1 Introduction

In a recent letter we have presented the bare finite temperature effective potential for regularized $SU(2)$ Yang-Mills theory taking the Haar measure into account\[1\]. Here we present the renormalized effective potential at the order $g^2$ and show that it develops a non-perturbative minimum for sufficiently strong coupling, i.e. below a critical temperature. We discuss that this minimum corresponds a phase which can be a candidate for the confining phase of the continuum theory.

As to confinement in pure Yang-Mills theories general arguments were given that the deconfining phase transition occurs with increasing temperature if the theory is confining at zero temperature \[2, 3\]. It was also shown that lattice gauge theory does not confine static quarks if the Haar measure is replaced by the Euclidean one \[4\]. Non-trivial phases for $SU(2)$ continuum theory has been predicted in Ref. \[6\].

The 2–loop contribution of the order $g^2$ to the effective potential of $SU(N)$ gauge theory is well-known in the perturbative regime \[8, 7\]. Now we determine the effective potential for $SU(2)$ up to this order by using the non-perturbative mean-field approach and treating the fluctuations around the mean field perturbatively. This mixture of the non-perturbative and perturbative approaches provides some insight in the mechanism producing the non-perturbative phase below the critical temperature. We assume that the component $A_{\mu=0}^{a=3}$ ($\mu$ Lorentz index, a $SU(2)$ colour index) of the gluon vector potential has the non-vanishing expectation value, $v \equiv \langle a^{-1} \beta g A^{03} \rangle$ ($a$ the lattice spacing for the regularized path integral, $\beta$ the inverse temperature). Doing so we cannot assume any more that the field $A^{03}$ is small and, consequently, we must not replace the Haar measure,

$$\prod_x d^3 \alpha_x^a \frac{\sin^2 \left( \frac{1}{2} (\alpha_x^a \alpha_x^b \alpha_x^c) \right)^{1/2}}{\alpha_x^a \alpha_x^b}$$

(1.1)

(with $\alpha_x^a = g \beta a^{-1} A_x^{0a}$, \[10, 11\]) by the Euclidean one in the path integral. This is in contrast to that one generally does in the perturbative approach.

Lattice results have shown that confinement and global center symmetry are strongly related \[10\]. Since global center symmetry of the path integral is guaranteed by the usage of the Haar measure, the question arises whether does the Haar measure influence the effective potential. There are general
arguments that it does not influence the physics in the perturbative regime [12, 13]. It was shown for SU(2) Yang-Mills theory that the tree level contribution of the Haar measure is cancelled by a piece of the 1-loop contribution of the longitudinal gluons [9]. Thus the Haar measure does not play any role in the effective potential at the order $g^0$ even if a finite background field is assumed. In [1] we have shown that the Haar measure reveals itself in the bare effective potential of the pure SU(2) Yang-Mills theory at the order $g^2$.

The authors of [6] found a cancellation of the Haar measure induced terms using a different approximation. Later we discuss the possible reason of this disagreement. Nevertheless, our results for the bare theory are in qualitative agreement with the findings for lattice gauge theory [5] as discussed in [1]. Here we show that the basic features of the renormalized effective potential are not influenced by the Haar measure. It can reveal itself only in the fine details of the effective potential (in the approximation used).

Our strategy of treating the IR and UV divergences is as follows. The bare theory is formulated in terms of three dimensional parameters: the inverse temperature $\beta$, the UV momentum cut-off $\Lambda$, and the IR momentum cut-off $\mu$. Thus UV divergences can occur as powers and logarithms of $\beta\Lambda$ and $\Lambda/\mu$. UV divergences of the first type are those controlled by power counting and are removed by subtracting loops in the limit $\beta \to \infty$ [14]. Of the second type UV divergences accompanying the IR divergences need some particular treatment. Their counterterms are introduced by choosing the IR momentum cut-off as a power series of $1/\Lambda$ and the logarithmically divergent terms are neglected. This removes both IR and UV divergences. The IR singularities are now removed in a way getting independent of the UV cut-off $\Lambda$ when approaching the continuum limit (the rather correct approach instead of our earlier one [1]), since the IR problem in the real world is solved via dynamical generation of electric and magnetic masses, defining a physical mass scale independent of the UV cut-off.

In [9] the effective potential was determined at the order $g^0$ allowing for the non-vanishing vacuum expectation value $C/g \equiv (a/\beta)(v/g)$. In the present paper we determine the terms of the order $g^2$ of the effective potential by using the general techniques of finite temperature field theory [14]. Following [9] we use a time independent, diagonal gauge, and periodic boundary conditions for the spatial components of the vector potential in the ‘time’ direction. On the other hand we take all the terms of the order $g^2$ with in the effective potential. In this gauge the Haar measure induced potential in the
tree level action takes the form:

\[ V_{H\text{ tree}} = -\frac{1}{a^3} \int d^3x \ln \left( 1 - \cos \alpha^3(\vec{x}) \right). \quad (1.2) \]

Our discussion starts with establishing the 2-loop corrections in terms of the gluon propagators. Then we determine their explicit expressions revealing their dependence on the IR momentum cut-off \( \mu \) and the UV momentum cut-off \( \Lambda \), and perform the first step of the renormalization procedure by subtracting the corresponding loops at zero temperature. As to the next we complete the renormalization procedure by removing the additional UV divergences together with the IR divergences. Finally, we discuss the phase structure of the continuum \( SU(2) \) Yang-Mills theory on the base of the renormalized effective potential obtained.

Throughout this paper we use the notations of Ref. [9] and use the same cut-off regularization. The UV cut-off \( \Lambda \) is interpreted in terms of the lattice spacing \( a \) used for the definition of the path integral via \( a^{-3} = (2\pi)^{-3} \int_0^\Lambda d^3k \) with \( \Lambda = (6\pi^2)^{1/3} a^{-1} \) and \( d^3k \) the volume element in 3-momentum space. The same spacing \( a \) is assumed in the ‘time’ direction.

## 2 The bare effective potential at 2-loop order

Let us allow for the non-vanishing expectation value \((a/\beta)(v/g)\) of the component \( A^{03} \) of the gluon vector potential and shift the field variable by it. Inserting \( A^{03}(\vec{x}) = (a/\beta)(v/g) + \delta \phi(\vec{x}) \) in the Euclidean action

\[ S_{YM} = -\frac{1}{4a^4\hbar} \int_0^\beta dx^0 \int d^3x F^{\mu \nu a} F_{\mu \nu a} \quad (2.1) \]

of the Yang-Mills system (\( F^{\mu \nu a} \) the field strength tensor), the tree level action takes the following form:

\[ S_{\text{tree}} = S_0 + S_1 + S_2. \quad (2.2) \]

The term

\[ S_0 = \frac{1}{2a^4\hbar} \int_0^\beta dx^0 \int d^3x a^2 \left[ (\nabla^i \delta \phi)^2 + \left( \partial^0 A^{ia} + (v/\beta)e^{a_3b} A^{ib} \right)^2 \right. \]

\[ \left. + \frac{1}{2} (\partial^i A^{ia} - \partial^i A^{ia})^2 \right] \]

\[ - \frac{1}{a^3} \int d^3x \ln(1 - \cos v) + \frac{1}{2a^2} \frac{g^2 a^{-2} \beta^2}{1 - \cos v} \int d^3x (\delta \phi)^2 \quad (2.3) \]
contains the action of the free gluon field of the order \((g\sqrt{\hbar})^0\) and the Haar
measure induced self-interaction of timelike gluons, a term of order \(g^2\hbar\) (the
fields are of order \(\hbar^{1/2}\)). The other Haar measure induced vertices are
neglected. \(S_1\) and \(S_2\) contain the usual 3-gluon and 4-gluon interaction vertices:

\[
S_1 = \frac{g}{a^4\hbar} \int d^4x a \left\{ \epsilon^{3d\delta} \delta \partial^0 A^{id} \left( \partial^0 A^{ia} + \left( v/\beta \right) \epsilon^{a3e} A^{ic} \right) + \epsilon^{abc} A^{ib} A^{jc} \partial^0 A^{ja} \right\},
\]

(2.4)

\[
S_2 = \frac{g^2}{2a^4\hbar} \int d^4x \left\{ \epsilon^{3ca} \epsilon^{3da} \partial^0 A^{ia} \left( \delta \phi \right)^2 + \frac{1}{2} \epsilon^{abc} \epsilon^{ade} A^{ib} A^{jc} A^{id} A^{je} \right\}
\]

(2.5)

(\(\epsilon^{abc}\) the completely antisymmetric tensor of rank 3). The self-interaction
terms \(S_1\) and \(S_2\) are of the order \(gh^{1/2}\) and \(g^2\hbar\), respectively.

The main goal of this section is to determine the effective potential as
function of the constant background field \(v\) including all the terms being of
the same order \(g^2\) as the one-loop contribution from the Haar measure. Then
the interaction \(S_1\) and \(S_2\) must be taken into account at 2-loop level. Our
procedure is the following:

- determine the partition function by treating the interaction terms \(S_1\)
  and \(S_2\) perturbatively;
  
- determine the effective potential from the 1PI diagrams.

In order to calculate the perturbative corrections we introduce the partition function in the presence of the external sources \(j^{ia}\) and \(q\):

\[
\mathcal{Z}[j^{ia}, q] = \exp \left\{ -\hat{S}_1 - \hat{S}_2 \right\} \mathcal{Z}_0[j^{ia}, q]
\]

(2.6)

with the generating functional of the free fields

\[
\mathcal{Z}_0[j, q] = \int \mathcal{D}\delta \phi \mathcal{D}A^{ia} e^{-S_0 + (jA) - (q\delta \phi)} = Z_0[0, 0] z[j, q],
\]

(2.7)

where

\[
z[j, q] = \exp \left\{ \frac{1}{2} \left( jDj \right) + \frac{1}{2} \left( qDq \right) \right\}
\]

(2.8)
and

\[ Z_0[0,0] = (\text{Det} \mathcal{M} \cdot \text{Det}(\mathcal{N} + \mathcal{N}_H))^{-1/2} \exp \left\{ a^{-3} \int d^3x \ln(1 - \cos v) \right\}. \] (2.9)

Here \( D = \mathcal{M}^{-1} \) and \( D = (\mathcal{N} + \mathcal{N}_H)^{-1} \) are the free propagators of the spatial and timelike gluons, respectively, defined through

\[ S_0 = -a^{-3} \int d^3x \ln(1 - \cos v) + \frac{1}{2}(A \mathcal{M} A) + \frac{1}{2}(\delta \phi (\mathcal{N} + \mathcal{N}_H) \delta \phi). \] (2.10)

The matrices \( D \) and \( \mathcal{M} \) (\( D \) and \( \mathcal{N}, \mathcal{N}_H \)) possess Lorentz, colour and space-time coordinate (spatial coordinate) indices, (\( \ldots \)) is a shorthand for contraction. The operator \( \hat{S}_{\text{int}} = \hat{S}_1 + \hat{S}_2 \) is defined by replacing the fields \( A^{ia} \) and \( \delta \phi \) by functional derivatives

\[ A^{ia} \rightarrow -\frac{\delta}{\delta j^{ia}}, \quad \delta \phi \rightarrow -\frac{\delta}{\delta \eta} \] (2.11)

with respect to the corresponding external sources in the expressions for \( S_1 \) and \( S_2 \).

We expand the exponential operator in powers of \( (g \hbar^{1/2}) \):

\[ e^{-\hat{S}_{\text{int}}} = 1 - \hat{S}_1 + \left[ -\hat{S}_2 + \frac{1}{2}(\hat{S}_1)^2 \right] + \mathcal{O} \left( (g \sqrt{\hbar}) \right)^3. \] (2.12)

After some algebra we obtain \( (\hbar = 1) \):

\[ \hat{S}_1 z[j,q] = -g \int dx \left\{ e^{3da}(Dq) \left[ (Dj)^{id} \left( \partial^0 (Dj)^{ia} + (v/\beta) \epsilon^{3ca} (Dj)^{ic} \right) \right. \right. \]
\[ + \partial^0 D^i_{iad} + (v/\beta) \epsilon^{3ca} D^idc \left. \right] + \epsilon^{abc} \left[ (Dj)^{ib} (Dj)^{ic} \partial_x^j (Dj)^{ja} + D^ijbc \partial^j(Dj)^{ja} \right. \]
\[ + (Dj)^{ib} \partial_x^j (Dj)^{ia} + (Dj)^{ja} \partial_x^i (Dj)^{ib} \left] \right\} z[j,q], \] (2.13)

\[ \hat{S}_1 z[j,q] \big|_0 = 0, \] (2.14)
and
\[ \hat{S}_2[j, q] \bigg|_0 = \frac{1}{2} g^2 \int \! dx \left\{ e^{3ca} e^{3da} D_{xx}^a D_{xx}^{icd} + e^{abc} e^{ade} \left( D_{xx}^{ibj} D_{xx}^{jde} + D_{xx}^{ibj} D_{xx}^{jicd} + D_{xx}^{ibj} D_{xx}^{jicd} \right) \right\} . \] (2.15)

The 1PI part of \( \hat{S}_1(\hat{S}_1 z) \big|_0 \) can easily be identified by noticing that (i) the terms containing first or third functional derivatives of \( z[j, q] \) with respect of the external sources vanish, (ii) the non-vanishing terms with second functional derivatives of \( z[j, q] \) are 1-particle reducible. Thus the terms not containing the derivatives of \( z \) are the only ones contributing to the 1PI part:
\[ \frac{1}{2} \hat{S}_1(\hat{S}_1 z) \bigg|_0 1\text{PI} = 'DDD' + 'DDD', \] (2.16)

where
\[ 'DDD' = e^{3da} e^{3eg} g^2 \int d^4 y d^4 x D_{xy} \cdot \]
\[ \cdot a^2 \left[ D_{xy}^{kied} \partial_y^0 D_{xy}^{kiga} + \partial_x^0 D_{xy}^{kicd} \cdot \partial_x^0 D_{xy}^{kigd} + (v/\beta)e^{3fg} \left( D_{xy}^{kied} \partial_y^0 D_{xy}^{kifa} + D_{xy}^{kifd} \partial_y^0 D_{xy}^{kidea} \right) + (v/\beta)e^{3ca} \left( D_{xy}^{kied} \partial_y^0 D_{xy}^{kifc} + D_{xy}^{kiec} \partial_y^0 D_{xy}^{kifd} \right) + (v/\beta)^2 e^{3ca} e^{3fg} \left( D_{xy}^{kied} D_{xy}^{kifc} + D_{xy}^{kiec} D_{xy}^{kifd} \right) \right], \] (2.17)

\[ 'DDD' = g^2 \frac{e^{abc} e^{gef}}{2 a^8} \int d^4 y d^4 x a^2 \left[ D_{xy}^{kib} D_{xy}^{lja} \partial_y^0 \partial_x^0 D_{xy}^{ljga} + D_{xy}^{kib} \partial_y^0 D_{xy}^{ljfa} \cdot \partial_x^0 D_{xy}^{ljgc} + D_{xy}^{kjec} D_{xy}^{lifb} \partial_y^0 \partial_x^0 D_{xy}^{ljga} + D_{xy}^{kjec} \partial_y^0 D_{xy}^{ljfa} \cdot \partial_x^0 D_{xy}^{ljgb} + D_{xy}^{lifb} \partial_y^0 D_{xy}^{kjea} \partial_x^0 D_{xy}^{ljga} + D_{xy}^{lifb} \partial_y^0 D_{xy}^{kjea} \partial_x^0 D_{xy}^{ljgb} \right]. \] (2.18)

The effective potential is defined by the 1PI part of the partition function.

The 1-loop contribution of the gluon self-interaction \( S_{\text{int}} \) to the effective potential vanishes identically due to Eq. (2.14). The 2-loop contribution \( \Delta V_{\text{eff}} \) is given by
\[ Z[0, 0]_{1\text{PI}} = Z_0[0, 0] \exp \{-\beta V \Delta V_{\text{eff}} \} \] (2.19)
Figure 1: 2–loop contributions to the effective potential.

\[ \Delta V_{\text{eff}} = (\beta V)^{-1} \left( \dot{S}_2 \right)_{[j,q]} \bigg|_0 - \frac{1}{2\beta V} \left( \dot{S}_1 \right)^2_{[j,q]} \bigg|_0 1\text{PI} \]
\[ \equiv f_1 + f_2 + f_3 + f_4 \quad (2.20) \]

(V \to \infty \text{ the 3-volume of the system}). The various terms on the r.h.s. of Eq. (2.20) correspond to the diagrams in Fig. 1 and represent the bare 2–loop contributions to the effective potential:

\[ f_{1B} = \frac{g^2}{2a^4} \epsilon^{abc} \epsilon^{ade} \left( D_{ijbc} D_{ijde} + D_{ijbe} D_{ijcd} + D_{iibd} D_{jjec} \right), \quad (2.21) \]

\[ f_{2B} = \frac{1}{2\beta V a^4} g^2 \int d^4x \left( \delta^{33} \delta^{cd} - \delta^{3c} \delta^{3d} \right) D_{xx} D_{ixcd} \]
\[ = \frac{g^2}{2a^4} \mathcal{D}_{xx} \left( D_{jicc} - D_{i33} \right), \quad (2.22) \]

\[ f_{3B} = \frac{1}{2\beta V a^8} g^2 \epsilon^{abc} \epsilon^{gef} a^2 \int d^4y d^4x \left[ D_{xy}^{kib} D_{xy}^{lje} \partial_x^i \partial_x^j \partial_x^k D_{xy}^{lga} \right] \]
\[ f_{AB} = -\frac{g^2}{2\beta V a^8} \varepsilon^{3da} \varepsilon^{3eg} \int d^4y d^4x D_{xy} \cdot \varepsilon^{2da} \varepsilon^{2eg} \sum_{i,j,k,l} D_{xy}^{kij} \partial_x^k D_{xy}^{lij} + D_{xy}^{kij} \partial_x^k D_{xy}^{lij} \partial_x^l D_{xy}^{kij} + D_{xy}^{kij} \partial_x^k D_{xy}^{lij} \partial_x^l D_{xy}^{kij} \cdot \partial_x^l D_{xy}^{kij} \right], \tag{2.23} \]

Later we shall discuss how to obtain the renormalized contributions \( f_1, f_2, f_3, \) and \( f_4 \) from the corresponding bare ones.

The 2-loop corrections (2.21)-(2.24) are given in terms of the free gluon propagators. After changing to momentum representation we read them off from the free action \( S_0 \). Then we obtain for the free propagator of timelike gluons

\[ D(\vec{k}) = \left[a^2(\vec{k}^2 + M^2)\right]^{-1} \tag{2.25} \]

with \( M^2 a^2 \equiv a^{-1} g^2 \beta (1 - \cos \nu)^{-1} \) and for that of the spatial gluons

\[ D^{ijab}(\omega_n, \vec{p}) = a^{-2} \left[ (\omega_n^2 + \vec{p}^2) \delta^{ij} \delta^{ab} + (v/\beta)^2 \delta^{ij}(\delta^{ab} - \delta^{a3} \delta^{b3}) \right. \]

\[ - p^i p^j \delta^{ab} - 2(v/\beta) \omega_n \varepsilon^{3ab} \delta^{ij} \right]^{-1} \tag{2.26} \]

with the bosonic Matsubara frequencies \( \omega_n = 2\pi n/\beta \ (n = 0, \pm 1, \pm 2, \ldots) \).

Separating the transverse and the longitudinal parts the matrix on the r.h.s. of Eq. (2.26) can be inverted:

\[ D^{ijab}(\omega_n, \vec{p}) = a^{-2} \Delta^{ab}(\omega_n, 0) \frac{p^i p^j}{\vec{p}^2} + a^{-2} \Delta^{ab}(\omega_n, \vec{p}) \left( \delta^{ij} - \frac{p^i p^j}{\vec{p}^2} \right), \tag{2.27} \]
with

\[
\Delta^{ab}(\omega_n, \vec{p}) = \begin{pmatrix}
1&1 \frac{1}{2} (d_n^+ (\vec{p}) + d_n^- (\vec{p})) & -\frac{1}{2} (d_n^+ (\vec{p}) - d_n^- (\vec{p})) & 0 \\
\frac{1}{2} (d_n^+ (\vec{p}) - d_n^- (\vec{p})) & 1 & 0 & 0 \\
0 & 0 & \frac{1}{2} (d_n^+ (\vec{p}) + d_n^- (\vec{p})) & 0 \\
& & & d_n(\vec{p})
\end{pmatrix}
\]

(2.28)

and 

\[d_n^\pm (\vec{p}) = [\left(\omega_n^2 + \vec{p}^2\right)^{-1}, d_n(\vec{p}) = (\omega_n^2 + \vec{p}^2)^{-1}, \omega_n^\pm = \omega_n \pm (v/\beta)].\]

In coordinate representation, the free propagators are then given by

\[
D_{\vec{x}\vec{y}} = a^3 \int \frac{d^3 k}{(2\pi)^3} \exp\{i\vec{k}(\vec{x} - \vec{y})\} = \frac{a}{2\pi} \exp\{-M|\vec{x} - \vec{y}|\},
\]

(2.29)

\[
D_{x^i x^j}^{ijab} = \beta^{-1} \sum_n \int \frac{d^3 p}{(2\pi)^3} D^{ijab}(\omega_n, \vec{p}) e^{i\vec{p}(\vec{x} - \vec{x}')} + i\omega_n(\tau - \tau').
\]

(2.30)

In order to avoid IR divergence we introduce the IR momentum cut-off \(\mu\) and replace \(\Delta^{ab}(\omega_n, 0)\) by \(\Delta^{ab}(\omega_n, \mu)\).

### 3 Explicit form of the 2–loop contributions

The explicit expressions for the various contributions \(f_\alpha (\alpha = 1, 2, 3, 4)\) to the effective potential were obtained by (i) performing the contraction of colour and spatial Lorentz indices; (ii) performing the Matsubara sums by contour integral technique; (iii) expressing the result in terms of the gluon occupation numbers. The first step of renormalization was carried out by subtracting from each Feynman diagram the similar ones with a loop taken in the limit \(\beta \to \infty\) in all possible ways, and neglecting all terms of the effective potential independent of the temperature and of the background field \(v\). Below we shall see that this renormalization procedure removes the terms which are linear in the gluon occupation numbers. Exceptions are some terms containing the occupation numbers of the zero mode (that for gluons with the IR cut-off momentum \(\mu\)). Therefore the ‘renormalized’ contributions \(f_\alpha\) obtained in this chapter still contain the UV divergences accompanying the IR divergences.

We analyze the contributions \(f_\alpha (\alpha = 1, 2, 3, 4)\) and separate the pieces \(f_\alpha \text{ gl}\) and \(f_\alpha \text{ vac}\) corresponding to the gluon gas and the vacuum, respectively: 

\[
f_\alpha = f_\alpha \text{ gl} + f_\alpha \text{ vac}.
\]

Each of these terms can be written as the sum: 

Figure 2: ‘Renormalized’ contribution $f_1$ to the effective potential.

\[ f_0^\alpha + f_\Delta^\alpha \] where $f_0^\alpha$ and $f_\Delta^\alpha$ are the contributions for $v = 0$ and the additional contributions due to $v \neq 0$, respectively. The terms depending on the occupation number of the zero mode are considered as part of the vacuum contribution.

Below we do not write out the terms of the order $\Lambda^s$ with $s = 0, 1, 2, 3$ for each contribution $f_\alpha$ separately. However, we shall include all these terms in our final expression for the bare effective potential in Sect. 5.

**Contribution $f_1$**

The ‘renormalized’ contribution $f_1$ corresponds to the diagrams in Fig. 2. Performing the subtractions on the r.h.s. of Eq. (2.21) we obtain for the ‘renormalized’ contribution

\[ f_1 = \frac{1}{4a^4} g^2 \epsilon^{abc} \epsilon^{ade} \left( \bar{D}_{xx}^{ijbc} D_{xx}^{ijde} + \bar{D}_{xx}^{ijbe} D_{xx}^{jicd} + \bar{D}_{xx}^{iibd} D_{xx}^{jiec} \right) \]  

(3.1)

with

\[ \bar{D}_{xx}^{ijab} = D_{xx}^{ijab} - 2 \lim_{\beta \to \infty} D_0^{ijab}. \]  

(3.2)

The calculation described in Appendix A gives the following expression for the renormalized glue gas contribution for $v = 0$ (see Eq. (A.14)):

\[ f_{1,gl}^0 = 4g^2 \int \frac{d^3p d^3q \, n_p n_q}{(2\pi)^6 \, pq} = \frac{g^2}{36\beta^4} \]  

(3.3)
with the occupation number \( n_p = \frac{1}{\left[\exp(\beta p) - 1\right]} \) of the gluon state with momentum \( \vec{p} \). Eq. (A.15), leads to the vacuum contribution for \( v = 0 \):

\[
\begin{align*}
 f_{1\ vac}^0 &= \frac{g^2}{\alpha^4} \left[ -\frac{\alpha^4}{16\pi^4} &- \frac{1}{\mu a} \frac{\alpha^2}{4\pi^2} \\
 &\quad + \frac{a^2}{3\beta^2} \frac{n_{\mu}}{\mu a} + \frac{1}{4\mu^2 a^2} (2n_{\mu} + 1)(2n_{\mu} - 1) \right]. \tag{3.4}
\end{align*}
\]

From Eq. (A.16) we find for the additional contributions due to \( v \neq 0 \):

\[
\begin{align*}
 f_{1\ gl}^\Delta &= \frac{g^2}{3\pi^2 \beta^4} \left| \sin \frac{v}{2} \right| \left( \left| \sin \frac{v}{2} \right| - \frac{2\pi}{3} \right), \tag{3.5}
\end{align*}
\]

\[
\begin{align*}
 f_{1\ vac}^\Delta &= \frac{g^2}{3} \left[ -\frac{2}{\pi a^2 \beta^2 (\mu a)} (N_{\mu}(v) + n_{\mu}) + \frac{2}{3a^2 \beta^2 (\mu a)} (N_{\mu}(v) - n_{\mu}) \\
 &\quad + \frac{1}{a^4 (\mu a)^2} (N_{\mu}(v) - n_{\mu}) (N_{\mu}(v) + 3n_{\mu}) \right] \tag{3.6}
\end{align*}
\]

in terms of the ‘generalized occupation number’ \( N_{\mu}(v) = N_{\mu=v}(v) \) defined via Eq. (A.7). The terms \( f_{1\ gl}^0 \) and \( f_{1\ gl}^\Delta \) contribute to the energy density of the blackbody radiation, whereas the vacuum contributions \( f_{1\ vac}^0 \) and \( f_{1\ vac}^\Delta \) are IR divergent and depend on the temperature. Their dependence on the IR momentum cut-off \( \mu \) is revealed using the expansions:

\[
\begin{align*}
 n_{\mu} &\approx \frac{1}{\beta \mu} - \frac{1}{2} + \frac{1}{12} \beta \mu - \frac{1}{720} (\beta \mu)^3 + \frac{1}{720 \cdot 42} (\beta \mu)^5 + \mathcal{O}(\mu) \tag{3.7}
\end{align*}
\]

\[
\begin{align*}
 N_{\mu}(v) &\approx -\frac{1}{2} + \frac{1}{2} A \beta \mu + \frac{1}{12} B (\beta \mu)^3 + \mathcal{O}(\mu^5) \tag{3.8}
\end{align*}
\]

with \( A = (1 - \cos v)^{-1} \), \( B = A - 3 A^2 \). Then we obtain:

\[
\begin{align*}
 f_{1\ vac}^0 + f_{1\ vac}^\Delta &= \frac{2g^2 \Lambda^6}{(2\pi)^4 27} \left[ -\frac{2}{\beta \mu^3} + \frac{2}{\mu^2 A} \beta \mu \left( \frac{1}{6} + 2A \right) + \beta^2 \left( \frac{2}{3} A - A^2 \right) \right] + \mathcal{O}(\Lambda^3) \tag{3.9}
\end{align*}
\]
Figure 3: ‘Renormalized’ contribution $f_2$ to the effective potential.

after removing the constants independent of the temperature and the background field $v$. Thus only the terms quadratic in the occupation numbers of the zero modes contribute.

• Contribution $f_2$

Making use of the expressions (2.25)-(2.30) of the free propagators, we rewrite the bare contribution (2.22) as follows:

$$f_{2B} = \frac{g^2}{2\pi a^4} e^{-Ma} [\sigma(\mu, v) + 2R(v)]$$  \hspace{1cm} (3.10)

where $\sigma(p, v)$ is given by Eq. (A.6) and from Eqs. (A.9), and (A.10) we get:

$$R(v) = a^2 \int \frac{dp}{(2\pi)^3} \sigma(p, v) \approx \frac{a^2}{12\beta^2} + \frac{\alpha^2}{8\pi^2} - \frac{a^2}{2\pi\beta^2} \left| \sin \frac{v}{2} \right|.$$ (3.11)

The ‘renormalized’ contribution is represented by the diagrams in Fig. 3 and is given by

$$f_2 = \frac{g^2}{2\pi a^4} e^{-Ma} \left[ a^{-1} (\sigma(\mu, v) - \sigma_\infty(\mu, v)) + 2 (R(v) - R_\infty(v)) \right]$$

$$= -\frac{g^2}{2\pi a^4} e^{-Ma} \left[ \frac{1}{\mu a} N_\mu(v) + \frac{a^2}{\beta^2} \left( \frac{1}{6} - \frac{1}{\pi} \right) \left| \sin \frac{v}{2} \right| \right]$$  \hspace{1cm} (3.12)

with $\sigma_\infty(\mu, v) = \lim_{\beta \to \infty} \sigma(\mu, v) = \sigma_\infty(\mu, 0)$, $R_\infty(v) = \lim_{\beta \to \infty} R(v) = R_\infty(0)$ given by Eqs. (A.13) and (A.12). Here we made use of the fact
that the loop with the propagator $D$ vanishes in the limit $\beta \to \infty$ due to $M \sim \beta \to \infty$. The contribution $f_2$ is IR divergent and depends on the temperature. It can be considered as part of the free energy density of the vacuum, $f_{2\,\text{vac}} = f_2$ and $f_{2\,\text{gl}} = 0$. For $v = 0$, i.e. $M \to \infty$ it vanishes, $f_{2\,\text{vac}}^0 = 0$, and for $v \neq 0$, i.e. finite $M$ we get $f_{2\,\text{vac}}^\Delta = f_2$. Using the expansion (3.8) we obtain:

$$f_{2\,\text{vac}}^\Delta = f_2 = -\frac{2g^2}{(2\pi)^4} \frac{1}{2} \Lambda^5 \beta A + O(\Lambda^3).$$

\section*{Contribution $f_3$}

After Fourier transformation the bare contribution $f_{3\,\text{B}}$ given by Eq. (2.23) takes the form:

$$f_{3\,\text{B}} = -\frac{1}{2g^2} \int \frac{d^3p d^3q d^3Q}{(2\pi)^9} \cdot (2\pi)^3 \delta(p + q + Q) \beta \delta_{n_1 + n_2 + n_3} \cdot a^2 \left[ (D^{kib}(\omega_{n_1}, \bar{q}) D^{jfc}(\omega_{n_2}, \bar{p})) Q^i Q^j Q^k D^{lig}(\omega_{n_3}, \bar{Q}) + (D^{kib}(\omega_{n_1}, \bar{q}) D^{jfa}(\omega_{n_2}, \bar{p})) p^i Q^j Q^k D^{lig}(\omega_{n_3}, \bar{Q}) + (D^{kje}(\omega_{n_1}, \bar{q}) D^{jfa}(\omega_{n_2}, \bar{p})) p^i Q^j Q^k D^{lig}(\omega_{n_3}, \bar{Q}) + (D^{kje}(\omega_{n_1}, \bar{q}) D^{jfa}(\omega_{n_2}, \bar{p})) p^i Q^j Q^k D^{lig}(\omega_{n_3}, \bar{Q}) \right].$$

The 'renormalized' contribution of the diagrams in Fig. 4 is given by (Appendix B):

$$f_3 = -\frac{1}{2} g^2 \int \frac{d^3p d^3Q}{(2\pi)^6} \mathcal{I},$$

where the integrand takes the form:

$$\mathcal{I} = \mathcal{I}_2 + \mathcal{I}_2' + \mathcal{I}_1 + \mathcal{I}_1' + \mathcal{I}_\beta + \mathcal{I}_\beta' + \mathcal{I}_0.$$
With the help of the momentum dependent functions $P_{\phi_3}(\vec{q}, \vec{p}, \vec{Q})$, $\Phi(\vec{q}, \vec{p}, \vec{Q})$, and $\phi_0(\vec{q}, \vec{p}, \vec{Q})$ defined in Appendix B by Eqs. (B.14), (B.13), and (B.9), respectively, and the ‘occupation numbers’ $n_q^\pm = [\exp(\beta E^\pm_q) - 1]^{-1}$ with $E^\pm_q = q \pm i v / \beta$ we find the expressions below for the various terms of the integrand. (For equalities holding only up to the terms containing odd powers of the scalar product $\vec{p}\vec{Q}$ stands $\sim$.) The terms quadratic in the occupation numbers give contributions to the blackbody radiation term:

$$I_2 = -(n_p^+ + n_p^-) \frac{4n_Q + n_Q^+ + n_Q^-}{4Qp} \left[ -\frac{1}{2} \frac{(\vec{p}\vec{Q}) P_{\phi_3}}{(\vec{p}\vec{Q})^2 - p^2 Q^2} \frac{p^2 + Q^2}{(p^2 - Q^2)^2} \Phi \right]$$

$$= -(n_p^+ + n_p^-) \frac{4n_Q + n_Q^+ + n_Q^-}{4Qp} \left( 1 + \frac{(\vec{p}\vec{Q})^2}{p^2 Q^2} \right)$$

$$\sim -(n_p^+ + n_p^-) \frac{4n_Q + n_Q^+ + n_Q^-}{4Qp} \frac{4}{3} \left[ \frac{p^2 + Q^2}{(p^2 - Q^2)^2} \right] \Phi(\vec{q}, \vec{p}, \vec{Q})$$

(3.17)

$$I_2' = -\frac{1}{8} (n_p^+ - n_p^-)(n_Q^+ - n_Q^-) \frac{P_{\phi_3}}{(\vec{p}\vec{Q})^2 - p^2 Q^2} \Phi(\vec{q}, \vec{p}, \vec{Q})$$

(3.18)

The term $I_1$ is linear in the occupation number at the IR cut-off and is IR divergent:

$$I_1 = \left[ \frac{2n_Q + n_Q^+ + n_Q^-}{2Q(p^2 - Q^2)} \frac{n_p^+ + n_p^-}{\mu} + \frac{n_Q^+ + n_Q^-}{Q(p^2 - Q^2)} \frac{n_p^+ + n_p^-}{\mu} \right] \Phi(\vec{q}, \vec{p}, \vec{Q})$$
The term $I'_1$ vanishes for vanishing background field identically; it is IR finite:

$$I'_1 = -\frac{1}{p^3\mu} \left[ (n^+_p n^-_p + 2n_-) (n^+ + n^-) + 2(n^+_p + n^-_p) n^+ \right] \phi_0(\vec{q}, \vec{p}, \vec{Q}) \sim -\frac{1}{\mu} \left[ (2nQ + n^+_Q + n^-_Q) (n^+ + n^-) + (n^+_Q + n^-_Q) n^+ \right] \cdot \left[ \frac{\phi_0(\vec{q}, \vec{Q}, \vec{p})}{Q^3} - \frac{\Phi(\vec{q}, \vec{p}, \vec{Q})}{2Q(p^2 - Q^2)} \right] ,$$  

(3.19)

The terms quadratic in the occupation number at the IR cut-off are given by:

$$I_\beta = \frac{1}{2p^2\mu^2} (n^+_\mu + n^-_\mu + 4n^-)(n^+_\mu + n^-_\mu) \phi_0(\vec{q}, \vec{p}, \vec{Q}),$$  

(3.21)

and

$$I'_\beta = -\frac{1}{p^4} (n^+_\mu - n^-_\mu)^2 \phi_0(\vec{q}, \vec{p}, \vec{Q}).$$  

(3.22)

The integrand $I_\beta$ is temperature dependent and IR divergent. Finally, there is the IR divergent, temperature independent part of the integrand:

$$I_0 = \frac{1}{2qpQ(p + q + Q)} \mathbf{P} \phi_3(\vec{q}, \vec{p}, \vec{Q}) - \frac{3\Phi(\vec{q}, \vec{p}, \vec{Q})}{2pQ(p + Q)} \left( \frac{1}{\mu} - \frac{1}{p + Q} \right) - \frac{3}{p^2\mu^2} \left( 1 - \frac{2\mu}{p} \right) \phi_0(\vec{q}, \vec{p}, \vec{Q}).$$  

(3.23)
The corresponding contribution to the effective potential is a part of the energy of the perturbative vacuum.

Introducing the notation

\[ f_3 \cdots = -\frac{1}{2} g^2 \int \frac{d^3pd^3Q}{(2\pi)^6} \mathcal{I} \cdots \]  

(3.24)

we write

\[ f_3 = f_{32} + f'_{32} + f_{31} + f'_{31} + f'_{3\beta} + f_{30}. \]  

(3.25)

Here the terms \( f'_{32}, f'_{31}, \) and \( f'_{3\beta} \) contribute only for \( v \neq 0 \), whereas the other terms can be cast into a piece for \( v = 0 \) and the rest vanishing for \( v = 0 \): \( f_{32} = f_{32}^0 + f_{32}^\Delta, f_{31} = f_{31}^0 + f_{31}^\Delta, \) and \( f_{3\beta} = f_{3\beta}^0 + f_{3\beta}^\Delta \).

The only terms not suffering from IR divergences are \( f_{32} \) and \( f'_{32} \), the contributions to the density of the free energy of the glue gas,

\[ f_{3\text{gl}} = f_{32}^0 + f_{32}^\Delta + f'_{32}. \]  

(3.26)

For \( v = 0 \) we find:

\[ f_{3\text{gl}}^0 \equiv f_{32}^0 = 2 g^2 \int \frac{d^3pd^3Q n_Qn_p}{(2\pi)^6} \frac{Q}{Qp} = \frac{g^2}{72\beta^4}. \]  

(3.27)

It is a contribution to the energy of the blackbody radiation.

For \( v \neq 0 \) we obtain the additional contribution to the density of the free energy of the glue gas, \( f_{3\text{gl}}^\Delta = f_{32}^\Delta + f'_{32} \) where (see Eqs. (B.36), (B.37), and (B.38)-(B.40) in Appendix B)

\[ f_{32}^\Delta = \frac{1}{24\pi^4 \beta^4} \left( \frac{4\pi^2}{3} + G(v) \right) G(v), \]  

(3.28)

\[ f'_{32} \approx \frac{7}{2e^{-2}(2\pi)^4 \beta^4} \frac{g^2}{(\cosh 1 - \cos v)^2} \frac{\sin^2 v}{(1 - (1/e))(2 - (5/e))} \]  

(3.29)

with \( G(v) \) defined by Eq. (B.37).
The IR divergent terms

\[ f_3 \text{\_vac} = f_{31} + f_{3\beta} + f_{30} + f'_{31} + f'_{3\beta} \]  \hfill (3.30)

contribute to the free energy of the vacuum. Their dependence on the IR cut-off \( \mu \) is revealed as described at the end of Appendix B. Thus we obtained ( \( A = (1 - \cos v)^{-1} \)):

\[
 f_{3 \text{\_vac}} = \frac{2g^2}{(2\pi)^4} \left\{ \begin{array}{l}
 1 \frac{\Lambda^7}{\mu^4 \beta} \frac{25}{128 \cdot 7} \\
 + \frac{1}{\mu^3} \left[ -\Lambda^7 \frac{93}{128 \cdot 3 \cdot 7} A + \frac{\Lambda^6}{\beta} \frac{2431}{16 \cdot 9 \cdot 5 \cdot 7} + \frac{\Lambda^6}{12\beta} \ln \frac{\Lambda}{\mu} \right] \\
 + \frac{1}{\mu^2} \left[ \frac{\Lambda^5}{\beta} \frac{33}{32 \cdot 5} \ln(\beta \mu) - \frac{\Lambda^6}{12} A \ln \frac{\Lambda}{\mu} \right] \\
 + \Lambda^7 \beta \left( \frac{17}{16 \cdot 9 \cdot 7} - \frac{1}{32e} - \frac{5}{64 \cdot 7} A - \frac{1}{64} A^2 \right) \\
 - \Lambda^6 \frac{5}{64 \cdot 9 \cdot 5 \cdot 7} A - \frac{\Lambda^5}{32 \cdot 5 \beta} \left( \frac{101}{3} + \frac{33}{e} - 33A \right) \\
 + \frac{1}{\mu} \left[ \frac{\Lambda^7 \beta^2}{64 \cdot 3} (A - 3A^2) - \frac{\Lambda^5}{64 \cdot 5} \ln(\beta \mu) \right] \\
 + \frac{1}{\mu} \frac{\Lambda^6 \beta}{24} \left( \frac{1}{6} + 2A \right) \ln \frac{\Lambda}{\mu} \\
 + \frac{1}{\mu} \left[ \frac{\Lambda^7 \beta^2}{64} \left( \frac{5}{3} - \frac{4}{e} - \frac{93}{9 \cdot 7} A - \frac{281}{4 \cdot 3 \cdot 7} A^2 \right) \right] \\
 + \Lambda^6 \beta \frac{2431}{32 \cdot 9 \cdot 5 \cdot 7} \left( \frac{1}{6} + 2A \right) - \Lambda^5 \frac{1423}{64 \cdot 3 \cdot 5} A - \frac{3\Lambda^4}{4\beta} \right] \ln(\beta \mu) \\
 + \left[ \frac{\Lambda^7 \beta^3}{64} \left( \frac{1}{360} - \frac{A}{3} + A^2 \right) + \frac{\Lambda^5 \beta}{30} \left( \frac{13}{3} + \frac{1531}{32} A \right) \right] \ln(\beta \mu) \\
 + \left[ \frac{\Lambda^6 \beta^2}{48} (-A + A^2) + \frac{\Lambda^4}{16} A \right] \ln \frac{\Lambda}{\mu} \\
 + \left[ \frac{\Lambda^7 \beta^3}{128} \left( \frac{101}{2 \cdot 9 \cdot 5 \cdot 7} - \frac{1}{3e} + \frac{593}{2 \cdot 9 \cdot 7} A - \frac{10}{e} A \right) \right. \\
 - \frac{411}{2 \cdot 3 \cdot 7} A^2 - A^3 \right) \right. 
\]
Figure 5: ‘Renormalized’ contribution $f_4$ to the effective potential

$$
\begin{align*}
\frac{\Lambda^6 \beta^2}{128 \cdot 9 \cdot 5 \cdot 7} & \left( \frac{10669}{2} - 5807 A \right) \\
+ \Lambda^5 \beta & \left( -\frac{4851}{128 \cdot 9 \cdot 5} + \frac{159}{16 \cdot 3 \cdot 5 \epsilon} + \frac{1587}{128 \cdot 9} A \right) \\
- & \left( \frac{77}{32 \cdot 3 \epsilon} A + \frac{23}{24 A^2} \right) \\
+ \Lambda^4 A & \left( \frac{551}{4 \cdot 3 \cdot 5 \cdot 7} \right) \\
+ O(\Lambda^3). 
\end{align*}
$$

(3.31)

The terms independent of $\beta$ and $\nu$ are neglected.

- **Contribution $f_4$**

As discussed in Appendix C we obtain the expression (C.12) for the contribution $f_4$ after carrying out the subtractions shown in Fig. 5.

In the perturbative regime with $\nu = 0$, i.e. $M \to \infty$ we get $f_4 = f_4^\Delta_{\text{vac}} = 0$ from Eq. (C.12). This is because the infinitely heavy timelike gluons do not propagate.

Following the procedure described at the end of Appendix C, we get

$$
\begin{align*}
\Delta f_4_{\text{vac}} = f_4 &= -\frac{2g^2}{(2\pi)^4} \frac{1}{3(6\pi^2)^{1/3}} \frac{\nu^2}{\beta^2} \left( \frac{\Lambda^5}{\mu^3} + 2 \frac{\Lambda^4}{\mu^2} \right) + O(\Lambda^3). 
\end{align*}
$$

(3.32)

Here the terms independent of $\beta$ and $\nu$ are neglected.
4 Blackbody radiation

The complete 2–loop contribution to the density of the free energy of the glue gas is given by \( f_{gl} = f_{gl}^0 + f_{gl}^\Delta \).

Consider at first \( f_{gl}^0 \) for the case \( v = 0 \). The renormalized diagrams 2 and 4 (Fig. 1) do not contribute to the glue gas as discussed in Sect. 3. The renormalized diagrams 1 and 3 (Fig. 1) give the gauge invariant contribution to the free energy density of the glue gas (see Eqs. (3.3) and (3.27)):

\[
f_{gl}^0 = f_{1,gl}^0 + f_{3,gl}^0 = (4 + 2)g^2 \int \frac{d^3p d^3Q}{(2\pi)^6} \frac{n_Q n_p}{Q_p} = \frac{g^2}{24\beta^4}. \tag{4.1}
\]

This result is just the same as that one obtains by adding the contribution of the gauge field and that of the ghosts in the usual perturbative calculation given in [8]:

\[
c_K g^2 \int \frac{d^3p d^3Q}{(2\pi)^6} \frac{n_Q n_p}{Q_p} \tag{4.2}
\]

with \( c_K = c_{ghost} + c_1 + c_3 \), where the weights of the ghosts, diagram 1 and diagram 3 are \( c_{ghost} = \frac{1}{4} \cdot 6 \), \( c_1 = \frac{12}{4} \cdot 6 \), and \( c_3 = -\frac{4}{3} \cdot 6 \), respectively, for \( SU(2) \), so that \( c_K = 6 \). The contributions of the single diagrams are different in our case, they are gauge dependent, but the sum of them is gauge invariant, as expected [8].

For \( v \neq 0 \) the additional contributions to the free energy of the glue gas are due to diagrams 1 and 3, as \( f_{2,gl}^\Delta = f_{4,gl}^\Delta = 0 \). According to Eqs. (3.3), (3.28), and (3.29) we find:

\[
f_{gl}^\Delta = f_{1,gl}^\Delta + f_{3,gl}^\Delta = \frac{g^2}{24\beta^4} \left[ \frac{8}{\pi^2} \left| \sin \frac{v}{2} \right| \left( \left| \sin \frac{v}{2} \right| - \frac{2\pi}{3} \right) + \frac{1}{\pi^4} \left( \frac{4\pi^2}{3} + G(v) \right) G(v) 
+ \frac{21}{4\pi^4 e^{-2}} \frac{\sin^2 v}{(\cosh 1 - \cos v)^2 (1 - e^{-1}) (2 - 5e^{-1})} \right] \tag{4.3}
\]

with \( e \) the basis of the natural logarithm, and the function \( G(v) \) given by Eq. (B.37). This contribution vanishes for \( v = 0 \).

The sum \( f_{gl} = f_{gl}^0 + f_{gl}^\Delta \) represents the complete contribution of the glue gas to the free energy density. This is an IR finite contribution, independent.
of the IR cut-off momentum $\mu$. The free energy of the vacuum, however, depends on the background field $v$ and the IR cut-off momentum, as well. In order to reveal its $v$ dependence, we have to get rid of the dependence on the IR momentum cut-off.

5 IR divergent contributions

Gathering the IR divergent pieces obtained in Sect. 3 we find the following contribution to the effective potential:

$$V_{IR} = f_1^0_{vac} + f_1^\Delta_{vac} + f_2^\Delta_{vac} + f_3_{vac} + f_4^\Delta_{vac}$$

$$= V_2 + V_0$$

(5.1)

with $A = (1 - \cos v)^{-1}$ and

$$V_2 \approx \frac{2g^2}{(2\pi)^4} \left\{ \mu^{-4} \Lambda^7 \beta^{-1} b_{41} 
+ \mu^{-3} \left[ -\Lambda^7 b_{31} A + \Lambda^6 \beta^{-1} b_{32} - \Lambda^5 \beta^{-2} b_{31} v^2 \right] 
+ \mu^{-3} \frac{\Lambda^6}{12 \beta} \ln(\Lambda/\mu) 
+ \mu^{-2} \left[ \Lambda^7 \beta(b_{21} - b_{22} A - b_{23} A^2) - \Lambda^6 b_{24} A 
+ \Lambda^5 \beta^{-1} (-b_{25} + b_{26} A) - \Lambda^4 \beta^{-2} b_{27} v^2 \right] 
+ \mu^{-2} \left[ \Lambda^5 \beta^{-1} b_{28} \ln(\beta \mu) - \Lambda^6 (A/12) \ln(\Lambda/\mu) 
+ \Lambda^3 \beta^{-3} \Gamma_{23}(v) + \Lambda \beta^{-5} \Gamma_{21}(v) + \beta^{-6} \Gamma_{20}(v) \right] 
+ \mu^{-1} \left[ \Lambda^7 \beta^2 (-b_{11} + b_{12} A + b_{13} A^2) 
+ \Lambda^6 \beta(b_{14} + b_{15} A) - \Lambda^5 b_{16} A - \Lambda^4 \beta^{-1} b_{17} 
- \Lambda^3 \beta^{-2} \Gamma_{13}(v) - \Lambda \beta^{-4} \Gamma_{11}(v) - \beta^{-5} \Gamma_{10}(v) \right] 
+ \mu^{-1} \left[ \Lambda^7 \beta^2 b_{18} (A - 3 A^2) - \Lambda^5 (b_{19} + b_{110} A) \right] \ln(\beta \mu) 
+ \mu^{-1} \left[ \Lambda^6 (\beta/144) (1 + 12 A) + \Lambda^4 b_{111} A \right] \ln(\Lambda/\mu) 
+ \Lambda^7 \beta^3 (b_{01} + b_{02} A - b_{03} A^2 - b_{04} A^3) + \Lambda^6 \beta^2 (-b_{05} A + b_{06} A^2) 
+ \Lambda^5 \beta (-b_{07} + b_{08} A + b_{09} A^2) + \Lambda^4 b_{010} A \right\} \ln(\Lambda/\mu)$$
\[ +\Lambda^3\beta^{-1}\Gamma_{03}(v) + \Lambda^2\beta^{-2}\Gamma_{02}(v) + \Lambda\beta^{-3}\Gamma_{01}(v) + \beta^{-4}\Gamma_{00}(v) \]
\[ + \left[ \frac{\Lambda^7\beta^3}{64} \left( \frac{1}{360} - \frac{1}{3}A + A^2 \right) + \Lambda^5\beta(b_{01} + b_{02}A) - \frac{\Lambda^3}{4\beta} \right] \ln(\beta\mu) \]
\[ + \left[ \frac{\Lambda^6\beta^2}{48} (A - A^2) + \Lambda^4b_{013}A - 0.52\Lambda^2\beta^{-2}v^2 \right] \ln(\Lambda/\mu) \]
\[ = 0 \]
\[ (5.2) \]

and
\[ V_0 \approx \frac{1}{4\pi^4(6\pi^2)^{1/3}\beta A} \left\{ \mu^{-2}\Lambda^3\beta^{-2}v^2 \right\} \ln(\Lambda/\mu) \]
\[ + \mu^{-1}\Lambda^2\beta^{-2}v^2 24\pi^4 + \pi^4 \left(-2.25\Lambda^3 + 78\Lambda\beta^{-2}v^2 \right) \ln(\Lambda/\mu) \]
\[ - 643\Lambda^3 + \Lambda\beta^{-2}(-130 + 1022v^2) \]
\[ (5.3) \]

where the constants \(b_{\_}\) are positive numbers:
\[ b_{41} = 0.028, b_{31} = 0.035, b_{32} = 0.408, b_{33} = 0.086, \]
\[ b_{21} = 0.0054, b_{22} = 0.011, b_{23} = 0.016, b_{24} = 0.424, b_{25} = 0.287, b_{26} = 0.206, \]
\[ b_{27} = 0.171, b_{28} = 0.206, \]
\[ b_{11} = 0.003, b_{12} = 0.023, b_{13} = 0.052, b_{14} = 0.033, b_{15} = 0.408, b_{16} = 1.48, \]
\[ b_{17} = 0.793, b_{18} = 0.0052, b_{19} = 1.65, b_{110} = 0.667, b_{111} = 0.063, \]
\[ b_{01} = 0.0003, b_{02} = 0.008, b_{03} = 0.076, b_{04} = 0.0078, b_{05} = 0.108, b_{06} = 0.107, \]
\[ b_{07} = 0.60, b_{08} = 0.66, b_{09} = 0.96, b_{10} = 1.31, b_{101} = 0.145, b_{102} = 1.59, \]
\[ b_{103} = 0.063, \]
and
\[ \Gamma_{23}(v) = 14.35 - 0.23A + 0.05A^2 - 2.79 \left| \sin \frac{v}{2} \right|, \]
\[ (5.4) \]
\[ \Gamma_{21}(v) = -3.48 + 0.28A - 0.054A^2, \]
\[ (5.5) \]
\[ \Gamma_{20}(v) = -0.60 + 0.048A - 0.016A^2, \]
\[ (5.6) \]
\[ \Gamma_{13}(v) = 17.66 - 0.23A + 0.05A^2 - 2.79 \left| \sin \frac{v}{2} \right|, \]
\[ (5.7) \]
\[ \Gamma_{11}(v) = -0.52 + 0.28A - 0.054A^2, \]
\[ (5.8) \]
\[ \Gamma_{10}(v) = -3.24 + 0.16A - 0.016A^2, \]
\[ (5.9) \]
\[ \Gamma_{03}(v) = 1.429 - 3.96A - 0.182A^2 + 0.025A^3 \]
\[ - (0.233 + 1.40A) \left| \sin \frac{v}{2} \right|, \]
\[ (5.10) \]
\[ \Gamma_{02}(v) = 1.94 v^2, \quad (5.11) \]
\[ \Gamma_{01}(v) = -0.064 - 0.904 A + 0.023 A^2 - 0.008 A^3. \quad (5.12) \]

The contribution \( V_0 \) of the order \( g^0 \) arises from the diagram \( f_4 \) through expansion in powers of \((\mu/M)^2 \sim \mu^2 \beta^{-1} \Lambda^{-3} g^{-2}\). Thus terms of the order \( g^2 (1/g^2) = 1 \) occur. The terms of higher order in \((1/g^2)\) vanish as some powers of \((1/\Lambda)\).

The bare finite temperature effective potential \( V_{\text{eff}}(\beta, v) \) as function of the constant background field \( A^{i3} = (a/\beta)(v/g) \), \( A^{01} = A^{02} = A^{ia} = 0 \) \((i = 1, 2, 3; a = 1, 2, 3)\) takes now the following form:

\[ V_{\text{eff}}(\beta, v) = V_W(\beta, v) + V_H(\beta, v) + \Delta V(\beta, v) + V_{IR}, \quad (5.13) \]

where

\[ V_W(\beta, v) = -\frac{2\pi^2}{\beta^4} \left\{ \frac{1}{45} - \frac{1}{24} \left[ 1 - \left( \frac{|v|}{\pi} - 1 \right)^2 \right] \right\}, \quad (5.14) \]

is the term obtained by Weiss [4],

\[ V_H = \frac{\alpha_0}{a^4 \sin^2 \frac{v}{\pi}} g^2 \quad (5.15) \]

with \( \alpha_0 = (6\pi^2)^{1/2} \) is the term induced by the Haar measure, the 2–loop contribution due to the usual non-abelian self-interaction is given by Eq. (4.3) as

\[ \Delta V(\beta, v) = f_{gl}^A = \]

\[ = \frac{g^2}{24\beta^4} \left\{ 1 + \frac{8}{\pi^2} \left| \sin \frac{v}{2} \right| \left( \left| \sin \frac{v}{2} \right| - \frac{2\pi}{3} \right) + \frac{1}{\pi^4} \left( \frac{4\pi^2}{3} + G(v) \right) G(v) \right. \]

\[ + \frac{21}{4\pi^4 e^{-2} (\cosh 1 - \cos v)^2} \left( 1 - e^{-1} \right) \left( 2 - 5e^{-1} \right) \left( 1 - e^{-1} \right) \left( 2 - 5e^{-1} \right) \right\}, \quad (5.16) \]

and the IR divergent piece is given by Eq. (5.4). As the variable \( A^{03} \) is compact, we have to set forth periodically the expression of the effective potential given for \( v \in [0, 2\pi] \).

The terms periodic in \( v \) arose due to the longitudinal (spatial) gluons. As observed in [4] their Matsubara frequencies are shifted by the background.
field, $\omega_n \pm (v/\beta)$. Having rewritten the Matsubara sums over the longitudinal modes as contour integrals on the complex energy plane, they are dominated by the complex poles at $|\vec{p}| \pm i(v/\beta)$. Thus the generalized occupation numbers $n^\pm_p$ and $N_p(v)$ arise and lead to terms periodic in $v$.

Here we recover our observation made in [1], that the Haar measure term $\sim +g^2\Lambda^4 A$ of the bare effective potential is not cancelled by the appropriate IR finite piece $\sim +g^2b_{010}\Lambda^4 A$ due to the longitudinal gluons as $b_{010} > 0$. Our finding that the Haar measure term, $g^2\Lambda^4 A$ of the bare effective potential is not cancelled by a corresponding piece of the contributions of longitudinal gluons is in disagreement with the results of [6]. This disagreement can be the consequence of the difference of the both approaches. We treat the quadratic piece of the Haar measure induced potential (the last term of Eq. (2.3)) non-perturbatively, including it into the free propagator of the field $\delta\phi$, and the first term of Eq. (2.5) perturbatively. On the other hand both terms are treated non-perturbatively in [6]. Below we show, however, that the Haar measure does not influence the main qualitative features of the renormalized effective potential in our approximation. Thus the question of cancellation of the Haar measure is not crucial as far as our conclusions on the basic features of the effective potential are considered.

The bare effective potential represents a sum of IR singular terms, each of them being a sum of powers of the UV cut-off. IR divergences appearing as powers of the dimensionless parameter $\Lambda/\mu$ lead to additional UV divergences of higher order than one would expect by power counting. These are yet present although the subtractions of loop integrals at zero temperature have been performed.

Now we remove the IR singularities. For this purpose we choose the IR momentum cut-off $\mu$ as a power series of the inverse of the UV cut-off (i.e. of the lattice spacing). Actually we introduce the power series for the dimensionless IR cut-off $\beta\mu$ in terms of $1/(\beta\Lambda)$:

$$ (\beta\mu)^{-1} = \sum_{k=0}^{7} \mu_k (\beta\Lambda)^{-k}. \quad (5.17) $$

The series can be terminated by $k = 7$, as the terms of higher order will vanish in the continuum limit when the series is inserted in the bare effective
potential. For later use we write:

\[(\beta \mu)^{-s} = \sum_{k=0}^{7} m_{sk}(\beta \Lambda)^{-k}, \quad (s = 1, 2, 3, 4) \quad (5.18)\]

where the coefficients are obtained by taking the powers of (5.17). The IR cut-off defined by Eq. (5.17) becomes independent of the UV cut-off for large values of \(\Lambda\) and proportional to the temperature \(T = 1/\beta, \mu \to (1/\mu_0)T\).

For the sake of simplicity we neglect the logarithmically divergent term \(s\). Then the coefficients on the r.h.s. of Eq. (5.17) are defined by requiring that the coefficients of \(\Lambda^7, \Lambda^6, \ldots, \Lambda\) vanish in the effective potential (5.13) having inserted the sum (5.17) in it. These renormalization conditions lead to highly non-linear equations for the determination of the coefficients \(\mu_k\) \((k = 0, 1, \ldots, 7)\). Assuming that our procedure gives a reasonable result, i.e. the inequality \(\mu_0 \gg 1\) holds, the coefficients \(m_{sk}\) in Eq. (5.18) take the rather simple form: \(m_{s0} = \mu_0^s, \; m_{sk} = s\mu_0^{s-1}\mu_k\) for \(k = 1, 2, \ldots, 7\), and \(s = 2, 3, 4\). Keeping only the terms of leading order in \(\mu_0\), the renormalization conditions give rise to the following equations linearized in \(\mu_k\) \((k = 1, 2, \ldots, 6)\):

\[0 = b_{41}m_{40} - b_{31}A m_{30} + \mathcal{O}(\mu_0^2), \quad (5.19)\]
\[0 = b_{41}m_{41} + b_{32} m_{30} + \mathcal{O}(\mu_0^2), \quad (5.20)\]
\[0 = b_{41}m_{42} - v^2 b_{33} m_{30} + \mathcal{O}(\mu_0^2), \quad (5.21)\]
\[0 = b_{41}m_{43} + \mathcal{O}(\mu_0^2), \quad (5.22)\]
\[0 = b_{41}m_{44} + \mathcal{O}(\mu_0^2), \quad (5.23)\]
\[0 = b_{41}m_{45} + \mathcal{O}(\mu_0^2), \quad (5.24)\]
\[0 = b_{41}m_{46} + \mathcal{O}(\mu_0^2). \quad (5.25)\]

The approximate solution of these equations is given by

\[\mu_0 = \rho_0 A, \quad \mu_1 = -\rho_1, \quad \mu_2 = v^2 \rho_2, \quad \mu_k \sim \mathcal{O}(1/\mu_0) \approx 0 \quad \text{for} \quad k = 3, 4, \ldots, 6 \quad (5.26)\]
with \( \rho_0 = b_{31}/b_{41} \), \( \rho_1 = b_{32}/(4b_{41}) \), and \( \rho_2 = b_{33}/(4b_{41}) \). Thus the coefficients \( \mu_k \) \((k \geq 3)\) are suppressed and can be set zero in our approximation. As the series terminates at \( k = 3 \) it seems to be reasonable to choose \( \mu_7 = 0 \).

In the continuum limit the IR momentum cut-off takes \( \mu = 0 \) for \( v = 0 \), whereas it has its maximum value \( \mu = 1.6/\beta \approx (\pi/2)T \) for \( v = \pi \). Thus the IR momentum cut-off is much less than the lowest non-vanishing Matsubara frequency \( 2\pi T \). This seems to be a justification for considering the coefficient \( \mu_0 \) large.

Inserting the limiting value of the IR momentum cut-off in the expression (5.13), we obtain for the renormalized effective potential:

\[
V_R = V_W + V_{0R} + V_{2R}
\]

with

\[
V_{0R} = 0.097 \, v^4 \beta^{-4}(1 - \cos v),
\]

\[
V_{2R} = \frac{g^2}{8\pi^4\beta^4} \left\{ P_0(v) + P_1(v) A^2 + P_2(v) A^4 - 0.40A^3 - 0.025A^4 \right. \\
\left. + \left(-55.5 + 15.7v^2\right)|\sin \frac{v}{2}| + 26.3 \, \sin^2 \frac{v}{2} \right\}
\]

where

\[
P_0(v) = 25.4 - 80.5v^2 + 0.46v^6,
\]

\[
P_1(v) = 35.5 + 1.29v^2,
\]

\[
P_2(v) = -3.9 - 0.28v^2 + 1.27\sin^2 v.
\]

Here we used approximately \( G(v) \approx -0.32A - 1.2 \). The purely polynomic part of the renormalized effective potential arises due to the decay of the \( A^{03} \) condensate into two transverse gluons and a timelike fluctuation \( \delta \Phi \) which then annihilate reproducing the condensate. This is a piece of the contribution \( f_4 \). The term \( \sim gv(A^1 A^2 + A^2 A^2)\delta \Phi \) in Eq. (2.4) for the cubic self-interaction is responsible for this subprocess.

Now we can come back to the question of the significance of the Haar measure as far as the renormalized effective potential is concerned. The Haar measure term \( V_H \) present in the bare effective potential explicitly as
found in [1] gives a contribution of the order one to the coefficient of $\Lambda^4$, i.e. to the r.h.s of Eq. (5.22). Thus it means a correction of the order $1/\mu_3^0$ to the leading term of $\mu_3$ being of the order $1/\mu_0$. Thus we conclude that the Haar measure term does not influence the basic features of the renormalized effective potential in our approximation. Nevertheless it reveals itself in the details explicitly.

The periodic terms of the renormalized effective potential occur due to the longitudinal gluons in our approximation, as discussed above. They are invariant under the global center transformation $v \to (2\pi - v)$. The polynomials break the global center symmetry explicitly. This explicit symmetry breaking is the consequence of the ansatz $\langle A^{03} \rangle \sim v$ inserted in the tree action. It would be a way to preserve the global center symmetry that one makes use of the Yang-Mills action derived in [10].

Here we observed renormalizability at the order $g^2$. Although the Haar measure generates infinitely many vertices, the general results for the nonlinear sigma model [12] and those for renormalizable lattice field theories [13] give the hope that the theory is renormalizable.

At the end of this section let us make some remarks on the validity of the expression obtained for the renormalized effective potential: (i) $V_2$ tends to infinity for $v = 0$, this must not be taken, however, seriously as the expansions used for the generalized occupation number are only valid for $v \neq 0$. It means that our effective potential is valid in the middle of the interval $v \in [0, 2\pi]$. At the ends of the interval we have to accept the perturbative results of [4, 5]. (ii) Our approach of linearizing the renormalization conditions and solving them for the coefficients $\mu_k$ of the expansion of the IR cut-off is rather crude as the coefficient $\mu_0$ becomes of the order one for $v \approx \pi$. (iii) It is unclear what effect the ‘less dangerous’ logarithmic divergences on the effective potential have.

6 Phase structure

Below we show that the renormalized effective potential exhibits a very remarkable feature. Namely, it has a minimum at non-vanishing background field $v \neq 0$ which becomes deeper than the minimum of $V_W$ at $v = 0$ describing the perturbative phase. Furthermore, the position of the new minimum is rather close to $v = \pi$ which corresponds to the vanishing Polyakov line.
\[ \cos(v/2) = 0, \text{ i.e. the confinement.} \]

It is easy to see that the polynomial \( P_0(v) \) has a minimum for \( v \approx 0.9\pi \) and the periodic functions in \( V_{2R} \) have their minima at \( v = \pi \). Therefore we seek the extremum of the effective potential at \( v = \pi - \varepsilon \) with \( \varepsilon \ll \pi \). The necessary condition for having a minimum at \( v = \pi - \varepsilon \) is given by:

\[
0 = \frac{\partial V_R}{\partial v} \bigg|_{v=\pi-\varepsilon} = \left[ \frac{\partial V_W}{\partial v} + \frac{\partial V_{0R}}{\partial v} + \frac{\partial V_{2R}}{\partial v} \right]_{v=\pi-\varepsilon}
\]

\[
= \beta^{-4} \left\{ \frac{1}{3} \varepsilon + (23.3 - 13.1\varepsilon) + \frac{g^2}{8\pi^4} (342.4 - 6534.3\varepsilon) \right\} . \quad (6.1)
\]

The solution is given by

\[
\varepsilon = 0.052 \left( g^2 + 53.0 \right) \left( g^2 + 1.5 \right)^{-1} \quad (6.2)
\]

leading to \( \varepsilon = 1.8, 1.1, \) and 0.05 for \( g^2 = 0, 1, \) and \( g^2 \to \infty \), respectively. For strong coupling the position of the extremum is defined by the term \( V_{2R} \). For weak coupling the term \( V_{0R} \) defines the position of the extremum in our approximation. However, then \( \pi - v \) gets large and our expression for \( V_R \) is not valid any more. It is easy to show (with accuracy of \( \mathcal{O}(g^2) \) ) that we obtain a minimum at \( v \neq 0 \) for any coupling, as

\[
\frac{\partial^2 V_R}{\partial v^2} \bigg|_{v=\pi-\varepsilon} \approx \beta^{-4} \left( 52.7 + 9.2g^2 \right) > 0. \quad (6.3)
\]

It is well-known that there is a minimum at \( v = 0 \), since for weak coupling, i.e. small values of \( v \) the formula \( V_W \) modified by the 2-loop correction (1.1) is valid [8]. Our expression is not valid for small values of \( v \). For sufficiently strong coupling the minimum found by us is in the validity range of our expression for the effective potential. Then there should be a maximum somewhere between the minima at \( v = 0 \) and \( v = \pi - \varepsilon \). The value of the effective potential at \( v = 0 \) is given by \( V_W(v = 0) + g^2/(24\beta^4) = \beta^{-4} (-0.44 + 0.04g^2) \). The value of the effective potential in the minimum at \( v = \pi - \varepsilon \) is given by

\[
V_R(v = \pi - \varepsilon) = \beta^{-4} \left( 18.0 - 0.26g^2 \right) \quad (6.4)
\]

for large \( g^2 \) (i.e. for \( \varepsilon = 0.052 \)). Thus we conclude that the minimum at \( v \neq 0 \) is deeper than that at \( v = 0 \) for sufficiently strong coupling: \( g^2 > g_c^2 \) with the
critical coupling \( \alpha_c = g_c^2/(4\pi) \approx 4.9 \). Using the running coupling constant \( g^2 = -G_2/\ln(\Lambda_2^2\beta^2) \) with the \( SU(N = 2) \) scale parameter \( \Lambda_2 \) and \( G_2 = 4\pi(11N/3)^{-1} \), the following estimate is obtained for the critical temperature: \( T_c = \Lambda_2 \exp(G_2/g_c^2) \approx \Lambda_2 \).

It is rather interesting that the non-trivial minimum of the effective potential due to the polynomic part \( P_0(v) \) is very close to \( v = \pi \) corresponding to vanishing Polyakov line \( W = \cos(v/2) \) \[9, 1\]. As discussed above, the basic subprocess leading to this non-perturbative minimum is the decay of the \( A^{03} \) condensate in two transverse gluons and a timelike fluctuation \( \delta \phi \), which annihilate once again into the condensate. This subprocess is suppressed for \( v \to 0 \) due to the infinitely increasing mass \( M \sim (1 - \cos v)^{-1/2} \to \infty \) of the fluctuations \( \delta \phi \), but it turns out to decrease the free energy of the vacuum for finite values of \( M \), i.e. for \( v \neq 0 \). The decrease of the free energy is expected to have a maximum for the smallest value of the mass \( M \), i.e. for \( v = \pi \). In our result the small shift \( \varepsilon \) of the minimum of the effective potential is most probably the consequence of the various approximations.

7 Summary

In the present article we have derived the renormalized finite temperature effective potential of the continuum \( SU(2) \) Yang-Mills theory as function of the non-vanishing expectation value \( v \) of the vector potential component \( A^{03} \), \( v = g(\beta/a)\langle A^{03} \rangle \). The quantum fluctuations around this mean field are taken into account up to the order \( g^2 \) perturbatively. The Haar measure has been taken into account in the definition of the partition function of the theory in order to treat large fields correctly. Due to the non-vanishing background field IR divergent terms occurred in the effective potential accompanied by UV divergences. Both are removed by choosing the IR momentum cut-off as a power series of the inverse of the UV momentum cut-off. The renormalization conditions requiring the cancellation of UV divergences have been solved in a linearized form. (Logarithmically divergent terms were neglected.) In the continuum limit the IR momentum cut-off turned out to be proportional to the temperature and takes values smaller than the first non-vanishing Matsubara-mode.

In our approximation the main qualitative features of the finite temperature effective potential are independent of the Haar measure. It reveals itself
only in the rather fine details. The renormalized effective potential has a minimum at non-vanishing background field very close to \( v = \pi \). For sufficiently strong coupling \( g^2 > g_c^2 \), i.e. below the critical temperature \( T < T_c \approx \Lambda_2 \) (with the SU(2) scale parameter \( \Lambda_2 \)) this non-trivial minimum is deeper than the minimum at \( v = 0 \) describing the perturbative phase of the theory. It is well-known that a minimum at \( v = \pi \) corresponds to vanishing Polyakov line, i.e. confinement. In our approach some important non-perturbative effects are already taken into account by the mean field, although our treating the field fluctuations around the mean field perturbatively is not really justified for strong coupling. Consequently, we cannot expect to obtain precise information on the details of the phase transition (the position of the non-perturbative minimum, the order of the phase transition, etc.) Nevertheless, we can conclude that the main mechanism responsible for the existence of the non-perturbative phase is the decay of the \( A^{03} \) condensate (the background field \( v \)) in two transverse gluons and a timelike one.

In order to get more reliable information on the position of the non-trivial minimum our treatment had to be improved at least by using the global center invariant tree level action proposed in [10, 11], not to break global center symmetry explicitly at the beginning. Then all polynomials of \( v \) would be replaced by some polynomials of periodic functions of \( v \) and the non-trivial extremum would occur exactly at \( v = \pi \).

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The contribution \( f_1 \) is given by the expression (3.1). Using Eqs. (2.27) and (2.30) for the propagator we see that it is diagonal in the spatial indices \((ij)\) for \( x = x' \), \( \tilde{D}_{xx}^{ijab} = \frac{1}{3} \delta^{ij} D^{ab} \), and, consequently, the matrix \( \tilde{D}^{ijab} = \frac{1}{3} \delta^{ij} D^{ab} \) defined by Eq. (B.2), as well. Here we introduced the traces with respect of the spatial indices:

\[
D^{ab} = \frac{a^2}{\beta} \sum_n \int \frac{d^3p}{(2\pi)^3} \left[ \Delta^{ab}(\omega_n, \mu) + 2\Delta^{ab}(\omega_n, \vec{p}) \right],
\]

(A.1)
and

\[
\bar{D}^{ab} = \frac{a^2}{\beta} \sum_n \int \frac{d^3p}{(2\pi)^3} \left[ \tilde{\Delta}^{ab}(\omega_n, \mu) + 2\tilde{\Delta}^{ab}(\omega_n, \vec{p}) \right]
\] (A.2)

with

\[
\frac{a^2}{\beta} \sum_n \tilde{\Delta}^{ab} = \frac{a^2}{\beta} \sum_n \Delta^{ab} - 2 \lim_{\beta \to \infty} \frac{a^2}{\beta} \sum_n \Delta^{ab}.
\] (A.3)

Performing the sums over the colour indices we obtain the following expression:

\[
f_1 = \frac{g^2}{3a^4} \left( \bar{D}^{33}D^{11} + \bar{D}^{11}D^{33} + \bar{D}^{11}D^{11} - \bar{D}^{12}D^{12} \right),
\] (A.4)

where the last term vanishes identically due to \( D^{12} = 0 \).

Further we make use of the following Matsubara sums:

\[
\sigma(p, 0) = \frac{1}{\beta} \sum_n d_n(p) = \frac{1}{2p}(2n_p + 1),
\] (A.5)

\[
\sigma(p, v) = \frac{1}{\beta} \sum_n d^+_n(p) = \frac{1}{2p} \left( 2N_p(v) + 1 \right)
\] (A.6)

with the Bose-Einstein occupation number \( n_p = \left( e^{\beta p} - 1 \right)^{-1} \) and the ‘generalized occupation number’ \( N_p(v) \) defined by

\[
2N_p(v) + 1 = \frac{\sinh(\beta p)}{\cosh(\beta p) - \cos v}.
\] (A.7)

Let us introduce the integral:

\[
R(v) = a^2 \int \frac{d^3p}{(2\pi)^3} \sigma(p, v).
\] (A.8)

For \( v = 0 \) we find:

\[
R(0) = a^2 \int \frac{d^3p}{(2\pi)^3} \left( \frac{n_p}{p} + \frac{1}{2p} \right) = \frac{a^2}{12\beta^2} + \frac{a^2}{8\pi^2}
\] (A.9)
with $\alpha = a\Lambda = (6\pi^2)^{1/3}$. For $v \neq 0$ let us write $R(v) = R(0) + \Delta R$ and $\sigma(\mu, v) = \sigma(\mu, 0) + \Delta \sigma$ with

$$
\Delta R = a^2 \int \frac{d^3p}{(2\pi)^3} \frac{\sinh(\beta p)}{2p} \left[ \frac{1}{\cosh(\beta p) - \cos v} - \frac{1}{\cosh(\beta p) - 1} \right]
$$

$$
\sim a^2 \int \frac{d^3p}{(2\pi)^3} \frac{\beta p}{2p} \left[ \frac{1}{1 - \cos v + \beta^2 p^2/2} - \frac{1}{\beta^2 p^2/2} \right]
$$

$$
= -\frac{a^2}{2\pi\beta^2} \left| \sin \frac{v}{2} \right|, \quad (A.10)
$$

and

$$
\Delta \sigma = \frac{1}{\mu} (N_\mu(v) - n_\mu). \quad (A.11)
$$

For the renormalization we need the limits

$$
R_\infty(0) = \lim_{\beta \to \infty} R(0) = a^2 \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p} = \frac{\alpha^2}{8\pi^2}, \quad (A.12)
$$

and

$$
a^{-1}\sigma_\infty(\mu, 0) = \lim_{\beta \to \infty} a^{-1}\sigma(\mu, 0) = \frac{1}{2\mu a}. \quad (A.13)
$$

Furthermore we find $\Delta_\infty R = 0$, i.e. $R_\infty(v) = R_\infty(0)$ and $\Delta_\infty \sigma = 0$.

Putting the above formulas together we obtain for $v = 0$:

$$
f_{1,gl}^0 = 4g^2 \int \frac{d^3pd^3q}{(2\pi)^6} \frac{n_p n_q}{pq} + 4g^2 \int \frac{d^3pd^3q}{(2\pi)^6} \frac{n_p}{pq} - 4g^2 \int \frac{d^3pd^3q}{(2\pi)^6} \frac{n_p}{pq}
$$

$$
= \frac{g^2}{36\beta^4}, \quad (A.14)
$$

and

$$
f_{1,\text{vac}}^0 = \frac{g^2}{a^4} \left[ -a^4 \int \frac{d^3pd^3q}{(2\pi)^6} \frac{1}{pq} \right]
$$
leading to Eq. (3.4). The terms linear in the occupation numbers cancel after renormalization in the glue gas contribution \( f_{1 \text{gl}}^0 \).

The additional terms due to \( v \neq 0 \) are given by:

\[
\begin{align*}
\Delta f_1 &= \frac{g^2}{3a^4} \left[ 4 \left( \frac{1}{a} \sigma(\mu, 0) + 2R(0) \right) \left( \frac{1}{a} \Delta \sigma + 2\Delta R \right) \\
&- 4 \left( \frac{1}{a} \sigma_\infty(\mu, 0) + 2R_\infty(0) \right) \left( \frac{1}{a} \Delta \sigma + 2\Delta R \right) \\
&+ \left( \frac{1}{a} \Delta \sigma + 2\Delta R \right)^2 \\
&- 2 \frac{1}{a} \Delta_\infty \sigma \left( a^{-1} \sigma(\mu, 0) + 2R(0) + \frac{1}{a} \Delta \sigma + 2\Delta R \right) \right]. \tag{A.16}
\end{align*}
\]

Separating the contribution of the glue gas and that of the vacuum, we arrive at Eqs. (3.3) and (3.4).

\section*{B \ Contribution \( f_3 \)}

\subsection*{B.1 \ Contracting the spatial indices}

By appropriate redefinition of the colour indices we obtain from Eq. (3.14):

\[
\begin{align*}
f_{3B} &= -\frac{1}{2} \epsilon^{abc} \epsilon^{gef} \frac{g^2}{a^8} \frac{1}{\beta^3} \sum_{n_1n_2n_3} \int \frac{d^3q d^3p d^3Q}{(2\pi)^9} \cdot \\
&\cdot (2\pi)^3 \delta(\vec{q} + \vec{p} + \vec{Q}) \beta \delta_{n_1+n_2+n_3} \cdot \\
&\cdot a^2 \left[ \left( D^{kieh}(\omega_{n_1}, \vec{q}) D^{ljfc}(\omega_{n_2}, \vec{p}) \\
- D^{kieh}(\omega_{n_1}, \vec{q}) D^{ljfc}(\omega_{n_2}, \vec{p}) \right) Q^k Q^l D^{ijg}(\omega_{n_3}, \vec{Q}) \\
+ \left( -D^{kieh}(\omega_{n_1}, \vec{q}) D^{ljfc}(\omega_{n_2}, \vec{p}) \right) \right].
\end{align*}
\]

33
\begin{align}
+ D^{k\mu\nu}(\omega_n, q) D^{ij}(\omega_n, p) & p^i Q^k D^{k\nu\gamma}(\omega_n, \bar{Q}) \\
+ (D^{k\mu\nu}(\omega_n, q) D^{ij}(\omega_n, p) & - D^{k\mu\nu}(\omega_n, q) D^{ij}(\omega_n, \bar{p})) p^i Q^k D^{k\nu\gamma}(\omega_n, \bar{Q}). \quad (B.1)
\end{align}

We start with contracting the spatial indices. For the sake of simplicity we write:

\begin{align*}
U &= \Delta_{eb}(\omega_n, \mu), \\
\tilde{U} &= \Delta_{eb}(\omega_n, \bar{q}), \\
V &= \Delta_{fc}(\omega_n, \mu), \\
\tilde{V} &= \Delta_{fc}(\omega_n, \bar{p}), \\
W &= \Delta_{ga}(\omega_n, \mu), \\
\tilde{W} &= \Delta_{ga}(\omega_n, \bar{Q}).
\end{align*}

$U$, $V$, and $W$ correspond to the longitudinal parts, $\tilde{U}$, $\tilde{V}$, and $\tilde{W}$ to the transverse parts of the propagators.

The expression

\begin{align}
&\left[ (U - \tilde{U}) \frac{q^k q^j}{q^2} + \tilde{U} \delta^{kj} \right] \left[ (V - \tilde{V}) \frac{p^l p^j}{p^2} + \tilde{V} \delta^{lj} \right] \\
- &\left[ (U - \tilde{U}) \frac{q^k q^j}{q^2} + \tilde{U} \delta^{kj} \right] \left[ (V - \tilde{V}) \frac{p^l p^j}{p^2} + \tilde{V} \delta^{li} \right] \\
= &\left( U - \tilde{U} \right) \left( V - \tilde{V} \right) \frac{q^k p^j}{q^2 p^2} (q^l p^j - q^l p^j) \\
+ &\tilde{U} \left( V - \tilde{V} \right) \frac{p^j}{p^2} (\delta^{kj} p^j - \delta^{kj} p^i) + \tilde{V} \left( U - \tilde{U} \right) \frac{q^k}{q^2} (\delta^{lj} q^j - \delta^{lj} q^j) \\
+ &\tilde{U} \tilde{V} (\delta^{kj} \delta^{lj} - \delta^{kj} \delta^{lj}) \quad (B.2)
\end{align}

is multiplied with

\begin{align}
Q^i Q^k \left[ (W - \tilde{W}) \frac{Q^l Q^j}{Q^2} + \tilde{W} \delta^{ij} \right] \quad (B.3)
\end{align}

in the first term of the bracket. The terms proportional to $(W - \tilde{W})$ vanish due to the contraction of a symmetric tensor with an antisymmetric one with indices $ij$, and we find:

\begin{align}
&\left( U - \tilde{U} \right) \left( V - \tilde{V} \right) \tilde{W} \left( \frac{(\bar{q} \bar{Q})^2}{q^2} - \frac{(\bar{q} \bar{Q})(\bar{q} \bar{p})(\bar{p} \bar{Q})}{q^2 p^2} \right) \\
+ &\tilde{U} \left( V - \tilde{V} \right) \tilde{W} \left( \bar{Q}^2 - \frac{(\bar{p} \bar{Q})^2}{p^2} \right) + \left( U - \tilde{U} \right) \tilde{V} \tilde{W} \frac{2(\bar{q} \bar{Q})}{q^2} \\
+ &\tilde{U} \tilde{V} \tilde{W} 2\bar{Q}^2. \quad (B.4)
\end{align}
Proceeding similarly in the second and third terms we find for the bracket in Eq. (B.1):

\[
\tilde{U} \tilde{V} \tilde{W} \phi_3(\vec{q}, \vec{p}, \vec{Q}) + \tilde{U} \tilde{V} \tilde{W} \phi_1(\vec{q}, \vec{p}, \vec{Q}) + \tilde{U} \tilde{V} \tilde{W} \phi_2(\vec{q}, \vec{p}, \vec{Q}) + \tilde{U} \tilde{V} \tilde{W} \phi_0(\vec{q}, \vec{p}, \vec{Q})
\]

(B.5)

with

\[
\phi_3(\vec{q}, \vec{p}, \vec{Q}) = \vec{Q}^2 - 2(\vec{p} \vec{Q}) + \frac{(\vec{p} \vec{Q})^2}{p^2} - \frac{(\vec{q} \vec{Q})^2}{q^2} + 2\left(\frac{\vec{q} \vec{p}}{q^2} (\vec{q} \vec{Q})\right) - \frac{(\vec{q} \vec{p}) (\vec{p} \vec{Q}) (\vec{q} \vec{Q})}{q^2 p^2},
\]

(B.6)

\[
\phi_2(\vec{q}, \vec{p}, \vec{Q}) = \frac{(\vec{q} \vec{Q})^2}{q^2} - 2\left(\frac{\vec{q} \vec{p}}{q^2} (\vec{q} \vec{Q})\right) + \frac{(\vec{q} \vec{p}) (\vec{p} \vec{Q}) (\vec{q} \vec{Q})}{q^2 p^2},
\]

(B.7)

\[
\phi_1(\vec{q}, \vec{p}, \vec{Q}) = \vec{Q}^2 - \frac{(\vec{p} \vec{Q})^2}{p^2} - \frac{(\vec{q} \vec{Q})^2}{q^2} + \frac{(\vec{q} \vec{p}) (\vec{p} \vec{Q}) (\vec{q} \vec{Q})}{q^2 p^2},
\]

(B.8)

\[
\phi_0(\vec{q}, \vec{p}, \vec{Q}) = \frac{(\vec{q} \vec{Q})^2}{q^2} - \frac{(\vec{q} \vec{p}) (\vec{p} \vec{Q}) (\vec{q} \vec{Q})}{q^2 p^2}.
\]

(B.9)

The contribution \(f_{3B}\) takes now the form:

\[
f_{3B} = -\frac{1}{2} g^2 \epsilon^{abc} \epsilon^{def} \frac{a^2}{\alpha^8 \beta^3} \sum_{n_1, n_2, n_3} \int \frac{d^3q d^3p d^3Q}{(2\pi)^6} \delta(\vec{q} + \vec{p} + \vec{Q}) \beta \delta_{n_1+n_2+n_3} \cdot
\]

\[
\cdot \left[ \Delta^{eb}(\omega_{n_1}, \vec{q}) \Delta^{fc}(\omega_{n_2}, \vec{p}) \Delta^{ga}(\omega_{n_3}, \vec{Q}) \phi_3(\vec{q}, \vec{p}, \vec{Q}) + \Delta^{eb}(\omega_{n_1}, \mu) \Delta^{fc}(\omega_{n_2}, \vec{p}) \Delta^{ga}(\omega_{n_3}, \vec{Q}) \phi_2(\vec{q}, \vec{p}, \vec{Q}) + \Delta^{eb}(\omega_{n_1}, \vec{q}) \Delta^{fc}(\omega_{n_2}, \mu) \Delta^{ga}(\omega_{n_3}, \vec{Q}) \phi_1(\vec{q}, \vec{p}, \vec{Q}) + \Delta^{eb}(\omega_{n_1}, \mu) \Delta^{fc}(\omega_{n_2}, \mu) \Delta^{ga}(\omega_{n_3}, Q) \phi_0(\vec{q}, \vec{p}, \vec{Q}) \right].
\]

(B.10)
B.2 Matsubara sums

Let us now introduce the sum:

\[
S(q, p, Q) = \frac{1}{\beta^3} \sum_{n_1n_2n_3} \beta \delta_{n_1+n_2+n_3} \epsilon^{abc} \epsilon^{gef} \Delta^{eb}(\omega_{n_1}, \vec{q}) \Delta^{fc}(\omega_{n_2}, \vec{p}) \Delta^{ga}(\omega_{n_3}, \vec{Q})
\]

\[= \sum_{\text{permutations of } (q,p,Q)} s_B(q, p, Q) \quad (B.11)\]

with

\[
s_B(q, p, Q) = \frac{1}{\beta^3} \sum_{n_1n_2n_3} \beta \delta_{n_1+n_2+n_3} d^{-}_{n_1}(\vec{q}) d^{+}_{n_2}(\vec{p}) d_{n_3}(\vec{Q}). \quad (B.12)
\]

With the appropriate change of the momentum variables we can write:

\[\]

\[f_{3B} = -\frac{1}{2a^2} \int \frac{d^3q d^3p d^3Q}{(2\pi)^6} \delta(\vec{q} + \vec{p} + \vec{Q}) \cdot \]

\[\cdot [S(q, p, Q) \phi_3(\vec{q}, \vec{p}, \vec{Q}) + S(\mu, p, Q) \phi_2(\vec{q}, \vec{p}, \vec{Q}) + S(q, \mu, Q) \phi_1(\vec{q}, \vec{p}, \vec{Q}) + S(\mu, \mu, Q) \phi_0(\vec{q}, \vec{p}, \vec{Q})]
\]

\[= -\frac{1}{2a^2} \int \frac{d^3q d^3p d^3Q}{(2\pi)^6} \delta(\vec{q} + \vec{p} + \vec{Q}) \cdot \]

\[\cdot [s_B(q, p, Q) \mathbf{P} \phi_3(\vec{q}, \vec{p}, \vec{Q}) + (s_B(\mu, p, \vec{Q}) + s_B(\vec{p}, \mu, \vec{Q}) + s_B(\vec{Q}, \vec{p}, \mu)) \Phi(\vec{q}, \vec{p}, \vec{Q}) + 2(s_B(\mu, \vec{Q}) + s_B(\vec{Q}, \mu, \mu) + s_B(\vec{Q}, \vec{p}, \vec{Q})) \phi_0(\vec{q}, \vec{p}, \vec{Q})], \quad (B.13)
\]

With

\[\mathbf{P} \phi_3 = 8(p^2 + Q^2) - 2p\vec{Q} - 6(p\vec{Q})^2 + 2\frac{p^2 + Q^2}{p^2 Q^2} - \frac{(p\vec{Q})^3}{p^2 Q^2}
\]

\[+ 2 \left( 2 - \frac{p\vec{Q}^2 + Q^2}{p^2 Q^2} \right) \cdot \]

\[p^2 Q^2 (p^2 + Q^2) + (p^4 + Q^4)(p\vec{Q}) - (p^2 + Q^2)(p\vec{Q})^2 - 2(p\vec{Q})^3 \]

\[\frac{(p^2 + Q^2)^2 - 4(p\vec{Q})^2}{(p^2 + Q^2)^2 - 4(p\vec{Q})^2}, \quad (B.14)
\]
\[ \Phi = \left( p^2 + Q^2 \right) \left( 1 - \left( \frac{\vec{p} \vec{Q}}{p^2 Q^2} \right)^2 \right) \\
+ 2 \left( \frac{\vec{p} \vec{Q}}{p^2 Q^2} (p^2 + Q^2) - 2 \right) \cdot \frac{(p^2 + Q^2) p^2 Q^2 + (p^4 + Q^4) \vec{p} \vec{Q} - (p^2 + Q^2) (\vec{p} \vec{Q})^2 - 2 (\vec{p} \vec{Q})^3}{(p^2 + Q^2)^2 - 4 (\vec{p} \vec{Q})^2}, \]

(B.15)

where we inserted \( \vec{q} = -\vec{p} - \vec{Q} \).

We write for the Kronecker-delta:

\[ \beta \delta_{n_1+n_2+n_3} = \frac{e^{\beta(q^0 + p^0 + Q^0)} - 1}{q^0 + p^0 + Q^0} \equiv I(q^0, p^0, Q^0), \quad \text{(B.16)} \]

and perform the sum \( s_B \) over the Matsubara frequencies by the contour integral technique of finite temperature field theory \([14]\):

\[ s_B(q, p, Q) = -\frac{1}{8qpQ} h(q, p, Q), \quad \text{(B.17)} \]

with

\[ h(q, p, Q) = \left[ n_q^- \right] n_q^+ + 1 \left[ n_p^+ \right] n_p^- + 1 \left[ n_Q \right] n_Q + 1 \left[ I(\pm E_q^\pm, \pm E_p^\pm, \pm E_Q) \right], \quad \text{(B.18)} \]

and \( E_q^\pm = q \pm \frac{i v}{\beta} \).

We express the sum \( h \) through the occupation numbers. Then we separate the vacuum term \( h_0 \), and the terms \( h_1 \) and \( h_2 \) linear and quadratic in the occupation numbers, respectively, \( h = h_0 + h_1 + h_2 \) with

\[ h_0(q, p, Q) = \frac{2}{q + p + Q}, \quad \text{(B.19)} \]

\[ h_1(q, p, Q) = 4n_Q \left[ \frac{q + p}{(q + p)^2 - Q^2} + 2(n_p^+ + n_p^-) \frac{q + Q}{(q + Q)^2 - p^2} \right. \]

\[ + 2(n_q^+ + n_q^-) \left. \frac{p + Q}{(p + Q)^2 - q^2} \right], \quad \text{(B.20)} \]
and

$$h_2(q, p, Q) = 4n_Q(n_p^+ + n_p^-)q \frac{q^2 - p^2 - Q^2}{q^4 + p^4 + Q^4 - 2q^2p^2 - 2q^2Q^2 - 2p^2Q^2}$$

$$+ 4n_Q(n_q^+ + n_q^-)p \frac{p^2 - q^2 - Q^2}{q^4 + p^4 + Q^4 - 2q^2p^2 - 2q^2Q^2 - 2p^2Q^2}$$

$$+ 2 \frac{(n_q^+ + n_q^-)(n_p^+ + n_p^-)Q(Q^2 - q^2 - p^2)}{q^4 + p^4 + Q^4 - 2q^2p^2 - 2q^2Q^2 - 2p^2Q^2}$$

$$- 4 \frac{(n_q^+ - n_q^-)(n_p^+ - n_p^-)Qq q}{q^4 + p^4 + Q^4 - 2q^2p^2 - 2q^2Q^2 - 2p^2Q^2}.$$  (B.21)

These expressions are symmetric in the momenta for vanishing background field $v = 0$, as expected.

### B.3 Subtraction of loop-integrals at zero temperature

The ‘renormalized’ contribution $f_3$ is given in terms of the diagrams in Fig. 4. As the momentum dependent functions $\phi_3, \phi_0$, and $\Phi$ are independent of the temperature, we can perform the subtractions on the sum $s_B$ directly. We have to subtract the limiting values of $s_B$ when two of the occupation numbers with all possible choices are taken in the limit $\beta \to \infty$.

If none of the momentum variables is equal to zero, we get:

$$h_{\infty}^{(pq)}(\vec{q}, \vec{p}, \vec{Q}) = \frac{2}{q + p + Q} + \frac{q + p}{(q + p)^2 - Q^2}n_Q,$$

$$h_{\infty}^{(qQ)}(\vec{q}, \vec{p}, \vec{Q}) = \frac{2}{q + p + Q} + \frac{q + Q}{(q + Q)^2 - p^2}(n_p^+ + n_p^-),$$

$$h_{\infty}^{(pQ)}(\vec{q}, \vec{p}, \vec{Q}) = \frac{2}{q + p + Q} + \frac{p + Q}{(p + Q)^2 - q^2}(n_q^+ + n_q^-)$$  (B.22)

and for the renormalized sum $s(q, p, Q)$:

$$s(q, p, Q) = s_B(q, p, Q) + \frac{1}{8pqQ} \left[ h_{\infty}^{(pq)}(\vec{q}, \vec{p}, \vec{Q}) + h_{\infty}^{(qQ)}(\vec{q}, \vec{p}, \vec{Q}) + h_{\infty}^{(pQ)}(\vec{q}, \vec{p}, \vec{Q}) \right]$$

$$\sim - \frac{n_Q(n_p^+ + n_p^-)}{pQ} \frac{q^2 - p^2 - Q^2}{q^4 + p^4 + Q^4 - 2q^2p^2 - 2q^2Q^2 - 2p^2Q^2}$$

$$- \frac{(n_q^+ + n_q^-)(n_p^+ + n_p^-)}{4pQ} \frac{q^2 - p^2 - Q^2}{q^4 + p^4 + Q^4 - 2q^2p^2 - 2q^2Q^2 - 2p^2Q^2}.$$
\[ + \left( \frac{n^+_p - n^-_p}{2Qp} \right) \left( n^+_Q - n^-_Q \right) \frac{Qp}{q^4 + p^4 + Q^4 - 2q^2p^2 - 2q^2Q^2 - 2p^2Q^2} \]
\[ + \frac{1}{2qpQ(q + p + Q)}. \]  

(B.23)

(The \( \sim \) sign means that \( s(q, p, Q) \) can be replaced in the integrand by the r.h.s. of Eq. (B.23).) Inserting \( \vec{q} = -\vec{p} - \vec{Q} \) we obtain:

\[
s(q, p, Q) = -\frac{1}{8Qp} \left( 4n_Q + n^+_Q + n^-_Q \right) \left( n^+_p + n^-_p \right) \frac{\vec{p} \vec{Q}}{(\vec{p} \vec{Q})^2 - p^2Q^2} \]
\[ + \frac{1}{8} \left( n^+_p - n^-_p \right) \left( n^+_Q - n^-_Q \right) \frac{1}{(\vec{p} \vec{Q})^2 - p^2Q^2} \]
\[ + \frac{1}{2qpQ(q + p + Q)}. \]  

(B.24)

The terms linear in the occupation numbers cancel due to renormalization, similarly as in the perturbative calculation without background field \(^8\).

If one or two of the momentum variables take the value of the IR cut-off \( \mu \), we make an expansion in its powers. The UV renormalization is performed by subtracting the corresponding limiting values with \( \beta \to \infty \) for fixed finite \( \mu \). It removes the terms linear in the occupation numbers. Thus we obtain:

\[
s(p, \mu, Q) + s(\mu, p, Q) + s(Q, p, \mu) = \]
\[ = \left( 4n_Q + n^+_Q + n^-_Q \right) \left( n^+_p + n^-_p \right) \frac{p^2 + Q^2}{4Qp} \frac{(p^2 - Q^2)^2}{p^2Q^2} \]
\[ - \frac{n_Q(n^+_p + n^-_p)}{Q(p^2 - Q^2)} \mu - \frac{n_p(n^+_Q + n^-_Q)}{2Q(p^2 - Q^2)} \mu \]
\[ + \frac{n_p(n^+_p + n^-_p)}{2p(p^2 - Q^2)} \mu + \frac{\left( n^+_p + n^-_p \right) \left( n^+_Q + n^-_Q \right)}{2p(p^2 - Q^2)} \mu \]
\[ + \frac{2(n^+_p - n^-_p) + n^+_Q - n^-_Q}{2(p^2 - Q^2)} \left( n^+_Q + n^-_Q \right) \]
\[ + \frac{3}{2pQ(p + Q)} \left( \frac{1}{\mu} - \frac{1}{p + Q} \right), \]  

(B.25)
\[ = \frac{1}{2p^3\mu} \left[ (2n_p + n_p^+ + n_p^-)(n_{\mu}^+ + n_{\mu}^-) + (n_p^+ + n_p^-)n_{\mu} \right] \\
- \frac{1}{4p^2\mu^2} (n_{\mu}^+ + n_{\mu}^- + 4n_{\mu})(n_{\mu}^+ + n_{\mu}^-) \\
+ \frac{1}{2p^4} \left[ (n_{\mu}^+ - n_{\mu}^-)^2 + 2(n_{\mu}^+ - n_{\mu}^-)(n_{p}^+ - n_{p}^-) \right] \\
+ \frac{3}{2p^2\mu^2} \left( 1 - \frac{2\mu}{p} \right). \]  

(B.26)

The ‘renormalized’ contribution \( f_3 \) can now be rewritten as

\[ f_3 = -\frac{g^2}{2} \int \frac{d^3p d^3Q}{(2\pi)^6} I, \]

(B.27)

with the integrand

\[ I = -s(q, p, Q) \Phi_3(\vec{q}, \vec{p}, \vec{Q}) \]

\[ - (s(\mu, p, Q) + s(p, \mu, Q) + s(Q, p, \mu)) \Phi(\vec{q}, \vec{p}, \vec{Q}) \]

\[ - 2 (s(\mu, \mu, Q) + s(\mu, Q, \mu) + s(Q, \mu, \mu)) \Phi_0(\vec{q}, \vec{p}, \vec{Q}) \]  

(B.28)

and \( \vec{q} = -\vec{p} - \vec{Q} \). This can be cast in the terms of the order \( n_{\mu}^0, n_{\mu}, \) and \( n_{\mu}^2 \) depending on the occupation numbers, and the term independent of the occupation numbers according to Eq. (3.16).

B.4 Momentum integrals

Here is appropriate to make some remarks on the calculation of the momentum integrals.

The momentum dependent functions multiplying the Matsubara sums in the integrands (3.17)-(3.23) can be replaced by more simple expressions. They are obtained by separating the terms of the integrands with odd powers of the scalar product \( \langle \vec{p} \vec{Q} \rangle \) and leaving them out because they do not contribute to the integrals. Equations valid only up to those terms are denoted by \( \sim \). Thus the following expressions are found:

\[ \phi_0(\vec{q}, \vec{Q}, \vec{p}) \sim \frac{p^2Q^2 - (\vec{p}\vec{Q})^2}{2Q^2} + \frac{p^4 - Q^4}{8Q^2} - \frac{(p^2 - Q^2)(p^4 - Q^4)}{8Q^2 \left( (p^2 + Q^2)^2 - 4(\vec{p}\vec{Q})^2 \right)}. \]

(B.29)
\[
\frac{(\vec{p}\vec{Q})\Phi_3(\vec{q}, \vec{p}, \vec{Q})}{(\vec{p}\vec{Q})^2 - p^2Q^2} \sim -2\frac{(\vec{p}\vec{Q})^2}{p^2Q^2} - \frac{1}{2} \left[4 + \frac{(p^2 + Q^2)^2}{p^2Q^2}\right] \\
+ \frac{1}{2} \left[4 + \frac{(p^2 + Q^2)^2}{p^2Q^2}\right] \frac{(p^2 + Q^2)^2}{(p^2 + Q^2)^2 - 4(\vec{p}\vec{Q})^2}, \quad (B.30)
\]

\[
\frac{P\Phi_3}{(\vec{p}\vec{Q})^2 - p^2Q^2} \sim -\frac{p^2 + Q^2}{p^2Q^2} \left[7 + \frac{p^4 + Q^4 + 6p^2Q^2}{2(p^2 + Q^2)pQ} \ln \left|\frac{p + Q}{p - Q}\right|\right], \quad (B.31)
\]

(obtained after performing the angle integration),

\[
\frac{2\Phi(\vec{q}, \vec{p}, \vec{Q})}{p^2 - Q^2} \sim 2(p^2 + Q^2)(p^2 - Q^2) \left\{-\frac{1}{4p^2Q^2} \right. \\
+ \left. \left[1 + \frac{(p^2 + Q^2)^2}{4p^2Q^2}\right] \frac{1}{(p^2 + Q^2)^2 - 4(\vec{p}\vec{Q})^2}\right\}. \quad (B.32)
\]

There is a remarkable cancellation of several terms in the integrand of \( I_2 \) (see Eq. (B.17)):

\[
\left[\frac{p^2 + Q^2}{(p^2 - Q^2)^2}\Phi(\vec{q}, \vec{p}, \vec{Q}) - \frac{1}{2} \frac{(\vec{p}\vec{Q})P\Phi_3(\vec{q}, \vec{p}, \vec{Q})}{(\vec{p}\vec{Q})^2 - p^2Q^2}\right] \\
\sim \frac{1}{2} \left\{2(p^2 + Q^2)^2 \left[-\frac{1}{4p^2Q^2} + \left(1 + \frac{(p^2 + Q^2)^2}{4p^2Q^2}\right) \frac{1}{(p^2 + Q^2)^2 - 4(\vec{p}\vec{Q})^2}\right] \\
+ \frac{(\vec{p}\vec{Q})^2}{p^2Q^2} + 2 \left[1 + \frac{(p^2 + Q^2)^2}{4p^2Q^2}\right] \\
- 2 \left[1 + \frac{(p^2 + Q^2)^2}{4p^2Q^2}\right] \frac{(p^2 + Q^2)^2}{(p^2 + Q^2)^2 - 4(\vec{p}\vec{Q})^2}\right\} \\
= 1 + \frac{(\vec{p}\vec{Q})^2}{p^2Q^2} \sim \frac{4}{3}, \quad (B.33)
\]

The last equation is obtained by performing the angle integration.

For evaluating the glue gas contribution \( f_{32}^A \) the following approximation is used:

\[
n_{p^+} + n_{p^-} - 2n_p = \frac{1}{e^{\beta_p e^{iw}} - 1} + \frac{1}{e^{\beta_p e^{-iw}} - 1} - \frac{2}{e^{\beta_p} - 1}
\]

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\[
\frac{2 \left( e^{\beta p} \cos v - 1 \right)}{e^{2\beta p} - 2e^{\beta p} \cos v + 1} - \frac{2}{e^{\beta p} - 1} \approx \begin{cases} 
0 & \text{for } p > 1/\beta \\
\nu(p) & \text{for } p < 1/\beta 
\end{cases}
\]  
(B.34)

where

\[
\nu(p) = \frac{2}{e^{\beta p}} \left[ \frac{\cos v - e^{-\beta p}}{1 - 2e^{-\beta p} \cos v + e^{-2\beta p}} - \frac{1}{1 - e^{-\beta p}} \right].
\]  
(B.35)

Then we obtain from Eqs. (3.17), and (3.24):

\[
f_{32}^\Delta \approx 2 \frac{g^2}{(2\pi)^4} \int_{1/\beta}^{1/\beta} dp \int_{\mu}^{\Lambda} dQ \int_{\mu}^{\Lambda} d\tilde{p} d\tilde{Q} \frac{pQ}{3} \left[ 8n_\alpha \nu(p) + \nu(p)\nu(Q) \right].
\]  
(B.36)

We use the saddle point approximation and replace the expression \( p\nu(p) \) by its value taken at \( p_0 = 1/\beta \):

\[
G(v) \equiv \beta p_0 \nu(p_0) = \frac{2}{e^\beta} \left[ \frac{\cos v - (1/e)}{1 - 2(1/e) \cos v + (1/e^2)} - \frac{1}{1 - (1/e)} \right].
\]  
(B.37)

This leads to the expression (3.28) for \( f_{32}^\Delta \).

For the evaluation of the integral

\[
f_{32}' = \frac{1}{2} g^2 \int d^3\tilde{p} d^3Q \left\{ (n^+_p - n^-_p)(n^+_Q - n^-_Q) \right\} \frac{\mathbf{P} \cdot \mathbf{Q}}{(\tilde{p} \cdot \tilde{Q})^2 - p^2 Q^2}
\]  
(B.38)

we make use of the approximation:

\[
n^+_p - n^-_p = \frac{1}{e^{\beta p + iv} - 1} - \frac{1}{e^{\beta p - iv} - 1} \approx \begin{cases} 
0 & \text{for } p > 1/\beta \\
\delta(p) & \text{for } p < 1/\beta 
\end{cases}
\]  
(B.39)

with

\[
\delta(p) \approx e^{-\beta p} \left\{ \frac{1}{e^{iv} - e^{-\beta p}} - \frac{1}{e^{-iv} - e^{-\beta p}} \right\} \approx e^{-\beta p} \frac{-i \sin v}{(1/e)(\cosh 1 - \cos v)}
\]  
(B.40)

Then we obtain the expression (3.29) for \( f_{32}' \).
The contribution

\[ f_3^{IR} = f_{31} + f_{33} + f_{30} + f'_{31} + f'_{33} \]  

was estimated in the following way: (i) the integrals containing occupation numbers in their integrands were calculated by using the low-momentum \((\mu \leq Q < 1/\beta)\) expansions

\[ n_Q \approx \frac{1}{\beta Q} - \frac{1}{2} + \frac{1}{12} \beta Q - \frac{1}{720} (\beta Q)^3 + \frac{1}{720 \cdot 42} (\beta Q)^5 + \ldots , \]  

\[ n^+_Q + n^-_Q \approx -1 + A \beta Q + \frac{1}{6} B (\beta Q)^3 + \frac{1}{120} C (\beta Q)^5 + \ldots , \]  

\[ n^+_Q - n^-_Q \approx -iA \sin v \left( 1 - \frac{1}{2} A (\beta Q)^2 + \frac{1}{24} D (\beta Q)^4 + \ldots \right) , \]  

with \( A = (1 - \cos v)^{-1} \), \( B = A - 3A^2 \), \( C = A - 15A^2 + 30A^3 \), \( D = -A + 6A^2 \); (ii) for large momenta \((1/\beta < Q < \Lambda)\) the estimates \( n_Q \approx e^{-\beta Q} \), \( n^+_Q + n^-_Q \approx -e^{-\beta Q} \), and \( n^+_Q - n^-_Q \approx -iA \sin v e^{-\beta Q} \) are used; (iii) slowly varying factors in the integrands were generally separated and taken out from the integral with their value at the maximum of the rapidly varying factor; (iv) all IR divergent terms were included.

### C Contribution \( f_4 \)

Calculating the bare contribution \( f_{4B} \) we performed the sums over colour indeces at first. The product of propagators was rewritten as

\[ D^{kiab}(\omega_n, \vec{p}) D^{kia'b'}(\omega_m, \vec{q}) = \]

\[ = \Delta^{ab}(\omega_n, \mu) \Delta^{a'b'}(\omega_m, \mu) \cos^2 \theta \]

\[ + \left[ \Delta^{ab}(\omega_n, \vec{p}) \Delta^{a'b'}(\omega_m, \mu) + \Delta^{ab}(\omega_n, \mu) \Delta^{a'b'}(\omega_m, \vec{q}) \right] \sin^2 \theta \]

\[ + \Delta^{ab}(\omega_n, \vec{p}) \Delta^{a'b'}(\omega_m, \vec{q}) \left( 1 + \cos^2 \theta \right) , \]  

where \( \theta \) is the angle of the vectors \( \vec{p} \) and \( \vec{q} \). Making use of expression (2.28) for the matrices \( \Delta^{ab}(\omega_n, \vec{p}) \) and the fact that the sums contain only the terms
with \( n_2 = -n_3 \), and \( d_0^\pm(\vec{p}) = d_3^\pm(\vec{p}) \) we obtained:

\[
f_{4B} = -\frac{2g^2a^5}{\beta^2} \sum_{n_2n_3} \beta \delta_{n_2+n_3} \int \frac{d^3p_2d^3p_3}{(2\pi)^6} D(-\vec{p}_2 - \vec{p}_3) \cdot \\
\cdot \left\{ (\omega_{n_3} - (v/\beta))^2 d_{-n_2}^+(0)d_{n_3}^+(0) \cos^2 \theta \\
+ 2(\omega_{n_3} - (v/\beta))^2 d_{-n_2}^+(\vec{p}_2)d_{n_3}^+(0) \sin^2 \theta \\
+ (\omega_{n_3} - (v/\beta))^2 d_{-n_2}^+(\vec{p}_2)d_{n_3}^+(\vec{p}_3)(1+\cos^2 \theta) \\
+ 2(v/\beta)\omega_{n_3}d_{n_2}^+(\vec{p}_2)d_{n_3}^+(\vec{p}_3)(1+\cos^2 \theta) \right\} \\
= -\frac{2g^2a^5}{\alpha^4} \int \frac{d^3pd^3q}{(2\pi)^6} D(-\vec{p} - \vec{q}) \cdot \\
\cdot \left\{ s_{2B}(\mu, \mu) \cos^2 \theta + 2s_{2B}(p, \mu) \sin^2 \theta \\
+ s_{2B}(p, q)(1+\cos^2 \theta) \right\} \tag{C.2}
\]

with the Matsubara sums:

\[
s_{1B}(p, q) = \frac{1}{\beta^2} \sum_{n_2n_3} \frac{I_1(p^0, q^0)}{(p^0 + i(v/\beta))^2 - \vec{p}^2} \left[ (q^0 + i(v/\beta))^2 - \vec{q}^2 \right] \\
= \frac{1}{4pq} \left\{ \begin{array}{ll} n_p^- & n_p^+ + 1 \\ n_q^- & n_q^+ + 1 \end{array} \right\} I_1(\pm E_p^\pm, \pm E_q^\pm), \tag{C.3}
\]

and

\[
s_{2B}(p, q) = \frac{1}{\beta^2} \sum_{n_2n_3} e^{\beta(p^0+q^0)} - 1 \left[ \frac{(-i\vec{q}^0 - (v/\beta))^2}{p^0 + q^0} \left[ (i\vec{p}^0 + (v/\beta))^2 + \vec{p}^2 \right] \left[ (-i\vec{q}^0 + (v/\beta))^2 + \vec{q}^2 \right] \right] \\
= -\frac{1}{\beta^2} \sum_{n_2n_3} \frac{I_2(p^0, q^0)}{(p^0 - i(v/\beta))^2 - \vec{p}^2} \left[ (q^0 + i(v/\beta))^2 - \vec{q}^2 \right] \\
= -\frac{1}{4pq} \left\{ \begin{array}{ll} n_p^+ & n_p^- + 1 \\ n_q^+ & n_q^- + 1 \end{array} \right\} I_2(\pm E_p^\pm, \pm E_q^\pm), \tag{C.4}
\]

where

\[
I_1(p^0, q^0) = \frac{e^{\beta(p^0+q^0)} - 1}{p^0 + q^0}(-i(v/\beta)q^0a^2), \tag{C.5}
\]
\[ I_2(p^0, q^0) = \frac{e^{\beta(p^0+q^0)} - 1}{p^0 + q^0} \left( q^0 - i(v/\beta) \right)^2 a^2, \]  

(C.6)

and \( n^\pm_p = (e^{\beta E^\pm_p} - 1)^{-1}, E^\pm_p = p \pm i(v/\beta) \). Let us express these sums in terms of the occupation numbers:

\[ 4pq s_{1B}(p, q) = \left( -\frac{1}{p+q-2iC} + \frac{1}{-p+q-2iC} \right) iC E_q^- a^2 n_q^- 
+ \left( \frac{1}{p-q+2iC} + \frac{1}{p+q+2iC} \right) iC E_q^+ n_q^+ 
- \left( \frac{1}{p+q-2iC} + \frac{1}{p-q-2iC} \right) iC E_q^+ a^2 n_q^- 
+ \left( \frac{1}{p-q+2iC} + \frac{1}{p+q+2iC} \right) iC E_q^+ n_q^+ 
+ iC a^2 \left( \frac{E_q^+}{p+q+2iC} - \frac{E_q^-}{p+q-2iC} \right), \]  

(C.7)

\[ 4pq s_{2B}(p, q) = \frac{2a^2 p}{p^2 - q^2} \left[ (q^2 - 4C^2)(n_q^++n_q^-) + 4qC(n_q^+-n_q^-) \right] 
- \frac{2a^2 q}{p^2 - q^2} (q^2 - 4C^2)(n_p^++n_p^-) 
- \frac{2a^2 p}{p^2 - q^2} 4qC(n_p^+-n_p^-) 
+ \frac{a^2}{p+q} (q^2 - 4C^2). \]  

(C.8)

The subtractions shown in Fig. 5 can be carried out on the sums \( s_{1B}(p, q) \) and \( s_{2B}(p, q) \) directly. Then the ‘renormalized’ contribution \( f_4 \) must be calculated by replacing the bare sums via

\[ s_\alpha = s_\alpha^{(p)} - s_\alpha^{(q)} - s_\alpha^{(pq)} \]  

(C.9)

(\( \alpha = 1, 2 \)), where the upper indices \((p), (q), \) and \((pq)\) denote that the occupation numbers \( n^\pm_p, n^\pm_q, \) and both \( n^\pm_p \) and \( n^\pm_q \) are taken in the limit \( \beta \to \infty \), respectively. As a result the terms linear in the occupation numbers cancel in the ‘renormalized’ expressions and we obtain:

\[ 4pq s_1(p, q) = 2(v/\beta)^2 a^2 \frac{p - q}{(p + q)^2 + 4(v/\beta)^2}, \]  

(C.10)
\[4pq s_2(p, q) = -2 \frac{a^2}{p+q} \left( q^2 - 4(v/\beta)^2 \right). \] (C.11)

Replacing the bare expressions by these ‘renormalized’ ones we obtain:

\[
f_4 = \frac{2g^2}{a^4} \int \frac{d^3p d^3q}{(2\pi)^6} \left[ M^2 + (\vec{p} + \vec{q})^2 \right]^{-1} \left[ -a^2 \frac{1}{4a\mu} \left( 1 - 4v^2 \frac{(a/\beta)^2}{(a\mu)^2} \right) \cos^2 \theta \right.
\]
\[
- \frac{a^2a\mu}{p^2} \left( 1 - 4v^2 \frac{(a/\beta)^2}{(a\mu)^2} \right) \sin^2 \theta
\]
\[
- a^3 \frac{1}{2pq(p+q)} \left( q^2 - 4(v/\beta)^2 \right) (1 + \cos^2 \theta)
\]
\[
+ a^3 \frac{p - q}{pq} \frac{(v/\beta)^2}{(p+q)^2 + 4(v/\beta)^2} \left( 1 + \cos^2 \theta \right) \right]. \] (C.12)

For \( v = 0 \) the contribution \((C.12)\) vanishes due to \( M \to \infty \). Therefore this is a vacuum contribution for \( v \neq 0 \), \( f_4 = f_4^\Delta_{\text{vac}} \).

The momentum integrals are treated similarly as those in Appendix B. The integral over the angle \( \theta \) is performed at first. Then estimates are obtained by casting the integrands in slowly and rapidly varying factors and taking the slowly varying factor out of the integral with its value at the maximum of the rapidly varying one. Keeping the terms up to the order \( g^2 \) and making use of \( M^2 \sim g^2 \Lambda^3 \), we obtain the final expression \((3.32)\).