Signal-Theoretic Characterization of Waveguide Mesh Geometries for Models of Two-Dimensional Wave Propagation in Elastic Media

Federico Fontana and Davide Rocchesso

Abstract: Waveguide mesh geometries are a brief and versatile models of wave propagation along a multidimensional ideal medium. The choice of the mesh geometry affects both the computational cost and the accuracy of simulations. In this paper, we focus on 2D geometries and use multidimensional sampling theory to compare the square, triangular, and hexagonal mesh geometries in terms of sampling efficiency and dispersion error under conditions of critical sampling. The analysis shows that the square mesh geometry exhibits the most desirable tradeoff between accuracy and computational cost.

Keywords: Waveguide mesh geometries, finite difference methods, wave propagation, dispersion error, multidimensional sampling.

I. Introduction

Among the techniques for modeling wave propagation in multidimensional media, the Digital Waveguide Meshes have been recently established [1], [2], [3], [4], [5], [6], [7] as intuitive and efficient formulations of multidimensional waveguide models.

A Waveguide Mesh (W M) is a discrete-time computational structure that is constructed by tiling a multidimensional medium into regular elements, each with a local description of wave propagation phenomena. This local description is lumped into a waveguide junction [3], [5], which is lossless by construction. Therefore, waveguide meshes are free of numerical losses, even though lumped passive elements can be explicitly inserted to simulate physical losses. However, wavefronts propagating along a multidimensional WM M are affected by dispersion error, i.e., different frequencies experience different propagation velocities. Numerical dispersion cannot be completely eliminated, but it can be arbitrarily reduced increasing the density of the elements, and the impact of choosing the least dispersive geometry [6]. Moreover, interpolation schemes [13] or offline warping techniques [14] can be applied to attenuate the effects of numerical dispersion.

In this paper, we investigate how the density of waveguide junctions and the sampling frequency affect the signal coming out from the model. As a result, the analysis depends on the geometry of the WM, and we focus on 2D media, noting properties for the square, triangular, and hexagonal WM (named respectively SW M, TW M, and HW M in the following). Such properties allow to calculate the bandwidth of a signal produced by a WM working at a given sampling frequency, once its geometry and the density of its junctions have been determined.

The paper is structured as follows. Section II provides a background material on waveguide meshes and their interpretation as multidimensional sampling (and their sampling properties for square, triangular, and hexagonal WMs) and illustrates the spatial sampling efficiency of the three waveguide mesh geometries for signals having circular spatial band shape. In Section III, we explain how the critical spatial sampling affects the choice of the temporal sampling frequency in non-aliasing conditions. In Section IV, we compare the computational performance of the three geometries under critical sampling conditions.

II. Background

A. Digital Waveguide Models and Waveguide Meshes

An ideal one-dimensional physical waveguide can be modeled in discrete time by means of a couple of parallel delay lines where two wave signals, s1 and s2, travel in opposite directions. Such a structure, based on spatial sampling (with interval D) and time sampling (with interval T), is called a digital waveguide [12], [13]. If wave propagation in the physical medium is lossless and nondispersive with speed c = D/T, no error is introduced by the discrete-time simulation as long as s1 and s2 are band limited to a band B = (2T)c. In this case, the signal s(x,t) along the physical waveguide can be reconstructed with no aliasing error from samples of the wave signals:

\[ s(mD; nT) = s_1(mD; nT) + s_2(mD; nT) \]  

N digital waveguide termination conditions can be connected by means of a lossless scattering junction [7], [8]. Preservation of the total energy in the form of a Kirchhoff’s node equations leads to the scattering equation

\[ s_i = \sum_{k=1}^{N} \frac{N}{\delta} s_{k+1} \]  

which allows for calculating the outgoing wave signal to the i-th waveguide branch from the N incoming wave signals \( s_1; s_2; \ldots; s_N \), under the assumption of equal wave impedances at the junction for all the waveguides.
A WM, as proposed by Van Duyne and Smith in 1993 [1], [2], is obtained by connecting unit-length digital waveguide branches by means of lossless scattering junctions. For the simulation of uniform and isotropic multidimensional media, a few kinds of geometries, all corresponding to tiling the multidimensional space into regular elements, have been proposed: square [4], triangular [5], [6], hexagonal [7] for 2D media such as membranes; rectilinear [3] and tetrahedral [4], [5] for 3D media. The WMs which are considered in this paper (SWM, TWM and HWM) are depicted in figure 1. Among the proposed geometries, the HWM is peculiar because its 3-port lossless scattering junctions exhibit two different orientations, and it can be interpreted as two interleaved TWMs (see the junctions marked with in figure 1). WMs introduce a basic relation, between signal $s(x_j; nT)$ taken from a junction located at position $x_j$ in space, and signals $s(x_j + D_k; nT)$, $k = 1, \ldots, N$, taken from the $N$ adjacent junctions connected to $x_j$, $D_k$ meters far from position $x_j$:

$$s(x_j; nT + T) + s(x_j; nT - T) = \frac{2}{N} \sum_{k=1}^{N} s(x_j + D_k; nT) \quad (3)$$

which is obtained from (2).

Equation (3) indicates that each WM behaves like a nine di erence scheme [9], the di erence being that the former has a state lim ped in the digital waveguide, and the latter has a state lim ped in the junctions. For the purpose of the analysis that follows, when the lossless scattering junctions have more than one orientation, as in the HWM, a di erence equation such as (3) should be written as many times as there are orientations, each one using a proper set of vectors $\mathbf{D}_1, \ldots, \mathbf{D}_N$ [7].

Following the lines of von Neumann stability analysis [6], equation (3) can be Fourier transformed with spatial variables $x$ and $y$, resulting in

$$S(x; y; nT + gT) + S(x; y; nT - gT) = b_g S(x; y; nT) \quad (4)$$

where $x$ and $y$ are spatial frequencies, $g$ takes the value 2 for the HWM and 1 for the other meshes, and $b_g$ is a geometric factor equal to

$$b_{g_1} = \cos(2D_x) + \cos(2D_y) \frac{P_1}{2} \# !$$
$$b_{g_2} = \frac{2}{3} \cos(2D_x) + \frac{2}{3} \cos(2D_y) \frac{P_2}{2} \frac{3}{2} \# !$$
$$+ \frac{2}{3} \cos(2D_y) \frac{P_3}{2} \frac{3}{2} \# !$$
$$+ \frac{2}{3} \cos(2D_x) \frac{P_4}{2} \frac{3}{2} \# !$$
$$b_{g_1} = \frac{8}{9} \cos(2D_x) \frac{P_3}{2} \frac{3}{2} \# !$$
$$+ \frac{8}{9} \cos(2D_y) \frac{P_3}{2} \frac{3}{2} \# !$$
$$+ \frac{8}{9} \cos(2D_x) \frac{P_3}{2} \frac{3}{2} \frac{2}{3} \quad (5)$$

for the SWM, TWM and HWM, respectively.

Solving equation (4) as a nine di erence equation in the discrete time variable, the spatial phase shift a ecting a traveling signal in one time sample is found to be

$$' = \frac{1}{g} \arctan \left( \frac{2}{x} + \frac{2}{y} \right)$$

We can compare the propagation speed of a signal traveling along a WM, versus the propagation speed of a signal traveling along an ideal membrane. If we consider membranes where relation $D = cT$ holds, meaning that signals, during a time period $T$, travel for a distance equal to the digital waveguide length, we can calculate the spatial phase shift of these signals, occurring during a time period:

$$' = \frac{2}{x}$$

where $c = \frac{2}{x}$ and $\frac{2}{y}$. By comparing (6) and (7), we nd the ratio $k_3$ between the propagation speed of a signal traveling
along a W M and along an ideal membrane $\hat{f}[i, j, k]$: 

$$k_{0}(x; y) = \frac{1}{g} \arctan \frac{b_{0}}{b_{y}} .$$

(8)

Since $k$ is a non-constant function of the spatial frequencies, W M s introduce a dispersion error, i.e., the signal traveling along a W M is acted upon by dispersion of its components. As a starting point, dispersion can be evaluated for spatial frequencies lower than the Nyquist limit $\frac{f_{s}}{2}$ that is, in the frequency domain:

$$(x; y): j_{x} < \frac{1}{2D} ; j_{y} < \frac{1}{2D} .$$

(9)

This domain will be used, according to the considerations to be presented in section 2.4.

Figures 2(a) and 2(b) show plots of $k_{x}$, $k_{y}$, and $k_{0}$, where $D$ has been set to unity. It can be noticed that the propagation speed decreases for increasing spatial frequencies. In particular, $k_{0}$ is maximum when $x = y$, and minimum for high values of the spatial frequencies located along the main axes, suggesting that dispersion in the SW M does not act on the diagonal component traveling along $\pm 45^\circ$. On the contrary, the HW M exhibits the same dispersion error on the region centered around dc. Finally, the TW M seems to have the most uniform behavior of the dispersion error.

The propagation speed in all the W M s has a maximum at dc:

$$k(0) = \lim_{i \to 0} k_{0}( ) = \frac{1}{2} .$$

(10)

It is worth noticing that this value corresponds with the nominal propagation speed of a signal traveling along a finite difference scheme $\hat{f}[i, j]$.

1 For the purpose of this paper, the Nyquist limit is defined as half the sample rate, both in time and space. In some previous works [9, 10, 11], the simulations were considered valid up to a quarter of the sam- ple rate because in the square mesh the frequency response repeats itself after that point. However, as it was pointed out in [12], this is due to the fact that all transfer functions do not have any one junction as functions of $\delta^2$. This does not imply that the response to a signal having all components up to half the sample rate will be allowed. However, for certain applications such as modal analysis of physical membranes, the frequency response at a quarter of the sample rate is of no interest.

B. Sampling Lattices

The evaluation of the dispersion error does not give a complete description of the constraints holding when an ideal membrane is modeled using W M s. In particular, a method is needed for computing the signal bandwidth with a W M is able to process. The theory of sampling lattices [14, 15, 16], that we are briefly reviewing in this section, gives the background for characterizing W M s from this viewpoint.

Let us sample a 2D continuous signal over a domain $L$, subset of $\mathbb{R}^2$, so defining a discrete signal $s_{x}(x) \times 2 \times 2$. If $L$ can be described by means of a non-singular matrix such that each element of the domain $L$ is a linear combination of the columns of $L$, the coefficients being signed integers:

$$x = L 1 \times 2 \times 2 ; u_{1} 2 \times 2 ;$$

(11)

then $L$ is called a sampling lattice, and $L$ is its basis.

The number of samples per unit area is $\frac{1}{\det(L)}$:

$$D_{L} = \frac{1}{\det(L)} .$$

(12)

as $L$ contains information about the distance between adjacent samples.

$s_{L}$, the Fourier transform of $s_{x}$, is defined over $\mathbb{R}^2$ and obtained by periodic imaging of $S$. These Fourier images are centered around the elements of the lattice $L$, which is described by the basis $L^\dagger$, inverse transposed of $L$. Notice that the denser the sampling lattice is, the sparser the image centers are. In formulas,

$$D_{L} = \frac{1}{\det(L^\dagger)} = \det(L) = \frac{1}{D_{L}} .$$

(13)

Let us consider an ideal, unlimited membrane traveled by a spatial bandwidth limited signals $x(y; t)$, and let us do a spatial sampling of the signal at a given time. However, the spatial sampling defines a sampling lattice $L$, so that $s_{L}$ is defined, we can calculate $s_{L}$ using the above results. The multidimensional sampling theorem [14] tells that if the Fourier images of $s$ do not intersect one
with each other, s can be recovered from s, with no aliasing error | indicates whether the chosen sampling scheme induces aliasing.

Equation (15) tells that for each choice of the sampling lattice geometry, the density $D_L$ can be increased until the images do not intersect. Conversely, given s, there exists a sampling lattice capable of capturing all the information needed to recover the original signal using the least density of samples. Clearly, it will exhibit the highest sampling efficiency.

III. Sampling efficiency of the WMs

Sampling efficiency will be calculated for signals having circular spatial band shape centered around the origin of the frequency axes, with radius equal to $B$. Even if the analysis procedure does not depend on the shape of the spatial domain, the circular band seems to include all the signals occurring in practical applications. Indeed, the procedure is independent of the system evolution | which in the WMs is controlled by equation (2) | so that it applies to any model discretizing distributed system s, and where signal information can be located over a sampling lattice. This is the case, for example, in the difference schemes.

A. WMs and Sampling Lattices

A SWM, having digital waveguides of length $D_s$, corresponds to a sampling scheme over the lattice $L_s(D_s)$, described by the basis

$$L_s(D_s) = \begin{bmatrix} D_s & 0 \\ 0 & D_s \end{bmatrix} : \quad (14)$$

A TWM, having digital waveguides of length $D_t$, corresponds to a sampling scheme over the lattice $L_t(D_t)$, described by the basis

$$L_t(D_t) = \begin{bmatrix} D_t & \frac{1}{2}D_t \\ 0 & \frac{1}{2}D_t \end{bmatrix} : \quad (15)$$

Notice that a triangular scheme is denser than a square scheme made with digital waveguides of the same length. This is evidenced by relation

$$\frac{D_L(D_t)}{D_L(D_s)} = \frac{\text{det}(L_s(D_s))}{\text{det}(L_t(D_t))} = \frac{D^2}{\frac{1}{4}D^2} = \frac{4}{3} : \quad (16)$$

The description of an HWM having digital waveguides of length $D_h$, again can be given in terms of sampling lattices, with some extra care. The HWM is obtained by subtracting a TWM, whose junctions lie on the sampling lattice $L_t(D_t)$, from a denser TWM having junctions on $L_t(D_h)$. Hence, the junctions of the HWM lie on

$$L_h(D_h) = L_t(D_h) \cap L_t(D_h) : \quad (17)$$

The basis of $L_t(D_h)$ is

$$L_t(D_h) = \begin{bmatrix} \frac{1}{2}D_h \\ D_h \end{bmatrix} : \quad (18)$$

The HWM, obtained by subtraction of TWMs, are elements belonging to $L_t(D_h)$, are elements belonging to $L_t(D_h)$.

Figure 3 shows the HWM over $L_h(D_h)$, obtained by subtracting the sampling lattice $L_t(D_h)$ (whose elements are depicted with ) from $L_t(D_h)$ (elements depicted with ).

B. TWM vs SWM

The image centers of the spectra belonging to signals traveling along a SWM and a TWM are respectively described by the basis matrices of $L_s$ and $L_t$:

$$L_s(D_s) = \begin{bmatrix} 1 & D_s \\ 0 & 1 & D_s \end{bmatrix} : \quad (19)$$

and

$$L_t(D_t) = \begin{bmatrix} 1 & \frac{1}{2}D_t \\ \frac{1}{2}D_t & \frac{1}{2}D_t \end{bmatrix} : \quad (20)$$

Figure 4 shows Fourier images for the SWM (empty circles in dashed line) and the TWM (dotted circles), located around the origin of the frequency plane. It can be noticed that the square sampling scheme induces a square positioning of the images and, consequently, a square tiling of the frequency plane, as emphasized by the square in dashed line. Similarly, sampling over a TWM results in a triangular positioning of the images, thus producing an hexagonal tiling of the frequency plane, as shown by the hexagon located over the center. Such hexagonal tiling allows $s$, as it appears quite evident in the figure, to "pack" the inages better than those coming from a square sampling.

When both a SWM and a TWM critically sample the same signal (i.e., the dotted circles touch each other without intersecting, and the same thing happens for the empty circles), it is easy to derive the following relation between the digital waveguide lengths:

$$\frac{D_t}{D_s} = \frac{2}{3} : \quad (21)$$

This class of signals encompasses the signals obtained when a membrane is excited, in one or several points, by a single shot or by a sequence of shots, using a stick with an approximately round tip.
We consider the two geometries: (square in dashed line and hexagon in solid line). The empty and the filled circles have the same radius and touch each other without intersecting, meaning that both the SWM and the TWM critically sample the same signal.

Under this condition we can relate the sample densities in the two geometries:

$$\frac{D_{L_s(D_s)}}{D_{L_s(D_h)}} = \frac{\det(L_s(D_s))}{\det(L_s(D_h))} = \frac{P}{2}; \quad (22)$$

concluding that the TWM exhibits a better sampling efficiency relative to the SWM. In other words, a signal can be spatially sampled with a triangular geometry using 13.4% less samples per unit area.

C. TWM vs HWM

Said $s_h$ and $s_t$ the signals sampled over the junctions of the HWM and the TWM, respectively, and said $s_T$ the signal sampled over the lattice $L_T(D_h)$, we can derive the zero-padded signals

$$s_T(x) = 0 \quad ; \quad x \in 2L_h(D_h)$$

and

$$s_h(x) = s_t(x) \quad ; \quad x \in 2L_T(D_h) \quad (23)$$

Since we cannot define a Fourier transform over $L_h(D_h)$, we consider $S_h$, Fourier transform of $s_h$ (which is defined over $L_T(D_h)$), as a description of $s_h$ in the frequency domain. $S_h$, according to its definition, can be obtained by subtracting $S_T$ from $S_t$:

$$S_h(x_i, y_i) = S_T(x_i, y_i) - S_T(x_i, y_i); \quad (24)$$

The result is shown in Figure 4, where some images of $S_T$ are depicted (filled circles plus circle in dashed line). The elements of $L_T(D_h)$ are marked with , while the elements of $L_T(D_h)$ are marked with .

The HWM induces a hexagonal positioning of the Fourier images (corresponding to the filled circles). In fact, their centers are elements of a set which, again, can be obtained by a subtraction between two sampling lattices, $L_T(D_h)$ and $L_T(D_h)$, which are reciprocals of $L_T(D_h)$ and $L_T(D_h)$, respectively. The consequent tiling geometry is triangular, as emphasized by the triangles depicted in Figure 4.

It can be observed that in each intersection does not occur in $L_T(D_h)$ if and only if $s$ is sampled without aliasing over $L_T(D_h)$. In other words, when $S_T$ exhibits superposition of its images, the same thing happens for the images of $S_h$, and vice versa.

Moreover, it must be noticed that equation

$$L_T(D_h) = L_T(P_h) \quad (25)$$

holds between $L_T$ and $L_T$, if a rotation is neglected. This relation, together with the considerations about image superposition made just above, allows us to say that an HWM does not perform better than a TWM, whose digital waveguides are times longer.

Another interesting consideration comes out by noticing
that, since
\[ \frac{D_{LT}(\Phi_h)}{D_{LT}(\Phi_h)} = \det(\mathcal{L}_T(\Phi_h)) = 3; \] (27)
then the TW M defined over the lattice \( \mathcal{L}_T(\Phi_h) \) is 3 times as sparse as the TW M defined over \( \mathcal{L}_T(\Phi_h) \), thus (directly from the definition of \( \Phi_h \)) 2 times as sparse as the HW M.

Hence, the number of lossless scattering junctions per unit area of the HW M is twice as large as that of the TW M defined over \( \mathcal{L}_T(\Phi_h) \), with no benefit on the accuracy of sampling. This relates with the fact that the hexagons tiling the frequency plane (as the inner hexagon depicted in the figure) contain twice the information needed to recover a signal correctly: the 6 partial images included in the slices inside the central hexagon can be composed into 2 Fourier images.

IV. Signal time evolution

In this section, we discuss the consequences of critical spatial sampling on the temporal sampling frequency.

First, let us consider the sampled version of a signal traveling at speed \( c \) along an ideal hexagon mesh that has been excited by a bandlimited signal \( \Phi(x,y) \) having Fourier transform equal to \( F_{\Phi}(x,y) \). It can be shown (see Appendix A) that a time sampling frequency
\[ F_s = 2c \max_{j \neq 0} f \Phi(j) \] (28)
is required to recover the original signal correctly.

When such a signal has a spatial circular band shape, thus belonging to the class seen in section IV, the relation
\[ \max_{j \neq 0} f \Phi(j) = B \] (29)
holds, so that equation (28) simplifies into
\[ F_s = 2cB \] (30)
Relation (30) has an immediate physical interpretation: waves having wavelength equal to \( 1 = B \), propagating at speed \( c \) along a medium, exhibit a temporal frequency equal to \( cB \). In order to preserve information in their temporal samples, they must be sampled above twice their frequency.

The value given by equation (30), if used as the sampling frequency of a 2D resonator model (realized by means of WMs having critical waveguide lengths), causes inaccurate positioning of the modal frequencies. A summed dispersion (see section IV) cannot in general be eliminated, as it is a consequence of the finite number of directions a 2D signal can propagate along a W M. The propagation speed can at least be set to its physical value in low frequency. This can be done by simply rescaling \( F_s \) according to the geometrical.

Recalling the procedure leading to equation (17), we can reformulate the spatial phase shift of a signal traveling along the ideal hexagon mesh, during a time period \( T = 1 = F_s \).

Hence, we find the following expression for the ratio \( \Phi(x,y) \), reformulated using for each geometry its respective critical waveguide length:
\[ K_g(x,y) = \frac{1}{2} \frac{1}{\gamma D} \arctan \left( \frac{4}{\gamma D} \right); \] (31)
where \( K_g \) has the structure of \( B \) (see equation (2)), but the waveguide length \( D \) has been replaced by its critical counterpart \( D_g \).

Recalculation of the limit in (3) gives:
\[ K_g(0) = \frac{D_q}{D} K_q(0) = \frac{1}{2} \frac{D_q}{D} \] (32)
that is, considering \( D_g \) as the reference waveguide length,
\[ K_g(0) = \frac{1}{2} ; K_t(0) = \frac{P}{3} \] (33)

This result shows that signals, under conditions of critical sampling, propagate in the WM s at different speeds, according to the geometry. The dispersion ratios can be set to unity at dc by using the temporal sampling frequencies
\[ F_{\Phi} = 2cB \] (34)
For the purpose of section IV, the dispersion ratio of a critically sampled 2D medium, adjusted to be one at dc, is called \( K_g(x,y) \).

V. Performance

In order to compare the three geometries under critical sampling conditions, we show in figure 5 the contour plots of \( K_g \), for a nominal spatial bandwidth \( B = 1 \) (corresponding to the circle in the figure).

It can be noted that the behaviors of the SW M and the TW M are not dramatically different in terms of averaged dispersion error: dispersion stands quite below 20% for most spatial frequencies in both the geometries. However, the TW M exhibits a more uniform behavior, and this uniformity can be exploited using frequency warping (2). Conversely, dispersion in the HW M stays below 10% almost everywhere, thus indicating that this geometry has the most uniform propagation speed under these test conditions.

The computational cost and the memory requirements in the different geometries can be calculated under the same conditions. They are based on equation (2) which shows that each N (port lossless scattering junction requires 2N operations to compute the wave signal coming out from the junctions, to be stored into \( N \) locations belonging to the adjacent digital waveguides. These operations amount to \( 2N - 1 \) additions, and 1 multiplication (which can be replaced by a bit shift in fixed point implementations of the SW M).

Table 1 summarizing the performance, when the reference sampling rate of the square mesh \( F_{\Phi} \) has been set to a...
Fig. 6
Contour plots of propagation speed ratios in the SWM (a), TWM (b) and HWM (c), when temporal sampling frequency has been set to $T_{\text{sam}}$. All the WMs process a signal having bandwidth (depicted with the circle) equal to 1.2.

**TABLE I**
Performance of the geometries in terms of memory requirement and computational cost (multiplications are pure bit shifts in the SWM). Two implementations are considered: as a waveguide mesh and as a finite difference scheme.

|                      | Waveguide Mesh | Finite Difference Scheme |
|----------------------|----------------|--------------------------|
|                      |   Sq. |  Tr. |  Hex. |      Sq. |  Tr. |  Hex. |
| additions per junction |  7   | 11  |  5   |   4     |  6   |  3   |
| multiplications per junction |  1   |  1  |  1   |   1     |  1   |  1   |
| memory locations per junction |  4   |  6  |  3   |   2     |  2   |  2   |
| density of junctions | 1 0.866 | 1.732 | 1 0.866 | 1.732 |
| density of memory locations | 4 52 | 52 | 2 1.732 | 3.464 |
| sample rate | 1 0.866 | 1.5 | 1 0.866 | 1.5 |
| additions per unit time and space | 7 8.25 | 12.990 | 4 4.5 | 7.794 |
| multiplications per unit time and space | 1 0.75 | 2.598 | 1 0.75 | 2.598 |

Nominal value equal to unity. Numbers are given for two implementations: as a waveguide mesh (with memory in the waveguide branches), and as a finite difference scheme (with memory in the junctions).

The numbers of operations and memory locations per junction for the WMs follow directly from equation (2). The numbers of operations and memory locations per junction for the finite difference scheme follow directly from equation (3). The densities of junctions result from equations (22) and (26). The densities of locations are obtained by multiplying the third row times the fourth row. The sample rates are a consequence of equation (34).

The results of Table I show that the triangular finite difference scheme uses the least quantity of memory. Among WMs implementations, the SWM is the most efficient in terms of memory occupation.

Finally, the number of additions (multiplications) per unit time and space results by multiplication of the number of additions (multiplications) per junction, density of junctions, and sample rate. Once again, the SWM has the least density of operations when a WMs model is required. Conversely, the square and the triangular geometries have about the same computational requirements in a finite difference implementation.
A numerical example

Let us design a WM, capable of modeling in real time an ideal 2D round resonator, of radius \( r = 0 \) m, where waves propagate at a speed \( c = 130 \) m/s. Let the signal contain information up to a frequency \( f = 10 \) kHz.

The spatial bandwidth \( B \) of the signal traveling along the resonator is:

\[
B = \frac{f}{c} = \frac{10000 \text{ Hz}}{130 \text{ m/s}} = 76.923 \text{ m}^{-1}; \quad (35)
\]

and consequently the critical waveguide lengths required to model the signal in the respective geometries are:

\[
D_s = \frac{1}{2B} = 65 \text{ mm};
\]

\[
D_t = \frac{1}{3B} = 75 \text{ mm}; \quad (36)
\]

\[
D_h = \frac{1}{3B} = 43 \text{ mm}.
\]

This means that the numbers of junctions \( N_s, N_t, \) and \( N_h \), respectively, needed in the three models are:

\[
N_s = \frac{r^2}{D_s} = 744;
\]

\[
N_t = \frac{r^2}{D_t} = 645; \quad (37)
\]

\[
N_h = \frac{2}{3} \frac{r^2}{D_h} = 1308.
\]

The time sampling frequencies needed to have \( k_0(0) = 1 \) are, from equation (34):

\[
\begin{align*}
F_{s,s} &= 2 \frac{p}{r} f = 28285 \text{ kHz}; \\
F_{s,t} &= 2 \frac{3}{2} f = 24495 \text{ kHz}; \quad (38) \\
F_{s,h} &= 2 \frac{3}{2} \frac{3}{2} f = 42427 \text{ kHz}.
\end{align*}
\]

VI. Conclusion

A novel wave propagation model has recently been introduced for the simulation of isotropic multidimensional media. It makes use of structures called waveguide meshes. Even if most of the waveguide mesh properties have already been understood, there was lack of literature about their performance from a signal sampling viewpoint.

In this paper, some properties of the most common 2D waveguide meshes \( \square, \triangle, \) and \( \diamond \) related with the bandwidth of the signal traveling on them, have been inspected. In particular, it has been shown that the triangular waveguide mesh is capable of processing a larger bandwidth than the square or hexagonal waveguide meshes having the same digital waveguide lengths.

Further on, when processing signals of the same bandwidth, the triangular waveguide mesh does not exhibit a computational and memory load much larger than the rest of waveguide meshes \( \square, \) the square waveguide mesh \( \square \) such that it can be considered, for the uniformity of its dispersion error, a good choice both in terms of simulation errors and computational cost.

The analysis presented in this article may be extended to 3D media, comparing the three geometries which most directly correspond to the waveguide meshes here reviewed: 3D rectilinear, dodecahedral, and tetrahedral.

VII. Acknowledgment

We would like to thank Lauri Saviola and Vesa Valimaki for any insightful discussions. We are also grateful to the anonymous reviewers for their constructive criticism.

Appendix A: Sampling of a signal traveling along an ideal membrane

An ideal membrane establishes a relation between the spatial Fourier transforms of the signals traveling on it, taken in correspondence of two times, \( t \) and \( t' \):

\[
S(x_i, y_i ; t) = e^{j2 \pi c t} S(x_i, y_i ; t') \quad \text{(39)}
\]

This relation has the following interpretation: each spatial component of the signal traveling along a distance equal to \( c t' \) during time \( c t \), has no magnitude variation, and has phase variation equal to \( c (t - t') \).

Let us excite, at time \( t \), the ideal membrane with a spatial band limited signal \( f(x,y) \) having Fourier transform \( F(x,y) \). The application of (39) gives the spatial Fourier transform of the signal on the membrane:

\[
S(x_i, y_i ; t = 0) = \left\{ \begin{array}{ll}
0 & \text{if } t < t' \\
S(x_i, y_i ; t) e^{j2 \pi c t} F(x_i, y_i) & \text{if } t > t'
\end{array} \right. \quad (40)
\]

From this relation, we can calculate the critical temporal sampling frequency required to sample, without loss of information, a signal traveling on an ideal membrane. In fact, the Fourier transform of (40),

\[
S(x_i, y_i ; t) = Z_+^{1} S(x_i, y_i ; t)e^{j2 \pi c t} dt; \quad (41)
\]

has a magnitude equal to

\[
\mathcal{S}(x_i, y_i ; j \omega) = Z_+^{1} S(x_i, y_i ; t)e^{j2 \pi c t} dt; \quad (42)
\]

\[
Z_+^{1} = \left\{ \begin{array}{ll}
S(x_i, y_i ; t) e^{j2 \pi c t} dt & \text{if } t > t' \\
0 & \text{if } t < t'
\end{array} \right. \quad (43)
\]

The values result by calculating the number of small areas, each one being associated with its own lossless scattering junction, which tessellate the resonator.
In particular, when \( t! 1 \):

\[
\mathcal{F}(x; y; f) = \lim_{\tau \to \infty} e^{-2\tau f} c \quad \text{df} (43)
\]

This means that some spectral power exists for any temporal frequency (equal to \( c \)) associated to a spatial component of frequencies \( (x, y) \) excited in the membrane. Invoking the sampling theorem, the critical time for sampling frequency \( F_s \) results to be equal to \( (22) \).

References

1. F. Fontana and D. Rocchesso, "A new formulation of the 2D -waveguide mesh for percussion instrument ensembles," in Proc. XI Colloquium Mus. Informat., Bologna, Italy, Nov. 1995, A&M, pp. 27(30).
2. F. Fontana and D. Rocchesso, "A new formulation of the 2D -waveguide mesh for percussion instrument ensembles," in Proc. International Computer Music Conference, Aarhus, Denmark, Sept. 1994, pp. 463(466).
3. S. A. Van Duyne and J. O. Smith, "The 2D digital waveguide mesh in Proc. IEEE Workshop on Applications of Signal Processing to Audio and Acoustics, M. Chonk, N. Y., 1993, IEEE.
4. S. A. Van Duyne and J. O. Smith, "The tetrahedral digital waveguide mesh in Proc. IEEE Workshop on Applications of Signal Processing to Audio and Acoustics, M. Chonk, N. Y., Oct. 1995, IEEE, p. 54(47).
5. S. A. Van Duyne and J. O. Smith, "The tetrahedral digital waveguide mesh with musical applications," in Proc. International Computer Music Conference, Hong Kong, Aug. 1996, IEEE, p. 19(24).
6. J. Stakem, Finite Di erence Schem es and Paritial Di erential Equations, W. adsworth & Brooks, Paci c Grove, CA, 1989.
7. J. O. Smith, "Musical applications of digital waveguides," in Report No. 39, CCRMA - Stanford University, Stanford, California, 1987.
8. J. O. Smith, "Musical applications of digital waveguides," in Proc. International Computer Music Conference, Hong Kong, Aug. 1996, IEEE, p. 19(24).
9. J. O. Smith, "Musical applications of digital waveguides," in Proc. International Computer Music Conference, Hong Kong, Aug. 1996, IEEE, p. 19(24).
10. S. A. Van Duyne and J. O. Smith, "The tetrahedral digital waveguide mesh with musical applications," in Proc. International Computer Music Conference, Hong Kong, Aug. 1996, IEEE, p. 19(24).
11. S. A. Van Duyne and J. O. Smith, "The tetrahedral digital waveguide mesh with musical applications," in Proc. International Computer Music Conference, Hong Kong, Aug. 1996, IEEE, p. 19(24).
12. J. O. Smith, "Physical modeling using digital waveguides," Computer Music J., vol. 16, no. 4, pp. 74(91), Winter 1992.
13. J. O. Smith, "Principles of digital waveguide modeling of musical instruments," in Proceedings of the Digital Signal Processing Conference, vol. 2, pp. 417(466), K. dier and E. dier, eds., Academic Press, 1989.
14. D. E. Dudgeon and R. M. Mersereau, Multidimensional Signal Processing, Prentice Hall, Englewood Cliffs, N. J., 1984.
15. P. P. Vaidyanathan, Multirate Systems and Filter Banks, Prentice Hall, Englewood Cliffs, N. J., 1993.
16. A. V. Oppenheim and R. W. Schafer, Discrete-Time Signal Processing, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1989.