On Distance Antimagic Graphs

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Abstract
For an arbitrary set of distances \( D \subseteq \{0, 1, \ldots, \text{diam}(G)\} \), a \( D \)-weight of a vertex \( x \) in a graph \( G \) under a vertex labeling \( f : V \rightarrow \{1, 2, \ldots, v\} \) is defined as \( w_D(x) = \sum_{y \in N_D(x)} f(y) \), where \( N_D(x) = \{y \in V | d(x, y) \in D\} \).

A graph \( G \) is said to be \( D \)-distance magic if all vertices has the same \( D \)-vertex-weight, it is said to be \( D \)-distance antimagic if all vertices have distinct \( D \)-vertex-weights, and it is called \((a, d)\)-\( D \)-distance antimagic if the \( D \)-vertex-weights constitute an arithmetic progression with difference \( d \) and starting value \( a \).

In this paper we study some necessary conditions for the existence of \( D \)-distance antimagic graphs. We conjecture that such conditions are also sufficient. Additionally, we study \( \{1\} \)-distance antimagic labelings for some cycle-related connected graphs: cycles, suns, prisms, complete graphs, wheels, fans, and friendship graphs.

1 Introduction

As standard notation, assume that \( G = G(V, E) \) is a finite, simple, and undirected graph with \( v \) vertices and \( e \) edges. By a labeling we mean a one-to-one mapping that carries a set of graph elements onto a set of numbers, called labels.

The notion of distance magic labeling was introduced separately in the PhD thesis of Vilfred [23] in 1994 and an article by Miller et. al [15] in 2003. A distance magic labeling is a bijection \( f : V \rightarrow \{1, 2, \ldots, v\} \) with the property that there is a constant \( k \) such that at any vertex \( x \), the vertex-weight of \( x \), \( w(x) = \sum_{y \in N(x)} f(y) = k \), where \( N(x) \) is the set of vertices adjacent to \( x \). This labeling was introduced due to two different motivations; as a tool in utilizing magic squares into graphs and as a natural extension of previously known graph labelings: magic labeling [20, 13] and radio labeling (which is distance-based) [10].

In the last decade, many results on distance magic labeling have been published. Several families of graphs have been showed to admit the labeling
and constructions of distance magic graphs have also been studied \cite{12, 15, 1, 19, 8, 3, 21}. It has also been showed that there is no forbidden subgraph characterization for distance magic graph \cite{23, 1, 18}. Additionally, an application of the labeling in designing incomplete tournament is introduced in \cite{7}. For more results in distance magic labeling, please refer to Gallian’s dynamic survey on graph labelings \cite{9}.

O’Neal and Slater \cite{16, 17} generalized the notion of distance magic labeling to an arbitrary set of distances $D \subseteq \{0, 1, \ldots, \text{diam}(G)\}$, where $\text{diam}(G)$ is the diameter of $G$. As in the previous distance magic labeling, the domain of this new labeling is the set of all vertices and the codomain is $\{1, 2, \ldots, v\}$. We define the $D$-vertex-weight of each vertex $x$ in $G$, $w_D(x) = \sum_{y \in N_D(x)} f(y)$, where $N_D(x) = \{y \in V | d(x, y) \in D\}$. If all vertices in $G$ have the same weight, we call the labeling a $D$-distance magic labeling.

Recently, Arumugam and Kamatchi \cite{2} considered an antimagic version of distance labeling. They defined an $(a, d)$-distance antimagic labeling of a graph $G$ as a bijection $f : V \to \{1, 2, \ldots, v\}$ such that the set of all vertex-weights is $\{a, a + d, a + 2d, \ldots, a + (v - 1)d\}$, where $a$ and $d$ are fixed integers with $d \geq 0$. Any graph which admits such a labeling is called an $(a, d)$-distance antimagic graph. The characterization of $(a, d)$-distance antimagic cycles and $(a, d)$-distance antimagic labelings for paths and prisms were also studied in \cite{2}. Froncek proved that disjoint copies of the Cartesian product of two complete graphs and its complement are $(a, 2)$-distance antimagic and $(a, 1)$-distance antimagic, respectively (see \cite{5} and \cite{6}). He also proved that disjoint copies of the hypercube $Q_3$ is $(a, 1)$-distance antimagic.

In addition to the $(a, d)$-distance antimagic labeling, we also consider the following three other labelings.

**Definition 1.1.** Let $G$ be a graph, $x$ a vertex in $G$, $f$ a bijection from $V$ onto $\{1, 2, \ldots, v\}$, and $D \subseteq \{0, 1, \ldots, \text{diam}(G)\}$.

The bijection $f$ is called distance antimagic labeling if all vertices have distinct vertex-weights. A graph is called distance antimagic if it admits a distance antimagic labeling.

The bijection $f$ is called a $D$-distance antimagic labeling if the $D$-vertex-weights are all different. The bijection $f$ is called an $(a, d)$-$D$-distance antimagic labeling if all $D$-vertex-weights constitute an arithmetic progression with difference $d$ and starting value $a$, for $a$ and $d$ fixed integers with $d \geq 0$. A graph $G$ is $D$-distance antimagic or $(a, d)$-D-distance antimagic if it admits a $D$-distance antimagic labeling or an $(a, d)$-D-distance antimagic labeling, respectively.

Note that if $D = \{1\}$, a $D$-distance antimagic labeling is a distance antimagic labeling and similarly an $(a, d)$-$D$-distance antimagic labeling is an $(a, d)$-distance antimagic labeling. If $d = 0$, an $(a, 0)$-D-distance antimagic labeling is a $D$-distance magic labeling. It is clear that if a graph is $(a, d)$-$D$-distance antimagic
for $d > 0$ then it is also $D$-distance antimagic, but not necessarily the other way around.

In this paper we study some necessary conditions for the existence of $D$-distance antimagic graphs. Additionally, we study distance antimagic labelings for some connected graphs containing one or more cycles: cycles, suns, prisms, complete graphs, wheels, fans, and friendship graphs. Finally, we conjecture that the necessary conditions for the existence of $D$-distance antimagic graphs are also sufficient.

2 Main Result

We start with a couple of obvious observations.

**Lemma 2.1.** If a graph contains two vertices with the same neighborhood then it is not distance antimagic.

**Proof.** If $G$ has two vertices with the same neighborhood, say $u$ and $v$, then $w(u) = w(v)$, a contradiction.

Let us define a $D$-neighborhood of a vertex $x$ as the set of all vertices at distance $k$ to $x$, where $k \in D$. Then Lemma 2.1 can be generalized in the following lemma.

**Lemma 2.2.** If a graph contains two vertices with the same $D$-neighborhood then it is not $D$-distance antimagic.

As a consequence of Lemma 2.1, we have

**Corollary 2.3.** All complete multipartite graphs are not distance antimagic.

The following lemma gives us an upper bound for $d$ of an $(a, d)$-distance antimagic labeling of a regular graph.

**Lemma 2.4.** Let $G$ be an $r$-regular graph. If $G$ is $(a, d)$-distance antimagic then $d \leq r \frac{v-r}{v-1}$ and $a = r \frac{(v+1)-d(v-1)}{2}$.

**Proof.** If we consider a particular vertex $x$, it contributes exactly $d(x)$ times to the sum of all vertex-weights, where $d(x)$ is the degree of $x$. Thus,

$$a + (a + d) + \ldots + (v-1)d = \sum_{x \in V(G)} d(x)f(x),$$

which leads to

$$va + d \frac{v(n-1)}{2} = \sum_{x \in V(G)} d(x)f(x).$$

Since $G$ is an $r$-regular graph, then

$$va + d \frac{v(n-1)}{2} = r \sum_{x \in V(G)} f(x) = r \frac{v(v+1)}{2}. $$
Therefore, \( d = \frac{r(v+1)-2a}{v-1} \) which gives us the second result.

Now, consider the least possible value of a vertex-weight. Obviously, it has to be equal to \( 1 + 2 + \ldots + d \), and so \( a \geq \frac{r(r+1)}{2} \). This gives the desired upper bound for \( d \).

\[
d \leq \frac{r(v+1) - 2\frac{r(r+1)}{2}}{v-1} = \frac{r(v-1)}{v-1}.
\]

Next we study distance antimagic labelings and \((a,d)-\)distance antimagic labelings for some families of graphs containing one or more cycles: cycles, suns, complete graphs, prisms, wheels, fans, and friendship graphs.

2.1 Cycle

In [2], Arumugam and Kamatchi gave a characterization of \((a,d)-\)distance antimagic cycles.

**Theorem 2.5.** [2] The cycle \( C_n \) is \((a,d)-\)distance antimagic if and only if \( n \) is odd and \( d = 1 \).

The characterization missed out a single case when \( n = 4 \) and \( d = 0 \) and so we rewrite the theorem as follow.

**Theorem 2.6.** A cycle \( C_n \) has an \((a,d)-\)distance antimagic labeling if and only if \( d = 0 \) and \( n = 4 \) or \( d = 1 \) and \( n \) is odd.

The previous theorem showed that only odd cycles have \((a,d)-\)distance antimagic labelings for \( d \geq 1 \). However in the next theorem we shall construct distance antimagic labelings for even cycles.

**Theorem 2.7.** All cycles are distance antimagic.

**Proof.** Consider a cycle \( C_n \) of order \( n \). For odd \( n \), it is already \((a,d)-\)distance antimagic by Theorem 2.6. For even \( n = 2k \), we define a vertex labeling \( f \) as follow. Suppose that \( V(C_n) = \{x_1, x_2, \ldots , x_n\} \) and \( E(C_n) = \{x_nx_1, x_ix_{i+1}, i = 1, 2, \ldots n\} \).

\[
f(x_i) = \begin{cases} 
1, & \text{for } i = 1, \\
i - 1, & \text{for odd } i, 3 \leq i \leq k + 1, \\
n + 2 - i, & \text{for odd } i, k + 2 \leq i \leq n - 1, \\
n + \frac{2}{2} - 1 + i, & \text{for even } i, 2 \leq i \leq k + 1, \\
n + \frac{2}{2} + 2 - i, & \text{for even } i, k + 2 \leq i \leq n.
\end{cases}
\]
Under the previous labeling, we obtain the following all distinct vertex-weights.

\[
w(x_i) = \begin{cases} 
  n + 3, & \text{for } i = 1, \\
  n - 2 + 2i, & \text{for odd } i, 3 \leq i \leq k, \\
  2n - 1, & \text{for odd } i = k + 1 \text{ or } k + 2, \\
  3n + 4 - 2i, & \text{for odd } i, k + 3 \leq i \leq n - 1, \\
  3, & \text{for } i = 2, \\
  2i - 2, & \text{for even } i, 4 \leq i \leq k, \\
  n - 1 + \frac{1+i}{2}, & \text{for even } i = k + 1 \text{ or } k + 2, \\
  2n + 4 - 2i, & \text{for even } i, k + 3 \leq i \leq n. 
\end{cases}
\]

Adding an edge to each vertex in a cycle results in a unicyclic graph called sun. While for cycle, \((a, d)\)-distance antimagic labelings do not exist for even cycles; for suns, the labelings do not exist for all suns.

### 2.2 Sun

A sun \(S_n\) is a cycle on \(n\) vertices with a leaf attached to each vertex on the cycle. Let the vertex set of sun \(V(S_n) = \{x_1, \ldots, x_n, y_1, \ldots, y_n\}\), where \(d(x_i) = 3\) and \(d(y_i) = 1\).

**Theorem 2.8.** All suns are not \((a, d)\)-distance antimagic.

**Proof.** Consider a sun \(S_n\) of order \(2n\). Since \(w(y_i) = f(x_i)\) then \(1 \leq w(y_i) \leq 2n\) and so \(d \leq \frac{2n-1+i}{n} = 2\). For \(d = 0\), it is obvious that a distance magic labeling does not exist. If \(d = 1\) then the labels of \(x_i\)s are \(c, c + 1, \ldots, c + n - 1\) for \(1 \leq c \leq n + 1\). Thus the smallest possible weight of \(x_i\) is \(c + (c + 1) + (c + n) = 3c + n + 1\) and so there is a gap in vertex-weights. If \(d = 2\) then the labels of \(x_i\) are either \(1, 3, \ldots, 2n - 1\) or \(2, 4, \ldots, 2n\). In both cases, there will be parity difference between \(w(x_i)\) and \(w(y_i)\).

Although all suns are not \((a, d)\)-distance antimagic, next we shall prove that they are otherwise distance antimagic.

**Theorem 2.9.** All suns are distance antimagic.

**Proof.** We define a vertex labeling \(f\) of \(S_n\) as follow.

\[
f(x_i) = n + i \quad \text{for } i = 1, 2, \ldots, n, \quad \text{and}
\]

\[
f(y_i) = i \quad \text{for } i = 1, 2, \ldots, n,
\]

and so

\[
w(y_i) = f(x_i) = n + i \quad \text{for } i = 1, 2, \ldots, n, \quad \text{and}
\]

\[
w(x_i) = \begin{cases} 
  3n + 3 & \text{for } i = 1, \\
  2n + 3i & \text{for } i = 2, 3, \ldots, n - 1, \\
  4n & \text{for } i = n.
\end{cases}
\]
When \( n \neq 0 \mod 3 \), all vertex-weights are distinct. Otherwise, \( 3n + 3 = 2n + 3i \) for \( i = \frac{2}{3} + 1 \) and \( 4n = 2n + 3i \) for \( i = \frac{n}{2} \). In that case, we exchange the labels of \( y_{\frac{2}{3}+1} \) with \( y_{\frac{2}{3}} \) and \( y_{\frac{2n}{2}+1} \) with \( y_{\frac{2n}{2}} \) to obtain distinct weights for all vertices.

We have studied the distance antimagic labelings for cycles, the 2-regular connected graphs, and next we will consider two families of regular connected graphs: prisms and complete graphs. Here we manage to characterize all \((a, d)\)-distance antimagic prisms and complete graphs.

### 2.3 Prism

A prism \( C_n \times P_2 \) is a 3-regular graph of order \( 2n \). Let \( V(C_n \times P_2) = \{x_1, \ldots, x_n, y_1, \ldots, y_n\} \) and \( E(C_n \times P_2) = \{x_i y_i, i = 1, \ldots, n\} \).

In [2], Arumugam and Kamatchi proved that prisms are \((a, 1)\)-distance antimagic.

**Theorem 2.10.** [2] The prism \( C_n \times K_2 \) is \((n+2, 1)\)-distance antimagic.

Next we will prove that \((a, d)\)-distance antimagic prisms only exist when \( d = 1 \).

**Theorem 2.11.** A prism is \((a, d)\)-distance antimagic if and only if \( d = 1 \).

**Proof.** By Lemma 2.4, \( d \leq \frac{2n-3}{2n-1} \leq 2 \). If \( d = 0 \) or \( d = 2 \), then \( a = \frac{3(2n+1)-d(2n-1)}{2} \) is not an integer. By Theorem 2.10, we have the desired labeling for \( d = 1 \). \( \Box \)

### 2.4 Complete Graph

**Theorem 2.12.** A nontrivial complete graph has an \((a, d)\)-distance antimagic labeling if and only if \( d = 1 \).

**Proof.** Consider a complete graph of order \( n \), \( K_n \). Since \( K_n \) is \((n-1)\)-regular, then by Lemma 2.4, \( d \leq 1 \). It is known that distance magic labelings do not exist for nontrivial complete graphs (see for example [15]), and so \( d = 1 \). Again, by applying Lemma 2.4 we obtain \( a = \frac{(n-1)n}{2} \).

Suppose that \( V(K_n) = \{x_1, x_2, \ldots, x_n\} \). We then define a vertex labeling of \( K_n \) as follow.

\[
 f(x_i) = i \quad \text{for} \quad i = 1, 2, \ldots, n. 
\]

Under the labeling \( f \), the vertex-weights are

\[
 w(x_i) = \frac{n(n+1)}{2} - i \quad \text{for} \quad i = 1, 2, \ldots, n, 
\]

which constitute an arithmetic progression with difference 1. \( \Box \)
The last families of graphs to be considered are wheels, fans, and friendship graphs. They are closely related since deleting one edge in a wheel results in a fan and deleting half of the edges results in a friendship graph. Not surprisingly, the graphs have similar distance antimagicness characteristics. We prove that all three graphs are not \((a, d)\)-distance antimagic in general, but distance antimagic instead.

\[ \text{Figure 1:} \quad (a, d)\text{-distance antimagic labelings for wheel-related graphs.} \]

### 2.5 Wheel

A wheel \(W_n\) is a graph obtained by joining all vertices of a cycle of order \(n\) to a further vertex called the center. Let \(V(W_n) = \{x_0, x_1, \ldots, x_n\}\) where \(v_0\) is the center and \(x_1, \ldots, x_n\) are the vertices of the cycle.

**Lemma 2.13.** A wheel \(W_n\) of order \(n + 1\) has an \((a, d)\)-distance antimagic labeling if and only if \(3 \leq n \leq 5\).

**Proof.** Since \(d(x_i) = 3\) then \(6 \leq w(x_i) \leq 3n\), for \(i = 2, \ldots, n - 1\). Thus, \(a + (n - 1)d \leq 3n\) or \(d \leq 3 - \frac{3}{n-1}\), and so \(d \leq 2\). On the other hand, \(\frac{n(n+1)}{2} \leq w(x_0) \leq \frac{n(n+3)}{2}\). This leads to \(w(x_0) - w(x_i) \geq \frac{n(n+1)}{2} - 3n\) for some \(i\) and so \(d \geq \frac{n^2-5n}{2}\). For \(n \geq 6\), we obtain \(d \geq 3\), a contradiction.

Now we need to consider \(W_n\) for \(n = 3, 4, 5\). For \(n = 3\), since \(W_3 \simeq K_4\), we have the desired labeling as in Theorem 2.12. For \(n = 4\), it is known that \(W_4\) is distance magic or \((10, 0)\)-distance antimagic (see [15]). For \(d > 0\), by Lemma 2.1 \(W_4\) is not \((a, d)\)-distance antimagic. To complete the proof, for \(n = 5\), consider a vertex labeling of \(W_5\) whose vertex-weights constitute an arithmetic progression with difference 1 as depicted in Figure 1(a).

Although only two small wheels are \((a, d)\)-distance antimagic for \(d \geq 1\), we could construct distance antimagic labelings for all wheels of order other than 5.

**Lemma 2.14.** All wheels of order other than 5 are distance antimagic.

**Proof.** By Lemma 2.1, \(W_4\) is not distance antimagic. For \(n \neq 4\), define a vertex labeling where for \(i = 1, \ldots, n\), \(x_i\) is labeled as vertex \(x_i\) of a cycle \(C_n\) in the
proof of Theorem 2.7 and $v_0$ is labeled with $n + 1$. Since the vertex-weights of vertices in the cycle are distinct, then the vertex-weights of vertices in the wheel are also distinct.

\section{2.6 Fan}

A fan $F_n$ is a graph obtained by joining all vertices of a path of order $n$ to a further vertex called the center. Let $V(F_n) = \{x_0, x_1, \ldots, x_n\}$ where $x_0$ is the center and $x_1, \ldots, x_n$ are the vertices of the path.

\textbf{Theorem 2.15.} The fan $F_n$ admits an $(a,d)$-distance antimagic labeling if and only if $n = 2$ or $n = 4$.

\textbf{Proof.} Since $d(x_1) = d(x_n) = 2$ and $d(x_i) = 3$ for $i = 2, \ldots, n - 1$ then $3 \leq w(x_i) \leq 3n$. Thus, $d \leq \frac{3n - 3}{n}$, and so $d \leq 2$. On the other hand, $\frac{n(n+1)}{2} \leq w(x_0) \leq \frac{n(n+3)}{2}$. This leads to $w(x_0) - w(x_i) \geq \frac{n(n+1)}{2} - 3n$ for some $i$ and so $d \geq \frac{n^2 - 5n}{2}$. For $n \geq 6$, $d \geq 3$, a contradiction.

Now we need to consider $F_n$ with $2 \leq n \leq 5$. For $n = 2$, since $F_2 \simeq K_3$, we have the desired labeling as in Theorem 2.12. For $n = 3$, $F_3$ has no $(a,d)$-distance antimagic labeling by Lemma 2.2. For $n = 4$ consider a vertex labeling of $F_4$ whose vertex-weights constitute an arithmetic progression with difference 1 as depicted in Figure 1(b). For $n = 5$, if we assign 1, 2, 3 or 6 as the label of vertex $x_0$ then the difference between the weight of $x_0$ and the largest weight of $x_i, i = 1, \ldots, n$ is greater than 2, a contradiction. If we assign 4 as the label of $x_0$ then $w(x_0) = 17$. Due to the impossibility to attain 16 as weight, we have $d = 2$ and so the weights of all $x_i$s are odd. This implies that the labels of $x_2$ and $x_4$ must be odd, causing the weight of $x_3$ to be even, a contradiction. If we assign 5 as the label of $x_0$ then $w(x_0) = 16$. If $d = 1$ then $w(x_i) \geq 11$, for $i = 1, 2, \ldots, 5$. However, the weights of $x_1$ and $x_3$ are summation of two labels, one of which is 5, and so the weight 12 is not achievable. If $d = 2$ then the weights of all $x_i$s are even; thus the labels of $x_2$ and $x_4$ must be odd and the weight of $x_3$ is also odd, a contradiction.

Again, we could prove that all fans, except for $F_3$, are distance antimagic.

\textbf{Lemma 2.16.} All fans of order other than 4 are distance antimagic.

\textbf{Proof.} By Lemma 2.1 $F_3$ is not distance antimagic. For $n \neq 3$, by defining the following vertex labeling $f$

$$f(x_i) = \begin{cases} \left\lfloor \frac{n+2}{2} \right\rfloor & \text{for } i = 0, \\ i & \text{for } i = 1, 2, \ldots, \left\lfloor \frac{n+1}{2} \right\rfloor, \\ 1 + i & \text{for } i = \left\lfloor \frac{n+1}{2} \right\rfloor + 1, \left\lfloor \frac{n+1}{2} \right\rfloor + 2, \ldots, n, \end{cases}$$

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we obtain all distinct vertex-weights below.

\[ w(x_i) = \begin{cases} 
\frac{1}{2}(n+1)(n+2) - \left\lfloor \frac{n+2}{2} \right\rfloor & \text{for } i = 0, \\
\left\lfloor \frac{n+2}{2} \right\rfloor + 2 & \text{for } i = 1, \\
\left\lfloor \frac{n+2}{2} \right\rfloor + 2i & \text{for } i = 2, 3, \ldots, \left\lfloor \frac{n+1}{2} \right\rfloor - 1 \text{ and } i = \left\lfloor \frac{n+1}{2} \right\rfloor + 2, \left\lfloor \frac{n+1}{2} \right\rfloor + 3, \ldots, n - 1, \\
2\left\lfloor \frac{n+2}{2} \right\rfloor + \left\lfloor \frac{n+1}{2} \right\rfloor + 2 & \text{for } i = \left\lfloor \frac{n+1}{2} \right\rfloor, \\
2\left\lfloor \frac{n+2}{2} \right\rfloor + \left\lfloor \frac{n+1}{2} \right\rfloor + 2 & \text{for } i = \left\lfloor \frac{n+1}{2} \right\rfloor + 1, \\
\left\lfloor \frac{n+2}{2} \right\rfloor + n & \text{for } i = n. 
\end{cases} \]

\[2.7 \text{ Friendship graph}\]

A friendship graph \( f_n \) is obtained by identifying a vertex from \( n \) copies of complete graphs of order 3. Let \( V(f_n) = \{x_0, x_1, \ldots, x_{2n}\} \) where \( x_0, x_{2i-1}, x_{2i} \) are the vertices in the \( i \)-th \( K_3 \), for \( i = 1, \ldots, n \).

**Theorem 2.17.** A friendship graph \( f_n \) is \((a, d)\)-distance antimagic if and only if \( n = 1 \) or \( n = 2 \).

**Proof.** For \( i = 1, 2, \ldots, 2n \), we have \( 3 \leq w(x_i) \leq 4n + 1 \) and so \( d \) is at most \( \frac{(4n+1)-3}{2n} = 2 - \frac{1}{n} \leq 2 \). On the other hand, \( n(2n+1) \leq w(x_0) \leq n(2n+3) \). Thus we have \( w(x_0) - w(x_i) \geq n(2n+1) - (4n+1) = 2n^2 - 3n - 1 \). For \( n \geq 3 \), \( w(x_0) - w(x_i) \geq 8 \), a contradiction.

To complete the proof, we need to consider \( f_1 \) and \( f_2 \). Since \( f_1 \simeq K_3 \) then \( f_1 \) has a \((3, 1)\)-distance antimagic labeling by Theorem 2.12. A \((6, 1)\)-distance antimagic labeling for \( f_2 \) is depicted in Figure 1(c).

Finally, a simple vertex labeling leads to the distance antimagicness of friendship graphs.

**Theorem 2.18.** All friendship graphs are distance antimagic.

**Proof.** We define a vertex labeling \( f \) of \( f_n \) as follow

\[ f(x_i) = \begin{cases} 
n + 1 & \text{for } i = 0, \\
i & \text{for } i = 1, 2, \ldots, n, 
\end{cases} \]

and so we obtain the following vertex-weights

\[ w(x_i) = \begin{cases} 
n(2n+1) & \text{for } i = 0, \\
n + 2 + i & \text{for } i = 1, 3, \ldots, 2n - 1, \\
n + i & \text{for } i = 2, 4, \ldots, 2n. 
\end{cases} \]

We can see that the weights are all distinct.
3 Final remark

Revisiting the necessary conditions for the existence of distance antimagic and D-distance antimagic graphs in Lemmas 2.1 and 2.2, we strongly believe that those conditions are also sufficient and propose the following conjectures.

Conjecture 3.1. A graph is distance antimagic if and only if it does not contain two vertices with the same neighborhood.

Conjecture 3.2. A graph is D-distance antimagic if and only if it does not contain two vertices with the same D-neighborhood.

As with the antimagic conjecture of Harstfield and Ringel [11], proving or disproving the afore-mentioned conjectures is likely to be a hard problem.

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