LÉVY MIXING RELATED TO DISTRIBUTED ORDER CALCULUS, SUBORDINATORS AND SLOW DIFFUSIONS

BRUNO TOALDO

ABSTRACT. The study of distributed order calculus usually concerns about fractional derivatives of the form \( \int_0^1 \frac{\partial^\alpha u}{\partial x^\alpha} m(ds) \) for some measure \( m \), eventually a probability measure. In this paper an approach based on Lévy mixing is proposed. Non-decreasing Lévy processes associated to Lévy triplets of the form \((a(y), b(y), \nu(ds, y))\) are considered and the parameter \( y \) is randomized by means of a probability measure. The related subordinators are studied from different point of views. Some distributional properties are obtained and the interplay with inverse local times of Markov processes is explored. Distributed order integro-differential operators are introduced and adopted in order to write explicitly the governing equations of such processes. An application to slow diffusions (delayed Brownian motion) is discussed.

CONTENTS

1. Introduction 1
1.1. Some background information on distributed and fractional order calculus 3
2. Distributed order subordinators 5
2.1. Definition of the process 5
2.2. Distributional properties of the process 6
3. Special distributed order subordinators and inverse local times 10
4. Distributed order integro-differential operators and governing equations 15
4.1. Distributed order integro-differential operators 15
4.2. The governing equations 17
5. An application to slow diffusions 19
References 22

1. INTRODUCTION

Distributed order calculus usually concerns about the distributed order fractional derivative i.e.

\[
\frac{p \frac{d^\beta}{dx^\beta} u(x)}{dx^\beta} = \int_0^1 \frac{d^\beta}{dx^\beta} u(x)p(d\beta),
\]

1.1. Some background information on distributed and fractional order calculus

Date: July 15, 2014.
2010 Mathematics Subject Classification. 60G51, 60J55, 45K05.
Key words and phrases. Subordinators, Lévy mixing, distributed order calculus, inverse local time, Bernstein functions, slow diffusions.
where \( \frac{d^\beta}{dx^\beta} \) can be a fractional derivative in the Riemann-Liouville sense as well as a Dzerbayshan-Caputo derivative and \( \rho(d\beta) \) is a measure (eventually a probability measure). For the definitions of fractional derivatives see, for example, Kilbas et al. [16].

In this paper we develop an approach to distributed-order calculus based on Lévy mixing (see Barndorff-Nielsen et al. [3] for Lévy mixing). We consider, under suitable assumptions, functions of the form

\[
f(\lambda, y) = a(y) + b(y)\lambda + \int_0^\infty (1 - e^{-\lambda s}) \nu(ds, y)
\]

(1.2)

such that \( \lambda \to f(\lambda, y) \) is a Bernstein function for all \( y \) in some polish space \( E \). The measure \( \nu(ds, y) \) is a Lévy measure parametrized by \( y \). We then consider the function

\[
\int_W f(\lambda, y)p(dy) = \int_W (a(y) + b(y)\lambda)p(dy) + \int_0^\infty (1 - e^{-\lambda s}) \int_W \nu(ds, y)p(dy)
\]

(1.3)

for a probability measure \( p \) on \( W \subseteq E \). Since \( p \) is a probability measure we will write for the sake of simplicity

\[
\int_W f(\lambda, y)p(dy) = Ef(\lambda, Y)
\]

(1.4)

for a r.v. \( Y \) with law \( p \) on \( W \subseteq E \). Such a procedure is a particular case of the so-called Lévy mixing which has been recently studied in a systematic way by Barndorff-Nielsen et al. [3] in case the measure \( p \) is not necessarily a probability measure. Among other things the authors pointed out that Lévy mixing arises naturally from stochastic integral representation of processes.

One of the goal of the present paper is to study subordinators with Laplace exponent as in (1.3) and we call such processes distributed order subordinators. We study the transition probabilities of such subordinators by relating them with the most important classes of convolution semigroups. We explore the connection with the inverse local times of Markov processes and we study the right-continuous inverses i.e. the hitting-times. Distributed order calculus comes into play for writing the governing equations of distributed order subordinators and of their hitting-times. From this point of view we have inserted the theory of distributed order calculus (based on operators of the form (1.1)) in a more general unifying framework.

We introduce the distributed order integro-differential operator

\[
ej D_t^{\lambda,p} u(t) = Eb(Y) \frac{d}{dt} u(t) + \frac{d}{dt} \int_c^t u(s)E\tilde{\nu}(t - s, Y)ds, \quad c > 0,
\]

(1.5)

where \( \tilde{\nu}(s, y) = a(y)+\nu((s, \infty), y) \) and \( Y \) a r.v. with law \( p \) on \( W \subseteq E \). The operator (1.5) turns out to have Laplace symbol

\[
L \left[ \ej D_t^{\lambda,p} u(t) \right] (\lambda) = Ef(\lambda, Y)\tilde{u}(\lambda) - Eb(Y)u(0)
\]

(1.6)

where \( \lambda \to Ef(\lambda, Y) \) is a Bernstein function as in (1.3), and permits us to write explicitly the governing equations of the above distributed order subordinators with Laplace exponent (1.3). Then we introduce the distributed order operator

\[
ej D_t^{\lambda,p} u(t) = Eb(Y) \frac{d}{dt} u(t) + \frac{d}{dt} \int_0^t u(s)E\tilde{\nu}(t - s, Y)ds - E\tilde{\nu}(t - c, Y)u(c)
\]

(1.7)
and we discuss an application to the diffusion equation
\[ 0D_t^{p,q}(x,t) = \Delta q(x,t), \quad x \in \mathbb{R}^d, t > 0. \] (1.8)

Under suitable assumptions we show that the mean square displacement
\[ \mathcal{M}(t) = \int_{\mathbb{R}^d} (x - u)^2 q(x - u, t) du, \] (1.9)
where \( q \) is the fundamental solution to (1.8), behaves as
\[ \frac{1}{\int_0^1 f(1/t, y)p(dy)} \] for \( t \to \infty. \) (1.10)

Usually a diffusion is said to be slow (or subdiffusion) if \( \mathcal{M}(t) \sim Ct^\alpha \) with \( \alpha < 1, C > 0. \) Here we study
\[ \lim_{t \to \infty} t^{-1}\mathcal{M}(t) \] (1.11)
by using (1.10) and we show that (1.11) can not be zero but may be either finite or infinite. Actually in the most common cases (1.11) is infinite proving that diffusions related to (1.8) are chiefly subdiffusive. We examine a particular case (related to distributed order fractional calculus) in which the mean square displacement is \( \sim C \log t. \) This last situation was discussed in Chechkin et al. [12, 13, 14]; Kochubei [18] and is related to the so-called ultraslow diffusions.

1.1. **Some background information on distributed and fractional order calculus.** Operators of the form (1.1) have been introduced for the first time in Caputo [11]. Successively a rigorous mathematical theory has been developed in Kochubei [18] and applied to diffusion equations of the form
\[ \frac{p\partial^\beta}{\partial t^\beta} u(x,t) = \Delta u(x,t) + f(x,t), \] (1.12)
which describe some particular anomalous diffusions (for anomalous diffusions the reader can consult Metzler and Klafter [32, 33]). In particular Kochubei [18] deals with the distributed order fractional derivative
\[ \frac{d}{dt} \int_0^t u(s) \int_0^1 (t - s)^{-\beta} \frac{p(d\beta)}{\Gamma(1 - \beta)} ds - \int_0^1 \frac{t^{-\beta}}{\Gamma(1 - \beta)} u(0)p(d\beta) \] (1.13)
which is defined for a continuous function \( u. \) For different forms of distributed order fractional operators the reader can consult Lorenzo and Hartley [21] and the references therein. The study of boundary value problem for time-fractional distributed order diffusion equations can be found in Luchko [22]. In physics the literature is vast. For example in Chechkin et al. [12, 13] different types of diffusions related to distributed order operators have been studied and a probabilistic description of slow diffusions can be found in Meerschaert and Scheffer [30]. In Mainardi and Pagnini [26] the relationship between distributed order diffusions and the Fox function have been explored. An application to the fractional relaxation can be found in Mainardi et al. [25] and the asymptotic behaviour of the solution to a distributed order relaxation equation has been studied in Kochubei [19].

Much attention on the probabilistic point of view has been adopted recently in Beghin [4]; Meerschaert et al. [27]. In particular in [4] an approach based on time-changed process is investigated. The authors take inspiration from the known
results concerning the interplay between fractional equations and time-changed processes and extend the results to the distributed order case. Such an interplay has been explored firstly in work such as Allouba and Zheng [1]; Baeumer and Meerschaert [2]; Orsingher and Beghin [34, 35]; Saichev and Zaslavsky [36]; Zaslavsky [43]. A classical result ([2]) states, roughly speaking, that a Lévy motion time-changed with the hitting-time process of a $\beta$-stable subordinator is the stochastic solution to a time-fractional Cauchy problem. A $\beta$-stable subordinator is a non-decreasing Lévy process with Laplace exponent $\lambda \beta$, $\beta \in (0, 1)$. Such result has been deeply investigated (see, for example, Meerschaert et al. [28] for fractional Cauchy problems in bounded domains) and has been successively extended as follows. In Meerschaert and Scheffer [31] the authors considered continuous time random walks (CTRW) time-changed with a renewal process. They pointed out that the limit of such a CTRW is a Lévy process time-changed with the hitting-time of a subordinator $\sigma_f(t)$, $t > 0$, and that its density is the fundamental solution to the abstract Cauchy problem

$$f(\partial_t) q(x, t) = L_A q(x, t) + \delta(x) \nu(t, \infty), \quad (1.14)$$

in the mild sense. In (1.14) $L_A$ is the generator of $A$, $f$ is the Laplace exponent of $\sigma_f$ with Lévy measure $\nu$ and

$$f(\partial_t) q(t) = \mathcal{L}^{-1} \left[ f(\lambda) \mathcal{L} \left[ q \right](\lambda) - \lambda^{-1} f(\lambda) q(0) \right](t). \quad (1.15)$$

If the subordinator considered is the $\beta$-stable subordinator then $f(\partial_t) q(t)$ reduces to the Dzerbayshan-Caputo fractional derivative.

In Kochubei [20] the author sheds lights on the explicit form of the operator (1.15). The author considered derivatives of the form

$$\mathbb{D}(k) u = \frac{d}{dt} \int_0^t k(t - \tau) u(\tau) d\tau - k(t) u(0). \quad (1.16)$$

under suitable assumptions on the Laplace transform $\mathcal{L}[k(\cdot)]$ and he relates such operators with the relaxation and heat equations

$$\mathbb{D}(k) u = -\lambda u, \quad \mathbb{D}(k) u = \Delta u. \quad (1.17)$$

He pointed out that the solution to (1.17) appears in the description given in Meerschaert et al. [29] of the process $N(L_f(t))$, $t > 0$, where $N$ is a Poisson process with rate $\lambda$ and $L_f$ is an inverse subordinator. In Toaldo [42] an approach based on operators of the form

$$\mathcal{D} f(t) u = b \frac{d}{dt} u(t) + \frac{d}{dt} \int_0^t u(s) \bar{\nu}(t - s) ds \quad (1.18)$$

$$\mathcal{D}^f u = b \frac{d}{dt} u(t) + \int_0^t u'(s) \bar{\nu}(t - s) ds \quad (1.19)$$

has been adopted. In (1.18) and (1.19) $\bar{\nu}(s) = a + \nu(s, \infty)$ for a Lévy triplet $(a, b, \nu)$ such that $f$ is the Bernstein function

$$f(\lambda) = a + b \lambda + \int_0^{\infty} \left(1 - e^{-\lambda s}\right) \nu(ds). \quad (1.20)$$

Note that $\mathcal{L} \left[ \mathcal{D}^f u(t) \right](\lambda) = f(\lambda) \bar{u}(\lambda) - \lambda^{-1} f(\lambda) u(0) = (1.15)$. 
2. DISTRIBUTED ORDER SUBORDINATORS

All throughout the paper we will deal with Bernstein functions (Bernstein [5]). A Bernstein function is a $C^\infty$ non-negative function $f : (0, \infty) \to \mathbb{R}$ for which
\begin{equation}
(-1)^k f^{(k)}(\lambda) \leq 0, \text{ for all } \lambda > 0 \text{ and } k \in \mathbb{N},
\end{equation}
and has the representation
\begin{equation}
f(\lambda) = a + b\lambda + \int_0^\infty \left(1 - e^{-\lambda s}\right) \nu(ds), \quad a \geq 0, b \geq 0.
\end{equation}
The measure $\nu$ is a non-negative $\sigma$-finite measure on $(0, \infty)$ such that the integrability condition
\begin{equation}
\int_0^\infty (s \wedge 1) \nu(ds) < \infty
\end{equation}
is fulfilled (see more on Bernstein functions in Schilling et al. [38]). In what follows we will denote by $\mathcal{BF}$ the family of all Bernstein functions. In this work we will deal with functions of the form
\begin{equation}
f(\lambda, y) = a(y) + b(y)\lambda + \int_0^\infty \left(1 - e^{-\lambda s}\right) \nu(ds, y), \quad y \in E,
\end{equation}
where $0 \leq a(y) < \infty$ and $0 \leq b(y) < \infty$ for all $y \in E$ and $\nu(ds, y)$ is a non-negative $\sigma$-finite measure on $(0, \infty)$ such that
\begin{equation}
\int_0^\infty (s \wedge 1) \nu(ds, y) = V(y) < \infty, \forall y \in E.
\end{equation}
We will also need the function
\begin{equation}
\frac{f(\lambda, y)}{\lambda} = b(y) + \int_0^\infty e^{-\lambda s} \bar{\nu}(s, y) ds, \quad y \in E,
\end{equation}
where
\begin{equation}
\bar{\nu}(s, y) = a(y) + \nu((s, \infty), y), \quad s > 0, y \in E.
\end{equation}
The representation (2.6) is obtained from (2.4) by performing an integration by parts. Note that for all fixed $y \in E$ the function $\lambda \to \lambda^{-1}f(\lambda, y)$ is completely monotone i.e. it is $C^\infty$ and such that
\begin{equation}
(-1)^k \frac{\partial^k}{\partial \lambda^k} \frac{f(\lambda, y)}{\lambda} \geq 0, \text{ for all } \lambda > 0 \text{ and } k \in \mathbb{N} \cup \{0\}.
\end{equation}

2.1. Definition of the process. In this section we introduce the concept of distributed order subordinator. This can be done by means of the Lévy-Itô decomposition. A classical subordinator $\sigma^f$ is a non-decreasing Lévy process and it should be defined as (Itô [15])
\begin{equation}
\sigma^f(t) = bt + \sum_{0 \leq s \leq t} \epsilon(s), \quad b \geq 0,
\end{equation}
where $\epsilon(s)$ is a Poisson point process with characteristic measure $a\delta_\infty + \nu(ds)$ (the reader can consult Bertoin [6, 7] for some details). Such a procedure is still valid if we consider a mixture approach. Let $A$ be a polish space and $\iota_y$ be a non-negative measure on $A$. Suppose $\iota_y(A) = \infty$, for all $y \in E$, but assume that it is $\sigma$-finite.
and thus exists a partition of $A$, say $A_n$, into Borel sets such that $t_y(A_n) < \infty$, for all $n$ and for all $y \in E$. Let $p$ be a probability measure on $W \subseteq E$. Clearly

$$
\int_W t_y(A_n)p(dy) \leq \sup_{y \in W} t_y(A_n) < \infty.
$$

(2.10)

Consider now the product space $A \times [0, \infty)$, the measure $\int_W t_y(\cdot)p(dy) \otimes dt$ and a Poisson measure $\psi$ on $A \times [0, \infty)$. Since $\psi(A \times \{t\}) = 0$ or 1, we can write

$$
\psi(\cdot) = \sum_{t \geq 0} \delta_{\psi(t),t}(\cdot)
$$

(2.11)

for $\psi(t)$ a Poisson point process with mixed characteristic measure $\int_W t_y(\cdot)p(dy)$.

**Definition 2.1.** We define the distributed order subordinator according to the Lévy-Itô decomposition as

$$
\sigma^{f,p}(t) = t \int_W b(y)p(dy) + \sum_{0 \leq s \leq t} e(s),
$$

(2.12)

where $e(s)$ is a Poisson point process such that, for functions $a(y)$ and $b(y)$ as in (2.4), the characteristic measure is $\int_W (\nu(dx,y) + a(y)\delta_\infty)p(dy)$ and $0 \leq b(y) < \infty$.

By using (2.5) we can write

$$
\int_0^\infty (s \land 1) \int_W \nu(ds,y)p(dy) \leq \sup V(y) < \infty
$$

(2.13)

and thus we may use Campbell theorem (see, for example, Kingman [17], page 28) for writing

$$
\mathbb{E}e^{-\lambda \sigma^{f,p}(t)} = \mathbb{E}\exp \left\{ -\lambda t \int_W b(y)p(dy) - \lambda \sum_{0 \leq s \leq t} e(s) \right\}
$$

$$
= \exp \left\{ -\lambda t \int_W b(y)p(dy) + \int_0^\infty (e^{-\lambda s} - 1) \mathbb{E}\psi(ds \times [0,t]) \right\}
$$

$$
= \exp \left\{ -t \int_W f(\lambda,y)p(dy) \right\}.
$$

(2.14)

**2.2. Distributional properties of the process.** The process $\sigma^{f,p}(t), t \geq 0$, is a subordinator generated by the Lévy triplet

$$
\left( \int_W a(y)p(dy), \int_W b(y)p(dy), \int_W \nu(ds,y)p(dy) \right), \quad W \subseteq E,
$$

(2.15)

and therefore has Laplace exponent

$$
\mathbb{E}e^{-\lambda \sigma^{f,p}(t)} = \exp \left\{ -t \left( \mathbb{E}a(Y) + \mathbb{E}b(Y)\lambda + \int_0^\infty (1 - e^{-\lambda s}) \mathbb{E}\nu(ds,Y) \right) \right\},
$$

(2.16)

where we denoted as $Y$ a r.v. with law $p$ on $W \subseteq E$. Here we study the distributional properties of distributed order subordinators, denoted by $\sigma^{f,p}(t), t \geq 0$, with Lévy triplet (2.15), by making assumptions on the subordinator $\sigma^{f,p}$ corresponding to the Lévy triplet $(a(y), b(y), \nu(ds,y))$.

The transition probabilities of subordinators are convolution semigroups supported on $[0, \infty)$. A family $\mu_t, t \geq 0$, of sub-probability measures on $[0, \infty)$ is said to be a convolution semigroup if
The explicit form of the transition probabilities of subordinators is known just in some particular cases. However since such distributions are characterized by the Laplace transform
\[ \mathbb{E} e^{-t\sigma_f(t)} = e^{-tf(\lambda)} \] (2.17)
something could be said by observing at the Laplace exponent \( f \). There exist indeed some special classes of subordinators with nice properties and such classes can be very often distinguished by observing at the Laplace exponent \( f \). This requires some theory of Bernstein functions. For a background on such a theory and on the relationships between family of Bernstein functions and convolution semigroups of sub-probability measures the reader can consult Schilling et al. [38] and the references therein.

In what follows we will always refer to \( \mu_{t} \) as the distributed order subordinator obtained by randomizing the Lévy triplet \((a(y), b(y), \nu(ds, y))\) associated to the subordinator \( \mu_{y} \).

2.2.1. Bondesson class. One of the most important class of sub-probability measures related to subordinators is the Bondesson class. Such class of measures is closed under convolution and vague convergence (see Lemma 9.2. of [38]) and is composed of infinitely divisible measures. A measure \( \mu \) is said to be of the Bondesson class, and we write \( \mu \in \text{BO} \), if
\[ \mathcal{L} [\mu] (\lambda) = e^{-f(\lambda)} \] (2.18)
where \( f \) is a complete Bernstein function. A Bernstein function \( f \) is said to be complete, and we write \( f \in \text{CBF} \), if has the representation
\[ f(\lambda) = a + b\lambda + \int_{0}^{\infty} (1 - e^{-\lambda s}) m(s)ds, \] (2.19)
where the density \( s \rightarrow m(s) \) is a completely monotone function.

**Proposition 2.2.** Let \( \mu^y_t(B) = \text{Pr}\{\sigma^{f,y}_t \in B\}, t \geq 0 \). Assume that \( \mu^y_t \in \text{BO}, \forall y \in E \). Let \( \mu^y_p(B) = \text{Pr}\{\sigma^{f,p}_t \in B\}, \) where \( \sigma^{f,p} \) is the corresponding distributed order subordinator. We have that \( \mu^y_t \in \text{BO} \).

**Proof.** The fact that \( \mu^y_t \in \text{BO} \) is equivalent to say that the Laplace exponent \( f : [0, \infty) \times E \rightarrow \mathbb{R} \) is a complete Bernstein function and therefore from (2.19)
\[ f(\lambda, y) = a(y) + b(y)\lambda + \int_{0}^{\infty} (1 - e^{-\lambda s}) \nu(ds, y) \]
\[ = a(y) + b(y)\lambda + \int_{0}^{\infty} (1 - e^{-\lambda s}) m_y(s)ds, \quad \text{for all } y \in E. \] (2.20)
The function
\[ \mathbb{E} f(\lambda, Y) = \mathbb{E} a(Y) + \mathbb{E} b(Y)\lambda + \int_{0}^{\infty} (1 - e^{-\lambda s}) \mathbb{E} m_y(s)ds \] (2.21)
is again a complete Bernstein function. This can be ascertained by using the fact that since \( m \) is completely monotone then it is the Laplace transform of a certain
measure \( m_y(dt) \) and thus

\[
E m_Y(s) = \int_0^\infty e^{-st} \int_W m_y(dt)p(dy)
\]  

(2.22)
is a completely monotone function representing the density of the Lévy measure \( E \nu(ds,Y) \).

\[\square\]

2.2.2. Mixture of exponential distributions. A measure \( \mu \) on \([0, \infty)\) is said to be a mixture of exponential distributions, and we write \( \mu \in ME \), if

\[
\mu[0,t] = \int_{(0,\infty]} (1 - e^{-t\alpha}) \rho(d\alpha)
\]  

(2.23)
for some sub-probability measure \( \rho \) on \((0, \infty]\). We recall that \( ME \subset BO \) (see [38] page 81 for further details) since it consists in the family of measures such that \( L[\mu](\lambda) = e^{-f} = \frac{1}{g} \) where \( 1/g \) is a Stieltjes function (we write \( 1/g \in S \), with \( 1/g(0+) \leq 1 \) for \( f \in CBF \). A Stieltjes function is a function \( h: (0, \infty) \rightarrow [0, \infty) \) with representation

\[
h(\lambda) = a + b + \int_0^\infty \frac{1}{\lambda + t} w(dt), \quad a, b \geq 0,
\]  

(2.24)
where \( w(dt) \) is a measure on \((0, \infty)\) such that

\[
\int_0^\infty \frac{1}{1 + t} w(dt) < \infty.
\]  

(2.25)
The reader can also consult Steutel and van Harn [41], Chapter VI, for detailed information on mixture of exponential distributions. Formally a measure \( \mu \) is said to be a mixture of exponential distributions if it is such that \( L[\mu](\lambda) \in S \) with \( L[\mu](\lambda)_{\lambda=0} \leq 1 \).

**Proposition 2.3.** If \( \sigma^{f,y} \) has transition probabilities \( \mu^y_t \in ME \), for all fixed \( y \in E \), we have that the corresponding distributed order subordinator \( \sigma^{f,p}(t), t \geq 0 \) also has transition probabilities \( \mu^p_t \in ME \).

**Proof.** From Theorem 9.5 of [38] we know that the condition (2.23) is equivalent to say that there exists a function \( \eta : (0, \infty) \rightarrow [0, 1] \) satisfying \( \int_0^1 \eta(t)t^{-1}dt \) and such that, for some \( \beta \geq 0 \),

\[
L[\mu](\lambda) = \exp\left\{ -\beta - \int_0^\infty \left( \frac{1}{t} - \frac{1}{\lambda + t} \right) \eta(t)dt \right\}
\]  

\[
= \exp\left\{ -\beta - \int_0^\infty (1 - e^{-\lambda t}) \nu(dt) \right\}
\]  

(2.26)
for some Lévy measure \( \nu \). Here we must have for all \( y \in E \)

\[
L[\mu^y_t](\lambda) = \exp\left\{ -t \left( a(y) + \int_0^\infty (1 - e^{-\lambda s}) \nu(ds,y) \right) \right\} \in S
\]  

(2.27)
and

\[
f(\lambda, y) = a(y) + \int_0^\infty (1 - e^{-\lambda s}) \nu(ds,y) \in CBF
\]  

(2.28)
Since \( f \in CBF \) the Lévy measure \( \nu(ds,y) \) has a density \( m_y(s) \) such that \( s \rightarrow m_y(s) \) is completely monotone for all fixed \( y \in E \). This implies that \( m_y(s) \) is the Laplace
transform of a certain measure $m_y(dt)$, i.e. $m_y(s) = \int_0^\infty e^{-st}m_y(dt)$. Thus we can write

$$Ef(\lambda, Y) = E_a(Y) + \int_0^\infty (1 - e^{-\lambda s}) E\nu(ds, Y)$$

$$= E_a(Y) + \int_0^\infty (1 - e^{-\lambda s}) \int_W \int_0^\infty e^{-st}m_y(dt) p(dy) ds$$

$$= E_a(Y) + \int_W \int_0^\infty \int_0^\infty (e^{-st} - e^{-s(\lambda + t)}) ds m_y(dt) p(dy)$$

$$= E_a(Y) + \int_W \int_0^\infty \left( \frac{1}{t} - \frac{1}{\lambda + t} \right) m_y(dt) p(dy)$$  \hspace{1cm} (2.29)

Since $\mu^y \in ME$, for all $y \in E$ we must have that $m_y(dt)$ has a density $\eta_y(t)$ such that

$$\eta_y : (0, \infty) \to [0, 1] \text{ with } \int_0^1 w^{-1} \eta_y(w) dw < \infty,$$  \hspace{1cm} (2.30)

and this permits us to conclude

$$L[\mu]^y_p(\lambda) = \exp \left\{ -E_a(Y) - \int_0^\infty \left( \frac{1}{t} - \frac{1}{\lambda + t} \right) E\eta_Y(t) dt \right\}.$$  \hspace{1cm} (2.31)

The fact that

$$E\eta_Y : (0, \infty) \to [0, 1]$$  \hspace{1cm} (2.32)

is verified since

$$0 \leq \int_W \eta_y(*) p(dy) \leq \sup_{y \in W} \eta_y(*) \text{, with } \sup_{y \in W} \eta_y \in [0, 1], \forall y \in E.$$  \hspace{1cm} (2.33)

Furthermore note that in view of (2.30)

$$\int_0^1 s^{-1} \int_W \eta_y(s) p(dy) ds < \infty.$$  \hspace{1cm} (2.34)

2.2.3. Generalized Gamma Convolution. A measure $\mu$ on $[0, \infty)$ is said to be a Generalized Gamma convolution, and we write $\mu \in GGC$ if

$$L[\mu](\lambda) = e^{-f(\lambda)},$$  \hspace{1cm} (2.35)

where $f$ is a Thorin-Bernstein function. A quite important subclass of Bernstein functions consists in those Bernstein functions whose derivative is a Stieltjes function. Such functions are called Thorin-Bernstein functions (we write $f \in TBF$) and are of the form

$$f(\lambda) = a + b\lambda + \int_0^\infty (1 - e^{-\lambda t}) \tau(t) dt$$  \hspace{1cm} (2.36)

where $\int_0^\infty (t \wedge 1) \tau(t) dt < \infty$ and $t \to t\tau(t)$ is a completely monotone function. The class of GGC is the smallest class of sub-probability measures on $[0, \infty)$ closed under convolution and vague limits containing all Gamma distributions (see Theorem 9.12 of [38]).

**Proposition 2.4.** Let $Pr\{\sigma^{f-w}(t) \in B\} = \mu^w_t(B) \in GGC$. For the distributed order subordinator $\sigma^{f-p}$ we have that

$$Pr\{\sigma^{f-p}(t) \in B\} = \mu^p_t(B) \in GGC.$$  \hspace{1cm} (2.37)
Proof. Since $\mu^y_t \in \text{GGC}$ we have that $\lambda \rightarrow f(\lambda, y) \in \text{TBF}$ for all $y \in E$ and thus
\[
f(\lambda, y) = a(y) + b(y)\lambda + \int_0^\infty (1 - e^{-\lambda t}) \tau_y(t) \, dt.
\] (2.38)

It is sufficient to prove that
\[
E f(\lambda, Y) = E a(Y) + E b(Y)\lambda + \int_0^\infty (1 - e^{-\lambda t}) E \tau_Y(t) \, dt \in \text{TBF}.
\] (2.39)

Note that since $\lambda \rightarrow f(\lambda, y) \in \text{TBF}$, one have
\[
t \rightarrow t \tau_y(t) = \int_0^\infty e^{-ts} \phi_y(ds), \text{ for all } y \in E,
\] (2.40)

for a certain measure $\phi_y(ds)$. Thus we can write
\[
t \rightarrow t E \tau_Y(t) = t \int W \tau_y(t) p(dy) = \int W \int_0^\infty e^{-ts} \phi_y(ds) p(dy) = \int_0^\infty e^{-ts} E \phi_Y(ds),
\] (2.41)

which is a completely monotone function. □

3. Special distributed order subordinators and inverse local times

In this section we point out the relationships between special subordinators, distributed order subordinators and inverse local times of Markov processes.

A subordinator with Laplace exponent $f$ is said to be special if $\lambda/f(\lambda) \in \text{BF}$. This is because a Bernstein function $f$ is said to be special if $\lambda/f(\lambda) \in \text{BF}$. The collection of all special Bernstein functions will be denoted by $\text{SBF}$. We recall that in case $f \in \text{SBF}$ we clearly have that $f \ast \lambda f \in \text{SBF}$. We call $f \ast \lambda f$ the conjugate of $f$. The most important property of special subordinators concerns their potential measures
\[
U^f(dx) = E \int_0^\infty 1_{\{\sigma^f(t) \in dx\}} dt
\] (3.1)

which is such that
\[
\mathcal{L}[U^f(dx)](\lambda) = \frac{1}{f(\lambda)}.
\] (3.2)

It is well-known that $\sigma^f$ is special if and only if
\[
U^f(dx) = c \delta_0(dx) + u(x) dx
\] (3.3)

for $c \geq 0$ and for some non-increasing function $u : (0, \infty) \rightarrow (0, \infty)$ satisfying $\int_0^1 u(x) dx < \infty$. The reader can consult Schilling et al. [38], Chapter 10, or Song and Vondraček [40], for further information on special Bernstein functions and the related subordinators.

It is well-known that the inverse local time at $y$ of a Markov process is a subordinator (see, for example, Bertoin [6], Chapter IV, or Blumenthal and Getoor [9], page 219), provided that such a local time exists. In the upcoming theorem we will need the following information on local times. Let $X = (\Omega, \mathcal{M}, \mathcal{M}_t, X_t, \theta_t, P^z)$ be a temporally homogeneous Markov process on the polish space $E$. As usual
given \((E, \mathcal{E})\) we have \(E_\Delta = E \cup \{\Delta\}\) and \(\mathcal{E}_\Delta\) is the \(\sigma\)-algebra in \(E_\Delta\) where \(\Delta\) is clearly a point not in \(E\) for which \(\forall t \in [0, \infty), \ X_t(\omega) = \Delta\) implies \(X_s(\omega) = \Delta, \ \forall s \geq t.\) \((\Omega, \mathcal{M})\) is a measurable space and \(\mathcal{M}_t\) an increasing family \(\{\mathcal{M}_t : t \in [0, \infty]\}\).

With \(\theta_t\) we denote the translation operator i.e. the map \(\theta_t : \Omega \to \Omega\) such that \(\theta_{t} \omega = \omega_{\Delta}, \ \forall \omega\) and where \(\omega_{\Delta}\) is a distinguished point of \(\Omega\). Due to homogeneity we write \(X_t \circ \theta_h = X_{t+h}, \ \forall t, h \in [0, \infty].\)

In what follows we suppose that \(X\) is in duality with \(\hat{X}\), i.e.

\[
R^\lambda u(z) = E^z \left( \int_0^\infty e^{-\lambda t} u(X_t) dt \right) = \int_E r^\lambda(z, x) u(x) \xi(dx) \tag{3.4}
\]

\[
\hat{R}^\lambda u(z) = \hat{E}^z \left( \int_0^\infty e^{-\lambda t} u(\hat{X}_t) dt \right) = \int_E r^\lambda(x, z) u(x) \xi(dx) \tag{3.5}
\]

for some \(\sigma\)-finite measure \(\xi\) on \(E\). See Blumenthal and Getoor [9, 10] for details on duality and potential theory. A necessary and sufficient condition for the existence of a local time in some \(y \in E\) is that \(y\) is regular for itself (see [9] Theorem 3.13) i.e.

\[
P^y(T_y = 0) = 1, \text{ where } T_y = \inf \{t \geq 0 : X(t) = y\}. \tag{3.6}
\]

However in the above framework Proposition 7.3 of [10] holds and thus one has the following equivalent conditions for a point \(y \in E\) to be regular for itself

1. \(y\) is regular for \(y\)
2. \(r^\lambda(z, y) \leq r^\lambda(y, y) < \infty, \forall z \in E\)
3. The function \(z \to r^\lambda(z, y)\) is bounded and continuous at \(z = y\)

Assume that the function \(z \to r^\lambda(z, y)\) is lower semi-continuous for all \(y \in E\). Denote by \(L(y, t)\) the local time at \(y\). In such a framework it is well known that the inverse local time at \(y\) of \(X_t\), say \(L^{-1}(y, t)\), exists and is a subordinator with Laplace exponent \(1/r^\lambda(y, y)\). Furthermore

\[
r^\lambda(x, y) = E^x \int_0^\infty e^{-\lambda t} dL(y, t) \tag{3.7}
\]

where \(dL(y, t)\) is meant in the sense that

\[
L(y, t) = \lim_{\epsilon \to 0} \int_0^t \delta_{y, y}^\epsilon(X_s) ds. \tag{3.8}
\]

With the symbol \(\delta_{y, y}^\epsilon(x)\) we denote a family of continuous functions on \(E\) with compact support \(S_{\epsilon}\) in a neighborhood of \(y\) such that \(\lim_{\epsilon \to 0} S_{\epsilon} = \{y\}\) and for which

\[
\int \delta_{y, y}^\epsilon(x) \xi(dx) = 1. \tag{3.9}
\]

**Theorem 3.1.** Let \(Y\) be a random variable on \(W \subseteq E\) with law \(p\). Let

\[
\lambda \to f(\lambda, y) = a(y) + b(y)\lambda + \int_0^\infty (1 - e^{-\lambda s}) \nu(ds, y) \tag{3.10}
\]

be a special Bernstein function for all fixed \(y \in E\) and let \(f^* = \lambda/f\) be the conjugate of \(f\). We have the following results.

i) The function \(\hat{f}^*(\lambda) = Ef^*(\lambda, Y)\) is a special Bernstein function and has representation

\[
\hat{f}^*(\lambda) = Ef^*(\lambda, Y) = Ea^*(Y) + Eb^*(Y)\lambda + \int_0^\infty (1 - e^{-\lambda s}) E\nu^*(ds, Y) \tag{3.11}
\]
where
\[ b^*(y) = \begin{cases} 
0, & b(y) > 0, \\
\frac{1}{a(y) + \nu_L((0,\infty),y)}, & b(y) = 0.
\end{cases} \]
\[ a^*(y) = \begin{cases} 
0, & a(y) > 0, \\
\frac{1}{b(y) + \int_0^\infty \nu(t,y) dt}, & a(y) = 0.
\end{cases} \]
(3.12)

ii) Suppose that \( f \) is the Laplace exponent of the inverse local time at \( y \) of \( X_t \), where \( X_t \) is the process above. Assume that \( X_t \) is space-inhomogeneous and that one of the conditions (1), (2), (3) holds for all \( y \in W \). Let \( P^p = \int_W P^p(dy) \). Let \( \bar{X} = (\bar{\Omega}, \bar{\mathcal{M}}, \bar{\mathcal{M}}, \bar{\mathcal{X}}, \bar{\mathcal{Y}}, P^p) \) be the Markov process on \( E \) with initial distribution \( p \) such that \( \bar{X} \) is identical in law at \( X \) under \( p(dy) = \delta_y \), for all \( y \in W \). The conjugate of \( \bar{f}^* \), i.e. the function \( \bar{f}^* : \lambda \mapsto \frac{\lambda}{\mathcal{E}^*_{\bar{X},\bar{Y}}(\lambda)} \), is the Laplace exponent of the inverse local time of \( \bar{X}_t \) at its random starting point \( \bar{X}_0 \).

iii) Denote the inverse local time of \( \bar{X}_t \) at \( \bar{X}_0 \) by \( L^{-1}(\bar{X}_0,t) \) and by \( U^{f,p}(dt) \) its potential measure. Let \( U^{f,g}(dt) \) be the potential measure of the inverse local time at \( y \) of \( X_t \). We have that
\[ U^{f,p}(dt) = \int_W U^{f,y}(dt) p(dy) \]
\[ = \int_W (b^*(y)\delta_0(dt) + \nu^*(t,y) dt) p(dy) \]
where \( \nu^*(s,y) = a^*(y) + \nu^*(s,(t,\infty),y) \).

Proof. i) Since \( f(\lambda, y) \in \text{SBF} \) for all fixed \( y \in E \) we have that \( f^*(\lambda, y) = \lambda/f(\lambda, y) \in \text{BF} \). In particular by applying Theorem 10.3 of [38] we may write
\[ f^*(\lambda, y) = a^*(y) + b^*(y)\lambda + \int_0^\infty (1 - e^{-\lambda s}) \nu^*(ds, y) \]
(3.14)
where
\[ b^* = \begin{cases} 
0, & b(y) > 0, \\
\frac{1}{a(y) + \nu_L((0,\infty),y)}, & b(y) = 0.
\end{cases} \]
\[ a^* = \begin{cases} 
0, & a(y) > 0, \\
\frac{1}{b(y) + \int_0^\infty \nu(t,y) dt}, & a(y) = 0.
\end{cases} \]
(3.15)
Let \( \sigma^{f,y} \) be the subordinator with Laplace exponent
\[ f(\lambda, y) = a(y) + b(y)\lambda + \int_0^\infty (1 - e^{-\lambda s}) \nu(ds, y). \]
(3.16)
Since \( \lambda \mapsto f(\lambda, y) \in \text{SBF} \) we have that the corresponding subordinator \( \sigma^{f,y} \) has potential measure (parametrized by \( y \))
\[ U^{f,y}(dt) = b^*(y)\delta_0(dt) + u_y(t)dt \]
(3.17)
for a function \( t \mapsto u_y(t) : (0, \infty) \to (0, \infty) \) non-increasing and satisfying
\[ \int_0^1 u_y(t)dt < \infty, \quad \text{for all } y \in E. \]
(3.18)
In particular we deduce from Theorem 10.3, page 94, of [38] that
\[ u_y(t) = \nu^*(t,y) = \nu^*((t,\infty),y) + a^*(y). \]
(3.19)
Now consider the measure
\[
\int_W U^f_y(dt)p(dy) = \mathbb{E}b^*(Y)\delta_0(dt) + \mathbb{E}\nu^*(t,Y)dt.
\] (3.20)
The function \( t \to \mathbb{E}\nu^*(t,y) : (0, \infty) \to (0, \infty) \) is non-increasing and satisfies (3.18) and thus (3.20) is the potential measure of some special subordinator.

In particular since

\[
\mathcal{L} \left[ \int_W U^f_y(dt)p(dy) \right] (\lambda) = \lambda^{-1}Ef^*(\lambda, Y) = \frac{1}{E_{f^*(\lambda,Y)}}
\] (3.21)
we use (3.2) and we deduce that \( \frac{\lambda}{Ef^*(\lambda,Y)} \in \text{SBF} \) which implies that \( Ef^*(\lambda,Y) \in \text{SBF} \).

ii) Now we consider the inverse local time \( L^{-1}(\bar{X}_0,t) \). Since we assume that one of the conditions (1), (2), (3) holds for all \( y \in W \subseteq E \) we have that the points \( y \in W \) are regular for theirselves since it is clear that

\[
P^y \{ T_{\bar{X}_0} = 0 \} = 1 \text{ where } T_{\bar{X}_0} = \inf \{ t \geq 0 : \bar{X}_t = \bar{X}_0 \},
\] (3.22)
and this guarantees that such a local time exists. We recall that

\[
\mathbb{E}^y \int_0^\infty e^{-\lambda L^{-1}(y,t)}dt = \mathbb{E}^y \int_0^\infty e^{-\lambda t}dL(y,t) \quad (3.8)
\] (3.23)
By using Proposition 2.3 Chapter V of [9] which states that

\[
L^{-1}(y,t+s) = L^{-1}(y,t) + L^{-1}(y,s) \circ \theta_{L^{-1}(y,t)}
\] (3.24)
and since \( \bar{X}(L^{-1}(\bar{X}_0,t)) = \bar{X}_0 \) we perform calculation similar to that of Proposition 3.17 Chapter V of [9] and we write

\[
\mathbb{E}^y e^{-\lambda L^{-1}(\bar{X}_0,t+s)} = \mathbb{E}^y e^{-\lambda L^{-1}(\bar{X}_0,t)}\mathbb{E}^\bar{X}(L^{-1}(\bar{X}_0,t))e^{-\lambda L^{-1}(\bar{X}_0,s)}
\] (3.25)
Now in view of (3.7) and (3.8) we may write

\[
\mathbb{E}^y \int_0^\infty e^{-\lambda L^{-1}(\bar{X}_0,t)}dt = \mathbb{E}^y \int_0^\infty e^{-\lambda t}dL(\bar{X}_0,t)
\]
\[
= \int_W \mathbb{E}^y e^{-\lambda L^{-1}(\bar{X}_0,t)}p(dy)
\] (3.26)
and from (3.25) we conclude that

\[
\mathbb{E}^y e^{-\lambda L^{-1}(\bar{X}_0,t)} = e^{-tk} \text{ with } k = \frac{1}{\int_W \mathbb{E}^y L^{-1}(\bar{X}_0,t)p(dy)}.
\] (3.27)
Now observe that since we suppose that \( f(\lambda,y) \) is the Laplace exponent of the inverse local time of \( X_t \) at \( y \) we must have

\[
f(\lambda,y) = a(y) + b(y)\lambda + \int_0^\infty (1 - e^{-\lambda s}) \nu(ds,y)
\]
\[
= \frac{1}{r^\lambda(y,y)}
\] (3.28)
and thus we get that

\[
\mathbb{E}^y e^{-\lambda L^{-1}(\bar{X}_0,t)} = \exp \left\{ -t \frac{1}{\int_W \frac{1}{r^\lambda(y,y)}p(dy)} \right\}
\]
\[
\exp \left\{ -t \int_W f^*(\lambda, y)p(dy) \right\} = \exp \left\{ -tf^*(\lambda) \right\}.
\]

Therefore
\[
\mathbb{E}^{\exp} e^{-\lambda L^{-1}(\bar{X}_0, t)} = e^{-tf^*(\lambda)}. \quad (3.30)
\]

Note that the process \( L^{-1}(\bar{X}_0, t) \) is clearly non-decreasing and has stationary independent increments in the ordinary sense if
\[
\lim_{\lambda \to 0} \int_W r^\lambda(y, y)p(dy) = \infty. \quad (3.31)
\]

By adapting Proposition 3.19 of [9] we may prove that
\[
P \left\{ \bigcap_{j=1}^n \left\{ L^{-1}(\bar{X}_0, t_j) - L^{-1}(\bar{X}_0, t_{j-1}) \right\} \in B_j; L^{-1}(\bar{X}_0, t_n) < \infty \right\}
= \prod_{j=1}^n P \left\{ L^{-1}(\bar{X}_0, t_j - t_{j-1}) \in B_j \right\} \quad (3.32)
\]

for \( B_j, j = 1, \ldots, n \), disjoints and \( 0 = t_0 < \cdots < t_n \) by using (3.24) and the fact that \( \bar{X} (L^{-1}(\bar{X}_0, t)) = \bar{X}_0 \) under \( L^{-1}(\bar{X}_0, t) < \infty \). From (3.27) we know that this last condition is verified under (3.31). If (3.31) holds we have
\[
\lim_{\lambda \to 0} f^*(\lambda) = \lim_{\lambda \to 0} \frac{1}{\mathbb{E}r^\lambda(Y, Y)} = \frac{1}{\mathbb{E}1^\lambda} = 0. \quad (3.33)
\]

If instead \( \lim_{\lambda \to 0} \int_W r^\lambda(y, y)p(dy) < \infty \) one has
\[
0 < \lim_{\lambda \to 0} f^*(\lambda) = \frac{1}{\mathbb{E}1^\lambda} < \infty \quad (3.34)
\]

and we get what in literature is known as a killed subordinator. We can apply Theorem 3.21 of [9] and state that there exists a subordinator \( \sigma(t) \) and an independent non-negative r.v. \( \zeta \) on some \((\Omega, \Sigma, P)\) with \( P \{ \zeta \geq t \} = \exp \left\{ -t \lim_{\lambda \to 0} \frac{1}{\lambda} \right\} \) such that
\[
L^{-1}(\bar{X}_0, t) = \begin{cases} \sigma(t), & t < \zeta, \\ \infty, & t \geq \zeta, \end{cases} \quad (3.35)
\]

and \( L^{-1}(\bar{X}_0, t) \) are stochastically equivalent.

iii) In view of (3.17) we can write
\[
U^{f^y}(dt) = b^y(t, y)\delta_0(dt) + \tilde{\nu}^y(t, y) dt \quad (3.36)
\]

and therefore
\[
\mathcal{L} \left[ \mathbb{E}b^y(Y)\delta_0(dt) + \mathbb{E}\tilde{\nu}^y(t, Y)dt \right] (\lambda) = \frac{\mathbb{E}f^y(\lambda, Y)}{\lambda} = \frac{1}{\mathbb{E}f^y(\lambda, Y)}. \quad (3.37)
\]

Now from (3.29) we can write
\[
\mathcal{L} \left[ \mathbb{E} \int_0^\infty \mathbb{1}_{\{L^{-1}(\bar{X}_0, x) \in dt\}} dx \right] (\lambda) = \frac{1}{\lambda} \mathbb{E}f^y(\lambda, Y), \quad (3.38)
\]
and the proof is complete.

4. Distributed order integro-differential operators and governing equations

In this section we introduce distributed order integro-differential operators by means of which we write the governing equations of the distributed order subordinators. Furthermore we are interested in the governing equations of their inverse processes. We will consider the inverse of \( \sigma_{f,p}(t), t>0 \), defined as

\[
L_{f,p}(t) = \inf \{s \geq 0 : \sigma_{f,p}(s) > t\}
\]

for which

\[
\{L_{f,p}(t) > x\} = \{\sigma_{f,p}(x) < t\}.
\]

4.1. Distributed order integro-differential operators. According to Theorem 4.1 of Meerschaert and Scheffer [31] we know that the density of the inverse of a subordinator is the fundamental solution in the mild sense of the pseudo-differential problem

\[
-\frac{\partial}{\partial t} u(x,t) = \frac{\partial}{\partial x} u(x,t) + \delta_x \nu(t,\infty),
\]

where \( f(\partial_t) \) is the following inverse Laplace transform

\[
f(\partial_t) = \mathcal{L}^{-1} \left[ f(\lambda) \mathcal{L} [u](\lambda) \right] (t).
\]

According to [31] a solution to a space-time pseudo-differential equation is said to be mild if its Fourier-Laplace of Laplace-Laplace transform solves the corresponding algebraic equation. Clearly if \( f(\lambda) = \lambda^\alpha \) we retrieve from (4.4) the Dzerbayshan-Caputo derivative.

In Kochubei [20] the author introduced operators of the form

\[
\mathbb{D}_{(k)} u = \frac{d}{dt} \int_0^t k(t-\tau)u(\tau)d\tau - k(t)u(0)
\]

under assumptions on the Laplace transform of \( k(t) \) (see formula (1.6) of [20]). Clearly if \( k(t) = \frac{t^\alpha}{\Gamma(1-\alpha)} \) one retrieves from (4.5) the regularized Riemann-Liouville fractional derivative having Laplace symbol \( \mathcal{L} [\mathbb{D}_{(k)} u](\lambda) = \lambda^\alpha \tilde{u} - \lambda^{\alpha-1} u(0) \). In the present work we follow the last approach (in a way similar to [42]) but we focus on the distributed order case. We introduce the following distributed order integro-differential operator.

**Definition 4.1.** Let \( u \) be an absolutely continuous function on the interval \([c,d]\). Let \( f \) be the Bernstein function as in (2.4) and assume that the function \( s \to \tilde{\nu}(s,y) \) is absolutely continuous on \((0,\infty)\) for all \( y \in E \). We define the generalized distributed order Riemann-Liouville derivative as

\[
_{c}D^{f,p}_{t} u(t) = \mathbb{E} b(Y) \frac{d}{dt} u(t) + \frac{d}{dt} \int_c^t u(s) \mathbb{E} \tilde{\nu}(t-s,Y)ds, \quad c < t < d,
\]

where

\[
\mathbb{E} \tilde{\nu}(s,Y) = \mathbb{E} (a(Y) + \nu((s,\infty),Y)),
\]
and $Y$ is a r.v. with law $p$ on $W \subseteq E$. If $E \bar{b}(Y) = 0$ formula (4.6) make sense for a continuous function $\nu$ as it happens for the classical distributed order fractional derivative ([27] formula (2.6) and (2.7)).

By comparing Definition 4.1 with the operator $f(\partial_t)$ ([31]) defined as in (4.4) and with (1.18), (1.19) ([42]) we note the following relationships.

**Lemma 4.2.** We have that

$$
\mathcal{L} \left[ \alpha \mathcal{D}^{p}_{t} u(t) \right] (\lambda) = E\bar{b}(Y)\tilde{u}(\lambda) - E\bar{b}(Y)u(0) \tag{4.8}
$$

$$
\mathcal{L} \left[ \alpha \mathcal{D}^{p}_{t} u(t) - E\bar{b}(t, Y)u(0) \right] (\lambda) = E\bar{b}(Y)\tilde{u}(\lambda) - \lambda^{-1}E\bar{b}(Y)u(0) \tag{4.9}
$$

**Proof.** The results follow by explicitly computing the transformation as

$$
\mathcal{L} \left[ \alpha \mathcal{D}^{p}_{t} u(t) \right] (\lambda) = \mathcal{L} \left[ E\bar{b}(Y) \frac{d}{dt} u(t) + \frac{d}{dt} \int_{0}^{t} u(s)E\bar{b}(t-s, Y)ds \right] (\lambda)
$$

$$
= E\bar{b}(Y) (\lambda\tilde{u}(\lambda) - u(0)) + \lambda \mathcal{L} [u \ast E\bar{b}] (\lambda)
$$

$$
= E\bar{b}(Y) (\lambda\tilde{u}(\lambda) - u(0)) + \lambda (\tilde{u}(\lambda) (\lambda^{-1}E\bar{b}(Y) - E\bar{b}(Y)))
$$

$$
= E\bar{b}(Y)\tilde{u}(\lambda) - E\bar{b}(Y)u(0) \tag{4.10}
$$

where we used (2.6), and this proves (4.8). With the symbol $u \ast v$ we mean the Laplace convolution. The expression (4.9) follows from (2.6) and (4.10) since

$$
\mathcal{L} [E\bar{b}(s, Y)u(0)](\lambda) = (E\lambda^{-1}f(\lambda, Y) - E\bar{b}(Y))u(0). \tag{4.11}
$$

**Remark 4.3.** Note that if $E \bar{b}(Y) = 0$, and

$$
\nu(ds, y) = \frac{\alpha(y)s^{-\alpha(y)-1}}{\Gamma(1-\alpha(y))} ds \tag{4.12}
$$

the integro-differential operator of Definition 4.1 becomes

$$
_{0} \mathcal{D}^{p}_{t} u(t) = \frac{d}{dt} \int_{0}^{t} u(s) \int_{W} \frac{(t-s)^{-\alpha(y)}}{\Gamma(1-\alpha(y))} p(dy) ds \tag{4.13}
$$

Since $0 < \alpha(y) \leq 1$, we obtain a form of the distributed order Riemann-Liouville derivative written as

$$
\int_{W} R_{t}^{\alpha(y)} \frac{\partial}{\partial t^{\alpha(y)}} u(t) p(dy) = \int_{W} \frac{1}{\Gamma(1-\alpha(y))} \frac{d}{dt} \int_{0}^{t} u(s) (t-s)^{-\alpha(y)} ds p(dy), \tag{4.14}
$$

for which

$$
\mathcal{L} \left[ \alpha \mathcal{D}^{p}_{t} u(t) \right] (\lambda) = E\lambda^{\alpha(Y)}\tilde{u}(\lambda) \tag{4.15}
$$

In order to obtain a more familiar expression of the distributed order fractional derivative one can set in (4.14) $\alpha(y) = \alpha y$, for a constant $0 < \alpha < 1$ and $p(dy)$ a probability measure on $(0, 1)$.

**Remark 4.4.** The operator

$$
\alpha \mathcal{D}^{p}_{t} u(t) - E\bar{b}(t, Y)u(0) \tag{4.16}
$$

has the form of a regularized Riemann-Liouville derivative with a different kernel (as in (4.5)) and Lemma 4.2 shows that it may be viewed as a generalized distributed
order Dzerbayshan-Caputo derivative. In the logic of (4.16) we note that if $Eb(Y) = 0$, $\tilde{v}(s, y) = s^{-\alpha}(y)/\Gamma(1 - \alpha(y))$ we get that

$$\mathcal{L} \left[ D^{f,p}_t u(t) - Eb(s, Y) \right] (\lambda) = E \lambda^\alpha(Y) \tilde{u}(\lambda) - \lambda^{-1} E \lambda^\alpha(Y) u(0).$$  \tag{4.17}$$

The above arguments and Lemma 4.2 inspire the following

**Definition 4.5.** Let $u$, $f$ and $\tilde{v}$ be as in Definition 4.1. The regularized distributed order Riemann-Liouville derivative may be generalized as

$$D^{f,p}_t u(t) = D^{f,p}_t u(t) - u(c)\tilde{v}(t - c, Y)$$

$$= Eb(Y) \frac{d}{dt} u(t) + \int_0^t u(s)\tilde{v}(t - s, Y)ds - u(c)\tilde{v}(t - c, Y).$$ \tag{4.18}$$

If $Eb(Y) = 0$ formula (4.18) make sense for a continuous function $u$, as it is for the classical distributed order fractional derivative (see [27] formula (2.6) and (2.7)).

4.2. The governing equations. In what follows we work with subordinators with a density. In view of Theorem 27.7 in Sato [37] we know that a sufficient condition for saying that a subordinator has a density is that $\nu(0, \infty) = \infty$ and that the function $s \rightarrow \tilde{v}(s) = a + \nu(s, \infty)$ is absolutely continuous on $(0, \infty)$. In such a framework we have the following.

**Theorem 4.6.** Let $\sigma^{f,p}(t)$ be a subordinator with Laplace exponent $f(\lambda, y)$. Suppose that $\nu((0, \infty), y) = \infty$, $\forall y \in E$, and that the function $s \rightarrow \tilde{v}(s, y) = a(y) + \nu((s, \infty), y)$ is absolutely continuous on $(0, \infty)$ for all $y \in E$. Let $\sigma^{f,p}$ be the corresponding distributed order subordinator with Laplace exponent $E f(\lambda, Y)$. Let $L^{f,p}(t)$, $t > 0$, be the inverse of $\sigma^{f,p}$ with $l_s(B) = \Pr \{ L^{f,p}(t) \in B \}$, B Borel and denote by $l_p(x, t)$ the corresponding density. We have that

1. The subordinator $\sigma^{f,p}(t)$ has a density
2. Denote the density of $\sigma^{f,p}(t)$ of point (1) by $\mu_p(x, t)$. We have that $l_p(x, t) = Eb(Y)\mu_p(t, x) + \int_0^t \mu_p(s, x)\tilde{v}(t - s, Y)ds.$
3. If $Eb(Y) > 0$ assume that $x \rightarrow \mu_p(x, t)$ is differentiable. We have that $\mu_p(x, t)$ solves

$$\frac{\partial}{\partial t} q(x, t) = -u(t)D^{f,p}_x q(x, t), \quad x > t Eb(Y), 0 < t < \infty,$$  \tag{4.19}$$

subject to

$$\begin{cases}
q(tEb(Y), t) = 0, \\
q(x, 0)dx = \delta_0(dx),
\end{cases}  \tag{4.20}$$

4. The density $l_p(x, t)$ of $L^{f,p}(t)$ solves

$$0 D^{f,p}_t q(x, t) = -\frac{\partial}{\partial x} q(x, t), \quad 0 < t < \infty, \begin{cases}
0 < x < \frac{t}{Eb(Y)}, & Eb(Y) > 0; \\
0 < x < \infty, & Eb(Y) = 0,
\end{cases}$$  \tag{4.21}$$

subject to

$$\begin{cases}
q(0, t) = \tilde{v}(t, Y), \\
q(t/Eb(Y), t) = 0, \text{ if } Eb(Y) > 0, \\
q(x, 0)dx = \delta_0(dx).
\end{cases}  \tag{4.22}$$
Proof. (1) Since $s \to \tilde{v}(s, y)$ is absolutely continuous we have that

$$
\int_{W} \tilde{v}(s, y)p(dy) = \int_{W} a(y)p(dy) + \int_{W}^{\infty} v(w, y)dw p(dy)
$$

$$
= \int_{W} a(y)p(dy) + \int_{s}^{\infty} \int_{W} v(w, y)p(dy) dw
$$

(4.23)

where we denote by $v$ the density of $\nu$. Since $v$ is a density we have that $Ev(s, Y)$ is a density and therefore $\int_{s}^{\infty} Ev(s, Y)ds$ is absolutely continuous. Furthermore if $\nu((0, \infty), y) = \infty$, $\forall y \in E$ we clearly have that $Ev((0, \infty), Y) = \infty$. By applying Theorem 27.7 of [37] we have proved this point.

(2) First note that

$$
\mathcal{L}[\sigma^{f,p}(x)](\lambda) = -\frac{\partial}{\partial x} \int_{0}^{\infty} e^{-\lambda t} \Pr \{\sigma^{f,p}(x) \leq t\} \, dt
$$

$$
= \left(\lambda^{-1} \int_{W} f(\lambda, y)p(dy)\right) \exp \left\{-x \int_{W} f(\lambda, y)p(dy)\right\}. \quad (4.24)
$$

Now by (2.6) we write

$$
\mathcal{L} \left[ Eb(Y)\mu_{t}(s, x) + \int_{0}^{s} \mu_{t}(s, x)Eb(\cdot - s, Y)ds \right](\lambda)
$$

$$
= Eb(Y)e^{-\lambda Ef(\lambda, Y)} + \mathcal{L} [\mu_{t}(s, x) * Eb(\cdot, Y)](\lambda)
$$

$$
= Eb(Y)e^{-\lambda Ef(\lambda, Y)} + e^{-\lambda Ef(\lambda, Y)} \left( \frac{Ef(\lambda, Y)}{\lambda} - Eb(Y) \right)
$$

$$
= \left(\lambda^{-1} \int_{W} f(\lambda, y)p(dy)\right) \exp \left\{-x \int_{W} f(\lambda, y)p(dy)\right\}. \quad (4.25)
$$

and since (4.25) coincides with (4.24) we have proved this point.

(3) Note that we need the differentiability of $x \to \mu_{t}(x, t)$ only if $Eb(Y) > 0$, indeed for $Eb(Y) = 0$ the operator

$$
\int_{0}^{t} \mu_{t}(s, t) \int_{W} \tilde{v}(t - s, y)p(dy)ds
$$

(4.26)

exists for a continuous function. Therefore we may conclude that (4.26) is well defined since $x \to \mu_{t}(x, t)$ is a density. In view of Lemma 4.2 it is easy to show that the Laplace-Laplace transform of the analytical solution to (4.19) is

$$
\tilde{q}(\phi, \lambda) = \mathcal{L} [\mathcal{L} [q(x, t)] (\phi)](\lambda) = \frac{1}{\lambda + Ef(\phi, Y)}. \quad (4.27)
$$

By observing that

$$
\int_{0}^{\infty} e^{-\lambda t} e^{-\phi f^{p}(t)} = \int_{0}^{\infty} e^{-t(\lambda + Ef(\phi, Y))} dt = \frac{1}{\lambda + Ef(\phi, Y)}, \quad (4.28)
$$

the proof is complete.

(4) In view of point (2) we know that

$$
l_{p}(x, t) = Eb(Y)\mu_{t}(x, t) + \int_{0}^{t} \mu_{t}(s, x)Eb(s, Y)ds
$$

(4.29)
and therefore the map
\[ t \to l_p(x,t) \]  
(4.30)
is differentiable. Note that if \( \mathbb{E}b(Y) = 0 \) we don’t need to use the differentiability of \( t \to \mu_p(t,x) \) since it is a density and the operator \( _aD^{f,p} \) exists for a continuous function, as well as the classical distributed order derivative (27 page 217).

Now consider the Laplace-Laplace transform of the solution to (4.21). In view of Lemma 4.2 and formula (2.6) we have
\[ \begin{cases} E_f(\lambda, Y)\tilde{q}(x, \lambda) + E_f(\lambda, Y)\frac{\lambda}{\lambda} \tilde{q}(x, \lambda) = -\frac{\partial}{\partial x} \tilde{q}(x, \lambda), \\ \tilde{q}(0, \lambda) = E_f(\lambda, Y) - E_b(Y) \\ \tilde{q}(x, 0) = \delta_0 \end{cases} \]
(4.31)
and thus
\[ \tilde{q}(\phi, \lambda) = \frac{1}{\phi + E_f(\lambda, Y)} E_f(\lambda, Y). \]
(4.32)
Now by considering (4.25) we have that
\[ \mathcal{L}[\mathcal{L} [l_p(x,t)](\lambda)](\phi) = \frac{1}{\lambda \phi + E_f(\lambda, Y)}, \]
(4.33)
and this completes the proof.

□

5. AN APPLICATION TO SLOW DIFFUSIONS

An important application of distributed order fractional calculus is to model ultraslow diffusions, i.e. diffusions with mean square displacement \( (\Delta x)^2 \sim C \log t, \ C > 0 \) (see, for example, Chechkin et al. [12, 13]; Kochubei [18]; Meerschaert and Scheffer [30]). Roughly a diffusion is said to be slow if the mean square displacement behaves like \( ct^\alpha \), for some \( c > 0 \) and \( \alpha < 1 \). For subdiffusion see also Magdziarz [23]. In Kochubei [18] the reader can find a rigorous mathematical treatment of the equation
\[ \int_0^1 \frac{\partial^{\beta}}{\partial \gamma^\beta} q(x,t)p(d\gamma) = \Delta q(x,t). \]
(5.1)
Among other things, the author studied the behaviour of
\[ (\Delta x)^2 = \int_{\mathbb{R}^n} (x - u)^2 Z^\beta(x - u, t)du \]
(5.2)
where \( Z^\beta(x,t) \) is the fundamental solution to (5.1). In this section we consider the fundamental solution to the equation
\[ _aD^{f,p}_t q(x,t) = \Delta q(x,t), \quad x \in \mathbb{R}^n, t > 0, \]
(5.3)
where \( _aD^{f,p}_t \) is defined in (4.18) and we study the mean square displacement. Our approach based on Lévy mixing permits to carry out the study of (5.3) by means of the so-called delayed Brownian motion (consult Magdziarz and Schilling [24] for recent developments on sample paths properties of delayed Brownian motion). We work as in the previous section with Lévy measures such that \( \nu((0, \infty), y) = \infty \) and \( s \to \tilde{\nu}(s, y) \) is absolutely continuous, for all \( y \in E \). In the following Theorem we study the behaviour of the mean square displacements related to the fundamental
solution to (5.3) under the additional assumption that $E f(\lambda, Y)$ is regularly varying at $0^+$. A function $f$ is said to be regularly varying at $0^+$ if $\lim_{\lambda \to 0^+} f(\lambda x)/f(\lambda)$ is convergent. In our case $E f(\lambda, Y) \in BF$ and due to the Lévy-Khintchine representation the limit is necessarily equal to $x^\alpha$ for $\alpha \in [0,1]$. The reader can consult Bingham et al. [8] for a self-contained text on regular variation.

**Theorem 5.1.** Let $L^{f,p}(t)$ be as in Theorem 4.6. Let

$$\mathcal{M}(t) = \int_{\mathbb{R}^n} (x-u)^2 q(x-u,t)du$$

(5.4)

where $q(x,t)$ is the fundamental solution to (5.3). Let $\alpha$ be such that

$$\lim_{\lambda \to 0} E f(\lambda x, Y)/E f(\lambda, Y) = x^\alpha$$

(5.5)

and $B_n$ be the $n$-dimensional Brownian motion. We have the following results.

1. $q(x,t) = \Pr \{ B_n (L^{f,p}(t)) \in dx \}$ /dx
2. $\frac{1}{2}\Gamma(1+\alpha) \mathcal{M}(t) \sim \frac{1}{t^{1+\alpha}}$ as $t \to \infty$
3. If $E a(Y) > 0$ then $\lim_{t \to \infty} \mathcal{M}(t) < \infty$
4. In general we have that

$$\lim_{t \to \infty} \frac{t}{2\Gamma(1+\alpha)} \mathcal{M}(t) = E b(Y) + \int_0^\infty \int_W \bar{\nu}(s,y)p(dy)ds > 0,$$

(5.6)

which is infinite or finite (but greater than zero), depending on the kernel $E \bar{\nu}(s,Y)$.

**Proof.** First we prove that the fundamental solution to (5.3) admits the representation (1) and thus coincides with the law of the $n$-dimensional time-changed Brownian motion $B_n (L^{f,p}(t))$, $t > 0$. The law of $B_n (L^{f,p}(t))$ is

$$\Pr \{ B_n (L^{f,p}(t)) \in dx \} /dx = \int_0^\infty e^{-\frac{|x|^2}{4s}} \frac{1}{(4\pi s)^{n/2}} l_p(s,t)ds.$$

(5.7)

In view of point (1) of Theorem 4.6 we have that (5.7) becomes

$$\Pr \{ B_n (L^{f,p}(t)) \in dx \} /dx$$

$$= \int_0^\infty e^{-\frac{|x|^2}{4s}} \left( b_{l_p}(t,s) + \int_0^t \mu_p(w,s)E \bar{\nu}(t-w,Y)dw \right)ds$$

(5.8)

where $\mu_p(w,s)$ is the density of the subordinator $\sigma^{f,p}$ such that

$$L^{f,p}(t) = \inf \{ s \geq 0 : \sigma^{f,p}(s) > t \}.$$

(5.9)

Furthermore $\Pr \{ B_n (L^{f,p}(t)) \in dx \} /dx \xrightarrow{t \to 0} \delta(x)$. By using Lemma 4.2 we can consider the Fourier-Laplace transform of the analytical solution $q$ to (5.3) (with $q(x,0) = \delta(x)$) which reads

$$\hat{q}(\xi, \lambda) = \frac{E f(\lambda, Y)}{\lambda} \frac{1}{E f(\lambda, Y) + |\xi|^2}$$

(5.10)

and by computing the Fourier-Laplace transform of (5.7) we get that

$$\mathcal{L} \left[ e^{iB_n \{ L^{f,p}(\cdot) \}} \right] (\lambda) = \mathcal{L} \left[ \int_0^\infty e^{-s|\xi|^2} l_p(s,\cdot)ds \right] (\lambda)$$

$$= \lambda^{-1} E f(\lambda, Y) \frac{1}{E f(\lambda, Y) + |\xi|^2}.$$

(5.11)
We have proved (1). The mean square displacement in this case becomes
\[
\mathcal{M}(t) = \langle (\Delta x)^2 \rangle = \int_{\mathbb{R}^n} (x - u)^2 q(x - u, t) du
\]
\[
= 2 \int_0^\infty s \lambda_p(s, t) ds
\]
\[
= 2 U^{f-p}(t) \tag{5.12}
\]
where
\[
U^{f-p}(t) = E^{f-p}(t) = E \int_0^\infty 1_{\{\sigma^{f-p}(x) < t\}} dx \tag{5.13}
\]
is known in literature as the renewal function. The behavior of such a function has been studied under the assumption of regular variation. We recall that if \( f(x) \) is a Bernstein function regularly varying at 0+ then for \( \alpha \in [0, 1] \)
\[
\Gamma(1 + \alpha) U_f(cx) \sim c^{\alpha} / f(1/x) \quad \text{as} \quad x \to \infty. \tag{5.14}
\]
Such a result may be found in Bertoin [7], Proposition 1.5. By applying (5.14) we write for (5.12)
\[
\frac{1}{2} \Gamma(1 + \alpha) \mathcal{M}(t) = \frac{1}{2} \Gamma(1 + \alpha) U^{f-p}(t) \sim \frac{1}{Ef(1/t, Y)} \quad \text{as} \quad t \to \infty, \tag{5.15}
\]
provided that the Bernstein function \( Ef(t, Y) \) is regularly varying at 0+. We have also proved (3).

We observe that in general from result (5.15) we can write
\[
\lim_{t \to \infty} \frac{t}{2} \Gamma(1 + \alpha) \mathcal{M}(t) = \mathbb{E}b(Y) + \lim_{t \to 0} \int_0^\infty e^{-st} \int_W \bar{v}(s, y) p(dy) ds \tag{5.16}
\]
which clearly can not be zero but can be either finite or infinite, depending from the Lévy measure and from \( \mathbb{E}a(Y) \). If \( \mathbb{E}a(Y) > 0 \) the limit (5.16) is clearly infinite. From (5.16) we can write
\[
\lim_{t \to \infty} \frac{t}{2} \Gamma(1 + \alpha) \mathcal{M}(t) = \mathbb{E}b(Y) + \int_0^\infty \int_W \bar{v}(s, y) p(dy) ds, \tag{5.17}
\]
which proves (4) and completes the proof of the Theorem. \( \square \)

**Remark 5.2.** Actually the limit (5.17) is infinite in most common cases and since it can not be zero we can state that the corresponding diffusions are subdiffusive or at most it can happen that \( \frac{1}{2} \Gamma(1 + \alpha) \mathcal{M}(t) \sim Ct, \ C > 0 \).

**Remark 5.3.** As pointed out in Remark 4.3, a fractional case can be for example \( f(t, y) = t^{\beta(y)}, \ 0 < \beta(y) < 1 \). Suppose that \( \beta(y) = \beta y \) with \( p(dy) \) uniform in \((0, 1/C), \ C > 1 \) and a constant \( 0 < \beta < 1 \). According to (5.15) we have for the mean square displacement
\[
\frac{1}{2} \Gamma \left( 1 + \frac{\beta}{C} \right) \mathcal{M}(t) \sim \frac{1}{\int_0^{1/C} \int_{t^{\beta/y}} dy} = \frac{\beta}{1 - t^{-\beta/C}} \tag{5.18}
\]
and this is in accord with Theorem 4.3 of Kochubei [18]. Suppose instead, for example, that
\[
f(t, y) = \log(1 + t/y) = \int_0^\infty (1 - e^{-st}) e^{-ys/s} ds, \quad y > 0, \tag{5.19}
\]
which is the Laplace exponent of a Gamma subordinator, parametrized by \( y \). If \( p(dy) = \delta_y \) one has from (5.17) that
\[
\lim_{t \to \infty} \frac{t}{2 \Gamma(2)} \mathcal{M}(t) = \int_0^\infty e^{-ys} ds = \frac{1}{y}.
\]
(5.20)

Suppose instead that \( p(dy) = \beta \gamma^{-1} y^{-\beta-1} dy \) for \( y \geq \gamma > 0, \beta > 0 \), formula (5.17) yields
\[
\lim_{t \to \infty} \frac{t}{2 \Gamma(1+\alpha)} \mathcal{M}(t) = \frac{1}{\gamma \beta + 1}.
\]
(5.21)

References

[1] H. Allouba and W. Zheng. Brownian-time processes: The PDE connection and the half-derivative generator. *The Annals of Probability*, 29(4): 1780 – 1795, 2001.
[2] B. Baeumer and M.M. Meerschaert. Stochastic solutions for fractional Cauchy problems. *Fractional Calculus and Applied Analysis*, 4(4): 481 – 500, 2001.
[3] O.E. Barndorff-Nielsen, V. Pérez-Abreu and S. Thorbjørnsen. Lévy Mixing. *ALEA, Lat. Am. J. Probab. Stat.*, 10(2): 1013 – 1062, 2013.
[4] L. Beghin. Random-time processes governed by differential equations of fractional distributed order. *Chaos, Solitons & Fractals*, 45: 1314 – 1327, 2014.
[5] S. Bernstein. Sur les fonctions absolument monotones (french). *Acta Mathematica*, 52; 1 – 66, 1929.
[6] J. Bertoin. Lévy processes. *Cambridge University Press*, Cambridge, 1996.
[7] J. Bertoin. Subordinators: examples and appications. *Lectures on probability theory and statistics (Saint-Flour, 1997)*, 1 – 91. *Lectures Notes in Math.*, 1717, Springer, Berlin, 1999.
[8] N.H. Bingham, C.M. Goldie and J.L. Teugels. Regular variation. *Cambridge University Press*, Cambridge, 1987.
[9] R.M. Blumenthal and R.K. Getoor. Markov processes and potential theory. *Pure and Applied Mathematics*, Vol. 29, *Academic press*, New York - London, 1968.
[10] R.M. Blumenthal and R.K. Getoor. Dual processes and potential theory. *Proc. 12th Biennial Seminar, Canad. Math. Congress*, 137-156, 1970.
[11] M. Caputo. Mean fractional-order derivatives, differential equations and filters. *Ann. Univ. Ferrara Sez. VII Sci. Mat.*, 41: 73 – 84, 1995.
[12] A.V. Chechkin, R. Gorenflo and I.M. Sokolov. Retarding subdiffusion and accelerating superdiffusion governed by distributed order fractional diffusion equations. *Physical Review E*, 66, 046129, 2002.
[13] A.V. Chechkin, V.Yu. Gonchar, R. Gorenflo, N. Korabel and I.M. Sokolov. Generalized fractional diffusion equations for accelerating subdiffusion and truncated Lévy flights. *Physical Review E*, 78, 021111, 2008.
[14] A.V. Chechkin, R. Gorenflo, I.M. Sokolov, V.Yu. Gonchar. Distributed order fractional diffusion equation. *Fractional Calculus and Applied Analysis*, 6: 259 – 279, 2003.
[15] K. Itô. On stochastic processes. I. (Infinitely divisible laws of probability). *Japanese Journal of Mathematics*, 18: 261 – 301, 1942.
[16] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo. Theory and Applications of Fractional Differential Equations. *North-Holland Mathematics Studies, 204.* Elsevier Science B.V., 2006.

[17] J.F.C. Kingman. Poisson Processes. *Oxford University Press,* 1993.

[18] A.N. Kochubei. Distributed order calculus and equations of ultraslow diffusion. *Journal of Mathematical Analysis and Applications,* 340: 252 – 281, 2008.

[19] A.N. Kochubei. Distributed order derivatives and relaxation patterns. *J. Phys. A,* 315203, 2009.

[20] A.N. Kochubei. General fractional calculus, evolution equations and renewal processes. *Integral Equations and Operator Theory,* 71: 583 – 600, 2011.

[21] C.F. Lorenzo and T.T. Hartley. Variable order and distributed order fractional operators. *Nonlinear Dynamics,* 29: 57 – 98, 2002.

[22] Y. Luchko. Boundary value problems for the generalized time-fractional diffusion equation of distributed order. *Fractional Calculus and Applied Analysis,* 12: 409–422, 2009.

[23] M. Magdziarz. Stochastic representation of subdiffusion processes with time-dependent drift. *Stoch. Proc. Appl.***, 119: 3238 – 3252, 2009.

[24] M. Magdziarz and R.L. Schilling. Asymptotic properties of Brownian motion delayed by inverse subordinators. *Preprint, arXiv:1311.6043,* 2013.

[25] F. Mainardi, A. Mura, R. Gorenflo and M. Stojanović. The two forms of fractional relaxation of distributed order. *Journal of Vibration and Control,* 13(9-10): 1249 – 1268, 2007.

[26] F. Mainardi and G. Pagnini. The role of the Fox-Wright functions in fractional sub-diffusion of distributed order. *Journal of Computational and Applied Mathematics,* 207: 245 – 257, 2007.

[27] M.M. Meerschaert, E. Nane and P. Vellaisamy. Distributed order fractional diffusions on bounded domains. *Journal of Mathematical Analysis and Applications,* 379: 216 – 228, 2011.

[28] M.M. Meerschaert, E. Nane and P. Vellaisamy. Fractional Cauchy problems on bounded domains. *The Annals of Probability,* 37(3):979 – 1007, 2009.

[29] M.M. Meerschaert, E. Nane and P. Vellaisamy. The fractional Poisson process and the inverse stable subordinator. *Electronic Journal of Probability,* 16(59): 1600–1620, 2011.

[30] M.M. Meerschaert and H.P. Scheffer. Stochastic model for ultraslow diffusion. *Stochastic Processes and their Applications,* 116: 1215 – 1235, 2006.

[31] M.M. Meerschaert and H.P. Scheffer. Triangular array limits for continuous time random walks. *Stochastic Processes and their Applications,* 118(9): 1606 – 1633, 2008.

[32] R. Metzler, J. Klafter. The random walks guide to anomalous diffusion: A fractional dynamics approach. *Phys. Rep.,* 339: 1–77, 2000.

[33] R. Metzler, J. Klafter. The restaurant at the end of the random walk: Recent developments in the description of anomalous transport by fractional dynamics. *J. Phys. A,* 339(37): R161–R208, 2004.

[34] E. Orsingher and L. Beghin. Time-fractional telegraph equations and telegraph process with Brownian time. *Probability Theory and Related Fields,* 128: 141 – 160, 2003.

[35] E. Orsingher and L. Beghin. Fractional diffusion equations and processes with randomly varying time. *The Annals of Probability,* 37(1):206 – 249, 2009.
[36] A.I. Saichev and G.M. Zaslavsky. Fractional kinetic equations: solutions and applications. *Chaos, 7*(4): 753 – 764, 1997.

[37] K. Sato. Lévy processes and infinitely divisible distributions. *Cambridge University Press*, 1999.

[38] R.L. Schilling, R. Song and Z. Vondraček. Bernstein functions: theory and applications. *Walter de Gruyter GmbH & Company KG*, Vol 37 of De Gruyter Studies in Mathematics Series, 2010.

[39] I.M. Sokolov, A.V. Chechkin and J. Klafter. Distributed-order fractional kinetics. *Acta Phys. Polon. B.*, 35: 1323 – 1341, 2004.

[40] R. Song and Z. Vondraček. Potential theory of subordinate Brownian motion. In: *Potential Analysis of Stable Processes and its Extensions*, P. Graczyk, A. Stos, editors, Lecture Notes in Mathematics 1980: 87–176, 2009.

[41] F.W. Steutel and K. van Harn. Infinite divisibility of probability distributions on the real line. *Volume 259 of Monographs and Textbooks in Pure and Applied Mathematics.*, Marcel Dekker Inc., New York, 2004.

[42] B. Toaldo. Convolution-type derivatives, hitting-times of subordinators and time-changed $C_0$-semigroups. *Potential Analysis*, online since 3rd July 2014.

[43] G. Zaslavsky. Fractional kinetic equation for Hamiltonian chaos. Chaotic advection, tracer dynamics and turbulent dispersion. *Phys. D, 76*: 110 – 122, 1994.

**Department of Statistical Sciences, Sapienza - University of Rome**

*E-mail address: bruno.toaldo@uniroma1.it*