3-manifolds which are spacelike slices of flat spacetimes

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Statement of Main Result

We will be considering spacetimes homeomorphic to $M^3 \times \mathbb{R}$ where $M^3$ is a compact, connected, topological 3-manifold without boundary. The slices $M^3 \times \{t\}$ will always be spacelike.

A fundamental question is to determine the possible topologies of the universe after imposing various (usually strong) conditions on the spacetime metric. Only in the last few years has 3-manifold topology advanced enough provide answers to this question in certain cases:

**Main Theorem.** $M$ is a spacelike slice of a flat spacetime if and only if $M$ is hyperbolic (admits a Riemannian metric of constant curvature $-1$) or $M$ is finitely covered by $\Sigma^2 \times S^1$ where $\Sigma$ is a closed orientable surface other than $S^2$.

The condition that $M$ is hyperbolic is a purely topological condition – all that is meant is that $M$ admits a hyperbolic metric, not that the induced Riemannian metric on some slice is hyperbolic (indeed this is usually not the case).
Related results

**Theorem** (UCLA thesis, 1996). $M^n$ is a spacelike slice of a de Sitter spacetime if and only if $M$ admits a conformally flat Riemannian metric.

**Remarks**

1. This (de Sitter) result works in all dimensions.

2. Admitting a conformally flat Riemannian metric is a non-trivial topological constraint when $n \geq 3$.

3. More is true: the moduli space of de Sitter domains of dependence $M \times \mathbb{R}$ is parameterized by the moduli space of conformally flat Riemannian metrics on $M$.

4. In contrast, the main theorem is special to $3 + 1$ and makes no statement about the moduli space of flat metrics on $M \times \mathbb{R}$. In the course of the proof, though, we will classify all holonomy representations $\pi_1(M) \rightarrow ISO(3,1)$.

5. The moduli space in the flat $2 + 1$ dimensional case was worked out by Geoff Mess in 1990.
Some perspective

Thurston discovered in the 1970’s that “most” 3-manifolds are hyperbolic. The expected characterization is that \( M \) is hyperbolic if and only if \( M \) is irreducible (i.e. every smoothly embedded sphere bounds a ball), \( \pi_1(M) \) is infinite, and \( \pi_1(M) \) contains no \( \mathbb{Z} \oplus \mathbb{Z} \) subgroup.

Nevertheless, there are lots of other possibilities for 3-manifolds which we must prove do not occur as slices:

1. Manifolds not covered by \( \mathbb{R}^3 \) (like \( S^3, S^2 \times S^1, \ldots \))

2. Some Seifert fiber spaces. These are 3-manifolds which are foliated by circles. Of these, some are finitely covered by \( \Sigma \times S^1 \) while others are not (e.g. the unit tangent bundle of a hyperbolic surface). A key element in the proof is to exclude these non-trivial Seifert fiber spaces.

3. Solv-manifolds, graph manifolds, etc.

While it is instructive to think about the main theorem in terms of excluding various families, one cannot prove it this way since 3-manifolds are not classified.
Realizing 3-manifolds as slices

This is the easier half of the theorem. The 3-manifolds which arise in the statement all admit nice Riemannian metrics with (metric) universal cover either $\mathbb{H}^3$, $\mathbb{E}^3$, or $\mathbb{H}^2 \times \mathbb{R}$.

It suffices to realize these “geometries” inside $\mathbb{R}^4_1$. 
Proof of the converse

A flat metric on $M \times \mathbb{R}$ yields a spacelike immersion $d : \tilde{M} \to \mathbb{R}^4_1$ and a holonomy representation $\phi : \pi_1(M) \to ISO(3, 1)$ with discrete image.

A result of S. Harris shows that $d$ is an achronal embedding, in particular that $\tilde{M}$ is homeomorphic to $\mathbb{R}^3$.

The rest of the proof amounts to classifying possibilities for the discrete groups $\phi(\pi_1(M))$ in $ISO(3, 1)$ and then using powerful “homotopy equivalence implies homeomorphism” results from 3-manifold topology. In other words, the 3-manifolds which arise as slices are all determined by their fundamental groups.
Holonomy representations

We have

\[ 1 \rightarrow \mathbb{R}_1^4 \rightarrow ISO(3,1) \rightarrow SO(3,1) \rightarrow 1 \]

and so the holonomy \( \Gamma = \phi(\pi_1(M)) \) has a translational part which is a discrete subgroup of \( \mathbb{R}_1^4 \), isomorphic to \( \mathbb{Z}^k \) for \( k = 0, 1, 2, 3 \). We have:

\[ 1 \rightarrow \mathbb{Z}^k \rightarrow \Gamma \rightarrow L(\Gamma) \rightarrow 1 \]

Statements below are “up to finite covers”:

**Case** \( k = 0 \): Either \( \Gamma = \mathbb{Z}^3 \) or \( \Gamma \) embeds as a discrete subgroup of \( SO(3,1) \). In the first case, \( M \) is a 3-torus (Waldhausen), and in the second case \( M \) is hyperbolic (Gabai, Meyerhoff, and N. Thurston).

**Case** \( k = 1 \): The Seifert Fiber Space Conjecture, recently resolved by Gabai, Casson-Jungreis, and Mess, implies in this case that \( M \) is a Seifert fiber space. The key step is to show it is covered by \( \Sigma \times S^1 \).

**Case** \( k = 2 \): A theorem of Stallings implies that \( M \) fibers over the circle with torus fibers. A straightforward argument shows that in fact \( M \) is a 3-torus.

**Case** \( k = 3 \): \( M \) is a 3-torus (Waldhausen).