Estimation of all parameters in the reflected Ornstein-Uhlenbeck process from discrete observations

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Abstract

Assuming that a reflected Ornstein-Uhlenbeck state process is observed at discrete time instants, we propose generalized moment estimators to estimate all drift and diffusion parameters via the celebrated ergodic theorem. With the sampling time step $h > 0$ arbitrarily fixed, we prove the strong consistency and asymptotic normality of our estimators as the sampling size $n$ tends to infinity.

This provides a complete solution to an open problem left in Hu et al. [5].

Keywords: Reflected Ornstein-Uhlenbeck process; Ergodic theorem; Spectral representation of transition density; Strong consistency; Asymptotic normality.

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1. Introduction

On a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0})$ let $W = \{W(t)\}_{t \geq 0}$ be a one-dimensional standard Brownian motion. All the processes mentioned in this paper will be adapted to $\{\mathcal{F}_t\}_{t \geq 0}$. We consider the (reflected) Ornstein-Uhlenbeck (ROU) process, reflected at zero, which is defined by the following
one-dimensional stochastic differential equation (SDE):

\[
\begin{cases}
    dX_t = \kappa(\theta - X_t)dt + \sigma dW_t + L_t, & t \in \mathbb{R}_+ = \{x, x \geq 0\}, \\
    X_0 = x \in \mathbb{R}_+
\end{cases}
\]

(1.1)

where \( \kappa, \theta, \sigma \in (0, \infty) \) are constants and \( L_t \) is the minimal continuous increasing process which ensures that \( X_t \geq 0 \) for all \( t \geq 0 \). The ROU process is a useful stochastic model in finance and queue theory (cf. Linetsky [7], Ward and Glynn [8] and the references therein).

This paper will concern with the statistical estimation problem for the parameters \( \kappa, \theta, \sigma \) from the observations. In most practical situations the observations of the process \( \{X_t, t \geq 0\} \) can be made only at discrete time instants \( t_k = kh, \ k = 1, 2, \cdots \), and usually the time interval \( h \) between consecutive observations cannot be made arbitrarily small. To deal with this situation an ergodic type of estimator to estimate \( \kappa \) and \( \theta \) is proposed in a previous work Hu et al. [5] and the strong consistency and asymptotic normality of the estimators are also obtained there. However, as pointed out in Hu et al. [5] they were unable to estimate \( \sigma \) (or \( \sigma^2 \)) by using the ergodic type estimator and instead they proposed to use \( \hat{\sigma}_{c,n} := \frac{1}{nh} \sum_{k=1}^{n} (X_{(k+1)h} - X_{kh})^2 \) as the estimator of \( \sigma^2 \). Let us also mention a work on the estimation of the parameters \( \kappa, \theta, \sigma \) for this ROU when continuous observation is available (Bo et al. [2]). This would require that \( h \to 0 \) to guarantee the strong consistency of the estimator (e.g. \( \hat{\sigma}_{c,n}^2 \to \sigma^2 \)). This paper will fill this gap. We shall introduce an ergodic type estimator to estimate \( \sigma^2 \) (and hence we can estimate all the parameters \( \kappa, \theta, \sigma^2 \) simultaneously) and prove the strong consistency and asymptotic normality for all estimators (including the estimator \( \hat{\sigma}_n \) for \( \sigma \) regardless of the (fixed) value of \( h \). This work is motivated by a recent work of Cheng et al. [4], where the Ornstein-Uhlenbeck process has no reflection, but the Brownian motion was replaced by a stable process.

Now let us describe our ergodic estimators for all parameters. It is well-known that there is a unique invariant probability density function \( \pi(x) \) of \( X_t \)
such that for any integrable function $f$ we have
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f(X_{tk}) = \int_{\mathbb{R}_+} f(x) \pi(x) dx, \quad (1.2) \]
and the invariant probability density function $\pi(x)$ has the following explicit expression (see Hu et al. [5]):
\[ \pi(x) = \frac{\sqrt{2\kappa}}{\sigma \left(1 - \Phi(-\sqrt{2\kappa}/\sigma)\right)}, \quad (1.3) \]
\[ \phi(x) = e^{-\frac{1}{2}x^2}/\sqrt{2\pi} \]
\[ \Phi(x) = \int_{-\infty}^{x} \phi(u) du \]
is the standard normal probability density function, and
\[ \Phi(x) = \int_{-\infty}^{x} \phi(u) du \]
is the standard normal distribution function. As observed in Hu et al. [5] the invariant measure $\pi(x)$ remains the same function if the quantities $\theta$ and $\frac{\kappa}{\sigma^2}$ remain unchanged. Thus, we cannot expect to use (1.2) to estimate $\kappa$ and $\sigma^2$ simultaneously. To this end and motivated by Cheng et al. [4] we shall use the ergodic theorem for $X_{kh}, X_{(k+1)h}$, which states that for any integrable function $f : \mathbb{R}_+^2 \to \mathbb{R}$,
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f(X_{kh}, X_{(k+1)h}) = \mathbb{E} f(\tilde{X}_0, \tilde{X}_h) = \int_{\mathbb{R}_+^2} f(x, y) \pi(x) p_h(x, y) dx dy, \quad (1.4) \]
where $\tilde{X}_0$ is a random variable independent of the Brownian motion $W$ and having the invariant probability density $\pi(x)$, $\tilde{X}$ is the solution to (1.1) with initial random variable $\tilde{X}_0$, and $p_h(x, y)$ is the transition density of $X$.

With some specific choices of $f$ in (1.2) and (1.4) we can obtain our ergodic estimators, whose detailed construction is given in the next section, where the strong consistency and asymptotic normality are also obtained.

Section 3 will provide a numerical example which demonstrates the convergence results of our estimators and which also demonstrates that $\hat{\sigma}_{c,n}$ does not converge.

2. Strong consistency and asymptotic normality

In this section, we aim to construct the estimators for all the parameters $\kappa, \theta, \sigma$ of the ROU process $\{X_t, t \geq 0\}$ given by (1.1) based on discrete observations $\{X_{t_1}, \cdots, X_{t_n}\}$, where $t_k = kh$ with the observation time interval $h$
arbitrarily fixed. We will also study their strong consistency and asymptotic normality. We begin with two crucial convergence results, which are adapted from Lemma 1 in Hu et al. [5] and Theorem 1.1 in Billingsley [1], respectively.

Lemma 2.1. The $h$-skeleton sampled chain $\{X_{kh} : k \geq 0\}$ is ergodic. Namely, for any initial value $x \in \mathbb{R}_+$ and $f \in L_1(\mathbb{R}_+)$ and $g \in L_1(\mathbb{R}_+^2)$ we have

$$
\begin{align*}
\lim_{N \to \infty} \frac{1}{n} \sum_{k=1}^{n} f(X_{kh}) &= \mathbb{E}[f(X_{\infty})] = \int_{\mathbb{R}_+} f(x)\pi(x)dx, \quad \text{a.s.,} \\
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} g(X_{kh}, X_{(k+1)h}) &= \mathbb{E}g(\tilde{X}_0, \tilde{X}_h) \\
&= \int_{\mathbb{R}_+^2} g(x, y)\pi(x)p_h(x, y)dxdy, \quad \text{a.s.,}
\end{align*}
$$

(2.1) (2.2)

where $\tilde{X}_0$ is a random variable independent of the Brownian motion $W$ and having the invariant probability density $\pi$, $\tilde{X}$ is the solution to (1.1) with initial random variable $\tilde{X}_0$, and $p_h(x, y)$ is the transition density of $X$.

As illustrated in Hu et al. [5] it is impossible to use (2.1) alone to estimate all the parameters $\kappa, \theta, \sigma$. So we take $f_1(x) = x$ and $f_2(x) = x^2$ in (2.1) and we take $g(x, y) = xy$ in (2.2) to obtain a system of three equations to determine the parameters $\kappa, \theta, \sigma$. Some elementary computations yield the following expressions for the stationary moments of the invariant measure.

$$
\begin{align*}
\mathbb{E}[X_{\infty}] &= \theta + \frac{\sigma}{\sqrt{2\kappa}} \frac{\phi\left(\frac{\sqrt{2\kappa} \theta}{\sigma}\right)}{1 - \Phi\left(-\frac{\sqrt{2\kappa} \theta}{\sigma}\right)}, \\
\mathbb{E}[X_{\infty}^2] &= \frac{\sigma^2}{2\kappa} + \theta^2 + \theta \frac{\sigma}{\sqrt{2\kappa}} \frac{\phi\left(\frac{\sqrt{2\kappa} \theta}{\sigma}\right)}{1 - \Phi\left(-\frac{\sqrt{2\kappa} \theta}{\sigma}\right)}, \\
\mathbb{E}(\tilde{X}_0 \tilde{X}_h) &= \int_{\mathbb{R}_+^2} xy\pi(x)p_h(x, y)dxdy.
\end{align*}
$$

(2.3)

However, to our best knowledge, there is no compact explicit form for the transition probability density $p_t(x, y)$. We shall use the following spectral representation for the transition density $p_t(x, y)$ derived in Linetsky [7]

$$
p_t(x, y) = \pi(y) + m(y) \sum_{i=1}^{\infty} e^{-\lambda_i t} \varphi_i(x)\varphi_i(y), \quad t > 0,
$$

where the notations are described as follows.
(1) $\pi(x)$ is the stationary density given by (1.3) and $m(x)$ is the speed measure defined by
\[ m(x) = \frac{2}{\sigma^2} e^{-\kappa(\theta-x)^2/\sigma^2}. \]

(2) The eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_i < \cdots$ are roots of
\[ H_{\lambda/\kappa}(-\sqrt{\kappa}\theta/\sigma) = 0, \]
where $H$ is the Hermite function (see Lebedev [6]).

(3) The normalized eigenfunctions $\varphi_i(x)$ are given by
\[ \varphi_i(x) = \pm \frac{\kappa^{3/4} \sigma^{1/2} e^{\kappa \theta^2/(2\sigma^2)} H_{\lambda/\kappa}(\sqrt{\kappa}(x-\theta)/\sigma)}{\sqrt{2\lambda \Delta_i H_{\lambda/\kappa}(-\sqrt{\kappa}\theta/\sigma)}}, \quad i = 1, 2, \cdots \]
where $\Delta_i = \frac{\partial H_{\nu}(-\sqrt{\nu\theta}/\sigma)}{\partial \nu}_{\nu = \lambda/\kappa}$.

Now we replace $E(X_{\infty}), E(X_{\infty}^2), E(\tilde{X}_n \tilde{X}_n)$ in (2.3) by their sample approximations to yield
\[
\begin{align*}
\frac{1}{n} \sum_{k=1}^{n} X_{kh} &= \theta + \frac{\sigma}{\sqrt{2\kappa}} \frac{\phi\left(\sqrt{2\kappa} u\right)}{1 - \Phi\left(-\sqrt{2\kappa} u/\sigma\right)}, \\
\frac{1}{n} \sum_{k=1}^{n} X_{kh}^2 &= \frac{\sigma^2}{2\kappa} + \theta^2 + \theta \frac{\sigma}{\sqrt{2\kappa}} \frac{\phi\left(\sqrt{2\kappa} u\right)}{1 - \Phi\left(-\sqrt{2\kappa} u/\sigma\right)}, \\
\frac{1}{n} \sum_{k=1}^{n} X_{kh} X_{(k+1)h} &= \int_{\mathbb{R}_2^+} x y \pi(x) p_h(x, y) dxdy.
\end{align*}
\]
This is a system of three equations for the three unknown parameters. We expect that it would give a unique solution $\hat{\kappa}_n, \hat{\theta}_n, \hat{\sigma}_n$, which we call the ergodic estimators of the parameters. The system is still complicated to analyze and to be solved. We will further simplify it. To this end we denote $u = \theta$ and $v = \sqrt{2\kappa \sigma}$. Then the first two equations in (2.4) depends only on $u$ and $v$ and they give a unique solution $\hat{u}_n$ and $\hat{v}_n$. We then write $p_h(x, y) = p_h(x, y; \kappa, \theta, \sigma)$ as a kernel $p_h(x, y; u, v, \sigma)$ depending on parameters $u, v, \sigma$. Finally, we replace the parameters $u$ and $v$ in kernel $p_h(x, y; u, v, \sigma)$ by the obtained values $\hat{u}_n$ and $\hat{v}_n$, then the third equation in (2.4) becomes one equation for one unknown $\sigma$. This greatly simplifies the computations.
To summarize the above discussion, we have transformed the system (2.4) into the following system of equations.

\[
\begin{aligned}
\frac{1}{n} \sum_{k=1}^{n} X_{kh} &= u + \frac{u}{v} \frac{\phi(v)}{1 - \Phi(-v)}, \\
\frac{1}{n} \sum_{k=1}^{n} X_{kh}^2 &= \frac{u^2}{v^2} + u^2 + \frac{u^2}{v} \frac{\phi(v)}{1 - \Phi(-v)}, \\
\frac{1}{n} \sum_{k=1}^{n} X_{kh} X_{(k+1)h} &= \int_0^{\infty} \int_0^{\infty} xy p_h(x,y) \pi(x) dx dy.
\end{aligned}
\]  

The right-hand side of the above third equation depends on \(u, v\) and \(\sigma\) by substituting \(\theta\) and \(\kappa\) by \(\theta = u\) and \(\kappa = \frac{v^2 \sigma^2}{2u^2}\) into the expression of \(\pi\) and \(p_h(x,y)\). Define \(\tilde{\lambda}_i = \lambda_i / \sigma^2\). For sake of the numerical computation we write the dependence explicitly as follows:

\[
p_h(x,y) = \pi(y) + m(y) \sum_{i=1}^{\infty} e^{-\tilde{\lambda}_i \sigma^2} \phi_i(x) \phi_i(y),
\]

where

1. \(\pi\) and \(m\) are given by

\[
\pi(x) = \frac{v}{u} \phi \left(\frac{v}{u} x - v\right) / [1 - \Phi(-v)],
\]

and

\[
m(x) = \frac{2}{\sigma^2} e^{-v^2/2 + v^2 x/u - v^2 x^2/(2u^2)},
\]

2. The eigenvalues \(0 < \tilde{\lambda}_1 < \tilde{\lambda}_2 < \cdots < \tilde{\lambda}_i < \cdots\) are roots of

\[
H_{2u^2 \tilde{\lambda}_i / u^2 - 1}(-v/\sqrt{2}) = 0.
\]

3. The eigenfunctions are given by

\[
\phi_i(x) = \pm \sigma \frac{(v/(\sqrt{2} u))^{3/2} e^{v^2/4} H_{2u^2 \tilde{\lambda}_i / u^2}((\frac{v}{u} x - v) / \sqrt{2})}{\sqrt{2 \lambda_i \Delta_i} H_{2u^2 \tilde{\lambda}_i / u^2}(-v/\sqrt{2})}, \quad i = 1, 2, \ldots
\]

with \(\Delta_i = \frac{\partial H_{\nu-1}(-v/\sqrt{2})}{\partial \nu} \bigg|_{\nu=2u^2 \tilde{\lambda}_i / v^2}\).

Now we summarize our discussion as follows.

**Construction of the ergodic estimators for all parameters \(\kappa, \theta, \sigma\):**
(i) Solve the first two equations in the system (2.5) to obtain \( \hat{u}_n \) and \( \hat{v}_n \).

(ii) Substitute the obtained \( \hat{u}_n \) and \( \hat{v}_n \) into the transition probability kernel \( p_h(x, y) \) according to (2.6)-(2.10) to obtain the third equation in the system (2.5), which now contains only one unknown \( \sigma \) and solve it to obtain \( \hat{\sigma}_n \).

(iii) Solve \( \hat{u}_n = \hat{\theta}_n \) and \( \hat{v}_n = \frac{\sqrt{2\pi} \hat{\sigma}_n}{\sigma_n} \) to obtain
\[
\hat{\theta}_n = \hat{u}_n \quad \text{and} \quad \hat{\kappa}_n = \frac{\hat{v}_n^2 \hat{\sigma}_n^{3}}{2\hat{\theta}_n^2}.
\]

(2.11)

Remark 2.1. In numerical computation, we shall need to take finite terms in the spectral representation of the transition probability function (in our numerical simulation we take about twelve terms and the results are satisfactory). The Hermite functions and the roots of the Hermite functions can be handled by the standard mathematical software package. The system (2.5) of algebraic equations does not give an explicit solution. There are many standard methods to solve it, such as the Newton-Raphson iteration method.

To study the strong consistency and the asymptotic normality, we denote the right-hand sides of the equation in the system (2.5) by \( g_1(u, v) \), \( g_2(u, v) \), and \( g_3(u, v, \sigma) \), respectively. Denote
\[
M_{1,n} = \frac{1}{n} \sum_{k=1}^{n} X_{kh}, \quad M_{2,n} = \frac{1}{n} \sum_{k=1}^{n} (X_{kh})^2, \quad M_{3,n} = \frac{1}{n} \sum_{k=1}^{n} X_{kh}X_{(k+1)h}.
\]

Then the equation (2.5) can be rewritten as
\[
g_1(u, v) = M_{1,n}, \quad g_2(u, v) = M_{2,n}, \quad g_3(u, v, \sigma) = M_{3,n}.
\]

(2.12)

\( g_3 \) also depends on \( h \) which is fixed) Or we write
\[
g(u, v, \sigma) = M_n, \quad \text{(2.13)}
\]

where
\[
g = (g_1, g_2, g_3)^T \quad \text{and} \quad M_n = (M_{1,n}, M_{2,n}, M_{3,n})^T.
\]

Denote by \( J(u, v, \sigma) \) the determinant of the Jacobian of \( g \). Then
\[
J(u, v, \sigma) = J(g_1, g_2) \frac{\partial}{\partial \sigma} g_3(u, v, \sigma), \quad \text{(2.14)}
\]
where \( J(g_1, g_2) \) is the determinant of the Jacobian of \( g_1 \) ad \( g_2 \). Hu et al. [5] proved that \( J(g_1, g_2) \) is never 0. If \( \frac{\partial}{\partial \sigma} g_3(u, v, \sigma) \) is not singular in some domain \( D \subseteq \mathbb{R}^3_+ \), then by the inverse function theorem, for any \( (u, v, \sigma) \in D \), \( g = (g_1, g_2, g_3) \) has a unique inverse in a neighbourhood \( (u, v, \sigma) \). If \( (u, v, \sigma) \) (or equivalently, \( (\kappa, \theta, \sigma) \)) are the true parameters, then by Lemma 2.1, we see when \( n \) is sufficiently large \( (M_{1,n}, M_{2,n}, M_{3,n}) \) will be in the neighbourhood of \( g(u, v, \sigma) \). This means when \( n \) is sufficiently large the equation (2.12) has a solution.

Thus, the critical question now is to find a domain \( D \) such that \( \frac{\partial}{\partial \sigma} g_3(u, v, \sigma) \) is not singular on \( D \). This is an elementary analysis problem. The explicit expression of the derivative of \( g_3(u, v, \sigma) \) with respect to \( \sigma \) can be obtained (see Remark 2.1 below). However, this expression is complicated and it is hard to obtain the domain of \( (u, v, \sigma) \) so that inside this domain this derivative is not singular. We shall proceed as follows to reduce the \( \frac{\partial}{\partial \sigma} g_3(u, v, \sigma) \) from a function of three variables \( u, v, \sigma \) to a function of one variable \( \sigma \).

Since the first two equations in (2.5) is independent of \( \sigma \), as indicated above we can solve them without considering the third equation in (2.5). Hu et al. [5] proved that there exist continuous inverse mapping \( (h_1, h_2) \) of \( (g_1, g_2) : \mathbb{R}^2_+ \rightarrow \mathbb{R}^2 \) such that the ergodic estimators defined by

\[
\hat{u}_n := h_1(M_{1,n}, M_{2,n}), \quad \hat{v}_n := h_2(M_{1,n}, M_{2,n})
\]  

(2.15)

converge almost surely to the true parameters

\[
u = h_1(g_1(u, v), g_2(u, v)) = \theta, \quad v = h_2(g_1(u, v), g_2(u, v)) = \frac{\sqrt{2}k\theta}{\sigma}.
\]

After the estimators \( \hat{u}_n \) and \( \hat{v}_n \) have been obtained, we can substitute them into the \( g_3 \). Thus \( g_3(\sigma) = g_3(\hat{u}_n, \hat{v}_n, \sigma) \) and \( g_3'(\sigma) = \frac{\partial}{\partial \sigma} g_3(\hat{u}_n, \hat{v}_n, \sigma) \) will be functions of single variable \( \sigma \). We can plot the derivative function \( g_3'(\sigma) \) in an interval \( D_\sigma \) that is as large as we believe it contains the true parameter \( \sigma \) (we shall plot \( g_3'(\sigma) \) for some value of \( u \) and \( v \) in next section). If \( g_3'(\sigma) \) is never equal to 0 on \( D_\sigma \), then the solution to \( g_3(\hat{u}_n, \hat{v}_n, \sigma) = M_{3,n} \) is unique on \( D_\sigma \) (if not then there are two different points \( \sigma_1 < \sigma_2 \) in \( D_\sigma \) such that
\( g_3(\hat{u}_n, \hat{v}_n, \sigma_1) = g_3(\hat{u}_n, \hat{v}_n, \sigma_2) = M_{3,n} \). By the mean value theorem there is a \( \sigma_0 \in [\sigma_1, \sigma_2] \subseteq D_\sigma \) such that \( \frac{\partial}{\partial \sigma} g_3(\hat{u}_n, \hat{v}_n, \sigma) = 0 \).

If \( g_3' \) is not singular on \( D_\sigma \), then the third equation (2.12) has a unique solution on \( D_\sigma \), which gives the ergodic estimator \( \hat{\sigma}_n = h_3(\hat{u}_n, \hat{v}_n, M_{3,n}) \) of \( \sigma \), where \( h_3(\hat{u}_n, \hat{v}_n, \cdot) \) is the continuous inverse of \( g(\hat{u}_n, \hat{v}_n, \cdot) \). By Lemma 2.1 it is easy to see that \( \hat{\sigma}_n \to \sigma \) a.s.

Now we summarize the above discussion as the following theorem.

**Theorem 2.1.** (i) The first two equations of the system (2.5) have a unique solution pair \( (\hat{u}_n, \hat{v}_n) = (h_2(M_{1,n}, M_{2,n}), h_2(M_{1,n}, M_{2,n})) \).

(ii) If \( \frac{\partial}{\partial \sigma} g_3(\hat{u}_n, \hat{v}_n, \sigma) \) (\( g_3 \) is defined by the right-hand side of the third equation in (2.5)) is not singular on some interval \( \sigma \subseteq D_\sigma \) which contains the true parameter \( \sigma \), then when \( n \) is sufficiently large the third equation of (2.5), namely,

\[
g_3(\hat{u}_n, \hat{v}_n, \sigma) = M_{3,n} = \sum_{k=1}^{n} X_{kh} X_{(k+1)h} \tag{2.16}
\]

has a unique solution \( \hat{\sigma}_n \).

(iii) \( (\hat{\kappa}_n, \hat{\theta}_n, \hat{\sigma}_n)^T \to (\kappa, \theta, \sigma)^T \) almost surely as \( n \to \infty \), where \( \hat{\theta}_n \) and \( \hat{\kappa}_n \) are given by (2.11).

Next, we study the joint asymptotic behavior of the all estimators \( (\hat{\theta}_n, \hat{\kappa}_n, \hat{\sigma}_n) \).

**Theorem 2.2.** Let \( \frac{\partial}{\partial \sigma} g_3(\hat{u}_n, \hat{v}_n, \sigma) \) be nonsingular on some interval \( \sigma \subseteq D_\sigma \) which contains the true parameter \( \sigma \). Then, the estimators \( (\hat{\theta}_n, \hat{\kappa}_n, \hat{\sigma}_n) \) satisfy the following asymptotic normality property:

\[
\sqrt{n}((\hat{\theta}_n, \hat{\kappa}_n, \hat{\sigma}_n)^T - (\theta, \kappa, \sigma)^T) \Rightarrow N(0, \Sigma),
\]

where \( \Sigma \) is a covariance matrix defined in (2.17) below.

**Proof.** For any nice function \( f \) and \( g \), denote

\[
\sigma_{fg} := \text{Cov}(f(\hat{X}_0, \hat{X}_h), g(\hat{X}_0, \hat{X}_h)) + \sum_{k=1}^{\infty} \text{Cov}(f(\hat{X}_0, \hat{X}_h), g(\hat{X}_{kh}, \hat{X}_{(k+1)h})) + \text{Cov}(g(\hat{X}_0, \hat{X}_h), f(\hat{X}_{kh}, \hat{X}_{(k+1)h})).
\]
Let \( f_1(x, y) = x \), \( f_2(x, y) = y \), \( f_3(x, y) = xy \) and denote
\[
\tilde{\Sigma}_3 := (\sigma_{f_k f_l})_{1 \leq k, l \leq 3}.
\]

Then an application of the multivariate Markov chain central limit theorem (e.g. Brooks et al. [3, Section 1.8.1]) yields
\[
\sqrt{n}((M_{1,n}, M_{2,n}, \tilde{M}_{2,n})^T - (E X_\infty, E X_\infty^2, E \tilde{X}_0 \tilde{X}_h)^T) \overset{d}{\to} N(0, \tilde{\Sigma}_3).
\]

To simplify notations, introduce the following two mappings:
\[
h : (x_1, x_2, x_3) \mapsto (h_1(x_1, x_2), h_2(x_1, x_2), h_3(x_1, x_2, x_3)),
\]
and
\[
\eta : (x_1, x_2, x_3) \mapsto (x_1, \frac{x_2^2 x_3^2}{2 x_1^2}, x_3),
\]
where \( \eta \) is the inverse transform of (2.11). Then from delta method, we have
\[
\sqrt{n}(h(M_{1,n}, M_{2,n}, \tilde{M}_{2,n})^T - h(\kappa, \theta, \sigma))^T \overset{d}{\to} N(0, \tilde{\Sigma})
\]
where \( \tilde{\Sigma} = \nabla h(\Theta) \tilde{\Sigma}_3 \nabla h(\Theta)^T \). Finally, applying the delta method again, we arrive at the asymptotic behavior of the ergodic estimators:
\[
\sqrt{n}(\eta h(M_{1,n}, M_{2,n}, \tilde{M}_{2,n}))^T - \eta(h(\Theta))^T \Rightarrow N(0, \Sigma)
\]
where
\[
\Sigma = \nabla \eta(h(\Theta)) \tilde{\Sigma}_3 \nabla \eta(h(\Theta))^T \tag{2.17}
\]
completing the proof of the theorem. \( \square \)

**Remark 2.1.** We mention that we do not know the monotonicity of \( g_3'(\sigma) \) in theory. But we can observe it numerically. Since \( \sigma > 0 \), to investigate the sign of \( g_3'(\sigma) \) is equivalent to discuss the sign of \( \frac{\partial}{\partial \sigma^2} g_3(u, v, \sigma) \).

\[
\frac{\partial}{\partial \sigma^2} g_3(u, v, \sigma) = -h \int_0^\infty \int_0^\infty xy|m(y)| \sum_{i=1}^\infty \tilde{\lambda}_i e^{-\tilde{\lambda}_i \sigma^2} h \varphi_i(x) \varphi_i(y) |dx dy.
\]

For fixed \( u \) and \( v \) an example of the values of \( \frac{1}{h} \frac{\partial}{\partial \sigma^2} g_3(u, v, \sigma) \) is plotted in Figure 1. It shows that the partial derivatives are always less than zero on the concerned interval.
3. Numerical experiments

In this section, we present a numerical experiment to illustrate our method. In Table 1 we set the following true parameters $\sigma = 0.5$, $\kappa = 1$, $\theta = 1$. The time step is fixed by $h = 0.5$. In the experiments, we use the truncation

$$p_{N,h}(x, y) = \pi(x) + \sum_{i=1}^{N} e^{-\tilde{\lambda}_i\sigma^2 h} \varphi_i(x) \varphi_i(y)$$

in (2.6). Here we take $N = 12$. It can be seen that our estimators for all the parameters, including $\hat{\sigma}_n$, are strongly consistent. On the other hand we also include variation estimator $\hat{\sigma}_{c,n} = \sqrt{\sum_{k=1}^{n}(X_{(k+1)h} - X_{kh})^2/(nh)}$, which is observed not consistent.

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Table 1: The estimators ($\hat{\kappa}, \hat{\theta}, \hat{\sigma}, \hat{\sigma}_c$) for different values of $n$

| n$(\times10^3)$ | 2   | 3   | 4   | 5   | 6   | 8   |
|------------------|-----|-----|-----|-----|-----|-----|
| $\hat{\kappa}$   | 0.963 | 0.953 | 1.134 | 0.994 | 0.956 | 0.966 |
| $\hat{\theta}$   | 0.992 | 1.001 | 0.996 | 0.998 | 0.989 | 0.997 |
| $\hat{\sigma}$   | 0.486 | 0.497 | 0.517 | 0.501 | 0.503 | 0.501 |
| $\hat{\sigma}_c$ | 0.431 | 0.443 | 0.451 | 0.444 | 0.446 | 0.444 |

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