DOUBLE SQUARE MOMENTS AND BOUNDS FOR RESONANCE SUMS OF CUSP FORMS

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Abstract. Let $f$ and $g$ be holomorphic cusp forms for the modular group $SL_2(\mathbb{Z})$ of weight $k_1$ and $k_2$ with Fourier coefficients $\lambda_f(n)$ and $\lambda_g(n)$, respectively. For real $\alpha \neq 0$ and $0 < \beta \leq 1$, consider a smooth resonance sum $S_X(f,g;\alpha,\beta)$ of $\lambda_f(n)\lambda_g(n)$ against $e(\alpha n^\beta)$ over $X \leq n \leq 2X$. Double square moments of $S_X(f,g;\alpha,\beta)$ over both $f$ and $g$ are nontrivially bounded when their weights $k_1$ and $k_2$ tend to infinity together. By allowing both $f$ and $g$ to move, these double moments are indeed square moments associated with automorphic forms for $GL(4)$. By taking out a small exceptional set of $f$ and $g$, bounds for individual $S_X(f,g;\alpha,\beta)$ will then be proved. These individual bounds break the resonance barrier of $X^{\frac{5}{8}}$ for $\frac{1}{6} < \beta < 1$ and achieve a square-root cancellation for $\frac{1}{3} < \beta < 1$ for almost all $f$ and $g$ as an evidence for Hypothesis S for cusp forms over integers. The methods used in this study include Petersson’s formula, Poisson’s summation formula, and stationary phase integrals.

1. Introduction

According to Iwaniec, Luo, and Sarnak [8, Appendix C], a general form of Hypothesis S over integers predicts a square-root cancellation in the sum

$$S_X(\{a_n\};\alpha,\beta) = \sum_n a_n e(\alpha n^\beta) \phi\left(\frac{n}{X}\right) \ll X^{\frac{3}{4} + \varepsilon},$$

where $0 < \beta \leq 1$ and $\alpha \neq 0$ are fixed real numbers, $\phi$ is a smooth function of compact support in $(1,2)$, and $\{a_n\}$ is an arithmetically defined sequence of complex numbers satisfying $a_n \ll n^{\varepsilon}$. The Case III in [8] of $a_n = \lambda_f(n)$ for a fixed holomorphic cusp form $f$ for $SL_2(\mathbb{Z})$, however, faces a resonance barrier when $\beta = \frac{1}{2}$ and $\alpha = \pm 2\sqrt{q}$ for a positive integer $q$. In other words,

$$S_X(f;\pm 2\sqrt{q},\frac{1}{2}) := \sum_n \lambda_f(n)e(\pm 2\sqrt{q}n)\phi\left(\frac{n}{X}\right)$$

has a main term of size $|\lambda_f(q)|X^{\frac{3}{2}}$. This resonance phenomenon has been further studied by Ren and Ye [12] – [17], Ernvall-Hytonen [4], Ernvall-Hytonen, Jääsaari, and Vesalainen [5], Czarnecki [2], Savala [19], and many other authors for fixed automorphic forms $f$.

It is believed that one might be able to break the resonance barrier if the cusp form $f$ is allowed to move. In this direction, Ye [22] proved the first known non-trivial bound for $S_X(f;\alpha,\beta)$ when the weight $k$ of $f$ tends to infinity with the summation length $X$. This bound, however, is far from reaching the square-root cancellation.

The goal of the present paper is to break the resonance barrier for

$$S_X(f,g;\alpha,\beta) = \sum_n \lambda_f(n)\lambda_g(n)e(\alpha n^\beta)\phi\left(\frac{n}{X}\right)$$

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for almost all holomorphic cusp forms \( f \) and \( g \) for \( SL_2(\mathbb{Z}) \) of even integer weights \( k_1 \) and \( k_2 \), respectively. Note here that \( \lambda_f(n)\lambda_g(n) \) in (1.1) corresponds to the Dirichlet coefficients coming from the Rankin-Selberg \( L \)-function

\[
L(s, f \times g) = \zeta(2s) \sum_{n \geq 1} \frac{\lambda_f(n)\lambda_g(n)}{n^s}.
\]

Consequently, \( S_X(f, g; \alpha, \beta) \) represents the interplay between the oscillation of the Dirichlet coefficients of \( L(s, f \times g) \) and that of a fractional exponential function.

By [16] and [2], the sum \( S_X(f, g; \alpha, \beta) \) has a main term of size \( |\lambda_f(q)\lambda_g(q)|X^{\frac{1}{4}+\frac{1}{2}} \) when \( \beta = \frac{1}{4} \) and \( \alpha \) is close or equal to \( \pm 4q^\frac{1}{4} \) for a positive integer \( q \) for fixed \( f \) and \( g \). The resonance barrier in this case is thus \( X^{\frac{1}{4}} \).

We will break this resonance barrier for almost all \( f \) and \( g \), in the sense that the number of exceptional pairs \((f, g)\) is a power less than the total number of pairs \((f, g)\) in consideration. In particular, the exceptional pairs have a probability tending to zero.

More precisely, let \( S_k(\Gamma) \) denotes the space of holomorphic cusp forms on \( \Gamma = SL_2(\mathbb{Z}) \) of even integral weight \( k \), and let \( H_k \) denote an orthogonal basis of \( S_k(\Gamma) \) consisting of Hecke eigenforms where each form is normalized to have the first Fourier coefficient equal to 1. Recall the dimension formula \( \dim S_k(\Gamma) = \frac{k}{12} + O(1) \). Consider parameters \( K_j \leq L_j \leq K_j^{1-\varepsilon} \) for \( j = 1, 2 \), and denote \( H_{K_j} = \bigcup_{K_j - L_j \leq k \leq K_j + L_j} H_k \).

We will bound the double square moment

\[
\sum_{f \in H_{K_1}, g \in H_{K_2}} |S_X(f, g; \alpha, \beta)|^2
\]

for \( K_1, K_2, L_1, L_2, X \) in various ranges. Let \( g_0 \in C^\infty(-1, 1) \) be a non-negative test function with \( g_0(0) = 1 \). We will equivalently bound the smooth double moment

\[
\sum_{2|k_1} \sum_{2|k_2} g_0\left(\frac{k_1 - K_1}{L_1}\right) g_0\left(\frac{k_2 - K_2}{L_2}\right) \sum_{f \in H_{k_1}, g \in H_{k_2}} |S_X(f, g, \alpha, \beta)|^2.
\]

In order to apply Petersson’s formula, we will actually bound a normalized sum

\[
\sum_{K_1L_1}^{K_2L_2} = K_1K_2 \sum_{2|k_1} \sum_{2|k_2} g_0\left(\frac{k_1 - K_1}{L_1}\right) g_0\left(\frac{k_2 - K_2}{L_2}\right) \sum_{f \in H_{k_1}} \frac{2\pi^2}{(k_1 - 1)L(1, Sym^2 f)}
\]

\[
\times \sum_{g \in H_{k_2}} \frac{2\pi^2}{(k_2 - 1)L(1, Sym^2 g)} |S_X(f, g, \alpha, \beta)|^2
\]

\[
= K_1K_2 \sum_{2|k_1} \sum_{2|k_2} g_0\left(\frac{k_1 - K_1}{L_1}\right) g_0\left(\frac{k_2 - K_2}{L_2}\right) \sum_{m} \sum_{n} e(\alpha n^\beta - \alpha m^\beta) \phi\left(\frac{n}{X}\right) \bar{\phi}\left(\frac{m}{X}\right)
\]

\[
\times \sum_{f \in H_{k_1}} \frac{2\pi^2\lambda_f(n)\lambda_g(m)}{(k_1 - 1)L(1, Sym^2 f)} \sum_{g \in H_{k_2}} \frac{2\pi^2\lambda_g(n)\lambda_f(m)}{(k_2 - 1)L(1, Sym^2 g)}.
\]

The factors

\[
\frac{2\pi^2K_1}{(k_1 - 1)L(1, Sym^2 f)} \quad \text{and} \quad \frac{2\pi^2K_2}{(k_2 - 1)L(1, Sym^2 g)}
\]

in (1.4) will result in a discrepancy bounded between \( K_i^{-\varepsilon} \) and \( K_i^\varepsilon, i = 1, 2 \), by Iwaniec [7] and Hoffstein and Lockhart [6].

For holomorphic cusp forms we have Deligne’s proof [3] of the Ramanujan conjecture which implies that \( \lambda_f(n) \ll n^\varepsilon \) and \( \lambda_g(n) \ll n^\varepsilon \) for \( \varepsilon > 0 \) arbitrarily small. Here the implied constants are independent of \( f \)
and $g$. Using this we obtain a trivial bound

\[ S_X(f, g; \alpha, \beta) \ll X^{1+\varepsilon}, \]

where the implied constant is independent of $f$ and $g$. Applying this to (1.4), we get a trivial bound $O(K_1L_1K_2L_2X^{2+\varepsilon})$ for $\sum_{K_1L_1}^{K_2L_2}$ in (1.4) and hence for (1.3) and (1.2). We seek to break this bound. Note that non-trivial bounds beyond (1.5) are known for $S_X(f, g; \alpha, \beta)$ for both $f$ and $g$ being fixed but not for $f$ and $g$ with their weights tending to infinity.

**Theorem 1.1.** For $j = 1, 2$ assume $K_j^\varepsilon \leq L_j \leq K_j^{1-\varepsilon}$. Then for $0 < \beta < 1$

\[ \sum_{K_1L_1}^{K_2L_2} \ll \begin{cases} K_1L_1K_2L_2X^{1+\varepsilon} & \text{if } K_1L_1 \geq X^{1+\varepsilon} \text{ and } K_2 \geq X^{1/2+\varepsilon}; \\ K_1L_1K_2X^{1+\varepsilon} + \frac{K_1L_1L_2}{K_2}X^{\varepsilon} & \text{if } K_1L_1 \geq X^{1+\varepsilon} \text{ and } K_2 \leq X^{1/2}; \\ K_1^2L_1L_2X^{1+\varepsilon} + \frac{X^{3+\varepsilon}}{K_1} & \text{if } K_1L_1, K_2L_2 \leq X^{1-\varepsilon}, K_1 = K_2, \text{ and } K_1^2L_1L_2 \geq X^{1+\beta+\varepsilon}. \end{cases} \]

When $\beta = 1$, bounds in (1.6) and (1.7) remain valid, while (1.8) is replaced by

\[ \ll \min(L_1, L_2)K_1X^{2+\varepsilon} + \frac{X^{3+\varepsilon}}{K_1} \text{ if } K_1L_1, K_2L_2 \leq X^{1-\varepsilon}, K_1 = K_2. \]

Since the number of terms in $\sum_{K_1L_1}^{K_2L_2}$ is $\asymp K_1L_1K_2L_2$, (1.6) show that the average size of $S_X(f, g; \alpha, \beta)$ is bounded by $O(X^{1/2+\varepsilon})$ when $K_1$ and $K_2$ are large. This represents a square-root saving on average for $S_X(f, g; \alpha, \beta)$.

If we take

\[ K_1 = K_2 = X^{\frac{1+\beta}{2}}, \quad L_1 = L_2 = X^{\frac{1-\varepsilon}{2}} \]

in (1.8), we have $K_1^2L_1L_2 = X^{1+\beta+\varepsilon}$ and the bound in (1.8) becomes

\[ \sum_{K_1L_1}^{K_2L_2} \ll X^{2+\beta+\varepsilon} + X^{\frac{2+\beta}{2}+\varepsilon}. \]

This is also a bound for (1.2) for $0 < \beta < 1$. Since the two terms on the right hand side of (1.11) are both larger than $X^2$, (1.11) cannot provide a non-trivial bound for all individual $S_X(f, g; \alpha, \beta)$. It can, nevertheless, allow us to break the resonance barrier $X^{\frac{5}{2}}$ for almost all forms $f$ and $g$.

To this end, define

\[ \Delta_{f,g}(\gamma) = \left\{ \text{pairs } (f, g) \bigg| f \in H_{K_1, L_1}, g \in H_{K_2, L_2} \text{ such that } |S_X(f, g; \alpha, \beta)| \geq X^{\gamma+\varepsilon} \right\} \]

for $0 < \gamma < 1$. By the bound (1.11) on (1.2) we get

\[ X^{2\gamma+2\varepsilon} |\Delta_{f,g}(\gamma)| \ll X^{2+\beta+\varepsilon} + X^{\frac{2+\beta}{2}+\varepsilon}. \]

Consequently,

\[ |\Delta_{f,g}(\gamma)| \ll X^{2+\beta-2\gamma-\varepsilon} + X^{\frac{2+\beta}{2}-2\gamma-\varepsilon}. \]

To make the exceptional set $\Delta_{f,g}(\gamma)$ a power smaller than the size $K_1^2L_1L_2 = X^{1+\beta+\varepsilon}$ of the averaging on $H_{K_1, L_1}$ and $H_{K_2, L_2}$, we require the two terms on the right hand side of (1.12) $\leq X^{1+\beta}$. In other words, we
need \(2 + \beta - 2\gamma \leq 1 + \beta\) and \(\frac{5-\beta}{2} - 2\gamma \leq 1 + \beta\), i.e.,

\[(1.13) \quad \max \left(\frac{1}{2}, \frac{3}{4}(1 - \beta)\right) \leq \gamma < 1.\]

**Corollary 1.2.** For parameters in (1.10) and \(\gamma\) in (1.13), we have

\[(1.14) \quad |S_X(f, g; \alpha, \beta)| \leq X^{\gamma + \varepsilon}\]

for all \(f \in H_{K_1, L_1}\) and \(g \in H_{K_2, L_2}\) except for at most \(O(X^{2+\beta-2\gamma} + X^{\frac{5-\beta}{2} - 2\gamma})\) pairs of \(f\) and \(g\). The exceptional set is a power smaller than the size of \(H_{K_1, L_1} \times H_{K_2, L_2}\).

When we take \(\gamma = \max\left(\frac{1}{2}, \frac{3}{4}(1 - \beta)\right)\), the bound (1.14) breaks the resonance barrier \(X^\frac{\beta}{2}\) when \(\frac{1}{6} < \beta < 1\) and reaches the square-root cancellation \(X^{\frac{\beta}{4} + \varepsilon}\) when \(\frac{1}{3} < \beta < 1\) for almost all \(f\) and \(g\). Similar corollaries can be formulated from (1.10), (1.7), and (1.9).

## 2. Petersson's Trace Formula

We recall Petersson’s formula (Petersson [11], cf. Liu and Ye [9]) for \(m, n \geq 1\),

\[
\sum_{f \in H_k \atop (k-1)L(1, Sym^2 f)} 2\pi^2 \frac{\lambda_f(n)}{\lambda_f(m)} = \delta(m, n) + 2\pi i^k \sum_{c \geq 1} S(m, n, c) J_{k-1} \left( \frac{4\pi \sqrt{mn}}{c} \right),
\]

where \(\lambda_f(n)\) is the Fourier coefficient of \(f\) and \(S(m, n, c)\) is the classical Kloosterman sum. Applying it to (1.4) we get

\[(2.1) \quad \sum_{K_1L_1}^{K_2L_2} K_1K_2 \sum_{2 | k_1 2 | k_2} g_0 \left( \frac{k_1 - K_1}{L_1} \right) g_0 \left( \frac{k_2 - K_2}{L_2} \right) \sum_n \sum_m e(\alpha n^\beta - \alpha m^\beta) \phi \left( \frac{n}{X} \right) \phi \left( \frac{m}{X} \right)
\]

\[\times \left( \delta(n, m) + 2\pi i^{k_1} \sum_{c_1 \geq 1} S(m, n, c_1) J_{k_1-1} \left( \frac{4\pi \sqrt{mn}}{c_1} \right) \right) \]

\[\times \left( \delta(n, m) + 2\pi i^{k_2} \sum_{c_2 \geq 1} S(m, n, c_2) J_{k_2-1} \left( \frac{4\pi \sqrt{mn}}{c_2} \right) \right)
\]

\[= D_{00} + D_{01} + D_{10} + D_{11}.
\]

Here

\[(2.2) \quad D_{00} = K_1K_2 \sum_{2 | k_1 2 | k_2} g_0 \left( \frac{k_1 - K_1}{L_1} \right) g_0 \left( \frac{k_2 - K_2}{L_2} \right) \sum_n \frac{\phi \left( \frac{n}{X} \right)}{|n|}^2
\]

\[(2.3) \quad D_{01} = K_1K_2 \sum_{2 | k_1 2 | k_2} g_0 \left( \frac{k_1 - K_1}{L_1} \right) g_0 \left( \frac{k_2 - K_2}{L_2} \right) \sum_n \frac{\phi \left( \frac{n}{X} \right)}{|n|}^2
\]

\[\times 2\pi i^{k_2} \sum_{c_2 \geq 1} S(n, n, c_2) J_{k_2-1} \left( \frac{4\pi n}{c_2} \right)
\]

\[(2.4) \quad D_{10} = K_1K_2 \sum_{2 | k_1 2 | k_2} g_0 \left( \frac{k_1 - K_1}{L_1} \right) g_0 \left( \frac{k_2 - K_2}{L_2} \right) \sum_n \frac{\phi \left( \frac{n}{X} \right)}{|n|}^2
\]

\[\times 2\pi i^{k_1} \sum_{c_1 \geq 1} S(n, n, c_1) J_{k_1-1} \left( \frac{4\pi n}{c_1} \right)
\]
\[
\begin{align*}
(2.5) \quad D_{11} &= K_1 K_2 \sum_{2 \mid k_1, 2 \mid k_2} g_0 \left( \frac{k_1 - K_1}{L_1} \right) g_0 \left( \frac{k_2 - K_2}{L_2} \right) \sum_n \sum_m e(\alpha n^2 - \alpha m^2) \phi \left( \frac{n}{N} \right) \phi \left( \frac{m}{N} \right) \\
&\times 4\pi^2 t_{k_1+k_2} \sum_{c_1 \geq 1} S(m, n, c_1) c_1 \sum_{c_2 \geq 1} S(m, n, c_2) c_2 \left( 4\pi \sqrt{mn} \right) J_{k_1-1} \left( \frac{4\pi \sqrt{mn}}{c_1} \right) J_{k_2-1} \left( \frac{4\pi \sqrt{mn}}{c_2} \right).
\end{align*}
\]

The diagonal term \( D_{00} \) in (2.5) can be estimated trivially for \( 0 < \beta \leq 1 \):

\[
(2.6) \quad D_{00} \ll K_1 L_1 K_2 L_2 X.
\]

3. The \( k_1^- \) and \( k_2^- \)-sums of Bessel functions

For off-diagonal terms \( D_{01}, D_{10}, \) and \( D_{11} \), we have the quantity

\[
V_{K_i, L_j}(x) = \sum_{2 \mid k_j} \frac{i^{k_j} g_0 \left( \frac{k_j - K_j}{L_j} \right) J_{k_j - 1}(x)}{c_j},
\]

with \( x = \frac{4\pi \sqrt{mn}}{c_j} \). Rewriting this sum as an oscillatory integral

\[
V_{K_i, L_j}(x) = \frac{1}{2t} \left( W_{K_i, L_j}(-x) - W_{K_i, L_j}(x) \right),
\]

where for \( \eta = \pm 1 \),

\[
W_{K_i, L_j}(\eta x) = \int_{-\infty}^{\infty} \tilde{g}_0(t) e \left( -\left( \frac{K_j - 1}{L_j} \right) t \right) \frac{\eta x}{2\pi} \cos \frac{2\pi t}{L_j} dt,
\]

then applying the method of stationary phase again gives an asymptotic expansion. Denote

\[
g(t) = \tilde{g}_0(t), \quad f(t) = -\left( \frac{K_j - 1}{L_j} \right) t - \frac{x}{2\pi} \cos \frac{2\pi t}{L_j}.
\]

Lemma 3.1. (Sun and Ye \cite{21} Lemma 4.1). cf. Salazar and Ye \cite{18} Suppose \( K_j^+ \leq L_j \leq K_j^{-1-\epsilon} \).

(1) If \( x \leq 8\pi K_j^{-1-\epsilon} L_j \), then \( W_{K_i, L_j}(\eta x) \ll \epsilon, A K_j^{-A} \) for any \( A > 0 \).

(2) If \( x \geq K_j^{-1-\epsilon} L_j \), then

\[
W_{K_i, L_j}(\eta x) = \sum_{\nu = 0}^{n_0} \tilde{W}_{j, \nu}(\eta x) + O \left( \frac{L_j^{2n_0+2}}{x^{n_0+1}} \right).
\]

Here

\[
\tilde{W}_{j, \nu}(\eta x) = e \left( \frac{1}{x} \right) (-1)^{\nu}(2\nu - 1)!! \frac{L_j^{2\nu+1}}{2^{2\nu+\nu^2+\nu^2} \pi^{2\nu+\nu} x^{2\nu} \pi^{2\nu+\nu} x^{2\nu}} \times G_{2\nu}(\eta x) e \left( -\frac{\eta_2}{2} \sqrt{x^2 - (K_j - 1)^2} - \frac{\eta(K_j - 1)}{x\pi} \sin^{-1} \left( \frac{K_j - 1}{x} \right) \right),
\]

\[
G_{2\nu}(t_0) = \frac{g(2\nu)(t_0)}{(2\nu)!} + \sum_{\ell = 0}^{2\nu - 1} \frac{g^{(\ell)}(t_0)}{\ell!} \sum_{k = 1}^{2\nu - \ell} \frac{C_{2\nu, \ell, k}}{2\pi \sqrt{x^2 - (K_j - 1)^2}^k} \times \sum_{3 \leq n_1, n_2, \ldots, n_k \leq 2\nu + 3, n_1 + n_2 + \cdots + n_k = 2\nu - \ell + 2k} f^{(n_1)}(t_0) f^{(n_2)}(t_0) \cdots f^{(n_k)}(t_0)
\]

where \( \gamma = \frac{\sqrt{x^2 - (K_j - 1)^2}}{2\pi} \) and \( C_{2\nu, \ell, k} \) are constants.

By Lemma 3.1 (1), \( W_{K_i, L_j}(\eta x) \) is negligible if \( c_j \geq \frac{x}{K_j^{-1-\epsilon} L_j} \). Consequently, (2.6) is negligible if \( K_2 L_2 \geq X^{1+\epsilon} \). (2.8) is negligible if \( K_1 L_1 \geq X^{1+\epsilon} \), and (2.7) is negligible if either \( K_1 L_1 \) or \( K_2 L_2 \geq X^{1+\epsilon} \). In all other cases, we may restrict the \( c_j \)-sum to \( c_j \leq \frac{x}{K_j^{-1-\epsilon} L_j} \).
We will summarize the general strategy we use to bound $\sum_{k_1}^{K_2} L_2 \sum_{k_1}^{K_1} L_1$ in each case. For the first step we use Lemma 3.1 to rewrite the sum over $k_1$ and $k_2$ as an asymptotic expansion of an oscillatory integral, and we focus on the main term in this expansion. We then open up the Kloosterman sum over $z$, interchange the summation and use the orthogonality relation for characters to get a relation between the sum over $n$ and $z$. Applying Poisson summation on $n$, we obtain another oscillatory integral and apply a weighted first derivative test as in McKee, Sun, and Ye [10] to shorten the sum over $n$. We follow a similar approach for the sum over $m$.

After applying Poisson summation for $n$ and $m$, we combine the two oscillatory integrals into a double integral that can be bounded using the following two-dimensional second derivative test. This is the final step in obtaining the non-trivial upper bound for $\sum_{k_1}^{K_2} L_2 \sum_{k_1}^{K_1} L_1$.

**Lemma 3.2.** (Aggarwal [1] and Srinivasan [20]) Suppose in a region $D \subset \mathbb{R}^2$ we have

$$\frac{\partial^2 \theta}{\partial u^2} \gg r_1^2, \quad \left| \frac{\partial^2 \theta}{\partial u^2} \right| \gg r_2^2, \quad \frac{\partial^2 \theta}{\partial u^2} \frac{\partial^2 \theta}{\partial v^2} - \left( \frac{\partial^2 \theta}{\partial u \partial v} \right)^2 \gg r_1^2 r_2^2,$$

for some $r_1, r_2 > 0$. Define

$$\text{var}(a) = \int \int_D \left| \frac{\partial^2 a}{\partial u \partial v} \right| \, du \, dv.$$

Then

$$\int \int_D a(u, v) e(\theta(u, v)) \, du \, dv \ll \frac{\text{var}(a)}{r_1 r_2}.$$

4. The $D_{01}$ and $D_{10}$ Terms

By Lemma 3.1 we have

$$D_{01}^\eta = K_1 K_2 \sum_{2|k_1} g_0 \left( \frac{k_1 - K_1}{L_1} \right) \sum_n \phi \left( \frac{n}{X} \right)^2 \sum_{c_2 \leq \frac{X}{L_1}} \sum_{\chi \mod \ell_2} \frac{S(n, n, c_2)}{c_2} W_{K_2, L_2}(\eta x).$$

Notice $c_2$ is a positive integer and so we can assume $K_2 L_2 \leq X^{1+\varepsilon}$, otherwise $\frac{X}{K_2^{-1} L_2} < 1$ and there will be no $D_{01}$ term. Focusing only on the leading term in the expansion for $W_{2,0}(\eta x)$, and expanding the Kloosterman sum, we obtain a term of the form

$$T_{01}^\eta = K_1 K_2 L_2 \sum_{2|k_1} g_0 \left( \frac{k_1 - K_1}{L_1} \right) \sum_n \phi \left( \frac{n}{X} \right)^2 \sum_{c_2 \leq \frac{X}{L_1}} \sum_{\chi \mod \ell_2} \frac{1}{c_2}$$

$$\times \sum_{z \mod c_2} e\left( \frac{nz + n\tilde{z}}{c_2} \right) h^\eta_2(n, n, c_2) e(\varphi^\eta_2(n, n, c_2)),$$

where

$$\varphi^\eta_2(m, n, c) = -\frac{\eta}{2\pi} \sqrt{\frac{16\pi^2 mn}{c^2}} - (K_j - 1)^2 - \frac{\eta (K_j - 1) c}{2\pi} \sin^{-1} \left( \frac{(K_j - 1)c}{4\pi \sqrt{mn}} \right),$$

$$h^\eta_2(m, n, c) = g_0 \left( \frac{\eta L}{2\pi} \sin^{-1} \left( \frac{(K_j - 1)c}{4\pi \sqrt{mn}} \right) \left( \frac{16\pi^2 mn}{c^2} - (K_j - 1)^2 \right)^{-\frac{3}{4}} \right).$$

Rewriting $n$ as $nc_2 + r$,

$$T_{01}^\eta = K_1 K_2 L_2 \sum_{2|k_1} g_0 \left( \frac{k_1 - K_1}{L_1} \right) \sum_{c_2 \leq \frac{X}{L_1}} \sum_{\chi \mod \ell_2} \frac{1}{c_2} \sum_{r \mod c_2} \sum_{r \mod c_2} e\left( \frac{r z + r \tilde{z}}{c_2} \right)$$

$$\times \sum_{n \geq 1} \phi \left( \frac{nc_2 + r}{X} \right)^2 h^\eta_2(nc_2 + r, nc_2 + r, c_2) e(\varphi^\eta_2(nc_2 + r, nc_2 + r, c_2)).$$
Applying the Poisson summation to the \( n \)-sum in (4.3), we get

\[
\sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} \left| \phi \left( \frac{y_c + r}{X} \right) \right|^2 h_2^\eta (y_c + r, y_c, c) e \left( \varphi_2^\eta (y_c + r, y_c + r, c) \right) e \left( -y n \right) dy
\]

\[
= \frac{1}{c_2} \sum_{n \in \mathbb{Z}} e \left( \frac{r n}{c_2} \right) \int_{\mathbb{R}} \left| \phi \left( \frac{t}{X} \right) \right|^2 h_2^\eta (t, t, c) e \left( \varphi_2^\eta (t, t, c) - \frac{t n}{c_2} \right) dt
\]

by changing variables to \( t = y_c + r \). Substituting (4.4) back to (4.3), we may evaluating the \( r \)-sum and get rid of the Kloosterman sum

\[
T_{01}^\eta = K_1 K_2 L_2 \sum_{2|k_1} \sum_{c_2 \leq K_2^{-1} L_2} \sum_{c_2 \equiv z} \sum_{n \equiv -z \mod c_2} \frac{1}{c_2} \sum_{e} \int_{\mathbb{R}} \left| \phi \left( \frac{t}{X} \right) \right|^2 h_2^\eta (t, t, c) e \left( \varphi_2^\eta (t, t, c) - \frac{t n}{c_2} \right) dt.
\]

We denote by \( I \) the integral on the right hand side of (4.3) and change variables from \( t \) to \( t X \), thus making it an integral over \([1, 2]\), the support of \( \phi \)

\[
I = X \int_{1}^{2} |\phi(t)|^2 h_2^\eta (X t, X t, c) e \left( \varphi_2^\eta (X t, X t, c) - \frac{t n X}{c_2} \right) dt.
\]

By the Riemann-Lebesgue lemma \( I \) is negligible for \( n \) outside a compact interval. Note that \( \frac{d \phi}{dt} \ll 1 \) and by (4.2)

\[
h_2^\eta (X t, X t, c) = \tilde{g}_0 \left( \frac{\eta L_2}{2 \pi} \sin^{-1} \left( \frac{(K_2 - 1) c_2}{4 \pi X t} \right) \right) \left( \frac{16 \pi^2 X^2 t^2}{c_2^2} - (K_2 - 1)^2 \right)^{-\frac{1}{4}}
\]

\[
\gg \left( \frac{X^2}{c_2^2} \right)^{-\frac{1}{4}} \gg c_2^\frac{1}{2} X^{-\frac{1}{4}} =: U
\]

as \( 16 \pi^2 X^2 t^2 c_2^{-2} \) always dominates \( (K_2 - 1)^2 \). Taking derivatives,

\[
\frac{d}{dt} \left( \frac{16 \pi^2 X^2 t^2}{c_2^2} - (K_2 - 1)^2 \right)^{-\frac{1}{4}} \ll U,
\]

\[
\frac{d}{dt} g_0 \left( \frac{\eta L_2}{2 \pi} \sin^{-1} \left( \frac{(K_2 - 1) c_2}{4 \pi X t} \right) \right) = \tilde{g}_0 \left( \frac{\eta L_2}{2 \pi} \sin^{-1} \left( \frac{(K_2 - 1) c_2}{4 \pi X t} \right) \right) \frac{\eta L_2}{2 \pi \sqrt{1 - \frac{(K_2 - 1) c_2}{4 \pi X t}^2}} \gg \frac{K_2 L_2 c_2}{X} \ll K_2^{-\frac{1}{2}}.
\]

Subsequent derivatives yield \( \ll K_2^{-\frac{1}{2}} \) too. Putting them together we have

\[
\frac{d}{dt} |\phi(t)|^2 h_2^\eta (X t, X t, c) \ll \frac{U}{X^l}, \ l \geq 0,
\]

for \( N = K_2^{-\varepsilon} \). Recall the definition of \( \varphi_2^\eta (X t, X t, c) \) in (4.1). Computing derivatives we get

\[
\frac{d}{dt} \varphi_2^\eta (X t, X t, c) - \frac{t n X}{c_2} = -\frac{\eta}{4 \pi} \sqrt{\frac{16 \pi^2 X^2 t^2 c_2^2}{c_2^2} - (K_2 - 1)^2} - \frac{\eta (K_2 - 1) c_2}{2 \pi \sqrt{1 - \frac{(K_2 - 1) c_2}{4 \pi X t}^2}} - \frac{n X}{c_2}
\]

\[
= -\frac{\eta}{2 \pi} \sqrt{\frac{16 \pi^2 X^2 t^2}{c_2^2} - (K_2 - 1)^2} - \frac{n X}{c_2}
\]

\[
\gg \frac{4 \eta X t}{2 \pi c_2} \left( 1 + O \left( \frac{c_2 K_2^2}{X^2} \right) \right) - \frac{n X}{c_2} = -(2 \eta + n) X \left( 1 + O \left( \frac{c_2 K_2^2}{X^2} \right) \right),
\]

\[
(4.7)
\]
since
\[ \frac{c_2^2 K_2^2}{X^2} \leq \frac{K_2^2}{K_2^{-\varepsilon} L_2} = \frac{K_2^\varepsilon}{L_2} \]
is small. Now from (4.7) using (4.6), if \( n \neq -2\eta \), we have
\[ \left| \frac{d}{dt} \Phi_2^n(Xt, Xt, c_2) - \frac{tnX}{c_2} \right| \geq \frac{X}{c_2} \geq K_2^{-\varepsilon} L_2. \]
Subsequent derivatives are all \( \ll X c_2^{-1} \). Thus we may take \( T = X c_2^{-1} \) and \( M = 1 \). By the first derivative test (cf. [10]), the integral
\[ I \ll U \left( \frac{c_2 K_2^2}{X} \right)^{n_0+1} \leq \frac{c_2^\varepsilon}{X^2 (K_2^{-1} L_2)^{n_0+1}} \]
is negligible for \( n_0 \) sufficiently large, when \( n \neq -2\eta \).

Excluding these negligible terms we get
\[ T_{01}^\eta = K_1 K_2 L_2 \sum_{2k_1} g_0 \left( \frac{k_1 - K_1}{L_1} \right) \sum_{c_2 \leq \kappa_2^{\varepsilon}} \frac{1}{c_2} \sum_{z + \bar{\varepsilon} \equiv 2\eta \text{ mod } c_2} \times \int_{\mathbb{R}} \left| \phi \left( \frac{t}{X} \right) \right|^2 h_2^2(t, t, c_2) c \left( \frac{\Phi_2 (t, t, c_2) + 2nX}{c_2} \right) dt + O(K_2^{-A}). \]
The congruence \( z + \bar{\varepsilon} \equiv 2\eta \text{ mod } c_2 \) is equivalent to \( z^2 + 1 \equiv 2\eta z \text{ mod } c_2 \) or equivalently to \( (z - \eta)^2 \equiv 0 \text{ mod } c_2 \). For \( c_2 = p^r \) with \( c \geq 1 \), this means \( p^r | (z - \eta)^2 \), i.e., \( p^r || z - \eta \text{ if } c \text{ is even, and } p^r || z - \eta \text{ if } c \text{ is odd. Consequently the number of solutions of } z + \bar{\varepsilon} \equiv 2\eta \text{ mod } c_2 \text{ for } z \text{ mod } c_2, (z, c_2) = 1, \text{ is } \ll \sqrt{c_2} \). Now we compute the integral \( I \) for \( n = -2\eta \).

By (4.7)
\[ \frac{d}{dt} \Phi_2^n(Xt, Xt, c_2) + \frac{2nX}{c_2} = -\frac{\eta}{2\pi i} \left( \frac{16\pi^2 X^2 t^2}{c_2^2} - (K_2 - 1)^2 + \frac{2nX}{c_2} \right)^{-1} \]
\[ \times \frac{K_2^{-1} \rho(X)}{\sqrt{4\pi X c_2}} \times \frac{c_2 K_2^\varepsilon}{X}. \]
Any subsequent differentiation yields a factor of \( \ll 1 \). By the first derivative test again,
\[ I \ll U \left( \frac{X}{c_2 K_2^{2\varepsilon}} \right)^{n_0+1} \]
which is negligible for \( n_0 \) sufficiently large if \( \frac{X}{c_2 K_2^{-\varepsilon}} \leq K_2^{-\varepsilon} \), i.e., if
\[ \frac{X}{K_2^{-\varepsilon}} \leq c_2 \leq \frac{X}{K_2^{1-\varepsilon} L_2}. \]
Therefore we can shorten the \( c_2 \)-sum and reduce \( T_{01}^\eta \) to
\[ T_{01}^\eta = K_1 K_2 L_2 \sum_{2k_1} g_0 \left( \frac{k_1 - K_1}{L_1} \right) \sum_{c_2 \leq \kappa_2^{\varepsilon}} \frac{1}{c_2} \sum_{z + \bar{\varepsilon} \equiv 2\eta \text{ mod } c_2} \times \int_{\mathbb{R}} \left| \phi \left( \frac{t}{X} \right) \right|^2 h_2^2(t, t, c_2) c \left( \frac{\Phi_2 (t, t, c_2) + 2nX}{c_2} \right) dt + O(K_2^{-A}), \]
if \( K_2 \leq X^{\frac{1}{2} + \varepsilon} \), while \( T_{01}^\eta \) is negligible if \( K_2 \geq T^{\frac{1}{2} + \varepsilon} \).
By trivial estimation, we have $I \ll e^{\frac{1}{2}X^{\frac{1}{2}}}$, Consequently,

\begin{equation}
T_{01}^n \ll K_1 L_1 K_2 L_2 X^{\frac{1}{2}} \sum_{c_2 \leq \frac{X}{3}} \frac{1}{c_2^2} + O(K_2^{-A})
\end{equation}

\begin{equation}
\ll \frac{K_1 L_1 L_2 X^{\frac{1}{2}}}{K_2^{1-\varepsilon}} \quad \text{if } K_2 \leq X^{\frac{1}{2}+\varepsilon}
\end{equation}

\begin{equation}
\ll K_2^{-A} \quad \text{if } K_2 \geq X^{\frac{1}{2}+\varepsilon}.
\end{equation}

Similarly,

\begin{equation}
T_{10}^n \ll \frac{K_2 L_1 L_2 X^{\frac{1}{2}}}{K_1^{1-\varepsilon}} \quad \text{if } K_1 \leq X^{\frac{1}{2}+\varepsilon}
\end{equation}

\begin{equation}
\ll K_1^{-A} \quad \text{if } K_1 \geq X^{\frac{1}{2}+\varepsilon}.
\end{equation}

These give upper bounds for the main terms in the expansions of $D_{01}$ and $D_{10}$ for $0 < \beta \leq 1$. Bounds in (1.6) and (1.7) then follow.

5. Poisson summation for $D_{11}$

To use Poisson summation, we rewrite for $0 < \beta \leq 1$

\begin{equation}
T_{11}^{n,n_2} = K_1 L_1 K_2 L_2 \sum_{n} \sum_{m} e(\alpha m^\beta - \alpha m^\beta) \phi\left(\frac{n}{X}\right) \frac{m}{X} \sum_{d \leq \min(\frac{X}{L_1}, \frac{X}{L_2}, \frac{X}{L_2}, \frac{X}{L_2}, \frac{X}{L_2}, \frac{X}{L_2})} \sum_{c_j \leq \frac{X}{L_j}, j \leq \frac{X}{L_j}, (c_1, c_2) = 1} \frac{1}{c_1 c_2 d^2} \times \sum_{z_1 \mod d_{c_1}} \sum_{z_2 \mod d_{c_2}} e\left(\frac{m z_1}{d c_1} + \frac{n z_1}{d c_1} + \frac{r z_2}{d c_2}ight) \times \sum_{r \mod d_{c_1} c_2} e\left(\frac{r z_1}{d c_1} + \frac{r z_2}{d c_2}ight) \times h_1^{n,n_2}(m, c_1 c_2 d + r, c_1 d) h_2^{n_2}(m, c_1 c_2 d + r, c_2 d).\end{equation}

Rewriting $n$ as $n c_1 c_2 d + r$, we have

\begin{equation}
T_{11}^{n,n_2} = K_1 L_1 K_2 L_2 \sum_{n} e(-\alpha m^\beta) \phi\left(\frac{n}{X}\right) \frac{m}{X} \sum_{d \leq \min(\frac{X}{L_1}, \frac{X}{L_2}, \frac{X}{L_2}, \frac{X}{L_2}, \frac{X}{L_2}, \frac{X}{L_2})} \sum_{c_j \leq \frac{X}{L_j}, j \leq \frac{X}{L_j}, (c_1, c_2) = 1} \frac{1}{c_1 c_2 d^2} \times \sum_{z_1 \mod d_{c_1}} \sum_{z_2 \mod d_{c_2}} e\left(\frac{m z_1}{d c_1} + \frac{n z_1}{d c_1} + \frac{r z_2}{d c_2}ight) \times \sum_{r \mod d_{c_1} c_2} e\left(\frac{r z_1}{d c_1} + \frac{r z_2}{d c_2}ight) \times h_1^{n,n_2}(m, n c_1 c_2 d + r, c_1 d) h_2^{n_2}(m, n c_1 c_2 d + r, c_2 d).\end{equation}

By Poisson summation on the $n$-sum, we get

\begin{equation}
\sum_{n} \int \phi\left(\frac{y c_1 c_2 d + r}{X}\right) h_1^{n,n_2}(m, y c_1 c_2 d + r, c_1 d) h_2^{n_2}(m, y c_1 c_2 d + r, c_2 d) \times e(\alpha(y c_1 c_2 d + r)\beta) \phi\left(\frac{v}{X}\right) h_1^{n,n_2}(m, v, c_1 d) h_2^{n_2}(m, v, c_2 d) = \sum_{n} \frac{1}{c_1 c_2 d} \left(\int \phi\left(\frac{v}{X}\right) h_1^{n,n_2}(m, v, c_1 d) h_2^{n_2}(m, v, c_2 d) - \frac{n v}{c_1 c_2 d}\right) dv.
\end{equation}
The $r$-sum in (5.1) becomes
\[
\sum_{r \mod c_1c_2d} e\left(\frac{r(n + c_2 \bar{z}_1 + c_1 \bar{z}_2)}{c_1c_2d}\right) = \begin{cases} 
 c_1c_2d & \text{if } n \equiv -c_2 \bar{z}_1 - c_1 \bar{z}_2 \mod c_1c_2d \\
 0 & \text{otherwise}.
\end{cases}
\]
Consequently
\[
T_{n_1n_2}^{m_1m_2} = K_1 L_1 K_2 L_2 \sum_m e(-am\beta) \phi\left(\frac{m}{X}\right) \sum_{d \leq \min \left(\frac{X}{K_1L_1}, \frac{X}{K_2L_2}\right)} \frac{1}{d^2} \sum_{c_j \leq \frac{X}{d K_1^j L_1}} \sum_{(c_1, c_2) = 1} \frac{1}{c_1c_2} 
\]
\[
\times \sum_{z_1 \mod d_{c_1}} \sum_{z_2 \mod d_{c_2}} e\left(\frac{mz_1}{d_{c_1}}\right) e\left(\frac{mz_2}{d_{c_2}}\right) \int_{-\infty}^{\infty} \phi\left(\frac{v}{X}\right) h_1^{n_1}(m, v, c_1d) 
\]
\[
\times h_2^{n_2}(m, v, c_2d) \left(\alpha v^\beta - \varphi_1^{n_1}(m, v, c_1d) + \varphi_2^{n_2}(m, v, c_2d) - \frac{mv}{c_1c_2d} \right) dv.
\]
Applying the same to the $m$-sum, we have
\[
(5.2) \quad T_{n_1n_2}^{m_1m_2} = K_1 L_1 K_2 L_2 \sum_m e(-am\beta) \phi\left(\frac{m}{X}\right) \sum_{d \leq \min \left(\frac{X}{K_1L_1}, \frac{X}{K_2L_2}\right)} \frac{1}{d^2} \sum_{c_j \leq \frac{X}{d K_1^j L_1}} \sum_{(c_1, c_2) = 1} \frac{1}{c_1c_2} 
\]
\[
\times \sum_{m \equiv -c_2z_1-c_1z_2 (mod c_1c_2d)} \sum_{n \equiv -c_2z_1-c_1z_2 (mod c_1c_2d)} \int_{-\infty}^{\infty} a(u, v) e(\theta^{n_1n_2}(u, v)) du dv,
\]
where
\[
a(u, v) = \phi\left(\frac{u}{X}\right) \phi\left(\frac{v}{X}\right) h_1^{n_1}(u, v, c_1d) h_2^{n_2}(u, v, c_2d),
\]
\[
\theta^{n_1n_2}(u, v) = \alpha u^\beta - \alpha u^\beta + \varphi_1^{n_1}(u, v, c_1d) + \varphi_2^{n_2}(u, v, c_2d) - \frac{mu}{c_1c_2d} - \frac{nv}{c_1c_2d} =: \theta(u, v).
\]
By (4.2), $a(u, v) \ll \frac{1}{d_{c_1}^{\frac{1}{2}}} X^{-1} =: U$, and each $\frac{\partial}{\partial u}$ produces a factor $\ll K_1^x K_2^x X^{-1} = \frac{1}{X}$ with $N = X K_1^x K_2^x$. Computing derivatives in the case of $0 < \beta < 1$ we get
\[
(5.3) \quad \frac{\partial \theta^{n_1n_2}}{\partial u} = -\alpha \beta u^{\beta-1} - \frac{m}{4\pi u} \sqrt{R_1} - \frac{n}{4\pi u} \sqrt{R_2} - \frac{m}{c_1c_2d},
\]
where $R_j = \frac{\alpha \beta + \sqrt{\alpha \beta + \sqrt{\alpha \beta + \sqrt{\alpha \beta}}}}{c_1c_2d} - (K_j - 1)^2$. Suppose that the absolute value of the sum of the first three terms on the right hand side of (5.3) is bounded by $\frac{\alpha \beta}{c_1c_2d}$ for some $0 < \delta < \tau$. Then for $|m| \geq \tau$, we have $|\frac{\partial \theta}{\partial u}| \geq \frac{\delta}{c_1c_2d}$. Moreover for $r \geq 2$, $\frac{\partial^r \theta}{\partial u^r} \ll \frac{T}{X^r}$, for $T = \frac{X}{c_1c_2d}$ and $M = X$. Denote by $J$ the double integral in (5.2). Then by the first derivative test [10] Theorem 1.1, $J$ is negligible for $|m| \geq \tau$. Therefore,
\[
(5.4) \quad T_{n_1n_2}^{m_1m_2} = K_1 L_1 K_2 L_2 \sum_m e(-am\beta) \phi\left(\frac{m}{X}\right) \sum_{d \leq \min \left(\frac{X}{K_1L_1}, \frac{X}{K_2L_2}\right)} \frac{1}{d^2} \sum_{c_j \leq \frac{X}{d K_1^j L_1}} \sum_{(c_1, c_2) = 1} \frac{1}{c_1c_2} 
\]
\[
\times \sum_{m \equiv -c_2z_1-c_1z_2 (mod c_1c_2d)} \sum_{n \equiv -c_2z_1-c_1z_2 (mod c_1c_2d)} J + O(X^{-A}).
\]

Lemma 5.1. The $z_1$, $z_2$, $m$, $n$-sums in (5.4) have at most $d(2\tau + 1)\left(\frac{\tau}{c_1c_2d}\right) + 1$ terms.

Proof. Reducing
\[
(5.5) \quad m \equiv -c_2z_1 - c_1z_2 (mod c_1c_2d)
\]
to congruences mod $c_1$ and mod $c_2$, we see that for each $m$, $z_j$ is uniquely determined mod $c_j$, $j = 1, 2$. But modulo $c_j d$, there are $d$ such $z_j$'s: $z_j + c_j k_j$ with $0 \leq k_j < d$. Then (5.5) becomes

$$k_1 + k_2 \equiv -\frac{m + c_1 z_2 + c_2 z_1}{c_1 c_2} \pmod{d}.$$  

Consequently, given $m$, there are at most $d$ such $z_1$ mod $dc_1$. Given $m$ and $z_1$, there is a unique $z_2$ mod $dc_2$. Given $m$, $z_1$, and $z_2$, there is a unique $n$ mod $dc_1 c_2$. The lemma then follows because there are at most $2\tau + 1$ $m$'s.

We want to apply the second derivative test to get an upper bound for the double integral $J$. We have the following second derivatives in the case of $0 < \beta < 1$:

$$\frac{\partial^2 \varrho}{\partial u \partial v} = -\frac{2\eta_1 \eta_2}{c_1^2 d^2 \sqrt{R_1}} - \frac{2\eta_2}{c_2^2 d^2 \sqrt{R_2}}. \tag{5.7}$$

$$\frac{\partial^2 \varrho}{\partial u^2} \frac{\partial^2 \varrho}{\partial v^2} = \left(\frac{\partial^2 \varrho}{\partial u \partial v}\right)^2 = (U_4 + U_5)(V_4 + V_5), \tag{5.8}$$

$$\frac{\partial^2 \varrho}{\partial u^2} - \frac{\partial^2 \varrho}{\partial v^2} = U_1 V_1 + U_5 V_1 + \sum_{i=2}^{5} U_i V_i,$$  

$$+ (U_2 + U_3) \sum_{i=2}^{5} V_i + (U_4 + U_5)(V_2 + V_3). \tag{5.9}$$

Note that (5.9) equals

$$\frac{(\eta_1 \sqrt{R_1} + \eta_2 \sqrt{R_2})^2}{16\pi^2 u^2 v^2} - \frac{\eta_1 \sqrt{R_1} + \eta_2 \sqrt{R_2}}{uvd^2} \left(\frac{\eta_1}{c_1^2 \sqrt{R_1}} + \frac{\eta_2}{c_2^2 \sqrt{R_2}}\right) \tag{5.10}$$

by

$$\frac{\sqrt{R_j}}{16\pi^2 u^2 v^2} = \frac{1}{u v c_1^2 d^2 \sqrt{R_j}} - \frac{16\pi^2 u^2 v^2 \sqrt{R_j}}{16\pi^2 u^2 v^2 \sqrt{R_j}} - \frac{1}{u v c_2^2 d^2 \sqrt{R_j}} = - \frac{(K_j - 1)^2}{16\pi^2 u^2 v^2 \sqrt{R_j}}. \tag{5.11}$$

6. CASE OF $\eta_1 = \eta_2$ FOR $D_{11}$ WHEN $0 < \beta < 1$

Assume $\eta_1 = \eta_2$ in this section. Then there is no cancellation in the middle two terms of (5.5), and hence their sum is equal to

$$-\frac{\eta_1 \sqrt{R_1}}{4\pi u} - \frac{\eta_2 \sqrt{R_2}}{4\pi u} = -\frac{\eta_1}{d} \sqrt{\frac{v}{u}} \left(\frac{1}{c_1} + \frac{1}{c_2}\right) \frac{c_1 d K^2}{X^2} + O \left(\frac{c_2 d K^2}{X^2}\right),$$

where the first term on the right hand side dominates. Thus,

$$\frac{c_1 + c_2}{\sqrt{2} dc_1 c_2} \leq \left| -\frac{\eta_1 \sqrt{R_1}}{4\pi u} - \frac{\eta_2 \sqrt{R_2}}{4\pi u} \right| \leq \frac{\sqrt{2}(c_1 + c_2)}{dc_1 c_2}. \tag{6.1}$$
We will assume \( K_1 = K_2 \) and
\[
K_1^2 L_1 L_2 \geq X^{1+\beta+\varepsilon}.
\]
Then \( K_1 L_1, K_1 L_2 \geq X^{\beta+\varepsilon} \) because \( K_1 L_1, K_1 L_2 \leq X^{1+\beta} \). Consequently, \(|-\alpha \beta u^{\beta-1}| \leq \alpha \beta X^{\beta-1} \) is a power smaller than \( \frac{1}{d(\frac{1}{c_1} + \frac{1}{c_2})} \). Thus, the absolute value of the sum of the first three terms on the right hand side of (5.3) is
\[
\leq \alpha \beta X^{\beta-1} + \frac{\sqrt{2}(c_1 + c_2)}{c_1 c_2 d} \leq 1.42(c_1 + c_2)
\]
and we may take \( \tau = 1.5(c_1 + c_2) \) and \( \delta = 0.08(c_1 + c_2) \) and apply Lemma 5.1 to get (5.4).

Now we bound \( J \) in (5.4). Because \( \eta_1 = \eta_2 \), there are no cancellations in (5.11). Consequently, (5.9) is
\[
(6.3) \qquad \sum_{i=1}^{5} U_i \simeq \frac{1}{dX} \left( \frac{X}{c_1} + \frac{1}{c_2} \right) \geq \frac{K_1^{1-\varepsilon} L_1 + K_1^{1-\varepsilon} L_2}{X^2} \gg X^{\beta-2+\varepsilon} \gg U_1
\]
by (6.2). Likewise, \( \sum_{i=1}^{5} V_i \) dominates \( V_1 \). Therefore, (5.6) and (5.7) are both \( \simeq \frac{1}{dX} \left( \frac{1}{c_1} + \frac{1}{c_2} \right) \). By the same reason,
\[
(6.5) \qquad U_1 \sum_{i=1}^{5} V_i \simeq V_1 \sum_{i=2}^{5} U_i \simeq \frac{X^{\beta-3}}{d} \left( \frac{1}{c_1} + \frac{1}{c_2} \right)
\]
Since (6.9) dominates (5.5), the left hand side of (6.8) is \( \approx (6.3) \). We observe that
\[
(6.6) \qquad \frac{1}{dX^2} \left( \frac{1}{c_1} + \frac{1}{c_2} \right)^2 \geq \frac{K_1^{1-\varepsilon} L_1 + K_1^{1-\varepsilon} L_2}{dX^3} \left( \frac{1}{c_1} + \frac{1}{c_2} \right)
\]
\[
(6.7) \qquad \frac{K_1^2(c_1 + c_2)}{X^4} \left( \frac{1}{c_1} + \frac{1}{c_2} \right) \leq \frac{1}{dX^3} \left( \frac{K_1^{1+\varepsilon}}{L_1} + \frac{K_1^{1+\varepsilon}}{L_2} \right) \left( \frac{1}{c_1} + \frac{1}{c_2} \right).
\]
Since (6.6) is a power larger than (6.7), we may choose
\[
r_1 = r_2 = \frac{K_1^\frac{1}{2}(c_1 + c_2)}{X} \left( \frac{1}{c_1} + \frac{1}{c_2} \right)^\frac{1}{2} = \frac{K_1^\frac{1}{2}(c_1 + c_2)^\frac{1}{2}}{(c_1 c_2)^\frac{1}{2}X}
\]
as in (5.1).

Note that \( a(u, v) \ll d(c_1 c_2)^\frac{1}{2} X^{-1} \) and each differentiation produces a factor \( \ll K_1^{-1} X^{-1} \). Consequently,
\[
(6.8) \qquad \frac{\partial^2 a}{\partial u \partial v} \ll \frac{K_1^{1} d\sqrt{c_1 c_2}}{X^3}, \quad \text{var}(a) \ll \frac{K_1^{1} d\sqrt{c_1 c_2}}{X}.
\]
By Lemma 3.2 for \( \eta_1 = \eta_2 \) we have
\[
J \ll \frac{d\sqrt{c_1 c_2} K_1^\varepsilon}{X} \left( \frac{\sqrt{c_1 c_2} X}{(c_1 + c_2)K_1} \right)^{1/2} \frac{c_1 c_2 dX}{(c_1 + c_2)K_1^{1-\varepsilon}},
\]
and hence by (5.4)
\[ T_{11}^{η,η} \ll K_1^{1+\varepsilon} L_1 L_2 X\]
\[ \sum_{d \leq \min\{\frac{x}{\kappa_1^{\varepsilon} L_1}, \frac{x}{\kappa_2^{\varepsilon} L_2}\}} \sum_{c_j \leq \frac{x}{dK_j^{\varepsilon} L_j}} \sum_{z_1 \mod dc_1} \sum_{z_2 \mod dc_2} \sum_* \sum^* \]
\[ \times \sum_{|m| \leq 1.5(c_1 + c_2)} \sum_{n \leq 1.5(c_1 + c_2)} \frac{1}{c_1 + c_2}, \]
\[ \text{if we use } 3d(c_1 + c_2) + d + 4.5\frac{(c_1 + c_2)^2}{c_1 c_2} \text{ as the number of terms in the } z_1, z_2, m, n \text{ sums as proved in Lemma 5.1, we have} \]
\[ (6.9) \quad T_{11}^{η,η} \ll K_1^{1+\varepsilon} L_1 L_2 \sum_{c_1 \leq \frac{x}{\kappa_1^{\varepsilon} L_1}} \sum_{c_2 \leq \frac{x}{\kappa_2^{\varepsilon} L_2}} \left(1 + \frac{c_1 + c_2}{c_1 c_2 d}\right) \]
\[ \ll K_1^{1+\varepsilon} L_1 L_2 X \sum_{c_1 \leq \frac{x}{\kappa_1^{\varepsilon} L_1}} \sum_{c_2 \leq \frac{x}{\kappa_2^{\varepsilon} L_2}} 1 \ll \frac{X^{3+\varepsilon}}{K_1}, \]
because \( \frac{1}{c_1} + \frac{1}{c_2} \leq 2. \)

7. Case of \( η_1 \neq η_2 \) for \( D_{11} \) when \( 0 < \beta < 1 \)

Going back to (5.3), we observe that for \( η_1 \neq η_2 \) and \( K_1 = K_2 \), we have
\[ \sqrt{R_1} - \sqrt{R_2} = \frac{R_1 - R_2}{\sqrt{R_1} + \sqrt{R_2}} \]
\[ (7.1) \]
\[ R_1 - R_2 = \frac{16\pi^2 uv}{d^2} \left(\frac{1}{c_1^2} - \frac{1}{c_2^2}\right) = \frac{16\pi^2 uv}{d^2} \left(\frac{1}{c_1} - \frac{1}{c_2}\right) \left(\frac{1}{c_1} + \frac{1}{c_2}\right), \]
\[ (7.2) \]
\[ \sqrt{R_1} + \sqrt{R_2} = \frac{4\pi \sqrt{uv}}{d} \left(\frac{1}{c_1} + \frac{1}{c_2}\right) \left(1 + O\left(\frac{dK_1^2}{X^2}(c_1^2 + c_2^2)\right)\right). \]
\[ (7.3) \]
Then (6.1) becomes
\[ \frac{|c_1 - c_2|}{\sqrt{2} c_1 c_2 d} \leq \left| -\frac{\eta_1 \sqrt{R_1}}{\sqrt{2}} - \frac{\eta_2 \sqrt{R_2}}{\sqrt{2}} \right| \leq \frac{\sqrt{2}|c_1 - c_2|}{c_1 c_2 d}\]
We will first consider the case of \( c_1 \neq c_2 \) with \( K_1 = K_2 \) and (6.2). Then
\[ \frac{|c_1 - c_2|}{\sqrt{2} c_1 c_2 d} \geq \frac{1}{\sqrt{2} c_1 c_2 d^2} \geq \frac{K_1^{1-\varepsilon} L_1 K_2^{1-\varepsilon} L_2}{\sqrt{2}X^2} \]
dominates \( -\alpha \beta u^{\beta - 1} \asymp X^{\beta - 1} \). Consequently, we may take \( \tau = 1.5|c_1 - c_2| \) and apply Lemma 5.1 to get (5.4) for this \( \tau \).

Now we bound \( J \) in (5.4) in the case at present. To compute (5.9) we note that (5.11) equals
\[ (7.4) \quad \frac{(K_1 - 1)^2(\sqrt{R_1} - \sqrt{R_2})^2}{16\pi^2 u^2 v^2 \sqrt{R_1 R_2}} = \frac{(K_1 - 1)(R_1 - R_2)^2}{16\pi^2 u^2 v^2 \sqrt{R_1 R_2}(\sqrt{R_1} + \sqrt{R_2})^2} \]
by (7.1). Using (7.2), (7.3) and
\[ \sqrt{R_1 R_2} = \frac{16\pi^2 uv}{c_1 c_2 d^2} \left(1 + O\left(\frac{dK_1^2}{X^2}(c_1^2 + c_2^2)\right)\right), \]
(7.3) and hence (5.11) and (5.9) are equal to
\[ \frac{c_1 c_2 K_1^2}{16\pi^2 u^2 v^2} \left(\frac{1}{c_1} - \frac{1}{c_2}\right)^2 \left(1 + O\left(\frac{dK_1^2}{X^2}(c_1^2 + c_2^2)\right)\right) = \frac{K_1^2 |c_1 - c_2|^2}{16\pi^2 u^2 v^2 c_1 c_2} \left(1 + O\left(\frac{dK_1^2}{X^2}(c_1^2 + c_2^2)\right)\right). \]
By (6.4) and (6.2), we know that
\[
\left| \sum_{i=2}^{5} U_i \right| = \frac{1}{2d} u \left( \frac{1}{c_1} - \frac{1}{c_2} \right) \left( 1 + O \left( \frac{d^2 K^2}{X^2} (c_1^2 + c_2^2) \right) \right) \gg X^{\beta - 2 + \varepsilon} \gg U_1 \sim V_1.
\]
Likewise, \( \sum_{i=2}^{5} U_i \) dominates \( U_1 \sim V_1 \). Then (6.3), (6.6), and (6.7) still hold and the left hand side of (5.8) is still \( \ll (6.3) \) if we replace \( \left( \frac{1}{c_1} + \frac{1}{c_2} \right) \) by \( \left| \frac{1}{c_1} - \frac{1}{c_2} \right| \). Since (6.8) is still a power larger than (6.7) with \( \left| \frac{1}{c_1} - \frac{1}{c_2} \right| \), we may choose
\[
r_1 = r_2 = \frac{|c_1 - c_2|^2 K_1^2}{(c_1 c_2)^3 X},
\]
and hence we may set \( T = X^{3 + \varepsilon} \), which is negligible for \( n \) sufficiently large. Therefore, the contribution of the case of \( \eta_1 \neq \eta_2 \) and \( c_1 = c_2 \) is negligible.

By (7.6) we obtain \( T_{11}^{\eta_1 \eta_2} \ll \frac{X^{3 + \varepsilon}}{K_1} \) when \( \eta_1 \neq \eta_2 \) and \( K_1 = K_2 \) under (6.2). Together with the bound in (6.9) for the case of \( \eta_1 = \eta_2 \), we finally prove
\[
D_{11} \ll \frac{X^{3 + \varepsilon}}{K_1}.
\]
when \( K_1 = K_2 \) under (6.2) when \( 0 < \beta < 1 \).

Recall that \( D_{01} \) and \( D_{10} \) are negligible when \( K_1 = K_2 \geq X^{\frac{1}{3} + \varepsilon} \) by (4.8) and (4.9). When \( K_1 = K_2 \leq X^{\frac{1}{4} + \varepsilon} \), we have \( \frac{X^3}{K_1} \geq X^{\frac{1}{3}} \geq L_1 L_2 X^{\beta} \) and hence the bound in (7.7) dominates those in (4.8) and (4.9). Back to (2.1), by collecting (2.6) and (7.7) we prove (1.8) for Theorem 1.1 when \( 0 < \beta < 1 \).
8. CASE OF $\beta = 1$ FOR $D_{11}$

When $\beta = 1$, (5.3) becomes

$$\frac{\partial^{\eta_1, \eta_2}}{\partial u} = -\alpha \beta - \frac{\eta_1}{4\pi u} \sqrt{R_1} - \frac{\eta_2}{4\pi u} \sqrt{R_2} - \frac{m}{c_1 c_2 d}.$$  

By the same arguments after (5.3), $J$ is negligible for $|n + \alpha c_1 c_2 d| \geq \tau$ and for $|n + \alpha c_1 c_2 d| \geq \tau$. Consequently, (5.4) holds after replacing summation conditions $|m| \leq \tau$ and $|n| \leq \tau$ by $|n + \alpha c_1 c_2 d| \leq \tau$ and $|n + \alpha c_1 c_2 d| \leq \tau$. Then Lemma 5.1 remains valid.

To use the second derivative test, we observe that $U_1 = V_1 = 0$ in (5.6) and (5.7). Then we don’t need to assume (6.2), and the same calculation leads to (7.6) for the case of $\eta_1 = \eta_2$ and (6.9) for the case of $\eta_1 \neq \eta_2$ with $c_1 \neq c_2$.

When $\eta_1 \neq \eta_2$ with $c_1 = c_2 = 1$, $\frac{\partial^{\eta_1, \eta_2}}{\partial u} = -\alpha - \frac{\eta_1}{\eta_2}$. Recall that

$$\frac{\partial}{\partial u} a(u, v) \ll \frac{U}{N^s}, \quad U = \frac{d \sqrt{2} c_2 \sqrt{X}}{X}, \quad N = \frac{X}{K_1}.$$

If $|\alpha + \frac{\eta_1}{\eta_2}| \geq X^{\epsilon - 1}$, then each integration by parts in $\int_R a(u, v)e(\theta(u, v))du$ produces a factor $\ll |\alpha + \frac{\eta_2}{\eta_1}|^{-1}K_1^{-1}X^{-1} \ll X^{-\epsilon}K_1^{-1}$. Hence the integral with respect to $u$ is negligible when $|\alpha + \frac{\eta_1}{\eta_2}| \geq X^{\epsilon - 1}$. Similarly the $v$-integral is negligible for $|\alpha - \frac{\eta_2}{\eta_1}| \geq X^{\epsilon - 1}$. Then the corresponding sum for (5.2) becomes

$$T_{c_1 = c_2 = 1}^{\eta_1, \eta_2} = K_1 L_1 K_2 L_2 \sum_{d \leq \min\left(\frac{X}{K_1^{1-\epsilon}L_1}, \frac{X}{K_2^{1-\epsilon}L_2}\right)} \frac{1}{d^2} \sum_{z_1, z_2 \bmod d} J.$$

Under the assumption of $\max(K_1 L_1, K_2 L_2) = X^\delta$ with $0 < \delta < 1$, we have

$$\min\left(\frac{X}{K_1^{1-\epsilon}L_1}, \frac{X}{K_2^{1-\epsilon}L_2}\right) \leq X^{1-\delta + \epsilon},$$

and hence

$$\left(\min\left(\frac{X}{K_1^{1-\epsilon}L_1}, \frac{X}{K_2^{1-\epsilon}L_2}\right)\right)^{1-\delta + \epsilon} \leq X^{1-\epsilon}.$$

Take $\epsilon < \frac{\delta}{2}$ so that the exponent in (8.2) becomes $\frac{1-\delta + \epsilon}{1-\epsilon} > 1$. Consequently, for a given $d$ as in (8.1), there is at most one $m$ and $n$ satisfying $|\alpha + \frac{\eta_1}{\eta_2}| \leq X^{\epsilon - 1}$ and $|\alpha - \frac{\eta_2}{\eta_1}| \leq X^{\epsilon - 1}$. For such a triple $d, m, n$, taking any $z_1 \mod d$ with $(z_1, d) = 1$, there is at most one $z_1 \mod d$ satisfying the congruences in (8.1). Thus, the multiple sums in (8.1) have at most $d \min\left(\frac{X}{K_1^{1-\epsilon}L_1}, \frac{X}{K_2^{1-\epsilon}L_2}\right)$ terms.

By $|a(u, v)| \leq \frac{d}{X}$, we bound $J \ll dX$ trivially. Then (8.1) is bounded by

$$\ll K_1 L_1 K_2 L_2 X \min\left(\frac{X}{K_1^{1-\epsilon}L_1}, \frac{X}{K_2^{1-\epsilon}L_2}\right) \ll \min(K_1 L_1, K_2 L_2)X^{2+\epsilon}.$$  

Collecting (8.3) for the case of $\eta_1 = \eta_2$, and (6.9) for the case of $\eta_1 \neq \eta_2$ with $c_1 \neq c_2$, we prove (6.10).

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