Resummation of Renormalon Chains for Cross Sections and Inclusive Decay Rates

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Abstract

Recently, we have developed a formalism to evaluate QCD loop diagrams with a single virtual gluon using a running coupling constant at the vertices. This corresponds to an all-order resummation of certain terms (the so-called renormalon chains) in a perturbative series and provides a generalization of the scale-setting prescription of Brodsky, Lepage and Mackenzie. In its original form, the method is applicable to Green functions without external gluons and to euclidean correlation functions. Here we generalize the approach to the case of cross sections and inclusive decay rates, which receive both virtual and real gluon corrections. We encounter nonperturbative ambiguities in the resummation of the perturbative series, which may hinder the construction of the operator product expansion in the physical region. The origin of these ambiguities and their relation to renormalon singularities in the Borel plane is investigated by introducing an explicit infrared cutoff. The ratios $R_{e^+e^-}$ and $R_{\tau}$ are discussed in detail.

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1 Introduction

Recently, an approach has been developed to analyse the momentum flow in Feynman diagrams containing a single virtual gluon line \[1\]. It was initiated by the observation that sometimes the ”typical” momenta in a loop diagram are different from the ”natural” scale of the process. This is indicated by the fact that in the perturbative series for some quantities depending on a single mass scale \( M \), there remain large two-loop corrections of order \( \beta_0 \alpha_s^2 \) when one uses the ”natural” scale \( M \) to evaluate the running coupling constant. Here \( \beta_0 = 11 - \frac{2}{3} n_f \) is the first coefficient of the \( \beta \)-function, and \( n_f \) denotes the number of light quark flavours. Brodsky, Lepage and Mackenzie (BLM) have argued that the appearance of such terms indicates an inappropriate choice of the renormalization scale, and that one should eliminate them by readjusting the scale in the one-loop running coupling constant \[2, 3\]. This prescription defines the BLM scale \( \mu_{BLM} \), which may be interpreted as the ”typical” scale of virtual momenta in Feynman diagrams. One of the goals of Ref. \[1\] was to understand the relation between the BLM scale and the ”natural” scale of a process, in particular in cases where \( \mu_{BLM} \ll M \).

Consider the calculation of a physical (i.e. renormalization-scheme invariant and infrared finite) quantity \( S(M^2) \) at order \( \alpha_s \) in perturbation theory. The BLM prescription is equivalent to using the average virtuality of the gluon as the scale in the running coupling constant. In Ref. \[1\] we have generalized this proposal by performing the calculation with a running coupling constant \( \alpha_s(-k^2) \) at the vertices, where \( k \) is the momentum flowing through the virtual gluon line. This idea is not new (see, for instance, the discussion in Ref. \[3\]), but because of its computational complexity it had not been pursued previously to the point of practical implementation. The result of such a calculation, which gives the average of the running coupling constant over the virtual momenta in Feynman diagrams, may be written as (for simplicity we normalize the lowest-order contribution to unity)

\[
S_{\text{res}}(M^2) = 1 + \int_0^\infty d\tau \; \tilde{w}(\tau) \frac{\alpha_s(\tau e^{C} M^2)}{4\pi},
\]

where the scheme-independent function \( \tilde{w}(\tau) \) describes the distribution of virtualities in the loop calculation. The constant \( C \) depends on the renormalization scheme in such a way that the value of the coupling constant \( \alpha_s(\tau e^{C} M^2) \) is scheme-independent (in MS-like schemes). This implies that the product

\[
\Lambda_V \equiv e^{-C/2} \Lambda_{\text{QCD}}
\]

is scheme-independent, where \( \Lambda_{\text{QCD}} \) is the scale parameter in the one-loop expression for the running coupling constant. We note that \( C = -5/3 \) in the \( \overline{\text{MS}} \) scheme, \( C = -5/3 + \gamma - \ln 4\pi \) in the MS scheme, and \( C = 0 \) in the so-called \( V \) scheme, in which the running coupling constant is defined in terms of the
heavy-quark potential \[3\]. Although the V scheme is particularly convenient for our considerations, we shall display the dependence on \(C\) in order to keep the discussion general.

Eq. (1) is equivalent to an all-order resummation of certain terms in the perturbative series for the quantity \(S(M^2)\). This explains the subscript “res”. Using the one-loop expression for the running coupling constant to relate \(\alpha_s(\tau e^C M^2)\) to \(\alpha_s(M^2)\), one finds that

\[
S_{\text{res}}(M^2) = 1 + \sum_{n=1}^{\infty} \left( \frac{\alpha_s(M^2)}{4\pi} \right)^n c_n \beta_0^{n-1},
\]

where the coefficients \(c_n\) are given by the integrals

\[
c_n = \int_0^\infty d\tau \, \hat{w}(\tau) \, (-C - \ln \tau)^{n-1}.
\]

In other words, our approach resums all terms of order \(\beta_0^{n-1} \alpha_s^n\) in the perturbative series for the quantity \(S(M^2)\). The resummation of such terms has also been considered by Beneke and Braun \[4\] (see also \[3\]), using however a different formalism. To understand what it corresponds to, consider a Feynman diagram with a single virtual gluon line. A particular set of higher-order graphs is obtained by inserting \((n - 1)\) light-quark loops on the gluon propagator. The resulting contributions are of order \(n_f \beta_0^n \alpha_s^n\). In an abelian theory they are obviously related to the renormalization of the coupling constant. In a non-abelian theory one may try to incorporate the effects of gauge- and ghost-field loops by replacing \(n_f\) by \(-\frac{3}{2} \beta_0\). After this replacement, these contributions are no longer related in an obvious way to subclasses of diagrams; however, they are called renormalon chains. It is the all-order resummation of renormalon chains that is accomplished in (3).

It is instructive to relate the approximation \(S_{\text{res}}(M^2)\) to the BLM scale-setting prescription \[4\]. We find

\[
S_{\text{res}}(M^2) = 1 + N \frac{\alpha_s(\mu_{\text{BLM}}^2)}{\pi} \left\{ 1 + \Delta \left( \frac{\beta_0 \alpha_s(\mu_{\text{BLM}}^2)}{4\pi} \right)^2 + \ldots \right\}
\]

\[
= S_{\text{BLM}}(M^2) + \frac{N \Delta}{16\pi^3} \beta_0^2 \alpha_s^3(\mu_{\text{BLM}}^2) + \ldots,
\]

where

\[
N = \frac{1}{4} \int_0^\infty d\tau \, \hat{w}(\tau),
\]

\[
\mu_{\text{BLM}}^2 = \exp \left( \langle \ln \tau \rangle + C \right) M^2,
\]

\[
\Delta = \sigma^2 = \langle \ln^2 \tau \rangle - \langle \ln \tau \rangle^2.
\]
We use the symbol
\[
\langle f(\tau) \rangle = \frac{\int_0^\infty d\tau \hat{w}(\tau) f(\tau)}{\int_0^\infty d\tau \hat{w}(\tau)}
\] (7)
for the average of a function \(f(\tau)\) over the distribution \(\hat{w}(\tau)\). The one-loop coefficient \(N\), the value of the coupling constant \(\alpha_s(\mu_{\text{BLM}}^2)\), and the parameter \(\Delta\) are renormalization-scheme invariant. The first correction to the BLM scheme appears at order \(\beta_0^2 \alpha_s^3\) and is related to the width (with respect to \(\ln \tau\)) of the distribution function.

Let us now investigate the infrared (IR) properties of the integral representation (1). The fact that the integration extends to \(\tau = 0\) indicates the appearance of nonperturbative effects, which arise because of the divergent behaviour of perturbative expansions in QCD. Because of a factorial growth of the expansion coefficients \(c_n\), a perturbative series such as the one on the right-hand side in (3) has a vanishing radius of convergence. It is useful to introduce a generating function for these coefficients,
\[
\hat{S}(u) = e^{Cu} \sum_{n=1}^{\infty} \frac{u^{n-1}}{\Gamma(n)} c_n,
\] (8)
which can be identified with the Borel transform of the series (3) with respect to the inverse coupling constant \(\hat{S}(\mu)\), in the limit where \(\beta_0 \to \infty\). The series on the right-hand side of (8) is usually convergent in a finite interval \(u \in ]-j,k[\) around the origin, where \(j\) and \(k\) are called the positions of the nearest UV and IR renormalons (see below). Note that we have defined the function \(\hat{S}(u)\) in a scheme-independent way. Its relation to the distribution function \(\hat{w}(\tau)\) is
\[
\hat{S}(u) = \int_0^\infty d\tau \hat{w}(\tau) \tau^{-u},
\]
\[
\hat{w}(\tau) = \frac{1}{2\pi i} \int_{u_0 - \infty}^{u_0 + \infty} du \hat{S}(u) \tau^{u-1}.
\] (9)
The choice of the parameter \(u_0\) in the inverse Mellin representation is arbitrary as long as it is inside the interval \([-j,k[\), where the first integral is well defined. Outside this interval the Borel transform \(\hat{S}(u)\) is defined by analytic continuation. The fact that the function \(\hat{w}(\tau)\) is well-defined and has a concrete physical interpretation, whereas the definition of \(\hat{S}(u)\) requires an analytic continuation, will become important later.

The Borel transformation can be inverted to give
\[
S_{\text{Borel}}(M^2) = 1 + \frac{1}{\beta_0} \int_0^\infty du \hat{S}(u) e^{-Cu} \exp \left( - \frac{4\pi u}{\beta_0 \alpha_s(M^2)} \right)
\]
\[ S_{\text{res}}(M^2) = 1 + \frac{1}{\beta_0 \lambda_V^2} \int_0^\infty du \hat{S}(u) \left( \frac{\Lambda_V^2}{M^2} \right)^u, \] (10)

where \( \Lambda_V \) is the scheme-independent parameter defined in (2). If the integral existed, it would define the Borel sum of the partial series on the right-hand side of (3). In general, however, the Borel transform \( \hat{S}(u) \) contains singularities on the real \( u \)-axis, and the result of the integration depends on how these singularities are regulated. Much of the nonperturbative structure of QCD can be inferred from a study of the Borel transform [3]–[17]. Its singularities on the negative axis arise from the large-momentum region in Feynman diagrams and are called ultraviolet (UV) renormalons. They are Borel summable and pose no problem to performing the integral in (10). The singularities on the positive axis arise from the low-momentum region in Feynman diagrams and are called IR renormalons. Their presence leads to an ambiguity in the evaluation of the Borel integral, reflecting the fact that in (3) one is attempting to sum up a series which is not Borel summable.

In our approach, IR renormalon ambiguities appear because the \( \tau \)-integral in (1) runs over the Landau pole in the running coupling constant. In order to regularize the integral, one may write

\[ \beta_0 \alpha_s(\tau e^C M^2) \frac{1}{4\pi} = \frac{1}{\ln \frac{\Lambda_V^2}{M^2} + \ln \tau} = P \left( \frac{1}{\ln \tau - \ln \tau_L} \right) + \eta \delta(\ln \tau - \ln \tau_L), \] (11)

where \( \tau_L = \frac{\Lambda_V^2}{M^2} \) is the position of the Landau pole, "P" denotes the principal value, and \( \eta \) is a complex parameter, which depends on the regularization prescription. A particularly useful regularization is to set \( \eta = i\pi \), corresponding to

\[ \beta_0 \alpha_s(\tau e^C M^2) \frac{1}{4\pi} = \frac{1}{\ln \tau - \ln \tau_L - i\epsilon}. \] (12)

Let us show that the resummation (11) together with this regularization prescription is equivalent to performing the Borel integral (10) along a contour above the real \( u \)-axis. We note that

\[ S_{\text{res}}(M^2) = 1 + \frac{1}{\beta_0 \lambda_V^2} \int_0^\infty du \hat{S}(u) \left( \frac{\Lambda_V^2}{M^2} \right)^u e^{i\epsilon u}, \] (13)

where we have used the integral representation (1) of the Borel transform \( \hat{S}(u) \) in terms of the distribution function \( \hat{\omega}(\tau) \). Note that interchanging the order
of integrations when passing from the second to the third line is justified, since both integrals converge. Let us now assume that the Borel transform $S(u)$ is an analytic function in the complex $u$-plane apart from singularities on the real axis. It is then allowed to rotate the integration contour in the last integral in (13) into a contour that runs above the real axis, and then to take the limit $\epsilon \to 0$. This proves the equivalence of our resummation with the Borel integral in (10). Using the same argument it follows that choosing a $+i\epsilon$ to regulate the Landau pole in (12) corresponds to performing the Borel integral below the real $u$-axis, and regulating the Landau pole with a principle value prescription is equivalent to taking the principle value of the Borel integral.

The dependence on the regularization prescription leads to an intrinsic ambiguity in the definition of $S_{\text{res}}(M^2)$. Inserting (11) into (1) and defining the renormalon ambiguity $\Delta S_{\text{ren}}$ as the coefficient of $\eta$, we find

$$\Delta S_{\text{ren}} = \frac{\tau_L}{\beta_0} \hat{w}(\tau_L) \simeq \frac{w_0}{\beta_0} \left( \frac{\Lambda^2}{M^2} \right)^k,$$

where we have used the fact that $\tau_L \ll 1$ to expand the distribution function:

$$\hat{w}(\tau) = w_0 \tau^{k-1} + \ldots \quad \text{for} \quad \tau \to 0.$$ (15)

It is the asymptotic behaviour of the function $\hat{w}(\tau)$ for small values of $\tau$ that determines the size of the renormalon ambiguity. The power $k$ coincides with the position of the nearest IR renormalon pole in the Borel plane [1]. Note that $k > 0$ in order for the integral in (1) to be IR convergent.

In Ref. [1] we have developed the resummation technique for QCD Green functions without external gluon fields and for euclidean correlation functions of currents. In these cases only virtual gluon corrections have to be considered. Both from the phenomenological and from the conceptual point of view it is interesting to extend the formalism to the case of cross sections and inclusive decay rates, which receive both virtual and real gluon corrections. This generalization is the subject of the present work. In particular, we shall consider the perturbative series for the cross section $\sigma(e^+e^- \to \text{hadrons})$ and for the decay rate $\Gamma(\tau \to \nu_\tau + \text{hadrons})$. In the $\overline{\text{MS}}$ scheme, one finds at order $\alpha_s^2 \sim [19, 25]$

$$\frac{1}{3} \left( \sum Q_i^2 \right) \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = 1 + \frac{\alpha_s(s)}{\pi} + (0.17\beta_0 + 0.08) \left( \frac{\alpha_s(s)}{\pi} \right)^2,$$

$$\frac{1}{3} \left( \Gamma(\tau \to \nu_\tau + \text{hadrons}) \right) = 1 + \frac{\alpha_s(m_\tau^2)}{\pi} + (0.57\beta_0 + 0.08) \left( \frac{\alpha_s(m_\tau^2)}{\pi} \right)^2,$$

where $Q_i$ denote the electric charges of the quarks, $s$ is the centre-of-mass energy, and quark mass effects are neglected. The terms proportional to $\beta_0$ give the main contribution to the two-loop coefficients. In the case of $R_\tau$ they are numerically quite important. If one uses the BLM prescription to absorb these terms into a
redefinition of the scale in the running coupling constant, one obtains \( \mu_{\text{BLM}}^{e^+e^-} \simeq 0.71 \sqrt{s} \) and \( \mu_{\text{BLM}}^{e^+e^-} \simeq 0.32 m_e \) (in the \( \overline{\text{MS}} \) scheme), respectively. A more striking example is provided by the parton model prediction for the semileptonic decay rate \( \Gamma (b \to u e \bar{\nu}_e) \). The corresponding perturbative series at two-loop order is

\[
\frac{\Gamma (b \to u e \bar{\nu}_e)}{\Gamma_{\text{tree level}}} = 1 - 2.41 \frac{\alpha_s(m_b^2)}{\pi} - (3.22 \beta_0 + k) \left( \frac{\alpha_s(m_b^2)}{\pi} \right)^2 \tag{17}
\]

where the constant \( k \) is yet unknown. The corresponding BLM scale is \( \mu_{\text{BLM}}^{b \to u} \simeq 0.07 m_b \). Here and in the case of \( R_\tau \), the BLM scales are so low that one may doubt the reliability of the perturbative expansion. It is therefore important to understand the origin of these low scales and to consider higher-order terms in the series. The fact that the contributions of order \( \beta_0 \alpha_s^2 \) in (16) give the dominant two-loop corrections justifies, to some extent, that we concentrate on the class of higher-order contributions provided by renormalon chains. Clearly, the resummation in (3) does not replace an exact higher-order calculation, but nevertheless it provides a nontrivial all-order resummation of a gauge-invariant subset of corrections. As such, it seems worth while to apply this approach in the cases considered above. However, quantities defined in the physical region (i.e. the region of time-like momenta) differ in two respects from those considered in Ref. [1], which were defined in the euclidean region. First, in the calculation of radiative corrections both virtual and real gluons contribute, and only the sum of their contributions is IR finite [27, 28]. Clearly, in such a situation one has to generalize the idea of performing a one-loop calculation with a running coupling constant \( \alpha_s(-k^2) \), since \( k^2 = 0 \) for real gluons. Second, the operator product expansion (OPE), which provides the framework for a systematic analysis of nonperturbative effects, can be justified in the euclidean region only, although it is sometimes argued that a “generalized OPE” based on quark–hadron duality holds in the physical region after applying a “smearing procedure” [29]. Below we shall discuss in detail the resummation of renormalon chain contributions for the ratios \( R_{e^+e^-} \) and \( R_\tau \), encountering some new features arising from the above-mentioned complications. The “linear” form of the integral representation in (1) is replaced by “non-linear” representations, in which instead of the coupling constant there appears a process-dependent function of the coupling constant. The choice of this function is not unique, leading to nonperturbative ambiguities in the definition of the resummed series that are not related to IR renormalons. Based on our results we cannot exclude that the one-to-one correspondence between the size of long-distance contributions and the position of renormalon singularities, which in the euclidean region establishes the link between the OPE and the Borel integral, may be lost in the physical region.

In Sect. 2 we review some results derived in Ref. [1] for the perturbative series for the correlator of two vector currents in the euclidean region. We discuss the connection between IR renormalons and nonperturbative effects. By introducing
to order $1/(Q^2)^2$. In Sect. 3 we perform the analytic continuation to the physical region and discuss the resummation of renormalon chains for the cross-section ratio $R_{e^+e^-}$. Section 4 is devoted to the analogous discussion for the decay-rate ratio $R_\tau$. In Sect. 5 we investigate the origin of the nonperturbative ambiguities in the definition of the resummed series. We study long-distance effects and their relation to the singularities in the Borel plane by introducing an IR cutoff to regularize the Borel integral. A numerical analysis of the results is presented in Sect. 6. In Sect. 7 we summarize our results and point out some phenomenological implications. The relation of our approach to the resummation procedure proposed by Beneke and Braun [4] is discussed in the Appendix.

2 Current correlator in the euclidean region

Consider the correlator $\Pi(Q^2)$ of two vector currents $j^\mu = \bar{q} \gamma^\mu q$ in the euclidean region ($Q^2 = -q^2 > 0$):

$$i \int d^4 x e^{i q \cdot x} \langle 0 | T \{j^\mu(x), j^\nu(0)\} | 0 \rangle = (q^\mu q^\nu - q^2 g^\mu\nu) \Pi(Q^2).$$

(18)

For simplicity we shall neglect the quark masses. The momentum transfer $Q^2 \gg \Lambda^2$ provides the large mass scale. The derivative of $\Pi(Q^2)$ with respect to $Q^2$ is UV convergent. We define

$$D(Q^2) = 4\pi^2 Q^2 \frac{d\Pi(Q^2)}{dQ^2}.$$

(19)

In Ref. [1] we have constructed the resummation of renormalon chains for the corresponding perturbative series. The result is

$$D_{\text{res}}(Q^2) = 1 + \int_0^\infty d\tau \tilde{w}_D(\tau) \frac{\alpha_s(r e C Q^2)}{4\pi},$$

(20)

where the distribution function is given by

$$\tilde{w}_D(\tau) = 8C_F \left\{ \left( \frac{7}{4} - \ln \tau \right) \tau + (1 + \tau) \left[ L_2(-\tau) + \ln \tau \ln(1 + \tau) \right] \right\}; \quad \tau < 1,$n

$$\tilde{w}_D(\tau) = 8C_F \left\{ 1 + \ln \tau + \left( \frac{3}{4} + \frac{1}{2} \ln \tau \right) \frac{1}{\tau} + (1 + \tau) \left[ L_2(-\tau^{-1}) - \ln \tau \ln(1 + \tau^{-1}) \right] \right\}; \quad \tau > 1.$$

(21)

Here $C_F = 4/3$ is a colour factor, and $L_2(x) = -\int_0^x \frac{dy}{y} \ln(1 - y)$ is the dilogarithm function. The function $\tilde{w}_D(\tau)$ and its first three derivatives are continuous
at $\tau = 1$, but higher derivatives are not. A graphical representation of the distribution function is shown in Fig. 1. We find it most useful to show the product $\tau \hat{w}_D(\tau)$ as a function of $\ln \tau$, since then the integrals $\langle \ln^a \tau \rangle$ have a direct graphical interpretation. In order to associate mass scales with the $\tau$-values in the figure, we note that $\ln(\mu^2/Q^2) = \ln \tau + C$, where $\mu$ is the scale in the running coupling constant. In the V scheme, the point $\ln \tau = 0$ corresponds to $\mu^2 = Q^2$; in the $\overline{MS}$ scheme, it corresponds to $\ln \tau = 5/3$. We observe that the distribution function is rather narrow and centred around $\ln \tau \simeq 1$. The long arrow indicates the average value $\langle \ln \tau \rangle$. In order to calculate the BLM scale and the width parameter $\Delta$ defined in (1), we compute

$$N = \frac{3}{4} C_F = 1,$$

$$\langle \ln \tau \rangle = 4\zeta(3) - \frac{23}{6} \simeq 0.975,$$

$$\langle \ln^2 \tau \rangle = 18 - 12\zeta(3) \simeq 3.575,$$  \hspace{1cm} (22)

where $\zeta(3) \simeq 1.20206$. This leads to

$$\frac{\mu_{BLM}^2}{\sqrt{Q^2}} \simeq 1.628 e^{C/2} \overline{MS} \rightarrow 0.708,$$

$$\Delta_D \simeq 2.625, \quad \sigma_D \simeq 1.620.$$  \hspace{1cm} (23)

Note that both in the $\overline{MS}$ and in the V scheme, the BLM scale is close to the “natural” scale $\sqrt{Q^2}$.

The asymptotic behaviour of the distribution function for $\tau \rightarrow 0$ is

$$\hat{w}_D(\tau) = C_F \left\{ 6\tau + (4 \ln \tau - 6) \tau^2 + O(\tau^3) \right\}.$$  \hspace{1cm} (24)

As mentioned in the introduction, the linear term signals that the nearest IR renormalon singularity in the Borel transform of the perturbative series is located at $u = 2$. Indeed, in the case of the function $D(Q^2)$ the Borel transform is [13, 14]

$$\hat{S}_D(u) = \frac{32 C_F}{2 - u} \sum_{k=2}^{\infty} \frac{(-1)^k}{[k^2 - (1 - u)^2]^2} = \frac{6 C_F}{2 - u} + \ldots ,$$  \hspace{1cm} (25)

where the ellipses represent terms that are regular at $u = 2$. The presence of IR renormalons leads to an ambiguity in the value of the resummed series, which according to [14] is given by

$$\Delta D_{\text{ren}} = \frac{\Lambda_V^2}{\beta_0 Q^2} \hat{w}_D(\Lambda_V^2/Q^2) \simeq \frac{6 C_F}{\beta_0} \left( \frac{\Lambda_V^2}{Q^2} \right)^2.$$  \hspace{1cm} (26)

The appearance of IR renormalons acts as a reminder that the result of any perturbative calculation in QCD is incomplete; it must be supplemented by non-perturbative corrections. Only the sum of all perturbative and nonperturbative
Figure 1: The distribution function $\tau \hat{w}_D(\tau)$ as a function of $\ln \tau$. The long arrow indicates the average value of $\ln \tau$, which determines the BLM scale. The short arrows show the point $\tau = \lambda^2/Q^2$ for $\lambda = 1$ GeV and $Q^2 = m^2_\tau$ (right) and $(20 \text{ GeV})^2$ (left).

contributions is unambiguous. Unlike any finite-order calculation, the representation (1) makes explicit that perturbative calculations contain long-distance contributions from the region of low momenta in Feynman diagrams. Moreover, it provides a convenient way to implement Wilson’s construction of the OPE [18]. Since the integration variable $\tau$ can be interpreted as a physical scale parameter, one can separate the contributions from different momentum scales by splitting up the integral into a short- and a long-distance piece. We define

$$D_{\text{res}}(Q^2) = 1 + \int_{\lambda^2/Q^2}^\infty d\tau \hat{w}_D(\tau) \frac{\alpha_s(\tau e^C Q^2)}{4\pi} + \int_{0}^{\lambda^2/Q^2} d\tau \hat{w}_D(\tau) \frac{\alpha_s(\tau e^C Q^2)}{4\pi} \equiv 1 + D_{\text{sd}}(Q^2, \lambda^2) + D_{\text{ld}}(Q^2, \lambda^2).$$

(27)

Here $\lambda$ acts as a factorization (or separation) scale, which should be chosen such that $\Lambda_V < \lambda \ll \sqrt{Q^2}$. The short-distance contribution can be reliably calculated in perturbation theory, and it is free of renormalon ambiguities. The long-distance contribution must be combined with other nonperturbative corrections. Only the sum of all long-distance contributions is well defined. The dependence on the arbitrary scale $\lambda$ must cancel in the final result. This $\lambda$-dependence can be controlled in perturbation theory by means of the renormalization-group equation

$$\lambda^2 \frac{d}{d\lambda^2} D_{\text{ld}}(Q^2, \lambda^2) = -\lambda^2 \frac{d}{d\lambda^2} D_{\text{sd}}(Q^2, \lambda^2) = \frac{\alpha_s(e^C \lambda^2)}{4\pi} \frac{\lambda^2}{Q^2} \hat{w}_D(\lambda^2/Q^2).$$

(28)

If $\lambda^2/Q^2 \ll 1$, the $Q^2$-dependence of the long-distance contribution is determined by the asymptotic behaviour of the distribution function for small values of $\tau$ and
coincides with the $Q^2$-dependence of the renormalon ambiguity. In the present case, it follows that $D_{ld}(Q^2, \lambda^2) \propto 1/(Q^2)^2$. At the same order in the OPE there appear nonperturbative effects parametrized by the gluon condensate \cite{30}, which has an UV renormalon ambiguity that compensates the ambiguity from the IR renormalon in the perturbative series \cite{8}–\cite{14}. Hence, to this order we may write for the physical correlator

$$D(Q^2) = 1 + D_{sd}(Q^2, \lambda^2) + D_{ld}(Q^2, \lambda^2) + \frac{2\pi}{3(Q^2)^2} \langle \alpha_s G^2 \rangle + \ldots$$

$$\equiv 1 + D_{sd}(Q^2, \lambda^2) + \frac{2\pi}{3(Q^2)^2} \langle \alpha_s G^2 \rangle(\lambda) + \ldots , \quad (29)$$

where the last equation defines a scale-dependent gluon condensate, which is free of renormalon ambiguities. From (28), it follows that

$$\lambda^2 \frac{d}{d\lambda^2} \langle \alpha_s G^2 \rangle(\lambda) = \frac{9C_F}{4\pi^2} \alpha_s(e^C \lambda^2)^4. \quad (30)$$

Let us now present a numerical analysis of our results. We choose $Q^2 = m^2_\tau$ as the momentum scale. In Table I we compare the following approximations for $D(m^2_\tau)$: the one- and partial two-loop expressions evaluated using the “natural” scale $Q^2$ in the running coupling constant, the one-loop expression evaluated using the BLM scale, the truncated series including the first correction to the BLM scheme (denoted by $D_{BLM}^*$) given by the term proportional to $\Delta$ in (5), and the partial resummation of the series provided by the integral over the distribution function regulated with the principle value prescription. We use the one-loop expression for the running coupling constant with $n_f = 3$ light quark flavours. We take $\alpha_s(m^2_\tau) = 0.32$ in the $\overline{\text{MS}}$ scheme, corresponding to $\Lambda_{3\overline{\text{MS}}} \simeq 201$ MeV. The corresponding value of the scale parameter in the V scheme is $\Lambda_V \simeq 461$ MeV. We observe that the effect of higher-order corrections is not negligible. The resummation of renormalon chains leads to a 3.3% increase in the value of $D(m^2_\tau)$ with respect to the two-loop result. This effect is twice as big as the two-loop correction itself. Most of the higher-order corrections are taken into account if one includes the first correction to the BLM scheme. A nice way to compare the all-order resummation of renormalon chains with the BLM approximation is to define a scale $\mu_*$ such that the one-loop correction evaluated at this scale reproduces the resummed series, i.e. $D_{\text{res}}(m^2_\tau) \equiv 1 + \alpha_s(\mu_*)/\pi$. We find $\mu_*/\mu_{BLM} \simeq 0.69$. The fact that $\mu_* < \mu_{BLM}$ means that higher-order corrections effectively decrease the average virtuality.

As mentioned above, the problem of IR renormalons can be avoided by introducing a factorization scale $\lambda$ to separate short- and long-distance contributions. For the two values $Q^2 = (20 \text{ GeV})^2$ and $Q^2 = m^2_\tau$, the factorization point

\footnote{To be consistent, we take into account only the part of the two-loop corrections proportional to $\beta_0 \alpha_s^2$. For the function $D(m^2_\tau)$ the remaining two-loop correction is $0.08 (\alpha_s/\pi)^2 \simeq 8 \times 10^{-4}$.}
Table 1: Comparison of various approximations for the euclidean correlator $D(Q^2)$ at $Q^2 = m_\tau^2$.

| Approximation | $D_1$ loop | $D_2$ loop | $D_{BLM}$ | $D_{BLM^*}$ | $D_{res}$ | $\Delta D_{ren}$ |
|---------------|------------|------------|------------|-------------|-----------|------------------|
|               | 1.102      | 1.118      | 1.121      | 1.145       | 1.151     | $3.3 \times 10^{-3}$ |

$\tau = \lambda^2 / Q^2$ for $\lambda = 1$ GeV is indicated by the short arrows in Fig. 1. Whereas long-distance contributions are completely negligible for $Q^2 = (20 \text{ GeV})^2$, they are important for $Q^2 = m_\tau^2$. This is reflected in Fig. 2, where we show the short-distance contribution $1 + D_{sd}(m_\tau^2, \lambda^2)$, defined in (27), as a function of the factorization scale. The solid line is used in the range $\Lambda_V < \lambda < m_\tau$, where $D_{sd}(m_\tau^2, \lambda^2)$ is well defined. The dotted line is used for the region below the Landau pole. Roughly speaking, perturbation theory can be trusted down to a value $\lambda_0 \simeq 0.93$ GeV (corresponding to $\mu \simeq 0.40$ GeV in the $\overline{\text{MS}}$ scheme), where $\alpha_s(e^C \lambda_0^2) = 1$. The variation of the short-distance contribution with $\lambda$ provides an estimate of the importance of long-distance effects. In the present case, these effects are of order a few per cent.

Figure 2: Short-distance contribution to $D(m_\tau^2)$ as a function of the factorization scale. The dotted line is used in the region below the Landau pole, where the short-distance contribution is no longer well defined.

---

2In the $\overline{\text{MS}}$ scheme, $\lambda = 1$ GeV corresponds to the rather low scale $\mu = e^{-5/6} \lambda \simeq 0.43$ GeV. The value of the ratio $\lambda / \Lambda_V = \mu / \Lambda_{QCD}$ is scheme independent.
3 Analytic continuation and $R_{e^+e^-}$

Let us now consider the analytic continuation of the euclidean correlator $\Pi(Q^2)$ to the physical region, where $s = -Q^2 > 0$. The imaginary part of the correlator is related to the total cross section for the process $e^+e^- \rightarrow$ hadrons at the centre-of-mass energy $s$ by

$$R_{e^+e^-}(s) = \frac{\sigma(e^+e^- \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)} = 12\pi \left( \sum Q_i^2 \right) \Im \Pi(-s + i\epsilon) \equiv 3 \left( \sum Q_i^2 \right) R(s),$$

(31)

where $Q_i$ denote the electric charges of the quarks. Our goal is to resum the renormalon chain contributions (i.e. terms of order $\beta_n^{-1} \alpha_s^n$) in the perturbative series for the quantity $R(s)$, and to derive a representation of the result as an integral over a distribution function. To this end, we integrate (20) with respect to $\ln Q^2$ and use the one-loop expression for the $\beta$-function to obtain [1]

$$4\pi^2 \left[ \Pi(Q^2) - \Pi(Q_0^2) \right]_{\text{res}} = \ln \frac{Q^2}{Q_0^2} + \frac{1}{\beta_0} \int_0^\infty d\tau \hat{w}_D(\tau) \ln \frac{\alpha_s(\tau e^C Q_0^2)}{\alpha_s(\tau e^C Q^2)}. $$

(32)

Here $Q_0^2$ is an arbitrary subtraction point. Next we perform the analytic continuation $Q^2 \rightarrow -s + i\epsilon$, using that for the one-loop running coupling constant

$$\frac{1}{\pi} \Im \ln \frac{1}{\alpha_s(-\tau e^C s + i\epsilon)} = F_1(a(\tau s)), $$

(33)

where

$$F_1(a) = \frac{1}{\pi} \arccot \left( \frac{1}{\pi a} \right) = \frac{1}{\pi} \arctan(\pi a) + \Theta(-a),$$

(34)

and

$$a(\mu^2) \equiv \frac{\beta_0 \alpha_s(e^C \mu^2)}{4\pi} = \frac{1}{\ln \mu^2 / \Lambda_V^2}. $$

(35)

Combining the above results we obtain an integral representation for the resummed perturbative series $R_{\text{res}}(s)$, in which there appears the same distribution function as in (20), but a non-linear function of the coupling constant. Using the notation

$$W_{e^+e^-}(\tau) = \hat{w}_D(\tau)$$

(36)

for the distribution function in this “non-linear” representation, we write the result as

$$R_{\text{res}}^{(1)}(s) = 1 + \frac{1}{\beta_0} \int_0^\infty d\tau W_{e^+e^-}(\tau) F_1(a(\tau s)) .$$

(37)

The need for the label “1” will become clear below. A similar integral representation has been suggested by Beneke and Braun [1]. The precise relation between their approach and ours is clarified in the Appendix. These authors suggest to
interpret $F_1(a)$ as an effective coupling constant, which has the attractive features that it agrees with the usual coupling constant in the perturbative region $0 < a \ll 1$, but it has a smooth behaviour in the IR region. In Fig. 3 we show the effective coupling constant $F_1(a)$ and the bare coupling constant $a$ as a function of $1/a = \ln \mu^2 / \Lambda^2$. 

$$
\begin{align*}
\mu_{\text{BLM}}^{e^+e^-} \sqrt{s} &\simeq 1.628 e^{C/2} \rightarrow 0.708, \\
\Delta_{e^+e^-} &= \sigma_D^2 - \frac{\pi^2}{3} \simeq -0.665 .
\end{align*}
$$

Since $\Delta_{e^+e^-}$ is negative, the leading correction to the BLM scheme effectively increases the value of the scale. In general, non-linear representations like (37), for which the growth of the running coupling constant in the IR region is damped, lead to an improved convergence of the perturbative series.

Let us now investigate the renormalon ambiguity of the resummed series (37). Expanding the function $F_1(a)$ in powers of the coupling constant and using the relation

$$
[\pi a(\tau s)]^{n+1} = \frac{\pi}{n!} (-\delta_\tau)^n a(\tau s) ; \quad \delta_\tau = \pi \frac{d}{d \ln \tau} ,
$$

Figure 3: Effective coupling constants $F_1(a)$ (dash-dotted line) and $F_2(a)$ (solid line) as a function of $1/a$. Both functions coincide for $a > 0$. The thin line shows the bare coupling constant $a$.

Since the perturbative series for $R(s)$ and $D(Q^2)$ agree to order $\alpha_s^2$, the BLM scales are the same in the two cases. Differences appear first at order $\alpha_s^3$, i.e. in the value of the parameter $\Delta$ in (3). In the case of $R(s)$, this parameter is no longer given by the square of the width $\sigma_D$ of the distribution function. Taking into account that $F_1(a) = a - \pi^2 a^3/3 + O(a^5)$, we find that
we find that
\[
\Delta R_{\text{ren}} = \frac{1}{\beta_0} \int_0^\infty d\tau W_{e^+ e^-}(\tau) \mathcal{O}[\delta_\tau] \delta(\ln \tau - \ln \tau_L) = \frac{1}{\beta_0} \mathcal{O}[\delta_\tau] \left. \left( \tau W_{e^+ e^-}(\tau) \right) \right|_{\tau = \tau_L},
\]
(40)
where \( \tau_L = \Lambda^2 / s \) is the position of the Landau pole, and \( \mathcal{O}[\delta_\tau] \) denotes the differential operator
\[
\mathcal{O}[\delta_\tau] = 1 - \frac{\delta_\tau^2}{3!} + \frac{\delta_\tau^4}{5!} \pm \ldots = \frac{\sin \delta_\tau}{\delta_\tau}.
\]
To proceed, we use the relations
\[
\mathcal{O}[\delta_\tau] \tau^k = \frac{\sin(k\pi)}{k\pi} \tau^k,
\]
\[
\mathcal{O}[\delta_\tau] \tau^k \ln^n \tau = \left( \frac{d}{dk} \right)^n \frac{\sin(k\pi)}{k\pi} \tau^k,
\]
(42)
which imply that integer powers of \( \tau \) in the distribution function do not contribute to \( \Delta R_{\text{ren}} \). Given the asymptotic behaviour shown in (24), we see that the leading contribution to the renormalon ambiguity comes from the term proportional to \( \tau^2 \ln \tau \). The result is
\[
\Delta R_{\text{ren}} = -\frac{4C_F}{3\beta_0} \left( \frac{\Lambda^2}{s} \right)^3 + O(1 / s^4),
\]
(43)
in accordance with the fact that the nearest IR renormalon pole in the Borel transform corresponding to the quantity \( R(s) \) is located at \( u = 3 \).\(^3\)
\[
\hat{S}_R(u) = \frac{32 C_F}{\pi} \frac{\sin(\pi u)}{u(2 - u)} \sum_{k=2}^\infty \frac{(-1)^k k}{[k^2 - (1 - u)^2]^2} = -\frac{4C_F}{3} \frac{1}{3 - u} + \ldots.
\]
(44)
The ellipses represent terms that are regular at \( u = 3 \).

At this point a word of caution related to the interpretation of the integral in (37) is in order. In the case of the linear integral representation (1), one can interpret the integration variable \( \tau \) as a physical scale parameter and introduce a factorization point \( \tau = \lambda^2 / M^2 \) to separate short- and long-distance contributions. If one would proceed in the same way in the present case, one would conclude that the long-distance contribution to \( R(s) \) is of order \( \lambda^4 / s^2 \). Such an interpretation may be misleading, however. In non-linear representations there is no physical significance to the integration variable; for instance, it is possible to obtain different non-linear representations by changing variables or integrating by parts. An example will be given in the following section. In the case of (37), one could try
\(^3\)However, it is known that a renormalon singularity at \( u = 2 \) appears when one goes beyond the large-\( \beta_0 \) approximation.\(^4\)
to obtain a linear representation by repeating the steps used to derive the result for the renormalon ambiguity given above. If $W_{e^+e^-}(\tau)$ was an analytic function, one would conclude that $(1/\tau)O[\delta\tau]\left[\tau W_{e^+e^-}(\tau)\right]$ is the distribution function for $R(s)$ in the linear representation. Since higher derivatives of $W_{e^+e^-}(\tau)$ are not continuous at $\tau = 1$, however, one cannot perform these steps. The resulting “distribution function” would contain an infinite number of $\delta$-function type singularities at $\tau = 1$. In other words, for the physical correlation function $R(s)$ a linear representation of the form (37) does not exist. As we will prove in Sect. 5, the same is true for all quantities defined in the physical region.

Let us now discuss an alternative method to resum renormalon chains for $R_{e^+e^-}$. It is based on a representation of the function $R(s)$ as an integral in the complex plane. Using that in the real world the discontinuities of $D(-s)$ are located on the positive real $s$-axis, and that $\text{Im} \Pi(0) = 0$, one obtains

$$R(s) = 4\pi \text{Im} \Pi(-s + i\epsilon) = \frac{1}{\pi} \text{Im} \int_{0}^{s} \frac{ds'}{s'} D(-s' + i\epsilon) = \frac{1}{2\pi i} \oint \frac{ds'}{s'} D(-s') \cdot (45)$$

Inserting the expression (20) for the correlation function and using the fact that

$$a(-x\tau s) = \frac{a(\tau s)}{1 + a(\tau s) \ln(-x)} \quad (46)$$

for the one-loop running coupling constant, we find

$$R_{\text{res}}^{(2)}(s) = 1 + \int_{0}^{\infty} d\tau \, \hat{w}_D(\tau) \frac{1}{2\pi i} \oint \frac{dx}{x} \frac{\alpha_s(-x\tau e^C s)}{4\pi}$$

$$= 1 + \frac{1}{\beta_0} \int_{0}^{\infty} d\tau \, W_{e^+e^-}(\tau) F_2(a(\tau s)) \cdot (47)$$

where

$$F_2(a) = \frac{1}{2\pi i} \oint \frac{dx}{x \, 1 + a \ln(-x)} = \frac{1}{\pi} \arctan(\pi a) \cdot (48)$$

In Fig. 3 the function $F_2(a)$ is compared to $F_1(a)$. It is surprising that the resummations (37) and (17) involve the same distribution function $W_{e^+e^-}(\tau)$ but different functions of the coupling constant. Clearly, both results would have to be equivalent if the perturbative approximation for the function $D(-s)$ would satisfy the analyticity properties of the physical correlation function. However, it does not. Consider, for example, the “one-loop” expression $D(-s) = 1 + \alpha_s(-s)/\pi$. The cut on the positive $s$-axis starts at $s = 0$ (not $s > 0$), and in addition there is the Landau pole on the negative $s$-axis. The same applies for the resummed series in (20); however, in this case the Landau pole of the running coupling constant leads to a cut on the negative $s$-axis. It is not difficult to show that these two
defects are responsible for the difference between the two resummations. Note that the effective coupling constants $F_1(a)$ and $F_2(a)$ are identical for $a > 0$, implying that (37) and (47) correspond to the resummation of one and the same perturbative series. The two functions differ only for negative values of $a$, i.e. below the Landau pole. The difference between the two resummations is thus a nonperturbative effect. We find

$$R_{\text{res}}^{(2)}(s) - R_{\text{res}}^{(1)}(s) = -\frac{1}{\beta_0} \int_0^{\tau_0} d\tau W_{e^+e^-}(\tau) = -\frac{3C_F}{\beta_0} \left( \frac{A_V^2}{s} \right)^2 + O(1/s^3),$$

(49)

where $\tau_L = \Lambda^2 / s$ is the position of the Landau pole. Note that this difference is parametrically larger than the IR renormalon ambiguity of the resummed series given in (43). Indeed, the effect is not related to renormalons; both resummations correspond to the same perturbative series and thus to the same Borel transform. We will come back to an analysis of this new type of ambiguity in Sect. 4.

4 Decay-rate ratio $R_\tau$

As a second example we consider the resummation of renormalon chains for the perturbative series corresponding to the ratio

$$R_\tau = \frac{\Gamma(\tau \to \nu_\tau + \text{hadrons})}{\Gamma(\tau \to \nu_\tau e^- \bar{\nu}_e)} \equiv 3T(m_\tau^2).$$

(50)

For simplicity we neglect quark mass effects and small electroweak corrections, which are however known [31, 32]. Our goal is to understand in physical terms the low value of the BLM scale given in the introduction, $\mu_{\text{BLM}} \approx 0.32 m_\tau$ (in the MS scheme), and to investigate the effect of higher-order corrections by performing the resummation of renormalon chains. As in the previous case, $R_\tau$ can be expressed in terms of the imaginary part of the correlator $\Pi(-s)$ in the physical region $s > 0$, or as a contour integral over the $D$-function in the complex plane. We will again find that the result of the resummation depends on which representation one chooses.

The first method is to relate $R_\tau$ to an integral over the function $R(s)$ introduced in (31). For fixed neutrino energy $E_\nu$, the invariant mass $s_{\text{had}}$ of the hadronic final state in the decay $\tau \to \nu_\tau + \text{hadrons}$ is given by

$$s_{\text{had}} = (p_\tau - p_\nu)^2 = m_\tau^2 x, \quad x = 1 - \frac{2E_\nu}{m_\tau} \in [0, 1].$$

(51)

In the limit where one neglects the light quark masses, and as long as one is interested only in terms of order $\beta_0^{n-1} \alpha_s^n$ in the perturbative series, the hadronic
dynamics is described by the function $R(s_{\text{had}})$ discussed in the previous section. One obtains \cite{24, 25}

\[
T(m_\tau^2) = 2 \int_0^1 dx \left(1 - 3x^2 + 2x^3\right) R(xm_\tau^2),
\]

(52)

where the function $2(1 - 3x^2 + 2x^3)$ is the normalized invariant mass spectrum at tree level. This expression allows us to understand the low value of the BLM scale. We expect that

\[
\ln \left( \frac{\mu_{\text{BLM}}^2}{m_\tau^2} \right) = \ln \left( \frac{\mu_{e^+e^-}^2}{s} \right) + \langle \ln x \rangle, \tag{53}
\]

\[
\langle \ln x \rangle = 2 \int_0^1 dx \left(1 - 3x^2 + 2x^3\right) \ln x = -\frac{19}{12} \tag{54}
\]

is the average of $\ln x$ over the invariant mass spectrum. Hence, the BLM scales corresponding to $R_{e^+e^-}(m_\tau^2)$ and $R_\tau$ should obey the relation [see (38)]

\[
\mu_{e^+e^-} = e^{-19/24} \mu_{\text{BLM}} \approx 0.738 e^{C/2} m_\tau. \tag{55}
\]

We will see below that this is indeed the exact result. The fact that the BLM scale for $R_\tau$ is lower than $m_\tau$ simply reflects the distribution of the hadronic mass in the decay.

Let us now proceed to construct the resummation of renormalon chains for $R_\tau$. Substituting either (37) or (47) into (52) and rescaling the integration variable ($\tau \rightarrow \tau/x$), we immediately obtain two different representations for the resummed series. They are $(n = 1$ or $2)$

\[
T_{\text{res}}^{(n)}(m_\tau^2) = 1 + \frac{1}{\beta_0} \int_0^\infty d\tau \, W_\tau(\tau) F_n(a(\tau m_\tau^2)), \tag{56}
\]

with $F_n(a)$ as given in (34) and (48), respectively. The new distribution function is given by

\[
W_\tau(\tau) = 2 \int_0^1 \frac{dx}{x} \left(1 - 3x^2 + 2x^3\right) \hat{w}_D \left(\frac{\tau}{x}\right). \tag{57}
\]

It is possible to perform this integral explicitly, with the result that

\[
W_\tau(\tau) = 16C_F \left\{ 4 - \frac{73}{12} \tau - \frac{23}{24} \tau^2 - \frac{259}{432} \tau^3 - 2L_3(-\tau) - 3\zeta(3) \right. \\
+ \left( \frac{17}{6} \tau + \frac{\tau^2}{3} + L_2(-\tau) \right) \ln \tau + \left( \frac{3}{4} \tau^2 + \frac{\tau^3}{6} \right) \ln^2 \tau \\
- \left( \frac{11}{6} + 3\tau + \frac{3}{2} \tau^2 + \frac{\tau^3}{3} \right) \left[ \ln \tau \ln(1 + \tau) + L_2(-\tau) \right] \right\}
\]
for $\tau < 1$, and
\[
W_{\tau}(\tau) = 16C_F \left\{ -\frac{575}{216} + \frac{37}{48\tau} - \frac{17}{12}\tau - \frac{\tau^2}{3} + 2L_3(-\tau^{-1}) \right.
\]
\[
- \left( \frac{85}{36} - \frac{1}{4\tau} + \frac{3}{2}\tau + \frac{3}{3}\tau^2 - L_2(-\tau^{-1}) \right) \ln \tau
\]
\[
+ \left( \frac{11}{6} + 3\tau + \frac{3}{2}\tau^2 + \frac{3^3}{3} \right) \left[ \ln \tau \ln(1 + \tau^{-1}) - L_2(-\tau^{-1}) \right] \right\} \tag{58}
\]
for $\tau > 1$. Here $L_3(x) = \int_x^\infty \frac{dy}{y} L_2(y)$ denotes the trilogarithm function. In Fig. 4 we compare the distribution functions appearing in the resummations (57) and (56) for $R_{e^+e^-}$ and $R_{\tau}$. Note that the area under the two curves is the same. The distribution function for $R_{\tau}$ is broader than that for $R_{e^+e^-}$ and shifted to lower scales, in accordance with the physical argument presented above.

![Figure 4: The distribution functions $\tau W_{e^+e^-}(\tau)$ (solid line) and $\tau W_{\tau}(\tau)$ (dash-dotted line) as a function of $\ln \tau$. The arrows indicate the average values of $\ln \tau$, which determine the BLM scales.]

With the function $W_{\tau}(\tau)$ we compute the integrals [cf. (5)]
\[
N = \frac{3}{4} C_F = 1,
\]
\[
\langle \ln \tau \rangle = 4\zeta(3) - \frac{65}{12} \simeq -0.608,
\]
\[
\langle \ln^2 \tau \rangle = \frac{2435}{72} - \frac{74}{3} \zeta(3) \simeq 4.169. \tag{59}
\]
For the BLM scale and the parameter $\Delta_{\tau}$, which determines the leading correction to the BLM scheme in (4), we obtain
\[
\frac{\mu_{\text{BLM}}}{m_{\tau}} \simeq 0.738 e^{C/2} \frac{\text{MeV}}{m_{\tau}} \Rightarrow 0.321,
\]
\[ \Delta_r = \langle \ln^2 \tau \rangle - \langle \ln \tau \rangle^2 - \frac{\pi^2}{3} \simeq 0.509. \] (60)

The asymptotic behaviour of the distribution function for \( \tau \to 0 \) is

\[ W_\tau(\tau) = 16 C_F \left\{ 4 - 3 \zeta(3) - \frac{9}{4} \tau + \left( \frac{4}{3} - \frac{3}{2} \ln \tau + \frac{3}{4} \ln^2 \tau \right) \tau^2 + O(\tau^3) \right\}. \] (61)

Using (62) we find that the renormalon ambiguity in the value of the resummed series is given by

\[ \Delta T_{\text{ren}} = \frac{8 C_F}{\beta_0} \left( \ln \frac{m^2}{\Lambda^2} + \frac{4}{3} \right) \left( \frac{\Lambda^2}{m^2} \right)^3 + O(1/m^8), \] (62)

corresponding to an IR renormalon singularity located at \( u = 3 \). This can also be seen from the Borel transform of the series \( T(m^2) \), for which we obtain

\[ \hat{S}_T(u) = \frac{384 C_F}{\pi} \frac{\sin(\pi u)}{u(1-u)(2-u)(3-u)(4-u)} \sum_{k=2}^{\infty} \frac{(-1)^k k}{[k^2 - (1-u)^2]^2}, \]

\[ = \frac{8 C_F}{3} \left\{ - \frac{3}{(3-u)^2} + \frac{4}{3-u} + \ldots \right\}. \] (63)

The ellipses represent terms that are regular at \( u = 3 \).

For later purposes it will be useful to rewrite the result (60) in another form. Instead of performing the \( x \)-integration over the distribution function, one can perform it over the function \( F_n(a) \) and write \( n = 1 \) or \( 2 \)

\[ T^{(n)}_{\text{res}}(m^2) = 1 + \frac{1}{\beta_0} \int_0^\infty d\tau \Omega_\tau(\tau) G_n(a(\tau m^2)) \] (64)

with

\[ G_n(a) = 2 \int_0^1 dx (1 - 3x^2 + 2x^3) F_n \left( \frac{a}{1 + a \ln x} \right), \] (65)

and

\[ \Omega_\tau(\tau) = \hat{w}_D(\tau). \] (66)

Once again we have introduced a new notation for the distribution function, since the form of the integral representation has changed. Eqs. (60) and (64) provide an example of the fact that the integration variable \( \tau \) has no significance as a physical scale if the integral representation is non-linear. Whereas the contribution from the region \( \tau < \lambda^2/m^2 \) scales like \( \lambda^2/m^2 \) in the case of (56), it scales like \( \lambda^4/m^4 \) in the case of (64). Nevertheless, both representations are equivalent and lead to the same numerical results.

We will now discuss a third resummation method, which uses the representation of \( T(m^2) \) as a contour integral in the complex plane. Since in the real world
Figure 5: Effective coupling constants $G_1(a)$ (dash-dotted line), $G_2(a)$ (dotted line) and $G_3(a)$ (solid line) as a function of (a) the inverse coupling constant $1/a$, and (b) the coupling constant $a$. For small positive values of $a$, the three functions have identical Taylor expansions.

the discontinuities of the function $D(-s)$ lie on the positive real $s$-axis and start at a value $s_0 > 0$, one can integrate by parts in (52) to obtain

\[
T(m_\tau^2) = \frac{1}{\pi} \text{Im} \int_0^1 \frac{dx}{x} (1 - 2x + 2x^3 - x^4) D(-xm_\tau^2 + i\epsilon)
\]

\[
= \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x} (1 - 2x + 2x^3 - x^4) D(-xm_\tau^2).
\]  

(67)

Inserting the expression (20) for the $D$-function we find

\[
T_{\text{res}}(m_\tau^2) = 1 + \frac{1}{\beta_0} \int_0^\infty d\tau \Omega_\tau(\tau) G_3(a(\tau m_\tau^2))
\]  

(68)
where

\[
G_3(a) = \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x} \left(1 - 2x + 2x^3 - x^4\right) \frac{a}{1 + a \ln(-x)}
\]

\[
= \frac{8a}{\pi} \int_0^\pi d\varphi \frac{\sin 2\varphi - a\varphi \cos 2\varphi}{1 + a^2\varphi^2} \sin \varphi \cos^2 \varphi \frac{\varphi}{2}
\]

(69)

is one of the functions encountered in Ref. [33]. Eqs. (64) and (68) provide three resummations of the perturbative series for \(T(m_\tau^2)\) that have the same distribution function \(\Omega_\tau(\tau)\), but different effective coupling constants \(G_n(a)\). These three functions are shown in Fig. 5. For small positive values of \(a\) they have identical Taylor expansions, meaning that all three resummations correspond to one and the same perturbative series. Outside the radius of convergence the functions \(G_n(a)\) are quite different, however, leading to differences in the value of the resummed series which have the form of nonperturbative power corrections. Again these nonperturbative effects have nothing to do with IR renormalons. It is not difficult to trace how the differences arise from the wrong analytic properties of the perturbative expression for the function \(D(-s)\), which has Landau pole singularities on the negative \(s\)-axis and a cut on the positive \(s\)-axis starting at \(s = 0\) (not \(s > 0\)). We find that

\[
G_2(a) - G_1(a) = \begin{cases} 
-2e^{-1/a} + 2e^{-3/a} - e^{-4/a} & ; a > 0 \\
-1 & ; a < 0 
\end{cases}
\]

\[
G_3(a) - G_2(a) = 4(e^{-1/a} - e^{-3/a}) \Theta(a).
\]

(70)

For the differences between the three resummations we obtain, using (66) and (70),

\[
T_{\text{res}}^{(2)}(m_\tau^2) - T_{\text{res}}^{(1)}(m_\tau^2) = -\frac{1}{\beta_0} \int_0^{\tau_L} d\tau W_\tau(\tau) = -\frac{16C_F}{\beta_0} \left[4 - 3\zeta(3)\right] \frac{\Lambda_V^2}{m_\tau^2} + O(1/m_\tau^4),
\]

\[
T_{\text{res}}^{\text{circle}}(m_\tau^2) - T_{\text{res}}^{(2)}(m_\tau^2) = \frac{4}{\beta_0} \int_{\tau_L}^{\infty} d\tau \Omega_\tau(\tau) \left(\frac{\tau_L}{\tau} - \frac{\tau_3}{\tau^3}\right)
\]

\[
= \frac{32C_F}{\beta_0} \left[4 - 3\zeta(3)\right] \frac{\Lambda_V^2}{m_\tau^2} + O(1/m_\tau^4),
\]

(71)

where \(\tau_L = \Lambda_V^2/m_\tau^2\) is the position of the Landau pole. As in the case of \(R_{e^+e^-}\), the differences are parametrically larger than the renormalon ambiguity given in (62). In the present case they are of order \(1/m_\tau^2\) and thus numerically quite significant; we note that the terms shown on the right-hand side in (71) equal \(-6.3\%\) and \(12.6\\%\). A new feature of the present case is that the effective coupling constants \(G_n(a)\) differ not only for \(a < 0\), but also in the region above the Landau
pole (although by exponentially small terms). This is what makes the various resummations differ by terms of order $1/m_c^2$. We will argue in the next section that such differences can be avoided. There is a physical argument why one should prefer one form of the resummation, $T_{\text{res}}$, over the others. However, as in the case of $R_{e^+e^-}$ there will remain ambiguities related to the treatment of the region below the Landau pole (for $a < 0$), which are of order $1/m_c^4$. These effects are still larger than the IR renormalon ambiguity.

5 Cutoff regularization of the Borel integral

The goal of this section is to understand better the origin of the ambiguities encountered in the resummation of renormalon chains for $R_{e^+e^-}$ and $R_\tau$. We recall that a surprising feature of these nonperturbative effects is that they are not related to IR renormalon singularities in the Borel plane; the differences between the various resummations are parametrically larger than the ambiguity resulting from the nearest IR renormalon. Therefore, it is tempting to conjecture that these differences reflect a failure of our resummation procedure, which could be avoided by defining the resummation in terms of the Borel integral (10), in which case only the usual IR renormalon ambiguities appear. This conjecture may be wrong, however. It has been noted already by 't Hooft that most likely the Borel integral is ill-defined because of strong singularities of the Borel transform at infinity [6]. The presence of resonances and multi-particle thresholds in the physical region is in conflict with Borel summation. It is argued in Ref. [6] that, as a consequence, the Borel transform $\hat{S}(u)$ must diverge at $u \to \infty$ stronger than any exponential of $u$. Discussions of the implications of this observation can be found in Refs. [34, 35]. We note that the strong singular behaviour for large values of $u$ is not seen if the Borel transform is calculated using dimensional regularization. The situation is similar to power-divergent Feynman integrals of the type $\int d^d k (k^2)^n$, which vanish by definition in dimensional regularization. Let us therefore study the singularities of the Borel transform in a more careful way.

5.1 Cutoff regularization in the euclidean region

The proper way to analyze long-distance effects is by introducing an IR cutoff $\lambda$, which eliminates the low-momentum contributions from perturbative calculations. Let us investigate the connection between this procedure and the Borel integral in the euclidean region, where the OPE provides a consistent framework to separate short- and long-distance contributions [18]. As shown in (19), the Borel transform $\hat{S}(u)$ can be defined in terms of an integral over a distribution function $\hat{w}(\tau)$, which has a concrete physical interpretation. This integral has a finite radius of convergence; it is well defined if $u \in ]-j, k [$, where $j$ and $k$ denote
the positions of the nearest UV and IR renormalon singularities. The standard way to proceed is to define $\hat{S}(u)$ outside this interval by analytic continuation, in which case the divergent behaviour of the integral for large or small values of $\tau$ is reflected in the appearance of renormalon singularities. The Borel integral \[10\] has an ambiguity of order $(\Lambda^2 V/M^2)\kappa$ due to the nearest IR renormalon. It is usually argued that the appearance of renormalon ambiguities signals the size of nonperturbative corrections, which have to be added to perturbative calculations in QCD \[8\]–\[12\]. In order to understand the link between the Borel integral and the OPE, let us define a regularized Borel transform as

$$\hat{S}_{\text{reg}}(u, \tau_0) = \int_{\tau_0}^{\infty} d\tau \hat{w}(\tau) \tau^{-u}; \quad \tau_0 = \frac{\lambda^2}{M^2}. \quad (72)$$

The dependence of $\tau_0$ on $M$ is dictated by the fact that, according to (1), the product $\sqrt{\tau M}$ has the interpretation of a physical scale. The introduction of this cutoff eliminates all IR renormalon singularities. In particular, if the asymptotic behaviour of the distribution function for $\tau \to 0$ is as shown in (15), the behaviour of the regularized Borel transform in the vicinity of $u = \kappa$ is

$$\hat{S}_{\text{reg}}(u, \tau_0) = \frac{w_0}{u - \kappa} (\tau_0^{\kappa-u} - 1) + \ldots = w_0 \ln \frac{M^2}{\lambda^2} + \ldots \quad (73)$$

instead of $\hat{S}(u) = -w_0/(u - \kappa) + \ldots$, which is obtained without using a cutoff. The regularization eliminates all singularities for $u > 0$, but at the same time it leads to a strong exponential growth of the Borel transform for $u \to \infty$:

$$\hat{S}_{\text{reg}}(u, \tau_0) \sim \exp \left( u \ln \frac{1}{\tau_0} \right) = \exp \left( u \ln \frac{M^2}{\lambda^2} \right). \quad (74)$$

This growth becomes arbitrarily strong as one tries to eliminate the IR cutoff, in accordance with 't Hooft’s conjecture.

The regularized form of the Borel integral reads

$$S_{\text{Borel}}(M^2, \tau_0) = 1 + \frac{1}{\beta_0} \int_0^\infty du \hat{S}_{\text{reg}}(u, \tau_0) \left( \frac{\Lambda^2 V}{M^2} \right)^u = 1 + \frac{1}{\beta_0} \int_0^\infty du \int_{\tau_0}^{\infty} d\tau \hat{w}(\tau) \left( \frac{\tau M^2}{\Lambda^2 V} \right)^{-u}. \quad (75)$$

The integral over $u$ is convergent if $\tau_0 > \Lambda^2 V/M^2$, or equivalently if $\lambda > \Lambda_V$, which we have to assume for any sensible IR regulator. It is then allowed to interchange the order of integration and perform the integral over $u$ first, which gives

$$\int_0^{\infty} du \left( \frac{\tau M^2}{\Lambda^2 V} \right)^{-u} = \frac{1}{\ln(\tau M^2/\Lambda^2 V)} = \frac{\beta_0 \alpha_s(\tau e^C M^2)}{4\pi}. \quad (76)$$

\[4\]A similar regularization can be applied to eliminate the UV renormalon singularities.
This leads to
\[
S_{\text{reg}}^\text{Borel} (M^2, \tau_0) = 1 + \int_{\tau_0}^{\infty} \frac{d \tau}{\tau} \frac{\alpha_s(\tau e^C M^2)}{4 \pi} = 1 + S_{\text{sd}}(M^2, \lambda^2) \bigg|_{\lambda^2 = \tau_0 M^2},
\] (77)
which is nothing but the short-distance contribution to resummed series defined in (77). The standard form of the Borel integral is recovered if one sets \( \tau_0 = 0 \) and defines the right-hand side of (76) to be the analytic continuation of the \( u \)-integral in the region below the Landau pole, i.e. for \( \tau < \Lambda^2_2 / M^2 \). Then the integration over the Landau pole leads to an ambiguity of order \( (\Lambda^2_2 / M^2)^k \) in (77), which is the usual IR renormalon ambiguity. However, the more physical way to proceed is to keep the IR cutoff. In the regularized version of the Borel integral it is the way in which the factorization scale appears that determines the size of long-distance contributions, which have to be added to the above result in order to cancel the dependence on \( \lambda \). From the renormalization-group analysis in Sect. 2, it follows that this long-distance contribution has the same \( M \)-dependence as the renormalon ambiguity [compare (26) with (28)]. Hence, the IR behaviour in the cutoff version of the Borel integral is in one-to-one correspondence with the appearance of renormalon singularities in the Borel plane. It is this fact which, a posteriori, gives a physical significance to these ambiguities, and which establishes the connection between the Borel integral and the OPE in the euclidean region.

Let us now investigate what changes if the quantity of interest is defined in the physical region. The crucial difference is that in this case the integral representation of the Borel transform shown in (9) does not exist. We will now prove this statement, which was already mentioned in the discussion of \( R_{e^+e^-} \) in Sect. 3. Consider the following alternative integral representation for the Borel transform \( \hat{S}(u) \) of a quantity \( S(M^2) \):
\[
\hat{S}(u) = \frac{\sin \pi u}{\pi u} \int_0^{\infty} d \tau W(\tau) \tau^{-u}. \tag{78}
\]
Beneke and Braun have shown that this representation always exists [4]. Let us assume that also the representation (9) exists. Then there is a simple relation between the functions \( \hat{w}(\tau) \) and \( W(\tau) \). Rewriting the sin-function in terms of \( \Gamma \)-functions, we find that
\[
\int_0^{\infty} d \tau W(\tau) \tau^{-u} = \Gamma(1 + u) \Gamma(1 - u) \int_0^{\infty} d \tau \hat{w}(\tau) \tau^{-u}
\]
\[
= \int_0^{\infty} d \tau \hat{w}(\tau) \tau^{-u} \int_0^1 dx x^u (1 - x)^{-u}, \tag{79}
\]
and introducing new variables $z = x/(1-x)$ and $t = \tau/z$ we obtain

$$\int_0^\infty d\tau \, W(\tau) \, \tau^{-u} = \int_0^\infty dt \int_0^\infty dz \, \frac{z}{(1+z)^2} \, \hat{w}(zt) \, t^{-u}. \quad (80)$$

From this equation we read off the relation ($\sigma = zt$)

$$W(\tau) = \int_0^\infty d\sigma \, \frac{\sigma \, \hat{w}(\sigma)}{(\sigma + \tau)^2} = \int_0^\infty d\sigma \, \frac{1}{\sigma + \tau} \, \frac{d}{d\sigma} \left[ \sigma \, \hat{w}(\sigma) \right], \quad (81)$$

which establishes the connection between the two functions. Provided a function $\hat{w}(\tau)$ satisfying the integral representation (8) exists, it follows that the function $W(\tau)$ is analytic in the complex $\tau$-plane with a branch cut along the negative axis.

We can now use the observation of Beneke and Braun that the function $W(\tau)$ can be calculated by performing loop integrals with a finite gluon mass $m_g^2 = \tau M^2$. In the euclidean region these integrals have discontinuities only for unphysical values $m_g^2 < 0$, and the function $W(\tau)$ has indeed the analyticity properties following from (81). However, in the physical region there is the possibility of real gluon emission if $m_g^2 < M^2$, and hence the function $W(\tau)$ must contain a contribution proportional to $\Theta(1-\tau)$, which is in conflict with (81). Therefore, a function $\hat{w}(\tau)$ satisfying the above integral relation does not exist in the physical region.

It follows then that a “linear” integral representation of the type (1) does not exist, too. One cannot avoid to use “non-linear” representations, such as the ones encountered in Sects. 3 and 4. Because the form of the non-linear representation is not unique, however, it is no longer obvious how to introduce a factorization scale in order to separate short- and long-distance contributions. A possible way to proceed is to define the resummation of renormalon chains in terms of the Borel integral. However, as in the euclidean region one should check that the resulting ambiguities from IR renormalons are in one-to-one correspondence with the size of long-distance contributions. To this end one should introduce an IR cutoff. A possibility which seems well motivated to us is first to regulate the Borel transform in the euclidean region, and then to perform the analytic continuation to the physical region. We shall now discuss this proposal for the quantities $R(s)$ and $T(m_g^2)$.

---

5In the last step we have assumed that the quantity $S(M^2)$ is at most logarithmically UV divergent, so that the distribution function $\hat{w}(\tau)$ vanishes for $\tau = \infty$. 

25
5.2 Cutoff regularization for $R(s)$

We start with the regularized form of the Borel integral for the correlator $D(Q^2)$ in the euclidean region, which reads

$$D_{\text{Borel}}^{\text{reg}}(Q^2, \tau_0) = 1 + \frac{1}{\beta_0} \int_0^\infty du \int_0^\infty d\tau \tilde{w}_D(\tau) \left(\frac{\tau Q^2}{\Lambda_V^2}\right)^{-u}.$$ \hspace{1cm} (82)

The integral over $u$ is convergent as long as $\tau_0 > \Lambda_V^2/Q^2$. Let us now consider the analytic continuation of the above result in the complex plane, which is well defined if $\tau_0 > \Lambda_V^2/|Q^2|$. Outside a circle of radius $\Lambda_V^2/\tau_0$ around the origin, the regularized function $D_{\text{Borel}}^{\text{reg}}(Q^2, \tau_0)$ is analytic in the complex $Q^2$-plane with a branch cut along the negative axis. Using the above expression, we find that both methods of calculating the function $R(s)$ – taking the imaginary part of $\Pi(-s+i\epsilon)$, or integrating $D(Q^2)$ around a circle of radius $s$ in the complex plane – lead to the same result:

$$R_{\text{reg}}(s, \tau_0) = 1 + \frac{1}{\beta_0} \int_0^\infty du \frac{\sin(\pi u)}{\pi u} \int_0^\infty d\tau \tilde{w}_D(\tau) \left(\frac{\tau s}{\Lambda_V^2}\right)^{-u}.$$ \hspace{1cm} (83)

As long as $\tau_0 > \Lambda_V^2/s$, both integrals are convergent and we can perform the integral over $u$ first. This gives

$$\int_0^\infty du \frac{\sin(\pi u)}{\pi u} \left(\frac{\tau s}{\Lambda_V^2}\right)^{-u} = \int_0^\infty du \frac{\sin(\pi u)}{\pi u} e^{-u/a} = F(a),$$ \hspace{1cm} (84)

where $a = 1/\ln(\tau s/\Lambda_V^2)$ agrees with the coupling constant $a(\tau s)$ defined in (35), and

$$F(a) = \frac{1}{\pi} \arccot \left(\frac{1}{\pi a}\right) = \frac{1}{\pi} \arctan(\pi a) \quad \text{for } a > 0.$$ \hspace{1cm} (85)

Note that $a > 0$ if $\tau > \Lambda_V^2/s$. Setting now $\tau_0 = \lambda^2/s$ with $\lambda > \Lambda_V$, and writing the final result as a function of $s$ and $\lambda^2$ (instead of $\tau_0$), we obtain

$$R_{\text{reg}}(s, \lambda^2) = 1 + \frac{1}{\beta_0} \int_{\lambda^2/s}^\infty d\tau W_{\text{e}^+\text{e}^-}(\tau) F(a(\tau s)).$$ \hspace{1cm} (86)

For $a > 0$, the effective coupling constant $F(a)$ coincides with the functions $F_1(a)$ and $F_2(a)$ in (34) and (48). However, if one wants to remove the cutoff one has to define $F(a)$ in the region $a < 0$, i.e. below the Landau pole, where the integral

$^6$Note that this is equivalent to using a representation of the form (78) for the Borel transform of $R(s)$, provided we identify $W_{\text{e}^+\text{e}^-}(\tau) = \tilde{w}_D(\tau)$ [see (30)]. This relation is in accordance with the fact that the Borel transform of $R(s)$ in (44) differs from the Borel transform of $D(Q^2)$ in (25) by a factor $\sin(\pi u)/\pi u$. 
in (84) is ill defined. Both $F_1(a)$ and $F_2(a)$ are “natural” continuations of $F(a)$; one is continuous as a function of $1/a$, the other is continuous as a function of $a$. The fact that the choice of the continuation below the Landau pole is not unique leads to an ambiguity in the definition of the resummed series. In the case of the physical correlator $R(s)$, we have seen in Sect. 4 that this ambiguity is parametrically larger than the IR renormalon ambiguity.

In this context it is interesting to note that neither $R_{\text{res}}^{(1)}(s)$ nor $R_{\text{res}}^{(2)}(s)$ coincide with the principle value of the Borel integral. This is different from the situation encountered in the euclidean region, where taking the limit $\lambda \to 0$ in (77) one reproduces the value of the Borel integral. In the present case, however, Beneke and Braun have shown that the principle value of the Borel integral is given by (see the Appendix) [4]

$$R_{\text{Borel}}(s) = 1 + \frac{1}{\beta_0} \int_0^\infty d\tau W e^{+e^-(\tau)} F_1(a(\tau s)) + \frac{1}{\beta_0} \text{Re} \int_{-\tau_L}^0 d\tau W e^{+e^-(\tau - i\epsilon)}, \quad (87)$$

where $\tau_L = \Lambda_V^2 / s$ is the position of the Landau pole. The imaginary part of the second integral determines the renormalon ambiguity:

$$\Delta R_{\text{ren}} = \frac{1}{\pi \beta_0} \text{Im} \int_{-\tau_L}^0 d\tau W e^{+e^-(\tau - i\epsilon)}. \quad (88)$$

Comparing (87) with (37) and (47), we find that

$$R_{\text{Borel}}(s) - R_{\text{res}}^{(1)}(s) = \frac{1}{\beta_0} \text{Re} \int_{-\tau_L}^{\tau_L} d\tau W e^{+e^-(\tau - i\epsilon)} = - \frac{3C_F}{\beta_0} \left( \frac{\Lambda_V^2}{s} \right)^2 + O(1/s^3),$$

$$R_{\text{Borel}}(s) - R_{\text{res}}^{(2)}(s) = \frac{1}{\beta_0} \text{Re} \int_{-\tau_L}^{\tau_L} d\tau W e^{+e^-(\tau - i\epsilon)}$$

$$= - \frac{8C_F}{3\beta_0} \left( \ln \frac{s}{\Lambda_V^2} + \frac{11}{6} \right) \left( \frac{\Lambda_V^2}{s} \right)^3 + O(1/s^4). \quad (89)$$

Moreover, we obtain

$$\Delta R_{\text{ren}} = \frac{4C_F}{\beta_0} \left( \tau_L - \frac{3}{2} \frac{\tau_L^2}{\tau_L} + (1 - \tau_L)^2 \ln(1 - \tau_L) \right), \quad (90)$$

which, when expanded in powers of $\tau_L = \Lambda_V^2 / s$, reproduces (43). The ambiguities arising from the choice of the continuation below the Landau pole lead to differences of order $1/s^2$, which are parametrically larger than the IR renormalon ambiguity.

It is not difficult to understand the size of these effects by considering the regularized form of the Borel integral given in (86). In fact, instead of taking the
limit $\lambda \to 0$, the more physical way to proceed is to keep the IR cutoff in the perturbative calculation, and to add to the regularized result a scale-dependent nonperturbative contribution, so that the dependence on the cutoff cancels in the sum. From the renormalization-group equation

$$\frac{\lambda^2}{d\lambda^2} R_{\text{reg}}(s, \lambda^2) = -\frac{1}{\beta_0} F(a(\lambda^2)) \frac{\lambda^2}{s} W_{e^+e^-}(\lambda^2/s)$$

$$= -\frac{3C_F}{\beta_0} F(a(\lambda^2)) \left(\frac{\lambda^2}{s}\right)^2 + O(1/s^3), \quad (91)$$

it follows that this nonperturbative contribution must scale like $1/s^2$. At this point we have to add a word of caution, however. Contrary to the euclidean region, one has to be careful when interpreting $\lambda$ as a physical scale parameter. Although the introduction of the IR cutoff certainly removes all soft contributions, we cannot exclude that it actually removes too much, i.e. contributions which may come from short distances as well. Unless one is able to construct a cutoff regularization for which the nonperturbative contribution scales like $1/s^3$, however, one should be conservative and assume that there are indeed long-distance contributions of order $1/s^2$.

### 5.3 Cutoff regularization for $T(m^2_{\tau})$

Let us now turn to the more complicated case of the function $T(m^2_{\tau})$. If we work with the representation of this quantity as an integral over $R(s)$ as shown in (52) and insert the regularized expression given in (86), we find

$$T_{\text{reg}}(m^2_{\tau}, \lambda^2) = 1 + \frac{1}{\beta_0} \int_0^1 dx \left(1-3x^2+2x^3\right) \int_0^\infty du \frac{\sin(\pi u)}{\pi u} \left(\frac{\tau x m^2_{\tau}}{\Lambda^2_V}\right)^{-u}.$$  

(92)

Note that in order for the $u$-integral to be convergent we have to introduce an $x$-dependent cutoff $\tau_0 = \lambda^2/(xm^2_{\tau}) = \lambda^2/s_{\text{had}}$ with $\lambda > \Lambda_V$, where $s_{\text{had}}$ is the invariant mass of the hadronic final state. It is thus convenient to introduce a new variable $\tau' = \tau x$ and to perform the integral over $x$ at fixed $\tau'$ to obtain

$$T_{\text{reg}}(m^2_{\tau}, \lambda^2) = 1 + \frac{1}{\beta_0} \int_0^\infty du \frac{\sin(\pi u)}{\pi u} \int_{\tau' m^2_{\tau}}^{\lambda^2} d\tau' W_\tau(\tau') \left(\frac{\tau' m^2_{\tau}}{\Lambda^2_V}\right)^{-u}$$

$$= 1 + \frac{1}{\beta_0} \int_{\lambda^2/m^2_{\tau}}^\infty d\tau W_\tau(\tau) F(a(\tau m^2_{\tau})), \quad (93)$$

where $W_\tau(\tau)$ is the distribution function defined in (57), and $F(a)$ has been given in (85). The freedom in the choice of the continuation to the region below
the Landau pole (where \( a < 0 \)) introduces an ambiguity of order \( 1/\tau^2 \), since

\[ W_\tau(\tau) \rightarrow \text{const. for } \tau \to 0. \]

Two examples of such continuations are provided by the quantities \( T^{(1)}_{\text{res}}(m_\tau^2) \) and \( T^{(2)}_{\text{res}}(m_\tau^2) \) in (93), which indeed differ by terms of order \( 1/\tau^2 \) [see (71)]. If the IR cutoff is kept, the ambiguity of order \( 1/\tau^2 \) is reflected in the fact that

\[
\lambda^2 \frac{d}{d\lambda^2} T_{\text{reg}}(m_\tau^2, \lambda^2) = -\frac{1}{\beta_0} F(a(\lambda^2)) \frac{\lambda^2}{m_\tau^2} W_\tau(\lambda^2/m_\tau^2) \]

\[ = -\frac{16C_F}{\beta_0} \left[ 4 - 3\zeta(3) \right] F(a(\lambda^2)) \frac{\lambda^2}{m_\tau^2} + O(1/m_\tau^4), \quad (94) \]

which means that one has to add a nonperturbative contribution of order \( 1/\tau^2 \) to cancel the dependence on the cutoff.

Let us now consider the representation of \( T(m_\tau^2) \) as a contour integral given in (67). Inserting there the regularized expression for the function \( D(Q^2) \) from (82), we find

\[
T^\circ_{\text{reg}}(m_\tau^2, \lambda^2) = 1 + \frac{1}{\beta_0} \int \frac{dx}{x} (1 - 2x + 2x^3 - x^4) \int_0^\infty du \int_0^\tau \hat{w}_D(\tau) \frac{(\tau_0 m_\tau^2)}{\Lambda_V^2}^{-u}, \quad (95) \]

where \( \tau_0 = \lambda^2/|x m_\tau^2| = \lambda^2/m_\tau^2 \). This means that the integral over \( x \) can be performed without affecting the value of the IR cutoff, which is determined by the radius of the contour in the complex plane. The result is

\[
T^\circ_{\text{reg}}(m_\tau^2, \lambda^2) = 1 + \frac{1}{\beta_0} \int_0^\infty du \frac{\sin(\pi u)}{\pi u} \frac{12}{(1-u)(3-u)(4-u)} \int_0^{\lambda^2/m_\tau^2} d\tau \hat{w}_D(\tau) \frac{(\tau m_\tau^2)}{\Lambda_V^2}^{-u}. \quad (96) \]

As long as \( \lambda > \Lambda_V \), the integral over \( u \) is convergent and can be performed first. Expressing the result in terms of an effective coupling constant \( G(a) \) as in (64), we find that

\[
T^\circ_{\text{reg}}(m_\tau^2, \lambda^2) = 1 + \frac{1}{\beta_0} \int_0^\infty d\tau \Omega_\tau(\tau) G(a(\tau m_\tau^2)), \quad (97) \]

where \( \Omega_\tau(\tau) = \hat{w}_D(\tau) \), and

\[
G(a) = \int_0^\infty du \frac{\sin(\pi u)}{\pi u} \frac{12}{(1-u)(3-u)(4-u)} e^{-u/a} = G_3(a); \quad \text{for } a > 0, \quad (98) \]

\footnote{The same result is obtained if one performs first the \( x \)-integral in (92), which is convergent for \( u < 1 \), and then analytically continues the result to arbitrary values of \( u \). However, this procedure involves an integration over regions where the Borel integral is ill defined.}
with \( G_3(a) \) as defined in (52). Now the freedom in the choice of the continuation to the region below the Landau pole introduces an ambiguity of order \( 1/m_\tau^4 \), since \( \Omega_\tau(\tau) \propto \tau \) for \( \tau \ll 1 \). An example of such a continuation is provided by the expressions for \( T_{\text{res}}(m_\tau^2) \) given in (68). If the IR cutoff is kept, the ambiguity is reflected in the fact that

\[
\lambda^2 \frac{d}{d\lambda^2} T_{\text{reg}}(m_\tau^2, \lambda^2) = \frac{1}{\beta_0} G(a(\lambda^2)) \frac{\lambda^2}{m_\tau^2} \Omega_\tau(\lambda^2/m_\tau^2)
\]

\[
= -\frac{6C_F}{\beta_0} G(a(\lambda^2)) \left( \frac{\Lambda_V}{m_\tau^2} \right)^2 + O(1/m_\tau^6) ,
\]

which means that one has to add a nonperturbative contribution of order \( 1/m_\tau^4 \) to cancel the dependence on the cutoff.

It is again instructive to compare our result for \( T_{\text{res}}(m_\tau^2) \) with the principle value of the Borel integral, which according to Refs. [4, 5] can be written in the form

\[
T_{\text{Borel}}(m_\tau^2) = 1 + \frac{1}{\beta_0} \int_0^\infty d\tau W_\tau(\tau) F_1(a(\tau m_\tau^2)) + \frac{1}{\beta_0} \text{Re} \int_{-\tau_L}^0 d\tau W_\tau(\tau - i\epsilon) ,
\]

with \( \tau_L = \Lambda_V^2/m_\tau^2 \). Using (71), we find that

\[
T_{\text{Borel}}(m_\tau^2) - T_{\text{res}}(m_\tau^2) = \frac{1}{\beta_0} \text{Re} \int_{-\tau_L}^{\tau_L} d\tau W_\tau(\tau) - \frac{4}{\beta_0} \int_{\tau_L}^{\infty} d\tau \Omega_\tau(\tau) \left( \frac{\tau_L}{\tau} - \frac{\tau_L^3}{\tau^3} \right)
\]

\[
= 48C_F \left( \frac{\Lambda_V^2}{m_\tau^2} \right)^2 + O(1/m_\tau^6) \approx 0.03 .
\]

Although of order \( 1/m_\tau^4 \), this difference is sizeable because of the large numerical coefficient. For completeness, we also quote the exact result for the renormalon ambiguity. It is

\[
\Delta T_{\text{ren}} = \frac{4C_F}{\beta_0} \left\{ \tau_L + \frac{47}{6} \frac{\tau_L^2}{3} - \frac{\tau_L^3}{3} - 4 L_2(\tau_L) - \left( 2\tau_L^3 - \frac{\tau_L^4}{3} \right) \ln \tau_L
\]

\[
+ \left( 1 + \frac{13}{3} \frac{\tau_L}{3} - \frac{5}{3} \frac{\tau_L^2}{3} + \frac{\tau_L^3}{3} \right) (1 - \tau_L) \ln(1 - \tau_L) \right\} ,
\]

which for \( \tau_L \ll 1 \) reproduces (62).

The discussion of this section illustrates that the choice of implementing the IR regulator requires some amount of ingenuity. In the first case, by introducing the regulator we have excluded a contribution of order \( 1/m_\tau^2 \) from the perturbative calculation of \( T(m_\tau^2) \), whereas in the second case we have found a better regularization scheme, in which this contributions is of order \( 1/m_\tau^4 \). It is important that there is a physical argument which favours the second method over the
first one. The integration along the circle in the complex plane involves larger momenta than the integration along the cut, and thus less is eliminated by introducing the IR cutoff. Clearly, the question arises whether there exists an even better representation, in which the excluded contribution is of order $1/m_6^6$ and thus of the same order as the renormalon ambiguity. Although we cannot give a definite answer to this question at present, we doubt that such a representation exists; we find it suggestive that a linear integral representation of the form (4), which would have this property, does not exist in the physical region. As long as one does not succeed in constructing such a regularized representation of the Borel integral, one has to take seriously the possibility of nonperturbative corrections to $R_\tau$ of order $1/m_4^4$.

To summarize this section, we repeat that the new kind of nonperturbative ambiguities encountered in our analysis is related to the singular behaviour of the Borel integral for $u \to \infty$. We have argued that this behaviour should be regulated by introducing an IR cutoff. To the perturbative calculation one then has to add a nonperturbative contribution in order to cancel the dependence on the cutoff in the final result. For quantities defined in the euclidean region, the power behaviour of the nonperturbative contribution is in one-to-one correspondence with the size of the ambiguities due to IR renormalons, which appear if the Borel integral is evaluated in dimensional regularization without using an IR cutoff. This implies that in these cases the singularities at infinity do not spoil the applicability of the OPE. For the cases of correlation functions in the physical region, however, we find that the singular behaviour at infinity leads to ambiguities that may be larger than those indicated by the positions of IR renormalon singularities. At least, we could not find a cutoff regularization for which these ambiguities are of the same order as those arising from renormalons. If this conjecture is correct, it would imply that the OPE does not provide a consistent framework for the analysis of nonperturbative corrections in the physical region.

6 Numerical analysis

We now turn to the numerical analysis of our results. In Tables 2 and 3 we show various approximations for the quantities $R(m_2^2)$ and $T(m_2^2)$. Compare first the results for $R(m_2^2)$ with those for $D(m_2^2)$ given in Table 1. The corresponding perturbative series are the same up to order $\alpha_s^2$. In the case of the euclidean correlation function, the resummation of renormalon chains leads to an increase of 3% with respect to the two-loop result, whereas the value of the physical correlation function is decreased by 1%. This reflects the smooth behaviour of the effective coupling constants $F_n(a)$ in the IR region. As a consequence, after the resummation the result lies significantly below the BLM prediction. Note that the difference between the two resummations $R^{(1)}_{\text{res}}$ and $R^{(2)}_{\text{res}}$ is numerically small, although larger than the renormalon ambiguity $\Delta R_{\text{ren}}$. We also quote the
Table 2: Comparison of various approximations for the quantity $R(s)$ at $s = m^2$.

| $R_{1 \text{ loop}}$ | $R_{2 \text{ loop}}$ | $R_{\text{BLM}}$ | $R_{\text{BLM}^*}$ | $R_{\text{res}}^{(1)}$ | $R_{\text{res}}^{(2)}$ | $R_{\text{Borel}}$ | $\Delta R_{\text{ren}}$ |
|----------------------|----------------------|-------------------|---------------------|----------------------|----------------------|----------------------|----------------------|
| 1.102                | 1.118                | 1.145             | 1.115               | 1.107                | 1.105                | 1.105                | $-6.2 \times 10^{-5}$ |
Generalizing the approach proposed in Ref. [1], we have investigated the resummation of renormalon chain contributions in the perturbative calculation of cross sections and inclusive decay rates, and in particular for the ratios $R_{e^+e^-}$ and $R_{\tau}$. Our approach is equivalent to an all-order resummation of the terms proportional to $\beta_0^{-n} \alpha_s^n$ in the perturbative series for these quantities. It provides a generalization of the BLM scale-setting prescription [2, 3], in which terms of order $\beta_0 \alpha_s^2$ are absorbed by a redefinition of the scale in the running coupling constant.

The discussion in Ref. [1] was devoted to QCD Green functions without external gluon fields and to euclidean correlation functions of currents, which receive only virtual gluon corrections. In the calculation of cross sections and inclusive decay rates, on the other hand, both virtual and real gluons have to be considered. As a consequence, in the expression for the resummed series, which has the form of a one-dimensional integral over the running coupling constant with some distribution function, there appears an effective coupling constant, which is screened in the low-momentum region. Such non-linear representations have also been derived, using a different formalism, by Beneke and Braun [4].

By summing an infinite set of diagrams our scheme reaches beyond perturbation theory. In any finite-order perturbative calculation nonperturbative effects are implicitly present due to low-momentum contributions in Feynman diagrams, but are not visible as they are exponentially small in the coupling constant. Yet perturbation theory “knows” about these contributions in the form of IR renormalons, which make a perturbative series non Borel summable. This means that
Figure 7: The regularized integral $T_{\text{reg}}(m^2_\tau, \lambda^2)$ as a function of the cutoff scale. For values $\lambda \lt \Lambda_V \approx 0.46$ GeV we define the integral by using the effective coupling constant $G_3(a)$. The arrow indicates the principle value of the Borel integral.

Attempts to resum the series will lead to unavoidable ambiguities. In our scheme these ambiguities arise from the integration over the Landau pole in the running coupling constant. They are a reminder of the fact that perturbation theory is incomplete; any perturbative calculation must be supplemented by nonperturbative contributions. For the quantities considered in Ref. [1], the OPE provides the framework for a consistent incorporation of nonperturbative contributions. In this case the IR renormalon ambiguities can be absorbed into the definition of other nonperturbative parameters, such as the vacuum condensates. A new feature of the present analysis is that it refers to quantities defined in the physical region (i.e. for time-like momenta), for which the applicability of the OPE is at least questionable. We have found a new source of nonperturbative ambiguities in the definition of the resummed series, which are not related to IR renormalons. These ambiguities are parametrically larger than the IR renormalon ambiguities and thus cannot be dealt with in the standard way.

The appearance of this new source of ambiguity is surprising and is the most striking observation of our analysis. In order to understand the origin and size of this effect, we have regularized the singular behaviour of the Borel integral by introducing an IR cutoff. We find that in the euclidean region there is a one-to-one correspondence between the power-like dependence on this cutoff and the position of IR renormalons, which establishes the link between the OPE and the structure of the singularities in the Borel plane. However, in the physical region we did not succeed to construct a cutoff regularization with this property. In other words, for the regularization schemes that we have investigated the cutoff dependence does not match with the position of IR renormalons. Although we
cannot prove at present that no better regularization exists, we cannot exclude the possibility that the ambiguities encountered in our analysis of \( R_{e^+e^-} \) and \( R_e \) hinder the standard application of the OPE in the physical region. This would imply that to the resummed perturbative series one would have to add nonperturbative power corrections that are not related to vacuum expectation values of local, gauge-invariant operators. The values of these nonperturbative terms would depend on the resummation prescription chosen for the perturbative series. It is not obvious to us which theoretical framework could be employed to accommodate these terms.

It is of great importance to settle the question if this pessimistic possibility can be ruled out. Let us illustrate some phenomenological implications with two examples. We have mentioned in the introduction that the inclusive \( B \to X e \bar{\nu}_e \) decay rate is subject to large two-loop corrections of order \( \beta_0 \alpha_s^2 \), and that the corresponding BLM scale is very low. In the case of \( b \to u \) transitions, for instance, one finds \( \mu_{BLM} \simeq 0.07 \, m_b \) in the \( \overline{MS} \) scheme \([26]\). This observation casts doubt on the reliability of the perturbative expansion and calls for an analysis of higher-order terms in the series. It is thus very interesting to investigate the resummation of renormalon chains in this case. The principle value of the Borel integral corresponding to the perturbative series for the inclusive semileptonic decay rate has recently been calculated by Ball et al. \([36]\). However, based on our analysis we expect that also in this case the result of the resummation will not be unique, i.e. there will be ambiguities of order \( 1/m_b^2 \) or even \( 1/m_b \). In the latter case this would limit the accuracy with which one can extract the elements \( V_{ub} \) and \( V_{cb} \) of the Cabibbo-Kobayashi-Maskawa matrix from the analysis of inclusive decays of \( B \) mesons.

Our second example concerns the quantity \( R_\tau \), measurements of which have been suggested as a way to extract with high precision the strong coupling constant \( \alpha_s(m_\tau^2) \) \([24, 33, 34, 44]\). From the most recent analysis of the experimental data, one has obtained the value \( R_\tau = 3.638 \pm 0.017 \) and extracted \( \alpha_s(m_\tau^2) = 0.367 \pm 0.018 \) \([41]\). The theoretical uncertainty in this result is quoted as \( \delta \alpha_s(m_\tau^2) = 0.014 \) \([33]\). To achieve such a high precision, it is essential to rely on the standard form of the OPE, in which the magnitude of nonperturbative contributions is linked to the structure of renormalon singularities in the Borel plane. Using the SVZ approach \([31]\) to relate these nonperturbative contributions to vacuum condensates, one finds that the contribution of the gluon condensate, which is proportional to \( \langle \alpha_s G^2 \rangle / m_\tau^4 \), is suppressed by two additional powers of \( \alpha_s \) (it is absent in the large-\( \beta_0 \) limit considered here), so that the dominant nonperturbative effects appear at order \( 1/m_\tau^6 \) \([24]\). Several authors have questioned this assumption and investigated, with different conclusions, whether \( R_\tau \) could receive larger nonperturbative corrections. Both IR and UV renormalons have received main attention in these studies \([11], [42] - [46] \). In the present work we have identified a new kind of nonperturbative effects, which are not related to vacuum condensates. We have seen that there exist different ways to resum renormalon
chain contributions. Some of these resummations are disfavoured for physical reasons, but others cannot be distinguished on physical grounds. In the case of \( R_\tau \), we see no physical argument to prefer one of the two resummations \( T^\text{circle} \) and \( T^\text{Borel} \), which differ by terms of order \( 1/m^4_\tau \). More generally, if the Borel integral is regulated by introducing an IR cutoff, one is forced to add a nonperturbative contribution of order \( 1/m^4_\tau \) to cancel the cutoff dependence of the perturbative calculation.

To illustrate the phenomenological impact of our results, let us compare the various predictions for the perturbative contribution to \( R_\tau \). We write

\[
R_\tau = 3 \left( |V_{ud}|^2 + |V_{us}|^2 \right) S_{\text{EW}} \left\{ 1 + \delta_{\text{EW}} + \delta_{\text{pert}} + \delta_{\text{nonpert}} \right\},
\]

(103)

where \( S_{\text{EW}} \simeq 1.0194 \) and \( \delta_{\text{EW}} \simeq 0.0010 \) are known electroweak corrections \([31, 32]\), \( \delta_{\text{pert}} \) is the perturbative correction, and \( \delta_{\text{nonpert}} \) denotes the sum of all power-suppressed corrections including quark-mass effects, which have been estimated to be \( \delta_{\text{nonpert}} = -0.019 \pm 0.007 \) \([24]\). Accepting this number, one obtains \( \delta_{\text{exp}}^{\text{pert}} = 0.21 \pm 0.01 \) from the experimental measurements. It is conventional to consider the perturbative correction as a function of the coupling constant \( \alpha_s(m^2_\tau) \) defined in the \( \overline{\text{MS}} \) scheme. We use the known exact values of the coefficients up to order \( \alpha^3 \) \([22, 23]\) to correct our result for contributions which are subleading in the large-\( \beta_0 \) limit and thus not included in our approach. This leads to

\[
\delta_{\text{pert}}[\alpha_s(m^2_\tau)] = T[\alpha_s(m^2_\tau)] - 1 + 0.0831 \frac{\alpha_s(m^2_\tau)}{\pi} - 2.4115 \left( \frac{\alpha_s(m^2_\tau)}{\pi} \right)^2,
\]

(104)

where for \( T \) we may substitute one of the resummations \( T^\text{circle} \) or \( T^\text{Borel} \), which depend on \( \alpha_s(m^2_\tau) \) through the QCD scale parameter \( \Lambda_V \).

In Fig. 8, we compare our results with the resummation of Pich and Le Diberder \([33]\), as well as with the exact order-\( \alpha^3 \) expression. There are significant differences between the various predictions. For instance, with the central “experimental” value \( \delta_{\text{pert}} = 0.21 \) one would obtain \( \alpha_s(m^2_\tau) \simeq 0.30 \) using the principle value resummation, \( \alpha_s(m^2_\tau) \simeq 0.34 \) using our resummation \( T^\text{circle} \), and \( \alpha_s(m^2_\tau) \simeq 0.37 \) using the resummation of Le Diberder and Pich. If \( \delta_{\text{pert}} \) is larger, the differences even increase. We conclude that the theoretical uncertainty in the perturbative contribution to \( R_\tau \) is probably larger than estimated in \([33]\). A reasonable value seems to be

\[
\delta \alpha_s(m^2_\tau) \simeq 0.05.
\]

(105)

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Appendix

Beneke and Braun have developed a different way to resum renormalon chains \( \Phi \) (see also \( \Phi \)). They start from the integral representation (78) for the Borel transform and show that the Borel integral performed above the real \( u \)-axis is equivalent to

\[
S_{\text{Borel}}(M^2) = 1 + \frac{1}{\beta_0} \int_0^\infty d\tau W(\tau) F_1(a(\tau M^2)) + \frac{1}{\beta_0} \int_{-\tau_L}^0 d\tau W(\tau - i\epsilon), \tag{106}
\]

where \( F_1(a) \) has been defined in (34). In this approach, the renormalon ambiguity is given by

\[
\Delta S_{\text{ren}} = \frac{1}{\pi \beta_0} \text{Im} \int_{-\tau_L}^0 d\tau W(\tau - i\epsilon). \tag{107}
\]

The function \( W(\tau) \) can be calculated by performing a one-loop calculation with a finite gluon mass \( m_g^2 = \tau M^2 \). Its relation to the distribution function \( \hat{w}(\tau) \) of our approach has been given in (81). Using this relation, one can recover our linear representation (1) from the non-linear representation (106) of Beneke and Braun. Inserting (81) into (106) and changing the order of integration (which is allowed since the integrals converge), one finds that

\[
S_{\text{Borel}}(M^2) = 1 + \frac{1}{\beta_0} \int_0^\infty d\tau \hat{w}(\tau) H(a(\tau M^2)), \tag{108}
\]
where
\[ H(a) = \int_0^\infty \frac{dz}{(1+z)^2} \frac{1}{\pi} \arccot \left( \frac{\ln z + 1/a}{\pi} \right) + \frac{1}{1 - e^{-1/a} - i\epsilon} - 1. \] (109)

After an integration by parts in the first term this can be written in the form
\[ H(a) = \frac{1}{1 - e^{-1/a} - i\epsilon} - \int_{-\infty}^{+\infty} \frac{dx}{(x^2 + \pi^2)(1 + e^{x-1/a})}. \] (110)

The \( x \)-integral can be performed using Cauchy’s theorem. There are simple poles at \( x = \pm i\pi \) and \( x = 1/a + (2n + 1)i\pi \). We find
\[
\int_{-\infty}^{+\infty} \frac{dx}{(x^2 + \pi^2)(1 + e^{x-1/a})} = \frac{1}{1 - e^{-1/a}} + a, \] (111)

which leads to
\[ H(a) = a + i\pi \delta(1/a) = \frac{1}{1/a - i\epsilon} = \frac{1}{\ln \tau - \ln \tau_L - i\epsilon}. \] (112)

This shows the equivalence of the two methods in the euclidean region.

Using the relation (81), let us derive some properties of the function \( W(t) \). We note that \( W(t) \) is analytic in the complex \( t \)-plane with a branch cut along the negative axis. For the discontinuity across the cut, we obtain
\[
\frac{1}{2\pi i} \left[ W(\tau + i\epsilon) - W(\tau - i\epsilon) \right] = -\Theta(-\tau) \frac{d}{d\tau} \left[ \tau \hat{w}(-\tau) \right]. \] (113)

Using this result, we find that our definition of the renormalon ambiguity in (14) is in fact equivalent to (107), since
\[
\Delta S_{\text{ren}} = \frac{1}{\pi \beta_0} \text{Im} \int_{-\tau_L}^{0} d\tau W(\tau - i\epsilon) = \frac{\tau_L}{\beta_0} \hat{w}(\tau_L). \] (114)

For the cases of \( R_{e^+e^-} \) and \( R_T \), the functions \( W_{e^+e^-}(\tau) \) and \( W_T(\tau) \) have been given in (36) and (57). These results agree with the corresponding expressions derived in Ref. [3]. For completeness, let us also quote the result for the function \( W_D(t) \) in the case of the \( D \)-function. Starting from relation (81) and using the result for the distribution function \( \hat{w}_D(\tau) \) given in (21), we obtain
\[
W_D(\tau) = 8C_F \left\{ \frac{7}{2} - \frac{5\tau}{2} \ln \frac{1+\tau}{\tau} + \left( \frac{1}{2\tau} - 2 \right) \ln(1+\tau) + 2L_2(-\tau) + \tau L_2(-1/\tau) 
\right. \\
\left. - (1 - 2\tau) \int_0^1 dx \frac{\ln x}{1+x} \ln \frac{x(x + \tau)}{(1+xt)} \right\} \\
= 8C_F \left\{ 4 - 3\zeta(3) + \left( 6\zeta(3) - \frac{17}{4} + \frac{3}{2} \ln \tau \right) \tau + O(\tau^2) \right\}. \] (115)
The first two terms in the expansion of this function have been calculated in Ref. [5]; the exact expression is new.

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