A Note on Entanglement Entropy for Primary Fermion Fields in JT Gravity

Chang-Zhong Guo
Department of Physics, Nanchang University, Nanchang, 330031, China

Wen-Cong Gan†
GCAP-CASPER, Physics Department, Baylor University, Waco, Texas 76798-7316, USA and
Department of Physics, Nanchang University, Nanchang, 330031, China

Fu-Wen Shu‡
Department of Physics, Nanchang University, Nanchang, 330031, China
Center for Relativistic Astrophysics and High Energy Physics, Nanchang University, Nanchang, 330031, China
GCAP-CASPER, Physics Department, Baylor University, Waco, Texas 76798-7316, USA and
Center for Gravitation and Cosmology, Yangzhou University, Yangzhou, China
(Dated: June 28, 2023)

In this paper we analyse and discuss 2D Jackiw-Teitelboim (JT) gravity coupled to primary fermion fields in asymptotically anti-de Sitter (AdS) spacetimes. We get a particular solution of the massless Dirac field outside the extremal black hole horizon and find the solution for the dilaton in JT gravity. Two dimensional JT gravity spacetime is conformally flat, we calculate the two point correlators of primary fermion fields under the Weyl transformations. The key point of this work is to present a standard technique which is called resolvent rather than CFT methods. We redefine the fields in terms of the conformal factor as the fermion fields, and we use the resolvent technique to derive the renormalized entanglement entropy for massless Dirac fields in JT gravity.

* chang.zhong.guo1997@gmail.com
† Wen-cong_Gan1@baylor.edu
‡ shufuwen@ncu.edu.cn; Corresponding author
I. INTRODUCTION

Two dimensional JT gravity [1–3] is a model of 2D dilaton gravity which admits AdS$_2$ holography [4], also it is the simplest nontrivial theory of gravity. In recent years, JT gravity has provided a simple and meaningful toy model for the study of black hole information loss problem. In particular, it has been able to describe the Page curve of black hole entropy, which is a key step towards solving black hole information paradox [5–7]. All these works suggest that after the Page time, there is a configuration that the entanglement wedge of Hawking radiation include an island inside the black hole interior, and the island configuration is the key to reproducing the Page curve. Therefore, it is of great significance to verify the validity of the island configuration. Motivated by this, there have been several proposals recently to show the existence of the island by proposing ways to extract information from the island to the radiation [8–11]. One of them is achieved by making use of the modular Hamiltonian and modular flow in entanglement wedge reconstruction and the equivalence between the boundary and bulk modular flow [12]. As a concrete example, they consider extremal black holes with modular flow in JT gravity coupled to baths. They claim that the explicit information extraction process can be observed in the case that the bulk conformal fields contain free massless fermion fields [12].

While the proposal in [12] shows a promising way to extract information from the island configuration in JT gravity, the details of this process have not been fully specified in the literature. In particular, the modular flow of the free massless fermion field considered in [12] is in two dimensional Minkowski spacetime. More details are needed on how to apply this flow to the conformally flat spacetime. Therefore, in this paper, we aim to fill this gap in the literature by providing detailed calculations of entanglement entropy for massless fermion fields with the help of the resolvent technique. Our goal is to provide a clear and comprehensive understanding of the proposed method and its implications for the black hole information paradox.

This paper is organized as follows. In Sec.II, we get the equations of motions in the background of JT gravity coupled to primary fermion fields and we find the particular solution of the wave function outside the extremal black hole horizon, and we also solve for the dilaton in JT gravity. In Sec.III, we calculate the two point correlators of primary fermion fields under Weyl transformations by CFT method. In Sec.IV, we review a standard technique called resolvent to derive the entanglement entropy in $n$ disjoint intervals for a massless Dirac field in two dimensional vacuum Minkowski spacetime [13, 14]. Correspondingly, we redefine the fields in terms of the conformal factor as the
fermion fields, and we use the resolvent technique as described in two dimensional vacuum Minkowski spacetime to derive the renormalized entanglement entropy for massless Dirac fields in JT gravity.

II. PRIMARY FERMION FIELDS IN JT GRAVITY BACKGROUND

The JT gravity model consists of 2D gravity coupled to a scalar \( \phi \) called the dilaton, with a classical bulk term action in Lorentzian signature on an asymptotically AdS spacetime,

\[
S_{JT} = \frac{1}{16\pi G_N} \int d^2x \sqrt{-g} \left( \phi R + 2\phi - 2\phi_0 \right),
\]

(2.1)

where \( R \) is the Ricci scalar and we have set the AdS\(_2\) length \( l_{AdS} = 1 \). The JT gravity action originates from a dimensional reduction of four dimensional near extremal magnetic charged black hole [15–17], the two-dimensional JT model is obtained by reduction of the spherically symmetric metric,

\[
ds_4^2 = g_{\mu\nu}(t, r)dx^\mu dx^\nu + \phi(t, r)d\Omega^2,
\]

(2.2)

where \( g_{\mu\nu} \) is the 2D part with coordinates \((t, r)\) and the dilaton \( \phi \) plays the role of the radius of the 2-sphere which we want to reduce, and \( \phi_0 \) is a constant which is proportional to the extremal entropy of the higher-dimensional black hole geometry.

In this paper, we consider the coupling of a massless Dirac field \( \Psi(x) \) to JT gravity. The massless Dirac field is also called the primary field which satisfies conformal invariance under conformal transformations in CFT method. The action of primary fermion fields in 2D curved spacetime is [18–22]:

\[
S_D = \frac{i}{2} \int d^2x \sqrt{-g} \bar{\Psi} \left( \mp \nabla_{\mu} \right) \Psi,
\]

(2.3)

where \( \mp \nabla_{\mu} \equiv \partial_{\mu} + \Gamma_{\mu} = \partial_{\mu} + \frac{1}{2} \eta_{abc} \omega^{c}_{\mu} [\gamma^{a}, \gamma^{b}] \) is the spinor covariant derivative, and the spin connection is \( \omega^{c}_{\mu} = -ie^{c}_{\mu} (\partial_{\nu} e^{c\nu} - \Gamma_{\mu}^{\lambda} e^{\lambda\nu}) \). Note that in eq.(2.3), \( \sqrt{g} \left( \mp \nabla_{\mu} \right) \bar{\Psi} \Psi \) is not real, so we should choose \( \frac{i}{2} \Psi \left( \pm \nabla_{\mu} \right) \bar{\Psi} \) as the Dirac Lagrangian, where \( \mp \nabla_{\mu} = \partial_{\mu} - \Gamma_{\mu} = \partial_{\mu} - \frac{1}{2} \eta_{abc} \omega^{c}_{\mu} [\gamma^{a}, \gamma^{b}] \) operates on \( \Psi \), and \( \mp \nabla_{\mu} \) is different from \( \pm D_{\mu} \).

We adopt the metric signature \((-+,+)\) and the anticommutator of the Dirac gamma metric is \( \{ \gamma^{a}, \gamma^{b} \} = 2\eta^{ab} \). The Dirac gamma matrices have this property: \( (\gamma^{0})^{2} = -1 \) and \( (\gamma^{1})^{2} = 1 \), we choose

\[
\gamma^{0} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^{1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

(2.4)

The Dirac adjoint in eq.(2.3) is defined as \( \bar{\Psi} = \Psi\gamma^{0} \), and \( \mp \nabla_{\mu} = e_{\nu}^{\mu} \gamma^{\nu} \), where \( e_{\nu}^{\mu} \) is the vierbein.

We define \( \alpha \) as the strength of the coupling between the massless Dirac field and JT gravity, and we also define \( \kappa^2 \equiv 8\pi G_N \), then the total action functional is

\[
S = S_{JT} + \alpha S_D = \int d^2x \sqrt{-g} \left[ \frac{1}{2\kappa^2} (\phi R + 2\phi - 2\phi_0) + \frac{i\alpha}{2} \bar{\Psi} \left( \mp \nabla_{\mu} \right) \Psi \right].
\]

(2.5)

By varying the total action (2.5) with respect to the metric field, then we get the classical equations of motion (see Appendix (A)):

\[
g_{\mu\nu}(\phi - \phi_0) + \nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \Box \phi = \frac{i\alpha}{8} \bar{\Psi} \left( \gamma_\nu D_{\mu}^{\gamma} + \gamma_\mu D_{\nu}^{\gamma} \right) \Psi,
\]

(2.6)

where \( \gamma_\nu \) is defined as \( \gamma_\nu = (g_{\mu\nu} e_\alpha^{\mu}) \gamma^\alpha = g_{\mu\nu} \gamma^{\mu} \), and \( \mp \nabla_{\mu} \left( \gamma_\nu D_{\mu}^{\gamma} \Psi \right) \) is defined as \( \sqrt{g} \left( \gamma_\nu D_{\mu}^{\gamma} \Psi \right) = \mp \nabla_{\mu} \left( \gamma_\nu D_{\mu}^{\gamma} \Psi \right) + \left( i\bar{\Psi} \gamma_\nu D_{\mu}^{\gamma} \Psi \right)^\dagger, \) with \( \left( i\bar{\Psi} \gamma_\nu D_{\mu}^{\gamma} \Psi \right)^\dagger = -i \left( \mp \nabla_{\mu} \Psi \right) \gamma_\nu \Psi. \)

A tetrad is a set of linearly independent vectors that can be defined at each point in a Riemannian spacetime, the tetrads by definition satisfy the relations: \( e^{\mu}_{a} e^{\nu}_{b} = \delta^{\nu}_{b}, \) \( e^{\mu}_{a} e^{\nu}_{b} = \delta^{\mu}_{\nu}. \) The choice of the tetrad field determines the metric: \( g_{\mu\nu} = e^{\mu}_{a} e^{\nu}_{b} \eta_{ab}, \) \( \eta_{ab} = e^{a}_{\mu} e^{b}_{\nu} g_{\mu\nu}. \)
A. Massless Dirac fields outside the extremal black hole horizon

In a generic conformal coordinate system $x^\pm$, the metric in two dimensional gravity is given by

$$ds^2 = -e^{2\rho(x^+, x^-)} dx^+ dx^-.$$  \hfill (2.7)

In this paper we consider a zero temperature black hole in the two-dimensional Jackiw-Teitelboim gravity, and we can use the Poincaré coordinates $x^\pm = t \pm z$ to describe the extremal black hole (see the Fig.1 for more details). The metric in the Poincaré patch is

$$ds^2 = -\frac{4dx^+ dx^-}{(x^+ - x^-)^2} = -\frac{dt^2 + dz^2}{z^2}, \quad z \leq 0.$$  \hfill (2.8)

The boundary of AdS$_2$ spacetime is at $z = 0$, the future horizon of the JT extremal black hole is at $x^- = +\infty$, while the past horizon is at $x^+ = -\infty$.

By varying the Dirac action $S_D$ with respect to the Dirac field, we can get the massless Dirac field equation in two dimensional conformally flat spacetime

$$i\gamma^\mu D_\mu \Psi = 0.$$  \hfill (2.9)

We can write the 2-component massless Dirac spinor $\Psi$ as

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} \psi_1 + i\psi_2 \\ \psi_3 + i\psi_4 \end{pmatrix}.$$  \hfill (2.10)

Any two dimensional spacetime is conformally flat, the massless Dirac field equation in the conformal gauge can be written as

$$2\partial_+ \Psi_1 - \frac{\Psi_1}{(x^+ - x^-)} = 0, \quad -2\partial_- \Psi_2 - \frac{\Psi_2}{(x^+ - x^-)} = 0.$$  \hfill (2.11)

---

2 A tetrad is a set of four linearly independent vectors that the direction can be arbitrarily selected, four vierbeins are constrained by three equations in light cone coordinates: $\eta_{00} = -1 = 2g_+ e^0 e^- $, $\eta_{11} = 1 = 2g_+ e^1 e^- $, $\eta_{01} = \eta_{10} = 0 = g_+ (e^0 e^1 + e^- e^0)$. We choose $e^0 = e^- = e^1 = -e^{-\rho(x^+, x^-)}$, and $e^1 = e^{-\rho(x^+, x^-)}$. 

---

FIG. 1. The Penrose diagram for the extreme black hole in JT gravity. The yellow region is the Poincaré patch where the wave function is distributed in. The blue null line is the future event horizon and the red null line is the past event horizon. Here $z$ ranges from $z \in (-\infty, 0]$, where $z = -\infty$ is the location of the horizon.
The wave function in JT gravity spacetime must satisfy the following two boundary conditions: the wave function is zero at the AdS$_2$ spacetime boundary and it is finite at the past event horizon or the future event horizon of the extreme black hole in JT gravity. Combine the two boundary conditions and the eq.(2.11), we can find a particular solution of the wave function distribution beyond the extremal black hole horizon:

\[
\Psi_1(x^+, x^-) = \frac{1}{\sqrt{x^-}} (x^- - x^+) \frac{1}{2} + i \frac{1}{\sqrt{x^-}} (x^- - x^+) \frac{1}{2},
\]

\[
\Psi_2(x^+, x^-) = \frac{1}{\sqrt{-x^+}} (x^- - x^+) \frac{1}{2} + i \frac{1}{\sqrt{-x^+}} (x^- - x^+) \frac{1}{2}.
\] (2.12)

B. The dilaton

In the conformal gauge, using the general metric in two dimensional gravity in eq.(2.7), and from eq.(2.6) we finally have

1. For the metric \(g_{+-} = e^{2\rho}/2\) \((\phi_0 - \phi) - \partial_+ \partial_- \phi = \frac{i \alpha^2}{8} \Psi (\gamma_- \Delta_+ - \Delta_- \gamma_+ + \gamma_+ \Delta_- - \Delta_- \gamma_+) \Psi\),

2. For the metric \(g_{++} = \partial_+ \partial_+ \phi - 2 \partial_+ \rho \partial_+ \phi = \frac{i \alpha^2}{4} \Psi (\gamma_+ \Delta_+ - \Delta_+ \gamma_+) \Psi\),

3. For the metric \(g_{--} = \partial_- \partial_- \phi - 2 \partial_- \rho \partial_- \phi = \frac{i \alpha^2}{4} \Psi (\gamma_- \Delta_- - \Delta_- \gamma_-) \Psi\).

(2.14)

(2.15)

(2.16)

The direction of the tetrad can be arbitrarily selected, we choose \(e_0^+ = e_0^- = -e^-\) and \(e_1^+ = e^-\). Then we can get the expression for the connection \(\Gamma^i_\mu\) and the matrix \(\gamma_\mu\) in the conformal gauge:

\[
\gamma_+ = \frac{e^\rho}{2} (\gamma^0 + \gamma^1), \quad \gamma_- = \frac{e^\rho}{2} (\gamma^0 - \gamma^1),
\]

\[
\Gamma_+ = \frac{\partial_+ e^\rho}{2} \gamma^0 \gamma^1, \quad \Gamma_- = \frac{\partial_- e^\rho}{2} \gamma^1 \gamma^0.
\] (2.17)

(2.18)

Next, we substitute the 2-component massless Dirac spinor (2.10) into the right hand side of eq.(2.14), eq.(2.15) and eq.(2.16). Using eq.(2.17) and eq.(2.18), then we have

\[
\Psi (\gamma_- \Delta_+ - \Delta_- \gamma_+ + \gamma_+ \Delta_- - \Delta_- \gamma_+) \Psi = 0,
\] (2.19)

\[
\Psi (\gamma_+ \Delta_+ - \Delta_+ \gamma_+) \Psi = \frac{e^\rho}{2} (-2 \Psi_1 \partial_+ \Psi_2 + 2 \Psi_2 \partial_+ \Psi_1),
\] (2.20)

\[
\Psi (\gamma_- \Delta_- - \Delta_- \gamma_-) \Psi = \frac{e^\rho}{2} (-2 \Psi_1 \partial_- \Psi_1 + 2 \Psi_1 \partial_- \Psi_1).
\] (2.21)

Substituting the particular solution of the 2-component massless Dirac spinor (2.12) back into the the right hand side of eq.(2.20) and eq.(2.21), we can find

\[
-2 \Psi_2 \partial_+ \Psi_2 + 2 \Psi_2 \partial_+ \Psi_2 = 0, \quad -2 \Psi_1 \partial_- \Psi_1 + 2 \Psi_1 \partial_- \Psi_1 = 0.
\] (2.22)

Finally, the equation of motion for the dilaton becomes

\[
\frac{2}{(x^+ - x^-)^2} (\phi_0 - \phi) - \partial_+ \partial_- \phi = 0,
\]

\[
\frac{2}{(x^+ - x^-)^2} \partial_+ \left( \frac{(x^+ - x^-)^2}{4} \partial_+ \phi \right) = 0,
\]

\[
\frac{2}{(x^+ - x^-)^2} \partial_- \left( \frac{(x^+ - x^-)^2}{4} \partial_- \phi \right) = 0.
\] (2.23)

\[\text{In conformal gauge} \, ds^2 = -e^{2\rho(x^+, x^-)} dx^+ dx^- \], we use the following identities to get the equations of motion.

\[
\nabla^\mu = \frac{1}{2} e^{2\rho} \partial_\mu, \quad g_{+-} = g_{-+} = -\frac{1}{2} e^{2\rho}, \quad \Box = g^{\mu\nu} \nabla_\mu \nabla_\nu = -4 e^{-2\rho} \partial_+ \partial_-,
\]

\[
\nabla_+ \nabla_+ \phi = \partial_+ \partial_+ \phi - 2 \partial_+ \partial_+ \phi, \quad \nabla_- \nabla_- \phi = \partial_- \partial_- \phi - 2 \partial_- \partial_- \phi, \quad \nabla_+ \nabla_- \phi = \partial_+ \partial_- \phi.
\] (2.13)
We can solve the equation for the dilaton
\[
\phi = \phi_0 + \frac{a + b (x^+ - x^-) + c x^+ x^-}{(x^+ - x^-)},
\]
(2.24)
where \(a, b\) and \(c\) are constants which determine the dilaton of JT gravity.

In particular, the dilaton diverges at the conformal boundary, and the location of this physical boundary is imposed by the boundary condition \[23\]:
\[
g_{uu} |_{\text{bdy}} = \frac{1}{\epsilon^2}, \quad \phi = \phi_b = \phi_r + \phi_0,
\]
(2.25)
where \(u\) is the physical boundary time, with \(\epsilon\) the UV cutoff.

The metric in JT gravity has \(SL(2, R)\) isometry. For the extreme black hole in JT gravity, under the \(SL(2, R)\) transformation the dilaton profiles can be recast as
\[
\phi = \phi_0 + \frac{2 \phi_r}{x^+ - x^-}.
\]
(2.26)

III. THE TWO POINT CORRELATORS

A. The primary fermion field correlator in two dimensional Minkowski spacetime

We consider a free Dirac field in two dimensions, it satisfies the Dirac equation and the canonical anticommutation relations :
\[
(i \gamma^\mu \partial_\mu - m) \Psi = 0, \quad \{ \Psi_\alpha (\vec{x}), \Psi^\dagger_\beta (\vec{y}) \} = \delta_{\alpha\beta} \delta(\vec{x} - \vec{y}),
\]
(3.1)
where \(x\) and \(y\) lie on the Cauchy surface with \(t\) = constant. And the two point field correlator in two dimensional Minkowski spacetime is \[4\]:
\[
C(\vec{x}, \vec{y}) = \langle 0 | \Psi(\vec{x}) \Psi^\dagger(\vec{y}) | 0 \rangle = \int \frac{dp}{2\pi} \frac{p_\mu \gamma^\mu + m}{\sqrt{p^2 + m^2}} \gamma^0 e^{-ip \cdot (x - y)}.
\]
(3.2)
The integral of the two point field correlator in eq.(3.2)is \[13\]:
\[
C(x, y) = \frac{1}{2} \delta (x - y) \mathbf{1} + \frac{m}{2\pi} K_0 \left( m |x - y| \right) \gamma^0 + \frac{im}{2\pi} K_1 \left( m (x - y) \right) \gamma^0 \gamma^1,
\]
(3.3)
where \(K_n(x)\) is the standard modified Bessel function, and in the massless limit this gives the two point correlator for the primary fermion field in two dimensional flat spacetime:
\[
C(x, y) = \frac{1}{2} \delta (x - y) \mathbf{1} + \frac{i}{2\pi} \frac{1}{(x - y)} \gamma^0 \gamma^1.
\]
(3.4)

B. The primary fermion field correlator in JT gravity

In general, the metric in 2D conformally flat spacetime is:
\[
ds^2 = -e^{2\rho(x^+, x^-)} dx^+ dx^- = -\Omega^{-2}(x^+, x^-) dx^+ dx^-,
\]
(3.5)
where \(\Omega = e^{-\rho}\) is the conformal factor. Two dimensional JT gravity is locally AdS\(_2\) spacetime, the conformal factor is \(\Omega = (x^+ - x^-)/2\).

---

\[4\] Casini first used \(\Psi^\dagger\) instead of \(\bar{\Psi}\) in their computation for the two point field correlator in \[13, 14\]. There exists local Lorentz boost transformations in 2D spacetime, for which \(\Psi \Psi\) is an invariant for fermions, the vacuum expectation value of \(\Psi \Psi\) called Feynman propagator is defined as \( \langle 0 | \Psi(x) \Psi(y) | 0 \rangle \) in QFT. In fact, Casini defined the two point field correlator as \( C(\vec{x}, \vec{y}) = \langle 0 | \Psi(\vec{x}) \Psi^\dagger(\vec{y}) | 0 \rangle \) in order to calculate the entanglement entropy of a massless Dirac field with the correlator trace formula (4.5).
In CFT method, the two point correlation function for primary operators on a curved manifold with Weyl rescaled metric $\Omega^{-2}g$ in terms of those with metric $g$ satisfies the following transformation relation under Weyl transformations [5, 24]:

$$\left\langle \Phi(x_1, \bar{x}_1) \Phi(x_2, \bar{x}_2) \right\rangle_{\Omega^{-2}g} = \Omega(x_1, \bar{x}_1)\Delta \Omega(x_2, \bar{x}_2)\Delta \left\langle \Phi(x_1, \bar{x}_1) \Phi(x_2, \bar{x}_2) \right\rangle_g,$$  

(3.6)

where $\Delta$ is the scale dimension for the twist field and $\left\langle \Phi(x_1, \bar{x}_1) \Phi(x_2, \bar{x}_2) \right\rangle_g$ is the two point correlation function for primary operators in two dimensional flat spacetime.

The entanglement entropy (von-Neumann entropy) provides us with a convenient way to measure the degree of entanglement between two quantum systems in QFT. We choose the total quantum system as a pure quantum state obtained by taking a partial trace over the subsystem $B$ of the total density matrix (see the Fig.2). The entanglement entropy for the subsystem $A$ is the corresponding von Neumann entropy:

$$S_A = -Tr (\rho_A \ln \rho_A).$$  

(4.1)

IV. ENTANGLEMENT ENTROPY

The entanglement entropy (von-Neumann entropy) provides us with a convenient way to measure the degree of entanglement between two quantum systems in QFT. We choose the total quantum system as a pure quantum state with the density matrix $\rho = |\Psi\rangle\langle\Psi|$. The reduced density matrix for the subsystem $A$ is $\rho_A = Tr_B |\Psi\rangle\langle\Psi|$, which is obtained by taking a partial trace over the subsystem $B$ of the total density matrix (see the Fig.2). The entanglement entropy for the subsystem $A$ is the corresponding von Neumann entropy:

$$S_A = -Tr (\rho_A \ln \rho_A).$$  

(4.1)

IV. ENTANGLEMENT ENTROPY

The entanglement entropy (von-Neumann entropy) provides us with a convenient way to measure the degree of entanglement between two quantum systems in QFT. We choose the total quantum system as a pure quantum state with the density matrix $\rho = |\Psi\rangle\langle\Psi|$. The reduced density matrix for the subsystem $A$ is $\rho_A = Tr_B |\Psi\rangle\langle\Psi|$, which is obtained by taking a partial trace over the subsystem $B$ of the total density matrix (see the Fig.2). The entanglement entropy for the subsystem $A$ is the corresponding von Neumann entropy:

$$S_A = -Tr (\rho_A \ln \rho_A).$$  

(4.1)

IV. ENTANGLEMENT ENTROPY

The entanglement entropy (von-Neumann entropy) provides us with a convenient way to measure the degree of entanglement between two quantum systems in QFT. We choose the total quantum system as a pure quantum state with the density matrix $\rho = |\Psi\rangle\langle\Psi|$. The reduced density matrix for the subsystem $A$ is $\rho_A = Tr_B |\Psi\rangle\langle\Psi|$, which is obtained by taking a partial trace over the subsystem $B$ of the total density matrix (see the Fig.2). The entanglement entropy for the subsystem $A$ is the corresponding von Neumann entropy:

$$S_A = -Tr (\rho_A \ln \rho_A).$$  

(4.1)

IV. ENTANGLEMENT ENTROPY

The entanglement entropy (von-Neumann entropy) provides us with a convenient way to measure the degree of entanglement between two quantum systems in QFT. We choose the total quantum system as a pure quantum state with the density matrix $\rho = |\Psi\rangle\langle\Psi|$. The reduced density matrix for the subsystem $A$ is $\rho_A = Tr_B |\Psi\rangle\langle\Psi|$, which is obtained by taking a partial trace over the subsystem $B$ of the total density matrix (see the Fig.2). The entanglement entropy for the subsystem $A$ is the corresponding von Neumann entropy:

$$S_A = -Tr (\rho_A \ln \rho_A).$$  

(4.1)

IV. ENTANGLEMENT ENTROPY
in the massless case. \textit{Resolvent} is a standard technique in complex analysis, the use of the resolvent technique for free massless fermions was first introduced in [13] to study the entanglement entropy in vacuum on the plane, and subsequently for the entanglement entropy of a chiral fermion on the torus [28–30]. In this section we will first review the derivation of the entanglement entropy for a massless Dirac field in two dimensional vacuum Minkowski spacetime in terms of the resolvent technique, and we can get the entanglement entropy of a single interval for a massless Dirac field in 2D conformally flat JT gravity by redefining the field in terms of the conformal factor as the Fermion field.

### A. Entanglement entropy for a massless Dirac field in two dimensional vacuum Minkowski spacetime

The two point function $C_V$ is related to the reduced density matrix of the region $V$ by the condition:

$$ C_V(x, y) = \langle \Psi(x) \Psi(y) \rangle = Tr(\rho_V \Psi(x) \Psi(y)). $$

Then the expression for the entanglement entropy of the region $V$ can be given by a propagator trace formula (see Appendix (D)) [13, 14, 31]:

$$ S_V = -Tr[(1 - C_V) \log(1 - C_V) + C_V \log C_V]. $$

The \textit{resolvent} of the two point function $C_V$ is defined as:

$$ R_V(\xi) := (C_V + \xi - 1/2)^{-1}. $$

Combining the the expression for the resolvent (4.6), the entanglement entropy can be rewritten as:

$$ S_V = -Tr \int_{1/2}^{1+\infty} d\xi [(\xi - 1/2)[R(\xi) - R(-\xi)] - \frac{2\xi}{\xi + 1/2}]. $$

In eq.(4.6), the inverse of an operator for the propagator is understood in the sense of a kernel that satisfies the following equation:

$$ \int_V dz R_V(\xi; x, z) R_V^{-1}(\xi; z, y) = \delta(x - y) = \int_V dz R_V(\xi; x, z)[C(z, y) + (\xi - 1/2) \delta(z, y)]. $$

Substituting (3.4) into (4.8) one obtains a singular integral equation [32]:

$$ \xi R_V(x, y) - \frac{i}{2\pi} \int_V \frac{R_V(x, z)}{z - y} dz = \delta(x - y). $$

Fortunately, we can solve the resolvent for this integral operator inside a region formed by $n$ disjoint intervals $(u_i, v_i)$ by the Plemelj formulae [32] in the theory of singular integral equations (see Appendix (B)). The resolvent of the two point function $C_V$ (see Appendix (C)):

$$ R_V(\xi) = (\xi^2 - 1/4)^{-1} \left( \xi \delta(x - y) + \frac{i}{2\pi} e^{-\frac{i}{\pi} \log(\frac{(\xi + 1/2)(z(x) - z(y))}{x - y})} \right), $$

where the function $z(x)$ is

$$ z(x) = \log \left( -\frac{\prod_{i=1}^{n}(x - u_i)}{\prod_{i=1}^{n}(x - v_i)} \right). $$

Substituting (4.10) into (4.7), then we have

$$ S_V = -\frac{2}{\pi} \int_{1/2}^{1+\infty} d\xi \int_V dx \lim_{y \to x} \sin \left[ \frac{1}{\pi} \log (\frac{(\xi + 1/2)(z(x) - z(y))}{(\xi + 1/2)(x - y)}) \right]. $$

Integrating over $\xi$ first, we can get the entanglement entropy in $n$ disjoint intervals for a massless Dirac field in two dimensional vacuum Minkowski spacetime:

$$ S_V = 2 \int_V dx \lim_{y \to x} \frac{z(x) - z(y)}{(x - y)(z(x) - z(y))} \coth \left( \frac{(z(x) - z(y))/2}{2} \right) - 1 = \frac{1}{6} \int_V dx \sum_{i=1}^{n} \left( \frac{1}{x - u_i} - \frac{1}{x - v_i} \right) $$

$$ = \frac{1}{3} \left( \sum_{i,j} \log |v_i - u_i| - \sum_{i < j} \log |u_i - u_j| - \sum_{i < j} \log |v_i - v_j| - n \log \epsilon \right), $$

where $\epsilon$ is a distance cutoff introduced in the last integration, and the Virasoro central charge of the primary fermion field is $c = 1$. For a single interval in 2D vacuum flat spacetime on the plane, we can verify the Cardy formula for the renormalized entanglement entropy $S = \frac{2}{3} \log \ell$. 

B. Entanglement entropy for a massless Dirac field in JT gravity

In this subsection, we apply the resolvent technique to 2D conformally flat spacetime. We begin by redefining the field in terms of the conformal factor as the Fermion field. Let us consider the rescaling field, which is given by:

$$\hat{\Psi}(\vec{x}) = \Omega^{\Delta}(\vec{x})\Psi(\vec{x}) = \Omega^{\hat{\Delta}}(\vec{x})\Psi(\vec{x}),$$  \hspace{1cm} (4.14)

Using this rescaling field, we can use the same approach as described in the previous subsection and obtain the same results as shown in equation (4.13). After performing the calculations using the original field $\Psi(\vec{x})$, one finally finds

$$S_V = \frac{1}{3} \left( \sum_{i,j} \log \frac{u_i - u_i}{(u_iu_j)^{1/2}} - \sum_{i,j} \log \frac{u_i - u_j}{(u_iu_j)^{1/2}} - \sum_{i,j} \log \frac{v_i - v_j}{(v_iu_j)^{1/2}} - n \log \epsilon \right),$$  \hspace{1cm} (4.15)

The renormalized entanglement entropy for a massless Dirac field of a single interval in JT gravity is:

$$S = \frac{1}{6} \log \frac{\ell^2}{\Omega_A\Omega_B} = \frac{1}{3} \log \frac{|x - y|}{(xy)^{1/2}},$$  \hspace{1cm} (4.16)

where the Virasoro central charge of the massless Dirac field is $c = 1$.

V. CONCLUSION AND DISCUSSION

In this paper we get the particular solution of the wave function outside the extremal black hole horizon in JT gravity, it is very important for the follow-up study of extracting extremal black hole information with modular flow in JT gravity. The specific expression for the modular flow of 2D free massless fermion depends on the wave function, other papers derived the modular flow formula for 2D free massless fermions but didn’t give us the specific expression for the wave function [12, 28, 33, 34].

In CFT$_2$ methods, a convenient way to compute entropies of intervals is using the replica trick to compute the Rényi entropy for integer index $n$:

$$S_n(V) = \frac{1}{1 - n} \log Tr \rho^n_V.$$  \hspace{1cm} (5.1)

Taking the limit $n \to 1$, we can derive the entanglement entropy of the primary fermion fields[5, 25, 26]. There is a simpler technique called resolvent to derive the entanglement entropy for 2D free massless fermions in comparison to the CFT method called replica trick. In this paper we calculate the two point correlators of primary fermion fields in JT gravity under Weyl transformations. And we redefine the fields in terms of the conformal factor as the fermion fields, then we use the resolvent technique as described in two dimensional vacuum Minkowski spacetime to derive the renormalized entanglement entropy for massless Dirac fields in JT gravity.

In this work, we have calculated the wave function and derived the entanglement entropy for the primary fermion fields outside the extremal black hole horizon in JT gravity. In this case, we only consider the quantum entanglement between the free massless fermions outside the extremal black hole horizon. In the future study, we will go a step further by paying attention to the following points:

(a) For the entanglement between the free massless fermions inside the horizon and outside the horizon, we should regard the whole spacetime as a total quantum entanglement system composed of the extremal black hole and Hawking radiation outside the horizon. The degrees of freedom for the free massless fermions located inside the horizon represent the degrees of freedom for the extremal black hole, and the degrees of freedom for the free massless fermions located outside the horizon represent the degrees of freedom for Hawking radiation particles.

(b) In order to calculate the entanglement entropy for the free massless fermions inside the horizon and outside the horizon, we should consider the entanglement island outside the extremal black hole interior in JT gravity. We may calculate the fine grained entropy of the extremal black hole and Hawking radiation via the semiclassical method called island rule.

---

5 We would like to thank Yiming Chen for bringing this point to our attention.

6 In 2D dilaton gravity, the generalized entropy of Hawking radiation is given by

$$S_\text{gen}(R) = S_\text{gravity} + S_\text{matter} = \frac{\phi}{4G_N} + \frac{c}{6} \log \frac{\ell^2}{\epsilon^2 (\Omega_A\Omega_B)}.$$  

We can omit the UV cutoff parameter $\epsilon$, since it can be absorbed in the renormalization of Newton constant $G_N$ [5, 27]. Then the renormalized entanglement entropy for a massless Dirac field of a single interval in JT gravity can be given by eq.(4.16).
ACKNOWLEDGMENTS

We thank Hong-An Zeng for helpful discussions on the resolvent of the primary fermion correlator in 2D vacuum Minkowski spacetime. This work is supported by the National Natural Science Foundation of China with Grant No. 11975116.

Appendix A: The equations of motions in the background of JT gravity coupled to primary fermion fields

The total action functional for JT gravity coupled to primary fermions is eq. (2.5), we can get the classical equation of motion by varying the metric of the total action:

$$\frac{\delta S}{\delta g^{\mu\nu}} = 0, \quad \implies \frac{\delta S_{JT}}{\delta g^{\mu\nu}} = \alpha \frac{\delta S_D}{\delta g^{\mu\nu}}. $$ (A.1)

The variation of (2.3) with respect to the frame vector indices $e^a_{\mu}$ is [18]:

$$\delta S_D = \int d^2x \frac{i}{16} \sqrt{-g} \Psi \left[ \gamma_\alpha \partial_\mu + \gamma_\mu e_\alpha D_\rho \right] \Psi e^{\alpha\mu},$$ (A.2)

where $\eta^{ab}e_b^\mu = e^{a\mu}$. We use $\delta e^{a\mu} = \frac{1}{2} e^a_{\nu} \delta g^{\mu\nu}$. By the variation of the metric $g^{\mu\nu}$, then eq. (A.2) can be written as:

$$\delta S_D = \int d^2x \frac{i}{16} \sqrt{-g} \Psi \left[ \gamma_\mu \partial_\rho + \gamma_\rho e_\mu D_\nu \right] \Psi \delta g^{\mu\nu},$$ (A.3)

where we have used the following contractions in (A.3),

$$\gamma_\alpha e^a_{\nu} = \gamma_\nu, \quad e_\alpha^\mu e^a_{\nu} = \delta_\nu^a.$$ (A.4)

For the classical bulk term action of JT gravity (2.1), using the standard relations[35],

$$\delta \sqrt{-g} = -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}, \quad \phi g^{\mu\nu} \delta R_{\mu\nu} = -[(\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box ) \phi] \delta g^{\mu\nu}. $$ (A.5)

By varying the metric $g^{\mu\nu}$ in 2D spacetime, we get:

$$\delta S_{JT} = \frac{1}{16\pi G_N} \int d^2x \left[ (\phi - 2\phi - 2\phi_0) + \sqrt{-g} \phi \delta (g^{\mu\nu} R_{\mu\nu}) \right]$$

$$= \frac{1}{16\pi G_N} \int d^2x \sqrt{-g} \left[ -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu} (\phi R + 2\phi - 2\phi_0) + \sqrt{-g} \phi R_{\mu\nu} \delta g^{\mu\nu} + \sqrt{-g} \phi g^{\mu\nu} \delta R_{\mu\nu} \right]$$

$$= \frac{1}{16\pi G_N} \int d^2x \sqrt{-g} \left[ -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu} (\phi R + 2\phi - 2\phi_0) + \sqrt{-g} \phi R_{\mu\nu} \delta g^{\mu\nu} + \sqrt{-g} [g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu] \phi \delta g^{\mu\nu} \right]$$

$$= \frac{1}{32\pi G_N} \int d^2x \sqrt{-g} \left[ 2g_{\mu\nu} (\phi_0 - \phi) + 2\phi (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) + 2g_{\mu\nu} \Box \phi - 2\nabla_\mu \nabla_\nu \phi \right] \delta g^{\mu\nu}. $$ (A.6)

In 2D gravity, we can easily calculate that the Einstein tensor is zero. In the last term in eq. (A.6), we have $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$. Then eq. (A.6) becomes

$$\delta S_{JT} = \frac{1}{32\pi G_N} \int d^2x \sqrt{-g} \left[ 2g_{\mu\nu} (\phi_0 - \phi) + 2g_{\mu\nu} \Box \phi - 2\nabla_\mu \nabla_\nu \phi \right] \delta g^{\mu\nu}. $$ (A.7)

Finally, substituting (A.3) and (A.7) into (A.1), then we get the classical equation of motion in JT gravity coupled to primary fermion fields:

$$g_{\mu\nu} (\phi - \phi_0) + \nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \Box \phi = \frac{i\alpha r^2}{8} \Psi \left[ \gamma_\mu \partial_\rho + \gamma_\rho e_\mu D_\nu \right] \Psi.$$

(A.8)
Appendix B: Singular integral equations and the Plemelj formulae

For the entire complex plane (see the Fig. 3), we can get the integral formula of the function $\varphi(t_0)$ by the Cauchy’s integral formula [32]:

$$\varphi(t_0) = \frac{1}{2\pi i} \int_{L_1-L_2} \frac{\varphi(t) dt}{t-t_0} = \frac{1}{2\pi i} \int_{L_1} \frac{\varphi(t) dt}{t-t_0} - \frac{1}{2\pi i} \int_{L_2} \frac{\varphi(t) dt}{t-t_0}. \quad \text{(B.1)}$$

From the eq.(B.1) we can easily see:

$$\frac{1}{2\pi i} \int_{L_1} \frac{\varphi(t) dt}{t-t_0} = \frac{1}{2} \varphi(t_0), \quad \frac{1}{2\pi i} \int_{L_2} \frac{\varphi(t) dt}{t-t_0} = -\frac{1}{2} \varphi(t_0). \quad \text{(B.2)}$$

Equations of the type

$$A(t_0)\varphi(t_0) + \frac{B(t_0)}{\pi i} \int_L \frac{\varphi(t) dt}{t-t_0} = f(t_0) \quad \text{(B.3)}$$

is called singular integral equations. We define the following functions:

$$\Phi(t_0) \equiv \frac{1}{2\pi i} \int_L \frac{\varphi(t) dt}{t-t_0} \quad \text{(B.4)}$$

$$\Phi^+(t_0) \equiv \frac{1}{2\pi i} \int_{L_1} \frac{\varphi(t) dt}{t-t_0} + \frac{1}{2\pi i} \int_{L_2} \frac{\varphi(t) dt}{t-t_0} = \frac{1}{2} \varphi(t_0) + \frac{1}{2\pi i} \int_{L_1} \frac{\varphi(t) dt}{t-t_0} \quad \text{(B.5)}$$

$$\Phi^-(t_0) \equiv \frac{1}{2\pi i} \int_{L_2} \frac{\varphi(t) dt}{t-t_0} + \frac{1}{2\pi i} \int_{L_1} \frac{\varphi(t) dt}{t-t_0} = -\frac{1}{2} \varphi(t_0) + \frac{1}{2\pi i} \int_{L_2} \frac{\varphi(t) dt}{t-t_0}. \quad \text{(B.6)}$$

Substituting eq.(B.4) into eq.(B.3), then we have

$$(A(t_0) + B(t_0)) \Phi^+(t_0) - (A(t_0) - B(t_0)) \Phi^-(t_0) = f(t_0) \quad \text{(B.7)}$$

$$\implies \Phi^+(t_0) = \frac{A(t_0) - B(t_0)}{A(t_0) + B(t_0)} \Phi^-(t_0) + \frac{f(t_0)}{A(t_0) + B(t_0)}. \quad \text{(B.8)}$$

We define $G(t_0) \equiv \frac{A(t_0) - B(t_0)}{A(t_0) + B(t_0)}$ and $g(t_0) = \frac{f(t_0)}{A(t_0) + B(t_0)}$, then eq.(B.7) is reduced to a simpler singular integral equation:

$$\Phi^+(t_0) = G(t_0) \Phi^-(t_0) + g(t_0). \quad \text{(B.9)}$$

We define a homogeneous equation :

$$X^+(t_0) = G(t_0)X^-(t_0), \quad G(t_0) = \frac{X^+(t_0)}{X^-(t_0)} = \frac{A(t_0) - B(t_0)}{A(t_0) + B(t_0)}. \quad \text{(B.10)}$$

```
Fig. 3. L is the line segment with two endpoints a and b, t_0 is the midpoint of the line segment L. L_1 is the blue semicircle which is going in the counterclockwise direction, and L_2 is the red semicircle which is going in the clockwise direction. L_1 + L represents the contour that contains t_0, L_2 + L represents the contour that doesn’t contain t_0. L_1 - L_2 represents a complete circle which is going in the counterclockwise direction.
```
By taking logarithms, we obtain
\[ \log X^+(t_0) - \log X^-(t_0) = \log G(t_0), \]  
where eq.(B.11) is the Plemelj formulae with the corresponding solution \[32]:
\[ \log X(t_0) = \frac{1}{2\pi i} \int_L \frac{\log G(t) dt}{t - t_0}, \quad \log X^\pm(t_0) = \pm \frac{1}{2} \log G(t_0) + \frac{1}{2\pi i} \int_L \frac{\log G(t) dt}{t - t_0}. \]  
And the solution to \( X^\pm(t_0) \) is
\[ X^\pm(t_0) = e^{\pm \frac{1}{2} \log G(t_0) + \frac{1}{\pi} \int_L \frac{\log G(t) dt}{t - t_0}}. \]  
Combining eq.(B.9) and eq.(B.10), then we have
\[ \Phi^+(t_0) X^+ - \Phi^-(t_0) X^- = g(t_0), \]  
\[ X^+(t_0) \Phi^+ - X^-(t_0) \Phi^- = g(t_0). \]  
Eq.(B.14) is also the Plemelj formulae, and the corresponding solution is
\[ \frac{\Phi^+(t_0)}{X^+(t_0)} = 1 + \frac{1}{2} g(t_0) + \frac{1}{2\pi i} \int_L g(t) dt X^+(t), \]  
\[ \frac{\Phi^-(t_0)}{X^-(t_0)} = -1 + \frac{1}{2} g(t_0) + \frac{1}{2\pi i} \int_L g(t) dt X^+(t). \]  

Appendix C: The resolvent of the primary fermion correlator in two dimensional vacuum Minkowski spacetime

To solve the singular integral equation of the resolvent (4.9), we define
\[ \Phi^\pm(x, y) = \pm \frac{1}{2} R(x, y) + \frac{1}{2\pi i} \int_L \frac{R(x, z)}{z - y} dz, \]  
then we have
\[ \Phi^+(x, y) - \Phi^-(x, y) = R(x, y), \quad \Phi^+(x, y) + \Phi^-(x, y) = \frac{1}{\pi i} \int_L R(x, z) dz. \]  
Then eq.(4.9) can be written as
\[ \left( \xi + \frac{1}{2} \right) \Phi^+(x, y) - \left( \xi - \frac{1}{2} \right) \Phi^-(x, y) = \delta(x - y). \]  
We define a homogeneous equation :
\[ X^+(x, y) = G(\xi) X^-(x, y), \quad G(\xi) = \frac{\xi + \frac{1}{2}}{\xi - \frac{1}{2}}. \]  
By taking logarithms, we obtain
\[ \log X^+(x, y) - \log X^-(x, y) = \log G(\xi), \]  
with the corresponding solution:
\[ \log X(x, y) = \frac{1}{2\pi i} \int_L \frac{\log G(\xi) dz}{z - y}, \quad \log X^\pm(x, y) = \pm \frac{1}{2} \log G(\xi) + \frac{1}{2\pi i} \int_L \frac{\log G(\xi) dz}{z - y}. \]  
For a single interval \( L = [a, b] \), the solution to \( X^\pm(x, y) \) is
\[ X^\pm(x, y) = e^{\pm \frac{1}{2} \log G(\xi) + \frac{1}{\pi} \log G(\xi) \log \frac{b - y}{a - y}}. \]  
Combining eq.(C.3) and eq.(C.4), then we can get the Plemelj formulae:
\[ \frac{\Phi^+(x, y)}{X^+(x, y)} - \frac{\Phi^-(x, y)}{X^-(x, y)} = f(x, y), \quad f(x, y) = \frac{\delta(x - y)}{\xi + \frac{1}{2}}. \]
Combining the solution to the Plemelj formulae (B.15), we can get the solution to (C.8):

\[
\Phi^+(x, y) = \frac{1}{2} f(x, y) + \frac{1}{2\pi i} \int_L \frac{f(x, z)dz}{X^+(x, z)(z - y)}
\]
\[
\Phi^-(x, y) = -\frac{1}{2} f(x, y) + \frac{1}{2\pi i} \int_L \frac{f(x, z)dz}{X^+(x, z)(z - y)}
\]
\[
\Rightarrow \Phi^+(x, y) = \frac{1}{2} f(x, y) + \frac{1}{2\pi i} X^+(x, y) \int_L \frac{f(x, z)dz}{X^+(x, z)(z - y)}
\]
\[
\Rightarrow \Phi^-(x, y) = -\frac{1}{2} f(x, y) + \frac{1}{2\pi i} X^-(x, y) \int_L \frac{f(x, z)dz}{X^+(x, z)(z - y)}.
\]

Then we can get the solution to the resolvent \(R(x, y)\) of a single interval \(L = [a, b]\):

\[
R(x, y) = (\xi^2 - 1/4)^{-1} \left( \frac{\xi \delta(x - y) + \frac{i}{2\pi} e^{-\frac{x+y}{\pi} \log(\frac{x-y}{\xi+i})}}{\xi - \frac{1}{2}} \right)
\]

When \(L\) contains \(n\) disjoint intervals, where \(L = (a_1, b_1) \cup (a_2, b_2) \cup \ldots \cup (a_n, b_n)\), the resolvent of the primary fermion correlator in multicomponent subsets of the \(L\) in two dimensional vacuum Minkowski spacetime can be written as

\[
R(x, y) = (\xi^2 - 1/4)^{-1} \left( \frac{\xi \delta(x - y) + \frac{i}{2\pi} e^{-\frac{x+y}{\pi} \log(\frac{x-y}{\xi+i})}}{x - y} \right)
\]

where the function \(z(x)\) is

\[
z(x) = \log \left( \frac{\prod_{i=1}^n (x - u_i)}{\prod_{i=1}^n (x - v_i)} \right)
\]

Appendix D: Entanglement entropy for primary fermion fields given by a correlator trace formula

The creation and annihilation operators \(\Psi_i^\dagger\) and \(\Psi_j\) for primary fermion fields satisfy the anticommutation relations: \(\{\Psi_i, \Psi_j^\dagger\} = \delta_{ij}\). Then the two point correlators are given as

\[
\{\Psi_i, \Psi_j^\dagger\} = C_{ij}, \quad \langle \Psi_i \Psi_j \rangle = \delta_{ij} - C_{ij}, \quad \langle \Psi_i \Psi_j^\dagger \rangle = \langle \Psi_i^\dagger \Psi_j^\dagger \rangle = 0
\]

The reduced density matrix of the fermion system can be written in the exponential form [14]:

\[
\rho_V = Ke^{-\mathcal{H}} = Ke^{-\Sigma_V H_i \Psi_i^\dagger \Psi_j},
\]

where \(\mathcal{H}\) is the modular Hamiltonian of the system and \(K\) is the normalization constant which satisfies \(Tr \rho_V = 1\). The two point correlators \(C_{ij}\) in the region \(V\) of space is related to the reduced density matrix \(\rho_V\) by the following equation:

\[
C_{ij} = Tr(\rho_V \cdot \Psi_i^\dagger \Psi_j^\dagger).
\]
We can diagonalize the exponent by the Bogoliubov transformation $d_\ell = U_{\ell m} \Psi_m$, with unitary operator $U$ in order to maintain the anticommutation relation $\{d_\ell, d_\ell^\dagger\} = \delta_{\ell j}$. We choose $U$ such that $UHU^\dagger = \{\epsilon_i\}$ is a diagonal matrix and $\epsilon_i$ is the eigenvalue of Hermitian matrix $H$. Using the normalization condition $Tr \rho_V = 1$ and the Bogoliubov transformation, then the reduced density matrix $\rho_V$ can be rewritten as

$$\rho_V = \prod_\ell \frac{e^{-\epsilon_\ell} d_\ell d_\ell^\dagger}{1 + e^{-\epsilon_\ell}}.$$  \hfill (D.4)

And the relation between $H$ and $C$ can be also rewritten as

$$K \cdot Tr(e^{-\Sigma_V H_{lm}} \Psi_m \cdot \Psi_j \Psi_j^\dagger) = K \cdot Tr(\prod_\ell \frac{e^{-\epsilon_\ell} d_\ell d_\ell^\dagger}{1 + e^{-\epsilon_\ell}} \cdot \Psi_j \Psi_j^\dagger) = C_{ij}.$$  \hfill (D.5)

Next we diagonalize the two point correlators $C_{ij}$ by Bogoliubov transformation, we can obtain

$$\text{diag}\{C_{ij}\} = \prod_{\ell=1}^N \frac{1}{1 + e^{-\epsilon_\ell}}.$$  \hfill (D.6)

We define $C_\ell$ as the eigenvalues of the matrix $\text{diag}\{C_{ij}\}$, then we have

$$\epsilon_\ell = -\log \left( \frac{1}{C_\ell} - 1 \right), \quad C_\ell \in (0, 1).$$  \hfill (D.7)

In terms of the definition of the von Neumann entropy (4.1), then the entanglement entropy for primary fermion fields of the region $V$ can be written as

$$S_V = -Tr(\rho_V \ln \rho_V) = -Tr \left( \prod_\ell \frac{e^{-\epsilon_\ell} d_\ell d_\ell^\dagger}{1 + e^{-\epsilon_\ell}} \cdot \log \left( \prod_\ell \frac{e^{-\epsilon_\ell} d_\ell d_\ell^\dagger}{1 + e^{-\epsilon_\ell}} \right) \right)$$

$$= \sum_\ell \left( \log (1 + e^{-\epsilon_\ell}) + \epsilon_\ell \cdot e^{-\epsilon_\ell} \right)$$

$$= -\sum_\ell \{(1 - C_\ell) \cdot \log (1 - C_\ell) + C_\ell \cdot \log C_\ell \}$$

$$= -Tr \left[ (1 - C_V) \log (1 - C_V) + C_V \log C_V \right],$$  \hfill (D.8)

where we have traced two quantum states such as $|0\rangle$ and $|1\rangle$ for primary fermion fields in the second line.

[1] C. Teitelboim, “Gravitation and Hamiltonian Structure in Two Space-Time Dimensions,” Phys. Lett. B 126 (1983), 41-45
[2] R. Jackiw, “Lower Dimensional Gravity,” Nucl. Phys. B 252 (1985), 343-356
[3] R. Jackiw, “Five lectures on planar gravity,” MIT-CTP-1936.
[4] A. Almheiri and J. Polchinski, “Models of AdS2 backreaction and holography,” JHEP 11 (2015), 014 [arXiv:1402.6334 [hep-th]].
[5] A. Almheiri, N. Engelhardt, D. Marolf and H. Maxfield, “The entropy of bulk quantum fields and the entanglement wedge of an evaporating black hole,” JHEP 12 (2019), 063 [arXiv:1905.08762 [hep-th]].
[6] A. Almheiri, R. Mahajan and J. Maldacena, “Islands outside the horizon,” [arXiv:1910.11077 [hep-th]].
[7] T. J. Hollowood and S. P. Kumar, “Islands and Page Curves for Evaporating Black Holes in JTGravity,” JHEP 08 (2020), 094 [arXiv:2004.14944 [hep-th]].
[8] T. Faulkner, M. Li and H. Wang, “A modular toolkit for bulk reconstruction,” JHEP 04, 119 (2019) [arXiv:1806.10560 [hep-th]].
[9] J. Cotler, P. Hayden, G. Penington, G. Salton, B. Swingle and M. Walter, “Entanglement Wedge Reconstruction via Universal Recovery Channels,” Phys. Rev. X 9, no.3, 031011 (2019) [arXiv:1704.05839 [hep-th]].
[10] C. F. Chen, G. Penington and G. Salton, “Entanglement Wedge Reconstruction using the Petz Map,” JHEP 01, 168 (2020) [arXiv:1902.02844 [hep-th]].
[11] G. Penington, S. H. Shenker, D. Stanford and Z. Yang, “Replica wormholes and the black hole interior,” JHEP 03, 205 (2022) [arXiv:1911.11977 [hep-th]].
[12] Y. Chen, “Pulling Out the Island with Modular Flow,” JHEP 03, 033 (2020) [arXiv:1912.02210 [hep-th]].
[13] H. Casini and M. Huerta, “Reduced density matrix and internal dynamics for multicomponent regions,” Class. Quant. Grav. 26, 185005 (2009) [arXiv:0903.5284 [hep-th]].
[14] H. Casini and M. Huerta, “Entanglement entropy in free quantum field theory,” J. Phys. A 42, 504007 (2009) [arXiv:0905.2562 [hep-th]].

[15] P. Nayak, A. Shukla, R. M. Soni, S. P. Trivedi and V. Vishal, “On the Dynamics of Near-Extremal Black Holes,” JHEP 09, 048 (2018) [arXiv:1802.09547 [hep-th]].

[16] A. Fabbri and J. Navarro-Salas, “Modeling black hole evaporation,”

[17] D. Grumiller, W. Kummer and D. V. Vassilevich, “Dilaton gravity in two-dimensions,” Phys. Rept. 369, 327-430 (2002) [arXiv:hep-th/0204253 [hep-th]].

[18] D. Z. Freedman and A. Van Proeyen, “Supergravity,” Cambridge Univ. Press, 2012, ISBN 978-1-139-36806-3, 978-0-521-19401-3

[19] P. Collas and D. Klein, “The Dirac Equation in Curved Spacetime: A Guide for Calculations,” Springer, 2019, ISBN 978-3-030-14825-6 [arXiv:1809.02764 [gr-qc]].

[20] S. Lippoldt, “Fermions in curved spacetimes,”

[21] E. Guendelman and R. Steiner, “Mach like principle from conserved charges,” Found. Phys. 43 (2013), 243-266 [arXiv:1201.5257 [hep-th]].

[22] L. E. Parker and D. Toms, “Quantum Field Theory in Curved Spacetime: Quantized Field and Gravity,” Cambridge University Press, 2009, ISBN 978-0-521-87787-9, 978-0-521-87787-9, 978-0-511-60155-2

[23] J. Maldacena, D. Stanford and Z. Yang, “Conformal symmetry and its breaking in two dimensional Nearly Anti-de-Sitter space,” PTEP 2016 (2016) no.12, 12C104 [arXiv:1606.01857 [hep-th]].

[24] P. Di Francesco, P. Mathieu and D. Senechal, “Conformal Field Theory,” Springer-Verlag, 1997, ISBN 978-0-387-94785-3, 978-1-4612-7475-9

[25] P. Calabrese and J. L. Cardy, “Entanglement entropy and quantum field theory,” J. Stat. Mech. 0406 (2004), P06002 [arXiv:hep-th/0405152 [hep-th]].

[26] P. Calabrese and J. Cardy, “Entanglement entropy and conformal field theory,” J. Phys. A 42 (2009), 504005 [arXiv:0905.4013 [cond-mat.stat-mech]].

[27] W. C. Gan, D. H. Du and F. W. Shu, “Island and Page curve for one-sided asymptotically flat black hole,” JHEP 07 (2022), 020 [arXiv:2203.06310 [hep-th]].

[28] D. Blanco, A. Garbarz and G. Pérez-Nadal, “Entanglement of a chiral fermion on the torus,” JHEP 09 (2019), 076 [arXiv:1906.07057 [hep-th]].

[29] P. Fries and I. A. Reyes, “Entanglement and relative entropy of a chiral fermion on the torus,” Phys. Rev. D 100 (2019) no.10, 105015 [arXiv:1906.02207 [hep-th]].

[30] S. He, Z. C. Liu and Y. Sun, “Entanglement entropy and modular Hamiltonian of free fermion with deformations on a torus,” JHEP 09 (2022), 247 [arXiv:2207.06308 [hep-th]].

[31] H. Casini and M. Huerta, “Analytic results on the geometric entropy for free fields,” J. Stat. Mech. 0801 (2008), P01012 [arXiv:0707.1300 [hep-th]].

[32] N. I. Muskhelishvili, Singular Integral Equations, Groningen-Holland (1953).

[33] J. Erdmenger, P. Fries, I. A. Reyes and C. P. Simon, “Resolving modular flow: a toolkit for free fermions,” JHEP 12 (2020), 126 [arXiv:2008.07532 [hep-th]].

[34] I. A. Reyes, “Moving Mirrors, Page Curves, and Bulk Entropies in AdS2,” Phys. Rev. Lett. 127 (2021) no.5, 051602 [arXiv:2103.01230 [hep-th]].

[35] S. M. Carroll, “Spacetime and Geometry,” Cambridge University Press, 2019, ISBN 978-0-8053-8732-2, 978-1-108-48839-6, 978-1-108-77555-7