Abstract. We study a class of algebras with non-Lie commutation relations whose symplectic leaves are surfaces of revolution: a cylinder or a torus. Over each of such surfaces we introduce a family of complex structures and Hilbert spaces of antiholomorphic sections in which the irreducible Hermitian representations of the original algebra are realized. The reproducing kernels of these spaces are expressed in terms of the Riemann theta-function and its modifications. They generate quantum Kähler structures on the surface and the corresponding quantum reproducing measures. We construct coherent transforms intertwining abstract representations of an algebra with irreducible representations, and these transforms are also expressed via the theta-function.

1. Introduction

A coherent transform is a linear mapping intertwining a given representation of an algebra or a group with some irreducible pseudodifferential model. By a pseudodifferential model we mean an algebra of pseudodifferential operators with symbols on a symplectic manifold. We call this manifold a base of a coherent transform.

The coherent transform is an important analytical and geometrical tool relating objects of classical geometry and quantum objects. Moreover, this relation is established at the level of underlying vector spaces (not only at the level of the correspondence symbol $\leftrightarrow$ operator).

The integral kernels of coherent transforms are called coherent states. In somewhat different form, such states appeared already at the dawn of quantum mechanics in the works due to Schrödinger, Heisenberg, Fock, and then were comprehended by Klauder [1] and Berezin [2, 3] from the viewpoint of the theory of quantization. Different generalizations, applications, and references to the literature devoted to coherent states can be found in [4, 5].

The above definition of the coherent transform is very general. For example, it involves constructions of geometric and asymptotic quantization [6–8]. The construction of coherent transforms in such a wide range over arbitrary base manifolds is still an open problem.

An important special class of coherent transforms corresponds to complex structures on the base manifold. In this case the pseudodifferential model can be realized by using the Wick or Toeplitz operators. Such operators act in the Hilbert space

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of antiholomorphic sections over the base manifold, and this space is characterized by its reproducing kernel (the Bergmann function) [9–11]. The coherent transform intertwines the space of antiholomorphic sections with the Hilbert space in which the original representation of an algebra is given.

Historically, the first example of coherent transform for the three-dimensional Heisenberg algebra originates precisely from the existence of a complex structure on the base manifold $\mathbb{R}^2$ (here the Gaussian exponential serves as the reproducing kernel, and the Bargmann coherent transform [12] intertwines the Schrödinger representation and the Fock representation of the Heisenberg algebra).

In such a complex framework, the coherent transforms were constructed for bounded homogeneous domains in $\mathbb{C}^n$ and for other Kählerian manifolds [4, 13–15]. In the nonhomogeneous case in which neither Lie algebras nor Lie groups are present and representations of certain general commutation relations are considered, the construction of coherent transforms is much less studied, e.g., see [16–24]. In the paper [23], where the approach of [20, 21] was developed, the authors systematically study the non-Lie algebras whose complex coherent transforms have, as base manifold, some surfaces of revolution (see also generalizations in [19, 22]).

It was shown that if the base surface (or its closure) is topologically a plane or a sphere, then an arbitrary hypergeometric or $q$-hypergeometric function can appear as the reproducing kernel over this surface. The correspondence between such special functions, various complex structures over the base manifold, and the representations of a class of algebras with non-Lie commutation relations were established. With respect to these algebras, the base manifolds serve as irreducible leaves.

In the present paper we continue the study of the algebras from [23], construct their coherent states over base manifolds diffeomorphic to the cylinder and the torus, and show their relation to the theta-functions.

It should be pointed out that the correspondence between complex structures on surfaces of revolution and special functions, which we have constructed, is universal and invariant (i.e., is independent of the choice of any bases in the spaces considered). This is a distinction between this correspondence and the well-known interpretations of special functions as matrix elements in the representation theory [25] (where the fixing of the basis is important). Moreover, we stress that the reproducing kernel determines the quantum Kähler structure over the base manifold and the quantum reproducing measure, which in the nonhomogeneous case essentially differ from the classical Kähler form and the classical Liouville measure. Thus under this approach the special functions (the hypergeometric and theta functions) generate new geometric objects, which play the key role in quantum geometry of surfaces of revolution.

Needless to say that, such a base manifold as a torus possessing a pair of noncontractible 1-cycles is always naturally associated with noncommutative discrete groups. In particular, the base manifolds is associated with discrete subgroups of the Heisenberg group, which, as is well-known, are closely related to the theory of theta-functions [26–28]. This fact has been used in a number of papers including those where the completeness of coherent states over elementary cells of the plane was studied [4] and those related to the Weyl quantization over the torus [29]. However, our results are significantly different. In particular, we associate the irreducible representations of a noncommutative algebra and the Riemann theta-function, first of all, with a cylinder on which there is only one cycle. In the case of a torus, our
correspondence implies already the product of two theta-functions of two different arguments\(^2\).

Finally, we mention one more (different) correspondence between special functions and noncommutative algebras which accompanies the Yang–Baxter and Knizhnik–Zamolodchikov equations and is related to the parametrization of the set of commutation relations (structure constants), e.g., see [16, 30].

The results of the present paper were partially presented in the report of one of the authors at the Euroconference dedicated to the memory of M. Flato (Dijon, September 2000); see [31] where a general discussion of quantum geometry and certain applications are given. The first of the authors is very grateful to A. Weinstein, P. Cartier, and A. A. Kirillov for useful remarks.

2. Commutation relations and surfaces of revolution

We consider a noncommutative algebra with involution and with Hermitian generators \(\hat{S}_1, \hat{S}_2, \hat{A}_1, \ldots, \hat{A}_k\) satisfying several commutation relations. To describe these relations, we introduce the following complex combinations of generators:

\[
\hat{B} = \hat{S}_1 - i\hat{S}_2, \quad \hat{C} = \hat{S}_1 + i\hat{S}_2
\]

and choose a one-parametric group of transformations

\[
\Phi_t : \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}, \quad -\infty < t < \infty
\]

(i.e., the flow of a vector field). For the coordinates in \(\mathbb{R}^{k+1}\) we write \(A_0, A_1, \ldots, A_k\) and denote the last \(k\) coordinates by \(A = (A_1, \ldots, A_k)\). Then \(\Phi_t\) has the form

\[
\Phi_t(A_0, A) = (\varphi^0_t(A_0, A), \varphi_t(A_0, A)). \quad (2.1)
\]

Here \(\varphi^0_t\) is a scalar function (the zero component of \(\Phi_t\)), and \(\varphi_t\) is a vector function ranging in \(\mathbb{R}^k\). In this notation we have

\[
\varphi^0_t(A_0, A)\big|_{t=0} = A_0, \quad \varphi_t(A_0, A)\big|_{t=0} = A. \quad (2.2)
\]

Into all components of the function (2.1), instead of the coordinates \(A_0, A\), we can substitute Hermitian elements of the algebra. We postulate the following relations between the generators of the algebra:

\[
\hat{C} \cdot \hat{B} = \varphi^0_t(\hat{B}\hat{C}, \hat{A}), \quad \hat{C} \cdot \hat{A} = \varphi_t(\hat{B}\hat{C}, \hat{A}) \cdot \hat{C}, \quad \hat{A}_j \cdot \hat{A}_l = \hat{A}_l \cdot \hat{A}_j, \quad (2.3)
\]

where \(\hbar > 0\) is a fixed number and \(j, l = 1, \ldots, k\). After conjugation, the second relation in (2.3) also implies

\[
\hat{A} \cdot \hat{B} = \hat{B} \cdot \varphi_t(\hat{B}\hat{C}, \hat{A}). \quad (2.4)
\]

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\(^2\)The latter correlates with Sklyanin’s hypothesis [16], which, though, deals with a somewhat different situation.
Note that, by (2.3) and (2.4), the element $\hat{B}\hat{C}$ commutes with all elements $\hat{A} = (\hat{A}_1, \ldots, \hat{A}_k)$ and hence the functions of $\hat{B}\hat{C}$ and $\hat{A}$ on the right-hand sides in (2.3) and (2.4) are well defined.

Formulas (2.2) show that the algebra with relations (2.3) is a deformation with respect to the parameter $\hbar$ of a commutative algebra (corresponding to $\hbar = 0$). Note that the parameter $\hbar$ replaces the “time”-argument $t$ in (2.3) and (2.4). Hence our noncommutative algebra is generated by the “deformation” flow $\Phi_t$. A similar idea to take $\hbar$ as time was used in [29] to derive the differential equation for the Weyl $*$-product, although the deformation flow was not considered there.

Obviously, the standard \textit{Casimir elements} of the algebra with relations (2.3) (which enter the center for any $\hbar > 0$) have the form\footnote{In general, the center of the algebra (2.3) is not exhausted by such elements; in what follows, see comments on the quantization condition (3.12) and Example 5.1.}

$$\kappa(\hat{B}\hat{C}, \hat{A}),$$  
(2.5)

where the function $\kappa$ is constant on the trajectories of the flow $\Phi_t$, i.e., $\kappa(\Phi_t) = \kappa$. The number of such independent functions $\kappa$ is equal to $k$. We are going to study operator irreducible representations of the algebra (2.3). In each such representation all Casimir elements are scalar, and we obtain the following $k$ relations:

$$\kappa_j(\hat{B}\hat{C}, \hat{A}) = \text{const}_j \cdot I, \quad j = 0, 1, \ldots, k-1.$$  
(2.6)

Instead of these equations, we can introduce the elements $\hat{B}\hat{C}$ and $\hat{A}$ as functions of a single Hermitian element $\hat{t} = t^\dagger$:

$$\hat{B}\hat{C} = \varphi_t^0(a_0, a), \quad \hat{A} = \varphi_t(a_0, a),$$  
(2.7)

where $(a_0, a)$ is a chosen point in $\mathbb{R}^{k+1}$ such that $a_0 \geq 0$ and $\kappa_j(a_0, a) = \text{const}_j, j = 0, \ldots, k-1$.

By using Eqs. (2.7) or (2.6), we assign to the quantum relations the classical surface $\mathcal{X} \subset \mathbb{R}^{k+2}$:

$$\mathcal{X} = \{BC = \varphi_t^0(a_0, a), \ A = \varphi_t(a_0, a)\}, \quad \text{or}$$  
$$\mathcal{X} = \{\kappa_j(BC, A) = \kappa_j(a_0, a), \ j = 0, \ldots, k-1\},$$  
(2.8)

where $B = \varphi_t^0(a_0, a)$ and $\varphi_t(0, a) = 0$ for all $t \in \mathbb{R}$. Since $BC = S_1^2 + S_2^2$, for each chosen $t$ Eqs. (2.8) determine a circle of radius $\varphi_t^0(a_0, a)^{1/2}$ in the plane of variables $S_1, S_2$. If $t$ varies, these circles fill a surface of revolution embedded in $\mathbb{R}^{k+2}$.

The geometry of the surface depends on the properties of the function $\varphi_t^0(a_0, a)$. For $a_0 = 0$, if $\varphi_t^0(0, a) > 0$ either for all $t > 0$ or for all $t < 0$, then the closure of the surface $\mathcal{X}$ is topologically equivalent to the plane. However, if either $\varphi_t^0(0, a) > 0$ only for $0 < t < t_0$ and $\varphi_t^0(0, a) = 0$ or $\varphi_t^0(0, a) > 0$ for $-t_0 < t < 0$ and $\varphi_{-t_0}^0(0, a) = 0$, then the closure of the surface $\mathcal{X}$ is topologically equivalent to the sphere. Both these cases are described in [23].

In the present paper we are interested in the case $a_0 > 0$ and $\varphi_t^0(a_0, a) > 0$ for all $t \in \mathbb{R}$. Then $\mathcal{X}$ is either a cylinder $S \times \mathbb{R}$ embedded in $\mathbb{R}^{k+2}$ or if, in addition, we have the periodicity

$$\exists T: \quad \varphi_T^0(a_0, a) = a_0, \quad \varphi_{-T}^0(a_0, a) = a,$$

(2.9)

then the surface $\mathcal{X}$ is a torus $\mathbb{T}^2$ embedded in $\mathbb{R}^{k+2}$.

We construct irreducible representations and coherent states of the algebra (2.3) which correspond to complex structures on the cylinder or on the torus $\mathcal{X} \subset \mathbb{R}^{k+2}$. 
3. REPRESENTATIONS AND COMPLEX STRUCTURES

We introduce another Hermitian generator $\hat{s} = \hat{s}^*$ whose canonical commutation relation with $\hat{t}$ is $[\hat{t}, \hat{s}] = -i\hbar \cdot I$ and which is related to $\hat{B}$ and $\hat{C}$ by the formulas

$$\hat{B} = \mu(t) \exp\{i\hat{s}\}, \quad \hat{C} = \exp\{-i\hat{s}\} \mu(\hat{t}).$$  (3.1)

Here the complex function $\mu$ (determining the modulus of $B$ and $C$ as well as the origin point of the argument $s$) is subject to the condition

$$|\mu(t)|^2 = \mathcal{F}(t), \quad \text{where } \mathcal{F}(t) \overset{\text{def}}{=} \varphi_0^\nu(a_0, a) > 0. \quad (3.2)$$

Thus formulas (3.1) introduce Darboux coordinates on the surface $\mathcal{X}$ (2.8). In addition, the surface is $2\pi$-periodic with respect to the coordinate $s$, i.e., $s$ corresponds to a noncontractible cycle on $\mathcal{X}$. We want to introduce a complex structure on $\mathcal{X}$ by using the Darboux coordinates.

Assume that we realized the operator $\hat{t}$ as the derivation by a complex coordinate, i.e., $\hat{t} = \hbar \partial/\partial \widetilde{z}$, $\widetilde{z} \in \mathbb{C}$. Then, in the space of functions of $\widetilde{z}$, relations (2.3) and (2.4) can be realized formally by the operators

$$\hat{B} = \mathcal{B}(\hat{t}) \cdot \exp\{-\tau \hat{t} + \mathcal{B}\}, \quad \hat{C} = \exp\{\tau \hat{t} - \mathcal{C}\} \cdot \mathcal{C}(\hat{t}), \quad \hat{A} = \varphi_\tau(a_0, a), \quad (3.3)$$

where $\mathcal{B}$ is the multiplication operator by $\mathcal{B}$, $\tau > 0$ is a constant, and the functions $\mathcal{B}$ and $\mathcal{C}$ satisfy the condition

$$\mathcal{F}(t) = \mathcal{B}(t) \mathcal{C}(t). \quad (3.4)$$

Relations (3.3) and (3.1) easily imply

$$\exp\{\mathcal{B}\} = \exp\{\tau \hat{t} + g(\hat{t}) + i\hat{s}\}. \quad (3.5)$$

Here $g$ is a solution of the equation

$$\exp\left\{\frac{1}{\hbar} \int_{t-h}^t g(t) \, dt \right\} = \nu(t), \quad \text{where } \nu \overset{\text{def}}{=} \frac{\mu}{\mathcal{B}}. \quad (3.6)$$

The quantum relation (3.5) prompts us to assign the complex coordinate $\mathcal{B}$ to the Darboux coordinates $t$ and $s$ as follows:

$$\mathcal{B} = \tau t + g(t) + is. \quad (3.7)$$

The mapping defined by this formula is one-to-one if

$$\tau > \tau_0, \quad \tau_0 \overset{\text{def}}{=} -\min_t \Re \left(\frac{g'(t)}{\tau + Re g'(t)}\right) < \infty. \quad (3.8)$$

It is easy to calculate that

$$\omega_{\text{class}} = dt \wedge ds = \frac{i}{2(\tau + Re g')^2} d\mathcal{B} \wedge dz. \quad (3.9)$$

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4 This is the solution that is regular as $\hbar \to 0$; the general solution of (3.6) allows us to add an arbitrary $\hbar$-periodic function with zero mean value.
Lemma 3.1. Suppose that the condition (3.8) is satisfied and in the periodic case
(2.9), in addition, the function \( g' \) is \( T \)-periodic. Then formulas (3.9) define a
complex structure and a Kähler form on \( X \).

First we study the case in which the surface \( X \) is homeomorphic to the cylinder.
The estimate \( \tau_0 < \infty \) (3.8) can be provided as follows. By \( S_m \) we denote the class
of smooth functions \( g \) on the straight line whose derivatives \( g^{(k)} \) satisfy the estimate
\[
\exists r \leq m \quad \forall k \geq 1 \quad \exists c: |g^{(k)}(t)| \leq c(1 + |t|)^{r-k}.
\]

Lemma 3.2. Let the function \( \nu = \mu/B \) be such that \( \nu'/\nu \in S_{-1} \). Then there exists
a unique solution \( \nu \in S_1 \) of (3.6). Since \( g'(t) \) is bounded for \( |t| \to \infty \), the
estimate (3.8) holds for some \( \tau_0 \).

Now we write explicit formulas for the solution of (3.6). Assume that \( \nu \) has the
asymptotics
\[
\nu(t) = |t|^b e^{pt+l}(1 + O(t^{-1}))
\]
either as \( t \to -\infty \) or as \( t \to +\infty \). Then the solution \( g \in S_1 \) of (3.6) has the form
\[
g(t) = \sum_{k=1}^{\infty} \left( \frac{h'_{\nu}}{\nu}(t-k\bar{h}) + \frac{\bar{b}}{k} - h_p \right) + \frac{\nu'(t)}{\nu(t)} + l + b(\ln h - \gamma) + p\left(t - \frac{\bar{h}}{2}\right)
\]
or respectively
\[
g(t) = \sum_{k=1}^{\infty} \left( -\frac{h'_{\nu}}{\nu}(t+k\bar{h}) + \frac{\bar{b}}{k} + h_p \right) + l + b(\ln h - \gamma) + p\left(t + \frac{\bar{h}}{2}\right).
\]
Here \( \gamma \) stands for the Euler constant \( \gamma = \lim_{m \to \infty} \left( 1 + \frac{1}{2} + \cdots + \frac{1}{m} - \ln m \right) = 0.577\ldots \).

Remark 3.1. Under the special choice of the factor \( B = \mu \) we obtain \( \nu = 1 \) and
\( g = 0 \). The following special complex structure corresponds to this case:
\[
\Omega = \tau t + is, \quad \omega_{\text{class}} = \frac{i}{2\tau} d\bar{\tau} \wedge dz. \quad (3.7a)
\]
Conversely, an arbitrary complex structure of the form (3.9) can be obtained by
formulas (3.6) by choosing an appropriate factor \( B \). Thus the family of complex
structures on \( X \) is always determined by the factorization choice (3.4) for the zero
component \( F(t) = \varphi'_{(a_0,a)} \) of the trajectory of the flow \( \Phi_t \).

Now we prove that the formal operator realization (3.3) becomes the actual
irreducible representation of relations (2.3).

For any nonzero function \( \nu \) on the real line we define
\[
\nu_n(n\bar{h}) = \begin{cases} \nu(h) \cdot \nu(2h) \cdots \nu(nh), & n \geq 1, \\ 1, & n = 0, \\ \nu(0)^{-1} \cdot \nu(-h)^{-1} \cdots \nu((n+1)\bar{h})^{-1}, & n \leq -1. \end{cases}
\]
Now let us consider the function \( \nu \) defined by (3.6), i.e., \( \nu = \mu/B \) and introduce the
Hilbert space \( \mathcal{L}_\nu \) of \( 2\pi i \)-periodic antiholomorphic functions on the complex plane
equipped with the norm
\[
\|\psi\| = \left( \sum_{n \in \mathbb{Z}} |\nu_n(n\bar{h})|^2 e^{-\rho n^2} |\psi_n|^2 \right)^{1/2} \quad \text{if} \quad \psi(\tau) = \sum_{n \in \mathbb{Z}} \psi_n e^{i\rho \tau}. \quad (3.10)
\]

Under the conditions of Lemma 3.2, if the inequality \( \tau > \tau_0 \) (3.8) holds, then the
space \( \mathcal{L}_\nu \) is identified with the space of antiholomorphic functions on the cylinder \( X \)
with complex structure (3.9).
Theorem 3.1. Suppose that the periodicity condition (2.9) does not hold (the cylinder case). Then formulas (3.3) determine an infinite-dimensional, irreducible in the operator sense, representation of relations (2.3) in the Hilbert space $L^\nu$ of antiholomorphic functions on the cylinder. A different choice of the factors $B$ and $C$ (and hence the choice of the complex structure on the cylinder), as well as a different choice of the parameter $\tau$ leads to equivalent representations.

Now we study the case of a torus. Suppose that the $T$-periodicity condition (2.9) is satisfied. It should be noted that the quantum generator $\hat{t}$ in the irreducible representation has a discrete spectrum consisting of the numbers $n\hbar$, where $n$ is integer. Hence, there are two versions in the periodic case: either the period $T$ with respect to the classical coordinate $t$ is not a multiple of $\hbar$, i.e.,

$$T \neq N\hbar \quad \text{not for any integer } N, \quad (3.11)$$

or $T$ is a multiple of $\hbar$ and then the condition $T = N\hbar$ can be written as an integral over the torus (or over its finite sheet covering):

$$\frac{1}{2\pi\hbar} \int_X \omega_{\text{class}} = N. \quad (3.12)$$

Let us explain this in more detail. It is not assumed that the period $T$ in (2.9) is minimal. Let $T_0$ be the minimal period. Then the nonresonance version (3.11) means that $T_0$ and $\hbar$ are incommensurable. Conversely, in the resonance version we have $T_0 = N\hbar/m$, where $N$ and $m$ are coprime integers. Then the $m$-multiple period $T = mT_0$ satisfies the condition $T = N\hbar$ and thus the integral in (3.12) must be taken over the $m$-sheet covering of the torus $X$.

In the nonresonance version (3.11) Theorem 3.1 remains valid, i.e, the representation (3.3) is irreducible in the operator sense. We point out that this representation is infinite-dimensional, although it corresponds to a compact manifold! The cylinder in Theorem 3.1 is the infinite sheet covering of the torus (the infinite winding proceeds along the $t$-axis).

In the resonance version (3.12), the representation (3.3) in the space $L^\nu$ is vector irreducible but already not operator irreducible. Indeed, in this case $\hat{B}^N$ is not a scalar operator but it commutes with all operators of the representation. In the representations theory, such “nonclassical” Casimir operators are not quite usual objects. For instance, in the paper [16] where the irreducible representations corresponding to tori were studied for the first time, these Casimir operators were not presented. We point out that, in the classical limit as $\hbar \to 0$, there does not exist any function on the Poisson manifold that corresponds to such elements, since $B^N = B^{T/\hbar}$ does not have any limit as $\hbar \to 0$. Later on this effect is considered in Example 5.1.

Remark 3.2. The complex structure on the torus $X$ is introduced by the general formulas (3.9), where $g'$ is now a $T$-periodic function. We also assume that the function $\mu$ in (3.2) is $T$-periodic and the factors $B$ and $C$ in (3.4) are determined by $g$ and $\mu$ by the formulas (3.6).

Here the inverse problem of constructing $g$ either from a given factor $B$ or from the function $\nu = \mu/B$ (the problem we considered in the nonperiodic case) is made more complicated in view of the problem of small denominators in (3.6) near the
resonance $T \approx Nh$. At the resonance $T = Nh$ we have additional conditions that the right-hand side is orthogonal to some Fourier harmonics.

Now let us consider the resonance version (3.12) and show how the operator irreducible representations of relations (2.3) can be constructed in this case. First we note that, by choosing appropriate constants to normalize the $Nh$-periodic functions $B$ and $C$ in (3.4), it is possible to make relation (3.4) still hold and, simultaneously, to have

$$|\nu(Nh)| = 1, \quad \sum_{n=1}^{N} \text{arg} B(nh) = \alpha \quad (\text{mod} 2\pi),$$

(3.13)

where $\alpha$ is a given real number.

Then we introduce a Hilbert space of functions over the torus $X$. The complex coordinate $z$ identifies the torus with the rectangle in the complex plane

$$0 \leq \text{Im} z < 2\pi, \quad 0 \leq \text{Re} z < \tau T = \tau Nh.$$

Since there do not exist double-periodic antiholomorphic functions with periods $2\pi i$ and $\tau Nh$ (except constants), we use quasiperiodic functions. We also consider the conditions of quasiperiodicity

$$\psi(z + 2\pi i) = \psi(z), \quad \psi(z + \tau Nh) = \exp\left\{ \frac{\tau h}{2} N^2 + N\bar{z} \right\} \psi(z).$$

(3.14)

These conditions can be rewritten as

$$\exp\left\{ \frac{2\pi i}{h} \right\} \psi(z) = \psi(z), \quad \exp\left\{ \frac{T}{h} (\bar{z} - i\tau) \right\} \psi(z) = \psi(z).$$

Thus the quasiperiodicity is the periodicity of powers of the exponents of the creation operator $\hat{z} - i\tau$ and the operator $\frac{2\pi i}{h}$.

The functions satisfying the quasiperiodicity conditions are the theta-functions. Let us choose one of them, e.g.,

$$\theta(\alpha, \varepsilon) = \sum_{n \in \mathbb{Z}} \exp\{-\varepsilon n^2 + in\alpha\}, \quad \varepsilon > 0.$$

(3.15)

Then system (3.14) has the solution $\psi(z) = \theta(N\frac{z}{i}, \tau Nh^2/2)$. The other solutions can be obtained by applying the powers of the exponent of the creation operator $\exp\{n(\bar{z} - i\tau)\}$, $n = 0, 1, \ldots, N - 1$.

Thus we arrive at the following description of antiholomorphic functions satisfying (3.14):

$$\psi(z) = \sum_{n=0}^{N-1} \psi_n e^{n\bar{z}} \theta\left(\frac{N(\bar{z} - \tau hn)}{i}, \frac{\tau hN^2}{2}\right).$$

(3.16)

We define their norm as

$$||\psi|| = \left( \sum_{n=0}^{N-1} |\nu(nh)|^2 e^{\tau h n^2} \psi_n|^2 \right)^{1/2}.$$

(3.17)

The Hilbert space thus obtained we denote by $\mathcal{L}_N^\ast$. Under the condition (3.12), the space $\mathcal{L}_N^\ast$ is identified with the space of antiholomorphic sections over the torus $X$. 

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Theorem 3.2. Suppose that the periodicity condition (2.9) holds for $T = N\hbar$ and not for $T = N'\hbar$ for any integer $0 < N' < N$. Suppose also that the $T$-periodic factors $B$ and $C$ in (3.4) satisfy (3.13). Then formulas (3.3) determine an $N$-dimensional operator irreducible representation of relations (2.3) in the Hilbert space $L_N^\nu$ of antiholomorphic sections over the torus and

$$\hat{B}^N = \mathcal{F}_t(N\hbar)^{1/2}e^{i\alpha} \cdot I. \quad (3.18)$$

The numbers $N$ and $\alpha$ are parameters of the representation, i.e., to different pairs $(N, \alpha)$ there correspond nonequivalent representations. A different choice of the factors $B$ and $C$ (and thus a choice of the complex structure on the torus) as well as a different choice of the parameter $\tau$ leads to equivalent representations.

It is natural to pose the question: does this construction include all irreducible representations of relations (2.3)?

Theorem 3.3. All operator irreducible representations of (2.3) for which the operators $\hat{B}\hat{C}$ and $\hat{A}_1, \ldots, \hat{A}_k$ have a nonempty point spectrum can be classified as:

(1°) either the eigenvalues $\hat{B}\hat{C}$ and $\hat{C}\hat{B}$ are positive and then the representation is equivalent to one of those in Theorem 3.1 or 3.2; this type of representation corresponds to the cylinder or the torus $X$;

(2°) or the operator $\hat{B}\hat{C}$ or $\hat{C}\hat{B}$ possess the zero eigenvalues and then the representation is equivalent to one of those given in [23]; this type of representation corresponds to the plane or the sphere $X$.

4. Reproducing kernels and coherent states

The Hilbert space of antiholomorphic functions can be characterized by its reproducing kernel. First, we study the case of a cylinder. The reproducing kernel corresponding to the space $L^\nu$ has the form $K_\nu = \sum_{n} |e^{(n)}|^2$, where $\{e^{(n)}\}$ is an orthonormal basis in $L^\nu$. Starting from (3.10), we choose the basis

$$e^{(n)}(z) = \nu(n\hbar)^{-1} \exp \left\{ -\frac{\tau n^2}{2} + n\bar{z} \right\}, \quad n \in \mathbb{Z}. \quad (4.1)$$

Then for $K_\nu$ we obtain the series

$$K_\nu(z, \bar{z}) = \sum_{n \in \mathbb{Z}} \frac{1}{|\nu(n\hbar)|^2} \exp \{-\tau n^2 + n(z + \bar{z})\} = \theta_{|\nu|^2} \left( \frac{z + \bar{z}}{\tau}, \tau \hbar \right). \quad (4.2)$$

Here by $\theta_\rho$ stands for the following modification of the theta-function:

$$\theta_\rho(\alpha, \varepsilon) \overset{\text{def}}{=} \sum_{n \in \mathbb{Z}} \frac{1}{\rho(n\hbar)} \exp \{-\varepsilon n^2 + i\rho n\alpha\} = \rho \left( -i\hbar \frac{d}{d\alpha} \right)^{-1} \theta(\alpha, \varepsilon),$$

where $\theta$ is the standard theta-function (3.15). Certainly, here and in (4.2) some estimates for $\rho$ or $|\nu|$ at $\pm \infty$, which ensure the uniform convergence of infinite series, are assumed to be satisfied.

Note that the modified theta-function $\theta_\rho$ in terms of which the reproducing kernel is expressed can be defined as the unique $2\pi$-periodic solution of the problem

$$\exp \left\{ i \left( \frac{2\varepsilon}{\rho} \frac{d}{d\alpha} + \alpha \right) \right\} y(\alpha) = \rho \left( -i\hbar \frac{d}{d\alpha} \right) y(\alpha), \quad \frac{1}{2\pi} \int_0^{2\pi} y(\alpha) d\alpha = 1. \quad (4.3)$$
Next, the imaginary Jacobi transformation is known for $\theta$:

$$
\theta(\alpha, \varepsilon) = \sqrt{\frac{\pi}{\varepsilon}} \exp \left\{ - \frac{\alpha^2}{4\varepsilon} \right\} \theta \left( \frac{i\pi\alpha}{\varepsilon}, \frac{\pi^2}{\varepsilon} \right),
$$

which generates some transformation of the function $\theta_\rho$. Hence for the reproducing kernel (4.2) we obtain the identity

$$
K_\nu(z|z) = \sqrt{\frac{\pi}{\tau\hbar}} \exp \left\{ \frac{(z + z)^2}{4\tau\hbar} \right\} q_\nu(z + z),
$$

where

$$
q_\nu(x) \overset{\text{def}}{=} |\nu\left( \frac{x}{2\tau} + \hbar \frac{d}{dx} \right)|^{-2} \theta \left( \frac{\pi x}{\tau\hbar}, \frac{\pi^2}{\tau\hbar} \right).
$$

Here we use the function $\nu(t)$ which is defined not only at the lattice points $t = n\hbar$ but also for arbitrary complex $t$ by the difference equation

$$
\nu(t + \hbar) = \nu(t + \hbar) \cdot \nu(t), \quad \nu(0) = 1
$$

(analog of the gamma-function).

Note that the function $\nu$ depends on the deformation parameter $\hbar$ in a singular way and can be expressed in terms of the solution $g$ of (3.6) as

$$
\nu(t) = \exp \left\{ \frac{1}{\hbar} \int_0^t g(\tau) \, d\tau \right\}
$$

$$
= \sqrt{\frac{\nu(t)}{\nu(0)}} \exp \left\{ \frac{1}{\hbar} \int_0^t \ln \nu(\tau) \, d\tau \right\} \prod_{k=1}^\infty \exp \left\{ \hbar^k b_{k+1} \left[ (\ln \nu)^{(k)}(t) - (\ln \nu)^{(k)}(0) \right] \right\},
$$

where $b_k$ are the Bernoulli numbers: $x/(1 - e^{-x}) = \sum_{k=0}^\infty x^k b_k$. More details formulas for $\nu(t)$ can be derived by using the series for $g(t)$ given after Lemma 3.2.

Now we construct one more function

$$
p_\nu(x) \overset{\text{def}}{=} |\nu\left( \frac{x}{2\tau} - \hbar \frac{d}{dx} \right)|^2 = \frac{1}{\sqrt{\pi\hbar\tau}} \int_{-\infty}^\infty \exp \left\{ - \frac{t^2}{\hbar\tau} \right\} |\nu| \left( \frac{x + 2it}{2\tau} \right)^2 \, dt. \quad (4.4)
$$

Here it is assumed that the integral in (4.4) converges. Unfortunately, at present we cannot present general sufficient conditions on $\nu(t)$ which guarantee the convergence. However, in examples the convergence is justified (e.g., see Example 5.2 below).

By using the functions $q_\nu$ and $p_\nu$, we specify the following measure on the cylinder $X$:

$$
dm_\nu \overset{\text{def}}{=} (q_\nu p_\nu)(\sigma + z) \frac{d\sigma dz}{2\tau}.
$$

**Lemma 4.1.** If the function $p_\nu$ satisfies the estimate $|p_\nu(x)| \leq c \exp\left\{ x^2 / (4\hbar\tau) \right\} (1 + |x|)^{-1-\varepsilon}$, where $\varepsilon > 0$, then the norm in the space $L^2_{\nu}$ (3.10) can be written in the integral form

$$
||\psi||^2 = \frac{1}{2\pi} \int_{0 \leq \text{Im } z \leq 2\pi} |\psi(\sigma)|^2 p_\nu(\sigma + z) \frac{\exp\left\{ - (\sigma + z)^2 / (4\hbar\tau) \right\}}{\sqrt{4\pi\hbar\tau}} \, d\sigma dz
$$

$$
= \frac{1}{2\pi\hbar} \int_X |\psi|^2 \, d\sigma_\nu. \quad (4.5)
$$
The last expression in (4.5) is the general geometric form of the norm in the space of antiholomorphic sections over the Kählerian manifold $X$ with Kähler form

$$\omega_\nu \overset{\text{def}}{=} i\hbar \bar{\partial}\partial (\ln K_\nu) d\overline{z} \wedge dz.$$ (4.6)

This form is called a quantum Kähler form, and the measure $dm_\nu$ in (4.5) is called a reproducing measure.

In particular, in the special case (3.7a), i.e., for $\nu = 1$ we obtain $q_1(x) = \theta(\pi x/\tau \hbar, \pi^2/\tau \hbar)$ and $p_1(x) = 1$. In this case the quantum Kähler form $\omega = \omega_1$ and the reproducing measure $dm = dm_1$ on the cylinder $X$ are determined by the standard theta-function as

$$\omega_1 = i \left( \frac{1}{2\pi} + \hbar \partial \partial \ln \theta \left( \frac{\pi (\overline{z} + z)}{\tau \hbar}, \frac{\pi^2}{\tau \hbar} \right) \right) d\overline{z} \wedge dz, \quad dm_1 = \theta \left( \frac{\pi (\overline{z} + z)}{\tau \hbar}, \frac{\pi^2}{\tau \hbar} \right) d\overline{z} dz. \tag{4.7}$$

Of course, in the limit $\hbar = 0$ these quantum geometric objects become the classical form and the classical measure on the cylinder:

$$\omega_{\text{class}} = \frac{i}{2\tau} d\overline{z} \wedge dz, \quad dm_{\text{class}} = \frac{1}{2\tau} d\overline{z} dz.$$

However, as $\hbar \neq 0$ the quantum Kähler form and the quantum measure significantly differ from the classical ones. Moreover, the quantum reproducing measure does not coincide with the Liouville measure generated by the quantum Kähler form. It is of interest that this distinction is exponentially small:

$$\omega = \omega_{\text{class}} + O(e^{-\pi^2/\tau \hbar}), \quad dm = dm_{\text{class}} + O(e^{-\pi^2/\tau \hbar}), \quad \hbar \to 0.$$

In conclusion, we write the formula for coherent states. The fiducial state is defined to be a vector $P^0$ in the space of the representation on which the operators $\hat{A}$ and $\hat{A}^0 = \hat{B}\hat{C}$ take the values $a$ and $a_0$ (which corresponds to the initial point $t = 0$), i.e.,

$$\hat{A}P^0 = aP^0, \quad \hat{B}\hat{C}P^0 = a_0P^0, \quad \|P^0\| = 1. \tag{4.8}$$

Note that the fiducial state of the representation (3.3) in the Hilbert space $L_\nu$ is the unit function $P^0(\overline{z}) = 1 \in L_\nu$. Then, by (3.5) and (3.6), we have

$$e^{in\overline{z}} = e^{n\overline{z}}1 = \nu(n\hbar) \exp\{n^2 \tau \hbar/2 + in\hat{s}\} P^0.$$

Substituting this expression into the formula (4.2) for $K_\nu$, we obtain

$$K_\nu = \sum_n \nu(n\hbar)^{-1} \exp\left\{ - \frac{\tau \hbar n^2}{2} + in\hat{s} + nz \right\} P^0 = \theta_\nu\left( \frac{\overline{z}}{t} + \hat{s}, \frac{\tau \hbar}{2} \right) P^0. \tag{4.9}$$

This formula determines the coherent states in the space $L_\nu$. If now we consider an abstract Hilbert space $L$ where some representation of the algebra (2.3) acts, determine the fiducial state $P^0 \in L$ by formulas (4.8), and introduce the operator $\hat{s}$ in $L$ by formulas (3.1), then the coherent states $P_z \in L$ are determined by the same formula (4.9):

$$P_z = \theta_\nu\left( \frac{\overline{z}}{t} + \hat{s}, \frac{\tau \hbar}{2} \right) P^0. \tag{4.10}$$

By $\Pi(\overline{z}|z)$ we denote the orthogonal projector in $L$ on the subspace generated by $P_z$. 

11
Theorem 4.1.

(a) The inner product of coherent states (4.10) over the cylinder \( \mathcal{X} \) yields the reproducing kernel \( K_\nu(z|z) = \langle \mathcal{P}^\nu_z, \mathcal{P}^\nu_z \rangle \).

(b) The partition of unity takes place:

\[
\frac{1}{2\pi \hbar} \int_\mathcal{X} \Pi \, dm_\nu = I^0,
\]

where \( I^0 \) is the unity operator in the invariant subspace \( L^0 \subset L \) which is generated by the representation of the algebra (2.3) from the fiducial state \( \mathcal{P}^0 \).

(c) The mapping

\[
\mathcal{L}_\nu \to L^0 \subset L, \quad \psi \mapsto \int_\mathcal{X} \psi(z) K_\nu(z|z) \mathcal{P}_z \, dm_\nu(z)
\]

defines a unitary isomorphism between Hilbert spaces. The inverse mapping is given by the formula

\[
L \to \mathcal{L}_\nu, \quad \mathcal{P} \mapsto \langle \mathcal{P}, \mathcal{P}_z \rangle.
\]

(d) The transformation (4.11) determined by the coherent states (4.10) intertwines the representation of the algebra (2.3) in the space \( L \) with the irreducible representation (3.3) in the space \( \mathcal{L}_\nu \) of antiholomorphic functions on the cylinder \( \mathcal{X} \).

Now let us consider the case of a torus. In the nonresonance version (3.11) Theorem 4.1 remains valid, but in this case the integral in statements (b) and (c) must be taken over an infinity sheet covering of the torus \( \mathcal{X} \).

Let us describe the reproducing kernel and the coherent states in the resonance case (3.12). The space of antiholomorphic functions \( \mathcal{L}_\nu^N \) is specified by (3.16) and (3.17). Hence we take the following orthonormal basis in \( \mathcal{L}_\nu^N \):

\[
e(n)(\tau) = \nu(n\hbar)^{-1} \exp \left\{ -\frac{\tau \hbar n^2}{2} + n\tau \right\} \theta \left( \frac{\tau \left( \tau - \frac{\tau \hbar}{2} \right)}{N} \right), \quad n = 0, 1, \ldots, N-1.
\]

This is the eigenbasis for the operator \( \exp \left\{ \frac{2\pi i}{N} \hat{s} \right\} = \exp \left\{ \frac{2\pi i n}{N} \right\} \). After the quantum Fourier transformation \( e(n) \to \tilde{e}(n) \), we obtain the new basis

\[
\tilde{e}(n)(\tau) = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} \exp \left\{ \frac{2\pi i mn}{N} \right\} e(m)(\tau) = \frac{1}{\sqrt{N}} \theta \left( \frac{\tau}{N} \right) \left( \frac{\tau \left( \tau - \frac{\tau \hbar}{2} \right)}{N} \right),
\]

which is the eigenbasis for the operator \( \exp \{ i\hat{s} \} = \nu \hbar^{-1} \exp \left\{ \frac{2\pi i n}{N} \right\} \) from (3.1).

Then the reproducing kernel is determined by the formula

\[
K_\nu^N(\tau|z) = \sum_{n=0}^{N-1} |e(n)(\tau)|^2 = \sum_{n=0}^{N-1} |\tilde{e}(n)(\tau)|^2 = |\nu|^{-2} K_N(\tau|z),
\]

where

\[
K_N^N(\tau|z) = \frac{1}{N} \sum_{n=0}^{N-1} \theta \left( \frac{\tau}{N} + \frac{2\pi n \hbar}{N} \right)^2.
\]

This is a modification of formula (4.2) for the case of a resonant torus. Note that the periodicity condition (3.13) allowed us to take the factor \( \nu \) outside the sum symbol in (4.12).
Lemma 4.2. In the special case (3.7a) (i.e., for \( \nu = 1 \) and \( g = 0 \)) the following representation for the reproducing kernel \( K^N \) over the resonant torus holds:

— if \( N \) is odd then
\[
K^N(z|z) = \theta\left(\frac{z + \tau h}{2}, \frac{r h N^2}{4}\right),
\]

— if \( N \) is even then
\[
K^N(z|z) = \theta\left(\frac{z + \tau h}{2}, \frac{r h N^2}{4}\right) + \theta\#\left(\frac{z + \tau h}{2}, \frac{r h N^2}{4}\right),
\]

where
\[
\theta\#(\alpha, \varepsilon) \overset{\text{def}}{=} \sum_{n \in \mathbb{Z}} \exp\{-\varepsilon n^2 + i n \alpha\}, \quad \varepsilon > 0.
\]

Thus in the case of a resonant torus, the reproducing kernel is determined by the product of two theta-functions (a similar assertion was proposed as a hypothesis in [16], but for a different space of antiholomorphic functions).

Note that, just as in (4.3), the reproducing kernel can be specified as a unique solution of a system of several (difference) equations. Then the function \( K^N(z|w) \) is a solution of the problem

\[
|\nu|^{-2} \left(\frac{\partial}{\partial z} + \frac{\partial}{\partial w}\right) \exp\left\{z + w - \tau h \left(\frac{\partial}{\partial z} + \frac{\partial}{\partial w}\right)\right\} K^N = K^N,
\]

\[
K^N(z + 2\pi i|w) = K^N(z|w), \quad \exp\left\{N\left(z - \tau h \frac{\partial}{\partial z}\right)\right\} K^N = K^N,
\]

\[
K^N(z + \frac{2\pi i}{N}|w) = K^N(z|w + \frac{2\pi i}{N}), \quad \frac{1}{(2\pi)^2} \int_0^{2\pi} d\alpha \int_0^{2\pi} d\beta \, K^N(i\alpha|i\beta) = 1.
\]

Next, just as in the case of a cylinder, we present the reproducing kernel in the form
\[
K^N(z|z) = \sqrt{\frac{\pi}{\tau h}} \exp\left\{-\frac{(z + z)^2}{4\tau h}\right\} q^N(z|z) \quad \text{(4.13)}
\]

and introduce the measure on the torus \( X \) as
\[
dm^N_{\nu} \overset{\text{def}}{=} q^N(z|z) p_{\nu}(z + z) \frac{dz \bar{dz}}{2\tau}, \quad \text{(4.14)}
\]

where \( p_{\nu} \) is the function (4.4).

Lemma 4.3. If the function \( p_{\nu} \) is bounded, the norm (3.17) in the Hilbert space \( L^N_{\nu} \) can be written in the integral form as

\[
\|\psi\|^2 = \frac{1}{2\pi} \int_{0 \leq \Im z \leq 2\pi} \int_{0 \leq \Re z \leq \tau T} |\psi(z)|^2 p_{\nu}(z + z) \frac{dz \bar{dz}}{4\pi h T} = \frac{1}{2\pi h} \int_X |\psi|^2 \, dm^N_{\nu}. \quad \text{(4.15)}
\]
The corresponding quantum Kähler form on the resonant torus $\mathfrak{X}$ has the form

$$\omega_{\nu}^N \overset{\text{def}}{=} i\hbar \partial \overline{\partial} (\ln \mathcal{K}_{\nu}^N) d\zeta \wedge dz.$$  \hspace{1cm} (4.16)

The quantization condition is satisfied:

$$\frac{1}{2\pi \hbar} \int_{\mathfrak{X}} \omega_{\nu}^N = N.$$  \hspace{1cm} (4.17)

In formulas (4.15) and (4.17) the operation $\int_{\mathfrak{X}}$ is understood, in general, as an integral over the m-sheet covering of the torus $\mathfrak{X}$ (see the comments on formula (3.12)).

This is an analog of Lemma 4.1 and formula (4.6). Note that the quantization condition (4.17) for the Kähler form $\omega_{\nu}^N$ does not differ from condition (3.12) for the form $\omega_{\text{class}} = \lim_{\hbar \to 0} \omega_{\nu}^N$ since the first Chern class of tori is trivial: $c_1(\mathfrak{X}) = 0$.

If the surface $\mathfrak{X}$ is homeomorphic to the sphere, then there is a distinction; for details see [23].

Now we can again follow the scheme for calculating coherent states. The fiducial state $P^0 \in \mathcal{L}_{\nu}^N$ is

$$P^0(\zeta) = \theta \left( -\frac{N\zeta}{i} + \frac{\tau \hbar N^2}{2} \right).$$

For any $T$-periodic function $f(t)$ the state $P^0$ is an eigenstate of the operator $f(\hat{t})$. Namely $f(\hat{t})P^0 = f(0) \cdot P^0$. Hence it follows from (3.5) and (3.6) that

$$\exp \left\{ -\frac{\tau \hbar n^2}{2} + n\zeta \right\} \theta \left( \frac{N(z - \tau \hbar n)}{i}, \frac{\tau \hbar N^2}{2} \right) = e^{n(\gamma(n) + i\lambda)} P^0(\zeta)$$

$$= e^{i\lambda} \exp \left\{ \frac{1}{\hbar} \int_{\tau + \frac{nT}{2}} \gamma(t) \, dt \right\} P^0 = e^{i\lambda} \nu(n\hbar + \hat{t})P^0 = \nu(n\hbar) \nu(n\hbar) e^{i\lambda} P^0.$$  

Substituting this expression into the formula for $e^{n(\gamma)}(\zeta)$ and then into (4.12), we obtain

$$\mathcal{K}_{\nu}^N = \nu(\hbar\theta)^{-1} \sum_{n=0}^{N-1} \exp \left\{ -\frac{\tau \hbar n^2}{2} + n(z + i\hat{s}) \right\} \theta \left( \frac{N(z - \tau \hbar n)}{i}, \frac{\tau \hbar N^2}{2} \right) P^0$$

$$= \nu(\hbar\theta)^{-1} \theta \left( \frac{z}{i} + \frac{\tau \hbar}{2} \right) P^0.$$  

This formula determines coherent states in $\mathcal{L}_{\nu}^N$. We see that this is identically the same formula as (4.9). Thus the coherent states over the resonant torus $\mathfrak{X}$ in an abstract Hilbert space $L$ are given by (4.10) if, along with (4.8), it is required that

$$\hat{B}^N P^0 = \beta P^0, \quad \beta = \text{const}.$$  

The value of $\beta$ can be found from (3.18): $\beta = F(\nu(\hbar))^{1/2} e^{i\lambda}$.

We have the following analog of Theorem 4.1.
Theorem 4.2. Under the assumptions of Theorem 3.2 the following assertions hold.

(a) The inner product of coherent states (4.10) over the resonant torus $X$ implies the reproducing kernel $K_\nu^N(\bar{z}|z) = (P_z, P_z)$.

(b) The following partition of unity holds:

$$\frac{1}{2\pi \hbar} \int_X \Pi dm^N_\nu = I^0,$$

where $I^0$ is the unity operator in the invariant subspace $L^0 \subset L$ generated by the representation of the algebra (2.3) from the fiducial state $P^0$.

(c) The mapping

$$\mathcal{L}_\nu^N \rightarrow L^0 \subset L, \quad \psi \mapsto \int_X \frac{\psi(\bar{z})}{K_\nu^N(\bar{z}|z)} P_z \, dm^N_\nu(\bar{z}|z) \quad (4.18)$$

defined a unitary isomorphism between Hilbert spaces. The inverse mapping is given by the formula

$$L \rightarrow \mathcal{L}_\nu^N, \quad P \mapsto (P, P_z).$$

(d) The transformation (4.18) determined by the coherent states (4.10) intertwines the representation of the algebra (2.3) in the space $L$ with the irreducible representation (3.3) in the space $\mathcal{L}_\nu^N$ of antiholomorphic sections over the torus $X$.

5. Examples: Sklyanin algebra and algebra $su(1,1)$

Example 5.1. We consider four Hermitian generators satisfying the quadratic relations

\begin{align*}
[S_1, S_2] &= i(S_0S_3 + S_3S_0), \quad [S_0, S_1] = -ir^2(S_2S_3 + S_3S_2), \\
[S_2, S_3] &= i(S_0S_1 + S_1S_0), \quad [S_0, S_2] = ir^2(S_3S_1 + S_1S_3), \\
[S_3, S_1] &= i(S_0S_2 + S_2S_0), \quad [S_0, S_3] = 0.
\end{align*}

In the paper [16] this algebra is numbered as the “degenerate case (2a).” We assume that $r > 0$ and introduce the number $q = \frac{1+ir}{1-ir} = e^{i\varphi}$, where $r = \tan \frac{\varphi}{2}$. We also introduce the new generators

$$\hat{A} = \sqrt{r} S_3 + \frac{i}{\sqrt{r}} S_0, \quad \hat{B} = S_1 - iS_2, \quad \hat{C} = S_1 + iS_2$$

satisfying the relations

\begin{align*}
[\hat{C}, \hat{B}] &= -i(\hat{A}^2 - \hat{A}^{*2}), \quad [\hat{A}, \hat{A}^{*}] = 0, \\
\hat{C} \hat{A} &= q \hat{A} \hat{C}, \quad \hat{A} \hat{B} = q \hat{B} \hat{A}, \quad \hat{B}^{*} = \hat{C}.
\end{align*}

(5.1)

Here, to simplify the notation, we use the non-Hermitian generator $\hat{A} = \hat{A}_1 + i\hat{A}_2$ instead of its real and imaginary parts.

These relations have the form (2.3) where we need to set $\hbar = 1$ and specify the flow $\Phi_t$ by the formula: $\Phi_t(A_0, A) = (A_0 + \frac{q^{2t} - 1}{i(q - \bar{q})} A^2 + q \frac{q^{2t} - 1}{i(q - \bar{q})} \bar{A}^2, \quad q^t A)$. Here the
Casimir elements of the form (2.5) are given by the functions \( \kappa_0 = A_0 - \frac{q A^2}{\sqrt{q^2 - q^2}} \) and \( \kappa_1 = A \overline{A} \).

Now we assume that the parameters \( a_0 \) and \( a \) of the surface \( X \) (2.8) are chosen so that
\[
a_0 > \frac{\kappa_1}{\sin \varphi} \left( 1 - \frac{\cos(\psi - \varphi)}{\sin \varphi} \right), \quad \text{where} \quad a = \sqrt{\kappa_1} e^{i \psi/2}. \tag{5.2}
\]
Then the function
\[
F(t) = a_0 + \frac{\kappa_1}{\sin \varphi} \left( \cos(\psi - \varphi) - \cos(\psi + (2t - 1)\varphi) \right)
\]
is strictly positive for all \( t \), and, obviously, the periodicity condition (2.9) holds with period \( T = 2\pi/\varphi \). Hence the surface \( X \) is embedded in \( \mathbb{R}^4 \) as a torus. However, the quantization condition (3.12) (for \( \hbar = 1 \)) holds if and only if
\[
\frac{\varphi}{2\pi} \quad \text{is rational.} \tag{5.3}
\]
This implies the following condition on the structural constant \( r \) in the original commutation relations: if \( \frac{1}{\pi} \arctan r \) is an irrational number, then the quantization condition does not hold on any torus, i.e., there are no resonant tori; but if \( \frac{1}{\pi} \arctan r \) is rational, then any torus is a resonant torus.

Following (3.2), we would like to introduce a function \( \mu(t) \) so that
\[
F(t) = a_0 + v_0(\varphi_t(a)) - v_0(a), \quad \text{where} \quad v_0(A) = \frac{q A^2}{\sqrt{q^2 - q^2}}, \quad \varphi_t(a) = q^t a.
\]
Thus we can set \( \mu(t) = M(\varphi_t(a)) \), where \( M(A)\overline{M}(A) = a_0 - v_0(a) + v_0(A) \). Let us seek the function \( M \) in the form \( M(A) = \zeta A - \xi \overline{A} \), where \( |A| = |a| \equiv \sqrt{\kappa_1} \). Then we obtain the following system for the coefficients \( \zeta \) and \( \xi \) (see the notation in (5.2)):
\[
\zeta \xi = \frac{e^{-i\varphi}}{2 \sin \varphi}, \quad |\zeta|^2 + |\xi|^2 = \frac{a_0}{\kappa_1} + \frac{\cos(\psi - \varphi)}{\sin \varphi}. \tag{5.4}
\]
This system is easily solved. So we choose
\[
\mu(t) = \zeta a e^{i\varphi t} - \overline{\zeta a} e^{-i\varphi t}, \tag{5.5}
\]
where \( \zeta \) and \( \xi \) are subject to (5.4).

Let us take the simplest factor \( B(t) \) in (3.4): \( B = \mu \). Then \( \nu = 1, \, g = 0 \), and the complex structure on \( X \) is determined by (3.7a): \( \tau = \tau t + is \), where \( \tau > 0 \).

(1) **Nonresonance version**: (5.3) does not hold. Here it is necessary to consider a cylinder infinitely wound on the torus \( \mathcal{X} \). In this case, by Lemma 4.1, the Hilbert space of \( 2\pi i \)-periodic antiholomorphic functions over the covering of the torus \( \mathcal{X} \) is endowed with the norm
\[
\|\psi\|^2 = \frac{1}{2\pi} \int_{0 \leq \text{Im} \, z \leq 2\pi} |\psi(\tau)|^2 \exp\left\{-\frac{-(\tau + z)^2}{4\tau}\right\} d\tau dz. \tag{5.6}
\]
By (4.2), the reproducing kernel of this space is determined by the theta-function:
\[ K(z|z) = \theta(z + z, \tau) \]. The quantum Kähler form and the reproducing measure are given by formulas (4.7) and, as is easily seen, are well defined only on the infinite sheet covering of the torus (on the cylinder).

The irreducible representation of relations (5.1) in the Hilbert space (5.6) is specified by operators of the form (3.3):
\[
\begin{align*}
\hat{A} &= a \exp\{i \varphi \partial \}, \\
\hat{A}^* &= \pi \exp\{-i \varphi \partial \}, \\
\hat{B} &= (\zeta \hat{A} - \xi \hat{A}^*) \exp\{\tau \partial\}, \\
\hat{C} &= \exp\{\tau \partial - \xi \hat{A} - \zeta \hat{A}^*\}.
\end{align*}
\] (5.7)

**Resonance version:** condition (5.3) is satisfied, i.e., \( \varphi = 2 \pi m/N \), where \( m \) and \( N \) are coprime integers. For the period we choose \( T = N \) (the minimal period is equal to \( N/m \)). Then over the \( m \)-multiple covering of the torus we construct the \( N \)-dimensional Hilbert space of functions satisfying the quasiperiodicity condition (3.14) (where \( \hbar = 1 \)). This Hilbert space is endowed with the norm
\[
\| \psi \|^2 = \frac{1}{2\pi} \int_{0 \leq \text{Re} z \leq \tau T} \int_{0 \leq \text{Im} z \leq 2\pi} |\psi(z)|^2 \exp\{((z + z)/4\tau)\} d\tau dz.
\]

The reproducing kernel of this space is given in Lemma 4.2 (where \( \hbar = 1 \)). Here the form \( \omega^N_\nu \) and the measure \( dm^N_\nu \) are determined by (4.13), (4.14), and (4.16) with \( \nu = 1 \) as geometric objects on the torus, more precisely, on its \( m \)-multiple covering.

Since we have \( \nu = 1 \) in this case, the first normalization condition (3.13) holds automatically. The second condition (3.13) can be ensured as follows: it is necessary to replace the originally chosen solution \( \zeta, \xi \) of system (5.4) by another solution \( \tilde{\zeta}, \tilde{\xi} \) according to the formulas
\[
\begin{align*}
\tilde{\zeta} &= \zeta \exp\left\{i \frac{\alpha - \delta}{N}\right\}, \\
\tilde{\xi} &= \xi \exp\left\{i \frac{\delta - \alpha}{N}\right\},
\end{align*}
\]
where \( \alpha \) is the parameter from (3.13), \( \delta = \sum_{n=1}^{N} \arg \mu(n\hbar) \), and the function \( \mu \) is defined in (5.5) by using the solution \( \zeta, \xi \). The operators of the irreducible representation of the algebra (5.1) are determined by the same formulas (5.7) with \( \zeta \) and \( \xi \) replaced by \( \tilde{\zeta} \) and \( \tilde{\xi} \).

Note that, in the resonance version, the algebra (5.1), in addition to two “classical” Casimir elements, also possesses two “nonclassical” elements \( \hat{B}^N, \hat{A}^N \) and their adjoints (which are scalars in the operator irreducible representation).

**Remark 5.1.** In [16] the nonresonance version was not studied. It should be noted that in this version infinite-dimensional representations are assigned to compact symplectic leaves of the corresponding Poisson algebra. In the resonance version, our representations (5.7) defined on antiholomorphic functions were also not studied in [16] (for this case the representations in [16] are constructed in the space of functions of a circle).

**Example 5.2.** Now we consider the Lie algebra \( su(1,1) \). Its three Hermitian generators satisfy the commutation relations
\[
[S_1, S_2] = i\hbar S_3, \quad [S_2, S_3] = -i\hbar S_1, \quad [S_3, S_1] = -i\hbar S_2.
\]
We denote \( \hat{B} = \hat{S}_1 - i \hat{S}_2, \hat{C} = \hat{S}_1 + i \hat{S}_2, \) and \( \hat{A} = \hat{S}_3. \) Then the relations become
\[
\hat{C}\hat{B} = \hat{B}\hat{C} + 2\hbar\hat{A}, \quad \hat{C}\hat{A} = (\hat{A} + \hbar)\hat{C}, \quad \hat{B}^* = \hat{C}, \quad \hat{A}^* = \hat{A}. \tag{5.8}
\]
This is a special case of relations (2.3) where the flow \( \Phi_t : \mathbb{R}^2 \to \mathbb{R}^2 \) has the form
\[
\Phi_t(x_0, A) = (t^2 + t(2A - h) + A_0, A + t).
\]
Assume that the parameters \( a_0 \) and \( \lambda \) are chosen so that \( a_0 - (a - h/2)^2 \equiv \lambda^2 > 0. \) Then the function (3.2) is positive:
\[
F(t) = t^2 + t(2a - h) + a_0 > 0 \quad \text{for all} \quad t \in \mathbb{R}.
\]
Thus the surface \( X \) is diffeomorphic to a cylinder embedded in \( \mathbb{R}^3 \) as a one-sheet hyperboloid
\[
X = \{ BC - (A - h/2)^2 = \lambda^2 \}.
\]
We choose the function \( \mu(t) \) from (3.2) as \( \mu(t) = t + a - h/2 - i\lambda. \) Now we consider two versions of choosing the factor \( \mathcal{B} \) in (3.4). Namely, we choose either \( \mathcal{B}(t) \equiv \mu(t) \) or \( \mathcal{B}(t) \equiv F(t). \)

**Version I.** Let us choose \( \mathcal{B} = \mu. \) Then \( \nu = 1, \) and the Hilbert space of antiholomorphic functions on the cylinder is determined by the norm (4.5) (with \( p_\nu \equiv 1). \) The irreducible representation of the Lie algebra \( su(1, 1) \) (i.e., the representation of relations (5.8)) is given by the following operators acting in this Hilbert space:
\[
\hat{A} = a + \hbar \theta, \quad \hat{B} = (\hat{A} - h/2 - i\lambda) e^{\pi - \tau \hbar \theta}, \quad \hat{C} = e^{\tau \hbar \theta} (\hat{A} - h/2 + i\lambda). \tag{5.9}
\]

**Version II.** Let us choose \( \mathcal{B} = F. \) Then \( \nu = \pi^{-1}, \) and the complex structure on \( X \) is determined by (3.7) with
\[
g(t) = -\frac{\hbar}{a + t - \frac{\hbar}{2} + i\lambda} - \frac{\hbar}{d} \ln \left( \frac{a + t}{\hbar} - \frac{1}{2} + i\lambda \right) - \ln \hbar,
\]
where \( \Gamma \) is the standard gamma-function. Hence we have
\[
\nu_\nu(t) = \frac{\Gamma\left(\frac{a}{\hbar} + \frac{1}{2} + \frac{i\lambda}{\hbar}\right)}{\Gamma\left(\frac{a + t}{\hbar} + \frac{1}{2} + \frac{i\lambda}{\hbar}\right)} \exp \left\{ - \frac{t}{\hbar} \ln \hbar \right\},
\]
which implies the following formula for the function (4.4):
\[
p_\nu(x) = \left| \frac{\Gamma\left(\frac{a}{\hbar} + \frac{1}{2} + \frac{i\lambda}{\hbar}\right)}{\sqrt{\pi \hbar}} \right|^2 
\times \int_{-\infty}^{\infty} \exp\left\{ -\frac{t^2}{\hbar} \right\} \exp\left\{ -\frac{x^2}{\hbar^2} + \frac{2x}{\hbar} \ln \hbar \right\} 
\times \frac{\Gamma\left(\frac{a}{\hbar} + \frac{1}{2} + \frac{x}{\hbar} + \frac{i\lambda}{\hbar}\right)}{\Gamma\left(\frac{a + t}{\hbar} + \frac{1}{2} + \frac{x}{\hbar} + \frac{i\lambda}{\hbar}\right)}\left( i \right) dt. \tag{5.10}\]

Thus, in this version, the Hilbert space \( \mathcal{L}_\nu \) of antiholomorphic functions on the cylinder is given by the norm (4.5), where \( p_\nu \) is defined in (5.10). The reproducing kernel of this space is given by the modified theta-function (4.2):
\[
\kappa_\nu(\pi|z) = \frac{\Gamma\left(\frac{a}{\hbar} + \frac{1}{2} + \frac{i\lambda}{\hbar} + \frac{\partial}{i}\right)}{\Gamma\left(\frac{a}{\hbar} + \frac{1}{2} + \frac{i\lambda}{\hbar}\right)} \theta\left( \frac{\pi + z + 2 \ln \hbar}{i}, \tau \hbar \right).
\]
The irreducible representations of the Lie algebra \( su(1, 1) \) in the Hilbert space \( \mathcal{L}_\nu \) has the form (3.3):
\[
\hat{A} = a + \hbar \theta, \quad \hat{B} = [(\hat{A} - h/2)^2 + \lambda^2] e^{\pi - \tau \hbar \theta}, \quad \hat{C} = e^{\tau \hbar \theta} - \pi. \tag{5.11}
\]
Remark 5.2. Apparently, the representations (5.9) and (5.11) have not been studied in the standard representation theory of Lie algebras. They are associated with the complex structure (the complex polarization), which is not invariant under the co-adjoint action of $su(1, 1)$ on the symplectic leaf $X$. The usual way is to study representations corresponding to the invariant real polarization of $X$ (the fibration by circles $A = \text{const}$).

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