On Secure Capacity of Multiple Unicast Traffic over Two-Layer Networks

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Abstract—This paper studies the problem of secure communication over two-layer networks, where a source is connected to a set of relays via direct edges. These relays are then connected to destinations, such that each destination has direct connections to a subset of relays. In multiple unicast traffic, the source wishes to transmit independent information to each of the destinations. This work studies the secure capacity region for this traffic over the defined two-layer networks, under the assumption of perfect information theoretic security criterion from an adversary who can access any edges of the network. In particular, the secure capacity region is characterized when either \( K = 1 \) or \( m = 3 \). Moreover, conditions sufficient to characterize the secure capacity region are provided for arbitrary values of \( K \) and \( m \).

I. INTRODUCTION

Today, a large portion of exchanged data over communication networks is inherently sensitive and private (e.g., banking, professional, health). This data is of the order of terabytes per seconds and hence it calls for efficient mechanisms for secure communication. Moreover, with the advent of quantum computing, we can no longer rely on cryptographic-based secure communication schemes. Thus, information theoretic security, which ensures that no information is leaked about the message exchanged between two or more trusted parties, is today more important than ever. However, information theoretic optimal schemes and secure capacity characterization of arbitrary networks with arbitrary traffic are not known.

In this paper, we consider wireline networks modeled as directed graphs of unit capacity edges. In particular, we consider two-layer networks, where a source is connected to a set of relays via direct edges. These relays are then connected to destinations, such that each destination is directly connected to a subset of relays. An example of such network with 6 relays and 3 destinations is shown in Fig. 1. A passive external adversary, Eve, wishes to learn the data exchanged over this network. Eve has unbounded computational capabilities (e.g., a quantum computer), but limited network presence, namely, she can wiretap at most \( K \) edges of her choice. Over such a network, information theoretic security seeks to design transmission schemes that are unconditionally/perfectly secure, i.e., no matter which \( K \) edges Eve wiretaps, she does not learn anything about the content of the exchanged information.

In [1], we recently designed an information theoretic secure communication scheme for two-layer networks, and showed that this scheme is indeed capacity achieving when \( m = 2 \). In particular, in [1], we showed that for \( m = 2 \) it is sufficient to ‘separate’ the network into two disjoint sub-networks, where: (i) one subnetwork is used to multicast the keys to the \( m = 2 \) destinations, and (ii) the other subnetwork is used to transmit the encoded messages. However, in [1] we also pointed out that such a ‘separation’ scheme is not optimal when \( m > 2 \). For instance, for the network in Fig. 1 we can achieve higher rates if we jointly use the network to transmit keys and encoded messages, i.e., multicasting the keys using a portion of the network, and sending the encoded messages using the remaining portion of the network does not achieve the capacity.

In this paper, we build on our recent result in [1] and extend it. In particular, we first characterize the secure rate region achieved by the scheme proposed in [1], and show that this scheme is capacity achieving for the following additional scenarios: (i) networks where \( m = 3 \) and \( K \) is arbitrary; (ii) networks where \( K = 1 \) and \( m \) is arbitrary; (iii) networks where \( K \) and \( m \) are arbitrary, but the network has some special structure in terms of minimum cut.

Related Work. Shannon [2] proved that the one-time pad can provide perfect information theoretic security with pre-shared keys. For degraded point-to-point channels, Wyner [3] showed that information theoretic security can be achieved without pre-shared keys. With feedback, Maurer [4] proved that secure communication is possible, even when the adversary has a channel of better quality than the legitimate receiver.

Multicast traffic over networks of unit capacity edges was analyzed by Cai et al. in [5], and followed by several other works, such as [6], [7]. In [5], the information theoretic secure capacity was characterized for networks where a source multi-
casts the exact same information to a number of destinations in the presence of a passive external adversary eavesdropping any $K$ edges of her choice. In [8], Cui et al. studied networks with non-uniform edge capacities when the adversary is allowed to eavesdrop only some specific subsets of edges.

In [9], we recently studied security over multiple unicast traffic, and we characterized the secure capacity region for networks with single source and $m = 2$ destinations. In [9], for networks with arbitrary number of destinations, we also provided a suboptimal scheme which first multicasts the keys to the $m$ destinations and then transmits the encoded messages.

Paper Organization. In Section II we define two-layer networks and formulate the problem of characterizing the secure capacity region. In Section III we review the secure scheme works and formulate the problem of characterizing the secure capacity region. In Section V and Section VI, we prove that the scheme achieves the capacity when $K, m$ and $m = 3$, respectively. In Section VI, we also provide sufficient conditions that ensure that the scheme is capacity achieving for two-layer networks with arbitrary values of $K$ and $m$.

II. SYSTEM MODEL AND PROBLEM FORMULATION

Notation: Calligraphic letters indicate sets; $\emptyset$ is the empty set and $|A|$ is the cardinality of $A$; for two sets $A_1, A_2, A_1 \subseteq A_2$ indicates that $A_1$ is a subset of $A_2$, $A_1 \cup A_2$ indicates the union of $A_1$ and $A_2$, $A_1 \cap A_2$ indicates the disjoint union of $A_1$ and $A_2$, $A_1 \triangle A_2$ is the set of elements that belong to $A_1$ but not to $A_2$; $[n]$ is the set of integers from 1 to $n \geq 1$; $[x]^+ := \max \{0, x\}$ for $x \in \mathbb{R}$; for a matrix $A$, $A^T$ is its transpose; $\dim(A)$ is the dimension of the rowspace spanned by rows of $A$; for two vector subspaces $V_1$ and $V_2$, we let $V_1 + V_2 := \{v_1 + v_2 : v_1 \in V_1, v_2 \in V_2\}$.

A two-layer network consists of one source $S$ that wishes to communicate with $m$ destinations, by hopping information through one layer of $t$ relays. As such, a two-layer network is parameterized by: (i) the integer $t$, which denotes the number of relays in the first layer; (ii) the integer $m$, which denotes the number of destinations in the second layer; (iii) $m$ sets $M_i$, $i \in [m]$, such that $M_i \subseteq [t]$, where $M_i$ contains the indexes of the relays connected to destination $D_i$. An example of a two-layer network is shown in Fig. 1 for which $t = 6$, $m = 3$, $M_1 = \{1, 2, 4\}$, $M_2 = \{3, 4, 5, 6\}$ and $M_3 = \{2, 3\}$.

We represent a two-layer wireline network with a directed acyclic graph $G = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the set of nodes and $\mathcal{E}$ is the set of edges. The edges represent orthogonal communication links, which are interference-free. In particular, these links are discrete noiseless memoryless channels over a common alphabet $\mathbb{F}_q$, i.e., they are of unit capacity over a $q$-ary alphabet. If an edge $e \in \mathcal{E}$ connects a node $i$ to a node $j$, we refer to node $i$ as the tail and to node $j$ as the head of $e$, i.e., $tail(e) = i$ and $head(e) = j$. For each node $v \in \mathcal{V}$, we define $\mathcal{I}(v)$ as the set of all incoming edges of node $v$ and $\mathcal{O}(v)$ as the set of all outgoing edges of node $v$.

Source $S$ has a message $W_i$ for destination $D_i$, $i \in [m]$. These $m$ messages are assumed to be independent. Thus, the network consists of multiple unicast traffic, where $m$ unicast sessions take place simultaneously and share the network resources. In particular, each message $W_i, i \in [m]$, is of $q$-ary entropy rate $R_i$. A passive external eavesdropper Eve is also present and can wiretap any $K$ edges of her choice. The symbol transmitted over $n$ channel uses on $e \in \mathcal{E}$ is denoted as $Y^n_e$. In addition, for $\mathcal{E}_i \subseteq \mathcal{E}$ we define $Y^n_{\mathcal{E}_i} = \{Y^n_e : e \in \mathcal{E}_i\}$.

We assume that $S$ has infinite sources of randomness $\Theta$, while the other nodes in the network do not have any randomness.

Over this network, we are interested in finding all possible feasible $m$-tuples $(R_1, R_2, \ldots, R_m)$ such that each $D_i, i \in [m]$, reliably decodes the message $W_i$ (with zero error) and Eve receives no information about the content of the messages. In particular, we are interested in ensuring perfect information theoretic secure communication, and we aim at characterizing the secure capacity region, which is next formally defined.

Definition 1. A rate $m$-tuple $(R_1, R_2, \ldots, R_m)$ is said to be securely achievable if there exist a block length $n$ and a set of encoding functions $f_e, \forall e \in \mathcal{E}$, with $Y^n_e = \begin{cases} f_e(W_{i}, \Theta) & \text{if tail}(e) = S, \\ f_e(\{Y^n_{\mathcal{E}_i} : e \in \mathcal{T}(tail(e))\}) & \text{otherwise}, \end{cases}$ such that each destination $D_i$ can reliably decode the message $W_i$ i.e., $H(W_i | \{X^n_{\mathcal{E}_i} : e \in \mathcal{T}(D_i)\}) = 0, \forall i \in [m]$. Moreover, $\forall E \subseteq \mathcal{E}, |E| \leq K$, $I(W_{m} ; Y^n_{E}) = 0$ (perfect secrecy requirement).

The secure capacity region is the closure of all such feasible $m$-tuples.

Definition 2. A cut is an edge set $\mathcal{E}_A \subseteq \mathcal{E}$, which separates the source $S$ from a set of destinations $D_A := \{D_i, i \in \mathcal{A}\}$. In a network with unit capacity edges, the minimum cut or min-cut is a cut that has the minimum number of edges.

With $M_A$ we denote the cardinality of a min-cut between the source and the set of destinations $D_i$, such that $i \in \mathcal{A} \subseteq [m]$. It is worth noting that for two-layer networks, we have that $M_A = |\cup_{i \in \mathcal{A}} M_i|$. For notational convenience, in the remainder of the paper, we let $M_{\mathcal{A}} = M_{\mathcal{A}} \cup \cup_{i \in \mathcal{A}} M_i$. Moreover, we also assume that $M_{\mathcal{A}} \geq K, \forall i \in [m]$ (otherwise secure communication is not possible) with $M_{\emptyset} := K$ for consistency.

III. SECURE TRANSMISSIONS SCHEME

We here review the secure transmission scheme for two-layer networks that we recently proposed in [1]. In Section V we will then derive its achieved rate region. The source $S$ encodes the message packets with $K$ random packets and transmits these packets on its outgoing edges to the $t$ relays. We can write the received symbols at the $t$ relays as

\[
\begin{bmatrix}
Y_1 \\
Y_2 \\
\vdots \\
Y_t
\end{bmatrix} =
\begin{bmatrix}
M & V \\
W_1 \\
W_2 \\
\vdots \\
W_m \\
X
\end{bmatrix},
\]

where: (i) $W_i, i \in [m]$ is a column vector of $R_i$ message packets for destination $D_i$, (ii) $X$ is a column vector which
contains the $K$ random packets, (iii) $M$ is a matrix of dimension $t \times (\sum_{i=1}^{m} R_i)$, and (iv) $V$ is a Vandermonde matrix of dimension $t \times K$. The matrix $V$ is chosen for security purposes, i.e., any set of $K$ rows of $V$ are linearly independent and hence, no matter which $K$ rows (i.e., edges) Eve wiretaps, she will learn nothing about the messages $W_{[m]}$.

Each relay $i \in [l]$ will then forward the received symbol $Y_i$ to the destinations to which it is connected. As such, each destination will subset a vector of symbols from $\{Y_1, Y_2, \ldots, Y_l\}$ (depending on which of the $t$ relays it is connected to). Finally, destination $D_i, i \in [m]$ selects a decoding vector and performs the inner product with $[Y_1, Y_2, \ldots, Y_l]$. In particular, this decoding vector is chosen such that it has components corresponding to the relays it is not connected to; this ensures that each destination uses only the symbols that it actually receives. Specifically, each destination will observe a subset of symbols from $\{Y_1, Y_2, \ldots, Y_l\}$ such that each destination uses only the symbols that it actually receives.

Theorem 2. For the two-layer network when Eve wiretaps any $K = 1$ edge of her choice, the secure capacity region is

$$\sum_{i \in A} R_i \leq M_A - C_A, \forall A \subseteq [m],$$

with $C_A$ being the number of connected components in an undirected graph where: (i) there are $|A|$ nodes, i.e., one for each $i \in A$; (ii) an edge between node $i$ and node $j$, $\{i, j\} \in A$, $i \neq j$, exists if $M_i \cap M_j \neq \emptyset$.

A. Outer Bound

Our outer bound on the secure capacity region from is

$$\sum_{i \in A} R_i \leq M_A - K, \forall A \subseteq [m].$$

We now show that the outer bound in can be equivalently written as in . Let $V_i, i \in [C_A]$, represent the set of nodes in the $i$-th component of the graph constructed as explained in Theorem Then, clearly $A = \bigcup_{i=1}^{C_A} V_i$ and we can write

$$\sum_{i \in A} R_i = \sum_{i \in V_1} R_i + \sum_{i \in V_2} R_i + \ldots + \sum_{i \in V_{C_A}} R_i \leq (M_{V_1} - K) + (M_{V_2} - K) + \ldots + (M_{V_{C_A} - K})$$

where: (i) the inequality in (a) follows by applying for each set $V_i, i \in [C_A]$, (ii) the equality in (b) follows since, by construction, $M_i \cap M_j = \emptyset$ for all $i \in V_x$ and $j \in V_y$ with $x \neq y$, and (iii) the equality in (c) follows since $A = \bigcup_{i=1}^{C_A} V_i$. Thus, implies . Moreover, since $C_A \geq 1$, implies . This shows that the rate region in Theorem is an outer bound on the secure capacity region when $K = 1$.

We now consider an example of a two-layer network and show how the upper bound derived above applies to it. Example: Let $A = \{2, 3, 4\}$, and assume that $M_1 = \{1, 2\}$, $M_2 = \{3, 4\}$, $M_3 = \{4, 5, 6\}$ and $M_4 = \{7, 8\}$. Then, we construct an undirected graph such that: (i) it has 3 nodes since $|A| = 3$ and (ii) it has an edge between node 2 and node 3 since $M_2 \cap M_3 = \{4\} \neq \emptyset$. It therefore follows that this graph has $C_A = 2$ components. In particular, we have

$$\sum_{i \in A} R_i = \sum_{i \in V_1} R_i + \sum_{i \in V_2} R_i \leq M_{\{2, 3, 4\}} - 2K = 4,$$

where $V_1 = \{2, 3\}$ and $V_2 = \{4\}$.

B. Computing Achievable Rate Region

We here show that the rate region in Theorem 2 is achieved by the scheme described in Section III. In particular, we show that,

$$\dim \left( \sum_{i \in A} N_i \right) \geq M_A - C_A, \forall A \subseteq [m],$$

IV. Achieved Secure Rate Region

In this section, we derive the rate region achieved by the secure scheme described in Section III. In particular, we have the following lemma, whose proof is in Appendix A.

Lemma 1. The secure rate region achieved by the proposed scheme is given by

$$0 \leq \sum_{i \in A} R_i \leq \dim \left( \sum_{i \in A} N_i \right), \forall A \subseteq [m],$$

where $N_i$ is the right null space of the matrix $V_i$ in (2).

In the remainder of this paper, we prove that the secure rate region in is indeed the secure capacity when there are $m = 3$ destinations (and arbitrary $K$), and when the eavesdropper wiretaps any $K = 1$ edge of her choice (and arbitrary $m$).

V. Secure Capacity for $K = 1$

In this section, we consider the case when Eve wiretaps any $K = 1$ edge of her choice, and characterize the secure capacity region. In particular, we prove the following theorem.
where recall that \( \dim \left( \sum_{i \in A} N_i \right) \) is the secure rate performance of our proposed scheme in Section III (see Lemma 1). The condition in (7) can be equivalently written as \( \forall A \subseteq [m], \)

\[
M_A - C_A \leq \dim \left( \sum_{i \in A} N_i \right) \overset{(g)}{=} \dim \left( \bigcap_{i \in A} V_i^c \right)
\]

\[
= t - \dim \left( \bigcap_{i \in A} V_i \right),
\]

where the equality in (a) follows by using the property of the orthogonal complement, and \( V_i, i \in A \) is defined in (2). In other words, we next show that

\[
\forall A \subseteq [m], \ \dim \left( \bigcap_{i \in A} V_i \right) \leq t - M_A + C_A. \tag{8}
\]

Towards this end, we would like to count the number of linearly independent vectors \( x \in \mathbb{F}_q^t \) that belong to \( \bigcap_{i \in A} V_i \).

We note that, by our construction: (i) \( V^T \) consists of one row of \( t \) ones, and (ii) \( C_i \) has zeros in the positions indexed by \( M_i \). Hence, if a vector belongs to \( V_i \), then all its components indexed by \( M_i \) have to be the same, i.e., either they are all zeros, or they are all equal to a multiple of one. Thus, we have \( q \) choices to fill these positions indexed by \( M_i \). Let \( F_i \) be the collection of all these vectors with equal components in \( M_i \).

Now, consider \( V_j \) with \( j \in A \) and \( j \neq i \). By using the same logic as above, if a vector belongs to \( V_j \), then all its components indexed by \( M_j \) have to be the same, and we also have \( q \) choices to fill these positions indexed by \( M_j \). Let \( \mathcal{F}_i \) be the collection of all vectors with equal components in \( M_i \).

Next, consider \( V_s \) with \( s \in A \) and \( s \notin \{i, j\} \). By using the same logic as above, there are \( q \) choices to fill these positions indexed by \( M_s \), which are collected inside \( \mathcal{F}_s \). We now need to count the number of these vectors so that they also belong to \( V_i \). Towards this end, we consider two cases:

1. **Case 1**: \( M_i \cap M_j = \emptyset \). In this case, there is no overlap in the elements indexed by \( M_i \) and \( M_j \), and hence we can select all the available \( q \) choices in \( F_{ij} \).

2. **Case 2**: \( M_i \cap M_j \neq \emptyset \). In this case, there is some overlap in the elements indexed by \( M_i \) and \( M_j \). Thus, since we have already fixed the elements indexed by \( M_i \), we do not have any choice for the elements indexed by \( M_j \). By iterating the same reasoning as above for all \( i \in A \), we conclude that we can fill all the positions indexed by \( \cup_{i \in A} M_i \) of a vector \( x \in \mathbb{F}_q^t \) and make sure that \( x \in (\bigcap_{i \in A} V_i) \) in \( q^{C_A} \) ways. This is because, there are \( C_A \) connected components, and for each of these components we have only \( q \) choices to fill the corresponding positions in the vector \( x \) (i.e., the positions that correspond to the relays to which at least one of the destinations inside that component is connected). Once we fix any position inside a component, in fact all the other positions inside that component have to be the same, and thus we have no more freedom in choosing the other positions. Moreover, the remaining \( t - M_A \) positions of \( x \) can be filled with any value in \( \mathbb{F}_q \) and for this we have \( q^{t-M_A} \) possible choices. Therefore, the number of vectors \( x \in \mathbb{F}_q^t \) that belong to \( (\bigcap_{i \in A} V_i) \) is at most \( q^{C_A+t-M_A} \), which implies

\[
\forall A \subseteq [m], \ \dim \left( \bigcap_{i \in A} V_i \right) \leq t - M_A + C_A.
\]

This proves that the secure scheme in Section III achieves the rate region in Theorem 2. We now illustrate our method of identifying vectors that belong to \( \bigcap_{i \in A} V_i \) through an example.

**Example:** Let \( t = 8, m = 4, M_1 = \{1, 2\}, M_2 = \{3, 4\}, M_3 = \{4, 5, 6\} \) and \( M_4 = \{7, 8\} \). Let \( A = \{2, 3, 4\} \). With this, we can construct \( V_i, i \in A \), as described in (2), where \( V_i^T \) consists of one row of \( 8 \) ones. For instance, we will have

\[
V_2 = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]

We want to count the number of the vectors \( x \in \mathbb{F}_q^8 \) such that \( x \in V_2 \cap V_3 \cap V_4 \). We use the following iterative procedure:

1. For \( x \) to belong to \( V_2 \) its elements in the 3rd and 4th positions have to be the same since \( M_2 = \{3, 4\} \). Thus, we have \( q \) choices for a vector \( x \) to belong to \( V_2 \).

2. For \( x \) to belong to \( V_3 \), its elements in the 4th, 5th and 6th positions have to be the same since \( M_3 = \{4, 5, 6\} \). However, the element in the 4th position has already been fixed in selecting vectors that belong to \( V_2 \). Thus, there is no further choice other than just repeating the value of the 4th position in the 5th and 6th positions.

3. For \( x \) to belong to \( V_4 \), its elements in the 7th and 8th positions have to be the same since \( M_4 = \{7, 8\} \). Since in the previous two steps, we have not filled yet the elements in these positions, then we have \( q \) possible ways to fill the elements in the 7th and 8th positions.

4. Moreover, we can fill the elements in the 1st and 2nd positions of \( x \) in \( q^2 \) possible ways.

With the above procedure we get that \( \dim \left( \bigcap_{i \in \{2,3,4\}} V_i \right) = 4 \), which is equal to the upper bound that we computed in (6) for the same example.

**VI. SECURE CAPACITY FOR \( m = 3 \)**

In this section, we consider the case \( m = 3 \), and we characterize the secure capacity region through the theorem below.

**Theorem 3.** For a two-layer network with \( m = 3 \) destinations, the secure capacity region is given by

\[
\sum_{i \in A} R_i \leq M_A - K, \ \forall A \subseteq [m]. \tag{9}
\]

Clearly the rate region in (9) is an outer bound on the secure capacity region [9] and can be equivalently written as

\[
\sum_{i \in A} R_i \leq \min_{Q \in \mathcal{P}} \left\{ \sum_{Q \in \mathcal{P}} M_Q - |P|K \right\}, \ \forall A \subseteq [m],
\]

where \( \mathcal{P} \) is a disjoint partition of \( A \). We will now show that for every \( A \subseteq [m] \),

\[
\dim \left( \sum_{i \in A} N_i \right) \geq \min_{Q \in \mathcal{P}} \left\{ \sum_{Q \in \mathcal{P}} M_Q - |P|K \right\}. \tag{10}
\]

We prove (10) by considering three different cases.
Towards this end, we would like to compute the number of linearly independent vectors $x$ of $V_i$ that belong to $V_1 \cap V_2 \cap V_3$. We start by noting that, similar to the case $K = 1$, we have $t - M_{\mathcal{M}(1,2,3)}$ degrees of freedom to fill the positions of $x$ corresponding to $t \setminus \cup_{i \in \mathcal{M}} A_i$. We now select a permutation $(i,j,k)$ of $(1,2,3)$. In order for $x$ to belong to $V_i$, the positions of $x$ corresponding to $\mathcal{M}_i$ can be filled with $K$ degrees of freedom. This is because: (i) $C_i$ in (2) has zeros in the positions specified by $\mathcal{M}_i$, and (ii) $V^T$ has $K$ rows. Then, to fill the positions of $x$ specified by $\mathcal{M}_j$ so that $x \in V_j$, we have at most $\lfloor K - M_{\mathcal{M}(i,j)} \rfloor^+$ degrees of freedom. This is because the positions of $x$ corresponding to $\mathcal{M}_i \cap \mathcal{M}_j$ have already been fixed (when filling the positions of $x$ specified by $\mathcal{M}_i$).

Finally, to fill the positions of $x$ corresponding to $\mathcal{M}_k$ such that $x \in V_k$, we have at most $\lfloor K - M_{\mathcal{M}(i,j)} \rfloor^+$ degrees of freedom. This is because, the positions of $x$ corresponding to $\mathcal{M}_k \cap (\mathcal{M}_i \cup \mathcal{M}_j)$ are already fixed. Thus, we obtain $\dim(V_1 \cap V_2 \cap V_3) = K + \lfloor K - M_{\mathcal{M}(i,j)} \rfloor^+ + \lfloor K - M_{\mathcal{M}(i,j)} \rfloor^+ + t - M_{\mathcal{M}(1,2,3)}$.

In Appendix B we further tighten $\dim(V_1 \cap V_2 \cap V_3)$ above and show that, when substituted in (12), it satisfies the condition in (10). This proves that the scheme described in Section III securely achieves the rate region in Theorem 3.

We now conclude this section by providing sufficient conditions for which the secure scheme in Section III is capacity achieving. In particular, we have the following lemma.

**Lemma 4.** The scheme in Section III achieves the secure capacity region of a two-layer network with arbitrary values of $K$ and $m$ whenever $\mathcal{M}_n \cap \{i,j\} \geq K$ for all $(i,j) \in [m]^2, i \neq j$.

**Proof.** We can compute $\dim(\cap_{i=1}^m V_i)$ as follows

$$\dim(\cap_{i=\mathcal{M}} V_i) \leq K + \lfloor K - \mathcal{M}_n \cap \{i,j\} \rfloor^+ + \lfloor K - \mathcal{M}_n \cap \{i,j\} \rfloor^+ + \cdots + \lfloor K - \mathcal{M}_n \cap \{m,1,2,\ldots,i_{m-1}\} \rfloor^+ + t - M_{\mathcal{M}}$$

where: (i) the inequality in (a) follows by extending to arbitrary values of $m$ the iterative algorithm proposed for Case 3 above to fill the vector $x$ so that $x \in V_1 \cap V_2 \ldots \cap V_m$, and (ii) the equality in (b) follows since

$$\mathcal{M}_n \cap \{i,j,i_2,\ldots,i_{m-1}\} \geq \mathcal{M}_n \cap \{i,j_{m-1}\} \geq K.$$  

By using the property of the orthogonal complement, we obtain $\dim(\cap_{i=\mathcal{M}} V_i) \leq M_{\mathcal{M}} - K$, which satisfies the condition in (10) $\forall A \in [m]$. This concludes the proof of Lemma 4.  

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APPENDIX A

PROOF OF LEMMA 1

We let $T$ be a matrix of dimension $(\sum_{i=1}^{m} R_i) \times t$ that, for each destination $D_i$, $i \in [m]$, contains $R_i$ decoding vectors that belong to $N_i$. Mathematically, we have

$$T = \begin{bmatrix}
  - & - & - & - & - & - & - & - & - & - \\
  - & - & - & - & - & - & - & - & - & - \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  - & - & - & - & - & - & - & - & - & - \\
  - & - & - & - & - & - & - & - & - & - \\
\end{bmatrix}, \quad (13)
$$

where $d_j^{(i)}$ denotes the $j$-th decoding vector (of length $t$) selected from the null space $N_i$, with $i \in [m], j \in [R_i]$. Note that, if for all $i \in [m]$, we can select $R_i$ decoding vectors from $N_i$ such that all the $d_j^{(i)}$ in (13) are linearly independent (i.e., such that $T$ has a full row rank), then it is possible to construct the matrix $M$ in (1) such that

$$TM = I,$$

which ensures that all the destinations are able to correctly decode their intended message as

$$[\hat{W}_1 \ldots \hat{W}_m] = T \begin{bmatrix}
  Y_1 \\
  \vdots \\
  Y_t \\
\end{bmatrix} = TM \begin{bmatrix}
  Y_1 \\
  \vdots \\
  Y_t \\
\end{bmatrix} + TVX = \begin{bmatrix}
  W_1 \\
  \vdots \\
  W_m \\
\end{bmatrix}.$$

We propose an iterative algorithm to select $R_i, i \in [m]$ decoding vectors from $N_i$ such that $T$ in (13) has indeed a full row rank. The performance of the proposed algorithm is provided in the following lemma.

Lemma 5. For any given permutation $\pi = \{\pi(1), \ldots, \pi(m)\}$ of $[m]$, it is possible to select

$$R_{\pi(i)} = \dim(\sum_{j=1}^{i} N_{\pi(j)}) - \dim(\sum_{j=1}^{i-1} N_{\pi(j)}), \quad i \in [m],$$

vectors from $N_{\pi(i)}$ so that all the $\sum_{i=1}^{m} R_i$ selected vectors are linearly independent.

Proof. We use an iterative algorithm that, for any permutation $\pi = \{\pi(1), \ldots, \pi(m)\}$ of $[m]$, allows to select $R_{\pi(i)}$ vectors from $N_{\pi(i)}$ (with $R_{\pi(i)}$ being defined in (14)) so that all the selected $\sum_{i=1}^{m} R_i$ vectors are linearly independent. We next illustrate the main steps of the proposed algorithm.

1) We select $R_{\pi(1)} = \dim(N_{\pi(1)})$ independent vectors from $N_{\pi(1)}$. Note that one possible choice for this consists of selecting the basis of the subspace $N_{\pi(1)}$.

2) Next we would like to select independent vectors from $N_{\pi(2)}$ that are also independent of the $R_{\pi(1)}$ vectors that we selected in the previous step. Towards this end, we note that a basis of the subspace $N_{\pi(1)} + N_{\pi(2)}$ is a subset of the union between a basis of $N_{\pi(1)}$ and a basis of $N_{\pi(2)}$. Therefore, we can keep selecting vectors from a basis of $N_{\pi(2)}$ as long as we select an independent vector. Since there are $\dim(N_{\pi(1)} + N_{\pi(2)})$ independent vectors in a basis of $N_{\pi(1)} + N_{\pi(2)}$, then we can select

$$R_{\pi(2)} = \dim(N_{\pi(1)} + N_{\pi(2)}) - \dim(N_{\pi(1)}),$$

independent vectors from $N_{\pi(2)}$ that are also independent of the $R_{\pi(1)}$ vectors that we selected in the previous step.

3) Similar to the above step, we now would like to select independent vectors from $N_{\pi(3)}$ that are also independent of the $R_{\pi(1)} + R_{\pi(2)}$ vectors that we selected in the previous two steps. Towards this end, we note that a basis of the subspace $N_{\pi(1)} + N_{\pi(2)} + N_{\pi(3)}$ is a subset of the union between a basis of $N_{\pi(1)} + N_{\pi(2)}$ and a basis of $N_{\pi(3)}$. Therefore, we can keep selecting vectors from a basis of $N_{\pi(3)}$ as long as we select an independent vector. Since there are $\dim(N_{\pi(1)} + N_{\pi(2)} + N_{\pi(3)})$ independent vectors in a basis of $N_{\pi(1)} + N_{\pi(2)} + N_{\pi(3)}$, then we can select

$$R_{\pi(3)} = \dim(N_{\pi(1)} + N_{\pi(2)} + N_{\pi(3)}) - \dim(N_{\pi(1)} + N_{\pi(2)}),$$

independent vectors from $N_{\pi(3)}$ that are also independent of the $R_{\pi(1)} + R_{\pi(2)}$ vectors that we selected in the previous two steps.

4) We keep using the iterative procedure above for all the elements in $\pi$, and we end up with $\sum_{i=1}^{m} R_i$ vectors that are linearly independent.

This concludes the proof of Lemma 5.

Remark 1. Note that, since there are $m!$ possible permutations of $[m]$, then Lemma 5 offers $m!$ possible choices for selecting $R_i, i \in [m]$ vectors from $N_i$ so that all the $\sum_{i=1}^{m} R_i$ selected vectors are linearly independent.

Remark 2. The result in Lemma 5 implies that rate $m$-tuple $(R_1, R_2, \ldots, R_m)$, with $R_i, i \in [m]$ being defined in (14), can be securely achieved by our proposed scheme.

We now leverage the result in Lemma 5 to prove Lemma 1. We start by noting that the rate region in (3) can be expressed as the following polyhedron:

$$P_f := \left\{ R \in \mathbb{R}^{|m|} : R \succeq \mathbf{0}, \sum_{i \in A} R_i \leq f(A), \quad \forall A \subseteq [m] \right\},$$

where $f(A) := \dim(\sum_{i \in A} N_i)$. We now prove the following lemma, which states that this function $f(\cdot)$ is a non-decreasing and submodular function over subsets of $[m]$. 


Lemma 6. The set function

\[ f(A) := \dim \left( \sum_{i \in A} N_i \right), \quad \forall A \subseteq [m] \]

is a non-decreasing and submodular function.

Proof. Let \( A \subseteq B \subseteq [m] \), then

\[ f(B) = \dim \left( \sum_{i \in B} N_i \right) = \dim \left( \sum_{i \in A} N_i + \sum_{j \in B \setminus A} N_j \right) \geq \dim \left( \sum_{i \in A} N_i \right) = f(A), \]

which proves that the function \( f(\cdot) \) is non-decreasing. For proving submodularity, consider two subsets \( C, D \subseteq [m] \). Then, we have

\[ f(C \cup D) = \dim \left( \sum_{i \in C \cup D} N_i \right) = \dim \left( \sum_{i \in C} N_i + \sum_{j \in D} N_j \right) = \dim \left( \sum_{i \in C} N_i \right) + \dim \left( \sum_{j \in D} N_j \right) - \dim \left( \sum_{i \in C} N_i \right) \cap \left( \sum_{j \in D} N_j \right) \leq \dim \left( \sum_{i \in C} N_i \right) + \dim \left( \sum_{j \in D} N_j \right) - \dim \left( \sum_{k \in C \cap D} N_k \right) = f(C) + f(D) - f(C \cap D), \]

which proves that the function \( f(\cdot) \) is submodular. \( \square \)

Since \( f(\cdot) \) is a submodular function, then the polyhedron defined in [15] is the polymatroid associated with \( f(\cdot) \). Moreover, since \( f(\cdot) \) is also non-decreasing, then the corner points of the polymatroid in [15] can be found as follows [10, Corollary 44.3a]. Consider a permutation \( \pi = \{\pi(1), \ldots, \pi(m)\} \) of \([m]\). Then, by letting \( S_\ell = \{\pi(1), \ldots, \pi(\ell)\} \) for \( 1 \leq \ell \leq m \), we get that the corner points of the polymatroid in [15] can be written as

\[ R_{\pi(\ell)} = f(S_\ell) - f(S_{\ell-1}). \]

Note that by using \( f(A) = \dim \left( \sum_{i \in A} N_i \right) \), the above corner points are precisely those in [14] in Lemma 5. Since each rate \( m \)-tuple \((R_1, R_2, \ldots, R_m)\), with \( R_i, i \in [m] \) being defined in [14], can be securely achieved by our proposed scheme, it follows that the secure rate region in (3) can also be achieved by our scheme. This concludes the proof of Lemma 1.

**APPENDIX B**

**ANALYSIS OF THE DIMENSION OF \((V_1 \cap V_2 \cap V_3)\)**

From our analysis, we have obtained

\[ \dim(V_1 \cap V_2 \cap V_3) \leq K + [K - M_{\cap(i,j)}]^+ + [K - M_{\cap(k,i,j)}]^+ + t - M_{\{1,2,3\}}. \]  

(16)

We now further consider two cases.

**Case 3A:** There exist \((i, j) \in [3]^2, i \neq j\), such that \( M_{\cap(i,j)} \geq K \). In this case, with the permutation \((i, j, k)\), the expression in (16) becomes

\[ \dim(V_1 \cap V_2 \cap V_3) \leq K + [K - M_{\cap(k,i,j)}]^+ + t - M_{\{1,2,3\}} \]

\[ = t - M_{\{1,2,3\}} + \max\{2K - M_{\cap(k,i,j)}, K\}. \]

From [12], this implies that

\[ \dim(N_1 + N_2 + N_3) \]

\[ \geq M_{\{1,2,3\}} - \max\{2K - M_{\cap(k,i,j)}, K\} \]

\[ = \min\{M_{\{1,2,3\}} - K, M_{\{k\}} + M_{\{i,j\}} - 2K\}, \]

where the last equality follows since \( M_{\{1,2,3\}} = M_{\{i,j\}} + M_{\{k\}} - M_{\cap(k,i,j)} \). With this, the condition in (10) is satisfied.

**Case 3B:** We have \( M_{\cap(i,j)} < K, \forall (i, j) \in [3]^2, i \neq j \). In this case, we compute \( \dim(V_1 \cap V_2 \cap V_3) \) as follows: we first fill the positions of \( x \) indexed by \( M_1 \) with \( K \) degrees of freedom, and then fill the positions of \( x \) indexed by \( M_2 \) with \( (K - M_{\cap(1,2)}) \) degrees of freedom as before. Now, we may have fixed more than \( K \) positions of \( x \) corresponding to indexes in \( M_3 \), which is not feasible. If that is the case, we backtrack (i.e., remove excess degrees of freedom) that we have used for filling positions of \( x \) indexed by \( M_2 \). Thus,

1) If \( M_{\cap(3,1,2)} \leq K \), then

\[ \dim(V_1 \cap V_2 \cap V_3) \leq t - M_{\{1,2,3\}} + K + (K - M_{\cap(1,2)}) \]

\[ + (K - M_{\cap(3,1,2)}). \]

This, from [12], implies

\[ \dim(N_1 + N_2 + N_3) \geq M_{\{1\}} + M_{\{2\}} + M_{\{3\}} - 3K, \]

which satisfies the condition in (10).

2) If \( M_{\cap(3,1,2)} > K \), then

\[ \dim(V_1 \cap V_2 \cap V_3) \leq t - M_{\{1,2,3\}} + K + (K - M_{\cap(1,2)}) - \min\{M_{\{3\}} - M_{\cap(1,2)}, M_{\cap(3,1,2)} - K\}. \]

This, from [12], implies

\[ \dim(N_1 + N_2 + N_3) \geq \min\{M_{\{1,2,3\}} - K, M_{\{1\}} + M_{\{2\}} + M_{\{3\}} - 3K\}, \]

which satisfies the condition in (10).