NOTE ON NAKAYAMA AUTOMORPHISMS OF
PBW DEFORMATIONS AND HOPF ACTIONS

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Abstract. PBW deformations of Artin-Schelter regular algebras are skew Calabi-Yau. We prove
that the Nakayama automorphisms of such PBW deformations can be obtained from their homog-
enizations. Some Calabi-Yau properties are generalized without Koszul assumption. We also show
that the Nakayama automorphisms of such PBW deformations control Hopf actions on them.

INTRODUCTION

Skew Calabi-Yau algebras, as a generalization of Calabi-Yau algebras, attract lots of researchers to
study. The latest paper [19] introduced a definition of graded skew Calabi-Yau category. Paralleling
with Calabi-Yau algebras, [19, Theorem 3.5] presented that certain full triangulated subcategories
of derived categories of a large class of algebras, which is called generalized Artin-Schelter Goren-
stein algebras, are graded skew Calabi-Yau categories. Such algebras possess a kind of specific
automorphisms called the Nakayama automorphisms. This is one of significant features of skew
Calabi-Yau algebras. When restricted on connected graded case, skew Calabi-Yau algebras are just
Artin-Schelter regular algebras. The recent papers [6, 16] showed that Nakayama automorphisms
control Hopf actions on Artin-Schelter regular algebras.

Since the Nakayama automorphism is a well behaved invariant, how to calculate becomes a pri-
mary issue. Fortunately, there are many homological identities to help obtain the Nakayama auto-
morphisms of noetherian Artin-Schelter regular algebras (see [18], for example). In [15], the authors
acquire the Nakayama automorphisms under Ore extensions. In fact, the Nakayama automorphisms
of noetherian algebras are related to rigid dualizing complexes. Using the particular dualizing com-
plexes, the paper [24, Proposition 6.18] proved that an algebra whose associated graded algebra
is connected noetherian Artin-Schelter regular for some filtration is endowed with a Nakayama
automorphism induced by the one of associated Rees algebra. PBW deformations of noetherian
Artin-Schelter regular algebras satisfy those conditions, such as the Weyl algebras and the ungraded
Down-Up algebras. The paper [8] told that such PBW deformations are skew Calabi-Yau algebras.
In the case of Koszul and low dimension, paper [9] has described the Nakayama automorphisms of
such PBW deformations explicitly. Those Nakayama automorphisms are induced by the ones of
the Rees algebras, but the Rees algebras are not easy to deal with in general. Our plan from the
beginning was to seek a handy method for calculating the Nakayama automorphisms of such PBW
deformations.

2000 Mathematics Subject Classification. Primary 16E65, 16S80; Secondary 16W30.

Key words and phrases. Nakayama automorphisms, PBW deformations, Artin-Schelter regular algebras, Homog-
enizations, Hopf actions.
We realize that homogenizations of PBW deformations will be one of the options that meet our requirements. By using the homogenizations, we give a different way to show, without noetherian assumption, that a PBW deformation of an Artin-Schelter regular algebra is skew Calabi-Yau with the Nakayama automorphism induced from the homogenization (see Theorem 2.3). The proof does not involve rigid dualizing complexes and localizations, and it is different from the method for the Rees algebras. This result is available and effective for computation in practice. Moreover, using the homogenization approach, we generalize some Calabi-Yau properties handily in Corollary 2.11.

In recent years, Hopf actions on low dimensional Artin-Schelter regular algebras have been made progress (see [6, 16]). An extensive question is Hopf actions on filtered Artin-Schelter regular algebras. The case of 2-dimension has been done in [5]. However, it becomes a tough and tedious work even for 3-dimensional case. Analogous to [16], we give some conditions to characterize Hopf algebras which act on PBW deformations of noetherian Artin-Schelter regular algebras. It implies their Nakayama automorphisms also control Hopf actions. We achieve it also with the help of homogenization.

Here is an outline of the paper. In Section 1, we recall definitions of skew Calabi-Yau algebras and Artin-Schelter regular algebras and review PBW deformations in briefly. Section 2 is devoted to calculating Nakayama automorphisms of PBW deformations of Artin-Schelter regular algebras. We reprove that the Weyl algebras are Calabi-Yau algebras and obtain the Nakayama automorphisms of ungraded Down-Up algebras and some other examples. We also generalize some Calabi-Yau properties without Koszul hypothesis. We mainly consider Hopf actions and give some conditions to determine whether Hopf algebras are group algebras in Section 3.

Throughout the paper, $k$ is an algebraically closed field of characteristic 0. All algebras and Hopf algebras are over $k$. Unless otherwise stated, tensor product $\otimes$ means $\otimes_k$.

1. Preliminaries

Let $A$ be an algebra, and let $A^e$ be the enveloping algebra $A \otimes A^\circ$ where $A^\circ$ is the opposite algebra of $A$. Let $M$ be a left (resp. right) $A^\circ$-module, so it is a left and right $A$-module (resp. $A^\circ$-module). A left $A^\circ$-module and a right $A^\circ$-module can be identified because of $A^\circ \cong (A^\circ)^\circ$. For two automorphisms $\mu, \nu$ of $A$, the twisted module $\mu M\nu$ is defined such that it is just $M$ as $k$-vector spaces, and the module structure becomes $a \ast m \ast b = \mu(a)m\nu(b)$ for any $a, b \in A$ and $m \in M$.

Suppose $A$ is also a graded algebra, that is, $A = \bigoplus_{i \in \mathbb{Z}} A_i$. For any $i \in \mathbb{Z}$, each element in $A_i$ is called homogeneous element with degree $i$. We say $A$ is connected if $A_i = 0$ for all $i < 0$ and $A_0 = k$. Let $M$ be a left graded $A$-module. For some integer $i$, shift of $M$ by degree $i$ is $M(i)$, defined by $M(i)_j = M_{i+j}$ for any $j$. Let $\sigma$ be a graded automorphism, then graded twisted algebra $A^\sigma$ is defined such that $A^\sigma = A$ as vector spaces, and the multiplication satisfying $a \ast b = \sigma^{\deg b}(a)b$.

**Definition 1.1.** An algebra $A$ is called skew Calabi-Yau (skew CY, for short) if

(a) $A$ is homologically smooth, that is, $A$ has a finite length projective resolution as left $A^\circ$-module such that each term is finitely generated; 
(b) there exists an automorphism $\mu$ of $A$ such that 

\[
\text{Ext}^i_{A^\circ}(A, A^\circ) \cong \begin{cases} 
1 A^\mu & \text{if } i = d, \\
0 & \text{if } i \neq d
\end{cases}
\]
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Remark 1.2. (a) If an algebra $A$ satisfies (1.1.1), then we say $A$ has a Nakayama automorphism $\mu$ (always denoted by $\mu_A$). It is unique up to inner automorphisms of $A$ if it exists.
(b) Suppose an algebra $A$ has a Nakayama automorphism $\mu_A$. The proof of [24, Corollary 3.6] still works and implies $\mu_A(z) = z$ for any central element $z \in A$.
(c) An algebra $A$ is Calabi-Yau (CY, for short) if and only if $A$ is skew CY and the identity map is a Nakayama automorphism.
(d) If $A$ is a graded algebra, then the definition of skew CY algebra should be in the category of graded $A^e$-modules, and (1.1.1) should be replaced by
\[
\text{Ext}^i_{A^e}(A, A^e) \cong \begin{cases} 
1A^e(l) & \text{if } i = d, \\
0 & \text{if } i \neq d
\end{cases}
\]
for some $l \in \mathbb{Z}$. The Nakayama automorphism $\mu$ is also a graded isomorphism.

Definition 1.3. A connected graded algebra $A$ is Artin-Schelter Gorenstein (AS-Gorenstein, for short) of dimension $d$ if the following conditions hold:
(a) $A$ has finite injective dimension $d$;
(b) $\text{Ext}^d_A(k, A) = k(l)$ and $\text{Ext}^i_A(k, A) = 0$ if $i \neq d$ for some $l \in \mathbb{Z}$. The integer $l$ is called the Gorenstein parameter.
(c) The right version of (b) holds.

In addition, $A$ has finite global dimension $d$, then $A$ is Artin-Schelter regular (AS-regular, for short).

Theorem 1.4. [18, Lemma 1.2] Let $A$ be a connected graded algebra. Then $A$ is AS-regular if and only if $A$ is skew CY.

The theorem shows that AS-regular algebras are naturally endowed with Nakayama automorphisms. Actually, any noetherian AS-Gorenstein algebras have Nakayama automorphisms which can be characterized by balanced dualizing complexes. Van den Bergh introduced rigid dualizing complexes in [22] especially for ungraded noetherian algebras. It is easy to check that a noetherian algebra $A$ has finite injective dimension with Nakayama automorphism $\mu_A$ if and only if $A$ has a rigid dualizing complex $1A^e\mu_A^{-1}[d]$ for some integer $d$ (consider $1A^e\mu_A^{-1}$ as a complex concentrated in degree 0. The symbol $[ ]$ means a shift of complex).

Now we review the definition of PBW deformations. Let $A = k\langle X \rangle/(R)$ be a connected graded $k$-algebra generated by degree 1, where $X = \{x_1, x_2, \cdots, x_n\}$ and $R = \{r_1, r_2, \cdots, r_s\}$. Let $X^*$ be the free monoid generated by $X$ including 1. Define a new algebra $U = k\langle X \rangle/(P)$, where
\[P = \{r_i + r'_i | r'_i \in k\langle X \rangle \text{ and } \deg r'_i < \deg r_i, \ i = 1, \cdots, s\}.
\]

There is a canonical filtration on $U$ induced by degree (or length) of monomials in $X^*$. Let
\[F_i U = \frac{F_i k\langle X \rangle + (P)}{(P)},\]
where $F_i k\langle X \rangle = \text{Span}_k \{u \in X^* | \deg u \leq i\}$ if $i \geq 0$, and $F_i U = 0$ if $i < 0$. The associated graded algebra $\text{gr} U = \bigoplus_{i \in \mathbb{Z}} F_i U/F_{i-1} U$, and the associated Rees algebra $\text{Rees}_F U = \bigoplus_{i \geq 0} F_i U t^i$. It is
well known $\operatorname{gr} U \cong \operatorname{Rees}_F U/(t)$ and $U \cong \operatorname{Rees}_F U/(1-t)$. And we have a natural graded surjective homomorphism

$$\phi : A \rightarrow \operatorname{gr} U.$$  

**Definition 1.5.** Retain the notations above. We say $U$ is a PBW deformation of $A$ if $\phi$ is a graded isomorphism.

Papers $[3, 7]$ presented the necessary and sufficient conditions on whether an algebra is a PBW deformation of Koszul algebra and of $N$-Koszul algebra respectively. In general, we can describe it by Gröbner basis. Let $x_1 < x_2 < \cdots < x_n$. Fix the ordering $<$ on $X^*$ to be deg-lex order, that is, $u < v$ if $\deg u < \deg v$ or $\deg u = \deg v$ and there exist factorizations $u = rx_is,v = rx_jt$ such that $i < j$ where $r,s,t \in X^*$ for any $u,v \in X^*$. Let $f = \sum_{i=1}^s f_i \in k(X)$ where each nonzero $f_i$ is a homogeneous polynomial and $\deg f_1 < \deg f_2 < \cdots < \deg f_s$, we call $f_s$ the leading homogeneous polynomial of $f$, which is denoted by $LH(f)$. And for any set $S$ of polynomials, $LH(S) = \{ LH(f) \mid f \in S \}$. Then $\operatorname{gr} U \cong k(X)/LH(G)$ by $[14]$ Chapter 4, Theorem 2.3], where $G$ is the Gröbner basis of $(P)$ with respect to $<$. In other words, $U$ is a PBW deformation of $A$ if and only if $LH(G)$ is a Gröbner basis of $(R)$ where $G$ is the Gröbner basis of $(P)$.

The main property we needed for PBW deformations is the following result.

**Theorem 1.6.** $[8]$ Corollary 3.5] Let $A$ be a connected graded noetherian skew CY algebra, and let $U$ be a PBW deformation of $A$. Then $U$ is a skew CY algebra.

**Remark 1.7.** The proof of this theorem in $[8]$ depends on rigid dualizing complexes. Hence it should be restricted on noetherian algebras which is lost in the original statement. We will give another proof in Theorem 2.3 without noetherian assumption.

### 2. Homogenizations and Nakayama Automorphisms

In this section, we introduce the definition of homogenization algebras. Then using the homogenization algebras to compute the Nakayama automorphisms of PBW deformations of AS-regular algebras.

Denote the free algebra $k(X)$ by $F$ where $X = \{x_1, x_2, \cdots, x_n\}$ with $\deg x_i = 1$. Consider the polynomial extension $F[t]$ of $F$ with $\deg t = 1$. Let $f$ be a polynomial of degree $s$ in $F$, then $f = \sum_{i=0}^s f_i$, where each $f_i$ is a homogeneous polynomial of degree $i$. Now we define a corresponding element $f^t = \sum_{i=0}^s f_i t^{s-i}$ in $F[t]$ which is homogeneous in $F[t]$. We say $f^t$ is the homogenization of $f$ and the central element $t$ is the homogenization element.

**Definition 2.1.** Retain the notations above. Let $U = F/(f_1, \cdots, f_m)$ be an algebra, where $f_1, \cdots, f_m \in F$. The graded algebra $H(U) = F[t]/(f_1^t, \cdots, f_m^t)$ is called the homogenization of $U$.

It is easy to know $H(U)/(1-t) \cong U$. If $U$ is a PBW deformation of a connected graded algebra $A$, then $H(U)/(t) \cong A \cong \operatorname{gr} U$. Firstly, we show that the Nakayama automorphisms of some algebras can be deduced form their regular central extension.

**Lemma 2.2.** Let $B$ be an algebra with a Nakayama automorphism $\mu_B$. Suppose $z$ is a regular central element in $B$ such that $B$ is a flat module over $k[z]$. Then $B/(z)$ has a Nakayama automorphism $\mu_{B/(z)}$ equals $\mu_B$ when induced on $B/(z)$.
Proof. Let \( d \) be the integer such that \( \text{Ext}^d_B(B, B^e) \cong \bar{B}^\mu_B \). Write \( \bar{B} = B/(z) \). There exists a well-defined automorphism \( \mu_B \) on \( B \) induced by \( \mu_B \) because of \( \mu_B(z) = z \). We have a free resolution of \( \bar{B} \) as left \( B \)-modules which is also a resolution as right \( B \)-modules:

\[
P_\mu: \quad 0 \rightarrow B \xrightarrow{z} B \rightarrow \bar{B} \rightarrow 0.
\]

Similarly, denote by \( P^\mu_B \) the analogous resolution of \( \bar{B} \) as left \( B \)-modules, and also as right \( B \)-modules. Thus \( \text{Ext}^i_B(B, B) = \text{Ext}^i_{B^e}(B, B^e) = 0 \) for all \( i \neq 1 \), \( \text{Ext}^1_B(B, \bar{B}) \cong \bar{B} \) as left \( B \)-modules, and \( \text{Ext}^1_{B^e}(\bar{B}, \bar{B}) \cong \bar{B} \) as left \( (B^e)^e \)-modules.

Notice that \( P_\mu \otimes P^\mu_B \) is a free resolution of \( \bar{B} \otimes \bar{B} \) as left \( B^e \)-modules and a free resolution as right \( B^e \)-modules. So the spectral sequence implies \( \text{Ext}^i_{B^e}(\bar{B}^e, B^e) = 0 \) if \( i \neq 2 \) and \( \text{Ext}^2_{B^e}(B^e, B^e) \cong \bar{B} \) as left \( (B^e)^e \) = \( B^e \otimes (B^e)^e \)-modules, also as left \( (B^e)^e \)-modules.

Take a free resolution \( Q \), of \( \bar{B} \) as left \( B \)-modules and an injective resolution \( I \) of \( B \) as left \( (B^e)^e \)-modules. The complex \( Q \), is also quasi-isomorphic to \( \bar{B} \) as left \( B \)-modules, and \( I \) is also an injective resolution of \( B \) when restricted to left \( B \)-modules. For all \( i \geq 0 \) and as right \( B \)-modules, also as left \( B \)-modules,

\[
\text{Ext}^i_{B^e}(\bar{B}, B^e) \cong H^i(Q_{C^e}, \text{Ext}^2_{B^e}(\bar{B}, B^e))
\]

\[
\cong H^i(Q_{C^e}, H^2(\text{Hom}_{B^e}(B^e, I^e)))
\]

\[
\cong H^{i+2}(Q_{C^e}, \text{Hom}_{B^e}(B, I^e))
\]

\[
\cong H^{i+2}(Q_{C^e}, \text{Hom}_{B^e}(\bar{B} \otimes \bar{B}, Q_{C^e}))
\]

\[
\cong \text{Ext}^{i+2}_{B^e}(\bar{B}, B^e).
\]

Write \( w = z \otimes 1 \in B^e \). Then \( w \) is a regular central element in \( B^e \), so \( B^e \) is a \( k[w] \)-algebra. With the natural right \( k[w] \)-module structure on \( B \), it is also a \( B^e \otimes k[w] \)-bimodule.

Let \( C \), be a free resolution of \( B \) as left \( B^e \)-\( k[w] \)-bimodules, and take a free resolution of \( k \) as \( k[w] \)-modules as follows,

\[
C^e: \quad 0 \rightarrow k[w] \xrightarrow{w} k[w] \rightarrow k \rightarrow 0.
\]

Tensoring \( B \) to \( C^e \), the regularity of \( w \) implies \( \bar{B} \cong B \otimes k[w] k \) as left \( B^e \)-modules.

Since \( B \) is a right flat module over \( k[w] \), \( B \) and \( B^e \) are right flat \( k[w] \)-modules. Thus complex \( D_e = C \otimes k[w] C^e \) is a free resolution of \( B \otimes k[w] k \) as left \( B^e \)-modules. Applying functor \( \text{Hom}_{B^e}(-, B^e) \) to \( D_e \), the flatness of \( B \) and \( B^e \) and the spectral sequence implies as left \( B^e \)-modules

\[
\text{Ext}^{d+1}_{B^e}(\bar{B}, B^e) \cong \text{Ext}^{d+1}_{B^e}(\bar{B} \otimes k[w] k, B^e \otimes k[w] k[w])
\]

\[
\cong \text{Ext}^{d}_{B^e}(B, B^e) \otimes k[w] \text{Ext}^{1}_{k[w]}(k, k[w])
\]

\[
\cong \text{Ext}^{1}_{B^e}(B^e \otimes k[w] k)
\]

\[
\cong B^{\mu_B} \otimes k[w] k.
\]

and \( \text{Ext}^i_{B^e}(\bar{B}, B^e) = 0 \) if \( i \neq d + 1 \).

Consequently, \( \text{Ext}^i_{B^e}(\bar{B}, B^e) = 0 \) if \( i \neq d - 1 \), and \( \text{Ext}^{d-1}_{B^e}(\bar{B}, B^e) \cong \text{Ext}^1_{B^e}(B^e) \cong B^{\mu_B} \).

If \( U \) is a PBW deformation of a connected graded algebra \( A \), then the homogenization \( H(U) \) is a regular central extension of \( A \) by \( \mathbb{A} \), Theorem 1.3]. That is to say, homogenization element \( t \)
is regular, so $H(U)$ is a flat $k[t^i]$-module for any $i \geq 1$ by [20 Chapter 3, Corollary 3.51]. So the condition flatness of Lemma 2.3 is always automatically satisfied for homogenizations.

The main result of this section is the following, which is a new version of Theorem 1.6 with no noetherian assumption on the algebra $A$.

**Theorem 2.3.** Let $U$ be a PBW deformation of an AS-regular algebra $A = k\langle X \rangle/(R)$, $H(U)$ be the homogenization algebra of $U$. Then $H(U)$ is skew CY with the Nakayama automorphism $\mu_{H(U)}$, and $U$ is skew CY with a Nakayama automorphism $\mu_U$ induced by $\mu_{H(U)}$.

**Proof.** Write $X = \{x_1, x_2, \ldots, x_n\}$ and $H = H(U)$. Suppose $A$ has global dimension $d$. By [4 Theorem 1.3], $t$ is a central regular element in $H$. So $H$ is AS-regular of dimension $d + 1$ by [25 Lemma 3.5] and skew CY with Nakayama automorphism $\mu_H$ by Theorem [14].

By Theorem [14] $A$ has a finite length of finitely generated projective resolution as left $A^e$-modules. Since $\text{gr } U \cong A$, $\text{gr } U^e \cong A^e$ for a natural filtration on $U^e$. Thus there is an associated finite length of finitely generated projective resolution of $U$ as left $U^e$-modules by [11 Chapter 2, Proposition 2.5]. Then it remains to show that $U$ has a Nakayama automorphism.

The element $t$ is regular central and $1 - t$ is central in $H$ with $\mu_H(1 - t) = 1 - t$. Write $z = 1 - t$. For any polynomial $f \in k[z]$ and any nonzero element $h \in H$, it is easy to check that $fh = 0$ implies $f = 0$. So $H$ is a flat $k[z]$-module by [20 Chapter 3, Corollary 3.51]. Thus $U \cong H/(1 - t)$ also has a Nakayama automorphism $\mu_U$ induced by $\mu_H$ from Lemma 2.3.

More precisely, if $\mu_H(x_i) = \sum_{j=1}^n a_{ij}x_j + b_it$ for some $a_{ij}, b_i \in k$, then $\mu_U(x_i) = \sum_{j=1}^n a_{ij}x_j + b_i$ for all $i = 1, \ldots, n$.

Immediately, we have a CY property about $U$.

**Corollary 2.4.** Let $U$ be a PBW deformation of an AS-regular algebra $A$, and let $H(U)$ be the homogenization of $U$. If $H(U)$ is CY, then $U$ and $A$ are CY.

The following lemma is proved for $N$-Koszul algebras in [9], we give a general situation.

**Lemma 2.5.** Let $U$ be a PBW deformation of an AS-regular algebra $A = k\langle X \rangle/(R)$ with the Nakayama automorphism $\mu_A$. Then there exists a Nakayama automorphism $\mu_U$ of $U$ preserving the canonical filtration on $U$ and $\text{gr}(\mu_U) = \mu_A$.

In addition, if $A$ is domain then the filtration-preserving Nakayama automorphism $\mu_U$ is unique.

**Proof.** Write $X = \{x_1, x_2, \ldots, x_n\}$ and $\mu_A(x_i) = \sum_{j=1}^n a_{ij}x_j$ for some $a_{ij} \in k$. Let $H$ be the homogenization of $U$ with the Nakayama automorphism of $\mu_H$. Since $H/(t) \cong A$ and Lemma 2.3 we have $\mu_H(x_i) = \sum_{j=1}^n a_{ij}x_j + b_it$ for some $b_i \in k$. Now using Theorem 2.3 $\mu_U(x_i) = \sum_{j=1}^n a_{ij}x_j + b_i$. It is a filtration preserving automorphism and $\text{gr}(\mu_U) = \mu_A$.

Assume $A$ is domain, so is $U$. Suppose $\mu_U$ and $\mu'_U$ are two Nakayama automorphisms of $U$ preserving the filtration such that $\text{gr}(\mu_U) = \text{gr}(\mu'_U) = \mu_A$. However, the Nakayama automorphism is unique up to inner isomorphisms, there exists an invertible elements $w \in U$ such that $\mu_U(x_i) = w\mu'_U(x_i)w^{-1}$. Consider the image of $w\mu'_U(x_i)w^{-1}$ in $A$ should be nonzero, it forces $w, w^{-1} \in F_0U = k$. Thus $\mu_U = \mu'_U$. 

□
Although the filtration-preserving Nakayama automorphism of PBW deformation $U$ is not unique in general, we always choose the Nakayama automorphism $\mu_U$ of $U$ as in Theorem 2.3 in the sequel. In this case, when the Nakayama automorphism $\mu$ of homogenization $H$ applies to the generators of $H$, $\mu_H(x_i)$ has the form $\mu(x_i)^t$ for every $x_i \in X$.

**Example 2.6.** Let quantum plane $k_q[x_1, x_2] = k\langle x_1, x_2 \rangle/(x_1 x_2 - q x_2 x_1)$ where $q$ is a scalar in $k^\times$ with the Nakayama automorphism sending $x_1 \mapsto q x_1$ and $x_2 \mapsto q^{-1} x_2$. We consider PBW deformations of it. Assume $q \neq 1$. By definition,

$$U = k\langle x_1, x_2 \rangle/(x_1 x_2 - q x_2 x_1 + ax_1 + bx_2 + c)$$

are all PBW deformations where $a, b, c \in k$. Let $H$ be the homogenization of $U$. Then $k_q[x_1, x_2] \cong H/(t)$. So by Lemma 2.2 the Nakayama automorphism $\mu_H$ of $H$ is

$$\mu_H(x) = q x_1 + d_1 t, \quad \mu_H(x_2) = q^{-1} x_2 + d_2 t,$$

for some $d_1, d_2 \in k$. By Theorem 2.3 $U$ has a Nakayama automorphism $\mu_U$ satisfying

$$\mu_U(x_1) = q x_1 + d_1, \quad \mu_U(x_2) = q^{-1} x_2 + d_2.$$

However, $\mu_U$ is a well-defined algebra homomorphism, so it needs $d_1 = -b$ and $d_2 = q^{-1} a$. Thus $U$ has a Nakayama automorphism sending $x_1 \mapsto q x_1 - b$ and $x_2 \mapsto q^{-1} x_2 + q^{-1} a$.

In the following, we will find some examples whose Nakayama automorphisms of AS-regular algebras and ones of PBW deformations of them have the same form. Here are some general conditions.

**Proposition 2.7.** Let $X = \{x_1, x_2, \cdots, x_n\}$ and let $U$ be a PBW deformation of an AS-regular algebra $A = k(X)/(R)$. Let $H$ be the homogenization of $U$. Assume the Nakayama automorphism $\mu_A$ of $A$ such that $\mu_A(x_i) = \sum_{j=1}^n a_{ij} x_j$ for some $a_{ij} \in k$ and all $i = 1, 2, \cdots, n$.

(a) If $H/(t^2) \cong A[t]/(t^2)$, then there exists a Nakayama automorphism $\mu_U$ of $U$ such that $\mu_U(x_i) = \sum_{j=1}^n a_{ij} x_j$ for any $i = 1, \cdots, n$.

(b) Let $J$ be the Jordan canonical form of the matrix $M = (a_{ij})$. Suppose $M$ has no eigenvalue 1, then there exists a basis $\{v_1, v_2, \cdots, v_n\}$ of $F_1 U = k \oplus k\langle X \rangle$ such that $\mu_U(v) = Jv$ where the column vector $v = (v_1, v_2, \cdots, v_n)^T$.

**Proof.** By Lemma 2.2 and Theorem 2.3 the Nakayama automorphism $\mu_H$ of $H$ satisfies $\mu_H(x_i) = \sum_{j=1}^n a_{ij} x_j + b_i t$ and there exists a Nakayama automorphism $\mu_U$ of $U$ such that $\mu_U(x_i) = \sum_{j=1}^n a_{ij} x_j + b_i$ where $b_i \in k$ and $i = 1, \cdots, n$.

(a) By Lemma 2.2 we have

$$\mu_H = \mu_H/(t^2) = \mu_A[t]/(t^2) = \mu_A[t]$$

when all maps are induced on $H/(t^2)$. Thus $\mu_H(x_i) = \sum_{j=1}^n a_{ij} x_j$. So $b_i = 0$ for all $i = 1, \cdots, n$.

(b) Write the matrix $N = \begin{pmatrix} 1 & 0 \\ b & M \end{pmatrix}$, which is the corresponding matrix of $\mu_H$ with respect to the basis $\{t, x_1, \cdots, x_n\}$, where $b = (b_1, b_2, \cdots, b_n)^T$. If $M$ has no eigenvalue 1, then $N$ has the Jordan canonical form $\text{diag}(1, J)$. The conclusion follows. \qed
By Example 5.3). The following are two special cases: 

Example 2.9. The Down-Up algebra $A(\alpha, \beta, \gamma) = k[x_1, x_2]/(f_1, f_2)$ was introduced in [2] firstly, where

$$f_1 = x_1^2 - 2x_2 x_1 - \alpha x_1 x_2 x_1 - \beta x_2 x_1^2 - \gamma x_1,$$

and $\alpha, \beta, \gamma \in k$. We assume $\beta \neq 0$, so $A(\alpha, \beta, \gamma)$ is noetherian and $A(\alpha, \beta, 0)$ is 3-dimensional AS-regular ([13]). One can easily check that $A(\alpha, \beta, \gamma)$ is a PBW deformation of $A(\alpha, \beta, 0)$. Write $B = A(\alpha, \beta, 0)[t]$. By Example A(6) in [10], the Nakayama automorphism $\mu_B$ of $B$ satisfies

$$\mu_B(x_1) = -\beta x_1, \quad \mu_B(x_2) = -\beta^{-1} x_2, \quad \mu_B(t) = t.$$

Let $H$ be the homogenization of $A(\alpha, \beta, \gamma)$. Then $H/(t^2) \cong B/(t^2)$. Now by Proposition 2.7(a), we know the Nakayama automorphism of $A(\alpha, \beta, \gamma)$ is

$$x_1 \mapsto -\beta x_1, \quad x_2 \mapsto -\beta^{-1} x_2.$$

Moreover, $A(\alpha, \beta, \gamma)$ is CY if and only if $\beta = -1$.

Following we investigate examples with non-diagonalizable Nakayama automorphisms.

Example 2.10. Starting with graded Down-Up algebras, there are four classes of Artin-Schelter regular algebras which are homogeneous PBW deformations of graded Down-Up algebras (see [21, Example 5.3]). The following are two special cases: $\mathfrak{B} = k[x_1, x_2]/(f_1, f_2)$ where

$$f_1 = x_1^2 x_2 - 2x_2 x_1 x_2 + x_2^2 x_1,$$

$$f_2 = x_1 x_2^2 - x_2 x_1,$$

and $\mathfrak{C} = k[x_1, x_2]/(g_1, g_2)$ where

$$g_1 = -x_1^2 x_2 - 2x_1 x_2 x_1 + x_2 x_1 x_2 + x_2^2 x_1,$$

$$g_2 = x_1 x_2^2 - 2x_2 x_1 x_2 + x_2^2 x_1.$$

The element $z = x_1 x_2 - x_2 x_1$ is normal in both $\mathfrak{B}$ and $\mathfrak{C}$. Then

$$\mathfrak{B}/(z) \cong \mathfrak{C}/(z) \cong k[x_1, x_2].$$

By [18, Lemma 1.5], we have the Nakayama automorphism of $\mathfrak{B}$ sending $x_1 \mapsto -x_1 - x_2$, $x_2 \mapsto -x_2$, and the Nakayama automorphism of $\mathfrak{C}$ sending $x_1 \mapsto x_1 + x_2$, $x_2 \mapsto x_2$. 

It has been proved that the Weyl algebras are CY in [1]. Also [3, 15] gave different approaches to the conclusion. However, the proof of [15, 5] used the Koszul dual. The authors of [11] defined a kind of special quadratic algebras and studied the CY property of their PBW deformations which contains the Weyl algebras. A more general way to obtain the Nakayama automorphisms of the Weyl algebras is using Ore extension (see [15, Remark 4.2]). Here we use homogenization to calculate the Nakayama automorphisms of the Weyl algebras.

**Corollary 2.8.** Weyl algebra $A_n(k)$ is CY.

Proof. Weyl algebra $A_n(k)$ is a PBW deformation of $B$, where $B = k[x_1, \ldots, x_n, y_1, \ldots, y_n]$. Let $H$ be the homogenization of $A_n(k)$. It is clear that $H/(t^2) \cong B[t]/(t^2)$. However the Nakayama automorphism of $B$ is identity, so $A_n(k)$ is CY by Proposition 2.7(a). 

$\square$

**Example 2.9.** The Down-Up algebra $A(\alpha, \beta, \gamma) = k[x_1, x_2]/(f_1, f_2)$ was introduced in [2] firstly, where

$$f_1 = x_1^2 x_2 - 2x_2 x_1 x_2 + x_2^2 x_1,$$

$$f_2 = x_1 x_2^2 - x_2 x_1,$$

and $\alpha, \beta, \gamma \in k$. We assume $\beta \neq 0$, so $A(\alpha, \beta, \gamma)$ is noetherian and $A(\alpha, \beta, 0)$ is 3-dimensional AS-regular ([13]). One can easily check that $A(\alpha, \beta, \gamma)$ is a PBW deformation of $A(\alpha, \beta, 0)$. Write $B = A(\alpha, \beta, 0)[t]$. By Example A(6) in [10], the Nakayama automorphism $\mu_B$ of $B$ satisfies

$$\mu_B(x_1) = -\beta x_1, \quad \mu_B(x_2) = -\beta^{-1} x_2, \quad \mu_B(t) = t.$$
Let $U_1 = k\langle x_1, x_2 \rangle/(f_1, f_2 + z)$ and $U_2 = k\langle x_1, x_2 \rangle/(g_1, g_2 + z)$. They are two PBW deformations of $\mathcal{B}$ and $\mathcal{C}$ respectively. Let $H_1, H_2$ be the homogenizations of $U_1, U_2$ respectively. Then $z$ is still a normal element in $H_1$ and $H_2$. Clearly,

$$H_1/(z) \cong H_2/(z) \cong k[x_1, x_2, t].$$

By [13] Lemma 1.5 and Theorem 2.3, we obtain a Nakayama automorphism of $U_1$ sending

$$x_1 \mapsto -x_1 - x_2 - 1, \quad x_2 \mapsto -x_2 - 1,$$

and a Nakayama automorphism of $U_2$ sending

$$x_1 \mapsto x_1 + x_2 + 1, \quad x_2 \mapsto x_2 + 1.$$

We conclude this section by utilizing homogenization to a CY properties to general setting. In the proof, there is a homological identity involving the notation homological determinant. We only give a brief introduction to the definition. For the details, we refer to [12, Section 3] and [13, Section 3].

Let $A$ be a noetherian AS-Gorenstein algebra, and let $K$ be a Hopf algebra. Suppose $A$ is a left $K$-module algebra. The balanced dualizing complex of $A$ is $R \cong {}^\ast A^1(-l)[d]$ where $d$ is injective dimension of $A$, $l$ is Gorenstein parameter and $\mu$ is the Nakayama automorphism of $A$. Then there is a left $K$-module structure on degree $(-d)$ part of $R$, so on $A^1(-l)$. Since the degree $l$ piece of $A^1(-l)$ is dimension one, each element of $K$ acts on it by a scalar. Then we use this point to define an algebra homomorphism $\text{hdet} : K \to k$ which is called homological determinant. On the other hand, assume $K$ is finite dimensional, then $\text{hdet}$ is an element in the finite dual $K^\circ$. Then it is called the homological codeterminant of the $K^\circ$-coaction on $A$ from right. If $\text{hdet}$ is just the counit of $K$, we say it is trivial. For example, Hopf algebra $K$ is just a group algebra where the group is a subgroup of automorphism group of $A$, so every automorphism has an image in $k$ through $\text{hdet}$.

If $A$ is a graded algebra, then denote by $\xi_c$ an endomorphism of $A$ sending $a$ to $c^{\deg a}a$ for some $c \in k^\times$ and all $a \in A$. With Koszul hypothesis, the next results have been proved in [23, Theorem 3.1(1,3)] and [10, Theorem 0.1 and 0.2].

**Corollary 2.11.** Let $A = k\langle X \rangle/(R)$ be an AS-regular algebra with Nakayama automorphism $\mu_A$. Let $U$ be a PBW deformation of $A$, and $H$ be the homogenization of $U$.

(a) If $A$ is a domain, then $H$ is CY if and only if $U$ is CY.

(b) If $A$ is noetherian. Choose the Nakayama automorphism $\mu_U$ of $U$ induced by the Nakayama automorphism $\mu_H$ of $H$. Then the skew extensions $A[z, \mu_A]$ and $U[z, \mu_U]$ are CY.

**Proof.** Set $X = \{x_1, x_2, \cdots, x_n\}$, $R = \{r_1, r_2, \cdots, r_m\}$ and $U = k\langle X \rangle/(P)$, where $P = \{\tilde{r}_1, \tilde{r}_2, \cdots, \tilde{r}_m\}$.

(a) It follows from Corollary 2.3 and Lemma 2.5 immediately.

(b) The first part can be deduced from [13, Corollary 0.6(1)] and [10, Theorem 5.3]. For the completeness, we prove it right on $Z$-graded twist here.

Consider the graded twisted algebras $B = \mathcal{A}^{\mu_A}[z]$ and $C = \mathcal{B}^\sigma$ where $\sigma$ is an automorphism of $B$ such that $\sigma|_{\mathcal{A}^{\mu_A}} = \mu_A^{-1}$ and $\sigma(z) = z$. Then $C \cong A[z, \mu_A]$. If the Gorenstein parameter of $A$ is $l$, then the one of $B$ and of $C$ are $l + 1$. Let $\mu_C$ be the Nakayama automorphism of $C$. Since
A is noetherian AS-regular, \( \text{hdet}_{A} = \text{hdet}_{A}^{-1} = 1 \) by \cite[Theorem 5.3]{19}. Thus \( \text{hdet}_{A} = 1 \) by \cite[Lemma 6.1]{20}. Using the homological identity of \cite[Theorem 5.4(a)]{20}, we have

\[
\mu_{B}(z) = z, \quad \mu_{B}(a) = \mu_{A} \circ \mu_{I_{A}}^{l} \circ \xi_{- \text{det}_{A}}(a) = \mu_{A}^{l+1}(a),
\]

for any \( a \in A \). Then

\[
\begin{align*}
\mu_{C}(a) &= \mu_{B} \circ \sigma^{l+1} \circ \xi_{- \text{det}_{A}}(a) = \mu_{A}^{l+1} \circ \xi_{- \text{det}_{A}}^{(l+1)}(a) = a, \quad \text{for any } a \in A, \\
\mu_{C}(z) &= \mu_{B} \circ \sigma^{l+1} \circ \xi_{- \text{det}_{A}}(z) = z.
\end{align*}
\]

Thus, we have \( \mu_{C} = id_{C} \) and \( A[z, \mu_{A}] \) is CY.

Now turn to the case of PBW deformation. Let \( H \) be the homogenization of \( U \), and let \( H(D) \) be the homogenization of \( D = U[z, \mu_{U}] \) whose generators are \( x_{1}, x_{2}, \ldots, x_{n}, z, t \) with the relations

\[
\begin{align*}
r_{1}, \quad r_{2}, \quad \ldots, \quad r_{n}, \\
x_{i}t - tzx_{i}, \quad zt - tz, \quad zt - \mu_{U}(x_{i})t z.
\end{align*}
\]

Since \( \mu_{H}(x_{i}) \) has the form \( \mu_{U}(x_{i})^{l} \) by Theorem \cite[2.3]{20}. So \( H(D) \cong H[z, \mu_{H}] \). By the last assertion, \( H(D) \) is CY. Notice that \( D \) is also a PBW deformation of AS-regular algebra \( A[z, \mu_{A}] \), it follows that \( D = U[z, \mu_{U}] \) is CY by Corollary \cite[2.4]{20}. \( \square \)

3. Hopf actions on PBW deformations

In this section, we consider finite dimensional Hopf algebras acting on PBW deformations of noetherian AS-regular algebras. Our main idea is transferring this question to their homogenizations. For the facts of Hopf algebras, we refer to \cite[17]{17}.

In the sequel, we always assume \( X = \{x_{1}, x_{2}, \ldots, x_{n}\} \), \( A = k(X)/(R) \) is a noetherian AS-regular algebra generated by degree 1, \( U = k(X)/(P) \) is a PBW deformation of \( A \), \( H \) is the homogenization of \( U \) and \( \mu_{A}, \mu_{U}, \mu_{H} \) are the Nakayama automorphisms of \( A, U, H \) respectively. Let \( K \) be a finite dimensional Hopf algebra. There is a standard hypothesis for the rest of this section.

**Hypothesis 3.1.** We assume that \( U \) is a left \( K \)-module algebra, \( K \)-action on \( U \) preserves the canonical filtration, and \( K \) acts on \( U \) inner faithfully, namely, there is no nonzero Hopf ideal \( I \subset H \) such that \( IU = 0 \).

We say that a right \( K^{o} \)-coaction on a \( K^{o} \)-comodule \( T \) is inner faithful, if for any proper Hopf subalgebra \( K' \) of \( K^{o} \) such that \( \rho(T) \nsubseteq T \otimes K' \). Let \( T \) be a left \( K \)-module, then \( K \)-action on \( T \) is inner faithful if and only if the right \( K^{o} \)-coaction on \( T \) is inner faithful (\cite[Lemma 1.3(a)]{20}).

**Definition 3.2.** Let \( K \) be a Hopf algebra, and \( B \) an algebra. Suppose \( K \) acts on \( B \), then the fixed subalgebra of \( B \) defined by

\[
B^{K} = \{ b \in B \mid g \cdot b = \epsilon(g)b \text{ for any } g \in K \}.
\]

Under the Hypothesis \cite[3.1]{17} it is clear that there exists a \( K \)-action on \( A \) naturally preserving grading. However, the \( K \)-action on \( A \) may not inner faithful. There is an additional condition for the \( K \)-action on \( U \) in \cite[8]{18}, which is automatically satisfied if \( K \) is semisimple, to guarantee the inner faithfulness. Our approach is to give a natural \( K \)-action on \( H \) and this action inherits the inner faithfulness.
Suppose $K$-action on $U$ satisfying $g \cdot x_i = \sum_{j=1}^{n} c_{ij}^g x_j + d_i^g$ where $g \in K$ and $c_{ij}^g, d_i^g \in k$. Then define a $K$-action on $H$ by, for any $g \in K$

$$g \triangleright x_i := \sum_{j=1}^{n} c_{ij}^g x_j + d_i^g t, \quad g \triangleright t := \epsilon(g)t.$$  

We distinguish the $K$-action on $U$ and $H$ by $\cdot$ and $\triangleright$. Since $g \triangleright f = (g \cdot f)^t$ for any homogeneous $f \in k(X)$, the $K$-action on $H$ is well-defined. It can be checked straightforward that $K$ acts on $H$ inner faithfully by definition. Conversely, any inner faithful Hopf action on $H$ preserves grading such that $t \in H^K$ can induce an inner faithful Hopf action on $U$ naturally. Hence, the problems of Hopf action on $U$ are equivalent to the problems of Hopf action on $H$ satisfying $t \in H^K$. We say $K$-action on $U$ has a trivial homological determinant if $K$-action on $H$ has a trivial homological determinant.

The following result may be considered a version of filtered AS-Gorensteiness of the fixed subalgebras of PBW deformations.

**Lemma 3.3.** Let $H(U^K)$ be the homogenization of $U^K$. Then $H^K \cong H(U^K)$.

**Proof.** Notice that $t \in H^K$ and $g \triangleright f^t = (g \cdot f)^t$ for any $f \in k(X), g \in K$. Others are clear. $\square$

We are interested in the question that when a Hopf algebra $K$ acting on $U$ is semisimple or a group algebra. In [16], the authors gave lots of conditions to determine what finite dimensional Hopf algebras that act on noetherian AS-regular algebras are. Here, we have some analogues results to judge the same questions on PBW deformations of noetherian AS-regular algebras.

Let $\{1, v_1, v_2, \cdots, v_n\}$ be a basis of $F_1U = k \oplus k\{X\}$. There exists a right $K^o$-coaction $\rho_U : U \to U \otimes K^o$ on $U$. Since the $K$-action preserves the filtration, then

$$\rho_U(v_i) = \sum_{j=1}^{n} v_j \otimes y_{ji} + 1 \otimes z_i, \quad \text{for all } i = 1, 2, \cdots, n,$$

for some $y_{ij}, z_i \in K^o$. It also induces $K^o$-coaction on $H$ satisfying

$$\rho_H(v_i^t) = \sum_{j=1}^{n} v_j^t \otimes y_{ji} + t \otimes z_i, \quad \text{for all } i = 1, 2, \cdots, n,$$

$$\rho_H(t) = t \otimes 1.$$

By coassocitation, $\Delta(y_{ij}) = \sum_{s=1}^{n} y_{is} \otimes y_{sj}$ and $\Delta(z_i) = \sum_{s=1}^{n} z_s \otimes y_{si} + 1 \otimes z_i, \epsilon(y_{ij}) = \delta_{ij}$ and $\epsilon(z_i) = 0$. The $K^o$-coaction on $U$ and on $H$ are inner faithful, and the set $\{y_{ij}, z_i\}$ generates $K$ as Hopf algebra by [16, Lemma 1.3(b)].

As the argument above Example 2.6, we choose the Nakayama automorphism $\mu_U$ of $U$ induced by the one of homogenization $H$. For convenience, suppose $\mu_U(x_i) = \sum_{s=1}^{n} a_{is} x_j + b_i$ and $\mu_H(x_i) = \sum_{s=1}^{n} a_{is} x_j + b_i t$ where $a_{ij}, b_i \in k$ for any $i = 1, \cdots, n$. Set the invertible matrix $M = (a_{ij})$, and

$$N = \begin{pmatrix} 1 & 0 \\ b & M \end{pmatrix}$$

where $b$ is the column vector $(b_1, b_2, \cdots, b_n)^T$. Let $J_M$ be the Jordan canonical form of $M$. 


3.1. Diagonalizable. Firstly, we consider $J_M$ is a diagonal matrix. Assume that $M$ has no eigenvalue 1. By Proposition 2.7, then $N$ has the Jordan canonical form $\text{diag}(1, J_M)$.

**Lemma 3.4.** Retain the notations above. Assume $J_M = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$ with $\lambda_i \neq 1$ for all $i = 1, \cdots, n$. Choose the basis $\{1, v_1, \cdots, v_n\}$ of $F_1 U$ such that $\mu_U(v_i) = \lambda_i v_i$ for $i = 1, \cdots, n$. Suppose either

1. $\lambda_i \lambda_j^{-1}$ is not a root of unity for all $i \neq j$ and $\lambda_i$ is not a root of unity for all $i$.
2. $K$ is semisimple, $K$-action on $U$ has trivial homological determinant (or $K^o$-coaction has trivial homological codeterminant) and $\lambda_i \neq \lambda_j$ for all $i \neq j$.

Then $y_{ij} = z_i = 0$ for all $i \neq j$. As a consequence, $K$ is a finite dual of a group algebra and $K$ is semisimple. In addition,

(a) If $U$ has a relation of form

$$ r = u_1 v_i v_j u_2 - s_0 u_1 u_2 + s_1 w_1 + s_2 w_2 + \cdots + s_m w_m, $$

where $s_0, s_1, \cdots, s_m \in k^*$, $u_1, u_2, w_1, w_2, \cdots, w_m \in X^*$ and $m$ is non-negative integer for some $i \neq j$. Assume that $\{1, w_1, w_2, \cdots, w_m\}$ are $k$-linearly independent. Then $y_{ii} = y^{-1}_{jj}$.

(b) If $U$ has relations of form

$$ r_{ij} = u_1^{ij} v_i v_j u_2^{ij} - s_{0ij} u_1^{ij} v_i v_j u_2^{ij} + s_{1ij} w_1^{ij} + s_{2ij} w_2^{ij} + \cdots + s_{m_{ij}} w_{m_{ij}}, $$

where $s_{0ij}, s_{1ij}, \cdots, s_{m_{ij}} \in k^*$, $u_1^{ij}, u_2^{ij}, w_1^{ij}, w_2^{ij}, \cdots, w_{m_{ij}}^{ij} \in X^*$ and $m_{ij}$ is non-negative integer for all $i < j$. Assume that $\{u_1^{ij}, v_i, u_2^{ij}, w_1^{ij}, w_2^{ij}, \cdots, w_{m_{ij}}^{ij}\}$ are $k$-linearly independent for all $i < j$. Then both $K$ and $K^o$ are commutative group algebras.

**Proof.** The corresponding matrix $N$ of $\mu_H$ has eigenvalues $\{1, \lambda_1, \cdots, \lambda_n\}$. Lifting $v_i$ to $H$, then $\mu_H(v_i^4) = \lambda_i v_i^4$. By [16] Lemma 2.3(1,2,4,5), $y_{ij} = z_i = 0$ for all $i \neq j$. So each $y_{ii}$ is a group-like element, and $\rho_U(v_i) = v_i \otimes y_{ii}$ for all $i$. Since $\{y_{ii}\}_{i=1}^n$ generates $K^o$ as Hopf algebra by inner faithfulness, $K$ is a finite dual of group algebra. As a consequence, $K$ is semisimple.

(a) Assume $u_1 = x_1^{p_1} y_1^{p_2} \cdots x_1^{p_n}$ and $u_2 = x_2^{q_1} y_2^{q_2} \cdots x_2^{q_n}$. Applying $\rho_U$ to $r$, we have

$$ 0 = \rho_U(r) = u_1 v_i v_j u_2 \otimes y_{h_1 h_2} y_{h_2 h_3} \cdots y_{h_i h_j} y_{y_1 y_2} y_{f_1 f_2} \cdots y_{f_i f_j} + \cdots $$

$$ = s_{0ij} u_1 u_2 \otimes y_{h_1 h_2} y_{h_2 h_3} \cdots y_{h_i h_j} y_{y_1 y_2} y_{f_1 f_2} \cdots y_{f_i f_j} + \cdots $$

for some $g_1, g_2, \cdots, g_m \in K^o$. By the $k$-linearly independent, $y_{ii} y_{jj} = 1$.

(b) Similar to the proof of (a). Apply $\rho_U$ to $r_{ij}$ for all $i < j$, then we get $y_{ii}$ commutes with $y_{jj}$. In other words, $K^o$ is commutative and cocommutative. so $K$ is a commutative group algebra. □

As an application, we consider the Hopf actions on PBW deformations of quantum plane. Partial result has been proved in [5] Corollary 5.8(a).

**Corollary 3.5.** Let $U = k[x_1, x_2]/(f)$ be a PBW deformation of quantum plane $k_q[x_1, x_2]$ with $q \neq 1$, where $f = x_1 x_2 - q x_2 x_1 + ax_1 + bx_2 + c$. Let $K$ act on $U$ satisfying Hypothesis 3.3. Then

(a) If $q$ is not a root of unity, then $K$ is a commutative group algebra.
Corollary 3.6. Let \( K \) is semisimple and \( K \)-action on \( U \) has trivial homological determinant. If \( q \neq -1 \), then \( K \) is a commutative group algebra.

In the setting of (a) or (b), if further \( c \neq abq(1-q)^{-1} \), we have \( K \cong k\mathbb{Z}_m \) for some \( m \).

Proof. Since \( q \neq 1 \), \( U \) is isomorphic to
\[
\mathbb{k}\langle x_1, x_2 \rangle / (x_1 x_2 - qx_2 x_1 - 1), \quad \text{or} \quad \mathbb{k}[x_1, x_2].
\]
So \( \mu_U(x_1) = qx_1 \) and \( \mu_U = q^{-1} x_2 \) by Example 2.6. Now (a) and (b) are obtained from Lemma 3.4 immediately. The right \( K^o \)-coaction on \( U \) should be
\[
\rho_U(x_1) = x_1 \otimes y_{11}, \quad \rho_U(x_2) = x_2 \otimes y_{22},
\]
for some commuting elements \( y_{11}, y_{22} \in K^o \).

If \( c \neq abq(1-q)^{-1} \), then \( U \cong \mathbb{k}\langle x_1, x_2 \rangle / (x_1 x_2 - qx_2 x_1 - 1) \). By Lemma 3.4(b), we have \( y_{11} = y_{22}^1 \).

So \( K^o = kG \), where \( G \) is a group generated by \( y_{11} \). Then \( K \) is also a group algebra of cyclic group. More precisely, \( g \cdot x_1 = \xi x_1 \), \( g \cdot x_2 = \xi^{-1} x_2 \) where \( \xi \) is a root of unity, if \( K = k(g) \).

\[ \square \]

Corollary 3.6. Let \( K \) act on Down-Up algebra \( A(\alpha, \beta, \gamma) \) where \( \beta \neq 0 \) satisfying Hypothesis 3.4.

(a) If \( \beta \) is not a root of unity, then \( K \) is a dual of a group algebra.
(b) If \( \alpha \neq 0 \) and \( \beta \) is not a root of unity, then \( K \) is a commutative group algebra.
(c) If \( \gamma \neq 0 \) and \( \beta \) is not a root of unity, then \( K \cong k\mathbb{Z}_m \) for some \( m \).
(d) Suppose \( K \) is semisimple and \( K \)-action on \( A(\alpha, \beta, \gamma) \) has trivial homological determinant.

If \( \beta \neq \pm 1 \), then \( K \) is a dual of a group algebra. In addition,

(i) if \( \alpha \neq 0 \), then \( K \) is a commutative group algebra.
(ii) if \( \gamma \neq 0 \), then \( K \cong k\mathbb{Z}_m \) for some \( m \).

Proof. Let \( U = A(\alpha, \beta, \gamma) \). By Example 2.3 we have that \( U \) is a PBW deformation of \( A = A(\alpha, \beta, 0) \) and \( \mu_U(x) = -\beta x_1, \mu_U(x_2) = -\beta^{-1} x_2 \). Now (a) is obtained from Lemma 3.4 immediately. The right \( K^o \)-coaction should be
\[
\rho_U(x_1) = x_1 \otimes y_{11}, \quad \rho_U(x_2) = x_2 \otimes y_{22},
\]
for some elements \( y_{11}, y_{22} \in K^o \).

(b, c) If \( \alpha \neq 0 \), there exists a relation of \( R \) satisfying Lemma 3.4(b). So \( K \) is a commutative group algebra.

If \( \gamma \neq 0 \), then Lemma 3.4(a) applies. Thus \( K^o \) is a cyclic group algebra, and \( K \cong k\mathbb{Z}_m \). More precisely, \( g \cdot x_1 = \xi x_1 \), \( g \cdot x_2 = \xi^{-1} x_2 \) where \( \xi \) is a root of unity, if \( K = k(g) \).

(d) It is similar to (a, b, c) by Lemma 3.4. \[ \square \]

3.2. Non-diagonalizable. Next, we consider the non-diagonalizable case of \( J_M \).

Lemma 3.7. Suppose \( A \) is generated by two generators \( \{ x_1, x_2 \} \).

(1) The Nakayama automorphism \( \mu_U \) satisfies \( x_1 \mapsto \lambda x_1 + b_1, \ x_2 \mapsto b_2 + ax_1 + \lambda x_2 \), for some \( \lambda, a \in k^\times, b_1, b_2 \in k \) and \( \lambda \neq 1 \). Suppose \( K \) is semisimple and \( K \)-action on \( U \) has trivial homological determinant. Then \( K \cong k\mathbb{Z}_m \) for some \( m \).
(2) The Nakayama automorphism \( \mu_U \) satisfies \( x_1 \mapsto x_1 + b_1, \ x_2 \mapsto b_2 + ax_1 + x_2 \) for some \( a, b_1 \in k^\times \) and \( b_2 \in k \). Then \( K \) is trivial (\( K \cong k \)).

Proof. (1) By Proposition 2.7 and \( \lambda \neq 1 \), there exists a basis \( \{1, v_1, v_2\} \) of \( F_1U \) such that \( \mu_U(v_1) = \lambda v_1 \) and \( \mu_U(v_2) = \lambda v_2 + v_1 \).

Let \( H \) be the homogenization of \( U \). The matrix corresponding to the Nakayama automorphism \( \mu_H \) of \( H \) with respect to the basis \( \{t, v^t_1, v^t_2\} \) is

\[
N = \begin{pmatrix}
1 & 0 & 0 \\
0 & \lambda & 0 \\
0 & 1 & \lambda
\end{pmatrix}
\]

If \( K \) is semisimple and \( K \)-action has trivial homological determinant, the conditions of [16, Lemma 3.4(2)] are satisfied. So \( y_{ij} = z_i = 0 \) for all \( i \neq j \) and \( y_{11} = y_{22} = 0 \). Then \( K \cong \mathbb{Z}_m \).

More precisely, \( g \cdot v_1 = \xi v_1, \ g \cdot v_2 = \xi v_2 \) where \( \xi \) is a root of unity, if \( K = k(g) \).

(2) There exists a basis \( \{1, v_1, v_2\} \) of \( F_1U \) such that \( \mu_U(v_1) = v_1 + c \) and \( \mu_U(v_2) = v_2 + cv_1 + c^2 \) for some \( c \in k^\times \). Now consider the \( K \)-action on homogenization of \( U \). By [16, Lemma 3.4(3)], \( y_{ij} = z_i = 0 \) for all \( i \neq j \) and \( y_{11} = y_{22} = 1 \). Thus \( K \cong k \).

Corollary 3.8. Let \( U_1 \) and \( U_2 \) be ones defined in Example 2.10 and let finite dimensional Hopf algebras \( K_1 \) and \( K_2 \) act on \( U_1 \) and \( U_2 \) satisfying Hypothesis 3.1 respectively. Suppose \( K_1 \) is semisimple and \( K_1 \)-action on \( U_1 \) has trivial homological determinant. Then \( K_1 \cong k\mathbb{Z}_m \) for some \( m \) and \( K_2 \) is trivial.

Proof. It follows from Examples 2.10 and Lemma 3.7. 

Acknowledgments. This research is supported by the NSFC (Grant No. 11271319). We thank the reviewers for careful reading and valuable suggestions.

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