Fundamentals for Symplectic $\mathcal{A}$-modules

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Abstract

In his [9], [10], the first author shows that the sheaf theoretically based Abstract Differential Geometry incorporates and generalizes all the classical differential geometry. Here, we undertake to partially explore the implications of Abstract Differential Geometry to classical symplectic geometry. The full investigation will be presented elsewhere.

Key Words: $\mathcal{A}$-module, vector sheaf, ordered $\mathbb{R}$-algebraized space, symplectic $\mathcal{A}$-structure, symplectic group sheaf.

Introduction

Here is an attempt of taking the theory of Abstract Differential Geometry (à la Mallios), [9], [10], to new horizons, such as those related to the classical symplectic geometry. In this endeavor, we show the extent to which tools provided by Abstract Differential Geometry help (re)capture in a sheaf-theoretic manner fundamental notions and results which characterize the standard symplectic algebra. This endeavor will pave the way to rewrite and/or recover a great deal of classical symplectic geometry, with no use at all of any notion of “differentiability” (differentiability is here understood in the sense of the standard differential geometry of $C^\infty$-manifolds). As a result of all this, we show that sheaf-theory
methods turn out to be an appropriate way for the algebraization of classical symplectic geometry. As pointed out by the first author in [10, 11], this algebraic approach to differential (and symplectic) geometry is of a particular interest to theoretical physicists, for it has been for long the demand and/or wish of many to “find a purely algebraic theory for the description of reality” (i.e. physics according to our understanding) (A. Einstein, [6], p.166).

Our main reference, throughout the present account, is the first author’s book [9], for which the reader is requested to have handy for we have skipped some necessary basics of Abstract Differential Geometry.

The paper is divided into four sections. §1 concerns with $\mathcal{A}$-tensors: these are the counterparts, in this framework, of classical tensors. Some results, pertaining to the standard multilinear algebra machinery, are hereby provided. $\mathcal{A}$-tensors constitute a precursor to the fundamental theory of exterior $\mathcal{A}$-k-forms, which are developed in §2. The exterior algebra sheaf is defined as the direct sum of sheaves of germs of exterior $\mathcal{A}$-k-forms on an $\mathcal{A}$-module $E$; this sum is endowed with the exterior product $\wedge$. In §3, we show that given a non-zero skew-symmetric non-degenerate $\mathcal{A}$-morphism $\omega$ on the standard free $\mathcal{A}$-module of rank $n$, defined on a topological space $X$, there is a basis $\mathcal{B}$ of $\mathcal{A}^n(X)$, relative to which the matrix of $\omega$ is

$$
\begin{bmatrix}
0 & I_n \\
-I_n & 0
\end{bmatrix},
$$

where $I_n$ is the $n \times n$ identity matrix (the diagonal entry 1 in $I$ is the global identity section). It also turns out that this result holds for vector sheaves as well. We further introduce in this same section the notion of symplectic group sheaf of an arbitrary $\mathcal{A}$-module. In §4, we deal with characteristic polynomial, eigenvector and eigenvalue sections of an $\mathcal{A}$-module, and prove the corresponding version of the Cayley-Hamilton theorem.

1 $\mathcal{A}$-Tensors

Throughout this paper, the pair $(X, \mathcal{A})$, or just $\mathcal{A}$ will denote a fixed $\mathbb{C}$-algebraized space, where $X$ is a topological space and $\mathcal{A}$ a sheaf (over $X$) of unital, commutative algebras. For more details about algebraized spaces, see [9].

Let

$$
\mathcal{E} \equiv (\mathcal{E}, \pi, X)
$$

be an $\mathcal{A}$-module on $X$. The (complete) presheaves of sections of sheaves $\mathcal{A}$ and $\mathcal{E}$ are denoted by

$$
\Gamma(\mathcal{A}) \equiv (\Gamma(U, \mathcal{A}), \tau_U^U) \quad \text{and} \quad \Gamma(\mathcal{E}) \equiv (\Gamma(U, \mathcal{E}), \tau_U^V),
$$
respectively. Let $\mathcal{E}^*$ be the dual $\mathcal{A}$-module of $\mathcal{E}$; so

$$\mathcal{E}^* = \text{Hom}_\mathcal{A}(\mathcal{E}, \mathcal{A})$$

(see [9], p. 298). It is easy (cf. [9], p. 129) to see that the correspondence that associates with every open subset $U$ of $X$ the $\Gamma(U, \mathcal{A})$-module

$$\otimes^p \Gamma(U, \mathcal{E}) \otimes_{\Gamma(U, \mathcal{A})} \otimes^q \Gamma(U, \mathcal{E}^*),$$

where

$$\otimes^p \Gamma(U, \mathcal{E}) := \Gamma(U, \mathcal{E}) \otimes_{\Gamma(U, \mathcal{A})} \ldots \otimes_{\Gamma(U, \mathcal{A})} \Gamma(U, \mathcal{E}),$$

does $p$-times

and

$$\otimes^q \Gamma(U, \mathcal{E}^*) := \Gamma(U, \mathcal{E}^*) \otimes_{\Gamma(U, \mathcal{A})} \ldots \otimes_{\Gamma(U, \mathcal{A})} \Gamma(U, \mathcal{E}^*),$$

does $q$-times

along with the obvious restriction morphisms, provides a presheaf of $\Gamma(\mathcal{A})$-modules on $X$. We denote this presheaf by

$$T^p_q \Gamma(\mathcal{E}) \equiv \otimes^p \Gamma(\mathcal{E}) \otimes_{\Gamma(\mathcal{A})} \otimes^q \Gamma(\mathcal{E}^*).$$

(1)

**Definition 1.1** Given an $\mathcal{A}$-module $\mathcal{E}$ on a topological space $X$, we denote by

$$\otimes^p \mathcal{E} \otimes_{\mathcal{A}} \otimes^q \mathcal{E}^*$$

the sheaf generated by the presheaf $T^p_q \Gamma(\mathcal{E})$, given in (1), i.e. one has

$$\otimes^p \mathcal{E} \otimes_{\mathcal{A}} \otimes^q \mathcal{E}^* := S(T^p_q \Gamma(\mathcal{E})) = S(\otimes^p \Gamma(\mathcal{E}) \otimes_{\Gamma(\mathcal{A})} \otimes^q \Gamma(\mathcal{E}^*)).$$

We extend the definition of $\otimes^p \mathcal{E} \otimes_{\mathcal{A}} \otimes^q \mathcal{E}^*$ to the cases $p = 0$ and $q = 0$ by setting

$$\otimes^0 \mathcal{E} \otimes_{\mathcal{A}} \otimes^q \mathcal{E}^* = \otimes^q \mathcal{E}^*,$$

and

$$\otimes^p \mathcal{E} \otimes_{\mathcal{A}} \otimes^0 \mathcal{E}^* = \otimes^p \mathcal{E}.$$

The elements of $\otimes^p \mathcal{E} \otimes_{\mathcal{A}} \otimes^q \mathcal{E}^*$ are called $\mathcal{A}$-tensors over $\mathcal{E}$, and are said to be contravariant of order $p$ and covariant of order $q$; or simply, of type $(p, q)$.

The next lemma shows the analog of a classical result of (ordinary) tensors of type $(p, q)$. Before we examine the result, let us recall that given an algebra sheaf $\mathcal{A}$ on a topological $X$, by a vector sheaf $\mathcal{E}$, of a rank $n$, on $X$, we mean a locally free $\mathcal{A}$-module of rank $n$ on $X$; that is for every $x \in X$, there exists an open neighborhood $U$ of $x \in X$ such that one has

$$\mathcal{E}|_U = \mathcal{A}^n|_U,$$

with the equality sign being actually an $\mathcal{A}|_U$-isomorphism of the $\mathcal{A}|_U$-modules $\mathcal{E}|_U$ and $\mathcal{A}^n|_U$. Any open set $U$ in $X$ for which (2) holds is called a local gauge of $\mathcal{E}$. 

3
Lemma 1.1 If $\mathcal{E}$ is a vector sheaf on a topological space $X$, then
\[
\otimes^p \mathcal{E} \otimes \mathcal{A} \otimes^q \mathcal{E}^* = \mathcal{H}om_{\mathcal{A}}(\otimes^p \mathcal{E}^* \otimes \mathcal{A} \otimes^q \mathcal{E}, \mathcal{A}).
\]

Proof. This is an easy verification. In fact, based on Mallios\[9\], Comment (5.27), p. 132, Theorems 5.1, p. 299, 6.1, p. 302, 6.2, p. 304, and Corollary 6.2, p. 305], one has
\[
\otimes^p \mathcal{E} \otimes \mathcal{A} \otimes^q \mathcal{E}^* = (\otimes^p \mathcal{E})^* \otimes \mathcal{A} \otimes^q \mathcal{E}^*
= \mathcal{H}om_{\mathcal{A}}((\otimes^p \mathcal{E})^*, \otimes^q \mathcal{E}^*)
= \mathcal{H}om_{\mathcal{A}}(\otimes^p \mathcal{E}^*, (\otimes^q \mathcal{E})^*)
= \mathcal{H}om_{\mathcal{A}}(\otimes^p \mathcal{E}^*, \mathcal{H}om_{\mathcal{A}}(\otimes^q \mathcal{E}, \mathcal{A}))
= \mathcal{H}om_{\mathcal{A}}(\otimes^p \mathcal{E}^* \otimes \mathcal{A} \otimes^q \mathcal{E}, \mathcal{A}).
\]

□

A corollary that one can derive from Lemma 1.1 requires the following definitions.

Definition 1.2 Let $\mathcal{E}_1, \ldots, \mathcal{E}_n, n \in \mathbb{N}$, and $\mathcal{F}$ be $\mathcal{A}$-modules on the same topological space $X$. The $\mathcal{A}$-morphism $\varphi : \mathcal{E}_1 \times \ldots \times \mathcal{E}_n \rightarrow \mathcal{F}$ is called an $\mathcal{A}$-multilinear morphism if, for all open subset $U \subseteq X$,
\[
\varphi_U : \Gamma(U, \mathcal{E}_1) \times_{\Gamma(U, \mathcal{A})} \cdots \times_{\Gamma(U, \mathcal{A})} \Gamma(U, \mathcal{E}_n) \rightarrow \Gamma(U, \mathcal{F})
\]
is a $\Gamma(U, \mathcal{A})$-multilinear morphism for the $\Gamma(U, \mathcal{A})$-modules concerned.

We are now set to generalize the functor $\mathcal{H}om_{\mathcal{A}}$ ($\mathcal{H}om_{\mathcal{A}}$ is a bifunctor $\mathcal{A}$-$\mathcal{M}od_X \rightarrow \mathcal{A}$-$\mathcal{M}od_X$, where $\mathcal{A}$-$\mathcal{M}od_X$ is the category of $\mathcal{A}$-modules on a topological space $X$, (see \[9\], p. 133)), to the functor $\mathcal{L}^n_{\mathcal{A}}: n \in \mathbb{N}$, which we define below.

In effect, suppose that we are given $\mathcal{A}$-modules $\mathcal{E}_i, i = 1, \ldots, n$, and $\mathcal{F}$ on the same topological space $X$. For any open set $U$ in $X$, let
\[
\mathcal{H}om_{\mathcal{A}|U} (\mathcal{E}_1|U \times \ldots \times \mathcal{E}_n|U, \mathcal{F}|U) \equiv \mathcal{L}^n_{\mathcal{A}|U} (\mathcal{E}_1|U, \ldots, \mathcal{E}_n|U; \mathcal{F}|U) \quad (3)
\]
be the set of $\mathcal{A}|U$-$n$-linear morphisms of the $\mathcal{A}|U$-module $\mathcal{E}_1 \times \ldots \times \mathcal{E}_n|U$ into the $\mathcal{A}|U$-module $\mathcal{F}|U$.

Lemma 1.2 The set, in (3), is a module over $\mathcal{A}(U)$; hence, in particular, a $\mathbb{C}$-vector space.
Proof. The proof is similar to the proof of Statement 6.1, p. 133, [9]. □

On the other hand, it is readily verified that, given $A$-modules $E_1, \ldots, E_n$, and $F$ on $X$ as above, the correspondence

$$U \mapsto L^n_A|U| \left( E_1|U|, \ldots, E_n|U|; F|U \right),$$

(4)

where $U$ runs over the open subsets of $X$, along with the obvious restriction maps yields a complete presheaf of $A$-modules on $X$.

Thus, we have

**Definition 1.3** Let $E_1, \ldots, E_n$ and $F$ be $A$-modules on a topological space $X$. By the sheaf of germs of $A$-linear morphisms of $E_1 \times \ldots \times E_n$ in $F$, we mean the sheaf, on $X$, generated by the (complete) presheaf, defined by (4). We denote the induced sheaf by $L^n_A(E_1, \ldots, E_n; F)$.

We may now state

**Corollary 1.1** Let $E$ be an $A$-module on $X$. Then,

$$\otimes^p E \otimes_A \otimes^q E^* = L^{p+q}_A(E^*, \ldots, E^*, E, \ldots, E; A) \equiv \mathcal{T}^{p+q}_A(E),$$

where $L^{p+q}_A(E^*, \ldots, E^*, E, \ldots, E; A)$ is the $A$-module of $A$-($p + q$)-linear morphisms.

Proof. Using Lemma 1.1 and Mallios [9], Lemma 5.1, p. 132 and Definition 6.1, p. 134, one has, for every open set $U$ in $X$,

$$\text{Hom}_A(\otimes^p E^* \otimes_A \otimes^q E, A)(U) = \text{Hom}_{A|U}(\otimes^p E^* \otimes_A \otimes^q E)|_{A|U}$$

$$= \text{Hom}_{A|U}(\otimes^p (E^*|U) \otimes_A \otimes^q |E|_U, A|U)$$

$$= L^{p+q}_{A|U}(E^*|U, \ldots, E^*|U, E|_U, \ldots, E|_U; A|U).$$

Therefore,

$$\text{Hom}_A(\otimes^p E^* \otimes_A \otimes^q E, A) = L^{p+q}_A(E^*, \ldots, E^*, E, \ldots, E; A).$$

So, we come now to the following definition.
**Definition 1.4** Let $\mathcal{E}$ be an $\mathcal{A}$-module on a topological space $X$, and let $t_1 \in \mathcal{T}^n_{f_1}(\mathcal{E})$ and $t_2 \in \mathcal{T}^n_{f_2}(\mathcal{E})$. The $\mathcal{A}$-tensor product of $t_1$ and $t_2$ is the $\mathcal{A}$-tensor $t_1 \otimes t_2 \in \mathcal{T}^{n+f_2}(\mathcal{E})$, defined by

$$t_1 \otimes t_2(s_1, \ldots, s_i, t_1, \ldots, t_{i_2}, u_1, \ldots, u_{j_1}, v_1, \ldots, v_{j_2}) = t_1(s_1, \ldots, s_i, u_1, \ldots, u_{j_1})t_2(t_1, \ldots, t_{i_2}, v_1, \ldots, v_{j_2})$$

where $s_\alpha, t_\alpha \in \mathcal{E}^*(U)$ and $u_\beta, v_\beta \in \mathcal{E}(U)$, and where, for all $k = 1, 2$, the $\mathcal{A}$-tensor $t_k$, viewed as a map on sections of the $\mathcal{A}$-modules of

$$\mathcal{E}^* \times_{\mathcal{A}} \cdots \times_{\mathcal{A}} \mathcal{E}^* \times_{\mathcal{A}} \mathcal{E} \times_{\mathcal{A}} \cdots \times_{\mathcal{A}} \mathcal{E}$$

is the $\mathcal{A}(U)$-$(i_k + j_k)$-linear morphism

$$\mathcal{E}^*(U) \times_{\mathcal{A}(U)} \cdots \times_{\mathcal{A}(U)} \mathcal{E}^*(U) \times_{\mathcal{A}(U)} \mathcal{E}(U) \times_{\mathcal{A}(U)} \cdots \times_{\mathcal{A}(U)} \mathcal{E}(U) \longrightarrow \mathcal{A}(U),$$

for all open subset $U \subseteq X$.

One can be assured that the standard multilinear algebra machinery can be appropriately reformulated within the present setting. For example, let us look at

**Proposition 1.1** Let $\mathcal{E}$ be a vector sheaf of rank $n$ on a topological space $X$. Then, for all $k, l \in \mathbb{N}$, the $\mathcal{A}$-module $\mathcal{T}^n_k(\mathcal{E})$ is a vector sheaf of rank $n^{k+l}$.

**Proof.** The proof is based on relations (5.25), p. 132, (6.23), p. 137, and Statement (5.19), p. 301, all found in [9].

Let $x \in X$ and $U$ an open neighborhood of $x$ such that $\mathcal{E}|_U = \mathcal{A}^n|_U = \mathcal{E}^*|_U$. Then, for all open subset $V \subseteq U$, one has

$$(\mathcal{T}^n_k(\mathcal{E})|_U)(V) = (\mathcal{L}^{k+l}_{\mathcal{A}}(\mathcal{E}^*, \mathcal{E}^*, \mathcal{E}, \mathcal{E}; \mathcal{A})|_U)(V)$$

$$= \mathcal{L}^{k+l}_{\mathcal{A}^V}(\mathcal{E}^*, \mathcal{E}^*, \mathcal{E}, \mathcal{E}; \mathcal{A}|_V)(V)$$

$$= \mathcal{L}^{k+l}_{\mathcal{A}^V}(\mathcal{E}^*|_V \times \cdots \times \mathcal{E}^*|_V \times \mathcal{E}|_V \times \cdots \times \mathcal{E}|_V \times \mathcal{A}|_V)$$

$$= \mathcal{L}^{k+l}_{\mathcal{A}^V}(\mathcal{E}^*|_V \times \cdots \times \mathcal{E}^*|_V \times \mathcal{E}|_V \times \cdots \times \mathcal{E}|_V \times \mathcal{A}|_V)$$

$$= \mathcal{H}_{\mathcal{A}|_V}(\mathcal{A}^n|_V \times \cdots \times \mathcal{A}^n|_V \times \mathcal{A}|_V)$$

$$= \mathcal{H}_{\mathcal{A}|_V}(\mathcal{A}^n \times \cdots \times \mathcal{A}^n|_V)$$

$$= (\mathcal{A}^{n^{k+l}}|_U)(V),$$
which shows that

\[ T^k_l(E)|_U = A^{n+k+l}|_U, \]

that is the \( A \)-module \( T^k_l(E) \) is a vector sheaf of rank \( n+k+l \), as desired. \( \square \)

**Corollary 1.2** Let \( E \) be a vector sheaf of rank \( n \) on a topological space \( X \), and \( \{s_i\}_{1 \leq i \leq n} \) a basis of the \( A(U) \)-module \( \Gamma(U, E) \), with \( U \) an open subset of \( X \) such that \( E|_U = A^n|_U \). Then, for all \( k, l \in \mathbb{N} \), a basis of the \( A(U) \)-module \( \Gamma(U, T^k_l(E)) \) is given by

\[ \{s_{i_1} \otimes \ldots \otimes s_{i_k} \otimes s^{*j_1} \otimes \ldots \otimes s^{*j_l} | i_p, j_p = 1, \ldots, n \}, \]

where \( \{s^{*j}\}_{1 \leq j \leq n} \) is the dual basis of \( \{s_i\}_{1 \leq i \leq n} \).

**Proof.** The proof is similar to the proof of Proposition 1.7.2, p. 53, [1]. \( \square \)

We close this section with the following important definition, which will be of use in the sequel. (See also [9]: p. 301, (5.22)- (5.24).)

**Definition 1.5** Let \( E \) and \( F \) be \( A \)-modules on a topological space \( X \). For \( \varphi \in \text{Hom}_A(E, F) \), in keeping with the classical notation, see [3], p. 234, or [5], p. 68, we define the **transpose of** \( \varphi \) by

\[ t\varphi \in \text{Hom}_A(F^*, E^*) \]

such that

\[ (t\varphi)(u) := u \circ \varphi, \quad u \in F^*(U) \] (5)

with \( U \) open in \( X \), i.e., in other words,

\[ t\varphi(u)(v) = u(\varphi(v)), \quad v \in E(U). \]

( In [3], we have used the \( t\varphi \)-corresponding map on sections of the \( A \)-modules \( E^* \) and \( F^* \). )

## 2 Exterior \( A \)-k-forms

As has been the case so far, we assume in this section as well that the triple \( (A, \tau, X) \) stands for the sheaf of commutative \( \mathbb{C} \)-algebras with an identity element on a topological space \( X \). Furthermore, we let

\[ \Gamma(A) \equiv (\Gamma(U, A), \tau^U_V) \]

be the corresponding (complete) presheaf of sections of \( A \).
Now, let \( E \) be an \( \mathcal{A} \)-module on \( X \). For any open set \( U \subseteq X \), let
\[
\Omega^k_{\mathcal{A}|U} (\mathcal{E}|_U \oplus \ldots \oplus \mathcal{E}|_U, \mathcal{A}|_U)
\]
be the set of all skew-symmetric \( \mathcal{A}|_U \)-linear morphisms of the \( \mathcal{A}|_U \)-modules \( \mathcal{E}|_U \oplus \ldots \oplus \mathcal{E}|_U \) and \( \mathcal{A}|_U \). It is obvious that we have
\[
\Omega^k_{\mathcal{A}|U} (\mathcal{E}|_U \oplus \ldots \oplus \mathcal{E}|_U, \mathcal{A}|_U) \subseteq \text{Hom}_{\mathcal{A}|U} (\mathcal{E}|_U \oplus \ldots \oplus \mathcal{E}|_U, \mathcal{A}|_U),
\]
where for every open set \( U \subseteq X \),
\[
\text{Hom}_{\mathcal{A}|U} (\mathcal{E}|_U \oplus \ldots \oplus \mathcal{E}|_U, \mathcal{A}|_U)
\]
is the set of all \( \mathcal{A}|_U \)-linear morphisms of \( \mathcal{E}|_U \oplus \ldots \oplus \mathcal{E}|_U \) into \( \mathcal{A}|_U \).

As is naturally expected, the correspondence
\[
U \mapsto \Omega^k_{\mathcal{A}|U} (\mathcal{E}|_U \oplus \ldots \oplus \mathcal{E}|_U, \mathcal{A}|_U),
\]
where \( U \) is open in \( X \), along with the obvious restriction maps, yields a complete presheaf of \( \mathcal{A} \)-modules on \( X \). The sheaf, on \( X \), generated by the presheaf defined by \( (6) \) is called the sheaf of germs of exterior \( \mathcal{A} \)-k-forms on \( \mathcal{E} \), and is denoted \( \Omega^k (\mathcal{E}) \equiv \mathcal{L}^k_0 (\mathcal{E} \oplus \ldots \oplus \mathcal{E}, \mathcal{A}) \equiv \mathcal{L}^k_0 (\mathcal{E}, \mathcal{A}) \).

It is clear that, for every open set \( U \subseteq X \), the set
\[
\Omega^k_{\mathcal{A}|U} (\mathcal{E}|_U \oplus \ldots \oplus \mathcal{E}|_U, \mathcal{A}|_U)
\]
is an \( \mathcal{A}(U) \)-module, i.e. a module over the \( \mathbb{C} \)-algebra \( \mathcal{A}(U) \); hence a \( \mathbb{C} \)-vector space. Thus, based on [9], Proposition 1.1, p. 104 and Theorem 9.1, p. 41, we conclude that the sheaf \( \Omega^k (\mathcal{E}) \) is an \( \mathcal{A} \)-module on \( X \). It follows, for every open set \( U \subseteq X \), that
\[
\Omega^k (\mathcal{E})(U) = \Omega^k_{\mathcal{A}|U} (\mathcal{E}|_U \oplus \ldots \oplus \mathcal{E}|_U, \mathcal{A}|_U),
\]
within an \( \mathcal{A}(U) \)-isomorphism of the \( \mathcal{A}(U) \)-modules concerned. In particular, one has
\[
\Omega^k (\mathcal{E})(X) = \Omega^k_{\mathcal{A}} (\mathcal{E} \oplus \ldots \oplus \mathcal{E}, \mathcal{A}),
\]
where the equality actually means an \( \mathcal{A}(X) \)-isomorphism of the \( \mathcal{A}(X) \)-modules \( \Omega^k (\mathcal{E})(X) \) and \( \Omega^k_{\mathcal{A}} (\mathcal{E} \oplus \ldots \oplus \mathcal{E}, \mathcal{A}) \).

Following the classical pattern, we set
\[
\Omega^0 (\mathcal{E}) = \mathcal{A}, \quad \text{and} \quad \Omega^1 (\mathcal{E}) = \mathcal{E}^*,
\]
as the sheaf of germs of exterior \( \mathcal{A} \)-0-forms and the sheaf of germs of \( \mathcal{A} \)-1-forms, on \( X \), respectively.
The standard exterior algebra of $k$-forms can also be repeated here to some significant extent. Consider for instance the analogue, in this setting, of the usual alternation or anti-symmetrizer map, [1], p. 101, or [8], p. 85, or [12], p. 196, which we define below. To this end, suppose given an $\mathcal{A}$-module $\mathcal{E} \equiv (\mathcal{E}, \pi, X)$ on a topological space $X$. Let

$$
\Gamma(\mathcal{E}) \equiv (\Gamma(U, \mathcal{E}), \pi_U^U)
$$

be the corresponding (complete) presheaf of sections of $\mathcal{E}$. Instead of considering the presheaf $T^0_k \Gamma(\mathcal{E})$, $k \in \mathbb{N}$, of $\Gamma(\mathcal{A})$-modules on $X$, in order to define the alternation morphism

$$
\mathcal{A} : \mathcal{T}^0_k(\mathcal{E}) \longrightarrow \mathcal{T}^0_k(\mathcal{E}),
$$

where $\mathcal{T}^0_k(\mathcal{E}) := \mathcal{S}(T^0_k \Gamma(\mathcal{E}))$, we deviate from this usual practice to defining $\mathcal{A}$ as the $\mathcal{A}_x$-morphism induced by maps

$$
\mathcal{A}_x : (\mathcal{T}^0_k(\mathcal{E}))(x) = \mathcal{E}^*_x \otimes_{\mathcal{A}_x} \cdots \otimes_{\mathcal{A}_x} \mathcal{E}^*_x \longrightarrow \mathcal{E}^*_x \otimes_{\mathcal{A}_x} \cdots \otimes_{\mathcal{A}_x} \mathcal{E}^*_x, \ x \in X, \quad (7)
$$
such that

$$
\mathcal{A}_x t_x(s_{1,x}, \ldots, s_{k,x}) = \frac{1}{k!} \sum_{\sigma \in S_k} (\text{sign}\sigma) t_x(s_{1,x}, \ldots, s_{k,x}),
$$

where $s_{1,x}, \ldots, s_{k,x} \in \mathcal{E}_x$, $t_x : \mathcal{E}_x \otimes_{\mathcal{A}_x} \cdots \otimes_{\mathcal{A}_x} \mathcal{E}_x \longrightarrow \mathcal{A}_x$ is $\mathcal{A}_x$-k-linear, and $S_k$ is the permutation group on $\{1, \ldots, k\}$.

The equality $(\mathcal{T}^0_k(\mathcal{E}))(x) = \mathcal{E}^*_x \otimes_{\mathcal{A}_x} \cdots \otimes_{\mathcal{A}_x} \mathcal{E}^*_x$, $x \in X$, holds within an $\mathcal{A}_x$-isomorphism; for this purpose see [9], relation 5.9, p. 130.

The reason for this approach comes from the observation that the $\Gamma(\mathcal{A})$-presheaf defined by

$$
U \longrightarrow T^0_k \Gamma(\mathcal{E})(U) := \Gamma(U, \mathcal{E}^*) \otimes_{\Gamma(U, \mathcal{A})} \cdots \otimes_{\Gamma(U, \mathcal{A})} \Gamma(U, \mathcal{E}^*), \quad (8)
$$

where $U \subseteq X$ is open, along with the restriction maps is not always complete, cf. [9], Statement 5.5, p. 129; therefore $T^0_k(\mathcal{E})(U)$ is not always $\mathcal{A}(U)$-isomorphic to the right hand side in the correspondence $[8]$ above, i.e., to $T^0_k \Gamma(\mathcal{E})(U)$. Thus, in order to circumvent this obstacle, we resort, by virtue of [9], Lemma 8.1, p. 36, to anti-symmetrizers

$$
\mathcal{A}_x : \mathcal{T}^0_k(\mathcal{E})(x) \longrightarrow \mathcal{T}^0_k(\mathcal{E})(x), \ x \in X,
$$

from which the sought anti-symmetrizer $\mathcal{A}$-morphism $\mathcal{A}$ is obtained.

We may now define the exterior product as follows.

**Definition 2.1** Let $\mathcal{E}$ be a vector sheaf of rank $n$ on a topological space $X$, and let $\xi$ and $\eta$ be elements of $\Omega^k(\mathcal{E})$ and $\Omega^l(\mathcal{E})$, respectively. The exterior product of $\xi$ and $\eta$, $x \in X$, is the germ $\xi_x \wedge \eta_x \in \Omega^{k+l}(\mathcal{E}_x)$, given by

$$
\xi_x \wedge \eta_x = \frac{(k + l)!}{k!l!} \mathcal{A}_x(\mathcal{E}_x \otimes \eta_x).$$
that is, for all $s_{1,x}, \ldots, s_{k+l,x} \in \mathcal{E}_x$,

$$
\xi_x \wedge \eta_x(s_{1,x}, \ldots, s_{k+l,x}) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \text{sign}(\sigma) \xi_x(s_{\sigma(1),x}, \ldots, s_{\sigma,k,x}) \eta_x(s_{\sigma(k+1),x}, \ldots, s_{\sigma(k+l),x}).
$$

In particular, for $\alpha_x \in \Omega^0(\mathcal{E}_x) \equiv \mathcal{A}_x$, $x \in X$, we put

$$
\alpha_x \wedge \xi_x \equiv \xi_x \wedge \alpha_x \equiv \alpha_x \xi_x.
$$

Finally, the $\mathcal{A}$-morphism

$$
\xi \wedge \eta \in \Omega^{k+l}(\mathcal{E}),
$$

obtained from germs $\xi_x \wedge \eta_x$, $x \in X$, above, by virtue of [9], Lemma 8.1, p. 36, is called the exterior product of $\xi$ and $\eta$. Like earlier, for $\alpha \in \Omega^0(\mathcal{E}) \equiv \mathcal{A}$, we put

$$
\alpha \wedge \xi \equiv \xi \wedge \alpha \equiv \alpha \xi.
$$

Note that we do not index $\wedge$, when considering the exterior product $\xi_x \wedge \eta_x$, $x \in X$, for given $\xi \in \Omega^k(\mathcal{E})$ and $\eta \in \Omega^l(\mathcal{E})$, in order to avoid unnecessary meticulousness.

With this product, we define the exterior algebra sheaf, or the Grassmann algebra sheaf of the vector sheaf $\mathcal{E}$ of rank $n$, to be the $\mathcal{A}$-module

$$
\Omega^*(\mathcal{E}) \equiv \Omega^0(\mathcal{E}) \oplus \Omega^1(\mathcal{E}) \oplus \ldots \oplus \Omega^n(\mathcal{E}),
$$

such that

$$
\Omega^*(\mathcal{E})_x \equiv \Omega^0(\mathcal{E}_x) \oplus \Omega^1(\mathcal{E}_x) \oplus \ldots \oplus \Omega^n(\mathcal{E})_x = \Omega^0(\mathcal{E}_x) \oplus \Omega^1(\mathcal{E}_x) \oplus \ldots \oplus \Omega^n(\mathcal{E}_x),
$$

for all $x \in X$, where the last relation is valid, of course, within an $\mathcal{A}_x$-isomorphism.

The $\mathcal{A}_x$-isomorphism in (10) can be obtained in the following manner. In fact, one has

$$
\Omega^1(\mathcal{E}) := \mathcal{L}^1(\mathcal{E}, \mathcal{A}) = \text{Hom}_\mathcal{A}(\mathcal{E}, \mathcal{A}) = \mathcal{E},
$$

which implies that

$$
\Omega^1(\mathcal{E})_x = \mathcal{E}_x, \quad x \in X.
$$

(For the last relation in [11], see [9], relation (6.18), p. 136.) On the other hand, for all $x \in X$,

$$
\Omega^1(\mathcal{E}_x) := \mathcal{L}^1(\mathcal{E}_x, \mathcal{A}_x) = \text{Hom}_{\mathcal{A}_x}(\mathcal{E}_x, \mathcal{A}_x) = \mathcal{E}_x,
$$

because $\mathcal{E}$ is a vector sheaf of finite rank on $X$. Thus,

$$
\Omega^1(\mathcal{E})_x = \Omega^1(\mathcal{E}_x),
$$

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for all $x \in X$. Likewise, for $k > 1$, we have

$$\Omega^k(E)_x := L^k_n(E_x \oplus \cdots \oplus E_x, \mathcal{A}_x) := \Omega^k(E_x).$$

Like in the classical theory, if $\alpha_i, i = 1, \ldots, k$, are elements of $\Omega^1(E)_x$, where $E$ is a vector sheaf of finite rank on a topological space $X$, then

$$(\alpha_1, \ldots, \alpha_k)(s_1, x, \ldots, s_k, x) = \sum_{\sigma} \text{sign}(\sigma)\alpha_{\sigma(1)}(s_{\sigma(1)}, x) \cdots \alpha_k(x, s_{\sigma k}),$$

where $s_i := s_i|E_x$, and $\alpha_i(x) = \alpha_i|\Omega^1(E_x)$, for all $x \in X$ and $i = 1, \ldots, k$, so that

$$(\alpha_1 \wedge \cdots \wedge \alpha_k)(s_1, \ldots, s_k) = \sum_{\sigma} \text{sign}(\sigma)\alpha_1(s_{\sigma(1)}) \cdots \alpha_k(s_{\sigma k}).$$

### 3 Skew-Symmetric $\mathcal{A}$-bilinear forms

**Definition 3.1** Let $X$ be a topological space and $(X, \mathcal{A}, \mathcal{P})$ an ordered $\mathbb{R}$-algebraized space on $X$, (cf. [3], p. 316). A section $s \in \mathcal{A}(U)$, with $U$ open in $X$, is called **strictly positive** if $s(0) \neq 0$.

For the purpose of what lays ahead, we need the following notion.

**Definition 3.2** Let $(X, \mathcal{A})$ be an algebraized space, and $E$ an $\mathcal{A}$-module on $X$. An **$\mathcal{A}$-bilinear sheaf morphism**

$$\omega \equiv (\omega_U)_{U \subseteq X, \text{open}} : E \oplus E \rightarrow \mathcal{A}$$

is called

- **skew symmetric** provided
  $$\omega_U(s, t) = -\omega_U(t, s),$$
  for all sections $s, t \in E(U)$ over any open set $U \subseteq X$.

- **nondegenerate** if
  $$\omega_U(s, t) = 0 \quad \text{for all } t \in E(U), \text{ with } U \text{ an arbitrary open set in } X,$$
  implies that $s = 0 \in E(U)$.
In this Definition 3.2, we have identified the sheaf morphism $\omega : \mathcal{E} \oplus \mathcal{E} \longrightarrow \mathcal{A}$ with the corresponding presheaf morphism $(\omega_U)_{U \text{ open}} : \Gamma(\mathcal{E} \oplus \mathcal{E}) \longrightarrow \Gamma(\mathcal{A})$ of (complete) presheaves of sections $\Gamma(\mathcal{E} \oplus \mathcal{E})$ and $\Gamma(\mathcal{A})$. This identification is based on the fact that, given a topological space $X$, we have

$$\Gamma : Sh_X \cong \text{CoPSh}_X,$$

(12)

where $Sh_X$ is the category of sheaves over $X$, $\text{CoPSh}_X$ is the category of complete presheaves over $X$, and $\Gamma$ is the section functor. For suitable details, see [9], Theorem 13.1, p. 73.

For the purpose of the following theorem, we assume the following condition, referred to in the sequel as the inverse-positive-section condition:

The ordered $\mathbb{R}$-algebraized $(X, \mathcal{A}, \mathcal{P})$ is such that all strictly positive sections of $\mathcal{A}$ are invertible. More explicitly, if $\mathcal{P}^*$ denotes the sub-sheaf of all strictly positive sections of $\mathcal{A}$, and $\mathcal{A}^*$ the sheaf on $X$, generated by the presheaf

$$U \mapsto \mathcal{A}(U)^* = \mathcal{A}^*(U),$$

where $U$ is open in $X$, and $\mathcal{A}(U)^*$ is the group of units of the unital $\mathbb{C}$-algebra $\mathcal{A}(U)$, then

$$\mathcal{P}^* \subset \mathcal{A}^*.$$

(13)

Section-wise, (13) would be understood in the following way: for any $s \in \mathcal{P}(U)$, where $U \subseteq X$ is open, such that $s(x) \neq 0_x \in \mathcal{A}_x$, $x \in U$, then there exists $s^{-1} \in \mathcal{P}(U)$ such that $s \cdot s^{-1} = s^{-1} \cdot s = 1_U \in \mathcal{A}(U)$.

Furthermore, we suppose that our ordered algebraized space $(X, \mathcal{A}, \mathcal{P})$, is also endowed with an absolute value, i.e., the following sheaf morphism,

$$| \cdot | : \mathcal{A} \longrightarrow \mathcal{A}^+ := \mathcal{P},$$

(14)

having the analogous properties of the classical function; hence, for instance, the property that

$$|s| = \alpha \in \mathbb{R}^+ \subseteq \mathcal{A}^+(X) \iff s = \pm \alpha \in \mathbb{R} \subseteq \mathcal{A}(X).$$

Now, the proof of the following theorem is based on the classical patterns, see e.g. [15], [1], [2], [14], within, of course, the present sheaf-theoretic context, for which we refer to [9], p. 316, Definition 8.1, along with p. 335 ff. So, we now have the following basic result.

**Theorem 3.1** Let $(X, \mathcal{A}, \mathcal{P}, | \cdot |)$ be an ordered $\mathbb{R}$-algebraized space, endowed with an absolute value, and $\mathcal{E}$ the standard free $\mathcal{A}$-module, $\mathcal{A}^n$, of rank $n$ on $X$. 

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Moreover let \( \omega : \mathcal{E} \oplus \mathcal{E} \rightarrow \mathcal{A} \) be a non-zero skew-symmetric and non-degenerate \( \mathcal{A} \)-bilinear sheaf morphism. Then, there exists an \( \mathcal{A}(U) \)-basis of \( \mathcal{A}^n(U) \), say,  
\[ s_1^U, \ldots, s_m^U, t_1^U, \ldots, t_m^U, \]
such that
\[
\begin{align*}
n &= 2m \\
\omega(s_i^U, s_j^U) &= 0 = \omega(t_i^U, t_j^U) \quad \text{for all } 1 \leq i, j \leq m \\
\omega(s_i^U, t_j^U) &= \delta_{ij}^U \quad \text{for all } 1 \leq i, j \leq m.
\end{align*}
\]

**Proof.** With no loss of generality, we assume that \( U = X \). Therefore, since \( \mathcal{A}^n \neq \{0\} \) (we already assumed that \( C \equiv C_X \subseteq \mathcal{A} \)), there exists an element
\[
0 \neq s_1 \in \mathcal{A}^n(X) \cong \mathcal{A}(X)^n
\]
(take e.g. an element from the canonical basis of (sections) of \( \mathcal{A}^n(X) \cong \mathcal{A}(X)^n \), see [2], p. 123). Next, consider the “\( \mathcal{A}(X) \)-line of \( s_1 \)”, i.e.,
\[
\mathcal{A}(X)[s_1] := \{ \alpha s_1 \in \mathcal{A}^n(X) : \alpha \in \mathcal{A}(X) \},
\]
which, by an obvious *abuse of notation*, we may still denote, for convenience, just, by
\[
\mathcal{A}(s_1) \subseteq \mathcal{A}^n(X).
\]
Now, it is also clear that there exists an element
\[
0 \neq \tilde{t}_1 \in \mathcal{A}^n(X) \setminus \mathcal{A}(s_1)
\]
(just, take e.g. another element of the previously considered basis of \( \mathcal{A}^n(X) \), *different from \( s_1 \)*). Furthermore, due to the hypothesis concerning \( s_1, \tilde{t}_1 \), and as well as, to that one for \( \omega \), one obtains that (see Lemma 3.1)
\[
\omega(s_1, \tilde{t}_1)(x) \neq 0_x \in \mathcal{A}_x,
\]
for all \( x \in X \). Hence, based also on our hypothesis for \( \mathcal{A} \), that is, the existence of the sheaf morphism \( | \cdot | : \mathcal{A} \rightarrow \mathcal{A}^+ \equiv \mathcal{P} \), we also obtain that,
\[
|\omega(s_1, \tilde{t}_1)| > 0,
\]
that is the section \( |\omega(s_1, \tilde{t}_1)| \in \mathcal{A}(X) \) is *strictly positive*; therefore, by assumption for \( (X, \mathcal{A}) \), see the inverse-positive-section condition above, it is also invertible in \( \mathcal{A}(X) \). Hence taking further \( t_1 := u^{-1} \tilde{t}_1 \), with \( u \equiv |\omega(s_1, \tilde{t}_1)| \in \mathcal{A}(X) \), one gets
\[
|\omega(s_1, t_1)| = 1,
\]
which implies that
\[
\omega(s_1, t_1) = \pm 1 \in \mathcal{A}(X).
\]
Now, let us consider
\[ S_1 := [s_1, t_1], \]
that is, the “flag” (alias, “\( \mathcal{A}(X) \)-plane”), defined by \( s_1 \) and \( t_1 \), in \( \mathcal{A}^n(X) \), in effect, an \( \mathcal{A}(X) \)-module, generated by \( s_1 \) and \( t_1 \), along with its “orthogonal complement” in \( \mathcal{A}^n(X) \), i.e.,
\[ S_1^\bot \equiv T_1 := \{ v \in \mathcal{A}^n(X) : \omega(v, z) = 0, \text{ for all } z \in S_1 \}. \]

Now, we first remark that \( s_1, t_1 \) are also “free generators” of \( S_1 \), for, if \( t_1 = \alpha s_1 \), then
\[ \pm 1 = \omega(s_1, t_1) = \omega(s_1, \alpha s_1) = \alpha \cdot \omega(s_1, s_1) = 0, \]
a contradiction. That is, \( \{s_1, t_1\} \) yields actually an \( \mathcal{A}(X) \)-basis of the flag \( S_1 \).
Furthermore, we prove that:
\begin{align*}
(i) & \quad S_1 \cap T_1 = \{0\}, \quad \text{and} \\
(ii) & \quad S_1 + T_1 = \mathcal{A}^n(X).
\end{align*}
Indeed, \((i)\) if \( z \equiv \alpha s_1 + \beta t_1 \in S_1 \cap T_1 \), with \( \alpha, \beta \in \mathcal{A}(X) \), one gets, by the very definition of \( S_1, T_1 \), and the fact that \( \omega(s_1, t_1) = 1 \), that,
\[ \omega(z, s_1) = \beta = 0, \quad \text{and} \quad \omega(z, t_1) = \alpha = 0, \]
that is, \( z = 0 \), which proves \((i)\). On the other hand, \((ii)\) for every \( z \in \mathcal{A}^n(X) \), one has,
\[ z = (-\omega(z, s_1)t_1 + \omega(z, t_1)s_1) + (z + \omega(z, s_1)t_1 - \omega(z, t_1)s_1), \]
with
\[ -\omega(z, s_1)t_1 + \omega(z, t_1)s_1 \in S_1, \]
and
\[ z + \omega(z, s_1)t_1 - \omega(z, t_1)s_1 \in T_1. \]
Thus,
\[ \mathcal{A}^n(X) = S_1 \oplus T_1. \]

Now, in a manner similar to the manner we found the elements \( s_1, t_1 \in S_1 \) with \( \omega(s_1, t_1) = 1 \), we conclude the existence of sections \( s_2, t_2 \in T_1 \setminus \{0\} \), such that
\[ \omega(s_2, t_2) = 1 \in \mathcal{A}(X); \]
while we further consider the flag
\[ S_2 := [s_2, t_2], \]
along with
\[ T_2 \equiv S_2^\bot := \{ v \in \mathcal{A}^n(X) : \omega(v, z) = 0, \quad z \in S_2 \}. \]
Yet, we still prove in a similar way, as before, that
\[ T_1 = S_2 \oplus T_2, \]
so that one obtains,
\[ A^n(X) = S_1 \oplus S_2 \oplus T_2, \]
and so on. Now, the above process stops eventually, due to the finite rank of \( A^n(X) \), so that one finally obtains
\[ A^n(X) = S_1 \oplus S_2 \oplus \ldots \oplus S_m \]
with the generators, \( s_i, t_i \), of \( S_i \) \( (1 \leq i \leq n) \) having the property that
\[
\begin{align*}
\omega(s_i, s_j) &= 0 = \omega(t_i, t_j) \\
\omega(s_i, t_j) &= \delta_{ij}.
\end{align*}
\]
Hence, the proof is finished. \( \square \)

In the proof of the previous Theorem 3.1, one still essentially applies the following standard fact of the classical theory, which for convenience we also formulate, within the present context:

**Lemma 3.1** Let \( E \) be a free \( A \)-module of rank \( n \in \mathbb{N} \) on a topological space \( X \). Then, a family \( \{s_i\}_{i \in I} \) of global sections of \( E \), i.e., \( \{s_i\}_{i \in I} \subseteq E(X) \), is \( A^n(X) \)-linearly independent if, and only if, the relation
\[
\sum_{i \in I} \alpha_i s_i = 0,
\]
with \( \{\alpha_i\}_{i \in I} \subseteq A(X) \), having finite support, implies \( \alpha_i = 0 \), for any \( i \in I \).

For the proof of Lemma 3.1, one can follow, for instance, the analogous argument in [3], Chap II; p. 25, remarks after Definition 10. Yet, for convenience, we recall that the term “finite support” for the family \( \{\alpha_i\}_{i \in I} \subseteq A(X) \), means that \( \alpha_i \neq 0 \), only for finitely many indices \( i \in I \), and the rest of the \( \alpha_i \) being 0; so that the sum used above acquires then a meaning.

When the skew-symmetric \( A \)-bilinear sheaf morphism \( \omega \) is not necessarily non-degenerate, then in place of Theorem 3.1, we have the theorem below. Let us first give the following definition:

**Definition 3.3** Let \( \omega \equiv (\omega^U) : A^n \oplus A^n \rightarrow A \) be an \( A \)-bilinear morphism on the standard free \( A \)-module \( A^n \). The rank of \( \omega \) is the rank of the matrix \( (\omega^U_{ij}) \), with \( \omega^U_{ij} = \omega^U(\varepsilon^U_i, \varepsilon^U_j) \), where \( \{\varepsilon^U_i\}_{1 \leq i \leq n} \) is the Kronecker gauge of \( A^n(U) \), and \( U \) is any open subset of \( X \).
By the methods of Linear Algebra, one can easily show that the rank of $\omega$ is independent of the basis considered.

Note the notation $\omega \equiv (\omega^U)$ instead of the usual $\omega \equiv (\omega_U)$; the reason being that we want to avoid, in the sequel, too many sub-indices.

**Theorem 3.2** Let $(X, \mathcal{A}, \mathcal{P}, |\cdot|)$ and $\mathcal{E}$ be as in Theorem 3.1. Let $\omega : \mathcal{E} \oplus \mathcal{E} \rightarrow \mathcal{A}$ be a skew-symmetric $\mathcal{A}$-bilinear sheaf morphism of rank $r$. Then, $r = 2m$, for some integer $m$, and for every $x \in X$ there are an open neighborhood $U \subseteq X$ of $x$ and a basis $s^U_1, \ldots, s^U_n \in \mathcal{A}^n(U) = \mathcal{A}(U)^n$ such that the matrix of $\omega^U \equiv \omega_U$ is

$$
\begin{bmatrix}
0 & 1_m & 0 \\
-1_m & 0 & 0 \\
0 & 0 & 0
\end{bmatrix},
$$

where $\omega_U$ is the component of $\omega$ relative to $U$.

**Proof.** Fix $x \in X$. Because of the continuity of $\omega_y : y \in X$, and of the fact that the rank of $\omega$ is $r$, there exist an open neighborhood $U_1$ of $x$ and $s^{U_1}_1, s^{U_1}_{m+1} \in \mathcal{A}^n(U_1)$ such that

$$
\omega^{U_1}(s^{U_1}_1, s^{U_1}_{m+1})(y) \neq 0_y \in \mathcal{A}_y,
$$

for all $y \in U_1$. Now, put $u^{U_1}_{1,m+1} := \omega^{U_1}(s^{U_1}_1, s^{U_1}_{m+1}) \in \mathcal{A}(U_1)$. Based on our hypothesis for $\mathcal{A}$, we have that $u_{1,m+1} := |u^{U_1}_{1,m+1}| > 0$; therefore by the inverse-positive-section condition, $u_{1,m+1} \in \mathcal{A}^*(U_1) = \mathcal{A}(U_1)^*$. Hence, taking further $s^{U_1}_{m+1} = u_{1,m+1}^{-1}s^{U_1}_1$, one gets

$$
|\omega^{U_1}(s^{U_1}_1, s^{U_1}_{m+1})| = 1.
$$

Assuming that $\omega^{U_1}(s^{U_1}_1, s^{U_1}_{m+1}) = 1_{U_1} =: 1$, the matrix of $\omega^{U_1}$ in the $\mathcal{A}(U_1)$-module $S^{U_1}_1 := [s^{U_1}_1, s^{U_1}_{m+1}]$, that is, the $\mathcal{A}(U_1)$-module spanned by $s^{U_1}_1$ and $s^{U_1}_{m+1}$, is

$$
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}.
$$

Let $(S^{U_1}_1)^\perp$ be the $\omega^{U_1}$-orthogonal complement of $S^{U_1}_1$ in $\mathcal{A}^n(U_1)$. As in the proof of Theorem 3.1, one shows that

$$
\mathcal{A}^n(U_1) = S^{U_1}_1 \oplus (S^{U_1}_1)^\perp.
$$

Now, we repeat the process on $(S^{U_1}_1)^\perp$ to get an open neighborhood $U_2 \subseteq X$ of $x$, and $s^{U_2}_2$ and $s^{U_2}_{m+2}$ such that $\omega^{U_2}(s^{U_2}_2, s^{U_2}_{m+2}) = 1$ and continue inductively.
Eventually, this process will stop because the rank of $\omega$ is finite. Certainly by taking $U = \bigcap_{i=1}^m U_i$, where $r = 2m$, one sees that $\omega^U$ has the stated matrix in the basis $\{s^U_1, \ldots, s^U_m\}$.

Note that if we denote by $\{(s^U_i)^*\}_{1 \leq i \leq m}$ the dual basis of $\{s^U_i\}_{1 \leq i \leq m}$, then

$$\omega^U = \sum_{i=1}^m (s^U_i)^* \wedge (s^U_{m+i})^*. \quad (15)$$

**Corollary 3.1** Let $(X, \mathcal{A}, \mathcal{P})$ be an ordered $\mathbb{R}$-algebraized space such that $\mathcal{P}^* \subseteq \mathcal{A}^*$. Let $\mathcal{E}$ be a vector sheaf of rank $n$ on $X$, and $\omega : \mathcal{E} \oplus \mathcal{E} \longrightarrow \mathcal{A}$ a skew-symmetric and non-degenerate $\mathcal{A}$-bilinear morphism. Then, given an open neighborhood $U$ of $x \in X$ such that $\mathcal{E}|_U = \mathcal{A}^n|_U$, where $x \in X$ is arbitrary, there exists a basis, $s^U_1, \ldots, s^U_m, t^U_1, \ldots, t^U_m$, of $\mathcal{A}^n(U)$ such that

$$n = 2m$$

$$\omega^U(s^U_i, s^U_j) = 0 = \omega^U(t^U_i, t^U_j) \quad \text{for all} \ 1 \leq i, j \leq m$$

$$\omega^U(s^U_i, t^U_j) = \delta^U_{ij} \quad \text{for all} \ 1 \leq i, j \leq m.$$ 

The pair $(\mathcal{E}, \omega)$ is called a **locally free symplectic $\mathcal{A}$-module**, alias symplectic vector sheaf, of rank $n$, on $X$.

**Proof.** One proceeds in the same manner as for the proof of Theorem 3.1, the small nuance being that one works locally. □

**Definition 3.4** Let $(X, \mathcal{A}, \mathcal{P})$ be an ordered $\mathbb{R}$-algebraized space, satisfying the inverse-positive-section condition. The non-degenerate skew-symmetric $\mathcal{A}$-bilinear morphism $\omega : \mathcal{A}^n \oplus \mathcal{A}^n \longrightarrow \mathcal{A}$ is called a **linear symplectic $\mathcal{A}$-structure** on the standard free $\mathcal{A}$-module $\mathcal{A}^n$, and the pair $(\mathcal{A}^n, \omega)$ is called a (standard) **free symplectic $\mathcal{A}$-module**. More generally, let $\mathcal{E}$ be an $\mathcal{A}$-module on $X$, and $\omega : \mathcal{E} \times \mathcal{E} \longrightarrow \mathcal{A}$ a non-degenerate, skew-symmetric $\mathcal{A}$-bilinear morphism. Then, the pair $(\mathcal{E}, \omega)$ is called a **symplectic $\mathcal{A}$-module** on $X$.

With respect to the notation of Theorem 3.1, the basis, $s_1, \ldots, s_m, t_1, \ldots, t_m$, is called a **symplectic basis** of the standard symplectic free $\mathcal{A}$-module $(\mathcal{A}^n, \omega)$.

It is clear that, with respect to a symplectic basis, the matrix representing $\omega$ is, as in the classical case, given by

$$J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$ \quad (16)

**Example 3.1** Let $\mathcal{E}$ be a vector sheaf of rank $n$ on $X$, and consider the direct sum $\mathcal{E} \oplus \mathcal{E}^*$. The $\mathcal{A}$-bilinear morphism

$$\omega : (\mathcal{E} \oplus \mathcal{E}^*) \oplus (\mathcal{E} \oplus \mathcal{E}^*) \longrightarrow \mathcal{A},$$
defined by
\[ \omega^U \left( (s^U_1, \alpha^U_1), (s^U_2, \alpha^U_2) \right) = \alpha^U_2(s^U_1) - \alpha^U_1(s^U_2), \]
where \( U \) is a local gauge of \( E \), and \( s^U_i \in \mathcal{E}(U) = \mathcal{A}^n(U) \) and \( \alpha^U_i \in \mathcal{E}^*(U) = \mathcal{A}^n(U) \), is a symplectic \( \mathcal{A} \)-morphism. This example shows that any vector sheaf \( \mathcal{F} \) of even rank admits a symplectic \( \mathcal{A} \)-structure, for one has
\[ \mathcal{F}|_U = \mathcal{A}^{2n}|_U = \mathcal{A}^n|_U \oplus \mathcal{A}^n|_U = \mathcal{A}^n|_U \oplus (\mathcal{A}^n)^*|_U, \]
where \( U \) is a local gauge of \( \mathcal{F} \). For the last equality of the previous line, see [9], relation (3.14), p.122.

We now would like to show that a useful criterion for non-degeneracy is also available in this setting. To this effect, we suppose as usual that our ordered algebraized space \((X, \mathcal{A}, \mathcal{P})\) is enriched with absolute value, see (14), and square root; the latter means a morphism of (complete) presheaves
\[ \sqrt{\cdot} : \Gamma(\mathcal{P}) \rightarrow \Gamma(\mathcal{P}), \quad s \mapsto \sqrt{s}, \quad \text{for all} \ s \in \mathcal{P}(U), \]
with \( U \) running over all the open subsets of \( X \). In addition, we assume that the pair \((\mathcal{A}, \rho)\) is a Riemann \( \mathcal{A} \)-module on \( X \), see [9], p. 320. We assume as well that \( \mathcal{P} \equiv \rho^n \) (cf. [9], p. 324) is the extension of \( \rho \) to the standard free \( \mathcal{A} \)-module \( \mathcal{A}^n \) on \( X \).

**Definition 3.5** Let \((X, \mathcal{A}, \mathcal{P})\) and \( \mathcal{P} \) be as above, and \( \{s_i\}_{1 \leq i \leq n} \subseteq \mathcal{A}^n(X) = \mathcal{A}(X)^n \) be a basis of \( \mathcal{A}^n(X) \). Denoting by
\[ \bigwedge^n(\mathcal{A}^n)^*, \]
the \( n \)-th exterior power of the \( \mathcal{A} \)-module \( (\mathcal{A}^n)^* \), see [9], p. 307, the section
\[ \Omega = \sqrt{\left| \det(\mathcal{P}(s_i, s_j)) \right|} s^*_1 \wedge \ldots \wedge s^*_n \in (\bigwedge^n(\mathcal{A}^n)^*) (X), \quad (17) \]
where \( \{s^*_i\}_{1 \leq i \leq n} \subseteq (\mathcal{A}^n)^*(X) \) is the dual \( \mathcal{A} \)-basis of \( \{s_i\}_{1 \leq i \leq n} \), is called a volume element of the \( \mathcal{A} \)-metric \( \mathcal{P} \). In the sequel, for the sake of brevity, the scaling factor \( \sqrt{\left| \det(\mathcal{P}(s_i, s_j)) \right|} \), above, will be denoted by \( \sqrt{\mathcal{P}(s)} \equiv \sqrt{\mathcal{P}} \), where \( s \equiv \{s_1, \ldots, s_n\} \).

It is clear that for an orthonormal gauge \( \{s_i\}_{1 \leq i \leq n} \) of \( \mathcal{A}^n(X) \) (see [9], p. 340), relation (17) becomes
\[ \Omega = s^*_1 \wedge \ldots \wedge s^*_n. \]
Like in the classical case, we have:
Corollary 3.2 Let \( \omega : \mathcal{A}^n \oplus \mathcal{A}^n \longrightarrow \mathcal{A} \) be a \( \mathcal{A} \)-bilinear and skew-symmetric \( \mathcal{A} \)-morphism on the standard free \( \mathcal{A} \)-module \( \mathcal{A}^n \), where \( (X, \mathcal{A}) \) is an enriched (with square root and absolute value) ordered algebraized space and \( (\mathcal{A}, \rho) \) a Riemannian \( \mathcal{A} \)-module. Then, \( \omega \) is non-degenerate if and only if \( n = 2m \), for some \( m \in \mathbb{N} \), and \( \omega^m = \omega \wedge \ldots \wedge \omega \) is a volume element on \( \mathcal{A}^n \).

Note that
\[
\omega^m \equiv (\omega^m) := (\omega^U \wedge \ldots \wedge \omega^U).
\]
Therefore, that \( \omega^m \) is a volume element on \( \mathcal{A}^n \) means that \( \omega^U \wedge \ldots \wedge \omega^U \) is a volume element on \( \mathcal{A}^n(U) \), for every open \( U \subseteq X \).

**Proof.** We refer to the proof in Abraham-Marsden [1], p. 165 for detail.

Suppose that \( \omega \) is non-degenerate. By Theorem 3.1, \( n = 2m \), for some \( m \in \mathbb{N} \). By virtue of Equation (15) and of induction, one sees that
\[
\omega^U m = m!(-1)^{\lfloor m/2 \rfloor} s_1^U \wedge \ldots \wedge s_{2m}^U,
\]
where \( \lfloor m/2 \rfloor \) is the largest integer in \( m/2 \). Thus, assuming that \( \sqrt{|\rho|} = m!(-1)^{\lfloor m/2 \rfloor} \), \( \omega^U m \) is a volume. The converse is clear. □

Following Abraham-Marsden [1], p. 167], we set that given a free symplectic \( \mathcal{A} \)-module \((\mathcal{A}^n, \omega)\), the volume element
\[
\Omega_\omega = \frac{(-1)^{\lfloor m/2 \rfloor}}{m!} \omega^m
\]
defines an orientation on \( \mathcal{A}^n \).

In all that precedes, by the \( \mathcal{A} \)-bilinear morphism \( \omega : \mathcal{E} \oplus \mathcal{E} \longrightarrow \mathcal{A} \), we meant the map on sections of the corresponding \( \mathcal{A} \)-modules. So, using the presheaf \((\Gamma(U, \mathcal{A}^n), \sigma_U')\) of sections of the free \( \mathcal{A} \)-module \( \mathcal{A}^n \) of Theorem 3.1 with \( U \) ranging over the topology \( T \) of \( \mathcal{A}^n \), we adopt the following classical terminology. Let \( S \) be an \( \mathcal{A}(X) \)-submodule of the \( \mathcal{A}(X) \)-module \( \Gamma(X, \mathcal{A}^n) \). Then,

- \( S \) is called symplectic if \( \omega|_S \) is non-degenerate. For example, \( S_1 \), in the proof of Theorem 3.1, is symplectic.
- \( S \) is called isotropic if \( \omega|_S \equiv 0 \). For instance, the span of \( s_1, s_2 \), in Theorem 3.1 is isotropic.

**Definition 3.6** Let \((\mathcal{E}, \omega)\) and \((\mathcal{E}', \omega')\) be symplectic \( \mathcal{A} \)-modules on the same topological space \( X \). An \( \mathcal{A} \)-morphism \( \varphi : \mathcal{E} \longrightarrow \mathcal{E}' \) is called symplectic if
\[
\varphi^* \omega' := \omega' \circ (\varphi \times \varphi) = \omega, \quad (18)
\]
that is, for any \( s, t \in \mathcal{E}(U), \) \( U \subseteq X \) open,
\[
(\varphi^*\omega')(s, t) := \omega'(\varphi(s), \varphi(t)) = \omega(s, t).
\] (19)

A symplectic \( \mathcal{A} \)-isomorphism is called an \( \mathcal{A} \)-symplectomorphism. Symplectic \( \mathcal{A} \)-modules \((\mathcal{E}, \omega)\) and \((\mathcal{E}', \omega')\) are called \( \mathcal{A} \)-symplectomorphic if there is an \( \mathcal{A} \)-symplectomorphism \( \varphi \) between them.

Strictly speaking, Equations (19) and (18) should be written as follows:
\[
(\varphi_U^*\omega'_U)(s, t) := \omega'_U(\varphi_U(s), \varphi_U(t)) = \omega_U(s, t),
\]
and
\[
\varphi_U^*(\omega'_U) := \omega'_U \circ (\varphi_U \times \varphi_U) = \omega_U,
\]
respectively, where \( U \) varies over the topology of \( X \).

It is clear that if \( (\mathcal{E}, \omega) \) and \( (\mathcal{E}', \omega') \) are symplectomorphic, then they are of the same rank, and their rank is an even positive integer. It is also clear that the set of symplectic \( \mathcal{A} \)-modules, defined over the same topological space, can be partitioned into equivalence classes. Furthermore, a symplectic \( \mathcal{A} \)-morphism is necessarily injective, since if \( \varphi := (\varphi_V)_{V \supseteq \text{open}} \) and \( \varphi_V(s) = 0 \), where \( s \in \Gamma(V, \mathcal{E}) \), then necessarily \( s = 0 \), for \( \omega \) is non-degenerate.

**Lemma 3.2** Let \( \mathcal{A} \) be a unital \( \mathbb{C} \)-algebra sheaf on a topological space \( X \), and let
\[
Sp(\mathcal{E}, \omega) \equiv Sp\mathcal{E}
\]
be the sheaf on \( X \), generated by the presheaf
\[
U \mapsto (Sp\mathcal{E})(U),
\] (20)
where \( U \) varies over the topology of \( X \), such that for every open set \( U \subseteq X \), \( (Sp\mathcal{E})(U) \) is the group (under composition) of all \( \mathcal{A}|_U \)-symplectomorphisms
\[
(\mathcal{E}|_U, \omega|_U \equiv \omega) \longrightarrow (\mathcal{E}|_U, \omega|_U \equiv \omega).
\]

Then, the correspondence, given by (20), yields a complete presheaf of groups on \( X \); so that one obtains
\[
(Sp\mathcal{E})(U) = (Sp\mathcal{E})(U),
\]
up to a group isomorphism, for every open set \( U \subseteq X \). The sheaf \( Sp\mathcal{E} \) is called the **symplectic group sheaf**, or even **group sheaf of symplectomorphisms** of \( \mathcal{E} \) (in fact, of \((\mathcal{E}, \omega)\)) on \( X \).
Proof. We first show that for all open set \( U \subseteq X \), \((Sp E)(U)\) is a group. In fact, let us consider the subset 
\[ \text{GL}_{A|U}(E|U, E|U) \subseteq \text{Hom}_{A|U}(E|U, E|U) \]
of all invertible elements of the \( A(U) \)-module \( \text{Hom}_{A|U}(E|U, E|U) \). For \( A|U \)-morphisms \( \varphi, \psi \in \text{GL}_{A|U}(E|U, E|U) \), we define 
\[ \varphi \circ \psi = (\varphi_V \circ \psi_V)_{U \supseteq V, \text{open}}, \]
and 
\[ \varphi^{-1} = (\varphi_V^{-1})_{U \supseteq V, \text{open}}. \]
It is easy to see that under the above law of composition \( \text{GL}_{A|U}(E|U, E|U) \) forms a group. Therefore, in order to show that \((Sp E)(U)\), where \( U \) is an open subset of \( X \), is a group, we need only show that if \( \varphi, \psi \in (Sp E)(U) \), then \( \varphi \circ \psi \) and \( \varphi^{-1} \in (Sp E)(U) \). To this end, let \( V \) be an open subset of \( U \). Then, we have 
\[ (\varphi_V \circ \psi_V)^*(\omega) = (\psi_V^* \circ \varphi_V^*)(\omega) = \psi_V^*(\varphi_V^*(\omega)) = \psi_V^*(\omega) = \omega, \]
from which we deduce that \( \varphi \circ \psi \in (Sp E)(U) \).

On the other hand, one also obtains 
\[ (\varphi_V^{-1})^*(\omega) = (\varphi_V^{-1})^*(\varphi_V^*(\omega)) = (\varphi_V \circ \varphi_V^{-1})^*(\omega) = \omega, \]
so that \( \varphi^{-1} \in (Sp E)(U) \), as well.

Let us now show that (20) yields a complete presheaf of groups. It is easy to see that Correspondence (20), along with the obvious restriction maps, defines a presheaf of groups on \( X \). Thus, we just prove that the presheaf of groups defined, on \( X \), by (20) is complete.

Indeed, let \( U \) be an open subset of \( X \) and \( \mathcal{U} = \{ U_\alpha \}_{\alpha \in I} \) an open covering of \( U \); let \( \varphi, \psi \in (Sp E)(U) \) such that 
\[ \rho_{U_\alpha}^U(\varphi) = \varphi_{U_\alpha} := \varphi_{U_\alpha} =: \psi_{U_\alpha} = \rho_{U_\alpha}^U, \]
for all \( \alpha \in I \), and where the \( \rho_{U_\alpha}^U \) are the restriction maps characterizing the presheaf \((Sp E)(U), V)\). Since 
\[ (Sp E)(U) \subseteq \text{GL}_{A|U}(E|U, E|U), \quad \text{GL}_{A|U}(E|U, E|U) \subseteq \text{Hom}_{A|U}(E|U, E|U), \]
and the \( \{ \rho_{U_\alpha}^U \}_{\alpha \in I} \) are also the restriction maps making the diagram 
\[ U \rightarrow \text{Hom}_{A|U}(E|U, E|U) \] (21)
into a presheaf, it follows that \( \varphi = \psi \). So the presheaf on \( X \), given by (20), satisfies Condition (S1) of presheaves, see [9], p. 46.
For axiom (S2), see [9], p. 47, let
\[(\varphi_\alpha)_{\alpha \in I} \in \prod_{\alpha \in I} (\text{Sp } \mathcal{E})(U_\alpha) \subseteq \prod_{\alpha \in I} \text{Hom}_{\mathcal{A}|U_\alpha}(\mathcal{E}|U_\alpha, \mathcal{E}|U_\alpha)\]
be such that
\[\rho^{U_\alpha}_{U_\alpha \cap U_\beta}(\varphi_\alpha) \equiv \varphi_\alpha|_{U_\alpha \cap U_\beta} = \varphi_\beta|_{U_\alpha \cap U_\beta} \equiv \rho^{U_\beta}_{U_\alpha \cap U_\beta}(\varphi_\beta)\]
for any \(\alpha, \beta \in I\), with \(U_\alpha \cap U_\beta \neq \emptyset\). Hence, since (21) yields a complete presheaf, there exists an element \(\varphi \in \text{Hom}_{\mathcal{A}|U}(\mathcal{E}|_U, \mathcal{E}|_U)\) such that one has
\[\varphi|_{U_\alpha} = \varphi_\alpha, \quad \alpha \in I.\]
It only remains to show that
\[\varphi^* \omega = \omega,\]
where \(\omega \equiv \omega|_U\) is a symplectic structure on \(\mathcal{E}|_U\).

To this end, we first observe that \(\varphi^* \omega, \omega \in \text{Hom}_{\mathcal{A}|U}((\mathcal{E} \times \mathcal{E})|_U, \mathcal{A}|_U)\), with
\[U \mapsto \text{Hom}_{\mathcal{A}|U}((\mathcal{E} \times \mathcal{E})|_U, \mathcal{A}|_U)\]
defining a complete presheaf of \(\mathcal{A}\)-modules on \(X\), see [9], p. 134. But,
\[\varphi^* \omega|_{U_\alpha} = (\varphi|_{U_\alpha})^* \omega|_{U_\alpha} = \varphi^* \omega|_{U_\alpha} = \omega|_{U_\alpha},\]
therefore
\[\varphi^* \omega = \omega,
\]
as desired. \(\square\)

On the other hand, the preceding notion of symplectic sheaf of groups can also be defined through an application of the isomorphism \(\Gamma\), which is given in (12). Namely, since by (12), \(\varphi \equiv (\varphi_U) : \Gamma(\mathcal{E}) \mapsto \Gamma(\mathcal{E})\) is an \(\mathcal{A}\)-symplectomorphism if and only if the corresponding \(\mathcal{A}\)-morphism is symplectomorphic, the symplectic sheaf of groups can be viewed as consisting of all \(\mathcal{A}\)-symplectomorphisms
\[\varphi : \mathcal{E} \mapsto \mathcal{E}.\]

Now, suppose that \(\mathcal{E}\) is the standard free \(\mathcal{A}\)-module \(\mathcal{A}^{2n}\) of rank \(2n\), and let \(\omega : \mathcal{A}^{2n} \oplus \mathcal{A}^{2n} \mapsto \mathcal{A}\) be a skew-symmetric, non-degenerate \(\mathcal{A}\)-bilinear morphism. Let \(\{s_i\}_{1 \leq i \leq 2n}\) be a basis of \(\mathcal{A}^{2n}(X)\) such that Theorem 3.1 holds, and let \(\varphi \in \text{Sp } \mathcal{A}^{2n}(X)\). Let’s consider the full matrix algebra sheaf \(M_{2n}\) (see [9], p. 280) induced by the presheaf
\[U \mapsto M_{2n}(\mathcal{A}(U)),\]
where \(U \subseteq X\) is open, and the range, \(M_{2n}(\mathcal{A}(U))\), of the preceding map consists of all \(2n \times 2n\)-matrices with entries in the \(\mathbb{C}\)-algebra (unital and commutative)
\[\mathcal{A}(U) \equiv \Gamma(U, \mathcal{A}).\]
Since
\[ \mathcal{S}p\mathcal{A}^{2n}(X) \subseteq \text{Hom}_\mathcal{A}(\mathcal{A}^{2n}, \mathcal{A}^{2n})(X), \]
and by Statement 3.17, [9], p. 293,
\[ \text{Hom}_\mathcal{A}(\mathcal{A}^{2n}, \mathcal{A}^{2n})(X) = M_{2n}(\mathcal{A}(X)) = M_{2n}(\mathcal{A})(X), \]
it follows that if \( M \) is the matrix representing \( \varphi \in \mathcal{S}p\mathcal{A}^{2n}(X) \) with respect to the basis \( \{ s_i \}_{1 \leq i \leq 2n} \) above, then Equation (18) becomes in matrix form
\[ ^t M J M = J, \]
where \( J \) is the matrix [110].

From (22), we deduce the following corollary.

**Corollary 3.3** The determinant of an \( \mathcal{A} \)-symplectomorphism
\[ \varphi : (\mathcal{A}^{2n}, \omega) \longrightarrow (\mathcal{A}^{2n}, \omega), \]
where \( \mathcal{A} \) is a unital and commutative \( \mathbb{C} \)-algebra sheaf on a topological space \( X \), is the global identity section \( 1 \in \Gamma(\mathcal{A}^{2n}) \). More explicitly, given \( M \) as the matrix representing the \( \mathcal{A} \)-symplectomorphism \( \varphi \), as in the paragraph preceding the corollary, one has
\[ \overline{\partial e t}(M) = 1 \in \Gamma(\mathcal{A}^{2n}). \]

We refer to [9], p. 294, for the definition of the determinant morphisms \( \partial e t \) and \( \overline{\partial e t} \).

### 4 The Characteristic Polynomial Section

Let \( \mathcal{A} \) and \( \mathcal{B} \) be sheaves of algebras on a topological space \( (X, \mathcal{T}) \), and let \( \varphi : \mathcal{A} \longrightarrow \mathcal{B} \) be a sheaf morphism such that, if
\[ \overline{\varphi} = (\varphi_U)_{U \in \mathcal{T}} : \Gamma(\mathcal{A}) \longrightarrow \Gamma(\mathcal{B}) \]
is the corresponding morphism between the associated complete presheaves of sections \( \Gamma(\mathcal{A}) \) and \( \Gamma(\mathcal{B}) \), then, for all \( U \in \mathcal{T} \),
\[ \overline{\varphi}_U(\mathcal{A}(U)) \subseteq C(\mathcal{B}(U)), \]
where \( C(\mathcal{B}(U)) \) stands for the center of the ring \( \mathcal{B}(U) \), cf. [7], p.121. Explicitly, Equation (24) means that
\[ \overline{\varphi}_U(s) t = t \overline{\varphi}_U(s). \]
for all \( s \in \mathcal{A}(U) \) and \( t \in \mathcal{B}(U) \). Now, given \( s \in \mathcal{A}(U) \) and \( t \in \mathcal{B}(U) \), with \( U \) an open set in \( X \), the assignment

\[
(s, t) \mapsto \varphi_U(s)t
\]

makes \( \mathcal{B}(U) \) into a module over \( \mathcal{A}(U) \). What more is that \( \mathcal{B}(U) \) is an algebra over \( \mathcal{A}(U) \); in effect,

\[
\varphi_U(s + s')t = \varphi_U(s)t + \varphi_U(s')t
\]

and

\[
\varphi_U(s)(t + t') = \varphi_U(s)t + \varphi_U(s)t'
\]

for all \( s, s' \in \mathcal{A}(U) \) and \( t, t' \in \mathcal{B}(U) \). On the other hand, since the sheafification functor preserves algebraic structures, cf. [9] (1.54), p.101, \( \mathcal{B} \) can be viewed as an \( \mathcal{A} \)-algebra sheaf. The \( \mathcal{A} \)-algebra sheaf \( \mathcal{B} \) thus obtained is called an \( \mathcal{A} \)-algebra sheaf with respect to the morphism \( \varphi : \mathcal{A} \rightarrow \mathcal{B} \), above. More accurately, we may yet say that \( \mathcal{B} \) is an \( \varphi(\mathcal{A}) \)-algebra sheaf, with \( \varphi(\mathcal{A}) \subseteq \mathcal{B} \), as above. Equivalently, by a \( \varphi(\mathcal{A}) \)-algebra sheaf (or \( \varphi(\mathcal{A}) \)-algebra, as a shorthand), we shall always mean a morphism \( \mathcal{A} \rightarrow \mathcal{B} \) of sheaves of algebras, as above.

Now, let \( \mathcal{A} \) be a sheaf of unital commutative \( \mathbb{C} \)-algebras, \( \mathcal{E} \) an \( \mathcal{A} \)-module, and \( \mathcal{R} \) a \( \varphi(\mathcal{A}) \)-algebra. A representation of \( \mathcal{R} \) in \( \mathcal{E} \) is an \( \mathcal{A} \)-morphism

\[
\Theta : \mathcal{R} \rightarrow \text{End}_\mathcal{A}(\mathcal{E}),
\]

which makes the diagram

\[
\begin{array}{ccc}
\mathcal{R} & \xrightarrow{\Theta} & \text{End}_\mathcal{A}(\mathcal{E}) \\
\downarrow & & \downarrow \\
\mathcal{A} & & 
\end{array}
\]

commutative; the morphism \( \mathcal{A} \rightarrow \text{End}_\mathcal{A}(\mathcal{E}) \) in the above diagram is given by

\[
a \mapsto aI|_U = aI_U = (a_V I_V)|_{U \supseteq V, \text{open}},
\]

where \( a \in \mathcal{A}(U) \) and \( I_U : \mathcal{E}(U) \rightarrow \mathcal{E}(U) \) denotes the identity \( \mathcal{A}(U) \)-morphism. Besides, for all open set \( V \) in \( U \), \( a_V \in \mathcal{A}(V) \), and \( s \in \mathcal{E}(V) \),

\[
(a_V I_V)(s) = a_V s \in \mathcal{E}(V).
\]

We observe that for all open \( U \subseteq X \), \( \mathcal{E}(U) \) may be viewed as a module over \( \text{Hom}_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{E}|_U) \). Indeed, let \( f \equiv (f_V) \in \text{Hom}_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{E}|_U) \), where \( V \) runs over the open subsets of \( U \), and \( s \in \mathcal{E}(U) \). The action

\[
(f, s) \mapsto f_U(s)
\]
defines a \( \text{Hom}_{\mathcal{A}[U]}(\mathcal{E}|_{U}, \mathcal{E}|_{U}) \)-module structure on \( \mathcal{E}(U) \), as was to be shown. So, by means of the sheafification process, \( \mathcal{E} \) may be viewed as an \( \text{End}_{\mathcal{A}} \mathcal{E} \)-module. Furthermore, given a representation \( \Theta : \mathcal{R} \rightarrow \text{End}_{\mathcal{A}} \mathcal{E} \) of a \( \varphi(\mathcal{A}) \)-algebra sheaf \( \mathcal{R} \) in \( \mathcal{E} \), it turns out that \( \mathcal{E} \) may be viewed as a \( \mathcal{R} \)-module, with the operation of \( \mathcal{R}(U) \) on \( \mathcal{E}(U) \) being given by

\[
(s, e) \mapsto \Theta_{U}(s)e \equiv \Theta_{U}(s)(e)
\]

for \( s \in \mathcal{R}(U) \) and \( e \in \mathcal{E}(U) \). Like in [7], p.554, we write \( se \) instead of the more accurate notation \( \Theta_{U}(s)(e) \).

**Definition 4.1** Let \( \mathcal{B} \) be a sheaf of \( \mathcal{C} \)-algebras over a topological space \( X \), and \( \mathcal{A} \) a subsheaf of \( \mathcal{B} \). For all open set \( U \) in \( X \), let

\[
\mathcal{A}(U)[t]
\]

denote the ring of polynomials, in the variable \( t \), whose coefficients are the (local) sections of \( \mathcal{A} \) on \( U \). Similarly, let

\[
\mathcal{A}(U)[s],
\]

where \( s \in \mathcal{B}(U) \), denote the subring of \( \mathcal{B}(U) \) of all polynomial values \( p(s) \), with \( p \in \mathcal{A}(U)[t] \). A local section \( s \in \mathcal{B}(U) \) is called **transcendental** over \( \mathcal{A}(U) \) if the evaluation map

\[
\mathcal{A}(U)[t] \rightarrow \mathcal{A}(U)[s], \quad p \mapsto p(s)
\]

is an **isomorphism**.

Now, suppose that \( \mathcal{A} \) is a sheaf of unital commutative of \( \mathcal{C} \)-algebras on the topological space \( X \). As above, we let \( \mathcal{A}(U)[t] \), where \( U \subseteq X \) is open, be the polynomial ring. It is clear that the correspondence

\[
U \mapsto \mathcal{A}(U)[t],
\]

(25)

where \( U \subseteq X \) is open, yields a **complete presheaf of \( \mathcal{A} \)-modules on \( X \)**. So, (25) defines a **complete \( \Gamma(\mathcal{A}) \)-presheaf on \( X \)**. The sheaf generated, on \( X \), by the complete presheaf defined by (25) is called the **sheaf of germs of polynomials** on \( \Gamma(\mathcal{A}) \), and is denoted by

\[
\mathcal{A}[t],
\]

where \( t \) is a variable. It is easy to verify that the sheaf \( \mathcal{A}[t] \) of germs of polynomials is an \( \mathcal{A} \)-module on \( X \). Thus, based on [9], Proposition 11.1, p. 51, one obtains, for every open set \( U \subseteq X \),

\[
\mathcal{A}[t](U) = \mathcal{A}(U)[t],
\]

up to an \( \mathcal{A}(U) \)-isomorphism.
Let \((X, \mathcal{A})\) be an ordered algebraized space, equipped with the \textit{inverse-positive-section condition}, \(\mathcal{E}\) a vector sheaf of rank \(n\), and \(\varphi \in \text{End}_{\mathcal{A}}\mathcal{E}\). Let \(t\) be transcendental over \(\mathcal{A}(U)\), with \(U\) open in \(X\), and

\[
\mathcal{A}(U)[t] \longrightarrow \mathcal{A}(U)[\varphi_U] \subseteq \text{Hom}_{\mathcal{A}(U)}(\mathcal{E}(U), \mathcal{E}(U))
\]

be a representation of the polynomial ring \(\mathcal{A}(U)[t]\) in \(\mathcal{E}(U)\). Like in [7], p. 561, we have for every open set \(U \subseteq X\), a homomorphism

\[
\mathcal{A}(U)[t] \longrightarrow \mathcal{A}(U)[\varphi_U],
\]

which is obtained by substituting \(\varphi_U\) for \(t\) in polynomials. The \(\mathcal{A}(U)\)-algebra \(\mathcal{A}(U)[\varphi_U]\) is the subalgebra of \(\text{End}_{\mathcal{A}(U)}\mathcal{E}(U)\), generated by \(\varphi_U\), and is commutative because

\[
\varphi^p_U \circ \varphi^q_U = \varphi^q_U \circ \varphi^p_U,
\]

for all \(p, q \in \mathbb{N}\). Thus, for all \(s \in \mathcal{E}(U)\) and \(f(t) \in \mathcal{A}(U)[t]\), where \(U\) is, as usual, an open subset of \(X\), we put

\[
f(t)s = f(t)(s) := f(\varphi_U)(s) = f(\varphi_U)s;
\]

consequently \(\mathcal{E}(U)\) turns out to be a module over \(\mathcal{A}(U)[t]\). Let \(M_U\) be any \(n \times n\) matrix in \(\mathcal{A}(U)\) (for instance the matrix representing the \(\mathcal{A}(U)\)-endomorphism \(\varphi_U\) relative to a canonical basis \(\{e^U_i\}\) of \(\mathcal{E}(U)\), where \(U\) is a local gauge of the vector sheaf \(\mathcal{E}\), and \(e^U_i = \varepsilon^U_i \circ \varphi^U_i\), with \(\varphi^U_i\) being the \(\mathcal{A}(U)\)-isomorphism [2]. The basis \(\{e^U_i\}\) is called a canonical gauge of \(\mathcal{E}(U)\).). We define the characteristic polynomial section \(P_{M_U}(t)\) of \(M_U\) or of \(\varphi_U\) to be the determinant

\[
\overline{\text{det}_U}(tI_U - M_U) := \text{det}_U(tI_U - M_U) \in \mathcal{A}(U)[t],
\]

where \(I_U\) is the unit \(n \times n\)-matrix (here, \(1 := 1_U\) is the (local) identity section over \(U\)). (We refer to [9], pp 294-298, for the sheaf-theoretic notation of the determinant morphism.) Next, we decree that the characteristic polynomial of an endomorphism \(\varphi \in \text{End}_{\mathcal{A}}\mathcal{E}\) is the endomorphism \(P_\varphi(t) \in \text{End}_{\mathcal{A}}\mathcal{A}\), given by

\[
P_\varphi(t) \equiv (P_{\varphi_U}(t)) = (\text{det}_U(tI_U - M_U)),
\]

where \(M_U\) represents \(\varphi_U \in \text{End}_{\mathcal{A}(U)}\mathcal{E}(U)\) with respect to the local gauge \(\{e^U_i\}\) of \(\mathcal{E}(U)\). Let

\[
M_x = M_U(x);
\]

its characteristic polynomial is, as obviously expected, given by

\[
\overline{\text{det}_U}(tI_U - M_U)(x) := \overline{\text{det}_x}(tI_x - M_x) = \text{det}_x(tI_x - M_x) \in \mathcal{A}_x[t],
\]

where \(\mathcal{A}_x[t] := \lim_{U \searrow x} \mathcal{A}(U)[t]\).

The characteristic polynomial section will also be referred to simply as characteristic polynomial.
We further look at the following correspondence

\[ U \mapsto ChP(A(U)), \quad (26) \]

where \( U \) ranges over the open subsets of \( X \), while the range of (26) is the set of all characteristic polynomials \( P_{M_U}(t) \) of \( n \times n \)-matrices \( M_U \), whose entries are (local) sections of \( A \) on \( U \). Now, the presheaf of full matrix \( \mathbb{C} \)-algebras on \( X \),

\[ U \mapsto M_n(A(U)), \quad (27) \]

where \( U \subseteq X \) is open, and \( M_n(A(U)) \) the (full) algebra of \( n \times n \)-matrices, with entries the (local) sections of \( A \) on \( U \), is complete. Hence, (26) yields a complete presheaf on \( X \); it is called presheaf of characteristic polynomials on \( X \). For the restriction maps of the presheaf defined by (27), see [9], pp 280-281.

So, we denote by

\[ ChP(A) \]

the sheaf of modules on \( X \), generated by the previous presheaf.

Now before we proceed over to the version of the Cayley-Hamilton theorem in this setting, we signal in passing (One might work through all the details to their own satisfaction!) that all the fundamental classical properties of the determinant morphism are also valid in our context. One of the properties, useful for the scope of the present paper, follows after this: Let \( (X, A, \mathcal{P}) \) be as usual an ordered \( \mathbb{R} \)-algebraized space satisfying the inverse-positive-section condition. Let \( A = (s_{ij}) \in M_n(A(X)) \), and \( A = (t_{ij}) \in M_n(A(X)) \) such that

\[ t_{ij} := (-1)^{i+j} \partial_{et_X}^{X}(A_{ji}) \equiv (-1)^{i+j} \det_X(A_{ji}); \]

\( A_{ji} \) is the \((n-1) \times (n-1)\) matrix obtained from \( A = (s_{ij})_{1 \leq i, j \leq n} \) by deleting the \( j \)-th row and \( i \)-th column.

**Proposition 4.1** *(Laplace decomposition), cf. [13]*. Let \( \det_X(A) = s \), with \( s \in A^\bullet(X) \). Then, \( \bar{A}A = AA = sI \). Furthermore, \( \det_X(A) \in A^\bullet(X) \) if and only if \( A \in M_n(A^\bullet(X)) \); consequently

\[ A^{-1} = s^{-1} \bar{A}. \]

**Proof.** The proof goes along similar lines as the proof of Proposition 4.16 in [7]. \( \square \)

**Theorem 4.1** *(Cayley-Hamilton)* Let \( \mathcal{E} \) be a vector sheaf of rank \( n \) on \( X \), and \( \varphi : \mathcal{E} \to \mathcal{E} \) an \( A \)-morphism. Then,

\[ P_{\varphi}(\varphi) \equiv (P_{\varphi U}(\varphi U)) = (0_U) \equiv 0. \]
Proof. Here as well, we base our proof on the proof of Theorem 3.1., [7], p.561. In fact, let $U$ be an open subset of $X$ such that $\mathcal{E}|_U = \mathcal{A}^n|_U$, and $\{e^U_i\}_{1 \leq i \leq n}$ be the gauge on $\mathcal{E}|(U) = \mathcal{E}(U)$, corresponding the Kronecker gauge of. Then, since $\mathcal{E}(U)$ may be viewed as a module over $\mathcal{A}(U)[t]$, one has

$$te^U_j = \sum_{i=1}^{n} s^U_{ij} e^U_i,$$

where $1 \leq j \leq n$, and $(s^U_{ij}) \equiv M_U$ is the matrix representing $\varphi_U$ with respect to $\{e^U_i\}$. Let $B_U(t) = tI_U - M_U$; then

$$\tilde{B}_U(t)B_U(t) = P_{\varphi_U}(t)I_U,$$

and

$$\tilde{B}_U(t)B_U(t) \left( \begin{array}{c} e^U_1 \\ \vdots \\ e^U_n \end{array} \right) = \left( \begin{array}{c} P_{\varphi_U}(t)e^U_1 \\ \vdots \\ P_{\varphi_U}(t)e^U_n \end{array} \right) = \left( \begin{array}{c} 0_U \\ \vdots \\ 0_U \end{array} \right)$$

because

$$B(t) \left( \begin{array}{c} e^U_1 \\ \vdots \\ e^U_n \end{array} \right) = \left( \begin{array}{c} 0_U \\ \vdots \\ 0_U \end{array} \right).$$

It follows that $P_{\varphi_U}(t)(\mathcal{E}(U)) = \{0_U\}$, and therefore $P_{\varphi_U}(\varphi_U)(\mathcal{E}(U)) = \{0_U\}$. This implies that $P_{\varphi_U}(\varphi_U)(\mathcal{E}(U)) = \{0_U\}$, as was to be shown. □

Now, suppose that the pair $(X, \mathcal{A})$ is an ordered algebraized space, satisfying the inverse-positive-section condition. Moreover, as above, we suppose that $\mathcal{E}$ is a vector sheaf of rank $n$ on $X$, and $\varphi : \mathcal{E} \rightarrow \mathcal{E}$ an $\mathcal{A}$-endomorphism; and we let $s \in \mathcal{E}(U) \equiv \Gamma(U, \mathcal{E})$, where $U$ is an open set in $X$. We further consider the associated endomorphism $\varphi \equiv (\varphi_U)_{U \in \mathcal{T}} : \Gamma(\mathcal{E}) \rightarrow \Gamma(\mathcal{E})$, where $\mathcal{T}$ is the assumed topology on $X$, and $\Gamma(\mathcal{E})$ is the presheaf of sections of $\mathcal{E}$. Now, we fix $U \in \mathcal{T}$ such that $\mathcal{E}|_U = \mathcal{A}^n|_U$. By an eigenvector section, or just eigenvector, of $\varphi_U \in \text{End}_{\mathcal{A}(U)}(\mathcal{E}(U))$, we mean a nowhere-zero (local) section $s \in \mathcal{E}(U)$, such that there exists a section $\lambda \in \mathcal{A}(U)$ for which we have

$$\varphi_U(s) = \lambda s,$$  \hspace{1cm} (28)

or equivalently

$$M_Us = \lambda s,$$

where $M_U$ is the matrix representing $\varphi_U$, with respect to the local Kronecker gauge on $\mathcal{E}(U)$. (A (local) section $s \in \mathcal{E}(U)$ is called a nowhere-zero section if $s_x \equiv s(x) \neq 0_x$ for all $x \in U$. The scalar section $\lambda$, in Equation (28), is called an eigenvalue section or simply eigenvalue of the morphism $\varphi_U$.

Next, we construct the following presheaf of sets, namely

$$U \mapsto \text{PV}(\mathcal{E}(U)), \quad \text{(29)}$$
where \( U \) varies over the open subsets of \( X \), and the range of (29), i.e. \( \text{PV}(E(U)) \), is the set of all eigenvectors of \( \mathfrak{f}_U \), with \( \mathfrak{f}_U \in \text{End}_{\mathcal{A}(U)}(E(U)) \).

**Proposition 4.2** Let \( \mathcal{E} \) be an \( \mathcal{A} \)-module on \( X \). The presheaf \( (\text{PV}(E(U)), \sigma_U^V) \), where for \( s \in \text{PV}(E(U)) \), \( \sigma_U^V(s) \equiv s|_V \), is a complete presheaf.

**Proof.** Indeed, let \( \mathcal{U} = \{ U_{\alpha} \}_{\alpha \in I} \) be an open covering of \( U \), and let \( s, t \) be two elements of \( \text{PV}(E(U)) \) such that

\[
\sigma_U^V(s) \equiv s_\alpha = t_\alpha = \sigma_U^V(t), \quad \alpha \in I,
\]

where the \( \sigma_U^V, U \supseteq V = \text{open} \), are the restriction maps of the aforementioned presheaf. Now, as before let \( \Gamma(\mathcal{A}) \equiv (\Gamma(U, \mathcal{A}), \rho_U^V) \) be the presheaf of sections of the sheaf \( \mathcal{A} \). Then, we have, assuming that \( \lambda \in \mathcal{A}(U) \) is the eigenvalue associated with the eigenvector \( s \in \text{PV}(E(U)) \), that

\[
\mathfrak{f}_{U_{\alpha}}(s_{\alpha}) = \sigma_U^V(s_{\alpha})(\mathfrak{f}_U(s)) = \sigma_U^V(s_{\alpha})(\lambda s) = \rho_U^V(\lambda)(\lambda s) = \rho_U^V(\lambda) s_\alpha \equiv \lambda s_\alpha.
\]

Likewise,

\[
\mathfrak{f}_{U_{\alpha}}(t_{\alpha}) = \mu t_{\alpha},
\]

with \( \mu \in \mathcal{A}(U) \) being the eigenvalue for the eigenvector \( t \). As, by hypothesis, \( s_\alpha = t_\alpha \), it follows that \( \mathfrak{f}_{U_{\alpha}}(s_{\alpha}) = \mu t_{\alpha} \); whence \( \rho_U^V(\lambda) \equiv \lambda s_\alpha \equiv \mu s_\alpha \). But \( \Gamma(\mathcal{A}) \) is a complete presheaf, therefore \( \lambda = \mu \); so that \( s = t \), as desired.

Now, let \( (s_{\alpha}) \in \prod_{\alpha} \text{PV}(E(U_{\alpha})) \) such that for any \( U_{\alpha \beta} \equiv U_{\alpha} \cap U_{\beta} \neq \emptyset \) in \( \mathcal{U} \), one has

\[
\sigma_{U_{\alpha \beta}}^U(s_{\alpha}) \equiv s_{\alpha}|_{U_{\alpha \beta}} = s_{\beta}|_{U_{\alpha \beta}} \equiv \sigma_{U_{\alpha \beta}}^U(s_{\beta}). \tag{30}
\]

The sequence \( (s_{\alpha}) \) of eigenvectors gives rise to a sequence

\[
(M_{n, \alpha}) \in \prod_{\alpha} \text{M}_n(E(U_{\alpha}))
\]

of \( n \times n \)-matrices whose entries are (local) sections of \( E \), and admitting the \( s_{\alpha} \) as eigenvectors correspondingly. It is clear that for any \( \alpha, \beta \in I \) such that \( s_{\alpha}, s_{\beta} \) fulfill (30), one has

\[
M_{n}(\sigma_{U_{\alpha \beta}}^U)(s_{ij}) := (\sigma_{U_{\alpha \beta}}^U(s_{ij}^\alpha)) = (\sigma_{U_{\alpha \beta}}^U(s_{ij}^\beta)) = M_{n}(\sigma_{U_{\alpha \beta}}^U)(s_{ij}^\beta),
\]

where \( s_{\alpha} \) and \( s_{\beta} \) are eigenvectors of matrices \( (s_{ij}^\alpha) \in M_n(E(U_{\alpha})) \) and \( (s_{ij}^\beta) \in M_n(E(U_{\beta})) \), respectively. But the presheaf

\[
(M_n(E(U)), M_{n}(\sigma_U^V)),
\]

29
cf. [9], p. 281, is a complete presheaf, therefore there exists a matrix $M \in M_{n}(\mathcal{E}(U))$ such that

$$M_{n}(\sigma^{U}_{\alpha})(M) = M_{n,\alpha}$$

for all $\alpha \in I$. Let $s \in \Gamma(U, \mathcal{E})$ such that $\sigma^{U}_{\alpha}(s) = s_{\alpha}$, $\alpha \in I$. It is easily seen that

$$\sigma^{U}_{\alpha}(Ms) = M_{n}(\sigma^{U}_{\alpha})(M)s = M_{n,\alpha}s_{\alpha} = s_{\alpha},$$

$\alpha \in I$, which implies that

$$Ms = \lambda s,$$

where $\lambda \in \mathcal{A}(U)$ is derived from the $\lambda_{\alpha}$, $\alpha \in I$, with $\lambda_{\alpha} \in \mathcal{A}(U_{\alpha})$ and $\sigma^{U}_{\alpha,\beta}(\lambda_{\alpha}) = \sigma^{U}_{\alpha,\beta}(\lambda_{\beta})$. Hence, axiom (S2), see [9], p46, is satisfied. \hfill $\square$

**Definition 4.2** Let $\mathcal{A}$ be a unital commutative $\mathbb{C}$-algebra sheaf on a topological space $X$, and let $\mathcal{E}$ be an $\mathcal{A}$-module on $X$. We denote by

$$PV(\mathcal{E})$$

the sheaf on $X$, generated by the presheaf defined by [29]. We call it the **eigenvector sheaf** of $\mathcal{E}$ or **sheaf of germs of eigenvectors** of $\mathcal{E}$.

**Proposition 4.3** Let $\mathcal{A}$ be a unital $\mathbb{C}$-algebra sheaf on a topological space $X$, $\omega : \mathcal{A}^{2n} \oplus \mathcal{A}^{2n} \to \mathcal{A}$, $n \in \mathbb{N}$, a symplectic structure on $\mathcal{A}$, and $\varphi \in Sp \mathcal{A}^{2n}(X)$, cf. Lemma 3.2 for the definition of $\varphi \in Sp \mathcal{A}^{2n}(X)$. Moreover, let $\lambda \in \mathcal{A}^{*}(X)$ be an eigenvalue of $\varphi$. Then, $\frac{1}{\lambda} \in \mathcal{A}(X)$ is an eigenvalue of $\varphi$ too.

**Proof.** Let $\{s_{i}\}_{1 \leq i \leq 2n}$ be a basis of $\mathcal{A}^{2n}(X)$ such that $(\omega_{ij})_{1 \leq i,j \leq 2n} = J$, where $J$ is given by (10) and $\omega_{ij} = \omega(s_{i}, s_{j})$, and let $M$ be the $2n \times 2n$-matrix representing the symplectomorphism $\varphi$ with respect to the aforementioned basis.

Consider the characteristic polynomial (section) of $M$

$$P(t) = \det_{\mathcal{A}}(M - tI),$$

where $I$ is understood as the $2n \times 2n$ identity matrix, and $t \in \mathcal{A}(X)$ a variable. Then, by virtue of (22) and (23), we have

$$P(t) = t^{2n}P\left(\frac{1}{t}\right),$$

as is done in [1] and [2]. Thus,

$$P(\lambda) = \lambda^{2n}P\left(\frac{1}{\lambda}\right).$$

But $P(\lambda) = 0$ by Cayley-Hamilton theorem, and $\lambda \in \mathcal{A}(X) = \mathcal{A}^{*}(X)$, so that $P\left(\frac{1}{\lambda}\right) = 0$. \hfill $\square$
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