A Geometric-Probabilistic problem about the lengths of the segments intersected in straights that randomly cut a triangle.

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Abstract. If a line cuts randomly two sides of a triangle, the length of the segment determined by the points of intersection is also random. The object of this study, applied to a particular case, is to calculate the probability that the length of such segment is greater than a certain value.

Let $ABC$ be an isosceles triangle, with $\overline{AB} = \overline{CB}$ and $\overline{OB} = \overline{AC}$, being $O$ the midpoint of $\overline{AC}$ (ie, $\overline{OB}$ is the height relative to the side $\overline{AC}$).

Through a randomly chosen point $P$ on $\overline{AC}$ is drawn a straight $r$ with also randomly chosen slope. Let $Q$ and $R$ be the points where $r$ intersects $\overline{AB}$ and $\overline{CB}$, respectively.

Calculate the probability for the following inequalities:

\[
PQ > AC \text{ or } PR > AC
\] (1)

Let us draw an arc of radius $\overline{AC}$ with center $P$. Let $P$ and $Q$ be the intersection points of this arc with the sides $\overline{AB}$ and $\overline{CB}$, respectively, as shown in the following picture, with the triangle represented in an orthonormal coordinate system, with origin at $O$, $x$-axis (abscissas) in the direction $OA$ and $y$-axis (ordinates) in the direction $OB$. 

![Diagram of the geometric problem](image)
A Geometric-Probabilistic problem about the lengths of the segments intersected in straights that randomly cut a triangle.

Clearly, all the straight lines of the bundle with vertex P in $\overline{AC}$ intersect the sides $\overline{AB}$ or $\overline{CB}$, and all the lines of the sub-bundle inner to the angle $\alpha = \angle QPR$, and only them, satisfies (1).

Since $x$ and $\alpha$ are continuous random variables uniformly distributed, for a differential of length $dx$ in $\overline{AC}$, the probability that the condition (1) is satisfied will be

$$dp = \frac{\alpha}{\pi} dx$$  \hspace{1cm} (2)

and therefore, the probability that the inequalities (1) are satisfied for a randomly chosen point in $\overline{AC}$ will be

$$p = \frac{1}{\pi} \int_{\frac{1}{2}}^{\frac{1}{2}} \alpha(x) \, dx$$  \hspace{1cm} (3)

where $\alpha(x)$ is the function relating the angle $\alpha$ with the abscissa $x$.

We have used the following facts:

- In (2):
  - In an infinitesimal length, $dx$, the limit angle $\alpha$ is constant.
  - The slope of the secant line is independent of the abscissa $x$.

- In (3):
  - The required probability $p$ is obtained by Riemann integration of the probability density function $\alpha(x)$ in the symmetric interval $\overline{AC} = [-\frac{1}{2}, \frac{1}{2}]$.

The limit angle $\alpha$ can be expressed in radians as:

$$\alpha = \pi - \angle QPA - \angle CPR$$  \hspace{1cm} (4)

But,

$$\triangle QPA: \quad \angle QPA = \pi - \hat{A} - \hat{Q}$$  \hspace{1cm} (5)

$$\triangle CPR: \quad \angle CPR = \pi - \hat{C} - \hat{R}$$  \hspace{1cm} (6)

So,

$$\alpha = \hat{Q} + \hat{R} + \hat{A} + \hat{C} - \pi$$  \hspace{1cm} (7)

Perhaps the easiest way to define $\alpha$ as a function of $x$ is trigonometrically:

$$\triangle QPA: \quad \frac{\sin \hat{A}}{\overline{PQ}} = \frac{\sin \hat{Q}}{\overline{AP}} \Rightarrow \sin \hat{Q} = \frac{\overline{AP}}{\overline{PQ}} \sin \hat{A}$$  \hspace{1cm} (8)

$$\triangle CPR: \quad \frac{\sin \hat{C}}{\overline{PR}} = \frac{\sin \hat{R}}{\overline{CP}} \Rightarrow \sin \hat{R} = \frac{\overline{CP}}{\overline{PR}} \sin \hat{C}$$  \hspace{1cm} (9)
A Geometric-Probabilistic problem about the lengths of the segments intersected in straights that randomly cut a triangle.

But \( \hat{C} = \hat{A} \) and \( \tan \hat{A} = 2 \). Moreover, without loss of generality, we can assume that

\[
\overline{AC} = \overline{OB} = 1
\]  

Hence,

\[
\sin \hat{A} = \sin \hat{C} = \frac{2}{\sqrt{5}}
\]

Applying (10) and (11) to (8) and (9), these reduce to

\[
\sin \hat{Q} = \frac{1 - 2x}{\sqrt{5}}
\]

\[
\sin \hat{R} = \frac{1 + 2x}{\sqrt{5}}
\]

Substituting in (7) this results we get the **probability density function** for the random variable \( \alpha \):

\[
\alpha(x) = \arcsin \left( \frac{1 - 2x}{\sqrt{5}} \right) + \arcsin \left( \frac{1 + 2x}{\sqrt{5}} \right) + 2 \arctan(2) - \pi
\]

Now, substituting in (3) the result given in (12), we obtain,

\[
p = \frac{1}{\pi} \left[ \int_{-\frac{1}{\sqrt{5}}}^{\frac{1}{\sqrt{5}}} \arcsin \left( \frac{1 - 2x}{\sqrt{5}} \right) dx + \int_{-\frac{1}{\sqrt{5}}}^{\frac{1}{\sqrt{5}}} \arcsin \left( \frac{1 + 2x}{\sqrt{5}} \right) dx \right] + \frac{2}{\pi} \arctan(2) - 1
\]

These integrals (in indefinite form) can be solved by *integration by parts*. Let

\[
I_1 = \int \arcsin \left( \frac{1 - 2x}{\sqrt{5}} \right) dx
\]

\[
I_2 = \int \arcsin \left( \frac{1 + 2x}{\sqrt{5}} \right) dx
\]

\[
I_1 = uv - \int v \, du
\]

\[
u = \arcsin \left( \frac{1 - 2x}{\sqrt{5}} \right) \quad du = \frac{-2}{\sqrt{5} \sqrt{1 - \left( \frac{1 - 2x}{\sqrt{5}} \right)^2}} dx
\]

\[
v = x \quad v = \frac{2}{\sqrt{5} \sqrt{1 - \left( \frac{1 - 2x}{\sqrt{5}} \right)^2}}
\]
A Geometric-Probabilistic problem about the lengths of the segments intersected in straights that randomly cut a triangle.

And applying the formula of integration by parts,

\[ I_1 = uv - \int vdu = x \arcsin \left( \frac{1-2x}{\sqrt{5}} \right) - \int \frac{2x}{\sqrt{1 - (\frac{1-2x}{\sqrt{5}})^2}} dx \]  

(16)

Let

\[ I_3 = \int \frac{\frac{2x}{\sqrt{1 - (\frac{1-2x}{\sqrt{5}})^2}}}{\sqrt{1 - (\frac{1-2x}{\sqrt{5}})^2}} dx \]  

(17)

After simplifying the sub-integral expression, through the elementary transformations shown below, \( I_3 \) is reduced to two quasi-immediate integrals (reducible to immediate integrals by simple adjustment of constants). Omitting integration constants, for simplicity:

\[ I_3 = \int \frac{-x dx}{\sqrt{-x^2 + x + 1}} = \int \frac{-2x + 1 - 1}{2\sqrt{-x^2 + x + 1}} dx = \int \frac{-2x + 1}{2\sqrt{-x^2 + x + 1}} dx - \int \frac{dx}{2\sqrt{-x^2 + x + 1}}, \]

(18)

Let

\[ I_4 = \int \frac{dx}{2\sqrt{-x^2 + x + 1}} \]  

(19)

\[ I_4 = \int \frac{dx}{\sqrt{5 - (1-2x)^2}} = \int \frac{1}{\sqrt{5}} \frac{dx}{\sqrt{1 - \left( \frac{1-2x}{\sqrt{5}} \right)^2}} = -\frac{1}{2} \int \frac{-2}{\sqrt{5}} \frac{dx}{\sqrt{1 - \left( \frac{1-2x}{\sqrt{5}} \right)^2}} \]

(20)

From (18) and (19), \( I_3 = \sqrt{-x^2 + x + 1} - I_4 \); substituting in this the result given by (20),

\[ I_3 = \sqrt{-x^2 + x + 1} - \frac{1}{2} \arcsin \left( \frac{1-2x}{\sqrt{5}} \right) \]  

(21)

From (16), \( I_1 = x \arcsin \left( \frac{1-2x}{\sqrt{5}} \right) - I_3 \), and substituting therein the result given by (21),

\[ I_1 = \left( x - \frac{1}{2} \right) \arcsin \left( \frac{1-2x}{\sqrt{5}} \right) - \sqrt{-x^2 + x + 1} \]  

(22)

And by a procedure completely analogously, we obtain

\[ I_2 = \left( x + \frac{1}{2} \right) \arcsin \left( \frac{1+2x}{\sqrt{5}} \right) + \sqrt{-x^2 - x + 1} \]  

(23)
A Geometric-Probabilistic problem about the lengths of the segments intersected in straights that randomly cut a triangle.

Substituting in (13) this results, we obtain the exact value of the requested probability:

\[
p = \frac{1}{\pi} \left[ l_1 + l_2 \right] \frac{1}{2} + \frac{2}{\pi} \arctan(2) - 1 = \frac{1}{\pi} \left[ 2 \arctan \left( \frac{1}{3} \right) + \frac{\pi}{2} + 1 - \sqrt{5} \right] + \frac{2}{\pi} \arctan(2) - 1
\]

\[
p = \frac{2}{\pi} \left[ \arctan \left( \frac{1}{3} \right) + \arctan(2) \right] - \frac{\sqrt{5} - 1}{\pi} - \frac{1}{2}
\]

(24)

The expression (24) can be simplified considering the definition of the golden ratio [1] and the following identity regarding tangent arcs (by the general shape established in [2] for the decomposition of \( \pi / 4 \) in two \( \arctan \)):

\[
\arctan(2) = \arctan \left( \frac{1}{3} \right) + \frac{\pi}{4}
\]

(25)

This identity can be proven easily by the formula of the tangent of a sum or through algebra of complex numbers, expressing the product of two complex numbers (suitably chosen) in two representation forms, binary form and polar form, as shown below.

Product in \textit{binary form} and its corresponding representation in \textit{polar form}:

\[
(3 + i)(1 + i) = (2 + 4i) \iff \sqrt{10} \arctan \left( \frac{1}{3} \right) \sqrt{2} = \sqrt{20} \arctan(2)
\]

After performed the product in polar form, the identity (25) is derived by identifying the arguments on both sides of the last equality:

\[
\sqrt{20} \arctan \left( \frac{1}{3} \right) + \frac{\pi}{4} = \sqrt{20} \arctan(2)
\]

(26)

Finally, the result (24) can be expressed in the following elegant form that involves two of the most remarkable numbers: the \textbf{number \( \pi \)} and the \textbf{Golden Ratio \( \Phi \)},

\[
\Phi = \frac{1 + \sqrt{5}}{2}
\]

(27)

As the number \( \pi \), it is surprising the ubiquity of this number, that emerge in the most diverse sceneries [1].

\[
p = \frac{2}{\pi} \left( 2 \arctan \left( \frac{1}{3} \right) - \frac{1}{\Phi} \right)
\]

(28)

The approximate value of \( p \) in ten thousandths is, \( p \approx 0.0162 \).
A Geometric-Probabilistic problem about the lengths of the segments intersected in straights that randomly cut a triangle.

References.

[1] Livio, Mario (2002). *The Golden Ratio: The Story of Phi, The World’s Most Astonishing Number*. New York: Broadway Books. ISBN 0-7679-0815-5.

[2] Tanton, James (2012). *Mathematics Galore! The First Five Years of the St. Mark’s Institute of Mathematics*. The Mathematical Association of America. Washington. ISBN 978-0-88385-776-2.