Abstract

We construct a new discrete analog of the Dirac-Kähler equation in which some key geometric aspects of the continuum counterpart are captured. We describe a discrete Dirac-Kähler equation in the intrinsic notation as a set of difference equations and prove several statements about its decomposition into difference equations of Duffin type. We study an analog of gauge transformations for the massless discrete Dirac-Kähler equations.

Key words and phrases: Dirac-Kähler equation, Dirac operator, difference equations, discrete models

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1 Introduction

The problem of finding discrete analogies of various continuous models is of considerable interest in physical theories. In this work we study a discrete counterpart of the Dirac-Kähler equation which describes fermion fields in terms of inhomogeneous differential forms. It is well-known that a very useful way to discretize quantum field equations is provided by a lattice approach. However, there are some difficulties in capturing fundamental properties of differential operators on the lattice, for example, the differential defined as a difference operator in the lattice formulation does not satisfy the Leibniz rule. We propose a discretization scheme based on the use of the differential form language in which the exterior derivative $d$, the Hodge star operator $\ast$ and the exterior product $\wedge$ of differential forms are replaced by their discrete analogies. The algebraic relations between these operators are captured in the proposed discrete model. We are interested in a discrete Dirac-Kähler equation only from the mathematics point of view. Our approach bases on the differential geometric formalism proposed by Dezin [9]. We adapt a combinatorial model of Minkowski space from \cite{23} and define discrete analogs of the operators $d$, $\ast$ and $\wedge$ in the
same way as in [22]. Alternative geometric discretisation schemes which use also a discrete exterior calculus has already been developed by several authors [3, 10, 19, 20, 24, 25]. In the lattice formulation the first attempts to construct a discrete model of the Dirac-Kähler equation based on differential geometric methods were done by Rabin [19] and by Becher and Joos [3]. In a sequel of papers [1, 2, 4, 6, 7, 8, 11] the Dirac-Kähler equation on the lattice has been extensively studied. Our approach is close to one proposed by Rabin [19]. The obtained discrete Dirac-Kähler and Duffin equations are formally the same as in [19]. However, our discrete analog of the Hodge star operator is different than that given in [19]. In [19], this operation is defined by using dual lattice. We define a star operation by using a discrete analog of the exterior product. This improvement allows us to preserve the Lorentz metric structure in our discrete model and introduce an inner product of discrete forms (cochains). We may therefore construct a discrete codifferential so as to be the formal adjoint of a coboundary operator. Consequently, we obtain more precise difference equations.

The main purpose of this paper is to construct a discrete model of the Dirac-Kähler equation which preserves the geometric structure of the continuum counterpart. We show that some key properties of the Dirac-Kähler system that hold in the smooth setting also hold in the discrete case. It is known [5] that the Dirac-Kähler equation decomposes uniquely into four uncoupled equations of Duffin type. We prove the same in the discrete case. Recently much attention has been directed to the study of the massless Dirac-Kähler field [14, 16, 18]. From the physics point of view the Dirac-Kähler system has three massless limits [14]. In [16, 18], it has shown that the Kalb-Ramond field (notoph) equations and the electromagnetic equations are partial cases of the Dirac-Kähler massless equation. We formulate a discrete version of these massless cases. It is well known that the electromagnetic field is invariant under the gauge transformations (see for instance [17]). We study analogies of these transformations for the discrete model.

2 Smooth settings

Let \( M = \mathbb{R}^{1,3} \) be Minkowski space with metric signature \((+,-,-,-)\). Denote by \( \Lambda^r(M) \) the vector space of smooth differential \( r \)-form, \( r = 0, 1, 2, 3, 4 \). Let \( \hat{\omega} \) and \( \hat{\varphi} \) be \( r \)-forms on \( M \). The inner product is defined by

\[
(\hat{\omega}, \hat{\varphi}) = \int_M \hat{\omega} \wedge \ast \hat{\varphi},
\]

where \( \wedge \) is the exterior product and \( \ast \) is the Hodge star operator \( \ast : \Lambda^r(M) \to \Lambda^{4-r}(M) \) (with respect to the Lorentz metric). Let \( d : \Lambda^r(M) \to \Lambda^{r+1}(M) \) be the exterior differential and let \( \delta : \Lambda^r(M) \to \Lambda^{r-1}(M) \) be the formal adjoint of \( d \) with respect to (2.1) (the codifferential). We have \( \delta = -\ast d \ast \). Then the Laplacian (Laplace-Beltrami operator) acting on \( r \)-forms is defined by

\[
\Delta \equiv -d\delta - \delta d : \Lambda^r(M) \to \Lambda^r(M).
\]
It is clear that \(-d\delta - \delta d = (d - \delta)^2\). For Minkowski space the operator \(\Delta\) is the d’Alembert (wave) operator. Then for any \(r\)-form \(\omega \in \Lambda^r(M)\) the Klein-Gordon equation can be written as

\[
\Delta^r \omega = m^2 \omega, \tag{2.3}
\]

where \(m\) is a mass parameter. Denote by \(\Lambda(M)\) the set of all differential forms on \(M\). We have \(\Lambda(M) = \Lambda^0(M) \oplus \Lambda^1(M) \oplus \Lambda^2(M) \oplus \Lambda^3(M) \oplus \Lambda^4(M)\). Let \(\Omega \in \Lambda(M)\) be an inhomogeneous differential form. This form can be expanded as

\[
\Omega = \sum_{r=0}^{4} \tilde{\omega}^r,
\]

where \(\tilde{\omega} \in \Lambda^r(M)\). The Dirac-Kähler equation is given by

\[
(d - \delta)\Omega = m\Omega. \tag{2.4}
\]

First proposed by Kähler [13], this equation is the generalization of the Dirac equation. It is easy to show, that Equation (2.4) is equivalent to the following equations

\[
-d^1 \omega^0 = m \omega^0,
\]
\[
d^0 \omega^2 - \delta^2 \omega = m \omega^2,
\]
\[
d^1 \omega^3 - \delta^3 \omega = m \omega^3,
\]
\[
d^2 \omega^4 - \delta^4 \omega = m \omega^4,
\]
\[
d^3 \omega = m \omega.
\]

There are three massless limits of this system [18]. Let us consider \(m = \sqrt{m_1 m_2}\). Then we obtain the massless Dirac-Kähler equation in the case if any of mass parameters \(m_1, m_2\) (or both simultaneously) is equal to zero.

### 3 Combinatorial model of Minkowski space

Following [9], let the tensor product \(C(4) = C \otimes C \otimes C \otimes C\) of a 1-dimensional complex be a combinatorial model of Euclidean space \(\mathbb{R}^4\). The 1-dimensional complex \(C\) is defined in the following way. Let \(C^0\) denotes the real linear space of 0-dimensional chains generated by basis elements \(x_\kappa\) (points), \(\kappa \in \mathbb{Z}\). It is convenient to introduce the shift operators \(\tau, \sigma\) in the set of indices by

\[
\tau \kappa = \kappa + 1, \quad \sigma \kappa = \kappa - 1. \tag{3.1}
\]

We denote the open interval \((x_\kappa, x_{\tau \kappa})\) by \(e_{\kappa}\). One can regard the set \(\{e_\kappa\}\) as a set of basis elements of the real linear space \(C^1\). Suppose that \(C^1\) is the space of 1-dimensional chains. Then the 1-dimensional complex (combinatorial real
line) is the direct sum of the introduced spaces $C = C^0 \oplus C^1$. The boundary operator $\partial$ in $C$ is given by

$$\partial x_\kappa = 0, \quad \partial e_\kappa = x_{\tau \kappa} - x_\kappa.$$ 

The definition is extended to arbitrary chains by linearity.

Multiplying the basis elements $x_\kappa, e_\kappa$ in various way we obtain basis elements of $C(4)$. Let $s_k$ be an arbitrary basis element of $C(4)$. Then we have $s_k = s_{k_0} \otimes s_{k_1} \otimes s_{k_2} \otimes s_{k_3}$, where $s_{k_i}$ is either $x_{k_i}$ or $e_{k_i}$ and $k_i \in \mathbb{Z}$. Here $k = (k_0, k_1, k_2, k_3)$ is a multi-index. The 1-dimensional basis elements of $C(4)$ can be written as

$$e_1^k = e_{k_0} \otimes e_{k_1} \otimes x_{k_2} \otimes x_{k_3}, \quad e_2^k = e_{k_0} \otimes x_{k_1} \otimes e_{k_2} \otimes x_{k_3}, \quad e_3^k = e_{k_0} \otimes x_{k_1} \otimes x_{k_2} \otimes e_{k_3}, \quad e_4^k = e_{k_0} \otimes x_{k_1} \otimes x_{k_2} \otimes x_{k_3},$$

where the superscript $i$ indicates a place of $e_k$ in $e_k$ and $i = 0, 1, 2, 3$. In the same way we will write the 2-dimensional basis elements of $C(4)$ as

$$e_1^k = e_{k_0} \otimes e_{k_1} \otimes e_{k_2} \otimes x_{k_3}, \quad e_2^k = e_{k_0} \otimes e_{k_1} \otimes x_{k_2} \otimes e_{k_3}, \quad e_3^k = e_{k_0} \otimes x_{k_1} \otimes e_{k_2} \otimes x_{k_3}, \quad e_4^k = e_{k_0} \otimes x_{k_1} \otimes x_{k_2} \otimes e_{k_3},$$

Denote by $e_k^{01}, e_k^{012}, e_k^{013}, e_k^{021}, e_k^{123}$ the 3-dimensional basis elements of $C(4)$.

Let $C(4) = C(p) \otimes C(q)$, where $p + q = 4$. If $a \in C(p)$ and $b \in C(q)$ are arbitrary chains, belonging to the complexes being multiplied, then we extend the definition of $\partial$ to chains of $C(4)$ by the rule

$$\partial(a \otimes b) = \partial a \otimes b + (-1)^r a \otimes \partial b,$$

where $r$ is the dimension of the chain $a$.

Suppose that the combinatorial model of Minkowski space has the same structure as $C(4)$. We will use the index $k_0$ to denote the basis elements of $C$ which correspond to the time coordinate of $M$. Hence the indicated basis elements will be written as $x_{k_0}, e_{k_0}$.

Let us now consider a dual complex to $C(4)$. We define its as the complex of cochains $K(4)$ with real coefficients. The complex $K(4)$ has a similar structure, namely $K(4) = K \otimes K \otimes K \otimes K$, where $K$ is a dual complex to the 1-dimensional complex $C$. We will write the basis elements of $K$ as $x^\kappa$ and $e^\kappa$, $\kappa \in \mathbb{Z}$. Then an arbitrary basis element of $K(4)$ can be written as $s^\kappa = s^{k_0} \otimes s^{k_1} \otimes s^{k_2} \otimes s^{k_3}$, where $s^{k_0}$ is either $x^{k_0}$ or $e^{k_0}$. Denote by $s^{k_0}_{(r)}$ an $r$-dimensional basis element of $K(4)$. Here the symbol $(r)$ contains the whole requisite information about the number and situation of $e^{k_0} \in K$ in $s^{k_0}_{(r)}$. For example, the 1-dimensional basic elements $e_k^1 \in K(4)$ and the 2-dimensional basic elements $e_k^{ij} \in K(4)$ have the form (3.2) and (3.3) respectively. We will call cochains forms, emphasizing their relationship with the corresponding continual objects, differential forms.
Denote by $K^r(4)$ the set of forms of degree $r$. We can represent $K(4)$ as

$$K(4) = K^0(4) \oplus K^1(4) \oplus K^2(4) \oplus K^3(4) \oplus K^4(4).$$

Let $\omega \in K^r(4)$. Then we have

$$0 \omega = \sum_k 0 \omega_k x^k,$$

where $x^k = x^{k_0} \otimes x^{k_1} \otimes x^{k_2} \otimes x^{k_3}$, \hspace{1cm} \hspace{1cm} (3.5)

$$1 \omega = \sum_k \sum_i 0 \omega_i^k e^i_k,$$

$$2 \omega = \sum_k \sum_{i<j} 0 \omega_{ij}^k e^i_k e^j_k,$$

$$3 \omega = \sum_k \sum_{i<j<l} 0 \omega_{ijl}^k e^i_k e^j_k e^l_k,$$

$$4 \omega = \sum_k 0 \omega^k e^k_k,$$

where $e^k_k = e^{k_0}_k \otimes e^{k_1}_k \otimes e^{k_2}_k \otimes e^{k_3}_k$. \hspace{1cm} \hspace{1cm} (3.6)

Here the components $0 \omega_k, 0 \omega_i^k, 0 \omega_{ij}^k, 0 \omega_{ijl}^k$ and $0 \omega^k$ are real numbers.

As in \cite{9}, we define the pairing (chain-cochain) operation for any basis elements $\varepsilon_k \in C(4), s^k \in K(4)$ by the rule

$$\langle \varepsilon_k, s^k \rangle = \begin{cases} 0, & \varepsilon_k \neq s_k \\ 1, & \varepsilon_k = s_k. \end{cases} \hspace{1cm} (3.8)$$

The operation (3.8) is linearly extended to arbitrary chains and cochains.

The coboundary operator $d^c : K^r(4) \rightarrow K^{r+1}(4)$ is defined by

$$\langle \partial a, \tilde{\omega} \rangle = \langle a, d^c \tilde{\omega} \rangle,$$

where $a \in C(4)$ is an $r + 1$ dimensional chain. The operator $d^c$ is an analog of the exterior differential. From the above it follows that

$$d^c \omega = 0 \quad \text{and} \quad d^c d^c \omega = 0 \quad \text{for any} \quad r.$$

Let us introduce for convenient the shifts operator $\tau_i$ and $\sigma_i$ as

$$\tau_0 k = (k_0, \ldots, \tau k_i, \ldots, k_3), \quad \sigma_0 k = (k_0, \ldots, \sigma k_i, \ldots, k_3), \quad i = 0, 1, 2, 3,$$

where $\tau$ and $\sigma$ are defined by (3.1). Using (3.8) and (3.9) we can calculate

$$d^c \omega = \sum_k \sum_{i=0}^3 (\Delta_i \omega_{k}^0) e^i_k,$$

$$d^c \omega = \sum_k \sum_{i<j} (\Delta_i \omega_{ij}^k) e^i_k e^j_k,$$

$$d^c \omega = \sum_k \sum_{i<j} \left[ (\Delta_{0} \omega_{k}^{12} - \Delta_{1} \omega_{k}^{02} + \Delta_{2} \omega_{k}^{01}) e^0_{12}^{k} \\
+(\Delta_{0} \omega_{k}^{13} - \Delta_{1} \omega_{k}^{03} + \Delta_{3} \omega_{k}^{01}) e^0_{13}^{k} \\
+(\Delta_{0} \omega_{k}^{23} - \Delta_{2} \omega_{k}^{03} + \Delta_{3} \omega_{k}^{02}) e^0_{23}^{k} \\
+(\Delta_{1} \omega_{k}^{23} - \Delta_{2} \omega_{k}^{13} + \Delta_{3} \omega_{k}^{12}) e^0_{123}^{k} \right].$$

(3.13)
\[ d^c \omega^3 = \sum_k (\Delta_0 \omega_k^{123} - \Delta_1 \omega_k^{023} + \Delta_2 \omega_k^{013} - \Delta_3 \omega_k^{012}) e^k, \]  
(3.14)

where \( \Delta_i \) is the difference operator defined by

\[ \Delta_i \omega_k = \omega_{r,k} - \omega_k \]  
(3.15)

for any components \( \omega_k \) of \( \omega \). For simplicity of notation we write here \( \omega_k \) instead of \( \omega_k^{(r)} \).

Let us now introduce in \( K(4) \) a multiplication which is an analog of the exterior multiplication for differential forms. Denote by \( K(r) \) the \( r \)-dimensional complex, \( r = 1, 2, 3 \). We define the \( \cup \)-multiplication by induction on \( r \). Suppose that the \( \cup \)-multiplication in \( K(r) \) has been defined. Then we introduce it for basis elements of \( K(r + 1) \) by the rule

\[ (s^k_{(p)} \otimes s^\kappa) \cup (s^k_{(q)} \otimes s^\mu) = Q(\kappa, q)(s^k_{(p)} \cup s^k_{(q)}) \otimes (s^\kappa \cup s^\mu), \]  
(3.16)

where \( s^k_{(p)}, s^k_{(q)} \in K(r) \), \( s^\kappa(s^\mu) \) is either \( x^\kappa(x^\mu) \) or \( e^\kappa(e^\mu) \), \( \kappa, \mu \in \mathbb{Z} \), and the signum function \( Q(\kappa, q) \) is equal to +1 if the dimension of both elements \( s^\kappa \), \( s^\mu \) is odd and to +1 otherwise. For the basis elements of \( K(1) = K \) the \( \cup \)-multiplication is defined as follows

\[ x^\kappa \cup x^\kappa = x^\kappa, \quad e^\kappa \cup x^\kappa = e^\kappa, \quad x^\kappa \cup e^\kappa = e^\kappa, \quad \kappa \in \mathbb{Z}, \]

supposing the product to be zero in all other case. To arbitrary forms the \( \cup \)-multiplication can be extended linearly.

**Proposition 3.1.** Let \( \varphi \) and \( \psi \) be arbitrary forms of \( K(4) \). Then

\[ d^c(\varphi \cup \psi) = d^c \varphi \cup \psi + (-1)^r \varphi \cup d^c \psi, \]  
(3.17)

where \( r \) is the degree of a form \( \varphi \).

The proof can be found in [9, p.147]. Note that Relation (3.17) is a discrete version of the Leibniz rule for differential forms.

By definition, the coboundary operator \( d^r \) and the \( \cup \)-multiplication do not depend on a metric. Hence they have the same structure in \( K(4) \) as in the case of the combinatorial Euclidean space. At the same time, to define a discrete analog of the Hodge star operator \( * \) we must take into account the Lorentz metric structure on \( K(4) \). Define the operation \( * : K^r(4) \to K^{4-r}(4) \) for an arbitrary basis element \( s^k = s^{k_0} \otimes s^{k_1} \otimes s^{k_2} \otimes s^{k_3} \) by the rule

\[ s^k \cup * s^k = Q(k_0) e^k, \]  
(3.18)

where \( Q(k_0) \) is equal to +1 if \( s^{k_0} = x^{k_0} \) and to \( -1 \) if \( s^{k_0} = e^{k_0} \). For example, for the 1-dimensional basis elements \( e^k_i \) we have \( e^k_0 \cup e^k_0 = -e^k \) and \( e^k_i \cup e^k_i = e^k \) for \( i = 1, 2, 3 \). Recall that \( e^k = e^{k_0} \otimes e^{k_1} \otimes e^{k_2} \otimes e^{k_3} \) is the 4-dimensional basic
element of $K(4)$. Relation (3.18) preserves the Lorentz signature of metric in our discrete model. From (3.18) we obtain

$$*x^k = *(x^{k_0} \otimes x^{k_1} \otimes x^{k_2} \otimes x^{k_3}) = e^k,$$

(3.19)

$$*e^k = -x^{\tau k_0} \otimes x^{\tau k_1} \otimes x^{\tau k_2} \otimes x^{\tau k_3} = -x^{\tau k},$$

(3.20)

$$*e_0 = -e_{123}, \quad *e_1 = -e_{023}, \quad *e_2 = e_{013}, \quad *e_3 = -e_{012},$$

(3.21)

$$*e_{01} = -e_{012}, \quad *e_{02} = e_{012}, \quad *e_{03} = -e_{012},$$

(3.22)

$$*e_{012} = -e_{012}, \quad *e_{013} = e_{012}, \quad *e_{023} = -e_{012}, \quad *e_{123} = -e_{012}. \quad (3.23)$$

Here we use $\tau_{ij}$ and $\tau_{ij}$ to denote the operators which shift to the right the indicated components of $k = (k_0, k_1, k_2, k_3)$, for example,

$$\tau_{12} k = (k_0, \tau k_1, \tau k_2, k_3), \quad \tau_{023} k = (\tau k_0, k_1, \tau k_2, k_3),$$

and $\tau k = (\tau k_0, \tau k_1, \tau k_2, k_3)$. It is easy to check that

$$**s^k_{(r)} = (-1)^{r+1}s^{\tau k}_{(r)},$$

where $s^k_{(r)}$ is an $r$-dimensional basic element of $K(4)$. Then if we perform the $*$ operation twice on any $r$-form $\omega \in K(4)$, we obtain

$$**\omega = ** \sum_k \omega^{(r)}_k s^k_{(r)} = (-1)^{r+1} \sum_k \omega^{(r)}_k s^k_{(r)} = (-1)^{r+1} \sum_k \omega^{(r)}_k s^k_{(r)},$$

where $\sigma k = (\sigma k_0, \sigma k_1, \sigma k_2, \sigma k_3)$.

Let $V \subset C(4)$ be some fixed "domain" of the complex $C(4)$. We can written $V$ as follows

$$V = \sum_k e_k, \quad k = (k_0, k_1, k_2, k_3), \quad k_i = 1, 2, ..., N_i,$$

(3.24)

where $e_k = e_{k_0} \otimes e_{k_1} \otimes e_{k_2} \otimes e_{k_3}$ is the 4-dimensional basis element of $C(4)$. We agree that in what follows the subscripts $k_i, i = 0, 1, 2, 3$, always run the set of values indicated in (3.24). Suppose that the forms (3.23) are vanished on $C(4) \setminus V$, i.e., if $k_l < 1$ or $k_l > N_l$ then $\omega^{(r)}_k = 0$ for any $r$-form $\omega \in K(4)$. For forms $\tilde{\varphi}, \tilde{\omega} \in K^r(4)$ of the same degree $r$ the inner product is defined by the relation

$$(\tilde{\varphi}, \tilde{\omega})_V = \langle V, \tilde{\varphi} \cup *\tilde{\omega} \rangle.$$

(3.25)

For the forms of different degrees the product (3.25) is set equal to zero. See also [21]. The definition imitates correctly the continual case (2.1). Using (3.28)
and (3.19–3.23) we obtain
\[
(\omega, \omega)_V = \sum_k (\omega_k^0)^2,
\]
\[
(\omega, \omega)_V = \sum_k [-(\omega_k^0)^2 + (\omega_k^1)^2 + (\omega_k^2)^2 + (\omega_k^3)^2],
\]
\[
(\omega, \omega)_V = \sum_k [-(\omega_k^{01})^2 - (\omega_k^{02})^2 - (\omega_k^{03})^2 + (\omega_k^{12})^2 + (\omega_k^{13})^2 + (\omega_k^{23})^2],
\]
\[
(\omega, \omega)_V = \sum_k [-(\omega_k^{012})^2 - (\omega_k^{013})^2 - (\omega_k^{023})^2 + (\omega_k^{123})^2],
\]
\[
(\omega, \omega)_V = -\sum_k (\omega_k^4)^2.
\]

**Proposition 3.2.** Let \( \varphi \in K^r(4) \) and \( \psi \in K^{r+1}(4) \). Then we have
\[
(d^c \varphi, \psi)_V = (\varphi, d^c \psi)_V,
\]
where
\[
d^c \psi = (-1)^{r+1} \ast^{-1} d^c \ast \psi
\]
is the operator formally adjoint of \( d^c \).

**Proof.** From (3.9, 3.17) and (3.23) we obtain
\[
(d^c \varphi, \psi)_V = \langle V, d^c \varphi \cup \ast \psi \rangle = \langle V, (d^c(\varphi \cup \ast \psi) - (-1)^r \varphi \cup d^c \ast \psi) \rangle
\]
\[
= \langle \partial V, \varphi \cup \ast \psi \rangle + (-1)^{r+1} \langle V, \varphi \cup \ast(\ast^{-1} d^c \ast \psi) \rangle
\]
\[
= \langle \partial V, \varphi \cup \ast \psi \rangle + (-1)^{r+1} \langle \varphi, \ast^{-1} d^c \ast \psi \rangle_V,
\]
where we used \( \ast \ast^{-1} = 1 \). It remains to prove that \( \langle \partial V, \varphi \cup \ast \psi \rangle = 0 \). Using (3.24) we can calculate
\[
\partial e_k = \partial(e_{k0} \otimes e_{k1} \otimes e_{k2} \otimes e_{k3})
\]
\[
= \epsilon^{123}_{r_0 k} - \epsilon^{123}_{r_1 k} e_{023} + \epsilon^{023}_{r_2 k} e_{k1} - \epsilon^{013}_{r_3 k} e_{012} + \epsilon^{012}_{r_3 k}.
\]
From this we obtain
\[
\partial V = \sum_k (\epsilon^{123}_{r_0 k} e_{k1} k_2 k_3 - \epsilon^{123}_{r_1 k} e_{k1} k_2 k_3 - \epsilon^{023}_{r_2 k} e_{k1} k_2 k_3 + \epsilon^{023}_{r_3 k} + \epsilon^{013}_{r_3 k} e_{k1} k_2 k_3 - \epsilon^{013}_{k_0 k_1} \tau_{N_k} k_2 - \epsilon^{013}_{k_0 k_1} \tau_{N_k} k_3 - \epsilon^{013}_{k_0 k_1} \tau_{N_k} k_3 + \epsilon^{012}_{k_0 k_1} \tau_{N_k} k_2 + \epsilon^{012}_{k_0 k_1} \tau_{N_k} k_2).
\]
Then if we compute the 3-form \( \varphi \cup \ast \psi \) on \( \partial V \), we obtain the expression which consists of only the terms
\[
\varphi_{k_0 \ldots k_3} \psi_{k_0 \ldots k_3}, \quad \varphi_{k_0 \ldots k_3} \psi_{k_0 \ldots k_3},
\]
where \( \varphi_{k_0 k_1 k_2 k_3} = \varphi_k \) and \( \psi_{k_0 k_1 k_2 k_3} = \psi_k \) are components of the forms \( \varphi \) and \( \psi \).
Since by assumption, we have \( \varphi_{k_0 \ldots k_3} = \psi_{k_0 \ldots k_3} = 0 \) for all \( i = 0, 1, 2, 3 \), it follows that \( \langle \partial V, \varphi \cup \ast \psi \rangle = 0 \). \( \square \)
The operator $\delta^r : K^{r+1}(4) \to K^r(4)$ is a discrete analog of the codifferential $\delta$. For the 0-form (3.3.5) we have $\delta^r \omega = 0$. Using (3.11)–(3.14) and (3.26) we can calculate

$$\delta^1 \omega = \sum_k (-\Delta_0 \omega_{\sigma_0 k}^0 + \Delta_1 \omega_{\sigma_1 k}^1 + \Delta_2 \omega_{\sigma_2 k}^2 + \Delta_3 \omega_{\sigma_3 k}^3) x^k,$$

(3.28)

$$\delta^2 \omega = \sum_k [(\Delta_1 \omega_{\sigma_1 k}^0 + \Delta_2 \omega_{\sigma_2 k}^0 + \Delta_3 \omega_{\sigma_3 k}^0) e_0^k + \Delta_0 \omega_{\sigma_0 k}^0] + \Delta_1 \omega_{\sigma_1 k}^0 - (\Delta_1 \omega_{\sigma_1 k}^0 + \Delta_3 \omega_{\sigma_3 k}^0) e_0^k + \Delta_0 \omega_{\sigma_0 k}^0 - \Delta_1 \omega_{\sigma_1 k}^0 + \Delta_3 \omega_{\sigma_3 k}^0 e_0^k + \Delta_0 \omega_{\sigma_0 k}^0 - \Delta_1 \omega_{\sigma_1 k}^0 - \Delta_3 \omega_{\sigma_3 k}^0 e_0^k],$$

(3.29)

$$\delta^3 \omega = \sum_k (\Delta_2 \omega_{\sigma_2 k}^0 + \Delta_3 \omega_{\sigma_3 k}^0) e_0^k + (\Delta_0 \omega_{\sigma_0 k}^0 + \Delta_3 \omega_{\sigma_3 k}^0) e_0^k + (\Delta_0 \omega_{\sigma_0 k}^0 + \Delta_3 \omega_{\sigma_3 k}^0) e_0^k + (\Delta_0 \omega_{\sigma_0 k}^0 + \Delta_3 \omega_{\sigma_3 k}^0) e_0^k + (\Delta_0 \omega_{\sigma_0 k}^0 + \Delta_3 \omega_{\sigma_3 k}^0) e_0^k,$$

(3.30)

$$\delta^4 \omega = \sum_k [ - (\Delta_3 \omega_{\sigma_3 k}^0) e_0^k + (\Delta_2 \omega_{\sigma_2 k}^0) e_0^k + \Delta_1 \omega_{\sigma_1 k}^0] e_0^k - (\Delta_1 \omega_{\sigma_1 k}^0) e_0^k - (\Delta_3 \omega_{\sigma_3 k}^0) e_0^k].$$

It is obvious that $\delta^r \delta^r \omega = 0$ for any $r = 1, 2, 3, 4$.

A discrete analog of the Laplace-Beltrami operator (2.22) is defined by

$$\Delta^c = -(d^c \delta^c + \delta^c d^c) : K^r(4) \to K^r(4).$$

(3.32)

We have

$$- (d^c \delta^c + \delta^c d^c) = (d^c - \delta^c)^2.$$

(3.33)

**Proposition 3.3.** The operator $\Delta^c$ is self adjoint for all forms supported in $V$, i.e.,

$$(\Delta^c \varphi, \psi)_V = (\varphi, \Delta^c \psi)_V.$$

(3.34)

**Proof.** By (3.26) it is obvious.

Due to the Lorentz signature of the metric of $K(4)$ the operator $\Delta^c$ is a discrete analog of the d’Alembert (wave) operator. Then for any $r$-form $\tilde{\omega}$ a discrete analog of the Klein-Gordon equation (2.3) can be written as

$$\Delta^c \tilde{\omega} = m^2 \tilde{\omega},$$

(3.35)

where $m$ is a mass parameter.
4 Discrete Dirac-Kähler equation

Define a discrete inhomogeneous form as follows

$$\Omega = \sum_{r=0}^{4} r \tilde{\omega},$$  \hspace{1cm} (4.1)

where $\tilde{\omega}$ is given by (3.5)–(3.7). Let us consider the discrete Klein-Gordon equation on these inhomogeneous forms:

$$\Delta^c \Omega = m^2 \Omega.$$  \hspace{1cm} (4.2)

It is clear that Equation (4.2) is equivalent to five equations (3.35) for $r = 0, 1, 2, 3, 4$. Using (3.33) it is possible to introduce a discrete analog of the Dirac-Kähler equation (2.4) by the rule

$$(d^c - \delta^c) \Omega = m \Omega.$$  \hspace{1cm} (4.3)

We can write this equation more explicitly by separating its homogeneous components as

$$-\delta^c 1 \omega = m^0 \omega,$$
$$d^c 0 \omega - \delta^c 2 \omega = m^1 \omega,$$
$$d^c 1 \omega - \delta^c 3 \omega = m^2 \omega,$$
$$d^c 2 \omega - \delta^c 4 \omega = m^3 \omega,$$
$$d^c 3 \omega = m^4 \omega.$$  \hspace{1cm} (4.4)
where $\Omega, \Phi$ are given by (4.1). Given the inner product of inhomogeneous forms, the operator

$$D_{\Omega} \not\ni \Delta \rightarrow K \ni \delta : K(4) \rightarrow K(4)$$

and its formally adjoint. Note that the operator (4.6) does not respect the degree of forms because $d^c : K^r(4) \rightarrow K^{r+1}(4)$ and $\delta^c : K^{r+1}(4) \rightarrow K^r(4)$.

Thus it is not possible to define $D_\Omega$ on $K^r(4)$, i.e., on homogeneous $r$-forms.

**Proposition 4.1.** The operator $D_{\Omega}$ is self adjoint and the operator $D_{\Omega}$ is anti self adjoint with respect to the inner product (4.5), i.e.,

$$\langle D_{\Omega} \Omega, \, \Phi \rangle_V = \pm \langle \Omega, \, D_{\Omega} \Phi \rangle_V.$$
Proof. We have
\[
(D^c_{\pm \Omega}, \Phi)_V = (d^c_0 \omega, \phi)_V + (d^c_1 \omega, \phi)_V + (d^c_2 \omega, \phi)_V + (d^c_3 \omega, \phi)_V \pm (\delta^c_1 \omega, 0 \phi)_V \pm (\delta^c_2 \omega, 1 \phi)_V \pm (\delta^c_3 \omega, 2 \phi)_V \pm (\delta^c_4 \omega, 3 \phi)_V.
\]
By (3.26) this implies (4.7) immediately.

5 Decomposition of the discrete Dirac-Kähler equation

Let us introduce a discrete inhomogeneous form of type
\[
\Omega = \omega + \tau \omega,
\]
where \(\tau\) is an \(r\)-form of type (3.5)–(3.7) and \(\tau\) is given by (3.1). By analogy with the continual case [5] we define a discrete analog of the Duffin type equation by the rule
\[
(d^c - \delta^c) \Omega = m \Omega.
\]
This equation is equivalent to the following two equations
\[
\begin{align*}
d^c \omega &= m \tau \omega, \\
-\delta^c \omega &= m \tau \omega.
\end{align*}
\]
For \(r = 0\) we have a discrete analog of the scalar Duffin equation and for \(r = 1\) we have a discrete analog of the vector Duffin equation (or the Proca equation). Similarly, if \(r = 2\) and \(r = 3\) we obtain discrete analogs of the pseudovector and pseudoscalar Duffin equations. On the Duffin-Kemmer-Petiau formulation of Klein-Gordon and field equations see [12, 15] and references therein.

**Theorem 5.1.** If the inhomogeneous form (5.1) is a solution of the discrete Duffin equation (5.2), then the \(r\)-form \(\omega\) satisfies the discrete Klein-Gordon equation (3.35) for any \(r = 0, 1, 2, 3\).

Proof. Let \(\Omega\) is a solution of (5.2). From (3.26) we obtain
\[
\delta^c d^c \omega = m \delta^c \tau \omega = -m \tau \omega.
\]
Since \(\delta^c \delta^c = 0\), from the second equation of (5.3) we have \(\delta^c \omega = 0\). Hence it follows that
\[
\delta^c d^c \omega + d^c \delta^c \omega = -m \tau \omega.
\]
Set
\[ r\omega = r\omega_1 + r\omega_2, \]  
(5.4)
where
\[ r\omega_1 = \omega + \frac{1}{m}\delta^c r\omega \quad \text{and} \quad r\omega_2 = -\frac{1}{m}\delta^c r\omega \]
for any \( r = 1, 2, 3 \). Let us construct the forms
\[ \begin{align*}
01\Omega &= \omega + \omega_1, \\
12\Omega &= \omega_2 + \omega, \\
23\Omega &= \omega_2 + \omega_1, \\
34\Omega &= \omega_2 + \omega. 
\end{align*} \]
(5.5)
According to (5.4) it turns out that
\[ 01\Omega + 12\Omega + 23\Omega + 34\Omega = \sum_{r=0}^{4} r\omega = \Omega. \]
(5.6)

**Theorem 5.2.** Let \( \Omega \) is given by (5.5). Then \( \Omega \) satisfies the discrete Duffin equation (5.2) for \( r = 0, 1, 2, 3 \), and any solution \( \Omega \) of the discrete Dirac-Kähler equation (4.3) can be uniquely represent as (5.6).

**Proof.** Since the differential operators \( d, \delta \) and their discrete counterparts \( d^c, \delta^c \) have the same properties, i.e., \( d^c d^c = 0 \) and \( \delta^c \delta^c = 0 \), the proof coincides with one in [5]. \( \square \)

## 6 Massless limits of the discrete Dirac-Kähler equation

We are interested in the massless discrete Dirac-Kähler equation. Let us suppose that the mass parameter \( m \) is equal to zero in equations (4.3) and (4.4). Consider the transformation
\[ \Omega \rightarrow \Omega + d^c \Phi, \]
(6.1)
where \( \Omega \) and \( \Phi \) are inhomogeneous forms in view (4.1). The transformation (6.1) is equivalent to
\[ \begin{align*}
0\omega &\rightarrow 0\omega, \\
r\omega &\rightarrow r\omega + d^c \sigma \varphi, 
\end{align*} \]
(6.2)
where \( r = 1, 2, 3, 4 \) and the shift operator \( \sigma \) is given by (3.1).

It should be noted that the transformation
\[ \alpha \rightarrow \alpha + d\varphi \]
is called in the continuum electromagnetic theory the gauge transformation. Here \( \alpha \) is a differential 1-form associated with the vector potential and \( \varphi \) is a scalar function (gauge function). The electromagnetic field is invariant under this gauge transformation. By analogy, the transformation (6.2) of discrete forms can be called gauge.
Proposition 6.1. The discrete wave equation
\[ \Delta^c \Omega = 0 \] (6.3)
is invariant under the gauge transformation (6.1), where the gauge form \( \tilde{\varphi} \) satisfies the condition \( \Delta^c \tilde{\varphi} = 0 \) for \( r = 0, 1, 2, 3 \).

Proof. Since \( \delta^c \delta^0 \varphi = 0 \) and \( \delta^c \delta^r \varphi = -\delta^c \delta^r \varphi \) for \( r = 1, 2, 3 \), we have
\[
\Delta^c (\tilde{\omega} + d^c \sigma^r \varphi) = - (d^c \delta^c + \delta^c d^c) (\tilde{\omega} + d^c \sigma^r \varphi) = \Delta^c \tilde{\omega} - d^c \delta^c d^c \sigma^r \varphi
\]
for any \( r = 1, 2, 3, 4 \). Hence
\[ \Delta^c (\Omega + d^c \Phi) = \Delta^c \Omega. \]

It is clear that Equation (6.3) is also invariant under the transformation
\[ \Omega \to \Omega + \delta^c \Phi, \] (6.4)
where the homogeneous components of \( \Phi \) satisfy the condition \( \Delta^c \tilde{\varphi} = 0 \) for \( r = 1, 2, 3, 4 \).

Proposition 6.2. The discrete massless Dirac-Kähler equation
\[ (d^c - \delta^c) \Omega = 0 \] (6.5)
is invariant under the following transformation
\[ \Omega \to \Omega + d^c \Phi - \delta^c \Phi, \] (6.6)
where \( \Phi \) satisfies equation (6.3).

Proof. Since, by assumption \( \Delta^c \Phi = 0 \) and by (3.33), one has
\[
(d^c - \delta^c) (\Omega + d^c \Phi - \delta^c \Phi) = (d^c - \delta^c) \Omega + (d^c - \delta^c)^2 \Phi = (d^c - \delta^c) \Omega.
\]

Note that the transformation (6.6) can be written more explicitly as
\[
0 \to 0 - \delta^c \varphi^1, \quad 4 \to 4 + d^c \varphi^3, \quad \tilde{\omega} \to \tilde{\omega} + d^c \sigma^r \varphi - \delta^c \tilde{\varphi},
\]
where \( r = 1, 2, 3 \).

It is known (see [13, 18]) that in the classical (continuous) theory there are three massless analogs of Equation (4.3). Now we describe discrete counterparts
of these cases. Let introduce two mass parameters \( m_1, m_2 \) and put \( m = \sqrt{m_1 m_2} \). Then the system \([4.4]\) with two mass parameters can be written as

\[
\begin{align*}
-\delta^c_1 \omega &= m_1 \omega, \\
\delta^c_0 \omega - \delta^c_2 \omega &= m_2 \omega, \\
\delta^c_1 \omega - \delta^c_3 \omega &= m_1 \omega, \\
\delta^c_2 \omega - \delta^c_4 \omega &= m_2 \omega, \\
\delta^c_3 \omega &= m_1 \omega.
\end{align*}
\]

(6.7)

It is easy to check that if the collection of all forms \( \tilde{\omega} \) is a solution of (6.7), then the inhomogeneous form \( \Omega \) (4.1) satisfies the equation

\[
\Delta^c \Omega = m_1 m_2 \Omega.
\]

(6.8)

Hence, if any the mass parameters \( m_1, m_2 \) (or both simultaneously) is equal to zero then the equation (6.8) corresponds to the massless case. Of course, in the case \( m_1 = m_2 = 0 \) the system (6.7) is equivalent to equation (6.5). Let us consider the case \( m_2 = 0 \) and \( m_1 \neq 0 \). In the continuous theory this is the so-called electromagnetic massless limit of the Dirac-Kähler equation [14] [18].

A discrete analog for this case has the form

\[
\begin{align*}
-\delta^c_1 \omega &= m_1 \omega, \\
\delta^c_0 \omega - \delta^c_2 \omega &= 0, \\
\delta^c_1 \omega - \delta^c_3 \omega &= m_1 \omega, \\
\delta^c_2 \omega - \delta^c_4 \omega &= 0, \\
\delta^c_3 \omega &= m_1 \omega.
\end{align*}
\]

(6.9)

Proposition 6.3. The system (6.9) is invariant under the following transformation

\[
\begin{align*}
\frac{1}{\omega} &\rightarrow \frac{1}{\omega} + d^c \varphi, & \frac{3}{\omega} &\rightarrow \frac{3}{\omega} + \delta^c \varphi,
\end{align*}
\]

(6.10)

where the gauge forms \( \varphi \) and \( \varphi \) satisfy the equations \( \Delta^c \varphi = 0 \) and \( \Delta^c \varphi = 0 \).

Proof. Under the gauge transformation (6.10), the left hand side of the corresponding equations of (6.9) becomes

\[
\begin{align*}
-\delta^c(1 + d^c \varphi) &= -\delta^c \omega - \delta^c d^c \varphi = -\delta^c \omega + \Delta^c \varphi = -\delta^c \omega, \\
\delta^c \omega - \delta^c \omega &= \delta^c \omega - \delta^c \omega = \delta^c \omega,
\end{align*}
\]

and

\[
\begin{align*}
\delta^c \omega - \delta^c \omega &= \delta^c \omega - \delta^c \omega = \delta^c \omega.
\end{align*}
\]

\qed
In the case \( m_1 = 0 \) and \( m_2 \neq 0 \) we obtain
\[
\begin{align*}
-\delta^c \omega^1 &= 0, \\
d^c \omega^0 - \delta^c \omega^2 &= m_2 \omega, \\
d^c \omega^1 - \delta^c \omega^3 &= 0, \\
d^c \omega^2 - \delta^c \omega^4 &= m_2 \omega, \\
d^c \omega^3 &= 0.
\end{align*}
\] (6.11)

This system is a discrete analog of the Kalb-Ramond field equations (notoph equations) \[16\] \[18\].

**Proposition 6.4.** The system (6.11) is invariant under the following transformation
\[
\begin{align*}
\omega^0 &\rightarrow \omega^0 - \delta^c \phi^1, \\
\omega^2 &\rightarrow \omega^2 + d^c \phi^1 - \delta^c \phi^3, \\
\omega^4 &\rightarrow \omega^4 + d^c \phi^3,
\end{align*}
\] (6.12)
where the gauge forms \( \phi^1 \) and \( \phi^3 \) satisfy the equations \( \Delta^c \phi^1 = 0 \) and \( \Delta^c \phi^3 = 0 \).

**Proof.** Substituting (6.12) into the left hand side of the second and fourth equations of (6.11) we obtain
\[
d^c (\omega^0 - \delta^c \phi^1) - \delta^c (\omega^2 + d^c \phi^1 - \delta^c \phi^3) = d^c \omega^0 - d^c \delta^c \phi^1 - \delta^c \omega^2 - \delta^c d^c \phi^1 - \delta^c \delta^c \phi^3 \\
= d^c \omega^0 - \delta^c \omega^2 + \Delta^c \phi^1 = d^c \omega^0 - \delta^c \omega^2
\]
and
\[
d^c (\omega^2 + d^c \phi^1 - \delta^c \phi^3) - \delta^c (\omega^4 + d^c \phi^3) = d^c \omega^2 + d^c d^c \phi^1 - \delta^c d^c \phi^3 - \delta^c \omega^4 - \delta^c d^c \phi^3 \\
= d^c \omega^2 - \delta^c \omega^4 + \Delta^c \phi^3 = d^c \omega^2 - \delta^c \omega^4.
\]

We should remark that in the continual case the two massless limits are related to self-dual and anti-self-dual Dirac-Kähler fields. The Dirac-Kähler equation decomposes on its self-dual and anti-self-dual parts by using the projections operators \( P_\pm = \frac{1}{2} (I \pm *) \), where \( * \) is the Hodge star operator. The reason that we do not discuss this relation here is that in our formalism the operation \( *^2 \) is equivalent to a shift with corresponding sign. This is somewhat different from the continual case, where the operation \( * \) is either an involution or antiinvolution, i.e., \( *^2 = \pm I \). Therefore there are difficulties in getting discrete counterparts of the projections operators.

**7 Conclusion**

Within a differential geometric discretisation approach a new discrete analog of the Dirac-Kähler equation has been proposed. In this discrete model some key
aspects of Hodge theory relevant to physics are captured. We have considered three massless limits of the discrete Dirac-Kähler equation. The intrinsic notation for the discrete equations, given in terms of the difference operators \( d^c \) and \( \delta^c \), allows us to discuss the intrinsic form of gauge transformations and the invariance of the discrete massless system under such transformations has been studied.

The Dirac-Kähler formalism can be formulated by the Clifford product of differential forms. It is important to clarify the relation between the Dirac-Kähler and Dirac equation. The identification of inhomogeneous differential forms and the gamma (Dirac) matrices is possible due to the Clifford product. It would be interesting to introduce a Clifford product which acts on the space of our discrete inhomogeneous forms. Giving this we should generalize the discrete Dirac operator introduced above. We also expect to give a better explanation to transformations properties of a discrete model. These directions must be investigated and we hope to treat them further in future work.

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