ON BASIC PROPERTIES OF JUMPING FINITE AUTOMATA

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We complete the initial study of jumping finite automata, which was started in a former article of Meduna and Zemek [6]. The open questions about basic closure properties are solved. Besides this, we correct erroneous results presented in the article. Finally, we point out important relations between jumping finite automata and some other models studied in the literature.

Keywords: Jumping Finite Automata, Insertion-Deletion Systems

1. Introduction

In 2012, Meduna and Zemek [6] introduced general jumping finite automata as a model of discontinuous information processing. A general jumping finite automaton (GJFA) is described by a finite set $Q$ of states, a finite alphabet $\Sigma$, a finite set $R$ of rules from $Q \times \Sigma^* \times Q$, an initial state $s \in Q$, and a set $F \subseteq Q$ of final states. In a step of the computation, the automaton switches from a state $q$ to a state $r$ using a rule $(q, v, r) \in R$ and deletes a factor equal to $v$ from any part of the input word. The choices of the rule used and of the factor deleted are made nondeterministically. A word can be accepted only if there is a computation resulting in the empty word.

There is an infinite hierarchy of GJFA according to the maximum length of factor deleted in a single step – a GJFA is of degree $n$ if $|v| \leq n$ for each $(q, v, r) \in R$. A GJFA of degree 1 is called a jumping finite automaton (JFA). Bold symbols $\text{JFA}$ and $\text{GJFA}$ denote the classes of languages accepted by these types of automata.

The present paper contains the following contributions:

(i) We correct erroneous claims from [6] and [7] about the closure properties of the class $\text{GJFA}$ – in fact it is neither closed under homomorphism and under inverse homomorphism.

(ii) We answer the open questions about closure properties of $\text{GJFA}$ formulated in these two publications. Specifically, we disprove the closure under shuffle, Kleene star, and Kleene plus, and prove the closure under reversal.

*Research supported by the Czech Science Foundation grant GA14-10799S and the GAUK grant No. 52215.
(iii) We relate the new models with the existing ones, pointing out that the expressive power of GJFA is equivalent to a basic type of graph-controlled insertion systems, and that the intersection emptiness of GJFA is undecidable.

2. Preliminaries
As described above, a GJFA is a quintuple $M = (Q, \Sigma, R, s, F)$. The following formal description of the computation performed by a GJFA was introduced in [6].

**Definition 1.** Any string from the language $\Sigma^*Q\Sigma^*$ is called a *configuration* of $M$. For $q, r \in Q$ and $t_1, t_2, t_1', t_2', v \in \Sigma^*$, we write

$$t_1qt_2 \rightsquigarrow_M t_1'rt_2'$$

if $t_1t_2 = t_1't_2'$ and $(q, v, r) \in R$. By $\rightsquigarrow^*_M$ we denote the reflexive-transitive closure of the binary relation $\rightsquigarrow_M$ over configurations, i.e. $c \rightsquigarrow^*_M c'$ for configurations $c, c'$ of $M$ if and only if $c = c'$ or

$$d_1 \rightsquigarrow_M \cdots \rightsquigarrow_M d_i$$

for some configurations $d_1, \ldots, d_i$ of $M$ with $c = d_1$, $c' = d_i$, $i \geq 2$. Finally,

$$L(M) = \{u_1u_2 \mid u_1, u_2 \in \Sigma^*, f \in F, u_1su_2 \rightsquigarrow^*_M f\}$$

is the language accepted by $M$. If $M$ is fixed, we write just $\rightsquigarrow$ and $\rightsquigarrow^*$.

The placement of the state symbol $q$ in a configuration $u_1qu_2$ marks the position of an imaginary tape head. Note that this information is redundant – the head is allowed to move anywhere in each step.

In the present paper we heavily use the natural notion of sequential insertion, as it was described e.g. in [3] and [4]:

**Definition 2.** Let $K, L \subseteq \Sigma^*$ be languages. The *insertion* of $K$ to $L$ is

$$L \leftarrow K = \{u_1vu_2 \mid u_1u_2 \in \Sigma^*, v \in K\}.$$ 

More generally, for each $k \geq 1$ we denote

$$L \leftarrow^k K = (L \leftarrow^{k-1} K) \leftarrow K,$$

$$L \leftarrow^* K = \bigcup_{i \geq 0} L \leftarrow^i K,$$

where $L \leftarrow^0 K$ stands for $L$. In expressions with $\leftarrow$ and $\leftarrow^*$, a singleton set $\{w\}$ may be replaced by $w$. A chain $L_1 \leftarrow L_2 \leftarrow \cdots \leftarrow L_d$ of insertions is evaluated from the left, e.g. $L_1 \leftarrow L_2 \leftarrow L_3$ means $(L_1 \leftarrow L_2) \leftarrow L_3$. According to [2], $L \subseteq \Sigma^*$ is a *unitary language* if $L = w \leftarrow^* K$ for $w \in \Sigma^*$ and finite $K \subseteq \Sigma^*$.
Next, we fix additional notation that turns out to be very useful in our proofs. The notions of paths and labels naturally correspond to graphical representations of GJFA, where vertices stand for states and labeled directed edges stand for rules.

**Definition 3.** Let \( M = (Q, \Sigma, R, s, F) \) be a GJFA. Each sequence of the form

\[
(q_0, v_1, q_1) (q_1, v_2, q_2) \cdots (q_{d-1}, v_d, q_d) \in R^*
\]

with \( d \geq 1 \) is a path from \( q_0 \in Q \) to \( q_d \in Q \). The path is accepting if \( q_0 = s \) and \( q_d \in F \). The labeling of the path is the sequence \( v_1, v_2, \ldots, v_d \) of words from \( \Sigma^* \).

The lemma below is very natural, its proof only has to deal with formal combination of different approaches. The symbol \( \epsilon \) stands for the empty word.

**Lemma 1.** Let \( M = (Q, \Sigma, R, s, F) \) be a GJFA. For each \( p, r \in Q \) and \( w, w'' \in \Sigma^* \) the following are equivalent:

\[(i)\ u_1p_2 \sim_M^* u''_1r''_2 \text{ for } u_1, u_2, u''_1, u''_2 \in \Sigma^* \text{ with } u_1u_2 = w \text{ and } u''_1u''_2 = w''.
(ii) p = r \text{ and } w = w'' \text{, or}
\]

\[
w \in w'' \leftarrow v_d \leftarrow v_{d-1} \leftarrow \cdots \leftarrow v_2 \leftarrow v_1,
\]

where \( v_1, v_2, \ldots, v_d \) is a labeling of a path from \( p \) to \( r \), \( d \geq 1 \).

**Proof.** First, assume that (i) holds and denote \( c = u_1p_2, c'' = u''_1r''_2 \). If \( c = c'' \), then \( p = r \) and \( w = w'' \), so we are done. Otherwise, \( d_1 \sim_M \cdots \sim_M d_i \) of \( M \) with \( c = d_1, c'' = d_i \), \( i \geq 2 \). We use induction by \( i \).

If \( i = 2 \), then \( c \sim_M c'' \) and according to the definition of \( \sim_M \), there are \( t_1, t_2, t''_1, t''_2, v \in \Sigma^* \) such that \( u_1 = t_1, u_2 = vt_2, u''_1 = t''_1, u''_2 = t''_2 \), and \( (p, v, r) \in R \). Thus, \( w \in w'' \leftarrow v \) and \( (p, v, r) \) is a path from \( p \) to \( r \).

If \( i \geq 3 \), then \( c \sim_M c' \sim_M c'' \) for some configuration \( c' = u'_1qu'_2 \). Denote \( w' = u'_1u'_2 \). According to the induction assumption applied to \( q, r, w', w'' \) we obtain a path denoted by

\[
(q_1, v_2, q_2) \cdots (q_{d-1}, v_d, q_d)
\]

with \( d \geq 2, q_1 = q, q_d = r \), and \( w' \in w'' \leftarrow v_d \leftarrow \cdots \leftarrow v_2 \). To conclude the proof it is enough to show that \( (p, v, q) \in R \) and \( w \in w' \leftarrow v \) for \( v = v_1 \), which both follow from \( c \sim_M c' \) according to the above analysis of the case \( i = 2 \).

Second, let (i) hold. If \( p = r \) and \( w = w'' \), then \( u_1p_2 = u''_1r''_2 \) for \( u_1 = u''_1 = \epsilon, u_2 = w, \) and \( u''_2 = w'' \). Otherwise, we fix the path

\[
(q_0, v_1, q_1) (q_1, v_2, q_2) \cdots (q_{d-1}, v_d, q_d)
\]

from \( q_0 = p \) to \( q_d = r \), \( d \geq 1 \), and use induction by \( d \). Denote \( v = v_1 \).

Let \( d = 1 \). We have \( (p, v, r) \in R \) and \( w \in w'' \leftarrow v \). According to the definition of \( \leftarrow \), there are \( t_1, t_2 \in \Sigma^* \) such that \( w = t_1vt_2 \) and \( w'' = t_1t_2 \). Obviously, \( t_1pvt_2 \sim_M t_2 \) follows from the definition of \( \sim_M \). As \( \sim_M \) is a special case of \( \sim^*_M \), we are done.
Let \( d \geq 2 \). From (1) and the definition of \( \leftarrow \) it follows that
\[
w \in w' \leftarrow v
\] (2)
for some \( w' \in \Sigma^* \) with
\[
w' \in w'' \leftarrow v_d \leftarrow v_{d-1} \leftarrow \cdots \leftarrow v_2.
\] (3)
Denote \( q = q_1 \). According to the induction assumption applied to \( q, r, w', w'' \) we obtain
\[
u_1' q u_2' \dashv_M u''_1 r u_2''
\]
for some \( u_1', u_2', u''_1, u''_2 \in \Sigma^* \) with \( u_1' u_2' = w' \) and \( u''_1 u''_2 = w'' \).
It remains to show that \( u_1 pu_2 \dashv_M u'_1 q u'_2 \) for some \( u_1, u_2 \in \Sigma^* \) with \( u_1 u_2 = w \). Due to (2) there are \( t_1, t_2 \in \Sigma^* \) such that \( w = t_1 vt_2 \) and \( w' = t_1 t_2 \). From the definition of \( \dashv_M \), together with \( u_1' u_2' = w' = t_1 t_2 \) and \( (p, v, q) \in R \) it follows that
\[
t_1 p v t_2 \dashv_M u'_1 q u'_2.
\]
We conclude by denoting \( u_1 = t_1 \) and \( u_2 = vt_2 \).

**Corollary 1.** Let \( M = (Q, \Sigma, R, s, F) \) be a GJFA and \( w \in \Sigma^* \). Then \( w \in L(M) \) if and only if \( w = \epsilon \) and \( s \in F \), or
\[
w \in \epsilon \leftarrow v_d \leftarrow v_{d-1} \leftarrow \cdots \leftarrow v_2 \leftarrow v_1,
\] (4)
where \( v_1, v_2, \ldots, v_d \) is a labeling of an accepting path in \( M \), \( d \geq 1 \).

**Proof.** If \( w \in L(M) \), then \( w = u_1 u_2 \) for \( u_1, u_2 \in \Sigma^* \) with \( u_1 u_2 \dashv_M f \) and \( f \in F \).
We apply the forward implication of Lemma 1 to \( s, f, w, \) and \( \epsilon \). On the other hand, an accepting path ends in some \( f \in F \) and we apply the backward implication of Lemma 1 to \( s, f, w, \) and \( \epsilon \).

The above corollary suggests a generative approach to GJFA – the computation of a GJFA may be equivalently described in terms of inserting factors instead of deleting them. A word is accepted by a GJFA if and only if it can be composed by inserting factors to the empty word according to labels of an accepting path. This characterization of \( L(M) \) will be used very frequently throughout the paper, so we omit explicit referring to Corollary 1.

Next, we give two simple lemmas that imply the membership in GJFA for each language that can be described using finite languages and insertions.

**Lemma 2.** Each finite language \( K \subseteq \Sigma^* \) lies in GJFA.

**Proof.** The language \( K \) is accepted by the two-state GJFA \( M \) with
\[
M = \{ \{ q, r \}, \Sigma, R, q, \{ r \} \},
\]
\[
R = \{ (q, w, r) \mid w \in K \}.
\]
Indeed, all the accepting paths in $M$ are of length 1 and their labels are exactly the words from $K$.

**Lemma 3.** Let $L \subseteq \Sigma^*$ lie in GJFA and $K \subseteq \Sigma^*$ be finite. Then

(i) $L \leftarrow K$ lies in GJFA and

(ii) $L \leftarrow^* K$ lies in GJFA.

**Proof.** Let $M_L = (Q_L, \Sigma, R_L, s_L, F_L)$ be a GJFA recognizing $L$. To obtain $M_1$ with $L(M_1) = L \leftarrow K$, we put

\[
M_1 = (Q_1, \Sigma, R_1, s, F_L),
\]

\[
Q_1 = Q_L \cup \{s\},
\]

\[
R_1 = \{(s, v, s_L) \mid v \in K\} \cup R_L.
\]

First, let $w \in L \leftarrow K$, which means $w \in w' \leftarrow v$ for $w' \in L$ and $v \in K$. As $w' \in L = L(M_L)$, we have $w' = \epsilon$ and $s_L \in F_L$, or

\[
w' \in \epsilon \leftarrow v_d \leftarrow \cdots \leftarrow v_1
\]

for some accepting path $p_L$ in $M_L$ labeled by $v_1, \ldots, v_d \in \Sigma^*$, $d \geq 1$. In the case of $w' = \epsilon$ and $s_L \in F_L$, we have $w = v$ and $w \in L(M_1)$ due to $sv \cap_{M_1} s_L$. Otherwise, observe that $(s, v, s_L) p_L$ is an accepting path in $M_1$ and $w$ lies in

\[
w' \leftarrow v \subseteq (\epsilon \leftarrow v_d \leftarrow \cdots \leftarrow v_1) \leftarrow v,
\]

which matches the labeling of $(s, v, s_L) p_L$. Thus, $w \in L(M_1)$.

Second, let $w \in L(M_1)$, whence $w = \epsilon$ and $s \in F_L$, or

\[
w \in \epsilon \leftarrow v_d \leftarrow v_{d-1} \leftarrow \cdots \leftarrow v_2 \leftarrow v_1,
\]

where $v_1, v_2, \ldots, v_d$ is a labeling of an accepting path in $M_1, d \geq 1$. As $s \notin F_L$, we assume the second case. We have $w \in w' \leftarrow v_1$ for some

\[
w' \in \epsilon \leftarrow v_d \leftarrow v_{d-1} \leftarrow \cdots \leftarrow v_2.
\]

According to the construction of $M_1$, $v_1 \in K$ and $v_2, \ldots, v_d$ is a labeling of an accepting path in $M_L$. Thus, $w' \in L(M_L) = L$ and hence $w \in L \leftarrow K$.

To obtain $M_2$ with $L(M_2) = L \leftarrow^* K$, we put

\[
M_2 = (Q_2, \Sigma, R_2, s, F_L),
\]

\[
Q_2 = Q_L \cup \{s\},
\]

\[
R_2 = \{(s, v, s) \mid v \in K\} \cup \{(s, \epsilon, s_L)\} \cup R_L.
\]

First, let $w \in L \leftarrow^* K$, which means $w \in w' \leftarrow^i K$ for $w' \in L$, $i \geq 0$.

(i) If $i = 0$, then $w = w' \in L$ and we show that $L \subseteq L(M_2)$. Indeed, due to Corollary\[\Box\] for each $w' \in L$ it holds that $w' = \epsilon$ and $s_L \in F_L$, or

\[
w' \in \epsilon \leftarrow v_d \leftarrow v_{d-1} \leftarrow \cdots \leftarrow v_2 \leftarrow v_1,
\]

(5)
for some accepting path $p_L$ in $M_L$ labeled by $v_1, \ldots, v_d \in \Sigma^*$, $d \geq 1$. If $w' = \epsilon$ and $s_L \in F_L$, then $w' \in L(M_2)$ due to Corollary 1 applied to the path $(s, \epsilon, s_L)$ together with $w' \in \epsilon \leftarrow \epsilon$. Otherwise, $w' \in L(M_2)$ due to Corollary 1 applied to the path $(s, \epsilon, s_L) p_L$ together with

$$w' \in \epsilon \leftarrow v_d \leftarrow v_{d-1} \leftarrow \cdots \leftarrow v_2 \leftarrow v_1,$$

$$= \epsilon \leftarrow v_d \leftarrow v_{d-1} \leftarrow \cdots \leftarrow v_2 \leftarrow v_1 \leftarrow \epsilon.$$

(ii) Let $i \geq 1$. According to the definition of $\ni$, we have

$$w \in w' \ni v_i \ni \cdots \ni v_1,$$

where $v_1, \ldots, v_i \in K$ and $w' \in L$. As $L \subseteq L(M_2)$ and $s \notin F_L$, there is an accepting path $p_L$ in $M_2$ labeled by $v'_1, \ldots, v'_d$ with

$$w' \in \epsilon \leftarrow v'_d \ni \cdots \ni v'_1.$$

It remains to observe that

$$(s, v_1, s) \cdots (s, v_i, s) (s, \epsilon, s_L) p_L$$

is an accepting path in $M_2$ and

$$w \in w' \ni v_i \ni \cdots \ni v_1,$$

$$\subseteq \epsilon \ni v'_d \ni \cdots \ni v'_1 \ni \epsilon \ni v_i \ni \cdots \ni v_1.$$

Second, let $w \in L(M_2)$, whence $w = \epsilon$ and $s \in F_L$, or

$$w \in \epsilon \ni v_d \ni v_{d-1} \ni \cdots \ni v_2 \ni v_1,$$

such that there is an accepting path

$$p = (q_0, v_1, q_1) \cdots (q_{d-1}, v_d, q_d)$$

in $M_2$, $d \geq 1$. As $s \notin F_L$, we assume the second case. According to the construction of $M_2$, there is a unique $1 \leq c \leq d$ such that $(q_{c-1}, v_c, q_d) = (s, \epsilon, s_L)$. Denote $p = p_K (s, \epsilon, s_L) p_L$, where $p_K$ is labeled by words from $K$ and $p_L$ is an accepting path in $M_L$. Both $p_K$ and $p_L$ may stand for empty strings, their lengths are $c - 1$ and $d - c$ respectively. We get

$$w \in w' \ni \epsilon \ni v_{c-1} \ni \cdots \ni v_2 \ni v_1,$$

where

$$w' \in \epsilon \ni v_d \ni \cdots \ni v_{c+1} \subseteq L.$$

Together, $w \in L \leftarrow^{c-1} K \subseteq L \leftarrow^{c} K$. 

Let us give a few examples of GJFA languages that are used later in this paper and follow easily from the above lemmas. Note that a GJFA over an alphabet $\Sigma$ can be seen as operating over any alphabet $\Sigma' \supseteq \Sigma$.

**Example 1.** The following languages lie in GJFA:
(i) The trivial language $\Sigma^* = \epsilon \leftarrow \Sigma$ over an arbitrary $\Sigma$.
(ii) The language $\Sigma^* u \Sigma^* = \Sigma^* \leftarrow u$ for $u \in \Sigma^*$ over an arbitrary $\Sigma$.
(iii) The Dyck language $D$ over $\Sigma = \{a, \overline{a}\}$. We have $D = \epsilon \leftarrow a \overline{a}$.
(iv) Any semi-Dyck language $D_k$ over $\Sigma = \{a_1, \ldots, a_k, \overline{a_1}, \ldots, \overline{a_k}\}$. We have $D_k \leftarrow \{a_1 \overline{a_1}, \ldots, a_k \overline{a_k}\}$.
(v) Any unitary language.

However, there are GJFA languages that cannot be simply obtained from finite languages by applying Lemma 3, such as the following classical language that is not context-free and lies even in JFA. By $|w|_x$ we denote the number of occurrences of a letter $x \in \Sigma$ in a word $w \in \Sigma^*$.

**Example 2.** The JFA $M$ with

$$M = (\{q_0, q_1, q_2\}, \Sigma, R, q_0, \{q_0\})$$

$$R = \{(q_0, a, q_1), (q_1, b, q_2), (q_2, c, q_0)\}$$

accepts the language $L = \{w \in \Sigma^* | |w|_a = |w|_b = |w|_c\}$ over $\Sigma = \{a, b, c\}^*$.

The above example shows that the class GJFA is not a subclass of context-free languages, but it was pointed out in [6] that each GJFA language is context-sensitive. The class GJFA does not stick to classical measures of expressive power – in the next section we give examples of regular languages that do not lie in GJFA. As for JFA languages, in [7] the authors show that a language lies in JFA if and only if it is equal to the permutation closure of a regular language.

### 3. A Necessary Condition for Membership in GJFA

In order to formulate our main tools for disproving membership in GJFA, the following technical notions remain to be defined.

**Definition 4.** A language $K \subseteq \Sigma^*$ is a composition if $K = \{\epsilon\}$ or $K = \epsilon \leftarrow v_d \leftarrow v_{d-1} \leftarrow \cdots \leftarrow v_2 \leftarrow v_1$, for some $v_1, \ldots, v_d \in \Sigma^*$, $d \geq 1$. A composition $K$ is of degree $n \geq 0$ if $K = \{\epsilon\}$ or $|v_i| \leq n$ for each $i \in \{1, \ldots, d\}$. For each $n \geq 0$, let $\mathcal{UC}_n$ denote the class of languages $L$ that can be written as

$$L = \bigcup_{K \in \mathcal{C}} K,$$

where $\mathcal{C}$ is any (possibly infinite) set of compositions of degree $n$. We also denote $\mathcal{UC} = \bigcup_{n \geq 0} \mathcal{UC}_n$.

The class $\mathcal{UC}$ itself does not seem to be of practical importance – we use the membership in $\mathcal{UC}$ only as a necessary condition for membership in GJFA. The acronym $\mathcal{UC}$ stands for union of compositions.

**Lemma 4.** GJFA $\subseteq \mathcal{UC}$. 

Proof. Let $M = (Q, \Sigma, R, s, F)$ be a GJFA. Let $\mathcal{P}$ be the set of all accepting paths in $M$. According to Corollary 1, we have

$$L(M) \setminus \{\epsilon\} = \bigcup_{p \in \mathcal{P}} (\epsilon \leftarrow v_{p,d_p} \leftarrow v_{p,d_p-1} \leftarrow \cdots \leftarrow v_{p,2} \leftarrow v_{p,1}),$$

where $v_{p,1}, \ldots, v_{p,d_p}$ is the labeling of $p$, $d_p \geq 1$. As $\{\epsilon\}$ is a composition, we have $L(M) \subseteq \text{UC}_n$, where $n = \max \{|v| \mid q, r \in Q, (q, v, r) \in R\}$. □

The following lemma deals with the language $L = \{ab\}^*$, which serves as a canonical non-GJFA language in the proofs of our main results.

Lemma 5. The language $L = \{ab\}^*$ does not lie in GJFA.

Proof. Assume for a contradiction that $L \in \text{GJFA}$. Due to Lemma 4, $L \in \text{UC}$ and thus $L \in \text{UC}_n$ for some $n \geq 0$. If $n = 0$, observe that $L = \{\epsilon\}$, which is a contradiction. Otherwise, fix $w = (ab)^{n+1}$. According to the definition of $\text{UC}_n$, $w$ lies in a composition $K \subseteq L$ of the form

$$K = \epsilon \leftarrow v_d \leftarrow v_{d-1} \leftarrow \cdots \leftarrow v_2 \leftarrow v_1$$

of degree $n$. Due to $w \neq \epsilon$, there exists the least $c$ with $d \geq c \geq 1$ and $v_c \neq \epsilon$. Moreover,

$$K = K' \leftarrow v$$

for suitable $K'$ and $v = v_c$. Thus, $w = u_1vu_2$ for $u_1u_2 \in K'$. As $|v| \leq n$, at least one of the following cases holds:

(i) Assume that $|u_1| \geq 2$ and write $u_1 = abu_1$. If $v$ starts by $a$, we have $avbuv_1u_2 \in K$. If $v$ starts by $b$, we have $abvu_1u_2 \in K$.

(ii) Assume that $|u_2| \geq 2$ and write $u_2 = v_2ab$. If $v$ starts by $a$, we have $u_1v_2abv \in K$. If $v$ starts by $b$, we have $u_1v_2abv \in K$.

In each case, $K$ contains a word having some of the factors $aa, bb$. Thus $K \not\subseteq L$, which is a contradiction. Informally, each $w$ coming from a language $L \in \text{UC}_n$ must contain a factor $v$ of length at most $n$ that can be inserted to any other place in $u_1u_2$ such that the result stays in $L$. In the case of $L = \{ab\}^*$ this property fails. □

4. Closure Properties of GJFA Languages

The table below lists various unary and binary operators on languages. The symbols +, − tell that a class is closed or is not closed under an operator, respectively. A similar table was presented in [6, 7], containing several question marks. In this section we complete and correct these results. The symbol ♦ marks answers to open questions and the symbol ♦ marks corrections.
Before proving the new results, let us deal with the closure under intersection. The authors of [6, 7] claim that the theorem below follows from an immediate application of De Morgan’s laws to the results about union and complement. We find this argument invalid and present an explicit proof of the claim.

**Theorem 1.** GJFA is not closed under intersection.

**Proof.** Let \( \Sigma = \{a, \overline{a}\} \) and \( L = L(M) \) for
\[
M = \{(q, r) , \Sigma, R, q, \{r\}\},
\]
\[
R = \{(q, \overline{a}a, q), (q, a\overline{a}, r)\},
\]
as depicted in Figure 1. For each \( d \geq 1 \) there is exactly one accepting path of length \( d \) in \( M \). According to Corollary 1 we have
\[
L = \bigcup_{d \geq 1} K_d,
\]
where \( K_1 = \epsilon \leftarrow a\overline{a} \) and \( K_{i+1} = K_i \leftarrow a\overline{a} \) for \( i \geq 1 \). We show that
\[
D \cap L = \{a\overline{a}\}^*,
\]
(6)
where \( D \in \text{GJFA} \) is the Dyck language from Example 1 and \( \{a\overline{a}\}^* \) does not lie in GJFA due to Lemma 5. The backward inclusion is easy. As for the forward one, we have
\[
D \cap L = D \cap \bigcup_{d \geq 1} K_d
\]
\[
= \bigcup_{d \geq 1} (D \cap K_d),
\]
Fig. 1. The GJFA $A$ with $D \cap L(M) = \{a \bar{a}\}^*$

Fig. 2. The GJFA $A$ with $L(M) = D_2 \bar{a} D_2$ so it is enough to verify that $D \cap K_d \subseteq \{a \bar{a}\}^*$ for each $d \geq 1$. The case $d = 1$ is trivial since $K_1 = \{a \bar{a}\}$. In order to continue inductively, fix $d \geq 2$. For any $w \in D \cap K_d$, we have $w = u_1 \bar{a} u_2$ for $u_1 u_2 \in K_{d-1}$. From $D = e \leftarrow^* a \bar{a}$ it follows that $u_1 \in DaD$, $u_2 \in D \bar{a} D$, and thus, $u_1 u_2 \in D$. By the induction assumption, $u_1 u_2 \in \{a \bar{a}\}^*$. Hence $w \in \{a \bar{a}\}^* (\bar{a} a) \{a \bar{a}\}^*$ or $w \in \{a \bar{a}\}^* a (\bar{a} a) \bar{a} \{a \bar{a}\}^*$. The first case implies $w \notin D$, which is a contradiction, and the second case implies $w \in \{a \bar{a}\}^*$.

The next theorem shows that some of our results actually follow very easily from Lemma 5, which claims that $\{ab\}^* \notin \text{GJFA}$. Theorems 3 and 4 provide special counter-examples for the closure under inverse homomorphism and shuffle.

**Theorem 2.** GJFA is not closed under:

(i) Kleene star,
(ii) Kleene plus,
(iii) $\epsilon$-free homomorphism,
(iv) homomorphism,
(v) finite substitution.

**Proof.** We have $\{ab\} \in \text{GJFA}$ and $\{ab\}^* \notin \text{GJFA}$ due to Lemma 5. As GJFA is closed under union, $\{ab\}^+ \notin \text{GJFA}$ as well. As for $\epsilon$-free homomorphism, consider $\varphi : \{a\}^* \rightarrow \{a,b\}^*$ with $\varphi(a) = ab$. We have $L = \{a\}^* \in \text{GJFA}$ and $\varphi(L) = \{ab\}^* \notin \text{GJFA}$. Trivially, $\varphi$ is also a general homomorphism and a finite substitution.

**Theorem 3.** GJFA is not closed under inverse homomorphism.

**Proof.** Let $\Sigma = \{a_1, \bar{a}_1, a_2, \bar{a}_2\}$ and
\[
M = (\{q, r\}, \Sigma, R, q, \{r\}), \quad R = \{(q, a_1 \bar{a}_1, q), (q, a_2 \bar{a}_2, q), (q, \bar{a}_1 a_1 r)\},
\]
see Figure 2. Let $L = L(M)$. Observe that $L = D_2 \bar{a}_1 D_2 a_1 D_2$, where $D_2$ is the semi-Dyck language with two types of brackets: $a_1, \bar{a}_1$ and $a_2, \bar{a}_2$. According to
Example 1. $D_2 \in \text{GJFA}$. Let $\varphi : \{a, b\}^* \rightarrow \Sigma^*$ be defined as

\[
\varphi(a) = \overline{a}_1a_2, \\
\varphi(b) = \overline{a}_2a_1,
\]

and let us claim that

\[
\varphi^{-1}(L) = \{ab\}^*,
\]

which is not a GJFA language according to Lemma 3.

The backward inclusion is easy – for each $v = (ab)^i$ with $i \geq 0$ we have $\varphi(v) = \overline{a}_1(a_2\overline{a}_2)^{-1}a_1$ if $i \geq 1$, and $\varphi(v) = \epsilon$ if $i = 0$. In both cases, $\varphi(v) \in L$ and thus $v \in \varphi^{-1}(L)$.

As for the forward inclusion, take any $v \in \varphi^{-1}(L)$ and fix $w \in L$ with $\varphi(v) = w$. As $w \in L$, we have $w = u_1u_2u_3$ for $u_1, u_3 \in D_2$ and $u_2 \in \overline{a}_1D_2a_1$. As $w \in \text{range}(\varphi)$, $w$ starts with $\overline{a}_1$ or $\overline{a}_2$ and ends with $a_1$ or $a_2$. Because $u_1 \in D_2$ cannot start with $\overline{a}_1$ nor $\overline{a}_2$ and end $u_3 \in D_2$ cannot end with $a_1$ nor $a_2$, we have $u_1 = u_3 = \epsilon$ and $w \in \overline{a}_1D_2a_1$. Denote $v = x_1 \cdots x_m$ for $x_1, \ldots, x_m \in \{a, b\}$. As $w \in \overline{a}_1D_2a_1$, $x_1 = a$, $x_m = b$, and

\[
w = \overline{a}_1a_2\varphi(x_2) \cdots \varphi(x_{m-1})\overline{a}_2a_1,
\]

where

\[
a_2\varphi(x_2) \cdots \varphi(x_{m-1})\overline{a}_2 \in D_2.
\]

None of the factors $a_2\overline{a}_1$ and $a_1\overline{a}_2$ can occur in $D_2$. It follows that $x_2 = b$ and for each $i = 2, \ldots, m - 2$ it holds that

\[
x_i = a \iff x_{i+1} = b,
\]

which together implies $v \in \{ab\}^*$.

For languages $K, L \subseteq \Sigma^*$, a word $w \in \Sigma^*$ belongs to $\text{shuffle}(K, L)$ if and only if there are words $u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_k \in \Sigma^*$ such that $u_1u_2 \cdots u_k \in K$, $v_1v_2 \cdots v_k \in L$, and $w = u_1v_1u_2v_2 \cdots u_kv_k$. Furthermore, we denote $L^R = \{w^R \mid w \in L\}$, where $w^R$ is the reversal of $w$.

Theorem 4. GJFA is not closed under shuffle.

Proof. Again, we fix $\Sigma = \{a_1, \overline{a}_1, a_2, \overline{a}_2\}$ and consider the semi-Dyck language $L = D_2 \in \text{GJFA}$ over $\Sigma$. We claim that $\text{shuffle}(D_2, D_2) \notin \text{GJFA}$. According to Lemma 4 we assume for a contradiction that $\text{shuffle}(D_2, D_2) \in \text{UC}_n$ for $n \geq 1$. Denote $w = a_1^n\overline{a}_1^n\overline{a}_2^n\overline{a}_2^n$. The word $w$ lies in a composition $K$ of degree $n$ having the form $K = K' \leftarrow v$, so $w = v_1v_2u_3$ for $u_1u_2 \in K'$. Clearly, there is $x \in \{a_1, a_2\}$ such that at least one of the following assumptions is fulfilled:

(i) Assume that $v$ contains $x$. As $|v| \leq n$, it cannot contain $\overline{a}$. The word $u_1u_2v$ lies in $K$ but it contains an occurrence of $x$ with no occurrence of $\overline{a}$ on the right, so it does not lie in $\text{shuffle}(D_2, D_2)$. 


(ii) Assume that $v$ contains $\overline{x}$. As $|v| \leq n$, it cannot contain $x$. The word $vu_1u_2$ lies in $K$ but it contains an occurrence of $\overline{x}$ with no occurrence $x$ on the left, so it does not lie in shuffle($D_2, D_2$).

Lemma 6. For each $K, L \subseteq \Sigma^*$ it holds that $(L \leftarrow K)^R = L^R \leftarrow K^R$.

Proof. First, let $w \in (L \leftarrow K)^R$, which means $w = (u_1vu_2)^R$ for $v \in K$ and $u_1u_2 \in L$. We have $(u_1vu_2)^R = u_2^R v^R u_1^R$, while $v^R \in K^R$ and $u_2^R u_1^R = (u_1u_2)^R \in L^R$. Thus, $w \in L^R \leftarrow K^R$. Second, let $w \in L^R \leftarrow K^R$, which means $w = u_1vu_2$ for $v \in K^R$ and $u_1u_2 \in L^R$. Thus, $v^R \in K$ and $(u_1u_2)^R = u_2^R u_1^R \in L$. As $w^R = u_2^R v^R u_1^R$, it follows that $w^R \in L \leftarrow K$ and thus $w \in (L \leftarrow K)^R$.

Theorem 5. GJFA is closed under reversal.

Proof. For arbitrary GJFA $M = (Q, \Sigma, R, s, F)$, we define a GJFA $\text{rev}(M)$ as follows:

$$\text{rev}(M) = (Q, \Sigma, R', s, F),$$

$$R' = \{(q, v^R, r) \mid (q, v, r) \in R\}.$$  

Trivially, $\text{rev}(\text{rev}(M)) = M$ for each GJFA $M$. We claim that $L(\text{rev}(M)) = L(M)^R$, which means that $L(M)^R$ is always a GJFA language.

First, we show $L(M)^R \subseteq L(\text{rev}(M))$. Let $w \in L(M)^R$, i.e. $w^R \in L(M)$, which means that $w^R = \epsilon$ and $s \in F$, or

$$w^R \in \epsilon \leftarrow v_d \leftarrow v_{d-1} \leftarrow \cdots \leftarrow v_2 \leftarrow v_1,$$  

(7)

where $v_1, v_2, \ldots, v_d$ is a labeling of an accepting path in $M$, $d \geq 1$. If $w^R = \epsilon$ and $s \in F$, it follows easily that $w \in L(\text{rev}(M))$. Otherwise, observe that

$$w \in (\epsilon \leftarrow v_d \leftarrow v_{d-1} \leftarrow \cdots \leftarrow v_2 \leftarrow v_1)^R$$

$$= \epsilon \leftarrow v_d^R \leftarrow v_{d-1}^R \leftarrow \cdots \leftarrow v_2^R \leftarrow v_1^R$$

according to Lemma \(\square\) and $v_1^R, v_2^R, \ldots, v_d^R$ is a labeling of an accepting path in $\text{rev}(M)$. Together, $w^R \in L(\text{rev}(M))$.

Second, we show $L(\text{rev}(M)) \subseteq L(M)^R$. According to the first inclusion applied to $\text{rev}(M)$ instead of $M$, it holds that

$$L(\text{rev}(M))^R \subseteq L(\text{rev}(\text{rev}(M))))$$

$$= L(M),$$

which is trivially equivalent to $L(\text{rev}(M)) \subseteq L(M)^R$. \(\square\)
5. Relations to the Other Models

An insertion-deletion system, as introduced in [8], generates words from a finite set of axioms by nondeterministic inserting and deleting factors according to insertion rules and deletion rules. An insertion or deletion rule may specify left and right contexts that are needed to perform the operation. Such a system is an insertion system if there are no deletion rules. The language accepted by an insertion system contains all the words that can be obtained from the axioms using the insertion rules. Generally, an insertion-deletion system may use nonterminals, but this does not make sense for insertion systems.

For each \( n, m, m' \geq 0 \), the term \( \text{ins}^{m,m'}_n \) denotes the class of languages accepted by insertion systems where each insertion rule inserts a factor of length at most \( n \) and depends on left and right contexts of lengths at most \( m, m' \) respectively. The symbol \( * \) says that a parameter is not bounded. For example, \( \text{ins}^{0,0}_* \) contains exactly finite unions of unitary languages (one unitary language for each axiom).

In [1] and [10], the authors introduce graph-controlled insertion systems. Such systems may be described by a set of axioms and a directed multigraph with edges labeled by insertion rules. The vertices are called components. An initial component and a final component are specified. A graph-controlled insertion system accepts a word if and only if the word is obtained from an axiom using a sequence of insertion rules that forms a path from the initial component to the final component.

For each \( n, m, m', k \geq 0 \), the term \( \text{LStP}_k^{\text{ins}^{m,m'}_n} \) denotes the class of languages accepted by graph-regulated insertion systems with at most \( k \) components where the properties of insertion rules are bounded by \( n, m, m' \) as above. It turns out that

\[
\text{GJFA} = \text{LStP}_*^{\text{ins}^{0,0}_*}.
\]

Indeed, with the generative approach to GJFA in mind, a GJFA may be transformed to a graph-regulated insertion system with the same structure, using only the axiom \( \epsilon \) and insertion rules with empty contexts. For the backward inclusion we just encode the axioms to rules specifying that the computation ends by deleting an axiom.

Other related models were introduced in 1980’s in the scope of Galiukschov semicontextual grammars, see [5]. For \( k \geq 0 \), a semicontextual grammar of degree \( k \) without appearance checking is actually an insertion system with left and right contexts of length at most \( k \). Moreover, such grammar may involve a regular control. In this case, a regular language \( C \) over the alphabet of insertion rules is specified, and a string is accepted only if it can be obtained from an axiom using a sequence of insertion rules that belongs to \( C \). The symbol \( C_k \) denotes the class of languages accepted by regular control semicontextual grammars of degree \( k \) without appearance checking. The equivalence of \( C_0 \) and \( \text{LStP}_*^{\text{ins}^{0,0}_*} \) is immediate. Thus

\[
\text{GJFA} = C_0.
\]

Finally, an interesting result follows from the fact that each unitary language lies in \( \text{GJFA} \). According the main result of the Haussler’s article [3], for a given
REFERENCES

a prefix-disjoint instance of the Post correspondence problem (PCP) over a range alphabet Σ, one can algorithmically construct sets S, T ⊆ Γ∗ over an alphabet Γ with |Γ| = 2|Σ| + 4 such that the PCP instance is positive if and only if the intersection of ϵ ←∗ S and ϵ ←∗ T contains a non-empty string. For given S and T, it is trivial to construct two GJFA over Γ accepting (ϵ ←∗ S) \ {ϵ} and T respectively. On the other hand, PCP can be easily reduced to a binary range alphabet with preserving prefix-disjointness (see, e.g., [9]). Together, we get the following:

**Theorem 6.** Given GJFA M1, M2 over an 8-letter alphabet, it is undecidable whether L(M1) ∩ L(M2) = ∅.

6. Conclusions

We have completed the list of closure properties of the class GJFA under classical language operators. Besides of that, we have pointed out that the class GJFA can be put into the frameworks of graph-controlled insertion-deletion systems and Galiukschov semicontextual grammars, and we have referred to a former result that implies undecidability of intersection emptiness for GJFA languages.

It is a task for future research to provide really alternative descriptions of the class GJFA. There also remain open questions about decidability, specifically regarding equivalence, universality, inclusion, and regularity of GJFA, see [7].

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aWe do not introduce PCP nor prefix-disjoint instances in this paper.
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