Radon transform and kinetic equations in tomographic representation

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Abstract

Statistical properties of classical random process are considered in tomographic representation. The Radon integral transform is used to construct the tomographic form of kinetic equations. Relation of probability density on phase space for classical systems with tomographic probability distributions is elucidated. Examples of simple kinetic equations like Liouville equations for one and many particles are studied in detail.

1 Introduction

The kinetic equations describing the behavior of classical system can be obtained using procedure suggested by Bogolyubov [1] where the chain of connected equations corresponding to \( N \) and \( N + 1 \) particles in the system was studied (see also [2]). The simplest Liouville equation for \( N + 1 \) particles probability density on phase-space provided the more complicated equation for \( N \) particles probability density after averaging over \((N + 1)\)th particle degree of freedom. The appropriate reduction procedure of the Bogolyubov chain of equations gives other kinetic equations, for example Boltzman equation with the collision term. The probability density on phase-space \( f(q,\vec{p},t) \equiv f(q_1,q_2,\ldots,q_N,p_1,p_2,\ldots,p_N,t) \) depends on positions and momenta of the particles. Recently it was suggested to use for description of quantum [3] and classical [4] states the tomographic probability distribution function (state tomogram). For classical particles the tomogram is Radon transform [5] of the probability density \( f(q,\vec{p},t) \) on the phase-space. The advantage of the tomographic probability representation is connected with the fact that in this representation both classical and quantum states are described by the identical objects-tomographic probability distributions. The physical meaning of the tomographic probability distribution is the following one. For one particle it is distribution of the particle position \( X \) but this position in measured is the reference frame in the system phase-space which rotated by angel \( \theta \) and before the rotation the axes \( q \rightarrow sq, \ p \rightarrow s^{-1}p \) where \( s \) it the scaling parameter. Then the scaled and rotated frame can be parameterized by two real parameters \( \mu = s \cos \theta \) and \( \nu = s^{-1} \sin \theta \). For one particle the tomogram is the probability
distribution function \( w(X, \mu, \nu) \) of random position \( X \) and two real reference-frame parameters \( \mu, \nu \).

For \( N \) particles the tomogram is the function \( w(\vec{X}, \vec{\mu}, \vec{\nu}) \) where \( \vec{X} \) is vector with particles positions \( X_j, \) \( j = 1, 2, ..., N \) and \( \vec{\mu} \) and \( \vec{\nu} \) vector components \( \mu_j = s_j \cos \theta_j \) and \( \nu_j = s_j^{-1} \sin \theta_j \) describe the scaling and rotation parameters of reference-frame axes associated with \( j \)th degree of freedom. The aim of this work is the study of classical kinetic equations in tomographic probability representation. We focus on the Bogolyubov chain written for the function \( f(\vec{q}, \vec{p}, t) \) and apply the Radon transform to write the chain of kinetic equations. On the other hand the consideration can be extended to quantum kinetic equations since in the tomographic picture the connection of tomograms for \( N + 1 \) and \( N \) particles is identical for classical and for quantum domains. This fact corresponds to analogous relations for Wigner functions [6] in quantum domain but the Wigner function can take negative values and it means that the function is not probability distribution function. Some aspects of the tomographic description of classical and quantum states were considered in [7], [8], [9], [10], [11], [12], [13], [14]. The paper is organized as follows. In Sec.2 we review the approach with the tomographic Radon map of classical probability distribution both for one particle and for \( N \) particles. In Sec.3 we obtain the Liouville equation in the tomographic probability representation. In Sec.4 we review the procedure with averaging the Liouville equation used in Bogolyubov chain approach. Also we consider the Radon representation. In Sec.5 the perspectives and conclusions are given.

2 Probability distribution and tomograms

Let us consider a particle with fluctuating momentum \( p \) and position \( q \). The state of such particle is associated with probability distributions \( f(q, p) \) on the particle phase space. The probability distribution is nonnegative and normalized function, i.e.

\[
f(q, p) \geq 0
\]

and

\[
\int f(q, p)dqdp = 1.
\]

The function \( f(q, p) \) can be considered also as generalized function, e.g. Dirac delta-function. For many particles the state of the system with fluctuations is described by the probability distributions \( f(\vec{q}, \vec{p}) \) where \( \vec{q} = (q_1, q_2, ..., q_N) \), \( \vec{p} = (p_1, p_2, ..., p_N) \) but this function is nonnegative and normalized as in one-dimensional case. The Radon transform of the probability distribution function \( f(q, p) \) is given by the formula

\[
w(X, \mu, \nu) = \int f(q, p)\delta(X - \mu q - \nu p)dqdp
\]

where the arguments \( X, \mu, \nu \) are real numbers. The function \( w(X, \mu, \nu) \) is nonnegative and normalized, i.e.

\[
\int w(X, \mu, \nu)dX = 1.
\]

This can be checked using integrations over \( X \) in [3] and property of Dirac delta-function. The function \( w(X, \mu, \nu) \) is called symplectic tomogram. The physical meaning of the symplectic tomogram is the
following one. If the parameters \( \mu \) and \( \nu \) are represented in the form \( \mu = s \cos \theta, \nu = s^{-1} \sin \theta \) the symplectic tomogram is the probability distribution of the particle position measured in the reference frame in the particle phase-space with following properties. Let us first scale the axes of the reference frame \( q \rightarrow sq = q', p \rightarrow s^{-1}p = p' \) and than rotate new rescaled axes \( q', p' \) by the angle \( \theta \). Then we get reference frame with axes where \( q'' = q' \cos \theta + p' \sin \theta \) and \( q'' = -q' \sin \theta + p' \cos \theta \). The coordinate \( X \) it the particle position measured in the reference frame with new coordinates \( q'' \) and \( p'' \). Thus taking \( \mu = s \cos \theta, \nu = s^{-1} \sin \theta \) we have

\[
X = \mu q + \nu p. \tag{5}
\]

The Radon transform (3) has inverse, i.e.

\[
f(q, p) = \frac{1}{4\pi^2} \int w(X, \mu, \nu)e^{i(X - \mu q - \nu p)} dXd\mu d\nu. \tag{6}
\]

The symplectic tomogram has the homogeneity property

\[
w(\lambda X, \lambda \mu, \lambda \nu) = \frac{1}{|\lambda|} w(X, \mu, \nu). \tag{7}
\]

This property follows from homogeneity property of Dirac delta-function \( \delta(\lambda y) = \frac{1}{|\lambda|} \delta(y) \). For many particles one has the multidimensional Radon transform

\[
w(\vec{X}, \vec{\mu}, \vec{\nu}) = \int f(\vec{q}, \vec{p}) \prod_{k=1}^{N} \delta(X_k - \mu_k q_k - \nu_k p_k) d\vec{q} d\vec{p}. \tag{8}
\]

This symplectic tomogram also determines the probability distribution \( f(\vec{q}, \vec{p}) \) on the phase-space of the system, i.e.

\[
f(\vec{q}, \vec{p}) = \frac{1}{(4\pi)^N} \int w(\vec{X}, \vec{\mu}, \vec{\nu}) \prod_{k=1}^{N} e^{i(X_k - \mu_k q_k - \nu_k p_k)} d\vec{X} d\vec{\mu} d\vec{\nu}. \tag{9}
\]

The symplectic tomogram (8) is nonnegative and normalized, i.e.

\[
w(\vec{X}, \vec{\mu}, \vec{\nu}) \geq 0 \tag{10}
\]

and

\[
\int w(\vec{X}, \vec{\mu}, \vec{\nu}) d\vec{X} = 1. \tag{11}
\]

The physical meaning of the symplectic tomogram \( w(\vec{X}, \vec{\mu}, \vec{\nu}) \) is the following one. It is the joint probability distribution function of \( N \) random variables (positions) \( X_k \) measured in specific reference frame in the system phase-space subjected to operation of rescaling \( q_k \rightarrow s_k q_k, p_k \rightarrow s_k^{-1} p_k \) and then rotated by the angle \( \theta_k \). Then the position \( X_k \) is measured in the reference frame where all the axes are transformed using this prescription. Important property which follows from the physical meaning of the tomogram \( w(\vec{X}, \vec{\mu}, \vec{\nu}) \) is the reduction property, i.e.

\[
\int w(\vec{X}, \vec{\mu}, \vec{\nu}) dX_N = \tilde{w}(\vec{X}', \vec{\mu}', \vec{\nu}'). \tag{12}
\]
Here \( \vec{X}' = (X_1, X_2, ..., X_{N-1}) \), \( \vec{\mu}' = (\mu_1, \mu_2, ..., \mu_{N-1}) \) and \( \vec{\nu}' = (\nu_1, \nu_2, ..., \nu_{N-1}) \). This reduction property corresponds to the reduction property of the probability distribution function

\[
\int f(\vec{q}, \vec{p}) dq_N dp_N = \tilde{f}(\vec{q}', \vec{p}')
\]

where \( \vec{q}' = (q_1, ..., q_{N-1}) \), \( \vec{p}' = (p_1, ..., p_{N-1}) \). The function \( \tilde{f}(\vec{q}', \vec{p}') \) is the probability distribution for the subsystem of \( N - 1 \) particles. This probability distribution determines the symplectic tomogram \( \tilde{w}(\vec{X}', \vec{\mu}', \vec{\nu}') \) by using the Radon transform

\[
\int f(\vec{q}', \vec{p}') \prod_{k=1}^{N-1} \delta(X_k - \mu_k q_k - \nu_k p_k) dq dp = \tilde{w}(\vec{X}', \vec{\mu}', \vec{\nu}').
\]

The symplectic tomogram of multidimensional system has the homogeneity property

\[
w(\lambda_1 X_1, \lambda_2 X_2, ..., \lambda_N X_N, \lambda_1 \mu_1, \lambda_2 \mu_2, ..., \lambda_N \mu_N, \lambda_1 \nu_1, \lambda_2 \nu_2, ..., \lambda_N \nu_N) = \left( \prod_{k=1}^{N} |\lambda_k| \right) w(\vec{X}, \vec{\mu}, \vec{\nu}).
\]

The function must satisfy the condition

\[
\int w(\vec{X}, \vec{\mu}, \vec{\nu}) \prod_{k=1}^{N} e^{i(X_k - \mu_k q_k - \nu_k p_k)} d\vec{X} d\vec{p} d\vec{q} \geq 0
\]

since this integral determines the probability distribution of the system on the phase-space.

### 3 Liouville equation in tomographic representation

The simplest kinetic equation in classical statistical mechanics is Liouville equation for the probability distribution \( f(q, p, t) \) of one particle on the phase-space. This equation reads

\[
\frac{\partial f(q, p, t)}{\partial t} + p \frac{\partial f(q, p, t)}{\partial q} - \frac{\partial U(q)}{\partial q} \frac{\partial f(q, p, t)}{\partial p} = 0.
\]

Here \( U(q) \) is potential energy and this equation corresponds to the Newton equation of motion

\[
\dot{p} = - \frac{\partial U(q)}{\partial q}.
\]

In the case of many degrees of freedom the Hamiltonian reads

\[
H(q_1, q_2, ..., q_N, p_1, p_2, ..., p_N) = \sum_{j=1}^{N} \frac{p_j^2}{2} + U(q_1, q_2, ..., q_N).
\]

The Newton equation of motion corresponding to this Hamiltonian has the form

\[
\dot{p}_j = - \frac{\partial U(q_1, ..., q_N)}{\partial q_j}.
\]
The Liouville kinetic equation in this case reads
\[
\begin{align*}
\frac{\partial f(q_1, q_2, ..., q_N, p_1, p_2, ..., p_N, t)}{\partial t} + \sum_{j=1}^{N} p_j \frac{\partial f(q_1, q_2, ..., q_N, p_1, p_2, ..., p_N, t)}{\partial q_j} \\
- \sum_{j=1}^{N} \frac{\partial f(q_1, q_2, ..., q_N, p_1, p_2, ..., p_N, t)}{\partial p_j} \frac{\partial U(q_1, q_2, ..., q_N, p_1, p_2, ..., p_N)}{\partial q_j} = 0.
\end{align*}
\] (21)

The Liouville kinetic equation can be rewritten in tomographic probability representation. It means that we replace the probability density \( f(q_1, q_2, ..., q_N, p_1, p_2, ..., p_N, t) \) by its Radon transform and write down equation for the tomographic probability density. For one degree of freedom we use the correspondence rules

\[
\begin{align*}
\phi(q, p, t) &\leftrightarrow w(X, \mu, \nu, t); \\
p \phi(q, p, t) &\leftrightarrow -\left(\frac{\partial}{\partial X}\right)^{-1} \frac{\partial}{\partial \nu} w(X, \mu, \nu, t); \\
\frac{\partial}{\partial q} \phi(q, p, t) &\leftrightarrow \mu \frac{\partial}{\partial X} w(X, \mu, \nu, t); \\
q \phi(q, p, t) &\leftrightarrow -\left(\frac{\partial}{\partial X}\right)^{-1} \frac{\partial}{\partial \mu} w(X, \mu, \nu, t); \\
\frac{\partial}{\partial p} \phi(q, p, t) &\leftrightarrow \nu \frac{\partial}{\partial X} w(X, \mu, \nu, t).
\end{align*}
\] (22)

Then we get the Liouville kinetic equation in the tomographic probability representation

\[
\begin{align*}
\frac{\partial w(X, \mu, \nu, t)}{\partial t} - \mu \frac{\partial w(X, \mu, \nu, t)}{\partial \nu} \\
- \frac{\partial U}{\partial q}(q - \left(\frac{\partial}{\partial X}\right)^{-1} \frac{\partial}{\partial \mu} w(X, \mu, \nu, t) \frac{\partial w(X, \mu, \nu, t)}{\partial X} = 0.
\end{align*}
\] (23)

For \( N \) degrees of freedom the correspondence rules have the form

\[
\begin{align*}
\phi(q_1, q_2, ..., q_N, p_1, p_2, ..., p_N, t) &\leftrightarrow w(X_1, X_2, ..., X_N, \mu_1, \mu_2, ..., \mu_N, \nu_1, \nu_2, ..., \nu_N, t); \\
p_j \phi(q_1, q_2, ..., q_N, p_1, p_2, ..., p_N, t) &\leftrightarrow -\left(\frac{\partial}{\partial X_j}\right)^{-1} \frac{\partial}{\partial \nu_j} w(X_1, X_2, ..., X_N, \mu_1, \mu_2, ..., \mu_N, \nu_1, \nu_2, ..., \nu_N, t); \\
\frac{\partial}{\partial q_j} \phi(q_1, q_2, ..., q_N, p_1, p_2, ..., p_N, t) &\leftrightarrow \mu_j \frac{\partial}{\partial X_j} w(X_1, X_2, ..., X_N, \mu_1, \mu_2, ..., \mu_N, \nu_1, \nu_2, ..., \nu_N, t); \\
q_j \phi(q_1, q_2, ..., q_N, p_1, p_2, ..., p_N, t) &\leftrightarrow -\left(\frac{\partial}{\partial X_j}\right)^{-1} \frac{\partial}{\partial \mu_j} w(X_1, X_2, ..., X_N, \mu_1, \mu_2, ..., \mu_N, \nu_1, \nu_2, ..., \nu_N, t); \\
\frac{\partial}{\partial p_j} \phi(q_1, q_2, ..., q_N, p_1, p_2, ..., p_N, t) &\leftrightarrow \nu_j \frac{\partial}{\partial X_j} w(X_1, X_2, ..., X_N, \mu_1, \mu_2, ..., \mu_N, \nu_1, \nu_2, ..., \nu_N, t).
\end{align*}
\] (24)

The operator \( \left(\frac{\partial}{\partial X}\right)^{-1} \) acting on a function \( \varphi(X) \) gives the function \( \Phi(X) \) satisfying the equality

\[
\frac{d\Phi(X)}{dX} = \varphi(X).
\] (25)
In case of representing the functions $\varphi(X)$ and $\Phi(X)$ in form of Fourier integrals, i.e.

$$\varphi(X) = \int \tilde{\varphi}(k)e^{ikX}dk$$  \hspace{1cm} (26)

and

$$\Phi(X) = \int \tilde{\Phi}(k)e^{ikX}dk$$  \hspace{1cm} (27)

we define the action of the operator $(\frac{\partial}{\partial X})^{-1}$ on the function $\varphi(X)$ by equality

$$(\frac{\partial}{\partial X})^{-1}\varphi(X) = \int \frac{1}{ik}\tilde{\varphi}(k)e^{ikX}dk$$  \hspace{1cm} (28)

which removes the ambiguity in choice of constant in the function $\Phi(X)$. Now we can write down the system with many degrees of freedom in the tomographic probability representation. To do this we make Radon transform in (21). Then we get

$$\frac{\partial w(\vec{X}, \vec{\mu}, \vec{\nu}, t)}{\partial t} - N \sum_{j=1}^{N} \mu_j \frac{\partial w(\vec{X}, \vec{\mu}, \vec{\nu}, t)}{\partial \nu_j} - \sum_{j=1}^{N} \frac{\partial U}{\partial q_j}(q_1 \to -\frac{\partial}{\partial X_1})^{-1} \frac{\partial}{\partial \mu_1}, q_2 \to -\frac{\partial}{\partial X_2})^{-1} \frac{\partial}{\partial \mu_2}, ..., \ q_N \to -\frac{\partial}{\partial X_N}^{-1} \frac{\partial}{\partial \mu_N}) \nu_j \frac{\partial w(\vec{X}, \vec{\mu}, \vec{\nu}, t)}{\partial X_j} = 0.$$  \hspace{1cm} (29)

The written equation describes evolution of tomographic probability distribution function which corresponds to standard Liouville equation for the probability density on the phase-space.

4 Reduced Liouville equation

The Bogolyubov chain of equations for probability distribution function of one particle can be obtained using Liouville equation for many particles. The ansatz is the following. One integrates the Liouville equation over $N-1$ pairs of variables $q_2, q_3, ..., q_N, p_2, p_3, ..., p_N$. Then one has the equation for one degree of freedom only with the contribution of the interaction of one particle with the rest. We provide this derivation considering system of two particles with the Liouville equation of the form

$$\frac{\partial}{\partial t} f(q_1, q_2, p_1, p_2, t) + p_1 \frac{\partial f(q_1, q_2, p_1, p_2, t)}{\partial q_1} + p_2 \frac{\partial f(q_1, q_2, p_1, p_2, t)}{\partial q_2} - \frac{\partial U(q_1, q_2)}{\partial q_1} \frac{\partial f(q_1, q_2, p_1, p_2, t)}{\partial p_1} - \frac{\partial U(q_1, q_2)}{\partial q_2} \frac{\partial f(q_1, q_2, p_1, p_2, t)}{\partial p_2} = 0.$$  \hspace{1cm} (30)

We integrate this equation over variables of second particle $q_2, p_2$. Let us assume that potential energy depends on distance between the particles, i.e.

$$U(q_1, q_2) = U(|q_1 - q_2|) = U(|X_{1,2}|),$$

$$X_{1,2} = q_1 - q_2.$$  \hspace{1cm} (31)
Then we get
\[
\frac{\partial}{\partial t} \tilde{f}(q_1, p_1, t) + p_1 \frac{\partial}{\partial q_1} \tilde{f}(q_1, p_1, t) + \int p_2 \frac{\partial}{\partial q_2} f(q_1, q_2, p_1, p_2, t) dq_2 dp_2 -
\]
\[- \int U'(|X_{1,2}|) \left( \frac{\partial}{\partial q_1} \right)_{q_1 = q_2} f(q_1, q_2, p_1, p_2, t) dq_2 dp_2 +
\]
\[+ \int U'(|X_{1,2}|) \left( \frac{\partial}{\partial p_1} \right)_{q_1 = q_2} f(q_1, q_2, p_1, p_2, t) dq_2 dp_2 = 0. \quad (32)
\]
We use the notation
\[
\tilde{f}(q_1, p_1, t) = \int f(q_1, q_2, p_1, p_2, t) dq_2 dp_2.
\quad (33)
\]
This equation can be transformed and analogous procedure can be done with Liouville equations \cite{21}. Now we make the reduction of tomographic Liouville equation \cite{20} on the example of two particles. The equation \cite{20} for two particles reads
\[
\frac{\partial w(X_1, X_2, \mu_1, \mu_2, \nu_1, \nu_2, t)}{\partial t} - (\mu_1 \frac{\partial}{\partial \nu_1} + \mu_2 \frac{\partial}{\partial \nu_2})w(X_1, X_2, \mu_1, \mu_2, \nu_1, \nu_2, t) -

-U'(\left( \frac{\partial}{\partial X_1} \right)^{-1} \frac{\partial}{\partial \mu_1} - (\frac{\partial}{\partial X_2})^{-1} \frac{\partial}{\partial \mu_2})\text{sgn}\left[ -\left( \frac{\partial}{\partial X_1} \right)^{-1} \frac{\partial}{\partial \mu_1} + (\frac{\partial}{\partial X_2})^{-1} \frac{\partial}{\partial \mu_2} \right]w(X_1, X_2, \mu_1, \mu_2, \nu_1, \nu_2, t) -

-U'(\left( \frac{\partial}{\partial X_1} \right)^{-1} \frac{\partial}{\partial \mu_1} - (\frac{\partial}{\partial X_2})^{-1} \frac{\partial}{\partial \mu_2})\text{sgn}\left[ -\left( \frac{\partial}{\partial X_2} \right)^{-1} \frac{\partial}{\partial \mu_2} + (\frac{\partial}{\partial X_1})^{-1} \frac{\partial}{\partial \mu_1} \right]w(X_1, X_2, \mu_1, \mu_2, \nu_1, \nu_2, t) = 0. \quad (34)
\]
We can get the reduction of this equation for tomogram describing the state of one particle integrating it over variable \(X_2\). One has
\[
\frac{\partial \tilde{w}(X_1, \mu_1, \nu_1, t)}{\partial t} - \mu_1 \frac{\partial}{\partial \nu_1} \tilde{w}(X_1, \mu_1, \nu_1, t) +

+ \int dX_2 \left[ -U'(\left( \frac{\partial}{\partial X_1} \right)^{-1} \frac{\partial}{\partial \mu_1} - (\frac{\partial}{\partial X_2})^{-1} \frac{\partial}{\partial \mu_2})\text{sgn}\left[ -\left( \frac{\partial}{\partial X_1} \right)^{-1} \frac{\partial}{\partial \mu_1} + (\frac{\partial}{\partial X_2})^{-1} \frac{\partial}{\partial \mu_2} \right]w(X_1, X_2, \mu_1, \mu_2, \nu_1, \nu_2, t) -

-U'(\left( \frac{\partial}{\partial X_1} \right)^{-1} \frac{\partial}{\partial \mu_1} - (\frac{\partial}{\partial X_2})^{-1} \frac{\partial}{\partial \mu_2})\text{sgn}\left[ -\left( \frac{\partial}{\partial X_2} \right)^{-1} \frac{\partial}{\partial \mu_2} + (\frac{\partial}{\partial X_1})^{-1} \frac{\partial}{\partial \mu_1} \right]w(X_1, X_2, \mu_1, \mu_2, \nu_1, \nu_2, t) \right] = 0. \quad (35)
\]
The obtained equation \cite{35} is consistent with taking Radon transform of the reduced Liouville equation \cite{32}. The Bogolyubov chain of equations in case of many particles can be obtained by analogous procedure applied to equation \cite{20}. In this case we integrate the equation over variables \(X_2, X_3, ..., X_N\).

5 Conclusion

We point out the main results of the work. We considered Liouville equation for \(N\) particles and using the integral Radon transform of the probability density on phase-space rewrote the equation in the tomographic probability representation. The Bogolyubov chain of the equations obtained by averaging the Liouville equations over subsystem degrees of freedom is also written in the tomographic probability representation. We hope to extend in future publications the tomographic representation analysis to the quantum kinetic equations using the Bogolyubov chain approach to Moyal equation \cite{15} for Wigner function.
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References

[1] N. N. Bogolyubov, Selected articles, three volumes, in Russian, Naukova Dumka, Kiev (1966)
[2] N. N. Jr. Bogolyubov and B. I. Sadovnikov, Some Aspects in the Statistical Mechanics, Higher School, Moscow (1975)
[3] S. Mancini, V. I. Man’ko, and P. Tombesi, Phys. Lett. A, 213, 1 (1996)
[4] Olga Man’ko and V. I. Man’ko, J. Russ. Laser Res., 18, 407 (1997)
[5] J. Radon, Ber. Sachs. Akad. Wiss., Leipzig, 69, 262 (1917)
[6] E. Wigner, Phys. Rev., 40, 749 (1932)
[7] V. N. Chernega, O. V. Man’ko, V. I. Man’ko, O. V. Pilyavets, and V. G. Zborovskii, J. Russ. Laser Res., 27, 132 (2006)
[8] V. N. Chernega and V. I. Man’ko, J. Russ. Laser Res., 28, 103 (2007)
[9] V. N. Chernega and V. I. Man’ko, J. Russ. Laser Res., 29, 505 (2008)
[10] V. N. Chernega and V. I. Man’ko, J. Russ. Laser Res., 29, 43 (2008)
[11] V. N. Chernega and V. I. Man’ko, J. Russ. Laser Res., 28, 535 (2007)
[12] V. N. Chernega and V. I. Man’ko, J. Russ. Laser Res., 29, 347 (2008)
[13] V. N. Chernega and V. I. Man’ko, J. Russ. Laser Res., 30, 359 (2009)
[14] V. I. Man’ko, V. I. Man’ko, and V. N. Chernega, in Fizika atomnogo yadra i elementarnikh chastic, Proceedings of XXXIXth and XLth winter school, Publishers PIYaF RAN, St.Petersburg, 261 (2007) [in Russian]
[15] J. E. Moyal, Proc.Camb.Phil.Soc, 45, 99 (1949)