WELL-POSEDNESS OF A ONE-DIMENSIONAL NONLINEAR KINEMATIC HARDENING MODEL

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Abstract. We investigate the quasistatic evolution of a one-dimensional elasto-plastic body at small strains. The model includes general nonlinear kinematic hardening but no nonlocal compactifying term. Correspondingly, the free energy of the medium is local but nonquadratic. We prove that the quasistatic evolution problem admits a unique strong solution.

1. Introduction. Elastoplasticity describes the behavior of solids undergoing permanent deformations under the effect of mechanical actions. In the setting of infinitesimally small deformation gradients, the strain of the material is additively decomposed in an elastic and an inelastic (plastic) part. The evolution of the body results from the coupling of the momentum balance with a flow rule for the plastic strain $p$. Along the evolution, the stress $\sigma$ is constrained to a fixed convex set $K$ in stress space. This encodes the occurrence of a yield threshold, which in turn defines the onset of plasticization [15, 35].

The mechanics of elastoplastic materials has attracted early attention. First observations are to be traced back to Tresca [37] in 1864 and an analysis in the plane-stress perfectly rigid case is due to St. Venant [2]. Moving from that, results in three dimensions have been obtained by Lévy [24], Von Mises [30, 31], and later Prandtl [33]. The mechanical response of solids often depends on the whole previous plastic deformation history. One specific instance in this direction is hardening, which occurs as the elastic stress range changes due to plasticization. A first modelization of hardening effects is due to Prandtl [33]. Later on, Melan [28] and Prager [32] formulated what is now referred to as linear kinematic hardening, with the stress constraint $\sigma \in K$ modified as $\sigma - \chi \in K$, where $\chi$ is a given back stress. In the linear case, the back stress is proportional to the plastic strain, namely $\chi = Bp$ for some given hardening tensor $B$.

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Moving from this first model, a variety of options have been proposed in order to better reproduce specific experimental evidences \[23\]. An early nonlinear kinematic hardening model was proposed by Armstrong & Frederick \[1\] by replacing the linear position $\chi = Bp$ with the differential relation $\dot{\chi} + |\dot{p}|B\chi = B\dot{p}$ where dots stand for time derivatives. The basic feature of this model is that plasticization saturates and very large stresses cannot be reached. The analysis of the constitutive relation for such model has been presented by Brokate & Krejčí \[4, 5, 6\]. Three-dimensional results have been obtained by Chelmiński \[8\], see also \[14\]. A variety of nonlinear hardening models, usually referred to as generalized or nonassociative stem as generalizations of the Armstrong & Frederick model \[25, 27\].

This paper is yet devoted to another instance of nonlinearity for kinematic hardening: the relation between back stress and plastic strain is nonlinear, namely $\chi = \phi'(p)$ for some convex but nonquadratic function $\phi$. This modelization is originally due to Kadashevich & Novozhilov \[21\], see also Eisenberg & Phillips \[13\], and delivers a quite accurate description of the accumulation of plastic strain, whereas predictions under unloading are questionable.

In order to clarify this discussion, let us now introduce the relations we will be dealing with. We investigate the elastoplastic behavior of a one-dimensional body with reference configuration $I := (0, 1)$. In particular, we are interested in determining the evolution of its displacement $u: I \times [0, T] \rightarrow \mathbb{R}$ and its plastic strain $p: I \times [0, T] \rightarrow \mathbb{R}$. These are asked to fulfill the equilibrium equation and the flow rule

$$\sigma_x + f = 0, \quad \partial R(\dot{p}) + \phi'(p) \ni \sigma.$$  

along with the initial conditions $u(0) = u_0$ and $p(0) = p_0$. Here, $R$ is a von-Mises-type dissipation potential with $\partial R$ denoting its subdifferential (in the sense of convex analysis) and $\phi$ is a uniformly convex hardening potential, see Section 2. The system is complemented by so-called hard-device boundary conditions

$$u(0, t) = 0, \quad u(1, t) = v(t).$$

The evolution is driven by the bulk force $f$ and the boundary displacement $v$. A discussion on this model is in Section 2 below.

The analysis of initial and boundary value problems in elastoplasticity is rather classic and the reader is referred to Duvaut & Lions \[12\] for some classical reference. The quasistatic evolution of an elastoplastic medium is usually formulated in terms of an evolutionary variational inequality, which can be tackled by discretization and approximation methods. In case $\phi$ is quadratic, the analysis of problem (1)-(2) is fairly classical, see the seminal papers by Johnson \[17, 18, 19, 20\] and refer to the monographs \[15, 23, 35\]. The case of nonquadratic $\phi$ is more involved and can be relatively easily tackled when one augments the flow rule in (1) by a compactifying term $p_{xx}$, modeling indeed nonlocal plastic effects \[26\].

In absence of such compactifying term, the problem becomes very challenging, for one is confronted with the lack of compactness to pass to the limit in the nonlinear term $\phi'(p)$. Indeed, quasistatic-evolution results with nonquadratic energies and no gradient terms are just a few. In the frame of brittle fractures, one can mention \[10, 11\], which take however advantage of the degenerate nature of the dissipation. An existence result for a two-phase problem featuring a scalar internal phase parameter is in \[16\]. A first result in the elastoplastic context is due to Conti & Theil \[9\], who however tackle the case of single-slip crystal plasticity and do not discuss evolution. At the macroscopic level, the reader is referred to Mielke & Müller \[29\] and \[36\].
for existence results for incremental problems under boundedness assumptions on the curl of the plastic strain. In the viscoplastic setting, Röger & Schweizer [34] prove that such bounds on the curl allow to find a suitably weak evolution notion.

To our knowledge, no existence result for quasistatic elastoplasticity with non-quadratic energy and no gradient regularization is presently available. The aim of our contribution is to provide a first result in this direction, albeit in a one-dimensional framework. Our main result is the strong well-posedness of system (1)-(2), see Theorem 3.1. Our argument builds up on the underlying variational structure of the flow rule in (1), especially the possibility of reformulating this nonlinear relation in terms of a hysteresis operator, see Subsection 3.2. This allows to obtain pointwise estimates on time and spatial translations, which ultimately deliver the necessary compactness. We first establish existence for a suitable viscous regularization of the system in Subsection 3.3. Then, we use the mentioned compactness in order to pass to the limit in this regularization in Subsection 3.4. Eventually, we show some improved time regularity of the solution in Subsection 3.5.

2. The model.

2.1. Notation. Given \( w : I \times (0, T) \to \mathbb{R}, (x, t) \mapsto w(x, t) \) we denote by \( \dot{w} \) and \( w_x \) its time and space derivative, respectively. We use the short-hand notation \( L^p = L^p(I), H^1 = H^1(I) \) etc. Moreover, \( L^p(0, T; X) \) is the space of \( L^p \)-Bochner-integrable functions with values in the normed Banach space \( (X, \| \cdot \|_X) \). The spaces \( W^{k,p}(0, T; X) \) (\( k \)-differentiable in the sense of distributions) and \( C([0, T]; X) \) (continuous functions) will be used as well. Given the convex function \( w : \mathbb{R} \to \mathbb{R} \) we indicate with \( \partial w(x) \subset \mathbb{R} \) its subdifferential at point \( x \), namely the set \( \partial w(x) = \{ y \in \mathbb{R} : y(w - x) \leq w(w) - w(x) \ \forall \ w \in \mathbb{R} \} \).

2.2. Derivation of the model. Recall that \( I = (0, 1) \) is the reference configuration of the body. We indicate with \( u : \tilde{I} \times [0, T] \to \mathbb{R} \) its displacement and by \( p : \tilde{I} \times [0, T] \to \mathbb{R} \) its plastic strain. Within the realm of small-strain elastoplasticity, the strain \( u_x \) is additively decomposed as

\[
    u_x = e + p
\]

where \( e \) is the elastic strain, directly related to the stress of the material, whereas \( p \) encodes the accumulated plastic history. Owing to the additive decomposition, \( e \) is obviously uniquely determined once we know \( u_x \) and \( p \).

The first ingredient of our model is the Helmholtz free energy \( W : \mathbb{R}^2 \to \mathbb{R} \), which describes the statics of the material. We assume \( W \) to be given by

\[
    W(e, p) := \frac{1}{2} E e^2 + \phi(p)
\]

where \( \phi : \mathbb{R} \to \mathbb{R} \) is a possibly nonquadratic (plastic) hardening potential and \( E > 0 \) denotes the Young modulus. From the internal energy we derive constitutive equations. We assume elastic response by defining the stress \( \sigma \) as

\[
    \sigma := W_e = E e = E(u_x - p).
\]

In the quasistatic-approximation regime, the divergence of the stress equilibrates the body force \( f \), namely

\[
    \sigma_x + f = 0.
\]
The evolution of the internal variable $p$ is driven by the \textit{thermodynamic force} $\xi$ which reads

$$\xi := -W_p = \sigma - \phi'(p) \quad (5)$$

where $\phi'(p)$ is the so-called \textit{back stress}.

The second ingredient of our model is the \textit{von Mises dissipation potential} $R : \mathbb{R} \to [0, \infty)$ defined as

$$R(\dot{p}) := r|\dot{p}| \quad (6)$$

for some yield stress $r > 0$. The evolution of the variable $p$ is then described by the normality \textit{flow rule}

$$\dot{p} \in \partial R^*(\xi) \quad (7)$$

where $R^*$ is the Fenchel conjugate of $R$ and is defined as $R^*(\xi) := \sup\{\xi \dot{p} - R(\dot{p}) : \dot{p} \in \mathbb{R}\}$. A straightforward calculation reveals that

$$R^*(\xi) = I_{[-r,r]}(\xi) := \begin{cases} 0 & \text{if } \xi \in [-r,r], \\ \infty & \text{else} \end{cases}$$

where $I_{[-r,r]}$ is the classical \textit{indicator function} of the interval $[-r,r]$. Correspondingly, the subdifferential relation in (7) reads

$$\dot{p} \leq 0 \text{ if } \xi = -r, \quad \dot{p} = 0 \text{ if } |\xi| < r, \quad \dot{p} \geq 0 \text{ if } \xi = r.$$ 

This can be equivalently recasted in the form of the so-called \textit{complementary conditions} as

$$\dot{p} = \dot{\xi} \frac{\xi}{|\xi|}, \quad |\xi| \leq r, \quad \dot{\xi} \geq 0, \quad (|\xi| - r)\dot{\xi} = 0.$$ 

By combining the constitutive equation (5) with the flow rule (7) we get the so-called \textit{Biot’s equation}

$$\partial R(\dot{p}) + \phi'(p) \ni \sigma \quad (8)$$

where the inclusion sign expresses the fact that $\partial R(\dot{p})$ is actually the whole interval $[-r,r]$ if $\dot{p} = 0$.

\textbf{2.3. Target problem.} Given a body force $f : I \times [0,T] \to \mathbb{R}$, a hard-device boundary displacement $v : [0,T] \to \mathbb{R}$ and initial values $u_0, p_0 : I \to \mathbb{R}$, we aim at finding $u : I \times [0,T] \to \mathbb{R}$ and $p, \xi : I \times [0,T] \to \mathbb{R}$ satisfying

\begin{align*}
\sigma_x + f &= 0 \quad \text{in } I \times (0,T), \\
\xi + \phi'(p) &= \sigma \quad \text{in } I \times (0,T), \\
\partial R(\dot{p}) &\ni \xi \quad \text{in } I \times (0,T), \\
u(0, \cdot) &= 0 \quad \text{in } (0,T), \\
u(1, \cdot) &= v \quad \text{in } (0,T), \\
u(\cdot, 0) &= u_0 \quad \text{in } I, \\
p(\cdot, 0) &= p_0 \quad \text{in } I. \quad (8a-8g)
\end{align*}

The assumptions on data $f, v, u_0$, and $p_0$ will be specified in Section 2.5 below.
2.4. On the choice of boundary conditions. We assume the hard-device Dirichlet boundary conditions (8d)-(8e). Correspondingly, the body is subject to no tension in \(x = 0\) and \(x = 1\). Alternatively, one could consider 

\[
u(0, t) = 0, \quad \sigma(1, t) = g(t), \quad t \in (0, T)
\]

for some given traction \(g : [0, T] \to \mathbb{R}\). In this case, the stress can be directly determined from (8a) as

\[
\sigma(x, t) = g(t) + \int_0^1 f(y, t) \, dy.
\]

In particular, the equation for \(p\) in (8) is parametrizable: for every fixed \(x \in I\) we can view it as an ordinary differential equation in \(t\). The corresponding quasistatic-evolution problem is then easily solvable: once \(p\) is obtained from solving (8b)-(8c), \(u_x = \sigma/E - p\) is also determined.

Our choice of boundary conditions (8d)-(8e) does not allow for such simplification. Indeed, \(\sigma\) turns out to be nonlocally-in-time dependent on \(p\), see (20) and Section 3.1.

2.5. Assumptions on the data. This subsection lists our assumptions on the parameters and data of the problem.

We assume \(\phi \in C^2(\mathbb{R})\) nonnegative, and normalized as \(\phi(0) = \phi'(0) = 0\). Moreover, \(\phi\) is asked to be \(\alpha\)-convex, meaning

\[
\phi'' \geq \alpha > 0.
\]

(9)

Note that no growth restriction is imposed on \(\phi\).

The body force \(f\) and the boundary displacement \(v\) are asked to be Lipschitz continuous, more precisely we ask

\[
f \in W^{1,\infty}(0, T; L^1) \cap W^{1,1}(0, T; L^2), \quad v \in W^{1,\infty}(0, T).
\]

(10)

For the initial values we assume

\[
u_0 \in H^1_D := \{w \in H^1 : w(0) = 0\}, \quad \phi(p_0) \in L^1.
\]

(11)

Note that assumption (11) for \(p_0\) is in line with the classical theory of linear hardening where \(\phi\) is quadratic and \(\phi(p_0) \in L^1\) means \(p_0 \in L^2\). In addition, we ask the initial stress \(\sigma_0 := E(u_{0,x} - p_0)\) to be at equilibrium at time 0, namely

\[
-\sigma_{0,x} = f(0).
\]

(12)

Moreover, \(\sigma_0\) needs to be compatible in the following sense

\[
\xi_0 := \sigma_0 - \phi'(p_0) \in [-r, r].
\]

(13)

Eventually, we assume \(u_0(1) = v(0)\).

3. Main result - Statement and proof. In this section, we state and prove the main result of this paper which reads as follows.

**Theorem 3.1** (Existence and uniqueness of strong solutions). Under the assumptions of Section 2.5, problem (8) admits a unique strong solution with regularity

\[
u \in W^{1,\infty}(0, T; H_D^1) \cap L^\infty(0, T; H^2),
\]

\[
\xi, p \in W^{1,\infty}(0, T; L^\infty) \cap L^\infty(0, T; H^1).
\]
Note that the plastic strain $p$ is bounded in $L^\infty(0,T; H^1)$, albeit no gradient term appears in (8b). Such additional regularity is obtained by directly controlling spatial translations, a possibility which is ensured by the one-dimensionality of the problem.

The proof of Theorem 3.1 will be carried out in several steps throughout this section. In Subsection 3.1 we reduce system (8) to a nonlocal-in-space evolutionary variational inequality for $p$. We introduce a viscous regularization in Section 3.3 and solve this regularized problem by a fixed point argument. Here, the Lipschitz-continuity of the underlying hysteresis operator is instrumental, see Subsection 3.2. In Section 3.4 we pass to the limit in the regularization. Eventually, regularity is further improved in Section 3.5.

### 3.1. Determination of the stress.

By integrating the equilibrium equation (8a) on $(0,x) \subset I$ we get

$$\sigma(x,t) = \sigma(0,t) - \int_0^x f(y,t) \, dy.$$  

This can be combined with the stress-strain relation (3) in the following way. We integrate (8a) over $I$ and use the boundary conditions (8d)-(8e) for $u$ obtaining

$$\int_I \sigma(x,t) \, dx = E \left(v(t) - \int_I p(x,t) \, dx \right).$$  

Now we integrate relation (14) over $I$ and combine it with equation (15) to get

$$\sigma(0,t) = E \left(v(t) - \int_I p(x,t) \, dx \right) + \int_I \int_0^x f(y,t) \, dy \, dx.$$  

By plugging (16) back into equation (14), the stress splits into two parts as follows

$$\sigma(x,t) = \tilde{\sigma}(x,t) - E \int_I p(x,t) \, dx$$  

where

$$\tilde{\sigma}(x,t) := E v(t) + \int_I \int_0^z f(y,t) \, dy \, dz - \int_0^x f(y,t) \, dy$$

is known in terms of data. Assumption (10) ensures that

$$\dot{\tilde{\sigma}} \in L^\infty(I \times (0,T)).$$  

Once the stress $\sigma$ and the plastic strain $p$ are known, we can compute the displacement $u$ via (3), getting

$$u(x,t) = \int_0^x \sigma(y,t)/E - p(y,t) \, dy.$$  

This shows that system (8b) is in fact decoupled: We can first solve the equation (8b) for $p$ without using $u$. Once $p$ is known, we use formula (19) to determine $u$. Thus, it remains to prove the existence of $p$ solving

$$\xi + \phi'(p) + E \dot{p} = \tilde{\sigma} \quad \text{in } I \times (0,T),$$  

$$\partial R(\dot{p}) \ni \xi \quad \text{in } I \times (0,T),$$  

$$p(\cdot,0) = p_0 \quad \text{in } I.$$  

(20a)  

(20b)  

(20c)
where \( \bar{\sigma} \) with Lipschitz regularity (18) is known and

\[
\bar{p} := \int_I p \, dx
\]
denotes the spatial mean of \( p \). The forthcoming discussion is hence focusing on proving well-posedness and regularity for the nonlocal-in-space evolutionary variational inequality (20).

We warn the reader that throughout the rest of the paper the symbol \( C \) is used to denote a generic positive constant which solely depends on the data and parameters (in particular \( E, \alpha, r, u_0, p_0, f, v, \) and \( T \)) and may change even within the same line. Occasionally, we explicitly indicate dependencies of a constant by subscripts.

### 3.2. Play operator and gradient estimates

Let us go back to the flow rule (8b)-(8c) \( \dot{p} \in \partial R^*(\sigma - \phi'(p)) \) which can equivalently be written as

\[
\sigma - \phi'(p) = -r \quad \Rightarrow \quad \dot{p} \leq 0,
\]

\[
-r < \sigma - \phi'(p) < r \quad \Rightarrow \quad \dot{p} = 0,
\]

\[
\sigma - \phi'(p) = r \quad \Rightarrow \quad \dot{p} \geq 0.
\]

Now, we change variables to

\[
q = \phi'(p)
\]

and note that the rates \( \dot{q} \) and \( \dot{p} \) have the same sign as \( \dot{q} = \phi''(p) \dot{p} \) and \( \phi \) is strongly convex. Therefore, we can rewrite the flow rule in terms of \( q \) as

\[
\dot{q} \in \partial R^*(\sigma - q).
\]  

The nonlinearity now appears in the nonlocal part of \( \sigma \) since

\[
\sigma = \bar{\sigma} - \bar{E}\bar{p} = \bar{\sigma} - E\psi(q)
\]

where we have used that \( \phi' \) is globally invertible and have indicated its inverse by \( \psi \).

Fix now a material point \( x \in I \). Given a trajectory \( \sigma(x, \cdot) \in C[0, T] \) with \( \sigma(x, 0) - \phi'(p_0) \in [-r, r] \), there exists a unique trajectory \( q \in C[0, T] \) with \( q(0) = \phi'(p_0) \) such that (21) is satisfied, see [22, Theorem I.3.1]. This defined a hysteresis operator

\[
P : C[0, T] \to C[0, T].
\]

The problem \( \dot{q} \in \partial R^*(\sigma - q) \) can hence be written as \( q = P(\sigma) \). This operator is the so-called play and is Lipschitz continuous in the following sense

\[
\sup_{s \in [0, t]} |q_1(s) - q_2(s)| \leq \sup_{s \in [0, t]} |\sigma_1(s) - \sigma_2(s)|,
\]

for all \( q_i \in P(\sigma_i), \, i = 1, 2, \) and for all \( t \in [0, T] \), see [7, Theorem 2.3.2].

Assume \( q \) to be a strong solution of problem (20) and let \( x, y \in I \). By using (22) with \( q_1 = q(x, \cdot) \) and \( q_2 = q(y, \cdot) \) one gets

\[
\sup_{t \in [0, T]} |q(x, t) - q(y, t)| \leq \sup_{t \in [0, T]} |\sigma(x, t) - \sigma(y, t)|.
\]

As \( x \) and \( y \) are arbitrary, we use equilibrium equation (8a) and assumption (10) to deduce the gradient control

\[
\|q_x\|_{L^\infty(0, T; L^2)} \leq \|\sigma_x\|_{L^\infty(0, T; L^2)} = \|f\|_{L^\infty(0, T; L^2)} \leq C(1 + \|f\|_{L^1(0, T; L^2)}) \leq C.
\]  

(23)
A control on \( p_x \) can then be obtained from \( \phi''(p)p_x = q_x \) by using the strong convexity (9), namely
\[
\alpha\|p_x\|_{L^\infty(0,T;L^2)} \leq \|q_x\|_{L^\infty(0,T;L^2)} \leq C.
\] (24)
This argument has to be rigorously implemented within a suitable approximation procedure, for which existence can be checked. We do this in the coming Subsection.

3.3. **Viscous regularization.** We regularize the problem by augmenting equation (20a) by the small-viscosity term \( \varepsilon \dot{p} \), for \( \varepsilon > 0 \). This changes system (20) into
\[
\varepsilon \dot{p} + \xi + \phi'(p) + E\dot{p} = \dot{\sigma} \quad \text{in} \quad I \times (0,T), \tag{25a}
\]
\[
\partial R(\dot{p}) \ni \xi \quad \text{in} \quad I \times (0,T), \tag{25b}
\]
\[
p(x,0) = p_0 \quad \text{in} \quad I. \tag{25c}
\]
We aim at showing that (25) admits a solution. To this end, we implement a fixed point procedure by solving first for some given trial nonlocal term \( E\tilde{p} \) and then iterating. Let \( \tilde{p} \in C([0,T];L^2) \) be given and consider instead of (25a)-(25b) the differential inclusion
\[
\varepsilon \dot{p} + \partial R(\dot{p}) + \phi'(p) \ni \tilde{\sigma} - E\tilde{p} \quad \text{in} \quad I \times (0,T),
\] or, equivalently,
\[
\dot{p} = (\varepsilon \text{id} + \partial R)^{-1}(\tilde{\sigma} - \phi'(p) - E\tilde{p}). \tag{27}
\]
Note that the operator \((\varepsilon \text{id} + \partial R)^{-1}\) is Lipschitz continuous with constant \(1/\varepsilon\). For all \( x \in I \) fixed, problem (27) is an ordinary differential equation \( \dot{p} = F_\varepsilon(t,p(t)) \) where \( F_\varepsilon : \mathbb{R} \to \mathbb{R} \) is locally Lipschitz continuous and depends on \( \varepsilon \). Thus, by the Picard-Lindelöf Theorem, it admits a unique local solution. This is indeed global, for one can prove that it remains uniformly bounded for all times. To see this we go back to equation (25a) and test it with \( \dot{p} \). Since \( \xi \in \partial R(\dot{p}) \) and \( R(0) = 0 \) we have that \( \xi \dot{p} \geq R(\dot{p}) \), leading to
\[
\varepsilon |\dot{p}|^2 + R(\dot{p}) + \frac{d}{dt} \phi(p) \leq (\tilde{\sigma} - E\tilde{p})\dot{p}. \tag{28}
\]
We use Young’s inequality to estimate
\[
(\tilde{\sigma} - E\tilde{p})\dot{p} \leq \frac{1}{2\varepsilon}(\tilde{\sigma} - E\tilde{p})^2 + \frac{\varepsilon}{2} |\dot{p}|^2.
\]
The second term is absorbed in the left-hand side of (28). Integrating over \((0,t)\) for \( t \in (0,T) \) we get
\[
\phi(p(x,t)) \leq \phi(p(x,0)) + \frac{1}{2\varepsilon} \int_0^T (\tilde{\sigma}(x,s) - E\tilde{p}(x,s)\tilde{\sigma}(x,s))^2 ds \leq C_{\varepsilon,x,\tilde{p}} \tag{29}
\]
which shows that \( p(x,\cdot) \in L^\infty(0,T) \) as \( \phi \) is strongly convex. As \( x \) is arbitrary, we have hence found a solution of (27). We now repeat the procedure from Section 3.2, by introducing \( q = \phi'(p) \), and getting
\[
\phi(q)q + q + \partial R(q) \ni \tilde{\sigma} - E\tilde{p} \tag{30}
\]
where
\[
\phi(q) = \frac{\varepsilon}{\phi''(\psi(q))} \in (0,\varepsilon/\alpha]
\]
and the right-hand side is known. We define the regularized play operator \( P_\varepsilon \) where \( q = P_\varepsilon(\tilde{\sigma} - E\tilde{p}) \) is the solution to (30). It is easy to see that the Lipschitz constant
of $P_\varepsilon$ is independent of $\varepsilon$ as $\phi$ is positive. Therefore, using the same arguments as in Section 3.2, we end up with
\[
\|p\|_{L^\infty([0,T];L^2)} \leq C
\]
where the constant neither depends on $\varepsilon$ nor on $\tilde{p}$. By the Sobolev embedding $H^1 \subset L^\infty$ the latter implies that
\[
\|p\|_{L^\infty([0,T])} \leq C_{\varepsilon,\tilde{p}}.
\]
Note that, albeit the bound (31) does not depend on $\tilde{p}$, the constant in estimate (32) does due to bound (29). More precisely, the constant $C_{\varepsilon,\tilde{p}}$ depends on the $L^\infty(0,T;L^2)$-norm of $\tilde{p}$.

Let us denote by $\mathcal{S} : C([0,\tau];L^2) \to C([0,\tau];L^2)$ the solution operator $\tilde{p} \mapsto p$, restricted to the small-time interval $[0,\tau]$ with $\tau \in (0,T)$. We shall prove that $\mathcal{S}$ has a unique fixed point when restricted (without changing notation) to a suitable ball $B := \{ p \in C([0,\tau];L^2) : \|p - p_0\|_{C([0,\tau];L^2)} \leq M \}$ for some given $M \geq \|p_0\|_{L^2}$ large and $\tau$ small, to be specified later. More precisely, we check that a choice of $M$ and $\tau$ can be made, so that $\mathcal{S}$ is a contraction.

Let $\tilde{p}_i \in B$, $i = 1,2$, be given and set $p_i := \mathcal{S}(\tilde{p}_i)$. We take the difference of the two equations satisfied by $p_1$ and $p_2$, respectively, and test it with $\tilde{p}_1 - \tilde{p}_2$. Upon integrating on $I \times (0,t)$ for $t \leq \tau$, the monotonicity of $\partial R$ gives
\[
\varepsilon \int_0^t \|\tilde{p}_1 - \tilde{p}_2\|^2_{L^2} ds \leq -\int_0^t \int_I (\phi'(p_1) - \phi'(p_2))(\tilde{p}_1 - \tilde{p}_2) dx \, ds \\
\quad \quad - \int_0^t \int_I E(\tilde{p}_2 - \tilde{p}_2)(\tilde{p}_1 - \tilde{p}_2) dx \, ds.
\]  
As $\tilde{p}_i$ belongs to $B$, bound (32) entails that
\[
|\phi'(p_1) - \phi'(p_2)|(x,t) \leq \sup \{ \phi''(\lambda) : |\lambda| \leq \tilde{C}_{\varepsilon,M} \} \|p_1 - p_2\|(x,t) \leq C_{\varepsilon,M} |p_1 - p_2|(x,t)
\]
for a.e. $x$ and $t$. We use this to estimate the first term on the right-hand side of (33) by
\[
\left| \int_0^t \int_I (\phi'(p_1) - \phi'(p_2))(\tilde{p}_1 - \tilde{p}_2) dx \, ds \right| \leq C_{\varepsilon,M} \int_0^t \|p_1 - p_2\|_{L^2} \|\tilde{p}_1 - \tilde{p}_2\|_{L^2} ds \\
\leq C_{\varepsilon,M} \int_0^t \|p_1 - p_2\|^2_{L^2} ds + \frac{\varepsilon}{4} \int_0^t \|\tilde{p}_1 - \tilde{p}_2\|^2_{L^2} ds
\]
and the last term can be absorbed by the left-hand side. The second term in the right-hand side of (33) can be analogously controlled leading to
\[
\int_0^t \|\tilde{p}_1 - \tilde{p}_2\|^2_{L^2} ds \leq C_{\varepsilon,M} \int_0^t \left( \|p_1 - p_2\|^2_{L^2} + \|\tilde{p}_1 - \tilde{p}_2\|^2_{L^2} \right) ds.
\]
By Jensen’s inequality
\[
\|p_1(s) - p_2(s)\|^2_{L^2} = \left\| \int_0^s (\dot{p}_1 - \dot{p}_2) dr \right\|^2_{L^2} \leq s^2 \int_0^s \|\dot{p}_1 - \dot{p}_2\|^2_{L^2} dr,
\]
which leads to
\[
\int_0^t \|\dot{p}_1 - \dot{p}_2\|^2_{L^2} ds \leq C_{\varepsilon,M} \int_0^t \left( \int_0^s \|\dot{p}_1 - \dot{p}_2\|^2_{L^2} dr \right) ds + C_{\varepsilon,M} \int_0^t \|\dot{p}_1 - \dot{p}_2\|^2_{L^2} ds.
\]
An application of Gronwall’s lemma entails
\[
\int_0^t \| \tilde{p}_1 - \tilde{p}_2 \|^2_{L^2} \, ds \leq C_{\varepsilon,M} \int_0^t \| \tilde{p}_1 - \tilde{p}_2 \|^2_{L^2} \, ds.
\]
By arguing as in (34) we conclude that
\[
\sup_{s \in [0,t]} \| p_1 - p_2 \|^2_{L^2} \leq \bar{C}_{\varepsilon,M} \int_0^t \left( \sup_{r \in [0,s]} \| \tilde{p}_1 - \tilde{p}_2 \|^2_{L^2} \right) \, ds. \tag{35}
\]
Take now \( M \geq 4\| S(p_0) - p_0 \|^2_{C([0,T];L^2)}, \kappa \in (0, 1) \), and
\[
\tau \leq \frac{\kappa}{2\bar{C}_{\varepsilon,M}}. \tag{36}
\]
We readily check that
\[
\| S(\tilde{p}) - p_0 \|^2_{C([0,T];L^2)} \leq 2\| S(\tilde{p}) - S(p_0) \|^2_{C([0,T];L^2)} + 2\| S(p_0) - p_0 \|^2_{C([0,T];L^2)} \leq \bar{C}_{\varepsilon,M} \tau \| \tilde{p} - p_0 \|^2_{C([0,T];L^2)} + 2\| S(p_0) - p_0 \|^2_{C([0,T];L^2)} \leq \bar{C}_{\varepsilon,M} \tau M + \frac{M}{2} \tag{36}
\]
This proves that \( S(B) \subset B \). On the other hand, we have that
\[
\| S(p_1) - S(p_2) \|^2_{C([0,T];L^2)} \leq \bar{C}_{\varepsilon,M} \tau \| \tilde{p}_1 - \tilde{p}_2 \|^2_{C([0,T];L^2)} \tag{35}
\]
showing that \( S : B \rightarrow B \) is a contraction. Its unique fixed point solves problem (25) on the time interval \((0, \tau)\). It is now a standard matter to argue as in (28) and prove that such solution can be extended to \((0, T)\). Indeed, the argument is here even simpler, for the term \( E\tilde{p}p \) yields a nonnegative contribution, whenever integrated on \( I \). Eventually, one readily checks that such a solution is unique by taking the difference of equations (25a) written for two possible solutions and testing it on their difference.

3.4. Limit passage. We have checked that, for all \( \varepsilon > 0 \), system (25) has a unique strong solution \((p_\varepsilon, \xi_\varepsilon)\). Our next aim is to pass to the limit in (25) as \( \varepsilon \) tends to 0.

Let us start by obtaining an estimate on the time derivatives \( \dot{p}_\varepsilon \). This follows by differentiating equation (25a) in time and testing by \( \dot{p}_\varepsilon \). Note that \( p_\varepsilon \) is \( C^1(0,T;L^2) \) as \( p_\varepsilon = (\varepsilon \text{id} + \partial D)^{-1}(\dot{\sigma} - \phi'(p_\varepsilon) - E\tilde{p}_\varepsilon) \), see reformulation (27), and we have \( p_\varepsilon \in C([0,T];L^2) \). In particular, \( \varepsilon \dot{p}_\varepsilon(0) = \dot{\sigma}_0 - \phi'(p_0) - E\bar{p}_0 - \xi_0 = 0 \), see definition (13). On the other hand, the regularity in time of \( \xi_\varepsilon \) is checked only in Subsection 3.5 below, so that this estimate is presently only formal and one would need to introduce here an additional approximation (time discretization, for instance). We get
\[
\frac{d}{dt} \int_I \left( \varepsilon \left| \frac{\dot{p}_\varepsilon}{2} + R^*(\xi_\varepsilon) \right| \right) \, dx + \int_I \phi''(p_\varepsilon)|\dot{p}_\varepsilon|^2 \, dx \leq \int_I \dot{\sigma} \dot{p}_\varepsilon \, dx.
\]
We first use the strong convexity of \( \phi \) together with Hölder’s and Young’s inequalities and get
\[
\frac{d}{dt} \int_I \left( \varepsilon \left| \dot{p}_\varepsilon \right|^2 \right) + \frac{\alpha}{2} \| \dot{p}_\varepsilon \|^2_{L^2} \leq \frac{1}{2\alpha} \| \dot{\sigma} \|^2_{L^2}.
\]
This shows that
\[
\| \dot{p}_\varepsilon \|^2_{L^2(0,T;L^2)} \leq C \tag{37}
\]
and \( \xi \in [-r,r] \) a.e. On the other hand, recall that
\[
\|p_\varepsilon\|_{L^\infty(0,T;H^1)} \leq C
\]  
from gradient estimate (31). Bounds (37)-(38) allow an application of the Aubin-Lions Lemma in order to extract (not relabeled) subsequences as \( \varepsilon \to 0 \) such that
\[
p_\varepsilon \to p \quad \text{strongly in } C([0,T];L^2) \quad \text{and weakly in } H^1(0,T;L^2),
\]
\[
\xi_\varepsilon \to \xi \quad \text{weakly-star in } L^\infty(0,T;L^\infty).
\]  
In particular, the latter implies that \( \xi \in [-r,r] \) a.e. By using the convergences (39) and (40) we can pass to the limit in equation (25a) and get
\[
\xi + \phi'(p) + E\ddot{p} = \dot{\sigma} \quad \text{a.e. in } I \times (0,T).
\]

It remains to check that \( \xi \in \partial \mathcal{R}(\dot{p}) \) almost everywhere. To show this it suffices to prove that
\[
\limsup_{\varepsilon \to 0} \int_0^T \int_I \xi_\varepsilon \dot{p}_\varepsilon \, dx \, dt \
\leq \int_0^T \int_I \xi \dot{p} \, dx \, dt,
\]  
and make use of the classical tool from [3, Proposition 2.5]. In order to check (41), we first test equation (25a) on \( \dot{p}_\varepsilon \) and write
\[
\int_0^T \int_I \xi_\varepsilon \dot{p}_\varepsilon \, dx \, dt = \int_0^T \int_I (\dot{\sigma} - E\ddot{p}_\varepsilon) \dot{p}_\varepsilon - \varepsilon |\dot{p}_\varepsilon|^2 - \phi'(p_\varepsilon) \dot{p}_\varepsilon \, dx \, dt.
\]
Convergence (39) allows us to pass to the limit in the first term on the right-hand side above, for we have that
\[
\dot{\sigma} - E\ddot{p}_\varepsilon \to \dot{\sigma} - E\ddot{p} \quad \text{strongly in } L^\infty(0,T;L^2),
\]
\[
\dot{p}_\varepsilon \to \dot{p} \quad \text{weakly in } L^2(0,T;L^2).
\]

On the other hand, bound (37) implies that \( \varepsilon |\dot{p}_\varepsilon|^2 \to 0 \) in \( L^1(I \times (0,T)) \). The remaining term can be handled as follows
\[
\limsup_{\varepsilon \to 0} \left( - \int_0^T \int_I \phi'(p_\varepsilon) \dot{p}_\varepsilon \, dx \, dt \right) = - \liminf_{\varepsilon \to 0} \int_I \phi(p_\varepsilon(x,T)) \, dx + \int_I \phi(p_0) \, dx
\]
\[
\leq - \int_I \phi(p(x,T)) \, dx + \int_I \phi(p_0) \, dx = - \int_0^T \int_I \phi'(p) \dot{p} \, dx \, dt
\]
where we used the fact that \( p_\varepsilon(\cdot,T) \to p(\cdot,T) \) in \( L^2 \), observe again the strong convergence (39). By collecting all terms, relation (41) follows. This proves that \( (p,\xi) \), along with the corresponding \( u \) given by (19), solve the original problem (8).

3.5. Improved regularity. We are left with the proof of the regularity of \( u, p, \xi \) stated in Theorem 3.1. Indeed, we still have to check that
\[
u \in L^\infty(0,T;H^2)
\]
\[
\xi \in L^\infty(0,T;H^1),
\]
\[
p, \xi \in L^\infty(I \times (0,T)).
\]  
The regularity of \( u \) in (42) follows directly from differentiating the stress-strain relation (3) with respect to space and using equilibrium equation (4), assumption (10), and estimate (38). On the other hand, by differentiating relation (20a) with respect to space and taking again (38) into account we get
\[
\xi_\varepsilon = f - \phi''(p)p_\varepsilon \in L^\infty(0,T;L^2)
\]
which shows (43). Analogously, differentiation in time shows that
\[ \dot{\xi} = \dot{\sigma} - \phi''(p)\dot{p} - E\dot{\phi}. \]
Therefore, the regularity (44) follows as soon as we check that \( \dot{p} \in L^\infty(I \times (0, T)) \).

Let us start by showing that \( \dot{p} \in L^\infty(0, T) \). Differentiate equation (20a) in time and test it with \( \dot{p} \). By using the strong convexity (9), the nonlocal term can be then bounded as
\[ \frac{\alpha}{2} ||\dot{p}||_{L^2}^2 + E|\dot{\phi}|^2 \leq \frac{1}{2\alpha} ||\ddot{\phi}||_{L^2}^2, \]
and one concludes that \( |\dot{p}| \leq C \) for all times as we have the bound (18).

In order to control \( \dot{p} \) we perform a Moser-type iteration. Fix \( q \in \mathbb{N}, q \geq 2 \), differentiate equation (20a) in time, and test it with \( \dot{p} |\dot{p}|^{q-2} \) getting
\[ \int I \phi''(p)|\dot{p}|^q + \tilde{\xi} \dot{p} |\dot{p}|^{q-2} \, dx = \int I (\ddot{\sigma} - E\ddot{\phi})\dot{p} |\dot{p}|^{q-2} \, dx. \]
Since \( \xi \in [-r, r] \) almost everywhere, we have that
\[ \dot{\xi} \dot{p} = \frac{d}{dt} R^*(\xi) = 0 \text{ a.e. in } I \times (0, T). \]
(45)

By the strong convexity (9) and (45) we have
\[ \alpha ||\dot{p}||_{q,s}^q \leq \int I |\ddot{\sigma} - E\ddot{\phi}||\dot{p}|^{q-1} \, dx \]
(46)

Use now Hölder’s and Young’s inequality with \( q' = q/(q - 1) \) to get
\[ \int I |\ddot{\sigma} - E\ddot{\phi}||\dot{p}|^{q-1} \, dx \leq ||\ddot{\sigma} - E\ddot{\phi}||_{L^q} ||\dot{p}||_{L^{q'} - 1}^q \leq \frac{1}{q} \left( \frac{1}{\alpha} \right)^{q/q'} ||\ddot{\sigma} - E\ddot{\phi}||_{L^q} + \frac{\alpha}{q'} ||\dot{p}||_{L^{q'}}^q . \]
Combining this with (46) leads to
\[ \left( 1 - \frac{1}{q} \right) \alpha ||\dot{p}||_{q,s}^q \leq \frac{1}{q} \left( \frac{1}{\alpha} \right)^{q/q'} ||\ddot{\sigma} - E\ddot{\phi}||_{L^q}^q \leq 2^{q/4} \frac{1}{q} \left( \frac{1}{\alpha} \right)^{q/q'} (||\ddot{\sigma}||_{L^q}^q + E^{q/2} ||\dot{p}||_{q'}) . \]
As \( |\dot{p}| \) is uniformly bounded, by bound (18) we conclude that \( ||p||_{L^q} \leq C \) almost everywhere in time, where the latter constant \( C \) does not depend on \( q \). It hence suffices to pass to the limit as \( q \to \infty \) in order to get that \( \dot{p} \) is almost everywhere bounded in \( I \times (0, T) \). This concludes the regularity proof.

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REFERENCES

[1] P. J. Armstrong and C. O. Frederick, A mathematical representation of the multiaxial Bauschinger effect, Materials at High Temperatures, 24 (2007), 11–26.
[2] J. Barré de Saint Venant, Mémoire sur l’établissement des équations différentielles des mouvements intérieurs opérés dans les corps solides ductiles au delà des limites où l’élasticité pourrait les ramener à leur premier état, J. Math. Pures Appl., 16 (1871), 308–316.
[3] H. Brezis, Opérateurs Maximaux Monotones, North Holland, 1973.
[4] M. Brokate and P. Krejčí, On the wellposedness of the Chaboche model, in Control and Estimation of Distributed Parameter Systems, Internat. Ser. Numer. Math., 126, Birkhäuser, Basel, 1998, 67–79.
[5] M. Brokate and P. Krejčí, Wellposedness of kinematic hardening models in elastoplasticity, RAIRO Math. Modél. Numer. Anal., 32 (1998), 177–209.
[6] M. Brokate and P. Krejčí, Maximum norm wellposedness of nonlinear kinematic hardening models, Contin. Mech. Thermodyn., 9 (1997), 365–380.
[7] M. Brokate and J. Sprekels, Hysteresis and Phase Transitions, Applied Mathematical Sciences, 121, Springer-Verlag, New York, 1996.
[8] K. Chelminski, Mathematical analysis of the Armstrong-Frederick model from the theory of inelastic deformations of metals. First results and open problems, Contin. Mech. Thermodyn., 15 (2003), 221–245.
[9] S. Conti and F. Theil, Single-slip elastoplastic microstructures, Arch. Ration. Mech. Anal., 178 (2005), 125–148.
[10] G. Dal Maso and G. Lazzaroni, Quasistatic crack growth in finite elasticity with noninterpenetration, Ann. Inst. H. Poincaré Anal. Non Linéaire, 27 (2010), 257–290.
[11] G. Dal Maso, G. Francfort and R. Toader, Quasistatic crack growth in nonlinear elasticity, Arch. Ration. Mech. Anal., 176 (2005), 165–225.
[12] G. Duvaut and J.-L. Lions, Inequalities in Mechanics and Physics, Springer-Verlag, Berlin-New York, 1976.
[13] M. A. Eisenberg and A. Phillips, On nonlinear kinematic hardening, Acta Mech., 5 (1968), 1–13.
[14] G. A. Francfort and U. Stefanelli, Quasi-static evolution for the Armstrong-Frederick hardening-plasticity model, Appl. Math. Res. Express. AMRX, 2013 (2013), 297–344.
[15] W. Han and B. D. Reddy, Plasticity: Mathematical Theory and Numerical Analysis, Interdisciplinary Applied Mathematics, 9, Springer-Verlag, New York, 1999.
[16] S. Heinz and A. Mielke, Existence, numerical convergence and evolutionary relaxation for a rate-independent phase-transformation model, Philos. Trans. Roy. Soc. A, 374 (2016), 23pp.
[17] C. Johnson, Existence theorems for plasticity problems, J. Math. Pures Appl., 55 (1976), 431–444.
[18] C. Johnson, On finite element methods for plasticity problems, Numer. Math., 26 (1976), 79–84.
[19] C. Johnson, A mixed finite element method for plasticity problems with hardening, SIAM J. Numer. Anal., 14 (1977), 575–583.
[20] C. Johnson, On plasticity with hardening, J. Math. Anal. Appl., 62 (1978), 325–336.
[21] I. Kadashevich and V. V. Novozhilov, The theory of plasticity which takes into account residual microstresses, J. Appl. Mech. Phys., 22 (1958), 104–118.
[22] P. Krejčí, Hysteresis, Convexity and Dissipation in Hyperbolic Equations, GAKUTō International Series, Mathematical Sciences and Applications, 8, Gakkotosho Col., Ltd., Tokyo, 1996.
[23] J. Lemaître and J.-L. Chaboche, Mechanics of Solid Materials, Cambridge University Press, Cambridge, 1990.
[24] M. Lévy, Extrait du mémoire sur les équations générales des mouvements intérieurs des corps solides ductiles au delà des limites où l’élasticité pourrait les ramener à leur premier état, J. Math. Pures Appl., 16 (1871), 369–372.
[25] J. Lubliner, Plasticity Theory, Macmillan Publishing Company, New York, 1990.
[26] A. Mainik and A. Mielke, Existence results for energetic models for rate-independent systems, Calc. Var. Partial Differential Equations, 22 (2005), 73–99.
[27] G. A. Maugin, The Thermomechanics of Plasticity and Fracture, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 1992.
[28] E. Melan, Zur Plastizität des räumlichen Kontinuums, Ingenieur Archiv, 9 (1938), 116–126.
[29] A. Mielke and S. Müller, Lower semicontinuity and existence of minimizers in incremental finite-strain elastoplasticity, ZAMM Z. Angew. Math. Mech., 86 (2006), 233–250.
[30] R. von Mises, Mechanik der festen Körper im plastisch deformablen Zustand, Nachr. Akad. Wiss. Göttingen Math. Phys. Kl., (1913), 582–592.
[31] R. von Mises, Mechanik der plastischen Formänderung von Kristallen, ZAMM Z. Angew. Math. Mech., 8 (1928), 161–185.
[32] W. Prager, Recent developments in the mathematical theory of plasticity, J. Appl. Phys., 20 (1949), 235–241.
[33] L. T. Prandtl, Ein Gedankenmodell zur kinetischen Theorie der festen Körper, ZAMM Z. Angew. Math. Mech., 8 (1928), 85–106.

[34] M. Röger and B. Schweizer, Strain gradient visco-plasticity with dislocation densities contributing to the energy, *Math. Models Methods Appl. Sci.*, 27 (2017), 2595–2629.

[35] J. C. Simo and T. J. R. Hughes, *Computational Inelasticity*, Interdisciplinary Applied Mathematics, 7, Springer-Verlag, New York, 1998.

[36] U. Stefanelli, Existence for dislocation-free finite plasticity, *ESAIM Control Optim. Calc. Var.*, 25 (2018), 20pp.

[37] H. E. Tresca, Mémoire sur l’écoulement des corps solides, Mémoire Présentés par Divers Savants, *Acad. Sci. Paris*, 20 (1872), 75–135.

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