ABSTRACT I construct an algebraic model for a typical fiber on a 1 + 1 dimensional spacetime. The vector space comprising the fiber is composed of elements $x \otimes x$ formed from the direct product of two copies of an element $x$ in the $D_2 = C_2 \otimes C_2$ finite group algebra over the real numbers. The fiber contains subspaces whose elements can be associated with the tangent and momentum vectors of trajectories in the manifold. The fiber also contains a subspace whose elements are associated with the local flow of action of each trajectory. The condition of minimum action translates into a constraint on the original vector $x$ in the direct product structure.

**Keywords:** tangent space, manifold, fiber, differential geometry, 1 + 1 space-time.
Fiber with Intrinsic Action on a 1 + 1 Dimensional Spacetime

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1 Introduction

In a recent paper I described a construction for a vector space with metric. In this construction one forms elements \( x \otimes x \) in a direct product algebra where \( x \) is an element in an underlying finite group algebra \([1]\). One uses a particular decomposition of the direct product algebra to obtain a direct sum of subspaces. One then observes that vectors in the various subspaces are interrelated. In particular examples that I considered there existed a 1d subspace whose measure was determined by the components of a second vector in a higher dimensional subspace. I proposed that the measure in the 1d subspace be identified with the norm of the vector in the higher dimensional subspace. In this way both a vector space and its norm are viewed as residing in particular subspaces of a given direct product algebra \( x \otimes x \). In this construction the signature of the metric arises intrinsically from the particular underlying finite group algebra.

The realization of Clifford algebras in terms of underlying finite group algebras is described by Salingaros \([2]\). The present work differs from the approaches of, e.g., Hestenes, Lounesto, and Greider \([3]\) by considering vector spaces obtained through the decomposition of a direct product algebra (having elements of form \( x \otimes x \)) rather than through a decomposition of a Clifford algebra. In particular cases that I consider, however, the underlying finite group algebra (containing elements \( x \)) does correspond to a Clifford algebra.

This work is motivated by the analogous construction in quantum mechanics where one forms observable vector spaces in terms of bilinear functions of an underlying state vector. The quantum mechanical 2-state problem provides one instructive example \([4]\). For this problem the underlying finite group is \( C_4 \otimes H \), the direct product of the cyclic group of order 4 and the quaternion group. The quantum wave function \( \psi \) is an element

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in a left ideal of the $C_4 \otimes H$ group algebra over the real numbers. The polarization vector and its norm, the total probability, reside in particular subspaces of the direct product algebra whose elements $\psi \otimes \psi$ are formed from the product of two copies of the quantum mechanical state $\psi$.

Vector spaces with metric that are constructed using this procedure are also of interest as models for typical fiber vector spaces residing at each location of a configurational manifold such as arises in the context of classical Lagrangian mechanics. In this paper I develop an algebraic model for a typical fiber at a location $P$ of a configurational manifold $M$ having 1 space and 1 time dimension. The fiber that I construct contains both the tangent and momentum vector spaces as subspaces. In addition, a subspace that can be associated with the flow of action at $P$ is included and arises in an intrinsic way. The construction for a configuration space with 2 space dimensions follows in a completely analogous way. The details for these two 2-dimensional cases are transparent. The extension of this construction to higher dimensional cases is also straightforward; however, the multiplicity of subspaces in the corresponding direct product algebras makes the interpretation of the overall structure more challenging, and it has not been fully addressed by this author.

In section 2, I review the construction of a vector space with metric signature $(p, q) = (1, 1)$ that corresponds to the tangent space at a point of a $1 + 1$ dimensional configurational manifold $M$. This construction uses the very simple $C_2$ group algebra. In section 3 I use the $D_2 = C_2 \otimes C_2$ group algebra to obtain the vector space that is the primary focus of this paper. In section 4 I summarize these results and indicate work that still remains to be done.

2 Algebraic Model for Tangent Space at a Point of a $1 + 1$ Dimension Spacetime

As a starting point for this paper let us review my earlier construction of an algebraic model for the vector space with signature $(p, q) = (1, 1)$. This vector space corresponds to the tangent space at each location of a $1 + 1$ dimensional spacetime $M$. The tangent space at an arbitrary location $(t_0, q_0) \in M$ consists of tangent vectors to curves passing through $(t_0, q_0)$. Here $t_0$ and $q_0$ denote, respectively, time and spatial location. For a curve

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1 This algebra is also termed the complex quaternion algebra. Elements in this algebra are linear combinations of the $C_4 \otimes H$ group elements with real coefficients. The $C_4 \otimes H$ group multiplication is used along with the distributive law to induce the algebra product rule.
1. Fiber with Intrinsic Action on a 1 + 1 Dimensional Spacetime

\( \gamma = \gamma(\lambda), \lambda \in \mathbb{R} \), the tangent vector is defined by

\[
\frac{d\gamma}{d\lambda} = \lim_{\lambda \to 0} \frac{\gamma(\lambda) - \gamma(0)}{\lambda} = (\frac{dt}{d\lambda}, \frac{dq}{d\lambda})
\]

where \( \gamma(0) = (t_0, q_0) \) and \( \gamma(\lambda) \in M \).

We begin the construction with the \( C_2 \) group which contains two elements \( C_2 = \{1, e\} \) with \( e^2 = 1 \), and then form the vector space \( V_{C_2} \) whose elements consist of formal sums of the elements of \( C_2 \) with real coefficients.

An arbitrary element \( x \in V_{C_2} \) can be written

\[
x = x_0 1 + x_1 e.
\]

where the second line follows from bilinearity of the direct product and in the third line we introduce the notation \( E = e \otimes e \) and \( e = 1 \otimes e \). \( Ee = e \otimes 1 \) follows from the product rule in the direct product algebra. Acting on \( x \otimes x \) with the projection operators \( P_\pm = \frac{1}{2}(1 \pm e \otimes e) \) we obtain

\[
x \otimes x = [P_+(\frac{dt}{d\lambda} + \frac{dq}{d\lambda} e) + P_- \frac{ds}{d\lambda}](1 \otimes 1),
\]

where

\[
\frac{dt}{d\lambda} = x_0^2 + x_1^2
\]
\[
\frac{dq}{d\lambda} = 2x_0 x_1
\]
\[
\frac{ds}{d\lambda} = x_0^2 - x_1^2 = (\frac{dt^2}{d\lambda} - \frac{dq^2}{d\lambda})^{\frac{1}{2}}
\]

The measure \( \frac{ds}{d\lambda} \) of the 1d \( P_- x \otimes x \) subspace is determined up to sign by the two components \( (\frac{dt}{d\lambda}, \frac{dq}{d\lambda}) \) of the 2d \( P_+ x \otimes x \) subspace and can be interpreted as their norm. We note that the measure \( x_0^2 + x_1^2 \) associated with the \( \frac{dt}{d\lambda} \) increment is positive definite and so has an intrinsic directionality. Also, since \((2x_0 x_1)^2 \leq (x_0^2 + x_1^2)^2\), we have \( |\frac{dq}{dt}| \leq 1 \) so that there is a maximum speed.

In this way the product \( x \otimes x \) provides a model for an element in the tangent space of a 1+1 dimensional spacetime. The set of all such elements \( x \otimes x \) can be identified with the tangent space itself.

Continuing further we find that rotations of the 2d vector space \( P_+ x \otimes x \) are induced by acting on \( x \) with an element \( u = u_0 1 + u_1 e \) and forming the
product \((x \otimes x)(u \otimes u) = xu \otimes xu\). For \(u\) such that \(u_0^2 - u_1^2 = 1\), \(P\cdot x \otimes x\) is unchanged while the \(P\cdot x \otimes x\) vector undergoes a proper orthochronous rotation.

A completely analogous treatment for the 2d Euclidean case is obtained by substituting the \(C_4\) group algebra for the \(C_2\) group algebra. This approach also extends to higher dimensional vector spaces, though in these cases we encounter a multiplicity of subspaces in the direct product algebra which make the interpretation of the overall structure more involved [1].

In the following section we extend this treatment to the case of a typical fiber at a location \(P\) of a configurational manifold \(M\) that contains both the tangent and momentum vector spaces.

3 Algebraic Model for Fiber on 1 + 1 Spacetime with an Intrinsic Action

Let us briefly review the classical mechanics that motivate this construction. The action function

\[
S_{t_0, q_0}(t, q) = \int_{\gamma} L dt
\]

is the integral of the Lagrangian \(L = L(q, \dot{q}, t)\) along an extremal path \(\gamma(\lambda), \lambda \in \text{real numbers}\), connecting an initial point \((t_0, q_0)\) with \((t, q)\) [4]. The action function for a free particle with mass \(m\) in a locally Minkowski coordinate system can be written as

\[
S_{t_0, q_0}(t, q) = \int_{\gamma} -m \frac{ds}{dt} dt
\]

where \(ds\) is the proper time with measure \(ds = (dt^2 - dq^2)^{\frac{1}{2}}\) and \(\gamma(\lambda)\) is a locally straight line [3]. This action corresponds to the Lagrangian \(L = -m \frac{ds}{dt}\). The rate of change of the action along the path \(\gamma\) for a fixed initial point is

\[
\frac{dS}{d\lambda} = p \frac{dq}{d\lambda} - H \frac{dt}{d\lambda}
\]

(3.1)

where

\[
p = \frac{\partial L}{\partial \dot{q}} = m\dot{q}/(1 - \dot{q}^2)^{\frac{3}{2}}
\]

and

\[
H = p\dot{q} - L = m/(1 - \dot{q}^2)^{\frac{1}{2}}.
\]
We now construct a model for the local tangent and momentum vectors of a trajectory $\gamma(\lambda)$ on a $1 + 1$ dimensional manifold for which such an action function arises intrinsically. For this construction we use the abelian group $(e_{12} := e_1 e_2)$

$$D_2 = C_2 \otimes C_2 = \{1, e_1, e_2, e_{12}\}$$

where $e_1^2 = e_2^2 = e_{12}^2 = 1$. A general element of the $D_2$ group algebra can be written

$$x = x_0 1 + x_1 e_1 + x_2 e_2 + x_3 e_{12}.$$

We decompose $x$ into a sum of two left ideals obtained by acting on the right with the projection operators $P_{\pm 2} = \frac{1}{2}(1 \pm e_2)$. We have

$$x = x(P_{+2} + P_{-2})$$

where

$$xP_{+2} = [(x_0 + x_2) 1 + (x_1 + x_3)e_1]P_{+2}$$

$$xP_{-2} = [(x_0 - x_2) 1 + (x_1 - x_3)e_1]P_{-2}.$$

We now form the tensor product of two copies of $x$,

$$x \otimes x = (x(P_{+2} + P_{-2}) \otimes x(P_{+2} + P_{-2})$$

Expanding out this expression and acting on the left side with the projection operator $P_{\pm 1} = \frac{1}{2}(1 \otimes 1 \pm E_1)$, where $E_1 = e_1 \otimes e_1$, we obtain, after a change of variables,

$$x \otimes x = x(P_{+1}(\frac{dt}{d\lambda} 1 + \frac{dq}{d\lambda} e_1) + P_{-1}(\frac{ds}{d\lambda})[(P_{+2} \otimes P_{+2})$$

$$+ [P_{+1}((\frac{1}{2}(H_{1} \frac{dt}{d\lambda} + p \frac{dq}{d\lambda} + m \frac{ds}{d\lambda}))^{\frac{3}{2}} 1$$

$$+ [\frac{1}{2}(H_{1} \frac{dt}{d\lambda} + p \frac{dq}{d\lambda} - m \frac{ds}{d\lambda}))^{\frac{3}{2}} e_1]$$

$$+ P_{-1}((\frac{1}{2}(H_{1} \frac{dt}{d\lambda} - p \frac{dq}{d\lambda} + m \frac{ds}{d\lambda}))^{\frac{3}{2}} 1$$

$$- [\frac{1}{2}(H_{1} \frac{dt}{d\lambda} - p \frac{dq}{d\lambda} - m \frac{ds}{d\lambda}))^{\frac{3}{2}} e_{12}])((P_{+2} \otimes P_{-2} + P_{-2} \otimes P_{+2})$$

$$+ [P_{+1}(H_{1} + pe_1) + P_{-1}m](P_{-2} \otimes P_{-2})$$

where $e_1 = 1 \otimes e_1$ and $e_{12} = 1 \otimes e_{12}$. In this expression

$$\frac{dt}{d\lambda} = (x_0 + x_2)^2 + (x_1 + x_3)^2$$

$$\frac{dq}{d\lambda} = 2(x_0 + x_2)(x_1 + x_3)$$

$$\frac{ds}{d\lambda} = (x_0 + x_2)^2 - (x_1 + x_3)^2 = \left(\frac{dt}{d\lambda} - \frac{dq}{d\lambda}\right)^{\frac{3}{2}}$$
while for the momenta variables we have,

\[ \begin{align*}
H &= (x_0 - x_2)^2 + (x_1 - x_3)^2 \\
p &= 2(x_0 - x_2)(x_1 - x_3) \\
m &= (x_0 - x_2)^2 - (x_1 - x_3)^2 = [H^2 - p^2]^{\frac{1}{2}}
\end{align*} \]

In analogy to the \( C_2 \) case discussed above, we associate the \( xP_{+2} \otimes xP_{+2} \) subspace with the tangent to a curve \( \gamma(\lambda) \) on the configurational manifold. The \( P_{+1}(xP_{+2} \otimes xP_{+2}) \) portion is identified with the tangent vector \( \left( \frac{dt}{d\lambda}, \frac{dq}{d\lambda} \right) \), while the \( P_{-1}(xP_{-2} \otimes xP_{-2}) \) portion with measure \( \frac{ds}{d\lambda} = \left[ \frac{dt^2}{d\lambda^2} - \frac{dq^2}{d\lambda^2} \right]^{\frac{1}{2}} \) is associated with the norm of the tangent vector.

Similarly, we identify the \( xP_{-2} \otimes xP_{-2} \) subspace with the momentum of the trajectory \( \gamma(\lambda) \) at \( \lambda \). The \( 2d \; P_{+1}(xP_{-2} \otimes xP_{-2}) \) projection is identified with the momentum vector \( (H, p) \) while the \( 1d \; P_{-1}(xP_{-2} \otimes xP_{-2}) \) projection with measure \( m = [H^2 - p^2]^{\frac{1}{2}} \) is associated with the norm of the momentum vector.

So far in this development the velocity tangent vector \( \left( \frac{dt}{d\lambda}, \frac{dq}{d\lambda} \right) \) is completely independent of the momentum vector \( (H, p) \). These two vectors contain the 4 degrees of freedom of the original vector \( x \) in the \( D_2 \) algebra.

Let us now consider the \( (x \otimes x)(P_{+2} \otimes P_{-2} + P_{-2} \otimes P_{+2}) \) subspace of this algebra. Both the \( P_{+1} \) and \( P_{-1} \) projections on this subspace are 2-dimensional. Taking the difference of the squares of the component measures for each of these two 2-vectors we find the resultant \( m \frac{ds}{d\lambda} \). The measure squared of the 1 component of the \( P_{-1} \) subspace

\[
\frac{1}{2} \left( H \frac{dt}{d\lambda} - \frac{dq}{d\lambda} + m \frac{ds}{d\lambda} \right)
\]

motivates its association with \( \frac{ds}{d\lambda} \) of Eq. (1). Minimizing this measure while keeping \( m \) and \( \frac{ds}{d\lambda} \) constant and their product \( m \frac{ds}{d\lambda} \) greater than zero requires setting the \( e_{12} \) component of the \( P_{-1} \) subspace to zero, since the difference of the squares of the two component measures is fixed. In this way we obtain the condition

\[
-m \frac{ds}{d\lambda} = \frac{dq}{d\lambda} - \frac{H dt}{d\lambda}
\]

that corresponds to eq. 1. This equation links the tangent vector \( \left( \frac{dt}{d\lambda}, \frac{dq}{d\lambda} \right) \) and the momentum vector \( (H, p) \). It is equivalent to the condition \( \frac{dt}{d\lambda} = \frac{H}{p} \) that holds for the trajectory of a free particle. Interestingly, this condition translates to the requirement

\[
x = \frac{1}{x_0}(x_0 + x_1 e_1)(x_0 + x_2 e_2) \quad \text{for} \quad x_0 \neq 0
\]

for the original vector \( x \) in the \( D_2 \) group algebra; or, stated differently, it ensures that \( x \) can be written as the tensor product of two vectors in the \( C_2 \) group algebra.
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Transformations that preserve the norms of the tangent and momentum vectors and the action differential \( \frac{ds}{d\lambda} = -m \frac{ds}{d\lambda} \) can be induced by acting on \( x \) with an element \( u = u_0 + u_1 e_1 + u_2 e_2 + u_3 e_3 \) in the \( D_2 \) algebra and forming the product \((x \otimes x)(u \otimes u) = xu \otimes xu\). For \( u \) such that \( u = u_0 + u_1 e_1 \) and \( u_0^2 - u_1^2 = 1 \), the \( P_- x \otimes x \) subspace that contains the three norms is unchanged. The tangent and momentum vectors undergo a proper orthochronous transformation under this rule.

Finally, we also note that if instead of using \( D_2 = C_2 \otimes C_2 \) in the construction above, we use \( C_2 \otimes C_4 = \{1, e_2\} \otimes \{1, e_1\} = \{1, e_1, e_2, e_{12}\} \) where \( e_{12} = e_1 e_2 \), \( e_{21} = e_{12}, \ e_2^2 = +1 \) and \( e_1^2 = e_{12}^2 = -1 \) then we obtain the 2d Euclidean counterpart to the above 1 + 1 spacetime case.

4 Summary

We associate the vector \( x \otimes x \) in the \( C_2 \otimes C_2 \) group algebra with an element in a typical fiber residing at a point in a 1 + 1 dimensional configurational manifold:

\[
x \otimes x = (P_{+1} \left( \frac{dt}{d\lambda} 1 + \frac{dq}{d\lambda} e_1 \right) + P_{-1} \left( \frac{ds}{d\lambda} e_1 \right)) (P_{+2} \otimes P_{+2})
\]

\[
+ \{P_{+1} \left( H \frac{dt}{d\lambda} \right) \} 1 + \left( p \frac{dq}{d\lambda} \right) e_1
\]

\[
+ P_{-1} \left( \frac{ds}{d\lambda} \right) \} (P_{+2} \otimes P_{-2} + P_{-2} \otimes P_{+2})
\]

\[
+ \{P_{+1} (H e_1 + p e_1) + P_{-1} m \} (P_{-2} \otimes P_{-2}).
\]

The collection of all such vectors \( x \otimes x \) comprise the typical fiber. The \((x \otimes x)(P_{+2} \otimes P_{+2})\) portion of \( x \otimes x \) is identified with the tangent vector and its norm. The \((x \otimes x)(P_{-2} \otimes P_{-2})\) portion is identified with the momentum vector and its norm. The \((x \otimes x)(P_{+2} \otimes P_{-2} + P_{-2} \otimes P_{+2})\) subspace is associated with the flow of action and its norm. The condition of minimum action translates into the condition that \( x \) has the form

\[
x = \frac{1}{x_0} (x_0 + x_1 e_1)(x_0 + x_2 e_2) \text{ for } x_0 \neq 0.
\]

It remains to determine how the fibers at different locations are connected. The description of this connection should also lead to a description of extended motions on this manifold.
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