Cosmic Force

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1 Abstract

The geodesics for classical particles in the gravitational field of a Schwarzschild black hole immersed in a vacuum of non-zero $\Lambda$ are found in their functional form. In particular, when $r$ is a function of $\phi$, the functional form is in terms of quadrupoly periodic double theta functions. The turning points are branch places on a one complex dimensional dissected Riemann surface. The four periods are found by integrating across the the period loops. It is shown that when $\Lambda$ is equal to zero, the quadrupoly periodic functions for the geodesics reduce to the functional form of doubly periodic elliptic functions.

2 Introduction

Steven Weinberg,[Weinberg,1], referred to the cosmological constant problem as crisis in physics, due to the discrepancy of some 120 orders of magnitude between the quantum field theory calculation and the cosmological calculation of the energy density in vacuum[Carroll,Ostlie,2]. Some progress is being made in astronomy with regards to this problem. The findings from the Type Ia Supernova data,[Filippenko,3], has created a new cosmology which is as revolutionary as Hubble’s finding’s of an expanding universe. Their observations suggest that the expansion of the universe is accelerating.

Presently there several investigations in progress[Efstathiou,4] which attempt to measure cosmic accelleration, also known as Dark Energy. If this is to be modelled by a cosmic force, $F = \frac{1}{3} \Lambda r m c^2$ where $c$ is the speed of light then the observations are in favor of a positive cosmological constant,$\Lambda$. This constant will be a fundamental constant of nature, and the cosmic force will be a fifth fundamental force of nature. Since the force is proportional to $r$, the method which probes the greatest distances will work the best, such as looking at anisotropies in the cosmic background radiation[Efstathiou,4]. The effects of the cosmological constant are neglible on the scale of the solar system[Neupane,5][Cardona,6]. It is vital to find independent experiments to measure cosmic accelleration[Boughn,Crittenden,7]. The equations in this paper...
may be tested on the scale of our galaxy, thus offering an independent measurement of the cosmological constant, in relatively well known setting. For example the work done with galactic velocity rotation curves [Whitehouse, 8] To do this consider the nucleus of the galaxy to be a black hole of radius \( r_g = 2GM/c^2 \), where \( M \) is the estimated mass of the galactic nucleus, and let the orbiting particle be a star or globular cluster. With such an experiment, one will have to make a reasonable estimate of the dark matter present. The mounting evidence in support of a cosmological constant makes it necessary to know the mathematical relations and the geometry associated with an object’s motion in space of intrinsic curvature and gravity. The development of the mathematical formula that determine the geodesics where formulated mostly by Riemann and Weierstrass, at least the general case [Baker, 9, 10].

To calculate the geodesics in the centrally symmetric gravitational field of a Schwarzschild black hole in a universe with a cosmological constant, \( \Lambda \neq 0 \), the Hamilton-Jacobi method is applied to the metric, [Witten, 11], [Hawking, 12], [Rindler, 13]

\[
ds^2 = g_{tt} \, dt^2 + g_{rr} \, dr^2 + g_{\phi\phi} \, d\phi^2 + g_{\theta\theta} \, d\theta^2
\]

(1)

\[
g_{tt} = (1 - \frac{1}{3} \sigma \, \Lambda \, r^2 - r_g/r) \, c^2
\]

\[
g_{rr} = -(1 - \frac{1}{3} \sigma \, \Lambda \, r^2 - r_g/r)^{-1}
\]

\[
g_{\phi\phi} = -\sin^2 \theta \, r^2
\]

\[
g_{\theta\theta} = -r^2
\]

Where \( \sigma = \pm 1 \) depending on whether or not the universe is de Sitter or anti-de Sitter, respectively. \( \Lambda \) is the cosmological constant and was first introduced by Einstein [Filippenko, 3] and \( r_g \) was introduced by Schwarzschild. The functional form of the geodesics when \( r_g = 0 \) have been studied, they are given in terms of genus zero circular functions. When \( \Lambda = 0 \) the geodesics are given by genus one elliptic functions [Bartlett, 14]. In this paper, the functional form of the geodesics when \( \Lambda \) and \( r_g \) are both non-zero are found, in terms of Riemann’s hyperelliptic theta functions [Baker, 9, 10].

### 3 Derivation of the Hyper-elliptic \( \phi \) Integral

Let a particle of mass \( m \) be constrained to move in the equatorial plane, \( \theta = 0 \), and let the action be given by [Landau, 15],

\[
S = -E \, t + L \, \phi + S_r
\]

(2)

where \( E \) is the total energy of the particle and \( L \) is it’s angular momentum. By the conservation of four momentum,

\[
g^{ij} \frac{dS}{dx^i} \frac{dS}{dx^j} = m^2
\]

(3)
we can insert $S$ and $g^i_j$ from (4) and (2) into (3) and obtain,

$$S_r = \int_{r_0}^{r} \frac{(E^2 - (1 - \frac{1}{3} \Lambda r^2 - r_g/r) (L^2/r^2 + m^2 c^2))^2}{(1 - \frac{1}{3} \Lambda r^2 - r_g/r)} \, dr$$

(4)

Differentiating $S_r$ w.r.t. $L$ and using

$$\phi + \frac{\partial S_r}{\partial L} = \text{const} = 0$$

(5)

$$\phi = \int_{r_0}^{r} \frac{L \, dr}{r^2 (E^2 - (1 - \frac{1}{3} \Lambda r^2 - r_g/r) (L^2/r^2 + m^2 c^2))^2}$$

(6)

This integral is a hyperelliptic integral of genus, $g = 2$. The problem is to invert this integral, i.e. express the upper limit $r$ as a function of $\phi$, this is known as the inversion problem.

4 The Solution to the Hyper-elliptic Integral of Arbitrary Genus

The following will be a review of the account given by H.F. Baker[9,10] and the references contained in these books. The general solution to the inversion problem for hyperelliptic integrals is given by solving $g$ of the $2g + 1$ equations, for the $g$ variable places $x_1, \ldots, x_g$,

$$\frac{\partial^2 (u|u^{b,a})}{\partial^2 (u|u^{b',a'})} = A(b) \ (b - x_1) \ldots (b - x_g)$$

(7)

The notation Baker uses for the generalized theta function is,

$$\vartheta(u|u^{b,a}) = \vartheta(u; \frac{1}{2} \Omega_{m,m'}) = \vartheta(u; \frac{1}{2} m, \frac{1}{2} m') = e^{a \cdot u} \Theta(v; \frac{1}{2} m, \frac{1}{2} m')$$

(8)

where $a$ is an arbitrary $g \times g$ symmetrical matrix, since putting (5) into (7) gives,

$$\frac{\partial^2 (u|u^{b,a})}{\partial^2 (u|u^{b',a'})} = \frac{\Theta^2(v; \frac{1}{2} m, \frac{1}{2} m')}{\Theta^2(v; \frac{1}{2} k, \frac{1}{2} k')}$$

(9)

where $m, m', k$ and $k'$ are integers. Riemann’s theta functions are defined as,

$$\Theta(v; \frac{1}{2} m, \frac{1}{2} m') = \sum \ e^{2 \pi i v (n + \frac{1}{2} m') + b (n + \frac{1}{2} m')^2} \pi m \ (n + \frac{1}{2} m')$$

(10)

with, $\sum_{n_1 = -\infty}^{n_1 = \infty} \ldots \sum_{n_g = -\infty}^{n_g = \infty}$. $h$ is a $g \times g$ matrix, in general non-symmetrical. $b$ is a $g \times g$ symmetrical matrix. In (7), $u$ denotes the $g$ quantities,

$$u^{x_1,a_1} + \ldots + u^{x_g,a_g} = u_i$$

(11)
with, \( i = 1 \ldots g \). This is known as Abel’s Theorem. Riemann’s Normal Integral of the first kind is defined as,

\[
u_x, a_i = \int_a^x \frac{(x, 1)_{i,g-1} \, dx}{y}
\]

(12)

where

\[(x, 1)_{i,g-1} = A_{i,g-1} \, x^{g-1} + A_{i,g-2} \, x^{g-2} + \ldots + A_{i,0}\]

(13)

and \( A_{i,g-1}, \ldots, A_{i,0} \) are arbitrary constants. The denominator in (12) is given by,

\[y^2 = 4 \, (x - a_1) \ldots (x - a_g) \, (x - c_1) \ldots (x - c_g) \, (x - c)\]

(14)

where \( a_1, \ldots, a_g, c_1, \ldots, c \) are branch places and \( x \) is a variable on the one complex dimensional Riemann surface of genus, \( g \), shown in figure 1.

From figure 1, the period table, table 1, is derived.

Table 1. Primative Normal Periods\[\text{Brioschi,19}\][\text{Bolza,20}\][\text{Konigsberger,21}]

| Int | \( \alpha_1 \) | \ldots | \( \alpha_g \) | \( \beta_1 \) | \ldots | \( \beta_g \) |
|-----|----------------|--------|----------------|----------------|--------|
| \( u_1 \) | \( \omega_{11} \) | \ldots | \( \omega_{1g} \) | \( \omega'_{11} \) | \ldots | \( \omega'_{1g} \) |
| \ldots | \ldots | \ldots | \ldots | \ldots | \ldots | \ldots |
| \( u_g \) | \( \omega_{g1} \) | \ldots | \( \omega_{gg} \) | \( \omega'_{g1} \) | \ldots | \( \omega'_{gg} \) |

Figure 1:
One Complex Dimensional Dissected Arbitrary Genus Riemann Surface
Now let,

\[ \frac{1}{2} \Omega_{m,m'} = m_1 \omega_{r,1} + \ldots + m_g \omega_{r,g} + m'_{r,1} \omega'_{r,1} + \ldots + m'_{g} \omega'_{r,g} \]  

(15)

where \( m \) and \( m' \) are equal to +1 when integrating across a period loop from right to left, -1 when crossing from left to right, and 0 when not crossing a period loop. Now define an integral of the first kind via,

\[ \pi i v^x,a = h_{r,1} u_{x,a}^1 + h_{r,2} u_{x,a}^2 + \ldots + h_{r,g} u_{x,a}^g \]  

(16)

with \( r = 1, 2, \ldots, g \) and where,

\[ 2 h w = \pi i \]  

(17)

\[ b = 2 h w' = \pi i \tau \]  

(18)

so with these definitions table 1 becomes table 2, below

Table 2. Hyperelliptic Moduli

| \( v^x,a \) | \( \alpha_1 \) | \( \alpha_2 \) | \ldots | \( \alpha_g \) | \( \beta_1 \) | \ldots | \( \beta_g \) |
|---|---|---|---|---|---|---|---|
| \( v^x,a \) | 1 | 0 | \ldots | 0 | \( \tau_{1,1} \) | \ldots | \( \tau_{1,g} \) |
| \ldots | \ldots | \ldots | \ldots | \ldots | \ldots | \ldots | \ldots |
| \( v^x,a \) | 0 | 0 | \ldots | 1 | \( \tau_{g,1} \) | \ldots | \( \tau_{g,g} \) |

with Riemann’s condition for hyperelliptic integrals,[Brioschi,13],[Bolza,14],

\[ \tau = \tau^t \]  

(19)

With this condition there are \( \frac{1}{2} n (n + 1) + 1 \) different non-zero periods. With the above equations, Riemann’s theta function becomes,

\[ \Theta(v; \frac{1}{2} m, \frac{1}{2} m') = \sum e^{2 \pi i v (n + \frac{1}{2} m') + i \pi \tau (n + \frac{1}{2} m')^2 + i \pi m (n + \frac{1}{2} m')} \]  

(20)

where the quadratic forms are given by,

\[ v (n + \frac{1}{2} m') = v n + \frac{1}{2} m' v = v_1 n_1 + \ldots + v_g n_g + \frac{1}{2} v_1 m_1' + \ldots + \frac{1}{2} v_g m_g' \]  

(21)

and similarly,

\[ \tau (n + \frac{1}{2} m')^2 = \tau n^2 + \tau n m' + \frac{1}{4} \tau m'^2 \]

\[ = (\tau_{1,1} n_1^2 + 2 \tau_{1,2} n_1 n_2 + \ldots + \tau_{g,g} (n_g)^2) \]

\[ + \sum_{r=1}^{g} \sum_{s=1}^{g} (\tau_{r,s}) n_r m_s' + \frac{1}{4} (\tau_{1,1} (m_1')^2 + 2 \tau_{1,2} m_1' m_2' + \ldots + \tau_{g,g} (m_g')^2) \]  

(22)
It has been proven by Riemann that for the integral of the first kind, i.e. \( \varphi^\omega_{x,a} \), when it has the period scheme as in Table 2, then the imaginary part of,

\[
\tau_{1,1} n_1^2 + 2 \tau_{1,2} n_1 n_2 + \ldots + \tau_{g,g} n_g^2
\]

is positive for all integer values of \( n_1 \) and \( n_2 \), with the exception of \( n_1 = n_2 = 0 \). Therefore the modulus of \( e^{i \pi \tau} \) is less than unity and the function (20) converges for all values of its argument, \( v \). The last thing to define in (23) is the constant \( A(b) \), it is defined as,

\[
A(b) = (\epsilon \frac{d}{dx} (x - a_1) \ldots (x - a_g) (x - c_1) \ldots (x - c_g)(x-c))|_{x=b} \cdot \frac{1}{2} \]

where, \( \epsilon = 1 \) when \( u^{b,a} \) is an odd half period and \( -1 \) when it is an even half period.

5 The Solution to the Hyperelliptic Integral of Genus Two

The solution of the hyper-elliptic integral (6), is given by (7), [Rosenhain, 22], [Gopel, 23], [Forsyth, 24], [Baker, 9], with \( g = 2 \) we have

\[
\frac{\Theta^2(v_1, v_2; \frac{1}{2} m_1, \frac{1}{2} m_2, \frac{1}{2} m'_1, \frac{1}{2} m'_2)}{\Theta^2(v_1, v_2; \frac{1}{2} k_1, \frac{1}{2} k_2, \frac{1}{2} k'_1, \frac{1}{2} k'_2)} = A(b) (b-x_1) (b-x_2)
\]

where by (10) we have, [Baker, 9, 10],

\[
\pi i v^x_{1,a} = h_{1,1} u^x_{1,a} + h_{1,2} u^x_{2,a} \\
\pi i v^x_{2,a} = h_{2,1} u^x_{1,a} + h_{2,2} u^x_{2,a}
\]

where \( h_{i,j} \) are the elements of the following matrix,

\[
h = \frac{\pi i}{2 \Delta} \begin{pmatrix}
\omega_{2,2} & -\omega_{1,2} \\
-\omega_{2,1} & \omega_{1,1}
\end{pmatrix}
\]

and,

\[
\Delta = det(h) \neq 0
\]

Abel’s Theorem [11], becomes

\[
\begin{align*}
u^x_{1,1} + u^x_{2,1} &= u_1 \\
u^x_{1,2} + u^x_{2,2} &= u_2
\end{align*}
\]

these become with,

\[
x_2 = a_2
\]
and with (30),

\[
\begin{align*}
  u_1^{x_1, a_1} &= u_1 \\
  &= \frac{\int_{a_1}^{x_1} (A_{1,0} + A_{1,1} x) \, dx}{y} \\
  u_2^{x_1, a_1} &= u_2 \\
  &= \frac{\int_{a_1}^{x_1} (A_{2,0} + A_{2,1} x) \, dx}{y}
\end{align*}
\]

where,

\[y^2 = 4 x^5 + \lambda_4 x^4 + \lambda_3 x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0 \]  \hspace{1cm} (31)

These coefficients may be substituted into the general solution to the quintic equation [Drociuk, 25] where the branch places, i.e. the roots of the quintic (31), are determined in their functional form. Then (31) maybe rewritten in the form,

\[y^2 = 4 (x - a_1) (x - a_2) (x - c_1) (x - c_2) (x - c) \]  \hspace{1cm} (32)

So the the Riemann Surface, figure 1, becomes a genus 2 Riemann surface figure 2.

From figure(2) the period table, table 3, is derived.

Table 3. Period Table for Genus Two Hyperelliptic Integrals

| Int  | \(\alpha_1\) | \(\alpha_2\) | \(\beta_1\) | \(\beta_2\) |
|------|-------------|-------------|-------------|-------------|
| \(\omega_1\) | \(\omega_1\) | \(\omega_{11}\) | \(\omega_{12}\) |
| \(\omega_2\) | \(\omega_2\) | \(\omega_{21}\) | \(\omega_{22}\) |
With definitions [26], table three becomes table four,

Table 4. Genus Two Hyperelliptic Moduli

| Int | $\alpha_1$ | $\alpha_2$ | $\beta_1$ | $\beta_2$ |
|-----|------------|------------|-----------|-----------|
| $\nu_1$ | 1         | 0          | $\tau_{1,1}$ | $\tau_{1,2}$ |
| $\nu_2$ | 0         | 1          | $\tau_{2,1}$ | $\tau_{2,2}$ |

So the four non-zero periods are, 1 and

\[
\tau_{1,1} = \frac{\omega_{2,2} \omega_{1,1}' - \omega_{1,2} \omega_{2,1}'}{\Delta}
\]

\[
\tau_{2,2} = \frac{-\omega_{2,1} \omega_{1,2}' + \omega_{1,1} \omega_{2,2}'}{\Delta}
\]

and by Riemann’s condition [23],

\[
\tau_{1,2} = \frac{-\omega_{2,1} \omega_{1,1}' + \omega_{1,2} \omega_{2,2}'}{\Delta} = \frac{\omega_{2,2} \omega_{1,2}' - \omega_{1,1} \omega_{2,2}'}{\Delta} = \tau_{2,1}
\]

The quadrupoly periodic Double Theta Function is given is from [20], [21], [22],

\[
\Theta(v_1, v_2; \frac{1}{2} m_1, \frac{1}{2} m_2, \frac{1}{2} m_1', \frac{1}{2} m_2') = \sum_{n_1 = -\infty}^{\infty} \sum_{n_2 = -\infty}^{\infty} e^{\pi i (v_1 (2 n_1 + m_1') + v_2 (2 n_2 + m_2'))} q^{(2 n_2 + m_2')^2} q'^{(2 n_1 + m_1')^2} r^{(2 n_1 + m_1) (2 n_2 + m_2)} (-1)^{m_1 n_1 + m_2 n_2} (i)^{m_1 m_1' + m_2 m_2'}
\]

where,

\[
q = e^{\frac{i}{4} i \pi \tau_{1,1}}
\]

\[
q' = e^{\frac{i}{4} i \pi \tau_{2,2}}
\]

\[
r = e^{\frac{i}{4} i \pi \tau_{1,2}}
\]

which is equal to Forsyth’s double theta function, [Forsyth,24], with the two characteristics related by the following,

\[
m_1 = \rho
\]

\[
m_2 = \rho'
\]

\[
m_1' = \sigma
\]

\[
m_2' = \sigma'
\]
so that \(a_r\) in Forsyth’s paper is equal to
\[
a_r = (i)^{\sigma'} r^{\rho} \sigma r^{(2n+\sigma')} (2m+\sigma)\]  
(39)

where \(n = n_1\) and \(m = n_2\) are the summation indices. Forsyth gives a review of Rosenhain’s theory, [Forsyth,26] of the fifteen ratio’s of the quadrupoly periodic theta function. From the list given in [Forsyth,24], the first ratio is selected and is equal to,
\[
\Theta^2(v_1, v_2; \frac{1}{2} (1), \frac{1}{2} (1), \frac{1}{2} (1), \frac{1}{2} (0)) = \sqrt{\frac{(a_1 - a_2)}{(a_1 - c_1) (a_1 - c_2) (a_1 - c)}} (x_1 - c) 
\]
(40)
where (24),(25),(29) and (35) have been used. The double theta functions used in (40) are
\[
\begin{align*}
\theta_{13} &= \sum_{n_1 = -\infty}^{\infty} \sum_{n_2 = -\infty}^{\infty} e^{\pi i (v_1 (2n_1+1)+v_2 (2n_2))} q^{(2n_2)^2} \\
\theta_{12} &= \sum_{n_1 = -\infty}^{\infty} \sum_{n_2 = -\infty}^{\infty} e^{\pi i (v_1 (2n_1)+v_2 (2n_2+1))} q^{(2n_2+1)^2} \\
\end{align*}
\]
(41)
where Forsyth’s \(\theta\) notation is adopted,
\[
\theta_{13} = \Theta(v_1, v_2; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0) 
\]
(42)
and in the denominator of (40), is
\[
\theta_{12} = \Theta(v_1, v_2; \frac{1}{2}, 0, 0, \frac{1}{2}) 
\]
(43)

6 The Solution to the Hyper-elliptic \(\phi\) Integral

The solution to the hyper-elliptic \(\phi\) integral is, with \(x_1 = \frac{r_1}{r}\),
\[
r = (r_2 \theta_{12}^2 \sqrt{a_1 - a_2})/(c \sqrt{a_1 - a_2} \theta_{12}^2 + \sqrt{(a_1 - c_1) (a_1 - c_2) (a_1 - c)}) \theta_{13}^2 
\]
(45)
where \(\theta_{13}\) and \(\theta_{12}\) are given in (41) and (43). The hyper-elliptic \(\phi\) integral may be rewritten as,
\[
\int_{a_1}^{x_1} \frac{x \, dx}{y} = -\frac{\phi}{2} 
\]
(46)
and define a new variable, \(\phi'\), given by
\[
\int_{a_1}^{x_1} \frac{dx}{y} = \frac{\phi'}{2} 
\]
(47)
so that the variables in the argument of the theta functions are,

\[ v_1 = \frac{((h_{1,1} A_{1,0} + h_{1,2} A_{2,0}) \phi' - (h_{1,1} A_{1,1} + h_{1,2} A_{2,1}) \phi)}{(2 \pi i)} \]

\[ v_2 = \frac{((h_{2,1} A_{1,0} + h_{2,2} A_{2,0}) \phi' - (h_{2,1} A_{1,1} + h_{2,2} A_{2,1}) \phi)}{(2 \pi i)} \]

where \( h \) is defined in \( (26) \) and \( (27) \). The constants are given by,

\[ A_{1,0} = -a_1 = A_{2,0} \]

\[ A_{1,1} = A_{2,1} = 1 \]

With this choice the equations of Baker and Forsyth are identical to these, in the case of \( g = 2 \). The roots of the quintic, \( a_1, a_2, c_1, c_2 \) and \( c \) are given by setting the coefficients in \( (31) \) equal to,

\[ \lambda_0 = \frac{(4 \Lambda m^2 c^2 r_4^4)/(3 L^2)}{3} \]

\[ \lambda_1 = 0 \]

\[ \lambda_2 = 4 \left( E^2 c^2 L^2 + \Lambda/3 - m^2 c^2 \right) r_g^2 \]

\[ \lambda_3 = \frac{(4 m^2 c^2 r_g^2)/L^2}{L} \]

\[ \lambda_4 = -4 \]

then substuting into the solution to the quintic [Drociuk, 25]. If one thinks in terms of the advancement of perihelia, i.e. Mercury, in the case \( \Lambda = 0 \), the orbits obey doubly periodic functions. In the case when \( \Lambda \neq 0 \) they are equal to quadrupoly periodic functions of two variables \( \phi \) and \( \phi' \), equations \( (46) \) and \( (47) \). \( \phi \) is the physical angular coordinate defined in equation \( (1) \). \( \phi' \) exist’s because of Abel’s theorem, equation \( (28) \). Simply taking \( \phi' \) equal to zero cannot work for then equation \( (27) \) will go to zero and the transformation \( (26) \) becomes singular. Therefore \( \phi' \) is an unknown parameter?

### 7 The Reduction of the Accellerating Universe to the General Relativistic Universe

When the Cosmic Force is set to zero and particles are subject to graviation alone, two roots of the quintic, see [Drociuk, 25] or by inspection, go to zero. Then \( \tau_{1,2} = \tau_{2,2} \Rightarrow 0 \), then \( q' = r = 1 \), and setting \( n = 0 \), gives

\[ \frac{\theta_{13}^2}{\theta_{12}^2} \Rightarrow -k \, \text{sn}^2(v_1) \]

Where \( \text{sn} \) is one of Jacobi’s four elliptic functions [Forsyth, 26]. Choosing to let \( a_2 = c_2 \Rightarrow 0 \), then \( (56) \) becomes

\[ -k \, \text{sn}^2(v_1) = (a_1 - x_1)/\sqrt{(a_1 - c_1) (a_1 - c)} \]
where,
\[ a_1 = \frac{r_2}{r_1} \]
\[ c_1 = \frac{r_2}{r_3} \]
\[ c = \frac{r_2}{r_5} \]
\[ k = \sqrt{\frac{(c_1 - a_1)}{(c - a_1)}} \] (58)

which gives the Schwarzschild black hole solution without the cosmological constant, as a genus 1 elliptic function, which is the same as [Bartlett,14], it is
\[ \frac{1}{r} - \frac{1}{r_1} = \left( \frac{1}{r_3} - \frac{1}{r_1} \right) sn^2(v_1) \] (59)

Since two roots are zero, (32) becomes with \( a_2 = c_2 = 0 \),
\[ y^2 = 4 (x - a_1) (x - c_1) (x - c) \] (60)

which gives rise to genus \( g = 1 \) functions whose Riemann Surface is, Figure 3

Table 5 Normal Primitive Periods Genus One

| \( \omega_1 \) | \( \alpha_1 \) | \( \beta_1 \) |
|--------------|-------------|-------------|
| \( \omega'_1 \) | \( \omega_1 \) | \( \omega'_1 \) |

then by (26), with \( h_{1,2} = h_{2,1} = h_{2,2} = 0 \), gives
\[ h = h_{11} = \pi \frac{i}{\omega_{11}} \] (61)
Table 6 Genus One Moduli

| Int | $\alpha_1$ | $\beta_1$ |
|-----|-------------|------------|
| $\tau_{11}$ | 1 | $\tau_{11}$ |

Where,

$$\tau_{11} = (\omega_{11}')/(\omega_{11}) = \left(\int_{a_1}^{c_1} dx/y\right)/\left(\int_{a_1}^{c} dx/y\right)$$  \hspace{1cm} (62)

and

$$\pi i v^{x,a}_1 = -\left(\frac{\pi i}{\omega_{11}} \int_a^x A_{1,1} \ dx/y\right)$$  \hspace{1cm} (63)

which becomes with (46),

$$v_1 = (A_{1,1} \phi)/(\omega_{11} 2)$$  \hspace{1cm} (64)

and choose,

$$A_{1,1} = \omega_{1,1} \left(r_g \left(\frac{1}{r_3} - \frac{1}{r_1}\right)\right)^{1/2}$$  \hspace{1cm} (65)

8 Conclusion

The geodesics of particles in the combined Schwarzschild black hole and a vacuum energy density, are given by ratios of quadrupoly periodic theta functions of two variables. The four moduli of the theta functions are given by ratios of periods of the hyperelliptic integrals of the first kind. These these primitive normal periods may be obtained by integrating between the branch places on the Riemann surface of genus two.

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