MUTUALLY INTERACTING SUPERPROCESSES WITH MIGRATION

LINA JI, HUILI LIU AND JIE XIONG

Abstract. A system of mutually interacting superprocesses with migration is constructed as the limit of a sequence of branching particle systems arising from population models. The uniqueness in law of the superprocesses is established using the pathwise uniqueness of a system of stochastic partial differential equations with non-Lipschitz coefficients, which is satisfied by the corresponding system of distribution-function-valued processes.

1. Introduction

Superprocesses, describing the evolution of large population undergoing random reproduction and spatial motion, were first constructed as high-density limits of branching particle systems by Watanabe (1968). The connection between superprocesses with stochastic evolution equations was investigated by Dawson (1975). Since then ample systematic research results have been published, see e.g., Dawson (1993); Etheridge (2000); Li (2011). The ones with immigration, a class of generalizations of superprocesses, have also attracted the interest of many researchers. We refer to Li (2011); Li and Shiga (1995); Li et al. (2010) and the references therein for immigration structure and related properties. Let $M_F(\mathbb{R})$ be the collection of all finite Borel measures on $\mathbb{R}$. Denote by $C^k_b(\mathbb{R})$ (resp. $C^k_0(\mathbb{R})$) the collection of all bounded (resp. compactly-supported) continuous functions on $\mathbb{R}$ with bounded derivatives up to $k$th order. We consider a continuous $M_F(\mathbb{R})$-valued process $(\mu_t)_{t \geq 0}$, solving the following martingale problem (MP): $\forall f \in C^2_b(\mathbb{R})$, the process

$$ M^f_t = \langle \mu_t, f \rangle - \langle \mu_0, f \rangle - \int_0^t \left( \frac{1}{2} \langle \mu_s, f'' \rangle + \langle \kappa, f \rangle \right) ds $$

is a continuous martingale with quadratic variation process

$$ \langle M^f \rangle_t = \gamma \int_0^t \langle \mu_s, f^2 \rangle ds $$

where $\gamma > 0$ and $\kappa \in M_F(\mathbb{R})$. The corresponding model is super-Brownian motion (SBM) when $\kappa = 0$. The uniqueness in law of SBM can be obtained by its log-Laplace equation. Xiong (2013) studied the stochastic partial differential equation (SPDE) satisfied by the distribution-function-valued process, and approached the...
uniqueness of SBM from a different point of view. Related work can also be found in [Dawson and Li (2012); He et al. (2014)].

For a finite measure $\kappa$, the corresponding model is superprocess with immigration, which was constructed in [Li (1992)] through the cumulant semigroup, see also [Li (1996); Li and Shiga (1995)]. In case of $\kappa$ being interactive, i.e., $\kappa = \kappa(\mu)$, the existence of solution to MP [1.1 1.2] has been verified in [Méleard and Roelly (1992)], where the result is applicable to the situation with interactive immigration, branching rate and spatial motion. By the approach of pathwise uniqueness for SPDEs satisfied by the distribution-function-valued process, Mytnik and Xiong (2015) established the well-posedness of MP for superprocess with interactive immigration. See also Xiong and Yang (2016) for the related work.

However, there exist some populations distributed in different colonies, such as the mutually catalytic branching model, see [Dawson and Perkins (1998); Dawson et al. (2002a,b); Méleard (1995); Mytnik (1998)]. The evolution of such a model can be depicted by interacting superprocesses. Some sudden event may induce mass migration and lead to an increment of population size in one colony while a decrement in the other. For instance, war makes large numbers of people move into neighboring country and radiation mutates normal cells and so on. Therefore, it is natural to study the superprocesses with interactive migration between different colonies. In this paper we consider a continuous $M_F(\mathbb{R})^2$-valued process $(\mu^1_t, \mu^2_t)_{t \geq 0}$, called as mutually interacting superprocesses with migration. It solves the following MP:

$$\forall f, g \in C_0^2(\mathbb{R}), \text{ the processes}$$

$$\left\{ \begin{array}{l}
M^f_t = \langle \mu^1_t, f \rangle - \langle \mu^2_t, f \rangle - \frac{1}{2} \int_0^t \langle \mu^1_s, f' \rangle ds - b_1 \int_0^t \langle \mu^1_s, \eta(\gamma, \cdot, \mu^1_s, \mu^2_s) f \rangle ds + \int_0^t \langle \mu^1_s, \eta(\cdot, \cdot, \mu^1_s, \mu^2_s) f \rangle ds \\
\hat{M}^g_t = \langle \mu^2_t, g \rangle - \langle \mu^2_t, g \rangle - \frac{1}{2} \int_0^t \langle \mu^2_s, g'' \rangle ds - b_2 \int_0^t \langle \mu^2_s, g \rangle ds - \langle \chi, g \rangle \int_0^t \langle \mu^1_s, \eta(\cdot, \cdot, \mu^1_s, \mu^2_s) \rangle ds
\end{array} \right.$$  

are two continuous martingales with quadratic variation and covariation processes

$$\left\{ \begin{array}{l}
\langle M^f \rangle_t = \gamma_1 \int_0^t \langle \mu^1_s, f^2 \rangle ds \\
\langle \hat{M}^g \rangle_t = \gamma_2 \int_0^t \langle \mu^2_s, g^2 \rangle ds \\
\langle M^f \rangle_t = \langle \hat{M}^g \rangle_t = 0
\end{array} \right.$$  

where $\chi$ is a finite measure on $\mathbb{R}$; $\gamma_1$ and $\gamma_2$ are positive constants; the migration intensity $\eta(\cdot, \cdot, \cdot) \in C^+_b(\mathbb{R} \times M_F(\mathbb{R})^2)$ is a nonnegative bounded continuous functions on $\mathbb{R} \times M_F(\mathbb{R})^2$.

The purpose of this paper is to establish the well-posedness of MP [1.3 1.4], i.e., the existence and uniqueness in law of such a mutually interacting superprocesses with migration. The process is constructed as the limit of a sequence of branching particle systems. To obtain the uniqueness in law of the superprocesses, we demonstrate the pathwise uniqueness of the solution to a system of mutually interacting SPDEs with non-Lipschitz coefficients, which are satisfied by the corresponding distribution-function-valued processes. As far as we know, this is the first attempt to discuss the well-posedness of mutually interacting superprocesses with migration.

Now we present some notation. Let $D(\mathbb{R}_+, M_F(\mathbb{R})^2)$ (resp. $C(\mathbb{R}_+, M_F(\mathbb{R})^2)$) denote the space of càdlàg (resp. continuous) paths from $\mathbb{R}_+$ to $M_F(\mathbb{R})^2$ furnished with the Skorokhod topology. Let $D(\mathbb{R}_+, \mathbb{R}^2)$ be the collection of càdlàg paths from $\mathbb{R}_+$ to $\mathbb{R}^2$. Recall that $C_b(\mathbb{R})$ is the collection of all bounded continuous functions on $\mathbb{R}$. Let $C_{b,m}(\mathbb{R})$ be the subset of $C_b(\mathbb{R})$ consisting of nondecreasing bounded continuous functions on $\mathbb{R}$. Write $\langle f, g \rangle$ as the integral of $f \in C_b^2(\mathbb{R})$ with respect to measure $\mu \in M_F(\mathbb{R})$. For any $f, g \in C_b^2(\mathbb{R})$, define $\langle f, g \rangle_1 =$...
\[ \int_{\mathbb{R}} f(x)g(x)dx. \] Let \( H_0 = L^2(\mathbb{R}) \) be the Hilbert space consisting of all square-integrable functions with Hilbertian norm \( \| \cdot \|_0 \) given by \( \| f \|_0^2 = \int_{\mathbb{R}} f^2(x)e^{-|x|}dx \) for any \( f \in H_0 \). Set \( v_i(x) = \nu_i((-,\infty, x]) \) as the distribution functions of \( \nu_i \in M_F(\mathbb{R}) \) for any \( x \in \mathbb{R} \) and \( i = 1, 2 \). Define distance \( \rho \) on \( M_F(\mathbb{R}) \) by
\[
(1.5) \quad \rho(v_1, v_2) = \int_{\mathbb{R}} e^{-|x|} | v_1(x) - v_2(x) | dx.
\]
It is easy to see that, under metric distance \( \rho \), \( M_F(\mathbb{R}) \) is a Polish space whose topology coincides with that given by weak convergence of measures. Moreover, we always assume that all random variables in this paper are defined on the same filtered probability space \( (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}) \). Let \( \mathbb{E} \) be the corresponding expectation.

The rest of this paper is organized as follows. In Section 2, a sequence of branching particle systems arising from population models is introduced. In Section 3, we show the existence of solution to MP (1.3, 1.4) by the convergence of the branching particle systems. In Section 4, the equivalence between MP (1.3, 1.4) and SPDEs satisfied by the distribution-function-valued processes is established and further we prove the pathwise uniqueness of the SPDEs by an extended Yamada-Watanabe argument. Throughout the paper, we let the letter \( K \) with or without subscripts to denote constants whose exact value is unimportant and may change from line to line.

2. A RELATED BRANCHING MODEL WITH MIGRATION

There exists a population living in two colonies with labels \{1, 2\}. Initially, each colony has \( n \) particles, spatially distributed in \( \mathbb{R} \). Write \( k \sim t \) to mean that the \( k \)th living particle at time \( t \) in each colony. For any \( s \geq t \), denote by \( X_{k \sim t}^{i, n}(s) \) the corresponding particle’s location at time \( s \) in colony \( i \) with \( i \in \{1, 2\} \) if it is alive up to time \( s \). The motions of the particles during their life times are modeled by independent Brownian motions. For any \( i \in \{1, 2\}, k \sim t \), accompanying the corresponding particle with standard Brownian motion \( \{B_{k \sim t}^{1}(s) : s \geq 0\} \), we have
\[
X_{k \sim t}^{i, n}(s) = X_{k \sim t}^{i, n}(t) + B_{k \sim t}^{i}(s) - B_{k \sim t}^{i}(t), \quad \forall s \geq t.
\]
If \( s < t \), the same notation \( X_{k \sim t}^{i, n}(s) \) represents the location of the corresponding particle’s mother at time \( s \). Denote by \([x]\) the integer part of \( x \). Let \( h \) be a positive constant which is small enough. For any exponential random variable \( Y \) with parameter \( \lambda \), we have
\[
\lim_{\Delta \to 0^+} h^{-1}\mathbb{P}(t \leq Y < t + \Delta) = \lambda.
\]

Now we are ready to introduce the branching particle systems. In colony 1, there exist independent branching and emigration, and there are also independent branching and immigration in colony 2. The branching mechanisms in two colonies are also independent. However, the emigration and immigration are interactive. The particles in colony 1 can move to colony 2 and the opposite direction is not allowed. During branching/emigration/immigration events, all the particles move according to independent Brownian motions.

\bullet (Measure-valued process in colony \( i \) with \( i = 1, 2 \)) Let \( \mu_{t}^{i, n} \) be the empirical distribution of particles living in colony \( i \), i.e., for any \( f \in C_0^2(\mathbb{R}) \), we have
\[
\left\langle \mu_{t}^{i, n}, f \right\rangle = \frac{1}{n} \sum_{k \sim t} f\left(X_{k \sim t}^{i, n}(t)\right).
\]
where the sum $k \sim t$ includes all those particles alive at $t$ in each colony.

- (Branching in colony $i$ with $i = 1, 2$) Let $\lambda_{n,i}$ be the branching rate of particles in colony $i$ with $\lambda_{n,i} \rightarrow \lambda_i$ as $n \rightarrow \infty$. At the time of a particle’s death, it gives birth to a random number $\xi^{n,i}$ of offspring with $\mathbb{E}\xi^{n,i} = 1 + \frac{\beta_{n,i}}{n}$, $\mathbb{V}\xi^{n,i} = \sigma^2_{n,i}$ satisfying $\beta_{n,i} \rightarrow \beta_i$ and $\sigma_{n,i} \rightarrow \sigma_i$ as $n \rightarrow \infty$. The offspring start moving from their mothers’ locations.

- (Emigration in colony 1) For a particle alive at time $t$ and position $x$ in colony 1, the conditional probability of emigration in the time interval $[t, t + h]$ is

$$1 - \exp \left\{ - \int_t^{t+h} \eta (x, \mu_s^{n,3}, \mu_s^{n,2}) \, ds \right\} \approx h \eta (x, \mu_t^{n,3}, \mu_t^{n,2}),$$

where $h$ is small enough and $\eta (x, \mu_s^{n,3}, \mu_s^{n,2}) \in C^+_b (\mathbb{R} \times M_F (\mathbb{R}^2)$ is the emigration intensity.

- (Immigration in colony 2) The emigration at time $t$ in colony 1 induces the immigration in colony 2, with total immigrants $\langle \mu_t^{n,1}, \eta (\cdot, \mu_t^{n,1}, \mu_t^{n,2}) \rangle$. Therefore, the probability of immigration to colony 2 in the time interval $[t, t + h]$ is

$$1 - \exp \left\{ - \int_t^{t+h} \langle \mu_s^{n,1}, \eta (\cdot, \mu_s^{n,1}, \mu_s^{n,2}) \rangle \, ds \right\} \approx \langle \mu_t^{n,1}, \eta (\cdot, \mu_t^{n,1}, \mu_t^{n,2}) \rangle h.$$ 

Moreover, we assume that the immigrants settle down in colony 2 according to a finite measure $\chi$ on $\mathbb{R}$.

### 3. Existence of solution to the martingale problem

In this section, we study the convergence of a sequence of measure-valued processes arising as the empirical measures of the proposed branching particle systems in previous section, where the limit is a weak solution to MP (1.3, 1.4).

Given any $t > 0$ and $i \in \{1, 2\}$, denote by $\tau_{k \sim t}^i$ the time of the first reproduction event in colony $i$ after $t$ induced by the $k$th living particle at $t$ in colony $i$. Let $\rho_{k \sim t}$ be the time of first migration after $t$ from colony 1 to colony 2 caused by the motion of the $k$th living particle at $t$ in colony 1. We denote that $\sum_{i=1}^{\rho_{k \sim t}} f_i = 0$ for any $f_i$. The total number of migration and reproduction during $[t, t + h]$ is at most once a.s. if $h$ is small enough. Without abuse of notation, these events $\{\rho_{k \sim t} < t + h\}, \{\tau_{k \sim t}^1 < t + h\}$, $k = 1, 2, \ldots$ are incompatible. It follows from the construction of the branching particle systems that

$$\langle \mu_{t+h}^{n,1}, f \rangle = \frac{1}{n} \sum_{k \sim t} \left( f (X_{k \sim t}^{n,1} (t + h)) + \Delta_{k \sim t}^{n,3} (f) - D_{k \sim t}^{n,3} (f) \right)$$

with

$$\Delta_{k \sim t}^{n,1} (f) = \left[ \sum_{i=1}^{\varepsilon_{k \sim t}^{n,1}} f (X_{k \sim t}^{n,1,i} (t + h)) - f (X_{k \sim t}^{n,1} (t + h)) \right] \mathbb{I}_{\{\tau_{k \sim t}^i < t + h\}}$$

and

$$D_{k \sim t}^{n,1} (f) = f (X_{k \sim t}^{n,1} (t + h)) \mathbb{I}_{\{\rho_{k \sim t} < t + h\}},$$

where $\varepsilon_{k \sim t}^{n,1}, k = 1, 2, \ldots$ are i.i.d. copies of $\varepsilon^{n,1}$; $X_{k \sim t}^{n,1,i} (t + h)$ has the same distribution as $X_{k \sim t}^{n,1} (t + h)$; $\mathbb{I}_{\{\cdot\}}$ is the indication function. Applying Itô’s formula, we
have
\[ f(X_{\bar{t}+h}^n(t+h)) = f(X_{\bar{t}+h}^n(t)) + \int_t^{t+h} f'(X_{\bar{t}+h}^n(s)) dB_{\bar{t}+h}^n(s) + \frac{1}{2} \int_t^{t+h} f''(X_{\bar{t}+h}^n(s)) ds. \]

Consequently,
\[ \langle \mu_{t+h}^n, f \rangle = \langle \mu_t^n, f \rangle + \frac{1}{n} \sum_{k=1}^{n} \int_t^{t+h} f'(X_{\bar{t}+h}^n(s)) dB_{\bar{t}+h}^n(s) + \frac{1}{n} \sum_{k=1}^{n} \left( \frac{\Delta_{\bar{t}+h}^n(f) - D_{\bar{t}+h}^{n,n}(f)}{h} \right). \]

Now we discretize the time interval according to step size \( h \). Assume that the random events (including branching, emigration and immigration events) only happen at the endpoints of each small subinterval. Then, for any \( jh \leq t < (j+1)h \) with \( j = 0, 1, 2, \ldots \), we have
\[
\langle \mu_t^n, f \rangle = \langle \mu_0^n, f \rangle + \langle \mu_{j\hbar}^n, f \rangle + \sum_{\ell = 0}^{j-1} \left( \langle \mu_{(\ell+1)\hbar}^n, f \rangle - \langle \mu_{\ell\hbar}^n, f \rangle \right)
\]
\[
= \langle \mu_0^n, f \rangle + \sum_{\ell = 0}^{j-1} \frac{1}{n} \sum_{k=1}^{n} \int_{(\ell+1)\hbar}^{(\ell+1)\hbar} f'(X_{(\ell+1)\hbar}^n(s)) dB_{(\ell+1)\hbar}^n(s) + \sum_{\ell = 0}^{j-1} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{2} \int_{(\ell+1)\hbar}^{(\ell+1)\hbar} f''(X_{(\ell+1)\hbar}^n(s)) ds.
\]

Let \( M_1^{n,f} \), \( M_2^{n,f} \) and \( M_3^{n,f} \) be martingales with
\[
M_1^{n,f}(t) = \sum_{\ell=0}^{j-1} \frac{1}{n} \sum_{k=1}^{n} \int_{(\ell+1)\hbar}^{(\ell+1)\hbar} f'(X_{(\ell+1)\hbar}^n(s)) dB_{(\ell+1)\hbar}^n(s),
\]
(3.2)

\[
M_2^{n,f}(t) = \sum_{\ell=0}^{j-1} \frac{1}{n} \sum_{k=1}^{n} \left\{ \Delta_{(\ell+1)\hbar}^n(f) - \mathbb{E}[\Delta_{(\ell+1)\hbar}^n(f) | F_{(\ell+1)\hbar}] \right\}
\]
and
(3.3)

\[
M_3^{n,f}(t) = \sum_{\ell=0}^{j-1} \frac{1}{n} \sum_{k=1}^{n} \left\{ D_{(\ell+1)\hbar}^{n,n}(f) - \mathbb{E}[D_{(\ell+1)\hbar}^{n,n}(f) | F_{(\ell+1)\hbar}] \right\}.
\]

Moreover, \( A^{n,f} \) is defined as
\[
A^{n,f}(t) = \sum_{\ell=0}^{[t/h]} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{2} \int_{(\ell+1)\hbar}^{(\ell+1)\hbar} f''(X_{(\ell+1)\hbar}^n(s)) ds
\]
\[
+ \sum_{\ell=0}^{[t/h]-1} \frac{1}{n} \sum_{k=1}^{n} \mathbb{E}[\Delta_{(\ell+1)\hbar}^n(f) - D_{(\ell+1)\hbar}^{n,n}(f) | F_{(\ell+1)\hbar}]
\]
\[
= \sum_{\ell=0}^{[t/h]} \frac{1}{2} \int_{(\ell+1)\hbar}^{(\ell+1)\hbar} \langle \mu_{\ell\hbar}^{n,1}, f'' \rangle ds + \sum_{\ell=0}^{[t/h]-1} \frac{\beta_{n,1}}{n} \chi_{n,1} \langle \mu_{\ell\hbar}^{n,1}, f \rangle h
\]
\[
- \sum_{\ell=0}^{[t/h]-1} \langle \mu_{\ell\hbar}^{n,1}, \eta (\cdot; \mu_{\ell\hbar}^{n,1}, \mu_{\ell\hbar}^{n,2}, f) \rangle h + O(h).
\]
One can see that

\[ \langle \mu^{n,1}_t, f \rangle = \langle \mu^{n,1}_0, f \rangle + M^{n,f}_1(t) + M^{n,f}_2(t) - M^{n,f}_3(t) + A^{n,f}(t). \]

Carrying out similar steps as above in colony 2, we get

\[ \langle \mu^{n,2}_t, g \rangle = \langle \mu^{n,2}_0, g \rangle + \hat{M}^{n,g}_1(t) + \hat{M}^{n,g}_2(t) + \hat{M}^{n,g}_3(t) + \hat{A}^{n,g}(t), \]

where

\[ \hat{M}^{n,g}_1(t) = \sum_{\ell=0}^{[t/h]} \frac{1}{n} \sum_{k_{-\ell}h} \int_{\ell h}^{(\ell+1)h\wedge t} g' \left( X^{n,2}_{k-\ell h}(s) \right) dB^2_{k-\ell h}(s) \]

and

\[ \hat{M}^{n,g}_2(t) = \sum_{\ell=0}^{[t/h]-1} \frac{1}{n} \sum_{k_{-\ell}h} \left\{ \Delta^{n,2}_{k-\ell h}(g) - \mathbb{E} \left[ \Delta^{n,2}_{k-\ell h}(g) | \mathcal{F}_{\ell h} \right] \right\} \]

are martingales with \( \Delta^{n,2}_{k-\ell h}(g) \) defined similar as \((3.1)\) and \( X^{n,2}_k(t+h) \) having the same distribution as \( X^{n,2}_k(t) \). Moreover,

\[ \hat{M}^{n,g}_3(t) = \sum_{\ell=0}^{[t/h]-1} \frac{1}{n} \sum_{k_{-\ell}h} \left\{ D^{n,2}_{k-\ell h}(g) - \mathbb{E} \left[ D^{n,2}_{k-\ell h}(g) | \mathcal{F}_{\ell h} \right] \right\} \]

with

\[ D^{n,2}_{k-\ell h}(g) = \langle \chi, g \rangle \mathbb{I}_{\{\rho_{k-\ell h} \leq (t+1)h\}} \]

is also a martingale and

\[ \hat{A}^{n,g}(t) = \sum_{\ell=0}^{[t/h]} \frac{1}{n} \sum_{k_{-\ell}h} \int_{\ell h}^{(\ell+1)h\wedge t} g'' \left( X^{n,2}_{k-\ell h}(s) \right) ds \]

\[ + \sum_{\ell=0}^{[t/h]-1} \frac{1}{n} \sum_{k_{-\ell}h} \mathbb{E} \left[ \Delta^{n,2}_{k-\ell h}(g) - D^{n,2}_{k-\ell h}(g) | \mathcal{F}_{\ell h} \right] \]

\[ = \sum_{\ell=0}^{[t/h]} \frac{1}{2} \int_{\ell h}^{(\ell+1)h\wedge t} \langle \mu^{n,2}_s, g'' \rangle ds + \sum_{\ell=0}^{[t/h]-1} \frac{1}{n} \beta^{n,2}_{\ell} \lambda^{n,2} \langle \mu^{n,2}_0, g \rangle h \]

\[ + \sum_{\ell=0}^{[t/h]-1} \langle \chi, g \rangle \langle \mu^{n,1}_{h}, \eta(\cdot, \mu^{n,2}_{h}, \mu^{n,2}_{h}) \rangle h + O(h). \]

From the construction of our model, one can check that \( M^{n,f}_1, M^{n,f}_2 \) and \( M^{n,f}_3 \) (resp. \( \hat{M}^{n,g}_1, \hat{M}^{n,g}_2 \) and \( \hat{M}^{n,g}_3 \)) are three mutually uncorrelated martingales. The interaction may only exist between \( M^{n,f}_3 \) and \( \hat{M}^{n,g}_3 \). Subsequently, we calculate the quadratic variation and covariation processes for these martingales. It follows from Itô’s integral that

\[ \langle M^{n,f}_1 \rangle_t = \sum_{\ell=0}^{[t/h]} \frac{1}{n^2} \sum_{k_{-\ell}h} \int_{\ell h}^{(\ell+1)h\wedge t} \left| f'(X^{n,1}_{k-\ell h}(s)) \right|^2 ds \]

\[ = \frac{1}{n} \int_0^t \langle \mu^{n,1}_s, f'' \rangle ds \to 0, \quad \text{as } n \to \infty. \]
Applying Lemma 8.12 in Xiong (2008), we obtain the quadratic variations of $M_2^{n,f}$ and $M_3^{n,f}$ as follows:

\[
\langle M_2^{n,f} \rangle_t = \sum_{\ell=0}^{[t/h]-1} \frac{1}{n^2} \mathbb{E} \left\{ \left[ \sum_{k=\ell h} \left( \Delta_{k-\ell h}^{n,1} f - \mathbb{E}(\Delta_{k-\ell h}^{n,1} f | \mathcal{F}_{\ell h}) \right) \right]^2 | \mathcal{F}_{\ell h} \right\} 
= \sum_{\ell=0}^{[t/h]-1} \frac{1}{n^2} \sum_{k=\ell h} \left[ f^2 \left( X_{k-\ell h}^{n,3} (\ell h) \right) \sigma_{n,1}^2 \lambda_{n,1} h + O(h^2) \right] 
= \frac{1}{n} \left( \sum_{\ell=0}^{[t/h]-1} \left( \mu_{\ell h}^{n,1} f^2 (\cdot) \sigma_{n,1}^2 \lambda_{n,1} h + O(h) \right) \right) 
\]

and

\[
\langle M_3^{n,f} \rangle_t = \sum_{\ell=0}^{[t/h]-1} \frac{1}{n^2} \mathbb{E} \left\{ \left[ \sum_{k=\ell h} \left( D_{k-\ell h}^{n,1} f - \mathbb{E}(D_{k-\ell h}^{n,1} f | \mathcal{F}_{\ell h}) \right) \right]^2 | \mathcal{F}_{\ell h} \right\} 
= \sum_{\ell=0}^{[t/h]-1} \frac{1}{n^2} \sum_{k=\ell h} \left[ f^2 (X_{k-\ell h}^{n,3} (\ell h)) \lambda (X_{k-\ell h}^{n,1} (\ell h), \mu_{\ell h}^{n,1}, \mu_{\ell h}^{n,2}) h + O(h^2) \right] 
= \frac{1}{n} \left( \sum_{\ell=0}^{[t/h]-1} \left( \mu_{\ell h}^{n,1} f^2 (\cdot) \lambda (\cdot, \mu_{\ell h}^{n,1}, \mu_{\ell h}^{n,2}) h + O(h) \right) \right) 
\]

The quadratic variations of $\hat{M}_1^{n,g}$, $\hat{M}_2^{n,g}$ and $\hat{M}_3^{n,g}$ can be similarly derived as follows:

\[
\langle \hat{M}_1^{n,g} \rangle_t = \frac{[t/h]}{n^2} \sum_{k=\ell h} \int_{\ell h}^{(\ell+1)h} \left| g^\prime \left( X_{k-\ell h}^{n,2} (s) \right) \right| ds 
= \frac{1}{n} \int_0^t \langle \mu_s^{n,2}, |g^\prime|^2 \rangle ds \rightarrow 0, \quad \text{as } n \rightarrow \infty, 
\]

\[
\langle \hat{M}_2^{n,g} \rangle_t = \sum_{\ell=0}^{[t/h]-1} \frac{1}{n^2} \sum_{k=\ell h} \left( g^2 \left( X_{k-\ell h}^{n,2} (\ell h) \right) \sigma_{n,2}^2 \lambda_{n,2} h + O(h^2) \right) 
= \frac{1}{n} \left( \sum_{\ell=0}^{[t/h]-1} \left( \mu_{\ell h}^{n,2} g^2 \sigma_{n,2}^2 \lambda_{n,2} h + O(h) \right) \right) 
\]

and

\[
\langle \hat{M}_3^{n,g} \rangle_t = \frac{1}{n} \sum_{\ell=0}^{[t/h]-1} \langle X; g \rangle^2 \langle \mu_{\ell h}^{n,1}, \eta (\cdot, \mu_{\ell h}^{n,1}, \mu_{\ell h}^{n,2}) \rangle h + O(h). 
\]

**Proposition 3.1.** The covariation process of $M_3^{n,f}$ and $\hat{M}_3^{n,g}$ is

\[
\langle M_3^{n,f}, \hat{M}_3^{n,g} \rangle_t = \sum_{\ell=0}^{[t/h]-1} \frac{1}{n} \langle X; g \rangle \langle \mu_{\ell h}^{n,1}, f (\cdot) \eta (\cdot, \mu_{\ell h}^{n,1}, \mu_{\ell h}^{n,2}) \rangle h 
- \frac{[t/h]-1}{n} \int_0^t \langle X; g \rangle \langle \mu_{\ell h}^{n,1}, f (\cdot) \eta (\cdot, \mu_{\ell h}^{n,1}, \mu_{\ell h}^{n,2}) \rangle \langle \mu_{\ell h}^{n,1}, \eta (\cdot, \mu_{\ell h}^{n,1}, \mu_{\ell h}^{n,2}) \rangle ds + \frac{O(h)}{n}. 
\]
Proof. This result shows the correlation between the processes \( (\mu_t^{n,3})_{t \geq 0} \) and \( (\mu_t^{n,2})_{t \geq 0} \). For simplicity of notation, we set

\[
I_1(\ell, k) = \frac{1}{n} \left\{ \mathbb{D}_{k\sim \ell h}^{n,1}(f) - \mathbb{E}[\mathbb{D}_{k\sim \ell h}^{n,1}(f) \mid \mathcal{F}_{th}] \right\}
\]

and

\[
I_2(\ell, k) = \frac{1}{n} \left\{ \mathbb{D}_{k\sim \ell h}^{n,2}(g) - \mathbb{E}[\mathbb{D}_{k\sim \ell h}^{n,2}(g) \mid \mathcal{F}_{th}] \right\}.
\]

It follows from Equations (3.3) and (3.8) that

\[
\mathbb{M}_3^{n,f}(t) = \sum_{\ell=0}^{[t/h]-1} I_1(\ell) = \sum_{\ell=0}^{[t/h]-1} I_1(\ell, k)
\]

and

\[
\hat{\mathbb{M}}_3^{n,g}(t) = \sum_{\ell=0}^{[t/h]-1} I_2(\ell) = \sum_{\ell=0}^{[t/h]-1} I_2(\ell, k).
\]

Since \( \mathbb{E}I_1(\ell) = \mathbb{E}I_2(\ell) = 0 \), we have

\[
\langle \mathbb{M}_3^{n,f}, \hat{\mathbb{M}}_3^{n,g}\rangle_t = \sum_{\ell=0}^{[t/h]-1} \mathbb{E} \left[ \sum_{k\sim \ell h} I_1(\ell, k) I_2(\ell, k) \mid \mathcal{F}_{th} \right] + \sum_{\ell=0}^{[t/h]-1} \mathbb{E} \left[ \sum_{k_1 \sim \ell h, k_2 \sim \ell h, k_1 \neq k_2} I_1(\ell, k_1) I_2(\ell, k_2) \mid \mathcal{F}_{th} \right].
\]

Note that

\[
\mathbb{E} \left[ I_1(\ell, k) I_2(\ell, k) \mid \mathcal{F}_{th} \right] = \left( \frac{\chi}{n^2} f(X_{k \sim \ell h}^{n,1}(\ell h)) \eta(X_{k \sim \ell h}^{n,2}(\ell h), \mu_{\ell h}^{n,1}, \mu_{\ell h}^{n,2}) h \right) + O(h^2/n^2).
\]

Similarly, for any \( k_1 \neq k_2 \) one can see that

\[
\mathbb{E} \left[ I_1(\ell, k_1) I_2(\ell, k_2) \mid \mathcal{F}_{th} \right] = \left( \frac{\chi}{n^2} f(X_{k_1 \sim \ell h}^{n,1}(\ell h)) \eta(X_{k_2 \sim \ell h}^{n,1}(\ell h), \mu_{\ell h}^{n,1}, \mu_{\ell h}^{n,2}) h \right) + O(h^3/n^2).
\]

Therefore, we proceed to have

\[
\langle \mathbb{M}_3^{n,f}, \hat{\mathbb{M}}_3^{n,g}\rangle_t = \sum_{\ell=0}^{[t/h]-1} \left( \frac{1}{n^2} \langle \chi, g \rangle f(X_{k \sim \ell h}^{n,1}(\ell h)) \eta(X_{k \sim \ell h}^{n,1}(\ell h), \mu_{\ell h}^{n,1}, \mu_{\ell h}^{n,2}) h + \frac{1}{n^2} O(h^2) \right).
\]
\[-\sum_{t=0}^{[t/h]-1} \sum_{k_{1,2} \sim \tilde{h}} \frac{1}{n^2} \langle \chi, g \rangle f \left( X_{k_1,2}^{n,1} (\hat{h}) \right) h^2 \prod_{i=1}^{2} \eta \left( X_{k_1,2}^{n,1} (\hat{h}) , \mu_{\hat{t}}^{n,1} , \mu_{\hat{t}}^{n,2} \right) \]

\[= \sum_{t=0}^{[t/h]-1} \frac{1}{n} \langle \chi, g \rangle \langle \mu_{\hat{t}}^{n,1} : f (\cdot) \eta \left( \cdot , \mu_{\hat{t}}^{n,1} , \mu_{\hat{t}}^{n,2} \right) \rangle h \]

\[= - \sum_{t=0}^{[t/h]-1} \langle \chi, g \rangle \langle \mu_{\hat{t}}^{n,1} , f (\cdot) \eta \left( \cdot , \mu_{\hat{t}}^{n,1} , \mu_{\hat{t}}^{n,2} \right) \rangle \langle \mu_{\hat{t}}^{n,1} , \eta \left( \cdot , \mu_{\hat{t}}^{n,1} , \mu_{\hat{t}}^{n,2} \right) \rangle h^2 + \frac{O(h)}{n} . \]

The result follows. \hfill \square

In fact, the tightness of \( \{ (\mu_{n,1}^{n,1} , \mu_{n,2}^{n,2}) : n \geq 1 \} \) in \( D(\mathbb{R}^+, M_f (\mathbb{R})^2) \) is equivalent to the tightness of \( \{ (\mu_{n,1}^{n,1} , f ) , (\mu_{n,2}^{n,2} , g ) : n \geq 1 \} \) in \( D(\mathbb{R}^+, \mathbb{R}^2) \) for any \( f, g \in C_b^\infty (\mathbb{R}) \). In the following we will make some estimations (see Lemmas 3.2, 3.4) and then prove the tightness of the empirical measure for the branching particle systems.

**Lemma 3.2.** Assume that \( \sup_n \mathbb{E} \left[ (\mu_{n,1}^{0,1} , 1)^{2p} + (\mu_{n,2}^{0,1} , 1)^{2p} \right] < \infty \) for any \( p \geq 1 \).

There exists a constant \( K = K(p, T) \) such that

\[ \sup_n \mathbb{E} \left[ \sup_{t \leq T} \left( (\mu_{n,1}^{0,1} , 1)^{2p} + (\mu_{n,2}^{0,1} , 1)^{2p} \right) \right] < K. \]

**Proof.** Replacing \( f \) and \( g \) with 1 in Equations (3.3) and (3.6), we have

\[ \langle \mu_{1}^{n,1} , 1 \rangle = \langle \mu_{0}^{n,1} , 1 \rangle + M_{1}^{n,1} (t) + M_{2}^{n,1} (t) - M_{3}^{n,1} (t) + A^{n,1} (t) , \]

and

\[ \langle \mu_{1}^{n,2} , 1 \rangle = \langle \mu_{0}^{n,2} , 1 \rangle + \tilde{M}_{1}^{n,1} (t) + \tilde{M}_{2}^{n,1} (t) + \tilde{M}_{3}^{n,1} (t) + \tilde{A}^{n,1} (t) . \]

Consequently, by (3.14), (3.10) and (3.12) we have

\[ \mathbb{E} \langle \mu_{1}^{n,1} , 1 \rangle^{2p} \leq K + K \mathbb{E} A_{1}^{n,1} (t)^{2p} + K \mathbb{E} \langle M_{1}^{n,1} , 1 \rangle^{p} + K \mathbb{E} \langle M_{2}^{n,1} , 1 \rangle^{p} + K \mathbb{E} \langle M_{3}^{n,1} , 1 \rangle^{p} \]

\[ \leq K + K \int_{0}^{t} \mathbb{E} \langle \mu_{1}^{n,1} , 1 \rangle^{2p} ds \]

for any \( t \in [0, T] \). Moreover, by (3.3), (3.4) and (3.15) we have

\[ \mathbb{E} \langle \mu_{1}^{n,2} , 1 \rangle^{2p} \leq K + K \mathbb{E} \tilde{A}_{1}^{n,1} (t)^{2p} + K \mathbb{E} \langle \tilde{M}_{1}^{n,1} , 1 \rangle^{p} + K \mathbb{E} \langle \tilde{M}_{2}^{n,1} , 1 \rangle^{p} + K \mathbb{E} \langle \tilde{M}_{3}^{n,1} , 1 \rangle^{p} \]

\[ \leq K + K \int_{0}^{t} \mathbb{E} \langle \mu_{1}^{n,1} , 1 \rangle^{2p} + \langle \mu_{1}^{n,2} , 1 \rangle^{2p} ds \]

for any \( t \in [0, T] \). Combining the above, by Gronwall’s inequality we have

\[ (3.16) \]

\[ \mathbb{E} \left( \langle \mu_{1}^{n,1} , 1 \rangle^{2p} + \langle \mu_{1}^{n,2} , 1 \rangle^{2p} \right) \leq Ke^{Kt} . \]

Moreover, it follows from Doob’s inequality that

\[ \mathbb{E} \left( \sup_{t \leq T} \langle \mu_{1}^{n,1} , 1 \rangle^{2p} \right) \leq K \mathbb{E} \langle \mu_{0}^{n,1} , 1 \rangle^{2p} + K \mathbb{E} \langle M_{1}^{n,1} (T) \rangle^{2p} + K \mathbb{E} \langle M_{2}^{n,1} (T) \rangle^{2p} \]

\[ + K \mathbb{E} \langle M_{3}^{n,1} (T) \rangle^{2p} + K \mathbb{E} \sup_{t \leq T} \langle A_{1}^{n,1} (t) \rangle^{2p} ; \]

\[ \mathbb{E} \left( \sup_{t \leq T} \langle \mu_{1}^{n,2} , 1 \rangle^{2p} \right) \leq K \mathbb{E} \langle \mu_{0}^{n,2} , 1 \rangle^{2p} + K \mathbb{E} \langle \tilde{M}_{1}^{n,1} (T) \rangle^{2p} + K \mathbb{E} \langle \tilde{M}_{2}^{n,1} (T) \rangle^{2p} \]

\[ + K \mathbb{E} \langle \tilde{M}_{3}^{n,1} (T) \rangle^{2p} + K \mathbb{E} \sup_{t \leq T} \langle \tilde{A}_{1}^{n,1} (t) \rangle^{2p} . \]
By Equations (3.4), (3.10)-(3.12), we have
\[
\mathbb{E}\left(\sup_{t \leq T} \langle \mu_t^{n,1}, 1 \rangle^{2p}\right) \leq K \mathbb{E}\langle \mu_0^{n,1}, 1 \rangle^{2p} + K \int_0^T \mathbb{E}\langle \mu_s^{n,3}, 1 \rangle^{2p} ds + K \int_0^T \mathbb{E}\langle \mu_s^{n,1}, 1 \rangle^{2p} ds
\]
\[
\leq K_T + K \int_0^T \mathbb{E}\langle \mu_s^{n,3}, 1 \rangle^{2p} ds,
\]
where the last inequality follows from Lemma 3.2. Similarly estimations can be carried out by Equations (3.4), (3.10)-(3.12), we have
\[
\mathbb{E}\left(\sup_{t \leq T} \langle \mu_t^{n,2}, 1 \rangle^{2p}\right) \leq K_T + K \int_0^T \mathbb{E}\langle \mu_s^{n,2}, 1 \rangle^{2p} ds + K \int_0^T \mathbb{E}\langle \mu_s^{n,1}, 1 \rangle^{2p} ds.
\]
By the above equations we have
\[
\mathbb{E}\sup_{t \leq T} \left(\langle \mu_t^{n,3}, 1 \rangle^{2p} + \langle \mu_t^{n,2}, 1 \rangle^{2p}\right) \leq K_T + K \int_0^T \mathbb{E}\left(\langle \mu_s^{n,1}, 1 \rangle^{2p} + \langle \mu_s^{n,2}, 1 \rangle^{2p}\right) ds.
\]
The result follows from Gronwall’s inequality.

Lemma 3.3. For any \(t \geq s \geq 0, p \geq 1\) and \(f, g \in C_b^2(\mathbb{R})\), we have
\[
\mathbb{E}\left[\left|\langle M_{2,f}^{n,f} \rangle_t - \langle M_{2,f}^{n,f} \rangle_s \right|^p + \left|\langle \tilde{M}_{2,g}^{n,g} \rangle_t - \langle \tilde{M}_{2,g}^{n,g} \rangle_s \right|^p\right] \leq K|t - s|^p.
\]

Proof. It follows from Equation (3.11) and Hölder inequality that
\[
\mathbb{E}\left|\langle M_{2,f}^{n,f} \rangle_t - \langle M_{2,f}^{n,f} \rangle_s \right|^p \leq \frac{1}{n^p} \mathbb{E}\left(\sum_{t=[s/h]}^{[t/h]-1} \left\langle \mu_{t,h}^{n,1}, f^2 \right\rangle \sigma_{n,1}^2 \lambda_{n,1} h + O(h) \right)^p
\]
\[
\leq K \mathbb{E}\left[\int_s^t \langle \mu_r^{n,1}, 1 \rangle dr\right]^p \leq K|t - s|^p.
\]
The last inequalities comes from Lemma 3.2. Similarly estimations can be carried out for \(\langle M_{2,g}^{n,g} \rangle_t\).

Lemma 3.4. For any \(t \geq s \geq 0, p \geq 1\) and \(f, g \in C_b^2(\mathbb{R})\), we have
\[
\mathbb{E}|A^{n,f}(t) - A^{n,f}(s)|^{2p} \leq K|t - s|^{2p}
\]
and
\[
\mathbb{E}|\tilde{A}^{n,g}(t) - \tilde{A}^{n,g}(s)|^{2p} \leq K|t - s|^{2p}.
\]

Proof. It follows from Equation (3.4) that
\[
\mathbb{E}|A^{n,f}(t) - A^{n,f}(s)|^{2p}
\]
\[
= \mathbb{E}\left[\sum_{t=[s/h]}^{[t/h]-1} \frac{1}{2} \int_{t/h}^{(t+1)/h\wedge t} \left\langle \mu_{r,h}^{n,1}, f'' \right\rangle dr + \sum_{t=[s/h]}^{[t/h]-1} \frac{\beta_{n,2}}{n} \lambda_{n,1} \left\langle \mu_{r,h}^{n,1}, f \right\rangle h \right.
\]
\[
\left. - \sum_{t=[s/h]}^{[t/h]-1} \left\langle \mu_{r,h}^{n,1}, f \right\rangle \eta_\|r_{\|h} \|_{\|h}^{n,2} \left\rangle h + O(h) \right]\right)^{2p}
\]
\[
\leq K \mathbb{E}\left[\int_s^t \langle \mu_r^{n,2}, 1 \rangle dr\right]^{2p} \leq K|t - s|^{2p}.
\]
The last inequality follows by Lemma 3.2. Similar estimation can be carried out for \(\tilde{A}^{n,g} (\cdot)\).
Theorem 3.5. Let \( h := h_n \rightarrow 0 \) as \( n \rightarrow \infty \). The sequence \( \{ (\mu_n^{1,1}, \mu_n^{2,2}) : n \geq 1 \} \) is tight in \( D(\mathbb{R}_+, M_F(\mathbb{R})^2) \). Furthermore, the limit \( (\mu^1, \mu^2) \) lies in \( C(\mathbb{R}_+, M_F(\mathbb{R})^2) \) and is a solution to MP (1.3, 1.4) with \( b_1 = \beta_1 \lambda_1, \ b_2 = \beta_2 \lambda_2, \gamma_1 = \sigma_1^2 \lambda_1 \) and \( \gamma_2 = \sigma_2^2 \lambda_2 \).

Proof. For any \( f, g \in C^2_b(\mathbb{R}) \) and \( t \geq 0 \) one can see that \( \langle M_{3,n}^{1,1} \rangle_t \rightarrow 0 \) and \( \langle \hat{M}_{3,n}^{1,1} \rangle_t \rightarrow 0 \) as \( n \rightarrow \infty \) by (3.12) and (3.15). For any \( t \geq s \geq 0 \) and \( p > 1 \), by Hölder inequality, Lemma 3.3 and Lemma 3.3 we have

\[
\mathbb{E}|\langle \mu_t, f \rangle - \langle \mu_s, f \rangle|^2 \leq K \mathbb{E}|A_t(f) - A_s(f)|^p + K \mathbb{E}|\langle M_t^{1,1} \rangle_t - \langle M_t^{1,1} \rangle_s|^p \leq K |t - s|^p + K |t - s|^2.
\]

The tightness of \( \{ (\mu_t^{1,1}, f) : 0 \leq t \leq T \}, \{ A_t(f) : 0 \leq t \leq T \}, \{ (\mu_t^{1,1})_t : 0 \leq t \leq T \} \) and \( \{ (\mu_t^{1,2}) : 0 \leq t \leq T \}, \{ A_t(f) : 0 \leq t \leq T \}, \{ (\mu_t^{1,2})_t : 0 \leq t \leq T \} \) on \( C([0, T], \mathbb{R}) \) follows from Theorem VI.4.1 in Jacod and Shiryaev (2003), which implies that \( \{ (\mu_t^{1,1}, \mu_t^{1,2}) : 0 \leq t \leq T \} \) is tight on \( C([0, T], M_F(\mathbb{R})^2) \). Hence there is a subsequence \( \{ (\mu_t^{1,1}, \mu_t^{1,2}) : 0 \leq t \leq T \} \) converges in law. Suppose that \( (\mu^1, \mu^2) \) is the weak limit of \( (\mu_{n,k}^{1,1}, \mu_{n,k}^{1,2}) \) as \( k \rightarrow \infty \). For any \( f, g \in C^2_b(\mathbb{R}) \), we have

\[
\left( \langle \mu_{n,k}^{1,1}, f \rangle, A_{n,k}^{1,1}, \langle \mu_{n,k}^{1,2}, f \rangle, A_{n,k}^{1,2}, \langle \mu_{n,k}^{2,1}, g \rangle, A_{n,k}^{2,1}, \langle \mu_{n,k}^{2,2}, g \rangle, A_{n,k}^{2,2}, \langle \hat{M}_{n,k}^{1,1} \rangle \right) \rightarrow \left( \langle \mu^1, f \rangle, A^1, \langle \mu^2, f \rangle, A^2, \langle \mu^1, g \rangle, A^2, \langle \mu^2, g \rangle, A^1, \langle \hat{M}_2 \rangle \right)
\]

weakly as \( k \rightarrow \infty \). Furthermore, \( \langle M_{2,n,k}^{1,1} \rangle_t \rightarrow \langle M_2^{1,1} \rangle_t, \langle M_{2,n,k}^{1,2} \rangle_t \rightarrow \langle M_2^{1,2} \rangle_t \) as \( k \rightarrow \infty \), and for \( t \leq T \),

\[
\langle M_{2,n,k}^{1,1} \rangle_t \leq K \left( 1 + \sup_{0 \leq t \leq T} \langle \mu^1_t, 1 \rangle \right), \quad \langle M_{2,n,k}^{1,2} \rangle_t \leq K \left( 1 + \sup_{0 \leq t \leq T} \langle \mu^2_t, 1 \rangle \right).
\]

By Lemma 3.2 we can pass to the limit to conclude that \( M_2^1(t) \) and \( \hat{M}_2^1(t) \) are martingales. Let \( b_1 = \beta_1 \lambda_1, \ b_2 = \beta_2 \lambda_2, \gamma_1 = \sigma_1^2 \lambda_1 \) and \( \gamma_2 = \sigma_2^2 \lambda_2 \). One can see that

\[
\langle M_2^1 \rangle_t = \gamma_1 \int_0^t \langle \mu^1_s, f^2 \rangle ds \quad \text{and} \quad \langle \hat{M}_2^1 \rangle_t = \gamma_2 \int_0^t \langle \mu^2_s, g^2 \rangle ds
\]

by (3.11) and (3.14). It implies that \( (\mu^1, \mu^2) \) is a solution to MP (1.3, 1.4). The result follows.

\[
\square
\]

4. Uniqueness of Solution to the Martingale Problem

In this section, we first derive the SPDEs satisfied by the distribution-function-valued processes of the mutually interacting superprocesses with migration, and then establish its equivalence with MP (1.3, 1.4). Moreover, the pathwise uniqueness of the SPDEs is proved by an extended Yamada-Watanabe argument.
4.1. A related system of SPDEs. For any \( y \in \mathbb{R} \), we write

\[
(4.1) \quad u^1_t(y) = \mu^1_t((\infty, y]) \quad \text{and} \quad u^2_t(y) = \mu^2_t((\infty, y])
\]
as the distribution-function-valued processes for the mutually interacting superprocesses with migration \((\mu^1_t, \mu^2_t)_{t \geq 0}\). For any \( x, y \in \mathbb{R} \cup \{\pm \infty\} \), \( \nu_1, \nu_2 \in M_F(\mathbb{R}) \) and \( \eta(\cdot, \nu_1, \nu_2) \in C^+_b(\mathbb{R} \times M_F(\mathbb{R})^2) \), denote by \( \xi(y, \nu_1, \nu_2) = \int_{-\infty}^y \eta(x, \nu_1, \nu_2) \nu_1(dx) \).

Let \( W^1(ds \, da) \) be independent space-time white noise random measures on \( \mathbb{R}_+ \times \mathbb{R} \) with intensity \( ds \, da \) and \( i \in \{1, 2\} \). We consider the following SPDEs: for any \( t \in \mathbb{R}_+ \) and \( y \in \mathbb{R} \),

\[
(4.2) \quad \begin{dcases}
       u^1_t(y) = u^1_0(y) + \sqrt{\gamma_1} \int_0^t \int_0^y f(y) W^1(ds \, da)
       + \int_0^t \left[ \frac{\partial}{\partial y} u^1_s(y) + b_1(u^1_s(y)) \right] ds,
       \\
u^2_t(y) = u^2_0(y) + \sqrt{\gamma_2} \int_0^t \int_0^y f(y) W^2(ds \, da)
       + \int_0^t \left[ \frac{\partial}{\partial y} u^2_s(y) + b_2(u^2_s(y)) \right] ds,
   \end{dcases}
\]

where \( \chi \) is the distribution function of \( \chi \), i.e., \( \chi(y) = \chi((\infty, y]) \).

**Definition 4.1.** The SPDEs (4.2) have a weak solution if there exists a \( C_{b,m}(\mathbb{R})^2 \)-valued process \((u^1_t, u^2_t)_{t \geq 0}\) on a stochastic basis such that for any \( f, g \in C^0(\mathbb{R}) \) and \( t \geq 0 \), the following holds:

\[
(4.3) \quad \begin{dcases}
       \langle u^1_t, f \rangle_1 = \langle u^1_0, f \rangle_1 + \sqrt{\gamma_1} \int_0^t \int_0^\infty f(y) \Pi_{\{a \leq u^2(s,y)\}} dy W^1(ds \, da)
       + \int_0^t \left[ \frac{\partial}{\partial y} u^1_s(y) \right] ds,
       \\
       \langle u^2_t, g \rangle_1 = \langle u^2_0, g \rangle_1 + \sqrt{\gamma_2} \int_0^t \int_0^\infty g(y) \Pi_{\{a \leq u^2(s,y)\}} dy W^2(ds \, da)
       + \int_0^t \left[ \frac{\partial}{\partial y} u^2_s(y) \right] ds.
   \end{dcases}
\]

**Proposition 4.2.** Suppose that \((u^1_t, u^2_t)_{t \geq 0}\) is a solution to the system of SPDEs (4.2). Then the corresponding measure-valued process \((\mu^1_t, \mu^2_t)_{t \geq 0}\) is a solution to MP (1.3, 1.4).

**Proof.** For a non-decreasing continuous function \( h \) on \( \mathbb{R} \), the inverse function is defined as \( h^{-1}(a) = \inf\{x : h(x) > a\} \). Then for any \( f, g \in C^0(\mathbb{R}) \), we have

\[
\langle \mu^1_t, f \rangle = -\langle u^1_t, f' \rangle_1
   = -\langle u^1_0, f' \rangle_1 - \sqrt{\gamma_1} \int_0^t \int_0^\infty f'(y) \Pi_{\{a \leq u^2(s,y)\}} dy W^1(ds \, da)
   - \int_0^t \left[ \frac{\partial}{\partial y} u^1_s(y) + b_1(u^1_s(y)) \right] ds
   = \langle \mu^1_0, f \rangle + \sqrt{\gamma_1} \int_0^t \int_0^\infty f \left( (u^2_s)^{-1}(a) \right) W^1(ds \, da)
   + \int_0^t \left[ \frac{\partial}{\partial y} u^1_s(y) + b_1(u^1_s, f) - \langle \mu^1_s, \eta(\cdot, \mu^1_s, \mu^2_s) f \rangle \right] ds.
\]
and
\[ \langle \mu_t^2, g \rangle = - \langle u_t^2, g' \rangle_1 \]
\[ = - \langle u_0^2, g' \rangle_1 - \sqrt{2} \int_0^t \int_0^\infty \int_{\mathbb{R}} g' (y) \mathbb{I}_{a \leq u_s^2 (y)} dy W^2 (ds \, da) \]
\[ - \int_0^t \left( \left( \frac{\Delta}{2} u_s^2, g' \right)_1 + b_2 (u_s^2, g' )_1 + \langle \chi, g' \rangle_1 (\chi + \mu_s^1, \mu_s^2) \right) ds \]
\[ = \langle u_0^2, g \rangle + \sqrt{2} \int_0^t \int_0^\infty g (u_s^2)^{-1} (a) W^2 (ds \, da) \]
\[ + \int_0^t \left( \langle \mu_s^2, \frac{1}{2} g'' \rangle + b_2 (\mu_s^2, g) + \langle \chi, g \rangle (\chi + \mu_s^1, \mu_s^2) \right) ds . \]
Thus, \( M_t^f \) and \( \tilde{M}_t^g \) are martingales with quadratic variation processes
\[ \langle M^f \rangle_t = \gamma_1 \int_0^t \int_0^\infty f^2 (u_s^2)^{-1} (a) ds \, da \]
\[ = \gamma_1 \int_0^t \int_{\mathbb{R}} f^2 (y) ds (u_s^2 (y)) = \gamma_2 \int_0^t \langle \mu_s^1, f^2 \rangle ds \]
and
\[ \langle \tilde{M}^g \rangle_t = \gamma_2 \int_0^t \int_0^\infty g^2 (u_s^2)^{-1} (a) ds \, da \]
\[ = \gamma_2 \int_0^t \int_{\mathbb{R}} g^2 (y) ds (u_s^2 (y)) = \gamma_2 \int_0^t \langle \mu_s^2, g^2 \rangle ds . \]
The independence of \( W^1 \) and \( W^2 \) leads to \( \langle M^f, \tilde{M}^g \rangle_t = 0 \). Therefore, \((\mu^1_t, \mu^2_t)_{t \geq 0}\) is a solution to MP \((1.3, 1.4)\). That completes the proof. \( \square \)

**Proposition 4.3.** Suppose that \((\mu^1_t, \mu^2_t)_{t \geq 0}\) is a solution to MP \((1.3, 1.4)\) and \( \eta (\cdot, \nu_1, \nu_2) \in C^1_b (\mathbb{R}) \) for any \( \nu_1, \nu_2 \in M_{P_1} (\mathbb{R}) \). Then the random field \((u^1_t, u^2_t)_{t \geq 0}\) defined by \((4.1)\) is a weak solution to SPDEs \((4.2)\).

**Proof.** Let \( f, g \in C^2_0 (\mathbb{R}) \) and set \( \tilde{f} (y) = \int_y^\infty f (x) \, dx, \tilde{g} (y) = \int_y^\infty g (x) \, dx \). Then we have
\[ \langle u_t^2, f \rangle_1 = \langle \mu_t^1, \tilde{f} \rangle \]
\[ = \langle u_0^2, \tilde{f} \rangle + \int_0^t \int_0^\infty \frac{1}{2} f'' ds + b_1 \int_0^t \int_0^\infty \mu_s^1 \tilde{f} ds - \int_0^t \int_0^\infty \mu_s^1 \eta (\cdot, \mu_s^1, \mu_s^2) \tilde{f} ds + M^f_t \]
\[ = \langle u_0^2, f \rangle_1 + \int_0^t \int_0^\infty \frac{1}{2} f'' ds + b_1 \int_0^t \int_0^\infty u_s^1 f \tilde{f} ds + \int_0^t \int_0^\infty \eta (\cdot, \mu_s^1, \mu_s^2) \eta' \tilde{f} ds + M^f_t . \]
Note that
\[ \langle u_s^1, (\eta (\cdot, \mu_s^1, \mu_s^2) \tilde{f})' \rangle_1 = \langle u_s^1, \eta' (\cdot, \mu_s^1, \mu_s^2) \tilde{f} - \eta (\cdot, \mu_s^1, \mu_s^2) \tilde{f} \rangle_1 \]
\[ = \langle u_s^1, \eta' (\cdot, \mu_s^1, \mu_s^2) \tilde{f} \rangle_1 - \langle u_s^1, \eta (\cdot, \mu_s^1, \mu_s^2) \tilde{f} \rangle_1 \]
\[ = \left( \int_0^\infty u_s^1 (x) \eta' (x, \mu_s^1, \mu_s^2) \, dx, f \right)_1 - \langle u_s^1, \eta (\cdot, \mu_s^1, \mu_s^2) \tilde{f} \rangle_1 \]
\[ = - \langle \eta (x, \mu_s^1, \mu_s^2) d u_s^2 (x), f \rangle_1 \]
\[ = - \langle \xi (\cdot, \mu_s^1, \mu_s^2), f \rangle_1 . \]
Therefore, we continue to have
\[
\langle u_t^1, f \rangle_1 = \langle u_0^1, f \rangle_1 + \int_0^t \langle u_s^3, 1/2 f'' \rangle_1 \, ds
\]  
(4.4)
\[+ b_1 \int_0^t \langle u_s^1, f \rangle_1 \, ds - \int_0^t \langle \xi (\cdot, \mu_{s}^{2}, \mu_{s}^{3}) , f \rangle_1 \, ds + M_t^f.
\]
Similarly, one shall have
\[
\langle u_t^2, g \rangle_1 = \langle u_0^2, g \rangle_1 + \int_0^t \langle u_s^2, 1/2 g'' \rangle_1 \, ds + b_2 \int_0^t \langle u_s^2, g \rangle_1 \, ds
\]  
(4.5)
\[+(\chi, g) \int_0^t \xi (+\infty, \mu_{s}^{1}, \mu_{s}^{2}) \, ds + \tilde{M}_t^g.
\]

Let \( S' (\mathbb{R}) \) be the space of Schwarz distribution and define the \( S' (\mathbb{R}) \)-valued processes \( \tilde{N}_t \) and \( \tilde{\dot{N}}_t \) by \( \tilde{N}_t (f) = M_t^f \) and \( \tilde{\dot{N}}_t (g) = \tilde{M}_t^g \) for any \( f, g \in C_0^\infty (\mathbb{R}) \). Then \( \tilde{N}_t \) and \( \tilde{\dot{N}}_t \) are \( S' (\mathbb{R}) \)-valued continuous square-integrable martingales with
\[
\langle \tilde{N} (f) \rangle_t = \langle M_t^f \rangle_t = \gamma_2 \int_0^t \int_\mathbb{R} \tilde{f}^2 (y) \, \mu_s^2 (dy) \, ds
\]  
\[= \int_0^t \int_0^\infty (\sqrt{\gamma_2})^2 \tilde{f}^2 \left( (u_s^1)^{-1} (a) \right) \, da \, ds
\]  
\[= \int_0^t \int_0^\infty (\sqrt{\gamma_2}) \int_\mathbb{R} \Pi_{\{a \leq u_s^1 (y)\}} f (y) \, dy \right)^2 \, da \, ds
\]
and
\[
\langle \tilde{\dot{N}} (g) \rangle_t = \langle \tilde{M}_t^g \rangle_t = \gamma_2 \int_0^t \int_\mathbb{R} \tilde{g}^2 (y) \, \mu_s^2 (dy) \, ds
\]  
\[= \int_0^t \int_0^\infty (\sqrt{\gamma_2})^2 \tilde{g}^2 \left( (u_s^1)^{-1} (a) \right) \, da \, ds
\]  
\[= \int_0^t \int_0^\infty (\sqrt{\gamma_2}) \int_\mathbb{R} \Pi_{\{a \leq u_s^2 (y)\}} g (y) \, dy \right)^2 \, da \, ds.
\]

Moreover, one can see that \( \langle N (f) , \tilde{N} (g) \rangle_t = \langle M_t^f , \tilde{M}_t^g \rangle_t = 0 \). By Theorem III-7 and Corollary III-8 in Karoui and Mélèard (1990), on some extension of the probability space, one can define two independent Gaussian white noises \( W^i (ds \, da) \), \( i = 1, 2 \) on \( \mathbb{R}_+ \times \mathbb{R} \) with intensity \( ds \, da \) such that
\[
\tilde{N}_t (f) = \int_0^t \int_\mathbb{R} \sqrt{\gamma_1} \Pi_{\{a \leq u_s^1 (x)\}} \tilde{f} (x) \, x \, W^1 (ds \, da)
\]
and
\[
\tilde{\dot{N}}_t (g) = \int_0^t \int_\mathbb{R} \sqrt{\gamma_2} \Pi_{\{a \leq u_s^2 (x)\}} \tilde{g} (x) \, x \, W^2 (ds \, da).
\]

Plugging back to (4.4) and (4.5), one can see that \( (u_t^1, u_t^2)_{t \geq 0} \) is a solution to (4.2).

\[\square\]

4.2. **Uniqueness for SPDEs.** This subsection is devoted to prove the pathwise uniqueness for the solution to the system of SPDEs (4.2). By Propositions 4.2 and 4.3, the uniqueness for the solution to MP (4.1) is then a direct consequence. We apply the approach of an extended Yamada-Watanabe argument to smooth function. This is an adaptation to that of Proposition 3.1 in Mytnik and Xiong (2013).
Before we go deep into the uniqueness theorem, let’s introduce some notation. Let \( \Phi \in C^\infty_c(\mathbb{R})^+ \) such that \( \text{supp}(\Phi) \subset (-1,1) \) and the total integral is 1. Define \( \Phi_m(x) = m\Phi(mx) \). Notice that \( \lim_{m \to \infty} \Phi_m(x) = \delta_0(x) \), which is the Dirac-\( \delta \) function on \( \mathbb{R} \). Let \( \{a_k\} \) be a decreasing sequence defined recursively by \( a_0 = 1 \) and \( \int_{a_k}^{a_{k-1}} z^{-1} dz = k \) for \( k \geq 1 \). Let \( \psi_k \) be non-negative functions in \( C^\infty_c(\mathbb{R}) \) such that \( \text{supp}(\psi_k) \subset (a_k, a_{k-1}) \) and \( \int_{a_k}^{a_{k-1}} \psi_k(z) \, dz = 1 \) and \( \psi_k(z) \leq 2(kz)^{-1} \) for all \( z \in \mathbb{R} \). Denote by

\[
\phi_k(z) = \int_0^{|z|} dy \int_0^y \psi_k(x) \, dx, \quad \forall \, z \in \mathbb{R}.
\]

Then, \( \phi_k(z) \uparrow |z|, \, |\phi_k'(z)| \leq 1 \) and \( |z| \phi_k''(z) \leq 2k^{-1} \). Let

\[
J(x) = \int_\mathbb{R} e^{-|x|} \rho(x-y) \, dy,
\]

where \( \rho \) is the mollifier given by \( \rho(x) = C \exp\{-1/\,(1-x^2)\} \mathbb{I}_{\{\,|x|<1\,\}} \), and \( C \) is a constant such that \( \int_\mathbb{R} \rho(x) \, dx = 1 \). Then, for any \( m \in \mathbb{Z}_+ \), there are positive constants \( c_m \) and \( C_m \) such that

\[
(4.6) \quad c_m e^{-|x|} \leq |J^{(m)}(x)| \leq C_m e^{-|x|}, \quad \forall \, x \in \mathbb{R}.
\]

Assume that \((u_1^1, u_2^1)_{t \geq 0}\) and \((\bar{u}_1^1, \bar{u}_2^1)_{t \geq 0}\) are two solutions to the system of SPDEs \([1.2]\) with the same initial values. \((\mu_1^1, \mu_2^1)_{t \geq 0}\) and \((\bar{\mu}_1^1, \bar{\mu}_2^1)_{t \geq 0}\) stand for their corresponding measure-valued processes, namely, \( u_1^1(y) = \mu_1^1(-\infty, y) \) and \( \bar{u}_1^1(y) = \bar{\mu}_1^1(-\infty, y) \) for \( \hat{a} = 1, 2 \). Let \( v_1^1(y) = u_1^1(y) - \bar{u}_1^1(y) \) and \( \bar{G}_1^1(a, y) = \mathbb{I}_{\{a \leq u_1^1(y)\}} - \mathbb{I}_{\{a \geq \bar{u}_1^1(y)\}} \). Moreover, we denote

\[
(4.7) \quad I_{1}^{m,k,i} = \frac{1}{2} \mathbb{E} \left[ \int_0^t \int_{\mathbb{R}} \phi_k' \left( \langle v_1^i, \Phi_m(x-\cdot) \rangle \right) \langle v_1^i, \Delta_y \Phi_m(x-\cdot) \rangle J(x) \, dx \, ds \right];
\]

\[
I_{2}^{m,k,i} = \mathbb{E} \left[ \int_0^t \int_{\mathbb{R}} \phi_k' \left( \langle v_1^i, \Phi_m(x-\cdot) \rangle \right) \langle v_1^i, \Phi_m(x-\cdot) \rangle J(x) \, dx \, ds \right];
\]

\[
I_{3}^{m,k,i} = \mathbb{E} \left[ \int_0^t \int_{\mathbb{R}} \phi_k'' \left( \langle v_1^i, \Phi_m(x-\cdot) \rangle \right) \left| \int \bar{G}_1^1(a, y) \Phi_m(x-y) \, dy \right|^2 \, da \, J(x) \, dx \, ds \right].
\]

Proposition 4.4. For \( i = 1, 2 \) we have

\[
\mathbb{E} \left[ \int_{\mathbb{R}} \phi_k \left( \langle v_1^i, \Phi_m(x-\cdot) \rangle \right) J(x) \, dx \right] = I_{1}^{m,k,i} + b_i I_{2}^{m,k,i} + \frac{\gamma_i}{2} I_{3}^{m,k,i} + I_{4}^{m,k,i},
\]

where \( I_{1}^{m,k,i}, I_{2}^{m,k,i} \) and \( I_{3}^{m,k,i} \) are given by \((4.7)\).

\[
(4.8) \quad I_{4}^{m,k,1} = -\mathbb{E} \left[ \int_0^t \int_{\mathbb{R}} \phi_k' \left( \langle v_1^1, \Phi_m(x-\cdot) \rangle \right) \Phi_m(x-y) \bar{\xi}_s(y) \, dy \, J(x) \, dx \, ds \right]
\]

and

\[
(4.9) \quad I_{4}^{m,k,2} = \mathbb{E} \left[ \int_0^t \int_{\mathbb{R}} \phi_k' \left( \langle v_2^1, \Phi_m(x-\cdot) \rangle \right) \langle \hat{\chi}, \Phi_m(x-\cdot) \rangle \bar{\xi}_s(\infty) \, J(x) \, dx \, ds \right]
\]

with \( \bar{\xi}_s(\cdot) = \xi(\cdot, \mu_1^1, \mu_2^1) - \xi(\cdot, \bar{\mu}_1^1, \bar{\mu}_2^1) \).

Proof. It follows from \((4.2)\) that

\[
v_1^1(y) = \sqrt{\gamma_1} \int_{0}^{t} \int_{0}^{\infty} \bar{G}_1^1(a, y) \, W^1 \, (ds \, da) + \int_{0}^{t} \left( \frac{\Delta_y}{2} v_1^1(y) + b_1 v_2^1(y) - \bar{\xi}_s(y) \right) \, ds
\]
and
\[ v_t^2(y) = \sqrt{2} \int_0^t \int_0^\infty \mathcal{G}_s^2(a,y) W_s^2(ds \, da) + \int_0^t \left( \frac{\Delta y v_s^2(y)}{2} + b_2 v_s^2(y) + \chi(y) \xi_s(\infty) \right) ds. \]

Consequently, we have
\[
\langle v_t^1, \Phi_m(x - \cdot) \rangle = \sqrt{\gamma_1} \int_0^t \int_{\mathbb{R}^+} \int_{\mathbb{R}} \mathcal{G}_s^1(a,y) \Phi_m(x - y) dy W^1(ds \, da) \\
+ \int_0^t \phi_k' \left( \langle v_s^1, \Phi_m(x - \cdot) \rangle \right) \left[ \frac{1}{2} \langle v_s^1, \Delta_y \Phi_m(x - \cdot) \rangle + b_1 \langle v_s^1, \Phi_m(x - \cdot) \rangle \right] ds \\
+ \gamma_1 \int_0^t \int_{\mathbb{R}^+} \phi_k'' \left( \langle v_s^1, \Phi_m(x - \cdot) \rangle \right) \mathcal{G}_s^1(a,y) \Phi_m(x - y) dy ds \\
- \int_0^t \phi_k \left( \langle v_s^1, \Phi_m(x - \cdot) \rangle \right) \Phi_m(x - y) \xi_s(y) dy ds.
\]

and
\[
\phi_k \left( \langle v_t^2, \Phi_m(x - \cdot) \rangle \right) = \sqrt{\gamma_2} \int_0^t \int_{\mathbb{R}^+} \phi_k' \left( \langle v_s^2, \Phi_m(x - \cdot) \rangle \right) \mathcal{G}_s^2(a,y) \Phi_m(x - y) dy W^2(ds \, da) \\
+ \int_0^t \phi_k' \left( \langle v_s^2, \Phi_m(x - \cdot) \rangle \right) \left[ \frac{1}{2} \langle v_s^2, \Delta_y \Phi_m(x - \cdot) \rangle + b_2 \langle v_s^2, \Phi_m(x - \cdot) \rangle \right] ds \\
+ \gamma_2 \int_0^t \int_{\mathbb{R}^+} \phi_k'' \left( \langle v_s^2, \Phi_m(x - \cdot) \rangle \right) \mathcal{G}_s^2(a,y) \Phi_m(x - y) dy ds \\
+ \int_0^t \phi_k' \left( \langle v_s^2, \Phi_m(x - \cdot) \rangle \right) \chi(\Phi_m(x - \cdot) \xi_s(\infty) ds.
\]

Applying Itô’s formula to (4.10) and (4.11), we can easily get
\[
\phi_k \left( \langle v_t^1, \Phi_m(x - \cdot) \rangle \right) = \langle v_t^1, \Phi_m(x - \cdot) \rangle \\
+ \frac{1}{2} \int_0^t \langle v_s^1, \Delta_y \Phi_m(x - \cdot) \rangle ds
\]
(4.10)
and
\[
\phi_k \left( \langle v_t^2, \Phi_m(x - \cdot) \rangle \right) = \langle v_t^2, \Phi_m(x - \cdot) \rangle \\
+ \frac{1}{2} \int_0^t \langle v_s^2, \Delta_y \Phi_m(x - \cdot) \rangle ds + b_2 \int_0^t \langle v_s^2, \Phi_m(x - \cdot) \rangle ds
\]
(4.11)

Taking the expectations of \( \langle \phi_k \left( \langle v_t^i, \Phi_m(x - \cdot) \rangle \right), J(x) \rangle \) with \( i = 1, 2, \) we obtain the desired results.

**Lemma 4.5.** For \( i = 1, 2 \) we have
\[
2I_{m,k,i}^1 \leq \mathbb{E} \left[ \int_0^t \int_{\mathbb{R}} \langle |v_s^i|, \Phi_m(x - \cdot) \rangle |J''(x)| dx ds \right].
\]
Proof. Note that
\[
2I_{1}^{m,k,i} = \mathbb{E} \left[ \int_{0}^{t} \int_{\mathbb{R}} \phi_{k}^{\prime} \left( \langle v_{s}^{i}, \Phi_{m}(x-\cdot) \rangle \right) \langle \nabla_{y}^{i} \Phi_{m}(x-\cdot) \rangle J(x) dx ds \right]
\]
\[
= \mathbb{E} \left[ \int_{0}^{t} \int_{\mathbb{R}} \phi_{k}^{\prime} \left( \langle v_{s}^{i}, \Phi_{m}(x-\cdot) \rangle \right) \Delta_{x} \langle v_{s}^{i}, \Phi_{m}(x-\cdot) \rangle J(x) dx ds \right]
\]
\[
= -\mathbb{E} \left[ \int_{0}^{t} \int_{\mathbb{R}} \left( \frac{\partial}{\partial x} \langle v_{s}^{i}, \Phi_{m}(x-\cdot) \rangle \right) \phi_{k}^{\prime} \left( \langle v_{s}^{i}, \Phi_{m}(x-\cdot) \rangle \right) J(x) dx ds \right]
\]
\[
\leq -\mathbb{E} \left[ \int_{0}^{t} \int_{\mathbb{R}} \left( \frac{\partial}{\partial x} \langle v_{s}^{i}, \Phi_{m}(x-\cdot) \rangle \right) \phi_{k} \left( \langle v_{s}^{i}, \Phi_{m}(x-\cdot) \rangle \right) J'(x) dx ds \right]
\]
\[
\leq -\mathbb{E} \left[ \int_{0}^{t} \int_{\mathbb{R}} \phi_{k} \left( \langle v_{s}^{i}, \Phi_{m}(x-\cdot) \rangle \right) J''(x) dx ds \right].
\]
(4.12) Use \( \phi_{k}(z) \leq |z| \) to get
\[
\phi_{k} \left( \langle v_{s}^{i}, \Phi_{m}(x-\cdot) \rangle \right) \leq |\langle v_{s}^{i}, \Phi_{m}(x-\cdot) \rangle| \leq \langle |v_{s}^{i}|, \Phi_{m}(x-\cdot) \rangle.
\]
That implies the result. \( \square \)

Lemma 4.6. For \( i = 1, 2 \) we have
\[
I_{3}^{m,k,i} \leq 4 \mathbb{E} \left[ \int_{0}^{t} \int_{\mathbb{R}} \phi_{k}^{\prime} \left( \langle v_{s}^{i}, \Phi_{m}(x-\cdot) \rangle \right) \langle |v_{s}^{i}|, \Phi_{m}(x-\cdot) \rangle J(x) dx ds \right].
\]
Proof. It is easy to see that
\[
I_{3}^{m,k,i} \leq \mathbb{E} \left[ \int_{0}^{t} \int_{\mathbb{R}} \phi_{k}^{\prime} \left( \langle v_{s}^{i}, \Phi_{m}(x-\cdot) \rangle \right) \int_{\mathbb{R}} \left( \tilde{G}_{s}(a,y) \right)^{2} \Phi_{m}(x-y) dy da J(x) dx ds \right]
\]
\[
\leq 4 \mathbb{E} \left[ \int_{0}^{t} \int_{\mathbb{R}} \phi_{k} \left( \langle v_{s}^{i}, \Phi_{m}(x-\cdot) \rangle \right) \int_{\mathbb{R}} |v_{s}^{i}(y)| \Phi_{m}(x-y) dy da J(x) dx ds \right].
\]
The result follows. \( \square \)

Theorem 4.7. Assume that there exists a constant \( K \) such that
\[
|\xi(x, \nu_1, \nu_2) - \xi(x, \tilde{\nu}_1, \tilde{\nu}_2)| \leq K \left[ \rho(\nu_1, \tilde{\nu}_1) + \rho(\nu_2, \tilde{\nu}_2) \right]
\]
for any \( x \in \mathbb{R} \cup \{ \pm \infty \} \) and \( \nu_i, \tilde{\nu}_i \in M_{F}(\mathbb{R}) \) with \( i = 1, 2 \). Then the pathwise uniqueness holds for SPDEs (4.2), namely, if (4.2) has two solutions defined on the same stochastic basis with the same initial values, then the solutions coincide almost surely.

Proof. The coincidence of \( (u_{1}^{i}, u_{2}^{i})_{t \geq 0} \) and \( (\tilde{u}_{1}^{i}, \tilde{u}_{2}^{i})_{t \geq 0} \) is sufficient to the pathwise uniqueness of SPDEs (4.2). Subsequently, we estimate the values of \( I_{\ell}^{m,k,i} \) with \( \ell = 1, 2, 3, 4 \) and \( i = 1, 2 \). Since
\[
\lim_{m \to \infty} \langle v_{s}^{i}, \Phi_{m}(x-\cdot) \rangle = v_{s}^{i}(x) \quad \text{and} \quad \lim_{m \to \infty} \langle |v_{s}^{i}|, \Phi_{m}(x-\cdot) \rangle = |v_{s}^{i}(x)|
\]
for Lebesgue-a.e. \( x \) and any \( s \geq 0 \) almost surely. By Lemma 4.5 and dominated convergence theorem, we have
\[
\limsup_{k,m \to \infty} 2I_{1}^{m,k,i} \leq \mathbb{E} \left[ \int_{0}^{t} \int_{\mathbb{R}} |v_{s}^{i}(x)| \cdot |J''(x)| dx ds \right].
\]
By (4.6), there exists a constant $K$ such that

$$
\limsup_{k,m \to \infty} 2I_{1}^{m,k,i} \leq K \mathbb{E} \left[ \int_{0}^{t} \int_{\mathbb{R}} \left| v_{s}^{i}(x) \right| \cdot \left| J(x) \right| dx ds \right].
$$

Using $|\phi_k'(z)| \leq 1$ and dominated convergence theorem, we can easily get

$$
\limsup_{k,m \to \infty} |I_{2}^{m,k,i}| \leq \mathbb{E} \left[ \int_{0}^{t} \int_{\mathbb{R}} \left| v_{s}^{i}(x) \right| \cdot \left| J(x) \right| dx ds \right].
$$

Recall that $\phi_k''(z)|z| \leq 2k^{-1}$, by Lemma 4.6 one shall check that

$$
\limsup_{m \to \infty} I_{3}^{m,k,i} \leq K \mathbb{E} \left[ \int_{0}^{t} \int_{\mathbb{R}} \phi_k''(v_{s}^{i}(x)) \left| v_{s}^{i}(x) \right| J(x) dx ds \right] = O(k^{-1}).
$$

Recall that $\chi$ is a finite measure on $\mathbb{R}$ and $|\phi_k'(z)| \leq 1$. By (4.8), (4.9) we have

$$
\limsup_{k,m \to \infty} |I_{4}^{m,k,i}| \leq \mathbb{E} \left[ \int_{0}^{t} \int_{\mathbb{R}} \rho \left( \mu_s^{n}, \tilde{\mu}_s^{n} \right) + \rho \left( \mu_s^{n}, \tilde{\mu}_s^{n} \right) ds \right] 
\leq K \mathbb{E} \left[ \int_{0}^{t} \int_{\mathbb{R}} \left( |v_{s}^{i}(x)| + |v_{s}^{j}(x)| \right) J(x) dx ds \right].
$$

By Proposition 4.4 and putting (4.13)-(4.16) together, one can see that

$$
\mathbb{E} \left[ \int_{\mathbb{R}} \left( |v_{t}^{1}(x)| + |v_{t}^{2}(x)| \right) J(x) dx \right] \leq K \mathbb{E} \left[ \int_{0}^{t} \int_{\mathbb{R}} \left( |v_{s}^{1}(x)| + |v_{s}^{2}(x)| \right) J(x) dx ds \right].
$$

Then Gronwall’s inequality implies that

$$
\mathbb{E} \left[ \int_{\mathbb{R}} \left( |v_{t}^{1}(x)| + |v_{t}^{2}(x)| \right) J(x) dx \right] = 0
$$

for any $t \geq 0$ and the pathwise uniqueness follows. \hfill \qed

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LINA JI: DEPARTMENT OF MATHEMATICS, SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY
*Email address*: jiln@sustch.edu.cn

HUILI LIU: SCHOOL OF MATHEMATICAL SCIENCES, HEBEI NORMAL UNIVERSITY, SHIJIAZHUANG, HEBEI, CHINA
*Email address*: liuhuili@hebtu.edu.cn

JIE XIONG: DEPARTMENT OF MATHEMATICS AND NATIONAL CENTER FOR APPLIED MATHEMATICS (SHENZHEN), SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY
*Email address*: xiongj@sustch.edu.cn