Two philosophical applications of algorithmic information theory

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Abstract

Two philosophical applications of the concept of program-size complexity are discussed. First, we consider the light program-size complexity sheds on whether mathematics is invented or discovered, i.e., is empirical or is a priori. Second, we propose that the notion of algorithmic independence sheds light on the question of being and how the world of our experience can be partitioned into separate entities.

1. Introduction. Why is program size of philosophical interest?

The cover of the January 2003 issue of La Recherche asks this dramatic question:

Dieu est-il un ordinateur? [Is God a computer?]

The long cover story [1] is a reaction to Stephen Wolfram’s controversial book A New Kind of Science [2]. The first half of the article points out Wolfram’s predecessors, and the second half criticizes Wolfram.

The second half of the article begins (p. 38) with these words:

Il [Wolfram] n’avance aucune raison sérieuse de penser que les complexités de la nature puissent être générées par des règles énonçables sous forme de programmes informatiques simples.

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The reason for thinking that a simple program might describe the world is, basically, just Plato’s postulate that the universe is rationally comprehensible \((\textit{Timaeus})\). A sharper statement of this principle is in Leibniz’s \textit{Discours de métaphysique} [3], section VI. Here is Leibniz’s original French (1686):

\begin{quote}
Mais Dieu a choisi celuy qui est le plus parfait, c’est à dire celuy qui est en même temps le plus simple en hypotheses et le plus riche en phenomenes, comme pourroit estre une ligne de Geometrie dont la construction seroit aisée et les proprietés et effects seroient fort admirables et d’une grande étendue.
\end{quote}

For an English translation of this, see [4].

And Hermann Weyl [5] discovered that in \textit{Discours de métaphysique} Leibniz also states that a physical law has no explicative power if it is as complicated as the body of data it was invented to explain.\footnote{See the Leibniz quote in Section 2.3 below.}

This is where algorithmic information theory (AIT) comes in. AIT posits that a theory that explains \(X\) is a computer program for calculating \(X\), that therefore must be smaller, much smaller, than the size in bits of the data \(X\) that it explains. AIT makes a decisive contribution to philosophy by providing a mathematical theory of complexity. AIT defines the \textit{complexity} or \textit{algorithmic information content} of \(X\) to be the size in bits \(H(X)\) of the smallest computer program for calculating \(X\). \(H(X)\) is also the complexity of the most elegant (the simplest) theory for \(X\).

In this article we discuss some other philosophical applications of AIT.

For those with absolutely no background in philosophy, let me recommend two excellent introductions, Magee [6] and Brown [7]. For introductions to AIT, see Chaitin [8, 9]. For another discussion of the philosophical implications of AIT, see Chaitin [10].

\section*{2. Is mathematics empirical or is it a priori?}

\subsection*{2.1. Einstein: math is empirical}

Einstein was a physicist and he believed that math is invented, not discovered. His sharpest statement on this is his declaration that “the series of
integers is obviously an invention of the human mind, a self-created tool which simplifies the ordering of certain sensory experiences.”

Here is more of the context:

In the evolution of philosophic thought through the centuries the following question has played a major rôle: What knowledge is pure thought able to supply independently of sense perception? Is there any such knowledge?... I am convinced that... the concepts which arise in our thought and in our linguistic expressions are all... the free creations of thought which can not inductively be gained from sense-experiences... Thus, for example, the series of integers is obviously an invention of the human mind, a self-created tool which simplifies the ordering of certain sensory experiences.2

The source is Einstein’s essay “Remarks on Bertrand Russell’s theory of knowledge.” It was published in 1944 in the volume [11] on The Philosophy of Bertrand Russell edited by Paul Arthur Schilpp, and it was reprinted in 1954 in Einstein’s Ideas and Opinions [12].

And in his Autobiographical Notes [13] Einstein repeats the main point of his Bertrand Russell essay, in a paragraph on Hume and Kant in which he states that “all concepts, even those closest to experience, are from the point of view of logic freely chosen posits.” Here is the bulk of this paragraph:

Hume saw clearly that certain concepts, as for example that of causality, cannot be deduced from the material of experience by logical methods. Kant, thoroughly convinced of the indispensability of certain concepts, took them... to be the necessary premises of any kind of thinking and distinguished them from concepts of empirical origin. I am convinced, however, that this distinction is erroneous or, at any rate, that it does not do justice to the problem in a natural way. All concepts, even those closest to experience, are from the point of view of logic freely chosen posits...

2[The boldface emphasis in this and future quotations is mine, not the author’s.]
2.2. Gödel: math is a priori

On the other hand, Gödel was a Platonist and believed that math is a priori. He makes his position blindingly clear in the introduction to an unpublished lecture Gödel *1961/?, “The modern development of the foundations of mathematics in the light of philosophy,” *Collected Works* [14], vol. 3:

I would like to attempt here to describe, in terms of philosophical concepts, the development of foundational research in mathematics . . . , and to fit it into a general schema of possible philosophical world-views [Weltanschauungen] . . . I believe that the most fruitful principle for gaining an overall view of the possible world-views will be to divide them up according to the degree and the manner of their affinity to or, respectively, turning away from metaphysics (or religion). In this way we immediately obtain a division into two groups: skepticism, materialism and positivism stand on one side, spiritualism, idealism and theology on the other . . . Thus one would, for example, say that apriorism belongs in principle on the right and empiricism on the left side . . . Now it is a familiar fact, even a platitude, that the development of philosophy since the Renaissance has by and large gone from right to left . . . It would truly be a miracle if this (I would like to say rabid) development had not also begun to make itself felt in the conception of mathematics. Actually, mathematics, by its nature as an a priori science, always has, in and of itself, an inclination toward the right, and, for this reason, has long withstood the spirit of the time [Zeitgeist] that has ruled since the Renaissance; i.e., the empiricist theory of mathematics, such as the one set forth by Mill, did not find much support . . . Finally, however, around the turn of the century, its hour struck: in particular, it was the antinomies of set theory, contradictions that allegedly appeared within mathematics, whose significance was exaggerated by skeptics and empiricists and which were employed as a pretext for the leftward upheaval . . .

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3The numbering scheme used in Gödel’s *Collected Works* begins with an * for unpublished papers, followed by the year of publication, or the first/last year that Gödel worked on an unpublished paper.
Nevertheless, the Platonist Gödel makes some remarkably strong statements in favor of adding to mathematics axioms which are not self-evident and which are only justified pragmatically. What arguments does he present in support of these heretical views?

First let’s take a look at his discussion of whether Cantor’s continuum hypothesis could be established using a new axiom [Gödel 1947, “What is Cantor’s continuum problem?”, Collected Works, vol. 2]:

...even disregarding the intrinsic necessity of some new axiom, and even in case it has no intrinsic necessity at all, a probable decision about its truth is possible also in another way, namely, inductively by studying its “success.” Success here means fruitfulness in consequences, in particular in “verifiable” consequences, i.e., consequences demonstrable without the new axiom, whose proofs with the help of the new axiom, however, are considerably simpler and easier to discover, and make it possible to contract into one proof many different proofs. The axioms for the system of real numbers, rejected by intuitionists, have in this sense been verified to some extent, owing to the fact that analytical number theory frequently allows one to prove number-theoretical theorems which, in a more cumbersome way, can subsequently be verified by elementary methods. A much higher degree of verification than that, however, is conceivable. There might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole field, and yielding such powerful methods for solving problems (and even solving them constructively, as far as that is possible) that, no matter whether or not they are intrinsically necessary, they would have to be accepted at least in the same sense as any well-established physical theory.

Later in the same paper Gödel restates this:

It was pointed out earlier... that, besides mathematical intuition, there exists another (though only probable) criterion of the truth of mathematical axioms, namely their fruitfulness in mathematics and, one may add, possibly also in
physics... The simplest case of an application of the criterion under discussion arises when some... axiom has number-theoretical consequences verifiable by computation up to any given integer.

And here is an excerpt from Gödel’s contribution [Gödel 1944, “Russell’s mathematical logic,” Collected Works, vol. 2] to the same Bertrand Russell festschrift volume [11] that was quoted above:

The analogy between mathematics and a natural science is enlarged upon by Russell also in another respect... axioms need not be evident in themselves, but rather their justifica-
tion lies (exactly as in physics) in the fact that they make it possible for these “sense perceptions” to be deduced... I think that... this view has been largely justified by subsequent developments, and it is to be expected that it will be still more so in the future. It has turned out that the solution of certain arithmetical problems requires the use of assumptions essentially transcending arithmetic... Furthermore it seems likely that for deciding certain questions of abstract set theory and even for certain related questions of the theory of real numbers new axioms based on some hitherto unknown idea will be necessary. Perhaps also the apparently insurmountable difficulties which some other mathematical problems have been presenting for many years are due to the fact that the necessary axioms have not yet been found. Of course, under these circumstances mathematics may lose a good deal of its “absolute certainty;” but, under the influence of the modern criticism of the foundations, this has already happened to a large extent...

Finally, take a look at this excerpt from Gödel *1951, “Some basic theo-
rems on the foundations,” Collected Works, vol. 3, an unpublished essay by Gödel:

I wish to point out that one may conjecture the truth of a universal proposition (for example, that I shall be able to verify a certain property for any integer given to me) and at the same time conjecture that no general proof for this fact exists. It is easy to imagine situations in which both these conjectures would
be very well founded. For the first half of it, this would, for example, be the case if the proposition in question were some equation $F(n) = G(n)$ of two number-theoretical functions which could be verified up to very great numbers $n$.\footnote{Such a verification of an equality (not an inequality) between two number-theoretical functions of not too complicated or artificial structure would certainly give a great probability to their complete equality, although its numerical value could not be estimated in the present state of science. However, it is easy to give examples of general propositions about integers where the probability can be estimated even now...} Moreover, exactly as in the natural sciences, this \textit{inductio per enumerationem simplicem} is by no means the only inductive method conceivable in mathematics. I admit that every mathematician has an inborn abhorrence to giving more than heuristic significance to such inductive arguments. I think, however, that this is due to the very prejudice that mathematical objects somehow have no real existence. \textbf{If mathematics describes an objective world just like physics, there is no reason why inductive methods should not be applied in mathematics just the same as in physics.} The fact is that in mathematics we still have the same attitude today that in former times one had toward all science, namely, we try to derive everything by cogent proofs from the definitions (that is, in ontological terminology, from the essences of things). Perhaps this method, if it claims monopoly, is as wrong in mathematics as it was in physics.

So Gödel the Platonist has nevertheless managed to arrive, at least partially, at what I would characterize, following Tymoczko [16], as a pseudo-empirical or a quasi-empirical position!

\section*{2.3. AIT: math is quasi-empirical}

What does algorithmic information theory have to contribute to this discussion? Well, I believe that AIT also supports a quasi-empirical view of mathematics. And I believe that it provides further justification for Gödel’s belief that we should be willing to add new axioms.

Why do I say this?

As I have argued on many occasions, AIT, by measuring the complexity (algorithmic information content) of axioms and showing that Gödel incompleteness is natural and ubiquitous, deepens the arguments that forced Gödel,
in spite of himself, in spite of his deepest instincts about the nature of mathematics, to believe in inductive mathematics. And if one considers the use of induction rather than deduction to establish mathematical facts, some kind of notion of *complexity* must necessarily be involved. For as Leibniz stated in 1686, a theory is only convincing to the extent that it is substantially *simpler* than the facts it attempts to explain:

> ... non seulement rien n’arrive dans le monde, qui soit absolument irrégulier, mais on ne scénauroit même rien feindre de tel. Car supposons par exemple que quelcun fasse quantité de points sur le papier à tout hazard, comme font ceux qui exercent l’art ridicule de la Geomance, je dis qu’il est possible de trouver une ligne géométrique dont la motion soit constante et uniforme suivant une certaine règle, en sorte que cette ligne passe par tous ces points... Mais *quand une règle est fort composée, ce qui lui est conforme, passe pour irrégulier*. Ainsi on peut dire que de quelque manière que Dieu auroit créé le monde, il auroit toujours esté régulier et dans un certain ordre general. Mais Dieu a choisi ceulx qui est le plus parfait, c’est à dire ceulx qui est en même temps *le plus simple en hypotheses et le plus riche en phenomenes*. ... [[Discours de métaphysique, VI]]

In fact Gödel himself, in considering inductive rather than deductive mathematical proofs, began to make some tentative initial attempts to formulate and utilize notions of complexity. (I’ll tell you more about this in a moment.) And it is here that AIT makes its decisive contribution to philosophy, by providing a highly-developed and elegant mathematical theory of complexity. How does AIT do this? It does this by considering the size of the smallest computer program required to calculate a given object \(X\), which may also be considered to be the most elegant theory that explains \(X\).

Where does Gödel begin to think about complexity? He does so in two footnotes in vol. 3 of his *Collected Works*. The first of these is a footnote to Gödel *1951*. This footnote begins “Such a verification...” and it was reproduced, in part, in Section 2.2 above. And here is the relevant portion of the second, the more interesting, of these two footnotes:

> ...Moreover, if every number-theoretical question of Goldbach type... is decidable by a mathematical proof, there *must* exist an
infinite set of independent evident axioms, i.e., a set \( m \) of evident axioms which are not derivable from any finite set of axioms (no matter whether or not the latter axioms belong to \( m \) and whether or not they are evident). Even if solutions are desired only for all those problems of Goldbach type which are simple enough to be formulated in a few pages, there must exist a great number of evident axioms or evident axioms of great complication, in contradistinction to the few simple axioms upon which all of present day mathematics is built. (It can be proved that, in order to solve all problems of Goldbach type of a certain degree of complication \( k \), one needs a system of axioms whose degree of complication, up to a minor correction, is \( \geq k \).)

This is taken from Gödel *1953/9–III, one of the versions of his unfinished paper “Is mathematics syntax of language?” that was intended for, but was finally not included, in Schilpp’s Carnap festschrift in the same series as the Bertrand Russell festschrift [11].

Unfortunately these tantalizing glimpses are, as far as I’m aware, all that we know about Gödel’s thoughts on complexity. Perhaps volumes 4 and 5, the two final volumes of Gödel’s Collected Works, which contain Gödel’s correspondence with other mathematicians, and which will soon be available, will shed further light on this.

Now let me turn to a completely different—but I believe equally fundamental—application of AIT.

3. How can we partition the world into distinct entities?

For many years I have asked myself, “What is a living being? How can we define this mathematically?!” I still don’t know the answer! But at least

\[ \addtocounter{footnote}{1}\text{[This is reminiscent of the theorem in AIT that } p_k = (\text{the program of size } \leq k \text{ bits that takes longest to halt}) \text{ is the simplest possible “axiom” from which one can solve the halting problem for all programs of size } \leq k. \text{ Furthermore, } p_k \text{’s size and complexity both differ from } k \text{ by at most a fixed number of bits: } |p_k| = k + O(1) \text{ and } H(p_k) = k + O(1). \]

Actually, in order to solve the halting problem for all programs of size \( \leq k \), in addition to \( p_k \), one needs to know \( k - |p_k| \), which is how much \( p_k \)’s size differs from \( k \). This fixed amount of additional information is required in order to be able to determine \( k \) from \( p_k \).]
I think I now know how to come to grips with the more general notion of “entity” or “being.” In other words, how can we decompose our experience into parts? How can we partition the world into its components? By what right do we do this in spite of mystics who like Parmenides insist that the world must be perceived as an organic unity (is a single substance) and cannot be decomposed or analyzed into independent parts?

I believe that the key to answering this fundamental question lies in AIT’s concept of algorithmic independence. What is algorithmic independence? Two objects \(X\) and \(Y\) are said to be algorithmically independent if their complexity is (approximately) additive. In other words, \(X\) and \(Y\) are algorithmically independent if their information content decomposes additively, i.e., if their joint information content (the information content of \(X\) and \(Y\)) is approximately equal to the sum of their individual information contents:

\[
H(X, Y) \approx H(X) + H(Y).
\]

More precisely, the left-hand side is the size in bits of the smallest program that calculates the pair \(X, Y\), and the right-hand side adds the size in bits of the smallest program that produces \(X\) to the size in bits of the smallest program that calculates \(Y\).

Contrariwise, if \(X\) and \(Y\) are not at all independent, then it is much better to compute them together than to compute them separately and \(H(X) + H(Y)\) will be much larger than \(H(X, Y)\). The worst case is \(X = Y\). Then \(H(X) + H(Y)\) is twice as large as \(H(X, Y)\).

I feel that this notion of algorithmic independence is the key to decomposing the world into parts, parts the most interesting example of which are living beings, particularly human beings. For what enables me to partition the world in this way? The fact that thinking of the world as a sum of such parts does not complicate my description of the world substantially and at the same time enables me to use separate subroutines such as “my wife” and “my cat” in thinking about the world. That is why such an analysis of the world, such a decomposition, works.

Whereas on the contrary “my left foot” and “my right hand” are not well thought of as independent components of the world but can best be understood as parts of me. A description of my right hand and its activities and history would not be substantially simpler than a description of me and my entire life history, since my right hand is a part of me whose actions express my intentions, and not its own independent desires.
Of course, these observations are just the beginning. A great deal more work is needed to develop this point of view...

For a technical discussion of algorithmic independence and the associated notion of mutual algorithmic information defined as follows

$$H(X : Y) \equiv H(X) + H(Y) - H(X, Y),$$

see my book Chaitin [17].

4. Conclusion and future prospects

Let’s return to our starting point, to the cover of the January 2003 issue of *La Recherche*. Is God a computer, as Wolfram and some others think, or is God, as Plato and Pythagoras affirm, a mathematician?

And, an important part of this question, is the physical universe discrete, the way computers prefer, not continuous, the way it seems to be in classical Newtonian/Maxwellian physics? Speaking personally, I like the discrete, not the continuous. And my theory, AIT, deals with discrete, digital information, bits, not with continuous quantities. But the physical universe is of course free to do as it likes!

Hopefully pure thought will not be called upon to resolve this. Indeed, I believe that it is incapable of doing so; Nature will have to tell us. Perhaps someday an experimentum crucis will provide a definitive answer. In fact, for a hundred years quantum physics has been pointing insistently in the direction of discreteness.6

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6Discreteness in physics actually began even earlier, with atoms. And then, my colleague John Smolin points out, when Boltzmann introduced coarse-graining in statistical mechanics.
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Chaitin’s papers are also available at [http://cs.auckland.ac.nz/CDMTCS/chaitin](http://cs.auckland.ac.nz/CDMTCS/chaitin)