Pair of null gravitating shells II. Canonically theory and embedding variables

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December 2001

Abstract

The study of the two shell system started in our first paper “Pair of null gravitating shells I” is continued. An action functional for a single shell due to Louko, Whiting and Friedman is generalized to give appropriate equations of motion for two and, in fact, any number of spherically symmetric null shells, including the cases when the shells intersect. In order to find the symplectic structure for the space of solutions described in paper I, the pull back to the constraint surface of the Liouville form determined by the action is transformed into new variables. They consist of Dirac observables, embeddings and embedding momenta (the so-called Kuchař decomposition). The calculation includes the integration of a set of coupled partial differential equations. A general method of solving the equations is worked out.
1 Introduction

The present paper is the second in a series dedicated to the two-shell system. The first paper, Ref. [1], will be referred to as I henceforth. The general motivation and aims are explained in I. Some solutions to the classical equations of motion containing two shells have been studied in I with two main results: First, certain coordinates have been chosen on the space of all solutions; they are candidates for a complete set of Dirac observables of the system. Second, all symmetries of the system that are associated with diffeomorphisms between solution spacetimes have been found. These symmetries will define the most interesting observables as well as the true Hamiltonian of the system.

To proceed, the space of solutions ought to be promoted to a physical phase space, which is a symplectic manifold. Without knowledge of the symplectic structure, not even the decision can be made whether some physical coordinates are differentiable. Therefore, we turn in this paper to the canonical theory.

As far as we know, no action functional for the two-shell system exists in literature. For a single spherically symmetric null shell, a Hamilton action principle has been studied by Kraus and Wilczek [2] and by Louko, Whiting and Friedman [3]. We shall start from this action and generalize it for two and more shells. The generalization is easy, and a proof is given that the resulting equations of motion admit the correct solutions for the system, including all solutions described in I. This, of course, is not a full justification for the choice of action. As it is well known, given dynamical equations admit many non-equivalent action principles from which they follow. An example for a single massive shell is investigated in [4]. A direct reduction of the Einstein-Hilbert plus a shell-matter action to spherical symmetry might lead to a non-equivalent theory. Ref. [5] is an example of such an action principle without symmetry, but it is valid only for a single shell. The generalization of the Louko-Whiting-Friedman action to more shells and intersections is simpler than that of the action in [3] so we have chosen to start from it. We assume that both actions lead to equivalent theories.

The plan of the paper is as follows. In Sec. 2, we propose our generalization of the Louko-Whiting-Friedman action principle to include two and, in fact, any number of in-going and out-going shells. We prove that the variation of this action gives the equations of motion that we need.

Sec. 3 begins with the calculation of Poisson brackets between the parameters that have been chosen as coordinates on the space of solutions in I. The strategy is to pull back the Liouville form defined by our action to the constraint surface, and

\footnote{These solutions are a subset of the solutions admitted by the equations of motion due to the additional assumption of the regular center on the left.}
to transform variables so that the new ones consist of the parameters, embeddings and embedding momenta. This is the so-called Kuchař decomposition studied in [4]. A similar transformation has been accomplished in [5] for the case of a single shell. The calculation includes integration of a set of partial differential equations of the first order. The integration method used in [5] seems to be rather closely associated with the particular problem studied and the gauge chosen there. We shall find a general method that works for any number of shells and in any double-null gauge.

2 Canonical formalism

2.1 The action

The action functional for a single light-like shell and its gravitational field has been written down in Ref. [3]. It is modified below so that it describes two such shells:

\[ S_2 = \int d\tau \left[ p_1 \dot{r}_1 + p_2 \dot{r}_2 + \int_0^\infty d\rho (P_{\Lambda} \dot{\Lambda} + P_R \dot{R} - H_2) \right]. \]  

The Hamiltonian has the same overall form as in [3]

\[ H_2 = N' H + N^\rho \mathcal{H}_\rho + N_\infty E_\infty, \]

but the new constraints are

\[ \mathcal{H} = \frac{\Lambda P_\Lambda^2}{2 R^2} - \frac{P_\Lambda P_R}{R} + \frac{R R''}{\Lambda} - \frac{R R' \Lambda'}{\Lambda^2} + \frac{R^2}{2 \Lambda} - \frac{\Lambda}{2}, \]

\[ \mathcal{H}_\rho = \frac{P_R R'}{\Lambda} - \frac{P_\Lambda \Lambda'}{\Lambda} - p_1 \delta(\rho - r_1) - p_2 \delta(\rho - r_2). \]

The momenta of the shells are \( p_1 \) and \( p_2 \), their radial coordinates are \( r_1 \) and \( r_2 \) and their radial directions are \( \eta_1 \) and \( \eta_2 \). The dot denotes the derivative with respect to \( \tau \) and the prime that with respect to \( \rho \). The “volume” variables \( \Lambda \), \( R \), \( P_\Lambda \), \( P_R \), \( N \) and \( N^\rho \) are the same as in [3] and [5]. The meaning of the variables \( \Lambda \), \( R \), \( N \) and \( N^\rho \) can be inferred from the spacetime metric

\[ ds^2 = -N^2 d\tau^2 + \Lambda^2 (d\rho + N^\rho d\tau)^2 + R^2 d\Omega^2. \]

The momenta conjugate to the configuration variables \( \Lambda \) and \( R \) are

\[ P_\Lambda = -\frac{R}{N} (\dot{R} - N^\rho R'), \]

and

\[ P_R = -\frac{\Lambda}{N} (\dot{R} - N^\rho R') - \frac{R}{N} [\dot{\Lambda} - (N^\rho \Lambda)']. \]
Compared to ref. [3], the only change is the presence of one more term in the Liouville form and of a corresponding term in each constraint. Each of these terms has the same form as in [3], but now, they carry indices so that the two shells can be distinguished.

It is important to state the differentiability conditions on the shells, especially at a possible shell crossing (if \( \eta_1 \neq \eta_2 \)). One can assume as in [3] that the gravitational variables are smooth functions of \( \rho \), with the exception that \( N', (N^\rho)' \), \( \Lambda' \), \( R' \), \( P_\Lambda \) and \( P_R \) may have finite discontinuities at isolated values of \( \rho \). The coordinate loci of the discontinuities are smooth functions of \( \tau \) for each shell. This follows from (i) the conditions at isolated single shell points which are the same as in [3], (ii) from the conditions at any shell crossing as described in Sec. 2.2 of I and (iii) from the corresponding choice of foliation: the metric with respect to coordinates \( \tau \) and \( \rho \) may be piecewise smooth and everywhere continuous.

### 2.2 Equations of motion

The rest of the section will be devoted to a check that the variation of the action (1) yields the proper dynamical equations for two shells. These equations have to imply (a) that the geometry outside the shells is the Schwarzschild one and (b) that the shells move along null surfaces. These two properties, together with the continuity of the metric everywhere and the assumption of a regular center on the left, have been used in Sec. 2 of I to construct all solutions. The continuity of the metric is part of the definition of the configuration space, and the existence of a left regular center is an assumption of our model, but the two other properties have to result from the variation of the action.

The continuity of the metric defines a \( C^1 \) class of coordinates; we shall work with such coordinates in this section. We also assume that the foliation coordinates \( \tau \) and \( \rho \) are \( C^1 \). The variations of the action (1) with respect to the gravitational variables \( N, N^\rho, R, \Lambda, P_R \) and \( P_\Lambda \) give the constraints,

\[
\mathcal{H} = 0, \quad \mathcal{H}_\rho = 0
\]
and the dynamical equations

\begin{align*}
\dot{\Lambda} &= \mathcal{N} \left( \frac{\Lambda P_\Lambda}{R^2} - \frac{P_R}{R} \right) + (\mathcal{N}' \rho \Lambda)', \\
\dot{R} &= -\frac{\mathcal{N} P_\Lambda}{R} + \mathcal{N}' \rho R', \\
\dot{P}_\Lambda &= \mathcal{N} \left[ \frac{P_\Lambda^2}{R^2} - \left( \frac{R'}{\Lambda} \right)^2 \right] + 1 + \frac{2 \eta_1 p_1}{\Lambda^2} \delta(\rho - r_1) + \frac{2 \eta_2 p_2^2}{\Lambda^2} \delta(\rho - r_2) \\
&\quad - \frac{\mathcal{N}' RR'}{\Lambda^2} + \mathcal{N}' \rho \rho', \\
\dot{P}_R &= \mathcal{N} \left[ \frac{\Lambda P_\Lambda^2}{R^3} - \frac{P_\Lambda P_R}{R^2} - \left( \frac{R'}{\Lambda} \right)' \right] - \left( \frac{\mathcal{N}' R}{\Lambda} \right)' + (\mathcal{N}' P_R)',
\end{align*}

Outside the shells ($\rho \neq r_1, \rho \neq r_2$), they coincide with the equations of [3] and imply that the geometry there is the Schwarzschild one corresponding to some, as yet arbitrary, value of Schwarzschild mass parameter.

### 2.2.1 Outside the crossing

Let us first consider a Cauchy surface that does not contain a crossing point of the two shells, $r_1 \neq r_2$. This includes data for crossing as well as for parallel shells.

The equations that we obtain at $\rho = r_i$, $i = 1, 2$, result partly from setting the coefficients at the delta functions $\delta(\rho - r_1)$ and $\delta(\rho - r_2)$ in Eqs. (8)-(11) equal to zero, and partly from the variations of the action (1) with respect to $r_i$ and $p_i$. If we use the notation $\delta_r(X)$ for the coefficient at the $\delta(\rho - r)$ in the expression $X$, then the equations read:

\begin{align*}
0 &= \delta_{r_i} \left( \frac{R R''}{\Lambda} \right) + \frac{\eta_i p_i}{\Lambda_{\rho = r_i}}, \\
0 &= \delta_{r_i} (\Lambda P_\Lambda') + p_i, \\
\delta_{r_i}(\hat{P}_\Lambda) &= \left( \frac{\mathcal{N}}{\Lambda^2} \right)_{\rho = r_i} \eta_i p_i + (\mathcal{N})_{\rho = r_i} \delta_{r_i}(P_\Lambda'), \\
\delta_{r_i}(\hat{P}_R) &= -(\mathcal{N})_{\rho = r_i} \delta_{r_i} \left( \left[ \frac{R'}{\Lambda} \right]' \right) - \delta_{r_i} \left( \left[ \frac{\mathcal{N}' R}{\Lambda} \right]' \right) + \delta_{r_i} \left( [\mathcal{N}' P_R]' \right), \\
\dot{r}_i &= \eta_i \left( \frac{\mathcal{N}}{\Lambda} \right)_{\rho = r_i} - (\mathcal{N}' \rho)'_{\rho = r_i}, \\
\dot{p}_i &= p_i \left( \mathcal{N}' - \eta_i \mathcal{N} \right)_{\rho = r_i}.
\end{align*}

By inspection, the equations with any fixed value of $i$ coincide with the one-shell equations of [3]. Thus, outside the shells, as well as at the shells outside the crossing,
we obtain equations that have been shown in [3] to imply the required properties (a) and (b).

To obtain expressions for the momenta $p_i$ conjugate to $r_i$, we can use the constraints (12) and (13) because we shall always work only at the constraint surface within the present series of papers. We obtain the equations

$$ R(r_i)\Delta r_i(R') + \eta_i p_i = 0 , $$
$$ \Lambda(r_i)\Delta r_i(P_\Lambda) + p_i = 0 . $$

Here, we use the notation $\Delta x(X)$ for the jump $\lim_{\rho \to x^+} X - \lim_{\rho \to x^-} X$ at the shell point $\rho = x$.

In any double-null coordinates $U$ and $V$, the metric has the form

$$ ds^2 = -AdUdV + R^2d\Omega^2 $$

and the transformation formulae are:

$$ \Lambda = \sqrt{-AU'V'} , $$
$$ \mathcal{N} = -\frac{\dot{U}V' - \dot{V}U'}{2UV'}\sqrt{-AU'V'} , $$
$$ \mathcal{N}^{\rho} = \frac{\dot{U}V' + \dot{V}U'}{2UV'} . $$

It follows that

$$ R' = R_UU' + R_VV' , $$
$$ P_\Lambda = \frac{R}{\Lambda}(R_UU' - R_VV') . $$

The last equation is derived from Eq. (5) and (21)–(23). We shall often need the following simple Lemma.

**Lemma 1** Let $\phi$ be a continuous function of $U$ and $V$ and let its derivatives have jumps only at $U = U(r)$ and $V = V(r)$, being themselves continuous elsewhere. Then $\phi_U$ is a continuous function of $V$ everywhere except for $U = U(r)$ where the jump $\Delta(\phi_U)$ is a continuous function of $V$ for all $V$ (even at $V = V(r)$). Similarly for $\phi_V$.

The Lemma follows immediately from the continuity of $\phi$.

Applying the Lemma to the function $R$, which is continuous, we find that $\Delta(R_U) \neq 0, \Delta(R_V) = 0$ for $\eta_i = +1$ and $\Delta(R_U) = 0, \Delta(R_V) \neq 0$ for $\eta_i = -1$.

Substituting this into Eqs. (18) and (19), one finds the desired expressions:

$$ p_i = -R(r_i)\Delta r_i(R_U)U' $$
for $\eta = 1$ and

$$p_i = R(r_i) \Delta_r(R,\tau)V'$$

(27)

for $\eta = -1$. The orientation of the functions $U$, $V$, $\tau$ and $\rho$ is such that $U' < 0$ and $V' > 0$ (see [3]).

### 2.2.2 At the crossing: the constraints

Let us now turn to the Cauchy surfaces that contain a crossing point of the shells, where we have $r_1 = r_2 = r$. Here, there is only one $\delta$-function, $\delta(\rho - r)$, and there is only one coefficient in all variation equations at $\delta(\rho - r)$. Thus, instead of two equations, we obtain only one in each case. A new analysis is in order.

Let us first consider the constraints (7). The coefficients of $\delta(\rho - r)$ in both constraints are

$$\delta_r(\mathcal{H}) = \frac{\eta_1 p_1}{\Lambda(r)} + \frac{\eta_2 p_2}{\Lambda(r)} + R(r) \Delta_r(R')$$

$$\delta_r(\mathcal{H}') = -p_1 - p_2 - \Lambda(r) \Delta_r(P\Lambda)$$

To study the equations $\delta_r(\mathcal{H}) = 0$ and $\delta_r(\mathcal{H}') = 0$, it is advantageous to choose some double-null coordinates in which the metric is continuous. Similar tactic has been pursued in the Appendix B of [3]. The expressions (24) and (25) for $R'$ and $P\Lambda$ in terms of general double-null coordinates yield now

$$\Delta_r(R') = \Delta_r(R,\tau)U'(r) + \Delta_r(R,\tau)V'(r)$$

$$\Delta_r(P\Lambda) = \frac{R(r)}{\Lambda(r)} [\Delta_r(R,\tau)U'(r) - \Delta_r(R,\tau)V'(r)]$$

Thus, the equation $\delta_r(\mathcal{H}) = 0$ becomes

$$R(r) \Delta_r(R,\tau)U'(r) + R(r) \Delta_r(R,\tau)V'(r) = -\eta_1 p_1 - \eta_2 p_2$$

(28)

and $\delta_r(\mathcal{H}') = 0$ is

$$R(r) \Delta_r(R,\tau)U'(r) - R(r) \Delta_r(R,\tau)V'(r) = -p_1 - p_2$$

(29)

Let $\eta = -\eta_2 = 1$. Then, we obtain from Eqs. (28) and (29)

$$p_1 = -R(r) \Delta_r(R,\tau)U'(r)$$

$$p_2 = R(r) \Delta_r(R,\tau)V'(r)$$

If $\eta = -\eta_2 = -1$, then

$$p_1 = R(r) \Delta_r(R,\tau)V'(r)$$

$$p_2 = -R(r) \Delta_r(R,\tau)U'(r)$$
This can be summarized by

\begin{align}
\mathbf{p}_{\text{out}} &= -R(r)\Delta_r(R,U)U'(r) , \\
\mathbf{p}_{\text{in}} &= R(r)\Delta_r(R,V)V'(r) .
\end{align}

(30) (31)

Notice that these equations are exactly the same as Eqs. (29) and (27) for shells outside the crossing. They can serve as definition of the momenta everywhere. From Lemma 1 and Eqs. (30) and (31) we can infer that \( \mathbf{p}_{\text{out}} \) is continuous along \( U = U(r) \) and \( \mathbf{p}_{\text{in}} \) along \( V = V(r) \).

The fact that we obtain only two relations from the four equations \( \delta r_1(H) = 0, \delta r_2(H) = 0, \delta r_1(H_\rho) = 0 \) and \( \delta r_2(H_\rho) = 0 \) outside the crossing is due to the mutual dependence of \( H \)- and \( H_\rho \)-equations (see [3]) in this case. At the crossing the \( H \)- and \( H_\rho \)-equations are independent.

### 2.2.3 At the crossing: the shell variables

Next, we study the variation of the action (1) with respect to the shell variables \( r_i \) and \( p_i \). The variation is to be done so that \( r_1 \) and \( r_2 \) are considered as independent variables even where \( r_1 = r_2 \); therefore, one must vary first and only then set \( r_1 = r_2 \). The resulting Euler-Lagrange equations then are

\begin{align}
\dot{r}_i &= \frac{\mathcal{N}(r)}{\Lambda(r)} \eta_i - \mathcal{N}_\rho(r) , \\
\dot{p}_i &= -\left( \frac{\mathcal{N}(\rho)}{\Lambda(\rho)} \eta_i - \mathcal{N}_\rho(\rho) \right)'_{\rho=r} \mathbf{p}_i .
\end{align}

(32) (33)

Eq. (32) is analogous to Eq. (16) and Eq. (33) to Eq. (17).

Let us assume that the index \( i \) is chosen in such a way that \( \eta_i \) remains constant along the trajectory of the \( i \)-th shell for each \( i = 1, 2 \). This is not the only possible assumption if the shells cross each other. For example, one could choose \( i = 1 \) for the innermost shell and \( i = 2 \) for the outermost one. Then the subsequent argument has to be modified.

The interpretation of Eq. (32) is the same as in [3], if we use our convention on the index \( i \). The right-hand side of Eq. (32) is then continuous along each shell because the functions \( \mathcal{N}, \mathcal{N}_\rho, \) and \( \Lambda \) are components of the metric in the coordinates \( \tau \) and \( \rho \). The vector tangential to the shell,

\[ l_i^\mu = (1, \dot{r}_i, 0, 0), \]

is a null vector as a consequence of Eq. (32). Hence, the equation completes the dynamical information that is necessary for construction of solutions. The remaining
equations for the shell have, therefore, to follow from Einstein’s equations outside the shells, Eqs. (30), (31) and (32).

Let us extend the definition of the quantity $l_i^\rho$ to a whole neighborhood of $i$-th shell by

$$l_i^\rho(\rho) := \frac{N(\rho)}{\Lambda(\rho)} \eta_i - N^\rho(\rho).$$

The function $l_i^\rho(\rho)$ can be calculated in terms of the metric and embeddings for an arbitrary double-null gauge. Using the transformation Eqs. (21)–(23), we obtain

$$l_i^\rho(\rho) = -\frac{\dot{U}}{U},$$

for $\eta_i = +1$ and

$$l_i^\rho(\rho) = -\frac{\dot{V}}{V},$$

for $\eta_i = -1$. At the shells, this follows also directly from the relations $U(\tau, r(\tau)) = \text{const}$ for $\eta_i = +1$ and $V(\tau, r(\tau)) = \text{const}$ for $\eta_i = -1$, and it is analogous to Eq. (B3) in [3]. The right-hand sides of Eqs. (34) and (35) are, of course, continuous functions in a neighborhood of $i$-th shell because we have required the foliation to be $C^1$.

Let us turn to Eq. (33). As in [3], the first question is whether the right-hand side is continuous across the shells; if it is not we have a bad ambiguity. Let $\eta_i = +1$ and let us consider the question of derivatives of the function $l_i^\rho(\rho)$ with respect to $\rho$ from both sides of $i$-th shell (the argument is completely analogous to that in Appendix B of [3]):

$$\left(\frac{\partial l_i^\rho}{\partial \rho}\right)_{\rho=r\pm} = -\frac{1}{U'} \left(\dot{U}' - \frac{\dot{U}}{U'} U''\right)_{\rho=r\pm}.$$

We have, however,

$$\left(\frac{\dot{U}'}{U'} U''\right)_{\rho=r\pm} = (\dot{U}')_{\rho=r\pm} + \dot{r}(U'')_{\rho=r\pm} = \frac{d}{d\tau} U'(\tau, r(\tau)),$$

and this is, of course, well defined. Hence, Eq. (33) is not ambiguous. The proof for $\eta_i = -1$ is similar.

Next, we show that Eq. (33) is satisfied if Eqs. (30) and (31) hold and the function $R$ is continuous at the crossing. As we have seen, Lemma 1 then implies that the jumps $\Delta(R,U)$ and $\Delta(R,V)$ are continuous along the corresponding shells, even
through the crossing point. Let $\eta_i = +1$ and let us substitute Eq. (26) for $p_i$ into Eq. (33). A simple calculation gives

$$\frac{d}{d\tau}[\Delta_r((R^2)_,U)]U'(\tau, r(\tau)) = 0$$

along the shell and also at the crossing point. However, outside the crossing, this equation follows from other equations (see [3]), so it does not contain any new information, even at the crossing: if $\Delta_r((R^2)_,U)$ is time independent from both sides and continuous everywhere, then it must be also constant everywhere.

2.2.4 At the crossing: $\Lambda$ and $R$

The remaining dynamical equations for the shells are obtained by varying the action (1) with respect to the variables $\Lambda$ and $R$ and setting the coefficients at $\delta(\rho - r)$ in the resulting equations equal to zero. The variations with respect to $P_\Lambda$ and $P_R$ give just definitions of $P_\Lambda$ and $P_R$, which do not contain $\delta(\rho - r)$. Thus, at the crossing point, we obtain the equations:

$$\delta_r(\dot{P}_\Lambda) - N^\rho(r)\delta_r(P'_\Lambda) = \frac{N(r)}{\Lambda^2(r)}(\eta_1 P_1 + \eta_2 P_2) ,$$

(36)

and

$$\delta_r(\dot{P}_R) - N^\rho(r)\delta_r(P'_R) + \frac{N(r)}{\Lambda(r)}\delta_r(R'') + \frac{R(r)}{\Lambda(r)}\delta_r(N'') = 0 .$$

(37)

while there are four equations outside the crossing, (14) and (15), that can be cast as follows:

$$\delta_r_i(\dot{P}_\Lambda) - N^\rho_i(r_i)\delta_r_i(P'_\Lambda) = \frac{N(r_i)}{\Lambda^2(r_i)}\eta_i P_i ,$$

(38)

and

$$\delta_r_i(\dot{P}_R) - N^\rho_i(r_i)\delta_r_i(P'_R) + \frac{N(r_i)}{\Lambda(r_i)}\delta_r_i(R'') + \frac{R(r_i)}{\Lambda(r_i)}\delta_r_i(N'') = 0$$

(39)

for each $i = 1, 2$.

Let us study the four discontinuous functions $R', N', P_\Lambda$ and $P_R$, the $\tau$ and $\rho$ derivatives of which feature under the $\delta_r$ signs in Eqs. (36)–(39). It is advantageous to express them as functions of $U$ and $V$ and of the foliation $U(\tau, \rho)$ and $V(\tau, \rho)$. For $R'$ and $P_\Lambda$, Eqs. (24) and (25) yield the desired expressions. For $N'$, we have immediately

$$N' = N'_U U' + N'_V V' .$$

(40)
For $P_R$, Eq. (6) and (21)–(23) yield
\[
P_R = R_u U' - R_v V' + \frac{RA_u}{2A} U' - \frac{RA_v}{2A} V' + \frac{RU''}{2U'} - \frac{RV''}{2V'}.
\]

We observe that
\[
\frac{U''}{U'} - \frac{V''}{V'} = \left[ \ln \left( -\frac{U'}{V'} \right) \right]' = \left[ \ln \left( -\frac{U'}{V'} \right) \right]_U U' + \left[ \ln \left( -\frac{U'}{V'} \right) \right]_V V',
\]
where $\ln(-U'/V')$ is considered as a (continuous) function of $U$ and $V$. Thus, we obtain, finally:
\[
P_R = \frac{1}{2AR} \left( R^2 A \right)_U U' - \frac{1}{2AR} \left( R^2 A \right)_V V'
+ \frac{R}{2} \left[ \ln \left( -\frac{U'}{V'} \right) \right]_U U' + \frac{R}{2} \left[ \ln \left( -\frac{U'}{V'} \right) \right]_V V'.
\]

Eqs. (24), (25), (40) and (41) show that each discontinuous function is a sum of terms of the form $\varphi \psi U$ or $\varphi \psi V$, where $\varphi$ and $\psi$ are continuous functions of $U$ and $V$. The jump structure of such terms is given by the Lemma 1.

Let us denote by $R'_\text{out}$, $N'_\text{out}$, $P_{\Lambda \text{out}}$ and $P_{R \text{out}}$ the sum of all terms in the right-hand sides of Eqs. (24), (25), (40) and (41) that contain only the $U$-derivatives, and similarly for $R'_\text{in}$ etc. Thus, e.g.,
\[
P_{R \text{out}} = \frac{1}{2AR} \left( R^2 A \right)_U U' - \frac{1}{2AR} \left( R^2 A \right)_V V'
+ \frac{R}{2} \left[ \ln \left( -\frac{U'}{V'} \right) \right]_U U',
\]

etc. Eqs. (36) and (37) can then be written as follows:
\[
[\delta_r(\langle P_{\Lambda \text{out}} \rangle') - \mathcal{N}^p(r) \delta_r(\langle P_{\Lambda \text{out}} \rangle')] + [\delta_r(\langle P_{\Lambda \text{in}} \rangle') - \mathcal{N}^p(r) \delta_r(\langle P_{\Lambda \text{in}} \rangle')]
= \frac{\mathcal{N}(r)}{\Lambda^2(r)} (\eta_{\text{out}} P_{\text{out}} + \eta_{\text{in}} P_{\text{in}}),
\]
and
\[
\delta_r(\langle P_{R \text{out}} \rangle') - \mathcal{N}^p(r) \delta_r(\langle P_{R \text{out}} \rangle')
+ \frac{\mathcal{N}(r)}{\Lambda(r)} \delta_r(\langle R'_{\text{out}} \rangle') + \frac{R(r)}{\Lambda(r)} \delta_r(\langle N'_{\text{out}} \rangle')
\]
\[
+ \delta_r(\langle P_{R \text{in}} \rangle') - \mathcal{N}^p(r) \delta_r(\langle P_{R \text{in}} \rangle')
+ \frac{\mathcal{N}(r)}{\Lambda(r)} \delta_r(\langle R'_{\text{in}} \rangle') + \frac{R(r)}{\Lambda(r)} \delta_r(\langle N'_{\text{in}} \rangle') = 0,
\]
while Eqs. (38) and (39) become, for the out-going shell,
\[
\delta_{r_{\text{out}}}(\langle P_{\Lambda \text{out}} \rangle') - \mathcal{N}^p(r_{\text{out}}) \delta_{r_{\text{out}}}(\langle P_{\Lambda \text{out}} \rangle')
= \frac{\mathcal{N}(r_{\text{out}})}{\Lambda^2(r_{\text{out}})} \eta_{\text{out}} P_{\text{out}}.
\]
\[
\delta r_{\text{out}} ((P_{\text{out}})'') - \mathcal{N}^p (r_{\text{out}}) \delta r_{\text{out}} ((P_{\text{out}})''') \\
+ \frac{\mathcal{N}(r_{\text{out}})}{\Lambda(r_{\text{out}})} \delta r_{\text{out}} ((R_{\text{out}})''') + \frac{R(r_{\text{out}})}{\Lambda(r_{\text{out}})} \delta r_{\text{out}} ((N_{\text{out}})''') = 0 \quad (45)
\]

and similarly for the in-going shell. The reason is that only the derivatives of the out-terms give contributions to \(\delta\)'s along the out-going shell because the in-terms are continuous, according to Lemma 1.

The left-hand sides of Eqs. (44) and (45) are obtained from the jumps of \(U\) derivatives of continuous functions in a way that is continuous along the out-going shell. The jumps themselves are continuous along the shell because of Lemma 1. However, the out-parts of Eqs. (42) and (43) are made in the same way from the jumps of \(U\) derivatives at \(r_{\text{out}} = r\). Because of the continuity of all terms along the shell, the out-part of the left-hand side of Eq. (42) is the \(r_{\text{out}} \to r\) limit of the left-hand side of Eq. (44). Analogous claims hold for left-hand sides of Eqs. (43) and (45) as well as for the in-terms. Moreover, because of the continuity of \(p_{\text{out}}\) and \(\eta_{\text{out}}\) along the out-going shell, the out-term on the right-hand side of Eq. (42) is the limit of the right-hand side of Eq. (44).

It follows that each of Eqs. (42) and (43) can be considered as sum of two equations, one being the limit \(r_{\text{out}} \to r\) of the corresponding out-going shell Eqs. (44) and (45), the other being the same limit of an in-going shell equation. Hence, if the out- and in-going shell equations hold for all values of \(r_{\text{out}}\) and \(r_{\text{in}}\) outside the crossing point, then Eqs. (42) and (43) are also valid. This is implied by the continuity conditions on the phase-space variables.

However, Eqs. (44) and (45) are satisfied because they follow from Eqs. (38) and (39), and these, in turn, follow from other dynamical equations. This has been shown in [3] for a single shell and the proof is, formally, the same in our case because all one-shell equations outside the shell crossing are formally identical with equations in [3].

To summarize: We have shown that the action (1) gives proper dynamical equations for the two-shell system.

### 3 The Liouville form at the constraint surface

Our final aim is to calculate the Poisson brackets between Dirac observables such as \(M_m, M_r, v_{m_2} - v_{m_1}\) and \(v_r\) defined in I. Our method will employ the property of the pull-back \(\Theta_\Gamma\) of the Liouville form \(\Theta\) to the constraint surface \(\Gamma\) that it depends only on the Dirac observables. Its external differential then defines the symplectic
form of the physical phase space. In the present section, we develop some general tools in this line.

The Liouville form of the action (1) can be written as follows:

\[ \Theta = p_1 \dot{r}_1 + p_2 \dot{r}_2 - N_\infty E_\infty + \int_0^\infty d\rho (P_\Lambda \dot{\Lambda} + P_R \dot{R}) . \]  

We have included the boundary part of the Hamiltonian into \( \Theta \); the form of this part justifies such inclusion (cf. [8], [7]). We can now start to transform (46) into the Kuchař variables corresponding to arbitrary double-null gauge. While \( \Theta \) does not depend of the gauge and dependent degrees of freedom, the old variables contain them and we must use the complete transformation; it goes from the variables \( R, \Lambda, P_R, P_\Lambda \) to observables, gauge variables and dependent variables. Still, the resulting \( \Theta \) contains then only the former and none of the latter.

To write down such a transformation, we shall use a particular Kuchař decomposition; for definition and existence, see [6]. To this aim, we shall choose an arbitrary double-null gauge, represented by the coordinates \( U \) and \( V \). We also let the choice of Dirac observables open, denoting them by \( o^k \), \( k = 1, 2, \ldots, 2N \). The complete set of final variables consists, therefore, of the physical variables \( o^k \), the gauge variables represented by the embeddings \( (U(\rho), V(\rho)) \) and the dependent variables represented by the embedding momenta \( P_U(\rho) \) and \( P_V(\rho) \); at the constraint surface, \( P_U(\rho) = P_V(\rho) = 0 \).

To start the calculation, we just need to know that the metric (20) depends on \( o^i \):

\[ A = A(U,V;o) , \quad R = R(U,V;o) . \]

Then we use the transformation formulae (21)–(23) and the definitions (5) and (6) of the momenta. Such calculation has been already carried out in [7] so we just take over the relevant general formulae from there.

The calculation in [7] then leaves this general stage and proceeds by making a particular choice of \( o^i \)'s as well as of the gauge \( U \) and \( V \). It grows rather complicated and can be accomplished by some miraculous tricks whose nature seems to be closely connected with the particular choices of gauge and observables made. The main purpose of the present section is to reveal a general structure that underlies the tricks and that is entirely general.

### 3.1 The volume part

The form (46) can be divided into a boundary part (the first three terms on the right-hand side) and the volume parts. Each volume part is associated with a particular component of the space between the shells; it has the form

\[ \int_a^b d\rho (P_\Lambda \dot{\Lambda} + P_R \dot{R}) . \]
where \( a \) and \( b \) are values of the coordinate \( \rho \) at the boundary of the volume. For example, \( a = 0 \) and \( b = r_1 \), or \( a = r_2 \), \( b = \infty \), etc.

The only “volume variables” among the final set are the embedding ones, \( U(\rho) \) and \( V(\rho) \), and the embedding momenta \( P_U(\rho) \) and \( P_V(\rho) \). However, \( \Theta_{\Gamma} \) cannot contain \( U(\rho) \) and \( V(\rho) \) because they are gauge variables; still less can it contain \( P_U(\rho) \) and \( P_V(\rho) \) because they vanish at \( \Gamma \). Hence, we expect that the volume parts of \( \Theta_{\Gamma} \) can all be reduced to some boundary terms. A general account of such a reduction is now given.

We can see that the form of the volume parts is independent of the system, more precisely, of the number of the shells in the system. Hence, we can use the methods of Ref. [7] for its transformation to the new variables.

For any double-null gauge, the equations

\[
4RR,UV + 4R,UR,V + A = 0 , \tag{47}
\]

\[
AR,UU - A,UR,U = 0 , \tag{48}
\]

\[
AR,VV - A,V,R,V = 0 \tag{49}
\]

represent the condition that the transformation is performed at the constraint surface \( \Gamma \) (cf. [7], Eqs. (32)–(34)).

For the transformation of any volume part into the new variables, we can make the ansatz

\[
\Theta^b_a |_{\Gamma} = \int_a^b d\rho \left[ (f \dot{U} + g \dot{V} + h_i \dot{o}^i)' + \dot{\varphi} \right] , \tag{50}
\]

as in [7]. In the expressions for the functions \( f \), \( g \), \( h_i \) and \( \varphi \), we can separate the terms with \( \rho \)-derivatives of the embeddings \( U(\rho), V(\rho) \) from terms in which \( U(\rho) \) and \( V(\rho) \) are not differentiated. The form of the transformation between the variables \( \Lambda, R, P_\Lambda \) and \( P_R \) on one side and \( U, V, P_U, P_V \) and \( o^i \) on the other then imply that

\[
f = \frac{RR,U}{2} \ln \left( -\frac{U'}{V'} \right) + F(U,V,o^i) , \tag{51}\]

\[
g = \frac{RR,V}{2} \ln \left( -\frac{U'}{V'} \right) + G(U,V,o^i) , \tag{52}\]

\[
h_i = \frac{RR,i}{2} \ln \left( -\frac{U'}{V'} \right) + H_i(U,V,o^i) , \tag{53}\]

\[
\varphi = RR,UU' - RR,VV' - \frac{R}{2}(R,UU' + R,VV') \ln \left( -\frac{U'}{V'} \right)
- FU' - GV' + \phi(U,V,o^i) , \tag{54}\]

13
and (see Eqs. (45)–(48) of [7])

\[ F_V - G_U = \frac{R}{2A} (2AR_{UV} - A_u R_V - A_v R_U) , \]  
(55)

\[ H_i U - F_i = -\frac{R}{2A} (2AR_{iU} - A_u R_U - A_v R_i) , \]  
(56)

\[ H_i V - G_i = \frac{R}{2A} (2AR_{iV} - A_u R_V - A_v R_i) , \]  
(57)

\[ \phi = 0 . \]  
(58)

Eqs. (55)–(58) are valid for any double-null gauge.

The ansatz (50) leads to the following transformation of the volume part:

\[ \Theta^b_a \big|_\Gamma = (f \dot{U} + g \dot{V} + h_i \dot{\phi} - \varphi \dot{b})_{\rho=b} - (f \dot{U} + g \dot{V} + h_i \dot{\phi} - \varphi \dot{a})_{\rho=a} + \frac{d}{d\tau} \left( \int_a^b d\rho \varphi \right) . \]

By ignoring the total time derivative we are thus left with an equivalent form that contains only boundary terms:

\[ \Theta^b_a \big|_\Gamma = (f \dot{U} + g \dot{V} + h_i \dot{\phi} - \varphi \dot{b})_{\rho=b} - (f \dot{U} + g \dot{V} + h_i \dot{\phi} - \varphi \dot{a})_{\rho=a} . \]  
(59)

The boundary values \( a \) and \( b \) of \( \rho \) depend, in general, on the time parameter \( \tau \) as they can also describe positions of shells. If the intermediate boundaries are chosen to coincide with the positions of shells, the total Liouville form \( \Theta \) reduces to the sum of contributions from (i) \( \rho = r \) (corresponding to points \( a, b \) at which the embedding \( U(\rho), V(\rho) \) intersects a shell), (ii) \( \rho = 0 \) (corresponding to the center \( R = 0 \)) and (iii) \( \rho \to \infty \) (corresponding to asymptotic infinity). The boundary between two adjacent spacetime regions \( \mathcal{M}_K \) and \( \mathcal{M}_{K+1} \) is either a light-like hypersurface (defined by an in-going or an out-going shell) or a crossing point (defined by the intersection of two shells of different \( \eta \)).

In this way, \( \Theta_\Gamma \) can be transformed to a sum of boundary terms, provided that a solution to the system of differential equations (55)–(57) can be found. Let us study these equations.

### 3.2 Properties of functions \( F, G \) and \( H_i \)

Let us first establish some general properties of the functions \( F, G \) and \( H_i \) that follow from Eqs. (55)–(57).

#### 3.2.1 Freedom in the functions \( F, G, H_i \)

Eqs. (55)–(57) form an inhomogeneous system of linear partial differential equations of first order. Eqs. (47)–(49) constitute the integrability condition for the system.
The general solution of the system can be written as a sum of a particular solution and a general solution of the homogeneous system. Let $F^0$, $G^0$ and $H^1_i$ be a particular solution, and let $F^1$, $G^1$ and $H^1_i$ be a solution of the homogeneous equations:

\begin{align*}
F^1_{,V} - G^1_{,U} &= 0, \\
H^1_{i,U} - F^1_{,i} &= 0, \\
H^1_{i,V} - G^1_{,i} &= 0.
\end{align*}

Let us define the function $c_{ij}$ by

\[ c_{ij} := H^1_{i,j} - H^1_{j,i} \]  

and study its properties. Immediately from the definition, we have

\[ c_{ij} = -c_{ji}, \]  

and

\[ c_{ijk} + c_{jki} + c_{kij} = 0. \]

The derivative of Eq. (61) with respect to $o^i$ with subsequent anti-symmetrization in the indices $i$ and $j$ yield

\[ c_{ij,U} = 0. \]

Using Eq. (62) in a similar way gives

\[ c_{ij,V} = 0. \]

Hence, $c_{ij}$ depends only on Dirac’s observables. Eqs. (64) and (63) imply then that there is a function $C_i(o)$ such that

\[ c_{ij} = C_{i,j} - C_{j,i}. \]

Let us choose an arbitrary $C_i(o)$. Then the functions $F^1$, $G^1$ and $H^1_i$ have to satisfy the equations

\begin{align*}
F^1_{,V} - G^1_{,U} &= 0, \\
H^1_{i,U} - F^1_{,i} &= 0, \\
H^1_{i,V} - G^1_{,i} &= 0, \\
H^1_{i,j} - H^1_{j,i} &= C_{i,j} - C_{j,i},
\end{align*}

Again, this is an inhomogeneous linear differential system. The following is clearly a particular solution:

\[ F^1 = G^1 = 0, \quad H^1_i = C_i. \]
Any solution of the corresponding homogeneous system has, however, the form

\[ F^1 = W, \quad G^1 = W, \quad H_i^1 = W, \]

where \( W \) is an arbitrary function of \( U, V, \) and \( o^i. \)

We have shown: Let \( F^0, G^0 \) and \( H_i^0 \) be a solution to the system (55)–(57). Then any other solution \( F, G, H_i \) has the form

\[ F = F^0 + W, \quad G = G^0 + W, \quad H_i = H_i^0 + W + C_i, \]

(70)

where \( W \) is an arbitrary function of the variables \( U, V \) and \( o^i, \) and \( C_i \) is an arbitrary function of Dirac’s observables.

If we substitute the solution (70) into Eq. (50) for \( \Theta^b_a | \Gamma \), we obtain for the terms containing the functions \( W \) and \( C_i \) (the sum of these terms is denoted by \( \delta \Theta^b_a | \Gamma \)):

\[ \delta \Theta^b_a | \Gamma = \int_a^b d\rho \left[ (W, \dot{U} + W, \dot{V} + W, \dot{o}^i + C_i \dot{o}^i)' + (-W, U' - W, V') \right]. \]

However, Dirac’s observables do not depend on \( \rho \), so \( (C_i \dot{o}^i)' = 0 \) and \( W, U' - W, V' = w' \), where we have defined

\[ w(\rho, \tau) := W(U(\rho, \tau), V(\rho, \tau), o^i(\tau)). \]

Using this, we have

\[ \delta \Theta^b_a | \Gamma = \int_a^b d\rho \left[ (\dot{w})' - (w')' \right] = 0 \]

because all functions are \( C^\infty \) in the space between the shells. Hence, each solution (70) leads to the same Liouville form.

The \( W \)- and \( C_i \)-part of the first parenthesis in Eq. (55) is, however, a total time derivative:

\[ (W, \dot{U} + W, \dot{V} + W, \dot{o}^i + C_i \dot{o}^i + W, U' \dot{b} - W, V' \dot{b})_{\rho=b} = \]

\[ \frac{\partial w(b, \tau)}{\partial b} \dot{b} + \frac{\partial w(b, \tau)}{\partial \tau} = \frac{d}{d\tau} w(b(\tau), \tau), \]

which, in general, is non zero. Similar result holds for the second parenthesis. Hence, different solutions lead to equivalent boundary Liouville forms.

### 3.2.2 Gauge transformation of \( F, G \) and \( H_i \)

Two different double-null gauges lead to two different transformation of a given volume part of the Liouville form. We can, therefore, ask how the functions \( F, G \) and \( H_i \) are transformed if the gauge changes. The leading idea of course is that the Liouville form itself does not change.
A general gauge transformation between two double-null gauges, $\tilde{U}, \tilde{V}$ and $U, V$, reads

$$U = X(\tilde{U}, o), \quad V = Y(\tilde{V}, o), \quad (71)$$

where $X$ and $Y$ are suitable functions; the inverse transformation can be written as

$$\tilde{U} = \tilde{X}(U, o), \quad \tilde{V} = \tilde{Y}(V, o). \quad (72)$$

The form $f\tilde{U} + g\tilde{V} + h_i\tilde{o}^i$ transforms under the change (72) of variables as follows

$$f\tilde{U} + g\tilde{V} + h_i\tilde{o}^i = \tilde{f}\tilde{U} + \tilde{g}\tilde{V} + \tilde{h}_i\tilde{o}^i,$$

where

$$f = \tilde{f}\tilde{X}, \quad g = \tilde{g}\tilde{Y}, \quad h_i = \tilde{h}_i + \tilde{f}\tilde{X}_i + \tilde{g}\tilde{Y}_i.$$

For the function $R(U, V, o) = \tilde{R}(\tilde{U}, \tilde{V}, o)$, we obtain

$$R_{\tilde{U}} = \tilde{R}_{\tilde{X}}\tilde{X}_U, \quad R_{\tilde{V}} = \tilde{R}_{\tilde{Y}}\tilde{Y}_V,$$

and

$$R_i = \tilde{R}_i + \tilde{R}_{\tilde{X}}\tilde{X}_i + \tilde{R}_{\tilde{Y}}\tilde{Y}_i.$$

The transformation of the logarithm is

$$\ln \left( -\frac{\tilde{U}'}{\tilde{V}'} \right) = \ln \left( -\frac{\tilde{X}_U U'}{\tilde{Y}_V V'} \right) = \ln \left( -\frac{U'}{V'} \right) + \ln \left( -\frac{\tilde{X}_U}{\tilde{Y}_V} \right).$$

Collecting all terms, we obtain

$$F = \frac{\partial}{\partial U} \left[ \frac{R^2}{4} \ln \left( \frac{\tilde{X}_U}{\tilde{Y}_V} \right) \right] - \frac{R^2}{4} \tilde{X}_{UU} + \tilde{F}\tilde{X}_U, \quad (73)$$

$$G = \frac{\partial}{\partial V} \left[ \frac{R^2}{4} \ln \left( \frac{\tilde{X}_U}{\tilde{Y}_V} \right) \right] + \frac{R^2}{4} \tilde{Y}_{VV} + \tilde{G}\tilde{Y}_V, \quad (74)$$

$$H_i = \frac{\partial}{\partial o^i} \left[ \frac{R^2}{4} \ln \left( \frac{\tilde{X}_U}{\tilde{Y}_V} \right) \right] - \frac{R^2}{4} \left( \frac{\tilde{X}_{Ui}}{\tilde{X}_U} - \frac{\tilde{Y}_{Vi}}{\tilde{Y}_V} \right) + \tilde{H}_i + \tilde{F}\tilde{X}_i + \tilde{G}\tilde{Y}_i. \quad (75)$$

The first terms on the right-hand sides of Eqs. (73)–(75) represent a divergence; according to the result of the previous section, these terms can be thrown away, and we obtain finally:

$$F = \tilde{F}\tilde{X}_U - \frac{R^2}{4} \tilde{X}_{UU}, \quad (76)$$

$$G = \tilde{G}\tilde{Y}_V + \frac{R^2}{4} \tilde{Y}_{VV}, \quad (77)$$

$$H_i = \tilde{H}_i + \tilde{F}\tilde{X}_i + \tilde{G}\tilde{Y}_i - \frac{R^2}{4} \left( \frac{\tilde{X}_{Ui}}{\tilde{X}_U} - \frac{\tilde{Y}_{Vi}}{\tilde{Y}_V} \right). \quad (78)$$
These equations yield the transformation of the functions $\tilde{F}$, $\tilde{G}$ and $\tilde{H}_i$ that solve Eqs. (55)–(57) for the gauge $\tilde{U}, \tilde{V}$ to the functions $F$, $G$, and $H_i$ that solve them for the gauge $U, V$.

### 3.3 Integration of equations (55)–(57)

The transformation equations (76)–(78) can be used to calculate the solutions $F$, $G$, and $H_i$ from some well-known solutions $\tilde{F}$, $\tilde{G}$ and $\tilde{H}_i$. In fact, one can always choose the gauge $\tilde{U}, \tilde{V}$ in such a way that the right-hand sides of Eqs. (55)–(57) simplify and become trivially solvable. Let us, for instance, choose the gauge as follows.

Let the volume term to be transformed correspond to the flat spacetime. Then we can choose $\tilde{U}$ and $\tilde{V}$ as the retarded and advanced time coordinates for Minkowski spacetime and

$$\tilde{A} = 1, \quad \tilde{R} = \frac{-\tilde{U} + \tilde{V}}{2}.$$ 

Eqs. (55)–(57) now read

$$\tilde{F}_{\tilde{V}} - \tilde{G}_{\tilde{U}} = 0,$$

$$\tilde{H}_{i,\tilde{U}} - \tilde{F}_{,i} = 0,$$

$$\tilde{H}_{i,\tilde{V}} - \tilde{G}_{,i} = 0,$$

and we guess a solution to be

$$\tilde{F} = \tilde{G} = \tilde{H}_i = 0 \quad (79)$$

for all $i$.

Let the volume term correspond to the $(\alpha\beta)$-quadrant (see I, Sec. 2.1) of the Schwarzschild spacetime with the mass parameter $M(o)$ (the mass parameter is Dirac’s observable and so it is, in general, a function of the chosen complete system of those observables). Then we can choose $\tilde{U}^\alpha$ and $\tilde{V}^\beta$ to be the double-null Eddington-Finkelstein coordinates defined by Eqs. (1) and (2) of I so that Eqs. (3), (4) and (5) of I hold:

$$\tilde{A} = \left| 1 - \frac{2M(o)}{\tilde{R}} \right| = \alpha\beta \left( 1 - \frac{2M(o)}{\tilde{R}} \right), \quad (80)$$

$$\tilde{R} = 2M(o)\kappa \left[ \alpha\beta \exp \left( -\alpha\tilde{U}^\alpha + \beta\tilde{V}^\beta \right) \right], \quad (81)$$

where $\kappa$ is defined by Eq. (5) of Ref. [1]. On substituting Eqs. (80) and (81) for $\tilde{A}$
and $R$ into Eqs. (53)–(57), we obtain

$$
\tilde{F}_{,\bar{V}} - \tilde{G}_{,\bar{U}} = 0 ,
$$

$$
\tilde{H}_{i,\bar{U}} - \tilde{F}_{,i} = -\frac{\alpha}{2} M_{,i} ,
$$

$$
\tilde{H}_{i,\bar{V}} - \tilde{G}_{,i} = -\frac{\beta}{2} M_{,i} .
$$

Again, we guess easily

$$
\tilde{F} = 0 ,
$$

$$
\tilde{G} = 0 ,
$$

$$
\tilde{H}_{i} = -\frac{\alpha \bar{U} + \beta \bar{V}}{2} M_{,i} .
$$

The double-null Eddington-Finkelstein gauge may be simple, but it is singular ($\bar{U} = \infty$ at the future and $\bar{V} = -\infty$ at the past horizon). The solution (83), (84) and (85) diverges at both horizons. This singularity can, however, be removed by subtracting a suitable $W$-term. To show this, let us transform to a regular gauge, for example to the Kruskal coordinates $U$ and $V$:

$$
U = -\exp \left( -\frac{\alpha \bar{U}}{4M} \right) , \quad V = \exp \left( \frac{\beta \bar{V}}{4M} \right) .
$$

Then

$$
\tilde{X}(U,o) = -4\alpha M(o) \ln(-U) ,
$$

$$
\tilde{Y}(U,o) = 4\beta M(o) \ln(V) .
$$

An easy calculation using Eqs. (76)–(78) yields

$$
F = \frac{R^2}{4U} , \quad G = -\frac{R^2}{4V} , \quad H_{i} = -2MM_{,i} \ln \left( -\frac{U}{V} \right) .
$$

These functions are singular at the horizons ($U = 0$ or $V = 0$). However, if we choose

$$
W(U,V,o) = -M^2(o) \ln \left( -\frac{U}{V} \right) ,
$$

then the equivalent solution defined by

$$
F_{\text{reg}} := F + W_{,U} , \quad G_{\text{reg}} := G + W_{,V} , \quad H_{i,\text{reg}} := H_{i} + W_{,i}
$$
is regular everywhere. We obtain (using some properties of the function $\kappa$, see Eq. (12) and (51) of [7])

\begin{align*}
F_{\text{reg}} &= -\frac{M(R + 2M)}{2\exp(R/2M)}V, \\
G_{\text{reg}} &= \frac{M(R + 2M)}{2\exp(R/2M)}U, \\
H_{\text{reg}} &= 0,
\end{align*}

where

\[ R = 2M\kappa(-UV). \]

The solutions (79) and (83)–(85) or (86)–(88) together with the formulae (76)–(78) can help us to calculate the functions $F$, $G$ and $H_i$ for the two-shell system. For the single shell, the resulting formulae of [7] can easily be reproduced by the new method.

**Acknowledgments**

The authors are thankful for useful discussions by C. Kiefer and K. V. Kuchař. The work has been supported by the Swiss Nationalfonds and by the Tomalla Foundation, Zurich.

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