On a decomposition lemma for positive semi-definite block-matrices

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January, 2012

Abstract

This short note, in part of expository nature, points out several new or recent consequences of a quite nice decomposition for positive semi-definite matrices.

Keywords: Operator inequalities, positive semi-definite matrix, unitary orbit, symmetric norm.

AMS subjects classification 2010: 15A60, 47A30, 15A42.

1 A decomposition lemma

Let \( \mathbb{M}_n \) denote the space of \( n \times n \) complex matrices, or operators on a finite dimensional Hilbert space, and let \( \mathbb{M}_n^+ \) be the positive (semi-definite) part. For positive block-matrices, 
\[
\begin{bmatrix}
A & X \\
X^* & B
\end{bmatrix} \in \mathbb{M}_{n+m}^+, \quad \text{with} \quad A \in \mathbb{M}_n^+, B \in \mathbb{M}_m^+,
\]
we have a remarkable decomposition lemma for elements in \( \mathbb{M}_{n+m}^+ \) noticed in [3]:

Lemma 1.1. For every matrix in \( \mathbb{M}_{n+m}^+ \) written in blocks, we have the decomposition
\[
\begin{bmatrix}
A & X \\
X^* & B
\end{bmatrix} = U \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} U^* + V \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix} V^*
\]
for some unitaries \( U, V \in \mathbb{M}_{n+m} \).

The motivation for such a decomposition is various inequalities for convex or concave functions of positive operators partitioned in blocks. These results are extensions of some classical majorization, Rotfel’d and Minkowski type inequalities. Lemma 1.1 actually implies a host of such inequalities as shown in the recent papers [2] and [3] where a proof of Lemma 1.1 can be found too.

*Supported by ANR 2011-BS01-008-01.

†Research supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2010-0003520)
This note aims to give in the next section further consequences of the above decomposition lemma, including a simple proof of a recent majorization shown in \[5\].

Most of the corollaries below are rather straightforward consequences of Lemma 1.1 except Corollary 2.4 which also requires some more elaborated estimates. Corollary 2.10 is the majorization given in \[5\], a remarkable extension of the basic and useful inequality

\[
\left\| \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \right\| \leq \left\| A + B \right\| \tag{1.1}
\]

for all \(A, B \in \mathbb{M}_n^+\) and all symmetric (or unitarily invariant) norms. The recent survey \[4\] provides a good exposition on these classical norms.

\section{Some Consequences}

If we first use a unitary congruence with

\[ J = \frac{1}{\sqrt{2}} \begin{bmatrix} I & -I \\ I & I \end{bmatrix} \]

where \(I\) is the identity of \(\mathbb{M}_n\), we observe that

\[
J \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} J^* = \begin{bmatrix} \frac{A+B}{2} + \text{Re}X & * \\ * & \frac{A+B}{2} - \text{Re}X \end{bmatrix}
\]

where \(*\) stands for unspecified entries and \(\text{Re}X = (X + X^*)/2\). Thus Lemma 1.1 yields:

**Corollary 2.1.** For every matrix in \(\mathbb{M}_{2n}^+\) written in blocks of the same size, we have a decomposition

\[
\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} = U \begin{bmatrix} \frac{A+B}{2} + \text{Re}X & 0 \\ 0 & 0 \end{bmatrix} U^* + V \begin{bmatrix} 0 & \frac{A+B}{2} \\ 0 & -\text{Re}X \end{bmatrix} V^*
\]

for some unitaries \(U, V \in \mathbb{M}_{2n}\).

This is equivalent to Corollary 2.2 below by the obvious unitary congruence

\[
\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \simeq \begin{bmatrix} A & iX \\ -iX^* & B \end{bmatrix}.
\]

In Corollary 2.1 if \(A, B, X\) have real entries, i.e., if we are dealing with an operator on a real Hilbert space \(\mathcal{H}\) of dimension \(2n\), then \(U, V\) can be taken with real entries, thus are isometries on \(\mathcal{H}\). Do we have the same for Corollary 2.2? The answer might be negative, but an explicit counter-example would be desirable.

**Corollary 2.2.** For every matrix in \(\mathbb{M}_{2n}^+\) written in blocks of same size, we have a decomposition

\[
\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} = U \begin{bmatrix} \frac{A+B}{2} + \text{Im}X & 0 \\ 0 & 0 \end{bmatrix} U^* + V \begin{bmatrix} 0 & \frac{A+B}{2} \\ 0 & -\text{Im}X \end{bmatrix} V^*
\]

for some unitaries \(U, V \in \mathbb{M}_{2n}\).
Here \( \text{Im}X = (X - X^*)/2i \). The decomposition allows to obtain some norm estimates depending on how the full matrix is far from a block-diagonal matrix. If \( Z \in \mathbb{M}_n \), its absolute value is \( |Z| := (Z^*Z)^{1/2} \). If \( A, B \in \mathbb{M}_n \) are Hermitian, \( A \leq B \) means \( B - A \in \mathbb{M}_n^+ \). Firstly, by noticing that \( \text{Im}X \leq |\text{Im}X| = \frac{1}{2}|X - X^*| \), we have:

**Corollary 2.3.** For every matrix in \( \mathbb{M}_{2n}^+ \) written in blocks of same size, we have

\[
\begin{bmatrix}
A & X \\
X^* & B
\end{bmatrix} \leq \frac{1}{2} \left\{ U \begin{bmatrix}
A + B + |X - X^*| & 0 \\
0 & 0
\end{bmatrix} + V \begin{bmatrix}
0 & A + B + |X - X^*| \\
0 & 0
\end{bmatrix} \right\} V^*
\]

for some unitaries \( U, V \in \mathbb{M}_{2n} \).

We may then obtain estimates for the class of symmetric norms \( \| \cdot \| \). Such a norm on \( \mathbb{M}_n \) satisfies \( \|UA\| = \|AU\| = \|A\| \) for all \( A \in \mathbb{M}_n \) and all unitaries \( U \in \mathbb{M}_n \). Since a symmetric norm on \( \mathbb{M}_{n+m} \) induces a symmetric norm on \( \mathbb{M}_n \) we may assume that our norms are defined on all spaces \( \mathbb{M}_n, n \geq 1 \).

**Corollary 2.4.** For every matrix in \( \mathbb{M}_{2n}^+ \) written in blocks of same size and for all symmetric norms, we have

\[
\left\| \begin{bmatrix}
A & X \\
X^* & B
\end{bmatrix}^p \right\| \leq 2^{p-1} \left\{ \| (A + B)^p \| + \| (X - X^*)^p \| \right\}
\]

for all \( p > 0 \).

**Proof.** We first show the case \( 0 < p < 1 \). From [1] (for a proof see also, [3, Section 3]) it is known that:

If \( S, T \in \mathbb{M}_n^+ \) and if \( f : [0, \infty) \to [0, \infty) \) is concave, then, for some unitary \( U, V \in \mathbb{M}_n^+ \),

\[
f(S + T) \leq U f(S) U^* + V f(T) V^*.
\]

(2.1)

Applying (2.1) to \( f(t) = t^p \) and the RHS of Corollary 2.3 with

\[
S = \frac{1}{2} U \begin{bmatrix}
A + B + |X - X^*| & 0 \\
0 & 0
\end{bmatrix} U^*, \quad T = \frac{1}{2} V \begin{bmatrix}
0 & A + B + |X - X^*| \\
0 & 0
\end{bmatrix} V^*
\]

we obtain

\[
\left\| \begin{bmatrix}
A & X \\
X^* & B
\end{bmatrix}^p \right\| \leq 2^{1-p} \left\{ \| (A + B + |X - X^*|)^p \| \right\}
\]

Applying again (2.1) with \( f(t) = t^p, S = A + B \) and \( T = |X - X^*| \) yields the result for \( 0 < p < 1 \).

To get the inequality for \( p \geq 1 \), it suffices to use in the RHS of Corollary 2.3 the elementary inequality, for \( S, T \in \mathbb{M}_n^+ \),

\[
\left\| \left( \frac{S + T}{2} \right)^p \right\| \leq \frac{\| S^p \| + \| T^p \|}{2}
\]

(2.2)

(see [3, Section 2] for much stronger results). With

\[
S = U \begin{bmatrix}
A + B + |X - X^*| & 0 \\
0 & 0
\end{bmatrix} U^*, \quad T = V \begin{bmatrix}
0 & A + B + |X - X^*| \\
0 & 0
\end{bmatrix} V^*
\]
we get from Corollary 2.3 and (2.2)

\[ \left\| \begin{bmatrix} A & X \\ X^* & B \end{bmatrix}^p \right\| \leq \left\| (A + B + |X - X^*|)^p \right\| \]

and another application of (2.2) with \( S = 2(A + B) \) and \( T = 2|X - X^*| \) completes the proof.

\[ \square \]

**Corollary 2.5.** For any matrix in \( \mathbb{M}_{2n}^+ \) written in blocks of same size such that the right upper block \( X \) is accretive, we have

\[ \left\| \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right\| \leq \left\| A + B \right\| + \left\| \text{Re}X \right\| \]

for all symmetric norms.

**Proof.** By Corollary 2.1, for all Ky Fan \( k \)-norms \( \left\| \cdot \right\|_k, k = 1, \ldots, 2n \), we have

\[ \left\| \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right\|_k \leq \left\| \begin{bmatrix} A + B + \text{Re}X & 0 \\ 0 & 0 \end{bmatrix} \right\|_k + \left\| \begin{bmatrix} 0 & 0 \\ 0 & \frac{A + B}{2} \end{bmatrix} \right\|_k . \]

Equivalently,

\[ \left\| \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right\|_k \leq \left\| \begin{bmatrix} A + B + \text{Re}X \end{bmatrix}^\dagger \right\|_k + \left\| \left( \frac{A + B}{2} \right)^\dagger \right\|_k \]

where \( Z^\dagger \) stands for the diagonal matrix listing the eigenvalues of \( Z \in \mathbb{M}_n^+ \) in decreasing order. By using the triangle inequality for \( \left\| \cdot \right\|_k \) and the fact that

\[ \left\| Z^\dagger_1 \right\|_k + \left\| Z^\dagger_2 \right\|_k = \left\| Z^\dagger_1 + Z^\dagger_2 \right\|_k \]

for all \( Z_1, Z_2 \in \mathbb{M}_n^+ \) we infer

\[ \left\| \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right\|_k \leq \left\| (A + B)^\dagger + (\text{Re}X)^\dagger \right\|_k . \]

Hence

\[ \left\| \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right\|_k \leq \left\| (A + B)^\dagger + (\text{Re}X)^\dagger \right\| \]

for all symmetric norms. The triangle inequality completes the proof.

\[ \square \]

**Corollary 2.6.** For any matrix in \( \mathbb{M}_{2n}^+ \) written in blocks of same size such that \( 0 \notin W(X) \), the numerical range the of right upper block \( X \), we have

\[ \left\| \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right\| \leq \left\| A + B \right\| + \left\| X \right\| \]

for all symmetric norms.
**Proof.** The condition $0 \notin W(X)$ means that $zX$ is accretive for some complex number $z$ in the unit circle. Making use of the unitary congruence

$$
\begin{bmatrix}
A & X \\
X^* & B
\end{bmatrix} \sim 
\begin{bmatrix}
A & zX \\
\overline{z}X^* & B
\end{bmatrix}
$$

we obtain the result from Corollary 2.5. \hfill \square

The condition $0 \notin W(X)$ in the previous corollary can obviously be relaxed to $0$ does not belong to the relative interior of $X$, denoted by $W_{int}(X)$. In case of the usual operator norm $\| \cdot \|_\infty$, this can be restated with the numerical radius $w(X)$:

**Corollary 2.7.** For any matrix in $M_{2n}^+$ written in blocks of same size such that $0 \notin W_{int}(X)$, the relative interior of the numerical range the of right upper block $X$, we have

$$
\left\| \begin{bmatrix}
A & X \\
X^* & B
\end{bmatrix} \right\|_\infty \leq \| A + B \|_\infty + w(X).
$$

In case of the operator norm, we also infer from Corollary 2.1 the following result:

**Corollary 2.8.** For any matrix in $M_{2n}^+$ written in blocks of same size, we have

$$
\left\| \begin{bmatrix}
A & X \\
X^* & B
\end{bmatrix} \right\|_\infty \leq \| A + B \|_\infty + 2w(X).
$$

Once again, the proof follows by replacing $X$ by $zX$ where $z$ is a scalar in the unit circle such that $w(X) = w(zX) = \| \text{Re}(zX) \|_\infty$ and then by applying Corollary 2.1.

**Example 2.9.** By letting

$$
A = \begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix}, \quad X = \begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
$$

we have an equality case in the previous corollary. This example also gives an equality case in Corollary 2.4 for the operator norm and any $p \geq 1$. (For any $0 < p < 1$ and for the trace norm, equality occurs in Corollary 2.4 with $A = B$ and $X = 0$.)

From Corollary 2.2 we also recapture in the next corollary the majorization result obtained in [5] for positive block-matrices whose off-diagonal blocks are Hermitian. Example 2.9 shows that the Hermitian requirement on the off-diagonal blocks is necessary.

**Corollary 2.10.** Given any matrix in $M_{2n}^+$ written in blocks of same size with Hermitian off-diagonal blocks, we have

$$
\left\| \begin{bmatrix}
A & X \\
X & B
\end{bmatrix} \right\| \leq \| A + B \|
$$

for all symmetric norms.
Letting \( X = 0 \) in the above corollary we get the basic inequality (1.1). The last two corollaries seem to be folklore.

**Corollary 2.11.** Given any matrix in \( \mathbb{M}_{2n}^+ \) written in blocks of same size, we have

\[
\left\| \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \oplus \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right\| \leq 2\|A \oplus B\|
\]

for all symmetric norms.

**Proof.** This follows from (1.1) and the obvious unitary congruence

\[
\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \oplus \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \simeq \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \oplus \begin{bmatrix} A & -X \\ -X^* & B \end{bmatrix}
\]

\( \square \)

Let \( \| \cdot \|_p, 1 \leq p < \infty \), denote the usual Schatten \( p \)-norms. The previous corollary entails the last one:

**Corollary 2.12.** Given any matrix in \( \mathbb{M}_{2n}^+ \) written in blocks of same size, we have

\[
\left\| \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right\|_p \leq 2^{1-1/p} (\|A\|_p^p + \|B\|_p^p)^{1/p}
\]

for all \( p \in [1, \infty) \).

Note that if \( A = X = B \) we have an equality case in Corollary 2.12.

**Remark 2.13.** Lemma 1.1 is still valid for compact operators on a Hilbert space, by taking \( U \) and \( V \) as partial isometries. A similar remark holds for the subadditivity inequality (2.1). Hence the symmetric norm inequalities in this paper may be extended to the setting of normed ideals of compact operators.

The lack of counter-example suggests that the following could hold:

**Conjecture 2.14.** Corollary 2.10 is still true when the off-diagonal blocks are normal.

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