Abstract

The paper studies the complex differentiable functions of double argument and their properties, which are similar to the properties of the holomorphic functions of complex variable: the Cauchy formula, the hyperbolic harmonicity, the properties of general $h$-conformal mappings and the properties of the mappings, which are hyperbolic analogues of complex elementary functions. We discuss the utility of $h$-conformal mappings to solving 2-dimensional hyperbolic problems of Mathematical Physics.

Keywords: double numbers, hyperbolic Cauchy problem, $h$-holomorphicity, conformal mappings.

1 Introduction

Double numbers (or, as they are called sometimes, hyperbolic-complex or splittable numbers) are known for quite a long time, and find applications both in Mathematics and Physics [1, 2]. Taking into account that their corresponding algebra is isomorphic to the direct sum of two real algebras, it was considered for a long time that the properties of double numbers are not interesting, and moreover, that these are even trivial, while compared to complex numbers.

In this paper we try to prove the wrongness of this opinion and to show that the possibilities offered by the algebra and the analysis (i.e., the $h$-analysis) of double numbers, considered together with the associated geometry of the 2-dimensional plane, are still far from being exhaustively studied.

We shall see that, in many respects, double numbers are far – in any sense, from being inferior to complex numbers. Considering the important circumstance that the geometry of double numbers is pseudo-Euclidean and hence corresponding to the 2-dimensional geometry of Space-Time, we obtain a new framework fruitful in ideas for Physics, which have a purely algebraic essence. A special particular feature of our study expresses the fact that the $h$-analytic mappings of double variable cover an infinite-dimensional space – which is exactly the case for the functions of one complex variable. Moreover, to each mapping defined on the complex plane, it corresponds a unique $h$-analytic mapping of double variable, and converse.

In Section 2 we concisely present properties of the plane of double variable and define the hyperbolic polar coordinate system, cones, $h$-conformal mappings and study their general properties. In Section 3 we define the $h$-holomorphic mappings and discuss several of their properties, which include analogs of the Cauchy formula and the Cauchy theorem, of the Cauchy-Riemann conditions, and of $h$-conformal mappings. In Section 4 we define and study the properties of standard hyperbolic elementary functions of double variable. At last, in the Conclusions section, we discuss the potential applications and the perspectives of further work in developing the present theory of functions of double variable.
2 Double numbers

By analogy with the algebra of complex numbers \( \mathbb{C} \), we define the algebra of double numbers \( \mathcal{H} \) by means of two generators \( \{1, j\} \) of the 2-dimensional \( \mathbb{R} \)-modules with the related multiplication table:

\[
\begin{array}{c|c|c}
1 & j \\ 
\hline 
1 & 1 & j \\ 
\hline 
\end{array}
\]

(2.1)

Having in view the further application of this algebra to describing the 2-dimensional Space-Time, we shall denote the elements of \( \mathcal{H} \) as:

\[ h = 1 \cdot t + jx \in \mathcal{H}, \text{ where } t, x \in \mathbb{R}. \]

Similarly to the complex numbers, the real number \( \text{Re} h \equiv t \) will be called the real part of the double number \( h \), and the real number \( \text{Im} h \equiv x \) will be called the imaginary part of the double number \( h \). The algebra of double numbers with the multiplication table (2.1) does not determine a numerical field, since it contains zero-divisors, i.e., the equation \( h_1 h_2 = 0 \) may be satisfied by nonzero elements \( h_1 \) and \( h_2 \). This is one of the reasons for which double numbers were not widely used in applications, as the complex numbers did. But exactly this feature shows in algebraic terms a very important circumstance of the 2-dimensional Space-Time – the occurrence of light-cones. The geometrical interpretation of double numbers is analogous to the interpretation of complex numbers: in the plane of double variable (in brief – on the hyperbolic plane), to each double number there corresponds a position-vector, whose coordinates are the real and the imaginary parts of this number. Here, the sum and the difference of double numbers is represented by the standard parallelogram rule for the corresponding position-vectors on the hyperbolic plane.

The involutive operation of complex conjugation for double numbers can be defined in the following way: \( h = t + jx \mapsto \bar{h} = t - jx \). Geometrically, this operation describes the reflection of the hyperbolic plane with respect to the axis \( \text{Im} h = 0 \). Like in the complex case, the couple \( (h, \bar{h}) \) can be regarded as independent double coordinates on the hyperbolic plane, which relate to the Cartesian coordinates by means of the formulas:

\[
x = \frac{h + \bar{h}}{2}, \quad y = \frac{h - \bar{h}}{2j}.
\]

(2.2)

The complex coordinate bilinear form \( G \equiv dh \otimes d\bar{h} \) splits into its symmetric \( \Xi \) and skew-symmetric \( \Omega \) irreducible components, as follows:

\[
B \equiv dh \otimes d\bar{h} = \Xi - j\Omega,
\]

(2.3)

where \( \Xi = dt \otimes dt - dx \otimes dx \) is a pseudo-Euclidean metric form, and \( \Omega \equiv dt \wedge dx = -jdh \wedge d\bar{h}/2 \) is the 2-dimensional volume form. We note, that the algebra of double numbers induces on the plane of double variable a 2-dimensional pseudo-Euclidean (hyperbolic) geometry endowed with the metric form \( \Xi \), which justifies the established by us denomination "hyperbolic plane".

The transition to hyperbolic polar coordinates and to the exponential form of representing double numbers exhibit a series of special features, which are absent in the case of complex numbers. The pair of lines \( t \pm x = 0 \) contains the subset of double numbers with zero squared norm\(^1\). For the brevity of terms and for preserving the partial analogy to the complex numbers, we shall call the quantity \( \sqrt{|hh|} \) the norm or the module of the double number (see further formula (2.5)). The lines of double numbers whose square of the norm vanishes, split the whole hyperbolic plane into four quadrant-type domains, which are represented on the drawing by the numbers I, II, III and IV (Fig.1). One can immediately verify that in each of the mentioned areas, the double numbers allow a hyperbolic polar representation of the form:

\[
h = t + jx = e^\varphi (\cosh \psi + j \sinh \psi),
\]

(2.4)

\(^1\)Strictly speaking, the occurrence of zero-divisors and the possibility of having negative values for the expression \( hh \) does not allow us to speak about the norm of a double number in its rigorous meaning.
Figure 1: The domain $\mathbb{R} \sqcup -\mathbb{R} \sqcup \mathbb{R} \sqcup -\mathbb{R}$ of the change of the angle $\psi$ on the plane $\mathcal{H}$. The orientation is synchronized in opposed quadrants and is opposed in the neighboring ones. For different angles in different quadrants one can enumerate the angle $\psi$ by using the index $k$: $\psi_k, (k = 1, 2, 3, 4)$.

where for each quadrant take place the following definitions of quantities:

\begin{align*}
\text{I} & : \quad \epsilon = 1, \quad \varrho = \sqrt{t^2 - x^2}, \quad \psi = \text{Arth}(x/t); \\
\text{II} & : \quad \epsilon = j, \quad \varrho = \sqrt{x^2 - t^2}, \quad \psi = \text{Arth}(t/x); \\
\text{III} & : \quad \epsilon = -1, \quad \varrho = \sqrt{t^2 - x^2}, \quad \psi = \text{Arth}(x/t); \\
\text{IV} & : \quad \epsilon = -j, \quad \varrho = \sqrt{x^2 - t^2}, \quad \psi = \text{Arth}(t/x). 
\end{align*}

The quantities $\varrho$ and $\psi$, defined in each of the quadrants by the formulas (2.5), will be called the module and respectively the argument of the double number $h$. In this way, in each of the quadrants we have $0 \leq \varrho < \infty$, and the quadrants themselves are parametrized by different copies of real lines, which together determine the manifold $\Psi$ of angular variables as a oriented disjoint sum $\mathbb{R} \sqcup -\mathbb{R} \sqcup \mathbb{R} \sqcup -\mathbb{R}$. Moreover, the manifold $\Psi$ can be suggestively represented by compactifying each copy of $\mathbb{R}$ into an open interval and further by gluing the intervals at their ends to obtain a circle with four pinched points.

We note, that the set of double numbers of zero norm is not described in any of the coordinate charts introduced above of the hyperbolic polar coordinate system. In the following, we shall call the subset of double numbers of the form

$$h_0 + h(1 \pm j),$$

(2.6)

(where $h_0$ — some fixed double number, $h$ is any double number) as the cone of the number $h_0$ and will be denoted as Con($h_0$). All the points which lie in Con($h_0$), have their hyperbolic distance to the point $h_0$ equal to zero. Sometimes we shall make a difference between Con$_+$(h$_0$) and Con$_-$($h_0$), depending accordingly on the signs in (2.6). In the same way, we can distinguish the sub-cones Con$^+_4$(h$_0$) and Con$^a_4$(h$_0$), relative to the cases Re$h + \text{Im}h > 0$ and Re$h + \text{Im}h < 0$ accordingly and the sub-cones Con$^+_4$(h$_0$) and Con$^a_4$(h$_0$) for the cases Re$h - \text{Im}h > 0$ and Re$h - \text{Im}h < 0$, accordingly. All the cones and subcones are shown in Fig.2.

The hyperbolic Euler formula: $\cosh \psi + j \sinh \psi = e^{j\psi}$ can be verified by expanding the left and right sides into formal Maclaurin series, and comparing their real and imaginary parts. The hyperbolic Euler formula leads to the exponential representation of double numbers:

$$h = t + jx = \epsilon \varrho e^{j\psi} = \epsilon e^{\Theta},$$

(2.7)

where in the last equality we passed to the ”complex hyperbolic angle”

$$\Theta = \ln \varrho + j\psi \equiv \ln h,$$

(2.8)
Figure 2: The definition of the cones and subcones of the point \( h_0 \). \( \text{Con}(h_0) = \text{Con}^+(h_0) \cup \text{Con}^-(h_0) = (\text{Con}^+(h_0) \cup \text{Con}^+_c(h_0)) \cup (\text{Con}^-(h_0) \cup \text{Con}^-_c(h_0)) \).

having sense in the first quadrant. So, the product of a pair of double numbers reduces to adding their hyperbolic angles and the product of the sign factors \( \epsilon \).

The formulas for computing lengths of curves and areas of domains in \( \mathcal{H} \) reads as follows:

\[
\text{Length}[\gamma] = \int_{\tau_A}^{\tau_B} \sqrt{|\dot{h} \bar{\dot{h}}|} \, d\tau; \quad \text{Area}[\Sigma] = -\frac{j}{2} \int_{\Sigma} dh \wedge d\bar{h} = -\frac{j}{4} \oint_{\partial \Sigma} (h \, d\bar{h} - \bar{h} \, dh), \tag{2.9}
\]

where in the last equality we used the double number variant of the fundamental Poincaré-Darboux Theorem regarding the integration of differential forms.

**Example.** We shall compute the length of the arc of Euclidean circle of Euclidean radius \( r \) whose center is zero, encompassed between the points 1 and \( j \) on the plane of double variable. Substituting in the Euclidean equation of the circle \( t^2 + x^2 = r^2 \) the polar hyperbolic coordinates: \( t = \rho \cosh \psi, \, x = \rho \sinh \psi \), we get the hyperbolic polar equation of the Euclidean circle:

\[
\rho(\psi) = \frac{r}{(\cosh^2 \psi + \sinh^2 \psi)^{1/2}}. \tag{2.10}
\]

Constructing the pseudo-Euclidean line-element: \( dt^2 = |dp^2 - \rho^2 \, d\psi^2| \) along the circle, by considering (2.10), we get after several elementary calculations and differentiation:

\[
dl = \frac{r \, d\psi}{(\cosh^2 \psi + \sinh^2 \psi)^{3/2}} \tag{2.11}
\]

Due to the symmetry of the arc relative to the first bisector \( t = x \), it suffices to compute the length of the half of the arc in which \( \psi \) varies from 0 to \( \infty \), and then we double the obtained result. The integral which provides this length is:

\[
L = 2 \int_0^\infty \frac{r \, d\psi}{(\cosh^2 \psi + \sinh^2 \psi)^{3/2}}.
\]

Using the substitution: \( \tanh \psi = \xi \), this integral reduces to a simpler form, and can be expressed in terms of elliptic integrals of first and second order:

\[
L = 2r \int_0^1 \frac{1 - \xi^2}{1 + \xi^2} \frac{d\xi}{1 + \xi^2} = 2\sqrt{2}r \left[ E(1/\sqrt{2}) - K(1/\sqrt{2})/2 \right] \approx 1.2r
\]
3 \( h \)-holomorphic functions of double variable

The function \( \ln h \), defined by formula (2.8), is a simple and important sample of the class of so-called \( h \)-holomorphic mappings of double variable, whose definition emerges from considerations similar to those which yield the definition of holomorphic functions of complex variable. Any smooth mapping \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) can be represented by a pair of real components, and we can pass (with using of (2.2)) to its representation by means of a pair of double variables \( \{ h, \bar{h} \} \), as follows:

\[
(h, \bar{h}) \mapsto (h', \bar{h}'): \quad h' = F_1(h, \bar{h}); \quad \bar{h}' = F_2(h, \bar{h}).
\]

If \( \mathbb{R}^2 \) is regarded now as the plane \( \mathcal{H} \) of double variable, then we can naturally limit ourselves to the mappings which preserve the hyperbolic complex structure of the plane, i.e., such mappings of the form \( h \in \mathcal{H} \to s = F(h) \in \mathcal{H} \). The differentiable functions \( \mathbb{R}^2 \to \mathbb{R}^2 \), which satisfy the condition:

\[
F_{\frac{\partial}{\partial h}} = 0
\]

are called \( h \)-holomorphic mappings of double variable \( h \). The functions which satisfy the condition:

\[
F_{\frac{\partial}{\partial \bar{h}}} = 0
\]

are called anti-\( h \)-holomorphic mappings of double variable.

By analogy to holomorphic functions of complex variable, the holomorphic functions of double variable can be defined by formal power series, whose convergence often follows from the convergence of the corresponding real series.

**Example.** The following identities can be straightforward verified by means of expansion into formal series:

\[
S(jx) = jS(x); \quad C(jx) = C(x);
\]

\[
S(h) = S(t + jx) = S(t)C(x) + jC(t)S(x); \quad C(h) = C(t + jx) = C(t)C(x) - jS(t)S(x),
\]

where \( x \in \mathbb{R} \), \( S \) is the sinus (elliptic or hyperbolic), \( C \) is the cosine (elliptic or hyperbolic) in the left and the right hand sides of the equalities, accordingly, which are defined by their standard series.

In fact, these equalities are particular cases of the more general identity:

\[
f(jx) = S_f(x) + jA_f(x),
\]

where \( S_f \equiv [f(x) + f(-x)]/2 \), \( A_f \equiv [f(x) - f(-x)]/2 \) are, respectively, the symmetric and the skew-symmetric parts of the arbitrary analytic function \( f \).

We shall show, that a holomorphic function always maps zero-divisors to zero-divisors. The proof relies on the following formal identity:

\[
(1 \pm j)^\alpha \equiv 2^{\alpha-1}(1 \pm j), \quad \alpha \in \mathbb{R}.
\]

For \( \alpha \in \mathbb{N} \), the identity immediately emerges from the simpler one: \( (1 \pm j)^2 = 2(1 \pm j) \). For arbitrary \( \alpha \), we need to use the expansion formulas into Maclaurin series. From one side, one has

\[
(1 \pm j)^\alpha = 1 \pm \alpha j + \frac{\alpha(\alpha - 1)}{2!} j + \frac{\alpha(\alpha - 1)(\alpha - 2)}{3!} j^3 + \frac{\alpha(\alpha - 1)(\alpha - 2)(\alpha - 3)}{4!} j^4 + \ldots
\]

On the other side,

\[
2^{\alpha-1} = (1 + 1)^{\alpha-1} = 1 + \alpha - 1 + \frac{(\alpha - 1)(\alpha - 2)}{2!} + \frac{(\alpha - 1)(\alpha - 2)(\alpha - 3)}{3!} + \frac{(\alpha - 1)(\alpha - 2)(\alpha - 3)(\alpha - 4)}{4!} + \ldots
\]

\[\text{The notion of derivative of a function } F(h, \bar{h}) \text{ relative to its arguments is similar to the one used in real analysis.}
\]

Namely, we can define the differentiability of a function \( F \) at the point \( (h, \bar{h}) \) as the following property of its variation: \( \Delta F = A(h, \bar{h}) \Delta h + B(h, \bar{h}) \Delta \bar{h} + o(\|\Delta h\|_H) \), where \( \|\Delta h\|_H \equiv \|\Delta x^2 - \Delta x^4 1/2 \) is the pseudo-Euclidean norm of the variation of the variable. Passing to different limits for \( \|\Delta h\|_H \to 0 \), we obtain the definition of partial derivatives or “directional derivatives”. We encounter several significant problems, related to the definition of the convergence and the limit by natural for the double numbers hyperbolic norm. In this paper we shall not deal with these purely mathematical questions and we use only those operations and properties, whose definitions are clear, though to a certain extent, formal.
Multiplying this line, term by term, with \((1 \pm j)\), we get:

\[
m_\alpha \frac{1}{2!} \pm \frac{(\alpha - 1)(\alpha - 2)}{2!} j + \ldots
\]

By combining in the expansion (3.8) all the successive pairs of even terms (which contain \(j\)), we obtain all the successive even terms of the series (3.6), and combining in (3.8) all the successive pairs of even terms (beginning with the pair "third-fifth"), we get all the successive odd terms of the series (3.6) (beginning with the third one). In this way, the formal series from the left and right sides of (3.5) coincide, q.e.d.

Considering now a holomorphic function \(F(h)\) given by a power series:

\[
F(h) = \sum_{k=0}^{\infty} c_k (h - h_0)^k,
\]

due to the identity (3.9), we get on the cone \(\text{Con}(h_0)\) of the arbitrary point \(h_0\):

\[
F(h)|_{h \in \text{Con}(h_0)} = F(h_0) + \sum_{k=1}^{\infty} c_k ((t - t_0) \pm j(t - t_0))^k = F(h_0) + (1 \pm j) \sum_{k=0}^{\infty} c_k 2^{k-1} (t - t_0)^k \subset \text{Con}(F(h_0)),
\]

which proves the claim. In fact, as we notice from the obtained expression, the holomorphic mapping can provide an inversion of the cone (i.e., it can maps the component \(\text{Con}_\pm\) into the corresponding component \(\text{Con}_\mp\) and converse), but it cannot map its branches \(\text{Con}_+\) and \(\text{Con}_-\) one into another. It is easy to check that the last property is achieved by means of anti-holomorphic mappings.

### 3.1 Hyperbolic Cauchy-Riemann conditions

Let’s write the condition (3.2) in Cartesian coordinates:

\[
F_{\bar{h}} = (U + jV)_\bar{h} = \frac{(U + jV)_t}{\bar{h}_t} + \frac{(U + jV)_x}{\bar{h}_x} = U_t - V_x + j(V_t - U_x) = 0.
\]

This implies the Cauchy-Riemann condition of hyperbolic analyticity:

\[
U_t = V_x; \quad U_x = V_t.
\]

If the mapping \(F\) is \(h\)-holomorphic in the sense of the former definition, then the mapping \(\ln F = \ln \varphi_F + j\psi_F + \ln \epsilon_F\) is \(h\)-holomorphic as well. This leads to the Cauchy-Riemann condition, written in terms of module and argument of the function of double variable\(^3\):

\[
(\ln \varphi_F)_t = (\psi_F)_x; \quad (\ln \varphi_F)_x = (\psi_F)_t.
\]

It is easy to check that from the conditions (3.10), it follows the hyperbolic harmonicity of the real and of the imaginary parts of the holomorphic function \(F\), which is expressed by the equations:

\[
\Box U = \Box V = 0,
\]

where \(\Box \equiv \partial_t^2 - \partial_x^2\) is the wave operator of second order (the "hyperbolic Laplacian").

\(^3\)We need to say that, in general, \(\epsilon_F\) is equal to a certain constant inside each quadrant on the image plane, which abruptly changes while passing over the borders of the quadrants. For this reason, the formulas (3.11) are correctly defined on \(\mathcal{H}\) minus the cross-shape lines of its zero-divisors. These considerations remove the question regarding the meaning of the expression \(\ln \epsilon_F\), which may arise while going beyond the framework of the double numbers algebra.
3.2 The hyperbolic analogues of the Cauchy theorem

In order to stress the independence of our proof of Cauchy theorem on the metrical properties of the double (or complex) plane, we shall provide it in terms of differential forms. For any simple closed contour $\Gamma$ which surrounds the domain $\Sigma \subset \mathcal{H}$ and for any holomorphic function of double variable $F = U + jV$, we have the following chain of equalities:

$$\oint_{\Gamma} F(h) \, dh = \oint_{\Gamma} U \, dt + V \, dx + j \oint_{\Gamma} U \, dx + V \, dt = \int_{\Sigma} [(V_t - U, x) + j(U_t - V, x)] \, dt \wedge dx = 0$$

as consequence of the conditions (3.10). The second sign of the equality expresses the Poincaré-Darboux Theorem regarding integration of 1-forms along closed paths. Using the complex analysis terminology, the proof looks even shorter:

$$\oint_{\Gamma} F(h) \, dh = \int_{\Sigma} F_{\bar{h}} \, \bar{h} \wedge dh = 0$$

if taking into account (3.2). From purely topological considerations, similar to the ones from the complex plane, the integral of a holomorphic function cancels on the border of a multi-connected domain.

![Figure 3: Towards setting the Cauchy Theorem on the plane of double variable](image)

For the Cauchy integral in its hyperbolic version, we have now the equality:

$$\oint_{\Gamma} \frac{F(h)}{h - h_0} \, dh = \oint_{S_r(h_0)} \frac{F(h)}{h - h_0} \, dh,$$

(3.13)

which follows from the hyperbolic Cauchy Theorem. Here $S_r(h_0)$ is the (Euclidean) circle of radius $r$ and center at the point $h_0$, and the integral does not depend on the radius of this circle (see Fig.3).

4We do not limit ourselves to defining curvilinear integrals on the plane of double variable, having in view their convergence to a pair of integrals of a 1-form on the Cartesian plane, whose definition is standard.

5We need to remark a certain ambiguity in the notation (3.13): the division in the integrand is meaningless on the intersection $\Gamma \cap \text{Con}(h_0)$. In this way, rigorously speaking, we need to remove the points $\text{Con}(h_0)$ from the domain of the integrant, and the integral should be regarded as the limit of the integral over a non-connected contour, whose discontinuities are concentrated in the neighborhood $\text{Con}(h_0)$ and their (Euclidean) measure tends to zero. Our results correspond to such an integral, considered in the sense of its principal value. For its existence, the contour has to transversally approach the lines of the cone. We do not insist on these purely mathematical questions in this paper, and postpone their more detailed investigation for a complementary paper.
We perform the change of variable: \( h = h_0 + \epsilon \rho(r, \psi)e^{j\psi} \), where the mapping \( \rho(r, \psi)e^{j\psi} = rf(\psi)e^{j\psi} \) is the polar parametrization of the Euclidean circle \( S_r(0) \) in terms of the hyperbolic polar coordinate system. We have, as follows from the example in Section 3, \( f = 1/\sqrt{\cosh^2 \psi + \sinh^2 \psi} \). We further need only the bijectivity of the function \( f \). Integrating by \( \psi \), we infer: \( h - h_0 = \epsilon rf(\psi)e^{j\psi}, \) \( dh = \epsilon df + \epsilon jfd\psi)e^{j\psi} \), and the Cauchy integral gets the form:

\[
\oint_{S_r(h_0)} F(h)(d\ln f + jd\psi).
\]

Using the \( r \)-independence of the integral and passing to limit for \( r \rightarrow 0 \), we obtain:

\[
\oint_{S_r(h_0)} F(h)(d\ln f + jd\psi) = \lim_{r \rightarrow 0} \oint_{S_r(h_0)} F(h)(d\ln f + jd\psi) = F(h_0) \int_{\Psi} (d\ln f + jd\psi).
\]

The integral of the first term cancels due to the bijectivity of the function \( \ln f \). In this way, we get the following formula of the hyperbolic version of the integral Cauchy formula:

\[
\oint_{\Gamma} \frac{F(h)}{h - h_0} dh = jF(h_0) \int_{\Psi} d\psi.
\]

In general, the integral obtained in the right hand side, is divergent. However, it can have a meaning, if we introduce the formal quantity \( \ell_H \) of the size of the hyperbolic space of directions by means of the formula:

\[
\frac{\ell_H}{2} = \int_{\mathbb{R}} d\psi.
\]

Taking into consideration the orientations of the pieces \( \mathbb{R} \) in \( \Psi \) (see Fig.1), we get:

\[
\int_{\Psi} d\psi = \ell_H/2 - \ell_H/2 + \ell_H/2 - \ell_H/2 = 0.
\]

In this way, the hyperbolic Cauchy formula in some (improper) sense has a more simple form than in the complex case:

\[
\oint_{\Gamma} \frac{F(h)}{h - h_0} dh = 0.
\]

We can obtain a more comprehensive analogue for the standard Cauchy formula, if we better examine the closed contour \( \Gamma_r \) of the form depicted in Fig.4. This contour consists of two arcs of arbitrary piecewise-smooth simple curves, which lie in the domains \( |t - t_0| \geq |x - x_0| \) and have their ends on the components of the cone \( \text{Con}(h_0) \), slices of this cone, and a pair of arcs of the Euclidean circle of radius \( r \) and center at \( h_0 \), which lie on the components of the cone \( \text{Con}(h_0) \). The Cauchy-type integral vanishes on the contour \( \Gamma_r \), with the same general meaning like \( (3.11) \), due to the fact that the contour \( \Gamma_r \) is homotopic to the initial contour \( \Gamma \) inside the domain of holomorphy of the function \( F(h)/(h - h_0) \). We have now

\[
0 = \oint_{\Gamma_r} \frac{F(h)}{h - h_0} dh = \oint_{S_r(h_0)} \frac{F(h)}{h - h_0} dh + \oint_{\Gamma'} \frac{F(h)}{h - h_0} dh,
\]

where \( \Gamma' \equiv \Gamma_r \setminus S_r(h_0) \). By introducing on \( S_r(h_0) \) hyperbolic polar coordinates, reiterating the previous reasoning, and using the properties of the function \( f(\psi) \) (its property of being even relative to \( \psi \)), which is provided by the polar equation of the Euclidean circle, we get

\[
\lim_{r \rightarrow 0} \oint_{S_r(h_0)} \frac{F(h)}{h - h_0} dh = -j\ell_H F(h_0),
\]
Figure 4: Towards the setting the integral Cauchy Theorem on the plane of double variable: the contour $\Gamma_r$.

whence from (3.16) we infer a more direct analogue of the Cauchy formula:

$F(h_0) = \frac{1}{\ell_H} \oint_{\Gamma_0} \frac{F(h)}{h - h_0} dh,$  \hspace{1cm} (3.18)

where $\Gamma_0 = \lim_{r \to 0} \Gamma_r$. Formally, the obtained formula has a totally equivalent shape to the standard complex Cauchy formula, while replacing the size of the space of Euclidean directions $\ell_E = 2\pi$ with the size of the space of hyperbolic directions $\ell_H$ in the pairs of quadrants whose signs $h\bar{h}$ coincide. The quantity $\ell_H$ may be considered as "the fundamental constant" of the geometry of double numbers. While using this constant in computations, we need to accurately consider its properties and to use the procedure of regularization of expressions.

The hyperbolic Cauchy formula (3.18) can be written in a more general form:

$F(h_0) = \pm \frac{1}{\ell_H} \oint_{\Gamma_{\pm}} \frac{F(h)}{h - h_0} dh,$  \hspace{1cm} (3.19)

considering the possibility of choosing the contour $\Gamma_{\pm}$ instead of $\Gamma_+ = \Gamma_0$ – obtained by rotating the latter one by the Euclidean angle $+\pi/2$, and the reverse orientation of the parameter $\psi$ in the domain $|t - t_0| \leq |x - x_0|$ relative to the general sense of positive (trigonometric, anti-clockwise) tracing of the contours in $\mathcal{H}$.

In a similar way one can obtain the following variants of the Cauchy formula:

$F(h_0) = (-1)^{n+1} \frac{2}{\ell_H} \oint_{\Gamma_n} \frac{F(h)}{h - h_0} dh,$  \hspace{1cm} (3.20)

where $n = 1, 2, 3, 4$, the contour $\Gamma_1$ is presented in Fig.5 and the contours $\Gamma_n$ emerge form this by rotations of angles $\pi(n - 1)/2$ around the point $h_0$.

**Example.** We illustrate the way in which the Cauchy formula works in its form (3.20) for $n = 1$, by means of explicitly computing the integral over the contour $\Gamma_1$. We obtain

$\frac{2}{\ell_H} \oint_{\Gamma_1} \frac{F(h)}{h - h_0} dh = \frac{2}{\ell_H} \oint_{\Gamma_1} \frac{F(h) - F(h_0)}{h - h_0} dh + \frac{2F(h_0)}{\ell_H} \oint_{\Gamma_1} \frac{dh}{h - h_0}.$  \hspace{1cm} (3.21)
Figure 5: Towards the setting the integral Cauchy Theorem on the plane of double variable: the contours \( \Gamma \).

The expression within the first integral is a holomorphic mapping in the domain which is bounded by the contour \( \Gamma \) and on the contour itself, and therefore this integral cancels. On the cone \( \text{Con}^+ \), we choose as integration variable \( t \in [t_0 + \tau_1, t_0] \), and on the cone \( \text{Con}^- \), \( t \in [t_0, t_0 + \tau_2] \), where \( \tau_1 \) and \( \tau_2 \) are the abscissae of the endpoints of the curvilinear part of the contour \( \Gamma \) (the upper and lower ends, respectively) in the system of coordinates having the origin at the point \( h_0 \). In this way, the main contribution of the integral is:

\[
\frac{2F(h_0)}{\ell_{HJ}} \left[ \int_{t_0}^{t_0 + \tau_2} \frac{(1 + j) dt}{(1 + j)(t - t_0)} + \int_{t_0}^{t_0 + \tau_1} \frac{(1 - j) dt}{(1 - j)(t - t_0)} \right] =
\]

\[
\frac{2F(h_0)}{\ell_{HJ}} (\ln 0 - \ln \tau_1 + \ln \tau_2 - \ln 0) = \frac{2F(h_0)}{\ell_{HJ}} \ln(\tau_2/\tau_1) = 0.
\]

In the prior to last equality we took into consideration the pairwise canceling of two logarithmic singular terms, and in the last one, we used the "unboundedness property" of the fundamental constant \( \ell_{H} \). In this way, the main contribution in the Cauchy integral is provided only by the segment \( \Gamma' \) of the contour located between the components of the cone \( \text{Con}(h_0) \). Passing to polar coordinate system with center at the point \( h_0 \), we get:

\[
\frac{2F(h_0)}{\ell_{HJ}} \int_{\Gamma'} \frac{dh}{h - h_0} = \frac{2F(h_0)}{\ell_{HJ}} \int_{\Gamma'} (d\ln \varrho + d\psi).
\]

The integral in the first term cancels, due to the fact that at the ends of the contour \( \Gamma' \) we have \( \varrho = 0 \). We integrate the second term by considering (3.14), and we get \( F(h_0) \), fact which confirms the validity of the hyperbolic Cauchy formula in the form (3.20).

We shall examine now the question of possibility of computing the coefficients of the Taylor series of a \( h \)-holomorphic function, by means of a formula similar to Cauchy one in complex analysis. To this aim, we consider the integral of the form:

\[
\oint_{\Gamma} (h - h_0)^\alpha dh, \quad \alpha \in R.
\]
We deform the contour $\Gamma$ in such a way, that it gets the form $\Gamma'$, shown in Fig. 6 (the value of the integral remaining, obviously, unchanged).

\[
q = t_0 + jx_0
\]

\[
h_0 = t_0 + jx_0
\]

\[
S_r(h_0)
\]

\[
I
\]

\[
t_0 + \tau
\]

\[
\text{Figure 6: Towards the question of finding the coefficients of the Taylor series for } h\text{-holomorphic mappings.}
\]

Splitting the integral into additive terms which correspond to different segments of the contour $\Gamma' = -\text{Con}^\uparrow_+ (h_0) \cup S_r(h_0) \cup \text{Con}^\downarrow_+ (h_0) \cup I$, we get for contributions over cones (the notations being similar to the ones used in the previous example, with $\tau_1 = \tau_2$):

\[
\int_{-\text{Con}^\uparrow_+ (h_0)} (h - h_0)^\alpha dh + \int_{\text{Con}^\downarrow_+ (h_0)} (h - h_0)^\alpha dh = \int_{t_0 + \tau}^{t_0 + \tau} (t - t_0)^\alpha (1 + j)^{\alpha + 1} dt + \int_{t_0 + \tau}^{t_0 + \tau} (t - t_0)^\alpha (1 - j)^{\alpha + 1} dt.
\]

Using the identity (3.5) and collecting the similar terms, we obtain the following result:

\[
\int_{\text{Con}^\downarrow_+ (h_0) - \text{Con}^\uparrow_+ (h_0)} (h - h_0)^\alpha dh = \int_{C} \frac{2^{\alpha + 1}}{\alpha + 1} [r^{\alpha + 1} - \tau^{\alpha + 1}].
\]  

(3.23)

The integral over the straight line segment is easy to compute as well:

\[
\int_{I} (h - h_0)^\alpha dh \big|_{h = h_0 + \tau + jsx} = \int_{-1}^{1} (\tau + jsx)^\alpha j\tau ds = \tau^{\alpha + 1} \int_{1-j}^{1+j} \xi^{\alpha} d\xi = j^{2^{\alpha + 1} - \tau^{\alpha + 1}}
\]  

(3.24)

Comparing (3.23) with (3.24), we conclude that

\[
\int_{\Gamma'} \frac{2^{\alpha + 1}}{\alpha + 1} r^{\alpha + 1}, \quad (\alpha \neq -1).
\]

For the integral over the circle $S_r(h_0)$, by means of the parametrization (2.5), we obtain the representation:

\[
\int_{S_r(h_0)} (h - h_0)^\alpha dh = \int_{S_r(h_0)} (h - h_0)^\alpha dh + \frac{2^{\alpha + 1}}{\alpha + 1} r^{\alpha + 1}, \quad (\alpha \neq -1).
\]

Comparing (4.28) with (3.24), we conclude that

\[
\int_{\Gamma'} \frac{2^{\alpha + 1}}{\alpha + 1} r^{\alpha + 1}, \quad (\alpha \neq -1).
\]

For the integral over the circle $S_r(h_0)$, by means of the parametrization (2.5), we obtain the representation:

\[
\int_{S_r(h_0)} (h - h_0)^\alpha dh = ([(-1)^{\alpha + 1} - j^{\alpha + 1} - (-j)^{\alpha + 1}] \int_{-\infty}^{+\infty} \xi^{\alpha + 1} e^{j(\alpha + 1)\psi} (d \ln \rho + j d\psi).
\]

By substituting $\rho = \rho(\psi) = rf(\psi)$ (for us, the concrete form of $f$ is of less importance, and what matters is the dependence of this mapping only on $\psi$, and its independence of $r$), we get the expression:

\[
\int_{S_r(h_0)} (h - h_0)^\alpha dh = r^{\alpha + 1} K_1(\alpha)
\]

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and hence the overall result:

$$\oint_{\Gamma_i} (h - h_0)^\alpha \, dh = r^{\alpha + 1} K_2(\alpha), \quad (\alpha \neq -1). \quad (3.25)$$

where $K_1(\alpha)$ and $K_2(\alpha)$ are certain functions, which depend only on $\alpha$. Due to the homotopicity of the contours having different values for $r$, the expression (3.25) should not depend on $r$. This is possible only if $K_2(\alpha) = 0$ for $\alpha \neq -1$. Taking into consideration the previous result for $\alpha = -1$, we infer the following final formula:

$$\oint_{\Gamma} (h - h_0)^\alpha \, dh = \begin{cases} 0, & \alpha \neq -1; \\
\oint_{\Gamma^H}, & \alpha = -1. \end{cases} \quad (3.26)$$

### 3.3 Conformal-analytic hyperbolic mappings

Our formerly introduced bilinear form $G$ behaves under $h$-holomorphic mappings $F(h)$ as a relative scalar:

$$G \mapsto G' = |F'(h)|^2 G, \quad (3.27)$$

where $F'(h) = dF/dh$, whence there follow the transformation laws for the hyperbolic elements of length and area:

$$dl' = |F'|' dl; \quad (dh \wedge d\bar{h})' = |F'|^2 (dh \wedge d\bar{h}). \quad (3.28)$$

Like in the complex case, the property of being a relative scalar of the area form takes place for any diffeomorphism, and the first equality in (3.28) means, that the $h$-holomorphic functions form the conformal mappings of the hyperbolic plane, i.e., they preserve the hyperbolic angles between curves at each point, where $|F'(h)|^2 \neq 0$. This fact is tightly related to the already proved invariance of cones Con relative to $h$-holomorphic mappings. We note, that $|F'|^2 = |\nabla u|^2 = |\nabla v|^2 = \Delta_F$, where $\nabla$ is the gradient operator for the pseudo-Euclidean metric, and $\Delta_F$ is the Jacobian of the mapping $F$, regarded as mapping $R^2 \rightarrow R^2$.

Like in the complex case, each diffeomorphism $f : R^2 \rightarrow R^2$ can be regarded as a smooth vector field. The vector fields which correspond to $h$-holomorphic mappings of double variable have certain interesting and important applicative properties. From the condition (3.10) it follows, that each component of the vector field $F(h) = U + iV$ is a hyperbolic $h$-harmonic function, i.e., it satisfies the wave-equation of second order (3.12). The $h$-harmonic functions which are mutually related by the Cauchy-Riemann conditions (3.10), will be called $h$-conjugate. Any $h$-harmonic mapping on the Cartesian plane defines (up to a constant) its hyperbolic conjugate partner. The hyperbolic Cauchy-Riemann conditions (3.10) have, in terms of vector analysis in hyperbolic space, the following geometric meaning: the vector field $\bar{F} = U - jV$ is $h$-potential and $h$-solenoidal, i.e., the components $\{U, -V\}$ of the vector field $F$ satisfy the relations:

$$\bar{F}_{,h} = 0 \Leftrightarrow \text{roth} F \equiv U_x - V_t = 0; \quad \text{divh} F \equiv U_{,t} - V_{,x} = 0. \quad (3.29)$$

The physical meaning of these conditions and the corresponding problems with initial/boundary conditions, which are naturally solved using hyperbolic conformal mappings, will be further considered in a separate section. We note here, that the family of curves $U = \text{const}$ and $V = \text{const}$ define on the Cartesian plane $R^2$ pseudo-orthogonal families of curves, where $\nabla U \cdot \nabla V \equiv U_{,t}V_{,x} - U_{,x}V_{,t} = 0$ holds everywhere true, due to the conditions (3.10).

### 4 Properties of some elementary functions of double variable

We examine now in detail the properties of the basic elementary functions of double variable.
4.1 Power functions $F(h) = h^n$

Unlike the power function of complex variable, the cases of even powers $n$ and odd powers $n$ are essentially different. Indeed, when passing to the exponential representation (2.7), we get:

$$h = \epsilon \rho e^{j\psi} \mapsto \epsilon^n \rho^n e^{jn\psi}$$

(4.1)

Since for any even $n$ we have $\epsilon^n = 1$, we conclude that the power mapping $h \mapsto h^n$, for $n = 2k$, $k \in \mathbb{Z}$ bijectively maps each of the quadrants I, II, III and IV on the first quadrant, and maps the cones $\text{Con}_+ \mapsto \text{Con}_+$. On the contrary, for odd $n$, each of the coordinate quadrants is bijectively mapped by the transformation $h \mapsto h^n$ $n = 2k + 1$, $k \in \mathbb{Z}$ into itself. We easily see from (4.1) that the net of coordinate lines $\rho = \text{const}$, $\psi = \text{const}$ maps to the net of coordinate lines $\rho' = \rho^n = \text{const}$, $\psi' = n\psi = \text{const}$ for all integer $n$. In the case of positive integers $n$, the radial lines extend for $\rho > 1$ and shrink for $\rho < 1$. Moreover, they rotate from the value $\psi = 0$ in the sense of the cone components which correspond to their signs. For integer negative $n$, there occur a complimentary inversion relative to the unit spheres $\rho = 1$ and a mirroring of the space of angles $\Psi \rightarrow -\Psi$. As example of function with even $n$ we examine the mapping $w = h^2 = x^2 + y^2 + 2jxy = \rho^2 e^{2\psi}$.

![Figure 7: The global structure of the mapping $h \mapsto h^2$.](image)

In Fig[7] there is represented the global structure of the mapping $h \mapsto h^2$: the quadrant 1-2 is mapped into itself (its borders map to the corresponding ones), and the mapping of the other quadrants into quadrant 1-2 is shown by the corresponding figures (accented figures which identify a quadrant, show how the corresponding quadrant maps to the quadrant 1-2). In this way, the mapping $h \mapsto h^2$ is 4-fold. A similar situation occurs with the mapping: $h \mapsto h^{2k}$ $k \in \mathbb{Z}$.

We present in Fig[8] an illustrative representation of mapping $h \mapsto h^2$.

![Fig. 8. The hyperbolic polar system of coordinates (left) and the image of its first quadrant under the mapping $h \mapsto h^2$.](image)
From the properties of power functions it is easy to derive the properties of roots of different orders and of the rational powers $h \mapsto h^{1/n}$ and $h \mapsto h^{m/n}$. Each root $\sqrt[n]{h}$ of even order is defined in quadrant I. Such a root is a 4-valued function. Each leaf of the hyperbolic Riemannian surface of this function represents a unit copy of the first quadrant I, as shown on Fig.7. On each leaf, the mapping is bijective. All the leaves glue together into a Riemannian surface, which represents $\mathbb{R}^2$, with the point $(0;0)$ belonging to all the leaves and being a hyperbolic analogue of the branching point. The Riemannian surface of the roots of even orders can be illustrated by means of a sheet of paper, folded as shown on Fig.9.

![Figure 9: The hyperbolic Riemannian surface of the 4-valued mapping $h \mapsto h^{1/2k}$, $k \in \mathbb{Z}$](image)

The roots of odd order are bijective in each of the 4 quadrants.

### 4.2 The exponential of double variable $w = e^h$

The relations $e^h = e^{t+jx} = e^t e^{jx}$ naturally lead to the global structure of the exponential mapping, which is represented in Fig.10.

![Fig. 10. The global structure of the mapping $h \mapsto e^h$.](image)

The rectangular pseudo-orthogonal net on the plane with variable $h$ is mapped by the exponential into the pseudo-orthogonal net, which consists of the rays and hyperbolas from the first quadrant and the vertex at the point $h = 0$. The mapping $h \mapsto e^h$ is bijective. Obviously, the inverse mapping $\ln h = \ln \rho + j\psi$ is defined inside the first quadrant. On its border (i.e., on the cone $\text{Con}^+(0)$) the polar coordinate system is ineffective and we need supplementary investigation regarding the behavior of the mapping $h \mapsto e^h$, which we shall not address here.
4.3 The trigonometric functions $\sin h$, $\cos h$ and their inverses

While writing the sine of double variable:

$$\sin h = \sin(t + jx) = \sin t \cos x + j \sin x \cos t,$$

we note that the lines $x = \text{const}$ and $t = \text{const}$ mapped into families of ellipses with their center at the point $(0;0)$. These lines are wrapped around these ellipses countless times. In Fig.11 (at right) there are shown the images of the squares with various sides, having the center at the point $(0;0)$ (at left).

![Fig. 11. To the properties of the mapping $h \mapsto \sin h$.](image)

Each square is mapped into a 4-ray star-shaped figure, and the square with the side $\pi/2$ is mapped into a circle, the square with the side $\pi$ is mapped into a coordinate cross with the vertices at the points $(1;0), (0;1), (-1;0), (0;-1)$.

In order to clear global structure of the mapping $h \mapsto \sin h$ it is more convenient to consider a system of fundamental squares. One of them is presented in Fig. 12 (from the left).

![Fig. 12. Global structure of the mapping $h \mapsto \sin h$. Fundamental square (the largest from the left) is mapped onto the largest square from the right. In each fundamental square function $\sin h$ is bijective.](image)

The whole plane of variable $h$ is covered by the squares such as in fig. 12 (from the left) by the shifts on vectors $(k \pm jm)\pi/2$, $k, m \in \mathbb{Z}$. Note, that the mapping $h \mapsto \sin h$ in two neighboring squares have opposite screw sense (i.e. opposite sign of Jacobian).
It is easy to see, that the mapping \( h \mapsto \cosh \) similarly works, with the difference that the whole family of "fundamental squares" is left-shifted on the plane of variable \( h \) with \( \pi/2 \) (since \( \cosh = \sinh(h + \pi/2) \)).

So, the function \( \arcsin \) (and \( \arccos \)) we can define on the square with vertices at the point \((1; 0), (0; 1), (-1; 0), (0; -1)\) (on such a square, left-shifted by \( \pi/2 \)). Explicit formulas for \( \arcsin \) and \( \arccos \) have the form:

\[
\arcsin h = \frac{1}{2} \left[ \arcsin((t + x)\sqrt{1 - (t - x)^2} + (t - x)\sqrt{1 - (t + x)^2}) + j \arcsin((t + x)\sqrt{1 - (t - x)^2} - (t - x)\sqrt{1 - (t + x)^2}) \right];
\]

\[
\arccos h = \frac{1}{2} \left[ \arccos((t^2 - x^2 - \sqrt{1 - (t - x)^2})\sqrt{1 - (t + x)^2}) + j \arccos((t^2 - x^2 + \sqrt{1 - (t - x)^2})\sqrt{1 - (t + x)^2}) \right].
\]

### 4.4 The trigonometric functions \( \tan h \), \( \cot h \) and its inverses

By identifying in the functions \( w = \tan h \) the real and the imaginary parts, after elementary transformations, we get:

\[ \tan h = \frac{\sin 2t + j \sin 2x}{\cos 2t + \cos 2x}. \]

This mapping transforms the square with the center in the point \((0; 0)\) and edge \( \pi/2 \) into the domain bounded by hyperbolas, and the rectangular net from the original square, into the symmetric hyperbolic net inside the domain (Fig. 13).

Globally, the mapping \( h \mapsto \tan h \) has an infinite number of leaves. These are the squares which are obtained from the fundamental square \((\pi/2; 0), (0; \pi/2), (-\pi/2; 0), (0; -\pi/2)\) (see left fig. 12) by means of translations with vectors multiple of \( \pi \) by \( t \) and by \( x \).

In view of the identity \( \cot h = -\tan(h - \pi/2) \), in a similar way behaves the mapping \( w = \cot h \). The functions \( \arctan \) and \( \arccot \) are multi-valued, their independent branches can be identified in each of the fundamental squares. E.g., the mapping \( \arctan h \) has in coordinates the following explicit form:

\[
\arctan h = \frac{1}{2} \left\{ \arctan \left( \frac{2t}{1 - t^2 + x^2} \right) + j \arctan \left( \frac{2x}{1 + t^2 - x^2} \right) \right\}.
\]
4.5 The hyperbolic functions $\sinh h$, $\cosh h$, $\tanh h$, $\coth h$ and their inverses

Separating, like in the case of the elliptic sine, in the function $w = \sinh h$ the real and the imaginary parts, we obtain the expression:

$$\sinh h = \sinh t \cosh x + j \sinh x \cosh t.$$  

It is easy to see that the rectangular coordinate net $(t, x)$ is mapped into the hyperbolic set on the plane of images $w$ (Fig. 14).

![Fig. 14. The structure of the mapping $h \mapsto \sinh h$.](image)

The mapping $h \mapsto \sinh h$ is bijective, and hence its inverse mapping $\text{Arsh}$ is defined on the whole double plane. Its explicit expression in coordinates is given by the formula:

$$\text{Arsh} h = \frac{1}{2} \left( \text{Arsh} [(t + x) \sqrt{1 + (t - x)^2} + (t - x) \sqrt{1 + (t + x)^2}] + j \text{Arsh} [(t + x) \sqrt{1 + (t - x)^2} - (t - x) \sqrt{1 + (t + x)^2}] \right).$$

Due to the duality of the hyperbolic cosine, the mapping

$$\cosh h = \cosh t \cosh x - j \sinh t \sinh x$$

is of a different kind.

The first quadrant with its vertex in zero is bijectively mapped by $\cosh$ into the first quadrant with the vertex at the point 1. Under these circumstances, the Cartesian net is mapped into a net of orthogonal hyperbolas. The remaining quadrants with the vertex at the point 0 are mapped as well to this quadrant (Fig. 15).

![Fig. 15. The structure of the mapping $h \mapsto \cosh h$.](image)
Fig. 15. The structure of the mapping $h \mapsto \cosh h$.

The global structure of the mapping $\cosh h$ is illustrated in Fig[7] from which the hashed quadrant is shifted to the right with one unit. In this way, the hyperbolic cosine is a 4-fold mapping, and the hyperbolic arcocose is 4-valent, with its Riemannian surface, represented in Fig[9] Its explicit coordinate representation is given by the formula:
\[
\operatorname{Arch} h = \frac{1}{2} \left( \operatorname{Arch} \left( t^2 - x^2 - \sqrt{(t + x)^2 - 1} \right) + j \operatorname{Arch} \left( t^2 - x^2 + \sqrt{(t + x)^2 - 1} \right) \right).
\]

The function $\tanh h \equiv \sinh h / \cosh h = \tanh t(1 - \tanh^2 x) + j \tanh x(1 - \tanh^2 t) / (1 - \tanh^2 t \tanh^2 x)$ maps the double plane into the interior of the square with vertexes $(1, 0), (0, 1), (-1, 0), 0, -1$, and the square with side 2 — into some interior domain near to the square-image of double plane (Fig[16]).

Fig. 16. The structure of the mapping $h \mapsto \tanh h$.

The hyperbolic tangent is a mapping with one leaf, and hence its inverse mapping $\operatorname{Arth}$ is bijective on its domain. Its coordinate expression is given by the formula:
\[
\operatorname{Arth} h = \frac{1}{2} \left( \operatorname{Arth} \left( t + x \right) + \operatorname{Arth} \left( t - x \right) + j \left( \operatorname{Arth} \left( t + x \right) - \operatorname{Arth} \left( t - x \right) \right) \right).
\]

The function $\coth h \equiv \cosh h / \sinh h = \coth t(1 - \coth^2 x) + j \coth x(1 + \coth^2 t) / (\coth^2 t - \coth^2 x)$ is, in a certain sense, complementary to the function $\tanh h$: it maps the whole plane $H$ into the exterior of the square in Fig[10] (at right).

Here, the rectilinear coordinate net is mapped to an orthogonal family of hyperbolas, which intersect at an infinity-point.

The function $\coth h$ has one leaf and its inverse mapping $\operatorname{Arcth}$ is univalent in its definition domain. Its coordinate expression is given by the formula:
\[
\operatorname{Arcth} h = \frac{1}{2} \left( \operatorname{Arcth} \left( t + x \right) + \operatorname{Arcth} \left( t - x \right) + j \left( \operatorname{Arcth} \left( t + x \right) - \operatorname{Arcth} \left( t - x \right) \right) \right).
\]
4.6 The homographic mapping: $h \mapsto (ah + b)(ch + d)^{-1}$

We define the homographic function by the relation:

$$w = \frac{ah + b}{ch + d} = D_{cd}^{ab}(h),$$

(4.3)

where $a, b, c, d$ are arbitrary double numbers which satisfy the condition $ad - bc \neq 0$. The special character of the definition (4.3) resides in the occurrence of the cone $\text{Con}(-d/c)$ on which this mapping is not well defined.

1. The homographic transformation (4.3) leads to a bijective and continuous $H \setminus \text{Con}(-d/c) \to H$ mapping in its domain of definition, which is conformal. Its inverse mapping is also homographic and has the form:

$$h = \frac{dw - b}{a - cw} = D_{c,d}^{-ab}(w) = (D_{cd}^{ab})^{-1}(h), \quad w \notin \text{Con}(a/c).$$

(4.4)

2. The composition of two homographic mappings $D_2 \circ D_1$ is also a homographic mapping. The set of all homographic mappings forms a group which is isomorphic to $\text{SL}(2, H)$.

3. The homographic mapping transforms the hyperbolic circle $H_{S_R}(h_0)$ given by the equation $|h - h_0|^2 = \pm R^2$ into a hyperbolic circle.

4. By introducing the definition of points $h$ and $h^\vee$, which are conjugate relative to the circle $H_{S_R}(h_0)$, similar to the corresponding definition on the complex plane, (conjugate points lie on the same ray which emerges from the center, one being located inside and the other outside the circle and their hyperbolic distances to the center satisfy the relation: $|h - h_0| \cdot |h^\vee - h_0| = R^2$), we can state another geometric property of homographic mappings: the points, which are conjugate relative to $H_{S_R}(h_0)$, are mapped to points which are conjugate relative to the circle $D(\text{HS}_R(h_0))$.

The proof of the latter assertions formally repeat the proof for the similar statements from the theory of homographic functions of complex variable.

![Fig. 17. The mapping of two arcs of hyperbolic circles into arcs of circles by means of the homographic mapping $h \mapsto (2h + j)/(1 - jh)$. The pair of straight lines from the left picture represent the cone $\text{Con}(j)$.

We shall examine, at last, the hyperbolic version of the Zhukovskiy function:

$$Z(h) \equiv \frac{1}{2} \left( h + \frac{1}{h} \right).$$
This transformation twicely maps double plane onto the plane with removed unit square such as depicted in Fig. 18.

Like in the case of complex variable, the Zhukowskiiy mapping has two leaves: the outer part of the unit square and its inner part, which it maps onto the exteriority of the such square. At points $\pm 1, \pm j$ the conformality of functions $Z(h)$ is broken, since at these points we have $Z'(h) = 0$.

5 Solving plane initial-boundary problems of the 2-dimensional field theory

By analogy to the applications of conformal transformations on the standard (elliptic) complex plane, the $h$-analytic functions can be used for solving problems of field theory, which are related to the 2-order wave equation: $\Box_2 \varphi = 0$. The lack of acknowledgement which the hyperbolic conformal mapping receive is firstly related by the non-traditional way in which are posed the initial-boundary problems, which can be solved by the method of hyperbolic conformal transformations. Indeed, the use of conformal transformations in the plane of complex variable for solving problems of elliptic type, which are related to the Laplace equation, relies on the circumstances which were formerly described: the holomorphic mapping $w = f(z)$, whose real part represents the solution of some boundary problem, maps the boundary from outside the sources of the field into the straight line $\text{Re} w = \text{const}$. This requirement reflects the condition of having a constant potential on the boundary of the domain, in which we solve the Laplace equation and guarantees the uniqueness of the solution up to an arbitrary choice of the value of potential on the boundary.

In initial-boundary problems of hyperbolic type we also use another setting of the problem: one examines the Space-Time domain as a multiple-boundary cylinder $D^3 \times R_+$ (or a topologically equivalent to it alias), and there are provided the initial conditions on the initial surface $D^3 \times \{0\}$ (the initial values for the field and its derivatives by time) and the initial-boundary conditions on the flank surface $\partial D^3 \times R_+$ (the values for the field and its derivatives by space coordinate). If the problem is well-posed, then these initial boundary condition data lead a unique solution with good properties at each moment of time $t > 0$. In the 2-dimensional Space-Time, the boundary of the domain represents a time-like rectangle $I \times R_+$ or a topologically equivalent to it figure.

The use of conformal transformations, which are $h$-holomorphic mappings, assumes passing from plane elliptic problems into plane hyperbolic problems. In other words, an $h$-holomorphic mapping $w = f(h)$ represents a solution of some initial-boundary problem, namely the one, for which this mapping transforms the 1-dimensional border from the domain located outside sources

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*In our case, $D^3$ is the 3-dimensional ball, $R_+ = [0, \infty)$*
into the line $\text{Re } w = \text{const}$. It is obvious, that such a posing of the problem is different from the standard one, since the providing of the initial-boundary conditions change to providing the form of the world-surface (lines) of constant potential. This surface has a Space-Time character. In principle, this can be obtained by means of measuring the wave field $\varphi$ at different points of the space at different moments of time by means of an appropriate, sufficiently large number of devices. The event points of Space-Time, given by the readings of the devices provide $\varphi = \text{const}$ and give the needed surface. According to the above considerations, the form of this surface uniquely defines the solution of the wave equation. However, such a posing of the problem is seldom used in practice, since the data are distributed in Space-Time.

As an example, we examine the problem of determining the wave field, which has the constant value $\varphi_0$ on the hyperbolic circle $t^2 - x^2 = R^2$. According to the results of section 4.2, the appropriate solution has the form: $\varphi = \varphi_0 + \ln[(t^2 - x^2)/R^2] = \text{Re}(\ln h + \varphi_0)$. Its 3-dimensional graph and the subsequent time slices are shown in Fig.19.

![Fig. 19. The wave solution, which is constant on the hyperbolic circle ($R = 1, \varphi_0 = 1$). In the right picture there are represented the subsequent time slices of the surface, which is shown in the left picture, for $t = 1, 2, 3, 4$.](image)

### 6 Conclusions

We showed the general features of the fundamental for physics analogy between complex and double numbers. The algebraic description of the geometry and physics in 2-dimensional Space-Time by means of double numbers is not less appealing than describing the problems on the Euclidean plane by means of of complex numbers. We have left non-addressed a series of problems (which essentially have a mathematical character), which appear along this path, giving priority to resemblance more than to rigor. The difficulties encountered in the analysis based on the natural metric (i.e., in the case examined by us, pseudo-Euclidean) topology are well known to the specialists in mathematical Relativity Theory and these (together with the difficulties of psychologic character) were the reason for rather low interest from physicists and mathematicians towards double numbers regarding their applications. In our opinion, these difficulties can be in principle removed, and moreover, what is usually considered to be "difficulty" is in fact the expression of several new fundamental concepts, which the geometry of Space-Time brings to Physics and Mathematics.

The formerly described tight correspondence between the complex and the double planes can be also extended to vector fields. Like for each holomorphic function on the complex plane there corresponds some potential or solenoidal vector field, related to sources, vortices, dipoles, quadrupoles, etc, at the points where the holomorphicity of a $h$-holomorphic function is lost, we can also give sense to some vector field in the 2-dimensional Space-Time, whose singularities now have not elliptic, but hyperbolic properties. Thus, we have the possibility to talk about hyperbolic positive and negative sources, hyperbolic vortices and vortex-wells, multipoles, Zhukowskii hyperbolic func-
tion, etc. The vector fields which are generated by these $h$-holomorphic mappings have remarkable properties, like potentiality and solenoidality, which are obviously apprehended in a hyperbolic meaning.

Another interesting consequence provided by the $h$-holomorphic functions of double variable is related to the possibility of regarding their associated conformal mappings, as changes from non-inertial reference systems in the flat 2-dimensional Space-Time. Then the study of the group of conformal symmetries naturally extends the frames of 2-dimensional Special Relativity Theory, in which there are usually considered only the isometric mappings, regarded as changes between inertial reference systems. Such a possibility (due to the absence of a corresponding infinite-dimensional conformal symmetry group) does not exist in three and four-dimensional pseudo-Euclidean spaces, but still, this exists in two dimensions and also in several three and four-dimensional flat Finsler geometries, e.g., in spaces with Berwald-Moor metric, whose particular case is, in fact, the space of double numbers itself (14-15). The idea of a fundamental symmetry group (linear isometric, and non-linear conformal) was widespread in physics, though the use of the infinite conformal groups was in its essence limited to the case of 2-dimensional geometry with quadratic metrics. While passing from quadratic metric forms of Space-Time to $n$-ary metric forms, this limitation is sometimes removed, and the problem of Finslerian extensions of SRT and GRT becomes actual and promising. We need to emphasize that we talk about extensions, and not about any rejection of the classic representations, since we assume that the symmetry group which lie at the basis of previous physical-mathematical constructions, appear as in some sense secondary (induced) fact of the new infinite-dimensional symmetry groups existence. Due to the limited space offered by a journal article, the authors plan to develop the present topic in a forthcoming paper.

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D.G. Pavlov
Research Institute of Hypercomplex Systems in Geometry and Physics,
Fryazino, Russia.
E-mail: geom2004@mail.ru

S.S. Kokarev
RSEC "Logos", Yaroslavl, Russia,
E-mail: logos-center@mail.ru