Solution of the equation \((pu')' + qu = \omega^2 u\) by a solution of the equation \((pu_0')' + qu_0 = 0\)

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February 1, 2008

Abstract

We give a simple solution of the equation \((pu')' + qu = \omega^2 u\) whenever a nontrivial solution of \((pu_0')' + qu_0 = 0\) is known. The method developed for obtaining this result is based on the theory of pseudoanalytic functions and their relationship with solutions of the stationary two-dimensional Schrödinger equation. The final result, that is the formula for the general solution of the equation \((pu')' + qu = \omega^2 u\) has a simple and easily verifiable form.

1 Introduction

The problem of solving the equation

\[
\partial_x (p(x)\partial_x u(x)) + q(x)u(x) = \omega^2 u(x)
\]

by a known nontrivial solution of the equation

\[
\partial_x (p(x)\partial_x u_0(x)) + q(x)u_0(x) = 0
\]

where \(p, q, u, u_0\) are complex valued functions of the real variable \(x\) and \(\omega\) is an arbitrary complex constant is of fundamental importance due to numerous
situations in mathematical physics where it arises. For example, when the method of separation of variables is applied to the equation

\[ \text{div}(P \nabla v) + Qv = 0 \]

where \(P\) and \(Q\) possess some symmetry sufficient for separating variables very often one can arrive at the equation \((1)\), and it is really desirable to have a possibility to solve only one equation \((2)\) and to derive from its solution the solution of \((1)\). Moreover, in many important cases the solution of \((2)\) is known. For example, consider the conductivity equation

\[ \text{div}(P \nabla v) = 0 \]

and suppose, e.g., that \(P\) is a function of one Cartesian variable (for a recent work motivating this example see, e.g., [4]). Separation of variables leads to the equation

\[ \partial_x (P(x) \partial_x u(x)) = \omega^2 u(x) \]

and the solution of the corresponding equation \((2)\) is given, a particular nontrivial solution can be chosen as \(u_0 \equiv 1\). Thus, to have a method allowing us to transform \(u_0\) into \(u\) would mean a complete solution of the original problem.

There are dozens of works dedicated to the construction of zero-energy solutions of the Schrödinger equation (see, e.g., [2], [3]). With the aid of the results of the present paper these solutions can be used for obtaining solutions for all other values of \(\omega\). These are just some immediate applications of the main result of the present work which can be also applied in spectral and scattering theories for the Schrödinger equation as well as for studying the Riccati equation.

Here we give a simple and general solution to the problem of solving \((1)\) by a solution of \((2)\). In obtaining it we used some classical results from pseudoanalytic function theory mainly developed by L. Bers in fourties and fifties of the last century as well as some recent results of the author about the relationship between pseudoanalytic functions and solutions of the stationary two-dimensional Schrödinger equation. Although the main result of this paper was obtained with the aid of the machinery of pseudoanalytic function theory it can be easily understood, verified and analized even without those tools. The reader not interested in the method derived for obtaining it can start reading this paper from equations \((42)\), \((43)\).
2 Some definitions and results from pseudo-analytic function theory

2.1 The Vekua equation and pseudoanalytic functions

The main object of study in the pseudoanalytic function theory is the following equation

\[ W_z = aW + b\overline{W} \tag{3} \]

which is usually called the Vekua equation. Here \( a, b \) and \( W \) are complex valued functions of a complex variable \( z = x + iy \), \( W_z = \partial_z W = \frac{1}{2}(\partial_x + i\partial_y)W \) and \( \overline{W} \) is a complex conjugate of \( W \).

Solutions of (3) are known as pseudoanalytic (=generalized analytic) functions. Basic results on them can be found in two well known monographs [1] and [9]. In particular, as a part of pseudoanalytic function theory, L. Bers created a well developed theory of formal powers which are solutions of (3) and locally, near a centre \( z_0 \), they behave as usual powers \((z - z_0)^n\), \( n = 0, 1, 2, \ldots \). Under quite general conditions he (joint with S. Agmon) proved the expansion theorem which establishes that any solution of (3) in a domain of interest \( \Omega \) can be developed into an infinite series of formal powers and that this series converges normally (uniformly on any compact subset of \( \Omega \)). In posterior works (e.g., [8]) stronger results were obtained guaranteeing the completeness of the infinite system of formal powers in the space of pseudoanalytic functions in the sense of the \( C(\overline{\Omega}) \)-norm.

In the present work we do not need to consider the general form (3) of a Vekua equation. Instead, a very special case is sufficient for our purposes. In this case the situation with expansion and convergence theorems is even simpler. We give corresponding details in subsection 2.3. Here we start with some basic definitions and facts from [1].

**Definition 1** A pair of complex functions \( F \) and \( G \) possessing in \( \Omega \) partial derivatives with respect to the real variables \( x \) and \( y \) is said to be a generating pair if it satisfies the inequality \( \text{Im}(FG) > 0 \) in \( \Omega \). The following expressions are known as characteristic coefficients of the pair \((F,G)\)

\[ a_{(F,G)} = -\frac{FG_\overline{z} - F_\overline{z}G}{FG - \overline{FG}}, \quad b_{(F,G)} = \frac{FG_\overline{z} - F_\overline{z}G}{FG - \overline{FG}}, \]
\[ A_{(F,G)} = -\frac{FG_z - F_zG}{FG - FG}, \quad B_{(F,G)} = \frac{FG_z - F_zG}{FG - FG}, \]

where the subindex \( z \) or \( z \) means the application of \( \partial_z \) or \( \partial_{\overline{z}} \) respectively.

Every complex function \( W \) defined in a subdomain of \( \Omega \) admits the unique representation \( W = \phi F + \psi G \) where the functions \( \phi \) and \( \psi \) are real valued. Sometimes it is convenient to associate with the function \( W \) the function \( w = \phi + i\psi \). The correspondence between \( W \) and \( \omega \) is one-to-one.

For \( W \in C^1(\Omega) \) the \( (F,G) \)-derivative \( \dot{W} = \frac{d_{(F,G)}W}{dz} \) exists and has the form

\[ \dot{W} = \phi_z F + \psi_z G = W_z - A_{(F,G)} W - B_{(F,G)} W \]  \quad (4)

if and only if

\[ \phi_{\overline{z}} F + \psi_{\overline{z}} G = 0. \]  \quad (5)

This last equation can be rewritten in the form (3):

\[ W_{\overline{z}} = a_{(F,G)} W + b_{(F,G)} \overline{W}. \]  \quad (6)

Solutions of this equation are called \( (F,G) \)-pseudoanalytic functions. It is said that \( (F,G) \) is a generating pair corresponding to the Vekua equation (4). If \( W \) is \( (F,G) \)-pseudoanalytic, the associated function \( w \) is called \( (F,G) \)-pseudoanalytic of second kind.

**Remark 2** The functions \( F \) and \( G \) are \( (F,G) \)-pseudoanalytic, and \( \dot{F} \equiv \dot{G} \equiv 0. \)

**Definition 3** Let \( (F,G) \) and \( (F_1,G_1) \) be two generating pairs in \( \Omega \). \( (F_1,G_1) \) is called successor of \( (F,G) \) and \( (F,G) \) is called predecessor of \( (F_1,G_1) \) if

\[ a_{(F_1,G_1)} = a_{(F,G)} \quad \text{and} \quad b_{(F_1,G_1)} = -B_{(F,G)}. \]

The importance of this definition becomes obvious from the following statement.

**Theorem 4** Let \( W \) be an \( (F,G) \)-pseudoanalytic function and let \( (F_1,G_1) \) be a successor of \( (F,G) \). Then \( W \) is an \( (F_1,G_1) \)-pseudoanalytic function.
**Definition 5** Let \((F,G)\) be a generating pair. Its adjoint generating pair \((F,G)^* = (F^*,G^*)\) is defined by the formulas

\[
F^* = -\frac{2F}{FG - FG}, \quad G^* = \frac{2G}{FG - FG}.
\]

The \((F,G)\)-integral is defined as follows

\[
\int_{\Gamma} W d_{(F,G)} z = F(z_1) \text{Re} \int_{\Gamma} G^* W dz + G(z_1) \text{Re} \int_{\Gamma} F^* W dz
\]

where \(\Gamma\) is a rectifiable curve leading from \(z_0\) to \(z_1\).

If \(W = \phi F + \psi G\) is an \((F,G)\)-pseudoanalytic function where \(\phi\) and \(\psi\) are real valued functions then

\[
\int_{z_0}^{z} W d_{(F,G)} z = W(z) - \phi(z_0) F(z) - \psi(z_0) G(z), \quad (7)
\]

and as \(\dot{F} = \dot{G} = 0\), this integral is path-independent and represents the \((F,G)\)-antiderivative of \(W\).

### 2.2 Generating sequences and Taylor series in formal powers

**Definition 6** A sequence of generating pairs \(\{(F_m, G_m)\}, m = 0, \pm 1, \pm 2, \ldots\), is called a generating sequence if \((F_{m+1}, G_{m+1})\) is a successor of \((F_m, G_m)\). If \((F_0, G_0) = (F, G)\), we say that \((F, G)\) is embedded in \(\{(F_m, G_m)\}\).

**Theorem 7** Let \((F, G)\) be a generating pair in \(\Omega\). Let \(\Omega_1\) be a bounded domain, \(\overline{\Omega_1} \subset \Omega\). Then \((F, G)\) can be embedded in a generating sequence in \(\Omega_1\).

**Definition 8** A generating sequence \(\{(F_m, G_m)\}\) is said to have period \(\mu > 0\) if \((F_{m+\mu}, G_{m+\mu})\) is equivalent to \((F_m, G_m)\) that is their characteristic coefficients coincide.

Let \(W\) be an \((F,G)\)-pseudoanalytic function. Using a generating sequence in which \((F,G)\) is embedded we can define the higher derivatives of \(W\) by the recursion formula

\[
W^{[0]} = W, \quad W^{[m+1]} = \frac{d_{(F_m,G_m)} W^{[m]}}{dz}, \quad m = 1, 2, \ldots
\]
Definition 9 The formal power \( Z_m^{(0)}(a, z_0; z) \) with center at \( z_0 \in \Omega \), coefficient \( a \) and exponent 0 is defined as the linear combination of the generators \( F_m, G_m \) with real constant coefficients \( \lambda, \mu \) chosen so that \( \lambda F_m(z_0) + \mu G_m(z_0) = a \). The formal powers with exponents \( n = 1, 2, \ldots \) are defined by the recursion formula

\[
Z_m^{(n)}(a, z_0; z) = n \int_{z_0}^{z} Z_m^{(n-1)}(a, z_0; \zeta) d(F_m, G_m) \zeta.
\] (8)

This definition implies the following properties.

1. \( Z_m^{(n)}(a, z_0; z) \) is an \( (F_m, G_m) \)-pseudoanalytic function of \( z \).

2. If \( a' \) and \( a'' \) are real constants, then \( Z_m^{(n)}(a' + ia'', z_0; z) = a' Z_m^{(n)}(1, z_0; z) + a'' Z_m^{(n)}(i, z_0; z) \).

3. The formal powers satisfy the differential relations

\[
\frac{d(F_m, G_m) Z_m^{(n)}(a, z_0; z)}{dz} = n Z_m^{(n-1)}(a, z_0; z).
\]

4. The asymptotic formulas

\[
Z_m^{(n)}(a, z_0; z) \sim a(z - z_0)^n, \quad z \to z_0
\]

hold.

Assume now that

\[
W(z) = \sum_{n=0}^{\infty} Z_m^{(n)}(a_n, z_0; z)
\] (9)

where the absence of the subindex \( m \) means that all the formal powers correspond to the same generating pair \((F, G)\), and the series converges uniformly in some neighborhood of \( z_0 \). It can be shown [1] that the uniform limit of a series of pseudoanalytic functions is pseudoanalytic, and that a uniformly convergent series of \((F, G)\)-pseudoanalytic functions can be \((F, G)\)-differentiated term by term. Hence the function \( W \) in (9) is \((F, G)\)-pseudoanalytic and its \( r \)th derivative admits the expansion

\[
W^{[r]}(z) = \sum_{n=r}^{\infty} n(n-1) \cdots (n-r+1) Z_m^{(n-r)}(a_n, z_0; z).
\]
From this the Taylor formulas for the coefficients are obtained

\[ a_n = \frac{W^{[n]}(z_0)}{n!}. \]  

(10)

**Definition 10** Let \( W(z) \) be a given \((F,G)\)-pseudoanalytic function defined for small values of \(|z - z_0|\). The series

\[ \sum_{n=0}^{\infty} Z^{(n)}(a_n, z_0; z) \]  

(11)

with the coefficients given by (10) is called the Taylor series of \( W \) at \( z_0 \), formed with formal powers.

### 2.3 An important special case

In the present work in fact we will need the formal powers in the case when the generating pair has the form

\[ F(x, y) = \frac{\sigma(x)}{\tau(y)} \quad \text{and} \quad G(x, y) = \frac{i\tau(y)}{\sigma(x)} \]

where \( \sigma \) and \( \tau \) are real-valued functions of their corresponding variables. For simplicity we assume that \( z_0 = 0 \) and \( F(0) = 1 \). In this case (see [1]) the formal powers are constructed in an elegant manner as follows. First, denote

\[ X^{(0)}(x) = \tilde{X}^{(0)}(x) = Y^{(0)}(y) = \tilde{Y}^{(0)}(y) = 1 \]

and for \( n = 1, 2, ... \) denote

\[
X^{(n)}(x) = \begin{cases} 
  n \int_{0}^{x} X^{(n-1)}(\xi) \frac{d\xi}{\sigma(\xi)} & \text{for an odd } n \\
  n \int_{0}^{x} X^{(n-1)}(\xi) \sigma^2(\xi) d\xi & \text{for an even } n 
\end{cases}
\]

\[
\tilde{X}^{(n)}(x) = \begin{cases} 
  n \int_{0}^{x} \tilde{X}^{(n-1)}(\xi) \sigma^2(\xi) d\xi & \text{for an odd } n \\
  n \int_{0}^{x} \tilde{X}^{(n-1)}(\xi) \frac{d\xi}{\sigma^2(\xi)} & \text{for an even } n 
\end{cases}
\]
\[
Y^{(n)}(y) = \begin{cases} 
0 \int_{0}^{y} Y^{(n-1)}(\eta) \frac{d\eta}{\tau^2(\eta)} & \text{for an odd } n \\
0 \int_{0}^{y} Y^{(n-1)}(\eta) \tau^2(\eta) d\eta & \text{for an even } n
\end{cases}
\]

\[
\tilde{Y}^{(n)}(y) = \begin{cases} 
0 \int_{0}^{y} \tilde{Y}^{(n-1)}(\eta) \tau^2(\eta) d\eta & \text{for an odd } n \\
0 \int_{0}^{y} \tilde{Y}^{(n-1)}(\eta) \frac{d\eta}{\tau^2(\eta)} & \text{for an even } n
\end{cases}
\]

Then for \( a = a' + ia'' \) we have

\[
Z^{(n)}(a, 0, z) = \frac{\sigma(x)}{\tau(y)} \text{Re} \ast Z^{(n)}(a, 0, z) + \frac{i\tau(y)}{\sigma(x)} \text{Im} \ast Z^{(n)}(a, 0, z)
\]

where

\[
\ast Z^{(n)}(a, 0, z) = a' \sum_{j=0}^{n} \binom{n}{j} X^{(n-j)} i^j Y^{(j)} \tag{12}
\]

\[
+ ia'' \sum_{j=0}^{n} \binom{n}{j} \tilde{X}^{(n-j)} i^j \tilde{Y}^{(j)} \quad \text{for an odd } n
\]

and

\[
\ast Z'^{(n)}(a, 0, z) = a' \sum_{j=0}^{n} \binom{n}{j} \tilde{X}^{(n-j)} i^j Y^{(j)} \tag{13}
\]

\[
+ ia'' \sum_{j=0}^{n} \binom{n}{j} X^{(n-j)} i^j \tilde{Y}^{(j)} \quad \text{for an even } n.
\]

Consider a rectangular domain containing the origin as an internal point. If \( \sigma \) and \( \tau \) are continuously differentiable and bounded together with \( 1/\sigma \) and \( 1/\tau \) on their respective intervals any \((F, G)\)-pseudoanalytic function can be represented in the form of a Taylor series in formal powers \( \binom{n}{j} \) with \( z_0 = 0 \) and the series converges normally on the domain of interest.
2.4 The main Vekua equation

The equation of the form

\[ W \frac{z}{f} = f \frac{z}{f} W \tag{14} \]

where \( f \) is a nonvanishing continuously differentiable real-valued function will be called the main Vekua equation. As was shown in [5] this equation is related to the stationary Schrödinger equation much in the same way as the Cauchy-Riemann system to the Laplace equation. We formulate two results which will be used throughout the paper.

**Theorem 11** [5] Let \( W = W_1 + iW_2 \) be a solution of (14). Then the function \( W_1 \) is a solution of the stationary Schrödinger equation

\[ -\Delta W_1 + q_1 W_1 = 0 \quad \text{in } \Omega \tag{15} \]

with \( q_1 = \Delta f / f \), and \( W_2 \) is a solution of the associated stationary Schrödinger equation

\[ -\Delta W_2 + q_2 W_2 = 0 \quad \text{in } \Omega \tag{16} \]

where \( q_2 = 2(\nabla f)^2 / f^2 - q_1 \) and \( (\nabla f)^2 = f_x^2 + f_y^2 \).

**Notation 12** Consider the equation

\[ \partial_\tau \varphi = \Phi \tag{17} \]

in a whole complex plane or in a convex domain, where \( \Phi = \Phi_1 + i\Phi_2 \) is a given complex valued function such that its real part \( \Phi_1 \) and imaginary part \( \Phi_2 \) satisfy the equation

\[ \partial_y \Phi_1 - \partial_x \Phi_2 = 0, \tag{18} \]

then as is well known there exist real-valued solutions \( \varphi \) to equation (17) which can be reconstructed up to an arbitrary real constant \( c \) in the following way

\[ \varphi(x, y) = 2 \left( \int_{x_0}^x \Phi_1(\eta, y) d\eta + \int_{y_0}^y \Phi_2(x_0, \xi) d\xi \right) + c \tag{19} \]

where \((x_0, y_0)\) is an arbitrary fixed point in the domain of interest.

By \( \mathcal{A} \) we denote the integral operator in (19):

\[ \mathcal{A}[\Phi](x, y) = 2 \left( \int_{x_0}^x \Phi_1(\eta, y) d\eta + \int_{y_0}^y \Phi_2(x_0, \xi) d\xi \right) + c. \]
Note that formula (19) can be easily extended to any simply connected domain by considering the integral along an arbitrary rectifiable curve $\Gamma$ leading from $(x_0, y_0)$ to $(x, y)$

$$\varphi(x, y) = 2 \left( \int_{\Gamma} \Phi_1 dx + \Phi_2 dy \right) + c.$$ 

**Theorem 13** \[5\] Let $W_1$ be a real valued solution of (15) in a simply connected domain $\Omega$. Then the real valued function $W_2$, solution of (16) such that $W = W_1 + iW_2$ is a solution of (14), is constructed according to the formula

$$W_2 = f^{-1} \overline{A}(if^2 \partial_z(f^{-1}W_1)).$$ (20)

Given a solution $W_2$ of (16), the corresponding solution $W_1$ of (15) such that $W = W_1 + iW_2$ is a solution of (14), is constructed as follows

$$W_1 = -fA(if^{-2}\overline{\partial_z(fW_2)}).$$ (21)

**Remark 14** Observe that the pair of functions

$$F = f \quad \text{and} \quad G = \frac{i}{f}$$

is a generating pair for (14).

### 3 The main result

Consider the equation

$$(-\partial_x^2 + q(x))g(x) = 0.$$ (23)

We suppose that $q$ and $g$ are real-valued functions and that on some interval $I_x$ of the independent variable $x$ there exists a bounded nonvanishing solution $g_0 \in C^2(I_x)$ such that $1/g_0$ is also bounded. For simplicity we suppose that the interval $I_x$ includes the point $x = 0$ and that $g_0(0) = 1$. Our first goal is to solve the equation

$$(-\partial_x^2 + q(x) \pm \omega^2)u(x) = 0$$

for any real constant $\omega$. We start with the “+”-case.
3.1 The “+”-case

Consider the equation
\[-\partial_x^2 + q(x) + \omega^2 u(x) = 0. \tag{24}\]

Let us notice that for the equation
\[-\Delta + q(x) + \omega^2 U(x,y) = 0 \tag{25}\]
where \(\Delta = \partial_x^2 + \partial_y^2\) we can immediately propose a particular solution, e.g.,
\[f(x,y) = g_0(x)e^{\omega y}. \tag{26}\]

This function does not have zeros on any rectangular domain \(\Omega = I_x \times I_y\) where \(I_y\) is an arbitrary finite interval of the variable \(y\). For simplicity we assume that the origin \(z = 0\) is an internal point of the domain \(\Omega\). According to theorem [13] any solution of (25) is a real part of a solution of the main Vekua equation (14) where \(f\) is defined by (26). Moreover, a generating pair for this Vekua equation has the form
\[F(x,y) = f(x,y) = g_0(x)e^{\omega y} \quad \text{and} \quad G(x,y) = i/f(x,y) = ig_0^{-1}(x)e^{-\omega y}.\]

Now using the results from subsection 2.3 we can construct the formal powers corresponding to equation (14) and to this generating pair. We have that
\[Z^{(n)}(a,0,z) = g_0(x)e^{\omega y} \text{Re} sZ^{(n)}(a,0,z) + ig_0^{-1}(x)e^{-\omega y} \text{Im} sZ^{(n)}(a,0,z)\]
where \(sZ^{(n)}(a,0,z)\) are constructed according to formulas (12) and (13) where \(\sigma(x) = g_0(x)\) and \(\tau(y) = e^{-\omega y}\).

We know that any solution \(W\) of (14) in \(\Omega\) can be represented in the form
\[W(z) = \sum_{n=0}^{\infty} Z^{(n)}(a_n,0,z)\]
and hence any solution \(U\) of (25) has the form
\[U(x,y) = \sum_{n=0}^{\infty} \text{Re} Z^{(n)}(a_n,0,z) \tag{27}\]
\[= g_0(x)e^{\omega y}\sum_{n=0}^{\infty} \left(a'_n \text{Re} sZ^{(n)}(1,0,z) + a''_n \text{Re} sZ^{(n)}(i,0,z) \right).\]
Observe that solutions of (24) are also solutions of (25). Consequently, for any solution \( u \) of (24) there exists such set of real numbers \( \{a'_n, a''_n\}_{n=0}^{\infty} \) that

\[
u(x) = g_0(x) e^{\omega y} \sum_{n=0}^{\infty} \left( a'_n \text{Re} \ast Z^{(n)}(1,0,z) + a''_n \text{Re} \ast Z^{(n)}(i,0,z) \right).
\]

(28)

In other words there exist such sets of coefficients \( \{a'_n, a''_n\}_{n=0}^{\infty} \) that

\[
\partial_y \left( e^{\omega y} \sum_{n=0}^{\infty} \left( a'_n \text{Re} \ast Z^{(n)}(1,0,z) + a''_n \text{Re} \ast Z^{(n)}(i,0,z) \right) \right) \equiv 0.
\]

(29)

Obviously, if this condition is fulfilled, the resulting function (28) is a solution of (24).

Let us analyse equation (29). First of all we have that for an odd \( n \),

\[
\ast Z^{(n)}(1,0,z) = \sum_{j=0}^{n} \binom{n}{j} X^{(n-j)} i^j Y^{(j)},
\]

\[
\ast Z^{(n)}(i,0,z) = i \sum_{j=0}^{n} \binom{n}{j} \tilde{X}^{(n-j)} i^j \tilde{Y}^{(j)}
\]

and for an even \( n \),

\[
\ast Z^{(n)}(1,0,z) = \sum_{j=0}^{n} \binom{n}{j} \tilde{X}^{(n-j)} i^j Y^{(j)},
\]

\[
\ast Z^{(n)}(i,0,z) = i \sum_{j=0}^{n} \binom{n}{j} X^{(n-j)} i^j \tilde{Y}^{(j)}.
\]

Thus, we obtain for an odd \( n \),

\[
\text{Re} \ast Z^{(n)}(1,0,z) = \sum_{\text{even } j=0}^{n} \binom{n}{j} X^{(n-j)} i^j Y^{(j)},
\]

\[
\text{Re} \ast Z^{(n)}(i,0,z) = \sum_{\text{odd } j=1}^{n} \binom{n}{j} \tilde{X}^{(n-j)} i^j \tilde{Y}^{(j)}
\]
and for an even $n$,

$$\text{Re} \ast Z^{(n)}(1, 0, z) = \sum_{\text{even } j=0}^{n} \binom{n}{j} \tilde{X}^{(n-j)} j \bar{Y}^{(j)},$$

$$\text{Re} \ast Z^{(n)}(i, 0, z) = \sum_{\text{odd } j=1}^{n} \binom{n}{j} X^{(n-j)} j \bar{Y}^{(j)}.$$

It is somewhat more convenient for what follows to rewrite these formulas in the following equivalent form

\[
\begin{cases}
\text{Re} \ast Z^{(n)}(1, 0, z) = \sum_{\text{odd } k=1}^{n} \binom{n}{k} X^{(k)} j^{n-k} \bar{Y}^{(n-k)} & \text{for an odd } n \\
\text{Re} \ast Z^{(n)}(i, 0, z) = \sum_{\text{even } k=0}^{n} \binom{n}{k} \tilde{X}^{(k)} j^{n-k+1} \bar{Y}^{(n-k)} & \text{for an even } n
\end{cases}
\]

and

\[
\begin{cases}
\text{Re} \ast Z^{(n)}(1, 0, z) = \sum_{\text{even } k=0}^{n} \binom{n}{k} \tilde{X}^{(k)} j^{n-k} \bar{Y}^{(n-k)} & \text{for an even } n \\
\text{Re} \ast Z^{(n)}(i, 0, z) = \sum_{\text{odd } k=1}^{n} \binom{n}{k} X^{(k)} j^{n-k+1} \bar{Y}^{(n-k)} & \text{for an odd } n
\end{cases}
\]

We remind that

\[
Y^{(n)}(y) = \begin{cases}
  n \int_{0}^{y} Y^{(n-1)}(\eta) e^{2\omega \eta} d\eta & \text{for an odd } n \\
  n \int_{0}^{y} Y^{(n-1)}(\eta) e^{-2\omega \eta} d\eta & \text{for an even } n
\end{cases}
\]

\[
\tilde{Y}^{(n)}(y) = \begin{cases}
  n \int_{0}^{y} \tilde{Y}^{(n-1)}(\eta) e^{-2\omega \eta} d\eta & \text{for an odd } n \\
  n \int_{0}^{y} \tilde{Y}^{(n-1)}(\eta) e^{2\omega \eta} d\eta & \text{for an even } n
\end{cases}
\]
Thus, from (30) and (31) we have

\[
\begin{cases}
\partial_y \text{Re}_s Z^{(n)}(1, 0, z) = e^{-2\omega y} \sum_{\text{odd } k=1}^{n} (n - k) \binom{n}{k} X^{(k)} i^{n-k} Y^{(n-k-1)} \\
\partial_y \text{Re}_s Z^{(n)}(i, 0, z) = e^{-2\omega y} \sum_{\text{even } k=0}^{n} (n - k) \binom{n}{k} \bar{X}^{(k)} i^{n-k+1} \bar{Y}^{(n-k-1)}
\end{cases}
\] for an odd \(n\)

and

\[
\begin{cases}
\partial_y \text{Re}_s Z^{(n)}(1, 0, z) = e^{-2\omega y} \sum_{\text{even } k=0}^{n} (n - k) \binom{n}{k} \bar{X}^{(k)} i^{n-k} Y^{(n-k-1)} \\
\partial_y \text{Re}_s Z^{(n)}(i, 0, z) = e^{-2\omega y} \sum_{\text{odd } k=1}^{n} (n - k) \binom{n}{k} X^{(k)} i^{n-k+1} \bar{Y}^{(n-k-1)}
\end{cases}
\] for an even \(n > 0\)

(32)

Now returning to equation (29) we observe that it is equivalent to the equation

\[
\sum_{n=0}^{\infty} (a'_n (\omega \text{Re}_s Z^{(n)}(1, 0, z) + \partial_y \text{Re}_s Z^{(n)}(1, 0, z)) + a''_n (\omega \text{Re}_s Z^{(n)}(i, 0, z) + \partial_y \text{Re}_s Z^{(n)}(i, 0, z))) = 0
\]

from which using (30), (31) and (32), (33) we obtain that (29) can be written as follows

\[
a'_0 \omega + \sum_{n=2}^{\infty} (a'_n \sum_{\text{even } k=0}^{n} i^{n-k} \binom{n}{k} \bar{X}^{(k)} (\omega Y^{(n-k)} + (n - k) Y^{(n-k-1)} e^{-2\omega y}) \\
+ a''_n \sum_{\text{odd } k=1}^{n} i^{n-k+1} \binom{n}{k} X^{(k)} (\omega \bar{Y}^{(n-k)} + (n - k) \bar{Y}^{(n-k-1)} e^{-2\omega y})) + \sum_{n=1}^{\infty} (a'_n \sum_{\text{odd } k=1}^{n} i^{n-k} \binom{n}{k} X^{(k)} (\omega Y^{(n-k)} + (n - k) Y^{(n-k-1)} e^{-2\omega y}) \\
+ a''_n \sum_{\text{even } k=0}^{n} i^{n-k+1} \binom{n}{k} \bar{X}^{(k)} (\omega \bar{Y}^{(n-k)} + (n - k) \bar{Y}^{(n-k-1)} e^{-2\omega y})) = 0.
\] (34)

In order that this equality hold identically the expressions corresponding to different \(X^{(n)}\) and \(\bar{X}^{(n)}\) for all \(n\) should vanish identically. Combining all
terms multiplied by $\tilde{X}^{(0)}$ we obtain the equation

$$a'_0 \omega + \sum_{\text{even } n=2}^{\infty} a'_n i^n (\omega Y^{(n)} + nY^{(n-1)} e^{-2\omega y})$$

$$+ \sum_{\text{odd } n=1}^{\infty} a''_n i^{n+1} (\omega \tilde{Y}^{(n)} + n\tilde{Y}^{(n-1)} e^{-2\omega y}) = 0. \tag{35}$$

Gathering all terms multiplied by $X^{(1)}$ we obtain the second equation

$$\sum_{\text{even } n=2}^{\infty} a''_n i^n n (\omega \tilde{Y}^{(n-1)} + (n-1)\tilde{Y}^{(n-2)} e^{-2\omega y})$$

$$+ \sum_{\text{odd } n=1}^{\infty} a'_n i^{n-1} n (\omega Y^{(n-1)} + (n-1)Y^{(n-2)} e^{-2\omega y}) = 0$$

which can be rewritten as follows

$$a'_1 \omega + \sum_{\text{even } n=2}^{\infty} a'_n i^n (n+1)(\omega Y^{(n)} + nY^{(n-1)} e^{-2\omega y})$$

$$+ \sum_{\text{odd } n=1}^{\infty} a''_n i^{n+1} (n+1)(\omega \tilde{Y}^{(n)} + n\tilde{Y}^{(n-1)} e^{-2\omega y}) = 0. \tag{36}$$

Gathering all terms multiplied by $\tilde{X}^{(2)}, X^{(3)}, \ldots$ we obtain an infinite system of equations which fortunately we do not need to solve. Here we are reasoning along the following lines. First of all we observe that if such sets of coefficients $\{a'_n, a''_n\}_{n=0}^{\infty}$ exist that all the equations derived from (34) are satisfied, they do not depend on functions $X^{(n)}$ and $\tilde{X}^{(n)}$ but only on $Y^{(n)}$ and $\tilde{Y}^{(n)}$. Thus, they can be constructed independently of the concrete form of $g_0$ and hence of the potential $q$. Second, we know that such sets of coefficients exist. This is due to our earlier observation that solutions of (24) are also solutions of (25) and hence they can be written in the form (27).

These two arguments lead to the following surprising solution. We can take any $q$, for example, $q \equiv 0$ and any pair of independent solutions of the resulting Schrödinger equation (24) and to obtain their corresponding sets of coefficients. These two sets will be universal in the sense that the general solution of (24) for any other $q$ will be constructed with the aid of this pair
of sets of coefficients just changing the generating function $g_0$ and obtaining a corresponding system of functions $X^{(n)}$ and $\tilde{X}^{(n)}$. On the first glance this conclusion can appear against the intuition, nevertheless its more detailed analysis as well as the final result convince that it is really natural. Thus, in the next subsection we construct such pair of sets of Taylor coefficients (in formal powers).

### 3.2 Two sets of Taylor coefficients

Here we consider the case $q \equiv 0$. Then the Schrödinger equation (24) becomes

$$(−\partial_x^2 + \omega^2)u(x) = 0. \tag{37}$$

Note that the corresponding equation (23) has the form $\partial_x^2 g(x) = 0$ and possesses a suitable particular solution satisfying all the requirements (see the beginning of section 3) $g_0 \equiv 1$. Then $f = e^{\omega y}$ and the main Vekua equation in this case has the form

$$W_z = \frac{i\omega}{2} \bar{W}. \tag{38}$$

Let us take two independent solutions of (37) $u^+(x) = e^{\omega x}$ and $u^-(x) = e^{-\omega x}$. First, we obtain the set of coefficients $\{a_n', a_n''\}_{n=0}^\infty$ for the function $u^+$. The first step consists in constructing the corresponding conjugate metaharmonic function $v^+$ (see theorem 13)

$$v^+ = e^{-\omega y}A(ie^{2\omega y} \partial_x e^{\omega(x-y)}) = e^{-\omega y}\frac{\omega}{2}(1+i)e^{\omega(x+y)}.$$

We have

$$A\frac{\omega}{2}(1+i)e^{\omega(x+y)} = e^{\omega(x+y)} + c.$$

We choose $c = 0$, then $v^+ = e^{\omega x}$. Thus, one of the solutions of the main Vekua equation (38) such that $u^+ = e^{\omega x}$ is its real part has the form

$$W^+ = (1+i)e^{\omega x}. \tag{39}$$

Now, in order to construct its corresponding Taylor coefficients (in formal powers) we notice that $A_{(F,G)} = 0$ and $B_{(F,G)} = -i\omega/2$ (here $F = e^{\omega y}$ and $G = ie^{-\omega y}$). Thus, the operation of the $(F,G)$-derivative has the form

$$\dot{W} = W_z + \frac{i\omega}{2} \bar{W}.$$
For the function (39) we have

\[ W^+ = \omega(1 + i)e^{\omega x}, \quad W^+ = \omega^2(1 + i)e^{\omega x}, \ldots \]

and it is easy to see that the \( n \)-th \((F, G)\)-derivative of \( W^+ \) has the form

\[ W^{+[n]} = \omega^n W^+. \]

We obtain that the Taylor coefficients (in formal powers) of the function (39) at the origin have the following simple form

\[ a^+_n = \frac{\omega^n}{n!}(1 + i). \] \quad (40)

In a similar way we study the case of the function \( u^- \). The corresponding pseudoanalytic function \( W^- \) has the form \( W^- = (1 - i)e^{-\omega x} \), and the corresponding Taylor coefficients at the origin are as follows

\[ a^-_n = \frac{(-\omega)^n}{n!}(1 - i). \] \quad (41)

Let us notice that from the fulfillment of (35) with the coefficients of the form (40) or (41) there follows the fulfillment of (36) and of all subsequent equations corresponding to \( \tilde{X}^{(2)}, X^{(3)}, \) etc. This is because of the fact that

\[ a^+_{n+1} = \frac{\pm \omega}{n+1} a^+_n. \]

### 3.3 General solution of (24)

Now with the aid of the sets of coefficients (40) and (41) we proceed in obtaining the general solution of (24) with any potential \( q \) for which a solution \( g_0 \) of (23) satisfying the nonzero and boundedness requirements exists. From (28) we have that the general solution of (24) has the form

\[ u = c_1 u_1 + c_2 u_2 \]

where \( c_1 \) and \( c_2 \) are arbitrary real constants and \( u_1, u_2 \) are defined as follows

\[ u_1(x) = g_0(x)e^{\omega y}\sum_{n=0}^{\infty} \frac{\omega^n}{n!} \left( \text{Re} \ *Z^{(n)}(1,0,z) + \text{Re} \ *Z^{(n)}(i,0,z) \right) \]
and

\[ u_2(x) = g_0(x)e^{\omega y}\sum_{n=0}^{\infty} \frac{(-\omega)^n}{n!} \left( \text{Re} Z^{(n)}(1, 0, z) - \text{Re} Z^{(n)}(i, 0, z) \right) \]

which according to (30) and (31) can be written in the following form

\[
\begin{align*}
  u_1(x) &= g_0(x)e^{\omega y}\left( \sum_{n=0}^{\infty} \frac{\omega^n}{n!} \left( \sum_{\text{even } k=0}^{n} i^{n-k} \binom{n}{k} \tilde{X}^{(k)} Y^{(n-k)} \right) \\
  &\quad + \sum_{\text{odd } k=1}^{n} i^{n-k+1} \binom{n}{k} X^{(k)} \tilde{Y}^{(n-k)} \right) \\
  &\quad + \sum_{\text{odd } n=1}^{\infty} \frac{\omega^n}{n!} \left( \sum_{\text{odd } k=1}^{n} i^{n-k} \binom{n}{k} X^{(k)} Y^{(n-k)} \right) \\
  &\quad + \sum_{\text{even } k=0}^{n} i^{n-k+1} \binom{n}{k} X^{(k)} \tilde{Y}^{(n-k)} \right) \\
  &\quad - \sum_{\text{odd } k=1}^{n} i^{n-k+1} \binom{n}{k} \tilde{X}^{(k)} \tilde{Y}^{(n-k)} \right) \\
  &\quad - \sum_{\text{odd } n=1}^{\infty} \frac{\omega^n}{n!} \left( \sum_{\text{odd } k=1}^{n} i^{n-k} \binom{n}{k} X^{(k)} Y^{(n-k)} \right) \\
  &\quad - \sum_{\text{even } k=0}^{n} i^{n-k+1} \binom{n}{k} \tilde{X}^{(k)} \tilde{Y}^{(n-k)} \right) \right) \\
\end{align*}
\]

and

\[
\begin{align*}
  u_2(x) &= g_0(x)e^{\omega y}\left( \sum_{n=0}^{\infty} \frac{\omega^n}{n!} \left( \sum_{\text{even } k=0}^{n} i^{n-k} \binom{n}{k} \tilde{X}^{(k)} Y^{(n-k)} \right) \\
  &\quad - \sum_{\text{odd } k=1}^{n} i^{n-k+1} \binom{n}{k} X^{(k)} \tilde{Y}^{(n-k)} \right) \\
  &\quad + \sum_{\text{odd } n=1}^{\infty} \frac{\omega^n}{n!} \left( \sum_{\text{odd } k=1}^{n} i^{n-k} \binom{n}{k} X^{(k)} Y^{(n-k)} \right) \\
  &\quad - \sum_{\text{even } k=0}^{n} i^{n-k+1} \binom{n}{k} \tilde{X}^{(k)} \tilde{Y}^{(n-k)} \right) \right) \\
\end{align*}
\]

As we know that both expressions are independent of \(y\) in order to simplify them we can substitute any value of \(y\). Of course, the easiest way is to substitute \(y = 0\) because by definition all \(Y^{(n)}(0)\) and \(\tilde{Y}^{(n)}(0)\) for \(n \geq 1\) are equal to zero, and \(Y^{(0)}(0) = \tilde{Y}^{(0)}(0) = 1\). Thus, finally we obtain

\[
u_1(x) = g_0(x)\left( \sum_{\text{even } n=0}^{\infty} \frac{\omega^n}{n!} \tilde{X}^{(n)} + \sum_{\text{odd } n=1}^{\infty} \frac{\omega^n}{n!} X^{(n)} \right)
\]
and
\[ u_2(x) = g_0(x) \left( \sum_{\text{even } n=0}^{\infty} \frac{\omega^n}{n!} \tilde{X}^{(n)} - \sum_{\text{odd } n=1}^{\infty} \frac{\omega^n}{n!} X^{(n)} \right). \] (43)

where
\[ \tilde{X}^{(0)}(0) \equiv 1, \quad X^{(0)}(0) \equiv 1, \] (44)

\[ \tilde{X}^{(n)}(x) = \begin{cases} \int_{0}^{x} \tilde{X}^{(n-1)}(\xi) g_0^2(\xi) d\xi & \text{for an odd } n \\ \int_{0}^{x} \tilde{X}^{(n-1)}(\xi) g_0^{-2}(\xi) d\xi & \text{for an even } n \end{cases} \] (45)

\[ X^{(n)}(x) = \begin{cases} \int_{0}^{x} X^{(n-1)}(\xi) g_0^{-2}(\xi) d\xi & \text{for an odd } n \\ \int_{0}^{x} X^{(n-1)}(\xi) g_0^2(\xi) d\xi & \text{for an even } n \end{cases} \] (46)

In the next subsection we validate this result by a direct substitution into equation (24).

### 3.4 Validating the result

In order to substitute (42) and (43) or equivalently
\[ v_1(x) = g_0(x) \sum_{\text{even } n=0}^{\infty} \frac{\omega^n}{n!} \tilde{X}^{(n)} \]
and
\[ v_2(x) = g_0(x) \sum_{\text{odd } n=1}^{\infty} \frac{\omega^n}{n!} X^{(n)} \]
into equation (24) we first make some helpful observations.

It is well known that a nonvanishing solution \( g_0 \) of (23) allows us to factorize the Schrödinger operator as follows
\[ \partial_x^2 - q(x) = \left( \partial_x + \frac{g_0'}{g_0} \right) \left( \partial_x - \frac{g_0'}{g_0} \right). \] (47)
The first order operators in their turn can be factorized as well, so we obtain
\[ \partial_x^2 - q = g_0^{-1} \partial_x g_0^2 \partial_x g_0^{-1}. \]

Now let us consider \( v_1 \). By definition, for an even \( n \) we have
\[ \tilde{X}^{(n)}(x) = n \int_0^x \tilde{X}^{(n-1)}(\xi) \frac{d\xi}{g_0^2(\xi)}. \]

Thus, application of the operator \( \partial_x^2 - q \) to \( g_0 \tilde{X}^{(n)} \) for an even \( n \) and \( n \geq 2 \) (for \( n = 0 \) the result is zero) gives us
\[
(\partial_x^2 - q) \left( g_0 \tilde{X}^{(n)} \right) = g_0^{-1} \partial_x g_0^2 \partial_x \tilde{X}^{(n)} = n g_0^{-1} \partial_x \tilde{X}^{(n-1)}
= (n - 1) n g_0 \tilde{X}^{(n-2)}.
\]

Then
\[
(\partial_x^2 - q) v_1 = g_0 \sum_{\text{even } n=2}^{\infty} \frac{\omega^n}{(n-2)!} \tilde{X}^{(n-2)}
= \omega^2 g_0 \sum_{\text{even } n=0}^{\infty} \frac{\omega^n}{n!} \tilde{X}^{(n)} = \omega^2 v_1.
\]

In a similar way one can verify that \( v_2 \) is a solution of (24) as well. Note that according to the general result formulated in subsection 2.3 both series in \( v_1 \) and \( v_2 \) are uniformly convergent on the interval \( I_\tilde{x} \). This fact can be quite easily verified as well by estimating the integrals in \( X^{(n)} \) and \( \tilde{X}^{(n)} \) by the supremum of the functions \( g_0^2 \) and \( g_0^{-2} \) multiplied by successive antiderivatives of \( x \).

### 3.5 The “-” case

Consider the equation
\[ (-\partial_x^2 + q(x) - \omega^2) u(x) = 0 \quad (48) \]
and the corresponding two-dimensional equation
\[ (-\Delta + q(x) - \omega^2) U(x, y) = 0. \quad (49) \]
Its particular solution can be chosen as

\[ f(x, y) = g_0(x) \cos \omega y \]

which is different from zero on the rectangular domain \( \Omega = I_x \times (-\frac{\pi}{2\omega}, \frac{\pi}{2\omega}) \).

In order to obtain the general solution of (48) in fact we should only obtain two sets of Taylor coefficients as in subsection 3.2. For this, once more we take \( q \equiv 0 \) and consider two linearly independent solutions of the equation

\[ (\partial_x^2 + \omega^2)u(x) = 0, \quad u^+(x) = \cos \omega x \text{ and } u^-(x) = \sin \omega x. \]

The next step is to construct \( v^+ \) and \( v^- \). We have

\[ v^+ = \frac{1}{\cos \omega y} \mathcal{A} \left( i \cos^2 \omega y \partial_x \left( \frac{\cos \omega x}{\cos \omega y} \right) \right) = -\sin \omega x \tan \omega y \]

(we have fixed the arbitrary constant as zero). Thus,

\[ W^+ = \cos \omega x - i \sin \omega x \tan \omega y. \]

In a similar way we obtain

\[ W^- = \sin \omega x + i \cos \omega x \tan \omega y. \]

Noting that the definition of the \((F,G)\)-derivative in this case has the form

\[ W = W_x - \frac{i\omega}{2} \tan \omega y \overline{W} \]

we obtain the following relations \( \dot{W}^+ = -\omega W^- \) and \( \dot{W}^- = \omega W^+ \) and hence the following formulas for the corresponding Taylor coefficients in formal powers in the origin

\[ a_n^+ = \frac{(i\omega)^n}{n!} \text{ for an even } n \quad \text{and} \quad a_n^+ = 0 \text{ for an odd } n, \]

\[ a_n^- = 0 \text{ for an even } n \quad \text{and} \quad a_n^- = -\frac{i(i\omega)^n}{n!} \text{ for an odd } n. \]

Thus we arrive at the following general solution of equation (48) for any potential \( q \) admitting a particular solution \( g_0 \) with the described above properties

\[ u = c_1 u_1 + c_2 u_2 \]
with
\[ u_1(x) = g_0(x) \sum_{\text{even } n=0}^{\infty} \frac{(i\omega)^n}{n!} \tilde{X}^{(n)} \]
and
\[ u_2(x) = g_0(x) \sum_{\text{odd } n=1}^{\infty} \frac{i(i\omega)^n}{n!} X^{(n)} \]
where \( X^{(n)} \) and \( \tilde{X}^{(n)} \) are defined by (44)-(46).

### 3.6 Complex potential

It is clear that the results obtained in the preceding subsections remain valid in the case of a complex valued potential \( q \) and an arbitrary complex number \( \omega \). Consider the equation
\[ (-\partial_x^2 + q(x) + \omega^2)u(x) = 0 \] (50)

where \( q \) and \( u \) are complex valued and \( \omega \) is any complex number. We assume that \( g_0 \) is a nonvanishing solution of the equation \((-\partial_x^2 + q(x))g_0 = 0\) satisfying the boundedness requirements. Then the general solution of (50) has the form
\[ u = c_1 u_1 + c_2 u_2 \] (51)

where \( c_1 \) and \( c_2 \) are arbitrary complex constants and \( u_1, u_2 \) are defined as follows
\[ u_1 = g_0 \sum_{\text{even } n=0}^{\infty} \frac{\omega^n}{n!} \tilde{X}^{(n)} \] (52)

and
\[ u_2 = g_0 \sum_{\text{odd } n=1}^{\infty} \frac{\omega^n}{n!} X^{(n)} \] (53)

where as before \( X^{(n)} \) and \( \tilde{X}^{(n)} \) are defined by (44)-(46).

### 3.7 Equation \((pu')' + qu = \omega^2 u\)

Having obtained the general solution of (50), it is easy to obtain the general solution of the more general equation
\[ \partial_x(p\partial_x u) + qu = \omega^2 u \] (54)

22
where $p \in C^2(I_x)$ is a nonvanishing complex valued function, $q$ and $u$ satisfy conditions from the preceding subsection. We assume that

$$\partial_x (p \partial_x g_0) + qg_0 = 0 \quad (55)$$

and observe that the following factorization holds

$$(\partial_x p \partial_x + q)u = p^{1/2} \left( \partial_x + \frac{g'}{g} \right) \left( \partial_x - \frac{g'}{g} \right) (p^{1/2}u) = g_0^{-1} \partial_x \left( g^2 \partial_x (g_0^{-1}u) \right)$$

where $g = p^{1/2}g_0$ and by analogy with (51)-(53) we obtain the following solution of (54)

$$u = c_1 u_1 + c_2 u_2 \quad (56)$$

where

$$u_1 = g_0 \sum_{\text{even } n=0}^{\infty} \frac{\omega^n}{n!} \tilde{X}^{(n)} \quad (57)$$

and

$$u_2 = g_0 \sum_{\text{odd } n=1}^{\infty} \frac{\omega^n}{n!} X^{(n)} \quad (58)$$

where the definition of $X^{(n)}$ and $\tilde{X}^{(n)}$ is slightly modified:

$$\tilde{X}^{(0)} \equiv 1, \quad X^{(0)} \equiv 1, \quad (59)$$

$$\tilde{X}^{(n)}(x) = \begin{cases} 
  n \int_0^x \tilde{X}^{(n-1)}(\xi) g_0^2(\xi) d\xi & \text{for odd } n \\
  n \int_0^x \tilde{X}^{(n-1)}(\xi) g^{-2}(\xi) d\xi & \text{for even } n 
\end{cases} \quad (60)$$

$$X^{(n)}(x) = \begin{cases} 
  n \int_0^x X^{(n-1)}(\xi) g^{-2}(\xi) d\xi & \text{for odd } n \\
  n \int_0^x X^{(n-1)}(\xi) g_0^2(\xi) d\xi & \text{for even } n 
\end{cases} \quad (61)$$
Let us verify that \( u_1 \) is indeed a solution of (54). We have

\[
(\partial_x p \partial_x + q)u_1 = g_0^{-1} \partial_x \left( g^2 \partial_x \sum_{\text{even } n=0}^{\infty} \frac{\omega^n}{n!} \tilde{X}^{(n)} \right)
\]

\[
= g_0^{-1} \sum_{\text{even } n=2}^{\infty} \frac{\omega^n}{(n-1)!} \partial_x \tilde{X}^{(n-1)}
\]

\[
= \omega^2 g_0 \sum_{\text{even } n=2}^{\infty} \frac{\omega^{n-2}}{(n-2)!} \tilde{X}^{(n-2)} = \omega^2 u_1.
\]

In a similar way the solution \( u_2 \) can be verified as well.

4 A remark on spectral problems

First, let us notice that at least in the case of real-valued coefficients \( p \) and \( q \) under the conditions that \( p, p' \) and \( q \) are continuous an appropriate nonvanishing particular solution \( g_0 \) of (55) always exists. In this case the equation \((pg')' + qg = 0\) possesses two linearly independent solutions \( g_1 \) and \( g_2 \) whose zeros do not coincide. Then \( g_0 \) can be chosen as follows \( g_0 = g_1 + ig_2 \). Thus, the proposed solution (56)-(58) can be obtained in a quite general situation and as we show in this section the corresponding spectral problems reduce to the problem of finding zeros of related analytic functions.

Sometimes it is slightly more convenient to consider the function (58) divided by \( \omega \), that is

\[
u_2 = g_0 \sum_{\text{odd } n=1}^{\infty} \frac{\omega^{n-1}}{n!} X^{(n)}.
\]

Then for the solutions of (54) \( u_1 \) and \( u_2 \) defined by (57) and (62) we obtain the following equalities

\[
u_1(0) = g_0(0), \quad u_1'(0) = g_0'(0),
\]

\[
u_2(0) = 0, \quad u_2'(0) = \frac{1}{g_0(0)p(0)}.
\]

Now consider a spectral problem for (54) on the interval \( I_x = (0, 1) \). For example,

\[
u(0) = 0 \quad \text{and} \quad u(1) = 0.
\]
Due to the first boundary condition the constant $c_1$ in (56) should be chosen as zero. Then the spectral problem reduces to finding such values of $\omega$ that

$$u_2(1) = g_0(1) \sum_{\text{odd } n=1}^{\infty} \frac{\omega^{n-1}}{n!} X^{(n)}(1)$$

vanish. In other words, this spectral problem reduces to the calculation of zeros of the complex analytic function

$$\kappa(\omega) = \sum_{m=0}^{\infty} a_m \omega^m$$

where $a_m$ are defined as follows

$$a_m = \begin{cases} 
0 & \text{for an odd } m \\
\frac{g_0(1)X^{(m+1)}(1)}{(m+1)!} & \text{for an even } m.
\end{cases}$$

Note that in some cases the function $\kappa$ will be even entire. For example, when the considered Sturm-Liouville problem is regular it is well known (see, e.g., [6]) that there exists an unboundedly increasing sequence of its eigenvalues which means that there exists an unboundedly increasing sequence of zeros of the function $\kappa$.

Let $\alpha$ and $\beta$ be arbitrary real numbers. Consider the following more general boundary conditions

$$u(0) \cos \alpha + u'(0) \sin \alpha = 0 \quad (66)$$

$$u(a) \cos \beta + u'(a) \sin \beta = 0 \quad (67)$$

together with equation (54). Taking the solutions $u_1$ and $u_2$ defined by (57) and (62) and using (63), (64) we obtain from (66) the following equation for $c_1$ and $c_2$

$$c_1 (g_0(0) \cos \alpha + g_0'(0) \sin \alpha) + c_2 \frac{\sin \alpha}{g_0(0)p(0)} = 0$$

which gives the relation

$$c_2 = \gamma c_1 \quad \text{for } \alpha \neq \pi n$$

where $\gamma = -g_0(0)p(0)(g_0(0) \cot \alpha + g_0'(0))$ and

$$c_1 = 0 \quad \text{for } \alpha = \pi n.$$
In this last case we arrive at a similar result as in the example considered above, thus let us consider the case $\alpha \neq \pi n$. From the definition of $u_1$ and $u_2$ we have

$$u'_1 = \frac{g'_0}{g_0} u_1 + \frac{1}{g_0 p} \sum_{\text{even } n = 2}^{\infty} \frac{\omega^n}{(n-1)!} \tilde{X}^{(n-1)}$$

and

$$u'_2 = \frac{g'_0}{g_0} u_2 + \frac{1}{g_0 p} \sum_{\text{even } n = 0}^{\infty} \frac{\omega^n}{n!} X^{(n)}.$$ 

Then the boundary condition (67) implies the following equation

$$(g_0(a) \cos \beta + g'_0(a) \sin \beta) \left( \sum_{\text{even } n = 0}^{\infty} \frac{\omega^n}{n!} \tilde{X}^{(n)}(a) + \gamma \sum_{\text{odd } n = 1}^{\infty} \frac{\omega^{n-1}}{n!} X^{(n)}(a) \right)$$

$$+ \frac{\sin \beta}{g_0(a) p(a)} \left( \sum_{\text{even } n = 2}^{\infty} \frac{\omega^n}{(n-1)!} \tilde{X}^{(n-1)}(a) + \gamma \sum_{\text{even } n = 0}^{\infty} \frac{\omega^n}{n!} X^{(n)}(a) \right) = 0.$$

Thus the spectral problem (54), (66), (67) reduces to the problem of finding zeros of the analytic function

$$\kappa(\omega) = \sum_{m=0}^{\infty} a_m \omega^m$$

where $a_m$ are defined as follows

$$a_0 = (g_0(a) \cos \beta + g'_0(a) \sin \beta) \left( 1 + \gamma X^{(1)}(a) \right) + \frac{\gamma \sin \beta}{g_0(a) p(a)}$$

and

$$a_m = \begin{cases} 
0 & \text{for an odd } m \\
(g_0(a) \cos \beta + g'_0(a) \sin \beta) \left( \frac{\tilde{X}^{(m)}(a)}{m!} + \gamma \frac{X^{(m+1)}(a)}{(m+1)!} \right) \\
+ \frac{\sin \beta}{g_0(a) p(a)} \left( \frac{\tilde{X}^{(m-1)}(a)}{(m-1)!} + \gamma \frac{X^{(m)}(a)}{m!} \right) & \text{for an even } m > 0.
\end{cases}$$
5 A remark on the Darboux transformation

The Darboux transformation is a very useful and important tool studied in dozens of works (see, e.g., [7]). It is closely related to the factorization of the Schrödinger operator (47). Consider the equation

\[
\left( \frac{\partial}{\partial x} + \frac{g_0'}{g_0} \right) \left( \frac{\partial}{\partial x} - \frac{g_0'}{g_0} \right) u = \omega^2 u.
\]

Applying the operator \(\left( \frac{\partial}{\partial x} - \frac{g_0'}{g_0} \right)\) to both sides and denoting \(v = \left( \frac{\partial}{\partial x} - \frac{g_0'}{g_0} \right) u\) one obtains that solutions of equation (50) are transformed into solutions of another Schrödinger equation

\[
\left( \frac{\partial}{\partial x} - \frac{g_0'}{g_0} \right) \left( \frac{\partial}{\partial x} + \frac{g_0'}{g_0} \right) v = \omega^2 v
\]

which can be written also as follows

\[
(-\partial_x^2 + r(x) + \omega^2) v(x) = 0,
\]

where \(r = 2 \left( \frac{\omega'}{g_0} \right)^2 - q\). Now, as we are able to construct the general solution of (50) by a known solution of (23) we can also obtain an explicit form of the result of the Darboux transformation. First, let us apply the operator \(\left( \frac{\partial}{\partial x} - \frac{g_0'}{g_0} \right)\) to \(u_1\) defined by (52). We have

\[
v_1 = \left( \frac{\partial}{\partial x} - \frac{g_0'}{g_0} \right) u_1 = g_0 \sum_{\text{even } n=0}^{\infty} \frac{\omega^n}{n!} \partial_x \tilde{X}^{(n)}
\]

\[
= g_0^{-1} \sum_{\text{even } n=2}^{\infty} \frac{\omega^n}{(n-1)!} \tilde{X}^{(n-1)} = \frac{\omega}{g_0} \sum_{\text{odd } n=1}^{\infty} \frac{\omega^n}{n!} \tilde{X}^{(n)}
\]

and in a similar way we obtain

\[
v_2 = \left( \frac{\partial}{\partial x} - \frac{g_0'}{g_0} \right) u_2 = \frac{\omega}{g_0} \sum_{\text{even } n=0}^{\infty} \frac{\omega^n}{n!} X^{(n)}
\]

Thus, the general solution of the Schrödinger equation (68) obtained from (50) by the Darboux transformation has the form

\[
v = \frac{c_1}{g_0} \sum_{\text{even } n=0}^{\infty} \frac{\omega^n}{n!} X^{(n)} + \frac{c_2}{g_0} \sum_{\text{odd } n=1}^{\infty} \frac{\omega^n}{n!} \tilde{X}^{(n)}
\]

where \(X^{(n)}\) and \(\tilde{X}^{(n)}\) are defined by (44)–(46).
6 Conclusions

The main result of this work allows us to find the general solution of (1) by a known solution of (2) as follows. By formulas (59)-(61) the functions $X^{(n)}$ and $\tilde{X}^{(n)}$ should be constructed and then the general solution of (1) has the form (56)-(58). An important feature of these formulas consists in the fact that the form of $X^{(n)}$ and $\tilde{X}^{(n)}$ does not depend on $\omega$. For given coefficients $p$ and $q$ they should be calculated only once and then the general solution for any $\omega$ is obtained by multiplying them by corresponding powers of $\omega$. This is very convenient for numerical calculations. Moreover, our numerical experiments show that $X^{(n)}$ and $\tilde{X}^{(n)}$ can be calculated up to high indices with a remarkable accuracy, thus we expect that among other possible applications the results of this work will be useful in numerical solving of a wide class of boundary value and spectral problems of mathematical physics.

Acknowledgement

The author wishes to express his gratitude to CONACYT for supporting this work via the research project 50424.

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