A Quantum Paradox of Choice and Purported Classical Analogues

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We recently considered the task of summoning an unknown quantum state and proved necessary and sufficient conditions for Alice to be able to guarantee to complete the task when there may be several possible calls, of which she need only respond to one. We showed that these are strictly stronger conditions than those previously established by Hayden and May for the case where Alice knows there will only be one call. We introduced the concept of a quantum paradox of choice to summarize the implications of these results: Alice is given more options to complete our version of the task, yet one can easily construct examples where our version is impossible and the apparently simpler version considered by Hayden-May is possible.

Finkelstein has argued that one can identify analogous classical paradoxes of choice in a relativistic setting. We examine Finkelstein’s proposed classical tasks and explain why they seem to us disanalogous.
INTRODUCTION

We introduced a recent paper \cite{1} with the following metaphor:

“A Holistic Magician (HM) repeatedly performs the following trick. He first asks you to give him an object that you are sure he cannot copy. After working behind a curtain, he presents you with $N$ boxes and asks you to choose one. Opening your chosen box, he reveals the original object inside.

You initially imagine that he has arranged some concealed mechanism that somehow passes the object sequentially through the boxes, allowing him to stop the mechanism and keep the object in one box if you select it. However, you are then puzzled to notice that he is unable to make the trick work if you select more than one box, even though you allow him to choose which of your selections to open. This argues against your mechanical explanation, and indeed seems to make any simple explanation problematic. How can giving the magician more freedom make him unable to complete the task?”

A comment by Finkelstein \cite{2} inspires another version:

“A Faux-magician (F) repeatedly performs the following trick. He first describes a signal that can easily be generated and copied: a light flash, for example. After working behind a curtain, he presents you with $N$ boxes with buttons and lights and asks you to choose one box and press its button. When you do so, its light flashes, and none of the other lights also flash. He stresses that he requires both of these conditions for the trick to have succeeded.

You are not at all intrigued or puzzled when F tells you that the trick (as he defines it) will not work if you press several buttons. This is because there seems to be an obvious explanation: each button is a switch for the corresponding light. Pressing several buttons would then cause several lights to flash, meaning that the trick as defined would not work. You thus tell F to get a better act and book HM for your next party.”

Notice that the two magicians’ tricks have something in common. In both cases you give them more ways of completing the trick by pressing several buttons, and in both cases this appears to prevent them from completing the task. But in the first case, this seems somewhat surprising, while in the second, it seems very unsurprising. The second story replicates one feature of the first while neglecting others. As a result, we think, no one is likely to call the second story puzzling or paradoxical.

RECAP

In Ref. \cite{1}, we considered the task of summoning an unknown quantum state. This task involves two agencies, Alice and Bob, who may have collaborating agents distributed throughout space-time. Bob secretly prepares a quantum state, and he (i.e. his local agent) hands it over to Alice (i.e. her local agent) at some point $s$ in space-time. Alice and Bob have agreed on a number of call points $c_i$ and corresponding return points $r_i$. Bob may request the state at any call point $c_i$, and Alice is then supposed to return it at the corresponding $r_i$. In the most interesting version of the task, $r_i > c_i$ and $r_i > s$ for each $i$, where $>$ denotes the causal relation between space-time points. We proved necessary and sufficient conditions for Alice to be able to guarantee to complete the task when there may be several possible calls, of which she need only respond to one. We showed that these are strictly stronger conditions than those previously established by Hayden and May \cite{6} for the case where Alice knows there will only be one call. We introduced the concept of a quantum paradox of choice to summarize the implications of these results: Alice is given more options to complete our version of the task, yet one can easily construct examples where our version is impossible and the apparently simpler version considered by Hayden-May is possible.

As we noted in Ref. \cite{1}, the discussion in that paper follows the tradition of using parables and apparent paradoxes to refine our understanding of quantum theory \cite{8-14}. The results of Refs. \cite{1,2} rely on relativistic causality as well as quantum theory, and we suggested in Ref. \cite{1} that the apparent tension between them may be the first intrinsically relativistic quantum paradox.

There is also a counter-tradition (e.g. \cite{3,4}) of criticizing these parables and paradoxes, usually on the grounds that they do not seem (to the critics) even superficially paradoxical, or that they are not intrinsically quantum theoretic, or both. This too can be interesting and valuable: it is certainly worth reflecting on exactly what any proposed example, such as ours, really teaches us about physical principles. As we understand it, Finkelstein \cite{2} follows this latter tradition by suggesting that there are precise classical analogues of our quantum paradox of choice.
FINKELSTEIN’S EXAMPLE

Finkelstein suggests the following purported classical analogue of our quantum paradox of choice ([2], emphasis added):

“Here is a simple example, for which quantum restrictions are not needed: Say there are two laboratories called $L$ and $R$; let $D$ be the distance between them, and $T = R/c$ the time required for a signal traveling at the speed of light to go either from $L$ to $R$ or from $R$ to $L$ (all times and distances as measured in the frame in which both $L$ and $R$ are at rest). There are two tasks which $B$ might request of $A$:

Task 1 Send a signal from $L$ to arrive at $R$ at time $T$, but do not send any signal from $R$ to $L$. If this task is requested, the request is submitted to $A$ in laboratory $L$ at time $t = 0$.

Task 2 Send a signal from $R$ to arrive at $L$ at time $T$, but do not send any signal from $L$ to $R$. If this task is requested, the request is submitted to $A$ in laboratory $R$ at time $t = 0$. Clearly it would not be possible for both requests to be fulfilled (just as in the [Adlam-Kent] example where the no-cloning restriction prevents more than one request from being fulfilled). The situation appears paradoxical because $B$, when making both requests, would be satisfied if either one were fulfilled.”

Now, as stated, this example does not work, without imposing further restrictions on $A$. In particular, it does not work in the framework we consider [3, 7] in which she may coordinate a network of collaborating agents distributed wherever she chooses in space-time. If $A$ is allowed this power, she may station an agent $A_i$ on a line between laboratories $L$ and $R$, and send all signals via this agent, who may instantaneously relay them. If $B$ makes both requests, then $A_i$ receives two signals, and can choose to relay the first (or, if she is at the midpoint, may choose to relay either one) and intercept the other. $B$ thus receives precisely one valid signal, at one of the two laboratories, whether he makes one request or two.

REFINING FINKELSTEIN’S EXAMPLE

However, Finkelstein’s underlying point is clear, and the example can be simply refined to make it work in our framework. Suppose that $A$ has two agents, $A_0$ and $A_1$, at spatially well separated sites distance $D$ apart, and $B$ has agents $B_0$ adjacent to $A_0$ and $B_1$ adjacent to $A_1$, so that each $B_i$ is separated from $A_i$ by distance $\epsilon \ll D$. All of these agents are stationary in some mutually agreed inertial frame. Now define the following non-local task. At time $t = 0$ in the agreed frame, each $B_i$ will send the corresponding $A_i$ a classical bit, 0 or 1. $A$ is guaranteed that at least one 1 will be sent. Her task is to return to the $B_i$, by time $t = 2\epsilon$, two classical bits, one 0 and one 1, ensuring that the 1 is sent to an agent $B_i$ who sent her a 1.

Now, if $A$ were also guaranteed that only one 1 will be sent, the task is trivial: each $A_i$ simply needs to return the bit they are sent – and this is the only way of satisfying the task. However, if it is possible that two 1’s will be sent, then $A$ has no way of ensuring that she completes the task. This is true although, if $A$ receives two 1’s, there are two possible valid ways of completing the task.

DISCUSSION

In the last example, $A$ may have more valid ways of completing the task, if two 1’s are sent, but nonetheless this possibility makes it impossible for her to guarantee completion of the task. But is this in any way paradoxical? Successfully completing the task involves returning to the $B_i$ appropriately anti-correlated bits at space-like separated points. If the $B_i$ promise to supply these bits – in other words, if they promise to give the $A_i$ the data that complete the task – then the $A_i$ can indeed complete the task. If they do not, then the $A_i$ can not. We understand the term paradox to imply a challenge to pre-existing intuitions, and we suspect few readers will feel such a challenge here.

Compare the summoning task in our original discussion [1]. There Alice needs to get a single quantum state to some requested point in space-time. Yet it turns out that this is strictly harder if she may be given several options for return points. That is, there are strictly fewer sets of request and return points for which it can be achieved. This seems to us, and at least to some others with whom we have discussed the problem, an interesting and initially surprising feature of relativistic quantum theory. As discussed in Ref. [1], it challenges our understanding of whether and how quantum states can be localized in space-time.

Of course, the results are explicable. The relevant theorems were proven in Refs. [8] and [1], and the paradox was resolved as well as presented in Ref. [1]. To summarize: it turns out that, in the version of the task where only one summons is allowed, the anti-correlated data given to Alice by the summonses constitute an exploitable resource.
However, to understand how and why this resource is relevant seems at present to require an intuitive understanding both of the possibilities given by iterative uses of teleportation and quantum secret sharing for transmitting quantum information in space-time and of the constraints implied by iterative uses of the no-signalling principle [1, 6].

It is perhaps worth emphasizing that, in our view, not every situation (classical or quantum) in which more options make a task harder deserves to be termed a paradox of choice. A single path from $A$ to $B$ is straightforward to navigate, however twisty it may be. Extending it into a maze generally makes things harder, even when there are several paths through the maze. But we would not call this a paradox: to deserve that term requires a challenge to the intuition, and we see none here.

That said, it should also be acknowledged that scientific paradoxes can only be characterized in terms of the limitations of human cognition and of pre-existing mental models, and these may perhaps be different for different readers. There may perhaps be readers whose pre-existing intuitions about summoning relativistic quantum information assured them that the precise results of Refs. [1, 6] must be true: if so, we salute them. Although we think it a bit less likely, there may perhaps also be readers whose pre-existing intuitions about classical information in space-time told them that the refined Finkelstein example above should not work. For any such readers (but, we would say, only for them), Finkelstein’s term “classical paradox of choice” would indeed be appropriate.

For us, the key point of Ref. [1] is that considering summoning quantum states with single and multiple calls reveals a new and surprising physical distinction between the two, encapsulated in the quantum paradox of choice described therein. This feature is intrinsic to relativistic quantum theory: nothing like it arises in summoning quantum states in non-relativistic quantum mechanics, nor in summoning classical states in relativistic classical mechanics.

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