The multicomponent KP hierarchy: differential Fay identities and Lax equations

Lee Peng Teo

Department of Applied Mathematics, Faculty of Engineering, University of Nottingham Malaysia Campus, Jalan Broga, 43500, Semenyih, Selangor Darul Ehsan, Malaysia

E-mail: LeePeng.Teo@nottingham.edu.my

Received 21 January 2011, in final form 12 April 2011
Published 3 May 2011
Online at stacks.iop.org/JPhysA/44/225201

Abstract
In this paper, we show that four sets of differential Fay identities of an $N$-component KP hierarchy derived from the bilinear relation satisfied by the tau function of the hierarchy are sufficient to derive the auxiliary linear equations for the wavefunctions. From this, we derive the Lax representation for the $N$-component KP hierarchy, which are equations satisfied by some pseudo-differential operators with matrix coefficients. Besides the Lax equations with respect to the time variables proposed in Date et al (1981 J. Phys. Soc. Japan 50 3806–12), we also obtain a set of equations relating different charge sectors, which can be considered as a generalization of the modified KP hierarchy proposed in Takebe (2002 Lett. Math. Phys. 59 157–72).

PACS number: 02.30.Ik

1. Introduction

The KP hierarchy [1] is one of the most extensively studied integrable hierarchies. It arises in many different fields of mathematics and physics such as enumerative algebraic geometry, hydrodynamics and string theory. It is an infinite set of coupled partial differential equations describing the evolution of infinitely many functions $u_1, u_2, \ldots$ with respect to the time variables $t_1, t_2, \ldots$. In terms of the pseudo-differential operator $L = \partial + u_1 \partial^{-1} + u_2 \partial^{-2} + \cdots$, where $\partial = \partial_{t_1}$, the partial differential equations can be expressed as

$$\frac{\partial L}{\partial t_n} = [B_n, L], \quad B_n := (L^n)_+, \quad n = 1, 2, \ldots,$$

where $(L^n)_+$ means the differential part of the operator $L^n$. One of the biggest breakthroughs in the study of the KP hierarchy is the group theoretical description of the solutions of the KP hierarchy [1], which is closely related to the infinite-dimensional Grassmann manifolds.
[5, 6]. To every solution of the KP hierarchy, there exists a tau function \( \tau(t) \) which satisfies the bilinear relation

\[
\oint \tau(t - [z^{-1}]) e^{\sum_{n=1}^{\infty} (t_n - [z^{-1}]) t_n} \tau(t' + [z^{-1}]) \, dz = 0.
\]  

(1.1)

Here, \( t = (t_1, t_2, \ldots) \) and \([z^{-1}] = (z^{-1}, z^{-2}/2, z^{-3}/3, \ldots)\). Such a tau function can be represented using the charge zero sector of a free fermion system.

In [2], Date, Jimbo, Kashiwara and Miwa extended their work [1] to the multicomponent KP hierarchy proposed by Sato in a lecture. For an \( N \)-component KP hierarchy, there are \( N \) infinite families of time variables \( t_{\alpha n}, \alpha = 1, \ldots, N, n = 1, 2, \ldots \). The coefficients \( u_1, u_2, \ldots \) of the Lax operator \( L = \partial + u_1 \partial^{-1} + u_2 \partial^{-2} + \cdots \) are \( N \times N \) matrices. The operator \( \partial \) now is equal to \( \partial_{t_1} + \cdots + \partial_{t_n} \). There are another \( N \) pseudo-differential operators \( R_1, \ldots, R_N \) of the form

\[
R_\alpha = E_\alpha + u_{\alpha 1} \partial^{-1} + u_{\alpha 2} \partial^{-2} + \cdots,
\]

where \( E_\alpha = (\delta_{\alpha j} \delta_{\beta i})_{j=1}^{N} \), and \( u_{\alpha 1}, u_{\alpha 2}, \ldots \) are \( N \times N \) matrices. The operators \( L, R_1, \ldots, R_N \) satisfy the following conditions:

\[
LR_\alpha = R_\alpha L, \quad R_\alpha R_\beta = \delta_{\alpha \beta} R_\alpha, \quad \sum_{\alpha = 1}^{N} R_\alpha = 1.
\]

The Lax equations are

\[
\frac{\partial L}{\partial t_{\alpha n}} = [B_{\alpha n}, L], \quad \frac{\partial R_\beta}{\partial t_{\alpha n}} = [B_{\beta n}, L], \quad B_{\alpha n} := (L^n R_\alpha)_+.
\]

The tau function of an \( N \)-component KP hierarchy can also be expressed in terms of fermions, but \( N \) components of free fermions are required. Moreover, one has to go beyond the charge zero sector. More precisely, let \( s_1, \ldots, s_N \) be the charge of each component of the free fermions. Then for a fixed \( s = (s_1, \ldots, s_N)^T \) with \( s_1 + \cdots + s_N = 0 \), the tau function of an \( N \)-component KP hierarchy can be written as \( \tau(s, t) \), where \( t \) is the collective notation for all the time variables. \( \tau(s, t) \) satisfies the following bilinear relations:

\[
\sum_{\gamma = 1}^{N} \epsilon_{\alpha \gamma} e_{\beta \gamma} e_{\gamma s} \oint dz z^{s_\gamma - s_\gamma'} \tau(s' + e_{\gamma}, t' + [z^{-1}]_\gamma) e^{(t_{\gamma} - t_{\gamma'})z} = 0
\]

(1.2)

for any \( 1 \leq \alpha, \beta \leq N \). Here, \( e_\alpha \) is the column vector whose \( \alpha \)th component is 1 and other components are zero, and \( e_{\alpha \gamma}(s) \) is a sign function. One problem that has not been explored in connection with the multicomponent KP hierarchy is the dependence of the operators \( L, R_1, \ldots, R_N \) on the charge variables \( s \). This will be considered in this paper.

In the seminal paper [8], Takasaki and Takebe derived the differential Fay identity for the KP hierarchy from the bilinear relation (1.1). It was shown that the differential Fay identity implies linear equations of the form

\[
\partial_t \Psi = B_n(\partial) \Psi,
\]

(1.3)

where \( \Psi \) is the wavefunction of the KP hierarchy, and \( B_n \) is a differential operator in \( \partial = \partial_t \) of order \( n \). From this, it was concluded that the differential Fay identity is equivalent to the KP hierarchy. In [9], Takasaki derived the differential Fay identities of BKP and DKP hierarchies and obtained the auxiliary linear equations of these hierarchies from their respective differential Fay identities. This further illustrates the importance of differential Fay identities as a set of identities that encode all the information of the integrable hierarchies. Differential
Fay identities also play important roles in studying the dispersionless limits of integrable hierarchies. In [7], Takasaki and Takebe derived four sets of differential Fay identities for the multicomponent KP hierarchy from the bilinear relation (1.2) and showed that their dispersionless limits give rise to the universal Whitham hierarchy. In fact, they considered an \((N+1)\)-component KP hierarchy, where one of the components is more special than the other components and is denoted by the zeroth component. Some auxiliary linear equations for \((N+1)\)-component of the matrix wavefunction were derived from the differential Fay identities but they do not directly lead to the Lax representation of the multicomponent KP hierarchy. In fact, for an \((N+1)\)-component KP hierarchy, the wavefunction is an \((N+1)\times(N+1)\)-matrix valued function, but only one row of this matrix, the zeroth row, was considered in [7]. The auxiliary linear equations derived in [7] only lead to linear evolution equations of the form

\[ \partial_{\alpha\nu} \Psi_0 = B_{\alpha\nu}(\partial_{\alpha\nu})\Psi_0, \]

where \(\Psi_0\) is the zeroth row of the matrix wavefunction, and \(B_{\alpha\nu}(\partial_{\alpha\nu})\) is a differential operator of order \(n\) in \(\partial_{\alpha\nu}\). One would expect that for a multicomponent KP hierarchy, the \(B_{\alpha\nu}\) is a differential operator of \(\partial = \partial_{t_1} + \cdots + \partial_{t_N}\) rather than of \(\partial_{\alpha\nu}\). In this paper, we show that without singling out a special component, one can derive the auxiliary linear equations for the matrix wavefunction of the multicomponent KP hierarchy from the differential Fay identities which have the expected form

\[ \partial_{\alpha\nu} \Psi = B_{\alpha\nu}(\partial)\Psi. \]

Besides these linear differential equations with respect to the time variables \(\ell_{\alpha\nu}\), we also obtain linear equations with respect to the charge variables. The latter is what we need for exploring the variations of the operators \(L, R_1, \ldots, R_N\) with respect to the charge variables \(s_1, \ldots, s_N\).

2. \(N\)-component KP hierarchy

Let \(N\) be a positive integer. The time variables of the \(N\)-component KP hierarchy are \(N\)-sequences of continuous variables \([t_{\nu j}]_{j=1}^{\infty}, [t_{\nu j}]_{j=1}^{\infty}\) collectively denoted by

\[ t = (t_1, t_2, \ldots, t_N)^T, \quad t_{\alpha\nu} = (t_{\alpha 1}, t_{\alpha 2}, \ldots), \quad \alpha = 1, 2, \ldots, N. \]

There are \(N\) additional discrete charge variables \(s_1, s_2, \ldots, s_N \in \mathbb{Z}\), collectively written as

\[ s = (s_1, \ldots, s_N)^T, \]

which satisfy the condition

\[ \sum_{\alpha=1}^{N} s_{\alpha} = 0. \]

An \(N\)-component KP hierarchy can be defined by the bilinear identities satisfied by its tau function \(\tau(s, t)\) [2–4, 7]:

\[ \sum_{y=1}^{N} \epsilon_{\alpha\nu}(y) \epsilon_{\beta\gamma}(y') \int d\zeta \zeta^{e_{\nu\gamma} - e_{\rho\nu}} \zeta^{e_{\rho\nu} - e_{\gamma\nu}} e^{(t_{\nu\gamma} - t_{\nu\rho} - \zeta)} \tau(s' - e_{\rho\nu}, t - [z^{-1}]_{\gamma}) \tau(s' - e_{\rho\nu}, t' + [z^{-1}]_{\gamma}) = 0 \quad (2.1) \]

for any \(1 \leq \alpha, \beta \leq N\). Here, \(e_{\alpha\nu}\) is the \(N \times 1\) column vector whose \(\alpha\)th component is 1 and other components are zero,

\[ \epsilon_{\alpha\nu}(s) = \begin{cases} (-1)^{s_{\alpha 1} + \cdots + s_{\alpha \nu}}, & \text{if } \alpha < \beta \\ 1, & \text{if } \alpha = \beta \\ (-1)^{s_{\alpha 1} + \cdots + s_{\alpha \nu}}, & \text{if } \alpha > \beta \end{cases} \quad (2.2) \]

for any \(1 \leq \alpha, \beta \leq N\). Here, \(e_{\alpha\nu}\) is the \(N \times 1\) column vector whose \(\alpha\)th component is 1 and other components are zero,
\[ \xi(t_\alpha, z) = \sum_{j=1}^{\infty} t_\alpha^j z^j \]

and
\[ (t - [z^{-1}]_\alpha)_{\alpha_j} = t_{\alpha_j} - \delta_{\alpha \gamma} \frac{z^{-j}}{j}. \]

As pointed out in [7], it is sufficient to consider the case where \( \alpha = \beta \), which gives
\[
\oint dz_\alpha^{\nu} e^{\xi(t_\alpha - t_\nu, z)} \tau(s, t - [z^{-1}]_\alpha) \tau(s', t' + [z^{-1}]_\alpha) + \sum_{1 \leq j \leq N, \gamma \neq \alpha} \epsilon_{\alpha \gamma}(s) \epsilon_{\alpha \gamma}(s') \\
\times \oint dz_\gamma^{\nu} e^{\xi(t_\gamma - t_\nu, z)} \tau(s + e_\alpha - e_\beta, t - [z^{-1}]_\gamma) \tau(s' - e_\alpha + e_\beta, t' + [z^{-1}]_\gamma) = 0.
\]

The general case (2.1) where \( \alpha \neq \beta \) can be recovered from (2.3) by replacing \( s' \) with \( s' + e_\alpha - e_\beta \) and using the identities
\begin{enumerate}
\item \( \epsilon_{\alpha \beta}(s + e_\alpha - e_\beta) = \epsilon_{\beta \alpha}(s) \),
\item \( \epsilon_{\alpha \gamma}(s + e_\alpha - e_\beta) = \epsilon_{\beta \gamma}(s) \epsilon_{\beta \alpha}(s). \)
\end{enumerate}

Here, \( 1 \leq \alpha, \beta, \gamma \leq N \) are distinct. These identities can be easily verified using (2.2).

The wavefunction \( \Psi(s, t, z) \) and the adjoint wavefunction \( \Psi^*(s, t, z) \) of the N-component KP hierarchy are \( N \times N \) matrix-valued functions with the \((\alpha, \beta)\) components defined respectively by
\[
\Psi_{\alpha \beta}(s, t, z) = \epsilon_{\alpha \beta}(s) \frac{\tau(s + e_\alpha - e_\beta, t - [z^{-1}]_\beta) e^{\xi(s, t, z)}}{\tau(s, t)}
\]
\[
\Psi_{\alpha \beta}^*(s, t, z) = \epsilon_{\alpha \beta}(s) \frac{\tau(s - e_\alpha + e_\beta, t + [z^{-1}]_\gamma) e^{-\xi(s, t, z)}}{\tau(s, t)}.
\]

The bilinear identity (2.1) is then equivalent to
\[
\oint dz \Psi(s, t, z) \Psi^*(s', t', z)^T = 0.
\]

3. Differential and difference Fay identities

3.1. Differential Fay identities

The differential Fay identities of a multicomponent KP hierarchy can be divided into four sets:

DFI: for any \( \alpha \),
\[
\partial_{\alpha \nu} \log \tau(s, t - [\mu^{-1}]_\nu) - \partial_{\nu \nu} \log \tau(s, t - [v^{-1}]_\nu)
\]
\[
= \frac{\tau(s, t) \tau(s, t - [\mu^{-1}]_\nu - [v^{-1}]_\nu)}{\tau(s, t - [\mu^{-1}]_\nu) \tau(s, t - [v^{-1}]_\nu)} - 1.
\]

DFII: for any distinct \( \alpha \) and \( \beta \),
\[
\partial_{\alpha \nu} \log \tau(s, t - [\mu^{-1}]_\nu) - \partial_{\nu \nu} \log \tau(s, t - [v^{-1}]_\nu)
\]
\[
= -\frac{\tau(s + e_\alpha - e_\beta, t + [\mu^{-1}]_\nu - [v^{-1}]_\nu) \tau(s - e_\alpha + e_\beta, t - [\mu^{-1}]_\nu - [v^{-1}]_\nu)}{\tau(s, t - [\mu^{-1}]_\nu) \tau(s, t - [v^{-1}]_\nu)}.
\]
DFII: for any distinct $\alpha$ and $\beta$,
\[
\partial_{s\alpha} \log \tau(s + e_\alpha - e_\beta, t + [v^{-1}]_\beta) - \partial_{s\beta} \log \tau(s, t - [\mu^{-1}]_\alpha) = \mu - \mu \frac{\tau(s, t)\tau(s + e_\alpha - e_\beta, t - [\mu^{-1}]_\alpha) - [v^{-1}]_\beta)}{\tau(s, t - [\mu^{-1}]_\alpha)\tau(s + e_\alpha - e_\beta, t - [v^{-1}]_\beta)}.
\]
(3.3)

DFIV: for any distinct $\alpha$, $\beta$ and $\kappa$,
\[
\partial_{s\alpha} \log \tau(s, t - [\mu^{-1}]_\alpha) - \partial_{s\kappa} \log \tau(s + e_\alpha - e_\beta, t - [v^{-1}]_\beta) = - \frac{\epsilon_{\beta\alpha}(s)\epsilon_{\beta\kappa}(s)\tau(s + e_\alpha - e_\kappa, t)\tau(s - e_\beta + e_\kappa, t - [\mu^{-1}]_\alpha) + \epsilon_{\alpha\beta}(s)\epsilon_{\beta\kappa}(s)\tau(s + e_\beta - e_\kappa, t - [\mu^{-1}]_\alpha)}{\epsilon_{\beta\alpha}(s)\tau(s + e_\alpha - e_\beta, t - [v^{-1}]_\beta)\tau(s, t - [\mu^{-1}]_\alpha)}.
\]
(3.4)

They are generalizations of identities (61), (62), (63) and (64) in [7]. For completeness, we give their derivations in appendix A.1.

3.2. Difference Fay identities

The difference Fay identities of a multicomponent KP hierarchy can be divided into two sets:

CFI: for any distinct $\alpha$, $\beta$, $\lambda$ and $\kappa$,
\[
\epsilon_{\beta\alpha}(s + e_\lambda - e_\kappa)\tau(s, t)\tau(s + e_\lambda - e_\beta - e_\kappa, t - [\mu^{-1}]_\kappa) + \epsilon_{\alpha\beta}(s)\tau(s + e_\lambda - e_\beta, t)\tau(s + e_\beta - e_\lambda, t - [\mu^{-1}]_\lambda) + \epsilon_{\alpha\kappa}(s)\epsilon_{\beta\kappa}(s)\tau(s + e_\beta - e_\kappa, t - [\mu^{-1}]_\kappa)\tau(s, t - [\mu^{-1}]_\alpha) = 0.
\]
(3.5)

CFII: for distinct $\alpha$, $\beta$ and $\lambda$,
\[
\epsilon_{\beta\alpha}(s)\tau(s, t)\tau(s - e_\beta + e_\alpha, t - [\mu^{-1}]_\alpha) + \epsilon_{\alpha\beta}(s)\tau(s + e_\alpha - e_\beta, t)\tau(s, t - [\mu^{-1}]_\alpha) + \epsilon_{\alpha\lambda}(s)\epsilon_{\beta\lambda}(s)\tau(s + e_\alpha - e_\lambda, t - [\mu^{-1}]_\lambda)\tau(s, t - [\mu^{-1}]_\alpha) = 0.
\]
(3.6)

The derivations of these identities from the bilinear identity (2.1) are given in appendix A.2.

Although these difference Fay identities are very different from the differential Fay identities, it can be shown that (see appendix A.3) they can be derived from the differential Fay identities DFII, DFIII and DFIV. Therefore, they do not contain any new information about the multicomponent KP hierarchy. Nevertheless, they are useful for establishing the auxiliary linear equations in the $s$ sector.

4. Auxiliary linear equations and Lax equations

In this section, we consider the auxiliary linear equations for the wavefunction $\Psi(s, t, z)$ (2.4).

Define the differential operator
\[
D_\alpha(z) = \sum_{j=1}^{\infty} \frac{z^{-j}}{j} \partial_{t\alpha}^j.
\]

First, we have

**Proposition 4.1.** For $1 \leq \alpha \leq N$, the wavefunction $\Psi(s, t, z)$ satisfies the following linear equation:
\[
(1 - e^{-D_\alpha(z)})\Psi(s, t, z) = \mathcal{D}(s, t, \partial, \lambda)\Psi(s, t, z)
= (\mathcal{B}(s, t, \partial, \lambda)E_\alpha + E_\alpha\mathcal{C}(s, t, \lambda))\Psi(s, t, z),
\]
(4.1)
where

\[ \mathcal{B}_{\beta\kappa}(s, t, \partial, \lambda) = \begin{cases} -\lambda^{-1}(\partial_t \log \Psi_{\kappa\kappa}(s, t, \lambda) - \partial), & \text{if } \beta = \kappa \\ -\epsilon_{\beta\kappa}(s) \frac{\tau(s, t)}{\tau(s + e_{\kappa} - e_{\beta}, t)} \partial_{t\lambda} \log \Psi_{\kappa\kappa}(s, t, \lambda), & \text{if } \beta \neq \kappa \end{cases} \]

\[ \mathcal{C}_{\beta\kappa}(s, t, \lambda) = \begin{cases} 0, & \text{if } \beta = \kappa \\ -\lambda^{-1} \epsilon_{\beta\kappa}(s) \frac{\tau(s + e_{\kappa} - e_{\beta}, t)}{\tau(s, t)}, & \text{if } \beta \neq \kappa. \end{cases} \]

Here, \( E_{\alpha} \) is the \( N \times N \) matrix whose \((\alpha, \alpha)\) component is 1 and other components are 0, and \( \partial \) is the operator

\[ \partial = \sum_{\kappa=1}^{N} \partial_{t\kappa}. \]

This proposition can be proved using the differential Fay identities. The proof is quite tedious and we leave it to appendix B.1.

By the definition of \( \Psi_{\alpha\beta}(s, t, z) \), we see that it can be written as

\[ \Psi_{\alpha\beta}(s, t, z) = \left( \delta_{\alpha\beta} + \sum_{j=1}^{\infty} (w_j)_{\alpha\beta}(s, t)z^{-j} \right) e^{\xi(t, z)} \]

\[ = W_{\alpha\beta}(s, t, \partial) e^{\xi(t, z)} \]

\[ = \hat{W}_{\alpha\beta}(s, t, \partial) \partial^\xi e^{\xi(t, z)}, \]

where

\[ W_{\alpha\beta}(s, t, \partial) = \left( \delta_{\alpha\beta} + \sum_{j=1}^{\infty} (w_j)_{\alpha\beta}(s, t)\partial^{-j} \right) \]

\[ = \hat{W}_{\alpha\beta}(s, t, \partial). \]

Now since

\[ \exp \left( \sum_{\alpha=1}^{N} E_{\alpha} \xi(t, z) \right) = \prod_{\alpha=1}^{N} \exp \left( E_{\alpha} \xi(t, z) \right) = \prod_{\alpha=1}^{N} \left\{ 1 + E_{\alpha} \left( \frac{\xi(t, z)}{1!} + \frac{\xi(t, z)^2}{2!} + \cdots \right) \right\} \]

\[ = 1 + \sum_{\alpha=1}^{N} E_{\alpha} \left( \frac{\xi(t, z)}{1!} + \frac{\xi(t, z)^2}{2!} + \cdots \right) \]

\[ = \sum_{\alpha=1}^{N} E_{\alpha} e^{\xi(t, z)}. \]

Therefore, \( 4.2 \) can be written in the matrix form

\[ \Psi(s, t, z) = W(s, t, \partial) \exp \left( \sum_{\alpha=1}^{N} E_{\alpha} \xi(t, z) \right) \]

\[ = \hat{W}(s, t, \partial) \begin{pmatrix} \partial^\xi & 0 & \cdots & 0 \\ 0 & \partial^\xi & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \partial^\xi \end{pmatrix} \exp \left( \sum_{\alpha=1}^{N} E_{\alpha} \xi(t, z) \right). \]

Using proposition 4.1, we can find the evolution of the wavefunction \( \Psi(s, t, z) \) with respect to the time variable.
Proposition 4.2. [Linear equations in the \( t \) sector] The wavefunction \( \Psi(s, t, z) \) satisfies the following linear equations:
\[
\frac{\partial \Psi(s, t, z)}{\partial t_{aj}} = B_{aj}(s, t, \partial) \Psi(s, t, z), \quad 1 \leq \alpha \leq N, \ j \in \mathbb{N},
\]  
(4.4)

where
\[
B_{aj}(s, t, \partial) = (W(s, t, \partial) E_{\alpha} \partial^{j} W(s, t, \partial)^{-1})_{+} = (\hat{W}(s, t, \partial) E_{\alpha} \partial^{j} \hat{W}(s, t, \partial)^{-1})_{+}.
\]  
(4.5)

Moreover,
\[
\left( \frac{\partial \hat{W}(s, t, \partial)}{\partial t_{ij}} \right) \hat{W}(s, t, \partial)^{-1})_{-} = - (\hat{W}(s, t, \partial) E_{\alpha} \partial^{j} \hat{W}(s, t, \partial)^{-1})_{-}.
\]  
(4.6)

The proof of this proposition is given in appendix B.2. Note that the equations in (4.4) are the natural generalizations of the auxiliary linear equation (1.3) for the KP hierarchy.

For the evolution of the wavefunction \( \Psi(s, t, z) \) with respect to the \( s \) variable, we have

Proposition 4.3. [Linear equations in the \( s \) sector] The wavefunction \( \Psi(s, t, z) \) satisfies the following linear equations: for any distinct \( \alpha \) and \( \beta \),
\[
\Psi(s + e_{\alpha} - e_{\beta}, t, z) = P_{\alpha, \beta}(s, t, \partial) \Psi(s, t, z) = (E_{\alpha} \partial + \Theta(s, t)) \Psi(s, t, z),
\]  
(4.7)

where
\[
\Theta(s, t) = \delta(s + e_{\alpha} - e_{\beta}, t) E_{\alpha} - E_{\alpha} \delta(s, t) + \sum_{1 \leq \gamma \leq N, \ \gamma \neq \alpha, \beta} E_{\gamma},
\]
\[
\delta_{\beta, \alpha}(s, t) = \begin{cases} 
- \partial_{\alpha} \log \tau(s, t), & \text{if } \lambda = \kappa \\
\epsilon_{\kappa}(s) \frac{\tau(s + e_{\kappa} - e_{\alpha}, t)}{\tau(s, t)}, & \text{if } \lambda \neq \kappa.
\end{cases}
\]

The proof of this proposition uses the difference Fay identities. It is also quite tedious and we leave it to appendix B.3.

In propositions 4.2 and 4.3, we have proved the following auxiliary linear equations:
\[
\frac{\partial \Psi(s, t, z)}{\partial t_{aj}} = B_{aj}(s, t, \partial) \Psi(s, t, z), \quad 1 \leq \alpha \leq N, \ j \in \mathbb{N},
\]
(4.8)

where \( B_{aj}(s, t, \partial) \) is a differential operator of order \( j \) in \( \partial \), and \( P_{\alpha, \beta}(s, t, \partial) \) is a first-order differential operator in \( \partial \). Define the \( N \times N \) matrix operators \( L \) and \( R_{\alpha}, \alpha = 1, \ldots, N \), by
\[
L(s, t, \partial) = \hat{W}(s, t, \partial) \partial \hat{W}(s, t, \partial)^{-1} = W(s, t, \partial) \partial W(s, t, \partial)^{-1} = \partial + \sum_{n=1}^{\infty} u_{n}(s, t) \partial^{-n},
\]
\[
R_{\alpha}(s, t, \partial) = \hat{W}(s, t, \partial) E_{\alpha} \hat{W}(s, t, \partial)^{-1} = W(s, t, \partial) E_{\alpha} W(s, t, \partial)^{-1}
\]
\[
= E_{\alpha} + \sum_{n=1}^{\infty} u_{an}(s, t) \partial^{-n}.
\]

Then it is straightforward to verify that
\[
LR_{\alpha} = R_{\alpha} L, \quad R_{\alpha} R_{\beta} = \delta_{\alpha \beta} R_{\alpha}, \quad \sum_{\alpha=1}^{N} R_{\alpha} = 1,
\]
$B_{\alpha j}(s, t, \partial) = (\hat{W}(s, t, \partial) E_{\alpha} \partial^j \hat{W}(s, t, \partial)^{-1})_+ = (L^j R_{\alpha})_+,$

and (4.3), (4.6) and (4.8) imply that

\[
\frac{\partial W(s, t, \partial)}{\partial t_n} W(s, t, \partial)^{-1} = -(W(s, t, \partial) \partial^l E_{\alpha} W(s, t, \partial)^{-1})_-, \\
W(s + e_{\alpha} - e_{\beta}, t, \partial) = P_{\alpha, \beta}(s, t, \partial) W(s, t, \partial).
\]

Therefore, we obtain the following equations:

\[
\frac{\partial L(s, t, \partial)}{\partial t_{ij}} = [B_{\nu j}(s, t, \partial), L(s, t, \partial)], \\
\frac{\partial R_{\beta}(s, t, \partial)}{\partial t_{ij}} = [B_{\nu j}(s, t, \partial), R_{\beta}(s, t, \partial)], \\
L(s + e_{\alpha} - e_{\beta}, t, \partial) P_{\alpha, \beta}(s, t, \partial) = P_{\alpha, \beta}(s, t, \partial) L(s, t, \partial), \\
R_{\nu j}(s + e_{\alpha} - e_{\beta}, t, \partial) P_{\alpha, \beta}(s, t, \partial) = P_{\alpha, \beta}(s, t, \partial) R_{\nu j}(s, t, \partial), \\
\frac{\partial R_{\beta}(s, t, \partial)}{\partial t_{j\nu}} = B_{\nu j}(s + e_{\alpha} - e_{\beta}, t, \partial) P_{\alpha, \beta}(s, t, \partial) - P_{\alpha, \beta}(s, t, \partial) B_{\nu j}(s, t, \partial).
\]

These are the Lax equations of the multicomponent KP hierarchy.

For a fixed $s$, the first two equations of (4.9) are the Lax equations of the multicomponent KP hierarchy proposed in [2]. The other three equations in (4.9) which determine the variations of $L$, $R_1, \ldots, R_N$ with respect to the charge variable $s$ are analogous to those proposed in the work on the modified KP hierarchy [10]. Therefore, the bi-linear relation formulation of the multicomponent KP hierarchy (2.1) contains more information than the Lax formulation proposed in [2]. It is essentially a multicomponent modified KP hierarchy.

Acknowledgments

We are grateful to K Takasaki and T Takebe for the useful discussion and helpful comments.

Appendix A. Derivations of the differential and difference Fay identities

A.1. Derivations of the differential Fay identities

Here, we give the derivations of the four sets of differential Fay identities in Section 3.1.

For DFI, differentiate (2.3) with respect to $t_{\nu j}$ and set $s' = s$, $t' = t - [\mu^{-1}]_0 - [v^{-1}]_0$, we have

\[
\int dz \frac{1}{1 - \frac{1}{\mu} - \frac{1}{v}} [\tau(s, t - [z^{-1}]_0) + \partial_{u_j} \tau(s, t - [z^{-1}]_0)] \\
\quad \times \tau(s, t - [\mu^{-1}]_0 - [v^{-1}]_0 + [z^{-1}]_0) = 0.
\]

Computing the residue, we find that

\[
\mu \tau(s, t - [\mu^{-1}]_0) + \partial_{u_j} \tau(s, t - [\mu^{-1}]_0) \tau(s, t - [v^{-1}]_0) - \mu \tau(s, t) \tau(s, t - [\mu^{-1}]_0) \\
\quad - [v^{-1}]_0 \tau(s, t - [v^{-1}]_0) - \mu \tau(s, t) \tau(s, t - [v^{-1}]_0) \\
\quad + v \tau(s, t) \tau(s, t - [\mu^{-1}]_0 - [v^{-1}]_0) = 0,
\]

which gives (3.1) after some rearrangement.
For DFII, differentiate (2.3) with respect to \( t_{\beta 1} \) and set \( s' = s, t' = t - [\mu^{-1}]_\alpha - [v^{-1}]_\beta \), we have
\[
\frac{d}{dt} \frac{1}{1 - \frac{1}{v}} \frac{1}{1 - \frac{1}{v}} \partial_{\gamma s_{\nu}} \tau(s, t - [z^{-1}]_\alpha) \tau(s, t - [\mu^{-1}]_\alpha - [v^{-1}]_\beta) + \frac{d}{dz} \frac{1}{1 - \frac{1}{v}} \partial_{\gamma s_{\nu}} \tau(s + e_\alpha - e_\beta, t - [z^{-1}]_\beta) + \frac{d}{dz} \frac{1}{1 - \frac{1}{v}} \partial_{\gamma s_{\nu}} \tau(s + e_\alpha - e_\beta, t - [z^{-1}]_\beta)) = 0.
\]
This gives
\[
\frac{d}{dt} \frac{1}{1 - \frac{1}{v}} \partial_{\gamma s_{\nu}} \tau(s, t - [\mu^{-1}]_\alpha) \tau(s, t - [v^{-1}]_\beta) - \frac{d}{dz} \frac{1}{1 - \frac{1}{v}} \partial_{\gamma s_{\nu}} \tau(s, t - [\mu^{-1}]_\alpha - [v^{-1}]_\beta) + \frac{d}{dz} \frac{1}{1 - \frac{1}{v}} \partial_{\gamma s_{\nu}} \tau(s, t - [\mu^{-1}]_\alpha - [v^{-1}]_\beta) = 0,
\]
which is equivalent to (3.2).

For DFII, differentiate (2.1) with respect to \( t_{\alpha 1} \) and set \( s' = s, t' = t - [\mu^{-1}]_\alpha - [v^{-1}]_\beta \), we have
\[
\epsilon_{\beta \alpha}(s) \int \frac{d}{dz} \frac{1}{1 - \frac{1}{v}} \partial_{\gamma s_{\nu}} \tau(s, t - [z^{-1}]_\alpha) \tau(s, t - [\mu^{-1}]_\alpha - [v^{-1}]_\beta) + \frac{d}{dz} \frac{1}{1 - \frac{1}{v}} \partial_{\gamma s_{\nu}} \tau(s + e_\alpha - e_\beta, t - [z^{-1}]_\beta) + \frac{d}{dz} \frac{1}{1 - \frac{1}{v}} \partial_{\gamma s_{\nu}} \tau(s + e_\alpha - e_\beta, t - [z^{-1}]_\beta)) = 0.
\]
Since
\[
\epsilon_{\beta \alpha}(s) = - \epsilon_{\alpha \beta}(s),
\]
this gives
\[
[(\mu \tau(s, t - [\mu^{-1}]_\alpha)) + \partial_{\gamma s_{\nu}} \tau(s, t - [\mu^{-1}]_\alpha)) \tau(s + e_\alpha - e_\beta, t - [v^{-1}]_\beta)] - [\mu \tau(s, t) \tau(s + e_\alpha - e_\beta, t - [\mu^{-1}]_\alpha - [v^{-1}]_\beta)]
\]
which is equivalent to (3.3).

For DFIV, differentiate (2.1) with respect to \( t_{\alpha 1} \) and set \( s' = s, t' = t - [\mu^{-1}]_\alpha - [v^{-1}]_\beta \), we have
\[
\epsilon_{\beta \alpha}(s) \int \frac{d}{dz} \frac{1}{1 - \frac{1}{v}} \partial_{\gamma s_{\nu}} \tau(s, t - [z^{-1}]_\alpha) \tau(s + e_\alpha - e_\beta, t - [\mu^{-1}]_\alpha - [v^{-1}]_\beta) + \frac{d}{dz} \frac{1}{1 - \frac{1}{v}} \partial_{\gamma s_{\nu}} \tau(s + e_\alpha - e_\beta, t - [z^{-1}]_\beta) + \frac{d}{dz} \frac{1}{1 - \frac{1}{v}} \partial_{\gamma s_{\nu}} \tau(s + e_\alpha - e_\beta, t - [z^{-1}]_\beta)) = 0.
\]
This gives
\[
\partial_{\gamma s_{\nu}} \tau(s, t - [\mu^{-1}]_\alpha) \tau(s + e_\alpha - e_\beta, t - [v^{-1}]_\beta)
\]
which is equivalent to (3.4).
A.2. Derivations of the difference Fay identities from the bilinear relation

Here, we use the bilinear relation (2.1) to derive the two sets of difference Fay identities in Section 3.2.

For CFI, set $s' = s + e_\alpha - e_\kappa$ and $t' = t - [\mu^{-1}]_\kappa$ in the bilinear identity (2.1). We have

\[
\epsilon_{\beta\alpha}(s + e_\alpha - e_\kappa) \oint dz^{-1}\tau(s, t - [z^{-1}]_\alpha)\tau(s + e_\alpha, t - [\mu^{-1}]_\kappa) + \epsilon_{\alpha\beta}(s) \oint dz^{-1}\tau(s + e_\alpha - e_\beta, t - [z^{-1}]_\beta)\tau(s + e_\alpha, t - [\mu^{-1}]_\alpha)
\]

\[
+ [z^{-1}]_\beta + \epsilon_{\alpha\beta}(s) \epsilon_{\beta\kappa}(s + e_\alpha - e_\kappa) \oint dz^{-1} \frac{1}{1 - \frac{z}{\mu}} \tau(s + e_\alpha - e_\kappa, t)
\]

\[
- [z^{-1}]_\kappa \tau(s + e_\kappa - e_\beta, t - [\mu^{-1}]_\kappa) + [z^{-1}]_\alpha \tau(s + e_\alpha - e_\beta, t - [\mu^{-1}]_\alpha) = 0,
\]

which gives (3.5). CFI is proved in the same way but by taking $\lambda = \kappa$.

A.3. Derivations of the difference Fay identities from the differential Fay identities

Here, we showed that the difference Fay identities can be derived from the differential Fay identities.

Taking $\nu \to \infty$ in (3.2), (3.3) and (3.4), we find that

\[
\mu(\partial_{s_\alpha} \log \tau(s, t - [\mu^{-1}]_\alpha) - \partial_{e_\alpha} \log \tau(s, t))
\]

\[
= - \frac{\tau(s + e_\alpha - e_\beta, t)\tau(s - e_\alpha + e_\beta, t - [\mu^{-1}]_\alpha)}{\tau(s, t - [\mu^{-1}]_\alpha)\tau(s, t)},
\]

(A.1)

\[
\partial_{s_\alpha} \log \tau(s + e_\alpha - e_\beta, t) - \partial_{e_\alpha} \log \tau(s, t - [\mu^{-1}]_\alpha)
\]

\[
= - \frac{\epsilon_{\alpha\beta}(s) \epsilon_{\beta\kappa}(s + e_\alpha - e_\kappa, t)\tau(s - e_\alpha + e_\kappa, t - [\mu^{-1}]_\alpha)}{\epsilon_{\beta\alpha}(s) \tau(s + e_\alpha - e_\beta, t)\tau(s, t - [\mu^{-1}]_\alpha)},
\]

(A.2)

\[
\partial_{s_\alpha} \log \tau(s, t - [\mu^{-1}]_\alpha) - \partial_{e_\alpha} \log \tau(s + e_\alpha - e_\beta, t)
\]

\[
= - \frac{\epsilon_{\alpha\beta}(s) \epsilon_{\beta\kappa}(s + e_\alpha - e_\kappa, t)\tau(s - e_\alpha + e_\kappa, t - [\mu^{-1}]_\alpha)}{\epsilon_{\beta\alpha}(s) \tau(s + e_\alpha - e_\beta, t - [\nu^{-1}]_\beta)\tau(s, t)}.
\]

(A.3)

Taking $\mu \to \infty$ in (3.4) and (A.3), we have

\[
\partial_{s_\alpha} \log \tau(s, t) - \partial_{e_\alpha} \log \tau(s + e_\alpha - e_\beta, t - [\nu^{-1}]_\beta)
\]

\[
= - \frac{\epsilon_{\alpha\beta}(s) \epsilon_{\beta\kappa}(s + e_\alpha - e_\kappa, t)\tau(s - e_\alpha + e_\kappa, t - [\nu^{-1}]_\beta)}{\epsilon_{\beta\alpha}(s) \tau(s + e_\alpha - e_\beta, t - [\nu^{-1}]_\beta)\tau(s, t)},
\]

(A.4)

\[
\partial_{s_\alpha} \log \tau(s) - \partial_{e_\alpha} \log \tau(s + e_\alpha - e_\beta, t)
\]

\[
= - \frac{\epsilon_{\alpha\beta}(s) \epsilon_{\beta\kappa}(s + e_\alpha - e_\kappa, t)\tau(s - e_\alpha + e_\kappa, t)}{\epsilon_{\beta\alpha}(s) \tau(s + e_\alpha - e_\beta, t)\tau(s, t)}.
\]

(A.5)

Now, we can prove CFI (3.6). First, the left-hand side of (3.6) can be rewritten as

\[
\epsilon_{\beta\alpha}(s)\tau(s + e_\alpha - e_\beta, t)\tau(s, t - [\mu^{-1}]_\alpha) \left( \frac{\tau(s, t)\tau(s - e_\alpha + e_\kappa, t - [\mu^{-1}]_\alpha)}{\tau(s + e_\alpha - e_\beta, t)\tau(s, t - [\mu^{-1}]_\alpha)} - 1 \right)
\]

\[
+ \epsilon_{\alpha\beta}(s) \epsilon_{\beta\kappa}(s) \mu^{-1} \frac{\tau(s + e_\alpha - e_\kappa, t - [\mu^{-1}]_\kappa)\tau(s - e_\alpha + e_\kappa, t)}{\tau(s + e_\alpha - e_\beta, t)\tau(s, t - [\mu^{-1}]_\alpha)}.
\]
From (A.3) with \( \alpha \to \lambda, \kappa \to \alpha \) and (A.1) with \( \alpha \to \lambda, \beta \to \alpha \), we have

\[
\frac{\tau(s, t) + \tau(s - e_\beta + e_\alpha, t - [\mu^{-1}]_\lambda)}{\tau(s + e_\alpha - e_\beta, t) \tau(s, t - [\mu^{-1}]_\lambda)} - 1
\]

\[
+ \frac{\epsilon_{\alpha \lambda}(s) \epsilon_{\beta \lambda}(s) \mu^{-1} \tau(s + e_\alpha - e_\lambda, t - [\mu^{-1}]_\lambda)}{\epsilon_{\beta \alpha}(s)} \frac{\tau(s + e_\alpha - e_\beta, t) \tau(s, t - [\mu^{-1}]_\lambda)}{\tau(s, t + e_\alpha - e_\beta, t)} - 1
\]

\[
= - \frac{\epsilon_{\alpha \lambda}(s) \epsilon_{\beta \alpha}(s) \mu^{-1} \tau(s + e_\alpha - e_\beta, t) \tau(s, t - [\mu^{-1}]_\lambda)}{\epsilon_{\beta \lambda}(s) \tau(s, t + e_\alpha - e_\beta, t)}
\]

\[
\times (\partial_{s_{\alpha \lambda}} \log \tau(s, t - [\mu^{-1}]_\lambda) - \partial_{s_{\alpha \lambda}} \log \tau(s + e_\lambda - e_\beta, t)) - 1
\]

\[
= - \frac{\epsilon_{\alpha \lambda}(s) \epsilon_{\beta \lambda}(s) \mu^{-1} \tau(s + e_\alpha - e_\beta, t) \tau(s, t - [\mu^{-1}]_\lambda)}{\epsilon_{\beta \alpha}(s) \tau(s, t + e_\alpha - e_\beta, t)}
\]

\[
\times (\partial_{s_{\alpha \lambda}} \log \tau(s, t - [\mu^{-1}]_\lambda) - \partial_{s_{\alpha \lambda}} \log \tau(s, t)).
\]

Using (A.5) with \( \alpha \to \lambda, \kappa \to \alpha \), we find that this is equal to

\[
\frac{\partial_{s_{\alpha \lambda}} \log \tau(s, t - [\mu^{-1}]_\lambda) - \partial_{s_{\alpha \lambda}} \log \tau(s + e_\lambda - e_\beta, t)}{\partial_{s_{\alpha \lambda}} \log \tau(s, t) - \partial_{s_{\alpha \lambda}} \log \tau(s + e_\lambda - e_\beta, t)} - 1
\]

\[
= - \frac{\partial_{s_{\alpha \lambda}} \log \tau(s, t - [\mu^{-1}]_\lambda) - \partial_{s_{\alpha \lambda}} \log \tau(s, t)}{\partial_{s_{\alpha \lambda}} \log \tau(s, t - [\mu^{-1}]_\lambda) - \partial_{s_{\alpha \lambda}} \log \tau(s, t)} = 0.
\]

This proves CFII (3.6). For CFII (3.5), the left-hand side can be rewritten as

\[
\epsilon_{\beta \alpha}(s) \tau(s + e_\alpha - e_\beta, t) \tau(s + e_\lambda - e_\alpha, t - [\mu^{-1}]_\lambda)
\]

\[
\times \left\{ 
\begin{array}{c}
\epsilon_{\beta \alpha}(s) \tau(s + e_\lambda - e_\alpha, t) \tau(s + e_\lambda - e_\alpha, t - [\mu^{-1}]_\lambda) - 1
\end{array}
\right. - 1
\]

\[
\quad + \frac{\epsilon_{\alpha \lambda}(s) \epsilon_{\beta \lambda}(s) \tau(s + e_\alpha - e_\lambda, t - [\mu^{-1}]_\lambda)}{\epsilon_{\beta \alpha}(s) \tau(s + e_\alpha - e_\beta, t) \tau(s, t + e_\alpha - e_\beta, t)}
\]

\[
\times (\partial_{s_{\alpha \lambda}} \log \tau(s + e_\alpha - e_\lambda, t - [\mu^{-1}]_\lambda) - \partial_{s_{\alpha \lambda}} \log \tau(s + e_\alpha - e_\beta, t)).
\]

(A.6)

Using (A.3) with \( \alpha \to \kappa, \kappa \to \alpha, s \to s + e_\lambda - e_\kappa \), we find that

\[
\epsilon_{\beta \alpha}(s) \tau(s + e_\lambda - e_\kappa, t) \tau(s + e_\lambda - e_\kappa, t - [\mu^{-1}]_\lambda)
\]

\[
\times \left\{ 
\begin{array}{c}
\epsilon_{\beta \lambda}(s) \epsilon_{\beta \alpha}(s) \tau(s + e_\lambda - e_\kappa, t - [\mu^{-1}]_\lambda)
\end{array}
\right. - 1
\]

\[
= - \frac{\epsilon_{\alpha \lambda}(s + e_\kappa - e_\alpha) \epsilon_{\beta \lambda}(s + e_\lambda - e_\kappa) \tau(s, t) \tau(s + e_\lambda - e_\beta, t)}{\epsilon_{\beta \alpha}(s) \tau(s + e_\lambda - e_\beta, t) \tau(s, t + e_\lambda - e_\beta, t)}
\]

\[
\times (\partial_{s_{\alpha \lambda}} \log \tau(s + e_\lambda - e_\kappa, t - [\mu^{-1}]_\lambda) - \partial_{s_{\alpha \lambda}} \log \tau(s + e_\lambda - e_\beta, t)).
\]

(A.7)

Applying (A.5) with \( \alpha \to \lambda, \kappa \to \alpha \), and using the fact that

\[
\frac{\epsilon_{\alpha \lambda}(s + e_\lambda - e_\kappa) \epsilon_{\beta \lambda}(s + e_\lambda - e_\kappa)}{\epsilon_{\beta \alpha}(s) \epsilon_{\alpha \lambda}(s)} = 1,
\]

we find that (A.7) is equal to

\[
\frac{\epsilon_{\alpha \lambda}(s + e_\lambda - e_\kappa) \epsilon_{\beta \lambda}(s + e_\lambda - e_\kappa)}{\epsilon_{\beta \alpha}(s) \epsilon_{\alpha \lambda}(s)}
\]

\[
\times \frac{\partial_{s_{\alpha \lambda}} \log \tau(s + e_\lambda - e_\kappa, t - [\mu^{-1}]_\lambda) - \partial_{s_{\alpha \lambda}} \log \tau(s + e_\lambda - e_\beta, t)}{\partial_{s_{\alpha \lambda}} \log \tau(s, t) - \partial_{s_{\alpha \lambda}} \log \tau(s + e_\lambda - e_\beta, t)}.
\]

On the other hand, using (A.3) with \( \alpha \to \kappa, \kappa \to \alpha, \beta \to \lambda, s \to s + e_\lambda - e_\kappa \), and the fact that

\[
\frac{\epsilon_{\alpha \lambda}(s + e_\lambda - e_\kappa)}{\epsilon_{\beta \lambda}(s + e_\lambda - e_\kappa)} = -1,
\]

we find that

\[
\frac{\epsilon_{\alpha \lambda}(s + e_\lambda - e_\kappa) \epsilon_{\beta \lambda}(s + e_\lambda - e_\kappa)}{\epsilon_{\beta \alpha}(s) \epsilon_{\alpha \lambda}(s)} = 1.
\]
we find that
\[
\frac{\epsilon_{\alpha\beta}(s)\epsilon_{\beta\alpha}(s + e_\lambda - e_\alpha)}{\epsilon_{\beta\alpha}(s + e_\lambda - e_\beta, t - [\mu^{-1}]_\lambda)\tau(s + e_\lambda - e_\beta, t)}
\]
\[
\frac{\epsilon_{\beta\alpha}(s + e_\lambda - e_\alpha)\epsilon_{\beta\lambda}(s + e_\lambda - e_\beta)}{\epsilon_{\beta\alpha}(s)\tau(s + e_\lambda - e_\beta, t)}
\]
\[
\tau(s) \tau(s + e_\lambda - e_\beta, t)
\]
\[
\times (\partial_{\alpha_i} \log \tau(s + e_\lambda - e_\beta, t - [\mu^{-1}]_\lambda) - \partial_{\alpha_i} \log \tau(s, t))
\]
\[
= - \frac{\partial_{\alpha_i} \log \tau(s + e_\lambda - e_\beta, t - [\mu^{-1}]_\lambda) - \partial_{\alpha_i} \log \tau(s, t)}{\partial_{\alpha_i} \log \tau(s, t) - \partial_{\alpha_i} \log \tau(s + e_\lambda - e_\beta, t)}.
\]

Therefore, (A.6) is equal to
\[
\frac{\partial_{\alpha_i} \log \tau(s + e_\lambda - e_\beta, t - [\mu^{-1}]_\lambda) - \partial_{\alpha_i} \log \tau(s + e_\lambda - e_\beta, t)}{\partial_{\alpha_i} \log \tau(s, t) - \partial_{\alpha_i} \log \tau(s + e_\lambda - e_\beta, t)} - 1
\]
\[
= \frac{\partial_{\alpha_i} \log \tau(s + e_\lambda - e_\beta, t - [\mu^{-1}]_\lambda) - \partial_{\alpha_i} \log \tau(s, t)}{\partial_{\alpha_i} \log \tau(s, t) - \partial_{\alpha_i} \log \tau(s + e_\lambda - e_\beta, t)}
\]
\[
= 0.
\]

This proves CFI (3.5).

Appendix B. Proofs of the propositions in section 4

B.1. Proof of proposition 4.1

Proof. Writing out the components of equation (4.1), we need to show that
\[
(1 - \epsilon^{-D_\alpha(s)}\Psi_{\alpha\beta}(s, t, z)) = -\lambda^{-1} (\partial_{\alpha_i} \log \Psi_{\alpha\beta}(s, t, \lambda) - \lambda - \beta) \Psi_{\alpha\beta}(s, t, z)
\]
\[
= -\lambda^{-1} \sum_{\substack{1 \leq \beta \leq N \beta \neq \alpha}} \epsilon_{\alpha\beta}(s) \frac{\tau(s + e_\beta - e_\beta, t)}{\tau(s, t)} \Psi_{\beta\alpha}(s, t, z).
\]

(B.1)

and if \( \beta, \kappa \neq \alpha, \)
\[
(1 - \epsilon^{-D_\alpha(s)}\Psi_{\alpha\beta}(s, t, z)) = -\epsilon_{\alpha\beta}(s) \frac{\tau(s, t)}{\tau(s + e_\beta - e_\beta, t)} \partial_{\beta_j} \log \Psi_{\alpha\beta}(s, t, \lambda) \Psi_{\alpha\beta}(s, t, z).
\]

(B.2)

\[
(1 - \epsilon^{-D_\beta(s)}\Psi_{\beta\lambda}(s, t, z)) = -\lambda^{-1} (\partial_{\beta_i} \log \Psi_{\beta\lambda}(s, t, \lambda) - \lambda - \beta) \Psi_{\beta\lambda}(s, t, z)
\]
\[
= -\lambda^{-1} \sum_{\substack{1 \leq \epsilon \leq N \epsilon \neq \alpha}} \epsilon_{\epsilon\alpha}(s) \frac{\tau(s + e_\epsilon - e_\epsilon, t)}{\tau(s, t)} \Psi_{\epsilon\beta}(s, t, z).
\]

(B.3)

\[
(1 - \epsilon^{-D_\alpha(s)}\Psi_{\beta\alpha}(s, t, z)) = -\epsilon_{\beta\alpha}(s) \frac{\tau(s, t)}{\tau(s + e_\beta - e_\beta, t)} \partial_{\alpha_j} \log \Psi_{\beta\alpha}(s, t, \lambda) \Psi_{\beta\alpha}(s, t, z).
\]

(B.4)

First, we note that for any \( \alpha, \beta, \kappa, \) if \( \alpha = \kappa, \)
\[
\epsilon^{-D_\alpha(s)}\Psi_{\beta\alpha}(s, t, z) = \epsilon^{-D_\alpha(s)} \left[ \epsilon_{\beta\alpha}(s) \frac{\tau(s + e_\beta - e_\beta, t - [\mu^{-1}]_\beta)}{\tau(s, t)} \right]
\]
\[
\epsilon_{\beta\alpha}(s) \frac{\tau(s + e_\beta - e_\beta, t - [\mu^{-1}]_\beta)}{\tau(s, t - [\mu^{-1}]_\beta)} \left( 1 - \frac{z}{\lambda} \right) e^{[t, z]},
\]
and if $\alpha \neq \kappa$,

$$
e^{-D_{\alpha}(\lambda)}\Psi_{\beta\alpha}(s, t, z) = e^{-D_{\alpha}(\lambda)} \epsilon_{\beta\alpha}(s) \frac{\tau(s + e\beta - e\kappa, t - [\lambda^{-1}]_\alpha)}{\tau(s, t)} \left[ e^{s\frac{\epsilon}{\lambda} - \delta_{\beta\alpha} - 1} \right] \epsilon^{(t, z)}.
$$

On the other hand, for any $\alpha$, $\beta$, $\kappa$, we have

$$
\partial_{\alpha\beta} \log \Psi_{\alpha\beta}(s, t, z) = \partial_{\alpha\beta} \log \tau(s + e\alpha - e\beta, t - [\lambda^{-1}]_\beta) - \partial_{\alpha\beta} \log \tau(s, t) + \delta_{\beta\alpha} z.
$$

Therefore with $\mu = z$, $\nu = \lambda$, DFI (3.1) implies that

$$
\partial_{\alpha\beta} \log \Psi_{\alpha\beta}(s, t, z) - \partial_{\alpha\beta} \log \Psi_{\alpha\beta}(s, t, \lambda) = (z - \lambda) \frac{\tau(s, t) \tau(s, t - [\lambda^{-1}]_\beta - [\lambda^{-1}]_\alpha)}{\tau(s, t - [\lambda^{-1}]_\beta) \tau(s, t - [\lambda^{-1}]_\alpha)}
$$

Therefore,

$$
(\partial_{\alpha\beta} \log \Psi_{\alpha\beta}(s, t, z)) \Psi_{\alpha\beta}(s, t, z)
$$

This is in fact equation (75) in [7]. Now (A.1) with $\mu = z$ implies that

$$
\partial_{\beta\alpha} \log \Psi_{\alpha\beta}(s, t, z) = [\partial_{\beta\alpha} \log \Psi_{\alpha\beta}(s, t, z)] \Psi_{\alpha\beta}(s, t, z)
$$

Using the fact that

$$
\partial \Psi_{\alpha\beta}(s, t, z) = \partial_{\beta\alpha} \Psi_{\alpha\beta}(s, t, z) + \sum_{1 \leq \beta \leq N, \beta \neq \alpha} \partial_{\beta\alpha} \Psi_{\alpha\beta}(s, t, z),
$$

we find that (B.5) and (B.6) together give (B.1).

To prove (B.2), note that with $\mu = z$, $\nu = \lambda$, DFI (3.2) implies that for $\beta \neq \alpha$,

$$
\partial_{\beta\alpha} \log \Psi_{\alpha\beta}(s, t, z) - \partial_{\beta\alpha} \log \Psi_{\alpha\beta}(s, t, \lambda)
$$

Therefore,

$$
(\partial_{\beta\alpha} \log \Psi_{\alpha\beta}(s, t, z)) \Psi_{\alpha\beta}(s, t, z)
$$

13
Together with (B.6), we find that
\[
\epsilon_{\alpha\beta}(s) \frac{\tau(s + e_{\alpha} - e_{\beta}, t)}{\tau(s, t)} \Psi_{\beta\alpha}(s, t, z) - \left[ \partial_{\alpha\beta} \log \Psi_{\alpha\alpha}(s, t, \lambda) \right] \Psi_{\alpha\alpha}(s, t, z)
\]
\[= \epsilon_{\alpha\beta}(s) \frac{\tau(s + e_{\alpha} - e_{\beta}, t)}{\tau(s, t)} e^{-D_{\beta}(\lambda)} \Psi_{\beta\alpha}(s, t, z),
\]
which is equivalent to (B.2).

Now to prove (B.3), DFIII (3.3) with \( \nu = z \) and \( \mu = \lambda \), shows that for \( \alpha \neq \beta \),
\[
(\partial_{\alpha\beta} \log \Psi_{\alpha\alpha}(s, t, \lambda)) \Psi_{\alpha\alpha}(s, t, z) = -\lambda e^{-D_{\beta}(\lambda)} \Psi_{\alpha\alpha}(s, t, z). \tag{B.7}
\]
Interchanging \( \alpha \) and \( \beta \), replacing \( s \) by \( s + e_{\alpha} - e_{\beta} \), and setting \( \mu = z \) in (A.2), we find that
\[
\partial_{\alpha\beta} \log \tau(s + e_{\alpha} - e_{\beta}, t - [\zeta^{-1}]_{\beta}) - \partial_{\alpha\beta} \log \tau(s, t + z)
\]
\[= -\lambda \frac{\tau(s + e_{\alpha} - e_{\beta}, t - [\zeta^{-1}]_{\beta})}{\tau(s, t - [\zeta^{-1}]_{\beta})} e^{\xi(t_{\beta}, z)}.
\]
This gives
\[
(\partial_{\alpha\beta} - \partial_{\beta\alpha}) \log \Psi_{\alpha\alpha}(s, t, \lambda)) \Psi_{\alpha\alpha}(s, t, z) = -\lambda e^{-D_{\beta}(\lambda)} \Psi_{\alpha\alpha}(s, t, z).
\tag{B.8}
\]
Now set \( \nu = z \) and let \( \mu \to \infty \) in DFIV (3.4), we find that for \( \kappa \neq \alpha, \beta \),
\[
\partial_{\alpha\beta} \log \tau(s + e_{\alpha} - e_{\beta}, t - [\zeta^{-1}]_{\beta}) - \partial_{\alpha\beta} \log \tau(s, t)
\]
\[= \epsilon_{\alpha\kappa}(s) \frac{\tau(s + e_{\alpha} - e_{\kappa}, t) \tau(s + e_{\alpha} + e_{\kappa}, t - [\zeta^{-1}]_{\beta})}{\tau(s + e_{\alpha} - e_{\beta}, t - [\zeta^{-1}]_{\beta}) \tau(s, t)}.
\tag{B.9}
\]
This gives
\[
\partial_{\alpha\beta} \Psi_{\alpha\beta}(s, t, z) = \epsilon_{\alpha\kappa}(s) \frac{\tau(s + e_{\alpha} - e_{\kappa}, t)}{\tau(s, t)} \Psi_{\alpha\beta}(s, t, z).
\tag{B.10}
\]
Combining together (B.7), (B.8) and (B.10) prove (B.3).

For (B.4), consider first the case \( \beta = \kappa \). Setting \( \mu = z \) and \( \nu = \lambda \), interchanging \( \alpha \) and \( \beta \) and replacing \( s \) by \( s + e_{\alpha} - e_{\beta} \) in DFIII (3.3), we find that
\[
\partial_{\alpha\beta} \log \tau(s, t - [\zeta^{-1}]_{\kappa}) - \partial_{\alpha\beta} \log \tau(s + e_{\alpha} - e_{\beta}, t - [\zeta^{-1}]_{\beta})
\]
\[= -\lambda \frac{\tau(s + e_{\alpha} - e_{\beta}, t - [\zeta^{-1}]_{\beta})}{\tau(s, t - [\zeta^{-1}]_{\beta})} e^{\xi(t_{\beta}, z)}.
\tag{B.11}
\]
This gives
\[
(\partial_{\alpha\beta} \log \Psi_{\alpha\alpha}(s, t, \lambda) - \partial_{\beta\alpha} \log \Psi_{\alpha\alpha}(s, t, \lambda)) \Psi_{\alpha\alpha}(s, t, z)
\]
\[= -\lambda e^{-D_{\beta}(\lambda)} \Psi_{\alpha\alpha}(s, t, z).
\]
Together with (B.8), (B.4) is proved when \( \beta = \kappa \). Finally, if \( \beta \neq \kappa \), let \( \mu = \lambda, \nu = z \) and interchange the role of \( \beta \) and \( \kappa \) in DFIV (3.4). This gives
\[
\partial_{\alpha\beta} \log \tau(s, t - [\zeta^{-1}]_{\kappa}) - \partial_{\beta\alpha} \log \tau(s + e_{\alpha} - e_{\beta}, t - [\zeta^{-1}]_{\beta})
\]
\[= -\lambda e^{-D_{\beta}(\lambda)} \Psi_{\alpha\alpha}(s, t, z).
\]
This shows that
\[
(\partial_{t_1} \log \Psi_{\alpha_1}(s, t, \lambda) - \partial_{t_2} \log \Psi_{\alpha_2}(s, t, z)) \Psi_{\alpha_k}(s, t, z) = -\epsilon_{\alpha\beta}(s) \frac{\tau(s + e_{\beta}, t) - \epsilon_{\alpha\beta}(s)}{\tau(s, t)} e^{-D_{\alpha}(\lambda)} \Psi_{\beta k}(s, t, z).
\] (B.12)

Interchanging \(\beta\) and \(\kappa\) in (B.10), we have
\[
\partial_{t_1} \Psi_{\alpha_k}(s, t, z) = \epsilon_{\alpha\beta}(s) \frac{\tau(s + e_{\beta}, t) - \epsilon_{\alpha\beta}(s)}{\tau(s, t)} \Psi_{\beta k}(s, t, z).
\] (B.13)

(B.12) and (B.13) together give (B.4) when \(\beta \neq \kappa\). \(\square\)

B.2. Proof of proposition 4.2

**Proof.** Note that
\[
\sum_{k=1}^{\infty} \frac{(1 - e^{-D_{\alpha}(\lambda)})^k}{k} = -\log(1 - [1 - e^{-D_{\alpha}(\lambda)}]) = \sum_{j=1}^{\infty} \frac{\lambda^{-j}}{j} \frac{\partial}{\partial t_{a_j}}.
\] (B.14)

By proposition 4.1,
\[
(1 - e^{-D_{\alpha}(\lambda)}) \Psi(s, t, z) = D(s, t, \partial, \lambda) \Psi(s, t, z) = ((\lambda^{-1} \partial) E_{\alpha} + D_{1,0}(s, t, \lambda)) \Psi(s, t, z),
\]
where \(D_{1,0}(s, t, \lambda)\) can be expanded as
\[
D_{1,0}(s, t, \lambda) = \sum_{j=1}^{\infty} D_{1,0,j}(s, t) \lambda^{-j}.
\]

Applying again the operator \((1 - e^{-D_{\alpha}(\lambda)})\), we find that
\[
(1 - e^{-D_{\alpha}(\lambda)})^2 \Psi(s, t, z) = ((\lambda^{-1} \partial) E_{\alpha} + D_{1,0}(s, t - [\lambda^{-1}]_{a_o}, \lambda))((1 - e^{-D_{\alpha}(\lambda)}) \Psi(s, t, z)
\]
\[
+ \{D_{1,0}(s, t, \lambda) - D_{1,0}(s, t - [\lambda^{-1}]_{a_o}, \lambda)\} \Psi(s, t, z)
\]
\[
= \{(\lambda^{-1} \partial) E_{\alpha} + D_{1,0}(s, t - [\lambda^{-1}]_{a_o}, \lambda)\}((\lambda^{-1} \partial) E_{\alpha} + D_{1,0}(s, t, \lambda))
\]
\[
+ \{D_{1,0}(s, t, \lambda) - D_{1,0}(s, t - [\lambda^{-1}]_{a_o}, \lambda)\} \Psi(s, t, z)
\]
\[
= \{(\lambda^{-1} \partial)^2 E_{\alpha} + D_{2,1}(s, t, \lambda) \partial + D_{2,0}(s, t, \lambda)\} \Psi(s, t, z),
\]
where for \(i = 0, 1,\)
\[
D_{2,i}(s, t, \lambda) = \sum_{j=2}^{\infty} D_{2,i,j}(s, t) \lambda^{-j}.
\]

By induction, one can show that
\[
(1 - e^{-D_{\alpha}(\lambda)})^k \Psi(s, t, z) = \left(\sum_{j=0}^{k} D_{k,i}(s, t, \lambda) \partial^i\right) \Psi(s, t, z),
\]
where
\[
D_{k,k}(s, t, \lambda) = \lambda^{-k} E_{\alpha}
\]
and \(D_{k,i}(s, t, \lambda) = \sum_{j=0}^{\infty} D_{k,i,j}(s, t) \lambda^{-j}, \quad 0 \leq i \leq k - 1.\)

Therefore, (B.14) gives
\[
\sum_{j=1}^{\infty} \frac{\lambda^{-j}}{j} \frac{\partial}{\partial t_{a_j}} \Psi(s, t, z) = \sum_{k=1}^{\infty} \frac{1}{k} \left(\sum_{j=0}^{k} D_{k,i,j}(s, t) \lambda^{-j} \partial^i\right) \Psi(s, t, z)
\]
\[
= \sum_{j=1}^{\infty} \lambda^{-j} \left(\sum_{k=1}^{\infty} \frac{1}{k} \sum_{i=0}^{k} D_{k,i,j}(s, t) \partial^i\right) \Psi(s, t, z).
\]
Comparing coefficients of $\lambda^j$, we find that
\[ \frac{\partial \Psi(s, t, z)}{\partial t_{\alpha j}} = j \left( \frac{1}{k} \sum_{i=0}^{k} \sum_{j=1}^{k} \mathcal{D}_{k,i,j}(s, t) \alpha_{ij} \right) \Psi(s, t, z) \]
\[ = j \left( \sum_{k=1}^{j} \frac{1}{k} \mathcal{D}_{k,0,j}(s, t) + \sum_{i=1}^{j} \left[ \sum_{k=1}^{j} \frac{\mathcal{D}_{k,i,j}(s, t)}{k} \alpha_{ij} \right] \right) \Psi(s, t, z). \]

Note that
\[ B_{\alpha j}(s, t, \partial) = j \left( \sum_{k=1}^{j} \frac{1}{k} \mathcal{D}_{k,0,j}(s, t) + \sum_{i=1}^{j} \left[ \sum_{k=1}^{j} \frac{\mathcal{D}_{k,i,j}(s, t)}{k} \alpha_{ij} \right] \right) \]
is a differential operator in $\partial$ with leading term
\[ \mathcal{D}_{j,j}(s, t) \partial_j = E_{\alpha} \partial_j. \]

On the other hand, (4.3) implies that
\[ \frac{\partial \Psi(s, t, z)}{\partial t_{\alpha j}} = \left( \frac{\partial \hat{W}(s, t, \partial)}{\partial t_{\alpha j}} \hat{W}(s, t, \partial)^{-1} \right) \Psi(s, t, z) \]
\[ + \hat{W}(s, t, \partial) \begin{pmatrix} \partial^{\alpha_1} & 0 & \ldots & 0 \\ 0 & \partial^{\alpha_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \partial^{\alpha_j} \end{pmatrix} E_{\alpha} \exp \left( \sum_{\alpha=1}^{N} E_{\alpha}(t, z) \right) \]
\[ = \left( \frac{\partial \hat{W}(s, t, \partial)}{\partial t_{\alpha j}} \hat{W}(s, t, \partial)^{-1} \right) \Psi(s, t, z) \]
\[ + (\hat{W}(s, t, \partial) E_{\alpha} \partial_j \hat{W}(s, t, \partial)^{-1}) \Psi(s, t, z). \]

Therefore,
\[ B_{\alpha j}(s, t, \partial) = \left( \frac{\partial \hat{W}(s, t, \partial)}{\partial t_{\alpha j}} \hat{W}(s, t, \partial)^{-1} \right) + (\hat{W}(s, t, \partial) E_{\alpha} \partial_j \hat{W}(s, t, \partial)^{-1}). \tag{B.15} \]

Since
\[ \left( \frac{\partial \hat{W}(s, t, \partial)}{\partial t_{\alpha j}} \hat{W}(s, t, \partial)^{-1} \right) \]
is a pseudo-differential operator in $\partial$ that only contains negative powers of $\partial$, but $B_{\alpha j}(s, t, \partial)$ is a differential operator, comparing both sides of (B.15) proves (4.5).

From (B.15), we also deduce that
\[ \left( \frac{\partial \hat{W}(s, t, \partial)}{\partial t_{\alpha j}} \hat{W}(s, t, \partial)^{-1} \right) = - (\hat{W}(s, t, \partial) E_{\alpha} \partial_j \hat{W}(s, t, \partial)^{-1}). \tag{B.16} \]

**B.3. Proof of proposition 4.3**

**Proof.** Writing out the components of (4.7), we need to show that
\[ \Psi_{\alpha y}(s + e_{\alpha} - e_{\beta}, t, z) = (\partial - \partial_{\alpha y} \log \tau(s + e_{\alpha} - e_{\beta}, t) + \partial_{\alpha y} \log \tau(s, t)) \Psi_{\alpha y}(s, t, z) \]
\[ = \sum_{1 \leq k \leq N, \alpha \neq \alpha} \epsilon_{\alpha k}(s) \frac{\tau(s + e_{\alpha} - e_{\alpha}, t)}{\tau(s, t)} \Psi_{\alpha y}(s, t, z); \tag{B.17} \]
and if $\lambda \neq \alpha$,
\[
\Psi_{x,y}(s + e_{\alpha} - e_{\beta}, t, z) = e_{x,x}(s + e_{\alpha} - e_{\beta}) \frac{\tau(s + e_{\alpha} - e_{\beta}, t)}{\tau(s + e_{\alpha} - e_{\beta}, t)} \Psi_{x,y}(s, t, z) + (1 - \delta_{\beta})\Psi_{x,y}(s, t, z).
\] (B.18)

If $\gamma = \alpha$, (B.17) is equivalent to
\[
\tau(s + e_{\alpha} - e_{\beta}, t - [z^{-1}]_{\alpha}) \tau(s, t) = \partial_{\alpha} \log \tau(s, t - [z^{-1}]_{\alpha}) + z
\]
\[
+ \sum_{1 \leq k \leq N \atop k \neq \alpha} \partial_{\alpha} \log \Psi_{x,\kappa}(s, t, z) - \partial_{\alpha} \log \tau(s + e_{\alpha} - e_{\beta}, t)
\]
\[
- \sum_{1 \leq k \leq N \atop k \neq \alpha} e_{x,\kappa}(s) \frac{\tau(s + e_{\alpha} - e_{\beta}, t)}{\tau(s, t)} \frac{\Psi_{x,\kappa}(s, t, z)}{\Psi_{x,\kappa}(s, t, z)}.
\] (B.19)

Setting $\mu = z$ in (A.2), we have
\[
\partial_{\alpha} \log \tau(s + e_{\alpha} - e_{\beta}, t) - \partial_{\alpha} \log \tau(s, t - [z^{-1}]_{\beta})
\]
\[
= z - \frac{\tau(s, t) \tau(s + e_{\alpha} - e_{\beta}, t - [z^{-1}]_{\beta})}{\tau(s, t - [z^{-1}]_{\beta}) \tau(s + e_{\alpha} - e_{\beta}, t)}.
\] (B.20)

On the other hand, (B.6) shows that
\[
\sum_{1 \leq k \leq N \atop k \neq \alpha} \partial_{\alpha} \log \Psi_{x,\kappa}(s, t, z) = \sum_{1 \leq k \leq N \atop k \neq \alpha} e_{x,\kappa}(s) \frac{\tau(s + e_{\alpha} - e_{\beta}, t)}{\tau(s, t)} \frac{\Psi_{x,\kappa}(s, t, z)}{\Psi_{x,\kappa}(s, t, z)}.
\] (B.21)

Equations (B.20) and (B.21) together prove (B.19).

If $\gamma = \beta$, (B.17) is equivalent to
\[
- z^{-1} \frac{\tau(s + 2e_{\alpha} - 2e_{\beta}, t - [z^{-1}]_{\beta}) \tau(s, t)}{\tau(s + e_{\alpha} - e_{\beta}, t - [z^{-1}]_{\beta}) \tau(s + e_{\alpha} - e_{\beta}, t)} = \partial_{\alpha} \log \tau(s + e_{\alpha} - e_{\beta}, t - [z^{-1}]_{\beta})
\]
\[
+ \sum_{1 \leq k \leq N \atop k \neq \alpha} \partial_{\alpha} \log \Psi_{x,\kappa}(s, t, z) - \partial_{\alpha} \log \tau(s + e_{\alpha} - e_{\beta}, t)
\]
\[
- \sum_{1 \leq k \leq N \atop k \neq \alpha} e_{x,\kappa}(s) \frac{\tau(s + e_{\alpha} - e_{\beta}, t)}{\tau(s, t)} \frac{\Psi_{x,\kappa}(s, t, z)}{\Psi_{x,\kappa}(s, t, z)}.
\] (B.22)

Interchanging $\alpha$ and $\beta$, replacing $s$ with $s + e_{\alpha} - e_{\beta}$ and setting $\mu = z$ in (A.1) gives
\[
z(\partial_{\alpha} \log \tau(s + e_{\alpha} - e_{\beta}, t - [z^{-1}]_{\alpha}) - \partial_{\alpha} \log \tau(s + e_{\alpha} - e_{\beta}, t))
\]
\[
= - \frac{\tau(s + 2e_{\alpha} - 2e_{\beta}, t - [z^{-1}]_{\beta}) \tau(s, t)}{\tau(s + e_{\alpha} - e_{\beta}, t - [z^{-1}]_{\beta}) \tau(s + e_{\alpha} - e_{\beta}, t)}.
\] (B.23)

On the other hand, (B.8) and (B.10) give
\[
\sum_{1 \leq k \leq N \atop k \neq \alpha} \partial_{\beta} \log \Psi_{x,\kappa}(s, t, z) = \sum_{1 \leq k \leq N \atop k \neq \alpha} e_{x,\kappa}(s) \frac{\tau(s + e_{\alpha} - e_{\beta}, t)}{\tau(s, t)} \frac{\Psi_{x,\kappa}(s, t, z)}{\Psi_{x,\kappa}(s, t, z)}.
\] (B.24)

Equations (B.23) and (B.24) together prove (B.22).
If \( \gamma \neq \alpha, \beta \), (B.17) is equivalent to
\[
\begin{align*}
\epsilon_{\alpha\gamma}(s + e_\alpha - e_\beta) \ & \frac{\tau(s + 2e_\alpha - e_\beta - e_\gamma, t - [-z^{-1}]) \tau(s, t)}{\epsilon_{\alpha\gamma}(s)} \\
= \ & \partial_{\alpha i} \log \tau(s + e_\alpha - e_\gamma, t - [-z^{-1}]) \tau(s + e_\alpha - e_\beta, t) \\
+ \ & \sum_{1 \leq i \leq N, \ k \neq a} \partial_{\alpha i} \log \Psi_{\alpha\gamma}(s, t, z) - \partial_{\alpha i} \log \tau(s + e_\alpha - e_\beta, t) \\
- \ & \sum_{1 \leq i \leq N, \ k \neq a} \epsilon_{\alpha}(s) \frac{\tau(s + e_\alpha - e_\alpha, t)}{\tau(s, t)} \Psi_{\alpha\gamma}(s, t, z).
\end{align*}
\] (B.25)

Replacing \( \alpha \) with \( \gamma, \kappa \) with \( \alpha, s \) with \( s + e_\alpha - e_\gamma \) and setting \( \mu = z \) in (A.3) gives
\[
\begin{align*}
\partial_{\alpha i} \log \tau(s + e_\alpha - e_\gamma, t - [-z^{-1}]) - \partial_{\alpha i} \log \tau(s + e_\alpha - e_\beta, t) \\
= \ & - \frac{\epsilon_{\beta\alpha}(s + e_\alpha - e_\gamma) \epsilon_{\gamma \alpha}(s + e_\alpha - e_\gamma)}{\epsilon_{\beta\gamma}(s + e_\alpha - e_\gamma)} \\
\times \ & \frac{\tau(s, t) \tau(s + 2e_\alpha - e_\beta - e_\gamma, t - [-z^{-1}])}{\tau(s + e_\alpha - e_\gamma, t - [-z^{-1}]) \tau(s + e_\alpha - e_\beta, t)}.
\end{align*}
\] (B.26)

Now, one can prove directly that
\[
- \frac{\epsilon_{\beta\alpha}(s + e_\alpha - e_\gamma) \epsilon_{\gamma \alpha}(s + e_\alpha - e_\gamma)}{\epsilon_{\beta\gamma}(s + e_\alpha - e_\gamma)} = \frac{\epsilon_{\alpha\gamma}(s + e_\alpha - e_\beta)}{\epsilon_{\alpha\gamma}(s)}.
\]

Therefore, (B.26) together with (B.24) (with \( \beta \) replaced by \( \gamma \)) implies (B.25). This completes the proof of (B.17).

If \( \lambda = \beta \), (B.18) is equivalent to
\[
\epsilon_{\beta\gamma}(s + e_\alpha - e_\beta) \frac{\tau(s + e_\alpha - e_\gamma, t - [-z^{-1}])}{\tau(s + e_\alpha - e_\beta, t)} \\
= \epsilon_{\beta\alpha}(s + e_\alpha - e_\beta) \epsilon_{\gamma \alpha}(s) \frac{\tau(s, t)}{\tau(s + e_\alpha - e_\beta, t)} \frac{\tau(s + e_\alpha - e_\gamma, t - [-z^{-1}])}{\tau(s, t)}.
\] (B.27)

This is a tautology since
\[
\epsilon_{\beta\gamma}(s + e_\alpha - e_\beta) = \epsilon_{\beta\alpha}(s + e_\alpha - e_\beta) \epsilon_{\gamma \alpha}(s).
\]

If \( \lambda \neq \alpha \) or \( \beta, \gamma = \alpha \), (B.18) is equivalent to
\[
\begin{align*}
\tau(s + e_\lambda - e_\beta, t - [-z^{-1}]) \\
\tau(s + e_\alpha - e_\beta, t) \\
= \ & \frac{\epsilon_{\beta\alpha}(s)}{\epsilon_{\lambda \alpha}(s + e_\alpha - e_\beta)} \frac{\tau(s + e_\lambda - e_\beta, t)}{\tau(s + e_\alpha - e_\beta, t)} \\
+ \ & \tau(s + e_\lambda - e_\alpha, t - [-z^{-1}]) \frac{\epsilon_{\lambda \alpha}(s + e_\alpha - e_\beta)}{\tau(s, t)}.
\end{align*}
\] (B.28)

To prove this, interchange \( \alpha \) and \( \lambda \) in CFI (3.6) and set \( \mu = z \). This gives
\[
\begin{align*}
\epsilon_{\beta\alpha}(s) \tau(s, t) \tau(s - e_\alpha + e_\lambda, t - [-z^{-1}]) + \epsilon_{\beta\alpha}(s) \tau(s + e_\lambda - e_\beta, t) \tau(s, t - [-z^{-1}]) \\
+ \epsilon_{\lambda \alpha}(s) \epsilon_{\lambda \alpha}(s) \tau(s - e_\beta + e_\alpha, t) = 0.
\end{align*}
\] (B.29)
Since
\[
\epsilon_{\beta\alpha}(s) = \frac{1}{\epsilon_{\lambda\alpha}(s + e_\alpha - e_\beta)},
\]
one finds that (B.29) is equivalent to (B.28).

If \(\lambda \neq \alpha\) or \(\beta \neq \gamma\), (B.18) is equivalent to
\[
\epsilon_{\lambda\beta}(s + e_\alpha - e_\beta)z^{-1} \frac{\tau(s + e_\alpha - e_\beta, t)}{\tau(s + e_\alpha - e_\beta, t)}
= \epsilon_{\lambda\alpha}(s + e_\alpha - e_\beta)e_{\alpha\beta}(s) \frac{\tau(s + e_\alpha - e_\beta, t)}{\tau(s + e_\alpha - e_\beta, t)}
+ \epsilon_{\lambda\beta}(s) \frac{\tau(s + e_\alpha - e_\beta, t)}{\tau(s + e_\alpha - e_\beta, t)}.
\]
(\text{B.30})

To prove this, interchange \(\beta\) and \(\lambda\), replacing \(s\) with \(s + e_\lambda - e_\beta\) in \(\text{CFII (3.6)}\), and set \(\mu = z\).

This gives
\[
\frac{\sigma(s)}{\tau(s, t)} = \frac{\epsilon_{\lambda\alpha}(s + e_\alpha - e_\beta)}{\epsilon_{\lambda\beta}(s + e_\alpha - e_\beta)}.
\]
(\text{B.31})

On can show that
\[
\frac{\epsilon_{\lambda\alpha}(s + e_\alpha - e_\beta)}{\epsilon_{\lambda\beta}(s + e_\alpha - e_\beta)} = \frac{\epsilon_{\lambda\alpha}(s + e_\alpha - e_\beta)}{\epsilon_{\lambda\beta}(s + e_\alpha - e_\beta)}.
\]
(\text{B.32})

Therefore, (B.31) is equivalent to (B.30).

If \(\lambda \neq \alpha\) or \(\beta \neq \gamma\), (B.18) is equivalent to
\[
\epsilon_{\lambda\beta}(s + e_\alpha - e_\beta)z^{-1} \frac{\tau(s + e_\alpha - e_\beta, t)}{\tau(s + e_\alpha - e_\beta, t)} = \epsilon_{\lambda\alpha}(s + e_\alpha - e_\beta)e_{\alpha\beta}(s) \frac{\tau(s + e_\alpha - e_\beta, t)}{\tau(s + e_\alpha - e_\beta, t)}
+ \epsilon_{\lambda\beta}(s) \frac{\tau(s + e_\alpha - e_\beta, t)}{\tau(s + e_\alpha - e_\beta, t)}.
\]
(\text{B.33})

The first identity in (B.32) implies that \(\text{CFII (3.6)}\) with \(\mu = z\) is equivalent to (B.33).

If \(\lambda \neq \alpha\) or \(\beta \neq \gamma\), (B.18) is equivalent to
\[
\epsilon_{\lambda\gamma}(s + e_\alpha - e_\beta)z^{-1} \frac{\tau(s + e_\alpha - e_\beta, t)}{\tau(s + e_\alpha - e_\beta, t)} = \epsilon_{\lambda\alpha}(s + e_\alpha - e_\beta)e_{\alpha\gamma}(s) \frac{\tau(s + e_\alpha - e_\beta, t)}{\tau(s + e_\alpha - e_\beta, t)}
+ \epsilon_{\lambda\gamma}(s) \frac{\tau(s + e_\alpha - e_\beta, t)}{\tau(s + e_\alpha - e_\beta, t)}.
\]
(\text{B.34})

One can show directly that
\[
\epsilon_{\lambda\alpha}(s + e_\alpha - e_\beta) = \epsilon_{\lambda\gamma}(s + e_\alpha - e_\beta) \epsilon_{\alpha\gamma}(s) - \epsilon_{\lambda\beta}(s + e_\alpha - e_\beta) \epsilon_{\alpha\beta}(s + e_\alpha - e_\beta).
\]
(\text{B.34})

Therefore, (B.34) is implied immediately by \(\text{CFI (3.5)}\) (with \(\kappa\) replaced by \(\gamma\)). This completes
the proof of (B.18) and also the proof of the proposition. □
References

[1] Date E, Kashiwara M, Jimbo M and Miwa T 1983 Transformation groups for soliton equations Nonlinear Integrable Systems—Classical Theory and Quantum Theory (Kyoto, 1981) (Singapore: World Scientific) pp 39–119

[2] Date E, Jimbo M, Kashiwara M and Miwa T 1981 Transformation groups for soliton equations III J. Phys. Soc. Japan 50 3806–12

[3] Dickey L 1991 Soliton Equations and Hamiltonian Systems (Singapore: World Scientific)

[4] Kac V and van de Leur J 1993 The $n$-component KP hierarchy and representation theory Important Developments in Soliton Theory ed A S Fokas and V E Zakharov (Berlin: Springer)

[5] Sato M 1981 Soliton Equations as Dynamical Systems on Infinite Dimensional Grassmann Manifolds (RIMS Kokyuroku) p 30

[6] Sato M 1989 The KP hierarchy and infinite-dimensional Grassmann manifolds Theta Functions—Bowdoin 1987, Part 1 (Proc. Symp. Pure Math. vol 49) (Providence, RI: American Mathematical Society) pp 51–66

[7] Takasaki K and Takebe T 2007 Universal Whitham hierarchy, dispersionless Hirota equations and multicomponent KP hierarchy Physica D 235 109–25

[8] Takasaki K and Takebe T 1995 Integrable hierarchies and dispersionless limit Rev. Math. Phys. 7 743–808

[9] Takasaki K 2007 Differential Fay identities and auxiliary linear problem of integrable hierarchies arXiv:0710.5356

[10] Takebe T 2002 A Note on the modified KP hierarchy and its (yet another) dispersionless limit Lett. Math. Phys. 59 157–72