The entropy of randomized network ensembles

Ginestra Bianconi

The Abdus Salam International Center for Theoretical Physics - Strada Costiera 11, 34014 Trieste, Italy

received 3 September 2007; accepted in final form 19 November 2007
published online 10 December 2007

PACS 89.75.-k – Complex systems
PACS 89.75.Fb – Structure and organization in complex systems
PACS 89.75.Hc – Networks and genealogical trees

Abstract – Randomized network ensembles are the null models of real networks and are extensively used to compare a real system to a null hypothesis. In this paper we study network ensembles with the same degree distribution, the same degree correlations and the same community structure of any given real network. We characterize these randomized network ensembles by their entropy, i.e. the normalized logarithm of the total number of networks which are part of these ensembles. We estimate the entropy of randomized ensembles starting from a large set of real directed and undirected networks. We propose entropy as an indicator to assess the role of each structural feature in a given real network. We observe that the ensembles with fixed scale-free degree distribution have smaller entropy than the ensembles with homogeneous degree distribution indicating a higher level of order in scale-free networks.

Copyright © EPLA, 2008

Introduction. – The complexity of a network [1] depends on its global structural organization which is linked to the functional constraints the network has to satisfy. Real networks show different levels of organization. To characterize their structure few different quantities have been proposed: i) the density of the links, ii) the degree sequence [2], iii) the degree-degree correlations [3–5], iv) the clustering coefficient [6,7], v) the k-core structure [8–10] and finally vi) the community structure [11–14].

To study the different information content retained by these structural quantities we will consider randomized network models which are best studied by statistical-mechanics methods. Out of different statistical-mechanics approaches of networks [15,16], one has been proposed [17,18] for networks with hidden variables \( \theta \) associated to each node \( i \) of the network. In the same framework it has been shown by [19] that the probability of a link should satisfy specific forms in order to guarantee good inference of the hidden variables.

Every real network can be considered as a specific instance of a particular network evolution compatible to its functional constraints. Nevertheless in many cases real networks are not determined exactly by their evolution. We propose here to consider a real network as belonging to an ensemble of networks which would perform the same task equally well. For example in the biological world we observe a certain variability of biological networks across different species with the same biological function. The complexity of a given ensemble of networks increases as the number of networks in the ensemble decreases. Consequently a high complexity of the network ensemble corresponds to a small variability of the networks in the ensemble. The entropy \( \Sigma \) of a given network ensemble [20] is proportional to the logarithm of the number of networks belonging to the ensemble. We expect that a very complex network is belonging to an ensemble of functionally equivalent networks of small entropy. Since it is difficult to characterize the minimal entropy ensemble a real network belongs to, we take successive approximations of the real network.

To characterize the complexity of a real network we consider a series of randomized network models which retain some characteristics of the real networks. In particular we consider networks with a given degree sequence, given degree-degree correlations and a given community structure. Degree-degree correlation [5] has been considered a signature of non-randomness in the topology of the networks. The correlations have been shown to be important in the Internet at the Autonomous System Level [3] and in biological networks [4] where the degree correlations are linked also to the modular structure [7] of the network.

In our approach we will first consider a particular real network to be part of the ensemble of networks with the same number of nodes \( N \) and links \( L \) the real network has. This network ensemble is the \( G(N,L) \) studied by the random graph community. Subsequently, we consider the configuration model of networks with given degree
sequence and we restrict the number of possible networks. Furthermore, we consider the ensemble of networks with a given degree sequence and with given degree correlations or with given community structure and we further restrict the space of possible networks. Finally, we will consider the ensemble of networks with given community structure and degree sequence. How much information is carried by each of these ensembles? This paper is trying to answer this question by calculating the entropies of these ensembles which subsequently approximate the real network.

The ensemble of networks with a given degree sequence falls in the class of hidden variable models \cite{17,18} with the hidden variable being nothing else than the Lagrangian multipliers of the connectivity of each node.

The ensemble of networks with given degree sequence and degree correlations, or given degree sequence and given community structure are a generalized hidden variable model and can also be used to generate networks with given degree-degree correlations/community structure.

**Undirected networks.** – Given a real network with \( N \) nodes and given adjacency matrix \((a_{ij})\), \( i = 1, \ldots, N \) we construct subsequent randomized networks ensembles. For an undirected network the first ensemble (zero order approximation) is the \( G(N, L) \) network ensemble of networks with given number of nodes \( N \) and links \( L = \sum_{i,j} a_{ij}/2 \). The first-order approximation is the configuration network of given degree sequence \( \{k_1, \ldots, k_N\} \) with \( k_i = \sum_j a_{ij} \). The second-order approximation is the ensemble with given degree sequence \( \{k_1, \ldots, k_N\} \) and given average nearest-neighbour connectivity \( k_{nn}(k) = (\delta(k_i - k)\sum_j a_{ij}k_j) \). Moreover, one can consider the partition function of the networks with given community structure, and fixed number of links within each community and between different communities. If the community \( q \) of node \( i \) is indicated with \( q_i \) we can consider graphs with given \( A(q, q') = \sum_{i,j} \delta(q_i - q)\delta(q_j - q')a_{ij} \). The partition functions of these network ensembles are given by

\[
Z_0 = \sum_{\{a_{ij}\}} \delta \left( L - \sum_{i<j} a_{ij} \right) \exp \left[ \sum_{i<j} h_{ij}a_{ij} \right],
\]

\[
Z_1 = \sum_{\{a_{ij}\}} \prod_i \delta \left( k_i - \sum_j a_{ij} \right) \exp \left[ \sum_{i<j} h_{ij}a_{ij} \right],
\]

\[
Z_2 = \sum_{\{a_{ij}\}} \prod_i \delta \left( k_i - \sum_j a_{ij} \right) \exp \left[ \sum_{i<j} h_{ij}a_{ij} \right] \times \prod_k \delta(k_{nn}(k)N_k - \sum_{ij} \delta(k_i - k)a_{ij}k_j),
\]

\[
Z_c = \sum_{\{a_{ij}\}} \prod_i \delta \left( k_i - \sum_j a_{ij} \right) \exp \left[ \sum_{i<j} h_{ij}a_{ij} \right] \times \prod_{q,q'} \delta(A(q, q') - \sum_{i<j} \delta(q_i - q)\delta(q_j - q')a_{ij}),
\]

where \( h_{ij} \) are auxiliary fields, \( N_k \) indicates the number of nodes of degree \( k \) in the network \( N_k = \sum_i \delta(k_i - k) \), the vector \( q_i \) indicates to which community a node belongs and \( A(q, q') \) indicates the number of links between the community \( q \) and the community \( q' \). The probability \( p_{ij}^{(k)} \) for a link between node \( i \) and node \( j \) (the probability for \( a_{ij} = 1 \)) is given by

\[
p_{ij}^{(k)} = \frac{\partial \ln(Z_k)}{\partial h_{ij}} |_{h_{ij} = 0 \forall i, j}.
\]

The number of undirected simple networks in each of these ensembles \( \kappa \) is consequently given by

\[
N_\kappa = Z_\kappa |_{h_{ij} = 0 \forall i, j}.
\]

We define the entropy per node \( \Sigma \) of the network ensemble \( \kappa \) as

\[
\Sigma_\kappa = \frac{1}{N} \ln N_\kappa.
\]

The number of undirected networks \( N_0 \) with given number of nodes \( N \) and links \( L \) is given by the binomial

\[
N_0 = \binom{N(N-1)/2}{L},
\]

for distinguishable nodes in the networks \cite{20}. The probability \( p_{ij} \) of a given link \((i,j)\) is given by \( p_{ij}^{(0)} = L/(N(N-1)/2) \) for every couple of nodes \( i,j \).

The volume of the network ensemble with given degree sequence. The first level of approximation is the one in which a given degree sequence is assumed. In the undirected simple case the partition function of the network ensemble with given degree distribution is given by

\[
Z_1 = \sum_{\{a_{ij}\}} \prod_i \delta \left( k_i - \sum_j a_{ij} \right) \exp \left[ \sum_{i<j} h_{ij}a_{ij} \right].
\]

Expressing the deltas in the integral form with Lagrangian multipliers \( \omega_i \), for every \( i = 1, \ldots, N \) we get

\[
Z_1 = \int \mathcal{D}\omega \, e^{-\sum_i \omega_i k_i \prod_{i<j} \left( 1 + e^{\omega_i + \omega_j + h_{ij}} \right)},
\]

where \( \mathcal{D}\omega = \prod_i d\omega_i/(2\pi) \). We solve this integral by saddle point equations. The entropy of this ensemble of networks can be approximated in the large network limit with

\[
N \Sigma_1^{und} \approx - \sum_i \omega_i k_i + \sum_{i<j} \ln \left( 1 + e^{\omega_i + \omega_j} \right) - \frac{1}{2} \sum_i \ln(2\pi\alpha_i)
\]

with the Lagrangian multipliers \( \omega_i \) satisfying the saddle point equations

\[
k_i = \sum_{j \neq i} \frac{e^{\omega_i + \omega_j}}{1 + e^{\omega_i + \omega_j}}.
\]
and the coefficients $\alpha_i$ defined as
\begin{equation}
\alpha_i = \sum_j \frac{e^{\omega_i + \omega_j^*}}{(1 + e^{\omega_i + \omega_j^*})^2}.
\end{equation}

The probability of a link $i, j$ in this ensemble is then given by
\begin{equation}
p_{ij}^{(1)} = \frac{e^{\omega_i + \omega_j^*}}{1 + e^{\omega_i + \omega_j^*}}
\end{equation}
recovering the hidden variable ensemble [17,19]. In particular in this ensemble $p_{ij} \neq f(\omega_i)f(\omega_j)$, consequently the model retains some “natural” correlations [19] given by the degree sequence and the constraint that we consider only simple networks. These in fact are nothing else than the correlations of the configuration model [21]. Nevertheless we can consider the case in which the network is sparse and there is a structural cutoff in the system, $k_i < \sqrt{\langle k \rangle N}$. In this case we can approximate eq. (9) by $e^{\omega_i} = k_i \sqrt{\langle k \rangle N}$, $\alpha_i = k_i$. In this limit the network is not correlated $p_{ij}^{(1), unc} = k_i k_j / (\langle k \rangle N)$, $\omega^*_i < 0$ and we can approximate the entropy of the ensemble as
\begin{equation}
N \Sigma_{1, unc}^{unc} \approx - \sum_i \ln k_i / \sqrt{\langle k \rangle N} k_i - \frac{1}{2} \sum_i \log (2\pi k_i)
\end{equation}
\begin{equation}
+ \frac{1}{2} \sum_{ij} k_i k_j / \langle k \rangle N - \frac{1}{2} \sum_{ij} k_i^2 / \langle k \rangle N^2 + \ldots
\end{equation}
\begin{equation}
= - \sum_i (\ln k_i - 1) k_i - \frac{1}{2} \sum_i \ln (2\pi k_i)
\end{equation}
\begin{equation}
+ \frac{1}{2} \langle k \rangle N \ln (\langle k \rangle N) - 1 - \frac{1}{2} \left( \frac{\langle k \rangle^2}{\langle k \rangle} \right)^2 + \ldots
\end{equation}
which approximately gives for the volume
\begin{equation}
N_{1, unc}^{unc} \approx \frac{(\langle k \rangle N)!}{\prod_i k_i!} \exp \left[ - \frac{1}{2} \left( \frac{\langle k \rangle^2}{\langle k \rangle} \right)^2 \right].
\end{equation}

The expression (13) was already derived in [22] by combinatorial considerations valid in a network with structural cutoff. In fact the term $(\langle k \rangle N)! / ((\langle k \rangle N - 1)!)$ gives the total number of different ways we can link the $2L = \langle k \rangle N$ half-edges associated to a degree sequence to form a network. In fact we can take a first half-edge of the network and we have $2L - 1$ choices to match it with one of the other half-edges. Then, we can take another half-edge and we have $(2L - 3)$ possible choices of other half-edges to link to, giving rise to $(2L - 1)!$ networks. Out of these networks only a part of them is simple providing for the correction exp $\left[ - \frac{1}{2} \left( \frac{\langle k \rangle^2}{\langle k \rangle} \right)^2 \right]$ [22]. Out of these simple networks for each distinct adjacency matrix there are $\prod_i k_i!$ networks that can be constructed by simply permuting the order of the edges at each node.

It can be shown that within sparse uncorrelated networks the scale-free networks with $\gamma \to 2$ are the ones which minimize $\Sigma_{1, unc}^{unc}$ [22]. For correlated networks with natural correlations the entropy of the configuration model $\Sigma_1$ decreases with the value of the power law exponent $\gamma$. In fig. 1 we plot the entropy of a scale-free network with natural cutoff and fixed average connectivity $\langle k \rangle = 6, 8, 10$. The entropy $\Sigma_1$ of the configuration model is decreasing with decreasing power law exponent $\gamma$ reaching its minimum at $\gamma \to 2$. This indicates that scale-free networks with low value of $\gamma$ presents higher level of ordering with respect to random homogeneous networks.

The volume of a network ensemble with fixed degree correlations. The second order of approximation is to take into consideration degree correlations behind the “natural correlations” of the configuration model. The partition function for this ensemble is given by
\begin{equation}
Z_2 = \sum_{\{a_{ij}\}} \prod_{i<j} \delta \left( k_i - \sum a_{ij} \right) \exp \left[ \sum_{i<j} \delta(k_i - k_j) \right],
\end{equation}
where $K$ is the maximal connectivity in the network. Expressing the deltas in the integral form we get for the partition function
\begin{equation}
Z_2 = \int \mathcal{D} \omega \int \mathcal{D} A e^{-\sum_{i} \omega_i k_i + \sum_{i,k} A_k k_{nn}(k) k} \times \prod_{i<j} (1 + e^{\omega_i + \omega_j + h_{ij} + k_i A_{ij} + k_j A_{ij}}).
\end{equation}
The expression can be evaluated as for the case of the calculation of $\Sigma_1$ where the Lagrange multipliers $\omega_i$ and $A_k$ satisfy
\begin{equation}
k_i = \sum_{j \neq i} \frac{e^{\omega_i + \omega_j^*} + k_j A_{ij}^* + k_i A_{ij}}{1 + e^{\omega_i + \omega_j^*} + k_j A_{ij}^* + k_i A_{ij}},
\end{equation}
\begin{equation}
k_{nn}(k) = \frac{1}{KN_k} \sum_i \delta(k_i - k) \sum_{j \neq i} k_j \frac{e^{\omega_i + \omega_j^*} + k_j A_{ij}^* + k_i A_{ij}}{1 + e^{\omega_i + \omega_j^*} + k_j A_{ij}^* + k_i A_{ij}}.
\end{equation}
If we solve this equation for a given real network degree sequence and nearest-neighbor average degree, we can then construct other networks in the same ensemble just by drawing a link $i,j$ with probability

$$p_{ij}^{(2)} = \frac{e^{\omega_i^* + \omega_j^* + k_i A_{ij}^* + k_j A_{ij}^*}}{1 + e^{\omega_i^* + \omega_j^* + k_i A_{ij}^* + k_j A_{ij}^*}}. \tag{17}$$

The entropy of this ensemble is approximately equal in the large network limit to

$$N \Sigma_2^{\text{rand}} \simeq -\sum_i \sum_{\langle i \rangle} \omega_i + \sum_i \ln (1 + e^{\omega_i^* + k_i A_{ii}^*}) - \frac{1}{2} \sum_i \ln (2 \pi \alpha_i) + \frac{1}{2} \sum_i \ln (2 \pi \alpha_k) \tag{18}$$

with $\alpha_i, \alpha_k$ defined as

$$\alpha_i = \sum_j \sum_{\langle j \rangle} \frac{e^{\omega_i^* + \omega_j^* + k_i A_{ij}^* + k_j A_{ij}^*}}{(1 + e^{\omega_i^* + \omega_j^* + k_i A_{ij}^* + k_j A_{ij}^*})^2},$$

$$\alpha_k = \sum_i \sum_{\langle i \rangle} \delta (k_i - k) \sum_{\langle j \rangle} \frac{e^{\omega_i^* + \omega_j^* + k_i A_{ij}^* + k_j A_{ij}^*}}{(1 + e^{\omega_i^* + \omega_j^* + k_i A_{ij}^* + k_j A_{ij}^*})^2}. \tag{19}$$

In table 1 we report the entropy for different undirected network ensembles at different level of approximation. We consider the Internet network at the Autonomous System Level, the Protein Interaction networks of S. cerevisiae (DIP database) and the partial map of protein interaction network of H. Sapiens [23]. We observe that for these networks taking into account the degree distribution strongly reduces the entropy of the randomized network ensemble.

The volume of a network ensemble with given community structure. A different ensemble of networks is the ensemble of networks with given community structure and degree sequence. Suppose that we have a network and we detect $Q$ communities such that each node $i = 1, \ldots, N$ belongs to the community $q_i = 1, \ldots, Q$ with $Q$ finite. To find a randomized ensemble of networks with the given community structure we impose that the nodes have fixed degree sequence and fixed number $A(q, q')$ of links in-between the communities $q$ and $q'$. In an undirected network, $A(q, q')$ is given by the following expression:

$$A(q, q') = \sum_{i < j} \delta (q_i - q) \delta (q_j - q') a_{ij}. \tag{20}$$

Following the same steps as in the previous case we find that the entropy for such an ensemble is given by

$$N \Sigma_c \simeq -\sum_i \sum_{\langle i \rangle} k_i \omega_i \sum_{q<q'} A(q, q') w_{q, q'} + \sum_{i < j} \ln (1 + e^{\omega_i^* + \omega_j^* + w_{q, q'}})$$

$$- \frac{1}{2} \sum_i \ln (2 \pi \alpha_i) - \frac{1}{2} \sum_{q<q'} \ln (2 \pi \alpha_{q, q'}) \tag{21}$$

with the Lagrangian multipliers $\{\omega_i\}$, $\{w_{q, q'}\}$ satisfying the saddle point equations

$$k_i = \sum_{i < j} \frac{e^{\omega_i^* + \omega_j^* + w_{q_i, q_j}}}{1 + e^{\omega_i^* + \omega_j^* + w_{q_i, q_j}}},$$

$$A(q, q') = \sum_{i < j} \delta (q_i - q) \delta (q_j - q') \frac{e^{\omega_i^* + \omega_j^* + w_{q, q'}}}{1 + e^{\omega_i^* + \omega_j^* + w_{q, q'}}}, \tag{22}$$

and $\alpha_i, \alpha_{q, q'}$ defined as

$$\alpha_i = \sum_j \frac{e^{\omega_i + \omega_j + w_{q_i, q_j}}}{1 + e^{\omega_i + \omega_j + w_{q_i, q_j}}},$$

$$\alpha_{q, q'} = \sum_{i, j} \delta (q_i - q) \delta (q_j - q') \frac{e^{\omega_i + \omega_j + w_{q, q'}}}{1 + e^{\omega_i + \omega_j + w_{q, q'}}}. \tag{23}$$

The probability for a link between node $i$ and $j$ is equal to

$$p_{ij}^{(c)} = \frac{e^{\omega_i + \omega_j + w_{q_i, q_j}}}{1 + e^{\omega_i + \omega_j + w_{q_i, q_j}}}. \tag{24}$$

In the case of the Zachary club [24] we were able to calculate $\Sigma_2^{\text{rand}} = 3.94$ and $\Sigma_2^{\text{rand}} = 3.25$ quantifying the amount of information present in the known community partition.

Table 1: Entropies of randomized network ensembles starting from real undirected networks with $N$ nodes and $L$ links. $\Sigma_0^{\text{rand}}$, $\Sigma_1^{\text{rand}}$, $\Sigma_2^{\text{rand}}$ indicate the entropy of an undirected network with assigned $N$ nodes and $L$ links, with given degree sequence and with given degree sequence and degree correlations, respectively. The data sets (see footnote $^*$) “AS-year-month” indicate different snapshot of the Internet at the Autonomous System level, the yeast DIP dataset is the protein interaction of S. cerevisiae and H. Sapiens PI is the partial human protein interaction map [23].

| Network     | $N$  | $L$  | $\Sigma_0^{\text{rand}}$ | $\Sigma_1^{\text{rand}}$ | $\Sigma_2^{\text{rand}}$ |
|-------------|------|------|--------------------------|--------------------------|--------------------------|
| AS-97-11    | 3015 | 5156 | 13.3                     | 7.5                      | 7.3                      |
| AS-98-10    | 4180 | 7768 | 14.9                     | 8.6                      | 8.4                      |
| AS-99-10    | 5816 | 11312| 16.1                     | 9.2                      | 9.0                      |
| AS-00-10    | 8836 | 17822| 17.5                     | 9.8                      | 9.6                      |
| AS-01-03    | 10515| 21455| 18.1                     | 10.1                     | 9.8                      |
| Yeast DIP   | 4135 | 8099 | 15.6                     | 12.3                     | 11.1                     |
| H. Sapiens PI | 3134 | 6726 | 16.3                     | 12.3                     | 12.2                     |

$^*$The network data are available at http://vlado.fmf.unilj.si/pub/networks/data/ (Chesapeake food web), www.cosinoproject.org/ (Littlerock foodweb), www.nd.edu/~networks/ (ND WWW), http://cdg.columbia.edu/cdg/ (C. elegans neural network), www.weizmann.ac.il/mcb/UriAlon/ (S. cerevisiae transcription network), www.ercot.com/ (Texas power-grid) at the Cosinoproject.org/ (E.coli metabolic network/DIP yeast protein networks) and as supplementary material [23].

28005-p4
The entropy of randomized network ensembles

Table 2: Entropy of randomized network ensembles starting from specific directed networks with \(N\) nodes and \(L\) links. \(\Sigma_0^{(dir/und)}\) is the entropy of the network ensembles with fixed number of nodes \(N\) and links \(L\) in the case of a directed ensemble or in the case of an undirected ensemble. \(\Sigma_1^{(dir/und)}\) is the entropy of the directed/undirected network ensemble with given degree sequence. The datasets (see footnote 1) indicate different foodwebs (FW), the metabolic network of \(E.\ coli\), the Texas power-grid, the Notre Dame University domain WWW, the neural network of \(C.\ elegans\).

| Network            | \(N\)   | \(L\)   | \(\Sigma_0^{dir}\) | \(\Sigma_0^{und}\) | \(\Sigma_1^{dir}\) | \(\Sigma_1^{und}\) |
|--------------------|---------|---------|--------------------|--------------------|--------------------|--------------------|
| Littlerock FW      | 183     | 2,494   | 48.4              | 38.4              | 13.28             | 23.44             |
| Seagrass FW        | 48      | 226     | 15.3              | 11.8              | 4.3               | 7.8               |
| Metabolic net.     | 896     | 964     | 8.3               | 7.5               | 3.2               | 4.3               |
| Neural net.        | 306     | 2,359   | 35.9              | 30.52             | 17.8              | 22.5              |
| Power-grid net.    | 4,888   | 5,855   | 11.1              | 10.3              | 7.5               | 8.7               |
| ND WWW             | 32,572  | 149,713 | 55.9              | 52.7              | 33.1              | 36.7              |

**Directed networks.** – An undirected network is determined by a symmetric adjacency matrix, while the matrix of a directed network is in general non-symmetric. Consequently the degrees of freedom of a directed network are more than the degrees of freedom of an undirected network. If we consider the number of directed networks \(N_0^{dir}\) with given number of nodes and of directed links we find

\[
N_0^{dir} = \frac{N!}{L!} \left( \frac{N(N-1)}{2} \right)^L. \quad (25)
\]

**Volume of randomized directed network ensembles with given degree sequence.** To calculate the volume of directed networks with a given degree sequence of in/out degrees \(\{k_i^{out}, k_i^{out}\}\) we just have to impose the constraints on the incoming and outgoing connectivities,

\[
Z_1^{dir} = \prod \delta \left( k_i^{out} - \sum_j a_{ij} \right) \prod \delta \left( k_i^{in} - \sum_j a_{ij} \right) \times \exp \left( \sum_{ij} h_{ij} a_{ij} \right). \quad (26)
\]

Following the same approach as for the undirected case, we find that the entropy of this ensemble of networks is given by

\[
N\Sigma_1^{dir} \simeq -\sum_i \omega_i^* k_i^{out} - \sum_i \bar{\omega}_i^* k_i^{in} + \sum_{i \neq j} \ln(1 + e^{\omega_i^* + \bar{\omega}_j^*}) - \frac{1}{2} \sum_i \ln \left( \frac{2\pi}{(2\pi)^2} \alpha_i^{(in)} \alpha_i^{(out)} \right) \quad (27)
\]

with the Lagrangian multipliers satisfying the saddle point equations

\[
k_i^{(out)} = \sum_{j \neq i} \frac{e^{\omega_i^* + \bar{\omega}_j^*}}{1 + e^{\omega_i^* + \bar{\omega}_j^*}},
\]

\[
k_i^{(in)} = \sum_{j \neq i} \frac{e^{\omega_j^* + \bar{\omega}_i^*}}{1 + e^{\omega_j^* + \bar{\omega}_i^*}}, \quad (28)
\]

with

\[
\alpha_i^{(out)} = \sum_{j \neq i} \frac{e^{\omega_i^* + \bar{\omega}_j^*}}{(1 + e^{\omega_i^* + \bar{\omega}_j^*})^2},
\]

\[
\alpha_i^{(in)} = \sum_{j \neq i} \frac{e^{\omega_j^* + \bar{\omega}_i^*}}{(1 + e^{\omega_j^* + \bar{\omega}_i^*})^2}. \quad (29)
\]

The probability for a directed link from \(i\) to \(j\) is given by

\[
P_{ij}^{(1, dir)} = \frac{e^{\omega_i^* + \bar{\omega}_j^*}}{1 + e^{\omega_i^* + \bar{\omega}_j^*}}. \quad (30)
\]

If the \(\omega_i^* + \bar{\omega}_j^* < 0\) \(\forall i, j = 1, \ldots, N\) the directed network becomes uncorrelated and we have \(P_{ij}^{(1, dir)} = k_i^{(out)} k_j^{(out)} / \sqrt{(k_i^{in})N}\). Given this solution the condition for having uncorrelated directed networks is that the maximal in-degree \(K^{(in)}\) and the maximal out-degree \(K^{(out)}\) should satisfy \(K^{(in)} K^{(out)} / \sqrt{(k_i^{in})N} < 1\). The entropy of the directed uncorrelated network is then given by

\[
N\Sigma_1^{unc} \simeq \ln((k_i^{in})N)! - \sum_i \ln(k_i^{(in)}!k_i^{(out)}!)
\]

\[
\simeq -\frac{1}{2} \frac{(k_i^{in}) (k_i^{out})}{(k_i^{in}) (k_i^{out})}, \quad (31)
\]

which has a clear combinatorial interpretation as it happens also for the undirected case. In table 2 we report the entropy of directed networks and their undirected version observing that different degree distributions reduce the entropy of randomized network ensembles by a different amount, some carrying more information than others.

**Conclusions.** – In conclusion we have studied the space of possible networks in randomized models of complex networks. We have found that random scale-free network ensembles with low power law exponent \(\gamma\) have a lower entropy than random networks with a homogenous degree distribution. The successive random approximations of a real graph characterize to which extent the degree sequence, the degree-degree correlations or the community structure constrain the network. We have
Ginestra Bianconi evaluated the entropy of randomized ensembles starting from a set of different real directed and undirected networks showing how much each structure feature reduces the space of possible networks. Future work will focus on extending these results to weighted networks and measurement of large deviations in ensembles of random networks with hidden variables.

***

This work was supported by IST STREP GENNETEC contract No. 034952; the author thanks D. Garlaschelli and M. Marsili for interesting discussions.

REFERENCES

[1] Newman M. E. J., Barabási A. L. and Watts D. J., Structure and Dynamics of Networks (Princeton University Press, Princeton) 2006.

[2] Barabási A.-L. and Albert R., Science, 286 (1999) 509.

[3] Pastor-Satorras R., Vázquez A. and Vespignani A., Phys. Rev. Lett., 87 (2001) 258701.

[4] Maslov S. and Sneppen K., Science, 296 (2002) 910.

[5] Berg J. and Lassig M., Phys. Rev. Lett., 89 (2002) 228701.

[6] Watts D. J. and Strogatz S. H., Nature, 4 (1998) 393.

[7] Ravasz E., Somera A. L., Mongru A. D., Oltvai Z. N. and Barabási A.-L., Science, 297 (2002) 1551.

[8] Carmi S., Havlin S., Kirkpatrick S., Shavitt S. and Shir E., Proc. Natl. Acad. Sci. U.S.A., 104 (2007) 11150.

[9] Dorogovtsev S. N., Goltsev A. V. and Mendes J. F. F., Phys. Rev. Lett., 96 (2006) 040601.

[10] Alvarez-Hamelin J. I., Dall’Asta L., Barrat A. and Vespignani A., cs.Ni/0511007 (2005).

[11] Gibian M. and Newman M. E. J., Proc. Natl. Acad. Sci. U.S.A., 99 (2002) 7821.

[12] Danon L., Díaz-Guilera A., Duch J. and Arenas A., J. Stat Mech. (2005) P09008.

[13] Newman M. E. J. and Leicht E. A., Proc. Natl. Acad. Sci. U.S.A., 104 (2007) 9364.

[14] Boccaletti S. et al., Phys. Rep., 424 (2006) 175.

[15] Burda Z., Correia J. D. and Krzywicki A., Phys. Rev. E, 64 (2001) 046118.

[16] Dorogovtsev S. N., Mendes J. F. F. and Samukhin A. N., Nucl. Phys. B, 666 (2003) 396.

[17] Park J. and Newman M. E. J., Phys. Rev. E, 70 (2004) 066117.

[18] Caldarelli G., Capocci A., De Los Rios P. and Angel Munóz M., Phys. Rev. Lett., 89 (2002) 258702.

[19] Garlaschelli D. and Loffredo M. I., cond-mat/0609015 (2006).

[20] Bogacz L., Burda Z. and Waclaw B., Physica A, 366 (2006) 587.

[21] Molloy M. and Reed B. A., Random. Struct. Algorithms, 6 (1995) 161.

[22] Bianconi G., Chaos, 17 (2007) 026114.

[23] Rual J. F. et al., Nature, 437 (2005) 1173.

[24] Zachary W. W., J. Anthropol. Res., 33 (1977) 452.