Fixed points of asymptotically nonexpansive mappings with center 0 and applications

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Abstract

In this paper, we investigate the existence of fixed points for asymptotically nonexpansive mappings with center 0 defined on closed convex subsets of various Banach spaces. Three applications are given. Firstly, we prove that our results refine those concerning alternate convexically nonexpansive (in short; ACN) mappings studied by P. N. Dowling in ”On a fixed point result of Amini-Harandi in strictly convex Banach spaces, Acta. Math. Hungar., 112 (1-2), (2006), 85-88”. Secondly, by using Lau’s result in ”Closed convex invariant subsets of $L_p(G)$, Trans. Amer. Math. Soc., 232, (1977), 131-142”, we give another characterization for the noncompactness of locally compact groups $G$. Finally, we discuss the existence of a solution for a nonlinear transport equation without using compactness results.

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1 Introduction

In 1922, S. Banach [6] established his metric fixed point theorem, named after him, for contraction mappings. In the case where the contraction constant equals 1 (the nonexpansive case), it is easy to construct examples that do not have fixed points. In fact, it suffices to take the complete metric space $(X, \tilde{d})$ where $X = \{0, 1\}$, $\tilde{d}$ is the discrete metric and $T : X \rightarrow X$ is a self-mapping defined by $T(0) = 1$ and $T(1) = 0$. Then $T$ satisfies $\tilde{d}(Tx, Ty) = \tilde{d}(x, y)$ for all $x, y \in X$. But $T$ does not have fixed points. So, since the situation does not work in the case of complete metric spaces, what about the framework of Banach spaces?

In 1965, F. E. Browder, D. G"ohde and W. A. Kirk ([11, 20, 22]) showed that every nonexpansive mapping defined on a bounded closed convex subset of a uniformly convex Banach space (or more generally reflexive Banach space having normal structure) has

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at least a fixed point. Their works were the foundation of the fixed point theory for nonexpansive mappings which illustrated the primary role of the geometry of Banach spaces in this axis of research. It was the birth of an interesting domain of nonlinear functional analysis which attracted the attention of many mathematician. For more details, see [2, 7, 18, 19].

A Banach space \( X \) is said to have the fixed point property (resp. the weak fixed point property) for nonexpansive mappings (in short; FPP) (resp. in short; w-FPP) if for all nonempty bounded closed (resp. weakly compact) convex subset \( C \) of \( X \), every nonexpansive mapping \( T : C \to C \) has at least a fixed point. According to this terminology, we can also define the fixed point property (resp. weak fixed point property) for generalized nonexpansive mappings (see [5, 8, 9, 10, 15, 18, 31, 32, 34, 35]).

One of the most passionate subjects is the link between the FPP and the FPP for generalized nonexpansive mappings. In 1976, C. S. Wong [40] proved that the w-FPP for Kannan mappings (mappings that satisfy \( \|Tx - Ty\| \leq \frac{1}{2}(\|Tx - x\| + \|Ty - y\|) \), for all \( x, y \in C \), see also [29]) characterizing the quasi-weak normal structure which is possessed by strictly convex and separable Banach spaces (see [39]). Consequently, Banach space \( L^1([0, 1]) \) has the w-FPP for Kannan mappings. However, according to the famous result due to D. Alspach (see [1]), \( L^1([0, 1]) \) does not have the w-FPP (resp. FPP). In the same direction, K. K. Tan [36] constructed a separable and reflexive Banach space that have FPP but fails to have the FPP for Kannan mappings.

In general, the link between the w-FPP and the w-FPP for generalized nonexpansive mappings is not known yet where \( X \) does not have neither the weak normal structure nor the quasi-weak normal structure. We have just a few contributions in this direction. For this, we cite for example, the works ([4, 8, 9, 15, 17, 27, 34, 35]). We remind that in the case of real Hilbert spaces, the FPP property characterizes the boundedness of closed convex subsets. This result was proved by W. O. Ray [28] and then simplified by R. Sine [33]. Also, if \( X = c_0 \) (the Banach space of real sequences that converge to zero), the FPP characterizes the weak compactness of bounded closed convex subsets. Recently, the authors in [4, 12, 13] gave an investigation on the FPP for \((c)\)-mappings in the bounded and unbounded cases. Particularly, by using the results of W. Takahashi et al [37]. A. Dehici and S. Atailia [12] proved a variant result of Ray for \((c)\)-mappings and the problem in the case of abstract Banach spaces, is still open.

In [16], J. García-Falset et al introduced the class of nonexpansive mappings with center as an extension of quasi-nonexpansive mappings. They proved by examples that a center is not necessarily a fixed point for these considered mappings. The advantage of the contributions in [15] concerning the study of fixed points, is that they are established for mappings which are not necessarily self-mappings but they are defined on nonempty subsets of various Banach spaces.

In this paper, we are interested in the class of asymptotically nonexpansive mappings that have zero as a center which contains the class of nonexpansive mappings with center 0. We study the existence of fixed points for these mappings defined on bounded closed convex subsets of reflexive strictly convex Banach spaces and spaces having Kadec-Klee property. Many illustrative examples are given. Afterwards, we materialize our results.
by three concrete applications.

For the first one, we refine the results of A. Amini-Harandi [3] and P. Dowling [14] which are established in the case of weakly compact convex subsets of strictly convex Banach spaces and we extend them to the case of closed convex (not necessarily bounded) subsets of reflexive strictly convex Banach spaces.

For the second one, with a result due to A. T-M. Lau [23] concerning the characterization of closed convex subsets invariant by modular isometries which are defined on $L_p(G)(1 < p < \infty)$ where $G$ is a locally compact noncompact group, we give a characterization for the noncompactness of $G$ using orbits associated with these isometries.

The last application is devoted to study the existence of solutions for a nonlinear transport equation with contractive boundary conditions. More precisely, we refine the results of [25] in this sense and we show following our assumptions that our results are independent of any use of the compactness argument established in the first section of [25].

# 2 Preliminaries and Preparatory Results

In this paper, we introduce a large class of asymptotically nonexpansive mappings with center 0 containing in particular that of nonexpansive mappings $T : C \rightarrow C$ having 0 as a fixed point (if $0 \in C$).

**Definition 2.1** Let $C$ be a nonempty subset of a Banach space $(X, \| \cdot \|)$ and let $T : C \rightarrow C$ be a self-mapping. $T$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \text{ for all } x, y \in C. \quad (2.1)$$

**Definition 2.2** Let $C$ be a nonempty subset of a Banach space $(X, \| \cdot \|)$ and let $T : C \rightarrow C$ be a self-mapping. $T$ is said to be asymptotically nonexpansive if

$$\limsup_{n \to +\infty} \|T^nx - T^ny\| \leq \|x - y\| \text{ for all } x, y \in C. \quad (2.2)$$

**Remark 2.1** It is easy to see that every nonexpansive mappings is asymptotically nonexpansive while the converse is not true in general as the following example shows:

**Example 2.1** Let $T : [0, 1] \rightarrow [0, 1]$ be defined by $Tx = \sqrt{x}$ if $x > 0$ and $T(0) = 1$. $T$ is asymptotically nonexpansive while $T$ is not nonexpansive since $T$ is not continuous at $x_0 = 0$.

**Remark 2.2** As it is indicated in [24], the notion of asymptotically nonexpansive given in Definition 2.2 is different and more general than the notion of nonexpansive mappings introduced by K. Goebel and W. A. Kirk in [17].

**Definition 2.3** Let $C$ be a nonempty subset of a Banach space $(X, \| \cdot \|)$ and let $T : C \rightarrow C$ be a self-mapping. $T$ is said to be nonexpansive mapping with center 0 if $\|Tx\| \leq \|x\|$ for all $x \in C$. 

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**Remark 2.3** Clearly if $0 \in C$ and $T$ is a nonexpansive mapping having 0 as a fixed point, then $T$ is nonexpansive with center 0 but the next examples show that there exist nonexpansive mappings with center 0 which fail to be nonexpansive.

**Example 2.2** Let

$$T : [0, 3] \to [0, 3]$$

$$x \mapsto \begin{cases} 0 & \text{if } x \neq 3; \\ 1 & \text{if } x = 3. \end{cases}$$

Obviously, we have $|Tx| \leq |x|$ for all $x \in [0, 3]$. However, $T$ is not nonexpansive since $T$ is not continuous at $x_0 = 3$.

**Example 2.3** Let

$$T : [0, 1] \to [0, 1]$$

$$x \to x^2$$

If $x \in [0, 1]$ then $|Tx| \leq |x|$, however $T$ is not nonexpansive. To see this, it suffices to take $x_1 = \frac{1}{2}$ and $x_2 = \frac{2}{3}$. For these values, we have

$$|Tx_1 - Tx_2| = |x_1^2 - x_2^2| = \left|\left(\frac{1}{2}\right)^2 - \left(\frac{2}{3}\right)^2\right| = \left|\frac{1}{2} + \frac{2}{3}\right|\frac{1}{2} - \frac{2}{3} > |x_1 - x_2|.$$  

**Remark 2.4** From Remark 2.1, we observe that nonexpansive mappings with center 0 are asymptotically nonexpansive with the same center.

**Definition 2.4** Let $C$ be a nonempty subset of a Banach space $X$ and let $T : C \to C$ be a self-mapping. $T$ is said to be a $(c)$-mapping if there exist $a, c \in [0, 1]$ with $(c > 0)$ and $a + 2c = 1$ such that

$$\|Tx - Ty\| \leq a\|x - y\| + c(\|x - Ty\| + \|Tx - y\|) \quad \text{for all } x, y \in C. \tag{2.3}$$

**Remark 2.5** A simple calculation shows that Example 2.2 is a $(c)$-mapping for $\frac{1}{3}$. So, $(c)$-mappings can be discontinuous.

**Remark 2.6** It is worth noting that there exit examples which are nonexpansive and $(c)$-mappings at the same time. To see this, it suffices to take $(X, \|\cdot\|) = (\mathbb{R}, |\cdot|)$ and $T : \mathbb{R} \to \mathbb{R}$ defined by $Tx = x + a$ with $a \neq 0$.

The following lemma due to the J. S. Bae [5] is a useful tool in the investigation of fixed points for $(c)$-mappings.

**Lemma 2.1** Let $C$ be a bounded subset of a Banach space $X$ and let $T : C \to C$ be a $(c)$-mapping. Then $T$ is asymptotically regular i.e.,

$$\lim_{n \to +\infty} \|T^{n+1}x - T^n x\| = 0 \quad \text{for all } x \in C.$$
Example 2.4 Let $C = [0, 1] \subseteq \mathbb{R}$ where $\mathbb{R}$ is equipped with its usual norm and let $T : [0, 1] \rightarrow [0, 1]$ be defined by $Tx = 1 - x$. It is easy to see that $T$ is nonexpansive. However, $T$ cannot be a $(c)$-mapping since $T^{2k+1}(0) = 1$ and $T^{2k}(0) = 0$ and the claim follows from Lemma 2.1.

Definition 2.5 Let $C$ be a weakly compact convex subset of a Banach space $X$. $C$ is said to have $(c)$-FPP if every $(c)$-self-mapping on $C$ has a fixed point.

Remark 2.7 It is an open problem whether $(c)$-FPP holds if FPP is satisfied.

Definition 2.6 A Banach space $X$ is said to be uniformly convex if for each $\epsilon \in (0, 2]$ there exists $\delta > 0$ such that for all $x, y \in X$, $\|x\| \leq 1$, $\|y\| \leq 1$, $\|x - y\| > \epsilon \implies \left\| \frac{x + y}{2} \right\| \leq \delta$.

Definition 2.7 A Banach space $X$ is said to be strictly convex if for all $x, y \in X$, $x \neq y$, we have

$$\|x\| = \|y\| = 1 \implies \left\| \frac{x + y}{2} \right\| < 1.$$

Definition 2.8 Every uniformly convex Banach space $X$ is strictly convex while the converse is not true in general. Recall that the Lebesgue spaces $L_p(\mu)$ are uniformly convex.

Example 2.5 Let $X = C([0, 1])$ be the Banach space of scalar continuous functions defined on $[0, 1]$ equipped with the following norm

$$\|f\| = \sup_{t \in [0, 1]} |f(t)| + \|f\|_{L^2([0, 1])}.$$ 

Then $(X, \|\cdot\|)$ is strictly convex but not uniformly convex Banach space (for more details, see the last paragraph in page 24 of [19]).

Definition 2.9 Let $C$ be a subset of a Banach space $X$ and let $T : C \rightarrow C$ be a self-mapping. $T$ is called generalized nonexpansive if there exists $a, b, c \in [0, 1]$ such that $a + 2b + 2c = 1$ and

$$\|Tx - Ty\| \leq a\|x - y\| + b(\|Tx - x\| + \|Ty - y\|) + c(\|Tx - y\| + \|Ty - x\|)$$

for all $x, y \in C$.

Remark 2.8 By a simple calculation, we infer that if $0 \in C$ and $T$ is a generalized nonexpansive mapping having 0 as a fixed point, then necessarily $T$ is nonexpansive with center 0.

Definition 2.10 Let $C$ be a nonempty subset of a Banach space $X$ and let $T : C \rightarrow C$ be a self-mapping, $T$ is said to be a Suzuki mapping if for all $x, y \in C$

$$\frac{1}{2}\|Tx - x\| \leq \|x - y\| \implies \|Tx - Ty\| \leq \|x - y\|.$$
The class of Suzuki mappings was introduced and studied in 2008 by T. Suzuki [35] who proved that this class contains strictly the set of nonexpansive mappings. Other interesting properties related to this class can be found in [27, 35].

**Remark 2.9** Similar to Remark 2.8, if \( 0 \in C \) and \( T \) is a Suzuki mapping having 0 as a fixed point then \( \|Tx\| \leq \|x\| \) for all \( x \in C \).

**Definition 2.11** (see Definition 2 in [15] and Definition (2) in [27]) Let \( C \) be a nonempty subset of a Banach space \( X \). \( T \) is said to satisfy the property \((E_{\mu_0})\) if there exists \( \mu_0 \geq 1 \) such that
\[
\|x -Ty\| \leq \mu_0\|Tx - x\| + \|x -y\|
\]
for all \( x, y \in C \).

Let us give now the following important useful lemma.

**Lemma 2.2** (see Lemma 7 in [35] and Proposition 3.6 in [27]) Let \( C \) be a nonempty subset of a Banach space \( X \). If \( T : C \rightarrow C \) is a Suzuki mapping or a generalized nonexpansive mapping. Then \( T \) satisfies the property \((E_{\mu_0})\).

**Example 2.6** Nonexpansive mappings with center 0 are not necessarily generalized nonexpansive or Suzuki mappings. Indeed, to see this, take \( T_{|[0, \frac{2}{3}] : [0, \frac{2}{3}] \rightarrow [0, \frac{2}{3}] \) defined by \( Tx = x^2 \). Then \( T \) is nonexpansive with center 0 (see Example 2.3) but it does not satisfy the property \((E_{\mu_0})\) (see Example 3.7 in [27]). So, by Lemma 2.2, \( T \) is neither a generalized nonexpansive mapping nor a Suzuki mapping.

### 3 Main results

We start this section by the following main result.

**Theorem 3.1** (see also page 1207 in [24]) Let \( C \) be a nonempty closed convex (not necessarily bounded) subset of a reflexive strictly convex Banach space. Then, there exists a unique element \( z_0 \in C \) such that
\[
\|z_0\| = \delta_0 = \inf\{\|y\| : y \in C\}.
\]

**Proof.** First case: \( 0 \in C \), then \( z_0 = 0 \).

Second case: \( 0 \notin C \). Let us define the following set
\[
C_0 = \{z \in C : \|z\| = \delta_0\}.
\]

It is easy to see that \( C_0 = \bigcap_{\epsilon > 0} C_{0,\epsilon}(\delta_0) \) where \( C_{0,\epsilon}(\delta_0) = \{z \in C : \|z\| \leq \delta_0 + \epsilon\} \). Since \( X \) is reflexive then for all \( \epsilon > 0, C_{0,\epsilon} \) is a nonempty weakly compact convex subset of \( X \) and by the finite intersection property, \( C_0 \) is a nonempty weakly compact convex subset of \( X \).
Now, let $z_1, z_2 \in C_0$, with $z_1 \neq z_2$. Since $X$ is strictly convex, then necessarily we obtain that
\[
\frac{\|z_1 + z_2\|}{2} < \frac{1}{2}(\|z_1\| + \|z_2\|) = \delta_0.
\]
which is a contradiction. Consequently, the set $C_0$ is reduced to a singleton $\{z_0\}$ which is the desired result.

**Remark 3.1** In the above theorem, the element $z_0$ is called the metric projection of $0$ on $C$ and is denoted by $P_C(0)$. In the case where $X$ is a real Hilbert space then $P_C(0)$ is characterized by the unique element $z_0 \in C$ such that
\[
Re < z_0, z - z_0 > \geq 0, z \in C.
\]

From the proof of Theorem 3.1, we can deduce the following

**Corollary 3.1** Let $C$ be a weakly compact convex subset of a strictly convex Banach space (not necessarily reflexive). Then, there exists a unique element $\tilde{z}_0 \in C$ such that
\[
\|\tilde{z}_0\| = \tilde{\delta}_0 = \inf\{\|y\| : y \in C\}.
\]

The first fixed point result concerning asymptotically nonexpansive mappings with center 0 is given in the following.

**Theorem 3.2** Let $C$ be a closed convex subset of a reflexive strictly convex Banach space and let $T : C \to C$ be an asymptotically nonexpansive mapping with center 0.

(i) If $0 \in C$ and $T$ is continuous then 0 is a fixed point for $T$.

(ii) If $0 \not\in C$ and $T$ is weakly continuous then $T$ has a fixed point in $C$.

**Proof.** Since $T$ is asymptotically nonexpansive, then
\[
\limsup_n \|T^n x\| \leq \|x\| \quad \text{for all} \quad x \in C.
\]

(i) If $0 \in C$, we have
\[
\limsup_n \|T^n 0\| \leq 0,
\]
which leads to
\[
\lim_{n \to +\infty} \|T^n 0\| = 0.
\]

So,
\[
\lim_{n \to +\infty} T^n 0 = 0,
\]
since $T$ is continuous, we get
\[
\lim_{n \to +\infty} T^{n+1} 0 = T(0),
\]
consequently, we have $T(0) = 0$. 

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(ii) If $0 \notin C$, following Theorem 3.1 there exists a unique $x_0 \in C$ such that
\[ \|x_0\| = \inf \{ \|x\| : x \in C \}. \]
So
\[ \limsup_n \|T^nx_0\| \leq \|x_0\|. \]
Assume that $y_0 \in \overline{(T^nx_0)_n}$ (the weak closure of the sequence $(T^nx_0)_n$). This implies the existence of a subsequence $(T^{nk}x_0)_k$ such that $T^{nk}x_0 \rightharpoonup y_0$ weakly, using the lower semicontinuity of the norm, it follows that
\[ \|y_0\| \leq \liminf_k \|T^{nk}x_0\| \leq \limsup_k \|T^{nk}x_0\| \leq \limsup_n \|T^nx_0\| \leq \|x_0\|. \]
So, necessarily
\[ y_0 \in \{ x \in C : \|x\| = \|x_0\| \}. \]
Hence, we deduce that $y_0 = x_0$. Thus every weakly convergent subsequence $(T^{nk}x_0)_k$ in $C$ weakly converges to $x_0$. Therefore the sequence $(T^nx_0)_n$ weakly converges to $x_0$. Since $T$ is weakly continuous, we get
\[ T(x_0) = T(wk - \lim_n T^nx_0) = wk - \lim_n T^{n+1}x_0 = x_0 \]
(where $wk - \lim_n$ is the weak limit) which is the desired result.

**Remark 3.2** Following its proof, it can be seen that the assertion 1 of Theorem 3.2 holds for a nonempty closed convex subset $C$ of an arbitrary Banach space $X$.

Let us state the following theorem due to P. Dowling [14] (see also [3]).

**Theorem 3.3** Let $C$ be weakly compact convex subset of a strictly convex Banach space and let $T : C \rightharpoonup C$ be a nonexpansive mapping with center 0. Then $T$ has a fixed point.

By using Theorem 3.2 and adapting the same techniques given in the proof of the above theorem (for more details, see Theorem 3 in [14]), we can derive the following

**Corollary 3.2** Let $C$ be a closed convex subset of a reflexive strictly convex Banach space and let $T : C \rightharpoonup C$ be a nonexpansive mapping with center 0. Then $T$ has a fixed point.

**Remark 3.3** We observe that Theorem 3.2 is more general than Theorem 3.3 in the case of reflexive strictly convex spaces since the boundedness of $C$ in Corollary 3.2 is dropped.
Corollary 3.3 Let $C$ be a closed convex subset of a reflexive strictly convex Banach space $X$. Assume that $T : C \to C$ and $S : C \to C$ are two self-mappings such that $S$ is into and $\|TS(x)\| \leq \|S(x)\|$ for all $x \in C$. Then $T$ has a fixed point.

Proof. The assumption that $\|TS(x)\| \leq \|S(x)\|$ for all $x \in C$ is equivalent to the fact that $T$ is a nonexpansive mapping with center 0. So the result follows immediately from Corollary 3.2.

Corollary 3.4 Let $C$ be a closed convex subset of a reflexive strictly convex Banach space $X$ and let $T : C \to C$ be a self-mapping satisfying that

$$\|T(x) + T(y)\| \leq \|x + y\| \text{ for all } x, y \in C. \quad (3.1)$$

Then $T$ has a fixed point in $C$.

Proof. By taking $x = y$, we observe that $T$ is necessarily nonexpansive with center 0. Now, the result is an immediate consequence of Corollary 3.2.

In the same way of Corollary 3.4 and using Theorem 3.3, we can derive the following.

Corollary 3.5 Let $C$ be a weakly compact convex subset of a strictly convex Banach space $X$ and let $T : C \to C$ be a self-mapping satisfying that

$$\|T(x) + T(y)\| \leq \|x + y\|, \text{ for all } x, y \in C. \quad (3.2)$$

Then $T$ has a fixed point in $C$.

Example 3.1 Let $X = \mathbb{R}^2$ equipped with the Euclidean norm and let

$$C = \{(x_1, x_2) \in \mathbb{R}^2, x_1, x_2 \geq 0, x_1^2 + x_2^2 \leq 1\}.$$ 

Define $T : C \to C$ by $T(x, y) = (x^2, y^2)$. So, obviously $X$ is strictly convex (since $X$ is uniformly convex) and $C$ is a bounded closed convex subset of $X$. Furthermore, for all $(x_1, y_1), (x_2, y_2) \in C$, we have

$$\|T(x_1, y_1) + T(x_2, y_2)\| = \sqrt{(x_1^2 + x_2^2)^2 + (y_1^2 + y_2^2)^2}$$

$$\leq \sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2}$$

$$= \|(x_1, y_1) + (x_2, y_2)\|$$

Now, all assumptions of Corollary 3.5 are satisfied and the existence of a fixed point for $T$ is ensured. Clearly, $(0, 0), (1, 0)$ and $(0, 1)$ are the fixed points of $T$ in $C$.

Corollary 3.6 Let $C$ be a weakly compact convex subset of a strictly convex Banach space $X$ and let $T : C \to C$ be a self-mapping such that there exists $k_0 > 1$ for which $T^{k_0}$ is nonexpansive with center 0. If $T$ is a $(c)$-mapping then $T$ has a fixed point in $C$.

Proof. Denote $A = T^{k_0}$. So by assumption, $A$ is nonexpansive with center 0. Then, by Theorem 3.3, $A$ has a fixed point $x_0 \in C$. Then, since $T$ is a $(c)$-mapping then $x_0$ is a fixed point in $C$ (see Proposition 4.1 in [12]).
Example 3.2 The mapping $T$ in Example 2.2 is a $(c)$-mapping and it is also nonexpansive with center 0. Obviously, we have $T^2 = 0$. Then, $T^2$ has 0 as a unique fixed point in $[0,3]$. But, it is easily seen that 0 is also the unique fixed point of $T$ in $[0,3]$.

Definition 3.1 A Banach space $X$ is said to have Kadec-Klee property (in short, KK property) if for every sequence $(x_n)_n$ in $X$ such that if $x_n$ converges weakly to $x$ and $\|x_n\| \to \|x\|$ then $x_n$ converges in norm to $x$.

Remark 3.4 It is easy to deduce that Kadec property (for which weak topology and norm topology are the same) implies KK property but the converse is not true in general (see [38]). In particular, spaces having Schur property satisfy KK property.

Example 3.3 Banach spaces $L_p(\mu)$ ($1 < p < \infty$) have KK property.

Remark 3.5 Recall that the classes of strictly convex Banach spaces and those having KK property are different. Indeed $l^1(\mathbb{N})$ has Kadec-Klee property but $l^1(\mathbb{N})$ is not strictly convex. In addition, the space $(c_0, \|\cdot\|)$ where $\|\cdot\|$ is defined by
\[
\|x\| = \|x\|_{\infty} + \sum_{n=1}^{\infty} (\frac{x_n}{n^2})^{\frac{1}{2}} 
\text{for all } x \in c_0
\]
is strictly convex but fails to have KK property (see Example 23 in [16]).

Definition 3.2 Let $C$ be a nonempty subset of a Banach space $X$ and let $T : C \to C$ be a self-mapping. Assume that $(x_n)_n$ is a sequence in $C$. $(x_n)_n$ is called an almost fixed point sequence (in short; a.f.p.s) for $T$ if
\[
\lim_{n \to +\infty} \|x_n - Tx_n\| = 0.
\]

It was proved (see Lemma 2.2 in [8]) that if $C$ is bounded convex and $T$ is a Suzuki self-mapping on $C$ then $T$ has an a.f.p.s in $C$. In particular every nonexpansive mapping $T : C \to C$ has an a.f.p.s.

Definition 3.3 Let $C$ be a bounded closed convex subset of a Banach space $X$ and let $T : C \to C$ be a self-mapping. $T$ is said to satisfy the condition $(L)$ if the following two conditions hold:

1. If a subset $C_0 \subset C$ is nonempty, closed, convex and $T$-invariant, then there exists an a.f.p.s for $T$ in $C_0$.

2. For any a.f.p.s of $T$ in $C$ and all $x \in C$
\[
\limsup_{n} \|x_n - Tx\| \leq \limsup_{n} \|x_n - x\|.
\]

Remark 3.6 If $C$ is a weakly compact convex subset of a Banach space $X$. Then every generalized nonexpansive self-mapping on $C$ with $a + c > 0$ and every Suzuki self-mapping on $C$ satisfies the condition $(L)$. (see Propositions 3.4 and 3.6 in [27]).
Remark 3.7 The converse of Remark 3.6 is not true in general. Indeed, it was proved (see page 9 in [27]) that the mapping \( T \) of Example 2.6 satisfies condition \((L)\) but fails to be generalized nonexpansive or a Suzuki mapping.

In the next result, we give a fixed point theorem concerning the class of mappings satisfying the condition \((L)\) which are nonexpansive with center 0 in the setting of Banach spaces having KK property.

Theorem 3.4 Let \( X \) be a Banach space having KK property and let \( C \) be a weakly compact convex subset of \( X \). Assume that \( T : C \rightarrow C \) is a nonexpansive mapping with center 0 which satisfies the condition \((L)\). Then \( T \) has a fixed point.

Proof. If \( 0 \in C \), then the result is trivial and 0 is a fixed point. Assume now that \( 0 \notin C \) and denote by \( \Gamma \) the set \( \{ x \in C : \|x\| = \theta_0 \} \) where \( \theta_0 = \inf \{ \|x\| : x \in C \} > 0 \). So, \( \Gamma \) is nonempty weakly compact convex subset of \( X \). The fact that \( \|Tx\| \leq \|x\| \) shows that \( T(\Gamma) \subset \Gamma \). So, since \( T \) satisfies the condition \((L)\) then \( T \) has an a.f.p.s in \( \Gamma \).

Denote by \((x_n)\) this a.f.p.s. But \( \Gamma \) is weakly compact, thus from \((x_n)\), we can extract a subsequence \((x_{n_k})\) in \( \Gamma \) which converges weakly to some \( y_0 \in \Gamma \). Furthermore, from the definition of \( \Gamma \), for all integer \( k \), we have \( \|x_{n_k}\| = \|y_0\| = \theta_0 > 0 \). In addition, since \( X \) satisfies KK property, we infer that \( x_{n_k} \) converges in norm to \( y_0 \). Now, by (2) of the condition \((L)\) and using the fact that \((x_{n_k})\) is also an a.f.p.s for \( T \), we get

\[
0 \leq \liminf_k \|x_{n_k} - Ty_0\| \leq \limsup_k \|x_{n_k} - Ty_0\| \leq \limsup_k \|x_{n_k} - y_0\| = 0.
\]

This leads to

\[
\|y_0 - Ty_0\| = \lim_k \|x_{n_k} - Ty_0\| = 0.
\]

and so, \( y_0 = Ty_0 \) which proves that \( y_0 \) is a fixed point for \( T \) in \( C \) and completes the proof.

Following Remark 3.6 and Theorem 3.4, we have

Corollary 3.7 Let \( X \) be a Banach space having KK property and let \( C \) be a weakly compact convex subset of \( X \). If \( T : C \rightarrow C \) is nonexpansive with center 0 satisfying one of the following assumptions:

(i) \( T \) is a generalized nonexpansive mapping with \( a + c > 0 \);

(ii) \( T \) is a Suzuki mapping.

Then \( T \) has a fixed point in \( C \).

Now, we are in position to state the following fixed point result.

Theorem 3.5 Let \( C \) be a weakly compact convex subset of a Banach space and let \( T : C \rightarrow C \) be a self-mapping. Then

(i) If \( 0 \in C \) and \( T \) is a continuous asymptotically nonexpansive with center 0. Then 0 is a fixed point for \( T \)
(i) If $X$ has KK property and $0 \notin C$. Set $\theta_0 = \inf\{\|x\| : x \in C\}$ and assume that $T$ satisfies the following assumptions:

(1') $T$ satisfies the condition $(L)$;

(2') $T$ is asymptotically regular;

(3') $T$ leaves the set $C_{\theta_0} = C \cap \{x \in X : \|x\| = \theta_0\}$ invariant.

Then $T$ has a fixed point in $C$.

Proof.

(i) This claim is trivial (see the part (i) in the proof of Theorem 3.2).

(ii) Assume that $0 \notin C$. Since $C$ is weakly compact convex subset of $X$, then

$$\inf\{\|x\| : x \in C\} = \theta_0 > 0$$

and the set $C_{\theta_0}$ is nonempty. On the other hand, by (3') it is easy to observe that $C_{\theta_0}$ is a $T$-invariant weakly compact convex subset of $X$. Let $x_0 \in C_{\theta_0}$, since $C_{\theta_0}$ is weakly compact, from the sequence $(T^n x_0)_n$, we can extract a subsequence $(T^{n_k} x_0)_k$ which converges weakly to some $y_0 \in C_{\theta_0}$. Next, since $X$ has KK property, then $T^{n_k} x_0 \rightarrow y_0$ in norm. On the other hand, by (2'), $T$ is asymptotically regular, then the sequence $(x_{n_k})$ defined by $x_{n_k} = T^{n_k} x_0$ is an a.f.p.s for $T$ in $C_{\theta_0}$. Now, from the fact that $T$ satisfies the condition $(L)$, we have

$$0 \leq \limsup_k \|x_{n_k} - T y_0\| \leq \limsup_k \|x_{n_k} - y_0\| = 0.$$

Then $T y_0 = y_0$ which is the desired result.

Remark 3.8 Let $0 \notin C$. If $X$ satisfies KK property and assuming that $T : C \rightarrow C$ is continuous and satisfies (3') then $T$ has a fixed point in $C$. Indeed, by our proof, the set $C_{\theta_0}$ is a nonempty compact convex subset of $X$ and the result follows immediately from Schauder fixed point theorem. In this case, assumptions (1') and (2') of Theorem 3.5 can be dropped.

Remark 3.9 It is worth noting that if

$$K = \{f \in L_1([0,1]) : 0 \leq f \leq 2 \text{ a.e. } \int_0^1 f(t) dt = 1\}$$

and $T$ is Alspach’s mapping on $K$ given by

$$T f(t) = \begin{cases} 2f(2t) \wedge 2, & \text{if } 0 \leq t \leq \frac{1}{2}, \\ (2f(2t-1) - 2) \vee 0, & \text{if } \frac{1}{2} < t \leq 1. \end{cases}$$

then $T$ is a free fixed point nonexpansive mapping which satisfies the assumption (3') of Theorem 3.5 and in this case $\theta_0 = 1$ with $K \cap C_1 = K$. It was proved (see [1]) that $T$ is a nonlinear isometry satisfying $\|Tx\| = \|x\|$ for all $x \in K$. So, $T$ is nonexpansive with center 0. But, Banach space $L_1([0,1])$ does not have KK property.
The following example shows that Theorem 3.5 is not true if we consider $C$, a bounded closed convex subset of an arbitrary Banach space even when assumptions $(1')$, $(2')$ and $(3')$ are satisfied.

**Example 3.4** Let $X = C([0, 1])$ equipped with the sup-norm. Let 

$$C = \{ x \in C([0, 1]) : 0 = x(0) \leq x(t) \leq x(1) = 1 \}$$

and let $T : C \rightarrow C$ be the self-mapping defined by $Tx(t) = tx(t)$. Then $C$ is bounded closed and convex subset of $X$ and $T$ is a $(c)$-mapping (see Example in [5]) (then it is asymptotically regular by Lemma 2.1). Obviously,

$$\|Tx\| \leq \sup_{t \in [0,1]} |tx(t)| \leq \sup_{t \in [0,1]} |x(t)| = \|x\|$$

and $T$ is nonexpansive with center 0 which satisfies the condition $(3')$ with $\theta_0 = 1$. However, $T$ is a free fixed point mapping. Notice that $X$ in this case does not have KK property.

### 4 Applications

#### 4.1 The case of alternate convexically nonexpansive mappings

In [3], Amini-Harandi studied the existence of fixed points for a class of mappings called alternate convexically nonexpansive mappings defined on weakly compact convex subsets of strictly convex Banach spaces. To prove his result, Amini-Harandi used the existence of an afps (almost fixed point sequences) for such mappings and some properties of minimal sets associated with them (see p. 52 in [2]). In [14], P. Dowling simplified Harandi’s result without using any classical tool linked to the nonexpansive case and he observed that alternate convexically nonexpansive mappings form a subclass of that of nonexpansive mappings with center 0.

**Definition 4.1** Let $K$ be a nonempty subset of a Banach space $X$. A self-mapping $T : K \rightarrow K$ is called alternate convexically nonexpansive if

$$\| \sum_{i=1}^{n} \frac{(-1)^{i+1}}{n}Tx_i - Ty \| \leq \| \sum_{i=1}^{n} \frac{(-1)^{i+1}}{n}x_i - y \|$$

for all $n \in \mathbb{N}$ and $x_i, y \in K$.

Let us state Amini-Harandi fixed point theorem (see [3]).

**Theorem 4.1** Let $K$ be a weakly compact convex subset of a strictly convex Banach space and let $T : K \rightarrow K$ be an alternate convexically nonexpansive mapping. Then $T$ has at least a fixed point.

**Remark 4.1** To see that every alternate convexically nonexpansive mapping is nonexpansive with center 0, it suffices to take $n = 2, x_1, x_2 \in K$ with $x_1 = x_2$. So, the case of alternate convexically nonexpansive mappings becomes a particular case of the setting of nonexpansive mappings with center 0. As a consequence, fixed point results associated with nonexpansive mappings with center 0 hold also for alternate convexically nonexpansive mappings (see [14, 16]).
Furthermore, the same author in [14] introduced the following weakening of the alternate convexically nonexpansiveness property.

**Definition 4.2** Let $K$ be a nonempty subset of a Banach space $X$. A self-mapping $T: K \rightarrow K$ is called $k$-alternate convexically nonexpansive if

$$\|\sum_{i=1}^{n} \frac{(-1)^{i+1}}{n} T x_i - T y\| \leq \|\sum_{i=1}^{n} \frac{(-1)^{i+1}}{n} x_i - y\|$$

for all $1 \leq n \leq k$ and $x_i, y \in K$.

It was observed (see Remark 3 in [14]) that if $T$ is $k$-alternate convexically nonexpansive then $T$ is nonexpansive with center 0. In addition, in the same paper it was proved that Alspach transformation (see Remark 3.9) is an example of 1-alternate convexically nonexpansive that is not 2-alternate convexically nonexpansive.

From Corollary 3.2, we can establish the following fixed point result concerning 2-alternate convexically nonexpansive mappings.

**Corollary 4.1** Let $K$ be a closed convex subset of a reflexive strictly convex Banach space and let $T: K \rightarrow K$ be a 2-alternate convexically nonexpansive mapping. Then $T$ has at least a fixed point.

**Remark 4.2** Corollary 4.1 extend Theorem 3 of [14] to the case of unbounded closed convex subsets of reflexive strictly convex Banach spaces.

### 4.2 The linear isometries $l_g$ and $r_g$

We start this section by investigating some exotic situations associated with linear isometries acting on $L_p(G), 1 < p < \infty$ where $G$ is a locally compact group.

Let $G$ be a locally compact group with a left Haar measure $v$ and modular function $\Delta$ defined by

$$\Delta(g) \int_G k(xg) dv(x) = \int_G k(x) dv(x)$$

for $k \in C_0^\infty(G)$ the space of continuous functions $k$ vanishing off compact subsets of $G$. The left and the right translations in $L_p(G), 1 < p < \infty$ by $g \in G$ are given respectively by

$$l_g f(x) = f(gx) \text{ and } (r_g f)(x) = \Delta^{\frac{1}{p}}(g) f(xg)$$

for all $x \in G$. These mappings satisfy $l_g l_{g_2} = l_{g_2 g_1}$ and $r_{g_1} r_{g_2} = r_{g_1 g_2}$ for all $g_1, g_2 \in G$. Furthermore each $l_g$ and $r_g$ is a linear isometry.

A subset $C_0 \subset L_p(G)$ is called left (resp. right) invariant if $l_a(C_0) \subset C_0$ (resp. $r_a(C_0) \subset C_0$) for each $a \in G$. 
In [23], A. T-M. Lau studied closed convex left or right invariant subsets of $L_p(G)$. He proved in particular that if $G$ is a locally compact noncompact group then every closed convex left invariant subset $C_0$ of $L_p(G)$ must contain $0$. In addition, if $C_0$ is assumed to be compact convex, then $C_0$ is reduced to the singleton $\{0\}$.

If $K$ is a nonempty closed convex subset of $L_p(G)(1 < p < \infty)$ which is invariant by every $l_g$ (resp. $r_g$) $(g \in G)$, we denote by $F(l_g), g \in G$ (resp. $F(r_g), g \in G$) the set of fixed points of $l_g$ (resp. $r_g$) in $K$.

First of all, we remark that the family of mappings $(l_g, g \in G)$ (resp. $r_g, g \in G$) are not commuting in general. But since for $1 < p < \infty$, $L_p(G)$ is strictly convex (see page 293, Corollary 20.14 in [21]) and the fact that $\|l_g(x)\| = \|x\|$ for all $x \in L_p(G)$ and $g \in G$ then we deduce that if $K$ is a closed convex invariant subset of $L_p(G)(1 < p < \infty)$ then each $l_g$ $(g \in G)$ is nonexpansive with center $0$. By using Corollary 3.2, we have $l_g(x_0) = x_0$ for all $g \in G$, so if

$$x_0 \in \bigcap_{g \in G} F(l_g)$$

and $x_0 \neq 0$, then by Lau’s result indicated above, necessarily $G$ is a compact group which is a contradiction. Thus, we can derive the following.

**Corollary 4.2** Let $G$ be a locally compact group. Assume that $G$ is noncompact and there exists a closed convex subset of $L_p(G), 1 < p < \infty$ which is invariant by each $l_g, g \in G$. Then

$$\bigcap_{g \in G} F(l_g) = \{0\}.$$ 

In the next result, we give another characterization of the noncompactness of a locally compact group $G$ by means of orbits associated with the mappings $l_g$ or $r_g$. We will restrict our proof to the case of $l_g(g \in G)$ mappings.

**Corollary 4.3** Let $G$ be a locally compact group. Assume that $K$ is an arbitrary nonempty weakly compact convex subset of $L_p(G), 1 < p < \infty$ which is $l_g$-invariant for all $g \in G$. Then the following assertions are equivalent:

(i) $G$ is noncompact;

(ii) For all fixed $h \in K$, we have $0 \in \overline{\text{co}}\{l_g h : g \in G\}$.

**Proof.**

(i) $\implies$ (ii) Assume that there exists $h_0 \in K$ such that $0 \notin \overline{\text{co}}\{l_g h_0 : g \in G\}$. Since each $l_a (a \in G)$ is continuous and affine (since it is linear), $\overline{\text{co}}\{l_g h_0 : g \in G\}$ is a closed convex subset which is invariant by each $l_a, a \in G$. This fact contradicts Lau’s result.

(ii) $\implies$ (i) Assume that $G$ is compact, then if we take $f_0 = 1$ (that is $f_0(x) = 1, \forall x \in G$), then $f \in L_p(G), 1 < p < \infty$, we have $\overline{\text{co}}\{l_g f_0 : g \in G\} = \{f_0\}$ which does not contain the origin. By taking $K = \{f_0\}$, this is a contradiction.
Remark 4.3 Recall that the paper [26] is an important investigation on the existence of fixed points for isometries defined on weakly compact convex subsets of Banach spaces. Indeed, in the indicated paper, the authors proved that isometries which are defined on bounded closed convex subsets of uniformly convex Banach spaces have the Chebychev center as a common fixed point. It is easily seen that in our setting related to the isometries $l_g$ or $r_g \ (g \in G)$ this Chebychev center is reduced to the set $\{0\}$.

4.3 The case of a nonlinear transport equation

Here, using our results in Section 1, we will investigate the existence of a solution for the following boundary problem

$$
\lambda \varphi (x, v) + v. \nabla_x \varphi (x, v) + \sigma (v) \varphi (x, v) = \int_V h(x, v, v') f(x, v', \varphi (x, v')) d\mu (v')
$$

(4.1)

$$
\varphi_- = H(\varphi_+)
$$

where $\lambda \in \mathbb{R}$, $f(., ., .)$ is a measurable nonlinear function of $\varphi$ and $h(., ., .)$ is a measurable function from $D \times V \times V$ to $\mathbb{R}$ where $D$ is a smooth open subset of $\mathbb{R}^n$ that represents the domain of positions and $V$ is the support of the Radon measure $\mu$ on $\mathbb{R}^n$ with $\mu(\{0\}) = 0$. Recall that $V$ is the velocities space. The unknown function $\varphi (x, v)$ is the number (or probability) density of gas particles having the position $x$ and the velocity $v$. The homogeneous function $\sigma(.)$ is called the collision frequency. The boundary conditions are modeled by

$$
\varphi_- = H(\varphi_+)
$$

where $\varphi_-$ (resp. $\varphi_+$) is the restriction of $\varphi$ to $\Gamma_-$ (resp. $\Gamma_+$) which is the incoming (resp. outcoming) part of the phase space boundary and $H$ is a bounded linear operator acting between suitable Lebesgue function spaces on $\Gamma_+$ and $\Gamma_-$, covering in particular the classical boundary conditions (vacuum boundary conditions corresponding to $H = 0$, periodic boundary conditions, reflexive boundary conditions,...)

In our setting, the function $h(., ., .)$ is chosen such that the linear operator

$$
R : L_p(D \times V) \longrightarrow L_p(D \times V) (1 < p < \infty)
$$

$$
\varphi \longrightarrow \int_V h(x, v, v') \varphi (x, v') d\mu (v')
$$

is bounded.

Definition 4.3 A function $g : D \times V \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Caratheodory function if the following condition is satisfied

for all $s \in \mathbb{R}, \ (t, s) \longrightarrow g(t, s)$ is measurable in $D \times V$

$s \longrightarrow g(t, s)$ is continuous on $\mathbb{R}$ a.e. $t \in D \times V$.

Remark 4.4 If $f$ is a Caratheodory function, then we can define the Nemytskii operator $N_f$ by

$$(N_f \varphi)(x, v) = f(x, v, \varphi (x, v))$$
for all \((x, v) \in D \times V\). In addition, if the operator \(N_f\) acts on \(L^p(D \times V)\), then \(N_f\) is continuous and it takes bounded sets into bounded sets.

**Remark 4.5** It is easily seen that if \(N_f\) is a nonexpansive mapping with center 0 on \(L^p(D \times V)\) then \(N_f\) takes every ball \(B((0, r))\) in \(L^p(D \times V)\) into itself (and consequently has 0 as a fixed point in \(B((0, r))\)).

In [25], the author studied the existence of solutions for the nonlinear equation (4.1) by using some compactness results in transport theory which require the boundedness and the convexity of \(D\) together with the regularity of the bounded linear operator \(R\) (that is the compactness on \(L^p(V)\), if the position \(x\) is fixed). So, the solution is derived from Schauder’s fixed point theorem for convenable mappings acting on balls with center 0.

In our main results below, compactness assumptions are not required.

Denote by \(T_H\) the unbounded linear operator defined on \(L^p(D \times V), 1 < p < \infty\) by

\[
T_H \varphi(x, v) = -v\nabla_x \varphi(x, v) - \sigma(v)\varphi(x, v),
\]

\[
\varphi_+ = H(\varphi_-).
\]

Our assumptions denoted by \((\mathcal{H})\) are the following:

- \(D\) is an open smooth subset of \(\mathbb{R}^n\).
- \(R\) is a bounded operator on \(L^p(D \times V)\).
- \(f\) is a Caratheodory function.
- \(N_f\) acts from \(L^p(D \times V)\) into itself \((1 < p < \infty)\).
- There exists \(r_0\) and \(x_0 \in L^p(D \times V)\) such that \(N_f\) is a nonexpansive self-mapping with center 0 on \(\overline{B}(x_0, r_0)\).
- For \(\lambda\) sufficiently large and \(\|H\| < 1\), the mapping \(B_{\lambda} = (\lambda - T_H)^{-1}R\) leaves \(\overline{B}(x_0, r_0)\) invariant.

**Theorem 4.2** Assume that \((\mathcal{H})\) is satisfied. Then there exists \(\lambda_0 > 0\) such that for all \(\lambda > \lambda_0\), the problem (4.1) has at least one solution in \(\overline{B}(x_0, r_0)\).

**Proof.** It is easy to observe that the problem (4.1) has a solution if and only if the mapping

\[
S_{\lambda} = (\lambda - T_H)^{-1}RN_f
\]

has a fixed point. By our assumption \(S_{\lambda}\) leaves \(\overline{B}(x_0, r_0)\) invariant. On the other hand, for all \(\varphi \in \overline{B}(x_0, r_0)\), since \(N_f\) is nonexpansive with center 0, we have

\[
\|S_{\lambda}\varphi\| \leq \|(\lambda - T_H)^{-1}RN_f\varphi\| \leq \|(\lambda - T_H)^{-1}\|\|R\|\|N_f\varphi\| \leq \|(\lambda - T_H)^{-1}\|\|R\|\|\varphi\|
\]

Following Lemma 2.2 in [25], we have

\[
\|\lambda - T_H\|^{-1} \leq \frac{1}{\lambda + \lambda^*}
\]
when \( \lambda^* = - \lim_{|v| \to 0} \sigma(v) \).

Now, since
\[
\lim_{\lambda \to +\infty} \frac{\|R\|}{\lambda + \lambda^*} = 0,
\]
there exists \( \lambda_0 \in \mathbb{R} \) such that for all \( \lambda \in \mathbb{R} \) satisfying \( \lambda \geq \lambda_0 \), we infer that
\[
\frac{\|R\|}{\lambda + \lambda^*} \leq 1
\]
which gives that
\[
\|S_\lambda \varphi\| \leq \|\varphi\|
\]
for all \( \varphi \in \overline{B}(x_0, r_0) \) and \( \lambda \in \mathbb{R} \) satisfying \( \lambda \geq \lambda_0 \). So, \( S_\lambda \) is a nonexpansive mapping with center 0. In addition, since for \( 1 < p < +\infty \), Banach spaces \( L_p(D \times V) \) are reflexive and strictly convex, then \( \overline{B}(x_0, r_0) \) is a closed convex subset of \( L_p(D \times V) \) and the fixed point for \( S_\lambda \) follows from Corollary 3.2.

**Remark 4.6** Theorem 4.2 does not require any compactness results or a specific geometry on the spaces of positions \( D \). In addition, following Corollary 3.2, in conditions (5) and (6) of \( (H) \), we can replace the closed ball \( \overline{B}(x_0, r) \) by any nonempty closed convex subset.

**Remark 4.7** In the case where \( D = \mathbb{R}^3 \) and \( V = \mathbb{R}^3 \) then \( D \times V = \mathbb{R}^3 \times \mathbb{R}^3 \) is a locally compact group. When \( \sigma(.) \equiv 0 \), then if \( H = 0 \), \( T_H \) generates a semigroup of contractions on \( L_p(\mathbb{R}^3 \times \mathbb{R}^3), 1 < p < +\infty \) given by
\[
U(t) \varphi(x, v) = \varphi(x - tv, v) \quad (x, v) \in \mathbb{R}^3 \times \mathbb{R}^3.
\]

It is interesting to investigate weakly compact (resp. compact) convex subsets which are invariant by the flow \( (U(t))_{t \geq 0} \) on \( L_p(\mathbb{R}^3 \times \mathbb{R}^3) \).

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