REVISITING CLIFFORD ALGEBRAS

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Abstract. This paper is divided in three parts. In the first part, I study the Clifford algebra associated to the hessian of a functional $f$ defined on an open subset of $\mathbb{R}^n$ and the Clifford algebra associated to the hessian of the Legendre transform of $f$. I give also the definition of a tensorial topology on a Clifford algebra. In the second part, I study the Clifford algebra $\mathcal{C}(E, q)$ of an infinite dimensional Banach space $E$ and its main properties. Finally, in the third part, I give the explicit formula of the hilbertian kernel of a Clifford algebra with examples.

1. Basic properties of a Clifford algebra

1.1. Quadratic space.

Definition 1. Let $K = \mathbb{R}$ or $\mathbb{C}$ and $E$ a vector space on $K$. A quadratic form $q$ on $E$ is a mapping from $E$ into $K$ such that:

(i) $\forall x \in E$, $\forall \lambda \in K$, $q(\lambda x) = \lambda^2 q(x)$

(ii) If $\forall x, y \in E$, $b(x, y) = q(x + y) - q(x) - q(y)$ then $b$ is bilinear and symmetric.

So, we say that $(E, q)$ is a quadratic space (on $K$).

1.2. Clifford algebra.

Problem 1. Let $(E, q)$ be a quadratic space (on $K$). We want to find if there exists an algebra $\mathbf{A}$ on $K$, and a linear mapping $\varphi$ from $E$ into $\mathbf{A}$ such that:

$$\forall x \in E, \left[ \varphi(x)^2 \right] = q(x) \cdot 1_{\mathbf{A}}$$

where $1_{\mathbf{A}}$ is the identity of $\mathbf{A}$.

So, if the problem P1 has a solution, then:

$$\forall x, y \in E, \varphi(x) \cdot \varphi(y) + \varphi(y) \cdot \varphi(x) = 2b(x, y) \cdot 1_{\mathbf{A}}.$$ 

In that case, we set:

$$W = \varphi(E)$$

and $\forall x, y \in E$, $B(\varphi(x) \cdot \varphi(y)) = b(x, y) \cdot 1_{\mathbf{A}}$.

Then, $W$ is a linear space on $K$ and $B$ is a bilinear and symmetric mapping from $W$ into $K$.

The problem is equivalent to the following:

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Problem 2. Find an algebra $W$ on $K$ such that:
$$\forall v, w \in W, v \cdot w + w \cdot v = 2B(v, w)$$
where $\cdot$ is the product in $W$.

Now, let us suppose that $E$ is finite dimensional. Let $n \in \mathbb{N}^*$ be the dimension of $E$. We can prove that:

The problem has a solution and there exists an algebra on $K$, denoted by $C(E, q)$ such that:

if $(e_1, ..., e_2)$ is an (orthogonal) basis of $E$ such that:
$$\forall j, l \in (1, ..., n), b(e_j, e_l) = 0 \text{ if } j \neq l \text{ and } b(e_j, e_j) = \pm 1$$
then the set
$$\{\varphi(e_{j_1}) \cdot \cdots \cdot \varphi(e_{j_k}) ; 1 \leq j_1 < j_2 < ... < j_k \leq n\}$$
is a basis of $C(E, q)$ and $\dim(C(E, q)) = 2^n$.

1.3. Clifford algebra and tensor analysis.
When $p \in \mathbb{N}, p \geq 2$, we set:
$$E \otimes p = E \otimes \ldots \otimes E.$$ Then $E \otimes p$ is generated by the following tensors:
$$x_1 \otimes \ldots \otimes x_p, \quad x_j \in E, \quad j \in (1, ..., p).$$
Let $E^\wedge p$ be the vector space generated by the following antisymmetric tensors:
$$\frac{1}{p!} \sum_{\sigma \in S_p} \varepsilon(\sigma) (x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(p)})$$
where $S_p$ is the symmetric group of order $p$, $x_{\sigma(j)} \in E$, $j \in (1, ..., p)$ and $\varepsilon(\sigma) = +1$ (resp. $-1$) if $\sigma$ is even (resp. odd).

So, when $p \in \mathbb{N}, p \geq 2$, we set:
$$E_p^\wedge = E \oplus E^\wedge 2 \oplus \ldots \oplus E^\wedge p$$
and
$$E^\ominus_p = E \oplus E^\ominus 2 \oplus \ldots \oplus E^\ominus p.$$ Then, we can prove the following

Proposition 1. With the same hypotheses as in the previous paragraph, $C(E, q)$ is (algebraically) isomorphic to $E^\wedge_p$.

1.4. Topology on $C(E, q)$.
Let us suppose that $E$ is a $n$-dimensional ($n \in \mathbb{N}^*$) vector space on $K$, and $\nu$ a norm on $E$. Let $\gamma$ be a tensorial (semi-) norm such that:
$$\pi \leq \gamma \leq \varepsilon$$
where $\pi$ (resp. $\varepsilon$) is the canonical projective (resp. injective) tensorial (semi-)norm.

We denote by $E^\otimes_p, \nu_{\gamma}^\otimes_p$ the vector space $E^\otimes_p$ imbedded with the (semi-)norm $\nu_{\gamma}^\otimes_p$ and $E^\wedge_p$ the direct sum of the spaces:
$$E^\otimes_p, \nu_{\gamma}^\otimes_p$$
Then $E^\wedge_p$ is a topological vector space.

Let $E^\wedge_p$ be the space $E^\wedge_p$ imbedded with the topology induced by that of $E^\otimes_p$. 

\[{P2}\]
Proposition 2. \( C(E,q) \) is a topological vector space when it is imbedded by the reciprocal image topology of that we have defined above on \( E^\wedge_{\gamma,p} \).

1.5. About second derivatives of a functional.

Definition 2. Let \( \Omega \) be an open set in \( \mathbb{R}^n, n \in \mathbb{N}^* \) and \( f \in C^2(\Omega) \). Then, if \( a \in \Omega, f'(a) \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}) \) and \( f''(a) \in \mathcal{L}(\mathbb{R}^n; \mathcal{L}(\mathbb{R}^n; \mathbb{R})) = \mathcal{B}(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}) \) which is the space of bilinear (and continuous) mappings from \( \mathbb{R}^n \times \mathbb{R}^n \) into \( \mathbb{R} \). Let \( \langle \cdot \mid \cdot \rangle \) be the canonical euclidean scalar product on \( \mathbb{R}^n \). Then \( \forall h = (h_1, \ldots, h_n) \) and \( \forall k = (k_1, \ldots, k_n) \in \mathbb{R}^n \) we let:

\[
f'(a) \cdot h = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(a) \cdot h_j = \langle f'(a) \mid h \rangle
\]

and

\[
(f''(a) \cdot h) \cdot k = \sum_{j,l=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_l}(a) \cdot h_j \cdot k_l = \langle f''(a) \cdot h \mid k \rangle = \langle f''(a) \mid h \otimes k \rangle \otimes.
\]

In the following, we shall identify:

(i) \( f'(a) \) with the vector \( t \left( \frac{\partial f}{\partial x_1}(a), \ldots, \frac{\partial f}{\partial x_n}(a) \right) \)

(ii) \( f''(a) \) with the matrix \( \left[ \frac{\partial^2 f}{\partial x_j \partial x_l}(a) \right]_{j,l=1 \ldots n} \)

which is symmetric.

1.5.1. Geometrical point of view.

Let \( graph(f) = \{ (x, z) \in \Omega \times \mathbb{R} : z = f(x) \} \) then \( f'(a), -1 \) is a vector in \( \mathbb{R}^{n+1} \) which is normal at the point \( (a, f(a)) \) to \( graph(f) \). Moreover

\[
\forall y \in \Omega, P_y = \{ (x, z) \in \mathbb{R}^n \times \mathbb{R} : z - xf'(y) - f(y) + \langle y \mid f'(y) \rangle = 0 \}
\]

is an hyperplane which is tangent to \( graph(f) \) at the point \( (y, f(y)) \).

In the following, we shall denote by \( q_f(\cdot, a) \) the quadratic form: \( h \rightarrow \langle f''(a) \mid h \otimes h \rangle \otimes \).

1.5.2. Legendre transform of a functional.

Definition 3. We call Legendre transform of \( graph(f) \) the set:

\[
(\text{graph}(f))^* = \{ (x^*, z^*) \in \mathbb{R}^n \times \mathbb{R} \mid x^* = f'(y) \text{ and } z^* = f(y) - \langle y \mid f'(y) \rangle, \ y \in \Omega \}
\]

Generally, \( (\text{graph}(f))^* \) is not the graph of a mapping from \( \mathbb{R}^n \) into \( \mathbb{R} \).

In the following we set:

\[
\forall x^* \in \mathbb{R}^n, \ f^*(x^*) = \{ z^* \in \mathbb{R} : x^* = f'(y) \text{ and } z^* = f(y) - \langle y \mid f'(y) \rangle, y \in \Omega \}.
\]
1.5.3. Examples.

(i) \( \forall x \in \mathbb{R}, \ f(x) = p^{-1}|x|^p \), when \( p \in \mathbb{N}^* \). Then:
\[
f^*(x^*) = -(p^*)^{-1}|x^*|^{p^*} \quad \text{with} \quad p^* \in \mathbb{N}^* \quad \text{and} \quad p^{-1} + (p^*)^{-1} = 1, \ x^* \in \mathbb{R}.
\]
(ii) \( \forall x \in \mathbb{R}, \ f(x) = (x^2 - 1)^2 \). Then:
\[
(\text{graph}(f))^* = \{(x^*, z^*) \in \mathbb{R} \times \mathbb{R}; \ x^* = 4y(y^2 - 1) \quad \text{and} \quad z^* = (y^2 - 1)(3y^2 + 1) \}, \ y \in \mathbb{R}\}
\]
(iii) Minkowski quadratic form.
Let \( \forall x = (x_1, \ldots, x_n), \ f(x) = x_1^2 + \cdots + x_p^2 - (x_{p+1})^2 + \cdots + (x_n)^2 \)
when \( 1 \leq p \leq n \). Then: \( \forall a \in \mathbb{R}^n, \ f''(a) \) is a diagonal matrix with elements are equal to \( (+1) \) (resp. \(-1 \)) and
\[
\forall a^* \in \mathbb{R}^n, \ (f^*)''(a^*) = \frac{1}{2} f''(a).
\]

1.5.4. The relation between \( f'' \) and \( (f^*)'' \).
Let us suppose that \( f' \) is a diffeomorphism from a (non void) open subset \( \Omega_0 \)
contained in \( \Omega \) onto an other (non void) open subset \( \Omega_0^* \subset \mathbb{R}^n \). With previous notations, we have:
\[
(x^* = f'(y), \ y \in \Omega_0) \implies \left( y = (f')^{-1}(x^*) \quad \text{def} \quad g(x^*) ; \ x^* \in \Omega_0^* \right).
\]
Then:
\[
z^* = f(g(x^*)) - \langle g(x^*) | f'(g(x^*)) \rangle = f''(x^*) ; \ x^* \in \Omega_0^*.
\]
It can be proved easily that:
when \( n \in \mathbb{N}, \ n \geq 2 \) and when \((f'')''(y)\) is an invertible operator, then:
\[
(f^*)''(x^*) = ((f'')''(y))^{-1}, \ y \in \Omega_0, \ x^* \in \Omega_0^*.
\]

1.6. Clifford algebra associated to the second derivative of a functional.
Let \( f \in C^2(\Omega) \). For all \( a \in \Omega \), we denote by \( B_f(\cdot, \cdot ; a) \) the bilinear form which
is associated to \( q_f(\cdot, a) \) and by \( M_f(a) \) the \( n \times n \) matrix \( \left[ \frac{\partial^2 f}{\partial x_j \partial x_l} (a) \right]_{j,l=1,\ldots,n} \).

Let us suppose that \( M_f(a) \) is invertible. As \( M_f(a) \) is a symmetric matrix, then
\( M_f(a) \) has \( n \) eigenvectors \( e_1, \ldots, e_n \) which are linearly independant such that:
\[
B_f(e_j, e_l ; a) = 0 \quad \text{if} \quad j \neq l, \ 1 \leq j, l \leq n \quad \text{and} \quad B_f(e_j, e_j ; a) \neq 0 \quad \text{if} \quad 1 \leq j \leq n.
\]
So,

**Proposition 3.** *To the functional \( f \) defined above, we can associate the Clifford algebras :\n\( C(\mathbb{R}^n, q_f(\cdot, a)) \), \( a \in \Omega_0 \) and \( C(\mathbb{R}^n, q_{f^*}(\cdot, a^*)) \), \( a^* \in \Omega_0^* \).*
2. Clifford algebra on Banach spaces

2.1. Definitions.

2.1.1. Banach spaces.
Let \( B \) be a linear space and \( \nu \) a norm on \( B \).
We say that \((B, \nu)\) is a normed space.
We denote by \( T(B, \nu) \) the vectorial topology on \( B \) associated to \( \nu \).
We say that \((B, \nu)\) is a Banach space when \((B, \nu)\) with \( T(B, \nu) \) is complete.
We shall denote by \( B^* \) the topological dual of \((B, \nu)\) and by \( \nu^* \) the dual norm of \( \nu \) on \( B^* \).
We denote, also, by \( \langle \cdot, \cdot \rangle \) the duality bracket between \( B \) and \( B^* \).

2.1.2. Schauder basis.
Let \((B, \nu)\) be an infinite dimensional Banach space. We say that a sequence \( \{e_j ; j \in \mathbb{N}^*\} \subset B \) is a Schauder basis in \((B, \nu)\) if for any \( x \in B \), there exists a unique sequence \( \{\alpha_j(x) ; j \in \mathbb{N}^*\} \subset \mathbb{K} \) such that:
\[
\lim_{n \to \infty} \nu\left( x - \sum_{j=1}^{n} \alpha_j(x) e_j \right) = 0.
\]
Below, we use the following notations:
\[
\beta_n(x) = \sum_{j=1}^{n} \alpha_j(x) e_j, \quad \rho_n(x) = \sum_{j=n+1}^{\infty} \alpha_j(x) e_j \quad \text{and} \quad B_n = \{\beta_n(x) ; x \in B\}.
\]
Then \( B_n \) is a linear subspace in \( B \). Moreover, we suppose that the topology on \( B_n \) is induced by the topology \( T(B, \nu) \).
It is clear that:
\[
\forall n \in \mathbb{N}^*, \quad B_n \subset B_{n+1} \subset B \quad \text{and} \quad \bigcup_{n} B_n = B \quad \text{(the sequence \( (B_n) \) pointwise converges to \( B \)).}
\]
The injection of \( B_n \) into \( B_{n+1} \) (or in \( B \)) is continuous.

2.2. Tensorial topological products of Banach spaces.

2.2.1. Definitions.
Let \((E, \lambda)\) and \((F, \mu)\) be two (semi-) normed linear spaces. The projective tensorial product of \((E, \lambda)\) and \((F, \mu)\) is the (semi-)normed linear space \((E \otimes F, (\lambda \pi \mu))\) where \( \lambda \pi \mu \) is the tensorial (semi-) norm on \( E \otimes F \) such that:
\[
\forall z \in E \otimes F, \quad (\lambda \pi \mu)(z) = \inf \left\{ \sum_{k} \lambda(x_k) \mu(y_k) ; \ z = \sum_{k} x_k \otimes y_k \right\}.
\]
Let \( E^* \) (resp. \( F^* \)) be the topological dual of \((E, \lambda)\) (resp. \((F, \mu)\)) and \( \lambda^* \) (resp. \( \mu^* \)) be the dual norm of \( \lambda \) (resp. \( \mu \)). Then:
the inductive tensorial product of \((E, \lambda)\) and \((F, \mu)\) is the (semi-) normed linear space \((E \otimes F, (\lambda \varepsilon \mu))\) where \( \lambda \varepsilon \mu \) is the tensorial (semi-) norm on \( E \otimes F \) such that:
\[
\forall z \in E \otimes F, \quad (\lambda \varepsilon \mu)(z) = \sup \left\{ \left| \sum_{k} \langle x_k, x^* \rangle \langle y_k, y^* \rangle \right| ; \ z = \sum_{k} x_k \otimes y_k \right\}.
\]
where $x^* \in E^*$ , $y^* \in F^*$ , $\lambda^* (x^*) \leq 1$ , $\mu^* (y^*) \leq 1$.

It can be proved that : $\varepsilon \leq \pi$ , and we say that a tensorial norm $\gamma$ on $E \otimes F$ is a reasonable tensor norm (r.t.n) if : $\varepsilon \leq \gamma \leq \pi$.

2.3. Schauder bases on topological tensorial products.

Let $(e_j)$ (resp. $(f_k)$) a Schauder basis of $(E, \lambda)$ (resp. $(F, \mu)$) . Then, if $x \in E$ (resp. $y \in F$) $x = \sum_{j=1}^{\infty} \zeta_j (x) e_j$ (resp. $y = \sum_{k=1}^{\infty} \eta_k (y) f_k$) we set :

$$\forall n \in \mathbb{N}^* , \forall x \in E , \varphi_n (x) = \sum_{j=1}^{n} \zeta_j (x) e_j$$

and

$$\chi_n (x) = \sum_{j=n+1}^{\infty} \zeta_j (x) e_j.$$  

(resp. $y \in F$ , $\psi_n (y) = \sum_{k=1}^{n} \eta_k (y) f_k$ and $\omega_n (y) = \sum_{k=n+1}^{\infty} \eta_k (y) f_k$).

So, we can easily prove that :

$$\lim_{n \to \infty} \lambda (\chi_n (x)) = \lim_{n \to \infty} \mu (\omega_n (y)) = 0.$$

2.3.1. First case: $E \otimes F$.

$\forall n \in \mathbb{N}^*, \forall x \in E , \forall y \in F$,

$$x \otimes y = (\varphi_n (x) + \chi_n (x)) \otimes (\psi_n (y) + \omega_n (y)) = A_n (x, y) + R_n (x, y)$$

where :

$$A_n (x, y) = \varphi_n (x) \otimes \psi_n (y)$$

and

$$R_n (x, y) = (\varphi_n (x) \otimes \omega_n (y)) + (\chi_n (x) \otimes \psi_n (y)) + ((\chi_n (x) \otimes \omega_n (y)))$$

Then, if $\gamma$ is a reasonable tensor norm on $E \otimes F$ , we have :

$$\gamma (R_n (x, y)) \leq \lambda (\varphi_n (x)) \mu (\omega_n (y)) + \lambda (\chi_n (x)) \mu (\psi_n (y)) + \lambda (\chi_n (x)) \mu (\omega_n (y))$$

and

$$\lim_{n \to \infty} (x \otimes y) - A_n (x, y) = 0.$$

We verify easily that : $A_n (x, y) = \sum_{j,k=1}^{n} \zeta_j (x) \eta_k (y) (e_j \otimes f_k)$ .

Now, we consider the infinite sequence $J = \{(m, n) ; m, n \in \mathbb{N}^* \}$ . We set :

$$J_1 = (1,1)$$

$$J_2 = (1,2) \ (2,1) \ (2,2)$$

$$J_3 = (1,3) \ (2,3) \ (3,3) \ (3,2) \ (3,1)$$

$$\vdots$$

$$J_l = (1,l) \ (2,l) \ (3,l) \ldots \ (l,l) \ (l, l-1) \ldots \ (l, 2) \ (l, 1) , \ l \in \mathbb{N}^*.$$  

Then : $J = \bigcup_{l \in \mathbb{N}^*} J_l$ .

Proposition 4. The sequence $\{(e_j \otimes f_k) , \ (j,k) \in J_l , l \in \mathbb{N}^* \}$ is a Schauder basis of the normed $(E \otimes F , \lambda \gamma \mu)$.
2.3.2. Second case : $E \wedge F$.

Let $E \wedge F$ be the linear space generated by the vectors such that

$$x \wedge y = \frac{1}{2!} (x \otimes y - y \otimes x)$$

Using previous notations, we have : $2 (x \wedge y) = A_n^\wedge (x, y) + R_n^\wedge (x, y)$ with :

$$A_n^\wedge (x, y) = \left( \sum_{j=1}^{n} \zeta_j (x) e_j \right) \otimes \left( \sum_{k=1}^{n} \eta_k (y) f_k \right) - \left( \sum_{k=1}^{n} \eta_k (y) f_k \right) \otimes \left( \sum_{j=1}^{n} \zeta_j (x) e_j \right)$$

and

$$\lim_{n \to \infty} R_n^\wedge (x, y) = 0.$$

As in the first case, we can prove the

**Proposition 5.** The sequence $\{(e_j \wedge f_k) , (j, k) \in J_1 , l \in \mathbb{N}^* \}$ is a Schauder basis of the normed space $(E \wedge F , \lambda \gamma \mu)$.

2.3.3. Third case : $E \vee F$.

Let $E \vee F$ be the linear space generated by the vectors such that

$$x \vee y = \frac{1}{2!} (x \otimes y + y \otimes x).$$

As in the second case, we can prove the

**Proposition 6.** The sequence $\{(e_j \vee f_k) , (j, k) \in J_1 , l \in \mathbb{N}^* \}$ is a Schauder basis of the normed space $(E \vee F , \lambda \gamma \mu)$.

2.3.4. Extension. Exercise.

Let $E_j$ be a Banach space with the Schauder basis $(e_j^k)_{k \in \mathbb{N}^*} \quad j = 1,...,n$. Deduce from the previous paragraphs what is the Schauder bases of the spaces $E_1 \otimes ... \otimes E_m$, $E_1 \wedge ... \wedge E_m$ and $E_1 \vee ... \vee E_m$.

2.4. Fock spaces and Schauder bases.

2.4.1. Definitions.

(i) Let $E$ a linear space on $\mathbb{K}$. The space

$$F^\otimes (E) = \mathbb{K} \times E \times E^\otimes 2 \times E^\otimes 3 \times ... \times E^\otimes p \times ...$$

is called a Fock space. This space is an algebra.

(ii) Let $p \in \mathbb{N} , p \geq 2$. We set :

$$\forall x_1,...,x_p \in E , \quad x_1 \vee ... \vee x_p = \frac{1}{p!} \sum_{\sigma \in G_p} (x_{\sigma(1)} \otimes ... \otimes x_{\sigma(p)})$$

and

$$x_1 \wedge ... \wedge x_p = \frac{1}{p!} \sum_{\sigma \in G_p} \varepsilon (\sigma) (x_{\sigma(1)} \otimes ... \otimes x_{\sigma(p)})$$
where $G_p$ is the symmetric group of order $p$ and $\varepsilon(\sigma)$ is equal to $(+1)$ $(\text{resp. } (-1))$ if $G_p$ is even $(\text{resp. odd})$. We shall denote by $E^{\otimes^p}$ $(\text{resp. } E^{\otimes^p})$ the linear subspace of $E^{\otimes^p}$ which is generated by the vectors of type $x_1 \vee \ldots \vee x_p$ $(\text{resp. } x_1 \wedge \ldots \wedge x_p)$.

So, we set:

$$\mathcal{F}^{\neg} (E) = \mathbb{K} \times E \times E^{\otimes 2} \times E^{\otimes 3} \times \ldots \times E^{\otimes^p} \times \ldots$$

and

$$\mathcal{F}^{\vee} (E) = \mathbb{K} \times E \times E^{\neg 2} \times E^{\neg 3} \times \ldots \times E^{\neg^p} \times \ldots$$

2.4.2. Topologies.

Let $E_j, \lambda_j$ $j = 1, 2, 3$ three Banach spaces (on $\mathbb{K}$) and $\gamma$ be a r.t.n. It can be proved (easily) that:

$$(\lambda_1 \gamma \lambda_2) \gamma \lambda_3 = \lambda_1 \gamma (\lambda_2 \gamma \lambda_3) = \lambda_1 \gamma \lambda_2 \gamma \lambda_3.$$ 

Now, let $(E, \lambda)$ be a Banach space (on $\mathbb{K}$).

We set: $\forall j \in \mathbb{N}, \ j \geq 2$, $\lambda^{(j, \gamma)} = \lambda_{\gamma, \ldots, \gamma_{\lambda}}$ where $\gamma$ appears $(j - 1)$ times.

We denote by $(E, \lambda)^{\otimes^j}$ $(\text{resp. } (E, \lambda)^{\vee^j}, (E, \lambda)^{\neg^j})$ the Banach space

$$\left( E^{\otimes^j}, \lambda^{(j, \gamma)} \right) \ (\text{resp. } \left( E^{\vee^j}, \lambda^{(j, \gamma)} \right), \left( E^{\neg^j}, \lambda^{(j, \gamma)} \right)).$$

With above hypotheses, we deduce from the results in paragraph 2 the

**Proposition 7.** Let $(E, \lambda)$ be a normed space (on $\mathbb{K}$) with a Schauder basis. Then: $(E, \lambda)^{\otimes^j}$ $(\text{resp. } (E, \lambda)^{\vee^j}, (E, \lambda)^{\neg^j})$ is a normed space with a Schauder basis.

Let us denote by $\perp$ one of the operators $\otimes, \vee, \wedge$. Below, we set:

$$\mathcal{F}^{\perp} \gamma (E, \lambda) = \mathbb{K} \times (E, \lambda) \times (E, \lambda)^{\perp 2} \times (E, \lambda)^{\perp 3} \times \ldots \times (E, \lambda)^{\perp^j} \times \ldots$$

We shall say that: $\mathcal{F}^{\perp} \gamma (E, \lambda)$ is a (tensorial) Fock space associated to $(E, \lambda)$ with the r.t.n. $\gamma$.

2.5. Limit of Fock spaces associated to a Banach space with a Schauder basis.

Let $(B, \nu)$ be a Banach space with a Schauder basis $\{e_j ; j \in \mathbb{N}^*\}$. We know (cf. paragraph 1) that:

$\forall n \in \mathbb{N}^*, \ B_n \subset B_{n+1} \subset B$ and that: $B = \bigcup_n B_n$.

From the three propositions stated in the previous paragraph, we deduce (easily) that:

$\forall n \in \mathbb{N}^*, \ \mathcal{F}^{\perp} \gamma (B_n, \nu) \subset \mathcal{F}^{\perp} \gamma (B_{n+1}, \nu) \subset \mathcal{F}^{\perp} \gamma (B, \nu)$

and that:

$$\mathcal{F}^{\perp} \gamma (B, \nu) = \bigcup_n \mathcal{F}^{\perp} \gamma (B_n, \nu).$$

Now, let $q$ be a quadratic form on $B$ and $q_n$ its restriction to $B_n$. We have:

$\forall x \in B_n, \ q_n (x) = q (x).$ Moreover,

$\forall x \in B, \ \exists n_0 \in \mathbb{N}^*$ such that: $\forall n \geq n_0, \ x \in B_n$ and $q_n (x) = q (x).$

Thus: $\lim q_n = q$. So,
Definition 4. We call Clifford algebra associated to \((B, q)\) the limit when \(n\) tends to infinity of the Fock space \(\mathcal{F}_q^\wedge ((B, \nu))\) which can be identify to \(\mathcal{F}_q^\wedge ((B, \nu))\) with the quadratic form \(q\) on \(B\).

The results of the paragraph 1.6 can be extended (easily) to the Hessian of a regular convex functional.

3. Hilbertian Clifford algebras.

3.1. Definitions.

3.1.1. Hilbert spaces.

Let \(K = \mathbb{R}\) or \(\mathbb{C}\). A couple \((E, \langle \cdot | \cdot \rangle)\) is a (pre-)Hilbert space if \(E\) is a linear space on \(K\) and \(\langle \cdot | \cdot \rangle\) is a scalar product on \(E \times E\) such that:

(i) \(\forall y \in E\) (resp. \(\forall x \in E\)), the mapping \(\langle \cdot | y \rangle\) (resp. \(\langle x | \cdot \rangle\)) from \(E\) into \(K\) is linear (resp. linear if \(K = \mathbb{R}\), antilinear if \(K = \mathbb{C}\)).

(ii) \(\forall x \in E\), \(\langle x | x \rangle \in \mathbb{R}_+\)

(iii) If \(x \in E\) and \(\langle x | x \rangle = 0\), then : \(x = 0\).

So, it can be proved that \(\sqrt{\langle x | x \rangle}\) is a norm on \(E\) denoted by \(\|\cdot\|\).

Below, we write equivalently : \((E, \langle \cdot | \cdot \rangle)\) or \((E, \|\cdot\|)\). If \((E, \|\cdot\|)\) is complete, then \((E, \langle \cdot | \cdot \rangle)\) is a Hilbert space. Thus a Hilbert space is a Banach space.

3.1.2. Basis of a Hilbert space.

Let \((E, \langle \cdot | \cdot \rangle)\) be a Hilbert space on \(K\) and \(J = \{1, ..., n\}\), \(n \in \mathbb{N}^*\) or \(J = \mathbb{N}^*\). A family of elements of \(E\), \(\{e_j ; j \in J\}\) is called a basis of \((E, \langle \cdot | \cdot \rangle)\) if for any \(x \in E\) there exists a unique family of scalar \(\{\alpha_j (x) ; j \in J\}\) such that:

\(x = \sum_{j=1}^{n} \alpha_j (x) e_j\)

when \(J = \{1, ..., n\}\) and

\[\lim_{k \to \infty} \left\| x - \sum_{j=1}^{k} \alpha_j (x) e_j \right\|\]

when \(J = \mathbb{N}^*\).

We say that \(\{e_j ; j \in J\}\) is an orthonormal basis of \((E, \langle \cdot | \cdot \rangle)\) if \(\langle e_j | e_l \rangle = \delta_{jl}\), \(j, l \in J\).

Remark 1. As any basis in an infinite dimensional Hilbert space is a Schauder basis in that space, all the results stated in the part B, for Banach spaces are true for Hilbert spaces.

3.2. Reproducing Kernel and Schwartz Kernel of a Hilbert space.

3.2.1. Riesz theorem.

Let \((E, \langle \cdot | \cdot \rangle)\) be a Hilbert space on \(K\) and \(E^* = \mathcal{L}((E, \|\cdot\|), \mathbb{K})\) its topological dual. Then:

\(\forall u \in E^*, \exists \) (only one) \(x^*(u)\) such that : \(\forall x \in E\), \(u(x) = \langle x | x^*(u) \rangle\).
3.3. Reproducing kernel or Aronszjan kernel.
Let $\Omega$ be a (non void) set. We say that the hilbertian space $(E, \langle \cdot | \cdot \rangle)$ is a hilbertian subspace of $K^\Omega$ and we set $(E, \langle \cdot | \cdot \rangle) \in Hilb(K^\Omega)$ if:

(i) $E$ is a linear subspace of $K^\Omega$
(ii) $\forall t \in \Omega$, $\exists M_t \in \mathbb{R}^*_+$ such that $\forall x \in E$, $|x(t)| \leq M_t \|x\|

Proposition 8. Let $(E, \langle \cdot | \cdot \rangle) \in Hilb(K^\Omega)$. Then:
$\forall t \in \Omega$, $\exists$ (only one) $E(\cdot, t) \in E$ such that $\forall x \in E$, $x(t) = \langle x | E(\cdot, t) \rangle$.

The mapping from $\Omega \times \Omega$ into $K$:
$(s, t) \rightarrow E(s, t)$ is called the reproducing kernel or the Aronszjan Kernel of $(E, \langle \cdot | \cdot \rangle)$.

3.3.1. Schwartz kernel.
Let $A$ be a lcs (locally convex separable space) on $K$, $A^*$ its topological dual and $\langle \cdot , \cdot \rangle$ the duality bracket between $A$ and $A^*$.

We say that the hilbertian space $(E, \langle \cdot | \cdot \rangle)$ is a hilbertian subspace of $A$ and we set $(E, \langle \cdot | \cdot \rangle) \in Hilb(A)$ if:

(i) $E$ is a linear subspace of $A$
(ii) $\forall a^* \in A^*$, $\exists M(a^*) \in \mathbb{R}^*_+$ such that $\forall x \in E$, $|\langle jx , a^* \rangle| \leq M(a^*) \|x\|$

where $j$ is the injection of $E$ into $A$.

Let $\Lambda$ be the duality mapping of $(E, \langle \cdot | \cdot \rangle)$. Then:

$A^* \rightarrow j^*E^* \rightarrow E \rightarrow jA$

where $j^*$ is the transpose of $j$ if $K = \mathbb{R}$ and $j^*$ is the conjugate transpose of $j$ if $K = \mathbb{C}$.

So, $E = j^*\Lambda j$ is called the Schwartz (hilbertian) Kernel of $(E, \langle \cdot | \cdot \rangle)$.

$E$ is characterized by the two following properties:

(i) $\forall a^*, b^* \in A^*$, $\langle E b^* , a^* \rangle = \begin{cases} \langle E a^* , b^* \rangle & \text{if } K = \mathbb{R} \\ \langle E a^* , b^* \rangle & \text{if } K = \mathbb{C} \end{cases}$

(ii) $\forall a^* \in A^*$, $\langle E a^* , a^* \rangle \geq 0$.

3.4. Examples of hilbertian kernels.

3.4.1. Example 1.
Let $a, b \in \mathbb{R}$, $-\infty < a < b < +\infty$ and $c \in (a, b)$. We denote by $H$ the linear space of polynomials of one variable with degree less or equal to $n \in \mathbb{N}^*$.

We set: $\forall \omega_1, \omega_2 \in H$, $\langle \omega_1 | \omega_2 \rangle = \sum_{j=0}^{n} \omega_1^{(j)}(c) \cdot \omega_2^{(j)}(c)$. Then:

$(H, \langle \cdot | \cdot \rangle) \in Hilb(\mathbb{R}^{(a,b)})$.

Let $H$ be the reproducing kernel of $H$. We have:

$\forall s, t \in (a, b)$, $H(s, t) = \sum_{j=0}^{n} \frac{(s-c)^j}{j!} \frac{(t-c)^j}{j!}$
3.4.2. Example 2.
Let $a, b \in \mathbb{R}$, $-\infty < a < b < +\infty$ and:
\[ \mathcal{D}'(a, b) \text{ the space of distributions on } (a, b), \]
\[ \mathcal{H}^1(a, b) = \{ x \in \mathcal{D}'(a, b) ; \ x \in L^2(a, b) \text{ and } x' \in L^2(a, b) \} . \]
Let \( \mathcal{H} = \{ x \in \mathcal{H}^1(a, b) ; \ x(a) = 0 \} \) and : \( \forall x, y \in \mathcal{H} \), \( \langle x \mid y \rangle = \int_a^b x'(t) \cdot y'(t) \, dt \).
Then
\[ (\mathcal{H}, \langle \cdot \mid \cdot \rangle) \in \text{Hilb} \left( \mathbb{R}^{(a, b)} \right) . \]
Let \( H \) be the reproducing kernel of \( \mathcal{H} \). We have:
\[ \forall s, t \in (a, b) \ , \ H(s, t) = (t - s)_+ + t - a = \text{Min} (t - a, s - a) \]

3.4.3. Example 3 (Fourier kernel).
Let \( \mathcal{H} = \left\{ x \in \mathcal{H}^1(0, 2\pi) ; \ x(0) = x(2\pi) \text{ and } \int_0^{2\pi} x(s) \, ds = 0 \right\} \) and:
\[ \forall x, y \in \mathcal{H} \ , \ \langle x \mid y \rangle = \int_0^{2\pi} x'(s) \cdot y'(s) \, ds \text{. Then :} \]
\[ (\mathcal{H}, \langle \cdot \mid \cdot \rangle) \in \text{Hilb} \left( \mathbb{R}^{(0, 2\pi)} \right) . \]
Let \( H \) be the reproducing kernel of \( \mathcal{H} \). We have:
\[ \forall s, t \in (0, 2\pi) \ , \ H(s, t) = (\pi)^{-1} \left( \sum_{p=1}^{\infty} \frac{\cos(p(t - s))}{\pi p^2} \right) \]

3.4.4. Example 4 (the kernel is an operator).
Let \( n \in \mathbb{N}^* \), \( \Omega \) an (non void) open subset of \( \mathbb{R}^n \) and \( \alpha \in \mathbb{N}^* \). Let \( \mathcal{H}^\alpha(\Omega) \) be the space of distributions which belong to \( L^2(\Omega) \) and which their derivatives of order less or equal to \( \alpha \) belong to \( L^2(\Omega) \). We set :
\[ \forall x, y \in \mathcal{H}^\alpha(\Omega) \ , \ \langle x \mid y \rangle_\alpha = \sum_{|p| \leq \alpha} \left( \int_{\Omega} a_p D^p x(\zeta) \cdot D^p y(\zeta) \, d\zeta \right) \]
where \( p = (p_1, ..., p_n) \) \( |p| = p_1 + ... + p_n \). Then :
\[ (\mathcal{H}^\alpha(\Omega), \langle x \mid y \rangle_\alpha) \in \mathcal{D}'(\Omega) . \]
We denote by \( \mathcal{H}^{-\alpha}(\Omega) \) the topological dual of \( (\mathcal{H}^\alpha(\Omega), \langle x \mid y \rangle_\alpha) \) which belongs to \( \mathcal{D}'(\Omega) \). So, it is proved that the hilbertian kernel of \( \mathcal{H}^{-\alpha}(\Omega) \) (resp. \( \mathcal{H}^0(\Omega) \), \( \mathcal{H}^\alpha(\Omega) \)) is the operator
\[ D = \sum_{|p| \leq \alpha} \left( (-1)^{|p|} a_p D^{2p} \right) \]
(resp. its Green operator for the Dirichlet or Neumann problem associated to \( D \) and the open set \( \Omega \).)
3.4.5. Example 5 (the complex case).
Let \( \Omega \) be a bounded (non void) open subset of \( \mathbb{C} \) simply connected. Given \( m \in \mathbb{N}^* \), we denote by \( \mathcal{A}_m(\Omega) \) the set of analytical functions \( f \) on \( \Omega \), such that:
\[
\int_{\Omega} \sum_{j=0}^{m} |f^{(j)}(z)|^2 \, dz < +\infty
\]
where \( z = x+iy \) and \( \mathcal{A}_0(\Omega) = L^2(\Omega) \).

Proposition 9. We suppose that \( \Omega \) is a disc centred at the origin with radius \( \rho \).

(i) We denote by \( H_0 \) the reproducing kernel of \( L^2(\Omega) \). Then:
\[
\forall t, z \in \Omega \ , \ H_0(t,z) = (\pi \rho^2)^{-1} \left( 1 - \frac{|z|^2}{\rho^2} \right)^{-1}
\]

(ii) We denote by \( H_1 \) the reproducing kernel of the set: \( \{ f \in \mathcal{A}_1(\Omega) ; f(\zeta_1) = 0 \, \text{ where } \zeta_1 \in \Omega \} \). Then:
\[
\forall t, z \in \Omega \ , \ H_1(t,z) = - (\pi)^{-1} \left[ \log \left( 1 - \frac{|z|^2}{\rho^2} \right) - \log \left( 1 - \frac{|\zeta_1|^2}{\rho^2} \right) - \log \left( 1 - \frac{|z|^2}{\rho^2} \right) - \log \left( 1 - \frac{|\zeta_1|^2}{\rho^2} \right) \right]
\]

3.4.6. Tensor product of Hilbert spaces.
Let \( (E_j, \langle \cdot | \cdot \rangle_j) \), \( j = 1, \ldots, m \), \( m \in \mathbb{N}^* \), be \( m \) Hilbert spaces and \( E = E_1 \otimes \ldots \otimes E_m \).
We denote by \( \langle \cdot | \cdot \rangle_\otimes \) the scalar product on \( E \) such that:
\[
\forall x_j, y_j \in E_j \ , \ j = 1, \ldots, m \ , \ \langle \otimes_{j=1}^m x_j | \otimes_{j=1}^m y_j \rangle_\otimes = \Pi_{j=1}^m \langle x_j | y_j \rangle.
\]
One can prove the

Proposition 10.

(i) \( (E, \langle \cdot | \cdot \rangle_\otimes) \) is a (pre-)Hilbert space.

(ii) If \( \left\{ e_j^k \ ; \ k \in J(j) \right\} \) is an (orthonormal) basis of \( (E_j, \langle \cdot | \cdot \rangle_j) \), \( j = 1, \ldots, m \), then:
\[
\left\{ e_1^{k_1} \otimes \ldots \otimes e_m^{k_m} ; k_1 \in J(1), \ldots, k_m \in J(m) \right\}
\]
is an (orthonormal) basis of \( (E, \langle \cdot | \cdot \rangle_\otimes) \).

3.5. (Hilbertian) Schwartz kernel of the tensor product of Hilbert spaces.

3.5.1. The tensor norm \( \sigma \).
Let \( \mathcal{A}_j \) be an e.l.c.s. which topology denoted \( \mathcal{T}(\mathcal{A}_j) \) is defined by the filtering family of (semi-)norms \( \{ p_{j,k} ; k \in J(j) \} \) for \( j = 1, 2 \).
We call projective (resp. inductive) tensor product of \( (\mathcal{A}_1, \mathcal{T}(\mathcal{A}_1)) \) by \( (\mathcal{A}_2, \mathcal{T}(\mathcal{A}_2)) \) the e.l.c.s.
\[
\mathcal{A}_1 \otimes \mathcal{A}_2 \overset{\text{def}}{=} (\mathcal{A}_1 \otimes \mathcal{A}_2 , \ \{(p_{k,1} \pi p_{l,2}) , (k,l) \in J(1) \times J(2))\})
\]
\( \text{(resp. } \mathcal{A}_1 \varepsilon \mathcal{A}_2 \overset{\text{def}}{=} (\mathcal{A}_1 \otimes \mathcal{A}_2 , \ \{(p_{k,1} \varepsilon p_{l,2}) , (k,l) \in J(1) \times J(2))\}) \)
We shall denote by $\mathcal{A}_1\hat{\pi}\mathcal{A}_2$ (resp.$\mathcal{A}_1\hat{\varepsilon}\mathcal{A}_2$) a completion of $\mathcal{A}_1\pi\mathcal{A}_2$ (resp.$\mathcal{A}_1\varepsilon\mathcal{A}_2$). It can be proved (cf M.A....) that there exists on $\mathcal{A}_1\otimes\mathcal{A}_2$ a reasonable tensor norm $\sigma$ which is the arithmetico-geometric of $\varepsilon$ and $\pi$.

We shall denote by $\mathcal{A}_1\sigma\mathcal{A}_2$ the linear space $\mathcal{A}_1\otimes\mathcal{A}_2$ with the tensor norm $\sigma$ and by $\mathcal{A}_1\hat{\sigma}\mathcal{A}_2$ a completion of $\mathcal{A}_1\sigma\mathcal{A}_2$.

So, we can easily prove the following injections:

$$\mathcal{A}_1\hat{\pi}\mathcal{A}_2 \rightarrow \mathcal{A}_1\hat{\sigma}\mathcal{A}_2 \rightarrow \mathcal{A}_1\hat{\varepsilon}\mathcal{A}_2.$$

That property can be extended to the tensor product $\mathcal{A}_1\otimes\mathcal{A}_2\otimes\ldots\otimes\mathcal{A}_m$ e.l.c.s., $m \in \mathbb{N}^*$, $m \geq 3$.

### 3.6. (Hilbertian) Schwartz kernel of a tensor product.

Let $(E_j,\langle \cdot | \cdot \rangle_j)$, $j=1,2$ two Hilbert spaces (on $\mathbb{K}$) and $E = E_1 \otimes E_2$. From the above paragraph we can deduce that : $(E,\langle \cdot | \cdot \rangle_\otimes) = E_1\sigma E_2$. Moreover, if $(E_j,\langle \cdot | \cdot \rangle_j) \in \text{Hilb}(A_j)$, $j=1,2$ then: $(E,\langle \cdot | \cdot \rangle_\otimes) \in \text{Hilb}(A_1\hat{\sigma}A_2)$. Thus, if $\mathcal{E}_j$ is the Schwartz kernel of $(E_j,\langle \cdot | \cdot \rangle_j)$, $j=1,2$, then:

the kernel of $(E,\langle \cdot | \cdot \rangle_\otimes)$ is equal to $\mathcal{E}_1 \otimes \mathcal{E}_2$ and

$$\forall a^*_k, b_k^* \in A_k^*, \quad k = 1,2, \quad \langle (\mathcal{E}_1 \otimes \mathcal{E}_2)(a^*_1 \otimes a^*_2),(b_1^* \otimes b_2^*)\rangle_\otimes = \langle \mathcal{E}_1 a_1^*, b_1^* \rangle_1 \cdot \langle \mathcal{E}_2 a_2^*, b_2^* \rangle_2.$$

That last property can be extended to the tensor product of the $m$ hilbertian subspaces $(E_j,\langle \cdot | \cdot \rangle_j)$ of the e.l.c.s. $\mathcal{A}_j$ for $j=1,\ldots,m$.

### 3.7. Hilbertian Fock spaces and their Schwartz kernels.

#### 3.7.1. Hilbertian Fock spaces.

Using the same notations as those in the paragraph B4, we set : Let $(E,\langle \cdot | \cdot \rangle)$ be a Hilbert space (on $\mathbb{K}$) and $\| \cdot \| = \sqrt{\langle \cdot | \cdot \rangle}$.

For $m \in \mathbb{N}^*$, $m \geq 2$. We denote by $(E^m,\langle \cdot | \cdot \rangle)$ (resp. $(E^\wedge m,\langle \cdot | \cdot \rangle\wedge)$) the (pre-) Hilbert space such that : $\forall x_j, y_j \in E$, $j=1,\ldots,m$,

$$\langle \bigvee_{j=1}^mx_j | \bigvee_{j=1}^my_j \rangle = \frac{1}{m!} \left( \sum_{\rho \in \mathcal{G}_m} \langle x_1 | y_{\rho(1)} \rangle \ldots \langle x_m | y_{\rho(m)} \rangle \right)$$

resp.

$$\langle \bigwedge_{j=1}^mx_j | \bigwedge_{j=1}^my_j \rangle = \frac{1}{m!} \left( \sum_{\rho \in \mathcal{G}_m} \varepsilon(\rho) \langle x_1 | y_{\rho(1)} \rangle \ldots \langle x_m | y_{\rho(m)} \rangle \right).$$

#### 3.7.2. Schwartz kernels of Hilbertian Fock spaces.

Let $\mathcal{A}$ be an e.l.c.s. We suppose that $(E,\langle \cdot | \cdot \rangle) \in \text{Hilb}(A)$ and that $\mathcal{E}$ is its Schwartz kernel. So, we denote by $\mathcal{E}^m$ and $\mathcal{E}^{\wedge m}$ the Schwartz kernels of $(E^m,\langle \cdot | \cdot \rangle)$ and $E^{\wedge m},\langle \cdot | \cdot \rangle\wedge$ respectively.

We know that : if $A_{\sigma}^{\otimes m} = A\sigma A\sigma \ldots\sigma A$ where $\sigma$ appears $(m-1)$ times, then :
\((E^\vee m, \langle \cdot | \cdot \rangle_\vee)\) and \(E^\wedge m, \langle \cdot | \cdot \rangle_\wedge\) belong to \(\text{Hilb}(\mathcal{A}^\otimes_m)\) and:

\[\forall a_k^*, b_k^* \in \mathcal{A}^*, \quad k = 1, \ldots, m,\]

\[
\langle E^\vee m(a_1^* \vee \ldots \vee a_m^*), (b_1^* \vee \ldots \vee b_m^*) \rangle_\wedge = \frac{1}{m!} \left( \sum_{\rho \in G_m} \langle E a_1^* | b_{\rho(1)}^* \rangle \ldots \langle E a_m^* | b_{\rho(m)}^* \rangle \right)
\]

and

\[
\langle E^\wedge m(a_1^* \vee \ldots \vee a_m^*), (b_1^* \vee \ldots \vee b_m^*) \rangle_\vee = \frac{1}{m!} \left( \det \left[ a_j^*, b_k^* \right]_{1 \leq j,k \leq m} \right).
\]

The following lemma can be proved easily:

**Lemma 1.** Let \(\mathcal{W}_j\) be an e.l.c.s, \((F_j, \langle \cdot | \cdot \rangle_j)\) an hilbertian subspace of \((\mathcal{W}_j)\)
and \(\Phi_j\) its Schwartz kernel for \(j = 1, \ldots, m\). Let \((F, \langle \cdot | \cdot \rangle)\) be the cartesian product:
\((F_1, \langle \cdot | \cdot \rangle_1) \times \ldots \times (F_m, \langle \cdot | \cdot \rangle_m)\). Then : \((F, \langle \cdot | \cdot \rangle) \in \text{Hilb}(\mathcal{W}_1 \times \ldots \times \mathcal{W}_m)\) and if \(\Phi\) is its Schwartz kernel we have:

\[
\Phi = \begin{bmatrix}
\Phi_1 & 0 & 0 & 0 \\
0 & \Phi_2 & 0 & 0 \\
0 & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & \Phi_m
\end{bmatrix}.
\]

So, with the same notations as at the beginning of this paragraph we deduce that:
if \(\mathcal{F}^\perp_\gamma(E, \|\|)\) is the (tensorial) Fock space associated to \((E, \|\|)\) with the r.t.n.
\(\gamma\) and \(\Gamma^\perp_\gamma(E)\) its Schwartz kernel then:
\(\Gamma^\perp_\gamma(E)\) is matrix diagonal block-matrix and its diagonal is equal to \(1_{\mathbb{R}}E E^\perp_2 \ldots E^\perp_m\).

**Exercise:**
Calculate \(\Gamma^\perp_\gamma(E)\) when \(E\) is one of the five examples of Schwartz kernels in the paragraph 4 above.

**Bibliography**

[1] P. Angles, The structure of the Clifford algebra. Advanced applied Clifford algebra, vol.19, n°3-4, p.585-610, 2009.
[2] M. Atteia, Hilbertian kernels and spline functions. Studies of computational Mathematics 4, North Holland,
[3] R. Deheuvels, Formes quadratiques et groupes classiques. Presses universitaires de France, 1981.
[4] A. Grothendieck, Produits tensoriels topologiques et espaces nucléaires. AMS 16, 1955.
[5] P. Lounesto, Clifford algebra and spinors. Cambridge University Press, 1997.
[6] J Neveu, Processus aléatoires gaussiens. Presses de l’Université de Montréal, n°34, 1968.
[7] R. Ryan, Clifford algebra in Analysis and related topics. Studies in Advanced Mathematics, CRC press, 1996.
[8] R. Ryan, Introduction to tensor products of Banach space. Springer, 2002.
[9] L. Schwartz, Sous-espaces hilbertiens d’espaces vectoriels topologiques et noyaux associés. Journal d’Analyse Mathématique, Jerusalem, vol.13, 1964.

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