A Hardy inequality in twisted waveguides

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15 December 2005

Abstract

We show that twisting of an infinite straight three-dimensional tube with non-circular cross-section gives rise to a Hardy-type inequality for the associated Dirichlet Laplacian. As an application we prove certain stability of the spectrum of the Dirichlet Laplacian in locally and mildly bent tubes. Namely, it is known that any local bending, no matter how small, generates eigenvalues below the essential spectrum of the Laplacian in the tubes with arbitrary cross-sections rotated along a reference curve in an appropriate way. In the present paper we show that for any other rotation some critical strength of the bending is needed in order to induce a non-empty discrete spectrum.
1 Introduction

The Dirichlet Laplacian in infinite tubular domains has been intensively studied as a model for the Hamiltonian of a non-relativistic particle in quantum waveguides; we refer to [5, 15, 12] for the physical background and references. Among a variety of results established so far, let us point out the papers [8, 9, 16, 5, 14, 4] where the existence of bound states generated by a local bending of a straight waveguide is proved. This is an interesting phenomenon for several reasons. From the physical point of view, one deals with a geometrically induced effect of purely quantum origin, with important consequences for the transport in curved nanostructures. Mathematically, the tubes represent a class of quasi-cylindrical domains for which the spectral results of this type are non-trivial.

More specifically, it has been proved in the references mentioned above that the Dirichlet Laplacian in non-self-intersecting tubular neighborhoods of the form

\[ \{ x \in \mathbb{R}^d \mid \text{dist}(x, \Gamma) < a \}, \quad d \geq 2, \tag{1} \]

where \( a \) is a positive number and \( \Gamma \) is an infinite curve of non-zero first curvature vanishing at infinity, always possesses discrete eigenvalues. On the other hand, the essential spectrum coincides as a set with the spectrum of the straight tube of radius \( a \). In other words, the spectrum of the Laplacian is unstable under bending. The bound states may be generated also by other local deformations of straight waveguides, e.g., by adding a “bump” [3, 2, 6].

On the other hand, the first two authors of this paper have shown recently in [6] (see also [1]) that a presence of an appropriate local magnetic field in a 2-dimensional waveguide leads to the existence of a Hardy-type inequality for the corresponding Hamiltonian. Consequently, the spectrum of the magnetic Schrödinger operator becomes stable as a set against sufficiently weak perturbations of the type considered above.

In this paper we show that in tubes with non-circular cross-sections the same stability effect can be achieved by a purely geometrical deformation which preserves the shape of the cross-section: twisting. We restrict to \( d = 3 \) and replace the definition (1) by a tube obtained by translating an arbitrary cross-section along a reference curve \( \Gamma \) according to a smooth moving frame of \( \Gamma \) (i.e. the triad of a tangent and two normal vectors perpendicular to each other). We say that the tube is twisted provided (i) the cross-section is not rotationally symmetric (cf. (9) below) and (ii) the projection of the derivative of one normal vector of the moving frame to the other one is not zero. The second condition can be expressed solely in terms of the difference between the second curvature (also called torsion) of \( \Gamma \) and the derivative of the angle between the normal vectors of the chosen moving frame and a Frenet frame of \( \Gamma \) (cf. (12) below); the latter determines certain rotations of the cross-section along the curve. That is, twisting and bending may be viewed as two independent deformations of a straight tube. In order to describe the main results of the paper, we distinguish two particular types of twisting.

First, when \( \Gamma \) is a straight line, then of course the curvatures are zero and the
twisting comes only from rotations of a non-circular cross-section along the line. In this situation, we establish Theorem containing a Hardy-type inequality for the Dirichlet Laplacian in a straight locally twisted tube. Roughly speaking, this tells us that a local twisting stabilizes the transport in straight tubes with non-circular cross-sections.

Second, when is curved, the torsion is in general non-zero and we show that it plays the same role as the twisting due to the rotations of a non-circular cross-section in the twisted straight case. More specifically, we use Theorem to establish Theorem saying that the spectrum of the Dirichlet Laplacian in a twisted, mildly and locally bent tubes coincides with the spectrum of a straight tube, which is purely essential. This fact has important consequences. For it has been proved in [4] that any non-trivial curvature vanishing at infinity generates eigenvalues below the essential spectrum, provided the cross-section is translated along according to the so-called Tang frame (cf. [11] below). We also refer to [11] for analogous results in mildly curved tubes. But the choice of the Tang frame for the moving frame giving rise to the tube means that the rotation of the cross-section compensates the torsion. Our Theorem shows that this special rotation is the only possible choice for which the discrete eigenvalues appear for any non-zero curvature of ; any other rotation of the cross-section will eliminate the discrete eigenvalues if the curvature is not strong enough. In the curved case, we also establish Theorem extending the result of Theorem to the case when also the torsion is mild.

After writing this paper, we discovered that Grushin has in [11] a result similar to our Theorem. Namely, using a perturbation technique developed in [10], he proves that there are no discrete eigenvalues in tubes which are simultaneously mildly curved and mildly twisted. We would like to stress that, apart from the different method we use, the importance of our results lies in the fact that the non-existence of discrete spectrum follows as a consequence of a stronger property: the Hardy-type inequality of Theorem.

The organization of the paper is as follows. In the following Section we present our main results; namely, the Hardy-type inequality (Theorem) and the stability result concerning the spectrum in twisted mildly bent tubes (Theorems and 2). The Hardy-type inequality and its local version (Theorems 3 and 4 respectively) are proved in Section 3. In order to deal with the Laplacian in a twisted bent tube, we have to develop certain geometric preliminaries; this is done in Section 4. Theorems 4 and 2 are proved at the end of Section 4. In the Appendix, we state a sufficient condition which guarantees that a twisted bent tube does not intersect.

The summation convention is adopted throughout the paper and, if not otherwise stated, the range of Latin and Greek indices is assumed to be 1, 2, 3 and 2, 3, respectively. The indices and are reserved for a function and a vector, respectively, and are excluded from the summation convention. If is an open set, we denote by the Dirichlet Laplacian in , i.e. the self-adjoint operator associated in with the quadratic form defined by .

\[ Q_D^2[\psi] := \int_U |\nabla \psi|^2, \psi \in D(Q_D^2) := H^1_0(U). \]
2 Main results

2.1 Twisted bent tubes

The tubes we consider in the present paper are determined by a reference curve $\Gamma$, a cross-section $\omega$ and an angle function $\theta$ determining a moving frame of $\Gamma$. We restrict ourselves to curves characterized by their curvature functions.

Let $\kappa_1$ and $\kappa_2$ be $C^1$-smooth functions on $\mathbb{R}$ satisfying

$$\kappa_1 > 0 \text{ on } I \quad \text{and} \quad \kappa_1, \kappa_2 = 0 \text{ on } \mathbb{R} \setminus I,$$

where $I$ is some fixed bounded open interval. Then there exists a unit-speed $C^3$-smooth curve $\Gamma : \mathbb{R} \to \mathbb{R}^3$ whose first and second curvature functions are $\kappa_1$ and $\kappa_2$, respectively (cf [13, Sec. 1.3]). Moreover, $\Gamma$ is uniquely determined up to congruent transformations and the restriction $\Gamma \upharpoonright I$ possesses a uniquely determined $C^2$-smooth distinguished Frenet frame $\{e_1, e_2, e_3\}$. Since the complement of $\Gamma \upharpoonright I$ is formed by two straight semi-infinite lines, we can extend the triad $\{e_1, e_2, e_3\}$ to a $C^2$-smooth Frenet frame of $\Gamma$. The components $e_1$, $e_2$ and $e_3$ are the tangent, normal and binormal vectors of $\Gamma$, respectively, and $\kappa_2$ is sometimes called the torsion of $\Gamma$.

Given a $C^1_0$-smooth function $\theta$ on $\mathbb{R}$, we define the matrix valued function

$$(\mathcal{R}_\theta^\mu) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}. \quad (3)$$

Then the triad $\{e_1, \mathcal{R}_\theta^\mu e_\mu, \mathcal{R}_\theta^\nu e_\nu\}$ defines a $C^1$-smooth moving frame of $\Gamma$ having normal vectors rotated by the angle $\theta(s)$ with respect to the Frenet frame at $s \in \mathbb{R}$. Later on, a stronger regularity of $\theta$ will be required, namely,

$$\dot{\theta} \in L^\infty(\mathbb{R}). \quad (4)$$

Let $\omega$ be a bounded open connected set in $\mathbb{R}^2$ and introduce the quantity

$$a := \sup_{t \in \omega} |t| . \quad (5)$$

We assume that $\omega$ is not rotationally invariant with respect to the origin, i.e.,

$$\exists \alpha \in (0, 2\pi), \quad \{ (t_\mu \mathcal{R}_\theta^\alpha, t_\mu \mathcal{R}_\theta^\mu e_\mu) \mid (t_2, t_3) \in \omega \} \neq \omega. \quad (6)$$

We define a twisted bent tube $\Omega$ about $\Gamma$ as the image

$$\Omega := \mathcal{L}(\mathbb{R} \times \omega), \quad (7)$$

where $\mathcal{L}$ is the mapping from $\mathbb{R} \times \omega$ to $\mathbb{R}^3$ defined by

$$\mathcal{L}(s, t) := \Gamma(s) + t_\mu \mathcal{R}_\theta^\mu(s) e_\mu(s). \quad (8)$$

We make the natural hypotheses that

$$a \|\kappa_1\|_\infty < 1 \quad \text{and} \quad \mathcal{L} \text{ is injective}, \quad (9)$$

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so that $\Omega$ has indeed the geometrical meaning of a non-self-intersecting tube; sufficient conditions ensuring the injectivity of $L$ are derived in the Appendix.

Our object of interest is the Dirichlet Laplacian in the tube, $-\Delta^D_\Omega$. In the simplest case when the tube is straight (i.e. $I = \emptyset$) and the cross-section $\omega$ is not rotated with respect to a Frenet frame of the reference straight line (i.e. $\dot{\theta} = 0$), it is easy to locate the spectrum:

$$\text{spec}(-\Delta^D_{\mathbb{R}^2 \times \omega}) = [E_1, \infty),$$

where $E_1$ is the lowest eigenvalue of the Dirichlet Laplacian in $\omega$.

Sufficient conditions for the existence of a discrete spectrum of $-\Delta^D_\Omega$ were recently obtained in [4, 11]. In particular, it is known from [4] that if the cross-section $\omega$ is rotated appropriately, namely in such a way that

$$\dot{\theta} = \kappa_2,$$

then any non-trivial bending (i.e. $I \neq \emptyset$) generates eigenvalues below $E_1$, while the essential spectrum is unchanged.

As one of the main results of the present paper we show that condition (11) is necessary for the existence of discrete spectrum in mildly bent tubes with non-circular cross-sections:

**Theorem 1.** Given $C^1_0$-curvature functions (2), a bounded open connected set $\omega \subset \mathbb{R}^2$ satisfying non-symmetricity condition (6) and a $C^1_0$-smooth angle function $\theta$ satisfying (4), let $\Omega$ be the tube as above satisfying (9). If

$$\kappa_2 - \dot{\theta} \neq 0,$$

then there exists a positive number $\varepsilon$ such that

$$\|\kappa_1\|_{\infty} + \|\kappa_1\|_{\infty} \leq \varepsilon \quad \Rightarrow \quad \text{spec}(-\Delta^D_{\Omega}) = [E_1, \infty).$$

Here $\varepsilon$ depends on $\kappa_2$, $\dot{\theta}$ and $\omega$.

An explicit lower bound for the constant $\varepsilon$ is given by the estimates made in Section 4.3 when proving Theorem 1; we also refer to Proposition 1 in the Appendix for a sufficient conditions ensuring the validity of (9).

Theorem 1 tells us that twisting, induced either by torsion or by a rotation different from (11), acts against the attractive interaction induced by bending. Its proof is based on a Hardy-type inequality in straight tubes presented in the following Section 2.2. The latter provides other variants of Theorem 1, e.g., in the situation when also the torsion is mild:

**Theorem 2.** Under the hypotheses of Theorem 1, with (12) being replaced by

$$\dot{\theta} \neq 0,$$

there exists a positive number $\varepsilon$ such that

$$\|\kappa_1\|_{\infty} + \|\kappa_1\|_{\infty} + \|\kappa_2\|_{\infty} \leq \varepsilon \quad \Rightarrow \quad \text{spec}(-\Delta^D_{\Omega}) = [E_1, \infty).$$

Here $\varepsilon$ depends on $\dot{\theta}$, $\omega$ and $I$.

We refer the reader to Section 5 for more comments on Theorems 1 and 2.
2.2 Twisted straight tubes

The proof of Theorems 1 and 2 is based on the fact that a twist of a straight tube leads to a Hardy-type inequality for the corresponding Dirichlet Laplacian. This is the central idea of the present paper which is of independent interest.

By the straight tube we mean the product set $\mathbb{R} \times \omega$. To any radial vector $t \equiv (t_2, t_3) \in \mathbb{R}^2$, we associate the normal vector $\tau(t) := (t_3, -t_2)$, introduce the angular-derivative operator

$$\partial_\tau := t_3 \partial_2 - t_2 \partial_3$$

and use the same symbol for the differential expression $1 \otimes \partial_\tau$ on $\mathbb{R} \times \omega$.

Given a bounded function $\sigma : \mathbb{R} \rightarrow \mathbb{R}$, we denote by the same letter the function $\sigma \otimes 1$ on $\mathbb{R} \times \omega$ and consider the self-adjoint operator $L_\sigma$ associated on $L^2(\mathbb{R} \times \omega)$ with the Dirichlet quadratic form

$$l_\sigma[\psi] := \|\partial_1 \psi - \sigma \partial_\tau \psi\|^2 + \|\partial_2 \psi\|^2 + \|\partial_3 \psi\|^2,$$

(15)

with $\psi \in D(L_\sigma) := \mathcal{H}^1_0(\mathbb{R} \times \omega)$, where $\| \cdot \|$ denotes the norm in $L^2(\mathbb{R} \times \omega)$.

The connection between $L_\sigma$ and a twisted straight tube is based on the fact that for $\sigma = \dot{\theta}$, $L_\sigma$ is unitarily equivalent to the Dirichlet Laplacian acting in a tube given by (7) for the choice $\Gamma(s) = (s, 0, 0)$, after passing to the coordinates determined by (8). This can be verified by a straightforward calculation.

If $\sigma = 0$, $L_0$ is just the Dirichlet Laplacian in $\mathbb{R} \times \omega$, its spectrum is given by (10) and there is no Hardy inequality associated with the shifted operator $L_0 - E_1$. The latter means that given any multiplication operator $V$ generated by a non-zero, non-positive function from $C^\infty_0(\mathbb{R} \times \omega)$, the operator $L_0 - E_1 + V$ has a negative eigenvalue. This is also true for non-trivial $\sigma$ in the case of circular $\omega$ centered in the origin of $\mathbb{R}^2$, since then the angular-derivative term in (15) vanishes for the test functions of the form $\varphi \otimes J_1$ on $\mathbb{R} \times \omega$, where $J_1$ is an eigenfunction of the Dirichlet Laplacian corresponding to $E_1$. However, in all other situations there is always a Hardy-type inequality:

**Theorem 3.** Let $\omega$ be a bounded open connected subset of $\mathbb{R}^2$ satisfying the non-symmetricity condition (7). Let $\sigma$ be a compactly supported continuous function with bounded derivatives and suppose that $\sigma$ is not identically zero. Then, for all $\psi \in \mathcal{H}^1_0(\mathbb{R} \times \omega)$ and any $s_0$ such that $\sigma(s_0) \neq 0$ we have

$$l_\sigma[\psi] - E_1 \|\psi\|^2 \geq c_h \int_{\mathbb{R} \times \omega} \frac{|\psi(s,t)|^2}{1 + (s-s_0)^2} \, ds \, dt,$$

(16)

where $c_h$ is a positive constant independent of $\psi$ but depending on $s_0$, $\sigma$ and $\omega$.

It is possible to find an explicit lower bound for the constant $c_h$; we give an estimate in (25).

The assumption that $\sigma$ has a compact support ensures that the essential spectrum of $L_\sigma$ coincides with (10). As a consequence of the Hardy-type inequality (16), we get that the presence of a non-trivial $\sigma$ in (15) represents a
repulsive interaction in the sense that there is no other spectrum for all small potential-type perturbations having \(O(s^{-2})\) decay at infinity.

As explained above, the special choice \(\sigma = \dot{\theta}\) leads to a direct geometric interpretation of \(L_\sigma\) in connection with the twisted straight tubes. As another application of Theorem 3, we shall apply it to the twisted bent tubes, namely, with the choice \(\sigma = \kappa_2 - \dot{\theta}\) to prove Theorem 1 and with \(\sigma = \dot{\theta}\) to prove Theorem 2 (cf. Section 4.3).

3 Hardy inequality for twisted straight tubes

It follows immediately from (15) and the inequality
\[
\|\nabla \varphi\|_{L^2(\omega)}^2 \geq E_1 \|\varphi\|_{L^2(\omega)}^2 + \|\partial_\tau \varphi\|_{L^2(\omega)}^2,
\]
that the spectrum of \(L_\sigma\) does not start below \(E_1\). In this section, we establish the stronger result contained in Theorem 3 in two steps. After certain preliminaries, we first derive a “local” Hardy inequality (Theorem 4). Then the local result is “smeared out” by means of a classical one-dimensional Hardy inequality.

3.1 Preliminaries

Definition 1. To any \(\omega \in \mathbb{R}^2\), we associate the number
\[
\lambda := \inf \frac{\|\nabla \varphi\|^2_{L^2(\omega)} - E_1 \|\varphi\|^2_{L^2(\omega)} + \|\partial_\tau \varphi\|^2_{L^2(\omega)}}{\|\varphi\|^2_{L^2(\omega)}},
\]
where the infimum is taken over all non-zero functions from \(H^1_0(\omega)\).

It is clear from (17) that \(\lambda\) is a non-negative quantity. Our Hardy inequality is based on the fact that \(\lambda\) is always positive for non-circular cross-sections.

Lemma 1. If \(\omega\) satisfies (7), then \(\lambda > 0\).

Proof. The quadratic form \(b\) defined on \(L^2(\omega)\) by
\[
b(\varphi) := \|\nabla \varphi\|^2_{L^2(\omega)} - E_1 \|\varphi\|^2_{L^2(\omega)} + \|\partial_\tau \varphi\|^2_{L^2(\omega)}, \quad \varphi \in D(b) := H^1_0(\omega),
\]
is non-negative (cf. (17)), densely defined and closed; the last two statements follow from the boundedness of \(\tau\) and from the fact that they hold true for the quadratic form defining the Dirichlet Laplacian in \(\omega\). Consequently, \(b\) gives rise to a self-adjoint operator \(B\). Moreover, since \(B \geq -\Delta^\omega_D - E_1\), the spectrum of \(-\Delta^\omega_D\) is purely discrete, the minimax principle implies that \(B\) has a purely discrete spectrum, too. \(\lambda\) is clearly the lowest eigenvalue of \(B\). Assume that \(\lambda = 0\). Then, firstly, the ground state \(\varphi\) of \(B\) and \(-\Delta^\omega_D\) coincide, hence \(\varphi\) is analytic and positive in \(\omega\); secondly, we have \(\partial_\tau \varphi = 0\). This implies that the angular derivative of \(\varphi\) is zero. Together with our assumption on \(\omega\) we can conclude that there is a point in \(\omega\) where \(\varphi\) vanishes. This contradicts the positivity of \(\varphi\).
Next we need a specific lower bound for the spectrum of the Schrödinger operator in a bounded one-dimensional interval with Neumann boundary conditions and a characteristic function of a subinterval as the potential.

Lemma 2. Let $\Lambda$ be a bounded open interval of $\mathbb{R}$. Then for any open subinterval $\Lambda' \subset \Lambda$ and any $f \in \mathcal{H}^1(\Lambda)$, the following inequality holds:

$$\|f\|_{L^2(\Lambda)}^2 \leq c(\Lambda, \Lambda') \left( \|f\|_{L^2(\Lambda')}^2 + \|f'\|_{L^2(\Lambda)}^2 \right),$$

where $c(\Lambda, \Lambda') := \max \left\{ 2 + 16 \left( \frac{|\Lambda|}{|\Lambda'|} \right)^2, 4 |\Lambda|^2 \right\}$.

Proof. Without loss of generality, we may suppose that $\Lambda' := (-b/2, b/2)$. Define a function $g$ on $\Lambda$ by

$$g(x) := \begin{cases} 2 \frac{|x|}{b} & \text{for } |x| \leq b, \\ 1 & \text{otherwise}. \end{cases}$$

Let $f$ be any function from $\mathcal{H}^1(\Lambda)$. Then $(fg)(0) = 0$ and the Cauchy-Schwarz inequality gives

$$|f(x)g(x)|^2 \leq |x| \int_0^x |(fg)'|^2 \leq |\Lambda| \|fg\|_{L^2(\Lambda)}^2 \quad (18)$$

for any $x \in \Lambda$. Now we write $f = fg + f(1 - g)$ to get

$$\|f\|_{L^2(\Lambda)}^2 \leq 2 \|fg\|_{L^2(\Lambda)}^2 + 2 \|f(1 - g)\|_{L^2(\Lambda)}^2 = 2 \|fg\|_{L^2(\Lambda)}^2 + 2 \|f\|_{L^2(\Lambda')}^2.$$}

Using the estimate $18$ and the fact that $|g'| = 2 |\Lambda'|^{-1}$ on $\Lambda'$, we obtain the statement of the lemma. \qed

### 3.2 A local Hardy inequality

Since $\sigma$ is continuous and has compact support there are closed intervals $A_j$ such that

$$\supp(\sigma) = \bigcup_{j \in K} A_j \quad \text{and} \quad |A_i \cap A_j| = 0, \ i \neq j,$$

where $K \subseteq \mathbb{N}$ is an index set. The main result of this subsection is the following local type of Hardy inequality:

Theorem 4. Let the assumptions of Theorem 3 hold. For every $j \in K$ there is a positive constant $a_j$ depending on $\sigma \upharpoonright A_j$ such that for all $\psi \in \mathcal{H}^1_0(\mathbb{R} \times \omega)$,

$$\int_{A_j \times \omega} \left( |\partial_2 \psi|^2 + |\partial_3 \psi|^2 + |\partial_1 \psi + \sigma \partial_r \psi|^2 - E_1 |\psi|^2 \right) \geq a_j \lambda \int_{A_j \times \omega} |\sigma \psi|^2, \quad (19)$$

where $\lambda$ is the positive constant from Definition 4 depending only on the geometry of $\omega$. 8
To prove Theorem 4, it will be useful to introduce the following quantities:

**Definition 2.** For any $M \subseteq \mathbb{R}$ and $\psi \in \mathcal{H}_0^1(\mathbb{R} \times \omega)$, we define

\[
I_1^M := \|\chi_M \nabla' \psi\|^2 - E_1 \|\chi_M \psi\|^2, \quad I_2^M := \|\chi_M \partial_\tau \psi\|^2, \quad I_3^M := \|\chi_M \sigma \partial_\tau \psi\|^2,
\]

where $\chi_M$ denotes the characteristic function of the set $M \times \omega$, $\nabla'$ denotes the gradient operator in the “transverse” coordinates $(t_2, t_3)$ and $(\cdot, \cdot)$ is the inner product generated by $\| \cdot \|$. 

Note that $I_1^M$ is non-negative due to (17) and that we have

\[
l_\sigma[\psi] - E_1 \|\psi\|^2 = I_1^R + I_2^R + I_3^{\supp(\sigma)} + I_2^{\supp(\sigma)}. \tag{20}
\]

Let $A$ be the union of any (finite or infinite) sub-collection of the intervals $A_j$.

The following lemma enables us to estimate the mixed term $I_{1,3}^A$.

**Lemma 3.** Let the assumptions of Theorem 3 be satisfied. Then for each positive numbers $\alpha$ and $\beta$, there exists a constant $\gamma_{\alpha, \beta}$ depending also on $\sigma \upharpoonright A$ such that for any $\psi \in \mathcal{H}_0^1(\mathbb{R} \times \omega)$,

\[
|I_{1,3}^A| \leq \gamma_{\alpha, \beta} I_1^A + \alpha I_2^B + \beta I_3^A,
\]

where $B := (\inf A, \sup A)$.

**Proof.** It suffices to prove the result for real-valued functions $\psi$ from the dense subspace $C_0^\infty(\mathbb{R} \times \omega)$. We employ the decomposition

\[
\psi(s, t) = J_1(t) \phi(s, t), \quad (s, t) \in \mathbb{R} \times \omega, \tag{21}
\]

where $J_1$ is a positive eigenfunction of the Dirichlet Laplacian on $L^2(\omega)$ corresponding to $E_1$ (we shall denote by the same symbol the function $1 \circ J_1$ on $\mathbb{R} \times \omega$), and $\phi$ is a real-valued function from $C_0^\infty(\mathbb{R} \times \omega)$, actually introduced by (21). Then

\[
I_1^A = \|\chi_A J_1 \nabla \phi\|^2, \quad I_3^A = \|\chi_A \sigma (J_1 \partial_\tau \phi + \phi \partial_\tau J_1)\|^2, \\
I_2^A = \|\chi_A J_1 \partial_\sigma \phi\|^2, \quad I_{1,3}^A = -2 (J_1 \partial_\sigma \phi, \chi_A \sigma (J_1 \partial_\tau \phi + \phi \partial_\tau J_1)),
\]

where we have integrated by parts to establish the identity for $I_1^A$. Using

\[
|\sigma \partial_\tau \phi|^2 \leq c_1 |\nabla' \phi|^2, \quad \text{with} \quad c_1 := \|[\sigma \upharpoonright A]\|^2_\infty a^2,
\]

and applying the Cauchy-Schwarz inequality and the Cauchy inequality with $\alpha > 0$, the first term in the sum of $I_{1,3}^A$ can be estimated as follows:

\[
|2 (J_1 \partial_\sigma \phi, \chi_A \sigma J_1 \partial_\tau \phi)| \leq 2 \sqrt{c_1} \sqrt{I_1^A} \sqrt{I_2^A} \leq \frac{2 c_1}{\alpha} I_1^A + \frac{\alpha}{2} I_2^A. \tag{22}
\]
In order to estimate the second term, we first combine integrations by parts to get
\[ |2(\mathcal{J}_1 \partial_t \phi, \chi A \sigma \phi \partial_r \mathcal{J}_1) = |(\phi, \chi A \dot{\mathcal{J}}_1^2 \partial_r \phi) |. \]
Using
\[ |\dot{\mathcal{J}}_1 \partial_r \phi| \leq c_2 |\nabla' \phi|^2, \quad \text{with} \quad c_2 := \|\dot{\mathcal{J}}_1\|= a^2, \]
and the Cauchy-Schwarz inequality, we have
\[ |(\phi, \chi A \dot{\mathcal{J}}_1^2 \partial_r \phi)|^2 \leq c_2 I_1^A \|\chi A \mathcal{J}_1 \phi\|^2. \]
Obviously, we can find an open interval \( A' \subset A \) such that there exists a certain positive number \( \sigma_0 \), for which
\[ \sigma(s) \geq \sigma_0, \quad \forall s \in A'. \]
Lemma 2 tells us that
\[ \|\chi A \mathcal{J}_1 \phi\|^2 \leq \|\chi B \mathcal{J}_1 \phi\|^2 \leq c(B, A') (I_2^B + \|\chi A' \mathcal{J}_1 \phi\|^2) \leq c(B, A') (I_2^B + \sigma_0^{-2} \|\chi A' \mathcal{J}_1 \phi\|^2). \]
Moreover, for each fixed value of \( s \in \mathbb{R} \) we have \( \sigma(s) \mathcal{J}_1 \phi(s, \cdot) \in \mathcal{H}_0^1(\omega) \), and therefore we can apply Lemma 1 to obtain
\[ \|\chi A' \mathcal{J}_1 \phi\|^2 \leq \|\chi A \mathcal{J}_1 \phi\|^2 \leq \lambda^{-1} (I_3^A + \|\sigma\|_\infty^2 I_1^A). \]
Writing \( c_3 := c_2 c(B, A') \lambda^{-1} \sigma_0^{-2} \), we conclude that
\[ |(\phi, \chi A' \dot{\mathcal{J}}_1^2 \partial_r \phi)|^2 \leq c_3 I_1^A (\|\sigma\|_\infty^2 I_1^A + \lambda \sigma_0^2 I_2^B + I_3^A) \leq \left( \frac{\gamma_{\alpha, \beta} I_1^A}{2} + \beta I_3^A \right)^2 \]
for any \( \beta > 0 \) and \( \gamma_{\alpha, \beta} := \max\{\sqrt{c_3} \|\sigma\|_\infty, c_3(2\beta)^{-1}, c_3 \lambda \sigma_0^2 \alpha^{-1}\} \). Finally, combining (22) with (23), the estimate for \( |I_{2,3}^A| \) follows by setting \( \gamma_{\alpha, \beta} := \gamma_{\alpha, \beta} + 2 c_1 \alpha^{-1}. \)

Now we are in a position to establish Theorem 1.

Proof of Theorem 4: We take \( A = A_j, \alpha = 1, \beta < 1 \) and keep in mind that \( \gamma_{1, \beta} \) in Lemma 3 depends on \( j \). We define \( \gamma(\beta, j) := \max\{1/2, \gamma_{1, \beta}\} \). Lemma 3 then gives
\[
\begin{align*}
\int_{A_j \times \omega} (&|\nabla' \psi|^2 + |\partial_t \psi + \sigma \partial_r \psi|^2 - E_1 |\psi|^2) \\
\geq & \frac{1}{2} I_1^{A_j} + \left( 1 - \frac{1}{2\gamma(\beta, j)} \right) (I_2^{A_j} + I_3^{A_j} - |I_{2,3}^{A_j}|) + \frac{1 - \beta}{2\gamma(\beta, j)} I_3^{A_j},
\end{align*}
\]
Since \( I_2^{A_j} + I_3^{A_j} - |I_3^{A_j}| \geq 0 \), we get from Lemma 1 that

\[
\int_{A_j \times \omega} (|\nabla'\psi|^2 + |\partial_1 \psi + \sigma \partial_\tau \psi|^2 - E_1|\psi|^2) \geq a_j (\|\sigma \upharpoonright A_j\|_\infty^2 I_1^{A_j} + I_3^{A_j}) \geq a_j \lambda \int_{A_j \times \omega} |\sigma \psi|^2,
\]

where

\[
a_j = \frac{1}{2} \min \left\{ \frac{1}{\|\sigma \upharpoonright A_j\|_\infty^2}, \frac{1 - \beta}{\gamma(\beta, j)} \right\}.
\]

\[\square\]

**Remark 1.** Note that the Hardy weight on the right hand side of (19) cannot be made arbitrarily large by increasing \( \sigma \), since the constant \( a_j \) is proportional to \( \|\sigma \upharpoonright A_j\|_\infty^2 \) if the latter is large enough. We want to point out that this degree of decay of \( a_j \) is optimal if the axes of rotation intersects \( \omega \). Assume there exists an \( \alpha < 2 \), such that \( a_j \) is proportional to \( \|\sigma \upharpoonright A_j\|_\infty^\alpha \) when \( \|\sigma \upharpoonright A_j\|_\infty \to \infty \).

Consider a test function \( \psi \) of the form \( \psi(s, t) = g(s)f(t) \), where \( g \in H^1(\mathbb{R}) \) is supported inside \( A_j \) and \( f \in H^1_0(\omega) \) is radially symmetric with respect to the intersection of \( \omega \) with the axes of rotation. Then \( \partial_\tau \psi = 0 \) on \( A_j \times \omega \) and therefore the left hand side of (19) is for this test function independent of \( \sigma \). Take \( \sigma = n \tilde{\sigma} \) with \( \tilde{\sigma} \) being a fixed function. The right hand side of (19) then tends to infinity as \( n \to \infty \) which contradicts the inequality.

### 3.3 Proof of Theorem 3

For applications, it is convenient to replace the Hardy inequality of Theorem 4 with a compactly supported Hardy weight by a global one. To do so, we recall the following version of the one-dimensional Hardy inequality:

\[
\int_{\mathbb{R}} \frac{|v(x)|^2}{x^2} \, dx \leq 4 \int_{\mathbb{R}} |v'(x)|^2 \, dx \tag{24}
\]

for all \( v \in C_0^\infty(\mathbb{R}) \) with \( v(0) = 0 \). Inequality (24) extends by continuity to all \( v \in H^1(\mathbb{R}) \) with \( v(0) = 0 \).

Without loss of generality we can assume that \( s_0 = 0 \). Let \( J = [-b, b] \), with some positive number \( b \), be an interval where \( |\sigma| > 0 \). Let \( \tilde{f} : \mathbb{R} \to \mathbb{R} \) be defined by

\[
\tilde{f}(s) := \begin{cases} 
1 & \text{for } |s| \geq b, \\
|s|/b & \text{for } |s| < b,
\end{cases}
\]

and put \( f := \tilde{f} \otimes 1 \) on \( \mathbb{R} \times \omega \). For any \( \psi \in C_0^\infty(\mathbb{R} \times \omega) \), let us write \( \psi = f\psi + (1 - f)\psi \). Applying (24) to the function \( s \mapsto (f\psi)(s, t) \) with \( t \) fixed, we
arrive at
\[
\int_{\mathbb{R} \times \omega} |\psi(s,t)|^2 \, ds \, dt \leq 2 \int_{\mathbb{R} \times \omega} \frac{|\tilde{f}(s)\psi(s,t)|^2}{s^2} \, ds \, dt + 2 \int_{J \times \omega} |(1-f)\psi|^2 \\
\leq 16 \, ((\partial_1 f)\psi)^2 + 16 \, f \partial_1 \psi \|^2 + 2 \, \chi_J (1-f)\psi \|^2 \\
\leq \left( \frac{16}{b^2} + 2 \right) \|\chi_J \psi\|^2 + 16 \, \|\partial_1 \psi\|^2,
\]
where \( \chi_J \) denotes the characteristic function of the set \( J \times \omega \). Theorem 4 then implies that there exists a positive constant \( c_0 \) depending on \( \sigma \) such that
\[
\|\chi_J \psi\|^2 \leq (c_0 \lambda \min_{J} |\sigma|)^{-1} (Q_0[\psi] - E_1 \|\psi\|^2).
\]
To estimate the second term we let \( A = \text{supp} (\sigma) \) and rewrite the inequality of Lemma 3 for \( \beta = 1 \) as
\[
\gamma^{-1} |I^A_{2,3}| \leq I^A_1 + \alpha \gamma^{-1} I^B_2 + \gamma^{-1} I^A_3,
\]
where \( \gamma = \max\{1, \gamma_0, 1\} \) and \( \alpha \in (0,1) \). Substituting this inequality into (20), writing \( I^A_{2,3} = \gamma^{-1} I^A_{2,3} + (1-\gamma^{-1}) I^A_{2,3} \) and employing \( I^A_2 + I^A_3 + I^A_{2,3} \geq 0 \), we obtain
\[
I^B_2 = \|\chi_B \partial_1 \psi\|^2 \leq \gamma \alpha (1-\alpha)^{-1} (Q_0[\psi] - E_1 \|\psi\|^2).
\]
On the complement of \( B \times \omega \) we have a trivial estimate
\[
\|\chi_{\mathbb{R} \setminus B} \partial_1 \psi\|^2 \leq Q_0[\psi] - E_1 \|\psi\|^2.
\]
Summing up, the density of \( C^\infty_0(\mathbb{R} \times \omega) \) in \( H^1_0(\mathbb{R} \times \omega) \) implies Theorem 4 with
\[
c_h \geq \left[ \frac{16}{b^2} + 2b^2 c_0 \lambda \min_{J} |\sigma| \right]^{-1} (\frac{\gamma_0}{1-\alpha} + 1).
\]

4 Twisted bent tubes

Here we develop a geometric background to study the Laplacian in bent and twisted tubes, and transform the former into a unitarily equivalent Schrödinger-type operator in a straight tube. At the end of this section, we also perform proofs of Theorems 1 and 2 using Theorem 3. We refer to Section 2.1 for definitions of basic geometric objects used throughout the paper.

While we are mainly interested in the curves determined by curvature functions of type 2, we stress that the formulae of Sections 3.1 and 3.2 are valid for arbitrary curves (it is only important to assume the existence of an appropriate Frenet frame for the reference curve of the tube, cf [4]).
4.1 Metric tensor

Assuming (9) and using the inverse function theorem, we see that the mapping \( L \) introduced in (8) induces a \( C^1 \)-smooth diffeomorphism between the straight tube \( \mathbb{R} \times \omega \) and the image \( \Omega \). This enables us to identify \( \Omega \) with the Riemannian manifold \( (\mathbb{R} \times \omega, G_{ij}) \), where \( (G_{ij}) \) is the metric tensor induced by the embedding \( L \), i.e.

\[
G_{ij} := (\partial_i L) \cdot (\partial_j L),
\]

with the dot being the scalar product in \( \mathbb{R}^3 \).

Recall the Serret-Frenet formulae (cf. [13, Sec. 1.3])

\[
\dot{e}_i = K_{ij} e_j, \quad i \in \{1, 2, 3\},
\]

where the matrix-valued function \( (K_{ij}) \) has the skew-symmetric form

\[
(K_{ij}) = \begin{pmatrix}
0 & \kappa_1 & 0 \\
-\kappa_1 & 0 & \kappa_2 \\
0 & -\kappa_2 & 0
\end{pmatrix}.
\]

Using (26) and the orthogonality conditions \( R^n_{\mu\nu} R^n_{\rho\nu} = \delta_{\mu\nu} \), we find

\[
(G_{ij}) = \begin{pmatrix}
h^2 + h_\mu h_\mu & h_2 & h_3 \\
h_2 & 1 & 0 \\
h_3 & 0 & 1
\end{pmatrix},
\]

where

\[
h(s, t) := 1 - \left[t_2 \cos \theta(s) + t_3 \sin \theta(s)\right] \kappa_1(s),
\]

\[
h_2(s, t) := -t_3 \left[\kappa_2(s) - \dot{\theta}(s)\right],
\]

\[
h_3(s, t) := t_2 \left[\kappa_2(s) - \dot{\theta}(s)\right].
\]

Furthermore,

\[
G := \det(G_{ij}) = h^2,
\]

which defines the volume element of \( (\mathbb{R} \times \omega, G_{ij}) \) by setting

\[
d\text{vol} := h(s, t) \, ds \, dt.
\]

Here and in the sequel \( dt \equiv dt_2 \, dt_3 \) denotes the 2-dimensional Lebesgue measure in \( \omega \).

The metric is uniformly bounded and elliptic in view of the first of the assumptions in (9); in particular, (5) yields

\[
0 < 1 - a \| \kappa_1 \|_\infty \leq h \leq 1 + a \| \kappa_1 \|_\infty < \infty.
\]

It can be directly checked that the inverse \( (G^{ij}) \) of the metric tensor (28) is given by

\[
(G^{ij}) = \frac{1}{h^2} \begin{pmatrix}
1 & -h_2 & -h_3 \\
-h_2 & h^2 + h_2^2 & h_2 h_3 \\
-h_3 & h_2 h_3 & h^2 + h_3^2
\end{pmatrix}.
\]
It is worth noticing that one has the decomposition
\[(G^{ij}) = \text{diag}(0, 1, 1) + (S^{ij}),\]  
(31)
where the matrix \((S^{ij})\) is positive semi-definite.

4.2 The Laplacian
Recalling the diffeomorphism between \(\mathbb{R} \times \omega\) and \(\Omega\) given by \(L\), we identify the Hilbert space \(L^2(\Omega)\) with \(L^2(\mathbb{R} \times \omega, d\text{vol})\). Furthermore, the Dirichlet Laplacian \(-\Delta^D\) is unitarily equivalent to the self-adjoint operator \(\tilde{Q}\) associated on \(L^2(\mathbb{R} \times \omega, d\text{vol})\) with the quadratic form
\[
\tilde{q}[\psi] := \int_{\mathbb{R} \times \omega} (\partial_i \psi, G^{ij} \partial_j \psi) \, d\text{vol}, \quad \psi \in D(\tilde{q}) := \mathcal{H}_0^1(\mathbb{R} \times \omega)\,.
\]  
(32)
We can write \(\tilde{Q} = -G^{-1/2} \partial_i G^{1/2} G^{ij} \partial_j\) in the form sense, which is a general expression for the Laplace-Beltrami operator in a manifold equipped with a metric \((G_{ij})\).

Now we transform \(\tilde{Q}\) into a unitarily equivalent operator \(Q\) acting in the Hilbert space \(L^2(\mathbb{R} \times \omega)\), without the additional weight \(G^{1/2}\) in the measure of integration. This is achieved by means of the unitary operator
\[
U : L^2(\mathbb{R} \times \omega, d\text{vol}) \to L^2(\mathbb{R} \times \omega) : \{\psi \mapsto G^{1/4} \psi\}.
\]
Defining \(Q := U \tilde{Q} U^{-1}\), it is clear that \(Q\) is the operator associated with the quadratic form
\[
q[\psi] := \tilde{q}[G^{-1/4} \psi], \quad \psi \in D(q) := \mathcal{H}_0^1(\mathbb{R} \times \omega).
\]
It is straightforward to check that
\[
q[\psi] = (\partial_i \psi, G^{ij} \partial_j \psi) + (\psi, (\partial_i F) G^{ij} (\partial_j F) \psi) + 2 \Re (\partial_i \psi, G^{ij} (\partial_j F) \psi), \quad (33)
\]
where
\[
F := \log(G^{1/4})\,.
\]

4.3 Proof of Theorems 1 and 2
Let us first prove Theorem 1. Putting \(\sigma := \kappa_2 - \dot{\theta}\), we observe that \(l_\sigma\) is equal to \(q\) after letting \(k := ||\kappa_1||_\infty + ||\kappa_1||_\infty\) equal to zero in the latter form. Hence, the proof of Theorem 1 reduces to a comparison of these quadratic forms and the usage of Theorem 3. Let \((G_{ij}^{30})\) be the matrix \((30)\) after letting \(\kappa_1 = 0\), i.e. with \(h\) being replaced by 1 while \(h_2\) and \(h_3\) being unchanged; then \(l_\sigma[\psi] = (\partial_i \psi, G_{ij}^{30} \partial_j \psi)\).

We strengthen the first of the hypotheses \(4\) to
\[
||\kappa_1||_\infty \leq 1/(2a),
\]
so that we have a uniform positive lower bound to \( h \), namely \( h \geq 1/2 \) due to (24). It is straightforward to check that we have on \( \mathbb{R} \times \omega \) the following pointwise inequalities:

\[
\max_{i,j \in \{1,2,3\}} |G^{ij} - G_0^{ij}| \leq C_1 k \chi_I,
\]

\[
\max_{i \in \{1,2,3\}} |\partial_i F| \leq C_2 k \chi_I,
\]

where \( \chi_I \) denotes the characteristic function of the set \( I \times \omega \) and

\[
C_1 := 6 a (1 + a \|\kappa_2 - \hat{\theta}\|_\infty)^2,
\]

\[
C_2 := 1 + a (1 + \|\hat{\theta}\|_\infty).
\]

At the same time,

\[
C_3^{-1} 1 \leq (G_0^{ij}) \leq C_3 1,
\]

in the matrix-inequality sense on \( \mathbb{R} \times \omega \), where \( 1 \) denotes the identity matrix and

\[
C_3 := 1 + a \|\kappa_2 - \hat{\theta}\|_\infty + a^2 \|\kappa_2 - \hat{\theta}\|_\infty^2.
\]

Consequently, we have the following matrix inequality on \( \mathbb{R} \times \omega \):

\[
(1 - C_4 k \chi_I)(G_0^{ij}) \leq (G^{ij}) \leq (1 + C_4 k \chi_I)(G_0^{ij}),
\]

where \( C_4 := 3 C_1 C_3 \). Finally, if we assume that \( k \leq 1 \) we have

\[
\|(\partial_i F)G^{ij}(\partial_j F)\| \leq C_5^2 k^2 \chi_I,
\]

(34)

where \( C_5 := C_2 \sqrt{3 C_3 (1 + C_4)} \).

Let \( \psi \) be any function from \( \mathcal{H}_0^1(\mathbb{R} \times \omega) \). First we estimate the term of indefinite sign on the right hand side of (33) as follows:

\[
2 |\Re (\partial_i \psi, G^{ij}(\partial_j F) \psi)| \leq 2 C_5 k \left( \chi_I \left[ \Re (\partial_i \psi) G^{ij}(\partial_j \psi) \right]^{1/2}, \chi_I |\psi| \right)
\]

\[
\leq C_5^2 k \|\chi_I |\psi|^2 + k (\partial_i \psi, \chi_I G^{ij} \partial_j \psi).
\]

Here the first inequality is established by applying the Cauchy-Schwarz inequality to the inner product induced by \( G^{ij} \) and using (34). The second inequality follows by the Cauchy-Schwarz inequality in the Hilbert space \( L^2(\mathbb{R} \times \omega) \) and by an elementary Cauchy inequality. Consequently,

\[
q[\psi] \geq (\partial_i \psi, (1 - C_6 k \chi_I) G_0^{ij} \partial_j \psi - C_7 k \|\chi_I \psi\|^2, \quad (35)
\]

where \( C_6 := 1 + C_4 \) and \( C_7 := 2 C_5^2 \).

Assume \( k < C_6^{-1} \), using the decomposition of the type (31) for the matrix \( (G_0^{ij}) \), neglecting the positive contribution coming from the corresponding matrix \( (S_0^{ij}) \), using the Fubini theorem and applying (17) to the function \( \varphi := \int_{\mathbb{R}} \sqrt{1 - C_6 k \chi_I(s) \psi(s, \cdot)} ds \), we may estimate (33) as follows:

\[
q[\psi] - E_1 \|\psi\|^2 \geq (1 - C_6 k) (l_\sigma[\psi] - E_1 \|\psi\|^2) - (C_6 E_1 + C_7) k \|\chi_I \psi\|^2.
\]
Applying Theorem 3 to the right hand side of the previous inequality, we have
\[ q|\psi - E_1 \|\psi\|^2 \geq \int_{\mathbb{R} \times \omega} \left( c_h \left( 1 - C_6 k \right) \frac{1}{1 + (s - s_0)^2} - (C_6 E_1 + C_7) k \chi_f(s) \right) |\psi(s, t)|^2 \, ds \, dt , \]
where \( c_h \) is the Hardy constant of Theorem 3. This proves that the threshold of the spectrum of \( Q \) (and therefore of \( -\Delta_D^\Omega \)) is greater than or equal to \( E_1 \) for sufficiently small \( k \).

In order to show that the whole interval \([E_1, \infty)\) belongs to the spectrum, it is enough to construct an appropriate Weyl sequence in the infinite straight ends of \( \Omega \). This concludes the proof of Theorem 1.

The proof of Theorem 2 is exactly the same, provided one chooses \( \sigma := \dot{\theta} \) and \( k := \|\kappa_1\|_\infty + \|\dot{\kappa}_1\|_\infty + \|\kappa_2\|_\infty \). Indeed, all the above estimates are valid with \((G^\theta)_{ij}\) being now the matrix \((G^{\dot{\theta}})_{ij}\) after letting both \( \kappa_1 \) and \( \kappa_2 \) equal to zero, and with \( C_1 \) and \( C_3 \) being replaced by
\[ C_1 := 6 a \left( 1 + a \|\kappa_2\|_\infty + a \|\dot{\theta}\|_\infty \right)^2 , \quad C_3 := \max \{ 2, 1 + 2 a^2 \|\dot{\theta}\|_\infty^2 \} , \]
respectively. Here \( C_1 \) can be further estimated by a constant independent of \( \kappa_2 \) provided one restricts, e.g., to \( \|\kappa_2\|_\infty < 1/a \).

5 Conclusions

We established Hardy-type inequalities for twisted 3-dimensional tubes. As an application we showed that the discrete eigenvalues of the Dirichlet Laplacian in mildly and locally bent tubes can be eliminated by an appropriate twisting. However, we would like to point out that for \( \sigma = \theta \), Theorems 3 and 4 can be used to prove certain stability of transport in straight twisted tubes also against other types of perturbations. For example against a local enlargement of the straight tube, mentioned in Introduction, or in principle against any potential perturbation which decays at least as \( O(s^{-2}) \) at infinity, where \( s \) is the longitudinal coordinate of the straight tube. The required decay at infinity is related to the decay of the Hardy weight in Theorem 3 and cannot be therefore improved by our method. The quadratic decay of the Hardy weight is determined by the classical inequality \((30)\) and is typical for Hardy inequalities for the Laplace operator.

For straight twisted tubes, the Hardy weight in the local inequality \((19)\) of Theorem 4 is given in terms of the the function \( \dot{\theta} \) and the constant \( \lambda \). Roughly speaking, the first tells us how fast the cross-section rotates, while the latter “measures” how much the cross-section differs from a disc. The actual value of \( \lambda \) depends of course on the geometry of \( \omega \).

The example of bent twisted tubes is of particular interest, since it shows the important role of the torsion. Namely, Theorem 1 tells us that, whenever \( \dot{\theta} \neq \kappa_2 \), the discrete eigenvalues in mildly curved tubes can be eliminated by torsion only. Note that Theorem 1 also provides a better lower bound to the spectrum in mildly bent tubes than that derived in 7.
Theorems 1 and 2 were proved for tubes about curves determined by (2). This restriction was made in order to construct the tube uniquely from given curvature functions by means of a uniquely determined Frenet frame. However, Theorems 1 and 2 will also hold for more general classes of tubes, namely, for those constructed about curves possessing the distinguished Frenet frame and with curvatures decaying as $O(s^{-2})$ at infinity, where $s$ is the arc-length parameter of the curve.

At least from the mathematical point of view, it would be interesting to extend Theorem 1 to higher dimensions. Here the main difficulty is that $\sigma$ in the form analogous to (15) will be in general a tensor depending also on the transverse variables $t$. Nevertheless, a higher dimensional analogue of Theorem 2 is easy to derive along the same lines as in the present paper, provided one restricts to rotations of the cross-section just in one hyperplane.

Summing up, the twisting represents a repulsive geometric perturbation in the sense that it eliminates the discrete eigenvalues in mildly curved waveguides. Regarding the transport itself, an interesting open question is whether this also happens to the singular spectrum possibly contained in the essential spectrum.

A Injectivity of the tube mapping

Let us conclude the paper by finding geometric conditions which guarantee the basic hypotheses (9). The first condition of (9) ensures that the mapping $L$ is an immersion due to (29). The second, injectivity condition requires to impose some global hypotheses about the geometry of the curve. Our approach is based on the following lemma:

**Lemma 4.** Let $\Gamma$ be determined by the curvature functions (2). Then for every $i \in \{1, 2, 3\}$ and all $s_1, s_2 \in \mathbb{R}$,

$$|e_i(s_2) - e_i(s_1)| \leq 2 k_i \min \{|s_2 - s_1|, |I|\},$$

where

$$k_i := \begin{cases} \|\kappa_1\|_{\infty} & \text{if } i = 1, \\ \|\kappa_1\|_{\infty} + \|\kappa_2\|_{\infty} & \text{if } i = 2, \\ \|\kappa_2\|_{\infty} & \text{if } i = 3. \end{cases}$$

**Proof.** It follows from the Serret-Frenet equations (26) and (2) that

$$|e_i(s_2) - e_i(s_1)| \leq 2 \int_{s_1}^{s_2} |\dot{e}_i| \, ds \leq 2 k_i \int_{s_1}^{s_2} \chi_I \, ds,$$

which immediately establishes the assertion. \qed

As a consequence of Lemma 4 we get the inequality

$$e_i(s_2) \cdot e_i(s_1) \geq 1 - 2 |I|^2 k_i^2, \quad i \in \{1, 2, 3\}. \quad (36)$$
In particular, since $e_1$ is the tangent vector of $\Gamma$, we obtain that the curve is not self-intersecting provided $|I| \|\kappa_1\|_\infty < 1$. A stronger sufficient condition ensures the injectivity of $L$:

**Proposition 1.** Let $\Gamma$ be determined by the curvature functions $\kappa_1$. Then the hypotheses hold true provided

$$\max \left\{ 4|I|^2 \|\kappa_1\|^2_\infty, \ 4a \left( \|\kappa_1\|_\infty + \|\kappa_2\|_\infty \right) \right\} < 1.$$

**Proof.** The idea is to observe that it is enough to show that the mapping $\Gamma_t$ from $\mathbb{R}$ to $\mathbb{R}^3$ defined by

$$\Gamma_t(s) := \Gamma(s) + t_\mu R_{\mu\nu}(s) e_\nu(s)$$

is injective for any fixed $t \in \mathbb{R}^2$ such that $|t| < a$ and arbitrary matrix-valued function $(R_{\mu\nu}) : \mathbb{R} \to SO(2)$. Let us assume that there exist $s_1 < s_2$ such that $\Gamma_t(s_1) = \Gamma_t(s_2)$. Then

$$0 = \Gamma(s_2) - \Gamma(s_1) + t_\mu \left\{ [R_{\mu\nu}(s_2) - R_{\mu\nu}(s_1)] e_\nu(s_1) + R_{\mu\nu}(s_2) [e_\nu(s_2) - e_\nu(s_1)] \right\}.$$

Taking the inner product of both sides of the vector identity with the tangent vector $e_1(s_1)$ and writing the difference $\Gamma(s_2) - \Gamma(s_1)$ as an integral, we arrive at the following scalar identity

$$0 = \int_{s_1}^{s_2} e_1(s_1) \cdot e_1(\xi) d\xi + t_\mu R_{\mu\nu}(s_2) [e_\nu(s_2) - e_\nu(s_1)] \cdot e_1(s_1).$$

Applying Lemma together with the first inequality of (36), recalling the orthonormality of $(R_{\mu\nu})$ and using obvious estimates, we obtain

$$0 \geq (s_2 - s_1) \left( 1 - 2|I|^2 k^2_1 - 2a k_2 \right).$$

This provides a contradiction for curves satisfying the inequality of Proposition, unless $s_1 = s_2$. \qed

**Remark 2.** The ideas of this Appendix do not restrict to the special class of tubes about curves determined by $\kappa_1$. Indeed, assuming only the existence of an appropriate Frenet frame for the reference curve (cf. (4)), more general sufficient conditions, involving integrals of curvatures, could be derived.

**Acknowledgment**

The authors are grateful to Timo Weidl for pointing out the presented problem to them. The work has partially been supported by the Czech Academy of Sciences and its Grant Agency within the projects IRP AV0Z10480505 and A100485051, and by DAAD within the project D-CZ 5/05-06. T.E. has partially been supported by the ESF European programme SPECT and D.K. has partially been supported by FCT/POCTI/FEDER, Portugal.
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