Quadratic Exponential Semimartingales and Application to BSDE’s with jumps

Nicole El Karoui†  Anis Matoussi ‡  Armand Ngoupeyou§

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Abstract

In this paper, we study a class of Quadratic Backward Stochastic Differential Equations (QBSDE in short) with jumps and unbounded terminal condition. We extend the class of quadratic semimartingales introduced by Barrieu and El Karoui [4] in the jump diffusion model. The properties of these class of semimartingales lead us to prove existence result for the solution of a quadratic BSDE’s.

Keywords: Backward stochastic differential equation, quadratic semimartingales, exponential inequality.

1 Introduction

Backward stochastic differential equations (in short BSDE’s) were first introduced by Bismut in 1973 [8] as equation for the adjoint process in the stochastic version of Pontryagin maximum principle. Pardoux and Peng [46] have generalized the existence and uniqueness result in the case when the driver is Lipschitz continuous. Since then BSDE’s have been widely used in stochastic control and especially in mathematical finance, as any pricing problem by replication can be written in terms of linear BSDEs, or non-linear BSDEs when portfolios constraints are taken into account as in El Karoui, Peng and Quenez [21]. Another direction which has attracted many works in this area, especially in connection with applications, is how to improve the existence/uniqueness conditions of a solution under weaker conditions on the driver. Particularly in those papers it is assumed that \( f \) is just continuous and satisfies a quadratic growth condition. Among them we can quote Kobylanski [34], Lepeltier and San Martin [38] and so on. All of those works are assumed that the terminal condition is bounded and they are based on an exponential change of variable, truncation procedure and comparison theorem of solutions of BSDE’s. Nonetheless, note that in general we do not have uniqueness of the solution. In [34] a uniqueness result is also

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†Laboratoire de Probabilités et Modèles Aléatoires, UPMC, nicole.elkaroui@cmmap.polytechnique.fr.

‡Université du Maine, Institut du Risque et de l’Assurance du Mans, anis.matoussi@univ-lemans.fr.

§Banque Centrale des États de l’Afrique de l’Ouest (BCEAO), ngoupeyouarmand@yahoo.fr.
given by adding a more stronger conditions on the coefficient. This latter model of BSDE’s is very useful in mathematical finance especially when we deal with exponential utilities or risk measure theory especially weather derivatives (see e.g. El Karoui and Rouge [22], Mania and Schweizer [42], Hu, Imkeller and Müller [33], Barrieu and El Karoui [4] and Becherer [5]). Actually it has been shown in [22] that in a market model with constraints on the portfolios, the indifference price is given by the resolution of a BSDE with quadratic growth coefficient. Finally let us point out that control risk-sensitive problems turn into BSDE’s which fall in the same framework in El Karoui and Hamadène [23].

Our work was also motivated by solving a utility maximization problem of terminal wealth with exponential utility function in models involving assets with jumps. Therefore we need to consider Backward Differential Equations with jumps of the form

\[
Y_t = \eta_T + \int_t^T f_s(Y_s, Z_s, U_s) ds - \int_t^T Z_s dW_s - \int_t^T \int_E U(s, x) \tilde{\mu}(ds, dx)
\] (1.1)

where \(\tilde{\mu}\) is a martingale random measure. A solution of such BSDE associated with \((f, \eta_T)\) is a triple of square integrable processes \((Y_t, Z_t, U_t)_{0 \leq t \leq T}\). The standard BSDE’s with jumps driven by Lipschitz coefficient was first introduced by Barles, Buckdahn and Pardoux [2] in order to give a probabilistic interpretation of viscosity solution of semilinear integral-Partial equations. Afterwards the case of BSDE’s with jumps and quadratic coefficient was studied by Becherer [7] and Morlais [45] in the context of exponential utility maximization problem in model involving jumps. In the both papers ([7], [45]), the authors have used in the case of bounded terminal condition the Kobylanski method in the jump setting. As a consequence, they obtain that the state process \(Y\) and the jump components \(U\) of the BSDE solution are uniformly bounded, and that the martingale component is a BMO-martingale. Moreover, the so-called Kobylanski method is based on analytical point view inspired from Boccardo, Murat and Puel paper [10] and it is based on the exponential change of variables, truncation procedure and stability theorem. Therefore, one of the main difficulty in this method is the proof of the strong convergence in the martingale part approximation. More recently Tevzadze [43] proposed a new different method to get the existence and uniqueness of the solution of quadratic BSDE’s. The method is based on a fixed point theorem but for only bounded terminal condition with small \(L^\infty\)-norm.

Our main task in this paper is to deal with quadratic BSDE’s with non-bounded terminal valued and jumps. Our point of view is inspired from Barrieu and El Karoui [4] for their study in the continuous case. By adopting a forward point of view, se shall characterize first a solution of BSDE’s as a quadratic Itô semimartingale \(Y\), with a decomposition satisfying the quadratic exponential structure condition \(q_{exp}(l, c, \delta)\), where the term exponential refers to the exponential feature of the jump coefficient which appears in the generator of the BSDE. More precisely, we assume that: there exists nonnegative processes constants \(c\) , \(\delta\) and \(l\) such

\[
-l_t - c_t|y| - \frac{1}{2}\delta |z|^2 - \frac{1}{\delta} j_t(-\delta u) \leq f(t, y, z, u) \leq l_t + c_t|y| + \frac{1}{2}\delta |z|^2 + \frac{1}{\delta} j_t(\delta u), \ a.s. \quad (1.2)
\]

where \(j_t(u) = \int_E (e^{u(x)} - u(x) - 1)\xi(t, x)\lambda(dx)\). The canonical structure \(q_{exp}(0, 0, \delta)\) will play a essential role in the construction of the solution associated to generate \(d_{exp}(l, c, \delta)\) structure
condition. The simplest generator of a quadratic exponential BSDE, called the canonical generator, is defined as \( f(t,y,z,u) = q_0(y,z,u) = \delta^2 |z|^2 + \frac{1}{2} f(\delta u) \). For a given random variable \( \psi_T \), we call entropic process, the process defined as \( \rho_{\delta,t}(\eta_T) = \frac{1}{\delta} \ln \mathbb{E} \left[ \exp(\delta \eta_T) \mid \mathcal{F}_t \right] \) which is a solution of the canonical BSDE’s associated to the coefficient \( q_0 \) and final condition \( \psi_T \). This is an entropic dynamic risk measure which have been studied, by Barrieu and El Karoui in \([5]\). The backward point of view of our approach permits to relate the quadratic BSDEs to a quadratic exponential semimartingale with structure condition \( q_{exp}(l,a,\delta) \), using the entropic processes. Namely, a semimartingale \( X \) with non bounded terminal condition \( \eta_T \) and satisfying the structure condition \( q_{exp}(l,a,\delta) \), yields the following dominated inequalities \( \rho_{\delta,t}(\underline{U}_T) \leq Y_t \leq \rho_{\delta,t}(\overline{U}_T) \), where \( \underline{U}_T \) and \( \overline{U}_T \) are two random variable depending only on \( l, a, \delta \) and \( \eta_T \). In the continuous setting, Briand and Hu \([11]\) prove implicitly the entropy inequalities in the proof of the existence of the solution of a quadratic BSDE, using Kobylianski method and localization procedure.

The main goal in our approach is then to deduce, from this dominated inequalities, a structure properties on the martingale part and the finite variation part of \( X \). Indeed, we obtain the canonical decomposition of an entropic quasimartingale which is a semimartingale which satisfies the entropy inequalities; as a canonical quadratic semimartingale part plus an predictable increasing process. This Doob type decomposition help us to define a general quadratic exponential semimartingale as a limit of a sequence of canonical quadratic semimartingale plus a sequence of an increasing process. Then, from the stability theorem for forward semimartingales given by Barlow and Protter \([3]\), we prove the existence of the solution of a quadratic exponential BSDE associated to \( (f, \eta_T) \) for a coefficient \( f \) satisfying the structure condition \( q_{exp}(l,a,\delta) \) and for non-bounded terminal condition \( \eta_T \). Finally, we have to mention that it is important to compare our approach with that used by Peng in \([47, 49, 50]\) within the representation theorem of small \( g \)-expectation in terms of a BSDE’s with coefficient \( g \) which admits a linear growth condition in \( z \). Peng’s approach is based on the notion of martingale associated to a nonlinear expectation, Monotonic limit theorem, a nonlinear Doob-Meyer’s decomposition Theorem (see e.g. \([48]\)). Moreover, Peng obtained the representation theorem for the nonlinear expectation which is dominated by a structure nonlinear expectation solution of BSDE’s with coefficient specially by \( g_\mu(y,z) = \mu(|y| + |z|) \). Barrieu and El Karoui in \([5]\) have extended this representation theorem for a dynamic convex risk measure in terms of quadratic BSDE’s with convex coefficient \( g \) which depends only in \( z \). Our approach is an extension of the Peng’s results in the more naturel framework of quadratic exponential semimartingale.

The paper is structured as follows: in a second section, we give a model and preliminary notation. In the third section, we define the quadratic exponential semimartingale and we study the entropic quadratic exponential semimartingale. In particular, we give the characterization of an entropic quasimartingale and its Doob decomposition. Then, a stability results of this class of semimartingale are given in the fourth section. The fifth section is dedicated to give application of the quadratic exponential semimartingale to prove existence result for a class of QBSDE’s associated to \( (f, \eta_T) \) where the coefficient \( f \) satisfies the structure condition \( q(l,a,\delta) \) and for non-bounded terminal condition \( \eta_T \).
2 Model and Preliminaries

This section sets out the notations and the assumptions that are supposed in the sequel. We start with a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with finite horizon time $T < +\infty$ and a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$ satisfying the usual conditions of right continuity and completeness such that we can take all semimartingales to have right continuous paths with left limits. For simplicity, we assume $\mathcal{F}_0$ is trivial and $\mathcal{F} = \mathcal{F}_T$. Without losing any generality we shall work with a random measure to characterize the jumps of any quasi-left continuous process $X$. Let $(\mathbb{E}, \mathcal{E})$ a measurable space, let define on the stochastic basis, a random measure left continuous $\mu(\omega, dt \times dx)$ on $(\Omega, \mathcal{F})$ in $([0,T] \times E, \mathcal{B}([0,T]) \otimes \mathcal{E})$:

$$\mu^X(\omega, dt, dx) = \sum_{s > 0} 1_{\{\Delta X_s(\omega) \neq 0\}} \delta_{(s, X_s(\omega))}(dt, dx).$$

where $\delta_a$ is the Dirac measure on $a$. Moreover the dual predictable projection $\nu^X$ of $\mu^X$ exists and it is called Lévy system of $X$ (see Yor [52] for more details). For simplicity we note $\mu = \mu^X$ and $\nu = \nu^X$. Define the measure $\mathbb{P} \otimes \nu$ on $(\widetilde{\Omega}, \widetilde{\mathcal{F}}) = (\Omega \times [0,T] \times E, \mathcal{F} \otimes \mathcal{B}([0,T]) \otimes \mathcal{E})$ by:

$$\mathbb{P} \otimes \nu(\widetilde{B}) = \mathbb{E} \left[ \int_{[0,T] \times E} 1_{\widetilde{B}}(\omega, t, e) \nu(\omega, dt, de) \right], \quad \widetilde{B} \in \widetilde{\mathcal{F}}.$$

Let $\mathcal{P}$ denote the predictable $\sigma$-field on $\Omega \times [0,T]$ and define $\mathcal{P} = \mathcal{P} \otimes \mathcal{E}$, for any $\mathcal{P}$-measurable function $y$ with values in $\mathbb{R}$; we define:

$$y.\mu_t = \int_0^t \int_E y(w, s, x) \mu(w, ds, dx), \quad \text{and} \quad y.\nu_t = \int_0^t \int_E y(w, s, x) \nu(w, ds, dx).$$

Let denote by $\mathcal{G}_{loc}(\mu)$, the set of $\mathcal{P}$- measurable functions $H$ with values in $\mathbb{R}$ such that

$$|H|^2.\nu_t < \infty, \quad a.s.$$

Moreover if $|H|.\nu_t < +\infty$ a.s, $H.\bar{\mu} := H.(\mu - \nu) = H.\mu - H.\nu$ is a local martingale and we assume the following representation theorem for any local martingale $M$:

$$M = M_0 + M^c + M^d.$$

where $M^c$ is the continuous part of the martingale and $M^d$ is the discontinuous part, moreover there exists $U \in \mathcal{G}_{loc}(\mu)$ such that $M^d = U.(\mu - \nu)$.

We first introduce the following notations:

- $\mathcal{M}_0^\mathcal{P}$ is the set of martingale $M$ such that $M_0 = 0$ and $\mathbb{E} \left[ \sup_{t \leq T} |M_t|^p \right] < +\infty$.
- $\mathcal{D}_{\text{exp}}$ is the set of local semimartingales $X$ such that $\exp(X) \in \mathcal{D}$ where $\mathcal{D}$ (see [13], [14] for the definition).

$\mathcal{U}_{\text{exp}}$ is the set of local martingales $M$ such that $\mathcal{E}(M)$ is uniformly integrable.
3 Quadratic exponential semimartingales

In all our work, we shall consider the class of quasi-left continuous semimartingales $X$ with canonical decomposition $X = X_0 - V + M$, with $V$ a continuous predictable process with finite total variation $|V|$, $M$ is a càdlàg local martingale satisfying the decomposition $M = M^c + M^d$ with $M^c$ is the continuous part of the martingale $M$ and $M^d = U \tilde{\mu}$ for some $U \in \mathcal{G}_{loc}(\mu)$ is the purely discontinuous part. The quadratic exponential semimartingales are the generalization of the quadratic semimartingales in jump diffusion models. The extra term in "exponential" comes from jumps and lead us to generalize the results given by [1].

Definition 3.1. The process $X$ is a local quadratic exponential semimartingale if there exists two positive continuous increasing processes $\Lambda$ and $C$ and a positive constant $\delta$ such that the processes $\delta M^c + (e^{\delta U} - 1)\tilde{\mu}$, $-\delta M^c + (e^{-\delta U} - 1)\tilde{\mu}$ are still local martingales and the the finite variation of $X$ satisfies the structure condition $Q(\Lambda, C, \delta)$:

$$\frac{\delta}{2}d\langle M^c \rangle_t - \frac{1}{\delta}d\Lambda_t - |X_t|dC_t - \frac{1}{\delta}d\tilde{\mu}_t(-\delta \Delta M^d_t) \ll dV_t \ll \frac{\delta}{2}d\langle M^c \rangle_t + d\Lambda_t + |X_t|dC_t + \frac{1}{\delta}d\tilde{\mu}_t(\delta \Delta M^d_t)$$

The process $j(\gamma \Delta M^d_t)$ represents the predictable compensator of the increasing process $A^\gamma := \sum_{s\leq t}(e^{\gamma \Delta M^d_s} - \gamma \Delta M^d_s - 1) < +\infty$ a.s. $B \ll A$ stands for $A - B$ is an increasing process.

Remark 3.1. (About the dual predictable compensator)

- Before studying the properties of this class of local semimartingale, let first remark that for all $\gamma \in \{-\delta, \delta\}$, the increasing càdlàg process $j(\gamma \Delta M)$ is continuous applying Chap IV $T[40]$ Dellacherie[13]. Moreover using representation theorem of the discontinuous martingale $M^d = U \tilde{\mu}$, then:

$$j_t(\gamma \Delta M^d_t) = (e^{\gamma U} - \gamma U - 1)\nu_t.$$

- Let remark that for $a = e^{\delta U} - 1$, $b = e^{-\delta U} - 1$ since $-2ab \leq a^2 + b^2$, we find

$$2[(e^{\delta U} - \delta U - 1) + (e^{-\delta U} + \delta U - 1)] \leq |e^{\delta U} - 1|^2 + |e^{-\delta U} - 1|^2$$

Since by assumption the processes $\delta M^c + (e^{\delta U} - 1)\tilde{\mu}$, $-\delta M^c + (e^{-\delta U} - 1)\tilde{\mu}$ are local martingales, the processes $|e^{\delta U} - 1|^2\nu_t$ a.s and $|e^{-\delta U} - 1|^2\nu_t < +\infty$ a.s, therefore the predictable compensator $j(\gamma \Delta M^d)$ of $A^\gamma$ is well defined for $\gamma \in \{-\delta, \delta\}$.

To understand better the class of local quadratic exponential semimartingales and their properties, we divide the class in three classes:

- The first class (The canonical class), where the finite variation part of $X$ satisfies:

$$V_t = \frac{1}{2}\langle M^c \rangle_t + j_t(\Delta M^d_t) \text{ or } V_t = -\frac{1}{2}\langle M^c \rangle_t - j_t(\Delta M^d_t)$$

- The second class (The class $Q(0,0,1)$), where the finite variation part of $X$ satisfies:

$$-j_t(-\Delta M^d_t) - \frac{1}{2}\langle M^c \rangle_t \ll V_t \ll \frac{1}{2}\langle M^c \rangle_t + j_t(\Delta M^d_t)$$
The third class (The general class $Q(\Lambda, C, \delta)$), where the finite variation part of $X$ satisfies:

$$-\delta^2 \langle M^c \rangle_t - \frac{1}{\delta} \Lambda_t - |X| \ast C_t - \frac{1}{\delta} j_t (-\delta \Delta M^d_t) \ll V_t \ll \frac{\delta}{2} \langle M^c \rangle_t + \frac{1}{2} \Lambda_t + |X| \ast C_t + \frac{1}{2} j_t (\delta \Delta M^d_t)$$

3.1 The canonical class

3.1.1 The exponential of Doléans-Dade

We describe the relation between the exponential transform of a first class of local quadratic exponential semimartingale and the exponential of Doléans-Dade. Let first recall that for any càdlàg local semimartingale $X$, the exponential of Doléans-Dade $Z$ solving the EDS $dZ_t = Z_{t-} dX_t$, $Z_0 = 1$ is given by:

$$Z_t = \mathbb{E}(X_t) = \exp(X_t - \langle X^c \rangle_t) \prod_{s \leq t} (1 + \Delta X_s) e^{-\Delta X_s}, \quad t \geq 0. \quad (3.3)$$

This formula is given by the Ito’s formula for discontinuous processes see Appendix (Theorem 6.8, Corollary 6.1) and Yor [52] for more details. We deduce that for a local martingale $M$, such that $\Delta M > -1$, the exponential of $M$ is a positive local martingale and there is some relation between exponential of a canonical quadratic exponential semimartingale and Doléans-Dade of some local martingale.

**Proposition 3.1.** (Doléans Dade martingale and canonical quadratic semimartingale). Let $\bar{M} = \bar{M}^c + \bar{U}, \bar{\mu}$ and $\bar{M} = \bar{M}^c + \bar{U}, \bar{\mu}$ two càdlàg local martingales such that $\bar{M}^c + (e^{\bar{U}} - 1), \bar{\mu}$ and $-\bar{M}^c + (e^{-\bar{U}} - 1), \bar{\mu}$ are still càdlàg local martingales. Let define the canonical local quadratic exponential semimartingale:

$$r(\bar{M}) = r(\bar{M}_0) + \bar{M}_t - \frac{1}{2} \langle \bar{M}^c \rangle_t - (e^{\bar{U}} - \bar{U} - 1) \nu_t,$$

$$r(\bar{M}) = r(\bar{M}_0) + \bar{M}_t + \frac{1}{2} \langle \bar{M}^c \rangle_t + (e^{-\bar{U}} + \bar{U} - 1) \nu_t$$

then we find the following processes:

$$\exp[r(\bar{M}) - r(\bar{M}_0)] = \mathcal{E}(\bar{M}^c + (e^{\bar{U}} - 1), \bar{\mu})$$

and

$$\exp[-r(\bar{M}) + r(\bar{M}_0)] = \mathcal{E}(-\bar{M}^c + (e^{-\bar{U}} - 1), \bar{\mu})$$

are positive local martingales.

**Proof.** We apply the Doléans-Dade exponential formula (3.3) with $\bar{X} = \bar{M}^c + (e^{\bar{U}} - 1), \bar{\mu}$ and $\underline{X} = -\bar{M}^c + (e^{-\bar{U}} - 1), \bar{\mu}$, and we find the expected results. \qed

**Definition 3.2.** ($Q$- local martingale) A local semimartingale $X$ is a $Q$-local martingale if $\exp(X)$ is a positive local martingale.

The canonical local quadratic exponential semimartingales $\bar{r}(\bar{M})$ and $-\underline{r}(\bar{M})$ defined above are $Q$- local martingales.
3.1.2 The entropic risk measure

The canonical local quadratic exponential semimartingales \( \tilde{r}(M) \) and \( r(M) \) are \( \mathcal{Q} \)-local martingales, we can find more conditions on the local martingales \( \tilde{M} \) and \( M \) to get the uniform integrability condition of these semimartingales. Let first denote by \( \mathcal{U}_{\text{exp}} \) the set of local martingales \( M \) such that \( \mathcal{E}(M) \) is uniformly integrable and \( \mathcal{D}_{\text{exp}} \) the set of local semimartingale such that \( \exp(X) \in \mathcal{D} \). The sufficient condition that a local martingale \( M = M^c + U,\tilde{\mu} \) belongs to \( \mathcal{U}_{\text{exp}} \) is given in Lepingle and Mémin \cite{Lepingle-Memin} Theorem IV.3:

\[
\mathbb{E} \left[ \exp \left\{ \frac{1}{2} (M^c)_\tau + ((1 + U) \ln(1 + U) - U) . \nu_\tau \right\} \right] < +\infty.
\] (3.4)

where \( \tau = \inf \{ t \geq 0, \mathcal{E}(M) = 0 \} \). This condition is sufficient and not necessary, another sufficient condition for a local semimartingale \( X \) to belong to \( \mathcal{D}_{\text{exp}} \) is satisfying if there exists a positive uniformly integrable martingale \( M \) such that \( \exp(X) \leq M \). In particular these sufficient conditions are satisfying for the dynamic entropic risk measure (see Barrieu and El Karoui for more details\cite{Barrieu-ElKaroui}).

**Proposition 3.2.** Let consider the fixed horizon time \( T > 0 \) and \( \psi_T \in \mathcal{F}_T \) such that \( \exp(|\psi_T|) \in L^1 \) and consider the two dynamic risk measures:

\[
\tilde{\rho}(\psi_T) = \ln \left[ \mathbb{E}(\exp(\psi_T) | \mathcal{F}_t) \right], \quad \text{and} \quad \rho(\psi_T) = -\ln \left[ \mathbb{E}(\exp(-\psi_T) | \mathcal{F}_t) \right]
\]

There exists local martingales \( \tilde{M} = \tilde{M}^c + \tilde{U},\tilde{\mu} \) and \( M = M^c + U,\tilde{\mu} \) such that:

\[
- d\tilde{\rho}(\psi_T) = -dM_t + \frac{1}{2} d(M^c)_t + \int_{\mathcal{E}} (e^{U(s,x) - \tilde{U}(s,x) - 1}) . \nu(dt, dx), \quad \tilde{\rho}_T(\psi_T) = \psi_T
\]

\[
- d\rho(\psi_T) = -dM_t - \frac{1}{2} d(M^c)_t - \int_{\mathcal{E}} (e^{-U(s,x) + \tilde{U}(s,x) - 1}) . \nu(dt, dx), \quad \rho_T(\psi_T) = \psi_T
\]

Moreover the local martingales \( \tilde{M}^c + (e^\tilde{U} - 1),\tilde{\mu} \) and \( -M^c + (e^{-\tilde{U}} - 1),\tilde{\mu} \) belong to \( \mathcal{U}_{\text{exp}} \). The dynamic risk measures \( \tilde{\rho}(\psi_T) \) and \( \rho(\psi_T) \) are uniformly integrable canonical quadratic exponential semimartingales.

**Proof.** We have \( \exp(\tilde{\rho}(\psi_T)) = \mathbb{E}(\exp(\psi_T) | \mathcal{F}_t) \) which is a positive uniform integrable martingale since \( \exp(|\psi_T|) \in L^1 \) then there exists a martingale \( \tilde{X} \in \mathcal{U}_{\text{exp}} \) satisfying \( \Delta \tilde{X} > -1 \) such that \( \exp(\tilde{\rho}(\psi_T)) = \mathcal{E}(\tilde{X}_t) \). Using martingale representation Theorem there exists a continuous martingale \( M^c \) and a process \( U \) satisfying \( e^U - 1 \in \mathcal{G}_{\text{loc}}(\mu) \) such that \( \tilde{X} = M^c + (e^U - 1),\tilde{\mu} \). Therefore we find \( \exp(\tilde{\rho}(\psi_T)) = \mathcal{E}(\tilde{M}_t^c + (e^U - 1),\tilde{\mu}_t) = \exp(\tilde{r}(M_t)) \). We use the same arguments to prove that there exists a martingale \( X = -M^c + (e^{-U} - 1),\tilde{\mu} \in \mathcal{U}_{\text{exp}} \) such that \( \exp(-\rho(\psi_T)) = \mathcal{E}(-M^c + (e^{-U} - 1),\tilde{\mu}_t) = \exp(-r(M_t)) \). \( \square \)

We adopt a forward and backward point of view to describe the canonical local quadratic exponential semimartingales class. In the forward point of view, we give condition of some martingales using Doléans-Dade exponential formula to find that for any canonical local quadratic exponential semimartingale \( X \), \( \exp(X) \) or \( \exp(-X) \) is a local martingale. In the backward point of view, we fix a terminal condition \( X_T \in \mathcal{F}_T \) such that \( \exp(|X_T|) \in L^1 \), then we can prove that some dynamic entropic risk measures of \( \psi_T \) belongs to canonical
quadratic exponential semimartingale class. In this point of view, we do not make assumption on the martingale part of the canonical semimartingale to satisfy the Lepingle and Mémin condition since the exponential condition on the terminal condition is sufficient to find uniform integrability condition.

3.2 The second class: \( Q(0, 0, 1) \)

3.2.1 The exponential transform

In the first part, we use the Doléans-Dade formula to explain how the canonical local quadratic exponential semimartingale can be represented using an exponential transform. The same technics can be developed for \( Q(0, 0, 1) \)-local semimartingale using the multiplicative decomposition Theorem studied by Meyer and Yoeurp \( [44] \) which stands that for any càdlàg positive local submartingale \( Z \) there exists an predictable increasing process \( A(A_0 = 0) \) and a local martingale \( M (\Delta M > -1, M_0 = 0) \) such that:

\[
Z_t = Z_0 \exp(A_t) \mathcal{E}(M_t), \quad t \geq 0.
\]

**Theorem 3.1.** Let \( X \) a càdlàg process, \( X \) is a \( Q(0, 0, 1) \)-local semimartingale if and only if \( \exp(X) \) and \( \exp(-X) \) are local submartingales. In both cases, \( X \) is a càdlàg process.

**Proof.** Let consider a \( Q(0, 0, 1) \)-local semimartingale \( X \) with canonical decomposition \( X = X_0 - V + M \) where \( V \) is the finite variation part of \( X \) (continuous) and \( M \) is a local martingale, then there exists \( U \in \mathcal{G}_{\text{loc}}(\mu) \) such that \( M = M^c + U.\tilde{\mu} \). Applying Ito’s formula, we find the decomposition of \( Z = \exp(X) \):

\[
d\tilde{Z}_t = \tilde{Z}_{t-} \left[ \frac{dM_t^c}{2} + \int_{E} (e^{U(t,x)} - 1) \tilde{\mu}(dt, dx) - dV_t + \frac{1}{2} d(M^c)_t + \int_{E} (e^{U(t,x)} - U(t,x) - 1) \nu(dt, dx) \right]
\]

Since \( X \) is a \( Q(0, 0, 1) \)-semimartingale then \( A = -V + \frac{1}{2} (M^c) + (e^U - U - 1) \nu \) is an increasing continuous predictable process. Therefore the process \( Z = \exp(X) \) is a positive local submartingale and satisfies the following Meyer and Yoeurp multiplicative decomposition:

\[
\exp(X_t - X_0) = \exp(A_t) \mathcal{E}(M_t^c + (e^U - 1)\tilde{\mu}_t), \quad t \geq 0.
\]

We use the same arguments to prove that \( \exp(-X) \) is a local positive submartingale. Let now assume that \( \exp(X) \) and \( \exp(-X) \) are local submartingales where \( X \) is a càdlàg process. Using Meyer and Yoeurp multiplicative decomposition, there exist local martingales \( \tilde{M}, \tilde{M} \) and increasing predictable processes \( \tilde{A}, \tilde{A} \) such that \( \exp(X_t - X_0) = \exp(\tilde{A}_t) \mathcal{E}(\tilde{M}_t) \) and \( \exp(-X_t + X_0) = \exp(\tilde{A}_t) \mathcal{E}(\tilde{M}_t) \). Using the representation martingale Theorem, there exist \( \tilde{U}, \tilde{U} \in \mathcal{G}_{\text{loc}}(\mu) \) and continuous local martingales \( \tilde{M}^c, \tilde{M}^c \) such that \( \tilde{M} = \tilde{M}^c + (e^\tilde{U} - 1)\tilde{\mu} \) and \( \tilde{M} = \tilde{M}^c + (e^\tilde{U} - 1)\tilde{\mu} \). Hence we find \( \exp(X - X_0) = \exp(\tilde{A}) \mathcal{E}(\tilde{M}) \) and \( \exp(-X + X_0) = \exp(\tilde{A}) \mathcal{E}(\tilde{M}) \) and we get

\[
X_t - X_0 = \tilde{A}_t + \tilde{M}_t - \frac{1}{2} (\tilde{M}^c) - (e^\tilde{U} - U - 1).\nu_t \quad \text{and} \quad -X_t + X_0 = \tilde{A}_t + \tilde{M}_t - \frac{1}{2} (\tilde{M}^c) - (e^\tilde{U} - U - 1).\nu_t.
\]

Using the uniqueness of the representation of the semimartingale \( X \), we deduce, \( \tilde{M} = -\tilde{M} \), then we find \( \tilde{A}_t + \tilde{A}_t = (\tilde{M}^c)_t + (e^\tilde{U} - U - 1).\nu_t + (e^{-U} + U - 1).\nu_t \). The process \( \tilde{A} \) and \( \tilde{A} \)
are continuous, moreover from Radon Nikodym’s Theorem, there exists a predictable process with \(0 \leq \alpha_t \leq 2\) such that 
\[
d\tilde{A}_t = \frac{\alpha_t}{2}d\langle M^c \rangle_t + (e^{\tilde{U}} - \tilde{U} - 1).\nu_t + (e^{-\tilde{U}} + \tilde{U} - 1).\nu_t
\]
Therefore the process \(X\) satisfies the dynamics 
\[
dX_t = dM_t - dV_t
\]
where:
\[
dV_t = \left(1 - \frac{\alpha_t}{2}\right)d\langle M^c \rangle_t + \left(\frac{2 - \alpha_t}{2}\right)[(e^{\tilde{U}} - \tilde{U} - 1).\nu_t] - \frac{\alpha_t}{2}d\left[(e^{-\tilde{U}} + \tilde{U} - 1).\nu_t\right]
\]
Since \(0 \leq \alpha_t \leq 2\), the local semimartingale \(X\) satisfies the structure condition \(Q(0,0,1)\). Moreover the finite variation part \(V\) of \(X\) is a predictable continuous process. We deduce \(X\) is \(Q(0,0,1)\)-local semimartingale and that all jumps of \(X\) come from the local martingale part which is càdlàg process.

**Definition 3.3.** Let consider a local semimartingale \(X\), if \(\exp(X)\) is a local submartingale then \(X\) is called \(Q\)-local submartingale.

From Theorem 3.1, any \(Q(0,0,1)\)-local semimartingale is a \(Q\)-local submartingale and the reverse holds true.

### 3.2.2 The entropic submartingales

We are interested to find uniform integrability condition for \(Q(0,0,1)\)-local semimartingales. Since \(Q(0,0,1)\)-local semimartingales are \(Q\)-local submartingales, we use the same technics developed for standard local submartingales. We recall that to prove \(X \in D_{exp}\), it is sufficient to prove there exists a positive martingale \(L \in D\) such that \(\exp(X) \leq L\). To construct the positive martingale \(L\), let first give some useful definitions.

**Definition 3.4.** A process \(X \in D_{exp}\) is called an entropic submartingale if for any stopping times \(\sigma \leq \tau\):

\[
X_\sigma \leq \tilde{\rho}_\sigma(X_\tau), \; \sigma \leq \tau.
\]

where \(\tilde{\rho}\) stands for the usual entropic risk measure defined above. In the same point of view, \(X\) is called a entropic supermartingale if \(-X\) is an entropic submartingale. If \(X\) and \(-X\) are entropic submartingales, \(X\) is called entropic quasi-martingale.

**Theorem 3.2.** Let \(T > 0\) the fixed horizon time and consider a semimartingale \(X = X_0 - V + M^c + U.\bar{\mu}\) such that \(\exp(|X_T|) \in L^1\) then \(X\) is a \(Q(0,0,1)\)-semimartingale \(\in D_{exp}\) if and only if \(X\) and \(-X\) are entropic submartingales. Moreover, in all cases the martingales \(M^c + (e^U - 1).\bar{\mu}\) and \(-M^c + (e^{-U} - 1).\bar{\mu}\) belong to \(U_{exp}\).

**Proof.** Let consider a \(Q(0,0,1)\)-semimartingale \(X = X_0 + M^c + U.\bar{\mu} - V \in D_{exp}\) such that \(\exp(|X_T|) \in L^1\). Since \(X\) is \(Q\)-submartingale we find:

\[
\exp(X_t) \leq E[\exp(X_T)|\mathcal{F}_t] \in D \quad \text{and} \quad \exp(-X_t) \leq E[\exp(-X_T)|\mathcal{F}_t] \in D
\]

and for any stopping times: \(\sigma \leq \tau \leq T\):

\[
X_\sigma \leq \ln(E[\exp(X_\tau)|\mathcal{F}_\sigma]) = \tilde{\rho}_\sigma(X_\tau) \quad \text{and} \quad -X_\sigma \leq \ln(E[\exp(-X_\tau)|\mathcal{F}_\sigma]) = \tilde{\rho}_\sigma(-X_\tau).
\]

then \(X\) and \(-X\) are entropic submartingales. Let prove the reverse, assume \(X\) and \(-X\) are entropic submartingales then for any stopping times \(\sigma \leq \tau\),

\[
\exp(X_\sigma) \leq E[\exp(X_\tau)|\mathcal{F}_\sigma] \leq E[\exp(X_\tau)|\mathcal{F}_\sigma]
\]
and \( \exp(-X_0) \leq \mathbb{E}[\exp(-X_t) | \mathcal{F}_0] \), then \( X \) is a uniformly integrable \( \mathbb{Q} \)-submartingale and from Theorem 3.1 \( X \) is a \( \mathbb{Q}(0, 0, 1) \)-semimartingale. Since \( X \) and \(-X\) belong to \( \mathcal{D}_{\exp} \) then for a fixed horizon time \( T \), \( \exp(X_T) \) and \( \exp(-X_T) \) belong to \( L^1 \) which lead to conclude \( \exp(|X_T|) \in L^1 \). Moreover since \( X \) and \(-X\) are \( \mathbb{Q} \)-submartingales using Meyer-Yoeurp multiplicative decomposition Theorem, there exist increasing processes \( \tilde{A} \) and \( \Lambda \) \((\tilde{A}_0 = 0 \) and \( \Lambda_0 = 0)) \) such that:

\[
\exp(X_t - X_0) = \exp(\tilde{A}_t)\mathcal{E}(M_t^e + (e^U - 1).\tilde{\mu}_t) \quad \text{and} \quad \exp(-X_t + X_0) = \exp(\tilde{A}_t)\mathcal{E}(-M_t^e + (e^{-U} - 1).\tilde{\mu}_t).
\]

Therefore we deduce that \( \mathcal{E}(M_t^e + (e^U - 1).\tilde{\mu}) \leq \exp(X_t - X_0) \) and \( \mathcal{E}(-M_t^e + (e^{-U} - 1).\tilde{\mu}) \leq \exp(-X_t + X_0) \). Since \(|X - X_0| \in \mathcal{D}_{\exp} \), we conclude the martingales \( M^e + (e^U - 1).\tilde{\mu} \) and \(-M^e + (e^{-U} - 1).\tilde{\mu} \in \mathcal{U}_{\exp} \).

To conclude this part, we can make some links with the sublinear \( g \)-expectation of Peng \[51\] since if we define the \( g \)-expectation of \( X \) by \( \mathbb{E}^g(X) \), we can define the submartingale under the \( g \)-expectation. Therefore, we deduce that if \( X \) is \( \mathbb{Q}(0, 0, 1) \)-semimartingale such that \(|X| \in \mathcal{D}_{\exp} \), \( X \) and \(-X\) are submartingales under \( \mathbb{E}^g = \ln[\mathbb{E}(\exp)] \).

### 3.3 General class: \( \mathbb{Q}(\delta, \Lambda, C) \)

#### 3.3.1 The exponential transform

We use some exponential transformations for general \( \mathbb{Q}(\Lambda, C, \delta) \) local quadratic exponential semimartingale such that the new transformed process belong to the class \( \mathbb{Q}(0, 0, 1) \). Therefore, we can apply the same methodology using in the previous sections to find general results for \( \mathbb{Q}(\Lambda, C, \delta) \) local semimartingales.

**Proposition 3.3.** Let consider a \( \mathbb{Q}(\Lambda, C, \delta) \)-local semimartingale \( X = X_0 - V + M^e + U.\tilde{\mu} \) then

1. For any \( \lambda \neq 0 \), the process \( \lambda X \) is a \( \mathbb{Q}(\Lambda, C, \frac{\lambda}{|\lambda|}) \)-local semimartingale and a \( \mathbb{Q}(\lambda \Lambda, C, \delta) \)-local semimartingale when \( \lambda > 1 \).

2. Let define the two transformations:

\[
Y^{\Lambda, C}(X) = X + \Lambda + |X| * C \quad \text{and} \quad \tilde{Y}^{\Lambda, C}(|X|) = e^C |X| + e^C * \Lambda.
\]

then the two processes \( Y^{\Lambda, C}(\delta X) \) and \( \tilde{Y}^{\Lambda, C}(|\delta X|) \) are \( \mathbb{Q} \)-local submartingales.

3. Exponential transformation: Let \( U^{\Lambda, C}(X) \) the transformation

\[
U^{\Lambda, C}_t(e^X) = e^{X_t} + \int_0^t e^{X_s}d\Lambda_s + \int_0^t e^{X_s}|X_s|dC_s.
\]

then \( U^{\Lambda, C}(e^{\delta X}) \) is a positive local submartingale.
Proof. 1. Let consider a $Q(\Lambda, C, \delta)$-local semimartingale $X = X_0 - V + M^c + M^d$ (where $M^d = U, \tilde{\mu}$) and consider $\lambda \neq 0$, hence $\lambda X = \lambda X_0 - \lambda V + \lambda M^c + \lambda M^d$ and $\lambda X$ satisfies the condition

$$
\begin{cases}
-|\lambda| \frac{\delta}{2} d(M^c)_t - \frac{|\lambda|}{\delta} d\Lambda_t - |\lambda| X_t dC_t - |\lambda| \frac{1}{\delta} d \mu_t [-\delta \text{sign} (\lambda) \Delta M^d_t] < \lambda dV_t, \\
\lambda dV_t < \frac{1}{|\lambda|} \frac{\delta}{2} d(M^c)_t + \frac{|\lambda|}{\delta} d\Lambda_t + |\lambda| X_t dC_t + \frac{|\lambda|}{\delta} d \mu_t [\delta \text{sign} (\lambda) \Delta M^d_t].
\end{cases}
$$

Since $j(\delta \Delta M^d) = j\left[\frac{\delta}{|\lambda|} \lambda \Delta M^d\right]$ then we find

$$
\begin{cases}
-\frac{1}{|\lambda|} \frac{\delta}{2} d(M^c)_t - \frac{|\lambda|}{\delta} d\Lambda_t - |\lambda| X_t dC_t - \frac{|\lambda|}{\delta} d \mu_t [-\delta \lambda \Delta M^d_t] < \lambda dV_t, \\
\lambda dV_t < \frac{1}{|\lambda|} \frac{\delta}{2} d(M^c)_t + \frac{|\lambda|}{\delta} d\Lambda_t + |\lambda| X_t dC_t + \frac{|\lambda|}{\delta} d \mu_t [\delta \lambda \Delta M^d_t].
\end{cases}
$$

then $\lambda X$ is a $Q(\Lambda, C, \delta)$-local semimartingale. Moreover for $\lambda > 1$:

$$
\frac{\delta}{2} d(M^c)_t < \frac{1}{\delta} d(M^c)_t \text{ and } \frac{|\lambda|}{\delta} d \mu_t [\lambda \Delta M^d_t] < \frac{1}{\delta} d \mu_t [\lambda \Delta M^d_t]
$$

see Lemma 6.3 in Appendix for more details for this inequality. We find that for $\lambda > 1$, $\lambda X$ is a $Q(|\lambda|\Lambda, C, \delta)$-semimartingale.

2. Let consider the $Y^{\Lambda, C}(\tilde{X}) = \delta X_0 + \tilde{M}_t - \tilde{V}_t$, where $\tilde{M}$ is the local martingale part given by $\tilde{M} = \delta M^c + \delta M^d$ and $\tilde{V}$ the finite variation part given by $\tilde{V} = \delta V - \Lambda - |\delta X| \cdot C$. Since $X$ is $Q(\Lambda, C, \delta)$-local semimartingale we have $d\tilde{V}_t < d \mu_t (\Delta M^d) + \frac{\delta}{2} d(M^c)_t$. We conclude $d\tilde{V}_t < d \mu_t (\Delta M^d) + \frac{\delta}{2} d(M^c)_t$ and the process $A$ defined by $dA_t = -d\tilde{V}_t + d \mu_t (\Delta \tilde{M}^c) + \frac{\delta}{2} d(M^c)_t$ is an increasing process and $Y^{\Lambda, C}(\delta X) = \delta X_0 + \tilde{M} - \frac{\delta}{2} (\Delta M^c) - j(\Delta M^d) + A$ then we conclude $\exp(Y^{\Lambda, C}(\delta X))$ is a local submartingale then it is $Q$-local submartingale. Let prove now the process $Y^{\Lambda, C}(|X|)$ belong to the $Q(0, 0, 1)$-class, let applying the Meyer-Ito’s formula, we find the decomposition:

$$
d e^{C_t} |X_t| = e^{C_t} \left[ |X_t| dC_t - \text{sign}(X_t) dV_t + d \left[ |X_+ - U| - |X_-| \right] \mu_t \right] + d \tilde{M}_t
$$

where $d \tilde{M}_t = \text{sign}(X_t) dM^c_t + d \left[ |X_+ - U| - |X_-| \right] \mu_t$ and $L^X$ stands for the local time of $X$ at $0$. Therefore the decomposition of the semimartingale $Y^{\Lambda, C}(|\delta X|)$ satisfies $d \tilde{Y}^{\Lambda, C}(|X|) = -d\tilde{V}_t + d \tilde{M}_t$ where $\tilde{M} = \delta \tilde{M}$ and

$$
d \tilde{V}_t = -e^{C_t} \left[ \delta X_t dC_t + d \Lambda_t - \text{sign}(X_t) dV_t + d \delta X_t + d[\delta |X_+ - U| - |X_-| \mu_t] \right].
$$

Since the process $X$ is a $Q(\Lambda, C, \delta)$-local semimartingale, the process $A$ defined by $dA_t = \delta |X_t| dC_t + \frac{\delta}{2} d \Lambda_t - \text{sign}(X_t) dV_t + \frac{\delta}{2} d(M^c)_t + \frac{\delta}{2} d \mu_t [\text{sign}(X_t) \delta |\Delta M^d|] + \frac{1}{\delta} d \delta X_t$ is an increasing process. Therefore we get:

$$
-d \tilde{V}_t = e^{C_t} \left[ -\frac{\delta^2}{2} d(M^c)_t - d \mu_t [\text{sign}(X_t) \Delta M_t] + d \left[ \delta |X_+ - U| - |X_-| \mu_t \right].
$$

From Lemma 6.3 (see Appendix for details), for any $k \geq 1$, $j(k \Delta M) \geq k j(\Delta M)$, therefore since $C$ is an increasing process with the initial condition $C_0 = 0$, we
get $j_s[\delta e^{C_i} \text{sign}(X_{s-}) \Delta M_t] - e^{C_i} j_s[\delta \text{sign}(X_{s-}) \Delta M_t] \geq 0$. Moreover for any $s \geq 0$,
$$\frac{\delta^2}{2} (e^{C_i} M^c)_t - \frac{\delta^2}{2} e^{C_i} (M^e)_t \geq 0,$$
then we obtain:
$$-d\tilde{V}_t = -\frac{1}{2} d(e^{C_i} \delta \text{sign}(X_{t-}) M^e)_t - d j_t[\delta e^{C_i} \text{sign}(X_{t-}) \Delta M_t] + d(\delta(|X_+ + U| - |X_-|) \nu_t) + d\tilde{A}_t$$
where $\tilde{A}$ is an increasing process. Finally we get:
$$d\tilde{Y}^{A,C}_t(|X|) = e^{C_i} \delta \text{sign}(X_{t-}) dM^c_t + \int_E e^{C_i} \delta(|X_{t-} + U(t, x)| - |X_{t-}|) \tilde{\mu}(dt, dx)$$
$$-\frac{1}{2} d(e^{C_i} \delta \text{sign}(X_{t-}) M^e)_t - d j_t[\delta e^{C_i}(|X_{t-} + \Delta M_t| - |X_{t-}|)] + d\tilde{A}_t$$
where
$$\tilde{A}_t = \tilde{A}_0 + \int_0^t \int_E [\exp\left(e^{C_i} \delta(|X_{s-} + U(s, x)| - |X_{s-}|)\right) - \exp\left(\delta \text{sign}(X_{s-}) U(s, x)\right)] \nu(ds, dx)$$
Since $|y + u| - |y| \geq \text{sign}(y) u$ we deduce $\tilde{A}$ is increasing then we get:
$$\tilde{Y}^{A,C}_t(|X|) = |\delta X_0| + \tilde{M} - \frac{1}{2} \langle \tilde{M} \rangle + j(\Delta \tilde{M}) + \tilde{A}.$$ 
Therefore, $\exp(\tilde{X}^{A,C})$ is a local submartingale then it is $\mathcal{Q}$-local submartingale.

3. Let apply Ito’s formula to find the decomposition of $U^{A,C}(e^{\delta X})$:
$$dU^{A,C}_t(e^{\delta X}) = e^{\delta X_{t-}} \left[ \delta dM^c_t + d[(e^{\delta U} - 1) \tilde{\mu}_t] - \delta dV_t + \frac{\delta^2}{2} d(M^e)_t + d j_t(\Delta M_t) + |\delta X_t| dC_t \right].$$
Since $X$ is $\mathcal{Q}(A, C, \delta)$-local semimartingale then the process $A$ defined by $dA_t = -dV_t + \frac{\delta}{2} d(M^e)_t + \frac{1}{2} d j_t(\Delta M_t) + |\delta X_t| dC_t$ is an increasing process, we deduce the process $U^{A,C}(e^{\delta X})$ is a positive local submartingale.

**Theorem 3.3.** Let $X$ a càdlàg optionnal process $X$. $X$ is a $\mathcal{Q}(A, C, \delta)$-local submartingale if and only if $\exp[Y^{A,C}(\delta X)]$ and $\exp[Y^{A,C}(-\delta X)]$ are submartingales or equivalently if the processes $U^{A,C}(e^{\delta X})$ and $U^{A,C}(e^{-\delta X})$ are local submartingales. In all cases; $X$ is a càdlàg process.

**Proof.** Let consider a $\mathcal{Q}(A, C, \delta)$-local submartingale $X$, using Proposition [3.3-2], we prove the process $\exp(Y^{A,C}(\delta X))$ is a local submartingale. The same arguments lead us to conclude also that $\exp(Y^{A,C}(-\delta X))$ is a local submartingale since $-X$ as the same structure condition as $X$. Let now consider that the both processes $\exp(Y^{A,C}(\delta X))$ and $\exp(Y^{A,C}(-\delta X))$ are positive submartingales then we can apply the Yoeurp-Meyer decomposition as Theorem [3.1] and conclude there exists continuous local martingales $\tilde{M}^c, \tilde{M}^e$, increasing processes $\tilde{A}, \tilde{A}$ and $\tilde{U}, \tilde{U} \in \mathcal{G}_{\text{loc}}(\mu)$ such that
$$\exp[Y^{A,C}_t(\delta X)] = \exp(\delta X_0) \exp(\tilde{M}_t - \frac{1}{2} \langle \tilde{M}^c \rangle_t - (e^{U} - \tilde{U} - 1) \nu_t + \tilde{A}_t)$$
$$\exp[Y^{A,C}_t(-\delta X)] = \exp(-\delta X_0) \exp(\tilde{M}_t - \frac{1}{2} \langle \tilde{M}^c \rangle_t - (e^{U} - \tilde{U} - 1) \nu_t + \tilde{A}_t)$$
then we find $\delta X_t + \Lambda_t + |X_t| \ast C_t = \delta X_0 + \bar{M}_t - \frac{1}{2}(\bar{M}^c)_t - (e^U - \bar{U} - 1)\nu_t + \bar{A}_t$ and $-\delta X_t + \Lambda_t + |X_t| \ast C_t = -\delta X_0 + \bar{M}_t - \frac{1}{2}(\bar{M}^c)_t - (e^U - \bar{U} - 1)\nu_t + \bar{A}_t$. Therefore $\bar{M} = -\bar{M}$ from uniqueness of the decomposition, moreover $\bar{A}_t + \bar{A}_t = (\bar{M}^c) + (e^U - \bar{U} - 1)\nu_t + (e^{-U} + \bar{U} - 1)\nu_t + 2\Lambda_t + 2|X_t| \ast C_t$. We deduce the both processes $\bar{A}$ and $\bar{A}$ are continuous and from Radon Nikodym Theorem, there exists a predictable process $0$ is a semimartingales. Since the predictable process $0$ is a semimartingales. From uniqueness of the decomposition, moreover $\bar{M} = -\bar{M}$ from uniqueness of the decomposition, moreover $\bar{A}_t + \bar{A}_t = (\bar{M}^c) + (e^U - \bar{U} - 1)\nu_t + (e^{-U} + \bar{U} - 1)\nu_t + 2\Lambda_t + 2|X_t| \ast C_t$. We deduce the both processes $\bar{A}$ and $\bar{A}$ are continuous and from Radon Nikodym Theorem, there exists a predictable process $0$ is a semimartingales. Since the predictable process $0$ is a semimartingales.

In all the rest of the paper, since from a multiplicative transformation (see Proposition 3.3), we can transform the general class $Q(\Lambda, C, \delta)$ to the class $Q(\Lambda, C, 1)$. We can give all results in the class $Q(\Lambda, C) := Q(\Lambda, C, 1)$ without losing any generality.

3.3.2 Uniform Integrable $Q(\Lambda, C)$- semimartingales

We use the entropic submartingales to characterize the integrability condition for $Q(0, 0, 1)$-class. Given an fixed horizon time, we find in this part sufficient condition on the terminal condition to have uniform integrability of general local quadratic exponential semimartingales. First, let give some generalization of entropic submartingales for general $Q(\Lambda, C)$-semimartingales.

**Theorem 3.4.** Let $X$ be a càdlàg process and $T$ a fixed horizon time.

1. Assuming, $\exp(|X_T|) \in L^1$, the process $X$ is a $Q(\Lambda, C)$-semimartingale which belongs to $D_{\exp}$ if and only if for any stopping times $\sigma \leq \tau \leq T$:

$$X_\sigma \leq \rho_{\sigma}(X_\tau + \Lambda_{\sigma, \tau} + |X| \ast C_{\sigma, \tau}) \text{ and } -X_\sigma \leq \rho_{\sigma}(-X_\tau + \Lambda_{\sigma, \tau} + |X| \ast C_{\sigma, \tau}). \quad (3.5)$$

2. Assuming $U^L X \in L^1$, the process $X$ is a $Q(\Lambda, C)$-semimartingale which belongs to $D_{\exp}$ if and only if for any stopping times $\sigma \leq \tau \leq T$:

$$X_\sigma \leq \rho_{\sigma}(X_\tau + \Lambda_{\sigma, \tau} + |X| \ast C_{\sigma, \tau}) \text{ and } -X_\sigma \leq \rho_{\sigma}(-X_\tau + \Lambda_{\sigma, \tau} + |X| \ast C_{\sigma, \tau}).$$
Proof. 1. Let $X$ a $Q(\Lambda, C)$-semimartingales which belongs to the class $D_{exp}$. From Theorem 3.3, $\exp(Y^{\Lambda,C}(X))$ and $\exp(Y^{\Lambda,C}(-X))$ are submartingales which belong to the class $D$. Therefore for any stopping times $\sigma \leq \tau \leq T$:

$$\exp(Y^{\Lambda,C}_\sigma(X)) \leq \mathbb{E} \left[ \exp(Y^\tau \Lambda,C(X)|\mathcal{F}_\sigma) \right] \text{ and } \exp(Y^{\Lambda,C}_\tau(-X)) \leq \mathbb{E} \left[ \exp(Y^\tau \Lambda,C(-X)|\mathcal{F}_\sigma) \right]$$

then the $Q(\Lambda, C)$ semimartingale $X$ satisfies the entropy inequalities. Let assume the inequalities are satisfied then we conclude $\exp(Y^{\Lambda,C}(X))$ and $\exp(Y^{\Lambda,C}(-X))$ are submartingales which belong to the class $D$ then from Theorem 3.3, $X$ is a $Q(\Lambda, C)$ semimartingales which belong to the class $D_{exp}$.

2. We use the same arguments with the positive submartingales $U^{\Lambda,C}(e^X)$ and $U^{\Lambda,C}(e^{-X})$.

The Theorem 3.4 gives sufficient integrable condition for $Q(\Lambda, C)$-semimartingale $X$ such that it belongs to the class $D_{exp}$. We can find another condition using the transformation $\tilde{Y}^{\Lambda,C}(|X|)$ since it is a $Q$-submartingale. Therefore, using the same arguments as assertions in Theorem 3.3, we find $\tilde{Y}^{\Lambda,C}(|X|) \leq \tilde{\rho} \{ \exp(\tilde{Y}^{\Lambda,C}(|X|)) \}$ which is equivalent to the condition given by [4] in the continuous case (see Hypothesis 2.8 [4]):

$$|X_t| \leq \tilde{\rho} \left[ e^{C_{t,T}}|Y_T| + \int_t^T e^{C_{s,s}}d\Lambda_s \right], \quad t \leq T. \quad (3.6)$$

This assumption is a necessary and sufficient condition for the process $\tilde{Y}^{\Lambda,C}(|X|)$ to be in class $D_{exp}$ (the proof is given in Lemma 2.9 of [4]). In the same way, assertions in Proposition 2.10 of [4] still hold since the authors give the result in the general case (without using the continuity of processes). Moreover using the same LLogL Doob-inequality, we can find the same sufficient condition on the terminal value $\tilde{Y}^{\Lambda,C}(|X|)$ such that $|X| \in D_{exp}$.

Proposition 3.4. Let consider an fixed horizon time $T > 0$ and let $L$ be a positive submartingale such that $\max L_T := \max_{t \in [0,T]} L_t \in (1, +\infty)$. For any $m > 0$, let $u_m$ the convex function defined on $\mathbb{R}^+$ defined by $u_m(x) = x - m - m \ln(x)$ and $u(x) := u_1(x)$, the following assertions are satisfied:

1. Using the Doléans Dade representation of positive martingale $L$, $L_t = \mathcal{E}(M^c_t + (e^U - 1)\tilde{\mu})$, $t \leq T$, we find:

$$H_{\text{ent}} := \mathbb{E}[L_T \ln(L_T)] = \mathbb{E} \left[ L_T \left( \frac{1}{2}(M^c)_T + (Ue^U - e^U + 1)\nu_T \right) \right].$$

2. The following sharp inequality holds true:

$$u(\mathbb{E}(\max L_T)) \leq \mathbb{E} \{ L_T \ln(L_T) \}.$$

Moreover, if $L$ is a positive $D$-submartingale, the previous inequality becomes:

$$u_m(\mathbb{E}(\max L_T)) - u_m(L_0) \leq \mathbb{E}[L_T \ln(L_T)] - \mathbb{E}(L_T) \ln(\mathbb{E}(L_T)),$$

where $m = \mathbb{E}(L_T)$.
Proof.:

1. To prove the assertion, let us first prove that the equality

\[ \mathbb{E}(\max L_T) - 1 = \mathbb{E}[L_T \ln(\max L_T)] \]

holds true in our case. From Dellacherie [15] p.375, \( \max L_t(\omega) = L_t(\omega) \) for every jump time \( t \) or every increasing of right of \( s \rightarrow \max L_s(\omega) \). Therefore \( L = \max L \) on the right support of \( d \max L \). Therefore we find \( \max L_t = 1 + \int_0^t d \max L_s = \int_0^t \frac{\omega_s}{\max L_s} d \max L_s \) then \( \mathbb{E}(\max L_T) - 1 = \mathbb{E}[L_T \ln(\max L_T)] \) holds true. From this equality, it is sufficient that \( \max L_T \in L^1 \) to find \( L_T \ln(\max L_T) \in L^1 \). Let assume, \( \max L_T \in L^1 \) and let define the stopping times \( T_K \) such that the positive local martingale \( L_t = \mathcal{E}(M_t + (e^U - 1)\bar{\mu}_t) \leq K \). The stopping times \( T_K \) is increasing and goes to infinity with \( K \). Let define the process \( \mathcal{N}^Q = M^c - \langle M^c \rangle + U.(\bar{\mu} - (e^U - 1)\nu) \) is a martingale with respect to \( \mathcal{Q} = L_T^\mathbb{P} \) and we get:

\[
\mathbb{E} \left[ L_T \left( \frac{1}{2} \langle M^c \rangle_T + (e^U - U - 1)\nu_T \right) \right] = \lim_K \mathbb{E} \left[ L_T \left( \frac{1}{2} \langle M^c \rangle_{T \wedge T_K} + (e^U - U - 1)\nu_{T \wedge T_K} \right) \right]
\]

\[
= \lim_K \mathbb{E} \left[ L_{T \wedge T_K} \left( \frac{1}{2} \langle M^c \rangle_{T \wedge T_K} + (e^U - U - 1)\nu_{T \wedge T_K} \right) \right]
\]

Since \( \mathbb{E}(L_{T \wedge T_K} \mathcal{N}^Q_{T \wedge T_K}) = 0 \), we find:

\[
\mathbb{E} \left[ L_{T \wedge T_K} \ln(L_{T \wedge T_K}) \right] = \mathbb{E} \left[ L_{T \wedge T_K} \left( \frac{1}{2} \langle M^c \rangle_{T \wedge T_K} + U(e^U - 1)\nu_{T \wedge T_K} + (e^U - U - 1)\nu_{T \wedge T_K} \right) \right]
\]

We have \( \mathbb{E} \left[ L_{T \wedge T_K} \ln(L_{T \wedge T_K}) \right] \leq \mathbb{E} \left[ L_{T \wedge T_K} \ln(\max L_{T \wedge T_K}) \right] \leq \mathbb{E} \left[ \max L_T \right] - 1 \leq +\infty \), then we get the result by taking the limit when \( K \) goes to infinity.

2. The proof is done in [4], since authors used the first assertion to prove the result.

\[\square\]

Let \( X \) be a \( Q(\Lambda, C) \)-semimartingale, applying the result of Proposition 3.3 to the positive submartingale \( \exp(Y^{\Lambda, C}(|X|)) \), we conclude if \( \mathbb{E} \left( Y^{\Lambda, C}_T(|X|) \exp[Y^{\Lambda, C}_T(|X|)] \right) \in L^1 \) then we have \( \max \mathbb{E} \left( Y^{\Lambda, C}_T(|X|) \exp[Y^{\Lambda, C}_T(|X|)] \right) \in L^1 \) and the inequality (3.6) is satisfied, therefore \( Y^{\Lambda, C}(|X|) \) belongs to class \( \mathcal{D}_{\exp} \). To conclude this part, let recall the definition of the class of \( Q(\Lambda, C) \)-semimartingales which belong to \( \mathcal{D}_{\exp} \) given by [4].

**Definition 3.5.** Let \( \eta_T \) be a \( \mathcal{F}_T \)-random variable such that

\[
\exp[\gamma Y^{\Lambda, C}_T(|\eta_T|)] = \exp[\gamma(e^{C_T} |\eta_T| + \int_0^T e^{C_s} d\Lambda_s)]
\]

belongs to \( L^1 \), for all \( \gamma > 0 \). We define a class of \( S_Q(|\eta_T|, \Lambda, C) \) of \( Q(\Lambda, C) \)-semimartingales \( X \) such that

\[
|X_t| \leq \bar{p}_t \left[ e^{C_{t,T}} |\eta_T| + \int_t^T e^{C_{s,s}} d\Lambda_s \right], \quad a.s.
\]
4 Quadratic-exponential variation and stability result

4.1 A priori estimates

We now focus on the estimate of the martingale part of a semimartingale \( X \in \mathcal{S}_Q(|\eta_T|, \Lambda, C) \). The estimates of the discontinuous martingales part allow us to conclude the predictable projection \( j(\gamma \Delta M^d) \), \( \gamma \in \{-1, 1\} \) is well defined when the semimartingale \( X \) lives in a suitable space.

**Proposition 4.5.** Let consider a semimartingale \( X \in \mathcal{S}_Q(|\eta_T|, \Lambda, C) \) which follows the decomposition \( X = X_0 - V + M^c + M^d \), where there exists a process \( U \in \mathcal{G}_{loc}(\mu) \) such that \( M^d = U \bar{\mu} \) then the martingales \( \bar{M} = M^c + (e^U - 1) \bar{\mu} \) and \( \underline{M} = -M^c + (e^{-U} - 1) \bar{\mu} \) belong to \( \mathcal{U}_{exp} \) and \( M_0^p \) for any \( p \geq 1 \).

Moreover if for any stopping times \( \sigma \leq T \) there exists a constant \( c > 0 \) such that

\[
\mathbb{E} \left[ \exp(e^{C^r} |\eta_T| + \int_0^T e^{C^r} d\Lambda_s) |\mathcal{F}_\sigma \right] \leq c,
\]

then the processes \( \bar{M} \) and \( \underline{M} \) are BMO martingales.

**Proof.** Let \( X \in \mathcal{S}_Q(|\eta_T|, \Lambda, C) \), from Proposition 3.3, \( Y^{\Lambda,C}(X) = X + \Lambda + |X| \ast C \) and \( Y^{\Lambda,C}(-X) \) are \( \mathcal{Q} \)-local submartingale. Moreover let recall the process \( \bar{Y}^{\Lambda,C}(X) = e^{C^r}.|X| + e^{C^r} \ast \Lambda \) satisfies \( Y^{\Lambda,C}(X) \leq \bar{Y}^{\Lambda,C}(X) \) and \( Y^{\Lambda,C}(-X) \leq \bar{Y}^{\Lambda,C}(X) \), therefore since \( X \in \mathcal{S}_Q(|\eta_T|, \Lambda, C) \), for any \( p \geq 1 \), we find:

\[
\exp(p|Y^{\Lambda,C}_t(|X|)|) \leq \exp[p\bar{Y}^{\Lambda,C}_t(X)] \leq \mathbb{E} \left[ \exp[p(e^{C^r} |\eta_T| + \int_0^T e^{C^r} d\Lambda_s)] |\mathcal{F}_t \right].
\]

We conclude:

\[
\mathbb{E} \left[ \sup_{t \leq T} \exp(p|Y^{\Lambda,C}_t(|X|)|) \right] < +\infty.
\]

From the submartingale property of \( \exp(Y^{\Lambda,C}(X)) \) and \( \exp(Y^{\Lambda,C}(-X)) \), from Yoeurp-Meyer decomposition, there exist increasing processes \( \tilde{A} \) and \( \underline{A} \) such that:

\[
\tilde{K}_t := \exp(Y^{\Lambda,C}_t(X)) = \exp(X_0)\mathcal{E}(\tilde{M}_t) \exp(\tilde{A}_t),
\]

\[
\underline{K}_t := \exp(Y^{\Lambda,C}_t(-X)) = \exp(-X_0)\mathcal{E}(\underline{M}_t) \exp(\underline{A}_t)
\]

Since \( \tilde{A} \) and \( \underline{A} \) are increasing, from (4.18) we conclude \( \bar{Z} := \mathcal{E}(\bar{M}) \) and \( \underline{Z} := \mathcal{E}(\underline{M}) \) are uniformly integrable then \( \bar{M} \) and \( \underline{M} \in \mathcal{U}_{exp} \). Moreover \( \bar{Z} \) and \( \underline{Z} \) belong to \( \mathcal{M}^p \), for any \( p \geq 1 \). Using integration by part formula we find \( d\tilde{K}_t = \tilde{K}_{t-} [d\tilde{A}_t + d\tilde{M}_t] \) and \( d\underline{K}_t = \underline{K}_{t-} [d\underline{A}_t + d\underline{M}_t] \), that leads to \( d[\bar{K}]_t = \tilde{K}_{t-}^2 d[\bar{M}]_t \) and \( d[\underline{K}]_t = \underline{K}_{t-}^2 d[\underline{M}]_t \). Therefore we find for any stopping times \( \sigma \leq T \), \( [\bar{M}]_{\sigma,T} = \int_\sigma^T \frac{d[\bar{K}]}{\tilde{K}_{t-}^2} \) and \( [\underline{M}]_{\sigma,T} = \int_\sigma^T \frac{d[\underline{K}]}{\underline{K}_{t-}^2} \) then we find:

\[
[M]_{\sigma,T} \leq \sup_{\sigma \leq t \leq T} \left( \frac{1}{\tilde{K}_t} \right) \times [\tilde{K}]_{\sigma,T} \quad \text{and} \quad \underline{[M]}_{\sigma,T} \leq \sup_{\sigma \leq t \leq T} \left( \frac{1}{\underline{K}_t} \right) \times [\underline{K}]_{\sigma,T}
\]
However we find a priori estimates of $\bar{K}_T$ and $\bar{K}_T$ using Ito’s decomposition of the submartingales $\bar{K}^2$ and $\bar{K}^2$:

$$d\bar{K}_t^2 = 2\bar{K}_t^2 d\bar{K}_t + d[\bar{K}]_t = 2\bar{K}_t^2 [d\bar{M}_t + d\bar{A}_t] + d[\bar{K}]_t$$

$$d\bar{K}_t^2 = 2\bar{K}_t^2 d\bar{K}_t + d[\bar{K}]_t = 2\bar{K}_t^2 [d\bar{M}_t + d\bar{A}_t] + d[\bar{K}]_t$$

Therefore for any stopping times $\sigma \leq T$, we find:

$$E[\bar{K}_{\sigma,T}^2 | \mathcal{F}_\sigma] \leq E[\bar{K}_T^2 | \mathcal{F}_\sigma]$$

and

$$E[\bar{K}_{\sigma,T}^2 | \mathcal{F}_\sigma] \leq E[\bar{K}_T^2 | \mathcal{F}_\sigma]$$

(4.10)

Since $\sup_{0 \leq t \leq T} \bar{K}_t$ and $\sup_{0 \leq t \leq T} \bar{K}_t$ belong to $L^p$, for any $p \geq 1$, from Garsia and Neveu Lemma (see [4] Lemma 3.3) we get:

$$E[\bar{K}_T^p] < +\infty \quad \text{and} \quad E[\bar{K}_T^p] < +\infty, \quad \forall p \geq 1$$

(4.11)

Since $\sup_{0 \leq t \leq T} \bar{M}_t$ and $\sup_{0 \leq t \leq T} \bar{M}_t$ belong to $L^p$ for any $p \geq 1$ and using 4.11 from 4.9 we conclude using Cauchy Schwartz inequalities that for any $p \geq 1$:

$$E[\bar{M}_T^p] \leq +\infty \quad \text{and} \quad E[\bar{M}_T^p] \leq +\infty$$

then using BDG inequalities, we conclude $\bar{M}$ and $\bar{M}$ belong to $\mathcal{M}_0^p$. Moreover if there exists a non negative constant $c$ such that

$$E\left[\exp(e^{C_T}|\eta_T| + \int_0^T e^{C_s} d\Lambda_s)|\mathcal{F}_\sigma\right] \leq c,$$

then from 4.7 the processes $\bar{K}$ and $\bar{K}$ are bounded, using 4.9 and 4.10 we conclude the martingales $\bar{M}$ and $\bar{M}$ are BMO-martingales.

\[ \square \]

4.2 Stability results of quadratic exponential semimartingale

Here, we present stability results for quadratic exponential semimartingales which we shall use for the construction of the maximal solution of a class of quadratic BSDE’s with jumps. We first recall a general stability theorem of Barlow and Protter [3] for a sequence of c` adl` ag special semimartingales converging uniformly in $L^1$. We denote by $X^*$ := $\sup_{0 \leq t \leq T} |X_t|$.

**Theorem 4.5.** Let $X^n$ be a sequence of special semimartingales which belongs to $\mathcal{H}^1$ with canonical decomposition $X^n = X^n_0 + M^n - V^n$, and satisfies:

$$E\left[\int_0^T |dV^n_s|\right] \leq C, \quad \text{and} \quad E\left[(M^n)^*\right] \leq C$$

(4.12)

for some positive constant $C$. Assume that:

$$E\left[(X^n - X)^*\right] \longrightarrow 0, \quad \text{as} \quad n \to \infty,$$

where $X$ is an adapted process, then $X$ is a semimartingale in $\mathcal{H}^1$ with canonical decomposition $X = X_0 + M - V$ satisfying:

$$E\left[\int_0^T |dV_s|\right] \leq C, \quad \text{and} \quad E\left[(M)^*\right] \leq C$$

(4.13)
and we have
\[
\lim_{n \to \infty} \mathbb{E} \left[ (V^n - V)^* \right] = 0 \quad \text{and} \quad \lim_{n \to \infty} \|M^n - M\|_{\mathcal{H}^1} = 0. \quad (4.14)
\]

**Lemma 4.1.** Let \( X^n \) a sequence of \( \mathcal{S}_Q(|\eta_T|, \Lambda, C) \) semimartingales which canonical decomposition \( X^n = X^0_0 - V^n + M^n \) which converge in \( \mathcal{H}^1 \) to some process \( X \). Therefore the process \( X \) which canonical decomposition \( X = X_0 - V + M \) is an adapted càdlàg process which belongs to \( \mathcal{S}_Q(|\eta_T|, \Lambda, C) \) such that:
\[
\lim_{n \to \infty} \mathbb{E} \left[ (V^n - V)^* \right] = 0 \quad \text{and} \quad \lim_{n \to \infty} \|M^n - M\|_{\mathcal{H}^1} = 0.
\]

**Proof.** Let consider \( X^n = X^0_0 - V^n + (M^n)^c + U^n.\dot{\mu} \) a sequence of \( \mathcal{S}_Q(|\eta_T|, \Lambda, C) \) semimartingales. Firstly let prove that if the sequence \( X^n \) converge to the process \( X \) then this limit belongs to the space \( \mathcal{S}_Q(|\eta_T|, \Lambda, C) \). The sequence \( X^n \in \mathcal{S}_Q(|\eta_T|, \Lambda, C) \), hence for each \( n \in \mathbb{N} \) and for any stopping times \( \sigma \leq T \):
\[
- \tilde{\rho}_\sigma (-\eta_T + \Lambda_{\sigma,T} + |X^n| * C_{\sigma,T}) \leq X^n_0 \leq \tilde{\rho}_\sigma (\eta_T + \Lambda_{\sigma,T} + |X^n| * C_{\sigma,T}), \quad a.s \quad (4.15)
\]
and we get also:
\[
|X^n_\sigma| \leq \tilde{\rho}_\sigma \left[ e^{C_{\sigma,T}}|\eta_T| + \int_\sigma^T e^{C_{\sigma,s}}d\Lambda_s \right], \quad a.s \quad (4.16)
\]

Since the sequence \( X^n \) converges in \( \mathcal{H}^1 \), we can extract a subsequence which converges uniformly almost surely to the limit \( X \), using the dominated convergence and taking the limit of this subsequence in \( (4.15) \) and \( (4.16) \) we conclude the limit \( X \) belongs to the space \( \mathcal{S}_Q(|\eta_T|, \Lambda, C) \). Since the limit \( X \) is a \( \mathcal{Q}(\Lambda, C) \)-semimartingale, from Theorem 3.3 the process \( X \) is càdlàg. Let now prove the convergence of the martingale part and finite variation part, the priori estimates of their limits. From the first assertion of Proposition 4.5 there exists a constant \( C_1 > 0 \) such that:
\[
\mathbb{E} \left[ \int_0^T |dV^n_\sigma| \right] \leq \mathbb{E} \left[ \int_0^T \frac{1}{2}d((M^n)^c)_s \right] + \int_E \left[ g(U^n(s,x)) + g(-U^n(s,x)) \right] \nu(ds,dx) \leq C_1.
\]

Moreover From the second assertion of Proposition 4.5 using BDG inequalities, there exists constants \( c_2 \) and \( C_2 > 0 \) such that:
\[
\mathbb{E} \left[ (M^n)^+ \right] \leq c_2 \mathbb{E} \left[ (M^n)^+ \right] \leq C_2.
\]

Therefore assuming the sequence \( X^n \) converges to a process \( X \) which canonical decomposition \( X = X_0 - V + M \), taking \( C = \max(C_1, C_2) \), from Barlow and Protter Theorem 4.5 we get the expected result. \( \square \)

### 5 Application of quadratic exponential semimartingales: quadratic BSDEs with jumps

In stochastic optimization problem with exponential utility (see [7], [15]), or robust optimization with entropy penalty see([9], [30]), using dynamic programing , the authors
solved the problem using quadratic Backward Stochastic Differential Equations (BSDE) with bounded terminal condition in most of these papers. In the papers [4] and [12], the authors dealt with the problem with unbounded terminal condition but they worked in continuous filtration. In this part, we use quadratic exponential semimartingales to find the solution of a quadratic BSDE with jumps in a general setup where the terminal condition is unbounded.

5.1 Quadratic Exponential BSDE

We consider the stochastic basis defined above \((Ω, F, F, P)\) with finite time horizon \(T < +∞\). On this basis, let \(W = (W_t)_{t≥0}\) be a \(d\)-dimensional standard Brownian motion and let \(µ\) the random measure defined above such that \(ν\) is equivalent to a product measure \(λ \otimes dt\) with density \(ξ\) satisfying \(ν(ω, dt, dx) = ξ(ω, t, x)λ(dx)dt\), where \(λ\) is a \(σ\)-finite measure on \((E, ℰ)\) satisfying \(∫_E |x|^2λ(dx) < +∞\) and where the density \(ξ\) is a measure, bounded nonnegative function such that for some constant \(C\):

\[
0 ≤ ξ(ω, t, x) ≤ Cν < +∞, \quad P \otimes λ \otimes dt - \text{a.e.}
\]

That implies in particular \(ν([0, T] \times E) ≤ C_T λ(E)\). We assume the following representation Theorem for any square integrable martingale \(M\):

\[
M = M_0 + ZW + U\tilde{µ},
\]

where \(Z\) and \(U\) are predictable processes such that \(E \left[ ∫_0^T (|Z_t|^2 + ∫_E |U(t, x)|^2ξ(t, x)λ(dx))dt \right] < +∞\). We note \(H^2\) the set of predictable process \(Z\) resp. predictable process \(U\) satisfying this square integrability condition. Let define the following norms \(||.||_{H^p}\) and \(||.||_{H^2_p}\), for \(p ≥ 1\):

\[
||Z||_{H^2_p} := \left( E \left[ ∫_0^T |Z_s|^2 ds \right]^{\frac{p}{2}} \right)^{\frac{1}{p}}, \quad \text{for any predictable process } Z,
\]

and

\[
||U||_{H^2_p} := \left( E \left[ ∫_0^T |U_{s, λ}|^2 ds \right]^{\frac{p}{2}} \right)^{\frac{1}{p}}, \quad \text{for any predictable process } U,
\]

where

\[
|U|^2_{s, λ} := ∫_E |U(s, x)|^2ξ(s, x)λ(dx).
\]

**Definition 5.6.** (Quadratic Exponential BSDE) We call quadratic exponential BSDE associated to \((f, η_T)\) and with parameters \((l, c, δ)\) (in short terms we note \(q_{exp}(l, c, δ)\) BSDE), the following stochastic differential equation:

\[
-dY_t = f(t, Y_t, Z_t, U_t) - Z_t dW_t - ∫_E U(t, x)\tilde{µ}(dt, dx), \quad Y_T = η_T, \tag{5.17}
\]

where the coefficient satisfying the following conditions:
1. (Continuity condition): for all \( t \in [0, T] \), \((y, z, u) \rightarrow f(t, y, z, u)\) is continuous.

2. (Growth condition): for all \((y, z, u) \in \mathbb{R} \times \mathbb{R}^d \times L^2, t \in [0, T]\): \(\mathbb{P}\)-a.s.,

\[
q(t, y, z, u) = \frac{1}{\delta} j_t (-\delta u) - \frac{\delta}{2} |z|^2 - l_t - c_t |y| \leq f(t, y, z, u) \leq \frac{1}{\delta} j_t (\delta u) + \frac{\delta}{2} |z|^2 + l_t + c_t |y| = \tilde{q}(t, y, z, u).
\]

3. \((A_\gamma)\) condition: there exists a process \(\gamma\) such that for all \((y, z) \in \mathbb{R} \times \mathbb{R}^d, t \in [0, T]\):

\[
f(t, y, z, u) - f(t, y, z, \bar{u}) \leq \int_E \gamma_t [u(x) - \bar{u}(x)] \xi(t, x) \lambda(dx), \quad (5.18)
\]

where the process \(\gamma, \bar{u}\) belongs to the space \(\mathcal{U}_{\text{exp}}\) and \(\gamma > -1\).

A solution of a Quadratic Exponential BSDE \((5.17)\) is a a triple \((Y, Z, U)\) of predictable processes satisfying:

\[
\int_0^T |Z_t|^2 ds < +\infty, \quad \int_0^T (e^{U(t,x)} - 1) \xi(t, x) \lambda(dx) < +\infty \quad \text{and} \quad \int_0^T (e^{-U(t,x)} - 1) \xi(t, x) \lambda(dx) \text{ a.s}
\]

In particular case, where the coefficient satisfies a Lipschitz condition and the condition \(A_\gamma\) with \(-1 < \gamma < c\), for some positive constant \(c\), there exists a unique solution of the BSDE see\([7, 45]\).

**Theorem 5.6.** i) Let consider the BSDE \((5.17)\) with terminal value \(Y_T = \eta_T \in L^2(\Omega, \mathcal{F}_T)\) where the coefficient satisfying the following conditions:

1. Continuity and Integrability condition: for all \( t \in [0, T] \), \((y, z, u) \rightarrow f(t, y, z, u)\) is continuous and satisfying the integrability condition:

\[
\mathbb{E} \left[ \int_0^T |f(t, 0, 0, 0)|^2 dt \right] < +\infty. \quad (5.19)
\]

2. Lipschitz condition: there exists a nonnegative constant \(C\) such that for all \( t \in [0, T]\):

\[
|f(t, y, z, u) - f(t, \bar{y}, \bar{z}, \bar{u})| \leq C [ |y - \bar{y}| + |z - \bar{z}| + |u - \bar{u}|_{L^1}]. \quad (5.20)
\]

3. The bound \((A_\gamma)\) condition: the coefficient \(f\) satisfies the \((A_\gamma)\) condition and there exists a nonnegative constant \(c\) such that \(-1 < \gamma \leq c\).

then there exists a unique triple \((Y, Z, U) \in S^2 \times \mathbb{H}^2 \times \mathbb{H}^2_{\gamma}\) solution of the BSDE \((5.17)\).

ii) Moreover for any BSDE with terminal value \(\bar{\eta}_T \leq \eta_T\) and coefficient \(\bar{f} \leq f\) satisfying the same last assumptions as \(f\), the solution \((\bar{Y}, \bar{Z}, \bar{U})\) associated to this BSDE satisfies \(\bar{Y} \leq Y\).

**Remark 5.2.** (Comparison result)

1. The comparison result holds true also in the case of the both coefficients satisfies the Lipschitz and the bound \((A_\gamma)\) conditions.

2. The bound \((A_\gamma)\) condition can be substituted to the \((A_\gamma)\) condition since we need only \(\gamma, \bar{u}\) should be in the space \(\mathcal{U}_{\text{exp}}\) to ensure the comparison result.
5.1.1 Existence of solution for quadratic BSDE with jumps

We prove the solution of the quadratic exponential BSDE using the properties of quadratic exponential semimartingales. We construct a sequence of quadratic exponential semimartingales which converges to some limit. Therefore using stability result, we find the convergence of the martingale part of the limit and conclude.

Remark 5.3. Let consider the triple \((Y, Z, U)\) solution of the \(q_{\text{exp}}(l, c, \delta)\) BSDE, then \(Y\) is a \(Q_{\text{exp}}(\Lambda, C, \delta)\) semimartingale with \(\Lambda_t = \int_0^t l_s ds, \ C_t = \int_0^t c_s ds, \ t \leq T\). In all the rest of paper without losing any generality, we assume \(\delta = 1\) and \(c\) is bounded.

Assumption 5.1. Let assume the integrability condition:

\[
\forall \gamma > 0, \quad \mathbb{E} \left[ \exp(\gamma(e^{C_T}|\eta_T| + \int_0^T e^{C_s}d\Lambda_s)) \right] < +\infty.
\]

Proposition 5.6. (Construction of sequences). Let assume \(5.1\) and consider the \(q_{\text{exp}}(l, c, 1)\) BSDE associated to \((f, \eta_T)\). We set \(\bar{f} = f1_{f \geq 0}\) and \(\bar{f} = f1_{f \leq 0}\) and we define for each \(n, m \in \mathbb{N}\) the sequences of coefficients \(\bar{f}^n = \bar{f} \vee \bar{b}^n, \ \bar{f}^m = \bar{f} \wedge \bar{b}^m\), \(\bar{q}^n = \bar{q} \vee \bar{b}^n\) and \(\bar{q}^m = \bar{q} \wedge \bar{b}^m\) where the regularizing functions \(\bar{b}^n\) and \(\bar{b}^m\) are the convex functions with linear growth defined by \(\bar{b}^n(w, r, v) = n|w| + n|r| + n|v|\) and \(\bar{b}^m(w, r, v) = -m|w| - m|r| - m|v|\). The symbols \(\vee\) and \(\wedge\) stands for inf-convolution and sup-convolution.

1. The sequence \(\bar{f}^n\) and \(\bar{q}^n\) resp( the sequence \(\bar{f}^m\) and \(\bar{q}^m\)) are increasing and converge to \(\bar{f}, \bar{q}\) resp ( are decreasing and converge to \(f, q\)). Moreover the sequence \(\bar{f}^n, \bar{q}^n, \bar{f}^m, \bar{q}^m\) satisfy the Lipschitz condition (5.20).

2. The sequences \((\bar{f}^n)_{n \in \mathbb{N}}\) resp( the sequence \((\bar{f}^m)_{m \in \mathbb{N}}\)) satisfies \(0 \leq \bar{f}^n \leq \bar{q}^n \leq \bar{q}\) (resp \(\underline{q} \leq \bar{q}^m \leq \underline{f}^m \leq 0\), for each \(n, m \in \mathbb{N}\). Moreover

\[
\underline{q} \leq \bar{f}^n + \bar{f}^m \leq \bar{q}.
\]

Proof. Let consider the coefficient \(f\) associated to the \(q_{\text{exp}}(l, c, 1)\) BSDE. Using the properties of infconvolution and supconvolution, we deduce all the sequences of coefficients defined above satisfy the Lipschitz condition, moreover we find the monotone property comparing using the definition of theses sequences. Let prove the lower and upper bound of \(\bar{f}^n\) and \(\bar{f}^m\) for each \(n, m \in \mathbb{N}\):

\[
0 \leq \inf_{w, r, v} \{ \bar{f}(t, w, r, v) + n|y-w| + n|z-r| + n|u-v| \} \leq \inf_{w, r, v} \{ \bar{q}(t, w, r, v) + n|y-w| + n|z-r| + n|u-v| \}.
\]

then we find \(0 \leq \bar{f}^n \leq \bar{q}^n \leq \bar{q}\). By similar arguments we find for each \(m \in \mathbb{N}\), \(0 \geq \bar{f}^m \geq \bar{q}^m \geq \underline{q}\), hence we conclude \(0 \leq \bar{f}^n \leq \bar{q}^n \leq \bar{q}\) and \(\underline{q} \leq \bar{q}^m \leq \bar{f}^m \leq 0\). Moreover, using the last inequalities we deduce \(\underline{q} \leq \bar{f}^n + \bar{f}^m \leq \bar{q}\), for each \(n, m \in \mathbb{N}\).

Theorem 5.7. Let assume \(5.1\) there exists a triple \((Y, Z, U)\) is \(\mathcal{S}_Q(|\eta_T|, \Lambda, C) \times \mathbb{H}^{2p} \times \mathbb{H}^{2p}_\Lambda\) solution of the \(q_{\text{exp}}(l, c, 1)\) BSDE associated to \((f, \eta_T)\).
**Proof.** We follow two steps to prove the existence. Firstly, we construct a sequence of quadratic exponential semimartingales which converges and secondly we find the convergence of the finite variation and martingale part using stability result.

**First step:** (Construction of the sequence of $\mathcal{S}_\mathcal{Q}(|\eta_T|, \Lambda, C)$ semimartingales). Let consider a $q_{\exp(l,c,1)}$ BSDE associated to $(f, \eta_T)$ and consider the sequence of coefficients $f^{n,m} = f^n + f^m$ which converges to $f$ when $n, m$ go to infinity from Proposition 5.6. We consider the BSDE associated to $(f^{n,m}, \eta_T)$:

$$-dY^{n,m}_t = f^{n,m}(t, Y^{n,m}_t, Z^{n,m}_t, U^{n,m}_t)dt - Z^{n,m}_tdW_t - \int_E U^{n,m}(t, x)\bar{\mu}(dt, dx), \quad Y^{n,m}_T = \eta_T.$$  

Since for each $n, m \in \mathbb{N}$ the coefficient $f^{n,m}$ satisfies the continuity, the integrability and the Lipschitz conditions of Theorem 5.6, there exists a solution $(\bar{q}^n, \bar{q}^m)$ for each $n, m$. For each $n, m \in \mathbb{N}$, consider the BSDE associated to $(f^{n,m}, \eta_T)$.

Since for each $n, m \in \mathbb{N}$ the BSDE associated to $(f^{n,m}, \eta_T)$, we deduce for each $n, m$ that $(\bar{q}^n, \bar{q}^m)$ satisfies the continuity, the integrability and the Lipschitz conditions for each $n, m$. Moreover since $\bar{q}^n \leq f^{n,m} \leq \bar{q}^m$ and the triples $(\bar{Y}^n, \bar{Z}^n, \bar{U}^n)$ and $(Y^{n,m}, Z^{n,m}, U^{n,m})$ solutions of the $q_{\exp(l,c,1)}$ BSDE associated to $(\bar{q}^n, |\eta_T|)$ and $(\bar{q}^m, |\eta_T|)$ exist and satisfy for all stopping times $\sigma \leq T$:

$$|\bar{Y}^n_\sigma| \vee |Y^{n,m}_\sigma| \leq \bar{\rho}_\sigma \left[ e^{C_{\sigma,}\eta_T} + \int_\sigma^T e^{C_{\sigma,s}}d\Lambda_s \right], \quad a.s$$

The existence of the triples is given by existence result of Theorem 5.6. Since $\bar{q}^n$ and $\bar{q}^m$ satisfy the continuity, the integrability and the Lipschitz conditions for each $n, m \in \mathbb{N}$, we can apply comparison result of Theorem 5.6 that $Y^{n,m} \leq Y^{n,m} \leq \bar{Y}^n$ and we conclude $(Y^{n,m})_{n,m \in \mathbb{N}}$ is a sequence of $\mathcal{S}_\mathcal{Q}(|\eta_T|, \Lambda, C)$ semimartingales since for each $n, m \in \mathbb{N}$, $Y^{n,m}$ is a $\mathcal{Q}(\Lambda, C)$ semimartingale satisfying:

$$|Y^{n,m}_\sigma| \leq \bar{\rho}_\sigma \left[ e^{C_{\sigma,}\eta_T} + \int_\sigma^T e^{C_{\sigma,s}}d\Lambda_s \right], \quad a.s$$  

for $n, m \geq c^* = \sup_{t \leq T} c_t$, see more details about the characteristics of the coefficient $\bar{q}^n$ and $\bar{q}^m$ in the Appendix Lemma 6.3. Let now prove the coefficient satisfies the $(A_\gamma)$-condition.

For $(y, z) \in \mathbb{R} \times \mathbb{R}^d$:

$$f^{n,m}(t, y, z, u) - f^{n,m}(t, y, z, \bar{u}) = [\bar{f}^n(t, y, z, u) - f^n(t, y, z, \bar{u})] = [\bar{f}^m(t, y, z, u) - f^m(t, y, z, \bar{u})].$$

Therefore since for any functions $\psi$ and $\bar{\psi}$:

$$\inf_x \psi(x) - \inf_x \bar{\psi}(x) \leq \sup_x \{\psi(x) - \bar{\psi}(x)\}, \quad \sup_x \psi(x) - \sup_x \bar{\psi}(x) \leq \sup_x \{\psi(x) - \bar{\psi}(x)\}$$

(5.22)

Since the coefficient $f^{n,m}$ satisfies the $(A_\gamma)$-condition, we can apply comparison result see Theorem 5.6, we deduce for each $n, m \in \mathbb{N}$:

$$Y^{n+1,m} \geq Y^{n,m} \geq Y^{n,m+1}.$$  

**Second step:** (Convergence of the semimartingale, the finite variation and the martingale part). For each $m \in \mathbb{N}$, $(Y^{n,m})_{n \geq 0}$ is an increasing sequence of bounded càdlàg
Moreover, for \( (Y^m)_m \) is a sequence of càdlàg \( S_Q(\eta_T, \Lambda, C) \) semimartingales, with canonical decomposition \( Y^{n,m} = Y^m_{0} - V^{n,m} - M^{n,m} \). Hence, this sequence converges, let denote \( Y^m \) its limit for each \( m \in \mathbb{N} \). From stability result, Lemma 4.1, \( (Y^m)_m \) is a decreasing sequence of bounded càdlàg \( S_Q(\eta_T, \Lambda, C) \) semimartingales. Let \( Y \) its limit, from stability result Lemma 4.1, \( Y \) is a càdlàg \( S_Q(\eta_T, \Lambda, C) \) semimartingale with canonical decomposition \( Y = Y^m_{0} - V + M \)

| \( \lim_{n \to \infty} E \left[ (V^{n,m} - V^m)^* \right] = 0 \) and \( \lim_{n \to \infty} \|M^{n,m} - M^m\|_{\mathcal{H}^1} = 0 \). |

For each \( n, m \geq e^s \), since \( Y^{n,m} \geq Y^{n,m+1} \) then \( (Y^m)_m \) goes to zero when \( n,m \) go to infinity, moreover for \( K > K_\epsilon \) large enough, \( \mathbb{P}(T_K < T) \leq \frac{1}{K} \). From Lemma 4.1, we find \( Y_{n,m}^{\infty} \) lives in a compact set and its convergence to the càdlàg process \( Y \) is uniform. The same property holds for \( M_{n,m}^{\infty} \) and \( V_{n,m}^{\infty} \). Let \( Z^{n,m,K}_t = Z^{n,m}_t 1_{T_K < T_t} \) and \( U^{n,m,K}_t = U^{n,m}_t 1_{T_K < T_t} \) in such that \( (Z^{n,m,W})_{n,m,K} \) and \( (U^{n,m,\mu})_{n,m,K} \) strongly converges, the sequence of orthogonal martingales \( (Z^{n,m,K}_t, W) \) and \( (U^{n,m,K}_t, \mu) \) also converge in their appropriate space. Therefore, we can extract a subsequence \( Z^{n,m,K} \) and \( U^{n,m,K} \) converging a.s. to some processes \( Z \) and \( U \).

For \( t \leq T_K \), the sequence \( f^{n,m}(t, Y^{n,m}_t, Z^{n,m}_t, U^{n,m}_t) \) converges to \( f(t, Y_t, Z_t, U_t) dt \otimes \mathcal{F} \) a.s. It remains to prove that \( \mathbb{E} \left[ \int_0^{T_K} |f^{n,m}(t, Y^{n,m}_t, Z^{n,m}_t, U^{n,m}_t) - f(t, Y_t, Z_t, U_t)| dt \right] \) goes to zero when \( n,m \) go to infinity. Firstly we have

\[
\mathbb{E} \left[ \int_0^{T_K} |f^{n,m}(t, Y^{n,m}_t, Z^{n,m}_t, U^{n,m}_t) - f(t, Y_t, Z_t, U_t)| 1_{\{Z^{n,m}_t + U^{n,m}_t \leq C\}} dt \right]
\]

goes to zero when \( n,m \) go to infinity, by dominated convergence since \( Y^{n,m} \) is bounded and \( |f^{n,m}(t, Y^{n,m}_t, Z^{n,m}_t, U^{n,m}_t) - f(t, Y_t, Z_t, U_t)| \) is uniformly bounded in \( L^1 \) by Lemma 4.1. Moreover for \( s \leq T_K \), \( \mathbb{P}(|Z^{n,m}_s| + |U^{n,m}_s| > C) \leq \frac{1}{2} \mathbb{E}(|Z^{n,m}_s|^2 + |U^{n,m}_s|^2) \), from Lemma 4.1 there exists a constants \( C_2 \) such that \( \mathbb{E}(\langle M^{n,m} \rangle_s) \leq C_2 \), therefore

\[
\mathbb{E} \left[ \int_0^{T_K} |f^{n,m}(t, Y^{n,m}_t, Z^{n,m}_t, U^{n,m}_t) - f(t, Y_t, Z_t, U_t)| 1_{\{Z^{n,m}_t + U^{n,m}_t > C\}} dt \right]
\]

goes to zero when \( C \) goes to infinity, uniformly in \( n,m \). As a consequence, the process \( V \) in the decomposition of the quadratic exponential semimartingale \( Y \) is given by \( dV_t = f(t, Y_t, z_t, U_t) dt \) on \([0, T_K]\) for any \( K \). We conclude the triple \((Y, Z, U)\) is a solution of the \( q_{exp}(l, c, 1) \) BSDE associated to \((f, \eta_T)\). Moreover since \( Y \) belongs to the space \( S_Q(\eta_T, \Lambda, C) \), then from Proposition 4.3, the martingales \( Z_W + (eU - 1), \tilde{\mu} \) and \(-Z_W + (e^{-U} - 1), \tilde{\mu} \) belongs to the space \( \mathcal{M}_p^0 \).

\[ \square \]

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6 Appendix

Lemma 6.2. For any $k \geq 1$ and any local martingale $M$:

\[ j_t(k \Delta M_t) \geq k j_t(\Delta M_t), \quad 0 \leq t \leq T \]

Proof. Let recall that for any local martingale $M = M^c + M^d$ from representation theorem, there exists $U \in \mathcal{G}_{loc}(\mu)$ such that $M^d = U - \tilde{\mu}$, then $j(\Delta M^d) = (e^U - U - 1)\nu$. Therefore from representation theorem it is sufficient to prove the following function $f_k$: $x \mapsto (e^{kx} - kx - 1) - k(e^x - x - 1)$ is positive to find the result. For any $x \in \mathbb{R}$, since $f'_k(x) = ke^x(e^{(k-1)x} - 1)$, then we conclude the function $f_k$ is increasing on $(0, +\infty)$ and decreasing on $(-\infty, 0)$. Therefore, for any $x \in \mathbb{R}$, $f_k(x) \geq f_k(0) = 0$. \qed

Lemma 6.3. Let us define the following quadractic exponential coefficients by $\bar{q}(y, z, u) = c|y| + |l| + \frac{1}{2}|z|^2 + j(u)$ and $\underline{q}(y, z, u) = -c|y| - |l| - \frac{1}{2}|z|^2 - j(-u)$ and define the sequence $\bar{q}^n$ and $\underline{q}^m$ by the inf-convolution and sup-convolution for $n, m \geq c^* = \sup_{t \in [0, T]} c$:

\[
\bar{q}^n(y, z, u) = \inf_{r, w, v} \{ \bar{q}(y, w, v) + n|y - r| + n|z - w| + n|u - v| \}
\]

\[
\underline{q}^m(y, z, u) = \sup_{r, w, v} \{ \underline{q}(y, w, v) - m|y - r| - m|z - w| - m|u - v| \}
\]

then:

i) The sequences $\bar{q}^n$ and $\underline{q}^m$ satisfy the structure condition $\mathcal{Q}_{exp}(\Lambda, C)$.

ii) There exists a unique solution $(\bar{Y}^n, \bar{Z}^n, \bar{U}^n)$ (resp. $(\underline{Y}^m, \underline{Z}^m, \underline{U}^m)$) of the BSDE’s associated to $(\bar{q}_n, |\xi_T|)$ (resp. to $(\underline{q}_m, -|\xi_T|)$).

iii) The processes $\bar{Y}^n$ and $\underline{Y}^m$ are values processes of the following robust optimization problem, for any $\sigma \leq T$:

\[
\bar{Y}^n_{\sigma} = \sup_{\{Q \in \mathcal{P}, |\beta| \leq m; -1 \leq \kappa \leq 1 \}} \mathbb{E}_{Q}^{\mathcal{G}_T} \left[ S^c_{\sigma, T}|\xi_T| + \int_{\sigma}^{T} S^c_{\sigma, t}|l_t|dt + \int_{\sigma}^{T} c_t S^c_{\sigma, t} \ln \left( \frac{Z^Q_t}{Z^\sigma_t} \right) dt + S^c_{\sigma, T} \ln \left( \frac{Z^Q_T}{Z^\sigma_T} \right) \right]
\]

\[
\underline{Y}^m_{\sigma} = \inf_{\{Q \in \mathcal{P}, |\beta| \leq m; -1 \leq \kappa \leq 1 \}} \mathbb{E}_{Q}^{\mathcal{G}_T} \left[ -S^c_{\sigma, T}|\xi_T| - \int_{\sigma}^{T} S^c_{\sigma, t}|l_t|dt - \int_{\sigma}^{T} c_t S^c_{\sigma, t} \ln \left( \frac{Z^Q_t}{Z^\sigma_t} \right) dt + S^c_{\sigma, T} \ln \left( \frac{Z^Q_T}{Z^\sigma_T} \right) \right]
\]

where $S^c_{\sigma, t} = \exp(\int_{\sigma}^{t} c_s ds)$ and the Radon Nikodym density of $Q$ with respect to $\mathbb{P}$ on $\mathcal{G}_T$ is $Z^Q_T$, the process $Z^Q_t := \mathbb{E}[Z^Q_T|\mathcal{G}_t] = \mathcal{E}(\beta.W + \kappa.\tilde{\mu})_t$.

Moreover, we have the following estimates:

\[
\underline{Y}^m_{\sigma} \leq \bar{Y}^{n,m}_{\sigma} \leq \bar{Y}^n_{\sigma}, \quad \sigma \leq T.
\]

and

\[
|\underline{Y}^m_{\sigma}| + |\bar{Y}^n_{\sigma}| \leq \rho_{\sigma} \left[ S^c_{\sigma, T}|\xi_T| + \int_{\sigma}^{T} S^c_{\sigma, s}|l_s|ds \right] , \quad \sigma \leq T.
\]
**Proof:** i) We compute explicitly the sequence of functions $\bar{q}^n$ and $\underline{q}^m$ which satisfy the structure condition $Q_{\text{exp}}(\Lambda, C)$ for every $n, m \geq c^*$. By definition we have

$$
\bar{q}_n(y, z, u) = \inf_{r, w, v}\{\bar{q}_n(y, w, v) + n|y - r| + n|z - w| + n|u - v|\}
$$

$$
= c|y| + |l| + \inf_w\left\{\frac{1}{2}|w|^2 + n|z - w|\right\} + \inf_{v}\{j_i(v) + n|u - v|\}
$$

Obviously one can find the explicit form of $\bar{q}_n$ which is given by

$$
\bar{q}^n(y, z, u) = c|y| + |l| + \frac{1}{2}|z|^21_{\{z \leq n\}} + n(|z| - \frac{n}{2})1_{\{|z| > n\}}
$$

$$
+ \int_E \left[ g(u(e))1_{\{e^{u(e)} - 1 \leq n\}} + (- (n + 1) \ln(n + 1) + n(u(e) + 1))1_{\{e^{u(e)} - 1 > n\}} \right] \zeta_t(e)\rho(de)
$$

where we recall $g(x) = e^x - x - 1$, using the similar arguments we find the explicitly form of $\underline{q}^m$:

$$
\underline{q}^m(y, z, u) = -c|y| - |l| - \frac{1}{2}|z|^21_{\{z \leq m\}} - m(|z| - \frac{m}{2})1_{\{|z| > m\}}
$$

$$
+ \int_E \left[ g(-u(e))1_{\{e^{-u(e)} - 1 \leq m\}} + ((m + 1) \ln(m + 1) + m(u(e) - 1))1_{\{e^{-u(e)} - 1 > m\}} \right] \zeta_t(e)\rho(de)
$$

then we conclude for each $n, m \geq c^*$, $\bar{q}^n$ and $\underline{q}^m$ satisfy the structure condition $Q_{\text{exp}}(\Lambda, C)$.

ii) Since the coefficients $\bar{q}^n$ and $\underline{q}^m$ are Lipschitz we deduce there exists a solution $(\bar{Y}^n, \bar{Z}^n, \bar{U}^n)$ resp. $(\underline{X}^m, \underline{Z}^m, \underline{U}^m)$ associated to $(\bar{q}^n, |\xi_T|)$ resp. $(\underline{q}^m, |\xi_T|)$. Let now prove these coefficients satisfy the $(A_\Lambda)$ condition. To prove this result let first remark that for all $x \in \mathbb{R}$:

$$
-(n + 1) \ln(n + 1) + n(x + 1) = g[\ln(n + 1)] + n(x - \ln(n + 1))
$$

Let $u, \bar{u}$, we set $E = \bigcup_{i=1}^4 A_i$ where

$$
A_1 = \{x \in \mathbb{R}, e^{u(x)} - 1 \leq n, e^{\bar{u}(x)} - 1 \leq n\}, \quad A_2 = \{x \in \mathbb{R}, e^{u(x)} - 1 \leq n, e^{\bar{u}(x)} - 1 > n\}
$$

$$
A_3 = \{x \in \mathbb{R}, e^{u(x)} - 1 > n, e^{\bar{u}(x)} - 1 \leq n\}, \quad A_4 = \{x \in \mathbb{R}, e^{u(x)} - 1 > n, e^{\bar{u}(x)} - 1 > n\}
$$

Therefore we find for all $y, z$:

$$
\bar{q}^n(y, z, u) - \bar{q}^n(y, z, \bar{u})
$$

$$
= \int_{A_1} [g(u(x)) - g(\bar{u}(x))]\zeta_t(x)\rho(dx) + \int_{A_2} [g(u(x)) - g(\ln(n + 1)) - n(\bar{u}(x) - \ln(n + 1))]\zeta_t(x)\rho(dx)
$$

$$
+ \int_{A_3} [n(u(x) - \ln(n + 1)) + g(\ln(n + 1)) - g(\bar{u}(x))]\zeta_t(x)\rho(dx)
$$

We now find different inequalities on every given subset of $E$:

**On the set $A_1$:** we use the convex property of the coefficient $g$ and we deduce:

$$
\int_{A_1} [g(u(x)) - g(\bar{u}(x))]\zeta_t(x)\rho(dx) \leq \int_{A_1} (e^{u(x)} - 1)(u(x) - \bar{u}(x))\zeta_t(x)\rho(dx)
$$
On the set \( A_2 \): Since on \( A_2 \), the function \( g \) is increasing we find \( \forall x \in A_2 \), \( g(u(x)) \leq g(\ln(n + 1)) \), moreover we have \( \bar{u}(x) - \ln(n + 1) \geq 0 \) for \( x \in A_2 \) then we conclude:

\[
\int_{A_2} \left[ g(u(x)) - g(\ln(n + 1)) - n(\bar{u}(x) - \ln(n + 1)) \right] \zeta_t(x) \rho(dx) \leq 0. \tag{6.24}
\]

On the set \( A_3 \): we use the convex property of the coefficient \( g \) and we find \( g(\ln(n + 1)) - g(\bar{u}(x)) \leq -n(\bar{u}(x) - \ln(n + 1)) \), \( \forall x \in A_3 \). Therefore, we find:

\[
\int_{A_3} \left[ n(\bar{u}(x) - \ln(n + 1) + g(\ln(n + 1)) - g(\bar{u}(x))) \right] \zeta_t(x) \rho(dx) \leq \int_{A_3} n(\bar{u}(x) - \bar{u}(x)) \zeta_t(x) \rho(dx)
\]

(6.25)

Hence using (6.23), (6.24) and (6.25), we find:

\[
\bar{q}^n(y, z, u) - \bar{q}^n(y, z, \bar{u}) = \int_E \gamma^n(u(x), \bar{u}(x))(u(x) - \bar{u}(x)) \zeta_t(x) \rho(dx)
\]

where \( \gamma^n(u(x), \bar{u}(x)) = n1\{ e^{u(x)} - 1 > n \} + (e^{\bar{u}(x)} - 1)1_{A_1} \), then \( -1 \leq \gamma^n \leq n \) hence \( \gamma^n \in \mathcal{U}^{\text{exp}} \). Therefore the sequence \( \bar{q}^n \) satisfies the \( \mathcal{A}_n \) condition. We use similar arguments to prove the sequence \( \bar{q}^m \) satisfies also the \( \mathcal{A}_n \) condition. We conclude from Comparison Theorem, the uniqueness of the triple \((\bar{Y}^n, \bar{Z}^n, \bar{U}^n)\) resp( \((\bar{V}^n, \bar{Y}^m, \bar{Z}^m, \bar{U}^m)\) solution of the BSDE associated to \((\bar{q}^n, \xi_T)\) resp(\((\bar{q}^m, -|\xi_T|)\)).

iii) Let define the cost functional of the robust optimization problems defined in Lemma 6.3, \( \mathcal{J}^{Q,n} \) and \( \mathcal{J}^{Q,m} \), and define the value processes \( \bar{V}^n \), \( \bar{V}^m \). Assume \( |\xi_T| \in \mathcal{L}^{\text{exp}}, |l| \in \mathcal{D}^{1}_{\text{exp}} \) and \( c \) bounded then the value processes of the robust optimization exist see Bordigoni, Matoussi and Schweizer [9] for more details. Moreover we deduce for any \( Q \ll P \):

\[
\begin{align*}
\mathcal{J}^{Q,n}_t &= S^n_t \bar{V}^n_t + \int_0^t S^n_s \nu_s ds + \int_0^t c_s S^n_s \ln(Z^Q_s) ds - S^n_t \ln(Z^Q_t) \\
\mathcal{J}^{Q,m}_t &= S^m_t \bar{V}^m_t - \int_0^t S^m_s \nu_s ds - \int_0^t c_s S^m_s \ln(Z^Q_s) ds + S^m_t \ln(Z^Q_t)
\end{align*}
\]

(6.26)

Moreover the value processes \( \bar{V}^n \) and \( \bar{V}^m \) are special semimartingales following the following the representation theorem there exist a predictable process \( Z^n, U^n \) and a predictable process \( A^n \) such that \( d\bar{V}^n_t = dA^n_t + Z^n_t dW_t + \int_E \bar{U}^n_t(e) \bar{\mu}(dt, de) \) resp( \( d\bar{V}^m_t = dA^m_t + Z^m_t dW_t + \int_E \bar{U}^m_t(e) \bar{\mu}(dt, de) \)). we define the dynamics of \( Z^Q \),

\[
dZ^Q_t = Z^Q_t \left( \beta_t dW_t + \int_E \kappa_t \bar{\mu}(dx, dt) \right)
\]

For every \( n, m \in \mathbb{N}^* \), for every \( Q \ll P \), \( \mathcal{J}^{Q,n} \) resp( \( \mathcal{J}^{Q,m} \) ) are submartingales and martingales for the optimal resp( submartingales and martingale for the optimal) then we get:

\[
A^n_t = - \max_{\{ |\beta| \leq n \}} \left\{ \int_0^t (|\nu_s| + c_s \bar{V}^n_s) + \int_0^t \left( \langle Z^n - \beta, \beta \rangle_s + \frac{1}{2} |\beta_s|^2 \right) ds \right\}
\]

\[
- \max_{\{-1 \leq \kappa \leq n \}} \left\{ \int_0^t \int_E (\kappa_s(x) (v_s(x) + 1) - (1 + \kappa_s(x)) \ln(1 + \kappa_s(x))) \zeta_s(x) \rho(dx) ds \right\}
\]

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resp

\[ A_m^V = - \min_{\{ |\beta| \leq m \}} \left\{ \int_0^t (|s| + c_s V_m) + \int_0^t \left( |Z_m^\beta + \beta_s - \frac{1}{2} |\beta_s|^2 \right) ds \right\} - \min_{\{ -1 \leq \kappa \leq m \}} \left\{ \int_0^t \left( (\kappa_s(x)(v_s(x) - 1) + (1 + \kappa_s(x))) \ln(1 + \kappa_s(x)) \right) \zeta_s(x) \rho(dx) ds \right\} \]

Using first order condition, we find for the first optimization problem \( \kappa^* = \left( e^{\bar{U}^n} - 1 \right) 1_{\{ e^{\bar{U}^n} \leq 1 \}} + n 1_{\{ e^{\bar{U}^n} > 1 \}} \), then we deduce \( (\bar{V}^n, \bar{Z}^n, \bar{U}^n) \) is the solution associated to the BSDE \( (\bar{h}^n, |\xi_T|) \):

\[ -d\bar{V}^n_t = h^n(\bar{V}^n_t, \bar{Z}^n_t, \bar{U}^n_t) dt - \bar{Z}^n_t dW_t - \bar{U}^n_t(x).\bar{\mu}(dt, dx), \quad \bar{V}^n_T = |\xi_T|. \]

where

\[
\begin{align*}
\bar{h}^n(y, z, u) &= cy + |l| + \frac{1}{2} |z|^2 1_{\{ |z| \leq m \}} + n(|z| - \frac{m}{2}) 1_{\{ |z| > m \}} \\
&+ \int_E \left[ (e^{u(e)} - u(e) - 1) 1_{\{ e^{u(e)} \leq 1 \}} + \left( (m + 1) \ln(n + 1) + n(u(e) - 1) \right) 1_{\{ e^{u(e)} > 1 \}} \right] \zeta_t(e) \rho(de)
\end{align*}
\]

We use the same arguments of first order condition to deduce the solution of the second optimization problem; We get the triple \( (\bar{V}^m, \bar{Z}^m, \bar{U}^m) \) is the solution of the BSDE associated to \( (\bar{h}^m, -|\xi_T|) \):

\[ -d\bar{V}^m_t = h^m(\bar{V}^m_t, \bar{Z}^m_t, \bar{U}^m_t) dt - \bar{Z}^m_t dW_t - \int_E \bar{U}^m(x).\bar{\mu}(dx, dt), \quad \bar{V}^m_T = -|\xi_T|. \]

where

\[
\begin{align*}
\bar{h}^m(y, z, u) &= cy - |l| - \frac{1}{2} |z|^2 1_{\{ |z| \leq m \}} - m(|z| - \frac{m}{2}) 1_{\{ |z| > m \}} \\
&+ \int_E \left[ g(-u(e)) 1_{\{ e^{-u(e)} \leq 1 \}} + \left( (m + 1) \ln(n + 1) + m(u(e) - 1) \right) 1_{\{ e^{-u(e)} > 1 \}} \right] \zeta_t(e) \rho(dx)
\end{align*}
\]

To finish the proof, we find \( \bar{Y}^n \geq 0 \) and \( \bar{Y}^m \leq 0 \) since \( |\xi_T| \geq 0 \) and \( -|\xi_T| \leq 0 \) by comparaison theorem in Lipschitz case. Then we conclude \( \bar{h}^n = \bar{q}^n \) and \( \bar{h}^m = \bar{q}^m \) for each \( n, m \in \mathbb{N} \). By uniqueness of the solution of the BSDE associated to \( (\bar{q}^n, |\xi_T|) \) and \( (\bar{q}^m, -|\xi_T|) \), we conclude \( \bar{V}^n = \bar{Y}^n \) and \( \bar{V}^m = \bar{Y}^m \) moreover since \( \bar{q}^m \leq \bar{q}^n \) by Comparison Theorem, we find \( \bar{Y}^m \leq \bar{Y}^n \). However, from the dual representation of \( \bar{Y}^n \) (resp \( \bar{Y}^m \)) given by \( \bar{V}^n \) resp( \( \bar{V}^m \)), we conclude by Proposition 4.2 of [4] that for any \( \sigma \leq T \):

\[
\bar{Y}^m_{\sigma} \leq \rho_{\sigma} \left[ S^c_{\sigma,T}|\xi_T| + \int_{\sigma}^{T} S^c_{\sigma,s} |s| ds \right] \quad \text{and} \quad \bar{Y}^m_{\sigma} \leq \rho_{\sigma} \left[ S^c_{\sigma,T}|\xi_T| + \int_{\sigma}^{T} S^c_{\sigma,s} |s| ds \right].
\]

\[ \Box \]

**Theorem 6.8.** Let \( X \) a semimartingale and \( f \in C^2 \), then \( f(X) \) is a semimartingale and we have:

\[
f(X_t) = f(X_0) + \int_0^t \frac{\partial f}{\partial x}(X_{s-}) dX_s + \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x^2}(X_{s-}) d(X^c)_s + \sum_{s \leq t} \left[ f(X_s) - f(X_{s-}) - \frac{\partial f}{\partial x}(X_{s-}) \Delta X_s \right]
\]

27
Moreover for any semimartingale $X = X_0 - V + M^c + M^d$, with continuous finite variation $V$, continuous martingale $M^c$ and jump martingale $M^d = U. (\mu - \nu)$ for some $U \in G_{loc}$, we find:

$$f(X_t) = f(X_0) + \int_0^t \frac{\partial f}{\partial x}(X_s^-)dX_s + \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x^2}(X_s^-)d\langle M^c \rangle_s$$
$$+ \int_0^t \int_\mathbb{R} \left[ f(X_s^- + U(s,x)) - f(X_s^-) - \frac{\partial f}{\partial x}(X_s^-)U(s,x) \right] \mu(ds,dx)$$

**Corollary 6.1.** Let $f \in C^2$ and consider the operators $D_1$ and $D_2$ defined by:

$$D_1 f(x,y) = \begin{cases} \frac{f(x+y)-f(y)}{y}, & \text{if } y \neq 0, \\ f'(x), & \text{if } y = 0 \end{cases}$$

and

$$D_2 f(x,y) = \begin{cases} \frac{2[f(x+y)-f(y)-yf'(y)]}{y^2}, & \text{if } y \neq 0, \\ f''(x), & \text{if } y = 0 \end{cases}$$

For all $x,y \in \mathbb{R}$. then for any semimartingale $X$, the Itô decomposition of $f(X)$ is given by:

$$f(X_t) = f(X_0) + \int_0^t D_1 f(X_s,0)dX_s + \frac{1}{2} \int_0^t D_2 f(X_s,\Delta X_s)d[X]_s.$$ 

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