Uncertainty on $w$ from large-scale structure

Valerio Marra,¹⋆ Mikko Pääkkönen²,³ and Wessel Valkenburg¹,⁴

¹Institut für Theoretische Physik, Universität Heidelberg, Philosophenweg 16, D-69120 Heidelberg, Germany
²Department of Physics, PL 35, 40014 University of Jyväskylä, Finland
³Helsinki Institute of Physics, PL 64, 00014 University of Helsinki, Finland
⁴Instituut-Lorentz for Theoretical Physics, Universiteit Leiden, Postbus 9506, 2333 CA Leiden, the Netherlands

ABSTRACT

We find that if we live at the centre of an inhomogeneity with total density contrast $|δ_0| \simeq 0.1 – 0.15$, dark energy is not a cosmological constant at 95 per cent confidence level. Observational constraints on the equation of state of dark energy, $w$, depend strongly on the local matter density around the observer. We model the local inhomogeneity with an exact spherically symmetric solution which features a pressureless matter component and a dark-energy fluid with constant equation of state and negligible sound speed, which reaches a homogeneous solution at finite radius. We fit this model to observations of the local expansion rate, distant supernovae and the cosmic microwave background. We conclude that the possible uncertainty from large-scale structure has to be taken into account if one wants to progress towards not just precision but also accurate cosmology.

Key words: cosmological parameters – cosmology: observations – large-scale structure of Universe.

1 INTRODUCTION

We have entered the so-called era of accurate cosmology (Peebles 2002). The aim is to understand the composition and expansion history of the Universe at the per cent level. In particular, one of the goals is to establish if the observed acceleration of the Universe (Riess et al. 1998; Perlmutter et al. 1999) is driven by the cosmological constant or by a fluid with negative pressure, dark energy. At the most basic level the question is if the observed equation of state $w$ is compatible with $–1$ or not, the value corresponding to the cosmological constant. It is therefore crucial to study all possible systematic effects on $w$ (Amendola et al. 2010; Marra & Paakkonen 2010; Sinclair, Davis & Haugbolle 2010; de Lavallaz & Fairbairn 2011; Romano & Chen 2011).

As our observations are confined to the light cone, there is an intrinsic degeneracy between temporal evolution and spatial variation around us. In particular, inhomogeneities around us are degenerate with the properties of dark energy, most importantly its equation of state. A clear example of how intertwined are attempts to detect any evolution of dark energy to large-scale structures is given by the so-called void models. An observer inside a spherical underdensity expanding faster than the background sees indeed apparent acceleration, thus, removing the need for dark energy (see e.g. Marra & Notari 2011, and references therein). Void models strongly violate the Copernican principle and have been ruled out – at least in their simplest incarnation – as they predict a too strong kinematic Sunyaev–Zel’Dovich effect (kSZ; Garcia-Bellido & Haugboelle 2008; Zhang & Stebbins 2011; Zibin 2011; Zibin & Moss 2011). While on one hand this strengthens the case for dark energy as the likely explanation for the acceleration of the Universe, on the other hand it illustrates how large-scale structure can alter the determination of cosmological parameters. Therefore, it is necessary to adequately model large-scale structures if one has to achieve the grand goal of accurately determining the composition of the Universe.

In Valkenburg (2012b) it was shown by means of mock data that a local inhomogeneity, of proportions similar to a structure on the surface of last scattering that could cause the cosmic microwave background (CMB) Cold Spot, can have strong effects on our perception of the equation of state of dark energy. Here, we extend that analysis to real data using the model of Marra & Paakkonen (2012). More precisely, we consider a $w$ cold dark matter ($w$CDM) model endowed with a local almost linear inhomogeneity surrounding the observer and test it against supernova (SN) observations, CMB anisotropies and local measurements of the Hubble parameter. By $w$CDM we mean a universe containing dark matter and a dark-energy fluid with equation of state $w$. In this way, we can show how the observed large-scale structure of the Universe could impact the reconstruction of the dark-energy parameters.

Following Valkenburg (2012b), we consider an inhomogeneity inspired by the observed Cold Spot in the CMB, which has a radius of roughly 5° and a temperature deviation of roughly $O(50 \sim 200) \mu K$ (Cruz et al. 2006; Zhang & Huterer 2010; Bennett et al. 2011).
The idea is that the Cold Spot is a primary CMB anisotropy due to an object on the surface of last scattering, and not a secondary effect caused by an object along the line of sight (Tomita 2005; Inoue & Silk 2006, 2007; Masina & Notari 2009). Such an inhomogeneity has a radius of roughly 1 Gpc and a density contrast today of roughly $-0.1$ (Valkenburg 2012b). It is therefore too shallow to bias the w/CDM model to the point of removing the need for dark energy, as in the void scenario. Nonetheless, as argued above, such a structure may bias the value of the dark-energy parameters to a level that may be important if one wants to determine whether dark energy is a cosmological constant or not. Moreover, as we briefly argue in the body of this paper, structures of radius $\sim 1$ Gpc and density contrast today $\sim 0.01$ – which still give an interesting effect – are not at more than three times the dispersion of the density perturbations arising from a close to scale-invariant primordial spectrum. Therefore, the setup considered in this paper is not in conflict with standard cosmology and, in particular, the Copernican principle (Valkenburg, Marra & Clarkson 2012).

We model the inhomogeneity with a particular case of the spherically symmetric solution presented in Marra & Paakkonen (2012), which features a pressureless matter component and a dark-energy fluid with constant equation of state and negligible sound speed $c_s$. The possibility of a dark-energy fluid with negligible sound speed has been investigated in the literature under various assumptions. This generally requires a non-canonical scalar field like $k$-essence and kinetic gravity braiding, as opposed to standard quintessence models with canonical scalar fields which always have $c_s = 1$ (see, e.g. Bertacca et al. 2008; Cremiellini et al. 2009; Bertacca, Bartolo & Matarrese 2010; Deffayet et al. 2010; Lim, Sawicki & Vikman 2010; Li et al. 2011, and references therein). Here, we choose this particular model because it significantly simplifies the dynamical equations and the numerical analysis as there are no pressure gradients that can generate peculiar velocities from an initially comoving motion. We use this model phenomenologically so as to minimally extend the parameter space of the w/CDM model by adding only two extra parameters: the radius of the inhomogeneity and its overall contrast. All the other initial conditions follow indeed rigidly.

The paper is organized as follows. In Section 2, we go briefly through the formalism of the model and its initial conditions, and in Section 3 we explain how the cosmological data analysis has been performed. We show in Section 4 that the effect of local inhomogeneity on the dark-energy parameters can be important and that the inhomogeneity is not unlikely to occur. We conclude in Section 5.

2 THE MODEL

We consider the case of an observer located at the centre of an inhomogeneous sphere embedded in a flat w/CDM universe. The inhomogeneities are given by a pressureless matter component and by a dark-energy fluid with constant equation of state $w_{\text{out}}$ and negligible sound speed (the subscript ‘out’ refers to values at $r > r_0$, where $r_0$ is the comoving radius of the inhomogeneity). To be more precise, the sound horizon is much smaller than the inhomogeneity scale considered so that we can set $c_s = 0$ throughout the paper. In terms of the radius dependent non-adiabatic equation of state this means that we consider

\[ w(r, t) = w_{\text{out}} \frac{\rho_{X, \text{out}}(t)}{\rho_X(r, t)}, \]

that is, the pressure is homogenous:

\[ p_X(r, t) = p_{X, \text{out}}(t) = w_{\text{out}} \rho_{X, \text{out}}(t), \]

where $r$ is the coordinate radius, $t$ is cosmic time, $\rho$ denotes energy density, $p$ denotes pressure and the label $X$ refers to the dark-energy fluid. Since pressure gradients are absent, matter and dark energy evolve along geodesics. Moreover, we set initial conditions such that dust and dark energy are initially comoving. Therefore, the absence of pressure gradients implies that peculiar velocities between the two fluids will never develop and that the matter and dark-energy reference frames always coincide. Next, we discuss the equations governing the dynamics of the model and the relevant initial conditions.

2.1 Dynamical equations

We adopt the exact spherically symmetric inhomogeneous solution with $n$ perfect fluids presented in Marra & Paakkonen (2012), which we limit to the case discussed above. We will now briefly introduce the relevant equations, and we refer to Marra & Paakkonen (2012) for the general equations and more details.

Using the reference frame of the dust and dark-energy components, the metric describing our model is

\[ ds^2 = -dt^2 + \frac{Y_{\prime}(r, t)^2}{1 - k(r)r^2} dr^2 + Y(r, t)^2 d\Omega^2, \]

where $Y(r, t)$ is the scale function, $k(r)$ is the curvature function, $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$ and we have set $c = 1$. A prime denotes partial derivation with respect to the coordinate radius $r$, whereas a dot denotes partial derivation with respect to the coordinate time $t$. The curvature function is time independent because of the adopted reference frame and sound speed (Marra & Paakkonen 2012). The metric (3) is written in the same form as the Lemaître–Tolman–Bondi (LTB) metric. However, with the inclusion of a dark-energy fluid, the dynamics is no longer that of the LTB metric. The metric of equation (3) reduces to the Friedman–Lemaître–Robertson–Walker (FLRW) metric, if $k(r) = \text{const.}$ and $Y(r, t) = r a(t)$, where $a(t)$ is the scalefactor.

We label the dust component with $M$ and the dark-energy component with $X$. The conservation equation for the dust source can be solved directly and gives

\[ \frac{\rho_{M}(r, t)}{\rho_{M}(r, t)} = \frac{Y_{\prime}(r, t)}{Y^2(r, t)} Y^\prime(r, t). \]

where $\bar{\iota}$ is the initial time at which we give the initial conditions. As explained before, there are no peculiar velocities between the two fluids and the remaining dynamical equations reduce to

\[ Y_{\prime}(r, t) = \frac{2 GF(r, t)}{Y(r, t)} - k(r)r^2, \]

\[ Y_{\prime}(r, t) = \frac{GF(r, t)}{Y(r, t)} Y^\prime(r, t) - \frac{G F(r, t) Y^\prime(r, t)}{Y^2(r, t)} - \frac{[k(r)r^2]^\prime}{2 Y(r, t)}, \]

\[ F(r, t) = -4\pi G Y^2(r, t) Y^\prime(r, t) p_X(r, t), \]

\[ \rho_{X}(r, t) = -[\rho_X(r, t) + p_{X, \text{out}}(t)][H_b(r, t) + 2H_A(r, t)], \]

where the radial and angular expansion rates are $H_b = Y^\prime / Y$ and $H_A = Y / Y$, and $F$ is the total effective gravitating mass which also satisfies the following consistency equation:

\[ F^\prime(r, t) = 4\pi G Y^2(r, t) Y^\prime(r, t) [p_{M}(r, t) + \rho_X(r, t)]. \]

Equation (6) is the r-derivative of equation (5) and allows us to solve directly for the unknown functions $Y$, $Y^\prime$, $F$ and $\rho_X$ without
having to take numerical derivatives. If $w = w_{\text{out}} = -1$, this solution becomes the usual LTB model which has been studied recently in, e.g., Enqvist & Mattsson (2007), Sinclair et al. (2010), Marra & Paakkonen (2010), Valkenburg (2012a) and Romano & Chen (2011).

Finally, in this particular case the light-cone equations have the same form as in the LTB model:

$$\frac{dt}{dz} = \frac{Y'}{(1+z)Y}, \quad \frac{dr}{dz} = \sqrt{1 - k(r)r^2/(1+z)Y}. \quad (10)$$

2.2 Initial and boundary conditions

We fix the flat $\omegaCDM$ background model by setting $h$, $\Omega_k$ and $w_{\text{out}}$, where $\Omega_k$ is the present-day background dark-energy density parameter and $h$ is the present-day dimensionless Hubble rate defined by $H_0 = 100h\, \text{km s}^{-1}\, \text{Mpc}^{-1}$.

We fix the gauge for the radial coordinate in equation (3) such that $Y(r, i) = r a(i)$ at some initial time $i$. We choose to parametrize the curvature by means of

$$k(r) = k_c W_3 \left( \frac{r}{r_o}, 0 \right), \quad (11)$$

where $r_o$ is the comoving radius of the spherical inhomogeneity and

$$W_3(x, 0) = \begin{cases} \frac{1}{4\pi^2} + \frac{1}{2} x^2 \frac{\cos(4\pi x)}{4\pi^2} & \text{for } 0 \leq x < \frac{1}{2} \\ \frac{1}{4\pi^2} + 2(1-x)^2 + \frac{\cos(4\pi x)}{4\pi^2} & \text{for } \frac{1}{2} \leq x < 1 \\ 0 & \text{for } x \geq 1 \end{cases}$$

is the third order of the function $W_3(x, \alpha)$, which has been defined in Valkenburg (2012a) and interpolates from 1 to 0 in the interval $\alpha < x < 1$ while remaining $C^0$ everywhere. Hence, $k(r)$ is $C^1$ everywhere, such that the metric is $C^2$ and the Riemann curvature is $C^3$. Although the function $W_3(x, 0)$ looks rather complicated, it has actually a very simple shape as one can see in Fig. 1. The curvature profile of equation (11) is exactly zero for $r \geq r_o$ and so the metric is correctly matched to the exterior spatially flat $wCDM$ model, which means that the central over- or under-density is automatically compensated by a surrounding under- or overdense shell. The constant $k_c$ gives the curvature at the centre of the inhomogeneity and will determine its density contrast. A spherical inhomogeneity depends crucially on only two physical parameters: the radius and the overall density contrast. Therefore, the precise shape of the density profile should not be essential and our analysis should be representative also of other possible curvature or density profiles.

Next, we have to give initial conditions at $t = i$ for $F(r, i) = F_M(r) + F_X(r)$. We choose $i$ such that matter is dominant over dark energy, $F_M(r) \ll F_W(r)$, and so the model becomes the standard dust LTB model. We can then link the curvature function $k(r)$ to the initial condition for $F_M(r)$ by demanding that the universe has the same age $i$ for any $r$. That is, we demand a homogeneous big bang, implying the absence of decaying modes in the matter density (Zibin 2008). In particular we can use the following analytic result of Van Acoleyen (2008) valid for a linear matter density contrast:

$$k(r) = \frac{4}{3} a^2(i) H^2_{\text{out}}(i) \delta_{F_M}(r), \quad (13)$$

which clearly relates the curvature at the centre $k_c$ to the matter contrast at the centre $\delta_M = \delta_{F_M}(0)$. The latter term is the contrast in the gravitating mass and is defined as

$$\delta_{F_M}(r) = \frac{F_M(r)}{F_{M,\text{out}}(r)} - 1, \quad (14)$$

where $F_{M,\text{out}} = \frac{4\pi}{3} a^3(i) r^3 \rho_{M,\text{out}}(i)$ is the corresponding background gravitating mass and the gauge $Y(r, i) = r a(i)$ has been used. The initial matter density is then

$$\rho_M(r, i) = \frac{F_M(r)}{4\pi a^3(i) r^2}. \quad (15)$$

In order to have initial conditions without decaying modes in the dark-energy component, we have to set its initial profile according to the following relation valid during matter domination and $\epsilon^2 \ll 1$ (Ballesteros & Lesgourgues 2010):

$$\frac{\delta_{F_X}(r)}{\delta_M} = \frac{\delta_X}{1 - 3 w_{\text{out}}}, \quad (16)$$

where, analogous to the matter contrast, we have

$$\delta_{F_X}(r) = \frac{F_X(r)}{F_{X,\text{out}}(r)} - 1, \quad (17)$$

$$F_{X,\text{out}} = \frac{4\pi}{3} a^3(i) r^3 \rho_{X,\text{out}}(i)$$

and $\delta_X = \delta_{F_X}(0)$. The initial dark-energy density is then

$$\rho_X(r, i) = \frac{F'_X(r)}{4\pi a^3(i) r^2}. \quad (18)$$

Hence, all the initial conditions relative to the inhomogeneous patch are indeed specified by a given curvature profile $k(r)$.

3 COSMOLOGICAL DATA ANALYSIS

In this section, we explain how we compare the predictions of this model with SNe, Hubble rate and CMB observations. We decided not to include baryon acoustic oscillations in the analysis as perturbation theory in an inhomogeneous background has not been thoroughly understood yet1 (Zibin 2008; Clarkson, Clifton & February 2009; Alonso et al. 2010).

---

1 See, however, Nishikawa, Yoo & Nakao (2012) for a recent development.
3.1 Hubble rate

The Hubble rate is obtained by measuring cosmological standard candles mostly within a redshift range with median value $z_b \sim 0.05$. We compare the observed value to the theoretical quantity,

$$H_{\text{obs}} = \frac{1}{z_{\text{max}} - z_{\text{min}}} \int_{z_{\text{min}}}^{z_{\text{max}}} H_A(r(z), t(z)) \, dz. \quad (19)$$

The values $z_{\text{max}}$ and $z_{\text{min}}$ depend on the redshift volume that is probed by a given experiment. The reason we compare an averaged expansion rate to the data is primarily observed expansion rate in fact is an averaged quantity, so this should be a fair comparison. Moreover, when the redshift $z_b$ of the boundary of the inhomogeneity is close in value to $z_b$, the averaged $H_{\text{obs}}$ may differ significantly from $H_A(r(z_b), t(z_b))$, which falsely would lead to a bad fit. The approach of equation (19) is to some extent arbitrary and one should instead reanalyze the raw data without assuming a FLRW fiducial model as it is usually done (see e.g. the discussion in Zumalacarregui, Garcia-Bellido & Ruiz-Lapuente 2012). However, as we will see in Section 4.1 our results depend weakly on the Hubble parameter constraint and so this caveat should not sizably affect our findings.

We mainly consider the determination of the Hubble rate from Riess et al. 2009 (R09). However, in order to study a possible sensitive dependence on this datum, we will also consider the results from Freedman et al. 2001 (F01) and Sandage et al. 2006 (S06). The three measurements are

$$H_{\text{R01}} = 72 \pm 8 \text{ km s}^{-1} \text{ Mpc}^{-1}, \quad 0.005 < z < 0.1, \quad (20)$$

$$H_{\text{S06}} = 62.3 \pm 6.3 \text{ km s}^{-1} \text{ Mpc}^{-1}, \quad 0.01 < z < 0.07, \quad (21)$$

$$H_{\text{R09}} = 74.2 \pm 3.6 \text{ km s}^{-1} \text{ Mpc}^{-1}, \quad 0.023 < z < 0.1. \quad (22)$$

3.2 Supernova observations

We use the Union2 SN compilation (Amanullah et al. 2010), which consists of 557 type Ia supernovae in the redshift range $z = 0.015-1.4$. As we are considering an almost linear inhomogeneity surrounding the observer, we are not departing strongly from the standard model. Therefore, it should be a good approximation to use the magnitude-redshift and correlation tables provided by Amanullah et al. (2010).

3.3 Cosmic microwave background

The metric of equation (3) is matched to the background FLRW metric at a redshift at which radiation is still negligible. In this way the last scattering surface, which is responsible for most of the CMB anisotropies, is outside the inhomogeneous patch and a standard analysis of the primordial CMB power spectrum is possible. One has to replace the inhomogeneous model with an effective FLRW metric, which accounts for the different angular diameter distance to the surface of last scattering of the CMB as compared to the homogeneous background model. This is done by placing an FLRW observer ($\theta_0 \equiv 0$) in the same coordinate system at $r = 0$ but at a different time than $t_0$, such that this observer’s angular diameter distance to the surface of last scattering, which lies at some constant time $t_{\text{LS}}$, agrees with the actual LTB observer’s angular diameter distance to the surface of last scattering. The physics at last scattering itself is unaffected, since the effective FLRW observer is placed in the same FLRW universe in which the LTB patch is embedded, albeit at a different time. The CMB spectrum is then calculated using CAMB (Lewis, Challinor & Lasenby 2000). See Biswas, Notari & Valkenburg (2010), Moss, Zibin & Scott (2011) and Marra & Paakkonen (2010) for more explicit details about how the effective model is obtained. Note that there are other contributions to the CMB coming from secondary effects, which in the inhomogeneity may differ from those in the effective FLRW metric, and are due to the photons travelling through inhomogeneities inside the void, such as the late-time integrated Sachs-Wolfe effect and weak lensing. For the same reason as for which we ignore the baryon acoustic oscillations, we ignore these secondary effects, since they are subdominant and studying them would require knowledge of the growth of perturbations in an inhomogeneous background. We fit the theoretical predictions of our model to the Wilkinson Microwave Anisotropy Probe (WMAP) 7-year data release (Komatsu et al. 2011).

3.4 Parameter estimation

We perform a Markov chain Monte Carlo (MCMC) likelihood analysis using COSMOMC (Lewis & Bridle 2002). We calculate all distance measures using an improved version of VOIDDISTANCES$^2$ (Biswas et al. 2010), which now acts as a wrapper around CAMB (Lewis et al. 2000), necessitating no changes to CAMB’s source code and minimal changes to COSMOMC’s source code. We combine this module with the LTB module COLLAB$^3$ (Valkenburg 2012a) for calculating all metric functions, which we extended to include the numerical solutions to the scenario discussed here, with $w_{\text{out}} \neq -1$ and $c_r = 0$.

With this setup, for every selected vector of parameter values, we calculate the theoretical predictions for SN distances, the local Hubble rate and the CMB power spectrum. For reference, next to the inhomogeneous model we analyse its homogeneous background model, $\Lambda$CDM, which is described by the same model but has $z_b \equiv 0$ and $\delta_b \equiv 0$.

We take flat priors on the parameters listed in Table 1. Most of these are the usual cosmological parameters: the background present-day Hubble rate $h$, the background baryon density $\Omega_b h^2$, the background dark matter density $\Omega_m h^2$, the equation of state of dark energy $w_{\text{out}}$, the optical depth to reionization $\tau$, the amplitude of primordial scalar perturbations $A_s$, the tilt of the spectrum of primordial scalar perturbations $n_s$ and its running $\alpha_s$. We set the spatial curvature outside the LTB patch, $\Omega_k$, as well as the amplitude of primordial tensor perturbations to zero.

As discussed in Section 2.2, two additional parameters describe the LTB patch: the curvature at the centre $k_c$ and the comoving radius $r_c$. As shown in Table 1, we will use as actual parameters the present-day total density contrast and the present-day proper size of the radius, respectively. The latter is given by

$$d(r) \equiv \int_0^{r_c} \frac{Y'(r, t_0)}{\sqrt{1 - k(r)r^2}} \, dr, \quad (23)$$

and we define the former as

$$\delta_0 \equiv \frac{\rho_{\text{in}} - \rho_{\text{out}}}{\max(\rho_{\text{in}}, \rho_{\text{out}})}, \quad (24)$$

where $\rho_{\text{in}} = \rho_{\text{M, in}}(t_0) + \rho_{\text{X, in}}(t_0)$ and $\rho_{\text{out}} = \rho_{\text{M, out}}(t_b) + \rho_{\text{X, out}}(t_b)$.

We calculate the contrast using the total density as in principle

$^2$ http://web.physik.rwth-aachen.de/download/valkenburg/

$^3$ http://web.physik.rwth-aachen.de/download/valkenburg/ColLambda/
Table 1. Priors imposed on the parameters in the numerical analysis. The size of the LTB patch $d(r_b)$ is defined in equation (23). The additional constraint $\Omega_x > 0$ with $\Omega_k = 0$ in fact implies a non-flat prior on both $\Omega_m h^2$ and $h$, as explained in Appendix A.

| Flat priors                      | Additional constraints       |
|----------------------------------|------------------------------|
| $0.4 < h < 1$                    | $\Omega_x > 0$               |
| $0.005 < \Omega_m h^2 < 0.1$     | $\Omega_k = 0$               |
| $0.001 < \Omega_m h^2 < 0.99$    |                              |
| $-0.4 < w_{\text{out}} < 0$      |                              |
| $0.01 < r < 0.8$                 |                              |
| $2.7 < \log 10^{10} A_S < 4$     |                              |
| $0.5 < n_S < 1.5$                |                              |
| $-0.2 < a_S < 0.2$               |                              |
| $-0.2 < \delta_0 < 0.2$          |                              |
| $100 \text{ Mpc} < d(r_b) < 3 \text{ Gpc}$ |                          |

Uncertainty on $w$ from large-scale structure

In Table 1, we have imposed priors on the parameters of interest. The size of the LTB patch $d(r_b)$ is defined in equation (23). The additional constraint $\Omega_x > 0$ with $\Omega_k = 0$ in fact implies a non-flat prior on both $\Omega_m h^2$ and $h$, as explained in Appendix A.

We discuss the priors in more detail in Appendix A. In the next section, where we discuss the results of the MCMC parameter estimation, we explore the full parameter space, never fixing the background parameters to some central value. Therefore, we can always marginalize over all parameters, and do not bias the result in any way. That is, any possible degeneracy, expected or unexpected, between the cosmological parameters and the LTB parameters will show up and will not influence the results without being noted.

4 RESULTS

The four left-hand panes in Fig. 2 show the parameters on which the main focus of this paper lies. This figure shows the two-dimensional marginalized posterior probability distributions of the background dark-energy density $\Omega_x$, the dark-energy equation of state $w_{\text{out}}$, the boundary of the inhomogeneity in redshift space $z_b$, and the total density contrast in the inhomogeneity $\delta_0$ (see equation 24). Alternatively, in the four right-hand panes of Fig. 2 we use as proxies for $z_b$ and $\delta_0$ the apparent size that the inhomogeneity would subtend if located at the last scattering surface and its corresponding temperature anisotropy, respectively. Here, we define the temperature perturbation $\Delta T/T$ as the relative difference in CMB temperature at the centre of the inhomogeneity and outside, in the homogeneous (average) background. This is not necessarily representative for the average temperature in the spot.

The most interesting result is the clear degeneracy between $\delta_0$ and $w_{\text{out}}$ in the lower-right pane in the left of Fig. 2, and the degeneracy between $\delta_0$ and $\Omega_x$ in the upper-right pane. This graph explicates the necessity of properly modelling the inhomogeneity of the local

![Figure 2](https://academic.oup.com/mnras/article-abstract/431/2/1891/1468314/1895/fig2)
Any $\delta_0$

$\delta_0 > 0$

$\delta_0 < 0$

$\delta_0 = -0.15$

$\delta_0 = 0.1$

$\Omega_X$

$w_{\text{out}}$

Figure 3. One-dimensional marginalized posterior probabilities for $\Omega_X$ and $w_{\text{out}}$, constrained by CMB, SNe and $H_{\text{loc}}$. The standard wCDM constraints on these parameters are shown as red dashed lines and are the same in the different panes. The blue vertical line serves as a guide for the eye, always going through the maximum of the $w_{\text{out}}$ CDF value. The constraints on the wCDM model endowed with a local inhomogeneity considered in this paper are given under different priors on $\delta_0$: $-0.2 < \delta_0 < 0.2$ (top), $0 < \delta_0 < 0.2$ (second row from top), $-0.2 < \delta_0 < 0$ (third row from top), and for two constraining priors on $\delta_0$ (bottom): $\delta_0 = 0.1$ (solid black line) and $\delta_0 = -0.15$ (dash–dotted black line). A prior on $\delta_0$ that averages around zero widens but does not shift the posterior distributions. If we know instead the sign of $\delta_0$, constraints on $w_{\text{out}}$ and $\Omega_X$ shift by as much as 5~10 per cent. In the bottom row, for $\delta_0 = 0.1$ the 95 per cent c.l. upper bound on $w_{\text{out}}$ is $-1.03$. For $\delta_0 = 0.15$ the 95 per cent c.l. lower bound on $w_{\text{out}}$ is $-0.98$. Both priors hence rule out the cosmological constant at 95 per cent c.l., given current observations.

In Fig. 3, we show the one-dimensional marginalized posterior probabilities for $\Omega_X$ and $w_{\text{out}}$, under different priors on $\delta_0$, imposed by means of importance sampling on the MCMC chains that explore the full range of $\delta_0$. The result is displayed in black, and for comparison the constraints on these parameters for the homogeneous wCDM model ($z_b \equiv 0$, $\delta_0 \equiv 0$) are displayed in dashed red lines. The top row shows $\Omega_X$ and $w_{\text{out}}$ marginalized over all values of $\delta_0$, both positive and negative. Since positive and negative values of $\delta_0$ have opposing effects on $w_{\text{out}}$ and $\Omega_X$, marginalizing over $\delta_0$ mostly widens the tails of the distributions. This is already an important result showing how inhomogeneities contribute to the error budget in the cosmological parameters. This uncertainty from large-scale structures is expected to become more important when future data will tighten the confidence regions of the parameters of interest. If we impose, however, the prior that $\delta_0 > 0$ (second row in Fig. 3) or $\delta_0 < 0$ (third row from top), we find even stronger results: we see indeed that both the tails and the central values of $\Omega_X$ and $w_{\text{out}}$ shift. This shift is significant if one wants to progress towards not just precision but also accurate cosmology. If we push the magnitude even further, pretending we know that the local density must be either $\delta_0 = 0.1$ or $\delta_0 = -0.15$, as in the bottom row in Fig. 3, then we find that $w = -1$ is excluded at 95 per cent c.l. in both cases: $w < -1.03$ and $w > -0.98$, respectively. However, these particular models are only included at 99.7 per cent c.l.
from large-scale structure hardly depend on $\Omega_1$ and $w$ for the standard $\Lambda$CDM that are far from $H_{0}$ and $\Omega_1$. Therefore, the inhomogeneity on $w_{\text{out}}$ are robust against different observational determinations of $H_{0}$, and the constraints hence mostly follow from the SNe and CMB.

4.1 Sensitivity to local Hubble rate constraints

In Fig. 5, we compare the effect of different Hubble rate observations on the resulting posterior probabilities of $\Omega_X$ and $w_{\text{out}}$, when we fit the $\Lambda$CDM model endowed with a local inhomogeneity to $H_{0}$, SNe and CMB. As explained in Section 3.1, we compare the three different values from $R_{09}$ (solid black), $F_{01}$ (dashed red) and $S_{06}$ (dashed dotted blue). We see that the constraints on $\Omega_X$ do depend on the chosen measurement for $H_{0}$, while the resulting constraints on $w_{\text{out}}$ hardly depend on $H_{0}$. This should be expected as SNe observations (constraining both $\Omega_X$ and $w_{\text{out}}$) are insensitive to $H_{0}$, while CMB observations (constraining $\Omega_X$ but weakly $w_{\text{out}}$) are instead sensitive to $H_{0}$. These results imply that our conclusions with regard to $w_{\text{out}}$ are robust against different observational determinations of $H_{0}$, and the constraints hence mostly follow from the SNe and CMB.

4.2 kSZ effect

In Fig. 6, we show the CMB dipole that observers at different radii would observe, for a given configuration; an overdensity on the left and an underdensity on the right, in both cases the LTB patch has a radius of 1 Gpc, but with different values for $w_{\text{out}}$. Both cases fit the data roughly as well as the standard $\Lambda$CDM, while still giving an interesting bias on $w_{\text{out}}$.

We obtained these figures by – at each radius – starting an integration of the geodesic equations in two directions (negative and positive $r$-direction), back to the surface of last scattering, which lies at constant time in the synchronous gauge of the LTB metric. The difference in redshift to the surface of last scattering in both directions is then translated into $\Delta T_{\text{CMB}}/T_{\text{CMB}} = (z_+ - z_-)/(2 + z_+ + z_-)$. Only for small radii this is to a good approximation equal to the dipole in a spherical harmonics expansion of the CMB temperature map. On the vertical axis on the right-hand side we list the corresponding peculiar velocity that an observed temperature difference corresponds to, if it were the effect of peculiar velocity alone. For both panes, left and right, the peculiar velocities do not exceed the magnitude of expected random peculiar velocities. Therefore, the kSZ effect that such velocities induce on CMB photons (Garcia-Bellido & Haugboelle 2008; Zhang & Stebbins 2011; Zibin & Moss 2011) should be at present undetectable. So as to strengthen this claim it is useful to look at the findings of Valkenburg et al. (2012), where constraints on the ALTB model from kSZ observations have been computed. While in this paper, dark energy is not the cosmological constant, the above analysis should give nevertheless an estimate of the kSZ signal. Therefore, this suggests that structures with a contrast of roughly $\sim 0.1$ extending for a radius of 1 to 2 Gpc are not excluded by present observations. A thorough study of the kSZ effect in these models is left to future work.
**4.3 FLRW observer’s \( w(z) \)**

Following Clarkson et al. (2007) one can, given a luminosity distance–redshift relation in a homogeneous universe, compute what the underlying \( w(z) \) of the dark-energy fluid is. In the homogeneous universe (described by the FLRW metric), one can find indeed an exact relation between \( w(z) \) and the first and second derivatives of the luminosity distance with respect to redshift and two more parameters, \( \Omega_m \) and \( \Omega_{\Lambda} \). If an observer knows the latter two parameters from other observations, and deduces the first and second derivatives of the luminosity distance from SN observations, the observer can derive \( w(z) \). In the scenario studied here, at background level \( w \) is not a function of time or redshift. However, the inhomogeneity comes into play in the luminosity distance–redshift relation. Therefore, an observer that falsely assumes that the metric surrounding him/her is FLRW will in fact see a redshift dependence on \( w \). We calculate the observed \( w_{\text{obs}}(z) \) (equation 3 in Clarkson et al. 2007) for the two example models of Fig. 6, and show the result in Fig. 7. Inside the inhomogeneity, \( w_{\text{obs}}(z) \) shows a very clear signature of the matter distribution: the contracting core and expanding compensating shell for the overdensity show corresponding effects on \( w_{\text{obs}}(z) \). The inverse holds for the underdensity.

**Figure 6.** The CMB dipole observed by observers at different radii \( d(r) \) in an LTB patch with a radius of 1 Gpc, for an overdensity (left) and an underdensity (right). On the right-hand vertical axis we list the corresponding peculiar velocity that is derived assuming that the observed CMB dipole is caused solely by the peculiar velocity of the observer. The magnitude of the velocities does not exceed the magnitude of expected random velocities. See Section 4.2 for more details.

**Figure 7.** The function \( w_{\text{obs}}(z) \) as defined in Clarkson, Cortes & Bassett (2007), which is the equation of state of dark energy that an observer thinks to see if the observer falsely assumes that the universe is described by the FLRW metric. The examples shown here are the same two cosmologies as in Fig 6; an overdensity (left) and an underdensity (right). In both cases, the observed \( w_{\text{obs}}(z) \) (solid red) matches closely with the fundamental \( w_{\text{out}} \) (dashed blue) once the radius is reached where the spherically symmetric metric matches to the surrounding FLRW metric. The inhomogeneity causes clear features in \( w_{\text{obs}}(z) \): the contracting core of the overdensity increases the magnitude of \( w_{\text{obs}}(z) \), while the underdense compensating shell has the opposite effect. For the underdense centre, the inverse holds. In this picture, \( w_{\text{obs}}(z) \) is computed using the exact solutions, while taking second derivatives of observed distances would not necessarily reveal these features.
Uncertainty on $w$ from large-scale structure

Figure 8. As in Fig. 7 but for a $\Lambda$CDM model endowed with a local underdensity of central contrast $\delta_0 = -0.06$ and redshift boundary $z_B = 0.4$ such that the observed $w_{\text{obs}}(z)$ appears qualitatively similar to the $w(z)$ reconstruction by Zhao et al. (2012, see e.g. the pane (A2) of fig. 1 in that reference). If interpreted within this framework, the results of Zhao et al. (2012) could indicate the presence of a large-scale underdensity around us, rather than a possibly time-dependent dark-energy equation of state.

Therefore, if one performs an analysis such as in Shafieloo, Sahni & Starobinsky (2009) and Zhao et al. (2012), one may find a significant deviation from a constant $w$, while fundamentally $w$ is constant at the background level. In particular, it is very interesting to note that the $w(z)$ reconstruction by Zhao et al. 2012 (see, e.g. the pane A2 of fig. 1 in that reference), if interpreted within this framework, could indicate the presence of a large-scale underdensity around us, and not of a possibly time-dependent equation of state. It is indeed worth noting the similarity of the result by Zhao et al. (2012) with the observed $w_{\text{obs}}(z)$ shown in Fig. 8, which corresponds to a $\Lambda$CDM model endowed with a local underdensity of central contrast $\delta_0 = -0.06$ and redshift boundary $z_B = 0.4$.

4.4 Other parameters

In Fig. 9, we show posterior probabilities of $H_0$, which is the expansion rate of the background universe (and does not correspond to the locally observed expansion rate), $\delta_0 = \frac{\rho_{\text{out}} - \rho_{\text{in}}}{\rho_{\text{in}}}$, $\delta_{M,0} = \frac{\rho_{\text{out}} - \rho_{M,0}}{\rho_{M,0}}$, and $\delta_{X,0} = \frac{\rho_{X,0} - \rho_{\text{in}}}{\rho_{\text{in}}} - 1$. All other parameters that we allowed to vary, listed in Table 1, show practically no deviation in their constraints in the presence and absence of the inhomogeneity. The constraint on $H_0$ is significantly weakened when one takes into account the possibility that we may live in a local inhomogeneity. This result is similar to the findings in Valkenburg & Bjoelde (2012), even if the local inhomogeneity considered is of the order of magnitude different. Moreover, the actual matter perturbation that is allowed by the data can be as large as $|\delta_{M,0}| \approx 0.5$, leaving the total density perturbation around $|\delta_0| \approx 0.1$, because the energy perturbation in the dark-energy fluid is generally small as $w_{\text{out}}$ is never very far from $-1$.

4.5 Probability under homogeneous initial conditions

Given the fact that we observe several large spots in the CMB, and the possibility that these spots are the result of density perturbations on the surface of last scattering, we can argue that living in such a perturbation must have a non-zero probability. If all perturbations arise from a smooth, close to scale invariant spectrum of primordial perturbations, it is not obvious that large cold and hot spots should exist, and there is an ongoing debate about this topic (Cruz et al. 2006; Ayaita, Weber & Wetterich 2010; Zhang & Huterer 2010).

Let us nonetheless quantify the probability of having the density perturbations that we took as examples for Figs 6 and 7. If these perturbations come from the same spectrum as the perturbations that we observe in the CMB, then the probability of their existence can be approximated by the variance of the Gaussian density field, smoothed by a top hat filter with a radius that corresponds to the radius of the density perturbation under consideration (Kolb & Turner 1990). It must be noted that, because of the compensated shape of the density profile that we consider, taking the full radius of the spherical patch, $r_s$, as the radius of the top hat filter would give almost zero density perturbation. Therefore, we choose the radius at which the density changes sign as the smoothing radius. This is roughly at $r_s/2$, but we use the numerically obtained exact value. Comparing at the time of decoupling when dark energy is negligible, and with the primordial spectrum of perturbations of the two models, respectively (since the models were fit to the CMB, they carry spectral parameters), we find that the overdense model considered in Figs 6 and 7 is at three times the dispersion of the smoothed density field of its cosmology (CMB spectrum), and the underdense model is at six times the dispersion of its cosmology. Notably the overdense model is not at all unlikely to occur, while it does give a large effect on both $w_{\text{out}}$ and $w_{\text{obs}}$ as shown in Fig. 7.

5 CONCLUSION

We have analysed present observations of the local expansion rate, distant SNe and the CMB within a flat $w$CDM model endowed with a local almost linear inhomogeneity surrounding the observer. We have found a significant impact on the dark-energy parameters, in particular on the equation of state, which is strongly degenerate with the inhomogeneity contrast. The implications of this degeneracy are twofold. On one hand we have shown that with prior knowledge on the inhomogeneity, to be obtained possibly with some future probe, it is already possible to rule out the case of the cosmological constant with current data. On the other hand, even if future probes exclude the case of the cosmological constant in a homogeneous universe, this still may be due to a poor modelling of the large-scale structure of the Universe. The same conclusions apply to constraints on the time variation of the equation of state.

The analysis in this paper is but a first step towards a more accurate reconstruction of the cosmological parameters. We have therefore chosen, for technical reasons, a very specific dark-energy model and inhomogeneity profile. Before drawing definitive conclusions, a more comprehensive analysis should be performed. First, it would be particularly interesting (even though perhaps challenging) to consider a non-zero dark-energy sound speed. However, since we found that the dark-energy component is only very mildly inhomogeneous, we do not expect our results to be strongly different.

\footnote{We would like to point out that for the almost linear models considered in this paper, the observer does not need be very close to the centre so as not to see a too large CMB dipole. For the cases of Fig. 6, for example, the dipole is never larger than the observed value of $\sim 10^{-3}$. For larger contrasts, there will be regions where observers would see a larger than measured dipole, but these regions will not occupy the majority of the inhomogeneity. Finally, we would like to stress that we have placed the observer at the centre simply to simplify the numerical calculations.}
dependent on the assumption of a negligible sound speed. Secondly, it would be interesting to consider more inhomogeneous patches with more general density profiles, possibly with the observer at randomized positions (Marra et al. 2007; Marra, Kolb & Matarrese 2008; Valkenburg 2009; Szybka 2011; Flanagan et al. 2012).

ACKNOWLEDGEMENTS

It is a pleasure to thank Luca Amendola and Ignacy Sawicki for useful comments and discussions. MP acknowledges financial support from the Magnus Ehrnrooth Foundation. VM and WV acknowledge funding from DFG through the project TRR33 ‘The Dark Universe’.

REFERENCES

Alonso D., Garcia-Bellido J., Haugboelle T., Vicente J., 2010, Phys. Rev. D, 82, 123530
Amanullah R. et al., 2010, ApJ, 716, 712
Amendola L., Kainulainen K., Marra V., Quartin M., 2010, Phys. Rev. Lett., 105, 121302
Ayaita Y., Weber M., Wetterich C., 2010, Phys. Rev. D, 81, 023507
Ballesteros G., Lesgourgues J., 2010, JCAP, 1010, 026
Bennett C. L. et al., 2011, ApJS, 192, 17
Bertacca D., Bartolo N., Diaferio A., Matarrese S., 2008, J. Cosmol. Astropart. Phys., 0810, 023
Bertacca D., Bartolo N., Matarrese S., 2010, Adv. Astron., 2010, 904379
Biswas T., Notari A., Valkenburg W., 2010, J. Cosmol. Astropart. Phys., 1011, 030
Clarkson C., Cortes M., Bassett B. A., 2007, J. Cosmol. Astropart. Phys., 0708, 011
Clarkson C., Clifton T., February S., 2009, J. Cosmol. Astropart. Phys., 0906, 025
Creminelli P., D’Amico G., Norena J., Vernizzi F., 2009, J. Cosmol. Astropart. Phys., 0902, 018
Cruz M., Tucci M., Martinez-Gonzalez E., Vielva P., 2006, MNRAS, 369, 57
de Lavallaz A., Fairbairn M., 2011, Phys. Rev. D, 84, 083005
Deffayet C., Pujolas O., Sawicki I., Vikman A., 2010, J. Cosmol. Astropart. Phys., 1010, 026
Enqvist K., Mattsson T., 2007, J. Cosmol. Astropart. Phys., 0702, 019
Flanagan E. E., Kumar N., Wasserman I., Vanderveld R., 2012, Phys. Rev. D, 85, 023510
Freedman W. L. et al., 2001, ApJ, 553, 47 (F01)
Garcia-Bellido J., Haugboelle T., 2008, J. Cosmol. Astropart. Phys., 0809, 016
Inoue K. T., Silk J., 2006, ApJ, 648, 23
Inoue K. T., Silk J., 2007, ApJ, 664, 650
Kolb E. W., Turner M. S., 1990, Front. Phys., 69, 1
Komatsu E. et al., 2011, ApJS, 192, 18
Lewis A., Bridle S., 2002, Phys. Rev., D, 66, 103511
Lewis A., Challinor A., Lasenby A., 2000, ApJ, 538, 473
Li M., Li X.-D., Wang S., Wang Y., 2011, Commun. Theor. Phys., 56, 525
Lim E. A., Sawicki I., Vikman A., 2010, J. Cosmol. Astropart. Phys., 1005, 012
Marra V., Notari A., 2011, Class. Quantum Grav., 28, 164004
Marra V., Paakkonen M., 2010, J. Cosmol. Astropart. Phys., 1012, 021
Marra V., Paakkonen M., 2012, J. Cosmol. Astropart. Phys., 1201, 025
Marra V., Kolb E. W., Matarrese S., Riotto A., 2007, Phys. Rev. D, 76, 123004
Marra V., Kolb E. W., Matarrese S., 2008, Phys. Rev. D, 77, 023003
Masina I., Notari A., 2009, J. Cosmol. Astropart. Phys., 0902, 019
Moss A., Zibin J. P., Scott D., 2011, Phys. Rev. D, 83, 103515
Nishikawa K., Yoo C.-M., Nakao K., 2012, Phys. Rev. D, 85, 103511
Peebles P., 2002, preprint (astro-ph/0208037)
Perlmutter S. et al., 1999, ApJ, 517, 565
Riess A. G. et al., 1998, AJ, 116, 1009
Riess A. G. et al., 2009, ApJ, 699, 539 (R09)
Romano A. E., Chen P., 2011, J. Cosmol. Astropart. Phys., 1110, 016
Sandage A., Tammann G. A., Saha A., Reindl B., Macchetto F. D., Panagia N., 2006, ApJ, 653, 843 (S06)
Shafielloo A., Sahni V., Starobinsky A. A., 2009, Phys. Rev. D, 80, 101301
Sinclair B., Davis T. M., Haugbolle T., 2010, ApJ, 718, 1445
Szybka S. J., 2011, Phys. Rev. D, 84, 044011
Tomita K., 2005, Phys. Rev., D, 72, 103506
Valkenburg W., 2009, J. Cosmol. Astropart. Phys., 0906, 010
Valkenburg W., 2012a, Gen. Relativ. Grav., 44, 2449
Valkenburg W., 2012b, MNRAS, 424, 495
Valkenburg W., 2012c, J. Cosmol. Astropart. Phys., 1201, 047
Van Acoleyen K., 2008, J. Cosmol. Astropart. Phys., 1006, 028
Zhang R., Huterer D., 2010, Astropart. Phys., 33, 69
Zhao G.-B., Crittenden R. G., Pogosian L., Zhang X., 2012, Phys. Rev. Lett., 107, 041301
Zibin J. P., 2008, Phys. Rev. D, 78, 123508
Zibin J. P., 2011, Phys. Rev. D, 84, 123508
Zibin J. P., Moss A., 2011, Class. Quantum Grav., 28, 164005
Zumalacarregui M., Garcia-Bellido J., Ruiz-Lapuente P., 2012, J. Cosmol. Astropart. Phys., 1210, 009

Figure 9. One-dimensional marginalized posterior probabilities of $H_0$, which describes the age of the universe and does not correspond to the locally observed expansion rate, $\delta_0 \equiv \frac{\Delta M_{\text{in}}}{M_{\text{in}}} - 1$ and $\delta_{X,0} \equiv \frac{\Delta M_{\text{out}}}{M_{\text{out}}} - 1$. The presence of the spherical structure weakens significantly the bounds on the background expansion rate (in red we show the constraint on $H_0$ in the homogeneous $\omega$CDM model). Secondly, the local energy perturbation consists of an almost negligible dark-energy perturbation and a significant dust perturbation, which is not apparent if one considers the total $\delta_0$ alone.
APPENDIX A: EFFECT OF CONSTRAINTS ON PRIORS

In Fig. A1, we show the prior probability of $\Omega_{dm}h^2$, $H_0$ and derived parameters, since they are dependent non-linearly on combinations of input parameters on which we take flat priors, and hence their prior probability is non-flat. We obtained these prior probabilities by running the MCMC analysis without any data, accepting all points in the parameter space with equal probability. Maybe surprisingly, the actual prior probabilities on $\Omega_{dm}h^2$ and $H_0$ are not flat, even though we do list their prior ranges in Table 1 as flat, and we indeed gave flat priors on these parameters in the input priors.

Figure A1. Priors on $\Omega_{dm}h^2$, $H_0$ and derived parameters, displaying those that are non-flat. The black (solid) lines represent the priors when the inhomogeneous metric is used, while the blue (dash-dotted) lines represent the priors in the homogeneous $w$CDM case. For comparison, in red (dashed) we show the constraints on $\Omega_{dm}h^2$ and $H_0$ when the homogeneous $w$CDM case is fitted to the data. The narrowness of the red curves shows that the data are very constraining on these parameters, and so the non-flat prior is of no importance. These prior distributions are obtained by running the MCMC analysis without any data, that is, accepting all points in the parameter space with equal probability.
of the MCMC simulation. The non-flatness stems from the additional constraint \( \Omega_\Lambda > 0 \), demanding that the energy density of the dark-energy fluid is positive (if the dark energy is a pure cosmological constant nothing prevents \( \Omega_\Lambda < 0 \) from happening). Because we set \( \Omega_k = 0 \), this condition is satisfied only when \( \Omega_m < 1 \) or \( \Omega_m h^2 < h^2 \).

To explain the relation, let us simplify the priors to \( 0 < \Omega_m h^2 < 1 \) and \( 0 < h < 1 \). The starting point is a flat prior on these parameters, i.e. that the probability \( P(\Omega_m h^2) = \text{constant} \) and \( P(h) = \text{constant} \), normalized such that \( \int_0^1 P(x) dx = 1 \) for \( x \) being both \( h \) and \( \Omega_m h^2 \). If we write shorthand notation \( C \) for the condition that \( \Omega_\chi > 0 \) and \( \Omega_k = 0 \), then imposing \( C \) we have the conditional joint probability

\[
P(\Omega_m h^2, h | C) \propto \begin{cases} 
P(\Omega_m h^2, h | C) = \text{constant} & \text{if } C \text{ is true} \\
0 & \text{if } C \text{ is false}
\end{cases},
\]

(A1)

normalized such that

\[
\int_0^1 \int_0^{1} P(\Omega_m h^2, h | C) d(\Omega_m h^2) dh = 1.
\]

(A2)

Then, we find for the probabilities of \( h \) and \( \Omega_m h^2 \), up to normalization constants

\[
P(h | C) = \int_0^1 P(\Omega_m h^2, h | C) d(\Omega_m h^2) \\
\propto \int_0^{h^2} d(\Omega_m h^2) \propto h^2,
\]

(A3)

\[
P(\Omega_m h^2 | C) = \int_0^1 P(\Omega_m h^2, h | C) dh \propto \int_{h = \sqrt{\Omega_m h^2}}^{1} dh \\
\propto 1 - \sqrt{\Omega_m h^2},
\]

(A4)

in agreement with the priors in Fig. A1, favouring large \( h \) and small \( \Omega_m h^2 \). However, as can be seen from the red dashed curves in Fig. A1, the data are constraining \( \Omega_m h^2 \) and \( H_0 \) so tightly, that the non-flatness of the prior has no effect on the final parameter estimation.

This paper has been typeset from a \TeX/LaTeX file prepared by the author.