Traveling Wave Solutions for Non-Newtonian Foam Flow in Porous Media

Wesley da Silva Pereira1 · Grigori Chapiro2

Received: 15 November 2022 / Accepted: 31 March 2023 / Published online: 19 April 2023
© The Author(s), under exclusive licence to Springer Nature B.V. 2023

Abstract
The injection and in situ generation of foam in porous media successfully control gas mobility and improve the fluids’ sweep efficiency inside porous media. Mathematical models describing this problem use two phases, foamed gas, and fluid, and usually have a term for foam generation and destruction. Moreover, the non-Newtonian foam behavior is frequently modeled using Hirasaki and Lawson’s formula for foamed gas viscosity. In this paper, we detail how the traveling wave analysis can be used to estimate the propagation profiles and velocity for a range of non-Newtonian foam models in porous media at constant total superficial flow velocity. We reformulate Hirasaki and Lawson’s formula in an explicit form allowing us to find traveling wave solutions for a foam model with non-Newtonian gas viscosity and a foam generation linearly dependent on the foam texture. Comparing the solution with the one for the Newtonian version allows us to analyze qualitatively and quantitatively the rheology of the foam flow in porous media.

Keywords Traveling waves · Non-Newtonian fluids · Foam flow · Porous media

Mathematics Subject Classification 35C07 · 76S05 · 76A05

1 Introduction
One technique to control gas mobility and improve fluid sweep efficiency in a porous medium consists of injecting foaming agents, e.g., surfactants and nanoparticles, in aqueous solutions that create bubbles in the gas phase. Several studies point out the favorable application of this method in enhanced oil recovery, e.g., (Casteel and Djabbarah 1988; Grassia et al. 2014), acid diversion during matrix stimulation, e.g., (Behenna 1995), and...
contaminated aquifer remediation, e.g., (Mamun et al. 2002), soil treatment (Boakye-Ansah and Grassia 2021).

The behavior of foam is frequently reported as shear-thinning (Fernø et al. 2016; Heller and Kuntamukkula 1987; Marsden and Khan 1966; Zitha 2006), which means that its apparent viscosity decreases with increased shear stress. Hirasaki and Lawson (1985) derived a shear-thinning expression for the apparent viscosity of foam flowing in capillary tubes and validated it through laboratory experiments. This non-Newtonian formula is consistent with the classical result of Bretherton (1961) using isolated bubbles and has application in several foam propagation models, e.g., (Izadi and Kam 2019; Kam 2008; Kovscek et al. 1995; Simjoo and Zitha 2015). Hirasaki and Lawson’s expression defines apparent gas viscosity as a function of the gas velocity, and gas velocity itself is a function of the viscosity. Other works report that, for particular regimes, it behaves as a Newtonian fluid, i.e., its viscosity does not depend on the gas velocity. For example, foam flow can be classified as Newtonian under nearly constant capillary pressure during steady radial flow (Rossen 1991). Following this idea, some models consider foam flow to be a Newtonian fluid (Ashoori et al. 2011; Lozano et al. 2021) and present results compatible with experimental data (Zavala et al. 2021). Hirasaki and Lawson’s formula is so important that numerous papers investigate the foam flow in porous media only focusing on the apparent viscosity (or, equivalently, the mobility reduction factor), see, e.g., (Kapetas et al. 2016, 2017; Ma et al. 2013a, b, 2014; Boeije and Rossen 2015).

The studies reported above motivate the current discussion on when to consider non-Newtonian behavior of the foam flow. Vassenden and Holt (1998) present and validate a model for the transition of foam flow behavior from Newtonian to shear-thinning according to the gas flow rate. Alvarez et al. (2001) conclude the foam behavior is shear-thinning in the low-quality regime and shear-thickening in high-quality regimes. Rossen (1991) highlights the change from Newtonian to non-Newtonian behavior of uniform texture foam for changing capillary pressure.

One may find several predictive mathematical models in the literature aiming to represent foam propagation inside porous media. Local Equilibrium foam models use algebraic expressions to compute the foam texture, which impacts the foam’s rheology. However, Local Equilibrium models can be inadequate when strong foam generation is not certain (Kam et al. 2007). Mechanistic foam models, introduced by Falls et al. (1988) and Patzek (1988), describe the foam texture dynamically using partial differential equations. They have the unique potential to describe both transient and stationary conditions for foam flow as they track the foam texture in space and time (Eide et al. 2020). Many works successfully report mechanistic models to match laboratory experiments, e.g., in Kovscek et al. (1995), Kam (2008) and Simjoo and Zitha (2015). Besides that some authors prefer these models when extracting analytical estimates of the foam flow, e.g., (Almajid et al. 2019; Eide et al. 2020; Simjoo and Zitha 2015). Moreover, several works suggest that, after a transient stage, foam travels inside the porous media under invariant water saturation profiles (similar to traveling wave solution), e.g., (de Vries and Wit 1990; Ettinger and Radke 1992; Simjoo and Zitha 2015; Chen et al. 2010). One strategy to obtain solution profiles, employed in (Izadi and Kam 2019; Kam 2008; Simjoo and Zitha 2015), consists of disregarding the capillary pressure gradient in each phase’s superficial velocity and applying the classical method of characteristics. Another strategy consists of seeking traveling wave solutions, which are profiles invariant to translation in space (Volpert et al. 2000). They are especially suited for analyzing nonlinear wave propagation problems at a constant speed. Ashoori et al. (2011) and Lozano et al. (2021) used this procedure to find traveling wave solutions for Newtonian foam models at fixed total superficial
velocity. Ashoori et al. (2011) also find traveling waves for the model from Kam (2008) which does not consider capillary pressure derivatives.

The present work shows how to compute traveling wave solutions that connect two equilibrium states of a generic mechanistic non-Newtonian foam model that consider capillary pressure derivatives. We assume the total superficial velocity is fixed, which is the case of predominant rectilinear flow. The non-Newtonian foamed gas viscosity is modeled by Hirasaki and Lawson’s formula. We rewrite this formula in terms of water saturation, foam texture, and total superficial velocity to avoid the direct dependency on the gas superficial velocity. The traveling wave profiles serve as a basis to compare the differences between using Newtonian and non-Newtonian gas viscosity expressions on the Linear Kinetics foam model [the name given by authors Ashoori et al. (2011)]. In the latter, foam generation and coalescence are linear functions of the foam texture.

We organize this paper as follows. Section 2 presents the general population-based foam model. Section 3 shows the Hirasaki and Lawson apparent gas viscosity and its reformulated version. Section 4 presents the procedure for obtaining traveling wave solutions for the non-Newtonian foam flow model. Section 5 applies the procedure to the Linear Kinetics model and uses the traveling wave profiles to compare Newtonian and non-Newtonian models. Section 6 presents the discussions and conclusions.

2 The Population-Balance Foam Model

Consider the two-phase flow of gas-water solution with foaming agents in a porous medium, where the flow occurs only in one direction, and all fluids are incompressible. The last assumption is undesirable but necessary for the mathematical analysis performed in this paper. The process is modeled by the mass balance of each phase as

$$\partial_t(\phi S_i) + \partial_x u_i = 0, \quad i \in \{w, g\},$$

where $t$ is the time variable, $x$ is the space variable, $\phi$ is the porosity, $S_i$ and $u_i$ are the saturation and superficial velocity of the phase $i$. In this paper, we use the notation $\partial_x$ and $\partial_t$ to indicate partial derivatives with respect to $x$ and $t$, respectively. The subscript ‘$w$’ represents the liquid (wetting) phase, and ‘$g$’ represents the gas (non-wetting) phase. The phase superficial velocities satisfy the generalized Darcy’s laws (Chen and Ewing 1997)

$$u_w = f_w \left( u + \lambda_g K \partial_x P_c \right), \quad u_g = f_g \left( u - \lambda_w K \partial_x P_c \right),$$

where $K$ is the permeability, $u = u_w + u_g$ is the (constant) total superficial velocity, and $P_c$ is the capillary pressure [often associated with a population-balance of bubbles within the foam, cf. (Hematpour et al. 2016)]. For each phase $i$, the fractional fluxes ($f_i$) and phase mobilities ($\lambda_i$) are given by

$$f_i := \frac{\lambda_i}{\lambda_w + \lambda_g}, \quad \lambda_i := \frac{k_{r_i}}{\mu_i}, \quad i \in \{w, g\},$$

where $k_{r_i}$ is the relative permeability, and $\mu_i$ is the apparent phase viscosity. The balance of foam texture $n_f$ (also known as population-balance of bubbles in the foam) in the gas phase is

$$\partial_t(\phi S_g n_f) + \partial_x (n_f u_g) = \phi S_g R,$$
where $R := r_g - r_c$, and $r_g$ and $r_c$ are the rates of foam generation and coalescence, respectively. It is useful to define the dimensionless foam texture ($n_D$) as follows

$$n_D := \frac{n_f}{n_{\text{max}}},$$

(5)

where $n_{\text{max}}$ is the reference foam texture.

Some auxiliary physical quantities are equally important in this work. We define the effective water saturation ($S_{\text{we}}$), with values in $[0,1]$, as

$$S_{\text{we}} := \frac{S_w - S_{\text{wc}}}{1 - S_{\text{wc}} - S_{\text{gr}}},$$

(6)

where $S_{\text{wc}}$ is the connate water saturation, and $S_{\text{gr}}$ is the residual gas saturation. The total apparent viscosity ($\mu_{\text{app}}$) is

$$\mu_{\text{app}} := \frac{1}{\lambda_w + \lambda_g},$$

(7)

and the mobility reduction factor (MRF) is

$$\text{MRF} := \frac{\mu_g}{\mu_{\text{g}}}.$$

(8)

where $\mu_{\text{g}}$ is the gas viscosity in the absence of foam. To avoid problems in the nomenclature, we emphasize using the word “total” when referring to the apparent viscosity $\mu_{\text{app}}$. Note that the total apparent viscosity represents the equivalent viscosity of the fluid system in the following Darcy’s law

$$u = -\frac{K}{\mu_{\text{app}}} \partial_x P,$$

(9)

where $P$ represents pressure. Finally, the phase-$i$ interstitial velocity is

$$v_i := \frac{|u_i|}{\phi S_i}, \quad i \in \{w, g\}.$$

(10)

It is worth mentioning that several models exist that state $P_c$ depending exclusively on the fluid phase saturation, and many of them are based on the classical works of Corey (1954), Brooks and Corey (1966) and van Genuchten (1980). The relative permeabilities $k_{rw}$ and $k_{rg}$ are either derived directly from the capillary pressure or fitted experimentally [see (Li and Horne 2006) and references therein]. Foam generation and coalescence expressions $r_g$ and $r_c$ are either inspired by microscopic mechanisms (Kam 2008; Kovscek et al. 1995), based on macroscopic observations (Ashoori et al. 2011; Simjoo and Zitha 2015), or fitted using laboratory experiments (Thorat and Bruining 2016). We postpone the definitions of the expressions $k_{rw}(S_w)$, $k_{rg}(S_w)$, $r_g(S_w, n_D, u_w)$, $r_c(S_w, n_D, u_w)$, and $P_c(S_w)$ to Sect. 5. Section 3 presents the formulas for $\mu_{\text{g}}$.

Some mechanistic (population-based) models use an expression for Local Equilibrium foam texture, $n_{\text{DL}}^{\text{LE}}$, that naturally arises from the constraint $R = 0$. For example:

1. The model from Simjoo and Zitha (2015) uses a constant value $n_{\text{DL}}^{\text{LE}}$, which is compatible with laboratory experiments.
2. In Ashoori et al. (2011), $R = 0$ if and only if $n_D = n_D^{LE}(S_w) = \tanh(400(S_w - 0.37))$, which approximates the Heaviside step function.

3. In Chen et al. (2010), $n_D^{LE}$ is the only real solution of

$$(n_D^{LE})^3 + \frac{k_{-1}v_{g}^{2/3}}{k_{1}^{0}v_{w}}n_D^{LE} - 1 = 0,$$  \hspace{1cm} (11)

where $k_{-1}$ and $k_{1}^{0}$ are positive coefficients possibly depending on $S_w$. Therefore, $n_D^{LE} = n_D^{LE}(S_w, u_w, u)$. In the examples above, $0 \leq n_D^{LE} \leq 1$. The expression $n_D^{LE}$ from Kovscek et al. (1995) can be also written as a function of $S_w$, $u_w$ and $u$ but it is not constrained to the interval $[0, 1]$. In (Kam 2008), $n_D^{LE}$ is also not constrained to the interval $[0, 1]$ and it is a function of $S_w$ and $\partial_x P$ (See Ashoori et al. (2011) for a traveling wave analysis of the model in Kam (2008)). In this work, we target models where $n_D^{LE} = n_D^{LE}(S_w, u_w, u)$. Sections 3 and 4 apply to models using all three kinds of $n_D^{LE}$ listed above. Section 5 shows examples of models using the expression $n_D^{LE}$ from Ashoori et al. (2011).

Remark 1 Some models make a distinction between trapped and flowing foam phases, e.g., (Izadi and Kam 2019; Kovscek et al. 1995; Kharabaf and Yortsos 1998). Other studies report the importance of understanding the gas trapping mechanism to model foam flow in porous media (Almajid et al. 2019; Jones et al. 2018). Simjoo and Zitha (2015) observe no trapping, arguing that foam could be moving slowly. On top of that, how to model trapped foam fraction still needs to be clarified, although some models try to explain it Kovscek et al. (1995), Cohen et al. (1997) and Tang and Kovscek (2006). The analysis presented in this paper may be applied to some trapped foam models, for instance, the model from Kovscek et al. (1995) and its derivatives.

3 The Hirasaki and Lawson’s Gas Viscosity

The apparent gas viscosity in the presence of foam is frequently modeled by Hirasaki and Lawson’s formula

$$\mu_g = \mu_g^0 + \alpha \frac{n_f}{v_{g}^{1/3}},$$  \hspace{1cm} (12)

when the foam is considered a shear-thinning fluid. The proportionality constant $\alpha$ depends on many factors, including the pore structure, the liquid viscosity, and the gas-liquid surface tension (Hirasaki and Lawson 1985). Due to the inherent difficulty of obtaining $\alpha$, many works use it as a fitting parameter, e.g., (Chen et al. 2010; Eide et al. 2020; Kam 2008; Kovscek et al. 1995; Simjoo and Zitha 2015; Thorat and Bruining 2016). Since $v_g$, $\mu_g$ (see (2), (3) and (10)), Eq. (12) defines $\mu_g$ implicitly, which hinders the analytical and numerical analysis of the model.

The first contribution of this work is to rewrite (12), eliminating the direct dependency on $v_g$. To do so, we replace (2), (3), and (10) into (12) obtaining a cubic equation $Q(X) = X^3 + 3AX - 2B = 0$ in the non-dimensional variable $X = (k_{rg} + \lambda_w \mu_g)^{1/3}$ where...
Since \( Q(0) < 0 < \lim_{X \to \infty} Q(X) \) and \( Q \) has a single inflection point \( X > 0 \), \( Q(X) = 0 \) admits a unique positive root \( X \), in which \( \mu_g \) is given by

\[
\mu_g = \mu_g^0 + \frac{3A}{\lambda_w} \left( 3\sqrt{B + \sqrt{B^2 - A^3}} + 3\sqrt{B - \sqrt{B^2 - A^3}} \right). 
\]

This means that (12) and (14) are equivalent expressions.

In the following sections, we need to evaluate \( \mu_g \) at equilibrium states where \( \partial_x P_c = 0 \). Therefore, it is useful to define

\[
\mu^\text{eq}_g := \mu_g^0 + \frac{3A}{\lambda_w} \left( 3\sqrt{B + \sqrt{B^2 - A^3}} + 3\sqrt{B - \sqrt{B^2 - A^3}} \right),
\]

where \( A \) is the same as \( A \) but disregards the term with \( \partial_x P_c \). Since \( \lambda_w \) and \( k_{tg} \) are functions of \( S_w \), and \( \mu^0_g, \alpha, n_{\text{max}}, \) and \( \phi \) are constant parameters, \( \mu^\text{eq}_g = \mu^\text{eq}_g(S_w, n_D, u) \).

### 4 Traveling Wave Solutions

Ashoori et al. (2011); Lozano et al. (2021) investigated traveling wave solutions for a foam model with Newtonian gas viscosity, where the apparent gas viscosity \( (\mu_g) \) depends only on the foam texture \( (n_D) \). Both works used parameter values from Alvarez et al. (2001). Below, we present the analysis for the generic non-Newtonian model described in Sect. 2 with the apparent viscosity formula detailed in Sect. 3. One of the examples we deal with uses the same parameter values as Ashoori et al. (2011) and Lozano et al. (2021).

#### 4.1 Traveling Wave Formulation

The traveling wave solutions of a system of evolutionary partial differential equations are the solutions invariant to translation in space [see (Volpert et al. 2000) and references therein]. Mathematically speaking, they depend on a traveling variable \( \xi := x - vt \), where \( v \in \mathbb{R} \) is the traveling wave velocity. For the system (1), (2) and (4) with step function initial conditions

\[
(S_w, n_D) = \begin{cases} 
(S_w^-, n_D^-), & x = 0 \\
[5pt] (S_w^+, n_D^+), & x > 0 
\end{cases}
\]

the traveling wave solution is represented by bounded and differentiable functions \( \tilde{S}_w, \tilde{n}_D : \mathbb{R} \to \mathbb{R} \) satisfying \( (\tilde{S}_w, \tilde{n}_D)(x - vt) = (S_w, n_D)(x, t) \) for some \( v \in \mathbb{R} \), all \( x \in (0, L) \), \( L > 0 \), and all \( t \in (0, +\infty) \). Moreover, the wave front \( (\tilde{S}_w, \tilde{n}_D) \) must satisfy the boundary conditions (Volpert et al. 2000):

\[
\lim_{\xi \to -\infty} (\tilde{S}_w, \tilde{n}_D)(\xi) = (S_w^-, n_D^-) \in \mathbb{R}^2, \quad \text{and} \quad \lim_{\xi \to +\infty} (\tilde{S}_w, \tilde{n}_D)'(\xi) = 0, 
\]

\[
\lim_{\xi \to -\infty} (\tilde{S}_w, \tilde{n}_D)(\xi) = (S_w^-, n_D^-) \in \mathbb{R}^2, \quad \text{and} \quad \lim_{\xi \to +\infty} (\tilde{S}_w, \tilde{n}_D)'(\xi) = 0, 
\]
where we use prime (′) to denote total derivatives. The (left) state \((S_w^-, n_D^-)\) corresponds to the inflow boundary conditions, and the (right) state \((S_w^+, n_D^+)\) corresponds to the initial conditions for the original system of partial differential equations.

Starting from (1), (2) and (4), one may verify that the wave front \((\bar{S}_w, \bar{n}_D)\) is a solution of the following system of ordinary differential equations (ODEs):

\[
\begin{aligned}
\{ (-v \phi \bar{S}_w + \bar{u}_w)'(\xi) &= 0, \\
[\bar{n}_D n_{\text{max}} (-v \phi (1 - \bar{S}_w) + (u - \bar{u}_w))]'(\xi) = \phi (1 - \bar{S}_w(\xi)) \bar{R}(\xi),
\end{aligned}
\]

for all \(\xi \in \mathbb{R}\), where \(\bar{R} := R(\bar{S}_w, \bar{n}_D, \bar{u}_w)\).

\[
\bar{u}_w := \bar{f}_w \left[ u + \frac{k_{tg}(\bar{S}_w)}{\bar{\mu}_g} K P'_c(\bar{S}_w) \bar{S}_w' \right], \quad \bar{f}_w := \frac{\lambda_w(\bar{S}_w)}{\lambda_w(\bar{S}_w) + \frac{k_{tg}(\bar{S}_w)}{\bar{\mu}_g}},
\]

and \(\bar{\mu}_g\) is given below in (23). The solution \((\bar{S}_w, \bar{n}_D)\) of (18) satisfying the limits (17) is known as a traveling wave solution.

Notice that, from (18), \(-v \phi \bar{S}_w + \bar{u}_w\) is a constant. Therefore, we can use the limits from (17) to conclude that there exists \(u_g^+ := u - \lim_{\gamma \to +\infty} \bar{u}_w\) such that

\[
-v \phi (1 - \bar{S}_w) + (u - \bar{u}_w) = -v \phi (1 - S^+_w) + u_g^+
\]

for all \(\xi \in \mathbb{R}\). Thus, we can rewrite (18)–(20) as follows

\[
\begin{aligned}
\{ \left( -\lambda_w \frac{k_{tg}}{k_{tg} + \lambda_w \bar{\mu}_g} K P'_c \right) \bar{S}_w' &= \left[ u_g^+ - \frac{k_{tg} u}{k_{tg} + \lambda_w \bar{\mu}_g} \right] - v \phi (\bar{S}_w - S^+_w), \\
n_{\text{max}} (u_g^+ - v \phi (1 - S^+_w)) \bar{n}_D' &= \phi (1 - \bar{S}_w) \bar{R}.
\end{aligned}
\]

One direct conclusion from (21) is that \(\lim_{\gamma \to +\infty} \bar{R}(\xi) = 0\). This property is used in Sect. 4.2 to obtain the equilibrium states and traveling wave velocity.

Note that (20) also implies that, for all \(\xi \in \mathbb{R}\),

\[
\frac{|u - \bar{u}_w|}{\phi (1 - \bar{S}_w)} = v \left( \frac{S_w^+ + \frac{u_G^+}{\phi v} - \bar{S}_w}{1 - \bar{S}_w} \right).
\]

Then, one may replace (22) into (12) to obtain

\[
\bar{\mu}_g = \bar{\mu}_g^0 \left[ 1 + \frac{\alpha n_{\text{max}}}{\bar{\mu}_g^0 \phi v} \left( \frac{1 - \bar{S}_w}{|S_w^+ + \frac{u_G^+}{\phi v} - \bar{S}_w|} \right)^{\frac{1}{2}} \bar{n}_D \right].
\]

Equation (23) is an alternative version of (12) valid only for the traveling wave profile.

Let us summarize these calculations. We start with the problem (1)–(4) defined in \((0, L) \times (0, +\infty)\), with initial condition \((S_w, n_D)(x, 0) = (S_w^+, n_D^+)\) and boundary condition \((S_w, n_D)(0, t) = (S_w^-, n_D^-)\). A traveling wave solution for this problem is given by the solution of (21), which satisfies limit conditions (17). We discuss how to obtain these limit conditions in Sect. 4.2.
The existence of the solution of (21) is not trivial. Lozano et al. (2021) investigate the existence of solutions for (21) using Newtonian foam flow and particular choice of $R = r_g - r_c$. They show that such solutions exist in specific parameter regions, depending on the model and the limit states. We do not perform such deep analysis here; however, we give evidence that the solutions exist for the examples from Sect. (5).

### 4.2 Traveling Wave Velocity and Equilibria

The usual procedure to determine the traveling wave velocity and equilibria is to apply limits (17) to the ODE system (21). On physical grounds, $k_{yw}(S_w^\pm)$, $k_{wg}(S_w^\pm)$, and $P'_c(S_w^\pm)$ are real numbers and $k_{wg}(S_w^\pm) > 0$. As $\lim_{\xi \to \pm \infty} \tilde{S}_w'(\xi) = 0$ from (17), it follows that $\lim_{\xi \to \pm \infty} \partial_x P_c(\xi) = 0$. Therefore, $\lim_{\xi \to \pm \infty} \tilde{R}_g(\xi) = \mu_{\text{eq}}(S_w^\pm, n_D^\pm, u)$ as defined in (15). Moreover, we use the fractional flux definition in (3), and apply the limits in (17) to (21) and (19) to obtain the algebraic system of nonlinear equations

$$\frac{k_{wg}(S_w^-)}{k_{wg}(S_w^-) + \lambda_w(S_w^-) \mu_{\text{eq}}(S_w^-, n_D^-, u)} = f_g^-,$$  
\hspace{1cm} (24)

$$\frac{k_{wg}(S_w^+)}{k_{wg}(S_w^+) + \lambda_w(S_w^+) \mu_{\text{eq}}(S_w^+, n_D^+, u)} = \frac{u_g^+}{u},$$  
\hspace{1cm} (25)

$$\left( u_g^+ - f_g^- u \right) - v \phi(S_w^- - S_w^+) = 0,$$  
\hspace{1cm} (26)

$$R(S_w^-, n_D^-, u(1 - f_g^-)) = 0,$$  
\hspace{1cm} (27)

$$R(S_w^+, n_D^+, u - u_g^+) = 0.$$
\hspace{1cm} (28)

As we have eight unknowns ($S_w^-, n_D^-, n_D^+, u_g^+, v, u, S_w^+, f_g^-$) and five algebraic Eqs. (24)–(28), one expects to find five of the unknowns as functions of the other three unknowns. From the experimental point of view, it is plausible to assume that we know the initial water saturation ($S_w^+$), we control the total superficial velocity ($u$), and we measure the fraction of the injected fluids ($f_g^-$). Therefore, we consider $u, S_w^+$ and $f_g^-$ to be the known values and ($S_w^-, n_D^-, n_D^+, u_g^+, v$) as the main unknowns in (24)–(28). In Sect. 5, we follow (Ashoori et al. 2011) and investigate cases with high initial water saturation ($S_w^- < S_w^+ \leq 0.72$) and injection of both high quality ($f_g^- = 0.268$) and low quality ($f_g^- = 0.9$) foam.

As discussed in Sect. 2, there is a particular case where the constraints $R = 0$ and $\partial_x P_c = 0$ result in the foam texture $n_D = n_D^{LE}(S_w, u_w, u)$. For instance, the models from Ashoori et al. (2011), Chen et al. (2010), Kam (2008), Kovscek et al. (1995) and Simjoo and Zitha (2015) allow for such an expression of local-equilibrium foam texture. For these models, one may use the following strategy to solve (24)–(28) given $u, S_w^+$ and $f_g^-$:

1. Solve (24) for $S_w^-$ using $n_D^- = n_D^{LE}(S_w^-, u(1 - f_g^-), u)$ consistently with (27);
2. Solve (25) for $u_g^+$ using $n_D^+ = n_D^{LE}(S_w^+, u - u_g^+, u)$ consistently with (28);
3. Solve (26) for $v$.

This algorithm involves the solution of two uncoupled scalar nonlinear equations, which can be done either analytically or by applying a classical numerical procedure, see (Mathews 1992). After obtaining all constant parameters, one must solve the ODE system (21) with limit conditions in (17) using usual ODE solvers for nonlinear systems to obtain the traveling wave profile.

Remark 2 The procedure to obtain the traveling wave limits and wave velocity described above can still be used for initial gas at residual saturation, i.e., $S_w^+ = 1 - S_{gr}$. In this case, the existence of traveling wave limits relies on the following limit

$$\lim_{\xi \to +\infty} \tilde{S}_w'(\xi) = \lim_{S_w \to 1 - S_{gr}} \frac{u - v \phi (1 - S_w - S_{gr}) k_{ig} \mu_w^{eq}}{k_{ig} \lambda_w \phi (1 - S_w - S_{gr})} = 0.$$ 

Ashoori et al. (2011) use a modified Brooks–Corey capillary pressure equation obtained by multiplying the original formula $P_b/S_{we}$ by the term $(1 - S_w - S_{gr})^{0.01}$, where $P_b$ is the non-wetting fluid (gas) entry capillary pressure head [see (Lenhard et al. 1989)]. Such modification does not affect the phenomenological conclusions but guarantees the limit above holds. In the present paper, we use the model from Ashoori et al. (2011) with the referred modification. We stress that one can propose other modifications to the capillary pressure, provided the limit above remains valid.

5 Application in The Linear Kinetics Model

In this section, we apply the methodology from the previous sections to obtain traveling wave solutions for a foam propagation model previously studied by Ashoori et al. (2011). We use them to compare the influence of the non-Newtonian apparent viscosity in the foam model. In this model, the relative permeabilities, $k_{rw}(S_w)$ and $k_{rg}(S_w)$, are generalized Brooks–Corey expressions fitted for the nitrogen-water flow in the Boise sandstone. The capillary pressure has an equivalent unitary pore size distribution index, also inspired in the work of Brooks and Corey (1966). Foam generation and coalescence are controlled by expressions $r_g(S_w)$ and $r_c(n_D)$ as follows

$$k_{rw}(S_w) := 0.2 S_{we}^{4.2},$$
$$k_{rg}(S_w) := 0.94 (1 - S_{we})^{1.3},$$
$$P_c(S_w) := \frac{P_b (1 - S_w - S_{gr})^{0.01}}{S_{we} (1 - S_{we} - S_{gr})},$$
$$r_g(S_w) := K_c n_{max} n_D^{LE}(S_w),$$
$$r_c(n_D) := K_c n_{max} n_D,$$
$$n_D^{LE}(S_w) := \begin{cases} 
\tanh(400 (S_w - 0.37)) & \text{if } S_w > 0.37, \\
0 & \text{if } S_w \leq 0.37,
\end{cases}$$

(29)
where $K_c$ is the kinetic generation/coalescence rate parameter. The coefficient $P_b = 330 \text{ Pa}$ is the gas entry capillary pressure head and stems from the Leverett J-function (Leverett 1941) with a gas-water surface tension of $0.03 \text{ N/m}$. Table 1 shows the remaining parameter values from Ashoori et al. (2011). To summarize, we analyze the model described by (1)–(4), (12) and (29) with parameters from Table 1.

Ashoori et al. (2011) studied the incompressible foam flow considering foam as a Newtonian fluid. They used the following apparent gas viscosity formula (hereafter, the superscript ‘N’ identifies the properties of the Newtonian model)

$$
\mu_g^N(n_D) := \mu_g^0 \text{MRF}^N(n_D), \quad \text{MRF}^N(n_D) := 1 + C_{\text{MRF}} n_D,
$$

where $C_{\text{MRF}}$ is a reference mobility reduction factor (MRF) for the strongest foam. One direct consequence of this simplification is that the fractional fluxes are algebraic expressions of the water saturation and foam texture as follows

$$
f_w^N(S_w, n_D) := \frac{k_w(S_w)}{\mu_w} \cdot \frac{k_w(S_w)}{\mu_w} + \frac{k_g(S_w)}{\mu_g} \cdot \frac{k_g(S_w)}{\mu_g}.
$$

Two well-known foam properties inspired the simplification: (1) foamed gas mobility reduction is large and nearly constant at high water saturation, and (2) there exists a limiting capillary pressure, where foam collapses abruptly.

To compare the influence of considering foam as a non-Newtonian fluid, we need to find a mapping between $\mu_g$ (defined in (12)) and $\mu_g^N$. We proceed by choosing a parameter $\alpha > 0$ such that it minimizes the mean squared difference between the gas mobilities under Local Equilibrium conditions and disregarding capillary pressure derivatives, i.e.,
Traveling Wave Solutions for Non-Newtonian Foam Flow in Porous…

\[
\min_{\alpha > 0} \int_{S_w}^{1-S_w} \left( \frac{k_{eq}(S_w)}{\mu_{eq}^N(S_w, n_{D}^{LE}(S_w), u)} - \frac{k_{eq}(S_w)}{\mu_{eq}^N(n_{D}^{LE}(S_w))} \right)^2 \, dS_w. \tag{32}
\]

using parameters from Table 1. We find \( \alpha = 2.44 \times 10^{-16} \text{ Pa s}^{2/3} \text{ m}^{-10} \) (See Remark 3). This mapping generalizes the one proposed by Zavala et al. (2021).

In what follows, we compute traveling wave profiles for the Linear Kinetics model using the non-Newtonian apparent gas viscosity (12). We compare those profiles with the ones obtained using the Newtonian apparent viscosity (30). We analyze two scenarios. The first one reproduces the example found in Ashoori et al. (2011). The second one shows a case where the non-Newtonian foam model brings significant information. Table 2 shows the equilibria computed using (24)–(28) for the Newtonian and non-Newtonian models in two scenarios. After those two scenarios, a third subsection shows how the total apparent viscosity \( \mu_{\text{app}} \) changes when we change the foam quality. We measure this quantity at equilibrium states, i.e., \( n_{D} = n_{D}^{LE} \) and \( \partial_x P_c = 0 \).

### 5.1 Scenario 1

Scenario 1 corresponds to the flow data from the literature (Ashoori et al. 2011) with the limit states \( S_w^+ = 0.72 \) and \( f_w^- = 0.268 \). The autonomous system (21) can be rewritten in the standard form:

\[
\begin{align*}
\min_{\alpha > 0} & \int_{S_w}^{1-S_w} \left( \frac{k_{eq}(S_w)}{\mu_{eq}^N(S_w, n_{D}^{LE}(S_w), u)} - \frac{k_{eq}(S_w)}{\mu_{eq}^N(n_{D}^{LE}(S_w))} \right)^2 \, dS_w. \tag{32}
\end{align*}
\]

### Table 2 Parameter values at limit states and traveling wave velocities in two scenarios used to compare Newtonian and non-Newtonian models

| Parameter | Scenario 1 | Scenario 2 |
|-----------|------------|------------|
| \( f_w^- = 0.268; S_w^+ = 0.72 \) | \( f_w^- = 0.9; S_w^+ = 0.819 \) |
| \( \text{Newtonian} \) | \( \text{non-Newt.} \) | \( \text{Newtonian} \) | \( \text{non-Newt.} \) |
| \( S_w^- [-] \) | 3.720 \times 10^{-1} | 3.719 \times 10^{-1} | 5.011 \times 10^{-1} | 4.672 \times 10^{-1} |
| \( n_{D}^+ [-] \) | 6.650 \times 10^{-1} | 6.503 \times 10^{-1} | 1.000 | 1.000 |
| \( n_{D}^- [-] \) | 1.000 | 1.000 | 1.000 | 1.000 |
| \( u_g^- \quad \text{[m/s]} \) | 7.251 \times 10^{-8} | 6.111 \times 10^{-9} | 8.782 \times 10^{-11} | 3.193 \times 10^{-13} |
| \( [\partial_x P^-] \quad \text{[Pa/m]} \) | 8.566 \times 10^{6} | 8.579 \times 10^{6} | 2.738 \times 10^{6} | 4.520 \times 10^{6} |
| \( [\partial_x P^+] \quad \text{[Pa/m]} \) | 3.059 \times 10^{5} | 3.066 \times 10^{5} | 1.475 \times 10^{5} | 1.475 \times 10^{5} |
| \( v \quad \text{[m/s]} \) | 2.457 \times 10^{-4} | 2.464 \times 10^{-4} | 3.687 \times 10^{-5} | 3.687 \times 10^{-5} |
| \( u_g^- / u \quad [-] \) | 2.475 \times 10^{-3} | 2.086 \times 10^{-4} | 2.997 \times 10^{-6} | 1.090 \times 10^{-8} |
| \( v/u \quad [-] \) | 8.385 | 8.410 | 1.258 | 1.137 |

We use \( u = 2.930 \times 10^{-6} \text{ m/s} \)
\[
\begin{bmatrix}
\tilde{S}_w \\
\tilde{n}_D
\end{bmatrix}' = F(\tilde{S}_w, \tilde{n}_D);
\]
where
\[
F(S_w, n_D) := \begin{bmatrix}
\left( u_g^+ - \frac{k_{ig} u}{k_{ig} + \lambda_w \mu} \right) - \nu \phi (S_w - S_w^+) \\
- \frac{k_{ig} u}{k_{ig} + \lambda_w \mu} \lambda_w K P_c' \\
- \phi (1 - S_w) R \\
n_{\text{max}} \left( u_g^+ - \nu \phi (1 - S_w^+) \right)
\end{bmatrix}^{1/3} n_D,
\]
and \( \mu := \mu_0 + \frac{\alpha n_{\text{max}}}{\nu^1} \left( \frac{1 - S_w}{S_w^+ + \frac{u_g^+}{\nu \phi} - S_w} \right)^{1/3} n_D, \)

where the latter comes from (23). One standard approach for proving the existence of the solution connecting two equilibria in an autonomous system is based on the analysis of the ODE’s phase space, as done by Lozano et al. (2021). We repeat some steps of their analysis.

Lozano et al. (2021) started by proving that:

1. \( F(S_w, n_D) = 0 \) if and only if \( n_D = n_{D}^{LE}(S_w) \) and \( u_g^+ - (k_{ig} u)/(k_{ig} + \lambda_w \mu) = \nu \phi (S_w - S_w^+) \);
2. The curve \( u_g^+ - (k_{ig} u)/(k_{ig} + \lambda_w \mu) \) constrained to \( n_D = n_{D}^{LE}(S_w) \) intersects the straight line \( \nu \phi (S_w - S_w^+) \) in at most two values of \( S_w \in (S_{wc}, 1 - S_{gr}) \).

Since \( F(S_w^+, n_D^+) = F(S_w^-, n_D^-) = 0 \), those are the only two roots of \( F \) in \( (S_{wc}, 1 - S_{gr}) \times [0, 1] \).

Second, let us comment on the dynamics in the neighborhood of the limit states of Scenario 1 in Table 2. In two-dimensional \((S_w, n_D)\) phase space, this analysis reduces to look at the real part’s sign of the eigenvalues of the Jacobian matrix \( F'(S_w, n_D) \); positive signs indicate source directions, and negative signs indicate sink directions. Since \( F(S_w, n_D) = 0 \) only over the equilibria, the orbit (solution of (33)) exits if there is at least one source direction in the left state and at least one sink direction in the right state. Table 3 shows the eigenvalues for states – and + using the non-Newtonian model. In all cases, the states—have eigenvalues with positive real parts (sources), and states + have eigenvalues with real parts with opposite signs (saddle points), which proves the traveling wave connection.

### Table 3 Eigenvalues of the Jacobian matrix \( F'(S_w, n_D) \) at the equilibrium states – and + of Scenario 1

| \( K_c \) [1/s] | State | \( \lambda_1 \) [1/m] | \( \lambda_2 \) [1/m] | Classification |
|---|---|---|---|---|
| 200 | – | 1.089×10^5 | 1.715×10^6 | Source |
| 200 | + | -1.020×10^6 | 8.092×10^5 | Saddle |
| 1 | – | 1091+2854i | 1091–2854i | Spiral source |
| 1 | + | -1.020×10^6 | 4.046×10^3 | Saddle |
| 0.01 | – | 1.209×10^4 | 7.726×10^2 | Source |
| 0.01 | + | -1.020×10^6 | 4.046×10^1 | Saddle |

The classification follows a typical nomenclature (Guckenheimer et al. 1984)
Table 2 shows that the relative difference between Newtonian and non-Newtonian viscosity models for the states $n_D$ is lower than 3%, and the values of $u^+$ are negligible relative to $u$. The remaining parameter values vary less than 1% between the models. Note that the Newtonian model’s values in scenario 1 coincide with those from Ashoori et al. (2011). The solution profiles for $S_w$ and $n_D$ for the non-Newtonian case are very similar to the ones of the Newtonian model found in Ashoori et al. (2011).

Figure 1 shows the MRF and $f_w$ profiles along the traveling wave using its coordinate $\xi$. The MRF ($\tilde{\mu}/\mu_0^*$) in the left panel of Fig. 1 is governed by (23) for the non-Newtonian case. This formula indicates that MRF increases indefinitely as $S_w$ approaches $S_w^+ + u^+/(u \phi)$, which we see when $\xi$ is close to 0.3. On the other hand, $\tilde{f}_w$ in the right panel of Fig. 1 presents a gradual change in the region where the non-Newtonian MRF changes abruptly. Notice that $f_w = 1/(1 + (\lambda_0^0/\lambda_w)(1/MRF))$, where $\lambda_0^0$ represents the gas mobility in the absence of foam. Thus, the gradual change in $\tilde{f}_w$ is due to the moderate magnitude of the ratio $\lambda_0^0/\lambda_w$ for $S_w > 0.7$.

**Remark 4** We recall that traveling waves are invariant to translation in space ($\xi$-axis). In Figs. 1, 2, we align the profiles to help visually compare the models.
5.2 Scenario 2

The second scenario explores higher differences between Newtonian and non-Newtonian behaviors by taking a higher initial water saturation, \( S^+_w = 0.819 \), and injecting a mixture with lower foam quality, \( f_w^- = 0.9 \). This scenario, usually denoted by the term low-quality regime, was not based on experimental data from the literature; it was built to enlarge the differences between Newtonian and non-Newtonian regimes describing foam flow in porous media. Scenario 2 addresses a fundamental scientific discussion on understanding the non-Newtonian nature of foam displacement. The analysis of the existence of traveling wave solutions for Scenario 2 is analogous to the one from Sect. 5.1; therefore, we omit it. We use \( K_c = 200 \text{ s}^{-1} \) for Scenario 2.

Table 2 shows the equilibria for the Newtonian and non-Newtonian models in Scenario 2. We note that the Newtonian water saturation \( S^-_w \) and pressure gradient \( \frac{\partial x P}{\partial x} \) differ from the corresponding non-Newtonian values 7% and 39%, respectively. As expected, the traveling wave velocity is larger than the total superficial velocity in all cases. However, in Scenario 1, the Newtonian model results in a wave 0.3% faster than the non-Newtonian one; while in Scenario 2, the Newtonian model results in a wave 11% slower than the non-Newtonian one.

Figure 2 shows the traveling wave profiles \( \ddot{S}_w \) and \( \ddot{n}_D \) and the absolute value of the pressure gradient along the traveling wave for both models. The pair \( (\ddot{S}_w, \ddot{n}_D) \) solve (21) with conditions (17). Besides the difference in the limit states, the non-Newtonian foam model profiles are sharper than the ones in the Newtonian model. Figure 3 shows the MRF and \( f_w \) along the traveling wave. As in scenario 1, MRF is qualitatively and quantitatively different between the models. In this case, however, the fractional flux profile changes between the models leading to different water saturation profiles; see left panel of Fig. 2.

5.3 Traveling Wave Equilibria for Different Flow Velocities

Figure 4 shows isolines of the superficial gas velocity \( u_g \) as a function of the foam quality \( f_g \) and the total apparent viscosity \( \mu_{\text{app}} \) defined in (7). To plot it, we consider equilibrium conditions, i.e., \( n_D = n_D^\text{LE} \) and \( \partial x P_c = 0 \); we use finite sets of values for total superficial velocity \( u \) (from 0.1 to 100 \( \mu \text{m/s} \)) and water saturation (from \( S_{wc} \) to \( 1 - S_{gr} \)). Those ranges cover laboratory and field experimental values (Chen et al. 2010; Ashoori et al. 2011; Farajzadeh et al. 2012). Using these sets of values, we compute \( \mu_{\text{app}} \) using (3), (5), (7), (15), (29) and (13). Next, \( u_g = u(1 - \lambda_w \mu_{\text{app}}) \) and \( f_g = 1 - \lambda_w \mu_{\text{app}} \) are generated using (29).

![Fig. 3 MRF and \( f_w \) along the traveling wave in the non-Newtonian and Newtonian models in Scenario 2](image-url)
As expected, at both ends in Fig. 4, all curves $\mu_{\text{app}}$ approach the same value. This happens because for the low value of the water saturation (corresponding to big $f_g$), the foam collapses, and all curves $\mu_g$ approach the same $\mu_g^0$. For high water saturation values (corresponding to low $f_g$), there is no gas phase to form foam, and $\mu_{\text{app}}$ approaches the water viscosity. One can see that the lower the superficial gas velocity, the stronger the foam (higher $\mu_{\text{app}}$). In agreement with Eide et al. (2020); Gassara et al. (2020), the high-quality regime region (after the peak of the strongest foam) shrinks when the superficial gas velocity increases.

In the Newtonian model, total apparent viscosity and fractional fluxes do not depend on $u_g$; however, they depend on $S_w$ (or equivalently on $f_g$) and the fitting constant $C_{\text{MRF}}$, see (30). In Fig. 4, we plot the apparent viscosity for the Newtonian model for $C_{\text{MRF}} = 37000$ and $C_{\text{MRF}} = 185000$. As one can see, varying $C_{\text{MRF}}$, it is possible to choose $C_{\text{MRF}}$ in such a way that the Newtonian model better represents the non-Newtonian gas viscosity accordingly to the flow regime.

**Remark 5** Scenario 2 evidences some differences between the traveling wave solutions of non-Newtonian and Newtonian models. The parameter values in this scenario assume the injection of a mixture with low gas concentration. Although this injection is uncommon, such configuration occurs in industrial applications, e.g., surfactant alternated gas (SAG) injection in the Enhanced Oil Recovery (EOR) (Farajzadeh et al. 2012). During this procedure, pure water (containing surfactant) and pure gas are injected into the reservoir periodically, resulting in alternated low and high gas concentrations inside the porous medium. The main challenge consists of avoiding the gas-fingering formation, e.g., not allowing the gas phase to penetrate the water phase. That is why several experimental and theoretical investigations (including the present one) concentrate on studying the two-phase flow in the presence of foam (Simjoo et al. 2012; Simjoo and Zitha 2015; Ashoori et al. 2011; Kapetas et al. 2016; Boeije and Rossen 2015).

**Remark 6** We used the explicit Runge–Kutta method RK5(4) (Shampine 1986) to solve (21) in the variable $\eta := -\xi$ on $[0, \eta_F]$ with initial conditions $(\bar{S}_w, \bar{n}_D)(0) = (S_w^+ - 10^{-6}, n_D^+)$.
and step size $\eta_F/1000$. This change of variables, $\xi$ to $-\eta$, helps the numerical solver to find the correct solution.

6 Discussion and Conclusions

In this paper, we detailed the procedure of obtaining traveling wave solutions for a range of models describing foam propagation in porous media. For that we reformulate Hirasaki and Lawson’s apparent viscosity in terms of the foam model’s unknowns $S_w$, $n_D$, and $u$. We restrict the analysis to the case of constant total superficial velocity $u$ as in previous works (Lozano et al. 2021, 2022). It is worth mentioning that the reformulated apparent viscosity formula can be used in other contexts. One example is to reconstruct fractional fluxes directly from the experimental data.

In order to evidence the potential of the proposed methodology, we find the traveling wave solutions for the Linear Kinetics model. We use the solutions to quantify the importance of considering the non-Newtonian apparent viscosity in the model. We proposed a mapping procedure to map one model in the other. This procedure involves solving a minimization problem for the mean square difference between gas mobilities in the two models. We conclude that:

- For the cases reproduced from Ashoori et al. (2011), the non-Newtonian viscosity brings no new relevant information. This is expected in such a high-quality regime, as observed in several other works, e.g., (Cheng et al. 2000; Salazar Castillo et al. 2020; Valdez et al. 2021).
- In the low-quality example, we identify a foam flow regime with stronger non-Newtonian fluid behavior and obtain more significant discrepancies between the two models. Our example showed a relative difference of 39% in the pressure gradient and 11% in the traveling wave velocity. We noticed that the Newtonian approximation might result in traveling wave velocities higher or lower than the non-Newtonian one. Estimating the traveling wave velocity is crucial to predicting the gas breakthrough during an enhanced recovery process in porous media. It is worth mentioning that we achieve such contrast even using the same constant total superficial velocity from Scenario 1.
- The mobility reduction factor gives significantly less information on the foam flow than the fractional flux curves. In the two scenarios we analyze, the mobility reduction factors between Newtonian and non-Newtonian presented qualitative and quantitative differences. However, the water saturation and foam texture wave profiles coincide qualitatively in the first scenario, while, in the second one, they do not.
- For different total flow velocities and quality regimes, one may still use the Newtonian gas viscosity formula, although the parameters should be adjusted so that the shear-thinning behavior of foam can be accurately represented.

Acknowledgements This research was carried out in association with the ongoing R &D project registered as ANP 20715-9, “Modelagem matemática e computacional de injeção de espuma usada em recuperação avançada de petróleo” (Universidade Federal de Juiz de Fora (UFJF) / Shell Brasil / ANP) - Mathematical and computational modeling of foam injection as an enhanced oil recovery technique applied to Brazil pre-salt reservoirs, sponsored by Shell Brasil under the ANP R &D levy as “Compromisso de Investimentos com Pesquisa e Desenvolvimento”. This project is carried out in partnership with Petrobras.
The author G.C. was supported in part by CNPq grants 306970/2022-8, 405366/2021-3, and FAPEMIG grant APQ-00405-21.

Data availability All data used to support this work are reported in the manuscript and the supporting information in the respective tables and figures.

Declarations

Conflict of interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

Almajid, M.M., Nazari, N., Kovscek, A.R.: Modeling steady-state foam flow: hysteresis and backward front movement. Energy Fuels 33(11), 11,353–11,363 (2019). https://doi.org/10.1021/acs.energyfuels.9b01842

Alvarez, J.M., Rivas, H.J., Rossen, W.R.: Unified model for steady-state foam behavior at high and low foam qualities. SPE J. 6(3), 325–333 (2001). https://doi.org/10.2118/74141-PA

Ashoori, E., Marchesin, D., Rossen, W.R.: Roles of transient and local equilibrium foam behavior in porous media: traveling wave. Colloids Surf A 377(1–3), 228–242 (2011). https://doi.org/10.1016/j.colsurfa.2010.12.042

Behenna, F.R.: Acid diversion from an undamaged to a damaged core using multiple foam slugs. In: SPE-European Formation damage conference, pp. 441–452, https://doi.org/10.2118/30121-ms (1995)

Boakye-Ansah, Y.A., Grassia, P.: Comparing and contrasting travelling wave behaviour for groundwater flow and foam drainage. Transp. Porous Media 137(1), 255–280 (2021)

Boeije, C.S., Rossen, W.: Fitting foam-simulation-model parameters data: I. To coinjection of gas and liquid. SPE Reserv. Eval. Eng. 18(02), 264–272 (2015)

Bretherton, F.P.: The motion of long bubbles in tubes. J. Fluid Mech. 10(2), 166–188 (1961). https://doi.org/10.1017/S0022112061000160

Brooks, R.H., Corey, A.T.: Properties of porous media affecting fluid flow. J. Irrig. Drain Div. 92(2), 61–88 (1966)

Casteel, J.F., Djabbarah, N.F.: Sweep improvement in the CO2 flooding by use of foaming agents. SPE Reserv. Eng. 3(4), 1186–1192 (1988). https://doi.org/10.2118/14392-PA

Chen, Z., Ewing, R.E.: Fully discrete finite element analysis of multiphase flow in groundwater hydrology. SIAM J. Numer. Anal. 34(6), 2228–2253 (1997). https://doi.org/10.1137/0036142995290063

Chen, Q., Gerritsen, M.G., Kovscek, A.R.: Modeling foam displacement with the local-equilibrium approximation: theory and experimental verification. SPE J. 15(1), 171–183 (2010). https://doi.org/10.2118/116735-pa

Cheng, L., Reme, A.B., Shan, D., et al.: Simulating foam processes at high and low foam qualities. In: SPE/DOE improved oil recovery symposium, society of petroleum engineers, https://doi.org/10.2118/59287-ms (2000)

Cohen, D., Patzek, T.W., Radke, C.J.: Onset of mobilization and the fraction of trapped foam in porous media. Transp. Porous Media 28(3), 253–284 (1997). https://doi.org/10.1023/A:1006552320036

Corey, A.T.: The interrelation between gas and oil relative permeabilities. Produc. Mon. 19(1), 38–41 (1954)

de Vries, A.S., Wit, K.: Rheology of gas/water foam in the quality range relevant to steam foam. SPE Reserv. Eng. 5(2), 185–192 (1990). https://doi.org/10.2118/18075-pa

Eide, Ø., Ferno, M., Bryant, S., et al.: Population-balance modeling of CO2 foam for CCUS using nanoparticles. J. Nat. Gas Sci. Eng. 80(103), 378 (2020). https://doi.org/10.1016/j.jngse.2020.103378

Ettinger, R.A., Radke, C.J.: Influence of texture on steady foam flow in Berea sandstone. SPE Reserv. Eng. 7(1), 83–90 (1992). https://doi.org/10.2118/19688-PA

Falls, A.H., Hirasaki, G.J., Patzek, T.W., et al.: Development of a mechanistic foam simulator: the population balance and generation by snap-off. SPE Reserv. Eng. 3(3), 884–892 (1988). https://doi.org/10.2118/14961-PA

Farajzadeh, R., Andrianov, A., Krastev, R., et al.: Foam-oil interaction in porous media: implications for foam assisted enhanced oil recovery. In: SPE EOR conference at oil and gas West Asia, OnePetro (2012)
Fernø, M.A., Jarand, G., Monrawee, P., et al.: Experimental study of foam generation, sweep efficiency, and flow in a fracture network. SPE J. 21(4), 1140–1150 (2016). https://doi.org/10.2118/170840-pa

Gassara, O., Douarche, F., Braconnier, B., et al.: Calibrating and scaling semi-empirical foam flow models for the assessment of foam-based EOR processes (in heterogeneous reservoirs). Transp. Porous Media 131(1), 193–221 (2020). https://doi.org/10.1007/s11242-018-01223-5

Grassia, P., Mas-Hernández, E., Shokri, N., et al.: Analysis of a model for foam improved oil recovery. J. Fluid Mech. 751, 346–405 (2014). https://doi.org/10.1017/jfm.2014.287

Guckenheimer, J., Holmes, P., Slemrod, M.: Nonlinear oscillations dynamical systems, and bifurcations of vector fields. J. Appl. Mech. 51(4), 947–947 (1984). https://doi.org/10.1115/1.3167759

Heller, J.P., Kuntamukkula, M.S.: Critical review of the foam rheology literature. Ind. Eng. Chem. Res. 26(2), 318–325 (1987). https://doi.org/10.1021/ie00062a023

Hematpour, H., Mahmood, S.M., Akbari, S., et al.: Foam modeling approaches in enhanced oil recovery: A review. Indian J. Sci. Technol. 9(22), (2016)

Hirasaki, G.J., Lawson, J.B.: Mechanisms of foam flow in porous media: apparent viscosity in smooth capillaries. SPE J. 25(2), 176–190 (1985). https://doi.org/10.2118/12129-pa

Heller, J.P., Kuntamukkula, M.S.: Critical review of the foam rheology literature. Ind. Eng. Chem. Res. 26(2), 318–325 (1987). https://doi.org/10.1021/ie00062a023

Hematpour, H., Mahmood, S.M., Akbari, S., et al.: Foam modeling approaches in enhanced oil recovery: A review. Indian J. Sci. Technol. 9(22), (2016)

Hirasaki, G.J., Lawson, J.B.: Mechanisms of foam flow in porous media: apparent viscosity in smooth capillaries. SPE J. 25(2), 176–190 (1985). https://doi.org/10.2118/12129-pa

Izadi, M., Kam, S.I.: Bubble-population-balance modeling for supercritical carbon dioxide foam enhanced-oil-recovery processes: from pore-scale to core-scale and field-scale events. SPE Reserv. Eval. Eng. 22(4), 1467–1480 (2019). https://doi.org/10.2118/191202-PA

Jones, S.A., Getrouw, N., Vincent-Bonnieu, S.: Foam flow in a model porous medium: II. The effect of trapped gas. Soft Matter 14(18), 3497–3503 (2018). https://doi.org/10.1039/c7sm02458d

Kam, S.I.: Improved mechanistic foam simulation with foam catastrophes theory. Colloids Surf A 318(1–3), 62–77 (2008). https://doi.org/10.1016/j.colsurfa.2007.12.017

Kam, S.I., Nguyen, Q.P., Li, Q., et al.: Dynamic simulations with an improved model for foam generation. SPE J. 12(1), 35–48 (2007). https://doi.org/10.2118/90938-PA

Kapetas, L., Vincent Bonnieu, S., Danelis, S., et al.: Effect of temperature on foam flow in porous media. J. Ind. Eng. Chem. 36, 229–237 (2016). https://doi.org/10.1016/j.jiec.2016.02.001

Kapetas, L., Vincent Bonnieu, S., Farajzadeh, R., et al.: Effect of permeability on foam-model parameters: an integrated approach from core-flood experiments through to foam diversion calculations. Colloids Surf. A Physicochem. Eng. Asp. 530, 172–180 (2017). https://doi.org/10.1016/j.colsurfa.2017.06.060

Kharabaf, H., Yortsos, Y.C.: Pore network model for foam formation and propagation in porous media. SPE J. 3(1), 42–53 (1998). https://doi.org/10.2118/36663-pa

Kovscek, A.R., Patzek, T.W., Radke, C.J.: A mechanistic population balance model for transient and steady-state foam flow in boise sandstone. Chem. Eng. Sci. 50(23), 3783–3799 (1995). https://doi.org/10.1016/0009-2509(95)00199-F

Lenhard, R.J., Parker, J.C., Mishra, S.: On the correspondence between brooks-corey and van genuchten models. J. Irrig. Drain. Eng. 115(4), 744–751 (1989). https://doi.org/10.1061/(ASCE)0733-9437(1989)115:4(744)

Leverett, M.C.: Capillary behavior in porous solids. Trans. AIME 142(01), 152–169 (1941). https://doi.org/10.2188/941152-g

Li, K., Horne, R.N.: Comparison of methods to calculate relative permeability from capillary pressure in consolidated water-wet porous media. Water Resour. Res. 42(6), 1–9 (2006). https://doi.org/10.1029/2005WR004482

Lozano, L.F., Zavala, R.Q., Chapiro, G.: Mathematical properties of the foam flow in porous media. Comput. Geosci. 25(1), 515–527 (2021). https://doi.org/10.1007/s10596-020-10020-3

Lozano, L.F., Cedro, J.B., Zavala, R.V.Q., et al.: How simplifying capillary effects can affect the traveling wave solution profiles of the foam flow in porous media. Int. J. Non-Linear Mech. 139(103), 867 (2022)

Ma, K., Farajzadeh, R., Lopez-Salinas, J.L., et al.: Estimation of parameters for the simulation of foam flow through porous media: part 3; non-uniqueness, numerical artifact and sensitivity. In: SPE enhanced oil recovery conference, OnePetro (2013a)

Ma, K., Lopez-Salinas, J.L., Puerto, M.C., et al.: Estimation of parameters for the simulation of foam flow through porous media. Part 1: the dry-out effect. Energy & fuels 27(5), 2363–2375 (2013)

Ma, K., Farajzadeh, R., Lopez-Salinas, J.L., et al.: Non-uniqueness, numerical artifacts, and parameter sensitivity in simulating steady-state and transient foam flow through porous media. Transp. Porous Media 102, 325–348 (2014)

Mamun, C., Rong, J., Kam, S., et al.: Simulating use of foam in aquifer remediation. In: Hassanizadeh SM, Schotting RJ, Gray WG, et al. (eds) Computational Methods in Water Resources, Developments in Water Science, vol 47. Elsevier, p 867–874, https://doi.org/10.1016/S0167-5648(02)80152-6 (2002)
Marsden, S.S., Khan, S.A.: The flow of foam through short porous media and apparent viscosity measurements. SPE J. 6(01), 17–25 (1966). https://doi.org/10.2118/1319-pa
Mathews, J.H.: Numerical Methods For Mathematics, Science, and Engineering. Prentice-Hall International, Englewood Cliffs, N.J. https://doi.org/10.2307/2282127 (1992)
Patzek, T.W.: Description of foam flow in porous media by the population balance method. In: ACS Symposium Series, vol 373. ACS, p 326–341, https://doi.org/10.1021/bk-1988-0373.ch016 (1988)
Rossen, W.R.: Rheology of foam in porous media at the "limiting capillary pressure. In: 6th European symposium on IOR. EAGE, https://doi.org/10.3997/2214-4609.201411236 (1991)
Salazar Castillo, R.O., Ter Haar, S.F., Ponners, C.G., et al.: Fractional-flow theory for non-newtonian surfactant-alternating-gas foam processes. Transp. Porous Media 131(2), 399–426 (2020). https://doi.org/10.1007/s11242-019-01351-6
Shampine, L.F.: Some practical Runge-Kutta formulas. Math. Comp. 46(173), 135–135 (1986). https://doi.org/10.1090/s0025-5718-1986-0815836-3
Simjoo, M., Zitha, P.L.J.: Modeling of foam flow using stochastic bubble population model and experimental validation. Transp. Porous Media 107(3), 799–820 (2015). https://doi.org/10.1007/s11242-015-0468-y
Simjoo, M., Nguyen, Q.P., Zitha, P.L.J.: Rheological transition during foam flow in porous media. Ind. Eng. Chem. Res. 51(30), 10225–10231 (2012). https://doi.org/10.1021/ie202218z
Tang, G.Q., Kovscek, A.R.: Trapped gas fraction during steady-state foam flow. Transp. Porous Media 65(2), 287–307 (2006). https://doi.org/10.1007/s11242-005-6093-4
Thorat, R., Bruining, H.: Foam flow experiments. I. Estimation of the bubble generation-coalescence function. Transp. Porous Media 112(1), 53–76 (2016). https://doi.org/10.1007/s11242-016-0632-z
Valdez, A.R., Rocha, B.M., Façanha, J.M.F., et al.: Foam-assisted water–gas flow parameters: from core-flood experiment to uncertainty quantification and sensitivity analysis. Transp. Porous Media pp 1–21 (2021)
van Genuchten, M.T.: A closed-form equation for predicting the hydraulic conductivity of unsaturated soils. Soil Sci. Soc. Am. J. 44(5), 892–898 (1980). https://doi.org/10.2136/sssaj1980.0361599500440005002x
Vassenden, F., Holt, T.: Experimental foundation for relative permeability modeling of foam. In: SPE/DOE IOR Symposium. SPE, https://doi.org/10.2118/39660-ms (1998)
Volpert, A.I., Volpert, V.A., Volpert, V.A.: Traveling Wave Solutions of Parabolic Systems, vol 140. AMS (2000)
Zavala, R.Q., Lozano, L.F., Chapiro, G., et al.: Analytical solution for the population-balance model describing foam displacement. Transp. Porous Media (2021). https://doi.org/10.1007/s11242-021-01589-z
Zitha, P.L.J.: A new stochastic bubble population model for foam in porous media. In: Proceedings–SPE symposium on IOR, vol 1. SPE, pp 116–129, https://doi.org/10.2118/98976-ms (2006)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.