Bounded time computation on metric spaces and Banach spaces

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Abstract

We extend the framework by Kawamura and Cook for investigating computational complexity for operators occurring in analysis. This model is based on second-order complexity theory for functionals on the Baire space, which is lifted to metric spaces by means of representations. Time is measured in terms of the length of the input encodings and the required output precision.

We propose the notions of a complete representation and of a regular representation. We show that complete representations ensure that any computable function has a time bound. Regular representations generalize Kawamura and Cook’s too restrictive notion of a second-order representation, while still guaranteeing fast computability of the length of the encodings.

Applying this notions, we investigate the relationship between purely metric properties of a metric space and the existence of a representation such that the metric is computable within bounded time. We show that a bound on the running time of the metric can be straightforwardly translated into size bounds of compact subsets of the metric space. Conversely, for compact spaces and for Banach spaces we construct a family of admissible, complete, regular representations that allow for fast computation of the metric and provide short encodings. Here it is necessary to trade the time bound off against the length of encodings.
1 Introduction

Computability theory provides a mathematical framework for algorithmic considerations about discrete structures. The merits and applications are well-known and appreciated by both computer scientists and mathematicians. One of the biggest impacts the development of computability theory had on computer science is that it became possible to give mathematical proofs of non-existence of algorithms (or real-world processes, assuming the Church-Turing hypothesis) to solve certain problems. This has concrete implications for programming practice: The halting problem is incomputable, which means that automatic checks for correctness are impossible in general and has seeded whole fields of research like model checking.

Complexity theory is the resource sensitive refinement of computability theory. For a decidable problem it investigates bounds on the number of steps or memory space a machine solving the problem needs to take or use. Again, in principal one of the main merits of a formal complexity theory are proofs that certain problems are inherently difficult to solve so that one does not need to continue looking for a better algorithm. However, while there exist hierarchy theorems, many interesting complexity classes are notoriously difficult to separate: The famous $P$ vs. $NP$ problem is one of the millennium problems and unsolved. Thus, the main focus in complexity theory has shifted to classifying problems as hard for certain classes. In practice, providing $NP$ hardness of a problem has similar implications as proving it to not be solvable in polynomial time as a provably better algorithm would solve a problem difficult enough for many people to give up before even trying.

In their traditional form, computability and complexity theory are only applicable to discrete structures. Many applications in modern computer science and numerical analysis require to model continuous structures as well. Engineers want to solve differential equations, mathematicians want to use computers to search for counter examples to the Riemann Hypothesis, etc. The most common way to model continuous structures on digital computers are floating point numbers. However, the use of floating point arithmetics leads to an offset between the mathematical description of an algorithm and its implementation: Floating point numbers do not provide many of the properties that mathematicians expect of real numbers, for instance distributivity or associativity of the operations. Also, the uncontrolled error propagation can lead to unreliable results of straight-forward implementations. Computer scientists and numerical analysts are well aware of these issues and circumvent the implications by hand using multiple precision arithmetic with correct rounding (MPFR), interval arithmetic, etc. For people only interested in computability issues that are willing to sacrifice a lot of efficiency, a full solution of the above problems is provided by computable analysis, Weihrauch’s TTE and his theory of representations. Computable analysis is a mathematically rigorous extension of computability theory to cover continuous spaces while still relying on a realistic machine model for computations.

While computable analysis sees its roots in one of the cornerstone papers of discrete computability theory [Tur36] and the main developments were done in the 1950’s [Grz55] and 1980’s [Wei00], its time restricted counterpart real complexity theory has until recently been stuck on the level of real functions and point-wise considerations about operators [Ko91, BBY07]. Only in 2010 the framework of second-order representations was introduced by Kawamura and Cook [KC10] and made it possible to consider uniform complexity of operators on spaces like the continuous functions on the unit interval. The framework heavily relies on second-order complexity theory for functions on the Baire space which was introduced by Mehlhorn [Meh76] and in particular on a fairly recent characterization of Mehlhorns basic feasible functionals by resource bounded oracle Turing machines by Kapron and Cook [KC96]. Among the early achievements of the framework of second-order representations are the characterization of the minimal representation of the continuous
functions on the unit interval that renders evaluation polynomial-time computable and a number of translations to the new setting of some earlier pointwise results about complexity theory that related certain operators on the continuous functions to important inclusions of discrete complexity classes. Prior to Kawamura and Cook, the first author introduced a slightly different, more restrictive framework for complexity [Sch04] and provided some applications [KS05]. Many of the results in this paper were found when attempting to unify these two approaches.

Computable analysis seems satisfactory in the sense that it is applicable to a big variety of problems that come up in practice: For separable metric spaces with a canonical sequence or more generally countably based \(T_0\) spaces with a canonical basis there exists a standard representation. It is possible to form products and disjoint unions of spaces, and there is a function space construction. Many results like the computable Weierstraß Approximation Theorem provide compatibility of the above constructions: Under reasonable assumptions on the spaces involved it does not matter whether a canonical sequence is chosen in the function space or the function space construction is used. The notion of admissibility provides a reasonable condition for whether or not a chosen computable structure is compatible with a topology on the space. While this still leaves some gaps on whose closure the community of computable analysis is actively working on [ZW03, Bra01], in the past years the focus has shifted to finding applications to problems that people in numerical analysis are interested in [Her99, BY06, WZ06, WZ07, SZZ15].

In comparison to computable analysis, the ground stock of problems that real complexity theory is applicable to is vanishingly small: The standard representation of metric spaces is known to only lead to a good complexity theory in very restricted cases [Wei03, KSZ16]. In particular it does not lead to the right result for the continuous functions on the unit interval. The construction for the representation of this space, in turn, can not be expected to generalize to a big class of spaces [KP14]. This situation is not acceptable and many efforts have been done to improve it. The most notable advances are the constructions of representations for the analytic functions on the unit disc and the levels of Gevrey’s hierarchy [KMRZ15] based on prior work by Norbert Müller [Müll95], the investigations of complexity of functionals [FGH14, FZ15] and advances on \(L^p\) and Sobolev spaces [Ste16, KSZ16]. However, all of the above examples do only cover very restricted areas and handle examples in a highly specialized and not generalizable way.

This paper aims to close the gap in applicability between computable analysis and real complexity theory where it is possible and to point out where this is impossible. It provides an in-depth investigation of the general restrictions of the framework of Kawamura and Cook when computations on metric spaces are carried out. It generalizes the Cauchy representation and in a way such that the constructions for continuous functions on the unit interval and \(L^p\) and Sobolev spaces can be seen as special cases. The construction is highly inspired by results from approximation theory and close to ideas frequently used in numerical analysis. We firmly believe that this provides a general recipe for constructing useful representations that are complexity theoretically well-behaved and opens the field for applications similar to those computable analysis has. For accessibility of the constructions the requirement of being second-order representation is relaxed to a weaker notion we call regularity. We believe this to be an important step as it removes the necessity of padding that seems to be an unnatural thing to do in real world applications. At the same time it maintains many of the advantages of second-order representations, like accessibility of the length function. Furthermore, we introduce the properties of completeness and computable completeness that guarantee that any computable operator allows a time bound. Computable completeness lifts computable admissibility to the bounded-time realm in that assuming it for a representation of a computable metric space implies bounded time equivalence
to the Cauchy representation, where without it only computable equivalence is guaranteed. We refrain from making this a definition as other authors have claimed that a resource restricted notion of admissibility should not exists [KP14].

1.1 The content of this paper

The first chapter contains the introduction, this description of the content of the paper and a short section listing the most basic notational conventions.

Section 2 gives a brief introduction into computable analysis formulated not in the standard way, but in an appropriate form such that second-order complexity theory is applicable. It also presents those results and definitions from second-order complexity theory that are of relevance for the content of the paper. The last part introduces the notion of a length of a representation which is new and important for the next chapter. It proves that many representations have a length.

In Section 3 the concept of metric entropy is presented. This concept is not new but has been used in approximation theory, constructive analysis and proof mining and computable analysis before. However, we skew the definitions a little in a way that make them fit in with the framework presented in Section 2. The most basic properties are listed for the convenience of the reader. Then the first major result of this paper is stated: Theorem 3.8 constructs bounds on the metric entropy of certain compact subsets from a running time of the metric. Let us state the result of the theorem, when applied to compact spaces in natural language: Whenever there exists an encoding of a compact metric space \( Z \) by string functions such that the metric can be obtained from the encodings in time \( T \) and each element has an encoding of length at most \( \ell \), then the metric entropy of the compact space is bounded polynomially in \( T \) and \( \ell \). More explicitly \( \text{ent}(Z) \in O(T(\ell, \cdot + 2)^2) \).

This is a general restriction of the framework. If a compact set is too big, we can not expect to find representations that both feature short names and fast computation of the metric. A trade-off is necessary. The rest of the paper constructs encodings whose combination of length of names and running times is close to optimal.

Section 4 prepares the construction of these representations by introducing some necessary restrictions on running times. We introduce a concept of time-constructibility for second-order running times that restricts to the usual notion of time-constructibility for usual running times and choose a notion of monotonicity. We then present the notion of regularity that relaxes being a second-order representation. This is mainly done to make the definitions of the standard representations in the next section more legible, however, we think that it introduces an important concept that makes the framework more accessible and brings it closer to actual implementations. Finally, we introduce completeness and computable completeness of representations. The notion is inspired by the notion of properness [Sch04] and we prove it to imply that any computable function is computable in bounded time in Theorem 4.13.

Theorem 4.14 proves that a separable metric space is complete if and only if every Cauchy representation is complete and that the Cauchy representation of a complete computable metric space is computably complete. This is the motivation for the name and shows that if a representation of a complete computable metric space is computably complete, it is bounded-time equivalent to the Cauchy representation.

The second to last Section 5 constructs standard representations for metric spaces. As can be expected from the results of Section 3 it is necessary to impose conditions on the relation between the length of names and the running time. We specify sets of conditions under which it is possible to construct such representations. However, in general it can not be guaranteed that the bounds from Theorem 3.8 are tight. To be able to provide lower bounds on the metric
entropy we have to impose further conditions on the metric space. The two conditions under which we succeeded are those of an infinite compact metric space (Theorem 5.5) and of an infinite dimensional Banach space that allows a Schauder basis (Theorem 5.10). Of course in the former case the bound is only valid if it is not bigger than the metric entropy of the compact space itself which makes the statement slightly more complicated. The merit of the compact case is that the existence of a uniformly dense sequence is automatic. This is in contrast to the situation for Banach spaces, where the existence of a Schauder basis has to be assumed. The last part mentions that additional assumptions have to be made to guarantee that the construction works well with a computable structure on the space. For Banach spaces it is known that this is for principal reasons, for metric spaces it is open whether these additional assumptions are indeed necessary.

The final Section 6 provides applications of the constructions and connections to other results. First it compares the results to the results from approximation that inspired the constructions for Banach spaces in the first place. These results in turn can be applied to the construction and improve some of the bounds drastically. This did not come as an surprise to the authors: While we consider the proofs within the framework of real complexity theory elegant, they are new and apply to a more general situation. The authors from approximation theory, in contrast, have had a lot of time and invested a lot of effort into optimizations. The last two parts of Section 6 go on to reconstruct different known representation by choosing the parameters in the constructions from Theorems 5.5 and 5.10 appropriately. Most notably the representation of the continuous functions on the unit interval that has been introduced and proven to be the minimal second-order representation to allow polynomial-time evaluation by Kawamura and Cook.

The paper ends in a conclusion and the bibliography.

1.2 Notational conventions

By Σ we denote the alphabet {0, 1} and by Σ∗ we denote the set of finite strings over Σ. The letters a, b, ... will be used for strings. By the Baire space we mean the space B := (Σ∗)Σ∗ of total functions ϕ, ψ, ... from finite strings to finite strings equipped with the product topology. It is topologically equivalent to the countable product of the discrete natural numbers. We assume familiarity with the notions of computability and complexity of total and partial string functions.

We identify the set N of natural numbers with the set {ε, 1, 10, 11, ...} of their binary representations. Another copy of the natural numbers is the set ω := {ε, 1, 11, 111, ...} of their unary representations. We denote elements of both N and ω by n, m, .... We identify the set of integers with the set Z := N \ {ε} ∪ 0N, where 0n is interpreted as −n (note that the union is disjoint).

By |·| : Σ∗ → ω the length mapping is denoted, which simply replaces all occurrences of 0 by 1s. For each k we fix some standard tuple function ⟨·, ..., ·⟩ : Nk → N. We require that it is a linear-time computable injection with linear-time computable projections and that it satisfies

\[ |⟨a_1, ..., a_k⟩| \leq k \left( \max_{1 \leq i \leq k} |a_i| + 1 \right) \land 0^n = (ε, ..., ε, 0^n) \]

On B we use the pairing function defined by ⟨φ, ψ⟩(a) := ⟨φ(a), ψ(a)⟩.

A multivalued function f :⊆ X ⇒ Y is a function from X to the power set of Y. The elements of f(x) are interpreted as the ‘acceptable return values’. The domain dom(f) of f is the set of x such that f(x) ≠ ∅. A multivalued function is total if dom(f) = X and this is indicated by writing f : X ⇒ Y. We choose the convention that a function is assumed to be total if it is not explicitly stated that it is allowed to be partial. A partial function
If \( f : \subseteq X \to Y \) is identified with the multivalued function \( \tilde{f} : \subseteq X \to Y \) with \( \tilde{f}(x) = \{ f(x) \} \) if \( x \in \text{dom}(f) \) and \( \tilde{f}(x) = \emptyset \) otherwise.

The reader is assumed to be familiar with the basic theory of metric spaces \((Y, d)\) such as the diameter of a set and its distance function. The closed ball of radius \( r \geq 0 \) around \( x \in Y \) is denoted by \( B^c_r(x) := \{ y \in Y \mid d(x, y) \leq r \} \) and the open ball by \( B^o_r(x) := \{ y \in Y \mid d(x, y) < r \} \). The open balls form a base of the topology induced by the metric. Note that \( B^o_r(x) \subseteq B^o_s(x) \subseteq B^c_r(x) \), but in general nothing further can be said about strictness of the inclusions.

In particular, the closed balls may in general be different from the closures of the open balls.

All vector spaces considered in this paper have the space \( \mathbb{R} \) of the real numbers as their underlying field.

## 2 Representations

Discrete computability theory encodes objects by finite strings to make them accessible to digital computers. This only works if the structure considered is countable. Most metric spaces appearing in practice, however, are not countable. For instance the real numbers, or, to mention a compact one, the unit interval are uncountable. Computable analysis removes the necessity of countability by encoding elements by infinite objects (infinite binary strings or string functions) instead of strings.

For us the Baire space is the space of all string functions \( B := (\Sigma^*)^\mathbb{N} \) equipped with the product topology.

### Definition 2.1

A represented space is a pair \( X = (X, \xi) \) of a set \( X \) and a partial surjective mapping \( \xi : \subseteq B \to X \). The elements of \( \xi^{-1}(x) \) are called the names (or \( \xi \)-names or \( X \)-names) of \( x \).

A space with a fixed representation is called a represented space. We denote represented spaces by \( X, Y, \ldots \), their underlying sets by \( X, Y, \ldots \) and their representations by \( \xi_X, \xi_Y, \ldots \). Like the topology of a topological space the representation of a represented space is only mentioned explicitly if necessary to avoid ambiguities. An element of a represented space is called computable if it has a computable name. It is said to lie within a complexity class if it has a name from that complexity class.

On the one hand, any represented space carries a natural topology: The final topology of the representation. On the other hand, one often looks for a representation suitable for a topological space. It is reasonable to require such a representation to induce the topology the space is equipped with. For this, continuity is necessary but not sufficient. Continuity together with openness is sufficient but not necessary.

The following is the standard representation for carrying out computability considerations on metric spaces [Wei00] and has also been used in different contexts for a long time [TvD88, Tro92].

### Definition 2.2

Let \( Z := (Z, d, (r_i)) \) be a triple such that \( (Z, d) \) is a separable metric space and \( (r_i)_{i \in \mathbb{N}} \) is a dense sequence in \( Z \). Define the Cauchy representation \( \xi_Z \) of \( Z \) with respect to \( (r_i) \) as follows: A string function \( \varphi \in B \) is a \( \xi_Z \)-name of \( x \in M \) if and only if for all \( n \in \mathbb{N} \)

\[
\varphi(n) \in \mathbb{N} \land d(x, r_{\varphi(n)}) \leq \frac{1}{n + 1}.
\]

This representation can be checked to induce the metric topology. Note that equivalently one could use integers in unary and require \( d(x, r_{\varphi(n)}) \leq 2^{-n} \), which is the more common convention in real complexity theory. A third equivalent option would have been to use the encodings of rational numbers.

The following representation is the standard representation of real numbers that is used throughout real complexity theory:
Definition 2.3 Let \( \varphi \in \mathcal{B} \) be an \( \mathbb{R} \)-name of \( x \in \mathbb{R} \) if and only if for all \( n \in \mathbb{N} \)
\[
\varphi(n) \in \mathbb{Z} \land \left| x - \frac{\varphi(n)}{n+1} \right| \leq \frac{1}{n+1}.
\]
This representation is in a very strong sense (see Definition 2.11) equivalent to the Cauchy representation, if the dense sequence is chosen as a standard enumeration of the dyadic numbers.

Recall the pairing \( \langle \cdot, \cdot \rangle : \mathcal{B} \times \mathcal{B} \to \mathcal{B} \) on string functions from the introduction.

Definition 2.4 Let \( X \) and \( Y \) be represented spaces. Define their product \( X \times Y \) by equipping \( X \times Y \) with the representation \( \xi_{X \times Y} \) defined by
\[
\xi_{X \times Y}((\varphi, \psi)) = (x, y) \iff (x = \xi_X(\varphi) \land y = \xi_Y(\psi)).
\]
That is: a string function is a name of the pair \( (x, y) \) if and only if it is the pairing of a name of \( x \) and a name of \( y \).

Computations of functions between represented spaces can be reduced to computations of functionals on the Baire space:

Definition 2.5 Let \( f : X \supseteq Y \) be a multivalued function between represented spaces. A partial function \( F: \subseteq \mathcal{B} \to \mathcal{B} \) is a realizer of \( f \), if
\[
\varphi \in \xi_X^{-1}(x) \implies F(\varphi) \in \xi_Y^{-1}(f(x)).
\]
That is: \( F \) translates each name of \( x \) into some name of \( f(x) \). No assumptions about the behavior on elements that are not names are made.

Computability of functionals on the Baire space was introduced in [Kle52] and is well established. An overview over the historical development can be found in [Lon05]. We use the model of computation by oracle Turing machines. A functional \( F : \subseteq \mathcal{B} \to \mathcal{B} \) is called computable if there is an oracle Turing machine \( M \) such that the computation of \( M \) with oracle \( \varphi \) and on input \( a \) halts with the string \( F(\varphi)(a) \) written on the output tape, or for short if \( M^\varphi(a) = F(\varphi)(a) \). A function between represented spaces is called computable if it has a computable realizer.

Example 2.6 (Comp. metric spaces) A triple \( \mathcal{Z} = (Z, d, (r_i)) \) is a computable metric space, if \((Z, d)\) is a separable metric space and \((r_i)\) is a dense sequence such that the discrete metric
\[
\tilde{d} : \mathbb{N} \times \mathbb{N} \to \mathbb{R}, \quad (i, j) \mapsto d(r_i, r_j)
\]
is computable. To be specific here, a representation of \( \mathbb{N} \) must be picked: Set \( \xi_\mathbb{N}(\varphi) = n \) if and only if \( \varphi(a) = n \) for all \( a \in \Sigma^* \). This is the Cauchy representation with respect to the discrete metric and \((n)_{n \in \mathbb{N}}\) as dense sequence.

The metric \( d : Z \times Z \to \mathbb{R} \) of a computable metric space is computable with respect to the Cauchy representation \( \xi_Z \) from Definition 2.2 and the representation from Definition 2.3 on \( \mathbb{R} \): To specify an oracle Turing machine computing a name of \( d(x, y) \) from a pair \( \langle \varphi, \psi \rangle \) of names of \( x \) and \( y \) as oracle, on input \( n \) do the computation of \( \tilde{d} \) on inputs \( i := \varphi(4n + 3), j := \psi(4n + 3) \) and string \( 2n + 1 \).

2.1 Second-order complexity theory
Complexity theory for functionals is called second-order complexity theory. It was originally introduced by Mehlhorn [Meh76]. This paper uses a characterization via resource bounded oracle Turing machines due to Kapron and Cook [KC96] as definition. Such a machine is granted time depending on the size of the input. The string functions are considered part of the input.
**Definition 2.7** For $\varphi \in B$ a string function define the length $|\varphi| : \omega \to \omega$ of $\varphi$ by

$$|\varphi|(n) := \max\{|\varphi(a)| \mid |a| \leq n\}.$$  
For instance: The length of each polynomial-time computable string function is bounded by an polynomial from above. Polynomials $\omega \to \omega$ have themselves as length. A polynomial as function $\mathbb{N} \to \mathbb{N}$ has linear length where the slope is the degree of the polynomial.

A running time is a mapping that assigns an allowed number of steps to sizes of the inputs. Thus, it should be of type $\omega \times \omega \to \omega$. The following conventions are used to measure the time of oracle interactions of an oracle Turing machine: Writing the query takes time. Reading the answer tape takes time. Writing the answer to an oracle query to the answer tape is done by the oracle and does not take time. It may very well happen that some of the content of the answer tape is not accessible to the machine due to running time restrictions.

**Definition 2.8** An oracle Turing machine $M^\varphi$ runs in time $T : \omega^\omega \times \omega \to \omega$, if for each oracle $\varphi \in B$ and input string $a$ it terminates within at most $T(|\varphi|, |a|)$ steps. It runs in time $O(T)$ if there is a $C \in \omega$ such that it runs in time $(l, n) \mapsto CT(l, n) + C$.

This definition, for instance also used in [FM13], is a straight-forward generalization of a characterization from [KC96] and its application to computable analysis from [KC10]. The latter sources only use the definition for a subclass of running times that are considered polynomial and called second-order polynomials. They are recursively defined as follows:

- Whenever $p$ is a positive integer polynomial, then the mapping $(l, n) \mapsto p(n)$ is a second-order polynomial.
- If $P$ and $Q$ are second-order polynomials then $P + Q$ and $P \cdot Q$ are second-order polynomials.
- If $P$ is a second-order polynomial then $(l, n) \mapsto l(P(l, n))$ is a second-order polynomial.

Second-order polynomials also turn up in other contexts [Koh96].

A functional $F : \subseteq B \to B$ is called computable in time $T$ resp. computable in polynomial time if there is an oracle machine that computes it and runs in time $T$ resp. in time bounded by some second-order polynomial.

**Definition 2.9** A function between represented spaces is polynomial-time computable resp. computable in time $T$, if it has a polynomial-time computable realizer (in the sense of Definition 2.5) resp. a realizer computable in time $T$.

A function is called computable in bounded time, if there is any running time $T$ such that it is computable in time $T$. While a total functional on the Baire space is always computable in bounded time, not every computable function between represented spaces is computable in bounded time as all realizers may be partial.

**Example 2.10 (2.6 continued)** Let $\mathcal{Z} = (Z, d, (r_i))$ be a computable metric space. Let $t : \omega \times \omega \to \omega$ be a running time of the discrete metric $d_{i,j} := d(r_i, r_j)$ (i.e. the function $S : \omega^\omega \times \omega \to \omega$ defined by $S(l, n) := t(l(0), n)$ is a running time, this is not a restriction). The algorithm to compute the metric from Example 2.6 terminates within $O(T)$ steps for the function $T(l, n) := t(l(n + 3), n + 1)$. In particular $d$ is polynomial-time computable if $d$ is polynomial-time computable.

Let $\xi$ and $\xi'$ be representations of the same space $X$.

A continuous functional $F : \subseteq B \to B$ is called a translation from $\xi$ to $\xi'$ if it is a realizer of the identity function, i.e. if it maps $\xi$-names into $\xi'$-names of the same element.
Definition 2.11 Two representations are called topologically equivalent if there exist translations in both directions. They are called computably, resp. bounded time or polynomial-time equivalent if there exists computable, resp. bounded time or polynomial-time computable translations in both directions.

The Cauchy representation from Definition 2.2 is polynomial-time equivalent to the open representation that arises if the inequality in its definition is made strict. The comment after Definition 2.3 points to polynomial-time equivalence when it says ‘equivalently’. In the following we identify representations if they are polynomial-time equivalent.

Definition 2.8 implies that any machine that runs in bounded time computes a total function. The parts of our results that assume bounded time computability remain valid under the weaker assumption that the machine runs in bounded time, whenever the oracle is from the domain of the computed function. Thus, we additionally use the following weaker notion:

Definition 2.12 We say that an oracle Turing machine $M$ runs in time $T$ on a subset $A \subseteq B$, if its run with oracle $\varphi \in A$ on input string $a$ terminates within $T(|\varphi|, |a|)$ steps.

A functional on the Baire space is computable in time $T$ on $A$ if there is a machine that computes it and runs in time $T$ on $A$. For functions between represented spaces the set $A$ is usually the domain of the representation of the domain of the function.

Definition 2.13 A multivalued function $f : X \Rightarrow Y$ between represented spaces is called computable in time $T$ relative to $X$ if it has a realizer (in the sense of Definition 2.5) that is computable in time $T$ on $\text{dom}(\xi_X)$.

Remark 2.14 Kawamura and Cook’s framework of second-order representations [KC10] (cf. also Section 4.1) can be seen to use this definition for its definition of polynomial-time computability. In this case, however, it turns out to be equivalent to the classical definition due to the existence of a polynomial-time retraction. Note, however, that some care has to be taken, as in other cases there are subtle differences between computing on the Baire space and computing on a subset even if there is such a retraction (cf. Definition 4.2 and the remark afterwards).

2.2 Admissibility, length and compactness

Recall that a set is called relatively compact if its closure is compact. The relatively compact subsets of the Baire space are easily classified. A subset of the Baire space is relatively compact if and only if it is contained in a set from the following family:

Definition 2.15 Define a family $(K_l)_{l \in \omega}$ of compact subsets of $B$ by $K_l := \{ \varphi \in B \mid |\varphi| \leq l \}$.

The mentioned property of this family of sets resembles hemi-compactness: Whenever $K$ is a compact subset of the Baire space, there is some $l \in \omega^\omega$ such that $K \subseteq K_l$. The only difference is that the index set $\omega^\omega$ is not countable but has the cardinality of the continuum.

Definition 2.16 We call a function $\ell \in \omega^\omega$ a length of a representation $\xi$ of a space $X$, if $\xi(K_l) = X$.

I.e. $\ell$ is a length of $\xi$ if every element has a $\xi$-name of length $\ell$. This condition does not imply that the domain of $\xi$ is included in $K_\ell$, but is strictly weaker.

Example 2.17 Let $\Omega$ be a bounded subset of $\mathbb{R}^d$. And let $C \in \omega$ be such that each element $x \in \Omega$ has supremum norm $|x|_\infty$ less than $2^C$. Let $\xi_\Omega$ denote the range restriction of the $d$-fold product of the standard representation from Definition 2.3. Then the function $\ell(n) := 2d(n + C + 4)$ is a length of $\xi_\Omega$. 

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Not every representation has a length, but representations of compact spaces usually do for the following reason:

**Theorem 2.18 (Compactness and length)** Any open representation of a compact space has a length.

**Proof** Let \( \xi \) be an open representation of a compact space \( K \). Let \( a_m \) denote the standard enumeration \( \mathbb{N} \to \Sigma^* \) that simply removes the first digit. Recursively define a function \( \ell : \omega \to \omega \) as follows:

For \( m \in \mathbb{N} \) set

\[
U_{0,m} := \{ \varphi \mid \varphi(\varepsilon) = a_m \} \subseteq \Sigma^* \Sigma^*.
\]

The sequence \( (U_{0,m})_{m \in \mathbb{N}} \) forms an open cover of the Baire space. Since \( \xi \) is open, the images of these sets compose an open cover of \( K \). Since \( K \) is compact, there exists a finite subset \( I \) of \( \mathbb{N} \) such that \( (\xi(U_{0,n}))_{n \in I} \) covers \( K \).

Define

\[
\ell(0) := \max \{|n| \mid n \in I\}
\]

Note that this means that each element of \( K \) has a name \( \varphi \) such that \( |\varphi| (0) = |\varphi(\varepsilon)| \leq \ell(0) \).

Now assume that \( \ell \) has been defined up to value \( n \) such that each element of \( K \) has a name \( \varphi \) such that for each \( k \leq n \) it holds that \( |\varphi|(k) \leq \ell(k) \). To construct \( \ell(n+1) \) let \( J \) be the set of all strings of length exactly \( n+1 \). Define a sequence \( (U_{n+1,m})_{m \in \mathbb{N}} \) as follows:

\[
U_{n+1,(m,b) \in J} := \{ \varphi \mid \forall k \leq n : |\varphi|(k) \leq \ell(k) \text{ and } \forall b \in J : \varphi(b) = m_b \}.
\]

Note that each of these sets is an open subset of Baire space and that the assumptions of the recursion make sure that the union of their images is \( K \). Thus, again from the compactness of \( K \) it follows that there is a finite subset \( I \) of \( \mathbb{N} \) such that \( (\xi(U_{n+1,m}))_{m \in I} \) covers \( K \).

Set

\[
\ell(n+1) := \max \{|m| \mid \exists (n_a)_{a \in J} \in I, \exists b \in J : m = n_b \}.
\]

By choice of \( I \) it is again the case that each element of \( K \) has a name \( \varphi \) such that for each \( k \leq n+1 \) it holds that \( |\varphi|(k) \leq \ell(k) \).

The constructed function is a length of the representation by its definition.

All representations considered so far were polynomial-time equivalent to open representations. This is due to the fact that only representations of metric spaces were considered and metric spaces are second countable. The abstract concept from computable analysis that remains appropriate in the general case is the following:

**Definition 2.19** A representation \( \xi \) of a topological space \( X \) is called admissible, if for any continuous representation \( \xi' \) there is a translation (that is: A continuous realizer of the identity) to \( \xi \).

A representation of a second countable Hausdorff space is admissible if and only if it is topologically equivalent to an open representation [BH02].

The following corollary can be obtained from Theorem 2.18 together with the observation, that any admissibly represented compact Hausdorff space is a metrizable space (for instance [Sch02, Proposition 3.3.2]).

**Corollary 2.20 (Length of restrictions)** Let \( \xi \) be an admissible representation of a Hausdorff space \( X \). Then the range restriction of \( \xi \) to any compact subset \( K \subset X \) has a length.
3 Metric entropy

Throughout this chapter let \( Y \) be a represented metric space, that is \( Y = (\mathcal{Y}, d) \) is a pair of a represented space \( \mathcal{Y} = (Y, \xi) \) and a metric \( d \). We do not impose any compatibility conditions, however, most results assume the metric to be computable in bounded time, this in turn implies continuity of the metric with respect to the topology that \( \mathcal{Y} \) has as a represented space.

It is well known that in a complete metric space a subset is relatively compact if and only if it is totally bounded. The following notion is a straightforward quantification of total boundedness and can be used to measure the ’size’ of compact subsets of metric spaces. It was first considered in [KT59], where some of the names originate from. It is also regularly used in proof mining [Koh08, KLN15] and constructive mathematics [Bis67], where other names are taken from. A comprehensive overview can be found in [Lor66].

**Definition 3.1** A function \( \nu : \omega \to \omega \) is called *modulus of total boundedness* of a subset \( K \) of a metric space, if for any \( n \in \omega \) the set \( K \) can be covered by \( 2^{\nu(n)} \) closed balls of radius \( 2^{-n} \). The smallest modulus of total boundedness of a set \( K \) is called the *metric entropy* of the set and denoted by \( \text{ent}(K) \).

Due to the use of closed balls it holds that \( \text{ent}(K) = \text{ent}(\overline{K}) \). The metric entropy of a compact set is often hard to get hold of while upper bounds are easily specified. In a complete metric space a closed set allows for a modulus of total boundedness if and only if it is compact.

A modulus of total boundedness should intuitively be understood as an upper bound on the size of a compact set. For providing lower bounds, a different is more convenient.

**Definition 3.2** A function \( \eta : \omega \to \omega \) is called a *spanning bound* of a subset \( K \subseteq \mathcal{Y} \) if for any \( n \) there exist elements \( x_1, \ldots, x_{2^{\eta(n)}} \) such that for all \( i, j \in \mathbb{N} \)

\[
i \neq j \implies d(x_i, x_j) > 2^{-n+1}.
\]

This means, that the closed \( 2^{-n} \)-balls around the points \( x_i \) are disjoint. It is not difficult to see that there is a biggest spanning bound if and only if the set \( K \) is relatively compact. The following states that spanning bound are lower bounds to the metric entropy:

**Proposition 3.3** Let \( K \) be a subset of a metric space. For any spanning bound \( \eta \) of \( K \) it holds that \( \eta(n) \leq \text{ent}(K)(n) \).

**Proof** Show by contradiction that \( \text{ent}(K)(n) + 1 \) is not a value of a spanning bound. Thus, assume it is a value of a spanning bound. This means that there are \( 2^{\eta(n)+1} \) elements \( x_i \) such that \( d(x_i, x_j) \geq 2^{-n+1} \). By definition of \( \text{ent}(K) \) there are \( 2^{\eta(n)} \) elements \( y_i \) such that the closed \( 2^{-n} \)-balls around the \( y_i \) cover \( K \). Since each \( x_i \) has to lie in one of the balls and there are more \( x_i \) than \( y_i \), there are indices \( j, l \) and \( m \) such that \( x_j \) and \( x_l \) both lie in the closed \( 2^{-n} \)-ball around \( y_m \). By triangle inequality it follows that

\[
d(x_j, x_l) \leq d(x_j, y_m) + d(y_m, x_l) < 2 \cdot 2^{-n},
\]

which is a contradiction to \( \text{ent}(K)(n) + 1 \) being a value of a spanning bound.\( \blacksquare \)

3.1 Metric entropy in normed spaces

In general, the metric entropy heavily depends on the metric chosen for a space, and not only on the topology it induces. For normed spaces, however, the situation is a lot better. Recall that norm \( \| \cdot \| \) on a vector space \( X \) induces a metric by \( d(x, y) := \| x - y \| \). But not every metric vector space can be equipped with a norm.
Proposition 3.4 Let \( \| \cdot \| \) and \( \| \cdot \|' \) be norms on a vector space \( X \) that induce the same topology. Then there exists a constant \( C \in \omega \) such that whenever \( \nu \) is a modulus of total boundedness of a subset \( K \) of \( X \) with respect to \( \| \cdot \| \), then \( \nu'(n) := \nu(n + C) \) is a modulus of total boundedness of \( K \) with respect to \( \| \cdot \|' \) and vice versa.

Proof Note that norms inducing the same topology are equivalent. This means that there exists a constant \( C \) such that for any \( x \in X \)
\[
2^{-C} \| x \| \leq \| x \|' \leq 2^C \| x \|.
\]
From these inequalities it is straightforward to compute the relation between the metric entropies. ■

Theorem 3.5 Whenever \((X, \| \cdot \|)\) is an infinite dimensional normed space and \( \mu : \omega \to \omega \) is a function, then there is a compact subset \( K \) of \( X \) such that \( \text{ent}(K) \geq \mu \).

Proof By Riesz’s Lemma there exists a sequence of elements \( x_i \) such that \( \| x_i \| = 1 \) and \( \| x_i - x_j \| > \frac{1}{2} \) for all \( i \neq j \). This means that a closed ball of radius 1 around zero contains all of the \( x_i \), but a closed ball of radius \( \frac{1}{2} \) around any point contains at most one \( x_i \). Define a set \( K \subseteq X \) by
\[
K := \{0\} \cup \bigcup_{i \in \omega} \bigcup_{j=1}^{2^{\mu(i)} - 2^{\mu(i-1)}} \{2^{-i+1} x_j\},
\]
where we use the convention \( 2^{\mu(-1)} = 0 \). To cover \( K \) with balls of radius \( 2^{-n} \) we need at least one ball for each of the \( 2^{\mu(n)} \) many elements \( x \in K \) with \( \| x \| \geq 2^{-n+1} \). Therefore \( |K| \) \( \| \mu(n) \| \geq \mu(n) \) for all \( n \in \omega \).

To argue that the constructed set \( K \) is compact, show sequential compactness. For metric spaces this is equivalent to compactness. Consider an arbitrary sequence in \( K \). From the construction it is clear that the ball of any radius around zero contains all but finitely many elements of \( K \). By the pigeonhole principle there is either an element of \( K \) that is visited infinitely often by the sequence, or there is an element of the sequence in any a ball of any radius around zero. In the first case there is a constant subsequence, in the second case there is a subsequence that is convergent to zero. Therefore the set is compact. ■

It is well known that for Banach spaces finite dimensionality and local compactness coincide. A similar characterization follows from the above:

Corollary 3.6 For a normed space the following are equivalent:
1. It is infinite dimensional.
2. For any \( \nu : \omega \to \omega \) there exists a compact subset \( K \) such that \( \text{ent}(K) \geq \nu \).
3. There exists a compact subset that has no linear modulus of total boundedness.

Proof
1. \( \Rightarrow 2. \): This is Theorem 3.5.
2. \( \Rightarrow 3. \): This is trivial.
3. \( \Rightarrow 1. \): By contradiction assume that the normed vector space was finite dimensional. Choose a basis to identify it with \( \mathbb{R}^d \). Since all norms on a finite dimensional vector space are equivalent, Proposition 3.4 shows that it is irrelevant which norm is used for whether or not all compact sets have a linear modulus of total boundedness. With respect to the supremum norm all sets have a linear modulus of total boundedness. This contradicts the third item. ■
3.2 Metric entropy and complexity

This chapter investigates connections between the concept of metric entropy and computational complexity. Recall the exhausting family \((K_l)_{l \in \omega}\) of compact subsets \(K_l\) of the Baire space indexed by functions \(l : \omega \to \omega\), namely:

\[
K_l := \{ \varphi \in B | \forall n : |\varphi| (n) \leq l(n) \}
\]

The following example is very instructive for the content of this section:

**Example 3.7 (A family of compacts)** Let \((Z, d, (r_i))\) be a complete separable metric space with a dense sequence. For the Cauchy representation \(\xi_Z\) of \(Z\) from Definition 2.2 it holds that

\[
\xi_Z(K_l) = \bigcap_{n \in \mathbb{N}} \bigcup_{i=0}^{\varphi(n)-1} B_{2^{-n}}(r_i).
\]

The set on the right hand side is closed as an intersection of finite unions of closed sets and its metric entropy is bounded in terms of \(l\). Due to the completeness of \(Z\) the set is compact and for all \(n \in \omega\) and non-decreasing \(l : \omega \to \omega\)

\[
\text{ent}(\xi_Z(K_l))(n) \leq l(n).
\]

Surprisingly, a similar connection exists for arbitrary representations of metric spaces. The goal of this section is to obtain from a bound of the running time of the metric a modulus of total boundedness of the sets \(\xi_Y(K_l)\) ⊆ \(Y\). For this we often consider the first order function that arises if the first order argument of a running time \(T: \omega \times \omega \to \omega\) is fixed to some \(l: \omega \to \omega\). We denote this function by \(T(l, \cdot)\).

**Theorem 3.8 (Complexity of metric entropy)** Let \((Y, d)\) be a represented metric space such that \(d\) is computable in time \(T\) relative to \(Y \times Y\). Then there exists some \(C \in \omega\) such that for all \(l \in \omega\) the function \(CT(l, \cdot + 2)^2 + C\) is a modulus of total boundedness of \(\xi_Y(K_l)\). That is:

\[
\text{ent}(\xi_Y(K_l))(n) \in O(T(l, \cdot + 2)^2).
\]

The proof is postponed to the next section. For now consider some implications of this theorem and some examples. For instance:

**Example 3.9 (2.6, 2.10, 3.7 continued)** Assume that \((Z, d, (r_i))\) is a computable metric space, where the representation \(\xi_Z\) of \(Z\) is the Cauchy representation from Example 2.6. Assume that the discrete metric (cf. Equation (d) on page 8) is polynomial-time computable. According to Example 2.10 the metric \(d\) is computable in time polynomial in \(l(n + 2)\) and \(n\). Thus, by Theorem 3.8 the metric entropy of \(\xi_Z(K_l)\) is bounded by a polynomial in \(l(n+4)\) and \(n\). This estimate is reasonably close to the estimate \(l(n)\) for non-decreasing \(l\) from Example 3.7.

Using Weihrauch’s TTE and Weihrauch’s representations (see [Wei00]) is equivalent to restricting to representations \(\xi\) such that all names fulfill \(\varphi(a) = \varphi(1^{[n]})\) and \(|\varphi| \equiv 1\). We call such representations Cantor space representations. For a Cantor space representation computability in time \(T(l, n)\) is equivalent to computability in time \(t : \omega \to \omega\) \(t(n) := T(1, n)\) within the TTE framework. From the previous theorem it follows that:

**Corollary 3.10** Let \((Y, d)\) be a Cantor space represented metric space such that the metric is computable in time \(t : \omega \to \omega\). Then \(\text{ent}(Y)\) is bounded by a polynomial in \(t(n + 2)\).
Since the norm of a continuous function can be computed in exponential time from the evaluations of a function, this in particular implies a version of the folklore fact from computable analysis that there is no Cantor space representation of the space of Lipschitz one functions on the unit interval that map zero to zero such that the evaluation is polynomial time computable (see for instance [Wei03]): Whenever there is a Cantor space representation of $C([0,1])$ such that the restriction of the evaluation operator to a set $F \subseteq C([0,1])$ is computable in time $t$, then $\text{ent}(F)$ is bounded exponentially in $t(n+2)$. This together with the fact that $C([0,1])$ as an infinite dimensional normed space has arbitrary big bounded subsets by Theorem 3.5 shows that there is no Cantor space representation of $C([0,1])$ or even the unit ball of this space such that evaluation is computable in bounded time. More generally:

**Corollary 3.11** No infinite dimensional normed space has a Cantor space representation such that the norm and the vector space operations are computable in bounded time when restricted to the closed unit ball.

**Proof** If the norm and the vector space operations are computable in bounded time, so is the metric. By the previous corollary this implies that there exists a modulus of total boundedness of the unit ball. Since the vector space is an infinite dimensional normed space, there exists arbitrary big compact subsets of the unit ball by Theorem 3.5. This is a contradiction as going to subsets increases the modulus of total boundedness by at most a shift by one. 

Recall from Corollary 2.20 that any range restriction of an admissible representation to a compact set has a length. The following corollary can be used to check whether a given representation has the minimal possible length.

**Corollary 3.12** Let $\xi$ be an admissible representation of a metric space $(Y,d)$ such that the metric is computable in time $T$. Then the range restriction of $\xi$ to any compact subset $K \subseteq M$ has a length $\ell$ that fulfills

$$\text{ent}(K) \in O(T(\ell, \cdot + 2)^2).$$

**Proof** Openness is preserved under taking range restrictions. Theorem 2.18 proves that the range restriction as an open representation of a compact set has a length $\ell$. Running times are also preserved under range restrictions. Apply Theorem 3.8 and get the assertion.

### 3.3 Proof of the main theorem

The goal of this section is to prove Theorem 3.8. The argument works in a very general setting, in particular the assumption about the space can be weaker than computability of the metric in bounded time. This is not very surprising, as the metric entropy only mentions small balls and does not use any information about the exact values of the distance of points far away from each other.

Equality in a metric space equipped with the Cauchy representation is usually not decidable. However, for two given elements $x, y$ of a represented metric space it is decidable whether or not the elements are far apart. This can be formalized as computability of the following multivalued function:

**Definition 3.13** Let $Y = (Y, d)$ be a represented metric space. Define the equality function $\text{eq}_Y : Y \times Y \rightrightarrows \mathcal{B}$ by:

$$\text{eq}_Y(x, y) := \left\{ \varphi \in \{0, 1\}^{X^2} \mid \forall n : d(x, y) \leq 2^{-n-1} \Rightarrow \varphi(1^n) = 1 \quad \text{and} \quad \forall n : d(x, y) > 2^{-n} \Rightarrow \varphi(1^n) = 0 \right\}.$$
Informal but more intuitive one might indicate this by
\[
eq_{\mathcal{Y}}(x, y)(1^n) = \begin{cases} 
1 & \text{if } d(x, y) \leq 2^{-n-1} \\
0 & \text{if } d(x, y) > 2^{-n} \\
0 \text{ or } 1 & \text{otherwise.}
\end{cases}
\]

Definition 3.14 Let \( \mathcal{Y} = (\mathcal{Y}, d) \) be a represented metric space. We say that equality is approximable in time \( T \) if its equality function \( \equiv_{\mathcal{Y}} \) is computable in time \( T \) relative to \( \mathcal{Y} \times \mathcal{Y} \).

This notion is indeed weaker than bounded time computability of the metric in a very precise sense:

Lemma 3.15 (From metric to equality) Let \( (\mathcal{Y}, d) \) be a represented metric space. If \( d \) is computable in time \( T \) relative to \( \mathcal{Y} \times \mathcal{Y} \), then equality is approximable in time \( O(T) \) for \( \tilde{T}(l, n) := T(l, n + 2) \).

Proof Let \( M^T \) be a machine that computes \( d \) in time bounded by \( T \) relative to \( \mathcal{Y} \times \mathcal{Y} \). Given a approximation requirement \( 1^n \), carry out the computation this machine does on input \( 2^{n+2} - 1 \) to produce an encoding of a \( 2^{n+2} \)-approximation to \( d(x, y) \). Return 0 if the return value encodes a dyadic number strictly bigger than \( 2^{-n-1} + 2^{-n-2} \), otherwise return 1. Carrying out the computations of \( M \) takes time \( T(l, n + 2) \), and checking the return value against \( 2^{-n-1} + 2^{-n-2} \) takes time less than twice that plus a constant. A triangle inequality argument proves that this machine approximates the equality function.

A running time bound restricts the access an oracle machine has to the oracle. The following Lemma describes this dependence in exactly as much detail as needed for our purposes. It assigns to a machine a function \( B \times \Sigma^* \to \Sigma^* \) that takes as input an oracle and a string, and whose return value is a description of the communication between the machine and the oracle. In particular, if the first argument is changed the return values only change if the machine can distinguish the oracles in a computation with the second input as input string.

Similar constructions have been done for oracles that only return 0 or 1 before [Wei87]. The function constructed is very closely related to moduli of sequentiality [BK02]. Our approach differs from the ones taken in these sources in that a ‘dialog’ only describes the return values of the oracle and not the queries.

Lemma 3.16 (Communication functions) For any oracle Turing machine \( M^T \) that runs in time \( T \) on a set \( A \subseteq B \) there exists a function \( L : A \times \Sigma^* \to \Sigma^* \) such that

(4) \( M^T(a) \) is determined by \( L(\varphi, a) \), that is for all \( \varphi, \psi \in A \) and strings \( a \) from \( L(\varphi, a) = L(\psi, a) \) it follows that \( M^T(a) = M^T(a) \).

(v) Each value of an oracle on a string either matters a lot or does not matter at all: For all \( \varphi, \psi, \phi \in A \) and strings \( b \), from \( L(\varphi, b) = L(\psi, b) \) and \( \phi(a) = \varphi(a) \), whenever \( a \) is such that \( \varphi(a) = \psi(a) \), it follows that \( M^T(b) = M^T(b) \).

(i) The length of \( L \) is bounded by the running time: For all strings \( a \) and \( \varphi \in A \)
\[
|L(\varphi, a)| \leq 2(T(|\varphi|, |a|) \cdot (T(|\varphi|, |a|) + 1) + 1).
\]

Note that (v) implies (d), however, since the meaning of (d) is a lot easier to grasp and (v) is only needed to guarantee that pairings work as expected, they are stated separately.
Proof Let \( L(\varphi, a) \) be an encoding of the number of oracle queries together with a list of the \( T(|\varphi|, |a|) \) first bits of the answers to the oracle calls during the run \( M^\varphi(a) \) of \( M^\varphi \) on input \( a \) with oracle \( \varphi \). I.e.
\[
L(\varphi)(a) = \langle N, \langle b_1, \ldots, b_N \rangle \rangle
\]
where \( b_i \) consist of the \( T(|\varphi|, |a|) \) first bits of \( \varphi(a_i) \) where \( a_i \) is the \( i \)-th of the \( N \) queries the machine asks to \( \varphi \).

As mentioned, the condition from (d) is implied by the one from (v). To see that the condition (v) holds note that the value \( L(\varphi, b) \) determines the number \( N \) of queries the machine asks the oracle \( \psi \) and also their values \( a_1, \ldots, a_N \). Now \( L(\varphi, b) = L(\psi, b) \) implies that \( \varphi(a_i) = \psi(a_i) \) for all \( i \). Therefore from the other assumption of (v) it follows that the run of \( M^\varphi \) on \( b \) with oracle \( \phi \) writes the same queries and gets the same answers. Thus \( M^\varphi(b) \) produces the same return value as both \( M^\varphi(b) \) and \( M^\psi(b) \).

From the restriction of the running time of \( M^\varphi \) it follows that the number \( N \) and each \( |b_i| \) can at most be \( T(|\varphi|, |a|) \). This put together with the length estimations for the pairing functions from the introduction leads to the bound on the length of \( L(\varphi, a) \).

The conclusion \( M^\varphi(b) = M^\psi(b) \) from item (v) cannot be replaced by the stronger \( L(\varphi, b) = L(\psi, b) \). This is due to the use of initial segments of the oracle answers. The length of these initial segments depend on the value of the running time, which we have no control over.

Recall that to each \( l : \omega \to \omega \) a compact subset \( K_l \) of the Baire space was assigned by
\[
K_l := \{ \varphi \in B \mid |\varphi| \leq l \}
\]
and that the family \( (K_l)_{l \in \omega} \) has the property that every compact subset of the Baire space is contained in some \( K_l \).

The proof of the following theorem is now a straightforward application of the previous lemma. Note that it does not require the metric space to be compact, but instead talks about certain relatively compact subsets of the space.

**Theorem 3.17** Let \( \mathcal{Y} \) be a represented metric space such that the equality is approximable in time \( T \). Then for all \( n \in \omega \)
\[
\text{ent}(\xi(K_i))(n) \leq 2(T(l, n) \cdot (T(l, n) + 1) + 1).
\]

Proof Let \( \mathcal{Y} = (Y, d) \) and fix some \( n \). Let \( M^\varphi \) be the machine computing the equality function in time \( T \) on the set
\[
A := \text{dom}(\xi_{\mathcal{X} \times \mathcal{Y}}) = \{ (\varphi, \psi) \in B \mid \varphi \in \text{dom}(\xi) \text{ and } \psi \in \text{dom}(\xi) \},
\]
and let \( L : A \times \Sigma^* \to \Sigma^* \) be the communication function assigned to \( M^\varphi \) by Lemma 3.16. Let \( I \) be the set of strings \( a \) such that there exists a \( \psi \in K_l \cap \text{dom}(\xi) \) such that \( L((\psi, \psi), 1^n+1) = a \). For each \( i \in I \) choose some \( \psi_i \in K_l \cap \text{dom}(\xi) \) such that \( L((\psi_i, \psi_i), 1^n+1) = i \). From the size limit for \( L((\psi, 1^n+1)) \) from Lemma 3.16 item (I) it follows that
\[
\# I \leq 2^{2(T(l, n) \cdot (T(l, n) + 1)) + 1}.
\]

Claim that the closed \( 2^{-n} \)-balls around the \( \xi(\psi_i) \) cover \( \xi(K_l) \): Indeed, take an arbitrary \( x \in \xi(K_l) \), that is \( x = \xi(\psi) \) for some \( \psi \in K_l \cap \text{dom}(\xi) \). Then for \( i := L((\psi, \psi), 1^n) \in I \) it holds that \( L((\psi_i, \psi_i), 1^n) = L((\psi, \psi), 1^n) \). Note that
\[
\langle \psi, \psi \rangle(b) = \langle \psi_i, \psi_i \rangle(b) \iff \psi(b) = \psi_i(b) \iff \langle \psi, \psi \rangle(b) = \langle \psi_i, \psi_i \rangle(b).
\]
Therefore, using the property of \( L \) from item (v) of Lemma 3.16 for the functions \( \langle \psi, \psi \rangle, \langle \psi_i, \psi_i \rangle \) and \( \langle \psi, \psi \rangle \) it follows that
\[
M^{\langle \psi, \psi \rangle}(1^n) = M^{\langle \psi, \psi \rangle}(1^n) = 1.
\]
Since $M'$ computes the function $eq_{\psi}$ from Definition 3.13, this implies that $d(x, \xi(\psi_i)) \leq 2^{-n}$. Thus, $x \in \mathcal{B}_{-n}^{\xi}(\psi_i)$ and, since $x \in \xi(K_l)$ was arbitrary, the closed $2^{-n}$-balls around the images of the $\psi_i$ cover $\xi(K_l)$. □

Finally, let us argue that the theorem indeed implies Theorem 3.8.

Proof (of Theorem 3.8) Let $Y = (Y, d)$ be a represented metric space and assume that $d$ is computable in time $T$ relative to $Y \times Y$. Lemma 3.15 provides a constant $\tilde{C} \in \omega$ such that equality on $Y$ is approximable in time $(l, n) \rightarrow \tilde{C}T(l, n + 2) + \tilde{C}$. Apply Theorem 3.17 to obtain
\[
\text{ent}(\xi(K_l))(n) \leq 2((\tilde{C}T(l, n + 2) + \tilde{C})(\tilde{T}(l, n + 2) + \tilde{C} + 1) + 1) \in O(T(l, n + 2)^2),
\]
where the constant $C$ can be chosen as $8\tilde{C}^2 + 4\tilde{C} + 2$. The independence of $l$ follows from the independence of $\tilde{C}$ of $l$ (by definition of $O(T)$). □

4 Regular and complete representations

Some functions $T : \omega^\omega \times \omega \rightarrow \omega$ are no reasonable candidates for running times. A running time should grant more time, if the input is bigger; therefore it should be monotone. We use the notion of monotonicity introduced by Kohlenbach [Koh05]. It restricts to monotonicity in the point-wise sense, if the function is only considered on increasing inputs and returns only increasing functions.

Definition 4.1 A function $T : \omega^\omega \times \omega \rightarrow \omega$ is monotone, if for all $l, l' \in \omega^\omega$ from $\forall n \leq m : T(l, n) \leq T(l', m)$ it follows that $\forall n \leq m : T(l, n) \leq T(l', m)$.

In Section 5 we encounter a situation, where a machine needs to compute a function that is similar to its running time. To guarantee that this can be done by the machine without taking too many steps, a notion of time-constructibility is needed for second-order running times.

Definition 4.2 We call $T : \omega^\omega \times \omega \rightarrow \omega$ time-constructible relative to $A \subseteq \mathcal{B}$, if there is an oracle Turing machine that computes the mapping $(\varphi, a) \mapsto T(|\varphi|, |a|)$ and runs in time $O(T)$ on $A$.

We call a function time-constructible, if it is time-constructible relative to the Baire space. This definition reproduces usual notion of time-constructibility for functions that are independent of the first argument.

Example 4.3 Recall that the length function $|\cdot| : \mathcal{B} \rightarrow \omega^\omega$ was defined by $|\varphi|(n) := \max \{ |\varphi(a)| \mid |a| \leq n \}$. Due to the evaluation of the maximum taking an exponential number of oracle queries, this function is not polynomial-time computable. Thus, the function $(l, n) \mapsto l(n)$ is not time-constructible. As a consequence, most second-order polynomials are not time-constructible. The function $(l, n) \mapsto 2^{l(n)}$ on the other hand is time-constructible.

The standard application of time-constructibility is increasing the domain of realizers:

Lemma 4.4 Let $F : \subseteq \mathcal{B} \rightarrow \mathcal{B}$ be a functional on the Baire space computable in time $T$ on some set $A$. If $T$ is time-constructible relative to $B \supseteq A$, then $F$ is computable in time $O(T)$ on $B$.

4.1 Regularity

The mapping $L(l, n) := l(n)$ plays a special role within the second-order polynomials. Indeed, if $L$ is time-constructible relative to a set $A$, then many second-order polynomials are time-constructible relative to $A$. Under additional assumptions about $A$, for instance that all elements are strictly increasing it is possible to prove that time-constructibility of $L$ relative to $A$ implies time-constructibility of all second-order polynomials relative to $A$. 

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Definition 4.5 We call a functional $F: \mathcal{B} \to \mathcal{B}$ linear-time computable, resp. linear-time computable on $A \subseteq \mathcal{B}$, if it is computable in time $O(L)$ for $L(l,n) := l(n)$ resp. computable in time $O(L)$ on $A$.

Using this concept the content of Example 4.3 can be formulated as ‘$L$ is time-constructible relative to $A$ if and only if the length function is linear-time computable on $A$’.

It often leads to problems in applications, if it is impossible to find an upper bound on the length of a name in a representation. The following regularity condition on representations removes most of these difficulties, while not restricting the freedom of choice of representations too much:

Definition 4.6 We call a representation $\xi$ regular, if there exists a linear-time computable upper bound of the length function. I.e. a linear-time computable function $m : \mathcal{B} \to \omega$ such that for all $\varphi \in \text{dom}(\xi)$ it holds that $m(\varphi)$ is monotone and $|\varphi| \leq m(\varphi)$.

If the length function is linear time computable relative on $\text{dom}(\xi)$, then $\xi$ is regular. Due to the use of an upper bound the converse does not hold. However, from the linear time computability of $m$ it follows that there exists some $C \in \omega$ such that $m(n) \leq C|\varphi|(n) + C$.

The same problem has been tackled by Kawamura and Cook before in a slightly less general way by introducing a restricted class of string functions that are allowed to be names (cf. [KC10] and [Kaw11]).

Definition 4.7 A string function $\varphi \in \mathcal{B}$ is called length-monotone, if $|\varphi(a)| \leq |\varphi(b)|$ for all $a, b$ such that $|a| \leq |b|$. The set of length-monotone string functions is denoted by $\Sigma^{**}$.

For length-monotone string functions it holds that $|\varphi|(n) = |\varphi(1^n)|$. Therefore, the length function is linear-time computable on $\Sigma^{**}$ and all second-order polynomials are time-constructible relative to $\Sigma^{**}$.

Definition 4.8 A representation is called a second-order representation, if its domain is contained in $\Sigma^{**}$.

The use of the term ‘second-order’ indicates applicability of second-order complexity theory, not the use of higher-order objects in the representation. The restriction to length-monotone names leads to excessive padding and technical complications that are avoided by regular representations. Since the length function is time constructible relative to $\Sigma^{**}$ any second-order representation is regular. More generally:

Proposition 4.9 (Regularity vs. second-order) Second-order representations are regular. For every regular representation there is a linear-time equivalent second-order representation.

Proof Let $\xi$ be a second-order representation. Set $m(\varphi)(n) := |\varphi(1^n)|$. This function is obviously linear-time computable and since any element of $\text{dom}(\xi)$ is length-monotone and therefore fulfills $|\varphi|(n) = |\varphi(1^n)|$, it is not only an upper bound to the length function, but restricts to the length function on $\text{dom}(\xi)$.

On the other hand assume that $\xi'$ is a regular representation. Let $m : \mathcal{B} \to \omega$ be the linear time computable upper bound on the length function. Define a padding function as follows: For a string $a$ denote by $\tilde{a}$ the string where each 0 in is replaced by 01 and each 1 is replaced by 11. Set

$$\text{pad}(\varphi)(a) := \varphi(\tilde{a})2^{\max(m(\varphi)(|a|) - |\varphi(a)|, 0)}.$$ 

Since $m(\varphi)$ is increasing and $m$ is an upper bound to the length function, $\text{pad}(\varphi)$ is a length-monotone function whenever $\varphi$ is an element of $\text{dom}(\xi')$. 

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Since \( m \) is linear time computable, the mapping \( \text{pad} \) is linear time computable. Thus the representation \( \xi \) can in linear time be translated to the second-order representation \( \xi^{\text{pad}} \), where \( \varphi \) is a \( \xi^{\text{pad}} \)-name of an element if and only if \( \varphi = \text{pad}(\psi) \) for a \( \xi \)-name \( \psi \) of the element.

A linear time translation in the opposite direction is easily written down by first removing pairs of \( 0s \) from the return-values of a string function and then undoing the encoding of single digits by pairs of digits.

\[ \square \]

## 4.2 Completeness

Usually bounded time computability is a more restrictive notion than computability. However, for total functions on complete spaces one might expect any computable function to be computable in bounded time. This is for instance true for the real numbers represented as in Definition 2.3.

### Lemma 4.10

Let \( A \subseteq \mathcal{B} \) be closed. Any computable total function \( F : A \to \mathcal{B} \) is computable in bounded time on \( A \).

**Proof** Let \( M' \) be an oracle Machine computing the total function. Consider the function \( \text{time}_{M'} : A \times \Sigma^* \to \omega \), where \( \text{time}_{M'}(\varphi, a) \) is the number of steps in the run \( M^* (a) \). To see this let \( (\varphi_n, a_n) \subseteq A \) be a sequence converging to \( (\varphi, a) \in A \). Since \( \Sigma^* \) is equipped with the discrete topology, there exists some \( N \) such that for all \( n \geq N \) it holds that \( a_n = a \). Furthermore, the run \( M^* (a) \) takes a finite number \( k \) of steps. Let \( N \) be bigger than \( N \) and such that the \( k \) beginning segments of the \( \varphi_n \) coincide whenever \( n \geq N \). Since \( M' \) is a deterministic machine, the runs \( M^* (a) \) and \( M^{\varphi_n} (a_n) \) coincide whenever \( n \) is bigger than \( N \). Thus, the sequence \( \text{time}_{M'} (\varphi_n, a_n) \) is eventually constant and converges in \( \omega \). Since \( A \) is closed, \( K_f \cap A \) is compact and since \( \text{time}_{M'} \) is continuous, the number

\[ T(l, n) := \max \{ \text{time}_{M'}(\varphi, a) \mid \varphi \in K_f \text{ and } |a| \leq n \} \]

exists as the maximum of a continuous function on a compact set. By the definition of \( \text{time}_{M'} \) the machine \( M' \) runs in time \( T \). Thus, the function is computable in bounded time.

\[ \square \]

In particular any computable total function \( F : \mathcal{B} \to \mathcal{B} \) is computable in bounded time.

A condition that guarantees that each computable function has a total computable realizer is the following: Recall that a subset \( A \) of the Baire space is called co-recursively enumerable closed or co-r.e. closed, if there exists an oracle Turing machine \( M' \) such that the run \( M^* (\varphi) \) terminates if and only if \( \varphi \notin A \).

### Definition 4.11

We call a representation complete, if its domain is closed and computably complete, if its domain is co-r.e. closed.

All of the representations we construct are complete. Typical examples of representations that are not complete are the ‘padded’ counter example representations constructed by Kawamura and Pauly to prove that a straightforward generalization of admissibility to the polynomial-time framework does not provide a reasonable notion [KP14, Section 7].

### Lemma 4.12

Let the representation of \( \mathbf{X} \) be complete. Then any computable function with domain \( \mathbf{X} \) has a total computable realizer.

**Proof** Let \( M' \) be a machine that computes a realizer of the function \( f \). Let \( M'' \) be a machine that witnesses the computable closedness of the domain of \( \xi \). Construct an oracle Turing machine \( N' \) as follows: When given access to an oracle \( \varphi \) and an input string \( a \) it will dovetail simulations of \( M^* (a) \) and \( M^* (\xi) \). If the simulation of \( M^* (a) \) terminates first, the machine returns
the return value of this computation. If the simulation of $\bar{M}^a(\varepsilon)$ terminates first, the machine stops and returns the empty string. On one hand, since $M^a$ computes a realizer of a function, $M^a(a)$ terminates whenever $a$ is an element of $\text{dom}(\varphi)$. On the other hand if $\varphi \notin \text{dom}(\xi)$ then $M^a(\varepsilon)$ is guaranteed to terminate. Thus $N^a$ indeed computes a total function on the Baire space. It is clear from its definition that this function is still a realizer of $f$.  

The following is the combination of the previous two lemmas:

**Theorem 4.13** Let $f$ be a computable function with domain $X$. If the representation of $X$ is complete, then $f$ is computable in time bounded relative to $X$. If it is computably complete, then $f$ is computable in bounded time.

An example of a situation where provably no reasonable complete representations exist is provided by Férré and Hoyrup: They prove that for a represented non $\sigma$-compact Polish space $X$ there is no representation of $C(X)$ such that the time complexity of the evaluation is well defined ([FM13, Theorem 3.1], a full proof can be found in the PhD thesis by Férré). By the above theorem, in this case there can not exists a complete representation of $C(X)$ such that the evaluation is computable. This is not surprising, as in fact it is known that the descriptive complexity of the domain of representations increases with the number of applications of the function space construction [SS15].

The following can be understood as the motivation for the names ‘complete’ resp. ‘computably complete’.

**Theorem 4.14 (Completenesses)** A Cauchy representation of a separable metric space is computable in time bounded relative to $X$ if and only if it is computable in time bounded relative to $X$. A Cauchy representation of a complete computable metric space is computably complete.

**Proof** Let $Z = (Z, d, (r_i))$ be the metric space with dense sequence. First assume that $Z$ complete and let $\varphi_n$ be a sequence of names of elements $x_n$ that converges to an element $\varphi$ of the Baire space. To see that $\varphi$ is a name, first note that the sequence $x_n$ is a Cauchy sequence: Given $n$ choose $N$ such that for all $k \geq N$ it holds that $\varphi_n(2n + 1) = \varphi(2n + 1)$. And therefore for $k, l \geq N$

$$d(x_k, x_l) \leq d(x_k, r_{\varphi_n(2n+1)}) + d(r_{\varphi_n(2n+1)}, x_l) \leq \frac{1}{n + 1}.$$  

Since the metric space is complete and $x_n$ is a Cauchy sequence, there exists a limit $x$ of the Cauchy sequence. That $\varphi$ is a name of $x$ follows from taking the limit $m \to \infty$ in

$$d(x, r_{\varphi(n)}) \leq d(x, x_m) + d(x_m, r_{\varphi(n)}).$$

Note that $\varphi(n) = \varphi_m(n)$ for $m$ big enough and $d(x_m, r_{\varphi_m(n)}) \leq \frac{1}{m+1}$.

Now assume that the domain of the Cauchy representation is closed and let $x_n$ be a sequence of elements in $Z$. From the Cauchy property it follows that it is possible to find a convergent sequence of names $\varphi_n$. Since the domain of the representation is closed the limit $\varphi$ is the name of some $x \in Z$ and since the Cauchy representation is continuous, $x$ is the limit of the sequence $x_n$. Therefore $Z$ is complete.

Finally assume that $Z$ is a computable metric space and let $\xi$ be the Cauchy representation. To see that the domain of the Cauchy representation is co-r.e. first prove that

$$\varphi \in \text{dom}(\xi) \iff \forall i, j \in \mathbb{N} : d(r_{\varphi(i)}, r_{\varphi(j)}) \leq \frac{1}{i + 1} + \frac{1}{j + 1}.$$  

For the first implication assume that $\varphi$ is a name of some element $x$ in the Cauchy representation. Thus, using the triangle inequality and the symmetry of the metric see that for all $i, j \in \mathbb{N}$

$$d(r_{\varphi(i)}, r_{\varphi(j)}) \leq d(r_{\varphi(i)}, x) + d(x, r_{\varphi(j)}) \leq \frac{1}{i + 1} + \frac{1}{j + 1}.$$
For the other implication assume that \( \varphi \) is such that the right hand side of the equivalence is fulfilled. It follows that \( r_{\varphi(i)} \) is a Cauchy sequence. Since \( Z \) was assumed to be complete, this Cauchy sequence converges to some element \( x \) and for any \( i, j \in \mathbb{N} \)

\[
d(x, r_{\varphi(i)}) \leq d(x, r_{\varphi(j)}) + d(r_{\varphi(j)}, r_{\varphi(i)}) \leq d(x, r_{\varphi(j)}) + \frac{1}{j+1},
\]

Taking the limit \( j \to \infty \) in this inequality proves that \( d(x, r_{\varphi(i)}) \leq \frac{1}{i+1} \) for all \( i \in \mathbb{N} \), i.e. that \( \varphi \) is a name of \( x \) in the Cauchy representation.

Thus, a machine that correctly identifies string functions that are not names in the Cauchy representation can be constructed by searching for values \( i \) and \( j \) such that \( d(r_{\varphi(i)}, r_{\varphi(j)}) \geq \frac{1}{i+1} \). This search can be done by a machine since the discrete metric is computable.

\[\square\]

## 5 Construction of standard representations

This chapter specifies families of representations for which it is possible to prove that the estimate obtained from Theorem 3.8 is close to optimal. This requires two steps. The first step is to eliminate the inherent computational difficulty of the metric, as this can not be reflected in the metric structure of the space. Recall from Example 2.10 that the bound on the metric entropy obtained from the Cauchy representation by Theorem 3.8 improves with the running time of the discrete metric \( \tilde{d} \). Within the framework of representations it can be simulated that the discrete metric has the lowest possible running time by considering a representation that additionally provides the discrete metric as oracle.

**Definition 5.1** Let \( Z = (Z, d, (r_i)) \) be a complete separable metric space with dense sequence. Define the relativized Cauchy representation \( \xi^Z_Z \) as follows: \( \varphi \in \mathcal{B} \) is a \( \xi^Z_Z \)-name of \( x \in Z \) if and only if for all \( n \in \mathbb{N} \) the string \( \varphi(0n) \) encodes a non-negative integer \( i \) with \( d(x, r_i) \leq \frac{1}{n+1} \) and for all \( k, m, n \in \mathbb{N} \)

\[
\left| d(r_k, r_m) - \frac{\varphi(1(k, m, n))}{n+1} \right| \leq \frac{1}{n+1}.
\]

That is: Any name comes with an oracle for the discrete metric \( \tilde{d} \). Note that \( Z \) was not assumed to be a computable metric space. In particular the discrete metric may not be computable. If it is incomputable then \( \xi^Z_Z \) does not have any computable string functions in its domain. In any case, \( \xi^Z_Z \) renders the metric computable in time \( O(T) \) for \( T(l, n) := \max\{l(n+2), n\} \) and the bound obtained by Theorem 3.8 is \( \text{ent}(\xi^Z_Z(K_i)) \leq C(n+2)^2 + C \). This is reasonably close to the bound \( \text{ent}(\xi_Z(K_i)) \leq l(n) \) that came from the concrete structure of the Cauchy representation (cf. Example 3.7). This bound itself, however, may still be far off, as the one point space as example shows.

Theorem 4.14 remains valid if the Cauchy representation is replaced by the relativized Cauchy representation:

**Proposition 5.2 (Relativized completenesses)** A relativized Cauchy representation of a separable metric space is complete if and only if the space is complete. The relativized Cauchy representation of a computable complete metric space is computably complete.

**Proof** If the relativized Cauchy representation is complete, it follows that the Cauchy representation with respect to the same dense sequence is complete, thus completeness of the metric space follows from Theorem 4.14.

Now assume that the space is complete and let \( \varphi_n \) be a sequence of names that converges to some \( \varphi \in \mathcal{B} \). The proof of Theorem 4.14 can be copied to see that the values \( \varphi(0n) \) encode indices of approximations to an element. That
the values of $\varphi(1(k,m,n))$ are valid approximations of the discrete metric follows from the use of non-strict inequality.

Finally assume that the metric space is computably complete. A machine that terminates on the empty string if and only if the oracle is not a name can be described as follows: It dovetails the procedure described in the proof of Theorem 4.14 and a search for incompatibilities between the values of the discrete metric and values of the names when they are interpreted as an oracle for the discrete metric.

Note that while the relativized Cauchy representation is a good starting point, for many applications from complexity theory it is not appropriate. For instance the standard representation of Lipschitz one functions that map zero to itself does not polynomial-time translate to a Cauchy representation. The examples right before the conclusion illustrate that the constructions below include these as special cases. It is not clear how to tackle the problem in the most general case of a separable metric space, thus we present two results for restricted classes of spaces: First compact metric spaces and then separable Banach spaces.

5.1 Compact metric spaces

The basic idea for compact metric spaces is to use a refinement of the relativized Cauchy representation for a sequence that is well-behaved in the following sense:

**Definition 5.3** A sequence $(r_i)$ in a compact metric space $(Y,d)$ is called uniformly dense, if both of the following hold:

- (c): $(r_i)$ has the covering property: The closed $2^{-n}$-balls around the first $2^{\text{ent}(Y)(n)+[\log_2(n+1)]}$ elements cover $Y$.
- (s): $(r_i)$ has the spanning property: For any $k \leq \text{ent}(Y)(n-1)$ there are at least $2^{k-1}$ elements of pairwise distance strictly more than $2^{-n}$ within $r_0, \ldots, r_{2^k-1}$.

Examples for uniformly dense sequences are the standard enumeration of the dyadic numbers in the unit interval (cf. Example 6.7) or standard enumerations of the piece-wise continuous functions with dyadic breakpoints in the Lipschitz one functions that map zero to zero.

Under the assumption of compactness, uniformly dense sequences do always exist:

**Lemma 5.4 (Uniformly dense sequences)** In every infinite compact metric space there is a uniformly dense sequence.

**Proof** Let $(Y,d)$ be the compact metric spaces. Let $I_n$ be a maximal set of elements $y_{i,n}$ of pairwise distance strictly more than $2^{-n+1}$. From the maximality it can be seen that the closed $2^{-n}$-balls around the $y_{i,n}$ cover $Y$ and therefore $\#I_n \leq 2^{\text{ent}(Y)(n)}$. Let $(r_i)$ be the sequence that arises by writing the tuples after one another.

To verify that $(r_i)$ has the covering property it suffices to note that all elements of $I_n$ are listed within the first

$$\sum_{m=0}^{n} \#I_m \leq \sum_{m=0}^{n} 2^{\text{ent}(Y)(m)} \leq (n+1)2^{\text{ent}(Y)(n)} \leq 2^{\text{ent}(Y)(n)+[\log_2(n+1)]}$$

elements.

The spanning property follows by induction: For a fixed $m \leq n-2$ at most one $y_{i,m+1}$ can lie in the closed $2^{-n}$-ball around $y_{i,m}$. Thus, starting from the beginning and always only adding those elements that are far enough away from the ones previously chosen one always ends up with at least $2^{k-1}$ elements.
Theorem 5.5 (Representing compact spaces) Let \((Y,d)\) be an infinite compact metric space. Let \(\ell : \omega \rightarrow \omega\) be a function such that there exists a time-constructible and monotone function \(S : \omega^2 \times \omega \rightarrow \omega\) such that
\[
\ell(n)S(\ell, n) \geq \text{ent}(Y)(n) + \lfloor \log(n + 1) \rfloor.
\]
Then there exists an admissible, regular, complete representation \(\xi\) of length \(\ell\) such that the metric is computable in time \(O(T)\) for
\[
T(l,n) := l(n + 2) \cdot S(l,n + 2)
\]
and for all strictly monotone \(l : \omega \rightarrow \omega\) and \(n \in \omega\)
\[
l(n) \cdot S(l,n) \leq \text{ent}(Y)(n - 1) \Rightarrow l(n) \cdot S(l,n) \leq \text{ent}(\xi(K_l))(n + 1) + 1.
\]

Proof Let \((r_i)_{i \in N}\) be a uniformly dense sequence like the one constructed in Lemma 5.4. Define \(\varphi\) as follows: A string function \(\varphi\) is a name of an element \(x \in Y\) if and only if the following conditions hold:

1. \(\varphi\) provides its length: For all \(n \in N\) it holds that \(|\varphi(0^n)| = |\varphi(n)|\).
2. \(\varphi\) provides indices of approximations: For all \(n \in N\) the strings \(\varphi(0(j,n))\) where \(j\) reaches from 0 to \(S(|\varphi|, |n|)\) have length less than \(|\varphi(|n|)|\) and their concatenation is a non-negative integer \(i\) with \(d(x,r_i) \leq \frac{1}{n+1}\).
3. \(\varphi\) provides an oracle for the metric: For any \(i,j,n \in N\) it holds that \(\varphi(1(i,j,n)) \in \mathbb{Z}\) and
\[
d(r_i, r_j) - \frac{\varphi(1(i,j,n))}{n+1} \leq \frac{1}{n+1}.
\]

This defines a second-order representation: For any two distinct elements \(x,y \in Y\) there is an \(n\) such that \(2^{-n} < d(x,y)\). Thus, if \(r_i\) resp. \(r_m\) are \(2^{-n-1}\)-approximations of \(x\) resp. \(y\), then \(i \neq m\). Now assume that \(\varphi\) and \(\psi\) fulfill the conditions to be names of \(x\) resp. \(y\). Then the indices from (a) must differ. Thus, \(\xi\) is single-valued. The choice of the sequence \((r_i)_{i \in N}\) and (1) make sure that condition (a) can be fulfilled by a function of length \(\ell\), thereby leaving enough freedom in the choice of the function to make it also fulfill (m). Thus, \(\xi\) has length \(\ell\) and is in particular surjective.

It is left to provide an appropriate algorithm for computing the metric. When given a name \((\varphi, \psi)\) of some element \((x,y) \in (Y,\xi) \times (Y,\xi)\) as oracle and a precision requirement \(n\) this algorithm proceeds as follows: From the time-constructibility of \(S\) it follows that \(N := S(|\varphi, \psi|, |n| + 2)\) can be computed in time \(O(T)\). Next the machine queries the oracle \(N\) times for \(0(j, 4n + 3)\), with the values of \(j\) going from 0 to \(N\) to obtain indices \(i\) and \(k\) of \(\frac{1}{4n+3}\) approximations to \(x\) and \(y\). This takes time less than \(O(N \cdot |\varphi, \psi| (|n| + 2))\). Finally, the machine queries the oracle input \(1(i, k, 2n + 1)\) for an approximation of \(d(r_i, r_k)\). Using the triangle inequality one verifies that the result leads to a valid dyadic approximation to \(d(x,y)\) being written on the output tape.

The lower bound on the metric entropy follows from the additional property of the uniformly dense sequence from Lemma 5.4. The admissibility follows since it is possible to translate back and forth between the representation \(\xi\) and the relativized Cauchy representation in time \(O(S)\). For the completeness of the representation use that compactness of a metric space implies completeness and the time constructibility of \(S\) implies continuity. These can be used to repeat the argument from the proof of Theorem 4.14 that the limit of a convergent sequence of names is a name of the limit of the named elements.

The representation depends heavily on the choice of the uniformly dense sequence. Note that there is no computability condition on the compact metric space. In particular there is no guarantee that the representation has any computable names at all.
5.2 Separable Banach spaces

The goal of this chapter is to obtain a similar result as in the previous section for the case where the metric space is not compact but instead a Banach space. The construction is highly inspired from ideas from approximation theory: The representation is chosen in such a way that the images of the sets $K_i$ are full approximation sets as introduced in [Lor66] (cf. also Section 6.1).

**Definition 5.6** Let $X = (X, \|\cdot\|)$ be a Banach space. A sequence $(e_i)$ in $X$ is called a fundamental system if $\|e_i\| = 1$ for all $i$ and the linear span of the $e_i$ is dense in $X$. That is: For each $x \in X$ and $n \in \mathbb{N}$ there exists an $N \in \mathbb{N}$ and some real numbers $\lambda_1, \ldots, \lambda_N$ such that

$$\left\| \sum_{i=1}^{N} \lambda_i e_i - x \right\| \leq 2^n.$$

A fundamental system should be regarded as the Banach space equivalent of a dense sequence in a compact metric space. Any separable Banach space has a fundamental system as a dense sequence can be normalized.

The following notion can be understood as a stronger version of the equivalent of being a uniformly dense sequence in a compact metric space:

**Definition 5.7** Let $X$ be a Banach space. A fundamental system $(e_i)_{i \in \mathbb{N}}$ is called a Schauder basis, if for each $x \in X$ there exists exactly one sequence $(\lambda_i)_{i \in \mathbb{N}}$ of real numbers such that

$$x = \lim_{n \to \infty} \sum_{i=0}^{n} \lambda_i e_i.$$

However, in contrast to the situation for compact metric spaces there is no way to construct a Schauder basis in an arbitrary separable Banach space. Quite to the contrary of Lemma 5.4, that constructs a uniformly dense sequence in any compact metric space, a famous example by Enflo construct a separable Banach space that does not have any Schauder basis [Enf73].

Recall that the norm of a linear continuous functional $f$ on a Banach space $X$ is defined by

$$\|f\| := \sup_{\|x\| \leq 1} \{|f(x)|\}.$$

We need the following well-known result for Schauder bases that follows directly from the Uniform Boundedness Principle:

**Theorem 5.8** Let $(e_i)_{i \in \mathbb{N}}$ be a Schauder basis in a Banach space $X$. Then there are continuous linear functionals $(f_i)_{i \in \mathbb{N}}$ such that for all $x$ in $X$ it holds that

$$x = \lim_{n \to \infty} \sum_{i=0}^{n} f_i(x) e_i.$$

Furthermore, there is a constant $C$ such that

$$\forall i \in \mathbb{N} : \|f_i\| \leq C.$$

The implication of this theorem that we need is that there is a finite lower bound on the pairwise distance of the elements of a Schauder basis.

**Corollary 5.9** Let $(e_i)_{i \in \mathbb{N}}$ be a Schauder basis in a Banach space. Then there exists a constant $\alpha > 0$ such that for all non-negative integers $i \neq j$

$$\|e_i - e_j\| > \alpha.$$
Proof Let \( f_i \) be the family of continuous coordinate functionals from the previous theorem and \( C \) the constant bounding their norms. Set \( \alpha := \frac{1}{\sqrt[n]{l+1}} \).

To see that \( \alpha \) is as required note that since \( \langle e_i \rangle \) is a Schauder basis \( e_i - e_j \neq 0 \) for \( i \neq j \). Therefore the vector can be normalized and it holds that

\[
C + 1 > \|f_i\| \geq \left| f_i \left( \frac{e_i - e_j}{\|e_i - e_j\|} \right) \right| = \frac{1}{\|e_i - e_j\|}.
\]

The assertion follows by taking reciprocals on both sides.

Now we are prepared to prove the main theorem of this section.

**Theorem 5.10 (Representing Banach spaces)** Let \( X \) be an infinite dimensional separable Banach space and let \( S : \omega^\omega \to \omega \) be monotone and time-constructible such that for all \( l \in \omega^\omega \) there exists an \( l' \in \omega^\omega \) such that \( S(l',.) \geq l \). Then there exists an admissible, regular, complete representation \( \xi \) of \( X \) such that the addition is computable in time \( O((l(n+1)), \) the scalar multiplication in time \( O(l(n)) \) and the norm in time \( O(T) \) for

\[
T(l, n) := l(n + \lfloor lb(S(l, n + 1) + 1) \rfloor) : S(l, n + 1)
\]

If the space \( X \) allows for a Schauder basis, then \( \xi \) can be chosen such that additionally there is a constant \( C \) such that for all strictly increasing \( l : \omega \to \omega \) and \( n \in \omega \)

\[
S(l, n) \leq \text{ent}(\xi(K_i))(n + C).
\]

Proof Since the space \( X \) is separable, there exists a fundamental system \( (e_i)_{i \in \mathbb{N}} \). If the space \( X \) allows a Schauder basis, choose \( (e_i)_{i \in \mathbb{N}} \) as a Schauder basis. Define \( \xi \) as follows: A string function \( \varphi \) is a name of \( x \in X \) if and only if all of the following conditions hold:

1. \( \varphi \) provides its length: For all \( n \in \mathbb{N} \) it holds that \( |\varphi(0^n)| = |\varphi(n)| \).
2. \( \varphi \) encodes linear combinations that approximate \( x \): For all \( n \in \mathbb{N} \) there exists a linear combination of the first \( S(|\varphi|,|n|) \) vectors \( e_i \) that approximates \( x \) with precision \( \frac{1}{n^l} \) and whenever \( m \in \mathbb{N} \) is bigger than \( S(|\varphi|,|n| + 1) + 1)(n + 1) \) it holds that

\[
\left\| \sum_{i=0}^{S(|\varphi|,|n|)} \varphi(O(i,n,m)) \frac{m}{m+1} e_i - x \right\| \leq \frac{2}{n + 1}.
\]

3. \( \varphi \) provides an oracle for the norm: I.e. for all \( n, m, N \in \mathbb{N} \) and integers \( z_0, \ldots, z_N \in \mathbb{Z} \)

\[
\left\| \sum_{i=0}^{N} \frac{z_i}{m+1} e_i \right\| \leq \frac{\varphi(1(\langle z_0, \ldots, z_N \rangle), N,n,m)}{n + 1} \leq \frac{1}{n + 1}.
\]

This defines a representation due to the assumptions: To see that each element has a name note that due to the assumptions about the pairing function and the the requirements (l), (a) and (o) do not interfere with one another: The growth condition on \( S \) and \( e_i \) being a fundamental system guarantee that it is always possible to find a string function long enough to fulfill (a). This function can always be modified to a longer one fulfilling (o). Since \( S \) is monotone, the first condition will still be fulfilled by this function. Afterwards its value on \( 0^n \) can be changed to have its length as length. To see that a name uniquely determines an element note that for a given precision requirement \( 2^n - 1 \) it suffices to choose \( m \) in (a) bigger than \( (S(|\varphi|,|n| + 1) + 1)(n + 1) \) to guarantee that there exists rational coefficients. These coefficients determine \( x \) up to precision \( 2^{-n} \) and this works for any \( n \). Therefore \( x \) is uniquely determined from any of its names and \( \xi \) is a representation.

The straight-forward algorithms for the vector space operations are easily seen to run in the specified times. The norm can be computed as follows: Given a name \( \varphi \) of some \( x \in X \) as oracle and a precision requirement \( n \),
first compute $S(|\varphi|, |n| + 1)$. Since $S$ is time-constructible this is possible in time $O(T)$. Next, query the given name $\varphi$ of some $x \in X$ to obtain integers $z_0, \ldots, z_{S(|\varphi|, |n| + 1)}$ such that

$$\left\| \sum_{i=0}^{S(|\varphi|, |n| + 1)} z_i \frac{s_i}{(S(|\varphi|, |n| + 1) + 1)(n + 1)} e_i \right\| \leq \frac{1}{n + 1}$$

to write each query time linear in $|n| + \lceil \log(S(|\varphi|, |n| + 1)) \rceil$ is needed. Since the length of each of the $S(|\varphi|, |n| + 1)$ answers is bounded by $|\varphi| (|n| + \lceil \log(S(|\varphi|, |n| + 1) + 1) \rceil)$, the final query to the oracle for the metric can be asked in time $O(T)$. This query obtains an approximation to the norm of this linear combination with precision $\frac{1}{n + 1}$. Writing the query can also be done in time $O(T)$ and results in a valid return value for the norm.

Let $(q_i)$ be a standard enumeration of the dyadic numbers. It is possible to translate back and forth between this representation and the relativized Cauchy representation with respect to the dense sequence $r_{(N, (i_1, \ldots, i_N))} := \sum_{i=1}^{N} q_{i_j} e_{i_j}$. The time needed to do this depends on $S$ but is bounded since $S$ is time-constructible. Therefore $\xi$ is admissible and complete.

To prove the lower bound on the size fix some $n$ and note that for all $i \in \mathbb{N}$ with $|i| \leq S(l, 1^n)$ the elements $2^{-n} e_i$ have names of length $l$. Here the strict monotonicity guarantees that (o) can be fulfilled. By Corollary 5.9 there exists a constant $\alpha > 0$ such that these elements are of pairwise distance more than $2^{-n} \alpha$. Set $C := \lceil \log(\alpha) \rceil$, then the elements are of pairwise distance more than $2^{-n} - C$. Thus,

$$n \mapsto \begin{cases} 0 & \text{if } n < C \\ S(l, n - C) & \text{otherwise} \end{cases}$$

is a spanning bound of $\xi(K_l)$ and by Proposition 3.3 the assertion follows.

### 5.3 Computability issues

The last sections completely neglected considering computability issues of the metric by always providing its values via an oracle in each name. However, in practice computable metric spaces are very important. Under the assumption that the sequence of a compact computable metric space is uniformly dense it is straightforward to see that representation defined in the proof of Theorem 5.5 is computably complete.

**Corollary 5.11** Additionally to the assumptions of Theorem 5.5 assume that $(r_i)_{i \in \mathbb{N}}$ is a uniformly dense sequence such that $Z := (Z, d, (r_i))$ is a computable metric space. Then there exists a representation $\xi$ as in the theorem that is additionally computably complete.

This result is unsatisfactory. It would be desirable to be able to construct the uniformly dense subsequence in a computable way from a dense sequence such that the corresponding discrete metric is computable. That is: It would be nice to have an effective version of version of Lemma 5.4. However, the authors failed to produce such a version of the result. As already mentioned at the beginning of the previous section the existence of a Schauder basis does not follow from the separability of a Banach space due to an example by Enflo. A version of this example provided by Bosserhoff proves that computability together with the existence of a Schauder basis does not guarantee the existence of a computable Schauder basis:

**Theorem 5.12** ([Bos09]) There exists a computable Banach space that has a Schauder basis but does not have a computable Schauder basis.

If a computable Schauder basis exists, however, the representation constructed as in Theorem 5.10 is computable complete.
6 Relations to other work and examples

Let \( Z = (Z, d, (r_i)) \) be a complete computable metric space. Recall from Example 3.7 that it was possible to write down the sets \( \xi_Z(K_l) \) of elements that have a short name in the Cauchy representation explicitly:

\[
\xi_Z(K_l) = \bigcap_{n \in \mathbb{N}} \bigcup_{i=0}^{2^{l(n)-1}} B_{2^{-n}}(r_i).
\]

From this, for non-decreasing \( l \), the bound \( \text{ent}(\xi_Z(K_l)) \leq l \) followed. If \( Z \) is compact and the sequence \( (r_i) \) is assumed to be uniformly dense in the sense of Definition 5.3 (actually it suffices that it has the spanning property (s)), then it is possible to also specify a lower bound:

\[
\text{ent}(\xi_Z(K_l))(n) \geq l(n) - 1 \quad \text{whenever} \quad l(n) \leq \text{ent}(Y)(n - 1).
\]

If we step up to the relativized Cauchy representation from Definition 5.1, the same relations remain valid as long as the metric space is bounded and \( l \) is strictly increasing with \( l(0) \geq \lceil \text{lb}(\text{diam}(Z)) \rceil \). If these additional assumptions are not made, then depending on the choice of the sequence \( (r_i) \) there may only exist long oracles for the discrete metric.

It is possible to give a similar description for the sets \( \xi(K_l) \), where \( \xi \) is the representation from the proof of Theorem 5.5:

**Lemma 6.1** Let \( (Y, d) \) be an infinite compact metric space and \( \ell \) and \( S \) fulfill the assumptions of Theorem 5.5 and let \( \xi \) be the representation from the proof. Whenever \( \ell \) is strictly increasing such that \( \ell(0) \geq \lceil \text{lb}(\text{diam}(Y)) \rceil \), then

\[
\xi(K_l) = \bigcap_{n \in \mathbb{N}} \bigcup_{i=0}^{2^{\ell(n)(l,n)-1}} B_{2^{l(n)}(l,n)}(r_i).
\]

An upper bound on the metric entropy of \( \xi(K_l) \) can be extracted from this like in Example 3.7:

\[
\text{ent}(\xi(K_l))(n) \leq \mu(n) + \lceil \text{lb}(n + 1) \rceil.
\]

On the other hand, the spanning property (s) of a uniformly dense sequence (cf. Definition 5.3) is equivalent being able to specify a lower bound on the set \( \xi(K_l) \) whenever \( l(n) \cdot S(l, n) \leq \text{ent}(Y)(n - 1) \). Here, the additional assumption about \( l \) is reasonable as it is necessary to guarantee that the candidate for the bound is not bigger than the compact space \( Y \) itself. Without the requirement that the sequence is uniformly dense it is impossible to specify a lower bound on the metric entropy and in non-compact spaces there does not seem to be any sensible notion of uniform density.

We saw that if a metric space is represented as in Theorem 5.5, then the sets \( \xi(K_l) \) have an explicit description and the bounds on the size of these sets break down when compactness fails. Thus, for non-compact spaces the representation \( \xi \) has to be changed in a way that \( \xi(K_l) \) is a different set. In the case of a Banach spaces, families of compact sets that are candidates for the sets \( \xi(K_l) \) have been investigated in approximation theory for a long time [Lor66].

### 6.1 Full approximation sets

This section compares the results of this paper to the results from approximation theory that inspired them in the first place [Lor66]. For the convenience of the reader all relevant definitions and results from the above source are repeated here and then compared to our definitions.

**Definition 6.2** ([Lor66]) For a subset \( A \) of a metric space \( Z \) and \( \varepsilon > 0 \) a real number, denote the minimal number of sets of diameter \( 2\varepsilon \) needed to cover \( A \) by \( N_\varepsilon(A) \). And define the entropy of \( A \) by

\[
H_\varepsilon(A) := \log(N_\varepsilon(A)).
\]
Our notion of metric entropy $\text{ent}(A)$ from Definition 3.1 used closed balls instead of sets of diameter $2\varepsilon$. Since balls of radius $2^{-n}$ are sets of diameter $2^{-n}$ it holds that $[\lambda_b(N_{2^{-n}}(A))] \leq \text{ent}(A)(n)$. Any set of diameter $2 \cdot 2^{-n-1}$ is contained in the closed $2^{-n}$-ball around any of its elements. Therfore, it also holds that $\lambda_b(\Delta \cdot H_{2^{-n}}(A)) \leq \text{ent}(A)(n)$. Note that the log in the definition of $H_{\nu}(A)$ denotes the natural logarithm, while our definition uses a power of two. It follows that

$$[\lambda_b(\epsilon)H_{2^{-n}}(A)] \leq \text{ent}(A)(n) \leq [\lambda_b(\epsilon)H_{2^{-n-1}}(A)].$$

**Definition 6.3 ([Lor66])** Let $X$ be a Banach space, let $\Phi = (\varepsilon_i)$ be a fundamental sequence, that is a sequence whose linear span is dense in $X$. Let $\Delta := (\delta_i)$ be a non-increasing zero sequence of positive real numbers. Define the full approximation set $A(\Delta, \Phi)$ by

$$A(\Delta, \Phi) := \{ f \in X \mid \forall n \in \mathbb{N} \exists \lambda_1 \ldots \lambda_n \in \mathbb{R} : \| \sum_{i=1}^{n} \lambda_i e_i - f \| \leq \delta_n \}.$$  

Note that a fundamental sequence is different from a fundamental system as introduced in Definition 5.6 as it is not required that $\|e_i\| = 1$. Lorentz provides upper and lower bounds for the entropy of the approximation sets:

**Theorem 6.4 (Theorem 2 from [Lor66])** Let $X$ be a Banach space and let $A(\Delta, \Phi)$ be a full approximation set. Define a sequence $N_i$ by $N_0 := 0$ and $N_i := \min\{k \mid \delta_k \leq 2^{-i} \}$ for $i \neq 0$. Set $\Delta N_i := N_{i+1} - N_i$. Let $\varepsilon$ be from the interval $(0, 4)$ and set $j := \lceil 2 - \lambda_b(\varepsilon) \rceil$. Then

$$\log(2) \sum_{i=1}^{j-3} N_i \leq H_{\nu}(A(\Delta, \Phi))$$

and

$$H_{\nu}(A(\Delta, \Phi)) \leq \log(2) \sum_{i=1}^{j} N_i + \sum_{i=0}^{j-1} N_i \log \frac{N_{j-1}}{\Delta N_i} + N_1 \log \delta_0 + N_j \log 9.$$  

Thus, these full approximation sets are compact and it is possible to specify both upper and lower bounds of their size. The representation $\xi$ from Theorem 5.10 is deliberately chosen such that the sets $\xi(K_i)$ of elements that have names of length $l$ are full approximation sets:

**Lemma 6.5** Let $X$ be a Banach space, $\Phi = (e_i)$ a Schauder basis, $C$ the constant from Theorem 5.8 and set $c := [\lambda_b(C)]$. Let $S$ be as in Theorem 5.10, let $\xi$ be the representation from the proof of Theorem 5.10 and for a strictly increasing $l \in \omega^\omega$ such that $l(0) \geq c$ denote by $\Delta \lambda := (\delta_{l,i})$, the sequence

$$\delta_{l,i} := \begin{cases} 2^{l(0) - c} & \text{if } i < S(l, 0) \\ 2^{-n} & \text{whenever } S(l, n) \leq i < S(l, n + 1). \end{cases}$$  

Then

$$\xi(K_i) = A(\Delta \lambda, \Phi).$$

**Proof** First note that due to the monotonicity and the bound condition on $S$, the function $S(l, \cdot)$ is non-decreasing and unbounded. Therefore $\delta_{l,i}$ is well-defined, and a non-increasing zero sequence as required.

The first inclusion $\xi(K_i) \subseteq A(\Delta \lambda, \Phi)$ follows straight-forwardly from comparing the definition of the representation $\xi$ from the proof of Theorem 5.10 with the definition of full approximation sets and the sequence $\Delta \lambda$.

To prove the second inclusion, namely $A(\Delta \lambda, \Phi) \subseteq \xi(K_i)$ assume that $x \in A(\Delta \lambda, \Phi)$. By the definition of the full approximation sets and the sequence elements $\delta_{l,i}$ for $i \geq S(l, 0)$, there exists appropriate real numbers such that the first condition of (a) from the proof of Theorem 5.10 can be fulfilled.
by a function of length \( l \). Recall that we used \( f_i \) to denote the coordinate functionals corresponding to the Schauder basis \( e_i \) and that \( c \) was chosen such that \( \| f_i \| \leq 2^c \). Therefore,

\[
|f_i(x)| \leq \| f_i \| \| x \| \leq 2^{l(0)}
\]

and since \( l \) is strictly increasing there is enough space to encode approximations of \( f_i(x) \) in a function of length \( l \). That is: the second condition of (a) in the proof of Theorem 5.10 can additionally be fulfilled. The conditions from (l) and (o) from the proof of Theorem 5.10 can easily be fulfilled by modifying this name without changing its length. That the oracle for the norm does not need to increase the length is guaranteed by the normalization of the \( (e_i) \).

Let us apply Lorentz’s Theorem 6.4 to the sets \( \xi(K_l) = A(\Delta_l, \Phi) \) and translate the result back to our definitions. The notations introduced in Theorem 6.4 simplify considerably in this context. Indeed, if \( \Delta = \Delta_l \) and \( \varepsilon = 2^{-n} \), then

\[
N_i = S(l, i) \quad \text{and} \quad j = n + 2.
\]

Also note that \( \log(2) = 1 \) and recall that

\[
\lceil \log(2) \rceil \leq \log(7) \cdot S(l, n + 3).
\]

Theorem 6.6 Let \( X \) be a Banach space with Schauderbasis \( (e_i) \), and \( S, l \) and \( c \) like in Lemma 6.5. Then

\[
\text{ent}(\xi(K_l)(n)) \leq \sum_{i=1}^{n+3} S(l, i) + \sum_{i=0}^{n+2} S(l, i) \log \left( \frac{S(l, n + 3)}{S(l, i + 1) - S(l, i)} + 2 \right) + \log(2) \cdot S(l, 1) \cdot (l(0) - c) + \log(7) \cdot S(l, n + 3) + \log(2) \cdot S(l, 1) \cdot (l(0) - c)
\]

and

\[
\sum_{i=1}^{n-1} S(l, i) \leq \text{ent}(\xi(K_l))(n)
\]

The lower bound improves the lower bound given in Theorem 5.10 considerably: Most prominently, the dependency on \( c \) is completely removed (only the condition \( l(0) \geq c \) remains). But also the bound from Theorem 5.10 only mentions one of the summands from the sum in the lower bound form the theorem above (i.e. \( S(l, n - c) \)).

The upper bounds are more difficult to compare. This is partly because we never stated the upper bound explicitly but it has to be obtained from combining Theorems 3.8 and 5.10. Also, these results do not state what the constants are. However, a superficial comparison indicates that the bound from Theorem 6.6 is superior as it does not contain squares of \( S(l, n + 4) \), does not apply of the length function to terms containing \( S \) and states the constants explicitly.

6.2 Compact spaces

Let us start with a couple of examples of well-known representations that can be recovered using the constructions presented in Section 5. First for the construction for compact metric spaces from Theorem 5.5:

Example 6.7 (The unit interval) Let \([0, 1]\) denote the computable metric space \(([0, 1], |\cdot|, (q_i))\), where the sequence \((q_i)\) is given by

\[
q_0 := 0, \quad q_1 := 1 \quad \text{and} \quad q_i := \frac{2(i - 2^{1/(i-1)}) - 3}{2^{1/(i-1)}} \quad \text{for} \ i \geq 2.
\]

I.e.

\[
(q_i) = \left(0, 1, \frac{1}{2}, \frac{1}{4}, \frac{3}{8}, \frac{1}{8}, \frac{3}{16}, \frac{7}{16}, \frac{1}{16}, \frac{3}{32}, \ldots\right).
\]
It is easy to check that \( \text{ent}([0,1]) = \max\{n-2,0\} \) and that the sequence \((q_i)\) is uniformly dense in the sense of Definition 5.3. Actually, \((q_i)\) fulfills even stronger conditions than (c) and (s) from this definition: (c) can be replaced by the closed \(2^{-n}\)-balls around the first first \(2^{n+1}\) elements cover \([0,1]\) and (s) can be improved to “For all \(n\) the first \(2^{n+1} - 1\) elements are of pairwise distance more than \(2^{-n}\).” Let \(\xi_{[0,1]}\) be the representation constructed in the proof of Theorem 5.5 with the choices

\[
\ell(n) := n \quad \text{and} \quad S(l,n) := 1.
\]

These do not fulfill the assumptions of the theorem but due to the improved uniform density of the sequence \((q_i)\) the construction still works and returns a representation which is equivalent to the range restriction of the representation from Definition 2.3.

Note that the constructed representation \(\xi_{[0,1]}\) differs from the range restriction in that it always specifies a \(2^{-|n|}\)-approximation. After Definition 2.2 we claimed that this does not make a difference. This is because the identity on the Baire space is a realizer of the function \(\omega \rightarrow \mathbb{N}, n \mapsto 2^n - 1\). And for the opposite direction the restriction of the length function is time-constructible with respect to the constant functions and thus a linear-time translation in the other direction.

We could also have used Theorem 5.5 directly by choosing the functions as \(\ell(n) := \text{ent}([0,1]) + [\text{lb}(n+1)]\) and \(S(l,n) := 1\). This leads to a polynomial-time equivalent representation.

Example 6.8 (Separable metric spaces) If \(Z = (Z,d,(r_i))\) is a separable metric space with a uniformly dense sequence, using \((r_i)\) as uniformly dense sequence and setting \(S(l,n) := 1\) and \(\ell(n) := \text{ent}(Z) + [\text{lb}(n+1)]\) produces the relativized Cauchy representation \(\xi_Z\) from Definition 5.1. This still works if the sequence is only dense, but in this case the lower bound for the size of the sets \(\xi_Z(K_i)\) may fail.

As a last example of a compact space consider a space of Lipschitz functions [Wei03]. Since the procedure to produce this is similar to what we do in the next section we skip the details:

Example 6.9 The standard enumeration of piecewise linear Lipschitz one functions with dyadic breakpoints is uniformly dense in the set of Lipschitz one functions that map 0 to itself. Using the construction from Theorem 5.5 for this sequence and setting \(S(l,n) := 2^{\ell(n)}\) and \(\ell(n) := n\) or \(S(l,n) := 2^{\ell(n)} + [\text{lb}(n+1)]\) and \(\ell(n) := 1\) produces the range restriction of the standard representation of the continuous functions as introduced in [KC10] (see also Definition 6.11). I.e. the two produced representations are polynomial time equivalent.

6.3 Continuous functions on the unit interval

This chapter demonstrates how to reconstruct the standard representation of the continuous functions on the unit interval (as introduced in [KC10]) using the construction from the proof of Theorem 5.10. For the convenience of the reader the relevant definitions from the above source are repeated. In the following \(C([0,1])\) is always considered a Banach space where the norm is the supremum norm and denoted by \(\|\cdot\|_{\infty}\).

A central notion for the standard representation is the modulus of continuity.

Definition 6.10 ([KC10]) A non-decreasing function \(\mu : \omega \rightarrow \omega\) is called a modulus of continuity of \(f \in C([0,1])\) if for all \(x,y \in [0,1]\) and \(n \in \omega\) it holds that

\[
|x - y| \leq 2^{-\mu(n)} \quad \Rightarrow \quad |f(x) - f(y)| \leq 2^{-n}
\]
To be exact, this notion of a modulus should be called modulus of uniform
continuity. Since functions on compact sets are uniformly continuous, any
function from $C([0, 1])$ has a modulus of continuity.

Definition 6.11 ([KC10]) Make $C([0, 1])$ a represented space by equipping
it with the standard second-order representation $\delta_\Box$ defined by: A length-
monotone string function $\varphi$ is a $\delta_\Box$-name of $f$ if and only if $\varphi = \langle \mu, \psi \rangle$
for some $\mu, \psi \in \Sigma^{**}$ such that both of the following are fulfilled:

(m) $n \mapsto |\mu(1^n)|$ a modulus of continuity of $f$.

(d) $\varphi$ provides approximations of the values of $f$ on dyadic arguments. I.e. for all $n, m \in \omega$ and $r \in \mathbb{N}$ whose absolute value does not exceed $2^{-m}$ it holds that $\varphi(0^n, r, 10^m) = (q, 10^k)$ for some $q \in \mathbb{N}$ and $k \in \omega$ and

$$|f(r \cdot 2^{-m}) - q \cdot 2^{-k}| \leq 2^{-n}.$$ 

Note that this definition differs subtly from the definitions used earlier in this
paper: We usually chose to use integers in binary instead of unary, but have
already remarked that this does not make a difference. Additionally we have
always chosen what is called $k$ in the above definition to be equal to what is
called $n$ in the above definition. This is impossible if one wants to work within
the realm of second-order representations: it must be possible to find length-
monotone names, so it has to be possible to choose the return values arbitrary
big. This does not make a difference up to polynomial-time translatability as
we can always just round the dyadic number to have denominator $2^{-n}$. In the
other direction the fraction can be extended for the return value to be long
enough as soon as we have access to an upper bound of the length function
(which is the case if the representation is regular). Finally, considering names
as pairs of a modulus of continuity and a string function encoding discrete
information is up to polynomial-time equivalence the same as just using the
function encoding the discrete information and requiring its length to be a
modulus of continuity.

We need the following important result about the representation $\delta_\Box$:

Theorem 6.12 (Lemma 4.9 from [KC10]) For a second-order represen-
tation $\delta$ of $C([0, 1])$ the following are equivalent:

- The evaluation operator $eval : C([0, 1]) \times [0, 1] \rightarrow \mathbb{R}, (f, x) \mapsto f(x)$ is
  polynomial time computable.
- $\delta$ can be translated to $\delta_\Box$ in polynomial-time.

Let us introduce the Schauder basis to use in the construction from The-
orem 5.10.

Definition 6.13 ([Fab10]) Let $(q_i)$ be the uniformly dense sequence in the
unit interval from Example 6.7 and choose the convention $\lceil \log (0) \rceil = 0$. The
Faber-Schauder system is the sequence $(e_i)$ of functions $e_i \in C([0, 1])$ defined
by

$$e_i(x) := \max \{1 - 2^{\lceil \log (i) \rceil} |x - q_i|, 0\}$$

It is well-known that the Faber-Schauder system is a Schauder basis. Since it
is instructive for the following, we repeat the proof of this fact:

Lemma 6.14 The Faber-Schauder system is a Schauder basis of the Banach
space $C([0, 1])$.

Proof First assume that $f$ is a function such that there are real numbers $\lambda_i$
that fulfill

$$f = \lim_{n \to \infty} \sum_{i=1}^{n} \lambda_i \cdot e_i.$$
Note that the sequences \((e_i)\) and \((q_j)\) fulfill that \(e_i(q_j) \neq 0\) implies \(i \leq j\). This implies that
\[
f(q_j) = \sum_{i=0}^{j} \lambda_i e_i(q_j)
\] (evl)
This allows us to recursively find the \(\lambda_i\):
\[
\lambda_0 = f(0) \quad \text{and} \quad \lambda_{j+1} = f(q_{j+1}) - \sum_{i=0}^{j} \lambda_j e_i(\lambda_j).
\] (rec)
It follows that the sequence is unique. On the other hand for an arbitrary function \(f\), the sequence defined above can easily be seen to always be such that
\[
f = \lim_{n \to \infty} \sum_{i=0}^{n} \lambda_i e_i
\]
Therefore, \((e_i)\) is a Schauder basis.

Note that \(e_i(q_j) \neq 0\) also implies \(|q_i - q_j| < 2^{-[\log(j)]}\) and therefore that in the sum that defining \(\lambda_{j+1}\) in the previous proof at most \(2 \cdot [\log(j)]\) summands are not zero. Furthermore, it can quickly be checked which are the ones that are not zero by evaluating \(|q_i - q_j|\). The last piece we need to see that the representation of \(C([0,1])\) is polynomial-time equivalent to the representation \(\xi\) that is produced by the proof of Theorem 5.10 is that \(\lambda_i \cdot e_i\) has the function \(n \mapsto n + [\log(i)] + [\log([\lambda_i])]\) as modulus of continuity. This can easily be checked.

**Theorem 6.15 (Reconstruction of the standard representation)** The standard second-order representation of the continuous functions on the unit interval is polynomial time equivalent to the representation \(\xi\) that is produced by the proof of Theorem 5.10 when choosing \((e_i)\) as the Faber-Schauder system and \(S(l, n) := 2^{|n|}\).

**Proof** To see that the representation \(\xi\) can be translated to the representation \(\delta_2\) first note that \(\xi\) as a regular representation is polynomial-time equivalent to a second-order representation by Proposition 4.9. Therefore using the minimality of \(\delta_2\) from Theorem 6.12 it suffices to specify an polynomial-time algorithm for evaluation.

This can be done using Equation (evl). Given a pair \((\varphi, \psi)\) of a \(\xi\)-name \(\varphi\) of a function \(f\) and an \(\mathbb{R}\)-name \(\psi\) of a number \(x\) as oracle and an integer \(n\) as input a machine computing the evaluation may proceed as follows. It obtain values of \(|\varphi|\) from \(\varphi\) using condition (l) of the definition of the representation \(\xi\): \(\varphi(0^n) > |\varphi|(n)\). Due to the choice \(S(l, n) = 2^{|n|}\), from condition (a) of the definition of \(\xi\) it follows that a linear combination of the first \(2^{[\log(n)]}\) elements of \((e_i)\) approximates the function \(f\) with precision \(\frac{1}{2^n}\) in supremum norm. Note that a modulus of continuity of the sum of two functions can be computed as the maximum of the moduli of continuity of the summands plus one. The sum approximating \(f\) has at most \(2[\log(2^{|n|})] = 2|\varphi|(n)\) summands. Furthermore, the machine can obtain rational numbers \(\lambda_i\) that can be used as coefficients and \([\log([\lambda_i])]\) is bounded in absolute value by \(|\varphi|([\varphi](n) + 1) + [\log(n+1)]\) (compare to the part of the proof of Theorem 5.10 that provides that a name determines an element). Therefore, a modulus of continuity of the sum can be specified as
\[
\mu_\varphi(m) := m + 3|\varphi|(n) + |\varphi|([\varphi](n) + 1) + [\log(n+1)]).
\]
Use the name \(\psi\) of the input \(x\) to obtain a rational approximation to \(x\) with precision requirement \(2^{-\mu_\varphi(m+1]}\) and round this to a dyadic approximation of at least half the quality. Note that the sequence \((q_j)\) lists all dyadic numbers, therefore the approximation is of the form \(q_i\) for some \(i \in \mathbb{N}\). The index \(i\) can easily be found from an encoding of \(q_i\) (this is called having a polynomial
time computable rounding function and proven for this sequence in [KSZ16]).

The machine now evaluates the sum
\[ 2^{\|\varphi\|(|n|)} \sum_{i=0}^{\|\varphi\|(|n|)} \lambda_i e_i(q_j) \]

By first listing the at most \( \left\lfloor \|\varphi\|(|n|) \right\rfloor \) values of \( i \) such that the corresponding summand is not zero, then getting the \( \lambda_i \) from \( \varphi \) and evaluating \( e_i(q_j) \). All of this can be done in polynomial time.

To see that a \( \xi \)-name can be written down using the information encoded in a \( \delta\xi \)-name first note that using a modulus of continuity as \( \|\varphi\| \), fulfills the first requirement of condition (a) from the proof of Theorem 5.10 and that (l) and (o) are unproblematic. To fulfill the second requirement of (a) valid coefficients \( \lambda_i \) have to be specified. Those can be found from a \( \delta\xi \)-name by using the recursion for the \( \lambda_i \) from Equation (rec) and the values on dyadic numbers that are encoded in a \( \delta\xi \)-name. Again, it is essential that in all the sums that show up at most a logarithmic number of summands are not equal to zero.

\[ \blacksquare \]

6.4 Spaces of integrable functions

**Example 6.16** The representations of the spaces \( L^p([0, 1]) \) discussed in [Ste16] can be reproduced by choosing the Haar system as Schauder basis and setting \( S(l, n) := 2^l(n) \).

7 Conclusion

In Theorem 3.8 there is still some space for improvements. It might be possible to remove the square in the bound of the metric entropy by further reducing the size bound from Lemma 3.16 by choosing the communication function \( L \) cleverer: Instead of saving initial segments of length of the value of the run-time, one could explicitly follow the computation and only save those entries that are touched by the reading head. However, we consider the payoff to be small in comparison to the additional technical complications encountered in the proof.

The question whether it is possible to computably construct a uniformly dense sequence from some additional information on the compact metric space, like a bound on the metric entropy or a spanning bound was left open. This is unsatisfactory, it would be desirable to either have a computable version of Lemma 5.4 or a counterexample as can be constructed for Banach spaces. The authors attempted only the former, but it seems like the covering property of a uniformly dense sequence is out of grasp of any computable procedure as it is notoriously difficult to verify it by a machine.

In the solution theory of partial differential equations the finite element methods provide a powerful tool. The formulation of these results within computable analysis is an active field of research [BY06]. Part of what makes finite element methods so valuable is that they can not only be applied to prove existence of solutions but also work well for implementations and provide fast algorithms for approximating the solutions. We consider a formulation of these methods within real complexity theory our long time goal and hope that the content of this paper is a first step towards achieving this.

Completeness may or may not lead to the right notion of bounded time admissibility. Even if it does, this does not end the quest for a good notion of polynomial-time admissibility.

Results of the kind presented in this paper often allow refinements that replace polynomial time bounds by logarithmic space bounds [KSZ16]. However, space-bounded computation in presence of oracles is a tricky field. Classes of sub-linear space restrictions have been defined for real complexity theory.
The results of this paper require classes with arbitrary space bounds and a considerably more elaborate machinery than introduced in this paper.

We believe that relaxing the condition of being a second-order representation to regularity is an important step. We furthermore believe that arbitrary representations should be studied. One reason for doing so is that one of the most popular attempts to implement the ideas of real complexity theory, namely $\text{iRRAM}$ [Mü], features polynomial-time evaluation of functions, while it seems impossible to efficiently find a modulus of uniform continuity. The minimality result for the standard representation of continuous functions on the unit interval proves that such a behavior cannot be modeled with second-order representations. A representation such that it takes exponential time to compute a modulus of uniform continuity, but evaluation is possible in polynomial time, however, is not difficult to write down. Note, that this can not simply be achieved by dropping the condition (l) in the proof of Theorem 5.10 as this leads to the loss of polynomial-time evaluation.

As mentioned before many of the results from this paper were produced in an attempt to unify two approaches to real complexity theory. It should be mentioned that there exist still different frameworks, for instance the model of analog computation has recently made huge advancements [BGP16]. It might be a good idea to look for interconnections.

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