Creator-annihilator domains
and the number operator

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Abstract

We show that for the bosonic Fock representation in infinite dimensions, the maximal common domain of all creators and annihilators properly contains the domain of the square-root of the number operator.

Introduction

A standard construction of the bosonic Fock representation of a complex Hilbert space $V$ is in terms of creators and annihilators on the symmetric algebra $SV$ in which the number operator $N$ scales elements by homogeneous degree; the symmetric algebra is completed relative to a canonical inner product and these various operators are extended to maximal domains in the resulting Fock space $S[V]$.

It is well-known that all creators and annihilators are defined on the domain of the square-root $N^{\frac{1}{2}}$. A natural question (raised on page 16 of [2] by Berezin, among others; compare page 65 of [4]) is whether the domain of $N^{\frac{1}{2}}$ coincides with the maximal common domain of all creators and annihilators. Here, we demonstrate that the answer to this question is affirmative when $V$ is finite-dimensional but negative when $V$ is infinite-dimensional.

As noted above, the specific question that is answered in this paper appears in [2]. Among many standard references concerning the bosonic Fock representation, we cite [1] and [3]. The particular approach taken here (involving the full antidual of the symmetric algebra) was introduced in [6] as a means to establishing a generalized version of the classic Shale theorem.
on the implementation of symplectic automorphisms, independently of the generalization previously presented in [5].

**Fock-lore**

We begin by recalling certain familiar elements of Fock-lore, pertaining to the construction of bosonic Fock space and the various operators defined therein. For traditional accounts, see a standard text such as [1] or [3]; for an account in line with the present paper, see [6].

To be explicit, let $V$ be a complex Hilbert space. Extend its inner product $\langle \cdot | \cdot \rangle$ to the symmetric algebra $SV = \bigoplus_{n \geq 0} S^n V$ by declaring that the homogeneous summands $(S^n V : n \geq 0)$ be perpendicular and that if $n \geq 0$ and $x_1, \ldots, x_n, y_1, \ldots, y_n \in V$ then

$$\langle x_1 \cdots x_n | y_1 \cdots y_n \rangle = \sum_p \prod_{j=1}^n \langle x_j | y_{p(j)} \rangle$$

where $p$ runs over all permutations of $\{1, \ldots, n\}$; the resulting complex Hilbert space completion is (by definition) the bosonic Fock space $S[V] = \bigoplus_{n \geq 0} S^n [V]$.

For our purposes, it is convenient to introduce also the full antidual $SV'$ comprising all antilinear functionals $SV \rightarrow \mathbb{C}$. This full antidual $SV'$ is naturally a commutative, associative algebra under the product defined by

$$\Phi, \Psi \in SV' \implies [\Phi \Psi](\theta) = (\Phi \otimes \Psi)(\Delta \theta)$$

where the coproduct $\Delta : SV \rightarrow SV \otimes SV$ arises when the canonical isomorphism $S(V \oplus V) \cong SV \otimes SV$ follows the homomorphism $SV \rightarrow S(V \oplus V)$ induced by the diagonal map $V \rightarrow V \oplus V$. The inner product on $SV$ engenders an algebra embedding

$$SV \rightarrow SV' : \phi \mapsto \langle \cdot | \phi \rangle$$

and $S[V]$ is identified with the subspace of $SV'$ comprising all bounded antilinear functionals.

Let $v \in V$. The creator $c(v)$ is defined initially on $SV$ as the operator of multiplication by $v$:

$$\phi \in SV \implies c(v)\phi = v\phi.$$
The annihilator $a(v)$ is defined initially on $SV$ as the linear derivation that kills the vacuum $1 \in C = S^0V$ and sends $w \in V = S^1V$ to $\langle v|w \rangle$: thus, if $v_1, \ldots, v_n \in V$ then
\[
a(v)(v_1 \cdots v_n) = \sum_{j=1}^{n} \langle v|v_j \rangle v_1 \cdots \widehat{v_j} \cdots v_n
\]
where the circumflex $\widehat{}$ signifies omission. As may be verified by direct calculation, $c(v)$ and $a(v)$ are mutually adjoint:
\[
\phi, \psi \in SV \implies \langle \psi|c(v)\phi \rangle = \langle a(v)\psi|\phi \rangle.
\]
Accordingly, the creator $c(v)$ and annihilator $a(v)$ extend to $SV'$ by antiduality: if $\Phi \in SV'$ and $\psi \in SV$ then
\[
[c(v)\Phi](\psi) = \Phi[a(v)\psi]
\]
and
\[
[a(v)\Phi](\psi) = \Phi[c(v)\psi].
\]
Finally, the corresponding (mutually adjoint) operators in Fock space $S[V] \subset SV'$ are defined by restriction to the (coincident) natural domains
\[
\mathcal{D}[c(v)] = \{ \Phi \in S[V] : c(v)\Phi \in S[V] \}
\]
and
\[
\mathcal{D}[a(v)] = \{ \Phi \in S[V] : a(v)\Phi \in S[V] \}.
\]
The number operator $N$ is defined initially on $SV$ by the rule
\[
n \geq 0 \implies N|S^nV = nI
\]
and extends to $SV'$ by antiduality:
\[
\Phi \in SV', \psi \in SV \implies [N\Phi](\psi) = \Phi[N\psi].
\]
The number operator in $S[V] \subset SV'$ is defined by restriction to the natural domain
\[
\mathcal{D}[N] = \{ \Phi \in S[V] : N\Phi \in S[V] \}
\]
which may be identified in terms of the decomposition $S[V] = \bigoplus_{n \geq 0} S^n[V]$ as
\[
\mathcal{D}[N] = \{ \sum_{n \geq 0} \Phi_n \in S[V] : \sum_{n \geq 0} \|n\Phi_n\|^2 < \infty \}.
\]
We remark that the number operator \( N \) in \( S[V] \) is selfadjoint (indeed, positive): \( \bigoplus_{n \geq 0} S^n[V] \) is its spectral decomposition, whence powers of \( N \) are readily described in concrete terms; in particular,
\[
\mathcal{D}[N^{\frac{1}{2}}] = \{ \Phi \in S[V] : \sum_{n \geq 0} n \|\Phi_n\|^2 < \infty \}.
\]

**Theorems**

Having established sufficient background, we now proceed to our primary task: that of relating \( \mathcal{D}[N^{\frac{1}{2}}] \) to the maximal common domain of all creators and annihilators in Fock space.

Let \( u \in V \) be (for convenience) a unit vector: the unitary decomposition \( V = C u \bigoplus u^\perp \) induces (for each \( n \geq 0 \)) a unitary decomposition
\[
S^n V = \bigoplus_{p+q=n} \{ S^p(C u) \otimes S^q(u^\perp) \}.
\]
Decomposing \( \phi \in S^n V \) as
\[
\phi = \sum_{p+q=n} u^p \otimes \psi_q
\]
we note (by perpendicularly) that
\[
\|\phi\|^2 = \sum_{p+q=n} \|u^p\|^2 \|\psi_q\|^2 = \sum_{p+q=n} p! \|\psi_q\|^2
\]
and that
\[
\|c(u)\phi\|^2 = \sum_{p+q=n} \|u^{p+1}\|^2 \|\psi_q\|^2 = \sum_{p+q=n} (p+1)! \|\psi_q\|^2
\]
whence it follows that
\[
\|c(u)\phi\|^2 \leq (n+1)\|\phi\|^2.
\]
This inequality continues to apply when \( \phi \) lies in the closure \( S^n[V] \); in particular, \( S^n[V] \subset \mathcal{D}[c(u)] \). Now, if \( \Phi \in \mathcal{D}[N^{\frac{1}{2}}] \) then
\[
\|c(u)\Phi\|^2 = \sum_{n \geq 0} \|c(u)\Phi_n\|^2 \leq \sum_{n \geq 0} (n+1)\|\Phi_n\|^2 = \|N^{\frac{1}{2}}\Phi\|^2 + \|\Phi\|^2
\]
whence \( \Phi \in \mathcal{D}[c(u)] = \mathcal{D}[a(u)] \). Lifting the convenient hypothesis that \( u \in V \) be a unit vector, we have justified the following result.
Theorem 1 If \( u \in V \) then \( D[N^\frac{1}{2}] \) is contained in \( D[c(u)] = D[a(u)] \).

Thus, \( D[N^\frac{1}{2}] \) is contained in the maximal common domain of all creators and annihilators in Fock space.

We now consider the reverse containment in case \( V \) is finite-dimensional, with \( (u_1, \ldots, u_m) \) a unitary basis. If the integers \( n_1, \ldots, n_m \geq 0 \) have sum \( n \) then for each \( j \in \{n_1, \ldots, n_m\} \)
\[
c(u_j)a(u_j)(u_1^{n_1} \cdots u_m^{n_m}) = n_j(u_1^{n_1} \cdots u_m^{n_m})
\]
so
\[
\sum_{j=1}^{m} c(u_j)a(u_j)(u_1^{n_1} \cdots u_m^{n_m}) = n(u_1^{n_1} \cdots u_m^{n_m}).
\]
By linearity, it follows that
\[
\sum_{j=1}^{m} c(u_j)a(u_j) \phi = n \phi
\]
whenever \( \phi \in S^n V \) and indeed whenever \( \phi \in S^n [V] \) by the boundedness of creators and annihilators on homogeneous elements of Fock space. Now, if \( \Phi \in D[a(u_1)] \cap \cdots \cap D[a(u_m)] \) then
\[
\infty > \sum_{j=1}^{m} \|a(u_j)\Phi\|^2 = \sum_{j=1}^{m} \sum_{n \geq 0} \|a(u_j)\Phi_n\|^2
\]
whence (valid) passage of annihilators across the inner product as creators yields
\[
\infty > \sum_{n \geq 0} \langle \Phi_n | \sum_{j=1}^{m} c(u_j)a(u_j)\Phi_n \rangle = \sum_{n \geq 0} n \|\Phi_n\|^2
\]
which places \( \Phi \) in \( D[N^\frac{1}{2}] \). This justifies the following result.

Theorem 2 If \( (u_1, \ldots, u_m) \) is a unitary basis for the finite-dimensional \( V \) then
\[
D[N^\frac{1}{2}] = D[a(u_1)] \cap \cdots \cap D[a(u_m)]
\]
and if \( \Phi \) lies in this domain then
\[
\|N^\frac{1}{2} \Phi\|^2 = \sum_{j=1}^{m} \|a(u_j)\Phi\|^2.
\]

In particular, if \( V \) is finite-dimensional then \( D[N^\frac{1}{2}] \) coincides with the maximal common domain of all creators and annihilators in Fock space.
The case in which $V$ is infinite-dimensional is different. To see this, let
\[ \Phi = \sum_{n>0} \Phi_n \in SV' \]
be defined by
\[ n > 0 \implies \Phi_n = \lambda_n (u_n)^n / n! \]
where $(\lambda_n : n > 0)$ is a complex sequence and where the unit vectors $(u_n : n > 0)$ in $V$ are perpendicular. From
\[ n > 0 \implies \|\Phi_n\|^2 = |\lambda_n|^2 \|u_n\|^2 / (n!)^2 = |\lambda_n|^2 / n! \]
it follows that
\[ \Phi \in S[V] \iff \sum_{n>0} \frac{|\lambda_n|^2}{n!} < \infty \]
and that
\[ \Phi \in D[N^{\frac{1}{2}}] \iff \sum_{n>0} \frac{|\lambda_n|^2}{(n-1)!} < \infty. \]
Further, let $v \in V$: if $n > 0$ then
\[ a(v)\Phi_n = \langle v|u_n \rangle \lambda_n (u_n)^{n-1} / (n-1)! \]
whence
\[ \|a(v)\Phi\|^2 = \sum_{n>0} |\langle v|u_n \rangle|^2 \frac{|\lambda_n|^2}{(n-1)!}. \]
Now, if $(\lambda_n : n > 0)$ is chosen so that
\[ \sum_{n>0} \frac{|\lambda_n|^2}{n!} < \infty = \sum_{n>0} \frac{|\lambda_n|^2}{(n-1)!} \]
then $\Phi \in S[V] \setminus D[N^{\frac{1}{2}}]$ while if also $(|\lambda_n|^2 / (n-1)! : n > 0)$ is bounded above by $K > 0$ then
\[ \|a(v)\Phi\|^2 \leq \sum_{n>0} |\langle v|u_n \rangle|^2 K \leq K \|v\|^2 \]
which places $\Phi$ in $D[a(v)] = D[c(v)]$. Of course, these conditions are easily satisfied: for example, when $n > 0$ simply take $|\lambda_n|^2 / (n-1)! = 1/n$. As the vector $v \in V$ is arbitrary, the following result is justified.

**Theorem 3** If $V$ is infinite-dimensional then $D[N^{\frac{1}{2}}]$ is properly contained in the maximal common domain of all creators and annihilators.

Without proof, we remark that similar results hold for higher powers of the number operator: thus, if $k > 0$ then $D[N^{\frac{k}{2}}]$ is contained in the maximal common domain of all degree $k$ polynomials in creators and annihilators, containment being strict precisely when $V$ is infinite-dimensional.
References

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