Convergence in Law for the Branching Random Walk Seen from Its Tip

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Abstract Consider a critical branching random walk on the real line. In a recent paper, Aïdékon (2011) developed a powerful method to obtain the convergence in law of its minimum after a log-factor translation. By an adaptation of this method, we show that the point process formed by the branching random walk seen from the minimum converges in law to a decorated Poisson point process. This result, confirming a conjecture of Brunet and Derrida (J Stat Phys 143:420–446, 2011), can be viewed as a discrete analog of the corresponding results for the branching Brownian motion, previously established by Arguin et al. (2010, 2011) and Aïdékon et al. (2011).

Keywords Branching random walk · Decorated Poisson point process · Convergence in law

Mathematics Subject Classification (2000) 60J80 · 60G55

1 Introduction

We consider a branching random walk on the real line $\mathbb{R}$. Initially, a single particle sits at the origin. At time 1, the particle gives birth to some children which form the first generation of the branching random walk and whose positions are given by a point process $L$ on $\mathbb{R}$. At time 2, each of the particles in the first generation gives birth to new particles that are positioned—with respect to their birth positions—according to the law of the same point process $L$; they form the second generation, and so on. We assume that each particle produces new particles independently of other particles in the same generation and of everything up to that generation.

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Let \( \mathbb{T} \) be the genealogical tree of the particles in the branching random walk, and then \( \mathbb{T} \) is a Galton–Watson tree. We write \( |z| = n \) if a particle \( z \) is in the \( n \)th generation and denote its position by \( V(z) \). The collection of positions \( (V(z), z \in \mathbb{T}) \) is our branching random walk.

The study of the minimal position \( M_n := \min_{|z| = n} V(z) \) has attracted many recent interests. The law of large numbers for the speed of the minimum goes back to the works of Hammersley \[14\], Kingman \[18\] and Biggins \[7\]. The second-order problem was recently studied separately by Hu and Shi \[15\] (a.s fluctuation), and Addario-Berry and Reed \[1\]. In \[1\], the authors computed the expectation of \( M_n \) up to \( O(1) \) and showed, under suitable assumptions, that the sequence of the minimum is tight around its mean. Through recursive equations, Bramson and Zeitouni \[9\] obtained the tightness of \( M_n \) around its median, assuming some hypotheses on the decay of the tail distribution. A definitive answer was recently given by Aïdékon \[2\], where he proved the convergence of the minimum \( M_n \) centered around \( \frac{3}{2} \) log \( n \) for the general class of critical branching random walks.

One problem of great interest in the study of branching random walk is to characterize its behavior seen from the minimal position, namely the asymptotic of the point process formed by \( \{V(z) - M_n, |z| = n\} \) as \( n \to \infty \). The corresponding problem for the branching Brownian motion (the continuous analog of branching random walk) was solved very recently by Arguin et al. \[5,6\] and in parallel by Aïdékon et al. \[3\].

The aim of this paper is to establish the analogous results for branching random walk. Our main result, summed up in Theorem 1.1, will give the existence of the limiting point process together with a partial description, which also confirms the prediction in Brunet and Derrida \[10\]. Our method, largely inspired by Aïdékon \[2\], consists in an analysis of the Laplace transform of the point process.

Following \[2\], we assume

\[
\begin{align*}
\mathbb{E} \left[ \sum_{|z| = 1} 1 \right] &> 1, \\
\mathbb{E} \left[ \sum_{|z| = 1} e^{-V(z)} \right] &= 1, \\
\mathbb{E} \left[ \sum_{|z| = 1} V(z)e^{-V(z)} \right] &= 0. 
\end{align*}
\tag{1.1}
\]

Every branching random walk satisfying mild assumptions can be reduced to this case by a linear transformation. We refer to the Appendix A in \[16\] for a precise discussion. Notice that we allow \( \mathbb{E} \left[ \sum_{|z| = 1} 1 \right] = \infty \) and even \( \mathbb{P} \left( \sum_{|z| = 1} 1 = \infty \right) > 0 \). The couple \((M_n, W_{n,\beta})\) is the most often encountered random variables in our work, with

\[
M_n := \min\{V(z), |z| = n\}, \quad W_{n,\beta} := \sum_{|z| = n} e^{-\beta V(z)}, \quad \beta > 1, \quad n \geq 1.
\]

We also need the derivative martingale

\[
Z_n := \sum_{|z| = n} V(z)e^{-V(z)}, \quad Z_\infty = \lim_{n \to \infty} Z_n. \tag{1.2}
\]

By \[8\] and \[2\], we know that \( Z_\infty \) exists almost surely and is strictly positive on the set of nonextinction of \( \mathbb{T} \). As in the continuous case \[3\], we introduce the point process formed by the particles of the recentered branching random walk.
\[ \mu_n := \sum_{|z|=n} \delta_{\{V(z) - \frac{3}{2} \log n + \log Z_n\}}, \quad n \geq 1. \]

As in the setting of Kallenberg [17, pp. 10], a point process is considered as a random map with value in \( \mathcal{N} \) the space of locally finite counting measure equipped with the vague topology. Then convergence in distribution of point processes will mean weak convergence of the corresponding distributions with respect to the vague topology (see [17, pp. 42]).

We will show the existence in distribution of a limiting point process of \( \mu_n \) as \( n \to \infty \), from which we deduce results on \( \mu'_n := \sum_{|z|=n} \delta_{\{V(z) - M_n\}}, n \geq 1. \)

Writing for \( y \in \mathbb{R} \cup \{\infty\} \), \( y^+ := \max(y, 0) \), we introduce the random variables

\[ X := \sum_{|z|=1} e^{-V(z)}, \quad \tilde{X} := \sum_{|z|=1} V(z)^+ e^{-V(z)}. \quad (1.3) \]

with the convention \( \infty e^{-\infty} = 0. \) We finally assume that

\[ \mathbf{E} \left[ \sum_{|z|=1} V(z)^2 e^{-V(z)} \right] < \infty, \quad \mathbf{E} \left[ X (\log_+ X)^2 \right] < \infty, \quad \mathbf{E} \left[ \tilde{X} (\log_+ \tilde{X}) \right] < \infty. \quad (1.4) \]

\[ \mathbf{E} \left[ \sum_{|z|=1} V(z)^2 e^{-V(z)} \right] < \infty, \quad \mathbf{E} \left[ X (\log_+ X)^2 \right] < \infty, \quad \mathbf{E} \left[ \tilde{X} (\log_+ \tilde{X}) \right] < \infty. \quad (1.5) \]

The main result of this paper is the following theorem:

**Theorem 1.1** Under (1.1), (1.4) and (1.5), as \( n \to \infty \), conditioning on the set of nonextinction, the pair \( (\mu_n, Z_n) \) converges jointly in distribution to \( (\mu_\infty, Z_\infty) \) where \( \mu_\infty \) and \( Z_\infty \) are independent and \( \mu_\infty \) is obtained as follows.

(i) Define \( \mathcal{P} \) to be a Poisson point process on \( \mathbb{R} \), with intensity measure \( \lambda e^x dx \) for some positive real constant \( \lambda \).

(ii) For each atom \( x \) of \( \mathcal{P} \), we attach a point process \( x + \mathcal{D}^{(x)} \) where \( \mathcal{D}^{(x)} \) are independent copies of a certain point process \( \mathcal{D} \) in \( \mathbb{R}_+ \).

(iii) The point process \( \mu_\infty \) is the superposition of all the point processes \( x + \mathcal{D}^{(x)} \), i.e., \( \mu_\infty := \{x + y : x \in \mathcal{P}, y \in \mathcal{D}^{(x)}\} \).

**Remark** The point process \( \mu_\infty \) is called “decorated Poisson point process with decoration \( \mathcal{D} \).” We refer to [22] for a more complete description.

**Corollary 1.2** Under (1.1), (1.4) and (1.5), conditioning on the set of nonextinction, seen from the leftmost particle, the point process \( \mu'_n \) formed by the particles \( \{V(u) - M_n, |u| = n\} \) converges in distribution to the point process \( \mu'_\infty \) obtained by replacing the Poisson point process \( \mathcal{P} \) in step (i) above by \( \mathcal{P}' \) described in step (i)' below:

(i)' Let \( e \) be a standard exponential random variable. Conditionally on \( e \), define \( \mathcal{P}' \) to be a Poisson point process on \( \mathbb{R}_+ \), with intensity measure \( e e^x 1_{\mathbb{R}_+} dx \) to which we add an atom in 0.

The decoration point process \( \mathcal{D} \) remains the same.
These two results imitate the corresponding results for the branching Brownian motion, in particular Theorem 2.1 and Corollary 2.2 of Aïdékon, Berestycki, Brunet and Shi [3] (and also those of [5] and [6]). However, we do not adopt the same method as in [3] because first, the spine decomposition for the branching random walk leads to a use of Palm measures, which is much more complicated than in the case of branching Brownian motion, and second, the path decomposition for a random walk is also more complex than in the Brownian case. Instead, we shall imitate the fine analysis of Aïdékon [2] to study the Laplace transform of $\mu_n$. More precisely, the main step in the proof of Theorem 1.1 is to establish the convergence in law of $(n^{3/2}\beta_1 W_{n,\beta_1}, \ldots, n^{3/2}\beta_k W_{n,\beta_k})$ for any $k \geq 1$ and any $\beta_k > \cdots > \beta_1 > 1$. A crucial observation, inspired from [2], is that the convergence in law of this $k$-dimensional random vector can be reduced to the study of its tail distribution. From this, we can prove the convergence in law stated in Theorem 1.1, and as a by-product, we shall also get an expression for the Laplace transform of the limiting point process. The later might have some independent interest for further analysis of $\mu_\infty$.

Note that the present paper provides a simplification of the method of [2] and could be used to recover Theorem 1.1 of [2]. Indeed, using the deep understanding induced by the work of [2], we are able to skip the use of the “killed branching random walk” (see [2] for a definition). However, this simplification necessitates to prove numerous lemmas which have very close analog in [2]. Most often, these lemmas are stated and proved in “Appendices 1, 2 and 3”.

The paper is organized as follows: Sect. 2 contains the main estimates on the tail of distribution of $(n^{3/2}\beta_1 W_{n,\beta_1}, \ldots, n^{3/2}\beta_k W_{n,\beta_k})$ for $k \geq 1$ and $\beta_k > \cdots > \beta_1 > 1$, from which we establish the convergence of some Laplace transforms of $\mu_n$ (Theorem 2.3) and give the proof of Theorem 1.1. Sect. 3 is devoted to the proof of Theorem 2.3 by admitting Proposition 2.2. Finally, we prove in Sects. 4 and 5, respectively, Propositions 2.1 and 2.2.

2 Main Steps of the Proof of Theorem 1.1

To shorten the statements, we introduce some notations: For $n \geq 1$, $\beta > 1$, define

$$\tilde{W}_{n,\beta} := n^{3/2}\beta W_{n,\beta}, \quad \tilde{\mu}_n(\beta) = n^{3/2}\beta \sum_{|z|=n} e^{-\beta(V(z)+\log Z_\infty)}.$$  

Remark that $\tilde{\mu}_n(\beta)$ is also equal to $\int_{\mathbb{R}} e^{-\beta x} d\mu_n(x)$. In a general context, many quantities with tilde are associated with the natural normalization $n^{3/2}\beta$ except for some obvious abuse of notation: For example, in the sequel, we will denote for brevity $\tilde{W}_{n,-|u|,\beta} := n^{3/2}\beta W_{n,-|u|,\beta}$. In a similar spirit, we write $\tilde{M}_n := M_n - \frac{3}{2} \log n$ and $\tilde{M}_{n,-|u|} := M_{n,-|u|} - \frac{3}{2} \log n$ for some vertex $|u| \leq n$ (we shall recall these notations to avoid any confusion). Throughout the paper, $\mathbb{N}^*$, $\mathbb{R}^+$ and $\mathbb{R}^*_+$ denote, respectively, $\mathbb{N}\setminus\{0\}$, $[0, \infty)$ and $(0, \infty)$. At last, we often encounter notations $\delta$, $\beta$ and $y$ for, respectively, $(\delta_1, \ldots, \delta_k)$, $(\beta_1, \ldots, \beta_k)$ and $(y_1, \ldots, y_k)$, with some $k \geq 1$ determined in the context.
2.1 Main Preliminary Results

In this section, we state the main results which will lead to the proof of Theorem 1.1 (deferring their proofs to the next sections).

The following proposition gives a uniform control on the tail of distribution of the process \((n^{\frac{3}{2}}W_{n,\beta})_{n\geq 1}\).

**Proposition 2.1** Under (1.1) and (1.5), there exists \(c_1 > 0\) such that for any \(n > 1\), and \(x \geq 1\),
\[
P \left( \widetilde{W}_{n,\beta} \geq e^{\beta x} \right) \leq c_1 (1 + x) e^{-x}. \tag{2.1}
\]

**Proposition 2.2** Under (1.1) and (1.5), for any \(d \in \mathbb{N}^*\) and \(\beta \in (1, \infty)^d\) there exists a function \(\rho_\beta : \theta \ni (\mathbb{R}^*_+)^d \rightarrow \rho_\beta(\theta) \in (0, \infty)\), which satisfies the following: For any \(\theta \in \mathbb{R}^d_+\), \(\epsilon > 0\), there exists \((A, N)_{(\epsilon)} \in \mathbb{R}_+ \times \mathbb{N}^*\) such that \(\forall n > N\) and \(x \in [A, \frac{3}{2} \log n - A]\), we have
\[
\left| e^x \mathbb{E} \left( 1 - \exp \left\{ - \sum_{i=1}^{d} \theta_i e^{-\beta_i x} \widetilde{W}_{n,\beta_i} \right\} \right) - \rho_\beta(\theta) \right| \leq \epsilon. \tag{2.2}
\]

Moreover, \(\lim_{\theta \rightarrow 0} \rho_\beta(\theta) = 0\) and for any \(y \in \mathbb{R}\), \(\theta \in (\mathbb{R}^*_+)^d\), \(\rho_\beta(\theta_1 e^{\beta_1 y}, \ldots, \theta_d e^{\beta_d y}) = e^y \rho_\beta(\theta)\) (this last equality is plain by a change of variable in (2.2)).

The key step in the proof of Theorem 1.1 is the following result:

**Theorem 2.3** Under (1.1) and (1.5), \(\forall d \in \mathbb{N}\), \(\beta \in (1, \infty)^d\),
\[
\forall \alpha \in \mathbb{R}_+, \lim_{n \rightarrow \infty} \mathbb{E} \left( e^{-\sum_{i=1}^{d} \theta_i \hat{\mu}_n(\beta_i) e^{-\alpha Z_{\infty}} 1_{\{Z_{\infty} > 0\}}} \right) = e^{-\rho_\beta(\theta)} \mathbb{E} \left( e^{-\alpha Z_{\infty}} 1_{\{Z_{\infty} > 0\}} \right). \tag{2.3}
\]

In particular as \(n \rightarrow \infty\), conditionally on \(\{Z_{\infty} > 0\}\), \((\hat{\mu}_n(\beta_1), \ldots, \hat{\mu}_n(\beta_d))\) converges in law to some random vector \((\hat{\mu}_{\infty}(\beta_1), \ldots, \hat{\mu}_{\infty}(\beta_d))\) independent of \(Z_{\infty}\).

2.2 Proof of Theorem 1.1 by Admitting Theorem 2.3

Let us introduce the conditional probability \(P^*(\cdot) := P(\cdot|\text{nonextinction})\). Recall that under \(P^*\), \(Z_{\infty} > 0\) a.s. To prove Theorem 1.1, we have to keep in mind two facts:

- According to Theorem 2.3, for any \(l \in \mathbb{N}^*\) and \(\beta \in ((2, 3, \ldots))^l\) the vector \((\hat{\mu}_n(\beta_1), \ldots, \hat{\mu}_n(\beta_l))\) converges in law under \(P^*\). We deduce that for any \(\theta \in \mathbb{R}^l\), \(\sum_{i=1}^{l} \theta_i \hat{\mu}_n(\beta_i)\) converges also in law. But \(\sum_{i=1}^{l} \theta_i \hat{\mu}_n(\beta_i)\) is the same as \(\int_{\mathbb{R}} Q(e^{-x}) \, d\mu_n(x)\) with \(Q(X) := \sum_{i=1}^{l} \theta_i X^{\beta_i}\). Hence, if \(Q\) is polynomial function such that \(Q(0) = Q'(0) = 0\), then \(\int_{\mathbb{R}} Q(e^{-x}) \, d\mu_n(x)\) converges in law in law under \(P^*\).
- In [2], Elie Aïdékon has proved that under \(P^*\), \(M_n - \frac{3}{2} \log n\) converges in law, but \(M_n - \frac{3}{2} \log n + Z_{\infty}\) is nothing but the smallest atom of the point process \(\mu_n\). Thus, it is clear that \(\lim_{b \rightarrow \infty n \epsilon \mathbb{N}} \sup B^* (\mu_n(-\infty, -b) > 0) = 0\).
Let \( C_c(\mathbb{R}) \) be the set of continuous functions in \( \mathbb{R} \) with compact support. The existence of \( \mu_\infty \) is now a consequence of Kallenberg [17]. Indeed Lemma 5.1 in [17] says that \( \mu_n \) converges in law to some \( \mu_\infty \) (for the vague topology, see [17] chapter 4), providing that \( \forall f \in C_c(\mathbb{R}), \left( \int_{\mathbb{R}} f(x) d\mu_n(x) \right)_{n \in \mathbb{N}} \) converges in law to some random variable \( \mu(f) \).

Without loss of generality, suppose that \( f(x) = g(x)e^{-2x} \) with \( g \in C_c(\mathbb{R}) \). Let \( \epsilon > 0 \). Let \( b \in \mathbb{R} \) such that \( g(x) = 0 \) for any \( x \in [-b, b]^c \) and \( \sup_{n \in \mathbb{N}} (\mu_n(-\infty, -b] > 0) \leq \epsilon \). According to Stone–Weierstrass theorem, there exists a sequence of polynomial function \( Q_q \in \mathbb{R}[x] \) such that \( \sup_{y \in [0, e^b]} \left| Q_q(y) - g(y) \right| \leq \frac{1}{q} \). By a change of variable, this is equivalent to \( \sup_{y \in [-b, +\infty)} \left| Q_q(e^{-y}) - g(y) \right| \leq \frac{1}{q} \).

For \( \theta \in \mathbb{R} \) and \( n, p, q \in \mathbb{N}^* \), the triangle inequality implies that

\[
\left| \mathbb{E}^* \left( e^{i\theta \int_{\mathbb{R}} g(x)e^{-2x} d\mu_n(x)} \right) - \mathbb{E}^* \left( e^{i\theta \int_{\mathbb{R}} g(x)e^{-2x} d\mu_p(x)} \right) \right| \leq (1)_{n,q} + (1)_{p,q} + (2)_{n,p,q},
\]

with

\[
(1)_{n,q} := \left| 1 - \mathbb{E}^* \left( e^{i\theta \int_{\mathbb{R}} g(x)Q_q(e^{-x})e^{-2x} d\mu_n(x)} \right) \right|,
\]

\[
(2)_{n,p,q} := \left| \mathbb{E}^* \left( e^{i\theta \int_{\mathbb{R}} Q_q(x)e^{-2x} d\mu_n(x)} \right) - \mathbb{E}^* \left( e^{i\theta \int_{\mathbb{R}} Q_q(x)e^{-2x} d\mu_p(x)} \right) \right|.
\]

For any \( y \in \mathbb{R} \), \( |1 - e^{iy}| \leq |y| \); therefore, for any \( n \) and \( q \in \mathbb{N}^* \),

\[
(1)_{n,q} \leq 2\mathbb{P}^* \left( \int_{\mathbb{R}} e^{-2x} d\mu_n(x) \geq \sqrt{q} \right) + 2\mathbb{P}^* (\mu_n(-\infty, -b] > 0)
\]

\[
+ \mathbb{E}^* \left( \left| e^{i\theta \int_{\mathbb{R}} g(x)Q_q(e^{-x})e^{-2x} d\mu_n(x)} - 1 \right| 1_{\{ \int_{\mathbb{R}} e^{-2x} d\mu_n(x) \leq \sqrt{q}, \mu_n(-\infty, -b]=0 \}} \right)
\]

\[
\leq 2\mathbb{P}^* (\mu_n(2) \geq \sqrt{q}) + 2\epsilon + |\theta| \sqrt{q}.
\]

Thanks to the tightness of \( (\mu_n(2))_{n \in \mathbb{N}} \), we can choose \( q_0 \) sufficiently large such that \( \sup (1)_{n,q_0} \leq 3\epsilon \). As \( \int_{\mathbb{R}} Q_q_0(e^{-x})d\mu_n(x) \) converges in law, for \( n \) and \( p \) sufficiently large \( (2)_{n,p,q} \leq \epsilon \) uniformly in \( \theta \) in any compact set. Thus, the sequence \( \mathbb{E}^* \left( e^{i\theta \int_{\mathbb{R}} f(x)e^{-2x}d\mu_n(x)} \right) \) satisfies Cauchy’s criterion and hence admits a limit that we denote \( \Psi_f(\theta) \). Moreover, the convergence is uniform on every compact in \( \theta \); thus, \( \Psi_f(\theta) \) is continuous at 0.

We have proved that the limiting law \( \mu_\infty \) exists. To obtain the description of this point process as a decorated Poisson point process, it suffices, according to Corollary 5.2 of Maillard [22] (see also [12]), to prove that \( \mu_\infty \) is superposable. See [10] for the origin of this idea. We recall what this notion means:
Let $\mathcal{N}$ be the space of locally finite counting measures on $\mathbb{R}$. For every $x \in \mathbb{R}$, define the translation operator $T_x : \mathcal{N} \to \mathcal{N}$, by $(T_x \mu)(A) = \mu(A + x)$ for every Borel set $A \subset \mathbb{R}$. Let $\mathcal{L}'$ be an independent copy on $\mathcal{L}$. We say that $\mathcal{L}$ is superposable, if

$$T_\alpha \mathcal{L} + T_\beta \mathcal{L}' \overset{(d)}{=} \mathcal{L},$$

for every $\alpha, \beta \in \mathbb{R}$ such that $e^{-\alpha} + e^{-\beta} = 1$.

In view of proving the superposability of $\mu_\infty$, for any $(a, b) \in \mathbb{R}^2$, we introduce $\mathbb{T}^a$ and $\mathbb{T}^b$ the genealogical trees formed by two independent branching random walks starting, respectively, at $a$ and $b$. Following section 1, we introduce all the objects related to $\mathbb{T}^a$ and $\mathbb{T}^b$ by adding an extra superscript $a$ or $b$ in the notation (i.e., $Z_n^a := \sum_{|z|=n, z \in \mathbb{T}^a} V(z)e^{-V(z)}$, $Z_n^b := \sum_{|z|=n, z \in \mathbb{T}^b} V(z)e^{-V(z)}$, $\mu_n^a, \mu_n^b$...). We also define

$$\mu_n^{a,b} := \mu_n^a + \mu_n^b = \sum_{u \in \mathbb{T}^a, |u|=n} \delta_{V(u)+Z_\infty^a} + \sum_{u \in \mathbb{T}^b, |u|=n} \delta_{V(u)+Z_\infty^b},$$

and

$$\hat{\mu}_n^{a,b}(\beta) := \int_{\mathbb{R}} e^{-\beta x} d\mu_n^{a,b}(x), \quad \beta > 1.$$  

We note that $\hat{\mu}_n^{a,b}(\beta) \overset{(d)}{=} e^{-\beta a} \hat{\mu}_n^{(1)}(\beta) + e^{-\beta b} \hat{\mu}_n^{(2)}(\beta)$ with $\hat{\mu}_n^{(i)}(\beta)$, $i \in \{1, 2\}$, two independent copy of $\hat{\mu}_n(\beta)$. The key point to prove that $\mu_\infty$ is superposable is the following corollary:

**Corollary 2.4** Under (1.1) and (1.5), for any $d \in \mathbb{N}^*$, $\beta \in (1, \infty)^d$, $a$ and $b \in \mathbb{R}$ such that $e^{-a} + e^{-b} = 1$ we have

$$\lim_{n \to \infty} \mathbb{E} \left( e^{-\sum_{i=1}^d \theta_i \hat{\mu}_n^{a,b}(\beta_i)} 1_{\{Z_{\infty}^a > 0, Z_{\infty}^b > 0\}} \right) = e^{-\rho_\beta(\theta)} \mathbb{P}(Z_{\infty}^a > 0, Z_{\infty}^b > 0), \quad (2.5)$$

where $\rho_\beta$ is the same function in (2.3) and (2.5). Therefore, when $n \to \infty$, the limit in law of $(\hat{\mu}_n^{a,b}(\beta_1), \ldots, \hat{\mu}_n^{a,b}(\beta_d))$ conditionally on $\{Z_{\infty}^a > 0, Z_{\infty}^b > 0\}$ is also the law of $(\hat{\mu}_\infty(\beta_1), \ldots, \hat{\mu}_\infty(\beta_d))$ (see Theorem 2.3).

**Proof of Corollary 2.4.** We apply Theorem 2.3 to obtain

$$\lim_{n \to \infty} \mathbb{E} \left( e^{-\sum_{i=1}^d \theta_i \hat{\mu}_n^{a,b}(\beta_i)} 1_{\{Z_{\infty}^a > 0, Z_{\infty}^b > 0\}} \right) = e^{-\rho_\beta(\theta)} \mathbb{P}(Z_{\infty}^a > 0, Z_{\infty}^b > 0)$$

for all $x \in \mathbb{R}$, $\rho_\beta(\theta) := \rho_\beta(e^{-\beta_1 x \theta_1}, \ldots, e^{-\beta_d x \theta_d})$. By Proposition 2.2, $\rho_\beta(\theta) = e^{-x} \rho_\beta(\theta)$. Applied to $x = a$ and $x = b$ this gives the result.

**Proof of the superposability of $\mu_\infty$.** Let $(a, b) \in \mathbb{R}^2$ be two constants such that $e^{-a} + e^{-b} = 1$, via the proof of the existence of $\mu_\infty$ and Corollary 2.4, it is clear that conditionally on $\{Z_{\infty}^a > 0, Z_{\infty}^b > 0\}$, $\mu_n^{a,b}$ converges also in law to $\mu_\infty$. Moreover, for any $n \in \mathbb{N}$, $\mu_n^{a,b} \overset{\text{law}}{=} \mu_n^a + \mu_n^b \overset{\text{law}}{=} T_a \mu_n^{(1)} + T_b \mu_n^{(2)}$ with $\mu_n^{(i)}$, $i \in \{1, 2\}$ two independent branching random walks. So if we reformulate in terms of superposability, we have

$$T_a \mu_\infty^{(1)} + T_b \mu_\infty^{(2)} \overset{(d)}{=} \lim_{n \to \infty} \mu_n^{a,b} \overset{\text{law}}{=} \mu_\infty.$$
thus $\mu_\infty$ is a point process superposable. □

Assuming Theorem 2.3, the proof of Theorem 1.1 is complete. □

3 Proof of Theorem 2.3 by Admitting Proposition 2.2

When $Y$ is a nonnegative random variable and $\Xi$ an event, we often will write $E(Y; \Xi)$ for $E(Y 1_\Xi)$.  

For any vertex $z \in T$, we denote by $[\emptyset, z]$ the unique shortest path relating $z$ to the root $\emptyset$, and $z_i$ (for $i < |z|$) the vertex on $[\emptyset, z]$ such that $|z_i| = i$. The trajectory of $z \in T, |z| = n$, corresponds to the ancestor’s positions of $z$, i.e., the vector $(V(z_1), \ldots, V(z_n))$. If $x$ is an ancestor of $y$, we will write $x < y$.

Let $d \in \mathbb{N}^*$, $(\beta, \theta, \alpha) \in (1, +\infty)^d \times (\mathbb{R}_+^*)^d \times \mathbb{R}_+^*$. Let $\mathcal{Z}[A]$ (and $\mathcal{F}_A$ the corresponding $\sigma$-field) be the set of particles absorbed at level $A \in \mathbb{R}_+$, i.e.,

$$\mathcal{Z}[A] := \{ u \in T : V(u) \geq A, V(u_k) < A \forall k < |u| \} \text{ and } \mathcal{F}_A := \sigma ((u, V(u)) ; \ u \in \mathcal{Z}[A]).$$

With a slight extension of [8], equality (5.2) pp. 46 in [2] affirms that $Z_A := \sum_{u \in \mathcal{Z}[A]} V(u) e^{-V(u)}$ satisfies that

$$\lim_{A \to \infty} Z_A = Z_\infty \quad \text{a.s.} \quad (3.1)$$

In a first step, we will show that

**Lemma 3.1** Under (1.1) and (1.5),

$$\lim_{A \to \infty} \lim_{n \to \infty} E \left( e^{-\sum_{i=1}^d \theta_i e^{-\beta_i \log Z_A} W_n, \beta_1 1_{(Z_A>0)} e^{-\alpha Z_A}} \right) = e^{-\beta_2(\theta)} E(e^{-\alpha Z_\infty}; Z_\infty > 0). \quad (3.2)$$

**Proof of Lemma 3.1** For every $n \in \mathbb{N}$ and $L > 0$, we define $\Xi_A(n, L) = \Xi_A$ by

$$\Xi_A(n, L) := \left\{ \max_{u \in \mathcal{Z}[A]} |u| \leq (\log n)^{10}, \max_{u \in \mathcal{Z}[A]} V(u) \leq A + \frac{1}{30} \log n, \log Z_A \in [-L, L] \right\}. \quad (3.3)$$

On the set of nonextinction $Z_\infty > 0$ and $M_n \to +\infty$ a.s, thus, conditional on $Z_\infty > 0$, the probability of $\Xi_A$ increases to 1 when $n$, then $A$ and then $L$ go to infinity.

At first we study

$$\mathcal{F}_\Xi(A, n, L) = \mathcal{F}_\Xi := E \left( e^{-\sum_{i=1}^d \theta_i e^{-\beta_i \log Z_A} \tilde{W}_{n, \beta_1} e^{-\alpha Z_A} 1_{(Z_A>0)}; \mathcal{F}_A} \right). \quad (3.4)$$

On $\Xi_A$, we have $\tilde{W}_{n, \beta} := \sum_{u \in \mathcal{Z}[A]} e^{-\beta V(u)} \tilde{W}_{n, \beta}^u$ with $W_{n, \beta}^u := \sum_{z > u, |z| = n} e^{-\beta (V(z) - V(u))}$ (recalling that $W_{n, \beta}^u = n^{-2} \beta W_{n, \beta}$). For any $\beta > 1$, $W_{n, \beta}^u$ has the same law as $W_{n-|u|, \beta}$ has, and then the Markov property leads to

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\[ \mathcal{F}_\mathcal{A} = E \left( e^{-\sum_{i=1}^{d} \theta_i \sum_{u \in \mathcal{Z}[A]} e^{-\beta_i (V(u) + \log Z_A)} \tilde{W}_{n,\beta_i} \mathbb{1}_{[Z_A > 0]} \}; \mathcal{A} \right) \]
\[ = E \left( \prod_{u \in \mathcal{Z}[A]} E \left( e^{-\sum_{i=1}^{d} \theta_i e^{-\beta_i (V(u) + \log Z_A)} \tilde{W}_{n,\beta_i} \mathbb{1}_{[Z_A > 0]} \}; \mathcal{A} \right) \right) \]
\[ = E \left( \prod_{u \in \mathcal{Z}[A]} E \left( e^{-\sum_{i=1}^{d} \theta_i e^{-\beta_i (V(u) + \log Z_A)} \tilde{W}_{n,\beta_i} \mathbb{1}_{[Z_A > 0]} \}; \mathcal{A} \right) \right). \]

On \( \mathcal{A} \), for any \( u \in \mathcal{Z}[A] \), \( V(u) + \log Z_A \in [A - L, L + A + \frac{1}{30} \log n] \), and then by Proposition 2.2, there exists \( A, N \) large enough such that \( \forall n > N \), we have for any \( u \in \mathcal{Z}[A] \),
\[ \left| 1 - E \left( e^{-\sum_{i=1}^{d} \theta_i e^{-\beta_i (V(u) + \log Z_A)} \tilde{W}_{n,\beta_i} \mathbb{1}_{[Z_A > 0]} \}; \mathcal{A} \right) \right| \]
\[ \leq \epsilon (V(u) + \log Z_A) \frac{e^{-V(u)}}{Z_A}. \] (3.5)

Moreover, with \( A \) large enough such that \( \rho_{\beta} (\theta) \frac{\log Z_A}{V(u)} \leq \rho_{\beta} (\theta) \frac{L}{A} \leq \epsilon \), (3.5) becomes
\[ \left| 1 - E \left( e^{-\sum_{i=1}^{d} \theta_i e^{-\beta_i (V(u) + \log Z_A)} \tilde{W}_{n,\beta_i} \mathbb{1}_{[Z_A > 0]} \}; \mathcal{A} \right) \right| \]
\[ \leq 3 \epsilon V(u) \frac{e^{-V(u)}}{Z_A}. \] (3.6)

We deduce that
\[ \mathcal{F}_\mathcal{A} \leq E \left( \prod_{u \in \mathcal{Z}[A]} (1 - [\rho_{\beta} (\theta) - 3\epsilon]) \frac{V(u)}{Z_A} e^{-V(u)} e^{-\alpha Z_A \mathbb{1}_{[Z_A > 0]}} \right) \]
\[ = E \left( e^{\sum_{u \in \mathcal{Z}[A]} \log \left( 1 - [\rho_{\beta} (\theta) - 3\epsilon] \frac{V(u)}{Z_A} e^{-V(u)} \right) \mathbb{1}_{[Z_A > 0]}} \right) \]
\[ \leq E \left( e^{-(1-\epsilon)(\rho_{\beta} (\theta) - 3\epsilon) \frac{1}{Z_A} \sum_{u \in \mathcal{Z}[A]} V(u) e^{-V(u)} \mathbb{1}_{[Z_A > 0]}} \right) \]
\[ = e^{-(1-\epsilon)(\rho_{\beta} (\theta) - 3\epsilon)} \mathbb{E} \left( e^{-\alpha Z_A \mathbb{1}_{[Z_A > 0]}} \right). \]

Since \( \lim_{A \to \infty} Z_A \overset{\text{a.s.}}{=} Z_\infty \), we get
\[ \lim_{A \to \infty} \lim_{n \to \infty} E \left( e^{-\sum_{i=1}^{d} \theta_i e^{-\beta_i \log Z_A} \tilde{W}_{n,\beta_i} \mathbb{1}_{[Z_A > 0]} \mathbb{1}_{[Z_A > 0]} e^{-\alpha Z_A} \mathbb{1}_{[Z_A > 0]}} \right) \leq e^{-\rho_{\beta} (\theta)} \mathbb{E} (e^{-\alpha Z_\infty} ; Z_\infty > 0). \]

The lower bound follows similarly. This completes the proof of Lemma 3.1. \( \square \)

**Proof of Theorem 2.3** Because of (3.2), it suffices to show that
\[ E \left( e^{-\sum_{i=1}^{d} \theta_i e^{-\beta_i \log Z_A} \tilde{W}_{n,\beta_i} \mathbb{1}_{[Z_A > 0]} \mathbb{1}_{[Z_A > 0]}} \right) \overset{A \to \infty}{\to} E \left( e^{-\sum_{i=1}^{d} \theta_i \hat{\mu}_n (\beta_i) e^{-\alpha Z_{\infty} \mathbb{1}_{[Z_{\infty} > 0]} \mathbb{1}_{[Z_{\infty} > 0]} \mathbb{1}_{[Z_{\infty} > 0]}} \right). \]
uniformly on \( n \in \mathbb{N} \). Let \( \varepsilon > 0 \). As \( Z_A \overset{p.s.}{\to} Z_\infty \) and the tightness of the sequence \((\tilde{W}_n, \beta)_{n \in \mathbb{N}}, \beta > 1 \) (see Proposition 2.1), there exists \( K > 0 \) such that for any \( n \geq 1, \)

\[
P\left( \max_{i \in [1,d]} \theta_i \tilde{W}_n, \beta_i \geq K \right) + P\left( \max_{i \in [1,d]} \left| \frac{1}{Z_A^{\beta_i}} - \frac{1}{Z_\infty^{\beta_i}} \right| + |Z_A - Z_\infty| \geq K, Z_\infty > 0 \right) \leq \frac{\varepsilon}{2}.
\]

Then by employing the inequality \(|e^{-x} - e^{-y}| \leq |x - y|\) for any \( x, y \in \mathbb{R}_+ \), uniformly in \( n \geq 1 \) we have

\[
\left| E\left( e^{-\sum_{i=1}^d \theta_i e^{-\beta_i \log Z_A} \tilde{W}_n, \beta_i e^{-\alpha Z_A} 1_{\{Z_A > 0\}} - e^{-\sum_{i=1}^d \theta_i \hat{\mu}_n(\beta_i) e^{-\alpha Z_\infty} 1_{\{Z_\infty > 0\}}} \right) \right| \\
\leq |P(Z_A > 0) - P(Z_\infty > 0)| \\
+ \frac{\varepsilon}{2} + E\left( K \sum_{i=1}^d \left| \frac{1}{Z_A^{\beta_i}} - \frac{1}{Z_\infty^{\beta_i}} \right| 1_{\left| \frac{1}{Z_A^{\beta_i}} - \frac{1}{Z_\infty^{\beta_i}} \right| \leq K} \right) \\
+ E\left( \alpha |Z_A - Z_\infty| 1_{\{|Z_A - Z_\infty| \leq K\}} \right).
\]

By dominated convergence, this amount converges to 0 when \( A \) goes to infinity. \( \Box \)

4 Estimation on the Tail of Distribution of \( \tilde{W}_n, \beta \)

4.1 The Many-to-One Formula and Lyons’ Change of Measure

For \( a \in \mathbb{R} \), we denote by \( P_a \) the probability distribution associated with the branching random walk starting from \( a \), and \( E_a \) the corresponding expectation. Under (1.1), we can define a random variable \( X \) such that for any nonnegative function \( f \),

\[
E(f(X)) = E\left( \sum_{|z|=1} e^{-V(z)} f(V(z)) \right). \tag{4.1}
\]

Moreover, via (1.5) we have \( \sigma^2 := E[X^2] < +\infty \). Let \((X_i)_{i \in \mathbb{N}} \) be a i.i.d sequence of copies of \( X \). Write for any \( n \in \mathbb{N}, S_n := \sum_{0 \leq i \leq n} X_i \), and then \( S \) is a mean-zero random walk starting from the origin.

Lemma 4.1 (Biggins-Kyprianou) Under (1.1), for any \( n \geq 1 \) and any measurable function \( g : \mathbb{R}^n \to [0, +\infty) \),

\[
E\left( \sum_{|z|=n} g(V(z_1), \ldots, V(z_n)) \right) = E\left( e^{S_n} g(S_1, \ldots, S_n) \right). \tag{4.2}
\]

We can see the so-called many-to-one formula (4.2) as a consequence of Proposition 4.2 below. Let introduce the additive martingale,
\[ W_n := \sum_{|z|=n} e^{-V(z)}, \]  

and the probability measure \( Q \) such that for any \( n \geq 0 \),

\[ Q|\mathcal{F}_n := W_n \cdot P|\mathcal{F}_n, \]  

where \( \mathcal{F}_n \) denotes the sigma-algebra generated by the positions \((V(z), |z| \leq n)\) up to time \( n \). Let \( \hat{L} \) be a point process which has Radon–Nikodym derivative \( \int e^{-x} L(dx) \) with respect to the law of \( L \). In [20], Russell Lyons gave the following description of the branching random walk under \( Q \):

- start with one particle \( w_0 \) at the origin. Generate offspring and displacements according to a copy \( \hat{L}_1 \) of \( \hat{L} \),
- choose \( w_1 \) among children of \( w_0 \) with probability proportional to \( e^{-(V(x))} \) when its displacement is \( V(x) \),
- the children other than \( w_1 \) give rise to ordinary independent branching random walks,
- \( w_1 \) gives birth to particles distributed according to \( \hat{L} \),
- again, choose one of the children of \( w_1 \) at random, call it \( w_2 \), with the others giving rise to ordinary independent branching random walks and so on.

We still call \( \mathbb{T} \) the genealogical tree of the process, so that \((w_n)_{n \in \mathbb{N}}\) is a ray of \( \mathbb{T} \), which we will call the spine. This change of probability was also used in [15]. We refer to [21] for the case of the Galton–Watson tree, to [11] for the analog for the branching Brownian motion and to [8] for the spine decomposition in various types of branching.

**Proposition 4.2** Under (1.1),

(i) for any \(|z| = n\), we have

\[ Q\left\{ w_n = z \mid \mathcal{F}_n \right\} = \frac{e^{-V(z)}}{W_n}; \]  

(ii) the spine process \((V(w_n), n \geq 0)\) has distribution of the centered random walk \((S_n, n \geq 0)\) under \( Q \) satisfying (4.2).

Before closing this subsection, we collect some useful facts about the centered random walks with finite variance \((S_n)_{n \in \mathbb{N}}\). We have taken these statements from Sect. 2 of [2]:

**Lemma 4.3** (i) There exists a constant \( \alpha_1 > 0 \) such that for any \( x \geq 0 \) and \( n \geq 1 \),

\[ P_x \left( \min_{j \leq n} S_j \geq 0 \right) \leq \alpha_1 (1 + x) n^{-\frac{1}{2}}. \]  

(ii) There exists a constant \( \alpha_2 > 0 \) such that for any \( b \geq a, x \geq 0 \) and \( n \geq 1 \),

\[ P_x \left( S_n \in [a, b], \min_{j \leq n} S_j \geq 0 \right) \leq \alpha_2 (1 + x)(1 + b - a)(1 + b) n^{-\frac{3}{2}}. \]
(iii) Let $0 < \Lambda < 1$. There exists a constant $\alpha_3 = \alpha_3(\Lambda) > 0$ such that for any $b \geq a, x \geq 0, y \in \mathbb{R}$

$$P_x \left( S_n \in [y + a, y + b], \min_{j \geq n} S_j \geq 0, \min_{\Lambda n \leq j \leq n} S_j \geq y \right) \leq \alpha_3 (1 + x)(1 + b - a)(1 + b)n^{-3/2}. \quad (4.8)$$

See [19] for (4.6). The estimates (4.7) and (4.8) are, for example, Lemmas A.1 and A.3 in [4]. (In our case, $(S_n)$ is the centered random walk under $P$, with finite variance $E[S_1^2] = \sigma$ which appears in the many-to-one formula.)

We introduce its renewal function $R(x)$ which is zero if $x < 0$, 1 if $x = 0$, and

$$R(x) := \sum_{k \geq 0} P \left( S_k \geq -x, \min_{0 \leq j \leq k-1} S_j \right), \quad \text{for } x > 0. \quad (4.9)$$

It is known that there exists $c_0 > 0$ such that

$$\lim_{x \to \infty} \frac{R(x)}{x} = c_0. \quad (4.10)$$

Similarly, we define $R_-(x)$ as the renewal function associated with $-S$. Finally according to Theorem 1a, section XII. 7, p. 415 of [13], there exists $C_-, C_+ > 0$ such that

$$P \left( \min_{1 \leq i \leq n} S_i \geq 0 \right) \sim \frac{C_+}{\sqrt{n}}, \quad P \left( \max_{1 \leq i \leq n} S_i \leq 0 \right) \sim \frac{C_+}{\sqrt{n}}, \quad \text{as } n \to \infty. \quad (4.11)$$

4.2 Notations

We will use the notations of Aïdékon in [2]. As the typical order of $M_n$ is $\frac{3}{2} \log n$, it will be convenient to use the following notation, for $x \geq 0$:

$$a_n(x) := \frac{3}{2} \log n - x,$$

$$I_n(x) := (a_n(x) - 1, a_n(x)).$$

Let us introduce for any $x_1, x_2, x_3 > 0$ the set $Z^{x_1,x_2,x_3}$ defined by

$$z \in Z^{x_1,x_2,x_3} \iff |z| = n, \min_{k \leq n} V(z_k) \geq -x_1,$$

$$\min_{k \in [\frac{n}{2}, n]} V(z_k) \geq a_n - x_2, V(z) \leq a_n - x_3.$$
4.3 On the Tail of Distribution of $W_{n,\beta}$

In this section, we will study the tail of distribution of $W_{n,\beta}$. As we will see, when $A$ is large enough, up to a negligible amount, $W_{n,\beta}$ is equal to $\sum_{|z|=n} e^{-\beta V(z)} \mathbb{1}_{\{V(z)\leq M_n+A\}}$. So we have to study the extremal particles, i.e., the particles $z \in T_n$ such that $V(z)$ such that $M_n \sim V(z)$.

Let us recall two known results. For any $x \geq 0$,

$$P(\exists u \in T : V(u) \leq -x) \leq \sum_{n \geq 0} E\left(\sum_{|u|=n} \mathbb{1}_{\{V(u)\leq -x, V(u_k)\geq -x, \forall k<n\}}\right)$$

$$= \sum_{n \geq 0} E(e^{S_n}, S_n \leq -x, S_k > -x, \forall k < n)$$

$$\leq e^{-x} \sum_{n \geq 0} P(S_n \leq -x, S_k > -x, \forall k < n) \leq e^{-x}. \quad (4.12)$$

where we have used (4.2) in the equality. From Corollary 3.4 [2] and (4.12), it is plain to deduce:

**Proposition 4.4** ([2]) Under (1.1) and (1.5), there exists $c_2 > 0$ such that for any $x \geq 0$ and any integer $n \geq 1$,

$$P(M_n \leq a_n - x) \leq c_2(1 + x)e^{-x}. \quad (4.13)$$

In this section, we will principally show three results:

**Proposition 4.5** Under (1.1) and (1.5), for any $\epsilon, K > 0$ there exists $L_0$ large enough such that for any $L > L_0$, $n \in \mathbb{N}$, $x \in \mathbb{R}$, $y > 0$ and $\delta \in [-\infty, K]$,

$$P\left(n^{\frac{3}{2}} \beta \sum_{|z|=n} e^{-\beta V(z)} \mathbb{1}_{\{|z| \notin \mathbb{Z}^n, x+y+L, x-L\}} \geq e^{\beta(x-\delta)}\right) \leq \epsilon(1 + y)e^{-x} + e^{-y}. \quad (4.14)$$

**Proposition 4.6** Under (1.1) and (1.5), there exist $c_3, c_4 > 0$ such that for any $n \in \mathbb{N}^*$, $j, x \geq 1$,

$$P\left(\widehat{W}_{n,\beta} \geq e^{\beta x}, M_n \in I_n(x-j)\right) \leq c_3(1 + x)e^{-x}e^{-c_4j}. \quad (4.15)$$

In particular, we see that

$$P(\widehat{W}_{n,\beta} \geq e^{\beta x}) \leq c_5 x e^{-x}, \quad \forall n \geq 1, x \geq 1. \quad (4.16)$$

And finally,
Corollary 4.7 (i) For any $\epsilon > 0$, there exists $L$, $A$, $N > 0$ large enough we have for any $n \geq N$ and $x \geq A$,

$$E\left(1 - \exp\left(-\sum_{|z|=n} e^{-\beta[V(z) - a_n + x]} \mathbb{1}_{\{V(z) \geq a_n - x + L\}}\right)\right) \leq \epsilon x e^{-x}. \quad (4.17)$$

(ii) There exist $c_6$, $c_7 > 0$ such that for any $A$, $N > 0$ large enough we have for any $n \geq N$ and $x \in [A, \frac{3}{2} \log n - A]$,

$$c_6(1 + x)e^{-x} \leq E\left(1 - \exp\left(-\sum_{|z|=n} e^{-\beta[V(z) - a_n + x]} \right)\right) \leq c_7(1 + x)e^{-x}. \quad (4.18)$$

Remark Only (4.17) and (4.18) will be used in the next section.

Proof of Corollary 4.7 The lower bound of (4.18) stems directly from

$$E\left(1 - \exp\left(-\sum_{|z|=n} e^{-\beta[V(z) - a_n + x]} \right)\right) \geq (1 - e^{-1}) P(M_n \leq a_n - x) \geq c_6(1 - e^{-1})(1 + x)e^{-x},$$

where in the last line, we have used Proposition 4.1 in [2]. For the upper bound of (4.18), notice that for any $w > 0$, $1 - e^{-w} = \int_0^{+\infty} e^{-x} \mathbb{1}_{\{w \geq x\}} dx$, and thus according to Proposition 2.1, we can write

$$E\left(1 - \exp\left(-\sum_{|z|=n} e^{-\beta[V(z) - a_n + x]} \right)\right) = \int_0^{+\infty} e^{-u} P(\widetilde{W}_{n, \beta} \geq e^{\beta y} u) du \leq e^{-\beta (x-1)} + c_1 \int_{e^{-\beta (x-1)}}^{+\infty} \frac{e^{-u}}{u^\beta} \left(x + \frac{1}{\beta} \log u\right) e^{-x} du \leq c_7(1 + x)e^{-x},$$

which proves (4.18). By using (4.14) instead of (4.16), we derive (4.17) identically; indeed, it suffices to observe that $V(u) > a_n - x + L$ implies $u \notin Z^{x,x+L,x-L}$. \qed

Now we will prove Proposition 4.5 and 2.1. First recall the important following lemma:

Lemma 4.8 (Aïdékon [2]) Under (1.1) and (1.5), there exist constants $c_8$, $c_9 > 0$ such that for any $n \geq 1$, $L \geq 0$ and $y \geq 0$, $x \in \mathbb{R}$,

$$P\left(\exists u \in \mathbb{T}_n, \min_{k \leq n} V(u_k) \geq -y, \min_{\frac{n}{2} < k \leq n} V(u_k) \leq a_n(x + L), V(u) \leq a_n(x)\right) \leq c_8(1 + y)e^{-c_9 L} e^{-x}. \quad (4.19)$$
(we have given a statement slightly stronger than in [2], but reader can see easily that actually this is an equivalent statement).

The proofs require two lemmas.

**Lemma 4.9** Under (1.1) and (1.5), there exist \( c_{10}, c_{11} > 0 \) such that \( \forall x \in \mathbb{R}, y, L \geq 0, n \geq 1, \)

\[
P_y \left( \sum_{|z| = n} \frac{1}{n^2} \sum_{|z|=n} e^{-\beta V(z)} \mathbb{1}_{\min_{k \leq n} V(z_k) \geq 0} \min_{\frac{m}{2} < k \leq n} V(z_k) \right.
\leq a_n (x + L) \right) \geq e^{\beta x} \right) \leq c_{10} (1 + y) e^{-c_{11} L} e^{-y} e^{-x}. \tag{4.20}
\]

**Proof of Lemma 4.9** For any \( a, b > 0 \), let \( P_{(4.19)}(a, b) \) and \( P_{(4.20)}(a, b) \) be the probability of, respectively, (4.19) and (4.20) when \( y = a \) and \( x = b. \) Observe that

\[
P_{(4.20)}(y, x) \leq P_y \left( \sum_{|z| = n} \frac{1}{n^2} \sum_{|z|=n} e^{-\beta V(z)} \mathbb{1}_{\min_{k \leq n} V(z_k) \geq 0} \min_{\frac{m}{2} < k \leq n} V(z_k) \right. 
\left. + P_{(4.19)}(y, y + x) \right)
\leq P (\ldots) + c_{8} (1 + y) e^{-c_{9} L} e^{-x-y}. \tag{4.21}
\]

We have to bound \( P (\ldots). \)

We need some notations: for \( |z| = n, j \geq 0, \frac{n}{2} < k \leq n \) and \( L' \geq L, \) we define the event

\[
E_{j, k, L'}(z) := \{ \min_{l \leq n} V(z_l) \geq 0, V(z_k) = \min_{\frac{n}{2} < l \leq n} V(z_l) \in I_n(x + L'), V(z_n) \in I_n(x) + j \}. \tag{4.22}
\]

For any integer \( a \in [0, \frac{n}{2}], \)

\[
F_{L'}^{[\frac{n}{2}, n-a]}(z) := \mathcal{U} E_{j, k, L'}(z), F_{L'}^{(n-a, n]}(z) := \mathcal{U} E_{j, k, L'}(z). \tag{4.23}
\]

Remark that for any sequence of integer \( (a_{L+p})_{p \geq 0}, \)

\[
\left. \sum_{|z| = n} \frac{1}{n^2} \sum_{|z|=n} e^{-\beta V(z)} \mathbb{1}_{\min_{k \leq n} V(z_k) \geq 0} \min_{\frac{m}{2} < k \leq n} V(z_k) \right\}_{\{z \geq a_{n}(x + L), V(z) \geq a_{n}(x)\}}
\leq \sum_{p \geq 0} \sum_{|z| = n} e^{-\beta V(z)} \left( \mathbb{1}_{F_{L'}^{[\frac{n}{2}, n-a_{L+p}]}(z)} + \mathbb{1}_{[F_{L'}^{(n-a_{L+p}, n]}(z)]} \right). \tag{4.24}
\]
Similarly we introduce, for the centered random walk \((S_n)_{n \geq 0}\),

\[
E_{j,k,L'} := \{ \min_{l \leq n} S_l \geq 0, \quad S_k = \min_{\frac{n}{2} < l \leq n} S_l \in I_n(x + L'), \quad S_n \in I_n(x) + j \},
\]

\[(4.25)\]

\[
F_{\left[\frac{n}{2}, n-a\right]} := \bigcup_{j \geq 0, k \in \left[\frac{n}{2}, n-a\right]} E_{j,k,L'}, \quad F_{\left[n-a, n\right]} := \bigcup_{j \geq 0, k \in \left(n-a, n\right]} E_{j,k,L'}.
\]

\[(4.26)\]

Let estimate \(P_y(E_{j,k,L'})\) for \(\frac{n}{2} < k \leq n - a\). By the Markov property at time \(k\),

\[
P_y(E_{j,k,L'}) \leq P_y \left( \min_{l \leq k} S_l \geq 0, \quad \min_{\frac{n}{2} < l \leq k} S_l \geq a_n(x + L'), \quad S_k \in I_n(x + L') \right)
\]

\[
\times P \left( S_{n-k} \in [L' - 1 + j, L' + 1 + j], \quad \min_{l \leq n-k} S_l \geq 0 \right).
\]

We know by \((4.7)\) that there exists a constant \(c_{12}\) such that

\[
P \left( S_{n-k} \in [L' - 1 + j, L' + 1 + j], \quad \min_{l \leq n-k} S_l \geq 0 \right) \leq c_{12}(n - k + 1)^{-\frac{3}{2}}(1 + L' + j).
\]

For the first term, we have to discuss on the value of \(k\). Suppose that \(\frac{3}{4}n \leq k \leq n\), then by \((4.8)\),

\[
P_y \left( \min_{l \leq k} S_l \geq 0, \quad \min_{\frac{n}{2} < l \leq k} S_l \geq a_n(x + L'), \quad S_k \in I_n(x + L') \right) \leq c_{13} \frac{1 + y}{n^2}.
\]

If \(\frac{1}{2}n \leq k \leq \frac{3}{4}n\), we simply write

\[
P_y \left( \min_{l \leq k} S_l \geq 0, \quad \min_{\frac{n}{2} < l \leq k} S_l \geq a_n(x + L'), \quad S_k \in I_n(x + L') \right)
\]

\[
\leq P_y \left( S_k \in I_n(x + L'), \quad \min_{l \leq k} S_l \geq 0 \right)
\]

\[
\leq c_{14}(1 + y)n^{-\frac{3}{2}} \log n.
\]

To summarize, we have obtained

\[
P_y(E_{j,k,L'}) \leq \begin{cases} 
  c_{15} \frac{(1+y)\log n}{n^2(n-k+1)^2} (1 + L' + j) & \text{if } \frac{n}{2} < k \leq \frac{3}{4}n, \\
  c_{15} \frac{1+y}{n^2(n-k+1)^2} (1 + L' + j) & \text{if } \frac{3}{4}n < k \leq n - a.
\end{cases}
\]

\[(4.27)\]
Now we can tackle the proof and study $P_y(\ldots)$. By Proposition 4.2,

\[
\frac{n^3\beta}{e^{\beta x}}E_y \left( \sum_{|z|=n} e^{-\beta V(z)} 1_{\{J_{L',n-a}^L(z)\}} \right) = \frac{n^3\beta}{e^{\beta x}}e^{-\beta x}E_y \left[ e^{(1-\beta)S_n} 1_{\{J_{L',n-a}^L(z)\}} \right] \\
\leq \frac{n^3\beta}{e^{\beta x}}e^{-\beta x} \sum_{k=n/2}^{n-a} e^{(1-\beta)(a_n(x)+j)}P_y(E_{j,k,L'}) \\
\leq c_{16}(1+y)(1+L')e^{-x-y}a^{-\frac{3}{2}}.
\]

We also get

\[
P_y \left( \sum_{|z|=n} 1_{\{J_{L',n-a,n}^L(z)\}} \geq 1 \right) \leq \sum_{k\in[n-a,n]} P_y \left( \sum_{|z|=n} \sum_{j\geq 0} 1_{\{E_{j,k,L'}(z)\}} \geq 1 \right) \\
\leq \sum_{k\in[n-a,n]} P_y \left( \exists |z| = k : \min_{l\leq k} V(z_l) \geq 0, \\
\min_{\frac{3}{2} \leq l \leq k} V(z_l) \geq a_n(x + L'), V(z_k) \in I_n(x + L') \right).
\]

Using Proposition 4.2 and then (4.8), we deduce that

\[
P_y \left( \sum_{|z|=n} 1_{\{J_{L',n-a,n}^L(z)\}} \geq 1 \right) \leq \sum_{k\in[n-a,n]} c_{17}(1+y)e^{-x-y-L'} \\
= c_{17}(1+a)(1+y)e^{-x-y-L'}.
\]

These estimations lead us to choose correctly the integer $a$. Let $\alpha \in (0, 1)$, for any $p \geq 0$ we define $a_{L+p} := \lfloor e^{\alpha(L+p)} \rfloor$. Recalling (4.24), to conclude we just assemble our previous inequalities and observe that

\[
P_y(\ldots) \leq P_y \left( \sum_{p \geq 0} \sum_{|z|=n} e^{-\beta V(z)} 1_{\{J_{L+p}^{\frac{p}{2},a-a_{L+p}}(z)\}} \geq \frac{e^{\beta x}}{2n^3\beta} \right) \\
+ P_y \left( \sum_{p \geq 0} \sum_{|z|=n} 1_{\{J_{L+p}^{a-a_{L+p},n}(z)\}} \geq 1 \right) \\
\leq \sum_{p \geq 0} \left( \frac{n^3\beta}{e^{\beta x}}E_y \left( \sum_{|z|=n} e^{-\beta V(z)} 1_{\{J_{L+p}^{\frac{p}{2},a-a_{L+p}}(z)\}} \right) \right) \\
+ P_y \left( \sum_{|z|=n} 1_{\{J_{L+p}^{a-a_{L+p},n}(z)\}} \geq 1 \right)
\]
\[ \begin{align*}
&\leq \sum_{p \geq 0} \left( c_{16}(1 + y)(1 + L + p)e^{-x - y}a^{\frac{1}{2}}_{L + p} + c_{17}(1 + a_{L + p})(1 + y)e^{-x - y - (L + p)} \right) \\
&\leq c_{10}(1 + y)e^{-c_{11}L}e^{-y - x}. 
\end{align*} \tag{4.28} \]

Combining (4.21) with (4.28), we obtain Lemma 4.9. \qed

**Lemma 4.10** Under (1.1) and (1.5), there exist \( c_{18} \geq 0, c_{19} > 0 \) (\( c_{19} = \beta - 1 \)) such that for any \( n \geq 0, x \in \mathbb{R}, A, y \geq 0 \) and \( L \geq 0 \),

\[ P_x \left( \sum_{|z| = n} e^{-\beta V(z)} \mathbb{I}_{\min_{k \leq n} V(z_k) \geq 0, \min_{\frac{n}{2} < k \leq n} V(z_k) \geq a_n(x + L), V(z) \geq a_n(x) + A} \geq \frac{e^{\beta x}}{n^{\frac{3}{2}} \beta} \right) \leq c_{18}(1 + y)e^{-x - y}(L + A + 1)e^{-c_{19}A}. \]

**Proof of Lemma 4.10** The Markov inequality \( P(X \geq 1) \leq E(X) \), for \( X \) positive, gives:

\[ P_y \left( \sum_{|z| = n} e^{-\beta V(z)} \mathbb{I}_{\min_{k \leq n} V(z_k) \geq 0, \min_{\frac{n}{2} < k \leq n} V(z_k) \geq a_n(x + L), V(z) \geq a_n(x) + A} \geq \frac{e^{\beta x}}{n^{\frac{3}{2}} \beta} \right) \leq \frac{n^{\frac{3}{2}} \beta}{e^{\beta x}} E_y \left( \sum_{|z| = n} e^{-\beta V(z)} \mathbb{I}_{\min_{k \leq n} V(z_k) \geq 0, \min_{\frac{n}{2} < k \leq n} V(z_k) \geq a_n(x + L), V(z) \geq a_n(x) + A} \right). \]

By Proposition 4.2, this is equal to

\[ \begin{align*}
&= \frac{n^{\frac{3}{2}} \beta}{e^{\beta x}} e^{-y} \sum_{k \in \mathbb{N}} E_y \left( e^{(1 - \beta)S_n} \mathbb{I}_{\min_{k \leq n} S_k \geq 0, \min_{\frac{n}{2} < k \leq n} S_k \geq a_n(x + L), S_n \in I_n(x - A - k)} \right) \\
&\leq e^{-x - y - A(\beta - 1)} \sum_{k \in \mathbb{N}} e^{(1 - \beta)k} c_{20}(1 + y)(L + A + k + 1) \\
&\leq c_{18}(1 + y)e^{-x - y}(L + A + 1)e^{-c_{19}A},
\end{align*} \]

where in the second inequality we have used (4.8). \qed

**Proof of Proposition 4.5** By noticing that \( Z^{y, x + L, x - L}_n = Z^{y, x - \delta + L + \delta, x - \delta - (L - \delta)}_n \subseteq Z^{y, x - \delta + L + K, x - \delta - (L + K)}_n \), it is sufficient to prove Proposition 4.5 for \( K = 0 \). Let \( z \in \mathbb{T}_n \), recall that

\[ \mathcal{S} \] Springer
\[ z \in \mathbb{Z}_n^{x+L,x-L} \iff \min_{k \leq n} V(z_k) \geq -y, \quad \min_{k \in [n/2,n]} V(z_k) \geq a_n - (x + L), \]
\[ V(z) \leq a_n - (x - L). \]  
(4.29)

As \( P(\exists z \in \mathbb{T}, V(z) \leq -y) \leq e^{-y} \), we deduce that the probability (4.14) is smaller than
\[ P\left(n^{3/2} \sum_{|z|=n} e^{-\beta V(z)} 1_{\{\min_{k \leq n} V(z_k) \geq -y, z \notin \mathbb{Z}_n^{y,x+L,x-L}\}} \geq e^{\beta x}\right) + e^{-y}. \]  
(4.30)

Then we observe that on \( \{\min_{k \leq n} V(z_k) \geq -y\}, \)
\[ \{z \notin \mathbb{Z}_n^{y,x+L,x-L}\} \subset \{\min_{k \leq n} V(z_k) \leq a_n(x + L)\} \cup \{\min_{k \leq n} V(z_k) \geq a_n(x + L), V(z) \geq a_n(x + L)\}. \]

According to Lemmas 4.9 and 4.10, by choosing \( L_0 \) large enough such that \( \forall L \geq L_0, c_{10}e^{-c_{11}L} + c_{18}(2L + 1)e^{-c_{19}L} \leq \epsilon \), we obtain for any \( n \in \mathbb{N}, x \in \mathbb{R} \) and \( y > 0 \),
\[ P\left(n^{3/2} \sum_{|z|=n} e^{-\beta V(z)} 1_{\{z \notin \mathbb{Z}_n^{y,x+j,x-j}\}} \geq e^{\beta x}\right) \leq \epsilon(1 + y)e^{-x} + e^{-y}, \]  
(4.31)

which proves (4.14). \( \square \)

**Proof of Proposition 4.6** If \( M_n \geq a_n(x) + j \), then clearly for any \( z \in \mathbb{T} \) such that \( |z| = n \) we have \( z \notin \mathbb{Z}_n^{x,x+j,x-j} \). Therefore,
\[ P\left(\tilde{W}_{n,\beta} \geq e^{\beta x}, M_n \in I_n(x - j + 1)\right) \leq P\left(n^{3/2} \sum_{|z|=n} e^{-\beta V(z)} 1_{\{z \notin \mathbb{Z}_n^{y+j},x+j,x-j\}} \geq e^{\beta x}\right) \]
\[ \leq c_{10}(1 + x + j)e^{-x}e^{-c_{11}j} + c_{18}(1 + x + j)e^{-x}j e^{-c_{19}j} + e^{-(x+j)}, \]
by Lemma 4.9, Lemma 4.10 and (4.12). Set \( c_{11} = \frac{1}{2} \min(c_{14}, c_{22}) \) to obtain (4.15). The estimate (4.16) follows easily from (4.15) and Proposition 4.4 applied to the equality
\[ P\left(\tilde{W}_{n,\beta} \geq e^{\beta x}\right) = P\left(\tilde{W}_{n,\beta} \geq e^{\beta x}, M_n \leq a_n(x)\right) + P\left(\tilde{W}_{n,\beta} \geq e^{\beta x}, M_n > a_n(x)\right). \]

This proves Proposition 4.6. \( \square \)

**5 Proof of Proposition 2.2**

We fix \( d \geq 1, \beta \in (1, \infty)^d, \theta \in (\mathbb{R}_+^*)^d \) and \( \epsilon > 0 \). Our aim is to study for \( n \) and then \( x \) large,
\[ E\left(1 - \exp\left(-\sum_{i=1}^d \sum_{|z|=n} \theta_i e^{-\beta_i[V(z)+x-a_n]}\right)\right). \]  
(5.1)
According to (4.17), for $L$ large enough we can restrain our study to the expectation of

$$
\Phi^{(L)}(x, n) := 1 - \exp \left\{ \sum_{i=1}^{d} \sum_{|z|=n} \theta_i e^{-\beta |V(z)+x-a_n|} \mathbb{1}_{|V(z)| \leq a_n-x+L} \right\}. \quad (5.2)
$$

Indeed as for any $u, v \geq 0$, $1 - e^{-u-v} \leq 1 - e^{-u} + 1 - e^{-v}$, we deduce that

$$
\left| \mathbb{E} \left( 1 - \exp \left\{ \sum_{i=1}^{d} \sum_{|z|=n} \theta_i e^{-\beta |V(z)+x-a_n|} \mathbb{1}_{|V(z)| \leq a_n-x+L} \right\} \right) - \mathbb{E}(\Phi^{(L)}(x, n)) \right| \\
\leq \mathbb{E} \left( 1 - \exp \left\{ \sum_{i=1}^{d} \sum_{|z|=n} \theta_i e^{-\beta |V(z)+x-a_n|} \mathbb{1}_{|V(z)| \geq a_n-x+L} \right\} \right).
$$

On the set $\{M_n > a_n - x + L\}$, $\Phi^{(L)}(x, n) = 0$; thus, we have

$$
\mathbb{E}(\Phi^{(L)}(x, n)) = \mathbb{E} \left( 1 - \exp \left\{ \sum_{i=1}^{d} \sum_{|z|=n} \theta_i e^{-\beta |V(z)+x-a_n|} \mathbb{1}_{|V(z)| \leq a_n-x+L} \right\} \right) \\
= \mathbb{E} \left( \sum_{|z|=n} \frac{\mathbb{1}_{|V(z)| = a_n-x+L}}{\sum_{|z|=n} \mathbb{1}_{|V(z)| = M_n}} \Phi^{(L)}(x, n) \right) := \mathbb{E}(5.3). \quad (5.3)
$$

According to Lemma 4.8 and (4.12), we can enhance $L$ such that for any $n, x \geq 0$ large enough,

$$
P\left( M_n \leq a_n - x + L, \exists z \notin \mathcal{Z}^{x, x+\frac{2}{c_0}, L, x-L} \right) \leq \epsilon x e^{-x}. \quad (5.4)
$$

where $c_0$ is the constant defined in (4.19). The random variable in (5.3) is smaller than 1, and then we deduce that for any $x, n \geq 1$ large enough,

$$
|\mathbb{E}(5.3) - \mathbb{E} \left( \sum_{|z|=n} \frac{\mathbb{1}_{|V(z)| = M_n \leq a_n-x+L, z \in \mathcal{Z}^{x, x+\frac{2}{c_0}, L, x-L}}{\sum_{|z|=n} \mathbb{1}_{|V(z)| = M_n}} \Phi^{(L)}(x, n) \right) | \leq \epsilon x e^{-x}. \quad (5.5)
$$

Combining (5.5) to

$$
\sum_{|z|=n} \frac{\mathbb{1}_{|V(z)| = M_n \leq a_n-x+L, z \in \mathcal{Z}^{x, x+\frac{2}{c_0}, L, x-L}}}{\sum_{|z|=n} \mathbb{1}_{|V(z)| = M_n}} \Phi^{(L)}(x, n)
$$
\[ \sum_{|z|=n} \frac{1}{\sum_{|z|=n} \mathbb{I}_{|z|=n} \mathbb{I}_{V(z)=M_n} \Phi(L)(x, n)} \]

and

\[ \mathbb{E} \left( |\Phi(L)(x, n) - \Phi(L^{\frac{2}{c_9}})(x, n)| \right) \]

\[ \leq \mathbb{E} \left( 1 - \exp \left( - \sum_{i=1}^{d} \sum_{|z|=n} \theta_i e^{-\beta_i [V(z)+x-a_n]} \mathbb{I}_{V(z)=a_n-x+L} \right) \right) \leq \epsilon xe^{-x}, \]

we finally deduce the following statement: there exists \( L' = \frac{2}{c_9} L \) large enough such that for any \( n, x \) large enough

\[ \left| \mathbb{E} \left( 1 - \exp \left( - \sum_{i=1}^{d} \sum_{|z|=n} \theta_i e^{-\beta_i [V(z)+x-a_n]} \right) \right) \right| \]

\[ \leq \mathbb{E} \left( \sum_{|z|=n} \frac{\mathbb{I}_{V(z)=M_n \leq a_n-x+L', z \in \mathbb{Z}^{x,x+L',x-L}'}}{\sum_{|z|=n} \mathbb{I}_{V(z)=M_n}} \Phi(L')(x, n) \right) \]

\[ \leq \epsilon xe^{-x}. \] (5.6)

From now we fix \( L \) large such (5.6) is true and study,

\[ \mathbb{E} \left( \sum_{|z|=n} \frac{\mathbb{I}_{V(z)=M_n \leq a_n-x+L, z \in \mathbb{Z}^{x,x+L,x-L}}}{\sum_{|z|=n} \mathbb{I}_{V(z)=M_n}} \Phi(L)(x, n) \right) \]

\[ = \mathbb{E} \left( e^{V(w_n)} \mathbb{I}_{V(w_n)=M_n, w_n \in \mathbb{Z}^{x,x+L,x-L}} \frac{\Phi(L)(x, n)}{\sum_{|z|=n} \mathbb{I}_{V(z)=M_n}} \right). \] (5.7)

Note that we have used Proposition 4.2 in the equality (5.7). We denote by \( \mathbb{E}(5.7) \) the expectation in (5.7).

**Definition 5.1** For \( b \) integer, we define the event \( \xi_n \) by

\[ \xi_n := \xi_n(x, b, L) := \{ \forall k \leq n-b, \forall v \in \Omega(w_k), \min_{u \geq v, |u|=n} V(u) > a_n-x+L \}, \] (5.8)

where \( \Omega(w_k) \) denotes the set of brothers of \( w_k \). On the event \( \xi_n \cap \{ M_n \leq a_n-x+L \} \), we are sure that any particle located at the minimum separated from the spine after the time \( n-b \).

**Definition 5.2** For \( y \in \mathbb{R}, L, > 0, \theta \in (\mathbb{R}^*_+)^d \) and \( b \in \mathbb{N}^* \), we define
(i) the function
\[
F_{L,b}(\theta, y) := E_Q\left[ \frac{e^{V_0(b,y) - L} 1_{\{V(0,y) = M_b\}}}{\sum_{|u| = b} 1_{\{V(u) = M_b\}}} 1_{\{V(0,y) \leq 2L, \min_{k \leq b} V(k) \geq 0\}} \times \left[ 1 - \exp\left(-\sum_{i=1}^d \sum_{|z| = b} \theta_i e^{-\beta_i [V(z) + y - L]} 1_{\{V(z) + y \leq 2L\}}\right)\right] \right].
\]

(iii) the constant \( \rho_{\beta, L, b}(\theta) := \frac{C-C_+ e^{-b}}{\sigma \sqrt{\pi}} \int_{y \geq 0} F_{L,b}(\theta, y) R_-(y) dy \), where \( C_-, C_+ \) and \( R_-(x) \) are defined in (4.11).

The proof of the following lemma (which is an extension of Lemma 3.8 in [2]) is postponed in “Appendix 3”.

**Lemma 5.3** Under (1.1) and (1.5), \( \forall \eta, L > 0 \exists D(L, \eta) > 0 \) and \( B(L, \eta) \geq 1 \) such that \( \forall b \geq B, n \geq e^{5b}, x \geq D \),

\[
Q\left( (\xi_n(b, x, L))^c, w_n \in \mathcal{Z}_{n}^{\times, x + L, x - L} \right) \leq \eta n^{-\frac{3}{2}} (1 + x). \tag{5.10}
\]

Lemma 5.3 justifies Definitions 5.1 and 5.2. Indeed observe that on \( \xi_n(b, x, L) \), \( \sum_{|z| = n} 1_{\{V(z) = M_n\}} = \sum_{|z| = n, z > w_{n-b}} 1_{\{V(z) = M_n\}} \) and \( \Phi^{(L)}(x, n) \) is equal to

\[
\Phi^{(L,b)}(x, n, w_{n-b}) = 1 - \exp\left\{-\sum_{i=1}^d \sum_{|z| = n, z > w_{n-b}} \theta_i e^{-\beta_i [V(z) - a_n + x]} 1_{\{V(z) \leq a_n - x + L\}}\right\}.
\]

The functions \( \Phi^{(L,b)}(\cdot, \cdot, \cdot) \) and \( \Phi^{(L)}(\cdot, \cdot) \) are bounded by 1, and then applying Lemma 5.3, there exists \( b > 0 \) large enough (associated with \( L, \frac{\eta}{c} \)) such that

\[
\left| E\left( e^{V(w_n)} 1_{\{V(w_n) = M_n, w_n \in \mathcal{Z}_{n}^{\times, x + L, x - L}\}} \frac{\Phi^{(L)}(x, n)}{\sum_{|z| = n} 1_{\{V(z) = M_n\}}} - \frac{\Phi^{(L,b)}(x, n, w_{n-b})}{\sum_{|z| = n, z > w_{n-b}} 1_{\{V(z) = M_n\}}} \right)\right|.
\]

\[
\leq E\left( e^{V(w_n)} \left[ \frac{\Phi^{(L,b)}(x, n, w_{n-b})}{\sum_{|z| = n, z > w_{n-b}} 1_{\{V(z) = M_n\}}} + \frac{\Phi^{(L)}(x, n)}{\sum_{|z| = n} 1_{\{V(z) = M_n\}}} \right] \right] \times V(w_n) = M_n, w_n \in \mathcal{Z}_{n}^{\times, x - L, x + L}, \ (\xi_n(b, x, L))^c \right) \leq e^{-x + L n^{-\frac{3}{2}}} Q\left( (\xi_n(b, x, L))^c, w_n \in \mathcal{Z}_{n}^{\times, x + L, x - L} \right) \leq \eta e^{-x} (1 + x).
\]
Moreover, using Definition 5.2 and the branching property at time \( n - b \), we have

\[
E\left( e^{V(w_n)} \mathbb{1}_{\{V(w_n) = M_n, w_n \in \mathbb{Z}_{x,x+L,x-L}\}} \frac{\Phi(L,b)(x, n, w_{n-b})}{\sum_{|z| = n, z > w_{n-b}} \mathbb{1}_{\{V(z) = M_n\}}} \right) \\
= n^\frac{3}{2} e^{-x} E_Q \left( \mathbb{1}_{\{\min_{k \leq n-b} V(w_k) \geq -x, \ min_{k \in \mathbb{Z}_{a-n}} V(w_k) \geq a_n - x - L\}} F_{L,b}(\theta, V(w_{n-b}) - a_n + x + L) \right) \\
= n^\frac{3}{2} e^{-x} E_Q \left( \mathbb{1}_{\{\min_{k \leq n-b} V(w_k) \geq 0, \ min_{k \in \mathbb{Z}_{a-n}} V(w_k) \geq a_n - L\}} F_{L,b}(\theta, V(w_{n-b}) - a_n + L) \right).
\]

By combining this equality with (5.6) and (5.7), we can affirm that there exists \( L, B, D \geq 0 \) such that for any \( b \geq B, n \geq e^{5b}, x \geq D, \)

\[
\left| E\left( 1 - \exp\left( - \sum_{i=1}^{d} \sum_{|z|=n} \theta_i e^{-\beta_i(V(z)+x-a_n)} \right) \right) \\
- n^\frac{3}{2} e^{-x} E_Q \left( \mathbb{1}_{\{\min_{k \leq n-b} V(w_k) \geq 0, \ min_{k \in \mathbb{Z}_{a-n}} V(w_k) \geq a_n - L\}} F_{L,b}(\theta, V(w_{n-b}) - a_n + L) \right) \right| \\
\leq \epsilon e^{-x}.
\] (5.11)

Keeping in mind this last display, we shall now state and prove yet two lemmas which will be used in the proof of Proposition 2.2:

**Lemma 5.4** For any \( \theta \in (\mathbb{R}^+)^d \), the function \( y \mapsto F_{L,b}(\theta, y) \) is Riemann integrable and there exists a nonincreasing function \( \bar{F} : \mathbb{R}^+ \to \mathbb{R} \) such that \( |F_{L,b}(\theta, y)| \leq \bar{F}(y) \) for any \( y \geq 0 \) and \( \int_{y \geq 0} y \bar{F}(y) dy < \infty \).

**Proof of Lemma 5.4** We recall that by Proposition 4.2, the spine has the law of \((S_n)_{n \geq 0}\). We see that \( -\sum_{|z|=b} \mathbb{1}_{\{V(z)=M_b\}} \) is smaller than 1, and \( e^{V(w_b)-L} \leq e^L \). Hence, \( |F_{L,b}(\theta, y)| \leq e^L P(S_b \leq L - y) =: \bar{F}(y) \) which is nonincreasing in \( y \), and \( \int_{y \geq 0} \bar{F}(y) dy = e^L \frac{1}{2} E[(L - S_b)^2 \mathbb{1}_{\{S_b \leq L\}}] < \infty \). Moreover, using the identity \( |1_E - a1_F| \leq 1 - a + |1_E - 1_F| \) for \( a \in (0, 1) \), it yields that for \( y_2 \geq 0, e > 0 \) and any \( y_1 \in [y_2, y_2 + \epsilon] \),

\[
\left| F_{L,b}(\theta, y_1) - F_{L,b}(\theta, y_2) \right| \leq E_Q \left[ \mathbb{1}_{\{\min_{k \leq b} V(w_k) + y_1 \geq 0, V(w_b) + y_2 \leq 2L\}} \\
- \mathbb{1}_{\{\min_{k \leq b} V(w_k) + y_2 \geq 0V(w_b) + y_2 \leq 2L\}} \right] \\
+ 1 - e^{-\epsilon} + \sum_{j=1}^{d} E_Q (e^{-\sum_{|z|=b} \theta_j e^{-\beta_j(V(z)+y_1-L)}} \mathbb{1}_{\{V(z) + y_1 \leq 2L\}} \\
- e^{-\sum_{|z|=b} \theta_j e^{-\beta_j(V(z)+y_2-L)}} \mathbb{1}_{\{V(z) + y_2 \leq 2L\}}).
\]

Then we easily deduce that for any \( \theta \in (\mathbb{R}^+)^d \), \( y \mapsto F_{L,b}(\theta, y) \) is Riemann integrable. \( \square \)
Lemma 5.5 (Aïdékon [2]) Let \((r_n)_{n \geq 0}\) and \((\lambda_n)_{n \geq 0}\) be two sequences of numbers resp. in \(\mathbb{R}_+\) and in \((0,1)\) and such that resp. \(\lim_{n \to \infty} \frac{r_n}{n^3} = 0\), and \(\frac{2}{5} < \lim \inf_{n \to \infty} \lambda_n \leq \lim \sup_{n \to \infty} \lambda_n < 1\). Let \(F : \mathbb{R}_+ \to \mathbb{R}\) be a Riemann integrable function. We suppose that there exists a nonincreasing function \(F_+ : \mathbb{R}_+ \to \mathbb{R}\) such that \(|F(x)| \leq F_+(x)\) for any \(x \geq 0\) and \(\int_{x \geq 0} x F_+(x) < \infty\). Then as \(n \to \infty\),

\[
E\left[F(S_n - y_2); \min_{k \leq n} S_k \geq -y_1, \min_{k \in [\lambda_n n, n]} S_k \geq y_2\right] 
\approx \frac{C_- C_{+} \sqrt{\pi}}{\sigma \sqrt{2}} R(y_1)n^{-\frac{3}{2}} \int_{x \geq 0} F(x) R_-(x) dx 
\tag{5.12}
\]

uniformly in \(y_1, y_2 \in [0, r_n]\).

Proof of Lemma 5.5 Lemma 5.5 is a simple extension of Lemma 2.3 in [2]. By the Markov property, observe that

\[
E\left[F(S_n - y_2); \min_{k \leq n} S_k \geq -y_1, \min_{k \in [\lambda_n n, n]} S_k \geq y_2\right] 
= \sum_{k=0}^{\lambda_n n} E\left[\Upsilon_n(y_2 - S_k, k); \min_{j \leq k-1} S_j \geq y_1\right],
\]

(when \(k \geq \lambda_n n\), \(S_{\lambda_n n} \in [-y_1, 0)\) is impossible because of \(\min_{k \in [\lambda_n n, n]} S_k \geq y_2 \geq 0\),

with \(\Upsilon_n(y, k) := E\left[F(S_{n-k} - y); \min_{j \leq n-k} S_j \geq 0, \min_{j \in [\lambda_n n-k, n-k]} S_j \geq y\right], y \in \mathbb{R}, k \in \{0, \ldots, \lambda_n n - 1\}\).

Let \((m_n)_{n \in \mathbb{N}}\) a sequence of integers such that \(\frac{n}{m_n}\) and \(\frac{m_n}{r_n}\) go to infinity. First, we will show that

\[
\sum_{k=m_n}^{\lambda_n n-1} E\left[\Upsilon_n(y_2 - S_k, k); \min_{j \leq k-1} S_j \geq y_1\right] = o(R(y_1)n^{-\frac{3}{2}}). \tag{5.14}
\]

We divide the proof of (5.14) in two steps:

- According to the proof of Lemma 2.3 in [2], for any \(k \in [m_n, \frac{2}{3} n]\), \(y \in [0, 2r_n]\), we have \(\Upsilon_n(y, k) \leq \frac{c}{n^2} \sum_{j \geq 0} \bar{F}(j) j\). Thus, we deduce that
By combining (5.14) and (5.15), we obtain Lemma 5.5.

Recall that ∑_{j ≥ 0} \bar{F}(j) j < ∞. Moreover, by (4.7) (after reversing the time)

\[ \sum_{k=m_n}^{n/3} \mathbb{E} \left[ \min_{j ≤ k-1} S_j > S_k \geq -y_1 \right] ≤ c(1 + y_1)^2 \sum_{k=m_n}^{n/3} k^{-1} ≤ c(1 + y_1) \frac{1 + y_1}{\sqrt{m_n}} = o(R(y_1)), \]

where we have used that y_1 ≤ r_n and \( \frac{m_n}{r_n} \) go to infinity. So ∑_{k=m_n}^{n/3} \mathbb{E}[\mathcal{Y}_n(y_2 - S_k, k); \min_{j ≤ k-1} S_j > S_k ≥ -y_1] = o\left(\frac{R(y_1)}{n^{\frac{3}{2}}}\right).

- For the second step, we notice that for any \( k ∈ \left\{ \frac{2}{3} \lambda_n n - 1 \right\}, y ∈ [0, 2r_n], \) we have
\[ \mathcal{Y}_n(y, k) ≤ \sum_{j ≥ 0} \bar{F}(j) \mathbf{P}(S_{n-k} - y ∈ [j - 1, j], S_{n-k} ≥ 0) ≤ c \frac{r_n}{n^{\frac{3}{2}}} \sum_{j ≥ 0} \bar{F}(j) j. \]

Thus, we deduce that
\[ \sum_{k=n^{\frac{2}{3}}}^{\lambda_n n - 1} \mathbb{E} \left[ \mathcal{Y}_n(y_2 - S_k, k); \min_{j ≤ k-1} S_j > S_k ≥ -y_1 \right] \]
\[ ≤ \sum_{k=n^{\frac{2}{3}}}^{\lambda_n n - 1} c \mathbf{P} \left[ \min_{j ≤ k-1} S_j > S_k ≥ -y_1 \right] \frac{r_n}{n^{\frac{3}{2}}} \]
\[ ≤ c' (1 + y_1) \frac{r_n}{n^{\frac{3}{2}}} \frac{1 + y_1}{n^{\frac{3}{2}}} = o \left( \frac{R(y_1)}{n^{\frac{3}{2}}} \right). \]

The proof of (5.14) is now completed. Via Lemma 2.3 in [2]: uniformly in y_1, y_2 ∈ [0, r_n],

\[ \sum_{k=0}^{m_n} \mathbb{E} \left[ \mathcal{Y}_n(y_2 - S_k, k); \min_{j ≤ k-1} S_j > S_k ≥ -y_1 \right] \]
\[ = \sum_{k=0}^{m_n} \mathbf{P} \left[ \min_{j ≤ k-1} S_j > S_k ≥ -y_1 \right] \left[ \frac{C - C_+ \sqrt{\pi}}{\sigma \sqrt{2}} n^{-\frac{3}{2}} \int_{x ≥ 0} F(x) R_-(x) dx + o(n^{-\frac{3}{2}}) \right] \]
\[ = \frac{C - C_+ \sqrt{\pi}}{\sigma \sqrt{2}} R(y_1)n^{-\frac{3}{2}} \int_{x ≥ 0} F(x) R_-(x) dx + o \left( \frac{R(y_1)}{n^{\frac{3}{2}}} \right). \quad (5.15) \]

By combining (5.14) and (5.15), we obtain Lemma 5.5.

Now we end the proof of Proposition 2.2. Recalling that \( R(x) \xrightarrow{x→∞} c_0 x \) and defining \( \rho_{\beta,L,b}(\theta) = c_0 \rho_{\beta,L,b}^n(\theta) \), via Lemma 5.4 and 5.5 and the inequality (5.11),
we have obtained that: for any $\theta \in (\mathbb{R}_+^*)^d$, $\epsilon > 0$, there exists $(B, L_0)$ such that for any $b \geq B$, $L \geq L_0$, there exists $(A, N)_\epsilon \in \mathbb{R}_+ \times \mathbb{N}$ such that $\forall n > N$ and $x \in [A, \frac{3}{2} \log n - A]$, we have

$$\left| \frac{e^x}{x} E \left( 1 - \exp \left\{ - \sum_{i=1}^{d} \theta_i e^{-\beta_i x} \tilde{W}_{n, \beta_j} \right\} \right) - \rho_{\beta, L, b}(\theta) \right| \leq \epsilon. \quad (5.16)$$

In addition, by (4.18): There exist $c_6^{(\theta)}$, $c_7^{(\theta)} > 0$ and $A$, $N > 0$ large such that: for any $n \geq N$, $x \in [A, \frac{3}{2} \log n - A]$,

$$c_6^{(\theta)} xe^{-x} \leq E \left( 1 - \exp \left\{ - \sum_{i=1}^{d} \theta_i e^{-\beta_i x} \tilde{W}_{n, \beta_j} \right\} \right) \leq c_7^{(\theta)} xe^{-x}. \quad (5.17)$$

For any $p > 0$, let $(L, b)_p$ such that (5.16) is true (in the sense that: there exists $(A, N)_{\frac{1}{p}} \in \mathbb{R}_+ \times \mathbb{N}$ such that $\forall n > N$ and...) with $\epsilon = \frac{1}{p}$, and then we clearly have $\rho^{(p)}_{\beta}(\theta) := \rho_{\beta, (L, b)_p}(\theta) \in \left[ \frac{c_6^{(\theta)}}{p}, 2c_7^{(\theta)} \right]$ for any $p > \frac{2}{c_6^{(\theta)}}$. Let $\phi : \mathbb{N} \to \mathbb{N}$ strictly increasing such that $\rho^{(\phi(p))}_{\beta}(\theta) \to \rho_\beta(\theta) \in \left[ \frac{c_6^{(\theta)}}{2}, 2c_7^{(\theta)} \right]$. We shall complete the proof of (2.2) by using the following observation: in the display (5.16), the expectation only depends on $x$ and $n$, whereas $\rho_{\beta, L, b}(\theta)$ is independent of $x$, $n$.

Fix $\epsilon > 0$. Let $p_0 > 0$ such that $|\rho^{(\phi(p_0))}_{\beta}(\theta) - \rho_\beta(\theta)| \leq \epsilon$ and $\frac{1}{\phi(p_0)} \leq \epsilon$. Then it suffices to choose (according to (5.16) and associated with $(L, b)_{\phi(p_0)}$) $(A, N)_{\frac{1}{\phi(p_0)}} > 0$ such that for any $n \geq N$, $x \in [A, \frac{3}{2} \log n - A]$,

$$\left| \frac{e^x}{x} E \left( 1 - \exp \left\{ - \sum_{i=1}^{d} \theta_i e^{-\beta_i x} \tilde{W}_{n, \beta_j} \right\} \right) - \rho_\beta(\theta) \right| \leq \epsilon.$$

This completes the proof of (2.2). We stress here that our argument shows that all the possible extractions of $\rho^{(p)}_{\beta}(\theta)$ converge, in the end, to the same limit.

To complete the proof of Proposition 2.2, it remains to prove that $\rho_\beta$ is a continuous function at 0. For any $\theta \in \mathbb{R}_+^d$, let $i^* \in \{1, \ldots, d\}$ such that $\theta_{i^*} := \max_{i \in [1, d]} \frac{1}{\beta_i}$. By applying $d$ times the inequality $1 - e^{x+y} \leq 1 - e^x + 1 - e^y$ with $x$, $y \geq 0$, and then Corollary 4.7, we deduce that there exists $A$, $N > 0$ such that for any $n \geq N$ and $x \in [A + \frac{1}{\beta_{i^*}} \log \theta_{i^*}, \frac{3}{2} \log n - A + \frac{1}{\beta_{i^*}} \log \theta_{i^*}]$,
Here we prove a slight extension of Lemma B.3 in [2]. It will be used to prove Lemma 6.1.

**Appendix 1: Auxiliary Estimates**

I wish to thank my supervisor Yueyun Hu for introducing me to this subject and constantly finding the time for useful discussions and advice. I would also like to thank the referees for their helpful comments.

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For $\alpha > 0$, $d_1$, $d_2$, $a \geq 0$, $n \geq 1$ and $0 \leq i \leq n$, we define

$$k_i^{(d_1,d_2)}(x, a) = k_i^{(d_1,d_2)} := \begin{cases} -d_1 + i^\alpha, & \text{if } 0 \leq i \leq \lfloor \log d_1 \rfloor, \\ i^\alpha - x, & \text{if } \lfloor \log d_1 \rfloor \leq i \leq \lfloor n/2 \rfloor, \\ a + (n-i)^\alpha - d_2, & \text{if } \lfloor n/2 \rfloor < i \leq n. \end{cases} \quad (6.1)$$

**Lemma 6.1** Let $\alpha \in (0, 1/6)$ and $\epsilon > 0$. There exist $d_1$, $d_2 > 0$ large enough such that for any $u$, $x \geq 0$, $a \in (0, 10 \log n)$ and $n \geq e^{d_1+d_2}$,

$$\begin{align*}
\mathbb{P}\left\{ \exists i \leq n : S_i \leq k_i^{(d_1,d_2)}, \min_{j \leq n} S_j \geq -x, \min_{\lfloor n/2 \rfloor < j \leq n} S_j \geq a, \ S_n \leq a + u \right\} \\
&\leq (1+x)(1+u)^2 \frac{\epsilon}{n^{3/2}}. 
\end{align*} \quad (6.2)
$$

**Proof** We treat $n/2$ as an integer. Let $E$ be the event in (6.2). We have $\mathbb{P}(E) \leq \sum_{i=1}^n \mathbb{P}(E_i)$ where

$$E_i := \{ S_i \leq k_i^{(d_1,d_2)}, \min_{j \leq n} S_j \geq -x, \min_{\lfloor n/2 \rfloor < j \leq n} S_j \geq a, \ S_n \leq a + u \}. $$

When $d$ is large enough, by the Markov property at time $i \in [1, \lfloor \log d_1 \rfloor]$ and (4.8),

$$\begin{align*}
\sum_{i=1}^{\lfloor \log d_1 \rfloor} \mathbb{P}(E_i) &\leq \sum_{i=1}^{\lfloor \log d_1 \rfloor} \frac{c_{54}(1+u)^2}{n^{3/2}} \mathbb{E}\left[ (1+S_i+x) + 1\{ S_i \leq -d_1/2 \} \right] \\
&\leq \sum_{i=0}^{\lfloor \log d_1 \rfloor} \frac{c_{54}(1+u)^2}{n^{3/2}}(1+x)\mathbb{P}\left( S_i \leq -\frac{d_1}{2} \right)
\end{align*}$$
Finally we treat the case \( i \in ([\log d_1], n/2] \). By the Markov property at time \( i \geq 1 \), (4.7) and (4.8), we have

\[
\mathbf{P}(E_i) \leq \begin{cases} 
\frac{c_{21}(1+u)^2}{n^{3/2}} \mathbf{E} \left[ (1 + S_i + x) + 1_{\{S_i \leq i^a - x, \min_j S_j \geq x\}} \right] & \text{if } \log d_1 \leq i \leq n/3,
\frac{c_{21}(1+u)^2}{n^{3/2}} \mathbf{E} \left[ (1 + S_i + x) + 1_{\{S_i \leq i^a - x, \min_j S_j \geq x\}} \right] & \text{if } n/3 \leq i \leq n/2.
\end{cases}
\]

By (4.8), recalling that \( a \leq \log n \), it yields that for \( d_1 \) large enough,

\[
\sum_{i=\lfloor \log d_1 \rfloor}^{n/3} \frac{c_{22}(1+u)^2}{n^{3/2}} (1 + i^a)^3 < (1 + u)^2 (1 + x) \frac{\varepsilon}{n^{3/2}}.
\]

Finally we treat the case \( i \in [n/2, n] \). By the Markov property at time \( i \) and (4.7), we have

\[
\mathbf{P}(E_i) \leq \frac{c_{24}(1+u)^2}{(n-i+1)^{3/2}} \mathbf{E} \left[ (1 + S_i - a) + 1_{\{S_i \leq a+(n-i)^a - d_2, \min_j S_j \geq -x, \min_{n-i \leq j \leq i} S_j \leq a\}} \right].
\]

If \( i \geq n - d_2^{1/3} \), clearly we have \( \mathbf{P}(E_i) = 0 \). If \( n - d_2^{1/3} \geq i \geq 2n/3 \), we use (4.7) to see that \( \mathbf{P}(E_i) \leq c_{25} (1 + x) (1 + u)^2 \frac{(1+n-i)^{3a-3/2}}{n^{3/2}} \). Therefore, for \( d_2 \) large enough we have

\[
\sum_{i=\lceil 2n/3 \rceil}^{n} \mathbf{P}(E_i) \leq \sum_{i=\lceil 2n/3 \rceil}^{n-d_2^{1/3}} c_{25} (1 + x) (1 + u)^2 \frac{(1+n-i)^{3a-3/2}}{n^{3/2}} \leq (1 + x) (1 + u)^2 \frac{\varepsilon}{n^{3/2}}.
\]

If \( n/2 < i < 2n/3 \), we simply write

\[
\mathbf{P}(E_i) \leq \frac{c_{26}(1+u)^2}{(n-i+1)^{3/2}} \mathbf{E} \left[ (1 + S_i - a) 1_{\{a \leq S_i \leq a+(n-i)^a, \min_j S_j \geq -x\}} \right] \leq c_{27} (1 + u)^2 \frac{(n-i)^a}{(n-i+1)^{3/2}} \mathbf{P}(a \leq S_i \leq a+(n-i)^a, \min_j S_j \geq -x) \leq c_{28} (1 + x) (1 + u)^2 \frac{n^a (a + n^a)^2}{n^3}.
\]

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by (4.7). We deduce that
\[
\sum_{i=\lfloor n/2 \rfloor}^{[2n/3]} \mathbb{P}(E_i) \leq c_{29}(1 + x)(1 + u)^2 \frac{(n^\alpha + a)^2}{n^{2-\alpha}}. \tag{6.7}
\]
Inequalities (6.5), (6.6) and (6.7) suffice to prove Lemma 6.1 \(\square\)

Appendix 2: Good Vertex

Lemma 7.1 below gives a control on the behavior of the different brothers of the spine. It will be used to prove the Lemma 5.3 of the “Appendix 3”.

Let us recall some notations. Let \(e_k, 0 \leq k \leq n\) such that
\[
e_k = e_k^{(n)} = \begin{cases} \frac{1}{12} k & \text{if } 0 < k \leq \frac{1}{2} n, \\ \frac{1}{12} (n-k) & \text{if } \frac{n}{2} < k \leq n, \end{cases} \tag{7.1}
\]
and denote
\[
d_k = d_k(n, x, L, B_1) := \begin{cases} -B_1 & \text{if } 0 \leq k \leq \log B_1, \\ -x & \text{if } \log B_1 \leq k \leq \frac{n}{2}, \\ \max(a_n - x - L - 1, 0) & \text{if } \frac{n}{2} < k \leq n. \tag{7.2}
\end{cases}
\]

We say that \(|u| = n\) is a \((x, L, B_1, B_2)\)-good vertex if \(u \in \mathcal{Z}_n^{x,x+L,x-L}\) and
\[
\sum_{w \in \Omega(\Omega_k)} e^{-(V(v)-d_k)} \{1 + (V(v) - d_k)_+\} \leq B_2 e^{-e_k}, \quad \forall 1 \leq k \leq n. \tag{7.3}
\]

**Lemma 7.1** (Aïdékon [2]) Fix \(L \geq 0\). For any \(\epsilon > 0\), we can find \(B_1(0), B_2(0)\) large enough such that for any \(B_1 \geq B_1(0), B_2 \geq B_2(0)\), as in (7.1) and (7.2), for any \(n \geq e^{B_1+B_2}\) and \(x \geq 0\)
\[
\mathbb{Q}\left(w_n \text{ is not a } (x, L, B_1, B_2)\text{-good vertex, } w_n \in \mathcal{Z}_n^{x,x+L,x-L}\right) \leq (1 + x)\epsilon n^{-\frac{3}{2}}. \tag{7.4}
\]

**Proof of Lemma 7.1.** From Lemma 6.1, there exists \(B(L), c(L) > 0\) large enough such that for any \(B_1 \geq B, n \geq e^{B_1+B_2}\), and \(x \geq 0\),
\[
\mathbb{Q}\left\{w_n \in \mathcal{Z}_n^{x,x+L,x-L} \cup \bigcup_{j=0}^{n/2+1} \{V(w_j) \leq d_j + 2e_j\}
\right\} \leq \frac{e}{n^2} (1 + x), \tag{7.5}
\]
(with according to the notation of Lemma 6.1, \( a \leftrightarrow a_n - x - L - 1, u \leftrightarrow 2L, x \leftrightarrow x, B_1 \leftrightarrow d_1, c(L) \leftrightarrow d_2, i^a \leftrightarrow 2e_i \). From now we fix \( c(L) > 0 \). For any \( n, B_1 > 0 \), we denote \( \mathcal{H}_n \) the event \( \bigcup_{j=0}^{n} \{ V(w_j) \leq d_{j+1} + 2e_{j+1} \} \cup \{ V(w_j) \leq d_{j+1} + 2e_{j+1} - c(L) \} \). Consequently, it is enough to show that for \( B_1, B_2 \) large enough,

\[
\sum_{k=1}^{n} Q \left( \sum_{v \in \Omega(w_k)} e^{-(V(v)-d_k)} \left( 1 + (V(v) - d_k)_{+} \right) \right) > B_2 e^{-c(L)/2} e^{V(w_{k-1})_{-} - d_k}, w_n \in \mathbb{Z}_n^{x+L,x-L}, \mathcal{H}_n \right) 
\leq \epsilon (1 + x) n^{\frac{-3}{2}}. \tag{7.6}
\]

We see that

\[
\sum_{v \in \Omega(w_k)} e^{-(V(v)-d_k)} \left( 1 + (V(v) - d_k)_{+} \right) 
\leq e^{-(V(w_{k-1})_{-} - d_k)} \sum_{v \in \Omega(w_k)} e^{-(V(v)-V(w_{k-1}))} \left( 1 + (V(w_{k-1}) - d_k)_{+} + (V(v) - V(w_{k-1}))_{+} \right) 
\leq e^{-(V(w_{k-1})_{-} - d_k)} \left( 1 + (V(w_{k-1}) - d_k)_{+} \right) \sum_{v \in \Omega(w_k)} e^{-(V(v)-V(w_{k-1}))} \left( 1 + (V(v) - V(w_{k-1}))_{+} \right).
\]

By denoting for any \( |v| \geq 1, \xi(v) := \sum_{w \in \Omega(v)} (1 + (V(w) - V(\bar{w}))) e^{-V(w) - V(\bar{w})} \), we have then

\[
\sum_{v \in \Omega(w_k)} e^{-(V(v)-d_k)} \left( 1 + (V(v) - d_k)_{+} \right) \leq e^{V(w_{k-1})_{-} - d_k} \left( 1 + (V(w_{k-1}) - d_k)_{+} \right) \xi(w_k).
\]

Equation (7.6) boils down to showing that for \( B_1, B_2 \) large enough,

\[
\sum_{k=1}^{n} Q \left( \xi(w_k) > B_2 e^{-c(L)/2} e^{V(w_{k-1})_{-} - d_k}, w_n \in \mathbb{Z}_n^{x+L,x-L}, \mathcal{H}_n \right) 
\leq \sum_{k=1}^{n} Q \left( \xi(w_k) > B_2 e^{-c(L)/2} e^{V(w_{k-1})_{-} - d_k}, w_n \in \mathbb{Z}_n^{x+L,x-L}, \mathcal{H}_n \right) \leq \epsilon (1 + x) n^{\frac{-3}{2}}. \tag{7.7}
\]

First, we deal with the case \( k \in [1, \frac{3n}{4}] \). By the Markov property at time \( k \), we get

\[
Q \left( \xi(w_k) > B_2 e^{-c(L)/2} e^{V(w_{k-1})_{-} - d_k}, w_n \in \mathbb{Z}_n^{x+L,x-L}, \mathcal{H}_n \right) 
\leq Q \left[ \lambda(V(w_k), k, n) \mathbb{I}_{\{ \xi(w_k) > B_2 e^{-c(L)/2} e^{V(w_{k-1})_{-} - d_k}, V(w_j) \geq -x, \forall j < k \} \right].
\]
where

$$
\lambda(r, k, n) := \mathbb{P}_r(S_j \geq d_{j+k} + e_{j+k}, \ \forall j \leq \frac{n}{2} - k, \ S_j \geq d_{j+k} + e_{j+k} - c(L),
$$

$$
\forall j \in \left(\frac{n}{2} - k, n - k\right],
$$

$$
S_{n-k} \leq a_n - x + L, \ \min_{j \in [k, \frac{n}{2}]} S_{j-k} \geq -x, \ \min_{j \in [n/2, n]} S_{j-k} \geq a_n - x - L - 1.
$$

When $k \in [1, n/3]$ by (4.8),

$$
\lambda(r, k, n) \leq \mathbb{P}_r(S_{n-k} \leq a_n - x + L, \ \min_{j \in [k, \frac{n}{2}]} S_{j-k} \geq -x,
$$

$$
\min_{j \in [n/2, n]} S_{j-k} \geq a_n - x - L - 1)
$$

$$
\leq c_{30}(1 + L)^2n^{-\frac{3}{2}}(1 + (r + x)_+).
$$

(7.8)

When $k \in (n/3, 3n/4)$ by (4.7) (recalling $d_{j+k} + e_{j+k} - c(L) \geq a_n - x - L - 1$ for any $k \in (n/3, 3n/4)$, $j \in [0, n - k - \frac{n}{5}]$),

$$
\lambda(r, k, n) \leq \mathbb{P}_r(S_{n-k} \leq a_n - x + L, \ \min_{j \in [k, n]} S_{j-k} \geq a_n - x - L - 1)
$$

$$
\leq c_{30}(1 + L)^2n^{-\frac{3}{2}}(1 + (r + x)_+).
$$

(7.9)

This yields that

$$
\mathbb{Q}\left(\xi(w_k) > B_2 e^{-c(L)/2} e^{\frac{V(w_{k-1}) - d_k}{3}}, \ w_n \in \mathbb{Z}^{x, x + L, x - L, \mathcal{H}_n}\right)
$$

$$\leq c_{30}(1 + L)^2n^{-\frac{3}{2}}\mathbb{Q}\left[\left(1 + (V(w_k) + x)_+\right)
$$

$$\times \mathbbm{1}_{\left\{\xi(w_k) > B_2 e^{-c(L)/2} e^{\frac{V(w_{k-1}) - d_k}{3}}, V(w_j) \geq -x + (e_j - c(L))_+, \forall j \leq k\right\}}\right]
$$

$$\leq c_{30}(1 + L)^2n^{-\frac{3}{2}}\mathbb{Q}\left[\left(1 + V(w_k)_+\right)
$$

$$\times \mathbbm{1}_{\left\{\xi(w_k) > B_2 e^{\frac{1}{4}(V(w_{k-1}) + (B_1 - 1)/4)(k - 1 \leq \log B_1 - c(L)/2)}, V(w_j) \geq 0, \forall j \leq k\right\}}\right],
$$

(7.10)

indeed, $V(w_k) + x \geq (e_k - c(L))_+$ implies $\frac{V(w_k) - d_k}{3} \geq \frac{V(w_k) + x}{4} \forall k \geq \log B_1$. On the other hand, we have

$$
1 + (V(w_k) + x)_+ \leq 1 + (V(w_{k-1}) + x)_+ + (V(w_k) - V(w_{k-1}))_+.
$$

Let $(\xi, \Delta)$ be generic random variable distributed as $(\xi(w_1), V(w_1)_+)$ under $\mathbb{Q}$ and independent of the other random variables. By the Markov property at time $k - 1$, we obtain that

\[\text{(skipping)}\]
\[ Q_x \left[ (1 + V(w_k)) \mathbb{I}_{\xi(w_k) > B_2 e^{\frac{1}{2}(V(w_{k-1}) + (B_1 - x) \mathbb{I}_{\text{min} \geq \log B_1)} - c(L)/2}, V(w_j) \geq 0, \forall j \leq k} \right] \]
\[
\leq Q_x \left[ \kappa_k (V(w_{k-1})) \mathbb{I}_{\{V(w_j) \geq 0, \forall j \leq k-1\}} \right],
\]

with, for any \( z \geq 0 \)

\[
\kappa_k(z) := \begin{cases} 
(1 + z) \mathbb{I}_{\xi > B_2 e^{-c(L)/2} e^{\frac{z-x-B_1}{4}}} + \Delta \mathbb{I}_{\xi \leq B_2 e^{-c(L)/2} e^{\frac{z-x-B_1}{4}}} & \text{if } k \leq \log B_1 \\
(1 + z) \mathbb{I}_{\xi > B_2 e^{-c(L)/2} e^{\frac{z-x-B_1}{4}}} + \Delta \mathbb{I}_{\xi \leq B_2 e^{-c(L)/2} e^{\frac{z-x-B_1}{4}}} & \text{if } k > \log B_1 .
\end{cases}
\]

In view of (7.10), it follows that

\[
\sum_{k=1}^{3n} Q\left( \xi(w_k) > B_2 e^{-c(L)/2} e^{\frac{V(w_{k-1}) - d_k}{3}}, w_n \in \mathcal{Z}_{n}^{x, x+L, x-L} \right) \leq c_{30} (1 + L)^2 n^{-\frac{3}{2}} (D_1 + D_2),
\]

(7.11)

where \( c_{30} \) is the constant defined in (7.8) and (7.9) and

\[
D_1 := \sum_{k=1}^{\log B_1} Q_x \left[ (1 + V(w_k)) \mathbb{I}_{\{V(w_k) + B_1 - x \leq 4(\log \xi - \log B_2) + 2c(L)}, \min_{j \leq k} V(w_j) \geq 0 \right]
\]
\[
+ \sum_{k=\log B_1 + 1}^{\frac{3n}{3}} Q_x \left[ (1 + V(w_k)) \mathbb{I}_{\{V(w_k) \leq 4(\log \xi - \log B_2) + 2c(L)}, \min_{j \leq k} V(w_j) \geq 0 \right].
\]

\[
D_2 := \sum_{k=1}^{\log B_1} Q_x \left[ \Delta \mathbb{I}_{\{V(w_k) + B_1 - x \leq 4(\log \xi - \log B_2) + 2c(L)}, \min_{j \leq k} V(w_j) \geq 0 \right]
\]
\[
+ \sum_{k=\log B_1 + 1}^{\frac{3n}{3}} Q_x \left[ \Delta \mathbb{I}_{\{V(w_k) \leq 4(\log \xi - \log B_2) + 2c(L)}, \min_{j \leq k} V(w_j) \geq 0 \right]
\]

When \( k \in [1, \log B_1] \), using the definition of \( d_k \) in (7.2), observe that (for \( B_2 \geq e^{10c(L)} \))

\[
Q_x \left[ (1 + V(w_k)) \mathbb{I}_{\{V(w_k) + B_1 - x \leq 4(\log \xi - \log B_2) + 2c(L)}, \min_{j \leq k} V(w_j) \geq 0 \right]
\]
\[
\leq E_Q \left[ (1 + 4 \log \xi + x) \mathbb{I}_{\{B_1 \leq 3 \log \xi - V(w_k) \}} \right]
\]
\[
\leq c_{30} (1 + x) Q(B_1 \leq 4 \log \xi - V(w_k)) + E_Q (\log \xi \mathbb{I}_{\{B_1 \leq 4 \log \xi - V(w_k) \}},
\]

(7.12)

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and

\[ Q\left[ \Delta_+ \mathbb{I}_{\{V(w_k) + B_1 \leq 4(\log \xi - \log B_2) + 2c(L), \; \min_{j \leq k} V(w_j) \geq -x\}} \right] \]

\[ \leq 16Q\left[ \Delta_+(4 \log(\xi)_+ + \mathbb{E}(|V(w_k)|)) \right]/B_1. \]  \hspace{1cm} (7.13)

Recalling that for any \( k \in [1, \log B_1] \), \( \mathbb{E}(V(w_k)^2) \leq \sigma^2 \log B_1 \), and \( \mathbb{E}Q((1 + \log(\xi)^2) < \infty \), we deduce that for \( B_1(c_30(1 + L)^2) \) large enough, we have

\[ \sum_{k=1}^{\log B_1} Q_x\left[ (1 + V(w_k)) \mathbb{I}_{\{V(w_k) + B_1 - x \leq 4(\log \xi - \log B_2) + 2c(L), \; \min_{j \leq k} V(w_j) \geq 0\}} \right] \]

\[ \leq \frac{e(1 + x)}{c_30(1 + L)^2} n^{-\frac{3}{2}}, \] \hspace{1cm} (7.14)

\[ \sum_{k=1}^{\log B_1} Q_x\left[ \Delta_+ \mathbb{I}_{\{V(w_k) + B_1 - x \leq 4(\log \xi - \log B_2) + 2c(L), \; \min_{j \leq k} V(w_j) \geq 0\}} \right] \]

\[ \leq \frac{e(1 + x)}{c_30(1 + L)^2} n^{-\frac{3}{2}}. \] \hspace{1cm} (7.15)

When \( k \in (\log B_1, 3n/4] \),

\[ \sum_{k=\log B_1}^{3n/4} E_{Q_x}\left[ (1 + V(w_k)) \mathbb{I}_{\{V(w_k) - d_k \leq 4(\log \xi - \log B_2) + 2c(L), \; \min_{j \leq k} V(w_j) \geq -x\}} \right] \]

\[ \leq \sum_{k=0}^{+\infty} E_{Q_x}\left[ (1 + V(w_k)) \mathbb{I}_{\{V(w_k) \leq 4(\log \xi - \log B_2) + 2c(L), \; \min_{j \leq k} V(w_j) \geq 0\}} \right]. \] \hspace{1cm} (7.16)

Notice that in (7.16), the term inside the expectation is 0 if \( \log B_2 > \log \xi + 3c(L)/2 \). Therefore, we can add the indicator that \( \log B_2 - 3c(L)/2 \leq \log \xi \). By Lemma B.2 (i), we get that

\[ \sum_{k=0}^{+\infty} E_{Q_x}\left[ (1 + V(w_k)) \mathbb{I}_{\{V(w_k) \leq 4(\log \xi - \log B_2) + 2c(L), \; \min_{j \leq k} V(w_j) \geq 0\}} \right] \]

\[ \leq c_{30} Q\left[ \mathbb{I}_{\{\log B_2 - 2c(L) \leq 4\log \xi\}} (1 + (\log \xi - \log B_2)^2) \right](1 + x) \]

\[ \leq c_{30} Q\left[ \mathbb{I}_{\{\log B_2 - 2c(L) \leq 4\log \xi\}} (1 + \log(\xi)^2) \right](1 + x). \] \hspace{1cm} (7.17)

Observe that \( \xi \leq X + \tilde{X} \) with the notation of (1.3). Going back to the measure \( Q \), we get
for $B_2$ large enough. Similarly,
\[
\sum_{k = \log B_1}^{3n/4} Q_x \left[ \min_{j \leq k} V(w_j) \leq 0 \right] \leq \frac{\epsilon (1 + x)}{c_{30}(1 + L)^2}.
\] (7.19)

In order to prove (7.6), by combining (7.7), (7.14), (7.15), (7.18) and (7.19), it remains to treat the case $\frac{3n}{4} \leq k \leq n$. Recalling (7.7), we want to show that for $B \leq B_2 e^{-(cL)/2}$ large enough, $n \geq e^B$ and $x \geq 1$,
\[
\sum_{k = \frac{3n}{4}}^{n} Q \left( \xi(w_k) > B e^{\frac{(w_{k+1}) - d_k}{3}}, w_n \in \mathcal{Z}_n^{x,x+L,x-L}, \mathcal{H}_n \right) \leq \epsilon (1 + x) n^{-\frac{3}{2}}. 
\] (7.20)

This case is quasi-identical to the proof of (C.8) (in the proof of Lemma C.1 [2]) in [2], and then we omit the details of the proof of (7.20).}

\section*{Appendix 3: Proof of Lemma 5.3}

\begin{lemma}
\label{lem5.3}
\forall \eta, L > 0 \exists D(L, \eta) > 0 \text{ and } B(L, \eta) \geq 1 \text{ such that } \forall b \geq B, n \geq e^{5b}, x \geq D,
\[
Q((\xi_n(x, b, L))^c, \omega_n \in \mathcal{Z}_n^{x,x+L,x-L}) \leq \eta n^{-\frac{3}{2}} (1 + x).
\] (8.1)
\end{lemma}

\begin{proof}
Let $L, \eta > 0$. According to Lemma 7.1, there exists $B_0(= B_0(L, \eta))$ such that for any $B_1, B_2 \geq B_0, n \geq e^{B_1 + B_2}$ and $x \geq 0$
\[
Q(w_n \in \mathcal{Z}_n^{x,x+L,x-L}, w_n \text{ is not a}(x, L, B_1, B_2)\text{-good vertex}) \leq \frac{\eta}{n^2} (1 + x). 
\] (8.2)

Now we fix $B_2 \geq B_0$. For $\xi_n$ to happen, every brother of the spine at generation less than $n - b$ must have its descendants at time $n$ greater than $a_n(x) + L$. In others words,
\[
Q((\xi_n)^c, \omega_n \text{ is a good vertex}) = Q \left[ 1 - \prod_{k=1}^{n-b} \prod_{u \in \Omega(\omega_k)} p(u, x - L), \omega_n \text{ is a good vertex} \right], 
\] (8.3)

where $p(u, x - L) = P(M_{n-|u|} \geq a_n - x - V(u) + L))$ is the probability that the branching random walk rooted at $u$ has its minimum greater $a_n - x + L$ at time $n - |u|$. From Proposition 4.4 [2], we see that

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$1 - p(u, x - L) \leq c_2 (1 + (x + V(u) - L)_+) e^{-(x + V(u) - L)}$, \quad |u| \leq \frac{n}{2}.

Moreover, as $w_n$ is a good vertex, we have

$\sum_{u \in \Omega(w_k)} (1 + (V(u) + x - L)_+) e^{-(x - L) - V(u)} \leq \begin{cases} B_2 e^{B_1} e^{-k \frac{1}{2}} (1 + x) e^{-(x - L)}, & \text{if } k \leq \log B_1, \\ B_2 e^{-k \frac{1}{2} + L}, & \text{if } k \in (\log B_1, n/2], \end{cases}$

where we have used that $d_k = -B_1$ when $k \leq \log B_1$ and $d_k = -x$ when $k \in (\log B_1, n/2].$ Using the inequality $p \geq e^{\frac{n}{2} - 1}$ for $p$ close enough to 1, it implies that for $x$ large enough and $1 \leq k \leq n/2,$

$$\prod_{k=1}^{n/2} \prod_{u \in \Omega(w_k)} p(u, x - L) \geq \exp\left((-2c_2B_2e^{L})\right) \times \left(e^{B_1 (1 + x) e^{-x \sum_{k=1}^{\log B_1} e^{-k \frac{1}{2}}} + \sum_{k=\log B_1}^{n/2} e^{-k \frac{1}{2}}}\right) \geq \exp(-2c_2B_2e^{L}(c_3 e^{B_1 (1 + x) e^{-x} + e^{-\log B_1 \frac{1}{2}}})).$$

Therefore, there exists $B_1 > 0$ and $D_1(L, B_1) > 0$ large enough such that for any $x \geq D_1$

$$\prod_{k=1}^{n/2} \prod_{u \in \Omega(w_k)} p(u, x - L) \geq \left(1 - \frac{\eta}{L^2}\right)^{1/2}.$$ (8.4)

From now $B_1$ and $D_1$ are fixed.

If $k > n/2,$ since $W_n$ (defined in (4.3)) is a martingale, we have $1 = E[W_i] \geq E[e^{-M_l}] \geq e^{-x} P(M_l \leq x)$ for any $l \geq 1$ and $x \in \mathbb{R}.$ We get that

$$1 - p(u, x - L) \leq \mathbb{P}(M_{n-|u|} \leq a_n - x + L - V(u)) \leq e^{a_n - x + L} e^{-V(u)}.$$

We rewrite it (we have $x - L \geq 0$), $1 - p(u, x - L) \leq n^3 e^{-V(u)} e^{-x + L} = e^{-(V(u) - d_k)} e^{L}$ for $n/2 < k \leq n.$ Since $w_n$ is a good vertex, we get that $\prod_{u \in \Omega(w_k)} p(u, x - A) \geq e^{-c_3(B_1, B_2) e^{2L}} = e^{-c_3(B_1, B_2)(n-k)^{1/2} e^{2L}}.$ Consequently,

$$\prod_{k=\lceil n/2 \rceil + 1}^{n-b} \prod_{u \in \Omega(w_k)} p(u, x - L) \geq e^{-c_3(B_1, B_2) e^{L}} \sum_{|n/2| + 1}^{n-b} e^{-(n-k) \frac{1}{2}}.$$
It yields that there exists \( B(L, \eta, c_{32}(B_1, B_2)) \geq B_0 \) large enough such that \( \forall b \geq B, n > b, \) we have,

\[
\prod_{k=[n/2]+1}^{n-b} \prod_{u \in \Omega_{(ok)}} p(u, x - L) \geq \left( 1 - \frac{\eta}{L^2} \right) ^{\frac{1}{2}}. \tag{8.5}
\]

In view of (8.4) and (8.5), we have for \( b \geq B, x \geq D_1 \) and \( n \geq e^{5b}, \prod_{k=1}^{n-b} \prod_{u \in \Omega(w_k)} p(u, x - L) \geq (1 - \frac{\eta}{L^2}) ^{\frac{1}{2}}. \) Plugging into (8.3) yields that

\[
Q((\xi_n)^c, w_n) \leq \eta \left( 1 + x \right) \left( \frac{1}{L^2} Q(w_n \in Z_n^{x+L,x-L}) + n^{-\frac{1}{2}} \right).
\]

It follows from (8.2) that

\[
Q((\xi_n)^c, w_n \in Z_n^{x+L,x-L}) \leq \alpha_3 \left( 1 + L \right) ^{2} n^{-\frac{1}{2}}, \]

Remember that the spine behaves as a centered random walk. Then applying (4.8) to see that \( Q(w_n \in Z_n^{x+L,x-L}) \leq \alpha_3 \left( 1 + L \right) ^{2} n^{-\frac{1}{2}}, \) it completes the proof of Lemma 5.3. \( \square \)

References

1. Addario-Berry, L., Reed, B.: Minima in branching random walks. Ann. Probab. 37(3), 1044–1079 (2009)
2. Aïdékon, E.: Convergence in law of the minimum of a branching random walk. Ann. Probab. 41, 1362–1426 (2013)
3. Aïdékon, E., Berestycki, J., Brunet, É., Shi, Z.: The branching Brownian motion seen from its tip. Probab. Theory Rel. 157(1), 405–451 (2013)
4. Aïdékon, E., Shi, Z.: Weak convergence for the minimal position in a branching random walk: a simple proof. Period. Math. Hungar. 61(1–2), 43–54 (2010)
5. Arguin, L.-P., Bovier, A., Kistler, N.: Poissonian Statistics in the Extremal Process of Branching Brownian Motion. ArXiv e-prints, October (2010)
6. Arguin, L.-P., Bovier, A., Kistler, N.: The Extremal Process of Branching Brownian Motion. Probab. Theory Relat. 157(3), 535–574 (2013)
7. Biggins, J.D.: Chernoff’s theorem in the branching random walk. J. Appl. Probab. 14(3), 630–636 (1977)
8. Biggins, J.D., Kyprianou, A.E.: Measure change in multitype branching. Adv. Appl. Probab. 36(2), 544–581 (2004)
9. Bramson, M., Zeitouni, O.: Tightness for a family of recursion equations. Ann. Probab. 37(2), 615–653 (2009)
10. Brunet, É., Derrida, B.: A branching random walk seen from the tip. J. Stat. Phys. 143, 420–446 (2011)
11. Chauvin, B., Rouault, A.: KPP equation and supercritical branching Brownian motion in the subcritical speed area. Application to spatial trees. Probab. Theory Relat. Fields 80(2), 299–314 (1988)
12. Davydov, Y., Molchanov, I., Zuyev, S.: Strictly stable distributions on convex cones. Electron J Probab. \textbf{13}, 259–321 (2008)
13. Feller, W.: An Introduction to Probability Theory and Its Applications, vol. II, 2nd edn. Wiley, New York (1971)
14. Hammersley, J.M.: Postulates for subadditive processes. Ann. Probab. \textbf{2}, 652–680 (1974)
15. Hu, Y., Shi, Z.: Minimal position and critical martingale convergence in branching random walks, and directed polymers on disordered trees. Ann. Probab. \textbf{37}(2), 742–789 (2009)
16. Jaffuel, B.: The critical random barrier for the survival of branching random walk with absorption. Ann. I. H. Poincare-Pr. \textbf{48}, 989–1009 (2012)
17. Kallenberg, O.: Random Measures. Akademie-Verlag, Berlin (1976)
18. Kingman, J.F.C.: The first birth problem for an age-dependent branching process. Ann. Probab. \textbf{3}(5), 790–801 (1975)
19. Kozlov, M.V.: The asymptotic behavior of the probability of non-extinction of critical branching processes in a random environment. Teor. Verojatnost. i Primenen. \textbf{21}(4), 813–825 (1976)
20. Lyons, R.: A simple path to Biggins’ martingale convergence for branching random walk. IMA Vol. Math. Appl. \textbf{84}, 217–221 (1997)
21. Lyons, R., Pemantle, R., Peres, Y.: Conceptual proofs of $L \log L$ criteria for mean behavior of branching processes. Ann. Probab. \textbf{23}(3), 1125–1138 (1995)
22. Maillard, P.: A note on stable point processes occurring in branching Brownian motion. Electron. Commun. Probab. \textbf{18}, 1–9 (2013)