Photon-instanton collider implemented by a superconducting circuit

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Instantons, spacetime-localized quantum field tunneling events, are ubiquitous in correlated condensed matter and high energy systems. However, their direct observation through collisions with conventional particles has not been considered possible. We show how recent advance in circuit quantum electrodynamics, specifically, the realization of galvanic coupling of a transmon qubit to a high-impedance transmission line, allows the observation of inelastic collisions of single microwave photons with instantons (phase slips). We develop the formalism for calculating the photon-instanton cross section, which should be useful in other quantum field theoretical contexts. In particular, we show that the inelastic scattering probability can significantly exceed the effect of conventional Josephson quartic anharmonicity, and reach order unity values.

**Introduction.**—Instantons are time-localized solutions to a system’s imaginary time equations of motion, describing quantum tunneling events. They typically bridge between symmetry-related configurations and carry nontrivial topological indexes [1]. Instantons play important roles in many areas of physics, ranging from single-particle quantum-mechanical tunneling [1], through transport in low dimensional superconductors and superfluids (where they are also known as “phase slips”, and can be thought of as vortices crossing the system) [2,9], to determining the phase diagram [10] and breaking of classical conservation laws [11,12] in gauge theories. Most of these studies concern thermodynamic or transport properties. A more direct way to probe such short-lived excitations would be through resonances they may induce in the scattering cross sections or decay rates of other more stable particles with which they interact. However, such questions received much less attention, in large part due to lack of relevant experiments.

Advances in the fabrication and control of superconducting circuits allow to monitor the dynamics of single microwave photons propagation. Recent experiments have exposed intriguing photon dynamics in a uniform Josephson junction array, in which phase slips may play an important role [13]. However, their interpretation is complicated due to the importance of disorder and charge fluctuations [14,16]. It has recently been realized theoretically [17,27] that controllable quantum simulation of many-body physics may be easier to achieve in “quantum impurity” setups, leading to initial experimental works [28,31]. In the current context, this points at a setup involving a single flux-tunable small Josephson junction (transmon qubit [32]) coupled to an array of large junctions. The array effectively screens the effects of unwanted offset charges on the small junction, and allows to controllably scatter photons off it. Superstrong transmon-array coupling has recently been demonstrated in such an array [33]. In this work we will show how this setup could be used to reach a regime of significant inelastic collisions of single microwave photons off phase slips in the transmon, which would dominate over other nonlinearities in the system. For this we develop an extension of the standard equilibrium instanton calculation [1] to a scattering scenario, which may be useful in other fields.

**Model.**—We concentrate on the setup realized in a recent experiment [33], corresponding to the electric circuit depicted in Fig. 1(a). It consists of a long (length $N \gg 1$) two-leg array of superconducting islands connect by strong Josephson junctions $E_j$ with large junction capacitance $C_{j\text{line}}$, negligible ground capacitance (not depicted), and intermediate inter-leg capacitance $C_g$. The large $C_{j\text{line}}$ suppresses phase slips along the arrays, allowing their treatment as classical transmission lines. Except for this $C_{j\text{line}}$ could be ignored below the array plasma frequency. The small ground capacitance pushes the even modes to high frequencies, decoupling them from the transmon. We may thus employ a simplified single-leg array model [Fig. 1(b)] for the leg-odd degrees of freedom. The array capacitance to the ground $C_g$ and inductance $L$ in Fig. 1(b) are the inter-leg capacitance and twice the intra-leg Josephson inductance in Fig. 1(a), leading to a Lagrangian

$$\mathcal{L} = \frac{C_0 \dot{\phi}_0^2}{2} + E_J \cos(2\phi_0) + \sum_{n=1}^{N} \frac{C_g \dot{\phi}_n^2}{2} - \frac{(\phi_n - \phi_{n-1})^2}{2L}, \quad (1)$$

where $\phi_n$ is in units of flux and we employ units where $e = 1$ and $\hbar = 1$, hence the flux quantum is $\Phi_0 = \hbar/2e = \pi$. The array spacing $a$ will serve as the unit of length.

**FIG. 1.** The studied system: (a) The full circuit; (b) A simplified version. See the text for details.
The array is terminated by a transmon qubit (node $n = 0$, blue elements in Fig. 1) whose Josephson energy $E_J$ is flux-tunable and much larger than its charging energy, $E_C = 1/2C_0$. Hence, to a leading approximation we may treat it as a linear device [33]. This gives rise to total system eigenmodes with dispersion $\omega_k = 2v \sin(k/2) \approx vk$, where $v = 1/\sqrt{LC_g}$, the array wave velocity divided by the array spacing, is much larger than all other energy scales, i.e., for all relevant modes $k \ll 1$. The eigenmodes are $\propto \sin(kn + \delta_k)$, where

$$\delta_k = \tan^{-1} \left( \frac{\Gamma_0 \omega_k}{\omega_0^2 - \omega_k^2} \right)$$  

(2)

is the phase shift. Here $\omega_0 = \sqrt{8E_JE_C}$ is the trasmon LC frequency and $\Gamma_0 = 1/ZC_0 = 4E_C/\pi z$ is its elastic broadening due to its coupling to the array, where $Z = \sqrt{L/C_g}$ is the array wave impedance and $z = Z/R_Q$ ($R_Q = h/(2e)^2 = \pi/2$ is the superconducting resistance quantum). For $N \gg 1$ the mode spacing is $\Delta = \pi v/N$, hence $\sum_k \rightarrow \int_{0}^{\infty} \omega d\omega/\Delta$.

Upon increasing $E_C/E_J$ the transmon nonlinearity starts becoming significant. We will concentrate on the regime where $\sqrt{E_C/E_J}$ is still small, and furthermore, $\Gamma_0/\omega_0 \ll 1$ (i.e., $\sqrt{E_C/E_J} \ll z$), so the transmon resonance is well-defined [33]. In this regime the nonlinearity manifests in two ways: (i) Expanding the Josephson cosine gives rise to quartic nonlinearity, shifting $\omega_0$ by $-E_C$. It could also induce photon inelastic scattering, but we will show later on that for realistic device parameters this effect could be subleading; (ii) The periodicity of the cosine allows for instants (phase slips). We will now study in detail the latter effect.

### Instanton calculation.

For a disconnected transmon the classical instanton solution in imaginary time, describing a phase slip between $\phi_0 = 0$ at $\tau \rightarrow -\infty$ and $\phi_0 = \pm \Phi_0 = \pm \pi$ at $\tau \rightarrow \infty$, is $\phi_0(\tau) = \pm 2 \tan^{-1}(e^{i\omega_0 \tau})$, or, in Fourier space, $\phi_0^0(\omega) = \pm \pi/\omega \cos(\pi \omega/2\omega_0)$. The classical action $S_0$ of the instanton, together with the contributions of Gaussian fluctuations around it, give rise to the transmon ground state charge dispersion, the half bandwidth of the lowest Bloch band of the corresponding Mathieu equation [32, 33]. In the WKB approximation,

$$\lambda_0 = \frac{8}{\sqrt{\pi}} \left( 2E_JE_C \right)^{1/4} e^{-\sqrt{8E_JE_C}}.$$  

(3)

We now incorporate the array to lowest order in $\Gamma_0/\omega_0$. Expanding the imaginary time action around the classical isolated instanton solution $[\phi_0^0(\tau)]$ as given above and $\phi_{\mu \nu}^0(\tau) = 0$ to second order in the deviation $\delta \phi_\mu = \sum_k \delta \phi_k \sin(kn + \delta_k)$, one finds:

$$S = S_0 + \int \frac{d\omega}{2\pi} \left[ \frac{\phi_0^0(\omega)}{2L} + \sum_k \frac{C_k}{2} (\omega^2 + \omega_k^2) |\delta \phi_k(\omega)|^2 - \sin(k + \delta_k) - \sin(\delta_k) \phi_{\mu \nu}^0(-\omega) \delta \phi_k(\omega) \right]$$  

- $\sin(k + \delta_k) - \sin(\delta_k) \phi_{\mu \nu}^0(-\omega) \delta \phi_k(\omega)$
- $\sum_k \sin(k + \delta_k) - \sin(\delta_k) \phi_{\mu \nu}^0(-\omega) \delta \phi_k(\omega)$

$$- \int d\tau \frac{8E_J}{\cosh^2(\omega_0 \tau)} \left[ \sum_k \sin(\delta_k) \delta \phi_k(\tau) \right]^2,$$

(4)

where the capacitance of mode $k$ is $C_k \approx \sqrt{N} \frac{C_0}{2}$ for $N \gg 1$. The very last term contributes to higher orders in $\Gamma_0/\omega_0$ and will be neglected henceforth. The classical equations of motion for $\delta \phi_k$ result in

$$\delta \phi_k(\omega) \approx \frac{1}{C_k(\omega^2 + \omega_k^2)} \frac{\omega_k \cos \delta_k}{Z} \phi_{\mu \nu}^0(\omega),$$

(5)

leading to a renormalization $\lambda_0 \rightarrow \lambda_0 e^{-\sum_k \frac{f_k^2}{2}}$. For $z > 1$ instantons are relevant, resulting in an emergent scale, $\lambda_* = \lambda_0(\lambda_0/\omega_0)^{1/(z-1)}$, below which instanton effects are nonperturbative [2]. We limit ourselves to higher energies.

Within the approximations we employ, the contribution of a single instanton to a multipoint correlation of the $\phi_k$ is given by the corresponding classical solution [33], multiplied by $\lambda_0 e^{-\sum_k \frac{f_k^2}{2}}$. The Lehmann–Symanzik–Zimmermann reduction formula [36, 37] may then be employed to find the $T$-matrix element between $N_{in}$ incoming photons with momenta $k'_1, k'_2, \ldots, k'_{N_{in}}$ and $N_{out}$ outgoing photons with momenta $k_1, k_2, \ldots, k_{N_{out}}$:

$$\mathcal{T}_{k_1, k_2, \ldots, k_{N_{out}}} = \Delta \frac{C_{k'_1}}{2\pi} \sqrt{2C_{k'_1} \omega_{k'_1}^2} \cdots \frac{C_{k'_{N_{in}}}}{2\pi} \sqrt{2C_{k'_{N_{in}}} \omega_{k'_{N_{in}}}} \frac{C_{k_1}}{2\pi} \sqrt{2C_{k_1} \omega_{k_1}} \cdots \frac{C_{k_{N_{out}}}}{2\pi} \sqrt{2C_{k_{N_{out}}} \omega_{k_{N_{out}}}} \times \left. \left\langle \phi_{k'_1}(\omega'_{k'_1}) \cdots \phi_{k'_{N_{in}}}(\omega'_{k'_{N_{in}}}) \phi_{k_1}(\omega_1) \cdots \phi_{k_{N_{out}}}(\omega_{k_{N_{out}}}) \right\rangle \right|_{1\text{-instanton}} \left| \begin{array}{l} \omega_{k'_1} \rightarrow -i\omega_{k'_1} \\ \omega_{k'_2} \rightarrow -i\omega_{k'_2} \\ \vdots \\ \omega_{k_{N_{out}}} \rightarrow -i\omega_{k_{N_{out}}} \end{array} \right.$$

$$= (\mp 1)^{N_{in}} (\pm 1)^{N_{out}} f_{k'_1} f_{k'_2} \cdots f_{k'_{N_{in}}} f_{k_1} f_{k_2} \cdots f_{k_{N_{out}}} \frac{\lambda_0}{2} e^{-\sum_k \frac{f_k^2}{2}}$$

(7)
with

\[ f_k = \sqrt{\frac{2\Delta}{z\omega_k}} \sin\left(\frac{\pi\omega_k - \omega_0}{2}\right) \sqrt{\left(\omega_0^2 - \omega_k^2\right)^2 + \left(\Gamma_0\omega_k\right)^2}, \]  

(8)

being the “form factor” of the instanton in the photon modes basis. Note that it is finite at the resonance frequency \( \omega_0 \) but still peaked there. It rises towards low frequencies (assumed higher than \( \lambda_\ast \)). Thus, processes in which a nearly resonant photon scatters into a nearly resonant photon and several low energy photons (whose number is controlled by \( z \)) will play an important role. Note also that \( f_k \) diverges at higher odd multiples of \( \omega_0 \), which are nonlinear resonances broadened only at higher frequency \( \omega \). It is dominated by an inelastic resonance at \( \omega \), which accounts for processes in which photons at \( \omega \) are, respectively, absorbed-emitted, emitted-absorbed, emitted-emitted, or absorbed-absorbed, with appropriate signs to obey an energy conservation sum rule, \( \omega_k \Gamma_k^{\text{in}} = \sum \omega_k \Gamma_k^{\text{in}}[\omega_k] \). The last couple of equations are the central results of this work. To recap, they apply for any \( \omega_k, \omega_k' \) between \( \lambda_\ast \) and \( 3\omega_0 \), provided that \( \lambda_\ast \ll \max(\Gamma_0, T) \ll \omega_0 \) and \( E_C \ll E_J \). The single-instanton approximation further requires \( 2\pi\Gamma_k^{\text{in}}/\Delta \ll 1 \).

We exemplify the parameter dependence of \( \Gamma_k^{\text{in}} \) in Fig. 2. It is dominated by an inelastic resonance at \( \omega_0 \). Some of its salient features are: (i) The inelastic scattering probability can approach order unity. The charge dispersion \( \lambda_0 \) decreases fast with \( \omega_0 \), masking the corresponding increase in the number of possible decay channels contributing on-resonance [Fig. 2(a,b)]; (ii) The latter increase is visible in an asymmetry of the inelastic resonance lineshape, especially for small \( \Gamma_0/\omega_0 \) and \( z \) [Fig. 2(c,d)]; (iii) Temperature suppresses coherent quantum phase slips (particularly for \( z > 1 \), when they are relevant [2]) but gives rise to scattering by thermal photons, hence could either decrease or increase the decay rate, depending on \( z \) [Fig. 2(e)].

**Limiting cases.**—The general expressions given above can be simplified further for nearly-resonant incoming photons, if in addition \( \Gamma_0/\omega_0 \rightarrow 0 \) and \( z > 1 \). The energy
where \( \lambda \) over \( \omega \) off where \( \Gamma \) is the gamma function [34], and the effective cut-and different \( \omega_k \) total rate \{ \omega \}

\[
\Pi_R(\omega) \approx \frac{\pi}{\Gamma(2/z)} \frac{\omega^2}{\omega_c(z)}^{2z} \frac{1}{1 + 2/z} \times F_2 \left( 1, \frac{1}{z}, 1, \frac{1}{z}, -\frac{4\omega_0^2}{\Gamma^2} \right),
\]

where \( \lambda_1 = -\sqrt{\omega_0^2 + \omega_c^2} \) and \( \Gamma_{\omega_0} = \omega_0 / \Delta \sim (\lambda_1 / \Gamma)(\Gamma / \omega_0)^2 / 2 \) is the charge dispersion of the first excited level of an isolated transmon [32] and \( \omega \) \( \omega_c \) \( \omega \). Hence, for large \( \omega \) the \( \Gamma_{\omega_0} \) \( \Delta \) \( \omega_0 \) \( \Delta \) \\( \omega \). An extension to finite \( T \) \( \omega_0 \) \( \Delta \) straightforward. Similar expressions can be obtained by an effective Hamiltonian tailored to describe this particular class of processes [39], though that approach cannot give the value of \( \omega_c \). The quality of this approximation is tested in Fig. 2(f).

**Dual cosine approach.**— The instanton approach accounts for the full imaginary time dynamics of the phases along the array during a phase slip. A common phenomenological approach is to approximate a phase slip as an instantaneous \( 2\pi \) shift of all the phases (\( \Phi_0 = \pi \) shift of the fluxes) along the array. In a Hamiltonian formalism, this could be accounted for by a term

\[
H^{PS} = \lambda^{PS} \cos \left( \pi \sum_{n=0}^{N} q_n \right) = \lambda^{PS} \cosh \left[ \sum_{k} f_k^{PS} (a_k - a_k^\dagger) \right],
\]

where \( q_0 = C_0 \dot{\phi}_0 \) and \( q_n > 0 = C_g \dot{\phi}_n \) denote the charge operators along the array, \( a_k \) is the annihilation operator of the array mode \( k \), and where

\[
f_k^{PS} = \sqrt{2\Delta \omega_k} \left( \frac{\omega_0^2}{\omega_k^2 - \omega_0^2 + (\Gamma_0 \omega_k)^2} \right)^{1/2},
\]

The coefficient \( \lambda^{PS} \) needs to be set by the value of a known observable. For the study of nearly-resonant photon scattering it is natural to choose it so that \( H^{PS} \) reproduces the charge displacement of the first excited level of an isolated transmon [32], that is, \( \lambda^{PS} = \lambda_1 / \langle \cos(\pi q_0) \rangle \), where \( \langle \cos(\pi q_0) \rangle \approx -\pi \sqrt{E_J / 2E_C} e^{-\pi^2 \sqrt{E_J / 8E_C}} \). It is then straightforward to calculate the photon self energy to second order in \( \lambda^{PS} \) [10]. The result has the same form as the instanton expression derived above, but with different coefficients, \( \lambda_0 \rightarrow \lambda^{PS} \) and \( f_k, \tilde{f}_k \rightarrow f_k^{PS} \). The

**FIG. 2.** Parameter dependence of \( 2\pi \Gamma_{\omega_0}^{PS} / \Delta \), the total inelastic scattering probability of a single incoming photon with frequency \( \omega_k \). (a,b) On-resonance (mode \( k_0 \) = \( \omega_0 / \nu \)) the probability as function of \( \omega_k / \Gamma_0 \) for several values of \( z \) at \( T = 0 \): (a) presents the total rate (using the full Mathieu expression for \( \lambda \) [32] [34]), rather than the approximate Eq. (3), (b) excludes the prefactor \( \lambda^2 \). (c,d) \( T = 0 \) resonance lineshape at \( z = 2 \) and different \( \Gamma_0 / \omega_0 \) or (d) \( \Gamma_0 / \omega_0 = 0.2 \) and different \( z \). A simple Lorentzian with width \( \Gamma_0 \) is also plotted for comparison. (e) Temperature dependence of the on-resonance probability for \( \Gamma_0 / \omega_0 = 0.05 \) and different \( z \). (f) Ratio between the \( T = 0 \) on-resonance probabilities given by the instanton calculation and either the dual cosine approximation (continuous lines) or the limiting expression, Eq. (13) (dashed lines). See the text for further details.
latter agree for $\omega_k \ll \omega_0$ but differ significantly near resonance, as expected from a sudden phase-slip approximation. However, the overall respective coefficients in Eq. (11), namely $(\lambda_0 f_k f_k^*)^2 \Sigma_{\nu} \omega_{\nu}^2 f_{\nu}^* f_{\nu}$ for the instanton calculation and $(\lambda_0^2 f_k^* f_k)^2$ for the dual cosine, actually agree to leading order in $\Gamma_0/\omega_0$. Since the contribution of the low energy photons is similar, as just noted, the dual cosine provides a surprisingly good approximation to the full instanton result, see Fig. 2(f).

**Quartic nonlinearity.** — Let us now briefly discuss inelastic photon scattering by more mundane nonlinearities, coming from the Taylor expansion of the transmon Josephson cosine. To leading order in $\sqrt{E_C/E_J}$ it is dominated by the Fermi golden rule contribution of the quartic term in the expansion, which at $T = 0$ allows an incoming photon at $k$ to split into three at $k_i$, $i = 1, 2, 3$. Expressing $\phi_0$ in terms of the array modes, one finds

$$\Gamma_k^{\text{in}} = \frac{4\pi^2 \omega_0^2 \Delta^4 \sin^2(\delta_k)}{\omega_k} \sum_{k_i} \frac{\sin^2(\delta_{k_i})}{\omega_{k_i}} \frac{\sin^2(\delta_{k_2})}{\omega_{k_2}} \frac{\sin^2(\delta_{k_3})}{\omega_{k_3}} \delta(\omega_k - \omega_{k_1} - \omega_{k_2} - \omega_{k_3}).$$

As opposed to the instanton contribution, where $f_k^2$ increases towards low energies [Eq. (8)], here the factors of $\sin^2(\delta_{k_i})/\omega_{k_i} \propto \omega_k$ [cf. Eq. (2)] suppress the contribution of low frequency photons. Summing over $k_i$ we find the resulting total inelastic rate near resonance to scale as $\sim \omega^2 \Delta^4 / \omega_k^4$. The suppression with $\Gamma_0/\omega_0$ can make it significantly smaller than the instanton contribution, provided $\lambda_1$ is not too small [cf. Eq. (13)], which can be realized with devices similar to those in Ref. 33.

**Conclusions.** — In this work we have developed a general formalism for the study of instanton-particle collisions, and applied it to a recently-realized superconducting circuit in which a transmon qubit is strongly-coupled to a high impedance transmission line. We have shown that significant inelastic single-photon scattering by instantons can be controllably initiated and identified in such a setup. Recent experimental results nicely match our theory 41. This paves the way towards the study of similar effects not only in various superconducting circuits 2 3 6 9 17 31, but also in other condensed matter 4 42 43 and particle physics 10 12 systems.

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