The QCD collisional energy loss revised

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It is shown that to leading order the QCD collisional energy loss reads \( dE/dx \sim \alpha(m_T^2)/T^2 \). Compared to prevalent expressions, \( dE^2/dx \sim \alpha^2 T^2 \ln(ET/m_T^2) \), which could be considered adaptions of the (QED) Bethe-Bloch formula, the rectified result takes into account the running coupling, as dictated by renormalization. As one significant consequence, due to asymptotic freedom, the collisional energy loss becomes independent of the jet energy \( E \). Some implications with regard to heavy ion collisions are pointed out.

One of the key arguments for the creation of a ‘new state of matter’ in heavy ion collisions at RHIC is the observed jet quenching [1], which inter alia probes the jet’s energy loss in the traversed matter. In a quark gluon plasma (QGP) there are two effects causing a jet to lose energy: elastic collisions with deconfined partons [2, 3], or induced gluon radiation [4, 5, 6]. Presuming a dominance of the second mechanism, experimental findings have often been interpreted in terms of a purely radiative loss. However, the data-adjusted parameters (either \( q \) or \( dN_q/dy \), depending on the approach) are found to be considerably larger than theoretically expected or even in conflict with a strong constraint from \( dS/dy \) (see e.g. [7]) – which calls for a collisional component of the energy loss. The effect of collisions (as estimated within the framework [2, 3]) might actually be larger than conceded for a long time [8, 9]. The fact that such estimates depend crucially on the \( \alpha \) value of the coupling should motivate us to scrutinize the principal question ‘What is \( \alpha \)?’. Aside from its phenomenological relevance it will also lead to interesting theoretical insight.

Following Bjorken’s intuitive considerations [2], consider the propagation of a jet through a static QGP at a temperature \( T \gg \Lambda \), where the coupling is small. Its mean energy loss per length can be calculated from the rate of binary collisions with partons of the medium, as determined by the flux and the cross section,

\[
\frac{dE_j}{dx} = \sum_s \int_{k^3} \rho_s(k) \Phi \int dt \frac{d\sigma_{js}}{dt} \omega.
\]

Here \( \rho_s = d_s n_s \) is the density of scatterers, with \( d_q = 16 \) and \( d_g = 12n_f \) being the gluon and quark degeneracies for \( n_f \) light flavors, and \( n_{\pm}(k) = [\exp(k/T) \pm 1]^{-1} \) in the ideal gas approximation. Furthermore, \( \Phi \) denotes a dimensionless flux factor, \( t \) the 4-momentum transfer squared, and \( \omega = E - E' \) the energy difference of the incoming and outgoing jet. Focusing on the dominating scatterings with small momentum transfer, the cross sections read

\[
\frac{d\sigma_{js}}{dt} = 2\pi C_{js} \frac{\alpha^2}{t^2},
\]

with \( C_{qq} = \frac{4}{3}, C_{qg} = 1 \), and \( C_{gg} = \frac{4}{3} \). For \( E \) and \( E' \) much larger than the typical momentum of the thermal scatterers, \( k \sim T \), the relation of \( t \) to the angle \( \theta \) between the jet and the scatterer simplifies to

\[
t = -2(1 - \cos \theta)k\omega,
\]

and the flux factor can be approximated by \( \Phi = 1 - \cos \theta \).

At this point, Bjorken integrated in Eq. (1)

\[
\Phi \int_{t_1}^{t_2} dt \frac{d\sigma_{js}}{dt} \omega = \frac{\pi C_{js} \alpha^2}{k} \int_{t_1}^{t_2} dt = \frac{\pi C_{js} \alpha^2}{k} \ln \frac{t_1}{t_2},
\]

imposing both an IR and UV regularization. The soft cut-off is related to the Debye mass, \( \omega = \mu^2 \sim m_D^2 \sim \alpha T^2 \), describing the screening of the exchanged gluon in the medium. The upper bound of \( t \) was reasoned to be given by the maximum energy transfer: very hard transfers, say \( \omega \sim E \), effectively do not contribute to the energy loss; in this case the energy is collinearly relocated to the scatterer. Assuming \( \omega_{\text{max}} = E/2 \) implies \( t_1 = -(1 - \cos \theta)kE \), hence \( dE_j^B/dx = \pi\alpha^2 \sum_s C_{js} \int_{k^3} k^{-1} \rho_s \ln((1 - \cos \theta)kE/\mu^2) \). Replacing now, somewhat pragmatically, the logarithm by \( \ln(2(kE/\mu^2)) \) and setting \( (k) \to 2T \), Bjorken obtained

\[
\frac{dE_B}{dx} = \left( \widehat{\frac{2}{3}} \right)^{-1} 2\pi \left( 1 + \frac{4}{3}n_f \right) \alpha^2 T^2 \ln \frac{4TE}{\mu^2},
\]

which differs for quark and gluon jets only by the prefactor. This expression can be regarded as a relativistic adaption of the (QED) Bethe-Bloch formula [10], which describes the ionization/excitation energy loss of charged particles in matter as determined by the scatterer density, and with the logarithm reflecting the kinematics and the long-range Coulomb-type interaction.

There are various refinements of Bjorken’s ‘practical’ calculation, aiming at the precise determination of the cut-offs. Worth accentuating is the approach of Braaten and Thoma [3] who studied, within HTL perturbation theory, the propagation of a fermion through a QED plasma (and applied their method also to QCD). For light quarks, the result reproduces the generic form (5), with \( \mu \to m_D \) in the logarithm and the factor 4 replaced by some function of \( n_f \) [15].

A remark concerning a pragmatic usage of such ‘Bjorken-type’ formulae seems apposite here. Applied to
experimentally relevant temperatures and rather low $E$, a formally resulting negative loss has been interpreted, at times, as an energy transfer of the thermal medium to the ‘soft jet’. This interpretation, however, is untenable in the given framework: the jet always loses energy in a collision, $\omega > 0$, cf. (3). A negative result for $dE_B/dx$ is de facto the consequence of interchanged boundaries in the integration (4). Since $\mu^2$ is the minimal $|t|$ by definition, $dE_B/dx$ should instead be set to zero for $4E < \mu^2/T$. This concern, though, will prove irrelevant by the following considerations.

In Bjorken’s derivation, $\alpha$ is a fixed parameter. Conventionally, such a tree-level approximation may be appropriate for QED. The strong interaction, however, is known to vary considerably over the range of scales probed, e.g., in heavy ion collisions. Thus, in QCD one should study quantum corrections to the tree level processes, whose renormalization will specify unambiguously the value of ‘the’ coupling.

For the sake of transparency of the argument, consider first the analogous case of electron scattering in massless QED. There are three types of (divergent) loop corrections to the $t$-channel tree-level process, see e.g. [11]. First, the exchanged photon is dressed by a self-energy. Then, encoded in the quantity $Z_1$, there are vertex corrections and finally, via the field strength renormalization $Z_2$, self-energy corrections to the external fermions. Yet, due to the identity $Z_1 \equiv Z_2$, in QED the renormalized coupling is determined only by the boson self-energy.

It is appropriate to renormalize the theory (i.e., to fix its parameters) by a scattering experiment at $T = 0$. The relevant part of the matrix element leading to the (vacuum) cross section corresponding to (2) is $\alpha/(P^2 - \Pi_{\text{vac}}(P^2))$. Here $\alpha$ denotes the bare coupling, and $\Pi_{\text{vac}}$ is the unrenormalized boson self-energy at $T = 0$. In dimensional regularization, $\Pi_{\text{vac}}(P^2) = \alpha b_0 \left[\epsilon^{-1} - \ln(-P^2/\mu^2\right)]P^2$, where $b_0 = 4\pi\beta_0$ and $\beta_0$ is the leading coefficient of the $\beta$-function. For a specific $P^2 = t_r$, the matrix element reads explicitly

$$\frac{1}{t_r} \frac{\alpha}{1 - \alpha b_0 \left[\epsilon^{-1} - \ln(-t_r/\mu^2\right)]} = \frac{\alpha(t_r)}{t_r}.$$  

A measurement then specifies the renormalized coupling $\alpha(t_r)$ at the scale $t_r$ (as introduced on the rhs) which is related to the (finite) bare coupling by

$$\alpha^{-1}(t_r) = \alpha^{-1} - b_0 \left[\epsilon^{-1} - \ln(-t_r/\mu^2\right)] \quad \text{(6)}.$$  

An equivalent relation holds for the coupling $\alpha(t)$ at an arbitrary scale $t$, consequently $\alpha^{-1}(t) = \alpha^{-1}(t_r) + b_0 \ln(t/t_r)$ or, in a common alternative form,

$$\alpha(t) = \left[b_0 \ln(|t|/\Lambda^2\right)]^{-1} \quad \text{(7)}.$$  

It is underlined that the momentum dependence of the renormalized ('running') coupling is fully specified by its value at a certain scale $t_r$ or, equivalently, by the parameter $\Lambda$.

In a (thermal) medium, the boson self-energy has the generic structure

$$\Pi^\text{I} = \alpha b_0 \left[(\epsilon^{-1} - \ln(-P^2/\mu^2\right)]P^2 + f(p_0, p),$$

where the finite 'matter' contributions $f'$ differentiate transverse and longitudinal modes ($i = t, l$). Then, utilizing (6), the in-medium scattering matrix can be rewritten in terms of the renormalized coupling,

$$\frac{\alpha}{P^2 - \Pi^\text{I}} = \frac{\alpha - b_0 \left[\epsilon^{-1} - \ln(-P^2/\mu^2\right) + f'/P^2]}{P^2 - \Pi_{\text{mat}}}.$$  

This distinct form, where the divergent vacuum contribution of the self-energy is 'absorbed' in the running coupling, elucidates that the matrix element depends only on the physical parameter $\Lambda$ and the matter part of the self-energy, $\Pi_{\text{mat}} = \alpha(P^2)b_0 f$. Thus the effective IR cut-off for the energy loss is, as expected, related to the Debye mass [19]. The main emphasis here, though, is on the renormalized coupling in Eq. (8) and, consequently, also in the resulting differential cross section: the scale of the running coupling is set by the virtuality $P^2 = t$.

It is physically intuitive that this fact is generic. Thus, instead of a detailed analysis of loop corrections in QCD (which is more complex), it is useful to invoke a more comprehensive argument. The vacuum differential cross section, as a quantity with an unambiguous normalization, obeys a Callan-Symanzik equation with $\gamma \equiv 0$ [11],

$$\left[M \frac{\partial}{\partial M} + \beta(g) \frac{\partial}{\partial g}\right] \frac{d\sigma(t; M, g)}{dt} = 0,$$  

where $g = (4\pi\alpha(M^2))^{1/2}$ is related to the coupling at a given renormalization point $M$. The general solution of this linear partial differential equation is a function $h(x)$, whose argument $x = g(M)$ satisfies $M\,dg/dM + \beta(g) = 0$. To investigate, in this line of thoughts, the dependence on the momentum scale $Q$, with $Q^2 = q^2 = t$, note that, in the limit of small $t$, the most general form of the cross section is $d\sigma/dt = S(Q/M, g)/t^2$. Consequently, the function $S$ obeys $-Q \partial \partial_s + \beta(g) \partial_s S = 0$ (mind the minus sign). The corresponding characteristic equation, with $\beta(g) = -\beta_0 g^3$ at leading order, then leads readily to the running coupling as introduced above. In other words, renormalization group invariance implies that loop corrections to the differential cross section can be 'absorbed' in the tree level expression by replacing $\alpha \to \alpha(t)$.

A running coupling, as given by Eq. (7) with $b_0 = (11 - \frac{2}{3} n_f)/(4\pi)$ in QCD, alters the integral (4) [20],

$$\Phi \int_{t_1}^{t_2} dt \frac{d\sigma_{js}}{dt} \omega = -\pi C_{js} \left|\frac{k b_0}{k b_0}\right| \int_{t_1}^{t_2} dt \ln(|t|/\Lambda^2)$$  

$$= \pi C_{js} \left|\frac{1}{k b_0} \ln(|t_1|/\Lambda^2)\right|^{t_2}_{t_1} = \pi C_{js} \left[\alpha(\mu^2) - \alpha(|t_1|)\right].$$  

(10)
Opposed to the expression (4), the weighted cross section is
UV finite – due to the asymptotic weakening of the
strong interaction, large-\(t\) contributions are suppressed.
For hard jets, the integral (10) becomes independent of
the energy \(E\), i.e., it is then controlled solely by the
coupling at the screening scale. The necessary condition
\(|t_1| \sim TE \gg \mu^2 \sim m_D^2 \sim \alpha T^2\), is, for weak coupling,
actually much less restrictive than the previous
assumption \(E \gg T\) to simplify the kinematics.
Anticipating an extrapolation of the final result for
d\(E/dx\) to larger coupling,
where the hierarchy of scales \(\sqrt{\alpha T} \ll T\) breaks down,
 it is mentioned that \(m_D \approx 3T\) (see, e.g., [12]).
Hence from this perspective, the collisional loss could
become \(E\)-independent already for jet energies exceeding
several GeV’s – provided, of course, that the perturbative
framework gives at least a semi-quantitative guidance at
larger coupling, which will be advocated below.

The jet’s collisional energy loss approaches, thus,

\[
\frac{dE_i}{dx} = \pi \frac{\alpha(\mu^2)}{b_0} \sum_s C_{js} \int_{k_3} \rho_s(k) \frac{\rho_s(k)}{k}.
\]

In the ideal gas limit, the remaining integration yields

\[
\frac{dE_{q,g}}{dx} = \left(\frac{\alpha}{\mu^2}\right)^{\pm 1} 2\pi \left(1 + \frac{1}{6} n_f\right) \frac{\alpha(\mu^2)}{b_0} T^2,
\]

which departs radically from previous expressions as
Bjorken’s (5). Aside from the modified cut-off dependence
discussed above, the QCD collisional loss is propor-
tional to \(\alpha\) (instead of \(\alpha^2\)). It is highlighted that
the considerations also show that the relevant scale for
the coupling is the (perturbatively soft) screening mass
rather than a ‘characteristic’ thermal (hard) scale \(\sim T\),
as commonly presumed.

In order to quantitatively compare Eq. (12) to previous
estimates it is necessary to specify parameters, namely \(\Lambda\)
in (7) and the cut-off \(\mu\), i.e., the Debye mass. Similar
renormalization arguments as above lead in [12] to
an implicit equation for \(m_D\); to leading order

\[
m_D^2 = (1 + \frac{1}{6} n_f) 4\pi \alpha(m_D^2) T^2, \tag{13}
\]

whose solution can be given in terms of Lambert’s function.
Obviously, this mended perturbative formula is just-
ified strictly only at temperatures \(T \gg \Lambda\). Notwith-
standing this, it is found in quantitative agreement with
lattice QCD calculations down to \(T \approx 1.2T_c\) [21].

It may come as a surprise that a leading order for-
mula reproduces non-perturbative results [22]. Thus,
 it is worth emphasizing that the adjusted parameter
\(\Lambda = 205\,\text{MeV}\) for \(n_f = 2\) is right in the expected ball-
park, enervating a possibility of an uninterpretable ‘over-
strained’ fit. Moreover, the 1-loop running coupling (7)
with the same \(\Lambda\) reproduces lattice calculations for
another quantity, namely the QCD vacuum potential \(V(r)\),
in fact up to large distances corresponding to \(\alpha(r^{-2}) \approx 1\)
[12]. Although at first sight rather different, the potential
at \(T = 0\) and the Debye mass in the medium are
closely related by the renormalized \(t\)-channel scattering
discussed above – as is the collisional energy loss. In
other words (tidying the order of the arguments): one
can renormalize the theory at \(T = 0\) (i.e., determine
once and for all \(\Lambda\) from \(V(r)\)), verify the applicability
of the perturbative approach for larger couplings as rele-
vant near \(T_c\) by successfully calculating \(m_D(T)\), and then
make predictions for \(dE/dx\).

With this justification, an extrapolation of Eq. (12)
as presented in Fig. 1 might be not too unreasonable.
Assumed here is \(T_c = 160\,\text{MeV}\) and \(\mu^2 = \frac{1}{2} m_D^2\) to

\[\text{FIG. 1: Light quark collisional energy loss: Eq. (12) vs. the}
\text{prevailing expression (5) (which yields negative values for very}
\text{large} \ T\] for representative jet energies. For details see text.

estimate the uncertainty due to the IR cut-off. Shown for
comparison are results from (5); here \(\mu = m_D\) (like-
wise from (13)) albeit \(\alpha\) in the prefactor (unjustified, but as
often presumed) fixed at the scale \(Q_T = 2\pi T\). It turns
out that already near \(T_c\), the estimates from formula (12)
exceed those from (5), even for rather large values of \(E\).

For phenomenological implications it is instructive to
take further into account a main effect of the strong in-
teraction near \(T_c\). From the distinct decrease of the QGP
entropy seen in lattice QCD calculations [14], one can, on
general grounds, infer a similar behavior for the number
densities \(\rho_s\). In the framework of the quasiparticle model
[15], the ideal distribution functions in Eq. (11) are to
be replaced by \(\rho_s = \rho_s(\sqrt{m_s^2(T) + k^2})\), where the effective
coupling in the quasiparticle masses, \(m_s^2 \propto \alpha_{\text{eff}}(T) T^2\),
is adjusted to lattice results for the entropy. As shown in
Fig. 2, near \(T_c\) – despite the strong interaction – the
plasma becomes transparent due to the reduced number of
‘active’ degrees of freedom. Qualitatively, this charac-
teristic behavior is in line with a rather sudden transition
from small to large jet quenching when going from SPS
to RHIC energies.

In conclusion, it has been demonstrated in the context
of thermal field theory that renormalization does not only
dictate the value of the coupling for a given quantity, but that the running of the coupling can also influence the parametric behavior of results. For the QCD collisional energy loss, the relevant scale in $\alpha(t)$ is the (perturbatively soft) screening mass $m_D \sim \sqrt{\alpha T}$ (instead of $T$ as commonly presumed). The increasing coupling at soft momenta leads to a parametric enhancement compared to previous calculations, see Eqs. (12) vs. (5). On the other hand, due to asymptotic freedom, the collisional energy loss becomes independent of the jet energy $E$. Thus, the asymptotic behavior of the underlying interaction makes the energy loss qualitatively different in QCD and QED (where an analog of Eq. (5) indeed holds).

Except very near $T_c$, Eq. (12) suggests a larger collisional energy loss than previously estimated [2, 3]. This finding can be interpreted as a facet of the 'strongly coupled' QGP (sQGP), which is characterized by large cross sections. In fact, $\sigma = \int dt \frac{d\sigma}{dt}$ with running coupling can actually be an order of magnitude larger than expected from the widely used expression $\sigma_{\text{fix}} \propto \alpha^2(Q_T^2) / \mu^2$. Thus, the present approach gives a consistent and simple explanation of phenomenologically inferred $\sigma \sim \mathcal{O}(10) \text{ mb}$ [16, 17].

Close to $T_c$, though, the particle density is known to be substantially reduced. This implies that, irrespective of the strong coupling, the sQGP becomes transparent near the (phase) transition. Such a distinct temperature dependence of the energy loss should be observable. The quantification of this effect (including a discussion implications of the running coupling for the radiative energy loss) will be the subject of a forthcoming study.

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[17] A. Peshier and W. Cassing, Phys. Rev. Lett. 94 (2005) 172301.
[18] As an aside, the HTL result is obtained in Bjorken’s approach (with a careful k-integration) with the IR cut-off $\mu^2 = m_D^2/2$.
[19] This is evident for the longitudinal contribution, while soft ‘magnetic’ gluons are screened dynamically at the same scale.
[20] Aiming here only at a leading order result, details related to the broken Lorentz invariance in the presence of a medium can be omitted.
[21] In contrast, the ‘conventional’ expression, $\sim \sqrt{\alpha(2\pi T)} T$, deviates from the lattice results by an almost constant factor of 1.5 in a large temperature range.
[22] As argued e. g. in [13], perturbative expansions could be asymptotic series (at best), implying an optimal truncation order inversely proportional to the coupling. Hence for strong coupling, a leading order result might be more adequate than sophisticated high-order calculations – somewhat opposed to naive intuition.