On integrability of one third-order nonlinear evolution equation

S. YU. SAKOVICH

Institute of Physics, National Academy of Sciences, 220072 Minsk, Belarus; sakovich@dragon.bas-net.by

Abstract

We study one third-order nonlinear evolution equation, recently introduced by Chou and Qu in a problem of plane curve motions, and find its transformation to the modified Korteweg–de Vries equation, its zero-curvature representation with an essential parameter, and its second-order recursion operator.

1 Introduction

In their recent work on the motions of plane curves [1], Chou and Qu found the following new third-order nonlinear evolution equation:

\[ u_t = \frac{1}{2} \left( (u_{xx} + u)^{-2} \right)_x. \] (1)

“We do not know if this equation arises from the AKNS- or the WKI-scheme”, wrote Chou and Qu in [1].

In the present paper, we study integrability of (1). In Section 2, we find a chain of Miura-type transformations, which relates the equation (1) with the modified Korteweg–de Vries equation (mKdV). In Section 3 using the obtained transformations and the well-known zero-curvature representation (ZCR) of the mKdV, we derive a complicated nontrivial ZCR of (1), which turns out to be neither AKNS- nor WKI-type one; and then we prove that simpler ZCRs of the equation (1) are trivial. In Section 4 we derive a second-order recursion operator of (1) from the obtained ZCR. Section 5 gives some concluding remarks.
2 Transformation to the mKdV

Let us try to transform the equation (1) into one of the well-known integrable equations. We do it, following the way described in [2]; further details on Miura-type transformations of scalar evolution equations can be found in [3].

First, we try to relate the equation (1) with an evolution equation of the form
\[ v_t = v^3 v_{xxx} + g(v, v_x, v_{xx}) \] (2)
by a transformation of the type
\[ v(x, t) = a(u, u_x, \ldots, u_{x\ldots x}) \] (3).

If there exists a transformation (3) between an evolution equation
\[ u_t = f(u, u_x, u_{xx}, u_{xxx}) : \quad \partial f / \partial u_{xxx} \neq \text{constant} \] (4)
and an equation of the form (2), then necessarily
\[ a = (\partial f / \partial u_{xxx})^{1/3}. \] (5)

Applying the transformation
\[ (x, t, u(x, t)) \mapsto (x, t, v(x, t)) : \quad v = -(u_{xx} + u)^{-1} \] (6)
to the equation (1), we find that (6) really works and relates (1) with the equation
\[ v_t = v^3 v_{xxx} + 3v^2 v_x v_{xx} + v^3 v_x. \] (7)

Second, we notice that (7) can be written in the form
\[ v_t = v^2 \left( vv_{xx} + v_x^2 + \frac{1}{2} v_x^2 \right)_x. \] (8)

Owing to this property, the equation (7) admits the transformation
\[ (y, t, w(y, t)) \mapsto (x, t, v(x, t)) : \quad x = w, \quad v = w_y, \] (9)
which turns out to relate (7) with
\[ w_t = w_{yyy} + \frac{1}{2} w^3_y. \] (10)

And, third, we make the transformation
\[ (y, t, w(y, t)) \mapsto (y, t, z(y, t)) : \quad z = w_y \] (11)
of (10) to the mKdV
\[ z_t = z_{yyy} + \frac{3}{2} z^2 z_y, \] (12)
for convenience in what follows, because \( z(y, t) = v(x, t) \).
3 Zero-curvature representation

3.1 Transformation of the mKdV’s ZCR

Using the chain of transformations (6), (9) and (11), we can derive a Lax pair for the equation (1), in the form of a ZCR containing an essential parameter, from the following well-known ZCR of the mKdV (12) [4, 5]:

\[ \Phi_y = A \Phi, \quad \Phi_t = B \Phi, \quad D_t A = D_y B - [A, B] \] (13)

with

\[ A = \begin{pmatrix} \alpha & \frac{i}{2} z \\ \frac{i}{2} z & -\alpha \end{pmatrix}, \quad B = \begin{pmatrix} \frac{i}{2}\alpha z^2 + 4\alpha^3 & \frac{i}{2} z_{yy} + \frac{i}{4} z^3 + i\alpha z_y + 2i\alpha^2 z \\ \frac{i}{2} z_{yy} + \frac{i}{4} z^3 - i\alpha z_y + 2i\alpha^2 z & -\frac{1}{2}\alpha z^2 - 4\alpha^3 \end{pmatrix} \] (14)

where \( \Phi (y, t) \) is a two-component column, \( D_t \) and \( D_y \) stand for the total derivatives, \([A, B]\) denotes the matrix commutator, and \( \alpha \) is a parameter.

First of all, we obtain a ZCR for the equation (7) through the transformations (9) and (11). Introducing the column \( \Psi : \Psi (x, t) = \Phi (y, t) \), we have \( \Phi_y = z \Psi_x \), which allows to rewrite the equation \( \Phi_y = A \Phi \) as

\[ \Psi_x = X \Psi, \] (16)

where \( X = z^{-1} A \) after substitution of \( z (y, t) = v (x, t) \),

\[ X = \begin{pmatrix} \alpha v^{-1} & \frac{i}{2} \\ \frac{i}{2} & -\alpha v^{-1} \end{pmatrix}. \] (17)

The equation \( \Phi_t = B \Phi \), due to \( \Phi_t = \Psi_t + w \Psi_x \) and \( w_t = z_{yy} + \frac{1}{2} z^3 \), leads to

\( \Psi_t = T \Psi, \) (18)

where \( T = B - \left( z^{-1} z_{yy} + \frac{1}{2} z^2 \right) A \) after substitution of \( z = v, z_y = vv_x \) and \( z_{yy} = v^2 v_{xx} + vv_x^2 \),

\[ T = \begin{pmatrix} -\alpha (vv_{xx} + v_x^2) + 4\alpha^3 & i\alpha vv_x + 2i\alpha^2 v \\ -i\alpha vv_x + 2i\alpha^2 v & \alpha (vv_{xx} + v_x^2) - 4\alpha^3 \end{pmatrix}. \] (19)

It is easy to check that the compatibility condition

\[ D_t X = D_x T - [X, T] \] (20)
of the equations (16) and (18), with the matrices $X$ (17) and $T$ (19), determines exactly the equation (7).

Then we can use the transformation (6). Substituting $v = - (u_{xx} + u)^{-1}$ into $X$ (17) and $T$ (19), we obtain a ZCR of the equation (11), in the sense that (20) is satisfied by any solution of (11). This ZCR, however, determines not the equation (11) itself, but a differential prolongation of (11),

$$u_{xxx} + u_t = \frac{1}{2} \left( (u_{xx} + u)^{-2} \right)_{xxx} + \frac{1}{2} \left( (u_{xx} + u)^{-2} \right)_x,$$

(21)
due to the structure of the transformed matrix $X$,

$$X = \begin{pmatrix} -\alpha (u_{xx} + u) & \frac{i}{2} \\ \frac{i}{2} & \alpha (u_{xx} + u) \end{pmatrix}. \quad (22)$$

The situation can be improved by a linear transformation of the auxiliary vector function $\Psi$,

$$\Psi \mapsto G \Psi, \quad \det G \neq 0, \quad (23)$$

which generates a gauge transformation of $X$ and $T$,

$$X \mapsto GXG^{-1} + (D_x G) G^{-1}, \quad T \mapsto GTG^{-1} + (D_t G) G^{-1}. \quad (24)$$

The choice of

$$G = \begin{pmatrix} \exp (\alpha u_x) & 0 \\ 0 & \exp (-\alpha u_x) \end{pmatrix} \quad (25)$$

leads through (24) to the following gauge-transformed matrix $X$, which does not contain $u_{xx}$:

$$X = \begin{pmatrix} -\alpha u & \frac{i}{2} \exp (2\alpha u_x) \\ \frac{i}{2} \exp (-2\alpha u_x) & \alpha u \end{pmatrix}. \quad (26)$$

Note that $u$ and $u_x$ are separated in (26), and a ZCR with such a matrix $X$ can determine an evolution equation exactly.

Now, from (19), (11), (21) and (25), we obtain the following matrix $T$, where $u_t$ and $u_{xt}$ have been expressed through (11) in terms of $x$-derivatives of $u$:

$$T = \begin{pmatrix} 4\alpha^3 & T_{12} \\ T_{21} & -4\alpha^3 \end{pmatrix}, \quad (27)$$
with
\[
T_{12} = -i \alpha \exp (2\alpha u_x) \left( \frac{2\alpha}{u_{xx} + u} + \frac{u_{xxx} + u_x}{(u_{xx} + u)^3} \right), \tag{28}
\]
\[
T_{21} = i \alpha \exp (-2\alpha u_x) \left( -\frac{2\alpha}{u_{xx} + u} + \frac{u_{xxx} + u_x}{(u_{xx} + u)^3} \right). \tag{29}
\]
It is easy to check that the matrices \(X\) \((26)\) and \(T\) \((27)-(29)\) constitute a ZCR of \((1)\), in the sense that the condition \((20)\) with these matrices determines exactly the equation \((1)\).

### 3.2 Simpler ZCRs are trivial

The obtained ZCR of \((1)\) is characterized by the complicated matrix \(X\) \((26)\) containing \(u_x\). Does the equation \((1)\) admit any simpler ZCR, with \(X = X(x, t, u)\), of any dimension \(n \times n?\) This problem can be solved by direct analysis of the condition \((20)\).

Substituting \(X = X(x, t, u)\) and \(T = T(x, t, u, u_x, u_{xx})\) into \((20)\) and replacing \(u_t\) by the right-hand side of \((1)\), we obtain the following condition, which must be an identity, not an ordinary differential equation restricting solutions of \((1)\):

\[
X_t - \frac{u_{xxx} + u_x}{(u_{xx} + u)^3} X_u = D_x T - [X, T] \tag{30}
\]

(here and below, subscripts denote derivatives, like \(T_{u_x} = \partial T/\partial u_x\)). Applying \(\partial/\partial u_{xxx}\) and \(\partial/\partial u_{xx}\) to the identity \((30)\), we obtain, respectively,

\[
T_{u_{xx}} = -(u_{xx} + u)^{-3} X_u, \tag{31}
\]
\[
T_{u_x} = (u_{xx} + u)^{-3} (D_x X_u - [X, X_u]). \tag{32}
\]

The compatibility condition \((T_{u_{xx}})_{u_x} = (T_{u_x})_{u_{xx}}\) for \((31)\) and \((32)\) is \(D_x X_u = [X, X_u]\), which is equivalent to

\[
X = P(x, t) u + Q(x, t): \quad P_x = [Q, P]. \tag{33}
\]

Now, we make use of gauge transformations \((24)\) with \(G = G(x, t)\), choose \(G\) to be any solution with \(\det G \neq 0\) of the system of ordinary differential equations \(G_x = -GQ\), and thus set \(Q = 0\) and \(P = P(t)\) in the gauge-transformed matrix \(X\) \((33)\). Then, \(T_u = T_{u_{xx}}\) follows from \(\partial/\partial u_x\) of \((30)\), and this leads through the identity \((30)\) to

\[
T = \frac{1}{2} P(t) (u_{xx} + u)^{-2} + K(t): \quad P_t = [K, P]. \tag{34}
\]
Finally, we make $K = 0$ by a gauge transformation (24) with $G = G(t)$ satisfying $G_t = -GK$ and $\det G \neq 0$, and thus obtain

$$X = Pu, \quad T = \frac{1}{2} P (u_{xx} + u)^{-2}, \quad P = \text{constant}, \quad (35)$$

with any matrix $P$ of any dimension $n \times n$. However, these matrices $X$ and $T$ (35) commute, $[X, T] = 0$, and the corresponding ZCR (20) is nothing but $n^2$ copies of the evident conservation law of the equation (1). In this sense, all the ZCRs sought, with $X = X(x, t, u)$, turn out to be trivial, up to gauge transformations (24) with arbitrary $G(x, t)$.

4 Recursion operator

Let us derive a recursion operator of the equation (1) from the matrix $X$ (26) of its ZCR. We do it, following the way described in [6] (see also references therein). The recursion operator comes from the problem of finding the class of evolution equations

$$u_t = f(x, t, u, u_x, \ldots, u_{x\ldots x}) \quad (36)$$

that admit ZCRs (20) with the predetermined matrix $X$ (26) and any $2 \times 2$ matrices $T(\alpha, x, t, u, u_x, \ldots, u_{x\ldots x})$ of any order in $u_{x\ldots x}$.

The characteristic form of the ZCR (20) of an equation (36), with $X$ given by (26), is

$$fC = \nabla_x S, \quad (37)$$

where $C$ is the characteristic matrix,

$$C = \frac{\partial X}{\partial u} - \nabla_x \left( \frac{\partial X}{\partial u_x} \right), \quad (38)$$

the operator $\nabla_x$ is defined as $\nabla_x H = D_x H - [X, H]$ for any (here, $2 \times 2$) matrix $H$, and the matrix $S$ is determined by

$$S = T - f \frac{\partial X}{\partial u_x}. \quad (39)$$

The explicit form of $C$ (38) for $X$ (26) is

$$C = \begin{pmatrix} 0 & -2i \alpha^2 e^{2\alpha u_x} (u_{xx} + u) \\ -2i \alpha^2 e^{-2\alpha u_x} (u_{xx} + u) & 0 \end{pmatrix}. \quad (40)$$
Under the gauge transformations (24) with any $G(\alpha, x, t, u, u_x, \ldots, u_{x...x})$, the characteristic matrix $C$ is transformed as $C \mapsto GC G^{-1}$ [7], and therefore $\det C$ is a gauge invariant. We have $\det C = 4\alpha^4 (u_{xx} + u)^2$ in the case of (40), as well as that the matrix $X$ [26] cannot be transformed by (24) into some $X$ containing no derivatives of $u$.

Computing $\nabla_x C$, $\nabla_x^2 C$ and $\nabla_x^3 C$, we find the cyclic basis to be three-dimensional, $\{C, \nabla_x C, \nabla_x^2 C\}$, with the closure equation

$$\nabla_x^3 C = c_0 C + c_1 \nabla_x C + c_2 \nabla_x^2 C,$$  

where

$$c_0 = \frac{u_{xxxx} + 2u_{xxx} + u_x}{u_{xx} + u} - 9 \frac{(u_{xxx} + u_x)(u_{xxxx} + u_x)}{(u_{xx} + u)^2} + 12 \frac{(u_{xxx} + u_x)^3}{(u_{xx} + u)^3},$$  

$$c_1 = 4 \frac{u_{xxxx} + u_{xx}}{u_{xx} + u} - 12 \frac{(u_{xxx} + u_x)^2}{(u_{xx} + u)^2} + 4\alpha^2 (u_{xx} + u)^2 - 1,$$  

$$c_2 = 5 \frac{u_{xxx} + u_x}{u_{xx} + u}.$$  

Setting $T$ to be traceless (without loss of generality), we decompose the matrix $S$ [39] over the cyclic basis as

$$S = s_0 C + s_1 \nabla_x C + s_2 \nabla_x^2 C,$$  

where $s_0$, $s_1$ and $s_2$ are unknown scalar functions of $x, t, u, u_x, \ldots, u_{x...x}$ and $\alpha$. Substitution of (45) into (37) leads us through (41) to

$$f = D_x s_0 + c_0 s_2, \quad s_0 = -D_x s_1 - c_1 s_2, \quad s_1 = -D_x s_2 - c_2 s_2,$$  

where the function $s_2$ remains arbitrary. Then, from (46) and (42)–(44), we obtain

$$f = (M - \lambda N) r,$$  

where $\lambda = 4\alpha^2$, $r(\lambda, x, t, u, u_x, \ldots, u_{x...x}) = s_2$ is any function, of any order
in \( u_{x...x} \), and the linear differential operators \( M \) and \( N \) are

\[
M = D_x^3 + 5\frac{u_{xxx} + u_x}{u_{xx} + u} D_x^2 + \left( 6\frac{u_{xxxx} + u_{xx}}{u_{xx} + u} + 2\frac{(u_{xxx} + u_x)^2}{(u_{xx} + u)^2} + 1 \right) D_x + \left( 2u_{xxxx} + 3u_{xxx} + u \right) \frac{u_{xx} + u}{u_{xx} + u} \\
+ 4\left( \frac{u_{xxx} + u_x}{(u_{xx} + u)^2} \left( u_{xxx} + u_{xx} \right) - 2\frac{(u_{xxx} + u_x)^3}{(u_{xx} + u)^3} \right),
\]

(48)

\[
N = (u_{xx} + u)^2 D_x + 2(u_{xx} + u) (u_{xxx} + u_x).
\]

(49)

Now, using the expansion

\[
r = r_0 + \lambda r_1 + \lambda^2 r_2 + \lambda^3 r_3 + \cdots,
\]

(50)

we obtain from (47) the expression for the right-hand side \( f \) of the represented equation (36), such that \( \partial f / \partial \lambda = 0 \) holds,

\[
f = Mr_0,
\]

(51)

as well as the recursion relations for the coefficients \( r_i(x, t, u, u_x, \ldots, u_{x...x}) \) of the expansion (50),

\[
Mr_{i+1} = Nr_i, \quad i = 0, 1, 2, \ldots
\]

(52)

The problem has been solved: for any set of functions \( r_0, r_1, r_2, \ldots \) satisfying the recursion relations (52), the expression (51) determines an evolution equation (36) admitting a ZCR (20) with the matrix \( X \) given by (26).

It only remains to notice that, if a set of functions \( r_0, r_1, r_2, \ldots \) satisfies the recursion relations (52), then the set of functions \( r_0', r_1', r_2', \ldots \) determined by \( r_i' = N^{-1}Mr_i \) (\( i = 0, 1, 2, \ldots \)) satisfies (52) as well. Therefore the evolution equation \( u_t = f' \) with \( f' = Mr_0' = MN^{-1}Mr_0 = MN^{-1}f = Rf \) admits a ZCR (20) with \( X \) (26) whenever an equation \( u_t = f \) does. Eventually, (48) and (49) lead us to the following explicit expression for the recursion operator \( R = MN^{-1} \) of the equation (1):

\[
R = \frac{1}{u_{xx} + u} D_x - \frac{1}{u_{xx} + u} \left( D_x + D_x^{-1} \right).
\]

(53)
5 Conclusion

Some remarks on the obtained results follow.

We succeeded in transforming the new Chou–Qu equation (1) into an integrable equation, the old and well-studied mKdV. The applicability of Miura-type transformations, however, is not restricted to integrable equations only. For instance, the original Miura transformation relates very wide (continual) classes of (mainly non-integrable) evolution equations [8].

We found the simplest nontrivial ZCR of the evolution equation (1). Its matrix \( X \) (26) contains \( u_x \). For this reason, such a ZCR cannot be detected by those existent methods, which assume, as a starting point, that \( X = X(x,t,u) \) must suffice in the case of evolution equations.

Of course, we could derive the obtained recursion operator (53) from the well-known recursion operator of the mKdV through the transformations found. However, we used a different method instead, mainly in order to illustrate, by this rather complicated example of \( X = X(\alpha,u,u_x) \) (26), how the method works algorithmically.

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