Torsion/nonmetricity duality in $f(R)$ gravity

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Torsion and nonmetricity are inherent ingredients in modifications of Einstein’s gravity that are based on affine spacetime geometries. In the context of pure $f(R)$ gravity we discuss here, in some detail, the relatively unnoticed duality between torsion and nonmetricity. In particular we show that for $R^2$ gravity torsion and nonmetricity are related by projective transformations. Since the latter correspond simply to redefining the affine parameters of autoparallels, we conclude that torsion and nonmetricity are physically equivalent properties of spacetime. As a simple example we show that both torsion and nonmetricity can act as geometric sources of accelerated expansion in a spatially homogenous cosmological model within $R^2$ gravity and we briefly discuss possible implications of our results.

I. INTRODUCTION

The study of modifications of Einstein’s gravity is a major theme in current gravitational physics research. The primary interest behind such searches is the quest for a purely geometrical description for the observed cosmological evolution, without the necessity to reify into the (as yet) elusive ingredients of dark matter and dark energy. Although this is a rather optimistic target, the study of modified theories of gravity is per se an interesting intellectual exercise which might also open alternative and unexpected windows into physics as the celebrated example of AdS/CFT and its numerous spinoffs have recently taught us.

In Einstein’s gravity spacetime is a four dimensional Riemannian manifold equipped with a Levi-Civita (i.e. metric compatible and torsionless) connection that is fully determined by the symmetric metric. Going beyond, the most economical yet fully geometrical modifications of Einstein’s theory come in the form of metric-affine theories of gravity [1]. In such theories the modification comes from the introduction of a non-symmetric connection (i.e. torsion) which is not necessarily compatible with the metric (i.e. nonmetricity). Although torsion and nonmetricity are inherent ingredients of any geometric description of spacetime, they were hastily dismissed from a physical theory of gravity. Indeed, a physical implication of torsion is that parallel transport along a closed path results in a translation [2]. On the other hand, nonmetricity would imply that the norm of a vector changes when it is parallel transported along itself [1]. These effects, if they exist, were deemed unobservable in simple gravitational systems. Nevertheless, since they are geometrical by nature, there does not appear to be any apriori deep reason that excludes them from the generic description of gravitation as spacetime geometry. In that sense, it is left to the various forms of matter that are coupled to gravity to probe whether or not torsion or nonmetricity have a physical relevance and possibly lead to observable implication [3] see e.g. the review [4].

Metric-affine gravity is often studied using the Palatini formalism where the spacetime metric and the connection are considered as independent dynamical variables in a first-order Lagrangian formalism. This way, the relationship between the connection and the metric emerges as a consequence of the choice of the gravitational Lagrangian, and hence the latter becomes a nice guiding principle to study modifications of Einstein’s gravity. Among the Lagrangian models used in studies of modified gravity, the so-called $f(R)$ theories have attracted an enormous amount of interest in recent years (for some recent reviews see [5-8]). However torsion and nonmetricity in modified gravity are less well studied and this is related to the fact that their effect in $f(R)$ gravity is mimicked by a particular Brans-Dicke model.
of Einstein gravity coupled to scalars \[9\]. Nevertheless, an interesting observation of \[9\] was that metric gravitational theories with torsion or nonmetric theories without torsion are equivalent to the \textit{same} Brans-Dicke theory. This indicates that there may be some kind of duality in the physical implications of torsion and nonmetricity.

In order to dwell further into this idea we revisit here the role of torsion and nonmetricity in \(f(R)\) pure gravity. In Section II we review the well-known fact that only vector torsion and nonmetricity are allowed in generic \(f(R)\) theories, and further that they are proportional to each other. Then we study in detail the role of projective invariance for \(f(R)\) theories and show that they can be used to always obtain the Levi-Civita connection as a solution of the \(f(R) \neq R^2\) equations of motion. This confirms the well-known equivalence between the metric and Palatini formulations of \(f(R) \neq R^2\) gravity. In Section III we move to the study of \(f(R) = R^2\) theory. In this case we show that there is always a non-zero contribution of torsion and nonmetricity to the affine connection. Moreover, we also demonstrate that projective transformations generically interchange the roles of torsion and nonmetricity. The only projective invariant quantities are those that depend on the \textit{affine vector} \(w_\mu\). This result implies that torsion and nonmetricity are physically equivalent as far as \(R^2\) gravity is concerned and we make an attempt to give a geometric interpretation of that result. In Section IV we present a simple FRW spatially homogenous model in the context of \(f(R)\) gravity and show that both torsion and nonmetricity can act as geometric sources of accelerating expansion. We further analyse in some detail the nonmetric expansion that we have found. We present our conclusions and outlook in Section V.

II. TORSION AND NOMMETRICITY IN \(f(R)\) GRAVITY

Consider the generic \(f(R)\) action in \(d = 4\)

\[
S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} f(R), \quad R = g^{\mu\nu} R_{\mu\nu}(\Gamma),
\]  

(1)

where \(\kappa^2 = 8\pi G_4\). The independent variations of the metric and the connection yield \[10\]

\[
f' R_{(\mu, \nu)} - \frac{1}{2} f g_{\mu\nu} = 0,
\]

(2)

\[-\nabla_\lambda (\sqrt{-g} f' g^{\mu\nu}) + \nabla_\sigma (\sqrt{-g} f' g^{\mu\sigma}) \delta^{\nu}_{\lambda} + 2 \sqrt{-g} f' (g^{\mu\nu} S_{\lambda} - g^{\mu\rho} S_\rho + g^{\rho\sigma} S_{\sigma\lambda}^{\nu}) = 0.
\]

(3)

Taking the trace of (2) one obtains

\[
f'(R) R - 2 f(R) = 0,
\]

(4)

where the prime denotes derivative wrt \(R\). Taking the \(\nu, \lambda\) trace in (3) and substituting it back we obtain

\[-\nabla_\lambda (\sqrt{-g} f' g^{\mu\nu}) + 2 \sqrt{-g} f' \left( g^{\mu\nu} S_{\lambda} - \frac{1}{3} g^{\mu\rho} S_\rho + g^{\nu\sigma} S_{\sigma\lambda}^{\nu} \right) = 0.\]

(5)

Then, if we take the \(\mu, \nu\) trace in (3) and use the general formula

\[
\nabla_\mu \sqrt{-g} = -\frac{1}{2} \sqrt{-g} Q_\lambda,
\]

(6)

we find

\[
w_\mu = \frac{1}{4} Q_\mu + \frac{4}{3} S_\mu = \partial_\mu \ln f'.
\]

(7)

We call the above linear combination of torsion and nonmetricity the \textit{affine vector}. With the above results, we can go back to (3) and obtain a general formula relating torsion and nonmetricity as

\[
Q_\lambda^{\mu\nu} - \frac{1}{4} Q_\lambda g^{\mu\nu} = \frac{2}{3} g^{\mu\nu} S_\lambda - \frac{2}{3} S_\rho g^{\rho\mu} \delta^{\nu}_{\lambda} + 2 g^{\mu\sigma} S_{\sigma\lambda}^{\nu}.
\]

(8)

Notice that taking the \((\mu, \nu)\) trace of (8) leads to an identity that leaves unrelated the vectors \(S_\mu\) and \(Q_\mu\). Finally, using the formula (72) in the Appendix that gives the generic decomposition of an affine connection we obtain from
connection by choosing an appropriate gauge parameter \( \xi \). Then, the generic constant curvature metrics \([12]\), except in the case when nonmetricity tensors associated to the connection (9) are the affine vector vanishes

\[
\Gamma^\lambda_{\mu\nu} = \hat{\Gamma}^\lambda_{\mu\nu} + \frac{1}{8} Q_{\nu} \delta^\lambda_\mu + \frac{1}{2} (w_\mu \delta^\lambda_\nu - g_{\mu\nu} g^{\lambda\sigma} w_\sigma).
\]

(9)

In (9) \( \hat{\Gamma}^\lambda_{\mu\nu} \) denotes the Levi-Civita connection i.e. the Christoffel symbols of the metric \( g_{\mu\nu} \). The torsion and nonmetricity tensors associated to the connection (9) are

\[
S^{\lambda}_{\mu\nu} = \frac{1}{3} (S^\mu_\nu \delta^\lambda_\chi - S^\nu_\mu \delta^\lambda_\chi), \quad Q_{\lambda\mu\nu} = \frac{1}{4} Q_\lambda g_{\mu\nu},
\]

(10)

namely are fully determined from the vectors \( S_\mu \) and \( Q_\mu \) respectively \([11]\), the latter being at this point totally independent quantities.

To proceed, we consider (4) which is an algebraic equation for the scalar curvature \( R \) yielding generically zero or constant curvature metrics \([12]\), except in the case when \( f(R) \propto R^2 \) which is the focus of our work later. In those cases the affine vector vanishes \( w_\mu = 0 \), which in turn implies that the torsion and nonmetricity vectors are proportional to each other. Hence the affine connection (9) becomes

\[
\Gamma^\lambda_{\mu\nu} = \hat{\Gamma}^\lambda_{\mu\nu} + \frac{1}{8} Q_{\nu} \delta^\lambda_\mu.
\]

(11)

A. Projective invariance

Next one notices that in \( f(R) \) theories the connection can only be determined up to a vectorial degree of freedom. Indeed, it is not hard to show, i.e. using the definition (62), that under transformations of the form

\[
\Gamma^\lambda_{\mu\nu} \rightarrow \Gamma^\lambda_{\mu\nu} + \delta^\lambda_\mu \xi_\nu,
\]

(12)

with \( \xi_\mu \) an arbitrary vector, the symmetric part of the Ricci tensor \( R_{(\mu,\nu)} \) and consequently the Ricci scalar \( R \) remain invariant. Then, the generic \( f(R) \) action is invariant under (12) and one can always arrive at the Levi-Civita connection by choosing an appropriate gauge parameter \( \xi_\mu \) in (12) to get rid of the terms proportional to \( \delta^\lambda_\mu \) in (11).

The projective transformations (12) are defined as those transformations of the affine connection that leave invariant the autoparallels of vectors up to reparametrizations of the affine parameter. Given the curve \( C : x^\mu = x^\mu(\lambda) \) parametrized by the affine parameter \( \lambda \), and its tangent vector \( u^\mu(\lambda) = dx^\mu/d\lambda \), we define the autoparallel curves as those satisfying

\[
u^\lambda \nabla_\lambda u^\alpha = \frac{d^2 x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0.
\]

(13)

If \( \Gamma^\alpha_{\mu\nu} \) is the Levi-Civita connection this is just the geodesic equation that arises as usually when we extremize the spacetime distance \( \sqrt{-g_{\mu\nu} dx^\mu dx^\nu} \). Clearly, only the symmetric part of the affine connection \( \Gamma^\lambda_{\mu\nu} \) contributes to the autoparallel equation (13), yet this part generically given by (16) receives contributions both from torsion and nonmetricity.

There is nevertheless a freedom to define the affine connection in (13) such that it only corresponds to a reparametrization of the affine parameter \( \lambda \). To this end consider the transformation (12), when the autoparallel equation becomes

\[
\frac{d^2 x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = -\xi_\nu \frac{dx^\nu}{d\lambda} \frac{dx^\alpha}{d\lambda} = f(\lambda) \frac{dx^\alpha}{d\lambda},
\]

(14)

where we have set

\[
f(\lambda) = -\xi_\nu \frac{dx^\nu}{d\lambda}.
\]

(15)

This implies that \( \lambda \) is not an affine parameter any more. However, using the change of variables \( s = s(\lambda) \) it follows

\[
\frac{dx^\alpha}{d\lambda} = \frac{dx^\alpha}{ds} \frac{ds}{d\lambda}, \quad d^2 x^\alpha = d^2 x^\alpha \frac{ds^2}{d\lambda^2} + \frac{dx^\alpha}{d\lambda} \frac{ds}{d\lambda}.
\]

(16)
where the dot denotes differentiation with respect to $\lambda$. Plugging these into (14) we obtain
\[
\frac{d^2 x^\alpha}{ds^2} + \tilde{\Gamma}^\alpha_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = \frac{1}{s^2} \left( f(\lambda) \dot{s} - \ddot{s} \right) \frac{dx^\alpha}{ds}.
\]
(17)

From the above we see that if we choose $s(\lambda)$ such that
\[
f(\lambda) \dot{s} - \ddot{s} = 0,
\]
(18)
the right hand side vanishes and $s(\lambda)$ becomes a new affine parameter of the autorarallel equation. The reparametrization that we need to perform is
\[
s(\lambda) = \int g(\lambda) d\lambda, \quad g(\lambda) = e^{\int f(\lambda) d\lambda} = e^{-\int \xi_\mu dx^\mu}
\]
(19)

The projective transformations (12) are the most general form of transformations that change the autoparallel curves by a reparametrization of their affine parameter. They depend on the arbitrary vector parameter $\xi_\mu$. Viewing gravity as a dynamical system, it can be shown that projective transformations are actually gauge transformations in the classical sense i.e. (13, 14). This can be used to show the equivalence of the metric and the Palatini formulations in Einstein-Hilbert $f(R) \sim R$ gravity (15). We have thus presented here the generalization of this result to $f(R)$ gravity, implicit in a number of works i.e. (9) showing that pure $f(R)$ theories of gravity lead to the Levi-Civita connection, up to projective transformations.

III. TORSION AND NONMETRICITY IN $R^2$ GRAVITY

As we have discussed above, pure $f(R)$ gravity admits generically solutions with constant scalar curvature $R$, which are not suitable for the description of cosmological evolution. However, in the special case when $f(R) \propto R^2$ we cannot use (4) to fix the scalar curvature and this allows for the possibility that torsion and nonmetricity drive nontrivial cosmological solutions. Consider the action
\[
S = \frac{1}{2\kappa^2} \int d^4 x \sqrt{-g} R^2,
\]
(20)
where $\alpha$ is a parameter with dimensions of inverse mass squared. The metric variation gives now
\[
2R \left( R_{(\mu,\nu)} - \frac{R}{4} g_{\mu\nu} \right) = 0,
\]
(21)
which implies that either $R = 0$ or
\[
R_{(\mu,\nu)} - \frac{R}{4} g_{\mu\nu} = 0
\]
(22)
Therefore, although Ricci flat metrics solve the e.o.m. coming from (20), there exist also solutions that satisfy (22). The trace of the latter equation vanishes identically and hence it does not impose an algebraic constraint on the scalar curvature, but rather gives by virtue of (7)
\[
\partial_\mu \ln R = w_\mu.
\]
(23)
This can be formally integrated to yield
\[
R = R_0 e^{\int w_\mu dx^\mu},
\]
(24)
with $R_0$ some constant.

Nevertheless, as before we should be able to use projective transformations to further restrict torsion and nonmetricity and through them the affine connection. To begin with, we notice that under (12) and by virtue of the first one in (10) the vector torsion transforms as
\[
S_\mu \rightarrow S_\mu \bigg|_{\text{new}} = S_\mu \bigg|_{\text{old}} - \frac{3}{2} \xi_\mu.
\]
(25)
Then, using the projective transformation of the distortion tensor \((73)\) and \((10)\) we find the transformation of the nonmetricity vector to be

\[ N^\lambda_{\mu
u} \rightarrow N^\lambda_{\mu
u} \bigg|_{\text{new}} = N^\lambda_{\mu
u} \bigg|_{\text{old}} + \xi_\nu \delta^\lambda_{\mu} \Rightarrow Q_\mu \rightarrow Q_\mu \bigg|_{\text{new}} = Q_\mu \bigg|_{\text{old}} + 8 \xi_\mu. \quad (26) \]

From \((25)\) and \((26)\) we then learn that under projective transformations the affine vector \(w_\mu\) remains invariant. This is of course consistent with the fact that \(w_\mu\) depends on the projective invariant scalar curvature \(R\). We conclude that unless we assume zero curvature solutions, metric affine \(R^2\) gravity gives rise to spacetimes with non-constant Ricci curvature. This effect is totally due to the presence of torsion and nonmetricity.

### A. Torsion/nonmetricity duality and its physical implication

The above analysis shows that although the presence of \(w_\mu\) cannot be gauged away by projective transformations, the individual roles of torsion and nonmetricity are actually gauge dependent. For example, we could choose \(\xi_\mu\) such that we eliminate either torsion or nonmetricity, namely

\[ \xi_\mu = \frac{2}{3} S_\mu \bigg|_{\text{old}} \Rightarrow S_\mu \bigg|_{\text{new}} = 0, \quad Q_\mu \bigg|_{\text{new}} = Q_\mu \bigg|_{\text{old}} + \frac{16}{3} S_\mu \bigg|_{\text{old}} \neq 0, \quad (27) \]

\[ \xi_\mu = -\frac{1}{8} Q_\mu \bigg|_{\text{old}} \Rightarrow Q_\mu \bigg|_{\text{new}} = 0, \quad S_\mu \bigg|_{\text{new}} = S_\mu \bigg|_{\text{old}} + \frac{3}{16} Q_\mu \bigg|_{\text{old}} \neq 0. \quad (28) \]

More intriguingly we may chose \(\xi_\mu\) such that we interchange torsion and nonmetricity! Explicitly,

\[ \xi_\mu = \frac{2}{3} S_\mu \bigg|_{\text{old}} - \frac{1}{8} Q_\mu \bigg|_{\text{old}} \Rightarrow S_\mu \bigg|_{\text{new}} = \frac{3}{16} Q_\mu \bigg|_{\text{old}}; \quad Q_\mu \bigg|_{\text{new}} = \frac{16}{3} S_\mu \bigg|_{\text{old}}. \quad (29) \]

Notice that in the latter case the transformation leaves invariant the direct product of the torsion and nonmetricity vectors

\[ S_\mu Q^\mu \rightarrow (S_\mu Q^\mu) \bigg|_{\text{new}} = (S_\mu Q^\mu) \bigg|_{\text{old}}. \quad (30) \]

Our main result is therefore that torsion and nonmetricity are gauge equivalent physical properties of metric affine \(R^2\) gravity in \(d = 4\). The role of gauge transformations is here played by the projective transformations \((12)\) of the connection.

We may try to unveil the physical implications of the torsion/nonmetricity duality by a simple geometric example. It is known (i.e. see \([2]\) and references therein) that torsion is related to some kind of “spacetime dislocation”. Consider two infinitesimal vectors \(u^\mu\) and \(v^\mu\). Let \(\Delta_u u^\mu\) by the infinitesimal change of \(u^\mu\) parallel transported along \(v^\mu\), and correspondingly \(\Delta_u v^\mu\) the infinitesimal change of \(v^\mu\) parallel transported along \(u^\mu\). Then, the difference

\[ T^\mu \equiv \Delta_u u^\mu - \Delta_u v^\mu = -(\Gamma^\mu_{\rho\sigma} - \Gamma^\mu_{\sigma\rho}) v^\rho u^\sigma = 2 S^\mu_{\rho\sigma} u^\rho v^\sigma, \quad (31) \]

is the the four-dimensional analog of the usual Burgers vector in the theory of elastic dislocations. Its physical interpretation is to quantify the failure in closing of infinitesimal rectangles in a system with a dislocation effect i.e. by going around a closed loop we reach a point translated by \(T^\mu\) with respect to the starting point.

On the other hand, the physical implication of nonmetricity is to alter the length of vectors that are parallel transported along spacetime trajectories, which in turn leads to an inherent inability to define the notion of constant norm vectors. For example, if the vector \(u^\mu\) is parallel transported along \(v^\sigma\) then its norm \(||u|| = (g_{\mu\nu} u^\mu u^\nu)^{1/2}\) changes as

\[ D_v ||u|| \equiv u^\mu \nabla_\mu (g_{\rho\sigma} u^\rho u^\sigma)^{1/2} = -\frac{1}{2 ||u||} u^\mu (\Gamma^\mu_{\mu\rho} - \Gamma^\mu_{\mu\sigma}) u^\rho u^\sigma = -\frac{1}{2 ||u||} u^\mu Q_{\mu\rho\sigma} u^\rho u^\sigma. \quad (32) \]

If we then consider the form of torsion and nonmetricity given in \((10)\) and we further assume that \(u^\mu v_\mu = 0\), then we can find after some calculations

\[ D_v ||u|| + \frac{1}{||u||} u_\mu T^\mu = -\frac{1}{2} (v_\mu w^\mu)||u||. \quad (33) \]
We therefore notice that in the affine geometry described by the $R^2$ gravity there exists a physical effect on the norm of vectors that can be attributed to either torsion or nonmetricity of to both of them. Namely, the combined effect of parallel transporting the vector $u^\mu$ along a direction normal to it (i.e. along the vector $v^\mu$ with $v^\mu u_\mu = 0$), together with the normalized projection of $u^\mu$ along the dislocation in the closed rectangle formed by $u^\mu$ and $v^\mu$, is projective invariant since it depends on the affine vector $w_\mu$. In that sense, systems with just spacetime dislocations are seen to be physically equivalent to systems with just Weyl nonmetricity.

IV. TORSION AND NONMETRICITY IN COSMOLOGY

The physical equivalence of torsion and nonmetricity can also be seen in a simple cosmological model. It was noticed in [16] that torsion can act as a geometric source of accelerated cosmology in $R^2$ affine gravity. Hence, by our results above we expect that nonmetricity can also act as a source for cosmological acceleration in a physically indistinguishable manner [17]. Consider a spatially flat FLRW universe equipped with the metric

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2), \tag{34}$$

for which the non-vanishing Christoffel symbols are

$$\tilde{\Gamma}^0_{ij} = a\dot{a} \delta_{ij}, \quad \tilde{\Gamma}^i_{j0} = \frac{\dot{a}}{a} \delta^i_j. \tag{35}$$

Denoting as $\tilde{R}_{\mu\nu}$ the Riemannian parts of the Ricci tensor we then find

$$\tilde{R}_{00} = -3\ddot{a}, \quad \tilde{R}_{ij} = 6 \left[ \frac{\dot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 \right] g_{ij}. \tag{36}$$

Since the metric (34) is spatially homogenous, it can only accommodate a nonzero temporal component of the affine vector as $w_0 = w(t)$ and $w_i = 0$. Then, from (24) we obtain

$$6(\dot{H} + 2H^2) + 3\dot{w} + 9Hw + \frac{3}{2}w^2 = R_{0e} \int w dt. \tag{37}$$

Furthermore, from the field equations (2) we find

$$\frac{\ddot{a}}{a} = \frac{1}{12} R_{0e} \int w dt - \frac{1}{2} \dot{w} - \frac{1}{2}Hw, \tag{38}$$

or equivalently

$$(\dot{H} + H^2) + \frac{1}{2} \dot{w} + \frac{1}{2}Hw = \frac{R_{0e}}{12} \int w dt. \tag{39}$$

To derive (39) we have used the fact that the affine Ricci tensor can be decomposed as [18]

$$R_{\mu\nu} = \tilde{R}_{\mu\nu} + \frac{1}{2} \left( \tilde{\nabla}_\mu w_\nu + \tilde{\nabla}_\nu w_\mu - (\tilde{\nabla}_\alpha w^\alpha) g_{\mu\nu} \right) - 2\nabla_\nu w_\mu + \frac{1}{2} \left( w_\mu w_\nu - (w_\alpha w^\alpha) g_{\mu\nu} \right), \tag{40}$$

where $\tilde{\nabla}$ is the Riemannian covariant derivative. From (40) we obtain

$$R_{00} = \tilde{R}_{00} - \frac{3}{2}(\dot{w} + Hw). \tag{41}$$

Then, upon combining (37) and (39) we arrive at

$$\left( H + \frac{w}{2} \right)^2 = \frac{1}{12} R_{0e} \int w dt, \tag{42}$$

which gives

$$H = H_0 e^{\frac{1}{2} \int w dt} - \frac{w}{2}. \tag{43}$$
We have denoted as $H_0 = \sqrt{\frac{12}{R_0}}$ a parameter that plays the role of the initial Hubble constant. This can be positive or negative. Our final result (43) implies that the cosmological expansion in our simple FRW model (34) is driven both from torsion and nonmetricity through the projective invariant affine vector $w_\mu$. Therefore, at the level of our very simple spatially homogenous system torsion and nonmetricity can both act as indistinguishable sources for cosmological acceleration. Similar cosmological solutions driven by torsion have been recently discussed in [19, 20].

In particular, by integrating (43) one finds the evolution of the scale factor

$$a(t) = C_0 e^{\int H_0 dt}.$$ (44)

For $w = w_0 = constant$ we arrive at

$$a(t) = C_0 e^{\frac{2H_0}{w_0} t - \frac{w_0 t}{2}}.$$ (45)

Now, examining (45) a little further we see that assuming $w_0 < 0$ we find for early times

$$a(t) \approx a_0 \left[ 1 - H_0 \left( 1 - \frac{|w_0|}{2H_0} \right) t \right],$$ (46)

where $a_0 = C_0 e^{\frac{2H_0}{w_0}}$. On the other hand, for late times we have

$$a(t) \approx C_0 e^{\frac{|w_0|}{2} t}.$$ (47)

Namely, we have accelerated expansion due to both torsion and non-metricity. In particular, if $|w_0| = 2H_0$ the universe starts in a static state $a = a_0$ and accelerates exponentially for late times. It is interesting to point out that from (43), a static universe solution with $H = 0$, $a = a_0 = constant$ exists as long as

$$w(t) = -\frac{2}{C_0 + t},$$ (48)

namely when the affine vector varies inversely with time. Notice though that for late times $w \rightarrow 0$.

A. A nonmetric cosmological expansion

As a simple example we can consider a constant affine vector that receives only contributions from nonmetricity as $w = w_0 = Q_0/4$. Then (43) can be immediately integrated to give

$$a(t) = a(t) = C_0 e^{\frac{Q_0}{8Q_0} t - \frac{Q_0 t}{4}},$$ (49)

which is a nonmetric cosmological expansion. Furthermore, we have

$$\dot{a} = a \left( H_0 e^{\frac{Q_0}{8Q_0} t - \frac{Q_0 t}{8}} \right),$$ (50)

from which we conclude that if $H_0 > Q_0/8$ we have an ever expanding universe. On the other hand, for $H_0 = Q_0/8$ we have $\dot{a}(t = 0) = 0$ and the universe starts as static and then expands. Furthermore, when $H_0$ and $Q_0$ have the same sign, there exists a time

$$t^* = \frac{8}{Q_0} \ln \left( \frac{Q_0}{8H_0} \right),$$ (51)

after which the acceleration changes sign. Again, notice that for early times we have

$$a(t) \approx a_0 \left[ 1 - H_0 \left( 1 - \frac{|Q_0|}{8H_0} \right) t \right],$$ (52)

given that $Q_0 < 0$, while for late times

$$a(t) \approx C_0 e^{\frac{|Q_0|}{8} t},$$ (53)
which is a non-metric accelerated expansion.

To complete our analysis we note that for our nonmetric cosmological model above the Ricci scalar decomposition in terms of its Riemannian and nonmetric parts is given by

\[ R = \tilde{R} - \frac{3}{4} \tilde{\nabla}_\mu Q^\mu - \frac{3}{32} Q_\mu Q^\mu \Rightarrow R = \tilde{R} + \frac{3}{4} \tilde{\nabla}_\mu Q^\mu + \frac{9}{4} H Q + \frac{3}{32} Q^2. \]  

This expression is dual to the one that has appeared in [16], which depend only on torsion and was

\[ R = \tilde{R} + 2 \tilde{T} + 6 H T + \frac{2}{3} T^2. \]  

Indeed, it can be easily seen that (54) and (55) are mapped into each other under

\[ T \leftrightarrow \frac{3}{8} Q. \]  

The self-duality of the Ricci scalar under the torsion/nonmetricity exchange (56) is a general result in metric affine \( f(R) \) gravity. Indeed, if \textit{either} only torsion \textit{or} only nonmetricity are present, the Ricci scalar decompositions are

\[ R = \tilde{R} - 2 \tilde{\nabla}_\mu T^\mu - \frac{2}{3} T_\mu T^\mu, \]  

and

\[ R = \tilde{R} - \frac{3}{4} \tilde{\nabla}_\mu Q^\mu - \frac{3}{32} Q_\mu Q^\mu. \]  

The above map to each other under (56).

V. DISCUSSION

The main message of our work is that one needs to be very careful in distinguishing torsion and nonmetricity effects in modified theories of gravity. In particular, we have shown that for pure \( R^2 \) gravity torsion and nonmetricity are physically equivalent, being related by projective transformations. Similar results can be deduced from the recent analysis in a number of works e.g. [19, 21–26]. Nevertheless, we believe that we have added a useful ingredient by explicitly demonstrating the role of projective transformations in the torsion/nonmetricity duality.

It is clear the physical applications of our result should involve the study of matter coupled to modified gravity. For example one can study how the torsion/nonmetricity duality is preserved or broken by particular forms of matter. It would also be interesting to present explicit observational signatures of the duality i.e. in early cosmology. In another potential application we notice that our geometrical analysis in Section IIIA bears some resemblance with the results in [27] and hence they may be used in phenomenological studies of the standard model. Finally, a potentially huge area for applications of the duality is in the context of AdS/CFT, where it is expected to give rise to relations between different strongly-coupled 3d systems in the boundary, perhaps on the same par with the holographic effects of electromagnetic and gravitational duality [28, 29].

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APPENDIX

Denoting with \( \Gamma^\rho_{\beta\mu} \) the affine connection, covariant derivatives (of e.g. a mixed tensor) are defined as

\[ \nabla_\mu T^\alpha_\beta = \partial_\mu T^\alpha_\beta + \Gamma^\alpha_{\rho\mu} T^\rho_\beta - \Gamma^\rho_{\beta\mu} T^\alpha_\rho. \]  

(59)
Notice the position of the various indices in (59). The Riemann $R^\mu_{\nu\alpha\beta}$ and torsion tensors $S_{\alpha\beta}^\nu$ are defined via the commutator of two covariant derivatives acting on a vector $u^\mu$ as

$$[\nabla_\alpha, \nabla_\beta]u^\mu = 2\nabla_{[\alpha} \nabla_{\beta]}u^\mu = R^\mu_{\nu\alpha\beta}u^\nu + 2S_{\alpha\beta}^\nu \nabla_\nu u^\mu \quad (60)$$

$$R^\mu_{\nu\alpha\beta} := 2\partial_{[\alpha} \Gamma^\mu_{\nu]\beta]} + 2\Gamma^\mu_{\nu[\rho} \Gamma^\rho_{\beta]\alpha]} \quad (61)$$

With a generic connection the Riemann tensor is antisymmetric in its last two indices. So, we can generically define the independent contractions giving the Ricci tensor

$$R_{\nu\beta} := R^\mu_{\nu\mu\beta} = 2\partial_{[\alpha} \Gamma^\mu_{\nu]\beta]} = \partial_\alpha \Gamma^\mu_{\nu\beta} - \partial_\beta \Gamma^\mu_{\nu\alpha} \quad (62)$$

The above discussion did not require a metric. If a symmetric metric $g_{\mu\nu}$ is present we can also define a third independent contraction of the Riemann tensor as

$$\hat{R}_{\nu\beta} = g_{\nu\alpha} R^\mu_{\nu\alpha\beta} := 2g^{\nu\alpha} \partial_{[\alpha} \Gamma^\mu_{\nu]\beta]} = 2g^{\nu\alpha} \Gamma^\mu_{\rho[\alpha} \Gamma^\rho_{\beta]\alpha]} \quad (64)$$

Moreover, the Ricci scalar is still uniquely defined since

$$\hat{R} = \hat{R}_\alpha = R^\alpha_{\beta\mu\nu} g^{\beta\mu} = -R^\alpha_{\beta\mu\nu} g^{\beta\mu} = -R_{\beta\mu} g^{\beta\nu} = -R \quad (65)$$

There are two independent contractions of the torsion tensor giving the torsion vector $S_\mu$ and the torsion pseudovector $\tilde{S}_\mu$ respectively as

$$S_\mu \equiv S^\lambda_\mu, \quad \tilde{S}_\mu \equiv \epsilon^{\mu\nu\rho\sigma} S_{\nu\rho\sigma} \quad (66)$$

The non-metricity tensor is defined as

$$Q_{\alpha\mu\nu} := -\nabla_\alpha g_{\mu\nu} \quad (67)$$

It depends both on the metric tensor and the connection i.e. using the definition of the covariant derivative we have

$$Q_{\alpha\mu\nu} := -\nabla_\alpha g_{\mu\nu} = -\partial_\alpha g_{\mu\nu} + \Gamma^\rho_{\mu\alpha} g_{\rho\nu} + \Gamma^\rho_{\nu\alpha} g_{\rho\mu} \quad (68)$$

from which, the dependence on $\Gamma^\lambda_{\mu\nu}$ and $g_{\mu\nu}$ is apparent. Raising the last two indices we obtain

$$Q^\alpha_{\rho\beta} = \nabla_\rho g^{\alpha\beta} \quad (69)$$

From the non-metricity tensor we can construct two independent vectors. The Weyl vector is defined as

$$Q_{\alpha} := g^{\mu\nu} Q_{\alpha\mu\nu} = Q^\alpha_{\mu} \quad (70)$$

A second nonmetricity vector can also be defined and it is given by

$$\hat{Q}_{\nu} := g^{\mu\alpha} Q_{\alpha\mu\nu} = Q^\alpha_{\mu\nu} = -g^{\mu\alpha} \nabla_\alpha g_{\mu\nu} \quad (71)$$

Finally, using the results above one can decompose the general affine connection as

$$\Gamma^\lambda_{\mu\nu} = \tilde{\Gamma}^\lambda_{\mu\nu} + N^\lambda_{\mu\nu} \quad (72)$$

where the distortion tensor $N^\lambda_{\mu\nu}$ is given by

$$N^\lambda_{\mu\nu} = \frac{1}{2} g^{\alpha\lambda} (Q_{\mu\nu\alpha} + Q_{\nu\mu\alpha} - Q_{\alpha\mu\nu}) - g^{\alpha\lambda} (S_{\alpha\mu\nu} + S_{\alpha\nu\mu} - S_{\mu\nu\alpha}) \quad (73)$$
where the Levi-Civita connection is given by the usual Christoffel symbols

\[ \tilde{\Gamma}^\lambda_{\mu\nu} := \frac{1}{2} g^{\lambda\mu} \left( \partial_\mu g_{\nu\alpha} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu} \right) \]  

(74)

Some useful identities are

\[ Q_{\nu\alpha\mu} = N_{(\alpha\mu)\nu} , \quad S_{\mu\nu\alpha} = N_{\alpha[\mu\nu]} , \quad N_{[\alpha\mu\nu]} = S_{[\alpha\mu\nu]} \]  

(75)

where the latter refers to the totally antisymmetric part of the distortion. as can be easily checked. Notice also only the symmetric part \( N^\lambda_{(\mu\nu)} \) contributes to the autoparallel equation \( u^\nu \nabla_\nu u_\mu = 0 \). This is equal to

\[ N^\lambda_{(\mu\nu)} = g^{\alpha\lambda} \left( Q_{(\mu\nu)\alpha} - \frac{1}{2} Q_{\alpha\mu\nu} \right) - 2 g^{\alpha\lambda} S_{\alpha(\mu\nu)} . \]  

(76)

For vector torsion and nonmetricity as in (10) this coincides with (9). Notice that a completely antisymmetric torsion \( S_{\alpha\mu\nu} = S_{[\alpha\mu\nu]} \) has no effect on autoparallels.

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