ON HERMITIAN MANIFOLDS WITH STROMINGER PARALLEL TORSION

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Abstract. In this article, we study Hermitian manifolds whose Strominger connection has parallel torsion tensor, which will be called Strominger parallel torsion manifolds, or SPT manifolds for short. We obtain a necessary and sufficient condition characterizing this class in terms of the curvature tensor. In particular, Strominger flat or Strominger Kähler-like manifolds are SPT manifolds, known by our earlier results. We also obtain structural results for non-balanced SPT manifolds, and a classification theorem for balanced SPT threefolds.

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1. Introduction

Given a smooth manifold equipped with a connection, the parallelness of torsion often forms strong geometric restrictions. For instance, by the result of Kamber and Tondeur in [18], if a manifold admits a complete connection that is flat and having parallel torsion, then its universal cover must be a Lie group (equipped with a flat left invariant connection). The converse is certainly true. Also, by the classic result of Ambrose and Singer [1], if a complete Riemannian manifold admits a metric connection with parallel torsion and curvature, then the universal cover is a homogeneous Riemannian manifold (and vice versa). The complex version of this is also true, proved by Sekigawa in [24], namely, if a complete Hermitian manifold admits a Hermitian connection (that is, a connection compatible with both the metric and the almost complex structure) which has parallel torsion and curvature, then the universal cover is a homogeneous Hermitian manifold (and vice versa).

On a given Hermitian manifold, Strominger connection (also known as Bismut connection, see [7, 25]) is the unique Hermitian connection with totally skew-symmetric torsion. It serves as a bridge between Levi-Civita connection, which is the unique metric connection that is torsion free, and Chern connection, the unique metric connection compatible with the complex structure. When the metric is Kähler, these three canonical connections coincide, but when the metric is not Kähler, they are mutually distinct, leaving us with three different kinds of geometry.

In this article, we will focus on Strominger connection, and we want to know when Strominger connection will have parallel torsion. For convenience, we will call the class of Hermitian manifolds whose
Strominger connection has parallel torsion simply as Strominger parallel torsion manifolds, or SPT manifolds in short.

Let \((M^n, g)\) be a Hermitian manifold. Denote by \(\nabla^s\) the Strominger connection and by \(T^s, R^s\) its torsion and curvature tensor, respectively. Let us introduce the notation

\[
Q_X Y Z W = R^s_{X Y Z W} - R^s_{Z Y X W}, \quad \text{Ric}(Q)_{X Y} = \sum_{i=1}^{n} (R^s_{X Y e_i e_i} - R^s_{e_i X e_i Y})
\]

where \(X, Y, Z, W\) are type \((1,0)\) tangent vectors on \(M\) and \(\{e_1, \ldots, e_n\}\) is any local unitary frame. Denote by \(\omega\) the Kähler form of \(g\). Recall that Gauduchon’s torsion 1-form is the global \((1,0)\)-form \(\eta\) on \(M\) defined by \(\partial(\omega^{n-1}) = -2\eta \wedge \omega^{n-1}\). Let \(\chi\) be the type \((1,0)\) vector field on \(M\) dual to \(\eta\), namely, \(\langle Y, \chi \rangle = \eta(Y)\) for any \(Y\) where \(g(\cdot, \cdot)\) is the (Riemannian) metric, extended bi-linearly over \(\mathbb{C}\).

With these notations specified, we can now state the first main result of this article, which says that the SPT condition \(\nabla^s T^s = 0\) is equivalent to some conditions involving only \(R^s\):

**Theorem 1.1.** Let \((M^n, g)\) be a Hermitian manifold. The SPT condition \(\nabla^s T^s = 0\) is equivalent to the combination of the following

\[
\begin{align*}
(1.1) & \quad R^s_{X Y Z W} = 0 \\
(1.2) & \quad R^s_{X Y Z W} = R^s_{Z Y X W} \\
(1.3) & \quad \nabla^s \text{Ric}(Q) = 0 \\
(1.4) & \quad \text{Ric}(Q)_{X Y} = 0
\end{align*}
\]

for any type \((1,0)\) tangent vectors \(X, Y, Z, W\).

Recall that the Strominger Kähler-like (SKL) condition (cf. \([3, 12, 13]\)) for a Hermitian manifold is defined as

\[
R^s_{X Y Z W} = 0, \quad R^s_{X Y Z W} = R^s_{Z Y X W}
\]

for any type \((1,0)\) tangent vectors \(X, Y, Z, W\). The main result of [33] says that

\[
\text{SKL} \iff \text{SPT + pluriclosed.}
\]

The main technical part of the proof in [33] is to show that SKL implies SPT. Note that by our Theorem 1.1 above, this implication is immediate, since the SKL condition simply means (1.1) and \(Q = 0\). In this case, all four conditions in Theorem 1.1 are obviously satisfied, hence SPT. So Theorem 1.1 can be regarded as a generalization of the main result of [33].

We note that in Theorem 1.1, the conditions (1.3) and (1.4) are automatically satisfied if \(\text{Ric}(Q) = 0\). Also, under the SPT assumption, one always has \(\nabla^s Q = 0\). But there are examples of non-Kähler SKL manifolds with \(\nabla^s R^s \neq 0\). In other words, for SPT manifolds, the Strominger connection \(\nabla^s\) is not always an Ambrose-Singer connection.

Recall that a Hermitian manifold is said to be Gauduchon if \(\partial\overline{\partial}(\omega^{n-1}) = 0\), and is said to be balanced if its Gauduchon’s torsion 1-form \(\eta\) vanishes, or equivalently, if \(d(\omega^{n-1}) = 0\). Also, the metric is said to be strongly Gauduchon in the sense of Popovici [21] if there exists a global \((n, n-2)\)-form \(\Omega\) such that \(\partial\omega^{n-1} = \overline{\partial}\Omega\). Clearly,

\[
\text{balanced} \implies \text{strongly Gauduchon} \implies \text{Gauduchon.}
\]

As a consequence to Theorem 1.1, we have

**Corollary 1.2.** Let \((M^n, g)\) be a Hermitian manifold satisfying \(\nabla^s T^s = 0\). Then \(g\) is Gauduchon, i.e., \(\partial\overline{\partial}(\omega^{n-1}) = 0\). If in addition \(M^n\) is compact and \(g\) is not balanced, then \(g\) is not strongly Gauduchon and the Dolbeault group \(H^{0,1}_\partial(M) \neq 0\). In particular, any compact non-balanced SPT manifold is not a \(\partial\overline{\partial}\)-manifold and \(M\) does not admit any Kähler metric.

It has been shown in [33, Theorem 7 and Proposition 4] that compact non-Kähler SKL manifolds, or equivalently non-balanced SKL manifolds, does not admit any strongly Gauduchon metric and the Dolbeault group \(H^{0,1}_\partial(M)\) is non-trivial. We speculate that the same should be true for compact non-balanced SPT manifolds as well:

**Conjecture 1.3.** Let \((M^n, g)\) be a compact non-balanced Hermitian manifold satisfying \(\nabla^s T^s = 0\). Then \(M\) does not admit any strongly Gauduchon metric.
By [33, Theorem 2], the following equivalence holds for \( n = 2 \)

\[
SKL \iff SPT \iff \text{Vaisman}
\]

where Vaisman means a locally conformal Kähler manifold whose Lee form (which is \(-2(\eta + \overline{\eta})\)) is parallel under the Levi-Civita connection. Compact Vaisman surfaces are fully classified by Belgun [4] and a structure theorem for Vaisman manifolds is proved by Ornea and Verbitsky [20].

In dimension \( n \geq 3 \), locally conformal Kähler manifolds can no longer be (non-Kähler) SKL, shown in [32], but still can be SPT. For instance, it is easy to check that for any \( n \) the classic Hopf manifold

\[
M^n = (\mathbb{C}^n \setminus \{0\})/(f), \quad f(z) = \lambda z, \quad |\lambda| > 1, \quad \omega = \sqrt{-1} \frac{\partial |z|^2}{|z|^2},
\]

is SPT. More generally, all Vaisman manifolds are SPT. In fact, a locally conformal Kähler manifold is SPT if and only if it is Vaisman, a result due to Andrada and Villacampa [2, Theorem 3.6]. Here we give a slightly more general statement:

**Proposition 1.4.** A locally conformal Kähler manifold \((M^n, g)\) is SPT if and only if it is Vaisman. Furthermore, if two conformal Hermitian manifolds \((M^n, g)\) and \((M^n, e^{2u}g)\) are both SPT, with \( du \neq 0 \) in an open dense subset of \( M \), then both manifolds are locally conformal Kähler (thus Vaisman).

In other words, for each conformal class of Hermitian metrics, there is at most one SPT metric (up to constant multiples) unless the metric is locally conformal Kähler. When the manifold is compact, this is clearly true since SPT metrics are Gauduchon.

Next we consider SPT manifolds amongst the class of complex nilmanifolds with nilpotent \( J \). Complex nilmanifolds form an important class of Hermitian manifolds, and are often used to test and illustrate theory in non-Kähler geometry. See [11], [15], [19], [22], [26], [29] as a sample of examples. Following the discussions of [34], we have

**Proposition 1.5.** Let \((M^n, g)\) be a complex nilmanifold, namely, a compact Hermitian manifold whose universal cover is a nilpotent Lie group \( G \) equipped with a left-invariant complex structure \( J \) and a compatible left-invariant metric \( g \). Assume that \( J \) is nilpotent in the sense of [10]. Then \( g \) is SPT if and only if there exists a unitary left-invariant coframe \( \varphi \) on \( G \) and an integer \( 1 \leq r \leq n \) such that

\[
\begin{cases}
  d\varphi_1 = 0, \\
  d\varphi_\alpha = \sum_{i=1}^r Y_{\alpha i} \varphi_1 \wedge \overline{\varphi}_i, \quad \forall 1 \leq \alpha \leq n.
\end{cases}
\]

Note that the metric \( g \) will be balanced when and only when \( \sum_{i=1}^r Y_{\alpha i} = 0 \) for each \( \alpha \).

In particular, \( G \) (when not abelian) is a 2-step nilpotent group and \( J \) is abelian. Let us take the \( n = 3 \) case as an example. In this case, either \( r = 3 \), where \( G \) is abelian and \( g \) is Kähler and flat, or \( r = 2 \) and there exists a unitary coframe \( \varphi \) such that

\[
d\varphi_1 = d\varphi_2 = 0, \quad d\varphi_3 = a \varphi_1 \wedge \overline{\varphi}_1 + b \varphi_2 \wedge \overline{\varphi}_2,
\]

where \( a, b \) are constants. By a unitary change if necessary, we may assume that \( a > 0 \). Under such a coframe, the metric \( g \) is balanced if and only if \( b = -a \), while \( g \) is SKL if and only if \( b \in \sqrt{-1} \mathbb{R} \) is pure imaginary. For any other choice of \( b \) values, \( g \) is a non-SKL, non-balanced SPT metric. We will denote the following example by \( N^3 \),

\[
d\varphi_1 = d\varphi_2 = 0, \quad d\varphi_3 = \varphi_1 \wedge \overline{\varphi}_1 - \varphi_2 \wedge \overline{\varphi}_2,
\]

which will appear in the study of Lie-Hermitian threefolds that are balanced SPT, in §9.2.

The examples above show that when \( n \geq 3 \), there are more SPT manifolds than SKL manifolds, and there are more non-balanced SPT manifolds than balanced ones. For complex nilmanifolds with \( J \) not nilpotent, when \( n = 3 \) one can show by a lengthy computation that they cannot be SPT, but for \( n \geq 4 \), we do not know if they can be SPT or not. More generally, it would be a very interesting question on how to classify all SPT Lie-Hermitian manifolds (see for example [29] for a discussion on Lie-Hermitian manifolds).

For \( n \geq 3 \), one can naturally divide the discussion of (non-Kähler) SPT manifolds into two cases: non-balanced ones and balanced ones. All (non-Kähler) SKL manifolds and all (non-Kähler) Vaisman manifolds belong to the first subset, where the \( \nabla^* \) vector field \( \chi \) (dual to \( \eta \)) will play an important role. In §5 we will discuss some general properties of such spaces following the ideas of [32]. The main result of §5 is the following

**Proposition 1.6.** Let \((M^n, g)\) be a non-balanced SPT manifold with \( n \geq 3 \). Then the following holds:
(1) Locally there always exist admissible frames, and the holonomy group of the Strominger connection is contained in $U(n-1)$.
(2) The metric $g$ satisfies the LP condition if and only if it has degenerate torsion. In particular, for $n=3$, non-balanced SPT threefolds coincide with GCE threefolds of Belgun.

For the special case when $(M^n, g)$ is Vaisman, the conclusion (1) is due to Andrada and Villacampa [2]. Recall that a local unitary frame $e$ on a non-balanced SPT manifold is said to be admissible, if $\eta = \lambda \varphi_n$ and $T^i_{kj} = \lambda a_i \delta_{ij}$ for any $i, j$, where $\lambda > 0$ and $a_i$ are globally defined constants, as in Definition 5.1. The metric $g$ is said to have degenerate torsion if under any admissible frame $e$ it holds $T^e_{ij} = 0$ for any $i < k < n$. The Lee parallel (LP) condition of Belgun [6] means that a Hermitian manifold satisfying the condition

$$\partial \eta = 0, \quad \partial \omega = c \eta \wedge \partial \eta,$$

where $c \neq 0$ is a constant, and a Hermitian manifold is generalized Calabi-Eckmann (GCE) if it is both SPT and LP.

The balanced SPT manifolds, on the other hand, form a rather restrictive but non-empty subset. In the following, we will discuss the classification problem for such manifolds in dimension 3, starting from a technical lemma obtained in [35] for special frames on balanced threefolds. Recall that the $B$ tensor of a Hermitian manifold $(M^n, g)$ is defined by $B_{ij} = \sum_{r,s} T^j_{rs} T^i_{rs}$ under any local unitary frame, where $T^i_{jk}$ are components of the Chern torsion under the frame.

**Proposition 1.7.** Let $(M^3, g)$ be a non-Kähler balanced SPT threefold. Then, for any $p \in M$, there exists a local unitary frame near $p$ such that

$$B = \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix},$$

or

$$B = \begin{bmatrix} c & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c \end{bmatrix}$$

for some constant $c > 0$.

Such a local unitary frame will be called a special frame in §6. We will carry out the classification of non-Kähler balanced SPT threefolds according to the rank of the tensor $B$.

**Definition 1.8.** A balanced SPT threefold $(M^3, g)$ is said to be of middle type, if rank $B = 2$, or equivalently, for any $p \in M$, under a special frame near $p$

$$B = \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

for some constant $c > 0$.

We would like to illustrate two non-middle type examples. First let us consider the complex Lie group $SO(3, \mathbb{C})$ consisting of all $3 \times 3$ complex matrices $A$ such that $\det(A) = 1$ and $t^\dagger A A = I$. Let $\varphi$ be a left-invariant coframe satisfying

$$(1.6) \quad d\varphi_1 = \varphi_2 \wedge \varphi_3, \quad d\varphi_2 = \varphi_3 \wedge \varphi_1, \quad d\varphi_3 = \varphi_1 \wedge \varphi_2.$$ 

Let $g$ be the Hermitian metric with $\varphi$ being a unitary coframe. Then it is easy to check that $g$ is balanced SPT and its $B$ tensor takes the form

$$B = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$ 

So any compact quotient of $SO(3, \mathbb{C})$ is a non-Kähler, balanced SPT threefold of non-middle type. Note that this example has trivial canonical line bundle and is Chern flat. We remark that for this Hermitian threefold, the holonomy group of $\nabla^g$ is reduced to $SO(3) \subseteq U(3)$, but not contained in $U(2) \times 1$ like in the non-balanced SPT case.

There is another example of balanced SPT threefold which is not of the middle type: the Wallach threefold $(X, g)$. As a complex manifold, $X$ is the flag threefold $X = \mathbb{P}(T_{\mathbb{P}^2})$. Let $\tilde{\omega}$ be the Kähler-Einstein metric on $X$ with Ric$(\tilde{\omega}) = 2\tilde{\omega}$. Then the Kähler form of $g$ is given by $\omega = \tilde{\omega} - \sigma$, where $\sigma$ is the $(1,1)$-form on $X$ given by a holomorphic section $\xi \in H^0(X, \Omega_X \otimes L) \cong \mathbb{C}$ with $\|\xi\|^2 = \frac{1}{2}$. Here $\Omega_X$ is the holomorphic cotangent bundle, and $L$ is the ample line bundle such that $L^{\otimes 2}$ is the anti-canonical line bundle of $X$. 


We will see in §8 that $(X, g)$ is non-Kähler, balanced $SPT$. We will also show that its Levi-Civita (Riemannian) connection has constant Ricci curvature $6$ and has non-negative sectional curvature. Its $B$ tensor takes the form

$$B = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

so $(X, g)$ is not of middle type. This example is Fano, namely, its anti-canonical line bundle is ample. The Chern connection of $g$ has nonnegative bisectional curvature and positive holomorphic sectional curvature.

As observed by Wallach in [27], all the homogeneous metrics with positive sectional curvature on $X$ are Hermitian. This set of metrics (after scaling) forms a peculiar plane region (see Figure 1 in [9]), and our metric $g$ lies in the boundary of this region.

We have the following classification theorem of balanced $SPT$ threefolds, which is the second main theorem of this article. The proofs of the three cases below will be given in §6, §7 and §9.1.

**Theorem 1.9.** Let $(M^3, g)$ be a compact non-Kähler, balanced $SPT$ threefold. After scaling the metric by a constant multiple, it holds that,

1. when rank $B = 1$, $M$ is holomorphically isometric to the Wallach threefold.
2. when rank $B = 2$, namely the middle type, then there is an unbranched cover $\pi: \hat{M} \to M$ of degree at most 2 on which there is a bi-Hermitian structure $(\hat{M}, g, J, I)$, where $(\hat{M}, g, J)$ is balanced $SPT$ while $(\hat{M}, g, I)$ is Vaisman, with $IJ = JI$ and the Strominger connections of the two Hermitian structures $(\hat{M}, g, J)$ and $(\hat{M}, g, I)$ coincide. In particular, $\hat{M}$ is a smooth fiber bundle over $S^1$ with fiber being a Sasakian 5-manifold. Furthermore, a compact non-Kähler, balanced $SPT$ threefold of middle type admits no pluriclosed metric.
3. when rank $B = 3$, $M$ is holomorphically isometric to a compact quotient of $SO(3, \mathbb{C})$ with global unitary coframe satisfying (1.6).

In §9.2, we will study balanced $SPT$ threefolds of middle type in the category of Lie-Hermitian manifolds and prove the following theorem

**Theorem 1.10.** Let $(M^3, g)$ be a non-Kähler, balanced $SPT$ Lie-Hermitian threefold of middle type, then after a scaling of the metric by a constant multiple, it is one of the members in $A_{a,b}, B^\pm_{u,v,w}, C_{u,v}$ or $D^\pm_{u,v,\rho}$.

Here the constant parameters $a, b \in \mathbb{R}, u, v, w, \rho \in \mathbb{C}$ with $|\rho| = 1$, and for $B^\pm_{u,v,w}$ the parameters also satisfy $w - \overline{w} = \varepsilon((|u - v|^2 - |u + v|^2)$, where $\varepsilon = \pm 1$. See §9.2 for the explicit descriptions of these Lie-Hermitian threefolds.

Finally, we remark that there are examples of complete non-compact Hermitian surfaces with zero Chern curvature but non-parallel Chern torsion, so the curvature characterization of torsion parallelness for Strominger connection does not hold for Chern connection or other Hermitian connections in general. Of course it would be a much harder problem if one adds the compactness assumption on the manifolds, then things get more delicate and we plan to investigate it in future projects.

The paper is organized as follows. In §2, we will set up notations and also collect some known results that will be used later. In §3 we will discuss the properties of $SPT$ manifolds, and in §4, we will prove Theorem 1.1 stated above. In §5, we will discuss the properties of non-balanced $SPT$ manifolds, following ideas of [32] where non-Kähler SKL manifolds were investigated, and will prove Proposition 1.6. In §6, we will discuss the types of the $B$ tensor of a balanced $SPT$ threefold and prove Proposition 1.7, where we will also show that the rank $B = 3$ case of Theorem 1.9 leads to the compact quotients of $SO(3, \mathbb{C})$. In §7, we will prove that the rank $B = 1$ case of Theorem 1.9 leads to a Fano threefold of even index, so as a complex manifold it is either $\mathbb{P}^3$ or a del Pezzo threefold. All cases will be ruled out by topological or algebro-geometric argument, except the flag threefold case: the Wallach threefold $(X, g)$. In §8, we will carry out the detailed computation which shows that the Wallach threefold is indeed balanced $SPT$, with nonnegative bisectional curvature and positive holomorphic sectional curvature. We will also show that its Levi-Civita connection has non-negative sectional curvature and constant Ricci curvature 6. In §9, we will prove the rank $B = 2$ case of Theorem 1.9 and investigate balanced $SPT$ Lie-Hermitian threefolds of middle type, as well as discuss some generalization in higher dimensions.
2. Preliminaries

Let \((M^n, g)\) be a Hermitian manifold of complex dimension \(n\). We will set up notations following [30, 31, 33, 36, 37]. Let \(g = \langle \cdot, \cdot \rangle\) be the metric, extended bi-linearly over \(\mathbb{C}\). The bundle of complex tangent vector fields of type \((1,0)\), namely, complex vector fields of the form \(v = \sqrt{-1} Jv\), where \(v\) is a real vector field on \(M\), is denoted by \(T^{1,0}M\). Let \((e_1, \ldots, e_n)\) be a local frame of \(T^{1,0}M\) in a neighborhood in \(M\). Write \(e = \{e_1, \ldots, e_n\}\) as a column vector. Denote by \(\varphi = \{\varphi_1, \ldots, \varphi_n\}\) the column vector of local \((1,0)\)-forms which is the coframe dual to \(e\).

Denote by \(\nabla\), \(\nabla^c\), \(\nabla^s\) the Levi-Civita, Chern, and Strominger connection, respectively. Denote by \(T^c = T^c, R^c\) the torsion and curvature of \(\nabla^c\), and by \(T^s, R^s\) the torsion and curvature of \(\nabla^s\). Under the frame \(e\), denote the components of \(T^c\) as

\[
T^c(e_i, e_j) = 0, \quad T^c(e_i, e_j) = 2 \sum_{k=1}^n T^c_{ij} e_k.
\]

For Chern connection \(\nabla^c\), let us denote by \(\theta, \Theta\) the matrices of connection and curvature, respectively, and by \(\tau\) the column vector of the torsion 2-forms, all under the local frame \(e\). Then the structure equations and Bianchi identities are

\[
\begin{align*}
d\varphi &= -\theta \wedge \varphi + \tau, \\
d\theta &= \theta \wedge \theta + \Theta, \\
d\tau &= -\theta \wedge \tau + \Theta \wedge \varphi, \\
d\Theta &= \theta \wedge \Theta - \tau \wedge \theta.
\end{align*}
\]

The entries of \(\Theta\) are all \((1,1)\) forms, while the entries of the column vector \(\tau\) are all \((2,0)\) forms, under any frame \(e\). Similar symbols such as \(\theta^s, \Theta^s\) and \(\tau^s\) are applied to Strominger connection. The components of \(\tau\) are just \(T^s_{ij}\) defined in (2.1):

\[
\tau_k = \sum_{i,j=1}^n T^s_{ij} \varphi_i \wedge \varphi_j = \sum_{1 \leq i < j \leq n} 2 T^s_{ij} \varphi_i \wedge \varphi_j,
\]

where \(T^s_{ij} = -T^s_{ji}\). Under the frame \(e\), express the Levi-Civita (Riemannian) connection \(\nabla\) as

\[
\nabla e = \theta e + \overline{\theta_2 e}, \quad \nabla \overline{e} = \theta_1 e + \overline{\theta_1 e},
\]

thus the matrices of connection and curvature for \(\nabla\) become:

\[
\dot{\theta} = \begin{bmatrix} \theta_1 & \overline{\theta_2} \\ \overline{\theta_2} & \theta_1 \end{bmatrix}, \quad \dot{\Theta} = \begin{bmatrix} \Theta_1 & \overline{\Theta_2} \\ \overline{\Theta_2} & \Theta_1 \end{bmatrix},
\]

where

\[
\Theta_1 = d\theta_1 - \theta_1 \wedge \theta_1 - \overline{\theta_2} \wedge \theta_2, \\
\Theta_2 = d\theta_2 - \theta_2 \wedge \theta_1 - \overline{\theta_1} \wedge \theta_2, \\
d\varphi = -\theta_1 \wedge \varphi - \theta_2 \wedge \overline{\varphi}.
\]

When \(e\) is unitary, both \(\theta_2\) and \(\Theta_2\) are skew-symmetric, while \(\theta, \theta_1, \theta^s, \Theta, \Theta_1, \Theta^s\) are all skew-Hermitian. Consider the (2, 1) tensor \(\gamma = \frac{1}{2}(\nabla^s - \nabla^c)\) introduced in [30]. Its representation under the frame \(e\) is the matrix of 1-forms, which by abuse of notation we will also denote by \(\gamma\), is given by

\[
\gamma = \theta_1 - \theta.
\]

Denote the decomposition of \(\gamma\) into \((1,0)\) and \((0,1)\) parts by \(\gamma = \gamma' + \gamma''\). As in [28, Lemma 2], it yields that

\[
\theta^s = \theta + 2\gamma = \theta_1 + \gamma.
\]

As observed in [30], when \(e\) is unitary, \(\gamma\) and \(\theta_2\) take the following simple forms

\[
\theta_2_{ij} = \sum_{k=1}^n T^c_{ij} \varphi_k, \quad \gamma_{ij} = \sum_{k=1}^n (T^s_{ik} \varphi_k - T^c_{ik} \varphi_k),
\]

while for general frames the above formula will have the matrix \((g^e_{ij}) = (\langle e_i, \overline{e_j} \rangle)\) and its inverse involved, which is less convenient. This is why we often choose unitary frames to work with. By (2.6), we get the expression of the components of the torsion \(T^s\) under any unitary frame \(e\):

\[
T^s(e_i, e_j) = -2 \sum_{k=1}^n T^c_{ij} e_k, \quad T^s(e_i, \overline{e_j}) = 2 \sum_{k=1}^n (T^s_{ik} \varphi_k - T^c_{ik} \varphi_k).
\]
Note that in the second equation above we assumed that the frame $c$ is unitary, otherwise the matrix $g$ needs to appear in the formula. From this, we deduce that

$$\nabla^s T^s = 0 \iff \nabla^s T^c = 0.$$  

This is observed in [33, the proof of Theorem 3]. More generally, if we denote by $\nabla^t = (1 - t)\nabla^c + t\nabla^s$ the $t$-Gauduchon connection, where $t$ is any real constant, then its torsion tensor is $T^t(x, y) = T^c(x, y) + 2t(\gamma_x y - \gamma_y x)$, whose components under $e$ are again given by $T_{ij}^k$, so it is easy to see that for any fixed $t \in \mathbb{R}$

$$\nabla^t T^t_1 = 0 \iff \nabla^t T^t_2 = 0 \quad \forall \ t_1, t_2 \in \mathbb{R}.$$  

So for a fixed $t$-Gauduchon connection $\nabla^t$, if the torsion of one Gauduchon connection is parallel with respect to $\nabla^t$, then the torsion of any other Gauduchon connection will be parallel under $\nabla^t$ as well.

Next let us discuss the curvature. As usual, the curvature tensor $R^D$ of a linear connection $D$ on a Riemannian manifold $M$ is defined by

$$R^D(x, y, z, w) = \langle R^D_{xy} z, w \rangle = \langle D_x D_y z - D_y D_x z - D_{\langle x, y \rangle} z, w \rangle,$$

where $x, y, z, w$ are tangent vectors in $M$. We will also write it as $R^D_{xyzw}$ for brevity. It is always skew-symmetric with respect to the first two positions, and is also skew-symmetric with respect to its last two positions if the connection is metric, namely, if $Dg = 0$. The first and second Bianchi identities are respectively

$$\mathcal{S}\{R^D_{xy} z - (D_x T^D)(y, z) - T^D(T^D(x, y), z)\} = 0$$

$$\mathcal{S}\{(D_x R^D)_{yz} + R^D_{T^D(x, y)z}\} = 0$$

where $\mathcal{S}$ means the sum over all cyclic permutation of $x, y, z$. When the manifold $M$ is complex and the connection $D$ is Hermitian, namely, satisfies $Dg = 0$ and $DJ = 0$, where $J$ is the almost complex structure, then $R^D$ satisfies

$$R^D(x, y, Jz, Jw) = R^D(x, y, z, w)$$

for any tangent vectors $x, y, z, w$, or equivalently,

$$R^D(x, y, Z, W) = R^D(x, y, Z, W) = 0$$

for any type $(1, 0)$ tangent vectors $Z, W$. From now on, we will use $X, Y, Z, W$ to denote type $(1, 0)$ vectors. For any Hermitian connection $D$, with the skew-symmetry of first two or last two positions in mind, the above equation (2.10) says that the only possibly non-zero components of $R^D$ are $R^D_{XYZ\overline{W}}$, $R^D_{X\overline{Y}Z\overline{W}}$, and their conjugations.

Let us now apply the Bianchi identities to the Chern connection $\nabla$, and use the fact that $T^c(e_i, \overline{e}_j) = 0$ and $R^c_{e_i e_j} x = 0$ for any $x$ where $e$ is any local frame of type $(1, 0)$ vectors, we have the following

$$T^c_{ijk} = T^c_{jki} + T^c_{kij} = 2 \sum_{r=1}^n (T^c_{i j} T^c_{k r} + T^c_{j k} T^c_{i r} + T^c_{k i} T^c_{j r})$$

$$R^c_{ik\overline{j}} - R^c_{ik\overline{j}} = 2T^c_{ik\overline{j}}$$

$$R^c_{ik\overline{j},m} - R^c_{ik\overline{j},m} = 2 \sum_{r=1}^n T^c_{ir} R^c_{jk\overline{r}}$$

for any indices $1 \leq i, j, k, \ell, m \leq n$. Here the indices after semicolon stand for covariant derivatives with respect to the Chern connection $\nabla^c$. Note that in formula (2.12) we assumed that the frame $e$ is unitary, otherwise the right hand side needs to be multiplied on the right by $g$, the matrix of the metric.

3. Torsion and curvature of Strominger connection

Since we will primarily be interested in Strominger connection $\nabla^s$, we would like to have formula involving the curvature and covariant differentiation of $\nabla^s$. Using the Bianchi identities for general metric connections, we could get formula for $\nabla^s$ which are similar to the ones for Chern connection mentioned in the previous section. But since we also need relations between $R^c$ and $R^s$ later, let us deduce them from the structure equations. Under a $(1, 0)$-frame $e$, the components of the Chern, Strominger and Riemannian curvature tensors are given by

$$R^c_{ijk\overline{m}} = \sum_{p=1}^n \Theta^p_{k p}(e_i, \overline{e}_j) g_{p\overline{m}}, \quad R^s_{ab\overline{m}} = \sum_{p=1}^n \Theta^p_{k p}(e_a, e_b) g_{p\overline{m}}, \quad R_{abcd} = \sum_{f=1}^{2n} \hat{\Theta}_f(e_a, e_b) g_{f\overline{d}},$$

where $\Theta^p_{k p}(e_i, \overline{e}_j)$ are the structure constants of the metric.
where $i, j, k, \ell, p$ range from 1 to $n$, while $a, b, c, d, f$ range from 1 to $2n$ with $\epsilon_{n+i} = \tau_i$, and $g_{ab} = g(e_a, e_b)$. It follows that, for any Hermitian connection $D$, $R^D_{ijkl} = R^D_{abij} = 0$ by the discussion above.

For convenience, we will assume from now on that our local $(1, 0)$-frame $e$ is unitary for the Hermitian metric $g$. We have

Lemma 3.1. Let $(M^n, g)$ be a Hermitian manifold. It follows that

\begin{align}
R^s_{ijkl} &= 2(T^e_{ik,j} - T^e_{kj,i}) + 4 \sum_r (T^r_{ij,k} T^r_{kr} + T^r_{jk,i} T^r_{ir} + T^r_{ik,j} T^r_{ri}), \\
R^s_{ijkl} - R^c_{ijkl} &= 2(T^e_{ik,j} + T^e_{jk,i}) - 4 \sum_r (T^r_{kr} T^r_{jr} + T^r_{jr} T^r_{ir}) + T^r_{ik,j} T^r_{ri} - T^r_{ik,j} T^r_{ri}), \\
T^j_{ik,\ell} + T^j_{jk,\ell,i} - T^j_{\ell,i,k} &= -2 \sum_r (T^r_{ik} T^r_{jr} + T^r_{jr} T^r_{ir} - T^r_{kr} T^r_{ir} - T^r_{ir} T^r_{jr}) + \frac{1}{2} (R^r_{ikj} - R^r_{jki}), \\
R^s_{ijkl} + R^s_{pilk,j} + R^s_{pkhl,i} &= -2 \sum_r (R^r_{ikr} T^r_{jp} + R^r_{jkr} T^r_{ip} + R^r_{pkj} T^r_{il}), \\
R^s_{ijkl} - R^s_{ijkl} + R^s_{pilk,j} + R^s_{pkhl,i} &= 2 \sum_r (R^r_{ikr} T^r_{ip} - R^r_{pkj} T^r_{il} + R^r_{jkr} T^r_{ip} + R^r_{jkr} T^r_{ip} + R^r_{ikr} T^r_{ip}) + R^r_{pklr} T^r_{jr},
\end{align}

for any $i, j, k, \ell, p, q$, where indices after comma mean covariant derivatives with respect to $\nabla^s$.

Proof. Fix a point $p \in M$. We will choose our local unitary frame $e$ near $p$ such that $\theta^s(p) = 0$, thus $\theta^c = -2\gamma$ at $p$. From the structure equation $\Theta^s = d\theta^s - \theta^s \wedge \Theta^s$, it yields that

$$\Theta^s = \Theta + 2(d\gamma + 2\gamma \wedge \gamma)$$

at $p$. By comparison for the types of the differential forms we get

$$\Theta^*_{s,0} = 2(\partial \gamma' + 2\gamma' \wedge \gamma'), \quad \Theta^*_{s,1} = \Theta + 2(\overline{\partial} \gamma' - \partial \overline{\gamma}' - 2\gamma' \wedge \overline{\gamma}' - 2\overline{\gamma}' \wedge \gamma'),$$

hence (3.1) and (3.2) are established. From the first Bianchi identity $d\tau = -\psi \wedge \tau + \Theta \wedge \varphi$, it yields at $p$ that

$$\partial \tau = 2\gamma' \wedge \tau, \quad \overline{\partial} \tau = -2\overline{\gamma}' \wedge \tau + \Theta \wedge \varphi,$$

which imply (3.3) and (3.4). By the second Bianchi identity, $d\Theta^s = \theta^s \wedge \Theta^s - \Theta^s \wedge \theta^s$, we have $d\Theta^s = 0$ at $p$ since $\theta^s = 0$ at $p$. This leads to

$$\partial \Theta^s_{2.0} = 0, \quad \overline{\partial} \Theta^s_{2.0} + \partial \Theta^s_{1.1} = 0$$

which yields (3.5) and (3.6).
Lemma 3.2. Let \((M, g)\) be a Hermitian manifold. It follows that

\begin{align*}
T^\ell_{ij,k} &= -\frac{1}{2} (R^r_{jki\ell} + R^r_{kij\ell}), \\
\sum_r (T^r_{ij,T^r_{rk}} + T^r_{jk,T^r_{ri}} + T^r_{ki,T^r_{rj}}) &= -\frac{1}{4} (R^s_{ijk\ell} + R^s_{jki\ell} + R^s_{kij\ell}), \\
T^\ell_{ik,\ell} &= -\frac{2}{3} \sum_r (T^r_{ik,T^r_{\ell j}} + T^r_{ir,T^r_{r\ell}} - T^r_{kr,T^r_{\ell r}} - T^r_{ir,T^r_{j\ell}} + T^r_{kr,T^r_{j\ell}}) + \frac{1}{3} (R^s_{ikj\ell} - R^s_{kij\ell}) + \frac{1}{6} (R^s_{ijk\ell} - R^s_{kij\ell}), \\
\eta_{i,j} &= -\frac{1}{2} \sum_r (R^s_{ijr\ell} + R^s_{jir\ell}), \\
\eta_{i,j} &= -\frac{2}{3} \sum_r (\eta_r T^r_{ij} + \eta_r T^r_{jr} - \sum_{i,r} T^r_{ji}) + \frac{1}{3} \sum_r (R^s_{rij\ell} - R^s_{rij\ell}) + \frac{1}{6} \sum_r (R^s_{rjj\ell} - R^s_{rjj\ell}).
\end{align*}

for any \(i, j, k, \ell\), where indices after comma mean the covariant derivative with respect to \(\nabla^s\).

Proof. Equations (3.1) and (3.3) imply (3.7) and (3.8), and equation (3.4) implies (3.9). Also, since \(\eta_k = \sum_i T^i_{ik}\), (3.10) and (3.11) are implied by (3.7) and (3.9), respectively. \(\square\)

Definition 3.3 (See also (27) and (34) in [33]). Let us introduce the following notations

\begin{align*}
P^\ell_{ik} &:= \sum_r T^r_{ik,T^r_{\ell j}} + T^r_{ir,T^r_{k\ell}} - T^r_{kr,T^r_{\ell r}} - T^r_{ir,T^r_{j\ell}} + T^r_{kr,T^r_{j\ell}}, \\
A^\ell_{ik} &:= \sum_{r,s} T^r_{ik,T^r_{s\ell}} + T^r_{is,T^r_{k\ell}}, \\
B^\ell_{ik} &:= \sum_{r,s} T^r_{ik,T^r_{s\ell}} + T^r_{is,T^r_{k\ell}}, \\
C^\ell_{ik} &:= \sum_{r,s} T^r_{ik,T^r_{s\ell}}, \\
\phi^\ell_k &:= \sum_r T^r_{ir,T^r_{k\ell}}, \\
Q_{ijk\ell} &:= R^r_{ijk\ell} - R^s_{kij\ell}, \\
\text{Ric}(Q)_{ij} &:= \sum_r R^s_{rij\ell} - R^s_{rjj\ell}
\end{align*}

under any unitary frame. It is clear that \(C\) is symmetric, \(A, B\) are Hermitian symmetric, and \(P^\ell_{ik}\) satisfies

\[ P^\ell_{ik} = -P^\ell_{ki} = -P^\ell_{ik} = P^\ell_{jk}. \]

Proposition 3.4. Let \((M^n, g)\) be a Hermitian manifold, denote by \(\omega\) the Kähler form of \(g\). Then

\[ \sqrt{-1} \partial \bar{\partial} \omega = \left\{ \frac{1}{2} (T^\ell_{ik,j} - T^\ell_{ik,j}) - P^\ell_{ik} \right\} \varphi_i \wedge \varphi_k \wedge \bar{\varphi}_j \wedge \bar{\varphi}_\ell. \]

Hence we have the following equivalence

\[ \partial \bar{\partial} \omega = 0 \iff T^\ell_{ik,j} - T^\ell_{ik,j} = -2P^\ell_{ik} \iff (R^s_{ikj\ell} - R^s_{kj\ell}) - (R^s_{jki\ell} - R^s_{kij\ell}) = -4P^\ell_{ik}. \]

Proof. By [33, Lemma 1], we have

\[ \sqrt{-1} \partial \bar{\partial} \omega = ^t\overline{\varphi} + ^t\overline{\varphi} \Theta \overline{\varphi} \]

\[ = \sum_{i,j,k,\ell,p} \left( T^p_{ik,j} + T^p_{jk,i} - \frac{1}{4} (R^s_{ij,\mathcal{K}_{j\ell}} + R^s_{kij\ell} + R^s_{ij,\mathcal{K}_{j\ell}} + R^s_{kij\ell}) \right) \varphi_i \wedge \varphi_k \wedge \bar{\varphi}_j \wedge \bar{\varphi}_\ell. \]

Note that, by [30, Lemma 7], for any \(i, j, k, \ell\),

\[ 2T^\ell_{ik,j} = R^s_{ikj\ell} - R^s_{kij\ell}. \]
where the index after the semicolon stands for covariant derivative with respect to the Chern connection \( \nabla^c \), and it is easy to obtain, since \( \nabla^c - \nabla^s = -2\gamma 
abla^c \),

\[
T_{ik,j}^\ell = \tau^\ell T_{ik}^j - \sum_q \left( T_{qk}^j g(\nabla^c_{\tau} e_i, \tau_q) + T_{iq}^j g(\nabla^c_{\tau} e_k, \tau_q) + T_{ik}^q g(\nabla^c_{\tau} e_j, e_q) \right)
= T_{ik,j}^\ell + 2 \sum_q \left( T_{qk}^j \gamma_{iq}(\tau^\ell) + T_{iq}^j \gamma_{kq}(\tau^\ell) - T_{ik}^q \gamma_{qj}(\tau^\ell) \right)
= T_{ik,j}^\ell - 2 \sum_q \left( T_{qk}^j T_{\bar{q} \ell}^i + T_{iq}^j T_{\bar{q} \ell}^k - T_{ik}^q T_{\bar{q} j}^\ell \right),
\]

(3.13)

where the index after the comma means covariant derivative with respect to \( \nabla^s \). Then the equalities (3.12) and (3.13) imply

\[
\sqrt{-1} \partial \bar{\partial} \omega = \left\{ \frac{1}{2} (T_{ik,j}^\ell - T_{ik,\bar{j}}^\ell) - P_{ik}^j \right\} \varphi_i \wedge \varphi_k \wedge \bar{\varphi}_j \wedge \bar{\varphi}_k.
\]

Therefore the equivalence in the proposition follows from the equality (3.9).

\[\square\]

**Proposition 3.5.** Let \((M^n, g)\) be a Hermitian manifold, with \( \omega \) its Kähler form. Then it holds that

\[
\nabla^1_0 T^c = 0 \iff R^c_{ij,k} = 0 \iff \sum_r (T^r_{ki} T^k_{rj} + T^r_{ji} T^k_{ri} + T^r_{ij} T^k_{r\ell}) = 0,
\]

(3.14)

\[
\nabla^s_0 T^c = 0 \iff R^s_{ij,k} - R^s_{kji} = -4P^s_{ik},
\]

(3.15)

where \( \nabla^1_0 T^c \) and \( \nabla^s_0 T^c \) are respectively the \((1,0)-\) and \((0,1)-\) components of \( \nabla^s T^c \).

**Proof.** It is clear that the condition \( \nabla^s_0 T^c = 0 \) (that is, \( T^s_{ij,k} = 0 \)) implies \( \sum_r (T^r_{ki} T^k_{rj} + T^r_{ji} T^k_{ri} + T^r_{ij} T^k_{r\ell}) = 0 \) by (3.3). Thus \( R^c_{ij,k} = 0 \) follows from (3.1). Conversely, by (3.7) and (3.8), \( R^c_{ij,k} = 0 \) implies that \( T^s_{ij,k} = 0 \) and \( \sum_r (T^r_{ki} T^k_{rj} + T^r_{ji} T^k_{ri} + T^r_{ij} T^k_{r\ell}) = 0 \). Therefore, the conclusion (3.14) is established.

In the mean time, by (3.4), we know that the condition \( \nabla^s_0 T^c = 0 \) (that is, \( T^s_{ik,\ell} = 0 \)) implies that

\[
R^s_{iklj} - R^s_{klij} = 4P^s_{ik},
\]

which yields

\[
R^s_{ij,k} - R^s_{kij} = 4P^s_{ik} = -4P^s_{ik}.
\]

Conversely, if \( R^s_{ij,k} - R^s_{kij} = -4P^s_{ik} \), then by the fact that \( P^s_{ik} = -P^s_{ik} \) we get

\[
R^s_{ij,k} - R^s_{kij} = (R^s_{iklj} - R^s_{klij}),
\]

so by (3.9) we have

\[
T^s_{ik,\ell} = -\frac{2}{3} P^s_{ik} - \frac{1}{6} (R^s_{ij,k} - R^s_{kij}) = -\frac{2}{3} P - \frac{1}{6} (-4P) = 0.
\]

This establishes the equivalence of (3.15).

\[\square\]

**Proposition 3.6.** Let \((M^n, g)\) be a Hermitian manifold with \( \nabla^s T^c = 0 \). Then it holds that

1. \( R^s_{ij,k} = 0 \),
2. \( R^s_{ij,k} = R^s_{kij} \),
3. \( \nabla^s Q = 0 \),
4. \( R^s_{x,y,w} = 0 \), where \( x, y, w \) are any tangent vector and \( \chi \) is the associated vector field of Gauduchon’s torsion 1-form \( \eta \),

for any \( i, j, k, \ell \). Also, under the assumption \( \nabla^s T^c = 0 \), we have

\[
\partial \bar{\partial} \omega = 0 \iff P^s_{ik} = 0 \iff R^s_{ij,k} = R^s_{kij},
\]

(3.16)

\[
\partial \eta = 0, \quad \overline{\partial} \eta = -2 \sum_{i,j,k} \eta_{ik} \bar{\eta}_{j,k} \wedge \bar{\eta}_j.
\]

(3.17)

In particular, the metric \( g \) is Gauduchon, namely, \( \partial \bar{\partial} \omega^{n-1} = 0 \).

**Proof.** Let \((M^n, g)\) be a Hermitian manifold satisfying the condition \( \nabla^s T^c = 0 \), which means \( \nabla^1_0 T^c = 0 \) and \( \nabla^s_0 T^c = 0 \). By (3.14) and (3.15), we get (1) and (3) immediately, as \( P^s_{ik} \) is \( \nabla^s \)-parallel. Since
Gauduchon’s torsion 1-form $\eta = \sum_k \eta_k \varphi_k = \sum_{i,k} T^k_{ij} \varphi_k$ is the trace of the Chern torsion $T^s$, its associated vector field $\chi = \sum_k \chi_k e_k$ is clearly $\nabla^s$-parallel and thus

$$R^s_{xywu} = g \left( (\nabla^s_x \nabla^s_y - \nabla^s_y \nabla^s_x - \nabla^s_{[x,y]}) \chi, \omega \right) = 0,$$

which yields (4). As to (2), from $R^s_{ij\ell k} - R^s_{kij\ell} = -4 P_{ik}^\ell$, it yields that

$$R^s_{ij\ell k} - R^s_{kij\ell} = R^s_{ij\ell k} - R^s_{kij\ell} = -4 P_{ik}^\ell + (R_{jk\ell i}^s - R_{jk\ell i}^s)$$

$$= -4 P_{ik}^\ell - \frac{4}{4} P_{ik}^\ell$$

$$= -4 P_{ik}^\ell - 4 P_{ik}^\ell = 0.$$

Under the assumption $\nabla^s T^s = 0$, the equivalence (3.16) follows from Proposition 3.4 and (3.15). For any given point $p \in M$, let $e$ be a local unitary frame near $p$ such that $\theta^s|_p = 0$. The structure equation gives us $d\varphi = -4 \theta \wedge \varphi + \tau = 2 \gamma \wedge \varphi + \tau$, hence $\partial \varphi = -\tau$ and $\bar{\partial} \varphi = 2 \gamma \wedge \varphi$ at $p$. Therefore

$$\partial \eta = \partial (\sum \eta_i \varphi_i) = -\sum_{i,j} \eta_{ij} \varphi_i \wedge \varphi_j - \sum \eta_{ik} T^k_{ij} \varphi_i \wedge \varphi_j,$$

$$\bar{\partial} \eta = \bar{\partial} (\sum \eta_i \varphi_i) = -\sum_{i,j} \eta_{ij} \varphi_i \wedge \bar{\varphi}_j - 2 \sum \eta_{ik} T^k_{ij} \varphi_i \wedge \bar{\varphi}_j.$$

Here the indices after comma mean the covariant derivative with respect to $\nabla^s$. It is clear that $\nabla^s \eta = 0$ and thus $\eta_{i,j} = 0$, $\eta_{i,j} = 0$. Also, (3.14) implies that

$$\sum_r (T^r_{ik} T^k_{rj} + T^r_{ij} T^k_{rk} + T^r_{ij} T^k_{rk}) = 0.$$

Let $k = \ell$ in the equation above and sum up, which yields that $\sum_r \eta T^r_{ij} = 0$. Hence,

$$\partial \eta = 0, \quad \bar{\partial} \eta = -2 \sum \eta_{ik} T^k_{ij} \varphi_i \wedge \bar{\varphi}_j.$$

From the definition of $\eta$, it follows that $\partial \omega^{n-1} = -2 \eta \wedge \omega^{n-1}$ and thus

$$-\sqrt{-1} \partial \partial \omega^{n-1} = -2 \sqrt{-1} (\partial \eta + 2 \eta \wedge \eta) \wedge \omega^{n-1} = 2 \sum \eta_{i,j} \omega_i^j \wedge \omega^{n-1} = 0.$$

This completes the proof of the proposition. \qed

4. Manifolds with Strominger parallel torsion

In this section, we will prove Theorem 1.1, which characterizes Hermitian manifolds with Strominger parallel torsion in terms of the behavior of the Strominger curvature $R^s$.

First we will give the following two technical lemmata, the latter has been already shown in the proof of [33, Theorem 2], but we include the proof here for readers’ convenience.

**Lemma 4.1.** Let $(M^s, g)$ be a Hermitian manifold. Then the following are equivalent

1. $R^s_{ij\ell k} = R^s_{kij\ell}$ holds. It follows that

$$Q_{ij\ell k} = R^s_{ij\ell k} - R^s_{kij\ell} = R^s_{kij\ell} - R^s_{i\ell kj} = -Q_{i\ell kj},$$

$$\bar{Q}_{ij\ell k} = R^s_{j\ell ik} - R^s_{ikj\ell} = R^s_{ikj\ell} - R^s_{i\ell kj} = Q_{i\ell kj}.$$

By equation (3.9), we get

$$T^s_{ik,\ell} = -\frac{2}{3} P_{ik}^\ell - \frac{1}{6} Q_{ij\ell k},$$
which implies that
\[ T^j_{ik,\ell} = -T^\ell_{ik,j} = T^i_{j\ell,ik}. \]
Conversely, assuming that the equality \( T^j_{ik,\ell} = -T^\ell_{ik,j} = T^i_{j\ell,ik} \) holds. Then by (3.4) we get
\[ Q_{ijk\ell} = R^s_{ijk\ell} - R^s_{kij\ell} = 6T^j_{ik,\ell} + 4P^j_{ik,\ell}, \]
which implies that \( Q_{ijk\ell} = -Q_{i\ell kj} \) and \( Q_{ijk\ell} = Q_{ji\ell k} \).

This completes the proof of the lemma. \( \square \)

**Lemma 4.2.** Let \((M^n, g)\) be a Hermitian manifold. If \( R^s_{ijk\ell} = 0 \) for any \( i, j, k, \ell \), then the following commutation formulae hold

1. \( \eta_i, \bar{\pi}_k \eta_j - \eta_j, \bar{\pi}_k \eta_i = 2 \sum_{\ell} T^\ell_{jk,p} \eta_{k,p} \),
2. \( T^i_{jk,\bar{\pi}k} - T^i_{jk,\bar{\pi}k} = 2 \sum_{\ell} T^\ell_{jk,p} T^i_{jk,\bar{\pi}k} \),

for any \( i, j, k, \ell, p \). Again indices after comma stand for covariant derivatives with respect to \( \nabla^s \).

**Proof.** First let us prove the second equality. From the definition of covariant derivatives,
\[ T^i_{jk,\bar{\pi}k} = \bar{\pi}_k T^i_{jk} + \sum_{\ell} T^i_{jk,\bar{\pi}k} \langle \nabla^s_{\bar{\pi}_k} \epsilon_j, \epsilon_r \rangle - T^i_{jk,\bar{\pi}k} \langle \nabla^s_{\bar{\pi}_k} \epsilon_r, \epsilon_r \rangle + T^i_{jk,\bar{\pi}k} \langle \nabla^s_{\bar{\pi}_k} \epsilon_r, \epsilon_r \rangle, \]
Fix any given point in \( M \), choose a local unitary frame \( e \) so that the matrix \( \theta^s = 0 \) at the point, then we have
\[ T^i_{jk,\bar{\pi}k} = \bar{\pi}_k T^i_{jk} + \sum_{\ell} T^i_{jk,\bar{\pi}k} \langle \nabla^s_{\bar{\pi}_k} \epsilon_j, \epsilon_r \rangle - T^i_{jk,\bar{\pi}k} \langle \nabla^s_{\bar{\pi}_k} \epsilon_r, \epsilon_r \rangle + T^i_{jk,\bar{\pi}k} \langle \nabla^s_{\bar{\pi}_k} \epsilon_r, \epsilon_r \rangle, \]
which implies that
\[ T^i_{jk,\bar{\pi}k} - T^i_{jk,\bar{\pi}k} = \bar{\pi}_k T^i_{jk} - \sum_{\ell} T^i_{jk,\bar{\pi}k} R^s_{jk,p\ell} - T^i_{jk,\bar{\pi}k} R^s_{jk,p\ell} + T^i_{jk,\bar{\pi}k} R^s_{jk,p\ell}. \]
At that point, we have
\[ \bar{\pi}_k = \nabla_{\bar{\pi}_k} - \nabla_{\bar{\pi}_k} = -2T^i_{jk,p\ell}, \]
where \( \nabla \) is the Riemannian connection. So the second equality in the lemma is established since \( R^s_{ijk\ell} = 0 \).

By letting \( i = j \) in the second equality and sum over, we get the first equality. \( \square \)

Now we are ready to prove the main result of this article, Theorem 1.1.

**Proof of Theorem 1.1:** By Proposition 3.6, it is clear that \( \nabla^s T^c = 0 \) implies the four conditions in Theorem 1.1, so we just need to prove the converse. We will follow the strategy of the proof of [33, Theorem 2]. Note that the first condition is equivalent to \( \nabla^s_{1,0} T^c = 0 \), by (3.14), which implies
\[ \sum_r (T^r_{jk} T^k_{ij} + T^r_{jk} T^k_{ij} + T^r_{jk} T^k_{ij}) = 0, \]
and the fourth condition means
\[ \sum_{i,j} \eta_i R^s_{ijrf} - R^s_{rijf} = 0, \]
for any \( i, j, \ell \). In the mean time, Ric\((Q)\) is Hermitian symmetric due to \( R^s_{ijk\ell} = R^s_{kij\ell} \), as
\[ \text{Ric}(Q)_{ij} = \sum_r R^s_{ijrf} - R^s_{rijf} = \sum_r R^s_{ijrf} - R^s_{rijf} = \sum_r R^s_{ijrf} - R^s_{rijf} = \text{Ric}(Q)_{ij}. \]

Firstly, we will show that \( |\eta|^2 \) is a constant. We have \( \eta_{i,j} = 0 \) since \( R^s_{ijk\ell} = 0 \). By (3.11) and the symmetry assumption \( R^s_{ijk\ell} = R^s_{kij\ell} \), we get
\[ |\eta|^2 = \sum_i \eta_{i,j} \bar{\eta}_i = -\frac{3}{2} \left( \sum_r \eta_{i,j} \bar{T}^r_{jk} + \eta_{i,j} T^r_{jk} - \sum_r T^r_{jk} T^r_{jk} \bar{\eta}_i \right) - \frac{1}{6} \sum_{i,j} (R^s_{ijrf} - R^s_{rijf}) \bar{\eta}_i = 0 \]
for any \( j \). Here we used (4.1) and the fact \( \sum_j \eta_j T^r_{ij} = 0 \), which follows from the identity \( \sum_r (T^r_i T^k_r + T^k_j T^r_i + T^r_j T^k_r) = 0 \). Similarly \( |\eta|^2_j = 0 \) also holds, thus \( |\eta|^2 \) is a constant.

Secondly, we will show that \( \nabla^* \eta = 0 \), which is equivalent to \( \nabla^* \eta_{0,1} = 0 \). As \( |\eta|^2 \) is a constant, it yields, after the covariant derivative in \( \bar{\ell} \) is taken, that

\[
\sum_k \eta_{k,\bar{\ell}} = 0.
\]

Take the covariant derivative in \( \ell \) and sum up \( \ell \), it follows that

\[
(4.2) \quad \sum_{k,\ell} \left( |\eta_{k,\ell}|^2 + \eta_{k,\ell} \overline{\eta_{k,\ell}} \right) = 0.
\]

The equality (3.11) and \( R^*_i j k \ell = R^*_k \ell i j \) imply that

\[
(4.3) \quad \eta_{i,j} = -\frac{2}{3} \left( \phi^{i}_{\ell} + \phi^{j}_{\ell} - B_{\ell ij} \right) - \frac{1}{6} \text{Ric}(Q)_{ij},
\]

thus \( \eta_{i,j} \) is Hermitian symmetric in \( i, j \) since \( \text{Ric}(Q)_{ij} \) is so. It yields

\[
\sum_{k,\ell} |\eta_{k,\ell}|^2 = \frac{4}{9} |\phi + \phi^* - B|^2 + \frac{1}{36} |\text{Ric}(Q)|^2 + \frac{2}{9} \text{Re} \left( \phi \text{Ric}(Q) + \overline{\phi \text{Ric}(Q)} - B \text{Ric}(Q) \right),
\]

\[
= \frac{4}{9} \left( |\phi + \phi^*|^2 + |B|^2 - 4 \text{Re}(\phi B) \right) + \frac{1}{36} |\text{Ric}(Q)|^2 + \frac{4}{9} \text{Re}(\phi \text{Ric}(Q)) - \frac{2}{9} \text{Re}(B \text{Ric}(Q)),
\]

where

\[
|\phi + \phi^* - B|^2 = \sum_{k,\ell} (\phi_{\ell}^k + \overline{\phi_{\ell}^k} - B_{\ell k})(\phi_{\ell}^k + \overline{\phi_{\ell}^k} - B_{\ell k}), \quad |\text{Ric}(Q)|^2 = \sum_{k,\ell} \text{Ric}(Q)_{k \ell} \overline{\text{Ric}(Q)_{k \ell}},
\]

\[
\phi \text{Ric}(Q) = \sum_{k,\ell} \phi_{\ell}^k \text{Ric}(Q)_{k \ell}, \quad B \text{Ric}(Q) = \sum_{k,\ell} B_{k \ell} \text{Ric}(Q)_{k \ell}, \quad \phi B = \sum_{k,\ell} \phi_{\ell}^k B_{k \ell}.
\]

Note that

\[
|\phi + \phi^*|^2 = \sum_{k,\ell} (\phi_{\ell}^k + \phi_{\ell}^\ell)(\phi_{\ell}^k + \phi_{\ell}^\ell) = 2 |\phi|^2 + 2 \text{Re}(\phi \cdot \phi),
\]

where

\[
\phi \cdot \phi = \sum_{k,\ell} \phi_{\ell}^k \phi_{\ell}^\ell.
\]

Then it follows from Lemma 4.2, the \( \nabla^* \)-parallelness of \( \text{Ric}(Q) \), (4.3) and Lemma 4.1 that

\[
\sum_{\ell} \eta_{\ell,\bar{\ell}} = \sum_{\ell} \left( \eta_{\ell,\bar{\ell}} \ell \right) = \sum_{\ell} \left( \eta_{\ell,\bar{\ell}} \ell \right) + 2 \sum_{r} \overline{T_{\ell k}^r \eta_{r,\bar{\ell}}}
\]

\[
= \left( \sum_{\ell} \eta_{\ell,\bar{\ell}} \right)_{\bar{\ell}} + 2 \sum_{r} T_{\ell k}^r \eta_{r,\bar{\ell}}
\]

\[
= \left( \frac{2}{3} |T|^2 - 2 |\eta|^2 \right)_{\bar{\ell}} - \frac{1}{6} \sum_{r} \text{Ric}(Q)_{r \bar{\ell}} + 2 \sum_{r} T_{\ell k}^r \eta_{r,\bar{\ell}}
\]

\[
= \frac{2}{3} \sum_{i,j,r} T_{i j}^r T_{j r,\bar{\ell}} - \frac{4}{3} \sum_{i,j,r} T_{i k}^r \left( \phi_{i \ell}^r + \phi_{i \ell}^\ell - B_{i \ell,\bar{\ell}} + \frac{1}{4} \text{Ric}(Q)_{r \bar{\ell}} \right)
\]

\[
= \frac{2}{3} \sum_{i,j,r} T_{i j}^r T_{j k,\bar{\ell}} - \frac{4}{3} \sum_{i,j,r} T_{i k}^r \left( \phi_{i \ell}^r + \phi_{i \ell}^\ell - B_{i \ell,\bar{\ell}} + \frac{1}{4} \text{Ric}(Q)_{r \bar{\ell}} \right),
\]
Hence, it yields that, from (3.9), $R^r_{ij\ell\ell} = R^r_{k\ell\ell j}$, and $\sum_r \eta_r T^r_{ij} = 0,$

$$\sum_{k,\ell} \eta_{k,\ell} \overline{\eta}_{k,\ell} = \frac{2}{3} \sum_{i,j,k,r} \overline{\eta}_k T^r_{ij} T^i_{k,\ell} - \frac{4}{3} \sum_{k,\ell,r} \overline{\eta}_k T^r_{i,k} \left( \phi^\ell_r + \overline{\phi}^\ell_r - B_{r,\ell} + \frac{1}{4} \text{Ric}(Q)_{r,\ell} \right)$$

$$= -\frac{4}{9} \sum_{i,j,k,r} \overline{\eta}_k T^i_{jr} \left( \sum_q T^r_{iq \overline{\eta}^q_{k\ell}} + T^r_{iq \overline{\eta}^q_{k\ell}} - T^r_{iq \overline{\eta}^q_{k\ell}} + T^r_{iq \overline{\eta}^q_{k\ell}} + \frac{1}{4} \left( R^r_{ijk\ell} - R^r_{k\ell ij} \right) \right)$$

$$= -\frac{4}{9} \sum_{i,j,k,r} \overline{\eta}_k T^i_{jr} \left( \sum_q T^r_{iq \overline{\eta}^q_{k\ell}} + T^r_{iq \overline{\eta}^q_{k\ell}} - T^r_{iq \overline{\eta}^q_{k\ell}} + T^r_{iq \overline{\eta}^q_{k\ell}} + \frac{1}{4} \left( R^r_{ijk\ell} - R^r_{k\ell ij} \right) \right)$$

where

$$\phi A = \sum_{k,\ell} \phi^k_r A_{k,\ell}.$$ 

Hence, the equality (4.2) implies

$$\sum_{k,\ell} |\eta_{k,\ell}|^2 + \text{Re}(\eta_{k,\ell} \overline{\eta}_{k,\ell}) = 0,$$

which yields

$$|B|^2 - 2\text{Re}(\phi B) + 2\text{Re}(\phi A) - \frac{1}{2} \text{Re}(BRic(Q)) = \frac{1}{2} |\phi + \phi^*|^2 - \frac{1}{4} \text{Re}(\phi \text{Ric}(Q))$$

(4.4)

$$= -\frac{1}{16} |\text{Ric}(Q)|^2 + \frac{1}{4} \text{Re} \left( \sum_{i,j,k,r} \overline{\eta}_k T^i_{jr} \text{Q}_{ij\ell\ell} \right).$$

As $\sum_r \eta_r T^r_{ij} = 0$ has already been shown, it follows that, after the covariant derivative in $\ell$ is taken,

$$\sum_{r} \eta_{r,\ell} T^r_{ij} + \eta_r T^r_{ij,\ell} = 0,$$

which yields

$$\sum_{r} \left( \phi^r_{\ell} + \overline{\phi}^r_{\ell} - B_{r,\ell} + \frac{1}{4} \text{Ric}(Q)_{r,\ell} \right) T^r_{ij}$$

$$+ \sum_{r} \eta_r \left( \sum_q T^r_{ij \overline{\eta}^q_{k\ell}} + T^r_{iq \overline{\eta}^q_{k\ell}} - T^r_{iq \overline{\eta}^q_{k\ell}} + T^r_{iq \overline{\eta}^q_{k\ell}} + \frac{1}{4} \left( R^r_{i\ell j\ell} - R^r_{i\ell j\ell} \right) \right) = 0,$$

or equivalently,

(4.5)

$$\sum_{r} \left( \phi^r_{\ell} - B_{r,\ell} \right) T^r_{ij} + \frac{1}{4} \text{Ric}(Q)_{r,\ell} T^r_{ij} + \phi^r_{\ell} T^r_{ir} - \overline{\phi}^r_{\ell} T^r_{jr} + \frac{1}{4} \eta_r \text{Q}_{i\ell j\ell} = 0.$$

Multiply $\overline{\eta}_j$ on both sides of (4.5) and sum up $j$, it yields

$$\sum_{r} \left( \phi^r_{\ell} - \phi^r_{\ell} B_{r,\ell} + \frac{1}{4} \phi^r_{\ell} \text{Ric}(Q)_{r,\ell} + \phi^r_{\ell} \overline{\phi}^r_{\ell} + \frac{1}{4} \sum_j \eta_j \overline{\eta}_j \text{Q}_{i\ell j\ell} \right) = 0,$$

where $\sum_j \eta_j T^j_{ij} = 0$ is used here. Let $i = \ell$ and sum up $i$, it implies from (4.1) that

$$\phi B = \phi \cdot \phi + \frac{1}{4} \phi \text{Ric}(Q) + |\phi|^2,$$

and thus

(4.6) \hspace{1cm} \text{Re}(\phi B) = \text{Re}(\phi \cdot \phi) + \frac{1}{4} \text{Re}(\phi \text{Ric}(Q)) + |\phi|^2 = \frac{1}{2} |\phi + \phi^*|^2 + \frac{1}{4} \text{Re}(\phi \text{Ric}(Q)).$$

Similarly, multiply $\overline{T^j_{ij}}$ on both sides of (4.5) and sum up $i, j, \ell$, it yields

$$\phi B - |B|^2 + \frac{1}{4} B \text{Ric}(Q) + 2\overline{\phi} A + \frac{1}{4} \sum_{i,j,\ell} \eta_r \overline{T^r_{ij}} \text{Q}_{ij\ell\ell} = 0,$$

where $\sum_{i,j,\ell} \eta_r \overline{T^r_{ij}} = 0$ has already been shown.
and thus, by Lemma 4.1,

\begin{equation}
|B|^2 - \Re(\phi B) - 2\Re(\overline{\phi A}) - \frac{1}{4} \Re(B\Ric(Q)) = -\frac{1}{4} \Re \left( \sum_{i,j,\ell} n_i T^i_{ij} Q_{\ell ij} \right).
\end{equation}

Note that the equality (4.1) implies that

\[ \sum_i \eta_i \Ric(Q)_{ij} = 0, \]

for any \( j \). Take the covariant derivative in \( k \) of the equality above, it yields

\[ \sum_i \eta_i, \overline{k} \Ric(Q)_{ij} = 0, \]

where \( \nabla^* \Ric(Q) = 0 \) is used here, and thus

\[ \sum_i \left( \phi^*_i k + \overline{\phi^*_k} - B_{ik} + \frac{1}{4} \Ric(Q)_{ik} \right) \Ric(Q)_{ij} = 0. \]

Let \( j = k \) and sum up \( j \), we get

\[ \phi \Ric(Q) + \overline{\phi \Ric(Q)} - B \Ric(Q) + \frac{1}{4} |\Ric(Q)|^2 = 0, \]

and thus

\begin{equation}
\Re(B\Ric(Q)) = 2\Re(\phi \Ric(Q)) + \frac{1}{4} |\Ric(Q)|^2.
\end{equation}

In the meantime, we will also use the equality

\[ \sum_r (T^k_r T^k_{\ell r} + T^\ell_r T^k_{\ell r} + T^k_r T^k_{\ell r}) = 0. \]

Multiply \( \overline{T^k_{\ell r}} \) on both sides and sum up \( j, k, \ell \), we get

\[ \sum_{r,\ell} T^r_{\ell i} A_{\ell \overline{r}} - \sum_{k, r} T^k_{\ell i} B_{k \overline{r}} + \sum_{j, r} T^r_{j i} A_{\overline{r} j} = 0. \]

Hence

\begin{equation}
2 \sum_{r,\ell} T^r_{\ell i} A_{\ell \overline{r}} = \sum_{r,\ell} T^r_{\ell i} B_{r \overline{r}}.
\end{equation}

Multiply \( \eta_i \) on both sides above and sum up \( i \), we get

\[ 2\phi A = \phi B, \]

and thus

\begin{equation}
2\Re(\phi A) = \Re(\phi B).
\end{equation}

Therefore, the equalities (4.4), (4.6), (4.7), (4.8) and (4.10) imply that

\[ |B|^2 = |\phi + \phi^*|^2 + \Re(\phi \Ric(Q)) + \frac{1}{16} |\Ric(Q)|^2, \quad \Re(\phi B) = \frac{1}{2} |\phi + \phi^*|^2 + \frac{1}{4} \Re(\phi \Ric(Q)), \]

\[ \Re(\phi A) = \frac{1}{4} |\phi + \phi^*|^2 + \frac{1}{8} \Re(\phi \Ric(Q)), \quad \Re(B\Ric(Q)) = 2\Re(\phi \Ric(Q)) + \frac{1}{4} |\Ric(Q)|^2, \]

\[ \Re \left( \sum_{i,j,k,r} n_k T^i_{jr} Q_{ijkr} \right) = 0. \]

which yields

\[ \sum_{k,\ell} |n_{k,\ell}|^2 = \frac{4}{9} \left( |\phi + \phi^*|^2 + |B|^2 - 4\Re(\phi B) \right) + \frac{1}{36} |\Ric(Q)|^2 + \frac{4}{9} \Re(\phi \Ric(Q)) - \frac{2}{9} \Re(B\Ric(Q)) = 0, \]

hence \( \nabla^* \eta = 0 \) is proved.

Thirdly, we will show that \( |T|^2, \phi^*_i \) and \( B_{ij} \) are \( \nabla^* \)-parallel. To this end, the equality (4.3) implies that

\[ 0 = \sum \eta_{i,\overline{a}} = \frac{2}{3} (|T|^2 - 2|\eta|^2) - \frac{1}{6} \sum \Ric(Q)_{i\overline{a}}, \]
and it follows that $|T|^2$ is $\nabla^s$-parallel, since $|\eta|^2$ and $\text{Ric}(Q)$ are. It is clear that $\phi^j_{i,k} = 0$ now due to $\nabla^s \eta = 0$, while it is easy to see

$$\phi^j_{i,k} = \sum_r \eta_r T^j_{ir,k} = - \sum_r \eta_r T^j_{ri,k} = - \sum_r \eta_r T^j_{rj,k} = 0,$$

where Lemma 4.1 and $\sum_r \eta_r T^j_{ij} = 0$ are used. Hence $\phi^j_i$ is $\nabla^s$-parallel. Note that the equality (4.3) implies

$$0 = \eta_{ij} = - \frac{3}{2} \left( \phi^j_i + \phi^j_j - B_{ij} + \frac{1}{4} \text{Ric}(Q)_{ij} \right),$$

which yields that $B_{ij}$ is $\nabla^s$-parallel, as $\phi^j_i$ and $\text{Ric}(Q)$ are.

Finally, we are ready to show that $\nabla^s T^c = 0$. It follows from (4.9) that

$$(4.11) \quad \sum_{i,j,r} (T^i_{i,j} A_{r,i}) = \frac{1}{2} \sum_{i,j,r} (T^i_{i,j} B_{r,i}) = \frac{1}{2} \sum_{i,j,r} T^i_{i,j} B_{r,i} = - \frac{1}{2} \sum_{i,j,r} T^i_{i,j} B_{r,i} = \frac{1}{2} \sum_{i,j,r} \eta_{i,j} B_{r,i} = 0,$$

where Lemma 4.1 is used. Then, since $|T|^2$ is $\nabla^s$-parallel, it follows, after the covariant derivative in $\bar{\ell}$ is taken,

$$0 = |T|^2_{i,\bar{\ell}} = \sum_{i,j,k} \bar{T}^j_{ik} T^j_{ik,\bar{\ell}} = 0.$$

Take the covariant derivative in $\bar{\ell}$ and sum up $\ell$, we get

$$\sum_{i,j,k,\ell} |T^j_{ik,\bar{\ell}}|^2 + \bar{T}^j_{ik} T^j_{ik,\bar{\ell}} = 0.$$

Note that, from Lemma 4.1, Lemma 4.2 and $\nabla^s \eta = 0$,

$$\sum_{i,j,k,\ell} \bar{T}^j_{ik} T^j_{ik,\bar{\ell}} = \sum_{i,j,k,\ell} \bar{T}^j_{ik} \left( \sum_{j,\ell} \bar{T}^j_{\ell} \right)_{ik,\bar{\ell}} = \sum_{i,j,k,\ell} \bar{T}^j_{ik} \left( \sum_{j,\ell} \bar{T}^j_{\ell} \right)_{ik,\bar{\ell}}_{\ell} = - \sum_{i,j,k,\ell} \bar{T}^j_{ik} T^j_{i,k,\ell} + 2 \sum_{i,j,k,\ell} \bar{T}^j_{ik} \eta_{j,k,\ell} + 2 \sum_{i,j,k,\ell} \bar{T}^j_{ik} A_{r,k,\ell,\bar{\ell}} = - 2 \sum_{k,j,\ell} \bar{T}^j_{k,l} A_{r,k,\ell} + 2 \sum_{k,j,\ell} \eta_{j,k,\ell} A_{r,k} = - 2 \sum_{k,j,\ell} \bar{T}^j_{k,l} A_{r,k,\ell} + 2 \sum_{k,j,\ell} \eta_{j,k,\ell} A_{r,k} = - 2 \sum_{k,j,\ell} \bar{T}^j_{k,l} A_{r,k,\ell} + 2 \sum_{k,j,\ell} \eta_{j,k,\ell} A_{r,k} = - 2 \sum_{k,j,\ell} \bar{T}^j_{k,l} A_{r,k,\ell} + 2 \sum_{k,j,\ell} \eta_{j,k,\ell} A_{r,k} = - 2 \sum_{k,j,\ell} \bar{T}^j_{k,l} A_{r,k,\ell} + 2 \sum_{k,j,\ell} \eta_{j,k,\ell} A_{r,k} = 0,$$

where the last equality is due to (4.11). Therefore, $T^j_{ik,\bar{\ell}} = 0$ and the proof is completed. \[ \square \]

Next we prove Corollary 1.2. The proof is the same as the SKL case in [33], and we include it here for the sake of completeness.

**Proof of Corollary 1.2:** By Proposition 3.6, we know that $\partial \eta = 0$. If $g$ is strongly Gauduchon, that is, there exists a $(n, n - 2)$-form $\Omega$ on $M^n$ such that $\partial \omega^{n-1} = \overline{\partial} \Omega$, then by the fact that $\partial \eta = 0$ and $\partial \omega^{n-1} = -2 \eta \omega^{n-1}$, we get

$$\int_M 2 \eta \overline{\partial} \omega^{n-1} = \int_M \overline{\partial} \partial \omega^{n-1} = \int_M \overline{\partial} \overline{\partial} \partial \Omega = \int_M \overline{\partial} \partial \Omega = 0.$$

Hence $|\eta|^2 = 0$, contradicting to our assumption that $g$ is not balanced. Similarly, if $\overline{\partial} = \partial f$ for some smooth function $f$ on $M^n$, then

$$\int_M \partial \overline{\partial} \omega^{n-1} = \int_M \partial \partial f \wedge \omega^{n-1} = 0$$

as $g$ is Gauduchon. However, Proposition 3.6 implies $\overline{\partial} \eta = 2 \sum_{i,j} \phi^i_j \phi^j_i \wedge \overline{\partial}$. This leads to the vanishing of $\eta$, hence $g$ is balanced, which is a contradiction. This completes the proof of Corollary 1.2. \[ \square \]

From the proof of Theorem 1.1 above, we know that on a SPT manifold it holds that

$$(4.12) \quad \phi + \phi^* = B - \frac{1}{4} \text{Ric}(Q).$$

The right hand side of (4.12) is not necessarily non-negative, so the proof of [33, Theorem 6] does not go through, that is why we cannot prove Conjecture 1.3 here for non-balanced SPT manifolds. If $\text{Ric}(Q) = 0$, or more generally if $4B \geq \text{Ric}(Q)$, then Conjecture 1.3 holds.
Now let us prove Proposition 1.4, where the first part is due to Andrade and Villacampa [2, Theorem 3.6]. For readers convenience we gave a proof here as well.

**Proof of Proposition 1.4:** Let \((M^n, g)\) be a locally conformal Kähler manifold. Then under any local unitary frame \(e\), the torsion components are given by

\[
T^j_{ik} = \frac{1}{n-1} (\eta_k \delta_{ij} - \eta_i \delta_{kj})
\]

and the torsion 1-form \(\eta\) is equal to \(\eta = (n-1)\partial u\) for some smooth real-valued local function \(u\). For any given point \(p \in M\), choose the local unitary frame \(e\) so that the matrix \(\theta^s\) of Strominger connection vanishes at \(p\). Denote by \(\varphi\) the local unitary coframe dual to \(e\). By [30], the Levi-Civita connection \(\nabla\) has connection matrix

\[
\nabla e = \theta^\perp e + \partial u \varepsilon, \quad \theta^\perp = \theta^s - \gamma,
\]

\(
\gamma_{ij} = \sum_k (T^j_{ik} \varphi_k - T^i_{jk} \varphi_k), \quad \theta^2_{ij} = \sum_k T^k_{ij} \varphi_k.
\)

Since \(\theta^s_p = 0\), we have at point \(p\) that

\[
\nabla \varphi_i = -\theta^1_{ji} \varphi_j - \theta^2_{ji} \varphi_j = \gamma_{ji} \varphi_j
\]

\[
= \sum_{j,k} (T^i_{jk} \varphi_k \otimes \varphi_j - T^i_{kj} \varphi_k \otimes \varphi_j - T^j_{ki} \varphi_k \otimes \varphi_j)
\]

\[
= \frac{1}{n-1} \left\{ \eta \otimes \varphi_i - \varphi_i \otimes \eta + \varphi_i \otimes \overline{\eta} - \overline{\eta} \otimes \varphi_i + \eta_i \sum_k (\overline{\varphi}_k \otimes \varphi_k - \varphi_k \otimes \overline{\varphi}_k) \right\}
\]

\[
= \frac{2}{n-1} \left\{ \eta \wedge \varphi_i + \varphi_i \wedge \overline{\eta} - \eta_i \sum_k \varphi_k \wedge \overline{\varphi}_k \right\}
\]

Therefore

\[
\nabla \eta = \nabla \sum_i \eta_i \varphi_i = \sum_{i,j} (\eta_{i,j} \varphi_j \otimes \varphi_i + \varphi_i \overline{\varphi}_j \otimes \varphi_i) + \sum_i \eta_i \nabla \varphi_i
\]

\[
= \sum_{i,j} (\eta_{i,j} \varphi_j \otimes \varphi_i + \eta_{i,j} \overline{\varphi}_j \otimes \varphi_i) + \frac{2}{n-1} \{ \eta \wedge \overline{\eta} - \sum_i |\eta_i|^2 \sum_k \varphi_k \wedge \overline{\varphi}_k \},
\]

where indices after comma denote covariant derivatives with respect to \(\nabla^s\). If \((M^n, g)\) is SPT, then \(\nabla^s \eta = 0\), hence \(\nabla (\eta + \overline{\eta}) = 0\) and \(g\) is Vaisman. Conversely, when \(g\) is Vaisman, which means the manifold is locally conformal Kähler with \(\nabla (\eta + \overline{\eta}) = 0\), then from the above calculation we get

\[
\sum_{i,j} \{ \eta_{i,j} \varphi_j \otimes \varphi_i + \eta_{i,j} \overline{\varphi}_j \otimes \varphi_i + \eta_{i,j} \varphi_j \otimes \varphi_i + \eta_{i,j} \overline{\varphi}_j \otimes \varphi_i \} = 0.
\]

Hence \(\nabla^s \eta = 0\), and \(\nabla^s T = 0\) by (1.13). So we have proved that for a locally conformal Kähler manifold, the SPT condition is equivalent to the Vaisman condition.

Next we consider the uniqueness problem for SPT metrics within a conformal class. Note that when the manifold is compact, such metrics are clearly unique (up to constant multiples) within each conformal class, if it exists at all, as SPT metrics are Gauduchon. So the statement in Proposition 1.4 is really about non-compact cases.

Suppose \((M^n, g)\) and \((M^n, e^{2u}g)\) are two SPT manifolds, where \(u\) is a smooth real-valued function on \(M\) with \(du \neq 0\) in an open dense subset \(U \subseteq M\). Let \(e\) be a local unitary frame for \(g\) with dual coframe \(\varphi\). Then \(\tilde{e} = e^{-u} e\) and \(\tilde{\varphi} = e^u \varphi\) are local unitary frame and dual coframe for \(\tilde{g} = e^{2u} g\). We have (cf. [32, the proof of Theorem 3])

\[
e^{u} T^j_{ik} = T^j_{ik} + u_i \delta_{kj} - u_k \delta_{ij}
\]

\[
e^{u} \eta_k = \eta_k - (n-1)u_k
\]

\[
T^i_{ik} = 2u_i \varphi_k - \partial u \delta_{ik} - 2u_k \varphi_i + \partial u \delta_{ik}
\]

where \(P = \tilde{\theta}^s - \theta^s\) and \(u_k = e_k(u)\). Since \(\nabla^s T = 0\) and \(\nabla^s \tilde{T} = 0\), by the analysis of the (1,0)-part, a straight-forward computation leads to

\[
u_i \partial \delta_{kj} - u_k \partial \delta_{ij} = 2u_k T^j_{i\ell} - 2u_i T^j_{k\ell} + 2u_i u_\ell \delta_{kj} - 2u_k u_\ell \delta_{ij} - 2\delta_{ij} \sum_r u_r T^r_{ik}
\]

for any indices \(1 \leq i, j, k, \ell \leq n\) under any local unitary frame \(e\). As \(du \neq 0\) on the open dense subset \(U\), we may choose \(e\) such that \(\varphi^n = \frac{\partial u}{\partial u_n}\), which implies that \(u_1 = \cdots = u_{n-1} = 0\) and \(u_n = |\partial u|_g = \lambda > 0\) are established where the local unitary frame \(e\) intersects with \(U\).
Assume $i < k$. If we take $j \notin \{i, k\}$ in (4.14), we get
\begin{equation}
\delta_{ik}T^j_{i\ell} = \delta_{j\ell}T^n_{ik}, \quad i < k, \ j \neq i, k.
\end{equation}
Take $k = n$ and $j \neq \ell$, we get $T^j_{i\ell} = 0$. Take $k < n$ and $j = \ell$, we get $T^n_{ik} = 0$. So under this particular unitary frame $e$, we have $T^n_{ik} = 0$ whenever all three indices are distinct. Now let us take $n = k$ and $j = \ell$ in (4.15), we get $T^j_{i\ell} = T^n_{ij}$, for all $i, j < n$ with $i \neq j$. Therefore $T^j_{i\ell}$ are equal for $\ell \neq i$, and $T^j_{i\ell} = \frac{1}{n-1}\eta_i$ whenever $i < n$ and $\ell \neq i$, as $\sum_\ell T^j_{i\ell} = \eta_i$.

Take $k = \ell = n$ and $i = j < n$ in (4.14), which yields that

$$-u_{n,n} = 2\lambda T^n_{in} - 2\lambda^2, \quad i < n.$$ 

This shows that all $T^n_{in}$ for $i < n$ are equal, hence equal to $\frac{1}{n-1}\eta_n$. Combining the above results, we conclude that under this particular frame $e$, it holds that
\begin{equation}
T^j_{i\ell} = \frac{1}{n-1}(\eta_k\delta_{ij} - \eta_i\delta_{kj}), \quad 1 \leq i, j, k \leq n.
\end{equation}

which implies that the above equation holds in the open dense subset $U \subseteq M$, hence in the entire $M$. From this equation, it follows that

$$\phi^i_j = \sum_r \eta_r T^j_{ir} = \frac{1}{n-1}\left(\delta_{ij} \sum_r |\eta_r|^2 - \eta_i \eta_j\right).$$

In particular, $\phi^i_j = \phi^j_i$ for any $i, j$. By Proposition 3.6, we know that the torsion 1-form $\eta$ for any SPT manifold satisfies $\partial \eta = 0$ and $\overline{\partial} \eta = -2\sum_{i,j} \phi^i_j \varphi_i \wedge \overline{\varphi}_j$. Therefore, it yields that

$$\overline{\partial} \eta + \partial \eta = 2 \sum_{i,j} (\phi^i_j - \phi^j_i) \varphi_i \wedge \overline{\varphi}_j = 0,$$

which amount to $d (\eta + \overline{\partial} \eta) = 0$. It implies that locally there exists smooth, real-valued function $v$ such that $\eta = (n-1)\partial v$. By (4.16), the torsion of the metric $e^{2v}g$ will vanish, hence $g$ is locally conformal Kähler. This completes the proof of Proposition 1.4.

Next let us prove Proposition 1.5.

**Proof of Proposition 1.5:** Let $(M^n, g)$ be a complex nilmanifold with nilpotent $J$, namely, its universal cover is a nilpotent Lie group $G$ equipped with a left-invariant complex structure $J$ and a compatible left-invariant metric $g$. Assume that $J$ is nilpotent in the sense of [10] and $\nabla^s T^c = 0$. By Lemma 1, Lemma 2 and the discussion right after the proof of Lemma 2 in [34], we know that $G$ is of step most 2 and $J$ is abelian (namely, $C = 0$), and there exists an integer $1 \leq r \leq n$ and a unitary coframe $\varphi$ such that

$$d \varphi_i = 0, \quad 1 \leq i \leq r; \quad d \varphi_\alpha = \sum_{i=1}^r Y_{\alpha i} \varphi_i \wedge \overline{\varphi}_i, \quad \forall \ r < \alpha \leq n.$$ 

Conversely, for such a complex nilmanifold, it is easy to check that $\nabla^s T^c = 0$. This completes the proof of Proposition 1.5.

\[\Box\]

5. **Non-balanced SPT manifolds**

When $n \geq 3$, as illustrated by Proposition 1.5, the set of all non-balanced SPT manifolds in complex dimension $n$ contains the set of all non-Kähler SKL manifolds in complex dimension $n$ as a proper subset.

Given any non-balanced SPT manifold $(M^n, g)$, the type $(1, 0)$ vector field $\chi$ dual to $\eta$ is also $\nabla^s$-parallel, hence is nowhere zero. The following notion of admissible frames introduced by the authors in [32, Definition 4] to study non-Kähler SKL manifolds can be generalized to non-balanced SPT manifolds:

**Definition 5.1.** Let $(M^n, g)$ be a non-balanced SPT manifold and $\chi$ the vector field dual to Gauduchon’s torsion 1-form $\eta$. A local unitary frame $e$ on $M^n$ is said to be **admissible**, if $\chi = \lambda e_n$ for some constant $\lambda > 0$ and under the matrix the frame $e_i$ is diagonal.

**Corollary 5.2** (See also Proposition 3.6 of [6] and Lemma 3 of [32]). Let $(M^n, g)$ be a non-balanced SPT manifold. Then around any given point there exist an admissible frame $e$ such that $\chi = \lambda e_n$ and
\[ \phi_i^j = \lambda a_i \delta_{ij}, \text{ where } \lambda > 0 \text{ and } a_i \text{ are global constants with } a_n = 0. \] Therefore, \( T^{1,0}M \) admits a \( \nabla^s \)-parallel, \( g \)-orthogonal decomposition

\[ T^{1,0}M = \bigoplus_{i=0}^k H_i, \]

where \( H_0 \) is generated by \( \chi \) and other distributions \( \{H_i\}_{i=1}^k \) are the eigenspaces of \( \phi \) corresponding to distinct eigenvalues of \( \{\lambda a_1, \ldots, \lambda a_{n-1}\} \).

**Proof.** Clearly, \( |\eta|^2 \) is a positive constant, which will be denoted by \( \lambda^2 \). Thus we may choose the unitary coframe \( \varphi \) such that \( \eta = \lambda \varphi_n \). This means that \( \eta_1 = \cdots = \eta_{n-1} = 0 \) and \( \eta_n = \lambda \), thus \( \chi = \lambda e_n \). From Proposition 3.6, \( R^*_x y w = 0 \) yields \( R^*_i j m = 0 \) for any \( i, j, \ell \), and by \( R^*_i j k = R^*_k i j \) we get

\[ R^*_n j k \ell = R^*_i n k \ell = R^*_i j k n = 0 \]

for any \( i, j, k, \ell \). Moreover, the conclusion (3.14) of Proposition 3.5 implies

\[ (5.1) \sum_r \eta_r T^r_{ij} = 0, \]

which yields \( T^r_{ij} = 0 \) for any \( i, j \). Hence \( \phi_i^j = \lambda T^j_{in} \), where \( \phi_i^n = \phi_n^i = 0 \) for any \( i \).

Since \( R^*_i j k = R^*_k i j \), the equality (3.9) amounts to

\[ 0 = \sum_r (T^r_{iu} T^r_{jt} + T^r_{ir} T^r_{jt} - T^r_{ir} T^r_{jt} - T^r_{ir} T^r_{jr} + T^r_{kr} T^r_{jr}) + \frac{1}{4} (R^*_i j k \ell - R^*_k i j \ell). \]

Take \( k = \ell = n \) above and multiply \( \lambda^2 \) on both sides, we get

\[ \sum_r \phi_i^r \phi_j^r - \phi_i^r \phi_j^r = 0, \quad \forall \ i, j. \]

That is, the matrix \( \phi = (\phi_i^j) \) satisfies \( \phi \phi^* = \phi^* \phi \) hence is normal. Here \( \phi^* \) stands for the conjugate transpose of \( \phi \). So around each given point, by a unitary change of \( \{\varphi_1, \ldots, \varphi_{n-1}\} \) with \( \varphi_n \) fixed, we could always make \( \phi \) diagonal. This gives us an admissible local unitary frame.

As \( \nabla^s T^r = 0 \), the tensor \( \phi_i^j \) is also \( \nabla^s \)-parallel, which implies the eigenvalues \( \{\lambda a_i\}_{i=1}^n \) of \( \phi \) are global constants with \( a_n = 0 \). It is also obvious that the eigenspaces of \( \phi \) with respect to all the distinct eigenvalues lead to a \( \nabla^s \)-parallel and \( g \)-orthogonal decomposition of \( T^{1,0}M \).

**Remark 5.3.** It is easy to verify that, for non-balanced SPT manifolds, the distribution \( H_0 \oplus \overline{H}_0 \) is a foliation since \( [\chi, \overline{\chi}] = 0 \), which is known as canonical foliation in the study of Vaisman manifolds.

The following notation of degenerate torsion, introduced in [32, Defintion 5], could also be applied to non-balanced SPT manifolds.

**Definition 5.4.** A non-balanced SPT manifold \((M^n, g)\) is said to have degenerate torsion, if under any admissible frame \( e \), \( T^e_{ik} = 0 \) for any \( i, k < n \).

The Lee parallel (LP) condition was introduced by Belgun [6] to study generalized Calabi-Eckmann (GCE) structures (see also [4, 5]).

**Definition 5.5** ([6]). A Hermitian manifold \((M^n, g)\) is LP if the Gauduchon torsion 1-form \( \eta \) satisfies

\[ \partial \eta = 0, \quad \partial \omega = e \eta \wedge \overline{\eta}, \]

where \( c \neq 0 \) is a constant. A Hermitian manifold is GCE if it is LP and SPT. As shown in [6, Proposition 3.2], both the standard Hermitian structure and the modified ones on the Sasakian product are non-Kähler GCE manifolds.

**Proof of Proposition 1.6:** By Corollary 5.2, we know that non-balanced SPT manifolds always admit local admissible frames. Under such a frame \( e \), the vector \( e_n = \frac{1}{\lambda} \chi \) is parallel under \( \nabla^s \), so the connection matrix \( \theta^s \) of \( \nabla^s \) satisfies \( \theta^s_{n*} = \theta^s_{*n} = 0 \). Therefore, the curvature matrix \( \Theta^s \) under \( e \) will also be block-diagonal:

\[ \Theta^s = \begin{bmatrix} * & 0 \\ 0 & 0 \end{bmatrix} \]

where \( * \) is an \((n-1) \times (n-1)\) matrix. As is well-known, the Lie algebra of the holonomy group \( \text{Hol}(\nabla^s) \) of \( \nabla^s \) is generated by conjugation class of \( R^*_x y \) = \( \Theta^s(x, y) \) under \( \nabla^s \)-parallel transports, all of them operates in the \( \nabla^s \)-parallel block \( e_n^e \), so \( \text{Hol}(\nabla^s) \subseteq U(n-1) \). This proves the first part of Proposition 1.6.
For the second part, under an admissible frame \( e \), it follows from Proposition 3.6 that

\[
\eta \wedge \bar{\eta} = 2\lambda^2 \varphi_n \wedge \sum_{i,j} T^j_{in} \varphi_i \wedge \overline{\varphi_j}
\]

\[
= -2\lambda^2 \sum_{i<n} T^i_{in} \varphi_i \wedge \varphi_n \wedge \overline{\varphi_i},
\]

\[-\sqrt{-1} \omega = \sum_{i,j,k} T^j_{ik} \varphi_i \wedge \varphi_k \wedge \overline{\varphi_j} = 2 \sum_{i<k} T^j_{ik} \varphi_i \wedge \varphi_k \wedge \overline{\varphi_j}.
\]

It is clear that the condition \( \partial \omega = c \eta \wedge \bar{\eta} \) implies \( T^j_{ik} = 0 \) for any \( i, k < n \). Conversely, the torsion degeneracy condition \( T^j_{ik} = 0 \) for any \( i, k < n \) yields

\[-\sqrt{-1} \omega = 2 \sum_{i<n} T^i_{in} \varphi_i \wedge \varphi_n \wedge \overline{\varphi_i} = 2 \sum_{i<n} T^i_{in} \varphi_i \wedge \varphi_n \wedge \overline{\varphi_i}
\]

by the definition of admissible frames, thus \( \partial \omega = c \eta \wedge \bar{\eta} \), which shows that the LP condition is equivalent to the torsion degeneracy condition. To prove the last statement, namely when \( n = 3 \) the set of non-balanced SPT threefolds is equal to the set of GCE threefolds, it suffices to show that any non-balanced SPT threefold \((M^3, g)\) always has degenerate torsion. Let \( e \) be a local admissible frame on \( M^3 \). By definition, it follows that \( \eta_1 = \eta_2 = 0, \eta_3 = \lambda > 0 \), and \( T^i_{13} = T^i_{23} = T^i_{31} = 0 \) since \( \phi \) is diagonal. The equality (5.1) implies \( T^3_{ij} = 0 \) for any \( i, j \). Then it yields that

\[ T^1_{12} = \eta_2 - T^3_{32} = 0, \quad T^2_{12} = -T^3_{21} = -\eta_1 + T^3_{31} = 0. \]

That is, \( T^7_{12} = 0 \), so the manifold has degenerate torsion by definition. This completes the proof of Proposition 1.6. \( \square \)

**Remark 5.6.** In light of the proposition above, a SKL manifold with degenerate torsion is exactly a Hermitian manifold which is both SKL and GCE. A complete non-Kähler SKL manifold with degenerate torsion will split off a Kähler factor, on the universal cover, of complex codimension 2 or 3, as in [32, Theorem 9].

### 6. Balanced SPT Threefolds

Let \((M^3, g)\) be a compact non-Kähler balanced SPT manifold. We will start with a technical observation in [35, Proposition 2] which says that any balanced threefold always admits a particular type of unitary frame under which the Chern torsion takes a simple form, namely, for any given \( p \in M^3 \), there exists a unitary frame \( e \) near \( p \) such that the Chern torsion components satisfy

\[
T^1_{1k} = T^2_{2k} = T^3_{3k} = 0, \quad \forall \ 1 \leq k \leq 3.
\]

Under such a frame, the only possibly non-zero torsion components are

\[
a_1 = T^1_{23}, \quad a_2 = T^2_{31}, \quad a_3 = T^3_{12}.
\]

By a permutation of \( \{e_1, e_2, e_3\} \) if needed, we may assume that \( |a_1| \geq |a_2| \geq |a_3| \).

First we claim that each \( |a_i| \) is a global constant on \( M^3 \). To see this, recall that tensor \( B \) is defined by \( B_{ij} = \sum_{r,s} T^r_{rs} T^s_{ri} \). Under our \( e \), we have

\[
B = 2 \begin{bmatrix}
|a_1|^2 & 0 & 0 \\
0 & |a_2|^2 & 0 \\
0 & 0 & |a_3|^2
\end{bmatrix}.
\]

By the SPT assumption, we have \( \nabla^* B = 0 \), so the eigenvalues of \( B \) are all global constants on \( M \), thus we know that each \( |a_i| \) is a global constant.

Then one can rotate \( e \) to make \( a_i = |a_i| \) for each \( i \). To see this, since \( |a_i| \) is a constant, we can write \( a_i = \rho_i|a_i| \) where \( \rho_i \) is a smooth local function with \( |\rho_i| = 1 \). We will simply let \( \rho_i = 1 \) if \( a_i = 0 \). Replace each \( e_i \) by \( \tilde{e}_i = \sqrt{(p_j p_k)^{1/2}} e_i \), where \((ijk)\) is a cyclic permutation of \((123)\). Hence, under the new frame we have \( a_i = |a_i| \) for each \( i \). An appropriate permutation of \( \{e_1, e_2, e_3\} \) yields

\[
T^1_{23} = T^2_{31} \geq T^3_{12} \geq 0.
\]
We will call a local unitary frame $e$ on $M^3$ a special frame if both (6.1) and (6.3) are satisfied.

Let us fix a special frame $e$ in a neighborhood of $p \in M^3$. We have $a_1 \geq a_2 \geq a_3 \geq 0$, with $a_1 > 0$ since $M^3$ is not Kähler. Denote by $\theta^s$ the matrix of the Strominger connection $\nabla^s$ under $e$. Since $\nabla^s T = 0$, and all $T^s_{ik}$ are constants, we have

\begin{equation}
0 = dT^s_{ik} = \sum_r (T^s_{kr} \theta^s_{ir} + T^s_{ir} \theta^s_{kr} - T^s_{ik} \theta^s_{jr})
\end{equation}

Since the only possibly non-zero components of $T$ are $a_1$, $a_2$ and $a_3$, if we take $i$, $j$, $k$ all distinct in (6.4), we get

\begin{equation}
\begin{cases}
a_1(\theta^s_{22} + \theta^s_{33} - \theta^s_{11}) = 0 \\
a_2(\theta^s_{12} + \theta^s_{33} - \theta^s_{22}) = 0 \\
a_3(\theta^s_{11} + \theta^s_{22} - \theta^s_{33}) = 0
\end{cases}
\end{equation}

Similarly, by taking $j = i \neq k$ in (6.4), we get

\begin{equation}
a_1 \theta^s_{i2} + a_2 \theta^s_{i1} + a_3 \theta^s_{i3} = a_2 \theta^s_{2i} + a_3 \theta^s_{3i} = 0.
\end{equation}

Denote by $\varphi$ the unitary coframe dual to the special frame $e$, and by $\theta$, $\tau$ the matrix of connection and column vector of torsion under $e$ for the Chern connection $\nabla^e$. By [31], we have $\theta = \theta^s - 2\gamma$ where

\begin{equation}
\gamma = \begin{bmatrix}
0 & -\overline{\psi}_3 & \psi_2 \\
\overline{\psi}_3 & 0 & -\overline{\psi}_1 \\
-\overline{\psi}_2 & \psi_1 & 0
\end{bmatrix}, \quad \tau = \begin{bmatrix}
2a_1 \varphi_2 \varphi_3 \\
2a_2 \varphi_3 \varphi_1 \\
2a_3 \varphi_1 \varphi_2
\end{bmatrix},
\end{equation}

\begin{equation}
\psi_1 = a_2 \varphi_1 + a_3 \overline{\varphi}_1, \quad \psi_2 = a_3 \varphi_2 + a_1 \overline{\varphi}_2, \quad \psi_3 = a_1 \varphi_3 + a_2 \overline{\varphi}_3.
\end{equation}

We shall divide the classification into the following three cases:

1. $a_1 > a_2 > a_3$;
2. $a_1 = a_2 = a_3$;
3. $a_1 = a_2 > a_3$ or $a_1 > a_2 = a_3$,

where the third case contains the middle type and is the main part of the discussion.

**Proof of Proposition 1.7.**

**Case 1:** $a_1 > a_2 > a_3$.

In this case, $B$ has distinct eigenvalues. Since its eigenspaces are all $\nabla^s$-parallel, we know that the matrix $\theta^s$ is diagonal.

If $a_3 > 0$, by (6.5) we get $\theta^s_{i3} = 0$ for each $i$, hence $\theta^s = 0$. This means that $M^3$ is Strominger flat. Such a manifold cannot be balanced unless it is Kähler, contradicting to our assumption that $M^3$ is balanced and non-Kähler, so we must have $a_3 = 0$, which implies $\psi_1 = a_2 \varphi_1$ and $\psi_2 = a_1 \overline{\varphi}_2$. From Equation (6.5), we get

\[ \theta^s_{33} = 0, \quad \theta^s_{11} = \theta^s_{22} = \alpha, \]

and

\[ \theta = \theta^s - 2\gamma = \begin{bmatrix}
0 & \alpha & -2a_1 \overline{\varphi}_2 \\
-2\overline{\psi}_3 & 0 & 2a_2 \varphi_1 \\
2a_1 \varphi_2 & -2a_2 \varphi_1 & 0
\end{bmatrix}. \]

Then the structure equation of Chern connection yields

\begin{equation}
d\varphi = -\theta \wedge \varphi + \tau = \begin{bmatrix}
-\alpha \varphi_1 + 2\overline{\psi}_3 \varphi_2 \\
-\alpha \varphi_2 - 2\overline{\psi}_3 \varphi_1 \\
2a_2 \varphi_1 \overline{\varphi}_2 - 2a_1 \varphi_1 \overline{\varphi}_2
\end{bmatrix}.
\end{equation}

By the first two equations of (6.9), we get

\[ d(\varphi_2 \overline{\varphi}_1) = d\varphi_2 \wedge \overline{\varphi}_1 - \varphi_2 \wedge d\overline{\varphi}_1 \]

\[ = (-\alpha \varphi_2 - 2\overline{\psi}_3 \varphi_1) \overline{\varphi}_1 - \varphi_2 (-\alpha \varphi_1 + 2\overline{\psi}_3 \varphi_2) \]

\[ = 2\overline{\psi}_3 (\varphi_2 \overline{\varphi}_2 - \varphi_1 \overline{\varphi}_1), \]

where $\alpha + \overline{\alpha} = 0$ is used. Complex conjugation of the equality above yields

\[ d(\varphi_1 \overline{\varphi}_2) = -d(\varphi_2 \overline{\varphi}_1) = 2\overline{\psi}_3 (\varphi_2 \overline{\varphi}_2 - \varphi_1 \overline{\varphi}_1). \]

The exterior differentiation of the third equation of (6.9) implies

\[ 0 = d^2 \varphi_3 = 2a_2 d(\varphi_2 \overline{\varphi}_1) - 2a_1 d(\varphi_1 \overline{\varphi}_2) = 4(a_2 \overline{\psi}_3 - a_1 \psi_3) \wedge (\varphi_2 \overline{\varphi}_2 - \varphi_1 \overline{\varphi}_1). \]
Note that
\[ a_2 \bar{\psi}_3 - a_1 \psi_3 = a_2 (a_1 \bar{\psi}_3 + a_2 \phi_3) - a_1 (a_1 \phi_3 + a_2 \bar{\psi}_3) = (a_2^2 - a_1^2) \phi_3, \]
which yields a contradiction. This shows that the case of distinct \( a_1, a_2 \) and \( a_3 \) cannot occur.

**Case 2:** \( a_1 = a_2 = a_3 \).

Let us denote by \( a > 0 \) the common value of those \( a_i \) in this case. Equalities (6.5) and (6.6) now imply that \( \theta^s \) is skew-symmetric. Since \( \psi_i = a (\phi_i + \bar{\psi}_i) = \bar{\psi}_i \) for each \( i \), the equality (6.7) shows that \( \gamma \) is also skew-symmetric. So \( \theta = \theta^s - 2\gamma \) is skew-symmetric. Then it yields that
\[
\theta = \begin{bmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{bmatrix}, \quad \tau = 2a \begin{bmatrix} \phi_2 \phi_3 \\ \phi_3 \phi_1 \\ \phi_1 \phi_2 \end{bmatrix},
\]
where \( x, y, z \) are real 1-forms. As a result, the Chern curvature matrix \( \Theta = d\theta - \theta \wedge \theta \) is also skew-symmetric. The structure equation of Chern connection gives us
\[
d\varphi = -\theta \wedge \varphi + \tau = \begin{bmatrix} x \phi_2 + y \phi_3 + 2a \phi_2 \phi_3 \\ -x \phi_1 + z \phi_3 + 2a \phi_3 \phi_1 \\ -y \phi_1 - z \phi_2 + 2a \phi_1 \phi_2 \end{bmatrix}.
\]

It follows that
\[
\begin{align*}
d(\phi_2 \phi_3) &= (x \phi_3 - y \phi_2)\phi_1 \\
d(\phi_3 \phi_1) &= (x \phi_2 + z \phi_1)\phi_2 \\
d(\phi_2 \phi_3) &= (-y \phi_2 + z \phi_1)\phi_3.
\end{align*}
\]

On one hand, if we let \( \xi = x \phi_3 - y \phi_2 + z \phi_1 \), then the above equations simply says \( d\tau = 2a \xi \wedge \varphi \). On the other hand, by (6.10) we get \( \theta \tau = 2a \xi \wedge \varphi \). So by the first Bianchi identity
\[
d\tau = 4\Theta \varphi - 4\theta \tau,
\]
we conclude that \( 4\Theta \varphi = 0 \). This means that the Hermitian threefold \( M^3 \) is Chern Kähler-like, that is, the Chern curvature tensor \( R^c \) obeys the Kähler symmetry \( R^c_{ijkl} = R^c_{kijl} \) for any \( i, j, k, l \).

We claim that \( R^c = 0 \). If \( \{i, k\} \cap \{j, \ell\} \neq \emptyset \), for instance 1 is contained in the intersection, then by the Kähler symmetry, \( R^c_{ijkl} \) can be written as \( R^c_{ab_{ij}l} \), which has to vanish since \( \Theta_{11} = 0 \) as \( \Theta \) is skew-symmetric. When \( \{i, k\} \cap \{j, \ell\} = \emptyset \), what we need to show are the equalities \( R^c_{ijk\ell} = 0 \) and \( R^c_{ij\ell k} = 0 \) where \( i, j, k \) are distinct, as the dimension is 3. From the skew-symmetric \( \Theta \), it yields that
\[
\begin{align*}
R^c_{ijk\ell} &= -R^c_{ij\ell k} = -R^c_{i\ell kj} = 0, \\
R^c_{ij\ell k} &= -R^c_{ik\ell j} = -R^c_{i\ell j k} = 0.
\end{align*}
\]

So \( M^3 \) is Chern flat. By [8], we know that the universal cover of \( M \) is a connected, simply-connected complex Lie group \( G \), and the lifting metric \( \tilde{g} \) of \( g \) is left-invariant and compatible with the complex structure of \( G \). Let \( \{e_1, e_2, e_3\} \) be a left-invariant unitary frame of \( G \). Denote by \( T^i_{ij} \) the components of the Chern torsion under the frame \( e \), which are all constants. As \( M \) is three-dimensional and balanced, by the proof of [35, Proposition 2], we know that one can make a constant unitary change of \( e \) so that \( T^i_{ij} = 0 \) for each \( 1 \leq i \leq 3 \). Under the case assumption, the tensor \( B \)
\[
B = \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix},
\]
where \( c = 2a^2 \). Similarly, by a suitable constant rotation \( \rho_i e_i \), where \( |\rho_i| = 1 \) for each \( i \), we may assume that \( T^2_{33} = T^3_{31} = T^1_{32} = a > 0 \) under the left-invariant unitary frame \( e \) of \( G \), hence the Lie algebra of \( G \) is \( \mathfrak{g} = \mathbb{C}\{X, Y, Z\} \), satisfying
\[
[X, Y] = 2aZ, \quad [Y, Z] = 2aX, \quad [Z, X] = 2aY.
\]
Therefore \( G \) is isomorphic to \( SO(3, \mathbb{C}) \). Quotients of \( SO(3, \mathbb{C}) \) give us the only non-Kähler balanced SPT threefolds that are Chern flat.

Note that since \( \theta^s \) is skew-symmetric under a special frame, so is the curvature matrix \( \Theta^s \), therefore the holonomy group of \( \nabla^s \) is contained in \( SO(3) \subseteq U(3) \), but unlike the case of non-balanced SPT manifolds, it is not contained in \( U(n - 1) \times 1 \) here.

**Case 3:** \( a_1 = a_2 > a_3 \) or \( a_1 > a_2 = a_3 \).
In this case $B$ has two distinct eigenvalues. First we will rule out the possibility of $a_3 > 0$. Assume that $a_3 > 0$, namely, either $a_1 = a_2 > a_3 > 0$ or $a_1 > a_2 = a_3 > 0$, we will derive at a contradiction.

Since the argument for these two situations are exactly analogous, we will just focus on the case $a_1 = a_2 > a_3 > 0$. Write $a_1 \neq a_2 = \alpha$ for simplicity. Since the eigenspaces of $B$ are $\nabla^s$-parallel, it follows that $\theta_{13}^* = \theta_{23}^* = 0$. By (6.5) and (6.6), it yields that

$$\theta^s = \begin{bmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $\bar{\alpha} = \alpha$. Write $\alpha' = \alpha + 2\psi_3$, where $\psi_3 = a(\varphi_3 + \bar{\varphi}_3)$ is real, and the structure equation of Chern connection amounts to

$$(6.11) \quad d\varphi = -((\theta^s - 2\gamma)\varphi + \tau = \begin{bmatrix} \alpha'\varphi_2 + 2a_3\varphi_3\varphi_2 \\ -\alpha'\varphi_1 - 2a_3\varphi_3\bar{\varphi}_1 \\ -2a_3\varphi_1\varphi_2 + 2a_2(\varphi_1\bar{\varphi}_1 - \varphi\bar{\varphi}_2) \end{bmatrix}.$$ 

Then we get, from the first two lines,

$$d(\varphi_1\varphi_2) = d\varphi_1 \wedge \varphi_2 - \varphi_1 \wedge d\varphi_2 = -2a_3\varphi_3(\varphi_1\bar{\varphi}_1 + \varphi_2\bar{\varphi}_2),$$

$$d(\varphi_1\bar{\varphi}_2) = d\varphi_1 \wedge \bar{\varphi}_2 - \varphi_1 \wedge d\bar{\varphi}_2 = \alpha'(\varphi_2\bar{\varphi}_2 - \varphi_1\bar{\varphi}_1).$$

Here the fact $\bar{\alpha}' = \alpha'$ is used. In particular,

$$d(\varphi_2\bar{\varphi}_1) = -d(\bar{\varphi}_1\varphi_2) = \alpha'(\varphi_2\bar{\varphi}_2 - \varphi_1\bar{\varphi}_1) = d(\varphi_1\bar{\varphi}_2).$$

Take the exterior differentiation of the third equation of (6.11) and we obtain

$$0 = d^2\varphi_3 = -2a_3d(\varphi_1\varphi_2) + 2a_3d(\varphi_1\bar{\varphi}_2) = 4a_3^2\varphi_3(\varphi_1\bar{\varphi}_1 + \varphi_2\bar{\varphi}_2),$$

which is a contradiction. This shows that the case $a_3 > 0$ cannot occur and either $a_1 > 0 = a_2 = a_3$ or $a_1 = a_2 > 0 = a_3$ yields, which implies

$$B = \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} c & 0 & 0 \\ 0 & 0 & 0 \\ 0 & c & 0 \end{bmatrix},$$

where $c = 2a_3^2$. This completes the proof of Proposition 1.7.

Therefore, we have shown the rank $B = 3$ case of Theorem 1.9. We will deal with the case rank $B = 1$ in §7, which leads to a Fano threefold and eventually end up with the Wallach threefold $(X,g)$, which will be computed in details in §8. The case rank $B = 2$ is the middle type, which will be discussed in §9. This will complete the proof of Theorem 1.9.

7. The Fano case

In this section, we deal with the case rank $B = 1$. Eventually, we will show that this leads us to a unique example, the Wallach threefold, up to a scaling of the metric by a constant.

Let $(M^3,g)$ be a compact, non-Kähler, balanced $SPT$ threefold with $B$ tensor

$$B = \begin{bmatrix} 2a_1^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

under a special frame $e$, where the only non-zero component of the Chern torsion tensor is $a_1 = T_{23}^1 > 0$. We may assume for simplicity that $a_1 = \frac{1}{2}$, after a scaling of the metric $g$ by a suitable constant.

From (6.5) and (6.6) we get $\theta_{13}^* = \theta_{31}^* = 0$ and $\theta_{11}^* = \theta_{22}^* = \theta_{33}^*$. Then it follows that

$$\theta^s = \begin{bmatrix} x + y & 0 & 0 \\ 0 & x & \alpha \\ 0 & -\bar{\alpha} & y \end{bmatrix}, \quad \gamma = \frac{1}{2} \begin{bmatrix} 0 & -\bar{\varphi}_3 & \varphi_2 \\ \varphi_3 & 0 & 0 \\ -\bar{\varphi}_2 & 0 & 0 \end{bmatrix}, \quad \tau = \begin{bmatrix} \varphi_2\varphi_3 \\ 0 \\ 0 \end{bmatrix},$$

where $\bar{x} = -x$ and $\bar{y} = -y$. From $\theta = \theta^s - 2\gamma$ and $d\varphi = -\theta \wedge \varphi + \tau$, it yields that

$$\theta = \begin{bmatrix} x + y & \varphi_3 & -\varphi_2 \\ -\varphi_3 & x & \alpha \\ \varphi_2 & -\bar{\alpha} & y \end{bmatrix}, \quad d\varphi = \begin{bmatrix} -(x + y)\varphi_1 - \varphi_2\varphi_3 \\ \varphi_1\bar{\varphi}_2 - x\varphi_2 + \bar{\alpha}\varphi_3 \\ -\varphi_1\bar{\varphi}_2 + \alpha\varphi_2 - y\varphi_3 \end{bmatrix}.$$
The matrix of the curvature of $\nabla^s$ is $\Theta^s = d\theta^s - \theta^s \wedge \theta^s$, whose entries are
\begin{align}
\begin{cases}
\Theta^s_{22} = dx + \alpha \overline{\alpha}, & \Theta^s_{33} = dy - \alpha \overline{\alpha}, \\
\Theta^s_{23} = dx - x \alpha - \alpha y, & \Theta^s_{12} = \Theta^s_{13} = 0, \\
\Theta^s_{11} = dx + dy = \Theta^s_{22} + \Theta^s_{33}.
\end{cases}
\end{align}
(7.1)
For convenience, we will use $\varphi_{ij}$ and $\varphi_{ij}^*$ as the abbreviation of $\varphi_i \wedge \varphi_j$ and $\varphi_i \wedge \overline{\varphi_j}$, respectively. From the exterior differentiation $d\varphi$ of $\varphi$, the first Bianchi identity of $\nabla^s$ amounts to
\begin{align}
\begin{cases}
0 = d^2 \varphi_1 = \{ \varphi_{23}^* + \varphi_{33} - \Theta^s_{11} \} \wedge \varphi_1, \\
0 = d^2 \varphi_2 = \{ \varphi_{13}^* - \varphi_{33} - \Theta^s_{22} \} \wedge \varphi_2 - \Theta^s_{32} \wedge \varphi_3, \\
0 = d^2 \varphi_3 = \{ \varphi_{12}^* - \varphi_{22}^* - \Theta^s_{33} \} \wedge \varphi_3 - \Theta^s_{23} \wedge \varphi_2
\end{cases}
\end{align}
(7.2)
It follows from Theorem 1.1 that $\nabla^s T = 0$ implies
$$R^s_{ijkl} = R^s_{ijlk} = 0$$
for any $i, j, k, l$. So by $\Theta^s_{12} = \Theta^s_{13} = 0$ and $\Theta^s_{11} = \Theta^s_{22} + \Theta^s_{33}$, we get
$$R^s_{11ij} = R^s_{11ji} = 0, \quad R^s_{12ij} = R^s_{12ji} = R^s_{22ij} + R^s_{23ij} = R^s_{32ij} + R^s_{33ij}$$
for any $i, j$ and any $b \in \{2, 3\}$. Write
\begin{align}
\Theta^s_{22} &= A \varphi_{22} + B \varphi_{33} + E \varphi_{23} + \overline{E} \varphi_{32} + (A + B) \varphi_{1T} \\
\Theta^s_{33} &= B \varphi_{22} + C \varphi_{33} + F \varphi_{23} + \overline{F} \varphi_{32} + (B + C) \varphi_{1T} \\
\Theta^s_{23} &= E \varphi_{22} + F \varphi_{33} + D \varphi_{23} + G \varphi_{32} + (E + F) \varphi_{1T} \\
\Theta^s_{11} &= (A + B) \varphi_{22} + (B + C) \varphi_{33} + (E + F) \varphi_{23} + (E + F) \varphi_{32} + (A + B + C) \varphi_{1T}
\end{align}
where $A, B, C, G$ are local real smooth functions. Then the first Bianchi identity (7.2) indicates
$$E + F = 0, \quad A + B = B + C = G - B = 1.$$
In particular,
$$\Theta^s_{11} = 2 \varphi_{1T} + \varphi_{22} + \varphi_{33}$$
(7.3)
Remark 7.1. The pattern of the Strominger curvature implies that the holonomy group of $\nabla^s$ in this case would have its Lie algebra contained in the subalgebra
$$h = \left\{ \begin{pmatrix} \text{tr}(X) & 0 \\ 0 & X \end{pmatrix} \mid X \in \mathfrak{u}(2) \right\} \subseteq \mathfrak{u}(3).$$
Denote by $\Theta$ the curvature matrix of the Chern connection $\nabla^c$ under $e$, and the balanced condition $\eta = 0$ implies
$$\text{tr} \Theta = \text{tr} \Theta^s = 2 \Theta^s_{11}.$$ Let $\omega = \sqrt{-1} (\varphi_1 \overline{\varphi}_1 + \varphi_2 \overline{\varphi}_2 + \varphi_3 \overline{\varphi}_3)$ be the Kähler form of the Hermitian metric $g$, and
$$\tilde{\omega} = \sqrt{-1} (2 \varphi_1 \overline{\varphi}_1 + \varphi_2 \overline{\varphi}_2 + \varphi_3 \overline{\varphi}_3).$$
be the Kähler form of another Hermitian metric $\tilde{g}$. Clearly $\tilde{\omega}$ is independent of the choice of special frames hence is globally defined on $M$. The Chern Ricci form of $g$ is $\text{Ric}(\omega) = -\text{I} \text{tr} \Theta = 2 \tilde{\omega}$. As $\text{Ric}(\omega)$ is always closed, we know $\tilde{\omega}$ is Kähler. Note that $\tilde{\omega}^3 = 2 \omega^3$ and thus
$$\text{Ric}(\tilde{\omega}) = \text{Ric}(\omega) = 2 \tilde{\omega}.$$ Therefor $\tilde{\omega}$ is a Kähler-Einstein metric with positive Ricci curvature and $M^3$ is a Fano threefold.

Denote by $E$ and $F$ the $C^\infty$ complex vector bundle on $M$ with fibers $E_x = \mathbb{C} \{ e_2(x), e_3(x) \}$ and $F_x = \mathbb{C} \{ e_1 \}$, respectively. They are both globally defined since $E$ is the eigenspace of $B$ corresponding to the eigenvalue 0, and $F$ is the orthogonal complement of $E$ in $T^{1,0}M$.

We claim that $E$ is a holomorphic subbundle of $T^{1,0}M$. To see this, for any $i \in \{2, 3\}$ and any $j$, we have
$$\left( \nabla^c_{\bar{\gamma}e_i} e_i \right) = \left( \nabla^c_{\bar{\gamma}e_i} - 2 \gamma e_i \bar{\gamma} e_i \right) = \theta^s_{11}(\bar{\gamma} e_i) - 2 \gamma \theta^s_{11}(e_i) = 0.$$ This means that $\nabla^c_{\bar{\gamma}} E \subseteq E$ for any type $(1, 0)$ vector field $X$, so $E$ is holomorphic. Note that the distribution $E$ is not a foliation, while $F$ on the other hand is a foliation, but is not holomorphic.

Equip $E$ with the restriction metric from $(M^3, g)$. By the formula of Chern connection matrix $\theta$ on $M$, the matrices of connection and curvature of the Hermitian bundle $E$ under the local frame $\{ e_2, e_3 \}$ are respectively
$$\theta^E = \begin{bmatrix} x & \alpha \\ -\overline{\alpha} & y \end{bmatrix}, \quad \Theta^E = d\theta^E - \theta^E \wedge \theta^E = \begin{bmatrix} \Theta^s_{22} & \Theta^s_{23} \\ \Theta^s_{32} & \Theta^s_{33} \end{bmatrix},$$
In particular, $\sqrt{-1}\text{tr}\Theta^E = \sqrt{-1}(\Theta_{22} + \Theta_{33}^*) = \frac{2\pi}{g} \text{tr}\Theta = \sqrt{2} \text{tr}\Theta = \ddot{\omega}$. This means that

$$\text{(7.6)} \quad c_1(E) = \frac{1}{2} c_1(M) = [\ddot{\omega}].$$

Denote by $L$ the holomorphic line bundle on $M$ which is the quotient of $E$ in $T^{1,0}M$, namely, the exact sequence follows

$$\text{(7.7)} \quad 0 \to E \to T^{1,0}M \to L \to 0.$$

Let $h$ be the first Chern class $c_1(L)$ of $L$. The above short exact sequence implies

$$c_1(E) + h = c_1(M), \quad c_2(E) + hc_1(E) = c_2(M), \quad c_2(E)h = c_3(M).$$

Then the equality (7.6) yields

$$c_1(E) = h, \quad c_1(M) = 2h, \quad c_2(E) = c_2(M) - h^2, \quad c_2(M)h - h^3 = c_3(M).$$

In particular, $L$ is an ample line bundle on $M$, the anti-canonical line bundle $K^{-1}_M = 2L$ as holomorphic line bundles are uniquely determined by their Chern classes on Fano manifolds, and the Chern numbers of $M^3$ satisfy

$$\text{(7.8)} \quad c_1(M)c_2(E) = 2h(c_2(E) + h^2) = 2c_3(M) + \frac{1}{4}c_1^3(M).$$

Recall that the index of a Fano manifold $X^n$ is the largest positive integer $r$ so that $K_X^{-1} = rA$ for an ample line bundle $A$. It is necessarily less than or equal to $n + 1$, where $r = n + 1$ if and only if $X = \mathbb{P}^n$ and $r = n$ if and only if $X = \mathbb{Q}^n$, the smooth quadric hypersurface in $\mathbb{P}^{n+1}$. Fano manifolds satisfying $r = n - 1$ are called del Pezzo manifolds, which are classified by Fujita [14] as one of the following seven types, according to their degree $d$, which is the self-intersection number $A^n$:

1. $d = 1$: $X^n_6 \subset \mathbb{P}(1^{n-1}, 2, 3)$, a degree 6 hypersurface in the weighted projective space.
2. $d = 2$: $X^n_4 \subset \mathbb{P}(1^n, 2)$, a degree 4 hypersurface in the weighted projective space.
3. $d = 3$: $X^n_3 \subset \mathbb{P}^{n+1}$, a cubic hypersurface.
4. $d = 4$: $X^n_{2,2} \subset \mathbb{P}^{n+2}$, a complete intersection of two quadrics.
5. $d = 5$: $Y^n$, a linear section of $\text{Gr}(2, 5) \subset \mathbb{P}^n$.
6. $d = 6$: $\mathbb{P} \times \mathbb{P} \times \mathbb{P}^1$, or $\mathbb{P}^2 \times \mathbb{P}^2$, or the flag threefold $\mathbb{P}(T_{\mathbb{P}^2})$.
7. $d = 7$: $\mathbb{P} \# \mathbb{P}^3$, the blow-up of $\mathbb{P}^3$ at a point.

For $n = 3$, del Pezzo threefolds were classified by Iskovskikh [16] earlier, and in Table 12.2 of [17] we can find the third betti number $b_3$, hence the Euler number $c_3 = 4 - b_3$ of del Pezzo threefolds of degree $1 \leq d \leq 5$:

$$\text{(7.9)} \quad c_3(X^n_6) = -38, \quad c_3(X^n_4) = -16, \quad c_3(X^n_3) = -6, \quad c_3(X^n_{2,2}) = 0, \quad c_3(Y^n) = 4.$$

Let us return to our manifold $M^3$, where it holds that $K^{-1}_M = 2L$ for the ample $L$, so the index of $M^3$ is either 4 or 2, which means $M^3$ is biholomorphic to either $\mathbb{P}^3$ or a del Pezzo threefold. It is well-known that $c_1c_2 = 24$ holds for any Fano threefold, so the equality (7.8) implies

$$\text{(7.10)} \quad c_3(M) = 12 - \frac{1}{8} c_1^3.$$

If $M^3$ is a del Pezzo threefold of degree $d$, then $c_1^3 = 8d$, hence the equality (7.10) yields $c_3 = 12 - d$. This rules out the possibility of $1 \leq d \leq 5$ by (7.9). The case $\mathbb{P}^3 \# \mathbb{P}^3$ of degree $d = 7$ has the Euler number $c_3 = 7$, which is not equal to $12 - 7$. Similarly, for the case $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ of degree $d = 6$, its Euler number $c_3 = 8$ is not equal to $12 - 6$, which indicates that neither can be our $M^3$. Therefore only two possibilities are left, namely, $M^3$ is either the flag threefold $\mathbb{P}(T_{\mathbb{P}^2})$ or $\mathbb{P}^3$. Note that in both of these two cases the short exact sequence (7.7) exists, for which we have to dig further.

Consider the short exact sequence (7.7) of holomorphic vector bundles on our threefold $M^3$, where $E$ has the fiber $\mathbb{C}\{e_2, e_3\}$ under a special frame $e$. Denote by $\Omega$ the holomorphic cotangent bundle of $M^3$ and write $\Omega(L) = \Omega \otimes L$. Let $\xi \in H^0(M, \Omega(L))$ be the nowhere zero holomorphic section which gives the map $T^{1,0}M \to L$ in (7.7). The Kähler-Einstein metric $\ddot{\omega}$ on $M^3$ naturally induces Hermitian metrics on $\Omega$ and $L = -\tfrac{i}{2} K_M$, hence on $\Omega(L)$. We have the following claim

**Claim 7.2.** The norm $||\xi||^2$ is a constant under the Kähler-Einstein metric $\ddot{\omega}$.

**Proof:** To see this, let $e$ be a local special frame of $(M^3, g)$, with dual coframe $\varphi$. Let $s$ be a local holomorphic frame of $L$. Since $\xi$ is a $L$-valued 1-form, we have $\xi = \psi \otimes s$ where $\psi$ is a nowhere-zero local holomorphic 1-form. The kernel of the map given by $\xi$ is $E$, so $\psi = f \varphi_1$ for some local smooth function
\[ f, \text{ which is nowhere zero. By the structure equation, it follows that } d\varphi = -(x+y)\varphi_1 - \varphi_2 \varphi_3, \text{ so the holomorphicity of } \psi \text{ gives us } \]

\[ 0 = \overline{\partial} \psi = \overline{\partial} f \wedge \varphi_1 - f(x+y)^{0,1} \wedge \varphi_1. \]

Hence \((x+y)^{0,1} = \overline{\partial} \log f.\) By the fact that \(x = -x \text{ and } y = -y,\) we get

\[ x + y = -\partial \log f + \overline{\partial} \log f, \quad \frac{1}{\sqrt{-1}} \omega = \Theta^*_1 = d(x + y) = \partial \overline{\partial} \log |f|^2. \]

On the other hand, \(L = -\frac{1}{2} K_M\) is equipped with the induced metric from \(\omega\), then it yields

\[ \frac{1}{\sqrt{-1}} \omega = \Theta^L = -\partial \overline{\partial} \log \| s \|^2. \]

Combine the above two equations and we get

\[ \partial \overline{\partial} \log (|f|^2 \| s \|^2) = 0. \]

Since \(\{\sqrt{2} \varphi_1, \varphi_2, \varphi_3\}\) is a local unitary coframe for \(\omega\), the norm square of \(\varphi_1\) under \(\omega\) is \(\frac{1}{2}\), hence

\[ \| s \|^2 = \frac{1}{2} |f|^2 \| s \|^2. \]

It is a global positive function on \(M^3\), and its log is pluriclosed by the above equation, hence it is a constant. This establishes the claim above. \(\square\)

Then we will use Claim 7.2 to rule out the possibility of \(M^3\) being \(\mathbb{P}^3\). Assume that \(M^3\) is \(\mathbb{P}^3\). In this case \(L = \mathcal{O}(2)\). Any nowhere-zero holomorphic section \(\xi \in V := H^0(\mathbb{P}^3, \mathcal{O}(2))\) gives the so-called \textit{null-correlation bundle} (see for example [23]), which is \(E(-1)\) in our notation. Let \(\omega\) be the (scaled) Fubini-Study metric of \(\mathbb{P}^3\) with Ricci curvature 2. It has constant holomorphic sectional curvature 1. Let \([Z_0: Z_1: Z_2: Z_3]\) be the standard unitary homogeneous coordinate of \(\mathbb{P}^3\). In the coordinate neighborhood \(U_0 = \{Z_0 \neq 0\},\) let \(z_i = \frac{Z_0}{Z_i}, 1 \leq i \leq 3\) and it follows that

\[ \sqrt{-1} \Theta^L = \omega = \frac{1}{2} \text{Ric}(\omega) = 2\sqrt{-1} \partial \overline{\partial} \log (1 + |z|^2), \quad \| Z_0^2 \|^2 = \frac{1}{(1 + |z|^2)^2} \]

where \(Z_0^2\) is a local frame of \(L\) in \(U_0\) and \(|z|^2 = |z_1|^2 + |z_2|^2 + |z_3|^2\). Under the coordinate \(z,\)

\[ \tilde{g}_{ij} = \frac{2}{1 + |z|^2} \delta_{ij} - \frac{2}{(1 + |z|^2)^2} \tau_{ij}, \quad \tilde{g}^{ij} = \frac{1}{2}(1 + |z|^2)(\delta_{ij} + \tau_{ij}). \]

As is well-known, \(V \cong \mathbb{C}^6\) has a basis \(\{\lambda_{ij}\}_{0 \leq i < j \leq 3},\) where \(\lambda_{ij} = Z_i dz_j - Z_j dz_i\). Suppose that \(\xi \in V\) is a nowhere zero section. It follows that \(\xi = \sum a_{ij} \lambda_{ij}\) for some constants \(a_{ij}.\) In \(U_0,\)

\[ \lambda_{i0} = Z_0^2 dz_i, \quad \lambda_{ij} = Z_0^2 (z_i dz_j - z_j dz_i) \]

for any \(i, j > 0,\) hence \(\xi = Z_0^2 (\ell_1 dz_1 + \ell_2 dz_2 + \ell_3 dz_3),\) where

\[ \ell_1 = a_{01} - a_{12} z_2 - a_{13} z_3 \]
\[ \ell_2 = a_{02} + a_{12} z_1 - a_{23} z_3 \]
\[ \ell_3 = a_{03} + a_{13} z_1 + a_{23} z_2 \]

It yields that

\[ \| \xi \|^2 = \| Z_0^2 \|^2 \sum_{i,j=1}^3 \ell_i \tau_{ij} \tilde{g}^{ij} \]

\[ = \| Z_0^2 \|^2 \frac{1}{2} (1 + |z|^2) \left( \sum_i |\ell_i|^2 + \sum_k |\ell_k|^2 \right) \]

\[ = \frac{1}{2(1 + |z|^2)} \left( |\ell_1|^2 + |\ell_2|^2 + |\ell_3|^2 + |\ell_1 z_1 + \ell_2 z_2 + \ell_3 z_3|^2 \right) \]

\textbf{Claim 7.3.} The expression \(\| \xi \|^2\) above can not be a constant. Hence it rules out the possibility of \(M^3 = \mathbb{P}^3\).

\textbf{Proof.} If \(\| \xi \|^2\) above were a constant, there exists a constant \(C > 0\) such that

\[ |\ell_1|^2 + |\ell_2|^2 + |\ell_3|^2 + |\ell_1 z_1 + \ell_2 z_2 + \ell_3 z_3|^2 = C(1 + |z_1|^2 + |z_2|^2 + |z_3|^2). \]

Then it is clear that the degree 4 part of the left hand side of the equality above has to vanish, which is exactly

\[ |\ell_1^{(1)} z_1 + \ell_2^{(1)} z_2 + \ell_3^{(1)} z_3|^2, \]
if we decompose $\ell_i$ into $\ell_i^{(0)} + \ell_i^{(1)}$ for $1 \leq i \leq 3$, where $\ell_i^{(0)}$, $\ell_i^{(1)}$ are the parts of the degree 0 and the degree 1 of $\ell_i$ respectively. This indicates
\[
|\ell_1^{(1)}z_1 + \ell_2^{(1)}z_2 + \ell_3^{(1)}z_3|^2 = 0,
\]
which implies
\[
a_{12} = a_{13} = a_{23} = 0.
\]
It follows that
\[
|a_{01}|^2 + |a_{02}|^2 + |a_{03}|^2 + |a_{01}z_1 + a_{02}z_2 + a_{03}z_3|^2 = C(1 + |z_1|^2 + |z_2|^2 + |z_3|^2),
\]
which implies
\[
\begin{bmatrix}
a_{01}a_{01} & a_{01}a_{02} & a_{01}a_{03} \\
a_{02}a_{01} & a_{02}a_{02} & a_{02}a_{03} \\
a_{03}a_{01} & a_{03}a_{02} & a_{03}a_{03}
\end{bmatrix}
= \begin{bmatrix} C & \quad & C \\
\quad & C \\
\quad & \quad & C
\end{bmatrix}.
\]
It is a contradiction by comparison of the rank of matrices above. □

Finally let us consider the flag threefold $X^3 = \mathbb{P}(T_{\mathbb{P}^2})$. It is the hypersurface in $N^4 = \mathbb{P}^2 \times \mathbb{P}^2$ defined by $Z_0W_0 + Z_1W_1 + Z_2W_2 = 0$, where $Z, W$ are the standard unitary homogeneous coordinate of the two factors of $N$. For $i = 1, 2$, denote by $\pi_i : X^3 \to \mathbb{P}^2$ the restriction on $X$ of the projection map from $N$ onto its $i$-th factor. The Picard group Pic($X^3$) $\cong \mathbb{Z}^2$ is generated by $L_1$ and $L_2$, where $L_i = \pi_i^*O_{\mathbb{P}^2}(1)$, and the anti-canonical line bundle of $X$ is $-K_X = 2L = 2(L_1 + L_2)$. The Kähler-Einstein metric $\tilde{\omega}$ on $X$ is the restriction of the product of Fubini-Study metric, and we have
\[
\sqrt{-1}\Theta^L = \tilde{\omega} = \frac{1}{2}\text{Ric}(\tilde{\omega}) = \omega_0|_X, \quad \omega_0 = \sqrt{-1}\partial\bar{\partial}\log |Z|^2 + \sqrt{-1}\partial\bar{\partial}\log |W|^2.
\]

Then the following claim yields

Claim 7.4.

\[
H^0(X, \Omega \otimes L) = \mathbb{C}\xi, \quad \xi = \sum_{i=0}^2 W_i dZ_i = -\sum_{i=0}^2 Z_i dW_i.
\]

Proof. It is clear that $\xi$ is a global holomorphic section of $\Omega \otimes L$, and is nowhere zero, hence give a surjective bundle map $T_X \to L$, which will lead to the Wallach space example as we shall see in the next section. Here we want to show that the vector space $H^0(X, \Omega \otimes L)$ is one-dimensional, hence any section is a constant multiple of $\xi$. To see this, let us denote by $T_{X|\mathbb{P}^2}$ the relative tangent bundle of the map $\pi_1 : X \to \mathbb{P}^2$, given by
\[
0 \to T_{X|\mathbb{P}^2} \to T_X \to \pi_1^*T_{\mathbb{P}^2} \to 0.
\]
Then we have $T_{X|\mathbb{P}^2} = 2L - 3L_1 = 2L_2 - L_1$. Taking the dual of the above short exact sequence and tensoring it with $L$, we get
\[
0 \to \pi_1^*\Omega_{\mathbb{P}^2} \otimes L \to \Omega_X \otimes L \to L' \to 0, \quad L' = -T_{X|\mathbb{P}^2} + L = 2L_1 - L_2.
\]
On the other hand, $X = \mathbb{P}(T_{\mathbb{P}^2})$, so the relative Euler sequence is
\[
0 \to \Omega_X \to \pi_1^*\Omega_{\mathbb{P}^2} \otimes L \to T_{X|\mathbb{P}^2} \to 0.
\]
Since $L_1^3 = 0$, $L_2^4L_2 = 1$, we have $L_2^3L_2' = -1$, so $H^0(X, L') = 0$ as $L_2$ is represented by the fibers of $\pi_1$ which will have non-negative intersection with any effective divisor in $X$. Similarly, $H^0(X, T_{X|\mathbb{P}^2}) = 0$. So by (7.12) and (7.13), we get
\[
H^0(X, \Omega_X \otimes L) \cong H^0(X, \pi_1^*\Omega_{\mathbb{P}^2} \otimes L) \cong H^0(X, \Omega_X) \cong \mathbb{C}.
\]
This establishes Claim 7.4. □

Consider the global $(1, 1)$-form on $X$ defined by
\[
\sigma = \sqrt{-1} W dZ \wedge \overline{W} dZ / |Z|^2 |W|^2
\]
where $Z, W$ are unitary homogeneous coordinate on the two factors of $N = \mathbb{P}^2 \times \mathbb{P}^2$, viewed as column vectors. It is not hard to see that the norm $\|\sigma\| = \frac{1}{2}$ with respect to the Kähler-Einstein metric $\tilde{\omega}$ of $X$. Consider the Hermitian metric $g$ on $X$ with Kähler form $\omega = \tilde{\omega} - \sigma$, which is clearly a homogeneous Hermitian metric on $X$. We will call the Hermitian manifold $(X, g)$ the Wallach threefold from now on, to honor the influential work [27] in geometry.

We will verify in the next section that $(X, g)$ is indeed balanced and SPT. We will also show that its Chern connection has non-negative bisectional curvature and positive holomorphic sectional curvature,
and all three Ricci tensors of the Chern connection are positive. We will compute the sectional curvature of the Levi-Civita (Riemannian) connection, and show that it is non-negative, and the Levi-Civita connection has constant Ricci curvature 6, thus g lies in the boundary of the set of metrics with positive sectional curvature discovered by Wallach in [27].

Note that homogeneous metrics on X with positive sectional curvature, which are all Hermitian as observed by Wallach in [27], form a moduli which depends on three real parameters. After scaling, these metrics form a peculiar planar region (see for example Figure 1 in [9]). It is not clear which metric in the set is the ‘best’ amongst its peers. Our metric g corresponds to the one where all three parameters are equal (that is, the metric given by the Killing form). It is Einstein and has non-negative sectional curvature but not strictly positive sectional curvature, but it is the unique (up to scaling) balanced and $SPT$ metric on X. Its Chern connection also has non-negative bisectional curvature, while the (unique) Kähler-Einstein metric $\tilde{g}$ of X does not have nonnegative bisectional curvature.

8. The Wallach threefold

Let $\omega_0$ be the product of Fubini-Study metric on $N^4 = \mathbb{P}^2 \times \mathbb{P}^2$, given by (7.11), where Z and W are unitary homogeneous coordinates, and the flag threefold X given as the smooth ample divisor \{ZW = 0\} in N. Here and below we will consider Z and W as column vectors. The restriction $\tilde{\omega} = \omega_0|_X$ is the Kähler-Einstein metric on X with Ric($\tilde{\omega}$) = $2\tilde{\omega}$, and our Hermitian metric g, which will be called the Wallach metric from now on, is defined by $\omega = \tilde{\omega} - \sigma$ where the global (1,1)-form $\sigma$ on X is defined by (7.14). We will verify that g is balanced and $SPT$, and compute its Chern and Riemannian curvature.

Fix any point $p \in X$. Notice that for any $A \in SU(3)$, the map $([Z], [W]) \mapsto ([AZ], [AW])$ is an isometry on $(X, g)$. So without loss of generality, we may assume that $p = ([1:0:0],[0:0:1])$. Let $U_{02} = \{Z_0 \neq 0\} \times \{W_2 \neq 0\}$ be a coordinate neighborhood in N, with holomorphic coordinate $(z_1, z_2, w_0, w_1)$ where $z_i = \frac{Z_i}{Z_0}$, $i = 1, 2$, and $w_j = \frac{W_j}{W_2}$, $j = 0, 1$. Within $U_{02}$, the hypersurface X is defined by

$$w_0 = -z_2 - z_1 w_1,$$

so $(z_1, z_2, w_1)$ becomes a local holomorphic coordinate in $U = X \cap U_{02}$. Let us write $|z|^2 = |z_1|^2 + |z_2|^2$ and $|w|^2 = |w_0|^2 + |w_1|^2$ as usual, then in $U_{02}$ we have

$$\frac{1}{\sqrt{-1}} \omega_0 = \sum_{i,j=1}^{2} \frac{(1 + |z|^2)\delta_{ij} - \overline{z}_i z_j}{(1 + |z|^2)^2} dz_i \wedge d\overline{z}_j + \sum_{i,j=0}^{1} \frac{(1 + |w|^2)\delta_{ij} - \overline{w}_i w_j}{(1 + |w|^2)^2} dw_i \wedge d\overline{w}_j,$$

and in U, $\tilde{w}$ is just the restriction of the equation on $U_{02}$ using the equation (8.1). For convenience, let us write $w_1 = z_3$, and define

$$\alpha = 1 + |z_1|^2 + |z_2|^2, \quad \beta = 1 + |z_3|^2 + |f|^2, \quad f = z_2 + z_1 z_3.$$

In U, $(z_1, z_2, z_3)$ gives local holomorphic coordinate for X, and p corresponds to the origin $(0, 0, 0)$. By (8.2) the metric $\tilde{g}$ has components

$$\tilde{g}_{ij} = \frac{\alpha \gamma}{\alpha} - \frac{\alpha \gamma}{\alpha^2} + \frac{\beta \gamma}{\beta} - \frac{\beta \gamma}{\beta^2}, \quad 1 \leq i, j \leq 3,$$

where subscripts stand for partial derivatives in $z_i$ or $\overline{z}_j$. Taking partial derivative in $z_k$, we obtain

$$\tilde{g}_{ik} = -\frac{1}{\alpha^2} (\alpha_k \alpha + \alpha_i \alpha_k) + \frac{2 \alpha_i \alpha_k \alpha}{\alpha^3} + \frac{\beta_k \beta}{\beta^3} - \frac{1}{\beta^2} (\beta_k \beta + \beta_i \beta_k + \beta_j \beta_k) + \frac{2 \beta \beta_k \beta}{\beta^3}.$$

Here we used the fact that $\alpha_k = 0$ and $\alpha_{ik} = 0$. At the origin, $\alpha(0) = \beta(0) = 1$, $\alpha_i(0) = \beta_i(0) = 0$, $\alpha_k(0) = 0$, and $\alpha_{ij}(0) = \delta_{i1} \delta_{j1} + \delta_{i2} \delta_{j2} + \delta_{i3} \delta_{j3}$, $\beta_{ij}(0) = \delta_{i3} \delta_{j3}$, so we have

$$\tilde{g}_{ik}(0) = \beta_{ik}(0) = f_{ik}\overline{f}(0) = (\delta_{i1} \delta_{k3} + \delta_{i3} \delta_{k1}) \delta_{j2},$$

$$\tilde{g}_{ik}(0) = 0,$$

$$\tilde{g}_{ik}(0) = - (\alpha \gamma + \alpha_i \alpha_k) + f_{ik} \overline{f} - (\beta \gamma + \beta_i \beta_k).$$

From this, we see that

$$\begin{cases}
\tilde{g}_{ik}(0) &= 0, \quad \text{if } \{i, k\} \neq \{j, \ell\}, \\
\tilde{g}_{ik}(0) &= -2(\alpha \gamma + \beta \gamma) = -2(1 + \delta_{i2}), \\
\tilde{g}_{ik}(0) &= \tilde{g}_{ik}(0) = - (\alpha \gamma \alpha_k + \alpha_i \alpha_k) + f_{ik} = \begin{cases}
-1, & \text{if } \{i, k\} = \{1, 2\} \text{ or } \{2, 3\}, \\
1, & \text{if } \{i, k\} = \{1, 3\}.
\end{cases}
\end{cases}$$
Similarly, since in \( U \) the \((1, 1)\)-form \( \sigma \) is given by
\[
\sigma = \sqrt{-1} \sum_{i,j=1}^{3} \sigma_{ij} dz_i \wedge d\overline{z}_j = \sqrt{-1} \alpha \beta (z_3 dz_1 + d\overline{z}_2) \wedge (\overline{z}_3 d\overline{z}_1 + d\overline{z}_2),
\]
therefore we have
\[
\sigma_{ij} = \frac{1}{\alpha \beta} (\delta_{i1}\delta_{j1}|z_3|^2 + \delta_{i1}\delta_{j2}z_3 + \delta_{i2}\delta_{j1}\overline{z}_3 + \delta_{i2}\delta_{j2}).
\]
From this we compute
\[
\begin{align*}
\sigma_{ij}(0) &= \delta_{i2}\delta_{j2}, \\
\sigma_{ij,k}(0) &= \delta_{i1}\delta_{j2}\delta_{k3}, \\
\sigma_{ij,kp}(0) &= 0, \\
\sigma_{ij,kp}(0) &= \delta_{ij}\delta_{k3}\delta_{k3} - \delta_{k\ell}(1 + \delta_{k2})\delta_{ij}\delta_{i2}.
\end{align*}
\]
By (8.4), we have \( \tilde{g}_{ij}(0) = \delta_{ij}(1 + \delta_{i2}) \), so at the origin \( g_{ij} = \tilde{g}_{ij} - \sigma_{ij} \) satisfies
\[
\begin{align*}
g_{ij}(0) &= \delta_{ij}, \\
g_{ij,k}(0) &= \delta_{ij}\delta_{k3}\delta_{k3}, \\
g_{ij,kp}(0) &= 0,
\end{align*}
\]
and
\[
\begin{align*}
g_{ij,k\ell}(0) &= 0 \text{ if } \{i, k\} \neq \{j, \ell\}, \\
g_{ij,k}\overline{\ell}(0) &= -2,
\end{align*}
\]
and
\[
\begin{align*}
g_{ij,k\overline{\ell}}(0) &= \begin{cases} 
-1, & \text{if } \{i, k\} = \{1, 2\} \text{ or } \{2, 3\}, \\
1, & \text{if } \{i, k\} = \{1, 3\}.
\end{cases}
\end{align*}
\]
(8.13)
\[
\begin{align*}
g_{ij,k\overline{\ell}}(0) &= \begin{cases} 
-1, & \text{if } \{i, k\} = \{12\} \text{ or } \{32\}, \\
1, & \text{if } \{i, k\} = \{31\}, \\
0, & \text{if } \{i, k\} = \{13\}, \{21\} \text{ or } \{23\}.
\end{cases}
\end{align*}
\]
(8.14)
The curvature components of the Chern connection \( \nabla^c \), defined by \( \Theta_{ij} = \sum_{k, \ell} R^c_{k\ell ij} dz_k \wedge d\overline{z}_\ell \) where \( \Theta = \overline{\partial} \theta = \overline{\partial}(\partial g g^{-1}) \), is given by
\[
R^c_{k\ell ij} = -g_{ij,k\overline{\ell}} + \sum_{p, q} g_{ij,pq} \overline{g}_{p\ell q} \overline{g}^j_p.
\]
At the origin, \( g_{ij}(0) = \delta_{ij} \), and all \( g_{ij,k}(0) = 0 \) except \( g_{ij,1}(0) = 1 \), so the second term on the right hand side of the above equality is \( \delta_{ij}\delta_{i3}\delta_{k3}\delta_{k1} \), and by (8.12), (8.13) and (8.14) we get at the origin that
\[
\begin{align*}
R^c_{jk \overline{\ell}} &= 0, \text{ if } \{i, k\} \neq \{j, \ell\}, \\
R^c_{13 \overline{1}} &= 2, \\
R^c_{11 \overline{3}} &= R^c_{32 \overline{3}} = 1, \\
R^c_{13 \overline{3}} &= -1, \\
R^c_{21 \overline{3}} &= R^c_{31 \overline{3}} = R^c_{12 \overline{3}} = 0, \\
R^c_{12 \overline{2}} &= R^c_{32 \overline{3}} = 1.
\end{align*}
\]
In other words, at the origin, the Chern curvature matrix is
\[
\Theta = \begin{bmatrix}
2 dz_1 + dz_2 + dz_3 \\
2 dz_1 + dz_2 + dz_3 \\
-2 dz_1 \\
-2 dz_1 \\
dz_2 + dz_3 + 2 dz_3
\end{bmatrix}
\]
Here we wrote \( dz_{\overline{i}} \) for \( dz_i \wedge d\overline{z}_j \). In particular, \( \sqrt{-1} \text{ tr } \Theta = 2\sqrt{-1}(dz_{1\overline{1}} + 2dz_{2\overline{2}} + dz_{3\overline{3}}) = 2\tilde{\omega} \) as expected.
The biholomorphic (Griffiths) curvature of \( g \) is \( R^c_{XX,YY} = iY\Theta(X, X)Y \), which is equal to
\[
2 \sum_{i=1}^{3} |X_i Y_i|^2 + |X_2 Y_3|^2 + |X_2 Y_2|^2 + 2 \text{ Re}(X_1 X_2 Y_1 Y_2) + 2 \text{ Re}(X_2 X_3 Y_2 Y_3) - 2 \text{ Re}(X_1 X_3 Y_1 Y_3)
\]
\[
= |X_2 Y_1|^2 + |X_2 Y_2|^2 + |X_1 Y_1 + X_2 Y_2|^2 + |X_1 Y_1 - X_3 Y_3|^2 + |X_2 Y_2 + X_3 Y_3|^2 \geq 0.
\]
This also indicates that \( g \) is not-Kähler since \( X \) is not a Hermitian symmetric space. Note that when \( X = Y \), the holomorphic sectional curvature is given by
\[
R^c_{XX,XX} = |X_1 X_2|^2 + |X_2 X_3|^2 + (|X_1|^2 + |X_2|^2)^2 + (|X_3|^2 - |X_3|^2)^2 + (|X_2|^2 + |X_3|^2)^2,
\]
which is positive for any \( X \neq 0 \), so \( (X, g) \) does have positive holomorphic sectional curvature. Also, the first, second and third Chern Ricci form of \( g \) are respectively
\[
\text{Ric}(\omega) = 2\tilde{\omega}, \quad \text{Ric}^{(2)}(\omega) = 4\omega - \tilde{\omega}, \quad \text{Ric}^{(3)}(\omega) = 2\tilde{\omega},
\]
which are all positive definite, with the first and third Ricci equal to each other.
Next we verify that \((X, g)\) is balanced and \(SPT\). First let us recall the formula under a natural frame. Suppose \((z_1, \ldots, z_n)\) is a local holomorphic coordinate on a Hermitian manifold \((M^n, g)\), and write \(\varepsilon_i = \frac{\partial}{\partial z_i}\). Under the frame \(\varepsilon\), which we view as a column vector, the Levi-Civita connection \(\nabla\), Chern connection \(\nabla^c\), and Strominger connection \(\nabla^s\) are given by
\[
\nabla^c\varepsilon = \theta\varepsilon, \quad \nabla^s\varepsilon = \theta^s\varepsilon, \quad \nabla\varepsilon = \theta^{(1)}\varepsilon + \theta^{(2)}\varepsilon.
\]
It is well-known that \(\theta = \partial gg^{-1}\), where \(g = (g_{ij})\). Denote by \(T\) the torsion tensor of \(\nabla^c\), and write \(T(\varepsilon_i, \varepsilon_k) = 2 \sum_j T^j_{ik} \varepsilon_j\), then we have
\[
(8.20) \quad T^j_{ik} = \frac{1}{2} \sum_k (g_{ik,\ell} - g_{i\ell,k}) g^{\ell j}.
\]
Since \(\nabla\) is torsion free, it holds
\[
2(\nabla_x y, z) = x(y, z) + y(x, z) - z(x, y) + \langle[x, y], z\rangle - \langle[y, z], x\rangle - \langle[x, z], y\rangle
\]
for any vector fields \(x, y, z\) on \(M\), so under the natural frame we have
\[
(8.21) \quad \theta^{(1)}_{ij} = \frac{1}{2} \sum_{k,\ell} (g_{ik,\ell} + g_{i\ell,k}) g^{\ell j} dz_k + \frac{1}{2} \sum_{k,\ell} (g_{i\ell,k} - g_{ik,\ell}) g^{\ell j} d\varepsilon_k.
\]
By the relation \(\theta^s = 2\theta^{(1)} - \theta\), we get
\[
(8.22) \quad \theta^s_{ij} = \sum_{k,\ell} g_{ik,\ell} g^{\ell j} dz_k + 2 \sum_{r,\ell,k} g_{\ell r,\ell r} g^{\ell j} d\varepsilon_k.
\]
The \(SPT\) condition is given by
\[
(8.23) \quad \nabla^s T = 0 \iff dT^j_{ik} = \sum_r \left( \theta^s_{ir} T^j_{r k} + \theta^s_{kr} T^j_{i r} - \theta^s_{ij} T^r_{r k} \right), \forall i, j, k.
\]
If \(g_{ij} = \delta_{ij}\) at the origin 0, then by (8.22) the \(SPT\) condition at 0 is given by
\[
(8.24) \quad \frac{\partial}{\partial z_k} T^j_{ik} = \sum_r \left( g_{\ell r,\ell r} T^j_{r k} + g_{\ell r,k} T^j_{i r} - g_{\ell r,\ell r} T^r_{r k} \right)
\]
\[
(8.25) \quad \frac{\partial}{\partial \varepsilon_k} T^j_{ik} = 2 \sum_r \left( T^j_{r k} T^r_{i \ell} - T^j_{i \ell} T^r_{r k} + T^r_{r k} T^j_{i \ell} \right).
\]
Now let us check for our Wallach space \((X, g)\). At the origin, we have \(g_{ij} = \delta_{ij}\), and all \(g_{ij,\ell} = 0\) except \(g_{i\ell,i} = 1\), so by (8.20) we know that all components of \(T\) vanishes except \(T^3_{13} = \frac{1}{2}\). In particular, the torsion 1-form \(\eta = 0\), as \(\eta_k = \sum T^j_{ij} \varepsilon_j\).

For (8.24), the right hand side is zero because for each of these three terms, one of the two factors is zero when \(r = 2\) or not 2. Its left hand side at 0 is equal to \(\frac{1}{2} (g_{ik,ij} - g_{ik,ki})\), which is zero by the last equality in (8.11). For (8.25), twice of its left hand side at 0 is given by
\[
(g_{ik,ij} - g_{ik,ji}) = (g_{ik,ij} - g_{ik,ji}) d\varepsilon_k.
\]
When \(\{i, k\} \neq \{j, \ell\}\), both sides of (8.25) are zero. The same is true when \(i = j = k = \ell\), so we just need to check the \(i \neq k\) and \(\{i, k\} = \{i, \ell\}\) case. Assume that \(i = j \neq k = \ell\). Then the twice of the left hand side of (8.25) at 0 is equal to
\[
g_{ik,ik} - g_{ik,ik} + \delta_{i3} \delta_{k1} = -\sigma_{ik,ik} + \sigma_{ik,ik} + \delta_{i3} \delta_{k1} = -\delta_{i2}(1 + \delta_{k2}) + \delta_{i1} \delta_{k3} + \delta_{i3} \delta_{k1}.
\]
Here we used the fact that \(\tilde{g}_{ik,ik} = \tilde{g}_{ik,ik}\) and \(\sigma_{ik,ik} = 0\). In the mean time, twice of the right hand side of (8.25) is
\[
4 \sum_r \left( T^r_{r i} T^r_{i \ell} - |T^r_{r i}|^2 + |T^r_{r i}|^2 \right) = -\delta_{i2}(\delta_{k1} + \delta_{k3}) + \delta_{i1} \delta_{k3} + \delta_{i3} \delta_{k1}.
\]
Note that \(\delta_{k1} + \delta_{k3} = 1 - \delta_{k2}\). The two sides are differed by \(\delta_{i2}\delta_{k2}\), which is zero since \(i \neq k\). So (8.25) holds in this case. The \(i = \ell \neq k = j\) case can be verified similarly. So the Wallach threefold \((X, g)\) is indeed non-Kähler, balanced, and \(SPT\).

In the remaining part of this section, we will verify that \((X, g)\) has constant Ricci curvature and non-negative sectional curvature for its Levi-Civita connection \(\nabla\).

First let us recall some general formula from existing literature. Let \(e\) be a local unitary frame on a Hermitian manifold \((M^n, g)\). We have
\[
\nabla^c T^j_{ik} - \nabla^c T^j_{ik} = 2 \sum_r \left( T^j_{r k} T^r_{i \ell} + T^r_{i k} T^j_{r \ell} - T^j_{i k} T^r_{r \ell} \right).
\]
By Proposition 3.5, we know that \( \nabla_{g}^{1,0} T = 0 \) implies that the right hand side of the above equality is zero. Therefore we always have \( \nabla_{g}^{1,0} T = 0 \implies \nabla_{g}^{1,0} T = 0 \). So if \( g \) is SPT, then the \((1,0)\)-part of Chern covariant differentiation of torsion vanishes: \( T^{i}_{jk;\ell} = 0 \). The \((0,1)\)-part of Chern covariant differentiation of torsion, on the other hand, is given by

\[
T^{i}_{jk;\ell} = \frac{1}{2} (R^{c}_{ik\ell} - R^{c}_{ikj}).
\]

Denote by \( R \) the curvature tensor of the Levi-Civita (Riemannian) connection of \( g \). By [30, Lemma 7], and using the above equality to replace the \((0,1)\)-derivatives of \( T \), we obtain

\[
R_{ijk\ell} = T^{\ell}_{ij;k} + \sum_{r} (T^{\ell}_{rj} T^{r}_{ik} - T^{\ell}_{rj} T^{r}_{ik})
\]

(8.26)

\[
R_{k\ell j} = \frac{1}{2} (R^{c}_{k\ell j} + R^{c}_{kj\ell}) + \sum_{r} (T^{c}_{ik} T^{r}_{jr} - T^{c}_{ik} T^{r}_{jr} - T^{c}_{ik} T^{r}_{jr})
\]

(8.27)

Now let us specialize to the Wallach threefold \((X, g)\) at the origin 0. We have all \( T^{i}_{jk} = 0 \) except \( T^{2}_{13} = \frac{1}{2} \), so the right hand side of (8.26) is zero, hence \( R_{ijk\ell} = 0 \). By the information on \( R^{c}_{k\ell j} \), we get \( R_{k\ell j} = 0 \) if \( \{i, k\} \neq \{j, \ell\} \), \( R_{\pi^{4}} = 2 \) for each \( i \), and

\[
R_{12\pi^{2}} = R_{23\pi^{2}} = -R_{13\pi^{2}} = \frac{3}{4}, \quad R_{12\pi^{2}} = R_{32\pi^{2}} = \frac{1}{2}, \quad R_{13\pi^{2}} = -\frac{1}{4}.
\]

Note that for Riemannian curvature \( R \) we always have \( R_{ijk\ell} = R_{k\ell j} \). Now we compute the sectional curvature of \( R \). Let \( x, y \) be any two real tangent vector of \( X \) at the origin 0, with \( x \wedge y \neq 0 \). Write \( x = X + \overline{X} \) and \( y = Y + \overline{Y} \) for type \((1,0)\) tangent vectors \( X \) and \( Y \). We have

\[
x \wedge y = X \wedge Y + \overline{X} \wedge \overline{Y} + (X \wedge \overline{Y} - Y \wedge \overline{X}).
\]

By Gray’s theorem, \( R_{XYZW} = 0 \) for any type \((1,0)\) tangent vectors \( X, Y, Z, W \). Also, we have shown that \( R_{XYZW} = 0 \) for our manifold. By the first Bianchi identity, we have

\[
-R_{XYXY} = -R_{XYY} + R_{XYY}.
\]

Thus we have

\[
R_{xyyx} = -R(x \wedge y, x \wedge y) = -2R(X \wedge Y, Y \wedge \overline{Y}) - R(X \wedge \overline{Y} - Y \wedge \overline{Y}, X \wedge \overline{Y} - Y \wedge \overline{X})
\]

\[
= -2R_{XYY} + 2R_{XYXY} - R(X \wedge \overline{Y} - Y \wedge \overline{Y}, X \wedge \overline{Y} - Y \wedge \overline{X})
\]

\[
= -2R_{XYY} + 4R_{XYXY} - 2Re\{R_{XYXY}\}
\]

For \( i \neq k \), let us write \( R_{i\ell k} = a_{ik} \) and \( R_{i\ell k} = b_{ik} \). We have

\[
2b_{ik} - a_{ik} = \frac{1}{4}, \quad 2a_{ik} - b_{ik} = \begin{cases} -\frac{1}{4}, & \text{if } \{i, k\} = \{1, 3\}, \\ 1, & \text{otherwise} \end{cases}
\]

(8.29)

Continue with the calculation of \( R_{xyyx} \) above, we have

\[
R_{xyyx} = \sum_{i,j,k,\ell} R_{\overline{R}_{i\ell k}} \{ -2X_{i} \overline{X}_{j} Y_{k} \overline{Y}_{\ell} + 4X_{i} \overline{Y}_{j} Y_{k} \overline{X}_{\ell} - 2Re(X\overline{Y}_{j} X_{k} \overline{Y}_{\ell}) \}
\]

\[
= \sum_{i} 2 \{ |X_{i} Y_{j}|^2 - 2Re(X_{i} \overline{Y}_{j}) \} + \sum_{i \neq k} a_{ik} \{ -2 |X_{i} Y_{k}|^2 + 4X_{i} \overline{Y}_{j} Y_{k} \overline{X}_{\ell} - 2Re(X_{i} \overline{Y}_{j} X_{k} \overline{Y}_{\ell}) \} +
\]

\[
+ \sum_{i \neq k} b_{ik} \{ -2X_{i} \overline{X}_{k} Y_{j} \overline{Y}_{\ell} + 4 |X_{i} Y_{k}|^2 - 2Re(X_{i} \overline{Y}_{j} X_{k} \overline{Y}_{\ell}) \}
\]

\[
= \sum_{i \neq k} \{ |X_{i} Y_{j}|^2 - 2Re(X_{i} \overline{Y}_{j}) \} + \sum_{i \neq k} \{ (2b_{ik} - a_{ik})(|X_{i} Y_{k}|^2 + |X_{k} Y_{i}|^2) \} +
\]

\[
+ \sum_{i < k} \{ (2a_{ik} - b_{ik}) 2Re(X_{i} \overline{X}_{k} \overline{Y}_{j} Y_{k}) \} - 2 \sum_{i < k} \{ (a_{ik} + b_{ik}) 2Re(X_{i} X_{k} \overline{Y}_{j} \overline{Y}_{k}) \}
\]

\[
= \sum_{i < k} F_{ik},
\]

where

\[
F_{ik} = |X_{i} Y_{j}|^2 - 2Re(X_{i} \overline{Y}_{j}) + |X_{k} Y_{j}|^2 - 2Re(X_{k} \overline{Y}_{j}) + \frac{1}{4}(|X_{i} Y_{k}|^2 + |X_{k} Y_{i}|^2) +
\]

\[
+ (2a_{ik} - b_{ik}) 2Re(X_{i} \overline{X}_{k} \overline{Y}_{j} Y_{k}) - (a_{ik} + b_{ik}) 2Re(X_{i} X_{k} \overline{Y}_{j} \overline{Y}_{k}).
\]
For \((ik) = (13)\), \((2a - b) = -\frac{5}{4}\) and \((a + b) = -1\), so we have
\[
F_{ik} = |X_i Y_j|^2 - \text{Re}(X_i^2 Y_j^2) + |X_k Y_k|^2 - \text{Re}(X_k^2 Y_k^2) + \frac{1}{4}(|X_i Y_k|^2 + |X_k Y_i|^2) + \\
- \frac{5}{4}2\text{Re}(X_i X_k Y_j Y_k) + 2\text{Re}(X_i Y_j X_k Y_k)
\]
\[= 2\{\text{Im}(X_i Y_j)\}^2 + 2\{\text{Im}(X_k Y_k)\}^2 + \frac{1}{4}|X_i Y_k - X_k Y_i|^2 - 4\text{Im}(X_i Y_j)\text{Im}(X_k Y_k)
\]
\[= 2\{\text{Im}(X_i Y_j) - \text{Im}(X_k Y_k)\}^2 + \frac{1}{4}|X_i Y_k - X_k Y_i|^2 \geq 0.
\]
Similarly, for \((ik) = (12)\) or \((23)\), \(2a - b = 1, a + b = \frac{5}{4}\), so
\[
F_{ik} = |X_i Y_i|^2 - \text{Re}(X_i^2 Y_i^2) + |X_k Y_k|^2 - \text{Re}(X_k^2 Y_k^2) + \frac{1}{4}(|X_i Y_k|^2 + |X_k Y_i|^2) + \\
+ 2\text{Re}(X_i X_i Y_k Y_k) - \frac{5}{4}2\text{Re}(X_i X_k Y_i Y_k)
\]
\[= 2\{\text{Im}(X_i Y_i)\}^2 + 2\{\text{Im}(X_k Y_k)\}^2 + \frac{1}{4}|X_i X_k Y_k - Y_i X_k|^2 + 4\text{Im}(X_i Y_i)\text{Im}(X_k Y_k)
\]
\[= 2\{\text{Im}(X_i Y_i) + \text{Im}(X_k Y_k)\}^2 + \frac{1}{4}|X_i X_k Y_k - Y_i X_k|^2 \geq 0.
\]
That is, \(R_{xyyx}\) is equal to
\[
4(I_1 + I_2)^2 + 4(I_2 + I_3)^2 + 4(I_1 - I_3)^2 + \frac{1}{2}|X_1 Y_2 - Y_1 X_2|^2 + \frac{1}{2}|X_2 Y_3 - Y_2 X_3|^2 + \frac{1}{2}|X_1 Y_3 - Y_1 X_3|^2,
\]
where \(I_i = \text{Im}(X_i Y_i)\). Therefore the Levi-Civita connection of \(g\) has non-negative sectional curvature. We note that the sectional curvature is not strictly positive here: if we take \(X_1 = X_2 = X_3 \in \mathbb{R} \setminus \{0\}\), and \(Y_1 = Y_2 = Y_3 = \rho \not\in \mathbb{R}\), then \(I_1 = I_3 = -I_2\) and the above expression vanishes, so we get \(R_{xyyx} = 0\) yet \(x \land y \neq 0\).

To see the Ricci curvature of \(g\), let \(Y = e_i\), then the above formula becomes
\[
R_{xyyx} = 4|X_i|^2 - 4\text{Re}(X_i^2) + |X_j|^2 + |X_k|^2,
\]
where \(\{i, j, k\} = \{1, 2, 3\}\). Similarly, if we let \(Y = \sqrt{-1}e_i\), then we get
\[
R_{xyyx} = 4|X_i|^2 + 4\text{Re}(X_i^2) + |X_j|^2 + |X_k|^2.
\]
Add up the above two equalities for \(i\) from 1 to 3, we get \(12|X|^2 = 6|x|^2\). Let \(\varepsilon_i = \frac{1}{\sqrt{2}}(e_i + \overline{e}_i)\) and \(\varepsilon_{i^*} = \frac{1}{\sqrt{2}}(e_i - \overline{e}_i)\). Then \(\{\varepsilon_i, \varepsilon_{i^*}\}\) form an orthonormal tangent frame, so the Ricci curvature of the Riemannian metric \(g\) is
\[
(8.30) \quad \text{Ric}(x) = \frac{1}{|x|^2} \sum_i \left(R_{x\varepsilon_i x\varepsilon_i} + R_{x\varepsilon_i^* x\varepsilon_i^*} \right) = \frac{1}{|x|^2} 6|x|^2 = 6.
\]
That is, \(g\) is an Einstein metric on \(X\) with Ricci curvature 6.

9. Balanced SPT threefolds of middle type

9.1. The structural theorem. In this subsection, let us consider the rank \(B = 2\) case. Throughout this subsection we will assume that \((M^3, g)\) is a compact, balanced SPT manifold of middle type. Denote by \(L\) the kernel of the \(B\) tensor.

Claim 9.1. \(L\) is a holomorphic line bundle on \(M^3\) satisfying \(L^{\otimes 2} \cong \mathcal{O}_M\).

Proof. Here and below \(\mathcal{O}_M\) denotes the trivial line bundle of \(M\). Let us scale the metric \(g\) by a suitable constant multiple to make \(a_1 = a_2 = \frac{1}{2}\) from now on. It follows from (6.5) and (6.6) that \(\theta_{13} = \theta_{23} = \theta_{33} = 0, \theta_{11} = \theta_{22}\) and \(\theta_{12} + \theta_{21} = 0\), which implies
\[
\gamma = \frac{1}{2} \begin{pmatrix}
0 & -\overline{\varphi_3} & \overline{\varphi_2} \\
\varphi_3 & 0 & -\overline{\varphi_1} \\
-\overline{\varphi_2} & \varphi_1 & 0
\end{pmatrix}, \quad \theta^s = \begin{pmatrix}
\alpha & \beta_0 & 0 \\
-\beta_0 & \alpha & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad \theta = \begin{pmatrix}
\alpha & -\beta & \overline{\varphi_3} \\
-\beta & \alpha & \overline{\varphi_2} \\
\varphi_2 & -\varphi_1 & 0
\end{pmatrix}, \quad \tau = \begin{pmatrix}
\varphi_2 \varphi_3 \\
\varphi_3 \varphi_1 \\
\varphi_1 \varphi_2
\end{pmatrix},
\]
where \(\overline{\alpha} = -\alpha, \overline{\beta}_0 = \beta_0, \text{ and } \beta = \beta_0 + \varphi_3 + \overline{\varphi_3}\). This means that \(\nabla^s e_3 = 0\) for any special frame \(e\). Since \(\nabla^s e_3 = \sum_i \theta_{3i}(e_i)e_j = 0, \text{ so } e_3\) is a local holomorphic vector field on \(M^3\) which is also a local section of \(L\), thus \(L\) is a holomorphic line subbundle of the holomorphic tangent bundle \(T^{1,0} M\).
To see that $L^\mathcal{O} = \mathcal{O}_M$, let us consider the structure equation $d\varphi = -\theta \wedge \varphi + \tau$. In our case it becomes

\begin{equation}
(9.1) \quad d\varphi = \begin{bmatrix}
-\alpha \varphi_1 + \beta \varphi_2 \\
-\beta \varphi_1 - \alpha \varphi_2 \\
\varphi_2 \varphi_1 - \varphi_1 \varphi_2
\end{bmatrix}
\end{equation}

For convenience, $\varphi_i \wedge \varphi_j$ is abbreviated as $\varphi_{ij}$. The curvature matrices of $\nabla^s$ and $\nabla^c$ under $e$ are

\begin{equation}
\Theta^s = \begin{bmatrix}
d\alpha & d\beta_0 & 0 \\
-\beta_0 & d\alpha & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad \Theta = \begin{bmatrix}
d\alpha - \varphi_{22} & d\beta + \varphi_3 & 0 \\
-\beta + \varphi_{22} & d\alpha - \varphi_3 & 0 \\
0 & 0 & \varphi_1 \end{bmatrix}.
\end{equation}

**Remark 9.2.** By the above formula for $\Theta^s$, we know that the holonomy group for Strominger connection of any balanced SPT threefold of middle type is always contained in the subgroup of $U(2) \times 1$ that commutes with $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \times 1$, in $U(3)$. By taking the exterior differentiation of (9.1), we get the first Bianchi identity

\begin{equation}
(9.3) \quad d\alpha \wedge \varphi_1 - d\beta \wedge \varphi_2 = 0, \quad d\alpha \wedge \varphi_2 + d\beta \wedge \varphi_1 = 0.
\end{equation}

It also holds that, from (9.1),

\begin{equation}
(9.4) \quad d(\varphi_1 \varphi_1) = -d(\varphi_2 \varphi_2) = \beta(\varphi_2 \varphi_1 + \varphi_1 \varphi_2), \quad d\beta - d\beta_0 = 2(\varphi_2 \varphi_1 - \varphi_1 \varphi_2).
\end{equation}

It follows that the Riemannian connection $\nabla$ is given by

\[
\nabla e = (\theta^s - \gamma)e + \theta_2 \varphi = \begin{bmatrix}
\beta_0 + \frac{1}{2}(\varphi_3 + \varphi_3) & \alpha & -\frac{1}{2} \varphi_1 \\
\frac{1}{2} \varphi_2 & -\varphi_1 & 0 \\
0 & 0 & \frac{1}{2} \varphi_2
\end{bmatrix} e + \begin{bmatrix}
0 & 0 & -\frac{1}{2} \varphi_2 \\
\frac{1}{2} \varphi_2 & -\varphi_2 & 0 \\
0 & 0 & \frac{1}{2} \varphi_2
\end{bmatrix} \varphi,
\]

since $(\theta_2)_{ij} = \sum_k T_{ij}^k \varphi_k$. In particular, $\nabla e_3 = \nabla e_3$ is real. So we know that $e_3$ can only vary by a sign. In other words, $\varphi_3 \otimes \varphi_3$ is globally defined on $M$, hence $L^\mathcal{O}$ is trivial. This proves Claim 9.1.

Note that since $T_{ij}^k = 0$ for any $i, k$, it yields $[e_3, \varphi_3] = \nabla e_3 \wedge \varphi_3 - \nabla \varphi_3 \wedge e_3 = 0$, so the distribution $L$ generated by $e_3$ is actually a holomorphic foliation, which is also a flat holomorphic bundle, using the restriction metric of $g$.

If $L = \mathcal{O}_M$, let $\hat{M} = M$. If $L \neq \mathcal{O}_M$, then by the cyclic covering lemma, it gives us an unbranched double cover $\pi : \hat{M} \rightarrow M$ so that $\pi^* L$ is trivial. Thus $\hat{M}$ is either $M$ itself, or a double cover of $M$. Lift the metric $g$ to $\hat{M}$, which we will still denote by $g$. Then $(\hat{M}, g, J)$ is a balanced SPT threefold. The kernel of the $B$ tensor for $\hat{M}$ is now the trivial line bundle $L = \mathcal{O}_{\hat{M}}$, or equivalently, we may choose special frames $e$ on $\hat{M}$ so that $e_3$ is globally defined. We claim that

**Claim 9.3.** There exists a complex structure $I$ on $\hat{M}$ which is compatible with $g$, so that $I \circ J = J \circ I$, and $(\hat{M}, g, I)$ is a Vaisman threefold.

**Proof.** Let $e$ be a special frame on $\hat{M}$, where $e_3$ is a global holomorphic vector field of unit length on $\hat{M}$. Write $e_3 = \frac{1}{\sqrt{2}}(\xi - \sqrt{-1} \xi')$, or equivalently,

\[
\xi = \frac{1}{\sqrt{2}}(e_3 + \varphi), \quad \xi' = \frac{-1}{\sqrt{2}}(e_3 - \varphi),
\]

Then $\xi$ and $\xi'$ are global real vector fields of unit length satisfying $J\xi = \xi'$. We have $\nabla \xi' = 0$ and

\begin{equation}
\begin{cases}
\nabla \xi = \frac{1}{\sqrt{2}}(\varphi e_1 - \varphi e_2 + \varphi_2 e_1 - \varphi_1 e_2) \\
\nabla e_1 = \alpha e_1 + (\beta_0 + \frac{1}{2}(\varphi_3 + \varphi_3)) e_2 - \frac{1}{2} \varphi_2 e_2 \\
\nabla e_2 = (-\beta_0 - \frac{1}{2}(\varphi_3 + \varphi_3)) e_1 + \alpha e_2 + \frac{1}{2} \varphi_1 e_2
\end{cases}
\end{equation}

Note that $e_1$, $e_2$ are still local here. Write

\[
e_1 = \frac{1}{\sqrt{2}}(\varepsilon_1 - \sqrt{-1} \varepsilon_2), \quad e_2 = \frac{1}{\sqrt{2}}(\varepsilon_3 - \sqrt{-1} \varepsilon_3).
\]

Then $\{e_1, e_2, e_3, e_4, \xi, \xi'\}$ becomes a local orthonormal frame of $(\hat{M}, g)$, and $J\varepsilon_1 = \varepsilon_2$, $J\varepsilon_3 = \varepsilon_4$, $J\xi = \xi'$. Let us also write

\[
\varphi_1 = \frac{1}{\sqrt{2}}(\phi_1 + \sqrt{-1} \phi_2), \quad \varphi_2 = \frac{1}{\sqrt{2}}(\phi_3 + \sqrt{-1} \phi_4), \quad \varphi_3 = \frac{1}{\sqrt{2}}(\phi_5 + \sqrt{-1} \phi_6).
\]
Then \( \{ \phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6 \} \) becomes the dual coframe. For any real vector field \( X \) on \( \hat{M} \), by (9.5) we have
\[
\sqrt{3} \nabla_X \xi = \phi_3(X) \varepsilon_1 + \phi_4(X) \varepsilon_2 - \phi_1(X) \varepsilon_3 - \phi_2(X) \varepsilon_4.
\]
Let us denote by \( V \) the distribution in \( \hat{M} \) spanned by \( \varepsilon_i, 1 \leq i \leq 4 \). Denote by \( E' \) the distribution spanned by \( \xi' \), and by \( E = V \oplus \mathbb{R} \xi \) the orthogonal complement of \( E' \). Then both \( E' \) and \( E \) are globally defined, parallel under \( \nabla \), and \( V \) is also globally defined since it is the orthogonal complement of \( \xi \) in \( E \).

Define an orthogonal almost complex structure \( I \) on \( (M, g) \) by
\[
I(\varepsilon_3) = \varepsilon_1, \quad I(\varepsilon_4) = \varepsilon_2, \quad I(\xi) = \xi'.
\]
Then \( I \) is globally defined since for any \( X \in V \) it is given by \( I(X) = \sqrt{3} \nabla_X \xi \), and clearly, \( I \circ J = J \circ I \). In order to show that \( I \) is integrable and \( (\hat{M}, g, I) \) is Vaisman, let us carry out the computation in complex terms. Note that a standard orthonormal frame for \( I \) becomes \( \{ \varepsilon_3, \varepsilon_1, \varepsilon_2, \varepsilon_4, \xi, \xi' \} \). In particular, the orientation induced by \( I \) is opposite to that by \( J \). Let us write \( \beta' = \beta_0 + \frac{1}{2} (\varphi_3 + \varphi_4) \). Since \( \mathfrak{m} = -\alpha \) and \( \overline{\beta'} = \beta' \), by (9.5) we obtain
\[
\begin{align*}
\nabla \varepsilon_1 &= -\sqrt{3} \alpha \varepsilon_2 + \beta' \varepsilon_3 - \frac{1}{\sqrt{3}} \phi_3 \xi, \\
\nabla \varepsilon_2 &= -\sqrt{3} \alpha \varepsilon_1 + \beta' \varepsilon_4 - \frac{1}{\sqrt{3}} \phi_4 \xi, \\
\nabla \varepsilon_3 &= -\beta' \varepsilon_1 - \sqrt{3} \alpha \varepsilon_4 + \frac{1}{\sqrt{3}} \phi_1 \xi, \\
\nabla \varepsilon_4 &= -\beta' \varepsilon_2 + \sqrt{3} \alpha \varepsilon_3 + \frac{1}{\sqrt{3}} \phi_2 \xi.
\end{align*}
\]
Let us denote by
\[
u_1 = \frac{1}{\sqrt{2}} (\varepsilon_3 - \sqrt{-1} \varepsilon_1), \quad \nu_2 = \frac{1}{\sqrt{2}} (\varepsilon_4 - \sqrt{-1} \varepsilon_2),
\]
then \( \{ \nu_1, \nu_2, \varepsilon_3 \} \) forms a local unitary frame of \( (\hat{M}, g, I) \). Its dual coframe is \( \{ \psi_1, \psi_2, \varphi_3 \} \), where
\[
\psi_1 = \frac{1}{\sqrt{2}} (\phi_3 + \sqrt{-1} \phi_1), \quad \psi_2 = \frac{1}{\sqrt{2}} (\phi_4 + \sqrt{-1} \phi_2).
\]
By a straightforward computation, we get
\[
\begin{align*}
\nabla \nu_1 &= -\sqrt{-1} \beta' \nu_1 - \sqrt{-1} \alpha \nu_2 + \frac{\sqrt{3}}{\sqrt{2}} \psi_1 \xi, \\
\nabla \nu_2 &= -\sqrt{-1} \alpha \nu_1 - \sqrt{-1} \beta' \nu_2 + \frac{\sqrt{3}}{\sqrt{2}} \psi_2 \xi, \\
\nabla \varepsilon_3 &= \frac{\sqrt{2}}{\sqrt{3}} (\psi_1 \nu_2 + \psi_2 \nu_1 - \psi_1 \nu_2 - \psi_2 \nu_1)
\end{align*}
\]
To show that \( I \) is integrable, we need to verify that \( \langle T^{1,0}_i, T^{1,0}_j \rangle \subseteq T_{\nu}^{1,0} \), or equivalently, each \( \{ u_i, u_j \} \) is a combination of \( u_k \) for \( 1 \leq k \leq 3 \), but without \( \varepsilon_k \) terms. Here we have written \( u_3 = \varepsilon_3 \). First, for \( \{ u_1, u_2 \} \), by (9.6) we have
\[
[u_1, u_2] = \nabla u_1 u_2 - \nabla u_2 u_1 = (\ast) u_1 + (\ast) u_2.
\]
Also, by (9.6), \( \nabla u_1 e_3 = \frac{\sqrt{-1}}{\sqrt{2}} u_1 \), so
\[
[u_1, e_3] = \nabla u_1 e_3 - \nabla e_3 u_1 = \frac{\sqrt{-1}}{\sqrt{2}} u_1 - (\ast) u_1 - (\ast) u_2.
\]
is a combination of \( u_1 \) and \( u_2 \). Similarly, \( \{ u_2, e_3 \} \) is a combination of \( u_1 \) and \( u_2 \). Therefore, the almost complex structure \( I \) is integrable.

Next let us show that the Hermitian threefold \( (\hat{M}, g, I) \) is Vaisman. Denote by \( u \) the column vector of the local unitary frame, where \( u_3 = e_3 \), and by \( \psi \) the column vector of its dual coframe, where \( \psi_3 = \varphi_3 \).

By (9.6) the Levi-Civita connection \( \nabla \) has connection matrices \( \nabla u = \hat{\theta}_1 u + \theta_2 \mathfrak{m} \) where
\[
\hat{\theta}_1 = \sqrt{-1} \begin{bmatrix}
-\beta' & -\alpha & \frac{1}{\sqrt{2}} \psi_1 \\
\alpha & -\beta' & \frac{1}{\sqrt{2}} \psi_2 \\
\frac{1}{\sqrt{2}} \psi_1 & \frac{1}{\sqrt{2}} \psi_2 & 0
\end{bmatrix}, \quad \theta_2 = \sqrt{-1} \begin{bmatrix}
0 & 0 & -\frac{1}{\sqrt{2}} \psi_1 \\
0 & 0 & -\frac{1}{\sqrt{2}} \psi_2 \\
\frac{1}{\sqrt{2}} \psi_1 & \frac{1}{\sqrt{2}} \psi_2 & 0
\end{bmatrix}.
\]
Under the frame \( u \), the only non-zero components of the Chern torsion of \( (\hat{M}, I, g) \) are \( T^2_{13} = T^2_{23} = \frac{\sqrt{-1}}{\sqrt{2}} \), so the Gauduchon’s torsion 1-form is \( \tilde{\eta} = \sqrt{-1} \varphi_3 \). By the structure equation, \( \nabla \psi = -\hat{\theta}_1 \psi - \theta_2 \overline{\psi} \), we get
\[
\nabla \varphi_3 = \sum_j \left\{ -(\hat{\theta}_1)_{j3} \psi_j - (\theta_2)_{j3} \overline{\psi_j} \right\} = \sqrt{-1} (\psi_1 \overline{\psi_1} + \psi_2 \overline{\psi_2}),
\]
Therefore,
\[
\nabla (\tilde{\eta} + \overline{\eta}) = \sqrt{-1} (\nabla \varphi_3 - \nabla \overline{\varphi_3}) = 0.
\]
That is, the Lee form of \((\hat{M}, g, I)\) is parallel under the Levi-Civita connection, which means that the Hermitian threefold is Vaisman.

Next we show that the Strominger connection of \((\hat{M}, g, J)\) and \((\hat{M}, g, I)\) coincide:

**Claim 9.4.** The Strominger connection of \((\hat{M}, g, J)\) and \((\hat{M}, g, I)\) coincide.

**Proof.** Let us denote by \(\hat{\nabla}^s\), \(\hat{\nabla}^g\) the Strominger connection of \((\hat{M}, g, J)\) and \((\hat{M}, g, I)\), respectively. We want to show that \(\nabla^s = \nabla^g\). Let \(e\) be a special frame with \(e_3\) global, and define \(u_1, u_2\) as in the proof of Claim 9.3, then we have

\[
\begin{align*}
    u_1 &= \frac{1}{2}(e_2 + \bar{e}_2 - \sqrt{-1}e_1 - \sqrt{-1}\bar{e}_1) \\
    u_2 &= \frac{1}{2}(e_1 - \bar{e}_1 + \sqrt{-1}e_2 - \sqrt{-1}\bar{e}_2)
\end{align*}
\]

By the expression of the matrix \(\theta^s\) of \(\nabla^s\) under \(e\), we know that

\[
\nabla^s e_3 = 0, \quad \nabla^s e_1 = \alpha e_1 + \beta_0 e_2, \quad \nabla^s e_2 = -\beta_0 e_1 + \alpha e_2.
\]

Using the fact \(\bar{\tau} = -\alpha, \bar{\beta}_0 = \beta_0\), and (9.7), we get

\[
2\nabla^s u_1 = (-\beta_0 e_1 + \alpha e_2) + (-\beta_0 \bar{e}_1 - \alpha \bar{e}_2) - \sqrt{-1}(\alpha e_1 + \beta_0 e_2) - \sqrt{-1}(\alpha \bar{e}_1 + \beta_0 \bar{e}_2)
\]

\[
= \alpha(e_2 - \bar{e}_2 - \sqrt{-1}e_1 + \sqrt{-1}\bar{e}_1) - \beta_0(e_1 + \bar{e}_1 + \sqrt{-1}e_2 - \sqrt{-1}\bar{e}_2)
\]

\[
= -\sqrt{-1}\alpha u_2 - \sqrt{-1}\beta_0 u_1,
\]

and \(\nabla^s u_2\) can be computed similarly, so we get

\[
\nabla^s u_1 = -\sqrt{-1}\beta_0 u_1 - \sqrt{-1}\alpha u_2, \quad \nabla^s u_2 = \sqrt{-1}\alpha u_1 - \sqrt{-1}\beta_0 u_2.
\]

On the other hand, under the unitary frame \(u\) for \((\hat{M}, g, I)\) with dual coframe \(\psi\), the only non-zero components of the Chern torsion are \(\hat{T}_{13} = \hat{T}_{23} = \frac{1}{\sqrt{2}}\), hence the \(\gamma\)-tensor is given by

\[
\hat{\gamma} = \frac{\sqrt{-1}}{2} \begin{bmatrix}
    \varphi_3 + \bar{\varphi}_3 & 0 & -\bar{\psi}_1 \\
    0 & \varphi_3 + \bar{\varphi}_3 & -\psi_2 \\
    -\bar{\psi}_1 & -\psi_2 & 0
\end{bmatrix}.
\]

So by the expression for \(\hat{\theta}_1\), we get the connection matrix \(\hat{\theta}^s\) of \(\hat{\nabla}^s\)

\[
\hat{\theta}^s = \hat{\theta}_1 + \hat{\gamma} = \sqrt{-1} \begin{bmatrix}
    -\beta_0 & -\alpha & 0 \\
    \alpha & -\beta_0 & 0 \\
    0 & 0 & 0
\end{bmatrix}.
\]

Here we used the fact that \(\beta' = \beta_0 + \frac{1}{\sqrt{2}}(\varphi_3 + \bar{\varphi}_3)\). This means that

\[
\hat{\nabla}^s u_1 = -\sqrt{-1}\beta_0 u_1 - \sqrt{-1}\alpha u_2, \quad \hat{\nabla}^s u_2 = \sqrt{-1}\alpha u_1 - \sqrt{-1}\beta_0 u_2, \quad \hat{\nabla}^s e_3 = 0.
\]

Compare this with (9.8), we conclude that \(\nabla^s = \hat{\nabla}^s\). This completes the proof of Claim 9.4. 

By the nice structure theorem of Ornea and Verbitsky [20] for compact Vaisman manifolds, we know that \(\hat{M}\) is a smooth fibration over the circle \(S^1\) with fiber being a compact Sasakian manifold \(N\). On the universal cover level, since \(\nabla \xi' = 0\), we know that the universal cover \(\hat{M}\) of \(M\) is a Riemannian product \(N \times \mathbb{R}\), and \(\hat{N}\) is a Sasakian manifold, since \(\xi\) is a unit vector field and \(\sqrt{2}\nabla \xi\) gives the complex structure \(I\) on the orthogonal complement \(V = \xi^\perp\) in \(T\hat{N}\). Since the 1-form dual to \(\xi\) is \(\phi_5 = \frac{1}{\sqrt{2}}(\varphi_3 + \bar{\varphi}_3)\), we have

\[
d\phi_5 = \sqrt{2} d\varphi_3 = \sqrt{2}(\varphi_{21} - \varphi_{2\bar{1}}).
\]

So \(\phi_5 \wedge (d\phi_5)^2 = 2\sqrt{2}(\varphi_3 + \bar{\varphi}_3)\varphi_{123}\) which is nowhere zero. The vector field \(\xi\) is clearly a Killing field as \(I\) is orthogonal with respect to \(g\). By definition, this means that \(\xi\) is the Reeb vector field and \((\hat{N}, g, \xi, \phi_5)\) is a Sasakian 5-manifold.

Then we will prove the last statement in the middle type case of Theorem 1.9.

**Claim 9.5.** Let \((M^3, g, J)\) be a compact non-Kähler threefold, which is balanced SPT of the middle type. Then \((M^3, J)\) does not admit any pluriclosed metric.

**Proof.** Assume on the contrary that there is a pluriclosed metric \(\tilde{g}\) on \((M^3, J)\). Its Kähler form \(\tilde{\omega}\) satisfies \(\delta\tilde{\omega} = 0\). Locally let \(e\) be a special frame and \(\varphi\) its dual coframe. As we have seen before, \(e_3\) and \(\varphi_3\) are
determined up to a sign, so in particular, $\varphi_3\varphi_3$ is globally defined on $M$. Since $d\varphi_3 = \varphi_{\bar{1}1} - \varphi_{1\bar{2}} = d\varphi_3$, we know that $d(\varphi_3\varphi_3) = 2\varphi_{1\bar{2}2}$ is a global $(2,2)$-form on $M$. We have
\[
\begin{align*}
(d(\varphi_3\varphi_3) \omega) &= d\varphi_3 d\varphi_3 \omega - d\varphi_3 \varphi_3 \partial \omega \\
(d(\varphi_3 \varphi_3) \partial \omega) &= d\varphi_3 \varphi_3 \partial \omega - \varphi_3 d\varphi_3 \partial \omega + \varphi_3 \varphi_3 \partial \partial \omega.
\end{align*}
\]

Note that the middle term on the right hand side of the second equality is zero by type consideration. So if we add up these two equalities and integrate it over $M$, we would get
\[
0 = \int_M d\varphi_3 d\varphi_3 \omega = \int_M 2\varphi_{1\bar{2}2} \omega = \int_M 2\sqrt{-1} \hat{g}_{33} \varphi_{1\bar{1}2\bar{3}} = -\frac{1}{3} \int_M \hat{g}_{33} \omega^3 < 0,
\]
a contradiction. Here we have written $\omega = \sqrt{-1} \sum_{i,j=1}^3 \hat{g}_{ij} \varphi_i \wedge \varphi_j$ under the frame $e$, so the matrix $(\hat{g}_{ij})$ is positive definite. The contradiction means that $M^3$ cannot admit any pluriclosed metric. This completes the proof of the claim. \(\square\)

On the balanced SPT threefold $(M^3, g, J)$, we know from Theorem 1.1 the SPT condition indicates $R^s_{ijkl} = 0$ and $R^{\perp}_{ijkl} = R^{\perp}_{kijl}$. Then it follows from (9.2) that the $(0,2)$ and $(2,0)$-parts of $d\alpha$ and $d\beta$ are zero, and there are real-valued local smooth functions $x$, $y$, and $z$ such that
\[
\begin{align*}
(9.9) & \quad d\alpha = x(\varphi_{1\bar{1}} + \varphi_{\bar{2}\bar{2}}) + \sqrt{-1} y(\varphi_{1\bar{2}} - \varphi_{\bar{1}\bar{2}}), \\
(9.10) & \quad d\beta = -\sqrt{-1} y(\varphi_{1\bar{2}} + \varphi_{2\bar{1}}) + z(\varphi_{1\bar{1}} - \varphi_{\bar{2}\bar{2}}).
\end{align*}
\]
It yields from (9.3) that $x = z + 2$, hence
\[
(9.11) \quad d\beta = -\sqrt{-1} y(\varphi_{1\bar{1}} + \varphi_{2\bar{2}}) + x(\varphi_{1\bar{2}} - \varphi_{\bar{1}\bar{2}}).
\]

Note that the Ricci form of the Chern curvature of $\omega_J$ is given by $\text{Ric}(\omega_J) = 2d\alpha$, which indicates $x = \frac{1}{2} \text{Scal}(\omega_J)$ where $\text{Scal}(\omega_J)$ is the Chern scalar curvature, hence $x$ is a global function. On the other hand, since $\varphi_3$ is determined up to a sign, so $\varphi_{33}$, hence $\sqrt{-1}(\varphi_{1\bar{1}} + \varphi_{2\bar{2}})$, are globally defined on $M^3$. The latter can be regarded as the Kähler form of $J_{\xi_{L^2}}$. As a consequence, $y d\varphi_3 = -\sqrt{-1}(d\alpha - x(\varphi_{1\bar{1}} + \varphi_{2\bar{2}}))$ is globally defined on $M$. This means that $y$ and $d\varphi_3 = \varphi_{2\bar{1}} - \varphi_{\bar{1}\bar{2}}$ are determined up to a sign. When $\varphi_3$ is not globally defined on $M$, by lifting to $M$, we know that both $y$ and $\varphi_{2\bar{1}} - \varphi_{\bar{1}\bar{2}}$ are globally defined. On $M^3$, $\varphi_{2\bar{1}} - \varphi_{\bar{1}\bar{2}}$ is the Kähler form of $I_{\xi_{L^2}}$, while $y = \frac{1}{2} \text{tr}_{\omega_J} \sqrt{-1} d\beta$. Since $\varphi_{1\bar{1}} + \varphi_{2\bar{2}}$ and $\varphi_{1\bar{2}} - \varphi_{\bar{1}\bar{2}}$ are $d$-closed, by taking the exterior differentiation of (9.9) or (9.11), we get
\[
(9.12) \quad x_3 = x_3 = 0, \quad x_1 = -\sqrt{-1} y_2, \quad x_2 = \sqrt{-1} y_1.
\]
Here $x_i = e_i(x)$, $y_i = e_i(y)$. We suspect that $x$ and $y$ must be constants, but at this point we do not know how to prove it from the above system of equations, where the usual Bochner technique does not seem to apply, and more geometric information might be needed.

9.2. Examples of Lie-Hermitian threefolds. In this subsection, we will deal with the special case when the balanced SPT threefold $(M^3, g)$ of the middle type is assumed to be a Lie-Hermitian manifold, meaning that its universal cover is a (connected and simply-connected) Lie group $G$ equipped with a left-invariant complex structure and a compatible left-invariant metric.

**Proof of Theorem 1.10.** Let us start from a global, unitary, left-invariant frame $e$ on $G$. An appropriate constant unitary change of the frame enables us to assume that $e$ is special. Under our assumption that $g$ is balanced $\text{SPT}$ of the middle type, the formulae for connection matrices $\theta^g$ and $\theta$ are all valid.

Since $e$ is left-invariant, the 1-form $\alpha$ and $\beta$ are linear combinations of $\varphi$ and $\varphi$ with constant coefficients:
\[
\alpha = \sum_i (a_i \varphi_i - \overline{a_i} \overline{\varphi}_i), \quad \beta = \sum_i (b_i \varphi_i + \overline{b_i} \overline{\varphi}_i).
\]
The structure equation (9.1) implies
\[
\begin{align*}
(d\alpha)^{2,0} &= (a_1 b_1 + a_2 b_2) \varphi_{12} + (a_1 a_3 + a_2 b_3) \varphi_{13} + (a_2 a_3 - a_1 b_3) \varphi_{23} \\
(d\beta)^{2,0} &= (a_1 b_1 - a_2 b_2 + b_1^2 + b_2^2) \varphi_{12} + (a_1 b_1 + b_2) \varphi_{13} + (a_2 b_3 - b_1 b_3) \varphi_{23} \\
(d\alpha)^{1,1} &= 2(\text{Re}(a_1 b_1) - |a_1|^2) \varphi_{1\bar{2}} - 2(\text{Re}(a_1 b_2) + |a_2|^2) \varphi_{\bar{1}\bar{2}} + P \varphi_{1\bar{1}} + P \varphi_{\bar{2}\bar{2}} + \\
&\quad + (a_1 b_3 - a_2 b_3) \varphi_{1\bar{3}} - (a_2 b_3 + a_1 b_3) \varphi_{2\bar{3}} + (a_2 b_3 - a_1 b_3) \varphi_{3\bar{3}} - (a_2 b_3 + a_1 b_3) \varphi_{3\bar{3}} \\
(d\beta)^{1,1} &= 2\sqrt{-1} \text{Im}(a_1 b_1 - b_1) \varphi_{1\bar{2}} + 2\sqrt{-1} \text{Im}(a_2 b_2 - b_1 b_2) \varphi_{2\bar{3}} + Q \varphi_{1\bar{1}} + \\
&\quad + (b_1 b_3 - a_1 b_3) \varphi_{1\bar{3}} - (b_2 b_3 + b_1 b_3) \varphi_{2\bar{3}} - (b_2 b_3 - b_1 b_3) \varphi_{3\bar{3}} + (b_2 b_3 + b_1 b_3) \varphi_{3\bar{3}}
\end{align*}
\]
where

\[
P = a_3 - \overline{a}_3 - a_1 \overline{b}_1 - 2a_2 \overline{a}_1 + b_2 \overline{a}_2,
\]
\[
Q = b_3 + \overline{b}_3 - |b_1|^2 - |b_2|^2 + a_2 \overline{b}_1 - b_2 \overline{a}_1.
\]

Then equalities (9.9) and (9.11) indicate

\[
a_1 b_1 + a_2 b_2 = 0, \quad a_2 b_1 - a_1 b_2 + b_1^2 + b_2^2 = 0,
\]
\[
a_1 a_3 + a_2 b_3 = -a_1 b_3 + a_2 a_3 = 0,
\]
\[
a_3 b_1 + b_2 b_3 = -b_1 b_3 + b_2 a_3 = 0,
\]
\[
x = 2 \text{Re}(a_2 \overline{b}_1) - 2|a_1|^2 = -2 \text{Re}(a_1 \overline{b}_2) - 2|a_2|^2 = Q,
\]
\[
y = \sqrt{-1} P = 2 \text{Im}(a_1 \overline{b}_1 - b_1 \overline{b}_2) = 2 \text{Im}(a_2 \overline{b}_2 - b_2 \overline{b}_1).
\]

**Case A:** \(a_3^2 + b_3^2 \neq 0\).

In this case, by (9.14) and (9.15), we get \(a_1 = a_2 = 0\) and \(b_1 = b_2 = 0\). By (9.16) and (9.17) we get \(x = y = a_3 - \overline{a}_3 = b_3 - \overline{b}_3 = 0\). So in this case we have \(a = a_3(\varphi_3 - \overline{\varphi}_3), \beta = b_3(\varphi_3 - \overline{\varphi}_3)\), where \(a_3 \in \mathbb{R}\) and \(b_3 \in \sqrt{-1} \mathbb{R}\), hence it yields that

\[
\begin{align*}
\frac{d\varphi_1}{\sqrt{3}} &= \frac{(a_3 \varphi_3 - b_3 \varphi_2) \wedge (\varphi_3 - \overline{\varphi}_3)}{\sqrt{3}}, \\
\frac{d\varphi_2}{\sqrt{3}} &= \frac{(b_3 \varphi_3 + a_3 \varphi_2) \wedge (\varphi_3 - \overline{\varphi}_3)}{\sqrt{3}}, \\
\frac{d\varphi_3}{\sqrt{3}} &= \frac{\varphi_3 \wedge \varphi_\beta}{\sqrt{3}}.
\end{align*}
\]

By a unitary change of \((\varphi_1, \varphi_2, \varphi_3)\) into \((\frac{1}{\sqrt{2}}(\varphi_1 + \sqrt{-1} \varphi_2), \frac{1}{\sqrt{2}}(\sqrt{-1} \varphi_1 + \varphi_2), \varphi_3)\), the above is equivalent to the following

\[
A_{a,b} : \begin{cases}
\frac{d\varphi_1}{\sqrt{3}} = a \varphi_1 \wedge (\varphi_3 - \overline{\varphi}_3), \\
\frac{d\varphi_2}{\sqrt{3}} = b \varphi_2 \wedge (\varphi_3 - \overline{\varphi}_3), \\
\frac{d\varphi_3}{\sqrt{3}} = \sqrt{-1}(\varphi_\beta - \varphi_\beta).
\end{cases}
\]

Here \(a, b \in \mathbb{R}\) are given by \(a = a_3 + \sqrt{-1} b_3, b = a_3 - \sqrt{-1} b_3\). We will denote this Lie-Hermitian threefold by \(A_{a,b}\), where \(a, b\) are arbitrary real numbers. We have \(x = y = 0\) in this case, so the Ricci form \(\text{Ric}(\omega_f) = 0\). Since \(d(\varphi_{123}) = -2a \wedge \varphi_{123}\), we know that \(A_{a,b}\) will have an invariant nowhere-zero holomorphic 3-form (hence any quotient will have trivial canonical line bundle) if and only if \(a = 0\), or equivalently, if \(a = -b\). When \(a = b = 0\), \(A_{0,0}\) is clearly isometric to the nilmanifold \(N^3\) given by (1.5).

**Case B:** \(a_3^2 + b_3^2 = 0\), where \((a_3, b_3) \neq 0\).

In this cases, it follows that \(a_3 = \epsilon ib_3\), where \(\epsilon = \pm 1, \varphi_3 \neq 0\). By equalities (9.13) through (9.16), we get \(a_1 = \epsilon ia_2, b_1 = \epsilon ib_2, b_3 + \overline{b}_3 = 2(|b_2|^2 - |a_2|^2), \) and

\[
x = 2 \text{Im}(a_2 \overline{b}_2) - 2|a_2|^2, \quad y = -2 \text{Im}(a_2 \overline{b}_2) + 2 \epsilon |b_2|^2.
\]

Let \(u = a_2 - \epsilon i b_2, v = a_2 + \epsilon i b_2\), and \(w = 2 \epsilon i b_3\). By a unitary change of \((\varphi_1, \varphi_2, \varphi_3)\) into \((\sqrt{2}(\varphi_1 + \epsilon i \varphi_2), \sqrt{2}(\epsilon \varphi_1 + \varphi_2), \varphi_3)\), we get the Lie Hermitian manifold

\[
B^\epsilon_{a,v,w} : \begin{cases}
\frac{d\varphi_1}{\sqrt{2}} = \sqrt{2} \nu \varphi_{12} - \sqrt{2} \nu \varphi_{12} + w \varphi_{13} - w \varphi_{13}, \\
\frac{d\varphi_2}{\sqrt{2}} = -\sqrt{2} \nu \varphi_{22}, \\
\frac{d\varphi_3}{\sqrt{2}} = \epsilon \nu \varphi_{12} - \varphi_{11}.
\end{cases}
\]

It is easy to see that \(u, v, w\) satisfy \(w - \overline{w} = \epsilon i(|u - v|^2 - |u + v|^2), \) and

\[
x = \frac{1}{2}(|v|^2 - |u|^2 - |u + v|^2), \quad y = \frac{\epsilon}{2}(|u|^2 + |u - v|^2 - |v|^2).
\]

It is clear that \(B^\epsilon_{0,0,0}\) is isomorphic to the nilmanifold \(N^3\) given by (1.5).

Next let us deal with the remaining case \(a_3 = b_3 = 0\). If \(b_1 = b_2 = 0\), then \(Q = 0\), hence by (9.16) we get \(x = 0\) and \(a_1 = a_2 = 0\). This gives us \(a = \beta = 0\) and we end up with the nilmanifold \(N^3\). Hence we may assume \((b_1, b_2) \neq (0, 0), \) which splits into two cases below, depending on whether \(b_1^2 + b_2^2\) is zero or not.

**Case C:** \((a_3, b_3) = (0, 0)\) and \(b_1^2 + b_2^2 \neq 0\).

In this case, the equality (9.13) gives us

\[
a_1 = b_2, \quad a_2 = -b_1.
\]
Hence, if we let \( u = -\vec{b}_2 \) and \( v = -\vec{b}_1 \), which are complex numbers satisfying \( |u|^2 + |v|^2 > 0 \), and we get the Lie-Hermitian manifold

\[
C_{u,v}: \quad \begin{cases} 
\frac{d\varphi_1}{u} = u(\varphi_1 + \varphi_2) + v(\varphi_2 - \varphi_1), \\
\frac{d\varphi_2}{u} = -v(\varphi_1 + \varphi_2) + u(\varphi_2 - \varphi_1), \\
\frac{d\varphi_3}{u} = \varphi_2 - \varphi_1.
\end{cases}
\]

Then it yields that \( x = -2|u|^2 - 2|v|^2 < 0 \) and \( y = 4 \text{Im}(u\bar{v}) \). In particular, the Chern scalar curvature \( x \) is a negative constant. The complex structure is abelian. Clearly, \( C_{0,0} \) is isometric to \( \mathbb{N}^3 \).

**Case D:** \( (a_3,b_3) = (0,0) \) and \( b_1^2 + b_2^2 = 0 \), where \( (b_1,b_2) \neq (0,0) \).

This means that \( a_3 = b_3 = 0 \) and \( b_2 = \varepsilon b_1 \neq 0 \), where \( \varepsilon = \sqrt{-1} \) and \( \varepsilon = \pm 1 \). By equalities (9.13) through (9.17), we get \( a_2 = \varepsilon i a_1 \), \( a_1 = \rho b_1 \) for some \( |\rho| = 1 \), and

\[
x = -2|b_1|^2(1 + \varepsilon \text{Im}(\rho)), \quad y = \varepsilon x.
\]

Let \( u = \vec{b}_1 \) and we get the Lie-Hermitian manifold

\[
D_{u,\rho}^\varepsilon: \quad \begin{cases} 
\frac{d\varphi_1}{u} = \bar{\varphi}_1 + \rho \varepsilon i \varphi_{12} + u\{(\varphi_1 + \varphi_2) + (\varphi_1 - \varphi_2)\}, \\
\frac{d\varphi_2}{u} = \bar{\varphi}_2 + \rho \varepsilon i \varphi_{12} + u\{(\varphi_1 + \varphi_2) - (\varphi_1 - \varphi_2)\},
\end{cases}
\]

where \( u, \rho \in \mathbb{C}, \ |\rho| = 1, \) and \( \varepsilon = \pm 1 \). The complex structure of \( D_{u,\rho}^\varepsilon \) is non-abelian except when \( \rho = \varepsilon i \), in which case we have \( D_{u,e}^\varepsilon = C_{u,e,-u} \). The Chern scalar curvature \( x \) of \( D_{u,\rho}^\varepsilon \) is always non-positive. When \( u = 0 \), we get the complex nilmanifold \( \mathbb{N}^3 \) again.

From the discussion above, we know that for any Lie-Hermitian threefold which is balanced \( SPT \) of middle type, it must be a member of one of these four families. On the other hand, it is easy to check that each of the above four families of Lie-Hermitian manifolds are balanced \( SPT \) of middle type. This completes the proof of the theorem.

\[\square\]

**9.3. Generalization to higher dimensions.** We have seen in the rank \( B = 2 \) case of Theorem 1.9 that balanced \( SPT \) threefolds \((M^3, g, J)\) admits a covering space \( \hat{M} \) of degree at most 2, such that on \((\hat{M}, g)\) there exists another orthogonal complex structure \( I \), so that \((\hat{M}, g, J)\) is Vaisman. The universal cover \( \hat{M} \) splits as a Riemannian product \( \mathbb{N}^5 \times \mathbb{R} \) where \( \mathbb{N}^5 \) is a Sasakian 5-manifold. The complex structures \( I \) and \( J \) are also closely related and satisfies the condition \( IJ = JI \). Therefore, the classification of compact balanced \( SPT \) threefolds of middle type amounts to understanding this special type Vaisman threefolds, or equivalently, this special type of Sasakian 5-manifolds with a ‘bi-Hermitian’ structure, namely, it admits two commutative orthogonal complex structures simultaneously.

In this subsection, we want to fully understand this special ‘bi-Hermitian’ structure. It turns out the phenomenon persists in higher dimensions as well, which will give us examples of balanced \( SPT \) manifolds in all dimensions \( n \geq 3 \). The 5-dimensional manifold \( \mathbb{N}^5 \) that we encountered before motivates us with the following definition.

**Definition 9.6.** Let \((N^{2m+1}, g, \xi)\) be a Sasakian manifold, that is, as defined in [32] for instance, an odd dimensional Riemannian manifold \((N, g)\) equipped with a Killing vector field \( \xi \) of unit length, such that:

1. The tensor field \( \frac{1}{c} \nabla \xi \) (where \( c > 0 \) is a constant), which sends a tangent vector \( X \) to the tangent vector \( \frac{1}{c} \nabla_X \xi \), gives an integrable orthogonal complex structure \( I \) on the distribution \( H \), where \( H \) is the perpendicular complement of \( \xi \) in the tangent bundle \( TN \).
2. Denote by \( \phi \) the 1-form dual to \( \xi \), namely, \( \phi(X) = g(X, \xi) \) for any \( X \), then \( \phi \wedge (d\phi)^m \) is nowhere zero. That is, \( \phi \) gives a contact structure on \( N \).

The manifold \((N^{2m+1}, g, \xi, J)\) is said to be an abelian Sasakian manifold, if \((N^{2m+1}, g, \xi)\) is a Sasakian manifold, which admits an endomorphism \( J \) of the tangent bundle \( TN \), such that:

1. The Killing vector field \( \xi \) is in the kernel of \( J \).
2. The endomorphism \( J \) defines an integrable orthogonal complex structure on the distribution \( H = \xi^\perp \) and the Kähler form \( \omega_J \), defined as \( \omega_J(X, Y) = g(JX, Y) \), is \( d \)-closed.
3. The space \( T^1_{1,0} \) is \( J \)-invariant and the eigenspaces of \( \sqrt{-1} \) and \( -\sqrt{-1} \) of \( J|_{T^1_{1,0}} \) have the same dimension, denoted by \( n \), where the space \( T^1_{1,0} \) is induced from the Sasakian structure

\[
TN \otimes_{\mathbb{R}} \mathbb{C} = (\xi) \oplus T^1_{1,0} \oplus T^0_{1,0}, \quad I = \frac{1}{c} \nabla \xi.
\]
Clearly we must have \( m = 2n \) here, so the dimension of an abelian Sasakian manifold is necessarily \( 4n + 1 \), and the two complex structures \( I \) and \( J \) on the distribution \( H = \xi^\perp \) satisfy

\[
I \circ J = J \circ I,
\]

which justifies the name of abelian Sasakian.

**Proposition 9.7.** Let \((N^{2m+1}, g, \xi)\) be a Sasakian manifold and \( J \) be an endomorphism of the tangent bundle \( TN \), which is an integrable orthogonal complex structure on the distribution \( \xi^\perp \) and satisfies \( J(\xi) = 0 \). Then \( J \) defines an abelian Sasakian structure on \((N^{2m+1}, g, \xi)\) if and only if the Kähler form \( \omega_1 \) of \( I = \frac{1}{2} \nabla \xi \) decomposes into two \( d \)-closed \( 1 \)-semi-positive \( 2 \)-forms \( \omega_1 \) and \( \omega_2 \), such that the space \( T^{1,0}_I \) admits a \( g \)-orthogonal decomposition of two kernel distributions of \( \omega_1 \) and \( \omega_2 \) with the same rank \( n \)

\[
T^{1,0}_I = \text{Ker}(\omega_1) \oplus \text{Ker}(\omega_2).
\]

Here an \( 1 \)-semi-positive \( 2 \)-form \( \omega \) on \( N \) is defined to satisfy \( \omega(X, Y) = \omega(IX, IY) \) for any \( X, Y \in TN \) and \( \omega(Z, \overline{Z}) \geq 0 \) for any \( Z \in T^{1,0}_I \), and the kernel distribution \( \text{Ker}(\omega) \) of an \( 1 \)-semi-positive \( 2 \)-form \( \omega \) is defined as \( \{ Z \in T^{1,0}_I | \omega(Z, \overline{Z}) = 0 \} \).

**Proof.** Let \((\xi, v, \overline{\tau})\) be a frame of the Sasakian manifold \((N^{2m+1}, g, \xi)\) such that \( v \) is a unitary frame of the distribution \( \xi^\perp \), which yields the following structure equation

\[
\nabla \begin{bmatrix} \xi \\ v \\ \overline{\tau} \end{bmatrix} = \begin{bmatrix} 0 & -v & -\overline{\tau} \\ v & K & \overline{T} \\ \overline{\tau} & \overline{T} & K \end{bmatrix} \begin{bmatrix} \xi \\ v \\ \overline{\tau} \end{bmatrix},
\]

where \( K \) is a skew Hermitian matrix of \( 1 \)-forms and \( L \) is a skew symmetric matrix of \( 1 \)-forms. The dual frame of \((\xi, v, \overline{\tau})\) is denoted by \((\phi, \zeta, \zeta)\). As \( I = \frac{1}{2} \nabla \xi \) induces an integrable orthogonal complex structure on the distribution \( \xi^\perp \), it yields that \( \nu = -c\sqrt{-1} \zeta \). The dual version of the equation above is

\[
(9.22) \quad \nabla \begin{bmatrix} \phi \\ \zeta \\ \overline{\zeta} \end{bmatrix} = \begin{bmatrix} 0 & -\overline{\tau} & -v \\ \overline{\tau} & K & L \\ -v & L & K \end{bmatrix} \begin{bmatrix} \phi \\ \zeta \\ \overline{\zeta} \end{bmatrix}.
\]

As \( \nabla \) is torsion free, it follows that

\[
d \begin{bmatrix} \phi \\ \zeta \\ \overline{\zeta} \end{bmatrix} = \begin{bmatrix} 0 & -\overline{\tau} & -v \\ \overline{\tau} & K & L \\ -v & L & K \end{bmatrix} \begin{bmatrix} \phi \\ \zeta \\ \overline{\zeta} \end{bmatrix},
\]

which implies that \( d\phi = 2c\sqrt{-1} \zeta \overline{\zeta} \). Note that \( I \) is integrable on \( \xi^\perp \) and thus \( d\zeta_0 \) has no component of \( \phi \wedge \overline{\zeta}_j \) and \( \zeta_k \wedge \overline{\zeta}_l \), which indicates \( L \) is a matrix of \((1, 0)\)-forms of the complex structure \( I \). Then it is clear that

\[
0 = d(\zeta \overline{\zeta}) = -\langle \zeta^\perp L \overline{\zeta} + \zeta \overline{\zeta} \rangle,
\]

which yields that \( \zeta \overline{\zeta} \zeta = 0 \) by the type of \( L \). Since \( L \) is skew symmetric, it follows that \( L = 0 \).

If \( J \) defines an abelian Sasakian structure on \((N, g, \xi)\), we may assume that the basis \( v \) of \( T^{1,0}_I \) splits into two column vectors, where the former one is still denoted by \( v \) and the latter one by \( v_{+n} \), both of which contains \( n \) entries, such that

\[
Jv = \sqrt{-1}v, \quad Jv_{+n} = -\sqrt{-1}v_{+n}.
\]

This forces that \( \zeta \) splits into two parts as \( \begin{bmatrix} \zeta \\ \zeta_{+n} \end{bmatrix} \) and the matrix \( K \) also splits into four submatrices of the same size, denoted by

\[
K = \begin{bmatrix} A & C \\ -\overline{C} & D \end{bmatrix}.
\]

Then a unitary frame of \( J \) on the distribution \( \xi^\perp \) can be defined as

\[
e = v, \quad e_{+n} = \overline{v}_{+n}.
\]

The dual frame of \((\xi, e, e_{+n}, \overline{\tau}, \overline{\tau}_{+n})\) is denoted by \((\phi, \varphi, \varphi_{+n}, \varphi, \varphi_{+n})\), which satisfies

\[
\varphi = \zeta, \quad \varphi_{+n} = \overline{\zeta}_{+n}.
\]
Then the equation (9.22) can be reformulated, in terms of \((\phi, \varphi, \varphi^+, \varphi^+)\), as

\[
\begin{pmatrix}
\phi \\
\varphi \\
\varphi^+ \\
\varphi^+
\end{pmatrix}
= \begin{bmatrix}
0 & -\frac{\mu}{A} & -\mu & -\mu^+ \\
\mu & A & 0 & 0 \\
\mu^+ & 0 & D & -\frac{C}{A} \\
\mu^+ & 0 & C & A \\
\mu^+ & -C & 0 & 0 \\
\mu^+ & 0 & 0 & D
\end{bmatrix}
\begin{pmatrix}
\phi \\
\varphi \\
\varphi^+ \\
\varphi^+
\end{pmatrix},
\]

where \(\mu = -c\sqrt{-1} \varphi, \mu^+ = c\sqrt{-1} \varphi^+ \). As \(\nabla\) is torsion free, it yields that

\[
d\begin{pmatrix}
\phi \\
\varphi \\
\varphi^+ \\
\varphi^+
\end{pmatrix}
= \begin{bmatrix}
0 & -\frac{\mu}{A} & -\mu & -\mu^+ \\
\mu & A & 0 & 0 \\
\mu^+ & 0 & D & -\frac{C}{A} \\
\mu^+ & 0 & C & A \\
\mu^+ & -C & 0 & 0 \\
\mu^+ & 0 & 0 & D
\end{bmatrix}
\begin{pmatrix}
\phi \\
\varphi \\
\varphi^+ \\
\varphi^+
\end{pmatrix},
\]

which implies that \(d\phi = 2c\sqrt{-1}(\varphi^+ - t\varphi^+ t\varphi^+)\). As the Kähler form \(\omega_J = \sqrt{-1}(\varphi^+ + t\varphi^+ t\varphi^+)\) is necessarily \(d\)-closed by the definition of the abelian Sasakian structure, it follows that

\[
d(\varphi^+) = d(\varphi^+) = 0.
\]

Note that \(J\) is an orthogonal integrable complex structure on the distribution \(\xi^\perp\), which indicates \(d\varphi_i\) and \(d\varphi_{i+n}\) has no component of \(\varphi^\perp\). \(\omega_J\) is a \(\xi^\perp\)-orthogonal decomposition with the same rank \(\xi^\perp\), respectively. It is clear that

\[
d(\varphi^+) = 0 = d(\varphi^+) = 0.
\]

Therefore \(\omega = 0\).

Note that the Kähler form \(\omega_J\) of \(I\) is \(\sqrt{-1}(\varphi^+ + t\varphi^+ t\varphi^+)\) and the Kähler form \(\omega_J\) of \(J\) is \(\sqrt{-1}(\varphi^+ + t\varphi^+ t\varphi^+)\), from which we may decompose \(\omega_J\) into

\[
\omega_J = \omega_1 + \omega_2, \quad \omega_1 = \sqrt{-1}(\varphi^+ + t\varphi^+ t\varphi^+), \quad \omega_2 = \sqrt{-1}(\varphi^+ + t\varphi^+ t\varphi^+).
\]

It is clear that \(\omega_1, \omega_2\) are \(d\)-closed \(I\)-semi-positive 2-forms on \(N\) and \(\text{Ker}(\omega_J), \text{Ker}(\omega_1)\) are distributions generated by \(v, v_+, v_n\), respectively, which are the eigenspaces of \(J\) with respect to \(\sqrt{-1}, -\sqrt{-1}\). Then it follows that \(T_J^{1,0}\) admits a \(g\)-orthogonal decomposition into \(\text{Ker}(\omega_1) \oplus \text{Ker}(\omega_2)\).

Conversely, when the Kähler form \(\omega_J\) of \(I = \frac{1}{\sqrt{-1}}\xi\) decomposes into two \(d\)-closed \(I\)-semi-positive 2-forms \(\omega_1\) and \(\omega_2\), such that the space \(T_J^{1,0}\) admits a \(g\)-orthogonal decomposition with the same rank \(n\)

\[
T_J^{1,0} = \text{Ker}(\omega_1) \oplus \text{Ker}(\omega_2).
\]

We may define \(J\) by designating \(\text{Ker}(\omega_2)\) and \(\text{Ker}(\omega_1)\) are the eigenspaces of \(J\) with respect to the eigenvalues \(\sqrt{-1}\) and \(-\sqrt{-1}\) respectively. It is clear that \(T_J^{1,0}\) is \(J\)-invariant and \(J\) defines an abelian Sasakian structure.

**Definition 9.8.** Let \((N^{4n+1}_1, g_1, \xi_1, J_1)\) and \((N^{4n+1}_2, g_2, \xi_2, J_2)\) be two abelian Sasakian manifolds. Motivated by [6], on the product Riemannian manifold \(M = N_1 \times N_2\), of even dimension \(2n = 2(2n_1 + 2n_2 + 1)\), for \(\lambda \in \mathbb{H} = \{ \lambda \in \mathbb{C} \mid \text{Im}(\lambda) > 0 \}\), we can consider the natural almost complex structure \(J_\lambda\) defined by

\[
J_\lambda \xi_1 = \text{Re}(\lambda) \xi_1 + \text{Im}(\lambda) \xi_2,
J_\lambda X_i = J_i X_i, \quad \forall X_i \in H_i, \ i = 1, 2.
\]

Define the associated metric \(g_\lambda\) as

\[
g_\lambda(\xi_1, \xi_1) = g_\lambda(J_\lambda \xi_1, J_\lambda \xi_1) = 1, \quad g_\lambda(\xi_1, J_\lambda \xi_1) = 0,
\]

\[
g_\lambda(\xi_1, X) = 0, \quad \forall X \in H_1 \oplus H_2, \ i = 1, 2.
\]

\[
g_\lambda(X_i, Y_i) = g_\lambda(X_i, Y_i), \quad g_\lambda(X_i, X_2) = 0, \quad \forall X_i, Y_i \in H_i, \ i = 1, 2.
\]

It is clear that \(g_\lambda\) is a \(J_\lambda\)-Hermitian metric. It is easy to verify that \(J_\lambda\) is integrable, as shown in the proof of the following proposition, and when \(\lambda = \sqrt{-1}\), the Hermitian manifold \((N_1 \times N_2, g_\lambda = g_1 \times g_2, J_\lambda)\) is the standard Hermitian structure on the product of two abelian Sasakian manifolds.

Abelian Sasakian manifolds arise naturally in the study of balanced \(SPT\) manifolds.

**Proposition 9.9.** Let \((N^{4n+1}_1, g_1, \xi_1, J_1)\) and \((N^{4n+1}_2, g_2, \xi_2, J_2)\) be two abelian Sasakian manifolds. Then \((N_1 \times N_2, g_\lambda, J_\lambda)\) is a non-Kähler manifold, which is balanced and \(SPT\), and the rank of the \(B\) tensor is \(n - 1\), for the complex dimension \(n \geq 3\).
In the smallest possible dimension \( n = 2n_1 + 2n_2 + 1 = 3 \), the example above is always a Riemannian product, which is essentially \( N^5 \times \mathbb{R} \) above, whenever \( \lambda \in \mathbb{H} \), as \( n_1 \) or \( n_2 \) has to be zero. When the dimension increases, it may not be a Riemannian product for \( \text{Re}(\lambda) \neq 0 \). It has been shown by Belgun [6] that \( N_1 \times N_2 \) above admits a non-Kähler GCE structure, which is both LP and SPT, when we use the Sasakian structures of \( N_1 \) and \( N_2 \).

**Proof.** We consider the Hermitian structure \((\varphi_1, J_1)\) on the product \( N_1 \times N_2 \). As notations applied in the proof of Proposition 9.7 above, we may assume that \((\xi_1, e, e_{n_1}, \varphi, \varphi_{n_1})\) is a frame on \((N_1, g_1, \xi_1, J_1)\), where \( e, e_{n_1} \) is a unitary frame of \( \xi_1^+ \) endowed with the complex structure \( J_1 \), and the dual frame is denoted by \((\phi_1, \varphi, \varphi^{+n_1}, \varphi_0, \varphi^{+n_1})\), which satisfies the equation \((9.24)\). And \((\xi_2, \bar{e}, \bar{e}_{n_2}, \bar{\varphi}, \bar{\varphi}^{+n_2})\) is a frame on \((N_2, g_2, \xi_2, J_2)\), where \( \bar{e}, \bar{e}_{n_2} \) is a unitary frame of \( \xi_2^+ \) with the complex structure \( J_2 \), and the dual frame is denoted by \((\bar{\phi}_2, \bar{\varphi}, \bar{\varphi}^{+n_2}, \bar{\varphi}, \bar{\varphi}^{+n_2})\).

For simplicity, we will write \( \lambda = a + b\sqrt{-1} \), where \( b > 0 \). Let

\[
e_0 = \frac{1}{\sqrt{2}}(\xi_1 - \sqrt{-1} J_3 \xi_1) = \frac{1}{\sqrt{2}}(1 - a\sqrt{-1}) \xi_1 - b\sqrt{-1} \xi_2, \\
\varphi_0 = \frac{1}{\sqrt{2}} \left( \phi_1 - \frac{a}{b} \phi_2 + \sqrt{-1} \frac{1}{b} \phi_2 \right) = \frac{1}{\sqrt{2}} \left( \phi_1 - \frac{a - \sqrt{-1}}{b} \phi_2 \right),
\]

where \( e_0 \) is a \((1, 0)\)-vector and \( \varphi_0 \) is a \((1, 0)\)-form of of the almost complex structure \( J_\lambda \). It is easy to verify that \( F = (e_0, \bar{e}, \bar{e}_{n_2}) \) is a unitary frame of \( J_\lambda \) on \( N_1 \times N_2 \) and \( \Phi = (\varphi_0, \varphi, \varphi^{+n_1}, \varphi, \varphi^{+n_2}) \) is the dual frame, where \( F \) and \( \Phi \) are regarded as column vectors. The equation \((9.24)\) and the one on \((N_2, g_2, \xi_2, J_2)\) imply that

\[
\begin{align*}
d\varphi_0 &= \frac{1}{\sqrt{2}} \left( 2c_1 \sqrt{-1} \left( \varphi_0 - \varphi^{n_1} \varphi_0^{n_1} \right) - \frac{2c_2(1 + a\sqrt{-1})}{b} \left( \varphi_0 - \varphi^{n_2} \varphi_0^{n_2} \right) \right), \\
\begin{bmatrix} \varphi \\ \varphi^{n_1} \end{bmatrix} &= \frac{c_1}{\sqrt{2}} \begin{bmatrix} (a + \sqrt{-1}) \varphi_0 - (a - \sqrt{-1}) \varphi_0 \\ \varphi_0 \end{bmatrix} + \begin{bmatrix} \bar{A} & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} \varphi \\ \varphi^{n_1} \end{bmatrix}, \\
\begin{bmatrix} \varphi \\ \varphi^{n_2} \end{bmatrix} &= \frac{c_2}{\sqrt{2}} \begin{bmatrix} (b \varphi_0 - b \varphi_0) \\ -b \varphi_0 \end{bmatrix} + \begin{bmatrix} \bar{A} & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} \varphi \\ \varphi^{n_2} \end{bmatrix},
\end{align*}
\]

where the matrices \( \bar{A} \) and \( \bar{D} \) come from the analogous equation on \((N_2, g_2, \xi_2, J_2)\) to the one \((9.24)\). It implies that \( J_\lambda \) is indeed an integrable complex structure, as \( d\varphi_0, d\varphi, d\varphi^{n_1}, d\varphi \) and \( d\varphi^{n_2} \) has no component of \((0, 2)\)-forms. The Riemannian connection \( \nabla \) under the unitary frame \( \Phi \) can be denoted by

\[
\nabla \Phi = \partial_1 \Phi + \partial_2 \Phi,
\]

where \( \partial_1 \) is skew Hermitian and \( \partial_2 \) is skew symmetric. As \( \nabla \) is torsion free, it yields

\[
d\Phi = \partial_1 \Phi + \partial_2 \Phi,
\]

which indicates

\[
\begin{bmatrix} \mu_1 \\ H_1 \\ \mu_2 \\ H_2 \\ \mu_3 \\ H_3 \\ \mu_4 \\ H_4 \end{bmatrix}, \quad \begin{bmatrix} 0 & -\mu_5 & -\mu_6 & -\mu_7 & -\mu_8 \\ \mu_5 & 0 & 0 & 0 & 0 \\ \mu_6 & 0 & 0 & 0 & 0 \\ \mu_7 & 0 & 0 & 0 & 0 \\ \mu_8 & 0 & 0 & 0 & 0 \end{bmatrix},
\]

where

\[
\begin{align*}
\mu_1 &= \frac{c_1 \sqrt{-1}}{\sqrt{2}} \varphi, \\
\mu_2 &= \frac{c_1 \sqrt{-1}}{\sqrt{2}} \varphi^{n_1}, \\
\mu_3 &= -\frac{c_2(1 + a\sqrt{-1})}{b \sqrt{2}} \varphi, \\
\mu_4 &= \frac{c_2(1 + a\sqrt{-1})}{b \sqrt{2}} \varphi^{n_2},
\end{align*}
\]

\[
\begin{align*}
\mu_5 &= -\frac{c_1 \sqrt{-1}}{\sqrt{2}} \varphi, \\
\mu_6 &= \frac{c_1 \sqrt{-1}}{\sqrt{2}} \varphi^{n_1}, \\
\mu_7 &= \frac{c_2(1 + a\sqrt{-1})}{b \sqrt{2}} \varphi, \\
\mu_8 &= -\frac{c_2(1 + a\sqrt{-1})}{b \sqrt{2}} \varphi^{n_2},
\end{align*}
\]

\[
\begin{align*}
H_1 &= A - \frac{c_1 a}{\sqrt{2}} (\varphi_0 - \varphi_0) E_{n_1}, \\
H_2 &= D + \frac{c_1 a}{\sqrt{2}} (\varphi_0 - \varphi_0) E_{n_1},
\end{align*}
\]

\[
\begin{align*}
H_3 &= \bar{A} - \frac{c_2}{\sqrt{2}} \left( (b - 1 - a\sqrt{-1}) \varphi_0 - (b - 1 + a\sqrt{-1}) \varphi_0 \right) E_{n_2}, \\
H_4 &= \bar{D} + \frac{c_2}{\sqrt{2}} \left( (b - 1 - a\sqrt{-1}) \varphi_0 - (b - 1 + a\sqrt{-1}) \varphi_0 \right) E_{n_2}.
\end{align*}
\]
Here $E_{n_1}$ and $E_{n_2}$ are the identity matrices of the sizes $n_1$ and $n_2$. Then the matrix $\theta_2$ determines the components of Chern torsion of the Hermitian metric $(g_\lambda, J_\lambda)$, which implies, for $1 \leq i, j \leq n_1$, $2n_1 + 1 \leq \kappa, \iota \leq 2n_1 + n_2$, $0 \leq p, q \leq 2(n_1 + n_2)$,

$$T^p_0 = -\frac{c_1 \sqrt{-1}}{\sqrt{2}} \delta_{ip}, \quad T^p_{0+i+n_1} = \frac{c_1 \sqrt{-1}}{\sqrt{2}} \delta_{i+1+n_1 p},$$

$$T^p_0 = -\frac{c_2(1 - a \sqrt{-1})}{b \sqrt{2}} \delta_{i p}, \quad T^p_{0+i+n_2} = \frac{c_2(1 - a \sqrt{-1})}{b \sqrt{2}} \delta_{i+1+n_2 p},$$

with other components all vanishing, such as

$$T^p_{ij} = 0, \quad T^p_{i+j+n_1} = 0, \quad T^p_{i\kappa} = 0, \quad T^p_{i+n_2} = 0, \quad T^p_{i+1+n_1 j+n_1} = 0,$$

$$T^p_{i+1+n_1 \kappa+n_2} = 0, \quad T^p_{i+n_2 \kappa+n_2} = 0, \quad T^p_{\kappa+i+n_1} = 0, \quad T^p_{\kappa+n_2} = 0, \quad T^p_{\kappa+n_2 i+n_2} = 0.$$

From the definition of the Gauduchon's torsion 1-form $\eta$, it yields that

$$\eta = \eta_0 \varphi_0 + \sum_i \eta_i \varphi_i + \sum_{i+1+n_1} \eta_{i+n_1} \varphi_{i+n_1} + \sum_\kappa \eta_\kappa \varphi_\kappa + \sum_{\kappa+n_2} \eta_{\kappa+n_2} \varphi_{\kappa+n_2},$$

where

$$\eta_0 = \sum_p T^p_{0p} = \sum_i (T^i_{i0} + T^i_{i+n_1 0}) + \sum_\kappa (T^\kappa_{\kappa 0} + T^\kappa_{\kappa+n_2 0}) = 0, \quad \eta_0 = 0, \quad \eta_{i+n_1} = 0, \quad \eta_\kappa = 0, \quad \eta_{\kappa+n_2} = 0,$$

and thus $g_\lambda$ is balanced.

Then the matrix $\gamma$ be calculated by the components of Chern torsion as

$$\gamma = \begin{bmatrix} 0 & \bar{\eta}_1 & \bar{\eta}_2 & \bar{\eta}_3 & \bar{\eta}_4 \\ -\mu_1 & H_5 & 0 & 0 & 0 \\ -\mu_2 & 0 & H_6 & 0 & 0 \\ -\mu_3 & 0 & 0 & H_7 & 0 \\ -\mu_4 & 0 & 0 & 0 & H_8 \end{bmatrix},$$

where $\mu_1, \mu_2, \mu_3, \mu_4$ are the entries in the matrix $\theta_1$ and

$$H_5 = \frac{c_1 \sqrt{-1}}{\sqrt{2}} (\varphi_0 + \bar{\varphi}_0) E_{n_1}, \quad H_6 = -\frac{c_1 \sqrt{-1}}{\sqrt{2}} (\varphi_0 + \bar{\varphi}_0) E_{n_1},$$

$$H_7 = \frac{c_2}{\sqrt{2}} \left( 1 - a \sqrt{-1} \right) \frac{1}{b} \varphi_0 - \frac{1}{b} \left( 1 + a \sqrt{-1} \right) \bar{\varphi}_0 E_{n_2},$$

$$H_8 = -\frac{c_2}{\sqrt{2}} \left( 1 - a \sqrt{-1} \right) \frac{1}{b} \varphi_0 - \frac{1}{b} \left( 1 + a \sqrt{-1} \right) \bar{\varphi}_0 E_{n_2}.$$

Note that the Strominger connection matrix $\theta^\ast = \theta_1 + \gamma$ and we have

$$\theta^\ast = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & H_1 + H_5 & 0 & 0 & 0 \\ 0 & 0 & H_2 + H_6 & 0 & 0 \\ 0 & 0 & 0 & H_3 + H_7 & 0 \\ 0 & 0 & 0 & 0 & H_4 + H_8 \end{bmatrix}.$$

By the equalities $d\Phi = -\varphi^\ast \Phi + \tau_s$ and $d\Phi = \bar{\theta_1} \Phi + \theta_2 \bar{\Phi}$, it yields that

$$\tau_s = \imath_\gamma \Phi + \theta_2 \bar{\Phi} = \begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \\ \nu_4 \\ \nu_5 \end{bmatrix},$$

where

$$\nu_1 = \frac{2c_1 \sqrt{-1}}{\sqrt{2}} \left( \imath \varphi_0 - \imath \varphi_0 \right) \varphi_0 + \frac{2c_2(1 + a \sqrt{-1})}{b \sqrt{2}} \left( \imath \varphi_0 - \imath \varphi_0 \right) \varphi_0,$$

$$\nu_2 = \frac{2c_1 \sqrt{-1}}{\sqrt{2}} \left( \varphi_0 + \bar{\varphi}_0 \right) \varphi_0, \quad \nu_3 = -\frac{2c_1 \sqrt{-1}}{\sqrt{2}} \left( \varphi_0 + \bar{\varphi}_0 \right) \varphi_0,$$

$$\nu_4 = \frac{2c_2(1 + a \sqrt{-1})}{b \sqrt{2}} \varphi_0 - \frac{2c_2(1 + a \sqrt{-1})}{b \sqrt{2}} \bar{\varphi}_0, \quad \nu_5 = -\frac{2c_2(1 + a \sqrt{-1})}{b \sqrt{2}} \varphi_0 - \frac{2c_2(1 + a \sqrt{-1})}{b \sqrt{2}} \bar{\varphi}_0 \varphi_0.$$
By the expression of $\theta^*$, it is easy to verify that
\[
\nabla^s \epsilon_0 = 0, \quad \nabla^s \varphi_0 = 0,
\]
\[
\nabla^s t^\varphi e = 0, \quad \nabla^s t^\varphi_n \varphi^+ e_1 = 0, \quad \nabla^s t^\varphi_n \varphi^+ e_2 = 0, \quad \nabla^s t^\varphi_n \varphi^+ n_2 = 0, \quad \nabla^s t^\varphi_n \varphi^+ n_2 = 0.
\]

Therefore,
\[
\nabla^s T^s = \nabla^s (\tau_\psi \otimes F + \eta_\psi \otimes F) = 0,
\]
which establishes that $g_\lambda$ is SPT. It is easy to verify the tensor $B$ under the frame $F$ is
\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & H_9 & 0 & 0 & 0 \\
0 & 0 & H_{10} & 0 & 0 \\
0 & 0 & 0 & H_{11} & 0 \\
0 & 0 & 0 & 0 & H_{12}
\end{bmatrix},
\]
where
\[
H_9 = H_{10} = 2 \left[ \frac{c_1 \sqrt{1-1}}{\sqrt{2}} \right]^2 E_{n_1}, \quad H_{11} = H_{12} = 2 \left[ \frac{c_2 (1-a\sqrt{-1})}{b\sqrt{2}} \right]^2 E_{n_2},
\]
and thus $\text{rank}B = 2(n_1 + n_2) = n - 1$. 

\[\square\]

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