The carry propagation of the successor function

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The carry propagation of the successor function,

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Part I

What is the carry propagation?
Adding machine

The Pascaline (1642)

featured the first carry propagation mechanism
Carry propagation

\[
\begin{array}{ccccccccc}
1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & \\
1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & \\
\end{array}
\]

203

102
Carry propagation

\[
\begin{array}{cccccccc}
1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
+ & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
\hline & 1 & 1 & 0 & 0 & 1 & 1 & 0
\end{array}
\]
Carry propagation

\[
\begin{array}{cccccccc}
1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
+ & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
\hline
& 1 & 1 & 0 & 0 & 1 & 1 & 0
\end{array}
\]

\[0 \ 1\]
Carry propagation

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\
+ & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
\hline
0 & 0 & 0 & 0 & 1
\end{array}
\]

203 + 102 = 305
Carry propagation

\[
\begin{array}{cccccccc}
1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
+ & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
\hline
1 & 0 & 0 & 0 & 0 & 1 \\
\end{array}
\]
Carry propagation

\[
\begin{array}{cccccccc}
1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
+ & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
\hline
1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
\end{array}
\]
Carry propagation

\[
\begin{array}{cccccccc}
1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
+ & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
\hline
0 & 1 & 1 & 0 & 0 & 0 & 1
\end{array}
\]

203 + 102 = 305
Carry propagation

\[
\begin{array}{cccccccccccc}
& & & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & \quad & 203 \\
+ & & & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & \quad & 102 \\
\hline
& & & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & \\
\end{array}
\]
Carry propagation

\[
\begin{array}{ccccccccccc}
1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & + \\
1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & \\
\hline
1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1
\end{array}
\]

203 102 305
Carry propagation prevents addition to be parallelisable
Theorem (von Neumann et al. 63, Knuth 78, Pippenger 02)

Average carry propagation length for addition of two uniformly distributed $n$-digit binary numbers =

$$\log_2(n) + O(1)$$
Carry propagation for successor function in base 2

1 1 0 0 1 0 1 1

203
Carry propagation for successor function in base 2

\[
\begin{array}{cccccccccc}
1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
\end{array}
\]

\[
+ \\
\hline
1 \\
\end{array}
\]

\[
\begin{array}{cccccccccc}
1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
\end{array}
\]

203
Carry propagation for successor function in base 2
Carry propagation for successor function in base 2

\[
\begin{array}{cccccccc}
1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
+ & & & & & & 1 \\
\hline
& & & & & 0 & 0 & 0 \\
\end{array}
\]
Carry propagation for successor function in base 2

\[
\begin{array}{cccccccccc}
7 & 7 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{cccccccccc}
+ & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{cccccccccc}
\hline
1 & 0 & 0 & 0 \\
\end{array}
\]
Carry propagation for successor function in base 2

\[ \begin{array}{cccccccc}
1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
+ & & & & & & & \\
\hline
1 & 1 & 0 & 0 & 0 & & & \\
\end{array} \]

203

\[ \begin{array}{cccc}
1 & 1 & 0 & 0 \\
\end{array} \]
Carry propagation for successor function in base 2

\[
\begin{array}{cccccccc}
1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
\hline
+ & & & & & & & 1 \\
\hline
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
\end{array}
\]
Carry propagation for successor function in base 2

$$\begin{array}{cccccccc}
1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
+ & & & & & & & 1 \\
\hline
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0
\end{array}$$

\( cp_2(203) = 3 \)
Amortized carry propagation (in base 2)

\[ CP_2 = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} cp_2(i) \]
Carry propagation for successor function in base 2

\[
\begin{array}{cccccccccccc}
1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & \quad 203 \\
+ & & & & & & & & 1 \\
\hline
1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & \quad 204
\end{array}
\]

cp_2(203) = 3

Amortized carry propagation (in base 2)

\[
CP_2 = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} cp_2(i)
\]

if it exists!
Carry propagation for successor function in base 2

|   | 0 |
|---|---|
| 1 | 1 |
| 1 0 | 2 |
| 1 1 | 3 |
| 1 0 0 | 4 |
| 1 0 1 | 5 |
| 1 1 0 | 6 |
| 1 1 1 | 7 |
| 1 0 0 0 | 8 |
| 1 0 0 1 | 9 |
| 1 0 1 0 | 10 |
| 1 0 1 1 | 11 |
| 1 1 0 0 | 12 |
| 1 1 0 1 | 13 |
| 1 1 1 0 | 14 |
| 1 1 1 1 | 15 |
| 1 0 0 0 0 | 16 |
| 1 0 0 0 1 | 17 |
Carry propagation for successor function in base 2

|   | 0 | 1 |
|---|---|---|
| 1 | 1 |   |
| 10| 2 |   |
| 11| 3 |   |
| 1 0 0 | 4 |   |
| 1 0 1 | 5 |   |
| 1 1 0 | 6 |   |
| 1 1 1 | 7 |   |
| 1 0 0 0 | 8 |   |
| 1 0 0 1 | 9 |   |
| 1 0 1 0 | 10|   |
| 1 0 1 1 | 11|   |
| 1 1 0 0 | 12|   |
| 1 1 0 1 | 13|   |
| 1 1 1 0 | 14|   |
| 1 1 1 1 | 15|   |
| 1 0 0 0 0 | 16|   |
| 1 0 0 0 1 | 17|   |
Carry propagation for successor function in base 2

```
. 0 1
1 1 2
10 2
11 3
100 4
101 5
110 6
111 7
1000 8
1001 9
1010 10
1011 11
1100 12
1101 13
1110 14
1111 15
10000 16
10001 17
```
## Carry propagation for successor function in base 2

|     | 0   | 1   |
|-----|-----|-----|
| 1   | 1   | 2   |
| 10  | 2   | 1   |
| 11  |     | 3   |
| 100 |     | 4   |
| 101 |     | 5   |
| 110 |     | 6   |
| 111 |     | 7   |
| 1000|     | 8   |
| 1001|     | 9   |
| 1010|     | 10  |
| 1011|     | 11  |
| 1100|     | 12  |
| 1101|     | 13  |
| 1110|     | 14  |
| 1111|     | 15  |
| 10000|    | 16  |
| 10001|    | 17  |
Carry propagation for successor function in base 2

.  0  1
1  1  2
10  2  1
11  3  3
100  4  1
101  5  2
110  6  1
111  7  4
1000  8  1
1001  9  2
1010 10  1
1011 11  3
1100 12  1
1101 13  2
1110 14  1
1111 15  5
10000 16  1
10001 17  2
Carry propagation for successor function in base 2

|   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|
|   | 0 | 1 |
| 1 | 1 | 1 |
| 10| 2 | 1 |
| 11| 3 | 1 1 1 |
| 100| 4 | 1 |
| 101| 5 | 1 1 |
| 110| 6 | 1 |
| 111| 7 | 1 1 1 1 |
| 1000| 8 | 1 |
| 1001| 9 | 1 1 |
| 1010| 10 | 1 |
| 1011| 11 | 1 1 1 |
| 1100| 12 | 1 |
| 1101| 13 | 1 1 |
| 1110| 14 | 1 |
| 1111| 15 | 1 1 1 1 |
| 10000| 16 | 1 |
| 10001| 17 | 1 1 |
Carry propagation for successor function in base 2

\[
CP_2 = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots = \frac{2}{2-1} = 2
\]
Carry propagation for successor function in base $p$

$\text{CP}_2 = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots = \frac{2}{2 - 1} = 2$

$\text{CP}_p = 1 + \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \cdots = \frac{p}{p - 1}$
Carry propagation for successor function in base Fibonacci

1 0 0 1 0 1 0 1
Carry propagation for successor function in base Fibonacci

\[
\begin{array}{ccccccccc}
55 & 34 & 21 & 13 & 8 & 5 & 3 & 2 & 1 \\
\end{array}
\]

\[
\begin{array}{ccccccccc}
1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
\end{array}
\]

46
Carry propagation for successor function in base Fibonacci

\[ \begin{array}{cccccccc}
55 & 34 & 21 & 13 & 8 & 5 & 3 & 2 & 1 \\
\end{array} \]

\[ \begin{array}{cccccccccc}
1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 46 \\
\end{array} \]

\[ + \]

\[ \begin{array}{cccccccc}
\hline
\end{array} \]
Carry propagation for successor function in base Fibonacci

```
   1 0 0 1 0 1 0 1 1
+     1
-----
   1 0 1 0 0 0 0 0 0
```

46

47
Carry propagation for successor function in base Fibonacci

\[
\begin{array}{cccccccc}
55 & 34 & 21 & 13 & 8 & 5 & 3 & 2 & 1 \\
\hline
1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
+ & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]
Carry propagation for successor function in base Fibonacci

\[
\begin{array}{cccccccc}
55 & 34 & 21 & 13 & 8 & 5 & 3 & 2 & 1 \\
\end{array}
\]

\[
\begin{array}{cccccccccccc}
1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 46 \\
\end{array}
\]

+ \\
\[
\begin{array}{cccccccccccc}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 47 \\
\end{array}
\]

\[
\begin{array}{cccccccccccc}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

\[
\text{cp}_F(46) = 6
\]
Carry propagation for successor function in base Fibonacci

\[ \begin{array}{ccccccccccc}
55 & 34 & 21 & 13 & 8 & 5 & 3 & 2 & 1 \\
\hline
1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
\hline
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
46 & 47
\end{array} \]

\[ \text{cp}_F(46) = 6 \]

Amortized carry propagation in base Fibonacci

\[ \text{CP}_F = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} \text{cp}_F(i) \]

if it exists!
Carry propagation for successor function in base Fibonacci

|   | 0    | 1    | 2    | 3    | 4    | 5    | 6    | 7    | 8    | 9    | 10   | 11   | 12   | 13   | 14   | 15   | 16   | 17   |
|---|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|
|   | .    | 1    | 10   | 100  | 101  | 1000 | 1001 | 1010 | 10000| 10001| 10010| 10010| 10100| 10101| 100000| 100001| 100010| 1000100| 1000101| 1000000| 1000001| 1000100| 1000101 |
|   |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |
Carry propagation for successor function in base Fibonacci

\[
\begin{array}{cccccc}
\cdot & 0 & 1 \\
1 & 1 & 2 \\
10 & 2 & 3 \\
100 & 3 & 1 \\
101 & 4 & 4 \\
1000 & 5 & 1 \\
1001 & 6 & 2 \\
1010 & 7 & 5 \\
10000 & 8 & 1 \\
10001 & 9 & 2 \\
10010 & 10 & 3 \\
10100 & 11 & 1 \\
10101 & 12 & 6 \\
100000 & 13 & 1 \\
100001 & 14 & 2 \\
100010 & 15 & 3 \\
100100 & 16 & 1 \\
100101 & 17 & 4 \\
\end{array}
\]
Carry propagation for successor function in base Fibonacci

$CP_F$ 

```
.    0    1
  1    1    2
 1 0    2    3
 1 0 0    3    1
 1 0 1    4    4
1 0 0 0    5    1
1 0 0 1    6    2
1 0 1 0    7    5
1 0 0 0 0    8    1
1 0 0 0 1    9    2
1 0 0 1 0   10    3
1 0 1 0 0   11    1
1 0 1 0 1   12    6
1 0 0 0 0 0  13    1
1 0 0 0 0 1  14    2
1 0 0 0 1 0  15    3
1 0 0 1 0 0  16    1
1 0 0 1 0 1  17    4
```
Carry propagation for successor function in base Fibonacci

\[ \text{CP}_F \]

\[ \text{CP}_F = \frac{\varphi}{\varphi - 1} \]
Part II

A first observation and its 3 consequences
A primal observation

The Fibonacci tree
A primal observation

\[ cp_F(8) = 1 \]

The Fibonacci tree
A primal observation

\[ cp_F(8) = 1 \]
\[ cp_F(9) = 2 \]

The Fibonacci tree
A primal observation

\[ cp_F(8) = 1 \]
\[ cp_F(9) = 2 \]
\[ cp_F(10) = 3 \]

The Fibonacci tree
A primal observation

\[ \begin{align*}
  \text{cp}_F(8) &= 1 \\
  \text{cp}_F(9) &= 2 \\
  \text{cp}_F(10) &= 3 \\
  \text{cp}_F(11) &= 1 \\
\end{align*} \]

The Fibonacci tree
A primal observation

\[ cp_F(8) = 1 \]
\[ cp_F(9) = 2 \]
\[ cp_F(10) = 3 \]
\[ cp_F(11) = 1 \]
\[ cp_F(12) = 6 \]

The Fibonacci tree
A primal observation

\[ cp_F(8) = 1 \]
\[ cp_F(9) = 2 \]
\[ cp_F(10) = 3 \]
\[ cp_F(11) = 1 \]
\[ cp_F(12) = 6 \]

\[ \sum_{i=8}^{i=12} cp_F(i) = 13 \]

The Fibonacci tree
A primal observation

\[ \begin{align*}
\text{cp}_{F}(8) &= 1 \\
\text{cp}_{F}(9) &= 2 \\
\text{cp}_{F}(10) &= 3 \\
\text{cp}_{F}(11) &= 1 \\
\text{cp}_{F}(12) &= 6
\end{align*} \]

\[
\sum_{i=8}^{12} \text{cp}_{F}(i) = 13
\]

The Fibonacci tree
What we learn from the primal observation

- A framework: the Abstract Numeration System model
- A general working hypothesis: Prefix-closed Extendable Languages
- An essential parameter: The local growth rate
A framework:
the Abstract Numeration System model

Definition (Lecomte & Rigo 2001)

- A finite \textit{totally ordered} alphabet e.g. $A = \{0, 1\}$

$\Rightarrow A^*$ equipped with the \textit{radix ordering}

\textit{i.e.} ordered first \textit{by length}, and then, for words of equal length, ordered \textit{lexicographically}

\textit{e.g.} $A^* = \varepsilon, 0, 1, 00, 01, 10, 11, 000, 001, \ldots$
What we learn from the primal observation: the ANS model

- A framework: the Abstract Numeration System model

Definition (Lecomte & Rigo 2001)

- $A$ finite *totally ordered* alphabet e.g. $A = \{0, 1\}$
- $A^*$ equipped with the *radix ordering*
- $L \subseteq A^*$ any *language* over $A^*$ ordered by radix ordering

  e.g. $F = \varepsilon \cup 1A^* \setminus A^*11A^*

  F = \varepsilon, 1, 10, 100, 101, 1000, \ldots$
What we learn from the primal observation: the ANS model

- A framework: the Abstract Numeration System model

Definition (Lecomte & Rigo 2001)

• A finite *totally ordered* alphabet e.g. $A = \{0, 1\}$

⇒ $A^*$ equipped with the *radix ordering*

• $L \subseteq A^*$ any *language* over $A^*$ ordered by radix ordering

⇒ Natural integers are given *representations* by means of words of $L$

i.e. $\langle n \rangle_L = (n + 1)$-th word of $L$ in the radix ordering

e.g. $\langle 6 \rangle_F = 1001$
What we learn from the primal observation: the ANS model

- A framework:
  the Abstract Numeration System model

Definition (Lecomte & Rigo 2001)

- A finite *totally ordered* alphabet e.g. $A = \{0, 1\}$
  $\Rightarrow$ $A^*$ equipped with the *radix ordering*

- $L \subseteq A^*$ any *language* over $A^*$ ordered by radix ordering
  $\Rightarrow$ Natural integers are given *representations* by means of words of $L$

$L \subseteq A^*$ (together with the order on $A$) defines an ANS

Fact: All ‘classical’ numeration systems are ANS
What we learn from the primal observation: the ANS model

All ‘classical’ numeration systems are ANS

\[ L_2 = 1(0, 1)^* \]
What we learn from the primal observation: the ANS model

All ‘classical’ numeration systems are ANS

\[
L_F = 1(0, 1)^* \setminus (0, 1)^*11(0, 1)^*
\]
What we learn from the primal observation: the ANS model

Any language can be seen as an ANS

Language \( O \) (for Oscillating)

\[ \langle 10 \rangle_O = d a d a \]
What we learn from the primal observation: the ANS model

Any language can be seen as an ANS

Language \( V \) (for Vibrating)

\[ \langle 10 \rangle_\mathcal{V} = b d a \]
What we learn from the primal observation: an hypothesis

- A framework:
  the Abstract Numeration System model

- A general working hypothesis:
  Prefix-Closed Extendable Languages
What we learn from the primal observation: an hypothesis

- A framework:
  the Abstract Numeration System model

- A general working hypothesis:
  Prefix-Closed Extendable Languages

\[ L \subseteq A^* \text{ an ANS} \]
What we learn from the primal observation: an hypothesis

- A framework:
  the Abstract Numeration System model

- A general working hypothesis:
  Prefix-Closed Extendable Languages

\[ L \subseteq A^* \text{ an ANS} \]
What we learn from the primal observation: an hypothesis

- A framework:
  the Abstract Numeration System model

- A general working hypothesis:
  Prefix-Closed Extendable Languages

$L \subseteq A^*$ an ANS

Notation

$u_L(\ell) = \text{card } (L \cap A^\ell)$

$v_L(\ell) = \text{card } (L \cap A^{\leq \ell}) = \sum_{i=0}^{\ell} u_L(i)$
A framework: the Abstract Numeration System model

A general working hypothesis: Prefix-Closed Extendable Languages

$L \subseteq A^*$ an ANS

Notation

\[
\begin{align*}
    u_L(\ell) &= \text{card } (L \cap A^\ell) \\
    v_L(\ell) &= \text{card } (L \cap A^{\leq \ell}) = \sum_{i=0}^{\ell} u_L(i)
\end{align*}
\]

The formula we want:

\[
\sum_{i=v_L(\ell-1)}^{v_L(\ell)-1} c_{p_L(i)} = v_L(\ell)
\]
What we learn from the primal observation: an hypothesis

- A framework: the Abstract Numeration System model
- A general working hypothesis: Prefix-Closed Extendable Languages

$L \subseteq A^*$  an ANS

Notation

$u_L(\ell) = \text{card} \left( L \cap A^\ell \right)$

$v_L(\ell) = \text{card} \left( L \cap A^{\leq \ell} \right) = \sum_{i=0}^{\ell} u_L(i)$

The formula we want:

$\sum_{i=v_L(\ell-1)}^{v_L(\ell)-1} cp_L(i) = v_L(\ell)$

requires $L$ prefix-closed and extendable i.e. to be a PCE language
What we learn from the primal observation: an hypothesis

- A framework:
  the Abstract Numeration System model

- A general working hypothesis:
  Prefix-Closed Extendable Languages

\[ L \subseteq A^* \text{ an ANS} \]

**Notation**

\[
\begin{align*}
  u_L(\ell) &= \text{card} \ (L \cap A^\ell) \\
  v_L(\ell) &= \text{card} \ (L \cap A^{\leq \ell}) = \sum_{i=0}^{\ell} u_L(i)
\end{align*}
\]

The formula we want:

\[
\sum_{i=v_L(\ell-1)}^{v_L(\ell)-1} \text{cp}_L(i) = v_L(\ell)
\]

**Fact:** ‘All’ ‘classical’ ANS are PCE
What we learn from the primal observation: an hypothesis

‘All’ ‘classical’ ANS are PCE

L₂
What we learn from the primal observation: an hypothesis

The ANS we consider are \( \text{PCE} \)
What we learn from the primal observation: an hypothesis

The ANS we consider are \textbf{PCE}
What we learn from the primal observation: a new parameter

- A framework: the Abstract Numeration System model
- A general working hypothesis: Prefix-Closed Extendable Languages
- An essential parameter: the local growth rate
What we learn from the primal observation: a new parameter

- A framework: the Abstract Numeration System model
- A general working hypothesis: Prefix-Closed Extendable Languages
- An essential parameter: the local growth rate

From
\[ \sum_{i=\nu_{L}(\ell-1)}^{\nu_{L}(\ell)-1} c_{\nu_{L}}(i) = \nu_{L}(\ell) \]
What we learn from the primal observation: a new parameter

- A framework: the Abstract Numeration System model

- A general working hypothesis:
  Prefix-Closed Extendable Languages

- An essential parameter: the local growth rate

From

\[ \sum_{i=0}^{v_L(\ell)-1} cp_L(i) = v_L(\ell) \]

follows

\[ \sum_{i=0}^{v_L(\ell)-1} cp_L(i) = \sum_{j=0}^{\ell} v_L(j) \]
What we learn from the primal observation: a new parameter

- A framework: the Abstract Numeration System model

- A general working hypothesis:
  Prefix-Closed Extendable Languages

- An essential parameter: the local growth rate

From

$$\sum_{i=v_L(\ell-1)}^{v_L(\ell)-1} cp_L(i) = v_L(\ell)$$

follows

$$\sum_{i=0}^{v_L(\ell)-1} cp_L(i) = \sum_{j=0}^{\ell} v_L(j)$$

hence, if

$$CP_L = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} cp_L(i)$$
exists
What we learn from the primal observation: a new parameter

- A framework: the Abstract Numeration System model

- A general working hypothesis: Prefix-Closed Extendable Languages

- An essential parameter: the local growth rate

\[
\sum_{i=v_L(\ell-1)}^{v_L(\ell)-1} cp_L(i) = v_L(\ell)
\]

From

\[
\sum_{i=0}^{v_L(\ell)-1} cp_L(i) = \sum_{j=0}^{\ell} v_L(j)
\]

follows

hence, if \( CP_L = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} cp_L(i) \) exists

then \( CP_L = \lim_{\ell \to \infty} \frac{1}{v_L(\ell)} \sum_{j=0}^{\ell} v_L(j) \) exists
Lemma

\[(x_\ell)_{\ell \in \mathbb{N}} \quad x_\ell \in \mathbb{R}_+ \quad \forall \ell \quad y_\ell = \sum_{j=0}^{\ell-1} x_j \quad \gamma > 1\]

TFAE

(i) \[\lim_{\ell \to \infty} \frac{x_{\ell+1}}{x_\ell} = \gamma\]

(ii) \[\lim_{\ell \to \infty} \frac{y_{\ell+1}}{y_\ell} = \gamma\]

(iii) \[\lim_{\ell \to \infty} \frac{y_\ell}{x_\ell} = \frac{\gamma}{\gamma - 1}\]
What we learn from the primal observation: a new parameter

▶ A framework: the Abstract Numeration System model

▶ A general working hypothesis: Prefix-Closed Extendable Languages

▶ An essential parameter: the local growth rate

Proposition

\[
\text{If } \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} \text{cp}_L(i) \quad \text{exists,}
\]
What we learn from the primal observation: a new parameter

- A framework: the Abstract Numeration System model
- A general working hypothesis: Prefix-Closed Extendable Languages
- An essential parameter: the local growth rate

Proposition

If \( CP_L = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} cp_L(i) \) exists,

then the local growth rate \( \lim_{\ell \to \infty} \frac{u_L(\ell + 1)}{u_L(\ell)} = \gamma_L \) exists.
What we learn from the primal observation: a new parameter

- A framework: the Abstract Numeration System model
- A general working hypothesis: Prefix-Closed Extendable Languages
- An essential parameter: the local growth rate

Proposition

If \( CP_L = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} cp_L(i) \) exists,

then the local growth rate \( \lim_{\ell \to \infty} \frac{u_L(\ell + 1)}{u_L(\ell)} = \gamma_L \) exists

and \( CP_L = \frac{\gamma_L}{\gamma_L - 1} \)
What we learn from the primal observation: a new parameter

\[ L_2 \lim_{\ell \to \infty} \frac{u_2(\ell + 1)}{u_2(\ell)} = \gamma_2 = 2 \]
What we learn from the primal observation: a new parameter

\[ L_F \lim_{\ell \to \infty} \frac{u_F(\ell + 1)}{u_F(\ell)} = \gamma_F = \varphi \]
What we learn from the primal observation: a new parameter

\[ \frac{u_O(2\ell + 1)}{u_O(\ell)} = 1 \quad \frac{u_O(2\ell + 2)}{u_O(2\ell + 1)} = 4 \]
What we learn from the primal observation: a new parameter

\[ V, \quad \mathbf{u}_V(\ell) = 32^{\ell-1}, \quad \gamma_V = 2 \]
A natural question

Proposition

If \( CP_L = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} c_p_L(i) \) exists,

then the local growth rate \( \lim_{\ell \to \infty} \frac{u_L(\ell + 1)}{u_L(\ell)} = \gamma_L \) exists

and \( CP_L = \frac{\gamma_L}{\gamma_L - 1} \)
A natural question

Proposition

If \( CP_L = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} c\mu_L(i) \) exists,

then the local growth rate \( \lim_{\ell \to \infty} \frac{u_L(\ell + 1)}{u_L(\ell)} = \gamma_L \) exists

and \( CP_L = \frac{\gamma_L}{\gamma_L - 1} \)

Question

Is the existence of the local growth rate sufficient to insure the existence of the carry propagation?
$U \subseteq \{a, b, c\}^*$

$u_U(\ell) = 2^\ell$

$\frac{u_U(\ell + 1)}{u_U(\ell)} = \gamma_U = 2$
An unbalanced tree

\[ U \subseteq \{a, b, c\}^* \]
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\[
\lim_{\ell \to \infty} \frac{1}{v_U(\ell)} \sum_{j=0}^{v_U(\ell)-1} c_{p_U(j)} = 2
\]
An unbalanced tree

$U \subseteq \{a, b, c\}^*$
An unbalanced tree

$U \subseteq \{a, b, c\}^*$
An unbalanced tree

\[ U \subseteq \{a, b, c\}^* \]
An unbalanced tree

$$U \subseteq \{a, b, c\}^*$$
An unbalanced tree

\[ U \subseteq \{a, b, c\}^* \]

\[
\lim_{\ell \to \infty} \frac{1}{v'_U(\ell)} \sum_{j=0}^{v'_U(\ell)-1} cp_U(j) = \frac{11}{6} \neq 2
\]
An unbalanced tree

\[ U \subseteq \{a, b, c\}^* \]

\[ \text{CP}_U = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} \text{cp}_U(i) \text{ does not exist} \]
The *existence* of the carry propagation is more difficult to prove than the *computation* of the carry propagation itself.
Roadmap
Part III

Algebra
\[
\mathbf{u}_\mathcal{V}(\ell) = 3 \cdot 2^{\ell-1}
\]
Surprise!

\[ u_V(\ell) = 3 \cdot 2^{\ell-1} \]

\[
\liminf_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} cp_V(i) \leq \frac{28}{15} < \frac{13}{6} \leq \limsup_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} cp_V(i)
\]
\( u_V(\ell) = 3 \cdot 2^{\ell-1} \)

\[
\liminf_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} cp_V(i) \leq \frac{28}{15} < \frac{13}{6} \leq \limsup_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} cp_V(i)
\]

\( \gamma_V \) exists but \( CP_V \) does not exists
Generating functions

Definition

$L \subseteq A^*$

$g_L(z)$ generating function of $L$

$$g_L(z) = \sum_{\ell=0}^{\infty} u_L(\ell) z^\ell$$
Generating functions

Definition

\( L \subseteq A^* \quad g_L(z) \) generating function of \( L \)

\[ g_L(z) = \sum_{\ell=0}^{\infty} u_L(\ell) z^\ell \]

\( L \) rational language \( \Rightarrow \) \( g_L(z) \) rational function

\[ g_L(z) = \frac{R(z)}{Q(z)} \]

\( R(z) \), \( Q(z) \) \( \in \mathbb{Z}[z] \)
Generating functions of rational languages

$L$ rational language $\implies g_L(z)$ rational function

g_L(z) \text{ uniquely written as }

g_L(z) = T(z) + \frac{S(z)}{Q(z)} \quad T(z), S(z), Q(z) \in \mathbb{Z}[z]

\text{with } \deg S < \deg Q \text{ and } Q(0) \neq 0
Generating functions of rational languages

$L$ rational language $\implies g_L(z)$ rational function

$g_L(z)$ uniquely written as

$$g_L(z) = T(z) + \frac{S(z)}{Q(z)}$$

$T(z), S(z), Q(z) \in \mathbb{Z}[z]$ with $\deg S < \deg Q$ and $Q(0) \neq 0$

$P_L$ is the *reciprocal polynomial* of $Q$:

$$P_L(z) = Q\left(\frac{1}{z}\right) z^{\deg Q}$$
Generating functions of rational languages

$L$ rational language $\implies g_L(z)$ rational function

\[ g_L(z) \text{ uniquely written as} \]

\[ g_L(z) = T(z) + \frac{S(z)}{Q(z)} \]

with \( \deg S < \deg Q \) and \( Q(0) \neq 0 \)

\( P_L \) is the \textit{reciprocal polynomial} of \( Q : P_L(z) = Q\left(\frac{1}{z}\right)z^{\deg Q} \)

The \textit{eigenvalues} of \( L \) are the zeroes \( \lambda_1, \lambda_2, \ldots, \lambda_t \) of \( P_L \) and

\[ \forall \ell \in \mathbb{N} \quad u_L(\ell) = \sum_{j=1}^{t} \lambda_j^\ell P_j(\ell) \]

where \( \deg P_j = \text{multiplicity of } \lambda_j \text{ in } P_L \) minus 1
Positive rational functions

Theorem (Berstel 71)

\( f(z) \) \( \mathbb{R}_+ \)-rational function (not a polynomial)

\( \lambda \) maximum of the moduli of its eigenvalues.

(i) \( \lambda \) is an eigenvalue of \( f(z) \) (hence an eigenvalue in \( \mathbb{R}_+ \))

(ii) Every eigenvalue of \( f(z) \) of modulus \( \lambda \)

is of the form \( \lambda e^{i\theta} \), where \( e^{i\theta} \) is a root of the unity

(iii) The multiplicity of any eigenvalue of modulus \( \lambda \)

is at most that of \( \lambda \)
Theorem (Berstel 71)

\( f(z) \) is a rational function (not a polynomial)
\( \lambda \) maximum of the moduli of its eigenvalues.

(i) \( \lambda \) is an eigenvalue of \( f(z) \) (hence an eigenvalue in \( \mathbb{R}_+ \)).

(ii) Every eigenvalue of \( f(z) \) of modulus \( \lambda \) is of the form \( \lambda e^{i\theta} \), where \( e^{i\theta} \) is a root of the unity.

(iii) The multiplicity of any eigenvalue of modulus \( \lambda \) is at most that of \( \lambda \).

Definition

(i) \( f(z) \) is DEV if \( \lambda \) is the only eigenvalue of modulus \( \lambda \).

(ii) \( f(z) \) is ADEV if the multiplicity of \( \lambda \) is greater than the multiplicity of the other eigenvalues of modulus \( \lambda \).
Some examples

\[ M_O = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix} \quad \text{P}_O = X^2 - 4 \]

\[ u_O(\ell) = \frac{3}{4} 2^\ell + \frac{1}{4} (-2)^\ell \]

\[ M_C = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \quad \text{P}_C = X^2 - 4 \]

\[ u_C(\ell) = 2^\ell \quad \text{P}_C = X - 2 \]

\[ M_D = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{P}_D = (X^2 - 4)(2 - X) \]

\[ u_D(\ell) = \left( \frac{1}{4} \ell + \frac{7}{8} \right) 2^\ell + \frac{1}{8} (-2)^\ell \]
Some examples

- $O$ is neither DEV nor ADEV
  
  \[ u_O(\ell) = \frac{3}{4} 2^\ell + \frac{1}{4} (-2)^\ell \]

- $V$ is DEV
  
  \[ u_V(\ell) = \frac{3}{2} 2^\ell \]

- $D$ is ADEV but not DEV
  
  \[ u_D(\ell) = \left( \frac{1}{4} \ell + \frac{7}{8} \right) 2^\ell + \frac{1}{8} (-2)^\ell \]
Theorem
A rational language $L$ is ADEV iff the local growth rate $\gamma_L$ exists. In this case, the modulus of $L$ is equal to $\gamma_L$. 
Theorem

Let $\lambda$ be rational PCE and $\lambda$ its modulus.

If every quotient of $L$ whose modulus is equal to $\lambda$ is ADEV, then $CP_L$ exists and $CP_L = \frac{\lambda}{\lambda - 1}$.
Theorem

$L$ is ADEV rational PCE and $\lambda$ its modulus.

If every quotient of $L$ whose modulus is equal to $\lambda$ is ADEV, then $\text{CP}_L$ exists and $\text{CP}_L = \frac{\lambda}{\lambda - 1}$.
Part IV

Ergodic Theory
An unmistakable fit

Our problem

Does \( \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} c_p L(i) \) exist?
An unmistakable fit

Our problem

Does \( \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} cp_L(i) \) exist?

A rewriting

Does \( \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} cp_L(Succ^i_L(\varepsilon)) \) exist?
An unmistakable fit

Our problem

Does \( \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} \text{cp}_L(\text{Succ}_L^i(\varepsilon)) \) exist?
An unmistakable fit

Our problem

Does \( \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} c p_L(Succ^i_L(\varepsilon)) \) exist?

The Ergodic Theorem

Theorem (Birkhoff 31)

Let \((\mathcal{K}, \tau)\) be a dynamical system, \(\mu\) a \(\tau\)-invariant measure on \(\mathcal{K}\) and \(f : \mathcal{K} \to \mathbb{R}\) in \(L^1(\mu)\) (\(f\) is absolutely \(\mu\)-integrable).

If \((\mathcal{K}, \tau)\) is ergodic, then, for \(\mu\)-almost all \(s\) in \(\mathcal{K}\),

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} f(\tau^i(s)) = \int_{\mathcal{K}} f \, d\mu .
\] (*)

If \((\mathcal{K}, \tau)\) is uniquely ergodic and if \(f\) and \(\tau\) are continuous,
then (*) holds for every \(s\) in \(\mathcal{K}\).
A bunch of definitions
A bunch of definitions

- Dynamical system \((\mathcal{K}, \tau)\) = compact set \(\mathcal{K}\) equipped with \(\tau: \mathcal{K} \rightarrow \mathcal{K}\)
A bunch of definitions

- Dynamical system $(\mathcal{K}, \tau) = \text{compact set } \mathcal{K}$ equipped with $\tau : \mathcal{K} \to \mathcal{K}$

- Probability measure $\mu$ on $\mathcal{K}$ is $\tau$-invariant if $\tau$ measurable and $\forall B$ measurable, $\mu(\tau^{-1}(B)) = \mu(B)$
A bunch of definitions

- Dynamical system $(\mathcal{K}, \tau) = \text{compact set } \mathcal{K}$ equipped with $\tau: \mathcal{K} \to \mathcal{K}$

- Probability measure $\mu$ on $\mathcal{K}$ is $\tau$-invariant if $\tau$ measurable and $\forall B$ measurable, $\mu(\tau^{-1}(B)) = \mu(B)$

- $(\mathcal{K}, \tau)$ is ergodic if $\tau^{-1}(B) = B$ implies $\mu(B) = 0$ or 1 for every $\tau$-invariant measure $\mu$
A bunch of definitions

- **Dynamical system** $(K, \tau) = \text{compact set } K$ equipped with $\tau: K \to K$

- **Probability measure** $\mu$ on $K$ is $\tau$-invariant if $\tau$ measurable and $\forall B$ measurable, $\mu(\tau^{-1}(B)) = \mu(B)$

- $(K, \tau)$ is ergodic if $\tau^{-1}(B) = B$ implies $\mu(B) = 0$ or 1 for every $\tau$-invariant measure $\mu$

- $(K, \tau)$ is uniquely ergodic if it admits a unique $\tau$-invariant measure
A bunch of definitions

- Dynamical system \((\mathcal{K}, \tau)\) = compact set \(\mathcal{K}\) equipped with \(\tau: \mathcal{K} \to \mathcal{K}\)

- Probability measure \(\mu\) on \(\mathcal{K}\) is \(\tau\)-invariant if \(\tau\) measurable and \(\forall B\) measurable, \(\mu(\tau^{-1}(B)) = \mu(B)\)

- \((\mathcal{K}, \tau)\) is ergodic if \(\tau^{-1}(B) = B\) implies \(\mu(B) = 0\) or \(1\) for every \(\tau\)-invariant measure \(\mu\)

- \((\mathcal{K}, \tau)\) is uniquely ergodic if it admits a unique \(\tau\)-invariant measure

Theorem (Birkhoff 31)

Let \((\mathcal{K}, \tau)\) be a dynamical system, \(\mu\) a \(\tau\)-invariant measure on \(\mathcal{K}\) and \(f: \mathcal{K} \to \mathbb{R}\) in \(L^1(\mu)\). If \((\mathcal{K}, \tau)\) is ergodic, then

for \(\mu\)-almost all \(s \in \mathcal{K}\) \(\lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} f(\tau^i(s)) = \int_{\mathcal{K}} f \, d\mu \quad (*\))

If \((\mathcal{K}, \tau)\) is uniquely ergodic and if \(f\) and \(\tau\) are continuous, then \((*)\) holds for every \(s\) in \(\mathcal{K}\).

Turning a numeration system into a dynamical system
Turning a numeration system into a dynamical system

- \[ A = \{0, 1, \ldots, r - 1\} \]
Turning a numeration system into a dynamical system

- \( A = \{0, 1, \ldots, r - 1\} \)

- \( L \subseteq (A \setminus \{0\}) A^* \): no word of \( L \) ‘begins’ with 0
Turning a numeration system into a dynamical system

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- \( L \subseteq (A \setminus \{0\}) A^* \): no word of \( L \) ‘begins’ with 0

- \( \omega A \) = set of left infinite words over \( A \)

- \( s = \cdots s_2 s_1 s_0 \) and \( s_{[\ell, j]} = s_{\ell} s_{\ell - 1} \cdots s_j \)

- \( w \mapsto \omega 0 w \) induces a bijection between \( L \) and \( \omega 0 L \)
Turning a numeration system into a dynamical system

- $A = \{0, 1, \ldots, r - 1\}$

- $L \subseteq (A \setminus \{0\}) A^* : \text{no word of } L \text{ ‘begins’ with } 0$

- $\omega A = \text{set of left infinite words over } A$
  \[ s = \cdots s_2 s_1 s_0 \quad \text{and} \quad s[\ell, j] = s_\ell s_{\ell-1} \cdots s_j \]
  \[ w \mapsto \omega 0 w \text{ induces a bijection between } L \text{ and } \omega 0 L \]

- $\omega A$ with the ‘right factor distance’ topology is a \textit{compact set}
  
  \textbf{Basis: cylinders} \quad [w] = \omega A w$
Turning a numeration system into a dynamical system

- \( A = \{0, 1, \ldots, r - 1\} \)

- \( L \subseteq (A \setminus \{0\}) A^* : \) no word of \( L \) ‘begins’ with 0

- \( \omega A = \) set of left infinite words over \( A \)
  \[
  s = \cdots s_2 s_1 s_0 \quad \text{and} \quad s[\ell, j] = s_\ell s_{\ell - 1} \cdots s_j
  \]
  \( w \mapsto \omega 0 w \) induces a bijection between \( L \) and \( \omega 0 L \)

- \( \omega A \) with the ‘right factor distance’ topology is a compact set
  Basis: cylinders \( [w] = \omega A w \)

- Compactification of \( L \) : \( \mathcal{K}_L = \overline{\omega 0 L} \)
  \[
  \mathcal{K}_L = \left\{ s \in \omega A \mid \forall j \in \mathbb{N} \quad \exists w^{(j)} \in 0^* L \quad s[j, 0] \text{ right factor of } w^{(j)} \right\}
  \]
Turning a numeration system into a dynamical system

Definition of the odometer
Definition of the odometer

- Since \( L \subseteq (A \setminus \{0\})A^* \), \( \text{Succ}_L \) defined on \( \omega_0 L \) by
  \[
  \text{Succ}_L(\omega_0 w) = \omega_0 \text{Succ}_L(w)
  \]
Turning a numeration system into a dynamical system

Definition of the odometer

- Since \( L \subseteq (A \setminus \{0\}) A^* \), \( \text{Succ}_L \) defined on \( \omega 0 L \) by
  \[ \text{Succ}_L(\omega 0 w) = \omega 0 \text{Succ}_L(w) \]

- **Odometer** \( \tau_L \) on \( K_L = \) any extension of \( \text{Succ}_L \)
Turning a numeration system into a dynamical system

Definition of the odometer

- Since \( L \subseteq (A \setminus \{0\}) A^* \), \( \text{Succ}_L \) defined on \( \omega \in L \) by
  \[
  \text{Succ}_L(\omega w) = \omega \text{Succ}_L(w)
  \]
- **Odometer** \( \tau_L \) on \( \mathcal{K}_L \) = any extension of \( \text{Succ}_L \)
- \( \text{Succ}_L \) continuous \( \implies \)
  \( \tau_L \) unique continuous extension of \( \text{Succ}_L \)
Turning a numeration system into a dynamical system

Definition of the odometer

- Since $L \subseteq (A \setminus \{0\}) A^*$, $\text{Succ}_L$ defined on $\omega 0 L$ by
  
  $\text{Succ}_L(\omega 0 w) = \omega 0 \text{Succ}_L(w)$

- **Odometer** $\tau_L$ on $\mathcal{K}_L = \text{any extension of } \text{Succ}_L$

- $\text{Succ}_L$ continuous $\Rightarrow \tau_L$ unique continuous extension of $\text{Succ}_L$

Extension of the carry propagation
Turning a numeration system into a dynamical system

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- Since \( L \subseteq (A \setminus \{0\}) A^* \), \( \text{Succ}_L \) defined on \( \omega 0 L \) by
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  \]
- **Odometer** \( \tau_L \) on \( \mathcal{K}_L = \text{any extension of} \ \text{Succ}_L 
  \)
- **Succ}_L continuous} \ \Rightarrow \ \tau_L \ \text{unique continuous extension of} \ \text{Succ}_L 
  \]

Extension of the carry propagation

\[
\Delta(s, t) = \begin{cases} 
\min \{j \in \mathbb{N} \mid s_{[\infty,j]} = t_{[\infty,j]}\} & \text{if such } j \text{ exist} \\
+\infty & \text{otherwise}
\end{cases}
\]
Turning a numeration system into a dynamical system

Definition of the odometer

- Since $L \subseteq (A \setminus \{0\})A^*$, $\text{Succ}_L$ defined on $\omega^0 L$ by
  $\text{Succ}_L(\omega^0 w) = \omega^0 \text{Succ}_L(w)$

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Extension of the carry propagation

$$\Delta(s, t) = \begin{cases} 
\min \{ j \in \mathbb{N} \mid s_{[\infty,j]} = t_{[\infty,j]} \} & \text{if such } j \text{ exist} \\
+\infty & \text{otherwise}
\end{cases}$$

$$\forall s \in \omega A \quad \text{cp}_L(s) = \Delta(s, \tau_L(s))$$
Turning a numeration system into a dynamical system

Definition of the odometer

- Since $L \subseteq (A \setminus \{0\})A^*$, $\text{Succ}_L$ defined on $\omega_0L$ by $\text{Succ}_L(\omega_0w) = \omega_0 \text{Succ}_L(w)$

- Odometer $\tau_L$ on $\mathcal{K}_L$ = any extension of $\text{Succ}_L$

- $\text{Succ}_L$ continuous $\implies \tau_L$ unique continuous extension of $\text{Succ}_L$

Extension of the carry propagation

Proposition

If $\tau_L$ is continuous,
then $\text{cp}_L$ is continuous at any point where it takes finite values.
Turning a numeration system into a dynamical system

Definition of the odometer

- Since \( L \subseteq (A \setminus \{0\}) A^* \), \( \text{Succ}_L \) defined on \( \omega 0 L \) by \( \text{Succ}_L(\omega 0 w) = \omega 0 \text{Succ}_L(w) \)
- **Odometer** \( \tau_L \) on \( \mathcal{K}_L = \) any extension of \( \text{Succ}_L \)
- \( \text{Succ}_L \) continuous \( \implies \) \( \tau_L \) unique continuous extension of \( \text{Succ}_L \)

Where we are

We write \( 0 = \omega 0 \)

\[
\text{CP}_L = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} \text{cp}_L(\tau^i_L(0))
\]
Turning a numeration system into a dynamical system

Definition of the odometer

- Since \( L \subseteq (A \setminus \{0\})A^* \), \( \text{Succ}_L \) defined on \( \omega 0 L \) by
  \[
  \text{Succ}_L(\omega 0 w) = \omega 0 \text{Succ}_L(w)
  \]

- **Odometer** \( \tau_L \) on \( \mathcal{K}_L = \) any extension of \( \text{Succ}_L \)

- \( \text{Succ}_L \) continuous \( \implies \) \( \tau_L \) unique continuous extension of \( \text{Succ}_L \)

Where Birkhoff Theorem leads us

We want to show that \( 0 \) is a point such that

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} \text{cp}_L(\tau^i_L(0)) = \int_{\mathcal{K}_L} \text{cp}_L \, d\mu
\]
Greedy numeration systems
Greedy numeration systems

- **Basis** = strictly increasing sequence of integers

\[ G = (G_\ell)_{\ell \in \mathbb{N}} \text{ with } G_0 = 1 \]
Greedy numeration systems

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- *Greedy algorithm* yields *greedy G-expansion* of integer \( n \)
Greedy numeration systems

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- **Greedy algorithm** yields **greedy G-expansion** of integer \( n \)
  
  - \( k \) defined by \( G_k \leq n < G_{k+1} \)
  
  set \( x_k = q(n, G_k) \) and \( r_k = r(n, G_k) \)
Greedy numeration systems

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  - for every \( i \), \( k - 1 \geq i \geq 0 \), set
  
    \[ x_i = q(r_{i+1}, G_i) \text{ and } r_i = r(r_{i+1}, G_i) \]
**Greedy numeration systems**

- **Basis** = strictly increasing sequence of integers
  \[ G = (G_\ell)_{\ell \in \mathbb{N}} \text{ with } G_0 = 1 \]

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    set \( x_k = q(n, G_k) \) and \( r_k = r(n, G_k) \)
  
  - for every \( i \), \( k - 1 \geq i \geq 0 \), set
    
    \( x_i = q(r_{i+1}, G_i) \) and \( r_i = r(r_{i+1}, G_i) \)
  
  - \( n = x_k G_k + x_{k-1} G_{k-1} + \cdots + x_0 G_0 \)
Greedy numeration systems

- **Basis** = strictly increasing sequence of integers
  \[ G = (G_\ell)_{\ell \in \mathbb{N}} \text{ with } G_0 = 1 \]

- **Greedy algorithm** yields **greedy G-expansion** of integer \( n \)
  
  - \( k \) defined by \( G_k \leq n < G_{k+1} \)
    set \( x_k = q(n, G_k) \) and \( r_k = r(n, G_k) \)
  
  - for every \( i \), \( k - 1 \geq i \geq 0 \), set
    \( x_i = q(r_{i+1}, G_i) \) and \( r_i = r(r_{i+1}, G_i) \)

  - \( n = x_k G_k + x_{k-1} G_{k-1} + \cdots + x_0 G_0 \)

- \( L_G = \{ \langle n \rangle_G \mid n \in \mathbb{N} \} \)
Greedy numeration systems

- **Basis** = strictly increasing sequence of integers
  \[ G = (G_\ell)_{\ell \in \mathbb{N}} \text{ with } G_0 = 1 \]

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- If \[ r = \limsup \left\lceil \frac{G_{\ell+1}}{G_\ell} \right\rceil \] is finite
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▶ \( 0^*L_G \) is closed under right factor and
\[ \mathcal{K}_G = \overline{0^*L_G} = \{ s \in \omega A \mid \forall j \in \mathbb{N} \quad s_{[j,0]} \in 0^*L_G \} \]
Ergodicity of greedy numeration systems

Let $G$ be a GNS.

For every $s$ in $\mathcal{K}_G$, $\lim_{j \to \infty} \text{Succ}_G(s[j,0])$ exists and defines the odometer $\tau_G : \mathcal{K}_G \to \mathcal{K}_G$:

$$\forall s \in \mathcal{K}_G \quad \tau_G(s) = \lim_{j \to \infty} \text{Succ}_G(s[j,0]).$$
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Theorem (Barat–Grabner 16, Grabner–Liardet–Tichy 95)

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Definition

A GNS $G$ is said to be exponential if there exist two real constants $\alpha > 1$ and $C > 0$ such that $G_\ell \sim C \alpha^\ell$ when $\ell$ tends to infinity.
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Theorem (Barat–Downarowicz–Liardet 02)

If $G$ is an exponential GNS, then the dynamical system $(\mathcal{K}_G, \tau_G)$ is uniquely ergodic.
Theorem

If $G$ is an exponential GNS, then $CP_G$ exists.
Theorem

If $G$ is an exponential GNS, then $\text{CP}_G$ exists.

Corollary

Let $G$ be an exponential GNS with $G_\ell \sim C \alpha^\ell$.

If $L_G$ is PCE, then $\text{CP}_G$ exists and $\text{CP}_G = \frac{\alpha}{\alpha - 1}$.
Theorem

If \( G \) is an exponential GNS, then \( \text{CP}_G \) exists.
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Proof essentially based on the work [Barat–Grabner 16]
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\( G \) exponential GNS implies

\[ \mathcal{K}_G, \tau_G \] is uniquely ergodic \( \mu_G \) the \( \tau_G \)-invariant measure.
Carry propagation in greedy numeration systems

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- $(\mathcal{K}_G, \tau_G)$ is uniquely ergodic $\mu_G$ the $\tau_G$-invariant measure.
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$$\lim_{N \to \infty} \frac{1}{N} \text{card} \left( \{ i \mid \tau_G^i(0) \in [w] \} \right) = \mu_G([w])$$
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$$\lim_{N \to \infty} \frac{1}{N} \text{card} (\{i \mid \tau_G^i(0) \in [w]\}) = \mu_G([w]) = \int_{\mathcal{K}_G} \chi_{[w]} \, d\mu_G$$
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$G$ exponential GNS implies

1. $(\mathcal{K}_G, \tau_G)$ is uniquely ergodic $\mu_G$ the $\tau_G$-invariant measure.
2. 0 is a generic point

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} \chi_w(\tau_G^i(0)) = \mu_G([w]) = \int_{\mathcal{K}_G} \chi_w \, d\mu_G
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\]

- Implies $\lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} f(\tau_G^i(0)) = \int_{\mathcal{K}_G} f \, d\mu_G$ for Riemann-integrable $f$
Carry propagation in greedy numeration systems

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- Does not imply $$\lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} f(\tau_G^i(0)) = \int_{\mathcal{K}_G} f \, d\mu_G$$ for any $f$
Theorem

If $G$ is an exponential GNS, then $\text{CP}_G$ exists.

How the proof goes
Theorem

If $G$ is an exponential GNS, then $\text{CP}_G$ exists.

How the proof goes

- $\text{cp}_G$ is not \textit{Riemann-integrable}
  
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How the proof goes

- $cp_G$ is not Riemann-integrable
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- $\forall k \quad f_k(s) = \begin{cases} 
    cp_G(s) & \text{if } cp_G(s) \leq k + 1 \\
    0 & \text{otherwise}
  \end{cases}$
Carry propagation in greedy numeration systems

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- $\forall k \quad f_k(s) = \begin{cases} \text{cp}_G(s) & \text{if } \text{cp}_G(s) \leq k + 1 \\ 0 & \text{otherwise} \end{cases}$

- $\int_{K_G} \text{cp}_G \, d\mu_G$ exists and $\lim_{k \to \infty} \int_{K_G} f_k \, d\mu_G = \int_{K_G} \text{cp}_G \, d\mu_G$
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- \( \text{cp}_G \) is in \( L^1(\mu_G) \)
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- \( \forall N < G_{k+1} \)
  \[
  \sum_{i=0}^{N-1} f_k(\tau^i_G(0)) = \sum_{i=0}^{N-1} f_{k-1}(\tau^i_G(0)) + \left\lfloor \frac{N}{G_k} \right\rfloor (k+1)
  \]
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- Key result
  \[ \forall N \in \mathbb{N} \quad \sum_{i=0}^{N-1} f_k(\tau_G^i(0)) = \sum_{i=0}^{N-1} f_{k-1}(\tau_G^i(0)) + \left\lfloor \frac{N}{G_k} \right\rfloor (k+1) \]
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- $\int_{K_G} f_k d\mu_G \leq \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} \text{cp}_G(\tau^i_G(0)) \leq \int_{K_G} f_k d\mu_G + M_{k+1}$
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Comme il y a une infinité de choses sages
qui sont menées de manière très folle,
il y a aussi des folies qui sont menées de manière très sage.

Montesquieu

Just as wise ends are oftentimes sought
in the most foolish way,
so foolishness is sometimes sought with great wisdom.

Translation by Reuben Thomas