Harmonic entanglement in a degenerate parametric down-conversion

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Abstract
A detailed study of the harmonic entanglement and two-mode squeezing of radiation generated in a degenerate parametric down-conversion process when the cavity is coupled to a two-mode vacuum reservoir is presented. It is found that there is a quadrature entanglement between the harmonically related fundamental and residual pump modes where the superimposed radiation exhibits a higher degree of two-mode squeezing. It turns out that the two-mode squeezing can exist when there is no entanglement, since the correlations leading to these phenomena are essentially different. It is also shown that the more the external pumping radiation is down-converted by the nonlinear crystal, the stronger the entanglement and intensity of the two-mode radiation would be; this condition is not generally true for squeezing.

1. Introduction
Optical degenerate parametric down-conversion is one of the second-order nonlinear processes in which a pump photon of frequency $2\omega$ is down-converted into a pair of signal photons each of frequency $\omega$. Due to the inherent two-photon nature of the interaction, the degenerate parametric oscillator is one of the most interesting and well-studied devices in the nonlinear quantum optics [1–8]. The generated radiation is generally found to exhibit nonclassical features where the quantum optical properties are significantly degraded by the leakage through the mirrors and amplification of the quantum fluctuations in the cavity. But if the ordinary vacuum reservoir is replaced by a squeezed vacuum, the nonclassical features can be enhanced [6–9] provided that the reservoir modes are biased in the right quadrature. Moreover, as legitimately argued elsewhere, when a nonlinear crystal is illuminated with external radiation of frequency $2\omega$ only some part of this radiation is down-converted into a pair of photons [10, 11]. As a result, the cavity actually contains the down-converted and residual pump radiations (see figure 1). Although considerable attention has been given only to a single-mode squeezed radiation previously, it has been reported most recently that a degenerate parametric oscillator can be a source of a two-mode radiation in which nonclassical features are characterized by a strong correlation between each mode separately as well as the superimposed state [10]. Taking the quantum properties of the residual pump mode into consideration undoubtedly modifies the quantum features of the cavity radiation, which for instance leads to a tripartite entanglement in a nondegenerate parametric oscillator [12].

Generation of macroscopic entangled states is currently receiving attention in connection with their potential role in quantum information and measurement theories [13–15]. At the microscopic level, correlations accountable for an entanglement are expected to arise due to the down conversion of a single high frequency photon into a pair of correlated lower frequency photons. The generation of the entangled beams in the nondegenerate parametric down-conversion has been demonstrated by Reid and Drummond [16–18] earlier. Studies have also shown the existence of the correlation between the fundamental (residual pump) and second-harmonic modes [19] that leads to entanglement in the second-harmonic generation [20]. In this respect, Lim and Saffman [21] have analysed the production of two beams with nonclassical intensity correlations and quadrature entanglement in the dual-ported reservoir of the second-harmonic generation and found that the output radiation exhibits strong quantum correlations. In addition, Grosse et al [22] have recently predicted a perfect entanglement between the fundamental and second-harmonic modes in the pump-depleted nondegenerate parametric amplification and defined such an entanglement between harmonically related fields as harmonic entanglement. However, in this paper the quantum features (including entanglement) of a two-mode cavity radiation generated in a degenerate parametric...
down-conversion process (which basically is the reverse process of the system in [19–22]) would be studied. Hence it is hoped that the idea of generating macroscopic entangled state from a degenerate parametric down-converter the system which is known to be a source of a single-mode squeezed radiation would be interesting, since this system has been thoroughly studied for a long time and it is also believed to be an efficient frequency converter.

The essential approach to realize the Einstein–Podolsky–Rosen (EPR)-type entanglement [23] is to introduce correlated states of at least two particles that persist even when the particles are spatially separated. As thoroughly discussed by Dechoom et al [24], without having to construct an experimentally impossible states with a perfect correlation as envisaged in the original paper, it is possible to demonstrate this type of correlation using the inferred Heisenberg principle.

In this regard, a direct and experimentally feasible qualitative criterion for such a correlation of continuous variables was first proposed by Reid [17] using quadrature phase amplitude for a nondegenerate parametric amplification closely related to the original version. In addition, making use of the quadrature variables, Lodahl [25] has shown the existence of EPR-type correlations in the second-harmonic generation. Most recently, Olsen [20] has considered a travelling-wave second-harmonic generation and found that this quantum system can be employed in the experimental verification of entanglement of continuous variables.

In this paper, the entanglement attributed to the strong correlation between the fundamental and residual pump modes in the system under consideration would be studied applying the approach introduced in [26] based on the criterion set by Duan et al [27]. The criterion employed has upper and lower limits which makes it more appealing and convenient in quantifying the entanglement. The nonclassical properties of the two-mode cavity radiation can be analysed by applying the linearization procedure when the cavity is coupled to a two-mode vacuum reservoir. This approximation remains valid as long as the quantum fluctuations are much smaller than the classical mean values [28]. The semiclassical approximation that has been found to work surprisingly well in the threshold region [29] is also considered. The harmonic entanglement and its relation with the two-mode squeezing is studied near threshold, since the relation between the two is an interesting issue in its own right [30]. Moreover, the efficiency with which a nonlinear crystal down-converts the light falling on it and the association of the down-conversion process with the degree of entanglement and squeezing is also investigated. In addition, the mean photon number is calculated to see how strong the generated light could be.

2. Equations of evolution

In order to avoid the involved mathematical complications, this study is solely restricted to the interaction picture. Thus the interaction of an external pumping radiation with a nonlinear crystal placed in a resonant cavity can be described in the rotating-wave approximation and in the interaction picture by a Hamiltonian of the form [29, 31]

$$\hat{H}_I = \frac{i\hbar}{2} [\hat{a}^\dagger \hat{b} - \hat{a} \hat{b}^\dagger] + i\varepsilon [\hat{b}^\dagger - \hat{b}],$$

(1)

where $\varepsilon$ is a parameter proportional to the amplitude of the pumping radiation, $\lambda$ is the measure of the coupling of a nonlinear crystal with the radiation, $\hat{a}$ and $\hat{b}$ are time-independent annihilation operators for the fundamental and residual pump modes. For convenience, $\lambda$ and $\varepsilon$ are chosen to be real-positive constants. In view of the fact that $\hat{a}$ and $\hat{b}$ are mutually commuting operators, the pertinent quantum Langevin equations are found to be

$$\frac{d\hat{a}}{dt} = \lambda \hat{b}^\dagger + \kappa \hat{a} + \hat{F}_\alpha(t),$$

(2)

$$\frac{d\hat{b}}{dt} = -\lambda \hat{a}^\dagger - \kappa \hat{b} + \varepsilon + \hat{F}_b(t),$$

(3)

where $\kappa$ is the cavity damping constant chosen to be the same for both modes and $\hat{F}_i(t)$, with $i = a, b$, are the Langevin noise operators. For a two-mode vacuum reservoir, these noise operators satisfy the correlation functions

$$\langle \hat{F}_i(t) \rangle = 0,$$

(4)

$$\langle \hat{F}_i(t) \hat{F}_j(t') \rangle = \langle \hat{F}_i(t) \hat{F}_j(t') \rangle = \langle \hat{F}_i(t) \hat{F}_j(t') \rangle = 0,$$

(5)

$$\langle \hat{F}_i(t) \hat{F}_j(t') \rangle = \kappa \delta(t - t'),$$

(6)

$$\langle \hat{F}_i(t) \hat{F}_j(t') \rangle_{\rho} = 0.$$  

(7)

It is not difficult to note that equations (2) and (3) are nonlinear coupled differential equations which can be solved employing the linearization procedure [22, 28, 29, 31]. In this approach, one can first take

$$\hat{a}(t) = \alpha + \hat{\Lambda}(t),$$

(8)

$$\hat{b}(t) = \beta + \hat{B}(t),$$

(9)

where $\hat{\Lambda}(t)$ and $\hat{B}(t)$ are very small variations about the mean values at steady state, $\alpha = \langle \hat{a}(t) \rangle_{ss}$ and $\beta = \langle \hat{b}(t) \rangle_{ss}$. This approximation remains valid as long as the quantum fluctuations about the mean values are much smaller than the classical mean values at steady state. Upon taking the statistical average of equations (2) and (3) and then using the semiclassical approximation, $\langle \hat{a}(t) \hat{b}(t) \rangle_{ss} = \langle \hat{a}(t) \rangle_{ss} \langle \hat{b}(t) \rangle_{ss}$, and classical correlation, $\langle \hat{a}^2(t) \rangle_{ss} = \langle \hat{a}(t) \rangle_{ss}^2$, one obtains
\[ \lambda \alpha^2 \beta + \frac{\kappa}{2} \alpha = 0, \]  
(10)  
\[ \lambda \alpha^2 + \kappa \beta = 2 \varepsilon. \]  
(11)

The semiclassical assumption is found to work for weak nonlinearity or weak coupling between the external radiation and nonlinear medium where the mean photon number at threshold is very large.

Now with the aid of equations (2), (3), (8)–(11) and the fact that \( \hat{A} \) and \( \hat{B} \) are small perturbations, it is possible to verify that

\[ \frac{d\hat{A}(t)}{dt} = \varepsilon_1^* \hat{B}(t) + \varepsilon_2 \hat{A}(t) - \frac{\kappa}{2} \hat{A}(t) + \hat{F}_a(t), \]  
(12)  
\[ \frac{d\hat{B}(t)}{dt} = -\varepsilon_1 \hat{A}(t) - \frac{\kappa}{2} \hat{B}(t) + \hat{F}_b(t), \]  
(13)

where \( \varepsilon_1^* = \lambda \alpha^* \) and \( \varepsilon_2 = \lambda \beta \) are taken. It is recently shown that (see appendix of [10]) \( \varepsilon_1 = \varepsilon_1^* \) and \( \varepsilon_2 = \varepsilon_2^* \) which implies that \( \alpha \) and \( \beta \) are real, since \( \lambda \) is assumed to be real. It then follows from equations (10) and (11) that

\[ \varepsilon_1 = \pm \sqrt{2 \lambda \varepsilon - \kappa \varepsilon_2} \]  
(14)

and \( \varepsilon_2 = \kappa / 2 \) for \( \varepsilon_1 \neq 0 \). It is not difficult to realize that there are two possible values for the fundamental amplitude and hence the system may be in a transient superimposed state of these amplitudes prior to detection [14].

The solutions of these coupled differential equations are found following straightforward algebra to be

\[ \hat{A}(t) = a_1(t) + [a_3(t) + pa_4(t)] \hat{A}(0) + [a_4(t) + pa_5(t)] \]  
\[ \times \hat{A}(0) + qa_6(t) \hat{B}(0) + qa_7(t) \hat{B}(0) + \hat{f}(t), \]  
(15)  
\[ \hat{B}(t) = a_2(t) + [a_5(t) - pa_4(t)] \hat{B}(0) + [a_4(t) - pa_5(t)] \]  
\[ \times \hat{B}(0) - qa_6(t) \hat{A}(0) - qa_7(t) \hat{A}(0) + \hat{g}(t), \]  
(16)

where

\[ \hat{f}(t) = \int_0^t \left[ (a_3(t - \tau') + pa_4(t - \tau')) \hat{F}_a(\tau') \right. \]  
\[ + (a_5(t - \tau') - pa_4(t - \tau')) \hat{F}_a^\dagger(\tau') \]  
\[ + qa_6(t - \tau') \hat{F}_b(\tau') + qa_7(t - \tau') \hat{F}_b^\dagger(\tau') \right] d\tau', \]  
(17)  
\[ \hat{g}(t) = \int_0^t \left[ (a_5(t - \tau') - pa_4(t - \tau')) \hat{F}_b(\tau') \right. \]  
\[ + (a_3(t - \tau') - pa_4(t - \tau')) \hat{F}_b^\dagger(\tau') \]  
\[ - qa_6(t - \tau') \hat{F}_a(\tau') - qa_7(t - \tau') \hat{F}_a^\dagger(\tau') \right] d\tau', \]  
(18)

in which

\[ p = \frac{\varepsilon_2}{\sqrt{\varepsilon_2^2 - 4 \varepsilon_1^*}}, \]  
(19)  
\[ q = \frac{2 \varepsilon_1}{\sqrt{\varepsilon_2^2 - 4 \varepsilon_1^*}}. \]  
(20)

Further scrutiny reveals that \( a_1(t) \) and \( a_2(t) \) in equations (15) and (16) are followed directly from the introduction of the constants \( \alpha \) and \( \beta \) in equations (8) and (9) which are fundamentally related to the linearization procedure. Since the cavity is initially taken to be in a vacuum state at \( t = 0 \) (the time at which the crystal is started to be pumped externally), it is imperative to observe that the parameters \( \varepsilon_1 \) and \( \varepsilon_2 \) would be different from zero only after the inception of the interaction (\( \tau > 0 \) in which \( \lambda \neq 0 \)). It may also be worth mentioning that equations (15) and (16) are applied to calculate various quantities of interest in the subsequent sections. In addition, it is not difficult to note that for \( \varepsilon_1 \) to be real \( 2 \lambda \varepsilon \geq \kappa \varepsilon_2 \). I hence denote \( 2 \lambda \varepsilon = \kappa \varepsilon_2 \) that corresponds to \( \varepsilon_1 = 0 \) as a threshold condition.

3. Entanglement quantification

One of the most interesting and intriguing phenomena associated with a composite quantum system is entanglement. It is believed that preparation and manipulation of these entangled states that have nonclassical and nonlocal properties may lead to a better understanding of the basic quantum principle. Hence the entanglement of the fundamental and residual pump modes in the cavity would be analysed. It is also a well-established fact that a quantum system is said to be entangled if it is not separable. That is, if the density operator
for the combined state cannot be expressed as a product of the density operators of the individual constituents,

\[ \hat{\rho} \neq \sum_j p_j \hat{\rho}_j^{(1)} \otimes \hat{\rho}_j^{(2)}, \]  

(27)

where \( p_j \geq 0 \) and \( \sum_j p_j = 1 \) to ensure the normalization of the combined density operator. On the other hand, entangled continuous variable states can be expressed as a co-eigenstate of a pair of EPR-type operators such as \( \hat{X}_a - \hat{X}_b \) and \( \hat{P}_a + \hat{P}_b \) [26]. The total variance of these two operators reduces to zero for maximally entangled continuous variable states. Nonetheless according to the criterion set by Duan et al. [27] quantum states of the system are claimed to be entangled, if the sum of the variances of a pair of EPR-like operators

\[ \hat{u} = \hat{X}_a - \hat{X}_b, \]  

(28)

\[ \hat{v} = \hat{P}_a + \hat{P}_b, \]  

(29)

where \( \hat{X}_a = \frac{1}{\sqrt{2}}(\hat{a} + \hat{a}^\dagger) \), \( \hat{X}_b = \frac{1}{\sqrt{2}}(\hat{b} + \hat{b}^\dagger) \), \( \hat{P}_a = \frac{1}{\sqrt{2}}(\hat{a}^\dagger - \hat{a}) \), and \( \hat{P}_b = \frac{1}{\sqrt{2}}(\hat{b}^\dagger - \hat{b}) \), satisfy

\[ \Delta u^2 + \Delta v^2 < 2. \]  

(30)

On the basis of the boson commutation relation, it is not difficult to see that

\[ \Delta u^2 + \Delta v^2 = 2(\langle \hat{a}^\dagger \hat{a} \rangle + 2(\langle \hat{b}^\dagger \hat{b} \rangle + 2(\langle \hat{a} \rangle \langle \hat{b} \rangle + 2(\langle \hat{a}^\dagger \rangle \langle \hat{b}^\dagger \rangle - 2(\langle \hat{a} \rangle \langle \hat{b}^\dagger \rangle - 2(\langle \hat{a}^\dagger \rangle \langle \hat{a} \rangle - 2(\langle \hat{b} \rangle \langle \hat{b}^\dagger \rangle + 2. \]  

(31)

Now the various correlations involved in equation (31) would be determined. To this end, assuming the cavity modes to be initially in the vacuum state and using the fact that the Langevin noise operators have zero mean along with equations (8), (9), (15) and (16), one gets

\[ \langle \hat{a}(t) \rangle = \alpha + a(t), \]  

(32)

\[ \langle \hat{b}(t) \rangle = \beta + a(t), \]  

(33)

\[ \langle \hat{a}^\dagger(t) \hat{a}(t) \rangle = (\alpha + a(t))^2 + \langle \hat{j}^\dagger(t) \hat{j}(t) \rangle, \]  

(34)

\[ \langle \hat{b}^\dagger(t) \hat{b}(t) \rangle = (\beta + a(t))^2 + \langle \hat{g}^\dagger(t) \hat{g}(t) \rangle, \]  

(35)

\[ \langle \hat{a}(t) \hat{b}(t) \rangle = (\alpha + a(t))(\beta + a(t)) + \langle \hat{j}(t) \hat{g}(t) \rangle. \]  

(36)

Moreover, in view of the correlations of the Langevin noise operators (5)–(7), it is possible to show at steady state that

\[ \langle \hat{j}^\dagger(t) \hat{j}(t) \rangle_{ss} = \frac{\kappa [(1 + p^2 + q^2) (\kappa - \varepsilon_2) + 2p\sqrt{\varepsilon_2^2 - 4\varepsilon_1^2}]}{8\kappa(\kappa - 2\varepsilon_2) + 4\varepsilon_1^2}, \]  

(37)

\[ \kappa [(1 + p^2 + q^2) (\kappa + \varepsilon_2) + 2p\sqrt{\varepsilon_2^2 - 4\varepsilon_1^2}]}{8\kappa(\kappa + 2\varepsilon_2) + 4\varepsilon_1^2}], \]  

\[ \kappa^2(1 - p^2 - q^2) - \frac{\kappa^2(1 - p^2 + q^2)}{4(\kappa^2 - \varepsilon_2^2)} - \frac{\kappa^2(1 - p^2 + q^2)}{4(\kappa^2 - (\varepsilon_2^2 - 4\varepsilon_1^2))}, \]  

\[ \frac{1 + p^2 - q^2}{4}. \]  

(38)

\[ \langle \hat{g}^\dagger(t) \hat{g}(t) \rangle_{ss} = \frac{-\kappa^2 pq (\kappa - \varepsilon_2)}{4\kappa^2 (\kappa - 2\varepsilon_2) + 4\varepsilon_1^2}, \]  

(39)

\[ \kappa (1 + p^2 + q^2) (\kappa + \varepsilon_2) + \frac{2p\sqrt{\varepsilon_2^2 - 4\varepsilon_1^2}}{4(\kappa^2 - 2\varepsilon_2)}, \]  

\[ - \frac{\kappa^2 pq}{2(\kappa^2 - \varepsilon_2^2)} + \frac{pq}{2}. \]  

(40)

Therefore, with the aid of equations (31)–(39), it can be obtained at steady state that

\[ \Delta u^2 + \Delta v^2 = 2 + \frac{\kappa (1 + p^2 + q^2) + 2pq (\kappa - \varepsilon_2)}{2\kappa(\kappa - 2\varepsilon_2) + 4\varepsilon_1^2}], \]  

\[ \kappa (1 + p^2 + q^2) (\kappa + \varepsilon_2) + \frac{2p\sqrt{\varepsilon_2^2 - 4\varepsilon_1^2}}{4(\kappa^2 - 2\varepsilon_2)}, \]  

\[ - \frac{\kappa^2 pq}{2(\kappa^2 - \varepsilon_2^2)} + \frac{pq}{2}. \]  

(41)

Finally, on account of equations (19), (20) and the fact that \( \varepsilon_2 = \kappa/2 \) for \( \varepsilon_1 \neq 0 \), one finds

\[ \Delta u^2 + \Delta v^2 = \frac{\kappa^3 (\kappa + 4\varepsilon_1)}{8\varepsilon_1^2(\kappa^2 - 16\varepsilon_1^2)} + \frac{3\kappa^3 (\kappa + 4\varepsilon_1)}{4(\kappa^2 - 16\varepsilon_1^2)(\kappa^2 + 2\varepsilon_1^2)}, \]  

\[ 16\varepsilon_1 (4\varepsilon_1 + \kappa) + 24\kappa \varepsilon_1, \]  

\[ \frac{1}{3(\kappa^2 - 16\varepsilon_1^2)}, \]  

\[ + \frac{16\varepsilon_1^3 \varepsilon_1}{(\kappa^2 - 16\varepsilon_1^2)(\kappa^2 + 2\varepsilon_1^2)}. \]  

(42)

It can readily be observed from equation (41) that \( \Delta u^2 + \Delta v^2 \) is very large when \( \varepsilon_1 = 0 \) and \( \varepsilon_1 = 0.25\kappa \). I hence plot \( \Delta u^2 + \Delta v^2 \) versus \( \varepsilon_1 \) only for \( \varepsilon_1 > 0.25\kappa \).

On the basis of criterion (30), the correlation between the fundamental and residual pump modes exhibits entanglement as clearly shown in figure 2. The entanglement exists except near certain values of \( \varepsilon_1 \), for instance near \( \varepsilon_1 = 0.13 \) for \( \kappa = 0.3 \) at steady state. Moreover, it can be observed that the entanglement does not exist near a threshold value (\( \varepsilon_1 = 0 \)). However, the entanglement occurs and it decreases with the damping constant in the other cases for which \( \varepsilon_1 > 0.25\kappa \). This is related to a well-known fact that the lesser the cavity damping constant, the more probable for the radiation to stay in the cavity which in turn strengthens the correlation that leads to entanglement. It is possible to realize that the dependence of the entanglement on damping mechanism through the mirrors is insignificant for larger values of \( \varepsilon_1 \). Furthermore,
as can readily be inferred from equations (31) and (36), the entanglement is attributed to the correlation between the involved Langevin noise operators associated with the down-conversion of high frequency radiation to the fundamental modes at microscopic level. Since \( \epsilon_1 = \lambda (\hat{a})_{ss} \), it corresponds to the degree at which the external radiation of frequency \( 2\omega \) is down-converted by the nonlinear crystal. Therefore, it is possible to see from figure 2 that the entanglement is stronger when the external radiation is down converted by the nonlinear crystal more efficiently. This indicates that even though the down-conversion process splits the external radiation into two, it is unable to destroy the coherence responsible for the correlation.

4. Quadrature variances

Nowadays the relation between the degree of entanglement and squeezing turns out to be interesting in connection to entanglement manipulation and quantification. In this respect, the two-mode squeezing of the cavity radiation of the system under consideration is analysed. It is common knowledge that the degree of two-mode squeezing can be described by annihilation operator defined by

\[
\hat{c} = \frac{1}{\sqrt{2}}(\hat{a} + \hat{b}), \tag{42}
\]

where \( \hat{a} \) and \( \hat{b} \) are the boson annihilation operators that represent the fundamental and residual pump modes. In view of the boson commutation relation for modes \( \hat{a} \) and \( \hat{b} \), one can easily see that \([\hat{c}, \hat{c}^\dagger] = 1 \) and \([\hat{c}, \hat{c}] = 0 \). It is known for long that the degree of squeezing of the two-mode cavity radiation can be investigated employing the quadrature operators corresponding to \( \hat{c} \),

\[
\hat{c}_{+} = \hat{c}^\dagger + \hat{c}, \tag{43}
\]

and

\[
\hat{c}_{-} = i(\hat{c}^\dagger - \hat{c}). \tag{44}
\]

On the basis of the boson commutation relation, the squeezing is said to exist when

\[
\Delta c_{\pm}^2 = 1 + \langle \hat{a} \hat{a}^\dagger \rangle + \langle \hat{b} \hat{b}^\dagger \rangle \pm (\langle \hat{a} \hat{b} \rangle + \langle \hat{a}^\dagger \hat{b}^\dagger \rangle) + (\langle \hat{a} \hat{b}^\dagger \rangle + \langle \hat{a}^\dagger \hat{b} \rangle)
\]

\[
+ \langle \hat{b} \hat{b}^\dagger \rangle \pm \frac{1}{4}[\langle \hat{a}^4 \rangle + \langle \hat{b}^4 \rangle + \langle \hat{a}^2 \rangle + \langle \hat{b}^2 \rangle + \langle \hat{a} \rangle^2]
\]

\[
+ \langle \hat{b} \rangle^2 + \langle \hat{a} \rangle^2 + \langle \hat{b} \hat{b}^\dagger \rangle \pm (\langle \hat{a} \hat{b} \rangle + \langle \hat{a}^\dagger \hat{b}^\dagger \rangle)
\]

\[
+ \langle \hat{a} \rangle (\langle \hat{a}^\dagger \hat{b}^\dagger \rangle + \langle \hat{a} \hat{b} \rangle) + \langle \hat{b} \rangle (\langle \hat{a}^\dagger \hat{b} \rangle + \langle \hat{a} \hat{b}^\dagger \rangle) \tag{45}
\]

is less than one. Following a similar approach as in section 3, it is possible to verify that

\[
\Delta c_{\pm}^2 = \frac{\kappa^2 (\kappa \mp 2\epsilon_2) + \kappa \epsilon_2^2 + 4\kappa \epsilon_2^2 \pm 2\kappa \epsilon_2 \epsilon_3}{(\kappa \mp \epsilon_2)(\kappa (\kappa \mp 2\epsilon_2) + 4\epsilon_2^2)}, \tag{46}
\]

which can also be put on the basis of the fact that \( \epsilon_2 = \kappa/2 \) for \( \epsilon_1 \neq 0 \) in the form

\[
\Delta c_{\pm}^2 = \frac{\kappa^2 (5 \mp 4) + 4\kappa (4\epsilon_1 \pm \kappa)}{2(2 \mp 1)(\kappa^2 (1 \mp 1) + 4\epsilon_2^2)}. \tag{47}
\]

In contrast to the usual trend of treating the residual pump mode classically, it turns out that the superimposed cavity radiation exhibits a significant degree of two-mode squeezing for certain values of \( \epsilon_1 \). As clearly indicated in figure 3 the degree of two-mode squeezing decreases with the cavity damping constant for smaller values of \( \epsilon_1 \), but it increases for larger values. It is not difficult to note that a maximum obtainable squeezing is witnessed slightly above the threshold value and it is independent of the cavity damping constant. It is found that a maximum two-mode squeezing of about 42% occurs at various values of \( \epsilon_1 \) for different damping constants. Most recently, the same result has been reported following a different approach and using different parameters [10]. Moreover, the value of \( \epsilon_1 \) for which a maximum squeezing occurs increases with the cavity damping constant. On the other hand, critical scrutiny of figure 3 reveals that the maximum squeezing is found at \( \epsilon_1 = \kappa \). Comparison between the results given in figures 2 and 3 shows that the two-mode squeezing exists even for values of \( \epsilon_1 \) for which
there is no corresponding entanglement. It goes without saying that the difference in the degree of entanglement and squeezing is essentially related to the correlations that lead to these phenomena. The correlations between similar states of a radiation (like \(\hat{a}^\dagger\) and \(\hat{b}^\dagger\)) at microscopic level do not contribute to the entanglement, since the entanglement characteristically requires two different states of radiation to be correlated; this requirement should not be true for squeezing.

5. Mean photon number and intensity difference

The mean photon number of the cavity radiation corresponding to the superposition of the two available modes in the cavity can be expressed as

\[
\bar{n} = \langle \hat{c}(t)\hat{c}(t) \rangle, 
\]

where \(\hat{c}(t)\) is the annihilation operator given by equation (42). Hence it is not difficult to see that

\[
\bar{n} = \frac{1}{2} \left[ \langle \hat{a}^\dagger(t)\hat{a}(t) \rangle + \langle \hat{b}^\dagger(t)\hat{b}(t) \rangle + \langle \hat{a}^\dagger(t)\hat{b}(t) \rangle + \langle \hat{b}^\dagger(t)\hat{a}(t) \rangle \right].
\]

In order to obtain the mean photon number, one needs to determine the involved cross correlations which by the way do not correspond to a photon number of the involved modes. To this effect, making use of equations (8), (9), (15)–(20) and the correlations of the Langevin noise operators, one finds at steady state

\[
\langle \hat{a}^\dagger(t)\hat{b}(t) \rangle_{ss} = \langle \hat{b}^\dagger(t)\hat{a}(t) \rangle_{ss} = -\frac{\kappa_\delta E_2}{4(\epsilon_2^2 - 4\epsilon_1^2)} \times \left[ \frac{\kappa - \epsilon_2}{\kappa(\kappa - 2\epsilon_2) + 4\epsilon_1^2} - \frac{\kappa + \epsilon_2}{\kappa(\kappa + 2\epsilon_2) + 4\epsilon_1^2} + \frac{2\epsilon_2}{\kappa^2 - \epsilon_2^2} \right].
\]

Thus applying equations (34), (35), (37), (38), (49) and (50), one can arrive at

\[
\bar{n} = \frac{1}{2} \left[ (\alpha + \beta)^2 + \frac{\kappa(\kappa - 2\epsilon_2)(1 + p^2 + q^2)}{8[\kappa(\kappa - 2\epsilon_2) + 4\epsilon_1^2]} \right. \\
+ \frac{\kappa(\kappa + 2\epsilon_2)(1 + p^2 + q^2)}{8[\kappa(\kappa + 2\epsilon_2) + 4\epsilon_1^2]} + \frac{\kappa^2(1 + p^2 - q^2)}{4(\epsilon_2^2 - 4\epsilon_1^2)} \\
- \frac{\kappa^2(1 + p^2 - q^2)}{4(\epsilon_2^2 - 4\epsilon_1^2)} - \frac{(1 + p^2 - q^2)}{4(\epsilon_2^2 - 4\epsilon_1^2)} - \frac{\kappa E_2}{4(\epsilon_2^2 - 4\epsilon_1^2)} \\
\left. \times \left[ \frac{\kappa - \epsilon_2}{\kappa(\kappa - 2\epsilon_2) + 4\epsilon_1^2} - \frac{\kappa + \epsilon_2}{\kappa(\kappa + 2\epsilon_2) + 4\epsilon_1^2} + \frac{2\epsilon_2}{\kappa^2 - \epsilon_2^2} \right] \right].
\]

which can also be put for \(\epsilon_1 \neq 0\) in the form

\[
\bar{n} = \frac{(2\epsilon_1 - \kappa^2 - 4\epsilon_2^2)}{8\lambda^2} + \frac{\kappa^3(\kappa - 2\epsilon_1)}{32\epsilon_1^2(\kappa^2 - 16\epsilon_1^2)} \\
+ \frac{3\kappa^3(\kappa + 2\epsilon_1)}{16(\kappa^2 - 16\epsilon_1^2)(\kappa^2 + 2\epsilon_1^2)} + \frac{2\epsilon_1(\kappa + 16\epsilon_1)}{3(\kappa^2 - 16\epsilon_1^2)}.
\]

It can be deduced from equation (52) that the mean photon number takes larger values where there is no entanglement. Fortunately, the mean photon number increases with \(\epsilon_1\) after certain value. Therefore, as one can see from figure 4, considerably strong entangled two-mode radiation can be generated in a degenerate parametric down conversion process. This is believed to be an encouraging result. It can also readily be observed that the mean photon number decreases with the cavity damping constant, since more photons escape from the cavity for larger damping constant.

On the other hand, the intensity difference can be defined as

\[
\Delta I = \hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}.
\]

Employing equations (34), (35), (37) and (38), the mean of the intensity difference at steady state turns out to be

\[
\Delta I = \alpha^2 - \beta^2 - \frac{\kappa p\sqrt{\epsilon_2^2 - 4\epsilon_1^2}}{2[\kappa(\kappa - 2\epsilon_2) + 4\epsilon_1^2]} \\
+ \frac{\kappa p\sqrt{\epsilon_2^2 - 4\epsilon_1^2}}{2[\kappa(\kappa + 2\epsilon_2) + 4\epsilon_1^2]},
\]

in which using equation (19) along with the fact that \(\epsilon_2 = \kappa/2\) for \(\epsilon_1 \neq 0\) leads to

\[
\Delta I = \frac{4\epsilon_1^2 - \kappa^2}{4\lambda^2} - \frac{\kappa^2}{16\epsilon_1^2} + \frac{\kappa^2}{8(\kappa^2 + 2\epsilon_1^2)}.
\]

As clearly shown in figure 5, the mean of the intensity difference is nonnegative for all values of \(\epsilon_1\). This indicates that in principle the nonlinear crystal can down-convert a significant amount of light falling on it. It can be noted that the number of down-converted photons available in the cavity as compared to the number of photons of the external pumping radiation depends on the damping constant. Moreover, it is not difficult to note that the mean intensity difference decreases for smaller values of \(\epsilon_1\) but it increases for larger values with damping constant. On the basis of equation (14), it is possible to realize that \(\epsilon_1\) increases with the amplitude of the figure 4. Plots of the mean of the photon number (\(\bar{n}\)) at steady state for \(\lambda = 0.5\) and different values of \(\kappa\).
external pumping radiation provided that the cavity damping and coupling constants are taken to be independent of the amplitude of the pumping radiation. Hence in view of this fact, it is observed that the process of parametric down-conversion decreases with the amplitude of the external pumping radiation close to the critical point but it increases after a particular value of $\varepsilon_1$. It is hoped that such an interpretation can lead to a new way of quantifying the process of down-conversion by the crystal in relation to various parameters.

6. Conclusion

A thorough study of the squeezing of the two-mode superimposed radiation and entanglement in the fundamental and residual pump modes of a degenerate parametric oscillator coupled to a two-mode vacuum reservoir is presented. It turns out that the cavity radiation exhibits a significantly strong two-mode squeezing in which the degree of squeezing depends on the rate at which the external pumping radiation is down-converted. A maximum squeezing of about 42% occurs slightly above a threshold value for different amplitudes of the external pumping radiation for various damping constants. On account of equation (53) it can be noted that the mean of the intensity difference is positive, provided that more than 33.3% of the external pumping radiation is down-converted. Therefore comparison between figures 3 and 5 reveals that a two-mode squeezing exists when at least 33.3% of the external pumping radiation is down-converted. This leads to the understanding that although the squeezing of the fundamental mode has attracted a great deal of attention in previous studies, the degree of squeezing of the two-mode radiation is also equally significant with additional possible applications of course. As recently discussed in detail, the two-mode squeezing increases with the amplitude of the external pumping radiation and the squeezing occurs above the critical point ($\varepsilon_1 > 0$) [10]. Moreover, the two-mode squeezing is found to be maximum at $\varepsilon_1 = \kappa$. Though the maximum squeezing is independent of the damping constant, generally, the degree of squeezing substantially depends on the damping constant.

The strong correlation at microscopic level via the involved Langevin noise operators between the fundamental and residual pump modes due to their harmonic relation leads not only to quadrature squeezing, but also to entanglement. Furthermore, since the radiation has more chance to oscillate back and forth in the cavity for smaller damping constant the probability that it can be entangled is higher. This must be the reason for the decrement of the degree of entanglement with the cavity damping constant, which is also true for the squeezing near threshold. Comparison between figures 2 and 5 shows that unlike the degree of squeezing, the entanglement is found to be stronger in a region where the down-conversion of the external pumping radiation is more efficient. These differences in the degree of squeezing and entanglement are attributed to the differences in the correlations leading to these phenomena. It is a well-established fact that occurrence of the entanglement requires a correlation between two different states of the radiation, a restriction that does not necessarily apply to squeezing in harmonically related radiations.

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