On matrix modified KP hierarchy

A. Zabrodin∗

February 2018

Abstract

Using the bilinear formalism, we consider multicomponent and matrix modified KP hierarchies. The main tool is the bilinear identity for the tau-function which is realized as an expectation value of a Clifford group element composed from multicomponent fermionic operators. We also construct the Baker-Akhiezer functions and derive auxiliary linear equations satisfied by them.

1 Introduction

Integrable hierarchies of non-linear partial differential and difference equations are in the basis of the theory of integrable systems. Non-linear equations of integrable hierarchies can be represented as compatibility conditions for certain auxiliary linear problems. Special solutions to these linear problems depending on a complex spectral parameter $z$ are called Baker-Akhiezer functions $\Psi = \Psi(z)$. The multicomponent integrable hierarchies (see [1, 2, 3, 4]) are usually formulated in a matrix form with matrix pseudodifferential operators and matrix-valued Baker-Akhiezer functions.

Among different known examples of integrable hierarchies an archetypal one is the Kadomtsev-Petviashvili (KP) hierarchy. The modified KP (mKP) hierarchy is a larger hierarchy than the KP one. The set of independent variables in the $N$-component mKP hierarchy consists in $N$ infinite sets of continuous time variables $t_{\alpha,m}$ ($\alpha = 1, \ldots, N$, $m = 1, 2, \ldots$) and a finite set of $N$ auxiliary discrete variables $p_1, \ldots, p_N$ ($p_\alpha \in \mathbb{Z}$). The restriction to the (multicomponent) KP hierarchy is achieved by fixing the $p$-variables to zero values. What is usually called matrix mKP hierarchy is a restriction of the multicomponent mKP hierarchy to the following values of the times: $t_{\alpha,m} = t_m$ for each $\alpha$ and $m$, $p_\alpha = p$ for each $\alpha$.

In this paper we are going to discuss some aspects of the theory of the multicomponent and matrix mKP hierarchies which seem to be missing in the literature. We introduce

∗National Research University Higher School of Economics, 20 Myasnitskaya Ulitsa, Moscow 101000, Russian Federation; ITEP, 25 B.Cheremushkinskaya, Moscow 117218, Russian Federation; Skolkovo Institute of Science and Technology, 143026 Moscow, Russian Federation; e-mail: zabrodin@itep.ru
the wave operator and derive the auxiliary linear problems in the form $\partial_{t_{\alpha,m}} \Psi = A_{\alpha m} \Psi$, where $A_{\alpha m}$ are matrix difference operators in $p$ of order $m$ (linear combinations of the shift operators $e^{k\partial_p}$ with $k = 0, \ldots, m$ with coefficients depending on the times). For the matrix mKP hierarchy, the linear problems become $\partial_{t_m} \Psi = A_m \Psi$. In particular, we obtain from the first principles the linear problem $\partial_{t_1} \Psi = e^{\partial_p} \Psi + u \Psi$, where $u$ is a certain matrix function. We also derive the auxiliary linear problems for the adjoint Baker-Akhiezer function $\Psi^\dagger$. What is more, we obtain the Lax representation of the matrix mKP hierarchy, in which the Lax operator is a pseudo-difference operator in $p$, i.e., an infinite linear combination of $e^{k\partial_p}$ with $k \leq 1$ with coefficients depending on the times.

Our starting point is the formalism of free fermions developed in [5, 6]. For the $N$-component hierarchy one needs $N$-component fermionic fields $\psi^{(\alpha)}(z)$, $\psi^{*(\alpha)}(z)$. The main object of the theory is the tau-function $\tau$ which is a vacuum expectation value of certain composite fermion operators $g$ belonging to the Clifford group. In the $N$-component case, it is natural to consider an array of tau-functions $\tau_{\alpha\beta}$ which are matrix elements of operators from the Clifford group between different vacua with the same total charge. The key identity that allows one to derive equations of the hierarchy is the bilinear identity in the operator form

$$\sum_{\gamma=1}^N \oint dz \psi^{(\gamma)}(z) g \otimes \psi^{*(\gamma)}(z) g = \sum_{\gamma=1}^N \oint dz g \psi^{(\gamma)}(z) \otimes g \psi^{*(\gamma)}(z),$$

which holds for any $g$ from the Clifford group. Taking matrix elements of the operator bilinear identity between appropriate states, one is able to derive the fundamental bilinear identity for the tau-functions which is the corner stone of our approach. The bilinear identity generates a number of bilinear equations for the tau-functions of the Hirota type.

The plan of the paper is as follows. In section 2 we introduce the multicomponent fermions $\psi^{(\alpha)}(z)$, $\psi^{*(\alpha)}(z)$, modes of the current operator $J_{m}^{(\alpha)}$ coupled with the time variables $t_{\alpha,m}$ and define the tau-function as an expectation value of a Clifford group element. Using the bosonization rules, the bilinear identity for the array of tau-functions of the multicomponent mKP hierarchy is derived and some of its important consequences are explicitly written. Next, we introduce the Baker-Akhiezer function $\Psi$ and its adjoint $\Psi^\dagger$ in terms of the tau-functions. In section 3 we consider the specialization to the matrix mKP hierarchy. We introduce the pseudo-difference wave operator and derive the auxiliary linear problems for the Baker-Akhiezer function and its adjoint. Section 4 is devoted to the special linear problem for the time $t_1$. By a direct calculation, we show that it follows from the bilinear identity for the tau-functions.

## 2 Multicomponent fermions and bilinear identity

### 2.1 The multicomponent fermions

Following [11, 12], we introduce the creation-annihilation multicomponent free fermionic operators labeled by $\alpha = 1, \ldots, N$ as $\psi_j^{(\alpha)}$, $\psi_j^{*(\alpha)}$ ($j \in \mathbb{Z}$). They obey the anti-commutation relations

$$[\psi_j^{(\alpha)}, \psi_k^{*(\beta)}]_+ = \delta_{\alpha\beta}\delta_{jk}, \quad [\psi_j^{(\alpha)}, \psi_k^{(\beta)}]_+ = [\psi_j^{*(\alpha)}, \psi_k^{*(\beta)}]_+ = 0.$$


The Fock and dual Fock spaces are generated by the vacuum states \(|0\rangle\), \langle 0| that satisfy the conditions
\[
\psi_j^{(a)}|0\rangle = 0 \quad (j < 0), \quad \psi_j^{*(a)}|0\rangle = 0 \quad (j \geq 0),
\]
\[
\langle 0|\psi_j^{(a)} = 0 \quad (j \geq 0), \quad \langle 0|\psi_j^{*(a)} = 0 \quad (j < 0),
\]
so \(\psi_j^{(a)}\) with \(j < 0\) and \(\psi_j^{*(a)}\) with \(j \geq 0\) are annihilation operators while \(\psi_j^{(a)}\) with \(j \geq 0\) and \(\psi_j^{*(a)}\) with \(j < 0\) are creation operators. Let \(p = (p_1, p_2, \ldots, p_N)\) be a set of integer numbers. We define the states
\[
|p\rangle = \Psi_p^{(N)} \ldots \Psi_p^{(2)} \Psi_p^{(1)}|0\rangle, \quad \langle p| = \langle 0| \Psi_p^{(1)} \Psi_p^{(2)} \ldots \Psi_p^{(N)},
\]
where
\[
\Psi_p^{(a)} = \left\{ \begin{array}{ll}
\psi_{p-1}^{(a)} \ldots \psi_0^{(a)} & (p > 0) \\
\psi_p^{(a)} \ldots \psi_{-1}^{(a)} & (p < 0),
\end{array} \right.
\]
\[
\Psi_p^{(a)} = \left\{ \begin{array}{ll}
\psi_0^{*(a)} \ldots \psi_{p-1}^{*(a)} & (p > 0) \\
\psi_{-1}^{(a)} \ldots \psi_{p}^{(a)} & (p < 0).
\end{array} \right.
\]

Let us introduce the operators
\[
J_k^{(a)} = \sum_{j \in \mathbb{Z}} \psi_j^{(a)} \psi_{j+k}^{*(a)},
\]
where the normal ordering is defined by moving the annihilation operators to the right and creation operators to the left with the minus sign emerging each time when two fermionic operators are permuted (in fact the normal ordering is essential only for \(J_0^{(a)}\)). They are Fourier modes of the current operator. The operators \(J_0^{(a)} = Q_\alpha\) are charge operators. Assuming that there are \(N\) infinite sets of the independent continuous time variables
\[
t = \{t_1, t_2, \ldots, t_N\}, \quad t_\alpha = \{t_{\alpha,1}, t_{\alpha,2}, t_{\alpha,3}, \ldots\}, \quad \alpha = 1, \ldots, N,
\]
we introduce the operator
\[
J(t) = \sum_{\alpha=1}^{N} \sum_{k \geq 1} t_{\alpha,k} J_k^{(a)}.
\]

The tau-function \(\tau(p, t)\) of the multicomponent mKP hierarchy is defined as the expectation value
\[
\tau(p, t) = \langle p| e^{J(t)} g |p\rangle,
\]
where \(g\) is a general element of the Clifford group whose typical form is
\[
g = \exp \left( \sum_{\alpha, \beta} \sum_{j,k} A_{jk}^{(\alpha \beta)} \psi_j^{(a)} \psi_{j+k}^{*(a)} \psi_{j}^{(\beta)} \right)
\]
with some infinite matrix \(A_{jk}^{(\alpha \beta)}\).
2.2 The bilinear identity

An important property of the Clifford group elements is the following operator bilinear identity:

$$\sum_{\gamma=1}^{N} \sum_{j \in \mathbb{Z}} \psi_j^{(\gamma)} g \otimes \psi_j^{*(\gamma)} g = \sum_{\gamma=1}^{N} \sum_{j \in \mathbb{Z}} g\psi_j^{(\gamma)} \otimes g\psi_j^{*(\gamma)}.$$  \hspace{1cm} (2)

Let us introduce the free fermionic fields

$$\psi^{(\alpha)}(z) = \sum_{j \in \mathbb{Z}} \psi_j^{(\alpha)} z^j, \quad \psi^{*(\alpha)}(z) = \sum_{j \in \mathbb{Z}} \psi_j^{*(\alpha)} z^{-j},$$

then the operator bilinear identity acquires the form

$$\sum_{\gamma=1}^{N} \oint_{\gamma} dz \psi^{(\gamma)}(z) g \otimes \psi^{*(\gamma)}(z) g = \sum_{\gamma=1}^{N} \oint_{\gamma} dz g\psi^{(\gamma)}(z) \otimes g\psi^{*(\gamma)}(z).$$  \hspace{1cm} (3)

Here the contour integral is understood to be an integral along the big circle $|z| = R$ with sufficiently large $R$, $\oint dz^n = 2\pi i \delta_{n,-1}$.

The key identity is

$$\psi_j^{(\gamma)} |p\rangle \otimes \psi_j^{*(\gamma)} |p'\rangle = 0 \quad \text{if } p_{\alpha} \geq p'_{\alpha} \text{ for all } \alpha,$$

which holds for all $j \in \mathbb{Z}$ because either $\psi_j^{(\gamma)}$ annihilates $|p\rangle$ or $\psi_j^{*(\gamma)}$ annihilates $|p'\rangle$. Therefore, applying both sides of (2) to $|p\rangle \otimes |p'\rangle$, we get

$$\sum_{\gamma=1}^{N} \sum_{j \in \mathbb{Z}} \psi_j^{(\gamma)} g |p\rangle \otimes \psi_j^{*(\gamma)} g |p'\rangle = 0 \quad \text{if } p_{\alpha} \geq p'_{\alpha} \text{ for all } \alpha$$

or

$$\sum_{\gamma=1}^{N} \oint_{\gamma} dz \psi^{(\gamma)}(z) g |p\rangle \otimes \psi^{*(\gamma)}(z) g |p'\rangle = 0 \quad \text{if } p_{\alpha} \geq p'_{\alpha} \text{ for all } \alpha.$$  \hspace{1cm} (4)

Now apply $\langle p + e_\alpha | e^{J(t)} \otimes |p' - e_\beta\rangle | e^{J(t')}$, where $e_\alpha$ denotes the vector with 1 on the $\alpha$th place and zeros elsewhere, to get

$$\sum_{\gamma=1}^{N} \oint_{\gamma} dz \langle p + e_\alpha | e^{J(t)} \psi^{(\gamma)}(z) g |p\rangle \langle p' - e_\beta| e^{J(t')} \psi^{*(\gamma)}(z) g |p'\rangle = 0 \quad \text{if } p_{\alpha} \geq p'_{\alpha} \text{ for all } \alpha.$$  \hspace{1cm} (5)

Here

$$\xi(t_\gamma, z) = \sum_{k \geq 1} t_{\gamma,k} z^k$$

and the commutation relations

$$e^{J(t)} \psi^{(\gamma)}(z) = e^{\xi(t_\gamma, z)} \psi^{(\gamma)}(z) e^{J(t)}, \quad e^{J(t)} \psi^{*(\gamma)}(z) = e^{-\xi(t_\gamma, z)} \psi^{*(\gamma)}(z) e^{J(t)}.$$
are used. Now we are ready to employ the multicomponent bosonization rules [2]

\[
\langle p + e_\alpha | \psi^{(\gamma)}(z) e^{J(t)} | p + e_\gamma \rangle = \epsilon_{\alpha \gamma} e_{\gamma}(p) z^{p_\gamma + \delta_{\alpha \gamma} - 1} \langle p + e_\alpha - e_\gamma | e^{J(t)[z^{-1}]} | p + e_\gamma \rangle,
\]

\[
\langle p' - e_\beta | \psi^{(\gamma)}(z) e^{J(t)} | p' - e_\beta + e_\gamma \rangle = \epsilon_{\beta \gamma} e_{\gamma}(p') z^{p_\gamma' + \delta_{\beta \gamma}} \langle p' - e_\beta + e_\gamma | e^{J(t)[z^{-1}]} | p' - e_\beta + e_\gamma \rangle,
\]

where

\[
(t \pm [z^{-1}])_{\alpha k} = t_{\alpha k} \pm \delta_{\alpha \gamma} \frac{z^{-k}}{k}
\]

and the sign factors \(\epsilon_{\alpha \beta}, \epsilon_{\gamma}(p)\) are: \(\epsilon_{\alpha \beta} = 1\) if \(\alpha \leq \beta\), \(\epsilon_{\alpha \beta} = -1\) if \(\alpha > \beta\), \(\epsilon_{\gamma}(p) = (-1)^{p_{\gamma+1} + \cdots + p_N}\). Multiplying (5) by \(\epsilon_{\alpha}(p)\epsilon_{\beta}(p')\), we arrive at the bilinear identity for the tau-function of the \(N\)-component mKP hierarchy

\[
\sum_{\gamma=1}^{N} \epsilon_{\alpha \gamma}(p)\epsilon_{\beta \gamma}(p') \int dzz^{p_\gamma - p_\gamma' + \delta_{\alpha \gamma} + \delta_{\beta \gamma} - 2} e^{\xi(t_\gamma - t_\gamma', z)} \tau_{\alpha \gamma}(p, t - [z^{-1}]_\gamma) \tau_{\beta \gamma}(p', t' + [z^{-1}]_\gamma) = 0
\]

valid for any \(t, t', p, p'\) if \(p_\alpha \geq p'_\alpha\) for all \(\alpha\). Here and below

\[
\tau_{\alpha \gamma}(p, t) = \langle p + e_\alpha - e_\gamma | e^{J(t)} g | p \rangle
\]

and

\[
\epsilon_{\alpha \gamma}(p) = \begin{cases} 
(-1)^{p_{\alpha + 1} + \cdots + p_N} & \text{if } \alpha < \gamma \\
1 & \text{if } \alpha = \gamma \\
(-1)^{p_{\gamma+1} + \cdots + p_\alpha} & \text{if } \alpha > \gamma
\end{cases}
\]

The integration contour around \(\infty\) is such that all singularities coming from the power of \(z\) and the exponential function \(e^{\xi(t_\gamma - t_\gamma', z)}\) are inside it and all singularities coming from the \(\tau\)-factors are outside it.

### 2.3 The Hirota equations

At \(p = p'\) the bilinear identity generates the \(N\)-component KP hierarchy. Choosing \(t'\) in (7) in a specific way, one can obtain, after calculating the residues, a number of differential and difference bilinear equations for the tau-function of the Hirota type (they are called Fay identities in [3]). The complete list of such equations is given in [3]. Below we present only the equations that are used in what follows.

Differentiating (7) (with \(p = p'\)) with respect to \(t_{\gamma,1}\) and setting \(t' = t - [\mu^{-1}]_\beta\), we have, for any distinct \(\alpha, \beta, \gamma\):

\[
\tau_{\alpha \beta}(t - [\mu^{-1}]_\beta) \partial_{t_{\gamma,1}} \tau(t) - \tau(t) \partial_{t_{\gamma,1}} \tau_{\alpha \beta}(t - [\mu^{-1}]_\beta) + \frac{\epsilon_{\alpha \gamma}(p)\epsilon_{\gamma \beta}(p)}{\epsilon_{\alpha \beta}(p)} \tau_{\alpha \gamma}(t) \tau_{\gamma \beta}(t - [\mu^{-1}]_\beta) = 0,
\]

where we have suppressed the dependence of the tau-function on \(p\) (since \(p\) is the same for all tau-functions).

Differentiating (7) with respect to \(t_{\beta,1}\) and setting \(t' = t - [\nu^{-1}]_\alpha - [\nu^{-1}]_\beta\), we have, for any distinct \(\alpha, \beta\):

\[
\partial_{t_{\beta,1}} \tau_{\alpha \beta}(t - [\nu^{-1}]_\beta) \tau(t - [\mu^{-1}]_\alpha) - \partial_{t_{\beta,1}} \tau(t - [\mu^{-1}]_\alpha) \tau_{\alpha \beta}(t - [\nu^{-1}]_\beta)

+ \nu \tau_{\alpha \beta}(t - [\nu^{-1}]_\beta) \tau(t - [\mu^{-1}]_\alpha) - \nu \tau_{\alpha \beta}(t - [\mu^{-1}]_\alpha) \tau(t - [\nu^{-1}]_\beta) = 0.
\]
In a similar way, differentiating (7) with respect to $t_{\alpha,1}$ and setting $t' = t - [\mu^{-1}]_{\alpha} - [\nu^{-1}]_{\beta}$, we have, for any distinct $\alpha, \beta$:

$$
\partial_{t_{\alpha,1}} \tau_{\alpha\beta}(t - [\nu^{-1}]_{\beta}) \tau(t - [\mu^{-1}]_{\alpha}) - \partial_{t_{\alpha,1}} \tau(t - [\mu^{-1}]_{\alpha}) \tau_{\alpha\beta}(t - [\nu^{-1}]_{\beta})
- \mu \tau_{\alpha\beta}(t - [\nu^{-1}]_{\beta}) \tau(t - [\mu^{-1}]_{\alpha}) + \mu \tau(t) \tau_{\alpha\beta}(t - [\mu^{-1}]_{\alpha} - [\nu^{-1}]_{\beta}) = 0.
$$

(10)

Differentiating (7) at $\beta = \alpha$ with respect to $t_{\gamma,1}$ ($\gamma \neq \alpha$) and setting $t' = t - [\mu^{-1}]_{\alpha}$, we have, for any distinct $\alpha, \gamma$:

$$
\partial_{t_{\gamma,1}} \tau(t - [\mu^{-1}]_{\alpha}) \tau(t) - \partial_{t_{\gamma,1}} \tau(t) \tau(t - [\mu^{-1}]_{\alpha}) + \mu^{-1} \tau_{\alpha\gamma}(t) \tau_{\alpha\alpha}(t - [\mu^{-1}]_{\alpha}) = 0.
$$

(11)

Consequences of the bilinear identity with $p \neq p'$ will be discussed in the next sections.

### 2.4 The Baker-Akhiezer functions

The matrix Baker-Akhiezer function $\Psi(p, t; z)$ and its adjoint $\Psi^\dagger(p, t; z)$ are $N \times N$ matrices with components defined by

$$
\Psi_{\alpha\beta}(p, t; z) = \epsilon_{\alpha\beta}(p) \frac{\tau_{\alpha\beta}(p, t - [z^{-1}]_{\beta})}{\tau(p, t)} z^{p_{\beta} + \delta_{\alpha\beta} - 1} e^{\xi(t_{\beta}, z)},
$$

$$
\Psi^\dagger_{\alpha\beta}(p, t; z) = \epsilon_{\beta\alpha}(p) \frac{\tau_{\alpha\beta}(p, t + [z^{-1}]_{\alpha})}{\tau(p, t)} z^{-p_{\alpha} + \delta_{\alpha\beta} - 1} e^{-\xi(t_{\alpha}, z)}
$$

(12)

(here and below $\Psi^\dagger$ does not mean the Hermitian conjugation). In terms of the matrix Baker-Akhiezer functions, the bilinear identity (7) acquires the form

$$
\oint dz \Psi(p, t; z) \Psi^\dagger(p', t'; z) = 0.
$$

(13)

Near $z = \infty$ the Baker-Akhiezer functions can be expanded into the series

$$
\Psi_{\alpha\beta}(p, t; z) = \left( \delta_{\alpha\beta} + \sum_{k \geq 1} w_{\alpha\beta}^{(k)}(p, t) z^{-k} \right) z^{p_{\beta}} e^{\xi(t_{\beta}, z)},
$$

$$
\Psi^\dagger_{\alpha\beta}(p, t; z) = \left( \delta_{\alpha\beta} + \sum_{k \geq 1} v_{\alpha\beta}^{(k)}(p, t) z^{-k} \right) z^{-p_{\alpha}} e^{-\xi(t_{\alpha}, z)}
$$

(14)

(15)

It is proved in [3] that the Baker-Akhiezer function and its adjoint satisfy the auxiliary linear equations

$$
\partial_{t_{\alpha,m}} \Psi(p, t; z) = B_{am} \Psi(p, t; z),
$$

$$
-\partial_{t_{\alpha,m}} \Psi^\dagger(p, t; z) = \Psi^\dagger(p, t; z) B_{am},
$$

(16)

where $B_{am}$ is a matrix differential operator in $\partial_t \equiv \sum_{a=1}^{N} \partial_{t_{a,1}}$. In the second equation here it is assumed that the operators $\partial_{t_{1}}$ entering $B_{am}$ act to the left as $f \partial_{t_{1}} = -\partial_{t_{1}} f$. 

6
3 The matrix mKP hierarchy

The matrix mKP hierarchy is obtained from the multicomponent one after the following restriction of the time variables:

\[ t_{\alpha,m} = t_m \quad \text{for each} \quad \alpha \quad \text{and} \quad m, \quad p_{\alpha} = p \quad \text{for each} \quad \alpha, \]

so the evolution with respect to each \( t_{\alpha,m} \) and \( p_{\alpha} \) is the same and is defined by \( t_m, p \) only. The corresponding vector fields are related as \( \partial_{t_m} = \sum_{\alpha=1}^N \partial_{t_{\alpha,m}}, \partial_{p} = \sum_{\alpha=1}^N \partial_{p_{\alpha}}. \) In what follows we denote

\[ \tau(p, t) := \tau^p(t), \quad \epsilon_{\alpha\beta}(p) := \epsilon_{\alpha\beta}(p). \]

The bilinear identity (7) acquires the form

\[ \sum_{\gamma=1}^N \epsilon_{\alpha\gamma}(p) \epsilon_{\beta\gamma}(p-n) \oint d\tau z^{n+\delta_{\alpha\gamma}+\delta_{\beta\gamma}-2} \epsilon(t, \tau, z) \tau_{\alpha\gamma}^p(t - [z^{-1}]_\gamma) \tau_{\gamma\beta}^{p-n}(t + [z^{-1}]_\gamma) = 0 \]  

(17)

with \( n \geq 0 \). The Baker-Akhiezer functions (12) for the matrix mKP hierarchy are

\[ \Psi_{\alpha\beta}^p = \epsilon_{\alpha\beta}(p) \frac{\tau_{\alpha\beta}^{p}(t - [z^{-1}]_\beta)}{\tau^p(t)} z^{p - \delta_{\alpha\beta} - 1} e^{\xi(t, z)}, \]  

(18)

\[ \Psi_{\alpha\beta}^{ip} = \epsilon_{\beta\alpha}(p) \frac{\tau_{\alpha\beta}^{p}(t + [z^{-1}]_\alpha)}{\tau^p(t)} z^{-p + \delta_{\alpha\beta} - 1} e^{-\xi(t, z)}, \]

where \( \xi(t, z) = \sum_{k \geq 1} t_k z^k \). Their expansions around \( z = \infty \) read

\[ \Psi_{\alpha\beta}^p = \left( \sum_{k \geq 0} w_{\alpha\beta}^{(k)}(p) z^{-k} \right) z^{p + \delta_{\alpha\beta} - 1} e^{\xi(t, z)}, \]  

(19)

\[ \Psi_{\alpha\beta}^{ip} = \left( \sum_{k \geq 0} v_{\alpha\beta}^{(k)}(p) z^{-k} \right) z^{-p + \delta_{\alpha\beta} - 1} e^{-\xi(t, z)}, \]  

(20)

where \( w_{\alpha\beta}^{(0)}(p) = v_{\alpha\beta}^{(0)}(p) = \delta_{\alpha\beta} \) and we have suppressed the dependence on \( t \).

In order to represent \( w_{\alpha\beta}^{(k)}(p) \) and \( v_{\alpha\beta}^{(k)}(p) \) explicitly for arbitrary \( k \) we need some notation. Introduce the Schur polynomials \( h_k(t) \) via the expansion

\[ \exp(\sum_{k \geq 1} t_k z^k) = \sum_{k \geq 0} h_k(t) z^k \]

(clearly, \( h_0(t) = 1 \) and \( h_k(t) = 0 \) for \( k < 0 \)). We also denote

\[ \tilde{\partial}_\alpha = \{ \partial_{t_{\alpha,1}}, \frac{1}{2} \partial_{t_{\alpha,2}}, \frac{1}{3} \partial_{t_{\alpha,3}}, \ldots \}, \]

so that \( h_1(\tilde{\partial}_\alpha) = \partial_{t_{\alpha,1}}, \) etc. Using the fact that \( \tau(t \pm [z^{-1}]_\alpha) = \exp\left( \pm \sum_{k \geq 1} \frac{z^{-k}}{k} \partial_{t_{\alpha,k}} \right) \tau(t), \)

we have from (18):

\[ w_{\alpha\beta}^{(k)}(p) = \frac{h_k(\tilde{\partial}_\beta) \tau^p(t)}{\tau^p(t)} \delta_{\alpha\beta} + \epsilon_{\alpha\beta}(p) \frac{h_{k-1}(\tilde{\partial}_\beta) \tau_{\alpha\beta}^p(t)}{\tau^p(t)} (1 - \delta_{\alpha\beta}), \]  

(21)
\begin{equation}
  v_{\alpha\beta}^{(k)}(p) = \frac{h_k(\tilde{\partial}_\alpha)\tau^p(t)}{\tau^p(t)} \delta_{\alpha\beta} + \epsilon_{\beta\alpha}(p) \frac{h_{k-1}(\tilde{\partial}_\alpha)\tau^p_{\alpha\beta}(t)}{\tau^p(t)} (1 - \delta_{\alpha\beta}).
\end{equation}

In particular,
\begin{equation}
  w_{\alpha\beta}^{(1)}(p) = \begin{cases}
    \epsilon_{\alpha\beta}(p) \frac{\tau^p_{\alpha\beta}(t)}{\tau^p(t)} & \text{if } \alpha \neq \beta \\
    - \frac{\partial_{\alpha\beta} \tau^p(t)}{\tau^p(t)} & \text{if } \alpha = \beta,
  \end{cases}
\end{equation}

\begin{equation}
  v_{\alpha\beta}^{(1)}(p) = -w_{\alpha\beta}^{(1)}(p) \quad \text{(the latter relation follows from the bilinear identity in the form \([13]\) at } t = t', \ p = p').
\end{equation}

Let us introduce the matrix pseudo-difference wave operator
\begin{equation}
  W(p) = I + \sum_{k \geq 1} w^{(k)}(p) e^{-k\partial_p},
\end{equation}

where \(I\) is the unity \(N \times N\) matrix and \(w^{(k)}(p)\) are the same matrix functions as in \([19]\). In matrix elements we have
\begin{equation}
  W_{\alpha\beta}(p) = \sum_{k \geq 0} w^{(k)}_{\alpha\beta}(p) e^{-k\partial_p}.
\end{equation}

Clearly, the Baker-Akhiezer function \(\Psi^p\) can be written as a result of action of the wave operator to the function \(z^p\) times an exponential function:
\begin{equation}
  \Psi^p = W(p) z^p \exp\left(\sum_{\alpha=1}^N E_{\alpha} \xi(t_{\alpha}, z)\right),
\end{equation}

where \(E_{\alpha}\) is the \(N \times N\) matrix with 1 on the diagonal \((\alpha, \alpha)\) component and zero elsewhere.

We are going to show that the inverse wave operator \(W^{-1}(p)\) is given by
\begin{equation}
  W^{-1}(p) = \sum_{k \geq 0} e^{-k\partial_p} v^{(k)}(p + 1)
\end{equation}

with the matrices \(v^{(k)}\) as in \([22]\). Indeed, we have:
\begin{align*}
  I &= W(p) W^{-1}(p) = \sum_{k, k' \geq 0} w^{(k)}(p) e^{-(k+k')\partial_p} v^{(k')}(p + 1) \\
  &= \sum_{m \geq 0} \sum_{k = 0}^m w^{(k)}(p) v^{(m-k)}(p + 1 - m) e^{-m\partial_p},
\end{align*}

so we need to show that
\begin{equation}
  \sum_{\gamma} \sum_{k = 0}^m w^{(k)}_{\alpha\gamma}(p) v^{(m-k)}_{\gamma\beta}(p + 1 - m) = \delta_{m0} \delta_{\alpha\beta}.
\end{equation}

At \(m = 0\) this equality is obvious. At \(m > 0\), substituting \([21]\), \([22]\), we get:
\begin{equation}
  \sum_{k = 0}^m \epsilon_{\beta\alpha}(p+1-m) h_k(-\tilde{\partial}_\alpha) \tau^p(t) h_{m-k-1}(\tilde{\partial}_\alpha) \tau^{p+1-m}_{\alpha\beta}(t)
\end{equation}
\[
+ \sum_{k=0}^{m} \epsilon_{\alpha \beta}(p) h_{k-1}(-\partial_{\beta}) \tau_{\alpha \beta}^{p}(t) h_{m-k}(\partial_{\beta}) \tau^{p+1-m}(t)
\]

\[
+ \sum_{\gamma \neq \alpha, \beta} \sum_{k=0}^{m} \epsilon_{\alpha \gamma}(p) \epsilon_{\beta \gamma} (p+1-m) h_{k-1}(-\partial_{\gamma}) \tau_{\alpha \gamma}^{p}(t) h_{m-k-1}(\partial_{\gamma}) \tau_{\gamma \beta}^{p+1-m}(t) = 0
\]

for \( \alpha \neq \beta \) and

\[
\sum_{k=0}^{m} h_{k}(-\partial_{\alpha}) \tau^{p}(t) h_{m-k}(\partial_{\alpha}) \tau^{p+1-m}(t)
\]

\[
+ \sum_{\gamma \neq \alpha} \sum_{k=0}^{m} \epsilon_{\alpha \gamma}(p) \epsilon_{\alpha \gamma} (p+1-m) h_{k-1}(-\partial_{\gamma}) \tau_{\alpha \gamma}^{p}(t) h_{m-k}(\partial_{\gamma}) \tau_{\gamma \gamma}^{p+1-m}(t) = 0
\]

for \( \alpha = \beta \). One can check that these relations are precisely the ones that follow from the bilinear identity \([17]\) at \( n = m - 1 \) and \( t = t' \), so the formula \([26]\) for the inverse wave operator is proved. Therefore, we can represent the adjoint Baker-Akhiezer function in the form

\[
\Psi^{bp} = \exp\left(-\sum_{\alpha} E_{\alpha} \xi(t_{\alpha}, z)\right) z^{-p} W^{-1}(p - 1),
\]

(27)

where the left action of the operators \( e^{-\partial_{p}} \) according to \( f e^{-\partial_{p}} = e^{\partial_{p}} f \) is implied.

In the next section we will show that the Baker-Akhiezer function \( \Psi \) satisfies the auxiliary linear problem

\[
\partial_{t_{1}} \Psi = e^{\partial_{p}} \Psi + u \Psi
\]

(28)

with some matrix function \( u \). Therefore, since \( B_{\alpha m} \) in the auxiliary linear problems \([16]\) are differential operators in \( \partial_{t_{1}}, \partial_{t_{\alpha, m}} \), \( \Psi \) can be expressed as a result of action of a difference operator \( A_{\alpha m} \) in \( p \) that contains only non-negative powers of the shift operator \( e^{\partial_{p}} \). This remark allows one to express \( A_{\alpha m} \) in terms of the wave operator. Indeed, differentiating equation \([25]\), we have:

\[
\frac{\partial \Psi^{p}}{\partial t_{\alpha, m}} = \frac{\partial W(p)}{\partial t_{\alpha, m}} W^{-1}(p) \Psi^{p} + W(p) E_{\alpha} z^{m+p} \exp\left(\sum_{\gamma} E_{\gamma} \xi(t_{\gamma}, z)\right)
\]

\[
= \frac{\partial W(p)}{\partial t_{\alpha, m}} W^{-1}(p) \Psi^{p} + W(p) E_{\alpha} e^{m\partial_{p}} W^{-1}(p) \Psi^{p}.
\]

Therefore,

\[
A_{\alpha m}(p) = \frac{\partial W(p)}{\partial t_{\alpha, m}} W^{-1}(p) + W(p) E_{\alpha} e^{m\partial_{p}} W^{-1}(p).
\]

Since the first term here contains strictly negative powers of \( e^{\partial_{p}} \), one concludes that

\[
\partial_{t_{\alpha, m}} \Psi^{p} = A_{\alpha m}(p) \Psi^{p}, \quad A_{\alpha m}(p) = \left(W(p) E_{\alpha} e^{m\partial_{p}} W^{-1}(p)\right)_{+}
\]

(29)

and

\[
\partial_{t_{\alpha, m}} W(p) = -\left(W(p) E_{\alpha} e^{m\partial_{p}} W^{-1}(p)\right)_{-} W(p),
\]

(30)

where \((\ldots)_{\pm}\) denotes the part of a pseudo-difference operator containing only non-negative (respectively, negative) powers of \( e^{\partial_{p}} \). In a similar way, differentiating \([27]\), we obtain the auxiliary linear problems for the adjoint Baker-Akhiezer function:

\[
-\partial_{t_{\alpha, m}} \Psi^{bp} = \Psi^{bp} A_{\alpha m}(p - 1).
\]

(31)
One can also introduce the Lax operator
\[ L(p) = W(p)e^{\partial_p W^{-1}(p)}, \] (32)
with \( \Psi \) being its eigenfunction:
\[ L(p)\Psi^p = z\Psi^p. \] (33)
The compatibility of (29) and (33) implies the Lax equation
\[ \partial_{t_{\alpha,m}} L(p) = [A_{\alpha m}(p), L(p)]. \] (34)

We have thus obtained the auxiliary linear problems and Lax equations for the multicomponent hierarchy. They simplify for the matrix hierarchy:
\[ \partial_{t_1} \Psi^p = A_1(p)\Psi^p, \] (35)
\[ -\partial_{t_1} \Psi^\dagger p = \Psi^\dagger p A_1(p - 1), \] (36)
where
\[ A_1(p) = \left( W(p)e^{m\partial_p W^{-1}(p)} \right)_+. \]
(we use the fact that \( \sum_{\alpha=1}^N E_\alpha = I \)). In particular, for \( m = 1 \) we have:
\[ \partial_{t_1} \Psi^p = \Psi^{p+1} + \left( w^{(1)}(p) - w^{(1)}(p + 1) \right) \Psi^p, \] (37)
\[ -\partial_{t_1} \Psi^\dagger p = \Psi^{\dagger p-1} + \Psi^\dagger p \left( w^{(1)}(p - 1) - w^{(1)}(p) \right). \]

In fact the very derivation of (29) and (35), (36) was based on equations (37) (see (28)). In the next section we will derive these equations in an independent way as a consequence of the bilinear identity. It should be noted that in [7] the linear problems (37) for Baker-Akhiezer functions on Riemann surfaces were obtained using algebro-geometric reasoning.

4 Auxiliary linear problem for derivative with respect to \( t_1 \)

In this section we show that the auxiliary linear problems (37) are in fact equivalent to a consequence of the bilinear identity (7). We start from the linear problem for \( \Psi \) writing it in components in the form
\[ \Psi_{\alpha\beta}^{p+1} = \partial_{t_1} \Psi_{\alpha\beta}^p + \sum_\gamma \left( w^{(1)}_{\alpha\gamma}(p + 1) - w^{(1)}_{\alpha\gamma}(p) \right) \Psi_{\gamma\beta}^p. \] (38)
We recall that \( w_{\alpha\gamma}^{(1)}(p) \) is given by (23). Consider first the case \( \alpha \neq \beta \). Substituting (18) for \( \Psi \), we write (38) in the form
\[ \epsilon_{\alpha\beta}(p + 1) \frac{\tau_{\alpha\beta}^{p+1}(t - [z^{-1}]_\beta)}{\tau^{p+1}(t)} = \epsilon_{\alpha\beta}(p) \frac{\tau_{\alpha\beta}^p(t - [z^{-1}]_\beta)}{\tau^p(t)} + \epsilon_{\alpha\beta}(p) \partial_{t_1} \left( \frac{\tau_{\alpha\beta}^p(t - [z^{-1}]_\beta)}{\tau^p(t)} \right) z^{-1} \]
we use the Hirota equation (8) with 
µ
After some obvious transformations, separating the terms with the denominator \((\tau^p(t))^2\), we rewrite this as
\[
\begin{align*}
-\epsilon_{\alpha\beta}(p) \left( \frac{\partial_{t_{\alpha,1}} \tau^{p+1}(t)}{\tau^{p+1}(t)} - \frac{\partial_{t_{\alpha,1}} \tau^p(t)}{\tau^p(t)} \right) \frac{\tau_{\alpha\gamma}(t) \tau_{\gamma\beta}(t - [z^{-1}]_\beta) \tau^{p}(t - [z^{-1}]_\beta)}{\tau^p(t)} \cdot z^{-1}.
\end{align*}
\]

In the first line, we use the Hirota equation [2] with \(\mu = \infty, \nu = z\). In the second line, we use the Hirota equation [3] with \(\mu = z\). The result is
\[
\begin{align*}
\left\{ \ldots \right\} &= -\tau^p(t) \left[ z^{-1} \sum_{\gamma \neq \alpha} \partial_{t_{\gamma,1}} \tau_{\alpha\beta}(t - [z^{-1}]_\beta) + \tau_{\alpha\beta}(t - [z^{-1}]_\beta) \right].
\end{align*}
\]

Substituting this back, we obtain, after some cancellations:
\[
\begin{align*}
-\epsilon_{\alpha\beta}(p + 1) \frac{\tau_{\alpha\beta}^{p+1}(t - [z^{-1}]_\beta)}{\tau^{p+1}(t)} + \epsilon_{\alpha\beta}(p) \frac{\partial_{t_{\alpha,1}} \tau_{\alpha\beta}(t - [z^{-1}]_\beta)}{\tau^p(t)} \cdot z^{-1}
+ \epsilon_{\alpha\beta}(p + 1) \frac{\tau_{\alpha\beta}^{p+1}(t) \tau^p(t - [z^{-1}]_\beta)}{\tau^{p+1}(t) \tau^p(t)} - \epsilon_{\alpha\beta}(p) \frac{\partial_{t_{\alpha,1}} \tau^{p+1}(t) \tau_{\alpha\beta}(t - [z^{-1}]_\beta)}{\tau^{p+1}(t) \tau^p(t)} \cdot z^{-1}
+ z^{-1} \sum_{\gamma \neq \alpha, \beta} \epsilon_{\alpha\gamma}(p + 1) \epsilon_{\gamma\beta}(p) \frac{\tau_{\alpha\gamma}(t) \tau_{\gamma\beta}(t - [z^{-1}]_\beta)}{\tau_{\alpha\beta}(t) \tau_{\alpha\gamma}(t) \tau_{\gamma\beta}(t - [z^{-1}]_\beta)} = 0.
\end{align*}
\]
or
\[
\begin{align*}
&z\epsilon_{\alpha\beta}(p+1)\tau^{p+1}(t)^2\tau^{p}(t - [z^{-1}]_\beta) - z\epsilon_{\alpha\beta}(p+1)\tau^{p}(t)^2\tau'^{p+1}(t - [z^{-1}]_\beta) \\
&+ \epsilon_{\alpha\beta}(p)\partial_{t_{\alpha\beta}}\tau^{p}(t - [z^{-1}]_\beta) - \epsilon_{\alpha\beta}(p)\partial_{t_{\alpha\beta}}\tau^{p+1}(t)\tau^{p}_{\gamma\alpha}(t - [z^{-1}]_\beta) \\
&+ \sum_{\gamma \neq \alpha, \beta} \epsilon_{\alpha\gamma}(p+1)\epsilon_{\gamma\alpha}(p)\tau^{p+1}(t)\tau^{p}_{\gamma\alpha}(t - [z^{-1}]_\beta) = 0.
\end{align*}
\]

One can see that this relation is equivalent to the bilinear identity \([17]\) taken at \(n = 1, \alpha \neq \beta, t' = t - [\mu^{-1}]_\beta \) (with \(\mu = z\) in the end).

We now pass to the case \(\alpha = \beta\) in \([38]\):

\[
\begin{align*}
z \frac{\tau^{p+1}(t - [z^{-1}]_\alpha)}{\tau^{p+1}(t)} = z \frac{\tau^{p}(t - [z^{-1}]_\alpha)}{\tau^{p}(t)} + \partial_{t_{\alpha\beta}} \left( \frac{\tau^{p}(t - [z^{-1}]_\alpha)}{\tau^{p}(t)} \right) \\
+ z^{-1} \sum_{\gamma \neq \alpha} \epsilon_{\alpha\gamma}(p+1)\epsilon_{\gamma\alpha}(p) \frac{\tau^{p+1}(t)\tau^{p}_{\alpha\gamma}(t - [z^{-1}]_\alpha)}{\tau^{p+1}(t)^2\tau^{p}(t)} + z^{-1} \sum_{\gamma \neq \alpha} \frac{\tau^{p}_{\alpha\gamma}(t)\tau^{p}_{\gamma\alpha}(t - [z^{-1}]_\alpha)}{(\tau^{p}(t)^2)}(\tau^{p}(t))^2 \\
- \left( \frac{\partial_{t_{\alpha\beta}}\tau^{p+1}(t)}{\tau^{p+1}(t)} - \frac{\partial_{t_{\alpha\beta}}\tau^{p}(t)}{\tau^{p}(t)} \right) \frac{\tau^{p}(t - [z^{-1}]_\alpha)}{\tau^{p}(t)} \\
+ \frac{1}{(\tau^{p}(t)^2)} \left( \sum_{\gamma \neq \alpha} \left( z^{-1} \tau^{p}_{\alpha\gamma}(t)\tau^{p}_{\gamma\alpha}(t - [z^{-1}]_\alpha) - \tau^{p}(t - [z^{-1}]_\alpha)\partial_{t_{\alpha\beta}}\tau^{p}(t) \right) \right) \right) = 0.
\end{align*}
\]

Separating the terms with the denominator \((\tau^{p}(t))^2\), we rewrite this as

\[
\begin{align*}
&-z \frac{\tau^{p+1}(t - [z^{-1}]_\alpha)}{\tau^{p+1}(t)} + z \frac{\tau^{p}(t - [z^{-1}]_\alpha)}{\tau^{p}(t)} + \partial_{t_{\alpha\beta}} \frac{\tau^{p}(t - [z^{-1}]_\alpha)}{\tau^{p}(t)} \\
&+ z^{-1} \sum_{\gamma \neq \alpha} \epsilon_{\alpha\gamma}(p+1)\epsilon_{\gamma\alpha}(p) \frac{\tau^{p+1}(t)\tau^{p}_{\alpha\gamma}(t - [z^{-1}]_\alpha)}{\tau^{p+1}(t)^2\tau^{p}(t)} - \partial_{t_{\alpha\beta}} \frac{\tau^{p+1}(t)\tau^{p}(t - [z^{-1}]_\alpha)}{\tau^{p+1}(t)^2\tau^{p}(t)} \\
&+ \frac{1}{(\tau^{p}(t)^2)} \left( \sum_{\gamma \neq \alpha} \left( z^{-1} \tau^{p}_{\alpha\gamma}(t)\tau^{p}_{\gamma\alpha}(t - [z^{-1}]_\alpha) - \tau^{p}(t - [z^{-1}]_\alpha)\partial_{t_{\alpha\beta}}\tau^{p}(t) \right) \right) \right) = 0.
\end{align*}
\]

Using the Hirota equation \([11]\), we obtain:

\[
\left\{ \ldots \right\} = -\tau^{p}(t) \sum_{\gamma \neq \alpha} \partial_{t_{\gamma\alpha}} \tau^{p}(t - [z^{-1}]_\alpha),
\]

so the previous expression acquires the form

\[
\begin{align*}
&-z \frac{\tau^{p+1}(t - [z^{-1}]_\alpha)}{\tau^{p+1}(t)} + z \frac{\tau^{p}(t - [z^{-1}]_\alpha)}{\tau^{p}(t)} + \partial_{t_{\alpha\beta}} \frac{\tau^{p}(t - [z^{-1}]_\alpha)}{\tau^{p}(t)} \\
&- \partial_{t_{\alpha\beta}} \frac{\tau^{p+1}(t)\tau^{p}(t - [z^{-1}]_\alpha)}{\tau^{p+1}(t)^2\tau^{p}(t)} + z^{-1} \sum_{\gamma \neq \alpha} \epsilon_{\alpha\gamma}(p+1)\epsilon_{\gamma\alpha}(p) \frac{\tau^{p+1}(t)\tau^{p}_{\alpha\gamma}(t - [z^{-1}]_\alpha)}{\tau^{p+1}(t)^2\tau^{p}(t)} = 0.
\end{align*}
\]

or

\[
\begin{align*}
z^2\tau^{p+1}(t)\tau^{p}(t - [z^{-1}]_\alpha) - z^2\tau^{p+1}(t - [z^{-1}]_\alpha)\tau^{p}(t) \\
+ z\tau^{p+1}(t)\partial_{t_{\alpha\beta}}\tau^{p}(t - [z^{-1}]_\alpha) - z\partial_{t_{\alpha\beta}}\tau^{p+1}(t)\tau^{p}(t - [z^{-1}]_\alpha) \\
+ \sum_{\gamma \neq \alpha} \epsilon_{\alpha\gamma}(p+1)\epsilon_{\gamma\alpha}(p)\tau^{p+1}(t)\tau^{p}_{\gamma\alpha}(t - [z^{-1}]_\alpha) = 0.
\end{align*}
\]
One can see that this relation is equivalent to the bilinear identity (17) taken at \(n = 1\), \(\alpha = \beta\), \(t' = t - \lfloor \mu^{-1} \rfloor\alpha\) (with \(\mu = z\) in the end).

The second equation in (37), which we write in components in the form

\[ \Psi^p_{\alpha \beta} = -\partial_t \Psi^{p+1}_{\alpha \beta} + \sum_\gamma \Psi^{p+1}_{\alpha \gamma} \left( w^{(1)}_{\gamma \beta} (p + 1) - w^{(1)}_{\gamma \beta} (p) \right) \]  

(39)
can be processed in a similar way, using the Hirota equations (8), (10), (11).

Finally, we would like to remark that one can interpret the linear problems (37) also in a different way. Namely, the equation \(\partial_t \Psi = e^{\partial_p} \Psi + u \Psi\) can be read as \(\Psi^{p+1} = \partial_t \Psi^p - u \Psi^p\), and in this form it can be regarded as describing an extension of the matrix KP hierarchy to the discrete flow \(\Psi^p\) generated by the first order differential operator \(\partial_t - u\).

5 Conclusion

In this paper we have studied the matrix and multicomponent mKP hierarchies from the point of view of fermionic formalism and bilinear identities. The pseudo-difference wave operator has been introduced and the auxiliary linear problems for the Baker-Akhiezer function and its adjoint have been derived. The Lax representation of the hierarchy in terms of pseudo-difference operators has been obtained.

It should be noted that even more general matrix hierarchy than mKP exists. It is the matrix or non-abelian (and multicomponent) Toda hierarchy, in which in addition to \(t_{\alpha,m}\) with \(m > 0\) there is yet another infinite set of continuous times \(t_{\alpha,m}\) with \(m < 0\). It would be interesting to extend the approach developed in this paper to this more general case. The algebro-geometric solutions to the matrix Toda hierarchy were discussed in [7, 8].

Acknowledgments

This work was funded by the Russian Academic Excellence Project ‘5-100’. This work was supported in part by RFBR grant 18-01-00461.

References

[1] E. Date, M. Jimbo, M. Kashiwara and T. Miwa, *Transformation groups for soliton equations III*, J. Phys. Soc. Japan **50** (1981) 3806-3812.

[2] V. Kac and J. van de Leur, *The n-component KP hierarchy and representation theory*, in: A.S. Fokas, V.E. Zakharov (Eds.), Important Developments in Soliton Theory, Springer-Verlag, Berlin, Heidelberg, 1993.

[3] L.-P. Teo, *The multicomponent KP hierarchy: differential Fay identities and Lax equations*, J. Phys. A: Math. Theor. **44** (2011) 225201.
[4] K. Takasaki and T. Takebe, *Integrable hierarchies and dispersionless limit*, Physica D 235 (2007) 109-125.

[5] E. Date, M. Jimbo, M. Kashiwara and T. Miwa, *Transformation groups for soliton equations*, in: M. Jimbo, T. Miwa (Eds.), Nonlinear Integrable Systems – Classical and Quantum, World Scientific, 1983, pp. 39-120.

[6] M. Jimbo and T. Miwa, *Solitons and infinite dimensional Lie algebras*, Publ. Res. Inst. Math. Sci. Kyoto 19 (1983) 943-1001.

[7] I. Krichever and A. Zabrodin, *Spin generalization of the Ruijsenaars-Schneider model, non-abelian 2D Toda chain and representations of Sklyanin algebra*, Uspekhi Mat. Nauk 50 (1995) 3-56 (in Russian) (English translation: Russ. Math. Surv., 50 (1995) 1101-1150).

[8] I. Krichever, *Periodic non-abelian Toda chain and its two-dimensional generalization*, Uspekhi Mat. Nauk 36 (1981) 72-77 (in Russian) (English translation: appendix to the paper by B.A. Dubrovin *Theta functions and non-linear equations*, Russ. Math. Surv., 36 (1981) 11-92).