Criticality and scaling corrections for two-dimensional Heisenberg models in plaquette patterns with strong and weak couplings

Xiaoxue Ran,¹ Nvsen Ma,²† and Dao-Xin Yao¹†

¹State Key Laboratory of Optoelectronic Materials and Technologies, School of Physics, Sun Yat-sen University, Guangzhou 510275, China
²Beijing National Laboratory of Condensed Matter Physics and Institute of Physics, Chinese Academy of Sciences, Beijing 100190, China

We use the stochastic series expansion quantum Monte Carlo method to study the Heisenberg models on the square lattice with strong and weak couplings in the form of three different plaquette arrangements known as checkerboard models C2 × 2, C2 × 4 and C4 × 4. The a × b here stands for the shape of plaquette consisting with spins connected by strong couplings. Through detailed analysis of finite-size scaling study, the critical point of C2 × 2 model is improved as gc = 0.548524(3) compared with previous studies with g to be the ratio of weak and strong couplings in the models. For C2 × 4 and C4 × 4 we give gc = 0.456978(2) and 0.314451(3). We also study the critical exponents ν, η, and the universal property of Binder ratio to give further evidence that all quantum phase transitions in these three models are in the three-dimensional O(3) universality class. Furthermore, our fitting results show the importance of effective corrections in the scaling study of these models.

I. INTRODUCTION

The S = 1/2 Heisenberg antiferromagnetic model with different interactions⁴,⁵ has always been a very interesting topic in both theoretical and experimental fields because of its rich ground states and close relations to cuprate superconductors⁶,⁷, Bose-Einstein condensation of magnons⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate superconductors⁵–⁷, Bose-Einstein condensation⁸,⁹, etc. One of the best studied cuprate su
the arrangements of $J_2$ shown in Fig. 1, we refer to these plaquette models as $C_2 \times 2$, $C_2 \times 4$ and $C_4 \times 4$ model. It is obvious that the quadrumerized Heisenberg model mentioned before is the $C_2 \times 2$ model. We set $J_1 = 1$ and define the ratio of weak and strong couplings to be $g = J_2 / J_1$. When $g = 1$ the model becomes an isotropic Heisenberg plane which has antiferromagnetic ground state with long-range order. When $g = 0$, the ground state turns into a disordered phase with no magnetism. It is a product state of singlets differ in different arrangements of $J_2$ (black thin bonds) to form the checkerboard pattern.

of the arrangements of $J_2$ shown in Fig. 1 we refer to these plaquette models as $C_2 \times 2$, $C_2 \times 4$ and $C_4 \times 4$ model. It is obvious that the quadrumerized Heisenberg model mentioned before is the $C_2 \times 2$ model. We set $J_1 = 1$ and define the ratio of weak and strong couplings to be $g = J_2 / J_1$. When $g = 1$ the model becomes an isotropic Heisenberg plane which has antiferromagnetic ground state with long-range order. When $g = 0$, the ground state turns into a disordered phase with no magnetism. It is a product state of singlets differ in different arrangements of $J_2$ (black thin bonds) to form the checkerboard pattern.

of the arrangements of $J_2$ shown in Fig. 1 we refer to these plaquette models as $C_2 \times 2$, $C_2 \times 4$ and $C_4 \times 4$ model. It is obvious that the quadrumerized Heisenberg model mentioned before is the $C_2 \times 2$ model. We set $J_1 = 1$ and define the ratio of weak and strong couplings to be $g = J_2 / J_1$. When $g = 1$ the model becomes an isotropic Heisenberg plane which has antiferromagnetic ground state with long-range order. When $g = 0$, the ground state turns into a disordered phase with no magnetism. It is a product state of singlets differ in different arrangements of $J_2$ (black thin bonds) to form the checkerboard pattern.

of the arrangements of $J_2$ shown in Fig. 1 we refer to these plaquette models as $C_2 \times 2$, $C_2 \times 4$ and $C_4 \times 4$ model. It is obvious that the quadrumerized Heisenberg model mentioned before is the $C_2 \times 2$ model. We set $J_1 = 1$ and define the ratio of weak and strong couplings to be $g = J_2 / J_1$. When $g = 1$ the model becomes an isotropic Heisenberg plane which has antiferromagnetic ground state with long-range order. When $g = 0$, the ground state turns into a disordered phase with no magnetism. It is a product state of singlets differ in different arrangements of $J_2$ (black thin bonds) to form the checkerboard pattern.

of the arrangements of $J_2$ shown in Fig. 1 we refer to these plaquette models as $C_2 \times 2$, $C_2 \times 4$ and $C_4 \times 4$ model. It is obvious that the quadrumerized Heisenberg model mentioned before is the $C_2 \times 2$ model. We set $J_1 = 1$ and define the ratio of weak and strong couplings to be $g = J_2 / J_1$. When $g = 1$ the model becomes an isotropic Heisenberg plane which has antiferromagnetic ground state with long-range order. When $g = 0$, the ground state turns into a disordered phase with no magnetism. It is a product state of singlets differ in different arrangements of $J_2$ (black thin bonds) to form the checkerboard pattern.

of the arrangements of $J_2$ shown in Fig. 1 we refer to these plaquette models as $C_2 \times 2$, $C_2 \times 4$ and $C_4 \times 4$ model. It is obvious that the quadrumerized Heisenberg model mentioned before is the $C_2 \times 2$ model. We set $J_1 = 1$ and define the ratio of weak and strong couplings to be $g = J_2 / J_1$. When $g = 1$ the model becomes an isotropic Heisenberg plane which has antiferromagnetic ground state with long-range order. When $g = 0$, the ground state turns into a disordered phase with no magnetism. It is a product state of singlets differ in different arrangements of $J_2$ (black thin bonds) to form the checkerboard pattern.

of the arrangements of $J_2$ shown in Fig. 1 we refer to these plaquette models as $C_2 \times 2$, $C_2 \times 4$ and $C_4 \times 4$ model. It is obvious that the quadrumerized Heisenberg model mentioned before is the $C_2 \times 2$ model. We set $J_1 = 1$ and define the ratio of weak and strong couplings to be $g = J_2 / J_1$. When $g = 1$ the model becomes an isotropic Heisenberg plane which has antiferromagnetic ground state with long-range order. When $g = 0$, the ground state turns into a disordered phase with no magnetism. It is a product state of singlets differ in different arrangements of $J_2$ (black thin bonds) to form the checkerboard pattern.

of the arrangements of $J_2$ shown in Fig. 1 we refer to these plaquette models as $C_2 \times 2$, $C_2 \times 4$ and $C_4 \times 4$ model. It is obvious that the quadrumerized Heisenberg model mentioned before is the $C_2 \times 2$ model. We set $J_1 = 1$ and define the ratio of weak and strong couplings to be $g = J_2 / J_1$. When $g = 1$ the model becomes an isotropic Heisenberg plane which has antiferromagnetic ground state with long-range order. When $g = 0$, the ground state turns into a disordered phase with no magnetism. It is a product state of singlets differ in different arrangements of $J_2$ (black thin bonds) to form the checkerboard pattern.

of the arrangements of $J_2$ shown in Fig. 1 we refer to these plaquette models as $C_2 \times 2$, $C_2 \times 4$ and $C_4 \times 4$ model. It is obvious that the quadrumerized Heisenberg model mentioned before is the $C_2 \times 2$ model. We set $J_1 = 1$ and define the ratio of weak and strong couplings to be $g = J_2 / J_1$. When $g = 1$ the model becomes an isotropic Heisenberg plane which has antiferromagnetic ground state with long-range order. When $g = 0$, the ground state turns into a disordered phase with no magnetism. It is a product state of singlets differ in different arrangements of $J_2$ (black thin bonds) to form the checkerboard pattern.

of the arrangements of $J_2$ shown in Fig. 1 we refer to these plaquette models as $C_2 \times 2$, $C_2 \times 4$ and $C_4 \times 4$ model. It is obvious that the quadrumerized Heisenberg model mentioned before is the $C_2 \times 2$ model. We set $J_1 = 1$ and define the ratio of weak and strong couplings to be $g = J_2 / J_1$. When $g = 1$ the model becomes an isotropic Heisenberg plane which has antiferromagnetic ground state with long-range order. When $g = 0$, the ground state turns into a disordered phase with no magnetism. It is a product state of singlets differ in different arrangements of $J_2$ (black thin bonds) to form the checkerboard pattern.
average values of the observables.

A. Obeservables

In order to study the criticality of the certain spin model we choose to measure several important physical quantities in our work. The first one is Binder ratio defined as

$$R_2 = \frac{\langle (m_x^2)^4 \rangle}{\langle (m_x^2)^2 \rangle^2},$$

where

$$m_x^2 = \frac{1}{N} \sum_{i}^N S_i^x (-1)^{x_i+y_i}$$

with $N = L \times L$ to be the total number of spins on the square lattice and $(x_i, y_i)$ the coordinate of corresponding spin $S_i$. The Binder ratio is dimensionless and universal regardless of the detailed structures and couplings of models. However it does depend on the boundary conditions and effective aspect ratios from previous experience. Here we use periodic boundary conditions on these three models and effective aspect ratio of time-space is related to the critical spin-wave velocity.

Another quantity is the uniform susceptibility

$$\chi_u = \chi(0,0) = \frac{\beta}{N} \left\langle \left( \sum_{i=1}^N S_i^x \right)^2 \right\rangle$$

whose scaling form at $g_c$ is $\chi_u \sim L^{-d}$, giving $\chi_u \sim L^{-1}$ and $\chi_u L$ to be dimensionless in our case.

The last physical observable calculated in our work is the spin stiffness $\rho_s$. Stiffness $\rho$ is covered in the calculation

$$\delta f = \frac{1}{2} \rho(\nabla \theta)^2 = \frac{1}{2} \rho(\Phi/L)^2$$

in the continuum field theory with $f$ to be the density of free energy, $\Phi$ the boundary twist and $\theta$ the order parameter field. In Heisenberg model $\rho_s$ is the spin stiffness determined by twist $\Phi$ directly to the Hamiltonian, which in SSE procedure can be obtained through the calculation

$$\rho_s^a = \frac{3}{2\beta N} \langle (N_a^+ - N_a^-)^2 \rangle,$$

where $N_a^+$ ($N_a^-$) represent the total number of $S_i^+ S_j^-$ ($S_i^- S_j^+$) operators in the sampling along a (x or y) direction of the square lattice. When the system is isotropic on the lattice $\rho_s^a$ is the same as $\rho_s^b$, while for the anisotropic they are different. So in C2 x 2 and C4 x 4 model we only calculate $\rho_s = (\rho_s^a + \rho_s^b)/2$ and for C2 x 4 model both $\rho_s^a$ and $\rho_s^b$ are recorded separately. However they all have the same scaling form at critical point as $\rho_s \sim L^{2-d-z}$ with $\rho_s \sim L^{-1}$ in our models, which means that $\rho_s L$ is a size-independent dimensionless quantity.

B. Finite-size scaling

After all the mean observable values mentioned above are obtained from the simulations, we need to deal with all these data using the finite-size scaling study method to estimate the critical properties in the thermodynamical limit. From the renormalization group theory we know that a physical quantity $Q$ near its critical point obeys

$$Q(g, L) = L^{\kappa/\nu} f(\delta L^{1/\nu}, \lambda_1 L^{-\omega_1}, \lambda_2 L^{-\omega_2}, \ldots),$$

with $\kappa$ to be the critical exponent of $Q$, $\nu$ the correlation length exponent, and $\delta = g - g_c$. The set $\{\lambda_i\}$ refers to all irrelevant fields with their correction exponents $\{\omega_i\}$, which is arranged as $\omega_{i+1} > \omega_i$. Usually at most one irrelevant field is supposed to be considered in the FSS analysis, but there are still some special cases where more than one field is necessary. Here we start with one correction exponent to the first order of the dimensionless quantities ($\kappa = 0$) so that Eq. (7) can be written as

$$Q(g, L) = f_Q^{(0)}(\delta L^{1/\nu}) + L^{-\omega_1} f_Q^{(1)}(\delta L^{1/\nu}),$$

in which $L^{-\omega_1}$ is regarded as a deviation value of theoretical scaling function $f_Q$ near critical point. Ignoring irrelevant items, dimensionless quantity $Q(g, L)$ does not depend on size of the system at critical point $g_c$ because $g = 0$ then. Thus, $Q(g, L)$ for different sizes cross at critical point in this simplified situation. But here we need to take irrelevant item into consideration and $Q(g, L)$ of different sizes would cross at $g_c(L)$, which is near to the real $g_c$ with a correction to the order $L^{-\omega_1}$.

For two different simulated sizes $L$ and $L'$, using Eq. (8) we have

$$f_Q^{(0)}(g^* L^{1/\nu}) + L^{-\omega_1} f_Q^{(1)}(g^* L^{1/\nu}) = f_Q^{(0)}(g L^{1/\nu}) + L^{-\omega_1} f_Q^{(1)}(g L^{1/\nu})$$

at cross point $g_c(L)$ with $g^* = g_c(L') - g_c$. Expanding $f_Q^{(0)}$ and $f_Q^{(1)}$ to the first order of $L^{-\omega_1}$ with $L' = bL$ we can get

$$g^* = \frac{f_Q^{(1)}(0)}{f_Q^{(0)}(0)} b^{-\omega_1}(b^{\omega_1} - 1) L^{-\omega_1 - 1/\nu},$$

which is more easily understood as

$$g_c(L) = g_c(\infty) + \frac{f_Q^{(1)}(0)}{f_Q^{(0)}(0)} b^{-\omega_1}(b^{\omega_1} - 1) L^{-\omega_1 - 1/\nu}.$$
Besides, with the definition of

$$\frac{1}{\nu(L)} = \frac{1}{\ln(b)} \left( \ln \frac{S(L)}{S(L)} \right)$$  \hspace{1cm} (13)$$

where

$$S(L) = \frac{dQ(g,L)}{dg} \bigg|_{g=g_c(L)}$$  \hspace{1cm} (14)$$

we can also get the scaling of critical exponent $\nu$ combining Eq. (13) and Eq. (15) as

$$\frac{1}{\nu(L)} = \frac{1}{\nu} + aL^{-\omega}$$  \hspace{1cm} (15)$$

with a free parameter $a$. For simplicity the scaling forms of the coordinates of crossing points $(g_c(L), Q_c(L))$ are written as

$$g_c(L) = g_c(\infty) + bL^{-\omega - 1/\nu}$$  \hspace{1cm} (16)$$

$$Q_c(L) = Q_c(\infty) + cL^{-\omega}$$  \hspace{1cm} (17)$$

with $b$ and $c$ to be fitted as free parameters. From Eq. (16) and Eq. (15), we know that using crossing points from $g$ dependence of dimensionless quantity for two sizes $(L, bL)$, the extrapolation value when $L \to \infty$ can give quantum critical point $g_c$ and critical exponent $\nu$ at the thermodynamical limit. In our work, we use $b = 2$ in obtaining all crossing points for different models.

III. SIMULATION RESULTS AND DATA ANALYSIS

We performed the SSE QMC simulations on $C2 \times 2$, $C2 \times 4$ and $C4 \times 4$ models and obtained the average values of all observables $R_2$, $\chi_u$ and $\rho_s$ (or $\rho_y$ and $\rho_{xy}$ especially for $C2 \times 4$). One example of the simulation results for $C2 \times 4$ is illustrated in Fig. 2 to show the crossings of different sizes for four dimensionless quantities $R_2$, $\chi_uL$, $\rho_{xy}L$ and $\rho_{y}L$. Similar figures can also be obtained from SSE data for $C2 \times 2$ and $C4 \times 4$ models. The obvious shift of crossings from different sizes implies that it is necessary to take the correction into account in the scaling analysis.

A. Critical points and corrections

After all crossing points are extracted from the raw data we use the finite-size scaling method to estimate the critical points of our models. Fitting all points in Fig. 3 with function in Eq. (10) separately for each quantity we give all $g_c$ results in Table I. In all the fits we use $1/\nu = 1.406$ from the O(3) universality class with one correction exponent $\omega$ to the first order. For $C2 \times 2$ model $g_c$ from each quantity is the same considering one error bar and agrees with the former result $g_c = 0.54854(1)$. It is also true for the other two models with $\chi^2/d.o.f$ close to one, implying the credibility of the fits. These results give a further evidence to show that plaquette models with different checkerboard patterns all belong to the O(3) universality class. From Table I we also find that with only one correction term included the correction exponents $\omega$ are not the same for different quantities in the

![Fig. 2](https://via.placeholder.com/150)
same model, while they are the same for same quantity in different models within at most two error bars if we take the average of $\rho^2_s$ and $\rho^2_p$ in C2 × 4 model. The difference shows that $\omega$ calculated here is more likely to be an “effective correction” including higher orders. However, fitting including 2$\omega$ or higher order is very difficult and challenging with too many free parameters. Here we did not find any nonmonotonic behaviour in the size dependence of all crossings in plaquette models as shown in Fig. 3, so that one correction term can also give convincing criticality analysis, which is also confirmed by $\chi^2$/d.o.f of each fit.

In order to obtain a better estimation of the critical points, we continue to deal with the crossing points by joint fits as all size dependencies of $g_c(L)$ for different quantities should converge to the same value in one system. Therefore we fix $g_c(\infty)$ to be the same in each curve and fit all data together with other parameters independent and 1/$\nu = 1.406$. The fitting results are shown in all curves in Fig. 3 with $g_c = 0.548524(3)$ in C2 × 2, 0.456978(2) in C2 × 4 and 0.314451(3) in C4 × 4. Our result of C2 × 2 model fully consists with the value in Ref. 32 with higher precision. By Comparing these critical point values we find $g_c$ gets smaller from C2 × 2 to C4 × 4 model, indicating that our model more easily turn into a disordered state with less strong couplings as expected. Therefore we deduce it is a universal rule of QPTs at checkerboard patterns with even strong Heisenberg interactions units. The effective correction exponents $\omega$ in each model using different quantities in the joint fitting results still share the same rule as the separate ones. Thus, we can estimate the effective $\omega$ by taking the average values of all three results as same quantity in all models gives same $\omega$. Taking the weighted average values of the results of $\omega$ from three models we have the effective correction exponent $\omega = 1.058(7)$ for $R_2$, for $\chi_u L$ $\omega = 0.834(7)$. For $\rho_s L$ we first get the average $\omega$ from the correlated results of $\rho^2_s$ and $\rho^2_p$ in C2 × 4 model and take the larger error of them as the error, which gives $\omega = 0.65(1)$ in the end. Then taking the weighted average of all three values gives $\omega = 0.66(1)$ for $\rho_s L$. Comparing with the standard correction exponent $\omega \approx 0.78(33)$ in the O(3) universality class, we can see that the system sizes included in our fits are still not large enough to rule out the affection of higher order corrections even with $L$ up to 160. However, the fit including higher order corrections would bring in much more uncertainty. Therefore the value of effective $\omega$ becomes very important in the FSS study to obtain the critical point and critical exponents.

### B. Universal quantities at critical points

As discussed above, in order to study the critical point we use fixed value 1/$\nu = 1.406$ as the QPTs in plaquette models are believed to be the O(3) universality class. The goodness of all fitting results also proves the theo-

| $g_c$ | $\omega$ | $\chi^2$/d.o.f |
|------|----------|---------------|
| $R_2$ | 0.548532(6) | 1.14(2) | 0.89 |
| $\chi_u L$ | 0.548522(8) | 0.83(4) | 0.88 |
| $\rho_s L$ | 0.548521(5) | 0.68(2) | 1.09 |
| $R_2$ | 0.456985(6) | 1.08(2) | 0.63 |
| $\chi_u L$ | 0.456972(8) | 0.86(3) | 0.67 |
| $\rho^2_s L$ | 0.456975(5) | 0.70(2) | 0.92 |
| $\rho^2_p L$ | 0.456983(6) | 0.60(3) | 0.95 |
| $R_2$ | 0.31446(1) | 1.08(5) | 1.11 |
| $\chi_u L$ | 0.314441(9) | 0.87(5) | 0.80 |
| $\rho_s L$ | 0.314449(6) | 0.67(3) | 0.92 |
Compared with the best estimate of $1/\nu$ in Eq. (13) for all models. All data are fitted with Eq. (15) and give $1/\nu = 1.406(6)$ in $C2 \times 2$, $1/\nu = 1.401(6)$ in $C2 \times 4$ and $1/\nu = 1.404(5)$ in $C4 \times 4$. The correction exponent $\omega$ in Eq. (16) is 1.8(1), 1.6(1) and 1.7(1) for $C2 \times 2$, $C2 \times 4$ and $C4 \times 4$ correspondingly. The $\chi^2/d.o.f$ of all fittings are close to one. The inset figure zooms in with the same data and fitted curves for only larger system sizes to show details of the convergence more clearly.

To begin with, the correlation length exponent is calculated using the scaling of $1/\nu(L)$, which is defined as Eq. (13), in Eq. (15). The simulation and scaling results are shown in Fig. 4 for all three models. Fitting values of $1/\nu$ are the same for all models considering error. The weighted average of all three $1/\nu$ is 1.404(4). Compared with the best estimate of $1/\nu = 1.4061(7)$ (reciprocal value of $\nu = 0.7112(5)$ in Ref. [33]) in O(3) it is proved again that the QPTs here are in the same universality class as the CDM, SDM and 3D classical Heisenberg. But the accuracy of the estimation using scaling of $1/\nu(L)$ in our work is much less compared to the previous results. Usually $\nu$ can be got from the data collapse together with critical point and corrections. Here we use the combination of two sizes together at once in order to lease the influence of the corrections. It does help as the fitting results of $\omega$ are very large in our study, which means that $1/\nu$ converges very fast with the increase of $L$. But the value of $1/\nu(L)$ obtained from simulation has much larger errors compared with other quantities studied before, which brings in larger error to the final extrapolation value. Much more computational effort is needed in order to obtain better estimation of $\nu$. We just stop here in this paper as it is not a key point of our work, but we want to point it out for other studies using this procedure.

Another critical exponent considered here is the anomalous dimension $\eta$. Once the critical point $g_c$ is obtained, we can study the scaling of order parameter at

\[
\langle m_s^2 \rangle \propto L^{-(1+\eta)}(1+aL^{-\omega}).
\]

Similar to $1/\nu(L)$ we can also define $\eta(L)$ from the scaling of two sizes $L$ and $2L$ as

\[
\eta(L) = \frac{\ln[\langle m_s^2(L) \rangle/\langle m_s^2(2L) \rangle]}{\ln(2)} - 1. \tag{19}
\]

In this way, the size dependence of $\eta(L)$ is

\[
\eta(L) = \eta + dL^{-\omega}. \tag{20}
\]

with correction to the first order. This time our fits use the best known estimation of $\eta = 0.0375(5)$ and leave the other parameters in Eq. (20) free. The fitting results shown in Fig. 5 again imply that it is correct to set $\eta = 0.0375(5)$ here as all QPTs are in the O(3) universality class. Furthermore, all corrections are the same in three models considering error and the weighted averaged $\omega = 0.78(1)$ is the same as the first correction exponent $\omega_1 = 0.782(13)$ in the O(3) model[33]. This shows that $\eta$ could be a good quantity in testing the correction exponent in the FSS study once a good estimation of $g_c$ is obtained. And this scaling results also give us more confidence on the accuracy of $g_c$ here for all three plaquette models.

At last we test the Binder ratio $R_2$ at the critical point, which is known as a universal quantity regardless of the details of the model. But it is not always the same in one universality class as it still changes in different boundary conditions or aspect ratios. With crossing points extracted from two different sizes ($L, 2L$) of the $g$ dependence for $R_2$ near critical point, we can obtain
IV. SUMMARY AND DISCUSSIONS

In this paper we carried out the FSS study on data with high-precision using the SSE QMC method. The criticality of three $S = 1/2$ Heisenberg models on the square lattice with strong and weak couplings in plaquette patterns $C2 \times 2, C2 \times 4$ and $C4 \times 4$ is studied using the Binder ratio, uniform susceptibility and spin stiffness. By the joint fits combining the scalings of crossing points from all three quantities we have obtained the most accurate estimates of critical points $g_c$ for three plaquette models up to now. Our scaling analysis implies the importance of corrections in FSS, and with only one correction term value of $\omega$ is more likely to be an effective one. The effective $\omega$ does not change in different models as long as it describes the scaling behavior of the same physical observable. The calculation of $1/\nu$, $\eta$ and $R_2$ at the critical point shows that QPTs in all three models are in the O(3) universality class as predicted. The scaling of $\eta$ using order parameter at critical point also gives $\omega \approx 0.78$ as same as $\omega_1$ determined in 3D classical Heisenberg model, which further support the estimate of the critical points.

The fitting results of $\omega$ using different quantities in these three models help us to understand the influence of corrections in the scalings. With system sizes up to $L = 160$ the correction exponent $\omega$ is still an effective one differs in different variables. However, fitting including higher order of correction terms would be quite challenging and difficult. Here we find that for models with detailed difference structure in our case the effective $\omega$ does not change for same quantity. This might be helpful in the further FSS studies on other similar models. We also obtain the value of the universal quantity $R_2$ at the critical point. The Binder ratios in systems with different aspect ratios would be different even if they are in the same universality class. The $R_2$ converges to the same value here in our three models and we suggest that any $CL_x \times L_y$ models might have the same $R_{2c}$ with $L_xL_y$ even and same $\beta/L$.

ACKNOWLEDGMENTS

We would like to thank A. W. Sandvik for useful discussions and a careful reading of the manuscript. The work of X.R and D.X.Y was supported by Grants NKRDP-2017YFA0206203, NSFC-11574404, NSFG-2015A030313176, National Supercomputer Center in Guangzhou and Leading Talent Program of Guangdong Special Projects.

TABLE II. Estimate results of the critical binder ratio $R_{2c}$ for $C2 \times 2$, $C2 \times 4$ and $C4 \times 4$ model. The fitted curves are shown in Fig. 6.

| Model      | $R_{2c}$     | $\omega$   | $\chi^2/d.o.f.$ |
|------------|--------------|------------|-----------------|
| $C2 \times 2$ | 2.2549(5)    | 1.163(9)   | 1.00            |
| $C2 \times 4$ | 2.2549(7)    | 1.17(1)    | 1.29            |
| $C4 \times 4$ | 2.2542(9)    | 1.02(2)    | 0.84            |

[1] M. Vojta, Rep. Prog. Phys. 66, 2069-2110 (2003).
[2] V. Murg, F. Verstraete, and J. I. Cirac, Phys. Rev. B 79, 195119 (2009).
[3] P. Horsch and W. von der Linden, Zeitschrift für Physik B Condensed Matter 72, 181 (1988).
[4] E. Manousakis, Rev. Mod. Phys. 63, 1 (1991).
[5] B. Keimer, N. Belk, R. J. Birgeneau, A. Cassanho, C. Y. Chen, M. Greven, M. A. Kastner, A. Aharony, Y. Endoh, R. W. Erwin, and G. Shirane, Phys. Rev. B 46, 14034 (1992).
[6] P. A. Lee, N. Nagaosa, and X.-G. Wen, Rev. Mod. Phys. 78, 17 (2006).
[7] T. Giamarchi, C. Rüegg, and O. Tchernyshyov, Nat. Phys. 4, 198 (2008).
[8] A. Rakhimov, S. Mardonov, E. Y. Sherman, and A. Schilling, New J. Phys. 14, 113010 (2012).
[9] M. Matsumoto, C. Yasuda, S. Todo, and H. Takayama, Phys. Rev. B 65, 014407 (2001).
[10] A. W. Sandvik, AIP Conf. Proc. 1297, 135 (2010).
[11] S. Yasuda and S. Todo, Phys. Rev. E 88, 061302(R) (2013).
[12] S. Sachdev, Quantum Phase Transition 2nd ed. (Cambridge University Press, Cambridge, U.K., 2011).
[13] S. Sachdev, Nat. Phys. 4, 173 (2008).
[14] L. Fritz, R. L. Doretto, S. Wessel, S. Wenzel, S. Burdin and M. Vojta, Phy. Rev. B 83, 174416 (2011).
[15] S. Wenzel, L. Bogacz and W. Janke, Phys. Rev. Lett. 101, 127202 (2008).
[16] L. Wang, K. S. D. Beach, and A. W. Sandvik, Phys. Rev. B 73, 014431 (2006).
[17] O. F. Syljuåsen, Phys. Rev. B 73, 245105 (2006).
[18] D.-X. Yao, J. Gustafsson, E. W. Carlson, and A. W. Sandvik, Phys. Rev. B 82, 172409 (2010).
[19] R. R. P. Singh, M. P. Gelfand and D. A. Huse, Phys. Rev. Lett. 61, 2484 (1988).
[20] N. Ma, P. Weinberg, H. Shao, W. Guo, D.-X. Yao and A. W. Sandvik, Phys. Rev. Lett. 121, 117202 (2018).
[21] A. Sen, H. Suwa, and A. W. Sandvik, Phys. Rev. B 92, 195145 (2015).
[22] A. Koga, S. Kumada, and N. Kawakami, J. Phys. Soc. Jpn. 68, 642 (1999).
[23] M. Mambrini, A. Läuchli, D. Poilblanc, and F. Mila, Phys. Rev. B 74, 144422 (2006).
[24] A. F. Albuquerque, M. Troyer, and J. Ottna, Phys. Rev. B 78, 132402 (2008).
[25] C. Stock, W. J. L. Buyers, R. A. Cowley, P. S. Clegg, R. Coldea, C. D. Frost, R. Liang, D. Peets, D. Bonn, W. N. Hardy, and R. J. Birgeneau, Phys. Rev. B 71, 024522 (2005).
[26] J. M. Tranquada, H. Woo, T. G. Perring, H. Goka, G. Gu, G. Xu, M. Fujita, and K. Yamada, J. Phys. Chem. Solids 67, 511 (2006).
[27] B. S. Shastry and B. Sutherland, Physica B&C 108, 1069 (1981).
[28] A. Läuchli, S. Wessel, and M. Sigrist, Phys. Rev. B 66, 014401 (2002).
[29] M. E. Zayed, C. Rüegg, J. Larrea J., A. M. Läuchli, C. Panagopoulos, S. S. Saxena, M. Ellerby, D. F. McMorrow, T. Strüssle, and S. Klotz, Nat. Phys. 13, 962 (2017).
[30] P. Corboz and F. Mila, Phys. Rev. B 87, 115144 (2013).
[31] B. Zhao, P. Weinberg, and A. W. Sandvik, arXiv:cond-mat/1804.07115 (2018).
[32] S. Wenzel and W. Janke, Phys. Rev. B 79, 014410 (2009).
[33] M. Campostrini, M. Hasenbusch, A. Pelissetto, P. Rossi, and E. Vicari, Phys. Rev. B 65, 144520 (2002).
[34] Y. Xu, Z. Xiong, H-Q. Wu, D-X. Yao, arXiv:cond-mat/1811.12753 (2018).
[35] M. Troyer, M. Imada, and K. Ueda, J. Phys. Soc. Jpn. 66, 2957 (1997).