ON A STACKELBERG SUBSET SUM GAME

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Abstract. This contribution deals with a two-level discrete decision problem, a so-called Stackelberg strategic game: A Subset Sum setting is address ed with a set $N$ of items with given integer weights. One distinguished player, the leader $L$, may alter the weights of the items in a given subset $L \subset N$, and a second player, the follower $F$, selects a solution $A \subseteq N$ in order to utilize a bounded resource in the best possible way. Finally, the leader receives a payoff from those items of its subset $L$ that were included in the overall solution $A$, chosen by the follower. We assume that $F$ applies a publicly known, simple, heuristic algorithm to determine its solution set, which avoids having to solve $NP$-hard problems.

Two variants of the problem are considered, depending on whether $L$ is able to control (i.e., change) the weights of its items (i) in the objective function or (ii) in the bounded resource constraint. The leader’s objective is the maximization of the overall weight reduction, for the first variant, or the maximization of the weight increase for the latter one. In both variants there is a trade-off for each item between the contribution value to the leader’s objective and the chance of being included in the follower’s solution set.

We analyze the leader’s pricing problem for a natural greedy strategy of the follower and discuss the complexity of the corresponding problems. We show that setting the optimal weight values for $L$ is, in general, $NP$-hard. It is even $NP$-hard to provide a solution within a constant factor of the best possible solution. Exact algorithms, based on dynamic programming and running in pseudopolynomial time, are provided. The additional cases, in which $F$ faces a continuous (linear relaxation) version of the above problems, are shown to be straightforward to solve.

1. Introduction

A special class of decision problems in which two competing agents act sequentially are the so-called Stackelberg Games, a name that has its origins in the early 1930s [13]. In particular one agent, the leader $L$, needs to optimally choose the values of some variables that a second agent, the follower $F$, uses as parameters of an optimization problem and the leader’s payoff depends on the solution determined by the follower. These games can be formulated using bilevel programming (BP). In a bilevel optimization problem some of the constraints specify that a subset of variables must be an optimal solution to another optimization problem. This paradigm is particularly appropriate to model Stackelberg games. From a computational point of view, bilevel programming is a hard task. Jeroslow [8] showed that even in presence of linear objective functions and constraints, BP is already
\(\mathcal{NP}\)-hard. The strong hardness of the same problem was later proved by Hansen et al. [7]. Further complexity results are presented in, e.g., [10] where the authors state that most solution techniques for BP have been developed focusing on special cases in which convenient properties, such as linearity or convexity, can be exploited in order to develop efficient solution methods.

In the classical Subset Sum Problem (SSP), we are given a set of \(n\) items \(N = \{1, 2, \ldots, n\}\), each having a positive integer weight \(w_i, i = 1, \ldots, n\), and a knapsack of capacity \(c\). The problem is to select a subset \(A \subseteq N\) such that the corresponding total weight is closest to \(c\) without exceeding \(c\). SSP is a well known and widely studied problem and can be considered as a special case of the classical binary knapsack problem (KP) arising when the profit and the weight associated with each item are identical. Although SSP is a special case of KP, it is still \(\mathcal{NP}\)-hard [6], but can be solved to optimality in pseudopolynomial time and admits a FPTAS (see [9] for a review on solution algorithms for SSP). SSP has also been studied in the context of game theory [3, 4, 5], social choice, and multi-decider systems [11, 12].

In this paper, we consider a special Stackelberg game arising in a SSP setting: The leader \(L\) may alter the weights of some items (the leader's items \(L \subseteq N\)), while the follower \(F\) selects a solution set \(A \subseteq N\) in order to utilize a bounded resource in the best possible way. In order to do so, \(F\) applies a simple, possibly suboptimal, strategy which is known to \(L\). The leader receives a payoff only from its own items that are included in the solution, i.e., from items in \(A \cap L\). We address two variants of the problem: In a first version the leader may control, i.e., change, the weights of its items in the objective function (but not in the knapsack constraint) and wants to maximize the overall reduction in the weights of its items included in the solution set as much as possible. A second variant considers pricing items in the constraint only and not in the objective. In this case, the leader aims at maximizing the total weight increase of its solution items. In both variants there is a trade-off for each item between the contribution value to the leader's objective and the chances of being included in the follower's solution set.

Our study can be motivated by the application scenarios we sketch hereafter. A financial trader \(L\) (e.g., a bank) offers a number of products (i.e., investments opportunities) to a client \(F\) who also has access to alternative forms of speculations but limited knowledge, bounded computational capacity, and restricted budget availability. All these products, either offered by the trader or not, are characterized by a given cost and ROI which are determined by the market and the client selects a portfolio using simple strategies based on the efficiency of the financial products at hand. Due to client's limited information, the trader may offer its own products at a reduced ROI, taking the difference as a profit. On the other hand, the trader has to make its investments attractive in order to increase chances that client purchases these instead of other products. In conclusion, the trader has to decide, for all its financial products, the (decreased) ROI offered to the client in order to maximize the overall profit which is in turn given by the overall discount on the actual ROI values associated to the products the trader has sold. In a slightly different setting, the trader tries to sell its products at prices larger than the actual costs, while maintaining their real market profitability valid for the clients. In this case, the trader's
profit is associated to the difference between the price set for those items included in the solution set and their genuine costs. The model presented in this work represents the special case of a market scenario in which ROI is proportional to the cost of an investment.

A different example, motivating our model in a production planning context, considers an agent $F$ that controls a processing resource with limited capacity (e.g., total available processing time) $c$. The agent performs its own jobs on the machine but also accepts external orders by a different agent $L$ in order to improve the utilization of its resource. $F$ wants to decide on a production plan satisfying the capacity constraint while maximizing its utility which is directly proportional to the overall resource consumption. In a first setting the leader wants to have some of its jobs processed by $F$ but could also have the jobs processed at a different workshop, where it has to pay the same proportional price for each unit of processing time. $L$ might gain something if it manages to have some of its jobs processed by $F$ at a cheaper price. Even if less profitable, it may be beneficial for $F$ to devote a fraction of—otherwise unused—resource capacity for processing the external jobs. In this case $L$ wants to set prices of its jobs to maximize the total saving obtained by having its jobs processed by $F$ at a cheaper cost than another workshop. An alternative situation arises if there is no possibility for $L$ to negotiate prices, but $F$ may be willing to let the leader use the machine for a bit longer than the original processing time of each job. Then the leader tries to extend this total extra time as much as possible (and use this time to do some additional work on these machines for free).

The paper is organized as follows. In the following Section 1.1 the addressed problem is rigorously defined. In Section 2 we consider the first variant of the problem where $L$ is pricing items in the objective and propose an algorithm that allows the leader to find the solution in pseudopolynomial time. We also show that, even though the problem admits a solution algorithm running in $O(nc^2)$, it is not possible to obtain an algorithm guaranteeing a constant as approximation ratio (unless $P=NP$). Section 3 addresses the second version of the problem where items are priced in the constraint. Even in this case constant-ratio approximation algorithms are not possible, however it is possible to find an optimal pricing for $L$ in a shorter time $O(n^{3/2}c)$. Oddly enough, a simple variant (quite similar at a first glance) of the latter problem can be easily solved as discussed in Section 3.1. Section 4 briefly sketches the straightforward solutions for the case in which the follower is allowed to include fractions of items in the solution set, i.e., $F$ solves the LP relaxation for the above problems. Finally, in Section 5 some conclusions are drawn.

### 1.1. Problem Definition.

For ease of presentation, we partition the items into two classes: The set $L$ of items controlled by the leader $L$ and the set $F = N \setminus L$ of the remaining items. For simplicity we also refer to $L$ and $F$ as the leader’s items, or $L$-items, and follower’s items or $F$-items. The original SSP posed to the agents can be written as an integer program:

\[
\max \left\{ \sum_{j \in N} w_j y_j : \sum_{j \in N} w_j y_j \leq c; \ y \in \{0, 1\}^n \right\}
\]
As discussed above, the leader $L$ is interested in filling the knapsack with its items (i.e., items $j \in L$) and, to this purpose, it may alter the parameters of Problem (1) above in two different ways, by changing its item weights—as perceived by the follower—either in the objective function or, alternatively, in the capacity constraint. Note that in this way we allow the item data to be separated and different in the objective and in the constraint. Hereafter, we refer to the original and revised values of the leader’s item $j$ by $w_j$ and $\tilde{w}_j$, respectively and limit our analysis to nonnegative weights $\tilde{w}_j \geq 0$.

In this paper we consider the following two models:

1. **Objective-control**: $L$ may change the weights of its items in the objective of Problem (1). Then $F$ is confronted with the following KP:

$$\max \left\{ \sum_{j \in L} \tilde{w}_j y_j + \sum_{i \in F} w_i y_i : \sum_{j \in N} w_j y_j \leq c ; \ y \in \{0, 1\}^n \right\}$$

Let $y^*$ be an optimal solution of (2). For each item $j \in L$, the leader $L$ would like to gain some of the value of $j$ by reducing the item profit (i.e., setting $\tilde{w}_j < w_j$) for the follower. The resulting leader decision problem may be expressed via the bilevel program below:

$$\max \sum_{j \in L} (w_j - \tilde{w}_j) x_j$$

s.t. $x \in \arg \max \limits_{y \in \{0, 1\}^n} \left\{ \sum_{j \in L} \tilde{w}_j y_j + \sum_{i \in F} w_i y_i : \sum_{j \in N} w_j y_j \leq c \right\}$

$$x \in \{0, 1\}^n ; \ \tilde{w} \in \mathbb{R}_+^{\left|L\right|}$$

where $x$ and $\tilde{w}$ are the variables of the leader’s problem: $x$ is restricted to be an optimal solution of the follower’s problem, i.e., $x$ components take the optimal values of the $y$ variables in the KP faced by $F$. In particular, the binary variables $y_j$, in the follower’s problem, indicate whether item $j$ is or is not included in the knapsack. Continuous variables $\tilde{w}_j$ indicate the chosen $L$-item weights.

2. **Constraint-control**: $L$ may change the weights of its items in the constraint of Problem (1). Then $F$ is confronted with the following KP:

$$\max \left\{ \sum_{j \in N} w_j y_j : \sum_{j \in L} \tilde{w}_j y_j + \sum_{i \in F} w_i y_i \leq c ; \ y \in \{0, 1\}^n \right\}$$
while $L$ wants to maximize the weight increase of its items relative to their original values, in the optimal solution set of $F$. The corresponding leader decision problem is therefore:

$$\max \sum_{j \in L} (\tilde{w}_j - w_j) x_j$$

subject to:

$$x \in \arg \max \left\{ \sum_{j \in N} w_j y_j : \sum_{j \in L} \tilde{w}_j y_j + \sum_{i \in F} w_i y_i \leq c \right\}$$

$$x \in \{0,1\}^n; \tilde{w} \in \mathbb{R}_+^{\left|L\right|}.$$  \hspace{1cm} (5)

Here, increased $\tilde{w}_j$ values correspond to a higher utility for $L$. At the same time however, this lowers the chance that $j \in L$ is selected by the follower and thus becomes relevant for $L$.

Note that, in both cases, the problem faced by the follower is a binary knapsack problem. KP is a (binary) $\mathcal{NP}$-hard optimization problem, however we will assume that the follower has limited computational capability as it was done in [1]. As a consequence, instead of finding the optimal solution, $F$ determines a feasible solution set of the resulting knapsack problem by applying the very natural and intuitively appealing Greedy strategy. This holds for both versions of our problem, namely (3) and (5).

In KP the efficiency $e_j$, measured as the ratio between the profit and the weight of an item $j$, indicates how well $j$ utilizes the capacity in a solution set. Greedy follows the natural idea of sorting the items in non-increasing order of efficiencies and then adding the items to the solution set in that order. Whenever an item exceeds the current residual capacity it is discarded. If an item can be added to the current solution it is inserted into the knapsack and never removed again. For the decision of $F$ on a leader’s item $j \in L$, in the first variant, i.e., Objective-control, we have $e_j = \tilde{w}_j / w_j$; while for the Constraint-control variant $e_j = w_i / \tilde{w}_j$. On the other hand, since all items $i \in F$ have the same efficiency $e_i = 1$, it makes sense for a subset sum setting to apply a tie-breaking rule where the follower selects items of equal efficiency in decreasing order of weights. With reference to the mechanism implemented by Greedy, we write that an item $j$ is positioned before $i$ if $e_j > e_i$ (or, after $i$ if $e_j < e_i$.)

In the next two sections, we consider the integer problem and derive solution strategies for the leader. In Section [4] we briefly consider a continuous model, in which fractions of items are allowed to be included in the knapsack, and show that both versions of the problem are easy to solve in this case.

2. **Objective-control model**

In general, $L$ has the following options for each of its items: (ii) Either $j \in L$ is positioned before all $F$-items, i.e. with efficiency slightly larger than 1 by setting $\tilde{w}_j := w_j + \varepsilon$ and thus incurring a marginal loss for $L$, if the capacity permits it to be packed; or (iii) an item $j$ of $L$ is positioned after all $F$-items. In this case, the sorting by efficiencies serves only to implement a certain order of the remaining $L$-items offered to the Greedy algorithm, which can also be reached by marginally small values of $\tilde{w}_j$. Thus, $L$ gains (almost) $w_j$ in
its objective for every packed item $j$. (iii) A third option is to set $\tilde{w}_j := w_j$ thus intermixing leader and follower items. This choice cannot increase the range of possibilities for $L$ or improve its objective either. Hence, one may avoid considering strategies for leader that leave unaltered the efficiencies of its items. As a consequence, an optimal solution can be partitioned into three (possibly empty) sets of items selected by Greedy: A first set of $L$-items, one block of $F$-items, and a third set of $L$-items.

Based on the above considerations, the optimal strategy for $L$ is characterized by the total weight $W_1$ of all items placed before the $F$-items. Recall that Greedy selects the $F$-items—all with equal unit efficiency—in non-increasing order of weights and simply packs whatever items fit: let $F(W_1)$ be the total weight of these included items. Now the residual capacity $\bar{c} = c - W_1 - F(W_1)$ is available for packing with the remaining $L$-items. The total weight of $L$-items packed in this step is denoted by $W_2 \leq \bar{c}$ and constitutes the objective value of $L$. It is important to point out that, given the residual capacity $\bar{c}$, the leader may actually choose the set $S$ of its items that will be packed by Greedy: This can be done by slightly increasing the efficiency of such items e.g., setting $\tilde{w}_j := \varepsilon$ for each $j \in S$.

Since the algorithm tries to pack each $F$-item, the residual capacity $\bar{c}$ remaining for $L$ is upper bounded (unless the follower is left with no items at all) by the weight of the smallest unpacked item of $F$. In order to maximize the residual capacity for its own items, it would be natural to think that $L$ would try to reach a capacity $\bar{c}$ strictly smaller than $w_{F_{\min}}$ for Greedy. However the following example shows that, if $F$ plays Greedy, $L$ may obtain better than that.

**Example 1.** Consider an instance of our problem with $c = 20$ and 8 items (sorted in non-increasing weight order) with $L = \{4, 5, 6, 8\}$ and $F = \{1, 2, 3, 7\}$. The item weights are \{w_4 = 9, w_5 = 8, w_6 = 5, w_8 = 3\} for the $L$-items and \{w_1 = 12, w_2 = 11, w_3 = 10, w_7 = 4\} for the items of $F$.

In Figure 1(a), a solution is depicted in which by sequencing $(w_4 = 9, w_5 = 8)$ we have $\bar{c} = 3$ which is the largest value smaller than $w_{F_{\min}} = 4$. $L$ may obtain this solution by setting $\tilde{w}_4 = w_4 + \varepsilon$ and $\tilde{w}_5 = w_5 + \varepsilon$ (with cost $+2\varepsilon$) so that $W_1 = 9 + 8 = 17$, $F(W_1) = 0$, and $\bar{c} = 3$ that eventually $F$ will pack with weight $w_8 = 3$ and $L$ sets $\tilde{w}_8 = 0$. The leader objective value is $3 - 2\varepsilon$. However, as illustrated in Figure 1(b), $L$ may improve its objective by including items $w_5 = 8$ and $w_8 = 3$ with $W_1 = 11$, then $F$ may only include $w_7 = 4$ so, eventually, $\bar{c} = 5$ can be exploited by packing $w_6 = 5$ and the final value for the leader objective is $5 - 2\varepsilon$.

![Figure 1](image-url)
From an algorithmic point of view, given $W_1$, the value $F(W_1)$ and thus $\bar{c}$ immediately follow and it remains for $L$ to solve an instance of a standard SSP with capacity $\bar{c}$ and the item set without those items included in $W_1$. In order to do so, we have to consider all possible candidate values $W_1$ and all weight values $W_2$ reachable after fixing $W_1$. By running a dynamic programming by reaching algorithm [9] and going through all $W_1$ values, the final optimal solution can be found easily in pseudopolynomial time.

**Theorem 2.** In the Objective-control Leader’s problem (3), the optimal values $\bar{w}_j$, $j \in L$ for $L$ can be determined in time $O(nc^2)$.

**Proof.** In order to consider all possible candidate values $W_1$ and we run a dynamic programming by reaching algorithm.

More formally, we define a two-dimensional array of reachable weight pairs $r(W_1, W_2)$ for $W_1, W_2 \in \{0, \ldots, c\}$, where $r(W_1, W_2) = 1$ iff there exist two disjoint subsets $S_1, S_2 \subseteq L$ with $\sum_{i \in S_1} w_i = W_1$ and $\sum_{i \in S_2} w_i = W_2$, and $r(W_1, W_2) = 0$ otherwise.

The entries of this array can be found by running a dynamic programming by reaching algorithm as follows: As an initialization we set all entries of $r$ to 0 except $r(0, 0) = 1$. Then we consider all items of $L$ in turn (in arbitrary order). For each item $j \in L$ we consider all pairs $(w', w'')$ with $w', w'' \in \{0, \ldots, c\}$: If $r(w', w'') = 1$, then we set $r(w' + w_j, w'') = 1$ and $r(w', w'' + w_j) = 1$, corresponding to the possibility of adding a weight $w_j$ before or after the $F$-items.

After $r$ is fully determined we go through all feasible choices of $W_1$, i.e. all values $W_1$ where $\sum_{w''=0}^c r(W_1, w'') \geq 1$. For each such candidate $W_1$ the value $F(W_1)$ is determined by executing GREEDY with item set $F$ and the value $\bar{c} = c - W_1 - F(W_1)$ is computed. Then, the best solution (given $W_1$) is determined as

$$W_2(W_1) := \max \{W_2 \mid W_2 \leq \bar{c} \text{ and } r(W_1, W_2) = 1\}.$$  

Finally, it remains to pick the optimal solution for $L$ by taking the best choice for $W_1$ as

$$\max \left\{W_2(W_1) \mid \sum_{w''=0}^c r(W_1, w'') \geq 1\right\}.$$  

The array $r$ has a size of $c^2$. Its computation requires each item in $L$ to be considered once for every array entry which gives a trivial pseudopolynomial running time bound of $O(nc^2)$.

Although an algorithm with pseudopolynomial running time exists, there is in fact no hope to find an efficient algorithm that at least guarantees a constant approximation ratio, as shown by the following theorem.

**Theorem 3.** Unless $P = NP$, the Objective-control Leader’s Problem (3) does not admit a constant approximation ratio.

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1With no loss of generality, in the computations we neglect the $\varepsilon$ values, needed to “guide” GREEDY in the selection of the $L$-items.
Proof. Hereafter, we show that it is \( \mathcal{NP} \)-complete to determine weights \( \tilde{w}_j \) in order to obtain a solution which is within a constant factor of the best possible solution for \( \mathcal{L} \). Consider an instance \( I \) of the decision problem PARTITION \([6]\), where \( m \) integer numbers \( a_1, \ldots, a_m \) are given and the question is whether there exists a subset of these numbers with total sum equal to \( b = \frac{1}{2} \sum_{i=1}^{m} a_i \).

Starting from \( I \), we define the following instance \( I' \) of the Objective-pricing Leader’s Problem, where \( M \) is a large enough number: The leader \( \mathcal{L} \) has \( m + 1 \) items with profits corresponding to the values of instance \( I \), i.e. \( w_j = a_j \), and one additional item with weight \( w_{m+1} = M \). The follower \( \mathcal{F} \) has only one item of weight \( w_{m+2} = M + 1 \). The knapsack capacity is set to \( c = M + b \).

As observed above, \( \mathcal{L} \) can only place its items before the \( \mathcal{F} \)-item, with a marginal positive cost in the objective, or after the \( \mathcal{F} \)-item, with a utility equal to their full weight.

If \( I \) is a YES-instance of PARTITION, then there exists a subset \( S \) with \( \sum_{i \in S} a_i = b \). The optimal strategy of \( \mathcal{L} \) will submit this set \( S \) first, before the \( \mathcal{F} \)-item of weight \( M + 1 \). Now the \( \mathcal{F} \)-item \( m + 2 \) is blocked since \( b + M + 1 > c \). However, the \( \mathcal{L} \)-item of weight \( w_{m+1} \) can still be packed and, by setting \( \tilde{w}_{m+1} := 0 \), \( \mathcal{L} \) gets a “gain” equal to \( M \).

Otherwise, if \( I \) is a NO-instance, there are two cases to consider:

1. \( \mathcal{L} \) lets \( \mathcal{F} \) packs its item: Hence, the optimal strategy of \( \mathcal{L} \) will not submit any items before \( w_{m+2} \) thus setting \( W = 0 \) and gaining at most \( b - 1 \).
2. \( \mathcal{L} \) blocks the \( \mathcal{F} \)-item by submitting, before \( m + 2 \), a subset of items with total weight \( W \geq b+1 \). Then the follower item \( m + 2 \) is blocked but also \( m + 1 \) of \( \mathcal{L} \) with weight \( w_{m+1} = M \) does not fit anymore. However, \( \mathcal{L} \) can submit all its remaining items and gets them packed by GREEDY. This results in a gain of at most \( 2b - W \leq b - 1 \).

Summarizing, the instance \( I \) of PARTITION has a YES answer if and only if the optimal weight obtained by the leader in the Stackelberg subset sum problem is equal to \( M \) while it is less than \( b \) for any NO answer. This rules out a polynomial time approximation algorithm with any constant approximation ratio \( \rho \), since we can choose, e.g., \( M > \rho b \). \( \square \)

3. Constraint-control model

We now turn to the Constraint-control model illustrated by Problem \([5]\). In this case the coefficients in the objective function for \( \mathcal{F} \) are fixed and equal to the original given weights \( w_j \). Instead, the leader can modify the weights of the \( \mathcal{L} \)-items in the constraint.

It is not hard to see that, for this version of the problem, the solution structure is somehow similar to that of Program \([3]\), however this variant is slightly less complex to solve. As before, recalling that here the efficiency of an item \( j \) in the follower’s problem is \( e_j = w_j/\tilde{w}_j \), the leader can position any item \( j \in \mathcal{L} \) before the \( \mathcal{F} \)-items by accepting a marginal loss, i.e., setting \( \tilde{w}_j := w_j - \varepsilon < 0 \). Denote the total weight of these items again by \( W_1 \leq c \). Then, \( \mathcal{F} \) will include some of its items following GREEDY with total weight \( F(W_1) \). Finally, when \( \mathcal{F} \) has no more items to include, the leader may set its objective value using the residual capacity \( \tilde{c} = c - W_1 - F(W_1) \). It suffices to pick the smallest remaining \( \mathcal{L} \)-item of weight \( w' \) (if it fits) and increase its weight up to \( c - W_1 - F(W_1) \) so that the capacity is fully consumed. All the remaining \( \mathcal{L} \)-item weights should be increased.
to a large enough value (e.g., $\bar{c} + 1$) in order to guarantee that Greedy actually picks the desired item $w'$.

A straightforward algorithm for determining the optimal solution for $L$ would consider each item $w' \in L$ as a candidate for the smallest remaining item. For each of these $|L|$ choices, one can run a simple one-dimensional dynamic programming by reaching to determine all weight values $W_1$ which can be reached by a subset of $L \setminus \{w'\}$. By a suitable implementation of the dynamic programming algorithm, we may state:

**Theorem 4.** In the Constraint-control Leader’s problem $L$, the optimal weights $\bar{w}_j$ for $j \in L$ can be determined in pseudopolynomial time of $O(n^{3/2}c)$.

**Proof.** For each possible candidate as smallest remaining item $w' \in L$, we may run a dynamic programming by reaching algorithm in order to determine all weight values $W_1$ that can be achieved by a subset of $L \setminus \{w'\}$. This can be done in $O(nc)$ time e.g. using a one-dimensional reduction similar to the algorithm devised for the objective-control model. For every reachable value $W_1$, Greedy can be applied for the $F$-items yielding $F(W_1)$ and requiring at most $O(nc)$ time (after sorting the $F$-items once). If $\bar{c} = c - W_1 - F(W_1) > w'$, we keep $\bar{c} - w'$ as a candidate for the objective function value.

In this way we consider every item in $L$ as a potential candidate for $w'$ and finally keep the best objective function value for the optimal solution. All together, this yields an $O(n^2c)$ pseudopolynomial running time.

It is possible to improve the running time of the algorithm by considering a slightly more involved implementation of the dynamic programming algorithm which works as follows (we omit the details on the rounding procedure to obtain integer valued quantities when required): Given a parameter $k$ (to be defined later) the set $L$ is partitioned into $k$ subsets $L_i$, $i = 1, \ldots, k$, of (roughly) equal size, i.e. $|L_i| \approx |L|/k$. Instead of running the dynamic programming iteration separately for each item $w' \in L$, we perform the computation in $k$ phases, considering all items $w' \in L_i$ in one phase.

In each such phase we first compute an initialization of the dynamic programming array taking all items in $L \setminus L_i$ into account which takes $O(nc)$ time. This array remains unchanged and is stored until the completion of the current phase. Then we pick one candidate $w' \in L_i$ and continue the computation from the above initialization by adding all items in $L_i \setminus \{w'\}$. This takes $O(n/k \cdot c)$ time and gives all values $W_1$ for the chosen $w'$.

Iterating this process for all candidates in $L_i$ (going back to the precomputed initialization of the array for each candidate) adds a total running time of $O((n/k)(n/k) \cdot c)$ for this phase.

Iterating this approach over all sets $L_i$ produces the dynamic programming array for all candidates $w' \in L$ and takes $O(k(n \cdot c + n^2/k^2 \cdot c))$ time. Choosing the parameter $k \approx \sqrt{|L|}$ we get an overall running time of $O(n^{3/2}c)$.

It remains to consider the outcome of Greedy and the resulting residual capacity $\bar{c}$, resp. the difference $\bar{c} - w'$, for each candidate $w'$. To do so, we can precompute the result of Greedy for every capacity value $W_1 = 0, 1, \ldots, c$ in $O(n \cdot c)$ time (after sorting the items of $F$) and store the outcome in an array $F(W_1)$. Now it is easy to determine the value
\[ \bar{c} - w' \text{ in constant time whenever we detect a new reachable value } W_1 \text{ for some candidate } w' \text{ and check for a new objective function value.} \]

The existence of a pseudopolynomial algorithm which is just linear in the capacity \( c \) would justify an even stronger hope for positive approximation results than for Objective-pricing. In fact, many discrete optimization problems with a linear pseudopolynomial algorithm permit even an FPTAS by rounding the objective value space. However, the following theorem shows that those expectations cannot be fulfilled.

**Theorem 5.** Unless \( P = NP \), the Constraint-control Leader’s Problem \([5]\) does not admit a constant approximation ratio.

**Proof.** Consider again an instance \( I \) of Partition, where \( m \) integer numbers \( a_1, \ldots, a_m \) are given and the question is whether there exists a subset of these numbers with total sum equal to \( b = \frac{1}{2} \sum_{i=1}^{m} a_i \).

In the following we will construct an instance of the Constraint-pricing model such that \( \mathcal{L} \) receives an objective function value of (almost) 1 if \( I \) is a YES-instance and 0 otherwise. Thus, any polynomial algorithm guaranteeing a constant approximation ratio for the Constraint-pricing Leader’s problem would also answer the decision problem for \( I \) in polynomial time.

Item set \( L \) consists of \( m \) items with \( w_j = 2a_j \) and another item \( w_{m+1} = \varepsilon \). Let \( k := \min\{j : 2^j > \sum_{i=1}^{m+1} w_i \} \) and add one additional item \( w_{m+2} = 2^k - 2b \). Thus, if \( I \) is a YES-instance then there is also a subset of \( L \) with total weight \( 2^k \). On the other hand, if \( I \) is a NO-instance then no subset of \( L \) can have a weight of \( 2^k \) (consider that \( w_{m+2} \) must be necessarily included in such a subset since \( \sum_{i=1}^{m+1} w_i < 2^k \)).

Item weights in \( F \) will be denoted for convenience by \( v_i \) and \( F \) consists of \( k - 1 \) items with weights \( v_i = 2^i \) for \( i = 1, \ldots, k-1 \). Clearly, the subsets of these items can reach every even weight value between 2 and \( 2^k - 2 \). Then we add an item \( v_k = 3 \) which means that now every weight value (even and odd) between 2 and \( 2^k + 1 \) can be reached. Finally, we add a large item \( v_{k+1} = 2^k + 2 \). This means that there are subsets of \( F \) with total weight for every value from 2 to \( 2^k + 3 \) except for the missing value \( 2^k + 3 \) which is larger than \( \sum_{i=1}^{k} v_i \) and larger than \( v_{k+1} + v_1 \).

Now we choose \( c = 2^{k+1} + 3 \). If \( I \) is a YES-instance, we argued above that \( \mathcal{L} \) can submit items before the \( F \)-items with total weight \( 2^k \) which leaves a residual capacity of \( 2^k + 3 \) for \( \mathcal{F} \). As shown above, there is no way for \( \mathcal{F} \) to fill this capacity completely but a residual capacity of \( \bar{c} = 1 \) will remain which \( \mathcal{L} \) can fill with item \( w_{m+1} \) thus gaining an objective function value of 1 - \( \varepsilon \).

If \( I \) is a NO-instance, then \( \mathcal{L} \) can place some subset of items before the \( F \)-items with total weight different from \( 2^k \) leaving a gap different from \( 2^k + 3 \). \( \mathcal{F} \) can fill any such gap completely with its items leaving an objective value of 0 for \( \mathcal{L} \). (In fact \( \mathcal{F} \) could not fill a gap of only 1, but \( \mathcal{L} \) can not enforce a gap of 1 since its sum of weights is < \( 2^{k+1} \)).

Observe that the above reduction is polynomial in the length of the encoded input since \( \sum_{i=1}^{m+1} w_i \leq 2m \cdot a_{\max} \) and thus \( k \) can be expressed as a logarithm of \( m \cdot a_{\max} \). \( \square \)
3.1. A simple variant. A variant of the Constraint-pricing model in which the leader’s objective is given by the actual weight values of its items can also be addressed. As we see below, this problem turns out to be easy to solve. The bilevel program associated with the leader problem is the following, with the usual variables:

\[
\max_{j \in L} \sum_{j \in L} \tilde{w}_j x_j \\
\text{s.t. } x \in \arg \max_{y \in \{0,1\}} \left\{ \sum_{j \in N} w_j y_j : \sum_{j \in L} \tilde{w}_j y_j + \sum_{i \in F} w_i y_i \leq c \right\} \\
x \in \{0,1\}^n; \ w \in \mathbb{R}^{|L|}
\]

As we observed above, if \( L \) decreases an item weight, i.e. \( \tilde{w}_j < w_j \), then \( j \) is positioned before all \( F \)-items since it has an efficiency larger than 1. In this case, reducing the weights by a marginal quantity gives \( L \) control over the items considered by \( F \) but with a minimal loss in the leader objective. So, by setting \( \tilde{w}_j := w_j - \epsilon \), and thus incurring a marginal loss for \( F \), item \( j \)—if the capacity allows it—will be included in the solution. Clearly, a \( L \)-item \( j \) is positioned after all \( F \)-items if \( \tilde{w}_j > w_j \).

This allows a very simple solution strategy: \( L \) positions all its items before \( F \) in arbitrary order. As soon as an item can not be packed, \( L \) reduces its weight to match exactly the remaining capacity. Thus, neglecting the marginal loss due to the \( \epsilon \), \( L \) is guaranteed a best possible objective function value \( \min\{c, \sum_{j \in L} w_j\} \).

4. LP relaxation

We now consider the special case in which \( F \) faces a continuous (linear relaxation) version of the above problems (3) and (5) so that variables \( x \) and \( y \) are continuous and contained in the interval \([0,1]\). \( F \) is therefore able to optimally solve the follower problem in polynomial (in fact linear) time. We now show that in this case both problems are trivial.

Let us first consider the Objective-control model (3). Since \( F \) can split items to be included in the solution set, if \( \sum_{j \in F} w_j \geq c \) then the follower can fill the entire knapsack capacity with its own items (or some of \( L \) with \( \tilde{w}_j = w_j \)) and \( L \) cannot gain anything. However, if \( w(F) < c \), i.e., the capacity exceeds the total weight of the \( F \)-items, then \( L \) can set \( \tilde{w}_j := 0 \) for all \( j \in L \), and pack as much as possible of its items in the residual capacity. In conclusion, the optimal solution value for \( L \) is always given by \( \min\{0, c - w(F)\} \).

Now let us consider the Constraint-control model (5). If \( \sum_{j \in F} w_j \geq c \) then the follower will again fill the entire knapsack capacity with its own items. \( L \) could only set \( \tilde{w}_j := w_j - \epsilon \) to have an item included in the knapsack before the \( F \)-items, which would yield a negative contribution. Thus, \( L \) remains at value 0. If \( \sum_{j \in F} w_j < c \) then \( L \) can pick an arbitrary item \( j' \in L \), preferably the item with minimal weight, and set \( \tilde{w}_{j'} := M \). All other items \( j \in L \) receive values \( \tilde{w}_j \) yielding an even lower efficiency. Now after packing all items in \( F \), \( F \) will choose a fractional part of \( j' \) by setting \( y_{j'} = \frac{c - w(F)}{M} \) to fill the knapsack completely.
The gain for \( L \) is \( (M - w_j) \frac{c - w(F)}{M} \) which tends to \( c - w(F) \) for \( M \to \infty \). Hence, the optimal solution value for \( L \) is again \( \min \{ 0, c - w(F) \} \).

For variant (6) of the Constraint-pricing model, in which the leader is interested in maximizing \( \sum_{j \in L} \tilde{w}_j \), the continuous model works exactly in the same way as the discrete case treated in Section 3.1. Only for the last item of \( L \) which cannot be packed completely, the continuous model does not require any special weight selection but guarantees the objective function value \( \min \{ c, \sum_{j \in L} w_j \} \) by default (disregarding \( \varepsilon \)).

5. Conclusions

In this paper we analyzed the complexity of a Stackelberg game for a Subset Sum pricing problem. We considered two variants of the problem in which the leader may revise the items weight in the follower objective or in the knapsack (capacity) constraint and the follower selects the solution items set using the natural Greedy algorithm heuristic. The objective function of the leader relates to the overall variation operated on the items weight. We showed that both versions of the problem are binary \( \mathcal{NP} \)-hard but can be solved by dynamic programming in pseudopolynomial time. Even though, both versions turn out to be non approximable within a constant factor. We also characterize some easy cases and show that the continuous relaxation versions of the problems permit straightforward solution procedures.

There are a number of open questions directly addressing the results presented here. In particular, a natural generalization of the addressed problem involves the binary knapsack problem, so that items are characterized by their weight and profit parameters. We also assumed that \( F \) adopts a computationally-bounded strategy. Removing this constraint and considering a follower able to optimally solve the resulting optimization problem makes the leader strategy much harder to be determined.

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References

[1] Briest, P., Hoefer, M., Gualà, L., Ventre, C.: On Stackelberg Pricing with Computationally Bounded Consumers, Networks 60(1) 31–44 (2012)
[2] Caprara, A., Carvalho, M., Lodi, A., Woeginger, G.J.: A study on the computational complexity of the bilevel knapsack problem, SIAM Journal on Optimization 24, 823–838 (2014)
[3] Chakravarty, S.R., Mitra, M., Sarkar, P.: A Course on Cooperative Game Theory, Cambridge University Press (2015)
[4] Darmann A., Nicosia, G., Pferschy, U., Schauer J.: The subset sum game, European Journal of Operational Research, 233(3), 539–549 (2014)
[5] Fearnley, J., Jurdziski, M.: Reachability in Two-Clock Timed Automata Is PSPACE-Complete, Information and Computation, 243, 26–36 (2015)
[6] Garey, M. R., Johnson D. S.: Computers and intractability: a guide to the theory of NP-completeness, W. H. Freeman (1979)
[7] Hansen, P., Jaumard, B., Savard, G.: A new branch-and-bound rules for linear bilevel programming, SIAM Journal on Scientific and Statistical Computing, 5(13), 1194–1217 (1992)
[8] Jeroslow, R. G.: The polynomial hierarchy and a simple model for competitive analysis, Mathematical Programming, 32(2), 146–164 (1985)
[9] Kellerer, H., Pferschy, U., Pisinger, D.: Knapsack Problems, Springer, (2004)
[10] Labbé, M., Violin, A.: Bilevel programming and price setting problems, Annals of Operations Research, 240, 141–169 (2016)
[11] Nicosia, G., Pacifici, A., Pferschy, U.: Price of Fairness for allocating a bounded resource, European Journal of Operational Research, 257, 933–943 (2017)
[12] Nicosia, G., Pacifici, A., Pferschy, U.: Brief announcement: On the fair subset sum problem, Proceedings of SAGT 2015, Springer Lecture Notes in Computer Science, 9347, 309–311 (2015)
[13] H. von Stackelberg. Marktform und Gleichgewicht (Market and Equilibrium). Verlag von Julius Springer (1934)