial x} + \frac{a_d^2 \rho_d}{r(1 - \frac{a_d}{r})} \frac{\partial \rho_d}{\partial x} &= -f_g.
\end{align*}
\]

Let us consider the solutions of a system with small deviations from the stationary, i.e. \( \rho_i = \rho_i^0 + \delta \rho_i, u_i = \delta u_i \) and \( f_g = \rho_d \delta u_d - \delta u_g \). We substitute them into the system (16) and
obtain a linear heterogeneous system in partial derivatives:

$$
\begin{cases}
\frac{\partial \rho_i}{\partial t} + \rho_i^0 \frac{\partial u_i}{\partial x} = 0, \\
\rho_g^0 \frac{\partial u_g}{\partial t} + a_g^2 \frac{\partial \rho_g^0}{\partial x} + \frac{a_g^2 \rho_g^0}{r(1 - \frac{\omega}{\rho^0})} \frac{\partial \rho_d}{\partial x} = \frac{\rho_d}{t_{stop}} \delta u_d - \delta u_g, \\
\rho_d \frac{\partial u_d}{\partial t} + (a_d^2 + \frac{a_g^2 \rho_g^0}{(r - \rho_d^0)^2}) \frac{\partial \rho_d}{\partial x} + \frac{a_d^2 \rho_d^0}{r(1 - \frac{\omega}{\rho^0})} \frac{\partial \rho_g}{\partial x} = -\frac{\rho_d}{t_{stop}} \delta u_d - \delta u_g.
\end{cases}
$$

(17)

We seek for small perturbations of density and velocity in the form $\delta \rho_i = \delta \rho_i \exp \imath kx - \omega t$ and $\delta u_i = \delta u_i \exp \imath kx - \omega t$, substituting them in (17) we obtain:

$$
\begin{cases}
-\omega \delta \rho_i + ik \rho_i^0 \delta u_i = 0, \\
(-\omega + \frac{\rho_d^0}{\rho_g^0 t_{stop}}) \delta u_g + ik \frac{a_g^2}{\rho_g^0} \delta \rho_g + ik \frac{a_g^2}{r(1 - \frac{\omega}{\rho^0})} \delta \rho_d - \frac{\rho_d^0}{\rho_g^0 t_{stop}} \delta u_d = 0, \\
(-\omega + \frac{1}{t_{stop}}) \delta u_d + ik \frac{\rho_d^0}{\rho_d^0} \delta u_g + ik \frac{a_g^2}{(r - \rho_d^0)^2} \delta \rho_d + ik \frac{\rho_d^0}{r(1 - \frac{\omega}{\rho^0})} \delta \rho_g - \frac{1}{t_{stop}} \delta u_g = 0.
\end{cases}
$$

(18)

The determinant of the system (18) have to set equal to zero for the existence of nontrivial solutions. We obtain the dispersion relation:

$$
P(\omega) \equiv \omega^4 - \omega^2 \left( \frac{1}{t_{stop}} + \frac{\rho_d^0}{\rho_g^0 t_{stop}} \right) + \omega^2 k^2 \left( a_d^2 + \frac{\rho_d^0}{\rho_g^0} \theta^2 a_g^2 + a_g^2 \right) - \frac{\omega^2}{t_{stop}} \left( \frac{\theta (a_d^2 + a_g^2)}{(1 - \theta)} \right) + a_d^2 + \frac{\rho_d^0}{\rho_g^0} a_d^2 + \frac{a_g^2 \rho_g^0}{(1 - \theta)^2} + k^4 \left( a_g^2 \frac{\rho_d^0}{10} + \frac{\theta^2 a_g^2 \rho_g^0}{(\rho_d^0)(1 - \theta)^2} - \frac{a_g^2 \theta^2}{(1 - \theta)^2} \right),
$$

(19)

where $\theta = \frac{\rho_d^0}{\rho_g^0}$. Let us consider limiting cases when $a_d = 0$ and denote $\omega_s = ka_g$, $\varepsilon = \frac{\rho_d^0}{\rho_g^0}$. If $t_{stop}$ tends to 0, then we get the equivalent equation:

$$
P(\omega) \equiv \omega \left( \omega^2 (1 + \varepsilon) + \omega_s^2 \left( \frac{\theta}{(1 - \theta)} + 1 + \frac{\theta^2 (1 - \theta)}{(1 - \theta)^2} \right) \right).
$$

Solutions of the equation are $\omega_{1,2} = \pm i \omega_s \sqrt{\frac{1 - \theta (1 - \theta)}{(1 - \theta)^2 (1 + \varepsilon)}}$. From this we obtain that the sound speed in a two-phase medium in the low-frequency limit is:

$$
a^s_g = v_{ph} = a_g \sqrt{\frac{1 - \theta (1 - \theta)}{(1 - \theta)^2 (1 + \varepsilon)}},
$$

(20)

Using the perturbation of the gas density $\delta \rho_g$ we determine all other perturbations.

$$
\begin{align*}
\delta \rho_g &= A \delta \rho_g \exp \imath kx - \omega t; \\
\delta u_g &= -A \frac{\imath \omega}{k \rho_g^0} \delta \rho_g \exp \imath kx - \omega t; \\
\delta \rho_d &= A \left( (-\omega + \frac{\rho_d^0}{\rho_g^0} \sqrt{\frac{ka_g^2}{\rho_g^0}}) \frac{\omega}{\frac{\rho_d^0}{\rho_g^0} + \sqrt{\frac{ka_g^2}{\rho_g^0}}} \right) \delta \rho_g \exp \imath kx - \omega t; \\
\delta u_d &= -A \frac{\imath \omega}{k \rho_d^0} \left( (-\omega + \frac{\rho_d^0}{\rho_g^0} \sqrt{\frac{ka_g^2}{\rho_g^0}}) \frac{\omega}{\frac{\rho_d^0}{\rho_g^0} + \sqrt{\frac{ka_g^2}{\rho_g^0}}} \right) \delta \rho_g \exp \imath kx - \omega t.
\end{align*}
$$

(21)
The real part of the functions (21) is the solution to the system (17). Numerical solution of the equation (19) and creating a particular solution (21) according to the given initial parameters were implemented in the program DustyWaveMonodispVolumeDust.sce. This program is available for download at the link presented in the Introduction.

4. Discussion and conclusions

The paper presents one-dimensional systems of equations describing the propagation of acoustic plane waves in a two-phase medium. A two-phase medium is a mixture of an isothermal gas with dispersed solid particles, in which the phases exchange momentum. Two particular cases are considered: a mixture of gas and polydisperse particles of an infinitesimal volume and a mixture of gas and monodisperse particles of a finite volume. In the second case the intrinsic pressure of the particles is taken into account. For each case, the main stages of obtaining the dispersion relation are given. For a gas-polydisperse dust medium, the dispersion relation obtained by us (9) coincides with the dispersion relation obtained by the authors [1]. To the best of our knowledge the dispersion relation for the second case of monodisperse mixture taking into account the volume of the dispersed phase and the intrinsic pressure of the particles (19) was obtained for the first time. In addition, formulas (10), (11) and (21) are presented for generating particular solutions to linearized problems.

We consider the limiting case when the impulse exchange time between phases is infinitely short. This limiting case corresponds to a mixture with small particle size when grains move with the velocity almost equal to the velocity of the gas. For this limiting case the phase velocity of the waves is derived from the dispersion relations. We found that in the limit considered, the phase velocity of wave propagation coincides with the group velocity and depends on the sound speed in a pure gas and the mass fraction of the dust. The greater is the mass fraction of the dust, the lower is the sound speed in the dusty gas. When the volume of the dispersed phase is taken into account, the sound speed of the mixture monotonically increases with increasing the dust volume fraction.

We note that the limiting value of the phase velocity that we found for the gas-polydisperse dust mixture (14) was confirmed by our numerical simulations. We simulate the propagation of a single acoustic wave and a shock tube problem with smoothed particle hydrodynamics [3]. On the other hand, for a gas-monodisperse mixture with finite dust volume fraction, the phase velocity found by us (20) does not coincide with the sound speed expression proposed by the authors of [4] (see their formula 5.2). Therefore, as a next step, we plan numerical simulation of acoustic waves and shock tube problem in a medium with finite volume fraction of small dust.

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