Induced Polyakov supergravity on Riemann surfaces of higher genus

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Abstract

An effective action is obtained for the $N = 1, 2D$–induced supergravity on a compact super Riemann surface (without boundary) $\hat{\Sigma}$ of genus $g > 1$, as the general solution of the corresponding superconformal Ward identity. This is accomplished by defining a new super integration theory on $\hat{\Sigma}$ which includes a new formulation of the super Stokes theorem and residue calculus in the superfield formalism. Another crucial ingredient is the notion of polydromic fields. The resulting action is shown to be well-defined and free of singularities on $\hat{\Sigma}$. As a by-product, we point out a morphism between the diffeomorphism symmetry and holomorphic properties.
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1 Introduction

In some recent publications \[1, 10\] results concerning the factorization of partition functions in superstring theories with non-vanishing central charge were obtained. This supersymmetric generalization of well-established facts on holomorphic factorization in string theory \[15\] consists in particular in subtracting from the effective action a local counterterm that allows one to go from the super Weyl to the superdiffeomorphism anomaly defined on a compact super Riemann surface (SRS) without boundary. The characteristic feature of the latter anomaly is that it appears as the sum of two contributions corresponding respectively to the holomorphic and antiholomorphic sectors of the theory. Furthermore it has been shown \[10\] that, for a generic value of the central charge, the holomorphic factorization of partition functions remains true for free superconformal fields when these functions are considered as functionals of the Beltrami coefficients and their fermionic partners.

Until now a generalization of these studies to a generic SRS has been worked out only for the supertorus \((g = 1)\) \[2\], providing thereby an explicit expression for the Polyakov action thereon. Here we extend these results to a compact SRS (without boundary) of arbitrary genus. The demonstration and formulation obtained are similar in spirit to those of the bosonic case \[23, 24\] but encounter considerable complications inherent to the superspace formulation. Some difficulties have been overcome through the systematic use of covariant derivatives and “tensor” notations. These technical tools are not mandatory in the bosonic case but become indispensable in the present more complicated situation. The other difficulty comes from the use of objects which can be singular on the SRS and require a careful study of their analytic properties. Obviously it is always possible, in principle, to consider their component expansion expressing these superfields in terms of usual fields which possess well-known analytic properties on the underlying Riemann surface. However this path can be technically very complicated and in fact renders the superfield approach useless and rather artificial. On the contrary, as we will show, it is possible to derive the general Polyakov action on a SRS of any genus entirely in terms of superfields throughout the calculations.

As a by-product, we have obtained a super analog of Stokes theorem (subsect.2.2) allowing us to perform any integral (when only integrable singularities are present) over a compact SRS by using super Cauchy theorems, without having recourse to component expansion. Indeed, this was possible by defining a new super integration procedure, which is not only suitable for our purpose but is interesting in its own right as well, and could be used for further generalizations. Furthermore, we have noticed a canonical morphism between the superdiffeomorphism transformations and holomorphy properties of the superfields on a SRS (sect.3).

Let us first review the properties of the \(N = 1\) compact SRS \(\hat{\Sigma}\) to establish
the framework in which we are working. This surface is locally described by a pair of complex coordinates \((z, \theta)\) with \(\theta\) anticommuting \(\circ\), and glued up from local charts by superconformal transition functions, i.e. two local coordinate charts are related by transformations that satisfy the superconformal condition \[ D_\theta \tilde{z} = \tilde{\theta} D_\theta \tilde{\theta}, \] where \(D_\theta = \partial_\theta + \theta \partial_z\) is the superderivative. Here \(\Sigma\) is of De Witt type, i.e. a SRS to which is associated a corresponding compact Riemann surface \(\Sigma\) of genus \(g\), called its body, with a particular spin structure. Such a SRS is adequate for a picture of a moving superstring in spacetime \([\circ]\).

In addition to the reference supercomplex structure \(\{(z, \theta)\}\), a SRS can be provided with the so-called projective structure. This is a collection of local homeomorphisms \((\hat{Z}_\alpha, \hat{\Theta}_\alpha)\) of \(\hat{\Sigma}\) into \(\mathbb{C}^{1|1}\), obeying the gluing laws \(\circ\) on overlapping domains \([\circ]\).

\[
\begin{align*}
\tilde{Z} & = a\hat{Z} + b \frac{c\hat{Z} + \beta}{c\hat{Z} + d}(c\hat{Z} + d)^2 \\
\tilde{\Theta} & = a\hat{\Theta} + \frac{\beta}{c\hat{Z} + d} + \hat{\Theta} \frac{1}{c\hat{Z} + d} (1 + \frac{1}{2}\alpha\beta)
\end{align*}
\] (1)

This structure is related to the reference structure \(\{(z, \theta)\}\) by a quasisuperconformal transformation on \(\hat{\Sigma}\) which is parametrized by a pair of fields, the super Beltrami coefficient \(H_{\hat{\theta}}\) (which contains the ordinary Beltrami coefficient) and the super Schwarzian derivative \(S(\hat{Z}, \hat{\Theta}; z, \theta)\) (and c.c) \([\circ]\). In fact the number of independent Beltrami coefficients is greater than one; the parametrization above corresponds to a particular choice which has been proved to be always possible \([\circ]\) and is adopted now. This choice corresponds to the special case \(H_{\hat{\theta}} = 0\) and is equivalent to the condition \(D_\theta \hat{Z} = \hat{\Theta} D_\theta \hat{\Theta}\). Henceforth, unless otherwise stated, holomorphy will be understood with respect to the projective structure \(\{(\hat{Z}, \hat{\Theta})\}\), this is sometimes referred to as the \(H\)–structure, while the reference structure will be referred to as the 0–structure. Objects that are holomorphic with respect to the latter structure are indexed by a subscript 0.

## 2 The Polyakov action on a SRS

The superspace generalization of Polyakov’s chiral gauge action proposed by Grundberg and Nakayama \([\circ]\) for the planar topology is

\[
\Gamma[H_{\hat{\theta}}^z] = \int_{\text{SC}} d^2 \lambda \partial_z \zeta_{\hat{\theta}} H_{\hat{\theta}}^z
\] (2)

\(\circ\) Obviously it is understood that the complex conjugate (c.c) conditions are also taken into account.

\(\circ\) The matrix \(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\) belongs to \(SL(2, \mathbb{C})\) whereas \(\alpha\) and \(\beta\) are odd Grassmann numbers.
where \( \zeta_\theta \) is the coefficient of a super affine connection built out of the solution of the super Beltrami equation \( (\hat{Z}, \hat{\Theta}) \) as follows

\[
\zeta_\theta = -D_\theta \ln D_\theta \hat{\Theta}.
\]

This allows us to build a superprojective connection \[9\]

\[
R_{z\theta} = -\partial_z \zeta_\theta - \zeta_\theta D_\theta \zeta_\theta
\]

which under a coordinate transformation behaves like a super Schwarzian derivative.

The measure in eq.\( (2) \) reads

\[
d^2\lambda = \frac{d\lambda \wedge d\bar{\lambda}}{2i}
\]

where \( d\lambda = (dz|d\theta) \) is the generator of the supercanonical line bundle over \( \hat{\Sigma} \) whose body is a spin bundle over the underlying Riemann surface \( \Sigma \) [17]. The generator \( d\bar{\lambda} \) is defined in a similar way.

The covariant superderivative \( \hat{\nabla}_\zeta \) associated to the superaffine connection \( \zeta_\theta \) is defined by its action on superfields or operators of conformal weights \( p \) corresponding to the \( (z, \theta) \) sector (for instance, \( p(H_{\bar{\theta}}) = -1, p(D_\theta) = \frac{1}{2} \)) by \[4\]

\[
\hat{\nabla}_\zeta = D_\theta + 2p\zeta_\theta
\]

and consequently the covariantization of \( \partial_z \) is given by

\[
\Delta_\zeta = \partial_z + 2pD_\theta \zeta_\theta + \zeta_\theta D_\theta.
\]

When acting on itself or on another super affine connection \( \zeta' \) the covariant derivative defines the corresponding “field strength”

\[
\Phi_{\zeta\zeta'} = D_\theta \zeta' + p(\zeta')\zeta'.
\]

Remembering that \( \zeta_\theta \) and \( \zeta'_\theta \) are superfields of type \( (\frac{1}{2}, 0) \) and by applying once more the covariant superderivative, we get \( (\zeta' = \zeta'_\theta) \)

\[
\Delta_\zeta \zeta' = \partial_z \zeta' + \frac{1}{2} \zeta_\theta D_\theta \zeta' + \frac{1}{2} \zeta' D_\theta \zeta_\theta
\]

which is the covariantization of \( \partial_z \zeta' \); here the derivative of the superaffine connection has been changed (up to a sign) into a superprojective connection.

So far, a proper generalization of eq.\( \text{(2)} \) to higher genus SRS has not been found except for the case of the supertorus \[2\]. The integrand of the globally defined super Polyakov action for genus \( g = 1 \) is given by

\[
A_{ST} = 4[(R_{z\theta} + \Delta_\zeta_\theta \zeta_\theta)H_{\bar{\theta}} + \frac{1}{2}(\zeta_\theta - \zeta_0)\Delta_{z\theta} H_{\bar{\theta}}]
\]
where \( \zeta_\theta \) is the coefficient of a superaffine connection on the supertorus, given by the same equation as (3) and \( \zeta_0 \equiv \zeta_{\theta 0} \) the holomorphic superaffine connection in the reference structure.

This expression solves the Ward identity

\[
 s(\Gamma[H, R_{z\theta}] + c.c) = \int_{ST} d^2\lambda [A(C^\theta, H^\theta, R_{z\theta}) + c.c]
\]

where \( s \) is the BRST operator \(^3\) and \( A(C^\theta, H^\theta, R_{z\theta}) \) is the chirally split form of the globally defined (non integrated ) superdiffeomorphism anomaly \(^7\)

\[
 A(C^\theta, H^\theta, R_{z\theta}) = C^\theta \partial^2_z D_\theta H^\theta + H^\theta \partial^2_\theta D_\theta C^\theta + 3R_{z\theta}(C^\theta \partial_\theta H^\theta - H^\theta \partial_\theta C^\theta) \\
 + D_\theta R_{z\theta}(C^\theta D_\theta H^\theta + H^\theta D_\theta C^\theta).
\]

\( C^\theta \) is the superdiffeomorphism ghost field and \( R_{z\theta} \) is a holomorphic superprojective connection which renders the expression above well-defined.

Now our starting point for the further generalization of (9) to a SRS of genus \( g > 1 \) is the following functional which is inspired from the one found in the bosonic case in ref.\(^{23}\).

\[
 \Gamma_{\hat{1}}[R_{z\theta}, H^\theta] = \frac{1}{4\pi} \int_{\hat{\Sigma}} d^2\hat{\lambda} \hat{A}_1
\]

where

\[
 \hat{A}_1 = 4R_{z\theta}H^\theta + 2\partial_\theta D_\theta H^\theta + 2\chi_0 D_\theta \chi H^\theta + 2\chi D_\theta \chi_0 H^\theta \\
 - D_\theta \chi_0 D_\theta H^\theta - D_\theta \chi D_\theta H^\theta - \chi_0 \partial_\theta H^\theta - \chi \partial_\theta H^\theta
\]

This expression can also be written in a compact form as

\[
 \hat{A}_1 = 4(R_{z\theta} - R_{\chi_0})H^\theta + D_\theta(\Delta \chi + \Delta \chi_0)H^\theta
\]

which thus becomes obviously globally defined.

Here, the coefficients \( \chi \) and \( \chi_0 \) are the superaffine connections related to the polydromic \( \frac{1}{2} \)-superdifferentials \( \Psi, \Psi_0 \) by

\[
 \chi = -D_\theta \ln \Psi, \quad \chi_0 = -D_\theta \ln \Psi_0.
\]

The coefficients \( \Psi, \Psi_0 \) are superholomorphic differentials in the H-structure and 0-structure respectively. They are free of zeros and consequently are multivalued objects on \( \hat{\Sigma} \). \( R_{\chi_0} \) is a particular holomorphic superprojective connection in the H-structure related to \( \chi, \chi_0 \) by

\[
 R_{\chi_0} = \frac{1}{2}(\Delta \chi \chi_0 + \Delta \chi_0 \chi)
\]

Let us now discuss these

\(^3\)In superspace the s-operator is assumed to act as an antiderivation from the right; the BRST algebra is graded by the ghost number, but does not feel the Grassman parity.
2.1 Polydromic fields

According to the super Riemann-Roch Theorem \[17, 20\], there are \( g \) holomorphic single-valued \( \frac{1}{2} \)–superdifferentials which have each globally \((g - 1)\) (counting multiplicity) zeros on a compact SRS of genus \( g \). We shall consider in the following two kinds of these differentials; the holomorphic ones w.r.t. the \( H \)–structure will be denoted by \( \eta \) and the holomorphic objects in the 0–structure by \( \eta_0 \). In contrast, the multivalued superdifferentials \( \Psi, \Psi_0 \) considered in eq.(15) are free of zeros. They can be written, in general, as \[12\]

\[
\Psi = (D_\theta \hat{\Theta})\xi(\hat{Z}, \hat{\Theta}), \quad \Psi_0 = (D_\theta \hat{\Theta}_0)\xi'_0(z, \theta) \quad (17)
\]

where \( \hat{\Theta}_0 \) is the Grassmann coordinate of another reference structure related to the 0–structure by a superconformal transformation. The expressions given above indeed transform as \( \frac{1}{2} \)–superdifferentials since \( D_\theta \hat{\Theta} \) does. The functions \( \xi(\hat{Z}, \hat{\Theta}), \xi'_0(z, \theta) \) are multivalued and nowhere vanishing \[12\].

However, to deal with these differentials, there are two equivalent methods. The first one considers these objects as defined on a universal covering of the SRS and invariant under the corresponding covering group. Then one chooses a fundamental domain where these differentials (in fact their coefficients) become single-valued. This insures that the whole expression they appear in is single-valued. However a second approach consists in choosing a branch for each of these differentials by working on a dissection of the SRS into its polygon. Mimicking the bosonic case \[23\] we cut the SRS \( \hat{\Sigma} \) into its polygon whose reduced domain is the polygon of the underlying Riemann surface. This dissection has \( 4g \) pairwise opposite sides and will be denoted by \( \hat{D} \). This is assumed to contain all zeros of \( \eta_0 \) while excluding those of \( \eta \). In the bosonic case, this was shown to be always possible \[23\], and even when zeros of the (corresponding) bosonic differentials \( \omega \) and \( \omega_0 \) overlap, the total Polyakov action is still continuous \[24\]. Using the trivial topology of De Witt, one can repeat the same demonstration on \( \hat{\Sigma} \). Next we cut infinitesimal superloops \( C_k \) around the zeros \( P_{hk} \) of \( \eta_0 \) with opposite orientation in regard to that of the boundary \( \partial \hat{D} \) of \( D \). These superloops are composed of their bodies (ordinary circles in \( \Sigma \)) which define the corresponding orientation and some “Grassmann circles” (defined below) over them.

The s-variation of a superaffine connection \[2\] is simple when expressed in terms of covariant derivatives of the ghost field \( C^z \)

\[
s_\chi = -\frac{1}{2} D_\theta \Delta_\chi C^z \quad (18)
\]

whereas other useful transformation laws are

---

\[4\]See in particular \[21\] for a proof in both cases of a split and non-split SRS.

\[5\]We are indebted to R. Zucchini for valuable discussions on this subject and in particular on polydromic differentials.
\[ s(\Delta_{\chi} H_\theta) = D_\theta \Delta_{\chi} C^z ; \quad s(\Delta_{\chi} \chi) = \frac{1}{2} \Delta_{\chi} D_\theta \Delta_{\chi} C^z. \quad (19) \]

Now since the superaffine connection \( \chi_0 \) is \( s \)-invariant, using the above laws and the BRST transformation of \( H_\theta \) given in Ref.[6] we obtain, after lengthy but straightforward calculations

\[ s \Gamma_1[R_{z\theta}, H_\theta] = \frac{1}{4\pi} \int d^2 \lambda \mathcal{A}(C^z, H_\theta^z, R_{z\theta}) + K_1 + K_2 + K_3 \quad (20) \]

where

\[
\begin{align*}
K_1 &= \frac{1}{2i\pi} \oint_{\partial \hat{D}(\eta_0)} d\lambda[(R_{z\theta} - R_{x0})C^z] \\
K_2 &= -\frac{1}{8i\pi} \oint_{\partial \hat{D}(\eta_0)} d\lambda(\Delta_{\chi} s H_\theta^z + \Delta_{\chi_0} D_\theta C^z + 2C^z D_\theta D_\theta \chi + D_\theta C^z D_\theta C^z) \\
K_3 &= -\frac{1}{8i\pi} \oint_{\partial \hat{D}(\eta_0)} d\lambda \left[ \frac{1}{2}(\chi_0 - \chi) D_\theta (D_\theta H_\theta^z D_\theta C^z) - \chi_0 \chi (D_\theta H_\theta^z D_\theta C^z) \right. \\
& \quad \left. - (\chi_0 D_\theta \chi + \partial_z \chi - \chi_0 \chi D_\theta + R_{z\theta})(C^z D_\theta H_\theta^z + H_\theta^z D_\theta C^z) \right] \\
\end{align*}
\]

and \( \mathcal{A}(C^z, H_\theta^z, R_{z\theta}) \) is the globally defined non-integrated anomaly already given by the same expression as in eq.(11) but now defined on \( \hat{\Sigma} \).

First we wish to stress that the terms in eq.(21) are separately globally defined. This is obvious for \( K_1 \) since its integrand is the product of the difference of two superprojective connections (i.e. a superquadratic differential) and \( C^z \) which transforms homogeneously. Concerning \( K_2 \) and \( K_3 \) this property has to be checked by hand. While \( K_2 \) contains the bosonic limit (see ref.[23], eq.(2.15)), the fourth term \( K_3 \) gathers the complications and novelties emerging from the supersymmetric formulation. Note that all these terms are in fact untractable because they all involve multivalued fields. Nevertheless, we will show later that these terms can be eliminated by adding other contributions to the Polyakov action, thus leaving the anomaly only.

To explain the line integrals in eqs.(21) we now present our integration procedure over a (compact) SRS with respect to which our solution to the superconformal Ward identity (10) will be defined. Most importantly, we give the definition of the boundary of a superdomain \( \hat{D} \) in \( \hat{\Sigma} \), and the analog of Stokes theorem to relate integration over superdomains on \( \hat{\Sigma} \) and integration on their boundaries.

### 2.2 Integration on \( \hat{\Sigma} \)

In our developments, the expression that we integrate over \( \hat{\Sigma} \) is a \((\frac{1}{2}, \frac{1}{2})\)-superdifferential which is, in general, meromorphic. More precisely, this expression happens to
be a function of singular objects like $\partial \log \eta_0$ (or $D_\theta \log \eta_0$) inside a domain $\hat{D}$ containing all zeros of $\eta_0$. To perform explicitly the corresponding integral and in particular the residue calculus, we first need an analog of Stokes theorem and a consistent procedure of integration on the boundary $\partial \hat{D}$ of $\hat{D}$. In fact, integration on $\partial \hat{D}$ reduces to the sum of integrals over small “circles” $C_k$ surrounding the zeros $P_{0k} = (z_{0k}, \theta_{0k})$ of $\eta_0$; the orientation of these circles being opposite to that of $\partial \hat{D}$. Since the remaining integration path is a sequence of pairs of geometrically coinciding but oppositely oriented arcs, this yields pairs of mutually cancelling contributions when the integrand is single-valued. Summing up, the only remnant is the contribution coming from the $C_k$’s. As explained above, these circles are composed of ordinary circles in $\Sigma$ and some “Grassmann circles” $C_\theta$ around the singular point $\theta = \theta_0$, $\bar{\theta} = \bar{\theta}_0$. $C_\theta$ is defined in such a way that the identity

$$\int_{C_\theta} d\theta f(z, \theta, \bar{z}, \bar{\theta}) = (\partial_\theta f)(z, \theta_0, \bar{z}, \bar{\theta}_0) \quad (22)$$

holds for every (locally) smooth function $f$ on $\hat{\Sigma}$. Thus any integration over Grassmann numbers is performed over $C_\theta$ instead of the whole space of Grassmann variables as is done in the standard Berezin integration. The usual Berezin rules obviously hold (in particular) on $C_\theta$

$$\int_{C_\theta} d\theta = 0, \quad \int_{C_\theta} d\theta(\theta - \theta_0) = 1, \quad \int_{C_\theta} d\theta(\bar{\theta} - \bar{\theta}_0) = 0. \quad (23)$$

However, our integration procedure (22) marks a crucial departure from that of Berezin by the result

$$\int_{C_\theta} d\theta(\theta - \theta_0)(\bar{\theta} - \bar{\theta}_0) = 0 \quad (24)$$

Indeed, in our point of view $\bar{\theta}$ is treated somehow as the complex conjugate of $\theta$ and hence $\theta$ (or $\bar{\theta}$) can not be taken out of the (line) integral over $\bar{\theta}$ (or $\theta$), since these variables are in fact linked on $C_\theta$. This super integration formalism reproduces the results obtained by the component expansion while avoiding the well-known difficulties of this procedure, especially when the superfields are singular. Obviously, the rules (22)-(24) are the same if $C_\theta$ is a circle around the point $\theta_0 = 0, \bar{\theta}_0 = 0$.

The rule (24) (in addition to others ) is, as we will show below, crucial to establish the super Stokes theorem, which thereby reads $^6$

$$\int_{\hat{\Sigma}} \hat{d}\Phi = \oint_{\partial \hat{D}} \Phi \quad (25)$$

where the coboundary operator $\hat{d}$ is defined by its action on a $(p/2, q/2)$–superdifferential $\Phi$ as follows $^7$

$$\hat{d}\Phi = (d\lambda D_\theta + (-1)^{(p+q)}d\bar{\lambda}D_{\bar{\theta}})\Phi. \quad (26)$$

$^6$See [4, 22] for other formulations of this theorem.

$^7$d will be denoted $d_+$ or $d_-$ when $(p + q)$ is even or odd respectively.
It is straightforward to see that the operator $\hat{d}$ is nilpotent as it must be, i.e. $\hat{d}^2 = 0$.

Now let us check that the theorem (25) is indeed based on the integration rules (22)-(24). First we recall that the expression that can be integrated over a SRS is a $(\frac{1}{2}, \frac{1}{2})$—superdifferential, and hence in (23) $\Phi$ is a linear combination of $(\frac{1}{2}, 0)$— and $(0, \frac{1}{2})$—superdifferentials. So, let us simply consider a $\frac{1}{2}$—superdifferential $\Phi = \Phi_\theta d\lambda$, then expand the even Grassmann coefficient $\Phi_\theta$ in its $\theta, \bar{\theta}$ components around the point $\theta = \theta_0, \bar{\theta} = \bar{\theta}_0$, that is

$$\Phi_\theta(z, \theta, \bar{z}, \bar{\theta}) = \phi_0 + (\theta - \theta_0)\phi_1 + (\bar{\theta} - \bar{\theta}_0)\phi_2 + (\theta - \theta_0)(\bar{\theta} - \bar{\theta}_0)\phi_3.$$  \hspace{1cm} (27)

This yields

$$\hat{d}\Phi = (d_-\Phi_\theta) \wedge d\lambda = (D_\theta\Phi_\theta)d\bar{\lambda} \wedge d\lambda$$

$$= [\phi_2 - (\theta - \theta_0)\phi_3 + (\bar{\theta} - \bar{\theta}_0)\bar{\partial}\phi_0 - (\theta - \theta_0)(\bar{\theta} - \bar{\theta}_0)\bar{\partial}\phi_1]d\bar{\lambda} \wedge d\lambda.$$  

Now due to the usual Berezin integration rules (23), we get

$$\int_{\mathcal{D}} \hat{d}\Phi = -\int_{\mathcal{D}} d\bar{\lambda} \wedge d\lambda(\theta - \theta_0)(\bar{\theta} - \bar{\theta}_0)\bar{\partial}\phi_1 = -\int_{\mathcal{D}} dz \wedge d\bar{z}\bar{\partial}\phi_1$$

$$= \oint_{\partial\mathcal{D}} \bar{\partial}\phi_1 dz$$

where in the last step we used the usual Stokes theorem on $\mathcal{D}$, the underlying domain of $\hat{D}$.

The right hand side of (25) is readily computed using now the rules (22)-(24),

$$\oint_{\partial\mathcal{D}} \Phi = \oint_{\partial\mathcal{D}} \phi_1 dz$$

thus showing that the left and right hand sides of (25) indeed coincide.

Here we wish to emphasize that this theorem is actually more subtle than it appears to be, since it allows us to compute integrals in the superfield formalism and thus spares us the generally cumbersome procedure of expanding superfields in their components, especially when these are singular, since there is no telling, in general, which components exhibit the corresponding singularities.

Now using these integration rules for Grassmann variables, the integral of a meromorphic $(\frac{1}{2}, 0)$— or $(0, \frac{1}{2})$—superdifferential $\Phi$ over the circles $\mathcal{C}_k$ is performed by using the (generic) local behaviour around $P_0$:

$$\Phi \sim \sum_j^N \frac{f_j}{(z - z_{0k} - \theta\theta_0)^j}.$$  \hspace{1cm} (28)

---

8Here $\Phi$ is explicitly written as $\Phi = \Phi_\theta(d\lambda)^p(d\bar{\lambda})^q$ and the operators $D_\theta$ and $D_{\bar{\theta}}$ act directly on the coefficient $\Phi_\theta$. We find $d_+d_- = -d_-d_+ = 0$.

9$N$ is the number of terms in $\Phi$. 

9
where the coefficient functions $f_j$ are superholomorphic around $P_0$; i.e. they do not depend on $(z, \bar{\theta})$ inside an open neighbourhood of $P_0$ contained in $C_k$. Then we get the final result with the help of the two super Cauchy theorems:

\[
\frac{1}{2i\pi} \oint_{C_k} d\lambda f(z, \theta)(z - z_k - \theta \theta_k)^{-n-1} = \frac{1}{n!} \partial_{z_k}^n D_{\theta_k} f(z_k, \theta_k)
\]

\[
\frac{1}{2i\pi} \oint_{C_k} d\lambda f(z, \theta)(\theta - \theta_k)(z - z_k - \theta \theta_k)^{-n-1} = \frac{1}{n!} \partial_{z_k}^n f(z_k, \theta_k)
\]

(29)

2.3 Other contributions to the Polyakov action and residue calculus

From the experience gained in the construction of the Polyakov action on an ordinary Riemann surface [23], we know that at least two other terms are necessary to describe the super Polyakov action for any genus $g$, namely one which has to be a $(0, \frac{1}{2})$-superdifferential and another which makes explicit some residues on the dissected SRS so as to cancel out the terms $K_1, K_2, K_3$ in eqs.(20).

Therefore we introduce the following contribution to the Polyakov action

\[
\Gamma_2[H_\psi^2] = \frac{1}{2i\pi} \oint_{\partial\mathcal{D}(\eta_0)} \ln(\Psi/\eta_0) \hat{d} \ln(\eta/\eta_0) \equiv \frac{1}{2i\pi} \oint_{\partial\mathcal{D}(\eta_0)} \hat{A}_2
\]

(30)

Then under the following BRST transformation laws

\[
s \ln \Psi = \frac{1}{2} \Delta^C \chi^C, \ s \Psi_0 = 0, \ s \ln \eta = \frac{1}{2} \Delta^C \chi_0^C, \ s \eta_0 = 0
\]

(31)

the response of the functional $\hat{A}_2$ reads

\[
s \hat{A}_2 = \hat{d}[s \ln \eta \ln(\Psi/\eta_0)] + d\lambda T_\lambda + d\bar{\lambda} \bar{T}_\lambda
\]

(32)

where

\[
T_\lambda = \frac{1}{2} D_\theta \phi + (R_{z\theta} - R_{\chi_0}) C^z - (R_{z\theta} - R_{\chi_0}) \chi_0 C^z
\]

(a)

\[
\bar{T}_\lambda = s \ln \Psi D_\bar{\theta} \ln(\eta/\Psi_0) - s \ln \eta D_\bar{\theta} \ln(\Psi/\eta_0)
\]

(b)

(33)

and

\[
\phi = D_\theta [C^z(\chi_0 + \chi - \zeta_0 - \zeta)] + (\zeta \zeta_0 - \chi \chi_0 - 2\zeta \chi) C^z
\]

(34)

is globally defined (i.e $\bar{\phi} \equiv \phi$). $\zeta_0$ and $\zeta$ are the superaffine connections built out of $\eta_0$ and $\eta$ respectively

\[
\zeta_0 = -D_\theta \ln \eta_0, \quad \zeta = -D_\theta \ln \eta.
\]

Note that the projective connection $R_{z\theta}$ in (33a) is needed to ensure the appropriate gluing of the integrand and may be replaced by any other superprojective connection which is holomorphic and single-valued on the integration domain.
Using the holomorphy equations satisfied by $\Psi$, $\Psi_0$, $\eta$, $\eta_0$

$$D_\theta \ln \Psi = \frac{1}{2} \Delta_z H^z_\theta \quad , \quad D_\theta \Psi_0 = 0 \quad , \quad D_\theta \ln \eta = \frac{1}{2} \Delta_z H^z_\theta \quad , \quad D_\theta \eta_0 = 0$$

we obtain

$$\bar T_\lambda = \frac{1}{4}(\Delta_z C^2 \Delta_z H^z_\theta - \Delta_z C^2 \Delta_z H^z_\theta).$$ (36)

We note that since the expression $\phi$ (which is globally defined ) appears in $T_\lambda$ as the holomorphic part of a total derivative, we can get rid of the first term in eq.(33a) by completing it in such a way that it becomes a total derivative. This is indeed achieved by adding and substracting the term $\frac{1}{2} d\lambda D_\theta \phi$ in $s\hat A_2$. Thus we obtain

$$s\hat A_2 = d\left[ \frac{\eta}{\eta_0} \ln(\Psi/\eta_0) \right] + d\lambda T'_\lambda + d\lambda T''_\lambda$$ (37)

where

$T'_\lambda = (R_{z\theta} - R_{\theta z})C^2 - (R_{z\theta} - R_{\theta z})C^2$

and

$$T''_\lambda = \overline{T_\lambda} - \frac{1}{2} D_\theta \phi.$$ (38)

Now the whole expression on which $\hat d$ acts is globally defined and single-valued and thus vanishes when we integrate along infinitesimal circles surrounding the zeros of $\eta_0$.

Explicitly $T'_\lambda$ reads

$$T'_\lambda = \frac{1}{4}\{(\Delta_z sH^z_\bar \theta + \Delta_{z\theta} D_\theta C^2 + 2C^2 D_\theta D_{\bar \theta} \chi + D_\theta C^2 D_{\bar \theta} \chi)$$

$$- (\Delta_z sH^z_\bar \theta + \Delta_{z\theta} D_\theta C^2 + 2C^2 D_\theta D_{\bar \theta} \zeta + D_\theta C^2 D_{\bar \theta} \zeta)$$

$$+ D_\theta (\chi - \zeta) D_{\bar \theta} H^z_\theta D_\theta C^2 + \frac{1}{2}(\zeta - \chi) D_{\bar \theta} (D_\theta H^z_\bar \theta D_{\bar \theta} C^2)$$

$$+ (\zeta - \chi) (C^2 \partial_z D_\theta H^z_\theta + H^z_\theta \partial_z D_\theta C^2) + D_{\bar \theta} C^2 (D_\theta \chi - D_{\bar \theta} \zeta)$$

$$+ (\chi + \zeta - \zeta - \zeta) D_\theta D_{\bar \theta} C^2 - 2D_{\bar \theta}[C^2 (\zeta \zeta - \chi \chi - 2\zeta \chi)]$$

$$+ 2(\chi D_{\bar \theta} \zeta - \zeta D_{\bar \theta} \chi) (C^2 D_\theta H^z_\theta + H^z_\theta D_\theta C^2) + 2\zeta \chi D_\theta H^z_\bar \theta D_{\bar \theta} C^2\}.$$ (39)

At this point we note that the second term in $T'_\lambda$ cancels out $K_1$ in eq.(20), and the first line in $T'_\lambda$ cancels out $K_2$. However, to get rid of $K_3$ and the other terms in $T'_\lambda$, we still need to add the following functional

\[\text{10} \text{The } s\text{-variation does not change the rules of gluing, since it leaves the variables } z, \theta \text{ invariant.}\]
\[
\Gamma_3[H_\theta^\pm] = \frac{1}{4i\pi} \oint_{\partial \mathcal{D}(\eta_0)} d\lambda \left[ (\zeta \zeta_0 - \chi \chi_0 - 2\zeta \chi) H_\theta^\pm - \frac{1}{2} (\chi + \chi_0 - \zeta - \zeta_0) D_\theta H_\theta^\pm \right]. \tag{40}
\]

It is easy to check that the integrand of this expression is globally defined. When expanded in components [2], this contribution disappears in the bosonic limit.

Therefore, the combination of all these contributions into \( \Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 \) yields after several (trivial) cancellations

\[
s\Gamma = \int \mathcal{A}(C^z, H_\theta^\pm, R_{z\theta}) + \frac{1}{8i\pi} \oint_{\partial \mathcal{D}(\eta_0)} \left\{ d\lambda \left[ R_{z\theta}(C^z D_\theta H_\theta^\pm + H_\theta^\pm D_\theta C^z) + 2(2R_{z\theta}C^z + \partial_z \zeta C^z) \right] \right. \\
+ \frac{1}{2}(\zeta - \zeta_0) D_\theta (D_\theta H_\theta^\pm D_\theta C^z) + \zeta_0 \zeta D_\theta H_\theta^\pm D_\theta C^z \right. \\
+ \left( \zeta_0 D_\theta \zeta - \zeta_0 \zeta D_\theta + \partial_z \zeta)(C^z D_\theta H_\theta^\pm + H_\theta^\pm D_\theta C^z) \right) \\
- \left( \Delta \zeta s H_\theta^\pm + \Delta \zeta_0 D_\theta C^z + 2C^z D_\theta \zeta + D_\theta C^z D_\theta \zeta \right) \\
+ \left. \frac{1}{2i\pi} \oint_{\partial \mathcal{D}(\eta_0)} d\lambda \left[ (R_{z\theta} - \zeta_0) C^z \right]. \right. \tag{41}
\]

A careful inspection shows that, since terms which are not singular at the zeros \( P_{0k} \) of \( \eta_0 \) do not contribute, expressions which are likely to give non-zero results are those containing \( \eta_0 \). These reduce to

\[
J_1 = \frac{1}{2i\pi} \oint_{\partial \mathcal{D}(\eta_0)} d\lambda \left[ (R_{z\theta} - \zeta_0) C^z \right] \tag{42}
\]

\[
J_2 = -\frac{1}{8i\pi} \oint_{\partial \mathcal{D}(\eta_0)} d\lambda \left[ \Delta \zeta_0 D_\theta C^z + \frac{1}{2} \zeta_0 \nabla_\zeta \left( D_\theta H_\theta^\pm D_\theta C^z \right) - \zeta_0 D_\theta \left( \zeta (C^z D_\theta H_\theta^\pm + H_\theta^\pm D_\theta C^z) \right) \right] \tag{43}
\]

The integrand of \( J_1 \), being in fact \( \frac{1}{2} [(2R_{z\theta} + \partial_z \zeta) C^z + (\partial_z \zeta + \zeta_0 D_\theta \zeta + \zeta D_\theta \zeta_0) C^z] \), reduces to

\[
J_1 = \frac{1}{4i\pi} \oint_{\partial \mathcal{D}(\eta_0)} d\lambda \left( \partial_z \zeta + \zeta_0 D_\theta \zeta + \zeta D_\theta \zeta_0 \right) C^z \tag{44}
\]

since the other terms are non-singular in the integration domain.

Now \( \eta_0 \) can be locally written in a neighborhood \( \mathcal{O}_k \) of \( P_{0k} \) as [12]

\[
\eta_0(P_k) = \beta(P_k)(z_k - z_{0k} - \theta_k \theta_{0k})^{\frac{1}{2}} \alpha_{0k}. \tag{45}
\]

Here \((z_k, \theta_k)\) are the coordinates of the point \( P_k \) belonging to \( \mathcal{O}_k \) and \((z_{0k}, \theta_{0k})\) those of \( P_{0k} \); \( \beta \) is an even superholomorphic function nowhere vanishing on \( \Sigma \). Therefore the local behaviour of \( \ln \eta_0 \) is

\[
\ln \eta_0 = \frac{1}{2} \alpha_{0k} \ln(z_{10}) + \ln \beta, \tag{46}
\]

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where the supercoordinate displacements $z_{10}, \theta_{10}$ are defined by

$$z_{10} = z_k - z_0 - \theta_k \theta_0, \quad \theta_{10} = \theta_k - \theta_0$$  \hfill (47)

This implies that locally around $P_{0k}$ we have\(^{11}\)

$$\zeta_0 \simeq -\frac{1}{2} \alpha_k \frac{\theta_{10}}{z_{10}} - D_\theta \ln \beta.$$  \hfill (48)

Note that since $\beta$ is nowhere vanishing, the term $D_\theta \ln \beta$ here is regular and single-valued and therefore it disappears upon integration over the infinitesimal circles $C_k$. Putting this behaviour back into $J_1$ in eq.\((44)\), we get\(^{12}\)

$$J_1 = -\sum_k \frac{\alpha_{0k}}{4} \frac{1}{2i\pi} \oint_{C_k} d\lambda \left( \frac{\theta_{10}}{z_{10}} C^z - \frac{\theta_{10}}{z_{10}} C^\zeta \right) \left( \frac{1}{2} \theta_{10} \zeta - \frac{1}{2} \zeta \theta_{10} \right)$$  \hfill (49)

and then using the super Cauchy’s theorems \((29)\) we finally obtain\(^{13}\)

$$J_1 = -\sum_k \frac{\alpha_{0k}}{4} \Delta_\zeta C^z(P_{0k}).$$  \hfill (50)

This term can then be readily cancelled out in \((41)\) by adding the last contribution to the Polyakov action

$$\Gamma_4[H_\theta^\zeta] = \frac{1}{2} \sum_k \alpha_{0k} \ln(\eta/\Psi_0)(P_{0k}),$$  \hfill (51)

since

$$s\Gamma_4[H_\theta^\zeta] = \sum_k \frac{\alpha_{0k}}{4} \Delta_\zeta C^z(P_{0k})$$

due to \((31)\).

Let us now turn to the remaining integral $J_2$. The first term

$$I = -\frac{1}{8\pi i} \oint_{\partial D(\eta_0)} d\bar{\lambda} \Delta_\zeta D_\theta C^z$$

is computed using the local behaviour \((15)\) of $\eta_0$ and discarding the non-singular term $\partial D_\theta C^z$

$$I = \frac{1}{8i\pi} \sum_k \alpha_{0k} \oint_{C_k} d\bar{\lambda} \left( \frac{D_\theta C^z}{z_{10}} - \frac{1}{2} \theta_{10} \zeta - \frac{1}{2} \zeta \theta_{10} D_\theta C^z \right).$$

\(^{11}\)In the neighborhood of any point $P$ we can pick a particular parametrization, defined by the normalization $z_{0k} = \theta_{0k} = 0$ \[2\]

\(^{12}\)Recall that a global (−) sign appears because the circles $C_k$ have opposite orientation with respect to $\partial D(\eta_0)$.

\(^{13}\)The use of these theorems here is justified by the fact that $C^\zeta$ and $\zeta$ are superholomorphic inside the $C_k$’s, i.e. they don’t depend on $(\bar{z}, \bar{\theta})$ therein.
To compute the first integral here we use the following identity
\[ d\bar{\lambda}D\bar{\theta}C^z \bigg|_{z_{10}} = d\bar{\lambda}D\bar{\theta}C^z = d\bar{\lambda}D\bar{\theta}C^z \bigg|_{z_{10}} \]
inside \( C_k \). The term \( d\bar{\lambda}D\bar{\theta}C^z \bigg|_{z_{10}} \) does not contribute to \( I \) since it is the total derivative of a single-valued function \( C^z \bigg|_{z_{10}} \) and the integration is performed along infinitesimal circles \( C_k \). Whereas the second term yields
\[ -\frac{1}{4} \sum_k \alpha_{0k} \frac{1}{2i\pi} \oint_{C_k} d\lambda(\bar{\theta}) \left( D\theta D\bar{\theta}C^z - \theta_{10}C^z \bigg|_{z_{10}} \right) \]
and this is zero by the super Cauchy’s theorems (29).

Thus the integral \( J_2 \) reduces to
\[ J_2 = -\frac{1}{16i\pi} \sum_k \alpha_{0k} \oint_{C_k} d\lambda(\bar{\theta}) \left( D\theta D\bar{\theta}C^z - \frac{1}{2} \nabla_{\bar{\theta}}(D_\theta H_\bar{\theta}D_\theta C^z) + D_\theta(\bar{\theta}(C^z D_\theta H_\bar{\theta} + H_\bar{\theta} D_\theta C^z)) \right) \]
Here the integrand comes with \( \theta_{10} \) as a global factor and hence \( J_2 \) vanishes whatever is the other factor in square brackets due to the rules (23) and (24), or more precisely their complex conjugates, and the fact that \( (\theta_{12})^2 = 0 \).

Summing up we have shown that a general Polyakov action on a SRS of genus \( g \) contains three kinds of terms, an integral on \( \hat{\Sigma} \), a line integral and a residue contribution
\[ \Gamma_a[H_\bar{\theta}, R_\bar{\theta}] = \frac{1}{4\pi} \int_{\hat{\Sigma}} d^2\chi \left[ 4(R_\bar{\theta} - R_{\chi_0})D_\theta^z \right] \]
\[ \Gamma_b[H_\bar{\theta}] = \frac{1}{2i\pi} \oint_{\partial D(\eta_0)} \ln(\Psi/\eta_0) d\ln(\eta/\Psi_0) \]
\[ \Gamma_c[H_\bar{\theta}] = \frac{1}{4i\pi} \oint_{\partial D(\eta_0)} d\lambda(\bar{\theta}) \left( \zeta(\chi_0 - \chi_0 - 2\zeta\chi)H_\bar{\theta}^z - \frac{1}{2} (\chi + \chi_0 - \zeta - \zeta_0)D_\theta H_\bar{\theta}^z \right) \]
\[ \Gamma_d[H_\bar{\theta}] = \frac{1}{2} \sum_k \alpha_{0k} \ln(\eta/\Psi_0)(P_{0k}). \] (52)

This solves the superconformal Ward identity (40) on a SRS of higher genus. The genus which characterizes the SRS appears explicitly in \( \Gamma_d \), since \( \sum_{k=1}^N \frac{\alpha_{0k}}{2} = g - 1 \) where \( N \) is the number (without counting multiplicity) of zeros of \( \eta_0 \).

Finally we wish to emphasize that by construction this solution is not unique due to the presence of zero modes, i.e. it is only defined up to addition of an arbitrary functional \( \mathcal{F}(H_\bar{\theta}) \) satisfying the condition \( s\mathcal{F} = 0 \).

\section{A morphism between the diffeomorphism symmetry and (super) holomorphy properties}

We note that one can go from eqs.(35) to eqs.(31) by substituting \( s \) and \( C^z \) for \( D_\theta \) and \( H_\bar{\theta} \) respectively. This is a trivial consequence of the fact that the projective
coordinates \((\hat{Z}, \hat{\Theta})\) obey the following holomorphic properties \[6\]

\[
D_{\bar{\theta}} \hat{Z} = \hat{\Theta} D_{\theta} \hat{\Theta} + H_{\bar{\theta}} (D_{\theta} \hat{\Theta})^2
\]

\[
D_{\bar{\theta}} \hat{\Theta} = -\frac{1}{2} D_{\theta} H_{\bar{\theta}} D_{\theta} \hat{\Theta} + H_{\bar{\theta}} \partial \hat{\Theta}
\]  

(53)

whereas their BRST transformations read

\[
s\hat{Z} = -\hat{\Theta} s \hat{\Theta} + C^z (D_{\theta} \hat{\Theta})^2
\]

\[
s\hat{\Theta} = \frac{1}{2} D_{\theta} C^z D_{\theta} \hat{\Theta} + C^z \partial \hat{\Theta}.
\]  

(54)

Thus substituting in eqs(53) \(s\) for \(D_{\bar{\theta}}\) and replacing \(H_{\bar{\theta}}\) by the superghost \(C^z\), we recover eqs.(54) up to a sign \[^{14}\]. Accordingly every function of the superprojective coordinates will exhibit this relation between its \(s\) transformation and its holomorphy equation. As an example of such objects consider the super affine connection \(\chi\) defined in eq.(15); a particular sample of which is given in eq.(3) (on the torus) where an explicit parametrization in terms of the superprojective variable \(\hat{\Theta}\) is given. The transformation law (18) and the holomorphic condition deduced from the first equation (35)

\[
D_{\bar{\theta}} \chi = \frac{1}{2} D_{\theta} \Delta_{\chi} H_{\bar{\theta}}^z,
\]  

(55)

exhibit the correspondence mentioned above. Other examples of such objects are the super Schwarzian derivative \[^{8}\] and the super Bol operators \[^{9}\] which are the covariant versions of the superderivatives \(\partial^n_{\theta} D_{\theta}\) on compact SRS.

Since this correspondence involves uniquely the classical fields of the BRST algebra and not the ghost sector, it is more accurate to speak about a morphism concerning the gauge symmetry underlying the BRST symmetry. For instance the nilpotency of the BRST algebra has no analog for the operator \(D_{\bar{\theta}}\). Of course there is an equivalent morphism in the antiholomorphic sector.

Obviously this property remains true in components and in the bosonic case as well, since the BRST law of the projective coordinate \(Z\) \[^{6}\]

\[
sZ = c \partial Z,
\]  

(56)

can be deduced from the Beltrami equation

\[
\bar{\partial} Z = \mu \partial Z
\]  

(57)

by replacing the Beltrami coefficient and the partial differential \(\bar{\partial}\) by the ghost \(c\) and the BRST operator \(s\) respectively.

Finally we mention a recent formulation of \(W\)-geometry in the light cone gauge \[^{25}\] where the same kind of morphism is present.

\[^{14}\]This sign difference follows from the fact that the operator \(D_{\bar{\theta}}\) acts from the left, whereas \(s\) acts from the right.
4 Projection onto component fields

In this section we give the expression in components of the super Polyakov action (52). This action involves superfields whose power series expansions in the Grassman variables \( \theta \) and \( \bar{\theta} \) have been given previously in refs.[6, 2]. However the analytic properties of the \( \frac{1}{2} \)-superdifferentials \( \eta_0 \) necessitate a particular discussion. In fact, these superfields admit a \( \theta \)-expansion of the form [2]

\[
\eta_0 = \sqrt{\omega_0} + i\theta \lambda_0, \tag{58}
\]

where \( \omega_0 \) is the 1-differential on the underlying Riemann surface \( \Sigma \), and \( \lambda_0 \) its supersymmetric partner. We recall that these fields are holomorphic with respect to the reference structure. The analytic behaviour (45) implies for \( \omega_0 \) the usual algebraic structure expected near the point \( z_{0k} \) from the Riemann-Roch theorem namely

\[
\omega_0(z_k) = \beta^2(z_k - z_{0k})^{\alpha_{0k}}, \tag{59}
\]

where \( \beta \) is the restriction to \( \Sigma \) of \( \beta(P_k) \) in (53), and \( \sum_k \alpha_{0k} = 2g - 2 \). On the other hand this yields the following singular behaviour of the field \( \lambda_0 \)

\[
\lambda_0(z_k) = i\theta_{0k} - \frac{1}{2} \alpha_{0k} \beta(z_k - z_{0k})^{\frac{3}{2} \alpha_{0k} - 1}. \tag{60}
\]

Obviously \( \lambda_0 \) and \( \omega_0 \) have to be linked since from (43) and (58) they share in common some analytical structure in the vicinity of \( z_{0k} \). In fact the 1-differential \( \lambda_0 \) behaves like \( iD_{\theta_{0k}}(\sqrt{\omega_0}) \) in this neighborhood, a behaviour which is compatible with the analytical properties of differentials on Riemann surfaces. Assuming the expansion:

\[
\zeta_0 = \zeta_0^0 + \theta \zeta_0^1, \tag{61}
\]

eqs (53), (60) imply

\[
\zeta_0^0 = \frac{1}{2} \alpha_{0k} \theta_{0k} \frac{1}{z_k - z_{0k}}, \quad \zeta_0^1 = -\frac{1}{2} \alpha_{0k} \frac{1}{z_k - z_{0k}}. \tag{62}
\]

Consequently, since \( \zeta_0^1 = -\frac{1}{2} \partial_z \ln \omega_0 \), the quantity \( \theta_{0k} \) can be interpreted as the ratio of the residues of the super component of the superaffine connection and the bosonic affine connection at the singular point \( z_{0k} \). The holomorphic superfield \( R_{z\theta} \) admits a \( \theta \)-expansion of the form

\[
R_{z\theta} = \frac{i}{2} \rho_{z\theta} + \theta \frac{1}{2} \mathcal{R}, \tag{63}
\]

where the bosonic projective connection \( \mathcal{R} \) and its supersymmetric partner \( \rho_{z\theta} \) depend only on the holomorphic variable \( z \). Moreover, we have
where the spacetime fields $\mu$ and $\alpha$ are the Beltrami coefficient and its fermionic partner respectively. The $\theta-$expansions of $\eta$, $\Psi$ and $\Psi_0$ are analogous to the expansion $[53]$ with coefficients denoted by $(\sqrt{\omega}; i\lambda)$, $(\sqrt{\Omega}; i\xi)$ and $(\sqrt{\Omega_0}, i\xi_0)$ respectively.

By substituting the above component field expansions in the first expression of (52) we find (in the following, we shall simplify the notation by suppressing all indices on the component fields)

$$
\Gamma_a(\mu; \alpha; \rho, \mathcal{R}) = \Gamma_{a1}(\mu; \mathcal{R}) + \Gamma_{a2}(\mu, \alpha; \rho)
$$

(65)

where $\Gamma_{a1}(\mu; \mathcal{R})$ and $\Gamma_{a2}(\mu, \alpha; \rho)$ are

$$
\Gamma_{a1}(\mu; \mathcal{R}) = \frac{1}{2\pi} \int_{\Sigma} d^2z [2\mathcal{R}\mu + 2 \nabla^2 \mu + \partial \ln(\Omega_0/\Omega) \nabla \mu - 2\mu \nabla (\partial \ln \Omega)]
$$

(66)

$$
\Gamma_{a2}(\mu, \alpha; \rho) = \frac{1}{2\pi} \int_{\Sigma} d^2z [2\rho \alpha + \frac{\xi}{\sqrt{\Omega}} \nabla_0 \alpha + \frac{\xi_0}{\sqrt{\Omega_0}} \nabla \alpha - \partial (\frac{\xi}{\sqrt{\Omega}})(\alpha + 2\mu \frac{\xi_0}{\sqrt{\Omega_0}}) - \partial (\frac{\xi_0}{\sqrt{\Omega_0}})(\alpha + 2\mu \frac{\xi}{\sqrt{\Omega}})].
$$

(67)

The functional $\Gamma_{a2}(\mu, \alpha; \rho)$ represents the contributions which are due to supersymmetry. In these formulae appears the covariant derivative $\nabla$ associated to the affine connection $\partial \ln \Omega$ and which is defined by

$$
\nabla \equiv \partial - p \partial \ln \Omega
$$

(68)

where $p$ is the conformal weight (relative to the $z-$index) of the field on which $\nabla$ is applied. When acting on the associated affine connection or another affine connection $\partial \ln \Omega'$, the covariant derivative defines the “field strength”

$$
\nabla (\partial \ln \Omega') \equiv (\partial - \frac{i}{2} p \partial \ln \Omega) \partial \ln \Omega'.
$$

(69)

The three other contributions of eq. (52) become in components

$$
\begin{align*}
\Gamma_b(\mu, \alpha) &= \frac{1}{2i\pi} \oint_{\partial \mathcal{D}(\omega_0)} \frac{1}{4} \ln(\Omega/\omega_0)d\ln(\omega/\Omega_0) - dz(\frac{\xi}{\sqrt{\Omega}} - \frac{\lambda_0}{\sqrt{\Omega_0}})(\frac{\xi_0}{\sqrt{\Omega_0}} - \frac{\lambda}{\sqrt{\omega}}) \nonumber \\
\Gamma_c(\mu, \alpha) &= \frac{1}{4i\pi} \oint_{\partial \mathcal{D}(\omega_0)} dz \{(2 \frac{\lambda}{\sqrt{\omega}} \frac{\xi}{\sqrt{\Omega}} - \frac{\lambda_0}{\sqrt{\omega_0}} \frac{\lambda}{\sqrt{\Omega}} - \frac{\xi_0}{\sqrt{\Omega_0}} \frac{\xi}{\sqrt{\Omega}})\mu \\
&+ \frac{i}{2} (\frac{\lambda}{\sqrt{\omega}} + \frac{\lambda_0}{\sqrt{\omega_0}} - \frac{\xi_0}{\sqrt{\Omega_0}} - \frac{\xi}{\sqrt{\Omega}})\alpha\}
\end{align*}
$$

(70)

$$
\begin{align*}
\Gamma_d(\mu, \alpha) &= \frac{1}{2} \sum_k \alpha_0 k \left[ \frac{1}{2} \ln(\omega/\Omega_0) + i\theta_0 k (\frac{\lambda}{\sqrt{\omega}} - \frac{\xi_0}{\sqrt{\Omega_0}}) \right] (z_0 k).
\end{align*}
$$

(70)
In the expressions above are included the results of the bosonic theory [23]. They are obtained by setting $\lambda = \lambda_0 = \rho = \alpha = \xi_0 = \xi = 0$. The dissection $\partial D(\omega_0)$ was introduced in [23]. The points used to define it are the zeros of $\omega_0$ and since this 1-differential is holomorphic in the reference conformal structure $\mu = \alpha = 0$, this dissection is independent of $\mu$ and $\alpha$; we recall that $\omega, \lambda, \omega_0$ and $\lambda_0$ are single-valued and have zeros on the underlying Riemann surface $\Sigma$. On the other hand $\Omega, \Omega_0, \xi, \xi_0$ are multivalued on $\Sigma$ and have no zeros. Explicit examples of such objects have been given in [24] in terms of the theta function and the prime form.

5 Concluding comments

In summary we have derived a general expression for the Polyakov action on an $N = 1$ SRS of arbitrary genus in the restricted geometry $H_\theta = 0$ (and c.c). The superfield formulation was obtained through a new formalism of integration rules reproducing the results of the component expansion while avoiding its technical complications, thus allowing a complete superfield treatment throughout the calculations. The main ingredient was the introduction of a supercontour (or as we called it “Grassmann circle”) over which the integration with respect to the Grassmann variables incorporates the usual Berezin rules and also implies the analytic structure of superfields on the underlying Riemann surface. In addition some technical difficulties inherent to the supersymmetric approach were circumvented by using the systematic method of covariant derivatives.

As mentioned at the end of paragraph 2, the action we have constructed is only defined up to a BRST invariant functional. In fact we can go further. Starting with the action $\Gamma_\alpha$ in (52) we replace the multivalued fields $\chi$ and $\chi_0$ by the single-valued ones $\zeta$ and $\zeta_0$ respectively. By doing this we end up with an expression which can be shown to be the first term for a second solution to the superconformal Ward identity (10). More precisely, let us denote the resulting expression by $\hat{A}_\alpha$ and then by using the holomorphy equations (35) we get

\[
\hat{A}_1 - \hat{A}_\alpha = -4\{D_\theta[\ln(\Psi/\eta_0)D_\theta\ln(\eta/\Psi_0)] + D_\theta[\ln(\Psi/\eta_0)D_\theta\ln(\eta/\Psi_0)]\} \quad (71)
- D_\theta\{(\zeta\zeta_0 - \chi\chi_0 - 2\zeta\chi)H_\theta - \frac{1}{2}(\chi + \chi_0 - \zeta - \zeta_0)D_\theta H_\theta}\]

where $\hat{A}_1$ is the integrand in (14). Next we integrate this expression over $\hat{\Sigma}$ which is now seen as the disjoint union of the domains $\hat{D}(\eta_0)$ and $\hat{D}(\eta)$ that contain all zeros of $\eta_0$ and $\eta$ respectively, and then use the Stokes theorem (25) to obtain the identity

\[
\Gamma_\alpha = \Gamma_\alpha + \frac{1}{2i\pi} \oint_{\partial \hat{D}(\eta_0)} \ln(\Psi/\eta_0)d\ln(\eta/\Psi_0) + \frac{1}{2i\pi} \oint_{\partial \hat{D}(\eta)} \ln(\Psi/\eta_0)d\ln(\eta/\Psi_0) + \frac{1}{4i\pi} \oint_{\partial \hat{\Sigma}} d\hat{\lambda}\{(\zeta\zeta_0 - \chi\chi_0 - 2\zeta\chi)H_\theta - \frac{1}{2}(\chi + \chi_0 - \zeta - \zeta_0)D_\theta H_\theta}\} \quad (72)
\]
Now we note that in the third term above we have integrable singularities and thus by doing the same calculation that led to the result (51) we get

\[ \frac{1}{2i\pi} \oint_{\partial \hat{D}(\eta)} \ln(\Psi/\eta_0)d\ln(\eta/\Psi_0) = -\frac{1}{2} \sum_j \alpha_j \ln(\Psi/\eta_0)(P_j) \]

Consequently, the sum \( \Gamma_a + \Gamma_b + \Gamma_c + \Gamma_d \) is equal to the sum of the following terms

\[
\Gamma_\alpha[H_{\hat{\theta}}, R_{z\theta}] = \frac{1}{4\pi} \int_\Sigma d^2 \lambda [4(R_{z\theta} - R_{\zeta_0})H_{\hat{\theta}}^z + D_{\theta}(\Delta \zeta + \Delta \zeta_0)H_{\hat{\theta}}^z]
\]

\[
\Gamma_\beta[H_{\hat{\theta}}] = \frac{1}{2} \sum_k \alpha_{0k} \ln(\eta/\Psi_0)(P_{0k}) + \frac{1}{2} \sum_j \alpha_j \ln(\Psi/\eta_0)(P_j)
\]

\[
\Gamma_\gamma[H_{\hat{\theta}}] = \frac{1}{4i\pi} \oint_{\partial \hat{D}(\eta)} d\bar{\lambda} \{(\zeta \zeta_0 - \chi \chi_0 - 2\zeta \chi)H_{\hat{\theta}}^z - \frac{1}{2}(\chi + \chi_0 - \zeta - \zeta_0)D_{\theta}
\]

This solution can be seen as the supersymmetric generalization of the second solution found by Zucchini in [24] on a Riemann surface. One of the advantages of this solution over the one in (52) is the fact that it can be easily related to the Polyakov action (9) we constructed on the supertorus [2]. Indeed, a simple calculation yields the following

\[
\hat{A}_1 = A_{ST} - D_{\theta}\{D_{\theta}H_{\hat{\theta}}^z D_{\theta} \ln(\eta/\eta_0) - 2H_{\hat{\theta}}^z D_{\theta} \ln \eta D_{\theta} \ln \eta_0\} + 4D_{\theta}D_{\bar{\theta}} \ln \eta
\]

which holds on a SRS. Then using the fact that \( \eta_0 \) is holomorphic in the reference structure i.e., \( D_{\bar{\theta}}\eta_0 = 0 \), we can rewrite \( I_2 \) as

\[ I_2 = 4D_{\theta}D_{\bar{\theta}} \ln(\eta/\eta_0) \]

thus yielding a well-defined expression since \( \eta/\eta_0 \) is now an (invariant) function. \( I_2 \) is therefore a total derivative of a single-valued, well-defined and singularity free \( (\frac{1}{2},0) \)–superdifferential \( D_{\theta} \ln(\eta/\eta_0) \), and hence vanishes upon integrating on small circles on the supertorus.

Similarly, \( I_1 \) is a total derivative of a single-valued and non-singular \( (0,\frac{1}{2}) \)–superdifferential, since the expression in brackets in (74) transforms with the factor \( D_{\bar{\theta}}\theta \) under conformal change of coordinates. Thus the integral of \( I_1 \) over the supertorus vanishes. Therefore the restriction of \( \hat{A}_1 \) onto the supertorus gives exactly \( A_{ST} \).

Now as the differentials on the supertorus have according to the super Riemann-Roch theorem no zeros, \( \Gamma_\beta \) vanishes trivially since \( \alpha_{0k} = \alpha_j = 0 \). As to \( \Gamma_\gamma \) the reasoning is as follows. Due to the fact that there is a unique holomorphic superdifferential (in a given structure) on the supertorus, \( \Psi \) and \( \Psi_0 \) become proportional to \( \eta \) and \( \eta_0 \) respectively. The corresponding factors are multivalued
functions, which must be holomorphic on the whole torus since we want this restriction to hold everywhere thereon; this implies that they are constant. In this case \( \chi \) and \( \chi_0 \) reduce exactly to \( \zeta \) and \( \zeta_0 \) respectively, and thereby \( \Gamma_\gamma \) vanishes indentically on the supertorus.

There are many issues that deserve serious study, namely the modular invariance of these solutions and their pertinence to resum the perturbative series provided by the renormalized field theory as an iterative solution to the superconformal Ward identity (10). This would provide a generalization of the work done on the supertorus [13].

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