Exact Testing of Many Moment Inequalities Against Multiple Violations

Nick Koning\textsuperscript{a} and Paul Bekker\textsuperscript{b}
Faculty of Economics and Business
University of Groningen
April 29, 2019

Abstract

This paper considers the problem of testing many moment inequalities, where the number of moment inequalities ($p$) is possibly larger than the sample size ($n$). Chernozhukov et al. (2018) proposed asymptotic tests for this problem using the maximum $t$ statistic. We observe that such tests can have low power if multiple inequalities are violated. As an alternative, we propose a novel randomization test based on a maximum non-negatively weighted combination of $t$ statistics. Simulations show that the test controls size in small samples ($n = 30, p = 1000$), and often has substantially higher power against alternatives with multiple violations.

Key words and phrases: Many moment inequalities; High-dimensional inference; Randomization test; Symmetry-based inference.

JEL classification: C12, C14, C55.

\textsuperscript{a}Corresponding author. Nick Koning, Faculty of Economics and Business, University of Groningen, PO Box 800, 9700 AV Groningen, The Netherlands. Email: n.w.koning@rug.nl.

\textsuperscript{b}Paul Bekker, Faculty of Economics and Business, University of Groningen, Postbus 800, 9700 AV Groningen, The Netherlands. Email: p.a.bekker@rug.nl.
1 Introduction

As discussed by Chernozhukov et al. (2018), henceforth CCK, the moments inequalities framework has developed into a powerful tool for inference on causal and structural parameters in partially identified models. In such models, the parameters of interest may be restricted to a subset of the parameter space defined by a collection of moment inequalities. The simultaneous testing of these moment inequalities provides inference about the true underlying parameter values. CCK provide an excellent review of the literature with detailed motivating examples.

They point out that many economic models give rise to problems where the number of moment inequalities \( p \) may be much larger than the number of observations \( n \). While there exists a large literature on testing moment inequalities, traditional methods are not well equipped for dealing with many moment inequalities. In order to test the moment inequalities against the alternative where at least one of them is violated, CCK use the maximum of \( p t \) values as a test statistic. They find critical values by using asymptotic theory and bootstrap methods. Additionally, a first-stage inequality selection step is included to improve the power of their tests. Allen (2018) suggest a refinement of the selection step. Bugni et al. (2016) consider a generalization of the same problem and use Lasso for the first-stage selection.

Our contribution to the framework of testing many moment inequalities is threefold. First, we propose a novel test statistic. Notice that the maximum of \( p \) statistics is invariant to the size of the second largest statistic. If the alternative hypothesis allows for no more than one violation, inference based on the maximum may be powerful, but it would discard power against alternatives where multiple moment inequalities are violated. In order to retain this power, we propose a test statistic that is a non-negatively weighted combination

\[ \sum_{i=1}^{p} w_i t_i \]

\footnote{For work on unconditional moment inequalities see, e.g., Canay (2010); Andrews and Barwick (2012); Andrews and Guggenberger (2009); Chernozhukov et al. (2007); Rosen (2008); Romano and Shaikh (2008). Andrews and Shi (2013) notice that conditional moment inequalities can be viewed as an infinite number of unconditional moment inequalities. Contributions to conditional moment inequalities are found in Chernozhukov et al. (2013); Lee et al. (2013, 2018); Armstrong (2014, 2015); Armstrong and Chan (2016).}
of the largest sample means normalized by the square-root of its combined sample variance. This test statistic can be viewed as a one-sided version of Hotelling’s $T^2$ statistic. An interesting feature of our statistic is that, unlike the Hotelling’s $T^2$ statistic, it can also be used in high dimensional settings due to the sparsifying properties of the non-negativity restriction (Meinshausen et al., 2013; Slawski et al., 2013). In a simulation study, we find substantial increases in power compared to the statistic proposed by CCK, against alternatives where multiple moment inequalities are violated. However, a limitation of the test seems to be that it loses power against alternatives where the number of violated moment inequalities is large compared to $n$.

Secondly, we propose using randomization tests in the moment inequalities framework, by imposing a symmetry assumption on the errors. Randomization tests originate with Fisher (1935) and have been widely used in the literature. See, for example, Maritz (1995), Romano (1990), Bekker (2002) and Bekker and Lawford (2008). More recently, randomization tests have also been studied in high dimensional testing (Chung and Romano, 2016; Wang and Xu, 2019).

The main advantage of randomization tests is that they are exact regardless of $p$ or $n$. This is especially fruitful in high dimensional settings, where asymptotic results typically limit the growth of $p$ in terms of $n$. In our simulation experiments we find that the randomization tests perform similarly or slightly better compared to the empirical bootstrap proposed by CCK in large samples, and better in small samples. While randomization tests work well in small samples, they do require the validity of a symmetry assumption. However, in the simulation experiments where the symmetry assumption does not hold, we find, to our surprise, that the randomization test has better control of size than the empirical bootstrap procedure proposed by CCK, even if the sample is large.

Finally, we describe a first stage selection step in order to improve power by eliminating non-binding moment inequalities. The approach is based on symmetry and it does not depend on a specific selection method. Simulation results show that the tests with selection
control size, and have increased power in the presence of non-binding moment inequalities compared to tests without selection.

2 The model and two test statistics

Let observations in the $n \times p$ matrix $X$ satisfy

$$X = \iota_n \mu' + E,$$

where $\iota_n$ is an $n$-vector of ones, $\mu$ is a $p$-vector of parameters and $E$ is a random matrix that satisfies a symmetry assumption. Specifically, we assume that any row of $E$ can be multiplied by $-1$ without affecting the distribution of $E$. This assumption is satisfied if the rows of $E$ are independently symmetrically distributed about zero. Moments need not exist and the rows of $E$ need not be identically distributed. Furthermore, there is no need to assume the sample size $n$ is “sufficiently large” to justify accurate asymptotic approximations.

Following CCK, we are interested in testing the hypothesis $H_0: \mu \leq 0$ against the alternative $H_1: \mu \not\leq 0$. That is, we want to test whether all elements of $\mu$ are non-positive, against the alternative that at least one is positive. We will refer to the null hypothesis as the moment inequalities. The $j$th moment inequality is said to violated if $\mu_j > 0$, it is binding if $\mu_j = 0$, and it is strictly satisfied if $\mu_j < 0$.

To define the test statistics, let $I_n = (e_1, \ldots, e_n)$ be the $n \times n$ identity matrix, and define $P_{t_n} = \iota_n \iota_n' / n$. Define $\hat{\mu} = \iota'_n X / n$ with elements $\hat{\mu}_j$, and $\hat{\Sigma} = X'(I_n - P_{t_n})X / n$ with diagonal elements $\hat{\sigma}_j^2$. Let $t$ have $t$ values as elements, $t_j = \sqrt{n} \hat{\mu}_j / \hat{\sigma}_j$ for $j = 1, \ldots, p$. In order to test $H_0$ against $H_1$, CCK use the maximum $t$ value. We will denote this statistic by

$$t_{\text{max}} = t_{\text{max}}(X) = \max_{1 \leq j \leq p} t_j = \max_{\lambda > 0, \|\lambda\|_0 \leq 1} \frac{\iota'_n X \lambda}{\sqrt{\lambda X' (I_n - P_{t_n}) X \lambda}},$$
where $\|\lambda\|_0 = \sum_{j=1}^{p} 1_{(\lambda_j \neq 0)}$.

By ignoring the sizes of all but the largest element of $t$, the $t_{\max}$ statistic may result in a test that lacks power when testing against alternatives with multiple violations. In order to retain power, we propose the following test statistic

$$T_+ = T_+ (X) = 0 \vee \max_{\lambda \geq 0} \sqrt{n} \frac{\hat{\mu}' \lambda}{\sqrt{\lambda' \hat{\Sigma} \lambda}} = 0 \vee \max_{\lambda \geq 0} \frac{\lambda' X \lambda}{\lambda' X' (I_n - P_n) X \lambda},$$

so that $T_+$ is nonnegative. The $T_+$ statistic can be viewed as a one-sided version of the square-root of Hotelling’s $T^2$-statistic. In particular, if the non-negativity restriction on $\lambda$ is dropped and $\hat{\Sigma}$ is invertible, then the statistic reduces to $\sqrt{T^2} = \sqrt{n \hat{\mu}' \hat{\Sigma}^{-1} \hat{\mu}}$.

An interesting feature of $T_+$ is its selection of moment inequalities: the maximizing weight vector $\lambda^*$ is often sparse in practice, which results in a selection of moment inequalities that are ‘suspected’ to violate the null. This selection is a consequence of the sparsity inducing properties of the non-negativity restriction (Meinshausen et al., 2013; Slawski et al., 2013). As a result, the statistic can be used even in cases where $\hat{\Sigma}$ is singular and $p > n$.

In order to compute $T_+$, we proceed as follows. Define

$$T^*_+ = 0 \vee \max_{\lambda \geq 0} \frac{\lambda' X \lambda}{\lambda' X' X \lambda} = \frac{T_+}{\sqrt{1 + T^2_/n}},$$

which is a monotone function of $T_+$. If $\lambda_n X \not\leq 0$, the maximization involved to compute $T^*_+$ reduces to a least squares problem with an inequality and equality restriction:

$$\lambda^* = \arg\max_{\lambda \geq 0} \frac{\lambda' X \lambda}{\sqrt{\lambda' X' X \lambda}} = \arg\min_{\lambda \geq 0} \frac{\lambda' X \lambda}{\lambda_n X \lambda = 1}.$$

This can be solved, as suggested by Haskell and Hanson (1981), by computing

$$\lambda^*_n = \arg\min_{\lambda \geq 0} \{ \lambda' X' X \lambda + \gamma^2 (\lambda_n X \lambda - 1)^2 \}$$

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for a large value of $\gamma$. The latter problem is solved by the non-negative least squares algorithm of [Lawson and Hanson (1995)]

3 Symmetry based inference

The randomization tests that we consider are based on the symmetry assumption. We explain how the assumption leads to exact randomization tests, and show how they can be applied in practice. For other descriptions of randomization tests, see e.g. [Romano (1990)], [Maritz (1995)] and [Bekker and Lawford (2008)].

Let $\mathcal{R} = \{R_1, R_2, \ldots, R_N\}$ be the set of $N = 2^n$ diagonal $n \times n$ matrices with diagonal elements in $\{-1, 1\}$, where $R_1$ denotes the identity matrix. This set constitutes a finite reflection group under matrix multiplication. The group $\mathcal{R}$ determines a partitioning $\mathcal{E}$ of $\mathbb{R}^{n \times p}$ into equivalence classes ("orbits") denoted by $\mathcal{R}_{E^*} = \{E^*, R_2 E^*, \ldots, R_N E^*\}$, where $E^* \in \mathbb{R}^{n \times p}$ acts as a representative of the class. For simplicity, we only consider orbits $\mathcal{R}_{E^*} \in \mathcal{E}$, for which the cardinality satisfies $|\mathcal{R}_{E^*}| = |\mathcal{R}| = N$. This is equivalent to assuming that we only consider the subset of orbits $\mathcal{E}^0 \subset \mathcal{E}$ where the elements of $\mathcal{R}_{E^*} \in \mathcal{E}^0$ have no rows that equal zero.

The basic assumption that permits the construction of randomization tests is a distributional invariance assumption on the error term under group transformations. In particular, we assume that $RE$ and $E$ have the same distribution, for all $R \in \mathcal{R}$. Equivalently, we assume the conditional distribution of $E$, given an orbit $E \in \mathcal{R}_{E^*}$, is uniform for all orbits $\mathcal{R}_{E^*} \in \mathcal{E}^0$.

Let $g : \mathcal{R}_{E^*} \rightarrow \mathbb{R} \cup \{\infty\}$ be a mapping. Notice that if $g$ is injective on all orbits $\mathcal{R}_{E^*} \in \mathcal{E}^0$, then the symmetry assumption implies that the conditional distribution of $g(E)$, given an orbit $\mathcal{R}_{E^*} \in \mathcal{E}^0$, is uniform just as well. Furthermore, the function $p(E) = |\{R \in \mathcal{R} \mid g(RE) \geq g(E)\}|/N$ has a conditional distribution, given an orbit, that is uniform over $\{1/N, 2/N, \ldots, 1\}$. As this holds for all $\mathcal{R}_{E^*} \in \mathcal{E}^0$, the function $p(E)$ has

\[^2\text{This algorithm is implemented in the R package ‘nnls’ and the MATLAB function ‘lsqnonneg’.}\]
an unconditional uniform distribution over \{1/N, 2/N, \ldots, 1\}. This leads to the following results.

**Proposition 1.** \( P(p(E) \leq \alpha) \leq \alpha, \text{ for } \alpha \in [0, 1]. \) If \( g \) is injective on all \( R_{E^*} \in E^0 \), then \( P(p(E) \leq \alpha) = \alpha, \text{ for } \alpha \in \{1/N, 2/N, \ldots, 1\}. \)

Following Bekker and Lawford (2008), we will refer to \( g \) as an inferential function. As the cardinality of \( R \) grows exponentially in \( n \), the computation of \( p(E) \) is intractable even in small samples. Therefore, we provide the following result to allow for sampling from \( R^0 \).

**Proposition 2.** Let \( R^M \) contain the identity matrix and \( M - 1 \) other elements drawn randomly without replacement from \( R \setminus \{I_1\} \). Let \( p_M(E) = |\{R \in \mathcal{R}^M \mid g(RE) \geq g(E)\}|/M \), then \( P(p_M(E) \leq \alpha) \leq \alpha, \text{ for } \alpha \in [0, 1] \) and, if \( g \) is injective, then \( P(p_M(E) \leq \alpha) = \alpha \) for \( \alpha \in \{1/M, 2/M, \ldots, 1\}. \)

### 4 Randomization tests for moment inequalities

To test the moment inequalities of model (1) we use randomization tests based on the test statistics \( t_{\max}(X) \) and \( T_+(X) \). Let the test statistic be generically denoted as \( T(X) \). The tests are exact if the moment inequalities are binding. In that case \( X = E \) and \( T \) may be seen as an inferential function \( g(E) = T(E) \). So, Proposition 2 can be used based on \( p_M(E) = p_M(X) = |\{R \in \mathcal{R}^M \mid T(R(X)) \geq T(X)\}|/M \). In particular, we use the following testing procedure.

**Algorithm 1** (Symmetry Randomization Test).

1. Create the set \( \mathcal{R}^M \) consisting of the \( I_n \) and \( M - 1 \) matrices drawn from \( \mathcal{R} \setminus \{I_n\} \) without replacement.

\[ \text{Bekker and Lawford (2008) also describe sampling with replacement.} \]
2. Generate the transformed dataset $RX$ and compute $T(RX)$, for all $R \in R_M$.

3. Compute $p_M(X)$ by counting the proportion that exceeds $T(X)$.

4. Reject the null if $p_M(X) \leq \alpha$.

If $\mu = 0$ and $g$ is injective and $\alpha \in \{1/N, 2/N, \ldots, 1\}$, this test is exact.

In case some moment inequalities are assumed to be strictly satisfied, the test is no longer exact, but we observe in all our simulations that the size is controlled if the symmetry assumption holds. To prove this formally is another matter. If, under the null hypothesis $\mu \leq 0$, it can be verified that $T(RX) \geq T(t_n \mu' + RE)$, then the test is conservative.\footnote{Notice that for the numerator of $T_+(RX)$ it holds that $\nu_n^\prime RX\lambda = \nu_n^\prime R t_n \mu' \lambda + \nu_n^\prime RE \lambda \geq \nu_n(t_n \mu' + RE)\lambda$.} In that case we find, that $T(X) = T(t_n \mu' + E)$ is an inferential function, $g(E) = T(t \mu' + E)$, so that

$$p^T(X) = |\{R \in R \mid T(RX) \geq T(X)\}| \geq |\{R \in R \mid g(RE) \geq g(E)\}| = p(E),$$

which implies $P(p^T(X) \leq \alpha) \leq P(p(E) \leq \alpha) \leq \alpha$. Unfortunately, we have not been able to formally prove whether or not the condition holds for our test statistics. Nonetheless, the simulation results do show that the size of the tests are well behaved.

5 Symmetry based inference with pre-selection

As the tests are found to be conservative in case some moment equalities are strictly satisfied, there may be room for power improvements. CCK propose an inequality selection step aiming at removal of such strict moment inequalities $j$. In particular, they remove moment inequalities $j$ for which $t_j < c$, where $c$ is some chosen cut-off value. They propose several techniques to select this cut-off value such that the asymptotic testing procedures remain applicable, up to a small correction of the significance level. A similar way to select the cut-off value using Lasso is proposed by Bugni et al. (2016).
Our approach for inference with pre-selection is similar to the approach of the previous section where there was no pre-selection. We use symmetry based inference. Given any inequality selection rule we describe tests that are exact or conservative if the moment inequalities are binding. That is to say, if $\mu = 0$, then $T(X) = T(E)$ and $p_T(X) = |\{R \in R \mid T(RX) \geq T(X)\}| \leq \alpha$.

Let $J(X) \subset \{1, \ldots, p\}$ be an index selection subset, and let $X_{J(X)}$ be the submatrix of $X$ consisting of columns with indexes in $J(X)$. As the selection depends on $X$, it is not guaranteed that $p_T(X_{J(X)}) = |\{R \in R \mid T(RX_{J(X)}) \geq T(X_{J(X)})\}| \leq \alpha$. However, if $\mu = 0$, then $T(X_{J(X)}) = T(E_{J(E)}) = g(E)$ is an inferential function and

$$p_{sel}^T(X_{J(X)}) = |\{R \in R \mid T(RX_{J(RX)}) \geq T(X_{J(X)})\}|
= |\{R \in R \mid g(RE) \geq g(E)\}| = p(E),$$

which implies $P(p_{sel}^T(X_{J(X)}) \leq \alpha) = P(p(E) \leq \alpha) \leq \alpha$, for $\alpha \in [0, 1]$ and, if $g$ is injective, then $P(p_M(E) \leq \alpha) = \alpha$ for $\alpha \in \{1/M, 2/M, \ldots, 1\}$, as in Proposition 2. Therefore, we use the following testing procedure.

**Algorithm 2** (Symmetry randomization test with inequality selection).

1. Create the set $R^M$ consisting of the $I_n$ and $M - 1$ matrices drawn from $R \setminus \{I_n\}$ without replacement.

2. Compute the selected inequalities $J(RX)$, for each $R \in R_M$.

3. Compute $T(RX_{J(RX)})$ for all $R \in R_M$.

4. Compute $p_{sel}^T(X)$, by counting the proportion that exceeds $T(X_{J(X)})$.

5. Reject the null if $p_{sel}^T(X) \leq \alpha$.

If $\mu \leq 0$ and $\mu \neq 0$, the test is conservative if $T(RX_{J(RX)}) \geq T((\mu \mu' + RE)_{J(\mu \mu' + RE)})$.

The power of the test varies with the value of $\mu$ and the selection method. We follow CCK
and use $\mathcal{J}(X) = \{ j \in \{1, \ldots, p\} \mid t_j > c \}$, where the selection constant is chosen using their empirical bootstrap procedure. The simulations show that the size of the tests is controlled when $\mu \leq 0$.

6 Simulations

In this section, we present Monte Carlo simulation results. We use setups based on the simulation experiments presented by CCK and Bugni et al. (2016). The tests described in this paper are compared to the Empirical Bootstrap (EB) tests described in CCK.\footnote{We only compare to the EB tests described in CCK, as they find that the EB tests perform similarly to the Multiplier Bootstrap tests and better than the self-normalized tests.} All experiments were implemented in R and code for the tests will be made available at https://github.com/nickwkoning/.

Data generation

The data is created as follows: we generate $n \times p$ matrix $X = \iota_n \mu' + EA$, where $\iota_n$ is an $n$-vector of ones, $n \times p$ matrix $E$ has i.i.d. elements drawn from a distribution $F$, and $A$ is defined such that $\Sigma = A'A$, where $\Sigma$ has elements $\sigma_{ij} = \rho^{|i-j|}$.

The parameters we use are $n \in \{30, 400\}$, $p \in \{200, 500, 1000\}$ and $\rho \in \{0, 0.5, 0.9\}$. For the vector $\mu$, we consider four different designs. The signs of the elements of $\mu$ are chosen to represent four general cases: all moment inequalities are binding (Design 1), most moment inequalities are strictly satisfied and some are binding (Design 2), all moment inequalities are violated (Design 3), and some moment inequalities are violated while some are strictly satisfied (Design 4). The values of $\mu$ for each combination of $n$ and $p$ were selected to ensure that the rejection probabilities are bounded away from 1.

For the errors, we use a symmetric and two asymmetric distributions: one left-skewed and one right-skewed. As symmetric distribution, we use $t(4)/\sqrt{2}$, where $t(4)$ is the Student’s $t$ distribution with 4 degrees of freedom and we divide by $\sqrt{2}$ so that the variance
is 1. As asymmetric distributions we use the skew-normal distribution with mean 0, variance equal to 1 and two configurations for the skewness: $\gamma = -0.667$ and $\gamma = 0.667$. A density plot comparing these skewed distributions to the standard normal distribution is provided in Figure 1. For the sake of brevity, the asymmetric error distributions were only considered for Design 1.

This setup is similar to the setup used by an earlier working paper of CCK\footnote{This earlier version can be found at \url{https://arxiv.org/abs/1312.7614v4}}. The differences are that we do not consider equicorrelated data or uniformly distributed errors, but instead consider small samples ($n = 30$) and asymmetric error distributions. In addition, the positive values of $\mu$ are substantially decreased to account for the higher power of tests based on the $T_+$ statistic.

**Tests**

We use symmetry based randomization (SR) tests based on the test statistics $t_{\text{max}}$ and $T_+$, with and without pre-selection. As a comparison, we also include the EB tests described by CCK with and without pre-selection.

For each setting, 1000 realizations of $X$ were generated and the proportion of rejections was recorded. For the EB tests 1000 bootstrap samples were used and for the SR tests we used 1000 reflection samples. The significance level was fixed at $\alpha = 0.05$. The selection constant for the tests that include pre-selection is chosen using the EB procedure with $\beta = 0.001$ and 1000 resamples as described in CCK. For the small sample experiments with $n = 30$, we do not use pre-selection as the selection constant depends on the asymptotic properties of the EB test.

**6.1 Results**

The results of the simulation experiments can be found in Tables 1 to 5 and are discussed for each design separately.
**Design 1:** $\mu = 0$.

In Design 1, all elements of $\mu$ are equal to zero. Therefore, the rejection probability for the symmetry randomization (SR) tests is at most equal to $\alpha = 0.05$, by Proposition 2. In addition, if the test statistic is injective on all orbits, then the rejection probability is equal to $\alpha$. The results for Design 1 are displayed in Table 1 for the symmetric error distribution, and Table 5 for the asymmetric error distributions.

The results from Table 1 show that if the number of observations is large ($n = 400$), then the size of the test is approximately $\alpha$. One exception to this is the configuration $p = 1000$ and $\rho = 0$, for the statistic $T_+$, where the proportion of rejections is 0. Further inspection shows that for this configuration without pre-selection, the test statistics $T_+(X)$ and critical values, defined as the $1-\alpha$ quantile of the set $\{T_+(R X) \mid R \in R_M\}$, are infinite. Therefore, the function $T_+$ is not injective on the orbit and the rejection probability can be strictly smaller than $\alpha$, according Proposition 2. A similar problem of non-injectivity occurs for the case with pre-selection. Occurrences of this phenomenon are marked in Tables 1, 3 and 5 by a * next to the rejection rate.

If the number of observations is small ($n = 30$), then the rejection rate for the SR tests based on the $t_{\text{max}}$ statistic remain approximately $\alpha$. In contrast, the EB tests over-reject. For the SR tests based on the $T_+$ statistic, the rejection rate is frequently zero due to the non-injectivity phenomenon described above.

Table 5 shows the rejection rate for Design 1 when the error distributions are left-skewed ($\gamma = -0.667$) and right-skewed ($\gamma = 0.667$). As the symmetry assumption is violated, it is not surprising that the SR tests have a rejection rate different from $\alpha$. In particular, we find that the rejection rate for the left-skewed error distribution is larger than $\alpha$, while the rejection rate for the right-skewed error distribution is smaller than $\alpha$.

Surprisingly, even though the sample size is large ($n = 400$), the EB tests over-reject even more than the SR tests under the left-skewed distribution. This suggests that the EB
tests may also require errors to be symmetrically distributed in finite samples. Although these simulation results are by no means exhaustive, they suggest that one may sometimes be better off using an SR tests than an EB test even if it is known that the true error distribution is symmetric.

**Design 2**: $\mu_j = 0$ for some $j$ and $\mu_j < 0$ for most $j$.

In Design 2, for $n = 400$, the first $0.1p$ elements of $\mu$ are equal to 0 and the remaining elements are equal to $-0.8$. For $n = 30$, the first 10 elements are 0 and the remaining elements are equal to $-5$. As the data for Design 2 is generated under $H_0$, the rejection rates should be smaller than $\alpha$ (up to sampling errors) if no pre-selection is used, and close to $\alpha$ if pre-selection is used. The results for Design 2 can be found in Table 2.

From Table 2, it can be observed that the rejection rates of the tests without pre-selection is smaller than $\alpha$. In addition, pre-selection lifts the rejection rates close to the nominal size $\alpha$.

**Design 3**: $\mu > 0$.

In Design 3, all elements of $\mu$ are positive. In particular, for $n = 400$ they are equal to 0.01 and for $n = 30$ they are equal to 0.15. As the data is generated under the alternative, the rejection probability should be as large as possible. The results for Design 3 are found in Table 3.

In Table 3, it can be seen that for $n = 400$ the power of the EB and SR tests are similar for the tests based on the $t_{\text{max}}$ statistic. The tests based on the $T_+$ statistic can result in much higher power, except for the cases marked by a *. The difference between the power of the tests based on the $t_{\text{max}}$ and $T_+$ statistics is largest under weak correlations.

For $n = 30$, the results for the EB tests should be ignored, as the tests do not control size. Although there are multiple violations, the tests based on the $t_{\text{max}}$ statistic often have higher power than the tests based on the $T_+$ statistic. This is caused by the same
phenomenon as observed for Design 1, where both the test statistic and critical value are infinite. Therefore, it is recommended that the test statistic and critical value are inspected before the outcome of the test is interpreted. If both are infinite, then the test based on the $T_+$ statistic should not be used. As we will observe for Design 4, the cause of the phenomenon does not seem to be that $p \gg n$, as it does not occur when the number of positive elements in $\hat{\mu}$ is small compared to $n$.

**Design 4**: $\mu_j > 0$ for some $j$ and $\mu_j < 0$ for most $j$.

In Design 4, for $n = 400$, the first $0.1p$ elements of $\mu$ are equal to 0.02 and the remaining elements are equal to $-0.75$. For $n = 30$, the first 10 elements of $\mu$ are equal to 0.3 and the remaining elements are equal to -5. The data is generated under the alternative, so the rejection rates should be as large as possible. The results for Design 4 are presented in Table 4.

The results in Table 4 for $n = 400$, show that the tests with pre-selection have substantially higher power than the tests without pre-selection. When considering the test with pre-selection, tests based on the $T_+$ statistic have substantially more power than the tests based on the $t_{\max}$ statistic. The power for the EB and SR tests based on the $t_{\max}$ statistic are similar if pre-selection is used, but the SR tests seem more powerful if pre-selection is not used.

Even though the EB tests do not control size for $n = 30$, the SR tests based on the $t_{\max}$ statistic are more powerful. Comparing the SR tests based on the $T_+$ and $t_{\max}$ statistics, it can be observed that the $T_+$ based tests perform better under weak correlations, while the $t_{\max}$ based tests can perform better under strong correlations. The phenomenon observed in Design 1 and 3 does not occur in Design 4, even in the case where $n = 30$ and $p = 1000$. 
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Figure 1: A density plot of the left-skewed (left) and right-skewed (right) normal distribution with mean 0, standard deviation 1 and skewness -0.667 and 0.667, respectively, overlaid on the density of the standard normal distribution.

Table 1: Monte Carlo rejection probabilities with 1000 repetitions for Design 1: $\mu = 0$, with symmetrically distributed errors ($t(4)/\sqrt{2}$). The columns represent the Empirical Bootstrap (EB) and Symmetry Randomization (SR) tests, based on the $t_{\text{max}}$ and $T_+$ statistics, both with pre-selection (sel) and without pre-selection. In the cases marked by *, both the test statistic and critical value were infinite in a large proportion of the 1000 simulations.
Table 2: Monte Carlo rejection probabilities with 1000 repetitions for Design 2: $\mu_j = 0$ for some $j$ and $\mu_j < 0$ for most $j$, with symmetrically distributed errors ($t(4)/\sqrt{2}$). The columns represent the Empirical Bootstrap (EB) and Symmetry Randomization (SR) tests, based on the $t_{\text{max}}$ and $T_+$ statistics, both with pre-selection (sel) and without pre-selection. For the cases that $n = 400$, $\mu_j = 0$ if $j \leq 0.1p$ and $\mu_j = -0.8$ if $j > 0.1p$, and for the cases where $n = 30$, $\mu_j = 0$ if $j \leq 10$ and $\mu_j = -5$ if $j > 10$.
Table 3: Monte Carlo rejection probabilities with 1000 repetitions for Design 3: $\mu > 0$, with symmetrically distributed errors ($t(4)/\sqrt{2}$). The columns represent the Empirical Bootstrap (EB) and Symmetry Randomization (SR) tests, based on the $t_{\text{max}}$ and $T_+$ statistics, both with pre-selection (sel) and without pre-selection. For the cases that $n = 400$, $\mu_j = 0.01$ and for the cases that $n = 30$, $\mu_j = 0.15$, for all $j$. In the cases marked by *, both the test statistic and critical value were infinite in a large proportion of the 1000 simulations.
Table 4: Monte Carlo rejection probabilities with 1000 repetitions for Design 4: \( \mu_j > 0 \) for some \( j \) and \( \mu_j < 0 \) for most \( j \), with symmetrically distributed errors \((t(4)/\sqrt{2})\). The columns represent the Empirical Bootstrap (EB) and Symmetry Randomization (SR) tests, based on the \( t_{\text{max}} \) and \( T_+ \) statistics, both with pre-selection (sel) and without pre-selection. For the cases that \( n = 400 \), \( \mu_j = 0.02 \) if \( j \leq 0.1p \) and \( \mu_j = -0.75 \) if \( j > 0.1p \), and for the cases that \( n = 30 \), \( \mu_j = 0.3 \) if \( j \leq 10 \) and \( \mu_j = -5 \) if \( j > 10 \).
Table 5: Monte Carlo rejection probabilities with 1000 repetitions for Design 1: $\mu = 0$, $n = 400$ observations, asymmetrically distributed errors (skew-normal with skewness parameter $\gamma$), for tests based on the $t_{\text{max}}$ and $T_+$ statistics. The columns represent the Empirical Bootstrap (EB) and Symmetry Randomization (SR) tests, based on the $t_{\text{max}}$ and $T_+$ statistics, both with pre-selection (sel) and without pre-selection. In the cases marked by *, both the test statistic and critical value were infinite in a large proportion of the 1000 simulations.