Theta Vectors and Quantum Theta Functions

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ABSTRACT

In this paper, we clarify the relation between Manin’s quantum theta function and Schwarz’s theta vector in comparison with the $kq$ representation, which is equivalent to the classical theta function, and the corresponding coordinate space wavefunction. We first explain the equivalence relation between the classical theta function and the $kq$ representation in which the translation operators of the phase space are commuting. When the translation operators of the phase space are not commuting, then the $kq$ representation is no more meaningful. We explain why Manin’s quantum theta function obtained via algebra (quantum tori) valued inner product of the theta vector is a natural choice for quantum version of the classical theta function ($kq$ representation). We then show that this approach holds for a more general theta vector with constant obtained from a holomorphic connection of constant curvature than the simple Gaussian one used in the Manin’s construction. We further discuss the properties of the theta vector and of the quantum theta function, both of which have similar symmetry properties under translation.

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I. Introduction

Classical theta functions can be regarded as state functions on classical tori, and have played an important role in the string loop calculation [1, 2]. Its quantum version on the noncommutative tori has been discussed mainly by Manin [3, 4, 5] and Schwarz [6, 7]. In the physics literature it has been discussed in the context of noncommutative soliton [8].

In noncommutative field theory, one can find nontrivial soliton solutions in terms of projection operators [9, 8, 10]. Before this development, Boca [11] has constructed projection operators on the $\mathbb{Z}_4$-orbifold of noncommutative two torus. There it was also shown that these projection operators can be expressed in terms of the classical theta functions, of which certain classical commuting variables are replaced with quantum operators. Hinted from and generalizing the Boca’s result, Manin [4, 5] explicitly constructed a quantum theta function, the concept of which he introduced previously [3]. In both Boca’s and Manin’s constructions, the main pillars were the algebra valued inner product that Rieffel [12] used in his classic work on projective modules over noncommutative tori. One major difference is that in Manin’s construction of quantum theta function, the so-called theta vector that Schwarz introduced earlier [6, 7] was used for the inner product, while in Boca’s construction the eigenfunctions of Fourier transform were used.

Both the classical theta function [13] and the $kq$ representation in the physics literature [14, 15] have been known for a long time. The $kq$ representation is a transformation of a wavefunction on (real $n$-dimensional) coordinate space to a function on (real $2n$-dimensional) phase space consisting of (quasi-)coordinates and (quasi-)momenta. However, the translation operators in the $kq$ representation acting on the lattice of the phase space are commuting. When the lattice of the phase space is periodic, one can identify functions possessing translational symmetry on the lattice with the classical theta functions on tori. When the translation operators of the coordinate and momentum directions are not commuting, the $kq$ representation and the classical theta function lose their meaning. One has to find other ways of representing periodic functions on the lattice of the non-commuting phase space. When the algebras are noncommutative, algebra valued inner product is a good fit for constructing operators out of state functions. In the case at hand, the coordinates of the phase...
space are non-commuting and so is the algebra based on them. And the functions on the non-commuting phase space can be regarded as operators.

Classical phase space variables are commuting variables, and thus they can be simply multiplied in front of a state function (wavefunction). Namely, we can simply put the values of observables in front of a state function. However, in the quantum case, we have to be very careful with observables. Quantum observables behave as operators acting on a state and in general they change the state.

In fact, the theta vector corresponds to a state on a quantum torus and the quantum theta function defined by Manin \[4, 5\] is an operator acting on the states (module) on a quantum torus. In quantum mechanics, one can build operators out of state vectors. In mathematics, this can be carried out via operator (algebra) valued inner product. Therefore, it is very natural to use algebra valued inner product to build the quantum theta functions from the theta vectors over noncommutative tori. The classical theta function possesses a certain symmetry property under the lattice translation, and Manin’s quantum theta function is constructed in such a way that this symmetry property is maintained as a functional relation which the quantum theta function should satisfy.

In this paper, we first review the classical theta function and the $kq$ representation briefly and discuss their relationship. We then proceed to the quantum case and explain why the Manin’s approach based on algebra valued inner product is a natural choice for quantum extension. As a support for this viewpoint, we show that the Manin’s construction also holds for a more general theta vector satisfying the holomorphicity condition. Namely, the quantum theta function built with our new theta vector also satisfies the Manin’s consistency requirement for the translational symmetry on the quantum lattice.

We also discuss how the theta vectors can be regarded as invariant state vectors under parallel transport over noncommutative tori equipped with complex structures, while quantum theta functions can be regarded as observables having translational symmetry on the quantum lattice.

The organization of the paper is as follows. In section II, we review the classical theta function briefly, then explain the relationship between the classical theta functions and the $kq$ representation. In section III, we first review the theta vectors on quantum tori, then
explain how the concept of Manin’s quantum theta function emerges from algebra valued inner product of a state function. In section IV, we first review Manin’s construction of quantum theta function in detail. Then, in order to provide a further support for the Manin’s approach we apply it to the case of a more general theta vector with constant satisfying the holomorphicity condition, and show that new quantum theta function also satisfies the Manin’s functional relation for consistency requirement. In section V, we conclude with discussion.

II. Classical complex tori and $kq$ representation

In this section, we discuss the relationship between the classical theta function and the so-called $kq$ representation [14, 15]. We first look into how the classical theta function emerges from Gaussian function via Fourier-like transformation. We then show that the transformed function is exactly equivalent to the $kq$ representation known in the physics literature.

We now recall the property of classical theta function briefly, then show how Gaussian function can be transformed into the classical theta function. The classical theta function $\Theta$ is a complex valued function on $\mathbb{C}^n$ satisfying the following relation.

\[ \Theta(z + \lambda') = \Theta(z) \quad \text{for} \quad z \in \mathbb{C}^n, \, \lambda' \in \Lambda', \quad (1) \]
\[ \Theta(z + \lambda) = c(\lambda)e^{q(\lambda,z)}\Theta(z) \quad \text{for} \quad \lambda \in \Lambda, \quad (2) \]

where $\Lambda' \oplus \Lambda \subset \mathbb{C}^n$ is a discrete sublattice of rank $2n$ split into the sum of two sublattices of rank $n$, isomorphic to $\mathbb{Z}^n$, and $c : \Lambda \to \mathbb{C}$ is a map and $q : \Lambda \times \mathbb{C} \to \mathbb{C}$ is a biadditive pairing linear in $z$.

The function $\Theta(z, T)$ satisfying (1) and (2) is defined as

\[ \Theta(z, T) = \sum_{k \in \mathbb{Z}^n} e^{\pi i (k^t Tk + 2k^t z)} \quad (3) \]

where $T$ is a symmetric complex valued $n \times n$ matrix whose imaginary part is positive definite. Let $f_T(x)$ be a Gaussian function defined as below using the same $T$ as above.

\[ f_T(x) = e^{\pi i x^t Tx} \quad \text{for} \quad x \in \mathbb{R}^n. \quad (4) \]
Then \( \tilde{f}_T(\rho, \sigma) \) is defined as \[ \tilde{f}_T(\rho, \sigma) \equiv \sum_{k \in \mathbb{Z}^n} e^{-2\pi i \rho^t k} f_T(\sigma + k) \] (5)

where \( \rho, \sigma \in \mathbb{R}^n \). When we fix \( \sigma \), this is a Fourier transformation between \( k \) and \( \rho \). Then from (5), we get \( \Theta(z, T) \) with a substitution \( z = T \sigma - \rho \) as follows.

\[
\tilde{f}_T(\rho, \sigma) = \sum_{k \in \mathbb{Z}^n} e^{\pi i((\sigma+k)^T(\sigma+k)-2\rho^t k)}
= e^{\pi i \sigma^t T \sigma} \sum_{k \in \mathbb{Z}^n} e^{\pi i(k^T k + 2 \rho^t (T \sigma - \rho))}
= e^{\pi i \sigma^t T \sigma} \Theta(T \sigma - \rho, T) \tag{6}
\]

We can do the same procedure for a general Gaussian function, \( f_{T,c}(x) \), as follows.

\[
f_{T,c}(x) = e^{\pi i(x^t T x + 2c^t x)} \tag{8}
\]

where \( c \in \mathbb{C}^n \). Then,

\[
\tilde{f}_{T,c}(\rho, \sigma) \equiv \sum_{k \in \mathbb{Z}^n} e^{-2\pi i \rho^t k} f_{T,c}(\sigma + k)
= \sum_{k \in \mathbb{Z}^n} e^{\pi i((\sigma+k)^t T(\sigma+k)+2c^t (\sigma+k)-2\rho^t k)}
= e^{\pi i(\sigma^t T \sigma + 2c^t \sigma)} \sum_{k \in \mathbb{Z}^n} e^{\pi i(k^T k + 2 \rho^t (T \sigma - \rho + c))}
= e^{\pi i(\sigma^t T \sigma + 2c^t \sigma)} \Theta(T \sigma - \rho + c, T). \tag{11}
\]

In this case we get \( \Theta(z, T) \) with a substitution \( z = T \sigma - \rho + c \).

The transformation (5) exactly matches the transformation used in defining the \( kq \) representation which already appeared in the physics literature \[14\] \[15\]. The \( kq \) representation is similar to the coherent states for a simple harmonic oscillator. The coherent states are the eigenstates of annihilation operator \( \hat{a} \), which is a linear combination of the position and momentum operators. Thus the eigenvalues of coherent states can be expressed in terms of expectation values of both position and momentum of the state. This is in contrast with a usual wavefunction in which position and momentum eigenvalues do not appear together.
The \( kq \) representation which defines symmetric coordinates \( k \) (quasimomentum) and \( q \) (quasicoordinate) is a transformation from a wavefunction in position space into a wavefunction in both \( k \) and \( q \), which we denote as \( C(k, q) \). \( C(k, q) \) is defined by

\[
C(k, q) = \left( \frac{a}{2\pi} \right)^{\frac{1}{2}} \sum_{l \in \mathbb{Z}} e^{i k a l} \psi(q - l a)
\]  

(12)

where \( a \) is a real number (lattice constant), and the “coordinates” of the phase space \( (k, q) \) run over the intervals \( -\frac{\pi}{a} < k \leq \frac{\pi}{a} \) and \( -\frac{a}{2} < q \leq \frac{a}{2} \). In this representation, the displacement operators \( e^{imbdx} \), \( e^{inap} \) in the \( x \) and \( p \) directions, where \([x, p] = i\), \( b = \frac{2\pi}{a} \), and \( m, n \in \mathbb{Z} \), are mutually commuting and thus they simply become simple multiplication by the function \( e^{im \frac{2\pi}{a} q} \) and \( e^{in a k} \), respectively [15].

Comparing (12) with (5), it is not difficult to see that \( C(k, q) \) corresponds to \( \tilde{f}_T(\rho, \sigma) \) in our previous discussion with a correspondence \((\rho \leftrightarrow k)\) and \((\sigma \leftrightarrow q)\). Furthermore, from (12) it can be easily checked that

\[
C(k + \frac{2\pi}{a}, q) = C(k, q),
\]

(13)

\[
C(k, q + a) = e^{ika} C(k, q).
\]

(14)

These exactly match (11) and (2), the property of the classical theta function. We can thus say that the classical theta function corresponds to the \( kq \) representation, \( C(k, q) \), while the pre-transformed Gaussian function \( f_T(x) \) for the classical theta function corresponds to the wavefunction \( \psi(x) \) for the \( kq \) representation. This correspondence is only valid when the translation operators of the phase space \((x, p)\) are mutually commuting.

Therefore, we can see from the above observation that the quantum theta functions on noncommutative tori cannot be obtained via this kind of Fourier-like transformation. Since the translation operators on noncommutative (quantum) tori are in general non-commuting, we need other ways of going from the position space representation (like a wavefunction) to the phase space representation (like \( C(k, q) \) or the classical theta function in the above correspondence) in the quantum case. Namely we have to find a way to transform a wavefunction (state vector) into an observable in a noncommuting phase space (consisting of operators \( x \) and \( p \)). This process can be done via the so-called algebra valued inner product demonstrated well in the Rieffel’s seminal work on noncommutative tori [12]. Manin [4, 5]
has demonstrated successfully how this machinery can be used to define the quantum theta function. We now turn to this subject in the next section.

III. Theta vectors on quantum tori and algebra valued inner product for a passage to quantum theta functions

In this section, we first discuss theta vectors on quantum tori and define algebra (quantum tori) valued inner product on the modules over the quantum tori. Then we introduce Manin’s quantum theta function \[5\] via algebra valued inner product.

A noncommutative \(d\)-torus \(T^d_\theta\) is a \(C^*\)-algebra generated by \(d\) unitaries \(U_1, \ldots, U_d\) subject to the relations

\[
U_\alpha U_\beta = e^{2\pi i \theta_{\alpha \beta}} U_\beta U_\alpha, \quad \text{for} \quad 1 \leq \alpha, \beta \leq d, \tag{15}
\]

where \(\theta = (\theta_{\alpha \beta})\) is a skew symmetric matrix with real entries.

Let \(L\) be all derivations on \(T^d_\theta\), i.e.,

\[
L = \{\delta : T^d_\theta \to T^d_\theta, \text{ which is linear, and } \delta(fg) = \delta(f)g + f\delta(g)\}.
\]

Then \(L\) has a Lie algebra structure since \([\delta_1, \delta_2] = \delta_1\delta_2 - \delta_2\delta_1 \in L\). We can also see that \(L\) is isomorphic to \(\mathbb{R}^d\). A noncommutative torus is said to have a complex structure if the Lie algebra \(L = \mathbb{R}^d\) acting on \(T^d_\theta\) is equipped with the complex structure that we explain below.

A complex structure on \(L\) can be considered as a decomposition of complexification \(L \oplus iL\) of \(L\) into a direct sum of two complex conjugate subspace \(L^{1.0}\) and \(L^{0.1}\). We denote a basis in \(L\) by \(\delta_1, \ldots, \delta_d\), and a basis in \(L^{0.1}\) by \(\tilde{\delta}_1, \ldots, \tilde{\delta}_n\) where \(d = 2n\). One can express \(\tilde{\delta}_\alpha\) in terms of \(\delta_j\) as \(\tilde{\delta}_\alpha = t_{\alpha j}\delta_j\), where \(t_{\alpha j}\) is a complex \(n \times d\) matrix.

Let \(\nabla_j\) (for \(j = 1, \ldots, d\)) be a constant curvature connection on a \(T^d_\theta\)-module \(\mathcal{E}\). A complex structure on \(\mathcal{E}\) can be defined as a collection of \(\mathbb{C}\) linear operators \(\tilde{\nabla}_1, \ldots, \tilde{\nabla}_n\) satisfying

\[
\tilde{\nabla}_\alpha(a \cdot f) = a\tilde{\nabla}_\alpha f + (\tilde{\delta}_\alpha a) \cdot f \tag{16}
\]

\[
[\tilde{\nabla}_\alpha, \tilde{\nabla}_\beta] = 0 \tag{17}
\]
where $a \in T^d_\theta$ and $f \in \mathcal{E}$ \[8\].

These two conditions are satisfied if we choose $\tilde{\nabla}_\alpha$ as

$$\tilde{\nabla}_\alpha = t_{\alpha j} \nabla_j \text{ for } \alpha = 1, \ldots, n, \ j = 1, \ldots, n.$$  \hspace{1cm} (18)

A vector $f \in \mathcal{E}$ is holomorphic if

$$\tilde{\nabla}_\alpha f = 0, \text{ for } \alpha = 1, \ldots, n.$$  \hspace{1cm} (19)

A finitely generated projective module over $T^d_\theta$ can take the form $S(\mathbb{R}^p \times \mathbb{Z}^q \times F)$ where $2p + q = d$ and $F$ is a finite Abelian group \[12\]. Here, $S(M)$ denotes the Schwartz functions on $M$ which rapidly decay at infinity.

Here, we consider the case that the module is given by $S(\mathbb{R}^n)$, and choose a constant curvature connection $\nabla$ on $S(\mathbb{R}^n)$ such that

$$(\nabla_\alpha, \nabla_{n+\alpha}) = (\frac{\partial}{\partial x^\alpha}, -2\pi i \sigma_\alpha x^\alpha) \text{ for } \alpha = 1, \ldots, n,$$  \hspace{1cm} (20)

where $\sigma_\alpha$ are some real constants, $x^\alpha$ are coordinate functions on $\mathbb{R}^n$ and repeated indices are not summed. Then the curvature $[\nabla_i, \nabla_j] = F_{ij}$ satisfies $F_{\alpha,n+\alpha} = 2\pi i \sigma_\alpha$, $F_{n+\alpha,\alpha} = -2\pi i \sigma_\alpha$ and all others are zero. Now, we change the coordinates such that $t = (t_{\alpha j})$ becomes

$$t = (1, \tau),$$  \hspace{1cm} (21)

where $1$ is an identity matrix of size $n$ and $\tau$ is an $n \times n$ complex valued matrix.

Then, the holomorphic vector $f$ satisfying \[19\] can be expressed as

$$\left(\frac{\partial}{\partial x^\alpha} - \sum_\beta 2\pi i T_{\alpha \beta} x^\beta\right)f = 0,$$  \hspace{1cm} (22)

where the $n \times n$ matrix $T = (T_{\alpha \beta})$ is given as follows. The condition \[17\] requires that the matrix $T$ be symmetric, $T_{\alpha \beta} = T_{\beta \alpha}$, and it is given by $T_{\alpha \beta} = \tau_{\alpha \beta} \sigma_\beta$, $\alpha, \beta = 1, \ldots, n$, with the repeated index $\beta$ not summed. Up to a constant we get,

$$f(x^1, \ldots, x^n) = e^{\pi i x^\alpha T_{\alpha \beta} x^\beta}.$$  \hspace{1cm} (23)

If $\text{Im}T$ is positive definite, then $f$ belongs to $S(\mathbb{R}^n)$. The vectors satisfying the holomorphicity condition \[19\] are called the theta vectors \[6\].
If a constant in $C^n$ is added to a given connection $\tilde{\nabla}$, it still yields the same constant curvature. Then the holomorphicity condition (19) becomes

$$(\tilde{\nabla}_\alpha - 2\pi i c_\alpha)f_c = 0, \quad \text{for } \alpha = 1, \ldots, n$$

(24)

for $f_c \in S(\mathbb{R}^n)$, giving the following condition

$$(\frac{\partial}{\partial x^\alpha} - \sum_\beta 2\pi i T_{\alpha \beta} x^\beta - 2\pi i c_\alpha) f_c = 0,$$

(25)

whose solution we get

$$f_c(x) = e^{\pi i x^\alpha T_{\alpha \beta} x^\beta + 2\pi i c_\alpha x^\alpha}. $$

(26)

Here, we would like to make an observation. The holomorphicity condition (19) means that the theta vector $f$ or $f_c$ is invariant under a parallel transport on a noncommutative torus with complex structure.

Now we turn to the concept of the quantum theta function introduced by Manin [3, 4, 5]. Recall that the classical theta function $\Theta(z)$ satisfies the conditions (1) and (2)

$$\Theta(z + \lambda') = \Theta(z), \quad z \in \mathbb{C}^n, \quad \forall \lambda' \in \Lambda',$$

$$\Theta(z + \lambda) = c(\lambda) e^{q(\lambda, z)} \Theta(z), \quad \forall \lambda \in \Lambda,$$

where $c : \Lambda \to \mathbb{C}$ is a map and $q : \Lambda \times \mathbb{C} \to \mathbb{C}$ is a biadditive pairing linear in $z$. This function can be written formally as follows [3],

$$\Theta(z) = \sum_{j \in J} a_j e^{2\pi i j(z)},$$

(27)

where $J = \text{Hom}(\Lambda', \mathbb{Z})$. The coefficients $a_j$ decay swiftly enough. Then this form satisfies the first condition (1) automatically and we impose a constraint for $a_j$ satisfying the second condition (2). If we define $T(J)(\mathbb{C}) = \text{Hom}(J, \mathbb{C}^*)$ where $\mathbb{C}^* = \mathbb{C} - \{0\}$. We have an isomorphism $e$ from $J$ to $\tilde{J} \equiv \text{Hom}(T(J)(\mathbb{C}), \mathbb{C}^*)$. We denote $e(j)$ the image of $j$ by this map $e$. Then

$$e(j + l) = e(j)e(l), \quad \text{for } j, l \in J.$$

We have an analytic map $P$ which is in fact an isomorphism up to $\Lambda'$,

$$P : \mathbb{C}^n \to T(J)(\mathbb{C}),$$

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inducing the pullback $P^*(e(j)) = e^{2\pi ij(\cdot)}$ where $j(\cdot)$ is the linear function on $\mathbb{C}^n$ extending $j$ as a function on $\Lambda'$. Then the classical theta function $\Theta$ can be expressed as

$$\Theta = P^*(\tilde{\Theta}), \text{ where } \tilde{\Theta} = \sum_{j \in J} a_j e(j).$$

Let $B$ be the image of $\Lambda$ under $P$, then $b^*(\tilde{\Theta})$, the translation of $\tilde{\Theta}$ by $b \in B$, is equal to $\sum_{j \in J} a_j j(b)e(j)$, where $j(b) = e(j)(b)$ is the value of $e(j)$ at the point $b \in B$:

$$b^*(\tilde{\Theta})(w) = \tilde{\Theta}(w \cdot b), \text{ where } \forall w \in T(J)(\mathbb{C}).$$

The second condition can be interpreted as

$$c_b e(j_b) b^*(\tilde{\Theta}) = \tilde{\Theta} \tag{28}$$

where $c_b \in \mathbb{C}$ and $j_b \in J$. To generalize this for $T^d$, the Heisenberg group $G(J)$ is defined. This is the group of linear endomorphisms of the space of functions $(\Phi)$ on algebraic torus $T(J)(\mathbb{C})$ generated by the following maps,

$$[c, x, j] : \Phi \rightarrow c e(j) x^*(\Phi), \tag{29}$$

where $c \in \mathbb{C}^*$, $x \in T(J)(\mathbb{C})$, $j \in J$ and $x^*(e(j)) = j(x)e(j)$, where $j(x)$ being the value of $e(j)$ at $x$. In these terms, a system consisting of a subgroup $B$ in $T(J)(\mathbb{C})$ and automorphy factors satisfying the second condition (28) become simply a homomorphism, which we will call a multiplier, $\mathcal{L}$,

$$\mathcal{L} : B \rightarrow G(J), \mathcal{L}(b) = [c_b, x_b, j_b], \tag{30}$$

where $b \rightarrow x_b$ is a bijection. Manin’s quantum theta function is invariant under the image of $\mathcal{L}$, the subgroup of the Heisenberg group $G(J)$.

Now, we consider the algebra valued inner product on a bimodule after Rieffel [12]. Let $M$ be any locally compact Abelian group, and $\hat{M}$ be its dual group and $\mathcal{G} \equiv M \times \hat{M}$. Let $\pi$ be a representation of $\mathcal{G}$ on $L^2(M)$ such that

$$\pi_x \pi_y = \alpha(x, y) \pi_{x+y} = \alpha(x, y) \overline{\alpha(y, x)} \pi_y \pi_x \text{ for } x, y \in \mathcal{G} \tag{31}$$
where $\alpha$ is a map $\alpha : G \times G \to \mathbb{C}^*$ satisfying
\[
\alpha(x, y) = \alpha(y, x)^{-1}, \quad \alpha(x_1 + x_2, y) = \alpha(x_1, y)\alpha(x_2, y),
\]
and $\overline{\alpha}$ denotes the complex conjugation of $\alpha$.

Let $D$ be a discrete subgroup of $G$. We define $S(D)$ as the space of Schwartz functions on $D$. For $\Phi \in S(D)$, it can be expressed as $\Phi = \sum_{w \in D} \Phi(w)e_{D,\alpha}(w)$ where $e_{D,\alpha}(w)$ is a delta function with support at $w$ and obeys the following relation.
\[
e_{D,\alpha}(w_1)e_{D,\alpha}(w_2) = \alpha(w_1, w_2)e_{D,\alpha}(w_1 + w_2)
\] (32)

For Schwartz functions $f, g \in S(M)$, the algebra $(S(D))$ valued inner product is defined as
\[
_D < f, g > \equiv \sum_{w \in D} D < f, g > (w) e_{D,\alpha}(w)
\] (33)

where
\[
_D < f, g > (w) = \langle f, \pi_w g \rangle.
\]

Here, the scalar product of the type $< f, p >$ used above for $f, p \in L^2(M)$ denotes the following.
\[
< f, p > = \int f(x_1)\overline{p(x_1)}d\mu_{x_1} \quad \text{for} \quad x = (x_1, x_2) \in M \times \hat{M},
\] (34)

where $\mu_{x_1}$ represents the Haar measure on $M$ and $\overline{p(x_1)}$ denotes the complex conjugation of $p(x_1)$. Thus the $S(D)$-valued inner product can be represented as
\[
_D < f, g > = \sum_{w \in D} < f, \pi_w g > e_{D,\alpha}(w).
\] (35)

For $\Phi \in S(D)$ and $f \in S(M)$, then $\pi(\Phi)f \in S(M)$ can be written as
\[
(\pi(\Phi)f)(m) = \sum_{w \in D} \Phi(w)(\pi_w f)(m)
\] (36)

where $m \in M$, $w = (w', w'') \in D \subset M \times \hat{M}$. For $f, g \in S(M)$ and $\Phi \in S(D)$, one can also check the following relation
\[
_D < \Phi f, g > = \Phi * D < f, g >
\] (37)
where $\ast$ denotes the convolution. This means the compatibility of the $S(D)$-valued inner product with the action of $S(D)$ on $S(M)$. Now one can define $D^\perp$, the set of $z$’s in $G$ such that $\pi_z$ commutes with $\pi_w$ for all $w \in D$,

$$D^\perp = \{ z \in G : \alpha(w, z)\pi(z, w) = 1, \ \forall w \in D \}.$$ 

Then the action of $\Omega \in S(D^\perp)$ on $f \in S(M)$ can be defined as,

$$f\Omega = \sum_{z \in D^\perp} (\pi_z^* f)\Omega(z),$$

(38)

and thus the $S(D^\perp)$-valued inner product can be expressed as

$$< f, g >_{D^\perp} = \sum_{z \in D^\perp} e_{D, \alpha}^*(z) < f, g >_{D^\perp}(z)$$

$$= \sum_{z \in D^\perp} e_{D, \alpha}^*(z) <\pi_z g, f >,$$

(39)

where $\ast$ denotes the adjoint operation. From the above definitions, the following relation holds [12].

$$D < f, g > h = f < g, h >_{D^\perp} \text{ for } f, g, h \in S(M).$$

(40)

Furthermore, if $< f, f >_{D^\perp} = 1$, then $D < f, f >$ is a projection operator [12] [4] [5].

The Manin’s quantum theta function $\Theta_D$ [4] [5] was defined via algebra valued inner product up to a constant factor,

$$D < f_T, f_T > \sim \Theta_D,$$

(41)

where $f_T$ used in the construction was a simple Gaussian theta vector

$$f_T = e^{\pi ix_1^T x_1}, \ x_1 \in M,$$

(42)

with $T$ be an $n \times n$ complex valued matrix. Manin required that the quantum theta function $\Theta_D$ defined in this way should satisfy the following condition under translation derived from the map [20]

$$\forall g \in D, \ C_g e_{D, \alpha}(g) x_g^*(\Theta_D) = \Theta_D$$

(43)
where $C_g$ is an appropriately given constant, and $x_g^*$ is a “quantum translation operator” defined as

$$x_g^*(e_{D,\alpha}(h)) = \mathcal{X}(g, h)e_{D,\alpha}(h)$$

with some commuting function $\mathcal{X}(g, h)$ for $g, h \in D$. The requirement (43) can be regarded as the quantum counterpart of the second property of the classical theta function, (2).

In physics language, the theta vector corresponds to a state vector (wavefunction) which can be expressed as a Dirac ket, say $|n>$, and the quantum theta function corresponds to an operator for an observable which in terms of the Dirac bra-ket notation can be represented as $\sum_n a_n |n><n|$ with $a_n \in \mathbb{C}$. In the case of algebra valued inner product, $D < f, f >$ corresponds to $\sum_n a_n |n><n| \not\approx 1$, and $< f, f >_{D\perp}$ corresponds to a case in which $\sum_n a_n <n|n> \approx 1$. Namely, the inner product in the latter case becomes a scalar which is equivalent to an identity operator. Furthermore, as we mentioned above, (43) represents the quantum version of the symmetry of the classical theta function under translation. Thus based on our above discussion in the Dirac’s notation and the symmetry property that we mentioned, we can deduce that the Manin’s quantum theta function constructed via algebra valued inner product is the quantum version of the classical theta function.

IV. Quantum theta functions - extended to holomorphic connections with constants

In this section, we review Manin’s construction of quantum theta function in detail starting from the algebra valued inner product of the Gaussian theta vector, and show that Manin’s approach for quantum theta function also holds for the case of a theta vector obtained from more general holomorphic connections with constants.

As in the classical theta function case, we first introduce an $n$-dimensional complex variable $\underline{x} \in \mathbb{C}^n$ with complex structure $T$ explained in the previous sections as

$$\underline{x} \equiv Tx_1 + x_2$$

(45)
where $x = (x_1, x_2) \in M \times \hat{M}$. Based on the defining concept for quantum theta function (11), Manin defined the quantum theta function $\Theta_D$ as

$$D < f_T, f_T > = \frac{1}{\sqrt{2^n \det(\text{Im} \; T)}} \Theta_D$$

(46)

with $f_T$ given by (12). Using (33) the $S(D)$-valued inner product in (41) can be expressed as

$$D < f_T, f_T > = \sum_{h \in D} < f_T, \pi_h f_T > e_{D, \alpha}(h).$$

(47)

Now, we define $\pi$ of $G$ on $L^2(M)$ as follows.

$$(\pi(y_1, y_2)f)(x_1) = e^{2\pi i x_1^t y_2 + \pi i y_1^t y_2} f(x_1 + y_1), \text{ for } x, y \in G = M \times \hat{M}$$

(48)

Then the cocycle $\alpha(x, y)$ in (31) is given by $\alpha(x, y) = e^{\pi i (x_1 y_2 - y_1 x_2)}$.

In [5], Manin showed that the quantum theta function defined in (46) is given by

$$\Theta_D = \sum_{h \in D} e^{-\frac{\pi}{2} H(g, h)} e_{D, \alpha}(h),$$

(49)

where

$$H(g, h) \equiv g^t (\text{Im} \; T)^{-1} h^*$$

with $h^* = \overline{T h_1 + h_2}$ denoting the complex conjugate of $h$, and satisfies the following functional equation.

$$\forall g \in D, \quad C_g e_{D, \alpha}(g) \; x_g^*(\Theta_D) = \Theta_D$$

(50)

where $C_g$ is defined by

$$C_g = e^{-\frac{\pi}{2} H(g, g)}$$

and the action of “quantum translation operator” $x^*_g$ is given by

$$x^*_g(e_{D, \alpha}(h)) = e^{-\frac{\pi}{2} H(g, h)} e_{D, \alpha}(h).$$

(51)

We now sketch the proof of the above statement. The scalar product inside the summation in (47) can be expressed as

$$< f_T, \pi_h f_T > = \int_{\mathbb{R}^n} d\mu_{x_1} e^{\pi i x_1^t T x_1 - \pi i (x_1 + h_1)^t T (x_1 + h_1) - 2\pi i x_1^t h_2 - \pi i h_1^t h_2}. $$

(52)
Denoting the exponent inside the integral sign as
\[ e^{-\pi(q(x_1)+l_h(x_1)+\tilde{C}_h)} \]
with
\[ q(x_1) = 2x_1^t(\text{Im}T)x_1 \]
\[ l_h(x_1) = 2ix_1^t(Th_1+h_2) \]
\[ \tilde{C}_h = ih_1^t(Th_1+h_2), \]
and using the relation
\[ q(x_1 + \lambda_h) - q(\lambda_h) = q(x_1) + l_h(x_1) \]
with
\[ \lambda_h \equiv \frac{i}{2}(\text{Im}T)^{-1}h^* \]
the integration now becomes
\[ \int_{\mathbb{R}^n} d\mu_{x_1}e^{-\pi(q(x_1)+l_h(x_1)+\tilde{C}_h)} = e^{-\pi(\tilde{C}_h-q(\lambda_h))} \int_{\mathbb{R}^n} d\mu_{x_1}e^{-\pi q(x_1+\lambda_h)} = \frac{1}{\sqrt{\det q}}e^{-\pi(\tilde{C}_h-q(\lambda_h))}. \]

With a straightforward calculation one can check that
\[ \tilde{C}_h - q(\lambda_h) = \frac{1}{2}H(h,h), \]
and with \[ \det q = 2^n \det(\text{Im} T), \] the expression for Manin’s quantum theta function [49] follows.

The functional relation for quantum theta function [50] can be shown by use of the definition of “quantum translation operator” [51] as follows.

\[ C_g e_{D,\alpha}(g) x_g^*(\sum_{h \in D} e^{-\frac{\pi}{2}H(h,h)}e_{D,\alpha}(h)) \]
\[ = e^{-\frac{\pi}{2}H(g,h)}e_{D,\alpha}(g) \sum_{h \in D} e^{-\frac{\pi}{2}H(h,h)-\pi H(g,h)}e_{D,\alpha}(h) \]
\[ = \sum_{h \in D} e^{-\frac{\pi}{2}H(g+h+g)}e_{D,\alpha}(g+h) \]

In the last step, the cocycle condition [52] with \[ \alpha(g,h) = e^{\pi i(g_1^th_2-h_1^t g_2)} = e^{\pi \text{Im}H(g,h)} \] was used. This proves the statement. □
In the rest of this section, we apply the Manin’s approach to a more general theta vector with constant obtained from a holomorphic connection of constant curvature. We do this to provide a further support for Manin’s quantum theta function approach based on the algebra valued inner product and to show that it is a natural choice for quantum extension of the classical theta function.

We begin again with $S(D)$-valued inner product (41) with a more general theta vector $f_{T,c}$ which appeared in [7,16].

$$D < f_{T,c}, f_{T,c} > = \sum_{h \in D} < f_{T,c}, \pi_h f_{T,c} > e_{D,\alpha}(h) \quad (53)$$

where

$$f_{T,c}(x_1) = e^{\pi i x_1^t T x_1 + 2\pi i c^t x_1}, \quad c \in \mathbb{C}^n, \quad x_1 \in M, \quad (54)$$

and $T$ is the complex structure mentioned before. From (34) and (48), the algebra valued inner product (53) can be written as

$$D < f_{T,c}, f_{T,c} > = \sum_{h \in D} \int_{\mathbb{R}^n} d\mu_{x_1} f_{T,c}(x_1) \overline{(\pi_h f_{T,c})(x_1)} e_{D,\alpha}(h)$$

$$\equiv \sum_{h \in D} \int_{\mathbb{R}^n} d\mu_{x_1} e^{-\pi[q(x_1)+l_{h,c}(x_1)+\tilde{C}_{h,c}]} e_{D,\alpha}(h) \quad (55)$$

where $q(x_1)$, $l_{h,c}(x_1)$, $\tilde{C}_{h,c}$ are defined by

$$q(x_1) = 2x_1^t (\text{Im } T)x_1,$$

$$l_{h,c}(x_1) = 2ix_1^t (Th_1 + h_2 - 2i(\text{Im } c)),$$

$$\tilde{C}_{h,c} = ih_1^t (\overline{Th}_1 + h_2 + 2\overline{c}). \quad (56)$$

Denoting

$$\lambda_{h,c} \equiv \frac{i}{2} (\text{Im } T)^{-1}(h^* - 2i(\text{Im } c)),$$

one can check that

$$q(x_1) + l_{h,c}(x_1) = q(x_1 + \lambda_{h,c}) - q(\lambda_{h,c}).$$
Thus, the algebra valued inner product (55) can be written as

$$D < f_{T,c}, f_{T,c} > = \sum_{h \in D} e^{-\pi (\tilde{C}_{h,c} - q(\lambda_{h,c}))} e_{D,\alpha}(h) \int_{\mathbb{R}^n} d\mu_{x_1} e^{-\pi q(x_1 + \lambda_{h,c})}. \quad (57)$$

Since \( \int_{\mathbb{R}^n} d\mu_{x_1} e^{-\pi q(x_1 + \lambda_{h,c})} = 1/\sqrt{\det q} \), the above expression can be rewritten as

$$D < f_{T,c}, f_{T,c} > = \frac{1}{\sqrt{2^n \det(\text{Im} T)}} \sum_{h \in D} e^{-\pi (\tilde{C}_{h,c} - q(\lambda_{h,c}))} e_{D,\alpha}(h) \quad (58)$$

and we define our quantum theta function \( \Theta_{D,c} \) as

$$D < f_{T,c}, f_{T,c} > \equiv \frac{1}{\sqrt{2^n \det(\text{Im} T)}} \Theta_{D,c}. \quad (59)$$

The quantum theta function defined above is evaluated as

$$\Theta_{D,c} = \sum_{h \in D} e^{-\pi (\tilde{C}_{h,c} - q(\lambda_{h,c}))} e_{D,\alpha}(h) = \sum_{h \in D} e^{-\pi \left[ \frac{i}{2} (h^t - 2i(\text{Im} c)^t)(\text{Im} T)^{-1}(h^* - 2i(\text{Im} c)) + 2ih_1^t(\text{Re} c) \right]} e_{D,\alpha}(h). \quad (60)$$

And the above defined quantum theta function \( \Theta_{D,c} \) satisfies the following.

**Theorem:** The quantum theta function \( \Theta_{D,c} \) defined by the following algebra valued inner product

$$D < f_{T,c}, f_{T,c} > \equiv \frac{1}{\sqrt{2^n \det(\text{Im} T)}} \Theta_{D,c} \quad (61)$$

with a theta vector \( f_{T,c} \) below, which is obtained from a holomorphic connection with constant \( c \in \mathbb{C}^n \),

$$f_{T,c}(x_1) = e^{\pi ix_1^t T x_1 + 2\pi i c^t x_1}, \quad (62)$$

satisfies the following identity

$$\forall g \in D, \quad C_{g,c} e_{D,\alpha}(g) x_{g,c}^*(\Theta_{D,c}) = \Theta_{D,c}. \quad (63)$$

Here \( C_{g,c} \) is a constant defined by

$$C_{g,c} \equiv e^{-\frac{\pi}{2} H_c(g,g)}$$

where \( H_c(g,g) \) is given by

$$H_c(g,g) = (g - 2i(\text{Im} c))^t(\text{Im} T)^{-1}(g^* - 2i(\text{Im} c)) + 4ig_1^t(\text{Re} c), \quad (64)$$
and $x_{g,c}^*$ is a “quantum translation operator” defined by

$$x_{g,c}^*(e_{D,\alpha}(h)) \equiv e^{-\pi X(g,h)}e_{D,\alpha}(h) \quad (65)$$

where $X(g,h)$ is given by

$$X(g,h) = g'(\Im T)^{-1}h^* + 2(\Im c)^t(\Im T)^{-1}(\Im c).$$

**Proof.** We first note that from (60) and (61) our quantum theta function $\Theta_{D,c}$ can be expressed as

$$\Theta_{D,c} = \sum_{h \in D} e^{-\pi H_c(h,h)} e_{D,\alpha}(h). \quad (66)$$

Thus the left hand side of the functional relation (63) can be written as

$$C_{g,c} e_{D,\alpha}(g) x_{g,c}^*(\Theta_{D,c}) = e^{-\frac{\pi}{2} H_c(g,g)} e_{D,\alpha}(g) x_{g,c}^*(e_{D,\alpha}(h))$$

$$= \sum_{h \in D} e^{-\frac{\pi}{2} H_c(g,g)} e^{-\frac{\pi}{2} H_c(h,h)} e_{D,\alpha}(g) x_{g,c}^*(e_{D,\alpha}(h))$$

$$= \sum_{h \in D} e^{-\frac{\pi}{2} H_c(g,g)} e^{-\frac{\pi}{2} H_c(h,h)} e^{-\pi X(g,h)} e_{D,\alpha}(g) e_{D,\alpha}(h).$$

Then using the cocycle relation (32)

$$e_{D,\alpha}(g)e_{D,\alpha}(h) = \alpha(g,h)e_{D,\alpha}(g + h) = e^{\pi i \Im(g'(\Im T)^{-1}h^*)} e_{D,\alpha}(g + h),$$

one can check that with a straightforward calculation

$$e^{-\frac{\pi}{2} H_c(g,g)} e^{-\frac{\pi}{2} H_c(h,h)} e^{-\pi X(g,h)} e^{\pi i \Im(g'(\Im T)^{-1}h^*)} = e^{-\frac{\pi}{2} H_c(g + h,g + h)},$$

proving the relation (63). □

The property of quantum theta function (63) represents the translational symmetry of the quantum lattice. This corresponds to the symmetry property (2) of the classical theta function on the complex tori:

$$\Theta(z + \lambda) = C(\lambda)e^{g(\lambda,z)}\Theta(z) \quad \text{for} \quad \lambda \in \Lambda.$$
where Λ is the period lattice for the complex tori. The relation is the same as in the case of
Manin’s construction expressed in (50). The only difference here is that the constant factor
$C_g$ and the action of “quantum translation operator” $x_g^*$ have been changed slightly due to
the constant $c \in \mathbb{C}^n$ appearing in our new theta vector $f_{T,c}$. The changes in these two were
possible due to quantum nature of the quantum theta functions which inherit the mapping
property (29) expressed as a multiplier $L$ in (30). For the multiplier $L$, we have a freedom
to select $c_b$ and $j_b$ in (30). The constant factor $C_g$ and the action of “quantum translation
operator” $x_g^*$ directly corresponds and is related to $c_b$ and $j_b$, respectively.

V. Conclusion

In this paper we explained how Manin’s quantum theta functions emerge naturally from the
state vectors on quantum (noncommutative) tori via algebra valued inner product.

As we discussed in section III, the theta vectors can be regarded as invariant state vec-
tors under parallel transport on the noncommutative tori equipped with complex structures.
However, they are not like the classical theta functions which are the state vectors (holomor-
phic sections of line bundles) over classical tori. This is because the classical theta functions
(complex $n$ dimensional) are equivalent to $kq$ representations (real $2n$ dimensional) which are
transformations of the functions over coordinates (real $n$ dimensional) only. Namely, these
are functions over the phase space (real $2n$ dimensional) consisting of coordinates and their
canonical momenta, while the theta vectors are more or less corresponding to the functions
over coordinates (real $n$ dimensional) only.

Therefore to build a quantum version of classical theta function, we need to build a
function over the quantum phase space (real $2n$ dimensional) via a transformation like $kq$
representation. However, a function over quantum phase space is necessarily an operator
since coordinates and their momenta are not commuting in general. As we discussed in
section III, the algebra valued inner product is a good fit for this purpose, since it transforms
a (commuting) function into an operator. Thus the quantum theta function obtained via
algebra valued inner product from the theta vector (a function over commuting variables)
can be regarded as a quantum version of $kq$ representation which corresponds to the classical theta function.

In conclusion, we can say that the quantum theta function is a quantum version of the classical theta function which is equivalent to the $kq$ representation, while the theta vector corresponds to a wavefunction over commuting coordinates, the pre-transformed function for the $kq$ representation.

Finally, we compare the characteristics of the quantum theta function and the theta vector. The theta vectors can be regarded as invariant state vectors under parallel transport on the noncommutative tori equipped with complex structures, since they are defined to vanish under the action of the holomorphic connection which can be regarded as the generator for parallel transport. While the quantum theta functions can be regarded as observables having translational symmetry on the quantum lattice. Thus it is not surprising that these two are related by algebra valued inner product which one can regard as a quantum version of the transformation for the $kq$ representation.

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