Induced Quadratic Modules

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Abstract

We give the constructions of pullback (or co-induced) and induced quadratic modules.

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Introduction

Algebraic models for homotopy connected 3-types can be thought as extended version of crossed modules which models for 2-types introduced by Whitehead in [19]. Some of these are 2-crossed modules [13], braided regular crossed modules [4], crossed squares [16] and quadratic modules [3]. For the categorical relations among these structures see [2].

Some universal constructions for crossed modules, for example, the notions of pullback and induced crossed modules have been worked in [5, 6, 7]. Furthermore, for Lie algebra cases of these constructions see [12], and for commutative algebras see [17]. By extending these constructions for two dimensional case of crossed modules, Arslan, Arvasi and Onarli in [1], have defined the notions of pullback and induced 2-crossed module. Brown and Sivera in [9] gave a construction of the induced crossed square. This gives another view of a presentation of the induced crossed square in [8] and which is applied to free crossed squares in [14] for homotopy type calculations. For another applications of higher homotopy van Kampen theorem see also [15]. In this work, by using a similar way given in these cited works, we have constructed the pullback and induced quadratic modules. More precisely, if \( \sigma : B \to C_0 \) is a monomorphism of groups, then there is a ‘pullback’ or restriction functor \( \sigma^* : \text{Quad}/C_0 \to \text{Quad}/B \), where \( \text{Quad}/C_0 \) is the subcategory of the category of quadratic modules \( \text{Quad} \) made up by the quadratic \( C_0 \)-modules. We have also constructed a functor, and the image by this functor of a quadratic \( Q \)-module is called the induced quadratic module.

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1 Quadratic Modules

Quadratic modules of groups were initially defined by Baues in [3] as models for connected 3-types. In this section we will give a construction of a pullback quadratic module. Firstly,
we recall some basic definitions from [3].

Recall that a pre-crossed module is a group homomorphism $\partial : M \to Q$ together with an action of $Q$ on $M$, written $m^q$ for $q \in Q$ and $m \in M$, satisfying the condition $\partial(m^q) = q^{-1}\partial(m)q$ for all $m \in M$ and $q \in Q$.

A nil(2)-module (cf. [3]) is a pre-crossed module $\partial : M \to Q$ with an additional "nilpotency" condition. This condition is $P_3(\partial) = 1$, where $P_3(\partial)$ is the subgroup of $M$ generated by Peiffer commutator $\langle x_1, x_2, x_3 \rangle$ of length 3. The Peiffer commutator in a pre-crossed module $\partial : M \to Q$ is defined by $\langle \partial \rangle = x^{-1}y^{-1}x(y)^{-1}\partial x$ for $x, y \in M$. For a pre-crossed module $\partial : M \to Q$, if $\langle \partial \rangle = 1$, then it is called a crossed module. That is, a nil(1)-module is a crossed module.

A morphism between two nil(2)-modules $\partial : M \to Q$ and $\partial' : M' \to Q'$ is a pair $(g, f)$ of homomorphisms of groups $g : M \to M'$ and $f : Q \to Q'$ such that $f\partial = \partial'g$ and the actions preserved, i.e. $g(m^q) = f(q)g(m)$ for any $m \in M, q \in Q$. We shall denote the category of nil(2)-modules by $\text{Nil}(2)$. Now we can give the following definition from [3].

**Definition 1.1** A quadratic module $(\omega, \partial_2, \partial_1)$ is a diagram

$$
\begin{array}{ccc}
C \otimes C & \xrightarrow{\omega} & C_0 \\
\downarrow{\partial_2} & & \downarrow{\partial_1} \\
C_2 & \xrightarrow{\omega} & C_1 \\
\end{array}
$$

of homomorphisms between groups such that the following axioms are satisfied.

**QM1** The homomorphism $\partial_1 : C_1 \to C_0$ is a nil(2)-module with Peiffer commutator map $w$ defined above. The quotient map $C_1 \to C = (C_1^{\text{cr}})^{\text{ab}}$ is given by $x \mapsto \{x\}$, where $\{x\} \in C$ denotes the class represented by $x \in C_1$ and $C = (C_1^{\text{cr}})^{\text{ab}}$ is the abelianization of the associated crossed module $C_1^{\text{cr}} \to C_0$.

**QM2** The boundary homomorphisms $\partial_2$ and $\partial_1$ satisfy $\partial_1 \partial_2 = 1$ and the quadratic map $\omega$ is a lift of the Peiffer commutator map $w$, that is $\partial_2 \omega = w$.

**QM3** $C_2$ is a $C_0$-group and all homomorphisms of the diagram are equivariant with respect to the action of $C_0$. Moreover, the action of $C_0$ on $C_2$ satisfies the formula $(a \in C_2, x \in C_1)$

$$a^{\partial_1 x} = \omega(\{x\} \otimes (\partial_2 a))((\partial_2 a) \otimes \{x\})a.$$

**QM4** Commutators in $C_2$ satisfy the formula $(a, b \in C_2)$

$$\omega(\{\partial_2 a\} \otimes \{\partial_2 b\}) = [b, a].$$

A morphism $\varphi : (\omega, \partial_2, \partial_1) \to (\omega', \partial'_2, \partial'_1)$ between quadratic modules is given by a commuta-
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A simplicial group \( G \) consists of a family of groups \( G_n \) together with face and degeneracy maps \( d^n_i : G_n \to G_{n-1}, 0 \leq i \leq n (n \neq 0) \) and \( s^n_i : G_n \to G_{n+1}, 0 \leq i \leq n \) satisfying the usual simplicial identities. In [2], the first and third authors have defined a functor from the category of simplicial groups with Moore complex of length 2 to that of quadratic modules. Therefore we can say that the Moore complex of a 2-truncated simplicial group gives a quadratic module.

1.1 Examples of Quadratic Modules

Porter in [18] has given the relations between 2-crossed complexes with the trivial Peiffer lifting map and crossed complexes. Here we give the similar relations about quadratic modules. The construction of quadratic modules from simplicial groups given in [2] gives a generic family of examples.

Example 1.2 ([3]) Any nil(2)-module \( \partial : M \to N \) yields a quadratic module \( \overline{\partial} : (1, w, \partial) \) given by

\[
\begin{array}{ccc}
C \otimes C & \xrightarrow{\omega} & C_2 \\
\downarrow{\varphi \otimes \varphi^*} & & \downarrow{f_2} \\
C' \otimes C' & \xrightarrow{\omega'} & C'_2
\end{array}
\]

where \( (f_1, f_0) \) is a morphism between nil(2)-modules which induces \( \varphi_* : C \to C' \) and where \( f_2 \) is an \( f_0 \)-equivariant homomorphism. We shall denote the category of quadratic modules by \( \text{Quad} \).

Example 1.3 A nil(2)-complex of groups is a positive chain complex of groups

\[
\begin{array}{ccc}
C & \xrightarrow{\partial_2} & C_2 \\
\downarrow{\partial_1} & & \downarrow{\partial_1} \\
C_1 & \xrightarrow{\partial_1} & C_0
\end{array}
\]

in which

(i) \( \partial_1 : C_1 \to C_0 \) is a nil(2)-module

(ii) For \( n \geq 2 \), \( C_n \) is Abelian and for \( n \geq 1 \), \( C_n \) is a \( C_0 \)-group and \( \partial_1(C_1) \) acts trivially on \( C_n \) for \( n \geq 2 \),
(iii) for $n \geq 1$ $\partial_n \partial_{n+1} = 1$.

Any nil(2)-complex of length 2, that is one of form

$$\cdots 1 \rightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

in which $\partial_1 : C_1 \rightarrow C_0$ is a nil(2)-module, gives us a quadratic complex

$$\cdots 1 \xrightarrow{\partial_1} L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N$$

on taking $L = C_2$, $M = C_1$, $C = (C^{cr})^{ab}$, $N = C_0$ with $\omega(\{x\} \otimes \{y\}) = 1$ for all $x, y \in M$.  

This is of course functorial and we can say that there is a functor from the category of nil(2)-complexes of length 2 to that of full subcategory of quadratic complexes of length 1 in which the quadratic modules with trivial quadratic map.

**Exploration of trivial quadratic map**

Suppose we have a quadratic module

$$\cdots 1 \xrightarrow{\partial_1} L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N$$

with the extra condition that $\omega(\{x\} \otimes \{y\}) = 1$ for all $x, y \in M$. The obvious thing to do is to see what each of the defining properties of a quadratic module give in this case.

(i) $\omega$ is a lifting of the Peiffer commutator, so if $\omega(\{x\} \otimes \{y\}) = 1$, the Peiffer identity holds for $\partial_1$, i.e., that is a crossed module. Indeed, from axiom QM2) we have

$$1 = x^{-1}y^{-1}xy^{\partial_1 x} = \omega(\{x\} \otimes \{y\}) = \partial_2 \omega(\{x\} \otimes \{y\})$$

for all $x, y \in M$.

(ii) From axiom QM4), if $l_0, l_1 \in L$, then

$$1 = [l_0, l_1] = \omega(\{\partial_2 l_0\} \otimes \{\partial_2 l_1\})$$

so $L$ is Abelian and

(iii) as $\omega$ is trivial, from QM3, we have $l^{\partial_1 (x)} = l$, so $\partial_1 (M)$ has trivial action on $L$.

This is functorial and we can say that there is a functor from the full subcategory of quadratic complexes of length 1 given by those quadratic complexes with trivial quadratic map to the category of crossed complexes of length 2. For further work about crossed and quadratic complexes see [8] and [7].
2 Pulback Quadratic Module

In this section we give a construction of a pullback quadratic module. Firstly, we should give the construction of a pullback nil(2)-module.

2.1 Pulback (Co-induced) Nil(2)-Module

Suppose that $\partial : M \rightarrow Q$ is a nil(2)-module and $\sigma : P \rightarrow Q$ is a homomorphism of groups. We give a construction of a pullback nil(2)-module hence we define a functor which changes the base of $\partial$ from $Q$ to $P$. That is, we shall define a functor

$$\lambda : \text{Nil}(2)/Q \longrightarrow \text{Nil}(2)/P$$

where $\text{Nil}(2)/Q$ is the subcategory of $\text{Nil}(2)$ whose objects are nil(2)-modules with the common codomain $Q$. Consider the following diagram

$$\begin{array}{ccc}
M & \xrightarrow{\partial} & Q \\
\downarrow & & \downarrow \\
P & \xrightarrow{\sigma} & Q
\end{array}$$

Take $\sigma^*(M) = \{(p, m) : \partial(m) = \sigma(p)\}$ as the fiber product of $\partial$ and $\sigma$. Thus we have the following pullback diagram

$$\begin{array}{ccc}
\sigma^*(M) & \xrightarrow{\sigma_1} & M \\
\downarrow & \downarrow & \downarrow \\
P & \xrightarrow{\beta_1} & Q
\end{array}$$

where $\sigma_1 : \sigma^*(M) \rightarrow P$ is given by $\sigma_1(p, m) = m$ and $\beta_1 : \sigma^*(M) \rightarrow P$ is given by $\beta_1(p, m) = p$ for all $(p, m) \in \sigma^*(M)$. The action of $p' \in P$ on $(p, m) \in \sigma^*(M)$ can be given by

$$(p, m)^{p'} = (p'^{-1}pp', m^{\sigma(p')}).$$

This action obviously is a group action of $P$ on $\sigma^*(M)$ and according to this action, $\beta_1$ becomes a nil(2)-module. Indeed, $\beta_1$ is a pre-crossed module since,

$$\beta_1((p, m)^{p'}) = \beta_1(p'^{-1}pp', m^{\sigma(p')})$$

$$= p'^{-1}pp'$$

$$= p'^{-1}\beta_1(p, m)p'.$$
for all \((p, m) \in \sigma^*(M)\) and for \((p_1, m_1), (p_2, m_2), (p_3, m_3) \in \sigma^*(M)\), we have

\[
\langle\langle (p_1, m_1), (p_2, m_2) \rangle, (p_3, m_3) \rangle = \langle\langle p_1^{-1}(p_2, m_2)^{-1}(p_1, m_1)(p_2, m_2)^{\partial_1(p_1, m_1)}, (p_3, m_3) \rangle
\]

\[
= \langle\langle 1, m_1^{-1}m_2^{-1}(m_1m_2^{\sigma_1(p_1)}), (p_3, m_3) \rangle
\]

\[
= \langle\langle 1, m_1^{-1}m_2^{-1}(m_1m_2^{\sigma_1(m_1)}), (p_3, m_3) \rangle
\]

\[
= \langle\langle 1, m_1^{-1}m_2^{-1}m_1m_2^{\sigma_1(m_1)}(1, m_1^{-1}m_2^{-1}m_1m_2^{\sigma_1(m_1)}) \rangle
\]

\[
= \langle\langle 1, m_1^{-1}m_2^{-1}m_1m_2^{\sigma_1(m_1)}(p_3, m_3) \rangle
\]

\[
= \langle\langle 1, 1 \rangle\rangle
\]

Since \(\partial : M \to Q\) is a nil(2)-module, we have \(\langle\langle m_1, m_2, m_3 \rangle, (p_3, m_3) \rangle = (1, 1) \in \sigma^*(M)\).

Similarly, it can be shown that \(\langle\langle p_1, m_1 \rangle, (p_2, m_2), (p_3, m_3) \rangle = (1, 1)\). Thus \(\beta_1 : \sigma^*(M) \to P\) is a nil(2)-module. In diagram (1), the pair of homomorphisms \((\sigma_1, \sigma)\) is a nil(2)-module morphism. This diagram is commutative since \(\partial\sigma_1(p, m) = \partial(m) = \sigma(p) = \sigma\beta_1(p, m)\) for \(p \in P\) and \(m \in M\). We have

\[
\sigma_1((p, m)^{p'}) = \sigma_1((p')^{-1}pp', m^{\sigma(p')})
\]

\[
= m^{\sigma(p')}
\]

\[
= \sigma_1(p, m)^{\sigma(p')}
\]

for all \((p, m) \in \sigma^*(M)\) and \(p \in P\). Thus we have a nil(2)-module with the base \(P\). Obviously this is functorial and we can define a functor by

\[
\lambda(\partial : M \to Q) = (\beta_1 : \sigma^*(M) \to P)
\]

which changes the base of the nil(2)-module \(\partial\) from \(Q\) to \(P\) and where \(\beta_1\) is the pullback nil(2)-module of \(\partial\) by the homomorphism \(\sigma\).

### 2.2 Construction of a Pullback Quadratic Module

Let

\[
\begin{array}{ccc}
C \otimes C & \overset{\omega}{\longrightarrow} & C_0 \\
\downarrow{\partial_2} & & \downarrow{\partial_1} \\
C_2 & \overset{\partial_2}{\longrightarrow} & C_1
\end{array}
\]
be a quadratic module of groups and \( \sigma : B \to C_0 \) a homomorphism of groups. We try to construct a pullback quadratic module by the homomorphism \( \sigma : B \to C_0 \). Given any homomorphisms of groups

\[
\begin{array}{c}
C_2 \\
\downarrow \partial_2 \\
\downarrow \partial_1 \\
C_1 \\
\downarrow \partial_1 \\
C_0
\end{array}
\]

and \( \sigma : B \to C_0 \), for the pullback \( \langle B_1, \beta_1, \sigma_1 \rangle \) of \( \partial_1 \) by \( \sigma \) and the pullback \( \langle B_2, \beta_2, \sigma_2 \rangle \) of \( \partial_2 \) by \( \sigma_1 \). Then \( \langle B_2, \beta_1 \beta_2, \sigma_2 \rangle \) is a pullback of \( \partial_1 \partial_2 \) by \( \sigma \). Now, consider the diagram

\[
\begin{array}{c}
C_2 \\
\downarrow \partial_2 \\
B/\ker \sigma \\
\downarrow \sigma^* \\
C_0
\end{array}
\]

where \( \sigma^* : B/\ker \sigma \to C_0 \) given by \( \sigma^*(b \ker \sigma) = \sigma(b) \) for \( b \in B \) and \( \partial_1 \partial_2 = 1 \). Thus the following diagram

\[
\begin{array}{c}
B_{21} \\
\downarrow p \\
B/\ker \sigma \\
\downarrow \sigma^* \\
C_0
\end{array}
\]

is a pullback diagram where

\[
B_{21} = \{ (b \ker \sigma, c_2) : \sigma(b) = \partial_1 \partial_2(c_2) = 1 \}
\]

\[
= \{ (b \ker \sigma, c_2) : b \in \ker \sigma \}
\]

\[
= \{ (\ker \sigma, c_2) : c_2 \in C_2 \}
\]

and where \( \beta \) and \( p \) are given by \( (\ker \sigma, c_2) \mapsto \ker \sigma \) and \( (\ker \sigma, c_2) \mapsto c_2 \) for \( (\ker \sigma, c_2) \in B_{21} \) respectively.

Now consider the following diagram

\[
\begin{array}{c}
C_1 \\
\downarrow \partial_1 \\
B/\ker \sigma \\
\downarrow \sigma^* \\
C_0
\end{array}
\]

in which \( \partial_1 \) is a nil(2)-module. The pullback nil(2)-module by the homomorphism \( \sigma^* : B/\ker \sigma \to C_0 \) can be given by a diagram

\[
\begin{array}{c}
B_1 \\
\downarrow \beta_1 \\
B/\ker \sigma \\
\downarrow \sigma^* \\
C_0
\end{array}
\]

where \( B_1 = \{ (b \ker \sigma, c_1) : \sigma^*(b \ker \sigma) = \sigma(b) = \partial_1(c_1) \} \) is the fiber product of \( \partial_1 \) and \( \sigma^* \). \( \beta_1 \), \( \overline{\sigma_1} \) are given by \( \beta_1 : B_1 \to B/\ker \sigma \), \( \beta_1(b \ker \sigma, c_1) = b \ker \sigma \) and \( \overline{\sigma_1} : B_1 \to C_1 \), \( \overline{\sigma_1}(b \ker \sigma, c_1) = \sigma_1 \).
$c_1$ for all $b \ker \sigma \in B/\ker \sigma$ and $c_1 \in C_1$. Then diagram (1) becomes a pullback diagram. Furthermore, from the following diagram

\[
\begin{array}{ccc}
C_2 & \xrightarrow{\partial_2} & C_1 \\
\downarrow & & \downarrow \\
B_1 & \xrightarrow{\pi_1} & C_1
\end{array}
\]

and since the pullback of a pullback is again a pullback, we can define a pullback of $\sigma_1$ and $\partial_2$ as given in the following diagram

\[
\begin{array}{ccc}
B_2 & \xrightarrow{\sigma_2} & C_2 \\
\downarrow & & \downarrow \\
B_1 & \xrightarrow{\pi_1} & C_1
\end{array}
\]

in which

\[
B_2 = \{ ((b \ker \sigma, c_1), c_2) : \sigma^*(b \ker \sigma) = \sigma(b) = \partial_1(c_1), \sigma_1(b \ker \sigma, c_1) = c_1 = \partial_2(c_2) \}
\]

\[
= \{ ((b \ker \sigma, c_1), c_2) : \sigma(b) = \partial_1(c_1) = \partial_1(\partial_2(c_2)) = 1 \}
\]

\[
= \{ (b \ker \sigma, c_1) : b \in \ker \sigma \}
\]

\[
= \{ (\ker \sigma, \partial_2(c_2), c_2) : c_2 \in C_2 \}
\]

and $\beta_2$ is given by $\beta_2(\ker \sigma, \partial_2(c_2), c_2) = (\ker \sigma, \partial_2(c_2))$ and $\sigma_2$ is given by $\sigma_2(\ker \sigma, \partial_2(c_2), c_2) = c_2$ for all $(\ker \sigma, \partial_2(c_2)) \in B_2$. Since for all $(\ker \sigma, \partial_2(c_2), c_2) \in B_2$

\[
\partial_2\sigma_2(\ker \sigma, \partial_2(c_2), c_2) = \partial_2(c_2) = c_1 = \sigma_1(\ker \sigma, c_1) = \sigma_1\beta_2(\ker \sigma, \partial_2(c_2), c_2),
\]

the diagram is commutative, and is also a pullback diagram. We can define an isomorphism $\Phi : B_2 \rightarrow B_{21}$ by $\Phi(\ker \sigma, \partial_2(c_2), c_2) = (\ker \sigma, c_2)$ for $(\ker \sigma, \partial_2(c_2), c_2) \in B_2$. By using this isomorphism, we have the following diagram

\[
\begin{array}{ccc}
B_2 & \xrightarrow{\phi} & B_{21} \\
\downarrow \beta_2 & & \downarrow p \\
B_{21} & \xrightarrow{\pi_1} & C_2 \\
\downarrow \beta_1 & & \downarrow \partial_2 \\
B/\ker \sigma & \xrightarrow{\sigma^*} & C_0
\end{array}
\]

where $\overline{\partial_2} : B_{21} \rightarrow B_1$ is given by $\overline{\partial_2}(\ker \sigma, c_2) = \beta_2(\ker \sigma, \partial_2(c_2), c_2) = (\ker \sigma, \partial_2(c_2))$ and $\beta_1(\ker \sigma, c_1) = b \ker \sigma$, for $(\ker \sigma, c_2) \in B_{21}$ and for $(b \ker \sigma, c_1) \in B_1$. 

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Then we have \( \beta_1 \sigma_2 (\ker \sigma, c_2) = \beta_1 (\ker \sigma, \partial_2 (c_2)) = \ker \sigma \in B / \ker \sigma \), hence,

\[
\begin{array}{ccc}
B_{21} & \xrightarrow{\sigma_2} & B_1 \\
\beta_1 & \downarrow & \downarrow \\
B / \ker \sigma & \xrightarrow{\partial_2} & B / \ker \sigma
\end{array}
\]

becomes a complex of groups.

But, we want to construct the pullback quadratic module by the homomorphism \( \sigma : B \to C_0 \) instead of \( \sigma^* : B / \ker \sigma \to C_0 \). To construct it by \( \sigma \), we must have \( B / \ker \sigma \cong B \). This is possible only if \( \ker \sigma = \{1\} \) which means \( \sigma \) is a monomorphism. That is, the homomorphism \( \sigma : B \to C_0 \) must be a monomorphism. We construct the pullback quadratic module by taking \( \sigma \) as a monomorphism. Then we have \( \sigma^* = \sigma \) and the following isomorphisms: \( B / \ker \sigma \cong B \) and

\[
B_1 = \{(b \ker \sigma, c_1) : \sigma^* (b \ker \sigma) = \sigma (b) = \partial_1 (c_1)\}
\cong \{(b, c_1) : \sigma (b) = \partial_1 (c_1)\} \quad (\because \ker \sigma = \{1\})
\cong \sigma^* (C_1)
\]

and where \( \sigma^* (C_1) \) is the usual fiber product. Thus we have the following commutative diagram

\[
\begin{array}{ccc}
B_1 & \xrightarrow{\sigma^* (C_1)} & C_1 \\
\beta_1 & \downarrow & \downarrow \beta_1 \\
B / \ker \sigma & \xrightarrow{\partial_2} & B / \ker \sigma \\
\end{array}
\]

in which the right square is a pullback square of \( \partial_2 \) by the monomorphism \( \sigma \) and where the maps \( \beta_1 \) and \( \mu_1 \) are given by \( \beta_1 (b, c_1) = b \) and \( \mu_1 (b, c_1) = c_1 \) for all \((b, c_1) \in \sigma^* (C_1)\) and then we have

\[
\partial_1 \mu_1 (b, c_1) = \partial (c_1) = \sigma (b) = \sigma \beta_1 (b, c_1).
\]

Thus we have that \( \beta_1 : \sigma^* (C_1) \to B \) is a pullback nil(2)-module by the homomorphism \( \sigma \) as constructed in section 2.1. Furthermore, since \( \ker \sigma = \{1\} \), we have an isomorphism

\[
B_{21} = \{(\ker \sigma, c_2) : c_2 \in C_2\} \cong \{(1, c_2) : c_2 \in C_2\} \cong \{1\} \times C_2 \cong C_2
\]

and we have the following diagram

\[
\begin{array}{ccc}
B_{21} & \xrightarrow{\sigma_2} & C_2 \\
\beta_2 & \downarrow & \downarrow \beta_2 \\
B_1 & \xrightarrow{\sigma^* (C_1)} & C_1 \\
\end{array}
\]

where \( \beta_2 : C_2 \to \sigma^* (C_1) \) is given by \( \beta_2 (c_2) = (1, \partial_2 (c_2)) \) for \( c_2 \in C_2 \). Thus the right square is a pullback square of \( \partial_2 \) by the homomorphism \( \mu_1 \). Consequently, we have the following
commutative diagram

\[
\begin{array}{ccc}
C_2 & \xrightarrow{id} & C_2 \\
\beta_2 & \downarrow & \partial_2 \\
\sigma^*(C_1) & \xrightarrow{\mu_1} & C_1 \\
\beta_1 & \downarrow & \partial_1 \\
B & \xrightarrow{\sigma} & C_0,
\end{array}
\]

where

\[
\begin{array}{ccc}
C_2 & \xrightarrow{\beta_2} & \sigma^*(C_1) \\
\beta_1 & \xrightarrow{\sigma} & B
\end{array}
\]

is a complex of groups, since for all \(c_2 \in C_2\)

\[
\beta_1 \beta_2(c_2) = \beta_1(1, \partial_2(c_2)) = 1.
\]

Now, we must define the quadratic map. Let \(C' = ((\sigma^*(C_1))^{cr})^{ab}\). The quadratic map \(\omega' : C' \otimes C' \rightarrow C_2\) can be given by

\[
\omega'((\{b_1, c_1\}) \otimes (\{b'_1, c'_1\})) = \omega((\{c_1\} \otimes \{c'_1\})
\]

for all \((b_1, c_1) \otimes (b'_1, c'_1) \in C' \otimes C'\) and \(c_1 \otimes c'_1 \in C \otimes C\) where \(C = (C_1^{cr})^{ab}\) and where \(\omega\) is the quadratic map of the first quadratic module.

Thus we have

**Proposition 2.1** The diagram

\[
\begin{array}{ccc}
C' \otimes C' & \xrightarrow{\omega'} & C_2 \\
\sigma^*(C_1) & \xrightarrow{\omega} & B \\
\beta_2 & \xrightarrow{\beta_1} & \beta_1
\end{array}
\]

is a quadratic \(B\)-module.

**Proof:**

**QM1:** In section 2.1 we have showed that \(\beta_1 : \sigma^*(C_1) \rightarrow B\) is a nil(2)-module and \(\beta_1 \beta_2 = 1\).

**QM2:** For all \((b_1, c_1), (b'_1, c'_1) \in \sigma^*(C_1)\), we have

\[
\beta_2 \omega'((\{b_1, c_1\}) \otimes (\{b'_1, c'_1\})) = \beta_2(\omega(c_1) \otimes c'_1)
= (1, \partial_2 \omega((\{c_1\} \otimes \{c'_1\}))
= (1, w((\{c_1\} \otimes \{c'_1\}))
\]
and since
\[
\langle (b_1, c_1), (b_2, c_2) \rangle = (b_1, c_1)^{-1}(b_2, c_2)^{-1}(b_1, c_1)(b_2, c_2)^h_i
\]
\[
= (b_1^{-1}, c_1^{-1})(b_2^{-1}, c_2^{-1})(b_1, c_1)(b_2, c_2)^h_i
\]
\[
= (b_1^{-1}b_2^{-1}b_1b_2, c_1^{-1}c_2^{-1}c_1c_2)^{\sigma(b_1)}
\]
\[
= (1, c_1^{-1}c_2^{-1}c_1c_2^{\sigma(b_1)})
\]
\[
= (1, w(c_1 \otimes c_1'))
\]
we have
\[
\overline{\beta}_2 \omega'((b_1, c_1) \otimes (b_1', c_1')) = (1, w((c_1 \otimes c_1')))
\]
\[
= w'((b_1, c_1) \otimes (b_1', c_1')).
\]

The verification of the other axioms of quadratic module is routine, so, we leave it to the reader. \hfill \Box

**Proposition 2.2** The constructed quadratic module

\[
\begin{array}{cccccc}
C' \otimes C' & \overset{\omega'}{\longrightarrow} & C_2 & \overset{\omega'}{\rightarrow} & \sigma^*(C_1) & \overset{\beta_1}{\longrightarrow} B \\
\end{array}
\]

is a pullback quadratic module of

\[
\begin{array}{cccccc}
C \otimes C & \overset{\omega}{\longrightarrow} & C_2 & \overset{\partial_2}{\rightarrow} & C_1 & \overset{\partial_1}{\rightarrow} C_0 \\
\end{array}
\]

by the monomorphism \(\sigma : B \to C_0\).

**Proof:** Firstly, we will show that in the following diagram

\[
\begin{array}{cccccc}
C_2 & \overset{id}{\longrightarrow} & C_2 & \overset{\partial_2}{\rightarrow} & C_1 & \overset{w}{\rightarrow} & C \otimes C \\
\end{array}
\]

\((id, \mu_1, \sigma)\) is a quadratic module morphism. In the above construction, we have showed that this diagram is commutative. Now, we show that \((id, \mu_1, \sigma)\) preserves the actions of \(B\). The
actions of $B$ on $\sigma^*(C_1)$ and $C_2$ are given by $(c_2)^b = (c_2)^{\sigma(b)}$ and $(b_1, c_1)^b = (b b_1^{-1}, c_1^{\sigma(b)})$ for $c_2 \in C_2$ and $(b_1, c_1) \in \sigma^*(C_1)$. We have $id(c_2)^b = c_2^b = c_2^{\sigma(b)}$,
\[
\mu_1((b_1, c_1)^b) = \mu_1(bb_1^{-1}, c_1^{\sigma(b)})
= c_1^{\sigma(b)}
= \mu_1(b_1, c_1)^{\sigma(b)}
\]
and
\[
\omega(\{\mu_1(b_1, c_1)\} \otimes \{\mu_1(b_1', c_1')\}) = \omega(\{c_1\} \otimes \{c_1'\})
= \omega'((\{b_1, c_1\} \otimes \{(b_1', c_1')\}
= id\omega'(\{b_1, c_1\} \otimes \{(b_1', c_1')\})
\]
for all $c_2 \in C_2, c_1 \in C_1$ and $b, b_1, b_1' \in B$. Thus $(id, \mu_1, \sigma)$ is a quadratic module morphism.

Now, we check the universal property.

Start with the quadratic module
\[
\begin{array}{c}
\xymatrix{ C \otimes C \ar[r]^\omega & \ar[d]^w \ar[r]^\partial_1 & \ar[d]^\partial_2 C_1 \ar[d]^\partial_1 & \ar[d]^\partial_2 C_0 \\
C_2 \ar[r]^\partial_2 & C_1 \ar[r]^\partial_1 & C_0 }
\end{array}
\]
and the homomorphism $\sigma : B \to C_0$, and the pullbacks $\langle \sigma^*(C_1), \beta_1, \mu_1 \rangle$ and $\langle C_2, \beta_2, id \rangle$ constructed above and any quadratic module
\[
\begin{array}{c}
\xymatrix{ C'' \otimes C'' \ar[r]^\delta & \ar[d]_{\beta_1} \ar[r]^\sigma^*(C_1) & B. \ar[d]_{\beta_1} \\
E \ar[r]^\delta & \ar[d]_{\beta_1} C_1 \ar[r]^\mu_1 & C_0. }
\end{array}
\]

Let $f : E \to B$ and $g : E \to C_2$ be two morphisms as given in the following diagram
\[
\begin{array}{c}
\xymatrix{ E \ar[r]^g & C_2 \ar[d]_{\beta_2} \ar[r]^\partial_2 & C_0 \ar[d]_{\beta_1} \\
B \ar[r]_{\sigma} & \sigma^*(C_1) \ar[ru]_{\mu_1} \ar[r]^\partial_1 & C_0 \ar[ru]_{\mu_1} \ar[r]^\partial_1 & C_0. }
\end{array}
\]
where $\sigma f = \partial_1 \partial_2 g = 1$. But then $\partial_2 g$ and the universal property of the pullback nil(2)-module $\langle \sigma^*(C_1), \beta_1, \mu_1 \rangle$ gives a unique morphism $h = \delta : E \to \sigma^*(C_1)$ with $f = \beta_1 h$ and $\partial_2 g = \mu_1 h$. 
There are two commutative diagrams

Using the isomorphism $B_{21} \cong C_2$ given above and the universal property for the pullback of a pullback $\langle C_2, id, \beta_2 \rangle$, for $g : E \to C_2$ and $h : E \to \sigma^*(C_1)$ with $\partial_2 g = \mu_1 h$ gives a map $\varepsilon : E \to C_2$ with $h = \beta_2 \varepsilon$ and $g = \varepsilon$. Thus, there are two commutative diagrams, where the second one is obtained by gluing two commutative diagrams together along $\mu_1 : \sigma^*(C_1) \to C_1$.

In particular $f = \beta_1 \beta_2 \varepsilon$ and $g = \varepsilon$, and the existence part of the proof has been accomplished.

Suppose now that $\eta : E \to C_2$ is another map with $f = \beta_1 \beta_2 \eta$. Then both $\beta_2 \varepsilon, \beta_2 \eta : E \to \sigma^*(C_1)$. Furthermore, the commutativity of the last diagram gives that $f = \beta_1 (\beta_2 \varepsilon) = \beta_1 (\beta_2 \eta)$ and $\partial_2 g = \partial_2 \varepsilon = \mu_1 (\beta_2 \varepsilon) = \mu_1 (\beta_2 \eta)$. By the uniqueness property for the pullback $\sigma^*(C_1)$, we have $\beta_2 \varepsilon = \beta_2 \eta$. We thus have two commutative diagrams:

Finally, the uniqueness for the pullback $\langle C_2, \beta_2, id \rangle$ constructed by using the isomorphism $B_{21} \cong C_2$ in the second diagram yields that $\eta = \varepsilon$. □

This construction can be expressed functorially

$$\sigma^* : \text{Quad}/C_0 \longrightarrow \text{Quad}/B$$
which is a pullback functor. This functor has a left adjoint

$$\sigma_* : \text{Quad}/B \to \text{Quad}/C_0$$

which gives an induced quadratic module as follows.

# 3 Induced Quadratic Module

In this section we give a construction of an induced quadratic module. We start by giving the construction of an induced nil(2)-module.

## 3.1 Induced Nil(2)-Module

Let \( \mu : M \to P \) be a nil(2)-module and \( f : P \to Q \) be a homomorphism of groups. Let \( f^*(M) = F(M \times Q) \) be a free group generated by the set \( M \times Q \). Let \( S \) be a subgroup of \( f^*(M) \) generated by the following relations: \( (m, m', q) \in M \times M \times Q \)

1. \((m, q)(m', q)(mm', q)^{-1} \in S \)
2. \((m^p, q)(m, f(p)q)^{-1} \in S \)

Now, consider the following diagram

\[
\begin{array}{ccc}
M & \xrightarrow{\theta} & f^*(M)/S \\
\downarrow{\mu} & \downarrow{\pi} & \\
P & \xrightarrow{f} & Q \\
\end{array}
\]

in which \( \pi : f^*(M)/S \to Q \) is given by \( \pi((m, q)S) = q^{-1}f\mu(m)q \) and \( \theta : M \to f^*(M)/S \) is given by \( \theta(m) = (m, 1)S \) for \( m \in M \) and \( q \in Q \). This diagram is commutative, since

\[
\pi\theta(m) = \pi((m, 1)S) = f\pi(m).
\]

for all \( m \in M \). The action of \( Q \) on \( f^*(M)/S \) can be given by

\[
((m, q)S)^q' = (m, qq')S
\]

for \( m \in M \) and \( q, q' \in Q \). By using this action, we have the following result.

**Theorem 3.1** The homomorphism \( \bar{\mu} : f^*(M)/S \to Q \) given by \( \bar{\mu}(m, q)S = q^{-1}f\mu(m)q \), as defined above, is an induced nil(2)-module by the homomorphism of groups \( f : P \to Q \) of the nil(2)-module \( \mu : M \to P \).
Proof: Since

\[ \mathfrak{p}((m, q)S) = \mathfrak{p}(m, q'')S \]
\[ = (qq')^{-1} f\mu(m)qq' \]
\[ = (q')^{-1}(q^{-1} f\mu(m)q)q' \]
\[ = (q')^{-1}\mathfrak{p}(m, q)S q', \]

for all \( m \in M \) and \( q, q' \in Q \), \( \mathfrak{p} \) is a pre-crossed module.

Further, for all \((m, q)S, (m', q)S, (m'', q)S \in f^*(M)/S, \)

\[ \langle((m, q)S, (m', q)S), (m'', q)S\rangle = \langle(m, q)S(m', q)S(m, q)S^{-1}((m', q)S^{-1}\mathfrak{p}(m,q)S), (m'', q)S\rangle \]
\[ = \langle(m, q)S(m', q)S(m^{-1}, q)S((m^{-1}, q)S)q^{-1}f\mu(m)qS, (m'', q)S\rangle \]
\[ = \langle(mm'm^{-1}, q)S((m^{-1}, q)S)q^{-1}f\mu(m)qS, (m'', q)S\rangle \]
\[ = \langle(mm'm^{-1}, q)(m^{-1})\mu(m), q)S, (m'', q)S\rangle \]
\[ = \langle(mm'm^{-1}(m^{-1})\mu(m), q)S, (m'', q)S\rangle \]
\[ = \langle((m', q), q)S, (m'', q)S\rangle \]
\[ = ((mm'm^{-1}, q)S(m^{-1}, q)S((m^{-1}, q)S)q^{-1}f\mu(m)qS) \]
\[ = ((mm'm^{-1}, q)(m^{-1})\mu(m), q)S, (m'', q)S \]
\[ = ((mm'm^{-1}(m^{-1})\mu(m), q)S, (m'', q)S) \]
\[ = (1, q)S \cong S \]

Similarly, it can be shown that

\[ \langle(m, q)S, ((m, q)S, (m'', q)S)\rangle = S. \]

Thus we have that \( \mathfrak{p} \) is a nil(2)-module. Now, we will show that \((\theta, f)\) is a nil(2)-module morphism. We have

\[ \theta(m^p) = (m^p, 1)S \]
\[ = (m, f\mu(p))S \]
\[ = ((m, 1)S)f(p) \]
\[ = \theta(m)f(p) \]
and \( \overline{\theta}(m) = \overline{\theta}((m,1)S) = f \mu(m) \) for all \( m \in M \) and \( p \in P \). Now we check the universal property. Consider the following diagram

\[
\begin{array}{c}
M \xrightarrow{\theta} F(M \times Q)/S \xrightarrow{h'} N \\
\downarrow \mu \downarrow \downarrow \\
P \xrightarrow{f} Q
\end{array}
\]

in which \( N \to Q \) is any nil(2) \( Q \)-module and

\[(h, f) : (M \to P) \to (N \to Q)\]

is any nil(2)-module morphism. The homomorphism \( h' : f^*(M)/S \to N \) given by \((m, q)S \mapsto h(m)q\) for \((m, q)S \in f^*(M)/S\) is the necessary unique morphism extending the commutativity of the diagram. Indeed, we have on generators

\[h'(m) = h'((m, 1)S) = h(m)\]

and

\[v h'((m, q)S) = v(h(m)q) = q^{-1} v(h(m))q = q^{-1} f \mu(m)q = \overline{\theta}((m, q)S).\]

Thus we have \( vh' = \overline{\theta} \). Therefore, \( \overline{\theta} \) is an induced nil(2)-module by the homomorphism of groups \( f : P \to Q \).

\[\square\]

### 3.2 Construction of an Induced Quadratic Module

**Definition 3.2** For any quadratic module

\[
\begin{array}{c}
C \otimes C \\
\downarrow \omega \downarrow w \\
L \xrightarrow{\omega} M \xrightarrow{w} P
\end{array}
\]

and a morphism \( \phi : P \to Q \), the induced quadratic module can be given by

(i) a quadratic module

\[
\begin{array}{c}
C' \otimes C' \\
\downarrow \omega' \downarrow w' \\
\phi_*(L) \xrightarrow{\omega'} \phi_*(M) \xrightarrow{w'} Q
\end{array}
\]

(ii) given a quadratic module morphism

\[(f_1, f_2, \phi) : (\sigma : (\omega, w, \partial_2)) \mapsto (\sigma'' : (\omega'', w'', \partial_2''))\]
then there is a unique quadratic module morphism
\[(f_1, f_2, id_Q) : \phi_* (\omega, w, \partial_2) \rightarrow (\sigma'' : (\omega'', w'', \partial_2'))\]
such that commutes the following diagram
\[
\begin{array}{ccc}
(f_1, f_2, \phi) & \rightarrow & (f_1, f_2, id_Q) \\
\downarrow & & \downarrow \\
(\sigma' : (\omega', w', \partial_2')) & \rightarrow & (f_1, f_2, id_Q) \phi_*(\omega, w, \partial_2).
\end{array}
\]

For a homomorphism of groups \(\phi : P \rightarrow Q\) and a quadratic module
\[
\begin{array}{ccc}
C \otimes C & \xrightarrow{\omega} & C \\
\downarrow \partial_2 & & \downarrow \partial_1 \\
P & \xrightarrow{w} & M & \xrightarrow{\partial_1} & P,
\end{array}
\]
let \(F(L \times Q)\) be a free group generated by elements of \(L \times Q\). Let \(S'\) be a normal subgroup of \(F(L \times Q)\) generated by following relations:

1. \((l, q)(m, q)(l^m, q)^{-1} \in S'\)
2. \((l, q)(l', q)(l l', q)^{-1} \in S'\)
3. \((l, q) p(l, \phi(p)q)^{-1} \in S'\)
4. \((l, q)q'(l, qq')^{-1} \in S'\)

for all \(l_1, l_2 \in L\), \(q_1, q_2 \in Q\) and \(p \in P\).

Recall from section 3.1 that if \(\partial_1 : M \rightarrow P\) is a nil(2)-module and \(\phi : P \rightarrow Q\) is a homomorphism of groups, we constructed the induced nil(2)-module \(\phi_*(M) \rightarrow Q\) where \(\phi_*(M) = F(M \times Q)/S\) and \(S\) is the normal subgroup of the free group \(F(M \times Q)\) generated by elements of the forms given in section 3.1.

We define \(\phi_*(L) = F(L \times Q)/S'\) and \(\phi_*(M) = F(M \times Q)/S\). From section 3.1 \(\overline{\partial}_1 : F(M \times Q)/S \rightarrow Q\) can be given by on generators \(\overline{\partial}_1((m, q)S) = q^{-1}(\phi \partial_1(m))q\) for all \(m \in M\) and \(q \in Q\). The map \(\overline{\partial}_2 : \phi_*(L) \rightarrow F(M \times Q)/S\) can be given by on generators \(\overline{\partial}_2((l, q)S') = (\partial_2 l, q)S\) for \(l \in L\) and \(q \in Q\). Since for all \(l, l' \in L\) and \(q \in Q\) and \(\partial_2(S') = S\), we have \(\overline{\partial}_2((l, q)S')(l', q)S') = (\partial_2 l, q)S\) so this is a well-defined group homomorphism.
Let \( C' = ((\phi_*(M))^\text{cr})^{ab} \). The quadratic map \( \omega' : C' \otimes C' \to \phi_*(L) \) can be given by on generators
\[
\omega'((m, q)S \otimes (m', q)S) = (\omega((m) \otimes (m')), q)S'
\]
where \((m, q)S, (m', q)S \in F(M \times Q)/S\), \{(m, q)S\}, \{(m', q)S\} \in C', \{m\}, \{m'\} \in C. Then we have

**Proposition 3.3** The diagram

\[
\begin{array}{ccc}
\phi_*(\omega; w, \partial_2) : & \phi_*(L) & \phi_*(M) \rightarrowdash Q \\
\omega' & & \omega' \\
\downarrow \partial_2 & & \downarrow \partial_1 \\
C' \otimes C' & & \\
\end{array}
\]

is a quadratic \( Q \)-module.

**Proof:**

**QM1:** In theorem 3.1 we have proven that \( \overline{\partial}_1 : \phi_*(M) \to Q \) is a nil(2)-module and we have \( \overline{\partial}_3 \overline{\partial}_2((l, q)S') = \overline{\partial}_1((\overline{\partial}_2l, q)S) = q^{-1}\phi \partial_1(\partial_2l)q = q^{-1}\phi(q) = 1 \), for all \((l, q)S' \in \phi_*(L)\).

**QM2:**

For all \(\{(m, q)S\} \otimes \{(m', q)S\} \in C' \otimes C',\) we have
\[
\overline{\partial}_2\omega'((m, q)S \otimes (m', q)S) = \overline{\partial}_2(\omega(m) \otimes (m'), q)S' \\
= (\overline{\partial}_2\omega(m) \otimes (m'), q)S' \\
= (w(m) \otimes (m'), q)S \\
= (mm'q^{-1}(m')^{-1}\partial_1(m), q)S.
\]

On the other hand,
\[
w'((m, q)S) \otimes (m', q)S) = (m, q)S(m', q)S((m, q)S)^{-1}((m', q)S)^{-1})\overline{\partial}_1((m, q)S) \\
= (mm'q^{-1}, q)(m^{-1}, q^{-1})\phi \partial_1(m)qS \\
= (mm'q^{-1}, q)((m^{-1})\partial_1(m), q)S \\
= (mm'q^{-1}(m')^{-1}\partial_1(m), q)S
\]
then we have
\[
\overline{\partial}_2\omega'((m, q)S \otimes (m', q)S) = (m^{-1}, q^{-1})(m^{-1}, q^{-1})(m, q)(m', q^{-1}\phi \partial_1(m)q)S \\
= w'((m, q)S) \otimes (m', q)S). \\
\]

**QM3:** For all \(\{(\overline{\partial}_2l, q)S' \otimes (m, q)S \in C' \otimes C'.\)
Now, we shall show that in this diagram (**φ** in which **σ**) 

\[ \omega'((Q_{2}(l, q)S')) \otimes \{(l, q)S\} \otimes \{(m, q)S\} \otimes \{(Q_{2}(l, q)S')\} \]

\[ = \omega'((\{Q_{2}(l, q)S\} \otimes \{(m, q)S\} \otimes \{(Q_{2}(l, q)S')\}) \]

\[ = (\omega\{Q_{2}(l)\} \otimes \{m\}, q)S' \cdot (\omega\{Q_{2}(l)\} \otimes \{m\}, q)S' \]

\[ = (\omega\{Q_{2}(l)\} \otimes \{m\} \cdot \{m\} \otimes \{Q_{2}(l)\}, q)S' \]

\[ = (l^{-1}l\partial_{m}, q)S' \]

\[ = (l^{-1}, q)S'(l\partial_{m}, q)S' \]

\[ = (l, q)S'(l, q^{-1}\partial_{l}mq)S' \]

\[ = (l, q)S'(l, q^{-1}\partial_{l}mq)S' \]

\[ = (l, q)S'(l, q^{-1})(l, q^{-1}), q^{-1})S'(l^{-1}, q^{-1})S' \]

\[ = (l, q)S'(l, q^{-1}) \]

\[ \square \]

**QM4:** For all \((\{Q_{2}(l, q)S')\} \otimes \{Q_{2}(l', q)S')\) \(\in C' \otimes C'.

\[ \omega'((Q_{2}(l, q)S')) \otimes \{Q_{2}(l', q)S')\} = \omega'((\{Q_{2}(l, q)S\} \otimes \{Q_{2}(l', q)S\}) \]

\[ = (\omega\{Q_{2}(l)\} \otimes \{Q_{2}(l')\}, q)S' \]

\[ = (l, q)S'(l, q^{-1}, q^{-1})S'(l^{-1}, q^{-1})S' \]

\[ = (l, q)S'(l, q^{-1}) \]

Now, we check the universal property for the constructed quadratic module \(\phi_{*}(\sigma) = \phi_{*}(\omega, w, \partial_{2})\). In the construction of the quadratic module \(\phi_{*}(\sigma)\), we have the following diagram

\[ \begin{array}{ccc}
C \otimes C & \xrightarrow{\omega} & L \\
\downarrow{w} & & \downarrow{\partial_{2}} \\
M & \xrightarrow{\phi_{*}(\sigma)} & Q_{2} \\
\downarrow{\partial_{1}} & & \downarrow{\phi_{1}} \\
P & \xrightarrow{\phi} & Q
\end{array} \]

in which \(\sigma_{1}\) and \(\sigma_{2}\) are given by \(\sigma_{1}(m) = (m, 1)S\) and \(\sigma_{2}(l) = (l, 1)S'\) for all \(l \in L\) and \(m \in M\).

Now, we shall show that in this diagram \((\phi, \sigma_{1}, \sigma_{2})\) is a quadratic module morphism.

We first show that the diagram (1) is commutative. For all \(m \in M\), we have

\[ \bar{Q}_{1}\sigma_{1}(m) = \bar{Q}_{1}((m, 1)S) = 1\phi\partial_{1}(m)1 = \phi\partial_{1}(m) \]
and, for all \( l \in L \)

\[
\bar{\partial}_2 \sigma_2(l) = \bar{\partial}_2((l, 1)S') \\
= ((\partial_2 l, 1)S) \\
= \sigma_1(\partial_2 l).
\]

Furthermore, for all \( p \in P \) and \( l \in L \), we have

\[
\sigma_2(l^p) = ((l^p, 1)S') \\
= (l, \phi(p))S' \quad (\because \text{generators of } S') \\
= ((l, 1)S')^{\phi(p)} \quad (\because \text{generators of } S') \\
= \sigma_2(l)^{\phi(p)},
\]

and for \( m \in M \) we have

\[
\sigma_1(m^p) = ((m^p, 1)S) \\
= (m, \phi(p))S \quad (\because \text{generators of } S) \\
= ((m, 1)S)^{\phi(p)} \quad (\because \text{generators of } S) \\
= \sigma_1(m)^{\phi(p)}.
\]

Thus \( \sigma_1 \) and \( \sigma_2 \) preserve the actions of \( P \) and diagram (1) is a commutative diagram.

Now we show that \( \sigma_1, \sigma_2 \) and \( \phi \) commute with the quadratic maps \( \omega \) and \( \omega' \). For \( m, m' \in M \), \( \sigma_1(m) = (m, 1)S \in \phi_*(M) \), \( \sigma_1(m') = (m', 1)S \in \phi_*(M) \) and \( \{\sigma_1(m)\} \otimes \{\sigma_1(m')\} \in C' \otimes C' \), we have

\[
\omega'(\{\sigma_1(m)\} \otimes \{\sigma_1(m')\}) = \omega'((m, 1)S \otimes (m', 1)S) \\
= (\omega\{m\} \otimes \{m'\}, 1)S' \\
= \sigma_2(\omega\{m\} \otimes \{m'\}).
\]

Consequently as shown in the following diagram

\[
\begin{array}{c}
C \otimes C \xrightarrow{\omega} L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} P \\
\downarrow \phi^* \quad \quad \downarrow \sigma_2 \quad \quad \downarrow \sigma_1 \quad \quad \downarrow \phi \\
C' \otimes C' \xrightarrow{\omega'} \phi_*(L) \xrightarrow{\partial_2} \phi_*(M) \xrightarrow{\partial_1} Q
\end{array}
\]

(\( \phi, \sigma_1, \sigma_2 \)) becomes a quadratic module morphism.

Now, suppose that

\[
(\sigma'': (\omega'', w'', \partial''_1)) : \quad B_2 \xrightarrow{\partial''_2} B_1 \xrightarrow{\partial''_1} Q
\]
is any quadratic $Q$-module and

$$(\phi, f_1, f_2) : (\sigma : (\omega, w, \partial_2)) \longrightarrow (\sigma'' : (\omega'', w'', \partial_2''))$$

is any quadratic module morphism. Since the constructions of $\phi_*(L)$ and $\phi_*(M)$, there is a unique morphism $(id_Q, f_1, f_2)$ as showed in the following diagram

![Diagram](image-url)

given by

$$f_2 : \phi_*(L) \rightarrow B_2; \quad f_2((l, q)S') = f_2(l)q$$

and

$$f_1 : \phi_*(M) \rightarrow B_1; \quad f_1((m, q)S') = f_2(m)q$$

for $l \in L, m \in M$ and $q, q' \in Q$. For example, we have

$$f_2((l, q)S')^{q'} = f_2((l, qq')S')$$

$$= f_2(l)qq'$$

$$= (f_2(l)^q)q'$$

$$= (f_2((l, q)S'))^q q'$$

and

$$f_1((m, q)S')^{q'} = f_1((m, qq')S')$$

$$= f_1(m)qq'$$

$$= (f_1(m)^q)q'$$

$$= (f_1((m, q)S'))^q q'$$

for $l \in L, m \in M$ and $q, q' \in Q$.

Consequently, the constructed quadratic module

$$\phi_*(\omega, w, \partial_2) : \phi_*(L) \longrightarrow \phi_*(M) \longrightarrow Q$$

is an induced quadratic module by the homomorphism $\phi : P \rightarrow Q$. 
Proposition 3.4 Let

\[ \begin{array}{c}
C \otimes C \\
\nu \downarrow \\
L \overset{\partial_2}{\longrightarrow} M \overset{\partial_1}{\longrightarrow} P
\end{array} \]

be a quadratic module, \( \phi : P \to Q \) be an epimorphism with \( \ker \phi = K \) then

\[ \phi_*(L) \cong L/[K, L] \quad \text{and} \quad \phi_*(M) \cong M/[K, M] \]

where \([K, L]\) is the subgroup of \(L\) generated by \(\{l^{-1}k : k \in K, l \in L\}\) and \([K, M]\) is the subgroup of \(M\) generated by \(\{m^{-1}k : k \in K, m \in M\}\).

Proof: As \( \phi : P \to Q \) is an epimorphism, \( Q \cong P/K \). Since \( Q \) acts on \( L/[K, L] \) and \( M/[K, M] \), \( K \) acts trivially on \( L/[K, L] \) and \( M/[K, M] \), \( Q \cong P/K \) acts on \( L/[K, L] \) by \( (l[K, L])^q = (l[K, L])^p = l^p[K, L] \) and \( M/[K, M] \) by \((m[K, M])^q = (m[K, M])^p = m^p[K, M] \) respectively.

\[ \begin{array}{c}
C' \otimes C' \\
\nu' \downarrow \\
L/[K, L] \overset{\partial_2}{\longrightarrow} M/[K, M] \overset{\partial_1}{\longrightarrow} Q
\end{array} \]

is a quadratic module where \( \partial_2 = \partial_2[l[K, M], \partial_1 = \partial_1 mK \), the action of \( M/[K, M] \) on \( L/[K, L] \) by \( (l[K, L])^m[K, M] = l^m[K, L] \). As \( \partial_1 \partial_2([l[K, L])] = \partial_1 \partial_2[l[K, M])] = \partial_1 \partial_2 lK = K \cong 1Q \),

\[ \begin{array}{c}
L/[K, L] \overset{\partial_2}{\longrightarrow} M/[K, M] \overset{\partial_1}{\longrightarrow} Q
\end{array} \]

is a complex of groups. The quadratic map \( \omega' : C' \otimes C' \to L/[K, L] \) is given by

\[ \omega'([m[K, M]]) = \omega([m[K, M]]) \]

QM1: We know that \( \partial_1 : M \to Q \) is a nil(2)-module for \( m[K, M], m'[K, M], m''[K, M] \in M/[K, M] \)

\[ \begin{align*}
\langle m[K, M], \langle m'[K, M], m''[K, M] \rangle \rangle &= \langle m[K, M], m'm''m'^{-1}[K, M](m''[K, M]^{-1})\partial_1(m'[K, M]) \rangle \\
&= \langle m[K, M], m'm''m'^{-1}[K, M](m''[K, M]^{-1})\partial_1 m' \rangle \\
&= \langle m[K, M], m'm''m'^{-1}[K, M](m'^{-1})\partial_1 m' [K, M] \rangle \\
&= \langle m[K, M], [m', m''][K, M] \rangle \\
&= m[K, M]\langle m', m''[K, M]m^{-1}[K, M](m', m'')^{-1}[K, M] \rangle \partial_1 m[K, M] \\
&= m[K, M]\langle m', m''[K, M]m^{-1}[K, M](m', m'')^{-1}[K, M] \rangle \partial_1 m[K, M] \\
&= \langle m, \langle m', m''[K, M]m^{-1}[K, M](m', m'')^{-1}[K, M] \rangle \partial_1 m[K, M] \rangle \\
&= [K, M]
\end{align*} \]
So, \( \partial_1 : M/[K, M] \to Q \) is a nil(2)-module.

**QM2:**

\[
\partial_2 \omega'([m[K, M]] \otimes [m'[K, M]]) = \partial_2 \omega([m] \otimes [m'])[K, L] = w([m] \otimes [m'])[K, M] = w'([m[K, M]] \otimes [m'[K, M]])
\]

**QM3:**

\[
\omega'([\partial_2 l[K, L]] \otimes [\partial_2 l'[K, L]]) = \omega'([\partial_2 l[K, M]] \otimes [\partial_2 l'[K, M]]) = [l, l'][[K, L]] = [l[K, L], l'[K, L]]
\]

**QM4:**

\[
\omega'([\partial_2 ([l[K, L]]) \otimes [m[K, M]]) \{m[K, M]\}) \otimes [\partial_2 ([l[K, L]])]) = \omega'([\partial_2 l[K, M]] \otimes [m[K, M]]) \otimes [\partial_2 [l[K, M]]] = (l[m][K, M])
\]

Additionally, universal property can be shown as in proposition 3.3

**Proposition 3.5** If \( \phi : P \to Q \) is an injection and

\[
\begin{array}{ccc}
C \otimes C & \xrightarrow{\omega} & C \\
\downarrow{\omega} & & \downarrow{\omega} \\
L & \xrightarrow{\partial_2} & M & \xrightarrow{\partial_1} & P
\end{array}
\]

is a quadratic module, let \( T \) be the right transversal of \( \phi(P) \) in \( Q \) and let \( B \) be the free product of groups \( L_T(t \in T) \) each isomorphic with \( L \) by an isomorphism \( l \mapsto l, (l \in L) \) and \( C \) be the free product of groups \( M_T(t \in T) \) each isomorphic with \( M \) by an isomorphism \( m \mapsto m, (m \in M) \). Let \( q \in Q \) acts on \( B \) by the rule \( (l_t)^q = (l^p)^u \) and similarly \( q \in Q \) acts on \( C \) by the rule \( (m_t)^q = (m^p)^u \), where \( p \in P, u \in T \) and \( qt = \phi(p)u \). Let

\[
\gamma : B \to C \quad \text{and} \quad \delta : C \to Q
\]

\[
l_t \mapsto \partial_2 (l_t) \quad \quad m_t \mapsto t^{-1} \phi_1 m_t
\]

and the action of \( C \) on \( B \) by \( (l_t)^m = (l^m)_t \). Then

\[
\phi_*(L) = B \quad \text{and} \quad \phi_*(L) = C \quad \text{and the quadratic map} \quad C \otimes C \to L \quad \text{is given by} \quad \omega(\{m\} \otimes \{m'\}) = (\omega(\{m\} \otimes \{m'\})_t)
\]
Remark 3.6 Since any $\phi : P \to Q$ is the composite of a surjection and an injection, an alternative description of the induced quadratic module can be obtained by using the construction methods of previous two propositions.

Now consider an arbitrary push-out square

$$\begin{align*}
&\{L_0, M_0, P_0, \omega, \partial_2, \partial_1\} \quad \{L_1, M_1, P_1, \omega, \partial_2, \partial_1\} \\
\downarrow & \quad \downarrow \\
&\{L_2, M_2, P_2, \omega, \partial_2, \partial_1\} \quad \{L, M, P, \omega, \partial_2, \partial_1\}
\end{align*}$$

(1)

of quadratic modules. In order to describe $\{L, M, P, \omega, \partial_2, \partial_1\}$, we first note that $P$ is the push-out of the group morphisms $P_1 \leftarrow P_0 \to P_2$. This is because the functor

$$\{L, M, P, \omega, \partial_2, \partial_1\} \to \{L/\omega(M/w(\{x, y\} \otimes \{z\}), P, \partial_1)\}$$

from quadratic module to nil(2)-module has a right adjoint $(N, P, \partial_1) \to (1, N, P, 1, 1, \partial)$ and the forgetful functor $(M/w(\{x, y\} \otimes \{z\}), P, \partial_1) \to P$ from nil(2)-module to group has a right adjoint $P \to (P, P, id)$. The morphisms $\phi_i : P_i \to P(i = 0, 1, 2)$ in (1) can be used to form induced quadratic $Q$-modules $B_i = (\phi_i)_* L_i$ and $C_i = (\phi_i)_* M_i$. Clearly $\{L, M, P, \omega, \partial_2, \partial_1\}$ is the push-out in $\text{Quad}/P$ of the resulting $P$-morphisms

$$(B_1 \to C_1 \to P) \leftarrow (B_0 \to C_0 \to P) \to (B_2 \to C_2 \to P)$$

can be described as follows.

**Proposition 3.7** Let

$$\begin{align*}
&D_i \otimes D_i \\
&B_i \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad C_i \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad P
\end{align*}$$

be a quadratic $P$-module for $i = 0, 1, 2$ where $D_i = (C_i^e)^{ab}$ and let

$$\begin{align*}
&C \otimes C \\
&L \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad M \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad P
\end{align*}$$

be the push-out in $\text{Quad}/P$ of $P$-morphisms

$$(B_1 \to C_1 \to P) \xleftarrow{(\alpha_1, \beta_1, id)} (B_0 \to C_0 \to P) \xrightarrow{(\alpha_2, \beta_2, id)} (B_2 \to C_2 \to P)$$
Let \((B \to M)\) be push-out of \((\alpha_1, \beta_1)\) and \((\alpha_2, \beta_2)\) in the category of \(\text{Nil}(2)\), equipped with the induced morphism \(B \xrightarrow{\mu} C \xrightarrow{\nu} P\), the quadratic map \(\omega : (C^{cr})^{ab} \otimes (C^{cr})^{ab} \to B\) and the induced action of \(P\) on \(B\) and \(C\). Then \(L = B/S\), where \(S\) is the normal subgroup of \(B\) generated by the elements of the form

\[
\omega(\{\mu b\} \otimes \{\mu b'\})[b, b']^{-1}
\]

\[
\omega(\{\mu b\} \otimes \{c\} \otimes \{\mu b'\})(b^{-1})^{\nu(c)}b
\]

and \(M = C/R\) where \(R\) is the normal subgroup of \(C\) generated by the elements of the form

\[
\mu \omega(\{c\} \otimes \{c'\})c^{\nu(c)}c'^{-1}c^{-1}
\]

for \(b, b' \in B, c, c' \in C\) and \(p \in P\).

In the case when \(\{L_2, M_2, P_2, \omega, \partial_2, \partial_1\}\) is the trivial quadratic module \(\{1, 1, 1, \text{id}, \text{id}, \text{id}\}\) the push-out quadratic module \(\{L, M, P, \omega, \partial_2, \partial_1\}\) in (1) is the cokernel of the morphism

\[
\{L_0, M_0, P_0, \omega, \partial_2, \partial_1\} \to \{L_1, M_1, P_1, \omega, \partial_2, \partial_1\}
\]

Cokernels can be described as follows

**Proposition 3.8** \(Q/P\) is the push-out of the group morphisms \(1 \leftarrow P \to Q\). Let \(\{A_s, G_s, Q/P, \omega, \partial_2, \partial_1\}\) be the induced from \(\{A, G, P, \omega, \partial_2, \partial_1\}\) by \(P \to Q/P\). If \(\{1, 1, Q/P, \text{id}, \text{id}, \text{id}\}\) and

\[
\{B/[P, B], H/[P, H], Q/P, \omega, \partial_2, \partial_1\}
\]

are induced from \(\{1, 1, 1, \text{id}, \text{id}, \text{id}\}\) and \(\{B, H, Q, \omega, \partial_2, \partial_1\}\) by \(1 \to Q/P\) and the epimorphism \(Q \to Q/P\) then the cokernel of a morphism

\[
(\beta, \lambda, \phi) : \{A, G, P, \omega, \partial_2, \partial_1\} \to \{B, H, Q, \omega, \partial_2, \partial_1\}
\]

is \(\{\text{coker}(\beta_s, \lambda_s), Q/P, \omega, \partial_2, \partial_1\}\) where \((\beta_s, \lambda_s)\) is a morphism of

\[
(A_s, G_s) \to (B/[P, B], H/[P, H]).
\]

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