Alpha-NML Universal Predictors

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Abstract—Inspired by Sibson’s alpha-mutual information, we introduce a new class of universal predictors that depend on a real parameter greater than one. This class interpolates two well-known predictors, the mixture estimator, that includes the Laplace and the Krichevsky-Trofimov predictors, and the Normalized Maximum Likelihood (NML) estimator. We point out some advantages of this class of predictors and study its performance in terms of known regret measures under logarithmic loss, in particular for the well-studied case of discrete memoryless sources.

I. INTRODUCTION

Universal prediction, see e.g. [1], refers to the problem of estimating the next symbols of a sequence given its past, and evaluating the confidence of such an estimate, when the true model of the source is any of the models belonging to a certain class. This problem found applications in a wide range of areas, such as compression [2], [3], gambling [4] and machine learning [5], [6].

For any \( n \geq 1 \), we assume that a sequence \( x^{n-1} = (x_1, x_2, \ldots, x_{n-1}) \) of \( n-1 \) symbols from a given discrete alphabet \( \mathcal{X} \) has been generated by some unknown (random or deterministic) source. We wish to design a predictor that assigns probabilities to all \( n \) values of the next outcome \( x_n \), that we denote by \( \hat{p}(x_n|x^{n-1}) \). The quality of the prediction is measured by the logarithmic loss. The cumulative loss for the entire sequence \( x^n \) equals

\[
L(\hat{p}, x^n) \triangleq \sum_{i=1}^{n} \log \frac{1}{\hat{p}(x_i|x^{i-1})} = \log \frac{1}{\hat{p}(x^n)} \tag{1}
\]

where \( \hat{p}(x^n) = \prod_{i=1}^{n} \hat{p}(x_i|x^{i-1}) \) can be seen as the joint estimated probability of the sequence \( x^n \).

Let us now consider a given class of distributions \( \mathcal{P} = \{p_\theta : \theta \in \Theta\} \) indexed by a parameter set \( \Theta \), and let us assume that this is the class that we want to compare our predictor to. A universal prediction needs to perform well for every sequence in \( \mathcal{X}^n \) and every source in \( \mathcal{P} \). The most widely-studied measure of goodness of a predictor, in the universal prediction setting, is the worst-case regret

\[
R_{\text{max}}(\hat{p}) \triangleq \sup_{\theta \in \Theta} \max_{x^n \in \mathcal{X}^n} \left[ \log \frac{1}{\hat{p}(x^n)} - \log \frac{1}{p_\theta(x^n)} \right], \tag{2}
\]

which is the maximum difference between the cumulative loss of the predictor and that of the source, over all sources in \( \Theta \) and all sequences in \( \mathcal{X}^n \). It is well known [7] that the predictor that minimizes the worst-case regret is the Normalized Maximum Likelihood (NML) estimator, whenever it exists. Its formula is

\[
\hat{p}_{\text{NML}}(x^n) = \frac{\sup_{\theta \in \Theta} p_\theta(x^n)}{\int_{\mathcal{X}^n} \sup_{\theta \in \Theta} p_\theta(x^n) \, dx^n}. \tag{3}
\]

Recently, an alternative Fourier-based formula for the NML was developed in [8]. Despite its nice closed-form expression, in general the NML has several disadvantages, including the fact that it may not exist since the integral in the denominator in (3) may not converge, the fact that the denominator involves exponentially many terms, and the necessity of computing the maximization over the parameter space at the numerator. For certain classes, efficient ways to compute NML have been developed [9], [10]. More generally, researchers looked for good alternatives to the NML predictor. For the class of discrete memoryless sources, such an alternative is the Krichevsky-Trofimov estimator [11]. This predictor achieves, for the class of discrete memoryless sources, the same asymptotic regret (up to a constant term) as the NML when \( n \to \infty \) [4], with none of the disadvantages that we pointed out above. However, no similar results are proved for other classes of distributions, and also, the NML estimator performs better in general when \( n \) is finite. For these reasons, the search for alternative predictors that perform well for finite-length sequences, with at the same time fewer drawbacks than the NML estimator, is still of interest.

The contribution of this paper is the introduction of a class of predictors, parametrized by \( \alpha \geq 1 \), whose definition is inspired by Sibson’s \( \alpha \)-mutual information. As an example, for DMS this class interpolates between the KT estimator and the NML. The KT assigns as a probability for the next symbol \( k \in \{1, 2, \ldots, m\} \) a value proportional to

\[
\hat{p}_{\text{KT}}(k|x^{n-1}) \propto n_k + \frac{1}{2}, \tag{4}
\]

where \( n_k \) is the number of \( k \)'s in the past sequence \( x^{n-1} \). For \( \alpha = 1 \), our predictor gives the same probability estimation. For \( \alpha = 2 \), e.g., it assigns a probability that is proportional to

\[
\hat{p}_{\alpha=2}(k|x^{n-1}) \propto \sqrt{\left(n_k + \frac{1}{4}\right) \left(n_k + \frac{3}{4}\right)}. \tag{5}
\]

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In both cases, the probabilities are normalized in such a way that \( \sum_{k=1}^{n} \hat{p}(x^n_{k-1}) = 1 \). The general formula is given in Equation (25). For the binary alphabet case, the performance improvement of the new predictor is illustrated numerically in Figure 1 below.

The remainder of the paper is organized as follows. In Section II, we introduce the class of the \( \alpha \)-NML predictors as a middle-way between mixture predictors and NML. In Section III, we apply \( \alpha \)-NML to the parametric family of discrete memoryless sources, deriving some simple closed-form formulae to compute the probabilities estimated by the predictor. Finally, in Section IV we study the performance of \( \alpha \)-NML for DMS in terms of worst-case regret.

II. THE CLASS OF \( \alpha \)-NML PREDICTORS

The worst-case regret defined in (2) is strongly related to information-theoretic metrics, namely the well-known Rényi divergence and maximal leakage. Rényi divergence is defined for any \( \alpha > 0, \alpha \neq 1 \), as

\[
D_\alpha(P||Q) = \frac{1}{\alpha-1} \log \sum_{x \in \mathcal{X}} P^\alpha(x) Q^{1-\alpha}(x) \tag{6}
\]

where \( P \) and \( Q \) are any two distributions defined over a common discrete alphabet \( \mathcal{X} \). The limiting case \( \alpha \to \infty \) gives

\[
D_\infty(P||Q) = \max_{x \in \mathcal{X}} \log \frac{P(x)}{Q(x)}. \tag{7}
\]

Maximal leakage is instead defined as

\[
\mathcal{L}(X \to Y) = \log \sum_{y \in \mathcal{Y}} \sup_{x \in \text{supp}(X)} P(y|x) \tag{8}
\]

where \( X \) is any random variable defined over an alphabet \( \mathcal{X} \), \( Y \) is any random variable defined over a discrete alphabet \( \mathcal{Y} \), and \( P(y|x) \) is the conditional probability of \( Y = y \) given \( X = x \). The next lemma (see, e.g., [12] Thm. 37) links the worst-case regret to Rényi divergence and maximal leakage.

**Lemma 1:** Whenever the NML predictor exists, the worst-case regret defined in (2) for any predictor \( \hat{p} \) is equal to

\[
R_{\max}(\hat{p}) = \mathcal{L}(\hat{p} \to X^n) + D_\infty(\hat{p}_{\text{NML}}||\hat{p}) \tag{9}
\]

where \( \phi \) is any random variable over \( \Theta \) such that \( \text{supp}(\phi) = \Theta \), and \( X^n = x^n \) is a random variable over \( \mathcal{X}^n \) such that the conditional probability of \( X^n = x^n \) given \( \phi = \theta \) is \( p_\theta(x^n) \).

Notice that \( D_\infty(\hat{p}_{\text{NML}}||\hat{p}) = 0 \) if and only if \( \hat{p}_{\text{NML}}(x^n) = \hat{p}(x^n) \) for every \( x^n \). Therefore, Lemma 1 shows that the NML predictor is the unique minimizer of the worst-case regret, whenever it exists, and that its worst-case regret is equal to maximal leakage \( \mathcal{L}(\phi \to X^n) \).

It is known that maximal leakage \( \mathcal{L}(X \to Y) \) is the limit as \( \alpha \to \infty \) of Sibson’s \( \alpha \)-mutual information \( I_\alpha(X,Y) \), which is defined as

\[
I_\alpha(X,Y) = \frac{\alpha}{\alpha-1} \log \sum_{y \in \mathcal{Y}} \left( \sum_{x \in \mathcal{X}} P(x) P^\alpha(y|x) \right)^{1/\alpha}. \tag{10}
\]

1We use the conventions \( \frac{0}{0} = 0 \) and \( \frac{\infty}{\infty} = \infty \) for \( \alpha > 0 \).

After noticing that the denominator of the NML predictor in (3) is \( \exp \mathcal{L}(\phi \to X^n) \), it is natural to generalize that predictor into a continuous class of estimators dependent on a parameter \( \alpha \geq 1 \), by replacing maximal leakage with Sibson’s \( \alpha \)-mutual information. This idea leads to the following definition of the \( \alpha \)-NML predictors.

**Definition 1:** For any \( \alpha \geq 1 \) and any probability distribution \( w \) on \( \Theta \), the \( \alpha \)-NML predictor is defined as

\[
\hat{p}_\alpha(x^n) = \frac{\int_\Theta w(\theta) p_\theta^n(x^n) d\theta}{\sum_{x^n} \{\int_\Theta w(\theta) p_\theta^n(x^n) d\theta\}^{1/\alpha}}. \tag{11}
\]

The \( \alpha \)-NML class is a continuous interpolation between the NML predictor and the class of mixture estimators, which is retrieved by choosing \( \alpha = 1 \) in (11). The NML predictor is instead retrieved in the limit \( \alpha \to \infty \), provided that for every \( x^n \in \mathcal{X}^n \), \( \sup_\theta p_\theta(x^n) \) is achieved for a \( \theta \) such that \( w(\theta) > 0 \). This condition is achieved in particular for a prior \( w \) such that \( w(\theta) > 0 \) for every \( \theta \in \Theta \).

A nice property of the class of \( \alpha \)-NML predictors is that they are not affected by some of the problems that the NML has. First of all, \( \alpha \)-NML predictors do not require any maximization over the parameter space \( \Theta \). The maximization is in fact replaced by a weighted average of the distributions \( p_\theta \) to the power of \( \alpha \). Furthermore, by choosing carefully the prior \( w \) and the parameter \( \alpha \), one is able to control the convergence of the integral at the denominator of (11). In this sense, the role of the prior \( w \) is similar to that of the luckiness function (13), an expedient that was introduced in the literature to overcome the convergence problem of the NML estimator. Finally, even if the \( \alpha \)-NML predictors are still horizon-dependent and do not solve the issue of the sum at the denominator that was already present in the NML, most of the tricks used to overcome these difficulties for the NML — for example by exploiting sufficient statistics for certain exponential families — can also be used for \( \alpha \)-NML.

The idea for the introduction of \( \alpha \)-NML is to find an alternative general predictor that is competitive with NML. Therefore, it is interesting to analyze the \( \alpha \)-NML predictor mainly in terms of worst-case regret, for which the NML is optimal. Putting the \( \alpha \)-NML formula (11) gives the following formula for the worst-case regret of \( \alpha \)-NML, which highlights the role of Sibson’s \( \alpha \)-mutual information.

**Lemma 2:** The worst-case regret of the \( \alpha \)-NML predictor with prior \( w \) can be written as

\[
R_{\max}(\hat{p}_\alpha) = \frac{\alpha-1}{\alpha} I_\alpha(\phi, X^n) + W_\alpha(P) \tag{12}
\]

where \( I_\alpha(\phi, X^n) \) is the \( \alpha \)-mutual information for \( (\phi, X^n) \sim w(\phi) p_\phi(X^n) \), and

\[
W_\alpha(P) = \max_{x^n \in \mathcal{X}^n} \log \frac{\max_{\theta \in \Theta} p_\theta(x^n)}{\sum_{x^n} \{\int_\Theta w(\theta) p_\theta^n(x^n) d\theta\}^{1/\alpha}}. \tag{13}
\]

In general, it is not clear neither from (9) nor from (12) what is the behavior of \( R_{\max}(\hat{p}_\alpha) \) as a function of \( \alpha \), and this might depend on the actual class of distributions that is considered. In
where the prior distribution on the parameter space is
\[ n(\theta) \sim \text{Dirichlet}(\theta_1, \theta_2, \ldots, \theta_m) \]
This class has been the focus of a large part of the literature on universal prediction and compression. The main reasons for this are that this class is the simplest non-trivial example for which one can get a sense of how a predictor behaves, and at the same time prove rigorously some results in terms of performance of a predictor compared to the optimal. The most important result on universal prediction for this class of distributions is possibly the Krichevsky-Trofimov estimator. Let the source alphabet be \( X = \{1, 2, \ldots, m\} \). Let also
\[ \Theta = \{\theta = (\theta_1, \theta_2, \ldots, \theta_m) : \sum_{i=1}^{m} \theta_i = 1 \text{ and } \theta_i \geq 0 \text{ for all } i \} \]
be the parameter set. For each parameter \( \theta \) and sequence \( x^n = (x_1, x_2, \ldots, x_n) \in X^n \), the source indexed by \( \theta \) generates the sequence \( x^n \) with probability
\[ p_\theta(x^n) = \prod_{i=1}^{m} \theta_{i}^{n_i}, \]
where \( n_i = |\{1 \leq j \leq n : x_j = i\}| \). For the class of discrete memoryless sources described above, the Krichevsky-Trofimov predictor is a simple mixture estimator,
\[ \hat{p}_{\text{KT}}(x^n) \triangleq \int_{\Theta} w_{\text{KT}}(\theta) p_\theta(x^n) \, d\theta \]
where the prior distribution on the parameter space is \( w_{\text{KT}} \sim D(\frac{1}{m}, \ldots, \frac{1}{m}) \), i.e., the Dirichlet distribution with parameters equal to \( \frac{1}{m} \).
\[ w_{\text{KT}}(\theta) = \frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m}{2}\right)} \prod_{i=1}^{m} \frac{1}{\sqrt{\theta_i}}. \]
This estimator has arguably three major advantages.

1) Its probability estimates \( \hat{p}_{\text{KT}}(x^n) \) can be computed easily in closed form using properties of the Gamma function.
2) Its worst-case regret is asymptotically optimal in \( n \) up to a constant term;
3) It is horizon independent, and simple formulae exist for the computation of the conditional probability of a new symbol given the previous ones.

Xie and Barron also devised an alternative predictor by modifying the prior distribution on the parameter space. With this modification, their predictor is shown to be asymptotically optimal — i.e., it has the correct dependence on \( n \) like the KT estimator, and also the correct constant term, — but it has two disadvantages: its prior distribution \( w \) depends on \( n \), and the predictor is horizon dependent. In any case, both this predictor and the Krichevsky-Trofimov have guarantees of optimality only when \( n \to \infty \), and are strictly worse than the optimal NML predictor when \( n \) is finite. Therefore, it is of practical interest to find a predictor that can be computed with simple closed-form formulae in an efficient way (in polynomial time with \( n \)), and that performs better than the above-mentioned predictors for finite-length sequences. It turns out that the \( \alpha \)-NML predictor satisfies these requirements, when the class of discrete memoryless sources is considered and the Dirichlet distribution \( D(\frac{1}{2}, \ldots, \frac{1}{2}) \) is chosen as the prior distribution \( w \) on the parameter space. In this case, the predictor takes the form
\[ \hat{p}_\alpha(x^n) = \frac{1}{Z_n(\alpha)} \left\{ \int_{\Theta} \prod_{i=1}^{m} \theta_{i}^{\alpha n_i - \frac{1}{2}} \, d\theta \right\}^{1/\alpha}, \]
where \( Z_n(\alpha) \) is a normalization constant equal to
\[ Z_n(\alpha) \triangleq \frac{\pi^{m/2}}{\Gamma\left(\frac{m}{2}\right)} \sum_{x^n} \left\{ \prod_{i=1}^{m} \theta_{i}^{\alpha n_i - \frac{1}{2}} \right\}^{1/\alpha}. \]
The integral on the right is known in the literature as the multivariate Beta function, and it can be written in closed-form as
\[ \int_{\Theta} \prod_{i=1}^{m} \theta_{i}^{\alpha n_i - \frac{1}{2}} \, d\theta = \prod_{i=1}^{m} \frac{\Gamma\left(\frac{\alpha n_i + \frac{1}{2}}{\alpha}\right)}{\Gamma\left(\frac{\alpha n_i + \frac{m}{2}}{\alpha}\right)}, \]
so that the probability estimates given by the \( \alpha \)-NML predictor can be written as
\[ \hat{p}_\alpha(x^n) = \frac{1}{Z_n(\alpha)} \left\{ \prod_{i=1}^{m} \frac{\Gamma\left(\frac{\alpha n_i + \frac{1}{2}}{\alpha}\right)}{\Gamma\left(\frac{\alpha n_i + \frac{m}{2}}{\alpha}\right)} \right\}^{1/\alpha}, \]
where
\[ Z_n(\alpha) = \frac{\pi^{m/2}}{\Gamma\left(\frac{m}{2}\right)} \sum_{x^n} \left\{ \prod_{i=1}^{m} \frac{\Gamma\left(\frac{\alpha n_i + \frac{1}{2}}{\alpha}\right)}{\Gamma\left(\frac{\alpha n_i + \frac{m}{2}}{\alpha}\right)} \right\}, \]
where the sum is over all possible vectors of symbol counts \( n = (n_1, n_2, \ldots, n_m) \) where \( n_i \geq 0 \) for every \( i \) and \( \sum_i n_i = n \).

We now want to briefly discuss the computational complexity of \( \alpha \)-NML. Written as in \[ (23) \], the sum in \( Z_n(\alpha) \) contains only a polynomial number of terms, since the number of different vectors \( n \) is upper-bounded by \( (n + 1)^m - 1 \). In particular, when the alphabet is binary — i.e., when \( m = 2 \), — the number of terms is linear in \( n \). Furthermore, the computation of the multinomial coefficients is also not a problem, since they can be computed recursively from the
previous ones with a constant number of operations. Finally, the Gamma terms in (21) and (22) can also be computed efficiently when $\alpha \geq 1$ is restricted to be an integer. In such a case, one can use the recurrence formula $\Gamma(z+1) = z\Gamma(z)$ to compute each of the Gamma terms in the two formulae, e.g.,

$$\Gamma\left(\alpha x_1 + \frac{1}{2}\right) = \left(\alpha x_1 - \frac{1}{2}\right) \left(\alpha x_1 - \frac{3}{2}\right) \cdots \frac{3}{2} \cdot \sqrt{\pi},$$

(24)

where we used the well-known fact that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. Similar computations can be used to calculate the denominator terms $\Gamma\left(\frac{\alpha x_1}{2}\right)$ and $\Gamma\left(\alpha x_1 + \frac{1}{2}\right)$. As one can see, the number of operations required for each term of the sum in (22) is linear in $\alpha n$. Therefore, for any positive integer $\alpha$, the number of operations required to compute $Z_n(\alpha)$ and $\hat{p}_o(x^n)$ is polynomial in $n$ and linear in $\alpha$. As we will see later on, a small value of $\alpha$ is already enough to improve significantly the worst-case regret of the $\alpha$-NML predictor, and to get close to the optimal regret achieved by the NML.

When $\alpha$ is a positive integer, one can also derive simple formulae for the conditional probability of the next symbol when a sequence of length $n-1$ is already given. Consider the setting where a fixed sequence $x^{n-1} \in \mathcal{X}^{n-1}$ has been revealed, and we want to estimate the conditional probability of symbol $k \in \mathcal{X}$ given $x^{n-1}$, where $\mathcal{X} = \{1, 2, \ldots, m\}$. As an intermediate step, let us compute the ratio $\hat{p}_o(x^{n-1}, k)/\hat{p}_o(x^{n-1})$.

$$\hat{p}_o(x^{n-1}, k) = \frac{1}{Z_n(\alpha)} \frac{\Gamma(\alpha(nk+\frac{1}{2})) \prod_{j \in k} \Gamma(\alpha+\frac{1}{2})}{\prod_{j \in k} \Gamma(\alpha(n-1)+\frac{1}{2})} \hat{p}_o(x^{n-1})$$

(25)

$$= \frac{Z_{n-1}(\alpha)}{Z_n(\alpha)} \left(\frac{\prod_{j=0}^{n-1} \frac{\alpha nk + \frac{1}{2} + j}{\alpha n - \alpha + \frac{m}{2} + j}}{\prod_{j=0}^{n-1} \frac{\alpha n - \alpha + \frac{m}{2} + j}{\alpha n - \alpha + \frac{m}{2} + j}}\right),$$

(26)

where in the last step we used the recurrence formula of the Gamma function recursively. Finally, we can obtain the conditional probability of $k$ given $x^{n-1}$ as

$$\hat{p}_o(k|x^{n-1}) \triangleq \frac{\hat{p}_o(x^{n-1}, k)}{\sum_{i=1}^{m} \hat{p}_o(x^{n-1}, i)}$$

(27)

$$= \frac{\hat{p}_o(x^{n-1}, k)}{\sum_{i=1}^{m} \hat{p}_o(x^{n-1}, i)}$$

(28)

$$= \prod_{j=0}^{n-1} \left(\frac{\alpha nk + \frac{1}{2} + j}{\alpha n - \alpha + \frac{m}{2} + j}\right)^{1/\alpha}$$

(29)

for any $k \in \mathcal{X}$. As one can see from (29), the computational complexity of each of these probabilities is linear in $\alpha$ and $n$ and does not depend on $n$. When $\alpha = 1$, one obtains the known formula for the conditional probabilities of the Krichevsky-Trofimov estimator:

$$\hat{p}_{KT}(k|x^{n-1}) = \frac{n_k + \frac{1}{2}}{n + \frac{m}{2} - 1}.$$ 

(30)

IV. WORST-CASE REGRET FOR DMS

We now want to discuss the performance of $\alpha$-NML for discrete memoryless sources in terms of worst-case regret, with the primary objective of analyzing how much the regret of $\alpha$-NML improves upon that of the Krichevsky-Trofimov estimator, and how it compares to the optimal NML. In order to do this, we start by finding the asymptotical value of the worst-case regret for $\alpha$-NML, starting from formula (12).

The main advantage of this formula is that the maximization over sequences in $\mathcal{X}^n$ in the $W_o(\mathcal{P})$ term that would be complicated to evaluate in general, can be resolved explicitly for this particular class of distributions.

**Theorem 1:** For the class of discrete memoryless sources, the $W_o(\mathcal{P})$ term defined in (13) is equal to

$$W_o(\mathcal{P}) = \frac{1}{\alpha} \log \frac{\Gamma(\alpha n + \frac{m}{2})}{\Gamma(\alpha n + \frac{1}{2})} + \frac{1}{2\alpha} \log \pi - \frac{1}{\alpha} \log \Gamma\left(\frac{m}{2}\right).$$

(31)

**Proof:** For the discrete memoryless sources case, one can rewrite (13) as

$$W_o(\mathcal{P}) = \max_n \frac{1}{\alpha} \log \frac{\prod_{i=1}^{m} \Gamma(\alpha n + \frac{m}{2})}{\prod_{i=1}^{m} \Gamma(\alpha n + \frac{1}{2})}$$

(32)

$$= \frac{1}{\alpha} \log \frac{\pi^{m/2}}{\Gamma\left(\frac{m}{2}\right)} - n \log n$$

(33)

$$+ \max_n \sum_{i=1}^{m} \left(\frac{n_i \log n_i - \frac{1}{\alpha} \log \Gamma\left(\alpha n_i + \frac{1}{2}\right)}{\alpha}\right)$$

(34)

where the maximization is over vectors $n = (n_1, n_2, \ldots, n_m)$ with integer entries such that $\sum_{i=1}^{m} n_i = n$ and $n_i \geq 0$ for every $i$. Notice that to prove the lemma, it suffices to show that the quantity

$$\sum_{i=1}^{m} \left(n_i \log n_i - \frac{1}{\alpha} \log \Gamma\left(\alpha n_i + \frac{1}{2}\right)\right)$$

(35)

is maximized for $n_m = n$ and $n_i = 0$ for every $i \neq m$, for every $n \geq 1$ and $m \geq 2$. We prove this by induction on $m$. For $m = 2$, let $t = \frac{m}{n}$, $0 \leq t \leq 1$. Then, we wish to prove that the function

$$f(t) = nt \log(nt) - \frac{1}{\alpha} \log \Gamma\left(\alpha nt + \frac{1}{2}\right) + \frac{1}{\alpha} \log \Gamma\left(\alpha nt + \frac{1}{2}\right)$$

(36)

$$- n(1-t) \log(n(1-t) - \frac{1}{\alpha} \log \Gamma\left(\alpha (1-t) + \frac{1}{2}\right)$$

is maximized at $t = 1$ for $0 \leq t \leq 1$. To prove this, it suffices to show that

$$g(t) = nt \log(nt) - \frac{1}{\alpha} \log \Gamma\left(\alpha nt + \frac{1}{2}\right)$$

(37)

is convex for $0 \leq t \leq 1$. Notice that

$$g'(t) = n - n \log \alpha + n h(\alpha nt)$$

where $\psi(x) = \frac{d}{dx} \log \Gamma(x)$ is the digamma function, and $h(x) = \log(x) - \psi(x + \frac{1}{2})$. By [16] Theorem 4.2, it follows that $h'(x) \geq 0$ for every $x \geq 0$. Therefore, one has
\[ \left(40\right) \]

where the first inequality follows from the case \(m = k\), and the second inequality follows from the case \(m = 2\). Thus, \(\left(41\right)\) shows that \(\left(38\right)\) is maximized for \(n_{k+1} = n\), as desired. Hence, the case \(m = k + 1\) is proved, and the lemma follows. \(\blacksquare\)

With the help of this result, we can prove the asymptotics of the worst-case regret for the \(\alpha\)-NML estimator.

**Theorem 2:** The worst-case regret of the \(\alpha\)-NML predictor is equal to
\[
R_{\text{max}}(\hat{\rho}_\alpha) = \frac{m - 1}{2} \log \frac{n}{2} + \frac{1}{2} \log \pi - \log \frac{m}{2} + \frac{m - 1}{2\alpha} \log 2 + o(1) \quad \left(42\right)
\]

where \(o(1) \to 0\) as \(n \to \infty\).

**Proof:** We start from equation \(\left(12\right)\). The asymptotics of the \(\alpha\)-mutual information term indirectly follows from the proof of Theorem 2 in \(\left[17\right]\): from \(\left[17\right]\) Equation (80) onwards, it is proved that
\[
I_\alpha(\phi, X^n) \geq \frac{m - 1}{2} \log \frac{n}{2} + \frac{1}{2} \log \pi - \log \frac{m}{2} - \frac{m - 1}{2(\alpha - 1)} \log \alpha + o(1), \quad \left(43\right)
\]
for \((\phi, X^n) \sim w(\phi) p_\theta(X^n)\) and \(w\) taken as the Dirichlet distribution \(\text{Dir}(1/2, \ldots, 1/2)\). However, \(\left[17\right]\) Equation (54) shows that the inequality in the opposite direction is also true. Thus, \(\left(43\right)\) is indeed an equality.

As for the \(W_\alpha(\mathcal{P})\) term, starting from \(\left(31\right)\), we want to find the asymptotics of the first logarithm, which is the only term dependent on \(n\). From \(\left(18\right)\) we have that
\[
\lim_{t \to \infty} t^{-b} \frac{\Gamma(t + a)}{\Gamma(t + b)} = 1 \quad \left(44\right)
\]
for all real numbers \(a\) and \(b\). Therefore, we also have
\[
\lim_{n \to \infty} \left[ \log \frac{\Gamma(\alpha n + \frac{m}{2})}{\Gamma(\alpha n + \frac{m}{2})} - \frac{m - 1}{2} \log(\alpha n) \right] = 0, \quad \left(45\right)
\]
or equivalently,
\[
\log \frac{\Gamma(\alpha n + \frac{m}{2})}{\Gamma(\alpha n + \frac{m}{2})} = \frac{m - 1}{2} \log(\alpha n) + o(1). \quad \left(46\right)
\]

Plugging this into \(\left(31\right)\) gives
\[
W_\alpha(\mathcal{P}) = \frac{m - 1}{2\alpha} \log(\alpha n) + \frac{1}{2\alpha} \log \pi - \frac{1}{\alpha} \log \left(\frac{m}{2}\right) + o(1). \quad \left(47\right)
\]

Finally, plugging this and \(\left(43\right)\) into \(\left(12\right)\) leads to \(\left(42\right)\). \(\blacksquare\)

From \(\left(42\right)\) it can be seen that the asymptotic behavior of the worst-case regret of \(\alpha\)-NML has the same dependence on \(n\) for every \(\alpha \geq 1\), while the terms that do not depend on \(n\) strictly decrease as \(\alpha\) increases. Therefore, the \(\alpha\)-NML has an asymptotic advantage with respect to the Krichevsky-Trofimov estimator only in the constant term. However, for finite length, computer evaluation of the worst-case regret show that the advantage of \(\alpha\)-NML over the KT estimator is larger. For example, Figure 1 shows some of these results for binary alphabet. Since asymptotically the difference of the regret of the \(\alpha\)-NML (and in particular the Krichevsky-Trofimov estimator) and that of the NML is a constant, one expects the percentage of increase of the regret to tend to zero as \(n\) goes to infinity, for every \(\alpha\). However, as one can see from Figure 1, this decrease appears to be very slow, an additional indication that the (almost) optimality of the Krichevsky-Trofimov estimator in terms of worst-case regret is only asymptotical, while for finite-length sequences the difference is actually substantial. Precise analysis of finite-length regret is difficult, and it is left for future work.

**REFERENCES**

[1] N. Merhav and M. Feder, “Universal prediction,” IEEE Trans. Inf. Theory, vol. 44, no. 6, pp. 2124–2147, 1998.
[2] J. Ziv and A. Lempel, “A universal algorithm for sequential data compression,” *IEEE Trans. Inf. Theory*, vol. 23, no. 3, pp. 337–343, 1977.

[3] F. Willems, Y. Shtarkov, and T. Tjalkens, “The context-tree weighting method: basic properties,” *IEEE Trans. Inf. Theory*, vol. 41, no. 3, pp. 653–664, 1995.

[4] Q. Xie and A. Barron, “Asymptotic minimax regret for data compression, gambling, and prediction,” *IEEE Trans. Inf. Theory*, vol. 46, no. 2, pp. 431–445, 2000.

[5] Y. Fogel and M. Feder, “Universal learning of individual data,” in *Proc. 2019 IEEE Int. Symp. Inf. Theory (ISIT)*, 2019, pp. 2289–2293.

[6] F. E. Rosas, P. A. M. Mediano, and M. Gastpar, “Learning, compression, and leakage: Minimising classification error via meta-universal compression principles,” in *Proc. 2020 IEEE Inf. Theory Workshop (ITW)*, 2021.

[7] Y. M. Shtarkov, “Universal sequential coding of single messages,” *Problems Inform. Transmission*, vol. 23, no. 3, pp. 175–186, 1987.

[8] A. Suzuki and K. Yamanishi, “Fourier-analysis-based form of normalized maximum likelihood: Exact formula and relation to complex bayesian prior,” *IEEE Trans. Inf. Theory*, vol. 67, no. 9, pp. 6164–6178, 2021.

[9] S. Hirai and K. Yamanishi, “Efficient computation of normalized maximum likelihood codes for gaussian mixture models with its applications to clustering,” *IEEE Trans. Inf. Theory*, vol. 59, no. 11, pp. 7718–7727, 2013.

[10] A. Barron, T. Roos, and K. Watanabe, “Bayesian properties of normalized maximum likelihood and its fast computation,” in *Proc. 2014 IEEE Int. Symp. Inf. Theory (ISIT)*, 2014, pp. 1667–1671.

[11] R. Krichevsky and V. Trofimov, “The performance of universal encoding,” *IEEE Trans. Inf. Theory*, vol. 27, no. 2, pp. 199–207, 1981.

[12] T. van Erven and P. Harremos, “Rényi divergence and kullback-leibler divergence,” *IEEE Trans. Inf. Theory*, vol. 60, no. 7, pp. 3797–3820, 2014.

[13] P. D. Grünwald, *The minimum description length principle*. Cambridge, MA: MIT Press, 2007.

[14] S. Verdú, “α-mutual information,” in *Proc. 2015 IEEE Inf. Theory Workshop (ITW)*, 2015.

[15] N. Cesa-Bianchi and G. Lugosi, *Prediction, learning, and games*. Cambridge, United Kingdom: Cambridge University Press, 2006.

[16] H. Alzer and C. Berg, “Some classes of completely monotonic functions, ii,” *The Ramanujan Journal*, vol. 11, no. 2, pp. 225–248, 2006.

[17] S. Yagli, Y. Altuğ, and S. Verdú, “Minimax rényi redundancy,” *IEEE Trans. Inf. Theory*, vol. 64, no. 5, pp. 3715–3733, 2018.

[18] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*. Gaithersburg, MD: National Bureau of Standards, 1970.