Possibly stable configurations of $\Theta^+_5$ in the flux-tube model

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Abstract

In this work, following the literature, we investigate the semi-classical picture where various spatial configurations for $\Theta^+_5$ were suggested by several authors. We use the flux tube model where the effective Column coefficient and string tension are extracted from the lattice QCD calculations, to obtain the interactions among the constituent quarks, which is later to be taken as an effective potential. It is reasonable to assume that keeping the simple spatial shape, only the breathing mode is considered, thus the whole system is much simplified and described by only one variable. Substituting the potential into the Schrödinger equation, we evaluate the binding energy of the system. We find that each of the configuration has higher symmetries and corresponds to a local minimum of energy and one of them results in a mass which is smaller than the others. It is interesting to notice that with such a very simplified semi-classical picture the minimal value corresponds to a mass of 1530 to 1554 MeV for $\Theta^+_5$. The good agreement with the LEPS’ data may indicate that such simplified picture indeed reflects physical reality in some sense.

**Key words** Pentaquark state, potential, flux-tube, toy model

Since the LEPS Collaboration [1] at the SPring8 announced the discovery of a sharp resonance with $M = 1.54 \pm 0.01$ GeV and $\Gamma < 25$ MeV in the $\gamma + n \rightarrow K^+K^-$ process in
2003, it has greatly attracted attentions of physicists. Because its strangeness is \( +1 \), the observed \( \Theta^+_5 \) cannot be a normal baryon with only three valence quarks, nowadays, it is suggested that the newly observed resonance is a five-quark state \( \Theta^+_5 \) \((uudd\bar{s})\) [2]. If it is true, it will open up a new epoch in the study of multi-quark states in hadron physics.

Since then, numerous experimental groups are involved in searching and confirming such a state and its family members through various reaction processes. Meanwhile, theorists have also been trying to explain that experimental finding by employing various theoretical models [2]. Even though remarkable progresses have been made, unfortunately, a common viewpoint is still not reached yet, in other words, the physical properties of the state, such as the structure, the mass, the decay width, the parity and etc. are still in puzzle.

The flux tube model [3] is a model based on the lattice QCD, indeed it establishes a theoretical framework where mesons, baryons and exotics are described in a unified way. Employing such a model to study the five-quark state is obviously reasonable. In fact, there have been several papers on this line. Okiharu et al. [4] suggested that for the lowest five-quark state, the effective potential should be composed of the two-body color-Coulomb interaction and a long-ranged confining potentia, and the five constituents are connected by flux tubes into the double Y-shape, shown in Fig.1(a). Alexandrou et al. [5] proposed an alternative structure that the quarks are connected by flux tubes and the tubes intersect at three Steiner points, shown in Fig.1(b). Song et al. [6] argued that the state should have a tetrahedron (they called it as a ”diamond”) structure depicted in Fig.1(c). Obviously, all the structures are simplified semi-classical scenarios, but they seem to reflect the physics reality in some sense. However, except a simple variational calculation on the \( \Theta^+_5 \) spectrum where a combined two-body interaction is employed[7] has been done, there are almost no quantitative calculations carried out with the above
mentioned models at the present moment. Therefore, it would be interesting and significant to judge what type of the flux tube structure corresponds to the stable one for $\Theta_5^+$ through a numerical evaluation of its mass. In the framework of the flux tube model which is a phenomenological version of the lattice QCD, the constituents in a hadron are connected by color-flux tubes to construct a pattern with the minimal total length of the tube, which provides a confining potential in the Hamiltonian. Meanwhile, the one-gluon-exchange (OGE) interaction is also included in the effective Hamiltonian to remedy the short range interaction. Substituting the Hamiltonian into the Schrödinger equation and solving it, we obtain the binding energy of the system and the mass of $\Theta_5^+$. Especially, we may find whether the tetrahedral structure suggested by Song et al. is the more stable one compared with the others.

In this letter, we suppose that the aforementioned semi-classical flux-tube structures shown in Figs.1(a),(c),(d) and (e) are relative stable, then estimate the mass of $\Theta_5^+$ with these structures.

In the figures, the structure of Fig.1(c) is in a regular tetrahedral form with four quarks sitting at four corners, respectively, and the anti-strange quark staying at the center of the tetrahedron. If twisting the triangle $\triangle_{CDE}$ around the axis $ab$, where $a$ and $b$ are the midpoints of lines $AB$ and $CD$, respectively, by $\pi/2$, one obtains Fig.1(d).

According to Okiharu’s suggestion, in our calculation, preserving the shape of the rectangle, the flux tubes join the quarks and anti-quark into the double Y form shown in Figs.1(a) and 1(e). The difference between these two figures is that in Fig.1(a), the quarks at two ends of the longer edges form color-$\bar{3}$ states, whereas in Fig.1(e), the two quarks at the ends of the shorter edges constitute color-$\bar{3}$ states.

For a 5-quark system, the non-relativistic Hamiltonian of the system can generally be written as $H = T + V$. For the pictures depicted in Fig.1, the four constituent quarks
of the same mass are residing at corners and the anti-strange quark occupies the position of the center of mass of the four quarks. By the simple picture, we further require that each configuration structure is invariant, namely only the breathing mode is considered. That can be understood from the classical angle, namely the relative rotation or twist may cause an drastic energy increase (larger tension), thus the main contribution to the energy of the ground state comes from the simplest, but most significant mode, i.e. the breathing mode. For convenience, the position of the anti-strange quark is set to be the origin of the coordinate system. Let the coordinate vectors of the four quarks be $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$, and $\vec{r}_4$, respectively, and obviously, $\vec{r}_1 + \vec{r}_2 + \vec{r}_3 + \vec{r}_4 = 0$. Then, the kinetic energy term of $H$ can be simplified as

$$
T = -\sum_{j=1}^{4} \frac{\hbar^2}{2m_j} \nabla^2_{\vec{r}_j}.
$$

(1)

Obviously, in this simplified form, we ignore the oscillation of the anti-strange quark about the center, i.e. keeping only the breathing mode. The general form of the potential term of $H$ reads $V = V^{\text{Coul}} + V^{\text{conf}} + C_{5q}$, where $V^{\text{Coul}}$ is the one-gluon-exchange (OGE) Coulomb potential which exists between quark-quark and quark-anti-quark, $V^{\text{conf}}$ is the confinement potential in the flux tube picture and $C_{5q}$ is a zero-point energy. $V^{\text{Coul}}$ for the configurations in Figs.1(a), (c) and (d) reads

$$
V^{\text{Coul}} = -A_{5q}[(\frac{1}{r_{12}} + \frac{1}{r_{34}}) + \frac{1}{2}(\frac{1}{r_{15}} + \frac{1}{r_{25}} + \frac{1}{r_{35}} + \frac{1}{r_{45}}) + \frac{1}{4}(\frac{1}{r_{13}} + \frac{1}{r_{14}} + \frac{1}{r_{23}} + \frac{1}{r_{24}})], \quad (2)
$$

where $r_{ij} = |\vec{r}_i - \vec{r}_j|$. $A_{5q}$ is the effective coefficient of the color Coulomb interaction, and its value can be extracted from the lattice QCD result [4]. For the configuration in Fig.1(e),

$$
V^{\text{Coul}} = -A_{5q}[(\frac{1}{r_{13}} + \frac{1}{r_{24}}) + \frac{1}{2}(\frac{1}{r_{15}} + \frac{1}{r_{25}} + \frac{1}{r_{35}} + \frac{1}{r_{45}}) + \frac{1}{4}(\frac{1}{r_{12}} + \frac{1}{r_{14}} + \frac{1}{r_{23}} + \frac{1}{r_{34}})]. \quad (3)
$$
The confinement term has the form of

$$V_{conf} = \sigma_{5q} L_{min},$$

where $\sigma_{5q}$ is the string tension and also extracted from the lattice QCD [4].

The key point is $L_{min}$ which is the minimal total length of the flux tube and has different values for different configurations. As mentioned above, only the breathing mode is considered in the calculation, we constrain that $|\vec{r}_1| = |\vec{r}_2| = |\vec{r}_3| = |\vec{r}_4| = r$, and the shape of the configuration of the system remains invariant, namely the system possesses only one degree of freedom. In this approximation, the expression of $L_{min}$ is greatly simplified

$$L_{min} = \begin{cases} 4r & \text{for Figs.1(a) and (c)} \\ \frac{2}{3}(\sqrt{3} + 3\sqrt{2})r & \text{for Fig.1(d)} \\ \frac{2}{3}(3 + \sqrt{6})r & \text{for Fig.1(e)} \end{cases}$$

In our calculation, we ask all the quarks staying in the s-wave and there is no any orbital excitation. The values of parameters $A_{5q}$ and $\sigma_{5q}$ are borrowed from Okiharu’s paper [4]

$$A_{5q} = 0.1366, \quad \sigma_{5q} = 0.165 GeV^2.$$  

Constant $C_{5q}$ is determined in the following way: By calculating the mass of nucleon with the same form of potential structure, namely only the Coulomb interaction and the Y-shape confining interaction are included, the Coulomb coefficient and the string tension of the Y form confinement are taken to be $A_{3q} = A_{5q}$ and $\sigma_{3q} = \sigma_{5q}$, respectively, we extract $C_{3q} = -1.801 GeV$. Therefore, we obtain $C_{5q} = \frac{5}{3}C_{3q} = -3.002 GeV$. Then there is no free parameter in the potential anymore. Substituting the potentials and kinetic piece into the Schrödinger equation and solving

$$(T + V)\psi = E\psi$$

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for the pentaquark system with \( m_q = 0.33 GeV \) and \( m_s = 0.42 GeV \) [9], we obtain the mass of \( \Theta_5^+ \) as

\[
M_{\Theta_5^+} = 4m_q + m_s + E. \tag{7}
\]

The resultant masses of \( \Theta_5^+ \) in various configurations are tabulated in Table 1.

| Configuration | Fig.1(a) | Fig.1(c) | Fig.1(d) | Fig.1(e) |
|---------------|----------|----------|----------|----------|
| \( M_{\Theta_5^+} \) (GeV) | 1.743 | 1.756 | 1.754 | 1.537 |

It is seen that although the calculation is based on a very simplified scenario, the obtained mass of the pentaquark is well consistent with the LEPS' data [1]. It is not simply an coincidence or obtaining a result by fitting data, the plausibility of the approach is obvious.

Among the four configurations considered in the text, the mass of the pentaquark with the structure shown in Fig 1(c), i.e. the tetrahedron, has the largest value, which is about 0.216 GeV above the central value of the mass measured by the LEPS collaboration [1]. As pointed by Okiharu [4], we find that the masses of the twisted non-planar configuration (Fig. 1(c)) and the corresponding planar configuration (Fig. 1(d)) are almost degenerate, and the planar one is slightly smaller. A 0.003 GeV difference comes from the difference of the color Coulomb energy. From the point of view that energy should take the minimal value, the planar structure is more favorite than others.

Indeed, one can note that the absolute value of the Coulomb energy is inversely proportional to the distance between constituents, whereas the confinement part is proportional
to the total length of the flux tubes, the two pieces compete with each other and minimization of their sum results in a minimal binding energy. When one investigates the configurations to determine which one would be the most favorite one, he cannot simply decide by comparing only $L_{\text{min}}$, but needs to consider the contributions from both.

The total length of the flux tube with double Y-shape shown in Fig.1(a) is shorter than that with the X-shape shown in Fig.1(d), the resultant mass of the former configuration (Fig.1(a)) is 0.011$GeV$ smaller than that of the later (Fig.1(b)). If we keep the shape of the configuration unchanged, the total length of the flux tube in Fig.1(e) is even shorter. In addition, the concerned color Coulomb energy is smaller compared with that resulted in by the configuration Fig.1(a), the mass of the configuration in Fig.1(e) becomes the smallest one among four calculated masses. This means that the most favorite configuration is the one in Fig.1(e). The obtained mass agrees with the LEPS’ data value.

Moreover, when the masses of quarks increase, the direct contribution from the constituent masses of quarks to $M_{\Theta}^{+}$ increases, but the eigen energy of the system decreases because then the kinetic energy contribution which is inversely proportional to the constituent mass, takes a leading role. Thus, there may exist an optimal value for the quark mass in the physical region. We present dependence of the mass of $\Theta_{5}^{+}$ on the quark mass in Table 2. From the result, one sees that when $m_q = 0.29$ GeV, the minimal $M_{\Theta}^{+}$

| $m_q (MeV)$ | 0.25 | 0.27 | 0.29 | 0.31 | 0.33 | 0.35 |
|-------------|------|------|------|------|------|------|
| $M_{\Theta}^{+} (GeV)$ | 1.534 | 1.524 | 1.523 | 1.527 | 1.539 | 1.554 |
is 1.523 GeV. Because the eigen-energy of the system is independent of the mass of the anti-strange quark, if $M_{\Theta^+_5} = 1.54$ GeV is confirmed, at this energy scale $m_{u(d)} = 0.29$ GeV and $m_s = 0.44$ GeV would be the best fit. In fact, from Table 2, one can note that as $m_q$ varies from 0.25 MeV to 0.35 MeV, the mass of $\Theta^+_5$ only changes by 1.3%, namely is insensitive to $m_q$. But, we still can see that there exists an optimal value for the mass of the light quark, which is slightly smaller than that in the three quark system. In our calculation, there is no fine tuning at all.

As mentioned above, in our model all the constituent quarks stay at $S$-wave, and no orbital excitation is considered. The parity of the pentaquark is negative which comes from the parity of the antiquark.

In summary, in the framework of the flux tube model where the effective coefficient of the Coulomb interaction and the string tension are extracted from the lattice QCD calculation, an estimation based on the simplified semi-classical configurations for the quark-antiquark disposal shows that $\Theta^+_5$ favors the planar configuration, where two quarks at the ends of the shorter edges of the rectangle form color-$\bar{3}$ states, over the tetrahedral configuration. And in this scenario, the parity of $\Theta^+_5$ is negative. Certainly, as a simplified version, a strong constraint is enforced on the possible motion mode of the pentaquark system, i.e. only the breathing mode is accounted for. Calculations based on more realistic structures of a five quark system would be much more complex. A more sophisticated calculation with the structure and the width of the pentaquark is our under-going project.

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Figure 1: Various configuration structures of pentaquarks.