ASYMPTOTIC CHOW SEMISTABILITY IMPLIES DING POLYSTABILITY
FOR GORENSTEIN TORIC FANO VARIETIES

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Abstract. In this paper, we prove that a Gorenstein toric Fano variety \((X, -K_X)\) is asymptotically Chow semistable, then it is Ding polystable with respect to toric test configurations (Theorem 1.3). This extends the known result obtained by others (Theorem 1.2) to the case where \(X\) admits Gorenstein singularity. We also show the additivity of the Mabuchi constant for the product toric Fano varieties in Proposition 1.5 based on the author’s recent work (Ono, Sano and Yotsutani in arXiv:2305.05924). Applying this formula to certain toric Fano varieties, we construct infinitely many examples that clarify the difference between relative K-stability and relative Ding stability in a systematic way (Proposition 1.4). Finally, we verify relative Chow stability for Gorenstein toric del Pezzo surfaces using the combinatorial criterion developed in (Yotsutani and Zhou in Tohoku Math. J. 71 (2019), 495-524.) and specifying the symmetry of the associated polytopes as well.

1. INTRODUCTION

Let \((X, L)\) be a polarized projective variety of complex dimension \(n\). One of the outstanding problem in Kähler geometry is to distinguish whether the first Chern class \(c_1(L)\) contains a Kähler metric \(\omega\) with constant scalar curvature (cscK metric). A parallel reasoning question in algebraic geometry is to study an appropriate notion of stability of \((X, L)\) in the sense of Geometric Invariant Theory (GIT). This leads us to investigate various notions of GIT-stability and study the relation among them. For example, Ross-Thomas clarified the following implications among GIT-stability in their paper [RT07]:

Asymptotic Chow stability \(\Rightarrow\) Asymptotic Hilbert stability
\Rightarrow Asymptotic Hilbert semistability \(\Rightarrow\) Asymptotic Chow semistability
\Rightarrow K-semistability.

In [Mab08], Mabuchi proved that Chow stability and Hilbert stability asymptotically coincide. Remark that for a fixed positive integer \(i \in \mathbb{Z}_+\), Chow stability for \((X, L^{\otimes i})\) implies Hilbert stability for \((X, L^{\otimes i})\) (i.e. not necessarily asymptotic stability case) by the classical result due to Forgaty [Fo69]. See also [KSZ92, Corollary 3.4] for more combinatorial description of this result in terms of GIT weight polytopes.

In order to describe our issue more precisely, we first recall that a complex normal variety \(X\) is said to be Fano if its anticanonical divisor \(-K_X\) is ample. It is called Gorenstein if \(-K_X\) is Cartier. Suppose that \(X\) is a smooth Fano variety (i.e. a Fano manifold) with a Kähler metric

\[
\omega = \sqrt{-1}g_{ij}dz_i \wedge d\bar{z}_j \in 2\pi c_1(X).
\]

We recall that \(\omega_{\varphi} = \omega + \sqrt{-1}\partial\bar{\partial} \varphi\) is Kähler-Einstein if and only if \(\varphi\) is a critical point of either the K-energy \(\nu_{\omega}\) or the Ding functional \(D_{\omega}\) where both functionals defined on the space of Kähler potentials \(H_{\omega} = \{ \varphi \in C^\infty(X) \mid \omega_{\varphi} > 0 \}\). It is known that these functionals satisfy the inequality \(D_{\omega} \leq \nu_{\omega}\) which turns out to be the Ding invariant is
less than or equal to the Donaldson-Futaki (DF) invariant \[ \text{Be16} \]. In the case where \( X \) is toric, Yao gave explicit description of the inequality between the Ding invariant and the DF invariant in terms of the associated polytope \[ \text{Yao21, Proposition 4.6} \]. In particular, Ding polystability implies K-polystability for a toric Fano manifold. Moreover, the converse direction has been also proved in \[ \text{Fu16} \] for (not necessarily toric) Fano manifolds.

**Theorem 1.1** (Fujita). Let \( X \) be a Fano manifold. Then Ding semistability is equivalent to K-semistability. Furthermore, Ding polystability (resp. Ding stability) is also equivalent to K-polystability (resp. K-stability).

On the one hand, in differential-geometrical point of view, Theorem 1.1 corresponds to the fact that cscK metrics in the anticanonical classes of Fano manifolds are Kähler-Einstein metrics. Recall that for a compact Kähler manifold \( X \) with a fixed Kähler class \( [\omega] \), \( \varphi \) is a critical point of \( \nu_\omega \) if and only if \( \omega_\varphi \) is a cscK metric. On the other hand, we conclude that if a Fano manifold \( X \) is asymptotically Chow semistable then it is Ding semistable according to the previous argument. In the case where \( X \) is a toric Fano manifold, it is known that \( X \) is K-semistable if and only if it is K-polystable \[ \text{Be16, BJ20, WZ04} \]. Summing up these arguments, we have the following.

**Theorem 1.2** (Berman, Ono, Yao). Let \( (X, -K_X) \) be a smooth toric Fano variety. If \( (X, -K_X) \) is asymptotically Chow semistable with respect to toric test configurations, then it is Ding polystable with respect to toric test configurations.

In this article, we show that a more general result by a combinatorial proof.

**Theorem 1.3.** Let \( (X, -K_X) \) be a Gorenstein toric Fano variety. If \( (X, -K_X) \) is asymptotically Chow semistable with respect to toric test configurations, then it is Ding polystable with respect to toric test configurations.

Essentially, the proof of Theorem 1.3 is based on the Ehrhart reciprocity law and the fact that any toric Fano variety is K-polystable if and only if the barycenter of the associated reflexive polytope \( \Delta \subseteq M_\mathbb{R} \) is the origin. As we mentioned above, another advantage of our combinatorial approach is that \( X \) may admit Gorenstein singularity (i.e. not necessarily smooth) in our main theorem. However, it does not work for a \( \mathbb{Q} \)-Gorenstein toric variety since the corresponding polytope \( \Delta \) contains not only the origin, but also other lattice points. It also should be noted that we only assume \( (X, -K_X) \) to be asymptotically Chow semistable and do not assume \( (X, -K_X) \) to be asymptotically Chow polystable in Theorem 1.3.

In the following Section 4, we discuss relative stability of toric Fano variety. Recently, we found that there are at least four examples of smooth toric Fano variety which clarify the difference between relative K-stability and relative Ding stability in \[ \text{NSY23} \]. In order to discover these four examples of a relatively K-polystable toric Fano variety, but which is relatively Ding unstable, the author focused on the geometrical description such that they are all \( \mathbb{P}^1 \)-bundles over \( \mathbb{P}^n \). In particular, we consider the case of Picard number one projective toric varieties. Based on a recent argument discussed in \[ \text{OSY23} \], we systematically construct such examples in arbitrary dimension.

**Proposition 1.4** (See, Corollary 4.9). Fixing a positive integer \( r \), we consider an extremal smooth toric Fano variety \( X_k \) with the associated polytope \( \Delta_k \), for \( 1 \leq k \leq r \). Suppose \( \theta_{\Delta_k}(x_k) \) be the potential function of \( \Delta_k \) defined in \[ \text{4.1} \] with \( \frac{1}{r} \leq \theta_{\Delta_k} < 1 \). For the product polytope \( \Delta = \prod_{k=1}^r \Delta_k \), the associated smooth toric Fano variety \( (X_\Delta, -K_{X_\Delta}) \) is relatively K-polystable, but it is relatively Ding unstable.
In order to prove Proposition 1.4, we shall use the following additive property of the Mabuchi constant $M_X$, for the products of toric Fano varieties.

**Proposition 1.5** (See, Corollary 4.6). For the product polytope $\Delta$ of reflexive polytopes $\Delta_k$ for $k = 1, \ldots, r$, let $M_{X_\Delta}$ and $M_{X_{\Delta_k}}$ be the Mabuchi constant defined in (4.9). Then we have the equality

$$M_{X_\Delta} = M_{X_{\Delta_1}} + \cdots + M_{X_{\Delta_r}}.$$ 

We give a purely combinatorial proof of Proposition 1.5 in Section 4.3. In the following Section 4.4 we classify Gorenstein toric del Pezzo surfaces in terms of (asymptotic) relative Chow polystability. We use the criteria (4.3) to verify asymptotic relative Chow stability of polarized toric variety. However, it is very difficult to verify asymptotic relative Chow stability of a given polarized toric variety because we have to prove that there exists $t_i \in \mathbb{R}$ satisfying the equality in (4.3) for any positive integer $i$. In order to solve this difficulty, we consider the symmetry of the associated polytopes $\Delta \subset M_\mathbb{R}$ which works very well for two dimensional reflexive polygons (16 types). Adapting the symmetry of reflexive polygons and a combinatorial criteria (4.3) investigated by Zhou and the author in [YZ19], we verify relative Chow stability of each Gorenstein toric del Pezzo surfaces.

**Proposition 1.6** (See, Proposition 1.7). Among all 16 isomorphism classes of Gorenstein toric del Pezzo surfaces, there are 5 isomorphism classes of asymptotically relatively Chow polystable surfaces and 4 isomorphism classes of asymptotically relatively Chow unstable surfaces. The remaining 7 classes are relatively Chow polystable with respect to the anticanonical polarization.

All the results are listed in Table 1. We also refer the reader to Table 2 for specifying symmetry of each reflexive polygon $\Delta \subset M_\mathbb{R}$.

This paper is organized as follows. Section 2 is a brief review of Gorenstein toric Fano varieties, Ding stability and asymptotic Chow stability. The proof of Theorem 1.2 is given in Section 3. Section 4 collects the results of relative algebro-geometric stability. In Sections 4.1 and 4.2, we recall the criteria of relative Chow stability of polarized toric variety investigated by the author and B. Zhou in [YZ19]. We prove Proposition 1.5 in Section 4.3 by applying the product formulas regarding convex polytopes which was also used in [OSY23]. See Lemma 4.4 and the proof of Proposition 4.5 for further details. Section 4.4 is devoted to verify asymptotic relative Chow stability of Gorenstein toric del Pezzo surfaces. All the results and practical values of invariants are summarized in Proposition 4.7 and Table 1.

2. Preliminary

2.1. Gorenstein toric Fano varieties. We first recall the standard notation and basic definitions of Gorenstein toric Fano varieties, as it can be found in [CLS11].

Let $N \cong \mathbb{Z}^n$ be a lattice of rank $n$, while $M = \text{Hom}(N, \mathbb{Z})$ is the $\mathbb{Z}$-dual of $M$. Let $P \subseteq N_\mathbb{R} \cong \mathbb{R}^n$ be a lattice polytope with $0 \in \text{Int}(P)$. We assume that all vertices of $P$ are primitive elements in $N$. For a subset $S$ of $N_\mathbb{R}$, we denote the positive hull of $S$ by $\text{pos}(S)$ i.e. $\text{pos}(S) = \sum_{v \in S} \mathbb{R}_{\geq 0} v$. Then

$$\Sigma_P := \{ \text{pos}(F) \mid F \text{ is a face of } P \}$$

forms the fan which is often called the normal fan of $P$. It is well-known that the fan $\Sigma = \Sigma_P$ associates a toric variety $X_\Sigma$ with the complex torus $T_M := \text{Spec} \mathbb{C}[M]$ action. Here and hereafter we denote the associated toric variety by $X$ for simplicity. Recall that the anticanonical divisor of $X$ is given by $-K_X = \sum_{\rho} D_\rho$ where $D_\rho$ is the torus invariant

\[ ... \]
Weil divisor corresponding to a ray \( \rho \in \Sigma(1) \). Then the dual polytope of \( P \) (w.r.t. \(-K_X\)) is defined by
\[
\Delta = \{ y \in M_\mathbb{R} \mid \langle x, y \rangle \geq -1 \text{ for all } x \in P \}
\]
which is also an \( n \)-dimensional (rational) polytope in \( M_\mathbb{R} \) with \( 0 \in \text{Int}(\Delta) \). Then \( \Delta \) is called reflexive if it is a lattice polytope. There is a bijective correspondence between isomorphism classes of reflexive polytopes and isomorphism classes of Gorenstein toric Fano varieties. For a fixed dimension \( n \), there are only finitely many isomorphism classes of \( n \)-dimensional reflexive polytopes \([KS98, KS00]\). They found 1, 16, 4319 and 473800776 isomorphism classes for \( n = 1, 2, 3 \) and 4. Throughout the paper, we assume that a (toric) Fano variety \( X \) admits at worst Gorenstein singularities.

2.2. Ding stability for Fano varieties. In this section, we briefly review a notion of Ding stability, see \([Be16, Fu16, Yao21]\) for more details.

Let \((X, \omega)\) be an \( n \)-dimensional Fano manifold with a Kähler metric \( \omega \in 2\pi c_1(X) \). We set \( V \) to be the volume \( V := \int_X \omega^n \) of the given Fano manifold \( X \). Recall that the Ding functional \( D_\omega : \mathcal{H}_\omega \to \mathbb{R} \) is given by
\[
D_\omega := -\frac{1}{V} \int_0^1 \int_X \varphi_t(1 - e^{\rho_\omega})\omega_t^{n-1} \, dt,
\]
where \( \varphi_t \) is a smooth path in \( \mathcal{H}_\omega \) joining 0 with \( \varphi \) and \( \rho_\omega \) is the function which satisfies
\begin{equation}
\text{Ric}(\omega) - \omega = \sqrt{-1} \partial \bar{\partial} \rho_\omega \quad \text{and} \quad \int_X (e^{\rho_\omega} - 1)\omega^n = 0.
\end{equation}
Then we readily see that \( \varphi \) is a critical point of \( D_\omega \) if and only if \( \omega_\varphi \) is a Kähler-Einstein metric.

Next we recall a notion of a test configuration. A test configuration for a Fano variety \((X, -K_X)\) is a polarized scheme \((\mathcal{X}, \mathcal{L})\) with:

- a \( \mathbb{C}^\times \)-action and a \( \mathbb{C}^\times \)-equivariant proper flat morphism \( \pi : \mathcal{X} \to \mathbb{C} \), where \( \mathbb{C}^\times \) acts on the base by multiplication.
- a \( \mathbb{C}^\times \)-equivariant line bundle \( \mathcal{L} \to \mathcal{X} \) which is ample over all fiber \( \mathcal{X}_z := \pi^{-1}(z) \) for \( z \neq 0 \), and \((X, -K_X)\) is isomorphic to \((\mathcal{X}_z, \mathcal{L}_z)\) with \( \mathcal{L}_z = \mathcal{L}|_{\mathcal{X}_z} \).

Taking a Hermitian metric \( h_0 \) on \( \mathcal{O}_X(-K_X) \) with positive curvature, we can construct the Phong-Sturm geodesic ray \( h_t \) which emanates from \( h_0 \) in \( \mathcal{H}_\omega \) \([PS07]\). In \([Be16]\), Berman defined the Ding invariant as the asymptotic slope of the Ding functional along the geodesic rays. Moreover he showed that
\[
DF(\mathcal{X}, \mathcal{L}) = \lim_{t \to \infty} \frac{1}{V} \frac{dD_\omega(h_t)}{dt} + q
\]
where the error term \( q \) is non-negative and \( DF(\mathcal{X}, \mathcal{L}) \) is the Donaldson-Futaki invariant. Then the Ding invariant \( \text{Ding}(\mathcal{X}, \mathcal{L}) \) is given by
\[
\text{Ding}(\mathcal{X}, \mathcal{L}) = \lim_{t \to \infty} \frac{1}{V} \frac{dD_\omega(h_t)}{dt}.
\]
A Gorenstein Fano variety \( X \) is said to be Ding-semistable if for any test configuration \((\mathcal{X}, \mathcal{L})\) for \((X, -K_X)\), we have \( \text{Ding}(\mathcal{X}, \mathcal{L}) \geq 0 \). Moreover \( X \) is said to be Ding polystable if \( X \) is Ding semistable and \( \text{Ding}(\mathcal{X}, \mathcal{L}) = 0 \) if and only if \((X, \mathcal{L})\) is equivariantly isomorphic to \((X \times \mathbb{C}, p_1^*(\mathcal{O}_X(-K_X)))\) where \( p_1 : X \times \mathbb{C} \to X \) is the projection.

Now we consider the toric case. Let \( X \) be an \( n \)-dimensional toric Fano variety and \( \Delta \subseteq M_\mathbb{R} \) the corresponding reflexive polytope with the coordinates \( x = (x_1, \ldots, x_n) \). Recall
that a piecewise linear convex function \( u = \max \{ f_1, \ldots, f_\ell \} \) on \( \Delta \) is called rational if \( f_\ell = \sum a_k x_i + c_k \) with \( (a_k,1, \ldots, a_k,n) \in \mathbb{Q}^n \) and \( c_k \in \mathbb{Q} \) for \( k = 1, \ldots, \ell \). A toric test configuration for \( (X, -iK_X) \), introduced by Donaldson [Do02], is a test configuration associated with a rational piecewise linear convex function \( u \) on \( \Delta \), so that \( iQ \) is a lattice polytope in \( \mathbb{M}_\mathbb{R} \times \mathbb{R} \cong \mathbb{R}^{n+1} \). Here \( Q \) is given by

\[
Q = \{ (x,t) \mid x \in \Delta, \ 0 \leq t \leq R - u(x) \}
\]

and \( R \) is an integer such that \( u \leq R \). Then \( iQ \) defines the \( n+1 \)-dimensional polarized toric variety \( (\overline{X}, \overline{L}) \) and a flat morphism \( \overline{X} \to \mathbb{C}P^1 \). Hence the family restricted to \( \mathbb{C} \) gives a torus equivariant test configuration \( (\mathcal{X}, \mathcal{L}) \) for \( (X, -iK_X) \).

The toric geodesic ray \( h_t \) associated to a toric test configuration was described by Song-Zeldich [SoZe12]. In [Yao21], Yao detected an explicit description of the Ding invariants of toric Fano varieties.

**Theorem 2.1 (Yao).** Let \( (X, -K_X) \) be a Gorenstein toric Fano variety with the associated reflexive polytope \( \Delta \). Let \( u \) be a piecewise linear convex function. The Ding invariant of the toric test configuration associated to \( u \) is given by

\[
\text{Ding}(\mathcal{X}, \mathcal{L}) = \lim_{t \to \infty} \frac{1}{\text{vol}(\Delta)} \frac{dD_\omega(h_t)}{dt} = -u(0) + \frac{1}{\text{vol}(\Delta)} \int_\Delta u(x) \, dv =: \mathcal{I}_\Delta(u).
\]

Then a reflexive polytope \( \Delta \subseteq \mathbb{M}_\mathbb{R} \) is said to be Ding polystable if \( \mathcal{I}_\Delta(u) \geq 0 \) for all convex piecewise linear functions \( u \) and the equality holds if and only if \( u \) is affine linear. One can observe that \( \mathcal{I}_\Delta(u) \) is invariant when we add affine linear functions to convex piecewise linear functions. Hence it suffices to consider normalized convex piecewise linear functions \( u \) on \( \Delta \) for our purpose, that is, \( u(x) \geq u(0) = 0 \). The following observation was given by Yao [Yao21], and we write down the detail for the reader’s convenience.

**Proposition 2.2 (Yao).** If \( \Delta \) is a reflexive polytope, then the associated Gorenstein toric Fano variety \( (X, -K_X) \) is Ding polystable if and only if the barycenter of \( \Delta \) is 0.

**Proof.** Suppose \( \Delta \) is Ding polystable. Hence

\[
\frac{1}{\text{vol}(\Delta)} \int_\Delta u(x) \, dv \geq 0
\]

for any normalized convex piecewise linear function \( u \). Applying (2.3) to linear functions, i.e. \( u = \pm x_i \) for \( i = 1, \ldots, n \) we conclude \( \int_\Delta x \, dv = 0 \).

Conversely we assume that \( \int_\Delta x \, dv = 0 \). Then for any normalized convex piecewise linear function \( u \), Jensen’s inequality implies that

\[
\int_\Delta u(x) \, dv \geq u(\int_\Delta x \, dv) = u(0) = 0.
\]

Hence \( \Delta \) is Ding polystable. \( \square \)

### 2.3. Asymptotic Chow stability of toric varieties.

In this section, let us briefly recall notion of Chow stability, see [On11] [Yo16] for more details.

Let \( X \subseteq \mathbb{C}P^n \) be an \( n \)-dimensional irreducible complex projective variety of degree \( d \geq 2 \). Recall that for a projectively embedded \( n \)-dimensional complex subvariety \( X \subseteq \mathbb{C}P^n \), the degree \( d \) of \( X \) is a number of intersection of \( X \) with a linear subspace \( L \) in general
position, such that \( n + \dim L = N \). Let us denote the Grassmann variety by \( \mathbb{G}(k, \mathbb{C}P^N) \).

We define the associated hypersurface of \( X \subset \mathbb{C}P^N \) by

\[
Z_X := \{ L \in \mathbb{G}(N - n - 1, \mathbb{C}P^N) \mid L \cap X \neq \emptyset \}.
\]

Remark that the construction of \( Z_X \) can be regarded as an analog of the projective dual varieties as in [GKZ94, Chapter 1]. In fact, it is well known that \( Z_X \) is an irreducible divisor in \( \mathbb{G}(N - n - 1, \mathbb{C}P^N) \) with \( \deg Z_X = d \) in the Plücker coordinates. Therefore there exists \( R_X \in H^0(\mathbb{G}(N - n - 1, \mathbb{C}P^N), \mathcal{O}_G(d)) \) such that \( Z_X = \{ R_X = 0 \} \). We call \( R_X \) the \( X \)-resultant. Since there is a natural action of \( SL(N + 1, \mathbb{C}) \) on \( H^0(\mathbb{G}(N - n - 1, \mathbb{C}P^N), \mathcal{O}_G(d)) \), we define GIT-stability for the \( X \)-resultant \( R_X \) as follows.

**Definition 2.3.** Let \( X \subset \mathbb{C}P^N \) be an \( n \)-dimensional irreducible complex projective variety. \( X \) is said to be Chow semistable if the closure of \( SL(N + 1, \mathbb{C}) \)-orbit of the \( X \)-resultant \( R_X \) does not contain the origin. \( X \) is said to be Chow polystable if the orbit \( SL(N + 1, \mathbb{C}) \cdot R_X \) is closed. We call \( X \) Chow unstable if it is not Chow semistable.

**Definition 2.4.** Let \((X, L)\) be a polarized projective variety. For \( i \gg 0 \), we denote the Kodaira embedding by \( \Psi_i : X \to \mathbb{P}(H^0(X, L^i))^* \). \((X, L)\) is said to be asymptotically Chow semistable (resp. polystable) if there is an \( i_0 \) such that \( \Psi_i(X) \) is Chow semistable (resp. polystable) for each \( i \geq i_0 \). \((X, L)\) is called asymptotically Chow unstable if it is not asymptotically Chow semistable.

Next we will give a quick review on Ono’s necessary condition for Chow semistability of polarized toric varieties. Let \( \Delta \) be an \( n \)-dimensional lattice polytope in \( M_\mathbb{R} \cong \mathbb{R}^n \). The Euler-Maclaurin summation formula for polytopes provides a powerful connection between integral over a polytope \( \Delta \) and summation of lattice points in \( \Delta \). More specifically, for any polynomial function \( \phi \) on \( \mathbb{R}^n \), we would like to see how the summation

\[
\sum_{\mathbf{a} \in \Delta \cap (\mathbb{Z}/i)\mathbb{R}^n} \phi(\mathbf{a}) =: I(\phi, \Delta)(i)
\]

will behave for a positive integer \( i \). If we take \( \phi \) to be 1, \( I(\phi, \Delta)(i) \) is so-called the Ehrhart polynomial which counts the number of lattice points in \( i \)-th dilation of a polytope \( \Delta \):

\[
I(1, \Delta)(i) = \#(\Delta \cap (\mathbb{Z}/i)\mathbb{R}^n).
\]

Recall that the Ehrhart polynomial has an expression

\[
E_\Delta(t) := I(1, \Delta)(t) = \vol(\Delta)t^n + \frac{\vol(\partial \Delta)}{2} t^{n-1} + \cdots + 1
\]

where \( \partial \Delta \) is the boundary of a lattice polytope \( \Delta \). Similarly if we take \( \phi \) to be the coordinate functions \( \mathbf{x} = (x_1, \ldots, x_n) \), then \( I(\phi, \Delta)(i) \) counts the weight of lattice points in \( i \)-th dilation of a polytope \( \Delta \):

\[
I(\mathbf{x}, \Delta)(i) = \sum_{\mathbf{a} \in \Delta \cap (\mathbb{Z}/i)\mathbb{R}^n} \mathbf{a}.
\]

Similar to the Ehrhart polynomial, it is also known that (2.5) gives the \( \mathbb{R}^n \)-valued polynomial satisfying

\[
s_\Delta(t) := I(\mathbf{x}, \Delta)(t)
= t^n \int_\Delta \mathbf{x} \, dv + \frac{t^{n-1}}{2} \int_{\partial \Delta} \mathbf{x} \, d\sigma + \cdots + c,
\quad s_\Delta(i) = \sum_{\mathbf{a} \in \Delta \cap (\mathbb{Z}/i)\mathbb{R}^n} \mathbf{a}
\]
for any positive integer $i$. We call $s_{\Delta}(t)$ the lattice points sum polynomial. The following necessary condition of Chow semistability of projective toric varieties was obtained in [On11].

**Theorem 2.5** (Ono). Let $\Delta$ be a lattice polytope, $E_{\Delta}(t)$ the Ehrhart polynomial and $s_{\Delta}(t)$ the lattice points sum polynomial. We fix a positive integer $i \in \mathbb{Z}_+$. If the associated toric variety $X$ with respect to $L^{\otimes i}$ is Chow semistable, then the following equality holds:

$$s_{\Delta}(i) = \frac{E_{\Delta}(i)}{\text{vol}(\Delta)} \int_{\Delta} x \, dv. \tag{2.6}$$

Suppose a projective polarized toric variety $(X, L)$ associated with a lattice polytope $\Delta$ is asymptotically Chow semistable. Then there is an $i_0 \in \mathbb{Z}_+$ such that (2.6) holds for any positive integer $i \geq i_0$. On the other hand, we observe that $E_{\Delta}(t)$ and $s_{\Delta}(t)$ are ($\mathbb{R}^n$-valued) polynomials. Hence polynomial identity theorem gives the following (see also [On11, Theorem 1.4]).

**Lemma 2.6.** Let $\Delta$ be a lattice polytope. If the associated projective polarized toric variety $(X, L)$ is asymptotically Chow semistable, then (2.6) holds for any (not-necessarily-positive) integer $i \in \mathbb{Z}$.

### 3. Proof of Theorem 1.3

#### 3.1. Ehrhart reciprocity law for polynomial functions.

Let $\Delta$ be an $n$-dimensional lattice polytope in $M_{\mathbb{R}} \cong \mathbb{R}^n$ and $\phi$ a polynomial function on $\mathbb{R}^n$. As in Section 2.3, we consider

$$I(\phi, \Delta)(i) = \sum_{a \in \Delta \cap (\mathbb{Z}/i)^n} \phi(a)$$

and

$$I(\phi, \text{Int}(\Delta))(i) = \sum_{a \in \text{Int}(\Delta) \cap (\mathbb{Z}/i)^n} \phi(a)$$

for a positive integer $i$. Remark that $I(1, \text{Int}(\Delta))(i) = \#(\text{Int}(\Delta) \cap (\mathbb{Z}/i)^n)$. The classical result of the Ehrhart reciprocity law says that the following equality holds for any positive integer $i \in \mathbb{Z}_+$:

$$I(1, \text{Int}(\Delta))(i) = (-1)^n I(1, \Delta)(-i).$$

Brion and Vergne gave the following beautiful generalization of this reciprocity law [BV97].

**Theorem 3.1** (Brion-Vergne). Let $\Delta$ be an $n$-dimensional lattice polytope. If $\phi$ is a homogeneous polynomial function of degree $d$ on $\Delta$, then the following reciprocity law

$$I(\phi, \text{Int}(\Delta))(i) = (-1)^{n+d} I(\phi, \Delta)(-i) \tag{3.1}$$

holds for any positive integer $i \in \mathbb{Z}_+$.

we use this result for proving the following.

**Lemma 3.2.** Let $\Delta$ be an $n$-dimensional reflexive polytope in $M_{\mathbb{R}}$. Let $E_{\Delta}(t)$ be the Ehrhart polynomial, $s_{\Delta}(t)$ the lattice point sum polynomial respectively. Then we have

$$E_{\Delta}(-1) = (-1)^n \quad \text{and} \quad s_{\Delta}(-1) = 0.$$

**Proof.** We note that $\text{Int}(\Delta) \cap \mathbb{Z}^n = \{0\}$ since $\Delta$ is a reflexive polytope. Taking $\phi = 1$ and $i = 1$ in (3.1), we have

$$E_{\Delta}(-1) = (-1)^n \cdot \#(\text{Int}(\Delta) \cap \mathbb{Z}^n) = (-1)^n.$$
Similarly, if we take \( \phi = x \) and \( i = 1 \), then (3.1) becomes
\[
\Delta (-1) = (-1)^{n+1} \sum_{a \in \text{Int}(\Delta) \cap \mathbb{Z}} a = 0.
\]

3.2. A combinatorial proof. Now we prove Theorem 1.3.

Proof of Theorem 1.3. If a Gorenstein toric Fano variety \((X, -K_X)\) is asymptotically Chow semistable, then (2.6) holds for any integer \( i \in \mathbb{Z} \), by Lemma 2.6. Taking \( i = -1 \) in (2.6), we have
\[
\int_{\Delta} x \, dv = 0
\]
by Lemma 3.2. Thus Proposition 2.2 implies that \((X, -K_X)\) is Ding polystable. This completes the proof.

3.3. Conclusion of the proof of Theorem 1.3. If \( \Delta \) is a simple reflexive polytope, then the corresponding toric Fano variety \((X, -K_X)\) may admit only orbifold singularities. Combining Theorem 1.3 and the result of [ShZh12, CS15], we conclude the following.

Corollary 3.3. Let \( X \) be a toric Fano orbifold. If \((X, -K_X)\) is asymptotic Chow semistable, then \( X \) admits a Kähler-Einstein metric in \( c_1(-K_X) \).

We finish this section with the following example which illustrates the combinatorial proof of Theorem 1.3 by using a Gorenstein toric del Pezzo surface.

Example 1. Let \( \Delta \) be the polygon in \( M_\mathbb{R} \cong \mathbb{R}^2 \) whose vertices are given by
\[
\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\},
\]
which is the polytope labeled with 9 in Table 2. Then the associated polarized toric variety \((X, L)\) is the cubic surface
\[
X = \{ [x : y : z : w] \in \mathbb{P}^3 \mid xyz = w^3 \}
\]
with the anticanonical line bundle \( L = \mathcal{O}_X(-K_X) \). It is known that \((X, L^0)\) is Chow polystable for any integer \( i > 0 \) by Theorem 1.2 (3) in [LLSW19]. Thus, \((X, -K_X)\) is asymptotically Chow semistable.

Let us compute the \( \mathbb{R}^2 \)-valued polynomial function \( s_\Delta(t) \). Firstly, the straight forward computation shows that
\[
\int_{\Delta} x \, dv = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]
for the standard volume form \( dv = dx \wedge dy \) of \( M_\mathbb{R} \). Secondary, we shall compute \( \int_{\partial \Delta} x \, d\sigma \) (which is equal to the second leading coefficient of \( s_\Delta(t) \) ) as follows: the polygon \( \Delta \) has three facets \( F_i = \{ x \in \Delta \mid \ell_i(x) = 0 \} \) for \( i = 1, 2, 3 \) whose defining equations are given by
\[
\ell_1(x) = 1 - x - y, \quad \ell_2(x) = 1 + 2x - y, \quad \text{and} \quad \ell_3(x) = 1 - x + 2y,
\]
respectively. Then the boundary measure \( d\sigma_i \) on each facet \( F_i \) is determined by
\[
dv = \pm d\sigma_i \wedge d\ell_i.
\]
Thus, (3.2) yields that we can take
\[
d\sigma_1 = -dx, \quad d\sigma_2 = -dx, \quad \text{and} \quad d\sigma_3 = \frac{1}{2} dx.
as the boundary measures on $\partial \Delta$. Consequently, the $x$-coordinates of the barycenter of each facet $F_i$ is given by

$$
\int_{F_1} x \, d\sigma_1 = \int_1^0 x(-dx) = \frac{1}{2}, \quad \int_{F_2} x \, d\sigma_2 = \int_{-1}^0 x \, dx = -\frac{1}{2}
$$

and

$$
\int_{F_3} x \, d\sigma_3 = \frac{1}{2} \int_{-1}^1 x \, dx = 0,
$$

respectively. By the symmetry of $\Delta$, we find that

$$
\int_{\partial \Delta} x \, d\sigma = \left(\frac{1}{2}, \frac{1}{2}\right) + \left(-\frac{1}{2}, 0\right) + \left(0, -\frac{1}{2}\right) = \left(0, 0\right).
$$

Hence, $s_\Delta(t)$ has the form of

\begin{equation}
(3.3) \quad s_\Delta(t) = \left(0, 0\right) t^2 + \left(0, 0\right) t + \left(c_1, c_2\right)
\end{equation}

for some constants $c_1$ and $c_2$. See (2.5). In order to determine $c_1$ and $c_2$, we plug the value of $s_\Delta(1) = \left(0, 0\right)$ into (3.3) which yields that $c_1 = c_2 = 0$. Thus, we see that $s_\Delta(t) \equiv 0$ and this is consistent with the Ehrhart reciprocity low

$$
s_\Delta(-1) = (-1)^3 \cdot \sum_{\mathbf{a} \in \text{Int}(\Delta) \cap \mathbb{Z}^2} \mathbf{a} = \left(0, 0\right).
$$

Moreover, we already see that $\int_{\Delta} x \, dv = \left(0, 0\right)$ in the above computation. Consequently, $(X, -K_X)$ is Ding polystable by Proposition 2.2.

4. Relative algebro-geometric stability

In order to deal with the existence problem of extremal Kähler metrics, the definition of K-stability was extended by Székelyhidi in $[Sz07]$ to Kähler classes with non-vanishing Futaki invariant which was called relative K-stability. Analogously, we can extend the notion of Chow stability to relative Chow stability which has been also investigated by many researchers $[Se17, Ha19]$.

In this section, we study relative Chow/K-stability of toric Fano varieties which were dealt with in $[YZ19, NSY23]$. The product formulas for potential functions $\theta_\Delta$ and the additivity of the constant $M_X$ defined in (4.9) are discussed in Section 4.3. Then in Section 4.4 we verify (asymptotic) relative Chow stability of Gorenstein toric del Pezzo surfaces, by applying our combinatorial criterion of relative Chow stability (see, Corollary 4.3) in the toric setting, and list the results in Table 1. In Section 4.5 we systematically construct examples of relatively K-polystable toric Fano manifolds, but which are relatively Ding unstable, building upon the works of $[NSY23]$ and $[OSY23]$. See Corollary 4.9 and Example 2 for more details.

4.1. Fundamental results on relative Chow stability. Firstly, we quick review notion of relative Chow stability and related results. See $[YZ19]$ for more detail.

Let us consider a reductive complex algebraic group $G$ with Lie algebra $\mathfrak{g}$. Suppose $G$ acts linearly on a finite dimensional complex vector space $V$. This induces a natural $G$-action on $\mathbb{P}(V)$. We will abbreviate $v \in \mathbb{P}(V)$ and its representatives in $V$. Let $T$ be a
torus in $G$ with Lie algebra $t$. We assume that $T$ fixes the point $v$. Using an inner product $\langle , \rangle$ and the Lie bracket $[ , ]$, we define subalgebras of $g$ by

$$g_T = \{ \alpha \in g \mid [\alpha, \beta] = 0 \text{ for all } \beta \in t \},$$

$$g_{T\perp} = \{ \alpha \in g_T \mid \langle \alpha, \beta \rangle = 0 \text{ for all } \beta \in t \}.$$  

Then the corresponding Lie group of $g_T$ (resp. $g_{T\perp}$) is denoted by $G_T$ (resp. $G_{T\perp}$). Following classical GIT (see Section 2.3), we call $v \in \mathbb{P}(V)$ is semistable relative to $T$ if the closure of $G_{T\perp}$ orbit $\mathcal{O}_{G_{T\perp}}(v)$ does not contain the origin. $v$ is polystable relative to $T$ if $\mathcal{O}_{G_{T\perp}}(v)$ is closed orbit. $v$ is said to be unstable relative to $T$ if it is not semistable relative to $T$.

Let us consider relative stability of the Chow form. For an irreducible complex projective variety $X \subset \mathbb{C}P^N$, we choose $G = SL(N + 1, \mathbb{C})$ and $T$ to be the $\mathbb{C}^\times$-action induced by extremal vector field.

**Definition 4.1.** A complex irreducible projective variety $X \subset \mathbb{C}P^N$ is said to be relatively Chow polystable (resp. semistable, unstable) if the $X$-resultant $R_X$ of $X$ is $SL(N + 1, \mathbb{C})$-polystable (resp. semistable, unstable) relative to $T$.

The definition of asymptotic relative Chow stability is analogous to Definition 2.4 hence we do not repeat the definition in this paper (see [YZ19 Definition 3.6]).

### 4.2. Toric reduction of relative Chow stability

We consider the toric case. In particular, we are interested in the case where $X$ is an $n$-dimensional Gorenstein toric Fano variety with the associated reflexive polytope $\Delta \subset M_{\mathbb{R}} \cong \mathbb{R}^n$. As in [Yao21], the Ricci affine function $\ell_\Delta$ associated to $\Delta$ is the unique function determined by $\int_\Delta \ell_\Delta u dv = u(0)$ for any affine linear function $u$, namely, one can solve the linear system

$$\int_\Delta \ell_\Delta(x) dv = 1, \quad \int_\Delta \ell_\Delta(x) \cdot x_i dv = 0 \quad \text{for} \quad i = 1, \ldots, n$$

in order to find $\ell_\Delta(x) = \sum a_i x_i + c$ with $a_i$ and $c$. Let us define the potential function of $\Delta$ by

$$\theta_\Delta := 1 - \text{vol}(\Delta) \ell_\Delta.$$  

Then, we consider its average

$$\bar{\theta}_\Delta = \frac{1}{N + 1} \sum_{j=1}^{N+1} \theta_\Delta(a_j),$$

where $\{a_1, \ldots, a_{N+1}\}$ are lattice points in $\Delta$. Denoting

$$d_\Delta = (1, \ldots, 1), \quad \tilde{\theta}_\Delta = ((\theta_\Delta(a_1) - \bar{\theta}_\Delta), \ldots, (\theta_\Delta(a_{N+1}) - \bar{\theta}_\Delta))$$

in $\mathbb{R}^{N+1}$, we can show the following.

**Theorem 4.2** (Theorem 3.8 in [YZ19]). Let $\text{Ch}(\Delta)$ be the Chow polytope of an $n$-dimensional Gorenstein toric Fano variety $X_\Delta \subset \mathbb{C}P^N$. Then $X_\Delta$ is relatively Chow polystable in the toric sense if and only if there exists $t \in \mathbb{R}$ such that

$$\frac{(n + 1)\text{vol}(\Delta)}{N + 1} (d_\Delta + t \bar{\theta}_\Delta) \in \text{Int}(\text{Ch}(\Delta)).$$

Let $\tilde{\theta}_{i\Delta} = \frac{1}{E_\Delta(i)} \sum_{a \in \Delta \cap (\mathbb{Z}/i) \mathbb{N}} \theta_\Delta(a)$.

Defining $d_{i\Delta}$ and $\tilde{\theta}_{i\Delta}$ by

$$d_{i\Delta}(a) = 1, \quad \tilde{\theta}_{i\Delta}(a) = \frac{\theta_\Delta(a) - \bar{\theta}_\Delta}{i}, \quad \text{for} \quad a \in \Delta \cap (\mathbb{Z}/i) \mathbb{N},$$

we have
we obtain a necessary condition for the associated polarized toric variety to be asymptotically relatively Chow semistable.

**Corollary 4.3** (Corollary 3.11 in [YZ19]). *If \((X_\Delta, -K_{X_\Delta})\) is asymptotically relatively Chow semistable, then for any \(i \in \mathbb{Z}_+\), there exists \(t_i \in \mathbb{R}\) satisfying*

\[
\sum_{a \in \Delta \cap (\mathbb{Z}/i)\mathbb{Z}} ia + t_i \sum_{a \in \Delta \cap (\mathbb{Z}/i)\mathbb{Z}} \tilde{\theta}_i\Delta(a) a = \frac{iE_\Delta(i)}{\text{vol}(\Delta)} \int_\Delta x \, dv.
\]

**4.3. Product formulas for potential functions.** Recently, Ono, Sano and the author proved that the only Bott manifolds such that the Futaki invariant vanishes for any Kähler class are isomorphic to the products of the projective lines [OSY23]. The key to prove the main theorem in [OSY23] is the analysis of the product of two polytopes. By applying this technique to potential functions in (4.1), we derive the product formula in this section.

Now let us discuss the *product* of two (or more) convex polytopes. For this, we consider the full dimensional polytopes \(\Delta_1 \subseteq \mathbb{R}^{n_1}\) and \(\Delta_2 \subseteq \mathbb{R}^{n_2}\), and define

\[
\Delta_1 \times \Delta_2 := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{n_1+n_2} \mid x = (x_1, \ldots, x_{n_1}) \in \Delta_1, \ y = (y_1, \ldots, y_{n_1}) \in \Delta_2 \right\}.
\]

Setting \(\Delta = \Delta_1 \times \Delta_2\), we see that \(\Delta\) is a polytope of dimension \(n_1 + n_2 (= n)\), whose any nonempty face is given by the product of a nonempty face \(F\) of \(\Delta_1\), and a nonempty face \(G\) of \(\Delta_2\). For \(i = 1, 2\), let \(dv_i\) be the standard volume form of \(\mathbb{R}^{n_i}\). Then, \(dv = dv_1 \wedge dv_2\) defines the volume form of \(\Delta\).

For a given arbitrary (not necessarily product) convex polytope \(P\) with \(\dim P = n\), we consider the functional \(\mathcal{L}_P(u)\) defined by

\[
\mathcal{L}_P(u) = \int_{\partial P} u \, d\sigma - \int_P \left( \frac{\text{vol}(\partial P)}{\text{vol}(P)} + \theta_P \right) u \, dv.
\]

Here \(u\) is a convex function, \(\theta_P\) is the potential function defined in (4.1), and \(d\sigma\) is the \((n-1)\)-dimensional Lebesgue measure of \(\partial P\) defined as follows: let \(\ell_j(x) = \langle x, v_j \rangle + c_j\) be the defining equation of a facet \(F_j\) of \(P\), where \(c_j \in \mathbb{Z}\) and \(v_j\) is a primitive vector. Recall that \(dv = dx_1 \wedge \cdots \wedge dx_n\) is the standard volume form of \(\mathbb{R}^n\). On each facet \(F_j = \{ x \in P \mid \ell_j(x) = 0 \} \subset \partial P\), we define the \((n-1)\)-dimensional Lebesgue measure \(d\sigma_j\) of \(\partial P\) by

\[
dv = \pm d\sigma_j \wedge d\ell_j.
\]

Then \(d\sigma\) is uniquely determined as the \((n-1)\)-dimensional Lebesgue measure of \(\partial P\) so that \(d\sigma_j = d\sigma|_{F_j}\), up to the sign.

Let us go back to the product polytope \(\Delta = \Delta_1 \times \Delta_2\). Let \(d\sigma_1\) (resp. \(d\sigma_2\)) be the \((n_1-1)\)-dimensional (resp. \((n_2-1)\)-dimensional) Lebesgue measure of \(\partial \Delta_1\) (resp. \(\partial \Delta_2\)) defined in (4.5). Since any nonempty face of \(\Delta\) is obtained by the product of a nonempty face \(F \subseteq \Delta_1\) and a nonempty face \(G \subseteq \Delta_2\), we see that the boundary of \(\Delta\) is written as

\[
\partial \Delta = \partial \Delta_1 \times \Delta_2 \cup \Delta_1 \times \partial \Delta_2.
\]

Also see, (4.18) in [OSY23]. In particular, we find the following equalities by direct computations.

**Lemma 4.4.** *Let \(\Delta = \Delta_1 \times \Delta_2\) be the product of two polytopes \(\Delta_k\) with \(\dim \Delta_k = n_k\) for \(k = 1, 2\). Let \(x = (x_1, \ldots, x_{n_1})\) and \(y = (y_1, \ldots, y_{n_2})\) be the coordinates of \(\Delta_1\) and \(\Delta_2\)*
respectively. We denote the volume form of \( \Delta \) (resp. \( \Delta_k \)) by \( dv \) (resp. \( dv_k \)), and the volume form of \( \partial \Delta \) (resp. \( \partial \Delta_k \)) by \( d\sigma \) (resp. \( d\sigma_k \)). For \( i = 1, \ldots, n_1 \) and \( j = 1, \ldots, n_2 \), we have

\[
\text{vol}(\Delta) = \text{vol}(\Delta_1)\text{vol}(\Delta_2),
\]

\[
\int_{\Delta} x_i\,dv = \text{vol}(\Delta_2)\int_{\Delta_1} x_i\,dv_1, \quad \int_{\Delta} y_j\,dv = \text{vol}(\Delta_1)\int_{\Delta_2} y_j\,dv_2,
\]

\[
\text{vol}(\partial \Delta) = \text{vol}(\partial \Delta_1)\text{vol}(\Delta_2) + \text{vol}(\Delta_1)\text{vol}(\partial \Delta_2),
\]

\[
\int_{\partial \Delta} x_i\,d\sigma = \text{vol}(\Delta_2)\int_{\partial \Delta_1} x_i\,d\sigma_1 + \text{vol}(\partial \Delta_2)\int_{\Delta_1} x_i\,dv_1, \quad \text{and}
\]

\[
\int_{\partial \Delta} y_j\,d\sigma = \text{vol}(\Delta_1)\int_{\partial \Delta_2} y_j\,d\sigma_2 + \text{vol}(\partial \Delta_1)\int_{\Delta_2} y_j\,dv_2.
\]

We finish this subsection with the following additive property of the potential functions \( \theta_\Delta \) and the Mabuchi constants \( M_{X_\Delta} \) for the product polytopes.

**Proposition 4.5.** Let \( \Delta = \Delta_1 \times \Delta_2 \) be the product of two polytopes as in Lemma 4.4. Then the potential function \( \theta_\Delta \) defined in (4.1) satisfies the equality

\[
\theta_\Delta(x, y) = \theta_{\Delta_1}(x) + \theta_{\Delta_2}(y).
\]

Moreover, for the product \( \Delta = \prod_{k=1}^r \Delta_k \), we see that \( \theta_\Delta(x_1, x_2, \ldots, x_r) = \sum_{k=1}^r \theta_{\Delta_k}(x_k) \).

**Proof.** As it was described in [YZ19, p.496], the potential function \( \theta_\Delta \) is uniquely determined by solving the \( n + 1 \)-linear system

\[
(4.7) \quad \mathcal{L}_\Delta(1) = 0, \quad \mathcal{L}_\Delta(x_i) = 0, \quad \mathcal{L}_\Delta(y_j) = 0 \quad \text{for} \quad i = 1, \ldots, n_1, \quad j = 1, \ldots, n_2,
\]

where \( \mathcal{L}_\Delta(u) \) is the function defined in (4.4). Since \( \theta_{\Delta_k} \) is the potential function of \( \Delta_k \) for each \( k = 1, 2 \), we have

\[
(4.8) \quad \mathcal{L}_{\Delta_1}(1) = \mathcal{L}_{\Delta_2}(1) = 0, \quad \mathcal{L}_{\Delta_1}(x_i) = 0, \quad \text{and} \quad \mathcal{L}_{\Delta_2}(y_j) = 0.
\]

In order to prove our assertion, it suffices to show that \( \theta_\Delta(x, y) := \theta_{\Delta_1}(x) + \theta_{\Delta_2}(y) \) satisfies the \( (n + 1) \)-equalities in (4.7) using our assumption (4.8).

Firstly, we find that

\[
\int_{\Delta} \theta_\Delta(x, y)dv = \int_{\Delta_1} \theta_{\Delta_1}(x)dv_1 + \int_{\Delta_2} \theta_{\Delta_2}(y)dv_2,
\]

which equals 0, by our assumption \( \mathcal{L}_{\Delta_1}(1) = \mathcal{L}_{\Delta_2}(1) = 0 \).

Secondly, for \( i = 1, \ldots, n_1 \), we prove that \( \mathcal{L}_{\Delta_1}(x_i) = 0 \). To see this, we compute that

\[
\int_{\Delta} \left( \frac{\text{vol}(\partial \Delta)}{\text{vol}(\Delta)} + \theta_\Delta(x, y) \right) x_i\,dv = \frac{\text{vol}(\partial \Delta)}{\text{vol}(\Delta)} \int_{\Delta} x_i\,dv + \int_{\Delta} \left( \theta_{\Delta_1}(x) + \theta_{\Delta_2}(y) \right) x_i\,dv
\]

\[
= \frac{\text{vol}(\partial \Delta_1)\text{vol}(\Delta_2) + \text{vol}(\Delta_1)\text{vol}(\partial \Delta_2)}{\text{vol}(\Delta_1)} \int_{\Delta_1} x_i\,dv_1 + \text{vol}(\Delta_2) \int_{\Delta_1} \theta_{\Delta_1}(x) x_i\,dv_1.
\]

By applying Lemma 4.4 into \( \int_{\partial \Delta} x_i\,d\sigma \), we find that

\[
\mathcal{L}_{\Delta_1}(x_i) = \text{vol}(\Delta_2)\mathcal{L}_{\Delta_1}(x_i) = 0,
\]

where we used (4.8) for the last equality.

Finally, for \( j = 1, \ldots, n_2 \), we have \( \mathcal{L}_{\Delta_2}(y_j) = \text{vol}(\Delta_1)\mathcal{L}_{\Delta_2}(y_j) = 0 \) in the same manner as the above computation. This completes the proof of \( \theta_\Delta(x, y) = \theta_{\Delta_1}(x) + \theta_{\Delta_2}(y) \).
In order to see the second assertion
\[ \theta_{\Delta}(x_1, x_2, \ldots, x_r) = \sum_{k=1}^{r} \theta_{\Delta_k}(x_k), \]
for the product polytope \( \Delta = \prod_{k=1}^{r} \Delta_k \), we use the inductive argument. Hence the assertion is verified. \( \square \)

For later use, we consider the value of constant
\[(4.9) \quad M_{X_{\Delta}} = \max_{x \in \Delta} \{ \theta_{\Delta}(x) \}, \]
which verifies relative Ding stability of the corresponding toric (Fano) variety. See Section 4.5 for further discussion. After posting this version of the paper on arXiv (version 5, arXiv:1711.10113v5), the author found that the following additivity of the constant \( M_{X_{\Delta}} \) is mentioned by Mabuchi in [Mab21, Theorem 9.9] for general (not necessarily toric) Fano manifold. However, it is worth to mention that we derive a direct combinatorial proof for the case of toric Fano manifolds from Proposition 4.5 and (4.9).

**Corollary 4.6.** Let \( \Delta = \prod_{k=1}^{r} \Delta_k \) be the product of (reflexive) polytopes. Then the constant of \( M_{X_{\Delta}} \) has the additive property such that
\[(4.10) \quad M_{X_{\Delta}} = M_{X_{\Delta_1}} + \cdots + M_{X_{\Delta_r}}. \]

### 4.4. Asymptotic relative Chow stability of Gorenstein toric del Pezzo surfaces.

As we mentioned in Section 2.1, there are 16 isomorphism classes of Gorenstein toric del Pezzo surfaces. See [N05] for more details. On the one hand, relative Ding stability of Gorenstein toric del Pezzo surfaces has been verified in [Yao21, Example 5.14]. On the other hand, it is difficult to verify asymptotic relative Chow stability of polarized toric variety because we have to show there exists \( t_i \in \mathbb{R} \) satisfying (4.3) for any positive integer \( i \) (cf. [LLSW19] for (not relative) Chow stability case). However, we can solve this difficulty in the case of 2 dimension, by using symmetry of the associated reflexive polytopes. See Case 3 in the proof of Proposition 4.7 below. As a consequence, we verify relative Chow stability of each Gorenstein toric del Pezzo surface. We list all the results in Table II.

**Proposition 4.7.** Among all 16 isomorphism classes of Gorenstein toric del Pezzo surfaces, there are 5 isomorphism classes of asymptotically relatively Chow polystable surfaces and 4 isomorphism classes of asymptotically relatively Chow unstable surfaces. The remaining 7 classes are relatively Chow polystable with respect to the anticanonical polarization \( (i = 1) \).

**Proof.** Case 1. Note that any toric surface has at worst orbifold singularities. There are 5 isomorphism classes of Kähler-Einstein Gorenstein toric del Pezzo surfaces with the vanishing Futaki character, that is, \( \mathbb{C}P^2, \mathbb{C}P^1 \times \mathbb{C}P^1, S_6, \mathbb{C}P^1 \times \mathbb{C}P^1 / \mathbb{Z}_2 \) and \( \mathbb{C}P^2 / \mathbb{Z}_3 \). Hence relative Chow stability coincides with Chow stability for these 5 classes of del Pezzo surfaces. In particular, the vanishing Futaki character i.e. \( \int_{\Delta} x \, dv = 0 \) implies \( \theta_{\Delta} \equiv 0 \). This means \( \theta_{i\Delta}(a) = 0 \) for any \( i \in \mathbb{Z}_+ \) and a necessary condition of asymptotic relative Chow semistability of a polarized toric variety (4.3) becomes
\[ \sum_{a \in \Delta \cap (\mathbb{Z}/i)^n} ia = \frac{iE_{\Delta}(i)}{\text{vol}(\Delta)} \int_{\Delta} x \, dv \]
for all \( i \in \mathbb{Z}_+ \). Hence we obtained the same equality in (2.6). Moreover, \( \int_{\Delta} x \, dv = 0 \) implies that \( \sum_{a \in \Delta \cap (\mathbb{Z}/i)^n} ia = 0 \) for any \( i \in \mathbb{Z}_+ \). Remark that this is equivalent to the
Table 1. Relative Chow stability of Gorenstein toric del Pezzo surfaces

| Label in [N05] | Stability | $t_1$ in (4.3) |
|----------------|-----------|----------------|
| $3$ ($\mathbb{C}P^2$) | Asymptotically relatively Chow polystable | No need |
| $4A$ ($\mathbb{C}P^1 \times \mathbb{C}P^1$) | Asymptotically relatively Chow polystable | No need |
| $4B$ ($dP_8$) | Relatively Chow polystable w.r.t $\mathcal{O}_X(-K_X)$ | $-65/828$ |
| $4C$ | Relatively Chow polystable w.r.t $\mathcal{O}_X(-K_X)$ | $-5/72$ |
| $5A$ ($dP_7$) | Relatively Chow polystable w.r.t $\mathcal{O}_X(-K_X)$ | $69/665$ |
| $5B$ | Asymptotically relatively Chow unstable | $-$ |
| $6A$ ($dP_6$) | Asymptotically relatively Chow polystable | No need |
| $6B$ | Relatively Chow polystable w.r.t $\mathcal{O}_X(-K_X)$ | $-259/1944$ |
| $6C$ | Asymptotically relatively Chow unstable | $-$ |
| $6D$ | Asymptotically relatively Chow unstable | $-$ |
| $7A$ | Relatively Chow polystable w.r.t $\mathcal{O}_X(-K_X)$ | $-409/2646$ |
| $7B$ | Asymptotically relatively Chow unstable | $-$ |
| $8A$ | Asymptotically relatively Chow polystable | No need |
| $8B$ | Relatively Chow polystable w.r.t $\mathcal{O}_X(-K_X)$ | $-33/200$ |
| $8C$ | Relatively Chow polystable w.r.t $\mathcal{O}_X(-K_X)$ | $-3/19$ |
| $9$ | Asymptotically relatively Chow polystable | No need |

vanishing of the obstruction for asymptotic Chow semistability defined in [Mab04] (see, [On11] p.1385). Since $X$ admits a Kähler-Einstein metric, it must be asymptotically Chow polystable for $X = \mathbb{C}P^2$, $\mathbb{C}P^1 \times \mathbb{C}P^1$ and $S_6$ due to the result in [Mab05, Main Theorem]. Hence we verified the assertion for these 3 classes.

For the remaining two orbifolds cases $X = \mathbb{C}P^2/\mathbb{Z}_3$ (labeled $9$ in Table 1) and $\mathbb{C}P^1 \times \mathbb{C}P^1/\mathbb{Z}_2$ (labeled $8A$ in Table 1), asymptotic Chow polystability of $(X, -K_X)$ has been verified in Theorem 1.2 (3) in [LLSW19]. We remark that the minimal embeddings of these del Pezzo surfaces are given by

$$\mathbb{C}P^2/\mathbb{Z}_3 = \{ [z_0 : z_1 : z_2 : z_3] \in \mathbb{C}P^3 | z_0^3 - z_1 z_2 z_3 = 0 \}$$

with three $A_2$ singularities, and

$$\mathbb{C}P^1 \times \mathbb{C}P^1/\mathbb{Z}_2 = \{ [z_0 : z_1 : z_2 : z_3 : z_4] \in \mathbb{C}P^4 | z_1 z_3 - z_2^2 = 0, z_2 z_4 - z_2^2 = 0 \}$$

with four $A_1$ singularities, respectively. See [KN09] for further details.

Case 2. Let $X$ be a Gorenstein toric del Pezzo surface labeled with $5B$ in Table 1. Then the associated reflexive polytope $\Delta \subseteq M_{\mathbb{R}}$ is given by

$$\Delta = \text{conv} \{ (-1, 0), (1, -2), (0, 1), (-1, 1) \} .$$

We claim that $X$ is asymptotically relatively Chow unstable by using Corollary 4.3. Hence it suffices to show that there is no $t_1 \in \mathbb{R}$ satisfying (4.3) for $i = 1$. See Remark 3.12 and
We prove that Proposition 5.4 in [YZ19]. We readily see that
\[ E_\Delta(i) = \frac{5}{2} i^2 + \frac{5}{2} i + 1, \quad \int_\Delta x \, dv = \left(-\frac{1}{3}, -\frac{1}{3}\right), \quad \sum_{a \in \Delta \cap \mathbb{Z}^2} a = (-1, -1), \]
\[ \theta_\Delta(x) = -\frac{1}{529} (1032x_1 + 648x_2 + 224) \quad \text{and} \quad \bar{\theta}_\Delta = \frac{56}{529}. \]
Therefore
\[ t_1 \sum_{a \in \Delta \cap \mathbb{Z}^2} \tilde{\theta}_\Delta(a)a = -t_1 \left(\frac{872}{529}, \frac{1160}{529}\right). \]
This yields that there is no \( t_1 \in \mathbb{R} \) satisfying (4.3).

**Case 3.** Let \( X \) be a weighted projective space \( \mathbb{C}P(1, 1, 2) \). This is a Gorenstein toric del Pezzo surface labeled with \( 8C \) in Table 1 and the corresponding reflexive polytope \( \Delta \) is
\[ \Delta = \text{conv} \{ (-1, 2), (1, 0), (-1, -2) \}. \]
We prove that \((X, -K_X)\) is relatively Chow polystable. A straightforward computation shows that
\[ E_\Delta(i) = 4i^2 + 4i + 1, \quad \int_\Delta x \, dv = \left(-\frac{4}{3}, 0\right), \quad \sum_{a \in \Delta \cap \mathbb{Z}^2} a = (-4, 0), \]
\[ \theta_\Delta(x) = -\frac{3}{2} x_1 - \frac{1}{2} \quad \text{and} \quad \bar{\theta}_\Delta = \frac{1}{6}. \]
Taking \( i = 1 \) in (4.3), we find that \( t_1 = -3/19 \) satisfies the equation
\[ \sum_{a \in \Delta \cap \mathbb{Z}^2} a + t_1 \sum_{a \in \Delta \cap \mathbb{Z}^2} \tilde{\theta}_\Delta(a)a = \frac{E_\Delta(1)}{\text{vol}(\Delta)} \int_\Delta x \, dv. \]
Moreover \( \Delta \) is invariant under unimodular transformation \( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) which gives the coordinate interchange \( x_2 \mapsto -x_2 \). By this symmetry we conclude that there exists \( t_i \) for **any** \( i \in \mathbb{Z}_+ \) such that (4.3) holds.

Next we verify (4.2). For \( t = -3/19 \), we readily see that the left hand side of (4.2) is given by \( p := \frac{4}{59} (11, 14, 17, 14, 11, 11, 11, 11, 14) \). On the other hand, the Chow polytope \( \text{Ch}(\Delta) \) is the 6-dimensional polytope in \( \mathbb{R}^9 \) with 296 vertices. In particular, \( \text{Ch}(\Delta) \) is determined by 3 defining equations \( f_i(x) = 0 \) \((i = 1, 2, 3)\) and 26 defining inequalities \( h_j(x) \geq 0 \) \((j = 1, \ldots, 26)\) in \( \mathbb{R}^9 \). By direct computation, one can see that \( f_i(p) = 0 \) and \( h_j(p) > 0 \) hold for all \( i, j \). This implies \( p \in \text{Int}(\text{Ch}(\Delta)) \) and the assertion is verified. Other cases are similar and further details are left to the reader. \( \square \)

**Remark 4.8.**

1. Using symmetry of polytopes, one can verify the existence of \( t_i \) for \( i \gg 0 \) satisfying (4.3) of each case \((4B, 4C, 5A, 6B, 7A, 8B \) and \( 8C \) in Table 1). We mention that this is only a necessary condition for \((X, L)\) to be asymptotically relatively Chow semistable (Corollary 4.3).

2. On the other hand, \( \text{Ch}(i\Delta) \) will be a huge number of vertices in a multidimensional Euclidean space if \( i > 0 \) is a sufficiently large positive integer. Hence, it is generally impossible to verify the condition
\[ \frac{i^n(n + 1)! \text{vol}(\Delta)}{E_\Delta(i)} (d_i\Delta + t_i\bar{\theta}_i\Delta) \in \text{Int}(\text{Ch}(i\Delta)) \]
\[ \text{We used package TOPCOM for the computation.} \]
for arbitrary positive integer $i$. See \cite{KSZ92} and \cite{GKZ94} for more combinatorial descriptions of $\text{Ch} (\Delta)$.

### 4.5. Relative Ding/K-stability

In \cite{NSY23}, we found that there are several examples of toric Fano manifolds which clarify the difference between relative K-stability and relative Ding stability. More specifically, we verified that if $X$ is one of

- a toric Fano 3-fold $B_1 = \mathbb{P}^2 (\mathcal{O} \oplus \mathcal{O} (2))$, or
- toric Fano 4-folds (which are all $\mathbb{P}^1$-bundles over $\mathbb{P}^3$) $B_1 = \mathbb{P}^3 (\mathcal{O} \oplus \mathcal{O} (3))$, $B_2 = \mathbb{P}^3 (\mathcal{O} \oplus \mathcal{O} (2))$, $L_1 = \mathbb{P}^3 (\mathcal{O} \oplus \mathcal{O} (1,1,1))$,

then $(X, -K_X)$ is relatively $K$-polystable, but it is relatively Ding unstable. In order to prove that these four examples ($B_1$, $B_1$, $B_2$ and $L_1$) admit extremal Kähler metrics in their first Chern classes, which in turn to be relatively K-polystable, we focused on their geometric structure such as projective bundles, Bott structures, etc \cite{ACGT-F08, BCT-F19, Gu95, Tw94}. On the one hand, relative Ding stability of toric Fano manifolds is determined by the value of constant $M_{X,\Delta}$ defined in \eqref{eq:M_X,Delta} is larger than 1 or not, due to the work of Yao \cite{Yao21}. On the other hand, Proposition \ref{prop:pl} implies that the products of (higher dimensional) toric extremal manifolds are more likely to be relatively Ding unstable, by the additive property of $M_{X,\Delta}$ (see, \eqref{eq:additive} and Corollary \ref{cor:prop}). In this section, we systematically construct examples of a relatively $K$-polystable toric Fano manifold, but it is relatively Ding unstable.

Let us quickly review the notion of relative $K$-stability and relative Ding stability for (smooth) toric Fano variety. Remark that we only consider toric (or $T$-equivariant) test configuration for the definitions of relative Ding/K-stability. This is because for polarized toric varieties, it suffices to check only toric test configurations of relative Ding/K-stability as in \cite{De20} and \cite{LL22}. We refer the reader to Section 2 in \cite{NSY23}, for more details.

Let $\Delta \subseteq \mathbb{R}^n$ be an $n$-dimensional reflexive Delzant polytope. In this case, the average of the scalar curvature, i.e., $\bar{S} = \text{vol}(\partial \Delta) / \text{vol}(\Delta)$ equals to $n$, and hence the functional defined in \eqref{eq:L_delta} will be

$$L_\Delta (u) = \int_{\partial \Delta} u \, d\sigma - \int_\Delta (n + \theta_\Delta) u \, dv,$$

where $u$ is a convex function of $\Delta$. A convex function $u : \Delta \to \mathbb{R}$ is called rational PL convex if $u$ has the form of

$$u(x) = \max \{ f_1(x), \ldots, f_m(x) \}$$

with each $f_k$ a rational affine function. The associated anticanonically polarized smooth toric Fano variety $(X_\Delta, -K_{X_\Delta})$ is relatively $K$-polystable if $L_\Delta (u) \geq 0$ for any rational PL convex function $u$, and the equality holds if and only if $u$ is affine linear. Let $M_{X_\Delta}$ be the Mabuchi constant defined in \eqref{eq:M_X,Delta}, $(X_\Delta, -K_{X_\Delta})$ is relatively Ding polystable if $M_{X_\Delta} \leq 1$. Conversely, it is called relatively Ding unstable if $M_{X_\Delta} > 1$. See, \cite{Yao21} and \cite{NSY23} Proposition 1.2] for further details.

On the other hand, Corollary \ref{cor:prop} implies that $X_\Delta$ is more likely to be relatively Ding unstable if the dimension of $X_\Delta$ is getting higher and higher. Meanwhile, for given extremal Kähler manifolds $(X_k, g_k)$ with $1 \leq k \leq r$, the product manifold $X = \prod_{k=1}^r X_k$ admits the product extremal Kähler metric $\prod_{k=1}^r g_k$. Thus, $(X, -K_X)$ must be relatively K-polystable. In particular, $X$ is Fano. From this observation, one can expect that there are more examples of toric Fano manifolds which clarify the difference between relative K-stability and relative Ding stability. As a consequence of \eqref{eq:additive}, we systematically construct infinitely many examples of relatively $K$-polystable extremal toric Fano manifolds which are relatively Ding unstable.
**Corollary 4.9.** For $1 \leq k \leq r$, let $X_k$ be an extremal toric Fano manifold with the associated polytope $\Delta_k$ and let $\theta_{\Delta_k}(x_k)$ be the potential function of $\Delta_k$ satisfying $\frac{1}{r} \leq \theta_{\Delta_k} < 1$. Let $\Delta$ be the product of polytopes $\Delta_k$ for $1 \leq k \leq r$. Then the associated anticanonically polarized toric Fano manifold $(X_\Delta, -K_{X_\Delta})$ is relatively $K$-polystable, but it is relatively Ding unstable.

Using Table 3 in [NSY23], we obtain the following examples.

**Example 2.** Let $dP_{9-i}$ denote a smooth del Pezzo surface with degree $(9-i)$ which is obtained by the blow-up of $\mathbb{P}^2$ at $i$ points. Fixing a positive integer $r$, we denote a copy of $dP_8$ by $X_k$ for $1 \leq k \leq r$. It is known that $X_k$ admits an extremal Kähler metric in every Kähler class [Ca82], and this yields that $X = \prod_{k=1}^{r} X_k$ also admits the extremal Kähler metric in its first Chern class. Hence $(X, -K_X)$ is relatively $K$-polystable for any positive integer $r$.

On the other hand, the direct computation shows that $M_{X_k} = \frac{5}{11}$. See, [NSY23, Table1, No.3]. Thus, we conclude that $M_{X} = \frac{5r}{11}$ by (4.10). Consequently, $(X, -K_X)$ is relatively (uniform) Ding polystable if $r = 2$, whereas it is relatively Ding unstable if $r \geq 3$. We note that the toric Fano 4-fold $dP_8 \times dP_8$ is denoted by $L_7$ (No. 55) in [NSY23, Table 3]. In particular, there are other examples such as $Q_{10} = dP_7 \times dP_8$ (No. 93) and $dP_7 \times dP_7$ (No. 119) in four dimensional case.

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**TABLE 2.** Combinatorial data and the delta invariant of Gorenstein toric del Pezzo surfaces

| Label | $\Delta \subseteq M_\mathbb{R}$ | Symmetry of $\Delta$ |
|-------|---------------------------------|-----------------------|
| 3     | conv $\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \end{pmatrix} \right\}$ | No need |
| 4A    | conv $\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ | No need |
| 4B    | conv $\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\}$ | $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ |
| 4C    | conv $\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right\}$ | $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ |
| 5A    | conv $\left\{ \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\}$ | $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ |
| 5B    | conv $\left\{ \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right\}$ | \(-\) |
| 6A    | conv $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$ | No need |
| 6B    | conv $\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right\}$ | $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ |
| 6C    | conv $\left\{ \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$ | \(-\) |
| 6D    | conv $\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ | \(-\) |
| 7A    | conv $\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$ | $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ |
| 7B    | conv $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\}$ | \(-\) |
| 8A    | conv $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\}$ | No need |
| 8B    | conv $\left\{ \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right\}$ | $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ |
| 8C    | conv $\left\{ \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \end{pmatrix} \right\}$ | $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ |
| 9     | conv $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right\}$ | No need |
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