Non-Eulerian Dehn-Sommerville relations

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Abstract

The classical Dehn-Sommerville relations assert that the $h$-vector of an Eulerian simplicial complex is symmetric. We establish three generalizations of the Dehn-Sommerville relations: one for the $h$-vectors of pure simplicial complexes, another one for the flag $h$-vectors of graded posets, and yet another one for the toric $h$-vectors of graded posets with restricted singularities. In all of these cases, we express any failure of symmetry in terms of “errors coming from the links.” For simplicial complexes, this further extends Klee’s semi-Eulerian relations.

1 Introduction

In this paper we generalize Dehn-Sommerville relations in three ways: the first one relates to the $h$-vectors of all pure simplicial complexes, the second one deals with the flag $h$-vectors of all graded posets, and the third one concerns the toric $h$-vectors.

In 1964, Klee defined Eulerian and Semi-Eulerian simplicial complexes and proved that their $h$-vectors are almost symmetric, see [Kle64a]. More precisely, the $h$-vector of a $(d-1)$-dimensional Eulerian simplicial complex $\Delta$ (for example, a simplicial sphere) satisfies $h_i(\Delta) = h_{d-i}(\Delta)$ for all $i$, while the $h$-vector of a $(d-1)$-dimensional semi-Eulerian complex $\Gamma$ (such as the boundary of a simplicial manifold) satisfies $h_{d-i}(\Gamma) = h_i(\Gamma) + (-1)^i \binom{d}{i} [\tilde{\chi}(\Gamma) - (-1)^{d-1}]$, where $\tilde{\chi}$ is the reduced Euler characteristic of $\Gamma$. Since then these relations have played a very important role in the $f$-vector theory, e.g., in the proof of the Upper Bound Theorem, see [Kle64b], [Sta75], and [Nov98]. In 2012, Novik and Swartz derived similar results for pseudo-manifolds with isolated singularities as defined in [NS12].

Stanley [Sta87] (see also [Sta94]) extended Klee’s definition of Eulerian and semi-Eulerian complexes to (finite) graded partially ordered sets (posets for short). The flag $f$-vector of a poset $P$, denoted $\alpha_P$, counts the number of chains in a poset according to the ranks they contain. The flag $h$-vector of $P$, denoted $\beta_P$, is the image of the flag $f$-vector
under a certain invertible linear transformation. Bayer and Billera proved the Dehn-Sommerville relations on flag $f$-vectors of Eulerian posets, see [BB84] (also see [Sta12, Thm. 3.16.6] for the proof of the flag $h$-vector version). The Bayer-Billera relations played an instrumental role in Fine’s definition of the cd-index (see [BK91, Sta94]).

Stanley also introduced a notion of toric $h$- and $g$-vectors of posets and proved that the toric $h$-vector of any Eulerian poset is symmetric. This result was extended by Swartz [Swa09] to semi-Eulerian posets.

Here we provide generalizations of these three types of Dehn-Sommerville relations. Our results can be summarized as follows; for all undefined terminology and notations, see Sections 2 and 4.

- For an arbitrary $(d-1)$-dimensional pure simplicial complex $\Delta$, we express $h_{d-i}(\Delta) - h_i(\Delta)$ in terms of the Euler characteristics of links of faces, see Theorem 3.1.

- For an arbitrary graded poset $P$ of rank $d+1$ and $S \subseteq [d]$, we express $\beta_P(S) - \beta_P([d] - S)$ in terms of the Möbius values of chains (see the definition in Section 5) that are disjoint from $S$, see Theorem 5.2.

- For finite posets, we define the notion of $j$-Singular posets (see Section 7) such that
  
  - $j = -1$ recovers Eulerian posets;
  - $j = 0$ recovers semi-Eulerian posets;
  - $j = 1$ is analogous to complexes with isolated singularities (see the definition in Section 6).

- Extending the results of Stanley and Swartz, for a 1-Singular poset $P$ of rank $d+1$, we express $\hat{h}(P, x) - x^d \hat{h}(P, \frac{1}{x})$ in terms of the Möbius functions of intervals $[s, t]$ in $P$ of length greater than or equal to $d-1$, see Theorem 6.6. Here $\hat{h}$ denotes the toric $h$-polynomial.

- We extend this result even further and obtain a similar formula for a $j$-Singular poset $P$ with $j < \lfloor \frac{d}{2} \rfloor$ (see Theorem 7.9 and 7.13).

The structure of this paper is as follows: Section 2 introduces some basic results and definitions pertaining to simplicial complexes. Section 3 is devoted to establishing the generalization of Dehn-Sommerville relations for pure simplicial complexes. Section 4 discusses basics of toric $h$- and $g$-vectors of posets as well as of flag $f$- and flag $h$-vectors. Section 5 proves the flag Dehn-Sommerville relations for graded posets. Sections 6 and 7 establish the toric generalizations of Dehn-Sommerville formulas. Our proofs build on methods used by Klee, Stanley, and Swartz.
2 Basics of simplicial complexes

In this section we review some definitions pertaining to simplicial complexes. Let $V$ be a finite set. A simplicial complex $\Delta$ with vertex set $V$ is a collection of subsets of $V$ that is closed under inclusion. We call each element of $\Delta$ a face of $\Delta$, and each face $F \in \Delta$ has a dimension defined by $\dim(F) = |F| - 1$. Similarly, the dimension of $\Delta$ is defined by $\dim(\Delta) = \max\{\dim F : F \in \Delta\}$. If all maximal faces of $\Delta$ (with respect to inclusion) have the same dimension, then $\Delta$ is called pure. We denote the collection of faces of $\Delta$ of a specific dimension $i$ by

$$\Delta_i := \{F \in \Delta : \dim(F) = i\}.$$ 

Lastly, the link of a face $F$ of $\Delta$, denoted $lk_\Delta F$, is defined by

$$lk_\Delta F := \{G \in \Delta : F \cup G \in \Delta \text{ and } F \cap G = \emptyset\}.$$ 

Let $\Delta$ be a simplicial complex of dimension $d - 1$. The $f$-vector of $\Delta$ is defined by $f(\Delta) := (f_{-1}(\Delta), f_0(\Delta), f_1(\Delta), \ldots, f_{d-1}(\Delta))$, where $f_i(\Delta) = |\Delta_i|$. We further define the $h$-vector of $\Delta$ by $h(\Delta) := (h_0(\Delta), h_1(\Delta), \ldots, h_d(\Delta))$, with entries determined by the equation

$$\sum_{i=0}^{d} h_i(\Delta) x^{d-i} = \sum_{i=0}^{d} f_{i-1}(\Delta) (x-1)^{d-i}.$$ 

For the remainder of this section, we will assume that $\Delta$ is a pure simplicial complex of dimension $d - 1$.

Each simplicial complex $\Delta$ admits a geometric realization $||\Delta||$ that contains a geometric $i$-simplex for each $i$-face of $\Delta$. We say that $\Delta$ is a simplicial sphere (manifold, respectively) if $||\Delta||$ is homeomorphic to a sphere (manifold, respectively).

The (reduced) Euler characteristic of $\Delta$ is

$$\tilde{\chi}(\Delta) := \sum_{i=-1}^{d-1} (-1)^i f_i(\Delta),$$

and by the Euler-Poincaré formula, $\tilde{\chi}(\Delta)$ is a topological invariant of $\Delta$, or more precisely, of its geometric realization $||\Delta||$. For instance, if $\Gamma$ is an $(i - 1)$-dimensional simplicial sphere, then $\tilde{\chi}(\Gamma) = (-1)^{i-1}$.

Given two simplicial complexes $\Delta_1$ and $\Delta_2$ on disjoint vertex sets, their simplicial join, $\Delta_1 \ast \Delta_2$, is defined as

$$\Delta_1 \ast \Delta_2 := \{F \cup G : F \in \Delta_1, G \in \Delta_2\}.$$ 

In particular, $\Delta_1 \ast \Delta_2$ is a simplicial complex of dimension $\dim \Delta_1 + \dim \Delta_2 + 1$.

Central to many classifications of simplicial complexes is the notion of the link of a face having the same Euler characteristic as that of a sphere of the appropriate dimension.
To that end, we measure potential failures of this condition by defining an error function \( \varepsilon_\Delta(F) \) on faces \( F \) of a pure \((d-1)\)-dimensional simplicial complex \( \Delta \) as

\[
\varepsilon_\Delta(F) := \tilde{\chi}(\text{lk}_\Delta F) - (-1)^{d-1-|F|}
\]

(note that \( \dim(\text{lk}_\Delta F) = d - 1 - |F| \), so \((-1)^{d-1-|F|}\) is the same as the reduced Euler character of a sphere of dimension \( \dim(\text{lk}_\Delta F) \)). In addition, we form the set of faces with non-trivial error as

\[
\mathcal{E}(\Delta) := \{ F \in \Delta : \varepsilon(F) \neq 0 \}.
\]

Lastly, we say that \( \Delta \) is **Eulerian** if \( \mathcal{E}(\Delta) = \emptyset \), and that it is **semi-Eulerian** if \( \mathcal{E}(\Delta) = \{\emptyset\} \). In line with these definitions, we refer to \( \mathcal{E}(\Delta) \) as the **non-Eulerian** part of \( \Delta \).

**Example 2.1.** If \( \Delta \) is a simplicial sphere, then \( \mathcal{E}(\Delta) = \emptyset \). More generally, if \( \Delta \) is a simplicial manifold, then \( \mathcal{E}(\Delta) \subseteq \{\emptyset\} \).

### 3 Dehn-Sommerville relations

The main result of this section is the following generalization of Dehn-Sommerville relations (see \[Kle64b\]) to all pure simplicial complexes.

**Theorem 3.1.** Let \( \Delta \) be a pure \((d-1)\)-dimensional simplicial complex. Then

\[
h_{d-j}(\Delta) - h_j(\Delta) = (-1)^j \sum_{F \in \Delta} \binom{d-1-|F|}{j} \varepsilon_\Delta(F) \quad \text{for } j = 0, \ldots, d.
\]

**Proof.** Note that \((-1)^{d-1-i} f_{i-1}(\Delta) - \sum_{F \in \Delta_{i-1}} (-1)^{d-1-|F|} = 0\), and hence

\[
\sum_{F \in \Delta_{i-1}} \tilde{\chi}(\text{lk}_\Delta F) = (-1)^{d-1-i} f_{i-1}(\Delta) + \sum_{F \in \Delta_{i-1}} \left[ \tilde{\chi}(\text{lk}_\Delta F) - (-1)^{d-1-|F|} \right]. \tag{3.1}
\]

On the other hand, since each \((j-1)\)-dimensional face of \( \Delta \) contains exactly \( \binom{j}{i} \) faces of dimension \( i-1 \),

\[
\sum_{F \in \Delta_{i-1}} \tilde{\chi}(\text{lk}_\Delta F) = \sum_{j=i}^d (-1)^{j-i-1} \binom{j}{i} f_{j-1}(\Delta). \tag{3.2}
\]

Setting the right-hand sides of (3.1) and (3.2) equal to each other and multiplying throughout by \((-1)^{d-1-i}\) yields

\[
\sum_{j=i}^d (-1)^{d-j} \binom{j}{i} f_{j-1}(\Delta) = f_{i-1}(\Delta) + (-1)^{d-1-i} \sum_{F \in \Delta_{i-1}} \varepsilon_\Delta(F). \tag{3.3}
\]
Now we multiply both sides of (3.3) by \((\lambda - 1)^{d-i}\) and sum the result over \(i\):

\[
\sum_{i=0}^{d} \left[ \sum_{j=i}^{d} (-1)^{d-j} \binom{j}{i} f_{j-1}(\Delta) \right] (\lambda - 1)^{d-i} = \sum_{i=0}^{d} \left[ f_{i-1}(\Delta) + (-1)^{d-1-i} \sum_{F \in \Delta_{i-1}} \varepsilon_{\Delta}(F) \right] (\lambda - 1)^{d-i}. \tag{3.4}
\]

The left hand-side of equation (3.4) may be rewritten as

\[
\sum_{i=0}^{d} \left[ \sum_{j=i}^{d} (-1)^{d-j} \binom{j}{i} f_{j-1}(\Delta) \right] (\lambda - 1)^{d-i} = \sum_{j=0}^{d} (-1)^{d-j}(\lambda - 1)^{d-j} f_{j-1}(\Delta) \left( \sum_{i=0}^{d} \binom{j}{i} (\lambda - 1)^{j-i} \right)
\]

\[
= \sum_{j=0}^{d} f_{j-1}(\Delta)(1 - \lambda)^{d-j}\lambda^j
\]

\[
= \sum_{j=0}^{d} h_j(\Delta)\lambda^j.
\]

The right hand-side of equation (3.4) can be broken up as

\[
\sum_{i=0}^{d} \left[ f_{i-1}(\Delta) + (-1)^{d-1-i} \sum_{F \in \Delta_{i-1}} \varepsilon_{\Delta}(F) \right] (\lambda - 1)^{d-i} = \sum_{i=0}^{d} f_{i-1}(\Delta)(\lambda - 1)^{d-i}
\]

\[
+ \sum_{i=0}^{d} (-1)^{d-1-i} \left( \sum_{F \in \Delta_{i-1}} \varepsilon_{\Delta}(F) \right) (\lambda - 1)^{d-i}.
\]

We will analyze each of these terms on the right independently. Firstly,

\[
\sum_{i=0}^{d} f_{i-1}(\Delta)(\lambda - 1)^{d-i} = \sum_{i=0}^{d} h_i(\Delta)\lambda^{d-i}.
\]

For the second term,

\[
\sum_{i=0}^{d} (-1)^{d-1-i} \left( \sum_{F \in \Delta_{i-1}} \varepsilon_{\Delta}(F) \right) (\lambda - 1)^{d-i}
\]

\[
= \sum_{i=0}^{d} (-1)^{d-1-i} \left( \sum_{F \in \Delta_{i-1}} \varepsilon_{\Delta}(F) \right) \left( \sum_{j=0}^{d-i} (-1)^{d-i-j} \binom{d-i}{j} \lambda^j \right)
\]

\[
= \sum_{i=0}^{d} \left( \sum_{F \in \Delta_{i-1}} \varepsilon_{\Delta}(F) \right) \left( \sum_{j=0}^{d-i} (-1)^{j-i} \binom{d-i}{j} \lambda^j \right)
\]

\[
= \sum_{j=0}^{d} (-1)^{j-i-1} \left( \sum_{i=0}^{d} \binom{d-i}{j} \left( \sum_{F \in \Delta_{i-1}} \varepsilon_{\Delta}(F) \right) \right) \lambda^j.
\]
By equating coefficients in equation (3.4) we obtain

\[ h_{d-j}(\Delta) - h_j(\Delta) = (-1)^j \left( \sum_{i=0}^{d} \binom{d-i}{j} \sum_{F \in \Delta_{i-1}} \varepsilon_\Delta(F) \right), \]

and the summations on the right may be re-written as in the statement of the theorem. \( \square \)

Since \( \varepsilon_\Delta(F) = 0 \) unless \( F \in \mathcal{E}(\Delta) \), we have the following corollary that phrases the relationship between \( h_j(\Delta) \) and \( h_{d-j}(\Delta) \) in terms of the non-Eulerian part of \( \Delta \).

**Corollary 3.2.** Let \( \Delta \) be a pure \((d-1)\)-dimensional simplicial complex. Then

\[ h_{d-j}(\Delta) - h_j(\Delta) = (-1)^j \sum_{F \in \mathcal{E}(\Delta)} \binom{d - |F|}{j} \varepsilon_\Delta(F). \]

**Example 3.3.** When \( \Delta \) is semi-Eulerian (so that \( \mathcal{E}(\Delta) = \{\emptyset\} \)),

\[ \sum_{F \in \mathcal{E}(\Delta)} \binom{d - |F|}{j} \varepsilon_\Delta(F) = \binom{d}{j} \left[ \tilde{\chi}(\Delta) - (-1)^{d-1} \right] \]

and

\[ h_{d-j}(\Delta) - h_j(\Delta) = (-1)^j \left( \binom{d}{j} \left[ \tilde{\chi}(\Delta) - (-1)^{d-1} \right] \right). \]

Thus, in this case Theorem 3.1 reduces to Klee’s Dehn-Sommerville equations in [Kle64b].

**Example 3.4.** As for \( h \)-vectors of complexes with \( \mathcal{E}(\Delta) \) containing faces of dimension larger than \(-1\), consider the case in which \( \|\Delta\| = S_1 \ast \|M\| \), where \( S_1 \) denotes the 1-dimensional sphere and \( M \) is some \((d-3)\)-dimensional simplicial manifold with \( \varepsilon_\Delta(\emptyset) \neq 0 \). Then \( \mathcal{E}(\Delta) \) forms a cycle (in the graph theory sense), say of length \( n \), and

\[
\sum_{i=0}^{d} \left( \sum_{F \in \mathcal{E}(\Delta)_{i-1}} \varepsilon_\Delta(F) \right) \binom{d-i}{j} = \binom{d}{j} \left( \tilde{\chi}(\Delta) - (-1)^{d-1} \right) + n \binom{d-1}{j} \left( \tilde{\chi}(\Delta) - (-1)^{d-2} \right) + n \binom{d-2}{j} \left( \tilde{\chi}(\Delta) - (-1)^{d-3} \right)
\]

\[ = (\tilde{\chi}(M) + (-1)^d) \left( \binom{d}{j} - n \binom{d-2}{j-1} \right), \]

and so

\[ h_{d-j}(\Delta) - h_j(\Delta) = (-1)^j \left( \tilde{\chi}(M) + (-1)^d \right) \left( \binom{d}{j} - n \binom{d-2}{j-1} \right) \]

for \( j = 0, \ldots, d \). In particular, if \( M \) is a triangulation of the torus, then \( d = 5 \) and \( \tilde{\chi}(M) = -1 \), and so

\[ h_{5-j}(\Delta) - h_j(\Delta) = (-1)^j (-2) \left( \binom{5}{j} - n \binom{3}{j-1} \right). \]
4 Preliminaries on posets and their invariants

The rest of the paper is devoted to Dehn-Sommerville relations for graded posets. In this section we review some relevant definitions, most notably of flag $h$-vectors and toric $h$-vectors.

Let $P$ be a finite graded poset with unique bottom and top elements $\hat{0}$ and $\hat{1}$. If $P$ has only one element, i.e., when $\hat{0} = \hat{1}$, then we call $P$ the trivial poset and denote it as $P = \mathbb{1}$. Let $\rho : P \to \mathbb{N}$ be the rank function. The rank of $P$, $\rho(P)$, is defined as $\rho(\hat{1})$.

Let $\tilde{P} = \{[\hat{0}, t] : t \in P\}$ be the poset of lower intervals of $P$ ordered by inclusion. Define two polynomials $\hat{h}(P, x)$ and $\hat{g}(P, x)$ recursively as follows.

$\hat{h}(\mathbb{1}, x) = \hat{g}(\mathbb{1}, x) = 1$.

If $P$ has rank $d + 1$, then $\deg \hat{h}(P, x) = d$. We first write

$$\hat{h}(P, x) = \hat{h}_d + \hat{h}_{d-1}x + \hat{h}_{d-2}x^2 + \cdots + \hat{h}_0x^d.$$  

We then define $\hat{g}(P, x)$ as

$$\hat{g}(P, x) := \hat{h}_d + (\hat{h}_{d-1} - \hat{h}_d)x + (\hat{h}_{d-2} - \hat{h}_{d-1})x^2 + \cdots + (\hat{h}_{d-m} - \hat{h}_{d-m+1})x^m,$$

where $m = \deg \hat{g}(P, x) = \lfloor \frac{d}{2} \rfloor$.

Finally, for a poset $P$ of rank $d + 1$, define

$$\hat{h}(P, x) := \sum_{Q \in \tilde{P}, Q \neq P} \hat{g}(Q, x)(x - 1)^{\rho(Q)}.$$

The coefficients of these polynomials, arranged as vectors, are called the toric $h$-vector and the toric $g$-vector, respectively.

Remark 4.1. We follow the convention of Swartz [Swa09], and so our $\hat{h}_i$ is $\hat{h}_{d-i}$ in Stanley’s paper [Sta87].

Let $\mu_P$ denote the Möbius function of poset $P$. If for all proper intervals $[s, t] \subsetneq P$, $\mu_P(s, t) = (-1)^{\rho(t) - \rho(s)}$, then $P$ is called semi-Eulerian. If in addition, $\mu_P(0, 1) = (-1)^{\rho(P)}$, then $P$ is Eulerian.

For each poset $P$, there is a simplicial complex associated to it which is called the (reduced) order complex of $P$ and denoted by $O(P)$, see for instance [Bjo95]. The complex $O(P)$ has the set $P \setminus \{\hat{0}, \hat{1}\}$ as its vertices, and the (finite) chains in the open interval $(\hat{0}, \hat{1})$ as its faces. Therefore if $\rho(P) = d + 1$, then $\dim O(P) = d - 1$.

Every face $F \in O(P)$ of size $k$ corresponds to a chain $C = \{t_1 < t_2 < \cdots < t_k\}$ in $P \setminus \{\hat{0}, \hat{1}\}$ (we sometimes denote $F \in O(P)$ as $F_C$ to emphasize this).

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Remark 4.2 (Relations between $\mu$ and $\tilde{\chi}$). For a poset $P$ and a chain $C$ in it, the following formulas hold:

$$\tilde{\chi}(O(P)) = \mu_P(\hat{0}, \hat{1}); \quad (4.1)$$

$$\tilde{\chi}(\text{lk}_{O(P)} F_C) = (-1)^{|F|} \mu_P(\hat{0}, t_1) \cdot \mu_P(t_1, t_2) \cdots \mu_P(t_{k-1}, t_k) \mu_P(t_k, \hat{1}). \quad (4.2)$$

The first formula is well known, for instance, see [Sta12]. To prove the second equality, note that $\text{lk}_{O(P)} F_C = O(\hat{0}, t_1) \ast O(t_1, t_2) \ast \cdots \ast O(t_{k-1}, t_k) \ast O(t_k, \hat{1})$, where $O(s, t)$ is the order complex of the open interval $(s, t) = \{x : s < x < t\}$. Using this together with the fact that $\tilde{\chi}(\Delta_1 \ast \Delta_2) = (-1) \tilde{\chi}(\Delta_1) \tilde{\chi}(\Delta_2)$ leads to

$$\tilde{\chi}(\text{lk}_{O(P)} F_C) = (-1)^{|F|} \tilde{\chi}(O(\hat{0}, t_1)) \cdot \tilde{\chi}(O(t_1, t_2)) \cdots \tilde{\chi}(O(t_{k-1}, t_k)) \tilde{\chi}(O(t_k, \hat{1})).$$

This together with (4.1) implies (4.2).

A certain refinement of $f$- and $h$-vectors called the flag $f$- and flag $h$-vectors, respectively, has received a lot of attention, see [BB84]. If $P$ is a graded poset of rank $d + 1$, and $S \subseteq [d]$, we define $P_S = \{x \in P : \rho(x) \in S \cup \{0, d + 1\}\}$ considered as a subposet of $P$, and we let $\alpha_P(S)$ be the number of maximal chains in $P_S$. The poset $P_S$ is called the $S$-rank selected subposet of $P$, and the function

$$S \mapsto \alpha_P(S)$$

is the flag $f$-vector of $P$. We also consider the function

$$S \mapsto \beta_P(S), \quad \beta_P(S) = \sum_{T \subseteq S} (-1)^{|S| - |T|} \alpha_P(T)$$

called the flag $h$-vector of $P$.

A few remarks are in order. By the inclusion-exclusion principle,

$$\alpha_P(S) = \sum_{T \subseteq S} \beta_P(T).$$

Also, by Phillip-Hall Theorem (see for example [Sta12 Prop 3,8,7]), for $S \neq \emptyset$,

$$\mu_{P_S}(\hat{0}, \hat{1}) = \sum_{C \in C(P_S)} (-1)^{|C| - 1} \quad (4.3)$$

where $C(P_S)$ denotes the set of all chains in $P_S$ that contain $\hat{0}$ and $\hat{1}$. In addition,

$$f_{i-1}(O(P)) = \sum_{|S| = i} \alpha_P(S), \quad h_i(O(P)) = \sum_{|S| = i} \beta_P(S),$$

so the flag $f$- and $h$-vectors of $P$ are refinements of the ordinary $f$- and $h$-vectors of its order complex $O(P)$. 

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5 Flag Dehn-Sommervile relations

The goal of this section is to generalize the Bayer-Billera theorem [BBS4] (see also Stanley’s book [Sta12, Cor. 3.16.6]) on flag $h$-vectors on Eulerian posets to all graded posets. This result states that if $P$ is an Eulerian poset of rank $d + 1$, then for all $S \subseteq [d],$

$$\beta_P(S) = \beta_P([d] - S).$$

Throughout this section, $P$ is a graded poset of rank $d + 1$.

Similar to the simplicial case, we need to define an error function. Let $C = \{ \hat{0} = t_0 < t_1 < \cdots < t_i = \hat{1} \}$ be a chain in $P$. Define

$$\mu_P(C) := \mu_P(\hat{0}, t_1)\mu_P(t_1, t_2) \cdots \mu_P(t_{i-1}, \hat{1}),$$

and

$$\varepsilon_P(C) := (-1)^{|C|}[\mu_P(C) - (-1)^{d+1}].$$

We call $C \mapsto \varepsilon_P(C)$ the error function for chains in a poset $P$. Comparing this new error function with $\varepsilon_{O(P)}(\cdot)$ and using (4.2), we obtain:

Remark 5.1. $\varepsilon_P(C) = \varepsilon_{O(P)}(F_C)$.

The main result of this section is the following.

Theorem 5.2. Let $P$ be a poset of rank $d + 1$. Let $S \subseteq [d]$ and $S^c = [d] - S$. Let $C(P)$ be the set of chains in $P$ that contain $\hat{0}$, $\hat{1}$, and let $P'_S = P_S - \{\hat{0}, \hat{1}\}$. Then

$$\beta_P(S) - \beta_P(S^c) = (-1)^{|S| - 1} \sum_{C \in C(P) \cap P'_S = \emptyset} \varepsilon_P(C).$$

Proof. By [Sta12, Prop 3.16.4],

$$\mu_{P_S}(\hat{0}, \hat{1}) = \sum_{C \in C(P) \cap P'_S = \emptyset} (-1)^{|C|}\mu_P(C)$$

$$= \sum_{C \in C(P) \cap P'_S = \emptyset} (-1)^{|C|+d+1} + \sum_{C \in C(P) \cap P'_S = \emptyset} \left[(-1)^{|C|}\mu_P(C) - (-1)^{|C|+d+1}\right].$$

Furthermore, by Equation (4.3),

$$\sum_{C \in C(P) \cap P'_S = \emptyset} (-1)^{|C|+d+1} = (-1)^d \mu_{P_S}(\hat{0}, \hat{1}).$$
Therefore (5.1) implies

$$
\mu_{PS}(\hat{0}, \hat{1}) = (-1)^d \mu_{P_{S^c}}(\hat{0}, \hat{1}) + \sum_{C \in \mathcal{C}(P) \atop C \cap P_S = \emptyset} \varepsilon_P(C).
$$

Hence

$$
\beta_P(S) = (-1)^{|S|-1} \mu_{PS}(\hat{0}, \hat{1}) = (-1)^{d+|S|-1} \mu_{P_{S^c}}(\hat{0}, \hat{1}) + (-1)^{|S|-1} \sum_{C \in \mathcal{C}(P) \atop C \cap P_S = \emptyset} \varepsilon_P(C)
$$

$$
= (-1)^{d+|S|+|S^c|} \beta_P(S^c) + (-1)^{|S|-1} \sum_{C \in \mathcal{C}(P) \atop C \cap P_S^c = \emptyset} \varepsilon_P(C)
$$

$$
= \beta_P(S^c) + (-1)^{|S|-1} \sum_{C \in \mathcal{C}(P) \atop C \cap P_S^c = \emptyset} \varepsilon_P(C),
$$

and the statement follows. \qed

The flag $f$- and $h$-vectors are also defined for a specific type of simplicial complexes, called the balanced simplicial complexes. These complexes were introduced by Stanley in [Sta79]. A $(d - 1)$-dimensional pure simplicial complex $\Delta$ is balanced if it has a vertex coloring $k : V \to [d]$ such that no two vertices of the same color are connected by an edge. For any subset of colors $S \subseteq [d]$, define $f_S(\Delta)$ as the number of faces in $\Delta$ with $\kappa(F) = S$. The numbers $f_S(\Delta)$ are called the flag $f$-numbers of $\Delta$ and the collection $(f_S(\Delta))_{S \subseteq [d]}$ is called the flag $f$-vector of $\Delta$. Similarly, the flag $h$-numbers of $\Delta$ are defined as

$$
h_T(\Delta) = \sum_{S \subseteq T} (-1)^{|T|-|S|} f_S(\Delta) \quad \text{for } T \subseteq [d].
$$

Note that the flag $f$- and $h$-numbers refine the ordinary $f$- and $h$-numbers:

$$
f_{i-1}(\Delta) = \sum_{S \subseteq [d], |S| = i} f_S(\Delta) \quad \text{and} \quad h_j(\Delta) = \sum_{T \subseteq [d], |T| = j} h_T(\Delta).
$$

Note also that if $\Delta = O(P)$ for a graded poset $P$ of rank $d + 1$, then $\Delta$ is balanced: it is equipped with the coloring that maps every vertex to the rank of the corresponding element in $P$. Furthermore, by definitions,

$$
h_S(O(P)) = \beta_P(S).
$$

Thus the following corollary holds.
Corollary 5.3. For all $S \subseteq [d],$
\[ h_S(O(P)) - h_{S^c}(O(P)) = (-1)^{|S|-1} \sum_{F \not\in O(P)S - \{\emptyset\}} \varepsilon_O(P)(F). \quad (5.2) \]

Observe that Corollary 5.3 refines Theorem 3.1 in the case that $\Delta$ is an order complex. Indeed, summing eq. (5.1) over all subsets $S \subseteq [d]$ of size $i$, we obtain
\[ \sum_{|S|=i} h_S(O(P)) = \sum_{|S|=i} h_{S^c}(O(P)) = (-1)^{i-1} \sum_{F \in O(P)} \sum_{S \subseteq [d]-\kappa(F), |S|=i} \varepsilon_O(P)(F) = (-1)^{i-1} \sum_{F \in O(P)} \left( d - \frac{|F|}{i} \right) \varepsilon_O(P)(F). \]

6 Posets with isolated singularities

Stanley extended Dehn-Sommerville relations for Eulerian simplicial complexes to the generality of toric $h$-vectors of Eulerian posets. The goal of this and the following sections is to further generalize these relations to more general posets.

We start by defining the error function for intervals in posets. Let $P$ be a graded poset of rank $(d + 1)$ and let $[s, t]$ be an interval in $P$. The error of $[s, t]$ is defined as
\[ e_P([s, t]) := \mu_P(s, t) - (-1)^{\rho(t) - \rho(s)}. \]

From now on we will use $e_P(s, t)$ as the abbreviation for $e_P([s, t])$.\footnote{We have already defined the error function for chains $\varepsilon_P(C)$, but to study toric vectors it is easier to use interval errors rather than link errors. The connection between the two will be discussed later in the proof of Corollary 7.11.}

Definition 6.1. A graded poset $P$ with $\rho(P) = d + 1$ has singularities of rank 1 or is $1$-$\text{Sing}$ if all intervals $[s, t]$ in $P$ of length $\rho(t) - \rho(s) \leq d - 1$ are Eulerian.

Proposition 6.2. A poset $P$ of rank $\rho(P) = d + 1$ is $1$-$\text{Sing}$ if and only if its reduced order complex $O(P)$ satisfies the following condition: for all faces $F \in O(P)$ with $\dim(F) \geq 1$, $\tilde{\chi}(lk_{O(P)} F) = (-1)^{d-1-|F|}$.

We omit the proof as we will prove a generalization of this result in Proposition 7.3. Our work in the rest of this section is motivated by the following theorem of Stanley \cite{Sta87} and its generalization due to Swartz (see \cite{Swa09} Theorem 3.15).

Theorem 6.3 (Stanley). Let $P$ be an Eulerian poset of rank $d + 1$. Then $\hat{h}_i(P) = \hat{h}_{d-i}(P)$ for all $0 \leq i \leq d$.\footnote{We have already defined the error function for chains $\varepsilon_P(C)$, but to study toric vectors it is easier to use interval errors rather than link errors. The connection between the two will be discussed later in the proof of Corollary 7.11.}
Theorem 6.4 (Swartz). Let $P$ be a semi-Eulerian poset of rank $d+1$ and let $O(\mathcal{P})$ be its reduced order complex. Then for all $0 \leq i \leq d$,

$$\hat{h}_{d-i}(P) - \hat{h}_{i}(P) = (-1)^{d-i+1} \binom{d}{i} \tilde{\chi}(O(\mathcal{P})) - (-1)^{d-1} = (-1)^{d-i+1} \binom{d}{i} \cdot e_{P}(0, \hat{1}).$$

Remark 6.5. The formula given by Swartz in [Swa09, Theorem 3.15] is equivalent to the statement above. Indeed, when $d$ is even, $P$ is Eulerian, and so the right hand-side is zero. If $d$ is odd, then $d - i + 1$ and $i$ have the same parity and the formula above agrees with the one in Swartz’s Theorem 3.15.

Using ideas from Swartz’s and Stanley’s proofs, we establish the following generalization for 1-Sing posets. For the rest of this section, we let $y = x - 1$.

Theorem 6.6. Let $P$ be a graded 1-Sing poset, and let $\rho(\mathcal{P}) = d + 1$. Then for $i > \lfloor \frac{d}{2} \rfloor$,

$$\hat{h}_{d-i}(P) - \hat{h}_{i}(P) = (-1)^{d-i+1} \left[ \left( \binom{d}{i} e_{P}(0, \hat{1}) + \binom{d}{i} \sum_{\rho(t) = d} e_{P}(0, t) + \binom{d-1}{i-1} \sum_{\rho(s) = 1} e_{P}(s, \hat{1}) \right) \right].$$

Proof. The left hand-side is the coefficient of $x^i$ in the polynomial $\hat{h}(P) - x^d \cdot \hat{h}(P, 1/x)$. We first prove the following lemma related to this polynomial. From now on we use $\hat{h}(P)$ to abbreviate $\hat{h}(P, x)$ and $\hat{g}(P)$ to abbreviate $\hat{g}(P, x)$.

Lemma 6.7. Let $P$ be a graded poset with $\rho(\mathcal{P}) = d + 1$. Then

$$\hat{h}(P) - x^d \cdot \hat{h}(P, 1/x) = -\left[ \mu_{P}(0, \hat{1}) - (-1)^{d+1} \right] y^d + \sum_{Q=\{0, q\} \in \mathcal{P}} \left[ \left( -y^{d-\rho(Q)} \left( \hat{g}(Q) + y \hat{h}(Q) \right) \cdot \mu_{P}(q, \hat{1}) \right) - \left( -y^{d-\rho(Q)} \hat{g}(Q, 1/x) \cdot x^{\rho(Q)} \right) \right].$$

(6.1)

Proof. By definitions, for $P \neq \mathbf{1}$,

$$x^d \cdot \hat{h}(P, 1/x) = \sum_{Q \in P, Q \neq P} (-y)^{d-\rho(Q)} \hat{g}(Q, 1/x) \cdot x^{\rho(Q)},$$

(6.2)

and equivalently,

$$\hat{h}(P) = \sum_{Q \in \mathcal{P}, Q \neq P} \hat{g}(Q) y^{d-\rho(Q)}.$$  

(6.3)

Multiplying equation (6.3) by $y$ and adding $\hat{g}(P)$ to both sides, we obtain that for $P \neq \mathbf{1}$,

$$\hat{g}(P) + y \hat{h}(P) = \sum_{Q \in \mathcal{P}} \hat{g}(Q) y^{\rho(P)-\rho(Q)}.$$
Therefore for $P \neq 1$,
\[ y^{-\rho(P)} \cdot (\hat{g}(P) + y\hat{h}(P)) = \sum_{Q \in \hat{P}} \hat{g}(Q) y^{-\rho(Q)}. \]

By Möbius inversion,
\[ \hat{g}(P) y^{-\rho(P)} = \mu_P(\hat{0}, \hat{1}) + \sum_{Q=[0,q] \in \hat{P}} y^{-\rho(q)} \cdot (\hat{g}(Q) + y\hat{h}(Q)) \cdot \mu_P(q, \hat{1}). \quad (6.4) \]

Multiplying \((6.4)\) by $y^{\rho(P)}$ and then subtracting $\hat{g}(P) + y\hat{h}(P)$ yields

\[ -y\hat{h}(P) = \mu_P(\hat{0}, \hat{1}) y^{\rho(P)} + \sum_{Q=[0,q] \in \hat{P}} y^{\rho(P)-\rho(q)} (\hat{g}(Q) + y\hat{h}(Q)) \cdot \mu_P(q, \hat{1}), \]

and so
\[ \hat{h}(P) = -\mu_P(\hat{0}, \hat{1}) y^d - \sum_{Q=[0,q] \in \hat{P}} y^{d-\rho(q)} (\hat{g}(Q) + y\hat{h}(Q)) \cdot \mu_P(q, \hat{1}). \]

This, together with equation \((6.2)\), proves the lemma. \( \square \)

Next we prove the following lemma, which helps us further simplify equation \((6.1)\) for semi-Eulerian posets.

**Lemma 6.8.** Let $Q$ be a semi-Eulerian poset with $\rho(Q) = r + 1$, let $s = \left\lceil \frac{r}{2} \right\rceil$, and let $y = x - 1$. Then

\[ \hat{g}(Q) + y\hat{h}(Q) = x^{\rho(Q)} \hat{g}(Q, 1/x) + \sum_{k=r-s+1}^{r+1} (-1)^{r-k} \binom{r+1}{k} e_Q(\gamma_k) \cdot x^k, \]

where $\sum^*$ means that, if $r$ is odd, then there is an extra summand, $\frac{1}{2} \gamma_k x^k$, for $k = r - s$.

**Proof.** Recall that $\hat{h}(Q) = \hat{h}_r + \hat{h}_{r-1} x + \cdots + \hat{h}_0 x^r$. This together with the definition of $\hat{g}(Q)$ implies

\[ \hat{g}(Q) + y\hat{h}(Q) = (\hat{h}_{r-s} - \hat{h}_{r-s-1}) x^{s+1} + (\hat{h}_{r-s-1} - \hat{h}_{r-s-2}) x^{s+2} + \cdots + (\hat{h}_1 - \hat{h}_0) x^r + \hat{h}_0 x^{r+1}, \]

while

\[ x^{\rho(Q)} \hat{g}(Q, 1/x) = (\hat{h}_{r-s} - \hat{h}_{r-s+1}) x^{r-s+1} + (\hat{h}_{r-s+1} - \hat{h}_{r-s+2}) x^{r-s+2} + \cdots + (\hat{h}_{r-1} - \hat{h}_r) x^r + \hat{h}_r x^{r+1}. \]

By Theorem \((6.4)\), if $Q$ is semi-Eulerian, then $\hat{h}_{r-k} = \hat{h}_k + (-1)^{r-k+1} \binom{r}{k} e_Q(\hat{0}, \hat{1})$. Hence for $k < r$,
\[ \hat{h}_{r-k} - \hat{h}_{r-k-1} = (\hat{h}_k - \hat{h}_{k+1}) + (-1)^{r-k+1} \left[ \binom{r}{k} + \binom{r}{k+1} \right] e_Q(\hat{0}, \hat{1}), \]

and since \( \binom{r}{k} + \binom{r}{k+1} = \binom{r+1}{k+1} \), we infer that

\[ \hat{h}_{r-k} - \hat{h}_{r-k-1} = (\hat{h}_k - \hat{h}_{k+1}) + (-1)^{r-k+1} \binom{r+1}{k+1} \cdot e_Q(\hat{0}, \hat{1}). \]

Comparing the coefficients of \( x^{k+1} \) in \( \hat{g}(Q) + y\hat{h}(Q) \) and \( x^{\rho(Q)}\hat{g}(Q, 1/x) \), yields the lemma. \( \square \)

Now we resume the proof of Theorem 6.6. For any lower interval \( Q = [\hat{0}, q] \) in \( P \), if \( 1 < \rho(q) < d \), then \( Q \) is Eulerian. If \( \rho(q) = d \), then \( Q \) is semi-Eulerian. By Swartz’s result and the lemma above,

\[ \hat{g}(Q) + y\hat{h}(Q) = \begin{cases} x^{\rho(q)}\hat{g}(Q, 1/x) & \text{if } 1 \leq \rho(q) < d \\ x^{\rho(q)}\hat{g}(Q, 1/x) + \sum_{\rho(q) = d}^{d} \gamma_i(Q) \cdot x^i & \text{if } \rho(q) = d. \end{cases} \]

We can now simplify equation (6.1):

\[ \hat{h}(P) - x^d \cdot \hat{h}(P, 1/x) = -e_P(\hat{0}, \hat{1})y^d \]

and when \( d \) is even, the last summation \( \sum_{\rho(q) = d}^{d} \gamma_i(Q) \cdot x^i \) on the right hand-side has an extra summand \( \frac{1}{2}\gamma_i x^i \) for \( i = \lfloor \frac{d}{2} \rfloor \). Comparing like-terms from both sides: for \( i > \lfloor \frac{d}{2} \rfloor \),

\[ \hat{h}_{d-i} - \hat{h}_i = (-1)^{d-i+1} \left[ \binom{d}{i} e_P(\hat{0}, \hat{1}) + \sum_{\rho(q) = 1}^{d} (-1)^{d-i+1} \binom{d-1}{i-1} e_P(q, \hat{1}) + \sum_{\rho(q) = d}^{d} (-1)^{d-i+1} \binom{d}{i} e_P(\hat{0}, q) \right] \]

and

\[ \hat{h}_{d-i} - \hat{h}_i = (-1)^{d-i+1} \left[ \binom{d}{i} e_P(\hat{0}, \hat{1}) + \sum_{\rho(q) = 1}^{d-1} \binom{d-1}{i-1} e_P(q, \hat{1}) + \sum_{\rho(q) = d}^{d} \binom{d}{i} e_P(\hat{0}, q) \right] \]

as desired. \( \square \)
The following special case is worth mentioning: if $d$ is even and $i = \left\lfloor \frac{d}{2} \right\rfloor = \frac{d}{2} = d - i$, the left hand-side is simply zero, and hence so is the right hand-side. This observation leads to the following Corollary. We will generalize it later in Corollaries 7.10 and 7.11.

**Corollary 6.9.** Let $P$ be a 1-Sing poset with odd rank $\rho(P) = d + 1$, and let $O(P)$ be the reduced order complex of $P$. Then

$$2(\tilde{\chi}(O(P)) + 1) = \sum_{\rho(q)=1,d} 1 - \sum_{\rho(q)=1,d} \tilde{\chi}(\text{lk}_{O(P)} v_q)$$

where $v_q$ is the vertex in $O(P)$ that corresponds to the element $q \in P$.

**Proof.** If $d$ is even, and $i = \left\lfloor \frac{d}{2} \right\rfloor$, then $d - i = i$. Equation (6.6) gives:

$$0 = \hat{h}_{\frac{d}{2}} - \hat{h}_{\frac{d}{2}} = (-1)^{\frac{d}{2}+1} \left( \frac{d}{2} \right) [\mu_P(\hat{0}, \hat{1}) - (-1)]$$

$$+ \sum_{\rho(q)=1} (-1)^{\frac{d}{2}+1} \left( \frac{d}{2} - 1 \right) [\mu_P(q, \hat{1}) - 1] + \frac{1}{2} \sum_{\rho(q)=d} (-1)^{\frac{d}{2}-1} \left( \frac{d}{2} \right) [\mu_P(\hat{0}, q) - 1]. \quad (6.7)$$

Since $(\frac{d-1}{2}) = \frac{1}{2}(\frac{d}{2})$, this can be simplified to

$$0 = [\mu_P(\hat{0}, \hat{1}) + 1] + \frac{1}{2} \sum_{\rho(q)=1} [\mu_P(q, \hat{1}) - 1] + \frac{1}{2} \sum_{\rho(q)=d} [\mu_P(\hat{0}, q) - 1].$$

Hence

$$2(\mu_P(\hat{0}, \hat{1}) + 1) = \sum_{\rho(q)=1} [1 - \mu_P(q, \hat{1})] + \sum_{\rho(q)=d} [1 - \mu_P(\hat{0}, q)]$$

$$= \sum_{\rho(q)=1,d} 1 + \sum_{\rho(q)=1,d} \mu_P(\hat{0}, q) \cdot \mu_P(q, \hat{1}) = \sum_{\rho(q)=1,d} 1 - \sum_{\rho(q)=1,d} \tilde{\chi}(\text{lk}_{O(P)} F_{\{q\}}).$$

**Remark:** The above corollary is a generalization of a result from [NS12] asserting that for a $(d - 1)$-dimensional simplicial psuedomanifold $\Delta$ with isolated singularities,

$$2(\tilde{\chi}(\Delta) + 1) = |V| - \sum_{v \in V} \tilde{\chi}(\text{lk}_\Delta v).$$

### 7 Posets with singularities of higher degrees

This section generalizes several results from Section 6, most notably Theorem 6.6.
7.1 $j$-Sing Posets

We start with a definition of posets with singularities of degree at most $j$. A similar notion was introduced in [Ehr01].

**Definition 7.1.** For a finite graded poset $P$ of rank $\rho(P) = d + 1$, we recursively define the notion of $j$-Sing:

- $P$ is ($-1$)-Sing if $P$ is Eulerian.
- $P$ is 0-Sing if $P$ is semi-Eulerian.
- $P$ is $j$-Sing if every interval of length $\leq d$ in $P$ is $(j - 1)$-Sing.

A few remarks are in order.

**Remark 7.2.** A poset $P$ is $j$-Sing if and only if for all $s \leq j + 1$, every interval of length $\leq d + 1 - s$ in $P$ is $(j - s)$-Sing. Also, $P$ is $j$-Sing if and only if every interval of length $\leq d - j$ in $P$ is Eulerian, i.e., every such interval $[s, t]$ has $\varepsilon_P(s, t) = 0$.

**Remark 7.3.** When $j$ is odd (even, resp.), all $j$-Sing posets $P$ with even (odd, resp.) rank are in fact $(j - 1)$-Sing. This follows from Definition 7.1 and the fact that every semi-Eulerian poset of odd rank is actually Eulerian.

**Definition 7.4.** A $(d - 1)$-dimensional pure simplicial complex $\Delta$ is called a $j$-singular complex if $\varepsilon_\Delta(F) = 0$ for every face $F \in \Delta$ of dim$(F) \geq j$, i.e., $\tilde{\chi}(lk_\Delta F) = (-1)^{d-1-|F|}$.

**Proposition 7.5.** The following are equivalent:

1. A poset $P$ is $j$-Sing.
2. The order complex of $P$, $O(P)$, is a $j$-singular complex.
3. For every chain $C = \{\hat{0} = t_0 < t_1 < \cdots < t_{i-1} < t_i = \hat{1}\}$ in $P$ such that $i > j + 1$, the (chain) error $\varepsilon_P(C) = 0$.

**Proof.** Assume $P$ has rank $d + 1$, and so dim $O(P) = d - 1$.

(2)$\iff$(3) is clear since by definitions, $\varepsilon_P(C) = \varepsilon_{O(P)}(F_C)$, and every chain $C$ in (3) corresponds to a face $F_C \in O(P)$ of dimension $\geq j$.

(1)$\implies$(3) $P$ is $j$-Sing if and only if every interval of length $\leq d - j$ in $P$ is Eulerian. Therefore, for any interval $[s, t]$ in $P$ with $\rho(t) - \rho(s) \leq d - j$, $\mu_P(s, t) = (-1)^{\rho(t) - \rho(s)}$. For any chain $C$ as in (3),

$$\rho(t_k) - \rho(t_{k-1}) \leq d - j \quad \text{for all } 1 \leq k \leq i,$$

therefore every interval $[t_{k-1}, t_k]$ is Eulerian. This implies $\mu_P(C) = (-1)^{d+1}$ and so $\varepsilon_P(C) = 0$. 

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(3)⇒(1) Pick any interval \([s, t]\) in \(\mathcal{P}\) with \(\rho(t) - \rho(s) \leq d - j\). Consider a maximal chain \((0 = t_0 < t_1 < \cdots < t_{\rho(s)} = s)\) in \([0, s]\) and a maximal chain \((t = t_{\rho(s)+1} < t_{\rho(s)+2} < \cdots < t_k = 1)\) in \([t, 1]\), and let \(C\) be the union of these two chains:

\[
C = (0 = t_0 < t_1 < \cdots < s < t < \cdots t_{k-1} < t_k = 1).
\]

In particular, \(\rho(t_i) - \rho(t_{i-1}) = 1\) unless \(i = \rho(s) + 1\). Then \(C\) has length \(k \geq j\), and so by (3),

\[
\mu_{\mathcal{P}}(0, t_1)\mu_{\mathcal{P}}(t_1, t_2)\cdots\mu_{\mathcal{P}}(s, t)\cdots\mu_{\mathcal{P}}(t_{i-1}, 1) = (-1)^{d+1}.
\]

Since intervals of length 1 always have the Möbius value \(-1\), this forces \(\mu_{\mathcal{P}}(s, t) = (-1)^{\rho(t) - \rho(s)}\). All intervals of length \(\leq d - j\) are Eulerian, therefore \(\mathcal{P}\) is \(j\)-Sing.

\[\square\]

To make the exposition cleaner, we introduce the following notation:

**Definition 7.6.** For any \(j\)-Sing poset \(\mathcal{P}\) of rank \(\rho(\mathcal{P}) = d + 1\), define

\[
A^{(j)}(\mathcal{P}) := \hat{h}_{d-k}(\mathcal{P}) - \hat{h}_k(\mathcal{P}).
\]

Note that any graded poset of rank \(d + 1\) is automatically \((d - 1)\)-Sing. The following claim on \(A^{(j)}(\mathcal{P})\) can be shown by an easy induction on \(j\).

**Proposition 7.7.** For any graded poset \(\mathcal{P}\) of rank \(d + 1\),

\[
\hat{h}_0(\mathcal{P}) - \hat{h}_d(\mathcal{P}) = (-1)^d \cdot e_{\mathcal{P}}(\hat{0}, \hat{1}).
\]

### 7.2 Dehn-Sommerville relations

We are now in a position to generalize Theorem 6.6. First, by comparing the polynomials \(\hat{g}(Q) + y\hat{h}(Q)\) and \(x^{\rho(Q)}\hat{g}(Q, \frac{1}{x})\) (as in the proof of Lemma 6.8), we obtain the following extension of Lemma 6.8. We omit the proof.

**Lemma 7.8.** Let \(Q\) be a \(j\)-Sing poset with rank \(r + 1\), and let \(y = x - 1\). Then

\[
\hat{g}(Q) + y\hat{h}(Q) = x^{\rho(Q)}\hat{g}(Q, \frac{1}{x}) + \sum_{k = \lceil \frac{r+1}{2} \rceil + 1}^{r+1} A^{(j)}_{k-1}(Q) - A^{(j)}_k(Q) x^k,
\]

where \(\sum^*\) means that, if \(r\) is odd, then there is an extra summand, \(\frac{1}{2}[A^{(j)}_{k-1}(Q) - A^{(j)}_k(Q)]x^k\) for \(k = \lceil \frac{r+1}{2} \rceil\), which equals \(-A^{(j)}_{\lceil \frac{r+1}{2} \rceil}(Q) \cdot x^{\lceil \frac{r+1}{2} \rceil}\).

The first main result of this section is the following generalization of Theorem 6.6.
Theorem 7.9. Let \( P \) be a \( j \)-Sing poset of rank \( \rho(P) = d + 1 \), where \(-1 \leq j \leq d\). For \( q \in P \), denote by \( Q \) the interval \([0, q]\). Then

\[
\hat{h}(P) - x^d \hat{h}(P, \frac{1}{x}) = - \sum_{\rho(q) \leq j} \hat{g}(Q, \frac{1}{x}) \cdot e_P(q, \hat{1}) \cdot y^{d-\rho(q)} \cdot x^{\rho(q)}
\]

\[
- \sum_{d-j<\rho(q) \leq d} \sum_{k=\lfloor \frac{\rho(q)}{2} \rfloor + 1}^{\rho(q)} \left( A_{k-1}^{(j+\rho(q)-d-1)}(Q) - A_k^{(j+\rho(q)-d-1)}(Q) \right) \cdot \mu(q, \hat{1}) \cdot y^{d-\rho(q)} x^k. \quad (7.1)
\]

Before proving this formula, we first work out several examples and show that for small \( j \)'s, equation (7.1) reduces to previous results.

Examples

- If \( P \) is \((-1)\)-Sing, then \( P \) is Eulerian. Hence by [Sta87], \( A_k^{(-1)}(P) = 0 \) for every \( k \). This matches equation (7.1) as both sums on the right hand-side of (7.1) are empty.

- If \( P \) is 0-Sing, then \( P \) is semi-Eulerian, so that all proper intervals \( Q \in \hat{P} \) are \((-1)\)-Sing (i.e., Eulerian). In this case, by [Swa09],

\[
A_k^{(0)}(P) = (-1)^{d-k+1} \left( \begin{array}{c} d \\ k \end{array} \right) e_P(0, \hat{1}).
\]

This coincides with equation (7.1) as the summand corresponding to \( Q \) of rank 0 is the only term showing up on the right hand-side of (7.1) for \( j = 0 \).

- If \( P \) is 1-Sing, then equation (7.1) implies that

\[
A_k^{(1)}(P) = (-1)^{d-k+1} \left( \begin{array}{c} d \\ k \end{array} \right) e_P(0, \hat{1}) + \sum_{\rho(q) = 1} (-1)^{d-k+1} \left( \begin{array}{c} d-1 \\ k-1 \end{array} \right) e_P(q, \hat{1}) + \sum_{\rho(q) = d} \left( A_k^{(0)}(Q) - A_{k-1}^{(0)}(Q) \right)
\]

\[
= (-1)^{d-k+1} \left( \begin{array}{c} d \\ k \end{array} \right) e_P(0, \hat{1}) + \sum_{\rho(q) = 1} (-1)^{d-k+1} \left( \begin{array}{c} d-1 \\ k-1 \end{array} \right) e_P(q, \hat{1}) + \sum_{\rho(q) = d} (-1)^{d-k+1} \left( \begin{array}{c} d \\ k \end{array} \right) e_P(0, q), \quad (7.2)
\]

where for the last step we used that, if \( \rho(q) = d \), then the interval \([0, q]\) is semi-Eulerian. This agrees with our formula in Theorem 6.6.

Proof of Theorem 7.9. By Lemma 6.7, the following equation holds for an arbitrary graded poset \( P \) of rank \( d + 1 \).

\[
\hat{h}(P, x) - x^d \hat{h}(P, \frac{1}{x}) = -y^d e_P(0, \hat{1})
\]

\[
+ \sum_{\begin{array}{c} Q=0, d \in \hat{P} \\ 1 \leq \rho(Q) \leq d \end{array}} \left[ \left( -y^{d-\rho(Q)} \hat{g}(Q) + y \hat{h}(Q) \right) \mu_P(q, \hat{1}) \right] - \left[ \left( -y^{d-\rho(Q)} \hat{g}(Q, \frac{1}{x}) x^{\rho(Q)} \right) \right].
\]
From now on we assume $P$ to be $j$-Sing of rank $d + 1$. We are interested in $A_k^{(j)}(P)$, which is the coefficient of the $x^k$-term on the left hand-side. Since $P$ is $j$-Sing, Remark 7.2 implies that each interval $Q = [0, q] \subseteq P$ of rank $r + 1$ is $(j + r - d)$-Sing. In particular,

$$\mu(q, \hat{1}) = (-1)^{d - \rho(q) + 1} \quad \text{for } \rho(q) > j.$$  

Using this observation together with Lemma 7.8 we conclude that the following holds for any interval $Q$ of rank $r + 1$:

Case 1: $j < \left\lfloor \frac{d}{2} \right\rfloor \leq d - j$.

$$C(Q) = \begin{cases} 
-\hat{g}(Q, \frac{1}{2}) \cdot e_P(q, \hat{1}) \cdot y^{d - \rho(Q)} x^{\rho(Q)} & \text{for } \rho(Q) \leq j \\
0 & \text{for } \rho(Q) \in (j, d - j] \\
- \sum_{k=\left\lfloor \frac{d}{2} \right\rfloor + 1}^{r+1} \left( A_{k-1}^{(j+r-d)}(Q) - A_k^{(j+r-d)}(Q) \right) x^k \cdot (-y)^{d - \rho(Q)} & \text{for } \rho(Q) > d - j.
\end{cases}$$

Case 2: $j \geq \left\lfloor \frac{d}{2} \right\rfloor$.

$$C(Q) = \begin{cases} 
-\hat{g}(Q, \frac{1}{2}) \cdot e_P(q, \hat{1}) \cdot y^{d - \rho(Q)} x^{\rho(Q)}, & \text{for } \rho(Q) \leq d - j \\
-\hat{g}(Q, \frac{1}{2}) \cdot e_P(q, \hat{1}) \cdot y^{d - \rho(Q)} x^{\rho(Q)} - \sum_{k=\left\lfloor \frac{d}{2} \right\rfloor + 1}^{r+1} \left( A_{k-1}^{(j+r-d)}(Q) - A_k^{(j+r-d)}(Q) \right) \cdot \mu(q, \hat{1}) y^{d - \rho(Q)} x^k, & \text{for } \rho(Q) \in (d-j, j] \\
- \sum_{k=\left\lfloor \frac{d}{2} \right\rfloor + 1}^{r+1} \left( A_{k-1}^{(j+r-d)}(Q) - A_k^{(j+r-d)}(Q) \right) \cdot \frac{\mu(q, \hat{1})}{(-1)^{d-r}} y^{d - \rho(Q)} x^k, & \text{for } \rho(Q) \geq j + 1.
\end{cases}$$

In both cases, comparing the coefficients on both sides, yields the statement.

This shows that the difference between $\hat{h}_{d-k}(P)$ and $\hat{h}_k(P)$ is a “weighted” sum of the error functions of the intervals in $P$. Unfortunately, as $j$ gets larger, the length of our formula expands very quickly. In the rest of this section, we will simplify this formula for $j > \frac{d}{2}$ and $k > \frac{d+1}{2}$, see Theorem 7.13. Our main tool is the following result, that might be of interest on its own.

**Corollary 7.10.** Let $P$ be a $j$-Sing poset of rank $d + 1$.

- If $d$ is even, then

$$2e_P(\hat{0}, \hat{1}) = - \sum_{1 \leq \rho(t) \leq j} e_P(t, \hat{1}) - \sum_{d-j+1 \leq \rho(t) \leq d} e_P(\hat{0}, t).$$
• If $d$ is odd, then
\[ \sum_{1 \leq \rho(t) \leq j} e_P(t, \hat{1}) = \sum_{d-j+1 \leq \rho(t) \leq d} e_P(\hat{0}, t). \]

Proof. We will only treat the case of $j \geq \lfloor \frac{d}{2} \rfloor$ since the case of $j < \lfloor \frac{d}{2} \rfloor$ is very similar. By Theorem 7.9
\[ \hat{h}_0(P) - \hat{h}_d(P) \]
\[ = \sum_{0 \leq \rho(t) \leq j} (-1)^{\rho(t)+1} \cdot \mu(\hat{0}, t) \cdot e_P(t, \hat{1}) + \sum_{d-j+1 \leq \rho(t) \leq d} (-1)^{d-\rho(t)} \cdot [\hat{h}_0(\hat{0}, t) - \hat{h}_{\rho(t)-1}(\hat{0}, t)] \]
\[ \hat{h}_0(P) - \hat{h}_d(P) \]
\[ = \sum_{0 \leq \rho(t) \leq j} (-1)^{\rho(t)+1} \cdot \mu(\hat{0}, t) \cdot e_P(t, \hat{1}) + \sum_{d-j+1 \leq \rho(t) \leq d} (-1)^{d-\rho(t)} \cdot (-1)^{\rho(t)+1} e_P(\hat{0}, t) \]
\[ = - \sum_{0 \leq \rho(t) \leq j} e_P(t, \hat{1}) + \sum_{d-j+1 \leq \rho(t) \leq d} [(-1)^{d+1} \mu(\hat{0}, t) - \mu(t, \hat{1})] + \sum_{j+1 \leq \rho(t) \leq d} (-1)^{d+1} e_P(\hat{0}, t) \]
\[ = - \sum_{0 \leq \rho(t) \leq j} e_P(t, \hat{1}) + \sum_{d-j+1 \leq \rho(t) \leq d} (-1)^{d+1} e_P(\hat{0}, t) \]

where the equality “\*” follows from Proposition 7.7. However by Proposition 7.7, the left hand-side should also equal $(-1)^d e_P(\hat{0}, \hat{1})$. Comparing it with the right hand-side of (7.3) yields the result. \[ \square \]

If $j < \lfloor d/2 \rfloor$, Corollary 7.10 is equivalent to the following geometric interpretation that generalizes Corollary 6.9.

**Corollary 7.11.** Let $P$ be a $j$-Sing poset of rank $d+1$ with $j < \lfloor d/2 \rfloor$. Let $\varepsilon_O(P)$ be the error function associated to $O(P)$.

• If $d$ is even, then
\[ 2\varepsilon_O(P)(\emptyset) = - \sum_{F \in O(P), |F| \leq j} \varepsilon_O(P)(F). \]

• If $d$ is odd, and for every face $F \in O(P)$ we let $F^\top$ and $F_\bot$ denote the top and the bottom element of $F$ (viewing $F$ as a chain in $P \setminus \{\hat{0}, \hat{1}\}$), then
\[ \sum_{F \in O(P)} \varepsilon_O(P)(F) = \sum_{\rho(F^\top) \leq j} \sum_{\rho(F_\bot) \geq d-j+1} \varepsilon_O(P)(F). \]

Proof. First notice that when $j < \lfloor d/2 \rfloor \leq d-j$, any face $F$ with $\rho(F^\top) > j$ or $\rho(F_\bot) < d-j+1$ will have $\varepsilon_O(P)(F) = 0$ since all the intervals defined by this chain are Eulerian. This means
\[ \sum_{F \in O(P), |F| \leq j} \varepsilon_O(P)(F) = \sum_{\rho(F^\top) \leq j} \sum_{\rho(F_\bot) \geq d-j+1} \varepsilon_O(P)(F). \]
Now it suffices to show the following claim: for \( q, t \in P \), \( \rho(q) \leq j \) and \( \rho(t) \geq d - j + 1 \),

\[
\sum_{F \in O(P)} \varepsilon_{O(P)}(F) = e_P(q, \hat{1}) \quad \text{and} \quad \sum_{F \in O(P)} \varepsilon_{O(P)}(F) = e_P(\hat{0}, t).
\]

Recall that every face \( F \) in \( O(P) \) corresponds to a chain in \( P \setminus \{ \hat{0}, \hat{1} \} \), therefore by abusing notation, we can write \( F = \{ t_1 < t_2 < \cdots < t_k = q \} \). If \( \rho(q) \leq j \), then all intervals \( [t_i, t_{i+1}] \) are Eulerian, and so

\[
\chi(\text{lk}_{O(P)} F) = (-1)^{|F|} \cdot \mu(\hat{0}, t_1) \mu(t_1, t_2) \cdots \mu(q, \hat{1}) = (-1)^{|F| + \rho(q)} \cdot \mu(q, \hat{1}).
\]

Hence \( \varepsilon_{O(P)}(F) = (-1)^{|F| + \rho(q)} e_P(q, \hat{1}) \). In particular, if \( F \) is a facet in \( O([\hat{0}, q]) \), i.e., a saturated chain in \([\hat{0}, q]\), then \( \rho(q) = |F| \) and \( \varepsilon_{O(P)}(F) = e_P(q, \hat{1}) \). If \( F \) a ridge in \( O([\hat{0}, q]) \), then \( \varepsilon_{O(P)}(F) = -e_P(q, \hat{1}) \), etc. This, together with the fact that \( O([\hat{0}, q]) \) is Eulerian, implies

\[
\sum_{F \in O(P)} \varepsilon_{O(P)}(F) = (-1)^{\rho(q)} \cdot \chi(O([\hat{0}, q])) \cdot e_P(q, \hat{1}) = e_P(q, \hat{1}).
\]

A symmetric argument takes care of the other half of the claim. Our statement now follows from Corollary 7.10. \( \square \)

**Remark 7.12.** The polynomial in (7.1) is a symmetric polynomial with half of its coefficients negated. In particular, it is the following (here \( A_k \) stands for \( A_k(P) \)):

\[
A_dx^d + A_{d-1}x^{d-1} + \cdots + A_kx^k + \cdots - A_kx^{d-k} - \cdots - A_{d-1}x - A_d.
\]

Therefore the coefficients of \( x^k \) and \( x^{d-k} \) add up to zero for every \( k \neq \frac{d}{2} \). The case of \( k = d \) was used to obtain Corollaries 7.10 and 7.11. A natural open problem is the following: can we make use of other equalities arising from comparing the coefficients of \( x^k \) and \( x^{d-k} \) for \( k \neq d \) of this polynomial?

The next main theorem in this section will give an explicit formula of \( A_k^{(j)}(P) \). Before stating the theorem, we first introduce some notation. Given a poset \( T = [\hat{0}, t] \) and integers \( u \) and \( v \), we define the following function.

\[
C(T, u, v) := \left( u - \rho(t) \right) - \hat{g}_{\rho(t)-2}(T) \cdot \left( u - \rho(t) \right) + \cdots + (-1)^m \hat{g}_{\rho(t)-1-m}(T) \cdot \left( u - \rho(t) \right)
\]

where \( m = \lfloor (\rho(t) - 1)/2 \rfloor \).

Note that by Pascal’s rule on binomial coefficients, for each \( u, v \), we have

\[
C(T, u, v) + C(T, u, v + 1) = C(T, u + 1, v + 1).
\] (7.4)
Theorem 7.13. Let $P$ be a $j$-Sing poset of rank $d+1$ (with $d > 2j$), and let $k > (d+j)/2$. For each element $t \in P$, let $T := [0, t]$, then

$$A_{k}^{(j)}(P) = (-1)^{k} \sum_{\rho(t) \leq j} e_{P}(t, 1) \cdot C(T, d, k) \quad \text{for } k > (d+j)/2. \quad (7.5)$$

Proof. The proof is by induction on $j$. The base cases of $j = -1$ and 0 are immediate from Stanley’s and Swartz’s results, see Theorems 6.3 and 6.4. The $j = 1$ case can be easily checked using (7.2) and Corollary 7.10. The inductive hypothesis is that for all $j' < j$, all $j'$-Sing posets $Q$ of rank $d' + 1$, and all $k > \frac{d'+j}{2}$,

$$A_{k}^{(j)}(Q) = (-1)^{k} \sum_{\rho(t) \leq j'} e_{P}(t, 1) \cdot C(T, d', k). \quad (7.6)$$

If $P$ is a $j$-Sing poset and $k > \frac{d+j}{2}$, then since $A_{k}^{(j)}(P)$ is the coefficient of $x^{k}$ in Equation (7.1),

$$A_{k}^{(j)}(P) = (-1)^{d-k+1} \sum_{\rho(t) \leq j} e(t, 1) \cdot C(T, d, d-k)
- \sum_{\rho(q) = d-j+b \quad 0 \leq b \leq j} (-1)^{k-a} \sum_{k-(d-q(q))-1 \leq a \leq k-1} \left( \begin{array}{c} d-ho(q) \\ k-(a+1) \end{array} \right) \cdot \left( A_{a}^{(b-1)}(Q) - A_{a+1}^{(b-1)}(Q) \right). \quad (7.7)$$

We now check that the inductive hypothesis applies to all of the summands in the second summation in (7.7). For each $q$ with $\rho(q) = d-j+b$, $Q = [0, q]$ is a $(b-1)$-Sing poset. By (7.6), for all $a > \frac{(d-j+b-1)+(b-1)}{2} = \frac{d+j}{2} + b - 1$,

$$A_{a}^{(b-1)}(Q) = (-1)^{a} \sum_{\rho(t) \leq b-1} e_{Q}(t, 1) \cdot C(T, \rho(Q)-1, a). \quad (7.8)$$

In (7.7), the second summation is a sum over $q \in P$ such that $\rho(q) = d-j+b$ and

$$a \geq k-d+\rho(q)-1 = k-j+b-1 > \frac{d+j}{2} - j + b - 1 = \frac{d-j}{2} + b - 1.$$

Therefore (7.8) holds for all $A_{a}^{(b-1)}(Q)$’s in (7.7), and so

$$A_{k}^{(j)}(P) = (-1)^{d-k+1} \sum_{\rho(t) = r+1 \quad m = [r/2]} e_{P}(t, 1) \cdot C(T, d, d-k)
- (-1)^{k} \sum_{\rho(q) = d-j+b \quad 0 < b \leq j} \sum_{a \in [k-(d-q(q))-1, k-1]} e_{Q}(t, 1) \left[ C(T, \rho(Q)-1, a) + C(T, \rho(Q)-1, a+1) \right].$$

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Note that $e_Q(t, \hat{1}) = e_P(t, q)$, and by recurrence relation (7.4), we obtain:

\[
A_k^j(P) = (-1)^{d-k+1} \sum_{\rho(t) \leq j} e_P(t, \hat{1}) \cdot C(T, d, d - k)
- (-1)^k \sum_{\rho(q) = d-j+b} \sum_{0 < b \leq j} e_P(t, q) \left( \frac{d - \rho(Q)}{k - (a + 1)} \right) \cdot C(T, \rho(Q), a + 1).
\]

Since each $C(T, \rho(Q), a + 1)$ is an alternating sum of multiples of the binomial coefficients $(\rho(Q) - \rho(T))$ (for some $c$'s), we can use the Chu-Vandermonde identity to conclude that:

\[
\sum_{a = k - (d - \rho(q)) - 1}^{k-1} \left( \frac{d - \rho(Q)}{k - (a + 1)} \right) \cdot \left( \frac{\rho(Q) - \rho(T)}{(a + 1) - c} \right) = \left( \frac{d - \rho(T)}{k - c} \right).
\]

This shows that (for $k > \frac{d+j}{2}$),

\[
A_k^j(P) = (-1)^{d-k+1} \sum_{\rho(t) \leq j} e_P(t, \hat{1}) C(T, d, d - k) - (-1)^k \sum_{\rho(q) = d-j+b} \sum_{0 < b \leq j} e_P(t, q) \cdot C(T, d, k).
\]

\[
= (-1)^{d-k+1} \sum_{\rho(t) \leq j} e_P(t, \hat{1}) \cdot C(T, d, d - k) + (-1)^{k+1} \sum_{\rho(t) \in [0, \rho(q) - (d-j+1)]} \sum_{\rho(q) \in [d-j+1, d]} e_P(t, q) \cdot C(T, d, k).
\]

The second equality holds because in the second summation, $e_P(t, q) = 0$ for all intervals $[t, q]$ of length $\leq d - j$.

The next step is to apply Corollary 7.10 to all intervals $[t, \hat{1}]$ with $0 \leq \rho(t) \leq j - 1$ to replace the summands in (7.9) that involve $e_P(t, q)$ with with sums of multiples of $e_P(t', \hat{1})$ where $0 \leq \rho(t') \leq j$.

The cases of even and odd $j$'s are slightly different because of the two cases in Corollary 7.10. Here we assume $j$ is odd (and hence $d$ is assumed to be even). The proof for the case of even $j$ is similar; we omit it.

As $j$ is odd and $d$ is even, Corollary 7.10 implies that

\[
(-1)^{k+1} \sum_{\rho(u) = i} \sum_{d-j+1 \leq \rho(q) \leq d} e_P(u, q) C(U, d, k)
\]

\[
= \begin{cases} 
(-1)^k \sum_{\rho(t) = i} 2e_P(t, \hat{1}) \cdot C(T, d, k) + (-1)^k \sum_{\rho(u) = i} e_P(t, \hat{1}) \cdot C(U, d, k) & \text{if } i \text{ is even} \\
(-1)^k \sum_{\rho(u) = i} e_P(t, \hat{1}) \cdot C(U, d, k) & \text{if } i \text{ is odd.}
\end{cases}
\]

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After these substitutions, all of the summands in equation (7.9) that involve $e_p(u, q)$ with $q \neq \hat{1}$ are replaced with sums of multiples of $e_p(t, \hat{1})$ for some $t \geq u$. Now we have:

\[
(-1)^{k+1} \sum_{0 \leq \rho(t) \leq j-1} \sum_{d-j+1 \leq \rho(q) \leq d} e(t, q)C(T, d, k)
\]

\[
= (-1)^{k} \sum_{i \text{ even}} \left[ \sum_{\rho(t) = i} 2e_p(t, \hat{1})C(T, d, k) + \sum_{\rho(u) = i, \rho(t) \leq j} e_p(t, \hat{1})C(U, d, k) \right] + (-1)^{k+1} \sum_{i \text{ odd}} \sum_{\rho(u) = i, i+1 \leq \rho(t) \leq j} e_p(t, \hat{1})C(U, d, k)
\]

Together with (7.9) and the assumption that $d$ is even, this shows

\[
A_k^{(j)}(P) = \sum_{\rho(t) \leq j} (-1)^{k} e_p(t, \hat{1}) \left[ -C(T, d, d-k) + \sum_{u < t} (-1)^{\rho(u)}C(U, d, k) \right] + (-1)^{k} \sum_{\rho(t) \leq j} e_p(t, \hat{1})2C(T, d, k).
\]

Comparing this with (7.5), it suffices to show that for each $t \in P$ with $\rho(t) \leq j$ and $T = [\hat{0}, t],

\[
(-1)^{k+1}C(T, d, d-k) + (-1)^{k} \sum_{u < t} (-1)^{\rho(u)}C(U, d, k) = \begin{cases} 
(-1)^{k}C(T, d, k) & \text{if } \rho(t) \text{ is odd,} \\
(-1)^{k+1}C(T, d, k) & \text{if } \rho(t) \text{ is even.} 
\end{cases}
\]

(7.10)

Observe that since $d$ is even,

- $(-1)^{k+1}C(T, d, d-k) = \text{the coefficient of } x^k \text{ in } -(x-1)^{d-\rho(t)} \cdot x^{\rho(t)} \cdot \hat{g}(T, \frac{1}{x}),$
- $(-1)^{k} \sum_{u < t} (-1)^{\rho(u)}C(U, d, k) = \text{the coefficient of } x^k \text{ in } \sum_{u < t} (x-1)^{d-\rho(u)} \cdot \hat{g}(U, x),$
- $(-1)^{k}C(T, d, k) = \text{the coefficient of } x^k \text{ in } (-1)^{\rho(t)}(x-1)^{d-\rho(t)} \cdot \hat{g}(T, x).$

Therefore, (7.10) is equivalent to:

\[
-(x-1)^{d-\rho(t)} \cdot x^{\rho(t)} \cdot \hat{g}(T, \frac{1}{x}) + \sum_{u < t} (x-1)^{d-\rho(u)} \cdot \hat{g}(U, x) = -(x-1)^{d-\rho(t)} \cdot \hat{g}(T, x).
\]

(7.11)

This equality holds since by the definition of $\hat{h}(T, x)$, the left hand-side is

\[
-(x-1)^{d-\rho(t)} \cdot x^{\rho(t)} \cdot \hat{g}(T, \frac{1}{x}) + (x-1)^{d-\rho(t)+1} \cdot \hat{h}(T, x)
\]

\[
= (x-1)^{d-\rho(t)} \left[ -x^{\rho(t)} \cdot \hat{g}(T, \frac{1}{x}) + (x-1) \cdot \hat{h}(T, x) \right]
\]

\[
= -(x-1)^{d-\rho(t)} \cdot \hat{g}(T, x).
\]
where the last equality holds since $T$ is Eulerian.

This completes the proof that $A^j_k = (-1)^k \sum_{0 \leq \rho(t) \leq j} e(t, \hat{1})C(t)$ when $j$ is odd. When $j$ is even, the proof is very similar; we omit it. \hfill \square

### 7.3 The lower Eulerian case

In this subsection we assume that $P$ is lower Eulerian, i.e., all intervals $[0, t]$ are Eulerian for $t \neq \hat{1}$. This is an important subclass of graded posets. For instance, the face posets of all regular CW complexes are lower Eulerian. If $P$ is $j$-Sing and lower Eulerian, the formula of Theorem 7.9 takes on the following simpler form:

$$A^{(j)}_k(P) = \sum_{0 \leq \rho(q) = r+1 \leq j} \sum_{l=0}^{\lfloor r/2 \rfloor} (-1)^{d-k-l+1} \left(\frac{d-\rho(q)}{k-\rho(q)} - l\right) \cdot e_P(q, \hat{1}) \cdot \hat{g}_{r-l}([\hat{0}, q]).$$

**Example** Let $P$ be a $j$-Sing poset of rank $d+1$, where $j \leq 2$. If $P$ is also lower Eulerian, then

$$\hat{h}_{d-k}(P) - \hat{h}_k(P) = (-1)^{d-k+1} \sum_{\rho(q) \leq 2} \left(\frac{d-\rho(q)}{k-\rho(q)} - 1\right) e_P(q, \hat{1}).$$

**Remark 7.14.** Unfortunately for larger $j$’s the situation becomes more complicated. For instance if $j = 3$, then using the fact (easy to check) that

$$\hat{g}(Q) = -\mu(Q) + [f_1(Q) + \mu(Q) - 2]x$$

for any poset $Q$ with $\rho(Q) = 3$ and the assumption that the poset $P$ is lower Eulerian, one can show that

$$A^{(3)}_k(P) = (-1)^{d-k+1} \sum_{\rho(q) \leq 2} \left(\frac{d-\rho(q)}{k-\rho(q)}\right) e(q, \hat{1})$$

$$+ (-1)^{d-k+1} \sum_{\rho(q) = 3} \left[\left(\frac{d-3}{k-3}\right) - \left(\frac{d-3}{k-4}\right) \cdot \left(f_1([0, q]) - 3\right)\right] e_P(q, \hat{1}).$$

Observe that “$\heartsuit$” equals $(d-3)_{k-3}$ if and only if $f_1(Q) = 3$, which is the case when $Q$ is the face poset of a simplex.

**Definition 7.15.** A pure graded poset $P$ is $k$-lower simplicial if for all $t \in P$ with $\rho(t) \leq k$, the interval $[0, t]$ is a Boolean lattice.

**Corollary 7.16.** Let $P$ be a $j$-Sing lower Eulerian poset $P$ of rank $d+1$. If $P$ is also $j$-lower simplicial, then for $k > \lfloor \frac{d}{2} \rfloor$,

$$\hat{h}_{d-k}(P) - \hat{h}_k(P) = (-1)^{d-k+1} \sum_{\rho(q) \leq j} \left(\frac{d-\rho(q)}{k-\rho(q)}\right) e_P(q, \hat{1}).$$

**Remark 7.17.** In the case that $P$ is the face poset of a simplicial complex $\Delta$ and $P$ is $j$-Sing, Corollary 7.16 agrees with the formula in Theorem 3.1.
7.4 Open problems

The most natural open problem is to find a “nice” formula for \( A_j^k(P) \) when \( j \geq \frac{d}{2} \). Here \( \rho(P) = d + 1 \) and \( \frac{d}{2} \leq k \leq \frac{d+j}{2} \).

To state the next problem, let \( P \) be a \( j \)-Sing poset, and let \( P^* \) be the dual poset of \( P \). By definition, \( P^* \) is also a \( j \)-Sing poset. Recall that

- When \( j = -1 \), \( A_{-1}^k(P) = A_{-1}^k(P^*) = 0 \).
- When \( j = 0 \),
  \[
  A_0^k(P) = (-1)^{d-k+1} \binom{d}{k} e_P(\hat{0}, \hat{1}) = A_0^k(P^*). 
  \]

When \( d \) is odd and \( k = \lfloor \frac{d}{2} \rfloor \), this means

\[
\hat{g}_{\lfloor \frac{d}{2} \rfloor}(P) = \hat{g}_{\lfloor \frac{d}{2} \rfloor}(P^*). 
\]

- When \( j = 1 \) (and assuming \( d \) is even), after cancellations, we obtain
  \[
  A_1^k(P) - A_1^k(P^*) = (-1)^{d-k} \binom{d-1}{k} \left[ \sum_{\rho(q)=1} e_P(q, \hat{1}) - \sum_{\rho(q)=d} e_P(\hat{0}, q) \right]. 
  \]

This leads to the following question: How do the numbers \( A_j^k(P) \) compare with \( A_j^k(P^*) \)?

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