THETA DIVISORS AND PERMUTOHEDRA

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ABSTRACT. We establish a surprising relation of the smooth theta-divisor $\Theta_n$ with permutohedron $\Pi^n$ and the corresponding toric variety $X^n_{\Pi}$. In particular, we show that the generalised Todd genus of the theta divisor $\Theta_n$ coincides with $h$-polynomial of permutohedron $\Pi^n$ and thus is different from the same genus of $X^n_{\Pi}$ only by the sign $(-1)^n$. As an application we find all the Hodge numbers of the theta divisors in terms of the Eulerian numbers. We reveal also interesting numerical relations between theta-divisors and Tomei manifolds from the theory of the integrable Toda lattice.

1. INTRODUCTION

Recently we have shown that the smooth theta divisors $\Theta^n \subset A^{n+1}$ of general principally polarised abelian varieties $A^{n+1}$ play an important role in the theory of complex cobordisms [4]. Namely, we proved that $\Theta^n$ can be chosen as irreducible algebraic representatives of the coefficients of the Chern-Dold character in complex cobordisms and described the action of the Landweber-Novikov operations on them in terms of the corresponding smooth intersections

$$\Theta^{n-k}_k = \Theta^n \cap \Theta^n(a_1) \cap \ldots \Theta^n(a_k)$$

of $\Theta^n$ with $k$ general translates $\Theta^n(a_i), a_i \in A^{n+1}$ of the theta divisor $\Theta^n$.

The aim of this paper is to establish a link of the theta-divisor $\Theta^n$ with combinatorics of permutohedron $\Pi^n$ and the corresponding permutohedral toric variety $X^n_{\Pi}$, which we found very intriguing.

Recall that permutohedron $\Pi^n$ is a simple $n$-dimensional lattice polytope, which we choose to be the standard one defined as the convex hull of the points $\sigma(\rho) \in \mathbb{R}^{n+1}, \sigma \in S_{n+1}, \rho = (1, 2, \ldots, n, n+1)$.

Our first result computes the Todd genus of $\Theta^{n-k}_k$ in terms of combinatorics of the permutohedron.

**Theorem 1.1.** The Todd genus of the intersection of $k$ theta divisors

$$Td(\Theta^{n-k}_k) = (-1)^{n-k}f_{n-k}(\Pi^n)$$

up to a sign coincides with the number $f_{n-k}(\Pi^n)$ of the codimension $k$ faces of permutohedron $\Pi^n$. 

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Since it is known that \( f_{n-k}(\Pi^n) = (k+1)!S(n+1, k+1) \), where \( S(n, k) \) are the Stirling numbers of second kind \[33\], we have the formula

\[
Td(\Theta^n_{k}) = (-1)^{n-k}(k+1)!S(n+1, k+1).
\]

Our second result reveals the relation of the two-parameter Todd genus \( Td_{s,t} \) of theta divisor \( \Theta^n \) with the \( h \)-polynomial of permutohedron \( \Pi^n \).

Recall that the \( h \)-polynomial \( h_{P^n}(s, t) \) of \( n \)-dimensional simple polytope \( P^n \) is related to \( f \)-polynomial \( f_{P^n}(s, t) = \sum_{k=0}^{n} f_{n-k}(P^n)s^{n-k}t^k \) by simple change

\[
h_{P^n}(s, t) := f_{P^n}(s - t, t).
\]

The two-parameter Todd genus \( Td_{s,t} \) is a homogeneous version of the Hirzebruch \( \chi_y \)-genus introduced by Krichever \[25\]. It corresponds to the generating series

\[
Q(x) = \frac{x e^{tx} - te^{sx}}{e^{sx} - e^{tx}}.
\]

When \( s = y, t = -1 \) it reduces to the \( \chi_y \)-genus \[23\].

**Theorem 1.2.** The two-parameter Todd genus \( Td_{s,t}(\Theta^n) \) of the theta divisor \( \Theta^n \) coincides with the \( h \)-polynomial of permutohedron \( \Pi^n \):

\[
(3) \quad Td_{s,t}(\Theta^n) = h_{\Pi^n}(s, t) = \sum_{k=0}^{n} A_{n+1,k} s^k t^{n-k},
\]

where \( A_{n+1,k} \) are the classical Eulerian numbers. In particular, the \( \chi_y \) genus

\[
(4) \quad \chi_y(\Theta^n) = h_{\Pi^n}(y, -1) = (-1)^nA_{n+1}(-y),
\]

where \( A_n(y) \) are the classical Eulerian polynomials.

Recall that the Eulerian number \( A_{n,k} \) is the number of permutations from \( S_n \) with \( k \) descents, which can be computed recursively, see \[33\] and section 3 below.

As an application we compute all the Hodge numbers \( h^{p,q}(\Theta^n) \).

**Theorem 1.3.** The Hodge numbers \( h^{p,q} \) of theta divisor \( \Theta^n \) with \( p + q \neq n \) are given explicitly by

\[
h^{p,q}(\Theta^n) = h^{n-p,n-q}(\Theta^n) = \binom{n+1}{p} \binom{n+1}{q}, \quad p + q \leq n - 1.
\]

When \( p + q = n \) we have \( h^{p,n-p}(\Theta^n) = A_{n+1,p} - S_{n,p} \), where \( A_{n,p} \) are the Eulerian numbers and

\[
S_{n,p} = (-1)^p \binom{n+2}{p+1} \left[ (-1)^p \frac{2p-n}{n+2} \binom{n+1}{p} + \sum_{k=0}^{p-1} (-1)^k \binom{n+1}{k} \right].
\]

In particular,

\[
h^{0,n}(\Theta^n) = n + 1, \quad h^{1,n-1}(\Theta^n) = 2^{n+1} - (n + 2) + \frac{n^2(n+1)}{2}.
\]
The explicit forms of the Hodge diamonds of $\Theta^n$ for $n = 2, 3, 4$ are shown in Section 4 below.

We establish also an interesting duality between theta divisor $\Theta^n$ and the permutohedral variety $X^n_{\Pi}$, which is the toric variety determined by $\Pi^n$ [18].

**Theorem 1.4.** The Betti number $b_{2k}(X^n_{\Pi})$ of the permutohedral variety coincides up to a sign with the Hirzebruch $\chi^k$-genus of the theta divisor $\Theta^n$:

$$b_{2k}(X^n_{\Pi}) = (-1)^{n-k}\chi^k(\Theta^n).$$

The same is true for the two-parameter Todd genus of these two varieties:

$$Td_{s,t}(X^n_{\Pi}) = (-1)^nTd_{s,t}(\Theta^n).$$

This suggests that the corresponding cobordism classes might be related by $[X^n_{\Pi}] = (-1)^n[\Theta^n]$. This indeed works for $n \leq 2$, but we show that already for $n = 3$ this is not the case. In fact we provide a formula expressing the cobordism class $[X^n_{\Pi}]$ in terms of the theta divisors (see Theorem 5.2 below).

In the rest of the paper we discuss the connection of $\Theta^n$ and $X^n_{\Pi}$ with two other manifolds appeared in relation with integrable Toda lattice and known to be related to permutohedra.

The first one is the Tomei manifold $M^n_T$, which is the set of the real symmetric tridiagonal matrices with given spectrum. Tomei [34] used the Toda flows to show that $M^n_T$ can be glued from $2^n$ copies of permutohedron and computed its Euler characteristic $\chi(M^n_T)$ in terms of the Bernoulli numbers. We show, in particular, that $\chi(M^n_T)$ equals the signatures of both $X^n_{\Pi}$ and $\Theta^n$.

We show also that the Hermitian version of Tomei manifold $M^n_{HT}$, studied by Bloch, Flaschka and Ratiu [2], is not diffeomorphic to any symplectic manifold $M^{4n}$ with Hamiltonian action of torus $T^{2n}$ and that $M^n_{HT}$ does not admit any almost complex (and hence, any symplectic) structure.

2. **Todd genera of intersection theta divisors and combinatorics of permutohedra**

We start with the results from [4], which will be used here.

Let $u \in U^2(\mathbb{C}P^\infty)$ and $z \in H^2(\mathbb{C}P^\infty)$ be the first Chern classes of the universal line bundle on $\mathbb{C}P^\infty$ in the complex cobordisms and cohomology theory respectively. The Chern-Dold character is uniquely defined by its action

$$ch_U : u \rightarrow \beta(z), \quad \beta(z) := z + \sum_{n=1}^{\infty} [B^{2n}] \frac{z^{n+1}}{(n+1)!},$$

where $B^{2n}$ are certain $U$-manifolds, characterised by their properties in [7]. In [4] we found an explicit form of this series as

$$\beta(z) = z + \sum_{n=1}^{\infty} [\Theta^n] \frac{z^{n+1}}{(n+1)!},$$

(5)
where \( \Theta^n \) is a smooth theta divisor of a general principally polarised abelian variety \( A_{n+1} \), considered as the complex manifold of real dimension \( 2n \). The cobordism class of the theta divisor does not depend on the choice of such abelian variety provided \( \Theta^n \) is smooth, which is true in generic case.

Let us introduce the generating function of the Todd genera of the intersections of theta divisors as

(6) \[
T_d(\Theta, x, b, t) := \sum_{k,n \geq 0, k \leq n} Td(\Theta^{n-k}) \frac{b^{n-k} t k x^{n+1}}{(n+1)!}.
\]

We can show now that it can be viewed also as the generating function of the \( K \)-theory Chern numbers of theta divisors. Indeed, Conner and Floyd \[12\] constructed the transformation \( \mu_c : U^*(X) \to K^*(X) \) of complex cobordisms to complex \( K \)-theory, related to Riemann-Roch theorem in algebraic geometry \[23\]. When \( X = pt \), we have \( \mu_c : \Omega^*_U \to \mathbb{Z}[b, b^{-1}] \), where \( b \) is the Bott periodicity operator with \( \text{deg} b = -2 \), defined by

(7) \[
\mu_c([M]) = Td(M) b^n.
\]

Using the complex cobordism theory one can define the \( K \)-theory Chern numbers \( c^K_\lambda(M^{2n}) \in \mathbb{Z}[b, b^{-1}] \) of any \( U \)-manifold \( M^{2n} \) as follows

(8) \[
c^K_\lambda(M^{2n}) := Td(S_\lambda[M^{2n}]) b^{n-|\lambda|},
\]

where \( \lambda \) is a partition with \( |\lambda| \leq n \) and \( S_\lambda \) is the Landweber-Novikov operation \[30\]. If \( \lambda = \emptyset \), then \( S_\lambda = Id \) and we have formula (7) for \( c^K_\emptyset = c^K_\emptyset \).

In [4] we have described explicitly the action of the Landweber-Novikov operations on the theta divisors.

**Theorem 2.1.** \([4]\) If \( \lambda \) is not a one-part partition, then \( S_\lambda[\Theta^n] = 0 \), while for \( \lambda = (k), k \leq n \) we have

(9) \[
S_{(k)}[\Theta^n] = [\Theta^{n-k}],
\]

where \( \Theta^{n-k} \) is the intersection of shifted theta divisors \[7\].

In combination with (8) this implies the following result.

**Proposition 2.2.** The generating function of the \( K \)-theory Chern numbers of the theta divisors

(10) \[
K_\Theta(x, t) := \sum_{k,n \geq 0, k \leq n} c^K_{(k)}(\Theta^n) \frac{t k x^{n+1}}{(n+1)!}
\]

coincides with the generating function \( Td_\Theta(x, b, t) \).

Now we give an explicit formula for both these generating functions.

**Theorem 2.3.** The generating functions \( Td(x, b, t) \) and \( K_\Theta(x, t) \) can be given explicitly as

(11) \[
Td_\Theta(x, b, t) = K_\Theta(x, t) = \frac{1 - e^{-bx}}{b - t(1 - e^{-bx})},
\]
Proof. We use the fact that Chern-Dold character $ch_U$ commutes with Landweber-Novikov operations:

\begin{equation}
S(k) \circ ch_U = ch_U \circ S(k)
\end{equation}

(see [7]) and that $S(k)u = u^{k+1}$, where $u \in U^2(\mathbb{C}P^\infty)$ as before is the first Chern class of the universal line bundle on $\mathbb{C}P^\infty$ in the complex cobordisms. Applying this to $u \in U^2(\mathbb{C}P^\infty)$ and using the relations (5) and (9) we have

\[
S(k) \circ ch_U(u) = S(k)(\beta(z)) = \sum_{n \geq 0} S(k)([\Theta^n]) \frac{z^{n+1}}{(n+1)!} = \sum_{n \geq 0} [\Theta^{n-k}] \frac{z^{n+1}}{(n+1)!}.
\]

On the other hand since $ch_U \circ S(k)(u) = ch_U(u^{k+1}) = \beta(z)^{k+1}$, we have

\begin{equation}
\sum_{n \geq 0} [\Theta^{n-k}] \frac{z^{n+1}}{(n+1)!} = \left( \sum_{n \geq 0} [\Theta^n] \frac{z^{n+1}}{(n+1)!} \right)^{k+1}.
\end{equation}

Applying now the Riemann-Roch transformation (7) to both sides of (13) and using the fact that $Td(\Theta^n) = (-1)^n$ (see [4]), we have

\[
\sum_{n \geq 0} Td(\Theta^{n-k}) b^{n-k} \frac{z^{n+1}}{(n+1)!} = \left( \sum_{n \geq 0} (-1)^n b^n \frac{z^{n+1}}{(n+1)!} \right)^{k+1} = \left( 1 - \frac{e^{-bz}}{b} \right)^{k+1}.
\]

Multiplying both sides by $t^k$ and adding over $k \leq n$ we have the relation (11) and the claim. \hfill \Box

Remarkably the same generating function describes the combinatorics of the permutohedron.

Recall that permutohedron $\Pi^n$ is simple convex polytope, which is a convex hull of the points $\sigma(x), \sigma \in S_{n+1}$, being the orbit of the symmetric group $S_{n+1}$, acting on a generic point $x \in \mathbb{R}^{n+1}$, which can be chosen to be $\rho = (1,2,\ldots,n,n+1)$. Its combinatorics is well-studied, see e.g. 20, 32, 35 and references therein.

![Permutohedra in dimension 1,2 and 3.](image)

Figure 1. Permutohedra in dimension 1,2 and 3.
In particular, it is known that the number \( f_{n-k}(\Pi^n) \) of faces of dimension \( n - k \) (or, codimension \( k \)) can be given as
\[
(14) \quad f_{n-k}(\Pi^n) = (k + 1)! S(n + 1, k + 1),
\]
where \( S(n, k) \) are the Stirling numbers of second kind \([33]\). These numbers can be computed recursively:
\[
S(n+1, k) = kS(n, k) + S(n, k-1),
\]
with \( S(0,0) = 1 \) and \( S(n,0) = S(0,n) = 0 \) for \( n > 0 \).

Consider the corresponding \( f \)-polynomial of permutohedron \( \Pi^n \)
\[
(15) \quad f_{\Pi^n}(s, t) := \sum_{k=0}^{n} f_{n-k}(\Pi^n) s^{n-k} t^k,
\]
where \( f_{n-k}(\Pi^n) \) is the number of faces of \( \Pi^n \) of dimension \( n - k \):
\[
\begin{align*}
 f_{\Pi^1}(s, t) &= s + 2t, \\
 f_{\Pi^2}(s, t) &= s^2 + 6ts + 6t^2, \\
 f_{\Pi^3}(s, t) &= s^3 + 14s^2t + 36st^2 + 24t^3, \ldots
\end{align*}
\]
Let
\[
F_{\Pi}(x, s, t) := \sum_{n\geq0} f_{\Pi^n}(s, t) \frac{x^{n+1}}{(n+1)!} = \sum_{k,n\geq0, k\leq n} f_{n-k}(\Pi^n) s^{n-k} t^k \frac{x^{n+1}}{(n+1)!}
\]
be their generation function, which can also be considered as the generating function of the face numbers of all permutohedra.

**Theorem 2.4.** (Buchstaber \([5]\)) The generating function of the face numbers of permutohedra can be given explicitly as
\[
(16) \quad F_{\Pi}(x, s, t) = \frac{e^{sx} - 1}{s - t(e^{sx} - 1)}.
\]

The proof uses the recursive formula for the boundary \( d\Pi^n \) of the permutohedron \( \Pi^n \)
\[
d\Pi^n = \sum_{i+j=n-1} \binom{n+1}{i+1} \Pi^i \Pi^j
\]
(see formula (18), Theorem 17 and Corollary 21 in \([5]\)).

Using this we have the relation
\[
F_{\Pi}(x, -b, t) = \frac{e^{-bx} - 1}{-b - t(e^{-bx} - 1)} = \frac{1 - e^{-bx}}{b - t(1 - e^{-bx})} = Td_{\Theta}(x, b, t)
\]
and the proof of the following result.

**Theorem 2.5.** The Todd generating function \( Td(x, b, t) \) of the intersections of theta divisors coincides with the permutohedral face generating function \( F_{\Pi}(x, s, t) \) after the substitution \( s = -b \):
\[
(17) \quad F_{\Pi}(x, -b, t) = Td_{\Theta}(x, b, t).
\]

In particular, the Todd genus of \( \Theta_k^{n-k} \)
\[
(18) \quad Td(\Theta_k^{n-k}) = (-1)^{n-k} f_{n-k}(\Pi^n)
\]
up to a sign coincides with the number of faces of permutohedron $\Pi^n$ of
codimension $k$.

In particular, using the explicit form of the Stirling numbers $[33]$

$$S(n + 1, n) = \binom{n+1}{2}, \quad S(n + 1, 2) = 2^n - 1,$$

we have

$$Td(\Theta_{n-1}^1) = -n\frac{(n + 1)!}{2}, \quad Td(\Theta_{1}^{n-1}) = (-1)^{n-1}(2^{n+1} - 2),$$

so $\Theta_{n-1}^1$ is a curve of genus

$$g = 1 + \frac{n(n + 1)!}{2}$$
in agreement with $[4]$.

3. The two-parameter Todd genus of theta divisors and
$h$-polynomials of permutohedra

Consider the formal group depending on two parameters $a$ and $b$:

$$x, y \rightarrow F(x, y) = \frac{x + y + axy}{1 - bxy}.$$  \hfill (19)

Its exponential can be given as

$$\beta(x) = \frac{e^{sx} - e^{tx}}{se^{tx} - te^{sx}},$$  \hfill (20)

where parameters $s$ and $t$ are related to $a$ and $b$ as

$$a = s + t, \quad b = st.$$  

When $a = -1, b = 0$ (corresponding to $s = -1, t = 0$) we have the formal
group with the operation

$$x, y \rightarrow F(x, y) = x + y - xy$$

with the exponential

$$\beta(x) = 1 - e^{-x},$$
corresponding to the classical Todd genus $[23]$.

Let $Td_{s,t}$ be the corresponding two-parameter Todd genus, corresponding
to the formal group (19) and consider the exponential generating function
of this genus for the theta divisors:

$$Td_{s,t}^\Theta(x) := \sum_{n \geq 0} Td_{s,t}(\Theta^n) \frac{x^{n+1}}{(n + 1)!}.$$  \hfill (21)

In $[4]$ we have proved that the exponential generating function of any
Hirzebruch genus $\Phi$ of theta divisors:

$$\Phi(\Theta, z) := \sum_{n=0}^\infty \Phi(\Theta^n) \frac{z^{n+1}}{(n + 1)!} = \frac{z}{Q(z)} = \beta(z)$$  \hfill (22)
where \(Q(x)\) is the generating power series of genus \(\Phi\) and \(\beta(x)\) is the exponential \(\beta\) of the corresponding formal group. In particular, in our case we have

\[
Td_{s,t}^\Theta(x) = \frac{e^{sx} - e^{tx}}{se^{tx} - te^{sx}}.
\]

Remarkably the same generating function describes the \(h\)-polynomials of permutohedra.

Recall that \(h\)-polynomial \(h_{P^n}(s,t)\) of \(n\)-dimensional simple polytope \(P^n\) is related to \(f\)-polynomial \(f_{P^n}(s,t)\) by simple change

\[
h_{P^n}(s,t) := f_{P^n}(s-t, t) = \sum_{k=0}^{n} h_{n-k}(P^n)s^{n-k}t^k.
\]

The \(h\)-polynomials are known to be symmetric (Dehn-Sommerville relations):

\[
h_{P^n}(s,t) = h_{P^n}(t,s),
\]

and their coefficients \(h_k(P^n) = h_{n-k}(P^n) = \dim H^{2k}(X^n_P)\) are even Betti numbers of the corresponding toric varieties, see [18].

**Theorem 3.1.** The two-parameter Todd genus \(Td_{s,t}^\Theta(\Theta^n)\) of the theta divisor \(\Theta^n\) coincides with the \(h\)-polynomial of permutohedron \(\Pi^n\):

\[
Td_{s,t}^\Theta(\Theta^n) = h_{\Pi^n}(s,t).
\]

In particular, for the \(\chi_y\)-genus we have

\[
\chi_y(\Theta^n) = (-1)^n A_{n+1}(-y),
\]

where \(A_{n+1}(y)\) is the classical Eulerian polynomial.

**Proof.** The proof follows from the results of [5], where it was shown that the generating function of the \(h\)-polynomials of the permutahedra

\[
H_{\Pi^n}(x,s,t) := \sum_{n\geq 0} h_{\Pi^n}(s,t) \frac{x^{n+1}}{(n+1)!} = \frac{e^{sx} - e^{tx}}{se^{tx} - te^{sx}}
\]

and thus \(H_{\Pi^n}(x,s,t) = Td_{s,t}^{\Theta^n}(x)\), implying (25).

To prove the second claim recall that the Eulerian number \(A_{n,k}\) is the number of permutations from \(S_n\) with \(k\) descents, see e.g. [33]. These numbers have the symmetry \(A_{n,k} = A_{n,n-k-1}\) and satisfy the recurrence

\[
A_{n,k} = (n-k)A_{n-1,k-1} + (k+1)A_{n-1,k}.
\]

They can be given also as the sum

\[
A_{n,m} = \sum_{k=0}^{m} (-1)^k \binom{n+1}{k} (m+1-k)^n.
\]
The corresponding polynomials $A_n(s) = \sum_{k=0}^{n-1} A_{n,k}s^k$ were introduced by Euler in 1755 by the relation
\[
\sum_{k=1}^{\infty} k^n t^n = \frac{t A_n(t)}{(1-t)^{n+1}}.
\]
They can be computed recursively by
\[
A_{n+1}(t) = [t(1-t)\frac{d}{dt} + nt + 1]A_n(t), \quad A_1 = 1 : \quad A_1 = 1, \quad A_2 = s + 1, \quad A_3 = s^2 + 4s + 1, \quad A_4 = s^3 + 11s^2 + 11s + 1, \quad A_5 = s^4 + 26s^3 + 66s^2 + 26s + 1, \quad A_6 = s^5 + 57s^4 + 302s^3 + 302s^2 + 57s + 1.
\]
The generating function of Eulerian polynomials is known after Euler to be
\[
\sum_{n\geq 0} A_n(s) \frac{x^n}{n!} = \frac{s - 1}{s - e^{(s-1)x}}.
\]
Consider
\[
A(x, s) := \sum_{n\geq 0} A_{n+1}(s) \frac{x^{n+1}}{(n+1)!} = \frac{s - 1}{s - e^{(s-1)x}} - 1 = \frac{e^{sx} - e^x}{se^x - e^{sx}}.
\]
Replacing here $x$ by $tx$ and $s$ by $s/t$ we have the equality (see [5])
\[
\sum_{k,n \geq 0, k \leq n} A_{n+1,k}s^k t^{n-k} \frac{x^{n+1}}{(n+1)!} = \frac{e^{sx} - e^{tx}}{se^{tx} - e^{sx}} = Td_{s,t}^{\Theta}(x).
\]
Setting now $s = y, t = -1$ we have formula (26). 

4. APPLICATION: HODGE NUMBERS OF THE THETA-DIVISORS

Let $H^{p,q}(X)$ be the Dolbeault cohomology group of a complex $n$-dimensional manifold $X$ and $h^{p,q}(X) = \dim H^{p,q}(X)$.

Following Hirzebruch [23] consider the index of the elliptic operator
\[
\bar{\partial} : \Omega^{p,q}(X) \to \Omega^{p,q+1}(X)
\]
for fixed $p$ and consider the corresponding index
\[
\chi^p(X) := \sum_{q=0}^{n} (-1)^{q} h^{p,q}(X).
\]
When $p = 0$ we have the holomorphic Euler characteristic, which is known to coincide with the Todd genus of $X$: $\chi^0(X) = Td(X)$ and is related to the arithmetic genus $\chi_a(X)$ by the formula
\[
\chi_a(X) = (-1)^{n}(\chi^0(X) - 1)
\]
(see [23]). To compute other $\chi^p(X)$ introduce the generating polynomial
\[
\chi_g(X) := \sum_{p=0}^{n} \chi^p(X) y^p.
\]
Theorem 4.1. (Hirzebruch [23]) The value of $\chi_y(X)$ can be given by the Hirzebruch genus with the generating power series

$$Q(x) = \frac{x(1 + ye^{-x(1+y)})}{1 - e^{-x(1+y)}}.$$  

Applying now our general formula (22) we have

$$\sum_{n=1}^{\infty} \chi_y(\Theta^n) \frac{z^{n+1}}{(n+1)!} = \frac{1 - e^{-x(1+y)}}{1 + ye^{-x(1+y)}}.$$  

Since

$$\frac{1 - e^{-x(1+y)}}{1 + ye^{-x(1+y)}} = \frac{eyx - e^x}{eyx + ye^{-x}}$$

we see that we have a particular case of two-parameter Todd genus $Td_{s,t}$ with $s = y, t = -1$. Thus we have the following result.

Theorem 4.2. The $\chi_y$-genus of the theta divisor $\Theta^n$ can be given as

$$\chi_y(\Theta^n) = (-1)^{n} A_{n+1}(-y),$$

where $A_n(s)$ is the Eulerian polynomial. In particular,

$$\chi^p(\Theta^n) = (-1)^{n-p} A_{n+1,p},$$

where $A_{n,p}$ are the Eulerian numbers.

When $y = 0$ we have the classical Todd genus

$$\chi^0(\Theta^n) = Td(\Theta^n) = A_{n+1,n}(-1)^n = (-1)^n$$

in agreement with [4]. When $y = -1$ we have the Euler characteristic

$$\chi(\Theta^n) = (-1)^n A_{n+1}(1) = (-1)^n(n + 1)!$$

again in agreement with [4]. Finally when $y = 1$ we have the formula for the signature of the theta divisor for even $n$

$$\tau(\Theta^n) = \sum_{k=0}^{n} (-1)^k A(n + 1, k) = \frac{2^{n+2}(2^{n+2} - 1)}{n + 2} B_{n+2},$$

where $B_n$ are the classical Bernoulli numbers, again in agreement with [4].

We can use this to compute the Hodge numbers $h^{p,q}(\Theta^n) = \dim H^{p,q}(\Theta^n)$, where

$$H^{p,q}(M) = H^p_\partial(M) = H^q(M, \Omega^p_M)$$

are the Dolbeault cohomology groups of complex variety $M$, see e.g. [21].

First we can apply the Lefschetz hyperplane theorem to the embedding $i : \Theta^n \subset A^{n+1}$, which claims that the homomorphism

$$i^* : H^{p,q}(A^{n+1}) \to H^{p,q}(\Theta^n)$$

is an isomorphism for $p + q \leq n - 1$ and injective for $p + q = n$ (see [21]).

Since the Hodge numbers of abelian variety $A^{n+1}$ are

$$h^{p,q}(A^{n+1}) = \binom{n + 1}{p} \binom{n + 1}{q}, \quad 0 \leq p, q \leq n + 1,$$
we have

$$(35) \quad h^{p,q}(\Theta^n) = h^{p,q}(A^{n+1}) = \binom{n+1}{p} \binom{n+1}{q}, \quad p + q \leq n - 1.$$ 

By Serre duality $h^{p,q}(\Theta^n) = h^{n-p,n-q}(\Theta^n)$, so this implies that

$$(36) \quad h^{p,q}(\Theta^n) = \binom{n+1}{n-p} \binom{n+1}{n-q} = \binom{n+1}{p+1} \binom{n+1}{q+1}, \quad p + q \geq n + 1.$$ 

To compute the remaining Hodge numbers $h^{p,q}(\Theta^n)$ with $p + q = n$ we can use now our formula (33):

$$\chi^p(\Theta^n) = \sum_{q=0}^{n} (-1)^q h^{p,q}(\Theta^n) = (-1)^{n+p} A_{n+1,p}. $$

In this sum the only unknown term is $h^{p,n-p}(\Theta^n)$. The straightforward calculations using the properties of binomial coefficients show that the sum $S_{n,p}$ of the known terms is

$$(37) \quad S_{n,p} = (-1)^p \binom{n+2}{p+1} \left[ (-1)^p \frac{2p-n}{n+2} \binom{n+1}{p} + \sum_{k=0}^{p-1} (-1)^k \binom{n+1}{k} \right].$$

Thus we have the following result, giving all the Hodge numbers of the theta divisors.

**Theorem 4.3.** The Hodge numbers $h^{p,q}(\Theta^n)$ of the theta divisor $\Theta^n$ with $p + q \neq n$ are given by (35), (36), while when $p + q = n$ we have

$$(38) \quad h^{p,n-p}(\Theta^n) = A_{n+1,p} - S_{n,p},$$

where $A_{n,p}$ are the Eulerian numbers and $S_{n,p}$ is given by (37).

In particular, using formula (28) for the Eulerian numbers we have

$$A_{n,1} = 2^n - (n + 1), \quad A_{n,3} = 3^n - 2^n(n + 1) + \frac{(n + 1)(n + 2)}{2},$$

and thus

$$h^{0,n}(\Theta^n) = n + 1, \quad h^{1,n-1}(\Theta^n) = 2^{n+1} - (n + 2) + \frac{n^2(n + 1)}{2},$$

$$h^{2,n-2}(\Theta^n) = 3^{n+1} - 2^{n+1}(n + 2) + \frac{(n + 1)(n + 2)}{2} + \frac{n^3(n^2 - 1)}{12}.$$ 

The Hodge diamonds of the theta divisors $\Theta^n$ for $n = 2, 3, 4$ have the following form (with Betti numbers shown in the right column):
\[
\begin{array}{cccc}
1 & 1 \\
3 & 3 & 6 \\
3 & 10 & 3 & 16 \\
3 & 3 & 6 \\
1 & 1 \\
4 & 4 & 8 \\
6 & 16 & 6 & 28 \\
4 & 29 & 29 & 4 & 66 \\
6 & 16 & 6 & 28 \\
4 & 4 & 8 \\
1 & 1 \\
5 & 5 & 10 \\
10 & 25 & 10 & 45 \\
10 & 50 & 50 & 10 & 120 \\
5 & 66 & 146 & 66 & 5 & 288 \\
10 & 50 & 50 & 10 & 120 \\
10 & 25 & 10 & 45 \\
5 & 5 & 10 \\
1 & 1 \\
\end{array}
\]
5. Relation with permutohedral variety

There is another natural algebraic variety related to the permutohedron, namely the corresponding toric variety $X^n_\Pi$ called permutohedral. Its normal fan corresponds to the standard $A_n$ hyperplane arrangement in $\mathbb{R}^{n+1}$ given by $x_i = x_j, 1 \leq i < j \leq n + 1$ with $x_1 + \cdots + x_{n+1} = 0$. In particular, $X^n_1 = \mathbb{C}P^1$, $X^n_2$ is the degree 6 del Pezzo surface.

Recall that toric variety can be constructed from any simple integer polytope $P^n$ (see [18]). The topology of the permutohedral variety is being discussed in the literature (see e.g. the recent papers [11, 26] and references therein). In particular, it is known that the Hodge numbers $h^{p,q}(X^n_\Pi) = 0$ if $p \neq q$ and $h^{p,p}(X^n_\Pi) = h_p(P_n) = A(n + 1, p)$, which is very different from what we have just seen for the theta divisors.

We claim that actually there is an interesting duality between the theta divisor $\Theta^n$ and permutohedral variety $X^n_\Pi$. Some evidence of such duality is given by the fact that the Todd genus $Td(X^n_\Pi) = 1 = (-1)^n Td(\Theta^n)$ and the Euler characteristic is the number of vertices of $\Pi^n$: $\chi(X^n_\Pi) = (n+1)! = (-1)^n \chi(\Theta^n)$ (see e.g. [10]). We extends this to the following result.

**Theorem 5.1.** The Betti number $b_{2k}(X^n_\Pi)$ of the permutohedral variety coincides up to a sign with the Hirzebruch $\chi^k$-genus of the theta divisor $\Theta^n$:

$$b_{2k}(X^n_\Pi) = (-1)^{n-k} \chi^k(\Theta^n),$$

so the Poincare polynomial $P(X^n_\Pi, s) = \sum_{i=0}^{2n} b_i(X^n_\Pi) s^i$ coincides up to a sign with $\chi_y$-genus of $\Theta^n$ with $y = -s^2$:

$$P(X^n_\Pi, s) = (-1)^n \chi_y(-s^2(\Theta^n)).$$

The two-parameter Todd genus $Td_{s,t}(\Theta^n)$ of the theta divisor $\Theta^n$ and of the permutohedral variety $X^n_\Pi$ are different only by a sign:

$$Td_{s,t}(X^n_\Pi) = (-1)^n Td_{s,t}(\Theta^n).$$

**Proof.** By the general theory of toric varieties [18] its even Betti number $b_{2k}(X^n_\Pi)$ equals the coefficient $h_k(P^n)$ of the $h$-polynomial of the corresponding polytope $P$ (odd Betti numbers are zero). In our case of permutohedron $P = \Pi^n$ we have

$$b_{2k}(X^n_\Pi) = h_k(\Pi^n) = A(n + 1, k).$$

Comparing this with Theorem 4.2 we have the relation (39) and thus (40).

To prove the second part we use the results of T. Panov [31], who computed the $\chi_y$-genus of toric variety $X^n_P$ related to any simple polytope $P^n$ as the sum over vertices $p \in P^n$

$$\chi_y(X^n_P) = \sum_p (-y)^{ind(p)},$$

where $ind(p)$ is the index of $p$ with respect to generic height function on $P^n$ (see Theorem 3.1 in [31]). Since it is known that the number of the vertices
of index $k$ equals the coefficient $h_k(P^n)$ (see Khovanskii [24]) we have that

$$x_y(X^n_P) = \sum_{k=0}^n h_k(P^n)(-y)^k.$$  

This implies that

$$Td_{s,t}(X^n_P) = h_{P^n}(-s, -t) = (-1)^n h_{P^n}(s, t),$$

where $h_{P^n}(s, t)$ is the $h$-polynomial of the polytope $P^n$. Applying this to $P^n = \Pi^n$ and using our Theorem 3.1 we have the relation (41). □

In particular, for even $n$ using (34) we have the explicit formula for the signature $\tau(X^n_\Pi)$ in terms of Bernoulli numbers:

$$\tau(X^n_\Pi) = \tau(\Theta^n) = \frac{2^{n+2}(2^{n+2} - 1)}{n + 2} B_{n+2}. \tag{43}$$

This suggests that the cobordisms classes of the permutohedral variety $X^n_\Pi$ and theta divisor $\Theta^n$ might be related by $[X^n_\Pi] = (-1)^n [\Theta^n]$. However, this turns out to be true only for $n = 1$ and $n = 2$. To see this we can use the results from the paper [9] by Buchstaber, Panov and Ray expressing the cobordism class of any toric variety in combination with our formula (5) for the Chern-Dold character [4]. In the case of the permutohedral variety we have the following formula.

**Theorem 5.2.** The cobordism class $X^n_\Pi$ of the permutohedral variety can be expressed in terms of the cobordism classes of the theta divisors as

$$[X^n_\Pi] = \sum_{\sigma \in S_{n+1}} \prod_{i=1}^n \frac{1}{\beta(t(z_{\sigma(i)} - z_{\sigma(i+1)}))}|_{t=0},$$

where $\beta(z) = z + \sum_{n=1}^{\infty} [\Theta^n]z^{n+1}/(n+1)!$.

In particular, this gives that $[X^1_\Pi] = -[\Theta^1]$, $[X^2_\Pi] = [\Theta^2]$, but for $n = 3$ the computer calculations using Wolfram Mathematica\(^1\) show that

$$[X^3_\Pi] = \frac{1}{2}[\Theta^1]^3 - \frac{2}{3}[\Theta^1][\Theta^2] - \frac{5}{6}[\Theta^3]. \tag{45}$$

Thus the link between these two classes of varieties does not go beyond the coincidence of generalised Todd genera, which looks even more mysterious.

There is another interesting parallel between the theta divisor $\Theta^n \subset A^{n+1}$ in abelian variety $A^{n+1}$ and open hypersurface $Z^n(\Pi^{n+1}) \subset T^{n+1}$ in the complex torus $T^n = (\mathbb{C} \setminus 0)^{n+1}$ given as the zero set $f(z) = 0$ of a generic Laurent polynomial $f$ with permutohedral Newton polytope. The corresponding Hodge-Deligne numbers were computed by Danilov and Khovanskii in [13]. It would be interesting to analyse their results in our context.

\(^1\)We are grateful to Misha Kornev for helping us with this.
6. Toda lattice and Tomei manifolds

The (open) finite Toda lattice \cite{16, 29} is the Hamiltonian system describing the interaction $n+1$ particles on the line with the Hamiltonian

$$H = \frac{1}{2} \sum_{i=1}^{n+1} p_i^2 + \sum_{j=1}^{n} e^{q_j-q_{j+1}},$$

In the Flaschka variables

$$a_j = -\frac{1}{2} p_j, \quad j = 1, \ldots, n+1, \quad b_k = \frac{1}{2} e^{\frac{1}{2}(q_k-q_{k+1})}, \quad k = 1, \ldots, n$$

the equations of motion take the algebraic form

(46) \begin{equation} \begin{aligned} \dot{a}_j &= 2(b_j^2 - b_j^{2-1}), \\ \dot{b}_k &= b_k(a_{k+1} - a_k) \end{aligned} \end{equation}

(we assume here that $b_0 = b_{n+1} = 0$).

A crucial observation due to Flaschka and Manakov is that the system (46) has the following Lax representation

(47) $\dot{L} = [B, L]$,

where

$$L = \begin{pmatrix} a_1 & b_1 & & & \\ b_1 & a_2 & b_2 & & \\ & \ddots & \ddots & \ddots \\ b_{n-1} & a_n & b_n & & \\ & b_n & a_{n+1} & & \end{pmatrix}, \quad B = \begin{pmatrix} 0 & b_1 & & & \\ -b_1 & 0 & b_2 & & \\ & \ddots & \ddots & \ddots \\ & & -b_{n-1} & 0 & b_n \\ & & & -b_n & 0 \end{pmatrix}.$$

This means that the eigenvalues of the matrix $L$ are preserved by the Toda flow. It is known that the coefficients of the characteristic polynomial $P_L(\lambda) = \det(L - \lambda I)$ Poisson commute, proving that the Toda lattice is integrable in Liouville sense. The corresponding set $M^\text{p}_n$ of the matrices $L$ with $b_i > 0$ (called Jacobi matrices) with given spectrum $\Lambda = \{\lambda_1, \ldots, \lambda_{n+1}\}$ is open and diffeomorphic to $\mathbb{R}^n$, so we do not have usual Liouville tori with quasiperiodic motion but instead the scattering (see the details in \cite{29}).

Following Tomei \cite{34} consider the corresponding compact isospectral set

(48) $M^\text{p}_n = \{L : \text{spec } L = \{\lambda_1, \ldots, \lambda_{n+1}\}\}$

of all symmetric tridiagonal matrices $L$ (without restrictions that $b_i$ are positive), which we will call Tomei manifold. For generic $\Lambda$ this is a smooth real manifold of dimension $n$, which is invariant under the (extended) Toda flow (46). Tomei used this flow to study the topology of this manifold, which turned out to be quite interesting\footnote{Later Gaifullin \cite{19} proved a remarkable fact that Tomei manifold can be used as a universal one in Steenrod’s cycle realisation problem.}. In particular, he had shown that it admits the cell decomposition into $2^n$ permutohedra, corresponding to different choices of the signs of $b_i$. For $n = 2$ we have a surface of genus 2 glued from 4 hexagons (see \cite{34}).
**Theorem 6.1.** (Tomei [31]) $M^n_T$ is an aspherical manifold with Euler characteristic

\[(49) \quad \chi(M^n_T) = B_{n+2} \frac{2^{n+2}(2^{n+2} - 1)}{n+2},\]

where $B_n$ is $n$-th Bernoulli number.

Comparing (49) with the formula (34) for the signature $\tau(\Theta^n)$ of the theta divisor, we see that they coincide.

We extend this observation to the following result, demonstrating interesting relation of the Tomei manifold with $\Theta^n$ and $X^n_\Pi$. Note that $M^n_T$ is real manifold of dimension $n$, while $\Theta^n$ and $X^n_\Pi$ are complex manifolds of real dimension $2n$.

Let $b_m(X) = \dim H^m(X, \mathbb{Z}_2)$ be the corresponding Betti numbers of a manifold $X$. When the cohomology group $H^m(X, \mathbb{Z})$ is torsion-free (which is the case for all three our manifolds), $b_m(X)$ is its rank.

**Theorem 6.2.** The numerical characteristics of the Tomei manifold $M^n_T$, theta divisor $\Theta^n$ and permutohedral variety $X^n_\Pi$ are related by

\[(50) \quad b_k(M^n_T) = b_{2k}(X^n_\Pi) = (-1)^{n-k} \chi^k(\Theta^n).\]

In particular, the Euler characteristic of $M^n_T$ equals the signatures of $X^n_\Pi$ and $\Theta^n$:

\[(51) \quad \chi(M^n_T) = \tau(X^n_\Pi) = \tau(\Theta^n).\]

**Proof.** The Betti numbers of the Tomei manifold were computed by Fried, who showed that $b_k(M^n_T) = A(n+1, k)$, where $A(n, k)$ are Eulerian numbers. Comparing this with (42) and (39), we have (50).

A more conceptual proof of this follows from the theory of *small covers of simple polytopes* from Davis and Januszkiewicz [14]. The Tomei manifold $M^n_T$ corresponds to the case when the polytope is permutohedron $\Pi^n$ for certain characteristic function, which can be interpreted as colouring the faces of permutohedron in $n$ colours (see [14] [19]). Theorem 3.1 from [14] says that the Betti number $b_k(M_P)$ (over $\mathbb{Z}_2$) of a small cover of simple polytope $P$ equals the coefficient $h_k(P)$ of the corresponding $h$-polynomial. In our case this implies that $b_k(M^n_T) = h_k(\Pi^n)$, and thus (50).

To prove that $\chi(M^n_T) = \tau(X^n_\Pi)$ we use the general result from the theory of toric varieties [31] (see also [27]), that the signature of such variety $X^n_\Pi$ is the alternating sum of the even Betti numbers:

\[(52) \quad \tau(X^n_\Pi) = \sum_{k=0}^{\frac{n}{2}} (-1)^k b_{2k}(X^n_\Pi).\]

The equality $\tau(X^n_\Pi) = \tau(\Theta^n)$ follows now from (43).  \(\square\)
Let us consider now the Hermitian Tomei manifold $M_{HT}^{2n}$ as the set of Hermitian tridiagonal matrices

$$L^H = \begin{pmatrix} a_1 & b_1 & \cdots & b_2 & \cdots & \cdots & b_n & \cdots & \cdots & \cdots & b_{n-1} & a_n & b_n & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & b_n & a_{n+1} \end{pmatrix},$$

with given spectrum $\text{Spec } L = \Lambda = (\lambda_1, \ldots, \lambda_{n+1})$ (known to be real), where $a_k \in \mathbb{R}$ and $b_j \in \mathbb{C}$. For generic $\Lambda$ this is a smooth submanifold of the set $O_\Lambda$ of all Hermitian matrices with spectrum $\Lambda$, which can be viewed as a coadjoint orbit $U(n+1)/T^{n+1}$ of the unitary group $U(n+1)$.

Note that the embedding $M^n_T \subset M_{HT}^{2n}$ is equivariant with respect to the natural actions of $\mathbb{Z}_2^n$ and $T^n$, where $T^n$ is the group of diagonal matrices from $SU(n+1)$ and $\mathbb{Z}_2^n \subset T^n$ is its subgroup with $\pm 1$ on the diagonal.

Bloch, Brockett and Ratiu [3] had shown that the Toda flow is gradient for some metric on $M^n_T$ and the height function $\text{tr}(\rho L)$, $\rho = \text{diag}(1, \ldots, n+1)$, so that Tomei results [34] can be interpreted within the classical Morse theory [28]. Using this one can obtain covering of $M^n_T$ by $(n+1)!$ open charts and check that they satisfy the properties of the small cover in terminology of Davis and Januszkiewicz [14].

In the Hermitian case one can use the results of Bloch, Flaschka and Ratiu [2] to deduce that $M_{HT}^{2n}$ is a toric manifold (in the sense of Davis and Januszkiewicz) with the same orbit space $\Sigma^n$ and the same characteristic function as in the real Tomei case (see [14, 19]). It is natural to compare it with the permutohedral variety $X^n$.

**Theorem 6.3.** Hermitian Tomei manifold $M_{HT}^{4n}$ is not homotopically equivalent (and hence not diffeomorphic) to the permutohedral variety $X^{2n}$.

In addition, $M_{HT}^{4n}$ is not equivariantly diffeomorphic to any symplectic manifold $M^{4n}$ with Hamiltonian action of torus $T^{2n}$.

**Proof.** Davis and Januszkiewicz [14] proved that $M_{HT}^{4n}$ is stably parallelisable, so due to Hirzebruch [23] the signature $\tau(M_{HT}^{4n}) = 0$. On the other hand, from (43) we see that $\tau(X^{2n}) \neq 0$. Since the signature is homotopic invariant, we conclude that $M_{HT}^{2n}$ and $X^{2n}$ are not homotopically equivalent.

To prove the second part, we use the results of Delzant [15], which imply that that every manifold $M^{4n}$ with Hamiltonian action of torus $T^{2n}$ is equivariantly diffeomorphic (but, in general, not symplectomorphic) to an algebraic complex toric variety $Y^{2n}$ with combinatorially equivalent moment polytope. Panov [31] (see also [27]) proved that the signature $\tau(Y^{2n})$ depends only on combinatorics of the corresponding polytope (which in our case is permutohedron), so $\tau(Y^{2n}) = \tau(X^{2n}) \neq 0$. Since the signature of $M_{HT}^{4n}$ is zero, it cannot be diffeomorphic to $M^{4n}$. \qed
Note that $M^2_{HT} = S^2$ is two-dimensional sphere with the standard symplectic structure and a natural Hamiltonian action of $T^1 = S^1$, so our result cannot be extended to all dimensions.

Our theorem explains why Bloch, Flaschka and Ratiu [2] considered the embedding into the coadjoint orbit $O_\Lambda$ only of the “isospectral set” $J_\Lambda$, but not of the “full isospectral manifold” $M^2_{HT}$. (see the comments at the end of Section 2.2 in [2]).

For $n = 2$ we can claim a stronger result (cf. Section 6 in Hirzebruch [22] and Chapter 9 in Buchstaber, Panov [10]).

**Theorem 6.4.** Hermitian Tomei manifold $M^4_{HT}$ does not admit any almost complex (and hence, any symplectic) structure.

In particular, there is no embedding of $M^4_{HT}$ into the coadjoint orbit $O_\Lambda$ with non-degenerate restriction of the canonical symplectic form on $O_\Lambda$.

**Proof.** Assume that $M^4 = M^4_{HT}$ has an almost complex structure, then we have the canonically defined orientation and thus the fundamental cycle $< M^4 > \in H_4(M^4, \mathbb{Z})$. For any almost complex manifold we have well defined Chern numbers of such manifold as the values of the corresponding Chern classes on the fundamental cycle $< M^4 >$. In terms of these numbers one can express the Euler characteristic, signature and Todd genus of any almost complex manifold $M^4$ as follows [23]

$$\chi(M^4) = c_2, \quad \tau(M^4) = \frac{c_1^2 - 2c_2}{3}, \quad Td(M^4) = \frac{c_1^2 + c_2}{12}.$$ 

As a result for any almost complex manifold $M^4$ we have the relation

$$Td(M^4) = \frac{1}{3}(\tau(M^4) + \chi(M^4)).$$

From the results of [14] the Euler characteristic $\chi(M^4) = (2 + 1)! = 6$ and since the signature $\tau(M^4) = 0$ we have $Td(M^4) = \frac{6 + 0}{4} = \frac{3}{2}$. This contradicts the classical Hirzebruch result [22] that any almost complex manifold must have integer Todd genus. Since any symplectic manifold admits an almost complex structure, we conclude that $M^4_{HT}$ has no symplectic structures. □

Finally, let us discuss the Hermitian Tomei manifold in the context of complex cobordisms. Recall that $U$-structure on a real manifold $M^m$ is an isomorphism of real vector bundles

$$TM^m \oplus (2N - m) \mathbb{R} \cong r\xi,$$

where $TM^m$ is the tangent bundle of $M^m$, $(2N - m)\mathbb{R}$ is trivial real $(2N - m)$-dimensional bundle over $M^m$, $\xi$ is a complex vector bundle over $M^m$ and $r\xi$ is its real form. Buchstaber and Ray [8] showed that any smooth toric manifold (in particular, $M^2_{HT}$) can be supplied with a canonical $U$-structure, which is invariant under the $T^n$-action ($BR$-structure).

**Theorem 6.5.** As a $U$-manifold with $BR$-structure $M^2_{HT}$ has the zero complex cobordism class and does not admit any $T^n$-invariant almost complex structure.
Proof. We use the results of Buchstaber, Panov and Ray [9], who provided a formula for the cobordism class of any smooth toric $U$-manifold with the BR-structure (see Theorem 5.16 and Corollary 4.9 in [9]). To apply formula (4.10) from that paper, we need to find the signs of the vertices of permutohedron, corresponding to BR-structure. Since the characteristic function in our case comes from colouring of the faces, it is easy to see that the neighbouring vertices of permutohedron have opposite signs. This means that the total sum in the right hand side of formula (4.10) (and hence the cobordism class of $M_{2n}^{2n}$) is zero: $[M_{2n}^{2n}] = 0$.

If $M_{2n}^{2n}$ would admit $T^n$-invariant almost complex structure then in formula (4.10) all signs would be plus, which leads to a contradiction. □

When $n = 1$ the manifold $M_{2n}^{2n}$ can be identified with $\mathbb{C}P^1$, but with different $U$-structure. As a complex manifold $\mathbb{C}P^1$ is $U$-manifold with $N = 2$ and $\xi = \bar{\eta} \oplus \bar{\eta}$, where $\eta$ is the tautological line bundle over $\mathbb{C}P^1$ and $\bar{\eta}$ is its dual, while the BR-structure on $M_{2n}^{2n}$ corresponds to $N = 2$ and different choice of $\xi = \eta \oplus \bar{\eta}$ in [33]. The BR-structure on $M_{2n}^{2n}$ comes naturally from the representation of $S^2$ as the quotient of the unit quaternion sphere $S^3 = \{q \in \mathbb{H}, |q| = 1\}$ by the action of $S^1 = \{z \in \mathbb{C}, |z| = 1\} \subset \mathbb{H}$ given by the left multiplication $q \to zq$. If we identify $\mathbb{H}$ with $\mathbb{C}^2$ using $q = z_1 + jz_2$ then $S^1$ acts with the matrix $\text{diag}(z, \bar{z})$ (in contrast with the multiplication by $z$ in the $\mathbb{C}P^1$ case).

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