Mixed Estimates for Degenerate Multilinear Oscillatory Integrals and their Tensor Product Generalizations

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Abstract
We prove that the degenerate trilinear operator $C_{3,1,1,-1}$ given by the formula

$$C_{3,1,1,-1}(f_1, f_2, f_3)(x) = \int_{x_1 < x_2 < x_3} \hat{f}_1(x_1) \hat{f}_2(x_2) \hat{f}_3(x_3) e^{2\pi i x(-x_1 + x_2 + x_3)} dx_1 dx_2 dx_3$$

satisfies the estimate

$$||C_{3,1,1,-1}(\vec{f})||_{p_1^{-1}, p_2^{-1}, p_3^{-1}} \lesssim ||\hat{f}_1||_{p_1'} ||f_2||_{p_2} ||f_3||_{p_3}$$

for all $f_1 \in L^{p_1}(\mathbb{R}) : \hat{f}_1 \in L^{p_1'}(\mathbb{R})$, $f_2 \in L^{p_2}(\mathbb{R})$, and $f_3 \in L^{p_3}(\mathbb{R})$ under the assumption that $p_1 > 2$, $\frac{1}{p_1} + \frac{1}{p_2} < 1$, and $\frac{1}{p_2} + \frac{1}{p_3} < 3/2$. Mixed estimates for some multilinear generalizations of $C_{3,1,1,-1}$ and for several tensor product operators such as $BHT \otimes BHT$ are also shown. As an application, we obtain the boundedness of special upper-triangular biparameter AKNS systems.

Keywords: Multilinear Singular Operators, Oscillatory Integrals

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1. Introduction

Many boundedness results have been proven for singular multilinear oscillatory integrals with nonstandard symbols, e.g., \cite{5, 6, 8, 11, 13, 14}. One of the earliest examples is a theorem attributed to Menshov and Zygmund which states that the bilinear operator $\tilde{C}_2$ defined pointwise by

$$\tilde{C}_2(\tilde{f})(x) = \int_{x_1 < x_2} f_1(x_1)f_2(x_2)e^{2\pi i x_1x_2}dx_1dx_2,$$

is a continuous map from $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ into $L^{\frac{1}{p_1} + \frac{1}{p_2}}(\mathbb{R})$ whenever $f \in L^{p_1}(\mathbb{R})$ and $g \in L^{p_2}(\mathbb{R})$ for $1 \leq p_1, p_2 < 2$, see \cite{8}. Much later, Lacey and Thiele generated renewed interest in the subject by proving a large range of $L^p$ estimates in \cite{8} for the bilinear Hilbert transform given by

$$BHT(\tilde{f})(x) = \int_{x_1 < x_2} \hat{f}_1(x_1)\hat{f}_2(x_2)e^{2\pi i x_1x_2}dx_1dx_2,$$

after which estimates were proved by Muscalu, Tao, and Thiele in \cite{11} for a trilinear variant of the BHT called the Biest, defined as

$$C_{3}^{1,1,1}(\tilde{f})(x) = \int_{x_1 < x_2 < x_3} \hat{f}_1(x_1)\hat{f}_2(x_2)\hat{f}_3(x_3)e^{2\pi i x_1x_2x_3}dx_1dx_2dx_3.$$

However, oscillatory integrals with sign degeneracies such as the operator

$$C_{3}^{1,1,1}(\tilde{f})(z) = \int_{x_1 < x_2 < x_3} \hat{f}_1(x_1)\hat{f}_2(x_2)\hat{f}_3(x_3)e^{2\pi i (-x_1x_2x_3)}dx_1dx_2dx_3$$

have been shown to satisfy no $L^p$ estimates, see \cite{8}. Despite this fact, we prove in theorem \cite{8} that there exists a constant $C_{p_1,p_2,p_3}$ such that for all $f_1 \in L^{p_1}(\mathbb{R})$ satisfying $\hat{f}_1 \in L^{p_1'}(\mathbb{R})$ along with $f_2 \in L^{p_2}(\mathbb{R})$ and $f_3 \in L^{p_3}(\mathbb{R})$,

$$||C_{3}^{1,1,1}(\tilde{f})||_{L^\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}} \leq C_{p_1,p_2,p_3}||\hat{f}_1||_{L^{p_1'}}||f_2||_{L^{p_2}}||f_3||_{L^{p_3}},$$

as long as $p_1 > 2$, $\frac{1}{p_1} + \frac{1}{p_2} < 1$ and $\frac{1}{p_2} + \frac{1}{p_3} < 3/2$. Because continuity holds on the restricted domain $\{f_1 \in L^{p_1}(\mathbb{R}) : \hat{f}_1 \in L^{p_1'}(\mathbb{R})\} \times L^{p_2}(\mathbb{R}) \times L^{p_3}(\mathbb{R})$ for some set of exponents, we say $C_{3}^{1,1,1}$ satisfies mixed estimates.
1.1. Structure of the Paper

In section 2, we establish a large variety of mixed boundedness results for degenerate multilinear oscillatory integrals motivated by the bilinear Hilbert transform by combining facts from time-frequency analysis with the Christ-Kiselev type martingale structure decomposition used to prove the Menshov-Zygmund theorem. For a detailed account of martingale structures, see the excellent paper by Christ and Kiselev [2]. In the proofs, we come across a generalized version of the Littlewood-Paley inequality of Rubio de Francia for \( L^p \) functions whose Fourier transforms live in Lebesgue spaces with low exponents, even though it is false for generic \( L^p \) functions whose Fourier transforms live in Lebesgue spaces with low exponents.

We later extend our study in section 4 to tensor products of several non-degenerate operators, which are known to satisfy no \( L^p \) estimates from [10], and prove new statements in this setting as well. For instance, theorem 7 establishes continuity for the bilinear Hilbert transform and its generalizations. To show our results are, in a sense, the best possible, we prove in section 3 that boundedness fails for almost all of the remaining exponent cases.

NOTE: \( A \lesssim B \) means there is a constant \( C \), which may change from line to line, such that \( A \leq CB \). The relations \( A \gtrsim B \) and \( A \simeq B \) are defined similarly. The space \( L^p = L^p(\mathbb{R}) \) unless otherwise stated.

2. Mixed Estimates

To introduce the martingale structure decomposition originally due to Christ and Kiselev, we recreate the proof of the Menshov-Zygmund theorem as found in [3]. Continuity of the map \( \tilde{\mathcal{C}}^{\alpha_1, \alpha_2}_2 : L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \to L^{p_1+1/2p_2}(\mathbb{R}) \) given by

\[
\tilde{\mathcal{C}}^{\alpha_1, \alpha_2}_2(f_1, f_2)(x) := \int_{x_1 < x_2} f_1(x_1) f_2(x_2) e^{2\pi i x(\alpha_1 x_1 + \alpha_2 x_2)} dx_1 dx_2
\]

for \( p_1 < 2 \), \( p_2 = 2 \) and \( \alpha_1, \alpha_2 \neq 0 \) is shown by first splitting the domain of integration \( \{(x_1, x_2) : x_1 < x_2 \} \) into disjoint sets depending on the weighted distance between \( x_1 \) and \( x_2 \). Specifically, define a map \( \gamma_{f_2} : \mathbb{R} \to [0, 1] \) given by

\[
\gamma_{f_2}(x) = \frac{\int_{-\infty}^{x} |f_2(x_2)|^2 dx_2}{\|f_2\|^2}
\]

and form for every \( m \in \mathbb{Z}^+ \cup \{0\} \) and \( 0 \leq j < 2^m \), the martingale structure \( E_j^m = \gamma_{f_2}^{-1}([j 2^{-m}, (j + 1) 2^{-m})) \). Then define \( E_{j,t}^m = \gamma_{f_2}^{-1}([j 2^{-m}, (j + 1/2) 2^{-m})) \) and \( E_{j,s}^m = \gamma_{f_2}^{-1}([(j + 1/2) 2^{-m}, (j + 1) 2^{-m})) \) and construct the partition
\[(x_1, x_2) \in \mathbb{R}^2 : x_1 < x_2 \] = \bigcup_{m \in \mathbb{Z}^+ \cup \{0\}} \bigcup_{0 \leq j < 2^m} E_{j,i}^m \times E_{j,r}^m. \] (2)

This decomposition separates points according to the smallest dyadic interval that contains both \(\gamma(x_1)\) and \(\gamma(x_2)\). A quick computation then yields the result:

\[
\|\tilde{C}_2(f_1, f_2)\|_{1/2 + 1/p'} = \left\| \sum_{m \geq 0} \sum_{j=0}^{2^m-1} \tilde{C}_2(f_1 \chi_{E_{j,r}^m}, f_2 \chi_{E_{j,l}^m}) \right\|_{1/2 + 1/p'} \quad \text{(using (2))}
\]

\[
\leq \sum_{m \geq 0} \sum_{j=0}^{2^m-1} \left\| \tilde{f}_1 \chi_{E_{j,r}^m} (\alpha_1 \cdot \tilde{f}_2 \chi_{E_{j,l}^m} (\alpha_2)) \right\|_{1/2 + 1/p'}
\]

\[
\lesssim \sum_{m \geq 0} \sum_{j=0}^{2^m-1} \|f_1 \chi_{E_{j,r}^m}\|_p \|f_2 \chi_{E_{j,l}^m}\|_2 \quad \text{(by Hölder and Hausdorff-Young)}
\]

\[
= \sum_{m \geq 0} 2^m \left( \frac{1}{2m} \sum_{j=0}^{2^m-1} \|f_1 \chi_{E_{j,l}^m}\|_p^p \right)^{1/p} \|f_2\|_2 2^{-m/2}
\]

\[
\leq \sum_{m \geq 0} 2^{m/2} \left( \frac{1}{2m} \sum_{j=0}^{2^m-1} \|f_1 \chi_{E_{j,l}^m}\|_p^p \right)^{1/p} \|f_2\|_2 \quad \text{(by concavity)}
\]

\[
\lesssim \sum_{m \geq 0} 2^{m(1/2-1/p)} \|f_1\|_2 \|f_2\|_p \quad \text{(by disjointness)}
\]

\[
\lesssim \|f_1\|_p \|f_2\|_2.
\]

Note that the above proof adapted the martingale structure to the \(L^2\) function. It is worth pointing out that one could just as well have adapted the martingale to the \(L^p\) function, with a slightly modified proof. This second approach to the Menshov-Zygmund turns out to be the right one to generalize to more complicated operators. Before we illustrate this idea in theorem 2, we first record the following generalized Rubio de Francia inequality for arbitrary dimensions:

**Theorem 1** ([7], [12]). Let \(\{I_j\}_{j \in \mathbb{Z}}\) be a collection of disjoint rectangles in \(\mathbb{R}^n\) for \(n \geq 1\). Then the modified square function \(\mathcal{S}_r : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)\) given by

\[
\mathcal{S}_r(f) = \left( \sum_{j \in \mathbb{Z}} |f \ast \tilde{\chi}_j|^r \right)^{1/r}
\]
is continuous provided \( r > p' \).

**Definition 1.** For \( n \geq 1 \) and \( \vec{\epsilon} \in \{\pm 1\}^n \),

\[
C_n^\vec{\epsilon}(f_1, \ldots, f_n)(x) = \int_{x_1 < \ldots < x_n} \hat{f}_1(x_1) \ldots \hat{f}_n(x_n)e^{2\pi i \vec{\epsilon} \cdot \vec{x}}d\vec{x}.
\]

**Definition 2.** For \( 1 \leq p \leq \infty \), the Wiener space \( W_p \) is defined by

\[
W_p(\mathbb{R}) = \{ f \in L^p(\mathbb{R}) : \hat{f} \in L^{p'}(\mathbb{R}) \}
\]

and given the structure of a normed vector space with \( ||f||_{W_p} = ||\hat{f}||_{p'} \).

Note that as sets, \( W_p \subset L^p \) is properly included for \( p > 2 \) while \( W_p = L^p \) in the case \( p \leq 2 \). Before arriving at the main result in this section, namely theorem 5, we prove a few special cases.

**Theorem 2.** The trilinear operator \( C_{ij}^{-1,1,1} : W_{p_1} \times L^{p_2} \times L^{p_3} \to L^{\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}} \) is continuous provided \( \frac{1}{p_1} + \frac{1}{p_2} < 1 \) and \( \frac{1}{p_2} + \frac{1}{p_3} < 3/2 \).

**Proof.** We introduce a martingale structure \( E_j^m \) ala Christ and Kiselev given by

\[
\gamma_{f_1}(x) = \frac{\int_{-\infty}^{x} \hat{f}_1|^{p_1'} dx}{||\hat{f}_1||_{p_1'}}
\]

\[
E_j^m = \gamma_{f_1}^{-1}(\{j2^{-m}, (j+1)2^{-m}\})
\]

so that \( ||\hat{f}_1 \chi_{E_j^m}||_{p_1'} = 2^{-m/p_1'} \) for all \( 0 \leq j < 2^m \). As before, it is helpful to define

\[
E_{j,l}^m = \gamma_{f_1}^{-1}(\{j2^{-m}, (j+1/2)2^{-m}\})
\]

\[
E_{j,r}^m = \gamma_{f_1}^{-1}(\{(j+1/2)2^{-m}, (j+1)2^{-m}\})
\]

and construct the partition

\[
\{(x_1, x_2) \in \mathbb{R}^2 : x_1 < x_2\} = \bigcup_{m \in \mathbb{Z}^+ \cup \{0\}} \bigcup_{j=0}^{2^m-1} E_{j,l}^m \times E_{j,r}^m.
\]

We next split the problem into two cases, depending on whether the target exponent lies above or below 1. Perhaps it is a little surprising that our quasi-Banach analysis is not much different from the Banach version.

**CASE 1:** \( q = \frac{1}{p_1 + \frac{1}{p_2} + \frac{1}{p_2}} \geq 1 \). Splitting the connection \( x_1 < x_2 \) gives
\[ \|C_3^{-1,1,1}(f_1, f_2, f_3)\|_q \]
\[ \leq \sum_{m \geq 0} \left( \sum_{j=0}^{2^m-1} |f_1 \ast \tilde{E}^{m}_{j,l}|^2 \right)^{1/2} \left( \sum_{j=0}^{2^m-1} |BHT(f_2 \ast \tilde{E}^{m}_{j,r}, f_3)|^2 \right)^{1/2} \]
\[ \leq \sum_{m \geq 0} \left( \sum_{j=0}^{2^m-1} |f_1 \ast \tilde{E}^{m}_{j,l}|^2 \right)^{1/2} \left( \sum_{j=0}^{2^m-1} |BHT(f_2 \ast \tilde{E}^{m}_{j,r}, f_3)|^2 \right)^{1/2} \cdot \]

As before, the idea is produce a convergent geometric sum over scales. Thus, it suffices to observe for \( \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{p_3} \) that

\[ \left\| \left( \sum_{j=0}^{2^m-1} |f_1 \ast \tilde{E}^{m}_{j,l}|^2 \right)^{1/2} \left( \sum_{j=0}^{2^m-1} |BHT(f_2 \ast \tilde{E}^{m}_{j,r}, f_3)|^2 \right)^{1/2} \right\|_{q_1} \]
\[ \leq \left\| \left( \sum_{j=0}^{2^m-1} |f_1 \ast \tilde{E}^{m}_{j,l}|^2 \right)^{1/2} \left( \sum_{j=0}^{2^m-1} |BHT(f_2 \ast \tilde{E}^{m}_{j,r}, f_3)|^2 \right)^{1/2} \right\|_{p_1} \]
\[ \leq \left\| \left( \sum_{j=0}^{2^m-1} |f_1 \ast \tilde{E}^{m}_{j,l}|^2 \right)^{1/2} \right\|_{p_1} \left\| \left( \sum_{j=0}^{2^m-1} |f_2 \ast \tilde{E}^{m}_{j,r}|^2 \right)^{1/2} \right\|_{p_2} \|f_3\|_{p_3}, \]

where the last line follows from an application of lemma 3 and the boundedness of bilinear Hilbert transform. The advantage in writing the sum as a product in this way is that one may use Hölder’s inequality even in the quasi-Banach case. Next, use convexity of \( x \mapsto |x|^{p_1/2} \) and the Hausdorff-Young inequality to see

\[ \left\| \left( \sum_{j=0}^{2^m-1} |f_1 \ast \tilde{E}^{m}_{j,l}|^2 \right)^{1/2} \right\|_{p_1} \leq 2^{m(1/2-1/p_1)} \left( \sum_{j=0}^{2^m-1} \|f_1 \ast \tilde{E}^{m}_{j,l}\|_{p_1}^{p_1} \right)^{1/p_1} \]
\[ \leq 2^{m(1/2-1/p_1)} \left( \sum_{j=0}^{2^m-1} \|\hat{f}_1 \chi^{m}_{j,l}\|_{p_1}^{p_1} \right)^{1/p_1} \]
\[ = 2^{m(1/2-1/p_1)} 2^{m(1/p_1-1/p')} \]
\[ = 2^{m(1/2-1/p')}, \]
The remaining factor is \( \left\| \left( \sum_{j=0}^{2^m-1} \left| f_2 \ast \tilde{\chi}_{E_j^m} \right|^2 \right)^{1/2} \right\|_{p_2} \). If \( p_2 \geq 2 \), we may pass this problem to the original Rubio de Francia inequality and conclude the theorem for CASE 1. So, we may assume without loss of generality that \( p_2 < 2 \). For this, we need to invoke the generalized Rubio de Francia estimate, which is stated as theorem 1, by first raising the \( l^2 \) norm at an acceptable cost. Specifically, for any \( r > p_2 \), we compute

\[
\left\| \left( \sum_{j=0}^{2^m-1} \left| f_2 \ast \tilde{\chi}_{E_j^m} \right|^2 \right)^{1/2} \right\|_{p_2} \leq 2^{m(1/2 - 1/r)} \left\| \left( \sum_{j=0}^{2^m-1} \left| f_2 \ast \tilde{\chi}_{E_j^m} \right|^r \right)^{1/r} \right\|_{p_2} \leq 2^{m(1/2 - 1/r)} \|f_2\|_{p_2}.
\]

One checks that this loss does not affect the convergence of the sum over scales because \( 1/2 - 1/p_1^2 + 1/2 - 1/r < 0 \) provided one chooses \( r \) close enough to \( p_2 \).

CASE 2: \( q < 1 \). Because one still has recourse to Hölder’s inequality, the only computational difference with CASE 1 is how one moves the sum over scales outside the \( L^q \) norm in the absence of the triangle inequality. For this, it suffices to observe

\[
\left\| \sum_{m \geq 0} \sum_{j=0}^{2^m-1} (f_1 \ast \tilde{\chi}_{E_j^m}) \cdot BHT(f_2 \ast \tilde{\chi}_{E_j^m}, f) \right\|_q \leq \left( \sum_{m \geq 0} \left\| \sum_{j=0}^{2^m-1} (f_1 \ast \tilde{\chi}_{E_j^m}) \cdot BHT(f_2 \ast \tilde{\chi}_{E_j^m}, f) \right\|_q \right)^{1/q}.
\]

To prove the next theorem, we will need the following fact:

**Lemma 1** (Bicarleson Estimates). The operator \( \sup BHT(f_1, f_2)(x) \) given by

\[
\sup_N \left| \int_{x_1 < x_2 < N} \tilde{f}_1(x_1) \tilde{f}_2(x_2) e^{2\pi i x_1 x_2} dx_1 dx_2 \right|
\]

is bounded from \( L^{p_1} \times L^{p_2} \) into \( L^{\frac{p_1 p_2}{p_1 + p_2}} \) provided \( \frac{1}{p_1} + \frac{1}{p_2} < 3/2 \).

**Theorem 3.** The operator \( C_{s_1, s_2}^{p_1, p_2} : L^{p_1} \times L^{p_2} \times W^{p_3} \times L^{p_4} \times L^{p_5} \rightarrow L^{\frac{1}{p_1} \sum_{i=1}^5 p_i} \) is continuous provided \( \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4} + \frac{1}{p_5} < 3/2 \) while \( \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4} + \frac{1}{p_5} < 1 \).
Proof. One may try on a first attempt to introduce two copies of the same martingale structure, namely $E_{f_1}^{m_1}$ and $E_{f_2}^{m_2}$ adapted to $f_3$ this time, and split $C_5^{1,1,-1,1,1}$ as follows:

$$
\sum_{m_1,m_2 \geq 0} \sum_{j_1,j_2} BHT(f_1, f_2 \ast \tilde{X}_{E_{j_1,l}})(f_3 \ast \tilde{X}_{E_{j_1,r}} \ast \tilde{X}_{E_{j_2,l}}) BHT(f_4 \ast \tilde{X}_{E_{j_2,r}}, f_5).
$$

A computation similar to theorem 2 yields in the Banach case, setting $\frac{1}{q_1} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q_2} = \frac{1}{p_1} + \frac{1}{p_5}$.

$$
\lesssim \sum_{m_1,m_2 \geq 0} \left\| C_5^{1,1,-1,1,1}(\tilde{f}) \right\|_q \left\| \left( \sum_{j_1} |BHT(f_1, f_2 \ast \tilde{X}_{E_{j_1,l}})|^2 \right)^{1/2} \right\|_{q_1} \times \left\| \left( \sum_{j_1,j_2} |f_3 \ast \tilde{X}_{E_{j_1,r}} \ast \tilde{X}_{E_{j_2,l}}|^2 \right)^{1/2} \right\|_{p_3} \times \left\| \left( \sum_{j_2} |BHT(f_4 \ast \tilde{X}_{E_{j_2,r}}, f_5)|^2 \right)^{1/2} \right\|_{q_2} = \sum_{m_1,m_2 \geq 0} A_{m_1} \times B_{m_1,m_2} \times C_{m_2}.
$$

As before, the goal is to produce a convergent geometric series over the scales $m_1$ and $m_2$. The factors $A_{m_1}$ and $C_{m_2}$ are both handled by the $l^2$ vector-valued inequality for the $BHT$ followed by the generalized Rubio de Francia estimate. If both $p_2, p_4 < 2$, this part can be bounded above by

$$
2^{m_1(1/2-1/p'_2)} 2^{m_2(1/2-1/p'_4)} ||f_1||_{p_1} ||f_2||_{p_2} ||f_4||_{p_4} ||f_5||_{p_5}.
$$

The decay which enables the geometric series to converge of course comes from the middle factor, namely $B_{m_1,m_2}$. Using the convexity of $x \mapsto |x|^{p_3/2}$ and the Hausdorff-Young inequality, we can bound $B_{m_1,m_2}$ by

$$
\left( \sum_{j_1,j_2} ||\hat{f}_3 \chi_{j_1,r} \chi_{j_2,l}||_{p_3}^{p_3} \right)^{1/p_3} = 2^{\max\{m_1,m_2\}(1/p_3-1/p'_3)},
$$

where the last equality follows from the nesting property of dyadic intervals. To finish in this case, we must require
\[
\sum_{m_1,m_2 \geq 0} = 2^{m_1(1/2-1/p'_3)}2^{m_2(1/2-1/p'_4)}2^{\max\{m_1,m_2\}(1/2-1/p'_5)}
\]

converge, which happens if and only if \(1/p'_3 + 1/p'_4 + 1/p'_5 < 3/2\). This condition implies \(1/p'_3 + 1/p'_5 + 1/p'_4 < 1\) by the assumption \(p_2, p_4 < 2\), so we have proven only a subset of the exponent range claimed in the theorem. What cost us is the fact that

\[
B_{m_1,m_2} = 2^{\max\{m_1,m_2\}(1/2-1/p'_5)},
\]

which does not decay fast enough for the sum over scales to converge. The key idea to get the full range is to adopt a different martingale structure decomposition that yields \(B_{m_1,m_2} = 2^{(m_1+m_2)(1/2-1/p'_5)}\), which \textit{will} be enough to conclude the result.

First construct \(E^m_{j_2} \) given by

\[
\gamma_{f_3}(x) = \frac{\int_{-\infty}^{x} |\hat{f}_3|^{p'_3} dx}{||\hat{f}_3||^{p'_3}}
\]

\[
E^m_{j_2} = \gamma_{f_3}^{-1}((j_22^{-m_1}, (j_2+1)2^{-m_1})).
\]

Next, define the \textit{restricted} martingale structure \(E^m_{j_1,j_2} \) by setting

\[
\gamma_{m_1,j_1,f_3}(x) = \frac{\int_{-\infty}^{x} |\hat{f}_3|^{p'_3}\chi_{E^m_{j_1}} dx}{||\hat{f}_3\chi_{E^m_{j_1}}||^{p'_3}}
\]

\[
E^m_{j_1,j_2} = \gamma_{m_1,j_1,f_3}^{-1}((j_22^{-m_2}, (j_2+1)2^{-m_2})).
\]

Therefore, \(||\hat{f}_3\chi_{E^m_{j_1}}\chi_{E^m_{j_1,j_2}}||^{p'_3} = 2^{-(m_1+m_2)} \forall 0 \leq j_1 < 2^{m_1}, 0 \leq j_2 < 2^{m_2} \).

We now partition the domain

\[
\{(x_2,x_3,x_4) : x_2 < x_3 < x_4\} = \bigcup_{m_1,m_2} \bigcup_{j_1,j_2} E^m_{j_1,j_2} \times \big(E^m_{j_1,r} \cap E^m_{j_1,j_2,l}\big) \times E^m_{j_1,j_2,r}.
\]

CASE 1: \(q := \frac{1}{\sum_{i=1}^{2} \frac{1}{p'_i}} \leq 1\). Then

\[
\left\|C^1_{1,1,1,1,1,1}(\hat{f})\right\|_q
\]

\[
\leq \left( \sum_{m_1,m_2 \geq 0} \left\|\sum_{j_1,j_2} BHT(f_{1}, f_2 * \hat{\chi}_{E^m_{j_1,l}}) \times f_3 * \hat{\chi}_{E^m_{j_1,r}} \times \hat{\chi}_{E^m_{j_1,j_2,l}} \times \hat{\chi}_{E^m_{j_1,j_2,r}} \times BHT(f_{4} * \hat{\chi}_{E^m_{j_1,j_2,r}}, f_3) \right\|_q \right)^{1/q}.
\]

9
We now want to split the sum over $j_1, j_2$ as

$$\sum_{j_1, j_2} = \sum_{j_1=0}^{2^{m_1}-1} \sum_{j_2 \neq 2^{m_2}-1} + \sum_{j_1=0}^{2^{m_1}-1} \sum_{j_2 = 2^{m_2}-1}$$

to reflect the fact that $\{E_{j_1,j_2,r}\} \cup_{j_1,j_2 \neq 2^{m_2}-1}$ is not a disjoint collection of intervals, while $\{E_{j_1,j_2,r}\} \cup_{j_1,j_2 \neq 2^{m_2}-1}$ is.

**CASE 1a:** We now deal with the first term, which corresponds to $\sum_{j_1=0}^{2^{m_1}-1} \sum_{j_2 \neq 2^{m_2}-1}$.

Now the computation becomes

$$\left( \sum_{m_1,m_2 \geq 0} \left| \sum_{j_1,j_2 \neq 2^{m_2}-1} BHT(f_1, f_2, \mathcal{X}_{E_{j_1}^{m_1}}) \times \mathcal{X}_{E_{j_2}^{m_2}} \right| \right)^{1/q}$$

$$\leq \left( \sum_{m_1,m_2 \geq 0} \left| \sup_{j_1} \left| BHT(f_1, f_2, \mathcal{X}_{E_{j_1}^{m_1}}) \right| \right| \right)^{1/q} \times$$

$$\left( \sum_{j_1,j_2 \neq 2^{m_2}-1} \left| f_3 \mathcal{X}_{E_{j_1}^{m_1}} \mathcal{X}_{E_{j_2}^{m_2}} \right|^{1/2} \right)^{1/2} \times$$

$$\left( \sum_{j_1,j_2 \neq 2^{m_2}-1} \left| BHT(f_4, \mathcal{X}_{E_{j_1,j_2}^{m_2}}, f_5) \right|^{1/2} \right)^{1/2} \times$$

$$\left( \left( \sum_{j_1,j_2 \neq 0,2^{m_2}-1} \left| f_3 \mathcal{X}_{E_{j_1}^{m_1}} \mathcal{X}_{E_{j_2}^{m_2}} \right| \right| \right)^{1/q} \times$$

$$\left( \left( \sum_{j_1,j_2 \neq 0,2^{m_2}-1} \left| BHT(f_4, \mathcal{X}_{E_{j_1,j_2}^{m_2}}, f_5) \right| \right| \right)^{1/q} \times$$

$$= \left( \sum_{j_1,j_2 \neq 2^{m_2}-1} A_{m_1} B_{m_1,m_2} C_{m_1,m_2} \right)^{1/q}$$

To deal with $A_{m_1}$, we will use estimates for the Bicarleson operator. For $B_{m_1,m_2}$, we use the martingale structure to obtain $B_{m_1,m_2} < 2^{q(m_1+m_2)(1/2-1/p_0)}$, and

$$2^{q(m_1+m_2)(1/2-1/p_0)} < 2^{q(m_1+m_2)}$$
$C_{m_1,m_2}$ can be passed to the $l^2$ vector-valued story combined with the generalized Rubio de Francia estimate. Following the same argument as before, the resulting geometric sum will be given for any $r > p'_4$ by

$$\sum_{m_1,m_2 \geq 0} 2^{q(m_1+m_2)(1/2-1/p'_3 + 1/2-1/r)}$$

which again converges provided one chooses $r$ close enough to $p'_4$ one recalls that $\frac{1}{p_3} + \frac{1}{p'_4} < 1$ by assumption.

CASE 1b: It only remains to tackle the endpoint case, i.e. the sum is over all $j_1$ for fixed $j_2 = 2^{m_2} - 1$. Here, it is important to realize that the intervals $\{E^{m_1,m_2}_{j_1,2^{m_2}-1,r}\}$ all overlap. The calculation is

$$\left( \sum_{m_1,m_2 \geq 0} \left| \sum_{j_1} BHT(f_1, f_2 \ast \tilde{\chi}_{E^{m_1}_{j_1,r}}) \times f_3 \ast \tilde{\chi}_{E^{m_1}_{j_1,r}} \ast \tilde{\chi}_{E^{m_2}_{j_1,2^{m_2}-1,l}} \right| \right)^{1/q}$$

$$\leq \left( \sum_{m_1,m_2 \geq 0} \left| \sum_{j_1} |BHT(f_1, f_2 \ast \tilde{\chi}_{E^{m_1}_{j_1,r}})|^2 \right| \right)^{1/2} \times \left( \sum_{j_1} |f_3 \ast \tilde{\chi}_{E^{m_1}_{j_1,r}} \ast \tilde{\chi}_{E^{m_1,m_2}_{j_1,2^{m_2}-1,l}}|^2 \right)^{1/2} \times \sup_{j_1} |BHT(f_4 \ast \tilde{\chi}_{E^{m_1,m_2}_{j_1,2^{m_2}-1,r}}, f_5)|^{1/q}$$

$$\leq \left( \sum_{m_1,m_2 \geq 0} \left| \sum_{j_1} |BHT(f_1, f_2 \ast \tilde{\chi}_{E^{m_1}_{j_1,r}})|^2 \right| \right)^{1/2} \times \left( \left| \sum_{j_1} |f_3 \ast \tilde{\chi}_{E^{m_1}_{j_1,r}} \ast \tilde{\chi}_{E^{m_1,m_2}_{j_1,2^{m_2}-1,l}}|^2 \right| \right)^{1/2} \times \sup_{j_1} |BHT(f_4 \ast \tilde{\chi}_{E^{m_1,m_2}_{j_1,2^{m_2}-1,r}}, f_5)|^{1/q}$$

$$= \left( \sum_{m_1,m_2} A_{m_1} \times B_{m_1,m_2} \times C_{m_1,m_2} \right)^{1/q}.$$

This time we pass $A_{m_1}$ to the $l^2$ vector-valued story followed by generalized Rubio de Francia, $B_{m_1,m_2} < 2^{qm_1(1/2-1/p'_3)}2^{-qm_2/p'_4}$, and $C_{m_1,m_2}$ can be handled
using the Bicarleson estimates. Therefore, the geometric sum one eventually faces is of the form

$$\sum_{m_1, m_2 \geq 0} q^{m_1(1/2-1/p_2')}q^{m_1(1/2-1/p_4')}2^{-q(m_2/p_4')}$$

which again converges because of the assumption $1/p_2' + 1/p_4' < 1$.

CASE 2: $q > 1$. One passes the sum over scales outside the $L^q$ norm using the triangle inequality before proceeding exactly as before.

Before proving the main result of this section, we record the following result:

**Theorem 4.** The operator $C^{1,1,-1,1,1,1,1,1,1}_8 : L^{p_1} \times L^{p_2} \times W^{p_3} \times L^{p_4} \times W_{p_6} \times L^{p_7} \times L^{p_8} \to L^{p_1}$ is bounded provided $1/p_2' + 1/p_4' + 1/p_6' + 1/p_7' < 3/2$ and $1/p_2' + 1/p_4' + 1/p_6' + 1/p_7' = 3/2$.

**Proof.** The idea is to construct 4 martingale structures given by $E^{m_1}_{j_1}, E^{m_1, m_2}_{j_1, j_2}, E^{m_3}_{j_3}$, and $E^{m_3, m_4}_{j_3, j_4}$, where the first two are adapted to $f_3$, and the last two are adapted to $f_6$. Specifically, we define

$$\gamma_{j_1}(x) = \frac{\int_{-\infty}^{x} |f_3|^2 dx}{\|f_3\|_{p_3}^2}$$

$$E^{m_1}_{j_1} = \gamma_{j_1}^{-1}([j_1 2^{-m_1}, (j_1+1)2^{-m_1}])$$

$$\gamma_{m_1, j_1, f_3}(x) = \frac{\int_{-\infty}^{x} |f_3|^2 \chi_{E^{m_1}_{j_1}} dx}{\|f_3\|_{E^{m_1}_{j_1}}^2}$$

$$E^{m_1, m_2}_{j_1, j_2} = \gamma_{m_1, j_1, j_2}^{-1}([j_2 2^{-m_2}, (j_2+1)2^{-m_2}])$$

$$\gamma_{j_2}(x) = \frac{\int_{-\infty}^{x} |f_3|^2 dx}{\|f_6\|_{p_6}^2}$$

$$E^{m_3}_{j_3} = \gamma_{j_3}^{-1}([j_3 2^{-m_3}, (j_3+1)2^{-m_3}])$$

$$\gamma_{m_3, j_3, f_3}(x) = \frac{\int_{-\infty}^{x} |f_3|^2 \chi_{E^{m_3}_{j_3}} dx}{\|f_3\|_{E^{m_3}_{j_3}}^2}$$

$$E^{m_3, m_4}_{j_3, j_4} = \gamma_{m_3, j_3, j_4}^{-1}([j_4 2^{-m_4}, (j_4+1)2^{-m_4}]).$$

The hardest case is when $p_2, p_4, p_5, p_7 < 2$, which places us in the quasi-Banach setting. We assume this now without loss of generality. Decomposing the operator $C^{1,1,-1,1,1,1,1,1}$ yields
The plan is now to bound each piece individually. To save space, it is helpful so that the corresponding estimate is broken into four pieces, $\sum_{m_1,m_2,m_3,m_4 \geq 0} \left| \sum_{j_1,j_2,j_3,j_4} BHT(f_1, f_2 * \tilde{\chi}_{E_{j_1,j_2}^m}) \right|_{q} \leq \left( \sum_{m_1,m_2,m_3,m_4 \geq 0} \right| \sum_{j_1,j_2,j_3,j_4} BHT(f_1, f_2 * \tilde{\chi}_{E_{j_1,j_2}^m}) \times f_3 * \tilde{\chi}_{E_{j_1,j_2}^m} \times BHT(f_4 * \tilde{\chi}_{E_{j_1,j_2}^m}, f_5 * \tilde{\chi}_{E_{j_1,j_2}^m}) \times (f_6 * \tilde{\chi}_{E_{j_1,j_2}^m} * \tilde{\chi}_{E_{j_1,j_2}^m}) \times BHT(f_7 * \tilde{\chi}_{E_{j_1,j_2}^m}, f_8) \right|_{q}^{1/q} \right)

We now separate the sum

\[
\sum_{j_1,j_2,j_3,j_4} = \sum_{j_1,j_2,j_3,j_4 \neq 2m^2 - 1, j_4 \neq 2m^4 - 1} + \sum_{j_1,j_2,j_3,j_4 \neq 2m^2 - 1, j_4 = 2m^4 - 1} + \sum_{j_1,j_2,j_3,j_4 \neq 2m^4 - 1, j_2 = 2m^2 - 1} + \sum_{j_1,j_3,j_4 = 2m^2 - 1, j_4 = 2m^4 - 1} = A + B + C + D,
\]

so that the corresponding estimate is broken into four pieces, $\tilde{A}, \tilde{B}, \tilde{C},$ and $\tilde{D}$. The plan is now to bound each piece individually. To save space, it is helpful to define $\frac{1}{q_1} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{q_2} = \frac{1}{p_4} + \frac{1}{p_5}, \frac{1}{q_4} = \frac{1}{p_7} + \frac{1}{p_8}$. First,

\[
\tilde{A} := \left( \sum_{m_1,m_2,m_3,m_4 \geq 0} \right| \sum_{j_1,j_2,j_3,j_4 \neq 2m^2 - 1, j_4 \neq 2m^4 - 1} BHT(f_1, f_2 * \tilde{\chi}_{E_{j_1,j_2}^m}) \times (f_3 * \tilde{\chi}_{E_{j_1,j_2}^m} * \tilde{\chi}_{E_{j_1,j_2}^m}) \times BHT(f_4 * \tilde{\chi}_{E_{j_1,j_2}^m}, f_5 * \tilde{\chi}_{E_{j_1,j_2}^m}) \times (f_6 * \tilde{\chi}_{E_{j_1,j_2}^m} * \tilde{\chi}_{E_{j_1,j_2}^m}) \times BHT(f_7 * \tilde{\chi}_{E_{j_1,j_2}^m}, f_8) \right|_{q}^{1/q} \right).
\]
This can be bounded above by

\[
\left( \sum_{m_1, m_2, m_3, m_4 \geq 0} \left| \frac{1}{q} \sum_{j_3, j_4 \neq 2m_4 - 1} |BHT(f_1^*, f_2^* \tilde{\chi}_{E_{j_1,j_2,l}}^m)| \times \right. \right.
\]

\[
\left. \left( \sum_{j_1, j_2 \neq 2m_2 - 1} |f_3 \ast \tilde{\chi}_{E_{j_1,j_2,r}}^{m_1} \ast \tilde{\chi}_{E_{j_1,j_2,l}}^{m_2}|^2 \right)^{1/2} \times \right.
\]

\[
\left( \sum_{j_1, j_2 \neq 2m_2 - 1} |BHT(f_4 \ast \tilde{\chi}_{E_{j_1,j_2,r}}^{m_1}, f_5 \ast \tilde{\chi}_{E_{j_1,j_2,l}}^{m_2})|^2 \right)^{1/2} \times
\]

\[
|f_6 \ast \tilde{\chi}_{E_{j_3,j_4,r}}^{m_3} \ast \tilde{\chi}_{E_{j_3,j_4,l}}^{m_4}| \times |BHT(f_7 \ast \tilde{\chi}_{E_{j_3,j_4,l}}^{m_3}, f_8)| \right)^{1/q} \frac{1}{q}
\]

\[
\left. \left( \sum_{j_3, j_4 \neq 2m_4 - 1} |BHT(f_7 \ast \tilde{\chi}_{E_{j_3,j_4,l}}^{m_3}, f_8)|^2 \right)^{1/2} \right) \frac{1}{q}.
\]

We now use Hölder’s inequality to deduce an upper bound of the form
\[
\left( \sum_{m_1, m_2, m_3, m_4 \geq 0} \left\| \sup_{j_1} \left| BHT(f_{11}, f_{2*} \check{\chi}_{E_{j_1,i}^m}) \right| \right\|_q^{q_1} \times \\
\left( \sum_{j_1, j_2 \neq 2^{m_2} - 1} \left| f_3 * \check{\chi}_{E_{j_1,i}^m} * \check{\chi}_{E_{j_1,i}^m} \right| \right)^{1/2}^{q_1} \times \\
\sup_{j_3} \left( \sum_{j_1, j_2 \neq 2^{m_2} - 1} \left| BHT(f_{41} * \check{\chi}_{E_{j_1,i}^m}, f_{5} * \check{\chi}_{E_{j_1,i}^m}) \right|^2 \right)^{1/2}^{q_1} \times \\
\left( \sum_{j_3, j_4 \neq 2^{m_4} - 1} \left| f_6 * \check{\chi}_{E_{j_3,i}^m} * \check{\chi}_{E_{j_3,i}^m} \right| \right)^{1/2}^{q_2} \times \\
\left( \sum_{j_3, j_4 \neq 2^{m_4} - 1} \left| BHT(f_{7} * \check{\chi}_{E_{j_3,i}^m}, f_{8}) \right| \right)^{1/2}^{q_3} \right) \right)^{1/q_4}
\]

The factor \(A_{m_1}\) is handled by the Bicarleson operator estimates. For the other factors, \(B_{m_1, m_2} < 2^{q(m_1 + m_2)(1/2 - 1/p_1)^q} \|f_1\|_{p_1}^q\), \(C_{m_1, m_2, m_3}\) is handled using lemma [5] in the appendix, \(D_{m_3, m_4} < 2^{q(m_3 + m_4)(1/2 - 1/p_2)^q} \|f_6\|_{p_2}^q\), and \(F_{m_3, m_4}\) is handled using lemma [4]. Since we are assuming \(p_2, p_4, p_5, p_7 < 2\), the geometric sum one eventually faces takes the form

\[
\sum_{m_1, m_2, m_3, m_4} 2^{q(m_1 + m_2)(1/2 - 1/p_1)^q + 1/2 - 1/p_1} 2^{q(m_3 + m_4)(1/2 - 1/p_2)^q + 1/2 - 1/p_2},
\]

which converges because \(1/p_1 > 1, 1/p_2 > 1\). The next term we face is

\[
\tilde{B} := \left( \sum_{m_1, m_2, m_3, m_4 \geq 0} \left| \sum_{j_1, j_2 \neq 2^{m_2} - 1} BHT(f_{11}, f_{2*} \check{\chi}_{E_{j_1,i}^m}) \times \\
(f_3 * \check{\chi}_{E_{j_1,i}^m} * \check{\chi}_{E_{j_1,i}^m}) \times BHT(f_{41} * \check{\chi}_{E_{j_1,i}^m}, f_{5} * \check{\chi}_{E_{j_1,i}^m}) \times \\
(f_6 * \check{\chi}_{E_{j_3,i}^m} * \check{\chi}_{E_{j_3,i}^m}) \times BHT(f_{7} * \check{\chi}_{E_{j_3,i}^m}, f_{8}) \right) \right)^{1/q_4}.
\]
It is readily seen that one has an upper bound of the form

$$
\hat{B} \leq \left( \sum_{m_1, m_2, m_3, m_4 \geq 0} \left| \sup_{j_1} |BHT(f_1, f_2 * \check{X}_{m_1}^{j_1})| \times \left( \sum_{j_1, j_2 \neq 2^{m_2} - 1} |f_3 * \check{X}_{m_1}^{j_1} * \check{X}_{m_2}^{j_1, j_2}|^2 \right)^{1/2} \times \left( \sum_{j_1, j_3 \neq 2^{m_2} - 1} |BHT(f_4 * \check{X}_{m_1}^{j_1, j_3}, f_5 * \check{X}_{m_3}^{j_3})|^2 \right)^{1/2} \times \left( \sum_{j_3} |f_6 * \check{X}_{m_3}^{j_3} * \check{X}_{m_4}^{j_3, 2^{m_4} - 1}|^2 \right)^{1/2} \sup_{j_3} |BHT(f_7 * \check{X}_{m_3}^{j_3, 2^{m_4} - 1}, f_8)| \right)^q \right)^{1/q},
$$

which is passed to Hölder’s inequality as before, giving the expression

$$
= \left( \sum_{m_1, m_2, m_3, m_4 \geq 0} A_{m_1} \times B_{m_1, m_2} \times C_{m_1, m_2, m_3} \times D_{m_3, m_4} \times F_{m_3, m_4} \right)^{1/q}.
$$

We pass $A_{m_1}$ and $F_{m_3, m_4}$ to the Bicarleson estimates, use the standard decay for $B_{m_1, m_2}$, use lemma 4 and generalized Rubio de Francia for $C_{m_1, m_2, m_3}$, and observe $D_{m_3, m_4} < 2^{q m_3 (1/2 - 1/p_6) - q m_4 / p_6}$. Doing this, gives the geometric series.
\[
\sum_{m_1, m_2, m_3, m_4 \geq 0} q_{p(m_1 + m_2)(1/2 - 1/p_3 + 1/2 - 1/p_4)} 2^{q_{m_3}(1/2 - 1/p_3 + 1/2 - 1/p_4)} 2^{-q_{m_4}/p_4},
\]
which converges because \(1/p_3 + 1/p_4 + \frac{1}{p_6} < 1\).

The analysis for \(C\) is the same as the analysis for \(B\) except that the roles of \(j_1, j_2\) and \(j_3, j_4\) are reversed. Thus, it only remains to bound \(\tilde{D}\) to obtain the result. For this, we observe

\[
\tilde{D} := \left( \sum_{m_1, m_2, m_3, m_4 \geq 0} \left| \sum_{j_1, j_3} BHT(f_{1,1}, f_{2}\times \tilde{X}_{E_{j_1,1}}^{m_1}) \times
\right.ight.
\]
\[
(f_3 \times \tilde{X}_{E_{j_1,1}^{m_1}} \times \tilde{X}_{E_{j_3,1}^{m_2}}) \times BHT(f_4 \times \tilde{X}_{E_{j_1,2}^{m_2}}, f_5 \times \tilde{X}_{E_{j_3,2}^{m_3}}) \times
\]
\[
(f_6 \times \tilde{X}_{E_{j_3,2}^{m_3}} \times \tilde{X}_{E_{j_4,1}^{m_4}}) \times BHT(f_7 \times \tilde{X}_{E_{j_3,2}^{m_4}}, f_8) \left|^{q/1} \right. \right)
\]
\[
\leq \left( \sum_{m_1, m_2, m_3, m_4 \geq 0} \left| \sum_{j_1} BHT(f_{1,1}, f_{2}\times \tilde{X}_{E_{j_1,1}}^{m_1}) \right|^{1/2}
\right.
\]
\[
|f_3 \times \tilde{X}_{E_{j_1,1}^{m_1}} \times \tilde{X}_{E_{j_3,2}^{m_2}}| \times \left( \sum_{j_3} |BHT(f_4 \times \tilde{X}_{E_{j_1,2}^{m_2}}, f_5 \times \tilde{X}_{E_{j_3,2}^{m_3}})|^2 \right)^{1/2}
\]
\[
\times \left( \sum_{j_3} |f_6 \times \tilde{X}_{E_{j_3,2}^{m_3}} \times \tilde{X}_{E_{j_4,1}^{m_4}}|^2 \right)^{1/2} \times \sup_{j_3} |BHT(f_7 \times \tilde{X}_{E_{j_3,2}^{m_4}}, f_8)|_{q}^{1/q}
\)
\[
\leq \left( \sum_{m_1, m_2, m_3, m_4 \geq 0} \left| \sum_{j_1} BHT(f_{1,1}, f_{2}\times \tilde{X}_{E_{j_1,1}}^{m_1}) \right|^2 \right)^{1/2}
\]
\[
\left( \sum_{j_1} \left| f_3 \times \tilde{X}_{E_{j_1,1}^{m_1}} \times \tilde{X}_{E_{j_3,2}^{m_2}} \right|^2 \right)^{1/2}
\]
\[
\times \left( \sum_{j_3} \left| f_6 \times \tilde{X}_{E_{j_3,2}^{m_3}} \times \tilde{X}_{E_{j_4,1}^{m_4}} \right|^2 \right)^{1/2} \times \sup_{j_3} \left| BHT(f_7 \times \tilde{X}_{E_{j_3,2}^{m_4}}, f_8) \right|_{q}^{1/q}
\).
\]

Using Hölder’s inequality once more, we obtain the upper bound.
reasonable to think that the same method of proof works for operators that
them for the purposes of this paper based on personal communica
tion with
While such results have not yet appeared in published form, we shall as
sume
which converges because
It is important to note that each BHT in the previous proof could have been
provided we had estimates for the maximal variant
The geometric series one eventually faces in this case is

\[
\sum_{j=1}^{\infty} \left( \left| BHT(f_1, f_2 * \tilde{f}_{m_1}^{m_1, i, j_1}) \right|^2 \right)^{1/2} \times \frac{1}{q_1}
\]

\[
\left( \sum_{j_1} \left| BHT(f_3 * \tilde{f}_{m_1}^{m_1, i, j_1}, f_5 * \tilde{f}_{m_2}^{m_2, i, j_1}) \right|^2 \right)^{1/2} \times \frac{1}{q_2}
\]

\[
\left( \sum_{j_3} \left| BHT(f_4 * \tilde{f}_{m_2}^{m_2, i, j_2}, f_5 * \tilde{f}_{m_3}^{m_3, i, j_2}) \right|^2 \right)^{1/2} \times \frac{1}{q_3}
\]

\[
\left( \sum_{j_3} \left| BHT(f_6 * \tilde{f}_{m_3}^{m_3, i, j_3}, f_5 * \tilde{f}_{m_4}^{m_4, i, j_3}) \right|^2 \right)^{1/2} \times \frac{1}{q_4}
\]

\[
\sup_{j_3} \left( \sum_{j_3} \left| BHT(f_7 * \tilde{f}_{m_4}^{m_4, i, j_3}, f_8) \right|^2 \right)^{1/2} \times \frac{1}{q_5}
\]

\[
= \left( \sum_{m_1, m_2, m_3, m_4} A_{m_1} B_{m_1, m_2} C_{m_1, m_2, m_3} D_{m_3, m_4} F_{m_3, m_4} \right)^{1/4}
\]

One handles $A_{m_1}$ using the $l^2$ vector-valued for the BHT and generalized Ru-
bio de Francia, $B_{m_1, m_2} < 2^{q_{m_1} (1/2 - 1/p_1')} 2^{-q_{m_2}/p_3}, C_{m_1, m_2, m_3}$ using lemma \[D_{m_3, m_4} < 2^{q_{m_3} (1/2 - 1/p_4')} 2^{-q_{m_4}/p_6}$, and $F_{m_3, m_4}$ using Bicarleson estimates.

The geometric series one eventually faces in this case is

\[
\sum_{m_1, m_2, m_3, m_4 > 0} 2^{q_{m_1} (1/2 - 1/p_1') + 1/2 - 1/p_2'} 2^{-q_{m_2}/p_3} 2^{q_{m_3} (1/2 - 1/p_4') + 1/2 - 1/p_5'} 2^{-q_{m_4}/p_6},
\]

which converges because $\frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_5} + \frac{1}{p_6} < 1$.

It is important to note that each BHT in the previous proof could have been
replaced by $C_{m_1}^{1, \ldots, 1}$ provided we had estimates for the maximal variant

\[
\sup_{M, N} C_{m_1}^{1, \ldots, 1} (f) := \sup_{M, N} \left| \int_{M < x_1 < \ldots < x_n < N} f_1(x_1) \cdots f_n(x_n) e^{2\pi i x_1 + \ldots + x_n} dx \right|.
\]

While such results have not yet appeared in published form, we shall assume
them for the purposes of this paper based on personal communication with
C. Muscalu. Also, having proved mixed estimates for $C_{m_1}^{1, 1, 1, 1, 1, 1, 1}$, it is
reasonable to think that the same method of proof works for operators that

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continue the sequence $1, 1, -1, 1, 1, -1, ...$ for arbitrarily long lengths. This is indeed the case as we prove in theorem but some care has to be taken with the order in which we use suprema and Cauchy-Schwarz inequalities. At this point, we need to introduce a few definitions.

**Definition 3.** A set of consecutive positive integers $\{i_1, ..., i_1 + m\} \subset [n]$ forms a Lebesgue block $B$ for the operator $C_n^\varepsilon$ provided $\varepsilon_{i_1} + \varepsilon_{i_1 + 1} \neq 0$ for all $0 \leq i < m$ and $\{i_1, ..., i_1 + m\}$ is maximal with respect to this property.

**Definition 4.** We say a sign degeneracy occurs between indices $i$ and $i + 1$ of the operator $C_n^\varepsilon$ if $\varepsilon_i + \varepsilon_{i + 1} = 0$.

**Definition 5.** Suppose $C_n^\varepsilon : \otimes_{i=1}^n X^i \to L^p$ where for each $1 \leq i \leq n$, $X^i \in \{L^p, W_{p_i}\}$. Then $W_{C_n^\varepsilon}^* := \{1 \leq i \leq n : X^i = W_{p_i}\}$.

**Theorem 5** (Main Theorem). For $n \geq 2$, fix $\varepsilon \in \{\pm 1\}^n$. Form the operator $C_n^\varepsilon$ with domain $\otimes_{i=1}^n X^i$ and assume for every $i : 1 \leq i \leq n$ either $X^i = L^p$ for some $1 < p_i < \infty$ or $X^i = W_{p_i}$ for some $p_i \neq 2$. In addition, suppose the following three conditions:

1) The restricted maximal operator on each Lebesgue block $B$ is bounded.

2) For all $i \in W_{C_n^\varepsilon}^*$, $\varepsilon_i + \varepsilon_{i - 1} = \varepsilon_i + \varepsilon_{i + 1} = 0$.

3) Whenever $\varepsilon_i + \varepsilon_{i + 1} = 0$, $\frac{1}{p_i} + \frac{1}{p_{i+1}} \neq 1$.

Then $C_n^\varepsilon : \otimes_{i=1}^n X^i \to L_{\sum_{i=1}^n \frac{1}{p_i}}$ is bounded if and only if

1) $p_i > 2$ for all $i \in W_{C_n^\varepsilon}^*$,

2) whenever $\varepsilon_{i - 1} + \varepsilon_i = 0$ or $\varepsilon_i + \varepsilon_{i + 1} = 0$, then $\{i - 1, i, i + 1\} \cap W_{C_n^\varepsilon}^* \neq \emptyset$,

3) $\varepsilon_i + \varepsilon_{i + 1} = 0$ implies $\frac{1}{p_i} + \frac{1}{p_{i+1}} < 1$.

**Proof.** The “only if” part of the theorem will follow from corollary in the next section on counterexamples and shows that our results are, in a sense, the best possible. To prove the “if” statement, it is enough by the remark at the end of the previous theorem to assume all Lebesgue blocks have length 1 or 2. In fact, we now specialize to the case where each Lebesgue block has length 2 and there is only 1 function in a Wiener space between each Lebesgue block. The same type of proof will work for the cases where a Lebesgue block has length one or there is more than one Wiener function separating a Lebesgue block. So, we will restrict our attention to the operator $C_{3n-1,1,1,1,1,1,1,1,1}$ and prove bounds for it. To this end, we introduce two martingale structures for each $f_{3i} \in W_{p_{3i}}$:

\[
\begin{align*}
\gamma_{f_{3i}}(x) &= \frac{\int_{-\infty}^{x} |\hat{f}_{3i}|^{p_{3i}} \, d\bar{x}}{||\hat{f}_{3i}||_{p_{3i}}}, \\
3i E_{j_1}^{m_{1}} &= \gamma_{f_{3i}}^{-1}((j_1 2^{-m_{1}}, (j_1 + 1) 2^{-m_{1}})) \\
\gamma_{m_{1}, j_1, f_{3i}}(x) &= \frac{\int_{-\infty}^{x} |\hat{f}_{3i} \chi_{X_{j_1}, E_{j_1}^{m_{1}}}|^{p_{3i}} \, d\bar{x}}{||\hat{f}_{3i} \chi_{X_{j_1}, E_{j_1}^{m_{1}}}||_{p_{3i}}}, \\
3i E_{j_1, j_2}^{m_{1}, m_{2}} &= \gamma_{m_{1}, j_1, f_{3i}}^{-1}((j_2 2^{-m_{2}}, (j_2 + 1) 2^{-m_{2}})).
\end{align*}
\]
Using the standard partition for each \( i : 1 \leq i \leq n - 1 \)

\[
\{ x_{3i-1} < x_{3i} < x_{3i+1} \} = \bigcup_{m_1,m_2} \bigcup_{j_1,j_2} 3iE_{j_1,j}^m \times (3iE_{j_1,j}^m \cap 3iE_{j_1,j_2}^{m_1,m_2}) \times 3iE_{j_1,j_2}^{m_1,m_2},
\]

we decompose \( C_n^{1,1,-1,1,1,-1,1,1} \). By moving the sum over \( n - 1 \) scales outside the \( L^2 \) norm as usual, we are left inside with a sum over \( 2(n - 1) \) indices \( j_1, \ldots, j_{2(n-1)} \). Of course, we can split the sum over \( j \) into \( 2^{n-1} \) pieces by restricting each even index \( j_{2k} \) either to \( 0 \leq j_{2k} < 2^{m_{2k}} - 1 \) or to the endpoint \( 2^{m_{2k}} - 1 \). We say a given scale \( m_{2k} : 1 \leq k \leq n - 1 \) is Type A if the corresponding \( j_{2k} \) is restricted to \( 0 \leq j_{2k} < 2^{m_{2k}} - 1 \) and \( m_{2k} \) is Type B if \( j_{2k} \) is restricted to \( 2^{m_{2k}} - 1 \). For example, in theorem 4 we broke apart the original sum into four smaller sums as follows:

\[
\sum_{j_1,j_2,j_3,j_4} = \sum_{j_1,j_2 \neq 2^{m_2} - 1,j_3,j_4 \neq 2^{m_4} - 1} + \sum_{j_1,j_2 \neq 2^{m_2} - 1,j_3,j_4 = 2^{m_4} - 1} + \sum_{j_1,j_3,j_4 \neq 2^{m_4} - 1,j_2 = 2^{m_2} - 1} + \sum_{j_1,j_3,j_4 = 2^{m_2} - 1,j_2 = 2^{m_4} - 1}.
\]

For the first sum on the right hand side, the scales \( m_2 \) and \( m_4 \) are both Type A. For the second sum, the \( m_2 \) is Type A and \( m_4 \) Type B. For the third sum, \( m_2 \) is Type B and \( m_1 \) is Type A. For the last sum, both \( m_1 \) and \( m_2 \) are Type B. For convenience, say that the first sum is type AA, the second type AB, the third type BA, and the fourth type BB. It is instructive to recall how sums of type AA in the decomposition of \( C_n^{1,1,-1,1,1,-1,1,1} \) were handled in theorem 4.

Setting

\[
\begin{align*}
A_{j_1}^{m_1} & = BHT(f_1, f_2 \ast \hat{X}_{E_{j_1}^{m_1}}) \\
B_{j_1,j_2}^{m_1,m_2} & = f_3 \ast \hat{X}_{E_{j_1}^{m_1}} \ast \hat{X}_{E_{j_2}^{m_2}} \\
C_{j_1,j_2,j_3}^{m_1,m_2,m_3} & = BHT(f_4 \ast \hat{X}_{E_{j_1}^{m_1}} \ast \hat{X}_{E_{j_2}^{m_2}} \ast \hat{X}_{E_{j_3}^{m_3}}) \\
D_{j_1,j_2}^{m_1,m_2} & = f_6 \ast \hat{X}_{E_{j_3}^{m_1}} \ast \hat{X}_{E_{j_4}^{m_2}} \\
F_{j_1,j_2}^{m_1,m_2} & = BHT(f_7 \ast \hat{X}_{E_{j_3,j_4}^{m_1,m_2}}, f_8),
\end{align*}
\]
we observed

\[
\left| \sum_{j_1, j_2: j_2 \neq 2^{m_2-1}, j_3, j_4 \neq 2^{m_4-1}} A_{j_1}^{m_1} B_{j_1, j_2}^{m_1, m_2} C_{j_1, j_2, j_3}^{m_1, m_2, m_3} D_{j_3, j_4}^{m_3, m_4} F_{j_3, j_4}^{m_3, m_4} \right| \\
\leq \sup_{j_1} \left| A_{j_1}^{m_1} \right| \left( \sum_{j_1, j_2: j_2 \neq 2^{m_2-1}} |B_{j_1, j_2}^{m_1, m_2}|^2 \right)^{1/2} \times \\
\sup_{j_3} \left( \sum_{j_1, j_2: j_2 \neq 2^{m_2-1}} |C_{j_1, j_2, j_3}^{m_1, m_2, m_3}|^2 \right)^{1/2} \times \\
\left( \sum_{j_3, j_4: j_4 \neq 2^{m_4-1}} |D_{j_3, j_4}^{m_3, m_4}|^2 \right)^{1/2} \times \\
\left( \sum_{j_3, j_4: j_4 \neq 2^{m_4-1}} |F_{j_3, j_4}^{m_3, m_4}|^2 \right)^{1/2}.
\]

Note that we used a supremum and the Cauchy-Schwarz inequality for the pair \((j_1, j_2)\) before proceeding to use a supremum and Cauchy-Schwarz inequality for the pair \((j_3, j_4)\). This order ensures that one takes the \(l^2\) norm over \(j_1, j_2\) for the cross factor \(C_{m_1, m_2, m_3}\) before the \(l^\infty\) norm over \(j_3\). That the supremum over \(j_3\) appears outside the sum over \(j_1\) and \(j_2\) is necessary for applying lemma \(\Box\) We summarize this observation by saying one resolves sums of type AA from “left to right.”

One quickly checks that for sums of type AB, the cross factor took the form \(\left( \sum_{j_1, j_2: j_2 \neq 2^{m_2-1}, j_3} |C_{m_1, m_2, m_3}|^2 \right)^{1/2}\), which we were able to handle using standard \(l^2\) vector-valued inequalities, while sums of type BA gave us cross factors like \(\sup_{j_1, j_3} |C_{j_1, 2^{m_2-1}, j_3}|\), which we could pass to the Bicarleson estimate. Neither sums of type AB nor sums of type BA required one to estimate the factors containing \((j_1, j_2)\) before those containing \((j_3, j_4)\) or the factors containing \((j_3, j_4)\) before those containing \((j_1, j_2)\). Lastly, sums of type BB required us to resolve from “right to left.” The cross factor here took the form \(\sup_{j_1} \left( \sum_{j_3} |C_{j_1, 2^{m_2-1}, j_3}|^2 \right)^{1/2}\). It is easy to check that resolving “left to right” gives a convergent sum for a block consisting of an arbitrary number of As, and similarly resolving “right to left” gives a convergent sum for a block consisting of an arbitrary number of Bs.

Now, for a given sum in the decomposition of \(\sum_{j_1, \ldots, j_2(n-1)}\), its type can be represented as a string of As and Bs of length \(n-1\). This string can be separated into blocks of As and blocks of Bs of varying lengths. For each block of As, one resolves each j pair from “left to right.” Then, for each block of Bs, one resolves each j pair from “right to left.” Doing this yields a convergent geometric series for each of the \(2^{(n-1)}\) pieces of the sum \(\sum_{j_1, \ldots, j_2(n-1)}\). \(\Box\)
3. Counterexamples

The following proposition is a well-known counterexample to the boundedness of $C_3^{\alpha,-1,1}$ originally due to C. Fefferman:

**Proposition 1 (8).** The operator $C_3^{\alpha,-1,1}$ satisfies no $L^p$ estimates.

**Proof.** We provide a sketch. It is a routine calculation to see that $C_3^{\alpha,-1,1}$ is up to harmless modifications the operator

$$C_3^{\alpha,-1,1}(f)(x) = \text{p.v.} \int_{\mathbb{R}^2} f_1(x-s)f_2(s-t-x)f_3(x+t) \frac{dsdt}{s \cdot t}.$$

Setting $f_1 = f_3 = e^{2\pi i x^2} \chi_{[-N,N]}$, $f_2 = e^{-2\pi i x^2} \chi_{[-N,N]}$, yields $|C_3^{\alpha,-1,1}(f)(x)|$

$$= \left| \int_{\mathbb{R}^2} e^{-4\pi ist} \chi_{[-N,N]}(x-s) \chi_{[-N,N]}(x-s+t) \chi_{[-N,N]}(x+t) \frac{dsdt}{s \cdot t} \right|.$$

For all $|x| < N/1000$, say, the support of the integrand contains a small box centered at the origin of side length $N/10$. Then one can compute

$$\left| \int_{-N/10}^{N/10} \int_{-N/10}^{N/10} e^{-4\pi ist} \frac{dsdt}{s \cdot t} \right| \gtrsim \log N$$

to conclude $||C_3^{\alpha,-1,1}||_r \gtrsim \log N^{1/r}$ while $||f_1||_{p_1} ||f_2||_{p_2} ||f_3||_{p_3} \approx N^{1/r}$. Taking $N$ large yields a contradiction to the boundedness of $C_3^{\alpha,-1,1}$.

**Corollary 1.** $||C_3^{\alpha,-1,1}(g_N^+, g_N^-, \check{g}_N^\pm)||_q \approx \log N^{\frac{1}{r}}$ for all $q > 0$.

Remark: The above proof also works if one takes $e^{\pm 2\pi i x^2} \tilde{\chi}_{[-N,N]}$ where $\tilde{\chi}_{[-N,N]} := \text{Dil}_N(\rho * \chi_{[-1,1]})$ is a dilated mollified characteristic function with the additional benefit of being in the Schwartz class. Thus, the proceeding proof shows that $T$ cannot be uniformly bounded on the family of functions $\{g_N^\pm\}_{N \in \mathbb{Z}^+}$ where $g_N^\pm = (e^{\pm 2\pi i x^2} \tilde{\chi}_{[-N,N]})$. For more detail in the proof using Fefferman’s counterexample, see [8].

**Definition 6.** $\mathcal{F}_c = \{ f \in \mathcal{S}(\mathbb{R}) : \text{supp}(\hat{f}) \text{ is compact} \}$.

**Definition 7.** The function $g_{N,M}^\pm = (e^{\pm 2\pi i x^2} \tilde{\chi}_{[-N,N]}) * \tilde{\chi}_{[-M,M]}$.

We can now prove the following:

**Proposition 2.** The operator $C_3^{\alpha,-1,1}$ is not bounded from $\mathcal{F}_c \cap L^p \times \mathcal{F}_c \cap L^q \times \mathcal{F}_c \cap L^r$ into $L^{\frac{p^*}{p} + \frac{q^*}{q} + \frac{r^*}{r}}$. 

In fact, for a given \( N \) there exists \( M \) such that

\[
||C_3^{1,-1,1}(g_{N,M}^+,\tilde{g}_{N,M}^+,\tilde{g}_{N,M}^+)||_q \asymp \log NN^{1/q}.
\]

Proof. Suppose for a contradiction that \( C_3^{1,-1,1} \) was bounded. Then for any \( f, g, h \in \mathcal{S}(\mathbb{R}) \),

\[
\lim_{N \to -\infty} C_3^{1,-1,1}(f(1-\chi([-N,N])),g,h)(x) = 0 \quad \forall \ x \in \mathbb{R},
\]

so by dominated convergence, Fatou’s lemma, our assumption on the boundedness of \( C_3^{1,-1,1} \), and Hausdorff-Young,

\[
\left|\left| \int_{x_1 < x_2 < x_3} \hat{f}(x_1) \hat{g}(x_2) \hat{h}(x_3) e^{2\pi i x(x_1-x_2+x_3)} d\vec{x} \right|\right|_{\frac{1}{p} + \frac{1}{q} + \frac{1}{r}}
\]

\[
= \left|\left| \lim_{N \to -\infty} \int_{-N < x_1 < x_2 < x_3 < N} \hat{f}(x_1) \hat{g}(x_2) \hat{h}(x_3) e^{2\pi i x(x_1-x_2+x_3)} d\vec{x} \right|\right|_{\frac{1}{p} + \frac{1}{q} + \frac{1}{r}}
\]

\[
\leq \lim\inf_{N \to -\infty} \left|\left| \int_{-N < x_1 < x_2 < x_3 < N} \hat{f}(x_1) \hat{g}(x_2) \hat{h}(x_3) e^{2\pi i x(x_1-x_2+x_3)} d\vec{x} \right|\right|_{\frac{1}{p} + \frac{1}{q} + \frac{1}{r}}
\]

\[
\leq \lim\inf_{N \to -\infty} ||f \ast \tilde{\chi}_{-N,N}||_p ||g \ast \tilde{\chi}_{-N,N}||_q ||h \ast \tilde{\chi}_{-N,N}||_r
\]

\[
\leq ||f||_p ||g||_q ||h||_r.
\]

Then \( C_3^{1,-1,1} \) would be continuous from \( \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R}) \), which contradicts our previous observation. \( \square \)

Corollary 2. The operator \( C_3^{1,1,-1} \) is not bounded from \( \mathcal{F}_c \cap L^p \times \mathcal{F}_c \cap L^q \times \mathcal{F}_c \cap L^r \) into \( L^{\frac{1}{p} + \frac{1}{q} + \frac{1}{r}} \) for any \( p, q, r \geq 1 \).

The usefulness in establishing such results for \( C_3^{1,-1,1} \) on the restricted spaces \( \mathcal{F}_c \) arises when one asks boundedness questions some multilinear generalizations. First, we need the following lemma:

Lemma 2. The following relation holds for \( p > 1 \):

\[
||e^{2\pi i x^2} \tilde{\chi}_{[-N,N]}||_p \asymp N^{1/p}.
\]

Proof. We compute explicitly

\[
\left|\left| \int_{\mathbb{R}} e^{2\pi i x^2} \chi_{[-N,N]}(x)e^{-2\pi i x \xi} dx \right|\right| = \left|\left| \int_{-N}^{N} e^{2\pi i x^2} dx \right|\right| = \left|\left| \int_{-N+\xi}^{N+\xi} e^{2\pi i x^2} dx \right|\right|.
\]
For $N$ is sufficiently large, there is $C, c > 0$ so that $C \geq \left| \int_{-N+\xi}^{N+\xi} e^{2\pi i x^2} \, dx \right| \geq c$
for all $|\xi| \leq N/2$. If $k \geq 0$ and $100N2^{k+1} \geq |\xi| \geq 100N2^k$, say, then we can estimate

$$\left| \int_{-N+\xi}^{N+\xi} e^{2\pi i x^2} \, dx \right| \approx \frac{1}{N2^k},$$

which ensures

$$N \lesssim \int_{-N+\xi}^{N+\xi} e^{2\pi i x^2} \, dx \approx N + \sum_{k \geq 0} \int_{100N2^k \leq |\xi| \leq 100N2^{k+1}} \frac{d\xi}{N^{p_2k} \rho} \approx N.$$  

The above proof also shows that $\| e^{2\pi i x^2} \chi_{[-N,N]} \|_1 = \infty$.

**Corollary 3.** The operator $C_n^{1,1,\ldots,-1,1} \in [L^{p_1} \times \ldots \times L^{p_n}]$ is not bounded from $L^{p_1} \times \ldots \times L^{p_n}$ into $L^{\frac{1}{p_1}+\ldots+\frac{1}{p_n}}$ for any $p_1, \ldots, p_n > 1$.

**Proof.** Fix $N \in \mathbb{Z}^+$ large. By the proceeding remark, there exists $M$ such that

$$\left| \int_{x_1 < x_2 < x_3} \tilde{g}_{N,M}(x_1) \tilde{g}_{N,M}(x_2) \tilde{g}_{N,M}(x_3) e^{2\pi i(x_1-x_2-x_3)} \, dx \right| \geq \log N \|g_{N,M}^+\|_{p_1} \|g_{N,M}^-\|_{p_2} \|g_{N,M}^\ast\|_{p_3}.$$  

Now pick $\{b_j\}_{j=1}^{n-3}$ and define $f_j(x) = e^{2\pi i b_j x} C_3^{1,1,\ldots,-1,1}(g_{N,M}^+,g_{N,M}^-,g_{N,M}^\ast)(x)$ in a way that ensures $\{\text{supp } f_j\}_{j=1}^{n-3}$ are disjoint sets and outside $[-M,M]$. Observe for $\frac{1}{p} = \sum_{i=1}^n \frac{1}{p_i}$

$$\|C_n^{+\ldots-+}(f_1, \ldots, f_n)\|_r \approx \|C_n^{+\ldots-+}(g_{N,M}^+,g_{N,M}^-,g_{N,M}^\ast)^{n-2}\|_r \geq (\log N)^{n-2} N^\frac{1}{p}.$$  

However, $\|C_3^{1,1,\ldots,1}(g_{N,M}^+,g_{N,M}^-,g_{N,M}^\ast)\|_{p_i} \approx \log N N^\frac{1}{p_i}$. Putting it all together,

$$\|C_n^{+\ldots-+}(f_1, \ldots, f_n)\|_r \gtrsim \log N \prod_{i=1}^n \|f_i\|_{p_i}.$$  

**Corollary 4.** $C_n^{(1,1,\ldots,1,-1)}$ is not bounded from $L^{p_1} \times \ldots \times L^{p_n}$ into $L^{\frac{1}{p_1}+\ldots+\frac{1}{p_n}}$ in the case where $\frac{1}{p_1} + \frac{1}{p_n} < 1$.  

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Proof. Set \( q = \frac{p_1 + p_2 + p_3}{p_3} \). The first claim is for sufficiently large \( M \)

\[
||C^{1,1,-1}_3(g^{+}_{N,M(N)}, g^{-}_{N,M(N}), g^0_{N,M(N)})||_q \leq q \log NN^{1/q}
\]

for \( \frac{1}{p_2} + \frac{1}{p_3} < 1 \). To see this, note

\[
\int_{x_1 < x_2} \hat{f}(x_1) \hat{g}(x_2) e^{2\pi i x_1 - x_2} dx_1 dx_2 = \frac{1}{2} f(x) \hat{g}(x) + \frac{1}{2} H((\hat{f} * \hat{g})^\cdot)(x)
\]

where \( \hat{g}(x) = g(-x) \) so by the boundedness of the Hilbert transform and the triangle inequality

\[
N^{1/q}
\]

\[
||g^0_{N,M(N)}(x) H(g^+_{N,M} \cdot g^-_{N,M}(x))||_q
\]

\[
= ||C^{1,1,-1}_3(g^0_{N,M} \cdot g^+_{N,M} \cdot g^-_{N,M})||_q + ||C^{1,1,-1}_3(g^0_{N,M} \cdot g^+_{N,M} \cdot g^-_{N,M})||_q
\]

\[
\leq ||C^{1,1,-1}_3(g^0_{N,M} \cdot g^+_{N,M} \cdot g^-_{N,M})||_q + \log NN^{1/q}.
\]

Thus, \( ||C^{1,1,-1}_3(g^0_{N,M} \cdot g^+_{N,M} \cdot g^-_{N,M})||_q \leq \log NN^{1/q} \).

The second claim is \( ||C^{1,1,-1}_3(g^0_{N,M} \cdot g^+_{N,M} \cdot g^-_{N,M})||_q \leq q \log NN^{1/q} \). For this, it suffices to observe

\[
||C^{1,1,-1}_3(g^0_{N,M} \cdot g^+_{N,M} \cdot g^-_{N,M})||_q
\]

\[
\leq ||g^0_{N,M} H(g^+_{N,M} \cdot g^-_{N,M})||_q + ||C^{1,1,-1}_3(g^0_{N,M} \cdot g^+_{N,M} \cdot g^-_{N,M})||_q
\]

\[
\leq q \log NN^{1/q}.
\]

Hence, \( ||C^{1,1,-1}_3(g^0_{N,M} \cdot g^+_{N,M} \cdot g^-_{N,M})||_q \leq q \log NN^{1/q} \), and by similar arguments as in corollary \( 3 \) the result follows.

3.1. \( C^p_n \) on domains containing \( W_p \) with \( p < 2 \)

Proposition 3. \( C^{1,1,1}_3(f, g, h)(x) := \int_{x_1 < x_2 < x_3} \hat{f}(x_1) \hat{g}(x_2) \hat{h}(x_3) e^{2\pi i x_1 + x_2 + x_3} d\vec{x} \)

satisfies no estimates of the form \( L^{p_1} \times L^{p_2} \times W_{p_3} \rightarrow L^{r_1} \times L^{r_2} \times W_{r_3} \) for any \( p_1, p_2, p_3 > 1 \) and \( p_3 < 2 \).

Proof. Recall \( g^0_{N,M}(x) = (e^{2\pi i x^2} \hat{\chi}_{[-N,N]} \ast \hat{\chi}_{[-M,M]})(x) \). By dominated convergence one may pick \( M \) such that for all \( p, ||g^0_{N,M}||_p \leq N^{1/p} \) for all \( p > 1 \) as well as for all \( q > 1, ||g^0_{N,M}||_q \leq q ||e^{2\pi i x^2} \ast (\hat{\chi}_{[-N,N]})(x)||_q \leq q N. \) Now modulate 3 copies of \( g^0_{N,M} \) to ensure they have mutually disjoint Fourier supports, label them \( f, g, h \), and observe
\[ ||C_3^{1,1}(f, g, h)|| \frac{1}{p_1 + \frac{1}{p_2} + \frac{1}{p_3}} = ||g_{N,M}^+||^3 \frac{1}{p_1 + \frac{1}{p_2} + \frac{1}{p_3}} \approx p_{1,2,3} N^{\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}} \]

whereas \[ ||g_{N,M}^+||_{p_1} ||g_{N,M}^+||_{p_2} ||g_{N,M}^+||_{p_3} \lesssim N^{\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}} \]. Boundedness requires \( \frac{1}{p_3} \leq \frac{1}{p_3} \), which is not the case for \( p_3 < 2 \).

**Proposition 4.** \( C_n^{1,1, \ldots, 1} \) is not bounded from \( W_{p_1} \times \ldots \times W_{p_n} \) into \( L^{\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}} \) if there exists some \( i \) for which \( p_i < 2 \).

**Proof.** Set \( F_{N,M}^+ = g_{N,M}^+ \) and \( F_{N,M}^2 = \tilde{\chi}_{[-N,N]} * \tilde{\chi}_{[-M,M]} \). Observe that for sufficiently large \( M \), \( ||(\tilde{\chi}_{[-N,N]} * \tilde{\chi}_{[-M,M]})(x') ||_{p'} \approx N^{\frac{1}{p'}} \) where \( ||(g_{N,M}^+)(x') ||_{p'} \approx N^{\frac{1}{p'}} \). By taking suitable modulations of \( F_1 \) for \( f_j \) with \( j \neq i \) and \( F_2 \) for \( f_i \), it is readily seen that

\[ ||C_n^{1,1, \ldots, 1}(f_1, \ldots, f_n)|| \frac{1}{p_1 + \frac{1}{p_2} + \frac{1}{p_3}} \approx N^{\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}} \]

\[ ||\hat{f_1}||_{p_1'} \ldots ||\hat{f_n}||_{p_n'} \approx N^{\frac{1}{p_1} + \sum_{j=1}^{n} \frac{1}{p_j}} \].

As \( \frac{1}{p_1} > \frac{1}{p_i} \) when \( p_i < 2 \), the claim follows. \( \Box \)

### 3.2. Connections between \( L^q \) and \( W_p \) where \( \frac{1}{p} + \frac{1}{q} > 1 \)

Recall \( \int_{x_1 < x_2} \hat{f}(x_1) \hat{g}(x_2) e^{2\pi i x_1} e^{2\pi i x^2} dx_1 dx_2 = \frac{1}{2} \int f(x) \hat{g}(x) + \frac{1}{2} H((\hat{f} * \hat{g}^-)(x)). \)

Therefore, \( C_2^{+,-} \) is essentially the Hilbert transform of a product, so we expect that for \( \frac{1}{p_1} + \frac{1}{p_2} \geq 1 \), it fails to be continuous. The next proposition proves that this is indeed the case even if one of the functions is in a Wiener space.

**Proposition 5.** Assume \( \frac{1}{p_1} + \frac{1}{p_2} \geq 1 \). Then

\( C_2^{+,-} : W_{p_1} \times L^{p_2} \not\rightarrow L^{\frac{1}{p_1} + \frac{1}{p_2}} \).

**Proof.** CASE 1: \( \frac{1}{p_1} + \frac{1}{p_2} \geq 1 \) and \( p_2 > 1 \). Take \( f_1(x) = f_2(x) = \tilde{\chi}_{[-1,1]}(x) \). Then, a brief calculation gives

\[ C_2^{+,-}(f_1, f_2)(x) = \int_{-1 < x_1 < x_2 < 1} e^{2\pi i x_1} dx_1 dx_2 \]

\[ = \int_{-1 < x_1 < 1} \frac{1}{2\pi i x} e^{2\pi i (-1 - x_2)} dx_2 \]

\[ = \frac{1}{\pi x} \left( \frac{e^{2\pi i x(-2)}}{4\pi^2 x^2} + \frac{1}{4\pi^2 x^2} \right) \]

\[ = \frac{1}{\pi x} + \frac{1 - e^{-4\pi x}}{4\pi^2 x^2} \]

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Therefore, by considering the large x behavior $C^{+, -}_2(f_1, f_2)(x) \sim \frac{1}{x}$, we conclude $||C^{+, -}_2||_{\frac{1}{p_1} + \frac{1}{p_2}} = \infty$. However, both $||f_1||_{p_1}, ||f_2||_{p_2} < \infty$ because $p_2 > 1$.

CASE 2: $p_1 > 2, p_2 = 1$. The only impediment to our previous method handling this case as well was the fact that $||\tilde{x}_{[-1, 1]}||_1 = \infty$. To fix this issue, we may let $\epsilon > 0$ and consider $f_1 = f_2 = (\chi_{[-1, 1]} * \rho_\epsilon)$ where $\rho$ is any standard smooth mollifier and $\rho_\epsilon = \frac{1}{\epsilon} \rho(x/\epsilon)$. Because $\chi_{[-1, 1]} * \rho_\epsilon$ is a Schwartz function, $||\chi_{[-1, 1]} * \rho_\epsilon||_{p_1}, ||(\chi_{[-1, 1]} * \rho_\epsilon)^{-1}||_1 < \epsilon \infty$. However,

$$C^{+, -}_2((\chi_{[-1, 1]} * \rho_\epsilon)^{-1}, (\chi_{[-1, 1]} * \rho_\epsilon)^{-1})(x) \sim \frac{1}{x}$$

for small enough $\epsilon$.

\[\square\]

**Corollary 5.** $C^{-1, 1}_2$ is bounded from $L^{p_1} \times W_{p_2}$ into $L^{\frac{1}{p_1} + \frac{1}{p_2}}$ if and only if $\frac{1}{p_1} + \frac{1}{p_2} < 1$ and $p_2 > 2$.

**Theorem 6.** If for some $1 \leq i \leq n - 1$, $\frac{1}{p_i} + \frac{1}{p_{i+1}} > 1$, $\epsilon_i = -\epsilon_{i+1}$, $X_i = W_{p_i}$, and $X^{i+1} = L^{p_i+1}$, then $C^{-1, 1}_n$ is not bounded from $\otimes_{i=1}^n X^i$ into $L^\infty$.

**Proof.** Fix $N \in \mathbb{Z}^+$ large. For each positive integer $M$, fix $\{b_j\}_{j \neq i, i+1}$ along with $f^M_i(x) = (\tilde{x} [1, N] * \tilde{x} [-M, M]) e^{2\pi i b_j x}$ and $f^M_{i+1} = \tilde{x} [-1, 1]$, so that the supports of the functions $\{f^M_j\}$ (except for $f^M_i$ and $f^M_{i+1}$) are mutually disjoint. By Fatou’s lemma, it follows that

$$\liminf_{M \to \infty} ||C_n(f^M_1, ..., f^M_n)||_{\frac{1}{p_1} + \frac{1}{p_2}} \geq \lim_{M \to \infty} |C_n(f^M_1, ..., f^M_n)|$$

$$\geq \left| \frac{\tilde{x}[1, N]}{x} \right|_{\frac{1}{p_1} + \frac{1}{p_2}}$$

$$= N^{\sum_{i \neq i, i+1} \frac{1}{p_i}}$$

$$> N^{\sum_{j \neq i, i+1} \frac{1}{p_i}}$$

$$\geq \lim_{M \to \infty} \prod_j ||f^M_i||_{X^i},$$

where $\sum_{j \neq i, i+1} \frac{1}{p_i} < \sum_i \frac{1}{p_i} - 1$ because $\frac{1}{p_i} + \frac{1}{p_{i+1}} > 1$.

\[\square\]

**Putting together all the results in this section, we obtain the following:**

**Corollary 6.** The “only if” part of theorem b holds.
4. Tensor Product Generalizations

A natural question to ask is whether one has estimates available for the bilinear operator $BHT \otimes BHT$ which is given pointwise by

$$\int_{x_1 < x_2 \atop y_1 < y_2} f(x_1, y_1)g(x_2, y_2)e^{2\pi i((x_1 + x_2) + (y_1 + y_2))} \, dx_1 \, dx_2 \, dy_1 \, dy_2.$$  

It is straightforward to verify that this expression is (up to harmless modifications) expressible as

$$p.v. \int \int f(z_1 + s - t, z_2 - s - t)g(z_1 + s + t, z_2 + s + t) \, ds \, dt,$$

and Muscalu, Tao, Pipher, and others observed that this principal value integral does not admit any $L^p$ estimates, see [10]. If we define $(f)\hat{1}(x, y) = \int f(\xi, y)e^{-2\pi i\xi \cdot x} \, d\xi$ as well as $(f)\hat{2}(x, y) = \int f(x, \xi)e^{-2\pi i\xi \cdot y} \, d\xi$ and in addition set

$$W^1_p[L^\frac{p}{2}](\mathbb{R}^2) = \left\{ f : \mathbb{R}^2 \to \mathbb{C} \left| \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |(f)\hat{1}(x, y)|^q \, dy \right)^{p'/q} \, dx \right)^{1/p'} \right. \right\}$$

$$L^2_p[W^1_p](\mathbb{R}^2) = \left\{ f : \mathbb{R}^2 \to \mathbb{C} \left| \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |(f)\hat{1}(x, y)|^p \, dx \right)^{q/p'} \, dy \right)^{1/q} \right. \right\}$$

$$W^2_p[L^\frac{p}{2}](\mathbb{R}^2) = \left\{ f : \mathbb{R}^2 \to \mathbb{C} \left| \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |(f)\hat{2}(x, y)|^q \, dx \right)^{p'/q} \, dy \right)^{1/p'} \right. \right\}$$

along with $W^2_p[L^{1/p}](\mathbb{R}^2)$, $L^{1/p}_p[W^1_p](\mathbb{R}^2)$ in the obvious way, one may ask whether any of the following estimates hold:

$$BHT \otimes BHT : \quad W^1_p[L^\frac{p}{2}] \otimes L^{p_\alpha}[L^{p_\alpha}] \to L^{\frac{p_\alpha}{1+\frac{p_\alpha}{p}}} [L^{\frac{p_\alpha}{p}+\frac{p_\alpha}{p}}](\mathbb{R}^2)$$

$$BHT \otimes BHT : \quad L^{p_\alpha}[W^{1/p} \otimes L^{p_\alpha}[L^{p_\alpha}] \to L^{\frac{p_\alpha}{1+\frac{p_\alpha}{p}}} [L^{\frac{p_\alpha}{p}+\frac{p_\alpha}{p}}](\mathbb{R}^2).$$

The answer is sometimes in the first case and never in the second. To see this, we first state and prove a lemma:

**Lemma 3.**

$$\| (e^{2\pi i x y} \chi_{[-N,N]}(x) \chi_{[-N,N]}(y)) \|_{W^p_{\alpha}[L^\gamma](\mathbb{R}^2)} \lesssim \max\{N^{\frac{p-\gamma}{p}}, N^\frac{1}{\alpha} \}$$

$$\| (e^{2\pi i x y} \chi_{[-N,N]}(x) \chi_{[-N,N]}(y)) \|_{L^\gamma[W^p_{\alpha}](\mathbb{R}^2)} \lesssim N^\frac{\gamma}{\alpha}.$$
Proof. First write down

\[
(e^{2\pi i \cdot x} \chi_{[-N, N]}(x) \chi_{[-N, N]}(y) \hat{\chi}(\xi) = \chi_{[-N, N]}(y) \left( \frac{e^{2\pi i N(y-\xi)} - e^{-2\pi i N(y-\xi)}}{2\pi i (y-\xi)} \right)
\]

Therefore, as long as \( p' > 1, \)

\[
\| (e^{2\pi i xy} \chi_{[-N, N]}(x) \chi_{[-N, N]}(y)) \hat{\chi} \|_{L^q W^p_1(\mathbb{R}^2)} \]

\[
= \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |(f) \hat{\chi}(\xi, y)|^{p'} \, dx \right)^{q/p'} \, dy \right)^{1/q}
\]

\[
= \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |\chi_{[-N, N]}(y) \left( \frac{e^{2\pi i N(y-\xi)} - e^{-2\pi i N(y-\xi)}}{2\pi i (y-\xi)} \right)|^{p'} \, dx \right)^{q/p'} \, dy \right)^{1/q}
\]

\[
\simeq N \left( \int_{-N}^{N} \left( \int_{\mathbb{R}} \left| \left( \frac{\sin(2\pi N(y-\xi))}{2\pi i N(y-\xi)} \right) \right|^{p'} \, dx \right)^{q/p'} \, dy \right)^{1/q}
\]

\[
\simeq NN^{-1/p'} \left( \int_{-N}^{N} \right)^{1/q}
\]

Similarly,

\[
\| (e^{2\pi i xy} \chi_{[-N, N]}(x) \chi_{[-N, N]}(y)) \hat{\chi} \|_{W^p_1 L^q(\mathbb{R}^2)} \]

\[
= \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |(f) \hat{\chi}(\xi, y)|^q \, dx \right)^{p/q} \, dy \right)^{1/p'}
\]

\[
= \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |\chi_{[-N, N]}(y) \left( \frac{e^{2\pi i N(y-\xi)} - e^{-2\pi i N(y-\xi)}}{2\pi i (y-\xi)} \right)|^q \, dx \right)^{p/q} \, dy \right)^{1/p'}
\]

\[
= NN^{-1/q} \left( \int_{\mathbb{R}} \left( \int_{-N}^{-2-N\xi} \left( \frac{e^{2\pi i y} - e^{-2\pi i y}}{2\pi i y} \right)^q \, dy \right)^{p'/q} \, dx \right)^{1/p'}
\]

\[
\simeq NN^{-1/q} \left( \int_{|\xi| \leq 100N} \left( \int_{-N}^{-2-N\xi} \left( \frac{\sin(2\pi y)}{2\pi y} \right)^q \, dy \right)^{p'/q} \, dx \right)^{1/p'}
\]

\[
+ NN^{-1/q} \left( \int_{|\xi| \geq 100N} \left( \int_{-N}^{-2-N\xi} \left( \frac{\sin(2\pi y)}{2\pi y} \right)^q \, dy \right)^{p'/q} \, dx \right)^{1/p'}
\]

\[
= I + II.
\]
Note that

\[
I \simeq NN^{-1/q} \left( \int_{|\xi| \leq 100N} \left( \int_1^{N^2} \left| \frac{1}{y} \right|^q d\bar{y} \right)^{p'/q} d\xi \right)^{1/p'}
\]

\[
\simeq NN^{-1/q} \left( \int_{|\xi| \leq 100N} \left( \int_1^{N^2} \left| \frac{1}{y} \right|^q d\bar{y} \right)^{p'/q} d\xi \right)^{1/p'}
\]

\[
\simeq N^{p'/p-q}
\]

as well as

\[
II \simeq NN^{-1/q} \left( \int_{|\xi| \geq 100N} \left( \int_{-N^2-N^2}^{N^2-N^2} \left| \frac{\sin(2\pi \bar{y})}{2\pi \bar{y}} \right|^q d\bar{y} \right)^{p'/q} d\xi \right)^{1/p'}
\]

\[
\simeq NN^{-1/q}N^{2/q} \left( \int_{|\xi| \geq 100N} \left( \frac{1}{N\xi} \right)^{p'} d\xi \right)^{1/p'}
\]

\[
\simeq N^{1/q}N^{1-\frac{1}{p'}} = N^\frac{p+q}{pq}.
\]

**Corollary 7.** If \(BHT \otimes BHT\) is continuous, then the factor \(L_q[W_p]\) may never appear as a factor in its domain. If \(\frac{1}{p'} + \frac{1}{q'} \leq \frac{1}{p} + \frac{1}{q}\), i.e. \(\frac{1}{p} + \frac{1}{q} \geq 1\), then \(W_p[L^q]\) cannot be a factor in its domain.

**Proof.** This follows immediately from the counterexample constructed by Muscalu, Pipher, Tao, and Thiele in [10], where they set \(h = e^{ixy} \chi_{[-N,N]}(x) \chi_{[-N,N]}(y)\) and show \(||BHT \otimes BHT(f, f)||_q \gtrsim \log N||f||_p, ||f||_p_2\) for \(\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2}\).

Remark: Observe by Minkowski’s integral inequality that \(L_q[W_p](\mathbb{R}^2) \supset W_p[L^q](\mathbb{R}^2)\) when \(q \geq p'\), i.e. \(\frac{1}{p'} \geq \frac{1}{q}\), or \(\frac{1}{p} + \frac{1}{q} \leq 1\). So, we may still have estimates of the form

\[
BHT \otimes BHT : W_{p_1}[L^{q_1}] \otimes L^{p_2}[L^{q_2}] \to L^{\frac{p_1 p_2}{p_1 + p_2}}[L^{\frac{q_1 q_2}{q_1 + q_2}}]
\]

whenever \(\frac{1}{q_1} + \frac{1}{p_2} < 1\) and at least sometimes do as the following theorems show. Of course, in the endpoint case when \(q = p'\), no mixed estimates can hold.
4.1. Statement and Proof of Estimates for Tensor Product Operators

**Theorem 7.** For any \( N \in \mathbb{N}_{even} \) and \( \varepsilon \in \{\pm 1\}^n \), the map \( C_N^\varepsilon \otimes C_N^\varepsilon \) is continuous from \( W_{p_1}[W_{p_1}] \otimes L^{p_2}[L^{p_2}] \otimes \cdots \otimes W_{p_{N-1}}[W_{p_{N-1}}] \otimes L^{p_N}[L^{p_N}] \) into \( L_{\frac{1}{1+\varepsilon}}^{\frac{1}{\frac{1}{p_1}+\varepsilon\frac{1}{p_2}+\cdots+\varepsilon\frac{1}{p_{N-1}}+\varepsilon\frac{1}{p_N}}} \) provided \( \frac{1}{p_i} + \max_{j \in \{i-1,i+1\}} \frac{1}{p_j} < 3/2 \) for all odd \( i \), where

\[
L^q[L^q](\mathbb{R}^2) := \left\{ f : \mathbb{R}^2 \to \mathbb{C} : \int_{\mathbb{R}^2} |f(x,y)|^q dx dy < \infty \right\}
\]

\[
W_q[W_q](\mathbb{R}^2) := \left\{ f : \mathbb{R}^2 \to \mathbb{C} : \int_{\mathbb{R}^2} |\hat{f}(x,y)|^q dx dy < \infty \right\}
\]

**Proof.** We first prove the theorem for the case \( N = 2 \) and \( \varepsilon = \bar{1} \) under slightly weaker conditions, namely \( \frac{1}{p_1} + \frac{1}{p_2} < 1 \). So, let \( f \in W_q[W_q](\mathbb{R}^2) \) and define the map \( \gamma_f : \mathbb{R} \to [0,1] \) given by

\[
\gamma_f(x) = \frac{\int_{-\infty}^{x} \int_{\mathbb{R}} |\hat{f}(x,y)|' dy dx}{||f||_{W_q[W_q](\mathbb{R}^2)}^q}.
\]

Then let \( E_{j_1}^{m_1} = \gamma_{j_1}^{-1}(j_1 2^{-m_1}, (j_1 + 1) 2^{-m_1}) \). Fix \( m, j \) and define another map \( \gamma_{m_1,j_1,j} : \mathbb{R} \to [0,1] \) by

\[
\gamma_{m_1,j_1,j}(y) = \frac{\int_{E_{j_1}^{m_1}} \int_{-\infty}^{y} |\hat{f}(x,y)|' dy dx}{||f * 1 \chi_{E_{j_1}^{m_1}}||_{W_q[W_q](\mathbb{R}^2)}^q}
\]

and set \( E_{j_1,j_2}^{m_1,m_2} = \gamma_{m_1,j_1,j}^{-1}(j_2 2^{-m_2}, (j_2 + 1) 2^{-m_2}) \). Note that by construction, for any \( 0 \leq j_1 < 2^{m_1} \) and \( 0 \leq j_2 < 2^{m_2} \),

\[
||f * 1 \chi_{E_{j_1}^{m_1}} * 2 \chi_{E_{j_2}^{m_1,m_2}}||_{W_q[W_q](\mathbb{R}^2)}^q = 2^{-(m_1 + m_2)} ||f||_{W_q[W_q](\mathbb{R}^2)}^q.
\]

The harder case occurs when \( p_2 < 2 \). Fix \( \epsilon > 0 \) such that \( \frac{1}{p_1} + \frac{1}{p_2} - 1 + \epsilon < 0 \). Choose \( r \) such that \( \frac{1}{r} = \frac{1}{p_2} - \epsilon \) and select \( \alpha \) such that \( \frac{1}{r} = \frac{1}{\alpha} + \frac{1}{\alpha} \). Set \( Q = \frac{p_1p_2}{p_1 + p_2} \).

We perform the decomposition of \( C_2^{1,1} \otimes C_2^{1,1} \) as follows:
The geometric sum converges because

\[ \sum_{m_1,m_2} \sum_{j_1,j_2} f_1 *_1 \tilde{X}_{m_1,j_1} *_2 \tilde{X}_{m_2,j_2} *_{1,2} f_2 *_1 \tilde{X}_{m_1,j_1} *_2 \tilde{X}_{m_2,j_2} = \mathbb{E} \left[ \left( \sum_{j_1,j_2} f_1 *_1 \tilde{X}_{m_1,j_1} *_2 \tilde{X}_{m_2,j_2} \right)^{1/2} \right] \leq \mathbb{E} \left[ \left( \sum_{j_1,j_2} f_2 *_1 \tilde{X}_{m_1,j_1} *_2 \tilde{X}_{m_2,j_2} \right)^{1/2} \right] \]

We then use generalized Rubio de Francia estimates in higher dimensions, see \( \tilde{f} \), together with the defining property of the martingale structures \( E_{m_1}^{m_1} \) and \( E_{j_1,j_2}^{m_1,m_2} \) to observe

\[ A \leq \sum_{m_1,m_2} \left( \sum_{j_1,j_2} \left| f_1 *_1 \tilde{X}_{m_1,j_1} *_2 \tilde{X}_{m_2,j_2} \right|^{(p_1-1)/p_1} \right)^{1/p_1} \times \left( \sum_{j_1,j_2} \left| f_2 *_1 \tilde{X}_{m_1,j_1} *_2 \tilde{X}_{m_2,j_2} \right|^{(p_2-1)/p_2} \right)^{1/p_2} \]

The geometric sum converges because \( \frac{1}{p_1} + \frac{1}{p_2} - 1 + \epsilon < 0 \) for small enough \( \epsilon \). The case when \( p_2 \geq 2 \) is easier because one can apply Rubio de Francia’s inequality directly.

**INDUCTION STEP:** Having proved the theorem for the \( N = 2 \) case, we suppose the \( N = n - 2 \) case has also been shown and prove the \( N = n \) case. For
CASE 1: \( m_1 + m_3 \leq m_2 + m_4 \).

Restricting our sum to all such \( \tilde{m} \), we compute

\[
A \leq \left( \sum_{m_1 \leq m_2; m_3 \leq m_4} \left\| \sum_{j \in \mathbb{Z}} \left| \sum_{j \in \mathbb{Z}} C_{n-2}^{1, \ldots, 1} \otimes C_{n-2}^{1, \ldots, 1} (f_1, \ldots, f_{j-2})^j \right| \right\|_{L^q_{1,1}} \right)^{1/q} \times
\|
\sum_{j \in \mathbb{Z}} C_{n-2}^{1, \ldots, 1} \otimes C_{n-2}^{1, \ldots, 1} (f_1, \ldots, f_{j-2})^j \|_{L^q_{1,1}} \right)^{1/q} \]

This fact follows from lemma \( 6 \) included in the appendix. Now we carve the 4 connections between \( f_{n-2}, f_{n-1}, \) and \( f_n \) using two copies of the two-dimensional grid martingale structure introduced earlier. This will introduce 4 scales, so we are naturally faced with the following situation:

\[
\| \sum_{j \in \mathbb{Z}} C_{n-2}^{1, \ldots, 1} \otimes C_{n-2}^{1, \ldots, 1} (f_1, \ldots, f_{j-2} \otimes \tilde{f}_{j-2}^2, \tilde{f}_{j-2})^j \|_{L^q_{1,1}} := A.
\]
Further computation given upper bounds of the form

\[
\sum_{m_1 \leq m_2; m_3 \leq m_4} 2^{Q(m_1+m_3) \max\{1/2-1/p'_{n-2}+\varepsilon,0\}} ||f_n-2||_{L^p_{n-2}}^{Q} \times \\
|| ||f_n-1\chi_{E_1}^{m_1} \chi_{E_{j_1;3}}^{m_1+m_3} \chi_{E_{j_2;4}}^{m_2} \chi_{E_{j_2;4}}^{m_2+m_4} ||W_{p_{n-1}}||_{j_1 \ldots j_4}^{Q} \times \\
2^{Q(m_2+m_4) \max\{1/2-1/p'_{n-2}+\varepsilon,0\}} ||f_n||_{L^p_{n-1}}^{Q} \right)^{1/Q}.
\]

Focusing just on factor inside the sum,

\[
\sum_{m_1 \leq m_2; m_3 \leq m_4} 2^{Q(m_1+m_3) \max\{1/2-1/p'_{n-2}+\varepsilon,0\}} ||f_n-1\chi_{E_1}^{m_1} \chi_{E_{j_1;3}}^{m_1+m_3} \chi_{E_{j_2;4}}^{m_2} \chi_{E_{j_2;4}}^{m_2+m_4} ||W_{p_{n-1}}||_{j_1 \ldots j_4}^{Q} \times \\
2^{Q(m_2+m_4) \max\{1/2-1/p'_{n-2}+\varepsilon,0\}} ||f_n||_{L^p_{n-1}}^{Q} \right)^{1/Q}.
\]

The correct upper bound is attained by the convergence of

\[
\sum_{m_1 \leq m_2; m_3 \leq m_4} 2^{Q(m_1+m_3) \max\{1/2-1/p'_{n-2}+\varepsilon,0\} + 1/2-1/p'_{n-2}+\varepsilon,0\} + \max\{1/2-1/p'_{n-2}+\varepsilon,0\}].
\]

CASE 2: \(m_1 + m_3 \geq m_2 + m_4\). This case is entirely symmetric to CASE 1, so it is omitted. As the above argument generalizes to any choice of sign \(\varepsilon \in \{\pm 1\}^n\), the theorem follows.

**Corollary 8.** We may take \(p_i = p\) for odd \(i\) and \(p_i = 2\) for even \(i\) in the previous theorem to obtain the boundedness of \(W_p[\omega] \otimes L^2[\omega] \otimes \cdots \otimes W_p[\omega] \otimes L^2[\omega]\) into \(L^{1 \pm p/2} [L^{1 \pm p/2}]\).

This result will be used later in our application to bi-parameter AKNS systems. Before moving on, we note that because we have a multi-dimensional generalized Rubio de Francia estimate along with a multiparameter carving designed to handle the tensor product, continuity results are attainable in this more complex setting. As in the original case, the sum over scales one gets in the decomposition can always be placed outside the integral norms in either the Banach or Quasi-Banach case without trouble.

It is perhaps a little surprising that one can also establish continuity results for \(BHT \otimes BHT\) in the case where the first function is Wiener only in first variable and the second function is Wiener only in the second.

**Theorem 8.** Let \(p_1, p_2 > 2\). Fix \(p'_{1} < q_1 < p_1; p'_{2} < q_2 < p_2\). Assume

\[
\frac{1}{p_1} + \frac{1}{q_1} < \min \left(1, \frac{3}{2} - \frac{1}{q_1}\right)
\]

\[
\frac{1}{p_2} + \frac{1}{q_1} < \min \left(1, \frac{3}{2} - \frac{1}{q_2}\right).
\]
Then the operator

\[ BHT \otimes BHT : W^1_{p_1}[L^{q_1}_2](\mathbb{R}^2) \times W^2_{p_2}[L^{q_2}_1](\mathbb{R}^2) \rightarrow L^{r_{1,2}}_{1/q_1 + 2/q_2} [L^{r_{1,2}}_{2/q_1 + 2/q_2}](\mathbb{R}^2) \]

is bounded.

**Proof.** One introduces the usual carving and computes as follows:

\[
\begin{align*}
\| \| \| (f_1 \ast 1 \tilde{X}_{E_{j_1}, l} \ast 2 \tilde{X}_{E_{j_2}, r}) (f_2 \ast 1 \tilde{X}_{E_{j_1}, l} \ast 2 \tilde{X}_{E_{j_2}, r}) ||_{L^{r_{1,2}}_1} ||_{L^{r_{1,2}}_2} & \\
\leq \| \| \| f_1 \ast 1 \tilde{X}_{E_{j_1}, l} \ast 2 \tilde{X}_{E_{j_2}, r} ||_{L^{r_{1,2}}_1} ||_{L^{r_{1,2}}_2} \times \| \| \| f_2 \ast 1 \tilde{X}_{E_{j_1}, l} \ast 2 \tilde{X}_{E_{j_2}, r} ||_{L^{r_{1,2}}_1} ||_{L^{r_{1,2}}_2} & \\
\leq \| \| \| f_1 \ast 1 \tilde{X}_{E_{j_1}, l} \ast 2 \tilde{X}_{E_{j_2}, r} ||_{L^{r_{1,2}}_1} ||_{L^{r_{1,2}}_2} \times \| \| \| f_2 \ast 1 \tilde{X}_{E_{j_1}, l} \ast 2 \tilde{X}_{E_{j_2}, r} ||_{L^{r_{1,2}}_1} ||_{L^{r_{1,2}}_2} & \\
\leq \| \| \| (f_1 \tilde{X}_{E_{j_1}, l} \ast 2 \tilde{X}_{E_{j_2}, r}) ||_{L^{r_{1,2}}_1} ||_{L^{r_{1,2}}_2} \times \| \| \| (f_2 \tilde{X}_{E_{j_1}, l} \ast 2 \tilde{X}_{E_{j_2}, r}) ||_{L^{r_{1,2}}_1} ||_{L^{r_{1,2}}_2} & \\
\end{align*}
\]

\[ := A \]

**CASE 1:** \( q_1 \leq 2, q_2 \leq 2. \) We compute as follows:

\[
\begin{align*}
A & \leq \| \| \| (f_1 \tilde{X}_{E_{j_1}, l} \ast 2 \tilde{X}_{E_{j_2}, r}) ||_{L^{r_{1,2}}_1} ||_{L^{r_{1,2}}_2} \times \| \| \| (f_2 \tilde{X}_{E_{j_1}, l} \ast 2 \tilde{X}_{E_{j_2}, r}) ||_{L^{r_{1,2}}_1} ||_{L^{r_{1,2}}_2} & \\
\leq \| \| \| (f_1 \tilde{X}_{E_{j_1}, l} \ast 2 \tilde{X}_{E_{j_2}, r}) ||_{L^{r_{1,2}}_1} ||_{L^{r_{1,2}}_2} \times \| \| \| (f_2 \tilde{X}_{E_{j_1}, l} \ast 2 \tilde{X}_{E_{j_2}, r}) ||_{L^{r_{1,2}}_1} ||_{L^{r_{1,2}}_2} & \\
\leq 2^{m_1(1/2 - 1/q_1 + \epsilon)} \| \| \| (f_1 \tilde{X}_{E_{j_1}, l} \ast 2 \tilde{X}_{E_{j_2}, r}) ||_{L^{r_{1,2}}_1} ||_{L^{r_{1,2}}_2} \times \| \| \| (f_2 \tilde{X}_{E_{j_1}, l} \ast 2 \tilde{X}_{E_{j_2}, r}) ||_{L^{r_{1,2}}_1} ||_{L^{r_{1,2}}_2} & \\
\leq 2^{m_1(1/2 - 1/q_1 + \epsilon)} \| \| \| (f_1 \tilde{X}_{E_{j_1}, l} \ast 2 \tilde{X}_{E_{j_2}, r}) ||_{L^{r_{1,2}}_1} ||_{L^{r_{1,2}}_2} \times \| \| \| (f_2 \tilde{X}_{E_{j_1}, l} \ast 2 \tilde{X}_{E_{j_2}, r}) ||_{L^{r_{1,2}}_1} ||_{L^{r_{1,2}}_2} & \\
\leq 2^{m_2(1/2 - 1/q_1 + \epsilon)} \| \| \| (f_1 \tilde{X}_{E_{j_1}, l} \ast 2 \tilde{X}_{E_{j_2}, r}) ||_{L^{r_{1,2}}_1} ||_{L^{r_{1,2}}_2} \times \| \| \| (f_2 \tilde{X}_{E_{j_1}, l} \ast 2 \tilde{X}_{E_{j_2}, r}) ||_{L^{r_{1,2}}_1} ||_{L^{r_{1,2}}_2} & \\
\leq 2^{m_2(1/2 - 1/q_1 + \epsilon)} \| \| \| (f_1 \tilde{X}_{E_{j_1}, l} \ast 2 \tilde{X}_{E_{j_2}, r}) ||_{L^{r_{1,2}}_1} ||_{L^{r_{1,2}}_2} \times \| \| \| (f_2 \tilde{X}_{E_{j_1}, l} \ast 2 \tilde{X}_{E_{j_2}, r}) ||_{L^{r_{1,2}}_1} ||_{L^{r_{1,2}}_2} & \\
\end{align*}
\]

The two geometric series over \( m_1 \) and \( m_2 \) are summable for small enough \( \epsilon \) when \( 1/2 - 1/q_2 - 1/p'_1 + 1/q_1, 1/2 - 1/q'_1 - 1/p'_2 + 1/q_2 < 0 \). These two conditions may be rephrased as
\[ \frac{1}{p_1} + \frac{1}{q_2} < \frac{3}{2} - \frac{1}{q_1}, \quad \frac{1}{p_2} + \frac{1}{q_1} < \frac{3}{2} - \frac{1}{q_2}. \]

CASE 2: If \( q_1 > 2 \) and \( q_2 \leq 2 \), we compute as follows:

\[
A \leq \| f_1 \| \| \mathcal{F}_1 \| \| f_2 \| \| \mathcal{F}_2 \| \| f_3 \| \| \mathcal{F}_3 \| \| f_4 \| \| \mathcal{F}_4 \|
\]

\[
\leq \left( \frac{1}{p_1} + \frac{1}{q_1} \right) \left( \frac{1}{p_2} + \frac{1}{q_2} \right) \left( \frac{1}{p_3} + \frac{1}{q_3} \right) \left( \frac{1}{p_4} + \frac{1}{q_4} \right)
\]

As \( \frac{1}{q_1} + \frac{1}{q_2} \) and \( \frac{1}{q_3} + \frac{1}{q_4} \) are small enough, the geometric series converge for small enough \( \varepsilon \).

CASE 3: if \( q_1 \leq 2 \) and \( q_2 > 2 \), we can apply CASE 2 with the roles of \( 1 \) and \( 2 \) interchanged.

CASE 4: If both \( q_1, q_2 > 2 \), then a similar calculation to the ones already described gives a product of two geometric series, both summable because

\[
\frac{1}{p_1} + \frac{1}{q_1} + \frac{1}{p_2} + \frac{1}{q_2} < 1.
\]

Remark: This argument does not depend on signs in the exponential appearing in \( C(f_1, f_2) \).

It is also interesting to consider the case where one faces one Wiener space and 3 Lebesgue spaces in the tensor product setting.

**Theorem 9.** The operator \( \text{BHT} \otimes \text{BHT} : W_{p_1}^{\mu_1} L_{q_1}^{\gamma_1} \times L_{q_2}^{\gamma_2} \rightarrow L_{p_2}^{\mu_2} L_{q_3}^{\gamma_3} \) is bounded provided \( \frac{1}{q_1} + \frac{1}{q_3} < \frac{3}{2}, p_1' < \min_i(q_i) \leq \max_i(q_i) < p_1, \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} + \frac{1}{q_i} < 2, \) and \( q_3 \leq q_2 \).
Proof. Begin by observing

\[
\|BHT \otimes BHT(f_1, f_2)\|_{L_1^p + q_2} \leq_{L_2^{q_1 + q_3}} \|BHT(f_1, f_2)\|_{L_1^p + q_2} \leq \left( \sum_{m \geq 0} \|BHT_2(f_1 \ast_1 \tilde{\chi}_{E_{j,m}^n}, f_2 \ast_1 \tilde{\chi}_{E_{j,m}^n})\|_{L_1^q} \right)^{1/Q}.
\]

Now split according to three cases.

CASE 1: \[ \frac{1}{q_1} + \frac{1}{q_2} \leq 1, q_3 \leq 2.

\[
\leq \|BHT_2(f_1 \ast_1 \tilde{\chi}_{E_{j,m}^n}, f_2 \ast_1 \tilde{\chi}_{E_{j,m}^n})\|_{L_1^{p_1} + q_3} \leq \|f_1 \ast_1 \tilde{\chi}_{E_{j,m}^n}\|_{L_2^{q_1}} \|f_2 \ast_1 \tilde{\chi}_{E_{j,m}^n}\|_{L_2^{q_3}} \leq \|f_1 \ast_1 \tilde{\chi}_{E_{j,m}^n}\|_{L_2^{q_1}} \|f_2 \ast_1 \tilde{\chi}_{E_{j,m}^n}\|_{L_2^{q_3}} \leq 2^{m(1/2 - 1/\tilde{p}_1)} \|f_1\|_{L_2^{q_1}} \|f_2\|_{L_2^{q_3}}.
\]

As \[ \frac{1}{q_1} + \frac{1}{q_2} < 1 \text{ and } p_1 > 2, \] this concludes CASE 1.

CASE 2: \[ \frac{1}{q_1} + \frac{1}{q_3} \leq 1, q_2 \geq 2. \] Then

\[
\leq \|BHT_2(f_1 \ast_1 \tilde{\chi}_{E_{j,m}^n}, f_2 \ast_1 \tilde{\chi}_{E_{j,m}^n})\|_{L_1^{p_1} + q_2} \leq \|f_1 \ast_1 \tilde{\chi}_{E_{j,m}^n}\|_{L_2^{q_1}} \|f_2 \ast_1 \tilde{\chi}_{E_{j,m}^n}\|_{L_2^{q_2}} \leq 2^{m(1/2 - 1/\tilde{p}_2)} \|f_1\|_{L_2^{q_1}} \|f_2\|_{L_2^{q_2}}.
\]

As \[ \frac{1}{q_1} < \frac{1}{q_2}, \] this concludes CASE 2.
CASE 3: $\frac{1}{q_1} + \frac{1}{q_2} > 1$. Then

$$\leq \|BHT_2(f_1 \ast_1 \tilde{X}_{E_{J,1}^m}, f_2 \ast_1 \tilde{X}_{E_{J,1}^m})\|_{L_2^{q_1+q_3}} \cdot \|f_1\|_{L_1^{p_1}} \cdot \frac{1}{q_2}$$

As the structure of the previous proof reveals, there are many possible generalizations to tensor products of more complicated operators and to an arbitrary number of tensor products of the BHT. We list one example of each and then provide proofs.

**Theorem 10.** Fix any $N \equiv 0 \mod 3$ and $\varepsilon \in \{\pm 1\}^n$. Then the operator

$$C_{N/3}^{\varepsilon} \otimes C_{N/3}^{1,\ldots,1} : W^{1}_{p_1}[L_{q_1}^1] \times L_{q_2}^2 [L_{q_3}^3] \times W^{1}_{p_2}[L_{q_4}^1] \times L_{q_5}^6 [L_{q_6}^8] \times \cdots \times L_{q_{N-1}}^{q_{N}} [L_{q_N}^{q_{N-1}}]$$

is continuous into $\prod_{i=0}^{N/3-1} L_{q_i}^1 \times \prod_{i=0}^{N/3-1} L_{q_i}^3 \mod 3 \frac{1}{q_i}$ provided the following conditions hold:

1. $C_{N/3}^{1,\ldots,1} : \otimes_{i=0}^{N/3-1} \mod 3 \frac{1}{q_i} \to L_{N/3}^{\otimes_{i=0}^{N/3-1} \mod 3 \frac{1}{q_i}}$ is bounded
2. $\max_i \{p_i\} < \min_i \{q_i\} \leq \min_i \{p_i\}$
3. $\sum_{i=0}^{N/3-1} \frac{1}{q_i} \leq 1$ for $q_{3j} \leq q_{3j-1} \ \forall \ j$
4. $\frac{1}{2} \left| \frac{1}{n} \right| - \frac{1}{2} \left( \frac{1}{2} - \frac{1}{q_{3j-3}} \right) < 1/2$
5. $\max \left\{ \frac{1}{2} - \frac{1}{q_{3j}} \left( 1 \right) \right\} + \frac{1}{2} \left( \frac{1}{2} - \frac{1}{q_{3j-3}} \right) < 1/2$

**Proof.** Set $Q_1 = \sum_{i=0}^{N/3-1} \frac{1}{q_i} \mod 3 \frac{1}{q_i}$ and $Q_2 = \sum_{i=0}^{N/3-1} \frac{1}{q_i} \mod 3 \frac{1}{q_i}$. Introduce a martingale structure for every connection in the first variable between a Wiener function and a Lebesgue function. First assume $\varepsilon = 1$. Then pass the sums...
over scales through the integration and compute with the $m_1$ scale (ignoring all other $m_i$ values) as follows:

\[
\|C_{N/3}^{1, \ldots, 1}(f_1, \ldots, f_{N/3})\|_{L^q_{1,1}} \leq \prod \|C_{N/3}^{1, \ldots, 1}(f_1*1 \tilde{X}_{E_{j_1}^{m_1}, q_1^*1*1 \tilde{X}_{E_{j_2}^{m_2}, q_2}^*1*1 \tilde{X}_{E_{j_{N-1}^{m_{N-1}}, q_{N-1}}})\|_{L^q_{1,1}}
\]

The reader will no doubt note that in an abuse of notation, $E_{j_1}^{m_1}$ and $E_{j_{N-1}^{m_{N-1}}}$ are martingale structures adapted to different functions.

CASE 1: $q_3, q_2 \leq 2$. Then we have

\[
A \lesssim \prod \|f_1\|_{L^{q_1}_{1,1}}^2 2^{-m_1/p_1^*} 2^{m_1/2} \|f_1\|_{L^{q_1}_{1,1}}^2 2^{-m_1/p_1^*} 2^{m_1/2} \|f_1\|_{L^{q_1}_{1,1}}^2 2^{-m_1/p_1^*} 2^{m_1/2} \|f_1\|_{L^{q_1}_{1,1}}^2 2^{m_1(1/2 - 1/p_1^*)}
\]

and since $\frac{1}{p_1^*} + \frac{1}{q_3} < 1$, the geometric sum over the $m_1$ scale converges.

CASE 2: $q_3 \leq 2, q_2 > 2$

\[
A \lesssim \prod \|f_1\|_{L^{q_1}_{1,1}}^2 2^{-m_1/p_1^*} 2^{m_1/2} \|f_1\|_{L^{q_1}_{1,1}}^2 2^{-m_1/p_1^*} 2^{m_1/2} \|f_1\|_{L^{q_1}_{1,1}}^2 2^{-m_1/p_1^*} 2^{m_1/2} \|f_1\|_{L^{q_1}_{1,1}}^2 2^{-m_1/p_1^*} 2^{m_1/2} \|f_1\|_{L^{q_1}_{1,1}}^2 2^{m_1(1/2 - 1/p_1^*)}
\]

and since $\frac{1}{p_1^*} + \frac{1}{q_3} < 1$, the geometric sum over the $m_1$ scale converges.

CASE 3: $q_3 > 2$. Then we have

\[
A \leq \prod \|f_1\|_{L^{q_1}_{1,1}}^2 2^{-m_1/p_1^*} 2^{m_1/2} \|f_1\|_{L^{q_1}_{1,1}}^2 2^{-m_1/p_1^*} 2^{m_1/2} \|f_1\|_{L^{q_1}_{1,1}}^2 2^{-m_1/p_1^*} 2^{m_1/2} \|f_1\|_{L^{q_1}_{1,1}}^2 2^{m_1(1/2 - 1/p_1^*)}
\]

As $q_3 < p_1$, the geometric sum over the $m_1$ scale again converges. For the $m_2$ and $m_3$ scales, the same decompositions leaves us with

\[
\prod \|f_2*1 \tilde{X}_{E_{j_1}^{m_1}, q_1}^*1*1 \tilde{X}_{E_{j_2}^{m_2}, q_2}^*1*1 \tilde{X}_{E_{j_{N-1}^{m_{N-1}}, q_{N-1}}})\|_{L^q_{2,2}}^2 \prod \|f_3*1 \tilde{X}_{E_{j_2}^{m_2}, q_2}^*1*1 \tilde{X}_{E_{j_{N-1}^{m_{N-1}}, q_{N-1}}})\|_{L^q_{2,2}}^2 \prod \|f_4*1 \tilde{X}_{E_{j_3}^{m_3}, q_3}^*1*1 \tilde{X}_{E_{j_{N-1}^{m_{N-1}}, q_{N-1}}})\|_{L^q_{2,2}}^2 \prod \|f_5*1 \tilde{X}_{E_{j_{N-1}^{m_{N-1}}, q_{N-1}}})\|_{L^q_{2,2}}^2
\]

The first factor contributes at most $2^{m_2}(\max\{\frac{1}{p_2}, \frac{1}{q_2} - \frac{1}{q_3}, 0\})$, the second factor contributes $2^{m_3}(\max\{m_2, m_3\}(1/2 - 1/p_2))$ and the third factor contributes at most
We introduce a martingale structure for every variable from $2m_3 \left( \max \left\{ \frac{1}{m_2}, \frac{1}{m_3}, 0 \right\} \right)$. For the sum over the $m_2$ and $m_3$ scales to be a convergent geometric series, we then require condition 5) in the statement of the theorem. The remaining scales behave similarly, so we omit the details. The above argument generalizes to arbitrary $\epsilon \in \{\pm 1\}^n$, and the theorem follows.

**Theorem 11.** Let $N \in \mathbb{Z}_{\text{odd}}^+$. Then, the operator

$$
\bigotimes_{N+1}^{\text{BHT}} : W_{P_1}^1 \left[ W_{P_2}^2 \left[ \cdots \left[ W_{P_N}^N \left[ L_{N/2+1}^{q_{N/2}} \left[ L_{N/2+2}^{q_{N/2+2}} \left[ \cdots \left[ L_{N+1}^{q_{N+1}} \right] \right] \right] \right] \right] \right] \right] \\
\times W_{P_2}^{N/2+1} \left[ W_{P_2}^{N/2+2} \left[ \cdots \left[ W_{P_2}^{N} \left[ L_{2}^{q_{2}} \left[ \cdots \left[ L_{N/2}^{q_{N/2}} \left[ L_{N+1}^{q_{N+1}} \right] \right] \right] \right] \right] \right] \right] \\
\rightarrow L_{1}^{\frac{1}{P_1} + \frac{1}{q_1}} \left[ L_{2}^{\frac{1}{P_2} + \frac{1}{q_2}} \left[ L_{3}^{\frac{1}{P_3} + \frac{1}{q_3}} \left[ \cdots \left[ L_{N+1}^{\frac{1}{q_{N+1}} + \frac{1}{q_{2N+2}}} \right] \right] \right] \right] 
$$

is bounded provided the following conditions hold:

1) $q_{2N+2} \leq q_{2N} \leq q_{2N-2} \leq \cdots \leq q_2 \leq P_2$

2) $q_{2N+1} \leq q_{2N-1} \leq q_{2N-3} \leq \cdots \leq q_1 \leq P_1$

3) $\frac{1}{q_{2N+1}} + \frac{1}{q_{2N+2}} \leq 1$

4) $\max \left\{ \frac{1}{q_{2N+2}} - \frac{1}{P_2'}, \frac{1}{2}, \frac{1}{P_2} \right\} + \max \left\{ \frac{1}{2} - \frac{1}{q_1}, \frac{1}{q_{2N+1}}, \frac{1}{q_1}, \frac{1}{q_{2N+1}} - \frac{1}{q_{2N+2}'} \right\} < 0$

5) $\max \left\{ \frac{1}{q_{2N+1}} - \frac{1}{P_2'}, \frac{1}{2}, \frac{1}{P_2} \right\} + \max \left\{ \frac{1}{2} - \frac{1}{q_2}, \frac{1}{q_{2N+2}}, \frac{1}{q_2}, \frac{1}{q_{2N+2}} - \frac{1}{q_{2N+2}'} \right\} < 0$.

**Proof.** We introduce a martingale structure for every variable from $x_1$ through $x_N$ and then pass the behavior in the $N+1$ variable to the story of the BHT.

For convenience, we set

$$
X_0 = W_{P_1}^1 \left[ W_{P_2}^2 \left[ \cdots \left[ W_{P_N}^N \left[ L_{N/2+1}^{q_{N/2}} \left[ L_{N/2+2}^{q_{N/2+2}} \left[ \cdots \left[ L_{N+1}^{q_{N+1}} \right] \right] \right] \right] \right] \right] \right] \\
X_1 = W_{P_2}^{N/2+1} \left[ W_{P_2}^{N/2+2} \left[ \cdots \left[ W_{P_2}^{N} \left[ L_{2}^{q_{2}} \left[ \cdots \left[ L_{N/2}^{q_{N/2}} \left[ L_{N+1}^{q_{N+1}} \right] \right] \right] \right] \right] \right] \right] 
$$

and inductively construct $N/2$ martingale structures based on $f_1$ and $N/2$ martingale structures based on $f_2$ as follows:
\[
\gamma_{f_1}(\bar{x}) = \frac{\|\chi(x_1)(-\infty,\bar{x})f_1\|_{X_0}}{\|f_1\|_{X_0}}
\]

\[
0E_{j_1}^{m_1} = \gamma_{f_1}^{-1}(j_12^{-m_1}, (j_1 + 1)2^{-m_1}))
\]

\[
\gamma_{m_1,j_1,f_1}(\bar{x}) = \frac{\|\chi_{E_{j_1}^{m_1}}(x_1)\chi(x_2)(-\infty,\bar{x})f_1\|_{X_0}}{\|\chi_{E_{j_1}^{m_1}}f_1\|_{X_0}}
\]

\[
0E_{j_1,j_2}^{m_1,m_2} = \gamma_{m_1,j_1,f_1}^{-1}(j_22^{-m_2}(j_2 + 1)2^{-m_2}))
\]

\[
\gamma_{m_1,j_1,\ldots,m_{k-1},j_{k-1},f_1}(\bar{x}) = \frac{\|\chi_{E_{j_1}^{m_1}}(x_1)\ldots\chi_{E_{j_{k-1}}^{m_{k-1}}}(x_{k-1})\chi(x_k)(-\infty,\bar{x})f_1\|_{X_0}}{\|\chi_{E_{j_1}^{m_1}}(x_1)\ldots\chi_{E_{j_{k-1}}^{m_{k-1}}}(x_{k-1})f_1\|_{X_0}}
\]

\[
0E_{j_1,\ldots,j_k}^{m_1,\ldots,m_k} = \gamma_{m_1,j_1,\ldots,m_{k-1},j_{k-1},f_1}^{-1}(j_k2^{-m_k}, (j_k + 1)2^{-m_k}))
\]

up to \(k = N/2\) along with

\[
\gamma_{f_2}(\bar{x}) = \frac{\|\chi(x_1)(-\infty,\bar{x})f_2\|_{X_1}}{\|f_2\|_{X_1}}
\]

\[
1E_{j_1}^{m_1} = \gamma_{f_2}^{-1}(j_12^{-m_1}, (j_1 + 1)2^{-m_1}))
\]

\[
\gamma_{m_1,j_1,f_2}(\bar{x}) = \frac{\|\chi_{E_{j_1}^{m_1}}(x_1)\chi(x_2)(-\infty,\bar{x})f_2\|_{X_1}}{\|\chi_{E_{j_1}^{m_1}}f_2\|_{X_1}}
\]

\[
1E_{j_1,j_2}^{m_1,m_2} = \gamma_{m_1,j_1,f_2}^{-1}(j_22^{-m_2}(j_2 + 1)2^{-m_2}))
\]

\[
\gamma_{m_1,j_1,\ldots,j_{k-1},f_2}(\bar{x}) = \frac{\|\chi_{E_{j_1}^{m_1}}(x_1)\ldots\chi_{E_{j_{k-1}}^{m_{k-1}}}(x_{k-1})\chi(x_k)(-\infty,\bar{x})f_2\|_{X_1}}{\|\chi_{E_{j_1}^{m_1}}(x_1)\ldots\chi_{E_{j_{k-1}}^{m_{k-1}}}(x_{k-1})f_1\|_{X_0}}
\]

\[
1E_{j_1,\ldots,j_k}^{m_1,\ldots,m_k} = \gamma_{m_1,j_1,\ldots,m_{k-1},j_{k-1},f_1}^{-1}(j_k2^{-m_k}, (j_k + 1)2^{-m_k}))
\]

up to \(k = N\).

For \(1 \leq j \leq N/2\), For convenience, define for \(j \leq N/2\), \(Q_j = \frac{1}{q^{2j}} + \frac{1}{q^{2(j-N/2)+1}}\), and for \(j = N + 1\), \(Q_j = \frac{1}{q^{2N+1}} + \frac{1}{q^{2N+2}}\).

CASE 1: \(q_{2N+1}, q_{2N+2} \geq 2\). Performing the standard decomposition and moving the sum over scales outside norms gives
where of the form and 5 in the statement of the theorem. Note that the resulting sum over each scale converges because of assumptions 4 and 5. This is again acceptable by assumptions 4 and 5.

$rally as systems of differential equations whose solutions can be expressed as \[ f_1 \] and \[ f_2 \] to be 1 gives us the upper bound

\[ q^{(m_{N+2}+\ldots+m_N)(1/2-1/q_{2N+2})} \] \[ q^{(m_1+\ldots+m_{N/2})(1/2-1/q_{2N+2})} \] \[ q^{(m_{N/2}+\ldots+m_N)(1/2-1/q_{2N+2})} q^{(m_1+\ldots+m_{N/2})(1/2-1/q_{2N+2})} \]

Note that the resulting sum over each scale converges because of assumptions 4 and 5 in the statement of the theorem.

CASE 2: \( q_{2N+1} < 2 \) and \( q_{2N+2} \geq 2 \). One eventually faces an upper bound of the form

\[ q^{(m_1+\ldots+m_{N/2})(1/2-1/q_{2N+2})} \] \[ q^{(m_{N/2}+\ldots+m_N)(1/2-1/q_{2N+2})} \] \[ q^{(m_1+\ldots+m_{N/2})(1/2-1/q_{2N+2})} \] \[ q^{(m_{N/2}+\ldots+m_N)(1/2-1/q_{2N+2})} \]

This is again acceptable by assumptions 4 and 5.

CASE 3: \( q_{2N+1} \geq 2 \) and \( q_{2N+2} < 2 \). This situation is similar to CASE 2 and is omitted.

CASE 4: \( q_{2N+1}, q_{2N+2} < 2 \). This cannot occur because of assumption 3.

5. An Application to Bi-Parameter AKNS Systems

AKNS systems play an important role in nuclear physics and arise naturally as systems of differential equations whose solutions can be expressed as multilinear oscillatory integrals, see [1, 3]. In this section, we will construct a biparameter generalization of the generic AKNS system and write down solutions as tensor products of singular integral operators. That a large subset of these multparameter solutions remain bounded for almost every parameter value will follow immediately from the continuity of a maximal variant of the tensor product of \( C_N \) with itself.
5.1. Setup for Bi-Parameter AKNS Systems

The 1-parameter generic upper triangular AKNS system is given in matrix form by

\[
\begin{pmatrix}
\frac{d}{dx}u_1 \\
\frac{d}{dx}u_2 \\
\vdots \\
\frac{d}{dx}u_N
\end{pmatrix} = \begin{pmatrix}
i\lambda & V_{12} & \cdots & V_{1n} \\
0 & i\lambda & \cdots & V_{2n} \\
0 & 0 & \ddots & \vdots \\
0 & 0 & \cdots & i\lambda
\end{pmatrix} \begin{pmatrix}
u_1 \\
u_2 \\
\vdots \\
u_N
\end{pmatrix}.
\]

We now want to extend this system to the biparameter setting. As initial data, suppose we are given \(N-1\) functions \((V_1, \ldots, V_{N-1})\) such that \(V_j \in W_p^p(W^p_p)\) for all odd \(j\) for some \(p > 2\), and \(V_j \in L^2(L^2)\) for all even \(j\), and in addition, suppose \(\{c_{j,1}, c_{j,2}\}_{j=1}^N\) is such that \(c_{j,1} \neq c_{j+1,1}\) along with \(c_{j,2} \neq c_{j+1,2}\). For each \(j\) and real parameters \((\lambda_1, \lambda_2)\), construct the differential operator

\[
D_j = D_j^{(\lambda_1, \lambda_2)} = \frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2} - ic_{j,1}\lambda_1 \frac{\partial}{\partial x_2} - ic_{j,2}\lambda_2 \frac{\partial}{\partial x_1}.
\]

Our goal is now to study the boundedness properties for solutions to the following biparameter AKNS system:

\[
\begin{pmatrix}
D_1 u_1 \\
D_2 u_2 \\
\vdots \\
D_N u_N
\end{pmatrix} = \begin{pmatrix}
c_{1,1}c_{1,2}\lambda_1\lambda_2 & V_1 & 0 & 0 & \cdots & 0 \\
0 & c_{2,1}c_{2,2}\lambda_1\lambda_2 & V_2 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & c_{N,1}c_{N,2}\lambda_1\lambda_2
\end{pmatrix} \begin{pmatrix}
u_1 \\
u_2 \\
\vdots \\
u_N
\end{pmatrix},
\]

that is \(D\tilde{u} = M\tilde{u}\) where \(M_{jj} = c_{j,1}c_{j,2}\lambda_1\lambda_2\) for \(1 \leq j \leq N\), \(M_{j-1,j} = V_{j-1}\) for \(2 \leq j \leq N\) and \(M_{ij} = 0\) otherwise. It is helpful to assign \(\tilde{u}_j(x_1, x_2) = \int (\lambda_1\chi_1 + \lambda_2\chi_2)u_j(x_1, x_2)\) along with \(\alpha_{j,1} = c_{j,1} - c_{j+1,1}\) and \(\alpha_{j,2} = c_{j,2} - c_{j+1,2}\), which yields

\[
\frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2} \tilde{u}_j(x_1, x_2) = V_j(x_1, x_2)\tilde{u}_{j+1}(x_1, x_2)e^{-i(\alpha_{j,1}\chi_1 + \alpha_{j,2}\chi_2)}.
\]

Therefore, a general solution of the above system will be given for each \(j\) by

\[
\tilde{u}_j(x_1, x_2) = \int_0^{\chi_2} C_j(\tilde{x}_2) d\tilde{x}_2 - D_j(x_1)
\]

in the quadrant \(S := \{(x_1, x_2) : x_1, x_2 \geq 0\}\). For simplicity, we choose the family \(C_j(x)\) satisfying \(C_j(x)D_j(x) = 0\) for \(j \neq N\). For \(j = N\), we set \(\tilde{u}_N(x_1, x_2) = \tilde{u}_N(0, 0) = 1\). By induction, \(\tilde{u}_j(x_1, x_2)\) is then

\[
C_N^{(\lambda_1, \lambda_2)} \otimes C_N^{(\lambda_1, \lambda_2)}(V_{N-1} \ast_1 \tilde{\chi}(0, x_1) \ast_2 \tilde{\chi}(0, x_2), \ldots, V_j \ast_1 \tilde{\chi}(0, x_1) \ast_2 \tilde{\chi}(0, x_2)).
\]
5.2. Boundedness of \( u_j \) for odd \( j \)

We now claim that whenever \( j \) is odd, the quantity \( \sup_{(x_1, x_2) \in S} |u_j(x_1, x_2)| \) is finite for almost every pair of parameter values \( (\lambda_1, \lambda_2) \). Since \( \sup_{(x_1, x_2) \in S} |u_j| \) is finite if and only if \( \sup_{(x_1, x_2) \in S} |\tilde{u}_j| \) is finite, it suffices to show that the \( L^q \) norm of \( \sup C_{N-1}^\varepsilon \otimes \sup C_{N-1}^\varepsilon \) is finite for some \( q > 0 \). Note that if we did not have to face the supremum, finiteness would follow directly from theorem 7.

However, we can adapt an argument from [8] to deal with this obstacle, which we do first for the case where one faces only a single function in \( W^p \[ W^p \] \).

**Proposition 6.** The operator \( \sup C_1^\varepsilon \otimes \sup C_1^\varepsilon : W^p [W^p] \rightarrow L^p [L^p] \) is bounded.

*Remark:* This proposition is interesting even by itself since the corresponding generalization of the Carleson-Hunt theorem is false, which is to say the map \( f \mapsto \sup I \times J \subset \mathbb{R}^2 |f \ast \hat{\chi}_{I \times J}| \) is not a bounded operator on any \( L^p \) space.

**Proof.** By definition, \( \sup C_1^\varepsilon \otimes \sup C_1^\varepsilon (f)(\xi_1, \xi_2) \) is given by

\[
\sup_{(x_1, x_2) \in \mathbb{R}^2} \int_{\bar{x}_1 < x_1} \int_{\bar{x}_2 < x_2} \hat{f}(\bar{x}_1, \bar{x}_2)e^{2\pi i (\xi_1 \bar{x}_1 + \xi_2 \bar{x}_2)} d\bar{x}_1 d\bar{x}_2.
\]

Let \((N_1(\xi_1, \xi_1), N_2(\xi_1, \xi_2))\) be the point where the supremum is attained. Construct

\[
\gamma_f(x) = \frac{\int_{-\infty}^{x} \int_{\mathbb{R}} |\hat{f}|(\bar{x}_1, \bar{x}_2) d\bar{x}_1 d\bar{x}_2}{||\hat{f}||_{W^p [W^p]}}
\]

\[
E_{j_1}^{m_1} = \gamma_f^{-1}(j_1 2^{-m_1}, (j_1 + 1) 2^{-m_1})
\]

\[
\gamma_{m_1, j_1, f}(y) = \frac{\int_{E_{j_1}^{m_1}} \int_{y}^{\infty} |\hat{f}|(\bar{x}_1, \bar{x}_2) d\bar{x}_1 d\bar{x}_2}{||f \ast \hat{\chi}_{E_{j_1}^{m_1}}||_{W^p [W^p]}}
\]

\[
E_{j_1, j_2}^{m_1, m_2} = \gamma_{m_1, j_1, f}^{-1}(j_2 2^{-m_2}, (j_2 + 1) 2^{-m_2})
\]
Then we decompose:

\[
\| \sup C^1_1 \otimes C^1_1(f) \|_{L^p[L^p]} \\
= \left\| \sum_{m_1, m_2 \geq 0} \sum_{j_1=0}^{2^{m_1-1} - 1} \sum_{j_2=0}^{2^{m_2-1} - 1} f *_1 \tilde{X}_{j_1, i}^{1, m_1} \ast_2 \tilde{X}_{j_2, i}^{1, m_2} \times \\
\chi(\xi_1, \xi_2) \{ N_1(\xi_1, \xi_2) \in E_{j_1, i}^{m_1}, N_2(\xi_1, \xi_2) \in E_{j_2, i}^{m_2} \} \right\|_{L^p[L^p]} \\
\leq \sum_{m_1, m_2 \geq 0} \left( \sum_{j_1=0}^{2^{m_1-1} - 1} \sum_{j_2=0}^{2^{m_2-1} - 1} \| f *_1 \tilde{X}_{j_1, i}^{1, m_1} \ast_2 \tilde{X}_{j_2, i}^{1, m_2} \|_{L^p[L^p]}^{p'} \right)^{1/p} \\
\leq \sum_{m_1, m_2 \geq 0} \left( 2^{m_1} 2^{m_2} 2^{-m_1 p/p'} 2^{-m_2 p/p'} \right)^{1/p} \\
= \sum_{m_1, m_2 \geq 0} \gamma(m_1 + m_2)(1/p - 1/p').
\]

As \( p > 2 \), the double geometric series is convergent. \( \Box \)

Since we may always assume each potential \( V_1, ..., V_{N-1} \) is supported on \( S \) along with \( \varepsilon = 1 \), it suffices to prove the following maximal result:

**Theorem 12.** The operator \( \sup C^1_{N-1} \otimes C^1_{N-1} : W_p[W_p] \times L^2[L^2] \times ... \times W_p[W_p] \rightarrow L^{\frac{N-1}{p-1} + \frac{N}{p-1} \frac{1}{L^2[L^2]}} \left[ L^\frac{N-1}{p-1} + \frac{N}{p-1} \frac{1}{L^2[L^2]} \right] := L^{\Omega_1[L^{\Omega_1}]} \) is bounded.

**Proof.** The proof uses our previous result without the supremum, i.e. theorem 7 along with four carvings and lemma 5 in the appendix. We introduce two carvings between the rightmost \( W_p[W_p] \) factor and the \( L^2[L^2] \) to its immediate left and two carvings between the rightmost \( W_p[W_p] \) and the supremum to its right. Passing sums over scales to the outside as usual, we need only estimate ...
Appendix A. Two Lemmas using Khintchine’s Inequality

The following lemma plays a crucial role in the above arguments and is more or less known, see e.g. [3]. We provide a proof here for completeness.

\[ \left\| \sum_{j_1,j_2,j_3} C_{N-2}^{j_1,j_2,j_3} \otimes C_{N-2}^{j_1,j_2,j_3} \left( f_N * \tilde{X}_{E_m^{j_1},r} * \tilde{X}_{E_m^{j_2},r} * \tilde{X}_{E_m^{j_3},r} \right) \right\|_{L^p[L^p]} \]

\[ \chi(\xi_1,\xi_2) \left( N_1(\xi_1,\xi_2) \in E_{m_3,r}, N_2(\xi_1,\xi_2) \in E_{m_4,r} \right) \]

\[ \left\| \sum_{j_1,j_2} \left( f_N * \tilde{X}_{E_m^{j_1},r} * \tilde{X}_{E_m^{j_2},r} * \tilde{X}_{E_m^{j_3},r} \right) \right\|_{L^p[L^p]} \]

CASE 1: \( m_1 + m_2 \geq m_3 + m_4 \). The relevant sum is

\[ \sum_{m_1 + m_2 \geq m_3 + m_4} 2^{(m_1+m_2)(1/2-1/p')} \leq \sum_{m_1,m_2,m_3,m_4} 2^{(m_1+m_2+m_3+m_4)(1/4-1/2p')} \]

CASE 2: \( m_3 + m_4 > m_1 + m_2 \). We have an upper bound of the form

\[ \sum_{m_3 + m_4 > m_1 + m_2} 2^{(m_3+m_4)(1/p-1/p')} 2^{m_1+m_2} \leq \sum_{m_3 + m_4 > m_1 + m_2} 2^{(m_3+m_4)(1/2-1/p')} \leq \sum_{m_1,m_2,m_3,m_4} 2^{(m_1+m_2+m_3+m_4)(1/4-1/2p')} \]

As the geometric sum over scales is convergent, the theorem follows.

Appendix A. Two Lemmas using Khintchine’s Inequality

The following lemma plays a crucial role in the above arguments and is more or less known, see e.g. [3]. We provide a proof here for completeness.

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Lemma 4. Fix $\sigma$-finite measure spaces $X$ and $Y$. Let $0 < q \leq p < \infty$ and fix $T : L^p(X) \to L^q(Y)$, a continuous linear operator. Then for any sequence $\{f_j\}$ of functions in $L^p(X)$,

$$\left\| \left( \sum_{j \in \mathbb{N}} |T(f_j)|^2 \right)^{1/2} \right\|_{L^q(Y)} \lesssim_{p,q} \|T\|_{p \to q} \left\| \left( \sum_{j \in \mathbb{N}} |f_j|^2 \right)^{1/2} \right\|_{L^p(X)}.$$

Proof. The proof linearizes using Khintchine as follows:

$$\left\| \left( \sum_{j \in \mathbb{N}} |T(f_j)|^2 \right)^{1/2} \right\|_{L^q(Y)} \lesssim \left\| \left( \mathbb{E} \left| \sum_{j \in \mathbb{N}} r_j(t)T(f_j) \right|^q \right)^{1/q} \right\|_{p/q}
= \left( \mathbb{E} \int_Y \left| T(\sum_{j \in \mathbb{N}} r_j(t)f_j) \right|^q \, dx \right)^{1/q}
\leq \|T\|_{p \to q} \left( \mathbb{E} \left( \int_X \left| \sum_{j \in \mathbb{N}} r_j(t)f_j \right|^p \, dx \right)^{q/p} \right)^{1/q}
= \|T\|_{p \to q} \left( \int_X \mathbb{E} \left( \sum_{j \in \mathbb{N}} |r_j(t)f_j|^p \right) \, dx \right)^{1/p}
\lesssim \|T\|_{p \to q} \left( \int_X \left( \sum_{j \in \mathbb{N}} |f_j|^2 \right)^{p/2} \, dx \right)^{1/p}.$$

\[ \square \]

Lemma 5. Let $\{f^1_j\}$ and $\{f^n_j\}$ be any two sequences of functions in $L^{p_1}(\mathbb{R})$ and $L^{p_2}(\mathbb{R})$ respectively. Moreover, let $\sup C_n^{1,...,1} : \prod_{i=1}^{n} L^{p_i}(\mathbb{R}) \to L^{1/p_1,...,1/p_n}$ be bounded. Then

$$\left\| \left( \sup_{j \in \mathbb{N}} \left( \sum_{j_1,j_2} C_n^{1,...,1}(f^1_{j_1} \ast f^1_{j_2} \ast f^2_{j_3} \ast f^n_{j_n}) \right)^2 \right)^{1/2} \right\|_q
\lesssim_{p,q} \left\| \sup C_n^{1,...,1} \right\|_{p \to q} \left\| \left( \sum_{j_1} |f^1_{j_1}|^2 \right)^{1/2} \right\|_{p_1} \ldots \left\| \left( \sum_{j_n} |f^n_{j_n}|^2 \right)^{1/2} \right\|_{p_n}.$$
Proof. For notational convenience, set $q = \frac{1}{\sum_{i=1}^{n} \frac{1}{p_i}}$. Now write down

$$\left\| \sup_{I \subset \mathbb{R}} \left( \sum_{j_1, j_n} C_{n}^{1, \ldots, 1} \left( f_{j_1} \ast \check{\chi}_I, f_{j_2}, \ldots, f_{j_n} \right) \right) \right\|_{q}^{1/2}$$

$$\leq \left( \int_{I} \sup_{E} \left( C_{n}^{1, \ldots, 1} \left( \sum_{j_1} r_{j_1} (t) f_{j_1} \ast \check{\chi}_I, f_{j_2}, \ldots, \sum_{j_n} f_{j_n} r_{j_n} (t) \right) \right) dx \right)^{1/q}$$

$$\leq \left( \mathbb{E} \int_{I} \sup_{E} \left( \sum_{j_1} r_{j_1} (t) f_{j_1} \ast \check{\chi}_I, f_{j_2}, \ldots, \sum_{j_n} f_{j_n} r_{j_n} (t) \right) dx \right)^{1/q}$$

$$\leq \left\| \sup_{I \subset \mathbb{R}} \mathbb{E} \left\| \sum_{j_1} r_{j_1} (t) f_{j_1} \right\|_{p_1} \sum_{j_n} r_{j_n} (t) f_{j_n} \right\|_{p_n}^{1/q}$$

$$\leq \left\| \sup_{I \subset \mathbb{R}} \mathbb{E} \left\| \sum_{j_1} r_{j_1} (t) f_{j_1} \right\|_{p_1} \sum_{j_n} r_{j_n} (t) f_{j_n} \right\|_{p_n}^{1/q}$$

Using Khintchine’s inequality once more, we arrive at the upper bound

$$|| \sup_{I \subset \mathbb{R}} \left( \sum_{j_1} | f_{j_1} |^2 \right) ||_{p_1} \ldots || \sum_{j_n} | f_{j_n} |^2 ||_{p_n}^{1/2}$$

\[ \square \]

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