CONVEX $C^1$ EXTENSIONS OF 1-JETS FROM COMPACT SUBSETS OF HILBERT SPACES

DANIEL AZAGRA AND CARLOS MUDARRA

ABSTRACT. Let $X$ denote a Hilbert space. Given a compact subset $K$ of $X$ and two continuous functions $f : K \to \mathbb{R}$, $G : K \to \mathbb{R}^n$, we show that a necessary and sufficient condition for the existence of a convex function $F \in C^1(X)$ such that $F = f$ on $K$ and $\nabla F = G$ on $K$ is that the 1-jet $(f,G)$ satisfies:

(1) $f(x) \geq f(y) + \langle G(y), x - y \rangle$ for all $x, y \in K$, and
(2) if $x, y \in K$ and $f(x) = f(y) + \langle G(y), x - y \rangle$ then $G(x) = G(y)$.

In [2], among other results, we showed the following.

Theorem 1. If $K$ is a compact subset of $\mathbb{R}^n$ and $f : K \to \mathbb{R}$, $G : K \to \mathbb{R}^n$ are continuous functions, then a necessary and sufficient condition for the existence of a convex function $F \in C^1(\mathbb{R}^n)$ such that $F = f$ on $K$ and $\nabla F = G$ on $K$ is that the 1-jet $(f,G)$ satisfies:

(C) $f(x) \geq f(y) + \langle G(y), x - y \rangle$ for all $x, y \in K$, and
(CW$^1$) if $x, y \in K$ and $f(x) = f(y) + \langle G(y), x - y \rangle$ then $G(x) = G(y)$.

Gilles Godefroy asked whether this statement should remain true if we replace $\mathbb{R}^n$ with a Hilbert space $X$. The purpose of this note is to give an affirmative answer to this question.

We refer to the introductions and the bibliography of [2] for motivation, insight and general reference about this kind of problems. Let us only mention that if one wants to replace $K$ with a closed set in Theorem 1 then it is necessary to introduce more sophisticated conditions, see [3].

Taking into account the difficulties that infinite dimensions add (such as the lack of local compactness and the existence of continuous convex functions which are not bounded on bounded sets), one can expect that even much more complicated conditions would be required to deal with the general case of a 1-jet $(f,G)$ defined on a noncompact closed set $E$ of a Hilbert space $X$. However, for a compact $E \subset X$, the result is as easy as in $\mathbb{R}^n$.

Theorem 2. Let $X$ denote a Hilbert space. Given a compact subset $K$ of $X$ and two continuous functions $f : K \to \mathbb{R}$, $G : K \to X$, a necessary and sufficient condition for the existence of a convex function $F \in C^1(X)$ such that $(F, \nabla F) = (f,G)$ on $K$ is that the 1-jet $(f,G)$ satisfies:

(C) $f(x) \geq f(y) + \langle G(y), x - y \rangle$ for all $x, y \in K$, and
(CW$^1$) if $x, y \in K$ and $f(x) = f(y) + \langle G(y), x - y \rangle$ then $G(x) = G(y)$.

Furthermore, whenever these conditions are satisfied, the extension $F$ can be taken to be Lipschitz, with $\text{Lip}(F) \leq 5 \max_{z \in K} |G(z)|$.

Proof. For each $y \in K$ let us define $\psi_y : X \to \mathbb{R}$ by

$$
\psi_y(x) = \sup_{z \in K} \{f(z) + \langle G(z), x - z \rangle - f(y) - \langle G(y), x - y \rangle \}.
$$

Date: November 8, 2019.

2010 Mathematics Subject Classification. 26B05, 26B25, 52A05, 52A20.

Key words and phrases. convex function, Whitney extension theorem, Hilbert space.
It is clear that \( \psi_y \) is convex and continuous. In fact the function \( X \times K \ni (x, y) \mapsto \psi_y(x) \) is continuous. Also we have

\[
\psi_y(y) = 0 \leq \psi_y(x) \leq C|x - y|
\]

for all \( x \in X \), as \( \psi_y \) is the supremum of a family of \( C \)-Lipschitz functions, where \( C := 2\|G\|_\infty = 2\max_{z \in K} |G(z)| \). Now let us consider the function \( \omega_0 : (0, \infty) \to [0, \infty) \) defined by

\[
\omega_0(t) = \sup \left\{ \frac{\psi_y(x)}{|x - y|} : 0 < |x - y| \leq t, x \in X, y \in K \right\}.
\]

It is obvious that \( \omega_0(s) \leq \omega_0(t) \) for all \( 0 < s < t \), and \( \omega(t) \leq C \) for all \( t \in [0, \infty) \). We also have the following.

**Lemma 3.** \( \lim_{t \to 0^+} \omega_0(t) = 0 \).

**Proof.** Suppose \( \limsup_{t \to 0^+} \omega_0(t) > 0 \). Then there exist \( \varepsilon > 0 \), a sequence of numbers \( (t_n) \searrow 0 \), and two sequences of points \( (y_n) \subset K \) and \((x_n) \subset X\) such that \( x_n \in B(y_n, t_n) \) and

\[
\frac{\psi_{y_n}(x_n)}{|x_n - y_n|} \geq \varepsilon
\]

for all \( n \in \mathbb{N} \). Since the supremum defining \( \psi_{y_n}(x_n) \) is attained there also exists a sequence \( (z_n) \subset K \) such that

\[
\psi_{y_n}(x_n) = f(z_n) + \langle G(z_n), x_n - z_n \rangle - f(y_n) - \langle G(y_n), x_n - y_n \rangle.
\]

By compactness of \( K \) we may assume, up to taking subsequences, that \( (y_n) \) and \( (z_n) \) converge, say

\[
\lim_{n \to \infty} y_n = y_0 \in K, \quad \lim_{n \to \infty} z_n = z_0 \in K.
\]

Thus we also have

\[
\lim_{n \to \infty} x_n = y_0,
\]

and by continuity of \( f, G \) and \((x, y) \mapsto \psi_y(x)\) we obtain

\[
f(z_0) + \langle G(z_0), x_n - z_0 \rangle - f(y_0) = \lim_{n \to \infty} f(z_n) + \langle G(z_n), x_n - z_n \rangle - f(y_n) - \langle G(y_n), x_n - y_n \rangle = \psi_{y_0}(y_0) = 0,
\]

which according to condition \((CW^1)\) implies

\[
G(z_0) = G(y_0).
\]

But then we have

\[
0 < \varepsilon \leq \frac{\psi_{y_n}(x_n)}{|x_n - y_n|} = \frac{f(z_n) + \langle G(z_n), x_n - z_n \rangle - f(y_n) - \langle G(y_n), x_n - y_n \rangle}{|x_n - y_n|} = \frac{f(z_n) + \langle G(z_n), y_n - z_n \rangle - f(y_n) + \langle G(z_n) - G(y_n), x_n - y_n \rangle}{|x_n - y_n|} \leq \frac{\langle G(z_n) - G(y_n), x_n - y_n \rangle}{|x_n - y_n|} \leq |G(z_n) - G(y_n)| \to |G(z_0) - G(y_0)| = 0
\]

as \( n \to \infty \), which is absurd. \( \square \)

Now let us set \( \omega_0(0) = 0 \). If \( \omega_0 : [0, \infty) \to [0, \infty) \) is constantly 0 then \( G \) is constant, and for any \( y_0 \in K \) the function \( F(x) = f(y_0) + \langle G(y_0), x - y_0 \rangle \) has the property that \((F, \nabla F) = (f, g)\) on \( K \).

Therefore we can assume that \( \omega_0 \) is not constant, and define \( \omega_1 : [0, \infty) \to [0, \infty) \) by

\[
\omega_1(t) = \inf \{g(t) \mid g : [0, \infty) \to \mathbb{R} \text{ is concave and } g \geq \omega_0\}
\]
Let us see that \((F, \in F \times h)\). As in [1, Lemma 4.14] it is not difficult to check that
\[
g(x) = \int_0^t \omega_1(s)ds, \quad \text{and} \quad \varphi_2(t) = \int_0^t \omega_2(s)ds, \quad t \in [0, \infty).
\]
The functions \(\varphi_1, \varphi_2\) are convex and \(C^1\), with uniformly continuous derivatives, and satisfy \(\varphi_1 \leq \varphi_2\) on \([0, \infty)\), \(\varphi_1(0) = \varphi_2(0) = 0\), and \(\varphi_1 = \varphi_2\) on \([0, t_1]\). According to [1 Lemma 4.6], for each \(y \in K\), the functions
\[
X \ni x \mapsto \Phi_y(x) = \varphi_2(|x - y|)
\]
are of class \(C^{1, \omega_2}(X)\), with
\[
|\nabla \Phi_y(x) - \nabla \Phi_y(z)| \leq 8\omega_2(|x - z|)
\]
for all \(x, z \in X\). On the other hand, since \(|\cdot|\) is of class \(C^{1, \omega_1}\) on \(X \setminus B(0, t_1/2)\), we have that the functions
\[
X \ni x \mapsto \varphi_y(x) = \varphi_1(|x - y|)
\]
are of class \(C^{1, \omega_1}(X \setminus B(0, t_1/2))\), with
\[
|\nabla \varphi_y(x) - \nabla \varphi_y(z)| \leq M\omega_1(|x - z|)
\]
for all \(x, z \in X \setminus B(y, t_1/2)\) and each \(y \in K\), where \(M > 0\) is a constant independent of \(y\). Since \(\varphi_1 = \varphi_2\) and \(\omega_1 = \omega_2\) on \([0, t_1]\), it follows that the functions \(\varphi_y : X \to [0, \infty)\) are of class \(C^{1, \omega_1}(X)\), with
\[
|\nabla \varphi_y(x) - \nabla \varphi_y(z)| \leq M\omega_1(|x - z|)
\]
for all \(x, z \in X\), where \(M\) is a constant independent of \(y \in K\).

Now consider the functions \(g : X \to \mathbb{R}\) defined by
\[
g(x) = \inf_{y \in K} \{f(y) + \langle G(y), x - y \rangle + 2\varphi_y(x)\},
\]
and \(F = \text{conv}(g)\) (the convex envelope of \(g\), that is to say, the largest convex function which is less than or equal to \(g\)). As in [1 Lemma 4.14] it is not difficult to check that
\[
g(x + h) + g(x - h) - 2g(x) \leq 2\varphi_1(2|h|)
\]
for all \(x, h \in X\), which implies, as in [1 Theorem 2.3], that
\[
F(x + h) + F(x - h) - 2F(x) \leq 2\varphi_1(2|h|)
\]
for all \(x, h \in X\). Since \(F\) is convex this inequality implies that \(F \in C^1(X)\) (in fact, we have that \(F \in C^{1, \omega_1}(X)\), although we do not need this).

Let us see that \((F, \nabla F) = (f, G)\) on \(K\). We first observe that, by concavity of \(\omega_1\), we have
\[
\frac{1}{2}\omega_1(t) \leq \int_0^t \omega_1(s)ds = \varphi_1(t),
\]
hence
\[
t\omega_0(t) \leq t\omega_1(t) \leq 2\varphi_1(t).
\]
Therefore, setting
\[ m(x) := \sup_{z \in K} \{ f(z) + \langle G(z), x - z \rangle \} \]
(the minimal extension of the jet \((f, G)\)) we have
\[
f(y) + \langle G(y), x - y \rangle + 2 \varphi(y)(x) = f(y) + \langle G(y), x - y \rangle + 2 \varphi_1(|x - y|) \geq f(y) + \langle G(y), x - y \rangle + \psi_y(x) = m(x),
\]
hence
\[ m(x) \leq g(x) \]
for all \( x \in X \), and since \( m \) is convex this implies that
\[ m \leq f \leq g \quad \text{on } X. \]

But we also have
\[ f \leq m \leq g \leq f \quad \text{on } K. \]

Therefore \( F = f \) on \( K \). On the other hand, since \( m \leq F \) on \( X \) and \( F = m \) on \( K \), where \( m \) is convex and \( F \) is differentiable on \( X \), we deduce that \( m \) is differentiable on \( E \) with \( \nabla m(x) = \nabla F(x) \) for all \( x \in K \). But it is clear, by definition of \( m \), that \( G(x) \in \partial m(x) \) (the subdifferential of \( m \) at \( x \)) for every \( x \in K \), so we must have \( \nabla F(x) = G(x) \) for every \( x \in K \).

Finally let us see that \( F \) is \( 5\|G\|_\infty \)-Lipschitz. It is clear that \( \psi_y \) is \( 2C \)-Lipschitz for all \( y \in K \), and this implies that \( g \) is \( 5\|G\|_\infty \)-Lipschitz. Besides, we have that
\[
F(x) = \text{conv}(g)(x) = \inf \left\{ \sum_{j=1}^{n} \lambda_j g(x_j) : \lambda_j \geq 0, \sum_{j=1}^{n} \lambda_j = 1, x = \sum_{j=1}^{n} \lambda_j x_j, n \in \mathbb{N} \right\}.
\]
Then, given \( x, h \in X \) and \( \varepsilon > 0 \), we can pick \( n \in \mathbb{N} \), \( x_1, \ldots, x_n \in X \) and \( \lambda_1, \ldots, \lambda_n > 0 \) such that
\[
F(x) \geq \sum_{i=1}^{n} \lambda_i g(x_i) - \varepsilon, \quad \sum_{i=1}^{n} \lambda_i = 1 \quad \text{and} \quad \sum_{i=1}^{n} \lambda_i x_i = x.
\]
Because \( x + h = \sum_{i=1}^{n} \lambda_i (x_i + h) \), we have \( F(x + h) \leq \sum_{i=1}^{n} \lambda_i g(x_i + h) \), which leads us to
\[
F(x + h) - F(x) \leq \sum_{i=1}^{n} \lambda_i (g(x_i + h) - g(x_i)) + \varepsilon \leq 5\|G\|_\infty |h| + \varepsilon,
\]
and since \( \varepsilon > 0 \) is arbitrary, we get \( F(x + h) - F(x) \leq 5\|G\|_\infty |h| \) for all \( x, h \in X \), which means that \( \text{Lip}(F) \leq 5\|G\|_\infty. \)

\[ \square \]

**References**

[1] D. Azagra, E. Le Gruyer and C. Mudarra, *Explicit formulas for \( C^{1,1} \) and \( C^{1,\omega}_e \) extensions of 1-jets in Hilbert and superreflexive spaces*, J. Funct. Anal. 274 (2018), 3003-3032.

[2] D. Azagra and C. Mudarra, *Whitney Extension Theorems for convex functions of the classes \( C^1 \) and \( C^{1,\omega} \)*, Proc. London Math. Soc. 114 (2017), no. 1, 133–158.

[3] D. Azagra and C. Mudarra, *Global geometry and \( C^1 \) convex extensions of 1-jets*, Analysis & PDE, 12 (2019) no. 4, 1065–1099.

ICMAT (CSIC-UAM-UC3-UCM), DEPARTAMENTO DE ANÁLISIS MATEMÁTICO Y MATEMÁTICA APlicada, FACULTAD CIENCIAS MATEMÁTICAS, UNIVERSIDAD COMPLUTENSE, 28040, MADRID, SPAIN.

E-mail address: azagra@imat.ucm.es

AALTO UNIVERSITY, DEPARTMENT OF MATHEMATICS AND SYSTEMS ANALYSIS, P.O. BOX 11100, FI-00076 AALTO, FINLAND

E-mail address: carlos.mudarra@aalto.fi