Optimal Control of Nonholonomic Systems via Magnetic Fields

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Abstract—Geometric optimal control utilizes tools from differential geometry to analyze the structure of a problem to determine the control and state trajectories to reach a desired outcome while minimizing some cost function. For a controlled mechanical system, the control usually manifests as an external force which, if conservative, can be added to the Hamiltonian. In this letter, we focus on mechanical systems with controls added to the symplectic form rather than the Hamiltonian. In practice, this translates to controlling the magnetic field for an electrically charged system. We develop a basic theory deriving necessary conditions for optimality of such a system subjected to nonholonomic constraints. We consider the representative example of a magnetically charged Chaplygin Sleigh, whose resulting optimal control problem is completely integrable.

Index Terms—Optimal control, symplectic geometry, nonholonomic systems.

I. INTRODUCTION

Magnetic fields are used in a variety of control problems. For example, magnetic fields in combination with electric quadrupoles are commonly used for confinement of particles in Penning traps, e.g., for both quantum trapping [1], [2], [3], [4], and for mass spectrometry [5]. Magnetic fields are also a popular external actuation tool as they allow fuel-free remote control and a high degree of programmability [6], [7], [8], [9]. While newer technologies have led to an increase in magnetic experimental work, the theory on optimal control of magnetic systems is scarce and underdeveloped.

In this letter, we build a geometric framework for dealing with magnetic controls, based on the observation that deforming the canonical symplectic form by incorporating a magnetic field results in a new symplectic form [10], [11].

Thus, introducing controls through the magnetic field is equivalent to controlling the symplectic form associated to the state space. Previous work on optimal control for Hamiltonian systems deals with controls implemented via an external force [12], [13], [14], [15], [16]. To the best of our knowledge, the case in which the symplectic form depends on the controls has not been rigorously analyzed before.

Additionally, we extend the framework by considering systems which are at the same time subject to nonholonomic constraints, such as the Chaplygin Sleigh (Fig. 1). This is a generalization of the unconstrained and the holonomic case, which can be obtained by turning off the Lagrange multipliers in the equation of motion.

Broadly, our results apply to any control that can be moved into the symplectic form. To see the applicability, consider Hamiltonian systems with the form

$$H(q, p) = \frac{p^2}{2m} + V(q),$$

where $V(q)$ is a potential, $q, p \in \mathbb{R}^n$ are the particle position and momentum respectively, and $m > 0$ is the particle mass. A valid symplectic control could be written as

$$m\ddot{q}_i = p_i, \quad \dot{p}_i = -\frac{\partial V}{\partial q_i} + \sum_j B_{ij}(u)\dot{q}_j.$$

Fig. 1. The Chaplygin sleigh subject to the left-invariant magnetic field, $B = B dx_c \wedge dy_c$. 

Digital Object Identifier 10.1109/LCSYS.2022.3226715

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where $B_{ij}$ is a closed skew-symmetric matrix depending on the control $u$. Modifying the symplectic form in this way is commonly called ‘magnetic,’ as a particle with charge $e$ traveling in an electric potential $V$ and magnetic field $B(u)$ can be written as

$$m\ddot{q} = p, \quad \dot{p} = -e\nabla V + e\dot{q} \times B(u).$$

Another example of ‘magnetic’ control is in planar normal-thrust dynamics of spacecraft in orbit around the earth [17], [18], [19]. When the thrust is assumed to be normal, the equations of motion can be written as

$$m\ddot{q} = -\nabla V(q) + u\left(\frac{\dot{q}_2}{\dot{q}_1}\right), \quad V(q) = -\frac{\mu}{|q|},$$

where $\mu$ is the gravity constant. While the above equation has no actual magnetic forces, the assumed form of the control restricts the dynamics to the distribution $D \subset TQ$ which contains the possible velocities that the system can have such that the constraints are satisfied. As these constraints are linear, we can (locally) find 1-forms $\eta_i$ for $i \in \{1, \ldots, k\}$ such that the span of $\eta_i$ annihilates $D$, i.e., for all vectors $v \in D$, $\eta_i(v) = 0$ [20].

Using the local constraints, Hamilton’s equations of motion for the forced nonholonomic system can be written as:

$$i_X\omega = dH + \lambda^i \pi_Q^\ast \eta_i + \pi_Q^\ast F. \tag{3}$$

where $\lambda^i$ are Lagrange multipliers chosen to such that the constraints are satisfied. As before, we lift the constraints to the phase space using $\pi_Q^\ast$.

The last ingredient is the magnetic field. Geometrically, this can be viewed as a closed 2-form $B \in \Omega^2(Q)$ on the state space. In coordinates, we can represent $B$ as a skew-symmetric $n \times n$ matrix with entries $B_{ij}$.

**Definition 1:** A control matrix $B = (B_{ij})$ is symplectic if it satisfies the following properties:

1. $B$ is skew symmetric: $B_{ij} = -B_{ji}$
2. $B$ is closed, i.e.,

$$\sum_{\sigma \in \Sigma} \text{sgn}(\sigma) \frac{\partial B_{\sigma(i)\sigma(j)}}{\partial x_{\sigma(k)}} = 0. \tag{4}$$

In the 2-dimensional case, as long as $B$ only depends on position, it is automatically closed. For the 3-dimensional case of magnetic fields, the closedness condition is equivalent to the requiring that the field be divergence free.

If the control matrix is symplectic then it generates a closed 2-form $B = \frac{1}{2}B_{ij}dq^i \wedge dq^j$ which will be added to the canonical symplectic form $\omega$ to generate the controlled equations of motion [21], [22]:

$$i_X\omega_B = dH + \lambda^i \pi_Q^\ast \eta_i + \pi_Q^\ast F,$$

$$\omega_B = \omega + \pi_Q^\ast B. \tag{5}$$

where $\omega = dq^i \wedge dp_i$ is the canonical symplectic form. Equation (5) can be written in coordinates as:

$$\dot{q}^i = \frac{\partial H}{\partial p_i},$$

$$\dot{p}_j = -\frac{\partial H}{\partial x_j} + \lambda^k (\eta_k)_j + B_{jk} \dot{q}^k + F_j \tag{6}$$
Equation (5) produces a vector field, $X_B = (\dot{q}, \dot{p})$ which lies tangent to the induced co-distribution $\mathcal{D}' = \mathcal{F}H^{-1}(\mathcal{D}) \subset T^*Q$, where $\mathcal{F}H$ is the fiber derivative. This means that the flow of this vector field will follow the equations of motion, and it will obey the constraints. In particular, we have that

$$\pi_Q^*(\eta)(X_B) = 0.$$  \hspace{1cm}(7)

**Proposition 1:** In the absence of external forcing, for any $B \in \Omega^2(Q)$, the Hamiltonian $H$ is preserved under the magnetic nonholonomic flow given by (5), i.e., if $X_B$ is given by (5) with $F = 0$, then $L_{X_B}H = 0$, where $L$ denotes the usual Lie derivative.

**Proof:** Using (5),

$$L_{X_B}H = dH(X_B) = i_{X_B}\omega_B(X_B) - \lambda^i\eta^*_Q\eta_i(X_B)$$

The second term vanishes by (7). Now use the definition of the interior product and the fact that $\omega_B$ is skew-symmetric to conclude that the first term also vanishes.

$$i_{X_B}\omega_B(X_B) = \omega_B(X_B, X_B) = 0.$$

### III. Optimal Control Problem

There are several ways in which one can introduce control in the system. One particular method, which was well studied before, is through controlling the external forcing or the Hamiltonian [13].

In this letter, we will take a different approach, and instead of manipulating $F$ or $H$ we will control the underlying geometry of the problem which is encoded in the symplectic form. To do this we define the controlled symplectic form to be the 2-form valued function $\omega : \mathcal{U} \rightarrow \Omega^2(T^*Q)$. Each different control $u$ in our control space $\mathcal{U}$ will produce a symplectic form, which in turns produces a different vector field. If we introduce controls in this manner, then we have a guarantee that independent of the form of the controls, the energy of the system given in the Hamiltonian is always preserved (Proposition 1). In particular, this will be the case for the optimal control.

In the following we will assume that there is no external forcing, and that the controls are introduced through the symplectic form.

The controlled $\omega$ should be symplectic for all times. As shown in [22], a symplectic form on $T^*Q$ different from the canonical one $\omega = dx \wedge dp$ can be obtained by adding a non-degenerate closed 2-form $B$ lifted from the state space by the pullback of the canonical projection $\pi_Q^*$. This is precisely what the magnetic field is.

We now turn to the optimal control problem. We wish to determine optimal controls for the system (5). Specifically, we want to solve a bounded horizon optimal control problem by minimizing the following cost functional:

$$J(B) = \int_0^T \ell(q, p; B) \, dt,$$

subject to the fixed endpoints,

$$q(0) = q_0, \quad q(T) = q_f,$$

$$p(0) = p_0, \quad p(T) = p_f,$$

whose dynamics evolves according to (5). $\ell : \mathcal{D}' \times \mathcal{U} \rightarrow \mathbb{R}$ is the running cost. Recall that $B$ refers to the matrix entries of the skew-symmetric 2-form $B$, and thus it can be parameterized as $B \in \mathbb{R}^N$ for $N = n(n - 1)/2$.

To solve the optimal control problem we construct the extended Hamiltonian [20].

**Definition 2:** For $(q, p) \in \mathcal{D}'$, let $(q, p, p_q, p_p)$ be the induced coordinates on $T^*\mathcal{D}'$. Then the extended Hamiltonian on $T^*\mathcal{D}'$ is given by

$$\tilde{H} : \mathcal{U} \times T^*\mathcal{D}' \rightarrow \mathbb{R}$$

$$(B, q, p, p_q, p_p) \mapsto \ell(q, p; B) + \langle (p_q, p_p), X_B \rangle.$$

To determine the optimal control we need to minimize $\tilde{H}$. We will assume that such a minimizer $B_{\text{opt}}$ exists and is attained at a critical point. If this is the case then:

$$\frac{\partial \tilde{H}}{\partial B_{\text{opt}}} = 0.$$  \hspace{1cm}(10)

We will call the optimal Hamiltonian $H_{\text{opt}} := \min_B \tilde{H} = \tilde{H}(B_{\text{opt}}, \cdot)$. An optimal trajectory is given be an integral curve of Hamilton’s equations

$$\dot{q} = \frac{\partial H_{\text{opt}}}{\partial p_q}, \quad \dot{p}_q = -\frac{\partial H_{\text{opt}}}{\partial q},$$

$$\dot{p} = \frac{\partial H_{\text{opt}}}{\partial p_p}, \quad \dot{p}_p = -\frac{\partial H_{\text{opt}}}{\partial p},$$

subject to the prescribed boundary conditions

$$q(0) = q_0, \quad q(T) = q_f,$$

$$p(0) = p_0, \quad p(T) = p_f.$$

In order to simplify notation, let $Y = (\dot{q}, \dot{p}, \dot{p}_q, \dot{p}_p)$ be the Hamiltonian vector field and $\Omega = dq \wedge dp_q + dp \wedge dp_p$ the canonical symplectic form on $T^*\mathcal{D}'$. Then, the optimal control equations can be written as:

$$i_Y \Omega = dH_{\text{opt}}.$$  

**Proposition 2:** The extended optimal control Hamiltonian flow for a magnetically controlled system has at least two constants of motion, i.e., there exists $E_1, E_2 : T^*\mathcal{D}' \rightarrow \mathbb{R}$ such that $L_Y E_i = 0, \ i \in \{1, 2\}$.

**Proof:** $E_1 = H_{\text{opt}}$ is conserved by the definition of $Y$. We now claim that $E_2 = \pi_{\mathcal{D}'}^*H$ is also a conserved quantity, where $\pi_{\mathcal{D}'} : T^*\mathcal{D}' \rightarrow \mathcal{D}'$ is the canonical projection.

$$L_Y(\pi_{\mathcal{D}'}^*H) = d(\pi_{\mathcal{D}'}^*H)(Y) = \pi_{\mathcal{D}'}^*dH(Y) = dH((\pi_{\mathcal{D}'}^*)_aY).$$

But $\pi_{\mathcal{D}'}Y = X_{B_{\text{opt}}}$, where $B_{\text{opt}}$ is the optimal magnetic field that satisfies (10). By Proposition 1 we know

$$L_{X_B}H = dH(X_B) = 0$$

for any $B$, so in particular it must vanish for $B_{\text{opt}}$. Hence,

$$L_Y(\pi_{\mathcal{D}'}^*H) = dH(X_{B_{\text{opt}}}) = 0.$$  

This result has an important consequence for 2-dimensional systems.
Corollary 1: Any magnetically controlled optimal control system on a 2-dimensional manifold is completely integrable, i.e., if (5) can be reduced to a 2-dimensional problem, the system $\left(T^*\mathcal{D}', H_{opt}, \Omega\right)$ is completely integrable.

Proof: By Proposition 2 $E_1 = \pi_T^*yH$ and $E_2 = H_{opt}$ are constants of motion. Moreover, they are linearly independent since $H_{opt}$ depends on $p_y$ and $p_q$ whereas $\pi_T^*yH$ does not. Hence, we can use the Arnold-Liouville theorem [23], [24] to conclude that the components $\left(T^*\mathcal{D}', \Omega, \{E_1, E_2\}\right)$ form an integrable system.

IV. CHAPLYGIN SLEIGH IN A MAGNETIC FIELD

We will apply the theory to the case of a magnetically controlled Chaplygin sleigh as shown in Fig. 1. Such a system can be thought of as an ice skater [25]. The state space is given by $Q = SE_2$. The coordinates $q = (x, y, \theta) \in Q$ represent the contact point of the sleigh and its orientation. We will assume that an electric charge $e$ is concentrated at the center of mass and is acted on by a vertical magnetic field of strength $B$. The coordinates of the center of mass are $x_c = x + a\cos \theta$ and $y_c = y + a\sin \theta$ for some length $a \neq 0$. The out-of-plane magnetic field can be written as:

$$B = eB \cdot dx_c \land dy_c$$

$$= eB(dx \land dy + a\cos \theta dx \land d\theta + a\sin \theta dy \land d\theta),$$

corresponding to a matrix

$$B = \begin{bmatrix}
0 & -eB & -aeB \cos \theta \\
eB & 0 & -aeB \sin \theta \\
aeB \cos \theta & aeB \sin \theta & 0
\end{bmatrix}.$$

In coordinates $(x, y, \theta)$, the Hamiltonian of the Chaplygin sleigh is $H = \frac{1}{2}M^\top p(p, q)$ with the mass matrix

$$M = \begin{bmatrix}
m & 0 & -ma \sin \theta \\
0 & m & ma \cos \theta \\
-ma \sin \theta & ma \cos \theta & I + ma^2
\end{bmatrix},$$

where $m$ is the mass of the sleigh and $I$ is the moment of inertia. The nonholonomic constraint is given by

$$\dot{y} \cos \theta - \dot{x} \sin \theta = 0 \iff \eta = \cos \theta dy - \sin \theta dx$$

and is equivalent to prohibiting movement perpendicular to the forward orientation. In this particular case, we can find an explicit formula for the terms in (5). The controlled symplectic form can be written as the matrix:

$$(\omega_B)_{ij} = \begin{bmatrix}
\frac{1}{2}B & I_3 \\
-I_3 & O_3
\end{bmatrix}$$

where $I_3$ and $O_3$ are the $3 \times 3$ identity and zero matrix respectively. The vector field can be written as the column vector $X_B = (x, \dot{x}, \dot{y}, \dot{\theta}, \dot{p}_x, \dot{p}_y, \dot{p}_\theta)^T$, whereas the right-hand side is the row vector $(-\lambda \sin \theta, \lambda \cos \theta, 0, (p_x, p_y, p_\theta)M^{-1})$. Thus, the equations of motion are written as

$$M \dot{q} = p, \quad \dot{p} = F_\eta - B \dot{q},$$

where $F_\eta = (-\lambda \sin \theta, \lambda \cos \theta, 0)$ is the force associated to the constraints.

Let $v = \dot{x} \cos \theta + \dot{y} \sin \theta$ be its forward velocity and let $\gamma = \dot{\theta}$ be the angular momentum of the ice skater. We then relate $(x, y, \theta)$ to this new coordinate system $(v, \gamma)$ through:

$$\dot{x} = v \cos \theta, \quad \dot{y} = v \sin \theta, \quad \dot{\theta} = \gamma.$$
If we use $\ell(B) = 1/2B^2$ as earlier, we find the optimal control

$$B^* = \arg \min_B \tilde{H}(\alpha, \alpha, p_\alpha) = -\sqrt{c}p_\alpha.$$  

Plugging this into the Hamiltonian gives out $H_{opt}$ in terms of $p_\alpha$ and $\alpha$. We scale $p_\alpha$ by a factor of $-\sqrt{c}$ in order to obtain a Hamiltonian in terms of $B$ and $\alpha$ so that we can solve the system of equations of motion (13) for the magnetic field directly. This gives

$$H_{opt}(\alpha, B) = \frac{1}{2c}B^2 - B \sin \alpha,$$

and the equations of motion

$$\dot{\alpha} = \frac{1}{c}B - \sin \alpha, \quad \dot{B} = B \cos \alpha.$$  

### V. Numerical Results

We begin with the phase portrait of the optimized Hamiltonian (12) (see Fig. 2). There are four critical points of the control for $\alpha \in S^1$ and $B \in \mathbb{R}$. Two stable critical points occur at $(\alpha, B) = (\pm \pi/2, \pm 1)$, corresponding to the Chaplygin sleigh rotating in place with a constant angular velocity $\omega = \pm c^{-1}$. Two unstable points occur at $(0, 0)$ and $(\pi, 0)$, corresponding to the sleigh moving in a straight line forwards or backwards respectively.

There are two heteroclinic orbits connecting the unstable points, defined by the equations $B = 0$ and $B = 2 \sin \alpha$. The $B = 0$ heteroclinic orbit reduces to the uncontrolled system where the Chaplygin is attracted to going straight, i.e., $\dot{\alpha} = -\sin \alpha$. The $B = 2 \sin \alpha$ orbit reverses time in that system, giving $\dot{\alpha} = \sin \alpha$. If we use this orbit, the optimal control orbit effectively makes the backwards direction stable and the forwards direction unstable.

In order for the system to be controllable, the value of the Hamiltonian (12) must exceed the Hamiltonian value of the connecting orbits, or

$$B^2 - 2c B \sin \alpha > 0.$$  

This implies that there is a minimum value of the magnetic field for controllability

$$|B| > \max_\alpha(2c \sin \alpha) = 2c. \quad (14)$$

Now, consider the problem

$$J = \min_B \int_0^T \frac{1}{2}B^2 \, dt,$$

$$\alpha(0) = 0, \quad \alpha(T) = \pi,$$

corresponding to the problem of turning the Chaplygin sleigh around from a straight forward trajectory in a fixed amount of time $T$. Solutions to this problem are symmetric with $(\alpha, B) \mapsto (\alpha, -B)$, so we only consider $B > 0$. In order to do this numerically, we begin by noting that (13) can be combined to find the equation for $\alpha$ as

$$\dot{\alpha} = \frac{1}{2} \sin(2\alpha),$$

i.e., the evolution of $\alpha$ in time exactly matches the nonlinear pendulum with frequency 2 and a $\pi/2$ phase shift. We note that this result holds for any value of $c$. This means that the time to get from our two end points can be given exactly by

$$T = \frac{K(-B(0)^{-2})}{\sqrt{B(0)}}, \quad (16)$$

where $K$ is the complete elliptic integral of the first kind. Then, using the fact that $T$ is monotonic in $B(0)$, we can solve (16) using a bisection search with a high-accuracy elliptic integral for an initial $B(0)$ with near machine precision. If we do not consider solutions that do a half turn, this gives a unique solution to the Pontryagin optimal control for (15).

In Fig. 3, we plot an example optimal trajectory for (15) in the $(x, y)$-plane. We solve for the initial magnetic field $B(0)$ for the time $T = 2$ using MATLAB’s $fzero$ and $elliptic\mathcal{K}$ functions for root finding and evaluating the elliptic integral respectively. The solution to this is evolved via a Runge-Kutta scheme (ode45) to obtain the spatial path, with initial conditions $(x, y, \theta) = (0, 0, 0)$.

In Fig. 4, we plot both the cost and the maximum magnetic field over the trajectories for a range of $T$ values between $T = 10^{-1.5}$ through $T = 10^{2.5}$. The cost was obtained by integrating over the trajectory using ode45, and the maximum...
magnetic field was found by noting that the Hamiltonian $B$ is maximized at $\alpha = \pi/2$ and using conservation of (12) to find
\[
\max B = 1 + \sqrt{1 + B(0)^2}.
\]
We see that the maximum magnetic field decreases as a function of $T$ and is asymptotic to the minimum value of 2 from (14). Additionally, we see that the cost is decreasing as $T$ increases. We can find the value it converges to by analytically evaluating the cost of the heteroclinic orbit $B = 2 \sin \alpha$:
\[
J_{\text{min}} = \int_{-\infty}^{\infty} \frac{1}{2} B^2 \, dt = 4.
\]

VI. CONCLUSION

We investigated the optimal control problem for nonholonomic Hamiltonian systems subject to a magnetic field. An example of an electrically charged Chaplygin sleigh was presented. Due to the energy-preserving property of the magnetic controls, the resulting optimal control problem is always completely integrable; in our specific example, the equations of motion were equivalent to the nonlinear pendulum and solutions were found via elliptic integrals.

In this letter, we assume that we can reach any location in a given time. That is, for any pair of points $x, y \in D'$ and time $T > 0$, does there exist a control law that drives the system from $x$ to $y$ while obeying the magnetic nonholonomic equations of motion? Whether this is possible for any given system is generally nontrivial.

Another possible research direction, specifically for the Chaplygin sleigh, is on trajectory-tracking as discussed in [25]. As energy is preserved, this places bounds on the maximum angular velocity which will make tracing an arbitrary path impossible.

A final immediate research direction is to extend this procedure to either relativistic or quantum systems.

ACKNOWLEDGMENT

The authors thank Mallory Gaspard for insightful conversations and her enthusiasm.

REFERENCES

[1] F. G. Major, V. N. Gheorghe, and G. Werth, Charged Particle Traps: Physics and Techniques of Charged Particle Field Confinement (Charged Particle Traps) New York, NY, USA: Springer, 2005.
[2] J. Pérez-Ríos and A. Sanz, “How does a magnetic trap work?” Am. J. Phys., vol. 81, no. 11, pp. 836–843, Nov. 2013.
[3] C. Brif, R. Chakrabarti, and H. Rabitz, “Control of quantum phenomena: Past, present and future,” New J. Phys., vol. 12, no. 7, Jul. 2010, Art. no. 75008.
[4] S. Deffner, “Optimal control of a Qubit in an optical cavity,” J. Phys. B Atomic, Mol. Opt. Phys., vol. 47, no. 14, Jul. 2014, Art. no. 145502.
[5] V. Martikyan, C. Beluffi, S. J. Glaser, M. Delsuc, and D. Sugny, “Application of optimal control theory to fourier transform ion cyclotron resonance,” Molecules, vol. 26, no. 10, p. 2860, May 2021.
[6] M. Kolosso, X. Feng, Y. Xue, Q. Li, T. Munshi, and X. Chen, “Micro/nanoscale magnetic robots for biomedical applications,” Mater. Today Bio., vol. 8, Sep. 2020, Art. no. 100085.
[7] L. Wang, Z. Meng, Y. Chen, and Y. Zheng, “Engineering magnetic micro/nanorobots for versatile biomedical applications,” Adv. Intell. Syst., vol. 3, no. 7, 2021, Art. no. 2000267.
[8] A. Ataka, H. Lam, and K. Althoefer, “Magnetic-field-inspired navigation for robots in complex and unknown environments,” Front. Robot. AI, vol. 9, Feb. 2022, Art. no. 834177.
[9] W. R. Johnson, S. J. Woodman, and R. Kramer-Bottiglio, “An electromagnetic soft robot that carries its own magnet,” in Proc. IEEE 5th Int. Conf. Soft Robot. (RoboSoft), vol. 5, 2022, pp. 761–766.
[10] A. M. Bloch, Nonholonomic Mechanics and Control. New York, NY, USA: Springer, 2003.
[11] J. E. Marsden, G. Misiolek, J. P. Ortega, M. Perlmutter, and T. S. Ratiu, Hamiltonian Reduction by Stages. Berlin, Germany: Springer, 2007.
[12] L. Pontryagin, “Optimal control process,” Uspekhi Matematicheskikh Nauk, vol. 14, pp. 3–20, Jul. 1959.
[13] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishechenko, “The mathematical theory of optimal processes,” ZAMM J. Appl. Math. Mechanics/Zeitschrift Angewandte Mathematik Mechanik, vol. 43, nos. 10–11, pp. 514–515, 1963.
[14] E. Lee and L. Markus, Foundations of Optimal Control Theory. New York, NY, USA: Wiley, 1986.
[15] A. M. Bloch and P. Croach, “Reduction of Euler Lagrange problems for constrained variational problems and relation with optimal control problems,” in Proc. 33rd IEEE Conf. Decis. Control, vol. 3, 1994, pp. 2584–2590.
[16] A. Bloch, L. Colombo, and F. Jiménez, “The variational discretization of the constrained higher-order Lagrange-Poincaré equations,” Discr. Continuous Dyn. Syst., vol. 39, no. 1, pp. 309–344, 2019.
[17] S. Hernandez and M. R. Akella, “Energy-conserving planar spacecraft motion with constant-thrust acceleration,” J. Guid., Control, Dyn., vol. 38, no. 12, pp. 2300–2323, Dec. 2015.
[18] S. Hernandez and M. R. Akella, “Energy preserving low-thrust guidance for orbit transfers in KS variables,” Celestial Mech. Dyn. Astron., vol. 125, no. 1, pp. 107–132, May 2016.
[19] P. Su, W. Feng, Y. Kun, and Z. Junfeng, “Optimal control of rapid cooperative spacecraft rendezvous with multiple specific-direction thrusts,” Proc. Inst. Mech. Eng. G, J. Aerosp. Eng., vol. 234, no. 16, pp. 2296–2322, Dec. 2020.
[20] L. J. Colombo, D. M. de Diego, A. Nayak, and R. T. S. M. de Almagos, “Geometric optimal control trajectory tracking of nonholonomic mechanical systems,” Soc. Ind. Appl. Math., vol. 8, no. 7, 2021, Art. no. 2000267.