Classical solutions from quantum regime for barotropic FRW model

J. Socorro

Instituto de Física de la Universidad de Guanajuato,
Apartado Postal E-143, C.P. 37150, León, Guanajuato, México

The quantization of gravity coupled to barotropic perfect fluid as matter field and cosmological constant is made and the wave function can be determined for any $\kappa$ in the FRW minisuperspace model. The meaning of the existence of the classical solution is discussed in the WKB semiclassical approximation.

_Keywords:_ Classical and quantum exact solutions.

PACS numbers: 04.20.Jb; 04.40.Nr; 98.80.Hw

**I. INTRODUCTION**

It is a general belief that the existence of an initial singularity can be remove through the employment of a quantum theory of gravity. However, there is no consistent theory of gravity until now, and in this sense the problem of the initial singularity remains of actuality.

It is well known that is possible to construct a quantum model for the universe as a whole, through the Wheeler-DeWitt (WDW) equation, based in the ADM decomposition of the gravity sector, which leads to a hamiltonian approach of general relativity, from which a canonical quantization procedure can be applied (Halliwell, 1991). Moreover, in the hamiltonian formalism, the notion of time is lost (Isham, 1992), but there are some proposals by which this notion of time can be recovered (Schutz, 1970, 1971) coupling of the gravity sector to a perfect fluid. In this scheme, called Schutz’s formalism, a quantization procedure is possible, and the canonical momentum associated with the perfect fluid appear linearly in the WDW equation, permitting to rewrite this equation in the form a Schrödinger equation with a time coordinate associated with the matter field.

Recently, a convincing quantum gravity origin for inflation should however be proved. According to Bojowald (Bojowald, 2002) a non-perturbative approach would produce the most reliable answer as to whether or not inflation can be derived from quantum gravity. At the less ambitious level of minisuperspace quantum cosmology the generality of inflation as well as the quantum creation of the universe should be treated by analyzing the Wheeler-DeWitt equation and its physical solutions, the so-called wave functions of the universe (Gibbons and Grishchuk, 1989).

In this way, the behaviour of the scale factor may be determined in two different forms from the quantum regime: when the wave function is time dependent, we can calculate the expectation value of the scale factor, in the spirit of the many worlds interpretation of quantum mechanics (Tipler, 1986)

$$< A >_t = \frac{\int_0^\infty W(A) \Psi^*(A, t) A(t) \Psi(A, t) dA}{\int_0^\infty W(A) \Psi^*(A, t) \Psi(A, t) dA}.$$  (1)
where \( W(A) \) is a weight function that normalizes the expectation value. The other way is to apply the WKB semiclassical approximation, in which case the system follows a real trajectory given by the equation

\[
\Pi_q = \frac{\partial \Phi}{\partial q},
\]

where the index \( q \) designates one of the degree of freedom of the system, and \( \Phi \) is the phase of the wave function when is written as

\[
\Psi = W e^{i\Phi},
\]

the functions \( W \) and \( \Phi \) are real functions. We follow this last procedure to obtain the classical behaviour for the scale factor, using the FRW model with a barotropic perfect fluid and cosmological constant.

The work is organized as follows. In next section we describe the quantum model with the solution for any \( \kappa \) case in the minisuperspace, considering a barotropic perfect fluid as matter field including the cosmological constant. In section III, we present the classical evolution for the scale factor derived following the semiclassical WKB procedure. Section IV is devoted to conclusions.

II. QUANTUM MODEL

The total Lagrangian for a barotropic perfect fluid coupled to gravity using the FRW geometry with the classical cosmological constant term is given by

\[
\mathcal{L}_{total} = \mathcal{L}_{geom} + \mathcal{L}_{matter},
\]

namely

\[
\mathcal{L}_{geom} = \sqrt{-\text{det}(g)} \left[ R - 2\Lambda \right] = -\frac{6A^2}{N} \frac{dA}{dt}^2 - \frac{6A}{N^2} \left( \frac{dA}{dt} \right)^2 + \frac{6A^2}{N} \frac{dA}{dt} \frac{dN}{dt} - 6\kappa NA - 2N\Lambda A^3
\]

\[
= \frac{d}{dt} \left( -\frac{6A^2}{N} \frac{dA}{dt} \right) + 6A \left( \frac{dA}{dt} \right)^2 - 6\kappa NA - 2N\Lambda A^3,
\]

and the Lagrangian matter density (Ryan, 1972; Pazos, 2000)

\[
\mathcal{L}_{matter} = \sqrt{-\text{det}(g)} \left[ 16\pi GN\rho \left\{ (\gamma + 1) \left( 1 + g^{km} U_k U_m \right)^{\frac{1}{2}} - \gamma \left( 1 + g^{km} U_k U_m \right)^{-\frac{1}{2}} \right\} 
- 16\pi G\rho (\gamma + 1) U_m N^m \right].
\]

These results are obtained by employing the ADM form of the FRW metric

\[
ds^2 = -N^2 dt^2 + A^2 \left[ \frac{dr^2}{1 - \kappa r^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right],
\]

where \( N \) is the lapse function, \( A \) is the scale factor of the model, and \( \kappa \) is the curvature index of the universe ( \( \kappa = 0, +1, -1 \) plane, close and open, respectively).
We also make use of the usual perfect fluid energy-momentum tensor
\[ T_{\mu\nu} = pg_{\mu\nu} + (p + \rho) U_\mu U_\nu, \tag{8} \]
where \( p, \rho, U_\mu \) are the pressure, energy density and the four-velocity of the cosmological fluid, respectively; and using the barotropic relationship \( p = \gamma \rho, \gamma = \text{constant} \), we have the solution for the energy density as a function of the scale factor of the FRW universe, in the usual way
\[ \rho = \frac{M_\gamma}{A^{3(\gamma+1)}}, \tag{9} \]
where \( M_\gamma \) is an integration constant. In \( \mathcal{L}_{\text{matter}} \) we also choose the comoving fluid (three-velocity \( U_k = 0 \)), and the gauge \( N^k = 0 \), obtaining
\[ \mathcal{L}_{\text{matter}} = 16\pi G M_\gamma A^{-3\gamma}. \tag{10} \]
Thus, the total lagrangian density has the following form
\[ \mathcal{L}_{\text{tot}} = \frac{d}{dt} \left( -6A^2 \dot{\Lambda} \right) + \frac{6A}{N} \left( \frac{dA}{dt} \right)^2 - 6\kappa N A - 2N\Lambda A^3 + 16\pi G M_\gamma A^{-3\gamma}. \tag{11} \]
Following the well-known procedure to get the canonical Hamiltonian function, we define the canonical momentum conjugate to the generalized coordinate \( A \) (scale factor) as
\[ \Pi_A \equiv \frac{\partial L}{\partial \dot{A}}, \tag{12} \]
\[ L = \Pi_A \dot{A} - N\mathcal{H} = \Pi_A \dot{A} - N \left[ \frac{\Pi_A^2}{24A} + 6\kappa A + 2\Lambda A^3 - 16\pi G M_\gamma A^{-3\gamma} \right], \tag{13} \]
where
\[ \mathcal{H} = \frac{\Pi_A^2}{24A} + 6\kappa A + 2\Lambda A^3 - 16\pi G M_\gamma A^{-3\gamma}. \tag{14} \]
Performing the variation of (13) with respect to \( N, \frac{d\Pi_A}{dN} = 0 \), implies the well-known result \( \mathcal{H} = 0 \). Imposing the quantization condition and applying this hamiltonian to the wave function \( \Psi \), we obtain the WDW equation in the minisuperspace
\[ \hat{\mathcal{H}}|\Psi\rangle = \frac{1}{24A} \left[ -\frac{d^2}{dA^2} + 144\kappa A^2 + 48\Lambda A^4 - 384\pi G M_\gamma A^{-3\gamma+1} \right]|\Psi\rangle = 0. \tag{15} \]
Notice that in principle the order ambiguity in equation (15) should be taken into account. This is quite a difficult problem to be treated in all its generality, since the hamiltonian operator in (15) must be written in a very general form in order to take into account all possible order, but, at least in the minimal case in which (Hartle and Hawking, 1983) (there are other possibilities depending on different considerations on the operators, see the works
of Christodoulakis and Zanelli, (Christodoulakis and Zanelli, 1984) or Lidsey and Moniz (Lidsey and Moniz, 2000)

\[ A^{-1} \frac{d^2 \Psi}{dA^2} \to A^{-1+p} \frac{d}{dA} A^{-p} \frac{d\Psi}{dA} = A^{-1} \left( \frac{d^2 \Psi}{dA^2} - p A^{-1} \frac{d\Psi}{dA} \right), \]  

(16)

where the real parameter \( p \) measures the ambiguity in the factor ordering. Therefore, the Wheeler-DeWitt equation can be written as follows

\[ -A \frac{d^2 \Psi}{dA^2} + p \frac{d\Psi}{dA} - V(A) \Psi = 0, \]  

(17)

where

\[ V(A) = -48 \Lambda A^5 + 384 \pi G M \gamma A^{-3\gamma+2} - 144 \kappa A^3. \]  

(18)

In the following we discuss some of the quantum solutions of equation (17) for particular values of the \( \gamma \) parameter and the parameter \( p \) of factor ordering.

A. Inflationary scenario

In the inflationary era, including the cosmological constant \( \Lambda \), we choose \( \gamma = -1 \). Notice that the WDW equation (17) can be written as

\[ A \frac{d^2 \Psi}{dA^2} - p \frac{d\Psi}{dA} + 144 A^3 \left( m^2 A^2 - \kappa \right) \Psi = 0, \]  

(19)

where

\[ m^2 = -\frac{\Lambda}{3} + \frac{8}{3} \pi G M. \]  

(20)

The latter relationship leads to three possible cases

1. \( m^2 > 0 \)

The differential equation for this subcase is

\[ A \Psi''_1 - Q \Psi'_1 + 144 \kappa A^3 \left( m^2 A^2 - \kappa \right) \Psi_1 = 0 \]  

(21)

which, after the substitutions \( v = m^2 A^2 - \kappa, \Psi_1 = v^{1/2} y(v) \), and \( z = \frac{4}{m^2} v^{3/2} \), turns for \( p = 1 \) into a Bessel equation with the general solution

\[ \Psi_1 = \left( m^2 A^2 - \kappa \right)^{1/2} \left[ a_0 J_+ (z) + b_0 J_+ (z) \right], \]  

where \( a_0 \) and \( b_0 \) are superposition constants.

\[ ^1 \text{In general } m^2 \text{ is not a positive constant, see the corresponding definition (20).} \]
2. $m^2 < 0$,

The differential equation for this subcase is

$$-A\Psi''_{-1} + p\Psi'_{-1} + 144\kappa A^3 \left(|m^2|A^2 + \kappa\right) \Psi_{-1} = 0. \quad (23)$$

For $p = 1$ we have a modified Bessel equation with the general solution

$$\Psi_{-1} = \left(|m^2|A^2 + \kappa\right)^{\frac{1}{4}} \left[a_1 I_{\frac{\nu}{4}}(z) + b_1 K_{\frac{\nu}{4}}(z)\right], \quad \text{where} \quad z = \frac{4}{|m^2|} \left(|m^2|A^2 + \kappa\right)^{3/2}, \quad (24)$$

where $a_1$ and $b_1$ are superposition constants.

3. For $m^2 = 0$, the differential equation for this situation is

$$-A\Psi''_{-1} + p\Psi'_{-1} + 144\kappa A^3 \Psi_{-1} = 0 \quad (25)$$

which for $\kappa = 1$, has as solution the modified Bessel functions of order $\nu = \frac{1 + p}{4}$

$$\Psi_{-1} = A^{2\nu} \left[A_0 I_{\nu}(6A^2) + B_0 K_{\nu}(6A^2)\right], \quad (26)$$

where $A_0$ and $B_0$ are superposition constants, while for $\kappa = -1$, the solution become to be the ordinary Bessel functions

$$\Psi_{-1} = A^{2\nu} \left[A_1 J_{\nu}(6A^2) + B_1 Y_{\nu}(6A^2)\right], \quad (27)$$

where $A_1$ and $B_1$ are superposition constants.

B. Inflationary like scenario, $\gamma = -\frac{1}{3}$

In this particular case, the WDW equation becomes

$$A \frac{d^2 \Psi}{dA^2} - p \frac{d\Psi}{dA} - 48A^3 \left(AA^2 - g\right) \Psi = 0, \quad \text{with} \quad g = 8\pi GM_{-1/3} - 3\kappa \quad (28)$$

which, after the substitutions $v = AA^2 - g, \Psi = v^{1/2}y(v)$, and $z = \frac{4\sqrt{3}}{3A}v^{3/2}$, yields for $p = 1$ a Bessel equation with the general solution

$$\Psi(A) = \left[AA^2 - g\right]^{1/2} \left\{C_0 I_{\frac{\nu}{4}} \left(\frac{4\sqrt{3}}{3A} \left[AA^2 - g\right]^{3/2}\right) + C_1 K_{\frac{\nu}{4}} \left(\frac{4\sqrt{3}}{3A} \left[AA^2 - g\right]^{3/2}\right)\right\}. \quad (29)$$

where $C_0$ and $C_1$ are superposition constants.
C. Dust case \((\gamma = 0, \kappa = 0)\)

For this case the WDW equation is

$$\Lambda \frac{d^2 \Psi}{dA^2} - p \frac{d\Psi}{dA} + 48A^2 \left(-\Lambda A^3 + 8\pi GM_0\right) \Psi = 0, \quad (30)$$

Making the transformations \(z = \frac{8}{3}\sqrt{3\Lambda} A^3\) and \(\Psi = e^{-\frac{1}{2}z} u(z)\), one gets for \(u(z)\) the confluent hypergeometric equation

$$z \frac{d^2 u}{dz^2} + (\gamma - z) \frac{du}{dz} - m u = 0 \quad (31)$$

where \(m = \frac{2-p}{6} - \frac{16\pi GM_0}{\sqrt{3\Lambda}}\) and \(\gamma = \frac{2-Q}{3}\). The linear independent solutions are (Gradshteyn and Ryzhik, 1980)

\[
\begin{align*}
  u_1(z) &= {}_1F_1(m, \gamma; z) \quad (32) \\
  u_2(z) &= z^{1-\gamma} {}_1F_1(m - \gamma + 1, 2 - \gamma; z) \quad (33)
\end{align*}
\]

where \(_1F_1\) is the degenerate hypergeometric function.

Thus, the exact solution for \(\Psi\) becomes

$$\Psi(A) = e^{-\frac{1}{2}z} \left[A_0 u_1(z) + B_0 u_2(z)\right], \quad (34)$$

where \(A_0\) and \(B_0\) are superposition constants, \(u_1\) and \(u_2\) are the functions given in (32) and (33), respectively.

D. Stiff fluid case \((\gamma = 1, \kappa = 0)\)

The WDW equation

$$\Lambda \frac{d^2 \Psi}{dA^2} - p \frac{d\Psi}{dA} + 48 \left(-\Lambda A^5 + 8\pi GM A^{-1}\right) \Psi = 0, \quad (35)$$

has the following exact solution

$$\Psi(A) = \Lambda^{\frac{1+p}{6}} Z_\nu \left(\frac{4\sqrt{-3\Lambda}}{3} A^3\right), \quad \text{with} \quad \nu = \frac{1}{3} \sqrt{\left(\frac{1+p}{2}\right)^2 - 384\pi GM_1}. \quad (36)$$

The exact expression of the wave function depends on the sign of the cosmological constant and the order \(\nu\):

- \(\Lambda > 0\) and \(\nu\) real, the functions \(Z_\nu\) become the modified Bessel functions, either \(I_\nu\) or \(K_\nu\), depending on the boundary conditions.
- \(\Lambda < 0\) and \(\nu\) real, the functions \(Z_\nu\) turn into the ordinary Bessel functions, either \(J_\nu\) or \(Y_\nu\), depending on the boundary conditions.
- \(\Lambda > 0\) and \(\nu\) pure imaginary, the functions \(Z_\nu\) become the modified Bessel functions of pure imaginary order (Dunster, 1990), either \(I_\nu\) or \(K_\nu\), depending on the boundary conditions.
- \(\Lambda < 0\) and \(\nu\) pure imaginary, the functions \(Z_\nu\) turn into the ordinary Bessel functions of pure imaginary order (Dunster, 1990), either \(J_\nu\) or \(Y_\nu\), depending on the boundary conditions.
III. THE CLASSICAL BEHAVIOUR FROM WKB REGIME

Interesting results can be obtained at the level of WKB method if one performs the transformation $\Pi_A \to \frac{d\Phi}{dA}$. Then, (14) becomes the Einstein-Hamilton-Jacobi equation, where $\Phi$ is the superpotential function that is related to the physical potential under consideration.

Introducing this ansatz in (14) we get

$$H = \frac{1}{24A} \left[ \left( \frac{d\Phi}{dA} \right)^2 + 144\kappa A^2 + 48\Lambda A^4 - 384\pi GM\gamma A^{-3\gamma+1} \right] = 0,$$

thus, we obtain

$$\frac{d\Phi}{dA} = \sqrt{-48\Lambda A^4 + 384\pi GM\gamma A^{-3\gamma+1} - 144\kappa A^2}$$

the superpotential $\Phi$ has the following form

$$\Phi = \pm \int \sqrt{-48\Lambda A^4 + 384\pi GM\gamma A^{-3\gamma+1} - 144\kappa A^2} dA,$$

This integral can be solved for particular cases of the $\gamma$ parameter.

Now, employing the equations (2,12) and (38) we will obtain the evolution for the scale factor. The classical equation of motion is

$$\frac{12A dA}{N dt} = \Pi_A = 12A \sqrt{-\frac{1}{3}\Lambda A^2 + \frac{8}{3}\pi GM\gamma A^{-3\gamma+1} - \kappa},$$

in term of the “cosmic time” $\tau$ defined by $d\tau = N(t)dt$, the equation (40) is read how (when we choose the gauge $N(t)=1$, this cosmic time is the physical time $t$).

$$d\tau = \frac{dA}{\sqrt{-\frac{1}{3}\Lambda A^2 + \frac{8}{3}\pi GM\gamma A^{-3\gamma+1} - \kappa}}.$$

A. Inflationary scenario, $\gamma = -1$

The equation (41) for the inflation regime is written as

$$\tau - \tau_0 = \int_0^A \frac{dx}{\sqrt{(-\frac{1}{3}\Lambda + \frac{8}{3}\pi GM\gamma)x^2 - \kappa}} = \frac{1}{\sqrt{a_1}} \ln \left[ A + \sqrt{A^2 - a_2^2} \right]$$

where $a_1 = -\frac{1}{3}\Lambda + \frac{8}{3}\pi GM\gamma$ and $a_2^2 = \frac{\kappa}{a_1}$.

The inverse of the equation (42) us give the following structure for the scale factor $A(\tau)$

$$A(\tau) = \frac{1}{2} \left[ e^{\sqrt{a_1}(\tau-\tau_0)} + a_2 e^{-\sqrt{a_1}(\tau-\tau_0)} \right].$$

The scale factor $A(\tau)$ will have an inflationary behaviour only if the parameter $a_1 \gg 1$, then the cosmological constant have the restriction value $\Lambda < 3 \left( \frac{8}{3}\pi GM_1 - 1 \right)$. If the parameter $0 < a_1 < 1$, the scale factor vanishes very fast.

For other scenarios and particular values in $\kappa$ and $\Lambda = 0$, we have the following:
1. For the dust case and \( \kappa = 0 \), we obtain the well-known solution \( A \propto t^{2/3} \) for \( N=1 \).

2. For \( \gamma = 1 \) and \( \kappa = 0 \), the scale factor have the traditional solution \( A \propto t^{1/3} \) for \( N=1 \).

3. For radiation case and \( \kappa = -1 \), the behaviour is

\[
A(\tau) = \sqrt{4 (\tau - \tau_0)^2 - \frac{8}{3} \pi GM_4^\frac{1}{3} \pi G M_4^\frac{1}{3}}. \tag{44}
\]

For the case of \( \kappa = 0 \), \( A \propto t^{1/2} \) for \( N=1 \).

On the other hand, the master equation (41) can be solved for any \( \gamma \) if the cosmological constant vanish, defining \( d\tau = A^{3\gamma+2}dT \), having the following general solution in the time \( T \)

\[
A(T) = \left( \frac{3}{8 \pi GM_\gamma} \right)^\frac{1}{3\gamma+1} \left\{ \frac{4}{3} \pi GM_\gamma (3\gamma + 1) (T - T_0) \right\}^2 + \kappa \right\}^\frac{1}{3\gamma+1}. \tag{45}
\]

The classical solutions obtained by this method perhaps coincide with these obtained solving the Einstein field equation, but is not possible to know if the quantum universe remains quantum forever for some case, due to the fact that the WDW equation is not time dependent.

**IV. CONCLUSIONS**

We discussed from the point of view of simple models of minisuperspace quantum cosmology. We found that the wave functions of the WDW equations in question are mostly Bessel functions. For particular values of the parameter \( p \) that measures the ambiguity of the factor ordering problem we obtained analytical results for some scenarios of the universe possessing a cosmological constant. Only in the case of stiff fluid, the cosmological constant can be positive or negative. In the other cases, it can be only positive for physical reasons related to the scale factor or the potential under consideration. It is well known that the Wheeler-DeWitt cosmological equation is not an evolution equation and therefore the associated quantum states do not evolve in time. A possible way out of this difficulty could be to connect some parameters of the ‘quantum’ WDW solutions with classical Einstein ones by phenomenological restrictions imposed on the superpotential function as we did in this work. Using this method, we find the classical behaviour for the scale factor, that are analogous to those found solving the Einstein equations.

**Acknowledgments**

We thank H.C. Rosu for critical reading of the manuscript.

**References**

Bojowald, M. (2002). *Physical Review Letters* **89**, 261301, gr-qc/0206054.

Christodoulakis, T. and Zanelli, J. (1984), *Physics Letters A* **102**, 227; (1984), *Physical Review D* **29**, 2738.

Dunster, T. M. Siam Journal Mathematical. Anal. **21**(4), 995 (1990).
Gibbons, G. W. and Grishchuk, L. P. (1989). *Nuclear Physics B* **313**, 736.
Gradsteyn, I. S. and Ryzhik, I. M. (1980), *Table of Integrals, Series, and Products*, (Academic Press), page 1059
Halliwell, J. A. (1991). in S. Coleman, J.B. Coleman, T. Piran, S. Weinberg (Eds), *Quantum cosmology and baby universes*, (World Scientific, Singapore).
Hartle, J. and Hawking, S. W. (1983). *Physical Review D* **28**, 2960.
Isham, C. J. (1992). gr-qc/9210011
Lidsey, J. E. and Moniz, P. V. (2000) *Classical and Quantum Gravity* **17**, 4823.
Pazos, E. (2000) *Aplicación del formalismo lagrangiano ADM a un modelo cosmológico*, Univ. de San Carlos de Guatemala, BS thesis in Spanish.
Ryan Jr., M. (1972). *Hamiltonian Cosmology*, (Springer Verlag).
Schutz, B. F. (1970). *Physical Review D* **2**, 2762; (1971). *Physical Review D* **4**, 3559.
Tipler, F. J. (1986). *Physics Report* **137**, 231.