Determinantal Correlations of Brownian Paths in the Plane with Nonintersection Condition on their Loop-Erased Parts

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Abstract

As an image of the many-to-one map of loop-erasing operation \( \mathcal{L} \) of random walks, a self-avoiding walk (SAW) is obtained. The loop-erased random walk (LERW) model is the statistical ensemble of SAWs such that the weight of each SAW \( \zeta \) is given by the total weight of all random walks \( \pi \) which are inverse images of \( \zeta \), \( \{ \pi : \mathcal{L}(\pi) = \zeta \} \). We regard the Brownian paths as the continuum limits of random walks and consider the statistical ensemble of loop-erased Brownian paths (LEBPs) as the continuum limits of the LERW model. Following the theory of Fomin on nonintersecting LERWs, we introduce a nonintersecting system of \( N \)-tuples of LEBPs in a domain \( D \) in the complex plane, where the total weight of nonintersecting LEBPs is given by Fomin’s determinant of an \( N \times N \) matrix whose entries are boundary Poisson kernels in \( D \). We set a sequence of chambers in a planar domain and observe the first passage points at which \( N \) Brownian paths \( (\gamma_1, \ldots, \gamma_N) \) first enter each chamber, under the condition that the loop-erased parts \( (\mathcal{L}(\gamma_1), \ldots, \mathcal{L}(\gamma_N)) \) make a system of nonintersecting LEBPs in the domain in the sense of Fomin. We prove that the correlation functions of first passage points of the Brownian paths of the present system are generally given by determinants specified by a continuous function called the correlation kernel. The correlation kernel is of Eynard-Mehta type, which has appeared in two-matrix models and time-dependent matrix models studied in random matrix theory. Conformal covariance of correlation functions is demonstrated.

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I. INTRODUCTION

The vicious walker model introduced by Fisher [1] is a one-dimensional system of simple symmetric random walks conditioned so that any pairs of trajectories of walkers in the 1+1 dimensional spatio-temporal lattice are nonintersecting. The transition probability of $N$ vicious walkers can be described by using a determinant of an $N \times N$ matrix, whose entries are transition probabilities of a single random walker with different initial and final positions. This determinantal expression for nonintersecting paths is called the Karlin-McGregor formula in probability theory [2] and the Lindström-Gessel-Viennot formula in enumerative combinatorics [3, 4]. Preserving the determinantal expression for transition probability densities, a continuum limit (the diffusion scaling limit) of vicious walkers can be taken and we have the system of one-dimensional Brownian motions conditioned never to collide with each other [5–7]. The important fact is that the obtained interacting particle systems defined in the continuous spatio-temporal plane, which can be called the noncolliding Brownian motion [8], is identified with Dyson’s Brownian motion model with $\beta = 2$ [9], which was originally introduced as a stochastic process of eigenvalues of an Hermitian-matrix valued Brownian motion in the random matrix theory [10, 11]. The notion of correspondence between nonequilibrium particle systems and random matrix theories is very useful [12] and spatio-temporal correlation functions of noncolliding diffusion processes have been determined explicitly not only for the systems with finite numbers of particles but also for the systems with infinite numbers of particles [8, 13–16].

In the present paper, we study a system of continuous paths not in the 1+1 dimensional spatio-temporal plane but in the two dimensional plane (i.e. the complex plane $\mathbb{C} = \{z = x + iy\}$ with $i = \sqrt{-1}$), which will be called the nonintersecting system of loop-erased Brownian paths (LEBPs) [17]. A version of nonintersection condition is imposed between the paths (see Eq.(7) below), and then the total weight of LEBPs is given by Fomin’s determinant [18, 19] instead of the determinant of Karlin-McGregor (and of Lindström-Gessel-Viennot). There the entries of matrix whose determinant is considered are the normal derivatives at boundary points of domain of the Green’s functions of the two-dimensional Poisson equation (the Poisson kernels and the boundary Poisson kernels) instead of the transition probability densities [17].

In Section II, we define the LERW model and briefly review Fomin’s theory of noninter-
secting LERWs.

As the continuum limit of LERW model, the statistical ensemble of LEBPs is introduced in Section III.A. There the Green’s function, the Poisson kernel, and the boundary Poisson kernel are defined for the Brownian motion in a domain in the two-dimensional plane or the complex plane \( \mathbb{C} \). Then for an \( L \times \pi \) rectangular domain \( R_L \) in \( \mathbb{C} \), Fomin’s determinant of the boundary Poisson kernels and of the Poisson kernels for \( N \)-tuples of Brownian paths are studied, and nonintersecting system of LEBPs are constructed in Section III.B (see Fig.3).

In Section III.C, we set two rectangular domains on \( \mathbb{C} \) adjacent to each other at a vertical line \( \text{Re} z = x \), in which \( N \) Brownian paths are running from the left rectangular domain to the right one through the line \( \text{Re} z = x \) (see Fig.4). We impose the condition that the loop-erased parts of Brownian paths are nonintersecting in the sense of Fomin as expressed by Eq.(7). Under this condition, the probability density function of the first passage points on the line \( \text{Re} z = x \), at which the \( N \) Brownian paths enter the right rectangular domain from the left domain, are given as Eqs.(36) for \( L < \infty \) and (39) for \( L \to \infty \), respectively.

In Section III.D, we consider a sequence of \( M + 1 \) rectangular domains on \( \mathbb{C} \), \( M \in \mathbb{N} \equiv \{1, 2, \ldots \} \), where the \( m \)-th domain and the \( (m+1) \)-th domain are adjacent at the line \( \text{Re} z = x_m, 1 \leq m \leq M \). \( N \)-tuples of Brownian paths are running from the left to the right (see Fig.5) under the condition that their loop-erased parts make a nonintersecting system of LEBPs. The probability density function of joint distributions of first passage points at \( M \) lines \( \text{Re} z = x_m, 1 \leq m \leq M \), of the Brownian paths are determined as Eqs.(43) for \( L < \infty \) and (44) for \( L \to \infty \), respectively.

In Section IV, a special initial condition is assumed when the \( N \)-tuples of Brownian paths start from the left boundary of the leftmost domain. In this special case, we can explicitly obtain all multiple correlation functions of first passage points on the lines \( \text{Re} z = x_m, 1 \leq m \leq M \) for any \( M \in \mathbb{N} \), in which they are given by determinants (Theorem 1). The correlation kernel, which completely specifies the determinants, are given by Eqs.(54) and (55). The statistical ensemble of points whose correlation functions are generally expressed by determinants with a correlation kernel is called a determinantal point process or a Fermion point process in probability theory [20–22]. It should be noted that the present correlation kernel is asymmetric \( K_{N/2}^\pi(x, \theta; x', \theta') \neq K_{N/2}^\pi(x, \theta; x, \theta) \) for \( x \neq x' \) as shown by Eqs.(54) and (55). This asymmetric correlation kernel is of Eynard-Mehta type [23], which has been studied in two-matrix models and time-dependent matrix models in random matrix theory.
Since the Brownian motions and their loop-erased parts on $C$ are conformally invariant \cite{17}, our correlation kernel is conformally covariant. In Section V, the determinantal correlation functions in the half-infinite-strip domain $R \equiv \lim_{L \to \infty} R_L = \{z \in \mathbb{C} : \text{Re} z > 0, 0 < \text{Im} z < \pi\}$ given by Theorem 1 is mapped to the domain $\Omega = \{z = re^{i\theta} \in \mathbb{C} : r > 1, 0 < \theta < \pi\}$ (Corollary 2). There numerical plots of the density function and the two-point correlation functions in the domain $\Omega$ are shown by figures.

Concluding remarks are given in Section VI. Appendix A is prepared to derive the formulas of the Poisson kernel and the boundary Poisson kernel used in the text.

II. LOOP-ERASED RANDOM WALKS AND FOMIN’S DETERMINANT

We consider an undirected planar lattice consisting of a set of vertices (sites) $V = \{v_j\}$ and a set of edges (bonds) $E = \{e_j\}$. Together with a set of the weight functions of the edges $W = \{w(e)\}_{e \in E}$, a network $\Gamma = (V, E, W)$ is defined.

For $a, b \in V$, let $\pi$ be a walk given by

$$\pi : a = v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} v_2 \xrightarrow{e_3} \cdots \xrightarrow{e_m} v_m = b$$

(1)

where the length of walk is $|\pi| = m \in \mathbb{N}$ and, for each $0 \leq j \leq m - 1$, $v_j$ and $v_{j+1}$ are nearest-neighboring vertices in $V$ and $e_j \in E$ is the edge connecting these two vertices. We will shorten (1) to $\pi : a \rightarrow b$, or $\pi \xrightarrow{\pi} b$. The weight of $\pi$ is given by $w(\pi) = \prod_{j=1}^{m} w(e_j)$. For any two vertices of $a, b \in V$, the Green’s function of walks $\{\pi : a \rightarrow b\}$ is defined by

$$W(a, b) = \sum_{m} \sum_{\pi : a \rightarrow b, |\pi| = m} w(\pi).$$

(2)

The matrix $W = (W(a, b))_{a, b \in V}$ is called the walk matrix of the network $\Gamma$.

The loop-erased part of $\pi$, denoted by $L(\pi)$, is defined recursively as follows. If $\pi$ does not have self-intersections, that is, all vertices $v_j, 0 \leq j \leq m$ are distinct, then $L(\pi) = \pi$. Otherwise, set $L(\pi) = L(\pi')$, where $\pi'$ is obtained by removing the first loop it makes. In other words, if $(k, \ell), k < \ell$ is the smallest pair in the index set $\{j : 0 \leq j \leq m\}$ of $\{v_j\}_{j=0}^{m}$ in the sequence (1) such that $v_k = v_\ell$, then the subsequence $v_k \xrightarrow{e_{k+1}} v_{k+1} \xrightarrow{e_{k+2}} \cdots \xrightarrow{e_{\ell}} v_{\ell}$ is removed from $\pi$ to obtain $\pi'$. 

4
The loop-erasing operator $\mathcal{L}$ maps arbitrary walks to ‘walks without self-intersections’, which are usually called *self-avoiding walks* (SAWs). Note that the map is many-to-one; if $\zeta$ is a SAW obtained by applying $\mathcal{L}$, the set of inverse images $\{\pi : \mathcal{L}(\pi) = \zeta\}$ has more than one element in general. For each SAW $\zeta$, the weight $\tilde{w}(\zeta)$ is given by

$$\tilde{w}(\zeta) = \sum_{\pi : \mathcal{L}(\pi) = \zeta} w(\pi).$$

(3)

We consider the statistical ensemble of SAWs with the weight (3) and call it *loop-erased random walks* (LERWs) [19, 24]. Note that the LERW model is different from the SAW model, in the sense that, though the configuration space of walks are the same, the weight of each walk is different from each other. In the SAW model we consider a statistical ensemble of SAWs with the weight $\hat{w}(\zeta) = e^{-\beta|\zeta|}$, where $e\beta$ is the SAW connective constant, while the weight of SAW in the LERW model is given by the sum of weights of all walks, which are the inverse images of the projection $\mathcal{L}$ as shown by (3).

Assume that $A = (a_1, a_2, \ldots, a_N) \subset V$ and $B = (b_1, b_2, \ldots, b_N) \subset V$ are chosen so that any walk from $a_j$ to $b_k$ intersects any walk from $a_{j'}, j' > j$, to $b_{k'}, k' < k$. The weight of $N$-tuples of independent walks $a_1 \pi_1 b_1, \ldots, a_N \pi_N b_N$ is given by the product of $N$ weights $\prod_{\ell=1}^{N} w(\pi_{\ell})$. Then we consider $N$-tuples of walks $(\pi_1, \pi_2, \ldots, \pi_N)$ conditioned so that, for any $1 \leq j < k \leq N$, the walk $\pi_k$ has no common vertices with the loop-erased part of $\pi_j$;

$$\mathcal{L}(\pi_j) \cap \pi_k = \emptyset, \quad 1 \leq j < k \leq N.$$  

(4)

See Fig.1. By definition, $\mathcal{L}(\pi_k)$ is a part of $\pi_k$, and thus nonintersection of any pair of loop-erased parts is concluded from (4);

$$\mathcal{L}(\pi_j) \cap \mathcal{L}(\pi_k) = \emptyset, \quad 1 \leq j < k \leq N.$$  

(5)

Fomin proved that total weight of $N$-tuples of walks satisfying such a version of nonintersection condition is given by the minor of walk matrix, $\det(W_{A,B}) \equiv \det_{a \in A, b \in B}(W(a, b))$ [18]. This minor is called *Fomin’s determinant* in the present paper and Fomin’s formula is expressed by the equality [17–19]

$$\det(W_{A,B}) = \sum_{\mathcal{L}(\pi_j) \cap \pi_k = \emptyset, j < k} \prod_{\ell=1}^{N} w(\pi_{\ell}).$$

(6)

It should be noted that the RHS of (6) seems to depend nontrivially on the ordering of the sets $A$ and $B$, but it is indeed invariant up to a sign under the change of ordering, since it is
FIG. 1: The situation \( \mathcal{L}(\pi_j) \cap \pi_3 = \emptyset, j = 1, 2 \) is illustrated in a planar domain \( D \), where \( A = (a_1, a_2, a_3) \) and \( B = (b_1, b_2, b_3) \) are all boundary points of \( \partial D \). In this figure, \( \mathcal{L}(\pi_1) \) and \( \mathcal{L}(\pi_2) \) denoted by solid curves are the loop-erased parts of the walks \( \pi_1 : a_1 \to b_1 \) and \( \pi_2 : a_2 \to b_2 \), respectively, where loops described by broken curves are erased. The third walk \( \pi_3 : a_3 \to b_3 \) can be self-intersecting, but it does not intersect with \( \mathcal{L}(\pi_1) \) nor \( \mathcal{L}(\pi_2) \). As a matter of course, \( \mathcal{L}(\pi_3) \) is a part of \( \pi_3 \), and thus \( \mathcal{L}(\pi_j) \cap \mathcal{L}(\pi_3) = \emptyset, j = 1, 2 \).

equal to the LHS of \( \mathbb{G} \), which is antisymmetric in exchanging any pair of rows or columns \( [18] \).

III. NONINTERSECTING SYSTEMS OF LOOP-ERASED BROWNIAN PATHS

A. Fomin’s determinants for Brownian paths

Random walks on a lattice can be regarded as discrete approximations of a Brownian path. For example, for a two-dimensional Brownian path \( \gamma \) starting from a point \( a \) and terminating at a point \( b \) in a domain \( D \) in the plane \( \mathbb{R}^2 \), we can consider a random walk on a planar lattice \( \Gamma \) embedded in \( \mathbb{R}^2 \) to approximate \( \gamma \). Assume that the lattice spacing of the planar lattice \( \Gamma^{(n)} \) is \( 1/n \), \( n \in \mathbb{N} \), choose two vertices \( a^{(n)} \) and \( b^{(n)} \) such that they are nearest vertices in \( \Gamma^{(n)} \) to \( a \) and \( b \), respectively, and denote the random walk on \( \Gamma^{(n)} \) from \( a^{(n)} \) to \( b^{(n)} \) by \( \pi^{(n)} \). The continuum limit can be taken by getting \( n \to \infty \) and \( \pi^{(n)} \) will converge to a Brownian path \( \gamma \) running from \( a \) to \( b \). When the points \( a \) and \( b \) are on the boundary of a domain \( D \), and especially when the boundary is not smooth, we need careful consideration
for continuum limit even for a single path. See [25] for rigorous argument.

Then, if we can perform the same limiting procedure $n \to \infty$ for Fomin’s determinants of $N$-tuples of random walks $\{\pi_1^{(n)}, \ldots, \pi_N^{(n)}\}$ on planar lattices $\Gamma^{(n)}$, the limit will give the total weight of the $N$-tuples of two-dimensional Brownian paths $(\gamma_1, \ldots, \gamma_N)$, which satisfy the following condition,

$$\mathcal{L}(\gamma_j) \cap \gamma_k = \emptyset, \quad 1 \leq j < k \leq N,$$

where $\mathcal{L}(\gamma_j)$ is considered to be the $n \to \infty$ limit of the sequence of loop-erased parts $\{\mathcal{L}(\pi_j^{(n)})\}_{n \in \mathbb{N}}$ of discrete approximations $\{\pi_j^{(n)}\}_{n \in \mathbb{N}}$ of $\gamma_j$, $1 \leq j \leq N$.

Recently Kozdron and Lawler [17] gave the mathematical justification of the above mentioned continuum-limit-procedure for simply connected planar domains $D$ in the complex plane $\mathbb{C}$, where the initial and the final points $A = \{a_j\}$ and $B = \{b_j\}$ of paths $\{\gamma_j\}$ can be put on the boundaries of the domains $\partial D$. We should note that the characteristics of Brownian motion look more similar to those of a surface than those of a curve (see, for example, Chapter 1 of [26]). It implies that the Brownian path has loops on every scale and then the loop-erasing procedure defined for random walks in Sect.II does not make sense for Brownian motion, since we can not decide which loop is the first one. Kozdron and Lawler proved explicitly, however, that the continuum limit of Fomin’s determinant of the Green’s functions of random walks converges to that of the Green’s functions of Brownian motions [17]. This will enable us to discuss nonintersecting systems of LEBPs without dealing with individual LEBP. See a remark put at the end of the present paper and [17, 27, 28] for more details.

One of the advantages of taking continuum limit is the fact that the Green’s function and its normal derivatives at boundary points (the Poisson kernel and the boundary Poisson kernel) are obtained by solving the Laplace equation with appropriate boundary conditions [17]. As a realization of the two-dimensional Brownian motion, we consider a complex Brownian motion $B_t = B_t^{(R)} + iB_t^{(I)}$, where $B_t^{(R)}$ and $B_t^{(I)}$ are independent one-dimensional standard Brownian motions satisfying $(dB_t^{(R)})^2 = (dB_t^{(I)})^2 = dt$ and $dB_t^{(R)}dB_t^{(I)} = 0$, $t > 0$. For a domain $D \subset \mathbb{C}$, let $p_D(t, a, b)$ be the transition probability density of the complex Brownian motion from $a \in D$ to $b \in D$ with duration $t \geq 0$ with the absorbing boundary condition at $\partial D$. The Green’s function for this Brownian motion is defined by

$$G_D(a, b) = \int_0^\infty p_D(t, a, b)dt.$$
(Note that the summation with respect to the length of walk \( m = |\pi| \) in [2] for the Green’s function of random walks is here replaced by the integral with respect to duration of time of Brownian motion.) It is also the Green’s function for the Poisson equation \( \Delta_a G_D(a, b) = \delta(a-b) \), where \( \Delta_a \) is the Laplacian with respect to the variable \( a \), with the Dirichlet boundary condition \( G_D(a, b) = 0 \) on \( a \in \partial D \). The complex Brownian motion is conformally invariant in the sense that the Green’s function in a domain \( G_D(a, b) \), \( a, b \in D \) has the property

\[
G_D(a, b) = G_{D'}(f(a), f(b))
\]

(9)

for any conformal transformation \( f : D \to D' \). Then we will select a suitable domain \( D \) in \( \mathbb{C} \) such that the Poisson equation can be analytically solved and explicitly determine the Green’s function \( G_D \), and then, though the equality (9), we can obtain \( G_{D'} \) for other domain \( D' \) by an appropriate conformal transformation.

For \( a \in D \) and \( b \in \partial D \), the Poisson kernel \( H_D(a, b) \) is defined by

\[
H_D(a, b) = \frac{1}{2} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} G_D(a, b + \varepsilon n_b),
\]

(10)

where \( n_b \) denotes the inward unit normal vector at \( b \in \partial D \). By this definition, we see that \( H_D(a, b) \) solves the Laplace equation \( \Delta_a H_D(a, b) = 0 \) for \( a \in D \), that is, \( H_D(a, b) \) is a harmonic function of \( a \in D \), and satisfies the boundary condition

\[
\lim_{a \to \alpha : a \in D} H_D(a, b) = \delta(\alpha - b) \quad \text{for} \quad \alpha \in \partial D.
\]

(11)

Moreover, we define the boundary Poisson kernel \( H_{\partial D}(a, b) \) for \( a, b \in \partial D \) by

\[
H_{\partial D}(a, b) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} H_D(a + \varepsilon n_a, b).
\]

(12)

From the conformal invariance (9) of the Green’s function \( G_D \), the following conformal covariance properties are derived; if \( f : D \to D' \) is a conformal transformation,

\[
H_D(a, b) = |f'(b)|H_{D'}(f(a), f(b)), \quad a \in D, b \in \partial D,
\]

(13)

\[
H_{\partial D}(a, b) = |f'(a)||f'(b)|H_{\partial D'}(f(a), f(b)), \quad a, b \in \partial D.
\]

(14)

See Section 2.6 of [17] and Chapters 2 and 5 of [29] for further information about Poisson kernels, boundary Poisson kernels and their conformal covariance. We will study Fomin’s determinants of the form \( \det_{1 \leq j, k \leq N}[H_D(a_j, b_k)] \) and \( \det_{1 \leq j, k \leq N}[H_{\partial D}(a_j, b_k)] \) in the following.
B. System in a rectangular domain and the crossing exponent

In order to give explicit expressions for Poisson kernel and boundary Poisson kernel, now we fix the domain $D$ as the following rectangular domain

$$R_L = \left\{ z = x + iy \in \mathbb{C} : 0 < x < L, 0 < y < \pi \right\}, \quad L > 0. \quad (15)$$

As shown in Appendix A, for $0 < x < L, 0 < \theta, \rho, \varphi < \pi$, the Poisson kernel connecting an inner point $x + i\theta \in D$ and a point $L + i\rho$ at the right boundary of $R_L$, $\partial R^R_L = \{ L + iy : 0 < y < \pi \}$, is given by

$$H_{R_L}(x + i\theta, L + i\rho) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sinh(nx) \sin(n\theta) \sin(n\rho)}{\sinh(nL)}, \quad (16)$$

and the boundary Poisson kernel connecting a boundary point $i\varphi$ on the left boundary of $R_L$, $\partial R^L_L = \{ iy : 0 < y < \pi \}$, and a boundary point $L + i\rho$ on the right boundary $\partial R^R_L$ is given by

$$H_{\partial R_L}(i\varphi, L + i\rho) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{n \sin(n\varphi) \sin(n\rho)}{\sinh(nL)}. \quad (17)$$

The definitions (8), (10) and (12) imply that $H_{R_L}(x + i\theta, L + i\rho)d\rho$ gives the total weight of the paths of complex Brownian motion starting from the inner point $x + i\theta \in R_L$, which make first exit from the domain $R_L$ at a point in the interval $[L + i\rho, L + i(\rho + d\rho)]$ on the right boundary $\partial R^R_L$, and that $H_{\partial R_L}(i\varphi, L + i\rho)d\varphi d\rho$ gives the total weight of the paths of complex Brownian motion, which enter $R_L$ at a point in $[i\varphi, i(\varphi + d\varphi)]$ on $\partial R^L_L$, and make first exit from $R_L$ at a point in $[L + i\rho, L + i(\rho + d\rho)]$ on $\partial R^R_L$. (See Fig.2.)

By the conservation of probability, the following equalities should be satisfied; for $0 < x < x' < L,

$$H_{R_L}(x + i\theta, L + i\rho) = \int_0^{\pi} d\theta' H_{R_L}(x + i\theta, x' + i\theta')H_{R_L}(x' + i\theta', L + i\rho), \quad (18)$$

$$H_{\partial R_L}(i\varphi, L + i\rho) = \int_0^{\pi} d\varphi' H_{\partial R_L}(i\varphi, x + i\varphi')H_{R_L}(x + i\varphi', L + i\rho), \quad (19)$$

$0 < \varphi, \rho < \pi$. As a matter of course, the validity of them can be directly confirmed by applying the orthogonality relation of the sine functions, $\int_0^{\pi} d\theta \sin(n\theta) \sin(m\theta) = (\pi/2)\delta_{nm}, n, m \in \mathbb{N}$, to the expressions (16) and (17). We note that in Eq.(18) the point $x' + i\theta'$ is regarded as the first passage point on the line $\text{Re}z = x'$ in $R_L$ of the Brownian path running from $x + i\theta \in R_L$ to $L + i\rho \in \partial R^R_L$, and similarly that in Eq.(19) $x + i\varphi'$ is regarded as the first
The broken curve denotes a path of complex Brownian motion starting from an inner point \( x + i\theta \in R_L \), which makes first exit from \( R_L \) at \( L + i\rho \in \partial R^R_L \). The solid curve does a path of complex Brownian motion, which enters \( R_L \) at \( i\varphi \in R^L_L \) and makes first exit from \( R_L \) at \( L + i\rho \in \partial R^R_L \). The former path contributes to \( H_{R_L}(x + i\theta, L + i\rho) \) and the latter does to \( H_{\partial R_L}(i\varphi, L + i\rho) \), respectively.

The point \( x' + i\theta' \) is the first passage point on the line \( \text{Re} z = x' \) in \( R_L \) of the former path, and the point \( x + i\theta'' \) is the first passage point on the line \( \text{Re} z = x \) in \( R_L \) of the latter path.

Passage point on the line \( \text{Re} z = x \) in \( R_L \) of the Brownian path running from \( i\varphi \in \partial R^L_L \) to \( L + i\rho \in \partial R^R_L \). See Fig. 2.

For \( N \in \mathbb{N} \), let \( W^\pi_N \equiv \{ \theta = (\theta_1, \theta_2, \ldots, \theta_N) : 0 < \theta_1 < \theta_2 < \cdots < \theta_N < \pi \} \). Then, for \( \varphi = (\varphi_1, \ldots, \varphi_N) \in W^\pi_N, \ \rho = (\rho_1, \ldots, \rho_N) \in W^\pi_N \), consider Fomin’s determinant

\[
f^\pi_N(L, \rho|\varphi) \equiv \det_{1 \leq j, k \leq N} \left[ H_{\partial R_L}(i\varphi_j, L + i\rho_k) \right]. \tag{20}
\]

If we write the path of \( j \)-th Brownian motion, \( 1 \leq j \leq N \), which enters \( R_L \) at \( i\varphi_j \in \partial R^L_L \) and makes first exit from \( R_L \) at \( L + i\rho_j \in \partial R^R_L \) as \( \gamma_j \), then (20) gives the total weight of the \( N \)-tuples of paths \((\gamma_1, \ldots, \gamma_N)\) satisfying the nonintersection condition with the loop-erased parts (7). We note again that it is a sufficient condition for

\[
\mathcal{L}(\gamma_j) \cap \mathcal{L}(\gamma_k) = \emptyset, \quad 1 \leq j < k \leq N. \tag{21}
\]

See Fig. 3.
FIG. 3: The condition (7) is illustrated for \( k = 3, N = 3 \). The Brownian path \( \gamma_3 : i\varphi_3 \to L + i\rho_3 \) in \( R_L \) can be self-intersecting, but it does not intersect with \( \mathcal{L}(\gamma_1) : i\varphi_1 \to L + i\rho_1 \) nor \( \mathcal{L}(\gamma_2) : i\varphi_2 \to L + i\rho_2 \). By definition \( \mathcal{L}(\gamma_3) \) is a part of \( \gamma_3 \), and thus \( \mathcal{L}(\gamma_1) \cap \mathcal{L}(\gamma_3) = \emptyset \) and \( \mathcal{L}(\gamma_2) \cap \mathcal{L}(\gamma_3) = \emptyset \).

By multilinearity of the determinant, we find that Eq.(20) with Eq.(17) is written as

\[
\frac{f_0^0(L, \rho|\varphi)}{2\pi} = \left(\frac{2}{\pi}\right)^N \prod_{j=1}^N \frac{n_j}{\sinh(n_j L)} \times \frac{1}{N!} \sum_{\sigma \in S_N} \det_{1 \leq j, k \leq N} \left[ \sin(n_{\sigma(j)} \varphi_j) \sin(n_{\sigma(j)} \rho_k) \right]
\]

\[
= \left(\frac{2}{\pi}\right)^N \det_{1 \leq j, k \leq N} \left[ \sin(j \varphi_k) \right] \det_{1 \leq \ell, m \leq N} \left[ \sin(\ell \rho_m) \right] \times \sum_{\lambda} a_\lambda \hat{s}_\lambda(\varphi) \hat{s}_\lambda(\rho),
\]  

(22)

where \( S_N \) is a set of all permutations \( \{\sigma\} \) of \( \{1, 2, \ldots, N\} \), \( \lambda = (\lambda_1, \ldots, \lambda_N) \) with \( \lambda_j = n_{N-j+1} - (N-j+1), 1 \leq j \leq N \), and

\[
a_\lambda = \prod_{j=1}^N \frac{\lambda_j + N - j + 1}{\sinh((\lambda_j + N - j + 1)L)},
\]

\[
\hat{s}_\lambda(\varphi) = \frac{\det_{1 \leq j, k \leq N} \left[ \sin((\lambda_j + N - k + 1)\varphi_j) \right]}{\det_{1 \leq j, k \leq N} \left[ \sin((N - k + 1)\varphi_j) \right]}. 
\]

(23)

(This is a modified version of ‘Schur function expansion’ used in [8, 12].) We note that

\[
\sin(\ell \theta) = \sin \theta \left[ 2^{\ell-1}(\cos \theta)^{\ell-1} + \sum_{s=1}^{[\ell/2]} (-1)^s \left( \frac{\ell-s-1}{s} \right) (2 \cos \theta)^{\ell-2s-1} \right],
\]

(24)

where, for \( r \in \mathbb{R}, \lfloor r \rfloor \) denotes the greatest integer not greater than \( r \). Then for \( \theta = (\theta_1, \ldots, \theta_N) \in W_N^\pi\)

\[
\det_{1 \leq \ell, m \leq N} \left[ \sin(\ell \theta_m) \right] = \det_{1 \leq \ell, m \leq N} \left[ 2^{\ell-1} \sin \theta_m \left\{ (\cos \theta_m)^{\ell-1} + O((\cos \theta_m)^{\ell-3}) \right\} \right] = 2^N(N-1)/2 \tilde{h}_N(\theta),
\]

(25)
with the function
\[
\hat{h}_N(\theta) = \prod_{j=1}^{N} \sin \theta_j \prod_{1 \leq k < \ell \leq N} (\cos \theta_\ell - \cos \theta_k).
\] (26)

Since
\[
a_\lambda \simeq 2^N \prod_{j=1}^{N} (\lambda_j + N - j + 1) e^{-L \sum_{j=1}^{N} (\lambda_j + N - j + 1)} \quad \text{as} \quad L \to \infty,
\]
and in particular for \( \emptyset = (0, 0, \ldots, 0) \)
\[
a_\emptyset \simeq 2^N N! e^{-LN(N+1)/2} \quad \text{as} \quad L \to \infty,
\]
we can conclude from the expansion (22) that
\[
f_\partial N(L, \rho | \varphi) \simeq 2^{N(N+1)} \pi^{-N} N! e^{-N(N+1)L/2} \hat{h}_N(\varphi) \hat{h}_N(\rho) \quad \text{as} \quad L \to \infty. \quad (27)
\]

Since the simple \( N \)-product of the boundary Poisson kernels \( \prod_{j=1}^{N} H_{\partial R_L}(i \varphi_j, L + i \rho_j) \simeq \prod_{j=1}^{N} 2/\{\pi \sinh L\} \simeq (4/\pi)^N e^{-NL} \) as \( L \to \infty \), the ratio behaves
\[
\Lambda_{RL}(\varphi, \rho) \equiv \frac{f_\partial N(L, \rho | \varphi)}{\prod_{j=1}^{N} H_{\partial R_L}(i \varphi_j, L + i \rho_j)} \simeq c_N(\varphi, \rho) e^{-\psi_N L} \quad \text{as} \quad L \to \infty \quad (28)
\]
with
\[
\psi_N = \frac{1}{2} N(N - 1), \quad (29)
\]
where \( c_N(\varphi, \rho) = 2^{N(N-1)} N! \hat{h}_N(\varphi) \hat{h}_N(\rho) \). The exponent (29) is called the \textit{crossing exponent} [17].

C. Probability density function for first passage points of Brownian paths underlying in nonintersecting LEBPs

We consider the integral of Fomin’s determinant (20) over all possible ordered sets of exits \( \rho \in W^N \) at the right boundary \( \partial R_L \),
\[
\mathcal{N}^0_N(L, \varphi) \equiv \int_{W^N} d\rho f_\partial^0 N(L, \rho | \varphi) = \int_{W^N} d\rho \det_{1 \leq j,k \leq N} \left[ H_{\partial R_L}(i \varphi_j, L + i \rho_k) \right]
\]
\[
= \int_{W^N} d\rho \det_{1 \leq j,k \leq N} \left[ \int_{0}^{\pi} d\theta H_{\partial R_L}(i \varphi_j, x + i \theta) H_{R_L}(x + i \theta, L + i \rho_k) \right], \quad (30)
\]
where \( d\rho = \prod_{j=1}^{N} d\rho_j, \) \( 0 < x < L \), and Eq. (19) has been used in the last equality. Applying the Heine identity
\[
\int dz \det_{1 \leq j,k \leq N} [\phi_j(z_k)] \det_{1 \leq \ell,m \leq N} [\psi_\ell(z_m)] = \det_{1 \leq j,k \leq N} \left[ \int dz \phi_j(z) \psi_k(z) \right], \quad (31)
\]
which is valid for square integrable functions $\phi_j, \psi_j, 1 \leq j \leq N$, it is written as

$$N^\theta_N(L, \varphi) = \int_{W^\pi_N} d\rho \int_{W^\pi_N} d\theta \det_{1 \leq j, k \leq N} \left[H_{\theta R_x}(i\varphi_j, x + i\theta_k)\right] \det_{1 \leq \ell, m \leq N} \left[H_{R_L}(x + i\theta_\ell, L + i\rho_m)\right]$$

$$= \int_{W^\pi_N} d\theta f_N^\theta(x, \theta|\varphi) \int_{W^\pi_N} d\rho \det_{1 \leq \ell, m \leq N} \left[H_{R_L}(x + i\theta_\ell, L + i\rho_m)\right]. \quad (32)$$

Then, if we introduce the integral

$$N_N(x, L, \theta) = \int_{W^\pi_N} d\rho f_N(x, \theta; L, \rho), \quad 0 < x < L, \quad (33)$$

of Fomin’s determinant for the Poisson kernels

$$f_N(x, \theta; L, \rho) = \det_{1 \leq j, k \leq N} \left[H_{R_L}(x + i\theta_j, L + i\rho_k)\right], \quad (34)$$

$\theta, \rho \in W^\pi_N$, and divide the both sides of (32) by $N^\theta_N(L, \varphi)$, we obtain the equality

$$1 = \int_{W^\pi_N} d\theta f_N^\theta(x, \theta|\varphi) \frac{N_N(x, L, \theta)}{N_N^\theta(L, \varphi)}. \quad (35)$$

Then, given $\varphi \in W^\pi_N$, if we put

$$p^L_N(x, \theta|\varphi) = f_N^\theta(x, \theta|\varphi) \frac{N_N(x, L, \theta)}{N_N^\theta(L, \varphi)}, \quad 0 < x < L, \quad (36)$$

for $\theta \in W^\pi_N$, it can be regarded as the probability density function. As illustrated in Fig.4, here we consider the rectangular domain (15) as a union of the two rectangular domains $R_x = \{z \in \mathbb{C} : 0 < \text{Re} z < x, 0 < \text{Im} z < \pi\}$ and $R_{(x, L)} = \{z \in \mathbb{C} : x \leq \text{Re} z < L, 0 < \text{Im} z < \pi\}$, which are adjacent to each other at a line $\text{Re} z = x$. We have considered $N$-tuples of Brownian paths $(\gamma_1, \ldots, \gamma_N)$ starting from $(i\varphi_1, \ldots, i\varphi_N)$, all of which run inside of the domain $R_L = R_x \cup R_{(x, L)}$ until arriving at the right boundary $\partial R^R_L$, under the condition that $(\mathcal{L}(\gamma_1), \ldots, \mathcal{L}(\gamma_N))$ makes a system of nonintersecting LEBPs in the sense of Fomin (7).

In an ensemble of such $N$-tuples of Brownian paths, Eq. (36) gives the probability density of the event such that the first passage points of $(\gamma_1, \ldots, \gamma_N)$ on the line $\text{Re} z = x$ are $(x + i\theta_1, \ldots, x + i\theta_N)$. That is, for $1 \leq j \leq N$, $\gamma_j$ makes a first exit from the left rectangular domain $R_x$ and enters the right rectangular domain $R_{(x, L)}$ at $x + i\theta_j$. The path $\gamma_j$, which made a first exit from $R_x$ at $x + i\theta_j$, can reenter $R_x$ at different point $x + i\theta'$ on the line $\text{Re} z = x$. As shown by the path $\gamma_2$ starting from $i\varphi_2$ in Fig.4, the path $\gamma_j$ can make a loop passing the line $\text{Re} z = x$, and if the first passage point $x + i\theta_j$ is included in such a loop, that point can not be included in $\mathcal{L}(\gamma_j)$. In other words, Eq. (36) gives the probability density for
FIG. 4: Three Brownian paths ($\gamma_1, \gamma_2, \gamma_3$) in $R_L = R_x \cup R_{\{x,L\}}$ starting from $(i\varphi_1, i\varphi_2, i\varphi_3)$ and arriving at $\partial R_L$ whose loop-erased parts ($\mathcal{L}(\gamma_1), \mathcal{L}(\gamma_2), \mathcal{L}(\gamma_3)$) are nonintersecting in the sense of Fomin. The first passage points on the line $\text{Re} z = x$ are $(x + i\theta_1, x + i\theta_2, x + i\theta_3)$. Since $x + i\theta_2$ is in a loop of $\gamma_2$ as shown by a broken curve, it is not included in $\mathcal{L}(\gamma_2)$. The probability density of points $(x + i\theta_1, x + i\theta_2, x + i\theta_3)$ is given by Eq.(36).

$N$ nonintersecting loop-erased Brownian paths ($\mathcal{L}(\gamma_1), \ldots, \mathcal{L}(\gamma_N)$) starting from the points $(i\varphi_1, \ldots, i\varphi_N)$ and satisfying the condition (7) in $R_L$, such that the underlying Brownian paths ($\gamma_1, \ldots, \gamma_N$) are realized so that they first arrive at the vertical line $\text{Re} z = x < L$ at $(x + i\theta_1, \ldots, x + i\theta_N)$.

Now we consider the system in the limit $L \to \infty$. By applying the asymptotics (27), we have

$$N_N^0(L, \varphi) = \int_{\mathcal{W}_N^0} d\rho \left[ f_N^0(L, \rho|\varphi) \right]$$

$$\approx 2^N(N+1)\pi^{-N}N!e^{-N(N+1)L/2}\hat{h}_N(\varphi) \times \int_{\mathcal{W}_N^0} d\rho \hat{h}_N(\rho), \quad (37)$$

as $L \to \infty$. Similarly, we can obtain the following asymptotics of (33)

$$N_N(x, L, \theta) = \int_{\mathcal{W}_N^0} d\rho \left[ \det_{1 \leq j, k \leq N} \left[ \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sinh(nx) \sin(n\theta_j) \sin(n\rho_k)}{\sinh(nL)} \right] \right]$$

$$\approx 2^N(N+1)\pi^{-N}e^{-N(N+1)L/2}\hat{h}_N(\theta) \prod_{j=1}^{N} \sinh(jx) \int_{\mathcal{W}_N^0} d\rho \hat{h}_N(\rho), \quad (38)$$
as $L \to \infty$. Then, given $\varphi \in \mathcal{W}_N^\pi$, for $\theta \in \mathcal{W}_N^\pi$,

$$p_N(x, \theta|\varphi) \equiv \lim_{L \to \infty} p^L_N(x, \theta|\varphi) = C_N(x) f^\theta_N(x, \theta|\varphi) \frac{\hat{h}_N(\theta)}{h_N(\varphi)}, \quad 0 < x < \infty,$$

where

$$C_N(x) = \frac{1}{N!} \prod_{j=1}^N \sinh(jx).$$

(40)

### D. Joint distribution of first passage points in a sequence of chambers

Let $M \in \mathbb{N}$ and $0 < x_1 < x_2 < \cdots < x_M < L < \infty$. Here we consider $M$ vertical lines in $R_N$ at $\text{Re } z = x_m, 1 \leq m \leq M$. For $(x_m, x_{m+1}), \theta^{(m)} = (\theta_1^{(m)}, \ldots, \theta_N^{(m)}) \in \mathcal{W}_N^\pi$, $\theta^{(m+1)} = (\theta_1^{(m+1)}, \ldots, \theta_N^{(m+1)}) \in \mathcal{W}_N^\pi$, $1 \leq m \leq M - 1$, we define

$$q^L_N(x_m, \theta^{(m)}; x_{m+1}, \theta^{(m+1)}) = \frac{N_N(x_m+1, L, \theta^{(m+1)})}{N_N(x_m, L, \theta^{(m)})} f_N(x_m, \theta^{(m)}; x_{m+1}, \theta^{(m+1)}).$$

(41)

Then by definition (36), we have

$$\int_{\mathcal{W}_N^\pi} d\theta^{(1)} p^L_N(x_1, \theta^{(1)}|\varphi) q^L_N(x_1, \theta^{(1)}; x_2, \theta^{(2)}) = \int_{\mathcal{W}_N^\pi} d\theta^{(1)} f^\theta_N(x_1, \theta^{(1)}|\varphi) \frac{N_N(x_1, L, \theta^{(1)}) N_N(x_2, L, \theta^{(2)})}{N^\theta_N(L, \varphi)} f_N(x_1, \theta^{(1)}; x_2, \theta^{(2)})$$

$$= \frac{N_N(x_2, L, \theta^{(2)})}{N^\theta_N(L, \varphi)} \int_{\mathcal{W}_N^\pi} d\theta^{(1)} f^\theta_N(x_1, \theta^{(1)}|\varphi) f_N(x_1, \theta^{(1)}; x_2, \theta^{(2)}).$$

By the Heine identity (31) and the equality (19),

$$\int_{\mathcal{W}_N^\pi} d\theta^{(1)} f^\theta_N(x_1, \theta^{(1)}|\varphi) f_N(x_1, \theta^{(1)}; x_2, \theta^{(2)})$$

$$= \int_{\mathcal{W}_N^\pi} d\theta^{(1)} \det_{1 \leq j,k \leq N} [H_{R_1}(i\varphi_j, x_1 + i\theta_j^{(1)})] \det_{1 \leq \ell,m \leq N} [H_{R_2}(x_1 + i\theta_\ell^{(1)}, x_2 + i\theta_m^{(2)})]$$

$$= \det_{1 \leq j,k \leq N} \left[ \int_0^\pi d\theta^{(1)} H_{R_1}(i\varphi_j, x_1 + i\theta^{(1)}) H_{R_2}(x_1 + i\theta^{(1)}, x_2 + i\theta^{(2)}_k) \right]$$

$$= \det_{1 \leq j,k \leq N} \left[ H_{R_1}(i\varphi_j, x_2 + i\theta^{(2)}_k) \right] = f^\theta_N(x_2, \theta^{(2)}|\varphi),$$

and thus we obtain the equality

$$p^L_N(x_2, \theta^{(2)}|\varphi) = \int_{\mathcal{W}_N^\pi} d\theta^{(1)} p^L_N(x_1, \theta^{(1)}|\varphi) q^L_N(x_1, \theta^{(1)}; x_2, \theta^{(2)}).$$
Obviously this equality can be generalized as
\[ p_N^L(x_{m+1}, \theta^{(m+1)}|\varphi) = \int_{\mathbb{W}_N^p} d\theta^{(m)} p_N^L(x_m, \theta^{(m)}|\varphi) q_N^L(x_m, \theta^{(m)}; x_{m+1}, \theta^{(m+1)}) \] (42)
for 1 ≤ m ≤ M − 1.

Then, given \( \varphi \in \mathbb{W}_N^p \), if we introduce a function of \( \theta^{(m)} = (\theta^{(m)}_1, \ldots, \theta^{(m)}_N) \in \mathbb{W}_N^\pi, 1 \leq m \leq M \) by
\[ p_N^L(x_1, \theta^{(1)}; x_2, \theta^{(2)}; \ldots; x_M, \theta^{(M)}|\varphi) \]
\[ = p_N^L(x_1, \theta^{(1)}|\varphi) \prod_{m=1}^{M-1} q_N^L(x_m, \theta^{(m)}; x_{m+1}, \theta^{(m+1)}) \]
\[ = \frac{N_N(x_M, L, \theta^{(M)})}{N_N^\pi(L, \varphi)} f_N^\theta(x_1, \theta^{(1)}|\varphi) \prod_{m=1}^{M-1} f_N(x_m, \theta^{(m)}; x_{m+1}, \theta^{(m+1)}), \] (43)
it can be regarded as the probability density function, since it is well normalized as
\[ \prod_{m=1}^{M} \int_{\mathbb{W}_N^p} d\theta^{(m)} p_N^L(x_1, \theta^{(1)}; \ldots; x_M, \theta^{(M)}|\varphi) = \int_{\mathbb{W}_N^p} d\theta^{(M)} p_N^L(x_M, \theta^{(M)}|\varphi) = 1 \]
by (42) and (35). As shown by Fig. 5 here we think that \( R_L \) has \( M + 1 \) ‘chambers’ \( R_{\{x_{m-1}, x_m\}}, 1 \leq m \leq M + 1; R_L = \bigcup_{m=1}^{M+1} R_{\{x_{m-1}, x_m\}} \) with \( x_0 = 0, x_{M+1} = L \), where \( R_{\{x_{m-1}, x_m\}} \) and \( R_{\{x_m, x_{m+1}\}} \) are adjacent to each other at the line \( \text{Re} z = x_m, 1 \leq m \leq M \). Under the initial condition that \( \gamma_j \) starts from \( i\varphi_j, 1 \leq j \leq N \) and the nonintersection condition of \( (\mathcal{L}(\gamma_1), \ldots, \mathcal{L}(\gamma_N)) \) in the sense of Fomin (7), Eq. (43) gives the probability density function for joint distributions of the first passage points \( (x_m + i\theta^{(m)}_1, \ldots, x_m + i\theta^{(m)}_N) \) at which \( \gamma_j \)’s first pass from the \( m \)-th chamber \( R_{\{x_{m-1}, x_m\}} \) to the \( (m+1) \)-th chamber \( R_{\{x_m, x_{m+1}\}} \) on the line \( \text{Re} z = x_m, 1 \leq m \leq M \).

By the asymptotics (37) and (38), the limit \( L \to \infty \) is taken as
\[ p_N(x_1, \theta^{(1)}; \ldots; x_M, \theta^{(M)}|\varphi) \]
\[ \equiv \lim_{L \to \infty} p_N^L(x_1, \theta^{(1)}; \ldots; x_M, \theta^{(M)}|\varphi) \]
\[ = C_N(x_M) \frac{\hat{h}_N(\theta^{(M)})}{\hat{h}_N(\varphi)} f_N^\theta(x_1, \theta^{(1)}|\varphi) \prod_{m=1}^{M-1} f_N(x_m, \theta^{(m)}; x_{m+1}, \theta^{(m+1)}), \] (44)
\( \theta^{(m)} \in \mathbb{W}_N^\pi, 1 \leq m \leq M \), given the initial condition \( \varphi \in \mathbb{W}_N^p \).
FIG. 5: Eq. (43) gives the probability density function for joint distributions of $M$ sets of first passage points on the lines $\text{Re } z = x_m, 1 \leq m \leq M$, of $N$-tuples of Brownian paths underlying in the nonintersecting LEBPs.

IV. DETERMINANTAL CORRELATION FUNCTIONS

A. Special initial condition

Let $A_\lambda(\varphi)$ be the numerator of $\hat{s}_\lambda(\varphi)$ given by (23). By the expansion formula (24), we find

$$A_\lambda(\varphi) = \det_{1 \leq j,k \leq N} \sin \varphi_j \left\{ 2^{\lambda_k + N - k} (\cos \varphi_j)_{\lambda_k + N - k} + O((\cos \varphi_j)_{\lambda_k + N - k - 2}) \right\}$$

$$= 2 \sum_{k=1}^N \lambda_k + N - k) \prod_{j=1}^N \sin \varphi_j \det_{1 \leq j,k \leq N} \left[ (\cos \varphi_j)_{\lambda_k + N - k} + O((\cos \varphi_j)_{\lambda_k + N - k - 2}) \right]$$

$$= 2 \sum_{k=1}^N \lambda_k + N - k) \hat{h}_N(\varphi) \tilde{A}_\lambda(\varphi),$$

(45)

where $\tilde{A}_\lambda(\varphi)$ is a symmetric function of $\{\cos \varphi_j\}_{j=1}^N$ with degree $\sum_{k=1}^N \lambda_k$. If we write $(\pi/2, \ldots, \pi/2)$ as $\pi/2$, $\lim_{\varphi \to \pi/2} \tilde{A}_\lambda(\varphi) = \delta_{\lambda, \emptyset}$. Then (22) and (25) give

$$\lim_{\varphi \to \pi/2} \frac{f_N(L, \rho|\varphi)}{\hat{h}_N(\varphi)} = \frac{2^{N^2}}{\pi^N C_N(L)} \hat{h}_N(\rho)$$

(46)

with the factor (40).

Applying (46) to (39), we obtain the limit distribution

$$p_{N/2}^\pi(x, \theta) \equiv \lim_{\varphi \to \pi/2} p_N(x, \theta|\varphi) = \frac{2^{N^2}}{\pi^N (\hat{h}_N(\theta))^2}.$$

(47)

The fact that it is well normalized, i.e. $\int_{W_N^{\pi/2}} p_{N/2}^\pi(\theta)d\theta = 1$, is directly confirmed by using
the $\gamma = 1$ case of ‘the Tchebichev version’ of the Selberg integral
\[
\int_{-1}^{1} \cdots \int_{-1}^{1} \left| \prod_{1 \leq \ell < m \leq N} (\xi_m - \xi_\ell) \right|^{2\gamma} \prod_{j=1}^{N} (1 - \xi_j^2)^{1/2} d\xi_j
= 2^{\gamma(N(N-1)+2N} \prod_{j=0}^{N-1} \frac{\Gamma(1 + \gamma + j\gamma)(\Gamma(j\gamma + 3/2))^2}{\Gamma(1 + \gamma)\Gamma(j\gamma + N + j - 1) + 3),
\]
which is given as Eq.(17.6.4) in [10]. Under this special initial condition $\pi/2$, the joint distribution function (44) for $L \to \infty$ becomes
\[
p_{N}^{\pi/2}(x_{1},\theta^{(1)};x_{2},\theta^{(2)};\ldots;x_{M},\theta^{(M)})
\equiv \lim_{\varphi \to \pi/2} p_{N}(x_{1},\theta^{(1)};x_{2},\theta^{(2)};\ldots;x_{M},\theta^{(M)}|\varphi)
= \frac{2^{N^2} C_{N}(x_{M})}{\pi^{N} C'_{N}(x_{1})} h_{N}(\theta^{(1)}) \prod_{m=1}^{M-1} f_{N}(x_{m},\theta^{(m)};x_{m+1},\theta^{(m+1)}) \hat{h}_{N}(\theta^{(M)}) \quad (48)
\]
for any $M \in \mathbb{N}, 0 < x_{1} < \cdots < x_{M} < \infty$. As a matter of course, if we set $M = 1$, [18] is reduced to be [17].

B. Multiple correlation functions

For $\theta^{(m)} \in \mathbb{W}_{N}$, $N'_{m} \in \{1,2,\ldots,N\}, 1 \leq m \leq M$, we put $\theta^{(m)} = (\theta^{(m)}_{1},\ldots,\theta^{(m)}_{N'})$, $1 \leq m \leq M$. For a sequence $\{N'_{m}\}_{m=1}^{M}$ of positive integers less than or equal to $N$, we define the $(N_{1},\ldots,N_{M})$-multiple correlation function by
\[
\rho_{N}^{\pi/2}(x_{1},\theta^{(1)}_{N_{1}};\ldots;x_{M},\theta^{(M)}_{N_{M}})
= \prod_{m=1}^{M} \prod_{j=N_{m+1}}^{N} \int_{0}^{\pi} d\theta_{j}^{(m)} P_{N}^{\pi/2}(x_{1},\theta^{(1)};\ldots;x_{M},\theta^{(M)}) \prod_{m=1}^{M} \frac{1}{(N - N_{m})!} \quad (49)
\]
For $0 < x < \infty, 0 < \theta < \pi$, we introduce two systems of functions as
\[
\phi_{n}(x,\theta) = \sqrt{\frac{2}{\pi}} \frac{\sin(n\theta)}{\sinh(nx)},
\hat{\phi}_{n}(x,\theta) = \sqrt{\frac{2}{\pi}} \frac{\sinh(nx)\sin(n\theta)}{\sin(n\theta)}, \quad n \in \mathbb{N}. \quad (50)
\]
It is easy to confirm the equalities
\[
\int_{0}^{\pi} \phi_{n}(x,\theta) H_{R_{x'}}(x+i\theta,x'+i\theta')d\theta = \phi_{n}(x',\theta'), \quad 0 < \theta' < \pi,
\int_{0}^{\pi} H_{R_{x'}}(x+i\theta,x'+i\theta')\hat{\phi}_{n}(x',\theta')d\theta' = \hat{\phi}_{n}(x,\theta), \quad 0 < \theta < \pi, \quad (51)
\]
0 < x < x' < ∞, n ∈ N. By using them, the probability density function of joint distributions \( p^{\pi/2}_N(x_1, \theta^{(1)}; \cdots; x_M, \theta^{(M)}) \) is rewritten as follows,

\[
p^{\pi/2}_N(x_1, \theta^{(1)}; \cdots; x_M, \theta^{(M)}) = \det_{1\leq j,k \leq N} \left[ \phi_j(x_1, \theta^{(1)}_k) \right] \prod_{m=1}^{M-1} \det_{1\leq \alpha, \beta \leq N} \left[ H_{R_{m+1}}(x_m + i\theta^{(m)}_{\alpha}, x_{m+1} + i\theta^{(m+1)}_{\beta}) \right] \\
\times \det_{1\leq p,q \leq N} \left[ \hat{\phi}_p(x_M, \theta^{(M)} q) \right].
\]

We have found that this product form of determinants is exactly the same as Eq.(4.5) in [8] given for the multitime distribution function of the noncolliding Brownian motion. Then following the argument given in Section 4 of [8], the following result is obtained.

**Theorem 1.** Any multiple correlation function \( \rho^{\pi/2}_N(x_1, \theta^{(1)}_N; x_2, \theta^{(2)}_N; \cdots; x_M, \theta^{(M)}_N) \) is given by a determinant

\[
\rho^{\pi/2}_N(x_1, \theta^{(1)}_N; x_2, \theta^{(2)}_N; \cdots; x_M, \theta^{(M)}_N) = \det_{1\leq j \leq N, 1\leq k \leq N, 1\leq m,n \leq M} \left[ K^{\pi/2}_N(x_m, \theta^{(m)}_j; x_n, \theta^{(m)}_k) \right]
\]

with the correlation kernel

\[
K^{\pi/2}_N(x, \theta; x', \theta') = \sum_{n=1}^{N} \phi_n(x, \theta) \hat{\phi}_n(x', \theta') \]

\[
= \frac{2}{\pi} \sum_{n=1}^{N} \frac{\sinh(nx')}{\sinh(nx)} \sin(n\theta) \sin(n\theta'), \quad \text{if} \quad x \leq x',
\]

\[
K^{\pi/2}_N(x, \theta; x', \theta') = \sum_{n=1}^{N} \phi_n(x, \theta) \hat{\phi}_n(x', \theta') - H_{R_{x'}}(x' + i\theta', x + i\theta)
\]

\[
= -\frac{2}{\pi} \sum_{n=N+1}^{\infty} \frac{\sinh(nx')}{\sinh(nx)} \sin(n\theta) \sin(n\theta'), \quad \text{if} \quad x > x',
\]

0 < x, x' < ∞, 0 < \theta, \theta' < \pi.

By using the terminology of probability theory, we can say that the ensemble of first passage points \( \{x_m + i\theta^{(m)}_j\} \) \( 1 \leq j \leq N, 1 \leq m \leq M \) is a determinantal point process (or a Fermion point process) \([20, 22]\). Note that the correlation kernel given by \((54)\) and \((55)\) is asymmetric in the ordering of \( x \) and \( x' \); \( K^{\pi/2}_N(x, \theta; x', \theta') \neq K^{\pi/2}_N(x', \theta'; x, \theta) \) for \( x \neq x' \). Such kind of asymmetric correlation kernel was first derived by Eynard and Mehta for two-matrix models in random matrix theory \([10, 23]\). See \([11, 30, 31]\) for recent study on the Eynard-Mehta type determinantal correlations.
The N-tuples of Brownian paths in \( R \) all starting from the point \( i \pi/2 \) are conformally transformed into the paths in \( \Omega \) all starting from the point \( i \).

V. CONFORMAL TRANSFORMATION TO OTHER DOMAIN

In the previous section, we considered a nonintersecting system of LEBPs in the half-infinite-stripe domain, which is divided into \( M \) rectangular chambers and one half-infinite strip by \( M \) straight lines on \( \text{Re} \, z = x_m, 1 < x_1 < \cdots < x_M < \infty \). For the underlying system of \( N \)-tuples of Brownian paths, whose loop-erased parts give the nonintersecting LEBPs, Theorem 1 gives the determinantal correlation functions for first passage points on the lines \( \text{Re} \, z = x_m, 1 \leq m \leq M \). In order to demonstrate that the result can be conformally mapped to other domain consisting of a sequence of chambers in different shapes, here we show a conformal transformation by an entire function \( w = f(z) = e^z \).

By this conformal transformation, the half-infinite-stripe domain \( R = \{z \in \mathbb{R} : \text{Re} \, z > 0, 0 < \text{Im} \, z < \pi \} \) is mapped to the domain \( \Omega = \{z = re^{i\theta} \in \mathbb{C} : r > 1, 0 < \theta < \pi \} \). The rectangular chambers \( R_{(x_{m-1},x_{m})}, 1 \leq m \leq M \), are mapped to the chambers \( \Omega_{(r_{m-1},r_{m})} = \{z = re^{i\theta} \in \Omega : r_{m-1} \leq r < r_m \} \) with \( r_m = e^{x_m}, 1 \leq m \leq M \), and the boundary lines \( \{z \in R : \text{Re} \, z = x_m \} \) are to the arcs of semicircles \( \{z \in \Omega : |z| = r_m \}, 1 \leq m \leq M \).

By this conformal transformation, the paths of complex Brownian motions in \( R \) all starting from the point \( i \pi/2 \) are mapped to those in \( \Omega \) all starting from the point \( i \) as shown by Fig.6.

The conformal invariance of the probability law of complex Brownian motions \( (9) \) implies
the following equality between the multiple correlation functions $\rho_N^{\pi/2}$ defined on $R$ and $\hat{\rho}_N^{\pi}$ defined on $\Omega$,

$$
\rho_N^{\pi/2}(x_1, \theta_{N_1}^{(1)}, \ldots; x_M, \theta_{N_M}^{(M)}) \prod_{m=1}^M dw_{Nm}^{(m)} = \hat{\rho}_N^{\pi}(w_N^{(1)}, \ldots; w_N^{(M)}) \prod_{m=1}^M dw_{Nm}^{(m)},
$$

(56)

where $w_{Nm}^{(m)} = (w_1^{(m)}, \ldots, w_N^{(m)})$ with $w_j^{(m)} = f(x_m + i\theta_j^{(m)}) = e^{x_m + i\theta_j^{(m)}}$, $r \equiv e^{x_m}$, $1 \leq m \leq M$. Since $0 < x_1 < \ldots < x_M < \infty$ are fixed,

$$
dw_j^{(m)} = \left| \frac{dw_j^{(m)}}{d\theta_j^{(m)}} \right| d\theta_j^{(m)} = r_m d\theta_j^{(m)}, \quad 1 \leq m \leq M, 1 \leq j \leq N_m,
$$

we have the following determinantal correlations for first passage points on the semicircles in $\Omega$.

**Corollary 2.** Any multiple correlation function in $\Omega$ is given by a determinant

$$
\hat{\rho}_N^{\pi}(\{r_1e^{i\theta_j^{(1)}}\}_{j=1}^{N_1}, \ldots; \{r_Me^{i\theta_j^{(M)}}\}_{j=1}^{N_M}) = \det_{1 \leq j \leq N_m, 1 \leq k \leq N_m, 1 \leq m, n \leq M} [\hat{K}_N^{i}(r_me^{i\theta_j^{(m)}}, r_ne^{i\theta_k^{(n)}})]
$$

(57)

with the correlation kernel

$$
\hat{K}_N^{i}(r_\theta e^{i\theta}, r_{\theta'} e^{i\theta'}) = \frac{2}{\pi r} \sum_{n=1}^{N} \frac{(r')^n - (r)^n}{r^n - r'^n} \sin(n\theta) \sin(n\theta'), \quad \text{if} \quad r \leq r',
$$

$$
\hat{K}_N^{i}(r_\theta e^{i\theta}, r_{\theta'} e^{i\theta'}) = -\frac{2}{\pi r} \sum_{n=N+1}^{\infty} \frac{(r')^n - (r)^n}{r^n - r'^n} \sin(n\theta) \sin(n\theta'), \quad \text{if} \quad r > r',
$$

(58)

$1 < r, r' < \infty, 0 < \theta, \theta' < \pi$.

From (58), we find that if we set $r = r' > 1$,

$$
\hat{K}_N^{i}(r_\theta e^{i\theta}, r_{\theta'} e^{i\theta'}) = \frac{2}{\pi r} \sum_{n=1}^{N} \sin(n\theta) \sin(n\theta')
$$

$$
= \frac{\sin((N+1)\theta) \sin(N\theta') - \sin(N\theta) \sin((N+1)\theta')}{\pi r (\cos \theta - \cos \theta')}
$$

(59)

for $\theta \neq \theta'$. From it the density function $\rho_N^{\pi}(r e^{i\theta}) = \lim_{e \to 0} \hat{K}_N^{i}(r e^{i\theta}, r e^{i(\theta + \epsilon)})$ is given as

$$
\rho_N^{\pi}(r e^{i\theta}) = \frac{1}{\pi r \sin \theta} \left[ N \sin \theta - \cos \theta \cos(N\theta) \sin(N\theta) + \sin \theta \sin^2(N\theta) \right].
$$

(60)

For $N = 3$, Fig.7 shows the dependence of $\rho_N^{\pi}(r e^{i\theta})$ on $x = \text{Re} (r e^{i\theta}) = r \cos \theta$ and $y = \text{Im} (r e^{i\theta}) = r \sin \theta$. There are $N = 3$ ridges in the plots.
FIG. 7: (Color online) The density function $\hat{\rho}^i_N(re^{i\theta})$ for $N = 3$, where $x = \text{Re}(re^{i\theta}) = r\cos\theta$, $y = \text{Im}(re^{i\theta}) = r\sin\theta$. There are three ridges.

On an arc of semicircle $|z| = r > 1, 0 < \arg(z) < \pi$, the two-point correlation function is given by

$$\hat{\rho}^i_N(re^{i\theta}, r'e^{i\theta'}) = \hat{\rho}^i_N(re^{i\theta})\hat{\rho}^i_N(r'e^{i\theta'}) - (\hat{K}^i_N(re^{i\theta}, r'e^{i\theta'})^2$$

(61)

with (59) and (60) for $0 < \theta, \theta' < \pi$. In Figs. 8 and 9 we set $r = 4$ and $\theta = \pi/2$ and plot (61) as a function of $\theta'$ for $N = 5$ and $N = 20$, respectively. Due to the nonintersection condition for loop-erased parts, the two-point correlation function becomes zero as $\theta' \to \theta = \pi/2$.

In general, for $1 < r < r', 0 < \theta, \theta' < \pi$, Corollary 2 gives the two-point correlation function as

$$\hat{\rho}^i_N(re^{i\theta}, r'e^{i\theta'}) = \hat{\rho}^i_N(re^{i\theta})\hat{\rho}^i_N(r'e^{i\theta'}) + \frac{4}{\pi^2rr'} \sum_{n=1}^{N} \frac{(r')^n - (r')^{-n}}{r^n - r^{-n}} \sin(n\theta) \sin(n\theta')$$

$$\times \sum_{m=N+1}^{\infty} \frac{r^m - r^{-m}}{(r')^m - (r')^{-m}} \sin(m\theta') \sin(m\theta)$$

(62)

with (60). For $N = 3$ we set $re^{i\theta} = 2e^{i\pi/2} = 2i$ and show by Fig. 10 the dependence of the two-point correlation function (62) on $x' = \text{Re}(r'e^{i\theta'}) = r'\cos\theta'$ and $y' = \text{Re}(r'e^{i\theta'}) = r'\sin\theta'$.
FIG. 8: For $N = 5$, the two-point correlation function (61) on an arc of semicircle $|z| = r = 4$ with $\theta = \pi/2$ is shown as a function of $\theta'$. There are $N - 1 = 4$ peaks in the plot.

FIG. 9: For $N = 20$, the two-point correlation function (61) on an arc of semicircle $|z| = r = 4$ with $\theta = \pi/2$ is shown as a function of $\theta'$.

VI. CONCLUDING REMARKS

Here we discuss the infinite number of paths limit, $N \to \infty$. When we take this limit in (60), we have

$$\lim_{N \to \infty} \frac{1}{N} \rho_N^i (re^{i\theta}) = \frac{1}{\pi r}.$$  

(63)
FIG. 10: (Color online) Two-point correlation function \( \hat{K}^i \) with \( N = 3, r = 2, \theta = \pi/2 \) is shown as a function of \( x' = \text{Re}(r'e^{i\theta'}) = r' \cos \theta' \) and \( y' = \text{Re}(r'e^{i\theta'}) = r' \sin \theta' \).

That is, the distribution of the first passage point becomes uniform on an arc of semicircle. If we set

\( r = N + u, \quad r' = N + u', \quad \theta = \frac{a}{N}, \quad \theta' = \frac{a'}{N}, \)

then the correlation kernel converges to the following as \( N \to \infty \),

\[
\hat{K}^i(u, a; u', a') = \lim_{N \to \infty} \hat{K}^i_N(((N + u)e^{ia/N}, (N + u')e^{ia'/N}))
\]

\[
= \begin{cases} \\
\frac{2}{\pi} \int_0^1 e^{-(u-u')s} \sin(as) \sin(as') ds, & \text{if } u < u', \\
\frac{2}{\pi} \int_1^\infty e^{-(u-u')s} \sin(as) \sin(as') ds, & \text{if } u > u'. 
\end{cases}
\]

(64)

In the present paper, we have imposed special initial conditions such that all Brownian paths start from a single point \( i\pi/2 \) for the domain \( R \) and from \( i \) for the domain \( \Omega \). Study for general initial condition will be reported elsewhere in the future.

At the end of this paper, we note the fact that the scaling limit of LERW is described by the SLE(2) path, a random continuous simple curve generated by the Schramm-Loewner evolution with a special value of parameter \( \kappa = 2 \) [29, 32, 33]. Kozdron [28] showed that \( 2 \times 2 \) Fomin’s determinant representing the event \( \mathcal{L}(\gamma_1) \cap \gamma_2 = \emptyset \) for two Brownian paths \((\gamma_1, \gamma_2)\) is proportional to the probability that \( \gamma_{\text{SLE}(2)} \cap \gamma = \emptyset \), where \( \gamma_{\text{SLE}(2)} \) and \( \gamma \) denote the SLE(2) path and a Brownian path (see also [27]). On the other hand, Lawler and Werner gave a method to correctly add Brownian loops to an SLE(2) path to obtain a Brownian
path \[34\]. Interpretation of the results reported in the present paper in terms of ‘mutually avoiding SLE paths’ will be an interesting future problem.

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Appendix A: Derivation of $H_{R_L}$ and $H_{\partial R_L}$

For $z = x + iy \in R_L, L > 0$, we solve the Laplace equation

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) H_{R_L}(x + iy, L + i\rho) = 0 \tag{A1}$$

for $0 < \rho < \pi$, by the method of separation of variables. We set $H_{R_L}(x + iy, L + i\rho) = X(x)Y(y)$, where description of dependence on $L$ and $\rho$ is omitted. Then we have a pair of ordinary differential equations

$$X''(x) = cX(x), \tag{A2}$$

$$Y''(y) = -cY(y) \tag{A3}$$

with a constant $c$, which does not depend on $x$ nor $y$. With the boundary condition $Y(0) = Y(\pi) = 0$, Eq.(A3) is solved as

$$Y(y) = a \sin(ny), \quad \sqrt{c} = n, \quad n \in \mathbb{N}, \quad 0 < y < \pi \tag{A4}$$

with a constant $a$. Then Eq.(A2) becomes $X''(x) = n^2X(x)$, which is solved under the condition $X(0) = 0$ as

$$X(x) = b \sinh(nx), \quad 0 < x < L. \tag{A5}$$

Then we have the form

$$H_{R_L}(x + iy, L + i\rho) = \sum_{n=1}^{\infty} c_n(L, \rho) \sinh(nx) \sin(ny) \tag{A6}$$
with a series of coefficients \( \{c_n(L, \rho)\} \), where dependence on \( L \) and \( \rho \) is now revealed. In this case the boundary condition for the Poisson kernel \([11]\) becomes

\[
\lim_{x \to L} H_{RL}(x + iy, L + i\rho) = \delta(y - \rho),
\]

(A7)

which uniquely determines the coefficients as

\[
c_n(L, \rho) = \frac{2}{\pi} \frac{\sin(n\rho)}{\sinh(nL)},
\]

(A8)

since the Fourier series of the Dirac delta function is known as

\[
\delta(y - \rho) = \frac{2}{\pi} \sum_{n=1}^{\infty} \sin(ny) \sin(n\rho)
\]

for \( y, \rho > 0 \).

Following \([12]\), the boundary Poisson kernel is obtained by taking the limit as

\[
H_{\partial RL}(iy, L + i\rho) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} H_{RL}(iy + \varepsilon, L + i\rho)
\]

\[
= \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{n \sin(ny) \sin(n\rho)}{\sinh(nL)}.
\]

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