The gravity of light

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Abstract The gravitational field of an idealized plane-wave solution of the Maxwell equations can be described in closed form. After discussing this particular solution of the Einstein-Maxwell equations, the motion of neutral test particles, which are sensitive only to the gravitational background field, is analyzed. This is followed by a corresponding analysis of the dynamics of neutral fields in the particular Einstein-Maxwell background, considering scalars, Majorana spinors and abelian vector fields, respectively.

1 Light and gravity

Light and gravity provide the main tools for studying the universe at large; gravity, as it determines the interactions and paths of celestial bodies, and light as it makes them visible to us and enables us to unravel their properties. The theoretical descriptions of light, as a form of electromagnetism, and gravity have much in common. The classical theories of electromagnetism and gravity are both local relativistic field theories; these fields carry physical degrees of freedom propagating energy, momentum and angular momentum at a finite speed \( c \), commonly referred to as the speed of light, even though gravity, and also the color charges of subatomic particles, propagate their interactions at the same universal speed as well.

Of course, at the microscopic level electromagnetism is more than a classical field theory, as quantum effects become essential to its propagation and interaction with matter in the form of electrons and other charged particles. A similar change in

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1 For a discussion of the role of the universal constant \( c \) characterizing the relations between inertial frames, see ref. [1]
the way gravity behaves in this domain is expected as well, although experimental confirmation of these ideas has as yet remained out of reach.

Even though the sources of gravity and electromagnetism are different, with gravity coupling to the local density of energy and momentum, and electromagnetism to the local density of electric charges and currents, the corresponding physical degrees of freedom (classical fields in the macroscopic world) do influence each other, but in an asymmetric way. The present chapter is dedicated to a discussion of some aspects of this mutual interaction. Unless specified otherwise (when numerical estimates are required) units are used in which the speed of light is unity: \( c = 1 \).

2 Einstein-Maxwell theory

General Relativity (GR), the classical theory of gravity, states that space-time is endowed with a geometry, encoded in the metric \( g_{\mu\nu} \), determined by the distribution of all combined energy- and momentum-densities. This geometry expresses itself in the motion of matter and light in the universe. For the interaction between gravity and electromagnetic fields this results in an Einstein equation specifying the Ricci curvature in terms of the electro-magnetic energy-momentum tensor:

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G T_{\mu\nu}[F],
\]

(1)

with the local energy-momentum density of electromagnetic fields given by

\[
T_{\mu\nu}[F] = F_{\mu\lambda} F^{\lambda}_{\nu} - \frac{1}{4} g_{\mu\nu} F_{\kappa\lambda} F^{\kappa\lambda}.
\]

(2)

At the same time the classical dynamics of the electromagnetic field is specified by the generalized Maxwell equations in a space-time with given dynamical metric:

\[
D_{\mu} F^{\mu\nu} = \partial_{\mu} F^{\mu\nu} + \Gamma_{\mu\lambda}^{\nu} F^{\lambda\sigma} + \Gamma_{\mu\nu}^{\sigma} F^{\mu\sigma} = -j^{\nu},
\]

(3)

where \( j^{\nu} \) is the electric charge-current density, and \( \Gamma_{\mu\nu}^{\lambda} \) the Riemann-Christoffel connection. In the absence of charges and currents: \( j^{\nu} = 0 \), equations (1)-(3) form a closed system describing gravity interacting with dynamical electromagnetic fields in otherwise empty space; this set-up applies in particular to the coupling of gravity with electromagnetic radiation.

In places where the energy-momentum density of the electromagnetic field is small compared to the Ricci-curvature determined by external sources, such as the sun or compact bodies like neutron stars or black holes, one can to first approximation neglect the contribution of the electromagnetic fields to the curvature and describe the electromagnetic fields outside the external source regions by Maxwell equations in the gravitational background of the external sources. This approach is usually taken in studies of gravitational lensing, which offered one of the first tests of GR.
The gravity of light observing the bending of light by the sun \[2\]; and in a more extreme case the recent observations of a black-hole shadow by the Event Horizon Telescope \[3\].

However, as equation (1) indicates, curvature can also be induced by electromagnetic fields themselves, even though this requires rather extreme electromagnetic energy densities. Indeed, according to this equation (temporarily reinstating the speed of light \(c\)) the curvature \(R\) measured in \(1/m^2\) corresponding to an energy flux \(\Phi\) in \(W/m^2\) is numerically of the order

\[
\frac{R}{1/m^2} \sim \frac{8\pi G}{c^5} \Phi \approx 2 \times 10^{-52} \frac{\Phi}{1 W/m^2}.
\]

(4)

Measuring the curvature due to even intense electromagnetic radiation will therefore be an even more extreme challenge than the curvature due to the collision of very distant black holes and neutron stars \[4\]. In the following sections a more precise analysis is presented.

3 Plane waves

An complete radiative solution of the Einstein-Maxwell equations is that of a plane electromagnetic wave of infinite width, which is accompanied by a parallel plane gravitational wave of \(p p\)-type \[5\]-\[11\]. With the wave propagating in the \(z\)-direction, it is convenient to use light-cone co-ordinates \(u = t - z\) and \(v = t + z\); the traveling plane-wave solution of the electromagnetic field is then expressed in terms of a transverse vector potential

\[
A_i(u) = \int_\infty^\infty \frac{dk}{2\pi} (a_i(k) \sin ku + b_i(k) \cos ku),
\]

(5)

where \(i = (1, 2)\) labels the directions in the transverse \(x\)-\(y\)-plane; the corresponding electric and magnetic field strengths are given by

\[
E_i(u) = -\varepsilon_{ij} B_j(u) = F_{ui}(u).
\]

(6)

Such solutions can take the form of wave packets of finite lengths carrying a finite energy flux per unit area. Specifically in the absence of external sources of curvature the energy density in the transverse plane is constant and the transverse geometry can be taken to be flat; the energy flux is then given in terms of the energy-momentum tensor by the only non-zero component

\[
T_{uu}(u) = F_{ui} F_u^i = \frac{1}{2} \left( E^2 + B^2 \right)(u),
\]

(7)

provided the metric is of the Brinkmann type

\[
ds^2 = -du dv - \Phi(u, x^i) du^2 + dx^i dx^i.
\]

(8)
which is flat in the $x$-$y$-plane as required. In this Brinkmann geometry the only non-zero components of the Riemann curvature and Ricci tensor are

$$R_{u1u} = -\frac{1}{2} \partial_i \partial_j \Phi, \quad R_{uu} = -\frac{1}{2} \left( \partial_x^2 + \partial_y^2 \right) \Phi. \tag{9}$$

The Einstein equation (1) then reduces to a single equation linking the $uu$-components of the Ricci and energy-momentum tensor:

$$\left( \partial_x^2 + \partial_y^2 \right) \Phi = 8\pi G \left( E^2 + B^2 \right). \tag{10}$$

The general solution of this inhomogeneous linear equation for the gravitational potential, the metric component $\Phi(u,x')$, is

$$\Phi = 2\pi G \left( x^2 + y^2 \right) \left( E^2 + B^2 \right) + \Phi_0, \tag{11}$$

where $\Phi_0(u,x')$ is an arbitrary solution of the homogeneous equation

$$\left( \partial_x^2 + \partial_y^2 \right) \Phi_0 = 0. \tag{12}$$

These solutions of the homogeneous equation represent pure gravitational waves of $pp$-type [5]:

$$\Phi_0 = \kappa_+(u) \left( x^2 - y^2 \right) + 2\kappa_x(u)xy. \tag{13}$$

Equation (9) then implies that the co-efficients $\kappa_+, \kappa_-$ represent the components of the corresponding Riemann tensor in the transverse plane:

$$R_{u1u}^{(0)} = -\begin{pmatrix} \kappa_+ & \kappa_-
\kappa_- & \kappa_+ \end{pmatrix}. \tag{14}$$

Under a rotation in the transverse plane over an angle $\varphi$ they transform as quadrupole components:

$$\begin{pmatrix} \kappa'_+ \\
\kappa'_- \end{pmatrix} = \begin{pmatrix} \cos 2\varphi & -\sin 2\varphi \\
\sin 2\varphi & \cos 2\varphi \end{pmatrix} \begin{pmatrix} \kappa_+ \\
\kappa_- \end{pmatrix}. \tag{15}$$

In contrast, the special solution (11) of the inhomogenous equation proportional to the energy density of the electromagnetic field: $\Phi - \Phi_0 \sim x^2 + y^2$, is of monopole type, being invariant under rotations in the transverse plane. Thus the plane electromagnetic wave is accompanied by a scalar gravitational wave, on which a free gravitational wave of quadrupole type can be superimposed.
4 Motion in the background of a plane wave

Classical motion of electrically neutral particles in the background of this specific gravitational wave is described in terms of geodesics [10, 11]. The worldline $X^\mu(\tau)$ of a massive particle parametrized by the proper time $\tau$ is restricted by the constraint

$$\dot{U} \dot{V} - \Phi(U, X^i) \dot{U}^2 + \dot{X}^i \dot{X}^i = 1,$$

with the overdot denoting a proper-time derivative. This constraint is one of the integrals of motion of the geodesic equation

$$\ddot{X}^\mu + \Gamma^\mu_{\lambda\nu}(X) \dot{X}^\lambda \dot{X}^\nu = 0,$$

The existence of a Killing vector of the metric defined by $\partial_i$ implies another constant of motion

$$\dot{U} = \gamma = \text{constant}.\quad (18)$$

As by definition of the laboratory velocity $\nu^a = dX^a/dT$ we get

$$\frac{dU}{dT} = 1 - v_z,\quad (19)$$

it follows from a rewriting of the constraint (16) that

$$\frac{1 - \nu^2}{(1 - v_z)^2} + \Phi = \frac{1}{\gamma^2}.\quad (20)$$

As $dU = \gamma d\tau$, the geodesic equations in the transverse plane can be written alternatively as

$$\frac{\partial^2 X^i}{\partial U^2} + \frac{1}{2} \frac{\partial \Phi}{\partial X^i} = 0.\quad (21)$$

In particular for the electromagnetic wave (11) these equations simplify to those of a 2-dimensional parametric oscillator:

$$\frac{\partial^2 X^i}{\partial U^2} + 2\pi G \left( E^2 + B^2 \right) X^i = 0.\quad (22)$$

In the special case $2\pi G (E^2 + B^2) = \mu^2 = \text{constant}$ the solutions take the form

$$X^i(U) = X^i_0 \cos \mu(U - U_0).\quad (23)$$

In this case the non-zero components of the Riemann tensor are

$$R_{\mu
u ij} = -\mu^2 \delta_{ij}.\quad (24)$$

Therefore the remarkable consequence is, that the geodesics oscillate in the transverse plane at a frequency proportional to the square root of the curvature. Scattering of
neutral test particles with a wavetrain of finite length has been discussed in this formalism in refs. \[10, 11\] and references therein.

5 Scalar fields in a plane-wave background

The gravitational field of the light wave can be probed by neutral test particles, as discussed in the previous section, or by electrically neutral fields of scalar, vector or spinor type, which at the classical level are sensitive only to the non-trivial gravitational background. At the quantum level these fields can describe e.g. pions, photons or neutrinos. As a first example we discuss a massive real scalar field \(\phi\), with action

\[
I[S] = -\frac{1}{2} \int d^4x \sqrt{-g} \left( g^{\mu\nu} \partial_\mu S \partial_\nu S + m^2 S^2 \right)
\]

(25)

where \(\nabla_\perp\) represents the gradient in the transverse plane. The corresponding field equation is

\[
\left( 4 \partial_\mu \partial_\nu - 4 \Phi \partial_\nu^2 - \partial_\mu^2 - \partial_\nu^2 + m^2 \right) S = 0.
\]

(26)

To solve this equation we introduce the expansion

\[
S(u, v, x_i) = \int \frac{dq ds}{2\pi} \kappa(q, s; x_i) e^{i(qv + su)},
\]

(27)

with \(\kappa^*(q, s; x_i) = \kappa(-q, -s; x_i)\). Note that in terms if standard space-time coordinates we get

\[
qv + su = (q + s)t + (q - s)z \equiv Et + pz \quad \leftrightarrow \quad q = \frac{1}{2}(E + p), \quad s = \frac{1}{2}(E - p).
\]

(28)

The field equation then constrains the amplitudes in the expansion to solutions of

\[
\left( -\partial_x^2 - \partial_y^2 + 4q^2 \Phi - 4sq + m^2 \right) \kappa = 0.
\]

(29)

For the special case \(24\) with constant energy density \(T_{\mu\nu}\) this equation becomes that of a 2-dimensional harmonic quantum oscillator:

\[
\left( -\partial_x^2 - \partial_y^2 + \omega_q^2 (x^2 + y^2) - 4sq + m^2 \right) \kappa = 0,
\]

(30)

where \(\omega_q = 2\mu|q|\). Introducing the notation \(\xi_i = \sqrt{\omega_q} x_i\) the solutions are of the form
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\[ \kappa(q, s; x_l) = \sum_{n_1, n_2=0}^{\infty} c_{n_1 n_2}(q, s) H_{n_1}(\xi_1) H_{n_2}(\xi_2) e^{-\left(\xi_1^2 + \xi_2^2\right)/2}, \quad (31) \]

where the \( H_{n}(\xi) \) are standard Hermite polynomials. It follows that the energy and momentum dispersion relation is quantized according to

\[ E^2 = p^2 + m^2 + 2(n_1 + n_2 + 1)\omega_q. \quad (32) \]

6 Spinor fields in a plane-wave background

Our second example is a Majorana spinor field \( \Psi = \Psi^c \equiv C\bar{\Psi}^T \), where \( C \) is the charge conjugation operator and \( T \) denotes transposition in spinor space (for our conventions on the Dirac algebra including charge conjugation, see the appendix); such a field can describe e.g. neutrinos of Majorana type.

As spinor fields are primarily defined in Minkowski space, we need to introduce the formalism of translating between the curved space-time manifold and the flat local tangent space-time; this is achieved by the use of vierbein-fields \( e^a_\mu \) such that

\[ g_{\mu\nu} = \eta_{abc} e^a_\mu e^b_\nu. \quad (33) \]

Here \( a \) labels vector components in the local Minkowski space, and \( \mu \) does the same thing in the curved space-time manifold. Using the vierbein fields one can define 1-forms \( E^a = e^a_\mu dx^\mu \), which for the metric (8) have the component form

\[ E^a = \left( \frac{1}{2} (dv + (\Phi + 1)du), dx, dy, \frac{1}{2} (dv + (\Phi - 1)du) \right). \quad (34) \]

Defining the inverse vierbein \( e'^a_\mu \) by

\[ e'^a_\mu e'^a_\nu = \delta^\mu_\nu, \quad (35) \]

there is a corresponding gradient operator

\[ \nabla_a = e'^a_\mu \partial_\mu = (\partial_u + (1 - \Phi)\partial_v, \partial_x, \partial_y, -(\partial_u + (1 + \Phi)\partial_v). \quad (36) \]

In order for the metric to be covariantly constant, the vierbein must satisfy the more general condition

\[ dE^a = \omega^a_\mu \wedge E^b, \quad (37) \]

where the antisymmetric-tensor valued 1-form \( \omega^{ab} = -\omega^{ba} = \omega^{ab}_\mu dx^\mu \) defines the spin connection. For the special vierbein (34) it is reduced to the form
\[
\omega_b^a = \omega_u^a \quad \text{du}, \quad \omega_u^a = -\frac{1}{2} \begin{pmatrix}
0 & \partial_x \Phi & 0 & 0 \\
\partial_x \Phi & 0 & 0 & -\partial_y \Phi \\
\partial_y \Phi & 0 & 0 & -\partial_x \Phi \\
0 & \partial_x \Phi & \partial_y \Phi & 0
\end{pmatrix},
\] (38)

modulo an arbitrary local Lorentz transformation in tangent space.

Defining \( \omega_a^{bc} = \omega_{\mu}^{bc} e^{\mu} \), the curved-space Dirac operator now is

\[
\gamma \cdot D = \gamma^a \left( \nabla_a - \frac{1}{2} \omega_a^{bc} \sigma_{bc} \right).
\] (39)

For our special metrics \( (8) \) or vierbeins \( (34) \) a great simplification is, that the spin-connection term \( \omega_a^{bc} \sigma_{bc} \) actually vanishes after contraction with \( \gamma^a \); therefore the Dirac operator simplifies to

\[
\gamma \cdot D = \gamma^a \nabla_a = -i \begin{pmatrix}
0 & \sigma_i \nabla_i + \nabla_0 \\
-\sigma_i \nabla_i + \nabla_0 & 0
\end{pmatrix},
\] (40)

and in the 2-component notation introduced in the appendix the Dirac equation becomes:

\[
2 \partial_v \chi_2 - (\partial_x + i \partial_y) \chi_1 = im \chi_1^*,
\] (41)

\[
(\partial_x - i \partial_y) \chi_2 - 2 (\partial_u - \Phi \partial_v) \chi_1 = im \chi_2^*.
\]

The complex conjugate components \( \chi^* \) can be eliminated by applying the complex conjugate Dirac equation to get

\[
\left( 4 \partial_u \partial_v - 4 \Phi \partial_v^2 - \partial_x^2 - \partial_y^2 + m^2 \right) \chi_1 = 0,
\] (42)

\[
\left( 4 \partial_u \partial_v - 4 \Phi \partial_v^2 - \partial_x^2 - \partial_y^2 + m^2 \right) \chi_2 = 2 (\partial_x + i \partial_y) \Phi \partial_v \chi_1.
\]

Therefore the general solution for \( \chi_1 \) is fully analogous to that for the scalar field \( S \), with the same spectrum of energy and momentum states; in contrast, the general solution for \( \chi_2 \) consists of a special solution, defined in terms of the solution for \( \chi_1 \) by the right-hand side of the second equation \( (42) \), plus an arbitrary solution of the homogeneous free Klein-Gordon equation, as for \( \chi_1 \). Therefore all solutions are found to have a structure similar to the scalar field, except that any non-trivial solution \( \chi_1 \) is accompanied by a special dependent solution for \( \chi_2 \) constructed from \( \chi_1 \) by

\[
\left[ - \left( \partial_x^2 + \partial_y^2 \right) + m^2 \right] \chi_2 = 2 (\partial_x + i \partial_y) \Phi \partial_v \chi_1 - 2 im (\partial_u - \Phi \partial_v) \chi_1^*. \] (43)
7 Massless abelian vector fields in a plane-wave background

Finally we describe the propagation of a massless abelian vector field in the plane-wave gravitational background. The Maxwell-action takes the form

\[
I[a] = \int dudvdx dy \left[ \left( \partial_\mu a_\nu - \partial_\nu a_\mu \right)^2 + \left( \partial_\mu a_i - \partial_i a_\mu \right) \left( \partial_\nu a_i - \partial_i a_\nu \right) \right. \\
\left. - \Phi (\partial_\nu a_i - \partial_i a_\nu)^2 - \frac{1}{8} \left( \partial_\nu a_j - \partial_j a_\nu \right)^2 \right].
\] (44)

The resulting field equations are

\[
4\partial_\mu \partial_\nu a_\nu = \Delta a_\nu - 2\partial_\nu \left[ \partial_\mu a_\nu + \partial_\nu a_\mu - \frac{1}{2} \partial_\mu a_i \right] = 0,
\]

\[
4\partial_\mu \partial_\nu a_\nu = \Delta a_\nu - 2\partial_\nu \left[ \partial_\mu a_\nu + \partial_\nu a_\mu - \frac{1}{2} \partial_\mu a_i \right] + 2\partial_\nu \left[ \Phi (\partial_\nu a_\nu - \partial_\nu a_i) \right] = 0,
\]

\[
-2\partial_\mu \partial_\nu a_i + \frac{1}{2} \Delta_\nu a_i + \partial_\nu \left[ \partial_\nu a_\nu + \partial_\nu a_\mu - \frac{1}{2} \partial_\nu a_i \right] - 2\partial_\nu \left[ \Phi (\partial_\nu a_\nu - \partial_\nu a_i) \right] = 0.
\] (45)

Gauge transformations \( a'_\mu = a_\mu + \partial_\mu \Lambda \) can be used to simplify these equations. First note that we can take

\[
\partial_\mu a'_\nu + \partial_\nu a'_\mu - \frac{1}{2} \partial_\mu a'_i = 0,
\] (46)

by taking \( \Lambda \) as the solution of

\[
\left( -4\partial_\mu \partial_\nu + \partial^2_\mu + \partial^2_\nu \right) \Lambda = 2 \left( \partial_\mu a_\nu + \partial_\nu a_\mu \right) - \partial_\mu a_i.
\] (47)

The remaining field equations are

\[
4\partial_\mu \partial_\nu a'_\nu - \left( \partial^2_\mu + \partial^2_\nu \right) a'_\nu = 0,
\]

\[
4\partial_\mu \partial_\nu a'_\mu - \left( \partial^2_\mu + \partial^2_\nu \right) a'_\mu + 2\partial_\nu \left[ \Phi (\partial_\nu a'_\nu - \partial_\nu a'_i) \right] = 0,
\] (48)

\[
4\partial_\mu \partial_\nu a'_i - \left( \partial^2_\mu + \partial^2_\nu \right) a'_i + 4\partial_\nu \left[ \Phi (\partial_\nu a'_\nu - \partial_\nu a'_i) \right] = 0.
\]

Next we can still make a residual gauge transformation to eliminate \( a'_\nu \) by taking \( \Lambda' \) restricted by

\[
\left( -4\partial_\mu \partial_\nu + \partial^2_\mu + \partial^2_\nu \right) \Lambda' = 0,
\]

\[
a''_\nu = a'_\nu + \partial_\nu \Lambda' = 0.
\] (49)

Then the gauge constraint (46) reduces to
\[
\partial_v a''_u = \frac{1}{2} \partial_i a''_i. \tag{50}
\]

Therefore we are left with

\[
4 \partial_u \partial_v a''_u - 4 \Phi \partial_v^2 a''_u - \left( \partial_v^2 + \partial_v^3 \right) a''_u = 2 \partial_i \Phi \partial_v a''_i, \tag{51}
\]

\[
4 \partial_u \partial_v a''_i - 4 \Phi \partial_i^2 a''_i - \left( \partial_i^2 + \partial_i^3 \right) a''_i = 0,
\]

This set of equations looks similar to that of the Majorana-Dirac equations in the previous section: the transverse components \(a''_i\) are solutions of the scalar Klein-Gordon equation, accompanied by a special fixed solution \(\bar{a}''_i\):

\[
- \left( \partial_i^2 + \partial_i^3 \right) \bar{a}''_i = -2 \partial_i \left[ (\partial_i - \Phi \partial_v) a''_i \right]. \tag{52}
\]

But in contrast to the Majorana-Dirac case there is no independent dynamical solution for \(a''_u\); as for vanishing \(a''_i = 0\) the homogeneous equation for \(a''_u\) implies its vanishing as well:

\[
\partial_v a''_u = 0 \quad \text{and} \quad \left( \partial_u^2 + \partial_u^3 \right) a''_u = 0. \tag{53}
\]

These conditions do not allow normalizable solutions for \(a''_u\); in fact, for \(a''_i = 0\) the longitudinal component \(a''_u\) can be gauged away by a third residual gauge transformation with a gauge function \(\Lambda''\) satisfying the constraints

\[
a''_u = a''_u + \partial_u \Lambda'' = 0, \quad \partial_v \Lambda'' = 0, \quad \left( \partial_v^2 + \partial_v^3 \right) \Lambda'' = 0. \tag{54}
\]

Therefore the transverse components are the only dynamical ones, taking the same form as solutions of the massless scalar wave equation, with the same spectrum of energy and momentum, whilst \(a''_u = \bar{a}''_u\) is a dependend field fixed entirely in terms of the transverse components by equation (52).

### 8 Conclusions

In summary, in the above it has been shown that the equations of motion of neutral test particles, and the field equations of neutral scalar fields, Majorana-Dirac spinor fields and abelian vector fields can all be solved in the background of gravitational \(pp\)-waves such as those accompanying infinite plane electromagnetic waves. For energy-momentum density of the source field constant in time, such as that of a circularly polarized plane light wave, a distinct signature is, that test particles oscillate in the transverse plane of the wave, whilst the spectrum of transverse momentum of the fields becomes discrete. Because of the small curvature to be expected from such waves, these effects will be difficult to observe; moreover, in realistic conditions beams of electromagnetic wave will be of finite width, introducing modifications to the above conclusions which still have to be considered. Nevertheless, as a matter
of principle the scattering of neutral particles by beams of electromagnetic waves will be another test in establishing the universality and dynamics of gravitational interactions.

Appendix: Spinors and the Dirac algebra

Spinor fields in curved space-time are most easily described in the tangent Minkowski space, using the vierbein formulation to translate the results to the curved space-time manifold. We use the flat-space representation of the Dirac algebra in which $\gamma_5$ is diagonal:

$$\gamma_0 = i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} 0 & -i\sigma_i \\ i\sigma_i & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

(55)

such that for $a = (0, 1, 2, 3)$

$$\{\gamma_a, \gamma_b\} = 2\eta_{ab} 1, \quad \gamma_5^2 = 1, \quad \{\gamma_5, \gamma_a\} = 0.$$  

(56)

The generators of the Lorentz transformations on spinors are defined by

$$\sigma_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left[ \gamma_a, \gamma_b \right],$$

(57)

with commutation relations

$$\left[ \sigma_{ab}, \sigma_{cd} \right] = \eta_{ad}\sigma_{bc} - \eta_{ac}\sigma_{bd} - \eta_{bd}\sigma_{ac} + \eta_{bc}\sigma_{ad}.$$  

(58)

Hermitean conjugation is achieved by

$$\gamma_a^\dagger = \gamma_0 \gamma_a \gamma_0.$$  

(59)

The charge conjugation operator $C$ is defined by

$$C = C^\dagger = C^{-1} = -C^T = \gamma_2 \gamma_0 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix},$$

(60)

such that

$$C^{-1} \gamma_a C = -\gamma_a^T.$$  

(61)

If the spinor $\Psi$ is a solution of the Dirac equation in Minkowski space

$$(\gamma \cdot \partial + m) \Psi = 0,$$

(62)

then this is also true for the charge-conjugate:

$$\Psi^c = C \overline{\Psi}^T = -\gamma_2 \Psi^T \Rightarrow (\gamma \cdot \partial + m) \Psi^c = 0.$$

(63)
The Majorana constraint $\Psi^c = \Psi$ reduces the number of independent spinor components from 4 to 2 complex ones. This makes it easy to work in terms of 2-component spinors $(\chi, \eta)$ which are eigenspinors of $\gamma_5$, by the decomposition

$$\Psi = \Psi^c = \begin{bmatrix} \chi \\ \eta \end{bmatrix} \quad \eta = -i\sigma_2 \chi^*.$$ (64)

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