Integrability of the $C_n$ and $BC_n$ Ruijsenaars-Schneider models

Kai Chen∗, Bo-yu Hou† and Wen-Li Yang‡
Institute of Modern Physics, Northwest University, Xian 710069, China

Abstract: We study the $C_n$ and $BC_n$ Ruijsenaars-Schneider (RS) models with interaction potential of trigonometric and rational types. The Lax pairs for these models are constructed and the involutive Hamiltonians are also given. Taking nonrelativistic limit, we also obtain the Lax pairs for the corresponding Calogero-Moser systems.

PACS: 02.20.+b, 11.10.Lm, 03.80.+r

I Introduction

Ruijsenaars-Schneider (RS) and Calogero-Moser (CM) models as integrable many-body models recently have attracted remarkable attention and have been extensively studied. They describe one-dimensional $n$-particle system with pairwise interaction. Their importance lies in various fields ranging from lattice models in statistics physics [1, 2], the field theory to gauge theory [3, 4], e.g. to the Seiberg-Witten theory [5] et al. Recently, the Lax pairs for the elliptic CM models in various root system have been given by Olshanetsky et al [6], Bordner et al [7, 8, 9, 10] and D’Hoker et al [11] respectively, while the commutative operators for RS model based on various type Lie algebra given by Komori [12, 13], Diejen [14, 15] and Hasegawa [1, 16] et al. An interesting result is that in Ref. [17], the authors show that for the $sl_2$ trigonometric RS and CM models exist the same non-dynamical $r$-matrix structure compared with the usual dynamical ones. On the other hand, similar to Hasegawa’s result that $A_{N-1}$ RS model is related to the $Z_n$ Sklyanin algebra, the integrability of CM model can be depicted by $sl_N$ Gaudin algebra [18].

As for the $C_n$ type RS model, commuting difference operators acting on the space of functions on the $C_2$ type weight space have been constructed by Hasegawa et al in Ref. [13]. Extending that work, the diagonalization of elliptic difference system of that type has been studied by Kikuchi in Ref. [19]. Despite of the fact that the Lax pairs for CM models have

∗e-mail :kai@phy.nwu.edu.cn
†e-mail :byhou@phy.nwu.edu.cn
‡e-mail :wlyang@phy.nwu.edu.cn
been proposed for general Lie algebra even for all of the finite reflection groups\cite{10}, however, the Lax integrability of RS model are not clear except only for \(A_{N-1}\) -type\cite{20,2,21,22,23,24} and for \(C_2\) by the authors by straightforward construction\cite{25}, i.e. the general Lax pairs for the RS models other than \(A_{N-1}\) -type have not yet been obtained.

Extending the work of Ref. \cite{25}, the main purpose of the present paper is to provide the Lax pairs for the \(C_n\) and \(BC_n\) Ruijsenaars-Schneider(RS) models with the trigonometric and rational interaction potentials. The key technique we used is Dirac's method on the system imposed by some constraints. We shall give the explicit forms of Lax pairs for these systems. It is turned out that the \(C_n\) and \(BC_n\) RS systems can be obtain by Hamiltonian reduction of \(A_{2n-1}\) and \(A_2\) ones. The characteristic polynomial of the Lax matrixes leads to a complete set of involutive Hamiltonians associated with the root system of \(C_n\) and \(BC_n\). In particular, taking their non-relativistic limit, we shall recover the systems of corresponding CM types.

The paper is organized as follows. The basic materials about \(A_{N-1}\) RS model are reviewed in Sec. II. We also give a Lax pair associating with Hamiltonian which has a reflection symmetry with respect to the particles in the origin. The main results are showed in Secs. III and IV. In Sec. III, we present the Lax pairs of \(C_n\) and \(BC_n\) RS models by reducing from that of \(A_{N-1}\) RS model. The explicit forms for the Lax pairs are given in Sec. IV. The characteristic polynomials, which gives the complete sets of involutive constant motions for these systems, will also be given there. Sec. V, is devoted to derive the nonrelativistic limits of these systems which coincide with the forms given in Refs. \cite{6} and \cite{7}. The last section is brief summary and some discussions.

II \(A_{N-1}\)-type Ruijsenaars-Schneider model

As a relativistic-invariant generalization of the \(A_{N-1}\)-type nonrelativistic Calogero-Moser model, the \(A_{N-1}\)-type Ruijsenaars-Schneider systems are completely integrable whose integrability are first showed by Ruijsenaars\cite{20,26}. The Lax pairs for this model have been constructed in Refs. \cite{24,2,21,22,23,24}. Recent progress have showed that the compactification of higher dimension SUSY Yang-Mills theory and Seiberg-Witten theory can be described by this model\cite{5}. Instanton correction of prepotential associated with \(sl_2\) RS system have been calculated in Ref. \cite{27}.

II.1 The Lax operator for \(A_{N-1}\) RS model

Let us briefly give the basics of this model. In terms of the canonical variables \(p_i, x_i (i, j = 1, \ldots, N)\) enjoying in the canonical Poisson bracket

\[
\{p_i, p_j\} = \{x_i, x_j\} = 0, \quad \{x_i, p_j\} = \delta_{ij}, \tag{II.1}
\]

we give firstly the Hamiltonian of \(A_{N-1}\) RS system
\[ H_{AN-1} = \sum_{i=1}^{N} \left( e^{p_i} \prod_{k \neq i} f(x_i - x_k) + e^{-p_i} \prod_{k \neq i} g(x_i - x_k) \right). \]  (II.2)

Notice that in Ref. [20] Ruijsenaars used another “gauge” of the momenta such that two are connected by the following canonical transformation:

\[ x_i \longrightarrow x_i, \quad p_i \longrightarrow p_i + \frac{1}{2}\ln \prod_{j \neq i}^{N} \frac{f(x_{ij})}{g(x_{ij})}. \]  (II.3)

The Lax operator for this model has the form (for the trigonometric case)

\[ L_{AN-1} = \sum_{i,j=1}^{N} \sin \gamma \sin(x_i - x_j + \gamma) \exp(p_j) b_j E_{ij}, \]

and for the rational case

\[ L_{AN-1} = \sum_{i,j=1}^{N} \frac{\gamma}{x_i - x_j + \gamma} \exp(p_j) b_j E_{ij}. \]  (II.4)

where

\[ b_j := \prod_{k \neq j} f(x_j - x_k), \quad b'_j := \prod_{k \neq j} g(x_j - x_k), \quad (E_{ij})_{kl} = \delta_{ik} \delta_{jl}, \]

\[ f(x) := \begin{cases} \frac{\sin(x-\gamma)}{\sin(x)}, & \text{trigonometric case,} \\ \frac{x-\gamma}{x}, & \text{rational case,} \end{cases} \]

\[ g(x) := f(x)|_{\gamma \rightarrow -\gamma}, \quad x_{ik} := x_i - x_k, \]  (II.5)

and \( \gamma \) denotes the coupling constant.

It is shown in Ref. [23] that the Lax operator satisfies the quadratic fundamental Poisson bracket

\[ \{ L_1, L_2 \} = L_1 L_2 a_1 - a_2 L_1 L_2 + L_2 s_1 L_1 - L_1 s_2 L_2, \]  (II.6)

where \( L_1 = L_{AN-1} \otimes 1, L_2 = 1 \otimes L_{AN-1} \) and the four matrices read as

\[ a_1 = a + w, \quad s_1 = s - w, \]
\[ a_2 = a + s - s^* - w, \quad s_2 = s^* + w. \]  (II.7)

The forms of \( a, s, w \) are

\[ a = \sum_{k \neq j} \cot(x_k - x_j) E_{jk} \otimes E_{kj}, \]
\[ s = -\sum_{k \neq j} \frac{1}{\sin(x_k - x_j)} E_{jk} \otimes E_{kk}, \]
\[ w = \sum_{k \neq j} \cot(x_k - x_j) E_{kk} \otimes E_{jj}, \]  (II.8)
for the trigonometric case and

\[
\begin{align*}
a &= \sum_{k \neq j} \frac{1}{x_k - x_j} E_{jk} \otimes E_{kj}, \\
s &= -\sum_{k \neq j} \frac{1}{x_k - x_j} E_{jk} \otimes E_{kk}, \\
w &= \sum_{k \neq j} \frac{1}{x_k - x_j} E_{kk} \otimes E_{jj},
\end{align*}
\]

for the rational case. The * symbol means \( r^* = \Pi r \Pi \) with \( \Pi = \sum_{k,j=1}^{N} E_{kj} \otimes E_{jk} \).

Noticing that

\[
(L - 1)_{ij} = \left\{ \begin{array}{ll}
\sum_{i,j=1}^{N} \frac{-\sin \gamma}{\sin(x_i - x_j - \gamma)} \exp(-p_i) b_j E_{ij}, & \text{for trigonometric case,} \\
\sum_{i,j=1}^{N} \frac{-\gamma}{x_i - x_j - \gamma} \exp(-p_i) b_j E_{ij}, & \text{for rational case,}
\end{array} \right.
\]

one can get the characteristic polynomials of \( L_{A_{N-1}} \) and \( L_{A_{N-1}}^{-1} \) [28]

\[
\det(L_{A_{N-1}} - v \cdot Id) = \sum_{j=0}^{N} (-v)^{n-j} (H_+^j)_{A_{N-1}},
\]

\[
\det(L_{A_{N-1}}^{-1} - v \cdot Id) = \sum_{j=0}^{N} (-v)^{n-j} (H_-^j)_{A_{N-1}},
\]

where \( (H_0^+)_{A_{N-1}} = (H_0^-)_{A_{N-1}} = 1 \)

\[
(H_+^j)_{A_{N-1}} = \sum_{J \subseteq \{1, \ldots, N\}} \exp \left( \sum_{j \in J} p_j \right) \prod_{j \in J, k \notin \{1, \ldots, N\} \setminus J} f(x_j - x_k),
\]

\[
(H_-^j)_{A_{N-1}} = \sum_{J \subseteq \{1, \ldots, N\}} \exp \left( \sum_{j \in J} -p_j \right) \prod_{j \in J, k \notin \{1, \ldots, N\} \setminus J} g(x_j - x_k).
\]

Define

\[
(H_i)_{A_{N-1}} = (H_i^+)_{A_{N-1}} + (H_i^-)_{A_{N-1}},
\]

from the fundamental Poisson bracket Eq.(II.6), we can verify that

\[
\{ (H_i)_{A_{N-1}}, (H_j)_{A_{N-1}} \} = \{ (H_i^\varepsilon)_{A_{N-1}}, (H_j^\varepsilon')_{A_{N-1}} \} = 0, \quad \varepsilon, \varepsilon' = \pm, \quad i, j = 1, \ldots, N.
\]
In particular, the Hamiltonian Eq.(II.2) can be rewritten as
\[
H_{A_{N-1}} = (H^+_1)_{A_{N-1}} + (H^-_1)_{A_{N-1}} = \sum_{j=1}^{N} (e^{p_j} b_j + e^{-p_j} b'_j) = Tr(L_{A_{N-1}} + L^{-1}_{A_{N-1}}). \quad (II.17)
\]

It should be remarked the set of integrals of motion Eq.(II.15) have a reflection symmetry which is the key property for the later reduction to $C_n$ and $BC_n$ cases. i.e. if we set
\[
p_i \longleftrightarrow -p_i, \quad x_i \longleftrightarrow -x_i, \quad (II.18)
\]
then the Hamiltonians flows $(H_i)_{A_{N-1}}$ are invariant with respect to this symmetry.

### II.2 The construction of Lax pair for the $A_{N-1}$ RS model

As for the $A_{N-1}$ RS model, a generalized Lax pair has been given in Refs. [20, 21, 22, 23, 24]. But there is a common character that the time-evolution of the Lax matrix $L_{A_{N-1}}$ is associated with the Hamiltonian $H_+$. We will see in the next section that the Lax pair can’t reduce from that kind of forms directly. Instead, we give a new Lax pair which the evolution of $L_{A_{N-1}}$ are associated with the Hamiltonian $H_{A_{N-1}}$

\[
\dot{L}_{A_{N-1}} = \{L_{A_{N-1}}, H_{A_{N-1}}\} = [M_{A_{N-1}}, L_{A_{N-1}}], \quad (II.19)
\]

where $M_{A_{N-1}}$ can be constructed with the help of $(r, s)$ matrices as follows

\[
M_{A_{N-1}} = Tr_2((s_1 - a_2)(1 \otimes (L_{A_{N-1}} - L^{-1}_{A_{N-1}}))). \quad (II.20)
\]

The explicit expression of entries for $M_{A_{N-1}}$ is

\[
(M_{A_{N-1}})_{ij} = \frac{\sin \gamma \cot(x_{ij})}{\sin(x_{ij} + \gamma)} e^{p_j} b_j + \frac{\sin \gamma \cot(x_{ij})}{\sin(x_{ij} - \gamma)} e^{-p_j} b'_j, \quad i \neq j,
\]

\[
(M_{A_{N-1}})_{ii} = -\sum_{l \neq i} \frac{\sin \gamma}{\sin(x_{il})} \frac{\sin(x_{il})}{\sin(x_{il} + \gamma)} e^{p_l} b_l + \frac{\sin \gamma}{\sin(x_{il})} \frac{\sin(x_{il})}{\sin(x_{il} - \gamma)} e^{-p_l} b'_l, \quad (II.21)
\]

for trigonometric case and

\[
(M_{A_{N-1}})_{ij} = \frac{\gamma}{x_{ij}(x_{ij} + \gamma)} e^{p_j} b_j + \frac{\gamma}{x_{ij}(x_{ij} - \gamma)} e^{-p_j} b'_j, \quad i \neq j,
\]

\[
(M_{A_{N-1}})_{ii} = -\sum_{l \neq i} \frac{\gamma}{x_{il}(x_{il} + \gamma)} e^{p_l} b_l + \frac{\gamma}{x_{il}(x_{il} - \gamma)} e^{-p_l} b'_l, \quad (II.22)
\]

for rational case.
III Hamiltonian reductions of $C_n$ and $BC_n$ RS models from $A_{N-1}$-type ones

Let us first mention some results about the integrability of Hamiltonian (II.2). In Ref. [26] Ruijsenaars demonstrated that the symplectic structure of $C_n$ and $BC_n$ type RS systems can be proved integrable by embedding their phase space to a submanifold of $A_{2n-1}$ and $A_{2n}$ type RS ones respectively, while in Refs. [14, 15] and [13], Diejen and Komori, respectively, gave a series of commuting difference operators which led to their quantum integrability. However, there are not any results about their Lax representations so far, i.e. the explicit forms of the Lax matrixes $L$, associated with a $M$ (respectively) which ensure their Lax integrability, haven’t been proposed up to now except for the special case of $C_2$ [25]. In this section, we concentrate our treatment to the exhibition of the explicit forms for general $C_n$ and $BC_n$ RS systems. Therefore, some previous results, as well as new results, could now be obtained in a more straightforward manner by using the Lax pairs.

For the convenience of analysis of symmetry, let us first give vector representation of $A_{N-1}$ Lie algebra. Introducing an $N$ dimensional orthonormal basis of $\mathbb{R}^N$

$$e_j \cdot e_k = \delta_{j,k}, \quad j, k = 1, \ldots, N.$$ (III.1)

Then the sets of roots and vector weights are:

$$\Delta = \{e_j - e_k : \quad j, k = 1, \ldots, N\}; \quad \Lambda = \{e_j : \quad j = 1, \ldots, N\}.$$ (III.2) (III.3)

The dynamical variables are canonical coordinates $\{x_j\}$ and their canonical conjugate momenta $\{p_j\}$ with the Poisson brackets of Eq.(II.1). In general sense, we denote them by $N$ dimensional vectors $x$ and $p$,

$$x = (x_1, \ldots, x_N) \in \mathbb{R}^N, \quad p = (p_1, \ldots, p_N) \in \mathbb{R}^N,$$

so that the scalar products of $x$ and $p$ with the roots $\alpha \cdot x$, $p \cdot \beta$, etc. can be defined. The Hamiltonian Eq.(II.2) can be rewritten as

$$H_{A_{N-1}} = \sum_{\mu \in \Lambda} \left( \exp(\mu \cdot p) \prod_{\Delta \ni \beta = \mu - \nu} f(\beta \cdot x) + \exp(-\mu \cdot p) \prod_{\Delta \ni \beta = -\mu + \nu} g(\beta \cdot x) \right),$$ (III.4)

in which $f(x)$ and $g(x)$ is given in Eq.(II.3) for various choices of potentials. Here, the condition $\Delta \ni \beta = \mu - \nu$ means that the summation is over roots $\beta$ such that for $\exists \nu \in \Lambda$

$$\mu - \nu = \beta \in \Delta.$$

So does for $\Delta \ni \beta = -\mu + \nu$. 6
### III.1 $C_n$ model

The set of $C_n$ roots consists of two parts, long roots and short roots:

$$\Delta_{C_n} = \Delta_L \cup \Delta_S,$$

in which the roots are conveniently expressed in terms of an orthonormal basis of $\mathbb{R}^n$:

$$\Delta_L = \{ \pm 2e_j : j = 1, \ldots, n \},$$

$$\Delta_S = \{ \pm e_j \pm e_k, : j, k = 1, \ldots, n \}. \quad (III.6)$$

In the vector representation, vector weights $\Lambda$ are

$$\Lambda_{C_n} = \{ e_j, -e_j : j = 1, \ldots, n \}. \quad (III.7)$$

The Hamiltonian of $C_n$ model is given by

$$H_{C_n} = \frac{1}{2} \sum_{\mu \in \Lambda_{C_n}} \left( \exp (\mu \cdot p) \prod_{\Delta_{C_n} \ni \beta = \mu - \nu} f(\beta \cdot x) + \exp (-\mu \cdot p) \prod_{\Delta_{C_n} \ni \beta = -\mu + \nu} g(\beta \cdot x) \right). \quad (III.8)$$

From the above data, we notice that either for $A_{N-1}$ or $C_n$ Lie algebra, any root $\alpha \in \Delta$ can be constructed in terms with vector weights as $\alpha = \mu - \nu$ where $\mu, \nu \in \Lambda$. By simple comparison of representation between $A_{N-1}$ or $C_n$, one can found that if replacing $e_j + n$ with $-e_j$ in the vector weights of $A_{2n-1}$ algebra, we can obtain the vector weights of $C_n$ one. Also does for the corresponding roots. This hints us it is possible to get the $C_n$ model by this kind of reduction.

For $A_{2n-1}$ model let us set restrictions on the vector weights with

$$e_j + n + e_j = 0, \quad \text{for} \quad j = 1, \ldots, n,$$

which correspond to the following constraints on the phase space of $A_{2n-1}$-type RS model with

$$G_i \equiv (e_i + n + e_i) \cdot x = x_i + x_{i+n} = 0,$$

$$G_{i+n} \equiv (e_i + n + e_i) \cdot p = p_i + p_{i+n} = 0, \quad i = 1, \ldots, n. \quad (III.10)$$

Following Dirac’s method, we can show

$$\{ G_i, H_{A_{2n-1}} \} \simeq 0, \quad \text{for} \quad \forall i \in \{ 1, \ldots, 2n \}, \quad (III.11)$$

i.e. $H_{A_{2n-1}}$ is the first class Hamiltonian corresponding to the above constraints Eq. (III.10). Here the symbol $\simeq$ represents that, only after calculating the result of left side of the identity, could we use the conditions of constraints. It should be pointed out that the most
necessary condition ensuring the Eq. (III.11) is the symmetry property Eq. (II.18) for the Hamiltonian Eq. (II.2). So that for arbitrary dynamical variable $A$, we have

$$
\dot{A} = \{A, H_{A_{2n-1}}\}_D = \{A, H_{A_{2n-1}}\} - \{A, G_i\} \Delta^{-1}_{ij} \{G_j, H_{A_{2n-1}}\}
$$

$$
\simeq \{A, H_{A_{2n-1}}\},
$$

where

$$
\Delta_{ij} = \{G_i, G_j\} = 2 \left( \begin{array}{cc} 0 & \text{Id} \\ -\text{Id} & 0 \end{array} \right),
$$

and the $\{,\}_D$ denote the Dirac bracket. By straightforward calculation, we have the nonzero Dirac brackets of

$$
\{x_i, p_j\}_D = \{x_{i+n}, p_{j+n}\}_D = \frac{1}{2} \delta_{i,j},
$$

$$
\{x_i, p_{j+n}\}_D = \{x_{i+n}, p_j\}_D = -\frac{1}{2} \delta_{i,j}.
$$

Using the above data together with the fact that $H_{A_{N-1}}$ is the first class Hamiltonian (see Eq. (III.11)), we can directly obtain Lax representation of $C_n$ RS model by imposing constraints $G_k$ on Eq. (II.19)

$$
\{L_{A_{2n-1}}, H_{A_{2n-1}}\}_D = \{L_{A_{2n-1}}, H_{A_{2n-1}}\}_{|G_k, k=1, \ldots, 2n},
$$

$$
= [M_{A_{2n-1}}, L_{A_{2n-1}}]_{|G_k, k=1, \ldots, 2n} = [M_{C_n}, L_{C_n}],
$$

(III.15)

$$
\{L_{A_{2n-1}}, H_{A_{2n-1}}\}_D = \{L_{C_n}, H_{C_n}\},
$$

(III.16)

where

$$
H_{C_n} = \frac{1}{2} H_{A_{2n-1}}_{|G_k, k=1, \ldots, 2n},
$$

$$
L_{C_n} = L_{A_{2n-1}}_{|G_k, k=1, \ldots, 2n},
$$

$$
M_{C_n} = M_{A_{2n-1}}_{|G_k, k=1, \ldots, 2n},
$$

(III.17)

so that

$$
\dot{L}_{C_n} = \{L_{C_n}, H_{C_n}\} = [M_{C_n}, L_{C_n}].
$$

(III.18)

Nevertheless, the $H_+$ is not the first class Hamiltonian, so the Lax pair given by many authors previously can’t reduce to $C_n$ case directly by this way.
III.2  BC\(_n\) model

The BC\(_n\) root system consists of three parts, long, middle and short roots:

\[\Delta_{BC\_n} = \Delta_L \cup \Delta \cup \Delta_S,\]  

(III.19)

in which the roots are conveniently expressed in terms of an orthonormal basis of \(\mathbb{R}^n\):

\[
\begin{align*}
\Delta_L &= \{\pm 2e_j : j = 1, \ldots, n\}, \\
\Delta &= \{\pm e_j \pm e_k : j, k = 1, \ldots, n\}, \\
\Delta_S &= \{\pm e_j : j = 1, \ldots, n\}.
\end{align*}
\]  

(III.20)

In the vector representation, vector weights \(\Lambda\) can be

\[\Lambda_{BC\_n} = \{e_j, -e_j, 0 : j = 1, \ldots, n\}.\]  

(III.21)

The Hamiltonian of BC\(_n\) model is given by

\[
H_{BC\_n} = \frac{1}{2} \sum_{\mu \in \Lambda_{BC\_n}} \left( \exp (\mu \cdot p) \prod_{\Delta_{BC\_n} \ni \beta = \mu - \nu} f(\beta \cdot x) + \exp (-\mu \cdot p) \prod_{\Delta_{BC\_n} \ni \beta = -\mu + \nu} g(\beta \cdot x) \right).
\]  

(III.22)

By similar comparison of representations between \(A_{N-1}\) or \(BC\_n\), one can found that if replacing \(e_j + n\) with \(-e_j\) and \(e_{2n+1}\) with 0 in the vector weights of \(A_{2n}\) Lie algebra, we can obtain the vector weights of \(BC\_n\) one. Also does for the corresponding roots. So by the same procedure as \(C_n\) model, it is expected to get the Lax representation of \(BC\_n\) model.

For \(A_{2n}\) model, we set restrictions on the vector weights with

\[
\begin{align*}
e_{j+n} + e_j &= 0, \quad \text{for} \quad j = 1, \ldots, n, \\
e_{2n+1} &= 0,
\end{align*}
\]  

(III.23)

which correspond to the following constraints on the phase space of \(A_{2n}\)-type RS model with

\[
\begin{align*}
G'_{i} &\equiv (e_{i+n} + e_i) \cdot x = x_i + x_{i+n} = 0, \\
G'_{i+n} &\equiv (e_{i+n} + e_i) \cdot p = p_i + p_{i+n} = 0, \quad i = 1, \ldots, n, \\
G'_{2n+1} &\equiv e_{2n+1} \cdot x = x_{2n+1} = 0, \\
G'_{2n+2} &\equiv e_{2n+1} \cdot p = p_{2n+1} = 0.
\end{align*}
\]  

(III.24)

Similarly, we can show

\[\{G_i, H_{A_{2n}}\} \simeq 0, \quad \text{for} \quad \forall i \in \{1, \ldots, 2n + 1, 2n + 2\}.\]  

(III.25)
i.e. $H_{A_{2n}}$ is the first class Hamiltonian corresponding to the above constraints Eq. (III.24). So $L_{BC_n}$ and $M_{BC_n}$ can be constructed as follows

\[
L_{BC_n} = L_{A_{2n}} |_{G_k, k=1,\ldots, 2n+2},
\]
\[
M_{BC_n} = M_{A_{2n}} |_{G_k, k=1,\ldots, 2n+2},
\]

(III.26)

while $H_{BC_n}$ is

\[
H_{BC_n} = \frac{1}{2} H_{A_{2n}} |_{G_k, k=1,\ldots, 2n+2},
\]

(III.27)

due to the similar derivation of Eq.(III.12-III.18).

IV Lax representations of $C_n$ and $BC_n$ RS models

IV.1 $C_n$ model

The Hamiltonian of $C_n$ RS system is Eq.(III.8), so the canonical equations of motion are

\[
\dot{x}_i = \{x_i, H\} = e^{p_i} b_i - e^{-p_i} b'_i, 
\]
\[
\dot{p}_i = \{p_i, H\} = \sum_{j \neq i} \left( e^{p_i} b_j \left( \frac{f'(x_{ji})}{f(x_{ji})} - \frac{f'(x_j + x_i)}{f(x_j + x_i)} \right) + e^{-p_i} b'_j \left( \frac{g'(x_{ji})}{g(x_{ji})} - \frac{g'(x_j + x_i)}{g(x_j + x_i)} \right) \right)
\]
\[+ e^{-p_i} b_i \left( 2 \frac{f'(2x_i)}{f(2x_i)} + \sum_{j \neq i} \left( \frac{f'(x_{ij})}{f(x_{ij})} + \frac{f'(x_j + x_j)}{f(x_j + x_j)} \right) \right) - e^{p_i} b_i \left( 2 \frac{g'(2x_i)}{g(2x_i)} + \sum_{j \neq i} \left( \frac{g'(x_{ij})}{g(x_{ij})} + \frac{g'(x_i + x_j)}{g(x_i + x_j)} \right) \right),
\]

(IV.2)

where

\[
f'(x) = \frac{df(x)}{dx}, \quad g'(x) = \frac{dg(x)}{dx},
\]
\[
b_i = f(2x_i) \prod_{k \neq i} (f(x_i - x_k) f(x_i + x_k)),
\]
\[
b'_i = g(2x_i) \prod_{k \neq i} (g(x_i - x_k) g(x_i + x_k)).
\]

(IV.3)
The Lax matrix for $C_n$ RS model can be written in the following form for the rational case

$$(L_{C_n})_{\mu\nu} = e^{\nu \cdot p_b \nu} \frac{\gamma}{(\mu - \nu) \cdot x + \gamma}, \quad (IV.4)$$

which is a $2n \times 2n$ matrix whose indices are labelled by the vector weights, denoted by $\mu, \nu \in \Lambda_{C_n}$, $M_{C_n}$ can be written as

$$M_{C_n} = D + Y, \quad (IV.5)$$

where

$$Y_{\mu\nu} = e^{\nu \cdot p_b \nu} \frac{\gamma}{((\mu - \nu) \cdot x)((\mu - \nu) \cdot x + \gamma)} + e^{-\mu \cdot p_b \nu} \frac{\gamma}{((\mu - \nu) \cdot x)((\mu - \nu) \cdot x - \gamma)}, \quad (IV.6)$$

$$D_{\mu\mu} = -\sum_{\nu \neq \mu} \left( e^{\nu \cdot p_b \nu} \frac{\gamma}{((\mu - \nu) \cdot x)((\mu - \nu) \cdot x + \gamma)} + e^{-\mu \cdot p_b \nu} \frac{\gamma}{((\mu - \nu) \cdot x)((\mu - \nu) \cdot x - \gamma)} \right) = -\sum_{\nu \neq \mu} Y_{\mu\nu}, \quad (IV.7)$$

and

$$b_\mu = \prod_{\Delta_{C_n} \ni \beta = \mu - \nu} f(\beta \cdot x),$$

$$b'_\mu = \prod_{\Delta_{C_n} \ni \beta = \mu - \nu} g(\beta \cdot x). \quad (IV.8)$$

For the trigonometric case, we have

$$(L_{C_n})_{\mu\nu} = e^{\nu \cdot p_b \nu} \frac{\sin \gamma}{\sin((\mu - \nu) \cdot x + \gamma)}, \quad (IV.9)$$

and

$$M_{C_n} = D + Y, \quad (IV.10)$$

where

$$Y_{\mu\nu} = e^{\nu \cdot p_b \nu} \frac{\sin \gamma \cot((\mu - \nu) \cdot x)}{\sin((\mu - \nu) \cdot x + \gamma)} + e^{-\mu \cdot p_b \nu} \frac{\sin \gamma \cot((\mu - \nu) \cdot x)}{\sin((\mu - \nu) \cdot x - \gamma)}, \quad (IV.11)$$

$$D_{\mu\mu} = -\sum_{\nu \neq \mu} \left( e^{\nu \cdot p_b \nu} \frac{\sin \gamma}{\sin((\mu - \nu) \cdot x)\sin((\mu - \nu) \cdot x + \gamma)} + e^{-\mu \cdot p_b \nu} \frac{\sin \gamma}{\sin((\mu - \nu) \cdot x)\sin((\mu - \nu) \cdot x - \gamma)} \right).$$
\[ - \sum_{\nu \neq \mu} \frac{Y_{\mu \nu}}{\cos((\mu - \nu) \cdot x)}, \]  

(IV.12)

where \( b_{\mu}, b'_{\mu} \) take the value as Eq. (IV.8) with the trigonometric forms of \( f(x) \) and \( g(x) \).

The \( L_{C_n}, M_{C_n} \) satisfies the Lax equation

\[ \dot{L}_{C_n} = \{ L_{C_n}, H_{C_n} \} = [ M_{C_n}, L_{C_n} ], \]  

(IV.13)

which equivalent to the equations of motion Eq.(IV.1) and Eq.(IV.2). The Hamiltonian \( H_{C_n} \) can be rewritten as the trace of \( L_{C_n} \)

\[ H_{C_n} = tr L_{C_n} = \frac{1}{2} \sum_{\mu \in \Lambda_{C_n}} (e^{\mu \cdot p} b_{\mu} + e^{-\mu \cdot p} b'_{\mu}). \]  

(IV.14)

The characteristic polynomial of the Lax matrix \( L_{C_n} \) generates the involutive Hamiltonians

\[ \det(L_{C_n} - v \cdot Id) = \sum_{j=0}^{n-1} (-1)^j (v^j + v^{2n-j})(H_j)_{C_n} + (-v)^n (H_n)_{C_n}, \]  

(IV.15)

where \((H_0)_{C_n} = 1\), and \((H_i)_{C_n}\) Poisson commute

\[ \{ (H_i)_{C_n}, (H_j)_{C_n} \} = 0, \quad i, j = 1, \ldots, n. \]  

(IV.16)

This can be deduced by verbose but straightforward calculation to verify that the \((H_i)_{A_2n-1}, i = 1, \ldots, 2n\) is the first class Hamiltonian with respect to the constraints Eq.(III.10), using Eq.(II.16), (III.12) and the first formula of Eq.(III.17).

The explicit form of \((H_l)_{C_n}\) are

\[ (H_l)_{C_n} = \sum_{J \subseteq \{1, \ldots, n\}, |J| \leq l} \exp(p_{\varepsilon J}) F_{\varepsilon J; l-|J|} U_{\varepsilon J; l-|J|}, \quad l = 1, \ldots, n, \]  

(IV.17)

with

\[ p_{\varepsilon J} = \sum_{j \in J} \varepsilon_j p_j, \]

\[ F_{\varepsilon J; K} = \prod_{j, j' \in J, j < j'} f^2(\varepsilon_j x_j + \varepsilon_{j'} x_{j'}) \prod_{j \in J} f(\varepsilon_j x_j) \prod_{j \in J} f(2\varepsilon_j x_j), \]

\[ U_{l, p} = \sum_{p' \subseteq p} \prod_{j \in p'} f(x_{jk}) f(x_{jk} + x_{kj}) g(x_{jk}) g(x_{jk} + x_{kj}) \begin{cases} 0, & (p \text{ odd}), \\ 1, & (p \text{ even}). \end{cases} \]  

(IV.18)

Here, \([p/2]\) denotes the integer part of \(p/2\). As an example, for \(C_2 RS\) model, the independent Hamiltonian flows \((H_1)_{C_2}\) and \((H_2)_{C_2}\) generated by the Lax matrix \(L_{C_2}\) are
\[(H_1)_{C_2} = H_{C_2} = e^{p_1} f(2x_1) f(x_{12}) f(x_1 + x_2) \\
+ e^{-p_1} g(2x_1) g(x_{12}) g(x_1 + x_2) \\
+ e^{p_2} f(2x_2) f(x_{21}) f(x_2 + x_1) \\
+ e^{-p_2} g(2x_2) g(x_{21}) g(x_2 + x_1), \quad (IV.19)\]

\[(H_2)_{C_2} = e^{p_1+p_2} f(2x_1) (f(x_1 + x_2))^2 f(2x_2) \\
+ e^{-p_1-p_2} g(2x_1) (g(x_1 + x_2))^2 g(2x_2) \\
+ e^{p_1-p_2} f(2x_1) (f(x_{12}))^2 f(-2x_2) \\
+ e^{p_2-p_1} g(2x_1) (g(x_{12}))^2 g(-2x_2) \\
+ 2f(x_{12}) g(x_{12}) f(x_1 + x_2)g(x_1 + x_2). \quad (IV.20)\]

### IV.2 BC\(_n\) model

The Hamiltonian BC\(_n\) model is expressed in Eq.\((III.22)\), so the canonical equations of motion are

\[
\dot{x}_i = \{x_i, H\} = e^{p_i} b_i - e^{-p_i} b_i', \quad (IV.21) \\
\dot{p}_i = \{p_i, H\} = \sum_{j \neq i}^n (e^{p_j} b_j (f'(x_{ji}) - f'(x_j + x_i) f(x_j + x_i)) \\
+ e^{-p_j} b_j' (g'(x_{ji}) - g'(x_j + x_i) g(x_j + x_i)) - e^{p_j} b_j (f(x_i) f(2x_i) f(2x_i) + \sum_{j \neq i}^n (f'(x_{ij}) f(x_{ij}) + f'(x_i + x_j))) \\
+ e^{-p_j} b_j' (g(x_i) g(2x_i) g(2x_i) + \sum_{j \neq i}^n (g'(x_{ij}) g(x_{ij}) + g'(x_i + x_j))) \\
- b_0 (f'(x_i) f(x_i) + g'(x_i) g(x_i)), (IV.22)\]

where

\[
b_i = f(x_i) f(2x_i) \prod_{k \neq i}^n (f(x_i - x_k) f(x_i + x_k)), \]

\[
b_i' = g(x_i) g(2x_i) \prod_{k \neq i}^n (g(x_i - x_k) g(x_i + x_k))
\]
\[ b_0 = \prod_{i=1}^{n} (f(x_i)g(x_i)). \] (IV.23)

The Lax pair for \( BC_n \) RS model can be constructed as the form of Eq. (IV.4)-(IV.12) where one should replace the matrices labels with \( \mu, \nu \in \Lambda_{BC_n} \), and roots with \( \beta \in \Delta_{BC_n} \).

The Hamiltonian \( H_{BC_n} \) can be rewritten as the trace of \( L_{BC_n} \)

\[ H_{BC_n} = tr L_{BC_n} = \frac{1}{2} \sum_{\mu \in \Lambda_{BC_n}} (e^{\mu p} b_{\mu} + e^{-\mu p} b_{\mu}'). \] (IV.24)

The characteristic polynomial of the Lax matrix \( L \) generates the involutive Hamiltonians

\[ \det(L_{BC_n} - v \cdot Id) = \sum_{j=0}^{n} (-1)^{j} (v^{j} - v^{2n+1-j})(H_{j})_{BC_n}, \] (IV.25)

where \( (H_0)_{BC_n} = 1 \) and \( (H_i)_{BC_n} \) Poisson commute

\[ \{ (H_i)_{BC_n}, (H_j)_{BC_n} \} = 0, \quad i, j = 1, \ldots, n. \] (IV.26)

This can be deduced similarly to \( C_n \) case to verify that the \( (H_i)_{A_{2n}} \) is the first class Hamiltonian with respect to the constraints Eq. (III.24).

The explicit form of \( (H_i)_{BC_n} \) are

\[ (H_l)_{BC_n} = \sum_{J \subset (1, \ldots, n), \mid J \mid \leq l} \exp(p_{\epsilon J}) F_{\epsilon J; l} U_{J, l-\mid J\mid}, \quad l = 1, \ldots, n, \] (IV.27)

with

\[ p_{\epsilon J} = \sum_{j \in J} \varepsilon_j p_j, \]
\[ F_{\epsilon J; K} = \prod_{j, j' \in J, j < j'} f^2(\varepsilon_j x_j + \varepsilon_j' x_{j'}) \prod_{j \in J} f(\varepsilon_j x_j + x_k) f(\varepsilon_j x_j - x_k) \prod_{j \in J} f(2\varepsilon_j x_j) \prod_{j \in J} f(\varepsilon_j x_j), \]
\[ U_{I, p} = \sum_{\mid J \mid \leq l} \prod_{j \in J} f(x_{jk}) f(x_j + x_k) g(x_{jk}) g(x_j + x_k) \left\{ \begin{array}{ll} \prod_{i \in I \setminus J} f(x_i) g(x_i), & (p \text{ odd}), \\ \prod_{i' \in I' \setminus J} f(x_{i'}) g(x_{i'}), & (p \text{ even}). \end{array} \right\} \] (IV.28)

V Nonrelativistic limit to the Calogero-Moser system

V.1 Limit to \( C_n CM \) model
The Nonrelativistic limit can be achieved by rescaling $p_i \mapsto \beta p_i$, $\gamma \mapsto \beta \gamma$ while letting $\beta \mapsto 0$, and making a canonical transformation

\[
\begin{align*}
    p_i \mapsto p_i + \gamma \left\{ \frac{1}{2x_i} + \sum_{k \neq i}^{n}(\frac{1}{x_{ik}} + \frac{1}{x_i + x_k}) \right\}, \\
    p_i \mapsto p_i + \gamma \left\{ \cot(2x_i) + \sum_{k \neq i}^{n}(\cot(x_{ik}) + \cot(x_i + x_k)) \right\},
\end{align*}
\]

such that

\[
\begin{align*}
    L &\mapsto \text{Id} + \beta L_{CM} + O(\beta^2), \quad \text{(V.2)} \\
    M &\mapsto 2\beta M_{CM} + O(\beta^2), \quad \text{(V.3)}
\end{align*}
\]

and

\[
H \mapsto 2n + 2\beta^2 H_{CM} + O(\beta^2). \quad \text{(V.4)}
\]

$L_{CM}$ can be expressed as

\[
L_{CM} = \left( \begin{array}{cc} A_{CM} & B_{CM} \\ -B_{CM} & -A_{CM} \end{array} \right), \quad \text{(V.5)}
\]

where

\[
\begin{align*}
    (A_{CM})_{ii} &= p_i, \\
    (B_{CM})_{ij} &= \frac{\gamma}{x_i + x_j}, \\
    (A_{CM})_{ij} &= \frac{\gamma}{x_{ij}}, \quad (i \neq j), \quad \text{(V.6)}
\end{align*}
\]

for the rational case, and

\[
\begin{align*}
    (A_{CM})_{ii} &= p_i, \\
    (B_{CM})_{ij} &= \frac{\gamma}{\sin(x_i + x_j)}, \\
    (A_{CM})_{ij} &= \frac{\gamma}{\sin(x_{ij})}, \quad (i \neq j), \quad \text{(V.7)}
\end{align*}
\]

for the trigonometric case.

$M_{CM}$ is

\[
M_{CM} = \left( \begin{array}{cc} A_{CM} & B_{CM} \\ B_{CM} & A_{CM} \end{array} \right), \quad \text{(V.8)}
\]

as for the rational case.
\[(A_{CM})_{ii} = -\sum_{k \neq i}^{n} \left( \frac{\gamma}{x_{ik}^2} + \frac{\gamma}{(x_i + x_k)^2} \right) - \frac{\gamma}{(2x_i)^2}, \quad (B_{CM})_{ij} = \frac{\gamma}{(x_i + x_j)^2}, \]

\[(A_{CM})_{ij} = \frac{\gamma}{x_{ij}^2}, \quad (i \neq j), \quad (V.9)\]

which identified with the results of Refs. [6] and [7], and for the trigonometric case

\[(A_{CM})_{ii} = -\sum_{k \neq i}^{n} \left( \frac{\gamma}{\sin^2 x_{ik}} + \frac{\gamma}{\sin^2(x_i + x_k)} \right) - \frac{\gamma}{\sin^2(2x_i)}, \quad (B_{CM})_{ij} = \frac{\gamma \cos(x_i + x_j)}{\sin^2(x_i + x_j)}, \]

\[(A_{CM})_{ij} = \frac{\gamma \cos(x_{ij})}{\sin^2 x_{ij}}, \quad (i \neq j), \quad (V.10)\]

which coincide with the form given in Ref. [6] up to a diagonalized matrix together with a suitable choose of coupling parameters.

The Hamiltonian of \(C_n\)-type \(CM\) model can be given by

\[H_{CM} = \frac{1}{2} \sum_{k=1}^{n} p_k^2 - \gamma^2 \sum_{k<i}^{n} \left( \frac{1}{x_{ik}^2} + \frac{1}{(x_i + x_k)^2} \right) - \frac{\gamma^2}{2} \sum_{i=1}^{n} \frac{1}{(2x_i)^2} \]

\[= \frac{1}{4} tr L^2, \quad \text{for the rational case}, \quad (V.11)\]

\[H_{CM} = \frac{1}{2} \sum_{k=1}^{n} p_k^2 - \gamma^2 \sum_{k \neq i}^{n} \left( \frac{1}{\sin^2 x_{ik}} + \frac{1}{\sin^2(x_i + x_k)} \right) - \frac{\gamma^2}{2} \sum_{i=1}^{n} \frac{1}{\sin^2(2x_i)} \]

\[= \frac{1}{4} tr L^2, \quad \text{for the trigonometric case}. \quad (V.12)\]

The \(L_{CM}, M_{CM}\) satisfies the Lax equation

\[\dot{L}_{CM} = \{L_{CM}, H_{CM}\} = [M_{CM}, L_{CM}]. \quad (V.13)\]

### V.2 Limit to \(BC_n\) CM model

The Nonrelativistic limit of \(BC_n\) model can also be achieved by rescaling \( p_i\rightarrow \beta p_i, \gamma\rightarrow \beta \gamma \) while letting \(\beta\rightarrow 0\), and making the following canonical transformation

\[
\begin{aligned}
\{ p_i \mapsto p_i + \gamma \left\{ \frac{1}{x_i} + \frac{1}{2x_i} + \sum_{k \neq i}^{n} \left( \frac{1}{x_{ik}} + \frac{1}{x_i + x_k} \right) \right\}, \\
\{ p_i \mapsto p_i + \gamma \{\cot(x_i) + \cot(2x_i) + \sum_{k \neq i}^{n} (\cot(x_{ik}) + \cot(x_i + x_k)) \}, \quad \text{rational case},
\end{aligned}
\]

\[
\begin{aligned}
\{ p_i \mapsto p_i + \gamma \{\cot(x_i) + \cot(2x_i) + \sum_{k \neq i}^{n} (\cot(x_{ik}) + \cot(x_i + x_k)) \}, \quad \text{trigonometric case}. \quad (V.14)
\end{aligned}
\]
such that

\[ L \mapsto \Id + \beta L CM + O(\beta^2), \quad (V.15) \]
\[ M \mapsto 2\beta M CM + O(\beta^2), \quad (V.16) \]

and

\[ H \mapsto (2n + 1) + 2\beta^2 H CM + O(\beta^2). \quad (V.17) \]

\( L CM \) can be expressed as

\[ L CM = \begin{pmatrix} A\text{CM} & B\text{CM} & E\text{CM} \\ -B\text{CM} & -A\text{CM} & -E\text{CM} \\ -(E\text{CM})^t & (E\text{CM})^t & G\text{CM} \end{pmatrix}, \quad (V.18) \]

where

\[ (A\text{CM})_{ii} = p_i, \quad (B\text{CM})_{ij} = \frac{\gamma}{x_i + x_j}, \quad (E\text{CM})_{i1} = \frac{1}{x_i}, \quad G\text{CM} = 0, \]

\[ (A\text{CM})_{ij} = \frac{\gamma}{x_{ij}}, \quad (i \neq j), \quad (V.19) \]

for the rational case, and

\[ (A\text{CM})_{ii} = p_i, \quad (B\text{CM})_{ij} = \frac{\gamma}{\sin(x_i + x_j)}, \quad (E\text{CM})_{i1} = \frac{1}{\sin x_i}, \quad G\text{CM} = 0, \]

\[ (A\text{CM})_{ij} = \frac{\gamma}{\sin(x_{ij})}, \quad (i \neq j), \quad (V.20) \]

for the trigonometric case.

\( M CM \) is

\[ M CM = \begin{pmatrix} A\text{CM} & B\text{CM} & E\text{CM} \\ B\text{CM} & A\text{CM} & E\text{CM} \\ (E\text{CM})^t & (E\text{CM})^t & G\text{CM} \end{pmatrix}, \quad (V.21) \]

where \( t \) denotes the transposition. As for the rational case, the forms of \( A, B, E, G \) are

\[ (A\text{CM})_{ii} = -\sum_{k \neq i}^{n} \frac{\gamma}{x_{ik}^2} + \frac{\gamma}{(x_i + x_k)^2} - \frac{\gamma}{(2x_i)^2} - \frac{\gamma}{(x_i)^2}, \]

\[ (B\text{CM})_{ij} = \frac{\gamma}{(x_i + x_j)^2}, \quad (E\text{CM})_{i1} = \frac{\gamma}{(x_i)^2}, \]

\[ (A\text{CM})_{ij} = \frac{\gamma}{x_{ij}^2}, \quad (i \neq j), \quad G\text{CM} = -\sum_{k=1}^{n} \frac{2\gamma}{x_k^2}, \quad (V.22) \]
which identified with the results of Refs. [6] and [7],
and for the trigonometric case

\[(A_{CM})_{ii} = -\sum_{k \neq i}^{n} \left( \frac{\gamma}{\sin^2 x_{ik}} + \frac{\gamma}{\sin^2(x_i + x_k)} \right) - \frac{\gamma}{\sin^2(2x_i)} - \frac{\gamma}{\sin^2(x_i)}, \]

\[(B_{CM})_{ij} = \frac{\gamma \cos(x_i + x_j)}{\sin^2(x_i + x_j)}, \quad (E_{CM})_{i1} = \frac{\gamma \cos x_i}{\sin^2 x_i}, \]

\[(A_{CM})_{ij} = \frac{\gamma \cos(x_{ij})}{\sin^2 x_{ij}}, \quad \text{(} i \neq j \text{)}, \quad G_{CM} = -\sum_{k=1}^{n} \frac{2\gamma}{\sin^2 x_k}, \quad \text{(V.23)}\]

which coincide with the form given in Ref. [6] up to a diagonalized matrix together with a suitable choose of coupling parameters.

The Hamiltonian of $BC_n$-type $CM$ model can be given by

\[H_{CM} = \frac{1}{2} \sum_{k=1}^{n} p_k^2 - \gamma^2 \sum_{k<i}^{n} \left( \frac{1}{x_{ik}^2} + \frac{1}{(x_i + x_k)^2} \right) - \frac{\gamma^2}{2} \sum_{i=1}^{n} \left( \frac{1}{(2x_i)^2} + \frac{2}{(x_i)^2} \right) \]

\[= \frac{1}{4} trL^2, \quad \text{for the rational case,} \quad \text{(V.24)}\]

\[H_{CM} = \frac{1}{2} \sum_{k=1}^{n} p_k^2 - \gamma^2 \sum_{k<i}^{n} \left( \frac{1}{\sin^2 x_{ik}} + \frac{1}{\sin^2(x_i + x_k)} \right) - \frac{\gamma^2}{2} \sum_{i=1}^{n} \left( \frac{1}{\sin^2(2x_i)} + \frac{2}{\sin^2(x_i)} \right) \]

\[= \frac{1}{4} trL^2, \quad \text{for the trigonometric case.} \quad \text{(V.25)}\]

The $L_{CM}, M_{CM}$ satisfies the Lax equation

\[\dot{L}_{CM} = \{L_{CM}, H_{CM}\} = [M_{CM}, L_{CM}]. \quad \text{(V.26)}\]

**VI Summary and discussions**

In this paper, we have proposed the Lax pairs for rational, trigonometric $C_n$ and $BC_n$ $RS$ models. Involutive Hamiltonians are showed to be generated by the characteristic polynomial of the corresponding Lax matrix. In the nonrelativistic limit, the system leads to $CM$ systems associated with the root systems of $C_n$ and $BC_n$ which are known previously. There are still many open problems, for example, it seems to be a challenging subject to carry out the Lax pairs with as many independent coupling constants as independent Weyl orbits in the set of roots, as done for the Calogero-Moser systems [3, 4, 5, 6, 10, 11]. What is also interesting may generalize the results obtained in this paper to the systems associated with all of other Lie Algebras even to those associated with all the finite reflection groups [10] which including models based on the non-crystallographic root systems and those based on crystallographic root systems.
Acknowledgement

One of the authors K. Chen is grateful to professors K. J. Shi and L. Zhao for their encouragement. This work has been supported financially by the National Natural Science Foundation of China.

References

[1] K. Hasegawa, Commun. Math. Phys. 187, 289 (1997).

[2] F.W. Nijhoff, V.B. Kuznetsov, E.K. Sklyanin and O. Ragnisco, J. Phys. A: Math. Gen. 29, L333 (1996).

[3] A. Gorsky, A. Marshakov, Phys. Lett. B375, 127 (1996).

[4] N. Nekrasov, Nucl. Phys. B531, 323 (1998).

[5] H.W. Braden, A. Marshakov, A. Mironov and A. Morozov, Nucl.Phys. B558, 371 (1999).

[6] M.A. Olshanetsky and A.M. Perelomov, Phys. Rep. 71, 314 (1981).

[7] A.J. Bordner, E. Corrigan and R. Sasaki, Prog. Theor. Phys. 100, 1107 (1998).

[8] A.J. Bordner, R. Sasaki and K. Takasaki, Prog.Theor.Phys. 101, 487 (1999).

[9] A.J. Bordner and R. Sasaki, Prog.Theor.Phys. 101, 799 (1999).

[10] A.J. Bordner, E. Corrigan and R. Sasaki, Prog.Theor.Phys. 102, 499 (1999).

[11] E. D’Hoker and D.H. Phong, Nucl.Phys. B530, 537 (1998).

[12] Y. Komori, K. Hikami, J. Math. Phys. 39, 6175 (1998).

[13] Y. Komori, “Theta Functions Associated with the Affine Root Systems and the Elliptic Ruijsenaars Operators,” e-print math.QA/9910003.

[14] J.F. van Diejen, J. Math. Phys. 35, 2983 (1994).

[15] J.F. van Diejen, Compositio. Math. 95, 183 (1995).

[16] K. Hasegawa, T. Ikeda, T. Kikuchi, J. Math. Phys. 40, 4549 (1999).

[17] K. Chen, B.Y. Hou, W.-L. Yang and Y. Zhen, Chin. Phys. Lett. 16, 1 (1999); High energy physics and nuclear physics. Vol.23, No. 9, 854 (1999).
[18] K. Chen, H. Fan, B.Y. Hou, K.J. Shi, W.-L. Yang and R. H. Yue, Prog. Theor. Phys. Suppl. 135, 149 (1999).

[19] T. Kikuchi, “Diagonalization of the elliptic Macdonald-Ruijsenaars difference system of type C₂,” e-print math/9912114.

[20] S.N.M. Ruijsenaars, Commun. Math. Phys. 110, 191 (1987).

[21] M. Bruschi and F. Calogero, Commun. Math. Phys. 109, 481 (1987).

[22] I. Krichever and A. Zabrodin, Usp. Math. Nauk, 50:6, 3 (1995).

[23] Y.B. Suris, “Why are the rational and hyperbolic Ruijsenaars-Schneider hierarchies governed by the same R-operators as the Calogero-Moser ones?” e-print hep-th/9602160.

[24] Y.B. Suris, Phys. Lett. A225, 253 (1997).

[25] K. Chen, B.Y. Hou and W.-L. Yang, “The Lax pair for C₂-type Ruijsenaars-Schneider model,” e-print hep-th/0004006.

[26] S.N.M. Ruijsenaars, Commun. Math. Phys. 115, 127 (1988).

[27] Y. Ohta, Instanton Correction of Prepotential in Ruijsenaars Model Associated with N=2 SU(2) Seiberg-Witten Theory. e-print, hep-th/9909196.

[28] S.N.M. Ruijsenaars and H. Schneider, Ann. Phys. 170, 370 (1986).

[29] Paul A.M. Dirac, Lectures on Quantum Physics(Yeshiva University, New York, 1964)