Research Article

Local Well-Posedness to the Cauchy Problem for an Equation of the Nagumo Type

Vladimir Lizarazo, Richard De la cruz, and Julio Lizarazo

1School of Geological Engineering, Universidad Pedagógica y Tecnológica de Colombia, Calle 4 Sur 15-134, Sogamoso, Colombia
2School of Mathematics and Statistics, Universidad Pedagógica y Tecnológica de Colombia, Av. Central Del Norte 39-115, Tunja, Colombia

Correspondence should be addressed to Julio Lizarazo; julio.lizarazo@uptc.edu.co

Received 30 December 2021; Accepted 16 April 2022; Published 27 May 2022

Copyright © 2022 Vladimir Lizarazo et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In this paper, we show the local well-posedness for the Cauchy problem for the equation of the Nagumo type (1) in the Sobolev spaces $H^s(\mathbb{R})$. If $D > 0$, the local well-posedness is given for $s > 1/2$ and for $s > 3/2$ if $D = 0$.

1. Introduction

In this paper, we show the local well-posedness for the following Cauchy problem:

$$
\begin{align*}
\frac{\partial u}{\partial t} &= D \frac{\partial^2 u}{\partial x^2} - u (u - \alpha) (u - 1) - \epsilon u \frac{\partial u}{\partial x}, & x \in \mathbb{R}, t > 0, \\
    u(x, 0) &= \psi(x),
\end{align*}
$$

(1)

where $D > 0$ is a constant diffusion coefficient, $\alpha \in (0, 1/2)$ and $\epsilon > 0$ is a small positive quantity. In [1], the equation (1) was used to model chemotaxis (see equation (55) in [1]). Organisms which use chemotaxis to locate food sources include amoebae of the cellular slime mold Dictyostelium discoideum, and the motile bacterium Escherichia coli [1]. Therefore, $u = u(x, t)$ models the population density, $n$ is a positive integer, and $\alpha$ is a parameter which determines the minimal required density for a population to be able to survive (for normalized population density, i.e., such that $u = 1$ is the maximum sustainable population). Balasuriya and Gottwald [1] studied the wave speed of travelling waves for the equation (1). Also, they have the numerical evidence for the wave speed of travelling waves for the equation (1). Other results related to the equation (1) can be found in [2].

When $\epsilon = 0$, the equation (1) is called a Nagumo equation or bistable equation [3–7] in which case the model describes an active pulse transmission line simulating a nerve axon. Also, we can see the equation (1) as a generalized viscous Burgers equation with a source term. Dix [8] proved local well-posedness of the viscous Burgers equation with a source term using a contraction mapping argument. Moreover, for the classical Burgers equation (without viscosity) it is well known that classical solutions cannot exist for all time, but weak global solutions can be established [9]. In addition, the uniqueness of the weak solution depends on some entropy condition. Observe that when $D = 0$, the equation (1) is a generalized Burgers equation (without viscosity) and nonlinear source term. Therefore, from the mathematical viewpoint, the case $D = 0$ is very interesting to study the existence and uniqueness of classical solution.

In this paper, we show the local well-posedness for the Cauchy problem to the equation of the Nagumo type (1) in the Sobolev spaces $H^s(\mathbb{R})$ for $s > 1/2$ if $D > 0$, and for $s > 3/2$ if $D = 0$. Our proof of local well-posedness is based on the results given in [10–12]. We use the Banach fixed point in a suitable complete space to guarantee the existence of local solutions to the problem (1) with $D > 0$. The Banach fixed point technique has been widely used to show existence and uniqueness of solutions to differential equations in Banach
spaces (for instance, see [10–14] for more details). When \( D = 0 \), we use the parabolic regularization method to show local well-posedness for the Cauchy problem (1) (e.g., [12,15]).

We will use the following notation: \( \mathbb{R} \) for the real numbers; \( \mathcal{S}(\mathbb{R}) \) for the Schwartz’s space usual; \( \hat{f} \) denotes the Fourier transform of \( f \); the inverse Fourier transform will be denoted by \( \check{f} \); by \( H^s(\mathbb{R}) \), \( s \in \mathbb{R} \), the set of all \( f \in \mathcal{S}'(\mathbb{R}) \) such that \( (1 + \xi^2)^{s/2} \hat{f} \in L^2(\mathbb{R}) \). \( H^s(\mathbb{R}) \) is called the Sobolev space and it is a Hilbert space with respect to the inner product \( \langle f, g \rangle = \int_\mathbb{R} (1 + \xi^2)^{s/2} \hat{f} \overline{\hat{g}} \, d\xi \); \( C(I; X) \) for the space of all continuous functions on an interval \( I \) into the Banach space \( X \); if \( I \) is compact, \( C(I; X) \) is seen as a Banach space with the sup norm; \( C_w(I; X) \) for the space of all weakly continuous functions on an interval \( I \) into Banach space \( X \); \( C^{\infty}_w(I; X) \) for the space of all weakly differentiable functions on an interval \( I \) into Banach space \( X \). We also denote by \( V(t) = e^{i(D^2 - \alpha l)d)}t, t \geq 0 \), the semigroup in \( H^s(\mathbb{R}) \) generated by the operator \( tQ \) where \( Q = (D^2 - \alpha l) \), i.e.,

\[
V(t)f = \left( e^{-t(D^2 + \alpha l)}f \right)^\vee, \quad f \in H^s(\mathbb{R}), t \geq 0,
\]

\( \{V(t)\}_{t \geq 0} \) is a \( C^0 \)-semigroup of contractions in \( H^s(\mathbb{R}), s \in \mathbb{R} \). Moreover, \( u(x, t) = V(t) \psi(x) \) is the unique solution to the linear problem associated with (1), i.e., \( u(x, t) = V(t) \psi(x) \) is the unique solution to the following problem.

\[
\begin{cases}
\frac{\partial u}{\partial t} = D_uxx - au, & x \in \mathbb{R}, t > 0, \\
u(x, 0) = \psi(x).
\end{cases}
\]

**Proposition 1.** Let \( \psi \in H^s(\mathbb{R}), s \in \mathbb{R}, \lambda \geq 0, D > 0 \) and \( t > 0 \). Then, there exists a constant \( C_\lambda, \lambda > 0 \), depending only on \( \lambda \), such that

\[
\|V(t)\psi\|_{s+\lambda} \leq C_\lambda \left( 1 + \left( \frac{\lambda}{2Dt} \right)^\lambda \right)^{1/2} \|\psi\|_{s},
\]

In particular, \( V(t)\psi \in \mathcal{S}(\mathbb{R}) \) for all \( t > 0 \).

When there is no risk of confusion, we will use the notations \( u(t) \) for \( u(x, t) \), \( \phi \) for \( \psi(x) \), and \( F(u) = etu'u_x + u^3 - (\alpha + 1)u^2 \).

**2. Local Well-Posedness of the Problem (1) with \( D > 0 \)**

In this section, we use the Banach fixed point in a suitable complete metric space to show the existence of local solutions for integral equation (9) in Sobolev space \( H^s(\mathbb{R}) \) for \( s > 1/2 \). In addition, the uniqueness of the solution and continuous dependence are established.

**Proposition 2.** Let \( s > 1/2 \) be fixed. Then, \( F(u) \) is a continuous map from \( H^s(\mathbb{R}) \) into \( H^{s-1}(\mathbb{R}) \) and satisfies the estimates as follows:

\[
\|F(u) - F(w)\|_{s-1} \leq L_s(\|u\|_{s}, \|w\|_{s})\|u - w\|_{s},
\]

for all \( u, v \in H^s(\mathbb{R}), \) where \( L_s(\cdot, \cdot, \cdot) \) is a continuous function, nonincreasing with respect to each of their arguments. In particular,

\[
\|F(u)\|_{s-1} \leq L_s(\|u\|_{s}, 0)\|u\|_{s}.
\]

**Proof.** Observe that \( F(u) = (\epsilon/n + 1)(u^{n+1})_x + u^3 - (\alpha + 1)u^2 \). Then, as \( H^s(\mathbb{R}) \) is a Banach algebra for \( s > 1/2 \), we have the following:

\[
\|F(u) - F(w)\|_{s-1} \leq \frac{\epsilon}{n + 1} \left\| \partial_x(u^{n+1} - w^{n+1}) \right\|_{s-1} + \left\| u^3 - w^3 \right\|_{s-1} + (\alpha + 1)\|u^2 - w^2\|_{s-1}
\]

\[
\leq \frac{\epsilon}{n + 1} \left\| u^{n+1} - w^{n+1} \right\|_{s} + \left\| u^3 - w^3 \right\|_{s} + (\alpha + 1)\|u^2 - w^2\|_{s}
\]

\[
= L_s(\|u\|_{s}, \|w\|_{s})\|u - w\|_{s},
\]

where

\[
L_s(\|u\|_{s}, \|w\|_{s}) = \frac{\epsilon}{n + 1} \sum_{k=0}^{n} \|\partial_x^{n-k}u\|_{s}^k + 2 \sum_{k=0}^{2} \|\partial_x^{n-k}u\|_{s}^k + (\alpha + 1)(\|u\|_{s} + \|w\|_{s}).
\]

**Proposition 3.** Let \( D > 0 \) be fixed, \( s > 1/2 \), \( \psi \in H^s(\mathbb{R}) \), and \( V(t) \) is defined by (2). Then, there exists \( T = T(\|\psi\|_{s}, M) > 0 \) and a unique function \( u \in C([0, T]; H^s(\mathbb{R})) \) satisfying the following integral equation:
\[ u(t) = V(t)\psi(t) - \int_{0}^{t} V(t - \tau)F(u(\tau))d\tau. \]  

**Sketch of proof.** Let \( M, T > 0 \) be fixed, but arbitrary. Consider the following:

\[ \mathcal{X}(M, T, \psi) = \left\{ u \in C([0, T]; H^s(\mathbb{R})) : \sup_{[0, T]} \| u(t) - V(t)\psi(t) \|_{s} \leq M \right\} , \]

which is a complete metric space with distance \( d(u, v) = \sup_{[0, T]} \| u(t) - v(t) \|_{s} \). Define on the space \( \mathcal{X}(M, T, \psi) \) the following map:

\[ (\mathcal{A}g)(t) := V(t)\psi(t) - \int_{0}^{t} V(t - \tau)F(g(\tau))d\tau. \]

We have the following:

1. If \( g \in \mathcal{X}(M, T, \psi) \) then \( \mathcal{A}g \in \mathcal{X}(M, T, \psi) \).
2. We can choose \( \bar{T} > 0 \) sufficiently small such that \( \mathcal{A}(\mathcal{X}(\bar{M}, \bar{T}, \psi)) \subset \mathcal{X}(M, \bar{T}, \psi) \).
3. There exists \( \bar{T} \in (0, T] \) such that \( \mathcal{A} \) is a contraction on \( \mathcal{X}(M, \bar{T}, \psi) \).

So, \( \mathcal{A} \) has a unique fixed point \( u \in \mathcal{X}(M, \bar{T}, \psi) \) which satisfies the integral equation (40) where \( \bar{T} = T(\|\psi\|_{s}, M) > 0 \).

**Proposition 4.** The problem (1) is equivalent to the integral equation (40). More precisely, if \( s > 1/2 \) and \( u \in C([0, T]; H^s(\mathbb{R})) \cap C^1((0, T]; H^{s-2}(\mathbb{R})) \) is a solution of (1), then \( u \) satisfies the integral equation (40). Conversely, if \( s > 1/2 \) and \( u \in C([0, T]; H^s(\mathbb{R})) \) is a solution of (40) then \( u \in C^1([0, T]; H^{s-2}(\mathbb{R})) \) and satisfies (1).

**proof.** Assume that \( u \in C([0, T]; H^s(\mathbb{R})) \cap C^1((0, T]; H^{s-2}(\mathbb{R})) \) is a solution of (1). Then, \( (d/d\tau)(V(t - \tau)u(\tau)) = -V(t - \tau)F(u(\tau)), 0 < \tau < t \). So, \( u \) satisfies the integral equation (40). Conversely, assume that \( u \in C([0, T]; H^s(\mathbb{R})) \) is a solution of (40). For \( t > 0 \), let \( \eta(t) := -\int_{0}^{t} V(t - \tau)F(u(\tau))d\tau \). Then, for \( h > 0 \) arbitrary,

\[
\begin{align*}
\left\| \eta(t + h) - \eta(t) / h - Q\eta(t) + F(u(t)) \right\|_{L^2} & \leq \int_{0}^{t} \left\| V(t - \tau)(V(h) - 1 / h - Q)F(u(\tau)) \right\|_{L^2} d\tau \\
& = \frac{1}{h} \int_{t}^{t + h} \left\| V(t - \tau)F(u(\tau)) - F(u(t)) \right\|_{L^2} d\tau.
\end{align*}
\]

However,

\[
\begin{align*}
\left\| V(t - \tau)(V(h) - 1 / h - Q)F(u(\tau)) \right\|_{L^2} & \leq C_1 \left( \frac{2D(t - \tau) + 1}{2D(t - \tau)} \right)^{1/2} \left\| V(h) - 1 / h - Q \right\|_{L^2} K\|F(u(\tau))\|_{L^2} \\
& \leq C_1 \left( \frac{2D(t - \tau) + 1}{2D(t - \tau)} \right)^{1/2} K\|F(u(\tau))\|_{L^2} \\
& \leq C_1 \left( \frac{2D(t - \tau) + 1}{2D(t - \tau)} \right)^{1/2} KL\left( \sup_{\tau \in [0, T]} \| u(\tau) \|_{L^2} \right) \sup_{\tau \in [0, T]} \| u(\tau) \|_{L^2},
\end{align*}
\]

and the right hand side of (57) is an integrable function of \( \tau \) in \([0, t]\). Thus, using the dominated convergence theorem, we have as follows:

\[
\lim_{h \to 0} \int_{0}^{t} \left\| V(t - \tau)(V(h) - 1 / h - Q)F(u(\tau)) \right\|_{L^2} d\tau = 0.
\]
Now, from the mean value theorem for integrals, there exists a value \( c \) on the interval \( (t, t + h) \) such that

\[
\begin{align*}
\frac{1}{h} \int_{t}^{t+h} \| V(t + h - \tau)F(u(\tau)) - F(u(t)) \|_{r - 2} d\tau &= \| V(t + h - c)F(u(c)) - F(u(t)) \|_{r - 2} \\
&= \frac{1}{h} \int_{t}^{t+h} \| V(t + h - \tau)F(u(\tau)) - F(u(t)) \|_{r - 2} d\tau = 0.
\end{align*}
\]

and therefore, \( \lim_{h \to 0} \frac{1}{h} \int_{t}^{t+h} \| V(t + h - \tau)F(u(\tau)) - F(u(t)) \|_{r - 2} d\tau = 0. \)

After, \( \delta^+_t \eta(t) = Q_t(t) - F(u(t)) \) in \( H^{s - 2} (\mathbb{R}) \), where \( \delta^+_t \eta(t) \) is the right derivative. In similar way, we can conclude that the left derivative is \( \delta^-_t \eta(t) = Q_t(t) - F(u(t)) \) in \( H^{s - 2} (\mathbb{R}) \). So, \( \eta \in C^1 ((0, T]; H^{s - 2} (\mathbb{R})) \) and \( \partial_t \eta(t) = Q_t(t) - F(u(t)) \). As \( \delta^+_t \eta(x) \) is the solution of the linear problem (3), we conclude that \( u(t) = V(t) \psi + \eta(t) \in C^1 ((0, T]; H^{s - 2} (\mathbb{R})) \) and satisfies (1).}

Lemma 1. Suppose \( \beta > 0, \gamma > 0, \beta + \gamma > 1, a \geq 0, b \geq 0, u \) is nonnegative and \( t^{-r} u(t) \) is locally integrable on \( [0, T] \). If
\[
u(t) \leq a + b \int_0^1 (t - s)^{\beta - 1} s^{\gamma - 1} u(s) ds,
\]
e.i., in \( (0, T) \), then
\[
u(t) \leq a \beta \Gamma(\beta) \bigg( (b \Gamma(\beta))^{1/\beta} \bigg).
\]
where \( \nu = \beta + \gamma - 1 > 0, E_{\beta, \gamma} (s) = \sum_{m=0}^{\infty} c_m s^m \) with \( c_0 = 1 \)
and \( c_{m+1}/c_m = \Gamma(\nu + \gamma)/\Gamma(\nu + \beta) \) for \( m \geq 0. \)

The proof of this lemma is given in Lemma 7.1.2 in [16].

Proposition 5. Let \( \psi, \phi \in H^s (\mathbb{R}) \) and \( u, v \in C ([0, T]; H^s (\mathbb{R})) \) be the corresponding solutions of equation (9). If \( s > 1/2, \)
\[
u(t) - v(t) \leq K \nu(t),
\]
where \( K = (1/2) ( (b \Gamma(1/2))^{1/2} T, \) \( b = L_c (L_c, L) C_1 (\sqrt{2 \Gamma(1/2)} + 1/\sqrt{2 D}) \) and \( L = \max \{ \sup_{[0,T]} \| u \|_s, \sup_{[0,T]} \| v \|_s \} \) (here \( E_{1/2,1} \) is given by previous lemma).

proof. Let \( \psi, \phi, u \) and \( v \) as in the statement of the proposition. Let \( s > 1/2 \). From (9) we have as follows:
\[
u(t) - v(t) = V(t) (\psi - \phi) - \int_0^t \nu(t) F(u(t)) - F(v(t)) dt.
\]
By Propositions 1 and 2, we obtain the following:
\[
u(t) - v(t) \leq K \nu(t),
\]
where \( L = \max \{ \sup_{[0,T]} \| u \|_s, \sup_{[0,T]} \| v \|_s \} \). Let \( b = L_c (L_c, L) C_1 (\sqrt{2 \Gamma(1/2)} + 1/\sqrt{2 D}) \). Observe that \( E_{1/2,1} ( (b \Gamma(1/2))^{1/2} T) \) is finite. In fact, \( E_{1/2,1} ( (b \Gamma(1/2))^{1/2} T) = \sum_{m=0}^{\infty} a_m \) where \( a_m = c_m (b^2 \pi T)^{m/2} \). Therefore, \( (a_{m+1}/a_m) = (c_{m+1}/c_m) (b^2 \pi T)^{1/2} \) and from Lemma 1 we have that
\[
\frac{c_{m+1}}{c_m} = \Gamma((m/2) + 1) / \Gamma((m + 1/2) + 1) = \frac{m \Gamma(m/2)}{(m + 1) \Gamma(m + 1/2)}
\]
As for all \( x > 0, \Gamma(x) = \sqrt{2 \pi x} e^{-(x^2)/2} \) with \( 0 < \theta < 1, \) we have that \( 1 < e^{\theta(x)} < e \) and \( e^{(m/2) \Gamma(m/2) - (m+1) \Gamma(m+1/2)} \) is bounded for \( m \geq 1. \) From (21), we obtain as follows:
\[
\frac{c_{m+1}}{c_m} = m + 1 \left( \frac{m/2}{m+1} \right)^{m+1/2 - 1} \left( \frac{m}{m+1} \right)^{m+1/2} e^{-1/2} e^{(6/m)(\theta m) - (6/m)(\theta m + 1/2)} \longrightarrow 0, \quad \text{as } m \longrightarrow \infty. \tag{22}
\]

3. Local Well-Posedness of the Problem (1) with \(D = 0\)

In this section, we show the local well-posedness of the problem (1) with \(D = 0\) using a priori estimate and the parabolic regularization method, the so-called vanishing viscosity method (for more details see [12]).

**Lemma 2.** Let \(\eta(t), a(t)\) and \(b(t)\) be real valued positive continuous functions defined on \([0, T] \subseteq [0, \infty)\). Let \(G(r)\) and \(H(r)\) be positive continuous functions for \(r \geq 0\), with \(G\) strictly increasing and \(H\) nondecreasing. Define \(A(t) = \sup_{0 \leq s \leq t} a(s)\) and \(B(t) = \sup_{0 \leq s \leq t} b(s)\). Then, the inequality

\[
G(\eta(t)) \leq a(t) + b(t) \int_0^t H(\eta(\tau)) d\tau, \quad 0 \leq t < T
\]

implies the inequality

\[
\eta(t) \leq G^{-1}(\Omega^{-1}(\Omega(A(t) + tB(t))), \quad 0 \leq t \leq T, \tag{27}
\]

where \(\Omega(r) = \int_0^r (dk/H(G^{-1}(\zeta)))\), \(0 < r \geq 0\), and \(T = \sup\{\tau \in [0, T]: \Omega(A(\tau) + \tau B(\tau) (\lim_{m \to \infty} G(r)))\}.

**Proof.** This is a particular case of the theorem given in [17] [pp. 78].

**Proposition 7.** Let \(s > 3/2\) be fixed. Then, \(F(u)\) satisfies the estimate

\[
\|u - w, F(u) - F(w)\|_0 \leq L_0(\|u\|_s, \|w\|_s) \|u - w\|_0, \tag{28}
\]

for all \(u, w \in H^s(\mathbb{R})\), where \(L_0(x, y) = c\sum_{k=0}^{s-1} x^k y^{s-1-k} + (en/2) x_2 + x y + y^2 + (\alpha + 1)(x + y).

**Proof.** We define \(q(u, w) = \sum_{k=0}^{s-1} x^k y^{s-1-k}. As s > 3/2 thus \(H^s(\mathbb{R})\) and \(H^{s-1}(\mathbb{R})\) are Banach algebras. Moreover, we have that \(H'(\mathbb{R}) \to H^{s-1}(\mathbb{R})\) and \(H'(\mathbb{R}) \to L^\infty(\mathbb{R})\). Thus, using the Cauchy–Schwartz inequality, we have as follows:

Theorem 1. Let \(s > 1/2\). The problem (1) is locally well-posed in \(H^s(\mathbb{R})\).
\[(u - w|F(u) - F(w))| \leq |(u - w(u^n - w^n))u, \xi| + |w - w(u - w)|0
\]
\[+ \left| (u - w(u^n - w^n)) \right| + (\alpha + 1) \left| (u - w(u^n - w^n)) \right| \]
\[\leq 0 \leq \left( (u - w)^2 \right) \cdot \left| (u - w)^2 \right| + \left( u + w + w^2 \right)_{L^\infty} + (\alpha + 1) \left| u + w \right|_{L^\infty}
\]
\[
||u - w||_0^2 \leq L_0(||u||_s, ||v||_s)||u - w||_s.
\]

**Lemma 3.** (T. Kato). Let \( r \geq 1 \) and \( s > 3/2 \) be fixed and \( h, v \) are real valued functions. Then, there exists a constant \( C = C(r, s) \) such that
\[
|((v, h, \partial_x v))| \leq C(||\partial_x h||_{s-1}||v||_s^2 + ||\partial_x h||_{s-1}||v||_s). \quad (30)
\]

In particular, \((v, h, \partial_x)\) \( \leq C||\partial_x h||_{s-1}||v||_s^2\).

**proof.** See Lemma A.5. in [13].

**Theorem 2.** Let \( s > 3/2 \) be fixed. For \( D > 0 \), consider the initial value problem (1) with initial data + and let \( u_D \in C([0, T^*_s]; H^s(\mathbb{R})) \) be the corresponding solution of (1) for some \( T^*_s > 0 \). Then, there exists a \( T_D^* = T_D(\psi) > 0 \), depending on \( ||\psi||_s \), such that \( u_D \) can be extended to the interval \([0, T_D^*(\psi)]\), and there is a function \( \rho \in C([0, T_D^*(\psi)]; [0, +\infty)) \) such that \( \rho(0) = ||\psi||_s^2 \) and \( ||u_D(t)||_s^2 \leq \rho(t) \), for all \( t \in [0, T_D^*(\psi)]. \)

**proof.** Using the inner product in \( H^s(\mathbb{R}) \) and Lemma 3 we have that
\[
\partial_t ||u_D||_s^2 = 2(u_D|\partial_t u_D|) + 2(u_D|D^2 u_D - u_D (u_D - \alpha)(u_D - 1) - e u_D^\alpha|) + (u_D|D^2 u_D - u_D (u_D - \alpha)(u_D - 1)|)
\]
\[= 2(u_D|D^2 u_D - u_D (u_D - \alpha)(u_D - 1)|) + (u_D|D^2 u_D - u_D (u_D - \alpha)(u_D - 1)|)
\]
\[= 2(u_D|D^2 u_D - u_D (u_D - \alpha)(u_D - 1)|) + (u_D|D^2 u_D - u_D (u_D - \alpha)(u_D - 1)|)
\]
\[\leq 2C_1(||u_D||_s + ||u_D||_s^2 + ||\partial_x u_D||_{s-1}||u_D||_s^2)
\]
\[\leq 2C_1(||u_D||_s^4 + (\alpha + 1)||u_D||_s^4 + ||D^2 u_D||_{s+2}^2), \quad \text{for all } t \in (0, T_D^*(D, \psi)).
\]

Then, \( ||u_D(t)||_s^2 \leq \rho(t) \) for all \( t \in [0, T_D^*] \), where \( \rho \in C([0, T^*_s]; [0, +\infty)) \) is the maximally extended solution of the following problem.
\[
\begin{aligned}
\frac{dp(t)}{dt} &= 2C_s \left( \rho^2 + (\alpha + 1)\rho^{3/2} + \epsilon \rho^{(n+2)/2} \right), \\
\rho(0) &= \|\psi\|^2.
\end{aligned}
\]  
\tag{32}

and integrating from 0 to \( t \) we have as follows:

\[
\rho(t) \leq \|\psi\|^2 + 2C_s (\alpha + \epsilon + 2)t + 8C_s \int_0^t \rho^{(n+2)/2}(r)dr, \quad 0 \leq t < T^*.
\]  
\tag{34}

For \( n \geq 2 \), from the problem (32) we obtain as follows:

\[
\rho(t) \leq \|\psi\|^2 + 2C_s (\alpha + \epsilon + 2)t + 8C_s \int_0^t \rho^{(n+2)/2}(r)dr, \quad 0 \leq t < T^*.
\]  
\tag{37}

\[
u_0(0) = \psi, \text{ and } \nu_0 \text{ satisfies (1) with } D = 0, \text{ in the weak sense, i.e.,}
\]

\[
\frac{d}{dt}(\nu_0(t)|\varphi|)_{s-2} = \left(-au_0 - cu_0^2 \partial_x u_0 - u^3 + (\alpha + 1)u_0^2|\varphi|\right)_{s-2},
\]  
\tag{39}

Moreover, \( \|\nu_0\|_s^2 \leq \rho(t) \) for all \( t \in [0, T_*] \), where \( \rho(t) \) is as in Theorem 2.

**proof.** Let \( T_* = T_* (\psi) \) be as in Theorem 2. Now, we will split the proof into four steps: \( \square \)

**Step 1.** First we will show that \( (\nu_D(t))_{D>0} \) is a net which converges to a function \( \nu_0 \in C([0, T_*] ; L^2(\mathbb{R})) \) in the \( L^2 \) - norm, uniformly over \( [0, T_*] \).

\[
\text{Let } D_1, D_2 \in (0, +\infty). \text{ Then,}
\]
\[ \frac{1}{2} \frac{d}{dt} \| u_{D_1} - u_{D_2} \|^2 = \left( u_{D_1} - u_{D_2} \right) \frac{d}{dt} (u_{D_1} - u_{D_2}) \]

\[ = \left( u_{D_1} - u_{D_2} \right) \left( D_1 \partial_x^2 u_{D_1} - D_2 \partial_x^2 u_{D_2} \right) - \left( u_{D_1} - u_{D_2} \right) \left[ u_{D_1}^2 - u_{D_2}^2 \right] \]

\[ + (\alpha + 1) \left( u_{D_1} - u_{D_2} \right) \left[ u_{D_1}^2 - u_{D_2}^2 \right] - \alpha \left( u_{D_1} - u_{D_2} \right) \left[ u_{D_1}^2 - u_{D_2}^2 \right] \]

\[ = -D_1 \left( \partial_x \left( u_{D_1} - u_{D_2} \right) \right)^2 - (D_1 - D_2) \left( \partial_x \left( u_{D_1} - u_{D_2} \right) \right)^2 \]

\[ \leq \left| D_1 - D_2 \right| \left( \left( u_{D_1} - u_{D_2} \right) \right)^2 \]

\[ + \left( \alpha + 1 \right) \left( \left( u_{D_1} - u_{D_2} \right) \right)^2 \]

\[ + \epsilon \left( \left( u_{D_1} - u_{D_2} \right) \right)^2, \quad \text{for all } t \in [0, T]. \]

(40)

Let \( M = \sup_{t \in [0, T]} \sqrt{\rho(t)} \), where \( \rho \) is the function defined in the proof of Theorem 2. We bound separately each term on the right-hand side of (40) as follows:

\[ \left| D_1 - D_2 \right| \left( \left( u_{D_1} - u_{D_2} \right) \right)^2 \]

\[ \leq \left| D_1 - D_2 \right| \left( \left( u_{D_1} - u_{D_2} \right) \right)^2 \]

\[ \leq 2M^2 \left| D_1 - D_2 \right|. \]

(41)

We can bound the second term by the following:

\[ \left( u_{D_1} - u_{D_2} \right)^2 \]

\[ \leq \left( u_{D_1} - u_{D_2} \right)^2 \left( u_{D_1}^2 + u_{D_2}^2 \right) \]

\[ \leq \left( u_{D_1} - u_{D_2} \right)^2 \left( u_{D_1}^2 + u_{D_2}^2 \right) \]

\[ \leq 3M^2 \left( u_{D_1} - u_{D_2} \right)^2. \]

(42)

The third term is bounded by \( \left( u_{D_1} - u_{D_2} \right)^2 \), \( u_{D_1}^2 \), and \( u_{D_2}^2 \).

Finally, we have
Step 3. We must show that \( L \in \text{Hs} \) is complete there exists the limit \( R \), \( T \), \( \text{Hs} \).

First of all, we will show that \( \left( u_{D_{1}}(t) - u_{D_{2}}(t) \right) \in \text{Hs} \in \text{Hs} \), uniformly with respect to \( \epsilon \in [0, T_{n}] \), i.e.,

\[
\lim_{D \to 0} + \sup_{t \in [0, T_{n}]} \| u_{D_{0}}(t) - u_{0}(t) \|_{0} = 0, \tag{46}
\]

Applying Gronwall’s inequality to the last relation, we show that there is a constant \( C > 0 \) satisfying \( \| u_{D_{n}}(t) - u_{D_{n}}(t) \|_{\text{Hs}} \leq C \| D_{1} - D_{2} \| \) for all \( t \in [0, T_{n}] \), and since \( L^{2}(\mathbb{R}) \) is compact there exists the limit \( u_{0}(t) = \lim_{D \to 0} u_{D_{n}}(t) \) in \( L^{2}(\mathbb{R}) \) uniformly with respect to \( t \in [0, T_{n}] \), i.e.,

\[
\lim_{D \to 0} + \sup_{t \in [0, T_{n}]} \| u_{D_{0}}(t) - u_{0}(t) \|_{0} = 0, \tag{46}
\]

We obtain by Fatou’s Lemma as follows:

\[
\| u_{0} \|_{\text{Hs}}^{2} = \int_{\mathbb{R}} \left( 1 + \xi^{2} \right)^{2} | \hat{u}_{0}(\xi) |^{2} d\xi \leq \liminf_{D \to 0} \int_{\mathbb{R}} \left( 1 + \xi^{2} \right)^{2} | \hat{u}_{D_{0}}(\xi) |^{2} d\xi \leq \rho(t). \tag{48}
\]

Step 3. We must show that \( u_{D_{n}} \to u_{0} \) in \( H^{s}(\mathbb{R}) \) for all \( t \in [0, T_{n}] \) as \( D \to 0 + \).

First of all, we will show that \( \left( u_{D_{0}}(t) \right)_{D \to 0} \) is a weak Cauchy net in \( H^{s}(\mathbb{R}) \), uniformly with respect to \( t \in [0, T_{n}] \).

\[
\left\| \left( u_{D_{1}}(t) - u_{D_{2}}(t) \right) \varphi \right\| = \left\| \left( u_{D_{1}}(t) - u_{D_{2}}(t) \right) \varphi - \varphi_{c} \right\| + \left\| \left( u_{D_{1}}(t) - u_{D_{2}}(t) \right) \varphi_{c} - \varphi_{c} \right\| \leq \left\| \left( u_{D_{1}}(t) - u_{D_{2}}(t) \right) \varphi - \varphi_{c} \right\| + \left\| \left( u_{D_{1}}(t) - u_{D_{2}}(t) \right) \left( 1 - \varphi_{c}^{*} \right) \varphi_{c} \right\| \leq 2M_{c} + \left\| \left( u_{D_{1}}(t) - u_{D_{2}}(t) \right) \left( 1 - \varphi_{c}^{*} \right) \varphi_{c} \right\| \leq 2M_{c} + C \| D_{1} - D_{2} \|, \tag{49}
\]

and therefore, we have \( \lim_{D_{1}, D_{2} \to 0} + \sup_{t \in [0, T_{n}]} \left( u_{D_{1}}(t) - u_{D_{2}}(t) \right) \varphi_{c} = 0 \).

Thus, we have that \( u_{D_{n}} \to u_{0} \) for all \( t \in [0, T_{n}] \), i.e.,

\[
\left( u_{D_{n}}(t) - u_{0}(t) \right) \varphi_{c} \to 0 \quad \text{as} \quad D \to 0 +, \tag{50}
\]

where \( q \) is defined in the proof of Proposition 7). From Theorem 2, we obtain as follows:

\[
\left\| \partial_{x} q \left( u_{D_{1}}, u_{D_{2}} \right) \right\|_{L^{\infty}} \leq \left\| \partial_{x} q \left( u_{D_{1}}, u_{D_{2}} \right) \right\|_{L^{2}} \leq \rho(t)^{m_{2}} \leq M^{n}, \tag{44}
\]

for all \( t \in [0, T_{n}] \).

Therefore, from the above bounds, we have as follows:

\[
1 \frac{d}{dt} \| u_{D_{1}} - u_{D_{2}} \|_{0}^{2} \leq 2M^{2} \| D_{1} - D_{2} \| + \left( 3M^{2} + 2(\alpha + 1)M + \frac{M^{n}}{2(n + 1)} \right) \| u_{D_{1}} - u_{D_{2}} \|_{0}^{2}, \tag{45}
\]

and so \( u_{0} \in C([0, T_{n}]; L^{2}(\mathbb{R})) \).

Step 2. Now we show that \( u_{0} \in H^{s}(\mathbb{R}) \). Let \( t \in [0, T_{n}] \). Since \( u_{D_{n}} \to u_{0} \) in \( L^{2}(\mathbb{R}) \), as \( D \to 0 + \), then there exists a subsequence \( \{ D_{n} \} \) such that

\[
\lim_{n \to \infty} u_{D_{n}}(t, \xi) = \tilde{u}_{0}(t, \xi), \xi - \text{a.e.} \tag{47}
\]

We obtain by Fatou’s Lemma as follows:

\[
\| u_{0} \|_{\text{Hs}}^{2} = \int_{\mathbb{R}} \left( 1 + \xi^{2} \right)^{2} | \hat{u}_{0}(\xi) |^{2} d\xi \leq \liminf_{n \to \infty} \int_{\mathbb{R}} \left( 1 + \xi^{2} \right)^{2} | \hat{u}_{D_{n}}(\xi) |^{2} d\xi \leq \rho(t). \tag{48}
\]

In fact, given \( \varphi \in H^{s}(\mathbb{R}) \) and \( \epsilon > 0 \), choosing \( \varphi_{\epsilon} \in H^{s}(\mathbb{R}) \) such that \( \| \varphi - \varphi_{\epsilon} \|_{s} \leq \epsilon \), then

\[
\left\| \left( u_{D_{1}}(t) - u_{D_{2}}(t) \right) \varphi_{\epsilon} \right\|_{s} = \left\| \left( u_{D_{1}}(t) - u_{D_{2}}(t) \right) \varphi - \varphi_{\epsilon} \right\|_{s} + \left\| \left( u_{D_{1}}(t) - u_{D_{2}}(t) \right) \varphi_{\epsilon} - \varphi_{\epsilon} \right\|_{s} \leq \left\| u_{D_{1}}(t) - u_{D_{2}}(t) \right\|_{s} \| \varphi - \varphi_{\epsilon} \|_{s} + \left\| \left( u_{D_{1}}(t) - u_{D_{2}}(t) \right) \left( 1 - \varphi_{\epsilon}^{*} \right) \varphi_{\epsilon} \right\|_{s} \leq 2M_{c} + \left\| \left( u_{D_{1}}(t) - u_{D_{2}}(t) \right) \left( 1 - \varphi_{\epsilon}^{*} \right) \varphi_{\epsilon} \right\|_{s} \leq 2M_{c} + C \| D_{1} - D_{2} \|, \tag{49}
\]

and therefore, we have \( \lim_{D_{1}, D_{2} \to 0} + \sup_{t \in [0, T_{n}]} \left( u_{D_{1}}(t) - u_{D_{2}}(t) \right) \varphi_{\epsilon} = 0 \).

Thus, we have that \( u_{D_{n}} \to u_{0} \) for all \( t \in [0, T_{n}] \), i.e.,
for all \( \varphi \in H^s(\mathbb{R}) \). Moreover, since the convergence is uniform for all \( \varphi \in H^s(\mathbb{R}) \), we can conclude that \( u_0 \in C_w([0, T_s]; H^{s-2}(\mathbb{R})) \).

\[
(u_D(t)|\varphi|)_{s-2} = (\varphi|\varphi|)_{s-2} + \int_0^t \left( D^2 u_D(t) - u_D(t) (u_D(t) - \alpha) (u_D(t) - 1) + e u_D(t) \partial_x (u_D(t) - \alpha) (u_D(t) - 1) - e u_D(t) \partial_x (u_D(t) - \alpha) (u_D(t) - 1) \right) dt,
\]

for all \( t \in [0, T_s] \). Since \( u_D \rightharpoonup u_0 \) in \( L^2(\mathbb{R}) \) and \( u_D \rightharpoonup u_0 \) in \( H^s(\mathbb{R}) \), we have \( \partial_x u_D \rightharpoonup \partial_x u_0 \) in \( H^{s-1}(\mathbb{R}) \) and \( D^2 u_D \rightharpoonup D^2 u_0 \) in \( H^{s-2}(\mathbb{R}) \) uniformly on \( [0, T_s] \). Observe that if \( r > 1/2 \), \( f \rightharpoonup f \) in \( H^r(\mathbb{R}) \) and \( g_n \rightharpoonup g \) in \( H^r(\mathbb{R}) \) then \( f_n g_n \rightharpoonup f g \) in \( H^r(\mathbb{R}) \).

After, we have

\[
(u_0(t)|\varphi|)_{s-2} = (\varphi|\varphi|)_{s-2} + \int_0^t (-u_0(t) (u_0(t) - \alpha) (u_0(t) - 1) - e u_0(t) \partial_x (u_0(t) - \alpha) (u_0(t) - 1)) dt.
\]

**Corollary 1.** Let \( u_0 \) be as in the preceding theorem, then \( u_0 \in AC([0, T_s]; H^{s-2}(\mathbb{R})) \).

**Proof.** Since \( t \in [0, T_s] \), \( \varphi \rightarrow u(u - \alpha) (u - 1) + e u^\alpha u_x \) is weakly continuous in \( H^{s-2}(\mathbb{R}) \) and the Sobolev space is separable, then applying the Bochner–Pettis theorem, it is a strongly measurable function in \( H^{s-2}(\mathbb{R}) \). Therefore,

\[
\int_0^t (u(u - \alpha) (u - 1) + e u^\alpha u_x) dt,
\]

exists as a Bochner integral. So, from (53) we conclude that

\[
u_0(t) = \psi + \int_0^t (u(u - \alpha) (u - 1) + e u^\alpha u_x) dt,
\]

and therefore, \( u_0 \in AC([0, T_s]; H^{s-2}(\mathbb{R})) \). \( \square \)

**Theorem 4.** Let \( s > 3/2 \) and \( T > 0 \) be fixed, \( \varphi_j \in H^s(\mathbb{R}) \), \( j = 1, 2 \), and \( \varphi_j \in C([0, T]; L^2(\mathbb{R})) \cap C_w([0, T]; H^s(\mathbb{R})) \cap AC([0, T]; H^{s-2}(\mathbb{R})) \) two weak sense solutions to (1) with \( D = 0 \) such that \( \psi_j(0) = \psi_j \), \( j = 1, 2 \). Then,

\[
\|v_1(t) - v_2(t)\|_0 \leq \|\psi_1 - \psi_2\|_{C^0([0,T];L^\alpha (\mathbb{R}))},
\]

where \( L_0 \) is as in the Proposition 7 and

\[
\partial_s w(t) = \lim_{h \to 0} \frac{w(t + h) - w(t)}{h} = -(v_1(t) (v_1(t) - \alpha) (v_1(t) - 1) - v_2(t) (v_2(t) - \alpha) (v_2(t) - 1))
\]

exists in the norm of \( H^{s-2}(\mathbb{R}) \rightarrow H^{-s}(\mathbb{R}) \), from (58) and (59) we have as follows:

**Step 4.** Finally, we show that \( u_0 \in C^1_w([0, T_s]; H^{s-2}(\mathbb{R})) \).

Let \( \varphi \in H^{s-2}(\mathbb{R}) \). Then,

\[
\|u_D(\varphi - \alpha) (u_D - 1) - u_0 (u_0 - \alpha) (u_0 - 1)\|_{H^s(\mathbb{R})},
\]

uniformly on \( [0, T_s] \). Therefore, taking the limit as \( D \to 0+ \) in (51), we obtain as follows:

\[
\int_0^t \left( u_D(t) (u_D(t) - \alpha) (u_D(t) - 1) + e u_D(t) \partial_x (u_D(t) - \alpha) (u_D(t) - 1) \right) dt.
\]
\[ \partial_t \|w(t)\|_0^2 = -2(v_1(t)(v_1(t) - \alpha)(v_1(t) - 1) - v_2(t)(v_2(t) - \alpha)(v_2(t) - 1) + \epsilon v_1(t)^\alpha \partial_x v_1(t) - \epsilon v_2(t)^\alpha \partial_x v_2(t)\|w(t)\|_s) \]

\[ = -2(v_1(t)(v_1(t) - \alpha)(v_1(t) - 1) - v_2(t)(v_2(t) - \alpha)(v_2(t) - 1) + \epsilon v_1(t)^\alpha \partial_x v_1(t) - \epsilon v_2(t)^\alpha \partial_x v_2(t)\|w(t)\|_0) \]

\[ = -2(v_1(t)^3 - (\alpha + 1)v_1(t)^2 - \epsilon v_1(t)^\alpha \partial_x v_1(t) - (v_2(t)^3 - (\alpha + 1)v_2(t)^2 - \epsilon v_2(t)^\alpha \partial_x v_2(t)) + \alpha (v_1(t) - v_2(t))\|w(t)\|_0) \]

From Proposition 7 and (60) we have as follows:

\[ \partial_t \|w(t)\|_0^2 \leq (L_0(R, R) + \alpha)\|v_1(t) - v_2(t)\|_0^2, \tag{61} \]

where \( R \) is given by (57). Applying Gronwall’s inequality to (61), and we have proved the theorem. \( \square \)

**Theorem 5.** Let \( \psi \in H^s(\mathbb{R}) \) with \( s > 3/2 \). Then, there exists a \( T_1 = T_1(\psi) > 0 \) and a unique \( u_0 \in C([0, T_1]; H^s(\mathbb{R})) \) such that

\[
\begin{align*}
\partial_t u_0(t) + u_0(t)(u_0(t) - \alpha)(u_0(t) - 1) + u_0(t)^\alpha \partial_x u_0 = 0, \\
u_0(0) = \psi.
\end{align*}
\tag{62}
\]

\[
\begin{align*}
|\langle \psi | \phi \rangle|_s = \lim_{t \to 0^+} \|u_0(t)\|_s & \leq \liminf_{t \to 0^+} \|u_0(t)\|_s \leq \liminf_{t \to 0^+} \|u_0(t)\|_s \\
& \leq \limsup_{t \to 0^+} \|u_0(t)\|_s \leq \limsup_{t \to 0^+} \|u_0(t)\|_s + \rho(t)^{1/2} = ||\psi||_s.
\end{align*}
\tag{63}
\]

for all \( \phi \in H^s(\mathbb{R}) \). As we have \( ||\psi||_s = \sup_{|\phi|_s = 1} |\langle \psi | \phi \rangle|_s \), then taking supremum over \( ||\phi||_s = 1 \) in (63) we have \( \liminf_{t \to 0^+} \|u_0(t)\|_s \leq \limsup_{t \to 0^+} \|u_0(t)\|_s \) and \( \lim_{t \to 0^+} \|u_0(t)\|_s = ||\psi||_s \). Since \( u(t) \to u \) weakly in \( H^s(\mathbb{R}) \) as \( t \to 0^+ \), it follows that \( \lim_{t \to 0^+} u(t) = u \) in the norm of \( H^s(\mathbb{R}) \). Let \( t' \in [0, T_1] \) be fixed. Then, there exists \( T > 0 \), with \( T > t' \), and a unique \( \psi \in C^1([0, T]; H^s(\mathbb{R})) \) satisfying \( \partial_t u(t) + u(t)(u(t) - \alpha)(u(t) - 1) + u(t)^\alpha \partial_x u = 0 \), with \( u(t') = u(t) \). We have noticed that the uniqueness of solutions implies that \( u(t) = u(t + t') \), for \( t \in [0, T] \). Since \( \psi \) is continuous from the right at \( t = 0 \), then \( u_0 \) is continuous from the right at \( t = t' \). Now, let \( t' \in (0, T) \) be fixed. Observe that the following problem

\[
\begin{align*}
-\partial_t w(t) + w(t)(w(t) - \alpha)(w(t) - 1) - w(t)^\alpha \partial_x w = 0, \\
w(0) = \bar{u}(t'),
\end{align*}
\tag{64}
\]

has a unique solution \( w(t, x) = u_0(t' - t, -x) \) with \( \bar{u}(t', x) = u_0(t', -x) \), because the equation in problem (64) is similar to the equation in problem (1) with \( D = 0 \) and it is easy to show similar results to those obtained for problem (1) with \( D = 0 \), specially the uniqueness results.

In particular, for the problem (64) there are results analogous to Theorems 3 and 4. Therefore, since \( w \) is continuous from the right at \( t = 0 \), then \( u_0 \) is continuous from the left at \( t = 0 \). So, \( u_0 \in C([0, T_1]; H^s(\mathbb{R})) \). Moreover, we have \( u_0(u_0 - \alpha)(u_0 - 1) + u_0^\alpha \partial_x u_0 \in C([0, T]; H^{s-2}(\mathbb{R})) \). From (55) we also conclude that \( u_0 \in C^1([0, T]; H^{s-2}(\mathbb{R})) \) and that it is the unique strong solution of (1) with \( D = 0 \). \( \square \)

**Theorem 6.** Let \( s > 3/2 \), \( \psi \in H^{s+1}(\mathbb{R}) \) and \( u_D \in C([0, T_{s+1}]; H^{s+1}(\mathbb{R})) \) and \( C^1([0, T_{s+1}]; H^{-1}(\mathbb{R})) \) be the corresponding solution of the problem (1) for \( D \geq 0 \), defined in the interval \( [0, T_{s+1}(\psi)] \) which is independent of \( D \). Then, \( u_D \) can be extended, if necessary, to the interval \( [0, T_s], \psi \) viewed as an element of \( H^s(\mathbb{R}) \).

**proof.** Applying (30) with \( r = s + 1 \) to obtain
\[
\frac{\text{d}}{\text{d}t}\|u_D\|_{s+1}^2 = 2\left(u_D[\partial u_D(t)]_{s+1}\right) \\
= 2\left[-D\|\partial x u_D\|_{s+1}^2 - (u_D\|u_D - \alpha\|_s)_{s+1} - \epsilon(u_D\|u_D\|_{s+1}^2)_{s+1}\right] \\
\leq 2C\left(\|u_D\|_{s+1}^2 + (\alpha + 1)\|u_D\|_s^2 + \|u_D\|_s^2\|u_D\|_{s+1}^2\right).
\]

(65)

Now, from inequality (3.12) and Theorem 3.2 in [14], we obtain as follows:

\[
\|u_D\|_{s+1}^2 \leq 2\|u_D\|_{L^2}^2\|u_D\|_{s+1} \leq \|u_D\|\|u_D\|_{s+1},
\]

(66)

and integrating from 0 to \(t\), we have as follows:

\[
\|u_D\|_{s+1}^2 \leq \|\psi\|_{s+1}^2 + 2C\int_0^t\left(\|u_D\|_s^2 + (\alpha + 1)\|u_D\|_s^2 + \|u_D\|_s^2\right)\|u_D\|_{s+1}^2\,d\tau.
\]

(69)

and applying the Gronwall’s inequality to obtain

\[
\|u_D\|_{s+1}^2 \leq \|\psi\|_{s+1}^2 \exp\left(2C\int_0^t\left(\|u_D\|_s^2 + (\alpha + 1)\|u_D\|_s^2 + \|u_D\|_s^2\right)\,d\tau\right).
\]

(70)

Observe that on the right-hand side of (70) is well-defined for \(t \in [0, T_s(\psi)]\) and therefore we can extend (if necessary) \(u = u(t)\) to \([0, T_s(\psi)]\) as a solution in \(H^{s+1}(\mathbb{R})\). Thus, we conclude that \(T_s(\psi) \leq T_{s+1}(\psi)\). So, \(u_D \in C([0, T_s(\psi)]; H^{s+1}(\mathbb{R}))\) for \(D > 0\). From (70) also we have that

\[
\|u_D\|_{s+1}^2 \leq \|\psi\|_{s+1}^2 \exp\left(2C\left(M^2 + (\alpha + 1)M + M^s\right)T_s(\psi)\right).
\]

(71)

Observe that the last inequality is independent of \(D > 0\) and since \(u_D\) weakly converges and uniformly to \(u_0\) in \(H^{s+1}(\mathbb{R})\), then we have \(u_0 \in C([0, T_s(\psi)]; H^{s+1}(\mathbb{R}))\).

Following Lemma 5 in [15] we have the next lemma.

**Lemma 4.** Let \(s > 3/2\). For \(\psi \in H^s(\mathbb{R})\) and \(\tau > 0\), we define

\[
\|\psi\|_{s+1}^2 = \int_0^{\tau} (1 + t^2)^s e^{-t(1 + t^2)^{s/2}}\,d\tau.
\]

(72)

Then, \(\lim_{\tau \to 0} \|\psi\|_{s+1}^2 = 0\) and there exists a constant \(C = C(s)\) such that

\[
\|\psi\|_{s+1} \leq C \left(1 + \left(\frac{1}{\tau s}\right)^{2s}\right)^{1/2} \|\psi\|_s.
\]

(73)

Moreover, \(\lim_{\tau \to 0} \|\psi\|_{s+1} \leq 0\) uniformly on compact subsets of \(H^s(\mathbb{R})\).

**Proof.** Notice that

\[
\|\psi\|_{s+1}^2 = \int_0^{\tau} (1 + t^2)^s e^{-t(1 + t^2)^{s/2}}\,d\tau.
\]

Then, using Lebesque’s dominated
convergence theorem, we obtain \( \lim_{t \to 0^+} \|\psi^T - \psi\|_s = 0 \). Now, to prove the uniformity on compact subsets, it is enough to show that \( \psi_n \to \psi \) in \( H^n(\mathbb{R}) \) implies \( \lim_{t \to 0^+} + \|\psi^T_n - \psi\|_s = 0 \) uniformly for \( n = 1, 2, \ldots \), since sequential compactness is equivalent to compactness in metric spaces. Thus, observe that

\[
\|\psi^T_n - \psi^T\|^2 = \int_{\mathbb{R}} \left(1 + \xi^2\right)^n e^{-2\xi(1+\xi^2)} \left|\overline{\psi}_n(\xi) - \overline{\psi}(\xi)\right|^2 d\xi
\]

\[
\leq \int_{\mathbb{R}} \left(1 + \xi^2\right)^n |\overline{\psi}_n(\xi) - \overline{\psi}(\xi)|^2 d\xi = \|\psi_n - \psi\|^2.
\]

(74)

Let \( \varepsilon > 0 \) be given and choose \( N \) such that if \( n \geq N \), then \( \|\psi^T_n - \psi\|_s < (1/3)\varepsilon \). Thus, for \( t_0 > 0 \) small enough that \( 0 < t < t_0 \) we have

\[
\|\psi^T_n - \psi\|_s < \varepsilon,
\]

for \( 1 \leq n \leq N \). Now, if \( n > N \) then we have

\[
\|\psi^T_n - \psi\|_s \leq \|\psi^T_n - \psi\|_s + \|\psi^T - \psi\|_s + \|\psi - \psi_n\|_s < \varepsilon.
\]

Hence (75) holds for all \( n \).

On the other hand, we have

\[
\|\psi^T\|_{s+1}^2 = \int_{\mathbb{R}} \left(1 + \xi^2\right)^{n+1} e^{-2\xi(1+\xi^2)} |\overline{\psi}(\xi)|^2 d\xi \leq \left(1 + \max_{\xi \in \mathbb{R}} \left(\xi^2 e^{-2\xi(1+\xi^2)}\right)\right) \|\psi\|_s^2.
\]

(77)

Finally, using the mean value theorem, \( |e^{-\tau(1+\xi^2)} - e^{-\xi(1+\xi^2)}| \leq |\tau - \xi| (1 + \xi^2)^{\eta/2} \), and then we have

\[
\|\psi^T - \psi\|^2 = \int_{\mathbb{R}} |\psi^T(x) - \psi(x)|^2 d\xi
\]

\[
= \int_{\mathbb{R}} \left|e^{-\tau(1+\xi^2)} - e^{-\xi(1+\xi^2)}\right|^2 |\overline{\psi}(\xi)|^2 d\xi \leq |\tau - \xi|^2 \|\psi\|_s^2.
\]

(78)

The proof is complete. \( \Box \)

**Proposition 8.** Let \( s > 3/2, \psi \in H^s(\mathbb{R}), \psi^T \) (for \( \tau > 0 \)) be as in the preceding lemma. If \( u^0_\tau \) is solution of the problem (62) with \( u^0_\tau(0) = \psi^T \), for all \( \tau > 0 \), then there are constants \( C = C(s, \|\psi\|_s, T) > 0 \) and \( \eta = \eta(s) \in (0, 1) \) such that

\[
\|u^\tau_\tau - u^\tau_0\|^2 \leq C\left[\|\psi^T - \psi\|^2 + \tau^{1-\eta}\right],
\]

(79)

for \( \tau \) sufficiently small and \( 0 \leq \theta \leq 2\tau \).

**Proof.** Let \( \tau_0 > 0 \) be such that \( u^\tau_\tau(t) \) is well-defined in \([0, T]\) for all \( 0 < \tau \leq \tau_0 \). Then,

\[
\frac{1}{2} \partial_s \|u^\tau_\tau - u^\tau_0\|^2 = \left(\frac{\partial_s}{\partial_s} \left(u^\tau_\tau - u^\tau_0\right) + \alpha \left(u^\tau_\tau - u^\tau_0\right)\right) + \frac{\partial_s^2}{\partial_s^2} \left(u^\tau_\tau - u^\tau_0\right)
\]

\[
\leq \left(\frac{\partial_s}{\partial_s} \left(u^\tau_\tau - u^\tau_0\right) + \alpha \left(u^\tau_\tau - u^\tau_0\right)\right) + \frac{\partial_s^2}{\partial_s^2} \left(u^\tau_\tau - u^\tau_0\right)
\]

\[
\leq \left(\frac{\partial_s}{\partial_s} \left(u^\tau_\tau - u^\tau_0\right) + \alpha \left(u^\tau_\tau - u^\tau_0\right)\right) + \frac{\partial_s^2}{\partial_s^2} \left(u^\tau_\tau - u^\tau_0\right)
\]

\[
+ \frac{\partial_s^2}{\partial_s^2} \left(u^\tau_\tau - u^\tau_0\right) + \frac{\partial_s^2}{\partial_s^2} \left(u^\tau_\tau - u^\tau_0\right)
\]

(80a)

(80b)

(80c)

Now, the right-hand side of the inequality (3) will be estimated. First, we will estimate (80a). Applying the Cauchy–Schwartz inequality to (80a) we have
\[
\left\| u_0^\tau - u_0^0 \right\|_2^2 \leq (\| u_0^\tau \|_2^2 - \| u_0^0 \|_2^2) \leq C M \| u_0^\tau - u_0^0 \|_2^2.
\]

(81)

Finally, we estimate (80c). As \( s > 3/2 \), there is \( s_0 \) such that \( 3/2 < s_0 + 1 < s \). From the Cauchy–Schwarz inequality, we obtain

\[
\left\| u_0^\tau - u_0^0 \right\|_2 \leq C \| u_0^\tau - u_0^0 \|_0.
\]

Now, we will estimate each term on the right-hand side of the last inequality. First, observe that

\[
\left\| \left( u_0^\tau - u_0^0 \right) \right\|_2 \leq C \| \left( u_0^\tau - u_0^0 \right) \|_1 \leq C \left\| \left( u_0^\tau - u_0^0 \right) \right\|_0.
\]

(82)

where \( q \) is defined in the proof of Proposition 7.

We also estimate \( \| u_0^\tau - u_0^0 \|_2 \| u_0^\tau \|_0 \| u_0^\tau \|_0 \). From Lemma 4 and the inequality (71), we have

\[
\left\| \left( u_0^\tau \right)^n - \left( u_0^0 \right)^n \right\|_2 \leq q(u_0^\tau, u_0^0) \| u_0^\tau \|_0 \| u_0^0 \|_0 \leq C \| u_0^\tau - u_0^0 \|_0 \| u_0^\tau - u_0^0 \|_0^{1+q}.
\]

(84)

(85)

for all \( \tau \leq \tau_0 \).

From Lemma 4, we have

\[
\| u_0^\tau \|_{\tau+1} \leq \| \psi \|_{\tau+1} e^{CT_\tau (M^2 + (a+1)M + \lambda^p)} \leq c \| \psi \|_0 \tau^{-(1/\alpha)},
\]

(86)

where \( q = (s_0/s) \). To estimate the term \( \| u_0^\tau - u_0^0 \|_0 \), observe that

\[
\partial_\tau \left\| u_0^\tau - u_0^0 \right\|_0^2 = 2 \left\| u_0^\tau - u_0^0 \right\|_0 \left\| u_0^\tau (u_0^\tau - \alpha) (u_0^\tau - 1) - u_0^0 (u_0^0 - \alpha) (u_0^0 - 1) \right\|_0
\]

\[
+ 2c \left\| \left( u_0^\tau - u_0^0 \right) \left( u_0^\tau - u_0^0 \right) \right\|_0 \leq C \left\| u_0^\tau - u_0^0 \right\|_2^2 + \frac{1}{n+1} \left\| \bar{q}(u_0^\tau, u_0^0) \right\|_{\infty} \left\| u_0^\tau - u_0^0 \right\|_2^2 \leq C \left\| u_0^\tau - u_0^0 \right\|_2^2.
\]

(87)

where \( \bar{q}(u, v) = \sum_j u^j v^{n-j} \), and from Gronwall inequality we have as follows:

\[
\left\| u_0^\tau - u_0^0 \right\|_0 \leq C \| \psi - \psi \|_0^2.
\]

From Lemma 4,
\[
\begin{align*}
\| (u_0^\tau)^n - (u_0^\theta)^n \|_{L^\infty} \leq C \| \psi - \psi_0^{(1-(s/2))} \|_{L^s} r^{-(1/s)} \\
\leq C|\tau - \theta|^{1-(s/2)} \| \psi \|_{L^s}^{1-(s/2)} \| \psi \|_{L^s} r^{-(1/s)} \\
\leq C \| \psi \|_{L^s}^{2-(s/2)} r^{1-(s+1/2)},
\end{align*}
\]

(89)

for \( 0 \leq \theta \leq r \). Therefore, we have

\[
\| u_0^\tau - u_0^\theta \|_{L^\infty} \leq 2MC \| \psi \|_{L^s} \leq 2MC \| \psi \|_{L^s} (r^{1-(s+1/2)}),
\]

(90)

and the term (80c) is bounded for

\[
\left| (u_0^\tau - u_0^\theta) \partial_x u_0 \right|_{L^2} \leq C \left( \| u_0^\tau - u_0^\theta \|_{L^2}^2 + r^{1-(s+1/2)} \right).
\]

(91)

Of the bounds that were found for (80a), (80b), (80c) we conclude that

\[
\partial_t \| u_0^\tau - u_0^\theta \|_{L^2}^2 \leq C \left( \| u_0^\tau - u_0^\theta \|_{L^2}^2 + r^{1-(s+1/2)} \right).
\]

(92)

and using Gronwall inequality, we obtain (79).

The following corollary follows immediately from Proposition 8 and Lemma 4.

\section*{Corollary 2}

Let \( F \) be a compact subset in \( H^s(\mathbb{R}) \). Suppose that \( \psi \in F \), \( \psi^\tau \) and \( u_0^\alpha \) are defined as in the preceding result.

Then, \( u_0^\tau \) converges uniformly to \( u_0 \), for all \( t \in [0, T] \), as \( \tau \to 0^+ \).

\section*{Theorem 7}

The map \( \psi \to u_0 \) is continuous in the following sense: let \( \psi_j \in H^s(\mathbb{R}) \), \( j = 1, 2, 3, \ldots \) such that \( \psi_j \to \psi \) in \( H^s(\mathbb{R}) \) and \( u_{0,j} \in C((0, T_s); H^s(\mathbb{R})) \cap C^1((0, T_s); H^{s-2}(\mathbb{R})) \) be the corresponding solutions of the problem (62) with initial condition \( u_{0,j}(0) = \psi_j \). Let \( T \in (0, T_{s,\infty}) \). Then, there exists a positive integer \( N_0 = N_0(\tau, s, \psi) \) such that \( T_{s,j} \geq T \) for all \( j \geq N_0 \) and

\[
\lim_{j \to \infty} \sup_{[0,T]} \| u_{0,j}(t) - u_0(t) \|_{L^\infty} = 0.
\]

(93)

\section*{Data Availability}

No data were used to support this study.

\section*{Conflicts of Interest}

The authors declare that they have no conflicts of interest.
References

[1] S. Balasuriya and G. A. Gottwald, "Wavespeed in reaction-diffusion systems, with applications to chemotaxis and population pressure," *Journal of Mathematical Biology*, vol. 61, no. 3, pp. 377–399, 2010.

[2] R. D. Benguria, M. C. Depassier, and V. Méndez, "Minimal speed of fronts of reaction-convection-diffusion equations," *Physics Review*, vol. 69, no. 3, Article ID 031106, 2004.

[3] B. H. Gilding and R. Kersner, *Travelling Waves in Nonlinear Diffusion-Convection Reaction*, Birkhauser, Basel, 2004.

[4] Z. Chen and B. Guo, "Analytic solutions of the Nagumo equation," *IMA Journal of Applied Mathematics*, vol. 48, no. 2, pp. 107–115, 1992.

[5] H. P. McKean, "Nagumo's equation," *Advances in Mathematics*, vol. 4, no. 3, pp. 209–223, 1970.

[6] J. Nagumo, S. Arimoto, and S. Yoshizawa, "An active pulse transmission line simulating nerve axon," *Proceedings of the IRE*, vol. 50, no. 10, pp. 2061–2070, 1962.

[7] J. Nagumo, S. Yoshizawa, and S. Arimoto, "Bistable transmission lines," *IEEE Transactions on Circuit Theory*, vol. 12, no. 3, pp. 400–412, 1965.

[8] D. B. Dix, "Nonuniqueness and uniqueness in the initial-value problem for burgers’ equation," *SIAM Journal on Mathematical Analysis*, vol. 27, no. 3, pp. 708–724, 1996.

[9] E. Hopf, "The partial differential equation $u_t + uu_x = \mu u_{xx}$," *Communications on Pure and Applied Mathematics*, vol. 3, pp. 201–230, 1950.

[10] B. Álvarez, "The Cauchy problem for a nonlocal perturbation of the KdV equation," *Differential and Integral Equations*, vol. 16, no. 10, pp. 1249–1280, 2003.

[11] R. José Iório Jr., "On the cauchy problem for the benjamin-ono equation," *Communications in Partial Differential Equations*, vol. 11, no. 10, pp. 1031–1081, 1986.

[12] R. Iório Jr. and V. Magalhães Iório, "Fourier analysis and partial differential equations," *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, vol. 70, 2001.

[13] T. Kato, "On the cauchy problem for the (generalized) korteweg-de vries equation," *Studies in Applied Mathematics, Advances in Math. Suppl. Studies*, Academic Press, vol. 8, pp. 92–128, 1983.

[14] F. Linares and G. Ponce, *Introduction to Nonlinear Dispersive Equations*, Springer, New York, NY, USA, 2009.

[15] J. L. Bona and R. Smith, "The initial-value problem for the korteweg-de vries equation," *Philosophical Transactions Royal Society London A*, vol. 278, pp. 555–601, 1975.

[16] D. Henry, "Geometric theory of semilinear parabolic equations," *Lecture Notes in Mathematics*, Vol. 840, Springer-Verlag, Berlin, Germnay, 1981.

[17] G. Butler and T. Rogers, "A generalization of a lemma of bihari and applications to pointwise estimates for integral equations," *Journal of Mathematical Analysis and Applications*, vol. 33, no. 1, pp. 77–81, 1971.