Regression Analysis for Multivariate Dependent Count Data Using Convolved Gaussian Processes

A’yunin Sofro\textsuperscript{1}, Jian Qing Shi \textsuperscript{*2}, and Chunzheng Cao\textsuperscript{3}

\textsuperscript{1}Department of Mathematics, Surabaya State University, Indonesia
\textsuperscript{2}School of Mathematics, Statistics and Physics, Newcastle University, UK
\textsuperscript{3}School of Mathematics \& Statistics, Nanjing University of Information Science and Technology, China

October 5, 2017

Abstract

Research on Poisson regression analysis for dependent data has been developed rapidly in the last decade. One of difficult problems in a multivariate case is how to construct a cross-correlation structure and at the meantime make sure that the covariance matrix is positive definite. To address the issue, we propose to use convolved Gaussian process (CGP) in this paper. The approach provides a semi-parametric model and offers a natural framework for modeling common mean structure and covariance structure simultaneously. The CGP enables the model to define different covariance structure for each component of the response variables. This flexibility ensures the model to cope with data coming from different resources or having different data structures, and thus to provide accurate estimation and prediction. In addition, the model is able to accommodate large-dimensional covariates. The definition of the model, the inference and the implementation, as well as its asymptotic properties, are discussed.

\textsuperscript{*}Corresponding author: j.q.shi@ncl.ac.uk
Comprehensive numerical examples with both simulation studies and real data are presented.

**Keywords:** Convolved Gaussian process, Cross-correlation, Multivariate dependent count data, Multivariate Poisson regression, Covariance functions.

1 Introduction

Regression analysis for dependent non-Gaussian data has been developed rapidly in the last several decades. We will focus on dependent count data in this paper. One way is to extend the conventional Poisson regression model by considering a covariance structure. However, the problem of modelling becomes more complex when there is more than one response variable. We illustrate the challenges using the example of dengue fever and malaria data that we will discuss in details later in this paper. The outputs are the number of cases of dengue fever and malaria occurred in different regions in East Java in Indonesia. Both diseases are transmitted by a virus via mosquitoes and occur often in tropical regions particularly in developing countries. They have similar signs and symptoms. The outbreak of the diseases depends on many factors such as living condition and healthy behaviour. The data is spatially correlated due to the movement of population, analogues of the environment and the healthy behaviour, etc. The study for such problems focuses on the following three aspects. First of all, we want to study how the count of cases depends on a set of covariates. A parametric model is usually used since it can provide a physical explanation on the relationship between the disease and the covariates. Secondly, we are interested in finding the structure of spatial correlation of the depend data for each disease and further to find the geographical patterns. This provides a tool in epidemic study. Due to the nature of the problem, it requires a flexible covariance model and ideally the covariance structure and the pattern can be learned from data rather than an assumption given in advance. Thirdly, we want to study similar diseases or response variables at the same time. We are interested in knowing if there are similar geographical patterns for those diseases and how they are spatially correlated and cross-correlated. The findings will provide important information for policy making on how to control the spread and transmission of the diseases.
Poisson regression analysis for an univariate count response variable with correlation structure has been studied by many researchers. The intrinsic conditional autoregressive (ICAR) model is one of the popular methods which was introduced by Besag and Kooperberg (1995). This method has been extended into a spatial or temporal correlated generalized linear mixed model (Sun et al., 2000; MacNab and Dean, 2001; Martínez-Beneito et al., 2008; Silva et al., 2008). A generalized linear mixed model using prior distribution for spatially structured random effect is an alternative way, see Banerjee et al. (2004). Rue and Held (2005) and Mohebbi et al. (2011) demonstrated how to apply the methods to analyse cancer data. However, based on extensive studies by Wall (2004), the spatially correlated structure of ICAR approach is too complicated, involving complex implementation and lack of physical explanation. Martínez-Beneito (2013) has also pointed out that preliminary knowledge and a good understanding are needed in determining and investigating the effect of the choice of precision for the covariance matrix. Thus, it is essential to develop a more flexible method to model the spatial correlation. One alternative is to use a Gaussian process (GP) prior (or kriging under spatial statistics, see Diggle et al. (1998)) to model the covariance structure (see e.g. Rasmussen and Williams (2006) and Shi and Choi (2011)). This is a nonparametric approach, providing a flexible method on modeling covariance structure. The Bayesian framework with GP priors with different covariance functions provides flexibility on fitting data with different degrees of nonlinearity and smoothness. It can also cope with multi-dimensional covariates. Some recent development can be found in e.g. Gramacy and Lian (2012) and Wang and Shi (2014).

For the problem involved multivariate response variables, we need to model covariance structure for each component as well as cross-covariance between them. The challenge here is how to find a model which can model the covariance and cross-variance flexibly, subject to the condition that the overall covariance function is positive definite. Several methods have been proposed, for example, two-fold CAR model (Kim et al., 2001) and multivariate CAR (MCAR) (Gelfand and Vounatsou, 2003). Jin et al. (2005) proposed a general framework for MCAR by using a conditional approach \( p(\tau_1, \tau_2) = p(\tau_1 | \tau_2)p(\tau_2) \), where \( \tau_1 \) and \( \tau_2 \) stand for the two components. As we pointed out before, the CAR model is useful for some problems but is less efficient for a general use. Crainiceanu et al. (2008) also used the idea of
conditional distribution but the covariance structure is modeled by a GP prior. It provides a promising result for some types of problem. However the covariance structure of $\tau_1$ depends on the covariance structure of $\tau_2$. If those two components have very different covariance structures, the model may be failed. The performance also depends on the ordering of the components. An additional problem is that it is not easy to extend it to cases with more than two components.

In this paper we propose to use convolved GP (CGP) \cite{Boyle2005} and provides a general framework on modeling individual covariance structure for each component and, at the same time, modeling cross-covariance for multivariate count data. The method can be easily extended to deal with multivariate case with any dimension. It inherits nice properties of GP model, for example, it offers a semiparametric regression model for Poisson data with multivariate responses; it models mean structure and covariance structure simultaneously; and it enables us to handle a large dimensional covariates.

This paper is organized as follows. In section 2, we will discuss how to construct multivariate dependent Gaussian processes using convolution. We will then explain the details how to define a multivariate CGP for dependent count data. The details of inference including estimation, prediction and asymptotic theory will also be provided in the section. Comprehensive simulation studies and real data applications will be discussed in Section 3. The final conclusive remarks will be given in Section 4.

2 Multivariate CGP model for Dependent Count Data

2.1 Multivariate Convolved Gaussian Processes

We first introduce Multivariate Convolved Gaussian Processes (MCGP) and defer the definition of the main model to the next subsection. Let $\gamma(\mathbf{x})$ be a Gaussian white noise $\gamma(\mathbf{x}) \sim N(0, \sigma^2)$ and $h(\mathbf{x})$ be a smoothing kernel for $\mathbf{x} \in \mathbb{R}^p$. We can construct a CGP $\eta(\mathbf{x})$ \cite{Boyle2005, Shi2011} as

$$\eta(\mathbf{x}) = h(\mathbf{x}) \star \gamma(\mathbf{x}) = \int h(\mathbf{x} - \mathbf{\alpha}) \gamma(\mathbf{\alpha}) d\mathbf{\alpha} = \int h(\mathbf{\alpha}) \gamma(\mathbf{x} - \mathbf{\alpha}) d\mathbf{\alpha},$$
where ‘⋆’ denotes convolution. We denote it by

$$\eta(x) \sim \text{CGP}(h(x), \gamma(x)).$$  \hfill (1)$$

For example, if we choose a smooth kernel $h(x)$ as $h(x) = v \exp \left\{ -\frac{1}{2} (x - \mu)^T A (x - \mu) \right\}$, then the CGP $\eta(x)$ defined in (1) is equivalent to a GP with zero mean and the following covariance function

$$k(x_i, x_j) = \pi^{p/2} v^2 |A|^{-1/2} \exp \left\{ -\frac{1}{4} (x_i - x_j)^T A (x_i - x_j) \right\},$$  \hfill (2)$$

for any $x_i, x_j \in \mathcal{X} \subset \mathbb{R}^p$, where $v$ and $A$ are parameters. This is the squared exponential covariance function.

To define a bivariate CGP, we first define three independent Gaussian white noises, namely $\gamma_0(x)$, $\gamma_1(x)$ and $\gamma_2(x)$. Using them, we construct four CGPs as follows:

$$\xi_1(x) \sim \text{CGP}(h_1(x), \gamma_0(x)), \quad \xi_2(x) \sim \text{CGP}(h_2(x), \gamma_0(x))$$  \hfill (3)$$

and

$$\eta_1(x) \sim \text{CGP}(g_1(x), \gamma_1(x)), \quad \eta_2(x) \sim \text{CGP}(g_2(x), \gamma_2(x)),$$  \hfill (4)$$

where $g_a(x)$ and $h_a(x)$ ($a = 1, 2$) are smoothing kernels. It is clear that $\eta_1(x)$ and $\eta_2(x)$ are independent, $\xi_1(x)$ and $\xi_2(x)$ are dependent but are independent from $\eta_1(x)$ and $\eta_2(x)$. Using those four CGPs we can define bivariate dependent GPs as

$$\tau_a(x) = \xi_a(x) + \eta_a(x), \quad a = 1, 2.$$  \hfill (5)$$

Based on equation in (5), the dependency between $\tau_1(x)$ and $\tau_2(x)$ is modeled by $\xi_1(x)$ and $\xi_2(x)$, while the individual characteristics are modeled by $\eta_1(x)$ and $\eta_2(x)$. Since the covariance structure can be modeled by different smoothing kernels $g_a(x)$ and $h_a(x)$, the multivariate CGP defined above provides a very flexible model and can model variant cross-correlation structures, and at the same time, can model the different correlation structure for each component. The covariance and cross-covariance at any two points $x_i, x_j \in \mathbb{R}^p$ can be calculated by

$$\text{Cov}(\tau_a(x_i), \tau_a(x_j)) = \text{Cov}(\xi_a(x_i), \xi_a(x_j)) + \text{Cov}(\eta_a(x_i), \eta_a(x_j)),$$

$$\text{Cov}(\tau_a(x_i), \tau_b(x_j)) = \text{Cov}(\xi_a(x_i), \xi_b(x_j)),$$

for $a, b = 1, 2 \ (a \neq b)$.
If we take \( h_a(x) = \nu_{a0} \exp\{-\frac{1}{2} x^T A_{a0} x\} \) and \( g_a(x) = \nu_{a1} \exp\{-\frac{1}{2} x^T A_{a1} x\} \) for \( a = 1, 2 \), the covariance in the first equation can be calculated by (2), and the cross-covariance in the second equation is given by

\[
\text{Cov}(\tau_a(x_i), \tau_b(x_j)) = (2\pi)^{p/2} \nu_{10} \nu_{20} |A_{10} + A_{20}|^{-1/2} \exp\{-\frac{1}{2} (x_i - x_j)^T \Sigma (x_i - x_j)\},
\]

where \( \Sigma = A_{10} A_{10}^{-1} + A_{20}^{-1} A_{20} \).

Now let us look at the specific covariance structure of (5) using a discrete form. Consider

\[
\tau = \{\tau_1(x_{1i}), i = 1, \ldots, n_1; \tau_2(x_{2j}), j = 1, \ldots, n_2\},
\]

where \( x_{1i}, x_{2j} \in \mathcal{X} \subset \mathcal{R}^p \). Then \( \tau \) is a realization of a multivariate CGP defined in (5). It has an \((n_1 + n_2)\)-dimensional Gaussian distribution with zero means. Let \( K \) be the \((n_1 + n_2) \times (n_1 + n_2)\) covariance matrix of \( \tau \). It includes elements of \( k_{ab}(x_{ai}, x_{bj}) = \text{Cov}(\tau_a(x_{ai}), \tau_b(x_{bj})) \) for \( a, b \in \{1, 2\} \) and \( i, j \) in either \( \{1, \ldots, n_1\} \) or \( \{1, \ldots, n_2\} \).

If we consider stationary processes, i.e. the covariance function depends only on the distance between two points \( d = x_{ai} - x_{bj} \), then the covariance function is defined by

\[
k_{11}(d) = k_{\xi_1}(d) + k_{\eta_1}(d), \quad k_{12}(d) = k_{\xi_2}(d),
\]

\[
k_{22}(d) = k_{\xi_2}(d) + k_{\eta_2}(d), \quad k_{21}(d) = k_{\xi_1}(d) - d,
\]

where, for example, \( k_{\xi_2}(d) \) stands for the covariance between \( \xi_1 \) and \( \xi_2 \). It is straightforward to get the formulas if we use the squared exponential covariance function in (2). This can also be applied to other types of covariance functions. We denote the multivariate GP defined above as a multivariate CGP (MCGP)

\[
(\tau_1(x), \tau_2(x))^T \sim \text{MCGP}(\xi_1(x), \xi_2(x), \eta_1(x), \eta_2(x)), \quad \text{or } \text{MGP}(0, k(\cdot, \cdot)), \quad (8)
\]

where \( \xi_a \) and \( \eta_a \) are defined in (3) and (4) respectively, and \( \text{MGP}(0, k(\cdot, \cdot)) \) stands for a multivariate GP with zero mean and covariance function \( k(\cdot, \cdot) \) which is determined by \( \xi_a \) and \( \eta_a \) in (7). It is not difficult to extend the above bivariate case to a general multivariate case.

The covariance function defined by the above way is positive definite.

**Proposition 1.** Assume that \( S(m) \) is an isotropic covariance function on \( \mathcal{R}^p \), for any \( p \in \mathcal{N} \). If the function of covariance \( k_{ab}(d) \) in (7) is given by

\[
k_{ab}(d) = \frac{\nu_a \nu_b (2\pi)^{p/2}}{|A_a + A_b|^{1/2}} S(\sqrt{Q_{ab}(d; A_a, A_b)}),
\]
where

\[ Q_{ab}(d; A_a, A_b) = d^T A_a (A_a + A_b)^{-1} A_b d \]

for any \( v_a, v_b \in \mathcal{R} \) and arbitrary positive matrices \( A_a, a = 1, 2 \), then the covariance function defined in (7) is positive definite.

The proof is similar to the one given in Andriluka et al. (2007) and the details can be found in Sofro (2016).

For the squared exponential covariance function (2), we have

\[
\begin{align*}
    k_{\xi a a}(d) &= \frac{v_{a0}^2 \pi^{p/2}}{|A_{a0}|^{1/2}} \exp\left\{ -\frac{1}{2} Q_{aa}(d; A_{a0}, A_{a0}) \right\}, \\
    k_{\xi a b}(d) &= \frac{v_a v_b (2\pi)^{p/2}}{|A_{a0} + A_{b0}|^{1/2}} \exp\left\{ -\frac{1}{2} Q_{ab}(d; A_{a0}, A_{b0}) \right\}, \\
    k_{\eta a a}(d) &= \frac{v_{a1}^2 \pi^{p/2}}{|A_{a1}|^{1/2}} \exp\left\{ -\frac{1}{2} Q_{aa}(d; A_{a1}, A_{a1}) \right\}
\end{align*}
\]

for \( a, b = 1, 2 \) and \( a \neq b \).

Similarly we can apply it to other covariance functions such as Matern and rational quadratics (Shi and Choi, 2011; Sofro, 2016).

2.2 The Model

Let \( z_1 \) and \( z_2 \) be two correlated response variables, for example the number of dengue fever and number of malaria cases in the example we discussed in Section 1. A general multivariate CGP model for dependent count data can be defined as follows.

\[
\begin{align*}
    z_a | \tau_a &\sim \text{Poisson}(\mu_a), \\
    \log(\mu_a) &= U_a^T \beta_a + \tau_a(x_a), \quad a = 1, 2,
\end{align*}
\]

where \((\tau_1, \tau_2) \sim \text{MCGP}(\xi_1, \xi_2, \eta_1, \eta_2)\), \(U_a\) is a set of covariates and a linear model is used here. Parametric \(\beta_a\) is used to describe the relationship between the response variable \(z_a\) and the covariates \(U_a\). The dependency of the observations for each component and the cross-correlation between components are modeled by \((\tau_1, \tau_2)\) via a MCGP. The cross-correlation or the cross-covariance is modeled by \(\xi_1\) and \(\xi_2\) in (3); while the covariance structure for each component is modeled by \(\eta_a\) and \(\eta_a\). Since different covariance functions can be used for \(\eta_a\) and \(\xi_a\) for \(a = 1, 2\), the model allows different covariance structures for each components.
This largely increase the flexibility of the model, enabling the model to cope with data coming from different resources, having different data form and/or having different degrees of nonlinearity and smoothness. Model (10) uses MCGP to model multivariate Poisson data; for convenience, we call it as MCGP for Poisson data, or MCGPP in short.

In the above model, \( U_a \) is a set of covariates to model the mean while \( x_a \) is to model the covariance. Some of those covariates may be the same. In (10), other parametric mean model can also be used. This will not add extra technical difficulty in the inference we will discuss next.

In model (10), \( \tau_a \) can be treated as a nonlinear random effect. The posterior distribution can be calculated and the information consistency we will prove later in this section will guarantee it approaches the underline true function if we have observations of sufficient large number.

Suppose that we have observed the following data \( D = \{ z_{ai}, U_{ai}, x_{ai} | a = 1, 2, \ i = 1, \ldots, n_a \} \), where \( n_1 \) and \( n_2 \) are the numbers of the observations for the two components respectively. Our model does not require the data is observed in pair, and those \( n_1 \) and \( n_2 \) could be different. Based on the model defined in (10), \( z = (z_{11}, \ldots, z_{1n_1}, z_{21}, \ldots, z_{2n_2})^T \) are conditional independent given \( \tau = (\tau_1^T, \tau_2^T)^T \), where \( \tau_a = (\tau_{a1}, \ldots, \tau_{an_a})^T \) for \( a = 1, 2 \). Thus,

\[
p(z | \tau) = \prod_{a=1}^{2} \prod_{i=1}^{n_a} p(z_{ai} | \tau_{ai})
\]

where \( p(z_{ai} | \tau_{ai}) \) is the probability density of the Poisson distribution with mean \( U_{ai}^T \beta_a + \tau_{ai} \).

Following the discussion in the last subsection, \( \tau \) is a realization of a MCGP. It has a \((n_1 + n_2)\)-dimensional Gaussian distribution with zero mean and covariance matrix \( K \). The element of \( K \) is calculated by equation (6) and depends on the kernels \( g_a \) and \( h_a \) \((a = 1, 2)\).

Under a Bayesian framework, this defines a prior distribution of the latent variable \( \tau \). The related covariance functions involve hyper-parameters, for example, the squared exponential covariance function defined in (9) depends on \( \{ v_{aj}, A_{aj}, a = 1, 2, j = 0, 1 \} \). Although the values of those hyper-parameters (denoted by \( \theta \)) can be given in advance based on prior knowledge, it is rather a difficult task if it is not impossible. This is because the physical meaning for some of them are not very clear, and the dimension of \( \theta \) is usually quite large. Among several different methods \cite{Shi2011}, we adopt an empirical Bayesian
approach in this paper, i.e. choosing the values of those hyper-parameters by maximising its marginal likelihood. Following the discussion in Wang and Shi (2014), we can estimate $\theta$ and other parameters, which are $\beta_a$ in model (10), at the same time.

### 2.3 Estimation and prediction

Given data $\mathcal{D}$, the marginal density of $z$ given $\beta$ and $\theta$ is given by

$$p(z | \beta, \theta, x) = \int p(z | \tau, \beta) p(\tau | \theta) d\tau = \int \{ \prod_{a=1}^{2} \prod_{i=1}^{n_a} p(z_{ai} | \tau_{ai}, \beta_a) \} p(\tau | \theta) d\tau,$$

and the marginal log-likelihood is

$$l(\beta, \theta) = \log \int \exp(\Phi(\tau)) d\tau$$

where

$$\Phi(\tau) = -\frac{1}{2} \log | K | - \frac{1}{2} \tau^T K^{-1} \tau - \frac{n_1 + n_2}{2} \log(2\pi) + \sum_{a=1}^{2} \sum_{i=1}^{n_a} \log[p(z_{ai} | \tau_{ai}, \beta_a)],$$

with $\log p(z_{ai} | \tau_{ai}, \beta_a) = z_{ai} \log(\mu_{ai}) - \mu_{ai} - \log(z_{ai}!)$ and $\mu_{ai} = \exp(U_{ai}^T \beta_a + \tau_{ai})$ for $a = 1, 2$.

The integral involved in the above marginal likelihood is analytical intractable since the dimension of $\tau$ is $n_1 + n_2$, the total sample size, which is usually very large. We use a Laplace approximation. Let $\tau_0$ be the maximiser of $\Phi(\tau)$, we have

$$\int \exp(\Phi(\tau)) d\tau \approx \exp \left\{ \Phi(\tau_0) + \frac{n_1 + n_2}{2} \log(2\pi) - \frac{1}{2} \log | H | \right\}$$

where $H$ is the second derivative of $-\Phi(\tau)$ respect to $\tau$ and evaluated at $\tau_0$. Thus, $H = C + K^{-1}(\theta)$ and $C$ is a diagonal matrix,

$$C = \text{diag}\{ \exp(U_{11}^T \beta_1 + \tau_{011}), ..., \exp(U_{1n_1}^T \beta_1 + \tau_{01n_1}), \exp(U_{21}^T \beta_2 + \tau_{021}), ..., \exp(U_{2n_2}^T \beta_2 + \tau_{02n_2}) \}.$$

We then estimate the parameters by maximising the likelihood function with Laplace approximation in equation (14).

We now turn to calculate prediction of $z^* = (z_1^*, z_2^*)^T$ at a new point with $U^* = (U_1^*, U_2^*)$ and $x^* = (x_1^*, x_2^*)$. We still use $\mathcal{D}$ to denote all the training data and assume that the model
itself has been trained (all unknown parameters have been estimated). We will calculate the predictive mean \( E(z^* \mid D) \) as well as the predictive variance \( \text{Var}(z^* \mid D) \).

Let \( \tau^* = \tau(x^*) = (\tau_1^*, \tau_2^*)^T \) be the underlying latent variable at \( x^* \). The expectation of \( z^* \) conditional on \( \tau^* \) is given by

\[
E(z^* \mid \tau^*, D) = \left( \begin{array}{c}
E(z_1^* \mid \tau_1^*, D) \\
E(z_2^* \mid \tau_2^*, D)
\end{array} \right) = \left( \begin{array}{c}
\exp(U_1^T \hat{\beta}_1 + \tau_1^*) \\
\exp(U_2^T \hat{\beta}_2 + \tau_2^*)
\end{array} \right) = \exp(U^T \hat{\beta} + \tau^*).
\]

It follows that

\[
E(z^* \mid D) = E[E(z^* \mid \tau^*, D)] = \int \exp(U^T \hat{\beta} + \tau^*) p(\tau^* \mid D) d\tau^*. \tag{15}
\]

Note that

\[
p(\tau^* \mid D) = \int p(\tau^* \mid \tau, D) p(\tau \mid D) d\tau = \int p(\tau^*, \tau \mid D) d\tau = \int p(z \mid \tau) p(\tau^*, \tau) d\tau. \tag{16}
\]

Hence, equation (15) above can be rewritten as

\[
E(z^* \mid D) = \frac{1}{p(z)} \int \int \exp(U^T \hat{\beta} + \tau^*) p(z \mid \tau) p(\tau^*, \tau) d\tau d\tau^*. \tag{17}
\]

For convenience we denote \( \tau_+ = (\tau^T, \tau^{*T})^T \), which is a realization of the MCGPP defined in (10). So its density function is a multivariate normal distribution with zero mean. The \((n_1 + n_2 + 2) \times (n_1 + n_2 + 2)\) covariance matrix is calculated similar to \( K \) in (13), and it is denoted by \( K_+ \). Thus, the above equation can be written as

\[
E(z^* \mid D) = \frac{1}{p(z)} \int \int \exp(U^T \hat{\beta} + \tau^*) p(z \mid \tau) p(\tau^*, \tau) d\tau d\tau^*. \tag{18}
\]

where

\[
\Phi(\tau_+) = U^T \hat{\beta} + \tau^* + \sum_{i=1}^{n_1} \log p(z_{1i} \mid \hat{\beta}_1, \tau_{1i}) + \sum_{i=1}^{n_2} \log p(z_{2i} \mid \hat{\beta}_2, \tau_{2i}) \\
- \frac{n_1 + n_2 + 2}{2} \log(2\pi) - \frac{1}{2} \log |K_+| - \frac{1}{2} \tau_+^T K_+^{-1} \tau_+. \tag{19}
\]
where \( p(z_{ai} | \beta_a, \tau_{ai}) \) is the density of the Poisson distribution with mean \( \mu_{ai} = \exp(U_{ai}^T \beta_a + \tau_{ai}) \) for \( a = 1, 2 \). The calculation of the integral is difficult and we also use a Laplace approximation:

\[
\int \exp(\Phi(\tau_+))d\tau_+ \approx \exp\{\tilde{\Phi}(\hat{\tau}_+) + \frac{n_1 + n_2 + 2}{2} \log(2\pi) - \frac{1}{2} \log |\tilde{K}-1 + \tilde{C}_+| \} \]

(20)

where \( \tilde{C}_+ \) is the second derivative of the first four items in (19) with respect to \( \tau_+ \) and evaluated at \( \hat{\tau}_+ \). It is an \((n_1 + n_2 + 2)\) dimensional diagonal matrix:

\[
\tilde{C}_+ = \text{diag}(exp(U_{11}^T \hat{\beta}_1 + \hat{\tau}_{11}), ..., exp(U_{1n_1}^T \hat{\beta}_1 + \hat{\tau}_{1n_1}), \exp(U_{21}^T \hat{\beta}_2 + \hat{\tau}_{21}), ..., \exp(U_{2n_2}^T \hat{\beta}_2 + \hat{\tau}_{2n_2}), 0, 0).
\]

Similarly, we can calculate the predictive variance, which is defined as

\[
\text{Var}(z^* | \mathcal{D}) = \begin{pmatrix}
\text{Var}(z^*_1 | \mathcal{D}) & \text{Cov}(z^*_1, z^*_2 | \mathcal{D}) \\
\text{Cov}(z^*_1, z^*_2 | \mathcal{D}) & \text{Var}(z^*_2 | \mathcal{D})
\end{pmatrix},
\]

(21)

where

\[
\text{Var}(z^* | \mathcal{D}) = \mathbb{E}[\text{Var}(z^* | \tau^*, \mathcal{D})] + \text{Var}[\mathbb{E}(z^* | \tau^*, \mathcal{D})]
\]

(22)

Here \( z \) could be either \( z_1 \) or \( z_2 \). Because \( \text{Var}(z^* | \tau^*, \mathcal{D}) = \mathbb{E}(z^* | \tau^*, \mathcal{D}) \) for a Poisson distribution, we have \( \mathbb{E}[\text{Var}(z^* | \tau^*, \mathcal{D})] = \mathbb{E}(z^* | \mathcal{D}) \). The second item can be calculated by

\[
\text{Var}[\mathbb{E}(z^* | \tau^*, \mathcal{D})] = \mathbb{E}[\text{Var}(z^* | \tau^*, \mathcal{D})]^2 - \mathbb{E}[\text{Var}(z^* | \tau^*, \mathcal{D})]^2
\]

\[
= \int (\exp(U^T \beta + \tau^*))^2 p(\tau^* | \mathcal{D})d\tau^* - [\mathbb{E}(z^* | \mathcal{D})]^2.
\]

(23)

The first item in (23) can be obtained by Laplace approximation using the similar way to calculate \( \mathbb{E}(z^* | \mathcal{D}) \) in (18).

The covariance \( \text{Cov}(z^*_1, z^*_2 | \mathcal{D}) \) is calculated by

\[
\text{Cov}(z^*_1, z^*_2 | \mathcal{D}) = \mathbb{E}[z^*_1 z^*_2 | \mathcal{D}] - \mathbb{E}[z^*_1 | \mathcal{D}]\mathbb{E}[z^*_2 | \mathcal{D}]
\]

\[
= \mathbb{E}\{\mathbb{E}[z^*_1 z^*_2 | \tau^*, \mathcal{D}]\} - \mathbb{E}[\mathbb{E}[z^*_1 | \mathcal{D}]\mathbb{E}[z^*_2 | \mathcal{D}]].
\]

(24)

The first item in (24) is similar to the first item in (23), and can be calculated by Laplace approximation.
2.4 Consistency

The prediction based on a GPR model is consistent when the sample size of the data collected from a certain curve is sufficiently large and the covariance function satisfies certain regularity conditions. The consistency does not depend on the common mean structure or the choice of the values of hyper-parameters involved in the covariance function.

In this section, we will discuss information consistency and extend it to a more general context than the result of Wang and Shi (2014). We focus on \( \tilde{z} \) to \( z \), where \( \tilde{z} = (\tilde{z}_{11}, \ldots, \tilde{z}_{1n_1}, \ldots, \tilde{z}_{1n_2}) \) are predicted observations and \( z = (z_{11}, \ldots, z_{1n_1}, z_{21}, \ldots, z_{2n_2}) \) are actual observations, and \( n_1 \) and \( n_2 \) are the number of observations of the first input and the second input respectively. The corresponding covariate are \( X_{n_1n_2} = \{ (x_{1i}, x_{2j}), i = 1, \ldots, n_1, j = 2, \ldots, n_2 \} \) where \( x_{ai} \in \mathcal{X} \subset \mathbb{R}^p \) are independently drawn from its distribution, and the latent variable is \( (\tau_{1i}, \tau_{2j}) \).

We assume that \( z_{1i} \) and \( z_{2j} \) is a set of samples and follow a bivariate Poisson distribution with \( \mu_{1i} = \exp(\mathbf{U}_i^T \beta_1 + \tau_{1i}(x_{1i})) \) and \( \mu_{2j} = \exp(\mathbf{U}_j^T \beta_2 + \tau_{2j}(x_{2j})) \) respectively and \( (\tau_{1i}(\cdot), \tau_{2j}(\cdot)) \sim \text{MGP}(0, k(\cdot, \cdot)) \) was discussed in the previous section. Therefore, the stochastic process \( \tau_1(\cdot) \) and \( \tau_2(\cdot) \) induces a measure on space \( \mathcal{F} : \{ f(\cdot) : \mathcal{X} \to \mathbb{R} \} \). For convenience, we can rewrite \( z = (z_{11}, \ldots, z_{1n_1}, z_{21}, \ldots, z_{2n_2}) = (z_1, \ldots, z_{n_1}, z_{n_1+1}, \ldots, z_{n_1+n_2}) \) and the covariate as \( X_{n_1n_2} = (x_1, \ldots, x_{n_1}, x_{n_1+1}, \ldots, x_{n_1+n_2}) \). Let \( \mathcal{D}_{n_1n_2} = \{ (x_i, z_i), i = 1, \ldots, n_1 + n_2 \} \), we have

\[
E(z|\tau) \triangleq \exp(\mathbf{U}^T \beta + \tau(x)).
\]

Suppose that the hyper-parameters \( \theta \) in the covariance function are estimated by an empirical Bayesian method and the estimator is denoted by \( \hat{\theta} \). Let \( \tau_0 \) be the true underlying function, i.e. the true mean of \( z_i \) is given by \( \mu_{0i} = \exp(\mathbf{U}_i^T \beta + \tau_0(x_i)) \). Denote

\[
p_{mgp}(z) = \int p(z_1, \ldots, z_{n_1}, z_{n_1+1}, \ldots, z_{n_1+n_2}|\tau(x))p_{n_1+n_2}(\tau)d\tau
\]

and

\[
p_0(z) = p(z_1, \ldots, z_{n_1}, z_{n_1+1}, \ldots, z_{n_1+n_2}|\tau_0(x)),
\]

then \( p_{mgp}(z) \) is the Bayesian predictive distribution of \( z \) based on a MCGPP model. Note that \( p_{n_1+n_2}(\tau) \) depends on the sample size \( n_1 + n_2 \) since the hyper-parameters of \( \tau \) are
estimated from the data. We say that $p_{mgp}$ achieves information consistency if

$$\frac{1}{n_1 + n_2} \mathbb{E}_{X_{n_1n_2}} \left( D[p_0(z), p_{mgp}(z)] \right) \to 0 \text{ as } n_1 \to \infty \text{ and } n_2 \to \infty,$$

where $\mathbb{E}_{X_{n_1n_2}}$ denotes the expectation under the distribution of $X_{n_1n_2}$ and $D[p_0(z), p_{mgp}(z)]$ is the Kullback-Leibler divergence between $p_0(\cdot)$ and $p_{mgp}(\cdot)$, i.e.,

$$D[p_0(z), p_{mgp}(z)] = \int p_0(z) \log \frac{p_0(z)}{p_{mgp}(z)} dz.$$

**Theorem 1.** Under the MCGPP model (10) and the condition given in Lemma 1 in Appendix, the prediction $\hat{z}$ is information consistent if the RKHS norm $\|\tau_0\|_{K_{n_1n_2}}^2$ is bounded and the expected regret term $\mathbb{E}_{X_{n_1n_2}}(\log |I + \delta K_{n_1n_2}|) = o(n_1 + n_2)$. The error bound is specified in (35).

The proof of the theorem is given in Appendix.

**Remark 1** The regret term $R = \log |I + \delta K_{n_1n_2}|$ depends on the covariance function $k(x_i, x_j)$ for a convolved bivariate GP and the distribution of $x$. We can use it to identify the upper bounds of the expected regret for some commonly used covariance functions by extending results in Wang and Shi (2014). The detailed discussion is given in Appendix.

### 3 Numerical Results

In this section, we demonstrate the performance of the proposed method by comprehensive simulation studies with two scenarios and also present results for two real data examples.

#### 3.1 Simulation Studies: Scenario 1

In the first scenario, we use a discrete bivariate Poisson regression model in (10) as the true model to generate data:

$$\begin{pmatrix} z_{1i}(x_i) \\ z_{2j}(x_j) \end{pmatrix} \sim \begin{pmatrix} \text{Poisson}(\mu_{1i}(x_i)) & i = 1, \ldots, n_1 \\ \text{Poisson}(\mu_{2j}(x_j)) & j = 1, \ldots, n_2 \end{pmatrix},$$

where

$$\begin{pmatrix} \mu_{1i}(x_i) = \exp(U_{i1}^T \beta_1 + \tau_{1i}(x_i)) \\ \mu_{2j}(x_j) = \exp(U_{2j}^T \beta_2 + \tau_{2j}(x_j)) \end{pmatrix}, \quad \begin{pmatrix} \tau_{1i}(\cdot) \\ \tau_{2j}(\cdot) \end{pmatrix} \sim \text{MGP}(0, k(\cdot, \cdot)), $$
and \( k(\cdot, \cdot) \) is defined by (6) and (7). We take \( \beta_{10} = 1, \beta_{11} = 2, \beta_{20} = 1 \) and \( \beta_{21} = 2 \).

Random processes \( \tau_{1i} \) and \( \tau_{2i} \) are generated from a MGP with a mixed covariance structure, the combination of two different covariance functions. Specifically, \( \eta_1 \) is generated from a GP with the squared exponential covariance function with \( v_{11} = 0.04 \) and \( A_{11} = 1 \), while \( \eta_2 \) from the Gamma exponential covariance function with \( v_{21} = 0.04 \) and \( A_{21} = 1 \). The shared processes \( \xi_a \)'s follow the squared exponential covariance function with \( v_{10} = 0.04, v_{20} = 0.04, A_{10} = 1 \) and \( A_{20} = 1 \). The covariates \( x_i \)'s are equally spaced in \([-5, 5]\). Recall that \( \tau_a = \xi_a + \eta_a \) for \( a = 1, 2 \). Thus \( \boldsymbol{\tau} = \{\tau_{1i}, \tau_{2j}\} \) is dependent GPs but have different covariance structure for each component. We set \( n_1 = n_2 = 20 \).

As we discussed in the previous section, the proposed MCGPP model allows different covariance structure for each component and thus it should be able to have a good fit for the data generated using the above way. To show the stability of the models, we considered the model (10) with the following covariance functions.

**Model 1** - \( \xi_1, \xi_2 \) and \( \eta_1 \) have squared exponential covariance functions and \( \eta_2 \) has a Gamma exponential covariance function, i.e. this model assumes the same covariance structure as the true model;

**Model 2** - all \( \eta_1, \eta_2, \xi_1 \) and \( \xi_2 \) have rational quadratic covariance functions;

**Model 3** - all \( \eta_1, \eta_2, \xi_1 \) and \( \xi_2 \) have Matern covariance functions;

**Model 4** - all \( \eta_1, \eta_2, \xi_1 \) and \( \xi_2 \) have squared exponential covariance functions.

As comparison, we also consider the model in Crainiceanu et al. (2008) (CDR), where \( \tau_1 \) is a GP with zero mean and a squared exponential covariance function, and \( \tau_2 \) is conditional on \( \tau_1 \), i.e. \( \tau_2 | \tau_1 \sim N(\alpha \tau_1, \sigma_\epsilon^2) \). The dependency is determined by \( \alpha \). It is a useful model but lack of flexibility on modelling covariance structures for multiple components since the covariance structure of the second component is determined by the first one.

We also compared them to the independent model (Indep). In this case, we assume that \( \tau_1 \) and \( \tau_2 \) are independent and each follows a GP with a squared exponential covariance function.

We use each of the six models to fit the data. To measure the performances of those models, we further generate a new set of test data (20 for each component) and use the fitted model to calculate the prediction of \( \mu_{ai}, a = 1, 2 \) and \( i = 1, \ldots, 20 \) for the test data. We then
calculate the root mean squared error (RMSE) between the predictions and the test data for \( \mu_{ai} \). Table 1 listed the average RMSEs based on 100 replications. As expected, Model 1 gives the best result. Models 2 to 4 also give reasonably good results although different covariance functions are used in those models. This shows that the proposed model is flexible to fit data with different covariance structure in each component, and is robust as well. Model CDR models the dependency using a conditional distribution, i.e. the covariance structure of the second component is dependent on the first one. When this model is applied to the data having different covariance structures for each component, the result is not satisfactory. Model Indep ignores the dependency between components and consequently has large errors.

Table 1: Average RMSEs between \( \mu \) and \( \hat{\mu} \) based on one hundred replications.

| Model | Average RMSE |
|-------|--------------|
| Model 1 | 0.02627 |
| Model 2 | 0.03841 |
| Model 3 | 0.03028 |
| Model 4 | 0.03459 |
| CDR | 0.10920 |
| Indep | 0.04628 |

We also calculate the difference between the estimation of \( \beta \) and its true values. The values of RMSE between \( \hat{\beta} \) and its true value and the sampling bias based on 100 replications are presented in Table 2. The findings are almost the same as those from Table 1.

Table 2: RMSEs between \( \hat{\beta} \) and their true values and the absolute value of the sampling bias (in parenthesis) based on one hundred replications.

| Model | RMSE (|bias|) |
|-------|-------|
|       | \( \beta_{11} \) | \( \beta_{12} \) | \( \beta_{21} \) | \( \beta_{22} \) |
| Model 1 | 0.03496 (.000) | 0.04547 (.004) | 0.03967 (.005) | 0.03739 (.007) |
| Model 2 | 0.03381 (.003) | 0.04130 (.000) | 0.03802 (.001) | 0.03626 (.004) |
| Model 3 | 0.03478 (.005) | 0.05156 (.002) | 0.04036 (.007) | 0.03279 (.000) |
| Model 4 | 0.04833 (.003) | 0.04560 (.001) | 0.04106 (.004) | 0.04066 (.000) |
| CDR | 0.13076 (.020) | 0.17025 (.026) | 0.13972 (.017) | 0.15640 (.005) |
| Indep | 0.09486 (.009) | 0.13251 (.018) | 0.14912 (.022) | 0.11417 (.013) |
3.2 Simulation Studies: Scenario 2

We now consider a scenario with multidimensional covariates and nonlinear mean function. The model is defined as

\[
\mu_{1i}(x_i) = \exp(y_{1i}(x_i)), \quad z_{1i}(x_i) \sim \text{Poisson}(\mu_{1i}(x_i)), \quad i = 1, \ldots, n_1,
\]

\[
\mu_{2j}(x_j) = \exp(y_{2j}(x_j)), \quad z_{2j}(x_j) \sim \text{Poisson}(\mu_{2j}(x_j)), \quad j = 1, \ldots, n_2,
\]

The latent variables \(y_{1i}(x_i)\) and \(y_{1j}(x_j)\) are generated by the following way

\[
y_{1i}(x_i) = 0.2x_{1i} | x_{1i} |^{\frac{1}{3}} + \log(x_{2i}) + \tau_{1i}(x_i), \quad i = 1, \ldots, n_1,
\]

\[
y_{2j}(x_j) = \sin(x_{2j}) + 0.4x_{2j} | x_{1j} |^{\frac{1}{4}} + \tau_{2j}(x_j), \quad j = 1, \ldots, n_2,
\]

where \((\tau_{1i}(\cdot), \tau_{2j}(\cdot)) \sim \text{MGP}(0, k(\cdot, \cdot))\) and \(k(\cdot, \cdot)\) is the same as the one in Scenario 1. \(x = \{x_{1i}, x_{2j}\}\) are equally spaced in \([-5, 10]\) and \([1, 2]\) respectively and \(\tau = \{\tau_{1i}, \tau_{2j}\}\) is dependent GP which is formed in the same way to Scenario 1 in Model 1, i.e. a mixed squared exponential covariance function and a Gamma exponential covariance function. Also the true values are the same as those used in Scenario 1.

In each replication, we generate \(n_1 = n_2 = 20\) observations as training data, and the further same numbers of observations as test data. We used all six models defined in Scenario 1 to fit the data. Bear in mind that, although we assumed the same covariance structures in Model 1 as those in the true model, Model 1 is different to the true model since nonlinear mean model is used in the true model while only linear mean model is assumed in the proposed model (i.e. Models 1 to 4). Shi et al. (2012) argued that the GPR is a flexible nonlinear Bayesian model and can fit nonlinear curves for continuous Gaussian data. We expect Models 1 to 4 can also fit the nonlinear latent curves, and thus they should provide a good fit to the non-Gaussian Poisson data in this scenario. The simulation study results presented in Table 3 confirm the expectation. The numbers in the table is the average RMSE between the generated value of \(\mu\) and its prediction \(\hat{\mu}\) based on 100 replications. The very small values of RMSE indicate that that GPR model is good on fitting the nonlinear data.

Different covariance functions are used in Models 2 to 4, but all of them provide reasonable good results and all are better than Models CDR and Indep, where CDR models the covariance...
structure by a conditional approach, and Model Indep assumed independence between two components.

Table 3: The average RMSE between $\mu$ and $\hat{\mu}$ based on one hundred replications.

| Model  | Average RMSE |
|--------|--------------|
| Model 1| 0.020587     |
| Model 2| 0.022521     |
| Model 3| 0.022453     |
| Model 4| 0.023001     |
| CDR    | 0.028196     |
| Indep  | 0.025159     |

3.3 Real Data Analysis

We will present results for two real sets of data. The first one is data relating to two type of cancers in Minnesota, USA. The second data concern Dengue fever and Malaria in Indonesia.

1. Lung and Oesophageal Cancer data

From information on the NHS web site (www.nhs.uk), one of the most dangerous and common types of cancer is lung cancer. Every year there are around 44,500 people diagnosed with this condition. The symptoms usually do not always appear in the early stages, although some symptoms develop in many people, such as blood or persistent coughing, breathlessness and weight loss. In over 85 percent of cases, the main cause of lung cancer is cigarette smoking although people who have never smoked can be diagnosed with this cancer. Smoking can cause other cancers, such as oesophageal cancer and mouth cancer.

There are more than 8,500 new cases of oesophageal cancer diagnosed each year in the UK which means that this cancer is uncommon but is not rare. As with lung cancer, smoking and drinking alcohol are the highest risk factors for this cancer.

Fig 1 in [Jin et al. (2005)] present the number of cases for each cancer in Minnesota, USA. The map shows clearly that the county-level maps of the age-adjusted standardized mortality ratios between lung and oesophageal have a positive correlation across region or area. Thus it is better to investigate those two cancers using a joint multivariate model.
Jin et al. (2005) analysed the relationship between lung cancer and oesophageal cancer using a generalized intrinsics autoregressive model which was based on neighbourhood for each region as the main effect of the model. In practice, this model may have difficulty in prediction due to problem of defining the neighbourhood for each area. Similar to CDR model in (Crainiceanu et al., 2008), a conditional approach is used in Jin et al. (2005) to define the cross-correlation between two components which is a less flexible model as we discussed in simulation studies.

We use MCGPP model here. The model can be written as

\[ z_{ia} \sim \text{Poisson}(E_{ia}e^{\tau_{ia}(x_i)}), \quad i = 1, \ldots, 87, \quad a = 1, 2, \]

where \( z_{ia} \) is the observed number of deaths due to cancer \( a \) in county \( i \), \( E_{ia} \) is the corresponding expected number of deaths (assumed known) and \( \tau_{ia}(\cdot) \sim MGP(0, k(\cdot, \cdot)) \) which is explained in equation (7). Here, \( x \) are defined from spaced point values of latitude and longitude, the location, of each county. The correlation of the mortalities between two areas depends on their locations. The nearer, the larger. This is similar to the assumptions in Jin et al. (2005), but it is straightforward to find the values of \( x \), and the covariance structure can be learned and adjusted from the data in MCGPP model.

As a comparison, we also used CDR model.

To measure the performance, we select data randomly from the whole data set to form training data consisting of two thirds of the data and the remainder is used for test data. We estimate parameters by an empirical Bayesian approach using the training data and then calculate prediction for the test data, and the value of error rate between the predictions and the actual observations. Table 4 reports the average ERs based on ten replications. It shows that the MCGPP model provides very accurate results and is better than CDR. AIC (using the full data) also support the MCGPP model.

| Method | Average ER | AIC        |
|--------|------------|------------|
| CDR    | 0.0149     | 1640.202   |
| MCGPP  | 0.0080     | 1399.822   |

2. Dengue Fever and Malaria data
We now analyse dengue fever and malaria data in Indonesia. Both of the diseases can be spread by two different types mosquitoes which are hard to distinguish from each other. Therefore, it is more sensible to analyse them together in a joint multivariate model. The data are also spatially correlated. We compared several methods to deal with this spatial effect, including MCGPP, an intrinsic autoregressive model (CAR), and a conventional Poisson regression model. Among all those models, we found MCGPP are the best for the data; the details can be found in (Sofro, 2016).

We present three models here taken from the different set of multidimensional covariates used in modelling covariance structure in MCGPP. The first model involves location (latitude and longitude) and all five observed covariates (health water \( x_1 \), healthy rubbish bin \( x_2 \), waste water disposal facilities \( x_3 \), clean and healthy behaviour \( x_4 \) and healthy house \( x_5 \)). The second model uses location and three covariates, \( x_1, x_2, x_3 \). The last model uses the location only.

| Models                  | Average ER | MCGPP   | CDR    |
|-------------------------|------------|---------|--------|
| Full (location and all covariates) | 0.000994   | 0.001374|
| Location and \( x_1, x_2, x_3 \) | 0.001018   | 0.002000|
| Location                | 0.001137   | 0.002252|

Similar to the previous example, we also calculate the error rate for the test data. The results based on fifteen replications are presented in Table 5. Not surprisingly, the first model provides the best result. However, the second model performs almost as well as the first one, indicating \( x_1, x_2 \) and \( x_3 \) are the most important facts related to both diseases. As a comparison, we also present the results by using CDR model. It gives less accurate results.

4 Conclusions

In this paper, we proposed a new method for multivariate Poisson regression analysis for dependent count data using convolved Gaussian processes. It is a very flexible model, can model nonlinear data, allow different covariance structure for each component, and also copy
with multidimensional covariates. The approach is also quite robust, providing reliable results even when different covariance functions are used.

We limited our discussion in this paper to the bivariate case, the idea can be used to general multivariate cases. However, it is worth a further investigation on how to define cross-correlation for multiple components and how to implement the method efficiently.

References

Andriluka, M., Weizsäcker, L and Hofmann, T. (2007). Multi-class classification with dependent Gaussian process. In proceedings of International Conference on Applied Stochastic Models and Data Analysis (ASMDA).

Banerjee, S., Carlin, B., and Gelfand, A. (2004). Hierarchical Modeling and Analysis for Spatial Data. Boca Raton: Chapman and Hall/CRC Press.

Besag, J and Kooperberg, C. (1995). On conditional and intrinsic autoregressions. Biometrika, 82(4), 733-746.

Boyle, P. and Frean, M. (2005). Dependent Gaussian Process. In Saul, L. K., Weiss, Y., and Bottou, L., editors, Advances in Neural Information Processing Systems, 17, 217-224, Cambridge: MIT Press.

Choi, T. (2005), Posterior Consistency in Nonparametric Regression Problems under Gaussian Process Priors, PhD thesis, Carnegie Mellon University, Pittsburgh, PA.

Crainiceanu, C. M., Diggle, P. J and Rowlingson, B. (2008). Bivariate binomial spatial modeling of Loa loa prevalence in Tropical Africa. Journal of the American Statistical Association, 103(481), 21-37.

Diggle, P. J., Tawn, J. A. and Moyeed, R. A. (1998). Model-based geostatistics. Journal of Royal Statistics Society, Series C, 47(3), 299-350.

Gelfand, A. E and Vounatsou, P. (2003). Proper multivariate conditional autoregressive models for spatial data analysis. BioStatistics, 4(1), 11-25.
Gramacy, R. and Lian, H. (2012). Gaussian process single-index models as emulators for computer experiments. *Technometrics*, 54(1): 30-41.

Jin, X., Carlin, B. and Banerjee, S. (2005). Generalized hierarchical multivariate CAR model for areal data. *Biometrics*, 61(4), 950-961.

Kim, H., Sun, D. and Tsutakawa, R. K. (2001). A bivariate Bayes method for improving the estimates of mortality rates with a twofold conditional autoregressive model. *Journal of American Statistical Association*, 96(456), 1506-1521.

MacNab, Y. C. and Dean, C. B. (2001). Autoregressive spatial smoothing and temporal spline smoothing for mapping rates. *Biometrics*, 57(3), 949-956.

Martínez-Beneito, M. A. (2013). A general modelling framework for multivariate disease mapping. *Biometrika*, 100(3), 539-553.

Martínez-Beneito, M. A., Quilez, A. L. and Botella-Rocamora, P. B. (2008). An autoregressive approach to spatio-temporal disease mapping. *Statistics in Medicine*, 27(15), 2874-2889.

Mohebbi, M., Wolfe, R., Jolley, D. et al. (2011). The spatial distribution of esophageal and gastric cancer in Caspian region of Iran: An ecological analysis of diet and socio-economic influences. *International Journal of Health Geographics*. 10, 1-13.

Rasmussen, C. E. and Williams, C. K. (2006). *Gaussian Process for Machine Learning*. Cambridge: MIT Press.

Rue, H. and Held, L. (2005). *Gaussian Markov Random Fields: Theory and Application*. Boca Raton: Chapman and Hall/CRC.

Seeger, M. W and Kakade, s. M and Foster, D. P. (2008) ”Information Consistency of Nonparameretic Gaussian Process Methods”, *IEEE Transactions on Information Theory*, 54, 2376-2382.

Shi, J. Q., Choi, T. (2011). *Gaussian Process Regression Analysis for Functional Data*. London: Chapman and Hall.
Shi, J. Q., Wang, B., Will, E. J. and West, R. M. (2012). Mixed-effects GPFR models with application to dose-response curve prediction. Statistics in Medicine. 31(26), 3165-77.

Silva, G. L., Dean, C. B., Niyonsenga, T. and Vanasse, A. (2008). Hierarchical Bayesian spatiotemporal analysis of revascularization odds using smoothing splines. Statistics in Medicine, 27(13), 2381-2401.

Sofro, A. (2016). Convolved Gaussian Process Regression Models for Multivariate Non-Gaussian Data. PhD thesis, Newcastle University, UK.

Sun, D., Tsutakawa, R. K., Kim, H. and He, Z. (2000). Spatio-temporal interaction with disease mapping. Statistics in Medicine, 19(15), 2015-2035.

Wall, M. M. (2004). A close look at the spatial structure implied by the CAR and SAR models. Journal of Statistical Planning and Inference. 121(2), 311-324.

Wang, B. and Shi, J. Q. (2014). Generalized Gaussian process regression model for non-Gaussian functional data Journal of American Statistical Association, 109(507), 1123-1133.
Appendix : Proof of information consistency

The proof presented below is an extension from consistency theorem in [Wang and Shi (2014)].

Lemma 1

Suppose \( z_{1i} \) and \( z_{2j} \) are conditional independent samples from a bivariate Poisson distribution given (10) and \( \tau_0 \in \mathcal{F} \) has a multivariate convolved Gaussian prior with zero mean and bounded covariance function \( k(\cdot, \cdot) \) for any covariate values in \( \mathcal{X} \). Suppose that \( k(\cdot, \cdot) \) is continuous in \( \theta \) and the estimator \( \hat{\theta} \rightarrow \theta \) almost surely as \( n_1 \rightarrow \infty \) and \( n_2 \rightarrow \infty \). Then

\[
- \log p_{mgp}(z_1, ..., z_{n_1+n_2}) + \log p_0(z_1, ..., z_{n_1+n_2}) \\
\leq \frac{1}{2} \| \tau_0 \|^2_{K_{n_1n_2}} + \frac{1}{2} \log |I + \delta K_{n_1n_2}| + C
\]

where \( \| \tau_0 \|^2_{K_{n_1n_2}} \) is the reproducing kernel Hilbert space (RKHS) norm of \( \tau_0 \) associated with \( k(\cdot, \cdot) \), \( K_{n_1n_2} \) is the covariance matrix of \( \tau_0 \) over the covariate \( X_{n_1n_2} \), \( I \) is the \( (n_1+n_2) \times (n_1+n_2) \) identity matrix, \( \delta \) and \( C \) are some positive constants.

Proof. In this proof, we use a covariance function to define a function on \( \mathcal{X} \). The space of such a function is known as a reproducing kernel Hilbert space (RKHS). Let \( \mathcal{H} \) be RKHS associated with covariance function \( k(\cdot, \cdot) \) e.g. the squared exponential covariance function defined in (2), \( \mathcal{H}_{n_1+n_2} \) be the linear span of \( \{k(\cdot, x_i), i = 1, ..., n_1+n_2\} \), i.e.

\[
\mathcal{H}_{n_1+n_2} = \left\{ f(\cdot) : f(x) = \sum_{i=1}^{n_1+n_2} \alpha_i k(x, x_i), \alpha_i \in \mathcal{R} \right\}.
\]

We first assume the true underlying function \( \tau_0 \in \mathcal{H}_{n_1+n_2} \) then \( \tau_0(\cdot) \) can be expressed as

\[
\tau_0(\cdot) = \sum_{i=1}^{n_1+n_2} \alpha_i k(\cdot, x_i) \triangleq K_{n_1n_2}(\cdot)\alpha.
\]

where \( K_{n_1n_2}(\cdot) = (k(\cdot, x_1), ..., k(\cdot, x_{n_1+n_2})) \) and \( \alpha = (\alpha_1, ..., \alpha_{n_1+n_2})^T \). By the properties of RKHS, \( \| \tau_0 \|^2_{K_{n_1n_2}} = \alpha^T K_{n_1n_2} \alpha \), and \( (\tau_0(x_1), ..., \tau_0(x_{n_1+n_2}))^T = K_{n_1n_2} \alpha \) where \( K_{n_1n_2} = (k(x_i, x_j)) \) is the covariance matrix over \( x_i, \ i = 1, ..., n_1+n_2 \).

Let \( P \) and \( \bar{P} \) be any two measures on \( \mathcal{F} \), then it yields by the Fenchel-Legendre duality relationship that, for any function \( g(\cdot) \) on \( \mathcal{F} \),

\[
E_P[g(\tau)] \leq \log E_P[e^{g(\tau)}] + D[\bar{P}, P].
\]

Now in the above inequality let
1. \( g(\tau) \) be \( \log p(z_1, \ldots, z_{n_1+n_2} | \tau) \) for any \( z_1, \ldots, z_{n_1+n_2} \) in \( Z \) and \( \tau \in F \).

2. \( P \) be the measure induced by \( \text{MGP}(0, k(\cdot, \cdot)) \), hence its finite dimensional distribution at \( \tau_1, \ldots, \tau_{n_1+n_2} \) is \( \mathcal{N}(0, K_{n_1n_2}) \) and

\[
E_P[e^{g(\tau)}] = \int p(z_1, \ldots, z_{n_1+n_2} | \tau)p_{n_1+n_2}(\tau)d\tau = p_{mgp}(z)
\]

where \( K_{n_1n_2} \) is defined in the same way as \( K_{n_1n_2} \) but the \( \theta \) being replaced by its estimator \( \hat{\theta} \).

3. \( \bar{P} \) be the posterior distribution of \( \tau(\cdot) \) on \( F \) which has a prior distribution \( \text{MGP}(0, k(\cdot, \cdot)) \) and normal likelihood \( \prod_{i=1}^{n_1+n_2} N(\hat{z}_i; \tau(x_i), \sigma^2) \), where

\[
\hat{z} \triangleq \begin{pmatrix}
\hat{z}_1 \\
\vdots \\
\hat{z}_{n_1+n_2}
\end{pmatrix} = (K_{n_1n_2} + \sigma^2 I)\alpha 
\]

and \( \sigma^2 \) is a constant to be specified. In other words, we assume a model \( z = \tau(x) + \epsilon \) with \( \epsilon \sim \mathcal{N}(0, \sigma^2) \) and \( \tau(\cdot) \sim \text{MGP}(0, k(\cdot, \cdot)) \), and \( \hat{z} \) defined by equation (30) is a set of observations at \( x_1, \ldots, x_{n_1+n_2} \). Thus, \( \bar{P}(\tau) = p(\tau | \hat{z}, X_{n_1n_2}) \) is a probability measure on \( F \). Therefore, by bivariate CGP regression, the posterior of \( (\tau_1, \ldots, \tau_{n_1+n_2}) \triangleq (\tau(x_1), \ldots, \tau(x_{n_1+n_2})) \) is

\[
\bar{p}(\tau_1, \ldots, \tau_{n_1+n_2}) \triangleq p(\tau_1, \ldots, \tau_{n_1+n_2} | \hat{z}, X_{n_1n_2}) \\
= \mathcal{N}(K_{n_1n_2}(K_{n_1n_2} + \sigma^2 I)^{-1}\hat{z}, K_{n_1n_2}(K_{n_1n_2} + \sigma^2 I)^{-1}\sigma^2) \\
= \mathcal{N}(K_{n_1n_2}\alpha, K_{n_1n_2}(K_{n_1n_2} + \sigma^2 I)^{-1}\sigma^2) \\
= \mathcal{N}(K_{n_1n_2}\alpha, K_{n_1n_2}B^{-1})
\]

where \( B = I + \sigma^{-2}K_{n_1n_2} \).
It follows that
\[
D[P, P] = \int_{\mathcal{F}} \log \frac{d\bar{P}}{dP} dP
\]
\[
= \int_{\mathbb{R}^{n_1+n_2}} \bar{p}(\tau_1, \ldots, \tau_{n_1+n_2}) \log \frac{\bar{p}(\tau_1, \ldots, \tau_{n_1+n_2})}{p(\tau_1, \ldots, \tau_{n_1+n_2})} d\tau_1 \cdots d\tau_{n_1+n_2}
\]
\[
= \frac{1}{2} \left[ \log |\bar{K}_{n_1+n_2}| - \log |K_{n_1+n_2}| + \log |B| + tr(\bar{K}_{n_1+n_2}^{-1} K_{n_1+n_2} B^{-1}) + (K_{n_1+n_2} \alpha)^T \bar{K}_{n_1+n_2}^{-1} (K_{n_1+n_2} \alpha) - (n_1 + n_2) \right]
\]
\[
+ \alpha^T K_{n_1+n_2} (\bar{K}_{n_1+n_2}^{-1} K_{n_1+n_2} - I) \alpha - (n_1 + n_2).
\]

On the other hand,
\[
E_P[g(\tau)] = E_P[\log p(z_1, \ldots, z_{n_1+n_2} | \tau)] = \sum_{i=1}^{n_1+n_2} E_P[\log p(z_i | \tau(x_i))].
\]

By Taylor’s expansion, expanding \(\log p(z_i | \tau(x_i))\) to the second order \(\tau_0(x_i)\) yields
\[
\log p(z_i | \tau(x_i)) = \log p(z_i | \tau_0(x_i)) + \frac{d[\log p(z_i | \tau(x_i))]}{d \tau(x_i)} \bigg|_{\tau(x_i) = \tau_0(x_i)} (\tau(x_i) - \tau_0(x_i))
\]
\[
+ \frac{1}{2} \frac{d^2[\log p(z_i | \tau(x_i))]}{[d \tau(x_i)]^2} \bigg|_{\tau(x_i) = \tilde{\tau}(x_i)} (\tau(x_i) - \tau_0(x_i))^2,
\]
where \(\tilde{\tau}(x_i) = \tau_0(x_i) + \lambda(\tau(x_i) - \tau_0(x_i))\) for some \(0 \leq \lambda \leq 1\).

For the canonical link function with Convolled GPR, we have
\[
\log p(z_i | \tau(x_i)) = z_i \log (U_i^T \beta + \tau(x_i)) - (U_i^T \beta + \tau(x_i)) - \log(z_i!). \tag{32}
\]

It follows that
\[
E_P[\log p(z_i | \tau(x_i))] = \log p(z_i | \tau_0(x_i)) + (z_i - \exp(U_i^T \beta + \tau_0(x_i))) E_P[(\tau(x_i) - \tau_0(x_i))] - \frac{1}{2} E_P[\exp(U_i^T \beta + \tilde{\tau}(x_i))(\tau(x_i) - \tau_0(x_i))^2].
\]

Since \(\bar{P}(\cdot)\) is the posterior of \(\tau(\cdot)\) which has prior \(MGP(0, k(\cdot, \cdot))\) and normal likelihood \(\prod_{i=1}^{n_1+n_2} \mathcal{N}(\tilde{z}_i; \tau(x_i), \sigma^2)\), where \(\tau(x_i)\) is normally distributed under \(\bar{P}\) and it follows from (31) that
\[
\tau(x_i) \sim \mathcal{N}(K_{n_1+n_2}^{(0)}, (K_{n_1+n_2} B^{-1})_{ii})
\]
\[
= \mathcal{N}(\tau_0(x_i), (K_{n_1+n_2} B^{-1})_{ii}) \Delta \mathcal{N}(\tau_0, k_{ii})
\]

25
where \( K_{n_1 n_2}^{(i)} \) denotes the \( i \)th row of \( K_{n_1 n_2} \) and \((K_{n_1 n_2} B^{-1})_{ii}\) is the \( i \)th diagonal element of \( K_{n_1 n_2} B^{-1} \). Therefore, \( E_{P}[\tau(x_i) - \tau_0(x_i)] = 0 \) and

\[
E_{P}[\exp(U_i^T \beta + \tilde{\tau}(x_i))(\tau(x_i) - \tau_0(x_i))^2] = \exp(U_i^T \beta + \tau_0(x_i))E_P[e^{\lambda(\tau(x_i) - \tau_0(x_i))}(\tau(x_i) - \tau_0(x_i))^2]
\]

\[
= \exp(U_i^T \beta + \tau_0(x_i)) + \frac{1}{2} \lambda^2 k_{ii} (\lambda^2 k_{ii} + 1) k_{ii} \leq \tilde{\delta} k_{ii}
\]

since the covariance function is bounded. Here \( \tilde{\delta} \) is a generic positive constant. Thus, we have

\[
- \sum_{i=1}^{n_1 + n_2} E_{P}[\log p(z_i | \tau(x_i))] \leq - \sum_{i=1}^{n_1 + n_2} \log p(z_i | \tau_0(x_i)) + \frac{\tilde{\delta}}{2} \text{tr}(K_{n_1 n_2} B^{-1}).
\]

i.e.

\[
\log p_0(z_1, \ldots, z_{n_1 + n_2}) \leq E_{P}[g(\tau)] + \frac{\tilde{\delta}}{2} \text{tr}(K_{n_1 n_2} B^{-1}).
\]

Combining the bounds gives

\[
- \log p_{mop}(z_1, \ldots, z_{n_1 + n_2}) + \log p_0(z_1, \ldots, z_{n_1 + n_2})
\]

\[
\leq - \log E_P[e^{g(\tau)}] + E_P[g(\tau)] + \frac{\tilde{\delta}}{2} \text{tr}(K_{n_1 n_2} B^{-1})
\]

\[
\leq D[\tilde{P}, P] + \frac{\tilde{\delta}}{2} \text{tr}(K_{n_1 n_2} B^{-1})
\]

\[
= \frac{1}{2} \left[- \log |\hat{K}_{n_1 n_2}^{-1} K_{n_1 n_2}| + \log |B| + \text{tr}(\hat{K}_{n_1 n_2}^{-1} K_{n_1 n_2} B^{-1} + \tilde{\delta} K_{n_1 n_2} B^{-1}) + \|\tau_0\|^2_{K_{n_1 n_2}} + \alpha^T K_{n_1 n_2} (\hat{K}_{n_1 n_2}^{-1} K_{n_1 n_2} - I) \alpha - (n_1 + n_2) \right]. \tag{33}
\]

Since the covariance function is continuous in \( \theta \) and \( \theta_{n_1 + n_2} \to \theta \) and we have \( \hat{K}_{n_1 n_2} K_{n_1 n_2} - I \to 0 \) as \( n_1 \to \infty \) and \( n_2 \to \infty \), hence \( n_1 + n_2 \to \infty \). Therefore there exist some positive constants \( C \) and \( \epsilon \) such that

\[
- \log |\hat{K}_{n_1 n_2}^{-1} K_{n_1 n_2}| < C,
\]

\[
\alpha^T K_{n_1 n_2} (\hat{K}_{n_1 n_2}^{-1} K_{n_1 n_2} - I) \alpha < C,
\]

\[
\text{tr}(\hat{K}_{n_1 + n_2}^{-1} K_{n_1 n_2} B^{-1}) < \text{tr}((I + \epsilon K_{n_1 n_2}) B^{-1}),
\]

since the covariance function is bounded.

Thus the right hand side (RHS) of (33)

\[
< \frac{1}{2} \|\tau_0\|^2_{K_{n_1 n_2}} + \frac{1}{2} [2C + \log |B| + \text{tr}((I + (\epsilon + \tilde{\delta}) K_{n_1 n_2}) B^{-1}) - (n_1 + n_2)].
\]
Note that the above inequality holds for all $\sigma^2 > 0$, thus letting $\sigma^2 = (\epsilon + \tilde{\delta})^{-1}$ and $\delta = \epsilon + \tilde{\delta}$ yields that the RHS of (33) becomes

$$\frac{1}{2} \| \tau_0 \|^2_{K_{n_1n_2}} + \frac{1}{2} \log(I + \delta K_{n_1n_2}) + C.$$ 

Thus we have

$$- \log p_{mgp}(z_1, ..., z_{n_1}, z_{n_1+1}, ..., z_{n_1+n_2}) \leq - \log p_0(z_1, ..., z_{n_1}) + \frac{1}{2} \| \tau_0 \|^2_{K_{n_1n_2}} + \frac{1}{2} \log(I + \delta K_{n_1n_2}) + C$$

for any $\tau_0(\cdot) \in H_{n_1+n_2}$.

Taking infimum on RHS of (34) over $\tau_0$ and applying Representer Theorem, we obtain

$$- \log p_{mgp}(z_1, ..., z_{n_1+n_2}) + \log p_0(z_1, ..., z_{n_1+n_2}) \leq \frac{1}{2} \| \tau_0 \|^2_{K_{n_1n_2}} + \frac{1}{2} \log(I + \delta K_{n_1n_2}) + C$$

for all $\tau_0(\cdot) \in H_{n_1+n_2}$. The proof is complete.

**Proof of Theorem 1.** It follows from the definition of information consistency that

$$D[p_0(z), p_{mgp}(z)] = \int p_0(z_1, ..., z_{n_1+n_2}) \log \frac{p_0(z_1, ..., z_{n_1+n_2})}{p_{mgp}(z_1, ..., z_{n_1+n_2})} dz_1 \cdots dz_{n_1+n_2}.$$

Applying Lemma 1 we obtain that

$$\frac{1}{n_1+n_2} \mathbb{E}_{X_{n_1n_2}}[D[p_0(z), p_{mgp}(z)]] \leq \frac{1}{2(n_1+n_2)} \| \tau_0 \|^2_{K_{n_1n_2}} + \frac{1}{2(n_1+n_2)} \mathbb{E}_{X_{n_1n_2}} \log(I + \delta K_{n_1n_2}) + \frac{C}{n_1+n_2},$$

where $\delta$ and $C$ are some positive constants. Theorem 1 follows from (35).

**Remark 2** Lemma 1 requires that the estimator of the coefficients $\beta$ and hyper-parameters $\theta$ are consistent. Yi et al. (2011) provided that the empirical Bayesian estimator of hyper-parameters $\theta$ as $n \to \infty$ under certain regularity. The estimator $\beta$ and $\theta$ for bivariate Poisson regression with CGP priors are consistent under certain regularity, if $n = n_1 + n_2$, where $n_1 \to \infty$ and $n_2 \to \infty$.

**Remark 3** Some specific results of the regret term $R = \mathbb{E}_{X_{n_1n_2}}[\log |I + \delta K_{n_1n_2}|]$ as follows:

27
1. if $k(x_i, x_j) = x_i^T x_j$, i.e. a linear covariance kernel, and the covariate distribution $u(x)$ has bounded support, then

$$
\mathbb{E}_{x_{n_1n_2}}(\log |I + \delta K_{n_1n_2}|) = O(\log(n_1 + n_2));
$$

2. if $u(x)$ is normal and the covariance functions are the squared exponential form, then

$$
\mathbb{E}_{x_{n_1n_2}}(\log |I + \delta K_{n_1n_2}|) = O((\log(n_1 + n_2))^{p+1});
$$

3. if $u(x)$ is bounded support and the covariance functions are Matern, then

$$
\mathbb{E}_{x_{n_1n_2}}(\log |I + \delta K_{n_1n_2}|) = O((n_1 + n_2)^{p/(2v+p)}(\log(n_1 + n_2)^{2v/(2v+p)}));
$$

4. if covariance functions are mixed between squared exponential and Matern, then

$$
\mathbb{E}_{x_{n_1n_2}}(\log |I + \delta K_{n_1n_2}|) = O((n_1 + n_2)^{p/(2v+p)}(\log(n_1 + n_2)^{2v/(2v+p)})).
$$

Thus the information consistency in the proposed model is achieved for all of the above cases.