Equality of Proofs for Linear Equality

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Abstract

This paper is about equality of proofs in which a binary predicate formalizing properties of equality occurs, besides conjunction and the constant true proposition. The properties of equality in question are those of a preordering relation, those of an equivalence relation, and other properties appropriate for an equality relation in linear logic. The guiding idea is that equality of proofs is induced by coherence, understood as the existence of a faithful functor from a syntactical category into a category whose arrows correspond to diagrams. Edges in these diagrams join occurrences of variables that must remain the same in every generalization of the proof. It is found that assumptions about equality of proofs for equality are parallel to standard assumptions about equality of arrows in categories. They reproduce standard categorial assumptions on a different level. It is also found that assumptions for a preordering relation involve an adjoint situation.

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1 Introduction

The purpose of this paper is to investigate in equational logic the hypothesis that two proofs are equal if and only if they have the same generality. Two proofs,
with the same premises and conclusions, have the same generality when after diversifying variables as much as possible without changing the rules of inference one ends up again with the same premises and conclusions, up to renaming of variables. This notion of generality of proof was investigated in [6] (see also [7]), [8] and [9]. However, individual variables (as opposed to propositional variables) and equality between them are mentioned only briefly, as an example, in [6] (end of Section 3). The present paper develops more fully these matters announced in that previous paper.

The results of this paper take the form of coherence theorems, understood as faithfulness results for functors from syntactically constructed categories to a category whose arrows correspond to diagrams. Edges in these diagrams join occurrences of variables that must remain the same in every generalization of the proof.

The central result of the paper is that assumptions about equality of proofs for equality induced by generality are parallel to standard assumptions about equality of arrows in categories. Usual categorial assumptions are reproduced on a different level. Reflexivity of equality corresponds to identity arrows, and transitivity corresponds to categorial composition of arrows. Equations involving reflexivity and transitivity are parallel to the categorial assumptions of omission of identity arrows in composition and associativity of composition. Similar correspondences hold for other postulates concerning equality.

Another result of the paper is that assumptions for a preordering relation involve an adjoint situation. This was foreshadowed in [14].

Besides [14], another paper about equality of proofs for equality is [12], which investigates the equivalence of various syntactical formulations in classical predicate logic with equality. There is a chapter on equational logic based on fibrations in [11] (Chapter 3), but, contrary to what we have in this paper, it is asserted there (p. 174) that there are no different proofs of the same proposition.

We restrict ourselves to equality added to multiplicative conjunctive propositional linear logic, with the multiplicative constant true proposition. We eschew going beyond this limited fragment of equational logic because generality of proofs involving equality, like that of proofs involving implication, prevents the arrows corresponding to the structural rules of contraction and thinning from making natural transformations (cf. [5], Section 1, [8], Section 14.3, and [10]). This requirement of naturality is otherwise quite natural, and proof-theoretically well motivated (see [8], Chapters 9-11). Contraction and thinning are required if we want to say that we deal with full equational logic. In their absence, we cannot pretend to cover more than a relation of equality appropriate for linear logic (such as the equality relations investigated in [1]). This explains the expression *linear equality* in the title of the paper.

We restrict ourselves in general to a context with minimal assumptions where our results can be obtained. So we do not assume the commutativity of multiplicative conjunction if this is not essential; i.e., we work also in noncommutative linear logic (which is related to the Lambek calculus; we stay however within
the multiplicative conjunctive fragment of this calculus).

As we indicated above concerning adjunction, our results are about relations more general than equality relations, such as preordering and equivalence relations. Most of the paper (Sections 2-7) is about such relations. Only in the last section we indicate how to deal with further assumptions, such as congruence. Within our syntactical systems, where the means of expression are limited, the equivalence relations involved amount however to equality relations. In the last section our proofs will be less formal.

We consider binary operational expressions in the last section, but we do not have anywhere predicate variables. So we cannot say that we deal yet with full linear equational logic, but only with fragments of it. In the context where we investigate equality of proofs involving equality, motivated by generality, we would have to enter into the question of what is a linear predicate (see [2]), and moreover we would have to restrict ourselves to the multiplicative fragment of linear logic without propositional constants (see [9]). Equations that can be expected in that context would be on the lines of [12] (Sections 2.3-4). We leave however these extensions of our approach for another occasion.

2 The category $M_{\leq}$

Let $\mathcal{V}$ be a set whose elements, for which we use the letters $x, y, z, \ldots$, perhaps with indices, are called variables. The cardinality of $\mathcal{V}$ is not restricted: $\mathcal{V}$ can be infinite or finite, and even empty. Let words of the form $x \leq y$ or $\top$ be called atomic formulae. The set of formulae is defined inductively as follows. Atomic formulae are formulae, and if $A$ and $B$ are formulae, then $(A \land B)$ is a formula. We use $A, B, C, \ldots$ for formulae, and we omit, as usual, the outermost parentheses of $(A \land B)$. (We proceed analogously for other similar expressions later on.)

The objects of the category $M_{\leq}$ are formulae. To define the arrows of $M_{\leq}$, we define first inductively the arrow terms of $M_{\leq}$ in the following way. We use $f, g, h, \ldots$, perhaps with indices, for arrow terms. Every arrow term $f$ has a type, which is an ordered pair $(A, B)$ of objects of $M_{\leq}$; that $f$ is of type $(A, B)$ is written $f : A \vdash B$.

For all formulae $A, B$ and $C$, and for all variables $x, y$ and $z$, the following are primitive arrow terms of $M_{\leq}$:

\[
\begin{array}{l}
1_A : A \vdash A, \\
b^-_{A, B, C} : (A \land (B \land C)) \vdash (A \land B) \land C, \\
\delta^-_A : A \land \top \vdash A, \\
\delta^+_A : A \vdash A \land \top, \\
\sigma^-_A : \top \land A \vdash A, \\
\sigma^+_A : A \vdash \top \land A, \\
\tau_x : \top \vdash x \leq x, \\
t_{x, y, z} : x \leq y \land y \leq z \vdash x \leq z.
\end{array}
\]
If $f : A \vdash B$ and $g : C \vdash D$ are arrow terms of $M_\leq$, then $f \land g : A \land C \vdash B \land D$ and $g \circ f : A \vdash D$ are arrow terms of $M_\leq$, provided that for $g \circ f$ we have that $B$ and $C$ are the same formula. This defines the arrow terms of $M_\leq$.

The arrows of $M_\leq$ are equivalence classes of arrow terms with respect to the smallest equivalence relation which guarantees that the following equations are satisfied:

categorial equations:

- (cat 1) $\mathbf{1}_B \circ f = f$, $f \circ \mathbf{1}_A = f$, for $f : A \vdash B$,
- (cat 2) $(h \circ g) \circ f = h \circ (g \circ f)$,

bifunctoriality equations:

- $(\land 1) \quad \mathbf{1}_A \land \mathbf{1}_B = \mathbf{1}_{A \land B}$, 
- $(\land 2) \quad (g_1 \circ f_1) \land (g_2 \circ f_2) = (g_1 \land g_2) \circ (f_1 \land f_2)$,

naturality equations:

- for $f : A \vdash D$, $g : B \vdash E$ and $h : C \vdash F$,
- (b nat) $(f \land g) \circ h \circ b_{A,B,C}^* = b_{D,E,F}^* (f \land (g \circ h))$,
- (δ nat) $f \circ \delta_A^* = \delta_D^* \circ (f \land 1_T)$,
- (σ nat) $f \circ \sigma_A^* = \sigma_D^* \circ (1_T \land f)$,

specific equations of monoidal categories:

- (bb) $b_{A,B,C}^* \circ b_{A,B,C}^* = 1_{A \land (B \land C)}$, $b_{A,B,C}^* \circ b_{A,B,C}^* = 1_{(A \land B) \land C}$,
- (b5) $b_{A,B,C,D}^* \circ b_{A,B,C,D}^* = (b_{A,B,C}^* \land 1_D) \circ b_{A,B,C,D}^* \circ (1_A \land b_{B,C,D}^*)$,
- (δδ) $\delta_A^* \circ \delta_A^* = 1_{A \land T}$, $\delta_A^* \circ \delta_A^* = 1_A$,
- (σσ) $\sigma_A^* \circ \sigma_A^* = 1_{T \land A}$, $\sigma_A^* \circ \sigma_A^* = 1_A$,
- (bδσ) $b_{A,T,C}^* = (\delta_A^* \land 1_C) \circ (1_A \land \sigma_C^*)$,

specific equations of $M_\leq$:

- (rtδ) $t_{x,y,z} \circ (1_{x \leq y} \land \tau_y) = \delta_x^* \leq y$,
- (rtσ) $t_{y,x,z} \circ (\tau_y \land 1_{y \leq x}) = \sigma_y^* \leq z$,
- (tb) $t_{x,y,z,u} \circ (1_{x \leq y} \land t_{y,z,u}) = t_{x,z,u} \circ (t_{x,y,z} \land 1_{z \leq u}) \circ b_{x,y,z,u}$

if $f_1 = g_1$ and $f_2 = g_2$, then $f_1 \land f_2 = g_1 \land g_2$ and $f_2 \circ f_1 = g_2 \circ g_1$, provided $f_2 \circ f_1$ and $g_2 \circ g_1$ are defined.

The category $M_\leq$ is a monoidal category in the sense of [16] (Section VII.1, see also [8], Section 4.6). The equations (rtδ) and (rtσ) are parallel to the equations (cat 1), and the equation (tb) is parallel in the same manner to the equation (cat 2). The equation (tb), for instance, says that a composition tied to the arrows $t$ is associative.

The following instance of (tb):
\[ t_{x,x,x} \cdot (1_{x \leq x} \land t_{x,x,x}) = t_{x,x,x} \cdot (t_{x,x,x} \land 1_{x \leq x}) \cdot b_{x \leq x,x \leq x,x \leq x} \]

is analogous to the equation \((\tilde{w} \tilde{u})\) of categories with coproducts (see [8], List of Equations), where \(t_{x,x,x} : x \leq x \land x \leq x \vdash x \leq x\) corresponds to the codiagonal arrow \(\tilde{w}_p : p \lor p \vdash p\) of these categories. The instances of \((rt\delta)\) and \((rta)\) with \(x\) and \(y\) the same variable correspond in an analogous manner to the equations \((\tilde{w} \tilde{k})\) of categories with coproducts (see [8], ibid.). The arrow \(r : \top \vdash x \leq x\) corresponds here to the arrow \(\tilde{k}_p : \bot \vdash p\) of categories with coproducts, \(\bot\) being the initial object.

The arrows \(r_x\) and \(t_{x,y,z}\) codify of course the reflexivity and transitivity of a relation corresponding to \(\leq\). So we deal here with a preorder relation.

### 3 The coherence of \(M_\leq\)

For every object \(A\) of \(M_\leq\) let \(G A\) be the number of occurrences of variables in \(A\). (One could modify the category \(Br\), mentioned below, so that its objects are formulae, instead of finite ordinals. In that case the object \(G A\) would be the formula \(A\). We draw diagrams in this paper in that spirit. The category \(Br\) abstracts from a formula just the position of occurrences of variables in a formula, which is the only thing relevant for drawing diagrams. We do not expect the functor \(G\) below to be one-one on objects. It suffices for our purposes that it be faithful.)

Let us assign the following diagrams to the primitive arrow terms of \(M_\leq\)

\[
\begin{array}{c}
A \\
\downarrow 1_A \\
A \\
\end{array}
\]

(where the line joining the two \(A\)'s stands for a family of parallel lines—one line for each occurrence of a variable in \(A\); for example, for \(A\) being \(x \leq y \land z \leq x\) we have

\[
\begin{array}{c}
x \leq y \land z \leq x \\
\mid \\
\mid \\
\mid \\
\mid \\
\mid \\
x \leq y \land z \leq x
\end{array}
\]

and analogously in other cases below)

\[
\begin{array}{c}
A \land (B \land C) \\
\downarrow b_{A,B,C} \\
(A \land B) \land C \\
\end{array}
\]

\[
\begin{array}{c}
(A \land B) \land C \\
\downarrow b_{A,B,C} \\
A \land (B \land C)
\end{array}
\]

5
In the diagram for \( r_x \) we have a cap joining the two occurrences of \( x \) and in the diagram for \( t_{x,y,z} \) we have a cup joining the two occurrences of \( y \). We use an analogous terminology in other cases.

These diagrams and the function \( G \) on objects serve to define a functor \( G \) from \( M \leq \) to the category \( Br \) of [9] (Section 2.3). Namely, \( G \) maps an arrow of \( M \leq \) to an arrow of \( Br \) that corresponds to a diagram. The composition of \( M \leq \) is mapped to composition in \( Br \), which corresponds to vertical composition of diagrams, while the operation \( \wedge \) on the arrows of \( M \leq \) is mapped to the operation of \( Br \) that corresponds to horizontal composition of diagrams (see [9], Section 2.3). We check by induction on the length of derivation that \( G \) is indeed a functor. Here is what we have in the basis of this induction for the specific equations of \( M \leq \):

\[(rt\delta)\]:

\[
\begin{align*}
\delta_{A} & \quad A \land \top \\
\sigma_{A} & \quad \top \land A \\
r_{x} & \quad x \leq x \\
t_{x,y,z} & \quad x \leq y \land y \leq z
\end{align*}
\]
(tb):

\[
\begin{array}{c}
\begin{array}{c}
1_{x \leq y} \land t_{y,z,u} \\
\end{array} \\
\begin{array}{c}
x \leq y \land (y \leq z \land z \leq u) \\
\end{array} \\
\begin{array}{c}
t_{x,y,u} \\
x \leq y \land y \leq u \\
x \leq u \\
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
b_{x \leq y, y \leq z, z \leq u} \\
\end{array} \\
\begin{array}{c}
x \leq y \land (y \leq z \land z \leq u) \\
\end{array} \\
\begin{array}{c}
t_{x,y,z} \land 1_{z \leq u} \\
x \leq z \land z \leq u \\
x \leq u \\
\end{array}
\end{array}
\]

(Note that there are no cups and caps in the diagrams corresponding to the left-hand sides of (rtδ) and (rtε); they were abolished by composing.)

So if \( f = g \) in \( M_{\leq} \), then \( Gf = Gg \). Our purpose is to show also the converse for \( f \) and \( g \) of the same type; we show, namely, that \( G \) is a faithful functor from \( M_{\leq} \) to \( Br \). We call this faithfulness result the coherence of \( M_{\leq} \).

An arrow term of \( M_{\leq} \) of the form \( f_n \circ \ldots \circ f_1 \), where \( n \geq 1 \), with parentheses tied to \( \circ \) associated arbitrarily, such that for every \( i \in \{1, \ldots, n\} \) we have that \( f_i \) is without \( \circ \) is called factorized. In a factorized arrow term \( f_n \circ \ldots \circ f_1 \) the arrow terms \( f_i \) are called factors.

If \( \beta \) is a primitive arrow term of \( M_{\leq} \) which is not of the form \( 1_B \), then \( \beta \)-terms are defined inductively as follows: \( \beta \) is a \( \beta \)-term; if \( f \) is a \( \beta \)-term, then for every object \( A \) of \( M_{\leq} \) we have that \( 1_A \land f \) and \( f \land 1_A \) are \( \beta \)-terms. In a \( \beta \)-term the subterm \( \beta \) is called the head of this \( \beta \)-term. For example, the head of the \( t_{x,y,z} \)-term \((1_{x \leq v} \land t_{x,y,z}) \land 1_{x \leq u}\) is \( t_{x,y,z} \).

We define \( 1 \)-terms as \( \beta \)-terms by replacing \( \beta \) in the definition above by \( 1_B \).

A factor that is a \( \beta \)-term for some \( \beta \) is called a headed factor. A factorized arrow term is called headed when each of its factors is either headed or a \( 1 \)-term. A headed arrow term \( f_n \circ \ldots \circ f_1 \) is called developed when \( f_1 \) is a \( 1 \)-term and if \( n > 1 \), then every factor of \( f_n \circ \ldots \circ f_2 \) is headed. Analogous definitions of factorized arrow term, factor, \( \beta \)-term, head, headed factor, headed factorized arrow term and developed arrow term can be given later for categories other than \( M_{\leq} \), and we will not dwell on these definitions any more.

By using the categorial equations (\( cat \; 1 \)) and (\( cat \; 2 \)) and the bifunctoriality equations we can easily prove by induction on the length of \( f \) the following lemma for \( M_{\leq} \).

**Development Lemma.** For every arrow term \( f \) there is a developed arrow term \( f' \) such that \( f = f' \).

An \( r \)-less arrow term of \( M_{\leq} \) is an arrow term of \( M_{\leq} \) in which \( r_x \) does not occur for any \( x \). A headed factorized arrow term each of whose factors is an \( r \)-term for some \( x \) or a \( 1 \)-term is called an \( r \)-factorized arrow term. We can easily prove the following lemma by applying the Development Lemma and the
equations \((rt\delta)\) and \((rt\sigma)\), besides bifunctoriality, naturality and other obvious equations.

**r-Normality Lemma.** For every arrow term \(f\) of \(M \leq \) there is a headed factorized arrow term of \(M \leq \) of the form \(f_r \cdot f'\) such that \(f\) is a developed \(r\)-less arrow term and \(f_r\) is an \(r\)-factorized arrow term and \(f = f_r \cdot f'\) in \(M \leq \).

The arrow term \(f_r \cdot f'\) of this lemma is called the \(r\)-normal form of \(f\).

Suppose \(f, g: A \vdash B\) are arrow terms of \(M \leq \) such that \(Gf = Gg\). Let \(f_r \cdot f'\) and \(g_r \cdot g'\) be the \(r\)-normal forms of \(f\) and \(g\) respectively. Then \(G(f_r \cdot f') = G(g_r \cdot g')\), and there is a bijection between the caps of \(G(f_r \cdot f')\) and \(G(g_r \cdot g')\). Moreover, there is a bijection between the caps in \(G(f_r \cdot f')\) and the \(r_x\)-factors of \(f_r\), and analogously for \(G(g_r \cdot g')\) and \(g_r\).

By using the bifunctoriality equations we can achieve that \(f_r\) and \(g_r\), which are \(r\)-factorized, are the same arrow term \(h\). Since

\[
G(h \cdot f') = Gh \cdot Gf',
\]
\[
G(h \cdot g') = Gh \cdot Gg',
\]
\[
G(h \cdot f') = G(h \cdot g'),
\]
and \(Gh\) has no cups, it is easy to conclude that

\[
Gf' = Gg'.
\]

(In the category \(Br\) the arrow \(Gh\), which has no cups, has a left inverse, which is its image in a horizontal mirror.)

There are no caps in \(Gf'\) and \(Gg'\), and there is a bijection between the caps of \(Gf'\) and the \(t_{x,y,z}\)-factors of \(f'\), and analogously for \(Gg'\) and \(g'\). A \(t_{x,y,z}\)-factor of \(f'\) and a \(t_{u,y,v}\)-factor of \(g'\) that correspond to each other according to these bijections are called coupled. Suppose \(Gf'\) and \(Gg'\) have at least one cup. Then by using the equation \((tb)\), besides the bifunctoriality, naturality and other obvious equations, we obtain \(f' = h_1 \cdot f''\) and \(g' = h_2 \cdot g''\) where \(f''\) and \(g''\) are developed arrow terms, while \(h_1\) is a \(t_{x,y,z}\)-factor coupled with the \(t_{u,y,v}\)-factor \(h_2\).

It is impossible that \(z\) coincides with \(v\) while \(x\) differs from \(u\). Otherwise, the targets of \(h_1\) and \(h_2\) would differ. If \(x\) differs from \(u\), and \(z\) differs from \(v\), then in the source of \(f''\) and \(g''\) we would have variables occurring in the following order:

\[
x \ldots z \ldots u \ldots v \quad \text{or} \quad u \ldots v \ldots x \ldots z,
\]
with a cup between \(x\) and \(z\) in \(G(h_1 \cdot f'')\) corresponding to \(h_1\) and a cup between \(u\) and \(v\) in \(G(h_2 \cdot g'')\) corresponding to \(h_2\). This is impossible because \(h_1\) and \(h_2\) are coupled. So \(t_{x,y,z}\) coincides with \(t_{u,y,v}\), and by using perhaps the bifunctoriality equation \((\wedge 1)\) we can achieve that \(h_1\) and \(h_2\) are the same arrow term. From \(G(h \cdot f'') = G(h \cdot g'')\) we conclude that \(Gf'' = Gg''\).
Then we proceed by induction on the number of cups in \( Gf' \), which is equal to \( Gg' \), to show that \( f' = g' \). In the basis of this induction we rely on Mac Lane’s monoidal coherence (see [15] and [16], Section VII.2, or [8], Section 4.6). From that it follows that if for \( f, g : A \vdash B \) arrow terms of \( M_\leq \) we have \( Gf = Gg \), then \( f = g \) in \( M_\leq \). This proves the coherence of \( M_\leq \).

Note that by omitting from the proof above the part involving \( r_x \) we would obtain an analogous coherence result for a category defined like \( M_\leq \) save that it lacks the arrows \( r_x \) and the specific equations \((rt\delta)\) and \((rt\sigma)\). We can also obtain coherence for a category defined like \( M_\leq \) but lacking the arrows \( t_{x,y,z} \) and all the specific equations of \( M_\leq \).

### 4 The coherence of \( S_\leq \)

The category \( S_\leq \) is defined like \( M_\leq \) save that we have for all formulae \( A \) and \( B \) the additional primitive arrow term

\[ c_{A,B} : A \land B \vdash B \land A, \]

which is subject to the following additional equations:

- **naturality equation:**
  
  \[ (c \text{ nat}) \quad (g \land f) \circ c_{A,B} = c_{D,E} \circ (f \land g), \]

- **specific equations of symmetric monoidal categories:**
  
  \[
  \begin{align*}
  (cc) & \quad c_{B,A} \circ c_{A,B} = 1_{A \land B}, \\
  (bc) & \quad c_{A,B,C} = b_{B,C,A} \circ (1_B \land c_{A,C}) \circ b_{A,C,B} \circ (c_{A,B} \land 1_C) \circ b_{A,B,C}.
  \end{align*}
  \]

The category \( S_\leq \) is a symmetric monoidal category (see [16], Section VII.7), in which \( \leq \) corresponds to a preordering relation.

We define the functor \( G \) from \( S_\leq \) to \( Br \) by extending the definition of \( G \) from \( M_\leq \) to \( Br \) with a clause corresponding to the following diagram:

\[
\begin{array}{ccc}
A \land B & \xrightarrow{c_{A,B}} & B \land A \\
\downarrow & & \downarrow \\
A & & A
\end{array}
\]

That \( G \) is indeed a functor follows from well-known facts about symmetric monoidal categories (which were established in [15]; see also [8], Chapter 5).

We can prove coherence for \( S_\leq \) with respect to \( G \), i.e., we can prove that \( G \) is faithful, by imitating the proof of coherence for \( M_\leq \) in the preceding section. The only difference is that we appeal to symmetric monoidal coherence (see [15], Section 5, second edition of [16], Section XI.1, and [8], Section 5.3) where we appealed before to monoidal coherence, and we replace the proof that \( h_1 \) and
$h_2$ can be taken to be the same arrow term by the following alternative proof. An analogous proof could have already been used in the preceding section, but there, in the absence of $c_{A,B}$, we also had a slightly simpler argument.

For $f: A \vdash B$ an $r$-less arrow term of $S\leq$, we say that a set $U$ of occurrences of variables in $A$ is $f$-closed when the following implication holds: if either $u \leq u'$ is a subformula of $A$ or $u$ and $u'$ are connected by a cup of $Gf$, and one of $u$ and $u'$ is in $U$, then the other is in $U$ too. It is easy to verify by induction on the complexity of $f$ that

\((\ast)\) for every $f$-closed set $U$ and for every atomic subformula $x \leq y$ of $B$, a member of $U$ is connected by $Gf$ to $x$ if and only if a member of $U$ is connected by $Gf$ to $y$.

This holds in particular for $f$-closed sets generated by a single occurrence of a variable in $A$. We call such $f$-closed sets maximal sequences. It is easy to see that a maximal sequence is a set $\{u_1, u_2, \ldots, u_{2n-1}, u_{2n}\}$ of occurrences of variables in $A$ for $n \geq 1$ such that $u_{2i-1} \leq u_{2i}$ is a subformula of $A$, for $1 \leq i \leq n$, while $u_{2j}$ and $u_{2j+1}$, for $1 \leq j \leq n - 1$, are connected by a cup of $Gf$. Note that it follows from $(\ast)$ that for a maximal sequence there must exist an atomic subformula $x \leq y$ of $B$ such that $u_1$ in $A$ is connected by $Gf$ to the occurrence of $x$ in $x \leq y$ in $B$ and $u_{2n}$ in $A$ is connected by $Gf$ to the occurrence of $y$ in $x \leq y$ in $B$.

Suppose, as in the preceding section, that for $r$-less arrow terms $f', g': A \vdash B$ of $S\leq$, we have $Gf' = Gg'$. Again, by using $(tb)$, besides bifunctoriality, naturality and other obvious equations, we obtain $f' = h_1 \circ f''$ and $g' = h_2 \circ g''$ where $h_1$ is a $t_{x,y,z}$-factor coupled with the $t_{u,y,v}$-factor $h_2$. Then for the same maximal sequence $u_1, \ldots, u_{2n}$ in $A$ we have that $u_1$ is connected by $Gf'$ to the $x$ of $t_{x,y,z}$ and by $Gg'$ to the $u$ of $t_{u,y,v}$. Analogously, we have that $u_{2n}$ is connected by $Gf'$ to the $z$ of $t_{x,y,z}$ and by $Gg'$ to the $v$ of $t_{u,y,v}$. This means that $t_{x,y,z}$ coincides with $t_{u,y,v}$, and by using perhaps $(\land 1)$ we can achieve that $h_1$ and $h_2$ are the same arrow term.

5 The coherence of $M_{\equiv}$

The category $M_{\equiv}$ is defined like $M_{\leq}$ save that $\leq$ is replaced everywhere by $\equiv$, and we have for all variables $x$ and $y$ the additional primitive arrow term

\[ s_{x,y} : x \equiv y \vdash y \equiv x, \]

which is subject to the following additional equations:

\[(ss)\quad s_{y,x} \circ s_{x,y} = 1_{x \equiv y},\]
\[(rs)\quad s_{x,x} \circ r_x = r_x.\]
The category $M=\equiv$ is a monoidal category in which $\equiv$ corresponds to an equivalence relation. Since the means of expression of $M=\equiv$ are limited, this equivalence relation is an equality relation.

As the specific equations of $M\leq$ are parallel to the categorial equations ($\text{cat} 1$) and ($\text{cat} 2$) (see Section 2), so the equations ($ss$) and ($rs$) are parallel to equations of groupoids, i.e. categories where every arrow $f$ has an inverse $f^{-1}$. The equation ($ss$) corresponds to $(f^{-1})^{-1} = f$ and ($rs$) corresponds to $1_{A}^{-1} = 1_{A}$.

We define the functor $G$ from $M=\equiv$ to $Br$ by extending $G$ from $M\leq$ to $Br$ with a clause corresponding to the following diagram:

$$
x \equiv y \quad \xymatrix{ s_{x,y} & x \equiv y \\ y \equiv x & y }
$$

Since for the equation ($ss$) we have

$$
x \equiv y \quad \xymatrix{ s_{x,y} & x \equiv y \\ s_{y,x} & y \equiv x \\ x \equiv y & x }
$$

and for ($rs$) we have

$$
x \equiv x \quad \xymatrix{ r_{x} & x \equiv x \\ s_{x,x} & x \equiv x \\ x \equiv x & x }
$$

we can conclude that $G$ is indeed a functor from $M=\equiv$ to $Br$. In the remainder of this section we prove the faithfulness of $G$; namely, the coherence of $M=\equiv$.

This proof is more complex than the other proofs of coherence in this paper. It involves a number of details. We will mention most of them, but not all, in order not to prolong the exposition excessively. We do not consider $M=\equiv$ as the most significant category of this paper. (We find $S=\equiv$ of the next section more important, and for it coherence is proved more easily.)

We can easily prove an analogue of the $r$-Normality Lemma of Section 3 for $M=\equiv$. To prove this analogue we apply also the equation ($rs$).

Then it is enough to prove coherence for $r$-less arrow terms of $M=\equiv$ to obtain coherence for the whole of $M=\equiv$. (For $r$-factorized arrow terms we proceed as in Section 3.)
An arrow term of $M_\equiv$ is called $\delta\sigma$-less when $\delta^+_A$, $\delta^-_A$, $\sigma^+_A$ or $\sigma^-_A$ does not occur in it for any $A$. We can establish the following.

**$\delta\sigma$-Normality Lemma.** For every $r$-less arrow term $f: A \vdash B$ of $M_\equiv$ such that $\top$ does not occur in $A \vdash B$ or both $A$ and $B$ are $\top$, there is a $\delta\sigma$-less arrow term $f': A \vdash B$ such that $f = f'$ in $M_\equiv$.

For the proof of this lemma we rely on bifunctoriality and naturality equations, and on monoidal coherence. Intuitively, we push every $\delta^+_C$-factor in a headed factorized arrow term towards the right (or $\delta^-_C$-factor towards the left), where it or its descendant will disappear in virtue of the equations $(\delta\delta)$ or $(\sigma\sigma)$.

For every formula $A$ we define a formula $A^\dagger$ in which $\top$ does not occur, or which is $\top$, in the following inductive manner: if $A$ is atomic, then $A^\dagger$ is $A$, and if $A$ is $B \land C$, then $(B \land C)^\dagger$ is either $B^\dagger \land C^\dagger$ when neither $B^\dagger$ nor $C^\dagger$ is $\top$, or $B^\dagger$ when $C^\dagger$ is $\top$, or $C^\dagger$ when $B^\dagger$ is $\top$. It is clear that there is an isomorphism $\varphi_A : A \vdash A^\dagger$. For every arrow $f : A \vdash B$ of $M_\equiv$, let $f^\dagger : A^\dagger \vdash B^\dagger$ be the arrow $\varphi_B \circ f \circ \varphi_A^{-1}$. We have that $Gf = Gf^\dagger$, and $f = g$ if and only if $f^\dagger = g^\dagger$.

A type $A \vdash B$ is diversified when every variable in it occurs exactly twice (once in $A$ and once in $B$, or twice in $A$, or twice in $B$). An arrow term whose type is diversified is also called diversified.

For every arrow term $f : A \vdash B$ of $M_\equiv$ there is a diversified arrow term $f' : A' \vdash B'$ of $M_\equiv$ such that $f$ is obtained from $f'$ by substitution in the variables of $f'$. (Here variables are uniformly replaced by variables.) This is clear from $Gf$, which dictates how the diversification is to be achieved. If $f, g : A \vdash B$ are diversified arrow terms of $M_\equiv$, then $Gf = Gg$. We also have that for $f, g : A \vdash B$, there are diversified arrow terms $f', g' : A' \vdash B'$ such that $f$ and $g$ are substitution instances of $f'$ and $g'$ respectively if and only if $Gf = Gg$.

For a headed factorized arrow term $f_n = \ldots = f_1$ of $M_\equiv$, whose factors are $f_1, \ldots, f_n$, we have that $Gf_i$, for $1 \leq i \leq n$, contains a crossing if and only if $f_i$ is an $s_{x,y}$-factor.

For $f : A \vdash B$ an arrow term of $M_\equiv$ we say that a cup in the diagram corresponding to $Gf$ covers an occurrence of $\land$ in $A$ when the ends of this cup are on different sides of this occurrence of $\land$. For example, in

$$
\begin{align*}
(t_{x,y,z} \land 1_{x\equiv u}) \land 1_{u\equiv v} & ((x \equiv y \land y \equiv z) \land x \equiv u) \land u \equiv v \\
(s_{x,z} \land 1_{x\equiv u}) \land 1_{u\equiv v} & ((x \equiv z \land x \equiv u) \land u \equiv v \\
t_{z,x,u} \land 1_{u\equiv v} & (z \equiv x \land x \equiv u) \land u \equiv v \\
& z \equiv u \land u \equiv v
\end{align*}
$$
the $y$-cup covers only the leftmost occurrence of $\wedge$ in $((x \equiv y \land y \equiv z) \land x \equiv u) \land u \equiv v$, and the $x$-cup covers the leftmost and middle occurrence of $\wedge$. The rightmost occurrence of $\wedge$ is uncovered; i.e., it is not covered by any cup.

Suppose now that for the headed factorized arrow term $f: A \vdash B$ of $M_\equiv$ we have that it is $r$-less, $\delta$-$\sigma$-less and diversified. Then there is an obvious one-to-one correspondence between occurrences of $\wedge$ in $B$ and uncovered occurrences of $\wedge$ in $A$. There is also an obvious one-to-one correspondence between the following sets:

- the set of $t_{x,y,z}$-factors of $f$,
- the set of cups in $Gf$,
- the set of variables occurring in $A$ and not in $B$,
- the set of occurrences of $\wedge$ in $A$ covered by a cup of $Gf$.

Note that all of these one-to-one correspondences that do not involve the first of these four sets do not depend on the arrow term $f$, but only on $Gf$.

An arrow term of $M_\equiv$ is called $s$-normal when for every pair of variables $(x, y)$ there is at most one occurrence of $s$ in this arrow term with the indices $x,y$ or $y,x$.

We can easily verify the following.

$s$-NORMALITY LEMMA. For every diversified arrow term $f$ of $M_\equiv$ there is a developed $s$-normal arrow term $f'$ of $M_\equiv$ such that $f = f'$ in $M_\equiv$. If $f$ is $r$-less, then $f'$ is $r$-less too.

This holds because, in a diversified developed arrow term, between two factors whose heads are $s_{x,y}$ or $s_{y,x}$ there can be no factor whose head is $t_{z,x,u}$ or $t_{z,y,u}$, which would be the only obstacle to bringing the two factors together, where they get cancelled.

In virtue of all that we have above it is enough to establish the following in order to prove coherence for $M_\equiv$.

AUXILIARY LEMMA. Suppose $f$ and $g$ are developed, $r$-less, $\delta$-$\sigma$-less, diversified and $s$-normal arrow terms of $M_\equiv$ of the same type. Then $f = g$ in $M_\equiv$.

Proof. We proceed by induction on the number $n$ of $s_{x,y}$-factors and $t_{z,u,v}$-factors in $f, g: A \vdash B$. This number must be the same in $f$ and $g$ because they are diversified and $s$-normal. (Note that $Gf = Gg$.) If $n = 0$, then we apply monoidal coherence. If $n > 0$, then there is in $B$ an atomic subformula $x \equiv y$ such that either (1) $y$ is in $A$ on the left-hand-side of $x$, or (2) $x$ is in $A$ on the left-hand side of $y$ and $x \equiv y$ is not a subformula of $A$.

In case (1) we have that

$$f = h \ast f' \quad \text{and} \quad g = h \ast g'$$

for an $s_{y,x}$-factor $h$, and we may apply the induction hypothesis to $f'$ and $g'$.
In case (2), we have that
\[ f = h \ast f' \]
for a \( t_{x,z,y} \)-factor \( h \). There must be a \( t_{u,z,v} \)-factor \( h' \) in \( g \). Note that the occurrence of \( \wedge \) in \( A \) corresponding to \( h \) is covered just by the cup of \( Gf \) corresponding to \( h \). This cup in \( Gg \), which is equal to \( Gf \), corresponds to \( h' \) in \( g \). The arrow term \( g \) is of the form
\[ g_m \ast \ldots \ast g_1 \ast h' \ast g' \]
for \( m \geq 0 \) (if \( m = 0 \), then we have just \( h' \ast g' \)); here \( g_1, \ldots, g_m \) are factors. We proceed by induction on \( m \) to show that \( g \) is equal to \( h'' \ast g'' \) for a \( t_{x',z,y'} \)-factor \( h'' \), which must be the same as \( h \), for reasons given in the preceding section. (We only replace \( \leq \) by \( \equiv \); moreover, “\( u_{2i-1} \leq u_{2i} \)” is replaced by “\( u_{2i-1} \equiv u_{2i} \) or \( u_{2i} \equiv u_{2i-1} \)” and \( \leq y \)” is replaced by \( \equiv y \) or \( y \equiv x \)). Note that there can be no factor in \( g_m \ast \ldots \ast g_1 \) whose head is \( s_{a,e} \), because, as we said above, the occurrence of \( \wedge \) in \( A \) corresponding to \( h \) is covered just by the cup of \( Gf \) corresponding to \( h \).

6 The coherence of \( S_\equiv \)

The category \( S_\equiv \) is defined like \( S_\leq \) save that, as when obtaining \( M_\equiv \) out of \( M_\leq \), the symbol \( \leq \) is replaced everywhere by \( \equiv \) and we have the additional primitive arrow terms \( s_{x,y} \) subject to the equation \((ss)\) and \((rs)\) of the preceding section, and the additional equation
\[ (ts) \quad s_{x,z} \ast t_{x,y,z} = t_{z,y,x} \ast (s_{y,z} \ast s_{x,y}) \ast c_{x\equiv y, y\equiv z}. \]

This equation is parallel to the following equation of groupoids (cf. Section 5): \((g \ast f)^{-1} = f^{-1} \ast g^{-1}\). It is analogous also to the equation \((\hat{c}w)\) of categories with coproducts (see [8], List of Equations, and cf. the end of Section 2 above).

Note that in the presence of \((ts)\) we can derive \((rt\delta)\) from \((rt\sigma)\), or vice versa. Here is a derivation of \((rt\delta)\) from \((rt\sigma)\):
\[
t_{x,y,z} \ast (1_{x \equiv y} \wedge r_y) = s_{y,z} \ast t_{y,y,z} \ast c_{y \equiv x, y \equiv y} \ast (s_{x,y} \ast s_{y,y}) \ast (1_{x \equiv y} \wedge r_y), \quad \text{with} \ (ts),
\]
\[
\equiv s_{y,z} \ast t_{y,y,z} \ast (r_y \wedge s_{x,y}) \ast c_{x \equiv y, \top}, \quad \text{with} \ (rs) \text{ and} \ (c \ \text{nat}),
\]
\[
\equiv s_{y,z} \ast \sigma_{y \equiv x} \ast (1_{\top} \wedge s_{x,y}) \ast c_{x \equiv y, \top}, \quad \text{with} \ (rt\sigma),
\]
\[
\equiv \delta_{x \equiv y}, \quad \text{with} \ (\sigma \ \text{nat}), \ (ss) \text{ and monoidal coherence.}
\]

The category \( S_\equiv \) is a symmetric monoidal category in which \( \equiv \) corresponds to an equivalence relation. Since the means of expression of \( S_\equiv \) are limited, this equivalence relation is an equality relation.

We define the functor \( G \) from \( S_\equiv \) to \( Br \) by combining what we had for \( G \) from \( M_\equiv \) to \( Br \) and for \( G \) from \( S_\leq \) to \( Br \). Since for the equation \((ts)\) we have
we can conclude that $G$ is indeed a functor.

To prove the faithfulness of $G$, i.e. coherence for $S_{\equiv}$, we proceed in principle as for $S_{\leq}$ in Section 4. Now that we have the equation $(ts)$, we can permute freely $t_{x,y,z}$-factors with $s_{x,z}$-factors, and eschew all the complications we had with $M_{\equiv}$ in the preceding section.

7 Preorder, equivalence and adjunction

In this section we will show that assumptions concerning $\leq$ and $\equiv$ in categories of the preceding five sections amount to assumptions about some adjoint situations. This matter is related to matters considered in [14].

Let $M_{\leq y}$ be the full subcategory of $M_{\leq}$ in whose objects a particular variable $y$ does not occur, and let $M_{\leq z}$ be the full subcategory of $M_{\leq}$ whose objects are of the form $y \leq u \land A$ for $y$ distinct from $u$ and not occurring in $A$. Note that if the generating set $\mathcal{V}$ of variables is infinite, then $M_{\leq y}$ and $M_{\leq}$ are isomorphic categories.

For every variable $z$ distinct from $y$ there is a functor $F^z$ from $M_{\leq y}$ to $M_{\leq z}$ defined as follows:

$$F^z A = df \ y \leq z \land A,$$

$$F^z f = df \ 1_{y \leq z} \land f.$$  

Conversely, there is a functor $G^z$ from $M_{\leq z}$ to $M_{\leq y}$ where

$$G^z(y \leq u \land A) = df \ z \leq u \land A,$$

and $G^z f$ is obtained from the arrow term $f$ by substituting $z$ for $y$, i.e. by uniformly replacing $y$ by $z$. (The function $G^z$ on objects is also substitution of $z$ for $y$, since $y$ is distinct from $u$ and does not occur in $A$.)

Let the arrow $\gamma_A^Z : A \vdash z \leq z \land A$ of $M_{\leq y}$, whose target is $G^z F^z A$, be defined by

$$\gamma_A^Z = df \ (r_z \land 1_A) \circ s_{A, A},$$
and let the arrow
\[ \varphi^z_{y \leq u \land A} : y \leq z \land (z \leq u \land A) \vdash y \leq u \land A \]
of \( M^y_x \), whose source is \( F^x G^x(y \leq u \land A) \), be defined by
\[ \varphi^z_{y \leq u \land A} = df \ (t_{y,z,u} \land 1_A) \circ b_{y \leq z,z \leq u,A}^{-1}. \]

Then we can verify easily by appealing to coherence for \( M \leq y \) that the functor \( F^x \) is left adjoint to \( G^x \); in this adjunction \( \gamma^z \) is the unit natural transformation, and \( \varphi^z \) the counit natural transformation (see [16], Section IV.1).

The “straightening of a sinuosity” involved in the equations \((rt\delta)\) and \((rt\sigma)\) (see the diagrams of Section 3) indicated that we have such an adjunction (cf. [3], Section 4.10, [9], Section 2.3, and [4], Section 7).

The arrow \( \gamma^z_x \) was defined in terms of \( r^z \), but in \( M \leq y \) we can define \( r^z \) in terms of \( \gamma^z \) as follows:
\[ r^z = df \ \delta^z_{z \leq z} \circ \gamma^z. \]

Analogously, the arrow \( \varphi^z_{y \leq u \land A} \) was defined in terms of \( t_{y,z,u} \), but we can define \( t_{v,z,u} \) in \( M \leq y \) in terms of \( \varphi^z \) as follows. Note first that in \( M \leq y \) we can take
\[ t_{y,z,u} = df \ \delta^z_{y \leq u} \circ \varphi^z_{y \leq u \land T} \circ (1_{y \leq z} \land \delta^z_{z \leq u}) \]
for \( y \) different from \( z \); in \( M \leq y \) we take
\[ t_{v,z,u} = df \ \delta^z_{v \leq u} \circ G^v \varphi^z_{y \leq u \land T} \circ (1_{v \leq z} \land \delta^z_{z \leq u}). \]

Then the specific equations of \( M \leq y \) can be derived from the assumption that we have the adjunction above between \( M \leq y \) and \( M \leq y \), together with the equations
\[ \gamma^z_x = ((\delta^z_{z \leq z} \circ \gamma^z) \land 1_A) \circ \sigma^z_{A}, \]
\[ \varphi^z_{y \leq u \land A} = ((\delta^z_{y \leq u} \circ \varphi^z_{y \leq u \land T} \circ (1_{y \leq z} \land \delta^z_{z \leq u})) \land 1_A) \circ b_{y \leq z,z \leq u,A}^{-1}. \]

These equations are obtained from the definition of \( \gamma^z_x \) in terms of \( r^z \) and the definition of \( \varphi^z_{y \leq u \land A} \) in terms of \( t_{y,z,u} \). They give a definition of \( \gamma^z_x \) for an arbitrary \( A \) in terms of \( \gamma^z \), and a definition of \( \varphi^z_{y \leq u \land A} \) for an arbitrary \( A \) in terms of \( \varphi^z_{y \leq u \land T} \).

Note that a category equivalent to \( M\leq y \) is the full subcategory of \( M \leq y \) whose objects have a single occurrence of \( y \) as the leftmost variable.

The adjunction we had above between \( M \leq y \) and \( M \leq y \) is obtained also when \( M \leq y \) is replaced by \( S \leq y \), \( M \equiv \) and \( S \equiv \). With \( M \equiv \) we can take instead of the category \( M \equiv \leq y \), defined like \( M \equiv \leq y \), the equivalent full subcategory \( M \equiv \leq y \) of \( M \equiv \leq y \).
whose objects are those of the form \( y \equiv u \land A \) or \( u \equiv y \land A \) for \( y \) distinct from \( u \) and not occurring in \( A \). We obtain an adjunction between \( M_{\leq} y \), defined like \( M_{\leq} y \) starting from \( M_{\equiv} \), and \( M_{\equiv y} \).

With the categories \( S_{\equiv y} \) and \( S_{\equiv y}^{*} \), defined analogously starting from \( S_{\equiv} \), an analogous adjunction obtains. The category \( S_{\equiv y}^{*} \) is equivalent to the full subcategory of \( S_{\equiv} \) in whose objects \( y \) occurs exactly once.

8 The coherence of \( \dot{S_{\leq}} \)

Let us now suppose that terms are built with the help of a symbol \( \cdot \) that stands for a binary operation. We suppose, namely, that terms are not only variables, but for \( t_1 \) and \( t_2 \) terms we have that \( t_1 \cdot t_2 \) is a term. We use \( t, s, r, \ldots \), also with indices, for terms. To define atomic formulae we suppose that if \( t_1 \) and \( t_2 \) are terms, then \( t_1 \leq t_2 \) is an atomic formula, as well as \( \top \). Formulae are defined otherwise as in Section 2.

The objects of the category \( \dot{S_{\leq}} \) are these new formulae; otherwise, \( \dot{S_{\leq}} \) is defined like \( S_{\leq} \) in Section 3 with the additional primitive arrow terms

\[
\begin{align*}
\alpha_{t_1, t_2, t_3, t_4} : t_1 \leq t_2 & \land t_3 \leq t_4 \vdash t_1 \cdot t_3 \leq t_2 \cdot t_4
\end{align*}
\]

for all terms \( t_1, t_2, t_3 \) and \( t_4 \), which are subject to the following additional equations:

\[
\begin{align*}
(ra) \quad a_{t_1, t_2, s_1, s_3} \cdot (r_{t_1} \land r_{s_3}) \cdot \delta_{t_1} = r_{t_1, s_1, s_3},
\end{align*}
\]

for \( c_{A, B, C, D} = df b_{A, B, C, D}^{\rightarrow} \cdot (1_A \land (b_{C, B, D}^\land \cdot (C_{B, C}^\land \cdot 1_D) \cdot b_{B, C, D}^{\rightarrow})) \cdot b_{A, B, C \land D}^{\rightarrow}

(A \land B) \land (C \land D) \vdash (A \land C) \land (B \land D),

(ta) \quad \alpha_{t_1, r_1, s_1, r_2, s_2, t_2, t_3, t_4} = \alpha_{t_1, t_2, s_1, t_3, r_1, r_2} = \alpha_{t_1, t_2, s_1, s_2, r_1, s_2, r_2} \cdot c_{t_1 \leq s_1, s_2 \leq r_1, t_2 \leq s_2, r_2 \leq r_2}.
\]

The equation \( ra \) is parallel to the equation \((\land 1)\) of Section 2, and \( ta \) is parallel in the same manner to the equation \((\land 2)\). In another manner, the equation \( ta \) is analogous to the equation \((\check{b} \check{c} \hat{a})\) of \([8]\) (see the List of Equations).

We define the functor \( G \) from \( \dot{S_{\leq}} \) to the category \( Br \) by extending the definition of \( G \) from \( S_{\leq} \) to \( Br \) with a clause corresponding to the following diagram:

\[
\begin{align*}
\alpha_{t_1, t_2, t_3, t_4} : t_1 \leq t_2 \land t_3 \leq t_4 & \vdash t_1 \cdot t_3 \leq t_2 \cdot t_4
\end{align*}
\]

(\text{where each line stands for a family of parallel lines; cf. Section 3). To prove coherence for \( \dot{S_{\leq}} \) we use essentially the equation}
and another analogous equation with $t_{t_1, r,s_1} \wedge 1_{t_2} \leq s_2$ on the left-hand side replaced by $1_{t_1} \leq r \wedge t_{t_2, r,s_2}$; we proceed otherwise using the ideas indicated in Section 3. This proof of coherence is parallel to the proof of the Development Lemma of Section 3; it formalizes the proof of this lemma on a different level.

When we take the category $\tilde{S}_{\equiv}$ obtained from $S_{\equiv}$ as $\tilde{S}_{\leq}$ was obtained from $S_{\leq}$, with the additional equation

$$s_{t_1,t_2,s_1} \cdot a_{t_1,t_2,s_2} = a_{t_1,t_1,s_1} \cdot (s_{t_1,t_1} \cdot s_{t_2,s_2}),$$

which is parallel to $(f \wedge g)^{-1} = f^{-1} \wedge g^{-1}$, we can prove coherence analogously. In $\tilde{S}_{\equiv}$ the relation corresponding to $\equiv$ is a congruence relation, which, due to the scarcity of the means of expression of $\tilde{S}_{\equiv}$, is an equality relation.

To obtain coherence results for various categories extending $\tilde{S}_{\leq}$, which would formalize fragments of the equational theory of semigroups, or of the equational theory of monoids, commutative or not, we would need additional arrows analogous to the arrows $b_{A,B,C}^\rightarrow$, $b_{A,B,C}^\leftarrow$, $\delta_{A}^\rightarrow$, $\delta_{A}^\leftarrow$, $\sigma_{A}^\rightarrow$, $\sigma_{A}^\leftarrow$ and $b_{A,B}$. For example, an arrow of type $\top \vdash t_1 \cdot (t_2 \cdot t_3) \equiv (t_1 \cdot t_2) \cdot t_3$ would correspond to $b^\rightarrow$. The additional equations for these new arrows would be parallel to the equations of monoidal or symmetric monoidal categories.

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