Quantized enveloping superalgebra of type $P$

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Abstract

We introduce a new quantized enveloping superalgebra $U_q p_n$ attached to the Lie superalgebra $p_n$ of type $P$. The superalgebra $U_q p_n$ is a quantization of a Lie bisuperalgebra structure on $p_n$, and we study some of its basic properties. We also introduce the periplectic $q$-Brauer algebra and prove that it is the centralizer of the $U_q p_n$-module structure on $C(n|n)^{\otimes \ell}$. We end by proposing a definition for a new periplectic $q$-Schur superalgebra.

Introduction

The simple finite-dimensional Lie superalgebras over $\mathbb{C}$ were classified by V. Kac in [K]. The list in loc. cit. contains three classes of Lie superalgebras: basic, strange and Cartan-type. There are two types of strange Lie superalgebras - $P$ and $Q$ - both of which are interesting due to the algebraic, geometric, and combinatorial properties of their representations. The study of the representations of type $P$ Lie superalgebras, which are also called periplectic in the literature, has attracted considerable attention in the last five years. Interesting results on the category $O$, the associated periplectic Brauer algebras, and related theories have been established in [BDEA+1], [BDEA+2], [CP], [Co], [CE1], [CE2], [DHIN], [EAS1], [EAS2], [HIR], [IN], [IRS], [KT], [Ser], among others.

The purpose of this paper is to introduce a quantum superalgebra of type $P$ via the FRT formalism [FRT]. A similar approach was used by G. Olshanski in [Ol] to define quantum superalgebras of type $Q$. We prove that our quantized enveloping superalgebra $U_q p_n$ quantizes a Lie bisuperalgebra structure on $p_n$, a periplectic Lie superalgebra.

Using a Manin triple, we find a solution $s$ of the classical Yang-Baxter equation. This element is similar but different from the fake Casimir element used in [BDEA+1], [BDEA+2]. The quantum version of $s$, denoted $S$, is a solution of the quantum Yang-Baxter equation which serves as an essential ingredient in the definition of $U_q p_n$. It follows that the tensor superspace $C(n|n)^{\otimes \ell}$ is a representation of $U_q p_n$ and the centralizer of the action of $U_q p_n$ is a quantum version of the periplectic Brauer algebra. The classical setting corresponding to $q = 1$ was studied in [Mo]. A similar result for type $Q$ Lie superalgebras was established in [Ol], where the centralizer of the action of the quantized enveloping superalgebra was proven to be the Hecke-Clifford superalgebra of the symmetric group $S_{\ell}$. Having at our disposal the periplectic $q$-Brauer algebra, we can introduce the periplectic $q$-Schur superalgebra in a natural way. We conjecture that these are mutual centralizers (that is, they satisfy a double-centralizer property).

One immediate problem is to define $U_q p_n$ in terms of Drinfeld-Jimbo generators and relations and study its category $O$. For type $Q$ Lie superalgebras, this problem was addressed in [GJKK]. Furthermore, in [GJKKK], a theory of crystal bases for the tensor representations of $U_q g$ was established. Unfortunately, it is unlikely that natural crystal bases exist in the type $P$ case due to the nonsemisimplicity of the category of tensor modules, contrary to what happens in type $Q$. Another natural direction is to construct, using also the FRT formalism, quantum affine superalgebras of type $P$. (See [ChGu] for the type $Q$ case.) Yangians of type $P$ and $Q$ appeared already many years ago in the work of M. Nazarov [Na1, Na2]. We hope to return to these questions in a future publication.

After setting up the notation and basic definitions in the first section, we introduce the “butterfly” Lie bisuperalgebra in Section 2 and define the quantized enveloping superalgebra of type $P$ in the following section. The main result of Section 3 is Theorem 3.3 which states that $S$, the $q$-deformation of $s$, is a solution of the quantum Yang-Baxter equation. In Section 4 we prove that $U_q p_n$ is a quantization of the
Lie bisuperalgebra structure from Section 2, see Theorem 4.3. The new periplectic $q$-Brauer algebra $\mathfrak{B}_q,\ell$ and the new periplectic $q$-Schur algebra are introduced in the last section, where we prove that $\mathfrak{B}_q,\ell$ can be defined equivalently either using generators and relations or as the centralizer of the action of $\mathfrak{U}_q(p_n)$ on the tensor space: see Theorem 5.5.

The proofs of our results require extensive computations: further details for all the computations can be found in [AGG].

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1 The Lie superalgebra of type $P$

Let $\mathbb{C}(n|n)$ be the vector superspace $\mathbb{C}^n \oplus \mathbb{C}^n$ spanned by the odd standard basis vectors $e_{-1}, \ldots, e_n$ and the even standard basis vectors $e_1, \ldots, e_n$. Let $M_{n|n}(\mathbb{C})$ be the vector superspace consisting of matrices $A = (a_{ij})$ with $a_{ij} \in \mathbb{C}$ and with rows and columns labelled using the integers $-n, \ldots, -1, 1, \ldots, n$, so $i, j \in \{\pm 1, \pm 2, \ldots, \pm n\}$. Set $p(i) = 1 \in \mathbb{Z}_2$ if $-n \leq i \leq -1$ and $p(i) = 0 \in \mathbb{Z}_2$ if $1 \leq i \leq n$. The parity of the elementary matrix $E_{ij}$ is $p(i) + p(j) \mod 2$. We denote by $\mathfrak{gl}_{n|n}$ the Lie superalgebra over $\mathbb{C}$ whose underlying vector space is $M_{n|n}(\mathbb{C})$ and which is equipped with the Lie superbracket

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - (-1)^{(p(i)+p(j))(p(k)+p(l))} \delta_{il} E_{kj}.$$ 

Recall that the supertranspose $(\cdot)^{st}$ on $\mathfrak{gl}_{n|n}$ is given by the formula $(E_{ij})^{st} = (-1)^{p(i)(p(j)+1)} E_{ji}$. The involution $\iota$ on $\mathfrak{gl}_{n|n}$ which will be relevant for this paper is given by $\iota(X) = -\pi(X^{st})$ where $\pi: \mathfrak{gl}_{n|n} \rightarrow \mathfrak{gl}_{n|n}$ is the linear map given by $\pi(E_{ij}) = E_{-i,-j}$.

Definition 1.1. The Lie superalgebra $p_n$ of type $P$, which is also called the periplectic Lie superalgebra, is the subspace of fixed points of $\mathfrak{gl}_{n|n}$ under the involution $\iota$, that is, $p_n = \{X \in \mathfrak{gl}_{n|n} | \iota(X) = X\}$.

If $X \in p_n$ with $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and $A, B, C, D \in M_n(\mathbb{C})$, then $D = -A^t, B = B^t$ and $C = -C^t$ where $t$ denotes the transpose with respect to the diagonal $i = -j$. For convenience, we set

$$E_{ij} = E_{ij} + \iota(E_{ij}) = E_{ij} - (-1)^{p(i)(p(j)+1)} E_{-j,-i}.$$

The superbracket on $p_n$ is given by

$$[E_{ji}, E_{lk}] = \delta_{jl} E_{ik} - (-1)^{(p(i)+p(j))(p(k)+p(l))} \delta_{jk} E_{il} - \delta_{il}(-1)^{p(i)(p(k)+1)} E_{j,-l} - \delta_{jk}(-1)^{p(j)(p(i)+1)} E_{i,-k}$$

(1)

A basis of $p_n$ is provided by all the matrices $E_{ij}$ with indices $i$ and $j$ respecting one of the following inequalities:

$$1 \leq |j| < |i| \leq n \text{ or } 1 \leq i = j \leq n \text{ or } -n \leq i = -j \leq -1.$$ 

Note that $E_{ij} = -(-1)^{p(i)(p(j)+1)} E_{-j,-i}$ for all $i, j \in \{\pm 1, \ldots, \pm n\}$, hence $E_{i,-i} = 0$ when $1 \leq i \leq n$. 

2
2 Lie bisuperalgebra structure

To construct a Lie bisuperalgebra structure on \( p_n \), we define a Manin supertriple. We follow the idea in [O] for the case of the Lie superalgebra of type \( Q \). Recall that a Manin supertriple \((a,a_1,a_2)\) consists of a Lie superalgebra \( a \) equipped with an ad-invariant supersymmetric non-degenerate bilinear form \( B \) along with two Lie subsuperalgebras \( a_1, a_2 \) of \( a \) which are \( B \)-isotropic transversal subspaces of \( a \). Note that such a bilinear form \( B \) defines a non-degenerate pairing between \( a_1 \) and \( a_2 \) and a supercobracket \( \delta : a_1 \to a_1^{\otimes 2} \) via

\[
B^{\otimes 2}(\delta(X), Y_1 \otimes Y_2) = B(X,[Y_1,Y_2]),
\]

where \( X \in a_1, Y_1, Y_2 \in a_2 \).

**Definition 2.1.** The “butterfly” Lie superalgebra \( b_n \) is the subspace of \( gl_{n|n} \) spanned by \( E_{ij} \) with \( 1 \leq |i| < |j| \leq n \) and by \( E_{ii} + E_{-i,-i}, E_{i,-i} \) for \( 1 \leq i \leq n \).

Note that after adding all diagonal matrices to \( b_n \) we obtain a Borel subalgebra of \( gl_{n|n} \) whose simple roots are all odd. Note also that \( gl_{n|n} = p_n \oplus b_n \). It is well-known that the bilinear form \( B(\cdot,\cdot) \) on \( gl_{n|n} \) given by the super-trace, \( B(A,B) = \text{Str}(AB) \), is ad-invariant, supersymmetric and non-degenerate.

One easily checks that \( B(X_1,X_2) = 0 \) if \( X_1, X_2 \in p_n \) or if \( X_1, X_2 \in b_n \). Hence we have the following result.

**Proposition 2.2.** \( (gl_{n|n},p_n,b_n) \) is a Manin supertriple.

**Remark 2.3.** A similar Manin supertriple is given in [LeSh], §2.2.

The quantum superalgebra that we will define in the next section will be a quantization of the Lie bisuperalgebra structure given by the Manin supertriple \((gl_{n|n},p_n,b_n)\).

We extend the form \( B(\cdot,\cdot) \) to a non-degenerate pairing \( B^{\otimes 2} \) on \( gl_{n|n} \otimes c gl_{n|n} \) by setting

\[
B^{\otimes 2}(X_1 \otimes X_2, Y_1 \otimes Y_2) = (-1)^{|X_2||Y_1|} B(X_1,Y_1) B(X_2,Y_2)
\]

for all homogeneous elements \( X_1,X_2,Y_1,Y_2 \in p_n \). The sign \((-1)^{|X_2||Y_1|}\) is necessary to make this form ad-invariant.

Let

\[
s = \sum_{1 \leq |j| < |i| \leq n} (-1)^{p(j)} E_{ij} \otimes E_{ji} + \frac{1}{2} \sum_{1 \leq i \leq n} E_{ii} \otimes (E_{ii} + E_{-i,-i}) + \frac{1}{2} \sum_{1 \leq i \leq n} E_{-i,i} \otimes E_{i,-i}
\]

**Remark 2.4.** We note that the fake Casimir used in [BDEA] is also defined using the sum of tensor product of basis vectors in \( p_n \), and their duals in \( p_n^\ast \), but the fake Casimir differs from the element \( s \) defined above. One crucial difference is that the space \( p_n^\ast \) used in [BDEA] is not a subalgebra of \( gl_{n|n} \), while \( b_n \) is.

**Proposition 2.5.** \( s \) is a solution of the classical Yang-Baxter equation: \([s_{12}, s_{13}] + [s_{12}, s_{23}] + [s_{13}, s_{23}] = 0\).

The proof of the above proposition follows from the lemma below, which should be well-known among experts.

**Lemma 2.6.** Let \( p \) be a finite dimensional Lie superalgebra and suppose that \((p,p_1,p_2)\) is a Manin triple with respect to a certain supersymmetric, invariant, bilinear form \( B(\cdot,\cdot) \). Let \( \{X_i\}_{i \in I}, \{X'_i\}_{i \in I} \) be bases of \( p_1 \) and \( p_2 \), respectively, dual in the sense that \( B(X'_i,X_j) = \delta_{ij} \). Set \( s = \sum_{i \in I} X_i \otimes X'_i \). Then \( s \) is a solution of the classical Yang-Baxter equation.
We next compute the supercobracket $\delta$ using the identity $B(X, [Y_1, Y_2]) = B(\delta(X), Y_1 \otimes Y_2)$ for all $X \in p_n$ and all $Y_1, Y_2 \in b_n$. The formula for $\delta$ is (assuming, without loss of generality, that $|j| \leq |i|$):

$$\delta(E_{ij}) = \sum_{k=-n}^{n} \frac{1}{2} \left( (-1)^{p(i)+1} E_{ik} \otimes E_{kj} - (-1)^{p(i)+p(k)} E_{kj} \otimes E_{ik} \right)$$

Finally, the super cobracket on $p_n$ is related to the element $s$. The following lemma is standard.

**Lemma 2.7.** The super cobracket can also be expressed as

$$\delta(X) = [X \otimes 1 + 1 \otimes X, s],$$

for $X \in p_n$.

### 3 Quantized enveloping superalgebra

In this section, we define the quantized enveloping superalgebra $U_q p_n$ following the approach used in [FRT] and [Ol]. We use a solution $S$ of the quantum Yang-Baxter equation such that $s$ is the classical limit of $S$.

For simplicity, denote by $C_q$ the field $\mathbb{C}(q)$ of rational functions in the variable $q$ and set $C_q(n|n) = C_q \otimes_{\mathbb{C}} C(n|n)$.

**Definition 3.1.** Let $S \in \text{End}_{C_q}(C_q(n|n)^{\otimes 2})$ be given by the formula:

$$S = 1 + \sum_{1 \leq i \leq n} \left( (q - 1) E_{ii} + (q^{-1} - 1) E_{-i,-i} \right) \otimes (E_{ii} + E_{-i,-i}) + \frac{q - q^{-1}}{2} \sum_{-n \leq i \leq -1} E_{i,-i} \otimes E_{-i,i}$$

$$+ (q - q^{-1}) \sum_{1 \leq |j| \leq |i| \leq n} (-1)^{p(j)} E_{ij} \otimes E_{ji}$$

(5)

**Remark 3.2.** If we define $S$ instead as an element of $\text{End}_{C[[\hbar]]}(C_h(n|n)^{\otimes 2})$ by the same formula as in definition 3.1 but with $q, q^{-1}$ replaced by $e^{\hbar/2}, e^{-\hbar/2}$ and $C_q(n|n)^{\otimes 2}$ replaced by $C_h(n|n)^{\otimes 2}$, which equals $C(n|n)^{\otimes 2}[[\hbar]]$, then $S = 1 + \hbar s + O(\hbar^2)$.

**Theorem 3.3.** $S$ is a solution of the quantum Yang-Baxter equation: $S_{12} S_{13} S_{23} = S_{23} S_{13} S_{12}$.

**Proof.** The proof consists of verifying long computations. To simplify them, we have used the following method. Set $f(q) = S_{12} S_{13} S_{23} - S_{23} S_{13} S_{12}$. The main idea is to consider $f(q)$ as a Laurent polynomial $\sum_{i=-3}^{i=3} f_i q^i$ with coefficients $f_i$ in $\text{End}_{C_q}(C_q^{n|n})$. Then one shows the eight relations $f(a) = 0$, $f'(b) = 0$, $f''(c) = 0$ for $a, b, c = \pm 1$ and $b = \pm \sqrt{-1}$. (Actually, just seven of those are enough.) We can then deduce that $f(q)$ is a scalar multiple of $(q - q^{-1})^3$ and we show that the coefficient of $q^3$ in $f(q)$ is zero.

Here are some more details.

Let us set

$$C = \sum_{1 \leq i \leq n} (E_{ii} + E_{-i,-i}) \otimes (E_{ii} + E_{-i,-i}).$$
Lemma 3.5. \[ \text{vanishes.} \]

Then
\[
S = 1 + (q - q^{-1})s + \left(\frac{q + q^{-1}}{2} - 1\right)C.
\]

For convenience, we introduce the following notation:
\[
[sC] = s_{12}C_{13} + s_{12}C_{23} + s_{13}C_{23} + C_{12}S_{13} + C_{12}S_{23} - S_{23}C_{13} - s_{13}C_{12} - C_{23}S_{12} - C_{13}S_{12}
\]
\[
[ssC] = s_{12}S_{13}C_{23} + C_{12}S_{13}S_{23} - S_{23}S_{13}C_{12} - C_{23}S_{13}S_{12} - S_{23}C_{13}S_{12}
\]

The relations \( f(a) = 0, f'(b) = 0, f''(c) = 0 \) for \( a, b, c = \pm 1 \) and \( b = \pm \sqrt{-1} \) follow from the next two lemmas and checking these involves explicit computations.

**Lemma 3.4.** \([sC] = 2[ssCC]\)

**Lemma 3.5.** \([ssC] = 0\)

For instance, \( f'(-1) = 0 \) follows from \( f'(-1) = -4[sC] + 8[ssCC] \) and the two lemmas. Furthermore,
\[
f''(-1) = -4[sC] + 8[ssCC] - 16[ssC] + 8([s_{12}, s_{13}] + [s_{12}, s_{23}] + [s_{13}, s_{23}]).
\]

Therefore, \( f''(-1) = 0 \) thanks to Lemmas 2.6 3.4 and 3.5. Similarly, the two lemmas above imply that
\[
f'(-1) = 2\sqrt{-1}[sC] - 4\sqrt{-1}[ssCC] - 4[ssC]
\]

vanishes.

The last step in the proof of Theorem 3.3 is to show the vanishing of the coefficient \( f_3 \) of \( q^3 \). We have
\[
f_3 = s_{12}S_{13}S_{23} - S_{23}S_{13}S_{12} + \frac{1}{4}[ssCC] + \frac{1}{2}[ssC] + \frac{1}{8}C_{12}C_{13}C_{23} - \frac{1}{8}C_{23}C_{13}C_{12},
\]
which simplifies to
\[
s_{12}S_{13}S_{23} - S_{23}S_{13}S_{12} + \frac{1}{4}[ssCC]
\]
thanks to Lemma 3.3 and \( C_{12}C_{13}C_{23} - C_{23}C_{13}C_{12} = 0 \). Verifying that (6) vanishes follows by direct and extensive computations.

With the aid of \( S \), we can now define the main object of interest in this paper.

**Definition 3.6.** The quantized enveloping superalgebra of \( p_n \) is the \( \mathbb{Z}_2 \)-graded \( \mathbb{C}_q \)-algebra \( \mathfrak{U}_q p_n \) generated by elements \( t_{ij}, t_{ii}^{-1} \) with \( 1 \leq |i| \leq |j| \leq n \) and \( i, j \in \{ \pm 1, \ldots, \pm n \} \) which satisfy the following relations:
\[
t_{ii} = t_{-i,-i}, \ t_{-i,i} = 0 \quad \text{if} \quad i > 0, \ t_{ij} = 0 \quad \text{if} \quad |i| > |j|; \quad (7)
\]
\[
T_{12}T_{13}S_{23} = S_{23}T_{13}T_{12} \quad (8)
\]
where \( T = \sum_{|i| \leq |j|} t_{ij} \otimes \mathbb{C} E_{ij} \) and the last equality holds in \( \mathfrak{U}_q p_n \otimes_{\mathbb{C}} \text{End}_{\mathbb{C}}(\mathbb{C}_q(n|n))^2 \). The \( \mathbb{Z}_2 \)-degree of \( t_{ij} \) is \( p(i) + p(j) \).

**Remark 3.7.** One immediate corollary of the definition above is that if \( t_{ij} \) is odd, then \( t_{ij}^2 = 0 \). This follows for example after taking \( i = k \) and \( j = l \) in (7).

\( \mathfrak{U}_q p_n \) is a Hopf algebra with antipode given by \( T \mapsto T^{-1} \) and with coproduct given by
\[
\Delta(t_{ij}) = \sum_{k=-n}^{n} (-1)^{(p(i)+p(k))(p(k)+p(j))} t_{ik} \otimes t_{kj}.
\]
4 Limit when \( q \rightarrow 1 \) and quantization

We want to explain how \( \U p_n \) can be viewed as the limit when \( q \rightarrow 1 \) of \( \U q p_n \) and how the co-Poisson Hopf algebra structure on \( \U p_n \), which is inherited from the cobracket \( \delta \) on \( p_n \), can be recovered from the coproduct on \( \U q p_n \).

Set \( \tau_{ij} = \frac{t_{ij}}{q - 1} \) if \( i \neq j \) and set \( \tau_{ii} = \frac{t_{ii} - t_{i-1}}{q - 1} \). Let \( A \) be the localization of \( \mathbb{C}[q, q^{-1}] \) at the ideal generated by \( q - 1 \). Let \( \U q p_n \) be the \( A \)-subalgebra of \( \U q p_n \) generated by \( \tau_{ij} \) when \( 1 \leq |i| \leq |j| \leq n \).

**Theorem 4.1.** The map \( \psi : \U p_n \rightarrow \U A p_n / (q - 1)\U A p_n \) given by \( \psi(E_{ji}) = (\psi)E_{ji} \) for \( |i| < |j| \), \( 1 \leq i = j \leq n \), and \( \psi(E_{-i, i}) = -2\tau_{i,-i} \) for \( 1 \leq i \leq n \), is an associative \( C \)-superalgebra isomorphism.

**Proof.** First, we need to write down explicitly the defining relation (8). Comparing coefficients of \( E_{ij} \otimes E_{kl} \) on both sides of relation (8), we obtain:

\[
(-1)^{(p(i)+p(j))(p(k)+p(l))}t_{ij}t_{kl} - t_{kl}t_{ij} + \theta(i, j, k)(\delta_{|i|<|j|} - \delta_{|k|<|i|})\epsilon t_{ij}t_{kl}
\]

\[
+ (-1)^{(p(i)+p(j))(p(k)+p(l))}\left((\delta_{i>0(q-1)} - \delta_{i<0(q-1)-1})\left(\delta_{i} + \delta_{i,-1}\right)\right)t_{ij}t_{kl}
\]

\[
- (\delta_{i>0(q-1)} + \delta_{i<0(q-1)-1})(\delta_{ik} + \delta_{i,-k})t_{ijkl}
\]

\[
+ \theta(i, j, k)\delta_{j<0\delta_{i,-k}t_{ijkl}} - (-1)^{(p(j))}\delta_{ij<0\delta_{i,-k}t_{ijkl}} - (-1)^{(p(j))}\epsilon \sum_{-n \leq \alpha \leq n} ((-1)^{(p(i)p(a)}\theta(i, j, k)\delta_{|i|<|j|}|i|)\epsilon t_{a}t_{-a,j} + (-1)^{(p(j)p(a)}\delta_{i,-k}\delta_{|i|<|j|})
\]

\[
= 0
\]

In the identity above, we set

\[
\theta(i, j, k) = \text{sgn}(\text{sgn}(i) + \text{sgn}(j) + \text{sgn}(k)) \text{ and } \epsilon = q - q^{-1}.
\]

In order to check that \( \psi([E_{ji}, E_{kl}]) = [\psi(E_{ji}), \psi(E_{kl})] \), we proceed as follows. We apply \( \psi \) on both sides of (8). To show that the resulting right hand side coincides with \( [\psi(E_{ji}), \psi(E_{kl})] \), we use (9) and pass to the quotient \( \U A p_n / (q - 1)\U A p_n \). This is done via a long case-by-case verification for \( i, j, k, l \).

From the way \( \U A p_n \) is defined, it follows that \( \psi \) is surjective. It remains to prove that it is injective. Since \( S \) is a solution of the quantum Yang-Baxter equation, the space \( C(n|n) \) is a representation of \( \U p_n \) via the assignment \( t_{ij} \mapsto s_{ij} \) (where \( S = \sum_{i,j=-n}^{n} s_{ij} \otimes E_{ij} \)), hence also of \( \U A p_n \) by restriction. More explicitly,

\[
\tau_{ij} \mapsto (-1)^{p(i)}E_{ij} \text{ if } |i| < |j|, \text{ and } \tau_{i,-i} \mapsto E_{i,-i}, \tau_{ii} \mapsto (E_{ii} - q^{-1}E_{i,-i}) \text{ if } 1 \leq i \leq n.
\]

Set \( C_A(n|n) = A \otimes C(n|n) \). The space \( C_A(n|n) \) is a \( \U A p_n \)-submodule and so are all the tensor powers \( C_A(n|n)^{\otimes \ell} \). We thus have a superalgebra homomorphism \( \phi_{\ell} : \U A p_n \rightarrow \text{End}_A(C_A(n|n)^{\otimes \ell}) \) for each \( \ell \geq 1 \).

Let \( \pi_{\ell} \) be the quotient homomorphism

\[
\text{End}_A(C_A(n|n)^{\otimes \ell}) \rightarrow \text{End}_A(C_A(n|n)^{\otimes \ell})/(q - 1)\text{End}_A(C_A(n|n)^{\otimes \ell}) \cong \text{End}_C(C(n|n)^{\otimes \ell}).
\]

The composite \( \pi_{\ell} \circ \phi_{\ell} \) descends to a homomorphism \( \overline{\pi_{\ell} \circ \phi_{\ell}} \) from \( \U A p_n / (q - 1)\U A p_n \) to \( \text{End}_C(C(n|n)^{\otimes \ell}) \). The composite \( \overline{\pi_{\ell} \circ \phi_{\ell}} \circ \psi \) is the superalgebra homomorphism \( \U p_n \rightarrow \text{End}_C(C(n|n)^{\otimes \ell}) \) induced by the natural \( p_n \)-module structure on \( C(n|n)^{\otimes \ell} \) twisted by the automorphism of \( p_n \) given by \( E_{ij} \mapsto (-1)^{p(i)+p(j)}E_{ij} \).

We can combine the homomorphisms \( \overline{\pi_{\ell} \circ \phi_{\ell}} \circ \psi \) for all \( \ell \geq 1 \) to obtain a homomorphism \( \U p_n \rightarrow \prod_{\ell=1}^{\infty} \text{End}_C(C(n|n)^{\otimes \ell}) \). This map is injective since \( C(n|n) \) is a faithful representation of \( p_n \). It follows that \( \psi \) is injective as well. 
\[\square\]
We next show that a PBW-type theorem holds for $\mathfrak{U}_q\mathfrak{p}_n$. For this, we first introduce a total order $\prec$ on the set of generators $t_{ij}$, $1 \leq |i| \leq |j| \leq n$, of $\mathfrak{U}_q\mathfrak{p}_n$ as follows. We declare that $t_{ij} \prec t_{kl}$ if

(i) $|i| > |k|$, or
(ii) $|i| = |k|$ and $|j| > |l|$, or
(iii) $i = k$ and $j = -l > 0$, or
(iv) $i = -k > 0$ and $|j| = |l|$.

This order leads to a total lexicographic order on the set of words formed by the generators $t_{ij}$. Namely, if $A = A_1 \cdots A_r$ and $B = B_1 \cdots B_s$ are two such words in the sense that each $A_k$ for $1 \leq k \leq r$ and each $B_l$ for $1 \leq l \leq s$ is equal to some generator $t_{ij}$, then $A \prec B$ if $r < s$ or if $r = s$ and there is a $p$ such that $A_k = B_k$ for $1 \leq k \leq p - 1$ and $A_p \prec B_p$. Note that, in this order, the generators $t_{ij}$ with $i = j$ or $i = -j$ are not grouped together. We call a generator of the form $t_{ij}$ diagonal. Also, a word $A_{k_1} \cdots A_{k_r}$ in the generators $t_{ij}$ is called a reduced monomial if $A_1 \prec \cdots \prec A_r$, and $k_1 \in \mathbb{Z}_{>0}$ if $A_1$ is not diagonal, $k_1 \in \mathbb{Z} \setminus \{0\}$ if $A_1$ is diagonal, and $k_1 = 1$ if $A_1$ is odd.

**Theorem 4.2.** The reduced monomials form a basis of $\mathfrak{U}_q\mathfrak{p}_n$ over $\mathbb{C}_q$.

**Proof.** We first show that the set of reduced monomials spans $\mathfrak{U}_q\mathfrak{p}_n$. Note that it is enough to show that all quadratic monomials are in the span of this set. Let $t_{ij}t_{kl}$ be a quadratic monomial which is not reduced. We have that either $t_{kl} \neq t_{ij}$, or $i = k$, $j = l$ and $t_{ij}$ is odd. In the latter case, as explained in Remark 3.7, $t_{ij}^2 = 0$. In the former case, we proceed with a case-by-case reasoning considering seven mutually exclusive subcases:

(a) $|i| < |k|$ and $|j| \neq |l|$.
(b) $|i| < |k|$ and $j = l$.
(c) $|i| < |k|$ and $j = -l$.
(d) $|i| = |k|$ and $|j| < |l|$.
(e) $i = k$ and $j = -l < 0$.
(f) $i = -k < 0$ and $j = l$.
(g) $i = -k < 0$ and $j = -l$.

Let’s consider in some details subcase (c). The remaining subcases are handled in a similar manner. In subcase (c), (9) simplifies to:

$$
(-1)^{(p(i)+p(j))(p(k)+p(\cdot))} \left( \delta_{j>0} t_{ij} t_{k,-j} - \delta_{j<0} t_{ij} t_{k,-j} + \theta(i,j,k) \delta_{j>0} \epsilon t_{i,-j} t_{kj} + \theta(i,j,k) \delta_{j<0} \epsilon t_{i,-j} t_{kj} \right)
$$

$$
+ (-1)^{p(j)(p(i)+1)} \epsilon \sum_{-n \leq a \leq n} (-1)^{p(a)} \theta(i,j,k) \delta_{j>0} \epsilon t_{i,-j} t_{ka} = 0
$$

Let us assume that $|j| = |l| = 1$. Then the previous equation reduces to

$$
(-1)^{(p(i)+p(j))(p(k)+p(\cdot))} \left( \delta_{j>0} t_{ij} t_{k,-j} + \delta_{j<0} t_{ij} t_{k,-j} + \theta(i,j,k) \delta_{j>0} \epsilon t_{i,-j} t_{kj} = t_{k,-j} t_{ij} \right)
$$

Replacing $j$ by $-j$ leads to the equation

$$
(-1)^{(p(i)+p(\cdot))(p(k)+p(j))} \left( \delta_{j<0} t_{ij} t_{k,j} + \delta_{j>0} t_{ij} t_{k,j} + \theta(i,-j,k) \delta_{j<0} \epsilon t_{ij} t_{k,-j} = t_{kj} t_{i,-j} \right)
$$
The monomials $t_{k,-j}t_{ij}$ and $t_{kj}t_{i,-j}$ are properly ordered and the previous two equations can be solved to express $t_{ij}t_{k,-j}$ and $t_{i,-j}t_{kj}$ in terms of the former.

We then proceed by descending induction on $|j|$ and show that $t_{ij}t_{k,-j}$ can be expressed as a linear combination of properly ordered monomials. The base case $|j| = 1$ was completed above. We use again (10) and the corresponding equation obtained after switching $j$ and $-j$. In these two equations, by induction, the monomials $t_{i,-a}t_{ka}$ with $|a| < |j|$ can be expressed as linear combinations of properly ordered monomials. Moreover, $t_{k,-j}t_{ij}$ and $t_{kj}t_{i,-j}$ are already correctly ordered. As in the case $|l| = |j| = 1$, we can then solve those two equations to express $t_{ij}t_{k,-j}$ and $t_{i,-j}t_{kj}$ in terms of properly ordered monomials.

It remains to show that the reduced monomials form a linearly independent set. We follow the approach in [1]. Let $M_1, \ldots, M_r$ be pairwise distinct reduced monomials in the generators $\tau_{ij}$ such that $a_1M_1 + \ldots + a_rM_r = 0$ for some $a_1, \ldots, a_r \in \mathbb{C}_q$. Without loss of generality, we can assume that $a_i \in A$. It is sufficient to prove that $a_1, \ldots, a_r \in \mathcal{A}$ implies $a_1, \ldots, a_r \in (q-1)A$.

Recall that there is a surjective homomorphism $\theta : \mathfrak{u}_A p_n \to \mathfrak{u}_p n$. More precisely, $\theta$ is the composite of $\psi^{-1}$ from Theorem 4.1 and the projection $\mathfrak{u}_A p_n \to \mathfrak{u}_n / (q-1)\mathfrak{u}_n p_n$ from Theorem 4.1. Let $\overline{M}_\ell = \theta(M_\ell)$ and denote by $\overline{a}_i$ the image of $a_i$ in $A/(q-1)A$. Since $M_1, \ldots, M_r$ are pairwise distinct reduced monomials, $\overline{M}_1, \ldots, \overline{M}_r$ are pairwise distinct monomials in $\mathfrak{u}_p n$. Then using that

$$\overline{a}_1\overline{M}_1 + \ldots + \overline{a}_r\overline{M}_r = \theta(a_1M_1 + \ldots + a_rM_r) = 0$$

and the (classical) PBW Theorem for $\mathfrak{u}_p n$, we obtain $\overline{a}_1 = \ldots = \overline{a}_r = 0$. Hence $a_1, \ldots, a_r \in (q-1)A$ as needed. \hfill $\square$

As mentioned in Remark 3.2, we may replace $\mathbb{C}(q)$ by $\mathbb{C}([q])$, $q$ by $e^{h/2}$, and $A$ by $\mathbb{C}[[h]]$, and an analog of Theorem 4.1 would hold true, implying that $\mathfrak{u}_{C[[h]]} p_n$ is a flat deformation of $\mathfrak{u}_p n$. Moreover, the next theorem states that $\mathfrak{u}_{C[[h]]} p_n$ is a quantization of the co-Poisson Hopf superalgebra structure on $\mathfrak{u}_p n$ induced by the Lie bisuperalgebra structure defined in Section 2. To be precise, the cobracket $\delta$ on $p_n$ extends to a Poisson co-bracket on $\mathfrak{u}_p n$, which we also denote by $\delta$. Let $(\cdot)^\circ$ be the involution on $(\mathfrak{u}_{C[[h]]} p_n) \otimes 2$ given by $A_1 \otimes A_2 \mapsto (-1)^{p(A_1)\cdot p(A_2)} A_2 \otimes A_1$, where $p(A_i)$ is the $\mathbb{Z}/2\mathbb{Z}$-degree of $A_i$, $i = 1, 2$.

For convenience, for $A \in \mathfrak{u}_{C[[h]]} p_n$, we denote by $\overline{A}$ both the image of $A$ in $\mathfrak{u}_{C[[h]]} p_n / h\mathfrak{u}_{C[[h]]} p_n$ and the corresponding element in $\mathfrak{u}_p n$ via the isomorphism of the $h$-analogue of Theorem 4.1. Similarly, we identify the corresponding elements in $\mathfrak{u}_{C[[h]]} p_n / h\mathfrak{u}_{C[[h]]} p_n \otimes \mathfrak{u}_{C[[h]]} p_n / h\mathfrak{u}_{C[[h]]} p_n$ and $\mathfrak{u}_p n \otimes \mathfrak{u}_p n$.

**Theorem 4.3.** If $A \in \mathfrak{u}_{C[[h]]} p_n$, we have $h^{-1}(\Delta(A) - \Delta(A)^\circ) = \delta(A)$. Hence, $\mathfrak{u}_{C[[h]]} p_n$ is a quantization of the co-Poisson Hopf superalgebra structure on $\mathfrak{u}_p n$.

**Proof.** We show that the identity above holds for the generators $\tau_{ij}$ of $\mathfrak{u}_{C[[h]]} g$, so let $A = \tau_{ij}$. We first note that the identity is trivially satisfied for $i = j$, as both sides are zero. Assume henceforth that $i \neq j$. Then:

$$h^{-1}(\Delta(\tau_{ij}) - \Delta(\tau_{ij})^\circ) = \left(\frac{e^{h/2} - e^{-h/2}}{h}\right) \sum_{k = |i| < |j|}^n \left((-1)^{p(i) + p(k)}(p(j) + p(k)) \tau_{ik} \otimes \tau_{kj} - \tau_{kj} \otimes \tau_{ik}\right)$$

$$+ \left(\frac{e^{h/2} - 1}{h}\right) (\tau_{ii} \otimes \tau_{ij} - \tau_{ij} \otimes \tau_{ii} + \tau_{ij} \otimes \tau_{jj} - \tau_{jj} \otimes \tau_{ij})$$

$$- \left(\frac{e^{h/2} - e^{-h/2}}{h}\right) \delta_{j>0} \left((-1)^{p(j)} \tau_{i,-j} \otimes \tau_{-i,j} + \tau_{-i,j} \otimes \tau_{i,-j}\right)$$

$$+ \left(\frac{e^{h/2} - e^{-h/2}}{h}\right) \delta_{j<0} \left((-1)^{p(i)} \tau_{i,-j} \otimes \tau_{-j,j} - \tau_{-j,j} \otimes \tau_{i,-j}\right)$$
Thus, in $\mathfrak{U}_{[\mathfrak{c}[\mathfrak{q}]][\mathfrak{g}][\mathfrak{h}]}$, we have:

$$\overline{h}^{-1}(\Delta(\tau_{ij}) - \Delta(\tau_{ij}))^\rho = \sum_{k=n}^{n} \left( (-1)^{(p(i)+p(k))(p(j)+p(k))} \tau_{ik} \otimes \tau_{kj} - \tau_{kj} \otimes \tau_{ik} \right)$$

$$+ \frac{1}{2} \left( \tau_{ii} \otimes \tau_{ij} - \tau_{ij} \otimes \tau_{ii} + \tau_{ij} \otimes \tau_{jj} - \tau_{jj} \otimes \tau_{ij} \right)$$

$$\delta_{i>j} \left( \tau_{i,-j} \otimes \tau_{j,-i} + (-1)^{p(j)} \tau_{i,-i} \otimes \tau_{i,-j} \right)$$

$$\delta_{j<i} \left( (-1)^{p(i)} \tau_{i,-j} \otimes \tau_{j,-i} - \tau_{j,-i} \otimes \tau_{i,-j} \right)$$

We next compute $\delta(\tau_{ij})$ using the isomorphism of Theorem 4.1 and 8.

$$\delta(\tau_{ij}) = (-1)^{p(j)} \delta(E_{ji})$$

$$\sum_{k=n}^{n} \left( (-1)^{p(j)+p(k)} \left( (-1)^{(p(i)+p(k))(p(j)+p(k))} E_{ki} \otimes E_{jk} - E_{jk} \otimes E_{ki} \right) \right)$$

$$- \frac{1}{2} (-1)^{p(j)} \left( (-1)^{p(j)} E_{jj} - (-1)^{p(i)} E_{ii} \right) \otimes E_{ji} + \frac{1}{2} (-1)^{p(j)} E_{jj} \otimes \left( (-1)^{p(j)} E_{jj} - (-1)^{p(i)} E_{ii} \right)$$

$$\delta_{j<i} \left( \tau_{j,-i} \otimes \tau_{i,-j} + \frac{1}{2} (-1)^{p(j)} \tau_{j,-i} \otimes \tau_{i,-j} \right)$$

as needed.

\[ \square \]

5 Periplectic $q$-Brauer algebra

In [Mo], D. Moon identified the centralizer of the action of $p_n$ on the tensor space $C^\otimes n$. This centralizer is called the periplectic Brauer algebra in the literature: see [CG, CP, CE1, CE2].

Since $S$ is a solution of the quantum Yang-Baxter equation, we have a representation of $U_q p_n$ on $C_q(n|n)$ via the assignment $t_{ij} \mapsto s_{ij}$ (where $S = \sum_{i,j=-n}^{n} s_{ij} \otimes E_{ij}$), and thus we also have a representation on each tensor power $C_q(n|n)^{\otimes l}$. In this section, we identify the centralizer of the action of $U_q p_n$ on $C_q(n|n)^{\otimes l}$ and call it the periplectic $q$-Brauer algebra. For the quantum group of type $Q$, this was done in [Ol] and the centralizer of its action is called the Hecke-Clifford superalgebra. Quantum analogs of the Brauer algebra were studied in [M] where they appear as centralizers of the action of twisted quantized enveloping algebras $U_q^{\nu} C_n$ and $U_q^{\nu} \mathfrak{sp}_n$ on tensor representations (here, $\mathfrak{sp}_n$ is the symplectic Lie algebra); see also [We].

Definition 5.1. The periplectic $q$-Brauer algebra $\mathfrak{B}_{q,i}$ is the associative $\mathbb{C}(q)$-algebra generated by elements $t_i$ and $c_i$ for $1 \leq i \leq l - 1$ satisfying the following relations:

$$ (t_i - q)(t_i + q^{-1}) = 0, \quad c_i^2 = 0, \quad c_i t_i = -q^{-1} c_i, \quad t_i c_i = q c_i \quad \text{for} \ 1 \leq i \leq l - 1; \quad (11)$$

$$ t_i t_j = t_j t_i, \quad c_i c_j = c_j c_i, \quad c_i c_j = c_j c_i \ \text{if} \ |i - j| \geq 2; \quad (12)$$

$$ t_i t_j t_i = t_j t_i t_j, \quad c_i c_{i+1} c_i = -c_{i+1}, \quad c_i c_{i+1} c_i = -c_i \ \text{for} \ 1 \leq i \leq l - 2; \quad (13)$$

$$ t_i c_{i+1} c_i = -t_i c_{i+1} c_i + (q - q^{-1}) c_{i+1} c_i, \quad c_{i+1} c_i t_i = -c_{i+1} c_i + (q - q^{-1}) c_{i+1} c_i \ \text{for} \ 1 \leq i \leq l - 2. \quad (14)$$

Remark 5.2. Setting $q = 1$ in this definition yields the algebra $A_l$ from Definition 2.2 in [Mo].

Lemma 5.3. Consider $\mathbb{C}(q)$ as purely odd $U_q p_n$-module. We have $U_q p_n$-module homomorphisms $\vartheta : C_q(n|n) \otimes C_q(n|n) \to \mathbb{C}(q)$ and $\epsilon : \mathbb{C}(q) \to C_q(n|n) \otimes C_q(n|n)$ given by $\vartheta(e_a \otimes e_b) = \delta_{a,-b}(-1)^{p(a)}$ and $\epsilon(1) = \sum_{a=-n}^{n} e_a \otimes e_a$. 

Proof. It is enough to check that, for all the generators \( t_{ij} \) of \( \mathfrak{u}_g \mathfrak{p}_n \) and any tensor \( \mathbf{v} \in C_q(n|n) \otimes C_q(n|n) \),

\[
\vartheta(t_{ij}(\mathbf{v})) = t_{ij}(\vartheta(\mathbf{v})) \quad \text{and} \quad \epsilon(t_{ij}(1)) = t_{ij}(\epsilon(1)).
\]

(15) Here is a brief sketch of some of the computations.

Using the formula for the coproduct, we have:

\[
t_{ij}(e_a \otimes e_{-a}) = \sum_{k=-n}^{n} (-1)^{(p(i)+p(k))(p(k)+p(j))+(p(k)+p(j))p(a)} t_{ik}(e_a) \otimes t_{kj}(e_{-a})
\]

(16) This can be made more explicit using

\[
t_{ii}(e_a) = \sum_{b=-n}^{n} q^{\delta_{a,b}(1-2p(i))+\delta_{a,-i}(2p(i)-1)} E_{bb}(e_a);
\]

\[
t_{i,-i}(e_a) = (q-q^{-1}) \delta_{i,>0} E_{-i,i}(e_a);
\]

\[
t_{ij}(e_a) = (q-q^{-1})(-1)^{p(i)} E_{ji}(e_a), \quad \text{if } |i| \neq |j|.
\]

We obtain, for instance,

\[
t_{ii}(e_a_1 \otimes e_{a_2}) = q^{\delta_{a_1,1}(1-2p(i))+\delta_{a_1,-i}(2p(i)-1)} q^{\delta_{a_2,1}(1-2p(i))+\delta_{a_2,-i}(2p(i)-1)} e_{a_1} \otimes e_{a_2}
\]

If \( a_2 = -a_1 = -a \), this simplifies to \( e_a \otimes e_{-a} \) and this allows us to check (15) quickly for \( i = j \).

Furthermore,

\[
t_{i,-i}(e_a \otimes e_{-a}) = (-1)^{p(a)} \delta_{i,>0} t_{ii}(e_a) \otimes t_{i,-i}(e_{-a}) + \delta_{i,>0} t_{i,-i}(e_a) \otimes t_{-i,-i}(e_{-a})
\]

It follows that \( t_{i,-i}(\sum_{a=-n}^{n} e_a \otimes e_{-a}) = 0 \), so the identity for \( \epsilon \) in (15) holds for \( j = -i \).

Suppose now that \( a_1 \neq -a_2 \). Then

\[
t_{i,-i}(e_{a_1} \otimes e_{a_2}) = \delta_{i,>0} \delta(a_1 = a_2 = i)(q-q^{-1}) q e_{i} \otimes e_{-i}
\]

\[
+ \delta_{i,>0} \delta(a_1 = a_2 = i)(q-q^{-1}) q e_{-i} \otimes e_{i}
\]

Observe that \( \vartheta(e_i \otimes e_{-i} + e_{-i} \otimes e_i) = 0 \), so we have shown that \( \vartheta(t_{i,-i}(e_{a_1} \otimes e_{a_2})) = t_{i,-i}(\vartheta(e_{a_1} \otimes e_{a_2})) \) and this proves (15) for \( \vartheta \) when \( j = -i \).

Next, we consider the case \( |i| \neq |j| \). To prove the identity for \( \epsilon \) in (15), we use again (16) and obtain that

\[
t_{ij} \left( \sum_{a=-n}^{n} e_a \otimes e_{-a} \right) = 0 \quad \text{by considering subcases } i = \pm a, \quad j = \pm a, \quad \text{and } k = \pm a.
\]

To show that (15) holds for \( \vartheta \) we also proceed with case-by-case verification. The case \( a_1, a_2 \notin \{ \pm i, \pm j \} \) is immediate. If \( a_1 \in \{ \pm i, \pm j \}, \quad a_2 \notin \{ \pm i, \pm j \}, \quad \text{and} \ a_1 \neq -a_2, \) then

\[
t_{ij}(e_{a_1} \otimes e_{a_2}) = (q-q^{-1})^2 (-1)^{(p(i)+p(a_2))(p(a_2)+p(j))+(p(a_2)+p(j))p(a_1)} (-1)^{p(i)+p(a_2)} E_{a_2 i}(e_{a_1}) \otimes E_{j a_2}(e_{a_2})
\]

\[
+ (q-q^{-1})(-1)^{p(i)} E_{ji}(e_{a_1}) \otimes E_{a_2 a_2}(e_{a_2}).
\]

This shows that \( \vartheta(t_{ij}(e_{a_1} \otimes e_{a_2})) = 0 = t_{ij}(\vartheta(e_{a_1} \otimes e_{a_2})) \). Similarly, we obtain the desired identity in the other cases. \( \square \)

By composing \( \vartheta \) and \( \epsilon \), we obtain a \( \mathfrak{u}_g \mathfrak{p}_n \)-module homomorphism \( \epsilon \circ \vartheta : C_q(n|n)^{\otimes 2} \to C_q(n|n)^{\otimes 2} \). In terms of elementary matrices, this linear map is given by \( \sum_{a,b=-n}^{n} (-1)^{p(a)p(b)} E_{ab} \otimes E_{-a,-b} \), which we abbreviate by \( \epsilon \). The super-permutation operator \( P \) on \( C_q(n|n)^{\otimes 2} \) is given by \( P = \sum_{a,b=-n}^{n} (-1)^{p(b)} E_{ab} \otimes E_{ba} \), so
\( c = P^{(\pi \circ st)} \) where \((\pi \circ st) \) stands for the map \( \pi \circ st \) applied to the second tensor in the previous formula for \( P \).

We can extend \( c \) to a \( \mathfrak{U}_q \mathfrak{p}_n \)-module homomorphism \( c_i : \mathbb{C}_q(n|n)^{\otimes l} \rightarrow \mathbb{C}_q(n|n)^{\otimes l} \) for \( 1 \leq i \leq l - 1 \) by applying \( c \) to the \( i^{th} \) and \((i + 1)^{th}\) tensors.

The linear map \( \mathbb{C}_q(n|n)^{\otimes l} \rightarrow \mathbb{C}_q(n|n)^{\otimes l} \) given by \( P_S i_{i+1} \) where \( P_i \) is the super-permutation operator acting on the \( i^{th} \) and \((i + 1)^{th}\) tensors is also a \( \mathfrak{U}_q \mathfrak{p}_n \)-module homomorphism: this is a consequence of the fact that \( S \) is a solution of the quantum Yang-Baxter relation.

**Proposition 5.4.** The tensor superspace \( \mathbb{C}_q(n|n)^{\otimes l} \) is a module over \( \mathfrak{B}_q \), if we let \( t_i \) act as \( P_S i_{i+1} \) and \( c_i \) act as \( c_i \).

**Proof.** That the linear operators \( P_S i_{i+1} \) satisfy the braid relation (the first relation in (13)) is a consequence of the fact that \( S \) is a solution of the quantum Yang-Baxter relation. The relations (12) for the operators \( P_S i_{i+1} \) and \( c_i \) can be easily verified. As for the other relations, they can be checked via direct computations. It is enough to check the relations (11) on \( \mathbb{C}_q(n|n)^{\otimes 2} \) and the relations (14) on \( \mathbb{C}_q(n|n)^{\otimes 3} \). We briefly sketch some of those computations below.

First, note that \( cP = -c \) and \( Pc = c \). Also, we easily obtain the following:

\[
\begin{align*}
   c \left( (q - 1) \sum_{i=1}^{n} E_{ii} \otimes E_{ii} \right) &= (q - 1) \sum_{i=1}^{n} E_{-i,i} \otimes E_{-i,i} = 0, \\
   c \left( (q - 1) \sum_{i=1}^{n} E_{ii} \otimes E_{-i,-i} \right) &= (q - 1) \sum_{a=-n}^{n} \sum_{b=1}^{n} E_{ab} \otimes E_{-a,-b}, \\
   c \left( (q^{-1} - 1) \sum_{i=1}^{n} E_{-i,-i} \otimes E_{ii} \right) &= (q^{-1} - 1) \sum_{a=-n}^{n} \sum_{b=-n}^{n} (-1)^{p(a)} E_{ab} \otimes E_{-a,-b}, \\
   c \left( \sum_{i=-n}^{-1} E_{i,-i} \otimes E_{-i,i} \right) &= - \sum_{a=-n}^{n} \sum_{b=1}^{n} E_{ab} \otimes E_{-a,-b}, \\
   c \left( \sum_{1 \leq |j| < |i| \leq n} (-1)^{p(j)} E_{ij} \otimes E_{ji} \right) &= \sum_{a=-n}^{n} \sum_{1 \leq |j| < |i| \leq n} (-1)^{p(a)(p(i)+1)+p(j)} E_{a,-i} \otimes E_{-a,i} = 0.
\end{align*}
\]

Therefore, we have that \( c(S - 1) = (q^{-1} - 1)c \), hence \( cS = q^{-1}c \). Now using that \( c = -cP \), we obtain the third relation in (11). Similarly, we prove \( (S - 1)c = (q - 1)c \), and then using \( Pc = c \), we obtain the fourth relation in (11).

For the remaining relations we use the following formula:

\[
\begin{align*}
   PS &= \sum_{i,j = -n}^{n} (-1)^{p(j)} E_{ij} \otimes E_{ji} + (q - 1) \sum_{i=1}^{n} (E_{-i,i} \otimes E_{i,-i}) \\
   &\quad + (q - 1) \sum_{i=1}^{n} (E_{ii} \otimes E_{ii}) - (q^{-1} - 1) \sum_{i=1}^{n} (E_{i,-i} \otimes E_{-i,i}) \\
   &\quad - (q^{-1} - 1) \sum_{i=1}^{n} (E_{-i,-i} \otimes E_{-i,-i}) + (q - q^{-1}) \sum_{i=-n}^{-1} (E_{i,-i} \otimes E_{ii}) \\
   &\quad + (q - q^{-1}) \sum_{|j| < |i|} (E_{ij} \otimes E_{ji}) + (q - q^{-1}) \sum_{|j| < |i|} (-1)^{p(i)p(j)} E_{ji} \otimes E_{j,-i}.
\end{align*}
\]

\( \square \)
As mentioned after the definition of $\mathfrak{B}_{q,t}$, the module structure given in the previous proposition commutes with the action of $\mathfrak{U}_q(\mathfrak{p}_n)$ on $C_q(n|n)^{\otimes l}$. We thus have algebra homomorphisms

$$\mathfrak{B}_{q,t} \longrightarrow \text{End}_{\mathfrak{U}_q(\mathfrak{p}_n)}(C_q(n|n)^{\otimes l})$$

and

$$\mathfrak{U}_q(\mathfrak{p}_n) \longrightarrow \text{End}_{\mathfrak{B}_{q,t}}(C_q(n|n)^{\otimes l}).$$

The main theorem of this section states that $\mathfrak{B}_{q,t}$ is the full centralizer of the action of $\mathfrak{U}_q(\mathfrak{p}_n)$ on $C_q(n|n)^{\otimes l}$ when $n \geq l$.

**Theorem 5.5.** The map $\mathfrak{B}_{q,t} \longrightarrow \text{End}_{\mathfrak{U}_q(\mathfrak{p}_n)}(C_q(n|n)^{\otimes l})$ is surjective and it is injective when $n \geq l$.

**Proof.** This is a $q$-analogue of Theorem 4.5 in [Mo]. The proof follows the lines of the proof of Theorem 3.28 in [BGJKW].

Recall that $\mathcal{A} = \mathbb{C}[q, q^{-1}]_{(q-1)}$ is the localization of $\mathbb{C}[q, q^{-1}]$ at the ideal generated by $q - 1$. The algebra $\mathfrak{U}_q(\mathfrak{p}_n)$ was defined at the beginning of Section 3 and it acts on $C_q(A(n|n)^{\otimes l}$. Let’s abbreviate it $\tilde{\mathfrak{U}}$ for the moment. Let $\text{End}_{\tilde{\mathfrak{U}}}(C_q(n|n)^{\otimes l})$ be the $\mathcal{A}$-subalgebra of $\text{End}_\mathcal{A}(C_q(A(n|n)^{\otimes l})$ that consists of all the $\mathcal{A}$-endomorphisms of $C_q(n|n)^{\otimes l}$ that commute with the action of $\tilde{\mathfrak{U}}$.

Let $\mathfrak{B}_{q,t}(\mathcal{A})$ be the $\mathcal{A}$-associative subalgebra of $\mathfrak{B}_{q,t}$ generated by $t_i$ and $c_i$ for all $i = 1, \ldots, l-1$. Theorem 5.5 will follow from the statement that the $\mathcal{A}$-homomorphism

$$\psi : \mathfrak{B}_{q,t}(\mathcal{A}) \longrightarrow \text{End}_{\tilde{\mathfrak{U}}}(C_q(n|n)^{\otimes l})$$

is surjective and is an isomorphism whenever $n \geq l$.

Let $A_l$ be the algebra given in Definition 2.2 in [Mo]. Proposition 5.6 below gives use an isomorphism $\rho : A_l \longrightarrow (\mathcal{A}/(q-1)\mathcal{A}) \otimes_\mathcal{A} \mathfrak{B}_{q,t}(\mathcal{A})$ which fits within the following diagram (see the proof of Theorem 3.28 in [BGJKW]).

The rest of the proof can proceed as in [BGJKW], using Theorem 4.5 in [Mo] along with Lemma 3.27 in [BGJKW], which can be applied in the present situation.

**Proposition 5.6.** The quotient algebra $\mathfrak{B}_{q,t}(\mathcal{A})/(q-1)\mathfrak{B}_{q,t}(\mathcal{A})$ is isomorphic to the algebra $A_l$ given in Definition 2.2 in [Mo].

**Proof.** It follows immediately from the definitions of both $A_l$ and $\mathfrak{B}_{q,t}(\mathcal{A})$ that we have a surjective algebra homomorphism $A_l \twoheadrightarrow \mathfrak{B}_{q,t}(\mathcal{A})/(q-1)\mathfrak{B}_{q,t}(\mathcal{A})$. That it is injective can be proved as in the proof of Proposition 3.21 in [BGJKW] using Theorem 4.1 in [Mo].

The $q$-Schur superalgebras of type $Q$ were introduced in [BGJKW] and [DuWa1, DuWa2]. Considering loc. cit. and the earlier work on $q$-Schur algebras for $\mathfrak{gl}_n$ (see for instance [Do]), the following definition is natural.

**Definition 5.7.** The $q$-Schur superalgebra $S_q(\mathfrak{p}_n, l)$ of type $P$ is the centralizer of the action of $\mathfrak{B}_{q,t}$ on $C_q(n|n)^{\otimes l}$, that is, $S_q(\mathfrak{p}_n, l) = \text{End}_{\mathfrak{B}_{q,t}}(C_q(n|n)^{\otimes l})$.

We have an algebra homomorphism $\mathfrak{U}_q(\mathfrak{p}_n) \longrightarrow S_q(\mathfrak{p}_n, l)$: it is an open question whether or not this map is surjective. We also have an algebra homomorphism $\mathfrak{B}_{q,t} \longrightarrow \text{End}_{S_q(\mathfrak{p}_n, l)}(C_q(n|n)^{\otimes l})$ and it is natural to expect that it should be an isomorphism, perhaps under certain conditions on $n$ and $l$. 


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