Ferromagnetic condensation in high density hadronic matter

Henrik Bohr
Department of Physics, B.307, Danish Technical University,
DK-2800 Lyngby, Denmark

Prafulla K. Panda
Department of Physics, C.V. Raman College of Engineering,
Vidya Nagar, Bhubaneswar-752054, India, and
CFC, Departamento de Física, Universidade de Coimbra,
P-3004-516 Coimbra, Portugal

Constança Providência, João da Providência
CFC, Departamento de Física, Universidade de Coimbra,
P-3004-516 Coimbra, Portugal

May 5, 2014

Abstract
We investigate the occurrence of a ferromagnetic phase transition in high density hadronic matter (e.g., in the interior of a neutron star). This could be induced by a four fermion interaction analogous to the one which is responsible for chiral symmetry breaking in the Nambu-Jona-Lasinio model, to which it is related through a Fierz transformation. Flavor SU(2) and flavor SU(3) quark matter are considered. A second order phase transition is predicted at densities about 5 times the normal nuclear matter density, a magnetization of the order of $10^{16}$ gauss being expected. It is also found that in flavor SU(3) quark matter, a first order transition from the so-called 2 flavor super-conducting (2SC) phase to the ferromagnetic phase arises. The color-flavor-locked (CFL) phase may be completely hidden by the ferromagnetic phase.

1 Introduction
The properties of a highly compressed fermion fluid, especially with respect to magnetization, are of great interest since they are relevant for stellar objects
such as neutron stars and, possibly, quark stars [1]. Recently, magnetars, which are a kind of neutron stars exhibiting extremely powerful magnetic fields, have been discovered [2, 3].

We investigate a model of the magnetization of a quark fluid which is characterized by a four fermion interaction [4] related to the interaction of the standard Nambu-Jona-Lasinio (NJL) model through a Fierz transformation. In the present model, a ferromagnetic transition is induced by an analogous mechanism to the one which is responsible for chiral symmetry breaking in the NJL model [5, 6, 7, 8]. This model accounts for the ferromagnetic transition at very high densities, after chiral symmetry restoration. This is at variance with ref.[9], where it is shown, in the framework of the chiral model of pions and quarks, also known as the linear sigma model, that spin polarization of hadronic matter may arise as a consequence of pion condensation, if chiral symmetry has not yet been restored.

It should be pointed out that, in terms of a Skyrme force, a ferromagnetic transition has been obtained for nuclear matter [10, 11, 12]. However, in that case, microscopic calculations do not predict a ferromagnetic transition, at least below $\rho = 7\rho_0$ [13]. A ferromagnetic phase of quark matter, has recently been investigated by several authors [14, 15, 16], who focus on its QCD origin.

2 Symmetrical quark matter

The model we are considering applies to high densities where chiral symmetry has been restored. Hence, the quark masses are reduced to their bare masses, which are assumed to vanish. Our model is of the NJL type [5, 6, 7, 8], but, for simplicity, we neglect the typical scalar-scalar and pseudoscalar-pseudoscalar interaction which is responsible for chiral symmetry breaking. This is because, at very high densities, it has no effect. Instead, we construct a pure fermionic QCD inspired model based on a chiral symmetric tensor-tensor interaction, so the Lagrangian reads

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - \frac{1}{4}G[(\bar{\psi}\gamma^\mu\gamma'^\nu\tau_k\psi)(\bar{\psi}\gamma_\mu\gamma_\nu\gamma_5\tau_k\psi) - (\bar{\psi}\gamma^\mu\gamma'^\nu\gamma_5\psi)(\bar{\psi}\gamma_\mu\gamma_\nu\gamma_5\psi)].$$

We stress that this interaction may be understood as arising from a the Fierz transformation of a standard NJL interaction. We consider $\mu, \nu$ of the form $i,j \in \{1, 2, 3\}$, $i \neq j$. In the following we will consider massless quarks and we will work in the no-sea and mean-field approximations. By mean-field approximation we mean that the operator $\bar{\psi}\gamma_i\gamma_j\tau_k\psi$ is replaced by $\langle \bar{\psi}\gamma_i\gamma_j\tau_k\psi \rangle + \bar{\psi}\gamma_i\gamma_j\tau_k\psi - \langle \bar{\psi}\gamma_i\gamma_j\tau_k\psi \rangle$ and second order terms (and higher) in $\bar{\psi}\gamma_i\gamma_j\tau_k\psi - \langle \bar{\psi}\gamma_i\gamma_j\tau_k\psi \rangle$ are neglect. In the Conclusions, the implications of the no-sea approximation will be briefly discussed. Let us assume polarization along the 3 axis. Then, $\langle \bar{\psi}\gamma_1\gamma_2\tau_k\psi \rangle^2$ is replaced by $2\langle \bar{\psi}\gamma_1\gamma_2\tau_k\psi \rangle\bar{\psi}\gamma_1\gamma_2\tau_k\psi - \langle \bar{\psi}\gamma_1\gamma_2\tau_k\psi \rangle^2$. We introduce the notation

$$F_k = G\langle \bar{\psi}\Sigma_3\tau_k\psi \rangle = iG\langle \bar{\psi}\gamma_1\gamma_2\tau_k\psi \rangle. \quad (1)$$
Manifestation of this interaction is seen only at very high densities. In the mean field approximation, the postulated interaction leads to the Lagrangian density

\[ \mathcal{L}_{MFA} = i \overline{\psi} \gamma^{\mu} \partial_{\mu} \psi - F_k \overline{\psi} \Sigma_3 \tau_k \psi - \frac{F_k^2}{2G}. \]  

In turn, this leads to the Dirac equation

\[ (-i \mathbf{\alpha} \cdot \nabla + \epsilon + \epsilon F \beta \Sigma_3) \psi = \varepsilon \psi, \]

where \( \epsilon = 1 \) for quarks \( u \) and \( \epsilon = -1 \) for quarks \( d \) denote the eigenvalues of \( \tau_3 \), and \( F_k \) is a parameter which is determined from (1). In the following, we take \( F_1 = F_2 = 0 \), and \( F_3 = F \).

In this section we will consider symmetric 2-flavor quark matter, the Fermi energy being the same for quarks \( u \) and \( d \). The single particle energy eigenvalues satisfy

\[ \varepsilon_p = \pm \sqrt{\left( |F| \pm \sqrt{p_1^2 + p_2^2}\right)^2 + p_3^2}, \]

where the \( \pm \) signs are such that the single particle energies of quarks \( u \) with momentum \( \mathbf{p} \) and spin \( s \) are the same as the energies of quarks \( d \) with the same momentum and spin \(-s\). By \( p_3 \) we denote the component of momentum along the direction of spin polarization, \( p_1 \) and \( p_2 \) being the transverse components. A no sea approximation shall be used. Thus, the quark Fermi surfaces are given by

\[ \mu^2 = 
\left( |F| \pm \sqrt{p_1^2 + p_2^2}\right)^2 + p_3^2, \]

the + sign corresponding to the Fermi surface of quarks \( u \) with spin up or quarks \( d \) with spin down, (in the direction of magnetization) and the – sign to the Fermi surface of quarks \( u \) with spin down or quarks \( d \) with spin up. Since the temperature is assumed to vanish, the Fermi energy coincides with the chemical potential \( \mu \).

We observe that the formalism behind the description of ferromagnetic condensation is analogous to the BCS theory (the parameter \( F \) playing the role of the gap \( \Delta \)), or to the NJL theory for chiral symmetry breaking (the order parameter \( F/G \) corresponding then to the quark condensate \( \langle \bar{q}q \rangle \)). The condition

\[ \frac{\partial \Phi}{\partial F} = 0 \]  

replaces the gap equation. Below the critical potential \( \mu_c \), this condition implies that \( F = 0 \), so that the state of equilibrium is the normal phase. Above \( \mu_c \), equilibrium occurs for \( F \neq 0 \) (cf. Fig 1). The critical chemical potential \( \mu_c \) is determined by the condition

\[ \frac{\partial^2 \Phi(\mu, F)}{\partial F^2} \bigg|_{F=0} = 0 \]
Figure 1: The thermodynamical potential $\Phi$ vs. $F$, for symmetric flavor $SU(2)$ quark matter, for fixed values of $\mu$, being $\mu_c = 0.41$ GeV.

and is given by

$$\mu_c = \frac{\pi}{\sqrt{3G}}. \tag{4}$$

The Fermi surface may be visualized as a circle of radius $\mu$ rotating around a line lying on its plane at the distance $F$ from its center. If the line lies outside the circle, the Fermi surface is doughnut-shaped and we have full polarization. If the line crosses the circle but is not a diameter, we have two Fermi surfaces, one for spin up and the other one for spin down, corresponding to partial polarization. If the line is a diameter, we have no polarization at all, since the Fermi surface is the same for particles with spin up or down. Thus, two different expressions are obtained for the thermodynamical potential $\Phi(F, \mu)$, one which is valid for $|F| > \mu$ (needed at higher densities), and another one for $|F| < \mu$ (needed at lower densities).

Taking into account that the color-flavor degeneracy is 6, and keeping in mind that $2\pi^2 \mu^2 F$ is the volume of the torus generated by a circle of radius $\mu$ whose center describes a circle of radius $F$, it is easily found that, for $|F| > \mu$, when full polarization is present, the particle number reads,

$$N = \frac{6V}{(2\pi)^3} \int d^3\vec{p}\theta(\varepsilon_p - \mu) = \frac{12V}{(2\pi)^3} \pi^2 |F| \mu^2, \tag{5}$$

and the energy reads,

$$E = \frac{6V}{(2\pi)^3} \int d^3\vec{p}\theta(\varepsilon_p - \mu)\varepsilon_p + \frac{VF^2}{2G} = \frac{8V}{(2\pi)^3} \pi^2 |F| \mu^3 + \frac{VF^2}{2G}, \tag{6}$$
Figure 2: The order parameter $F/G$ versus the chemical potential $\mu$, for symmetrical matter (SM). In the normal phase, $F/G=0$.  

where the term $VF^2/(2G)$ comes from the standard mean field approximation. The limits of integration over the transverse momentum $\sqrt{p_1^2 + p_2^2}$ in Eqs. (5) and (6) are $F-\mu$ and $F+\mu$. The thermodynamical potential becomes

$$\Phi = E - N\mu = -\frac{4V}{(2\pi)^3}|F|\mu^3 + \frac{VF^2}{2G}.$$ 

This is the expression which is valid for $F \geq \mu$. The physical $F$ value minimizes this expression and is equal to 

$$F = 4G\frac{\pi^2 \mu^3}{(2\pi)^3},$$ 

so that in the fully polarized phase the thermodynamical potential reads

$$\Phi = -\frac{V}{\pi^2} G\mu^6.$$ 

However, this expression is valid only for $\mu \geq \mu_p = \sqrt{2\pi/G}$, since $\sqrt{2\pi/G}$ is the $\mu$ value such that minimum of $\Phi$ occurs precisely for $F = \mu$. For $\mu < \mu_c$, the condition (4) implies that $F = 0$, so that the state of equilibrium is the normal phase for which the thermodynamical potential equals

$$\Phi = -\frac{V}{\pi^2} \frac{\mu^4}{2}.$$
Figure 3: The pressure $P$ vs. the chemical potential $\mu$ for $G = 20 \text{ GeV}^{-2}$ comparing the ferromagnetic phase (FP) and the normal phase (NP) of symmetric matter.

In order to describe the system for $\mu_c < \mu < \mu_p$, we need $\Phi(\mu, F)$ for $0 < F < \mu$. This is the case of partial polarization, which we now briefly discuss. Then, the limits of integration over the transverse momentum $\sqrt{p_1^2 + p_2^2}$ are $0, \mu - F$ (spin up) and $0, \mu + F$ (spin down), so that the quark number equals,

$$N(\mu, F) = \frac{6V}{(2\pi)^3} \left( \int_\uparrow d^3p \theta(\varepsilon_p - \mu) + \int_\downarrow d^3p \theta(\varepsilon_p - \mu) \right)$$

$$= \frac{V}{\pi^2} \left( \sqrt{\mu^2 - F^2}(F^2 + 2\mu^2) + 3F\mu^2 \tan^{-1} \frac{F}{\sqrt{\mu^2 - F^2}} \right). \quad (7)$$

For the energy we find

$$E = E_1 + \frac{VF^2}{2G}$$

where

$$E_1(\mu, F) = \frac{6V}{(2\pi)^3} \left( \int_\uparrow d^3p \varepsilon_p \theta(\varepsilon_p - \mu) + \int_\downarrow d^3p \varepsilon_p \theta(\varepsilon_p - \mu) \right)$$

$$= \int_0^\mu d\mu \frac{\partial N}{\partial \mu} = \mu N - \int_0^\mu d\mu N$$

$$= \mu N - \int_0^F d\mu N - \int_0^\mu d\mu N = \mu N - \frac{4V}{8\pi^2}F^4 + \Phi_1, \quad (8)$$

In the case $0 \leq |F| \leq \mu$, the thermodynamical potential $\Phi$ is given by

$$\Phi(\mu, F) = \Phi_1 - \frac{4V}{8\pi}F^4 + \frac{VF^2}{2G}. \quad (9)$$
Figure 4: The pressure $P$ vs. the quark density $\rho$ for $G = 20 \text{ GeV}^{-2}$, for symmetrical quark matter, in the ferromagnetic (FP) and in the normal (NP) phases. Partial polarization is described by the portion of the graph with lowest slope, which occurs for $4.87\rho_0 \leq \rho \leq 9.99\rho_0$ and for $0.00175 < P < 0.00273 \text{ (GeV}^4\text{)}$.

where,

$$
\Phi_1(\mu, F) = -\int_{F}^{\mu} d\mu N = -\frac{V F^4}{\pi^2} \left( \frac{2\mu^3 + 3\mu F^2}{4F^4} \sqrt{\mu^2 - F^2} \right.
+ \left. \frac{\mu^3}{|F|^3} \tan^{-1} \frac{|F|}{\sqrt{\mu^2 - F^2}} - \frac{1}{4} \log \frac{\mu + \sqrt{\mu^2 - F^2}}{|F|} - \frac{\pi}{2} \right). \tag{10}
$$

In (9), $|F|$ is determined by minimizing $\Phi$ for fixed $\mu$. It is found that for $\mu < \mu_c$, the minimum of $\Phi(\mu, F)$ occurs for $F = 0$, so that, for low densities, the normal phase prevails. For $\mu_c < \mu < \sqrt{2\pi/G} = \mu_p$ the minimum of $\Phi(\mu, F)$ occurs for $0 \leq |F| \leq \mu$, and then partial polarization is realized.

Numerical results for $G = 20 \text{ GeV}^{-2}$, arbitrarily chosen but of the order of the coupling constant of the chirally symmetric four-fermion interaction of the NJL model which takes values $\sim 5 - 10 \text{ GeV}^{-2}$ [8], are presented in Figs. 2 to 4.

In Fig. 2, the order parameter $F/G$ is plotted versus the chemical potential $\mu$, for symmetrical matter. The ferromagnetic phase occurs for a quark chemical potential above $\mu_c = 0.41 \text{ GeV}$. A transition of second order, characterized by a continuous behavior of the magnetization and of the baryonic density, is found.

In Fig. 3, the pressure $P$ is plotted versus the chemical potential $\mu$ for both the ferromagnetic and the normal phases of symmetric quark matter. The most stable phase is the one having the largest pressure for a given chemical potential. Therefore, for the lowest chemical potentials the normal phase is the equilibrium phase for both symmetric and neutral matter, while at high chemical potential the ferromagnetic phase is the stable one.
Figure 5: The critical values of the pressure and of the chemical potentials versus the coupling constant $G$. By $\mu_p$ we denote the critical chemical potential for the onset of full polarization.

In Fig. 4, the pressure $P$ is plotted versus $\rho/\rho_0$, where $\rho$ is the baryonic density and $\rho_0$ is the nuclear saturation density, comparing the ferromagnetic and the normal phases of symmetrical quark matter. Partial polarization is described by the portion of the graph with lowest slope. For symmetrical matter, this region occurs for $4.87\rho_0 \leq \rho \leq 9.99\rho_0$, and $0.00175 < P < 0.00273$ (GeV$^4$). The critical quark densities are quite high. A larger coupling constant would induce a transition at lower densities. In Fig. 9, the dependency of the critical values of the pressure and of the chemical potential on the coupling constant $G$ is displayed. A phase transition at densities comparable to the ones expected to exist in the interior of neutron stars, twice to four times the equilibrium density of nuclear matter, would require that a $G$ of the order $25$ GeV$^{-2}$ to $30$ GeV$^{-2}$. Although these couplings are probably too large, they may be justified, as explained below, if the vacuum polarization is taken into account. Only values of $\rho$ up to $10\rho_0$ are relevant.

3 Magnetization

As suggested in [4], magnetization of a relativistic fermion fluid may be associated with the lowest Landau level (LLL). The energy of the Landau level $\nu, p_3$ is obtained through the replacement $p_1^2 + p_2^2 \rightarrow 2\nu|Q|B$ in the expression (3) of the single particle energy, so that, in the presence of a magnetic field $B$, the energy of a quark with charge $Q$ is equal to

$$\varepsilon_{\nu}(p_3) = \sqrt{(F \pm \sqrt{2}|Q|B\nu)^2 + p_3^2}.$$  

In our model, the effective LLL does not arise for $\nu = 0$, but for $\nu = F^2/(2|Q|B)$, so that its energy is $\varepsilon_{LLL}(p_3) = |p_3|$. By analogy with [4], we postulate, some-
For $G=20$ GeV$^2$

Figure 6: Magnetization versus the chemical potential $\mu$, for symmetrical matter, $\mu_p$ being the critical $\mu$ for the onset of full polarized matter. In the normal phase, $\mathcal{M} = 0$.

what ad hoc, that the contribution of the effective LLL to the thermodynamical potential reads

$$-\mathcal{M}B = \frac{(|Q_u| + |Q_d|)}{2\pi} \sum_{p_3} \frac{1}{2} (\varepsilon_{LLL}(p_3) - \mu),$$

where $|Q_\tau|$ is the quark charge and $|Q_\tau|B/(2\pi)$ the Landau degeneracy. The magnetization of fully polarized matter is then equal to

$$\mathcal{M} = \frac{3\mu^2(|Q_u| + |Q_d|)}{8\pi^2} = \frac{3\mu^2\left(\frac{2}{3}|e| + \frac{1}{3}|e|\right)}{8\pi^2}, \quad \mu > \mu_p.$$

For $\mu_c < \mu < \mu_p$ we will have,

$$\mathcal{M} = \frac{3\mu^2(|Q_u| + |Q_d|)}{16\pi^2} \left(2\mu(\mu - \sqrt{\mu^2 - F^2}) + F^2 \log \frac{\mu - \sqrt{\mu^2 - F^2}}{\mu + \sqrt{\mu^2 - F^2}}\right).$$

In fig. 6, the magnetization is plotted vs. the chemical potential, for symmetric quark matter.

4 Neutral quark matter

In electrically neutral quark matter, the density $\rho_d$ of quarks $d$ is twice the density $\rho_u$ of quarks $u$. In the normal phase $|F| = 0$ and the numbers of quarks
$u$ and $d$ are given by

$$N_u = \frac{V}{\pi^2} \mu_u^3, \quad N_d = \frac{V}{\pi^2} \mu_d^3,$$

where $\mu_u$, $\mu_d$ are the Fermi energies of quarks $u$, $d$, respectively. Since $\rho_d = 2\rho_u$, we have $\mu_d = 2^{1/2} \mu_u$. The thermodynamical potential is written $\Phi = K_u + K_d - \mu(N_u + N_d)$. Here $\mu$ is the chemical potential which fixes the quark number independently of flavor and the kinetic energies are equal to $K_u = \frac{3V}{4\pi^2} \mu_u^4$, and $K_d = \frac{3V}{4\pi^2} \mu_d^4$, so that

$$\Phi = \frac{3V}{4\pi^2} (\mu_u^4 + \mu_d^4) - \frac{V}{\pi^2} (\mu_u^3 + \mu_d^3).$$

The values of $\mu_u$, $\mu_d$ are not fixed a priori but are determined by minimization of $\Phi$ under the condition $\mu_d = 2^{1/2} \mu_u$, which leads to

$$\mu_u = \frac{3}{1 + 2^{1/2} \mu}, \quad \mu_d = \frac{3 \cdot 2^{1/2}}{1 + 2^{1/2} \mu},$$

so that

$$\Phi = -\frac{3V}{4\pi^2} \left( \frac{3}{1 + 2^{1/2}} \right)^3 \mu^4. \quad (11)$$

In the fully polarized ferromagnetic phase, $|F| > \mu_d$ (since $\mu_d > \mu_u$) we have (cf. eq. (5))

$$N_u = \frac{3V}{4\pi} |F| \mu_u^2, \quad N_d = \frac{3V}{4\pi} |F| \mu_d^2.$$  

Therefore, for electrically neutral matter we obtain $\mu_d = \sqrt{2} \mu_u$, and

$$K_u = \frac{V}{2\pi} |F| \mu_u^3, \quad K_d = \frac{V}{2\pi} |F| \mu_d^3.$$  

The thermodynamical potential reads

$$\Phi = \frac{V}{2\pi} |F| (\mu_u^3 + \mu_d^3) - \mu \frac{3V}{4\pi} |F| (\mu_u^2 + \mu_d^2) + \frac{VF^2}{2G}.$$  

Under minimization with respect to $\mu_u, \mu_d$ subject to the condition $\mu_d = \sqrt{2} \mu_u$, we find

$$\mu_u = \frac{3}{1 + 2^{1/2} \mu},$$

so that

$$\Phi = -\frac{27V|F|}{4\pi (1 + 2\sqrt{2})^2 \mu^3} + \frac{VF^2}{2G}. \quad (12)$$

Upon minimization of the previous expression with respect to $F$ we obtain

$$\Phi = -\frac{3^6 VG}{2^5 (1 + 2\sqrt{2})^4 \pi^2 \mu^6}.$$
In order to describe the regime of partial polarization, the condition that for each quark \( u \) there are two quarks \( d \) is imposed by requiring that
\[
\frac{1}{2}N(\mu_u, F) = N(\mu_d, F),
\]
where the function \( N(\mu, F) \) is given by (7). The thermodynamical potential of neutral quark matter equals
\[
\Phi(\mu, F) = \frac{1}{2} \left( E_1(\mu_u, F) + E_1(\mu_d, F) \right) + \frac{VF^2}{2G} - \mu \left( \frac{1}{2} \left( N(\mu_u, F) + N(\mu_d, F) \right) \right),
\]
where the functions \( N(\mu, F) \), \( E_1(\mu, F) \) are given by (7), (8). The factor \( \frac{1}{2} \) accounts for the flavor degeneracy which was included in the definitions of these functions. The equilibrium thermodynamical potential is obtained by minimizing (14), for fixed \( \mu \), with respect to \( \mu_u, \mu_d, \) and \( F \), under the constraint (13).

We find that the behaviors of neutral quark matter and symmetric quark matter are qualitatively very similar, the pressure of neutral quark matter being slightly smaller, for a given \( \mu \), due to the charge neutrality constraint.

5 Strange quark matter

In this Section, we consider strange quark matter at densities which are so high that chiral symmetry has been restored, in the sense that the masses of quarks \( u, d \) and \( s \) are reduced to the bare masses. Moreover, it is assumed for simplicity that the bare masses vanish. For quarks \( s \), this is admittedly a very drastic assumption. However, we will analyze its consequences and comment at the end. As in Section 2, we postulate a QCD inspired tensor-tensor interaction, so that the Lagrangian reads
\[
L = i\bar{\psi}\gamma_\mu \partial_\mu \psi - \frac{1}{4}G(\bar{\psi}\gamma_\mu \gamma_\nu \Lambda_k \psi)(\bar{\psi}\gamma_\mu \gamma_\nu \Lambda_k \psi),
\]
where \( \Lambda_k \) denote the \( SU(3) \) flavor Gell-Mann matrices. This Lagrangian is not chiral invariant. The terms which insure chiral invariance have been omitted, for simplicity. We assume polarization along the 3 axis, consider the mean-field approximation, and define
\[
F_k = iG(\bar{\psi}\gamma_1 \gamma_2 \Lambda_k \psi).
\]
In the mean field approximation, for \( k \neq 3, 8 \), we have \( F_k = 0 \), so that the Lagrangian density reduces to
\[
L_{MFA} = i(\bar{\psi}\gamma_\mu \partial_\mu \psi) - \sum_{k \in \{3, 8\}} F_k(\bar{\psi}\Sigma_3 \Lambda_k \psi) - \sum_{k \in \{3, 8\}} \frac{F_k^2}{2G}.
\]
In turn, this leads to the Dirac equation

\[-i \partial_t + i \alpha_j \nabla_j - \beta \Sigma_3 \sum_{k \in \{3, 8\}} F_k \Lambda_k \psi = 0.\]

The operator \(\sum_{k \in \{3, 8\}} F_k \Lambda_k\) has different values for quarks \(u, d,\) and \(s\). Its eigenvalues are equal to

\[\mathcal{F}_\tau = \left( F_3 + \frac{1}{\sqrt{3}} F_8 \right) \delta_{\tau, u} - \left( F_3 - \frac{1}{\sqrt{3}} F_8 \right) \delta_{\tau, d} - \frac{2}{\sqrt{3}} F_8 \delta_{\tau, s} \tag{15}\]

so that the Dirac equations reduce to

\[-i \alpha \cdot \nabla + \mathcal{F}_\tau \beta \Sigma_3) \psi_\tau = \varepsilon_\tau \psi_\tau, \quad \tau \in \{u, d, s\}.\]

We assume that \(\mathcal{F}_u\) is positive and \(\mathcal{F}_d, F_s\) are negative. The single particle energy eigenvalues read

\[\varepsilon_{p, \tau} = \pm \sqrt{\left( |\mathcal{F}_\tau| \pm \sqrt{p_1^2 + p_2^2} \right)^2 + p_3^2},\]

where the + sign refers to quarks \(u\) with momentum \(p\) and spin up and to quarks \(d, s\) with the same momentum and spin down, while the − sign refers to quarks \(u\) with momentum \(p\) and spin down and to quarks \(d, s\) with the same momentum and spin up. The quark Fermi surfaces are given by

\[\mu^2 = \left( |\mathcal{F}_\tau| \pm \sqrt{p_1^2 + p_2^2} \right)^2 + p_3^2,\]

the + sign corresponding to the Fermi surface of quarks \(u\) with spin up or quarks \(d, s\) with spin down, and the − sign to the Fermi surface of quarks \(u\) with spin down or quarks \(d, s\) with spin up.

Several different analytic expressions, each one with its own range of validity, are obtained for the thermodynamical potential. One which is valid when \(|\mathcal{F}_\tau| > \mu, \tau \in \{u, d, s\}\), (needed at high densities), and another for \(|\mathcal{F}_\tau| < \mu\) (needed at low densities). Another one which is valid when the largest \(|\mathcal{F}_\tau|\) is bigger than \(\mu\) (needed at high densities), and the remaining \(|\mathcal{F}_\tau|\)'s are smaller than \(\mu\). Here, \(\mu\) is the Fermi energy, which is identified with the chemical potential. Taking into account that the color degeneracy is 3, in the first case, when full polarization is present, the \(\tau\)-quark number reads,

\[N_\tau = \frac{3V}{(2\pi)^3} \int d^3 p \, \theta(\varepsilon_{p, \tau} - \mu) = \frac{6V}{(2\pi)^3} \pi^2 |\mathcal{F}_\tau| \mu^2, \tag{16}\]

and the \(\tau\)-quark kinetic energy reads,

\[K_\tau = \frac{3V}{(2\pi)^3} \int d^3 p \, \theta(\varepsilon_{p, \tau} - \mu) \varepsilon_{p, \tau} = \frac{4V}{(2\pi)^3} \pi^2 |\mathcal{F}_\tau| \mu^3. \tag{17}\]
Figure 7: \( \Phi \) vs, \( F_3 \), for strange quark matter. The curvature at the origin turns upside-down for \( \mu \) about 0.410 GeV.

The total quark number and total energy read, respectively

\[
N = \frac{6V}{(2\pi)^3} \pi^2 \mu^2 \left( |F_u| + |F_d| + |F_s| \right),
\]

\[
E = \frac{4V}{(2\pi)^3} \pi^2 \mu^3 \left( |F_u| + |F_d| + |F_s| \right) + \frac{V}{2G} \frac{F_3^2 + F_8^2}{(2G}),
\]

where \( \mathcal{F}_\tau \) is given by (15) and the term \( V(F_3^2 + F_8^2)/(2G) \) comes from the standard mean field approximation. For full polarization, the thermodynamical potential becomes

\[
\Phi = E - N\mu = -\frac{2V}{(2\pi)^3} \pi^2 (|F_u| + |F_d| + |F_s|) \mu^3 + \frac{V}{2G} (F_3^2 + F_8^2).
\]

As a simplifying ansatz, let us assume that

\[
F_8 = \frac{1}{\sqrt{3}} F_3. \tag{18}
\]

This assumption places the quarks \( d \) and \( s \) on an equal footing. Then,

\[
\mathcal{F}_u = \frac{4}{3} F_3, \quad \mathcal{F}_d = \mathcal{F}_s = -\frac{2}{3} F_3. \tag{19}
\]
Figure 8: \( P \) vs \( \mu \) for strange matter. The crossing point of the 2SC and spin polarized phases is lower for the higher \( G \).

Under this ansatz, for \( \frac{2}{3} F_3 \geq \mu \) (which insures full polarization, i.e., \( |F_u|, |F_d|, |F_s| > \mu \)) the thermodynamical potential equals

\[
\Phi = -\frac{16V}{3(2\pi)^3} \pi^2 |F_3| \mu^3 + \frac{2V}{3G} F_3^2 , \quad (20)
\]

implying that

\[
F_3 = G \frac{\mu^3}{2\pi} ,
\]

so that in the ferromagnetic phase, which is realized if \( \mu \) is above the critical value \( \mu_p \), the thermodynamical potential reads

\[
\Phi = -\frac{V}{6\pi^2} G \mu^6 . \quad (21)
\]

This should be compared with the expression of the thermodynamical potential in the normal phase which is realized if \( \mu \) is below the critical value \( \mu_c \) and equals

\[
\Phi = -\frac{3V}{4\pi^2} \mu^4 . \quad (22)
\]

Then, in the regime of partial polarization, the thermodynamical potential of strange quark matter equals

\[
\Phi(\mu, F_3, F_8) = \frac{1}{2} (\Phi_1(\mu, F_u) + \Phi_1(\mu, F_d) + \Phi_1(\mu, F_s)) - \frac{3V}{8\pi} ((F_u)^4 + (F_d)^4 + (F_s)^4) + V \frac{F_3^2 + F_8^2}{2G} . \quad (23)
\]
Figure 9: The pressure versus the baryonic density, comparing the 2SC phase with the spin-polarized phase.

where $\Phi_1(\mu, F)$ is given by (10). Under the ansatz (18), (19), for $0 \leq \frac{4}{3} F_3 \leq \mu$ (or $0 \leq \frac{2}{3} F_3 \leq \frac{1}{2} \mu$), the thermodynamical potential reduces to

$$
\Phi(\mu, F_3) = \frac{1}{2} \left( \Phi_1 \left( \mu, \frac{4}{3} |F_3| \right) + 2 \Phi_1 \left( \mu, \frac{2}{3} |F_3| \right) \right) - \frac{8V}{9\pi} |F_3|^4 + \frac{2F_3^2}{3G}.
$$

(24)

For $\frac{2}{3} F_3 < \mu < \frac{4}{3} F_3$ (or $\frac{1}{2} \mu \leq \frac{2}{3} F_3 \leq \mu$), the thermodynamical potential reduces to

$$
\Phi(\mu, F_3) = \Phi_1 \left( \mu, \frac{2}{3} |F_3| \right) - \frac{8V}{81\pi} |F_3|^4 - \frac{V}{3\pi} F_3 \mu^3 + \frac{2F_3^2}{3G}.
$$

(25)

Therefore, we are faced with three partial expressions. Of course, the equilibrium thermodynamical potential is obtained by optimizing these partial expressions in their respective domains of validity. We find that $\mu_c = \pi/\sqrt{3G}$ is the critical $\mu$ for the spin polarization phase transition, and $\mu_p = \sqrt{3\pi/G}$ is the critical $\mu$ for the onset of absolute full polarization. The expression of the critical chemical potential $\mu_p$ for the onset of full polarization of quarks $u$ (but not of quarks $d, s$) is not given because it is cumbersome.

Color superconductivity in quark matter has been investigated in [17], where a color-flavor symmetric version of the so-called 2 flavor superconducting (2SC) phase has been proposed and its transition to so-called color-flavor locked phase (CFL) was analyzed. In [17], color superconductivity has been described by a mean-field thermodynamical potential operator (i.e., the Hamiltonian constrained
Figure 10: The minimizing $F_3$ vs $\mu$, for strange quark matter, comparing $G = 20$ GeV$^{-2}$ with $G = 25$ GeV$^{-2}$. The critical $\mu$ is lower for the higher $G$.

by the quark number conservation) of the form,

$$\hat{K}_{MFA} = \sum_{iap} \epsilon_p c_{iap}^\dagger c_{iap} + \sum_{ijkabcp} (\Delta_{kc}^* c_{jbp}^\dagger c_{iap} \epsilon_{ijk} \epsilon_{abc} \eta_p + \Delta_{kc} c_{jbp}^\dagger c_{iap} \epsilon_{ijk} \epsilon_{abc} \eta_p) + \frac{V}{2G_c} \sum_{jc} \Delta_{jc} \Delta_{jc}^*,$$

where $\epsilon_p = \sqrt{p^2 + M^2} - \mu$, $M$ being the current mass. In the 2SC phase presented in [17], the gaps $\Delta_{jc}$ are real and independent of the color-flavor indices, that is, $\Delta_{jc} = \Delta$, $j \in \{u, d, s\}, c \in \{r, g, b\}$, so that the two specific flavors or the two specific colors which are paired are not pure colors or flavors, but are related to pure colors and flavors through a certain rotation in color-flavor space, which has the great advantage of insuring automatically color-flavor symmetry. In the CFL phase, the indices $j, c$ are coupled in a prescribed way, for instance, ($u$ with $r$), ($d$ with $g$), ($s$ with $b$) (cf. [17] for further details). For the model considered in [17], the 2SC phase is stable for $\mu < 0.505$ GeV, while the CFL phase is stable for $\mu > 0.505$ GeV, for $G_c = 5$ GeV$^{-2}$. In that model, the quarks have all the same helicity (either positive or negative). If we wish to take into account both
helicities, the gaps reported in [17] are valid and the pressure becomes twice the pressure reported there, under the condition that half the value of $G_c$ is used, that is, for $G_c = 2.5\text{GeV}^{-2}$.

It is of interest to investigate the interplay between color superconductivity and spin polarization as is described here. The transition between the 2SC phase and the ferromagnetic phase is depicted in Fig. 8, where the pressure $P$ is plotted vs the chemical potential $\mu$ for the normal phase, for the ferromagnetic phase, for the 2SC phase and for the CFL phase. The curves corresponding to the spin polarized F phase, to the 2CS phase and to the CFL phase seem to cross at the same point. This is by pure accident. If instead of $G = 20\text{ GeV}^{-2}$ a slightly larger value is considered, the F curve will cross the 2CS and the CFL curves before these curves cross each other. This is clearly seen in fig 8 (b), where the spin polarized phase for $G = 25\text{ GeV}^{-2}$ is plotted. In our model, the CFL phase seems to be completely hidden by the ferromagnetic phase, but, if a slightly lower $G$ is used, the CFL phase will be realized for a short while. Spin polarization and color superconductivity has recently been considered in [18]. The relevance of the interplay between the CFL phase and the 2SC phase for the stability of hybrid stars is discussed in [19].

Suppose we have an unperturbed Lagrangian $L_0$ and two types of perturbations, $L_I$ and $L_{II}$, so that $L_0 + L_I$ describes the phase I associated with some condensate and $L_0 + L_{II}$ describes the phase II associated with another condensate. The interplay between the two phases is described by the Lagrangian $L_0 + L_I + L_{II}$. However, in order to treat approximately this interplay we may apply the Gibbs criterion to the first order transition between the phases described by the truncated Lagrangians. Clearly, it would be more correct to use the full Lagrangian, following [20], and then the first order transition may be replaced by a second order one. We have used the approximate method to describe the interplay between the 2SC phase and the spin polarized phase.

6 Conclusions

Spin polarization of a high density hadronic fluid of quarks has been studied. We describe a spin polarized phase in hadronic matter arising from a four Fermion interaction which may be regarded as the relativistic analogue of the interaction considered in [10, 11, 12] and is related to the interaction of the standard NJL model through a Fierz transformation. A no sea approximation has been used. Flavor $SU(2)$ and flavor $SU(3)$ quark matter are investigated. A second order phase transition at densities about 5 times the normal nuclear matter density is predicted and a magnetization of the order of $10^{16}$ gauss is expected. Numerical results for $G = 20\text{ GeV}^{-2}$, which is arbitrary but of the order of the coupling constant in the NJL model, predict a phase transition at $5-8$ times the saturation density $\rho_0$, which is probably too high. Increasing the coupling to $G = 30\text{ GeV}^{-2}$
would decrease the transition density to $3\rho_0$. It is also found that in flavor $SU(3)$ quark matter, a first order transition from the 2SC phase [17] to the spin polarized phase arises. The spin polarized phase is stable for $\mu > 0.51$ GeV. The 2SC phase is stable for $\mu < 0.51$ GeV. In our model, the CFL phase [17] is completely overshadowed by the spin polarized phase.

We have taken $G = 20$ GeV$^{-2}$. The critical chemical potential for symmetric quark matter and for strange matter turns out to be $\mu_c = 0.41$ GeV. If lower $G$ values are considered, $\mu_c$ increases and in that case the transition from the 2SC to the CFL phase occurs before the transition to the spin polarized phase, so that the CFL phase is realized. If higher $G$ values are considered, $\mu_c$ decreases, so that the transition to the spin polarized phase occurs before the transition from the 2SC phase to the CFL phase, which is then completely overshadowed. Due to the vacuum polarization, it is still meaningful to consider slightly higher $G$ values. Indeed, the main effect of taking the vacuum polarization into account, is the appearance, in the expression of the thermodynamical potential, of a new term of the form $-\kappa\Lambda^2 F^2$, where $\kappa > 0$ is a certain numerical factor and $\Lambda$ is the regularization cutoff. But this will only renormalize the coupling constant $G$, so that we may disregard the vacuum polarization provided we replace the unrenormalized $G_0$ by a new $G$ such that $-\kappa\Lambda^2 + 1/(2G_0) = 1/(2G)$, implying that $G_0 < G$. This may justify the use of a large $G$. For simplicity, we have assumed that the strange quark current mass vanishes. This is an acceptable assumption for $\mu_c = 0.41$ GeV.

This has been an exploratory work where the possibility of a phase transition to a magnetized phase in quark matter was investigated. The application to a physical system like a compact star may require the careful consideration of the vacuum contribution and of finite quark current masses.

Acknowledgements

The present research was partially supported by the projects FCOMP-01-0124-FEDER-008393 with FCT reference PTDC/FIS/113292/2009.

References

[1] P. Haensel, A.Y. Potekhin and D.G. Yakovlev, Neutron Stars 1 Springer, ISBN 10-0-387-33543-9 (2007).

[2] R. C. Duncan and C. Thompson, The Astrophys. J. 392 (1992) L9; C. Thompson and R. C. Duncan, MNRAS 275(1995) 255 ; V. V. Usov, Nature 357(1992) 472 ; B. Paczynski, Acta Astron. 42 (1992) 145 .

[3] E. Göğüs et al., The Astrophysical J. 718 (2010) 331.
[4] H. Bohr, C. Providência and J. da Providência, Braz. J. Phys. 42 (2012) 68.

[5] Y. Nambu and G. Jona-Lasinio, Phys Rev. 122 (1961) 345.

[6] S.P. Klevanisky, Rev. Mod. Phys. 64 (1992) 649.

[7] T. Hatsuda and T. Kunihiro, Phys. Rep. 247 (1994) 221.

[8] M. Buballa, Phys. Rep. 407 (2005) 205.

[9] M. Kutschera, W. Broniowski and A. Kotlorz, Nucl. Phys. A516 (1990) 566.

[10] A. Vidaurre, J. Navarro and J. Bernabéu, Astron. Astrophys. 135 (1984) 361.

[11] J. Margueron, J. Navarro, and Nguyen Van Giai, Phys. Rev. C 66 (2002) 014303.

[12] A. Rios, A. Polls and I. Vidaña, Phys. Rev. C71 (2005) 055802.

[13] I. Vidaña, A. Polls, and A. Ramos, Phys. Rev. C 65 (2002) 035804.

[14] T. Tatsumi, Phys. Lett B489 (2000) 132.

[15] A. Iwazaki, Phys. Rev. D 72 (2005) 114003.

[16] D. Ebert, V. Zhukovsky, O.V. Tarasov, Phys. Rev. D 72 (2005) 096007.

[17] H. Bohr, P. K. Panda, C. Providência and J. da Providência, Braz. J. Phys. 42 (2012) 59.

[18] E. Nakano, T. Maruyama and T. Tatsumi, Phys. Rev. D 68 (2003) 1051001.

[19] M. Buballa, F. Neumann, M. Oertel and I. Shovkovy Phys. Lett. B595 (2004) 36.

[20] M. Kitazawwa, T. Koide, T. Kunihiro and Y. Nemoto, Prog. Theor. Phys. 108 (2002) 929.