3D Jackiw–Pi model: (anti-)chiral superfield approach to BRST formalism

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Abstract We discuss and derive the continuous Becchi–Rouet–Stora–Tyutin (BRST) and anti-BRST symmetry transformations for the Jackiw–Pi (JP) model of three (2 + 1)-dimensional (3D) massive non-Abelian 1-form gauge theory by exploiting the standard technique of (anti-)chiral superfield approach (ACSA) to BRST formalism where a few appropriate and specific sets of (anti-)BRST invariant quantities (i.e., physical quantities at quantum level) play a very important role. We provide the explicit derivation of the nilpotency and absolute anticommutativity properties of (anti-)BRST conserved charges and existence of Curci–Ferrari (CF)-condition within the realm of ACSA to BRST formalism where we take only a single Grassmannian variable into account. We also provide the clear proof of (anti-)BRST invariances of the coupled (but equivalent) Lagrangian densities within the framework of ACSA to BRST approach where the emergence of the CF-condition is observed.

1 Introduction

Gauge theories describe the dynamics of the elementary particles under local transformations according to certain operations which is an important theory because many successful field theories explaining three (i.e., electromagnetic, strong, weak) out of four fundamental interactions of nature can be theoretically described. A gauge theory is endowed with first-class constraints in the terminology of Dirac’s prescription for the classification scheme of constraints [1, 2]. The key signatures of the gauge invariant theory are the invariance and singularity of the Lagrangian density for a given theory. Quantum electrodynamics is an example of Abelian gauge theory with the symmetry group $U(1)$ which has one gauge field (i.e., the electromagnetic four-potential) with the photon being the gauge boson. The standard model of particle physics is based on the non-Abelian gauge theory with the symmetry group $U(1) \times SU(2) \times SU(3)$ which has twelve gauge bosons (i.e., the photon, three weak bosons and eight gluons). The non-Abelian gauge theory is very important in the different prospectives of field theories as this theory is the foundation of electroweak and strong interactions. Our present investigation is based on the massive non-Abelian 1-form gauge theory.

The covariant quantization of a gauge-field system has a long-time history started from the famous works of Feynman [3], Faddeev and Popov [4] and DeWitt [5, 6]. For the covariant canonical quantization of a given gauge invariant theory, the Becchi–Rouet–Stora–Tyutin (BRST) formalism plays an important role where infinitesimal local gauge parameter is replaced by the ghost and anti-ghost fields [7–10] to respect the unitarity of the given theory. Thus, we have two supersymmetric-type global BRST ($s_b$) and anti-BRST ($s_{ab}$) symmetry transformations at the quantum level. These symmetry transformations have two important and key features. Mathematically, first is the nilpotency of order two (i.e., $s_b^2 = 0$, $s_{ab}^2 = 0$), and second is the absolute anticommutativity (i.e., $s_b s_{ab} + s_{ab} s_b = 0$) of symmetry transformations. The first property signifies that both the quantum BRST and anti-BRST symmetry transformations are fermionic in nature, whereas the second property, physically, shows that both symmetries are linearly independent of each other. The key signature of a BRST-quantized non-Abelian gauge theory is the existence of the Curci–Ferrari (CF)-condition at the quantum level [11]. The CF-condition is (anti-)BRST invariant quantity which shows that it is a physical condition at the quantum level. On the other hand, the advanced methods of covariant quantization for general gauge theories are based either on the BRST symmetry principle realized in the well-known quantization scheme [12–14] by Batalin and Vilkovisky (BV) or on the extended BRST symmetry principle realized within the quantization method [15–17] proposed by Batalin, Lavrov and Tyutin (BLT).

The mass generation in gauge theory is an important aspect [18, 19]. However, it has been observed that the 4D topological massive (non-)Abelian gauge theories have been studied [20–23] within the ambit of BRST formalism where (non-)Abelian 1-form gauge field acquires a mass in a very natural way without taking any help of the Higgs mechanism. However, these models have issues with renormalization, consistency and unitarity. The BRST analysis of these models has been performed with the hope of reducing these issues [24–29]. But, it remains still an open problem to construct a theory that is free from the above issues.

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The topology of odd-dimensional spacetime allows the construction of gauge theories with novel and attractive features. Therefore, in lower dimension, it is interesting to propose an odd-dimensional massive non-Abelian 1-form gauge invariant model which is free from the above issues (i.e., renormalization, consistency and unitarity). One such massive non-Abelian 1-form model is the Jackiw–P i (JP) model, proposed by Jackiw and Pi, in three (2 + 1)-dimensions (3D) of spacetime where the parity is respected due to the presence of 1-form vector field [30]. This three-dimensional (3D) model allows the construction of gauge theory with the mass term in a natural fashion. The JP model has been studied earlier in the different prospective of theoretical interest [30–34]. This model is endowed with the set of two interesting continuous symmetry transformations which are Yang–Mills (YM) and non-Yang–Mills (NYM) symmetry transformations [33, 34].

The usual superfield approach (USFA) to BRST formalism [35–39] exploits the idea of horizontality condition (HC) for the derivation of off-shell nilpotent (anti-)BRST symmetry transformations for the gauge, ghost, and anti-ghost fields where full super- expansions of the superfield [with two Grassmannian variables (θ, ̄θ)] have been taken into account. The USFA does not explain the derivation of the matter fields in an interacting theory. This approach is generalized to derive the (anti-)BRST symmetries for the matter fields where the idea of HC and gauge invariant restriction(s) (GIR) are used together [40–42]. This extended version of USFA is known as augmented version of the superfield approach (AVSA). The superfield approach has also been extensively discussed for the general gauge theories which provide the geometrical interpretations of the BRST quantization within the framework of superfield approach to BRST formalism [43–46]. The BRST analysis of the non-Abelian JP model has been performed within the framework of AVSA to BRST formalism [33, 34]. Against the backdrop of the above discussions, we have applied the newly proposed (anti)-chiral superfield formalism (ACSA) [47–51] to derive the complete set of (anti-)BRST symmetries as well as CF-condition of the theory where (anti)-chiral super-expansions of the superfields (i.e., only one Grassmannian variable in expansions) have been taken into account. The combination of ACSA and modified Bonora–Tonin superfield approach (MBTSA) has been utilized to derive the complete set of (anti-)BRST symmetry transformations for the various reparameterization invariant models [52, 53]. In our present endeavor, for the first time, the BRST analysis of the odd-dimensional theory (i.e., massive 3D non-Abelian JP model) is discussed within the realm of ACSA to BRST formalism.

In the present paper, the subject matters of different sections are organized as follows. In Sect. 2, we discuss a couple of local gauge symmetry transformations (i.e., YM and NYM symmetries) for the JP model. Our Sect. 3 deals with the coupled (but equivalent) Lagrangian densities and its (anti-)BRST symmetry transformations. Section 4 is fully devoted to the derivation of the conserved (anti-)BRST charges, nilpotency and absolute anticommutativity properties of charges in ordinary spacetime. Our Sect. 5 contains the explicit derivation of the complete set of (anti-)BRST symmetry transformations within the realm of ACSA to BRST formalism. In Sect. 6, we express the conserved (anti-)BRST charges onto the (3, 1)-dimensional super-sub-manifolds [of the general (3, 2)-dimensional supermanifold] on which our theory is generalized and provides the proof of nilpotency and absolute anticommutativity properties of the charges within the ambit of ACSA to BRST formalism. In Sect. 7, we discuss the (anti-)BRST invariances of the Lagrangian densities within the realm of ACSA. Finally, in Sect. 8, we highlight the most important findings and conclusions of present endeavor, as well as a few future issues and research prospects.

Convention and Notations: We acquire, in the present investigation, the convention and notations of the spacetime Minkowskian metric as: ημν = diag(−1, +1, +1, +1), totally antisymmetric 3D Levi-Civita tensor εμνρ satisfies εμνρ εστρ = −3!, εμνρ ερστ = −2! δνσ, etc., and ε012 = +1 = ε012. The Greek indices μ, ν, η, ..., = 0, 1, 2 denote the time and space directions. We take the dot and cross-products as: L·M = Lα Mα, L×M = f abc L a M b T c between a set of two non-null vectors (Lα, Mα) in the SU(N) Lie algebraic space where the generators T a satisfy the commutation relation [T a, T b] = if abc T c with a, b, c, ..., = 1, 2, 3, ..., N2 − 1. The structure constants f abc are taken to be totally antisymmetric in a, b, c indices for the SU(N) Lie algebra. We have also acquired the convention of left-tensor definition in all relevant calculations with respect to the fermionic fields (C, ̄C).

2 Preliminaries: gauge symmetries

We begin with the gauge invariant Lagrangian density of the three (2 + 1)-dimensional (3D) massive non-Abelian 1-form gauge theory proposed by Jackiw and Pi [30, 33, 34]

\[
\mathcal{L} = -\frac{1}{4} F^{\mu\nu} \cdot F_{\mu\nu} - \frac{1}{4} \left( G^{\mu\nu} + g \, F^{\mu\nu} \times \rho \right) \cdot \left( G^{\mu\nu} + g \, F_{\mu\nu} \cdot \phi_{\eta} \right) + \frac{m}{2} \, \varepsilon^{\mu\nu\rho} F_{\mu\nu} \cdot \phi_{\eta},
\]

(1)

where 2-form [\mathcal{F}^{(2)} = \frac{1}{2} (dx^\mu \land dx^\nu) F_{\mu\nu} \cdot T] field strength tensor F_{\mu\nu} = δ_{\mu} A_{\nu} - δ_{\nu} A_{\mu} - g \, (A_{\mu} \times A_{\nu}) corresponds to the 1-form [\mathcal{A}^{(1)} = dx^\mu A_{\mu} \cdot T] vector field A_{\mu}, is derived from the Maurer–Cartan equation F^{(2)} = dA^{(1)} + i \, g \, (A^{(1)} \land A^{(1)}). Similarly, other field strength tensor G_{\mu\nu} = D_{\mu} \phi_{\nu} - D_{\nu} \phi_{\mu}, corresponding to the 1-form [\phi^{(1)} = dx^\mu \phi_{\mu} \cdot T] vector field \phi_{\mu}, is obtained from G^{(2)} = d\phi^{(1)} + i \, g \, [\phi^{(1)} \land A^{(1)} + A^{(1)} \land \phi^{(1)}] = \varepsilon^{\mu\nu\rho} (dx^\mu \land dx^\nu) G_{\mu\nu} \cdot T where the covariant derivative is defined as: D_{\mu} \phi_{\nu} = \partial_{\mu} \phi_{\nu} - g \, (A_{\mu} \times \phi_{\nu}). In the above, vector fields A_{\mu} and \phi_{\mu} have opposite parity to conserve the parity of the Lagrangian density, ρ is a scalar field (auxiliary field), m is the mass parameter and g is a coupling constant.
The above Lagrangian density \( (L) \) obeys a couple of local and continuous gauge symmetry transformations, namely YM gauge transformation \( (\delta_L) \) and NYM gauge transformations \( (\delta_R) \). The infinitesimal version of these symmetries is given as

\[
\begin{align*}
\delta_L \phi \mu & = -g (\phi \mu \times \Sigma), & \delta_L \rho & = -g (\rho \times \Sigma), & \delta_L A_\mu & = D_\mu \Sigma, \\
\delta_R F_{\mu \nu} & = -g (F_{\mu \nu} \times \Sigma), & \delta_R G_{\mu \nu} & = -g (G_{\mu \nu} \times \Sigma), \\
\delta_L A_\mu & = 0, & \delta_R \phi \mu & = D_\mu \Lambda, & \delta_L \rho & = \Lambda, \\
\delta_R F_{\mu \nu} & = 0, & \delta_R G_{\mu \nu} & = -g (F_{\mu \nu} \times \Lambda), \\
\end{align*}
\]

(2)

where \( \Sigma = \Sigma \cdot T \equiv \Sigma^a T^a \) and \( \Lambda = \Lambda \cdot T \equiv \Lambda^a T^a \) are the infinitesimal gauge parameters. It is straightforward to check that the Lagrangian density (1) transforms under the local and continuous gauge transformations (2) and (3) as

\[
\delta_L L = 0, \quad \delta_R L = \partial_\mu \left[ \frac{m}{2} \epsilon^{\mu \nu \rho} F_{\nu \rho} \cdot \Lambda \right].
\]

(3)

Hence, the action integral \( S = \int d^3 x L \) remains invariant under both the YM and NYM gauge transformations (i.e., \( \delta_L S = 0, \delta_R S = 0 \)). These two gauge transformations \( \delta_L \) and \( \delta_R \) are independent of each other. Therefore, in the present endeavor, we shall focus only on the BRST analysis of the YM gauge transformations within the ambit of ACSA.

3 Coupled Lagrangian densities: off-shell nilpotent quantum (anti-)BRST symmetries

In this section, we discuss the construction of the (anti-)BRST invariant coupled (but equivalent) Lagrangian densities for the three \((2 + 1)\)-dimensional (3D) massive non-Abelian 1-form JP and its off-shell nilpotent and absolutely anticommuting (anti-)BRST symmetry transformations corresponding to the YM gauge transformations. The Lagrangian densities for this model are;

\[
\begin{align*}
\mathcal{L}_B & = -\frac{1}{4} F_{\mu \nu} \cdot F_{\mu \nu} - \frac{1}{4} (G_{\mu \nu} + g F_{\mu \nu} \times \rho) \cdot (G_{\mu \nu} + g F_{\mu \nu} \times \rho) + \frac{m}{2} \epsilon^{\mu \nu \rho} F_{\mu \nu} \cdot \phi_\eta \,, \\
& \quad + B \cdot (\partial_\mu A_\mu) + \frac{1}{2} (B \cdot B + \bar{B} \cdot \bar{B}) - i \bar{\partial}^\mu \bar{\bar{C}} \cdot D_\mu C, \\
\end{align*}
\]

\[
\mathcal{L}_{\bar{B}} = -\frac{1}{4} F_{\mu \nu} \cdot F_{\mu \nu} - \frac{1}{2} (G_{\mu \nu} + g F_{\mu \nu} \times \rho) \cdot (G_{\mu \nu} + g F_{\mu \nu} \times \rho) + \frac{m}{2} \epsilon^{\mu \nu \rho} F_{\mu \nu} \cdot \phi_\eta \,, \\
& \quad - \bar{B} \cdot (\bar{\partial}_\mu A_\mu) + \frac{1}{2} (B \cdot B + \bar{B} \cdot \bar{B}) - i \bar{\partial}^\mu \bar{C} \cdot \bar{\partial}_\mu C,
\]

(5)

where \( B \) and \( \bar{B} \) are the Nakanishi–Lautrup type auxiliary fields that have been introduced for linearizing the gauge fixing terms which are connected with the Cacciari–Ferrari condition: \( B + \bar{B} + ig (\bar{C} \times C) = 0 \) where the (anti-)ghost fields \( \bar{C} \) and \( C \) are fermionic [i.e., \((C^a)^2 = (\bar{C}^a)^2 = 0, \ C^a C^b + C^b C^a = 0, \ C^a \bar{C}^b + \bar{C}^b C^a = 0, \ C^a \bar{C}^b + \bar{C}^b \bar{C}^a = 0, \ bar{C}^a C^b + C^b \bar{C}^a = 0, \text{ etc.} \) in nature. In the above, we have the covariant derivatives \([D_\mu C] = \partial_\mu C - g (A_\mu \times C) \) and \( D_\mu \bar{C} = \partial_\mu \bar{C} - g (A_\mu \times \bar{C}) \) in the adjoint representation.

The complete set of off-shell nilpotent and absolutely anticommuting (anti-)BRST symmetry transformations corresponding to the above coupled (but equivalent) Lagrangian densities Eqs. (5) and (6) are as follows:

\[
\begin{align*}
\delta_{sab} A_\mu & = D_\mu \bar{C}, & \delta_{sab} \bar{C} & = \frac{g}{2} (\bar{C} \times C), & \delta_{sab} B = -g (B \times \bar{C}), \\
\delta_{sab} \rho & = -g (\rho \times \bar{C}), & \delta_{sab} \phi_\mu & = -g (\phi_\mu \times \bar{C}), & \delta_{sab} C = i \bar{B}, \\
\delta_{sab} \bar{B} & = 0, & \delta_{sab} F_{\mu \nu} & = -g (F_{\mu \nu} \times \bar{C}), & \delta_{sab} G_{\mu \nu} & = -g (G_{\mu \nu} \times \bar{C}), \\
\delta_{sab} A_\mu & = D_\mu C, & \delta_{sab} C & = \frac{g}{2} (C \times C), & \delta_{sab} \bar{B} = -g (\bar{B} \times C), \\
\delta_{sab} \rho & = -g (\rho \times C), & \delta_{sab} \phi_\mu & = -g (\phi_\mu \times C), & \delta_{sab} \bar{C} = i B, \\
\delta_{sab} B & = 0, & \delta_{sab} F_{\mu \nu} & = -g (F_{\mu \nu} \times C), & \delta_{sab} G_{\mu \nu} & = -g (G_{\mu \nu} \times C).
\end{align*}
\]

(7)

The Lagrangian densities Eqs. (5) and (6) can be written in term of the starting Lagrangian density (1) plus (anti-)BRST symmetry transformations of the some specific terms as:

\[
\begin{align*}
\mathcal{L}_B = \mathcal{L}_0 + \sum_{sab} \left[ \frac{i}{2} A_\mu \cdot A^\mu - \bar{C} \cdot C + \frac{1}{2} \phi_\mu \cdot \phi^\mu \right], \\
\mathcal{L}_{\bar{B}} = \mathcal{L}_0 - \sum_{sab} \left[ \frac{i}{2} A_\mu \cdot A^\mu - \bar{C} \cdot C + \frac{1}{2} \phi_\mu \cdot \phi^\mu \right].
\end{align*}
\]

(9)
The anticommutators of the (anti-)BRST symmetries \([s_{ab}]\) for the various fields and field strength tensors present in the Lagrangian densities Eqs. (5) and (6) are given as:

\[
\begin{align*}
[s_b, s_{ab}] A_\mu &= i \frac{\partial_\mu}{\partial \epsilon} [B + \bar{B} + i g (\bar{C} \times C)] + i g [B + \bar{B} + i g (\bar{C} \times C)] \times A_\mu, \\
&= i \partial_\mu [B + \bar{B} + i g (\bar{C} \times C)], \\
[s_b, s_{ab}] \phi_\mu &= i g [B + \bar{B} + i g (\bar{C} \times C)] \times \phi_\mu, \\
[s_b, s_{ab}] \rho &= i g [B + \bar{B} + i g (\bar{C} \times C)] \times \rho, \\
[s_b, s_{ab}] \mathcal{F}_{\mu\nu} &= i g [B + \bar{B} + i g (\bar{C} \times C)] \times \mathcal{F}_{\mu\nu}, \\
[s_b, s_{ab}] \mathcal{G}_{\mu\nu} &= i g [B + \bar{B} + i g (\bar{C} \times C)] \times \mathcal{G}_{\mu\nu}.
\end{align*}
\]

Hence, it is clear that absolute anticommutativity of (anti-)BRST symmetries for various fields is satisfied if and only if CF-condition \([B + \bar{B} + i g (\bar{C} \times C) = 0]\) is satisfied. The anticommutators for the rest of the fields directly come out to be zero. Thus, ultimately, we have the following relationships

\[
\begin{align*}
[s_b, s_{ab}] A_\mu &= 0, & \{s_b, s_{ab}\} \phi_\mu &= 0, & \{s_b, s_{ab}\} \rho &= 0, \\
\{s_b, s_{ab}\} \mathcal{F}_{\mu\nu} &= 0, & \{s_b, s_{ab}\} \mathcal{G}_{\mu\nu} &= 0, & \{s_b, s_{ab}\} C &= 0, \\
\{s_b, s_{ab}\} \bar{C} &= 0, & \{s_b, s_{ab}\} \bar{B} &= 0.
\end{align*}
\]

It can be checked that the preceding Lagrangian densities \(L_B\) and \(L_{\bar{B}}\) transform under the off-shell nilpotent (anti-)BRST transformations, as

\[
\begin{align*}
s_b L_B &= \partial_\mu [B \cdot (\mathcal{D}^{\mu} C)], & s_b L_{\bar{B}} &= - \partial_\mu [\bar{B} \cdot (\mathcal{D}^{\mu} \bar{C})], \\
s_{ab} L_B &= \partial_\mu [B \cdot \partial^{\mu} \bar{C}] - \mathcal{D}_\mu [B + \bar{B} + i g (\bar{C} \times C)] \cdot \partial^{\mu} \bar{C}, \\
s_{ab} L_{\bar{B}} &= - \partial_\mu [\bar{B} \cdot \partial^{\mu} C] + \mathcal{D}_\mu [B + \bar{B} + i g (\bar{C} \times C)] \cdot \partial^{\mu} C.
\end{align*}
\]

Thus, the corresponding actions (i.e., \(S_B = \int d^2x \; L_B\) and \(S_{\bar{B}} = \int d^2x \; L_{\bar{B}}\)) remain invariant under the (anti-)BRST symmetries in the three \((2 + 1)\)-dimensional (3D) ordinary spacetime manifold where the CF-condition is satisfied.

### 4 (Anti-)BRST currents and charges: nilpotency and absolute anticommutativity properties

In this section, we discuss the conserved currents and conserved charges corresponding to the (anti-)BRST symmetry transformations using the Noether theorem. We prove the nilpotency and absolute anticommutativity properties of the conserved charges within the framework of BRST formalism. Toward this aim in mind, first of all, we derive the Noether (anti-)BRST currents \(J^{\mu}_{(a)b}\) for the 3D JP model as:

\[
\begin{align*}
J^{\mu}_{b} &= B \cdot (\mathcal{D}^{\mu} C) - \left[ F^{\mu\nu} - g \left((G^{\mu\nu} + g \mathcal{F}^{\mu\nu} \times \rho) \times \rho \right) - m \varepsilon^{\mu\nu\eta} \phi_\eta \right] \cdot (\mathcal{D}_\nu C) \\
&\quad + g \left(G^{\mu\nu} + g \mathcal{F}^{\mu\nu} \times \rho \right) \cdot (\phi_\nu \times C) + \frac{i}{2} g \partial^{\mu} \bar{C} \cdot (C \times C), \\
J^{\mu}_{ab} &= - \bar{B} \cdot (\mathcal{D}^{\mu} \bar{C}) - \left[ F^{\mu\nu} - g \left((G^{\mu\nu} + g \mathcal{F}^{\mu\nu} \times \rho) \times \rho \right) - m \varepsilon^{\mu\nu\eta} \phi_\eta \right] \cdot (\mathcal{D}_\nu \bar{C}) \\
&\quad + g \left[G^{\mu\nu} + g (F^{\mu\nu} \times \rho) \right] \cdot (\phi_\nu \times \bar{C}) + \frac{i}{2} g \partial^{\mu} C \cdot (\bar{C} \times \bar{C}).
\end{align*}
\]

These expressions for the Noether (anti-)BRST currents can be re-expressed for the algebraic convenience in the following form:

\[
\begin{align*}
J^{\mu}_{b} &= B \cdot (\mathcal{D}^{\mu} C) - \partial^{\mu} \bar{B} \cdot (C \times C) - \frac{i}{2} g \partial^{\mu} \bar{C} \cdot (C \times C) \\
&\quad - \partial_\nu (F^{\mu\nu} - g \left((G^{\mu\nu} + g \mathcal{F}^{\mu\nu} \times \rho) \times \rho \right) - m \varepsilon^{\mu\nu\eta} \phi_\eta \cdot C), \\
J^{\mu}_{ab} &= - \bar{B} \cdot (\mathcal{D}^{\mu} \bar{C}) + \partial^{\mu} B \cdot \bar{C} + \frac{i}{2} g \partial^{\mu} C \cdot (\bar{C} \times \bar{C}) \\
&\quad - \partial_\nu (F^{\mu\nu} - g \left((G^{\mu\nu} + g \mathcal{F}^{\mu\nu} \times \rho) \times \rho \right) - m \varepsilon^{\mu\nu\eta} \phi_\eta \cdot \bar{C}),
\end{align*}
\]

The conservation law [i.e., \(\partial_\mu J^{\mu}_{(a)b} = 0\)] of the above (anti-)BRST currents can be proven by exploiting the Euler-Lagrange equations of motion (EL-EoMs) derived from the Lagrangian densities \(L_B\) and \(L_{\bar{B}}\), respectively, as:

\[
\mathcal{D}_\mu F^{\mu\nu} - g \mathcal{D}_\mu [(G^{\mu\nu} + g \mathcal{F}^{\mu\nu} \times \rho) \times \rho] + m \varepsilon^{\mu\nu\eta} \mathcal{D}_\mu \phi_\eta - \partial^{\nu} B
\]

According to the Noether theorem whenever any Lagrangian density or its corresponding action remains invariant under any continuous symmetry transformations, there exits conserved currents and charges corresponding to that given continuous symmetries.
It is crystal clear to prove the nilpotency and absolute anticommutativity properties for the conserved (anti-)BRST charges \( Q \) on the conserved charges Eqs. (19) and (20), whereas absolute anticommutativity property is satisfied if and only if CF-condition \( \{ Q_s, Q_b \} = 0 \) is satisfied 2.

Now the expressions of the Noether conserved currents lead to the derivation of the following conserved (anti-)BRST charges using \( Q_{(ab)} = \int d^2x \mathcal{J}^0_{(ab)} \):

\[
Q_{ab} = - \int d^2x \left[ \mathcal{B} \cdot D^0 \mathcal{C} - \mathcal{B} \cdot \mathcal{C} - \frac{i}{2} \mathcal{g} \mathcal{C} \cdot (\mathcal{C} \times \mathcal{C}) \right], \\
Q_b = \int d^2x \left[ \mathcal{B} \cdot D^0 \mathcal{C} - \mathcal{B} \cdot \mathcal{C} - \frac{i}{2} \mathcal{g} \mathcal{C} \cdot (\mathcal{C} \times \mathcal{C}) \right].
\]

It is straightforward to check that the above-conserved charges \( Q_{(ab)} \) are nilpotent of order two (i.e., \( Q_b^2 = Q_{ab} = 0 \)) and they obey absolute anticommutativity property (i.e., \( Q_b Q_{ab} + Q_{ab} Q_b = 0 \)) in the ordinary \((2 + 1)\)-dimensional (3D) spacetime. These two properties are captured by the definition of generator expressed as follows:

\[
s_b Q_b = -i \{ Q_b, Q_b \} = 0 \implies Q_b^2 = 0, \\
s_{ab} Q_{ab} = -i \{ Q_{ab}, Q_{ab} \} = 0 \implies Q_{ab}^2 = 0, \\
s_{ab} Q_b = -i \{ Q_b, Q_{ab} \} = 0 \implies \{ Q_b, Q_{ab} \} = 0, \\
s_{ab} Q_{ab} = -i \{ Q_{ab}, Q_b \} = 0 \implies \{ Q_{ab}, Q_b \} = 0.
\]

It is evident that, from the above expressions, the nilpotency property is satisfied on the direct application of (anti-)BRST symmetry transformations on the conserved charges Eqs. (19) and (20), whereas absolute anticommutativity property is satisfied if and only if CF-condition \([\mathcal{B} + \mathcal{B} + i \mathcal{g} (\mathcal{C} \times \mathcal{C}) = 0]\) is satisfied 2.

The (anti-)BRST conserved charges can be written in terms of the (anti-)BRST exact form with respect to the (anti-)BRST symmetries \((s_{(ab)})\) as:

\[
Q_b = \int d^2x \ s_b \left[ \mathcal{B}(x) \cdot \mathcal{A}_0(x) + i \mathcal{g} \mathcal{C}(x) \cdot \mathcal{C}(x) \right], \\
Q_{ab} = \int d^2x \ s_{ab} \left[ \mathcal{B}(x) \cdot \mathcal{A}_0(x) + i \mathcal{g} \mathcal{C}(x) \cdot \mathcal{C}(x) \right], \\
Q_b = s_b \int d^2x \left[ i \mathcal{C} \cdot \dot{\mathcal{C}} - \frac{1}{2} \mathcal{C} \cdot (\mathcal{A}_0 \times \mathcal{C}) \right], \\
Q_{ab} = s_{ab} \int d^2x \left[ -i \mathcal{C} \cdot \dot{\mathcal{C}} + \frac{1}{2} \mathcal{C} \cdot (\mathcal{A}_0 \times \mathcal{C}) \right].
\]

It is crystal clear to prove the nilpotency and absolute anticommutativity properties for the conserved (anti-)BRST charges \( Q_{(ab)} \) in quite straightforward manner using the above (anti-)BRST exact form Eqs. (22) and (23) which demonstrate the alternate proof of these two properties for the (anti-)BRST charges.

5 Off-shell nilpotent (anti-)BRST symmetry transformations: (anti-)chiral superfield approach

In this section, we derive the off-shell nilpotent (anti-)BRST symmetry transformations by exploiting the (anti-)chiral superfields approach (ACSA) to BRST formalism where we use the (anti-)chiral super-expansions of the (anti-)chiral superfields. Toward this aim in mind, first of all, we generalize the ordinary fields of Lagrangian densities Eqs. (5) and (6) (of three \((2 + 1)\)-dimensional (3D) spacetime) onto the (3, 1)-dimensional (anti-)chiral super-sub-manifold of the suitably chosen general (3, 2)-dimensional supermanifold.

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2 We obtain the following expressions after the application of \( s_b \) on \( Q_{ab} \) and \( s_{ab} \) on \( Q_b \) for the proof of absolute anticommutativity as: \( s_b Q_{ab} = -i \int d^2x \left[ \mathcal{B} \cdot \phi^0 (\mathcal{B} + \mathcal{B} + i \mathcal{g} (\mathcal{C} \times \mathcal{C})) \right] = -i \{ Q_b, Q_{ab} \} = 0 \) and \( s_{ab} Q_b = i \int d^2x \left[ \mathcal{B} \cdot \phi^0 (\mathcal{B} + \mathcal{B} + i \mathcal{g} (\mathcal{C} \times \mathcal{C})) \right] = -i \{ Q_{ab}, Q_b \} = 0 \).
5.1 BRST symmetry transformations: ACSA

In this subsection, we concentrate on the derivation of the BRST symmetry transformations for all the fields of Lagrangian densities Eqs. (5) and (6) using the anti-chiral super-expansions of the superfields. For this, we generalize three (2 + 1)-dimensional basic and auxiliary ordinary fields onto (3, 1)-dimensional super-sub-manifold as

\[
A_\mu(x) \rightarrow B_\mu(x, \bar{\theta}) = A_\mu(x) + \bar{\theta} R_\mu(x),
\]
\[
\phi_\mu(x) \rightarrow \bar{\Phi}_\mu(x, \bar{\theta}) = \phi_\mu(x) + \bar{\theta} S_\mu(x),
\]
\[
C(x) \rightarrow F(x, \bar{\theta}) = C(x) + i \bar{\theta} B_1(x),
\]
\[
\bar{C}(x) \rightarrow \bar{F}(x, \bar{\theta}) = \bar{C}(x) + i \bar{\theta} B_2(x),
\]
\[
B(x) \rightarrow \bar{B}(x, \bar{\theta}) = B(x) + \bar{\theta} f_1(x),
\]
\[
\bar{B}(x) \rightarrow \bar{\bar{B}}(x, \bar{\theta}) = \bar{B}(x) + \bar{\theta} f_2(x),
\]
\[
\rho(x) \rightarrow \rho(x, \bar{\theta}) = \rho(x) + \bar{\theta} f_3(x),
\]

(24)

where the coefficients of \( \bar{\theta} \) (i.e., \( R_\mu, S_\mu, B_1, B_2, f_1, f_2, f_3 \)) are the secondary fields which have to determine by using the key ideas of ACSA. The fermionic nature of \( \bar{\theta} \) ensures that \( (B_1, B_2) \) are bosonic in nature and \( (R_\mu, S_\mu, f_1, f_2, f_3) \) are fermionic in nature.

To this aim of determining the values of secondary fields, we note very usefully and interesting BRST invariant quantities\(^3\) which are specific combinations of the basic and auxiliary fields of the Lagrangian densities, namely;

\[
s_b B = 0, \quad s_b (D_\mu C) = 0, \quad s_b (C \times C) = 0, \quad s_b (A_\mu \cdot \partial_\mu B + i \partial_\mu \bar{C} \cdot D^\mu C) = 0, \quad s_b (\bar{B} \times C) = 0, \quad s_b (\rho \times C) = 0, \quad s_b (\bar{B}_\mu \times C) = 0.
\]

(25)

According to the basic principle of the ACSA to BRST formalism, all the above BRST invariant restrictions must be independent of the Grassmannian coordinate \( \bar{\theta} \) when these BRST invariant restrictions are generalized onto the (3, 1)-dimensional anti-chiral super-sub-manifold of the most general (3, 2)-dimensional supermanifold as

\[
\bar{B}(x, \bar{\theta}) = B(x), \quad \partial_\mu F(x, \bar{\theta}) - g B_\mu(x, \bar{\theta}) \times F(x, \bar{\theta}) = \partial_\mu C - g A_\mu(x) \times C(x),
\]
\[
F(x, \bar{\theta}) \times F(x, \bar{\theta}) = C(x) \times C(x), \quad B_\mu(x, \bar{\theta}) \cdot \partial_\mu B(x, \bar{\theta}) + i \partial_\mu F(x, \bar{\theta}) \cdot [\partial^\mu F(x, \bar{\theta})
\]
\[
- g B^\mu(x, \bar{\theta}) \times F(x, \bar{\theta})] = A_\mu(x) \cdot \partial_\mu B(x) + i \partial_\mu \bar{C}(x) \cdot [\partial^\mu C(x) - g (A^\mu(x) \times C(x))],
\]
\[
\bar{B}(x, \bar{\theta}) \times F(x, \bar{\theta}) = \bar{B}(x) \times C(x), \quad \rho(x, \bar{\theta}) \times F(x, \bar{\theta}) = \rho(x) \times C(x),
\]
\[
\bar{\Phi}_\mu(x, \bar{\theta}) \times F(x, \bar{\theta}) = \phi_\mu(x) \times C(x), \quad \bar{B}(x, \bar{\theta}) \times \bar{F}(x, \bar{\theta}) = \bar{B} \times \bar{C}(x).
\]

(26)

From the above-generalized quantities, we are able to find out the values of secondary fields present in the expansions of the anti-chiral superfields (24) as

\[
s_b B = 0 \Rightarrow \bar{B}(x, \bar{\theta}) = B(x) \Rightarrow f_1(x) = 0,
\]
\[
s_b (C \times C) = 0 \Rightarrow F(x, \bar{\theta}) \times F(x, \bar{\theta}) = C(x) \times C(x) \Rightarrow B_1 \times C = 0.
\]

(27)

The latter condition \( B_1 \times C = 0 \) implies that one of the possible solution is \( B_1 \propto (C \times C) \). Thus, we have \( B_1 = \kappa (C \times C) \) where \( \kappa \) is proportionality constant which implies the modified form of anti-chiral superfield \( F(x, \bar{\theta}) \) as

\[
F(x, \bar{\theta}) \rightarrow F^{(m)}(x, \bar{\theta}) = C(x) + i \bar{\theta} \kappa (C \times C),
\]

(28)

where superscript \( (m) \) denotes the modified form of anti-chiral superfield. Now, we focus on the generalization of the \( s_b (D_\mu C) = 0 \), we have the following

\[
\partial_\mu F^{(m)}(x, \bar{\theta}) - g B_\mu(x, \bar{\theta}) \times F^{(m)}(x, \bar{\theta}) = \partial_\mu C - g A_\mu(x) \times C(x)
\]
\[
\Rightarrow R_\mu = \frac{2i \kappa}{g} D_\mu C(x).
\]

(29)

As a consequence, we have the modified form of the superfield \( B_\mu(x, \bar{\theta}) \) as

\[
B_\mu(x, \bar{\theta}) \rightarrow B^{(m)}_\mu(x, \bar{\theta}) = A_\mu(x) + \frac{2i \kappa}{g} \bar{\theta} D_\mu C(x).
\]

(30)

\(^3\) The (anti-)BRST invariant quantities are obtained using nilpotency property of (anti-)BRST symmetry transformations Eqs. (7) and (8) and some of them are determined by the hit and trial method.
Now, we use BRST invariant quantity $s_b(B \times \tilde{C}) = 0$ which implies the following
\[
\tilde{B}(x, \tilde{\phi}) \times \tilde{F}(x, \tilde{\phi}) = B(x) \times C(x) \implies B_2 \propto B.
\] (31)

The above relationship (i.e., $B_2 \propto B$) leads to the value of $B_2$ as: $B_2 = B$ where numerical constant is taken to be unit for the sake of simplicity.

In order to determine the value of constant $\kappa$, we use now the generalization of the BRST invariant quantity $s_b(A_\mu \cdot \partial_\mu B + i \partial_\mu \tilde{C} \cdot D^\mu C) = 0$ as
\[
B_\mu(x, \tilde{\phi}) \cdot \partial_\mu B(x, \tilde{\phi}) + i \partial_\mu \tilde{C}(x) \cdot [\partial^\mu F(x, \tilde{\phi}) - g(B^\mu(x, \tilde{\phi}) \times F(x, \tilde{\phi}))]
= A_\mu(x) \cdot \partial_\mu B(x) + i \partial_\mu \tilde{C}(x) \cdot [\partial^\mu C(x) - g(A^\mu(x) \times C(x))],
\] (32)
after substitutions of the modified anti-chiral super-expansions from (28) and (30) into (32), we get $\kappa = -i g/2$. Thus, finally, we have the values of secondary fields as:
\[
\mathcal{R}_\mu = D_\mu C, \quad B_1 = -i g/2 (C \times C), \quad B_2 = B.
\] (33)

We now focus on the generalization of the BRST invariant quantity $s_b (\tilde{B} \times C) = 0$, $s_b(\phi_\mu \times C) = 0$ and $s_b(\tilde{B} \times \rho) = 0$ which lead to the derivation of secondary fields as:
\[
\tilde{B}(x, \tilde{\phi}) \times F(x, \tilde{\phi}) = \tilde{B}(x) \times C(x) \implies f_2 = -g(\tilde{B} \times C),
\]
\[
\Phi_\mu(x, \tilde{\phi}) \times F(x, \tilde{\phi}) = \Phi_\mu(x) \times C(x) \implies S_\mu = -g(\Phi_\mu \times C),
\]
\[
\tilde{\rho}(x, \tilde{\phi}) \times F(x, \tilde{\phi}) = \rho(x) \times C(x) \implies f_3 = -g(\rho \times C).
\] (34)

The substitution of the values of secondary fields from (33) and (34) into the expansions of superfields (24) leads to the following
\[
A_\mu(x) \rightarrow B^{(b)}_\mu(x, \tilde{\phi}) = A_\mu(x) + \tilde{\phi} \left[D_\mu C(x) = A_\mu(x) + \tilde{\phi} \left[s_b A_\mu(x)\right]\right],
\]
\[
\phi_\mu(x) \rightarrow \Phi^{(b)}_\mu(x, \tilde{\phi}) = \phi_\mu(x) + \tilde{\phi} \left[-g(\phi_\mu(x) \times C(x))\right] = \phi_\mu(x) + \tilde{\phi} \left[s_b \phi_\mu(x)\right],
\]
\[
C(x) \rightarrow F^{(b)}(x, \tilde{\phi}) = C(x) + \tilde{\phi} \left[\frac{g}{2} (C(x) \times C(x))\right] = C(x) + \tilde{\phi} \left[s_b C(x)\right],
\]
\[
\tilde{C}(x) \rightarrow \tilde{F}^{(b)}(x, \tilde{\phi}) = \tilde{C}(x) + \tilde{\phi} \left[iB(x)\right] = \tilde{C}(x) + \tilde{\phi} \left[s_b \tilde{C}(x)\right],
\]
\[
B(x) \rightarrow \tilde{B}^{(b)}(x, \tilde{\phi}) = B(x) + \tilde{\phi} \left[B_1(x)\right] = B(x) + \tilde{\phi} \left[s_b B(x)\right],
\]
\[
\tilde{\rho}(x) \rightarrow \tilde{\rho}^{(b)}(x, \tilde{\phi}) = \rho(x) + \tilde{\phi} \left[-g(\rho(x) \times C(x))\right] = \rho(x) + \tilde{\phi} \left[s_b \rho(x)\right],
\] (35)

where the superscript (b) on the anti-chiral superfields denotes the anti-chiral superfields that have been obtained after the application of the BRST invariant restrictions (25). Here, the coefficients of the $\tilde{\phi}$ are nothing but the BRST symmetry transformations for various fields [28–32, 35–39]. Thus, we have derived all the BRST symmetry transformations for coupled (but equivalent) Lagrangian densities and shown the sanctity of BRST symmetry transformations within the ambit of ACSA. We also conclude that there is a connection between BRST symmetry ($s_b$) and the translational generator ($\partial_\tilde{\phi}$) along the $\tilde{\phi}$-direction of the (3, 1)-dimensional anti-chiral super-sub-manifold with the following relationship and mapping:
\[
\partial_\tilde{\phi} \Omega^{(b)}(x, \tilde{\phi}) = s_b \omega(x), \quad s_b \leftrightarrow \partial_\tilde{\phi}.
\] (36)

Here it is clear that the BRST symmetry transformations for any ordinary generic field $\omega(x)$ are nothing but the translation of the generalized anti-chiral generic superfield $[\Omega^{(b)}(x, \tilde{\phi})]$ onto the (3, 1)-dimensional super-sub-manifold along the $\tilde{\phi}$-direction.

5.2 Anti-BRST symmetry transformations: ACSA

In this subsection, we derive all the anti-BRST symmetry transformations for various fields of the Lagrangian densities within the ambit of ACSA to BRST formalism. Toward this aim in mind, first of all, we generalize our basic and auxiliary fields onto (3, 1)-dimensional chiral super-sub-manifold of the most general (3, 2)-dimensional supermanifold as;
\[
A_\mu(x) \rightarrow B_\mu(x, \phi) = A_\mu(x) + \phi \tilde{R}_\mu(x),
\]
\[
\phi_\mu(x) \rightarrow \Phi_\mu(x, \phi) = \phi_\mu(x) + \phi \tilde{S}_\mu(x),
\]
\[
C(x) \rightarrow F(x, \phi) = C(x) + i \phi \tilde{B}_1(x),
\]
\[
\tilde{C}(x) \rightarrow \tilde{F}(x, \phi) = \tilde{C}(x) + i \phi \tilde{B}_2(x),
\]
\[
B(x) \rightarrow \tilde{B}(x, \phi) = B(x) + \phi \tilde{f}_1(x).
\]
where the coefficients of \( \vartheta \) (i.e., \( \bar{\rho}_\mu, \bar{S}_\mu, \bar{B}_1, \bar{B}_2, \bar{f}_1, \bar{f}_2, \bar{f}_3 \)) are nothing but the secondary fields which we have to determine by exploiting the ideas of ACSA to BRST formalism. The fermionic nature of \( \vartheta \) verifies that the secondary fields (\( \bar{B}_1, \bar{B}_2 \)) are bosonic in nature and (\( \bar{\rho}_\mu, \bar{S}_\mu, \bar{f}_1, \bar{f}_2, \bar{f}_3 \)) are fermionic in nature.

For the derivation of the above secondary fields, we use the very important and interesting anti-BRST invariant quantities which are the combinations of the basic and auxiliary fields of the Lagrangian densities, namely:

\[
\begin{align*}
  s_{ab} \bar{B} &= 0, \quad s_{ab} (D_\mu \bar{C}) = 0, \quad s_{ab} (\bar{C} \times \bar{C}) = 0, \quad s_{ab} (A_\mu \cdot \partial_\mu \bar{B} + i D_\mu \bar{C} \cdot \vartheta^\mu C) = 0, \\
  s_{ab} (\rho \times \vartheta) &= 0, \quad s_{ab} (\phi_\mu \times \bar{C}) = 0, \quad s_{ab} (\bar{B} \times C) = 0.
\end{align*}
\]

According to the basic principle of the ACSA to BRST formalism, above set of BRST invariant restrictions must be independent of the coordinate \( \vartheta \) when these invariant quantities are generalized onto the (3, 1)-dimensional chiral super-sub-manifold as:

\[
\begin{align*}
  \hat{\bar{B}}(x, \vartheta) &= \bar{B}(x), \quad \partial_\mu \hat{\bar{F}}(x, \vartheta) = -g B_\mu (x, \vartheta) \times \bar{F}(x, \vartheta) = \partial_\mu \bar{C} = -g A_\mu (x) \times \bar{C}(x), \\
  \hat{\bar{\rho}}(x, \vartheta) &= \bar{\rho}(x, \vartheta) = \rho(x) \times \bar{\vartheta} = \rho(x) \times \bar{C}(x), \\
  \hat{\bar{f}}_1 &= -g (B \times \bar{C}), \quad \hat{\bar{f}}_2 = -g (\phi_\mu \times \bar{C}), \quad \hat{\bar{f}}_3 = -g (\rho \times \bar{C}).
\end{align*}
\]

Using the above generalized quantities, as in the similar manner to anti-chiral, we find out the values of chiral secondary fields of the expansions of the chiral superfields (37) as:

\[
\begin{align*}
  \hat{\bar{\rho}}_\mu &= D_\mu \hat{\bar{C}}, \quad \hat{\bar{B}}_1 = \bar{B}, \quad \hat{\bar{B}}_2 = -\frac{i g}{2} (\hat{\bar{C}} \times \hat{\bar{C}}), \quad \hat{\bar{f}}_2 = 0 \\
  \hat{\bar{f}}_1 &= -g (B \times \hat{\bar{C}}), \quad \hat{\bar{S}}_\mu = -g (\phi_\mu \times \hat{\bar{C}}), \quad \hat{\bar{f}}_3 = -g (\rho \times \hat{\bar{C}}).
\end{align*}
\]

We have the following super-expansions of the chiral secondary field after the substitutions of the above secondary fields into Eq. (37):

\[
\begin{align*}
  A_\mu (x) &\rightarrow B^{(ab)}_\mu (x, \vartheta) = A_\mu (x) + \vartheta [D_\mu \bar{C}(x)] = A_\mu (x) + \vartheta [s_{ab} A_\mu (x)], \\
  \phi_\mu (x) &\rightarrow \hat{\phi}^{(ab)}_\mu (x, \vartheta) = \phi_\mu (x) + \vartheta [-g (\phi_\mu (x) \times \bar{C}(x))] = \phi_\mu (x) + \vartheta [s_{ab} \phi_\mu (x)], \\
  C(x) &\rightarrow \hat{C}^{(ab)}(x, \vartheta) = C(x) + \vartheta [i B\bar{C}(x)] = C(x) + \vartheta [s_{ab} C(x)], \\
  \bar{C}(x) &\rightarrow \hat{\bar{C}}^{(ab)}(x, \vartheta) = \bar{C}(x) + \vartheta [\frac{g}{2} (\bar{C}(x) \times \bar{C}(x))] = \bar{C}(x) + \vartheta [s_{ab} \bar{C}(x)], \\
  B(x) &\rightarrow \hat{B}^{(ab)}(x, \vartheta) = B(x) + \vartheta [-g (B(x) \times \bar{C}(x))] = B(x) + \vartheta [s_{ab} B(x)], \\
  \bar{B}(x) &\rightarrow \hat{\bar{B}}^{(ab)}(x, \vartheta) = \bar{B}(x) + \vartheta [0] = \bar{B}(x) + \vartheta [s_{ab} \bar{B}(x)], \\
  \rho(x) &\rightarrow \hat{\rho}^{(ab)}(x, \vartheta) = \rho(x) + \vartheta [-g (\rho(x) \times \bar{C}(x))] = \rho(x) + \vartheta [s_{ab} \rho(x)],
\end{align*}
\]

where the superscript \( (ab) \) on the superfields denotes the chiral superfields that have been obtained after the application of the anti-BRST invariant restrictions (38). Here, the coefficients of the \( \vartheta \) are nothing but the anti-BRST symmetry transformations for various fields of Lagrangian densities which have been listed in Eqs. (5) and (6). Thus, we have derived all the anti-BRST symmetry transformations for coupled Lagrangian densities and shown the sanctity of anti-BRST symmetry transformations within the ambit of ACSA.

We end this subsection with the remarks that we have derived the complete set of (anti-) BRST symmetry transformations for Lagrangian densities listed in Eqs. (5) and (6). We have also established that anti-BRST symmetry \( (s_{ab}) \) is connected with the translational generator \( (\partial_\vartheta) \) with the following relationship and mapping:

\[
\partial_\vartheta \Omega^{(ab)}(x, \vartheta) = s_{ab} \omega(x), \quad s_{ab} \longleftrightarrow \partial_\vartheta.
\]

It is clear from the above relationship that the anti-BRST symmetry transformation \( (s_{ab}) \) for the ordinary generic field \( \omega(x) \) gives the translation of the generalized generic chiral superfield \( [\Omega^{(ab)}(x, \vartheta)] \) onto (3, 1)-dimensional super-sub-manifold along the \( \vartheta \)-direction.
6 Nilpotency and absolute anticommutativity properties of conserved (anti-)BRST charges: ACSA

In this section, we discuss about the off-shell nilpotency and absolute anticommutativity properties of the (anti-)BRST conserved charges within the framework of ACSA to BRST formalism. For the proof of the nilpotency properties of the conserved (anti-)BRST charges \([Q_{ab}]\), we express the charges in terms of the (anti-)chiral superfields as:

\[
Q_b = \frac{\partial}{\partial \vartheta} \int d^2x \left[ B^{(b)}(x, \bar{\vartheta}) \cdot B_0^{(b)}(x, \bar{\vartheta}) + i \dot{F}^{(b)}(x, \bar{\vartheta}) \cdot \varphi^{(b)}(x, \bar{\vartheta}) \right]
\]

\[
Q_{ab} = \frac{\partial}{\partial \vartheta} \int d^2x \left[ B^{(ab)}(x, \vartheta) \cdot B_0^{(ab)}(x, \vartheta) + i \dot{F}^{(ab)}(x, \vartheta) \cdot \varphi^{(ab)}(x, \vartheta) \right]
\]

It is now crystal clear that we have the following interesting relationships:

\[
\partial_\vartheta Q_b = 0 \iff \partial_b^2 = 0, \quad s_b Q_b = -i \{Q_b, Q_b\} = 0 \iff s_b^2 = 0.
\]

\[
\partial_\vartheta Q_{ab} = 0 \iff \partial_{ab}^2 = 0, \quad s_{ab} Q_{ab} = -i \{Q_{ab}, Q_{ab}\} = 0 \iff s_{ab}^2 = 0.
\]

In other words, it is clear that the nilpotency of the conserved (anti-)BRST charges (i.e., \(Q_b^2 = 0\) and \(Q_{ab}^2 = 0\)) is deeply connected with the nilpotency of the translational generators (\(\partial_\vartheta\), \(\partial_b\), \(\partial_{ab}\)), respectively, along Grassmannian directions and the nilpotency of the (anti-)BRST symmetry transformations (\(\Delta_{(ab)}\)), too.

We now focus on the proof of absolute anticommutativity property of the nilpotent conserved (anti-)BRST charges within the realm of ACSA to BRST formalism where the CF-condition play major role. The conserved (anti-)BRST charges can be expressed in terms of the (anti-)chiral superfields of the (3, 1)-dimensional super-sub-manifold as:

\[
Q_b = \frac{\partial}{\partial \vartheta} \left[ \int d^2x \left\{ i \varphi^{(ab)}(x, \bar{\vartheta}) \cdot \dot{F}^{(ab)}(x, \bar{\vartheta}) \right\} \right]
\]

\[
Q_{ab} = \frac{\partial}{\partial \vartheta} \left[ \int d^2x \left\{ -i F^{(ab)}(x, \vartheta) \cdot \dot{F}^{(ab)}(x, \vartheta) \right\} \right]
\]

From the above expressions, it is clear that:

\[
\partial_\vartheta Q_b = 0 \iff \partial_b^2 = 0 \iff s_b Q_b = -i \{Q_b, Q_b\} = 0.
\]

\[
\partial_\vartheta Q_{ab} = 0 \iff \partial_{ab}^2 = 0 \iff s_{ab} Q_{ab} = -i \{Q_{ab}, Q_{ab}\} = 0.
\]

Thus, we note that the absolute anticommutativity of the BRST charge \(Q_b\) with the anti-BRST charge \(Q_{ab}\) is deeply connected with the nilpotency \(\partial_b^2 = 0\) of the translational generator \(\partial_\vartheta\) along the \(\vartheta\)-direction. Similarly, absolute anticommutativity of the anti-BRST charge with BRST charge is deeply connected with the nilpotency of the \(\partial_{ab}^2 = 0\) of the translational generator \(\partial_\vartheta\) along the \(\bar{\vartheta}\)-direction of the (3, 1)-dimensional super-sub-manifold of the general (3, 2)-dimensional supermanifold.
7 Invariances of Lagrangian densities: ACSA

In this section, we capture the (anti-)BRST invariances of the Lagrangian densities $\mathcal{L}_B$ and $\mathcal{L}_{\bar{B}}$ of Eqs. (5) and (6) within the realm of ACSA to BRST formalism. Toward this aim in mind, foremost, we generalize the ordinary Lagrangian densities $\mathcal{L}_B$ and $\mathcal{L}_{\bar{B}}$ to (anti-)chiral super-Lagrangian densities $\mathcal{L}^{(ac)}_B$ and $\mathcal{L}^{(c)}_{\bar{B}}$ onto the (3, 1)-dimensional super-sub-manifold as:

\[
\mathcal{L}_B \rightarrow \mathcal{L}^{(ac)}_B = - \frac{1}{4} \tilde{\mathcal{F}}^{\mu\nu}(x, \tilde{\theta}) \cdot \tilde{\mathcal{F}}^{(ac)}_{\mu\nu}(x, \tilde{\theta})
- \frac{1}{4} \left[ \tilde{g}^{\mu\nu}(x, \tilde{\theta}) \cdot \tilde{\mathcal{F}}^{(ac)}_{\mu\nu}(x, \tilde{\theta}) \right] \cdot \left[ \tilde{g}^{(ac)}_{\mu\nu}(x, \tilde{\theta}) \right]
+ g \tilde{F}^{(ac)}_{\mu\nu}(x, \tilde{\theta}) \times \tilde{\rho}^{(ab)}(x, \tilde{\theta}) + \frac{m}{2} \varepsilon^{\mu\nu\eta} \tilde{F}^{(ac)}_{\mu\nu}(x, \tilde{\theta}) \times \tilde{\Phi}^{(ab)}_{\eta}(x, \tilde{\theta})
+ \tilde{B}^{(ab)}(x, \tilde{\theta}) \cdot \left[ \tilde{C}^{\mu\nu}(x, \tilde{\theta}) \right] + \frac{1}{2} \left[ \tilde{B}^{(ab)}(x, \tilde{\theta}) \cdot \tilde{B}^{(ac)}(x, \tilde{\theta}) + \tilde{B}^{(ab)}(x, \tilde{\theta}) \cdot \tilde{B}^{(ac)}(x, \tilde{\theta}) \right]
- i \partial_{\mu} \tilde{F}^{(ab)}(x, \tilde{\theta}) \cdot \partial_{\mu} \tilde{F}^{(ab)}(x, \tilde{\theta}) + \frac{1}{2} \left[ \tilde{B}^{(ab)}(x, \tilde{\theta}) \times \tilde{F}^{(ab)}(x, \tilde{\theta}) \right]
\]

where the superscripts $(ac), (c)$ on the super-Lagrangian densities and field strength tensors denote the anti-chiral and chiral super-Lagrangian densities and super-field strength tensors, respectively, which are generalized on the (3, 1)-dimensional super-submanifolds. It is straightforward to check that after the application of the translational generators $\partial_\theta$ and $\partial_\bar{\theta}$ on the above super-Lagrangian densities $\mathcal{L}^{(ac)}_B$ and $\mathcal{L}^{(c)}_{\bar{B}}$, respectively, we have the following

\[
\frac{\partial}{\partial \bar{\theta}} \left[ \mathcal{L}^{(ac)}_B \right] = \partial_\mu \left[ B \cdot D^{\mu} C \right] \quad \iff \quad s_B \mathcal{L}_B = \partial_\mu \left[ B \cdot D^{\mu} C \right]
\]

\[
\frac{\partial}{\partial \bar{\theta}} \left[ \mathcal{L}^{(c)}_{\bar{B}} \right] = \partial_\mu \left[ - B \cdot D^{\mu} \bar{C} \right] \quad \iff \quad s_{ab} \mathcal{L}_{\bar{B}} = - \partial_\mu \left[ B \cdot D^{\mu} \bar{C} \right].
\]

The above relationships establish that the invariances of the Lagrangian densities within the realm of ACSA to BRST formalism which demonstrate the sanctity of the (anti-)BRST invariance [cf. Eq. (12)] of the Lagrangian densities. It should also be noted that (anti-) BRST symmetries ($s_{(ac)b}$) are connected with the translational generators ($\partial_\bar{\theta}, \partial_\theta$).

To capture the anti-BRST invariance of the of Lagrangian density $\mathcal{L}_B$ and BRST invariance of the Lagrangian density $\mathcal{L}_{\bar{B}}$ within the framework of (anti-)chiral superfields formalism, first of all, we generalize ordinary Lagrangian densities $\mathcal{L}^a$ and $\mathcal{L}^b$ onto the (3, 1)-dimensional super-manifold of the most general (3, 2)-dimensional supermanifold as

\[
\mathcal{L}_B \rightarrow \mathcal{L}^{(ac)}_B = - \frac{1}{4} \tilde{\mathcal{F}}^{\mu\nu}(x, \tilde{\theta}) \cdot \tilde{\mathcal{F}}^{(ac)}_{\mu\nu}(x, \tilde{\theta})
- \frac{1}{4} \left[ \tilde{g}^{\mu\nu}(x, \tilde{\theta}) \cdot \tilde{\mathcal{F}}^{(ac)}_{\mu\nu}(x, \tilde{\theta}) \right] \cdot \left[ \tilde{g}^{(ac)}_{\mu\nu}(x, \tilde{\theta}) \right]
+ g \tilde{F}^{(ac)}_{\mu\nu}(x, \tilde{\theta}) \times \tilde{\rho}^{(ab)}(x, \tilde{\theta}) + \frac{m}{2} \varepsilon^{\mu\nu\eta} \tilde{F}^{(ac)}_{\mu\nu}(x, \tilde{\theta}) \times \tilde{\Phi}^{(ab)}_{\eta}(x, \tilde{\theta})
+ \tilde{B}^{(ab)}(x, \tilde{\theta}) \cdot \left[ \tilde{C}^{\mu\nu}(x, \tilde{\theta}) \right] + \frac{1}{2} \left[ \tilde{B}^{(ab)}(x, \tilde{\theta}) \cdot \tilde{B}^{(ab)}(x, \tilde{\theta}) \right]
- i \partial_{\mu} \tilde{F}^{(ab)}(x, \tilde{\theta}) \cdot \partial_{\mu} \tilde{F}^{(ab)}(x, \tilde{\theta}) + \frac{1}{2} \left[ \tilde{B}^{(ab)}(x, \tilde{\theta}) \times \tilde{F}^{(ab)}(x, \tilde{\theta}) \right]
\]

\[
\mathcal{L}_{\bar{B}} \rightarrow \mathcal{L}^{(c)}_{\bar{B}} = - \frac{1}{4} \tilde{\mathcal{F}}^{\mu\nu}(x, \tilde{\theta}) \cdot \tilde{\mathcal{F}}^{(c)}_{\mu\nu}(x, \tilde{\theta})
- \frac{1}{4} \left[ \tilde{g}^{\mu\nu}(x, \tilde{\theta}) \cdot \tilde{\mathcal{F}}^{(c)}_{\mu\nu}(x, \tilde{\theta}) \right] \cdot \left[ \tilde{g}^{(c)}_{\mu\nu}(x, \tilde{\theta}) \right]
+ g \tilde{F}^{(c)}_{\mu\nu}(x, \tilde{\theta}) \times \tilde{\rho}^{(ab)}(x, \tilde{\theta}) + \frac{m}{2} \varepsilon^{\mu\nu\eta} \tilde{F}^{(c)}_{\mu\nu}(x, \tilde{\theta}) \times \tilde{\Phi}^{(ab)}_{\eta}(x, \tilde{\theta})
+ \tilde{B}^{(ab)}(x, \tilde{\theta}) \cdot \left[ \tilde{C}^{\mu\nu}(x, \tilde{\theta}) \right] + \frac{1}{2} \left[ \tilde{B}^{(ab)}(x, \tilde{\theta}) \cdot \tilde{B}^{(ab)}(x, \tilde{\theta}) \right]
- i \partial_{\mu} \tilde{F}^{(ab)}(x, \tilde{\theta}) \cdot \partial_{\mu} \tilde{F}^{(ab)}(x, \tilde{\theta}) + \frac{1}{2} \left[ \tilde{B}^{(ab)}(x, \tilde{\theta}) \times \tilde{F}^{(ab)}(x, \tilde{\theta}) \right]
\]
where superscripts \((c)\) and \((ac)\) have been explained earlier in Eq. (47). We apply the translational generators \((\delta_{\bar{\beta}}, \delta_{\beta})\) on the above super-Lagrangian densities \([\bar{\mathcal{L}}^{(c)}_{\bar{B}}, \bar{\mathcal{L}}^{(ac)}_{\bar{B}}]\), respectively, as follows:

\[
\begin{align*}
\frac{\partial}{\partial \bar{\theta}} \left[ \bar{\mathcal{L}}^{(c)}_{\bar{B}} \right] &= - \partial_{\mu} \left[ B + (\bar{C} \times C) \right] \cdot \partial^{\mu}\bar{C} - D_{\mu} \left[ B + \bar{B} + ig (\bar{C} \times C) \right] \cdot \partial^{\mu}\bar{C}, \\
\frac{\partial}{\partial \bar{\theta}} \left[ \bar{\mathcal{L}}^{(ac)}_{\bar{B}} \right] &= \partial_{\mu} \left[ B + (\bar{C} \times C) \right] \cdot \partial^{\mu}C + D_{\mu} \left[ B + \bar{B} + ig (\bar{C} \times C) \right] \cdot \partial^{\mu}C.
\end{align*}
\]

The above relationships lead to the invariances of Lagrangian densities if and only if CF-condition \([i.e., B + \bar{B} + i g (\bar{C} \times C) = 0]\) is satisfied. Therefore, we prove the sanctity of the invariance of coupled (equivalent) Lagrangian densities \([cf. Eq. (12)]\), as in ordinary spacetime, within the realm of the ACSA to BRST formalism.

8 Conclusions

In the present investigation, first of all, we have talked about the Yang–Mills (YM) and non-Yang–Mills (NYM) gauge symmetry transformations for (2 + 1)-dimensional (3D) non-Abelian massive 1-form Jackiw–Pi model (parity conserving odd-dimensional theory). We have discussed the construction of coupled (but equivalent) Lagrangian densities, BRST and anti-BRST symmetry transformations for various basic and auxiliary fields present in the Lagrangian densities, and invariances of Lagrangian densities under (anti-)BRST symmetry transformations in ordinary spacetime (cf. Sect. 3). We have also discussed the Noether conserved (anti-)BRST currents as well as charges and shown the conservation law using the Euler-Lagrange equation of motion. Along with these discussions, we have also examined the nilpotency and absolute anticommutativity properties of the (anti-)BRST conserved charges in the 3D ordinary spacetime. One of the key signatures of any non-Abelian gauge theory is the existence of the Curci–Ferrari (CF)-condition which is observed, for the present model, through the various mathematical techniques (i) equivalence of both the Lagrangian densities \([i.e., \mathcal{L}_{\bar{B}} = \mathcal{L}_{\bar{B}}]\) (ii) absolute anticommutativity of (anti-)BRST symmetries for the various fields (iii) invariances of coupled Lagrangian densities under the off-shell nilpotent (anti-)BRST symmetry transformations, and (iv) absolute anticommutativity property of the conserved (anti-)BRST charges. Moreover, CF-condition has played a crucial role in the deduction of coupled (but equivalent) Lagrangian densities.

We have discussed, for the first time, all the above properties (of ordinary spacetime) within the realm of (anti-)chiral superfield approach (ACSA) to BRST formalism and shown the obvious connections of (anti-)BRST symmetry transformations with the Grassmannian derivatives \((i.e., s_{\bar{b}} \leftrightarrow \delta_{\bar{\beta}}, s_{ab} \leftrightarrow \delta_{\theta})\). One of the important and novel result of the ACSA to BRST formalism for the JP model is that absolute anticommutativity property of the (anti-)BRST charges is satisfied even though we have taken only one Grassmannian coordinate in the superfields, whereas in Bonora–Tonin (BT) superfield formalism both the Grassmannian coordinate are taken into account for the proof of absolute anticommutativity of charges. We have observed that there is a deep connection between the nilpotency of the translational generator \(\delta_{\bar{\beta}}\) and anticommutativity of the BRST conserved charge \(Q_{\bar{b}}\), whereas there is deep connection between the nilpotency of translational generator \(\delta_{\theta}\) and anticommutativity of the anti-BRST charge \(Q_{ab}\) with BRST charge \(Q_{\bar{b}}\). We have also demonstrated the nilpotency properties of (anti-)BRST conserved charges within the realm of ACSA to BRST formalism where nilpotency of the translational generators \((\delta_{\theta}, \delta_{\bar{\beta}})\) is deeply connected with the nilpotency of the conserved charges \((Q_{ab}, Q_{\bar{b}})\) along the \((\theta, \bar{\theta})\)-directions of the (3, 1)-dimensional (anti-)chiral super-sub-manifold of the general (3, 2)-dimensional supermanifold, respectively \([cf. Eq. (44)]\). We have also shown the invariance of the coupled (but equivalent) Lagrangian densities where the emergence of the CF-condition is verified within the ambit of the ACSA to BRST formalism.

We would extend our standard techniques of (anti-)chiral superfield approach to BRST formalism for the various BRST-quantized models and theories. The techniques of the ACSA with modified Bonora–Tonin superfield approach (MBTSA) would be very interesting to discuss the various reparameterization invariant models. We also plan to discuss the massive Abelian 3-form gauge theory in 6D within the realm of ACSA where the possible candidate of dark matter and dark energy would be discussed within the realm of BRST formalism.

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Data Availability No data were used to support this study.

Declarations

Conflicts of interest The author declares that there are no conflicts of interest.
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