Some remarks about the time-dependent Schrödinger equation with damping

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Abstract
The missing derivation of the time-dependent Schrödinger equation following Schrödinger’s original description of the time-independent Schrödinger equation. Also, this description is extended to derive the Caldirola–Kanai, the Schuch–Schrödinger, and the Gisin–Schrödinger equation. In the second part, the Gisin–Schrödinger equation will be derived once more using the Ito formalism of stochastic differential equations. Furthermore, we discuss the extension to larger spin-system using the cluster mean-field theory.

1. Introduction
There are a few essential differential equations which describe the Universe and daily life. At the atomic level, this is the Dirac equation or if we ignore the effects of the special relativity the Schrödinger equation. The new quantum mechanics starts in 1925 with the work of W. Heisenberg, who has quantified the classical canonical variables $q$ and $p$ and introduced the corresponding matrix equations [1]. These equations are called now the Heisenberg equation and the formalism the matrix mechanics. However, this description of quantum mechanics, even if correct, is too complicated for practical purposes. For instance, Heisenberg itself was not able to solve the problem of the energy spectra of the Hydrogen atom using his theory. This was done one year later 1926 by Pauli [2]. The same year Schrödinger came up with another description of quantum mechanics which bases on the well-known formalism of differential equations: The Schrödinger equation [3]. However, and this is surprising, Schrödinger has not derived the time-dependent Schrödinger equation. Schrödinger has derived the time-independent version and wrote down a differential equation for the evolution in time of the wave function

$$\frac{\partial}{\partial t} \psi = -i\hbar \frac{\partial}{\partial t} \psi$$

Not surprising is the order of the first two publications (communications) in *Annalen der Physik* [3, 4] even if the second communication is undoubtedly in the timeline of the finding process before the first communication. This also explains, at least in parts, why Schrödinger has not derived the time-dependent version of his wave equation even if it is an easy task using the same way Schrödinger went to derive the time-independent Schrödinger equation.

It seems a more difficult task is to include friction or damping into quantum mechanics. The first steps in this direction go back to Caldirola in 1941 [6] and Kanai in 1948 [7]. The corresponding Schrödinger equation is called Caldirola–Kanai equation. However, neither Caldirola nor Kanai have seriously derived this differential equation. Caldirola shows how to include friction in the Lagrange formalism and thereby, his work is based on a description given by Levi-Civita in 1896 [8]. The result is what we call Caldirola–Kanai Lagrangian, respectively Caldirola–Kanai transformations nowadays. Bateman found the same transformation of the canonical variables in 1931 using another way [9]. In a second step, Caldirola wrote down the corresponding Schrödinger equation just using the two transformations: first, the transformation which includes the friction $\lambda$:

$$p \rightarrow p e^{-\lambda t}, \ V(x) \rightarrow V(x)e^{-\lambda t}$$

and furthermore the standard transformation to derive the corresponding operators $p \rightarrow -i\hbar \nabla, \ x \rightarrow \dot{x}$. Kanai has followed this example and has introduced the Heisenberg equations
using transformed momenta $R_{\text{CK}} = pe^{-\lambda t}$ and potential $V_{\text{CK}} = Ve^{-2\lambda t}$. It stays open from where Kanai has taken the idea of the transformed momenta $R_{\text{CK}}$ and potential $V_{\text{CK}}$. However, in 1948, the idea of this transformation to include friction in the Lagrangian or Hamilton formalism was already known.

Then for a while, this topic was without interest, however, became a topic of interest again in the 1970s and '80s. During this period several non-linear Schrödinger equations describing quantum systems with friction respectively damping have been proposed by, e.g., Kostin, Haase, Schuch, and Gisin [10–13]. Then the focus shifted towards the description of decoherence processes. During this time Gisin and Percival proposed the description of quantum state diffusion (QSD) [14] and Mølmer the Monte Carlo wavefunction method (MCWF) [15].

Nowadays, the focus has shifted again, away from the description of Markovian towards Non-Markovian processes. However, this shall not be the topic of this publication. Within this paper, we stay with the description of friction and damping in quantum mechanics. This publication provides some remarks and derivations which might be trivial but to the best of our knowledge have not been described or published so far. This lack of missing information as simple it appears shall be closed.

Furthermore, we present a new way to derive the Schrödinger equation proposed by Gisin starting from the formalism of stochastic differential equations. This way might look strange in the first moment because of the Gisin–Schrödinger equation is not a stochastic differential equation. The Lindblad equation, which is a Liouville-von-Neumann equation is also not a stochastic differential equation. However, the corresponding wave function is a stochastic differential equation. The corresponding Liouville-von-Neumann equation to the Gisin-Schrödinger equation is the quantum mechanical analog to the classical Landau–Lifshitz equation [16]. At this point, we can ask the question if the corresponding wave equation to this Liouville-von-Neumann equation is a stochastic differential equation or not. As we have mentioned it already, the corresponding Schrödinger equation is not a stochastic differential equation; this is easy to see without the complicated Ito formalism of stochastic differential equations. Nevertheless, strange or unusual ways offer new perspectives and are the basis for new ideas and descriptions. Therefore, additional descriptions to an already given one as strange they appear in the first moment should not be rejected immediately.

The publication is organized as follow: After a brief reminder of the variation calculus, we describe in detail the way Schrödinger has derived the time-independent Schrödinger equation and extend this description to derive as well the time-dependent Schrödinger equation. In the next step, we extend this description once more to get the Caldirola–Kanai equation. In section 4, we comment on the non-linear Schrödinger equations proposed by Schuch and N. Gisin and derive them in the before given context of the Hamilton–Jacobi equation. In section 5, we derive the Gisin–Schrödinger equation once more; however, this time using the theory of stochastic differential equations. The last section discusses the practical problems which occur when dealing with the Gisin–Schrödinger equation to describe a quantum mechanical spin system. Within this section, we give some ideas on how to overcome these problems, e.g., by using the cluster mean field theory (CMFT) [17].

2. Derivation of the time-dependent Schrödinger equation

Within this section, we close the lack of derivation of the time-dependent Schrödinger equation. The way will be the same as Schrödinger has used to derive the time-independent Schrödinger equation. However, we will extend the description to derive the time-dependent Schrödinger equation and several non-linear Schrödinger equations which take into account friction and damping. Schrödinger has started his description with the Hamilton–Jacobi equation where he inserted a wave ansatz for the action $S$. Then, he derived the time-independent Schrödinger equation by performing a calculus of variation. Therefore, the first step is to give a short reminder of the variational calculus.

2.1. Calculus of variation

The main idea of the calculus of variation is to find the extrema of a given functional, e.g., the Lagrangian $\mathcal{L}$ or the action $S$. Therefore, the calculus of variation is strongly connected to the principle of least action of the Hamilton mechanics, but also to other principles, e.g., Fermat, Euler, Maupertuis, or Dirichlet. Central element of the calculus of variation is the Euler–Lagrange equation which derivation shall briefly be described in the following.

Let us assume; we are searching for the extrema of the action $S$ which is defined after Hamilton as the integral of the Lagrangian $\mathcal{L}(q, \dot{q}, t)$ between time $t_1$ and $t_2$:

$$ S = \int_{t_1}^{t_2} dt \mathcal{L}(q, \dot{q}, t). \quad (1) $$
The Lagrangian itself is a functional of the generalized coordinates $\mathbf{q} = (q_1, q_2, \ldots, q_n)$, and their time derivatives $\dot{\mathbf{q}} = \frac{d\mathbf{q}}{dt}$. During the variational calculus, we vary the functions, in this case, $\mathbf{q}$ and $\dot{\mathbf{q}}$, within the functional $L(\mathbf{q}, \dot{\mathbf{q}}, t)$ and search for the $\delta \mathbf{q}$ and $\delta \dot{\mathbf{q}}$ which fulfill the following condition:

$$\delta S = \int_{t_1}^{t_2} dt \left( L(\mathbf{q} + \delta \mathbf{q}, \dot{\mathbf{q}} + \delta \dot{\mathbf{q}}, t) - L(\mathbf{q}, \dot{\mathbf{q}}, t) \right) = 0. \tag{2}$$

Caused by the fact that we assume small changes we can Taylor expand the first functional:

$$L(\mathbf{q} + \delta \mathbf{q}, \dot{\mathbf{q}} + \delta \dot{\mathbf{q}}, t) \approx L(\mathbf{q}, \dot{\mathbf{q}}, t) + \frac{\partial L}{\partial \mathbf{q}} \delta \mathbf{q} + \frac{\partial L}{\partial \dot{\mathbf{q}}} \delta \dot{\mathbf{q}},$$

and find the following condition which has to be fulfilled:

$$\delta S = \int_{t_1}^{t_2} dt \left( \frac{\partial L}{\partial \mathbf{q}} \delta \mathbf{q} + \frac{\partial L}{\partial \dot{\mathbf{q}}} \frac{d}{dt} \delta \mathbf{q} \right) = 0. \tag{4}$$

We can further write the second summand, using partial integration, as:

$$\int_{t_1}^{t_2} dt \left( \frac{\partial L}{\partial \mathbf{q}} \delta \mathbf{q} + \frac{\partial L}{\partial \dot{\mathbf{q}}} \frac{d}{dt} \delta \mathbf{q} \right) = \left[ \frac{\partial L}{\partial \mathbf{q}} \delta \mathbf{q} \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} dt \left( \frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{q}}} \delta \mathbf{q} \right). \tag{5}$$

The term in the middle is zero when we assume that we do not change the Lagrangian at the starting and final time $\delta L(t_1) = 0$. Then, all together leads to:

$$\delta S = \int_{t_1}^{t_2} dt \delta \mathbf{q} \left( \frac{\partial L}{\partial \mathbf{q}} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{q}}} \right) = 0. \tag{6}$$

Here, the expression in the round brackets, if set equal to zero, is the Euler–Lagrange equation.

There are some points we have to mention here. Most textbooks give the impression that the Euler–Lagrange equation is a fundamental equation of mechanics where we vary the Lagrangian, which is a functional of the functions $\mathbf{q}(t)$ and $\dot{\mathbf{q}}(t)$. This idea is not wrong but also not the whole truth. The Euler–Lagrange equation is the second-order partial differential equation whose solutions are the functions for which a given functional is stationary. This means the functional has not necessarily to be the Lagrangian $L(\mathbf{q}, \dot{\mathbf{q}}, t)$ and the included functions not necessarily $\mathbf{q}$ and $\dot{\mathbf{q}}$ or restricted to just two functions as in the example above. The Euler–Lagrange equation, and therefore the calculus of variation is quite general.

The second remark is concerning the integral. For the principle of the least action the integral is one-dimensional. However, the calculus of variation is not restricted to one-dimensional integrals, as we will see for the derivation of the Schrödinger equation. In that case, the boundary condition is not the Lagrangian $L(\mathbf{q}, \dot{\mathbf{q}}, t)$ at times $t_1$ and $t_2$. Here, we set the wave function $\psi$ equal to zero. The wave function itself is not necessarily one- and can be three-dimensional, and so the boundary condition and the integrals. This gives us a new challenge. Within the description above, the integral over time $t$ is one-dimensional, and therefore the partial integration is easy to handle. However, what if the integral is of higher dimension? The general concept of partial integration is for one-dimensional integrals and cannot be used for two- or three-dimensional integrals. However, in that case, there are theorems, e.g., in the three-dimensional case, the theorem given by G. Green:

$$\int_{\Omega} d^3 \mathbf{r} (\nabla u \cdot \nabla \nu) = \int_{\partial \Omega} d^2 \mathbf{r} \left( \frac{\partial \nu}{\partial \mathbf{n}} \right) - \int_{\Omega} d^3 \mathbf{r} (u \Delta \nu), \tag{7}$$

where $\nabla u = \text{grad } u$, the same for $\nu$, $\Delta \nu = \nabla^2 \nu = \text{div}(\text{grad } \nu)$, and

$$\frac{\partial \nu}{\partial \mathbf{n}} = \text{grad } \nu \cdot \mathbf{n}. \tag{8}$$

$n$ is the normal vector of the surface corresponding to the boundary condition. $u$ and $\nu$ functions of $\mathbf{r} = (x, y, z)^T$ and the integrals are over the total volume $\Omega$ respectively the surface $\partial \Omega$. This theorem replaces the partial integration within the derivation of the Schrödinger equation.

### 2.2. Derivation of the time-independent Schrödinger equation

Within this subsection, we describe Schrödinger’s derivation of the time-independent Schrödinger equation. It is a known fact that Schrödinger has first tried to find a full-relativistic wave equation but failed. Schrödinger found the Klein–Gordon equation which is correct in case of spin-less particles but was not able to give a complete relativistic description. This has been done by Dirac in 1928 (Dirac equation) [18]. Therefore, Schrödinger started to look for a non-relativistic wave equation and found the two differential equations named after him. As in many cases, Schrödinger has not started from zero and was able to build his description on the work of de Broglie [19]. de Broglie mentions two things in his Ph.D. thesis: First that the particle-wave duality of light holds in general. Moreover, second, the principles like the one of Maupertuis or Fermat shall also hold for
matter waves. These ideas have been the fuel for Schrödinger, and the starting point of his derivation is the following ansatz for the action $S$ in the framework of the Hamilton-Jacobi formalism:

$$S = K \ln \psi.$$

($9$)

$K$ is a constant and has the dimension of an action. In principle, it is clear that $K = -i\hbar$. It is also clear that the mysterious ansatz which Schrödinger has not further commented in the first communication is $\psi = \exp(S/K)$ [3], $\psi$ describes a wave with the action $S$ as the phase. This information, we get in the following (second) communication [4]. At this moment, it also becomes clear that $K$ is imaginary. However, within this publication, we follow the description of Schrödinger and handle $K$ as a constant and assume $\psi$ as a real function.

As already mentioned Schrödinger uses the framework of the Hamilton-Jacobi theory to find the wave equation which describes the central-potential problem of hydrogen. Caused by the fact that Schrödinger was looking for stationary orbits Schrödinger used the time-independent Hamilton-Jacobi equation:

$$H \left( r, \frac{\partial S}{\partial r} \right) = E.$$  

(10)

Within this differential equation, $H$ is the Hamilton function and $E$ the energy of the system. We get equation (10) from the time-dependent Hamilton-Jacobi equation if we assume that we can separate the time-dependent action $W(t)$ in two parts $W(t) = S - E t$: the time-independent action $S$ and the time-dependence $E t$. The idea here is that the action $W$ in three-dimensional space is described by a wave function with iso-surfaces of constant $S$ moving with constant velocity $v$. This means especially that we have $\psi(t) = \exp(W/K) = \psi \exp(-iEt/\hbar)$, with $\psi = \exp(S/K)$ and $K = -i\hbar$ defined before. In other words, this is the same separation we deal with in case of time-independent quantum mechanical problems where the time-dependence occurs just like an additional phase which can be separated.

Then, if we insert the ansatz for $S$, equation (9), in the time-independent Hamilton-Jacobi equation (10), we find:

$$\left(\nabla \psi\right)^2 - \frac{2m}{K^2} \left( E + \frac{e^2}{r} \right) \psi^2 = 0.$$  

(11)

Here, we have used that $H = T + V(r)$, with $T = p^2/2m$ the kinetic energy and $V(r) = -e^2/r$ the potential of the hydrogen atom. Thereby, the momentum $p$ is replaced by $p = \nabla S$, which comes from the canonical transformation $F_2$, which we need to derive the Hamilton-Jacobi formalism from the Hamilton theory. For completeness, $e$ is the elementary charge and $r = |r|$.

The modified Hamilton-Jacobi equation (11) is the functional $\mathcal{F}[\psi(r), \nabla \psi(r)]$ which we have to treat with the variational calculus to find the stationary solutions for the wave function $\psi(r)$. This means we can use the description given in section 2.1. For that purpose, we replace the one-dimensional integral over $t$ by the three-dimensional integral over the volume $\Omega$, $\mathbf{q}$ by $\psi$, $\mathbf{q}$ by $\nabla \psi$, and the Lagrangian $\mathcal{L}$ by the functional $\mathcal{F}$. Then, equation (4) becomes:

$$\delta J = \int_{\Omega} d^3 \mathbf{r} \left( \frac{\partial \mathcal{F}}{\partial \psi} \delta \psi + \frac{\partial \mathcal{F}}{\partial \nabla \psi} \nabla \delta \psi \right) = 0.$$  

(12)

As before in section 2.1, we modify the second summand using Green’s theorem

$$\int_{\Omega} d^3 \mathbf{r} \left( \nabla \delta \psi \frac{\partial \mathcal{F}}{\partial \nabla \psi} \right) = (*)$$

$$(*) = \int_{\partial \Omega} d^2 \mathbf{f} \left( \delta \psi \frac{\partial \mathcal{F}}{\partial \nabla \psi} \cdot \mathbf{n} \right) - \int_{\Omega} d^3 \mathbf{r} \left( \delta \psi \nabla \frac{\partial \mathcal{F}}{\partial \nabla \psi} \right).$$  

(13)

As said before, equation (11) is the functional $\mathcal{F}$. Therefore, we can write:

$$\frac{\partial \mathcal{F}}{\partial \nabla \psi} = 2 \nabla \psi$$

(14a)

$$\nabla \frac{\partial \mathcal{F}}{\partial \nabla \psi} = 2 \nabla^2 \psi = 2 \Delta \psi$$

(14b)

$$\frac{\partial \mathcal{F}}{\partial \nabla \psi} \cdot \mathbf{n} = 2 \nabla \psi \cdot \mathbf{n} = 2 \frac{\partial \psi}{\partial \mathbf{n}},$$

(14c)

and therefore:

$$2 \int_{\Omega} d^3 \mathbf{r} (\nabla \delta \psi \cdot \nabla \psi) = 2 \int_{\partial \Omega} d^2 \mathbf{f} \left( \delta \psi \frac{\partial \psi}{\partial \mathbf{n}} \right) - 2 \int_{\Omega} d^3 \mathbf{r} (\delta \psi \Delta \psi).$$  

(15)
This is the second summand, let us take a look at the first one too. There, we have:

\[
\frac{\partial \mathcal{F}}{\partial \psi} = -\frac{4m}{K^2} \left( E + \frac{\alpha^2}{r} \right) \psi,
\]

(16)

Then, putting all the pieces together equation (12) appears as:

\[
\frac{1}{2} \delta J = \int_{\partial \Omega} df \left( \dot{\psi} \frac{\partial \psi}{\partial \alpha} - \int_{\Omega} d^3r \dot{\psi} \left[ \Delta \psi + \frac{2m}{K^2} \left( E + \frac{\alpha^2}{r} \right) \psi \right] \right) = 0.
\]

(17)

The surface integral (first integral on the right-hand side) corresponds to our boundary condition. This integral shall be zero. Under this condition, \( \delta J \) becomes zero if the remaining volume integral (second integral on the right-hand side) is zero. This integral definitely becomes zero if the expression in the square brackets is zero:

\[
\Delta \psi + \frac{2m}{K^2} \left( E + \frac{\alpha^2}{r} \right) \psi = 0.
\]

(18)

This is the time-independent Schrödinger equation, for the hydrogen atom. Schrödinger solves this differential equation with a transformation to the Weierstrass differential equation and the Laplace transformation [3, 20].

2.3. Derivation of the time-dependent Schrödinger equation (TDSE)

Within this subsection, we derive the time-dependent Schrödinger equation. The procedure is the same as for the time-independent Schrödinger equation. Here, we start with the Hamilton-Jacobi equation

\[
H + \frac{\partial S}{\partial t} = 0,
\]

(19)

where we have not separated the time as before in the previous subsection. Here, the action \( S = S(t) \) is time-dependent, the same is true for the wave function \( \psi = \psi(\mathbf{r}, t) \). Then, with \( H = T + V \) and \( T = \frac{p^2}{2m} \), as well as \( p = \nabla S \), we get:

\[
\frac{1}{2m} (\nabla S)^2 + V + \frac{\partial S}{\partial t} = 0.
\]

(20)

Some readers might have already noticed the problem with the given description in section 2.2. Within this subsection, we have followed the original description of Schrödinger, which is inaccurate in one point. The wave function has been treated as a real function. The same is true for the action \( S \). Correct is that the wave function \( \psi \) and also the action \( S \) are complex functions. This means we have not only \( \psi \) but also the complex conjugate \( \psi^* \) and the same for the action \( S \). And this means, we have to replace equation (20) by the following two differential equations:

\[
\frac{1}{2m} |\nabla S|^2 + V + \frac{\partial S}{\partial t} = 0,
\]

(21a)

and

\[
\frac{1}{2m} |\nabla S|^2 + V + \frac{\partial S^*}{\partial t} = 0.
\]

(21b)

Here, \(|\nabla S|^2\) is equal to \((\nabla S^*)(\nabla S)\). Then, as ansatz for the wave functions, we take \( S = K \ln \psi \) respectively \( S^* = K^* \ln \psi^* \), with \( K = -i \hbar \) respectively \( K^* = +i \hbar \). With \( \nabla S^* = K (\nabla \psi^*) / \psi^* \) and \( \partial S^*/\partial t = K \dot{\psi}^* / \psi^*, \) where the dot stands for the time derivative and the star in parentheses (*) stands for complex conjugate or not, we find for equation (21b)

\[
\mathcal{F}^* = \frac{|K|^2}{2m} |\nabla \psi|^2 + V|\psi|^2 + K^* \dot{\psi} = 0.
\]

(22)

Analog, from equation (21a), we find \( \mathcal{F}[\psi, \nabla \psi, \dot{\psi}] \). Now, the variation of \( \mathcal{F}^*[\psi, \nabla \psi, \dot{\psi}] \) has to be extended to take the time derivative \( \dot{\psi} = \partial \psi / \partial t \) into account. Therefore, equation (12) becomes:

\[
\delta J^* = \int_{\Omega} d^3r \left( \frac{\partial \mathcal{F}^*}{\partial \psi} \dot{\psi}^* \psi + \frac{\partial \mathcal{F}^*}{\partial (\nabla \psi)^*} \nabla \psi^* + \frac{\partial \mathcal{F}^*}{\partial \psi} \dot{\psi}^* \right) = 0.
\]

(23)

Cause the second and third term is not handy, we treat them as before with Green’s theorem. The resulting surface integral with a now time-dependent surface corresponds to the boundary condition. We set these integrals equal to zero and focus on the remaining volume integral:

\[
\delta J^* = \int_{\Omega} d^3r \dot{\psi}^* \left( \frac{\partial \mathcal{F}^*}{\partial \psi} - \nabla \left( \frac{\partial \mathcal{F}^*}{\partial (\nabla \psi)^*} \right) \right) = 0,
\]

(24)
Then, with
\[
\frac{\partial F^*}{\partial \psi^*} = V\psi, \quad \text{(25a)}
\]
\[
\nabla \left( \frac{\partial F^*}{\partial \psi} \right) = \frac{|K|^2}{2m} \Delta \psi, \quad \text{(25b)}
\]
\[
\frac{\partial}{\partial t} \left( \frac{\partial F^*}{\partial \psi^*} \right) = K^* \frac{\partial \psi}{\partial t}, \quad \text{(25c)}
\]
we find from equation (24):
\[
\delta f^* = \int \delta \mathbf{r} \delta \psi^* \left[ -\frac{|K|^2}{2m} \Delta \psi + V\psi - K^* \frac{\partial \psi}{\partial t} \right] = 0. \quad \text{(26)}
\]
\(\delta f^*\) is zero if the expression in the square brackets is zero. The expression in the square brackets itself is the time-dependent Schrödinger equation:
\[
-\frac{\hbar^2}{2m} \Delta \psi + V\psi = i\hbar \frac{\partial \psi}{\partial t}. \quad \text{(27)}
\]
If we vary \(\delta f\), which means we deal with equation (21a), we find the corresponding complex conjugate time-dependent Schrödinger equation:
\[
-\frac{\hbar^2}{2m} \Delta \psi^* + V\psi^* = -i\hbar \frac{\partial \psi^*}{\partial t}. \quad \text{(28)}
\]

3. Derivation of the Caldirola-Kanai equation

So far, we have derived the time-independent and the time-dependent Schrödinger equations starting from the corresponding Hamilton-Jacobi equations. The main idea here is that there is a wave mechanics which corresponds to the point mechanics. A similar behavior we know from the optics where we have the wave and ray picture. Both types of the Schrödinger equation are quite successful. However, they describe systems without friction or damping. In principle, there are two ways to include these effects: microscopic or phenomenological. Microscopic means next to the process we want to describe, we deal with the process of friction or damping. This way is possible but increases the complexity of the treatment. In some cases, we also have no idea about the underlying friction or damping process. Alternatively, we are just interested in the effect of additional friction or damping without taking care of the underlying friction or damping processes and to describe the primary process. In such cases, our choice is the phenomenological description.

Now the question is, how can we describe such scenarios? First in classical mechanics: Here, we follow the description given by P. Caldirola [6]: In the manner of Lagrange, we can write
\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \mathbf{q}} \right) - \frac{\partial T}{\partial \mathbf{q}} = \frac{\partial V}{\partial \mathbf{q}} + \mathbf{F}. \quad \text{(29)}
\]
This is the Euler–Lagrange equation introduced in section 2.1 with the Lagrangian \(\mathcal{L} = T - V\), however for a nonconservative system and therefore with an additional external force \(\mathbf{F}\) which describes the friction.

The differential equations that govern the movement of a material system when individual points meet resistances proportional to the respective speed, can be obtained from the equations related to the free movement of the same by the change of independent variable \(t\). In other words, we perform the transformation:
\[
dt = \varphi(t) \, dt', \quad \text{(30)}
\]
and search for the function \(\varphi(t)\) which transforms equation (29) to:
\[
\frac{d}{dt'} \left( \frac{\partial T'}{\partial \mathbf{q}'} \right) - \frac{\partial T'}{\partial \mathbf{q}'} = \frac{\partial V'}{\partial \mathbf{q}'} \quad \text{(31)}
\]
Here, \(\mathbf{q}'\) stands for \(\mathbf{q}' = \frac{\partial \mathbf{q}}{\partial \mathbf{r}} = \frac{\partial \mathbf{q}}{\partial t'} = \frac{\partial \mathbf{q}}{\partial \varphi(t)} \). Furthermore, \(T'\) is the abbreviation for \(T' = T\varphi^2(t)\), which follows from \(T' \propto (\mathbf{q}')^2\). Then, we can write the first term on the left-hand side as:
\[
\frac{d}{dt'} \left( \frac{\partial T'}{\partial \mathbf{q}'} \right) = \varphi^2(t) \left[ \frac{d}{dt} \left( \frac{\partial T}{\partial \mathbf{q}} \right) + \frac{\varphi}{\varphi} \frac{\partial T}{\partial \mathbf{q}} \frac{\partial \varphi}{\partial \mathbf{q}} \right]. \quad \text{(32)}
\]
with $\dot{\varphi} = \partial \varphi / \partial t$ and $\varphi = \partial q / \partial t$. Moreover, the second term as:

$$\frac{\partial T'}{\partial q} = \varphi^2(t) \frac{\partial T}{\partial q}. \tag{33}$$

Therefore, equation (31) becomes:

$$\frac{d}{dr} \left( \frac{\partial T}{\partial q'} \right) + \frac{\varphi}{\varphi q} \frac{\partial T}{\partial q} - \frac{\partial T}{\partial q} = \frac{\partial V'}{\partial q} \varphi^{-2}. \tag{34}$$

Then, if we compare equation (34) with equation (29), we get $V' = V\varphi^2$ and

$$F = -\frac{\varphi}{\varphi q} \frac{\partial T}{\partial q}. \tag{35}$$

equation (35) is a differential equation:

$$\frac{\varphi'}{\varphi} = -1 \frac{d}{dr} \left( \frac{\partial T}{\partial q} \right)^{-1}, \tag{36}$$

and can be easily solved:

$$\varphi(t) = \varphi(0) e^{\int dr \left( \frac{\partial T}{\partial q} \right)^{-1}}. \tag{37}$$

The result is the general transformation. However, this transformation is quite complex, and in many cases it is already enough to assume that $f(t) = F \left( \frac{\partial T}{\partial q} \right)^{-1}$ is a constant, meaning $f(t) = \lambda$. In such a case, we can write:

$$\varphi(t) = e^{-\lambda t}, \tag{38}$$

where $\lambda$ rules the strength of the friction or damping. This transformation was the first proposed by Levi-Civita in 1896 [8]. In summary, we have the following transformation: $q' = q e^{-\lambda t}$, and therefore $p' = m q' = pe^{-\lambda t}$, and $T' = \frac{m}{T}(q')^2 = T e^{-2\lambda t}$. Furthermore, $V' = Ve^{-2\lambda t}$, $L' = (T - V)e^{-2\lambda t} = Le^{-2\lambda t}$, and $H' = (T + V)e^{-2\lambda t} = He^{-2\lambda t}$, and last but not least, $d\tau = dt e^{3\lambda t}$.

Then, with this in mind, we can write the transformed Hamilton-Jacobi equation as:

$$H' + \frac{\partial S}{\partial \tau} = 0, \tag{39}$$

and, with the ansatz $S = -i \hbar \ln \psi$, we get:

$$\frac{\partial S}{\partial \tau} = -\frac{\hbar}{i} \frac{\partial \psi}{\psi} \frac{\partial \psi}{\partial \tau} = -H'. \tag{40}$$

Furthermore, after transformation back to the original time $t$:

$$i \hbar \frac{\partial \psi}{\partial t} = e^{-\lambda t} H \psi, \tag{41}$$

we derive the Caldirola-Kanai equation, just without explicit representation of the Hamilton operator $H$.

### 4. Derivation of the Schuch- and Gisin-Schrödinger equation

Up to now, we have derived the time-independent and the time-dependent Schrödinger equation and extended the description to include friction. However, there are even more proposals of Schrödinger equations with the goal to include friction and damping in the dynamics of quantum mechanics. Within this section, we will derive and discuss two of these Schrödinger equations. We start with the Schrödinger equation proposed by Schuch, followed by the proposal of Gisin.

#### 4.1. The Schuch-Schrödinger equation

Motivated by the descriptions of Caldirola and Kanai, primarily inspired by equation (41), we start with the following Hamilton-Jacobi equation:

$$\hat{H} = -\frac{\partial \hat{S}}{\partial \tau}, \tag{42}$$

with $\hat{H} = He^{-\lambda t}$, and similar $\hat{S} = Se^{-\lambda t}$. As usual, $H$ is the Hamiltonian, $S$ the action, and $\lambda$ is the friction constant. This ansatz is in principle the reversal of the final result of Caldirola and Kanai.
Now, we can write the right-hand side of equation (42) as:
\[
\frac{\partial \tilde{S}}{\partial t} = \frac{\partial}{\partial t} (Se^{-\lambda t}) = e^{-\lambda t} \frac{\partial S}{\partial t} - \lambda \tilde{S},
\]
and therefore:
\[
e^{-\lambda t} \mathcal{H} = e^{-\lambda t} \left( \frac{\partial S}{\partial t} + \lambda \tilde{S} \right).
\]
This means the official starting point to derive the Schuch-Schrödinger equation is the modified Hamilton-Jacobi equation:
\[
\left( \frac{\partial}{\partial t} - \lambda \right) S + \mathcal{H} = -\lambda \langle S \rangle.
\]
Here, an additional term \(-\lambda \langle S \rangle\) has been added on the right-hand side to guarantee later that the norm of the wave function \(\psi\) is conserved. Different to the Cladirola-Kanai equation the norm is not conserved, and the additional term needed. The brackets \(\langle \ldots \rangle\), here, mean that we deal with the expectation value.

The last step of the procedure is to insert the ansatz \(S = -i\hbar \ln \psi\) in equation (45) which immediately leads to the following non-linear Schrödinger equation:
\[
i \hbar \frac{\partial \psi}{\partial t} = \mathcal{H} \psi + i\hbar \lambda (\ln \psi - \langle \ln \psi \rangle) \psi.
\]
Schuch has proposed this differential equation [12, 21]. However, there are also similar looking non-linear Schrödinger equations proposed by other physicists, e.g., Kostin [10].

### 4.2. The Gisin-Schrödinger equation

While Schuch had in mind to find a differential equation which describes the motion of a particle with friction which also fulfills the Ehrenfest theorem, the intention of Gisin was more general. The question which Gisin has tried to answer in his work was how does a quantum mechanical description with damping look like, in general. His ansatz is quite unusual for textbook quantum mechanics because Gisin starts with a non-Hermitian Hamilton operator \(\mathcal{H} = \mathcal{H} - i\lambda \Gamma\).

Now, we make the ansatz \(S = -i\hbar \ln \psi\) and insert it in the Hamilton-Jacobi equation:
\[
\frac{\partial S}{\partial t} + \mathcal{H} = \phi.
\]
The result of this step is the following Schrödinger equation
\[
i \hbar \frac{\partial \psi}{\partial t} = \mathcal{H} \psi - \phi \psi.
\]
As before, during the derivation of the Schuch-Schrödinger equation the additional term \(\phi\) is included to guarantee the norm. At this point, we switch to the Dirac bracket description, which means, we replace \(\psi\) by \(|\psi\rangle\). The bracket description makes the calculation of the norm \(n = \langle \psi|\psi\rangle\) easier to follow because we do not need to write out the integrals. From equation (48), we have the two differential equations for the bra- and the ket-vector:
\[
\frac{d}{dt} \langle \psi| = + \frac{i}{\hbar} \langle \psi|(\mathcal{H} + i\lambda \Gamma) - \frac{i}{\hbar} \phi^* \langle \psi|,
\]
\[
\frac{d}{dt} |\psi\rangle = - \frac{i}{\hbar} (\mathcal{H} - i\lambda \Gamma)|\psi\rangle + \frac{i}{\hbar} \phi |\psi\rangle.
\]
As before, the star * stands for the complex conjugate. The Hamilton operator \(\mathcal{H}\) and the operator \(\Gamma\) itself are assumed to be Hermitian. Then, we can write:
\[
\frac{dn}{dt} = \left( \frac{d}{dt} \langle \psi| \right) |\psi\rangle + \langle \psi| \left( \frac{d}{dt} |\psi\rangle \right) = \frac{i}{\hbar} (\phi - \phi^* + 2i\lambda \langle \psi|\Gamma|\psi\rangle) = 0.
\]
Now, we are searching for that specific function \(\phi\) which guarantees us that the norm \(n = 1\) is constant in time \(t\): \(dn/dt = 0\). This condition is fulfilled if the expression in the round brackets is equal to zero. Inserting \(\phi = \phi_R + i\phi_I\) respectively \(\phi^* = \phi_R - i\phi_I\) lead to the fact that the real parts \(\phi_R\) get neglected, which means that \(\phi = i\phi_I\) is purely imaginary:
\[
\phi = -i\lambda \langle \psi|\Gamma|\psi\rangle.
\]
With that, we finally find the Gisin-Schrödinger equation:
\[
\frac{d}{dt} |\psi\rangle = i\hbar [H, |\psi\rangle] - i\lambda (\Gamma - \langle \Gamma \rangle) |\psi\rangle.
\] (52)

Here, we have used the acronym \( \langle \Gamma \rangle = \langle \psi | \Gamma | \psi \rangle \). In the original description given by Gisin \( \Gamma \) is equal to the Hamilton operator \( H \).

It is known that if \( \Gamma = H \), and \( H \) a Heisenberg Hamiltonian, the Gisin-Schrödinger equation describes the dynamics of quantum spins [22–25]. We can further renormalize the time [22] to get:
\[
\frac{d}{dt} (1 + \lambda^2 t^2) |\psi\rangle = H |\psi\rangle - i\lambda (H - \langle H \rangle) |\psi\rangle.
\] (53)

In that case the spin expectation value of a single spin \( S \) follows the classical Landau–Lifshitz–Gilbert equation if no anisotropies are involved [26]. The originally proposed Gisin-Schrödinger equation follows the Landau–Lifshitz equation, which means it is valid only for small damping values \( \lambda \) [27]. In the case of several spins, we find quantum effects like entanglement [22]. The Schrödinger equations proposed by Schuch respectively Caldirola and Kanai are well suited to describe quantum spin systems. Therefore, the combined use of these Schrödinger equations to describe a single particle with a spin in an environment which provokes friction and damping appears conceivable.

5. Derivation of the Gisin-Schrödinger equation using the Ito concept of stochastic differential equations

In the previous section, we have derived the Gisin-Schrödinger equation starting from the Hamilton-Jacobi equation and making an ansatz for the action \( S \). Here, we derive the same equation in another way. The description might appear strange or useless. However, the description offers a new view or perspective on the underlying physics.

It is easy to see that the following Liouville-von-Neumann equation:
\[
\frac{d\rho}{dt} = \frac{i}{\hbar} [\rho, H] - \frac{\lambda}{\hbar} [\rho [\rho, H]],
\] (54)

offers similar dynamics as the corresponding Landau–Lifshitz equation:
\[
\frac{dM}{dt} = -\gamma M \times \mathbf{B} - \gamma \lambda M \times (M \times \mathbf{B}),
\] (55)

if \( M = \text{Tr}(\rho S) \), \( \gamma = \frac{g\mu_B}{\hbar} \), and \( H = -g\mu_B / \hbar (\mathbf{B} \cdot \mathbf{S}) \). Of course, we could now assume that the density operator is defined for a pure state \( \rho = |\psi\rangle \langle \psi| \). With this definition it is easy to split the Liouville-von-Neumann equation (54) into the Gisin-Schrödinger and its corresponding conjugate transpose Schrödinger equation. However, this is not our intention here. Here, we want to ignore or oversee this fact and ask ourselves is there a stochastic differential equation of the kind:
\[
\frac{d\rho}{dt} = |\nu\rangle \langle \nu| + |f\rangle \langle f| \frac{d\xi}{\gamma},
\] (56)

which corresponds to the Liouville-von-Neumann equation (54)? Equation (56) is a stochastic differential equation as used for instance to describe unstable stock prices or physical systems subject to thermal or quantum fluctuations. \( |d\psi\rangle \) gives the time evolution of \( |\psi\rangle \) which is described by a drift term \( |\nu\rangle \langle \nu| \) and a fluctuation term \( |f\rangle \langle f| \). \( d\xi \) is the time increment of the dynamics and \( \xi(t) \) is white noise.

Within this description, the density operator is defined by \( \rho = \mathbb{E}|\psi\rangle \langle \psi| \), where \( \mathbb{E} \) means, we have to perform an ensemble average. Then, the corresponding time evolution of the density operator is given by:
\[
\frac{d\rho}{dt} = \mathbb{E}(|d\psi\rangle \langle \psi| + |\psi\rangle \langle \psi| |d\psi\rangle + |d\psi\rangle \langle d\psi|).
\] (57)

Using the Ito rules:
\[
\mathbb{E}(d\xi) = 0 \quad \text{and} \quad \mathbb{E}(d\xi^2) = dt,
\] (58)

and ignoring any term with \( dt^2 \), we get:
\[
\mathbb{E}(d|\psi\rangle) = |\nu\rangle \langle \nu| dt
\] (59a)
\[
\mathbb{E}(d|\psi|) = |\nu| \langle \nu| dt
\] (59b)
\[
\mathbb{E}(d|\psi\rangle \langle d\psi| + |f\rangle \langle f| dt).
\] (59c)
We can use these expressions in equation (57), and find:

\[
\frac{d\rho}{dt} = |\psi\rangle \langle \psi| + \langle \psi| + |f\rangle \langle f|.
\]  

(60)

To preserve the norm, the differential change in the state vector due to the fluctuation must be orthogonal to the state vector itself. This means,

\[
\langle \psi| f\rangle = \langle f|\psi\rangle = 0.
\]  

(61)

Then, the fluctuation term \( f \) can be determined by considering the component of \( |dr\rangle \) in the space orthogonal to the projection operator \( P = |\psi\rangle \langle \psi| \) meaning \( Q = 1 - P \), where \( I \) is the unity matrix. This can be seen if we multiply equation (60) from both sides with the projection operator \( Q \) and use equation (61). The result is:

\[
|f\rangle \langle f| = (I - |\psi\rangle \langle \psi|) \frac{d\rho}{dt} (I - |\psi\rangle \langle \psi|)
\]

\[
= \frac{d\rho}{dt} - |\psi\rangle \langle \psi| \frac{d\rho}{dt} - \frac{d}{dt} |\psi\rangle \langle \psi| + |\psi\rangle \langle \psi| \frac{d}{dt} |\psi\rangle \langle \psi|.
\]  

(62)

This is \( |f\rangle \langle f| \). However, we are looking for the fluctuation term \( f \). We can get \( |f\rangle \) by symmetrical dividing expression (62) after, we have included the Liouville-von-Neumann equation (54). With \( \rho^2 = \rho \) and \( \rho = |\psi\rangle \langle \psi| \) because we have already performed the ensemble average, we find after a lengthy but straightforward calculation that \( |f\rangle = f |\psi\rangle = 0 \). This means no fluctuations. In other words, the wave equation corresponding to the Liouville-von-Neumann equation (54) is no stochastic differential equation. This is the result we have expected. Nevertheless, let us look for the drift term \( |\nu\rangle \). If our description is correct, we should at least reproduce the Gisin-Schrödinger equation. Therefore, we should look for the drift term \( |\nu\rangle \).

The drift term \( |\nu\rangle \), we get by multiplying equation (60) from the right with \( |\nu\rangle \). The result is:

\[
|\nu\rangle = \frac{d\rho}{dt} |\psi\rangle - \langle \psi| |\nu\rangle.
\]  

(63)

For that expression, we need knowledge about \( \langle \psi| \psi\rangle \). For that, we multiply equation (60) from the left with \( |\psi\rangle \) and from the right with \( |\psi\rangle \):

\[
\langle \psi| \frac{d\rho}{dt} |\psi\rangle = \langle \psi| \psi\rangle + \langle \psi| |\nu\rangle = 2 \text{Re}(\langle \psi| |\nu\rangle).
\]  

(64)

Here, we have used the fact that \( \langle \psi| \psi\rangle \) is a complex number and the conjugate transpose \( \langle \psi| |\nu\rangle = (|\nu\rangle \langle \psi|)^\dagger \) therefore its conjugate complex. Then, this leads to the fact that only the real parts survive, which means, we have:

\[
\langle \psi| |\nu\rangle = \text{Re}(\langle \psi| |\nu\rangle) - i \text{Im}(\langle \psi| |\nu\rangle) = \frac{1}{2} \langle \psi| \frac{d\rho}{dt} |\psi\rangle - i\chi(t).
\]  

(65)

The first term on the right-hand side is no problem, we can easily determine it. The second term contains \( \chi(t) \) an arbitrary phase, which we are free to choose. If we now insert the Liouville-von-Neumann equation (54) into equation (65), we find:

\[
\frac{1}{2} \langle \psi| \frac{d\rho}{dt} |\psi\rangle = 0,
\]  

(66)

and therefore \( \langle \psi| |\nu\rangle = -i\chi(t) \). If we insert this result together with the Liouville-von-Neumann equation, we get after some algebra:

\[
|\nu\rangle = -i \frac{\lambda}{\hbar} |H|\psi\rangle - \frac{\lambda}{\hbar} (\langle H| - \langle \psi| |\nu\rangle) + i \frac{\lambda}{\hbar} (|H| - \lambda |\psi\rangle |H| - \lambda |\psi\rangle |H|)
\]

\[
|\nu\rangle = \chi(t)|\psi\rangle.
\]  

(67)

where \( \langle H| = \langle \psi| |H| \psi\rangle \). Now, we still have the phase \( \chi(t) \) which we can choose. If we set \( \lambda \chi(t) = \langle H| \), the last term disappears, and we find as a final result the differential equation:

\[
|d\psi\rangle = |\psi\rangle dt + |f\rangle d\xi = -i \frac{\lambda}{\hbar} (|H|\psi\rangle - \lambda (\psi\rangle |H|) d\tau.
\]  

(68)

This means we have reproduced the Gisin-Schrödinger equation and have shown that this differential equation is not a stochastic differential equation. That still holds up if we add a stochastic Zeeman term in the Hamiltonian, as proposed in [22, 28] and as it is usual for the Landau–Lifshitz–Gilbert equation [29]. Gisin has shown that if, we add a stochastic term, in the sense of Ito, then the time evolution of the spin expectation values follow the trajectories of the Bloch equation [30]. The same is true for the Landau–Lifshitz equation as has been shown by Kubo [31, 32]. This means, in those cases, we have a connection to the Lindblad equation and decoherence.
6. Comment on the practical use

Until this point, we have derived several Schrödinger equations with friction, respectively damping. Within this subsection we discuss the practical usability of these Schrödinger equations.

In case of the equations proposed by Caldirola–Kanai and Schuch, we deal with wave functions $\psi(r, t)$, which depend on the space coordinate $r$ and the time $t$. In that case, the system can be discretized, and we can perform finite element or finite difference calculations using several methods to solve the Schrödinger equation. Here, we are restricted by computational power. However, we can calculate reliable system sizes and several problems like [22, 33, 34].

The situation is different in case of a spin system. The actual limit in case of a spin $S = 1/2$ system is 40 spins using exact diagonalization. In comparison to the 40 spins, we have approximately 4 million spins in case of the classical description [35–39]. The reason for the restriction to a small number of quantum spins are the large matrices we have to deal with. In general, for a spin system with $N$ spins with spin quantum number $S$ the dimension of the Hamilton operator matrix is $(2S + 1)^N \times (2S + 1)^N$. However, within one of our previous publications, we have shown how to extend the description of 40 spin $S = 1/2$ system to 40000 spins $S = 1$ [23]. Furthermore, this system size is not even the computational limit. In principle, with the given method, we can treat system sizes similar to sizes of classical spin dynamics simulations.

In our previous publication, we have described the system with the aid of the quantum mean-field theory. This means the spins do not interact directly as usual in case of the Heisenberg or Ising model. Instead, the spins experience only a mean-field additional to the external field. This means, we have individual spins, and we can write the Hamilton operator of our system as:

$$ H = \sum_{m=1}^{N} h_m, $$

with, e.g., the single spin Hamilton operator

$$ h_n = \bigotimes_{k<n} h \otimes \bigotimes_{l>n} \mathbf{1} $$

of the $n$th spin. Here,

$$ h = -g\mu_B / \hbar (\mathbf{B}_{\text{ext}} + \mathbf{B}_{\text{MF}}) \cdot \mathbf{S} $$

is the mean-field Hamilton operator, and the $\mathbf{1}$ are unity matrices, e.g., $h_1 = h \otimes \mathbf{1} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}$, $h_2 = \mathbf{1} \otimes h \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}$, etc.. Due to the mean-field theory ansatz the Hamilton operator matrix is of the size of a single spin with spin quantum number $S$:

$$ (2S + 1)^2 \times (2S + 1)^2. $$

Then, caused by this individuality, the corresponding wave function $|\Psi(t)\rangle$ is a product state:

$$ |\Psi(t)\rangle = \bigotimes_m |\psi_m(t)\rangle. $$

And, the Gisin–Schrödinger equation

$$ i\hbar (1 + \lambda^2) \frac{d}{dt} |\Psi(t)\rangle = (1 - i\lambda)H|\Psi(t)\rangle + i\lambda \langle \mathcal{H} | \Psi(t)\rangle $$

becomes

$$ \sum_n \left\{ \bigotimes_{k<n} |\psi_k(t)\rangle \otimes \mathcal{L}_n \otimes \bigotimes_{l>n} |\psi_l(t)\rangle \right\} = (*) $$

$$ (*) = \sum_n \left\{ \bigotimes_{k<n} |\psi_k(t)\rangle \otimes \mathcal{R}_n \otimes \bigotimes_{l>n} |\psi_l(t)\rangle \right\}, $$

with

$$ \mathcal{L}_n = \left[ i\hbar \frac{d}{dt} |\psi_n(t)\rangle \right], $$

$$ \mathcal{R}_n = \left[ (1 - i\lambda)h_n|\psi_n(t)\rangle + i\lambda \langle h_n | \psi_n(t)\rangle \right]. $$

Equation (74) means that due to the mean-field Hamilton operator $H$ the product state $|\Psi(t)\rangle$ stays a product state. Moreover, the dynamics of all spins can be calculated separately:

$$ i\hbar (1 + \lambda^2) \frac{d}{dt} |\psi_n(t)\rangle = (1 - i\lambda)h_n|\psi_n(t)\rangle + i\lambda \langle h_n | \psi_n(t)\rangle. $$
This means we have not to deal with all spins at the same time, and also not with large matrices. In that case, we can treat one spin after the other. Thereby, the matrix of the Hamilton operator \( h_n \) has just the size of a single spin, and therefore is reliable.

The disadvantage of this method is that we deal with a product state, which stays a product state during the dynamics. This means, we have no entanglement, and the spin dynamics is the same as for classical spins. Our simulations have also shown that the classical description using solving the Landau–Lifshitz–Gilbert equation is faster because there we immediately calculate the spin expectation values \( \langle S \rangle \). In the case of the quantum mechanical calculation, we first calculate the wave functions and in a second step the spin expectation values. Nevertheless, the described treatment also offers a chance. If \( |\psi_i\rangle \) describes a cluster, which is described without mean-field approximation. In other words, we treat the cluster as usual with the standard Heisenberg operator. Then, \( |\psi_i\rangle \) is a normal quantum state which also shows entanglement. The clusters themselves are coupled by mean-field Hamiltonians as described above. This means we have individual Schrödinger equations for each cluster, but we treat all \( N_C \) spins within the cluster accurately with \( |\psi_i\rangle \). This offers the chance to treat larger spin systems than 40 spins, and we include quantum effects like entanglement in our system, even if just confined within the clusters. Therefore, we have on the one site, the one spin cluster treatment described above, where we have classical spin dynamics. On the other site we have the complete quantum mechanical description without mean-field approximation, which restricts, us to small system sizes. Moreover, we have the cluster mean-field description which combines both descriptions. However, the results will be ‘semi-quantum’, which means semi-classical, but closer to the correct quantum mechanical description than other semi-classical methods.

7. Comment on the PT-symmetry

During the beginning of the twentieth century, several famous physicists have published papers on the subject of quantization in physics, and the foundations of quantum mechanics were established [1–3, 40–42]. The rigorous mathematically formulation of quantum mechanics goes back to Dirac, von Neumann, Jordan, and Weyl [43–46]. One main feature of the formulation made by Dirac et al is that for a system to be physically viable, the corresponding Hamiltonian needs to be Hermitian \( H = H^{\dagger} \). The Hermiticity of \( H \) is a postulate, but the reasons for this is that it guarantees that the eigenvalues of \( H \) are real and that the time evolution operator \( U \) is unitary, \( U^2 = U^{-1} \). The unitarity of \( U \) is necessary for the probability of the system to be conserved. In other words, the unitarity of \( U \) is essential to keep the norm \( n = \langle \psi | \psi \rangle = 1 \) conserved \( dn/dt = 0 \).

In 1928, Gamow made the first striking application of quantum mechanics to the \( \alpha \)-decay of nuclei [47]. For that purpose, Gamow introduced a non-Hermitian Hamilton operator who is, of course, contradictory to postulate above. However, since that publication and the more fundamental work of Wigner and Weisskopf in 1930 non-Hermitian Hamilton operators are established to describe unstable or open quantum systems [48]. As said before the Hermiticity of \( H \) guarantees that the eigenvalues of \( H \) are real. However, in 1959, Wu [49] and 1978, Brower et al [50] have shown that non-Hermitian Hamiltonians can provide real energy eigenvalues, although we expect to find complex energy eigenvalues caused by the non-Hermiticity. In its benchmark paper form 2007, Bender and Boettcher showed that the reason that the spectra of the Hamiltonians are real is the fact that the Hamiltonians are invariant under the inversion of parity \( P \) and time \( T \) [51]. In other words, they are PT-symmetric.

In a serious of publications, Mostafazadeh has shown that possessing PT-symmetry is equivalent to a quantum system having a Hermitian Hamiltonian[52–55]. In other words, we can replace Hermiticity postulate by the postulate that the Hamilton operator \( \mathcal{H} \) needs to be PT-symmetric which includes the Hermitian Hamiltonian operators \( H \).

At this point, we have to ask ourselves how does the Gisin-Schrödinger equation fit into this picture. It is apparent that we can write the Gisin-Schrödinger equation (53) in the form of the initially proposed time-dependent Schrödinger equation

\[
H_\phi = \frac{H - i\lambda(H - \langle H \rangle)}{1 + \lambda^2}
\]  

(76)

where \( \mathcal{H}_\phi \) is a non-Hermitian Hamiltonian operator. The reason why the Gisin-Schrödinger equation enables us to describe quantum spin-systems with damping is that we are bypassing the problems discussed above. We use the Heisenberg Hamilton operator \( H \) to calculate the energy \( E = \langle H \rangle \) of the system. \( H \) itself is assumed to be Hermitian which guarantees us a real energy value. Furthermore, we bypass the conservation of the norm \( n = \langle \psi | \psi \rangle \) by the fact that we normalize \( |\psi\rangle \) during every time step [56]. The normalization procedure leads to the additional \( \langle H \rangle \) term on the right-hand side of the Gisin-Schrödinger equation, which makes this equation a self-consistent equation.
We think that the dynamics still delivers reliable results if Hamilton operator H is not Hermitian but instead PT-symmetric. It seems to be interesting to investigate such scenarios and also scenarios where H is neither Hermitian or PT-symmetric.

8. Summary

Within this publication, we have explained the way Schrödinger has derived the time-independent Schrödinger equation. Also, we have extended this description to derive the time-dependent Schrödinger equation as well. We further proceed to include friction and damping into the Schrödinger equation. Therefore, we first discussed the way how to include friction in Lagrange, respectively Hamilton mechanics. In a second step, we have discussed how to use these transformations to derive the Caldirola-Kanai equation, the first proposed Schrödinger equation with friction. Furthermore, we then have used this idea to get the Schuch-Schrödinger equation. The Schuch-Schrödinger equation can be used to describe, e.g., the motion of a point particle under the influence of friction quantum mechanical.

Additional to the Schuch-Schrödinger equation, we have derived the Gisin-Schrödinger equation, in the same way, starting from the Hamilton-Jacobi equation and using an appropriate ansatz for the action S. At this point, we have left the description of quantum systems in real space and went over to spin systems. Besides the other derivation, we have presented another way to derive the same differential equation using the description of stochastic differential equations. The question, we have answered here was: is the wave equation which corresponds to the Liouville-von-Neumann equation which is the quantum mechanical analog to the classical Landau–Lifshitz equation a stochastic differential equation? The answer is as expected: NO. It is the Gisin-Schrödinger equation, and this is not a stochastic differential equation. At the end of this article, we have commented on the practical use of the Gisin-Schrödinger equation. Here, we have shown that a mean-field description leads to a classical spin dynamics if we use the 'traditional' mean-field description, which means a single spin which sees the mean-field created by the surrounding spins. In that case, we cannot expect quantum effects like entanglement. However, if we use a cluster mean field instead quantum effects are included and at the same time, we can describe spin systems which contain more than 40 spins.

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