Global stability of traveling waves with oscillations for Nicholson’s blowflies equation

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Abstract

For Nicholson’s blowflies equation, a kind of reaction-diffusion equations with time-delay, when the ratio of birth rate coefficient and death rate coefficient satisfies $\frac{b}{d} > e$, the large time-delay $r > 0$ usually causes the traveling waves to become oscillatory. In this paper, we are interested in the global stability of these oscillatory traveling waves, in particular, the challenging case of the critical traveling waves with oscillations. We prove that, the critical oscillatory traveling waves are globally stable with the algebraic convergence rate $t^{-1/2}$, and the non-critical traveling waves are globally stable with the exponential convergence rate $t^{-1/2}e^{-\mu t}$ for a positive constant $\mu$, where the initial perturbations around the oscillatory traveling wave in a weighted Sobolev can be arbitrarily large. The approach adopted is the technical weighted energy method with some new development in establishing the boundedness estimate of the oscillating solutions, which, with the help of optimal decay estimates by deriving the fundamental solutions for the linearized equations, can allow us to prove the global stability and to obtain the optimal convergence rates.

Keywords: Nicholson’s blowflies equation, time-delayed reaction-diffusion equation, critical traveling waves, oscillation, stability

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1 Introduction and main result

This is a continuation of the previous studies \[5, 27\] on the stability of oscillatory traveling waves for Nicholson’s blowflies equation, a class of non-monotone reaction-diffusion equations with time-delay, which describes the population dynamics of a single species like the Australian blowflies \[15, 16, 31, 32, 35, 45\]:

\[
\begin{aligned}
\frac{\partial v(t, x)}{\partial t} - D \frac{\partial^2 v(t, x)}{\partial x^2} + d(v(t, x)) &= b(v(t - r, x)), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}, \\
v(s, x) &= v_0(s, x), \quad s \in [-r, 0], \quad x \in \mathbb{R}.
\end{aligned}
\]  

(1.1)

Here, \(v(t, x)\) is the mature population at time \(t\) and location \(x\), \(D > 0\) the spatial diffusion rate of the mature blowflies. \(d(v)\) and \(b(v)\) represent Nicholson’s death rate function and Nicholson’s birth rate function, respectively, in the form

\[
d(v) = \delta v, \quad b(v) = pv e^{-av},
\]  

(1.2)

where \(\delta > 0\) is the death rate coefficient, \(p > 0\) is the maximal egg daily production rate per blowfly, \(a > 0\) is a constant, \(r > 0\) is the matured age of blowflies, the so-called time-delay.

Clearly, the equation (1.1) possesses two constant equilibria

\[v_- = 0 \quad \text{and} \quad v_+ = \frac{1}{a} \ln \frac{p}{\delta}.
\]  

When \(\frac{p}{\delta} > 1\), then \(v_+ > v_- = 0\). Throughout this paper, naturally we assume that

\[
\lim_{x \to \pm \infty} v_0(s, x) = v_\pm \quad \text{uniformly in} \quad s \in [-r, 0].
\]  

(1.3)

A traveling wave for (1.1) is a special solution to (1.1) of the form \(\phi(x + ct) \geq 0\) with \(\phi(\pm \infty) = v_\pm:\n
\[
\begin{aligned}
&c \phi'(\xi) - D \phi''(\xi) + \delta \phi(\xi) = b(\phi(\xi - cr)), \\
\phi(\pm \infty) &= v_\pm,
\end{aligned}
\]  

(1.4)

where \(\xi = x + ct\), \(c = \frac{d}{\delta}\), and \(c\) is the wave speed.

The main purpose of the paper is to prove the global stability of the oscillatory traveling waves as well as the optimal convergence rates.

First of all, let us review the progress in the existence of traveling waves. When \(1 < \frac{p}{\delta} \leq e\), by using the upper-lower solutions method, So and Zou \[46\] first proved that there exists a minimal wave speed \(c_* = c_*(r) > 0\), the so-called the critical wave speed, such that when \(c \geq c_*\), the equation (1.4) possesses monotone traveling waves \(\phi(x + ct)\), and no traveling waves exist for \(c < c_*\). The uniqueness (up to shift) of the traveling waves was proved by Aguerrea-Gomez-Trofimchuk \[1\] by means of the Diekmann-Kaper theory. The wave \(\phi(x + c_*t)\) with \(c = c_*\) is called the critical traveling wave, and the wave with \(c > c_*\) is usually called the non-critical traveling wave. When \(\frac{p}{\delta} > e\), the birth rate function \(b(v)\) is non-monotone under consideration of \(v \in [0, v_+]\), which causes that the equation (1.1) loses its monotonicity and the comparison principle does not hold. So the upper-lower solutions method cannot be applied to this case. By using the Lyapunov-Schmidt reduction method, Faria-Huang-Wu \[8\] showed that, when the time-delay is small...
enough $r ≪ 1$ and the wave speed is large enough $c ≫ c_*$, the monotone traveling waves $\phi(x + ct)$ exist. Later then, Faria-Trofimchuk [9] by analyzing heteroclinic solutions, and Ma [28] by constructing auxiliary functions, both showed that, when $e < p_\delta \leq e^2$, the traveling waves $\phi(x + ct)$ exist for all $c \geq c_*$. No restriction is on the time-delay $r > 0$ in this case. But, when the time-delay $r$ is big: $r > r_0$, where $r_0$ given by

$$r_0 = \frac{\pi - \arctan \sqrt{(\ln \frac{p_\delta}{\delta} - 2) \ln \frac{p_\delta}{\delta}}}{\delta \sqrt{(\ln \frac{p_\delta}{\delta} - 2) \ln \frac{p_\delta}{\delta}}} > 0$$

(1.7)

is the Hopf-bifurcation point to the equation (1.6).

In order to determine the minimal wave speed $c_* > 0$, let us linearize the equation (1.4) around $v_\xi = 0$ for $\xi \sim -\infty$, and test the eigenfunction by $\phi(\xi) = e^{\lambda \xi}$, we then have the characteristic equation for $\lambda > 0$:

$$c\lambda - D\lambda^2 + \delta = pe^{-\lambda r c}.$$  

The left-hand-side functions $F_c(\lambda) := c\lambda - D\lambda^2 + \delta$ and the right-hand-side function $G_c(\lambda) := pe^{-\lambda rc}$ of the above characteristic equation have a unique tangent point $(c_*, \lambda_*)$ with $c_*>0$ and $\lambda_*>0$, which is our minimal wave speed. For details, we refer to the graphs shown in [31]. Namely, $(c_*, \lambda_*)$ is uniquely determined by

$$c_*\lambda_* - D\lambda_*^2 + \delta = pe^{-\lambda_* c_* r} \quad \text{and} \quad c_* - 2D\lambda_* = -c_* r pe^{-\lambda_* c_* r},$$

(1.8)

and when $c > c_*$, there exist two numbers $\lambda_2 > \lambda_1 > 0$ such that

$$c\lambda_i - D\lambda_i^2 + \delta = pe^{-\lambda cr}, \quad \text{for} \quad i = 1, 2,$$

(1.9)

and

$$c\lambda - D\lambda^2 + \delta > pe^{-\lambda cr}, \quad \text{for} \quad \lambda \in (\lambda_1, \lambda_2).$$

(1.10)

As showed in [7, 9, 14, 28, 49, 50], see also the summary in [13, 27], we have the following existence and uniqueness of the traveling waves, as well as the property of oscillations:

1. When $e < \frac{p}{\delta} \leq e^2$, the traveling wave $\phi(x + ct)$ exists uniquely (up to a shift) for every $c \geq c_* = c_*(r)$, where the time-delay $r$ is allowed to be any number in $[0, \infty)$. If $0 \leq r < r_0$, where $r_0$ given by (1.5), is the critical point for the traveling waves to possibly occur oscillations, then these traveling waves are monotone [14]; while, if the time delay $r \geq r_0$, then the traveling waves are still monotone for
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\[(c, r) \in [c_*, c^*] \times [r_0, r]\], where \(c_* = c_*(r)\) is the minimum wave speed as mentioned before, \(c^* = c^*(r)\) is given by the characteristic equation for (1.4) around \(v_+\), namely, the pair of \((c^*, \lambda^*)\) is determined by

\[-c^* \lambda^* - D(\lambda^*)^2 + d = b'(v_+)e^{\lambda^*c^*r},\]  

(1.11)

and \(r_0(> r)\) is the unique intersection point of two curves \(c_*(r)\) and \(c^*(r)\); and the traveling waves are oscillating around \(v_+\) for \((c, r) \notin [c_*, c^*] \times [r_0, r]\), namely, either \(c > c^*\) or \(r > r_0\) (c.f. [14, 27]).

2. When \(\frac{p}{2} > e^2\), on the other hand, the traveling wave \(\phi(x + ct)\) with \(c \geq c_*\) can exist only when \(r < r\), and no traveling wave can exist for \(r \geq r\), where \(r\) is the Hopf-bifurcation point given in (1.7), and the waves are monotone for \(0 < r < r\) and oscillating for \(r \in (r, r)\) (c.f. [14, 27]).

To investigate the global stability of these oscillatory traveling waves will be our main target in this paper, including the critical wavefronts. The study on the critical traveling waves in the biological invasions is particularly interesting but also quite challenging, because the critical wave speed is usually the spreading speed for all solutions with initial data having compact supports [26, 47]. In what follows, we are going to review the progress on the stability of traveling waves for the type of mono-stable equations like Fisher-KPP equations without/with time-delay.

For regular mono-stable reaction-diffusion equations without time-delay \((r = 0)\) such as the classic Fisher-KPP equation, the existence of traveling waves and their stability have been one of the hot research spots. In 1976, by using the spectral analysis method, Sattinger [43] first proved that, for given non-critical waves with \(c > c_*\), when the initial perturbations around the waves are space-exponentially decay at the far field \(-\infty\), then these non-critical waves are time-exponentially stable. Since then, the study on stability of non-critical traveling waves has been intensively studied, for example, see [2, 4, 10, 11, 12, 17, 23, 24, 30, 44] and the references therein, see also the textbook [52] and the survey paper [54]. However, the study on stability of critical traveling waves with \(c = c_*\) is very limited, because this is critical case with special difficulty. In 1978, by using the maximum principle method, Uchiyama [51] proved the local stability for the traveling waves including the critical waves, but no convergence rate for the critical waves case was related. Later then, Bramson [3] derived the sufficient and necessary condition for the stability of noncritical and critical waves (no convergence rates issued) by probability method. Lau [24] obtained the same results in a different way. Regarding the convergence rates to the critical traveling waves, Moet [37] first obtained the algebraic convergence rate \(O(t^{-1/2})\) by using the Green function method. Kirchgässner [23] then showed the algebraic stability for the critical waves in the form \(O(t^{-1/4})\) by the spectral method. Gallay [11] further improved the algebraic rate to \(O(t^{-3/2})\) by using the renormalization group method, when the corresponding initial data converges to the critical wave much fast like \(O(e^{-x^2/4})\) as \(x \to -\infty\). More general case for parabolic equations was investigated by Eckmann and Wayne in [6].

For the mono-stable reaction-diffusion equations with time-delay, in 1987 Schaaf [42] first proved the linear stability for the non-critical traveling waves by the spectral analysis method. This topic was not touched until in 2004 Mei-So-Li-Shen [35] proved the nonlinear
stability of fast traveling waves with $c \gg c_*$ by the weighted energy method. When the equation is monotone (namely, $b(v)$ is increasing), Mei and his collaborators [31, 32, 33, 34, 36] further showed that all non-critical traveling waves are exponentially stable and all critical traveling waves are algebraically stable. When $b(v)$ is non-monotone, the equation (1.1) loses its monotonicity. The solution is usually oscillating as the time-delay $r$ is large, and the traveling waves may occur oscillations around $v_+$, as theoretically proved in [14, 50] and numerically reported in [5, 27]. Such oscillations of the traveling waves are interesting and important from both physical and mathematical points of view. Recently, by using the weighted energy method with the help of nonlinear Hanalay’s inequality, Lin-Lin-Lin-Mei [27] proved that when the initial perturbation is small, then all non-critical oscillatory traveling waves are locally stable with exponential convergence rate. Furthermore, by analyzing the decay rate of the (oscillatory) critical traveling waves at the unstable node $v_− = 0$, and applying the anti-weighted energy method, Chern-Mei-Yang-Zhang [5] obtained the local stability for the critical oscillatory traveling waves. But the convergence rate to the critical waves is still open. The interesting but also challenging questions are whether these oscillatory wavefronts are globally stable, and what will be the optimal convergence rates, particularly the convergence rate for the critical wavefronts. Note that, the existing methods cannot be applied to our case, due to the lack of monotonicity of the equation and the waves, and the bad effect of time-delay. Of course, the critical wave case is always challenging as we know.

In this paper, the main targets are to prove that all critical or non-critical oscillatory traveling waves are globally stable, and further to derive the optimal convergence rates to the critical/non-critical oscillatory traveling wave. That is, for all oscillatory traveling waves, including the critical traveling waves, the original solution of (1.1) will time-asymptotically converge to the targeted wave, even if the corresponding initial perturbation in a certain weighted space is big. The optimal convergence rates for the critical/non-critical wavefronts are $O(t^{−\frac{1}{2}})$ and $O(t^{−\frac{1}{2}}e^{−\mu t})$, respectively. As mentioned in [5], the usual approaches for deriving the convergence rate are either the monotonic method with the help of the decay estimates for linearized equations [31, 33], or Fourier transform [20, 36], or the multiplier method [41, 48], or the method of approximate Green function [40, 53]. Since our equation is lack of monotonicity and the traveling waves are oscillating, and the bad effect of time-delay, it seems that we could not be able to adopt these methods mentioned before. However, we have some key observation, that is, although the equation and the waves are non-monotone, and we lose the comparison principle, due to the structure of the governing equation, we realize that, after using the anti-weight technique, the perturbed equation is reduced to a new equation, and the absolute value of the oscillating solution for the new equation can be bounded by the positive solution of linear delayed heat equation with constant coefficients. This can make us possibly to reach our goal by deriving the optimal decay estimate for the linearized solution, where, by using Fourier transform, we derive the fundamental solution for the time-delayed linear heat equation and its optimal convergence rate. Based on the boundedness estimates for the oscillatory solution, we further prove the global stability for the oscillatory traveling waves. To our best knowledge, this is the first framework to show the global stability for the oscillatory traveling waves with the optimal convergence rates, particularly, for the critical wave case.

Before stating our main results, we first introduce some notations on the solution spaces. Throughout this paper, $C > 0$ denotes for a generic constant, and $C_i > 0$ ($i =$
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0, 1, 2, ⋅⋅⋅) for specific positive constants. $L^2(\mathbb{R})$ is the space of the square integrable functions, $H^k(\mathbb{R})$ is the Sobolev space, $C(\mathbb{R})$ is the space of bounded continuous functions, $C([0, T]; \mathcal{B})$ is the space of the $\mathcal{B}$-valued continuous functions on $[0, T]$, where $\mathcal{B}$ is a Banach space, and $T > 0$ is a number. Similarly, $L^2([0, T]; \mathcal{B})$ is the space of the $\mathcal{B}$-valued $L^2$-functions on $[0, T]$.

To handle delay equation with delay $r$, as denoted in [5, 27], we define the uniformly continuous space $C_{unif}[-r, T]$, for $0 < T \leq \infty$, by

$$C_{unif}[-r, T] := \{v(t, x) \in C([-r, T] \times \mathbb{R}) \text{ such that}$$

$$\lim_{x \to +\infty} v(t, x) \text{ exists uniformly in } t \in [-r, T], \text{ and}$$

$$\lim_{x \to +\infty} v_x(t, x) = \lim_{x \to +\infty} v_{xx}(t, x) = 0$$

uniformly with respect to $t \in [-r, T]$.}

For perturbed equation around the traveling waves, we now define the solution space. First, we introduce a weight function. For $c \geq c_*$, we define a weight function

$$w(\xi) := \begin{cases} e^{-2\lambda_c \xi}, & \xi \in \mathbb{R}, \text{ for } c > c_*, \lambda \in (\lambda_1, \lambda_2), \\ e^{-2\lambda_c \xi}, & \xi \in \mathbb{R}, \text{ for } c = c_*, \end{cases}$$

where $\xi = x + ct$ for $(t, x) \in [-r, \infty) \times \mathbb{R}$, and the number $\lambda$ for the case of $c > c_*$ is selected between $\lambda_1$ and $\lambda_2$, and $\lambda$ and $\lambda_2$ are specified in [13]. Notice that, for $c \geq c_*$, $\lim_{\xi \to +\infty} w(\xi) = +\infty$ and $\lim_{\xi \to -\infty} w(\xi) = 0$, because $\lambda > 0$ and $\lambda_* > 0$. We also denote the weighted Sobolev space $W^{2,1}_w(\mathbb{R})$ by

$$W^{2,1}_w(\mathbb{R}) = \{u | w^i u \in L^1(\mathbb{R}), w^i \partial_\xi^j u \in L^1(\mathbb{R}), i = 1, 2\}.$$

Our main stability theorems are as follows.

**Theorem 1.1 (Global stability with optimal convergence rates)** For any given traveling wave $\phi(x + ct) = \phi(\xi)$ with $c \geq c_*$, no matter it is oscillatory or not, when the initial perturbation satisfies $v_0(s, \xi) - \phi(\xi) \in C_{unif}(-r, 0) \cap C([-r, 0]; W^{2,1}_w(\mathbb{R}))$ and $\partial_s(v_0 - \phi) \in L^1([-r, 0]; L^1(\mathbb{R}))$, then the following global stability holds.

1. When $e < \frac{\mu}{2} \leq e^2$, for any time-delay $r > 0$, then

$$\left\{ \begin{array}{ll} \sup_{t \in \mathbb{R}} |v(t, x) - \phi(x + ct)| \leq Ct^{-\frac{1}{2}}e^{-\mu t} & \text{for } c > c_*, \\
\sup_{t \in \mathbb{R}} |v(t, x) - \phi(x + c_* t)| \leq Ct^{-\frac{1}{2}} & \text{for } c = c_* \end{array} \right.$$ (1.14)

where $\mu$ is a positive number satisfying

$$0 < \mu < \min \{\delta, c_\lambda + \delta - D\lambda^2 - pe^{-\lambda c r}\}, \lambda \in (\lambda_1, \lambda_2).$$ (1.15)

2. When $\frac{\mu}{2} > e^2$ but with a small time-delay $0 < r < \bar{r}$, where $\bar{r}$ is defined in [1.7], then the stability (1.14) holds.
Remark 1.1  
1. When \( 1 < \frac{p}{\delta} \leq e \), the traveling waves \( \phi(x + ct) \) for all \( c \geq c_\ast \) are monotone, and the global stability of these monotone critical/non-critical waves as well as the optimal convergence rates were shown by Mei-Ou-Zhao in \([33]\).

2. When \( e < \frac{p}{\delta} \leq e^2 \) with any size of time-delay \( r > 0 \), or \( \frac{p}{\delta} > e^2 \) but with small time-delay \( r \in (0, \bar{r}) \), the local stability for the critical/non-critical wavefronts was proved in \([?, ?]\), respectively. Here in Theorem 1.1, we obtain the global stability of these oscillatory traveling waves, where the initial perturbations around the traveling waves in the weighted space \( C_{\text{unif}}(-r, 0) \cap C([-r, 0]; W^{2,1}_w(\mathbb{R})) \) can be arbitrarily large. To our best knowledge, this is the first work to show the global stability for the oscillatory waves.

3. When \( c = c_\ast \), we obtain the optimally algebraic convergence rate \( O(t^{-\frac{1}{2}}) \) to the critical oscillating waves. This answers the open question left in \([5]\). The algebraic convergence rate also matches what shown for the critical monotone traveling waves for Fisher-KPP equations with or without time-delay \([33, 36, 37]\).

4. When \( c > c_\ast \), the obtained exponential convergence \( O(e^{-\mu t^{-\frac{1}{2}}}) \) improves the previous study for the non-critical oscillating waves in \([27]\).

2 Proof of main theorem

Let \( \phi(x + ct) = \phi(\xi) \) be any given monotone/non-monotone traveling wave with \( c \geq c_\ast \), and define

\[
 u(t, \xi) := v(t, x) - \phi(x + ct), \quad u_0(s, \xi) := v_0(s, x) - \phi(x + cs), \quad c \geq c_\ast.
\]

Then, from \((1.1)\), \( u(t, \xi) \) satisfies, for \( c \geq c_\ast \),

\[
\begin{aligned}
\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial \xi} - D\frac{\partial^2 u}{\partial \xi^2} + \delta u &= P(u(t - r, \xi - cr)), \quad (t, \xi) \in \mathbb{R}_+ \times \mathbb{R}, \\
u(s, \xi) &= u_0(s, \xi), \quad s \in [-r, 0], \quad \xi \in \mathbb{R}, \tag{2.1}
\end{aligned}
\]

where

\[
P(u) := b(\phi + u) - b(\phi) = b'(\tilde{\phi})u,
\]  

for some \( \tilde{\phi} \) between \( \phi \) and \( \phi + u \), with \( \phi = \phi(\xi - cr) \) and \( u = u(t - r, \xi - cr) \) for \( c \geq c_\ast \).

We first prove the existence and uniqueness of solution to the initial value problem \((2.1)\) in the uniformly continuous space \( C_{\text{unif}}[-r, \infty) \).

**Proposition 2.1 (Existence and Uniqueness)** When \( e < \frac{p}{\delta} \leq e^2 \) for any \( r > 0 \), or \( \frac{p}{\delta} > e^2 \) but for \( r \in (0, \bar{r}) \), if the initial perturbation \( u_0 \in C_{\text{unif}}[-r, 0] \) for \( c \geq c_\ast \), then the solution \( u(t, \xi) \) of the perturbed equation \((2.1)\) is unique and time-globally exits in \( C_{\text{unif}}[-r, \infty) \).
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**Proof.** When \( t \in [0, r] \), since \( t - r \in [-r, 0] \) and \( u(t - r, \xi - cr) = u_0(t - r, \xi - cr) \), then (2.1) reduce to the following linear equation

\[
\begin{align*}
\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial \xi} - D \frac{\partial^2 u}{\partial \xi^2} + \delta u &= P(u_0(t - r, \xi - cr)), \quad (t, \xi) \in \mathbb{R}_+ \times \mathbb{R}, \\
u(0, \xi) &= u_0(0, \xi), \quad \xi \in \mathbb{R},
\end{align*}
\]

(2.3) and it possesses a unique solution \( u(t, \xi) \) for \( t \in [0, r] \) in the integral form of

\[
u(t, \xi) = \begin{cases} 
 e^{-\delta t} \int_{-\infty}^{\xi} G(t, \eta)u_0(0, \eta)d\eta \\
 + \int_{0}^{t} e^{-\delta(t-s)} \int_{-\infty}^{\infty} G(t-s, \eta)P(u_0(s-r, \xi - cr))d\eta ds, \quad (2.4)
\end{cases}
\]

where \( G(t, \eta) \) is the heat kernel

\[
G(t, \eta) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{(\eta - \xi)^2}{4Dt}}, \quad c \geq c_*.
\]

Since \( u_0 \in C_{unif}[-r, 0] \), namely, \( \lim_{\xi \to \infty} u_0(t, \xi) = u_0(t, \infty) \) and \( \lim_{\xi \to \infty} u_0, \xi(t, \xi) = 0 \) and \( \lim_{\xi \to \infty} u_0, \xi(t, \xi) = 0 \) uniformly in \( t \in [-r, 0] \), we immediately prove the following uniform convergence

\[
\lim_{\xi \to \infty} u(t, \xi) = \begin{cases} 
 e^{-\delta t} \int_{-\infty}^{\xi} G(t, \eta) \lim_{\xi \to \infty} u_0(0, \eta)d\eta \\
 + \int_{0}^{t} e^{-\delta(t-s)} \int_{-\infty}^{\infty} G(t-s, \eta) \lim_{\xi \to \infty} P(u_0(s-r, \xi - cr))d\eta ds
\end{cases}
\]

(2.5)

and

\[
\lim_{\xi \to \infty} \frac{\partial^k u(t, \xi)}{\partial \xi^k} = \begin{cases} 
 e^{-\delta t} \int_{-\infty}^{\infty} \frac{\partial^k G(t, \eta)}{\partial \eta^k} \lim_{\xi \to \infty} u_0(0, \eta)d\eta \\
 + \int_{0}^{t} e^{-\delta(t-s)} \int_{-\infty}^{\infty} \frac{\partial^k G(t-s, \eta)}{\partial \eta^k} \lim_{\xi \to \infty} P(u_0(s-r, \xi - cr))d\eta ds
\end{cases}
\]

(2.6)

\[
= 0, \quad \text{for } k = 1, 2, \quad \text{uniformly in } t \in [0, r].
\]
Thus, we have proved $u \in C_{\text{unif}}[-r, r]$.

Now we consider (2.1) for $t \in [r, 2r]$. Since $t - r \in [0, r]$ and $u(t - r, \xi - cr)$ is solved already in last step for (2.9), thus $P(u(t - r, \xi - cr))$ is known for (2.1) with $t \in [0, 2r]$, namely, the equation (2.1) is linear for $t \in [0, 2r]$. As showed before, we can similarly prove the existence and uniqueness of the solution $u$ to (2.1) for $t \in [0, 2r]$, and particularly $u \in C_{\text{unif}}[-r, 2r]$.

By repeating this procedure for $t \in [nr, (n + 1)r]$ with $n \in \mathbb{Z}_+$ (the set of all positive integers), we prove that there exists a unique solution $u \in C_{\text{unif}}([-r, (n + 1)r]$ for (2.1), and step by step, we finally prove the uniqueness and time-global existence of the solution $u \in C_{\text{unif}}[-r, \infty)$ for (2.1). The proof is complete. □

The most important part of the paper is to prove the following global stability with the optimal convergence rates.

**Proposition 2.2 (Stability with optimal convergence rates)** When $e < \frac{P}{2} \leq e^2$ for any $r > 0$, or $\frac{P}{2} > e^2$ but for $r \in (0, \tilde{r})$, if $u_0 \in C_{\text{unif}}(-r, 0) \cap L^1([-r, 0]; W^{2,1}_w(\mathbb{R}))$ and $\partial_\xi u_0 \in L^1([-r, 0]; L^1_w(\mathbb{R}))$ for $c \geq c_*$, no matter how large the initial perturbation $u_0$ is, then

- when $c = c_*$, it holds
  
  \[ \sup_{\xi \in \mathbb{R}} |u(t, \xi)| \leq Ct^{-\frac{1}{2}}. \tag{2.7} \]

- when $c > c_*$, it holds
  
  \[ \sup_{\xi \in \mathbb{R}} |u(t, \xi)| \leq Ct^{-\frac{1}{2}}e^{-\mu t} \tag{2.8} \]

for a positive constant $\mu$ specified in (1.15).

In order to prove Proposition 2.2, we need several lemmas to complete it. Since $b'(0) = p > \delta = d'(0)$, namely, $u_- = 0$ is the unstable node of (2.1), heuristically, for a general initial data $u_0$, we cannot expect the convergence $u \to 0$ as $t \to \infty$. But, inspired by (12) for the equation (2.1) with or without time-delay, we expect the solution $u$ to decay to zero when the initial perturbation is only exponentially decay at the far field $\xi = -\infty$. We need the anti-weighted energy technique to treat this problem. Let us define

\[ u(t, \xi) = [w(\xi)]^{-\frac{1}{2}} \tilde{u}(t, \xi), \quad \text{i.e.,} \quad \tilde{u}(t, \xi) = \sqrt{w(\xi)}u(t, \xi) = e^{-\lambda \xi}u(t, \xi), \tag{2.9} \]

where $\lambda \in (\lambda_1, \lambda_2)$ for $c > c_*$ and $\lambda = \lambda_*$ for $c = c_*$, we get the following equations for the new unknown $\tilde{u}(t, \xi)$ with $c \geq c_*$:

\[
\begin{aligned}
\frac{\partial \tilde{u}}{\partial t} - D \frac{\partial^2 \tilde{u}}{\partial \xi^2} + a_0(c) \frac{\partial \tilde{u}}{\partial \xi} + a_1(c) \tilde{u} &= \tilde{P}(\tilde{u}(t - r, \xi - cr)), \quad (t, \xi) \in \mathbb{R}_+ \times \mathbb{R}, \\
\tilde{u}(s, \xi) &= \sqrt{w(\xi)}u(s, \xi) =: \tilde{u}_0(s, \xi), \quad s \in [-r, 0], \quad \xi \in \mathbb{R},
\end{aligned} \tag{2.10}
\]
where
\[ a_0(c) := c - 2D\lambda, \quad a_1(c) := c\lambda + \delta - D\lambda^2, \quad c \geq c_*, \] (2.11)
which satisfy (by (1.8) and (1.10))
\[ a_1(c) = c\lambda + \delta - D\lambda^2 > pe^{-\lambda cr}, \quad \text{for } c > c_*, \lambda \in (\lambda_1, \lambda_2), \] (2.12)
\[ a_1(c_*) = c_*\lambda_+ + \delta - D\lambda_+^2 = pe^{-\lambda_* c_* r}, \quad \text{for } c = c_*. \] (2.13)

Here,
\[ \tilde{P}(\tilde{u}) = e^{-\lambda \xi} P(u) \] (2.14)
satisfies (by Taylor’s expansion formula)
\[ \tilde{P}(\tilde{u}(t-r, \xi - cr)) = e^{-\lambda \xi} P(u(t-r, \xi - cr)) \]
\[ = e^{-\lambda \xi} b'((\tilde{\phi})u(t-r, \xi - cr)) \]
\[ = e^{-\lambda cr} b'((\tilde{\phi})u(t-r, \xi - cr)), \quad c \geq c_*, \] (2.15)
for some function \( \tilde{\phi} \) between \( \phi \) and \( \phi + u \) with \( c \geq c_* \) (see (2.2)). We further have
\[ |\tilde{P}(\tilde{u}(t-r, \xi - cr))| \leq pe^{-\lambda cr}|\tilde{u}(t-r, \xi - cr)|. \] (2.16)

Since \( b'(s) \) can be negative for \( s \in (0, v_+) \), then the solution \( \tilde{u} \) for the equation (2.10) with the nonlinear term (2.14) will be oscillating around \( v_+ \), when the time-delay \( r \) is large, as numerically reported in [5, 27], and the comparison principle doesn’t hold in this case. So the monotonic technique cannot be applied. On the other hand, in (2.15), since the coefficient \( b'((\tilde{\phi}) \) is variable, we are unable to derive the decay rate directly by applying Fourier’s transform. However, by a deep observation we may establish a crucial boundedness estimate for the oscillating solution \( \tilde{u}(t, \xi) \). In order to look for such a boundedness, let us consider the following linear delayed reaction-diffusion equation, for \( c \geq c_*, \)
\[
\begin{align*}
\frac{\partial u^+}{\partial t} - D\frac{\partial^2 u^+}{\partial \xi^2} + a_0(c)\frac{\partial u^+}{\partial \xi} + a_1(c)u^+ \\
= pe^{-\lambda cr}u^+(t-r, \xi - cr), \quad (t, \xi) \in \mathbb{R}_+ \times \mathbb{R},
\end{align*}
\] (2.17)
\[ u^+(s, \xi) = u_0^+(s, \xi) \geq 0, \quad s \in [-r, 0], \xi \in \mathbb{R}. \]

**Lemma 2.1** When \( u_0^+(s, \xi) \geq 0 \) for \( (s, \xi) \in [-r, 0] \times \mathbb{R} \), then \( u^+(t, \xi) \geq 0 \) for \( (t, \xi) \in [-r, \infty) \times \mathbb{R} \).

**Proof.** As showed in [31], we can similarly prove the positiveness of the solution \( u^+(t, \xi) \). In fact, when \( t \in [0, r] \), we have \( t-r \in [-r, 0] \) and
\[ pe^{-\lambda cr}u^+(t-r, \xi - cr)) = pe^{-\lambda cr}u_0^+(t-r, \xi - cr)) \geq 0. \]
Then (2.17) implies
\[ \frac{\partial u^+}{\partial t} - D\frac{\partial^2 u^+}{\partial \xi^2} + a_0(c)\frac{\partial u^+}{\partial \xi} + a_1(c)u^+ \geq 0, \]
which gives \( u^+(t, \xi) \geq 0 \) for \( t \in [0, r] \). Repeating this procedure step by step, we can prove \( u^+(t, \xi) \geq 0 \) for \( t \in [nr, (n+1)r] \), and further \( u^+(t, \xi) \geq 0 \) for \( t \in \mathbb{R}_+ \). The proof is complete. □

Next we establish the following crucial boundedness estimate for the oscillating solution \( \tilde{u}(t, \xi) \) of (2.1).
Lemma 2.2 (Boundedness estimate) Let \( \tilde{u}(t, \xi) \) and \( u^+(t, \xi) \) be the solutions of (2.1) and (2.17), respectively. When
\[
|\tilde{u}_0(s, \xi)| \leq u^+_0(s, \xi), \quad (s, \xi) \in [-r, 0] \times \mathbb{R},
\] (2.18)
then
\[
|\tilde{u}(t, \xi)| \leq u^+(t, \xi), \quad (t, \xi) \in \mathbb{R}_+ \times \mathbb{R}.
\] (2.19)

Proof. First of all, we prove \( |\tilde{u}(t, \xi)| \leq u^+(t, x) \) for \( t \in [0, r] \). In fact, when \( t \in [0, r] \), namely, \( t - r \in [-r, 0] \), from (2.15), we have
\[
|\tilde{u}(t-r, \xi - cr)| = |\tilde{u}_0(t-r, \xi - cr)| \leq u^+_0(t-r, \xi - cr) = u^+(t-r, \xi - cr), \quad \text{for } t \in [0, r] \text{ and } c \geq c_*.
\] (2.20)

Let 
\[
U^-(t, \xi) := u^+(t, \xi) - \tilde{u}(t, \xi) \quad \text{and} \quad U^+(t, \xi) := u^+(t, \xi) + \tilde{u}(t, \xi),
\]
we are going to estimate \( U^\pm(t, \xi) \) respectively.

From (2.10), (2.15) and (2.17), then \( U^-(t, \xi) \) satisfies, for \( c \geq c_* \),
\[
\frac{\partial U^-}{\partial t} + a_0(c)\frac{\partial U^-}{\partial \xi} - D\frac{\partial^2 U^-}{\partial \xi^2} + a_1(c)U^- = b'(0)e^{-\lambda cr}u^+(t-r, \xi - cr) - b'(\hat{\phi})e^{-\lambda cr}\tilde{u}(t-r, \xi - cr)
\geq b'(0)e^{-\lambda cr}u^+(t-r, \xi - cr) - |b'(\hat{\phi})|e^{-\lambda cr}|\tilde{u}(t-r, \xi - cr)|
\geq 0, \quad t \in [0, r],
\] (2.21)
where \( \lambda \in (\lambda_1, \lambda_2) \) for \( c > c_* \) and \( \lambda = \lambda_* \) for \( c = c_* \). Here we used (2.20) for \( |\tilde{u}(t-r, \xi - cr)| \leq |u^+(t-r, \xi - cr)| \) and the fact
\[
|b'(\hat{\phi})| = p|1 - a_\hat{\phi}|e^{-a_\hat{\phi}} \leq p = b'(0) \quad \text{for any } \hat{\phi} > 0.
\] (2.22)

Thus, (2.21) with the initial data \( U^-_0(s, \xi) = u^+_0(s, \xi) - \tilde{u}_0(s, \xi) \geq 0 \) reduces to
\[
\begin{cases}
\frac{\partial U^-}{\partial t} + a_0(c)\frac{\partial U^-}{\partial \xi} - D\frac{\partial^2 U^-}{\partial \xi^2} + a_1(c)U^- \geq 0, \\
U^-_0(0, \xi) \geq 0,
\end{cases}
\]
which, by the regular comparison principle for parabolic equation without delay, implies
\[
U^-(t, \xi) = u^+(t, \xi) - \tilde{u}(t, \xi) \geq 0 \quad \text{for } (t, \xi) \in [0, r] \times \mathbb{R} \text{ and } c \geq c_*.
\] (2.23)

On the other hand, \( U^+(t, \xi) \) satisfies
\[
\frac{\partial U^+}{\partial t} + a_0(c)\frac{\partial U^+}{\partial \xi} - D\frac{\partial^2 U^+}{\partial \xi^2} + a_1(c)U^+ = b'(0)e^{-\lambda cr}u^+(t-r, \xi - cr) + b'(\hat{\phi})e^{-\lambda cr}\tilde{u}(t-r, \xi - cr)
\geq b'(0)e^{-\lambda cr}u^+(t-r, \xi - cr) - |b'(\hat{\phi})|e^{-\lambda cr}|\tilde{u}(t-r, \xi - cr)|
\]
The proof is complete. □

Repeating this procedure, we then further prove which implies

\[ u_0^+(s, \xi) + \tilde{u}_0(s, \xi) \geq 0 \quad \text{implies} \]
\[ U^+(t, \xi) = u^+(t, \xi) + \tilde{u}(t, \xi) \geq 0 \quad \text{for} \quad (t, \xi) \in [0, r] \times \mathbb{R} \quad \text{and} \quad c \geq c_* . \]  

Combining (2.24) and (2.25), we prove

\[ |\tilde{u}(t, \xi)| \leq u^+(t, \xi) \quad \text{for} \quad (t, \xi) \in [0, r] \times \mathbb{R} \quad \text{and} \quad c \geq c_* . \]  

Next, when \( t \in [r, 2r] \), namely, \( t - r \in [0, r] \), based on (2.26) we can similarly prove

\[ \left\{ \begin{array}{ll}
U^-(t, \xi) = u^+(t, \xi) - \tilde{u}(t, \xi) \geq 0, \\
U^+(t, \xi) = u^+(t, \xi) + \tilde{u}(t, \xi) \geq 0,
\end{array} \right. \quad \text{for} \quad (t, \xi) \in [r, 2r] \times \mathbb{R} \quad \text{and} \quad c \geq c_* ,
\]

namely,

\[ |\tilde{u}(t, \xi)| \leq u^+(t, \xi) \quad \text{for} \quad (t, \xi) \in [r, 2r] \times \mathbb{R} \quad \text{and} \quad c \geq c_* . \]  

Repeating this procedure, we then further prove

\[ |\tilde{u}(t, \xi)| \leq u^+(t, \xi) \quad \text{for} \quad (t, \xi) \in [nr, (n + 1)r] \times \mathbb{R} \quad \text{and} \quad c \geq c_* , \quad n = 1, 2, \cdots , \]  

which implies

\[ |\tilde{u}(t, \xi)| \leq u^+(t, \xi) \quad \text{for} \quad (t, \xi) \in [0, \infty) \times \mathbb{R} \quad \text{and} \quad c \geq c_* . \]  

The proof is complete. □

Next we derive the global stability with the optimal convergence rates for the linear equation (2.17) by using the weighted energy method and by carrying out the crucial estimates on the fundamental solutions. In order to derive the optimal decay rates for the solution of (2.17) in the cases of \( c > c_* \) and \( c = c_* \), respectively, we first need to derive the fundamental solution, then show the optimal decay rates of the fundamental solution. Now let us recall the properties of the solutions to the delayed ODE.

**Lemma 2.3** ([22]) Let \( z(t) \) be the solution to the following linear time-delayed ODE with time-delay \( r > 0 \) and two constants \( k_1 \) and \( k_2 \)

\[ \frac{d}{dt} z(t) + k_1 z(t) = k_2 z(t - r) \]
\[ z(s) = z_0(s), \quad s \in [-\tau, 0]. \]  

Then

\[ z(t) = e^{-k_1 (t + r)} e^{\bar{k}_2 \tau} z_0(-r) + \int_{-\tau}^{0} e^{-k_1 (t - s)} e^{\bar{k}_2 (t - r - s)} [z_0(s) + k_1 z_0(s)] ds , \]  

where

\[ \bar{k}_2 := k_2 e^{k_1 \tau} , \]  

\[ \bar{k}_2 := k_2 e^{k_1 \tau} , \]  

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\[ \bar{k}_2 := k_2 e^{k_1 \tau} , \]
and \( e^{\tilde{k}_2 t} \) is the so-called delayed exponential function in the form

\[
ed^{\tilde{k}_2 t} = \begin{cases} 
0, & -\infty < t < -r, \\
1, & -r \leq t < 0, \\
1 + \frac{\tilde{k}_2 t}{1!}, & 0 \leq t < r, \\
1 + \frac{\tilde{k}_2 t}{1!} + \frac{\tilde{k}_2^2 (t-r)^2}{2!}, & r \leq t < 2r, \\
\vdots & \\
1 + \frac{\tilde{k}_2 t}{1!} + \frac{\tilde{k}_2^2 (t-r)^2}{2!} + \cdots + \frac{\tilde{k}_2^m |t-(m-1)r|^m}{m!}, & (m-1)r \leq t < mr, \\
\vdots & 
\end{cases}
\]  

(2.33)

and \( e^{\tilde{k}_2 t} \) is the fundamental solution to

\[
\begin{aligned}
\frac{d}{dt}z(t) &= \tilde{k}_2 z(t-r) \\
z(s) &= \begin{cases} 
1, & s \in [-r, 0]. 
\end{cases}
\end{aligned}
\]  

(2.34)

The property of the solution to the delayed linear ODE (2.30) is well-known [36].

**Lemma 2.4** [36] Let \( k_1 \geq 0 \) and \( k_2 \geq 0 \). Then the solution \( z(t) \) to (2.30) (or equivalently (2.31)) satisfies

\[
|z(t)| \leq C_0 e^{-k_1 t} \tilde{k}_2 t,
\]  

(2.35)

where

\[
C_0 := e^{-k_1 r}|z_0(-r)| + \int_{-r}^{0} e^{k_1 s}|z_0(s) + k_1 z_0(s)|ds,
\]  

(2.36)

and the fundamental solution \( e^{\tilde{k}_2 t} \) with \( \tilde{k}_2 > 0 \) to (2.31) satisfies

\[
e^{\tilde{k}_2 t} \leq C(1 + t)^{-\gamma} e^{\tilde{k}_2 t}, \quad t > 0
\]  

(2.37)

for arbitrary number \( \gamma > 0 \).

Furthermore, when \( k_1 \geq k_2 \geq 0 \), there exists a constant \( 0 < \varepsilon_1 < 1 \) such that

\[
e^{-k_1 t} e^{\tilde{k}_2 t} \leq C e^{-\varepsilon_1 (k_1-k_2) t}, \quad t > 0
\]  

(2.38)

and the solution \( z(t) \) to (2.30) satisfies

\[
|z(t)| \leq C e^{-\varepsilon_1 (k_1-k_2) t}, \quad t > 0.
\]  

(2.39)

Let us take Fourier transform to (2.17), and denote the Fourier transform of \( u^+(t, \xi) \) by \( \hat{u}^+(t, \eta) \), that is,

\[
\begin{aligned}
&\frac{d}{dt}\hat{u}^+(t, \eta) + A(\eta)\hat{u}^+(t, \eta) = B(\eta)\hat{u}^+(t-r, \eta), \\
&\hat{u}^+(s, \eta) = \hat{u}_0^+(s, \eta), \quad s \in [-r, 0], \quad \eta \in R,
\end{aligned}
\]  

(2.40)

where

\[
\begin{aligned}
A(\eta) &:= D|\eta|^2 + a_1(c) + ia_0(c)\eta, \\
B(\eta) &:= b'(0)e^{icr}e^{-\lambda cr}, \quad \text{for } c \geq c_*
\end{aligned}
\]  

(2.41)
From (2.31), the linear time-delayed ordinary differential equation (2.40) can be solved by
\[ u^+(t, \eta) = e^{-A(\eta)(t+r)} e_r^B(\eta)^t \hat{u}^+_0 (-r, \eta) \]
where
\[ B(\eta) := B(\eta) e^{\lambda(\eta)^r}, \] (2.43)
Taking the inverse Fourier transform to (2.42), we have
\[ u^+(t, \xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi \eta} e^{-A(\eta)(t+r)} e_r^B(\eta)^t \hat{u}^+_0 (-r, \eta) d\eta \]
\[ \leq \int_{-\infty}^{\infty} e^{-A(\eta)(t+s)} e_r^B(\eta)^t \hat{u}^+_0 (-r, \eta) d\eta \]
\[ \leq 1 \int_{-\infty}^{\infty} e^{-A(\eta)(t-r)} e_r^B(\eta)^{t-r} \hat{u}^+_0 (-r, \eta) d\eta \]
Next, we are going to estimate the decay rates for the solution \( u^+(t, \xi) \).

**Lemma 2.5 (Optimal decay rates for linear delayed equation)** Let the initial data \( u_0^+(s, \xi) \) be such that \( u_0^+ \in L^1([-r, 0]; W^{2,1}(\mathbb{R})) \) and \( \partial_s u_0^+ \in L^1([-r, 0]; L^1(\mathbb{R})) \), then
\[ \| u^+(t) \|_{L^\infty(\mathbb{R})} \leq \begin{cases} C(1 + t)^{-\frac{\mu_1}{2}} e^{-\mu_1 t}, & \text{for } c > c_s, \\ C(1 + t)^{-\frac{1}{2}}, & \text{for } c = c_s, \end{cases} \] (2.45)
for some \( 0 < \mu_1 < c\lambda + \delta - c\lambda^2 - pe^{-\lambda r} \) for \( c > c_s \).

**Proof.** Using Parseval’s inequality, from (2.42) we have
\[ \| u^+(t) \|_{L^\infty(\mathbb{R})} \leq \| \hat{u}^+(t) \|_{L^1(\mathbb{R})} \]
\[ \leq \int_{-\infty}^{\infty} e^{-A(\eta)(t+s)} e_r^B(\eta)^{t-r} \hat{u}^+_0 (-r, \eta) d\eta \]
\[ \leq 1 \int_{-\infty}^{\infty} e^{-A(\eta)(t-r)} e_r^B(\eta)^{t-r} \hat{u}^+_0 (-r, \eta) d\eta \]
\[ \leq 1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-A(\eta)(t-r)} e_r^B(\eta)^{t-r} \hat{u}^+_0 (-r, \eta) d\eta ds \]
\[ \leq 1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-A(\eta)(t-r)} e_r^B(\eta)^{t-r} \hat{u}^+_0 (-r, \eta) d\eta ds \]
\[ \leq 1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-A(\eta)(t-r)} e_r^B(\eta)^{t-r} \hat{u}^+_0 (-r, \eta) d\eta ds \]
\[ =: I_1(t) + I_2(t). \] (2.46)
To estimate \( I_i(t) \) for \( i = 1, 2 \), from the definitions of \( B(\eta) \) (see (2.43) and (2.41)) and the delayed-exponential function \( e_r^{k_1(\eta)} \) (see (2.35)), we first note
\[ |e^{-A(\eta)(t+r)}| = e^{-(D\eta^2 + a_1(c))(t+r)} = e^{-k_1(c, \eta)(t+r)}, \] (2.47)
with
\[ k_1(c, \eta) := D\eta^2 + a_1(c), \] (2.48)
Applying (2.52), we derive the optimal estimate for

\[ |\hat{B}(\eta)| \leq |B(\eta)|e^{A(\eta)r} \leq b'(0)e^{-\lambda cr}e^{k_1(c, \eta)} \]

\[ =: \bar{k}_2(c, \eta), \quad \text{for } c \geq c_*, \]  

(2.49)

where

\[ \bar{k}_2(c, \eta) := k_2(c)e^{k_1(c, \eta)}, \quad \text{and } k_2(c) := b'(0)e^{-\lambda cr}, \]  

(2.50)

and

\[ |e_r B(\eta)t| \leq e_r |B(\eta)t| = e_r \bar{k}_2(c, \eta)t, \quad \text{for } c \geq c_. \]  

(2.51)

Noting (1.8) and (1.10), we have

\[ k_1(c, \eta) = D\eta^2 + a_1(c) = D\eta^2 + c\lambda + \delta - D\lambda^2 \geq D\eta^2 + pe^{-\lambda cr} = D\eta^2 + k_2(c), \]  

and

\[ k_1(c, \eta) - k_2(c) = D\eta^2 + c\lambda + \delta - D\lambda^2 - pe^{-\lambda cr} = D\eta^2 + \mu_0, \quad \text{for } c > c_*, \]

where \( \mu_0 := c\lambda + \delta - D\lambda^2 - pe^{-\lambda cr} > 0 \) for \( c > c_*, \) and

\[ k_1(c_*, \eta) - k_2(c_*) = D\eta^2 + c\lambda_\ast + \delta - D\lambda^2 - pe^{-\lambda c_\ast r} = D\eta^2, \quad \text{for } c = c_. \]

Then, from (2.38) we have

\[ |e^{-A(\eta)(t+r)}e_r \hat{B}(\eta)t| \leq e^{-k_1(c, \eta)(t+r)}e_r \bar{k}_2(c, \eta)t \]

\[ \leq C e^{-\varepsilon_1(k_1(c, \eta) - k_2(c))}t \]

\[ = \begin{cases}Ce^{-\varepsilon_1(D\eta^2 + \mu_0)t} & \text{for } c > c_*, \\Ce^{-\varepsilon_1D\eta^2t} & \text{for } c = c_. \end{cases} \]  

(2.52)

Applying (2.52), we derive the optimal estimate for \( I_1(t): \)

\[ I_1(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |e^{-A(\eta)(t+r)}e_r B(\eta)t u^+_0(-r, \eta)| \ d\eta \]

\[ \leq \begin{cases}C \int_{-\infty}^{\infty} e^{-\varepsilon_1D\eta^2t} e^{-\varepsilon_1\mu_0t} |\hat{u}^+_0(-r, \eta)| d\eta & \text{for } c > c_*, \\C \int_{-\infty}^{\infty} e^{-\varepsilon_1D\eta^2t} |\hat{u}^+_0(-r, \eta)| d\eta & \text{for } c = c_*. \end{cases} \]

\[ \leq \begin{cases}C e^{-\varepsilon_1\mu_0t} \|\hat{u}^+_0(-r)\|_{L^\infty} \int_{-\infty}^{\infty} e^{-\varepsilon_1D\eta^2t} d\eta & \text{for } c > c_*, \\C \|\hat{u}^+_0(-r)\|_{L^\infty} \int_{-\infty}^{\infty} e^{-\varepsilon_1D\eta^2t} d\eta & \text{for } c = c_. \end{cases} \]

\[ \leq \begin{cases}C e^{-\mu_1 t} t^{-\frac{1}{2}} \|u^+_0(-r)\|_{L^1} & \text{for } c > c_*, \\C t^{-\frac{1}{2}} \|u^+_0(-r)\|_{L^1} & \text{for } c = c_. \end{cases} \]  

(2.53)

for \( \mu_1 := \varepsilon_1\mu_0 < c\lambda + \delta - D\lambda^2 - pe^{-\lambda cr}. \)

Similarly, we may estimate \( I_2(t). \) We note that

\[ \sup_{\eta \in R} |A(\eta)\hat{u}^+_0(s, \eta)| = \sup_{\eta \in R} \left| D\eta^2 + a_1(c) + ia_0(c)\eta \hat{u}^+_0(s, \eta) \right| \]
Thus, we can derive the decay rate for $I_2(t)$ as follows

$$I_2(t) = \int_{-r}^{r} \int_{-\infty}^{\infty} e^{-A(\eta)(t-s)} e^{-A'(\eta)(t-s)} \left[ \frac{d}{ds} \hat{u}_0^+(s, \eta) + A(\eta) \hat{u}_0^+(s, \eta) \right] \eta ds dt$$

$$\leq C \|u_0^+(s)\|_{W^{2,1}(\mathbb{R})}.$$  \hfill (2.54)

Substituting (2.53) and (2.55) to (2.44), we obtain the decay rates:

$$\|u^+(t)\|_{L^\infty(\mathbb{R})} \leq \begin{cases} C(1 + t)^{-\frac{1}{2}} e^{-\mu_1 t}, & \text{for } c > c_*, \\ C(1 + t)^{-\frac{1}{2}}, & \text{for } c = c_* \end{cases}$$

The proof is complete. \hfill \square
Let us choose \( u_0^+(s, \xi) \) such that \( u_0^+ \in L^1([-r, 0]; W^{2,1}(\mathbb{R})) \) and \( \partial_s u_0^+ \in L^1([-r, 0]; L^1(\mathbb{R})) \) and
\[
\exists \text{ large number for some } 0 < \mu.
\]
Combining Lemmas 2.2 and 2.5, we immediately get the convergence rates for \( \tilde{u}(t, \xi) \).

**Lemma 2.6** When \( \tilde{u}_0 \in L^1([-r, 0]; W^{2,1}(\mathbb{R})) \) and \( \partial_s \tilde{u}_0 \in L^1([-r, 0]; L^1(\mathbb{R})) \), then
\[
\| \tilde{u}(t) \|_{L^\infty(\mathbb{R})} \leq \begin{cases} C(1 + t)^{-\frac{3}{2}} e^{-\mu_1 t}, & \text{for } c > c_*, \\ C(1 + t)^{-\frac{1}{2}}, & \text{for } c = c_*. \end{cases}
\] (2.56)

Since \( \tilde{u}(t, \xi) = \sqrt{w(\xi)} u(t, \xi) = e^{-\lambda \xi} u(t, \xi) \) for \( c \geq c_*, \) and \( e^{-\lambda \xi} \to 0 \) as \( \xi \to \infty \), from (2.56) we cannot guarantee any decay estimate for \( u(t, \xi) \) at far field \( \xi \to \infty \). In order to get the optimal decay rates for \( u(t, \xi) \) in cases of \( c > c_* \) and \( c = c_* \), let us first investigate the convergence of \( u(t, \xi) \) at far field of \( \xi = \infty \).

**Lemma 2.7** When \( e < \frac{p}{\mu} \leq e^2 \) for any \( r > 0 \), or \( \frac{p}{\mu} > e^2 \) but for \( r \in (0, r) \), then there exists a large number \( x_0 \geq 1 \) such that the solution \( u(t, \xi) \) of (2.11) satisfies
\[
\sup_{[x_0, \infty)} |u(t, \xi)| \leq Ce^{-\mu_2 t}, \quad t > 0, \quad c \geq c_*
\] (2.57)
for some \( 0 < \mu_2 = \mu_2(p, \delta, r, b'(v_+)) < d \).

**Proof.** Since \( u \in C_{\text{unif}}(0, \infty) \), namely \( \lim_{\xi \to +\infty} u(t, \xi) = u(t, \infty) =: z^+(t) \) exists uniformly for \( t \in [-r, \infty) \) and \( \lim_{\xi \to +\infty} u_\xi(t, \xi) = \lim_{\xi \to +\infty} u\xi(t, \xi) = 0 \) are uniformly for \( t \in [-r, \infty) \), let us take the limits to (2.11) as \( \xi \to \infty \), then we have
\[
\begin{align*}
\begin{cases}
\frac{d}{dt} z^+(t) + \delta z^+(t) - b'(v_+) z^+(t - r) = Q(z^+(t - r)), \\
Q(z^+) = b(v_+ + z^+) - b(v_+) - b'(v_+) z^+.
\end{cases}
\end{align*}
\] (2.58)

As shown in [27], it is well-known that, when \( e < \frac{p}{\mu} \leq e^2 \) for any \( r > 0 \), or \( \frac{p}{\mu} > e^2 \) but with a small time-delay \( 0 < r < \tilde{r} \), then the above equation (2.58) satisfies
\[
|u(t, \infty)| = |z^+(t)| \leq Ce^{-\mu_2 t}, \quad t > 0, c \geq c_*,
\] (2.59)
for some \( 0 < \mu_2 = \mu_2(p, \delta, r, b'(v_+)) < d \), provided with \( |z^+_0| \ll 1 \).

From the continuity and the uniform convergence of \( u(t, \xi) \) as \( \xi \to +\infty \), there exists a large \( x_0 \geq 1 \) such that (2.59) implies the following convergence immediately
\[
\sup_{\xi \in [x_0, +\infty)} |u(t, \xi)| \leq Ce^{-\mu_2 t}, \quad t > 0, c \geq c_*,
\] (2.60)
provided \( \sup_{\xi \in [x_0, +\infty)} |u_0(s, \xi)| \ll 1 \) for \( s \in [-r, 0] \). Such a smallness for the initial perturbation \( u_0 \) near \( \xi = +\infty \) can be automatically verified, because \( \lim_{\xi \to +\infty} v_0(s, x) = v_+ \), which implies \( \lim_{\xi \to +\infty} u_0(s, \xi) = \lim_{\xi \to +\infty} [v_0(s, \xi) - \phi(\xi)] = v_+ - v_+ = 0 \) uniformly for \( s \in [-r, 0] \). Thus, the proof is complete. \( \square \)
Lemma 2.8 It holds
\[
\sup_{\xi \in (-\infty, x_0]} |u(t, \xi)| \leq \begin{cases} 
C(1 + t)^{-\frac{1}{2}} e^{-\mu_1 t}, & \text{for } c > c_*, \\
C(1 + t)^{-\frac{1}{2}}, & \text{for } c = c_*,
\end{cases}
\] (2.61)

Proof. Since
\[
w(\xi) = \begin{cases} 
e^{2\lambda |\xi|}, & \text{for } c > c_*, \\
e^{2\lambda_* \xi}, & \text{for } c = c_*, \xi \leq x_0,
\end{cases}
\]
and \(\tilde{u}(t, \xi) = \sqrt{w(\xi)} u(t, \xi)\), then from (2.56) we get
\[
\sup_{\xi \in (-\infty, x_0]} |u(t, \xi)| \leq \begin{cases} 
C(1 + t)^{-\frac{1}{2}} e^{-\mu_1 t}, & \text{for } c > c_*, \\
C(1 + t)^{-\frac{1}{2}}, & \text{for } c = c_*,
\end{cases}
\]
The proof is complete. □

Proof of Proposition 2.2 Based on Lemmas 2.7 and 2.8 we immediately prove (2.7) and (2.8) for the convergence rates of \(u(t, \xi)\) for \(\xi \in \mathbb{R}\), where \(0 < \mu < \min\{\mu_1, \mu_2\}\). □

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