Algebraic games

Martin Brandenburg

May 13, 2012

Abstract

Two players alternate moves in the following game: Given a finitely generated abelian group $A$, a move consists of picking some $0 \neq a \in A$. The game then continues with $A/\langle a \rangle$. Under the normal play rule, the player with the last possible move wins. In this note we prove that the second player has a winning strategy if and only if $A$ is a square, i.e. $A \cong B^2$ for some finitely generated abelian group $B$. Under the misère play rule, only minor modifications concerning elementary abelian groups are needed. In principle, this game can be played with arbitrary algebraic structures. We also study some examples of non-abelian groups and commutative rings.

Contents

1 Introduction 2

2 The game in general 3
  2.1 Basics in game theory 3
  2.2 The game on algebraic structures 4
  2.3 Selective compound games 6

3 The game on abelian groups 8
  3.1 The finite case 9
  3.2 The Finitely generated case 12

4 The game on groups 14
  4.1 Some classes of groups 14
  4.2 Small orders 15

5 The game on subgroups 17

6 The game on rings 19
  6.1 First examples 19
  6.2 Zero-dimensional rings 20

∗WWU Münster, Germany, brandenburg@uni-muenster.de
†revised May 1, 2014
1 Introduction

Consider the following two-person game: Given a finitely generated abelian group $A$, a move consists of picking some $0 \neq a \in A$ and replacing $A$ by $A/\langle a \rangle$. Under the normal/misère play rule, the player with the last possible move wins/loses: When $A = 0$, the next player cannot move and therefore wins under the misère play rule and loses under the normal play rule. For which $A$ does the first player have a winning strategy, i.e. $A$ is an $N$-position? And when is it a $P$-position, i.e. the second player has a winning strategy? This question can be asked both for the normal as well as for the misère play rule.

More generally, given some algebraic structure $A$, i.e. a (non-empty) set equipped with various functions with various arities, we can play the following game: A move identifies two non-equal elements $a \neq b$ of $A$, i.e. replaces $A$ by the algebraic structure of the same type $A/\langle a \sim b \rangle$, which is defined to be $A/\sim$, where $\sim$ is the congruence relation on $A$ generated by $(a,b)$. The game ends when $A$ has only one element.

In generality we cannot expect any non-trivial analysis of this game. However, this comes into reach if we consider specific types of algebraic structures, for example sets, vector spaces, (abelian) groups and rings. We analyze the game in these cases. This was inspired by the game on rings as proposed by Will Sawin on mathoverflow ([M]). The author is aware of the irrelevance of these games concerning academic mathematical research, but is convinced of their recreational value. These games might be called algebraic games in contrast to the well-studied topological games.

For example, we may start with a group $G$. A move consists of replacing $G$ by $G/\langle \langle a \rangle \rangle$, where $1 \neq a \in G$ and $\langle \langle a \rangle \rangle$ denotes the normal subgroup generated by $a$. The game can be also seen as a sequence of elements $a_1, a_2, \ldots$ such that $a_{i+1} \notin \langle \langle a_1, \ldots, a_i \rangle \rangle$. This is similar to the game considered in [AH] with the weaker condition $a_{i+1} \notin \{a_1, \ldots, a_i\}$. When we consider some abelian group $A$, every subgroup is normal, so that a move replaces $A$ by $A/\langle a \rangle$. Our main theorem states:

**Theorem.** Let $A$ be a finitely generated abelian group.

- Under the normal play rule, $A$ is a $P$-position if and only if $A$ is a square, i.e. $A \cong B^2$ for some finitely generated abelian group $B$.

- Under the misère play rule, $A$ is a $P$-position if and only if $A$ is
  - either a square, but not elementary abelian of even dimension
  - or elementary abelian of odd dimension.
Actually our proof will include a winning strategy. For example, $\mathbb{Z}/4 \oplus \mathbb{Z}/8$ is a normal $\mathcal{N}$-position: Player I mods out $0 \oplus 4$ to optain $\mathbb{Z}/4 \oplus \mathbb{Z}/4$. No matter what Player II does, he will produce something isomorphic to $\mathbb{Z}/4$ or $\mathbb{Z}/2 \oplus \mathbb{Z}/4$. In the first case Player I mods out the generator of $\mathbb{Z}/4$ and wins. In the second case Player I mods out $0 \oplus 2$, so that Player II gets $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. He can only react with something isomorphic to $\mathbb{Z}/2$, whose generator Player I mods out and therefore wins. It is also a misère $\mathcal{N}$-position: From $\mathbb{Z}/4$ Player I mods out 2 and from $\mathbb{Z}/2 \oplus \mathbb{Z}/4$ he mods out $0 \oplus 1$. In each case Player II has to play with $\mathbb{Z}/2$ and does the last move, so that he loses under the misère play rule.

Acknowledgements. For various discussions and suggestions I would like to thank Will Sawin and Kevin Buzzard. Special Thanks goes to Diego Montero who corrected some errors in the first version and simplified the proof of Proposition 3.2.

# 2 The game in general

## 2.1 Basics in game theory

In this subsection we recall briefly some basic notions in combinatorial game theory. We only sketch them and refer the reader to textbooks such as [Fe] for details. We will be quite informal because this suffices for our purposes. Readers familiar with the general theory may skip this subsection.

We consider impartial two-person combinatorial games. This means that Player I (who starts) and Player II alternate making moves, each having the same set of options for a given position in the game. No chance or probability is involved, the game is purely combinatorial. Every game has a set of terminal positions, in which it ends. We require the ending condition, which asserts that the game has to end after some finite number of moves. The first player who cannot move loses under the normal play rule. He wins under the misère play rule. Thus, under the normal play rule one wants to be the last one to move, whereas under the misère play rule one actually wants to prevent this. Often misère games are more complicated than normal ones. The reader who is only interested in the game on algebraic structures under the normal play rule can safely ignore every remark about misère games (however, see Example 2.5).

We call a position in the game an $\mathcal{N}$-position if the next player to move has a winning strategy. If the previous player has a winning strategy, we call it a $\mathcal{P}$-position. This definition applies to both play rules. Every terminal position is obviously a normal $\mathcal{P}$-position and a misère $\mathcal{N}$-position. One of the first basic observations in combinatorial game theory is the following: Under both play rules, every position is either an $\mathcal{N}$-position or a $\mathcal{P}$-position. In fact, we can declare a position to be $\mathcal{N}$ or $\mathcal{P}$ by recursion as follows:

1. Every terminal position is a normal $\mathcal{P}$-position / misère $\mathcal{N}$-position.
2. A non-terminal position is normal/misère $N$, when some move from it yields a normal/misère $P$-position.

3. A position is normal/misère $P$, when every move yields a normal/misère $N$-position.

Intuitively, 1. declares the play rule, 2. asserts the existence of a winning move for $N$-positions, and 3. denies it for $P$-positions. The ending condition easily implies:

**Proposition 2.1.** Under either play rule, the set of $P$- and $N$-positions are characterized by the above three properties.

**Example 2.2.** Consider the game Nim with just two piles: We have two piles of counters. A move consists in reducing the number of counters in one of the piles. Under the normal play rule, $(x, y)$ is a $P$-position iff $x = y$, i.e. it is a "square". In fact, 1. the terminal position $(0, 0)$ is a square, 2. every non-square can be moved to some square, and 3. squares cannot move to squares. Under the misère play rule, the $P$-positions are almost the same: $(0, 0), (1, 1)$ are misère $N$ and $(1, 0), (0, 1)$ are misère $P$, but the rest is as before. For more interesting and classical examples, we refer the reader to [Fe]. We have mentioned this example since the game on abelian groups will be quite similar.

### 2.2 The game on algebraic structures

Now let us introduce the game on algebraic structures:

**Definition 2.3.** Given some algebraic structure $A$, i.e. a (non-empty) set equipped with various functions with various arities ([BS, II, §1]), a player chooses two non-equal elements $a \neq b$ in $A$ and replaces $A$ by the algebraic structure of the same type $A/(a \sim b)$, which is defined to be $A/\sim$, where $\sim$ is the congruence relation on $A$ generated by $(a, b)$ ([BS, II, §5]). The game ends when $A$ has only one element. One might call $A/(a \sim b)$ a principal quotient of $A$.

**Example 2.4.** Given a group $G$, a move consists of replacing $G$ by $G/\langle\langle a\rangle\rangle$ for some element $1 \neq a \in G$, where $\langle\langle a\rangle\rangle$ is the normal subgroup generated by $a$. We only need one element since identifying $a$ with $b$ is the same as identifying $ab^{-1}$ with 1. If we play with an abelian group $A$ (written additively), we replace $A$ by $A/(a)$ for some $0 \neq a \in A$. More generally, given an $R$-module $M$ over a commutative ring $R$, a move replaces $M$ by $M/(m)$ for some $0 \neq m \in M$.

**Example 2.5.** Given a commutative ring $R$, a move consists of replacing $R$ by $R/(a)$ for some element $0 \neq a \in R$, where $(a)$ is the principal ideal generated by $a$. In this game every ring $\neq 0$ is a normal $N$-position since Player I just has to mod out $(1) = R$. Therefore the misère play rule is the correct and interesting one in this case. Note that fields are the first examples of misère $P$-positions. It is rather unwise to choose some unit $a \in R$, since one then loses immediately. For this reason one could just forbid units and work throughout with nontrivial rings. Under the normal play rule, this game is equivalent to the given one.
In the case of a general algebraic structure $A$, a sequence of moves consists of elements $(a_1, b_1), (a_2, b_2), (a_3, b_3), \ldots, (a_n, b_n)$ in $A^2$ such that

1. $a_i \neq b_i$
2. $a_i \sim b_i$ cannot be derived from $a_1 \sim b_1, \ldots, a_{i-1} \sim b_{i-1}$

More formally, 2. means that $a_i \neq b_i$ holds in $A/(a_1 \sim b_1, \ldots, a_{i-1} \sim b_{i-1})$. Even more formally, let $R_i$ be the congruence relation generated by $(a_1, b_1), \ldots, (a_i, b_i)$. Then we have proper inclusions

$$\Delta_A \subseteq R_1 \subseteq R_2 \subseteq \ldots \subseteq R_n \subseteq A^2.$$ 

Here, $\Delta_A$ is the diagonal. As a consequence, we have:

**Proposition 2.6.** The game on an algebraic structure $A$ satisfies the ending condition if and only if the partial order of congruence relations on $A$ satisfies the ascending chain condition.

**Example 2.7.** The game on a group satisfies the ending condition if and only if the partial order of normal subgroups satisfies the ascending chain condition. A similar statement is true for abelian groups, where every subgroup is already normal. More generally, the game on an $R$-module $M$ satisfies the ending condition if and only if $M$ noetherian. When $R$ is noetherian, this means that $M$ is finitely generated.

**Example 2.8.** The game on a commutative ring $R$ satisfies the ending condition if and only if $R$ is noetherian.

However, note that the outcome of the game does not only depend on the partial order of congruence relations: We cannot characterize principle ones in this language. At least the following is true and easy to prove:

**Proposition 2.9.** If $A, B$ are isomorphic algebraic structures, the corresponding games have the same outcome. In other words, $A$ is $P$ or $N$ if and only if $B$ is.

We will often use this result. The following two examples illustrate that the game is easy to understand when some dimension or size classifies the whole structure:

**Example 2.10.** We play with a set, without any additional structure. The ending condition holds if and only if the set is finite. Since only the isomorphism type matters, we only have to look at the cardinality. Every move reduces the cardinality exactly by one. The only terminal position is the set with one element. By induction it follows that a finite set is a normal $P$-position if and only if its cardinality is odd. Otherwise, it is a normal $N$-position. The misère positions are vice versa.

**Example 2.11.** We play with a vector space $V$ over a fixed field. The ending condition holds if and only if $V$ is finite-dimensional. Everything only depends on the dimension of $V$. Every move reduces it by one. The only terminal position is the 0-dimensional vector space. By induction it follows that $V$ is a normal $P$-position if and only if its dimension is even. Otherwise it is a normal $N$-position. The misère positions are vice versa.
2.3 Selective compound games

We review the concept of selective compound games. The general theory is due to Smith ([S], Sect. 7.8). Our application to the game on algebraic structures (Corollary 2.15) will become useful later when dealing with products of structures.

If $G_1, \ldots, G_n$ are games (as above, we only consider impartial ones), we can play a new game $G_1 \vee \ldots \vee G_n$, called the selective compound of $G_1, \ldots, G_n$: A position in that game is a tuple of positions in the games $G_i$. A move consists of picking a non-empty subset of $G_1, \ldots, G_n$ and making a move in each of the chosen games. If $G_i$ is already over, i.e. happens to be a terminal position, then of course we continue with $G_1 \vee \ldots \vee \hat{G}_i \vee \ldots \vee G_n$.

**Proposition 2.12.** The selective compound $G_1 \vee \ldots \vee G_n$ is a normal P-position if and only if every $G_i$ is a normal P-position.

**Proof.** It is clear that the terminal P-position satisfies this. If every $G_i$ is a normal P-position, then $G_1 \vee \ldots \vee G_n$ can only move to $G'_1 \vee \ldots \vee G'_n$, where either $G'_i = G_i$ is normal $P$ or $G'_i$ is a move in $G_i$, which is therefore normal $N$. And the latter happens for at least one $i$. Thus, it doesn’t consist entirely of normal $P$-positions. If, on the other hand, $G_1 \vee \ldots \vee G_n$ is such some $G_i$ are normal $N$, the winning move is to choose some move $G'_i$ there which makes it normal P. If $G_i$ is already normal $P$, we just let $G'_i = G_i$. Thus, we can move $G_1 \vee \ldots \vee G_n$ to $G'_1 \vee \ldots \vee G'_n$ where each $G'_i$ is normal P.

**Proposition 2.13.** The selective compound $G_1 \vee \ldots \vee G_n$ is a misère P-position if and only if either

- all games except one, say $G_i$, are over (i.e. terminal), and $G_i$ is a misère $P$-position,
- at least two of the games are not over yet, and each $G_i$ is a normal $P$-position.

The reader is invited to read the example 2.17 first as a piece of motivation.

**Proof.** Let us call $G_1 \vee \ldots \vee G_n$ a $P'$-position if it satisfies the condition in the proposition, i.e. every $G_i$ is normal $P$ when at least two are not over yet, or only one $G_i$ is still playing and misère $P$. We have to prove that $P'$ satisfies the defining properties of $P$ in Proposition 2.1. First of all, the terminal position is not $P'$. Next, every non-terminal position which is not $P'$ has some (winning) move to a position which is $P'$: If all games except for $G_i$ are over, then we continue to play only with $G_i$, which is misère $N$ and therefore has some move to a misère $P$ position, which is therefore $P'$. If at least two games are not over yet, then some of the games is normal $N$. Now move in every one of these normal $N$ games to some normal $P$-position and leave the normal $P$ games untouched. We obtain the game $G'_1 \vee \ldots \vee G'_n$ where each $G'_i$ is normal $P$. If at least two $G'_i$ are not over yet, this is $P'$ and we are done. Otherwise, every $G_i$ which is normal $N$ can be ended in one move and the other ones are already over. Now we choose the following winning move: Pick some $G_j$ which is normal $N$. If it is misère $P$, end all the other $G_i$ and keep $G_j$. If it is
misère $\mathcal{N}$, choose some move $G_j \rightarrow G'_j$ such that $G'_j$ is misère $\mathcal{P}$, and combine this move with ending all other $G_i$. In each case, we arrive at a single active game which is misère $\mathcal{P}$ and therefore $\mathcal{P}'$.

Finally, we have to prove that a $\mathcal{P}'$ position cannot move to a $\mathcal{P}'$ position. This is clear when only one game is active. When at least two games are active, then every active $G_i$ is normal $\mathcal{P}$ and therefore cannot be ended in one move, but rather can only be moved to some normal $\mathcal{N}$ position $G'_i$. Thus, after every move in the selective compound we still have at least two active games, but one of them must be normal $\mathcal{N}$ and therefore cannot be $\mathcal{P}'$.

**Proposition 2.14.** Given some type of an algebraic structure, assume that for all $A_1, \ldots, A_n$ and all $a, b \in A := A_1 \times \ldots \times A_n$ the canonical homomorphism

$$A/(a \sim b) \rightarrow A_1/(a_1 \sim b_1) \times \ldots \times A_n/(a_n \sim b_n)$$

is an isomorphism. Then, for all $A_1, \ldots, A_n$, the game on $A_1 \times \ldots \times A_n$ is the selective compound of the games on the $A_1, \ldots, A_n$.

More generally, assume that there are classes of algebras $\mathcal{T}_1, \ldots, \mathcal{T}_n$, each being stable under quotients and containing the terminal algebra. Assume that for all $A_i \in \mathcal{T}_i$ and $a, b \in A := A_1 \times \ldots \times A_n$ the canonical homomorphism

$$A/(a \sim b) \rightarrow A_1/(a_1 \sim b_1) \times \ldots \times A_n/(a_n \sim b_n)$$

is an isomorphism. Then, the game on $A$ is the selective compound game of the games on the $A_1, \ldots, A_n$.

**Proof.** This follows from the definitions. The requirement $a \neq b$ in the definition of a move means that $a_i \neq b_i$ for at least one $i$, i.e. that we move in at least one factor. ■

**Corollary 2.15.** In the situation of Proposition 2.14 we have that $A = A_1 \times \ldots \times A_n$ is

- normal $\mathcal{P}$ if and only if every $A_i$ is normal $\mathcal{P}$
- misère $\mathcal{P}$ if and only if all factors except one, say $A_i$, are terminal, and $A_i$ is misère $\mathcal{P}$, or at least two factors are non-terminal, and every $A_i$ is normal $\mathcal{P}$.

**Example 2.16.** Let $R_1, \ldots, R_n$ be commutative rings and let $R := R_1 \times \ldots \times R_n$ denote their product. Then for every $a \in R$ the induced homomorphism

$$R/(a) \rightarrow R_1/(a_1) \times \ldots \times R_n/(a_n)$$

is an isomorphism. Namely, $(a)$ also contains $e_i a = a_i e_i$, where $e_i = (\delta_{ij})_{1 \leq j \leq n}$. Therefore, Corollary 2.15 and Example 2.5 tell us that $R$ is a misère $\mathcal{P}$-position if and only if $R = R_i$ for some $i$ and $R_i$ is a misère $\mathcal{P}$-position. This implies that every misère $\mathcal{P}$-position $R$ in the game on rings is connected, in the sense that $\text{Spec}(R)$ is connected.
Example 2.17. Let $R_1, \ldots, R_n, R$ be as in the previous example. Consider the game on $R$-modules and let $T_i$ consist of those $R$-modules on which $e_i$ acts as the identity. These correspond to $R_i$-modules. The assumptions in Proposition 2.15 are easily verified. Every $R$-module $M$ decomposes as $M \cong \prod_i e_i M$ canonically, with $M_i := e_i M \in T_i$. Thus, we can apply Corollary 2.15 to this situation. For example, let $K, L$ be two fields and let us play with $K \times L$-modules. The noetherian ones have the form $K^n \times L^m$ for unique natural numbers $n, m$. A move decrements $n, m$, or both. Now, $K^n \times L^m$ is a normal $P$-position if and only if both $n$ and $m$ are even. It is a misère $P$-position if and only if one of the following three cases occurs:

- $n = 0$ and $m$ is odd
- $m = 0$ and $n$ is odd
- $n \geq 2$ and $m \geq 2$ are even

The following table shows the misère game on $K \times L$-modules.

| $n \backslash m$ | 0 | 1 | 2 | 3 | 4 | 5 |
|------------------|---|---|---|---|---|---|
| 0                | N | P | N | P | N | P |
| 1                | P | N | N | N | N | N |
| 2                | N | N | P | N | P | N |
| 3                | P | N | N | N | N | N |
| 4                | N | N | P | N | P | N |
| 5                | P | N | N | N | N | N |

3 The game on abelian groups

In this section we analyze the game on abelian groups. The ending condition is satisfied precisely for the finitely generated ones (Example 2.7). The structure theorem for finitely generated abelian groups $A$ will come to our rescue:

- The torsion subgroup $\text{tor}(A)$ is a direct summand von $A$ and the quotient is free abelian.
- If $A$ is torsion, then $A$ is finite, and there are unique natural numbers $s \geq 0$ and $n_1, \ldots, n_s \geq 2$ satisfying $n_i|n_{i+1}$ for $1 \leq i < s$ such that $A \cong \mathbb{Z}/n_1 \oplus \ldots \oplus \mathbb{Z}/n_s$.
- In this case, $s$ is the smallest integer such that $A$ can be generated by $s$ elements.
- If $A$ is finite, then $A = \oplus_p A_p$, where $A_p$ is the $p$-Sylow subgroup of $A$, which is given by
  $$A_p = \cup_n \ker(p^n : A \to A).$$

This theorem enables us find a beautiful characterization of the $P$-positions.
3.1 The finite case

Proposition 3.1. If $A, B$ are finite abelian groups of coprime orders, then the game on $A \times B$ is the selective compound of the games on $A$ and $B$. In particular, if $A$ is a finite abelian group, then the game on $A$ is the selective compound of the games on the $p$-Sylow subgroups $A_p$.

Proof. It is enough to verify the conditions of Proposition 2.15 i.e. that for every pair $A, B$ as in the claim the canonical homomorphism

$$(A \times B)/\langle (a, b) \rangle \rightarrow A/\langle a \rangle \times B/\langle b \rangle$$

is an isomorphism for all $a \in A$ and $b \in B$. Equivalently, $\langle (a, b) \rangle = \langle a \rangle \times \langle b \rangle$. Since $\subseteq$ is obvious, it suffices to check that the order of $(a, b)$ is the product of the orders of $a$ and $b$. But this is clear since the orders of $a$ and $b$ are coprime. $\blacksquare$

Thus, we may restrict to abelian $p$-groups. However, some aspects of the game are better formulated without this restriction. So let us stay with arbitrary finite abelian groups for the moment. The first step is to characterize all moves in the game. This is quite laborious, but the rest will be rather formal.

Proposition 3.2. Let $A$ be a finite abelian group, say $A \cong \mathbb{Z}/n_1 \oplus \ldots \oplus \mathbb{Z}/n_s$ with $n_i | n_{i+1}$. A finite abelian group $B$ is isomorphic to $A/\langle x \rangle$ for some $x \in A$ if and only if there are natural numbers $(m_i)_{1 \leq i \leq s}$ satisfying $m_1 | n_1 | m_2 | n_2 | \ldots | m_s | n_s$ such that

$$B \cong \mathbb{Z}/m_1 \oplus \ldots \oplus \mathbb{Z}/m_s.$$ If $(m_i)_{1 \leq i \leq s}$ is such a sequence, then we may choose

$$x = m_1 + m_1 \cdot \frac{m_2}{n_1} \oplus \ldots \oplus m_1 \cdot \frac{m_2}{n_1} \cdot \ldots \cdot \frac{m_s}{n_{s-1}}.$$ 

Proof. Let us first verify the easy direction. For $x$ defined as above, we want to show $A/\langle x \rangle \cong \mathbb{Z}/m_1 \oplus \ldots \oplus \mathbb{Z}/m_s$. Let us make an induction on $s$, the cases $s = 0$ and $s = 1$ being trivial. The quotient is given by generators $e_1, \ldots, e_s$ and relations $n_i e_i = 0$ for $1 \leq i \leq s$ as well as the relation

$$m_1 e_1 + m_1 \cdot \frac{m_2}{n_1} e_2 + \ldots = 0.$$ This can be also written as $m_1 e_1' = 0$, where

$$e_1' = e_1 + \frac{m_2}{n_1} e_2 + \frac{m_2}{n_1} \cdot \frac{m_3}{n_2} + \ldots.$$ We find a new presentation with the generator $e_1$ replaced by $e_1'$, and the relation $n_1 e_1 = 0$ replaced by

$$n_1 e_1' = m_2 e_2 + m_2 \cdot \frac{m_3}{n_2} + \ldots.$$ The left hand side vanishes because of $m_1 e_1' = 0$ and $m_1 | n_1$. Hence, the relation does not contain $e_1'$ anymore and we can split off $\langle e_1' : m_1 e_1' = 0 \rangle \cong \mathbb{Z}/m_1$, the
rest being isomorphic to \( \mathbb{Z}/m_2 \oplus \ldots \oplus \mathbb{Z}/m_s \) by the induction hypothesis. Thus, we optain \( \mathbb{Z}/m_1 \oplus \mathbb{Z}/m_2 \oplus \ldots \oplus \mathbb{Z}/m_s \).

Now for the other direction, assume that \( x \in A \) is an arbitrary element. We claim that there are natural numbers \( m_1 | n_1 | m_2 | m_2 | \ldots \) such that \( A/\langle x \rangle \) is isomorphic to \( \mathbb{Z}/m_1 \oplus \ldots \oplus \mathbb{Z}/m_s \). This will be done by induction on \( s \). By the structure theorem, or essentially the Chinese Remainder Theorem, we may assume that everything is a power of a prime \( p \). Technically, this isn’t an important ingredient for our proof, but it simplifies the complicated relation \( \mid \) to the simple relation of \( \leq \). Write \( n_i = p^{k_i} \) with \( k_i \geq 0 \). Then we claim that there are natural numbers \( m_i \geq 0 \) such that \( m_1 \leq k_1 \leq m_2 \leq k_2 \leq m_3 \leq \ldots \) such that \( A/\langle x \rangle \) is isomorphic to \( \mathbb{Z}/p^{m_1} \oplus \ldots \oplus \mathbb{Z}/p^{m_s} \). Now consider \( x_i \in \mathbb{Z}/p^{k_i} \) and lift it to some natural number, also denoted by \( x_i \). We may write \( x_i = p^{r_i} u_i \) for some unique \( 0 \leq r_i \leq k_i \) and \( u_i \) with \( p \not| u_i \). Since multiplication with \( u_i \) induces an automorphism of \( \mathbb{Z}/p^{k_i} \), we may even assume that \( x_i = p^{r_i} \).

Next, we give a recursive description of the quotient

\[
A_{k,r} := (\mathbb{Z}/p^{k_1} \oplus \ldots \oplus \mathbb{Z}/p^{k_s})/\langle (p^{r_1}, \ldots, p^{r_s}) \rangle.
\]

This can be also written as the abelian group defined by generators \( e_1, \ldots, e_s \), relations \( p^{k_i} e_i = 0 \) for \( 1 \leq i \leq s \), as well as the relation

\[
p^{r_1} e_1 + \ldots + p^{r_s} e_s = 0.
\]

Choose \( 1 \leq l \leq s \) in such a way that \( r_l \) becomes minimal. If we define

\[
e'_l := \sum_i p^{r_i - r_l} e_i = e_l + \sum_{i \neq l} p^{r_i - r_l} e_i,
\]

the above relations becomes \( p^{r_l} e'_l = 0 \). In terms of \( e'_l \), the relation \( p^{k_i} e_i = 0 \) becomes

\[
p^{k_i} e'_l = \sum_{i \neq l} p^{k_i + r_l - r_l} e_i.
\]

The left hand side vanishes because of \( p^{r_l} e'_l = 0 \) and \( r_l \leq k_l \). Thus, we can split off \( \langle e'_l \rangle \cong \mathbb{Z}/p^{r_l} \). Also, since \( p^{k_i} e_i = 0 \), we could equally well replace the coefficient of \( e_i \) by \( p^{r_i} \), where

\[
r'_i := \min(k_i + r_i - r_l, k_i).
\]

For \( i < l \) we have \( r'_i = k_i \), so that we may split off \( \langle e_i \rangle \cong \mathbb{Z}/p^{k_i} \). Thus, if we define \( k'_i = k_i \) for \( i > l \), we optain the recursive expression

\[
A_{k,r} \cong \mathbb{Z}/p^{r_l} \oplus \mathbb{Z}/p^{k_1} \oplus \ldots \oplus \mathbb{Z}/p^{k_{l-1}} \oplus A_{k',r'}.
\]

Let us add to the induction hypothesis that \( r_l \) is the smallest exponent in the decomposition, i.e. \( r_l = m_1 \). Applying the induction hypothesis to \( A_{k',r'} \), we get numbers \( m_{l+1} \leq k_{i+1} \leq m_{l+2} \leq \ldots \leq k_s \) such that \( A_{k',r'} \cong \mathbb{Z}/p^{m_{l+1}} \oplus \ldots \oplus \mathbb{Z}/p^{m_s} \). Besides, \( m_{l+1} \) is the minimum of the \( r'_i \), which is \( \geq k_1 \). Now let us define \( m_1 = r_l \) and \( m_i = k_{i-1} \) for \( 1 < i \leq l \). Then \( A_{k,r} \cong \mathbb{Z}/p^{m_1} \oplus \ldots \oplus \mathbb{Z}/p^{m_s} \) and we have

\[
m_1 \leq k_1 = m_2 \leq k_2 = m_3 \leq \ldots \leq k_{l-1} = m_l \leq k_l \leq m_{l+1} \leq k_{l+1} \leq \ldots \leq k_s,
\]

as required. \( \blacksquare \)
Proposition 3.3. The game on finite abelian groups is equivalent to the following number-theoretic game: Positions are divisor sequences $n_1 \ldots | n_s$ of natural numbers $\geq 1$, where those $n_i = 1$ may be removed. There is a move from $n_1 \ldots | n_s$ to $m_1 \ldots | m_s$ if and only if $n_1 | n_2 | n_3 | \ldots | m_s | n_s$ and for at least one $1 \leq i \leq s$ we have $m_i < n_i$. The only terminal position is the empty sequence.

Proof. This follows from the Proposition 3.2

Proposition 3.4. In the number-theoretic game described above, $n_1 | \ldots | n_s$ is a normal $\mathcal{P}$-position if and only if it is a square in the following sense: Either $s$ is even and $n_1 = n_2$, $n_3 = n_4$, \ldots, $n_{s-1} = n_s$, or $s$ is odd and $n_1 = 1$, $n_2 = n_3$, \ldots, $n_{s-1} = n_s$.

Proof. Clearly the terminal position, which is $\mathcal{P}$, is a square with $s = 0$. We have to prove that every non-square moves to some square, and that a square cannot move to another square.

Assume that a square $n_1 | \ldots | n_s$ moves to some square $m_1 | \ldots | m_s$. We may assume that $s$ is even; otherwise add 1 on the left. For even $i \geq 2$ we have $n_i = n_{i-1} | m_i | n_i$, thus $m_i = n_i$. Since both sequences are squares, this already implies $m_i = n_i$ for all $i$. This is a contradiction.

Now assume that $n_1 | \ldots | n_s$ is not a square. If $s$ is even, define $m_i := m_{i+1} := n_i$ for all odd $i$. Then we have $m_1 = n_1 = m_2 | n_2 | m_3 = n_3 = m_4 | \ldots$. Then $m$ is a square. In particular, $m \neq n$. Hence, $m$ is a winning move. The case that $s$ is odd can be reduced to this case by adding 1 on the left. The winning move is here $m_1 := 1$ and $m_i := m_{i+1} := n_i$ for all even $i > 1$.

Theorem 3.5. Let $A$ be a finite abelian group.

1. $A$ is a normal $\mathcal{P}$-position if and only if $A$ is a square, i.e. $A \cong B^2$ for some finite abelian group $B$.

2. If $A = \mathbb{Z}/n_1 \oplus \ldots \oplus \mathbb{Z}/n_s$ with $n_i | n_{i+1}$ is not a square, then a winning move is $x = \left\{ \begin{array}{ll} 0 \oplus n_1 \oplus 0 \oplus 0 \oplus n_1 \cdot \frac{n_3}{n_2} \oplus \ldots \oplus 0 \oplus \frac{n_s}{n_{s-1}} \cdot \ldots \cdot \frac{n_3}{n_2} & \text{if } s \text{ is even} \\ 1 \oplus 0 \oplus \frac{n_3}{n_1} \oplus 0 \oplus \ldots \oplus 0 \oplus \frac{n_s}{n_{s-1}} \cdot \ldots \cdot \frac{n_3}{n_2} & \text{else} \end{array} \right.$

In fact, we then have $A/\langle x \rangle \cong \left\{ \begin{array}{ll} (\mathbb{Z}/n_1 \oplus \mathbb{Z}/n_3 \oplus \ldots \oplus \mathbb{Z}/n_{s-1})^2 & \text{if } s \text{ is even} \\ (\mathbb{Z}/n_2 \oplus \mathbb{Z}/n_4 \oplus \ldots \oplus \mathbb{Z}/n_{s-1})^2 & \text{else} \end{array} \right.$

Proof. 1. follows from Propositions 3.3 and 3.4 and 2. follows from an inspection of the proofs of Propositions 3.4 and 3.2

Example 3.6. For example, $\mathbb{Z}/4 \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/40$ is a normal $\mathcal{N}$-position. Player I mods out $1 \oplus 0 \oplus 2$, since $8/4 = 2$. The quotient is isomorphic to the square $\mathbb{Z}/8 \oplus \mathbb{Z}/8$. Player II has many choices, but he loses in any case. Let us demonstrate this for the element $4 \oplus 0$. Then Player I gets $\mathbb{Z}/4 \oplus \mathbb{Z}/8$ and of course he mods out $0 \oplus 4$. Now
Player II has the smaller square $\mathbb{Z}/4 \oplus \mathbb{Z}/4$. If he wants to postpone his inevitable defeat, he could try $2 \oplus 2$ with quotient $\cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$. The next moves are $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ by Player I, $\mathbb{Z}/2$ by Player II and finally 0 by Player I, who wins.

**Theorem 3.7.** Let $A$ be a finite abelian group. Then $A$ is a misère $\mathcal{P}$-position if and only if $A$ is

- either elementary abelian of odd dimension,
- or a square, but not elementary abelian

Thus, the only difference to the normal $\mathcal{P}$-positions are the elementary abelian groups $(\mathbb{Z}/p)^s$, which become misère $\mathcal{P}$ iff $s$ is odd.

**Proof.** According to Proposition 3.1 and Corollary 2.15, it suffices to treat the case that $A$ is a finite abelian $p$-group, say $A = \mathbb{Z}/p^{k_1} \oplus \ldots \oplus \mathbb{Z}/p^{k_s}$.

We say that $A$ is $\mathcal{P}'$ if it is either elementary abelian of odd dimension, or it is a square, without being elementary abelian. We have to show the three properties characterizing misère $\mathcal{P}$-positions (Proposition 2.1). The terminal position is elementary abelian of dimension 0, thus it is not $\mathcal{P}'$. Next, we have to show that if $A \neq 0$ is not $\mathcal{P}'$, then there is some move which makes it $\mathcal{P}'$. If $A$ is elementary abelian, then its dimension is even $\neq 0$, and in fact every move reduces the dimension by one, so that we end up with something which is $\mathcal{P}'$. If $A$ is not elementary abelian, then it is not a square. By Theorem 3.5, there is some $0 \neq x \in A$ such that $A/\langle x \rangle$ is a square, namely isomorphic to $(\mathbb{Z}/p^{k_1} \oplus \ldots \oplus \mathbb{Z}/p^{k_s-1})^2$ if $s$ is even, and otherwise to $(\mathbb{Z}/p^{k_1} \oplus \ldots \oplus \mathbb{Z}/p^{k_s-1})^2$. If these are not elementary abelian, they are $\mathcal{P}'$ we are done. Now assume that they are elementary abelian, which means $k_{s-1} = 1$.

Thus, $A = (\mathbb{Z}/p)^{s-1} \oplus \mathbb{Z}/p^{k_s}$. We have $k_s > 1$. If $s$ is even, the winning move is now $0 \oplus \ldots \oplus 0 \oplus 1$, since the quotient is $(\mathbb{Z}/p)^{s-1}$, which is elementary abelian of odd dimension and therefore $\mathcal{P}'$. If $s$ is odd, the winning move is $0 \oplus \ldots \oplus 0 \oplus p$, since the quotient is $(\mathbb{Z}/p)^s$, therefore also $\mathcal{P}'$.

Finally, we have to show that if $A$ is $\mathcal{P}'$, it cannot move to something $A' = A/\langle x \rangle$ which is $\mathcal{P}'$. This is clear if $A$ is elementary abelian. Otherwise, $A$ is a square, $s$ is even, and $A'$ cannot be a square by Theorem 3.5. For a contradiction, assume that $A'$ is $\mathcal{P}$. Then $A'$ is elementary abelian of odd dimension. Since $pA' = 0$, we have $pA \subseteq \langle x \rangle$. Thus, $pA$ is cyclic. On the other hand, it contains $p(\mathbb{Z}/p^{k_{s-1}} \oplus \mathbb{Z}/p^{k_s}) \cong (\mathbb{Z}/p^{k_{s-1}})^2$, which is only cyclic when $k_s = 1$. But this implies $k_i = 1$ for all $i$, i.e. $A$ is elementary abelian. This contradiction finishes the proof. ■

### 3.2 The Finitely generated case

**Theorem 3.8.** Let $A$ be a finitely generated abelian group. Then $A$ is a normal $\mathcal{P}$-position if and only if $A$ is a square, i.e. $A \cong B^2$ for some finitely generated abelian group $B$.  

---

12
Proof. In the finite case, we may use our Theorem 3.5. In general, we may write $A \cong A_t \oplus \mathbb{Z}^r$, where $A_t$ is the finite torsion subgroup of $A$ and $r \geq 0$ is the rank of $A$. It is easy to see that $A$ is a square if and only if $A_t$ is a a square and $r$ is even.

As before, it is enough to prove that every non-square moves to some square and that every square cannot move to another square.

Assume that $A$ is not a square. If $r$ is even, then $A_t$ is not a square and by the finite case there is some $0 \neq x \in A_t$ such that $A_t/\langle x \rangle$ is a square. But then $A/\langle (x \oplus 0) \rangle \cong A_t/\langle x \rangle \oplus \mathbb{Z}^r$ is a square. If $r$ is odd, it is enough to consider the case $r = 1$ by ignoring the direct summand $\mathbb{Z}^{r-1}$ which is already a square. If $A_t$ is generated by $s$ elements, say $A_t \cong \mathbb{Z}/n_1 \oplus \ldots \oplus \mathbb{Z}/n_s$, let $n_{s+1} := 0$ and apply the winning strategy of Theorem 3.5 to $A \cong \mathbb{Z}/n_1 \oplus \ldots \oplus \mathbb{Z}/n_s \oplus \mathbb{Z}/n_{s+1}$. This works since we never divided through the last number $n_{s+1}$; in fact we didn’t use it at all! Thus, if $s$ is even, there is a move from $A$ to the square $(\mathbb{Z}/n_2 \oplus \mathbb{Z}/n_4 \oplus \ldots \oplus \mathbb{Z}/n_s)^2$. If $s$ is odd, there is a move to the square $(\mathbb{Z}/n_1 \oplus \mathbb{Z}/n_3 \oplus \ldots \oplus \mathbb{Z}/n_s)^2$.

Now assume that $A$ is a square of rank $r$ and there is some move to a square $B$. In other words, we have an exact sequence

$$0 \to C \to A \to B \to 0,$$

where $C \neq 0$ is cyclic. When $C$ is finite, the map $C \to A$ factors through $A_t$ and we obtain $B \cong \mathbb{Z}^r \oplus A_t/C$. Since $B$ is a square, it follows that $A_t/C$ is a square, which is impossible by the finite case since also $A_t$ is a square. Now assume that $C$ is infinite. Then $B$ is of rank $r - 1$, which is odd, a contradiction.  

Theorem 3.9. Let $A$ be a finitely generated abelian group. Then $A$ is a misère $\mathcal{P}$-position if and only if $A$ is

- either finite elementary abelian of odd dimension,
- or a square, but not finite elementary abelian

In particular, if $A$ is infinite and a square, then $A$ is misère $\mathcal{P}$.

Proof. Let $\mathcal{P}'$ be the class of groups described in the Theorem. Clearly $0 \notin \mathcal{P}'$. Again we have to verify that $A \in \mathcal{P}'$ cannot move to some $B \in \mathcal{P}'$, and that every $0 \neq A \notin \mathcal{P}'$ moves to some $B \in \mathcal{P}'$. If $A$ is finite, both follow from Theorem 3.7. Now assume that $A$ is infinite.

If $A \in \mathcal{P}'$, then $A$ is a square, and for every move $B := A/\langle x \rangle$ it follows from Theorem 3.8 that $B$ is not a square. If $B \in \mathcal{P}'$, it would follow that $B$ is finite, in fact elementary abelian of odd dimension and therefore of rank $0$. It follows $1 \leq \text{rank}(A) = \text{rank}(\langle x \rangle) \leq 1$, thus $\text{rank}(A) = 1$. But this contradicts $A$ being a square. Thus, $B \notin \mathcal{P}'$.

If $0 \neq A \notin \mathcal{P}'$, then $A$ is not a square, and by Theorem 3.8 there is some $0 \neq x \in A$ such that $B := A/\langle x \rangle$ is a square. If $B \in \mathcal{P}'$, we would be done. Otherwise, $B$ is finite
and elementary abelian of even dimension. It follows once again \( \text{rank}(A) = 1 \) and we may write \( A = \mathbb{Z}/n_1 \oplus \ldots \oplus \mathbb{Z}/n_s \oplus \mathbb{Z} \) for some \( n_1, \ldots, n_s \). The proof of Theorem 3.8 shows that we can choose \( x \) in such a way that \( B \cong (\mathbb{Z}/n_s \oplus \mathbb{Z}/n_{s-2} \oplus \ldots)^2 \). Since this is elementary abelian, it follows that \( n_s \) is some prime number \( p \). But then we even have \( n_1 = \ldots = n_s = p \), i.e. \( A = (\mathbb{Z}/p)^s \oplus \mathbb{Z} \). Now, if \( s \) is odd, we mod out \( 0^s \oplus 1 \) to obtain \( (\mathbb{Z}/p)^s \), which is \( \mathcal{P}' \). If \( s \) is even, we mod out \( 0^s \oplus p \) to obtain \( (\mathbb{Z}/p)^s+1 \), which is again \( \mathcal{P}' \). This finishes the proof.

\[ \square \]

4 The game on groups

4.1 Some classes of groups

Let’s play the game on groups under the normal play rule. In every step, a group \( G \) is replaced by \( G/\langle \langle a \rangle \rangle \) for some \( 1 \neq a \in G \). In general, the normal subgroup \( \langle \langle a \rangle \rangle \) generated by \( a \) is quite large compared to the cyclic subgroup \( \langle a \rangle \). This will be responsible for a variety of \( \mathcal{N} \)-positions.

For example, there are many groups which can be normally generated by a single element, which are therefore \( \mathcal{N} \) (\([\text{B]}\)). Every knot group has this property. For example, the Wirtinger presentation of the trefoil knot is

\[
G = \langle a, b, c : a^{-1}ca = b, c^{-1}bc = a, b^{-1}ab = c \rangle
\]

and we see \( G = \langle \langle a \rangle \rangle \).

Now let us treat some classes of finite non-abelian groups. We may use our results on abelian groups.

**Proposition 4.1.** For every \( n \geq 3 \) the following groups are \( \mathcal{N} \):

1. the symmetric group \( S_n \)
2. the alternating group \( A_n \)
3. the dihedral group \( D_n \)
4. the group \( D_n \times \mathbb{Z}/2 \)

**Proof.** 1. Since \( S_n \) is generated by 2-cycles and every 2-cycle is conjugated to \( (12) \), it follows that \( S_n = \langle \langle (12) \rangle \rangle \).

2. The abelian group \( A_3 \cong \mathbb{Z}/3 \) is \( \mathcal{N} \). The only nontrivial normal subgroup of \( A_4 \) is the Klein Four Group \( V_4 \), which does not contain \( (123) \), so that \( A_4 = \langle \langle (123) \rangle \rangle \). For \( n \geq 5 \), we even have \( A_n = \langle \langle \sigma \rangle \rangle \) for every \( 1 \neq \sigma \in A_n \) since \( A_n \) is simple.

3. The dihedral group has the presentation

\[
D_n = \langle r, s : r^n = s^2 = (rs)^2 = 1 \rangle.
\]
If \( n \) is even, then \( D_n/\langle r^2 \rangle \cong D_2 \cong (\mathbb{Z}/2)^2 \), which is square abelian and therefore \( \mathcal{P} \). Therefore, \( D_n \) is \( \mathcal{N} \). Now let us assume that \( n \) is odd. Then

\[
D_n/\langle \langle s \rangle \rangle = \langle r : r^n = r^2 = 1 \rangle = \{1\}.
\]

4. This follows from

\[
(D_n \times \mathbb{Z}/2)/\langle \langle (r,0) \rangle \rangle \cong D_n/\langle \langle r \rangle \rangle \times \mathbb{Z}/2 \cong (\mathbb{Z}/2)^2.
\]

Next, recall that the \textit{dicyclic group} \( \text{Dic}_n \) is defined by the presentation

\[
\text{Dic}_n = \langle a, x : a^{2n} = 1, a^n = x^2, axa = x \rangle.
\]

It has order \( 4n \). For \( n = 2 \) this is the Quaternion group \( Q = \{ \pm 1, \pm i, \pm j, \pm k \} \).

**Proposition 4.2.** For every \( n \geq 2 \), the dicyclic group \( \text{Dic}_n \) is \( \mathcal{N} \). The same is true for \( \text{Dic}_n \times \mathbb{Z}/2 \).

**Proof.** From the presentation we see \( \text{Dic}_n/\langle \langle x \rangle \rangle = \langle a : a^n = a^2 = 1 \rangle \), which is trivial when \( n \) is odd. When \( n \) is even, mod out \( a^2 \). The quotient is

\[
\langle a, x : a^2 = x^2 = (ax)^2 = 1 \rangle \cong (\mathbb{Z}/2)^2.
\]

This shows that \( \text{Dic}_n \) is \( \mathcal{N} \). As for the product, observe that

\[
(\text{Dic}_n \times \mathbb{Z}/2)/\langle \langle (a,0) \rangle \rangle \cong \text{Dic}_n/\langle \langle a \rangle \rangle \times \mathbb{Z}/2 \cong (\mathbb{Z}/2)^2.
\]

**Proposition 4.3.** Let \( p, q \) be two primes and \( G \) a non-abelian group of order \( pq \). Then \( G \) is \( \mathcal{P} \).

**Proof.** There is a well-known classification of groups \( G \) of order \( pq \). When \( G \) is not abelian, we must have \( p \neq q \), say \( p < q \), and in fact \( p|q - 1 \). Then there is some \( r \in (\mathbb{Z}/q)^* \) of order \( p \) and \( G \) is isomorphic to the semi-direct product \( \mathbb{Z}/q \rtimes \mathbb{Z}/p \), which actually does not depend on \( r \). It has the presentation

\[
G = \langle x, y : x^q = y^p = 1, yxy^{-1} = x^r \rangle.
\]

We have

\[
G/\langle \langle y \rangle \rangle = \langle x : x^q = x^{r-1} = 1 \rangle,
\]

which is cyclic of order \( d = \gcd(q,r-1) \). Since \( r \in (\mathbb{Z}/q)^* \) has order \( p \), it follows \( q \nmid r - 1 \) and therefore \( d = 1 \).

**4.2 Small orders**

There are various online resources of the classification of groups of small order, for example [G]. For the general theory and development of this classification, see for example [HBE].
Proposition 4.4. Every non-abelian group of order $\leq 15$ is $\mathcal{N}$.

Proof. We have already dealt with groups of orders $pq$ for primes $p, q$, and groups of prime order are cyclic. This only leaves the orders 8, 12. There are 2 non-abelian groups of order 8, namely the dihedral group $D_4$ and the quaternion group $Q$, which are $\mathcal{N}$ (Propositions 4.1 and 4.2). There are 3 non-abelian groups of order 12, namely $A_4$, $D_6$ and $Dic_3$, which are also $\mathcal{N}$ according to the same Propositions. ■

Next, there are 14 non-abelian groups of order 16 (see [W]). We denote them via their ID in GAP’s SmallGroup library. Thus, $G_n$ is $\text{SmallGroup}(16,n)$. Since $G_1, G_2, G_5, G_{10}, G_{14}$ are abelian, we only need to consider the other 9 non-abelian ones.

$G_3 = \langle a, b, c : a^4 = b^2 = c^2 = 1, ab = ba, bc = cb, cac^{-1} = ab \rangle 
\cong (\mathbb{Z}/4 \times \mathbb{Z}/2) \rtimes_{\phi} \mathbb{Z}/2$ with $\phi(c) = (a \mapsto ab, b \mapsto b)$.

$G_4 = \langle a, b : a^4 = b^4 = 1, ab = ba^3 \rangle \cong \mathbb{Z}/4 \rtimes_3 \mathbb{Z}/4$

$G_6 = \langle s, t : a^8 = b^2 = 1, ab = ba^5 \rangle = \mathbb{Z}/8 \rtimes_5 \mathbb{Z}/2$

$G_7 = D_8$

$G_8 = \langle a, b : a^8 = b^2 = 1, ab = ba^3 \rangle = \mathbb{Z}/8 \rtimes_3 \mathbb{Z}/2$

$G_9 = Dic_4$

$G_{11} = D_3 \times \mathbb{Z}/2$

$G_{12} = Dic_2 \times \mathbb{Z}/2$

$G_{13} = \langle a, x, y : a^4 = x^2 = 1, a^2 = y^2, xax = a^{-1}, ay = ya, xy = yx \rangle$

We already know that $G_7, G_9, G_{11}, G_{12}$ are $\mathcal{N}$ (Propositions 4.1 and 4.2). Now, observe that $G_6/\langle a^2 \rangle = \langle a, b : a^4 = b^2 = 1, ab = ba \rangle \cong (\mathbb{Z}/2)^2$. The same argument shows $G_8/\langle a^2 \rangle \cong (\mathbb{Z}/2)^2$. We also have

$G_{13}/\langle \langle a \rangle \rangle = \langle x, y : x^2 = y^2 = 1, xy = yx \rangle \cong (\mathbb{Z}/2)^2$.

Thus, $G_6, G_8, G_{13}$ are $\mathcal{N}$. However, $G_3, G_4$ turn out to be $\mathcal{P}$. This can be verified by computing all quotients by hand. In order to avoid this, we have written the following program in GAP ([GAP]). It has a small group $G$ as an input and returns the structural description of all quotients $G/\langle \langle g \rangle \rangle$ for $g \in G$ as a list, beginning with $G/\langle \langle 1 \rangle \rangle = G$ itself.

```gap
Quotients := function(G)
  local s,g,N,Q; s := [];
  for g in Elements(G) do
    N := NormalClosure(G,Subgroup(G,[g]));
    Q := FactorGroup(G,N);
    Add(s,StructureDescription(Q));
  od; return s; end;
```

Now let us compute the quotients of $G_3$ and $G_4$:
gap> Quotients(SmallGroup(16,3));
["(C4 x C2) : C2", "C2", "C4", "C4 x C2", "D8", "C2", "C2", "C2",
 "C4", "D8", "C2", "C2", "C2", "C2"]
gap> Quotients(SmallGroup(16,4));
[ "C4 : C4", "C2", "C4", "C4 x C2", "D8", "C2", "C2", "C2", "C4",
 "C4", "Q8", "C2", "C2", "C2", "C4", "C2"]

In our notation, these quotients are \(\mathbb{Z}/2\), \(\mathbb{Z}/4\), \(\mathbb{Z}/2 \times \mathbb{Z}/4\), \(D_4\), \(Q\), which have already been verified to be \(N\). Thus, \(G_3\) and \(G_4\) are \(P\). Hence:

**Proposition 4.5.** Among the 9 non-abelian groups of order 16, there are exactly two which are \(P\), namely \(G_3 = (\mathbb{Z}/4 \times \mathbb{Z}/2) \rtimes \phi \mathbb{Z}/2\) and \(G_4 = \mathbb{Z}/4 \rtimes_3 \mathbb{Z}/4\).

In the same way we may proceed with larger group orders. In the Appendix [A.1] you can find the GAP program which does the whole work for us. The function **Quotients** computes the ID-List of all quotients, where this time the group itself is excluded. The function **PTest** tests whether a given group is \(P\), i.e. if all quotients are known to be \(N\). The function **order** adds the IDs of all groups of order \(n\) to one of the lists \(P, N\) and returns \([P, N]\). The function **orders** does the same, but even for all groups of order \(\leq n\). Finally, the function **P** displays the IDs and structure description of all non-abelian groups of order \(\leq n\) which are \(P\). For example, we may enter: \texttt{gap> P(200);} This prints a lot of groups; we only summarize the output.

- 2 groups of order 16 with IDs 3, 4 already mentioned,
- 1 group of order 36 with ID 13,
- 68 groups of order 64 with IDs 3, \ldots, 16, 56, 193, \ldots, 245,
- 2 groups of order 81 with IDs 3, 4,
- 1 group of order 100 with ID 15,
- 2 groups of order 128 with IDs 175, 476,
- 9 groups of order 144 with IDs 92, \ldots, 95, 100, 102, 103, 194, 196,
- 1 group of order 196 with ID 11.

There 6065 groups of order \(\leq 200\), but only 105 of them are \(P\), of which 86 are non-abelian. It is rather unlikely that some pattern will emerge here.

5 The game on subgroups

The game on groups produced disproportionally many \(N\)-positions because the normal closure is rather large. A more interesting and balanced game could be the following one, which we will only sketch here:

Start with a group \(G\). A position in the game is a subgroup \(U \leq G\). The initial position is the trivial subgroup, the terminal position is the whole group. A move picks some \(g \in G \setminus U\) and replaces \(U\) by the subgroup \(\langle U, g \rangle\).
This game satisfies the ending condition if and only if $G$ is noetherian, i.e. the partial order of subgroups of $G$ satisfies the ascending chain condition. Equivalently, every subgroup of $G$ is finitely generated. For example, this happens when $G$ is finite. Let us restrict to the normal play rule. When is $G$ a $\mathcal{P}$-position?

Remark that this resembles the game proposed in [AH], but still differs from that. Actually this game is a special case of the game on algebraic structures, namely $G$-sets, starting with the torsor $G$: For a subgroup $U \leq G$, a move from the $G$-set $G/U$ picks some $g \in G \setminus U$ and replaces $G/U$ by $G/\langle U, g \rangle$.

Observe that when $G$ is abelian, we have just the game on the abelian group $G$ and we may use Theorem 3.5. More generally, when $G$ is Hamiltonian (i.e. every subgroup is normal), we have the game on the group $G$. But for arbitrary $G$, these games differ dramatically. Many more $\mathcal{P}$-positions arise. Some examples include $D_3 = S_3$, $D_5$ and $A_4$. However, $D_4$ and $D_6$ are $\mathcal{N}$. Let us verify this for $S_3$: If Player I starts with some 2-cycle (resp. 3-cycle), then Player II responds with any 3-cycle (resp. 2-cycle). Since a 2-cycle and a 3-cycle already generate $S_3$, Player II wins. The quaternion group $Q$ is $\mathcal{N}$ as before because it is Hamiltonian.

Again we may use GAP to compute examples of small orders; see Appendix A.2 for the program. However, we could not find any general pattern except for dihedral groups.

**Proposition 5.1.** For $n \geq 1$, the dihedral group $D_n$ is $\mathcal{P}$ in the game of subgroups if and only if $n$ is a prime number.

**Proof.** Clearly, $D_1 = \mathbb{Z}/2$ is $\mathcal{N}$ and $D_2 \cong (\mathbb{Z}/2)^2$ is $\mathcal{P}$. Now let us assume $n \geq 3$. If $r$ denotes the rotation and $s$ denotes the reflection, we have the following list of subgroups of $D_n$:

- $U_d := \langle r^d \rangle$ for $d|n$
- $U_{d,i} := \langle r^d, r^i s \rangle$ for $d|n$ and $0 \leq i < n$

One checks $U_d \cong \mathbb{Z}/(n/d)$ and $U_{d,i} \cong D_{n/d}$. In particular, $U_{d,i}$ is cyclic only for $n = d$, in which case $U_{n,i} = \langle r^i s \rangle \cong \mathbb{Z}/2$.

Now let us suppose first that $n$ is a prime number. Then Player I can only make the moves $U_1 = \langle r \rangle$ or $U_{n,i} = \langle r^i s \rangle$. In the first case, Player II answers with $s$; in the second case he answers with $r$. In each case, Player II arrives at $\langle r, s \rangle = D_n$ and wins.

Now let $n$ be not a prime number. Choose some prime factor $p$. The winning move for Player I is $U_p = \langle r^p \rangle$: If Player II chooses some $U_d$, only $d = 1$ is possible and Player I wins with $s$. If Player II chooses some $U_{d,i}$, we still have $d \in \{1, p\}$ because of $U_{d,i} \cap \langle r \rangle = U_d$. However, $U_{1,i} = D_{n}$ is not generated by a single element over $U_p$, since this has to be some reflection $r^i s$ and $\langle r^p, r^i s \rangle$ has order $2(n/p) < 2n$. So this wasn’t a legal move anyway, and Player II is forced to choose $d = p$. But then Player I wins with $r$. ■
Experiments with the GAP program suggest that $S_n$ is $\mathcal{P}$ for $n \neq 2$; however we cannot prove this.

6 The game on rings

6.1 First examples

Let’s play the game on rings or $k$-algebras for some fixed field $k$. This only makes sense under the misère play rule (see Example 2.5) and satisfies the ending condition precisely for noetherian rings. So we start with some noetherian ring $R$ and a move consists of choosing some $0 \neq a \in R$ and replacing $R$ by $R/(a)$, where $(a)$ is the ideal generated by $R$. The algebro-geometric picture will be quite useful: Given some noetherian ring, we replace it by some closed subscheme cut out by some single non-trivial equation in the coordinate ring.

Observe that the zero ring is $\mathcal{N}$ and therefore that fields are $\mathcal{P}$. Hence:

**Lemma 6.1.** If a ring $R$ has some principal maximal ideal $\neq 0$, then $R$ is $\mathcal{N}$.

This applies in particular to principal ideal rings (not necessarily domains) which are no fields, such as $k[x]$, $\mathbb{Z}$, or quotients thereof. It also shows that, for example, the union of the coordinate axis $k[x,y]/(xy)$ is $\mathcal{N}$ (mod out $x - 1$), as well as the hyperbola $k[x,y]/(xy - 1)$ (mod out $x - 1$).

We had already seen in Example 2.16:

**Lemma 6.2.** If a ring $R$ is $\mathcal{P}$, then $\text{Spec}(R)$ is connected.

The 1-dimensional smooth case is rather easy to understand:

**Proposition 6.3.** Let $R$ be a Dedekind domain. If $R$ has some principal maximal ideal, then $R$ is $\mathcal{N}$. Otherwise, $R$ is $\mathcal{P}$.

*Proof.* The first part is clear. Now assume that $R$ has no principal maximal ideal. Recall the fact that for every ideal $I \subseteq R$ and every $0 \neq a \in I$ there is some $b \in R$ such that $I = (a, b)$. If $0 \neq a \in R$ is not a unit, then there is some maximal ideal $I$ containing $a$. Now $R/(a)$ is $\mathcal{N}$ because if we choose $b$ as above, we have $b \notin (a)$, so that we may move to $R/(a, b) = R/I$, which is a field. ■

**Corollary 6.4.** Let $E$ be an elliptic curve over some algebraically closed field $k$. Then, for every closed point $x \in E$, the coordinate ring of the affine curve $E \setminus \{x\}$ is $\mathcal{P}$. For example, $k[x,y]/(y^2 - x^3 + x - 1)$ is $\mathcal{P}$.

**Corollary 6.5.** Let $k$ be an algebraically closed field. Then $k[x,y]$ is $\mathcal{N}$.
6.2 Zero-dimensional rings

We will prove that the coordinate ring of the cusp \( k[x, y]/(y^2 - x^3) \) is also \( \mathcal{P} \), which gives an alternative reason why \( k[x, y] \) is \( \mathcal{N} \). However, we will need some results on zero-dimensional rings first, which appear as intersections of the cusp with curves through the origin.

The following observation is due to Kevin Buzzard.

**Lemma 6.6.** Let \( V \) be a vector space over some field \( k \) of finite dimension. Then the ring \( k[V] := k \oplus V \) with multiplication \( V^2 = 0 \) is \( \mathcal{N} \) if and only if \( \dim(V) \) is odd; otherwise it is \( \mathcal{P} \).

*Proof.* For \( V = 0 \) this is true. Now induct on \( \dim(V) \). If \( \dim(V) \) is odd, choose some \( 0 \neq v \in V \). The ideal generated by \( 0 \oplus v \) equals \( 0 \oplus kv \) and the quotient is \( k \oplus V/kv \), which is \( \mathcal{P} \) by the induction hypothesis. Now assume that \( \dim(V) \) is even and \( 0 \neq a + v \in k[V] \) is some element. If \( a \neq 0 \), then \( a + v \) is invertible, so that the quotient is zero, which is \( \mathcal{N} \). Otherwise, \( a = 0 \) and the quotient is \( k \oplus V/kv \), which is \( \mathcal{N} \) by induction hypothesis. \( \blacksquare \)

**Corollary 6.7.** If \( k \) is a field, then \( k[x, y]/(x^2, xy, y^2) \) is \( \mathcal{P} \).

**Lemma 6.8.** Let \( k \) be a field and \( u \geq 0 \). Then \( k[x, y]/(y^2 - x^3, x^{u+1}, x^uy) \) is \( \mathcal{P} \).

*Proof.* Let us call this ring \( B_u \). Then \( B_0 = k \) and \( B_1 = k[x, y]/(x^2, y^2, xy) \) are \( \mathcal{P} \). Now let \( u \geq 2 \) and assume that the Lemma is true for all \( u < u \). Observe that \( 1, \ldots, x^u, y, xy, \ldots, x^{u-1}y \) is a \( k \)-basis of \( B_u \). Choose some non-zero element \( b \in B_u \). We want to show that \( Q := B_u/\langle b \rangle \) is \( \mathcal{N} \). If \( b \) has some \( 1 \)-coefficient, it is a unit and we are done. Otherwise write

\[
b = r_1x + \ldots + r_u x^u + s_1 y + \ldots + s_u x^{u-1} y
\]

with \( r_i, s_j \in k \), not all zero. Choose some minimal \( 1 \leq d \leq u \) with \( r_i = s_i = 0 \) for all \( 1 \leq i < d \). Thus, we have

\[
b = r_d x^d + \ldots + r_u x^u + s_d x^{d-1} y + \ldots + s_u x^{u-1} y,
\]

and at least \( r_d \) or \( s_d \) are non-zero. Now let us consider the case \( d = u \), so that \( b = r_u x^u + s_u x^{u-1} y \). When \( r_u = 0 \), we have \( Q = k[x, y]/(y^2 - x^3, x^{u+1}, x^{u-1}y) \) and therefore \( Q/\langle x^u \rangle = B_{u-1} \), which is \( \mathcal{P} \) by the hypothesis. This proves that \( Q \) is \( \mathcal{N} \). Now assume \( r_u \neq 0 \). Then \( 0 \neq x^{u-1} y \) in \( Q \) and \( Q/\langle x^{u-1} y \rangle = B_{u-1} \), which is \( \mathcal{P} \) by the hypothesis.

So let us assume \( d \leq u - 1 \). In \( B_u \) we compute:

\[
x^{u-d-1}b = r_d x^{u-1} + r_{d+1} x^u + s_d x^{u-2}y + s_{d+1} x^{u-1} y
\]

\[
x^{u-d}b = r_d x^u + s_d x^{u-1} y
\]

\[
x^{u-d-1}yb = r_d x^{u-1} y
\]
Now compute in the quotient $Q$, where $b = 0$. When $r_d \neq 0$, the third equation shows $x^{u-1} y = 0$ in $Q$, which in turn gives $x^u = 0$ by the second equation. But then we can consider $b$ as an element $b' \in B_{u-1}$ and obtain $Q \cong B_{u-1}/(b')$, which is $N$ by the hypothesis. When $r_d = 0$, we have $s_d \neq 0$, so that the second equation gives again $x^{u-1} y = 0$, and the first equation reads as $r_{d+1} x^u + s_d x^{u-2} y = 0$. We see $x^{u-1} \neq 0$ in $Q$ and $Q/(x^{u-1}) \cong B_{u-2}$, which is $\mathcal{P}$ by the hypothesis, so that $Q$ is $N$.

**Corollary 6.9.** Let $k$ be a field and $u \geq 0$. Then $k[x,y]/(y^2 - x^3, x^u y)$ and $k[x,y]/(y^2 - x^3, x^{u+1})$ are $N$.

For example for $u = 3$ we get that $k[x,y]/(x^3, y^2)$ is $N$.

**Proposition 6.10.** Let $k$ be an algebraically closed field. Then the coordinate ring of the cusp $k[x,y]/(y^2 - x^3)$ is $\mathcal{P}$.

**Proof.** Let $R := k[x,y]/(y^2 - x^3)$ and consider some $0 \neq f \in R$, represented by $f \in k[x,y] \setminus (y^2 - x^3)$. Assume first that $f \notin (x,y)$ and write $f = a_0 + a_1 x + a_2 x^2 + \ldots + b_0 y + b_1 xy + b_2 x^2 y + \ldots$ with $a_0 \neq 0$. We claim that $x$ is invertible in $R/f$. This is clear if $b_0 = 0$. Otherwise, let $g$ be the same polynomial as $f$, but with $a_0$ replaced by $-a_0$. In $R/f$ we have $0 = fg$ and in that product the $y$ has disappeared, but the constant term is still invertible. Thus, we may repeat the argument. Since $x$ is invertible, we get an isomorphism $R/f \cong (R_x)/f$. But the normalisation map $\pi : R \to k[t]$ defined by $x \mapsto t^2$ and $y \mapsto t^3$ induces an isomorphism on localizations, so that $R/f \cong k[t]/\pi(f) = k[t]/\pi(f)$ and $\pi(f)$ is some polynomial of degree $\geq 2$. Now apply Lemma 6.1 to conclude that $R/f$ is $N$.

Now let us assume $f \in (x,y)$. The intersection $V(f) \cap V(y^2 - x^3) \subseteq A^2(k)$ is zero-dimensional. Thus, $R/(f)$ is a direct product of local artinian rings. In order to show that it is $N$, we may even assume that it is local by Lemma 6.2. This means that there is a unique $\alpha \in k$ such that $\pi(f)(\alpha) = f(\alpha^2, \alpha^3) = 0$. Since $f(0,0) = 0$ it follows $\pi(f) = t^d$ for some $d \geq 2$, which means $f = x^u y$ or $f = x^{u+1}$ for some $u \geq 0$.

Now the claim follows from Lemmas 6.9.

With the same method of Lemma 6.8 we obtain more $\mathcal{P}$ rings:

**Lemma 6.11.** Let $k$ be a field and $a \geq 1$. Then $k[x,y]/(x^a, xy, x^a)$ is $\mathcal{P}$.

**Proof.** The cases $a = 1, 2$ are clear. Now assume that the claim is true for some $a \geq 2$ and consider the ring $k[x,y]/(x^{a+1}, xy, x^{a+1})$. As a vector space it is generated by $1, x, \ldots, x^a, y_1, \ldots, y^a$. Every non-zero non-unit has the form $r_1 x + \ldots + r_a x^a + s_1 y + \ldots + s_a y^a$ for some $r_i, s_i \in k$, not all zero. Now mod this out; we want to show that the quotient is $N$. If we multiply the relation with $x$, we get $r_1 x^2 + \ldots + r_{a-1} x^a = 0$, or $x^2(r_1 + \text{nilpotents})$. Thus, from $r_1 \neq 0$ we would already get $x^2 = 0$ and the relation becomes $r_1 x + s_1 y + \ldots + s_a y^a$, which enables us to replace $x$. Thus, the quotient is isomorphic to $k[y]/(y^{a+1}, y(s_1 y + \ldots + s_a y^a))$. If $y^l$ denotes the gcd of these two polynomials, we have $l \geq 2$ and the quotient is $k[y]/(y^l)$, which is $N$ according to Lemma 6.1. This settles the case $r_1 \neq 0$. By symmetry, we are also
done when $s_1 \neq 0$. Now assume $r_1 = s_1 = 0$. Choose the largest $1 < i \leq a$ such that $r_{i-1} = s_{i-1} = 0$. By symmetry, we may then assume $r_i \neq 0$. Our relation reads $x^i(r_i + \text{nilpotents}) + y^i(s_i + \text{nilpotents}) = 0$. If we mod out $x^i$, we then mod out $y^i$ automatically, and we obtain $k[x, y]/(x^i, xy, y^i)$, which is $\mathcal{P}$ by the induction hypothesis. Thus, the quotient is $\mathcal{N}$. ■

**Corollary 6.12.** Let $k$ be a field. The following zero-dimensional rings are $\mathcal{N}$:

- $k[x, y]/(x^a, y^a)$ for all $a \geq 1$
- $k[x, y]/(x^a, x^u y^v, y^a)$ for all $a, u, v \geq 1$
- $k[x, y]/(x^a, xy, y^b)$ for all $a, b \geq 1$ with $a \neq b$

### 7 Further questions

Several questions remain open:

- What about other algebraic structures, such as monoids, modules, lattices, Lie algebras?

- In the game of rings, is there any algebro-geometric characterization of affine curves, surfaces, etc. whose coordinate ring is $\mathcal{P}$?

- Some experiments with GAP indicate that $S_n$ is $\mathcal{P}$ for $n \neq 2$ in the game of subgroups. Is this true? The same question for $A_n$ for $n \geq 3$.

- What about the nimbers aka Sprague-Grundy values? For example, what is the nimber of a finitely generated abelian group in terms of the decomposition into cyclic groups?

- The game on subgroups may be easily generalized to arbitrary closure operators. In each step, a closed subset $U$ is replaced by $\langle U, a \rangle$ for some $a \notin U$. For example, who wins some finite extension of number fields, where a move replaces $L/K$ by $L/K(a)$ for some $a \in L \setminus K$? The Primitive element Theorem shows that we should better play misère here.

- We may even add topologies to our algebraic structures. Who wins the game on the closed subgroups of $\text{GL}_n(\mathbb{C})$?
A Appendix: Program Codes

A.1 Game on groups

Quotients := function(G)
local s,g; s := [];
for g in Elements(G) do
if not(g=Identity(G)) then
Add(s,IdGroup(FactorGroup(G,NormalClosure(G,Subgroup(G,[g])))));
fi;od; return s; end;

PTest:= function(G,P,N)
if ForAll(Quotients(G),r -> r in N)
then return true;
else return false;
fi; end;

order := function(n,L)
local P,N,A,G;
P := L[1]; N := L[2];
A := AllSmallGroups(n);
for G in A do
if PTest(G,P,N)=true then
Add(P,IdGroup(G));
else Add(N,IdGroup(G));
fi; od; return [P,N]; end;

orders := function(n)
local P,N,m;
P:=[[1,1]]; N:=[];
if n=1
then return [P,N];
else return order(n,orders(n-1));
fi; end;

P := function(n)
local I,i,G;
Print("\n");
I := orders(n)[1];
for i in I do
G:=SmallGroup(i);
if IsAbelian(G)=false
then Print(i);
Print(" ", StructureDescription(G), "\n");
fi; od; end;

23
A.2 Game on subgroups

#Compute all conj. classes of subgroups <U,g>, g not in U
Quotients := function(G,U)
local q,g,V; q := []
for g in Elements(G) do
  if (g in U) = false
  then V:=Subgroup(G,Concatenation(Elements(U),[g]));
  Add(q,ConjugacyClassSubgroups(G,V));
  fi; od; return q; end;

#Create all N and P information about subgroups of G
Create := function(G)
local N,P,S,U,Q,R;
N:=[]; P:=[]; S := Reversed(ConjugacyClassesSubgroups(G));
for U in S do
  R := Representative(U);
  Q := Quotients(G,R);
  if ForAll(Q,V -> V in N)
  then Add(P,U);
  else Add(N,U);
  fi; od; return [P,N]; end;

#Test if G is P
IsP := function(G)
local L,T;
L := Create(G);
T := ConjugacyClassSubgroups(G,Subgroup(G,[]));
if T in L[1] then return true; else return false;
fi; end;
References

[Fe] T. S. Ferguson, *Game Theory*, electronic textbook: http://www.math.ucla.edu/~tom/Game_Theory/Contents.html

[AH] M. Anderson, F. Harary, *Achievement and avoidance games for generating abelian groups*, International Journal of Game Theory 16 (4): 321-325. 1987

[B] A. J. Berrick, *Torsion Generators for All Abelian Groups*, J. Algebra, 139, 190-194 (1991)

[BS] S. N. Burris, H. P. Sankappanavar, *A Course in Universal Algebra*, electronic textbook: http://www.math.uwaterloo.ca/~snburris/htdocs/ualg.html

[G] The Group Properties Wiki, *Groups of a particular order* http://groupprops.subwiki.org/wiki/Category:Groups_of_a_particular_order

[GAP] GAP - Groups, Algorithms, Programming - a System for Computational Discrete Algebra http://www.gap-system.org

[HBE] H. U. Besche, B. Eick, E.A. O’Brien, *A millennium project: constructing Small Groups*, Internat. J. Algebra Comput. 12, 623 - 644 (2002)

[M] Question on mathoverflow by Will Sawin, *A game on noetherian rings*, http://mathoverflow.net/questions/93276

[S] C.A.B. Smith, *Graphs and composite games*, J. Combinatorial Theory 1 (1966) 51-81.

[W] M. Wild, *Groups Of Order Sixteen Made Easy*, Amer. Math. Monthly Vol. 112, No. 1 (Jan., 2005), pp. 20-31