STABILITY OF GROUND STATE FOR THE SCHRÖDINGER-POISSON EQUATION

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Abstract. We are concerned with the stability of the ground state for the Schrödinger-Poisson equation

\[ i\partial_t \psi + \Delta \psi - (|x|^{-1} \ast |\psi|^2)\psi + |\psi|^{p-1}\psi = 0, \quad x \in \mathbb{R}^3. \]

If \( 2 < p < \frac{7}{3} \) and the frequency is sufficiently large, we show that the ground state is orbitally stable.

1. Introduction. We consider the following so called Schrödinger-Poisson system

\[
\begin{cases}
  i\hbar \frac{\partial \psi(t,x)}{\partial t} + \frac{\hbar^2}{2m} \Delta \psi(t,x) - V(x)\psi(t,x) + C_s n^{\frac{p-1}{2}}(t,x)\psi(t,x) = 0, \quad x \in \mathbb{R}^3, t > 0, \\
  \Delta V(t,x) = -\chi \int_{\mathbb{R}^3} |\psi(t,y)|^2 |x-y| dy.
\end{cases}
\]

where \( p \in (1, 5), \chi > 0, C_s > 0 \), \( \hbar \) and \( m \) are the Plank constant and the particle mass, respectively. \( \psi : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{C} \) is the wave function, and \( n(t,x) \) denotes the charge density. This model is a local one particle approximation of the time dependent Hartree-Fock system [5, 6, 25], which can be solved by a deterministic global optimization method, see [20]. The local nonlinear term \( C_s n^{\frac{p-1}{2}} \psi \) in (1) appears as a local correction to the Hartree-Fock system ([30]). It describes the time evolution of electrons in a quantum model with respect to the Pauli principle in an approximate fashion([21]). The self-consistent potential \( V \) can be given by

\[ V(t,x) = \chi \int_{\mathbb{R}^3} \frac{|\psi(y,t)|^2}{|x-y|} dy. \]

Using this formula, (1) becomes a single nonlinear Schrödinger equation with a nonlinear nonlocal interaction term, that is,

\[ i\hbar \frac{\partial \psi}{\partial t} + \frac{\hbar^2}{2m} \Delta \psi - \chi(|x|^{-1} \ast |\psi|^2)\psi + C_s |\psi|^{p-1}\psi = 0, \quad x \in \mathbb{R}^3, t > 0. \]

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where “∗” denotes the convolution of two functions, e.g., \((f \ast g)(x) := \int_{\mathbb{R}} f(x - y)g(y)dy\). When \(\chi = 0\), the system (1) is reduced to the classical Schrödinger equation
\[
\imath \hbar \frac{\partial \psi}{\partial t} + \frac{\hbar^2}{2m} \Delta \psi + C_s |\psi|^{p-1} \psi = 0, \quad x \in \mathbb{R}^3, t > 0.
\]

Without loss of generality, we assume that \(\hbar = m = \chi = C_s = 1\) in (1), and consider the Cauchy system
\[
\imath \frac{\partial \psi}{\partial t} + \Delta \psi - (|x|^{-1} \ast |\psi|^2)\psi + |\psi|^{p-1}\psi = 0, \quad x \in \mathbb{R}^3, \quad \psi(0, x) = \psi_0(x), \quad x \in \mathbb{R}^3.
\]

Perturbation analysis to the Cauchy systems arising from various fields have been widely studied, see for example, the optimal control of Bose-Einstein condensates [26, 27] and the orbital stability for a standing wave of the Schrödinger system [13]. It is shown in [8] that the Cauchy problem (3)-(4) is locally well-posed in the energy space \(H^1(\mathbb{R}^3)\) for \(1 < p < 5\). Based on the idea of Cazenave and Lions [9], the sufficient conditions for the stability of the collection of standing waves to (3) in terms of the charge has been investigated widely, see, for example [2, 3, 4, 14, 30]. Roughly speaking, these work studied firstly the existence of the minimizers to the following constrained variational problem were studied in these work:
\[
d_N := \inf_{u \in H^1, \|u\|_{L^2} = N} E(u),
\]
\[
E(u) := \frac{1}{2} \int |\nabla u|^2 + \frac{1}{4} \int (\frac{1}{|x|} \ast |u|^2)|\psi|^2 - \frac{1}{p+1} \int |u|^{p+1}.
\]

Then orbital stability to the collection of minimizers
\[
S_N := \{e^{i\theta}u(x) : \theta \in [0, 2\pi), \|u\|_{L^2} = N, E(u) = d_N\}
\]
was obtained. Here the orbital stability of collection \(S_N\) means that, for every \(\varepsilon > 0\), there exists \(\delta > 0\) such that for any \(\psi_0 \in H^1\) with \(\inf_{v \in S_N} \|v - \psi_0\|_{H^1} < \delta\), it holds that \(\inf_{v \in S_N} \|\psi(t) - v\|_{H^1} < \varepsilon\) for all \(t > 0\).

In this paper, we consider the orbital stability for a standing wave of (3). To give a strict expression, we introduce the definition of orbital stability and instability as follows.

**Definition 1.1.** *(Orbital stability and instability)* A standing wave \(e^{i\omega t}u(x)\) is said to be orbitally stable under the flow of equation (3), if for any \(\varepsilon > 0\), there exists \(\delta > 0\) such that for any \(\psi_0 \in H^1\) with \(\|\psi_0 - u\|_{H^1} < \delta\), the solution \(\psi(t)\) of (3) with initial value \(\psi(0) = \psi_0\) exists for all \(t \in [0, \infty)\) and satisfies
\[
\sup_{0 \leq t < \infty} \inf_{\theta \in \mathbb{R}, y \in \mathbb{R}^3} \|\psi(t) - e^{i\theta}(\cdot + y)\|_{H^1} < \varepsilon.
\]

Otherwise, the standing wave \(e^{i\omega t}u(x)\) is said to be orbitally unstable.

In fact, we do not know whether there exists a unique minimizer in the collection \(S_N\). Hence, the orbital stability for a standing wave in Definition 1.1 is stronger than the conception about orbital stability of collection \(S_N\) in [2, 3, 4, 14, 30].

The existence of standing waves of (3) has been studied extensively, see, for example [1, 15, 16, 17, 18] and their references. Indeed, the standing wave of (3)
is a solution of the form \( \psi(t, x) = e^{i\omega t} u(x) \), where \( u \) is a nontrivial solution of the following elliptic equation
\[
- \Delta u + \omega u + (|x|^{-1} * |u|^2)u = |u|^{p-1}u, \quad x \in \mathbb{R}^3.
\] (5)

One could obtain a solution \( u \) of (5) by looking for the critical point of the action functional \( S_\omega : H^1(\mathbb{R}^3) \to \mathbb{R} \) defined by
\[
S_\omega(v) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla v|^2 + \frac{\omega}{2} \int_{\mathbb{R}^3} |v|^2 + \frac{1}{4} \int_{\mathbb{R}^3} \left( \frac{1}{|x|} * |v|^2 \right)|v|^2 - \frac{1}{p+1} \int_{\mathbb{R}^3} |v|^{p+1}. \] (6)

Due to the lack of compactness, it is not trivial to look for the existence of critical point of functional \( S_\omega(v) \) when \( 1 < p < 3 \), see, for example [1, 15, 17, 18, 29]. By applying truncation technique to the nonlocal part \( \int_{\mathbb{R}^3} \left( \frac{1}{|x|} * |v|^2 \right)|v|^2 \) of \( S_\omega(v) \), H. Kikuchi proved in [18] that there exists a nontrivial solution to (5) with a priori estimate uniformly with respect to the parameter \( \omega \). Based on a priori estimate to the infinite norm of solutions, orbital stability of this special nontrivial solution in the sense of Definition 1.1 was also obtained in [18]. Since the nontrivial solution of (5) in [18] was obtained via the critical point of a truncation functional, we do not know whether the nontrivial solution given in [18] is the ground state solution, which have the minimum action functional. On the other hand, the estimate to the infinite norm of the ground state solutions can not be obtained by applying the argument in [18]. Therefore, the argument in [18] can not be applied directly to obtain the stability of ground state solutions. Motivated by those facts, we develop a new perturbation method to study the stability of ground state solution in the sense of Definition 1.1 for the Schrödinger-Poisson equation.

In [1], the authors proved that there exists a ground state solution to (5) when \( 2 < p < 5 \). For convenience, we denote by \( \mathcal{X}_\omega \) and \( \mathcal{G}_\omega \) the sets of nontrivial solutions and ground state solution of (5), respectively, as follows,
\[
\mathcal{X}_\omega = \{ u \in H^1(\mathbb{R}^3) : S'_\omega(u) = 0, u \neq 0 \},
\]
\[
\mathcal{G}_\omega = \{ v \in \mathcal{X}_\omega : S_\omega(v) \leq S_\omega(u), \text{for all } u \in \mathcal{X}_\omega \}.
\]

Similarly, we denote by
\[
\mathcal{X}^*_\omega = \{ u : S'_\omega(u) = 0, u \neq 0, u(x) = u(|x|) \},
\]
\[
\mathcal{G}^*_\omega = \{ v \in \mathcal{X}^*_\omega : S_\omega(v) \leq S_\omega(u), \text{for all } u \in \mathcal{X}^*_\omega \}.
\]

For each \( \omega > 0 \), it follows from [1, 29] that \( \mathcal{G}^*_\omega \neq \emptyset \) and \( \mathcal{G}^*_\omega \neq \emptyset \) when \( 2 < p < 5 \). It is clear to see that \( \mathcal{X}^*_\omega \subseteq \mathcal{X}_\omega \) and \( S(v) \leq S(w) \) for any \( v \in \mathcal{G}_\omega, w \in \mathcal{G}^*_\omega \). Due to the repulsive influence of the nonlocal terms \((|x|^{-1} * |\psi|^2)\psi \) in (5), it is difficult to prove the uniqueness and radial symmetry of the ground state, see [24]. Until now, there is less conclusion about the uniqueness and radial symmetry of the ground state of (5). Hence, it is still not clear whether \( S_\omega(v) \) is equal to \( S_\omega(w) \) for \( v \in \mathcal{G}_\omega \) and \( w \in \mathcal{G}^*_\omega \). When \( p \in (\frac{7}{4}, 5) \), it is shown in [22] that the ground state \( e^{i\omega t} \phi_\omega \) is strongly unstable, for each \( \phi_\omega \in \mathcal{G}_\omega \cup \mathcal{G}^*_\omega \); that is, there exists an initial data nearby which leads to blow-up in finite time. For the case \( p \in (2, \frac{7}{4}) \), we prove in this paper that the ground state is stable as follows.

**Theorem 1.2.** Let \( 2 < p < \frac{7}{3} \) and \( \phi_\omega \in \mathcal{G}_\omega \cup \mathcal{G}^*_\omega \), then there exists \( \omega_* > 0 \) such that the standing wave \( e^{i\omega t} \phi_\omega \) of (3) is orbitally stable for any \( \omega \in (\omega_*, \infty) \).

If there is a small perturbation for the initial data of (3) at \( \psi(x, 0) = \phi_\omega(x) \), it follows from Theorem 1.2 that the solution \( \psi(x, t) \) of (3) is located at a small neighborhood of set \( \{ e^{i\omega t} \phi_\omega : \phi_\omega \in \mathcal{G}_\omega \cup \mathcal{G}^*_\omega \} \) for each \( t > 0 \), which is the perturbation
analysis for the initial data of the system (3), Compare with the optimal control of Schrödinger system \([26, 27, 28]\), it is interesting to study the perturbation analysis for the coefficient variation of (3), such as optimal control problem to the Cauchy systems related to (3).

The rest of this paper is organized as follows. Section 2, some preliminaries are stated. Section 3, Theorems 1.2 is proved. Throughout the paper, the norm of \(L^q := L^q(\mathbb{R}^3)\) \((1 \leq q < 5)\) is denoted by \(\| \cdot \|_{L^q}\), and the norm of \(H^1 := H^1(\mathbb{R}^3)\) by \(\| \cdot \|_{H^1}\). The integral \(\int_{\mathbb{R}^3} \cdot \, dx\) is simplified by \(\int \cdot\). \(\Im(z)\) denotes the imaginary part of a complex number \(z\) and \(\Re(z)\) denotes the real part. \(B(0, R) = \{x \in \mathbb{R}^3, |x| < R, R > 0\}\).

2. Preliminaries. We begin this section by recalling the local well posedness of the Cauchy problem (3)-(4) given by Cazenave \([8]\).

**Proposition 1.** [Theorem 4.3.1 in \([8]\)] Let \(1 < p < 5\). For any initial data \(\psi_0 \in H^1\), there exists a unique solution \(\psi(t, x)\) of the Cauchy problem (3)-(4) in \(C([0, T); H^1)\) for time \(t \in (0, +\infty)\) (maximal existence time) with the following property: either \(T = +\infty\) or else \(T < +\infty\) and \(\lim_{t \to T} \|\nabla \psi(t, \cdot)\|_{L^2} = +\infty\). Moreover, the following mass and energy conservation laws hold for every \(t \in [0, T)\)

\[
\|\psi(t, \cdot)\|_{L^2} = \|\psi_0\|_{L^2} \quad \text{and} \quad E(\psi(t)) = E(\psi(0)),
\]

where

\[
E(\psi) := \frac{1}{2} \int |\nabla \psi|^2 + \frac{1}{4} \int \left(\frac{1}{|x|} \ast |\psi|^2\right) |\psi|^2 - \frac{1}{p + 1} \int |\psi|^{p + 1}. \quad (8)
\]

By \([29]\), we know that a solution \(u\) of (5) satisfies two identities,

\[
\int |\nabla u|^2 + \omega \int |u|^2 + \int \left(\frac{1}{|x|} \ast |u|^2\right) |u|^2 - \int |u|^{p + 1} = 0, \quad (9)
\]

\[
\frac{1}{2} \int |\nabla u|^2 + \frac{3}{2} \omega \int |u|^2 + \frac{5}{4} \int \left(\frac{1}{|x|} \ast |u|^2\right) |u|^2 - \frac{3}{p + 1} \int |u|^{p + 1} = 0. \quad (10)
\]

It follows from (9) and (10) that the solution \(u\) of (5) satisfies

\[
N(u) := \frac{3}{2} \int |\nabla u|^2 + \frac{1}{2} \omega \int |u|^2 + \frac{3}{4} \int \left(\frac{1}{|x|} \ast |u|^2\right) |u|^2 - \frac{2p - 1}{p + 1} \int |u|^{p + 1} = 0. \quad (11)
\]

For \(2 < p < 5\) and \(\omega > 0\), it is proved in \([1, 29]\) that there exist solutions \(u\) and \(v\) of equation (5), which are minimizers of the following constrained variational problems, respectively,

\[
d_N := \inf_{w \in \mathcal{N}} S_\omega(v), \quad \mathcal{N} = \{w \in H^1; N(w) = 0, w \neq 0\}, \quad (12)
\]

\[
d_{N_r} := \inf_{w \in \mathcal{N}_r} S_\omega(w), \quad \mathcal{N}_r = \{w \in H^1_r; N(w) = 0, w \neq 0\}. \quad (13)
\]

Since \(N(w) = 0\) for \(w \in \mathcal{N}_r\), it is obvious that \(u \in \mathcal{G}_\omega\) and \(v \in \mathcal{G}_\omega^r\). On the other hand, it is easy to check that \(S_\omega(\phi) = d_N\) (or \(S_\omega(\varphi) = d_{N_r}\)) if \(\phi \in \mathcal{G}_\omega\) (or \(\varphi \in \mathcal{G}_\omega^r\)).

Recalling the definitions of \(\mathcal{G}_\omega, \mathcal{G}_\omega^r\) mentioned above, we can describe the elements of \(\mathcal{G}_\omega, \mathcal{G}_\omega^r\) by using (12) and (13). To make the new characterization more clear, we give the following lemma.

**Lemma 2.1.** Let \(2 < p < 5\) and \(\omega > 0\). Then,

(i) \(\phi \in \mathcal{G}_\omega\) if and only if \(\phi \in \mathcal{N}\) and \(S_\omega(\phi) = d_N\);

(ii) \(\varphi \in \mathcal{G}_\omega^r\) if and only if \(\varphi \in \mathcal{N}_r\) and \(S_\omega(\varphi) = d_{N_r}\).
Finally, we recall two lemmas due to Lions [23] and Brezis-Leib [7], which are useful in the following arguments.

**Lemma 2.2.** [Lemma 1.1 in [23]]. Let \( r > 0 \) and \( 2 \leq q < 6 \). If \( \{ u_n \} \) is bounded in \( H^1 \) and if

\[
\sup_{y \in \mathbb{R}^3} \int_{B(y,r)} |u_n|^q \to 0,
\]

then \( u_n \overset{\mathcal{L}}{\to} 0 \) in \( L^q \) for \( 2 < q < 6 \).

**Lemma 2.3.** [Theorem 1 in [7]]. Let \( 1 \leq q < \infty \) and \( \{ f_j \} \) be a bounded sequence in \( L^q \). Assume that \( f_j \overset{\mathcal{L}}{\to} f \) a.e. in \( \mathbb{R}^3 \), then we have

\[
\| f_j \|^q_{L^q} - \| f_j - f \|^q_{L^q} - \| f \|^q_{L^q} \overset{\mathcal{L}}{\to} 0.
\]

### 3. Stability of the standing wave

We prove Theorem 1.2 in this section only for the case \( \phi_\omega \in \mathcal{G}_\omega \). The same argument can apply for the case \( \phi_\omega \in \mathcal{G}_0^\omega \). For \( \phi_\omega \in H^1 \), define \( \tilde{\phi}_\omega \) by

\[
\phi_\omega(x) = \omega^{1/(p-1)} \tilde{\phi}_\omega(\sqrt{\omega}x).
\]

(14)

It is easy to check that \( \tilde{\phi}_\omega \) is a solution of (5) if and only if the rescaled function \( \tilde{\phi}_\omega(x) \) satisfies

\[
-\triangle \tilde{\phi}_\omega + \tilde{\phi}_\omega + \omega^{2(p-1)} (|x|^{-1} \ast \tilde{\phi}_\omega^2) \tilde{\phi}_\omega - |\tilde{\phi}_\omega|^{p-1} \tilde{\phi}_\omega = 0.
\]

(15)

Similar to the proof of Lemma 11 in [13], we show that as \( \omega \to \infty \), the rescaled function \( \tilde{\phi}_\omega \) converges in \( H^1 \) to the unique positive radial solution \( Q(x) \) of the following equation

\[
-\triangle Q + Q - |Q|^{p-1} Q = 0, \quad x \in \mathbb{R}^3
\]

(16)

for \( 2 < p < 5 \). Actually, we have the following proposition.

**Proposition 2.** Let \( 2 < p < 5 \), \( \phi_\omega \in \mathcal{G}_\omega \) and \( \tilde{\phi}_\omega(x) \) be given by (14). Then, for any sequence \( \{ \omega_j \} \subset \mathbb{R} \) with \( \omega_j \to \infty \), there exists a subsequence \( \{ \tilde{\phi}_{\omega_{j_k}} \} \) and a sequence \( \{ y_k \} \subset \mathbb{R}^3 \) such that

\[
\lim_{k \to \infty} \| \omega_{j_k} \tilde{\phi}_{\omega_{j_k}}(x + y_k) - Q \|_{H^1} = 0,
\]

(17)

where \( Q \) is the unique radial positive solution of (16).

**Remark 1.** This proposition provides a detailed description on the behavior of the ground state \( \phi_\omega \) of (5) as \( \omega \) is large enough. Sobolev embedding theorem and the definition of (14), (17) implies that for \( q \in [2, 6) \)

\[
\lim_{k \to \infty} \| \omega_{j_k} \tilde{\phi}_{\omega_{j_k}}(x + y_k) / \sqrt{\omega_{j_k}^q} - Q \|_{L^q} = 0.
\]

As \( \omega \to \infty \), the ground state \( \phi_\omega \) behaves like

\[
\phi_\omega(x) \approx \omega \frac{1}{p-1} Q(\sqrt{\omega}(x + x_\omega)).
\]

We can deduce that

\[
\lim_{\omega \to \infty} \| \phi_\omega \|_{L^q} = \omega^{\frac{1}{p-1} - \frac{q}{2}} \| Q \|_{L^q}.
\]

For \( r > 0 \), it holds that

\[
\| \phi_\omega \|_{L^q(|x-x_\omega|<r)} \approx \omega^{\frac{1}{p-1} - \frac{q}{2}} \| Q \|_{L^q(|x-x_\omega|<\sqrt{\omega}r)}.
\]
which shows a concentration phenomenon. More precisely, for any \( r > 0 \), we have
\[
\lim_{\omega \to \infty} \| \phi_\omega \|_{L^q(|x-x_\omega|<r)} = \infty, \quad \text{if } q > \frac{2}{3}(p-1); \\
\lim_{\omega \to \infty} \| \phi_\omega \|_{L^q(|x-x_\omega|<r)} = \| Q \|_{L^q}, \quad \text{if } q = \frac{2}{3}(p-1).
\]
In particular, if \( p = \frac{7}{3} \), we have \( \lim_{\omega \to \infty} \| \phi_\omega \|_{L^2} = \| Q \|_{L^2} \).

Let \( 2 < p < 5 \) and \( \phi_\omega \in \mathcal{G}_\omega \), then the rescaled function \( \tilde{\phi}_\omega \) in (14) is a minimizer of the constrained variational problem
\[
\tilde{S}_0 := \inf \left\{ \tilde{S}_\omega(v) : v \in H^1 \setminus \{0\}, \tilde{N}(v) = 0 \right\},
\]
where \( \tilde{S}_\omega(\cdot) \), \( \tilde{N}(\cdot) \) and \( \tilde{J}_\omega(\cdot) \) are defined by
\[
\tilde{S}_\omega(v) := \frac{1}{2} \int |\nabla v|^2 + \frac{1}{2} \int |v|^2 + \frac{2(2-p)}{4} \int \left( \frac{1}{|x|} * |v|^2 \right) |v|^2 - \frac{1}{p+1} \int |v|^{p+1}, \\
\tilde{N}(v) := \frac{3}{2} \int |\nabla v|^2 + \frac{1}{2} \int |v|^2 + \frac{3(2-p)}{4} \int \left( \frac{1}{|x|} * |v|^2 \right) |v|^2 - \frac{2p-1}{p+1} \int |v|^{p+1}, \\
\tilde{J}_\omega(v) := \frac{p-2}{2p-2} \int |\nabla v|^2 + \frac{p-1}{2p-2} \int |v|^2 + \frac{(p-2)}{2p-2} \int \left( \frac{1}{|x|} * |v|^2 \right) |v|^2.
\]
In fact, by using (14) we see that
\[
S_\omega(\phi_\omega) = \omega^{2/(p-1)-1/2} \tilde{S}_\omega(\tilde{\phi}_\omega), \quad N(\phi_\omega) = \omega^{2/(p-1)-1/2} \tilde{N}(\tilde{\phi}_\omega).
\]
For \( 2 < p < 5 \), if \( \phi_\omega \in \mathcal{G}_\omega \) we know from Lemma 2.1 that \( \phi_\omega(x) \) is the minimizer of
\[
\inf \{ S_\omega(\phi) : \phi \in H^1 \setminus \{0\}, N(\phi) = 0 \}.
\]
Thus it follows from (20) that \( \tilde{\phi}_\omega(x) \) is a minimizer of (18).

It is well known that for any ground state \( u \) of (16), we should have \( u = e^{i\theta}Q(-x_0) \) for some \( \theta \in [0, 2\pi) \) and \( x_0 \in \mathbb{R}^3 \), where \( Q \) is the unique radial positive solution of (16). Note that \( Q \) is also the minimizer of
\[
S_0 := \inf \{ S^0(v) : v \in H^1 \setminus \{0\}, N^0(v) = 0 \} = \inf \{ J^0(v) : v \in H^1 \setminus \{0\}, N^0(v) = 0 \},
\]
where the functionals \( S^0(\cdot), N^0(\cdot) \) and \( J^0(\cdot) \) are defined by
\[
S^0(v) := \frac{1}{2} \| \nabla v \|^2_{L^2} + \| v \|^2_{L^2} - \frac{1}{p+1} \| v \|^{p+1}_{L^{p+1}}, \\
N^0(v) := \frac{3}{2} \| \nabla v \|^2_{L^2} + \| v \|^2_{L^2} - \frac{2p-1}{p+1} \| v \|^{p+1}_{L^{p+1}}, \\
J^0(v) := \frac{p-2}{2p-2} \| \nabla v \|^2_{L^2} + \frac{p-1}{2p-2} \| v \|^2_{L^2} + \frac{(p-2)}{2p-2} \| v \|^2_{L^2}.
\]

**Lemma 3.1.** For \( S_1^0 \), which is defined by (21), we have
\[
S_1^0 = \inf \{ J^0(v) : v \in H^1 \setminus \{0\}, N^0(v) \leq 0 \}.
\]

**Proof.** It follows from the definition of \( S_1^0 \) that
\[
S_1^0 \geq \inf \{ J^0(v) : v \in H^1 \setminus \{0\}, N^0(v) \leq 0 \}.
\]
On the other hand, for any \( v \in H^1 \) satisfying \( N^0(v) < 0 \), there exists \( \mu \in (0, 1) \) such that \( N^0(\mu v) < 0 \). Since \( v \neq 0 \), we have \( S_1^0 \leq J^0(\mu v) = \mu^2 J^0(v) < J^0(v) \). \( \square \)

**Lemma 3.2.** Let \( Q \) be a ground state of (16) with \( 2 < p < 5 \). Then \( Q \) is a minimizer of (21). Moreover, if \( \{ v_j \} \) is a minimizing sequence of (21), that is \( N^0(v_j) = 0 \) and \( J^0(v_j) \to S_1^0 \) as \( j \to \infty \). Then passing a subsequence, there exists a sequence \( \{ y_j \} \subset \mathbb{R}^3 \) such that
\[
\lim_{j \to \infty} \| v_j(\cdot + y_j) - Q(\cdot) \|_{H^1} = 0.
\]
Proof. We claim that there exists a minimizer \( w \in H^1 \setminus \{0\} \) of (21) such that \( S_1^0 = S^0(w) \) and \( N^0(w) = 0 \). In fact, let \( \{v_j\} \subset H^1 \) be a minimizing sequence of (21), then for sufficiently large \( j \), we have

\[
2S_1^0 \geq \frac{p-2}{2p-1} \|\nabla v_j\|_{L^2}^2 + \frac{p-1}{2p-1} \|v_j\|_{L^2}^2,
\]

which yields that \( \{v_j\} \) is bounded in \( H^1 \). Moreover, by \( N^0(v_j) = 0 \) and the Sobolev inequality, there exist positive constants \( C_1, C_2 \) and \( C_3 \) such that

\[
C_1 \|v_j\|_{H^1}^2 \leq C_2 \|v_j\|_{L^{p+1}}^{p+1} \leq C_3 \|v_j\|_{H^1}^{p+1}.
\]

Since \( v_j \neq 0 \), we have \( \frac{C_1}{C_2} \leq \|v_j\|_{H^1}^{p+1} \) and \( \limsup_{j \to \infty} \|v_j\|_{L^{p+1}} > 0 \). By this fact and Lemma 2.2, we deduce that \( \sup_{y \in \mathbb{R}^3} \int_{B(y,1)} |v_j|^2 dx \geq \mu \) for some \( \mu > 0 \). In this case, we can choose \( \{y_j\} \subset \mathbb{R}^3 \) such that \( \sup_{y \in \mathbb{R}^3} \int_{B(0,1)} |v_j(\cdot + y_j)|^2 dx \geq \mu > 0 \). Hence, by the compactness of the embedding \( H^1(B(0,1)) \subset L^2(B(0,1)) \), we see that the weak limit of the sequence \( v_j(\cdot + y_j) \) cannot be zero. Thus, there exists a subsequence \( \{v_j(\cdot + y_j)\} \) and \( w \in H^1 \setminus \{0\} \) such that

\[
v_j(\cdot + y_j) \overset{j}{\rightharpoonup} w \text{ weakly in } H^1.
\]

By the weakly lower semicontinuity of \( J^0 \), we have

\[
J^0(w) \leq \liminf_{j \to \infty} J^0(v_j(\cdot + y_j)) = S_1^0.
\]

Moreover, by Lemma 2.3, we deduce that

\[
J^0(v_j(\cdot + y_j)) - J^0(v_j(\cdot + y_j) - w) - J^0(w) \overset{j}{\rightharpoonup} 0, \tag{23}
\]

\[
N^0(v_j(\cdot + y_j)) - N^0(v_j(\cdot + y_j) - w) - N^0(w) \overset{j}{\rightharpoonup} 0. \tag{24}
\]

We claim that \( N^0(w) \leq 0 \). Otherwise, if \( N^0(w) > 0 \), then \( N^0(v_j(\cdot + y_j) - w) < 0 \) for large \( j \), since \( N^0(v_j(\cdot + y_j)) = 0 \). By Lemma 3.1 and (23), we have

\[
J^0(v_j(\cdot + y_j) - w) > S_1^0 \quad \text{and} \quad J^0(w) = \limsup_{j \to \infty} J^0(v_j(\cdot + y_j)) - J^0(v_j(\cdot + y_j) - w) \leq 0,
\]

which contradicts to the facts that \( J^0(w) > 0 \) for \( w \neq 0 \).

By Lemma 3.1 and \( J^0(w) \leq S_1^0 \), we have \( J^0(w) = S_1^0 \). We claim that \( N^0(w) = 0 \). Otherwise, we have \( N^0(w) < 0 \). Then there exists \( \mu \in (0,1) \) such that \( N^0(\mu w) = 0 \). Moreover, we have

\[
J^0(\mu w) = \mu^2 J^0(w) < J^0(w) = S_1^0.
\]

This is a contradiction and \( w \) is a minimizer of problem (21) and

\[
v_j(\cdot + y_j) \overset{j}{\rightharpoonup} w \text{ strongly in } H^1.
\]

It is easy to check that \( w \), the minimizer of (21), is the ground state of (16). Kato’s inequality \( |\nabla w| \leq |\nabla w| \) a.e. implies that \(|w|\) is also a minimizer of (21). Moreover, by the uniqueness of the ground state of (16), we get that \(|w|(x + y) = Q(x)\) for some \( y \in \mathbb{R}^3 \).

Now, we are ready to prove Proposition 2.
Proof of Proposition 2. Let
\[ \mu(\omega)^{p-1} = \frac{3}{2} \| \nabla \tilde{\omega} \|_{L^2}^2 + \frac{1}{2} \| \tilde{\omega} \|_{L^2}^2. \]
Then we claim that
(i) \( \{ \mu(\omega) \} \subset \mathbb{R} \) such that \( \mu(\omega) \to 1 \) as \( \omega \to \infty \),
(ii) \( \{ \mu(\omega)\tilde{\omega} \} \) is a minimizing sequence of \( S_1^0 \) as \( \omega \to \infty \).
If the two properties mentioned above hold, then proof of Proposition 2 is completed by Lemma 3.2. In the following, we prove (i) and (ii).

For any \( \mu > 1 \), By \( N^0(Q) = 0 \) and a simple caculation, we have
\[ \tilde{N}(\mu Q) = \mu^2 \left( \frac{3}{2} \int |\nabla Q|^2 + \frac{1}{2} \int |Q|^2 \right) + \mu^{p+1} \frac{2p+1}{p+1} \int |Q|^{p+1} \]
\[ = \mu^2 (1 - \mu^{p-1}) \left( \frac{3}{2} \int |\nabla Q|^2 + \frac{1}{2} \int |Q|^2 \right) + \mu^{p+1} \frac{2p+1}{p+1} \int |Q|^{p+1}. \]
Hence, there exists a constant \( \omega_1(\mu) > 0 \) such that
\[ \tilde{N}(\mu Q) < 0 \text{ for } \omega > \omega_1(\mu). \quad (25) \]
For any \( \omega \in (\omega_1(\mu), \infty) \), it follows from (25) and the fact \( \tilde{N}(Q) > 0 \) that there exists \( \mu^* \in (1, \mu) \) such that
\[ \tilde{N}(\mu^* Q) = 0. \]
Since \( \{ \tilde{\omega} \} \) is the minimizer of (18), we have
\[ \tilde{J}(\tilde{\omega}) \leq \tilde{J}(\mu^* Q) \leq \tilde{J}(\mu Q), \]
which implies that \( \{ \tilde{\omega} \} \) is bounded in \( H^1 \). By \( N^0(Q) = 0 \) and a similar argument for (25), we get that \( \tilde{N}(Q) > 0 \). Let \( u^0 = \theta^2 u(\theta x) \), then \( \tilde{N}(Q^0) \to -\infty \) as \( \theta \to +\infty \).
Thus there exists \( \overline{\theta} > 1 \) such that \( \tilde{N}(Q^{\overline{\theta}}) = 0 \). It follows that \( \tilde{S}_\omega(Q^{\overline{\theta}}) = \max_{\theta > 0} \tilde{S}_\omega(Q^\theta) \) and
\[ \left| \tilde{S}_\omega(Q^{\overline{\theta}}) - S^0(Q^{\overline{\theta}}) \right| = \frac{3}{2} \omega^{\frac{2(2-p)}{p+1}} \int (|\theta|^{-1} * |Q|^2) |Q|^2 \leq C \overline{\theta}^3 \omega^{\frac{2(2-p)}{p+1}} \|Q\|_{H^1} \to 0. \]
For any \( \varepsilon > 0 \), there exists \( \omega_0 > 0 \) large enough such that
\[ \tilde{S}_\omega(Q^{\overline{\theta}}) \leq S^0(Q^{\overline{\theta}}) + \varepsilon, \text{ for } \omega > \omega_0. \]
Hence, the fact that \( \tilde{\omega} \) is the minimizer of (18) implies
\[ \tilde{S}_\omega(\tilde{\omega}) = \tilde{S}_1^0 = \inf_{\tilde{S}_\omega(\theta u) = \max_{\theta > 0} \tilde{S}_\omega(\theta Q^\theta) \leq \tilde{S}_\omega(Q^{\overline{\theta}}) + \varepsilon. \quad (26) \]
Since \( N^0(Q) = 0 \), we have that
\[ S^0(Q^\theta) \leq S^0(Q) \text{ for } \theta \in (0, \infty). \quad (27) \]
Using (26), (27) and \( S^0(\tilde{\omega}) \leq \tilde{S}_\omega(\tilde{\omega}) \), we deduce that
\[ \limsup_{\omega \to \infty} S^0(\tilde{\omega}) \leq S^0(Q). \quad (28) \]
By \( \tilde{N}(\tilde{\omega}) = 0 \), we get
\[ N^0(\tilde{\omega}) = \frac{3}{2} \| \nabla \tilde{\omega} \|_{L^2}^2 + \frac{1}{2} \| \tilde{\omega} \|_{L^2}^2 - \frac{2p-1}{p+1} \| \tilde{\omega} \|_{L^{p+1}}^{p+1} = -\frac{3}{4} \omega^{\frac{2(2-p)}{p+1}} \int (\frac{1}{|x|^2} * |\tilde{\omega}|^2) |\tilde{\omega}|^2. \]
Since $p > 2$ and $\|\tilde{\phi}_\omega\|_{H^1}$ is bounded, it follows from $\int(\frac{1}{|x|^4} * |\tilde{\phi}_\omega|^2)|\tilde{\phi}_\omega|^2 \leq C\|\tilde{\phi}_\omega\|_{H^1}^4$ that

$$N^0(\tilde{\phi}_\omega) \to 0 \text{ as } \omega \to \infty. \quad (29)$$

On the other hand, $\tilde{N}(\tilde{\phi}_\omega) = 0$ also implies that

$$\frac{2p-1}{p+1}\|\tilde{\phi}_\omega\|_{L^{p+1}}^{p+1} \geq \frac{1}{2}\|\tilde{\phi}_\omega\|_{H^1}^2 \geq C\|\tilde{\phi}_\omega\|_{L^{p+1}}^2.$$ 

Therefore, there exists a constant $C_1 > 0$ such that

$$\|\tilde{\phi}_\omega\|_{L^{p+1}} \geq C_1. \quad (30)$$

Recalling the definition of $\mu(\omega)$, it follows from (29) and (30) that

$$|\mu(\omega)^{p-1} - 1| = \frac{N^0(\tilde{\phi}_\omega)}{\frac{2p-1}{p+1}\|\tilde{\phi}_\omega\|_{L^{p+1}}^{p+1}} \leq \frac{N^0(\tilde{\phi}_\omega)}{C} \to 0 \text{ as } \omega \to \infty.$$

Hence (i) holds.

By the continuity of $S^0$ in $H^1$ and (28), we have

$$\limsup_{\omega \to \infty} S^0(\mu(\omega)\tilde{\phi}_\omega) = \limsup_{\omega \to \infty} S^0(\tilde{\phi}_\omega) \leq S^0(Q). \quad (31)$$

By the definition of $\mu(\omega)$, we know that

$$N^0(\mu(\omega)\tilde{\phi}_\omega) = \mu(\omega)^2\left(\frac{3}{2}\|\nabla \tilde{\phi}_\omega\|_{L^2}^2 + \frac{1}{2}\|\tilde{\phi}_\omega\|_{L^2}^2 - \mu(\omega)p^{-1}\frac{2p-1}{p+1}\|\tilde{\phi}_\omega\|_{L^{p+1}}^{p+1}\right) = 0.$$

Since $Q$ is a minimizer of (21), we deduce that

$$S^0(Q) \leq S^0(\mu(\omega)\tilde{\phi}_\omega). \quad (32)$$

It follows from (31) and (32) that (ii) holds.

By using the unique positive radial solution $Q$ of (16), we define the operators $L^0_+$ and $L^0_-$, respectively, by

$$\langle L^0_+ v, v \rangle = \|\nabla v\|_{L^2}^2 + \|v\|_{L^2}^2 - p \int Q^{p-1}|v|^2, \quad (33)$$

$$\langle L^0_- v, v \rangle = \|\nabla v\|_{L^2}^2 + \|v\|_{L^2}^2 - \int Q^{p-1}|v|^2. \quad (34)$$

Then, we have the following lemma.

**Lemma 3.3.** ([10]). Let $1 < p < \frac{7}{4}$, and $Q$ is the unique positive radial solution of (16). Then,

(i) there exists $\delta_1 > 0$ such that

$$\langle L^0_+ v, v \rangle \geq \delta_1 \|v\|_{H^1}^2,$$

for any $v \in H^1$ with $(v, Q)_{L^2} = 0$ and $(v, \partial_x Q)_{L^2} = 0$ for $i = 1, 2, 3$;

(ii) there exists $\delta_2 > 0$ such that

$$\langle L^0_- v, v \rangle \geq \delta_2 \|v\|_{H^1}^2,$$

for any $v \in H^1$ with $(v, Q)_{L^2} = 0$. 

Using $\tilde{\phi}_\omega$ defined by (14), we define the rescaled operators $\tilde{L}_{+,\omega}$ and $\tilde{L}_{-,\omega}$ as follows.

\[
\langle \tilde{L}_{+,\omega} v, v \rangle = \|\nabla v\|^2_{L^2} + \|v\|^2_{L^2} - p \int \tilde{\phi}_\omega^{-1}|v|^2 \\
+ \omega \frac{2(p+1)}{p-1} \left(2 \int (|x|^{-1} * (\tilde{\phi}_\omega v) \tilde{\phi}_\omega v) + \int (|x|^{-1} * \tilde{\phi}_\omega^2)|v|^2\right),
\]

(35)

\[\langle \tilde{L}_{-,\omega} v, v \rangle = \|\nabla v\|^2_{L^2} + \|v\|^2_{L^2} - \int \tilde{\phi}_\omega^{-1}|v|^2 dx + \omega \frac{2(p+1)}{p-1} \int (|x|^{-1} * \tilde{\phi}_\omega^2)|v|^2. \] (36)

By Proposition 2 and Lemma 3.3, it follows from the same argument of Lemma 4.1 in [18] that,

**Lemma 3.4.** Let $2 < p < \frac{7}{4}$, $\phi_\omega \in \mathcal{G}_\omega \cup \mathcal{G}_\omega^c$. $\tilde{\phi}_\omega$ is defined by (14) and the operators $\tilde{L}_{+,\omega}$ and $\tilde{L}_{-,\omega}$ are defined by (35) and (36), respectively. Then,

(i) there exists $\omega_1^*>0$ such that for any $\omega \in (\omega_1^*, \infty)$ and some $\delta_1 = \delta_1(\omega) > 0$, there holds

\[\langle \tilde{L}_{+,\omega} \tilde{v}, \tilde{v} \rangle \geq \delta_1 \|\tilde{v}\|^2_{H^1},\]

for all $\tilde{v} \in H^1$ with $(\tilde{v}, \tilde{\omega}_\omega)_{L^2} = 0$ and $(\tilde{v}, \partial_x, \tilde{\omega}_\omega)_{L^2} = 0$ for $i = 1, 2, 3$;

(ii) there exists $\omega_2^*>0$ such that for any $\omega \in (\omega_2^*, \infty)$ and some $\delta_2 = \delta_2(\omega) > 0$, there holds

\[\langle \tilde{L}_{-,\omega} \tilde{v}, \tilde{v} \rangle \geq \delta_2 \|\tilde{v}\|^2_{H^1},\]

for any $\tilde{v} \in H^1$ with $(\tilde{v}, \tilde{\omega}_\omega)_{L^2} = 0$.

To prove Theorem 1.2, we need some results due to Fukuizumi [10, 11], where a sufficient condition for the stability of the standing wave was established, based upon the idea of Grillakis, Shatah and Strauss [12].

**Proposition 3.** (Fukuizumi [10, 11]) Assume $2 < p < \frac{7}{4}$. Let $\phi_\omega \in \mathcal{G}_\omega$. If there exists $\delta > 0$ such that

\[\langle S''_\omega(\phi_\omega) v, v \rangle \geq \delta \|v\|^2_{H^1}, \] (37)

for any $v \in H^1$ satisfying $\Re(\phi_\omega, v)_{L^2} = 0$, $\Re(i\phi_\omega, v)_{L^2} = 0$ and $\Re(\partial_x, \phi_\omega, v)_{L^2} = 0$ for $i = 1, 2, 3$, then the standing wave solution $e^{it\omega(t)} \phi_\omega(x)$ is stable in $H^1$.

**Proof of Theorem 1.2.** By Proposition 3, we only need to check the condition (37). For $v \in H^1$ with $v_1(x) = \Re v(x)$ and $v_2(x) = \Im v(x)$, we have

\[\langle S''_\omega(\phi_\omega) v, v \rangle = \langle \tilde{L}_{+,\omega} v_1, v_1 \rangle + \langle \tilde{L}_{-,\omega} v_2, v_2 \rangle, \] (38)

where $L_{+,\omega}$ and $L_{-,\omega}$ are unbounded and self-adjoint operators, defined by

\[\langle L_{+,\omega} v_1, v_1 \rangle = \|\nabla v_1\|^2_{L^2} + \|v_1\|^2_{L^2} - p \int \tilde{\phi}_\omega^{-1}|v_1|^2 dx \\
+ 2 \int (|x|^{-1} * (\phi_\omega v_1) \phi_\omega v_1 + \int (|x|^{-1} * \tilde{\phi}_\omega^2)|v_1|^2, \] (39)

\[\langle L_{-,\omega} v_2, v_2 \rangle = \|\nabla v_2\|^2_{L^2} + \|v_2\|^2_{L^2} + \int (|x|^{-1} * \phi_\omega^2)|v_2|^2 - \int \tilde{\phi}_\omega^{-1}|v_2|^2 dx. \] (40)

Let $v(x) = \omega^{3/(p-1)} \tilde{v}(\sqrt{\omega} x)$. Then, by (35) and (36), we have

\[\langle L_{+,\omega} v, v \rangle = \omega^{2/(p-1)-1/2} \langle \tilde{L}_{+,\omega} \tilde{v}, \tilde{v} \rangle, \]

\[\langle L_{-,\omega} v, v \rangle = \omega^{2/(p-1)-1/2} \langle \tilde{L}_{-,\omega} \tilde{v}, \tilde{v} \rangle, \]

\[\langle \phi_\omega, v \rangle_{L^2} = \omega^{2/(p-1)-3/2} \langle \tilde{\phi}_\omega, \tilde{v} \rangle_{L^2}, \]

\[\langle \partial_x, \phi_\omega, v \rangle_{L^2} = \omega^{2/(p-1)-1} \langle \partial_x, \tilde{\phi}_\omega, \tilde{v} \rangle_{L^2}, \] (41)

Let $2 < p < \frac{7}{4}$, $\phi_\omega \in \mathcal{G}_\omega \cup \mathcal{G}_\omega^c$. It follows from Lemma 3.4 and (41) that
(i) there exists $\omega_1^* > 0$ such that for any $\omega \in (\omega_1^*, \infty)$, there is a $\delta_1 > 0$ satisfying
$$\langle L_{+, \omega} v, v \rangle \geq \delta_1 \|v\|^2_{H^1},$$
for any $v \in H^1$ with $(v, \phi_\omega)_{L^2} = 0$ and $(v, \partial_{x_i} \phi_\omega)_{L^2} = 0$, $i = 1, 2, 3$;
(ii) there exists $\omega_2^* > 0$ such that for any $\omega \in (\omega_2^*, \infty)$, there is a $\delta_2 > 0$ satisfying
$$\langle L_{-, \omega} v, v \rangle \geq \delta_2 \|v\|^2_{H^1},$$
for any $v \in H^1$ with $(v, \phi_\omega)_{L^2} = 0$.

These conclusions imply that (37) holds. Proof of Theorem 1.2 is completed. \qed

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