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Stabilization of photon-number states via single-photon corrections: a first convergence analysis under an ideal set-up

H. B. Silveira    P. S. Pereira da Silva    P. Rouchon

Abstract—This paper presents a first mathematical convergence analysis of a Fock states feedback stabilization scheme via single-photon corrections. This measurement-based feedback has been developed and experimentally tested in 2012 by the cavity quantum electrodynamics group of Serge Haroche and Jean-Michel Raimond. Here, we consider the infinite-dimensional Markov model corresponding to the ideal set-up where detection errors and feedback delays have been disregarded. In this ideal context, we show that any goal Fock state can be stabilized by a Lyapunov-based feedback for any initial quantum state belonging to the dense subset of finite rank density operators with support in a finite photon-number sub-space. Closed-loop simulations illustrate the performance of the feedback law.

I. INTRODUCTION

In [8], a photon-number states (Fock state) feedback stabilization scheme via single-photon corrections was described and experimentally tested. Such control problem is relevant for quantum information applications [6], [4]. The quantum state \( \rho \) corresponds to the density operator of a microwave field stored inside a superconducting cavity and described as a quantum harmonic oscillator. At each sample step \( k \in \mathbb{N} \), a probe atom is launched inside the cavity. The measurement outcome \( y_k \) detected by a sensor is the energy-state of this probe atom after its interaction with the microwave field. Each probe atom is considered as a two-level system: either it is detected in the lowest energy state \( |g⟩ \), or the highest energy state \( |e⟩ \). Consequently, the measurement outcomes correspond to a discrete-valued output \( y_k \) with only two distinct possibilities: \( g \) or \( e \). Similarly, the control inputs \( u_k \) are also discrete-valued with 3 distinct possibilities: \( -1, 0, +1 \). The open-loop value \( u_k = 0 \) corresponds to a dispersive atom/field interaction: it achieves in fact a Quantum Non-Demolition measurement of Fock states [2]. The two other values \( u_k = \pm 1 \) correspond to resonant atom/field interactions where the probe atom and the field exchange energy quanta: these values achieve single-photon corrections.

Although the feedback law proposed and implemented in [8] considered imperfect detections on \( y_k \) and delays in the control, here we focus on an ideal-set up, that is, detection errors and control delays have been disregarded. Theorem 2 shows that, by adding an arbitrarily small term to the Lyapunov function used in [8], one ensures almost sure global stabilization of any goal Fock state for the closed-loop ideal set-up. This is achieved by relying on an infinite-dimensional Markov model of the ideal set-up that takes into account the back-action of the measurement outcome \( y_k \) on the quantum state \( \rho_{k+1} \).

Loosely speaking, in [8], the control value \( u_k \) at each sampling step \( k \) was chosen so as to minimize the conditional expectation of the Lyapunov function \( V(\rho_k) = \text{Tr}(d(N)|\rho_k) \), where \( N \) is the photon-number operator, \( d(n) = (n - \bar{n})^2 \) and \( \bar{n} = \langle \bar{n} \rangle \) is the goal Fock state. However, in closed-loop, the difference between such \( V \) and its conditional expectation is not strictly positive: such \( V \) does not become a strict Lyapunov function in closed-loop and additional arguments have to be considered to prove convergence. These additional arguments are related to Lasalle invariance. They are well established in a smooth context where the control \( u \) is a smooth function of the state \( \rho \). This cannot be the case here since \( u \) is a discrete-valued control. In order to overcome such technical difficulties, we propose, similarly to [1], to add the arbitrarily small term \( -\epsilon \sum_{n=0}^{\infty} \langle n | \rho_k | n \rangle^2 \) to \( V(\rho_k) \), where \( \epsilon > 0 \). This slightly modified control-Lyapunov function becomes then a strict-Lyapunov function in closed-loop that simplifies notably the convergence analysis. Moreover, the developed convergence analysis is done in the infinite-dimensional setting in the following sense: we show that, for any initial density operator \( \rho_0 \) with a finite photon-number support \( \rho_0 |n⟩ = 0 \) for \( n \) large enough), the closed-loop trajectory \( k \mapsto \rho_k \) remains also with a finite photon-number support with a uniform bound on the maximum photon-number. This almost finite-dimensional behavior simplifies the convergence analysis despite the fact that such condition on \( \rho_0 \) is met on a dense subset of density operators (Hilbert-Schmidt topology on the Banach space of Hilbert-Schmidt self-adjoint operators).

The paper is organized as follows. Section II presents the ideal Markov model of the experimental set-up of the controlled microwave super-conducting cavity reported in [8] and precisely formulates the Fock state stabilization problem here treated (see Definition 1). Section III establishes the proposed solution to the control problem in two distinct parts. Firstly, Section III-A considers the case where the initial condition \( \rho_0 \) is a diagonal density operator (see Theorem 1).
Only the main ideas of the convergence proof are outlined. The technical details are given in Section [III]-[V]. Afterwards, in Section [III-B] the main result of the paper is presented: the general solution is obtained from Theorem [I] for \(\rho_0\) belonging to a dense subset (see Theorem [II]). The simulation results are exhibited in Section [IV]. The proof of some intermediate results and computations required in Sections [III] and [V] are presented in Appendices [B]-[G]. Finally, the concluding remarks are given in Section [VI].

II. IDEAL MARKOV MODEL

Denote by \(\mathcal{H}\) the separable complex Hilbert space \(L_2(\mathbb{C})\) with orthonormal basis \(\{ |n\rangle, n \in \mathbb{N} \}\) of Fock states (phonon-number). Hence, \(\mathcal{H} = \{ \sum_{n \in \mathbb{N}} \psi_n |n\rangle, (\psi_0, \psi_1, \ldots) \in l_2(\mathbb{C}) \}\). Let \(D\) be the set of all density operators on \(\mathcal{H}\), that is, the set of trace-class, self-adjoint, non-negative operators. Consider the set \(D = \{ \rho \in \mathbb{C}^{N \times N} : \rho = \rho^\dagger, \rho \geq 0, \sum_{n \in \mathbb{N}} |\rho_{nn}| = 1 \}\), where the measurements outcomes \(y_k = g\) and \(y_k = e\) occur with probabilities \(p_{g,k} = \text{Tr} (M_g(u_k) \rho_k M_g^\dagger (u_k))\) and \(p_{e,k} = \text{Tr} (M_e(u_k) \rho_k M_e^\dagger (u_k)) = 1 - p_{g,k}\), respectively. The control problem here treated is given as follows:

Definition 1: For the ideal Markov process \((1)-(4)\), the control problem is to find a feedback law \(u_k = f(p_k)\) such that, given an initial condition \(\rho_0\) and \(\pi \in D\), the closed-loop trajectory \(\rho_k\) converges almost surely towards the goal Fock state \(\pi = |\pi\rangle \langle \pi|\) as \(k \to \infty\).

The almost sure convergence above is with respect to the probabilities amplitudes \(P_n(\rho) = \text{Tr} (|n\rangle \langle n| \rho) = \langle n| \rho |n\rangle\) of \(\rho\), that is, \(\lim_{k \to \infty} P_n(\rho_k) = P_n(\pi)\) for each \(n \in \mathbb{N}\). In other words, \(\lim_{k \to \infty} P_n(\rho_k) = 0\) when \(n \neq \pi\). The solution proposed in this paper for the control problem above is developed in the next section.

III. STABILIZATION OF FOCK STATES

Given any operator \(A: \mathcal{H} \to \mathcal{H}\), let \(A_{mn} = \langle m|A|n\rangle\) for \(m, n \in \mathbb{N}\). Hence, \(A_{mn}\) is the \(n\)-th diagonal element of \(A\), while \(A_{mn}\) with \(m \neq n\) correspond to its “off-diagonal” elements. One says that the operator \(A\) is diagonal when \(A_{mn} = 0\) for all \(m, n \in \mathbb{N}\) with \(m \neq n\). One shall begin by solving the control problem given in Definition [I] in the particular case where the initial condition \(\rho_0\) is diagonal (see Theorem [I] in Section [III-A]). Afterwards, in Section [III-B] the solution to the general non-commutative case is presented (see Theorem [II]): its solution relies essentially on the diagonal case.

A. Diagonal case

For each \(n^* \in D\), define

\[ D_{n^*} = \{ \rho \in D \mid \rho \text{ is diagonal and } \rho |n\rangle = 0, \forall n > n^* \}. \]

Consider the set \(D_{n^*} = \bigcup_{n^* \in D_{n^*}} D_{n^*} \subset D\). Note that \(D_{n^*} \subset D_{n^*+1}\), and that each element \(\rho\) of \(D_n\) is “finite dimensional” in the following sense: \(\rho \in D\) if and only if \(\rho = \sum_{n=0}^{\infty} \rho_{nn} |n\rangle \langle n|\), and \(\rho \in D_{n^*}\) may be considered as an operator from \(\mathcal{H}\) to the finite-dimensional space \(\mathcal{H}_{n^*} = \text{span}\{ |0\rangle, \ldots, |n^*\rangle \}\), or as a density matrix on \(\mathcal{H}_{n^*}\). One defines the functions \(n_{\min}: D_{n^*} \to \mathbb{N}, n_{\max}: D_{n^*} \to \mathbb{N}\) and \(n_{\text{length}}: D_{n^*} \to \mathbb{N}\) respectively by:

- \(n_{\min}(\rho)\) is the smallest \(n \in \mathbb{N}\) such that \(\rho |n\rangle \neq 0\);
- \(n_{\max}(\rho)\) is the greatest \(n \in \mathbb{N}\) such that \(\rho |n\rangle \neq 0\);
- \(n_{\text{length}}(\rho) = n_{\max}(\rho) - n_{\min}(\rho)\).

It is clear that, given \(\rho \in D_{n^*}\), one has \(\rho \in D_{n^*}\) if and only if \(n_{\max}(\rho) \leq n^*\). The next result exhibits the properties of the state \(\rho_{k}\) of \((1)-(4)\) with respect to these functions.

1As usual in quantum physics, it is here assumed that the measurement outcome \(y_k = y\) cannot occur when \(\text{Tr} (M_y(u_k) \rho_k M_y^\dagger (u_k)) = 0\), for \(y = g, e\).

2For instance, \(M_g(1)|n\rangle = \left( \sin\left(\frac{\theta_0}{2}\sqrt{N}\right)/\sqrt{N} \right) \sqrt{n+1}|n+1\rangle = \sin\left(\frac{\theta_0}{2}\sqrt{N}\right)/\sqrt{N} \sqrt{n+1}|n+1\rangle\). In order for the definition of \(M_e(-1)\) to be consistent, it is assumed \(\sin(0)/0 = 1\).
Proposition 1: For every realization of the ideal Markov process \([1-3]\) with initial condition \(\rho_0 \in D_*\), one has that \(\rho_k \in D_*\) for all \(k \in \mathbb{N}\):

- If \(u_k = 0\) or \(u_k = -1\), then \(n_{\max}(\rho_{k+1}) \leq n_{\max}(\rho_k)\) and \(n_{\text{length}}(\rho_{k+1}) \leq n_{\text{length}}(\rho_k)\);
- If \(u_k = +1\), then \(n_{\max}(\rho_{k+1}) \leq n_{\max}(\rho_k) + 1\) and \(n_{\text{length}}(\rho_{k+1}) \leq n_{\text{length}}(\rho_k)\).

Proof: See Appendix B

Take a goal photon-number \(\pi \in \mathbb{N}\). As in [1], consider the following Lyapunov function \(V_\epsilon: D_* \to \mathbb{R}\) defined as

\[
V_\epsilon(\rho) = \text{Tr}(d(N)\rho) - \epsilon \sum_{n \in \mathbb{N}} \rho_{nn}, \quad \text{for } \rho \in D_*,
\]

where \(\epsilon > 0\) is a real number and \(d(n) = (n - \pi)^2\) as defined in [8]. The feedback law \(u: D_* \to \{-1, 0, 1\}\) is given by

\[
u = f(\rho) \triangleq \text{Argmin}_{u \in \{-1, 0, 1\}} E[V_\epsilon(\rho_{k+1}) | \rho_k = \rho, u_k = u].
\]

Note that for each \(\rho \in D_*\) and \(n^* \geq n_{\max}(\rho)\), \(d(N)\rho\) is a well-defined self-adjoint, non-square, trace-class operator on \(H\), by considering \(d(N)\) as an operator on \(H_n\), and \(\rho\) as an operator from \(H\) to \(H_n\). Indeed, \(d(N)\rho = \sum_{n=0}^{n^*} \rho_{nn} (n - \pi)^2 |n\rangle\langle n|\). Thus, \(\{\rho\}\) is well-defined. Moreover, since \(H_n\) is invariant under \(D_*\) for \(n^* \geq n_{\max}(\rho)\), it is clear that \(\text{Tr}(d(N)\rho) = \text{Tr}_{H_{n^*}}(d(N)\rho)\), where the right-hand side one considers \(\rho\) as an operator on the finite-dimensional space \(H_{n^*}\) and the trace is taken over \(H_{n^*}\).

We have the following convergence result when \(\rho_0 \in D_*\):

Theorem 1: Let \(\pi \in \mathbb{N}\) and \(\epsilon > 0\). In [2-4], assume that \(\phi_0/\pi\) and \((\theta_0/\pi)^2\) are irrational numbers, and take \(\phi_R = \pi/2 - \pi \phi_0\). Consider the closed-loop Markov process \([1-3]\) with \(u_k = f(\rho_k)\), where the feedback law \(f\) is as in (6).

Then, given any initial condition \(\rho_0 \in D_*\), one has that \(\rho_k\) converges almost surely towards \(\overline{\rho} = [\|\pi\|/|\pi|]k\) as \(k \to \infty\).

Its proof is decomposed into two steps:

First Step. Choose \(\pi \in \mathbb{N}\) and \(\epsilon > 0\). Let \(n_0 = n_{\text{length}}(\rho_0)\), \(r_0 = n_{\text{min}}(\rho_0)\). Then, there exists an integer \(n_0 > n_0 + r_0 + \pi - 1\) (depending on \(n_0, r_0, \pi\) and \(\epsilon\)) such that, for all closed-loop realizations \(\rho_k\), one has \(\rho_k \in D_{n_0}\) for \(k \in \mathbb{N}\).

Second Step. Choose irrational numbers \(\phi_0/\pi\) and \((\theta_0/\pi)^2\) in [2-4], and take \(\phi_R = \pi/2 - \pi \phi_0\). In \(D_{n_0}\), \(V_\epsilon\) is a strict super-martingale: for all density operators \(\rho \in D_{n_0}\), one has

\[
E[V_\epsilon(\rho_{k+1}) | \rho_k = \rho, u_k = f(\rho)] = V_\epsilon(\rho) - Q_{V_\epsilon}(\rho, f(\rho)),
\]

where \(Q_{V_\epsilon}(\rho, f(\rho)) \geq 0\), and \(Q_{V_\epsilon}(\rho, f(\rho)) = 0\) if and only if \(\rho = \overline{\rho}\). The almost sure convergence follows then from usual results on strict super-martingales for Markov processes with compact state spaces.

The complete proof of the two steps above is presented in Section V. The general case where the initial condition \(\rho_0\) is not necessarily diagonal is treated in the next subsection.

B. General case

Consider, for each \(n^* \in \mathbb{N}\),

\[
D_{n^*} = \{\rho \in D_* | \rho |n\rangle = 0, \forall n > n^*\} \subset D_{n^*+1},
\]

and let \(D^* = \bigcup_{n^* \in \mathbb{N}} D_{n^*} \supset D_*\). It is clear that \(\rho \in D^*\) if and only if \(\rho = \sum_{m,n=0}^{m^*} \rho_{mn} |m\rangle\langle n|\). Consequently, \(D^*\) is a dense subset of \(D\) when \(D\) is endowed with the subspace topology induced from the Hilbert-Schmidt norm. Indeed, let \(J_2\) be the complex Banach space of all Hilbert-Schmidt operators on \(H\) with the Hilbert-Schmidt norm \(\|B\|_2 = (\sum_{m,n=0}^{m^*} |B_{mn}|^2)^{1/2}\), for \(B \in J_2\) [7], [3]. Since \(D \subset J_2\) and \(\rho \in D_{n*}\) has the form \(\rho = \sum_{m,n=0}^{m^*} \rho_{mn} |m\rangle\langle n|\), the density property of \(D^*\) in \(D\) is clear.

One has that \(\rho \in D_{n^*}\) may be considered as an operator from \(H\) to the finite-dimensional space \(H_{n^*}\), or as a density matrix on \(H_{n^*}\). Hence, \(d(N)\rho\) is a well-defined trace-class operator on \(H\), by considering \(d(N)\) as an operator on \(H_{n^*}\) and \(\rho \in D_{n^*}\) as an operator from \(H\) to \(H_{n^*}\). Indeed, \(d(N)\rho = \sum_{n=0}^{n^*} \rho_{nn} (n - \pi)^2 |m\rangle\langle n|\), and it is trace-class because its range is finite-dimensional [7], [3]. Consequently, the Lyapunov function \(V_\epsilon\) in (5), the feedback in (6) and \(n_{\max}\) can be extended to \(D^*\).

Define the map \(\Delta: D_* \to D_* \subset D_*\) as \(\Delta \rho = \sum_{n=0}^{n^*} \rho_{nn} |m\rangle\langle m|\). Note that \(\Delta\) extracts the diagonal of \(\rho \in D_*\). It is easy to see that \(n_{\max}(\Delta \rho) = n_{\max}(\rho)\) and \(\|\Delta \rho\|_2 = \|\rho\|_2\). Moreover, \(\Delta \rho = \rho\) when \(\rho \in D_*\). Other properties of the map \(\Delta\) are given in the next result:

Proposition 2: Let \(\rho \in D^*\), \(u \in \{-1, 0, 1\}\), \(y = y_\rho\). Take \(\alpha = \text{Tr}(M_\rho(u_\rho)M_\rho^\dagger(u_\rho))\). Then:

- \(\text{Tr}(\Delta \rho) = \text{Tr}(\Delta \rho(y_\Delta \rho))\), for every diagonal bounded operator \(A: H \to H\);
- \(\text{Tr}(\Delta \rho) = \text{Tr}(\Delta \rho(y_\Delta \rho))\), for \(\epsilon > 0\);
- \(\alpha^{-1} M_\rho(y_\rho)M_\rho^\dagger(u)\) belongs to \(D^*\) with \(\Delta(\alpha^{-1} M_\rho(y_\rho)M_\rho^\dagger(u)) = \alpha^{-1} M_\rho(y_\rho)(\Delta \rho)(\Delta \rho)^\dagger(u)\);
- \(M_\rho(y_\rho)(\Delta \rho)(\Delta \rho)^\dagger(u)\) belongs to \(D^*\) with \(\Delta(M_\rho(y_\rho)(\Delta \rho)(\Delta \rho)^\dagger(u)) = \Delta(M_\rho(y_\rho)(\Delta \rho)(\Delta \rho)^\dagger(u))\), for all \(n \in \mathbb{N}\). In particular, \(\alpha = \text{Tr}(M_\rho(u_\rho)M_\rho^\dagger(u_\rho))\).
IV. SIMULATION RESULTS

This section presents the closed-loop simulation results concerning the application of Theorem 2 above to the ideal Markov process (1)–(4). The quantum experimental results exhibited in [8] used the following control parameter values in (2)–(4): \( \phi_0/\pi = 0.252 \) and \( \theta_0/\pi \approx 2/\sqrt{3} + 1 \). However, according to the assumptions in Theorem 2, \( \phi_0/\pi \) and \( (\theta_0/\pi)^2 \) should be irrational numbers. Hence, here one chooses \( \phi_0/3.14 \approx 0.252 \) and \( \theta_0/3.14 \approx 2/\sqrt{3} + 1 \). One takes \( \rho_0 = \sum_{n=0}^{15} |n\rangle\langle n|/16 \in \mathbb{D} \), as the initial condition, \( \pi = 10 \) for the goal Fock state \( \rho = |\pi\rangle\langle \pi| \), and \( \epsilon = 10^{10} \) as the gain for the feedback \( u_k = f(\rho_k) \) in (5)–(6). Figure 1 exhibits the simulation results for one closed-loop realization with such choices and a final sample step of 120. It shows: the dynamics of the populations of \( \rho_k \) (top), the controls \( u_k \) (middle) and the simulated outcomes \( y_k \) (bottom). The populations of \( \rho_k \) correspond to the following observables: \( A_1 = \sum_{n=1}^{\pi} |n\rangle\langle n| (n < \pi) \), \( A_2 = |\pi\rangle\langle \pi| (n = \pi) \), \( A_3 = \sum_{n=\pi+1}^{\infty} |n\rangle\langle n| (n > \pi) \). Therefore, one sees from the dynamics of the populations that \( \rho_k \) converges to \( \rho \) as \( k \to \infty \), which is in accordance with Theorem 2. Let \( \rho, \mu, \sigma \) be irrational numbers. Hence, here one has that \( |\rho\rangle|\pi\rangle|\pi| \approx 1 \) and \( u_k = 0 \) for all \( k > 45 \).

Recall that Theorem 2 assumes that \( \epsilon > 0 \). In order to further analyze the performance of the Lyapunov-based feedback law here proposed, we now make a comparison with the one used experimentally in [8], which corresponds to take \( \epsilon = 0 \) in (5), i.e. to disregard the term \( -\epsilon \sum_{n\in\mathbb{N}} \rho_n^2 \). Figure 2 presents the simulation results for one closed-loop realization of such case. The control parameters, \( \rho_0 \) and \( \pi = 10 \) are the same as above. Note that \( |\rho\rangle|\pi\rangle|\pi| \approx 1 \) and \( u_k = 0 \) for all \( k > 78 \). In order to make a comparison in terms of the speed of convergence, define the settling time \( k_s \) to be the smallest \( k \in \mathbb{N} \) such that \( |\rho\rangle|\pi\rangle|\pi| > 0.9 \) for all \( k \geq k_s \). One has \( k_s = 45 \) for the case \( \epsilon = 10^{10} \) above, and \( k_s = 78 \) for \( \epsilon = 0 \). Therefore, in the two realizations here simulated, the choice of \( \epsilon = 10^{10} \) reduced the settling time \( k_s \) by nearly \( 42\% \) with respect to \( \epsilon = 0 \). This behavior is typical on an average basis, thereby justifying the term \( -\epsilon \sum_{n\in\mathbb{N}} \rho_n^2 \) in (5).

Table 1 shows the average value \( k_s \) and the standard deviation \( \sigma \) of \( k_s \) for \( \epsilon \in \{0, 0.1, 1, 10, 10^2, 10^3, 10^4, 10^5\} \), of a total of 5000 realizations were simulated for each \( \epsilon \). Notice that when \( \epsilon \) is relatively large or relatively small in comparison to \( \epsilon = 10^3 \), the average settling time \( k_s \) deteriorated. Furthermore, although for \( \epsilon = 10^5 \) one has that \( k_s \) increased by nearly \( 22\% \) in comparison to \( \epsilon = 10^3 \), the standard deviation \( \sigma \) decreased by nearly \( 62\% \). Computer simulations have suggested that a choice of \( \epsilon > 0 \) which may perhaps significantly improve \( k_s \) generally depends on the initial condition \( \rho_0 \) and on the goal Fock state \( \rho = |\pi\rangle\langle \pi| \), and it has to be determined heuristically.

V. PROOF OF THEOREM 1 (DIAGONAL CASE)

Proof of the First Step:

Let \( \epsilon > 0 \). Define \( V : D_s \to \mathbb{R} \) and \( W : D_s \to \mathbb{R} \) as

\[
V(\rho) = \text{Tr}(d(N)\rho), \quad W(\rho) = -\sum_{n \in \mathbb{N}} \rho_n^2, \quad (7)
\]

respectively. Note that \( V_\epsilon = V + \epsilon W \). Define:

- \( Q_W(\rho, u) = W(\rho) - \mathbb{E}[W(\rho_{k+1}) | \rho_k = \rho, u_k = u] \),
- \( Q_V(\rho, u) = V(\rho) - \mathbb{E}[V(\rho_{k+1}) | \rho_k = \rho, u_k = u] \),
- \( Q_{V_\epsilon}(\rho, u) = V_\epsilon(\rho) - \mathbb{E}[V_\epsilon(\rho_{k+1}) | \rho_k = \rho, u_k = u] \),

TABLE 1

| \( \epsilon \)   | \( |\rho\rangle|\pi\rangle|\pi| \) | \( k_s \) | \( \sigma \) |
|-----------------|-----------------|-------|------|
| 0               | 79.94           | 164.97| 119.39|
| 0.1             | 79.95           | 166.61| 119.39|
| 1               | 81.24           | 174.29| 119.39|
| 10              | 73.33           | 150.95| 119.39|
| \( 10^2 \)      | 60.41           | 44.18 | 44.12|
| \( 10^3 \)      | 44.18           | 37.37 | 37.37|
| \( 10^4 \)      | 44.18           | 37.37 | 37.37|
| \( 10^5 \)      | 44.18           | 37.37 | 37.37|

Fig. 1. Simulation of one closed-loop realization with gain \( \epsilon = 10^{10} \), convergence of \( \rho_k \) towards \( |\pi\rangle\langle \pi| \), controls \( u_k \) (middle), and outcomes \( y_k \) (bottom). Notice that \( |\rho\rangle|\pi\rangle|\pi| \approx 1 \) and \( u_k = 0 \) for all \( k > 45 \).

Fig. 2. Simulation of one closed-loop realization with gain \( \epsilon = 0 \), convergence of \( \rho_k \) towards \( |\pi\rangle\langle \pi| \), controls \( u_k \) (middle), and outcomes \( y_k \) (bottom). Notice that \( |\rho\rangle|\pi\rangle|\pi| \approx 1 \) and \( u_k = 0 \) for all \( k > 78 \).
for \( \rho \in D_\star \) and \( u \in \{-1,0,1\} \). The proof of Theorem 1 is a straightforward consequence of the next proposition:

**Proposition 3.** Let \( \epsilon > 0 \) and \( n_0,r_0,\pi \in \mathbb{N} \). There exists an integer \( m_0 > n_0 + r_0 + \pi + 1 \) (depending on \( \epsilon, n_0, r_0, \pi \)) such that, for each \( \rho \in D_\star \) with \( n_{\text{length}}(\rho) \leq n_0 \), if \( n_{\max}(\rho) = m_0 \), then

\[
Q_V(\rho, -1) > \max \{ Q_V(\rho, 0), Q_V(\rho, +1) \}.
\]

In fact, given \( \rho_0 \in D_\star \), let \( n_0 = n_{\text{length}}(\rho_0) \) and \( r_0 = n_{\min}(\rho_0) \). Note that \( n_{\max}(\rho_0) = m_0 \leq r_0 < m_0 \). By Proposition 1, \( \rho_k \in D_\star \) with \( n_{\text{length}}(\rho_k) \leq n_0 \), for all \( k \in \mathbb{N} \). Since \( u = f(\rho) \) maximizes \( Q_V(\rho, f(\rho)) \), Proposition 3 implies that when \( n_{\max}(\rho_k) = m_0 \) for some \( k \in \mathbb{N} \), then the input \( u_k \) will be always be equal to \(-1\), and hence Proposition 1 ensures that \( n_{\max}(\rho_{k+1}) \leq n_{\max}(\rho_k) = m_0 \). Therefore, \( n_{\max}(\rho_k) \leq m_0, k \in \mathbb{N} \), showing the First Step.

The following two lemmas are instrumental for showing Proposition 3. Their proofs are given in Appendix D and Appendix E respectively.

**Lemma 1.** Given an arbitrary nonzero \( \theta_0 \in \mathbb{R} \), fix any \( a \in \mathbb{R} \) such that \( 0 < a < 1/2 \). For all nonzero \( N_0, N \in \mathbb{N} \), there exists an integer \( N > N \) big enough such that,

\[
0 < 1/2 - a \leq \sin^2 \left( \frac{\theta}{2 \sqrt{N}} \right) \leq 1/2 + a,
\]

for \( N = N_0, N_0 + 1, \ldots, N + N_0 - 1 \).

**Lemma 2.** Let \( \rho \in D_\star \). Then:

- \( |Q_V(\rho, u)| \leq 1 \), for each \( u \in \{-1,0,1\} \);
- \( Q_V(\rho, 0) = 0 \);
- \( Q_V(\rho, +1) = -\sum_{n \in \mathbb{N}} \rho_{nn}[2(n-\pi) + 1] \sin^2 \left( \frac{\theta}{2 \sqrt{n+1}} \right) \);
- \( Q_V(\rho, -1) = \sum_{n \in \mathbb{N}} \rho_{nn}[2(n-\pi) - 1] \sin^2 \left( \frac{\theta}{2 \sqrt{n+1}} \right) \).

The proof of Proposition 3 is shown in the sequel.

**Proof:** Let \( \epsilon > 0 \) and \( n_0, r_0, \pi \in \mathbb{N} \). One has to show that there exists \( m_0 > n_0 + r_0 + \pi + 1 \) such that, if \( \rho \in D_\star \) with \( n_{\text{length}}(\rho) \leq n_0 \), then \( u = -1 \) always maximizes \( Q_V(\rho, u) \) whenever \( n_{\max}(\rho) = m_0 \). From Lemma 2 and the fact that \( Q_V = Q_V + \epsilon Q_W \), to complete the proof it suffices to show that:

- If \( \rho \in D_\star \) is such that \( n_{\text{length}}(\rho) \leq n_0 \) and \( n_{\max}(\rho) \geq n_0 + \pi \), then \( Q_V(\rho, +1) \leq 0 \);
- There exists \( m_0 > n_0 + r_0 + \pi + 1 \) such that \( Q_V(\rho, -1) \geq 2\epsilon \), whenever \( \rho \in D_\star \) is such that \( n_{\text{length}}(\rho) \leq n_0 \) and \( n_{\max}(\rho) = m_0 \).

Note that

\[
Q_V(\rho, +1) = -\sum_{n = n_{\min}(\rho)}^{n_{\max}(\rho)} \rho_{nn}[2(n-\pi) + 1] \sin^2 \left( \frac{\theta}{2 \sqrt{n+1}} \right),
\]

for any \( \rho \in D_\star \). Thus, if \( n_{\text{length}}(\rho) \leq n_0 \) and \( n_{\max}(\rho) \geq \pi + n_0 \), then \( n_{\min}(\rho) \geq \pi \), and hence the first claim is shown.

Now, fix \( 0 < a < 1/2 \) and let \( N \geq \frac{2 - \pi^2}{2} \). Applying Lemma 1 for \( N_0 = n_0 + r_0 + 1 \) and such choice of \( N \), one gets \( N > \frac{N}{a} \leq a \leq 1/2 - a \leq \sin^2 \left( \frac{\theta}{2 \sqrt{\pi + 1}} \right) \).

As \( N \) is an integer, it follows that \( N \geq \pi + 1 \).
Theorem 3: [5, Theorem 1, p. 195] Let \( \Omega \) be a probability space and let \( W \) be a measurable space. Consider that \( X_k: \Omega \to W, k \in \mathbb{N} \), is a Markov chain with respect to the natural filtration. Let \( Q: W \to \mathbb{R} \) and \( V: W \to \mathbb{R} \) be measurable non-negative functions with \( V(X_k) \) integrable for all \( k \in \mathbb{N} \). If \( \mathbb{E}[V(X_{k+1})|X_k] - V(X_k) = -Q(X_k) \), for \( k \in \mathbb{N} \), then \( \lim_{k \to \infty} Q(X_k) = 0 \) almost surely.

Indeed, let \( J_1 \) be the complex Banach space of all trace-class operators on \( \mathcal{H} \) with the trace norm \( |\cdot|_1 \), that is, \( |B|_1 = \text{Tr}(|B|) \), where \( |B| \triangleq \sqrt{B^*B} \), for \( B \in J_1 \). Recall that \( |B| \leq |B|_1 \) and \( |\text{Tr}(AB)| \leq \|A\||B| \), for every \( B \in J_1 \) and each bounded operator \( A: \mathcal{H} \to \mathcal{H} \), where \( \|\cdot\| \) is the usual operator norm (sup norm of bounded operators) [7], [9]. Consider the subspace topology on \( D_{ma} \) with respect to \( J_1 \). One has that the closed-loop trajectory \( \rho_k, k \in \mathbb{N} \), is a Markov chain with respect to the natural filtration and the Borel algebra on \( D_{ma} \). It is clear that \( D_{ma} \) is compact, and that \( Q_e \) and \( V_e - \alpha_e \) are non-negative and continuous on \( D_{ma} \), for all \( e > 0 \), where \( \alpha_e \triangleq \min_{\rho \in D_{ma}} V_e(\rho) \). The theorem above implies that \( \rho_k \) converges almost surely towards \( \overline{\rho} \) as \( k \to \infty \) (with respect to the trace norm). This completes the proof of Theorem 1.

VI. CONCLUDING REMARKS

This paper provided a convergence analysis of Fock states stabilization via single-photon corrections under an ideal set-up, that is, assuming perfect measurement detection and no control delays. In terms of convergence speed, the simulation results here presented have justified the inclusion of the term \( -\epsilon \sum_{n \in \mathbb{N}} \frac{1}{n^2} \) in the Lyapunov-based feedback law (5)-(6). It is straightforward to verify that the convergence analysis developed in this paper remains valid for: (i) any other function \( d(n) \) in (5) satisfying \( d(\overline{n}) = 0 \), \( d(n) \) is increasing for \( n > \overline{n} \) and \( d(n) \) is decreasing for \( n < \overline{n} \); and (ii) \( \epsilon > 0 \) dependent on \( n \), that is, to take the term \( -\sum_{n \in \mathbb{N}} \epsilon_n \rho_{\overline{n}} \). However, it is an open problem how to choose the function \( d(n) \) and the gains \( \epsilon_n > 0 \) so as to achieve the best convergence speed.

Finally, the feedback law used in [8], which corresponds to \( \epsilon = 0 \), was tailored for an experimental set-up with measurement imperfections and control delays. The convergence analysis of such realistic situation will be investigated in the future.

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APPENDIX

A. Basic properties of the operators \( \mathcal{N} \), \( \alpha \) and \( \alpha^\dagger \)

Fix \( n^* \in \mathbb{N} \) and let \( \mathcal{H}_{n^*} = \text{span}\{0, \ldots, [n^*]\} \). Consider the (linear) operators \( \mathcal{N}: \mathcal{H}_{n^*} \to \mathcal{H}_{n^*}, \alpha: \mathcal{H}_{n^*} \to \mathcal{H}_{n^*+1} \) defined respectively as \( \mathcal{N}|n\rangle = n|n\rangle \), \( \alpha|0\rangle = 0 \), \( \alpha|n\rangle = \sqrt{n}|n-1\rangle \) for \( n \geq 1 \).

One also recalls that if \( A \) is a bounded operator on \( \mathcal{H} \) and \( B \in J_1 \), then \( AB, BA \in J_1 \) with \( \text{Tr}(AB) = \text{Tr}(BA) \).

Finally, the feedback law used in [8], which corresponds to \( \epsilon = 0 \), was tailored for an experimental set-up with measurement imperfections and control delays. The convergence analysis of such realistic situation will be investigated in the future.

B. Proof of Proposition 2

Fix any \( \rho \in D_{\ast} \) and let \( n \in \mathbb{N} \). In particular, \( \rho|n\rangle = \rho_{nn}|n\rangle \). It then follows from (2)–(4) that:

\[
M_g(0)\rho M_g^\dagger(0)|n\rangle = \rho_{nn}\cos^2\left(\frac{\phi_{n}+\phi_{m}}{2}\right)|n\rangle, \\
M_c(0)\rho M_c^\dagger(0)|n\rangle = \rho_{nn}\sin^2\left(\frac{\phi_{n}+\phi_{m}}{2}\right)|n\rangle, \\
M_g(1)\rho M_g^\dagger(1)|n\rangle = \begin{cases} 
0, & \text{for } n = 0, \\
(\rho_{n+1,n-1}\sin^2\left(\frac{\phi_{n}}{\sqrt{n}}\right)|n\rangle, & \text{for } n \geq 1.
\end{cases}
\]

This completes the proof of Proposition 2.

C. Computation of \( Q_V(\rho, u) \)

Fix any \( \rho \in D_{\ast} \) and \( \overline{n} \in \mathbb{N} \). Recall that \( V(\rho) = \text{Tr}(d(\mathcal{N})\rho) \), where \( d: \mathbb{N} \to \mathbb{R} \) be given by \( d(n) = (n-\overline{n})^2 \). Note that (10) implies that, for each \( u \in \{-1, 0, 1\} \),

\[
\mathbb{E}[V(\rho_{k+1})|\rho_k = \rho, u_k = u] = \text{Tr}\left(d(\mathcal{N})M_g(u)\rho M_g^\dagger(u)\right) + \text{Tr}\left(d(\mathcal{N})M_c(u)\rho M_c^\dagger(u)\right).
\]

(14)
Take $u = 0$. From (13–9) in Appendix [B] one has
\[ \mathbb{E}[V(\rho_{k+1}) | \rho_k = \rho, u_k = 0] = \text{Tr} (d(N)M_y(0) \rho M_y^*(0) + d(N)M_y(0) \rho M_y^*(0)) = \text{Tr} \left( d(N) \left[ M_y(0) \rho M_y^*(0) + M_y(0) \rho M_y^*(0) \right] \right) = \text{Tr} (d(N) \rho) = V(\rho). \]
In particular,
\[ Q_V(\rho, 0) = 0. \tag{15} \]

Now, take $u = +1$. Then, (14) above and (10–11) in Appendix [B] provide that
\[ \mathbb{E}[V(\rho_{k+1}) | \rho_k = \rho, u_k = +1] = \text{Tr} \left( \sin^2 \left( \rho_0 \frac{\sqrt{N+1}}{2} (N+1) \rho \right) + \cos^2 \left( \rho_0 \frac{\sqrt{N+1}}{2} (N+1) \rho \right) \right). \]
By summing and subtracting $\text{Tr} (\sin^2 \left( \rho_0 \frac{\sqrt{N+1}}{2} (N+1) \rho \right)) d(N) \rho$, \[ \mathbb{E}[V(\rho_{k+1}) | \rho_k = \rho, u_k = +1] = \text{Tr} (d(N) \rho) + \text{Tr} \left( \sin^2 \left( \rho_0 \frac{\sqrt{N+1}}{2} (N+1) \rho \right) \right) d(N) \rho = V(\rho) + \text{Tr} \left( \sin^2 \left( \rho_0 \frac{\sqrt{N+1}}{2} (N+1) \rho \right) \right) d(N) \rho. \]
In particular,
\[ Q_V(\rho, +1) = -\text{Tr} \left( \sin^2 \left( \rho_0 \frac{\sqrt{N+1}}{2} (N+1) \rho \right) \right) d(N-1) \rho = \sum_{n \in N} \rho_{nn} \left[ 2(n - n_1) + 1 \right] \sin^2 \left( \frac{\rho_0 \sqrt{N+1}}{2} n \right). \tag{16} \]
Finally, take $u = -1$. Using (14) above and (12–13) in Appendix [B] \[ \mathbb{E}[V(\rho_{k+1}) | \rho_k = \rho, u_k = -1] = \text{Tr} \left( \sin^2 \left( \rho_0 \frac{\sqrt{N}}{2} (N-1) \rho \right) \right) d(N) \rho + \text{Tr} \left( \cos^2 \left( \rho_0 \frac{\sqrt{N}}{2} (N-1) \rho \right) \right) d(N) \rho. \]
By summing and subtracting $\text{Tr} (\cos^2 \left( \rho_0 \frac{\sqrt{N}}{2} (N-1) \rho \right)) d(N) \rho$, \[ \mathbb{E}[V(\rho_{k+1}) | \rho_k = \rho, u_k = -1] = \text{Tr} (d(N) \rho) + \text{Tr} \left( \sin^2 \left( \rho_0 \frac{\sqrt{N}}{2} (N-1) \rho \right) \right) d(N) \rho = V(\rho) + \text{Tr} \left( \sin^2 \left( \rho_0 \frac{\sqrt{N}}{2} (N-1) \rho \right) \right) d(N) \rho. \]
In particular,
\[ Q_V(\rho, -1) = -\text{Tr} \left( \sin^2 \left( \rho_0 \frac{\sqrt{N}}{2} (N-1) \rho \right) \right) d(N) \rho = \sum_{n \in N} \rho_{nn} \left[ 2(n - n_1) - 1 \right] \sin^2 \left( \frac{\rho_0 \sqrt{N}}{2} n \right). \tag{17} \]

D. Proof of Lemma [7]

Assume that $N_0$ is even (otherwise one may take $N_0 + 1$ instead of $N_0$ in this proof). Define the function $\eta: \mathbb{N} \to \mathbb{R}$ by
\[ \eta(\ell) = \left[ \frac{2}{\rho_0} \left( \frac{\ell \pi}{2} + \frac{\pi}{4} \right) \right]^2. \tag{18} \]
By definition, one has $\frac{\rho_0}{\sqrt{\eta(\ell)}} = \ell \frac{\pi}{2} + \frac{\pi}{4}$ for all $\ell \in \mathbb{N}$. Let $h = \pi/4 - \arcsin \left( \sqrt{1/2 - a} \right)$. Using the definition of $h$ and the symmetric\(^6\) of the function $\sin^2(\cdot)$, it is easy to show that
\[ 1/2 - a \leq \sin^2(x + \pi/2) \leq a + 1/2, \quad \forall x \in [-h, h]. \tag{19} \]
Let $\ell \in \mathbb{N}$ be even and big enough such that the following two conditions are simultaneously met:
\[ \eta(\ell) > N_0/2 + N, \quad \frac{1}{8} \theta_0 \eta_0 / \sqrt{\eta(\ell) - N_0/2} \leq h. \tag{20} \]
Now, take $N = \left\lfloor \eta(\ell) \right\rfloor - N_0/2 + 1 > N$, where $\lfloor \eta \rfloor$ denotes the greatest integer which is less or equal to $\eta$. By construction, $\eta(\ell)$ is in-between the points $N_0/2 - 1$ and $N_0/2 + N$, and hence it is in the interval $[N, N_0/2 - 1]$. Then, for $n = N, \ldots, N_0/2 - 1$, one has that $n_0 - n < N_0/2$. Consider the function $\phi(x) = \frac{\theta_0}{4} \sqrt{x}$. From the fact that $\phi'(x) = \frac{\theta_0}{4\sqrt{x}}$, by the mean value theorem applied to the function $\phi$ and the second inequality in (20), one obtains
\[ \left| \frac{\theta_0}{2} \sqrt{h} - \frac{\theta_0}{2} \sqrt{\eta(\ell)} \right| < h, \quad \text{for } n = N, \ldots, N_0/2 - 1. \]
Then, the proof follows easily from (13), (19) and the fact that $\sin^2(x - \ell \pi/2) = \sin^2(x)$, for every even $n \in \mathbb{N}$.

E. Proof of Lemma [2]

Proof of the first claim: Let $u \in \{-1, 0, 1\}, \rho \in D_x$. Recall that $W(\rho) = -\sum_{n \in \mathbb{N}} \rho_{nn}$. Since $\text{Tr} (\rho) = \sum_{n \in \mathbb{N}} \rho_{nn} = 1$, then $-1 = -\sum_{n \in \mathbb{N}} \rho_{nn} \leq W(\rho) \leq 0$. Now, by (1), \[ \mathbb{E}[W(\rho_{k+1}) | \rho_k = \rho, u_k = u] = p_{g,k} W(M_y(0)) + p_{e,k} W(M_y(0)) = p_{g,k} + p_{e,k} \]
where $p_{g,k}, p_{e,k} \geq 0$ with $p_{g,k} + p_{e,k} = 1$. Thus $-1 \leq \mathbb{E}[W(\rho_{k+1}) | \rho_k = \rho, u_k = u] \leq 1$. Since $Q_W(\rho, u)$ is the difference of two numbers that are in-between $-1$ and $0$, one concludes that $|Q_W(\rho, u)| \leq 1$.

The second, third and fourth claims, are immediate from (15), (16) and (17) in Appendix [C] respectively.

F. Proof of Lemma [3]

Proof of the first claim: Let $\rho \in D_{m_0}$. By (8–9) in Appendix [B] $M_y(0) \rho M_y^*(0) + M_e(0) \rho M_e^*(0) = \rho$. Taking $\rho_k = \rho$ in $u_k = 0$ in (1), define
\[ \rho^g \triangleq \rho^g_{k+1} = \frac{M_y(0) \rho M_y^*(0)}{\text{Tr} \left( M_y(0) \rho M_y^*(0) \right)}, \quad \text{for } g = y, e. \]
Hence, $\alpha \rho^g + (1 - \alpha) \rho^e = \rho$, where $\alpha \triangleq p_{g,k} = \text{Tr} \left( M_y(0) \rho M_y^*(0) \right)$. In particular, $\alpha \rho_{nn} + (1 - \alpha) \rho_{nn} = \rho_{nn}$, for $n \in \mathbb{N}$. Note that, if $\alpha = 0$, then $M_y(0) \rho M_y^*(0) = 0$, and so $\rho^g = \rho$. Similarly, $\alpha = 1$ implies $\rho^e = \rho$. Thus, the identity $\alpha \rho_{nn} + (1 - \alpha) \rho_{nn} = \rho_{nn}$, for $n \in \mathbb{N}$, still holds when $\alpha = 0$ or $\alpha = 1$. From (1), (7) and $\alpha = p_{g,k}$, one has
\[ Q_W(\rho, 0) = W(\rho) - \left[ p_{g,k} W(M_y(0)) + p_{e,k} W(M_y(0)) \right] = \sum_{n \in \mathbb{N}} \alpha \left( \rho_{nn}^g \right)^2 + (1 - \alpha) \left( \rho_{nn}^e \right)^2 - \left[ \alpha \rho_{nn}^g + (1 - \alpha) \rho_{nn}^e \right]^2 \]
\[ = \alpha (1 - \alpha) \sum_{n \in \mathbb{N}} \left( \rho_{nn}^g - \rho_{nn}^e \right)^2 \geq 0, \tag{21} \]
\[ ^{\text{More precisely, } \sin^2(\pi/2 - x) = 1 - \cos^2(\pi/2 - x) = 1 - \sin^2(x).} \]
thereby showing the first part of the first claim.

If $\rho = |m\rangle\langle m|$ for some $m \in \mathbb{N}$ with $0 < \alpha < 1$, then (8)–(2) in Appendix B imply that $\rho^\beta = \rho^\alpha = \rho$, and so $Q_W(\rho, 0) = 0$. Now, one shows that $\rho = |m\rangle\langle m|$ for some $m \in \mathbb{N}$ whenever $Q_W(\rho, 0) = 0$. Suppose $Q_W(\rho, 0) = 0$. Then, (21) implies that $\alpha = 0$, or $\alpha = 1$, or $\rho^\alpha_n = \rho^\alpha_m$ for all $m \in \mathbb{N}$ with $0 < \alpha < 1$. Assume that $\alpha = 0$. Hence, $M_0(0)M_0^\dagger(0) = \sum_{n \in \mathbb{N}} \rho^\alpha_n \cos^2(\frac{\omega n + \phi_n}{2})|n\rangle\langle n| = 0$ by (8) in Appendix B. Suppose that $\rho \neq |m\rangle\langle m|$ for every $m \in \mathbb{N}$. Thus, there exists $n_1, n_2 \in \mathbb{N}$ with $n_1 \neq n_2$, $\rho_{n_1, n_1} > 0$, $\rho_{n_2, n_2} > 0$. Recall that $\sin(x_1) = \pm \sin(x_2)$ if and only if $x_1 + x_2 = \ell \pi$ or $x_2 - x_1 = \ell \pi$, where $\ell$ is an integer. Therefore, $\sin(\frac{\omega n_1 + \phi_n}{2}) = \pm \sin(\frac{\omega n_2 + \phi_n}{2})$, which contradicts the assumptions that $\phi_0/\pi$ is an irrational number and $\phi_R = \pi/2 - \tilde{n}\phi_0$. One has shown that $\rho = |m\rangle\langle m|$ for some $m \in \mathbb{N}$ whenever $\alpha = 0$. If $\alpha = 1$, or $\rho^\alpha_n = \rho^\alpha_m$ for all $n \in \mathbb{N}$ with $0 < \alpha < 1$, then from similar arguments and computations one also concludes that $\rho = |m\rangle\langle m|$ for some $m \in \mathbb{N}$.

Proof of the second claim: Let $m \in \mathbb{N}$ and take $\rho = |m\rangle\langle m| \in D_s$. It is clear that $W(\rho) = -\sum_{n \in \mathbb{N}} \rho^\alpha_n = -1$. From (10)–(13) in Appendix B one has that:

\[
\begin{align*}
(M_{g}(+1)\rho M_{g}^\dagger(+1))_{nn} &= \delta(n, m + 1) \sin^2 \left(\frac{\theta_n}{2} \sqrt{m + 1}\right), \\
(M_{c}(+1)\rho M_{c}^\dagger(+1))_{nn} &= \delta(n, m) \cos^2 \left(\frac{\theta_n}{2} \sqrt{m + 1}\right), \\
(M_{g}(-1)\rho M_{g}^\dagger(-1))_{nn} &= \delta(n, m) \cos^2 \left(\frac{\theta_n}{2} \sqrt{m}\right), \\
(M_{c}(-1)\rho M_{c}^\dagger(-1))_{nn} &= \delta(n + 1, m) \sin^2 \left(\frac{\theta_n}{2} \sqrt{m}\right),
\end{align*}
\]

(22) where $\delta(n, m)$ is the usual Kronecker delta: $\delta(n, m) = 0$ if $n \neq m$, and $\delta(n, m) = 1$ if $n = m$. In particular:

\[
\begin{align*}
\text{Tr} (M_{g}(+1)\rho M_{g}^\dagger(+1)) &= \sin^2 \left(\frac{\theta_n}{2} \sqrt{m + 1}\right), \\
\text{Tr} (M_{c}(+1)\rho M_{c}^\dagger(+1)) &= \cos^2 \left(\frac{\theta_n}{2} \sqrt{m + 1}\right), \\
\text{Tr} (M_{g}(-1)\rho M_{g}^\dagger(-1)) &= \cos^2 \left(\frac{\theta_n}{2} \sqrt{m}\right), \\
\text{Tr} (M_{c}(-1)\rho M_{c}^\dagger(-1)) &= \sin^2 \left(\frac{\theta_n}{2} \sqrt{m}\right),
\end{align*}
\]

\[
\sum_{n \in \mathbb{N}} \left(\frac{M_{g}(u)\rho M_{g}^\dagger(u)}{\text{Tr}(M_{g}(u)\rho M_{g}^\dagger(u))}\right)^2_{nn} = 1, \quad \text{for } u = g, c
\]

(assuming no division by 0). Now, using (1) and the above computations, one gets

\[
\mathbb{E}[W(p_{k+1}) | p_k = \rho, u_k = \pm 1] = p_{g,k} W(p_{k+1}^g) + p_{c,k} W(p_{k+1}^c) = - \sum_{y=g,c} \left[\text{Tr} \left( M_g(\pm 1)\rho M_g^\dagger(\pm 1) \right) \times \right] \\
\times \sum_{n \in \mathbb{N}} \left( \frac{M_g(\pm 1)\rho M_g^\dagger(\pm 1)}{\text{Tr}(M_g(\pm 1)\rho M_g^\dagger(\pm 1))}\right)^2_{nn} = -1 = W(\rho).
\]

Therefore, $Q_W(|m\rangle\langle m|, \pm 1) = 0$.

---

**G. Proof of Proposition 2**

Fix $\rho \in D_s$. Since $\text{Tr}(d(N)\rho) = \text{Tr}(d(N)(\Delta \rho))$ and $\rho_{mn} = (\Delta \rho)_{mn}$ for $n \in \mathbb{N}$, the first two assertions are immediate from the definitions. As for the third and fourth assertions, let $|\psi\rangle = \sum_{m=0}^{\infty} (m|\psi\rangle)|m\rangle \in \mathcal{H}$. Note that $|m\rangle = \sum_{n=0}^{n_{\max}(\rho)} \rho_{mn}|n\rangle$, for $m \in \mathbb{N}$. Using (2)–(4):

\[
\begin{align*}
M_g(0)\rho M_g^\dagger(0)\langle \psi | &= \sum_{m,n=0}^{\infty} \rho_{mn} \cos^2 \left(\frac{\theta_n}{2} \sqrt{m + 1}\right) \langle m|\psi\rangle \langle n|, \\
M_c(0)\rho M_c^\dagger(0)\langle \psi | &= \sum_{m,n=0}^{\infty} \rho_{mn} \sin^2 \left(\frac{\theta_n}{2} \sqrt{m + 1}\right) \langle m|\psi\rangle \langle n|, \\
M_g(+1)\rho M_g^\dagger(+1)\langle \psi | &= \sum_{m=0}^{n_{\max}(\rho)-1} \rho_{m+1,n} \sin \left(\frac{\theta_n}{2} \sqrt{m + 1}\right) \langle m|\psi\rangle \langle n| + 1, \\
M_c(+1)\rho M_c^\dagger(+1)\langle \psi | &= \sum_{m=0}^{n_{\max}(\rho)-1} \rho_{m,n+1} \sin \left(\frac{\theta_n}{2} \sqrt{m}\right) \langle m|\psi\rangle \langle n|, \\
M_g(-1)\rho M_g^\dagger(-1)\langle \psi | &= \sum_{m=0}^{n_{\max}(\rho)-1} \rho_{m,n} \cos \left(\frac{\theta_n}{2} \sqrt{m + 1}\right) \langle m|\psi\rangle \langle n|, \\
M_c(-1)\rho M_c^\dagger(-1)\langle \psi | &= \sum_{m=0}^{n_{\max}(\rho)-1} \rho_{m+1,n} \cos \left(\frac{\theta_n}{2} \sqrt{m}\right) \langle m|\psi\rangle \langle n| - 1. \\
\end{align*}
\]

Since $\Delta \rho \in D_s \subset D_s$, $n_{\max}(\Delta \rho) = n_{\max}(\rho)$ and $(\Delta \rho)_{mn} = \rho_{mn}$, the proof is straightforward from (8)–13 in Appendix B.

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