Maxwell’s Multipole Vectors and the CMB

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The recently re-discovered multipole vector approach to understanding the harmonic decomposition of the cosmic microwave background traces its roots to Maxwell’s Treatise on Electricity and Magnetism. Taking Maxwell’s directional derivative approach as a starting point, the present article develops a fast algorithm for computing multipole vectors, with an exposition that is both simpler and better motivated than in the author’s previous work.

Tests show the resulting algorithm, coded up as a Mathematica notebook, to be both fast and robust. This article serves to announce the free availability of the code.

The algorithm is then applied to the much discussed anomalies in the low-ℓ CMB modes, with sobering results. Simulations show the quadrupole-octopole alignment to be unusual at the 98.7% level, corroborating earlier estimates of a 1-in-60 alignment while showing recent reports of 1-in-1000 (using non-unit-length normal vectors) to have unintentionally relied on the near orthogonality of the quadrupole vectors in one particular data set. The alignment of the quadrupole and octopole vectors with the ecliptic plane is confirmed at better than the 2σ level.

PACS numbers: 98.80.-k

I. INTRODUCTION

The Wilkinson Microwave Anisotropy Probe (WMAP) provides an unprecedented view of the Cosmic Microwave Background (CMB) sky. WMAP’s first-year observations [1], while generally matching researchers’ expectations in the high-ℓ portion of the CMB’s spherical harmonic decomposition, yield suspicious results in the low-ℓ portion of the spectrum. Almost immediately after the first-year data release, Tegmark and colleagues noticed that the reported quadrupole (ℓ=2) and octopole (ℓ=3) align [2], and Eriksen et al. found statistical inconsistencies between the northern and southern ecliptic hemispheres [3]. Eriksen and colleagues later found that while the ℓ = 4 component is generic, the ℓ = 5 component is unusually spherical at the 3σ level while the ℓ = 6 component is unusually planar at the 2σ level [4].

To make sense of these anomalies in a systematic way, Copi et al. introduced “a novel representation of cosmic microwave anisotropy maps, where each multipole order ℓ is represented by ℓ unit vectors pointing in directions on the sky and an overall magnitude” [5]. The beauty of this scheme is that the CMB alone determines the multipole vectors, without reference to the coordinate system. This contrasts sharply with the more common aℓm representation of spherical harmonics, which depends strongly on the coordinate system. The coordinate-free, geometrical nature of the multipole vectors makes it trivially easy to test for alignments between modes (for example the quadrupole and octopole align with each other) and with external reference points (the quadrupole and octopole align with the CMB dipole) [6].

Inspired by these results, Katz and Weeks went on to recast the traceless tensor methods of Refs. [6] into the language of homogeneous harmonic polynomials, and within that context to prove the existence and uniqueness of the multipole vectors and to devise an algorithm for computing them [7].

Even though Copi et al. fancied themselves the discoverers of multipole vectors [8] and Katz-Weeks fancied themselves to be breaking new ground in the polynomial approach, the truth is much older. The true discover of multipole vectors was James Clerk Maxwell in his 1873 Treatise on Electricity and Magnetism [9]! Maxwell’s starting point was, of course, electromagnetism. He began with the 1 potential surrounding a point charge (a monopole) and asked what happens when two monopoles of opposite sign get pushed together. Elementary reasoning shows the potential of the resulting dipole to be given (up to rescaling) by the directional derivative ∇ui/ℓ, where ui is the direction along which the two monopoles approach one another. Similarly, pushing together two such dipoles with opposite signs gives (up to rescaling) a quadrupole with potential ∇u1,∇ui/ℓ, where u2 is the direction along which the dipoles approach, and so on. Maxwell recognized these potentials as solutions of Laplace’s equation, i.e. spherical harmonics. Because his expression λ∇ui,⋯∇ui/ℓ contains 2ℓ + 1 degrees of freedom, and because a simple dimension counting argument indicates that a generic harmonic polynomial of degree ℓ also has 2ℓ + 1 degrees of freedom, Maxwell concluded – with less than perfect rigor but nevertheless correctly – that his method yields all spherical harmonics.

Section II A establishes, by purely elementary means, the equivalence between Maxwell’s multipole vectors and the vectors used in more recent work [6, 7]. Section II B sheds fresh light on the fast algorithm for computing multipole vectors, providing a simple, natural motivation for a construction that felt unnatural and ad hoc in Ref. [7]. Section III announces the release of a Mathematica package implementing the fast algorithm, and analyzes its ef-

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ficiency and stability. Finally, Section IV applies the fast algorithm to the first-year WMAP data, finding both the quadrupole-octopole alignment and the alignments with the ecliptic plane to be significant only at the 99% level, not at the higher levels claimed in Refs. [3, 4].

II. THEORY

A. Proof of equivalence

While Maxwell’s directional derivatives provide a welcome breath of fresh air for researchers toiling with more difficult approaches, namely traceless tensors [2] and factored polynomials [5], one must nevertheless prove that Maxwell’s multipole vectors are indeed the same vectors that appear in those other approaches. Mark Dennis has successfully applied Fourier methods [11, Appendix A] to prove the equivalence of the polynomial and Maxwell interpretations. Here we obtain the same result by more elementary means, keeping to the spirit of Maxwell’s original work [3, Chap. IX].

Maxwell expresses a spherical harmonic as

$$f_\ell(x, y, z) = \nabla u_\ell \cdots \nabla u_2 \nabla u_1 \frac{1}{r},$$

(1)

where $$r = \sqrt{x^2 + y^2 + z^2}$$. Observe the simple pattern that emerges as we apply the directional derivatives one at a time,

$$f_0 = \frac{1}{r}$$

$$f_1 = \nabla u_1 f_0 = \frac{(-1)(u_1 \cdot r)}{r^3}$$

$$f_2 = \nabla u_2 f_1 = \frac{(3 \cdot 1)(u_1 \cdot r)(u_2 \cdot r) + r^2(-u_1 \cdot u_2)}{r^5}$$

$$f_3 = \nabla u_3 f_2 = \frac{(-5 \cdot 3 \cdot 1)(u_1 \cdot r)(u_2 \cdot r)(u_3 \cdot r) + r^2(...)}{r^7}$$

(2)

where boldface $$r = (x, y, z)$$ while plain $$r = \sqrt{x^2 + y^2 + z^2}$$ as before. The ellipsis (…) marks a polynomial whose form does not interest us.

Let $$P_\ell$$ denote the polynomial in the numerator of each $$f_\ell$$. The action of $$\nabla u_\ell$$ may be written explicitly as

$$f_\ell = \nabla u_\ell f_{\ell-1} = \nabla u_\ell \left( \frac{P_{\ell-1}}{r^{2\ell-1}} \right)$$

$$= -\frac{(2\ell - 1)(u_\ell \cdot r)P_{\ell-1} + r^2(u_\ell \cdot \nabla P_{\ell-1})}{r^{2\ell+1}}.$$  

(3)

It is now obvious at a glance that formula [6] takes

$$f_{\ell-1} = \frac{(-1)^{\ell-1}(2\ell - 3)!!(u_1 \cdot r) \cdots (u_{\ell-1} \cdot r) + r^2(...)}{r^{2\ell-1}}$$

(4)

to

$$f_\ell = \frac{(-1)^\ell (2\ell - 1)!!(u_1 \cdot r) \cdots (u_\ell \cdot r) + r^2(...)}{r^{2\ell+1}}$$

(5)

thus establishing by induction the validity of the latter for all $$\ell$$.

We would now like to understand more deeply the relationship between the rational function $$f_\ell$$ and the polynomial $$P_\ell$$ appearing in its numerator. Following the method of electrical inversion, which Maxwell [3, art. 162] credits to Thomson (Lord Kelvin) and Tait [12], but Axler [12] traces back to Kelvin’s work two decades earlier [13], define the Kelvin transform $$\tilde{f}$$ of a function $$f$$ to be

$$\tilde{f}(x, y, z) = \frac{1}{r} f(\frac{x}{r^2}, \frac{y}{r^2}, \frac{z}{r^2}).$$

(6)

The Kelvin transform in effect reflects a function across the unit sphere and then adjusts its amplitude to keep it harmonic (to be proved below).

**Lemma 1.** The functions $$f$$ and $$\tilde{f}$$ agree on the unit sphere $$r = 1$$.

**Lemma 2.** The Kelvin transformation $$\tilde{}$$ is an involution, that is,

$$f(x, y, z) \rightarrow \frac{1}{r} f(\frac{x}{r^2}, \frac{y}{r^2}, \frac{z}{r^2})$$

$$\rightarrow \frac{1}{r^3} f(\frac{x}{r^4}, \frac{y}{r^4}, \frac{z}{r^4}) = f(x, y, z).$$

(7)

**Lemma 3.** If $$f$$ is harmonic, then so is $$\tilde{f}$$. Maxwell calls this the theorem of electrical inversion [3, art. 129]. It is easily proved by a direct calculation

$$\nabla^2 \left( \frac{1}{r^2} f(\frac{x}{r^2}, \frac{y}{r^2}, \frac{z}{r^2}) \right) = \frac{1}{r^5}(\nabla^2 f)(\frac{x}{r^2}, \frac{y}{r^2}, \frac{z}{r^2}) = 0.$$  

(8)

The converse, that $$\tilde{f}$$ harmonic implies $$f$$ harmonic, is automatically true by Lemma 2.

Each multipole function $$f_\ell = \frac{P_\ell}{r^{2\ell+1}}$$ has a polynomial numerator $$P_\ell$$ of homogeneous degree $$\ell$$, so its Kelvin transform is easy to compute,

$$\tilde{f}_\ell = \frac{1}{r} \frac{P_\ell}{r^{2\ell+1}} = P_\ell.$$  

(9)

In other words, the Kelvin transformation of $$f_\ell$$ extracts the numerator $$P_\ell$$. Lemma 3 now implies that $$P_\ell$$ is a harmonic polynomial in its own right.

Equation (6) gives $$P_\ell$$ explicitly as

$$P_\ell = \frac{(-1)^\ell (2\ell - 1)!!(u_1 \cdot r) \cdots (u_\ell \cdot r) + r^2(...)}{r^{2\ell+1}}.$$  

(10)

Happily this is exactly the “factored form” whose existence and uniqueness were established in Ref. [2] and...
elsewhere. Thus we have proven

**Proposition 4.** Each harmonic function

\[
\nabla u_1 \cdots \nabla u_2 \nabla u_1 \frac{1}{r}
\]

(11)

defined by Maxwell agrees on the unit sphere with the “factored form”

\[
\lambda(u_1 \cdot r) \cdots (u_\ell \cdot r) + r^2 Q
\]

(12)
of Ref. [7]. In particular, a given function employs Proposition 4.

Inspired by Sylvester’s paper, the following paragraphs bring abundant clear, he leaves his algorithm and the motivation behind it decidedly unclear. Inspired by Sylvester’s paper, the following paragraphs lay out the algorithm in elementary terms, speculating on what Sylvester’s reasoning might have been.

Sylvester’s starting point was Maxwell’s directional derivative formulation of a spherical harmonic, which reduces to a polynomial, for present purposes may be written as

\[
P(x, y, z) = \lambda \prod_{i=1}^{\ell} (u_i \cdot (x, y, z)) + (x^2 + y^2 + z^2) \cdot Q.
\]

(15)

Let us take stock of the computational task that lies before us. We are given the harmonic polynomial \(P(x, y, z)\) of homogeneous degree \(\ell\), and we are asked to compute the linear factors \(u_i(x, y, z)\), but with no prior knowledge of the remainder term \(Q\).

If we could find a common solution to \(P(x, y, z) = 0\) and \(x^2 + y^2 + z^2 = 0\), then such a solution would also satisfy at least one of the linear equations \(u_i(x, y, z) = 0\). Of course \(x^2 + y^2 + z^2 = 0\) has no nontrivial solutions over the real numbers, which motivates us to consider complex solutions instead. Postponing the computational details, assume for the moment that we have found a common solution \((\hat{x}, \hat{y}, \hat{z})\) to \(P(x, y, z) = 0\) and \(x^2 + y^2 + z^2 = 0\), which is therefore a solution to \(u_i(x, y, z) = 0\) for some \(i\). Moreover, because \(u_i\) is real, the real and imaginary parts of \((\hat{x}, \hat{y}, \hat{z})\) individually satisfy the condition

\[
u_i \cdot (\text{Re} \, \hat{x}, \text{Re} \, \hat{y}, \text{Re} \, \hat{z}) = 0
\]

(16)

Furthermore, the real vectors \((\text{Re} \, \hat{x}, \text{Re} \, \hat{y}, \text{Re} \, \hat{z})\) and \((\text{Im} \, \hat{x}, \text{Im} \, \hat{y}, \text{Im} \, \hat{z})\) must be linearly independent, because otherwise \((\hat{x}, \hat{y}, \hat{z})\) could not be a nontrivial solution of \(x^2 + y^2 + z^2 = 0\). Therefore their cross product, when normalized to unit length, gives the desired multipole vector

\[
u_i \sim \text{Re} \, \hat{x}, \text{Re} \, \hat{y}, \text{Re} \, \hat{z} \times \text{Im} \, \hat{x}, \text{Im} \, \hat{y}, \text{Im} \, \hat{z}.
\]

(17)

If we can find enough distinct solutions \((\hat{x}, \hat{y}, \hat{z})\), we will get all the multipole vectors \(\nu_i\).

So how may one find the common solutions \((\hat{x}, \hat{y}, \hat{z})\) to \(P(x, y, z) = 0\) and \(x^2 + y^2 + z^2 = 0\)? Sylvester applied his own resolving equation method, and Mathematica might be employing a similar algorithm in its proprietary \texttt{NSolve} function. Unfortunately \texttt{NSolve} is slow. If we give Mathematica a little help we can speed the computation up enormously. Following Ref. [7], we parameterize the graph of \(x^2 + y^2 + z^2 = 0\), which topologically is a Riemann sphere \(\mathbb{CP}^1 = \mathbb{S}^2\) sitting inside \(\mathbb{CP}^2\), via the parameterization \(\alpha \mapsto (x(\alpha), y(\alpha), z(\alpha)) = (\alpha^2 - 1, 2\alpha, i(\alpha^2 + 1))\), for \(\alpha \in \mathbb{C} \cup \{\infty\}\). Evaluating \(P\) on the image of that parameterization yields a single equation \(P(\alpha^2 - 1, 2\alpha, i(\alpha^2 + 1)) = 0\) of degree \(2\ell\) in the single variable \(\alpha\), which Mathematica solves quickly and accurately for the \(2\ell\) values of \(\alpha\) (counting multiplicities). Applying the parameterization gives \(2\ell\) solutions \((\hat{x}_i, \hat{y}_i, \hat{z}_i) = (\alpha_i^2 - 1, 2\alpha_i, i(\alpha_i^2 + 1))\). Because the original equations are real, whenever a point \((\hat{x}_i, \hat{y}_i, \hat{z}_i)\) satisfies them, the complex conjugate \((\bar{x}_i, \bar{y}_i, \bar{z}_i)\) will satisfy
them as well. Thus the $2\ell$ solutions comprise $\ell$ conjugate pairs $\{\hat{x}_i, \hat{y}_i, \hat{z}_i\}$. Recalling from above that the real and imaginary parts of each solution are linearly independent, no nontrivial solution may equal its complex conjugate (or even a scalar multiple thereof) so the solution pairs are all non-degenerate. Each solution pair yields a single multipole vector $\pm \textbf{u}_i$ via the cross product of its real and imaginary parts (Eqn. (17)).

Must this procedure find all the multipole vectors $\textbf{u}_i$, as opposed to finding some $\textbf{u}_i$ several times each while missing others entirely? Yes. Pick an arbitrary but fixed $\textbf{u}_i$, and let $(\hat{x}, \hat{y}, \hat{z})$ be a common solution of $\textbf{u}_i \cdot (x, y, z) = 0$ and $x^2 + y^2 + z^2 = 0$. More concretely, any scalar multiple of $(\hat{x}, \hat{y}, \hat{z}) = (u_y^2 + u_z^2, -u_x u_y i u_z - u_x u_z i u_y)$ is such a solution, where $(u_x, u_y, u_z)$ are the components of the given $\textbf{u}_i$. A glance at (15) shows that $(\hat{x}, \hat{y}, \hat{z})$ must satisfy $P(x, y, z) = 0$ as well, so our algorithm will find scalar multiples of $(\hat{x}, \hat{y}, \hat{z})$ and $(\hat{x}, \hat{y}, \hat{z})$ among the common solutions to $P(x, y, z) = 0$ and $x^2 + y^2 + z^2 = 0$. Substituting either $(\hat{x}, \hat{y}, \hat{z})$ or $(x, y, z)$ into (17) then gives the desired multipole vector $\textbf{u}_i$, thus proving that our algorithm does indeed find all of them.

In the non-generic case that some of the $\textbf{u}_i$ occur multiple times each in (15), does our algorithm get the multiplicities right? The easy way to see that it does is to slightly perturb the polynomial $P(x, y, z)$ so that the $\ell$ multipole vectors $\textbf{u}_i$ become distinct. For the perturbed polynomial, exactly one pair of roots $\{(\hat{x}, \hat{y}, \hat{z}), (\hat{x}, \hat{y}, \hat{z})\}$ leads to each $\textbf{u}_i$. If we gradually unperturb the polynomial, the $\textbf{u}_i$ fuse back into their original groupings, and we see that each $\textbf{u}_i$ gets mapped to by the right number of root pairs.

### III. IMPLEMENTATION

The algorithm, implemented as a Mathematica notebook, is freely available at [http://www.geometrygames.org/Maxwell/](http://www.geometrygames.org/Maxwell/). The user specifies a spherical harmonic as an array $\{a_{00}, \ldots, a_{\ell\ell}\}$ of coefficients in the usual spherical harmonic decomposition $\sum_{m=-\ell}^{+\ell} a_{\ell m} Y_{\ell m}^m$. Note that the $a_{\ell m}$ are given explicitly only for $m \geq 0$; the values for $m < 0$ the defined implicitly via $a_{\ell,-m} = (-1)^m a_{\ell,m}^*$. As an example, the $a_{2,0}$ for the quadrupole component of Eriksen et al.’s Lagrange Internal Linear Combination (LILC) cleansing of the first-year WMAP data $\ell^2$ are

$$a_{2,0}^{\text{LILC}} = \{16.30, -4.257 + 4.98 i, -18.80 - 20.02 i\}. \quad (18)$$

Our Mathematica function $\text{AlmToPolynomial}$ converts the $a_{\ell m}^{\text{LILC}}$ to a polynomial

$$P(x, y, z) = -19.67 x^2 + 9.39 y^2 + 10.28 z^2 + 30.94 xy + 7.71 yz + 3.98 zx. \quad (19)$$

The main algorithm, described in Section II B and implemented in the Mathematica function $\text{PolynomialToMultipoleVectors}$, yields the Maxwell multipole vectors

$$u_1 = \{0.975, 0.043, 0.220\}$$

$$u_2 = \{0.634, -0.737, -0.234\} \quad (20)$$

As an error check, one may feed the multipole vectors (20) to the function $\text{MultipoleVectorsToPolynomial}$ to recover the harmonic polynomial (19).

### A. Speed

The algorithm of Section II B as implemented in the function $\text{PolynomialToMultipoleVectors}$, runs exceedingly fast. On a 2.4 GHz personal computer, the algorithm computes the multipole vectors $\textbf{u}_i$ for randomly generated $a_{\ell m}$ almost instantaneously for very small $\ell$, and within a fraction of a second for relevant larger $\ell$ (Table II middle column).

The algorithm’s bottleneck lies in the built-in Mathematica function $\text{Expand}$, with which $\text{PolynomialToMultipoleVectors}$ reduces the polynomial in $\alpha$ (discussed in Section II B) to its canonical form $c_0 + c_1 \alpha + c_2 \alpha^2 + \cdots + c_{2\ell} \alpha^{2\ell}$ before solving for the roots. In particular, simplifying the polynomial to standard form is far slower than extracting the roots. When using Monte Carlo methods to compare the observed WMAP data to large numbers of simulated CMB skies, one may speed up the multipole computation by pre-computing the spherical harmonics, making the substitutions $\{x \rightarrow \alpha^2 - 1, y \rightarrow 2\alpha, z \rightarrow i(\alpha^2 + 1)\}$ directly into the spherical harmonics, simplifying the polynomials to the canonical form $b_0 + b_1 \alpha + b_2 \alpha^2 + \cdots + b_{2\ell} \alpha^{2\ell}$, and caching the result. With pre-cached spherical harmonics, the runtime drops to a very manageable $O(\ell^2)$ (Table II rightmost column).

| $\ell$ | run time without caching | run time with caching |
|-------|-------------------------|-----------------------|
| 8     | 0.03 s                  | 0.02 s                |
| 16    | 0.2 s                   | 0.03 s                |
| 32    | 0.6 s                   | 0.1 s                 |
| 64    | 3 s                     | 0.4 s                 |
| 128   | 20 s                    | 2 s                   |

**TABLE I:** The implementation of the $\text{PolynomialToMultipoleVectors}$ algorithm runs quickly for low values of $\ell$. As $\ell$ increases, the call to the built-in Mathematica function $\text{Expand}$ for simplifying a polynomial becomes a bottleneck (middle column). For Monte Carlo simulations, which require many computations for the same $\ell$, pre-simplifying the spherical harmonics and caching them greatly improves performance (righthand column).
The multipole algorithm is quite stable. To test it,

1. begin with a random set of multipole vectors \( \{u_1, \ldots, u_\ell\} \),
2. convert the multipole vectors \( \{u_1, \ldots, u_\ell\} \) to a homogeneous polynomial \( P \), and then
3. use \( P \) to recompute the multipole vectors \( \{u'_1, \ldots, u'_\ell\} \).

Table II shows that the re-computed multipole vectors \( \{u'_1, \ldots, u'_\ell\} \) agree with the original multipole vectors \( \{u_1, \ldots, u_\ell\} \) to high precision, even for fairly large values of \( \ell \).

### IV. APPLICATIONS

#### A. Quadrupole-octopole alignment

Schwarz, Starkman, Huterer and Copi offer two variations on their approach to measuring the quadrupole-octopole alignment \( \theta \). Both variations begin with the two multipole vectors \( \{u_{2,1}, u_{2,2}\} \) for the quadrupole and the three multipole vectors \( \{u_{3,1}, u_{3,2}, u_{3,3}\} \) for the octopole.

According to the unnormalized version, a cross product \( w_2 = u_{2,1} \times u_{2,2} \) defines a normal vector to the quadrupole plane. Similarly

\[
\begin{align*}
  w_{3,1} &= u_{3,2} \times u_{3,3} \\
  w_{3,2} &= u_{3,3} \times u_{3,1} \\
  w_{3,3} &= u_{3,1} \times u_{3,2}
\end{align*}
\]

define normal vectors to the three octopole planes. The three dot products \( A_i = |w_2 \cdot w_{3,i}| \), for \( i = 1, 2, 3 \), then measure the alignment of the quadrupole plane with each of the three octopole planes. Katz and Weeks take the sum \( S = A_1 + A_2 + A_3 \), which they find to be unusually high at the 99.9% level when compared to \( 10^5 \) Monte Carlo simulations.

The normalized version is identical to the unnormalized one, except that it replaces the previous cross products with unit length normal vectors \( w'_2 = |w_2| = |u_{2,1} \times u_{2,2}| \) and \( w'_{3,i} = |w_{3,i}| \).

So which version should we use? If our goal is to measure how well the quadrupole plane aligns with each octopole plane, then the normalized version works perfectly. The unnormalized version, on the other hand, resists easy interpretation. By retaining the length of each cross product \( u_{i,j} \times u_{i,j'} \), it puts more weight on “well defined planes” (for which \( u_{i,j} \) and \( u_{i,j'} \) are nearly orthogonal and which therefore robustly determine a normal direction) and less weight on “poorly defined planes” (for which \( u_{i,j} \) and \( u_{i,j'} \) are nearly parallel or anti-parallel and which therefore only weakly determine a normal direction). Nevertheless it’s not clear what overall significance the resulting statistic carries, nor whether this version is a good choice for the quadrupole-octopole comparison. For the moment let us remain neutral and compute statistics relative to both versions.

The multipole algorithm of Section II.B as implemented in the Mathematica code of Section III generated a million Monte Carlo simulations, whose quadrupole-octopole alignments were then compared to (a) the DQ-corrected Tegmark (DQT) cleaning of the first-year WMAP data \( \theta \) and (b) the Lagrange Internal Linear Combination (LILC) cleaning of the first-year WMAP data \( \theta \). Table IV shows the results.
TABLE IV: How unusual is the quadrupole-octopole alignment observed in the first-year WMAP data? For each data set (DQT, LILC) and each version of the algorithm (normalizing the lengths of normal vectors or not), the table reports what fraction of a million Monte Carlo skies had weaker quadrupole-octopole correlations.

|        | unnormalized version | normalized version |
|--------|----------------------|--------------------|
| DQT    | 99.9%                | 98.7%              |
| LILC   | 97.3%                | 98.7%              |

TABLE V: Comparing the quadrupole ($\ell = 2$) to the hexadecapole ($\ell = 4$) finds the normalized statistic to be more stable than the unnormalized statistic, just as in the quadrupole-octopole comparison of Table IV.

|        | unnormalized version | normalized version |
|--------|----------------------|--------------------|
| DQT    | 34.6%                | 19.5%              |
| LILC   | 17.9%                | 21.0%              |

Thus the near orthogonality of the quadrupole vectors $\{u_{2,1}, u_{2,2}\}$ in the DQT data fully explains the difference between the two entries in the middle column of Table IV. The relative quality of the quadrupole-octopole alignments plays no role there.

One concludes that the 99.9% value in Table IV measures the combined unlikelihood of the quadrupole-octopole alignment and the near orthogonality of the DQT quadrupole vectors. To measure the quadrupole-octopole alignment alone, one should use the normalized statistic (Table V, right-hand column), which for the first-year WMAP data finds the alignment to be unusual at the 98.7% level. While this contradicts one recent claim [6, 7], it agrees well with an earlier estimate (using other methods) that the quadrupole-octopole alignment is unusual at the 1-in-60 level [13].

B. Ecliptic plane alignment

Schwarz et al. point out the alignment of the quadrupole and octopole vectors with the ecliptic plane. Here we test the statistical significance of that alignment and obtain results inconsistent with the 99.8% confidence level claimed in the preprint version of Ref. 6 but consistent with the revised ~99% confidence level appearing in the final published version of Ref. 6 if one considers only a one-tailed distribution.

First consider the normal to the quadrupole plane, which in galactic coordinates is (−106°, 57°) for the DQT data or (−112°, 62°) for the LILC data. Taking the dot product with the ecliptic pole ±(96.4°, 29.8°) gives 0.027 or 0.080, nominally implying a confidence level of 97.3% or 92.0%, depending on the data set (Endnote 13). However, assuming we would have been equally pleased with a dot product near 1 (meaning the quadrupole plane aligned with the galactic plane), a two-tailed distribution is appropriate, dropping the confidence level to 94.6% or 84.0%.

Applying the same method to the three octopole planes gives dot products of 0.523, 0.045 and 0.179 (for the DQT data, in agreement with Ref. 6) or 0.555, 0.030 and 0.146 (for the LILC data). At this point the Preprint 6 computes a raw score which cannot be interpreted as a confidence level; to avoid that trap we take a different approach and examine the sum of the three dot products, namely 0.747 (DQT) or 0.731 (LILC). Monte Carlo simulation of $10^5$ Gaussian random skies finds the sum to be larger than that 95.7% of the time (DQT) or 96.1% of the time (LILC).

Finally, for more direct comparison with the results of Ref. 6, consider the joint sum of the single quadrupole dot product along with the three octopole dot products. Monte Carlo simulation of $10^5$ Gaussian random skies finds this combined sum to be larger than the observed sum 99.0% of the time (DQT) or 98.9% of the time (LILC). This 99% result is weaker than the erroneous 99.8% claim of Preprint 6 but consistent with...
TABLE VI: For each data set (DQT, LILC) and each value of \( \ell \) (3–10), the table shows what percentage of 10^5 Monte Carlo simulations yielded a more planar set of multipole vectors.

| \( \ell \) | 3 4 5 6 7 8 9 10 |
|---|---|---|---|---|---|---|---|---|
| DQT | 28% 80% 89% 24% 25% 58% 82% 69% |
| LILC | 30% 58% 97% 33% 12% 72% 48% 34% |

V. CONCLUSIONS

Maxwell’s multipole vector construction sheds much light on spherical harmonics and offers a clean and simple approach to quantifying alignments among the low-\( \ell \) CMB modes. In the case of the quadrupole-octopole alignment, simulations convincingly show the alignment to be unusual at the 98.7% level, corroborating earlier estimates of a 1-in-60 alignment \([15]\) while showing recent reports of 1-in-1000 \([6, 7]\) to have depended on the near orthogonality of the quadrupole vectors in the DQT data. The alignment of the quadrupole and octopole vectors with the ecliptic plane is confirmed at better than the 2\( \sigma \) level.

Acknowledgments

I thank Mark Dennis whose paper \([10]\) brought Maxwell’s work to my attention and whose subsequent preprint \([16]\) offers additional insights, and I thank Ben Lotto and John McCleary for helpful conversations.

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[17] To prove that no nonzero multiple of \( r^2 = x^2 + y^2 + z^2 \) is harmonic, consider a polynomial \((r^2)^n S\), where \( n \geq 1 \) and \( S \) contains no further factors of \( r^2 \). If \((r^2)^n S\) were harmonic, then computing \( \nabla^2 ((r^2)^n S) \) and setting it equal to zero would lead to \( r^2 \nabla^2 S + (4n \deg(S) + 2n(2n+1)) S = 0 \), which would imply that \( S \) contained another factor of \( r^2 \), contrary to assumption. Therefore \((r^2)^n S\) cannot be harmonic.

[18] Using the dot product to infer a confidence level is surprisingly straightforward. Starting from Archimedes’ observation that axial projection of a cylinder onto an inscribed sphere preserves areas, it follows, for example, that precisely 5% of the vectors on the unit sphere yield a magnitude of 0.95 or greater when dotted with a fixed unit vector.