On one generalization of finite $\mathcal{U}$-critical groups

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Abstract

A proper subgroup $H$ of a group $G$ is said to be: $P$-subnormal in $G$ if there exists a chain of subgroups $H = H_0 < H_1 < \cdots < H_n = G$ such that $|H_i : H_{i-1}|$ is a prime for $i = 1, \ldots, n$; $P$-abnormal in $G$ if for every two subgroups $K \leq L$ of $G$, where $H \leq K$, $|L : K|$ is not a prime. In this paper we describe finite groups in which every non-identity subgroup is either $P$-subnormal or $P$-abnormal.

1 Introduction

Throughout this paper, all groups are finite, $G$ denotes a finite group and $p$ is a prime. We use $\mathcal{N}$ and $\mathcal{U}$ to denote the classes of all nilpotent and of all supersoluble groups, respectively. A subgroup $H$ of $G$ is said to be a Gaschütz subgroup of $G$ (Shemetkov [1, p. 170]) if $H$ is supersoluble and $|L : K|$ is not a prime whenever $H \leq K \leq L \leq G$.

Let $\mathfrak{F}$ be a class of groups. If $1 \in \mathfrak{F}$, then we write $G^\mathfrak{F}$ to denote the intersection of all normal subgroups $N$ of $G$ with $G/N \in \mathfrak{F}$. The class $\mathfrak{F}$ is said to be a formation if either $\mathfrak{F} = \varnothing$ or $1 \in \mathfrak{F}$ and every homomorphic image of $G/G^\mathfrak{F}$ belongs to $\mathfrak{F}$ for any group $G$. The formation $\mathfrak{F}$ is said to be: saturated if $G \in \mathfrak{F}$ whenever $G^\mathfrak{F} \leq \Phi(G)$; hereditary if $H \in \mathfrak{F}$ whenever $G \in \mathfrak{F}$ and $H$ is a subgroup of $G$.

A group $G$ is said to be $\mathfrak{F}$-critical if $G$ is not in $\mathfrak{F}$ but all proper subgroups of $G$ are in $\mathfrak{F}$ [2, p. 517]. An $\mathcal{N}$-critical group is also called a Schmidt group.

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\end{itemize}
A proper subgroup \( H \) of \( G \) is said to be: \( \mathfrak{F} \text{-subnormal} \) in \( G \) if there exists a chain of subgroups \( H = H_0 < H_1 < \cdots < H_n = G \) such that \( H_{i-1} \) is a maximal subgroup of \( H_i \) and \( H_i/(H_{i-1})H_i \in \mathfrak{F} \) for all \( i = 1, \ldots, n; \) \( \mathfrak{F} \text{-abnormal} \) in \( G \) if \( L/K \not\in \mathfrak{F} \) whenever \( H \leq K < L \leq G \) and \( K \) is a maximal subgroup of \( L \). A group \( G \not\in \mathfrak{F} \) is said to be an \( \mathcal{E}\mathfrak{F} \)-group \([3]\) if every non-identity subgroup of \( G \) is either \( \mathfrak{F} \text{-subnormal} \) or \( \mathfrak{F} \text{-abnormal} \) in \( G \).

In \([4]\), Fattahi described groups in which every subgroup is either normal or abnormal. As a generalization of this result, Ebert and Bauman classified the \( \mathcal{E}\mathfrak{F} \)-groups in the case when \( \mathfrak{F} = \mathfrak{N} \) (in this case \( G \) is a group in which every subgroup is either subnormal or abnormal), and in the case when \( \mathfrak{F} \) is the class of all soluble \( p \)-nilpotent groups, for odd prime \( p \) \([3]\). In the future, the \( \mathcal{E}\mathfrak{F} \)-groups were studied for some other \( \mathfrak{F} \) (see for example \([5, 6, 7, 8]\)). Nevertheless, it should be noted that a complete description of the \( \mathcal{E}\mathfrak{F} \)-groups was obtained only for such cases \( \mathfrak{F} \) when every \( \mathfrak{F} \)-critical group is a Schmidt group \([4, 6, 7, 8]\)). Thus, for example in the case, where \( \mathfrak{F} = \mathfrak{U} \), the structure of \( \mathcal{E}\mathfrak{U} \)-groups has not been known since the methods in \([4, 5, 6, 7, 8]\) could not be used in the analysis of this case.

Note, in passing, that if \( G \) is soluble and \( H \) is a subgroup of \( G \), then \( H \) is \( \mathfrak{U} \text{-subnormal} \) in \( G \) if and only if there exists a chain of subgroups \( H = H_0 < H_1 < \cdots < H_n = G \) such that \( |H_i : H_{i-1}| \) is a prime for \( i = 1, \ldots, n; \) \( \mathfrak{U} \text{-abnormal} \) in \( G \) if and only if \( |L : K| \) is not a prime whenever \( H \leq K < L \leq G \).

If \( G \) is supersoluble, then clearly every subgroup of \( G \) is \( \mathfrak{U} \)-subnormal in \( G \). A full description of \( \mathcal{E}\mathfrak{U} \)-groups, for the non-supersoluble case, gives the following our result.

**Theorem A.** Let \( G \) be an \( \mathcal{E}\mathfrak{U} \)-group and \( D = G^\mathfrak{U} \) the supersoluble residual of \( G \). Then \( G = D \rtimes H \), where:

(i) \( H \) is a Hall Gaschütz subgroup of \( G \). Hence if \( H \) is nilpotent, then it is a Carter subgroup of \( G \).

(ii) Every chief factor of \( G \) below \( D \) is non-cyclic. Hence \( H \) is a supersoluble normalizer (\( \mathfrak{U} \)-normalizer, in other words) of \( G \) in the sense of \([9]\).

(iii) \( |G : DG'| \) is a prime power number.

(iv) If \( H \) is not a cyclic group of prime power order \( p^n \), where \( n > 1 \), then \( D \) is nilpotent.

(v) \( H\Phi(G)/\Phi(G) \) is either a Miller-Moreno group or an abelian group of prime power order.

(vi) Every proper subgroup of \( G \) containing \( D \) is supersoluble.

Conversely, any group satisfying the above conditions is an \( \mathcal{E}\mathfrak{U} \)-group.

**Corollary 1.1.** Let \( G = D \rtimes H \) be an \( \mathcal{E}\mathfrak{U} \)-group, where \( D = G^\mathfrak{U} \). If \( H \) is nilpotent, then it is a system normalizer of \( G \).

From the description of \( \mathfrak{U} \)-critical groups \( G \) \([10, 11]\) it follows that every subgroup of \( G \) containing \( \Phi(G) \cap G^\mathfrak{U} \) is either \( \mathfrak{U} \)-subnormal or \( \mathfrak{U} \)-abnormal in \( G \) (see Lemma 2.6 below). Another application
of Theorem A is the following result, which classifies all groups with such a property.

**Theorem B.** Let $G$ be a non-supersoluble group and $\Phi = \Phi(G) \cap G^d$. Then every non-identity subgroup of $G$ containing $\Phi$ is either $\Omega$-subnormal or $\Omega$-abnormal in $G$ if and only if $G = D \rtimes H$ is a soluble group, where $H$ is a Hall subgroup of $G$ such that $H\Phi/\Phi$ is a Gaschütz subgroup of $G$, and with respect to $G$ Assertions (iii)–(vi) in Theorem A hold.

All unexplained notation and terminology are standard. The reader is referred to [12], [2], [13], or [14] if necessary.

# 2 Preliminaries

The following lemma collects some well-known properties of $\mathfrak{F}$-subnormal subgroups which will be used in our proofs.

**Lemma 2.1.** Let $\mathfrak{F}$ be a hereditary saturated formation, $H$ and $K$ the subgroups of $G$ and $H$ is $\mathfrak{F}$-subnormal in $G$.

1. $H \cap K$ is $\mathfrak{F}$-subnormal in $K$ [14, 6.1.7(2)].
2. If $N$ is a normal subgroup in $G$, then $HN/N$ is $\mathfrak{F}$-subnormal in $G/N$. [14, 6.1.6(3)].
3. If $K$ is an $\mathfrak{F}$-subnormal subgroup of $H$, then $K$ is $\mathfrak{F}$-subnormal in $G$ [14, 6.1.6(1)].
4. If $G^\mathfrak{F} \leq K$, then $K$ is $\mathfrak{F}$-subnormal in $G$ [14, 6.1.7(1)].
5. If $K \leq H$ and $H \in \mathfrak{F}$, then $K$ is $\mathfrak{F}$-subnormal in $G$.

A minimal normal subgroup $R$ of $G$ is called $\mathfrak{F}$-central in $G$ provided $R \rtimes (G/C_G(R)) \in \mathfrak{F}$, otherwise it is called $\mathfrak{F}$-eccentric in $G$.

**Lemma 2.2.** Let $\mathfrak{F}$ be a formation and $M$ a maximal subgroup of $G$. Let $R$ be a minimal normal subgroup of $G$ such that $MR = G$. Then $G/M_G \in \mathfrak{F}$ if and only if $R$ is $\mathfrak{F}$-central in $G$.

**Proof.** In view of the $G$-isomorphism $R \simeq RM_G/M_G$ we can assume without loss of generality that $M_G = 1$. Let $C = C_G(R)$.

If $G \simeq G/M_G \in \mathfrak{F}$, then $R$ is $\mathfrak{F}$-central in $G$ by the Barnes-Kegel’s Theorem [2, IV, 1.5]. Now let $R \rtimes (G/C) \in \mathfrak{F}$. First assume that $R$ is non-abelian. If $R$ is the unique minimal normal subgroup of $G$, then $C = 1$ and so $G \in \mathfrak{F}$. Now let $G$ have a minimal normal subgroup $L \neq R$. Then, since $M_G = 1$, $G = R \rtimes M = L \rtimes M$ and $C = L$ by [2, A, 15.2]. Hence $M \simeq G/L \simeq G/R \in \mathfrak{F}$ and so $G \in \mathfrak{F}$ since $\mathfrak{F}$ is a formation.

Finally, if $R$ is an abelian $p$-group, then $C = R$ by [2, A, 15.2] and so $G \simeq G/M_G \simeq R \rtimes (G/R) \in \mathfrak{F}$. The lemma is proved.

**Lemma 2.3** (See Lemma 2.15 in [15]). Let $E$ be a normal non-identity quasinilpotent subgroup of $G$. If $\Phi(G) \cap E = 1$, then $E$ is the direct product of some minimal normal subgroups of $G$. 

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Lemma 2.4. Let $\mathcal{F}$ be a non-empty hereditary saturated formation, $G$ an $E_\mathcal{F}$-group and $D = G^\mathcal{F}$.

(i) Every $\mathcal{F}$-subnormal subgroup of $G$ belongs to $\mathcal{F}$.

(ii) $F^*(G) \leq D\Phi(G)$.

Proof. (i) Let $H$ be any $\mathcal{F}$-subnormal subgroup of $G$ and $K$ a maximal subgroup of $H$. Then $K$ is not $\mathcal{F}$-abnormal, so it is $\mathcal{F}$-subnormal in $G$ by hypothesis. Hence $K$ is $\mathcal{F}$-subnormal in $G$ by Lemma 2.1(1), that is, $H/K_K \in \mathcal{F}$. Therefore, since $\mathcal{F}$ is a saturated formation, $H \in \mathcal{F}$.

(ii) Without loss of generality we can assume that $\Phi(G) = 1$. In this case $F^*(G) = N_1 \times \cdots \times N_t$ for some minimal normal subgroups $N_1, \ldots, N_t$ of $G$ by Lemma 2.3. Let $N = N_t$ and $M$ be a maximal subgroup of $G$ such that $G = MN$. Assume that $M$ is $\mathcal{F}$-subnormal in $G$. Then $D \leq M_G$, so $N$ is $\mathcal{F}$-central in $G$ by Lemma 2.2. On the other hand, Assertion (i) implies that $M \in \mathcal{F}$. Thus $G/N \simeq M/M \cap N \in \mathcal{F}$ and so $G \simeq G/N \cap M_G \in \mathcal{F}$. This contradiction shows that $M$ is $\mathcal{F}$-abnormal in $G$, so $G/M_G \not\in \mathcal{F}$. Hence $N \leq D$ by Lemma 2.2. Therefore $F^*(G) \leq D$. The lemma is proved.

Lemma 2.5. Let $\mathcal{F}$ be a non-empty formation, $G$ an $\mathcal{F}$-critical group and $D = G^\mathcal{F}$.

(i) If $G$ is soluble, then $D$ is a $p$-group for some prime $p$.

(ii) If $\mathcal{F}$ is saturated and $D$ is soluble, then the following statements hold:

(a) $D$ is a $p$-group for some prime $p$.

(b) $D/\Phi(D)$ is a chief factor of $G$.

Proof. (i) Since $G$ is soluble, $\Phi(G) < F(G)$. Hence for some prime $p$ we have $O_p(G) \not\leq \Phi(G)$. Let $M$ be a maximal subgroup of $G$ such that $G = O_p(G)M$. Then $G/O_p(G) = O_p(G)M/O_p(G) \simeq M/M \cap O_p(G) \in \mathcal{F}$ since $G$ is an $\mathcal{F}$-critical group. Thus $D \leq O_p(G)$.

(ii) See Theorem 24.2 in [1, V] or [2, VII, 6.18]. The lemma is proved.

Lemma 2.6. Let $\mathcal{F}$ be a hereditary saturated formation and $G$ an $\mathcal{F}$-critical soluble group. Then every subgroup of $G$ containing $\Phi(G) \cap G^\mathcal{F}$ is either $\mathcal{F}$-subnormal or $\mathcal{F}$-abnormal in $G$.

Proof. It is enough to consider the case when $\Phi(G) \cap G^\mathcal{F} = 1$. By Lemma 2.5, $D$ is a minimal normal subgroup of $G$. Let $A$ be any non-identity subgroup of $G$. First assume that $DA < G$. Then $DA \in \mathcal{F}$, and $DA$ is $\mathcal{F}$-subnormal in $G$ by Lemma 2.1(4). Hence $A$ is $\mathcal{F}$-subnormal in $G$ by Lemma 2.1(3). Now assume that $DA = G$. Then $A$ is a maximal subgroup of $G$, so $A$ is $\mathcal{F}$-abnormal in $G$. The lemma is proved.

Lemma 2.7 (Friesen [17, 4, 3.4]). If $G = AB$, where $A$ and $B$ are normal supersoluble subgroups of $G$ and $(|G:A|, |G:B|) = 1$, then $G$ is supersoluble.

We shall need the following special case of Theorem C in [15].

Lemma 2.8. Let $\mathcal{F}$ be a hereditary saturated formation containing all nilpotent groups and $E$ a normal subgroup of $G$. If $E/E \cap \Phi(G) \in \mathcal{F}$, then $E \in \mathcal{F}$.

A subgroup $H$ of $G$ is said to be: $\mathcal{P}$-subnormal in $G$ if there exists a chain of subgroups
\[ H = H_0 < H_1 < \cdots < H_n = G \] such that \( |H_i : H_{i-1}| \) is a prime for \( i = 1, \ldots, n \); \( \mathbb{P} \)-abnormal in \( G \) if \( |L : K| \) is not a prime whenever \( H \leq K \leq L \leq G \). We say that \( H \) satisfies the \( \mathbb{P} \)-property in \( G \) if \( H \) is either \( \mathbb{P} \)-subnormal or \( \mathbb{P} \)-abnormal in \( G \).

**Lemma 2.9.** (i) If every non-identity subgroup of \( G \) of prime order satisfies the \( \mathbb{P} \)-property in \( G \), then \( G \) is not a simple non-abelian group.

(ii) If every non-identity cyclic subgroup of \( G \) of prime power order satisfies the \( \mathbb{P} \)-property in \( G \), then \( G \) is soluble.

**Proof.** (i) Suppose that this is false and let \( p \) be the smallest prime dividing \( |G| \). Then a Sylow \( p \)-subgroup \( P \) of \( G \) is not cyclic. Let \( H \) be a subgroup of order \( p \) in \( P \). Then \( H < P \), so by hypothesis, \( G \) has a maximal subgroup \( M \) such that \( H \leq M \) and \( |G : M| = q \) for some prime \( q \). Since \( G \) is a simple non-abelian group, \( M_G = 1 \) and by considering the permutation representation of \( G \) on the right cosets of \( H \), we see that \( G \) is isomorphic to some subgroup of the symmetric group \( S_q \) of degree \( q \). Hence \( q \) is the largest prime divisor of \( |G| \) and \( |Q| = q \), where \( Q \) is a Sylow \( q \)-subgroup of \( G \). It follows that \( q \neq p \). It is clear that \( G \) is not \( q \)-nilpotent, so it has a \( q \)-closed Schmidt subgroup \( H \) such that \( Q \leq H \) by [13, IV, 5.4]. Since \( Q \) is normal in \( H \), it is \( P \)-subnormal in \( G \) by hypothesis. Hence \( G \) has a maximal subgroup \( T \) such that \( Q \leq T \) and \( |G : T| = r \) is a prime. But then \( r \) is the largest prime dividing \( |G| \) and so \( r = q \), a contradiction. Hence we have (i).

(ii) Since the hypothesis clearly holds for every quotient of \( G \) and every normal subgroup of \( G \), this assertion is a corollary of Assertion (i). The lemma is proved.

3 **Proofs of Theorems A and B**

**Proof of Theorem A.** Necessity. Suppose that this is false and let \( G \) be a counterexample of minimal order. Let \( \pi = \pi(D) \).

(1) The hypothesis holds on \( G/R \) for every normal subgroup \( R \) of \( G \) not containing \( D \).

First note that \( G/R \notin \mathcal{U} \) since \( D \notin R \). Therefore this claim is a corollary of Lemma 2.1(2).

(2) Every subgroup \( E \) of \( G \) containing \( D \) is supersoluble. Hence \( G \) is soluble.

First note that the hypothesis holds for \( D \), so \( D \) is soluble by Lemma 2.9. On the other hand, \( E \) is \( \mathcal{U} \)-subnormal in \( G \) by Lemma 2.1(4) and so \( E \) is supersoluble by Lemma 2.4(i). Hence we have (2).

(3) \( D \) is a Hall subgroup of \( G \).

Suppose that this is false and let \( P \) be a Sylow \( p \)-subgroup of \( D \) such that \( 1 < P < G_p \), where \( G_p \) is a Sylow \( p \)-subgroup of \( G \). Then \( |G_p| > p \). Let \( R \) be a minimal normal subgroup of \( G \).

(a) \( R \) is a \( p \)-group. Hence \( O_p'(G) = 1 \).

Since \( G \) is soluble by Claim (2), \( R \) is a \( q \)-group for some prime \( q \). Moreover, \( DR/R = (G/R)^{\mathcal{U}} \) is
a Hall subgroup of $G$ by the choice of $G$ since the hypothesis holds for $G/R$ by Claim (1). Therefore every Sylow $r$-subgroup of $D$, where $r \neq q$, is a Sylow subgroup of $G$. Hence $q = p$ and so $O_{p'}(G) = 1$.

(b) If $R \leq D$, then $R$ is a Sylow $p$-subgroup of $D$. If $R \cap D = 1$, then $G_p = R \rtimes P$.

Assume $R \leq D$. Then $R \leq P$ and $P/R$ is a Sylow $p$-subgroup of $D/R$. If $P/R \neq 1$, then Claim (1) and the choice of $G$ imply that $P/R = G_p/R$ and so $P = G_p$. This contradiction shows that $P = R$ is a Sylow $p$-subgroup of $D$.

Now assume that $R \nleq D$. Then $RP/R = G_p/R$ since $DR/R = (G/R)^{\Phi}$ is a Hall subgroup of $G/R$ by Claim (1), so $G_p = R \rtimes P$ since $R \cap D = 1$.

(c) $R \nleq \Phi(G)$. Hence $\Phi(G) = 1$.

Assume that $R \leq \Phi(G)$. If $R \cap D = 1$, then $G_p = R \rtimes P$ by Claim (b) and so $R \nleq \Phi(G)$ by the Gaschütz Theorem [16, I, 17.4]. This contradiction shows that $R \leq D$, so $R$ is the Sylow $p$-subgroup of $D$ by Claim (b). Hence $D/R$ is a $p'$-group, so a $p$-complement $S$ of $D$ is normal in $G$ by Lemma 2.8. But then $S \leq O_{p'}(G) = 1$. Hence $D = R$ and so $G$ is supersoluble since $D = G^\Phi$, a contradiction. Thus we have (c).

(d) $G_p$ is normal in $G$.

Let $E$ be any normal maximal subgroup of $G$ containing $D$ with $|G:E| = q$. Then $O_{p'}(E) \leq O_{p'}(G) = 1$, so $p$ is the largest prime dividing $|E|$ since $E$ is supersoluble by Claim (2). If $q \neq p$, then $G_p \leq E$ and so $G_p$ is normal in $G$ since in this case $G_p$ is a characteristic subgroup of $E$.

Finally, assume that $q = p$. Then $p$ is the largest prime dividing $|G|$ and so $DG_p$ is normal in $G$ since $G/D = G/G^{\Phi}$ is supersoluble. If $DG_p \neq G$, we can get as above that $G_p$ is normal in $G$. Now assume that $DG_p = G$. Since $R \nleq \Phi(G)$ by Claim (c), it has a complement in $G$ and so $R$ has a complement $V$ in $G_p$. It is clear that $V$ is not $\Phi$-subnormal in $G$, so for some maximal $\Phi$-subnormal subgroup $M$ of $G$ we have $V \leq M$, which implies that $G = DV \leq M$. This contradiction shows that the case under consideration is impossible. Hence $G_p$ is normal in $G$.

Final contradiction for (3). In view of Claim (d), $\Phi(G_p) \leq \Phi(G) = 1$. Therefore, by the Maschke’s Theorem, $G$ has a minimal normal subgroup $L \nleq \Phi(G)$ such that $L \leq G_p$ and $L \nleq D$. Then $|L| = p$ and for some maximal subgroup $M$ of $G$ we have $G = LM$. Hence $G/M_G$ is supersoluble by Lemma 2.3, which implies that $D \leq M$ and so $M$ is supersoluble by Claim (2). But then $G$ is supersoluble, a contradiction. Hence we have (3).

(4) $F(G) \leq D\Phi(G)$ (This directly follows from Lemma 2.4(ii)).

(5) $|G:DG'|$ is a prime power number (Since $G$ is not supersoluble, this directly follows from Claim (2) and Lemma 2.7).

(6) If $\Phi(G) = 1$, then $O_{p'}(G) = 1$.

Assume that $O_{p'}(G) \neq 1$ and let $R$ be a minimal normal subgroup of $G$ contained in $O_{p'}(G)$. Then $R \nleq D$, so in view of the $G$-isomorphism $RD/D \simeq R$, $R$ is cyclic. Since $\Phi(G) = 1$, for some
maximal subgroup $M$ of $G$ we have $RM = G$ and so $G/M_G$ is supersoluble by Lemma 2.3. It follows that $D ≤ M$ and hence $M$ is supersoluble by Claim (2). But then $G = RM$ is supersoluble. This contradiction shows that we have (6).

(7) If $H$ is a complement to $D$ in $G$, then $H\Phi(G)/\Phi(G)$ is either a Miller-Moreno group or an abelian group of prime power order.

Without loss of generality we can assume that $\Phi(G) = 1$. First we shall show that every proper subgroup $A$ of $H$ is abelian. Let $C = C_G(F(G))$. Then $C ≤ F(G)$ since $G$ is soluble. On the other hand, Claim (4) implies that $F(G) ≤ D$, so $C ≤ D$. It follows that $F(DA) = F(G)$, so $AF(G)/F(G) ≃ A$ is abelian since $DA$ is supersoluble by Claim (2). Therefore, if $H$ is not abelian, then $H$ is a Miller-Moreno group.

Finally, suppose that $H$ is abelian. Then $G' ≤ D$ by Claim (3), so Claim (5) implies that $|G : DG'| = |G : D| = |H|$ is a prime power number.

(8) $H$ is a Gaschütz subgroup of $G$.

Since $D = G^4$ and $G = D \rtimes H$, $H$ is supersoluble. It is clear also that $H$ is not $\U$-subnormal in $G$. Hence $H$ is $\U$-abnormal in $G$ by hypothesis. Therefore $H$ is a Gaschütz subgroup of $G$ since $G$ is soluble by Claim (2).

(9) If $H$ is not a cyclic group of prime power order $q^n$, where $n > 1$, then $D$ is nilpotent.

Suppose that this is false and let $R ≤ O_p(G)$ be a minimal normal subgroup of $G$.

(*) $|H|$ is a prime.

Indeed, assume that $H = \langle a \rangle$, where $|a|$ is a prime. Since $H$ is a Gaschütz subgroup of $G$, $N_G(H) = H$ and hence $a$ induces a regular automorphism on $D$. Hence $D$ is nilpotent by the Thompson’s theorem [20] V, 8.14, a contradiction. Hence we have (*).

(**) If $R ≤ D$ or $R ≤ \Phi(G)$, where $R$ is a minimal normal subgroup of $G$, then $DR/R$ is nilpotent. Hence $\Phi(G) = 1$, $R = F(D) = C_D(R)$ is the unique minimal normal subgroup of $G$ contained in $D$ and $R$ is the Sylow $p$-subgroup of $G$ for some prime $p$.

The choice of $G$ and Claims (1) and (*) imply that in order to prove that $DR/R$ is nilpotent, it is enough to show that $HR/R$ is not a cyclic group of order $q^n$, where $n > 1$ and $q$ is a prime. In the case when $R ≤ D$ it is evident. Now assume that $R ≤ \Phi(G) \cap H$. Then $R ≤ D$ and hence $|R| = p$ for some prime $p$. Let $G_p$ be a Sylow $p$-subgroup of $H$. Then $G_p$ is a Sylow $p$-subgroup of $G$ since $H$ is a Hall subgroup of $G$ by Claim (3). Suppose that $R ≤ \Phi(H)$. Then for some maximal subgroup $M$ of $H$ we have $H = R \rtimes M$, so $G_p = R \rtimes (M \cap H)$. But then $R$ has a complement in $G$ by Gaschütz’s Theorem [18] I, 17.4. This contradiction shows that $R ≤ \Phi(H)$. Suppose that $H/R$ is cyclic. Then $H$ is nilpotent and so $\Phi(H)$ is a maximal subgroup of $H$. It follows that $H$ is a cyclic group of order $p^n$, where $n > 1$, a contradiction. Therefore the hypothesis holds for $G/R$.

If $R ≤ \Phi(G)$, then from Lemma 2.8 we deduce that $D$ is nilpotent since $DR/R ≃ D$ is nilpotent,
a contradiction. Hence $\Phi(G) = 1$. Therefore $F(D)$ is the direct product of some minimal normal subgroups of $G$ by Lemma 2.3 since $\Phi(F(D)) \leq \Phi(G) = 1$. If $G$ has a minimal normal subgroup $L \neq R$ such that $L \leq D$, then $D \cong D/1 = D/R \cap L$ is nilpotent. Therefore $R$ is the unique minimal normal subgroup of $G$ contained in $D$. Therefore $R = F(D) = C_D(R)$. Finally, since $D$ is supersoluble, a Sylow $p$-subgroup $P$ of $G$, where $p$ is the largest prime dividing $|D|$, is normal in $D$ and so $P \leq C_D(R) = R$. Hence $R$ is the Sylow $p$-subgroup of $D$, so $R$ is the Sylow $p$-subgroup of $G$ since $D$ is a Hall subgroup of $G$ by Claim (3).

(***) $R = C_G(R)$. Hence $F(G) = R$.

Let $C = C_G(R)$ and $S$ be a $p$-complement of $C$. Then, in view of Claim (**), $C = R \times S$ is normal in $G$ and so $S$ and $S \cap D$ are normal in $G$. Therefore Claim (**) implies that $S \cap D = 1$. Therefore $S \leq O_{p'}(G) = 1$ by Claim (6). Hence $C = C_G(R)$.

Final contradiction for (9). First assume that $H$ is a $q$-group for some prime $q$ and $V$ and $W$ are different maximal subgroups of $H$. Then $DV$ and $DW$ are supersoluble by Claim (2) and $G = DVW = (DV)(DW)$. Hence $G$ is metanilpotent and then $G/R$ is nilpotent by Claim (**). Hence $D = R$ is nilpotent. This contradiction shows that $H = AB$, where $A$ is a Sylow $q$-subgroup of $H$ for some prime $q$ dividing $|H|$ and $B \neq 1$ is a $q$-complement of $H$. Let $S$ be a $p$-complement of $D$ such that $SB = BS$. Then $DB/F(DB) = DB/R \cong SB$ is abelian. Hence $[G : C_G(S)]$ is a $\{p, q\}$-number. Similarly, one can obtain that $[G : C_G(S)]$ is $\{p \cup \{q\}\}$-number. Hence $[G : C_G(S)]$ is a power of $p$. Therefore a $p$-complement of $G$ is supersoluble, which implies that $D = R$, a contradiction. Hence we have (9).

(10) Every chief factor $K/L$ of $G$ below $D$ is non-cyclic.

If $L \neq 1$, it is true by Claim (1) and the choice of $G$. On the other hand, in the case when $L = 1$ $K$ is not cyclic by Claim (8).

From Claims (1)–(10) it follows that Assertions (i)–(vi) are true for $G$, which contradicts the choice of $G$. This completes the proof of the necessity.

Sufficiency. Let $A$ be a non-identity subgroup of $G$. We shall show that $A$ is either $\mathcal{U}$-subnormal or $\mathcal{U}$-abnormal in $G$. It is clear that $A = V \times W$, where $V = A \cap D$ and $W$ is a Hall $\pi'$-subgroup of $H$. Moreover, since $G$ is soluble and $H$ is a Hall $\pi'$-subgroup of $G$, we can assume without loss of generality that $W \leq H$ and so $A = (A \cap D)(A \cap H)$. If $H \leq A$, then $A$ is $\mathcal{U}$-abnormal in $G$ since $H$ is a Gaschütz subgroup of $G$ by hypothesis. Assume that $A \cap H < H$ and let $E = D(A \cap H)$. Then $E$ is $\mathcal{U}$-subnormal in $G$ and $E$ is supersoluble by Assertion (vi). Hence $A$ is $\mathcal{U}$-subnormal in $G$ by Lemma 2.1(3).

Proof of Theorem B. Necessity. Suppose that this is false and let $G$ be a counterexample of minimal order. Then $\Phi \neq D$ and, in view of Theorem A, $\Phi \neq 1$.

(1) Assertions (i)–(vi) in Theorem A are true for $G/\Phi$. Moreover, Assertions (iii)–(vi) in Theorem A are true for $G/R$ for any non-identity normal subgroup $R$ of $G$ not containing $D$. 

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First note that \( \Phi(G/\Phi) \cap (G/\Phi)^{\mu} = (\Phi(G)/\Phi) \cap (D/\Phi) = (\Phi(G) \cap D)/\Phi = 1 \), so \( G/\Phi \) is an \( E_3 \)-group by Lemma 2.1(2). Therefore Assertions (i)–(vi) in Theorem A are true for \( G/\Phi \). In order to prove the second assertion of (1), it is enough to show that the hypothesis holds for \( G/R \). First note that \( G/R \notin \mathfrak{A} \) since \( D \notin R \) by our hypothesis on \( R \). Let \( A/R \) be a subgroup of \( G/R \) containing \( \Phi(G/R) \cap (G/R)^{\mu} = \Phi(G/R) \cap (DR/R) \). Then, since \( \Phi R/R \leq \Phi(G/R) \), \( A \) contains \( \Phi \) and so \( A \) is either \( \mathfrak{A} \)-subnormal or \( \mathfrak{A} \)-abnormal in \( G \). Hence \( A/R \) is either \( \mathfrak{A} \)-subnormal or \( \mathfrak{A} \)-abnormal in \( G/R \) by Lemma 2.1(2). Therefore the hypothesis holds for \( G/R \) for any normal subgroup \( R \) of \( G \) not containing \( D \).

(2) \( G = D \rtimes H \) is a soluble group, where \( H \) is a Hall subgroup of \( G \) such that \( H \Phi/\Phi \) is a Gaschütz subgroup of \( G/\Phi \).

From Claim (1) we get that \( D/\Phi = (G/\Phi)^{\mu} \) is a Hall subgroup of \( G \). Therefore \( G \) is soluble. It follows also that \( \Phi \) is the Sylow \( p \)-subgroup of \( D \) for some prime \( p \) and a \( p \)-complement \( S \) of \( D \) is a Hall subgroup of \( G \).

Since \( \Phi \leq \Phi(G) \), from Lemma 2.8 it follows that \( D = R \times S \). Hence for some minimal normal subgroup \( L \) of \( G \) we have \( L \leq S \) since \( \Phi \neq D \). Then \( D/L = (G/L)^{\mu} \) is a Hall subgroup of \( G \) by Claim (1) and hence \( \Phi \) is a Sylow subgroup of \( G \). But this is impossible since \( \Phi \leq \Phi(G) \). Hence \( D \) is a Hall subgroup of \( G \), so \( D \) has a complement \( H \) in \( G \). It is clear that \( H \Phi/\Phi \) is a complement to \( D/\Phi = (G/\Phi)^{\mu} \) in \( G/\Phi \), so \( H \Phi/\Phi \) is a Gaschütz subgroup of \( G/\Phi \) since Assertion (i) in Theorem A is true for \( G/\Phi \) by Claim (1).

(3) Assertion (iii) in Theorem A is true for \( G \).

Indeed, from Claim (1) we get that \(|(G/\Phi) : (D/\Phi)(G'/\Phi)| = |G : DG'\Phi| = |G : DG'|\) since \( \Phi \leq D \).

(4) Assertion (iv) in Theorem A is true for \( G \).

By Claim (1), \( D/\Phi \) is nilpotent. But then \( D \) is nilpotent by Lemma 2.8.

(5) Assertion (v) in Theorem A is true for \( G \) (This directly follows from Claim (1)).

(6) Assertion (vi) in Theorem A is true for \( G \).

Let \( E \) be any proper subgroup of \( G \) containing \( D \). We shall show that \( E \) is supersoluble. Suppose that this is false. Let \( R \) be a minimal normal subgroup of \( G \) contained in \( D \). Then \( D/R = (G/R)^{\mu} \leq E/R < G/R \). Hence \( E/R \) is supersoluble by Claim (1). Moreover, \( E/\Phi \) is supersoluble by the same Claim, so in the case when \( H \) is abelian \( E \) is supersoluble by Lemma 2.8. Hence \( H \) is not abelian. In this case \( D \) is nilpotent by Claim (1). If \( G \) has a minimal normal subgroup \( L \neq R \) such that \( L \leq D \), then \( E \simeq E/R \cap L \) is supersoluble. Therefore \( R \) is the only minimal normal subgroup of \( G \) contained in \( D \). Hence \( D \) is a Sylow \( p \)-group of \( G \) for some prime \( p \). Suppose that \( R \leq \Phi(D) \). Then \( E/\Phi(D) \) is supersoluble. Moreover, \( \Phi(D) \leq \Phi(E) \) since \( D \) is normal in \( E \) and hence \( E \) is supersoluble. Therefore \( \Phi(D) = 1 \), so \( R = D \) by the Maschke’s Theorem. But then \( \Phi = 1 \), a contradiction. Hence we have (6).
From Claims (2)–(6) it follows that the necessity conditions of the theorem are true for $G$, which contradicts the choice of $G$. This completes the proof of the necessity.

The sufficiency condition in the theorem directly follows from Theorem A.

4 Final remarks

1. The structure of $U$-critical groups are well-known [10, 11]. In particular, the supersoluble residual of $G^U$ of an $U$-critical group $G$ is a Sylow subgroup of $G$. This observation and Theorem A are motivations for the following question: Let $G$ be an $E_U$-group. Is it true then that $G^U$ is a Sylow subgroup of $G$ or, at least, the number $|\pi(G^U)|$ is limited to the top?

The following elementary example shows that the answer to this question is negative.

**Example 4.1.** Let $p_1 < p_2 < \cdots < p_n < p$ be a set of primes, $B$ a group of order $p$ and $P_i$ a simple $F_{p_i}$-module which is faithful for $B$. Let $A_i = P_i \times B$ and $G = (\cdots (A_1 \times A_2) \times A_3) \times \cdots ) \times A_n$ (see [16, p. 50]). Then $G$ is an $E_U$-group, $G^U = P_1 P_2 \cdots P_n$ and $|G/G^U| = p$.

2. The following example shows that the subgroup $D$ in Theorem A is not necessary nilpotent.

**Example 4.2.** Let $H = H_2 \times H_3$ is a 2-closed Schmidt group, where $H_2$ is a Sylow 2-subgroup of $G$ and $H_3 = \langle a \rangle$ a cyclic Sylow 3-subgroup of $G$. Then, by [2, B, 10.7], there exists a simple $F_7H$-module $P$ which is faithful for $H$. Let $G = P \times H$. It is no difficult to show that $G$ is an $E_U$-group and $G^U = PH_2$ is non-nilpotent.

3. It is also not difficult to show that the subgroup $H$ in Theorem A is not necessary cyclic.

4. We do not know the answer to the following question: What is the structure of the group $G$ provided that each nontrivial nilpotent subgroup of $G$ is either $U$-subnormal or or $U$-abnormal in $G$?

5. Partially, the results of this paper were announced in [21].

References

[1] L.A. Shemetkov, *Formations of finite groups*, Moscow, Nauka, Main Editorial Board for Physical and Mathematical Literature, 1978.

[2] K. Doerk, T. Hawkes, *Finite Soluble Groups*, Walter de Gruyter, Berlin, New York, 1992.

[3] G. Ebert, S. Bauman, A note on subnormal and abnormal chains, *J. Algebra*, 36(2) (1975), 287–293.

[4] A. Fattahi, Groups with only normal and abnormal subgroups, *J. Algebra*, 28(1) (1974), 15–19.

[5] P. Förster, Finite groups all of whose subgroups are $\mathfrak{S}$-subnormal or or $\mathfrak{S}$-subabnormal, *J. Algebra*, 103(1) (1986), 285–293.
[6] V.N. Semenchuk, The structure of finite groups with the $\mathfrak{F}$-abnormal or $\mathfrak{F}$-subnormal subgroups, in "Questions of Algebra", Minsk: Publishing House "University", 2 (1986), 50–55.

[7] V.N. Semenchuk, Finite groups with $f$-abnormal and $f$-subnormal subgroups, *Mat. Zametki*, 55(6) (1994), 111-115.

[8] V.N. Semenchuk, S.H. Shevchuk, Finite groups whose primary subgroups are $\mathfrak{F}$–abnormal or $\mathfrak{F}$–subnormal, *Izv. Vyssh. Uchebn. Zaved. Mathematics*, 11 (2011), 46–55.

[9] R. Carter, T. Hawkes, The $\mathfrak{F}$-normalizers of a finite soluble group, *J. Algebra*, 5(2) (1967), 175–202.

[10] B. Huppert, Normalteiler and maximale Untergruppen endlicher Gruppen, *Math. Z.*, 60 (1954), 409–434.

[11] K. Doerk, Minimal nicht uberauflosbare, endliche Gruppen, *Math. Z.*, 91 (1966), 198–205.

[12] L.A. Shemetkov, A.N. Skiba, Formations of Algebraic Systems, Nauka, Main Editorial Board for Physical and Mathematical Literature, Moscow, 1989.

[13] W. Guo, *The Theory of Classes of Groups*, Science Press-Kluwer Academic Publishers, Beijing–New York–Dordrecht–Boston–London, 2000.

[14] A. Ballester-Bolinches, L.M. Ezquerro, *Classes of Finite groups*, Springer, Dordrecht, 2006.

[15] W. Guo, A.N. Skiba, On $\mathfrak{F}^*$-hypercentral subgroups of finite groups, *J. Algebra*, 372 (2012), 275–292.

[16] B. Huppert, *Endliche Gruppen I*, Springer-Verlag, Berlin, Heidelberg, New York, 1967.

[17] M. Weinstein (ed.), et al., *Between Nilpotent and Solvable*, Polygonal Publishing House, Passaic N. J., 1982.

[18] A.F.Vasil’ev, T.I. Vasil’eva, V.N.Tyutyyanov, On the finite groups of supersoluble type, *Sib. Math. J.*, 51(6) (2010), 1004–1012.

[19] A.F.Vasil’ev, T.I. Vasil’eva, V.N.Tyutyyanov, On the rproducts of $\mathfrak{P}$-subnormal subgroups of finite groups, *Sib. Math. J.*, 51(6) (2012), 47–54.

[20] B. Huppert, N. Blackburn, *Finite Groups II*, Springer-Verlag, Berlin, New-York, 1982.

[21] V.N. Semenchuk, Finite groups with generalized subnormal formation subgroups, *Proc. Fr. Skorina Gomel St. Univ.*, 84(3) (2014), 104–107.