Abstract We address the occurrence of narrow planetary rings and some of their structural properties, in particular when the rings are shepherded. We consider the problem as Hamiltonian scattering of a large number of non-interacting massless point particles in an effective potential. Using the existence of stable motion in scattering regions in this set up, we describe a mechanism in phase space for the occurrence of narrow rings and some consequences in their structure. We illustrate our approach with three examples. We find eccentric narrow rings displaying sharp edges, variable width and the appearance of distinct ring components (strands) which are spatially organized and entangled (braids). We discuss the relevance of our approach for narrow planetary rings.

Keywords Narrow planetary rings · scattering systems · strands · braids

1 Introduction

The accidental discovery of the rings of Uranus in 1977 by stellar-occultation measurements (Elliot et al., 1977) led to a renewed interest in planetary ring systems. The main reason was that the Uranian rings turned to be extremely different from those of Saturn: They are narrow, opaque, sharp-edged, inclined and eccentric (Elliot and Nicholson, 1984; Esposito, 2002). To quote some figures (Murray and Dermott, 1999), the widest ring of Uranus, the $\varepsilon$ ring, is 20–96 km wide with a nominal semi-major axis at 51,149 km; in comparison, the main rings of Saturn are a few thousand kilometers wide. The eccentricity of the $\varepsilon$ ring is 0.0079; Saturn’s F ring has an eccentricity 0.0026. The structural properties have raised a number of questions, most of which remain unanswered (Esposito, 2002; Sicardy, 2005). For instance, an eccentric inclined narrow ring like the $\varepsilon$
ring is expected to spread in rather short time scales, $t_{\text{max}} = 6 \times 10^8$ years, which is “considerably smaller than the age of the solar system” (Esposito, 2002). This is due to interparticle collisions, drag and differential precession. This suggests an efficient confinement mechanism that maintains these structural properties of the ring over longer time scales (Esposito, 2002).

To explain these structural features, new models were introduced where the confinement was induced by nearby moons. Among these models we mention in particular the shepherding model introduced by Goldreich and Tremaine (1979), where two moons around the ring were proposed to bound it. Soon afterwards, the Pioneer and Voyager missions detected the shepherd moons around Saturn’s F ring, and later the shepherds of the outermost $\epsilon$ ring of Uranus. The shepherding confinement involves angular momentum transfer between the shepherd moons and the ring particles, self-gravity and viscous damping due to interparticle collisions (Borderies et al., 1983). While the full scenario for shepherding has not been fully understood (Esposito, 2002; Sicardy, 2005), the presence of dissipation seems to be essential. In addition, the formulation assumes the ring boundary located at a lower-order resonance.

While the discoveries of the shepherd moons around the F Saturn ring and the $\epsilon$ ring of Uranus represented some confirmation for the shepherd theory, other questions still remain unanswered. For instance, most Uranian rings have no associated shepherd moons around them (Murray and Thomason, 1990), nor the narrow eccentric rings of Saturn, which among others would provide an explanation for their sharp-edges (Murray and Dermott, 1999). Thus, either the shepherds are there but are too small to be detected, or “some physics is yet to be understood” (Sicardy, 2005). Other problems are related with the effects of collisions, drag or keplerian differential velocity, which define too short life-times for the narrow rings, and cast doubts on the confinement mechanism which maintains the eccentricity and sharp edges of these rings. In addition, Saturn’s F ring turned out to have a very rich dynamical structure (Smith et al., 1981, 1982; Murray et al., 1997): besides the non-zero eccentricity, it displays multiple components entangled in a complicated way, known as strands and braids, showing further puzzling features like kinks and clumps.

Numerical simulations have investigated a variety of physical interactions, like the gravitational perturbations of shepherd moons on circular and eccentric orbits (Showalter and Burns, 1982; Giuliani Winter et al., 2000), effects due to the action of embedded moonlets (Lissauer and Peale, 1986), and ring interparticle collision effects (Hänninen, 1993; Lewis and Stewart, 2000). The central questions investigated have been the formation of structure (strands, braids, clumps) and their short-term stability. While these studies are extremely valuable and have led to interesting predictions, e.g. the formation of channels and streamers (Giuliatti Winter et al., 2000) which have been recently observed (Murray et al., 2005), there is no self-consistent approach for the confinement of narrow rings and their radial and azimuthal structure. The F Saturn ring remains as the most fascinating example of these possibilities.

The present paper addresses this point, namely, a self-consistent scenario for the occurrence of narrow rings and the appearance of structure. The main idea is to consider phase space regions where scattering dominates the dynamics in systems with some intrinsic rotation. Scattering provides a simple scenario where confine-
ment and escape are clearly distinguished. Existence of stable periodic orbits or tori in such scattering regions implies effective (over long times) dynamically bounded motion where otherwise there would be escaping trajectories. The intrinsic rotation, which is associated to an external potential due to the shepherds or by the full many–body problem, defines a mechanism for creating the ring: the trapped region, if it exists, rotates around the central planet, thus the confined particles form a ring. While at first sight it may appear a strange idea to focus on scattering dynamics, close approaches to shepherds illustrate this situation. In plain words, what does not escape to infinity (due to scattering) builds a ring. The bifurcation scenario yielding the stable periodic orbits or tori, and other considerations supported by theorems of Hamiltonian systems with many degrees of freedom, allow us to understand the narrowing and the appearance of strands and braids. Within this approach, which at this stage is mainly qualitative, we succeed in explaining the confinement and also some of the questions described above, such as the occurrence of narrow eccentric rings with sharp edges, apse alignment, the appearance of different ring components (strands) and their spatial entanglement (braids). The narrow rings we obtain for the examples treated are stable although they may display dynamical evolution.

To illustrate our approach we consider three examples. The first, a planar billiard system (a disk rotating on a circular or Kepler ellipse) with an arbitrary large number of non-interacting massless particles, is very simple, yet the relevant steps can be followed to a good extent analytically. So it is an interesting system that allows a pedagogical insight on many properties of the scattering approach to narrow rings. The second example in addition includes a central attractive $1/r$ interaction; it helps to establish the connection to the full gravitational problem, showing the genericity and robustness of our approach. The third example is the circular restricted three-body problem with an extremely small mass parameter. With these examples we show the occurrence of rings, the appearance of radial and azimuthal structures, and the genericity and robustness of the whole approach.

The paper is organized as follows: in the next section we describe our approach in rather general terms, avoiding formalities. In Sec. 3 we illustrate our theory with the first example, a scattering billiard system. The main steps in the construction of a (narrow) ring are described in detail, discussing separately circular and elliptic motion of the billiard disk. For the elliptic case we present new results related to the appearance of structure within the rings, in particular, strands and braids. In Sec. 4 we present results for the rotating disk with the addition of a central attractive $1/r$ interaction. This example permits us to understand the connection to the gravitational problem, which we exemplify with the circular restricted three-body problem in Sec. 5. Finally, in Sec. 6 we present conclusions and some outlook.

2 Hamiltonian scattering approach to rings of non-interacting particles

We begin by considering the planar circular restricted three-body problem (RTBP) of point particles with a very small mass parameter (Szebehely, 1967); more complicated and realistic cases will be considered below. The larger mass (primary) is thus associated to the central planet, the smaller one (secondary) to a confining moon, and the massless particle to a particle of the ring, and therefore it does not
influence the motion of the other two. The motion of the massive bodies is circular about their common center of mass. In an inertial frame with the origin at the central planet, the Hamiltonian describing the motion of the massless particle takes the general form

$$H = \frac{1}{2} (P_X^2 + P_Y^2) + V_0(|X|) + V_1(|X - X_d(\phi)|).$$

(1)

Here, \(X = (X, Y)\) denotes the position of the massless particle, \(P_X\) and \(P_Y\) are the canonically conjugate momenta, \(X_d(\phi)\) denotes the position of the secondary, and \(V_0(|X|)\) and \(V_1(|X - X_d(\phi)|)\) are the interaction potentials due to the primary and secondary, respectively. The Hamiltonian is explicitly time-dependent through the dependence of \(V_1\) upon \(\phi = \omega t\), with \(\phi\) the angular position of the secondary defined with respect to the \(X\)-axis and \(\omega\) its constant angular velocity. By canonically changing to a rotating frame (with constant angular velocity \(\omega\)), the new Hamiltonian becomes time-independent and is thus a constant of motion (cf. Eq. (2) below). This is the well-known Jacobi constant (Szebehely, 1967).

We observe that the Hamiltonian (1) has a quite generic form. It consists of an integrable part \(H_0 = H - V_1(X, Y, \phi)\) and a periodic time-dependent perturbation \(V_1(X, Y, \phi)\). Following the standard terminology of scattering theory (Newton, 1966), we shall denote by \(H_0\) the free part of the interaction, which is completely integrable. In the case of the RTBP, the potential \(V_0\) is proportional to the mass of the central planet, while \(V_1\), the disturbing potential, is proportional to the mass parameter \(\mu\). For the known shepherd moons we have \(\mu \sim 10^{-8} - 10^{-10}\) (Ural'skaya, 2003). It is thus legitimate to consider the motion as small corrections about a Kepler solution (\(V_0\)), i.e., in a series expansion. In this case, we say that \(V_0\) defines the geodesics of the free interaction \(H_0\) (the conic solutions for the Kepler problem), while \(V_1\) acts as a small perturbation.

A large variety of periodic orbits (in the rotating frame) exist for the circular RTBP with small \(\mu\) (Hénon, 1997). Considering the measured eccentricities of the planetary narrow rings (\(\leq 0.01\), cf. Murray and Dermott, 1999), we are thus interested in those which have a very small eccentricity, and therefore are close to the circular or elliptic orbits of the two-body Kepler problem. In addition, these orbits must occur in regions in phase space where scattering motion and thus the possibility to escape to infinity dominates the dynamics, i.e., no zero-velocity curves may exist (Benet et al., 1998). That is, we focus on phase space regions with no “potential barriers” that confine the motion of the massless particle. It can be shown that such periodic orbits indeed exist, and some are linearly stable in certain intervals of the Jacobi constant (Hénon, 1997; Benet et al., 1998). Moreover, these stable periodic orbits appear through saddle–center bifurcations by changing the value of the Jacobi integral (e.g., by decreasing it): a pair of new periodic orbits is created at certain value of the Jacobi constant (Hénon, 1997; Benet et al., 1998). Eventually, further reducing the Jacobi integral induces a period-doubling bifurcation cascade on the stable periodic orbit, which finally turns it into an unstable one (cf. Fig. 3b); further reduction yields locally a hyperbolic horseshoe. The key property here is that, in the intervals of the Jacobi integral where there is one stable periodic orbit, the manifolds of the unstable partner define barriers
which *confine dynamically* the motion of the massless particle (Fig. 3b). Such dynamical confinement actually takes place all around the central planet due to the intrinsic rotation induced by the motion of the shepherd. Hence, there is a region in phase space where the solutions remain close to the stable periodic orbit, not escaping, irrespective of whether the actual motion is periodic, quasi-periodic or even chaotic. This behavior is generic for autonomous two degrees-of-freedom scattering Hamiltonian systems of the type considered; systems with more degrees of freedom are discussed below.

We turn now to the occurrence of rings (Benet and Seligman, 2000). Consider an arbitrarily large number (ensemble) of massless particles whose interaction is given by (1), but which do not interact between themselves. While the latter assumption is very restrictive, it is a first step in the theory. It essentially allows us to separate the many-body problem into a collection of independent one-particle problems, which is the first term in the series expansion of the full many-body Hamiltonian in terms of the mass of the ring particles. The initial conditions of such non-interacting particles are chosen close to the phase space location of the stable periodic solution described above. Yet, they differ slightly, in particular in the value of the Jacobi integral. We emphasize that we are not interested in collision orbits with a shepherd but in periodic orbits very close to it.

We restrict now to one interval of Jacobi integrals where the whole bifurcation scenario takes place (from the creation of the pair of periodic orbits, until both define a hyperbolic structure). For a fixed value of the Jacobi integral, all particles whose initial conditions are outside the phase space region enclosed by the manifolds of the unstable periodic orbit rapidly escape to infinity along a scattering trajectory. In contrast, all those within the manifolds are *dynamically confined*, moving around the central planet. The distinction among these two cases is sharp. Thus, letting the (non-interacting) system evolve from an initial time $t = 0$ a large number of particles escape, while the rest move close to the reference stable solution. These statements hold for all values of the Jacobi constant within the considered interval. The ring is then obtained, in the $X$–$Y$ space for a given time, from the position of all the ring particles of the ensemble that remain close to the reference periodic orbit, i.e., that are dynamically confined (Figs. 4, 7 and 8). Note that this argument does not require a priori any resonance condition.

Some important properties of the ring are the following. First, the ring displays sharp edges. This follows from considering only scattering regions in phase space. Indeed, the distinction between dynamically trapped particles and those with unbounded motion is rather clear typically after a few periods of revolution of the secondary mass. Yet, for very small values of the mass parameter certain subtleties arise, as we shall discuss in sections 4 and 5 due to the overwhelming attraction of the central planet. Second, the rings are in general eccentric. This can be understood by observing that the motion of each particle of the ring is close to the originating stable periodic orbit, which in turn is close to a keplerian ellipse of small eccentricity in the case of the RTBP, as we have chosen it. In the rotating frame, the latter has a well-defined periapse and apoapse. In fact, the ring resembles the shape of the periodic orbit pictured in the rotating frame. Notice that this is a collective property reflected in the $X$–$Y$ space, in the sense that it is defined by the ensemble of ring particles which are trapped. Put differently, apse alignment follows from the proximity and resemblance of the trapped trajectories to
the organizing reference orbit (stable periodic orbit). Third, the narrowness of the
ring can be understood from the fact that the region in phase space (for a specific
Jacobi integral) corresponding to dynamically bounded motion is typically quite
small (Benet et al., 1998), as well as the interval of values of the Jacobi integral
where the reference periodic orbit is stable. This is usually so also because of
the presence of other moonlets. We emphasize that, despite the qualitative nature
of these results, these properties are observed in real narrow planetary rings, and
some of them remain as open questions (Esposito, 2002; Sicardy, 2005).

Above, we have used the fact that the Hamiltonian (1) is autonomous with
two degrees of freedom. This permits us to compute directly the stability prop-
erties of the periodic orbits from the trace and the determinant of the linearized
dynamics (Yakubovich and Starzhinskii, 1975). Moreover, in this case the man-
ifolds of the unstable periodic orbit indeed define a region of bounded motion if
the partner is stable. One consequence is that, for the interval of Jacobi constant
of interest, the probability of being trapped is small but strictly non–zero; this fol-
lows from the KAM structure (Arnold et al., 1988) for a two degree-of-freedom
system (Fig. 3a). Furthermore, the scenario is structurally stable, i.e., there are
no qualitative changes under small generic perturbations: the organizing centers
(periodic orbits) will only be shifted in phase space. Therefore, oblateness of the
planet or other effects can also be included in this scenario. The whole approach,
so far, is robust.

The question now is whether this approach can be extended to systems with
more degrees of freedom, for instance, when other shepherd moons are also taken
into account, as in the case of Saturn’s F ring or the ε ring of Uranus. For more
degrees of freedom new phenomena appear, such as Arnold diffusion (Arnold,
1964). Furthermore, the larger dimensionality of phase space allows for other
possibilities as regards the stability of the periodic orbits (Skokos, 2001). These
properties seem to be a real constraint for extending our approach to more degrees
of freedom. Yet, for many degrees of freedom, the same line of reasoning can be
followed with some minor changes: instead of presenting the theory on the stable
periodic orbits, we do it on the stable quasi-periodic tori. According to a theorem
of Jorba and Villanueva, (1997a; 1997b), around lower-dimensional normally el-
liptic tori there is a region of effective stability. Moreover, if the corresponding
unstable solution is of center × center × · · · × center × saddle type, recent results
by Wiggins and collaborators assert that the scattering dynamics can be under-
stood similarly to the invariant construction of the autonomous two-degree-of-
freedom system (Wiggins et al., 2001; Uzer et al., 2002; Waalkens et al., 2004).
These results suggest that, at least locally and in the sense of effective stability,
i.e. for long enough times but perhaps not infinitely long, the underlying phase
space structure is essentially the same as for two degrees of freedom. In this case,
the construction of the ring can be carried out, even though we cannot assure in-
finity long confinement. A simple case where these statements can be checked
is the planar elliptic RTBP for small but non-zero eccentricity. In this case the
explicit quasi-periodic time-dependence of the potential $V_1$ cannot be removed,
leading to a phase space of larger dimensionality. As we shall illustrate below, a
quasi-periodic perturbation $V_1$ has important consequences for the ring structure:
The ring turns out to have many components (see Fig. 51) which are entangled in
a complicate way (cf. Figs. 5b–d).
We finish this section by emphasizing that, despite the fact that we have considered the case of a shepherd moon for concreteness, the intrinsic rotation defined by the circular orbit above, or any other rotating quasi-periodic motion, may be associated with other moons or with the effect of the whole many-body interacting problem. If this intrinsic rotation allows for the appearance of trapped motion embedded in scattering regions, which are somewhat localized, then narrow rings will occur. This is therefore a possible explanation, not requiring undiscovered shepherd moons, for the non-shepherded narrow rings of Uranus for instance.

3 Example 1: A scattering billiard system

We illustrate our theory first with a planar billiard system. While this system is a simple toy model, we avoid the numerical complications of dealing with $1/r$ potentials where the relevant masses differ significantly (many orders of magnitude). Yet, the connection to the gravitational case can be established rigorously (Sec. 2).

This toy model, as we shall show, has the great advantage that it provides insight into the scattering properties on which our approach relies, since the fact that there is no global confining potential enhances the possibility of escape.

Consider the planar motion of a massless point particle bouncing off one circular hard disk of radius $d$, which moves on a Kepler orbit. The center of the disk $X_d(\phi)$ describes a circular or elliptic Kepler orbit with one focus at the origin (Fig. 1). We denote its radial position by $R(\phi)$, with $\phi$ the angular position along the Kepler orbit measured from the pericenter. Then, $R(\phi) = a(1 - \varepsilon^2)/(1 + \varepsilon \cos(\phi))$ with $a = 1$ is the semi-major axis and $\varepsilon$ the eccentricity. The Hamiltonian expressed in an inertial frame can thus be written as Eq. (1), with $V_0 = 0$, $V_1$ zero for $|X - X_d(\phi)|^2 > d^2$ and infinite otherwise. We shall focus on the case of small or zero eccentricity. For non-zero $\varepsilon$, due to the explicit time dependence the system has two and a half degrees of freedom and no constant of motion.

By performing a canonical transformation of (1) to appropriate rotating–pulsating coordinates (Merlo, 2004), the disk is left at rest at $x_d = (\bar{R}, 0)$, where $\bar{R} = a$ is the major semi-axis of the orbit of the center of the disk. The new Hamiltonian $J$ is then given by

$$J = \frac{1}{2} \frac{R^2}{|R(\phi)|^2} (p_x^2 + p_y^2) + V_0 \left( \frac{R(\phi)}{R} |x| \right) + V_1 \left( \frac{R(\phi)}{R} |x - x_d| \right) - \dot{\phi} (xp_y - yp_x) - \dot{\phi} \frac{1}{R(\phi)} \frac{dR(\phi)}{d\phi} (xp_x + yp_y). \quad (2)$$

For $\varepsilon = 0$ (circular orbit), $J$ is time independent and thus a constant of motion (the Jacobi integral). We shall refer here to the numerical value of $J$ in (2) as the Jacobi integral, even for non-zero eccentricity where it is not a constant of motion.

The dynamics of the billiard is straightforward. The particle moves freely on a rectilinear trajectory of constant velocity (the potential $V_0 = 0$) until it encounters the disk; if it never collides then it escapes to infinity. When a collision takes place, the particle is specularly reflected with respect to the local (moving) frame of the disk. This defines the outgoing conditions at the disk, and the motion is rectilinear and uniform again. The precise result of a collision depends on the position on
the disk where it occurs, the relative velocities, and for non-vanishing $\varepsilon$, on the position along the ellipse ($\phi$). Collisions taking place on the front of the disk increase the (outgoing) kinetic energy of the particle, while collisions on the back reduce it.

In the context of rings, we are interested in the particles which are dynamically trapped, the ring particles. In the present example, this can only be obtained through consecutive collisions with the disk. It is important to emphasize here that collisions with the disk do not model or imply physical collisions with the actual shepherds; they model interactions with the external potential that allows for confinement. A convenient description of this situation is given by introducing the following quantities, defined at the collision point (Fig. 1). The angle $\phi$ defines the position of the center of the disk with respect to the $X$ axis (inertial frame), and $v$ is the magnitude of the velocity after the collision. The angle $\alpha$ denotes the angle formed by $X_d(\phi)$ and the position vector of the collision point referred to the center of the disk. Finally, the angle $\theta$ is the outgoing direction of the velocity.

3.1 Occurrence of rings: The circular case

For $\varepsilon = 0$, Eq. (2) is an autonomous two-degree-of-freedom Hamiltonian. The Hamiltonian flow defines a map, e.g., at the collision point with the disk. This map is open, in the sense that certain initial conditions may not have an associated image, as due to escape. Trapped trajectories are associated with consecutive collisions with the disk. In particular, simple periodic orbits can be worked out analytically. The Jacobi integral is given by $J = v^2/2 - v(R\sin \theta + d\sin(\theta - \alpha))$ ($\omega = 1$). Due to the circular symmetry, all radial collisions ($\alpha = \pi$) conserve the kinetic energy. It is easy to calculate the condition that these orbits must fulfill to
Strands and braids in narrow rings

Fig. 2 (a) Radial (consecutive collision) periodic orbits of the circular rotating billiard for \( n = 0, 1, 2, 3 \), in terms of \( J/(R - d)^2 \). (b) Stability of the radial periodic orbits: the orbits are stable if \( |\text{Tr} D\mathcal{P}_J| < 2 \).

be fixed points of the map (Meyer et al., 1995),

\[
\frac{J_n}{(R - d)^2} = \frac{2\cos^2 \theta + \Delta \phi \sin(2\theta)}{(\Delta \phi)^2} = \frac{2\cos^2 \theta (1 + \Delta \phi \tan \theta)}{(\Delta \phi)^2}.
\]

Here, \( n = 0, 1, \ldots \) denotes the number of full turns completed by the disk between consecutive collisions, and \( \Delta \phi = (2n - 1)\pi + 2\theta \) is the corresponding change in the angle \( \phi \). In Fig. 2a we plot \( J_n/(R - d)^2 \) for some values of \( n \). However, the periodic orbits are a set of measure zero of the initial conditions. Therefore, we must also consider the behavior of nearby trajectories. To this end, we note that the characteristic curves in Fig. 2a display one maximum and one minimum for each value of \( n \). The occurrence of maxima and minima on these curves points to the appearance of consecutive collision periodic orbits through saddle–center bifurcations.

The stability properties of a time-independent two-degree-of-freedom system can be obtained from the trace and the determinant of the linearized Poincaré map at the fixed points (Yakubovich and Starzhinskii, 1975). The transformation \( (\alpha_{k+1}, p_{k+1}) = \mathcal{P}_J(\alpha_k, p_k) \) with \( p_k = -d - R \cos \alpha_k - v \sin(\alpha_k - \theta_k) \) defines an iterative canonical map. Here, \( \alpha_k \in [0, 2\pi] \) and \( p_k \in [-p_{\text{max}}, p_{\text{max}}] \), where \( p_{\text{max}} = (2J + R^2 + d^2 + 2Rd \cos \alpha)^{1/2} \). Then, the information on the stability is contained in the trace of the linearized matrix \( D\mathcal{P}_J \). One finds (Merlo, 2004; Benet and Merlo, 2004)

\[
\text{Tr} D\mathcal{P}_J = 2 + \frac{(\Delta \phi)^2(1 - \tan^2 \theta) - 4(1 + \Delta \phi \tan \theta)}{d/R}.
\]

Changes in the stability of the periodic orbits, i.e. bifurcations, occur at \( \text{Tr} D\mathcal{P}_J = \pm 2 \). Equating the r.h.s. of Eq. (4) to 2 turns out to be equivalent to the condition \( dJ_n/d\theta = 0 \) (Benet and Seligman, 2000), i.e., the condition defining the position of the minima and maxima of \( J_n \). The corresponding result for \(-2\) yields a condition related to period-doubling bifurcations. Figure 2b shows the behavior of \( \text{Tr} D\mathcal{P}_J \) in terms of \( \theta \). Note that a simple semi-analytical estimate of the width of the rings is obtained by projecting the curves \( \text{Tr} D\mathcal{P}_J = \pm 2 \) onto the \( X-Y \) plane.

Figure 2b shows connected intervals in \( \theta \), and therefore in \( J \), where the radial collision periodic orbits are strictly stable, i.e. \( |\text{Tr} D\mathcal{P}_J| < 2 \). For \( J \) within these
Fig. 3  Phase space structure of the scattering billiard on a circular orbit near the maximum of the $n = 0$ curve ($R = 1$ and $d = 1/2$). (a) The horseshoe displays an elliptic and a hyperbolic fixed point ($J/(R - d)^2 = 0.29325$). (b) Both fixed points are unstable and the dynamics is dominated by scattering events, although the horseshoe is not yet completely hyperbolic ($J/(R - d)^2 = 0.29218$).

Fig. 4  (a) Stable ring of non-interacting particles of the scattering billiard system on a circular orbit corresponding to the $n = 0$ stable periodic orbit. (b) Zoom of a region of the ring. The black lines are the analytical estimates given by $\text{Tr}D \mathcal{P}_j = \pm 2$.

Intervals, the phase space structure of the scattering system displays one elliptic fixed point, surrounded by typical KAM tori and some chaotic layers (Fig. 3a). This region in phase space is strictly bounded by the stable and unstable invariant manifolds of the companion hyperbolic fixed point. Particles whose initial conditions lie within this region display infinite consecutive collisions and are thus dynamically trapped. Initial conditions that do not belong to any region of trapped motion eventually escape to infinity. Consider the case $n = 0$ for concreteness. After the appearance of the pair of periodic orbits (when $\text{Tr}D \mathcal{P}_j = 2$), by further
reducing the value of $J$, there occurs a value where $\text{Tr} \mathcal{D} \mathcal{J} = -2$. There, the elliptic fixed point becomes inverse hyperbolic, and a period-doubling bifurcation sets in: the region of trapped motion vanishes rapidly after further reducing $J$. In Fig. 3b we illustrate this case, plotting the phase space structure corresponding to an incomplete Smale horseshoe [Rückerl and Jung, 1994]. The dependence of $\text{Tr} \mathcal{D} \mathcal{J} = -2$ upon $d/R$ implies that the actual parameters of the billiard influence the width of the ring. Physically, this means that the parameters related with the motion and the mass ratios of the shepherd moons do indeed influence the width of the ring. Further reducing the value of $J$ yields a hyperbolic Smale horseshoe.

We turn now to the rings (Benet and Seligman, 2000). Consider an arbitrary large number of non-interacting particles. Their initial conditions are completely arbitrary except for the value of their Jacobi integral, which satisfies the condition $|\text{Tr} \mathcal{D} \mathcal{J}| < 2$ for the $n = 0$ curve. Then, the particles may collide with the disk at any value $\phi$ along the circular orbit of the disk. Letting the system evolve, many particles escape after a few collisions; the remaining ones are dynamically trapped. Note that the measure of the latter set is strictly positive, that is, there is a non-vanishing probability of finding such initial conditions. The motion of the particles belonging to this set, the ring particles, may be periodic, quasi-periodic or chaotic. The ring is the pattern formed by these trapped particles in the $X$–$Y$ plane (inertial frame). Figure 4 shows the pattern obtained after a few thousand revolutions of the disk, resulting from a large number of particles whose initial conditions are close to the maximum of the $n = 0$ curve. The plot shows the $(X, Y)$ position of all particles, in an inertial frame, which have not escaped at the time when the “photograph” was taken.

In Fig. 4 we also plot the curves corresponding to $\text{Tr} \mathcal{D} \mathcal{J} = \pm 2$, which serve as estimates of the ring shape. They do not coincide perfectly with the boundaries of the ring, since they are related to the appearance and destruction of the central elliptic fixed point and carry no information on the size of the island around the elliptic points. Yet, the nature of these curves and the scattering dynamics implies a sharp-edged ring. The narrowness of the ring is a consequence of the relatively small region, in phase space, of trapped motion; this also takes into account the (small) variations induced by changing $J$. Furthermore, the ring can be characterized as eccentric, in the sense that there are two points, the periapsis and apoapsis, forming a line that intersects the origin, the line of apsides. Finally, we mention that the same ring may be obtained by an ensemble of particles whose initial conditions, while different, are very localized in a region in the $X$–$Y$ plane. The time evolution in this case not only selects the particles of the ring, but also spreads them over the whole region of trapped motion (in phase space) on a short time scale.

3.2 Rings in the case of elliptic motion

In the case of elliptic motion of the disk, Eq. (2) is explicitly time-dependent. The organizing centers in phase space are now stable tori, which are indeed found at least for small enough values of the eccentricity. Close to these stable tori, the structure of phase space is somewhat similar to that discussed above for circular motion; details are given in Benet et al., (2005). Hence, rings occur just as before. For very small values of the eccentricity, the rings have essentially the same
structure as in the circular case. Increasing $\varepsilon$ slightly rapidly shrinks the width of the ring. Eventually, we reach a value of the eccentricity for which the ring displays gaps, i.e., regions of extremely low density of particles. This is illustrated in Fig. 5a. In this case ($\varepsilon = 0.00167$), the gaps divide the ring into three components, which we call strands. They are a direct consequence of the stable tori that yield the ring, which arise from the quasi-periodic character of $V_1$ in Eq. (1). We observe that particles belonging to a given strand stay in the same strand, i.e., they do not move along other components of the ring.

In Fig. 5b we show a different representation of the whole ring for a specific time (given as a fraction of the disk’s period $T_d$). We plot the azimuthal position $\Phi$ of the ring particles, measured anti-clockwise from the collision point with the disk as origin, and $r' = r - r_{\text{fit}}$, where $r$ is their radial position and $r_{\text{fit}}$ an appropriate fit to the mean value. This figure shows clearly the intricate entanglement of the strands, reminiscent of the beautiful braids of Saturn’s F ring. We emphasize that
the different strands and braids we observe are consequence of the fact that a stable quasi-periodic torus is now the organizing center of the trapped region in phase space. Figures 5c–d display the ring at other fractions of the period of the disk. Together these plots are a representation of the orbital motion of the ring, i.e., the dynamical behavior of the braids.

We emphasize that the strands and braids we have obtained for this toy model, which display a dynamical evolution, are not short-lived structural properties. Thus, stable tori allow for the occurrence of structured narrow rings. Moreover, and as we shall show in the next sections, the ideas described here can be extended to more realistic situations. This is so because of the robustness of relying on considerations in phase space. The above description of the appearance of strands and braids should therefore hold in more realistic situations.

4 Example 2: Rotating billiard with a central gravitational interaction

In the previous section we illustrated, using a billiard system, our approach to the occurrence of narrow rings and how some observed structural properties arise. This example, while showing the robustness of the ideas of the approach, constitutes the simplest scattering system that is able to produce narrow rings, and as such has a pedagogical value. However, it lacks of the gravitational potential with the central planet. In this section we include a central attractive 1/r interaction. As we shall show below, the present case establishes the connection to the gravitational restricted three-body problem, which we consider in Sec. 5.

Consider the billiard system on a circular orbit with the addition of a central attractive gravitational interaction, i.e., \( V_0 = -1/r \) between the (ring) particle and the central planet. In distinction to the pure billiard, where the free motion is rectilinear and uniform (the potential is \( V_0 = 0 \)), the particles now move along solutions of the Kepler two-body problem. Besides such (collisionless) solutions, there are also trajectories which collide with the disk. We focus on the latter, as they model rings close to shepherds; the former would trivially form a two-body keplerian ring, which is circular and wide. The collision solutions are constructed by segments of keplerian trajectories, glued together at the collision points on the disk, where the precise outcome of the collisions is taken into account.

We proceed as before and compute the consecutive-collision periodic orbits in the rotating frame. To this end, we observe that the radial collisions conserve, as before, energy and angular momentum, and occur only at the innermost \( r_- = R - d \) or outermost \( r_+ = R + d \) points of the disk. As mentioned above, in the present case we must also account for the keplerian free flight between the collisions. The present problem is thus equivalent to finding periodic orbits in a central gravitational field which collide consecutively with a moving point singularity on a circular orbit of radius \( r_- \) and \( r_+ \), respectively. This singularity influences the motion of the third particle only through collisions. This is so as long as the trajectory avoids other collisions with the disk; if not the periodic orbit is simply destroyed. We note that this is essentially the same as the circular RTBP with \( \mu = 0 \), as defined by Hénon.\cite{H68}

These statements can be formulated quantitatively as follows. The solutions of the keplerian free flight are precisely conic sections. We consider for concreteness the elliptic motion. Using symmetry arguments, it is easy to show that consecutive
periodic orbits can be obtained from initial conditions (of the moving singularity and the third particle, see Hénon, 1968) on the X-axis of the inertial frame. This axis is parallel to the major axis of the ellipse. The elliptic motion of the third particle, in an inertial frame, is parameterized by

\[
X_p = \sigma_0 a_e (\sigma_2 \cos E - \epsilon_e),
\]

\[
Y_p = \sigma_0 \sigma_1 a_e (1 - \epsilon_e^2) \sin E,
\]

\[
t = a_e^{3/2} (E - \sigma_2 \epsilon_e \sin E).
\]

Here, \(a_e\) and \(\epsilon_e\) are respectively the major semi-axis and eccentricity of the elliptic orbit, and \(E\) is the eccentric anomaly. The parameter \(\sigma_0\) defines if the periapse lies on the positive (1) or negative (−1) X-axes, \(\sigma_1\) determines if the trajectory is direct (1) or retrograde (−1) and \(\sigma_2\) whether the initial condition is at periapse (1) or apoapse (−1). Then, in order to obtain the periodic orbit one needs to match \((X_p, Y_p)\) with \((r_\pm \cos t, r_\pm \sin t)\) and solve for \(a_e\) and \(\epsilon_e\). With \(\sigma = \sigma_0 \sigma_2\), \(\sigma_\eta = \text{sgn}(\sin \eta)\) and \(\rho = \sigma_\eta (1 - \sigma \cos \tau \cos \eta)^{1/2}\), the timing condition reads

\[
F_0(\tau, \eta; \sigma, \sigma_\eta) = \rho [\eta \rho^2 - \sin \eta (\cos \eta - \sigma \cos \tau)] - \frac{\tau \sin^2 \eta}{r_\pm^{3/2}} = 0.
\]

Hénon, (1968) computed first the solutions of (8) for \(r_+ = r_-\) (\(d = 0\)). In Fig. 6a we show the consecutive collision periodic orbits for a disk of radius \(d = 10^{-3}\); in Fig. 6b the case \(d = 0\) is plotted for comparison. At first sight, no difference can be observed. The case \(d = 0\) corresponds precisely to the circular RTBP with mass parameter \(\mu = 0\) (Hénon, 1968). This naive remark has important consequences. We emphasize that we are interested not in the billiard problem with a central gravitational potential, which is only another example, but on the full gravitational problem, with its simpler version, the RTBP. Hitzl and Hénon, (1977) proved that the limit, in the case of vanishing \(\mu\), of the second species periodic orbits of the circular RTBP with small but non-zero \(\mu\) are the critical points of the consecutive collision periodic orbits for \(\mu = 0\) (Fig. 6b). This statement proves the validity of our approach for the gravitational circular RTBP, and
establishes the connection of our naive examples to the gravitational case. Using the arguments of Section 2, this validity can be extended to problems with other interactions as well as more degrees of freedom.

Coming back to the billiard with the central gravitational interaction, we are interested in the regions of phase space where scattering may take place. In this example, scattering is possible for positive keplerian energy. In the $\theta - J$ plane, scattering motion occurs if the Jacobi integral satisfies, in some $\theta$ interval, the inequality $J \geq \max(J_-, J_+)$, with $J_+ = -(2r_0)^{1/2} \sin \theta$ and $J_- = -1/r_0$ ($r_0 = r_\pm$, depending on the value of $\theta$). We notice that for $J > 0$ scattering is possible.

To construct the rings, we proceed as before, requiring that the periodic orbits where scattering is achievable are, in addition, stable. Knowing the periodic orbits it is easy to compute their stability. The same procedure then yields the rings, though some subtleties arise. These are related to the fact that the particle’s energy must be positive for the particle to escape. The only way of changing its energy is through collisions with the disk (which is why non-colliding orbits form trivially a wide circular ring). However, the change in the energy may not be so large as to lead to escape after a single collision, so that repeated collisions are typically required for escape. Since between collisions the particle moves along a Kepler ellipse, and we are interested in a tiny disk, the time between consecutive collisions can be rather long. In a Poincaré surface of section, defined when the particle crosses the $x$ synodic axis (e.g., $x$ vs. $p_x$), such long motions along a Kepler ellipse will be represented as something resembling a torus. The reason to consider such a Poincaré surface of section is inspired by the RTBP. However, these are fake tori, not true tori, since collisions will abruptly change the particle into a different one (if the energy is still negative) or lead to escape. Therefore, long integrations (about $10^5$ periods of the disk or even longer) are required in order to distinguish whether the trajectory is truly trapped, and not to “misguide the eye” with the fake tori described above.
In Fig. 7a we present the ring generated by the stable collision periodic orbit, denoted as $C_{12}$ by Hénon (1968), for a disk of radius $d = 0.0006$. The disk radius was chosen as large as possible but such that the characteristic curve in Fig. 6a displays a local maximum and therefore the saddle–center scenario that yields one stable and one unstable fixed points takes place. As observed in Fig. 7a, the ring displays multiple components through the inner loop, reminiscent of a strand structure, as shown in Fig. 5 for the rotating billiard on an elliptic orbit. The observed structure is inherited directly from the shape of the organizing periodic orbit in the synodic (rotating) frame; there, the organizing periodic orbit is closed after a few full turns of the disk. Figure 7b shows an enlargement of a region of the ring. As expected, the rings are narrow and eccentric, and display sharp edges.

5 Example 3: The circular restricted three-body problem

We now present results of our approach to the occurrence of narrow planetary rings in the circular restricted three-body problem. We concentrate on the case where the mass of the secondary (shepherd) is $10^{-6}$, in units of the mass of the central planet, i.e. $\mu \approx 10^{-6}$. While this value is much larger than the actual masses of the known shepherd satellites, it yields qualitatively the same results with an enormous saving in computing time.

The advantage of considering the rotating billiard disk with a central $1/r$ potential is that we know the precise location of the consecutive collision periodic orbits. Moreover, its finite size and the curvature of the disk enhance the instabilities which finally lead to escape. In the case of the RTBP, even for very small mass parameters, this knowledge has to be obtained numerically, despite the fact that Hénon’s work for the case $\mu = 0$ gives a starting point. The numerics are nevertheless non-trivial, since the orbits of interest are quite close to collisions with the secondary, which not only complicates the numerics but requires a rather high level of accuracy. As we shall see below, we also require rather long integrations.

We shall focus on non–collision orbits generated by the continuation of the $C_{12}$ consecutive collision periodic orbit of Hénon for $\mu = 0$. This illustrates that indeed the rotating billiard with a $1/r$ central attractive interaction provides the link between the naive rotating billiard system and the RTBP. Moreover, this also illustrates the result of Hitzl and Hénon (1977) that the generating periodic orbits for $\mu = 0$ are the second species periodic orbits for finite but small values of $\mu$. We emphasize that, the organizing centers and the trapped orbits we consider, and therefore the ring, do not display collisions with the secondary. We first use Fig. 6b to locate approximately the Jacobi integral interval where the continuation of the $C_{12}$ periodic orbit is stable. From here, numerically, we obtain the interval where the periodic orbit has a stable branch. We find that such a stable orbit exists for $0.400 \leq J \leq 0.453$. Note that these values of the Jacobi integral are much larger than the associated with the zero-velocity curves (Benet et al., 1998). Therefore, scattering dominates the dynamics unless the proposed mechanism (existence of stable periodic orbits) creates regions in phase space of dynamically trapped motion. That is, escaping trajectories largely fill phase space for these values of $J$, except when trapping occurs near a stable periodic orbit. Below, we shall focus in the dynamics close to, and including, one specific region of bounded motion. For the RTBP with very small values of $\mu$ the dynamics consists
of almost-elliptic keplerian motion, that takes place during long times, followed by an eventual close approach that finally leads to escape (Benet et al., 1998). Therefore, the numerical simulations must distinguish the fake tori described in the previous section, which again requires long and accurate integrations. In our simulations we have checked for about $10^6$ periods of the secondary that no abrupt tori exchange takes place.

Once the stability region (bounded motion) around the periodic orbit is located and also the approximate interval where it is stable, the construction of the ring is straightforward, following the ideas we have described above. Figures 8 show the whole ring and an enlargement of a region. All qualitative features described in the previous sections are observed once again: sharp edges and a narrow eccentric ring with many components. Again, these many components are inherited from the shape of the organizing periodic orbit in the synodic frame, which closes after a few full turns of the disk. Another important observation is that there is a variable width of the ring, with respect to the position closest to the shepherd. This property is observed in real narrow planetary rings (Esposito, 2002). The explanation within our framework is simple: the region of trapped motion is very small close to the shepherd, for any value of $J$; away from it, the two-body Kepler interaction allows for longer excursions due to the different values of $J$ involved. The projection onto the $X$–$Y$ plane is wider as we move away from the neighborhood of the shepherd.

We note that, for the interval of the Jacobi constant considered, other regions of dynamically trapped motion exist in phase space. In particular, there is a comparatively large stable region associated with a period-one fixed point, and another one related to a period-two fixed point, both belonging to phase space regions where scattering dominates the dynamics. From our construction, these regions would also yield rings. Yet, these stable regions are extremely robust, in the sense that they are present for an extremely wide interval of $J$. Focusing on the period-one stable region, it essentially sweeps out every value of $x$ in the synodic frame.
Therefore, except for a tiny gap which is opened by the shepherd, it yields a wide circular ring; this ring is of no interest in the context of the present work. It is the result of the dominance of the central $1/r$ keplerian interaction. This result implies that the RTBP is not a realistic model, at least to describe the Uranus rings or the narrow rings of Saturn. We believe that a realistic model should not only consider the planet, a shepherd and the ring particle as we have done, but at least another body corresponding to one of the major moons of the planet. For Saturn’s F ring as well as for the ε ring of Uranus, perhaps a fifth body is also required to fully account for the two shepherd moons.

6 Conclusions and outlook

In this paper we have presented a general and self-consistent approach for the occurrence of shepherded narrow planetary rings, and some consequences for their structural properties. Our approach is based on the local structure of phase space around stable (periodic or quasi-periodic) solutions in regions where scattering dominates the dynamics. The corresponding structure in phase space allows for a set of positive measure to exhibit dynamical trapped motion, i.e. a non-zero probability to find such trapped motion; particles with initial conditions outside this small region escape to infinity along scattering trajectories. The intrinsic rotation provides a mechanism for dynamical confinement around the central planet. Here we have considered that such rotation is induced by the perturbations of a shepherd moon, though our approach is more general since the intrinsic rotation can also be associated with interactions with other, perhaps more massive, moons, or with the combined evolution of the interacting many-body problem. A possible consequence is the explanation of the non-shepherded narrow rings of Uranus: the many-body problem is such that it defines several regions of trapped motion, one for each specific narrow ring. An ensemble of (non-interacting) particles will form a well-defined ring. Focusing in regions where scattering dominates the dynamics allows to sharply distinguish between trapped motion related to the occurrence of the ring, and open motion manifested as regions with essentially no material due to the escape of the particles.

Within our approach, the rings will be sharp-edged, as a consequence of the scattering dynamics. The size of the stability region in the extended phase space determines how narrow the rings are. For comparatively small regions of trapped motion, the narrow ring will typically be eccentric, since the particles will be close to the organizing center, which in general displays such eccentricity. Semi-analytical estimates of the edges of the ring can be obtained by analyzing the bifurcation conditions that define the region of bounded motion in the extended phase space (within the sea of scattering trajectories) around the organizing center, at least for models with two-degrees of freedom. Typically, the bifurcation scenario that yields the regions of dynamically trapped motion corresponds to the saddle–center bifurcation, and by consequence of its fast development, the rings are narrow. We have illustrated our considerations with three distinct examples, aiming to show explicitly the robustness of the approach and its extension to and consequences for a case with many degrees of freedom, and have proved that our considerations hold also in the gravitational case as illustrated by the RTBP without physical collisions. More realistic interactions, e.g. including the oblateness
Strands and braids in narrow rings of the planetary, can be easily incorporated within our theory. While this does not yet explicitly prove the connection to real narrow planetary rings, which are complicated collisional many-body systems with gravitational and other interactions, such an extension seems to us feasible, since our ideas are based on general phase space considerations, and on results and theorems which apply to Hamiltonian systems with more than two degrees of freedom.

We emphasize that our construction of the rings is self-consistent, in the sense that we do not introduce any “cell” or periodic boundary conditions on a segment of the ring to make the problem amenable in any sense. Therefore, radial and azimuthal structures do appear. Moreover, the theoretical framework developed is not restricted to short time scales, at least as long as the model considered is believed to be applicable for longer time scales. It is in this scope that our results may be relevant and contribute to the understanding of real planetary narrow rings: the mechanism of confinement is very efficient. We have explicitly showed which phase space structures allow for the occurrence of such narrow rings. Moreover, we have showed how rings develop further radial and azimuthal structure: rings may display variable width and many components which are entangled in a way that, at least qualitatively, resembles the observed braids of Saturn’s F rings. Within our approach, the appearance of different strands and braids is not related to the ring interparticle collisions, as proposed (Hanninen, 1993), nor to the existence of other moonlets (Lissauer and Peale, 1986); they are related to the quasi-periodic character of the perturbing force of the rotating interaction, as induced by the shepherd’s explicitly eccentric motion. This non-circular motion makes the Hamiltonian system effectively have many degrees of freedom, which complicates its analysis, but allows for other bifurcations as well as resonances. As far as we are aware, this is a new explanation for the occurrence of strands and braids, although it is somewhat in agreement with the idea of associating periodic clumps with the eccentricity of shepherd moons (Showalter and Burns, 1982). The strands and braids that we obtain display a periodic dynamical evolution, where the relevant time scale is related to the time for repeated interactions, or close approaches, with the shepherd. In our opinion, this points out the importance of the quasi-periodic perturbation, i.e. the non-circular motion of the intrinsic potential. This observation is in agreement with the results of Giuliani Winter et al., (2000), although these authors focus on certain perturbations induced by the repeated interactions of a group of particles with Prometheus within a cell of the ring. Whenever the organizing periodic orbit closes after more than one full turn of the secondary, the ring displays non-independent multiple components, reflecting the shape of the organizing periodic or quasi-periodic orbit in the rotating frame. Despite the fact that we have considered only planar motion, and that this is again only a qualitative comparison, the generalization of these “higher periodic” orbits to a non-planar case serves as an interpretation of recent observational findings (Charnoz et al., 2005). Moreover, preliminary results (Benet and Merlo in preparation) show that our theory can also yield other observed properties of narrow rings, such as arcs qualitatively similar to those of Neptune (Namouni and Porco, 2002).

Our ultimate goal is to provide quantitative comparisons with observed planetary rings. The results presented here are only qualitative due to the fact that, so far, the examples are analogous to restricted three-body like problems. Actually, the
most realistic example we considered is the circular (gravitational) RTBP. As we
have discussed, non-collision phase-space structures of the RTBP with extremely
small values of the mass parameter, which originate from the overwhelming dom-
inance of the central planetary attraction, make such simple modeling definitely
not realistic, since they yield two-body keplerian wide circular rings. We have
proposed to consider the interaction of other planetary satellites and, perhaps too,
other shepherds as a more realistic model. The shepherds can be included in
a first step as zero-mass moonlets which break certain solutions of the many-body
problem (planet including oblateness, other moons and a particle of the ring). Following
the ideas of [Hitzi and Henon, (1977)] for the RTBP with $\mu = 0$, periodic
or quasi-periodic collision solutions with respect to the shepherds can be generat-
ing orbits for the case of small masses of the shepherds. Considering the stability
regions in the extended phase space when the shepherds have finite mass allows
a straightforward construction of the associated rings. Work illustrating a many-
body problem is in progress.

Our considerations may also be used to extend the applicability of current
models to longer time scales, once it has been showed that the ring particles con-
sidered do belong to some regions of dynamically bounded motion. Conversely,
they may allow to qualify certain rings or some of their structural properties as
temporary phenomena, if a realistic model does not have dynamically trapped
particles. An interesting question is whether chaotic motion of the shepherds may
still yield long-lived rings within our framework, that is, whether they allow for
bounded motion, and what the consequences are for their structure. This is related
to recent investigations where the motion of Prometheus and Pandora seems to be
chaotic [Goldreich and Rappaport, [2003a,b], Renner et al., [2005].

The strongest assumption within our approach is the non-interacting character
of the ring particles. This essentially allows us to separate the many-body prob-
lem into a collection of independent one-particle problems. This assumption must
be taken as a first approximation in the series expansion of the full many-body
Hamiltonian in terms of the mass of the ring particles. Here, the particle–particle
interactions among the ring particles will be orders of magnitude smaller than
other interactions, except maybe when they collide. It is precisely the inelastic
collisions among the ring particles what we neglect. A possibility to include these
collisions in a simple way is with a random-walk model, taking place in all the
coordinates of the full phase space of the ring particles. Such a model could also
be used to describe dissipative effects like drag, depending on the particular im-
plementation. This random-walk approach would certainly introduce an erosion
mechanism of the ring, i.e., the possibility of escape from the ring and a typical
life-time; yet it will preserve the properties that we discuss, such as the sharp-
edges, the eccentricity and the many components of the ring. This is because,
even when collisions are included, once a particle is outside the region of trapped
motion it escapes along a scattering trajectory. Work in this direction is also in
progress.

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References

Arnold, V.: 1964, “Instability of dynamical systems with several degrees of freedom”, Sov. Math. Dokl. 5, 581–585.
Arnold, V., Kozlov, V., and Neishtadt, A.: 1988, Dynamical Systems III, Vol. 3 of Encyclopedia Math. Sci., Springer, Berlin.
Benet, L., and Merlo, O.: “Phase-space structure for narrow planetary rings”, Regular and Chaotic Dynamics 9, 373–384, arXiv:nlin.CD/0410028.
Benet, L., and Merlo, O., in preparation.
Benet, L., and Seligman, T. H.: 2000, “Generic occurrence of rings in rotating scattering systems”, Phys. Lett. A 273, 331–337, arXiv:nlin.CD/0001018.
Benet, L., Seligman, T. H., and Trautmann, D.: 1998, “Chaotic Scattering in the Restricted Three-Body Problem II: Small Mass Parameters”, Celest. Mech. Dynam. Astron. 71, 167–189.
Benet, L., Broch, J., Merlo, O., and Seligman, T. H.: 2005, “Symmetry breaking: A heuristic approach to chaotic scattering in many dimensions”, Phys. Rev. E 71, 036225.
Borderies, N., Goldreich, P., and Tremaine, S. D.: 1983, “The dynamics of elliptical rings”, Astron. J. 88, 1560–1568.
Charnoz, S., et al.: 2005, “Cassini discovers a kinematic spiral ring around Saturn”, Science 310, 1300–1304.

Elliot, J. L., and Nicholson, P. D.: 1984, “The rings of Uranus”, In R. Greenberg and A. Brahic (eds.), Planetary Rings, University of Arizona Press, 25–72.
Elliot, J. L., Dunham, E. W., and Millis, R. L.: 1977, “Discovering the rings of Uranus”, Sky and Telescope 53, 412–416, 430.
Esposito, L. W.: 2002, “Planetary rings”, Rep. Prog. Phys. 65, 1741–1783.
Giuliatti Winter, S. M., Murray, C. D., and Gordon, M.: 2000, “Perturbations of Saturn’s F–ring strands at their closest approach to Prometheus”, Planetary and Space Science 48, 817–827.
Goldreich, P., and Rappaport, N.: 2003a, “Chaotic motions of Prometheus and Pandora”, Icarus 162, 391–399.
Goldreich, P., and Rappaport, N.: 2003b, “Origin of chaos in the Prometheus–Pandora system”, Icarus 166, 320–327, arXiv:astro-ph/0307259.
Goldreich, P. and Tremaine, S. D.: 1979, “Towards a theory for the Uranian rings”, Nature 277, 97–99.

Hénon, M.: 1968, “Sur les Orbits Interplanétaires qui Rencontrent Deux Fois la Terre”, Bull. Astron. (serie 3) 3, 337-402.
Hénon, M.: 1997, Generating Families in the Restricted Three-Body Problem, Springer-Verlag, Berlin.
Hénon, M., and Hénon, M.: 1977, “Critical generating orbits for second species periodic solutions of the restricted problem”, Celest. Mech. 15, 421–452.

Jorba, A., and Villanueva, J.: 1997a, “On the Normal Behaviour of Partially Elliptic Lower Dimensional Tori of Hamiltonian Systems”, Nonlinearity 10, 783–822.
Jorba, A., and Villanueva, J.: 1997b, “On the persistence of lower dimensional invariant tori under quasi-periodic perturbations”, Journal of Nonlinear Science 7, 427–473.

Lewis, M. C., Stewart, G. R.: 2000, “Collisional dynamics of perturbed planetary rings. 1”, Astron. J. 120, 3295–3310.
Lissauer, J. J., and Peale, S. J.: 1986, “The production of "braids" in Saturn’s F ring”, Icarus 67, 358–374.

Meyer, N., et al.: 1995, “Chaotic scattering off a rotating target”, J. Phys. A: Math. Gen. 28, 2529–2544.

Murray, C. D., and Dermott, S. F.: 1999, “Solar System Dynamics”, Cambridge University Press, Cambridge.

Murray, C. D., and Thompson, R. P.: 1990, “Orbits of shepherd satellites deduced from the structure of the rings of Uranus”, Nature 348, 499–502.
Murray, C. D., Gordon, M. K., and Giuliatti Winter, S. M.: 1997, “Unraveling the Strands of Saturn’s F Ring”, Icarus 129, 304–316.
Murray, C. D., et al.: 2005, “How Prometheus creates structure in Saturn’s F ring”, Nature 437, 1326–1329.
Namouni, F., and Porco, C.: 2002, “The confinement of Neptune’s ring arcs by the moon Galatea”, Nature 417, 45–47.
Newton, R. G.: 1966, Scattering theory of waves and particles, McGraw–Hill, New York.
Renner, S., Sicardy, B., and French, R. G.: 2005, “Prometheus and Pandora: masses and orbital positions during the Cassini tour”, Icarus 174, 220–240.
Rückerl, B. and Jung, C.: 1994, “Scaling properties of a scattering system with an incomplete horseshoe”, J. Phys. A: Math. Gen. 27, 55–77.
Showalter, M. R., and Burns, J. A.: 1982, “A numerical study of Saturn’s F ring”, Icarus 52, 526–544.
Sicardy, B.: 2005, “Dynamics and Composition of Rings”, Space Science Reviews 116, 457–470.
Skokos, Ch.: 2001, “On the stability of periodic orbits of high dimensional autonomous Hamiltonian systems”, Physica D 159, 155–179.
Smith, B. A., et al.: 1981, “Encounter with Saturn —Voyager 1 imaging science results”, Science 212, 163–191.
Smith, B. A., et al.: 1981, “A new look at the Saturn system —Voyager 2 images”, Science 215, 504–537.
Szebehely, V.: 1967, Theory of Orbits, Academic Press, New York.
Ural’skaya, V. S.: 2003, “Dynamics of Planetary Satellites in the Solar System”, Solar System Research, 37, 337–365.
Uzer, T., Jaffé, C., Palacín, J., Yanguas, P., and Wiggins, S.: 2002, “The geometry of reaction dynamics”, Nonlinearity 15, 957–992.
Waalkens, H., Burbanks, A., and Wiggins, S.: 2004, “A computational procedure to detect a new type of high-dimensional chaotic saddle and its application to the 3D Hill’s problem”, J. Phys. A: Math. Gen. 37, L257–L265, [arXiv:nlin.CD/0403059]
Wiggins, S., Wiesenfeld, L., Jaffé, C., and Uzer, T.: 2001, “Impenetrable Barriers in Phase-Space”, Phys. Rev. Lett. 86 5478–5481.
Yakubovich, V. A., Starzhinskii, V. M.: 1975, Linear Differential Equations with Periodic Coefficients, Vol. 1, Wiley, New York.