Exact ABJM Partition Function from TBA

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Abstract

We report on the exact computation of the $S^3$ partition function of $U(N)_k \times U(N)_{-k}$ ABJM theory for $k = 1, N = 1, \ldots, 19$. The result is a polynomial in $\pi^{-1}$ with rational coefficients. As an application of our results we numerically determine the coefficient of the membrane 1-instanton correction to the partition function.
1 Introduction and Summary

It has recently been discovered that the partition function of a Chern–Simons–matter (CSM) theory with \( \mathcal{N} \geq 2 \) supersymmetry on a three-dimensional sphere reduces to a matrix integral [1, 2, 3]. These matrix integrals are powerful quantitative tools to analyze CSM theories, and has lead to a number of important results, including the successful derivation of the \( N^{3/2} \) behavior [4] and various precise checks of the AdS\(_4\)/CFT\(_3\) correspondence (see [5, 6, 7] and subsequent works).

In this paper we study the CSM theory with the highest amount of supersymmetry (\( \mathcal{N} \geq 6 \)), namely the ABJM theory [8]. It has gauge group \( U(N)_k \times U(N)_{-k} \), where \( k \) is the level of the Chern-Simons term. Since ABJM theory is the worldvolume theory of multiple M2-branes, it is natural to ask if we could extract any useful data about M-theory from the three-sphere partition function of the ABJM theory.

In M-theory we have non-perturbative corrections from membrane instantons. This is reflected in the three-sphere partition function as an expansion of the terms of order \( e^{-\sqrt{N/k}} \) [9]. However, this expansion is not directly captured in most of the previous analysis of the three-sphere partition function, where we take the t’ Hooft limit \( N, k \) large with \( N/k \) kept finite. Instead we need to take the M-theory limit, with \( N \) large and \( k \) kept finite. The leading \( N \) contribution in this limit is determined by [7] and the all order \( 1/N \) expansion in [10]. Moreover the paper [10] discuss the non-perturbative instanton correction in an expansion around \( k = 0 \). However the for the most interesting case of \( k \) finite, the general results on the non-perturbative corrections are still lacking. To answer this question it will be of great help to systematically compute the behavior of the three-sphere partition function for finite \( N \) and \( k \).

In this brief note we report on the exact computation of \( S^3 \) partition function \( Z(N) \) of the \( k = 1 \) (\( \mathcal{N} = 8 \)) ABJM theory for \( N = 1, \ldots, 19 \), based on the Fermi gas approach of [10] and the TBA-like equations of [11, 12].

\(^{1}\)See [13] for exact computation for \( N = 2 \) and general \( k \), and [14] for numerical calculations.
Our results are given as follows:

\[ Z(1) = \frac{1}{4}, \quad Z(2) = \frac{1}{16\pi}, \quad Z(3) = \frac{-3 + \pi}{64\pi}, \quad Z(4) = \frac{-\pi^2 + 10}{1024\pi^2}, \quad Z(5) = \frac{26 + 20\pi - 9\pi^2}{4096\pi^2}, \]
\[ Z(6) = \frac{78 - 121\pi^2 + 36\pi^3}{147456\pi^3}, \quad Z(7) = \frac{-126 + 174\pi + 193\pi^2 - 75\pi^3}{196608\pi^3}, \]
\[ Z(8) = \frac{876 - 4148\pi^2 + 2016\pi^3 + 1053\pi^4}{18874368\pi^4}, \quad Z(9) = \frac{4140 + 8880\pi - 15348\pi^2 - 13480\pi^3 + 5517\pi^4}{7549742\pi^4}, \]
\[ Z(10) = \frac{16860 - 136700\pi^2 + 190800\pi^3 + 207413\pi^4 - 81000\pi^5}{7549742\pi^5}, \]
\[ Z(11) = \frac{-122580 + 381900\pi + 837300\pi^2 - 1289300\pi^3 - 1091439\pi^4 + 447525\pi^5}{3019988800\pi^5}, \]
\[ Z(12) = \frac{626760 - 8856300\pi^2 - 18446400\pi^3 + 35287138\pi^4 + 30204000\pi^5 - 12504375\pi^6}{4348654387200\pi^6}, \]
\[ Z(13) = \frac{1563480 + 6714000\pi - 17252100\pi^2 - 40746000\pi^3 + 49141894\pi^4 + 45780780\pi^5 - 18083925\pi^6}{5798205849600\pi^6}, \]
\[ Z(14) = (21382200 - 421152600\pi^2 + 191835000\pi^3 + 2614227910\pi^4 - 5654854800\pi^5 - 366519223\pi^6 + 1732468500\pi^7)/(340934503956480\pi^7), \]
\[ Z(15) = (-222059880 + 1271579400\pi + 3613033620\pi^2 - 12266517900\pi^3 - 17757814914\pi^4 + 28941378130\pi^5 + 21727092861\pi^6 - 9162734175\pi^7)/(13637380158259200\pi^7), \]
\[ Z(16) = (288454320 - 8196441240\pi^2 + 54540622080\pi^3 + 83379537976\pi^4 + 337956998400\pi^5 - 310977507352\pi^6 - 35445849984\pi^7 + 132764935275\pi^8)/(87279233012858800\pi^8), \]
\[ Z(17) = (3171011760 + 2355595200\pi - 71723746080\pi^2 - 33319960800\pi^3 + 54288555064\pi^4 + 135526162352\pi^5 - 1384280129304\pi^6 - 133978574000\pi^7 + 518021476875\pi^8)/(34911693205143550200\pi^8), \]
\[ Z(18) = (4970745360 - 18063186480\pi^2 + 2270513495520\pi^3 + 2448401550408\pi^4 - 1825113215200\pi^5 - 1359044330584\pi^6 + 35949047130993\pi^7 + \]
\[ + 20671882502409\pi^8 - 9607707219600\pi^9)/(377046286615550361600\pi^9), \]
\[ Z(19) = (-2636096400 + 24895105200\pi + 79219113120\pi^2 - 487774106400\pi^3 - 852843285000\pi^4 + 3053792290360\pi^5 + 36304396183136\pi^6 - 6122444513560\pi^7 - \]
\[ - 4288974330849\pi^8 + 184038432075\pi^9)/(55858709128229683200\pi^9). \]

(1)

Section 2 of this paper is devoted to the derivation of this result. Similar methods could be applied to \( k > 1 \). It would be interesting to find an analytic expression for general \( k \) and \( N \).

The knowledge of the exact values of \( Z(N) \) in this paper allows one to perform various numerical tests with high precision. As an example, we compute in Section 3 the coefficient of the membrane 1-instanton contribution to the partition function (see (30)).

**Note:** During the preparation of this manuscript we received a paper [16], which has substantial overlap with our paper. The paper contains the exact results up to \( N = 9 \), which is consistent with ours.
2 Derivation

Let us consider the grand canonical partition function

\[ \Xi(z) = 1 + \sum_{N \geq 1} Z(N) z^N. \]  

(2)

As is shown in [10], this is given by a Fredholm determinant

\[ \Xi(z) = \text{Det} \left( 1 + \frac{z\hat{K}}{4\pi} \right), \]  

(3)

with \( \hat{K} \) defined by an integral kernel

\[ K(x, y) := \langle x|\hat{K}|y \rangle = e^{-u(x) - u(y)} \cosh \left( \frac{z-y}{2} \right), \]  

(4)

and

\[ u(x) = \frac{1}{2} \log \left( 2 \cosh \frac{kx}{2} \right). \]  

(5)

In practice, it is useful to use the following relation:

\[ \Xi(z) = \exp \left( \text{Tr} \log \left( 1 + \frac{z\hat{K}}{4\pi} \right) \right) = \exp \left( -\sum_{\ell} Z_\ell \frac{(-z)^\ell}{\ell} \right), \]  

(6)

with

\[ Z_\ell = \frac{1}{(4\pi)^\ell} \text{Tr} \left( \hat{K}^\ell \right) = \frac{1}{(4\pi)^\ell} \int dx_1 \ldots dx_\ell K(x_1, x_2)K(x_2, x_3) \ldots K(x_{\ell-1}, x_\ell)K(x_\ell, x_1). \]  

(7)

The problem thus reduces to the computation of \( Z_\ell \).

Let us define the kernel for the operator \( \hat{K}(I - \lambda^2\hat{K}^2)^{-1} \) by \( R_+(x, y) \) and for \( \lambda^2 \hat{K}^2(I - \lambda^2\hat{K}^2)^{-1} \) by \( R_-(x, y) \), respectively. We also denote \( R_+(x) := R_+(x, x) \), \( R_-(x) := R_-(x, x) \). As is clear from the definition, the integral of \( R_\pm(x) \) gives \( Z_\ell \):

\[ \frac{1}{4\pi} \int dx R_+(x) = \sum_{n \geq 0} (4\pi \lambda)^{2n} Z_{2n+1}, \quad \frac{1}{4\pi} \int dx R_-(x) = \sum_{n \geq 0} (4\pi \lambda)^{2n+1} Z_{2n+2}. \]  

(8)

Let us further define \( \epsilon(\theta), \eta(\theta) \) by

\[ e^{-\epsilon(\theta)} = R_+(\theta), \quad \eta(\theta) = 2\lambda \int_{-\infty}^{\infty} \frac{e^{-\epsilon(\theta')}}{\cosh(\theta - \theta')} d\theta'. \]  

(9)

It was conjectured in [11] and later proven in [12] that these functions satisfy the following two TBA-like equations:

\[ \epsilon(\theta) = 2u(\theta) - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\log \left( 1 + \eta^2(\theta') \right)}{\cosh(\theta - \theta')} d\theta', \]  

(10)

\[ R_-(\theta) = \frac{1}{\pi} R_+(\theta) \int_{-\infty}^{\infty} \frac{\arctan(\eta(\theta'))}{\cosh^2(\theta - \theta')} d\theta'. \]  

(11)
Let us define
\[ \epsilon(\theta) = \sum_{n \geq 0} \epsilon_n(\theta) \lambda^{2n}, \quad \eta(\theta) = \sum_{n \geq 0} \eta_n(\theta) \lambda^{2n+1}, \]
\[ R_+(\theta) = \sum_{n \geq 0} R_{+,n}(\theta) \lambda^{2n}, \quad R-(\theta) = \sum_{n \geq 0} R_{-,n}(\theta) \lambda^{2n+1}. \]
(12) \hspace{1cm} (13)

Suppose \( \epsilon_n(\theta), n = 0 \ldots j \) are known. We can then find \( \eta_n(\theta), n = 0 \ldots j \) by performing the integration in (9), and then \( \epsilon_{j+1}(\theta) \) from (10). Thus one can solve the TBA-like equations recursively, order by order in \( \lambda \), starting from
\[ \epsilon_0(\theta) = 2u(\theta). \]
(14)

Once we know \( \epsilon_n(\theta) \) and \( \eta_n(\theta) \) for \( n = 0 \ldots N \) we can find \( R_{+,n}(\theta) \) and \( R_{-,n}(\theta) \) for \( n = 0 \ldots N \) and, therefore, \( Z_{2n+1} \) and \( Z_{2n+2} \) for \( n = 0 \ldots N \) from (8).

Practically, it is useful to make the following change of variables: \( e^{\theta/2} = t \). Then the equations (9-10) read
\[ \eta(t) = 8\lambda \int_0^\infty ds \frac{t^2 s e^{-\epsilon(s)}}{s^4 + t^4}, \]
\[ \epsilon(t) = \log \left( t^k + \frac{1}{t^k} \right) - \frac{2}{\pi} \int_0^\infty ds \frac{t^2 s \log(1 + \eta^2(s))}{s^4 + t^4}. \]
(15) \hspace{1cm} (16)

Let us specialize to \( k = 1 \) for simplicity. One can show that the functions \( \epsilon_n(t), \eta_n(t) \) have rather simple structure:
\[ \epsilon_n(t) = \sum_{j=0}^{n-1} G_{j}^{(n)}(t)(\log t)^j, \quad n \geq 2, \]
\[ \eta_n(t) = \sum_{j=0}^{n} H_{j}^{(n)}(t)(\log t)^j, \quad n \geq 0, \]
(17) \hspace{1cm} (18)

where \( G_{j}^{(n)}(t) \) are rational functions with poles allowed at the roots of \( t^4 - 1 \) and \( H_{j}^{(n)}(t) \) are rational functions with poles allowed at the roots of \( t^4 + 1 \). This observation allows easy calculation of the integrals (15-16) order by order in \( \lambda \) with the help of the residue theorem and the following formula:
\[ \int_0^\infty dt \ C(t) \ (\log t)^j = - \frac{(2\pi i)^j}{j+1} \int_{\gamma} dt \ C(t) \ B_{j+1} \left( \frac{\log t}{2\pi i} \right), \]
(19)

where in the right hand side \( \log t \) has a brunch cut from 0 to \( +\infty \), the contour \( \gamma \) goes from \( +\infty \) to 0 below the cut and then to \( +\infty \) above the cut, \( C(t) \) is a rational function and \( B_{j+1}(x) \) is Bernoulli polynomial.
Using the described procedure we find
\[
Z_1 = \frac{1}{4},
\]
\[
Z_2 = \frac{-2 + \pi}{16\pi},
\]
\[
Z_3 = \frac{\pi - 3}{16\pi},
\]
\[
Z_4 = \frac{-4 - 8\pi + 3\pi^2}{128\pi^2},
\]
\[
Z_5 = \frac{10 - \pi^2}{256\pi^2},
\]
\[
Z_6 = \frac{36 - 2\pi - 3\pi^2}{1536\pi^2},
\]
\[
Z_7 = \frac{-42 + 126\pi + 49\pi^2 - 27\pi^3}{9216\pi^3},
\]
\[
Z_8 = \frac{-96 + 96\pi + 64\pi^2 - 27\pi^3}{18432\pi^3},
\]
\[
Z_9 = \frac{12 - 96\pi - 20\pi^2 + 5\pi^4}{32768\pi^4},
\]
\[
Z_{10} = \frac{1200 - 2400\pi - 1400\pi^2 + 226\pi^3 + 135\pi^4}{1474560\pi^4},
\]
\[
Z_{11} = \frac{-660 + 23100\pi - 12100\pi^2 - 25300\pi^3 - 6303\pi^4 + 4725\pi^5}{29491200\pi^5},
\]
\[
Z_{12} = \frac{-720 + 3600\pi + 1200\pi^2 - 2560\pi^3 - 1536\pi^4 + 675\pi^5}{7372800\pi^5},
\]
\[
Z_{13} = \frac{4680 - 561600\pi + 978900\pi^2 + 655200\pi^3 + 10114\pi^4 - 30375\pi^6}{4246732800\pi^6},
\]
\[
Z_{14} = \frac{141120 - 1693440\pi + 764400\pi^2 + 1764000\pi^3 + 625436\pi^4 - 162882\pi^5 - 70875\pi^6}{14863564800\pi^6},
\]
\[
Z_{15} = \frac{-2520 + 1076040\pi - 4024860\pi^2 - 1425900\pi^3 + 2429714\pi^4 + 2860522\pi^5 + 527265\pi^6 - 509355\pi^7}{55490641920\pi^7},
\]
\[
Z_{16} = \frac{-40320 + 1128960\pi - 1599360\pi^2 - 1646400\pi^3 - 238336\pi^4 + 1136128\pi^5 + 663552\pi^6 - 297675\pi^7}{52022476800\pi^7},
\]
\[
Z_{17} = \frac{(85680 - 124750080\pi + 931227360\pi^2 - 303878400\pi^3 - 1054571336\pi^4 - 405544384\pi^5 + 45621608\pi^6 + 19348875\pi^7) / \left(53271016243200\pi^8\right)}{1},
\]
\[
Z_{18} = \frac{(80640 - 5160960\pi + 15617280\pi^2 + 6397440\pi^3 - 10554208\pi^4 - 11079488\pi^5 - 2895216\pi^6 + 1060922\pi^7 + 385875\pi^8) / \left(1479750451200\pi^8\right)}{1},
\]
\[
Z_{19} = \frac{(-287280 + 1414279440\pi - 20169928800\pi^2 + 24409032480\pi^3 + 31396649256\pi^4 + 1177819272\pi^5 - 19209555560\pi^6 - 17783325576\pi^7 - 2533741371\pi^8 + 3094331625\pi^9) / \left(5753269754265600\pi^9\right)}{1}.
\]

From (2), (6) we obtain our main results (1).

3 Numerical Applications

It is easy to check numerically (cf. [14, 16]) that the exact results obtained in the previous section are in agreement with the Airy function asymptotics. According to [15, 10, 14] the perturbative part of the partition function for \( k = 1 \) is given by

\[ Z = \frac{1}{4} \]

This result agrees perfectly with the numerical result for \( N \leq 16 \) in [16].
Figure 1: In this figure, the dots represent the sequence $Z/N - 1$.

\begin{align*}
Z(N) &= C e^A \text{Ai} \left( C \left( N - \frac{1}{3} - \frac{1}{24} \right) \right), \\
A &= -\frac{\zeta(3)}{8\pi^2} + \frac{1}{6} \log \frac{\pi}{2} + 2\zeta'(-1) + \frac{1}{2} \log 2 - \frac{1}{3} \int_0^\infty dx \frac{1}{e^x - 1} \left( \frac{3}{x^3} - \frac{1}{x} - \frac{3}{x \left( \sinh x \right)^2} \right).
\end{align*}

Figure 2: In these figures, the dots represent the sequence $Z^{\text{pert}}(N)$ (left) and its 9-th Richardson-like transform $\tilde{F}^{(9)}(N)$ (right).

\begin{align*}
Z^{\text{pert}}(N) &= C e^A \text{Ai} \left( C \left( N - \frac{1}{3} - \frac{1}{24} \right) \right), \\
A &= -\frac{\zeta(3)}{8\pi^2} + \frac{1}{6} \log \frac{\pi}{2} + 2\zeta'(-1) + \frac{1}{2} \log 2 - \frac{1}{3} \int_0^\infty dx \frac{1}{e^x - 1} \left( \frac{3}{x^3} - \frac{1}{x} - \frac{3}{x \left( \sinh x \right)^2} \right).
\end{align*}

The Fig. 1 shows that indeed $Z(N)$ approaches $Z^{\text{pert}}(N)$ exponentially fast. In [16] it was checked that the non-perturbative part $Z^{\text{np}} = Z - Z^{\text{pert}}$ is suppressed by $e^{-2\pi\sqrt{2N}}$ which agrees with the previous analytical results [6, 10, 9]. One can go further and find the leading behavior of the prefactor. Namely, let us consider the following sequence:

\begin{align*}
\tilde{F}(N) &= Z^{\text{np}}(N) / N e^{-2\pi\sqrt{2N}} = \left( \frac{Z(N)}{Z^{\text{pert}}(N)} - 1 \right) / N e^{-2\pi\sqrt{2N}}. \\
\end{align*}

From the previous works one expects that this sequence has an asymptotic expansion of the following form:

\begin{align*}
\tilde{F}(N) = c_0 + \frac{c_1}{N^{1/2}} + \frac{c_2}{N} + \frac{c_3}{N^{3/2}} + \ldots.
\end{align*}
This assumption will be verified numerically \textit{a posteriori}. The graph of $\tilde{F}(N)$ is shown on the left of Fig. 2. One can accelerate convergence of the sequence $\tilde{F}(N)$ by performing Richardson-like transforms. Let us define the Richardson-like transform $R_\gamma$ of a sequence $S(N)$ as
\begin{equation}
R_\gamma[S(N)] \equiv (N/\gamma + 1) S(N + 1) - NS(N)/\gamma .
\end{equation}
Its crucial property is that
\begin{equation}
R_\gamma[c + O\left(\frac{1}{N}\right)] = c + o\left(\frac{1}{N}\right).
\end{equation}
In particular, if we define
\begin{equation}
\tilde{F}^{(n)}(N) \equiv R_\frac{1}{2}\left[\tilde{F}^{(n-1)}(N)\right] \quad (n \geq 1), \quad \tilde{F}^{(0)}(N) = \tilde{F}(N),
\end{equation}
then one can show that
\begin{equation}
\tilde{F}^{(n)}(N) = c_0 + O\left(\frac{1}{N^{2n+1}}\right).
\end{equation}
The graph of $\tilde{F}^{(9)}(N)$ is shown on the right of Fig. 2. The sequence converges very fast, which verifies the self-consistency of the assumption (25). Our numerical result suggests that $c_0 = 2$ exactly. One can also numerically obtain $c_1, c_2, \ldots$ using similar techniques. On the M-theory side of the AdS/CFT correspondence this gives the 1-instanton contribution from M2-branes:
\begin{equation}
\frac{Z^{(1\text{-inst})}}{Z^{(\text{pert})}} = \left(2N + O(\sqrt{N})\right) e^{-2\pi \sqrt{2N}}.
\end{equation}
It would be interesting to check this by a direct calculation of 1-instanton contribution in M-theory. Let us note that the prefactor in (30) cannot be obtained by previously developed techniques since they provide the non-perturbative part of the partition function as non-trivial asymptotic expansions either for large $k$ [6] or for small $k$ [10], whereas the result (30) is for $k = 1$.

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\textsuperscript{3}The smallest instanton action is twice the action of the D2-brane considered in [9] for $k = 1$ because there is no M2 in $S^7$ which is a degree one cover of the D2-brane wrapping $\mathbb{R}P^3 \subseteq \mathbb{C}P^3$. However, there is an M2 wrapping $S^3 \subseteq S^7$ which is a degree two cover.
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