A PERTURBED KANTOWSKI-SACHS COSMOLOGICAL MODEL

H. V. Fagundes
Instituto de Física Teórica, Universidade Estadual Paulista,
Rua Pamplona, 145, São Paulo, SP 01405-900, Brazil
e-mail: helio@ift.unesp.br

M. A. M. González
Rua Homero Sales, 333, São Paulo, SP 05126-000, Brazil

November 30, 2018

Abstract

A numerical integration is made, starting with a Kantowski-Sachs (KS) model with an added initial perturbation at time one-tenth the age of the universe, and evolving up to the present. According to a prediction by Barrow (1989), the ratio between the fluctuations of the average matter density and those of the metric tensor varies with the inverse squared wavelength of the fluctuation. In this paper we take an extravantly large value for this wavelength (20% of the horizon’s radius), to see that even then that ratio remains greater than unity. So we may be reasonably confident of models assuming density homogeneity, despite the fact that the scale of observed structure in the universe has been increasing lately.

1 Introduction

This paper is a sequel to a study on the evolution of small perturbations on cosmological models, either of the Friedmann-Lemaître-Robertson-Walker
(FLRW) family or closely related to it. The leading idea is to show that small perturbations of the spacetime metric are compatible with not-so-small perturbations of the average matter density in the universe - cf. Fagundes & Kwok (1991). Here we start with a Kantowski-Sachs (KS) model, which is similar to the spherical FLRW model, and is easier than the latter to deal with in a numerical study: while the space section for the latter is the hypersphere $S^3$, for KS it is $S^2 \times \mathbb{R}$, where $S^2$ is the ordinary sphere and $\mathbb{R}$ is the real axis. Then we introduce deviations from the KS metric at an initial time $t = 0$, taking the age of the universe as $t_0 = 1$. The idea is to compare this KS model with perturbations which evolve in time with the unperturbed KS universe.

In Fagundes & Kwok (1991), the scale (wavelength) of the perturbation was much smaller than the radius of observational horizon, and so, as expected from Barrow’s (1989) heuristic result, the fluctuations of the metric components turned out to be much smaller than the fluctuation of the matter density. Here we choose a perturbation with a very large wavelength ($1/5$ of the horizon’s radius), with the result that one of the metric components still varies less than (about one-half as much as) the density. This result reinforces our current confidence in models based on the assumption of homogeneity of matter distribution, such as those of the FLRW family, since the scale of observed inhomogeneity is believed to be smaller than that of the perturbation assumed in this investigation.

2 Einstein equations

We choose units such that $c = G = t_0 = 1$, and assume a metric in the form

\[ ds^2 = dt^2 - a^2(t, r)dr^2 - b^2(t, r)\sin^2 r \, d\varphi^2 - c^2(t, r)d\zeta^2 , \]  

with $0 \leq r \leq \pi, 0 \leq \varphi \leq 2\pi, -\infty \leq \zeta \leq +\infty$, which is to be compared with the KS metric (Kantowski & Sachs 1966)

\[ ds^2 = dt^2 - a_{KS}^2(t)(dr^2 + \sin^2 r \, d\varphi^2) - c_{KS}^2(t)d\zeta^2 . \]  

For the metric given by Eq. 1 Einstein equations are (a dot means $\partial/\partial t$, a prime $\partial/\partial r$):

\[ \frac{\dot{a}\dot{b}}{ab} + \frac{\dot{a}\dot{c}}{ac} + \frac{\dot{b}\dot{c}}{bc} + \frac{1}{a^2} + \frac{a'b'}{a^3b} + \frac{a'c'}{a^3c} - \frac{b'c'}{a^2bc} + \left( \frac{a'}{a^3} - \frac{2b'}{a^2b} - \frac{c'}{a^2c} \right) \cot r - \frac{b''}{a^2b} - \frac{c''}{a^2c} = 8\pi\rho \]  

(3)
\[
\frac{\ddot{a}b'}{ab} - \frac{\dot{b}'}{b} + \frac{\dot{a}c'}{ac} - \frac{\dot{c}'}{c} + \left(\frac{\dot{a}}{a} - \frac{\dot{b}}{b}\right) \cot r = -8\pi \rho v \\
\frac{\ddot{b}}{b} + \frac{\ddot{c}}{c} + \frac{\dot{b}\dot{c}}{bc} - \frac{b'c'}{a^2bc} - \frac{c'}{a^2c} \cot r = 0 \\
\frac{\ddot{a}}{a} + \frac{\ddot{c}}{c} + \frac{\dot{a}\dot{c}}{ac} - \frac{a'c'}{a^3c} - \frac{c'}{a^2c} = 0 \\
\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\ddot{a}b'}{ab} + \frac{1}{a^2} + \frac{a'b'}{a^3b} + \left(\frac{a'}{a^3} - \frac{2b'}{a^2b}\right) \cot r = 0
\]

where \(\rho = \rho(t, x)\) is the average (pressureless) matter density and \(v = v(t, x)\) is the velocity field with respect to the Hubble flow, assumed for simplicity to have only the \(x\)-component. Terms of the order of \(v^2\) are discarded.

As is well known (see, for example, Weinberg 1972, p. 163), Eqs. (3) and (4) are constraints on the solutions of the second order quasi-linear Eqs. (5) and (7). To solve the latter we define new variables \(f = \dot{a}, g = \dot{b},\) and \(h = \dot{c}\), in order to get a system of six first-order (in \(t\)) differential equations for the dependent variables \(a, b, c, f, g,\) and \(h\). It is convenient also to define \(f_{KS} = \dot{a}_{KS}, h_{KS} = \dot{c}_{KS}\).

We want our solution to be as close as possible to FLRW’s spherical model with a dustlike matter distribution, so we take as our comparison KS solution one with this property, namely that in Shikin’s (1967) equations (16) and (20), with \(A = D/3 = \bar{a}, B = E = 0\). (See also Fagundes 1982.) The functions \(a_{KS}, c_{KS}\) are obtained in terms of the parameter \(\eta(t)\), which is the inverse of \(t(\eta) = \bar{a}(\eta - \sin \eta)\), as in the FLRW models. Here

\[
a_{KS}(t) = \bar{a}[1 - \cos \eta(t)], \\
c_{KS}(t) = 3\bar{a}\{2 - \eta(t) \cot[\eta(t)/2]\},
\]

and we choose \(\bar{a} = [\pi/3 - \sin(\pi/3)]^{-1}\), which corresponds to \(\eta(1) = \pi/3, \Omega_{0}^{E\text{LRW}} = 4/3,\) and \(\rho_{KS}(1) = [(4/3)a_{KS}(1)/c_{KS}(1)]\rho_{0}^{\text{crit}} \approx 1.1934557651\rho_{0}^{\text{crit}},\) where \(\rho_{0}^{\text{crit}}\) is the usual critical density (present density in Einstein-de Sitter’s model).
3 Numerical integration

The system was then numerically integrated, in the interval $t = 0.1$ to $t = 1.0$. (An attempt was made to start at $t = 2 \times 10^{-5}$, the recombination time; but the result was not satisfactory.) The initial conditions are

\[
\begin{align*}
    a(0.1, r) &= a_{KS}(0.1)[1 + \alpha \cos(10r)] \\
    b(0.1, r) &= a_{KS}(0.1)[1 + \alpha \cos(10r)] \\
    c(0.1, r) &= c_{KS}(0.1)[1 + \beta \cos(10r)] \\
    f(0.1, r) &= f_{KS}(0.1)[1 + \alpha \cos(10r)] \\
    g(0.1, r) &= f_{KS}(0.1)[1 + \alpha \cos(10r)] \\
    h(0.1, r) &= h_{KS}(0.1)[1 + \beta \cos(10r)]
\end{align*}
\]

To get the desired effects we put $\alpha = \beta = 0.02$. Note also the large value of the initial perturbation comoving wavelength, $L = \pi/5$. Despite this the perturbation remains sub-horizon sized: $\lambda(t) \approx t^{2/3}L < 3t$ for $t \geq 0.1$; so we need not worry about gauge ambiguities - see, for example, Kolb & Turner (1990).

The region of integration was divided into 1000 parts for the $r$ variable, and 9000 parts for $t$, which means a spatial interval $\delta r = 10^{-3}\pi$ and an integration step $\delta t = 10^{-4}$. First and second spatial derivatives were obtained by the usual rules, with a shift of $0.5\delta r$ to soften the effect of the coordinate singularities at $r = 0, \pi$: for example, for each $t = 0.1 + n\delta t$, $0 \leq n \leq 9000$, putting $r = (k + 0.5)\delta r$, $0 \leq k \leq 1000$, we defined $A[k] = a(t, r)$, $B[k] = b(t, r)$, and so on; and took $a'(t, r)$, $b''(t, r)$ respectively as

\[
AR[k] = (A[kPlus] - A[kMinus])/2\delta r,
\]

\[
bRR[k] = (B[kPlus] - 2B[k] + B[kMinus]) / \delta r^2,
\]

where

\[
(kPlus, kMinus) = \begin{cases} 
(1, 0) & \text{for } k = 0 \\
(k + 1, k - 1) & \text{for } 0 < k < 1000 \\
(1000, 999) & \text{for } k = 1000
\end{cases}
\]

After each increment $t \rightarrow t + \delta t$ we make $a(t, r) \rightarrow a(t, r) + f(t, r)\delta t$, and similarly for $b, c, f, g, h$. As for $\rho(t, r)$ and $v(t, r)$ they are given by the
constraint equations. Stability of the solution was checked by redoing the integration with \( \delta \alpha, \delta \beta = \pm 0.002 \).

Calculations followed the simple rules, as given, for example, by Smith (1978), and were programmed in the C language on an HP 750 workstation.

\section{Results}

The results at \( t = 1.0 \) are shown in Table 1, for uniformly spaced values of \( r/\pi \). The entries are

- \( a_{rel} = a(1, r)/a_{KS}(1) \),
- \( b_{rel} = b(1, r)/a_{KS}(1) \),
- \( c_{rel} = c(1, r)/c_{KS}(1) \),
- \( v(1, r) \) in km/sec, and \( \rho_{rel} = \rho(1, r)/\rho_{KS}(1) \).

Except for \( v(1, r) \), they were fitted to the following short Fourier series, where \( x = r/\pi \):

\begin{align*}
    a_{rel}(x) & = 1.0008 - 0.1276 \cos(10\pi x) + 0.0051 \cos(20\pi x), \quad (8) \\
    b_{rel}(x) & = 0.9932 + 0.0200 \cos(10\pi x) + 0.0069 \cos(20\pi x), \quad (9) \\
    c_{rel}(x) & = 1.0008 + 0.0202 \cos(10\pi x) - 0.0008 \cos(20\pi x), \quad (10) \\
    \rho_{rel}(x) & = 1.0236 + 0.2256 \cos(10\pi x) - 0.0041 \cos(20\pi x), \quad (11)
\end{align*}

These fits have the approximate periodicity of the initial perturbation, and are symmetrical about \( \rho = \pi/2 \). Not counting the suspicious values near \( r = 0, \pi \), note the fluctuations of about 13\% for \( a_{rel} \), 2\% for \( b_{rel} \) and \( c_{rel} \), and 25\% for \( \rho_{rel} \). Further details are given in González (1994).

The fluctuation of \( a_{rel} \) is one-half that of \( \rho_{rel} \), qualitatively confirming Barrow’s (1989) result, that \( \delta \rho/\rho_0 \sim (ct_0/L)^2 \delta \phi/\phi_0 \), where \( \phi \) is the Newtonian potential (which corresponds to the metric tensor in general relativity), and \( L \) is the scale of the perturbation. Here we have \( L = (\pi/15) \times 3ct_0 \), or about 20\% of the horizon’s radius; this makes Barrow’s estimate \( \delta \rho/\rho_0 \sim 2.53 \times \delta \phi/\phi_0 \). In Fagundes & Kwok (1991) we had an initial perturbation with wavelength \( L = 300h^{-1} \) Mpc, and a ratio of the order of 100 was obtained between \( \delta \rho/\rho_0 \) and \( \delta \phi/\phi_0 \).

The fact that \( \delta a_{rel} \) is still smaller than \( \delta \rho_{rel} \), even with our exaggerated assumption for the size of \( L \) suggests that as long as the scale of observed inhomogeneities is smaller than, say, \( 600h^{-1} \) Mpc, models based on the homogeneity of matter distribution are reasonably correct as far as the metric tensor is concerned. However, this is not an absolute conclusion. In Fagundes & Mendonça da Silveira (1995), where initial conditions were contrived to illustrate another problem, we got \( \delta \rho/\rho_0 \sim 2.5 \times \delta \phi/\phi_0 \) for \( L = 120h^{-1} \) Mpc.
5 Acknowledgements

One of us (M.A.M.G.) thanks Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq - Brazil) for a scholarship; H.V.F. is grateful to CNPq for partial financial support.

REFERENCES

Barrow, J.D. 1989, Quart. J.R.A.S., 30, 163
Fagundes, H.V. 1982, Lett. Math. Phys., 6, 417
Fagundes, H.V., & Kwok, S.F. 1991, Ap.J., 368, 337
Fagundes, H.V., & Mendonça da Silveira, F.E. 1995, Brazil. J. Phys., 25, 219
González, M.A.M. 1994, Universo de Kantowski-Sachs com Perturbações, MS dissertation (São Paulo: IFT/UNESP); in Portuguese
Kantowski, R., & Sachs, R.K. 1966, J. Math. Phys., 7, 443
Kolb, E.W., & Turner, M.S. 1990, The Early Universe (Reading, MA: Addison-Wesley)
Shikin, I.S. 1967, Sov. Phys. Doklady, 11, 944
Smith, J.D. 1978, Numerical Solutions of Partial Differential Equations (2nd ed; Oxford: Clarendon Press)
Weinberg, S. 1972, Gravitation and Cosmology (New York: Wiley)
TABLE 1. Results of the integration for the functions defined in the beginning of Section 4.

| $r/\pi$ | $a_{rel}$ | $b_{rel}$ | $c_{rel}$ | $\rho_{rel}$ | $v(1,r)$ |
|---------|-----------|-----------|-----------|-------------|---------|
| 0.0005  | 0.87197   | 0.87197   | 1.02292   | 1.56328     | 1077.29 |
| 0.0504  | 0.99773   | 0.90637   | 1.0027    | 1.15528     | -69.32  |
| 0.1004  | 1.13339   | 0.98062   | 0.97988   | 0.79305     | -20.79  |
| 0.1503  | 0.99464   | 1.03031   | 1.00157   | 0.96130     | 82.13   |
| 0.2003  | 0.87825   | 1.01964   | 1.0214    | 1.24580     | 5.93    |
| 0.2502  | 0.99636   | 0.98646   | 1.00129   | 1.02331     | -69.62  |
| 0.3002  | 1.13342   | 0.98029   | 0.97970   | 0.79368     | -7.77   |
| 0.3501  | 0.99506   | 1.00894   | 1.00148   | 0.99168     | 73.20   |
| 0.4001  | 0.87831   | 1.01973   | 1.02015   | 1.24548     | 1.47    |
| 0.4500  | 0.99572   | 0.99899   | 1.00138   | 1.00534     | -70.74  |
| 0.5000  | 1.13343   | 0.98022   | 0.97969   | 0.79376     | 0.00    |
| 0.5500  | 0.99572   | 0.99899   | 1.00138   | 1.00534     | 70.74   |
| 0.5999  | 0.87831   | 1.01973   | 1.02015   | 1.24548     | -1.47   |
| 0.6498  | 0.99506   | 1.00894   | 1.00148   | 0.99168     | -73.20  |
| 0.6998  | 1.13342   | 0.98029   | 0.97970   | 0.79368     | 7.77    |
| 0.7498  | 0.99636   | 0.98646   | 1.00129   | 1.02331     | 69.62   |
| 0.7997  | 0.87825   | 1.01964   | 1.02014   | 1.24580     | -5.93   |
| 0.8497  | 0.99464   | 1.03031   | 1.00157   | 0.96130     | -82.13  |
| 0.8996  | 1.13339   | 0.98062   | 0.97988   | 0.79305     | 20.79   |
| 0.9496  | 0.99773   | 0.90637   | 1.00227   | 1.15528     | 69.32   |
| 0.9995  | 0.87197   | 0.87197   | 1.02292   | 1.56328     | -1077.45 |