Relation Between Holonomy Groups in Superstrings, M and F-theories

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June 26, 2008

Abstract

We consider manifolds with special holonomy groups $SU(3)$, $G_2$ and $Spin(7)$ as suitable for compactification of superstrings, M-theory and F-theory (with only one time) respectively. The relations of these groups with the octonions are discussed, reinforcing their role in the physics of string theory and duality. We also exhibit three triple exact sequences explaining the connections between the mentioned special holonomy groups.

\textbf{KEYWORDS:} Superstrings, M-theory, F-theory, Duality, Octonions, Holonomy Groups.

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1 Introduction

Once one includes strings and/or other extended objects, extra dimensions became unavoidable, and thence the necessity of dimensional reduction, introducing tiny compactifying spaces, as we see only four extended dimensions. Over the few past years, there has been an increasing interest in studying duality in superstring theory and supersymmetric models, related to compactification. Interesting examples are mirror symmetry in four dimensions between pairs of Calabi-Yau threefolds in type II superstrings [1] and their strong/weak coupling dualities with heterotic superstrings on $K3 \times T^2$ [2]. The most important consequence of the study of string duality is that all (five) superstring models are equivalent in the sense that they correspond to different limits in the moduli space of the same theory, called M-theory by Witten [3, 4, 5, 8, 9]. M-theory is considered nowadays as the best candidate for the unification of the weak and strong coupling sectors of superstring models, and is described, at low energies, by eleven dimensional supergravity theory. In this paper, we shall also consider F-theory in 12 dimensions, initiated by C. Vafa [13], with metric of signature $(2,10)$ (the original idea) but also $(1,11)$.

To preserve adequate supersymmetry (Susy) in our 4d these intermediate spaces have to behave as special holonomy manifolds. In particular, to obtain $\mathcal{N} = 1$ supersymmetry down to four dimensions, which is necessary to hold a phenomenologically acceptable chiral theory, we need a manifold with $G_2$ holonomy if coming from 11d M-theory [3, 4, 5, 6, 7, 8, 9, 11, 10], or a Calabi-Yau three-fold with SU(3) holonomy if starting from heterotic superstrings [12]. The first case is due to the fact that the $G_2$ group is the maximal subgroup of $SO(7)$, for which the eight dimensional spinor representation of $SO(7)$ can be decomposed as the fundamental representation of $G_2$ and one singlet. In the second case, as $Spin(6) = SU(4)$, the $SU(3)$ subgroup shows again the $4 = 1 + 3$ reducible representation, guaranteeing $\mathcal{N} = 1$ down to 4d; the further break, e.g, from one factor of $E_8 \times E_8$ to $E_6 \times SU(3)$ is well studied for example in [12]. Now the conventional F-theory of Vafa [13] has signature $(2,10)$; one can consider however an F-theory with conventional metric $(1,11)$. In this case, the holonomy of the compactifying manifold might be $Spin(7)$, the largest of the exceptional holonomy groups. The alternative case is $SU(4)$ corresponding to Calabi-Yau four-folds. Very recently, a GUT realization has been given in the context of F-theory for such manifolds with elliptic K3 fibration [16, 17].

As a whole, there are many ways to get four dimensional models using different compactifications as intermediates. These constructions could be related to each other due to different sort of dualities, which appear in the process. As an example, we point out here several
equivalences in seven dimensions giving rise to a web of dualities (with F and M for F-theory, M-theory, ...) [1]

\[ F/K3 \times S^1 \sim \text{het}/T^3 \sim IIB/S^2 \times S^1 \sim M/K3 \sim IIA/S^2 \times S^1. \]  

(1)

This relation of special holonomy manifolds and dualities can be pursued, of course, to lower dimensions. In particular, one might naturally ask for similar relations in four dimensions involving the special holonomy manifold compactifications previously mentioned. In this work, we address the problem of dualities and semi-realistic compactifications, as regards the different holonomy groups. We shall focus mainly in three groups, \(SU(3), G_2\) and \(Spin(7)\) suitable for superstrings, M-theory and F-theory. We shall not be so much concerned with the manifolds themselves [22]. The relation of the above groups with octonions should be apparent; we shall devote some space to study it. We shall exhibit also some exact triple sequences, which we believe illuminate the relations between these groups and several subgroups.

The organization of this paper is as follows. In section 2 we recall the classification of special holonomy manifolds (Berger 1955). In section 3, we review different ways of constructing four dimensional models with minimal number of supercharges from higher dimensional supersymmetric theories. The exact sequences mentioned above are explained in section 4. Background on toroidal M-theory compactification, division algebras and related group manifolds are included in the two appendices.

2 Special holonomy manifolds

2.1 Generalities on holonomy groups

Extended objects, unification of forces and supersymmetry, all suggest extra dimensions of space-time. However we see only \(4 = (1 + 3)\) dimensions: Some mechanism has to be advocated to prevent the extra size of the space to be visible. The compactification is the most accepted ingredient, namely making the extra dimensions too small to be observable. In the original Kaluza-Klein type of theories, the observable gauge group in 4 dimensions came from the isometry groups of the compactifying space (this is why the U(1) gauge group of electromagnetism came from circle compactification). But when supersymmetry is present, it was realized in the early 80s that the holonomy groups of intermediate spaces respond of the number of supercharges surviving in four dimensions [18].
Here we recall briefly the classification of special holonomy groups and manifolds of Berger (1955) in a form suitable for all later physical consideration. Let $\mathcal{M}$ be any $n$ dimensional differentiable manifold. The structure group of the tangent bundle is a subgroup of the general linear group, $GL(n, \mathbb{R})$. Now the maximal compact group of the linear group is $O(n)$. So the quotient homogenous space $\frac{GL(n, \mathbb{R})}{O(n)}$ is a contractible space; hence, a manifold admits always a Riemannian metric $g$, whose tangent structure group is (a subgroup of) the orthogonal group. The isometry group $Iso(\mathcal{M})$ is the set of diffeomorphisms $\sigma$ leaving $g$ invariant. For spheres we have $Iso(S^n) = O(n + 1)$; for torii $Iso(T^n) = U(n)$.

For an arbitrary $n$ dimensional Riemannian manifold $\mathcal{M}$, the structure group of the tangent bundle is, as said, a subgroup of $O(n)$. Carrying a orthonormal frame $\epsilon$ of $n$ vectors in a point $P$ through a closed loop $\gamma$ in the manifold

$$\gamma : P \rightarrow P' \rightarrow P$$

it becomes another frame $\epsilon' = o \cdot \epsilon$ which is shifted by certain element $o$ of $O(n)$. This is the holonomy element of the loop. All elements of all possible loops on the manifold from $P$ make up the holonomy group of the manifold $Hol_P(\mathcal{M})$, which is always a subgroup of $O(n)$, and is easily seen to be independent of the point $P$ for an arcwise-connected manifold. A generic Riemannian manifold would have holonomy $O(n)$, or $SO(n)$ if it is orientable, whereas the isometry group generically is just the identity; in a way isometry and holonomy are complementary.

For any vector bundle with connection, the structure group reduces to the holonomy group (reduction theorem). The corresponding Lie algebra of the holonomy group is generated by the curvature of the connection (the Ambrose-Singer theorem)[19].

### 2.2 Classification of special holonomy manifolds

Only special groups can act as holonomy groups; the classification of possible holonomy groups was carried by M. Berger in 1955. If the manifold is irreducible, $Hol(\mathcal{M})$ should be in $O(n)$. Its Lie algebra, as we said, is generated by the curvature. For the irreducible non symmetric case, there are three double series corresponding to the numbers $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$, and two isolated cases related to the octonion numbers $\mathbb{O}$. For each number domain there are the generic case

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1 Note that the book of Joyce [22] is the best modern reference for this subject.
and the unimodular subgroup restriction. The classification is given in the following Table [15]

| Numbers | Group | Unimodular Form |
|---------|-------|-----------------|
| R       | $O(n)$ | $SO(n)$         |
|         | generic case | orientable, $w_1 = 0$ |
| C       | $U(n)$ | $SU(n)$         |
|         | Kähler, $d\omega = 0$ | Calabi-Yau, $c_1 = 0$ |
| H       | $Sp(1) \times /_2Sp(n)$ | $Sp(n)$ |
|         | Quaternionic | Hyperkähler |
| O       | $Spin(7)$ in 8d spaces | $G_2$ in 7d spaces |
|         | $Oct(1)$ | $\text{AutO}$ |

We provide some explanations. An arbitrary $n$-dimensional Riemannian manifold $\mathcal{M}$ has $O(n)$ as the maximal holonomy group. The obstruction to orientability is measured by the first Stiefel-Witney class of the tangent bundle, $w_1(\mathcal{M}) = w_1(T\mathcal{M}) \in H^1(\mathcal{M}, \mathbb{Z}_2)$.

A $n$-dimensional complex Kähler manifold parameterized by $z_i, i = 1, ..., n$ has a closed regular real Kähler two form $\omega$ given in a local chart by

$$\omega = iw_{ij}dz_i \wedge d\bar{z}_j, \quad d\omega = 0$$

and its holonomy group is $U(n)$. Now as $\frac{U(n)}{SU(n)} = U(1)$, we have the diagram

$$
\begin{array}{ccc}
\text{SU}(n) & \downarrow & \\
\text{U}(n) & \longrightarrow & \text{B} \quad \longrightarrow \quad \text{M} \\
\text{det} \downarrow & & \\
\text{U}(1) & \longrightarrow & \text{B'} \quad \longrightarrow \quad \text{M}
\end{array}
$$

where the middle line is the frame bundle: $B$ is the principal bundle of orthogonal unitary frames. The last bundle is mapped to an element of $H^2(\mathcal{M}, \mathbb{Z})$; hence, the determinant map
defines the first Chern class of the bundle as \( c_1(\mathcal{M}) \in H^2(\mathcal{M}, \mathbb{Z}) \). It turns out that when \( c_1 = 0 \), the Kähler manifold becomes a Calabi-Yau manifold with \( SU(n) \) holonomy group and it is Ricci flat, \( Ric = 0 \). Note that one-dimensional Calabi-Yau manifold is nothing but a complex elliptic curve. Then its Hodge diagram is given by

\[
\begin{array}{ccc}
  h^{0,0} & 1 \\
  h^{1,0} \quad h^{0,1} & 1 & 1 \\
  h^{1,1} & 1 \\
\end{array}
\]  

(5)

The second example of such geometries is the K3 surface with \( SU(2) \) as holonomy group. Its Hodge diagram reads [30] as

\[
\begin{array}{ccc}
  h^{0,0} & 1 \\
  h^{1,0} \quad h^{0,1} & 0 & 0 \\
  h^{2,0} \quad h^{1,1} \quad h^{0,2} & 1 & 20 & 1. \\
  h^{2,1} \quad h^{1,2} & 0 & 0 \\
  h^{2,2} & 1 \\
\end{array}
\]  

(6)

Since \( SU(2) = Sp(1) \), the K3 surface is also a hyperkähler (Calabi-Yau) manifold. Notice that Hyperkähler manifolds are also Calabi-Yau, but the quaternionic manifolds in general are not. Quaternionic manifolds have for holonomy \( Sp(1) \times Sp(n)/\mathbb{Z}_2 \), abbreviated as \( Sp(1) \times /_2Sp(n) \) in the Table.

Finally, the two cases related to the octonions are \( G_2 = Aut(Octonions) \) and \( Spin(7) \). The former is well known and we shall elaborate on it later; as for the "Oct(1)" label for the later, this will also be justified in the Appendix 2.

Note that, in general, a manifold with a specific holonomy group \( Hol(\mathcal{M}) = G \) implies the manifold carries an additional structure, preserved by the group \( G \). For example, an orientable manifold, with holonomy within \( SO(n) \), has an invariant volume element, indeed \( SO(n) = O(n) \cap SL(n, \mathbb{R}) \). A Kähler manifold, with holonomy inside \( U(n) \) has an invariant closed 2-form as \( U(n) = O(2n) \cap Sp(n) \). A manifold with \( G_2 \) holonomy will carry an invariant 3-form, etc.
3 Semi-realistic compactifications

Consider first superstring theory, which lives in ten dimensions [21]; down to four dimensions we want only $\mathcal{N} = 1$, i.e. four supercharges, as to allow for parity violation. We know that compactification on a $SU(n)$-holonomy manifold would reduce the supercharges by a factor of $1/2^{n-1}$, so $SU(3)$-holonomy (i.e., a Calabi-Yau 3-fold) would be just right to descend from the heterotic string (16 supercharges) to a four dimensional model with only four supercharges. Indeed, the search for CY$_3$ manifolds was a big industry in the 80s [20]. This choice is also natural, as $SU(3) \subset SU(4) = Spin(6) \rightarrow SO(6)$, and obviously $4 = 3 + 1$ leaves one surviving spinor.

In M-Theory with 11 dimensions the preferable compactifying manifold would be one with $G_2$ holonomy: now the inclusions are $G_2 \subset SO(7) \leftarrow Spin(7)$, and $8 = 1 + 7$, as $2^{(7-1)/2} = 8$, type real. $G_2$-holonomy manifolds are also Ricci flat; they were first proposed for the M-Theory in [5]. Indeed, $G_2$ holonomy manifolds would preserve $1/2^2$ supercharges, and in 11d there are $2^{(11-1)/2} = 32$, type real again as $10 - 1 = 9 \equiv 1 \text{ mod } 8$.

We can also consider 8d compactifying manifolds in at least two context: 1) Descend 11 → 3 just for illustrative purposes, and 2) F-Theory with metric (1,11); the original sugestion of Vafa was $12 = (2,10)$, see [13, 14]. Here the manifolds of choice would be either CY$_4$, that is, $SU(4)$-holonomy manifolds, preserving 4 charges out of 32 (which is what we want), or $Spin(7)$, the last of the exceptional holonomy groups; $Spin(7)$ does the job as it has an irreducible 8d representation, same as $Spin(8)$ and $Spin(7) \subset Spin(8)$.

Finally, starting from the conventional F-Theory with signature (2,10) it is necessary to compactify in a manifold with signature (1,6). The following Table sums up the situation:

| Theory       | Dim Change | Holonomy                          |
|--------------|------------|-----------------------------------|
| Heter. String| 10d → 4d   | $SU(3)$ (CY$_3$ manifold)         |
| M-theory     | 11d → 4d   | $G_2$ (Ricci flat)                |
| M-theory     | 11d → 3d   | $Spin(7)$ (Ricci flat)            |
| F-theory (1,11) | 12d → 4d | $Spin(7), SU(4)(CY_4)$           |
| F-theory (2,10) | 12d → 4d | Indefinite form of $Spin(7)$      |
Let us comment this table. There is clearly a double inclusion

\[ SU(3)(\text{dim } 8, \text{ rep. } 6) \subset G_2(\text{dim } 14, \text{ rep. } 7) \subset Spin(7)(\text{dim } 21, \text{ rep. } 8) \]  

linked with the dimensions (10, 11 and 12) of the physical (!) theories. It is remarkable that these three groups have a neat relation with octonions, which we shall elaborate now. Also, the related question arises: given the many duality relations extant in string (and M) theories, is there any connection between the different holonomy groups? We shall take up this question in the next section 4.

3.1 Relation with octonions

The necessary properties we need of the division algebra of the octonions \( O \) are described in the Appendix 2. Here we recall that \( G_2 \) is the automorphism group of the octonions (as \( SO(3) \) is \( Aut(H) \) and \( Z_2 = Aut(C) \)); the reals \( R \) have not autos, hence the representation 8 of \( G_2 \) in the octonions split naturally in \( 8 = 1 + 7 \). Note that \( G_2 \) acts transitively in the \( S^6 \) sphere of unit imaginary octonions. This implies the 6-sphere acquires a quasi-complex structure (Borel-Serre). The sequence reads as follows

\[ SU(3) \to G_2 \to S^6 \quad (8 + 6 = 14) \]

Now the octonionic product preserved by \( G_2 \), as any algebra \( (xy = z) \), defines an invariant \( T_2^1 \) tensor and the conservation of the norm is like preserving a quadratic form. The \( T_2^1 \) tensor can be seen then as a \( T_3^0 \) tensor. Now the alternating property of the octonionic product is equivalent to this \( T_3^0 \) tensor to become a 3-form in \( R^7, \wedge T_3^0 \), which is generic, (i.e., they make up an open set). This implies

\[ \dim G_2 = \dim GL(7, R) - \dim \wedge T_3^0 = 49 - 35 = 14. \]

Besides, the dual form \( \wedge T_3^0 \) is also invariant, implying \( G_2 \) is unimodular, i.e. lies inside \( SO(7) \). The dimension 14 of this \( G_2 \) can of course be proved directly [24].

As with respect to \( Spin(7) \), it has a real 8-dimensional representation as we said, and hence it acts in \( S^7 \), indeed transitively. The little group acts in the \( S^6 \) equator, and it is certainly \( G_2 \). In fact, there is some sense, as explained in the Appendix 2, to call \( G_2 \) and \( Spin(7) \) respectively \( SOct(1) \) and \( Oct(1) \).
4 Relation between holonomy groups

We know that strings, M and F theories are related by different sorts of dualities and dimensional reductions. As a consequence we expect that also the different holonomy groups used in the different compactifications should be connected. The aim of this section is to address this question using exact sequences and commutative diagrams for these groups. To start, we note the following. If $H \subset G$ with (left)-coset space $X$, we write $H \to G \to X$ for $G/H = X$; when $H$ is normal, $X$ becomes the quotient group. Some of these diagrams have been already given in [25].

The first diagram that we present here comes from the inclusion of the exceptional holonomy group $G_2 \subset Spin(7)$. The later acts transitively in all units in $O$ (octonions of norm one and octonions with imaginary part of norm one), whereas $G_2 = Aut(O)$ obviously leaves 1 invariant (the real part of the octonion). So the main cross of the diagram takes the following form

$$
\begin{array}{cccc}
\text{Spin}(6) & \downarrow & \text{Spin}(7) & \to \ S^7 \\
G_2 & \to & \text{Spin}(7) & \to \ S^7 \\
& \downarrow & & \downarrow \\
& & S^6 & \equiv S^6
\end{array}
$$

where the vertical line is elemental$^2$. With the $A_3 = D_3$ isomorphism $Spin(6) = SU(4)$ and the fact that $SU(3) \subset G_2 \cap (Spin(6) = SU(4))$, we can complete the previous diagram. The result is given by

$$
\begin{array}{cccc}
SU(3) & \to & SU(4) = Spin(6) & \to \ S^7 \\
\downarrow & & \downarrow & \parallel \\
G_2 & \to & Spin(7) & \to \ S^7 \\
\downarrow & & \downarrow \\
S^6 & \equiv & \equiv & S^6
\end{array}
$$

From this picture we can see in particular the octonionic nature of $SU(3)$. It is a group of automorphism of octonions, fixing the product $(ij)k$. There is a suspicion, still conjectural,
that this is the reason why the gauge group of the strong forces is $SU(3)$ color.

To get the second diagram, we start by another obvious cross, since $Spin(7)$ is the (universal) double cover of $SO(7)$. In this way, we have the following diagram

$$
\begin{array}{ccc}
Z_2 & \longrightarrow & Spin(7) \\
\downarrow & & \downarrow \\
G_2 & \longrightarrow & S^7
\end{array}
$$

(12)

It is known that $G_2$ does not have a centre, so $Z_2 = Z_2$ must be the upper row. The rest is easy to complete since $S^7/Z_2$ is the real projective space $RP^7$. We end up with the following picture

$$
\begin{array}{ccc}
Z_2 & \longrightarrow & Spin(7) & \longrightarrow & S^7 \\
\downarrow & & \downarrow & & \downarrow \\
G_2 & \longrightarrow & SO(7) & \longrightarrow & RP^7
\end{array}
$$

(13)

From this diagram one can learn just the lower row, somewhat unexpected, until one sees the middle row. The lower row is also a remainder that the orthogonal groups have torsion [20].

The third, final, diagram is obtained by asking the question that $Spin(7)$ acts transitively and isometrically in the seventh sphere $S^7$. Indeed, it must be a subgroup of $SO(8)$. What about the quotient (homogeneous space)? To answer this question, we start first with the following incomplete cross

$$
\begin{array}{ccc}
Spin(7) & \longrightarrow & S^7 \\
\downarrow & & \downarrow \\
SO(7) & \longrightarrow & SO(8) & \longrightarrow & S^7
\end{array}
$$

(14)

??
and then try to finish it. Indeed, $G_2$ lies inside both $Spin(7)$ and $SO(7)$, then it must be their intersection and must appear in the upper left corner. The rest of the diagram can be obtained easily, and the result is

$$
\begin{array}{ccc}
G_2 & \rightarrow & Spin(7) \rightarrow S^7 \\
\downarrow & & \downarrow || \\
SO(7) & \rightarrow & SO(8) \rightarrow S^7 \\
\downarrow & & \downarrow \\
RP^7 & \Longrightarrow & RP^7
\end{array}
$$

The new result we learn is just the middle column.

This completes our study of the holonomy groups which are suitable for the compactification of superstrings, M, F-theories respectively. In particular, we have studied their relations with the divisor algebra of octonions. This may reinforce the role of the octonions in physics of strings and higher dimensional objects moving on manifolds with non trivial holonomy groups. We have also found three triple exact sequences explaining some links between these groups. One of the nice results that one gets from the diagrams is that one can also see the possible connections between the corresponding geometries. Indeed, from the following sequence of inclusions

$$SU(3) \rightarrow G_2 \rightarrow S^6,$$

one can see that the manifold with $G_2$ holonomy can be constructed in terms of Calabi-Yau three folds with the $SU(3)$ holonomy group. This has been already discussed in [22]. We can suppose the same thing for the manifold with $Spin(7)$ holonomy. It can be constructed either from manifold with $G_2$ holonomy or Calabi-Yau fourfolds. This can be easily seen from the subdiagram (11).

5 Appendix I: Toroidal Compactification

String theory lives in ten dimensions, maximal supergravity and M-theory in eleven, the original F-theory of Vafa in 12d. As we have seen, if we want to get models with minimal supercharges, we need compact manifolds with special holonomy groups.
To complete the study we give here some information about toroidal compactification. In particular we consider the case of M-theory. We start by recalling the particle content from 11 dimensional supergravity down to 4 or 3 dimensions. We suppose a step-by-step reduction, so the intermediate manifold is always a circle. This was first found by Cremmer (ca. 1980), who also showed that scalars make up a sigma model type of manifold, as homogeneous quasi-euclidean spaces. In particular, M-theory compactified on $T^k$ has U-duality group $E_k(Z)$ and scalars taking values in $E_k/H_k$, where $H_k$ is the maximal compact subgroup of $E_k$. In eleven dimensions, one has a graviton, a 3-form gauge field and the gravitino. We give the moduli space of the toroidal compactification from this theory; starting in dim 9, they are given in the following Table:

\[
\begin{align*}
\text{d=9} & \quad \frac{SL(2,R)}{SO(2)} \times SO(1,1) \text{ (3 scalars)} \\
\text{d=8} & \quad \frac{SL(3,R)}{SO(3)} \times \frac{SL(2,R)}{SO(2)} \text{ (7 scalars)} \\
\text{d=7} & \quad \frac{SL(5,R)}{SO(5)} \text{ (14 scalars)} \\
\text{d=6} & \quad \frac{SO(5,5,R)}{SO(5) \times SO(5)} \text{ (25 scalars)} \\
\text{d=5} & \quad \frac{E_6}{USp(8)} \text{ (42 scalars)} \\
\text{d=4} & \quad \frac{E_7}{SU(8)} \text{ (70 scalars)} \\
\text{d=3} & \quad \frac{E_8}{SO(16)} \text{ (128 scalars)}
\end{align*}
\]

Notice for $d = 3, 4$ and 5 the large group is "reconstructed" by a distinguished representation of the subgroup, as

\[
\begin{align*}
E_8 \sim (adj + spin) \text{ of } O(16) : 120 + 128 = 248 \\
E_7 \sim (adj + [1^4]) \text{ of } SU(8) : 63 + 70 = 133 \\
E_6 \sim (adj + [1^4]') \text{ of } Sp(4) : 36 + (70 - 28) = 78
\end{align*}
\]

Beyond this, the descent from $E_6$ has two branches: the $E$-branch $E_6 \rightarrow D_5 \rightarrow A_4$ and $Sp(4) \rightarrow Sp(2)^2 \rightarrow Sp(2)$ and the octonionic branch $E_8 \rightarrow E_7 \rightarrow E_6 \rightarrow F_4$ with subgroups $O(16) \rightarrow O(12) \times Sp(1) \rightarrow O(10) \times U(1) \rightarrow O(9)$.

6 Appendix II: The octonions

We recall here some properties of division algebras in relation with special holonomy groups and manifolds. Starting with the real numbers $\mathbb{R}$, the space $\mathbb{R}^2$ becomes an algebra $i \equiv \ldots$
\{0, 1\} and \(i^2 = -1\); we get the complex number \( C \). It is a commutative and distributive division algebra. Adding a second unit \( j \), \( j^2 = -1 \) a third \( ij \) is necessary, with \( ij = -ji \), and we obtain the division algebra of quaternions \( H \) in \( R^4 \). It is anticommutative but still distributive. Adding another independent unit \( k \) to \( i \) and \( j \), with \( k^2 = -1, ik = -ki, jk = -kj \), we have to complete with \( e_7 = (ij)k \) to the algebra of octonions \( O \) in \( R^8 \), with units \( 1, i, j, k; ij, jk, ki; (ij)k = -i(jk) \). It is neither commutative nor associative, but still a division algebra: if \( o = u_0 + \sum_{i=1}^{7} u_ie_i \) we have

\[
\bar{o} := u_0 - \sum_{i=1}^{7} u_ie_i \quad \mathcal{N}(o) = \text{norm}(o) := \bar{o}o; \text{inverse } o^{-1} = \frac{\bar{o}}{\mathcal{N}(o)}. \tag{19}
\]

The associator \( \{o_1, o_2, o_3\} := (o_1o_2)o_3 - o_1(o_2o_3) \) is completely antisymmetric. The four algebras \( \mathbb{RCHO} \) are composition algebras, that is, we have \( \mathcal{N}(xy) = \mathcal{N}(x)\mathcal{N}(y) \). The continuous automorphism groups are easily seen to be

\[
\text{Aut}(\mathbb{R}) = 1, \quad \text{Aut}(\mathbb{C}) = Z_2, \quad \text{Aut}(\mathbb{H}) = SO(3), \quad \text{Aut}(\mathbb{O}) := G_2. \tag{20}
\]

The norm-one elements form, for \( \mathbb{RCHO} \), respectively

\[
O(1) = Z_2 = S^0; \quad U(1) = SO(2) = S^1; \quad Sp(1) = SU(2) = Spin(3) = S^3; \quad \text{and } S^7. \tag{21}
\]

Now \( S^7 \) has an invertible product structure, in particular is parallellizable, but is not a group, because nonassociativity. Let us name jokingly \( 'Oct(1)' = S^7 \). One obtains a \textit{bona fide} group by stabilizing \( S^7 \) by the octonion automorphism group \( G_2 \) \cite{27}. The result is \( Spin(7) \approx G_2(\times S^7) \); we shall name \( Spin(7) := Oct(1) \).

We recall now the description of compact Lie groups as finitely twisted products of odd dimensional spheres (Hopf 1941); for details see \cite{28}. For example in the quaternion case one gets the sequence

\[
Sp(1) = Spin(3) = S^3, \quad Sp(1)^2 = Spin(4) = S^3 \times S^3, \quad Sp(2) = Spin(5) = S^3(\times S^7). \tag{22}
\]

There are analogous results for the octonions, after \( G_2 \) stabilization. The series goes up to dim 3, but not beyond. This is due to the lack of associativity. We just write the results, adding the sphere exponents

\[
G_2 = SOct(1) \approx S^3(\times S^{11}); \quad Spin(8) = Oct(1)^2 \approx S^3(\times S^7(\times S^7(\times S^{11}) \tag{23}
\]

\[
Spin(9) := Oct(2) \approx S^3(\times S^7(\times S^{11}(\times S^{15}); \quad F_4 := SOct(3) \approx S^3(\times S^{11}(\times S^{15}(\times S^{23}).
\]
where by the prefix "S" we mean the unimodular restriction (no $S^7$ factors). This is similar to $SO$ and $SU$ for $\mathbb{R}$ and $\mathbb{C}$ respectively. The usefulness of the notation can be seen e.g. in the projective line and plane:

$$HP^1 = S^4 = Sp(2)/Sp(1)^2 \text{ corresponds to } OP^1 = S^8 = Spin(9)/Spin(8),$$
$$CP^2 = S^5/S^1 = SU(3)/SU(2) \text{ corresponds to } OP^2 = SOct(3)/Oct(2) = F_4/Spin(9).$$

The later is called the Moufang plane (Moufang 1933; to call it the Cayley plane is historically inaccurate).

In any case, this is just a notational convention, that we find useful, if carefully employed. We finish by remarking that no use has been made so far of the fundamental triality property of the $O(8)$ group and the octonions, namely $Out[Spin(8)] \equiv Aut/Inner = S_3$, the order three symmetric group. Perhaps in a deeper analysis this triality will show up in particle physics.

**Acknowledgments.** AB would like to thank R. Ahl Laamara, B. Belhorma, P. Diaz, L.B. Drissi, M. P. Garcia de Moral, J. Rasmussen, E.H. Saidi, A. Sebbar and M. B. Sedra for collaboration on related subject. He would also thank GNPHE and UFR-PHE-Rabat for hospitality (the end of 2007). AS acknowledges discussions with P. Diaz, M.P. Garcia de Moral and E.H. Saidi. This work has been supported by CICYT (grant FPA-2006-02315) and DGIID-DGA (grant 2007-E24/2), Spain. We thank also the support by Fisica de altas energias: Particulas, Cuerdas y Cosmologia, A9335/07.

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