ON COUNTING CERTAIN ABELIAN VARIETIES OVER FINITE FIELDS

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Abstract. This paper contains two parts toward studying abelian varieties from the classification point of view. In a series of papers [30, 29, 31, 32], the current authors and T.-C. Yang obtain explicit formulas for the numbers of superspecial abelian surfaces over finite fields. In this paper, we give an explicit formula for the size of the isogeny class of simple abelian surfaces with real Weil number \( \sqrt{q} \). This establishes a key step that one may extend our previous explicit calculations of superspecial abelian surfaces to those of supersingular abelian surfaces. The second part is to introduce the notion of genera and ideal complexes of abelian varieties with additional structures in a general setting. The purpose is to generalize the results of [38] on abelian varieties with additional structures to similitude classes, which establishes more results on the connection between geometrically defined and arithmetically defined masses for further investigation.

1. Introduction

Throughout this paper, \( p \) denotes a prime number and \( q \) is a power of \( p \). Recall that an abelian variety over a field \( k \) of characteristic \( p \) is said to be supersingular if it is isogenous to a product of supersingular elliptic curves over an algebraic closure \( \overline{k} \) of \( k \); it is said to be superspecial if it is isomorphic to a product of supersingular elliptic curves over \( \overline{k} \). It is known that any supersingular abelian variety is isogenous to a superspecial abelian variety. Thus, studying superspecial abelian varieties is a vital step for studying supersingular abelian varieties.

By the work of Deuring [8] and the new input of Waterhouse [28] using the elegant theory of Honda and Tate [26], we have an explicit formula for the size of each isogeny class of elliptic curves over a finite field \( \mathbb{F}_q \) (also see [24]). The main part of these formulas are discussing various cases of supersingular elliptic curves. It is desire to have similar results for abelian surfaces over \( \mathbb{F}_q \). Abelian surfaces over \( \mathbb{F}_q \) are divided into ordinary, almost ordinary and supersingular ones. As in the case of elliptic curves, the classifications of ordinary and almost ordinary abelian surfaces are simpler; and the simple classes have been studied by Waterhouse [28].

In a series of papers [30, 29, 31, 32], the current authors and T.-C. Yang obtain an explicit formula for the number of \( \mathbb{F}_q \)-isomorphism classes of superspecial abelian surfaces in each isogeny class over \( \mathbb{F}_q \). Our next step is to compute explicitly that for each isogeny of supersingular abelian surfaces. These are of course special cases of the general question of how to compute explicitly the size of an \( \mathbb{F}_q \)-isogeny class.

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Suppose that $q = p^a$ be a power of a prime number $p$. For any Weil $q$-number $\pi$, let $A_\pi$ be the set of $\mathbb{F}_q$-isomorphism classes of abelian varieties in the $\mathbb{F}_q$-isogeny class associated to $\pi$ by the Honda-Tate theorem. Let $A_\pi$ be a member in $A_\pi$, and denote by $E_\pi := \text{End}^0(A_\pi) = \text{End}_{\mathbb{F}_q}(A_\pi) \otimes \mathbb{Q}$ its endomorphism algebra over $\mathbb{F}_q$, which depends only on the Weil number $\pi$. In Section 2, we give a description for $A_\pi$ in terms of double coset spaces; see Theorem 2.1. This can be viewed as a special case of the simple mass formula [38, Theorem 2.2], and also a generalization of some main results of Waterhouse [28]. The proof we give here is more conceptual than that in [38] as for finite ground fields we have convenient tools of Tate modules and Dieudonné modules. This method already appears in [17] where Kottwitz expresses the size of each isogeny class in $A_g(\mathbb{F}_q)$ in terms of orbital integrals. The main difference is that the description given here is in terms of a sum of class numbers, for which one may further compute their values using computer for each specific case or for some special cases. When the center $F = \mathbb{Q}(\pi)$ is a CM field, we show that $|A_\pi|$ is the sum of certain explicit ray class numbers of $F$ (see Proposition 2.2).

For the other case where $F$ is totally real, we discuss the classification of “genera” in details, and one can apply the generalized Eichler trace formula in [29] to compute the class numbers of these genera. For the reader’s convenience, we describe the extended trace formula in Section 3.

Our first main result is Theorem 4.4, which gives an explicit formula for $|A_\pi|$, where $\pi = \pm \sqrt{q}$ with odd exponent $a$. Note that the case where the exponent $a$ is even is a classical result of Deuring and Eichler. So Theorem 4.4 completes the explicit calculation of $|A_\pi|$ when $F$ is totally real. The main tool is the extended version of Eichler’s trace formula we just mention. We use this to calculate the number of superspecial members in $A_\pi$ first. For computing the non-superspecial members we use the Morel-Bailly family [14, Section 1], which may be also viewed as a special case of minimal isogenies introduced in [15, Lemma 1.8]. In the course of our computation, the Drinfeld period domain of rank two over $\mathbb{F}_q$ shows up and plays an interesting role. The method by minimal isogenies paves a way to reduce the computation of non-superspecial part to that of superspecial ones, and we have already established explicit formulas for the latter. Some of our arguments in the proof applies to other isogeny classes of supersingular abelian surfaces as well.

The second part of this paper is to discuss analogies between abelian varieties and lattices in a general framework. Based on this idea of analogy with lattices, we introduce the notion of genera and ideal complexes of abelian varieties with additional structures in a general setting. As a result, we extend some results of [38] to similitude classes, and establish more connections between algebraically defined and geometrically defined masses. Though our general result is still rather abstract, more explicit results may be established under this framework. As an example, the second author gives an explicit formula for the number of the superspecial locus in the good reduction modulo $p$ of a general type C Shimura varieties; see [37]. At the end of this paper we give one example as the application of the main result (Theorem 5.8).
In this section we discuss certain structures that may be used in the problem of
(more) explicit computation of isomorphism classes in an isogeny class of abelian
varieties over finite fields. As the problem of determining explicitly the size of the
number of isomorphism classes is quite difficult at this moment, we first consider
simple isogeny classes for simplicity. Theorem 2.1 may be viewed as a common
generalization of main results of Waterhouse (see the first part of Theorem 5.1 and
Theorems 6.1 and 7.2 of [28]). The method already appears in [17] where Kottwitz
expresses the size of each isogeny class in \( A_q(\mathbb{F}_q) \) in terms of orbital integrals.
The main difference is that the description given here is in terms of a sum of class
numbers, for which one may further compute the values using computer for each
specific case. Based on the similar description, Lipnowski and Tsimerman [19,
Section 3] further give a nice bound for the size of any isogeny class over \( \mathbb{F}_p \)
of arbitrary dimension \( g \).

2.1. Let \( k \) be a finite field of cardinality \( q \), where \( q = p^e \) is a power of a prime
number \( p \). Let \( \pi \) be a Weil \( q \)-number. By the Honda-Tate theory [11] [26], there is
a \( k \)-simple abelian variety \( A_\pi \) over \( k \), uniquely determined up to \( k \)-isogeny, so
that the Frobenius endomorphism of \( A_\pi \) over \( k \) is conjugate to \( \pi \). Let \( F := \mathbb{Q}(\pi) \)
and \( \mathcal{O}_F \) be the ring of integers. The field \( F \) is either a CM field or a totally real field,
because it is stable under a positive involution (Rosati involution). Let \( A_\pi \) be the
set of \( k \)-isomorphism classes of abelian varieties in the \( k \)-isogeny class of \( A_\pi \). In
this section we present a method toward computing the cardinality of \( A_\pi \).

Let \( E_\pi \) be the endomorphism algebra of \( A_\pi \) over \( k \). It is a central division algebra
over \( F \). The local invariants of \( E_\pi \) are given by [25]:

\[
\text{inv}_v(E_\pi) = \begin{cases} \frac{1}{2} & \text{if } v \text{ is real}, \\ \frac{\nu(\pi)}{\nu(q)} [F_v : \mathbb{Q}_p] & \text{if } v \mid p, \\ 0 & \text{otherwise}. \end{cases}
\]

Let \( [E_\pi : F] = e^2 \). Let \( P(t) \) be the characteristic polynomial of \( \pi \) on \( A_\pi \). By
definition this is the characteristic polynomial of the Frobenius endomorphism \( \pi \)
on the Tate module \( T_\ell(A_\pi) \) for any prime \( \ell \neq p \), and one has \( P(t) \in \mathbb{Z}[t] \). Let \( m(t) \)
be the minimal polynomial of \( \pi \) on \( A_\pi \); this is the same as that of \( \pi \) in \( E_\pi \),
over \( \mathbb{Q} \) and hence it is a irreducible polynomial in \( \mathbb{Z}[t] \). We have \( P(t) = m(t)^e \). Let \( R := \mathbb{Z}[\pi] = \mathbb{Z}[t]/(m(t)) \subset F \), and for any prime \( \ell \) (including \( p \)), write \( R_\ell := R \otimes \mathbb{Z}_\ell \).

For any prime \( \ell \neq p \), let \( T_\ell(A_\pi) \) denote the Tate module of \( A_\pi \) and put \( V_\ell(A_\pi) := T_\ell(A_\pi) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \). The Galois invariant \( \mathbb{Z}_\ell \)-lattices in \( V_\ell(A_\pi) \) are nothing but \( R_\ell \)-
lattices. Note that \( V_\ell(A_\pi) \) is a free \( F_\ell \)-module of rank \( e \), where \( F_\ell := F \otimes \mathbb{Q} \mathbb{Q}_\ell \).

Define

\[
\mathfrak{x}_{\pi,\ell} := \{ R_\ell \text{-lattices in } V_\ell(A_\pi) \}, \quad \text{and } \mathfrak{x}_{\pi,1} := \mathfrak{x}_{\pi,\ell}/\sim,
\]

where for any two \( R_\ell \)-lattices \( L_1 \) and \( L_2 \) in \( \mathfrak{x}_{\pi,\ell} \) we write \( L_1 \simeq L_2 \) if they are
isomorphic as \( R_\ell \)-modules. Since the order \( R_\ell \) is maximal for almost all primes \( \ell \),
the set \( \mathfrak{x}_{\pi,\ell} \) is a singleton for almost all primes \( \ell \).

At the prime \( p \), let \( M(A_\pi) \) denote the (covariant) Dieudonné module of \( A_\pi \) and
let \( N(A_\pi) := M(A_\pi) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \) be the associated (\( F \)-)isocrystal. Define

\[
\mathfrak{x}_{\pi,p} := \{ \text{Dieudonné submodules in } N(A_\pi) \text{ of full rank} \}, \quad \text{and } \mathfrak{x}_{\pi,1} := \mathfrak{x}_{\pi,p}/\sim,
\]
Define a compact subgroup $U$ admits a natural left action on the set $X$ where for any two Dieudonné submodules $M_1$ and $M_2$ in $\tilde{X}_{\pi,p}$ we write $M_1 \simeq M_2$ if they are isomorphic as Dieudonné modules. Taking the product over all primes, we obtain a finite set $X := \tilde{X}_{\pi,p} \times \prod_{\ell \neq p} \tilde{X}_{\pi,\ell}$. The association to each abelian variety $A$ over $k$ its Dieudonné module and all Tate modules defines a map $\Phi : A_\pi \to X$.

Recall that any quasi-isogeny $\varphi : A_1 \to A_2$ of abelian varieties over $k$ is an element $\varphi \in \text{Hom}_k(A_1, A_2) \otimes \mathbb{Q}$ such that $N\varphi$ is an isogeny for some integer $N$. Let us put

$$\tilde{X}_{\pi} := \tilde{X}_{\pi,p} \times \prod_{\ell \neq p} \tilde{X}_{\pi,\ell} = \{ (M, (T_{\ell})_{\ell \neq p}) \mid T_{\ell} = T_{\ell}(A_\pi), \forall \ell \},$$

and

$$\tilde{A}_\pi := \{ \text{quasi-isogenies } \varphi : A \to A_\pi \text{ over } k \}.$$

Two quasi-isogenies $\varphi_1 : A_1 \to A_\pi$ and $\varphi_2 : A_2 \to A_\pi$ over $k$ are regarded as the same element in $A_\pi$ if there exists a $k$-isomorphism $\alpha : A_1 \to A_2$ such that $\varphi_2 \circ \alpha = \varphi_1$. Define the map $\Phi : A_\pi \to \tilde{X}_{\pi}$ by

$$\tilde{\Phi}(\varphi : A \to A_\pi) := (\varphi_*(M(A)), \varphi_*(T_{\ell}(A))_{\ell \neq p}),$$

which is a bijection due to a theorem of Tate [25]. Then we have a natural commutative diagram:

$$\begin{array}{ccc}
\tilde{A}_\pi & \xrightarrow{\sim} & \tilde{X}_{\pi} \\
\downarrow \text{pr}_A & & \downarrow \text{pr}_X \\
A_\pi & \xrightarrow{\Phi} & X_{\pi},
\end{array}$$

(2.1)

where the vertical maps are natural surjective maps. It follows that the map $\Phi : A_\pi \to X_{\pi}$ is also surjective. We consider the fibers of this map.

Let $G_\pi$ be the algebraic group over $\mathbb{Q}$ associated to the multiplicative group $E^*_\pi$. By Tate’s homomorphism theorem on abelian varieties, one has canonical isomorphisms

$$(2.2) \quad G_\pi(\mathbb{Q}_\ell) = \text{Aut}_{E_\ell}(V_\ell(A_\pi)), \quad \text{and} \quad G_\pi(\mathbb{Q}_p) = \text{Aut}_{\text{DM}}(N(A_\pi)).$$

Let $\mathbb{A}_f$ be the finite adele ring of $\mathbb{Q}$, and $X = ([M], (T_{\ell})_{\ell \neq p})$ an element of $\tilde{X}_{\pi}$. Define a compact subgroup $U_X \subset G_\pi(\mathbb{A}_f)$ by

$$U_X := \text{Aut}_{\text{DM}}(M) \times \prod_{\ell \neq p} \text{Aut}_{\text{R}_\ell}(T_{\ell});$$

this is uniquely determined by $X$ up to conjugation by $G_\pi(\mathbb{A}_f)$. The group $G_\pi(\mathbb{A}_f)$ admits a natural left action on the set $\tilde{X}_{\pi}$ of Dieudonné and Tate modules. It also acts on $A_\pi$ from the left through the isomorphisms $G_{\varphi}$ (cf. Lemma 5.2) so that the map $\Phi$ is $G_\pi(\mathbb{A}_f)$-equivariant. Denote by $A_{\pi,X}$ and $\tilde{A}_{\pi,X}$ the fibers over $X$ in $A_\pi$ and $\tilde{A}_\pi$, respectively. The set $\tilde{A}_{\pi,X}$ is a single $G_\pi(\mathbb{A}_f)$-orbit and it is isomorphic to $G_\pi(\mathbb{A}_f)/U_X$ after choosing a base point. The projection map $\text{pr}_A$ is simply modulo the left action of $G_\pi(\mathbb{Q})$. Therefore, we obtain a bijection

$$A_{\pi,X} \xrightarrow{\sim} G_\pi(\mathbb{Q})/G_\pi(\mathbb{A}_f)/U_X.$$

Running over all elements $X$ in $X_{\pi}$, we obtain the following result.
Theorem 2.1. There is a bijection
\[ A_\pi \xrightarrow{\sim} \prod_{X \in \mathcal{X}_\pi} G_\pi(\mathbb{Q}) \backslash G_\pi(\mathbb{A}_f)/U_X. \]

For each \( X = ([M], [T_\ell]_{\ell \neq p}) \in \mathcal{X}_\pi \), there is an \( \mathbb{Z} \)-order \( O_X \) in \( E_\pi \) such that \( O_X \otimes \mathbb{Z}_p \cong \text{End}_{DM}(M) \) and \( O_X \otimes \mathbb{Z}_\ell \cong \text{End}_{R_\ell}(T_\ell) \) for all \( \ell \neq p \). We may choose the orders \( O_X \) so that they are contained in a common maximal order \( O_{\text{max}} \), because any two maximal orders are locally conjugate. Let \( \text{Cl}(O_X) \) denote the set of isomorphism classes of locally free right \( O_X \)-ideals in \( E_\pi \), and \( h(O_X) := |\text{Cl}(O_X)| \) the class number of \( O_X \). We have \( \text{Cl}(O_X) \cong G_\pi(\mathbb{Q}) \backslash G_\pi(\mathbb{A}_f)/U_X \). Theorem 2.1 then gives
\[ |A_\pi| = \sum_{X \in \mathcal{X}_\pi} h(O_X). \]

Thus, to compute \( |A_\pi| \) one needs to
(i) Classify members in \( \mathcal{X}_\pi \);
(ii) Compute the order \( O_X \) for each \( X \in \mathcal{X}_\pi \). This is again a local computation;
(iii) Compute the class number \( h(O_X) \) for each \( X \in \mathcal{X}_\pi \).

The first step is the most complicated part. For \( \mathcal{X}_{\pi, \ell} \), this is to classify \( R_\ell \)-lattices in \( F_\ell^\times \). The most generality of this problem is very difficult. We refer to the fundamental paper of Dade, Taussky and Zassenhaus [6] and subsequent work for detailed studies. For \( \mathcal{X}_{\pi, p} \), one can identify \( \mathcal{X}_{\pi, p} \) with the set of \( \mathbb{F}_p \)-points in a Rapoport-Zink space, which could be used to estimate the size of \( \mathcal{X}_{\pi, p} \). Once the first step is done, the second step is more or less straightforward, because one can compute \( O_X \) by Tate’s theorem for homomorphisms of abelian varieties over finite fields. The calculation of \( h(O_X) \) differs whether or not the endomorphism algebra \( E_\pi \) satisfies the Eichler condition. For our case, this is the case if and only if \( F \) is a CM field. If the Eichler condition holds for \( E_\pi \), then the computation of \( h(O_X) \) is much simpler by a result of Jacobinski [12] Theorem 2.2; see Section 2.2 for a simplified argument using Galois cohomology.

Recall that a central simple algebra \( \mathfrak{A} \) over a number field \( K \) satisfies the Eichler condition (respect to the Archimedean places) if the group of reduced norm one \( (\mathfrak{A} \otimes \mathbb{R})^\times_1 \) is not compact. If \( \mathfrak{A} \) does not satisfy the Eichler condition, then \( K \) must be totally real and \( \mathfrak{A} \) is a totally definite quaternion \( K \)-algebra. In this case, the standard tool for computing \( h(O_X) \) is Eichler’s trace formula. Note that the order \( O_X \) appearing in the content of abelian varieties may not be an \( O_F \)-order. Thus, the Eichler trace formula developed in [15] for an arbitrary \( O_F \)-order is not sufficiently general for computing \( h(O_X) \). In [23] we generalize the Eichler trace formula for an arbitrary \( \mathbb{Z} \)-order in any totally definite quaternion over a totally real field. For the reader’s convenience, we describe this extended formula in Section 3.

2.2. Assume that the center \( F \) of \( E_\pi \) has no real place, i.e. \( \pi \) is not \( \pm \sqrt{7} \). Let \( \text{Nr} : G_\pi \to \text{Res}_{F/\mathbb{Q}} \mathbb{G}_m \) be the reduced norm map with kernel \( G_{\pi, 1} \), and \( \bar{O}_F \) the profinite completion of \( O_F \). Define
\[ N(\pi) := \sum_{X \in \mathcal{X}_\pi} n_X, \quad \text{where} \quad n_X := [\bar{O}_F^\times : O_F^\times \text{Nr}(U_X)]. \]

Proposition 2.2. Assume that the field \( F = \mathbb{Q}(\pi) \) has no real place. Then \( |A_\pi| = N(\pi)h(F) \), where \( h(F) \) denotes the class number of the number field \( F \).
Proof. Since the group $G_{\pi,1}$ is semi-simple and simply-connected, we have $H^1(\mathbb{Q}_p, G_{\pi,1}) = 1$ for all primes $p$ by Kneser’s Theorem [20, Theorem 6.4, p. 284]. It then follows that $\text{Nr}(G_{\pi}(\mathbb{A}_f)) = \mathbb{A}_{F,f}^\times$. On the other hand, since $F$ has no real place, the Lie group $G_{\pi,1}(\mathbb{R})$ is not compact and strong approximation holds for $G_{\pi,1}$. It then follows (see [41, Lemma 14] for the argument) that the reduced norm map

$$\text{Nr} : G_{\pi}(\mathbb{Q}) \backslash G_{\pi}(\mathbb{A}_f)/U_X \overset{\sim}{\longrightarrow} \text{Nr}(G(\mathbb{Q})) \backslash \mathbb{A}_{F,f}^\times / \text{Nr}(U_X)$$

is bijective. By the Hasse-Schilling-Maass norm theorem [22, Theorem 33.15] that every element in $F^\times$ is a norm if and only if it is a local norm everywhere, one gets $\text{Nr}(G(\mathbb{Q})) = F^\times$ by Kneser’s Theorem again. This proves

$$|G_{\pi}(\mathbb{Q}) \backslash G_{\pi}(\mathbb{A}_f)/U_X| = |\mathbb{A}_{F,f}^\times / F^\times \cdot \text{Nr}(U_X)|.$$

On the other hand, consider the short exact sequence

$$1 \longrightarrow \frac{\hat{O}_F^\times}{(\hat{O}_F^\times \cap F^\times \text{Nr}(U_X))} \longrightarrow \frac{\mathbb{A}_{F,f}^\times}{F^\times \text{Nr}(U_X)} \longrightarrow \text{Pic}(O_F) \longrightarrow 1.$$

It is easy to check $\hat{O}_F^\times \cap F^\times \text{Nr}(U_X) = O_F^\times \text{Nr}(U_X)$. Thus, we have $|G_{\pi}(\mathbb{Q}) \backslash G_{\pi}(\mathbb{A}_f)/U_X| = n_X h(F)$. The proposition then follows from Theorem 2.1. \qed

2.3. An example. Take $\pi = \sqrt{-p}$. We have $E_{\pi} = F = \mathbb{Q}(\sqrt{-p})$ and $R = \mathbb{Z}[\sqrt{-p}]$. The corresponding abelian variety $A_{\pi}$ is a supersingular elliptic curve over $\mathbb{F}_p$. The $F$-isocrystal $N(A_{\pi})$ is a free $F$-module of rank one, and Dieudonné modules in $N(A_{\pi})$ are simply $R_p$-submodules. Since $R_p$ is a maximal order, $\mathfrak{X}_{\pi,p}$ has one element.

When $\ell \neq 2$ or $p \equiv 1 \pmod{4}$, the order $R_\ell$ is maximal and hence the set $\mathfrak{X}_{\pi,\ell}$ has one element. Therefore, when $p \equiv 1 \pmod{4}$ the set $\mathfrak{X}_{\pi}$ consists of one element $X$ with $n_X = 1$, and $N(\pi) = 1$.

Suppose that $p \equiv 3 \pmod{4}$. The set $\mathfrak{X}_{\pi,2}$ consists of two elements:

$$T_2 \simeq R_2, \quad \text{or} \quad T_2 \simeq O_{F,2}.$$

Therefore, the $\mathfrak{X}_{\pi}$ has two elements $X_1$ and $X_2$, corresponding to the two elements in $\mathfrak{X}_{\pi,2}$ above. We have $n_{X_2} = 1$ and $n_{X_1} = \lfloor \hat{O}_E^\times : O_E^\times \hat{R}\rfloor$. One computes that [39]

$$\lfloor \hat{O}_E^\times : O_E^\times \hat{R}\rfloor = \begin{cases} 1, & \text{if } p \equiv 7 \pmod{8} \text{ or } p = 3, \\ 3, & \text{if } p \equiv 3 \pmod{8} \text{ and } p \neq 3. \end{cases}$$

Therefore, $|A_{\pi}| = N(\pi) h(\mathbb{Q}(\sqrt{-p}))$, where

$$N(\pi) = \begin{cases} 1, & \text{if } p \equiv 1 \pmod{4} \text{ or } p = 2, \\ 2, & \text{if } p \equiv 7 \pmod{8} \text{ or } p = 3, \\ 4, & \text{if } p \equiv 3 \pmod{8} \text{ and } p \neq 3. \end{cases}$$

In [23] one finds more formulas for numbers of elliptic curves in an isogeny class over $\mathbb{F}_q$. 


2.4. Proposition 2.2 basically says that when $F$ has no real place, the computation of $|A_\pi|$ is reduced to the problem of computing the class numbers of certain orders $R_X$ in $F$ for each $X \in \mathcal{X}_\pi$, and we have shown the relation $h(R_X) = n_X h(F)$. Let 
\[ \tilde{n}_X := [\tilde{O}_E^X : \text{Nr}(U_X)], \]
then we have 
\[ \tilde{n}_X = \prod \tilde{n}_{X_t}, \]
where $\tilde{n}_{X_t} := [\tilde{O}_{E_t}^X : \text{Nr}(U_{X_t})]$. Put 
\[ \tilde{N}_\ell(\pi) := \sum_{X_t \in \mathcal{X}_\pi} \tilde{n}_{X_t} \quad \text{and} \quad \tilde{N}(\pi) := \sum_{X \in \mathcal{X}_\pi} \tilde{n}_X. \]
Then we have an upper bound which can be computed locally
\[
(2.5) \quad N(\pi) \leq \tilde{N}(\pi) \quad \text{and} \quad \tilde{N}(\pi) = \prod_\ell \tilde{N}_\ell(\pi).
\]
We make a few remarks on the classification of $\mathcal{X}_{\pi,\ell}$, i.e. classifying $R_\ell$-modules in a free $F_\ell$-module of rank $n$.

First observe that $R = \mathbb{Z}[\pi]$ is a complete intersection and Gorenstein. It might be possible to study $R_\ell$-modules through (co)-homological algebra.

In some special cases, it is possible to describe the finite set $\mathcal{X}_{\pi,\ell}$ more explicitly. For example if $R[1/p]$ is a Bass (\mathbb{Z}[1/p]-order, then one has a more explicit description of $R_\ell$-modules (see [5, Section 37] for the definition of Bass orders).

In this case any $R_\ell$-module $T_\ell$ in $F_\ell^n$ is isomorphic to $R_1 \oplus R_2 \oplus \cdots \oplus R_n$ for orders $R_1 \subset \cdots \subset R_n$ containing $R_\ell$ in $F_\ell$, and the set $\{ R_1, \ldots, R_n \}$ of orders with multiplicities is completely determined by $T_\ell$.

Examples of Bass orders include Dedekind domains and quadratic orders over a Dedekind domain. Bass orders share the local property in the sense that an order is Bass if and only if so are all its completions.

We illustrate the idea by an example. Assume that $F = \mathbb{Q}(\pi)$ is an imaginary quadratic field and let $n = \dim A_\pi$. For any integer $d \geq 1$, denote by $R_d$ the order in $F$ of conductor $d$. Suppose the conductor of $R = \mathbb{Z}[\pi]$ is $q_1 D$, where $q_1$ is a $p$-power and $(p, D) = 1$. For any $\mathbb{Z}$-lattice $L$ write $\tilde{L}(\mathbb{Z}) := L \otimes \mathbb{Z}(\mathbb{Z})$ with $\mathbb{Z}(\mathbb{Z}) = \prod_{\ell \neq p} \mathbb{Z}_\ell$. Then any $\tilde{R}(\mathbb{Z})$-module $T(\mathbb{Z})$ of rank $n$ is isomorphic to $\tilde{R}(d_1) \oplus \tilde{R}(d_2) \oplus \cdots \oplus \tilde{R}(d_n)$ for uniquely determined divisors $d_1, \ldots, d_n$ of $D$ with $d_n | d_{n-1} | \cdots | d_1$.

One computes directly that $\text{Nr}(\text{End}(T(\mathbb{Z}))) = \tilde{R}(d_n)$. Suppose that we have found the representatives $M_1, \ldots, M_r$ for $\mathcal{X}_{\pi,p}$. Then $\text{Nr}(\text{End}_{DM}(M_i)) = (R_{p^{e_i}})_p$ for some non-negative integers $e_i$. The cardinality of $A_\pi$ is given by
\[
|A_\pi| = \sum_{i=1}^r \sum_{d_1, \ldots, d_n} h(R_{p^{e_1}, d_n}),
\]
where $d_i$’s run over the positive divisors of $D$ with the condition $d_n | d_{n-1} | \cdots | d_1$.

2.5. Assume that $F$ has a real place. Then $\pi = \pm \sqrt{p}$, $F = \mathbb{Q}(\sqrt{p})$ and $R = \mathbb{Z}[\sqrt{p}]$. We separate the discussion into two cases depending on the parity of $a$.

When $a$ is even, $F = \mathbb{Q}$ and $E_\pi$ is isomorphic to the quaternion $\mathbb{Q}$-algebra $D_{p,\infty}$ ramified exactly at $\{ p, \infty \}$. The set $\mathcal{X}_\pi$ consists of one element $X$ and the corresponding group $U_X$ is maximal in $G_\pi(A_f)$. Thus, $|A_\pi|$ is the class number of a maximal order of $D_{p,\infty}$.

The class number formula for $D_{p,\infty}$ due to Deuring, Eichler and Igusa gives
\[
|A_\pi| = \frac{p-1}{12} + \frac{1}{3} \left( 1 - \left( -\frac{3}{p} \right) \right) + \frac{1}{4} \left( 1 - \left( -\frac{4}{p} \right) \right).
\]
Now consider the case where \(a\) is odd. The endomorphism algebra \(E_\pi\) is isomorphic to the quaternion algebra \(D_{\infty_1, \infty_2}\) over \(F = \mathbb{Q}(\sqrt{p})\) ramified exactly at the two Archimedean places \(\{\infty_1, \infty_2\}\). Moreover, the corresponding abelian variety \(A_\pi\) is a supersingular abelian surface over \(\mathbb{F}_q\).

Note that \([O_F : \mathbb{R}]\) is a divisor of \(2p(a-1)/2\). So for \(\ell \mid 2p\), the ring \(R_\ell\) is the maximal order and \(\mathfrak{x}_{\pi, \ell}\) has one element \(T_\ell\) whose endomorphism ring is isomorphic to \(\text{Mat}_2(O_{F, \ell})\), where \(O_{F, \ell} := O_F \otimes \mathbb{Z}_\ell\). For an odd prime \(p\), the projection gives natural identification \(\mathfrak{x}_\pi = \mathfrak{x}_{\pi, p} \times \mathfrak{x}_{\pi, 2}\). Similarly, if \(p = 2\), then \(\mathfrak{x}_\pi = \mathfrak{x}_{\pi, p}\).

When \(p \equiv 3 \pmod{4}\), the order \(R_2\) is maximal and hence \(\mathfrak{x}_{\pi, 2}\) has one element whose endomorphism ring is \(\text{Mat}_2(O_{F, 2})\). When \(p \equiv 1 \pmod{4}\), the set \(\mathfrak{x}_{\pi, 2}\) has three elements \([L_1], [L_2], [L_4]\) where
\[
L_1 = O_{F, 2}^\otimes, \quad L_2 = R_2 \oplus O_{F, 2}, \quad \text{and} \quad L_4 = R_{2, 2}^\otimes.
\]
The corresponding endomorphism rings are \(\text{End}_{R_2}(L_1) = \text{Mat}_2(O_{F, 2})\),
\[
\text{End}_{R_2}(L_2) = \left( \begin{array}{cc} R_2 & 2O_{F, 2} \\ O_{F, 2} & O_{F, 2} \end{array} \right), \quad \text{and} \quad \text{End}_{R_2}(L_4) = \text{Mat}(R_2).
\]
The latter two rings have index 8 and 16 respectively in \(\text{Mat}_2(O_{F, 2})\). Let us fix a maximal ring \(\mathcal{O}_1 \subseteq D_{\infty_1, \infty_2}\) and identify \(\mathcal{O}_1 \otimes \mathbb{Z}_2\) with \(\text{End}_{R_2}(L_1)\). If \(p \equiv 1 \pmod{4}\), then we define \(\mathcal{O}_8 \subset \mathcal{O}_1\) to be the unique suborder of index 8 such that \(\mathcal{O}_8 \otimes \mathbb{Z}_2 = \text{End}_{R_2}(L_2) \subset \text{End}_{R_2}(L_1)\), and \(\mathcal{O}_{16} \subset \mathcal{O}_1\) to be the unique suborder of index 16 such that \(\mathcal{O}_8 \otimes \mathbb{Z}_2 = \text{End}_{R_2}(L_4)\).

We partition set \(\mathfrak{x}_{\pi, p}\) into disjoint subsets \(\mathfrak{x}_{\pi, p, 1} \cup \mathfrak{x}_{\pi, p, 2}\) according to the \(a\)-number of its members (see [13] Section 1.5), where \(\mathfrak{x}_{\pi, p, i} := \{[M] \in \mathfrak{x}_{\pi, p} \mid a(M) = i\}\) for \(i = 1, 2\). Accordingly, \(\mathfrak{x}_{\pi} = \mathfrak{x}_{\pi, p} \cup \mathfrak{x}_{\pi, 2}\) and \(A_{\pi} = A_{\pi, p} \cup A_{\pi, 2}\). The subset \(A_{\pi, 2}\) is none other than the set of superspecial abelian surfaces up to isomorphism in \(A_{\pi}\) (Section 1.7, ibid.), so we also denoted it by \(\text{Sp}(\pi)\). Since \(a\) is odd, \(\mathfrak{x}_{\pi, p, 1}\) is a singleton \(\{[M_0]\}\), with endomorphism ring \(\text{End}_{\text{DM}}(M_0)\) isomorphic to the maximal order \(\text{Mat}_2(O_{F, p})\). More explicitly, \(M_0 = R_{p, 2}^\otimes \otimes_{\mathbb{Z}_p} W(\mathbb{F}_q)\), and the Frobenius map acts as \(\sqrt{p}\sigma\), where \(\sigma\) is the Frobenius morphism of the ring of Witt vectors \(W(\mathbb{F}_q)\).

Let \([A]\) be a member of \(\text{Sp}(\pi)\). If \(p = 2\) or \(p \equiv 3 \pmod{4}\), then \(\text{End}_{\mathcal{O}_p}(A)\) is maximal in \(D_{\infty_1, \infty_2}\). If \(p \equiv 1 \pmod{4}\), then \(\text{End}(A)\) is locally isomorphic to \(\mathcal{O}_1\), \(\mathcal{O}_8\) or \(\mathcal{O}_{16}\), depending on the component of \(\Phi([A])\) in \(\mathfrak{x}_{\pi, 2}\). We conclude that
\[
|\text{Sp}(\pi)| = \begin{cases} h(\mathcal{O}_1) & \text{if } p = 2 \text{ or } p \equiv 3 \pmod{4}, \\ h(\mathcal{O}_1) + h(\mathcal{O}_8) + h(\mathcal{O}_{16}) & \text{if } p \equiv 1 \pmod{4}. \end{cases}
\]

The cardinality of \(\text{Sp}(\pi)\) is explicitly calculated in [11], and that of \(A_{\pi, 2}\) will be calculated in Section 4.

3. Eichler’s Trace Formula for Brandt Matrices

In this section we extend the classical notion of Brandt matrices [27] Exercise III.5.8] to arbitrary \(\mathbb{Z}\)-orders in totally definite quaternion algebras and provide a trace formula for them. As a result, we obtain a class number formula for all such \(\mathbb{Z}\)-orders. Details of the proofs may be found in [29].

Throughout this Section, \(F\) denotes a totally real number field, \(A \subseteq O_F\) a \(\mathbb{Z}\)-order in \(F\), and \(D\) a totally definite quaternion \(F\)-algebra. A \(\mathbb{Z}\)-order \(\mathcal{O} \subset D\) is said to be a proper \(A\)-order if \(\mathcal{O} \cap F = A\). Similarly, we define the notion of proper
A-orders in finite field extensions $K/F$. For any $A$-lattice $I \subset D$, the norm $\text{Nr}_A(I)$ of $I$ over $A$ is definite to be the $A$-submodule of $F$ generated by the reduced norms of elements of $I$. If the multiplication $IJ$ of two $A$-lattices $I$ and $J$ is coherent [27 Section I.4] and one of $I$ and $J$ is locally principal with respect to its associated left (or right) order, then $\text{Nr}_A(IJ) = \text{Nr}_A(I) \text{Nr}_A(J)$.

When $I = \mathcal{O}$, the norm $\text{Nr}_A(\mathcal{O})$ is an $A$-order in $F$, denoted by $\tilde{A}$. It is known that $A = A$ if and only if $\mathcal{O}$ is closed under the canonical involution $x \mapsto \text{Tr}(x) - x$ ([29 Lemma 3.1.1]). If $I$ is a locally principal right $\mathcal{O}$-ideal, then $\text{Nr}_A(I)$ is a locally principal $\tilde{A}$-ideal. Let $\text{Cl}(\mathcal{O})$ be the set of isomorphism classes of locally principal right $\mathcal{O}$-ideals in $D$, and $h = h(\mathcal{O}) = |\text{Cl}(\mathcal{O})|$ the class number of $\mathcal{O}$. We fix a complete set of representatives $I_1, \ldots ,I_h$ for $\text{Cl}(\mathcal{O})$, and set

$$\mathcal{O}_i := \mathcal{O}_i(I_i), \quad w_i := |\mathcal{O}^\times_i : A^\times|.$$ 

Each $\mathcal{O}_i$ is a proper $A$-order uniquely determined up to $D^\times$-conjugation, and $w_i$ depends only on the ideal class of $I_i$. Note that $\text{Nr}_A(\mathcal{O}_i) = \tilde{A}$ for all $1 \leq i \leq h$. The mass of $\mathcal{O}$ is defined as the weighted sum

$$\text{Mass}(\mathcal{O}) := \sum_{i=1}^{h} \frac{1}{|\mathcal{O}_i^\times : A^\times|} = \sum_{i=1}^{h} \frac{1}{w_i}.$$ 

**Definition 3.1.** Let $n$ be a locally principal integral $\tilde{A}$-ideal. The Brandt matrix associated to $n$ is defined to be the matrix $\mathfrak{B}(n) := (\mathfrak{B}_{ij}(n)) \in \text{Mat}_h(\mathbb{Z})$, where $\mathfrak{B}_{ij}(n)$ is the cardinality of the set of right $O^\times_j$-orbits of elements $b \in I_i I_j^{-1}$ such that $\text{Nr}_A(b\mathcal{O}_j) = n \text{Nr}_A(I_i) \text{Nr}_A(I_j)^{-1}$.

It is clear from the Definition 3.1 that $\mathfrak{B}_{ij}(n) \neq 0$ only if $n$ is principal and generated by a totally positive element. So for the rest of this section we assume that $n = \tilde{A}\beta$ is generated by a totally positive element $\beta \in \tilde{A}$. Moreover, we define the symbol

$$\delta_n = \begin{cases} 1 & \text{if } n = \tilde{A} a^2 \text{ for some } a \in A; \\ 0 & \text{otherwise.} \end{cases}$$

A finite $A$-order (i.e. a finite $A$-algebra with no $\mathbb{Z}$-torsion) $B$ is said to be a CM proper $A$-order if $K := B \otimes \mathbb{Z} \mathbb{Q}$ is a CM-extension of $F$ ([3 Section 13]), and $B$ is a proper $A$-order in $K$. We set $\delta(B) = 1$ if $B$ is closed under the complex conjugation of $K/F$, and $\delta(B) = 0$ otherwise. Denote by $\text{Emb}(B, \mathcal{O})$ the set of optimal $A$-embeddings from $B$ into $\mathcal{O}$, i.e.,

$$\text{Emb}(B, \mathcal{O}) := \{ \varphi \in \text{Hom}_F(K, D) \mid \varphi(K) \cap \mathcal{O} = \varphi(B) \}.$$ 

This is a finite set equipped with a right action of $\mathcal{O}^\times$ sending $\varphi \mapsto g^{-1} \varphi g$ for all $\varphi \in \text{Emb}(B, \mathcal{O})$ and $g \in \mathcal{O}^\times$. We denote

$$m(B, \mathcal{O}, \mathcal{O}^\times) := |\text{Emb}(B, \mathcal{O})/\mathcal{O}^\times|, \quad \text{and} \quad w(B) := |B^\times : A^\times|.$$ 

Similarly for each prime $p$, we put

$$m_p(B) := m(B_p, \mathcal{O}_p, \mathcal{O}^\times_p) = |\text{Emb}(B_p, \mathcal{O}_p)/\mathcal{O}_p^\times|,$$

where $B_p$ and $\mathcal{O}_p$ denote the $p$-adic completions $B \otimes \mathbb{Z}_p$ and $\mathcal{O} \otimes \mathbb{Z}_p$, respectively. Note that $m_p(B) = 1$ for all but finitely many $p$. Choose a complete set $S = \{ \varphi \mid \varphi \in \text{Emb}(B, \mathcal{O}) \}$. 

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the identity and Tr

\[ T_{B,n} := \{ x \in B \setminus A \mid N_{K/F}(x) = \varepsilon \beta \text{ for some } \varepsilon \in S \}. \]

For each fixed \( n \), there are only finitely many CM proper \( A \)-orders \( B \) with \( T_{B,n} \notin \emptyset \).

**Theorem 3.2** (Eichler’s Trace Formula). Suppose that \( n = \tilde{A} \beta \) is generated by a totally positive element \( \beta \in \tilde{A} \). Then the trace of the Brandt matrix \( \mathcal{B}(n) \) is given by

\[
\Tr \mathcal{B}(n) = \delta_n \cdot \Mass(O) + \frac{1}{4} \sum_B \frac{(2 - \delta(B))h(B)|T_{B,n}|}{w(B)} \prod_p m_p(B),
\]

where \( B \) runs over all mutually non-isomorphic CM proper \( A \)-orders with \( T_{B,n} \neq \emptyset \).

The proof of this theorem follows closely Eichler’s original proof [9]; see also Vignéras’s book [27]. When \( n = (1) = A \), the Brandt matrix \( \mathcal{B}(A) \in \Mat_h(\mathbb{Z}) \) is the identity and \( \Tr \mathcal{B}(A) = h(O) \).

**Corollary 3.3** (Eichler’s Class number formula). Let \( O \) be a proper \( A \)-order in a totally definite quaternion algebra \( D/F \). Then

\[
h(O) = \Mass(O) + \frac{1}{2} \sum_{w(B) > 1} (2 - \delta(B))h(B)(1 - w(B)^{-1}) \prod_p m_p(B),
\]

where \( B \) runs over all mutually non-isomorphic CM proper \( A \)-orders with \( w(B) = [B^\times : A^\times] > 1 \).

Let \( h(\sqrt{d}) \) denotes the class number of \( \mathbb{Q}(\sqrt{d}) \) for any square free \( d \in \mathbb{Z} \). Applying formula (3.6) to the orders \( O_8 \) and \( O_{16} \) defined in Subsection 2.5 when \( p \equiv 1 \) (mod 4), we obtain

\[
h(O_8) = \varpi_p h(\sqrt{p}) \left[ \frac{4 - \left( \frac{2}{p} \right)}{2} \zeta_F(-1) + \left( 2 - \left( \frac{2}{p} \right) \right) \frac{h(\sqrt{-p})}{24} + \frac{\delta_{1,\varpi_p} h(\sqrt{-3p})}{3} \right],
\]

\[
h(O_{16}) = \varpi_p h(\sqrt{p}) \left[ \frac{3 - 2 \left( \frac{2}{p} \right)}{2} \zeta_F(-1) + \left( 2 - \left( \frac{2}{p} \right) \right) \frac{h(\sqrt{-p})}{12} + \frac{1}{6} h(\sqrt{-3p}) \right].
\]

Here \( \varpi_p := 3|\mathbb{O}_{\mathbb{Q}(\sqrt{-p})}^\times : \mathbb{Z}[\sqrt{p}]^\times |^{-1} \in \{1, 3\} \), and \( \delta_{1,\varpi_p} = 1 \) or \( 0 \) depending on whether \( \varpi_p = 1 \) or 3. When \( p \equiv 1 \) (mod 8), we always have \( \varpi_p = 3 \) by [30, Lemma 4.1], and hence \( \delta_{1,\varpi_p} = 0 \) in this case. The special value \( \zeta_F(-1) \) of the Dedekind zeta-function can be calculated by Siegel’s formula [42, Table 2, p. 70]:

\[
\zeta_F(-1) = \frac{1}{60} \sum_{b^2 + 4ac = \varphi_F \atop a,c > 0} a,
\]

where \( b \in \mathbb{Z} \) and \( a,c \in \mathbb{N}_{>0} \). We refer to [29] for the details of the calculation of the above formulas.

For the sake of completeness, we also list the class number of the maximal order \( O_1 \). If \( p \equiv 1 \) (mod 4) and \( p > 5 \), then

\[
h(O_1) = h(\sqrt{p}) \left[ \frac{\zeta_F(-1)}{2} + \frac{h(\sqrt{-p})}{8} + \frac{h(\sqrt{-3p})}{6} \right].
\]
If \( p \equiv 3 \pmod{4} \) and \( p \geq 7 \), then
\[
h(\mathcal{O}_1) = h(\sqrt{p}) \left[ \frac{\zeta_p(-1)}{2} + \left( \frac{13}{8} - \frac{5}{8} \left( \frac{2}{p} \right) \right) h(\sqrt{-p}) + \frac{h(\sqrt{2p})}{4} + \frac{h(\sqrt{3p})}{6} \right].
\]
Lastly, \( h(\mathcal{O}_1) = 1, 2, 1 \) for the primes \( p = 2, 3, 5 \), respectively. These formulas can be calculated directly using the classical Eichler class number formula [27, Corollaire 2.5].

4. Supersingular abelian surfaces in an isogeny class

In this section we compute \( |A_\pi| \) for \( \pi = \sqrt{q} \), where \( q = p^a \) is an odd power of \( p \). Recall that two orders \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \) in a quaternion algebra are said to be in the same genus if their \( p \)-adic completions are isomorphic for all primes \( p \). Let \( D = D_{\infty_1, \infty_2} \) be the totally definite quaternion algebra over \( F = \mathbb{Q}(\sqrt{p}) \) that splits at all finite places. Recall that the orders \( \mathcal{O}_8 \) and \( \mathcal{O}_{16} \) in \( D \) are defined only when \( p \equiv 1 \pmod{4} \). Let \( \mathcal{O} \) be an order in the genus of \( \mathcal{O}_r \) with \( r = 1, 8 \) or \( 16 \) and \( \mathfrak{R} := \mathcal{O} \cap F \) the center of \( \mathcal{O} \). Then \( \mathfrak{R} = O_F \) if \( \mathcal{O} \) is maximal, and \( \mathfrak{R} = \mathbb{Z}[\sqrt{p}] \) otherwise. Nevertheless, \( \mathfrak{R}_p \) is always the maximal order in \( F_p \). The \( p \)-adic completion \( \mathfrak{O}_p := \mathfrak{O} \otimes \mathbb{Z}_p \cong \text{Mat}_2(\mathbb{Z}_p[\sqrt{p}]) \) is a maximal order in \( D \otimes \mathbb{Z}_p \), and the natural projection
\[
\mathcal{O} \to \mathcal{O}/\sqrt{p}\mathcal{O} \cong \mathcal{O}_p/\sqrt{p}\mathcal{O}_p \cong \text{Mat}_2(\mathbb{F}_p)
\]
duces a group homomorphism
\[
\rho : \mathcal{O}^\times /\mathfrak{R}^\times \to \text{PGL}_2(\mathbb{F}_p).
\]
Let \( \tilde{u} \in \mathcal{O}^\times /\mathfrak{R}^\times \) be a nontrivial element, and \( u \in \mathcal{O}^\times \) a representative of \( \tilde{u} \). Both the field \( F(u) \subset D \) and its suborders \( \mathfrak{R}[u] \subseteq F(u) \cap \mathcal{O} \) depends only on \( \tilde{u} \), not on the choice of \( u \). For simplicity, we set
\[
K_\tilde{u} := F(u), \quad \mathfrak{R}[\tilde{u}] := \mathfrak{R}[u], \quad \text{and} \quad B_\tilde{u} := K_\tilde{u} \cap \mathcal{O}.
\]
Then both \( B_\tilde{u} \) and \( \mathfrak{R}[\tilde{u}] \) are CM proper \( \mathfrak{R} \)-orders with \( |B_\tilde{u} : \mathfrak{R}^\times| \geq |\mathfrak{R}[\tilde{u}]^\times : \mathfrak{R}^\times| > 1 \). All such orders \( \mathfrak{R}[\tilde{u}] \) have been classified in [30].

**Lemma 4.1.** If the index \( |B_\tilde{u} : \mathfrak{R}[\tilde{u}]| \) is coprime to \( p \), then \( \tilde{u} \notin \ker(\rho) \).

**Proof.** Suppose that \( p \nmid |B_\tilde{u} : \mathfrak{R}[\tilde{u}]| \). Then \( B_\tilde{u} \otimes \mathbb{Z}_p \cong \mathfrak{R}_p[u] \), and
\[
\mathcal{O}/\sqrt{p}\mathcal{O} \supset B_\tilde{u}/\sqrt{p}B_\tilde{u} = \mathfrak{R}_p[u] \otimes_{\mathfrak{R}_p} (\mathfrak{R}_p/\sqrt{p}\mathfrak{R}_p) = \mathbb{F}_p[\tilde{u}],
\]
where \( \tilde{u} \) denotes the image of \( u \) modulo \( \sqrt{p}B_\tilde{u} \). Since \( \mathfrak{R}_p \) is maximal in \( F_p \), \( \mathfrak{R}_p[u] \) is a free \( \mathfrak{R}_p \)-module of rank 2, and hence \( \mathbb{F}_p[\tilde{u}] \) is an \( \mathbb{F}_p \)-space of dimension 2. In particular, \( \tilde{u} \notin \mathbb{F}_p \).

Recall the \( h(D) = 1, 2, 1 \) when \( p = 2, 3, 5 \) respectively. In particular, when \( p = 2 \) or \( 5 \), \( \mathcal{O}_1 \) is the unique maximal order in \( D \) up to conjugation. Using the Magma Algebra System [24], one checks that
\[
\mathcal{O}_1^\times /\mathcal{O}_1^\times \simeq S_4 \quad \text{if } p = 2, \quad \text{and} \quad \mathcal{O}_1^\times /\mathcal{O}_1^\times \simeq A_5 \quad \text{if } p = 5.
\]
When \( p = 3 \), there are two maximal orders \( \mathcal{O}, \mathcal{O}' \) up to conjugation. We have
\[
\mathcal{O}^\times /\mathcal{O}^\times \simeq S_4 \quad \text{and} \quad \mathcal{O}'^\times /\mathcal{O}'^\times \simeq D_{12}, \quad \text{the dihedral group of order 24}.
\]
Proposition 4.2.

1. The morphism \( \rho \) is injective for any order \( \mathcal{O} \) in the genus of \( \mathbb{O}_r \) with \( r = 1, 8, 16 \) when \( p \geq 5 \).

2. When \( p = 2 \), \( \rho \) is surjective.

3. When \( p = 3 \), \( \rho \) is an isomorphism for \( \mathcal{O} = \mathbb{O} \), and \( \text{img}(\rho) \simeq D_4 \) for \( \mathcal{O} = \mathbb{O}' \).

Proof. Let \( \tilde{u} \in \mathcal{O}^\times/\mathbb{R}^\times \) be a nontrivial element. According to the classification in [30], if \( \text{ord}(\tilde{u}) \) is a power of 2, then \( \text{ord}(\tilde{u}) = 2 \) or 4, and \( [O_K : \mathbb{R}[\tilde{u}]] \in \{1, 2, 4\} \).

By Lemma 4.1 \( \rho \) is injective on the Sylow 2-subgroups of \( \mathcal{O}^\times/\mathbb{R}^\times \) when \( p \) is odd. Similarly, if \( p \neq 3 \) and \( \text{ord}(\tilde{u}) = 3 \), then \( [O_K : \mathbb{R}[\tilde{u}]] \in \{1, 2, 4\} \), so once again \( \tilde{u} \notin \ker(\rho) \).

This exhaust the list of all nontrivial elements of \( \mathcal{O}^\times/\mathbb{R}^\times \) when one of the following conditions holds: (i) \( p \geq 7 \); (ii) \( p = 5 \) and \( \mathcal{O} \) is not maximal. Hence \( \rho \) is injective in these two cases. Moreover, if \( p = 5 \) and \( \mathcal{O} \) is maximal, then \( \rho \) is injective since \( \mathcal{O}^\times/\mathcal{O}_p^\times \) is the simple group \( A_5 \). Part (1) of the proposition follows.

Suppose that \( p = 2 \). Then \( \mathcal{O}^\times/\mathcal{O}_p^\times \simeq S_4 \), and for any element \( \tilde{u} \in \mathcal{O}^\times/\mathcal{O}_p^\times \) of order 4, \( O_F[\tilde{u}] \) coincides with the maximal order \( O_K[\tilde{u}] \simeq \mathbb{Z}[\{5\}] \). Thus \( \rho(\tilde{u}) \in \text{PGL}_2(\mathbb{F}_2) \simeq S_3 \) is a nontrivial element of order 2. Since \( \rho \) does not vanish on any order 3 elements as remarked, it maps \( \mathcal{O}^\times/\mathcal{O}_p^\times \) surjectively onto \( \text{PGL}_2(\mathbb{F}_2) \).

Suppose that \( p = 3 \) and \( \mathcal{O}^\times/\mathcal{O}_p^\times \simeq S_4 \). In this case we have \( D = \left( \frac{-1}{3\sqrt{3}} \right) \), and \( \mathcal{O} \) is \( D \)-conjugate to \( B+B(1+i+j+k)/2 \), where \( B = \mathbb{Z}[\sqrt{3}, (1+\sqrt{3})(1+i)/2] \subset F(i) \). If \( \tilde{u} \in \mathcal{O}^\times/\mathcal{O}_p^\times \) is an element of order 3, then \( O_F[\tilde{u}] = B_0 \simeq \mathbb{Z}[\sqrt{3}, \zeta_6] \), and hence \( \tilde{u} \notin \ker(\rho) \). Since \( \rho \) is also injective on the Sylow 2-subgroups of \( \mathcal{O}^\times/\mathcal{O}_p^\times \), it induces an isomorphism between \( \mathcal{O}^\times/\mathcal{O}_p^\times \) and \( \text{PGL}_2(\mathbb{F}_3) \simeq S_4 \).

Lastly, suppose that \( p = 3 \) and \( \mathcal{O}^\times/\mathcal{O}_p^\times \simeq D_{12} \), whose Sylow 2-subgroups are isomorphic to \( D_4 \). Note that \( D_{12} \) has a unique Sylow 3-subgroup, which is cyclic of order 3, and \( D_{12} = C_3 \rtimes D_4 \). Necessarily, \( \ker(\rho) \) is nontrivial because \( D_{12} \neq \text{PGL}_2(\mathbb{F}_3) \). Since \( \rho \) is injective on the Sylow 2-subgroups, we must have \( \ker(\rho) = C_3 \) and \( \text{img}(\rho) \simeq D_4 \). \( \square \)

Let \( \pi = \sqrt{q} \) be the Weil \( q \)-number with \( a \) odd. Recall that \( A_\pi \) denotes the set of isomorphism classes in the \( \mathbb{F}_p \)-isogeny class of \( A_\pi \). The goal is to evaluate \( |A_\pi| \). We decompose \( A_\pi = \text{Sp}(\pi) \coprod A_{\pi}^1 \) as the union of the set \( \text{Sp}(\pi) \) of superspecial abelian surfaces and \( A_{\pi}^1 \) consisting of non-superspecial abelian surfaces. The case \( q = p \) is done in [29] and we may assume that \( q > p \) for which \( A_{\pi}^1 \) is non-empty. Let \( A \in A_{\pi}^1 \) be a member with covariant Dieudonné module \( M \). There is a unique isogeny \( \varphi : A \to \tilde{A} \) of degree \( p \) with \( \tilde{A} \) superspecial: \( \tilde{A} \) is constructed with the Dieudonné module \( M + F^{-1}VM \) [30] Lemma 4.4]. Consider the map \( \text{pr} : A_{\pi}^1 \to \text{Sp}(\pi) \) sending \( A \) to \( \tilde{A} \).

Proposition 4.3.

1. Let \( A_0 \in \text{Sp}(\pi) \) be a superspecial member. The fiber \( \text{pr}^{-1}(A_0) \) in \( A_{\pi} \) has \( (q-p)/|\rho(\text{Aut}(A_0))| \) elements, where \( \rho \) is the map defined in (1.1) by viewing the endomorphism ring \( \text{End}(A_0) \) as an order in \( D \).

2. We have \( |A_{\pi}| = |\text{Sp}(\pi)| + \sum_{A_0 \in \text{Sp}(\pi)} (q-p)/|\rho(\text{Aut}(A_0))| \).

Proof. (1) Let \( M_0 \) be the Dieudonné module of \( A_0 \). If \( A \in \text{pr}^{-1}(A_0) \), then the Dieudonné module \( M \) of \( A \) satisfies (i) \( VM_0 \subset M \subset M_0 \), (ii) \( \overline{M} := M/VM_0 \) is 1-dimensional \( \mathbb{F}_p \)-subspace of \( \overline{M}_0 := M_0/VM_0 \), and (iii) \( M \) is non-superspecial. Conversely, such a Dieudonné module gives rise to a member \( A \) in \( A_{\pi} \) with \( \tilde{A} \simeq A_0 \).
If two members $\varphi_i : A_i \to A_0$ ($i = 1, 2$) of $\text{pr}^{-1}(A_0)$ are isomorphic, then any isomorphism $\alpha : A_1 \to A_2$ lifts uniquely to an automorphism $\widetilde{\alpha} : A_0 \to A_0$. Thus, $\text{pr}^{-1}(A_0)$ is in bijection with the set $X$ of Dieudonné modules $M$ satisfying (i), (ii) and (iii) modulo the action of $\text{Aut}(A_0)$. The set $X$ consists of points $\mathcal{M}$ of $\mathbf{P}^1_{\mathcal{M}_0}(\mathbb{F}_q)$ satisfying (iii). By [36, Lemma 6.1], that a point $\mathcal{M} \in \mathbf{P}^1_{\mathcal{M}_0}(\mathbb{F}_q)$ is superspecial if and only if it is contained in $\mathbf{P}^1_{\mathcal{M}_0}(\mathbb{F}_{p^2})$. It follows that $X \simeq \mathbf{P}^1_{\mathcal{M}_0}(\mathbb{F}_q) - \mathbf{P}^1_{\mathcal{M}_0}(\mathbb{F}_p)$. Since $A_0$ is superspecial, $VM_0 = \sqrt{p}M_0$ and the action of $\text{Aut}(A_0)$ on $X$ factors through the action on $\mathcal{M}_0 = M_0/\sqrt{p}M_0$. This shows that

\[
\text{pr}^{-1}(A_0) \simeq \text{Aut}(A_0) \backslash X \simeq \rho(\text{Aut}(A_0))\backslash [\mathbf{P}^1_{\mathcal{M}_0}(\mathbb{F}_q) - \mathbf{P}^1_{\mathcal{M}_0}(\mathbb{F}_p)].
\]

The action of $\text{PGL}_2(\mathbb{F}_p)$ on $\mathbf{P}^1(\mathbb{F}_q)$ is faithful and it has no fixed points in $\mathbf{P}^1(\mathbb{F}_q) - \mathbf{P}^1(\mathbb{F}_p)$. Therefore, $|\text{pr}^{-1}(A_0)| = (q - p)/|\rho(\text{Aut}(A_0))|$. (2) This follows from (1). \hfill \Box

Note that the formula for $|A_\pi|$ holds when $q = p$.

**Theorem 4.4.** Let $\pi = \sqrt{q}$, where $q$ is an odd power of a prime $p$.

1. For $p = 2$, $|A_\pi| = 1 + (q - 2)/6$.
2. For $p = 3$, $|A_\pi| = 2 + (q - 3)/6$.
3. For $p = 5$, $|A_\pi| = 3 + 4(q - 5)/15$.
4. For $p > 5$ and $p \equiv 1 \pmod{4}$, one has

\[
|A_\pi| = h(\sqrt{p}) \left[ (q - p + 1) \frac{\zeta_p(-1)}{2} + \left( \frac{13}{8} - \frac{5}{8} \left( \frac{2}{p} \right) \right) h(\sqrt{-p}) + \frac{h(\sqrt{-2p})}{4} + \frac{h(\sqrt{-3p})}{6} \right].
\]

For $p > 5$ and $p \equiv 1 \pmod{4}$, one has

\[
|A_\pi| = h(\sqrt{p}) \left[ (q - p + 1)(1 + \beta_p) \frac{\zeta_p(-1)}{2} + (1 + \beta_p) \frac{h(\sqrt{-p})}{8} + \frac{2h(\sqrt{-3p})}{3} \right],
\]

where $\beta_p = \frac{\pi}{2}(2 - \left( \frac{2}{p} \right))$.

**Proof.** (1) This follows form Propositions 4.2 (2) and 4.3 (2), $|\text{Sp}(\pi)| = 1$ and $|\text{PGL}_2(\mathbb{F}_2)| = 6$. (2) This follows from Propositions 4.2 (3) and 4.3 (2).

(3) and (4): By Propositions 4.2 (1) and 4.3 (2), we have $|A_\pi| = |\text{Sp}(\pi)| + (q - p)\text{Mass}(\text{Sp}(\pi))$. The desired formulas follow from (2.7) and the class number formulas in Section 3. \hfill \Box

We observe that the set $X$ in the proof is the set of $\mathbb{F}_q$-rational points of the Drinfeld period domain $\Omega^2$ over $\mathbb{F}_p$.

5. Analogies between abelian varieties and lattices

5.1. Local analogues. In this section we discuss analogies between abelian varieties and lattices. There are several descriptions of abelian varieties in terms of ideal classes or lattices. The prototype of such descriptions is the Deuring-Eichler correspondence, which establishes the following bijection

\[
\{\text{isomorphism classes of supersingular elliptic curves over } \mathbb{F}_p\} \longleftrightarrow \{\text{ideal classes of a maximal order in } D_{p,\infty}\},
\]

where $p$ is a prime number and $D_{p,\infty}$ is the quaternion $\mathbb{Q}$-algebra ramified exactly at $\{p, \infty\}$. We refer to Waterhouse [28], Deligne [7], Ekedahl [10], Katsura and
The basic analogy between abelian varieties and lattices may start as follows. For any \( \mathbb{Z} \)-lattice \( \Lambda \), the completion \( \Lambda \otimes \mathbb{Z}_\ell \) at a prime \( \ell \) is a local analogue of \( \Lambda \) at \( \ell \). If \( A \) is an abelian variety over an arbitrary field \( k \), then the \( \ell \)-divisible group \( A(\ell) := A[\ell^\infty] = \varprojlim A[\ell^n] \) associated to \( A \) can be viewed as its local analogue at \( \ell \).

If \( \ell \neq \text{char } k \), then there is an equivalence of categories
\[
\begin{align*}
(\text{\( \ell \)-divisible groups over } k) & \leftrightarrow (\text{finite free } \mathbb{Z}_\ell\text{-modules with continuous } \Gamma_k\text{-action}),
\end{align*}
\]
where \( \Gamma_k := \text{Gal}(k_s/k) \) is the absolute Galois group of \( k \) and \( k_s \) is a separable closure of \( k \).

When \( k \) is a perfect field of characteristic \( p > 0 \), the covariant Dieudonné theory establishes an equivalence of categories
\[
\begin{align*}
(\text{\( p \)-divisible groups over } k) & \leftrightarrow (\text{finite } W\text{-free Dieudonné modules over } k),
\end{align*}
\]
Thus, when the ground field \( k \) is finite, the local analogues of \( A \) can be described using its Tate modules and Dieudonné module. This description is utilized in Section 2. We shall extend some results of Section 2 to more general ground fields.

**5.2. Genera of abelian varieties.** We shall set up a long list of definitions, in order to make everything precise. Several of them also appear in [38] and Section 2. A field that is finitely generated over its prime field is called a finitely generated field. We write \( \text{Hom}(A_1, A_2) \) for the abelian group of \( k \)-homomorphisms between abelian varieties \( A_1 \) and \( A_2 \) over \( k \). In the following definition, the prime \( \ell \) is not necessarily distinct from \( \text{char } k \).

**Definition 5.1.** (1) Let \( A_1 \) and \( A_2 \) be two abelian varieties over a field \( k \).

(i) Denote by \( \text{Qisog}(A_1, A_2) \) the set of quasi-isogenies \( \varphi : A_1 \to A_2 \) over \( k \), i.e. \( N\varphi \) is an isogeny for some integer \( N \).

(ii) For any prime \( \ell \), denote by \( \text{Qisog}(A_1(\ell), A_2(\ell)) \) the set of quasi-isogenies \( \varphi : A_1(\ell) \to A_2(\ell) \) over \( k \). Denote by \( \text{Isom}(A_1(\ell), A_2(\ell)) \) the set of isomorphisms \( \varphi : A_1(\ell) \cong A_2(\ell) \) of \( \ell \)-divisible groups over \( k \).

(2) Let \( x = A_0 \) be an abelian variety over \( k \).

(i) Denote by \( G_x \) the automorphism group scheme over \( \mathbb{Z} \) associated to \( A_0 \). It is the group scheme over \( \mathbb{Z} \) which represents the functor
\[
\begin{align*}
R \mapsto G_x(R) = (\text{End}(A_0) \otimes \mathbb{Z}_R)^\times,
\end{align*}
\]
for any commutative ring \( R \).

(ii) Write \( x_\ell := A_0(\ell) \) for the associated \( \ell \)-divisible group. We define the automorphism groups scheme \( G_{x_\ell} \) over \( \mathbb{Z}_\ell \) associated to \( A_0(\ell) \) in the same way.

(3) Any abelian variety \( A \) in the isogeny class \([A_0]_{\text{isog}} \) of \( A_0 \) can be represented by a pair \((A, \varphi)\), where \( \varphi \in \text{Qisog}(A, A_0) \).
(i) Denote by \( \tilde{A}_x \) the set of all quasi-isogenies \((A, \varphi)\) to \(A_0\), where we regard two quasi-isogenies \((A_1, \varphi_1)\) to \(A_0\), for \(i = 1, 2\), as the same member if there is an element \(\alpha \in \text{Isom}(A_1, A_2)\) such that \(\varphi_1 \cdot \alpha = \varphi_2\).

(ii) Let \(A_x\) be the set of isomorphism classes \([A]\) of all members \((A, \varphi)\) \(\in \tilde{A}_x\).

For any prime \(\ell\), we define the sets \(\tilde{A}_x, A_x\) in the same way.

(4) Let \((A_1, \varphi_1)\) and \((A_2, \varphi_2)\) be two members in \(\tilde{A}_x\).

(i) \((A_i, \varphi_i)\) \((i = 1, 2)\) are said to be in the same genus if \(\text{Isom}(A_1(\ell), A_2(\ell)) \neq \emptyset\) for all primes \(\ell\).

(ii) \((A_i, \varphi_i)\) \((i = 1, 2)\) are said to be \((\text{globally})\) equivalent if there is an element \(\alpha \in G_x(\mathbb{Q})\) such that \((A_2, \varphi_2) = (A_1, \alpha \cdot \varphi_1)\). It is easy to see that they are equivalent if and only if their images in \(A_x\) are the same.

(iii) For any member \((A, \varphi)\) \(\in \tilde{A}_x\), let \(\tilde{A}(A, \varphi)\) denote the genus in \(\tilde{A}_x\) that contains \((A, \varphi)\), and

\[
A(A, \varphi) := \{[A'] \mid \exists \varphi' \text{ such that } (A', \varphi') \in \tilde{A}(A, \varphi)\}.
\]

(iv) Let

\[
G_{x, \lambda_f} := \prod_{\ell} G_{x, \ell}(\mathbb{Q}_\ell)
\]

be the restricted product of \(G_{x, \ell}(\mathbb{Q}_\ell)\) with respect to the open compact subgroups \(G_{x, \ell}(\mathbb{Z}_\ell)\). For any member \((A, \varphi)\) \(\in \tilde{A}_x\) and any element \(\alpha = (\alpha_\ell) \in G_{x, \lambda_f}\), there is a unique member \((A_1, \varphi_1) \in \tilde{A}_x\) such that for all primes \(\ell\), one has \((A_1(\ell), \varphi_1(\ell)) = (A(\ell), \alpha_\ell \varphi(\ell))\) (see Lemma 5.2). We shall write \(\alpha \cdot (A, \varphi) = (A_1, \varphi_1)\).

(v) Let \(H \subset G_{x, \lambda_f}\) be a subgroup. Say \((A_i, \varphi_i)\) \((i = 1, 2)\) are \(H\)-equivalent if there is an element \(\alpha \in H\) such that \(\alpha(A_1, \varphi_1) = (A_2, \varphi_2)\). It follows from the definition that \((A_i, \varphi_i)\) \((i = 1, 2)\) are in the same genus if and only if they are \(G_{x, \lambda_f}\)-equivalent.

(vi) Let \(H\) be an algebraic group over \(\mathbb{Q}\) together with an inclusion \(H(\mathbb{A}_f) \subset G_{x, \lambda_f}\). Two members \((A_i, \varphi_i)\) \((i = 1, 2)\) are in the same \(H\)-genus if they are \(H(\mathbb{A}_f)\)-equivalent.

Thus, we have the notion of \(G_x\)-genus since \(G_x(\mathbb{A}_f) \subset G_{x, \lambda_f}\).

**Lemma 5.2.** For any member \((A, \varphi)\) \(\in \tilde{A}_x\) and any \(\alpha = (\alpha_\ell) \in G_{x, \lambda_f}\), there is a unique member \((A_1, \varphi_1) \in \tilde{A}_x\) such that for all primes \(\ell\), one has

\[
(A_1(\ell), \varphi_1(\ell)) = (A(\ell), \alpha_\ell \cdot \varphi_\ell).
\]

**Proof.** We first prove the uniqueness of \((A_1, \varphi_1)\). Note that two members \((A_1, \varphi_1)\) and \((A_2, \varphi_2)\) of \(\tilde{A}_x\) are the same if and only if \(\ker(N \varphi_1)^\ell = \ker(N \varphi_2)^\ell\) as subgroup scheme of \(A_0^N\) for an integer \(N\) such that \(N \varphi_1\) \((i = 1, 2)\) are isogeny. The uniqueness then is characterized by (5.7).

Now we construct \((A_1, \varphi_1)\). Assume first that \(\alpha_\ell \varphi_\ell\) are isogenies for all \(\ell\). Let \(K \subset A_0^N\) be the product of \(\ker(\alpha_\ell \varphi_\ell)^\ell\) for all \(\ell\). Let \(A_1\) be the abelian variety such that \(A_1^1 = A_0^N/K\). Put \(\varphi_1 := \text{pr}^1 : A_1 \to A_0\), where \(\text{pr} : A_0^N \to A_1^1 = A_0^N/K\) is the natural projection. Then \((A_1, \varphi_1)\) has the desired property.

Now we choose an integer \(N\) such that \(N \alpha_\ell \varphi_\ell\) is an isogeny for all \(\ell\). By what we just proved, there is a unique member \((A_1, \varphi'_1)\) such that \((A_1(\ell), \varphi'_1(\ell)) = (A(\ell), N \alpha_\ell \varphi_\ell)\) for all \(\ell\). Then \((A_1, \varphi'_1/N)\) satisfies (5.7). \(\square\)
Theorem 5.3. We have $G_x(A_f) = G_x(A_f)$ if one of the following conditions holds:

(a) $k$ is a finitely generated field.
(b) $\text{char } k = p > 0$, $A_0$ is supersingular and $k$ is sufficiently large for $A_0$.

We say that the ground field $k$ is sufficiently large for an abelian variety $A$ if $\text{End}(A) = \text{End}(A \otimes k_s)$.

Proof. Statement (a) follows from Tate’s theorem on homomorphisms of abelian varieties, due to Tate, Zarhin, Faltings and de Jong (cf. [38, Theorem 2.1]). Statement (b) follows from the validity of Tate’s theorem for supersingular abelian varieties, which is well known. □

Proposition 5.4. Let $\Lambda_x := \Lambda(x)$ denote the set of isomorphism classes in the genus containing $x = (A_0, \text{id}) \in \tilde{A}_x$. Assume that one of the conditions in Theorem 5.3 holds. Then there is a natural isomorphism

$$
\Lambda_x \simeq G_x(\mathbb{Q})/G_x(A_f)/G_x(\mathbb{Z})
$$

which sends $[A_0]$ to the identity class.

Proof. According to Definition 5.1 (4) (iii)-(v), the genus containing $x$ is $G_{x,A_f} / \text{Stab}(x)$. Under the condition of (a) or (b), we have $G_{x,A_f} = G_x(A_f)$ and $\text{Stab}(x) = G_x(\mathbb{Z})$ by Theorem 5.3. It is clear that two members in the genus are isomorphic if and only if they are $G_x(\mathbb{Q})$-equivalent. This proves the proposition. □

5.3. Genera and ideal complexes of abelian varieties with additional structures. We consider a PEL-type setting. Let $(B, \ast)$ be finite-dimensional semisimple $\mathbb{Q}$-algebra with positive involution $\ast$, and $O_B \subset B$ be a $\mathbb{Z}$-order stable under the involution $\ast$. When $O_B = \mathbb{Z}$, this setting specializes to the consideration of polarized abelian varieties.

Definition 5.5. (1)

(i) A polarized $O_B$-abelian variety is a triple $\mathcal{A} = (A, \lambda, \iota)$, where $(A, \lambda)$ is a polarized abelian variety and $\iota : O_B \to \text{End}(A)$ is a ring monomorphism such that $(a \lambda)^I = \lambda(a^*)$ for all $a \in O_B$.

(ii) A fractional polarization or $\mathbb{Q}$-polarization is an element $\text{Hom}(A, A^I) \otimes \mathbb{Q}$ such that $N \lambda$ is a polarization for some positive integer $N$; we call a fractional polarization integral if $N$ can be chosen to 1, i.e. it is a polarization.

(iii) A fractionally polarized (or $\mathbb{Q}$-polarized) $O_B$-abelian variety is a triple $(A, \lambda', \iota')$ such that $(A, N \lambda', \iota)$ is a polarized $O_B$-abelian variety for some positive integer $N$.

(iv) For any prime $\ell$, write $\mathcal{A}(\ell) = (A(\ell), \lambda_\ell, \iota_\ell)$, where $\lambda_\ell : A(\ell) \to A^I(\ell) = A(\ell)^I$ is induced morphism for $\ell$-divisible groups, and $\iota_\ell : O_B \otimes \mathbb{Z}_\ell \to \text{End}(A) \otimes \mathbb{Z}_\ell \to \text{End}(A(\ell))$ is the $\mathbb{Z}_\ell$-linear extension of $\iota$.

(2) Let $\mathcal{A}_1 = (A_1, \lambda_1, \iota_1)$ and $\mathcal{A}_2 = (A_2, \lambda_2, \iota_2)$ be two polarized $O_B$-abelian varieties over a field $k$.

(i) Denote by $\text{Qisog}(\mathcal{A}_1, \mathcal{A}_2)$ (resp. $\text{Gisom}(\mathcal{A}_1, \mathcal{A}_2)$) the set of all $O_B$-linear quasi-isogenies (resp. isomorphism) $\varphi : A_1 \to A_2$ such that $\varphi^* \lambda_1 = \lambda_2$ (resp. $\varphi^* \lambda_1 = c \lambda_2$ for some $c \in \mathbb{Q}^\times$).
(ii) For any prime \(\ell\), denote by \(\text{Qisog}(\underline{A}_1(\ell), \underline{A}_2(\ell))\) (resp. \(\text{Gisom}(\underline{A}_1(\ell), \underline{A}_2(\ell))\)) the set of all \(O_B \otimes \mathbb{Z}_\ell\)-linear quasi-isogenies (resp. isomorphism) \(\varphi : A_1(\ell) \to A_2(\ell)\) such that \(\varphi^* \lambda_2,\ell = \lambda_1,\ell\) (resp. \(\varphi^* \lambda_2,\ell = c_\ell \lambda_1,\ell\) for some \(c_\ell \in \mathbb{Q}_\ell^\times\)).

(iii) We say \(\underline{A}_1\) and \(\underline{A}_2\) (resp. \(\underline{A}_1(\ell)\) and \(\underline{A}_2(\ell)\)) are isogenous if \(\text{Qisog}(\underline{A}_1, \underline{A}_2) \neq \emptyset\) (resp. \(\text{Qisog}(\underline{A}_1(\ell), \underline{A}_2(\ell)) \neq \emptyset\)).

(iv) We say \(\underline{A}_1\) and \(\underline{A}_2\) (resp. \(\underline{A}_1(\ell)\) and \(\underline{A}_2(\ell)\)) are similar if \(\text{Gisom}(\underline{A}_1, \underline{A}_2) \neq \emptyset\) (resp. \(\text{Gisom}(\underline{A}_1(\ell), \underline{A}_2(\ell)) \neq \emptyset\)).

(3) Let \(\underline{x} = \underline{A}_0 = (A_0, \lambda_0, \iota_0)\) be a polarized \(O_B\)-abelian variety over \(k\).

(i) Denote by \(\tilde{G}_{\underline{x}}\) the automorphism group scheme over \(\mathbb{Z}\) associated to \(\underline{A}_0\). It is a group scheme which represents the functor

\[
R \mapsto \tilde{G}_{\underline{x}}(R) = \{g \in (\text{End}_{O_B}(A_0) \otimes \mathbb{Z})^\times \mid g^* \lambda g = \lambda\}.
\]

(ii) Denote by \(\tilde{G}_{\underline{x}}\) the group scheme of similitudes over \(\mathbb{Z}\) associated to \(\underline{A}_0\).

It is a group scheme which represents the functor

\[
R \mapsto \tilde{G}_{\underline{x}}(R) = \{g \in (\text{End}_{O_B}(A_0) \otimes \mathbb{Z})^\times \mid \exists c \in R^\times \text{ s.t. } g^* \lambda g = \lambda c(g)\}.
\]

(iii) For any prime \(\ell\), write \(\underline{x}_\ell = \underline{A}_0(\ell)\) and define the group schemes \(G_{\underline{x}_\ell}\) and \(\tilde{G}_{\underline{x}_\ell}\) over \(\mathbb{Z}_\ell\) in the same way.

(4) Let \(x = A_0\) be the underlying abelian variety of \(\underline{x} = \underline{A}_0\).

(i) \(\tilde{A}_{\underline{x}} := \left\{ (\underline{A}, \varphi) \mid \underline{A} \text{ is a } \mathbb{Q}\text{-polarized } O_B\text{-abelian variety over } k \text{ and } \varphi \in \text{Qisog}(\underline{A}, \underline{A}_0) \right\} \subseteq \tilde{A}_x\).

(ii) Let \(G_{\underline{x}, A_\ell} := \prod \text{G}_{\underline{x}}(\mathbb{Q}_\ell)\) and \(\tilde{G}_{\underline{x}, A_\ell} := \prod \tilde{G}_{\underline{x}, A_\ell}(\mathbb{Q}_\ell)\) be the restricted products of local groups \(G_{\underline{x}}(\mathbb{Q}_\ell)\) and \(\tilde{G}_{\underline{x}}(\mathbb{Q}_\ell)\) with respect to \(G_{\underline{x}}(\mathbb{Z}_\ell)\) and \(\tilde{G}_{\underline{x}}(\mathbb{Z}_\ell)\), respectively. The group action of \(G_{\underline{x}, A_\ell}\) on \(\tilde{A}_x\) induces a group action of \(G_{\underline{x}, A_\ell}\) and of \(\tilde{G}_{\underline{x}, A_\ell}\) on \(\tilde{A}_{\underline{x}}\).

(5) Let \((\underline{A}_1, \varphi_1)\) and \((\underline{A}_2, \varphi_2)\) be two members of \(\tilde{A}_{\underline{x}}\).

(i) \((\underline{A}_i, \varphi_i)\) (\(i = 1, 2\)) are said to be in the same genus if \(\text{Isom}(\underline{A}_1, \underline{A}_2) \neq \emptyset\) for all primes \(\ell\). It is easy to see that two members are in the same genus (resp. are isomorphic) if and only if they are \(G_{\underline{x}, A_\ell}\)-equivalent (resp. \(G_{\underline{x}}(\mathbb{Q})\)-equivalent). The isomorphism class of \(\underline{A}_1\) is denoted by \([\underline{A}_1]\).

(ii) \((\underline{A}_i, \varphi_i)\) (\(i = 1, 2\)) are said to be in the same ideal complex if \(\underline{A}_1(\ell)\) and \(\underline{A}_2(\ell)\) are similar for all primes \(\ell\). It is easy to see that two members are in the same ideal complex (resp. are similar) if and only if they are \(G_{\underline{x}, A_\ell}\)-equivalent (resp. are \(G_{\underline{x}}(\mathbb{Q})\)-equivalent). The similitude class of \(\underline{A}_1\) is denoted by \([\underline{A}_1]\).

(iii) \((\underline{A}_i, \varphi_i)\) (\(i = 1, 2\)) are said to be in the same similitude genus if there is an element \(\alpha \in G_{\underline{x}}(\mathbb{Q})\) such that the members \(\alpha \cdot (\underline{A}_1, \varphi_1)\) and \((\underline{A}_2, \varphi_2)\) are in the same genus.

(6) Let \(\underline{x}' = (\underline{A}', \varphi')\) be a member of \(\tilde{A}_{\underline{x}}\).

(i) Let \(\tilde{A}_{\underline{x}'} \subseteq \tilde{A}_{\underline{x}}\) be the genus containing \((\underline{A}', \varphi')\), and \(\Lambda_{\underline{x}'} := [\tilde{A}_{\underline{x}'}]\) the set of isomorphism classes of all members in \(\tilde{A}_{\underline{x}'}\). One has \(\Lambda_{\underline{x}'} = G_{\underline{x}}(\mathbb{Q}) \setminus \Lambda_{\underline{x}'}\).

(ii) Let \(\tilde{I}_{\underline{x}'} \subseteq \tilde{A}_{\underline{x}'}\) be the ideal complex containing \((\underline{A}', \varphi')\), and \(\tilde{I}_{\underline{x}'} := [\tilde{I}_{\underline{x}'}]\) the set of similitude classes of all members in \(\tilde{I}_{\underline{x}'}\). One has \(\tilde{I}_{\underline{x}'} = G_{\underline{x}}(\mathbb{Q}) \setminus \tilde{I}_{\underline{x}'}\).
(iii) Let \( \tilde{\Lambda}_s^* \subset \tilde{\Lambda}_x^* \) be the similitude genus containing \((A', \varphi')\), and \( \Lambda_{s'}^* := [\tilde{\Lambda}_s^*]_s \) the set of similitude classes of all members in \( \tilde{\Lambda}_s^* \). One has \( \Lambda_{s'}^* = GU_{s'}(Q) \tilde{\Lambda}_s^* \).

We now introduce the notion of basic abelian varieties with additional structures. The concept is originally defined in Kottwitz \([16]\) in the content of isocrystals with \(G\)-structures and the framework of Tannakian categories. Basic abelian varieties with additional structures are first introduced in Rapoport and Zink \([21]\) which rely on the construction of integral models of PEL-type Shimura varieties. A convenient definition which does not require the background on Shimura varieties is as follows (see \([38]\)).

**Definition 5.6.** Let \((B, \ast)\) and \(O_B\) remain as above.

1. Let \((V_p, \psi_p)\) be a \(Q_p\)-valued non-degenerate skew-Hermitian \(B_p\)-module, where \(B_p := B \otimes_Q Q_p\). A polarized \(O_B\)-abelian variety \(A = (A, \lambda, \iota)\) over an algebraically closed field \(k\) of characteristic \(p\) is said to be related to \((V_p, \psi_p)\) if there is a \(B_p \otimes_{Q_p} L\)-linear isomorphism \(\alpha : M(A) \otimes_W L \simeq (V_p, \psi_p) \otimes_{Q_p} L\) which preserves the pairings up to a scalar in \(L^\times\), where \(W\) is the ring of Witt vectors over \(k\), \(L\) the fraction field of \(W\), and \(M(A)\) is the covariant Dieudonné module with additional structures associated to \(A\).

Let \(G' := GU_{B_p}(V_p, \psi_p)\) be the algebraic group over \(Q_p\) of \(B_p\)-linear similitudes on \((V_p, \psi_p)\). By transporting the structure of the Frobenius map on \(V_p \otimes L\), we obtain an element \(b \in G'(L)\) such that

\[
(5.11) \quad \alpha : M(A) \otimes L \simeq (V_p \otimes L, \psi_p, b(id \otimes \sigma))
\]
is an isomorphism of isocrystals with additional structures. The decomposition of \(V_p \otimes L\) into isoclinic components induces a \(Q\)-graded structure, and thus defines a (slope) homomorphism \(\nu_b : D \to G'\) over \(L\), where \(D\) is the pro-torus over \(Q_p\) with character group \(Q\).

2. A polarized \(O_B\)-abelian variety \(A\) over an algebraically closed field \(k\) of characteristic \(p\) is said to be basic with respect to \((V_p, \psi_p)\) if

   (a) \(A\) is related to \((V_p, \psi_p)\), and
   (b) the slope homomorphism \(\nu_b\) is central.

Note that this is independent of the choice of \(\alpha\) and \(b\). We call \(A\) basic if it is basic with respect to \((V_p, \psi_p)\) for some skew-Hermitian space \((V_p, \psi_p)\).

3. An polarized \(O_B\)-abelian variety \(A\) over an arbitrary field \(k\) of characteristic \(p\) is said to be basic if the base change \(A \otimes_k \bar{k}\) over its algebraic closure \(\bar{k}\) is basic.

One can choose a suitable isomorphism \(\alpha\) in \((5.11)\) such that the slope homomorphism \(\nu_b : D \to G'\) is defined over \(Q_{p^s}\) for some positive integer \(s\) \([16]\) Section 4.3. However, this fact is not needed in Definition 5.6. It is clear that the notion of basic abelian varieties with additional structures is preserved under isogeny. By \([16]\) Theorem 1.1, any basic abelian varieties with additional structures over an arbitrary algebraically closed field of characteristic \(p\) is isogenous to another one which is defined a finite field. It follows that the notion of basic abelian varieties with additional structures does not depend on the ground field over which they are defined.

**Theorem 5.7.** Assume one of the following conditions holds:
(a) \( k \) is a finitely generated field.
(b) \( \text{char } k = p > 0 \), \( A_0 \) is basic and \( k \) is sufficiently large for \( A_0 \).
We have \( G_{\tilde{x}}(A_f) = G_{\tilde{x},A_f} \) and \( GU_{\tilde{x}}(A_f) = GU_{\tilde{x},A_f} \).

**Proof.** Statement (a) follows directly from Theorem 5.3. Statement (b) is proved in \[38, Theorem 3.12 and Proposition 4.5\]. \( \square \)

Let \( G \) be a linear algebraic group over \( \mathbb{Q} \) and \( U \subset G(A_f) \) an open compact subgroup. Denote by \( DS(G,U) \) for the double coset space \( G(\mathbb{Q}) \setminus G(A_f)/U \). This is a finite set by a finiteness result of Borel and Harish-Chandra \[1\]. Assume that any arithmetic subgroup of \( G(\mathbb{Q}) \) is finite. Let \( c_1, c_2, \ldots, c_h \) be a complete coset representatives for \( DS(G,U) \) and put \( \Gamma_i := G(\mathbb{Q}) \cap c_i U c_i^{-1} \) for \( i = 1, \ldots, h \). Define the mass of \( (G,U) \) by

\[
\text{Mass}(G,U) := \sum_{i=1}^{h} |\Gamma_i|^{-1}.
\]

For any finite set \( S \) consisting of objects with finite automorphism groups, define the mass of \( S \) by

\[
\text{Mass}(S) := \sum_{s \in S} |\text{Aut}(s)|^{-1}.
\]

**Theorem 5.8.** Let \( \Lambda_{\tilde{x}} \) (resp. \( I_{\tilde{x}} \); resp. \( \Lambda_{\tilde{x}}^* \)) the set of isomorphism classes (resp. similitude classes) in the genus (resp. the ideal complex; resp. the similitude genus) containing the base member \( \tilde{x} = (A_0, \text{id}) \in \tilde{\mathcal{A}}_{\tilde{x}} \). Assume that one of conditions in Theorem 5.7 holds.
(1) There are natural isomorphisms sending the base class to the identity class:

\[
\Lambda_{\tilde{x}} \simeq DS(G_{\tilde{x}}, G_{\tilde{x}}(\hat{\mathbb{Z}})),
\]
\[
I_{\tilde{x}} \simeq DS(GU_{\tilde{x}}, GU_{\tilde{x}}(\hat{\mathbb{Z}})).
\]
\[
\Lambda_{\tilde{x}}^* \simeq GU_{\tilde{x}}(\mathbb{Q}) \setminus GU_{\tilde{x}}(\mathbb{Q})G_{\tilde{x}}(A_f)/G_{\tilde{x}}(\hat{\mathbb{Z}}).
\]

(2) We have

\[
\text{Mass}(\Lambda_{\tilde{x}}) = \text{Mass}(G_{\tilde{x}}, G_{\tilde{x}}(\hat{\mathbb{Z}})),
\]
\[
\text{Mass}(I_{\tilde{x}}) = \text{Mass}(GU_{\tilde{x}}, GU_{\tilde{x}}(\hat{\mathbb{Z}})),
\]
\[
\text{Mass}(\Lambda_{\tilde{x}}^*) = \sum_{i=1}^{h} |\Gamma_i|^{-1},
\]

where \( c_1, \ldots, c_h \) are coset representatives for \( GU_{\tilde{x}}(\mathbb{Q}) \setminus GU_{\tilde{x}}(\mathbb{Q})G_{\tilde{x}}(A_f)/U \), \( U = G_{\tilde{x}}(\hat{\mathbb{Z}}) \), and \( \Gamma_i := GU_{\tilde{x}}(\mathbb{Q}) \cap c_i U c_i^{-1} \).

**Proof.** Statements (5.14) and (5.17) are proved in \[38, Theorems 2.2 and 4.6\]. The same proofs also prove (5.15) and (5.18), because we have \( GU_{\tilde{x},A_f} = GU_{\tilde{x}}(A_f) \) for cases (a) and (b) in Theorem 5.7. By definition,

\[
\tilde{\Lambda}_{\tilde{x}}^* = GU_{\tilde{x}}(\mathbb{Q}) \cdot \Lambda_{\tilde{x}} = GU_{\tilde{x}}(\mathbb{Q})G_{\tilde{x}}(A_f)/G_{\tilde{x}}(\hat{\mathbb{Z}}),
\]

and we get (5.16). Formula (5.19) follows from (5.16) and (5.18) because \( \Lambda_{\tilde{x}}^* \) is a subset of \( I_{\tilde{x}} \). \( \square \)
5.4. **Principally polarized abelian varieties over finite fields.** Let $\mathcal{A}_0 = (A_0, \lambda_0)$ be a principally polarized abelian variety over $F_q$. Then the group schemes $GU_{\mathcal{A}_0}$ and $G_{\mathcal{A}_0}$ represent the functor

\begin{align}
R \mapsto \{ x \in (\text{End}(A_0) \otimes R)^\times | xx' \in R^\times \}, \quad \text{and} \\
R \mapsto \{ x \in (\text{End}(A_0) \otimes R)^\times | xx' = 1 \},
\end{align}

respectively, where $'$ is the Rosati involution on $\text{End}(A_0)$ induced by $\lambda_0$. We denote by $I_{\mathcal{A}_0}$ the set of similitude classes of the ideal complex containing $(A_0, \lambda_0)$, and by $\Lambda_{\mathcal{A}_0}$ the set of isomorphism classes of the genus containing $(A_0, \lambda_0)$. By Theorem 5.8, we have natural bijections

\begin{align}
\Lambda_{\mathcal{A}_0} &\simeq G_{\mathcal{A}_0}(\mathbb{Q})\backslash G_{\mathcal{A}_0}(\mathbb{A}_f)/G_{\mathcal{A}_0}(\mathbb{Z}), \\
I_{\mathcal{A}_0} &\simeq GU_{\mathcal{A}_0}(\mathbb{Q})\backslash GU_{\mathcal{A}_0}(\mathbb{A}_f)/GU_{\mathcal{A}_0}(\mathbb{Z}).
\end{align}

We call an abelian variety $A$ over a finite field principal if the endomorphism algebra $\text{End}^0(A)$ of $A$ is commutative and the endomorphism ring $\text{End}(A)$ is the maximal order. A genus of abelian varieties which contains a principal abelian variety is called a principal genus.

Suppose that $A_0$ is principal. Then $\text{End}^0(A_0)$ is a CM algebra $K = \prod_{i=1}^r K_i$ with each $K_i$ a CM field, and $'$ is the canonical involution on $K$. Let $K^+ = \prod_{i=1}^r K_i^+$ be the maximal totally real subalgebra, which is the fixed subalgebra of $'$. We denote the maximal orders of $K$ and $K^+$ by $O_K$ and $O_{K^+}$, respectively. Let $T = T_{K, \mathbb{Q}}$ be the algebraic “torus” over $\mathbb{Z}$ defined by $T(R) = \{ x \in (O_K \otimes \mathbb{Z})^\times | N_{K/K^+}(x) = x \cdot x' \in R^\times \}$ for any commutative ring $R$. Let $T_{K, \mathbb{Q}}$ be the $K/K^+$-norm one subtorus of $T$. Recall the class number $h_T$ of $T$ is the cardinality of $T(\mathbb{Q})\backslash T(\mathbb{A}_f)/U_T$, where $U_T$ is the maximal open compact subgroup of $T(\mathbb{A}_f)$. Since $T(\mathbb{Z})$ and $T_{K, \mathbb{Q}}(\mathbb{Z})$ are maximal open compact subgroups in the adelic groups, by (5.21), we get

\begin{align}
|\Lambda_{\mathcal{A}_0}| = h_{T_{K, 1}} \quad \text{and} \quad |I_{\mathcal{A}_0}| = h_T.
\end{align}

**Proposition 5.9.** Let $K = \prod_{i=1}^r K_i$ be a CM algebra with maximal totally real subalgebra $K^+ = \prod_{i=1}^r K_i^+$.

1. We have

\begin{align}
h_{T_{K, 1}} &= \frac{h_K}{h_{K^+}} \cdot \frac{1}{Q \cdot 2^{t - r}},
\end{align}

where $h_K := |\text{Pic}(O_K)|$ (resp. $h_{K^+}$) is the class number of $K$ (resp. $K^+$), $Q := [O_K^\times : \mu_K \cdot O_{K^+}^\times]$ is the Hasse unit index, and $t$ is the number of finite places of $K^+$ ramified in $K$.

2. We have

\begin{align}
h_{T_{K, \mathbb{Q}}} &= \frac{h_K}{h_{K^+}} \cdot \frac{1}{Q \cdot 2^{t - r}},
\end{align}

that is, $h_{T_{K, \mathbb{Q}}} = h_{T_{K, 1}}$.

The formula (5.23) is proved when $K$ is a CM field; see [24] (16, p. 375). When $K$ is a CM algebra, the formula then follows from $h_{T_{K, 1}} = \prod_{i=1}^r h_{T_{K_i, 1}}$ and loc. cit. The formula (5.24) is proved in the same way by the class number relation [24].

\[2\] More general than in [28, Section 5], $A$ needs not be simple here.
Theorem 4, p. 375. The details of the computation will be published elsewhere. By Proposition 5.9 and (5.22), we obtain the following result.

**Theorem 5.10.** Let \( (A_0, \lambda_0) \) be a principally polarized principal abelian variety over \( \mathbb{F}_q \) with endomorphism algebra \( \text{End}^0(A_0) = K = \prod_{i=1}^r K_i \). Then the genus \( \Lambda_{x_0} \) of \( x_0 \) has

\[
\frac{h_K}{h_{K^+}} \cdot \frac{1}{Q \cdot 2^r-r}
\]

isomorphism classes, and the ideal complex \( I_{x_0} \) of \( x_0 \) has the same number of similarity classes.

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