INVERSE FUNCTION THEOREMS FOR ARC-ANALYTIC
HOMEOMORPHISMS

TOSHIZUMI FUKUI, KRZYSZTOF KURDYKA, ADAM PARUSIŃSKI

Abstract. We call a local homeomorphism $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ blow-analytic if it becomes real analytic, as a map, after composing with a finite number blowings-up with smooth nowhere dense centers. If the graph of $f$ is semi-algebraic then, by a theorem of Bierstone and Milman, $f$ is blow-analytic if and only if it is arc-analytic: the image by $f$ of a parametrized real analytic arc $\gamma : (\mathbb{R}, 0) \to (\mathbb{R}^n, 0)$ is again a real analytic arc.

We show that if $f$ is blow-analytic, the inverse $f^{-1}$ of $f$ is Lipschitz, and the graph of $f$ is semialgebraic, then $f$ is Lipschitz and $f^{-1}$ is blow-analytic. The proof is by a motivic integration argument, using additive invariants on the spaces of arcs.

1. Introduction.

Let $M$, $N$ be real analytic manifolds. We say that $f : M \to N$ is blow-analytic via $\pi$ if $\pi : \tilde{M} \to M$ is a locally finite composition of blowings-up with nonsingular nowhere dense centers and $f \circ \pi$ is analytic. We say that $f$ is blow-analytic if there is such $\pi : \tilde{M} \to M$, that $f$ is blow-analytic via $\pi$. We say that $f$ is semialgebraic if the graph of $f$ is semi-algebraic.

In this paper we show the following result.

**Theorem 1.1.** Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a semialgebraic homeomorphism such that $f$ is Lipschitz and $f^{-1}$ is blow-analytic. Then $f^{-1}$ is Lipschitz and $f$ is blow-analytic.

Theorem 1.1 gives a negative answer to Question 7.8 of [8]. As a corollary we obtain the following Inverse Function Theorem. By $C^\omega$ we mean real analytic.

**Corollary 1.2.** Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a semialgebraic homeomorphism such that $f^{-1}$ is blow-analytic. If $f$ is $C^k$, $k = 1, 2, \ldots, \infty, \omega$, then so is $f^{-1}$.

The proof of Theorem 1.1 uses the jet spaces of real analytic arcs and additive invariants of real algebraic sets. First we show the following theorem whose proof is given by a classical motivic integration argument on the jet spaces.

**Theorem 1.3.** Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a semialgebraic homeomorphism such that $f$ and $f^{-1}$ are blow-analytic. If the jacobian determinant $\det(df)$ is bounded, then there is a constant $c_1 > 0$ such that $c_1 < |\det(df)|$. 

Date: March 2, 2010.

2010 Mathematics Subject Classification. Primary: 14P99, 32S15. Secondary: 32B20.

Key words and phrases. Real analytic, subanalytic, arc-analytic, Lipschitz, motivic integration, equivalence of singularities.

Research partially supported by a Mathématiques en Pays de la Loire (MATPYL) grant.
1.1. Blow-analytic and arc-analytic maps. A map between real analytic manifolds $f : M \to N$ is arc-analytic if for every real analytic arc germ $\gamma : (\mathbb{R}, 0) \to M$ the composition $f \circ \gamma$ is analytic, see [14], [15]. A blow-analytic map is always arc-analytic. There is a partial reciprocal statement for $f$ subanalytic, see [1] and [19], where the blow-analyticity is replaced by a similar notion expressed in terms of local blowings-up. In the semialgebraic case the blow-analyticity and arc-analyticity are equivalent.

**Theorem 1.4.** (Bierstone & Milman,[1])

If $M$ and $N$ are real algebraic manifolds and the graph of $f$ is semi-algebraic then the arc-analyticity of $f$ is equivalent to the blow-analyticity. Moreover, we may require that the blowings-up are along nonsingular real algebraic subvarieties.

The blow-analytic maps were introduced in [13], see also surveys [6], [8], in the context of blow-analytic equivalence of real analytic function germs, that is the equivalence induced by blow-analytic homeomorphisms. Neither blow-analytic equivalence implies the bi-lipschitz one, nor the vice-versa, cf. [11], [12]. Nevertheless there is clear evidence that there is a relation between blow-analytic and Lipschitz property for homeomorphisms, that should be further investigated and better understood. The following Inverse Function Theorem for arc-analytic homeomorphism holds for homeomorphisms with subanalytic graphs, a class significantly bigger than the semialgebraic ones.

**Theorem 1.5.** ([7]) Let a subanalytic homeomorphism $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be bi-Lipschitz and arc-analytic. Then $f^{-1}$ is also arc-analytic.

Note: It is a widely accepted, [6], [8], to call a homeomorphism $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ a blow-analytic homeomorphism if both $f$ and $f^{-1}$ are blow-analytic. We avoid this terminology in this paper in order not to confuse it with a homeomorphism that is blow-analytic as a map (but maybe its inverse is not blow-analytic).

1.2. Open problems. The proof of Theorem 1.1 and Theorem 1.3 is based on the motivic integration method on the space of real analytic arcs. The essential point is the use of virtual Poincaré polynomial, an additive and multiplicative invariant of real algebraic varieties, that distinguishes their dimensions. This part of the proof cannot be carried out, at the moment this paper is being written, to the subanalytic case, since such an invariant it is not known in this case.

We conjecture that Theorems 1.1 and 1.3 hold without assumption of semialgebraicity of the graph. Then the graph has to be subanalytic since every blow-analytic map is subanalytic.

We conjecture also that the following property holds:

Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a (semialgebraic) homeomorphism such that $f^{-1}$ is blow-analytic. If the jacobian determinant $\det(df)$ is bounded from above, then there is a constant $c_1 > 0$ such that $c_1 < |\det(df)|$, and $f$ is blow-analytic.

2. Lipschitz and bi-Lipschiz maps.

In this section we show how Corollary 1.2 can be deduced from Theorem 1.3. The argument is elementary.
Let \( U \) be an open subset of \( \mathbb{R}^n \). A map \( f : U \to \mathbb{R}^p \) is said to be Lipschitz if there is a positive constant \( L \) so that
\[
|f(x) - f(y)| \leq L|x - y| \quad \forall x, y \in U.
\]

Let \( U \) be a convex open subset of \( \mathbb{R}^n \) and let \( f : U \to \mathbb{R} \) be a continuous function with subanalytic graph. Then there is an nowhere dense closed subanalytic subset \( Z \) so that \( f \) is analytic on \( U - Z \).

**Lemma 2.1.** The function \( f \) is Lipschitz if and only if all partial derivatives of \( f \) are bounded on \( U - Z \).

**Proof.** If \( f \) is Lipschitz, then the following inequality implies all directional derivatives are bounded whenever they exist.
\[
\left| \frac{f(x + tv) - f(x)}{t} \right| \leq L|v|, \quad v \in \mathbb{R}^n.
\]

Conversely, we assume that there is a positive constant \( M \) so that
\[
\left| \frac{\partial f}{\partial x_i}(x) \right| \leq M, \quad x \in U - Z, \; i = 1, \ldots, n.
\]

For \( x, x' \in U - Z \), we set \( v = x' - x \) and write \( v = (v_1, \ldots, v_n) \). Then the mean value theorem implies that there is \( \theta \) so that
\[
f(x') - f(x) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x + \theta v)v_i, \quad 0 < \theta < 1.
\]

This makes sense when \( x + \theta v \in U - Z \). This implies that
\[
|f(x') - f(x)| \leq \sum_{i=1}^{n} M|v_i| \leq M\sqrt{n}|x' - x|.
\]

Since \( f \) is continuous, this means that \( f \) is Lipschitz. \( \Box \)

**Corollary 2.2.** Let \( f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0) \) be a subanalytic map. If \( f \) is Lipschitz, then \( \det(df) \) is bounded.

**Corollary 2.3.** Suppose that \( f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0) \) is a subanalytic such that there are positive constants \( c_1, c_2 \) with
\[
c_1 \leq |\det(df)| \leq c_2
\]
If \( f \) is Lipschitz, then \( f^{-1} \) is Lipschitz.

**Proof.** The Jacobian matrix of \( f^{-1} \) equals
\[
\frac{1}{\det(df)} (\text{cofactor matrix of the Jacobian matrix of } f)
\]
in the complement of a nowhere dense subanalytic set. Since \( f \) is Lipschitz, each coefficient of the Jacobian matrix of \( f^{-1} \) is bounded. We conclude by Lemma 2.1. \( \Box \)

Thus Corollary 1.2 follows from Theorem 1.3 and Corollaries 2.2, 2.3.
3. Reduction to the normal crossing case.

A real modification is a classical notion introduced in [13]. It is a natural generalization of the notion of modification in algebraic geometry. For instance, a locally finite composition of blowing-ups with nonsingular nowhere dense centers is a real modification in the sense of [13]. It was shown in [11], [8], that every real modification satisfies the unique lifting of generic arc property:

**Theorem 3.1.** ([11], [8]) Let \( \tau : M \to N \) be a real modification. Then there is a closed subanalytic nowhere dense \( A \subset N \) such that for every real analytic curve germ \( \gamma : (\mathbb{R}, 0) \to (N, p) \), if the image of \( \gamma \) is not entirely contained in \( A \), then there is a unique real analytic lift \( \tilde{\gamma} : (\mathbb{R}, 0) \to (M, \tilde{p}) \), such that \( \tau \circ \tilde{\gamma} = \gamma \).

Suppose that, as in the assumption of Theorem 1.3, the mapping \( f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0) \) and its inverse \( f^{-1} \) are blow-analytic. Then by argument of [13], proof of Proposition 2, there is a real analytic manifold \( M \), and real modifications \( \sigma : (M, \sigma^{-1}(0)) \to (\mathbb{R}^n, 0) \), \( \sigma' : (M, \sigma'^{-1}(0)) \to (\mathbb{R}^n, 0) \), such that \( f \circ \sigma = \sigma' \).

Let \( \pi \) be a sequence of blowing-ups with non-singular centers, so that \( \det(d\sigma) \circ \pi \), \( \det(d\sigma') \circ \pi \) are simultaneously normal crossings. We may assume that the new centers are in normal crossings with all old exceptional divisors. Then the jacobian of \( \pi \), \( \det(df) \circ \pi \), is normal crossing. Therefore the jacobian determinants \( \det(d(\sigma \circ \pi)) = ((\det(d\sigma)) \circ \pi) \cdot \det(d\pi) \), \( \det(d(\sigma' \circ \pi)) = ((\det(d\sigma')) \circ \pi) \cdot \det(d\pi) \), are normal crossings and, moreover, after composing with blowing-ups with nonsingular centers, if necessary, we may assume that they are normal crossings in the same system of coordinates.

We thus may assume that the critical loci of \( \sigma \) and \( \sigma' \) are simultaneously normal crossing and denote them respectively by

\[
E = \sum_{i \in I} \nu_i E_i, \quad \text{and} \quad E' = \sum_{i' \in I'} \nu'_i E_i.
\]

Since \( (\det(df) \circ \sigma) \cdot \det(d\sigma) = \det(d\sigma') \), for a generic real analytic arc \( \gamma \) at a generic \( p \in E_i \)

\[
\text{ord}_0(\det(df) \circ \sigma \circ \gamma) = \nu'_i - \nu_i.
\]

Thus if the jacobian determinant \( \det(df) \) is bounded, then \( \nu'_i \geq \nu_i \) for all \( i \in I \).

Under assumptions of Theorem [13] we show in the next section that this implies that \( \nu'_i = \nu_i \) for all \( i \in I \).

4. Arc spaces and additive invariants.

4.1. Constructible sets. For the proof we need the virtual Poincaré polynomial, an additive and multiplicative invariant, first introduced for real algebraic varieties in [17], and then for a wider class of \( \mathcal{AS} \) sets in [4] and [18]. A semialgebraic subset \( X \) of a compact real algebraic variety \( V \) is called an \( \mathcal{AS} \) set if it is a finite set-theoretic combination of semi-algebraic arc-symmetric subsets of \( V \), cf. [14], [15]. The virtual Poincaré polynomial of \( X \)

\[
\beta(X) = \sum \beta_i(X) u^i \in \mathbb{Z}[u]
\]

satisfies the following properties, see [17], [4].
(1) Additivity: For finite disjoint union $X = \sqcup X_i$, $\beta(X) = \sum \beta(X_i)$.

(2) Multiplicativity: $\beta(X \times Y) = \beta(X) \beta(Y)$.

(3) Degree: For $X \neq \emptyset$, $\deg \beta(X) = \dim X$ and the leading coefficient $\beta(X)$ is strictly positive.

If $X$ is compact and nonsingular then $\beta_i(X) = \dim H_i(X; \mathbb{Z}_2)$.

The virtual Poincaré polynomial is an invariant of real analytic isomorphisms with semialgebraic graph, see [4], and, more generally, of bijections with $\mathcal{AS}$ graphs, see [18].

In the rest of this section we mean by constructible set, an $\mathcal{AS}$ set, and by constructible map, a map with constructible graph. We refer the reader to [15], and also to [18], [20], for more precise discussion.

By morphism we mean a Nash map $f : M \to N$ that is a real analytic map with semi-algebraic graph. For our purpose we may suppose that $M$ and $N$ are nonsingular real algebraic varieties, but, in general, we may consider $M$ and $N$ to be Nash manifolds, [2]. By a modification we mean a real modification that is a Nash map, i.e. a Nash modification in the sense of [5]. A regular proper birational map is a standard example of such modification.

4.2. Arc spaces. We use the technique developed in [3], and adapted to the real analytic set-up in [10], [4], [5]. Let $M$ be a real analytic manifold and let $S$ be a subset of $M$. Consider the arc space $L(M, S) := \{ \gamma : (\mathbb{R}, 0) \to (M, S), \text{analytic}\}$.

For a real analytic map $\sigma : M \to \mathbb{R}^n$, set $L := L(\mathbb{R}^n, 0)$, $\tilde{L} := L(M, \sigma^{-1}(0))$, $\tilde{L}_k := \bigcup_{x \in \sigma^{-1}(0)} L_k(M, x)$, where $L_k(M, x)$ denotes the set of $k$-jets of elements of $L(M, x)$. Setting $L_k = L_k(\mathbb{R}^n, 0)$, we have the following commutative diagram of natural maps:

$$
\begin{array}{ccc}
\tilde{L} & \xrightarrow{\sigma_*} & \mathcal{L} \\
p_k \downarrow & & \downarrow p_k \\
\tilde{L}_k & \xrightarrow{\sigma_{*, k}} & L_k
\end{array}
$$

where $p_k$ denote the maps defined by taking the $k$-jets. Consider $B_e(\sigma) = \{ \gamma \in \tilde{L} : \text{ord}_t \det(d\sigma) = e \}$, $B_{k,e}(\sigma) = p_k(B_e(\sigma))$, where $\text{ord}_t \det(d\sigma)$ is defined as the order of $\det(d\sigma)(\gamma(t))$ at $t = 0$. It is clear that $B_{k,e}(\sigma)$ is a difference of two analytic sets. By Lemma 2.11 of [5] we have the following.

**Lemma 4.1.** Let $\sigma : (M, E_0) \to (\mathbb{R}^n, 0)$ be a Nash modification. Assume $k \geq 2e$. Then $\sigma_{*, k}(B_{k,e}(\sigma))$ is constructible and $B_{k,e}(\sigma) \to \sigma_{*, k}(B_{k,e}(\sigma))$ is a piecewise trivial fibration with fiber $\mathbb{R}^e$.

**Remark 4.2.**

(i) The statement of Lemma 4.1 means that there is a finite partition of $\sigma_{*, k}(B_{k,e}(\sigma))$ into constructible sets so that $B_{k,e}(\sigma) \to \sigma_{*, k}(B_{k,e}(\sigma))$ over each piece is isomorphic to a trivial fibration by a constructible homeomorphism.
Lemma 4.1 does not hold, in general, if one assumes only that \( \sigma \) is a proper Nash map. In the statement of Lemma 2.11 of [5] the assumption that \( h \) is a Nash modification is missing.

Let \( \sigma : (M, E_0) \to (\mathbb{R}^n, 0) \), \( E_0 = \sigma^{-1}(0) \), be a Nash modification. Assume that the critical locus of \( \sigma \), \( E := \{ \det(d\sigma) = 0 \} \), is a divisor with normal crossings. Let

\[
(\det(d\sigma))_0 = \sum_{i \in I} \nu_i E_i,
\]

where \( E_i \) are components of \( E \), and \( \nu_i > 0 \) for each \( i \in I \). Denote \( \nu_{\max} = \max\{\nu_i : i \in I\} \). We also assume that \( E_0 = \sigma^{-1}(0) \) is a union of components of \( E \)

\[
\sigma^{-1}(0) = \bigcup_{i \in I_0} E_i.
\]

For a vector \( j = (j_i)_{i \in I} \), \( j_i \in \mathbb{N} \), we set \( J = J(j) = \{ i : j_i \neq 0 \} \subset I \), \( E_J = \bigcap_{i \in J} E_i \) and \( \tilde{E}_J = \bigcap_{i \in \bar{J} \setminus J} E_i \). We only consider such \( j \) that \( \sigma(E_J) = \{0\} \). For such \( j \) we denote

\[
B_j = \{ \gamma \in \mathcal{L}(M, \tilde{E}_j) : \text{ord}_i E_i = j_i, \ i \in J \} \subset \tilde{L}
\]

and for \( k \in \mathbb{N} \)

\[
B_{k,j} := p_k(B_j), \quad X_{k,j}(\sigma) = \sigma_{*k}(B_{k,j}).
\]

Finally we set

\[
A_k(\sigma) = \{ j : \sigma(E_J) = \{0\} \text{ and } (\nu, j) \leq k/2 \},
\]

where \( (\nu, j) := \sum_{i \in I} \nu_i j_i \).

**Lemma 4.3.** The sets \( X_{k,j}(\sigma) \), \( j \in A_k(\sigma) \), are constructible subsets of \( L_k \) and \( \dim X_{k,j}(\sigma) = n(k + 1) - s_j - \langle \nu, j \rangle \), where \( s_j = \sum_{i \in J} j_i \). We have a disjoint union

\[
L_k = Z_k(\sigma) \sqcup \bigcup_{j \in A_k(\sigma)} X_{k,j}(\sigma),
\]

and the constructible set \( Z_k(\sigma) \) satisfies \( \dim Z_k(\sigma) < n(k + 1) - k/2\nu_{\max} \).

**Proof.** Fix \( j \) such that \( \tilde{E}_J \neq 0 \), \( \sigma(E_J) = \{0\} \), and \( 0 \leq j_i \leq k + 1 \) for \( i \in I \). Since the fiber of the natural projection \( B_{k,j}(\tilde{E}_J) \to \tilde{E}_J \) is

\[
\prod_{i \in J}(\mathbb{R}^* \times \mathbb{R}^{k-j_i}) \times (\mathbb{R}^k)^{n-|J|} \simeq (\mathbb{R}^*)^{|J|} \times \mathbb{R}^{n-k-s_j},
\]

we conclude that

\[
\dim B_{k,j} = n(k + 1) - s_j.
\]

Assume that \( j \in A_k(\sigma) \). The sets \( X_{k,j}(\sigma) \) are constructible. Indeed, \( X_{k,j}(\sigma) \) is the image of a constructible set \( B_{k,j} \) and, by Lemma 4.1, \( B_{k,j} \to X_{k,j}(\sigma) \) has all the fibers of constant Euler characteristic with compact supports equal to \( \pm 1 \). Therefore, the characteristic function \( 1_{X_{k,j}(\sigma)} \) is Nash constructible in the sense of [15], see also [16] Section 5 or [18], that implies that \( X_{k,j}(\sigma) \) is constructible. Its dimension is given by

\[
\dim X_{k,j}(\sigma) = n(k + 1) - s_j - e.
\]
We also have
\[ X_{k,j}(\sigma) \cap X_{k,j'}(\sigma) = \emptyset \quad \text{if} \quad j \neq j', \quad j, j' \in A_k(\sigma). \]
Assume that \( j \not\in A_k(\sigma) \), that is \( k < 2\langle \nu, j \rangle \). Then
\[ \dim X_{k,j}(\sigma) \leq \dim B_{k,j}(k + 1) - s_j. \]
Since \( k/2 < \langle \nu, j \rangle \leq \nu_{\max}s_j \), we have
\[ \dim X_{k,j}(\sigma) < n(k + 1) - \frac{k}{2\nu_{\max}}, \]
as claimed. \( \square \)

**Corollary 4.4.** By Lemma 4.1, \( \beta(X_{k,j}(\sigma)) = \beta(\tilde{E}_j)((u - 1)^{|j|}u^{nk-s_j-(\nu,j)} \) and hence
\[ u^{nk} = \beta(Z_k(\sigma)) + \sum_{j \in A_k(\sigma)} \beta(\tilde{E}_j)((u - 1)^{|j|}u^{nk-s_j-(\nu,j)}). \]

**Theorem 4.5.** Let \( \sigma : (M, E_0) \to (\mathbb{R}^n, 0) \) and \( \sigma' : (M', E'_0) \to (\mathbb{R}^n, 0) \) are two Nash modifications. Suppose that the critical loci \( E \), \( E' \), of \( \sigma \) and \( \sigma' \) are normal crossing divisors
\[ (\det(d\sigma))_0 = \sum_{i \in I} \nu_i E_i, \quad (\det(d\sigma'))_0 = \sum_{i' \in I'} \nu'_i E'_i, \]
and that \( \sigma^{-1}(0) \), respectively \( (\sigma')^{-1}(0) \), is a union of components of \( E \), resp. of \( E' \).

Let \( F : M \to M' \) be a Nash isomorphism such that \( F(\sigma^{-1}(0)) = (\sigma')^{-1}(0) \) and \( F(E_i) = E'_{\varphi(i)} \), for \( i \in I \) and \( \varphi : I \to I' \) is a bijection. If \( \nu_i \leq \nu'_{\varphi(i)} \) for all \( i \in I \), then \( \nu_i = \nu'_{\varphi(i)} \) for all \( i \in I \).

**Proof.** We identify \( M' \) with \( M, E'_{\varphi(i)} \) with \( E_i \), and \( I' \) with \( I \), and denote them by the same letters. By Corollary 4.4
\[ P = Q' - Q + \beta(Z_k(\sigma')) - \beta(Z_k(\sigma)) \]
where
\[ P = \sum_{j \in A_k(\sigma) \setminus A_k(\sigma')} \beta(\tilde{E}_j)((u - 1)^{|j|}u^{nk-s_j-(\nu,j)}(u^{(\nu'-\nu,j)} - 1), \]
\[ Q = \sum_{j \in A_k(\sigma) \setminus A_k(\sigma')} \beta(\tilde{E}_j)((u - 1)^{|j|}u^{nk-s_j-(\nu,j)}, \]
\[ Q' = \sum_{j \in A_k(\sigma) \setminus A_k(\sigma')} \beta(\tilde{E}_j)((u - 1)^{|j|}u^{nk-s_j-(\nu',j)}. \]

The assumption \( \nu_i \leq \nu'_i \) gives \( A_k(\sigma) \supset A_k(\sigma') \) and therefore \( Q' \equiv 0 \).

Let
\[ C_k = \{ s_j + \langle \nu, j \rangle : j \in A_k(\sigma'), \langle \nu' - \nu, j \rangle > 0 \} \]
and suppose that for \( k \) big enough \( C_k \) is nonempty. The minimum \( c_k = \min C_k \) stabilizes. Thus denote \( c = c_k \) for \( k \) big enough, say \( k \geq k_0 \). Then, for \( k \geq k_0 \),
\[ \deg P = \max \{ \dim B_{k,j} : j \in A_k(\sigma') \} = n(k + 1) - c. \]

But this, for \( k \) big enough, contradicts the following lemma.
Lemma 4.6. We have the following degree bounds:

\[ \deg \beta(Z_k(\sigma)) < n(k + 1) - \frac{k}{2\nu'_{\text{max}}}, \quad \deg \beta(Z_k(\sigma')) < n(k + 1) - \frac{k}{2\nu'_{\text{max}}}, \]

\[ \deg Q < n(k + 1) - \frac{k}{2\nu'_{\text{max}}}, \]

where \( \nu'_{\text{max}} = \max\{\nu_i : i \in I\} \) and \( \nu''_{\text{max}} = \max\{\nu'_i : i \in I'\} \).

Indeed, the degree bounds for \( \beta(Z_k(\sigma)), \beta(Z_k(\sigma')) \) are a consequence of Lemma 4.3. If \( j \in A_k(\sigma) \setminus A_k(\sigma') \), then \( k/2 < (\nu', j) \leq \nu'_{\text{max}}s_j \). We thus have

\[ s_j + (\nu; j) \geq s_j > \frac{k}{2\nu'_{\text{max}}} \]

This implies the degree bound for \( Q \) and ends the proof of Theorem 4.3. \( \square \)

4.3. Proof of Theorem 1.3. If \( f \) is arc-analytic and its graph is semialgebraic then, by [1], \( f \) is blow-analytic via a sequence of blowings-up with nonsingular algebraic centers. Similarly for \( f^{-1} \). Therefore, by [13], there are Nash modifications \( \sigma, \sigma' : M \to \mathbb{R}^n \) such that \( f \circ \sigma = \sigma' \). Consequently, Theorem 1.3 follows from Theorem 4.3 and section 3. \( \square \)

Remark 4.7. Theorem 1.3 holds true if we require \( F : M \to M' \) to be only a homeomorphism satisfying \( F(\sigma^{-1}(0)) = \sigma'^{-1}(0) \) and \( F(E_i) = E'_{\varphi(i)} \). Indeed, in this case \( \beta(E_i) = \beta(E'_{\varphi(i)}) \) and hence by additivity \( \beta(E_j) = \beta(E'_{\varphi(j)}) \).

Suppose that \( f \) is given by a commutative diagram

\[
\begin{array}{ccc}
(M, E_0) & \xrightarrow{F} & (M', E'_0) \\
\sigma \downarrow & & \downarrow \sigma' \\
(R^n, 0) & \xrightarrow{f} & (R^n, 0)
\end{array}
\]

with \( \sigma \) and \( \sigma' \) Nash modifications (e.g. compositions of regular algebraic blowings-up) and \( F \) a homeomorphism satisfying the above mentioned properties. In this case, if the jacobian determinant \( \det(df) \) is bounded, then there is a constant \( c_1 > 0 \) such that \( c_1 < |\det(df)| \). In particular, if \( f \) is Lipschitz then so is its inverse \( f^{-1} \).

This shows that Theorems 1.3 and 1.4 hold without the assumption of semialgebraicity in the two variable case.

5. Proof of Theorem 1.1

We may suppose that there is a composition of blowings-up with non-singular nowhere dense centers \( \sigma : (M, E_0) \to (R^n, 0) \) such that \( \sigma' = f \circ \sigma : (M, E_0) \to (R^n, 0) \) is analytic. Moreover, by performing additional blowings-up, as in section 3, we may assume that the critical loci of \( \sigma \) and \( \sigma' \) are normal crossings as in (1). We also assume that that \( E_0 = \sigma^{-1}(0) = \sigma'^{-1}(0) \) is a union of components of \( E \).

We denote by \( \mathcal{L}(\sigma) \) and \( \mathcal{L}(\sigma') \) the space of real analytic arcs of the source and, respectively, of the target of \( f \). Thus both are equal to \( \mathcal{L}(R^n, 0) \). Since \( f \) is arc-analytic it induces a map

\[ f_* : \mathcal{L}(\sigma) \to \mathcal{L}(\sigma'). \]
In general, this map does not factor to a map $f_{s,k} : L_k(\sigma) \to L_k(\sigma')$. This is the case, for all $k$, if $f_s$ preserves the order of contact of parametrized curves. Therefore, by the curve selection lemma, $f_{s,k}$ is well defined for all $k \in \mathbb{N}$ if and only if $f$ is Lipschitz, that we would like show.

Note that $f_s$ is injective since $f$ is a homeomorphism, and therefore $f_s^{-1}$ is well-defined on the image of $\sigma'_s$. Moreover, $f^{-1}$ is Lipschitz and therefore the jacobian determinant of $f^{-1}$ is bounded and $f_{s,k}^{-1}$ factors to the jet spaces. More precisely we have the following result.

**Lemma 5.1.** We have $\nu_i' \leq \nu_i$ for all $i \in I$. Moreover, let $\gamma_1, \gamma_2 \in \tilde{L}_k$. If $\sigma'_{s,k}(\gamma_1) = \sigma'_{s,k}(\gamma_2)$ then $\sigma_{s,k}(\gamma_1) = \sigma_{s,k}(\gamma_2)$. Hence, $f_{s,k}^{-1}$ is well-defined on $\text{Im}(\sigma'_{s,k})$.

Consider the induced maps on $k$-jets

$$
\begin{array}{cccc}
\tilde{L}_k & \xrightarrow{\sigma_{s,k}} & \tilde{L}_k \\
\xrightarrow{\sigma_{s,k}} & \xrightarrow{\sigma'_{s,k}} & \\
L_k(\sigma) & \leftarrow f_{s,k}^{-1} & \text{Im}(\sigma'_{s,k})
\end{array}
$$

Then, similarly to Subsection 4.2 we have a decomposition

$$\text{Im}(\sigma'_{s,k}) = Z_k(\sigma') \sqcup \bigcup_{j \in A_k(\sigma)} X_{k,j}(\sigma').$$

Note that $X_{k,j}(\sigma') = (f_{s,k}^{-1})(X_{k,j}(\sigma)) = \sigma'_{s,k}(B_{k,j})$, and $Z_k(\sigma') = (f_{s,k}^{-1})(Z_k(\sigma))$. We have $\sigma_{s,k} = f_{s,k}^{-1} \circ \sigma'_{s,k}$ and since $\sigma_{s,k}$ is epi so is $f_{s,k}^{-1}$. Therefore

$$\dim Z_k(\sigma') \geq \dim Z_k(\sigma).$$

By Lemma 5.1 $A_k(\sigma') \supset A_k(\sigma)$. We divide $A_k(\sigma)$ into two pieces

$$A'_k(\sigma) = \{ j \in A_k(\sigma) : \langle \nu, j \rangle = \langle \nu', j \rangle \},$$

$$A''_k(\sigma) = \{ j \in A_k(\sigma) : \langle \nu, j \rangle > \langle \nu', j \rangle \}.$$

If $\sigma'$ is not a Nash modification then the statement of Lemma 5.1 may not hold for $\sigma'$. Nevertheless, the computations of the proofs of Lemma 4.2 and Lemma 3.4 of 3 give the following local result. We denote by $p_{k, k-e} : \tilde{L}_k \to \tilde{L}_{k-e}$ the truncation.

**Lemma 5.2.** Assume $k \geq 2e$. Let $\gamma \in B_{k,e}(\sigma')$, $\gamma(0) \in E_0$. Then the fibre of

$$p_{k, k-e}^{-1}(p_{k, k-e}(\gamma)) \cap B_{k,e}(\sigma') \to \sigma'_{s,k}(B_{k,e}(\sigma')),$$

containing $\gamma$, is an affine subspace of $p_{k, k-e}^{-1}(p_{k, k-e}(\gamma))$ of dimension $e$.

This lemma holds without assuming that $\sigma'$ is birational or a Nash modification. Thus any fibre of $B_{k,e}(\sigma') \to \sigma'_{s,k}(B_{k,e}(\sigma'))$ contains an affine space of dimension $e$.

**Corollary 5.3.** If $j \in A_k(\sigma)$ then $\dim X_{k,j}(\sigma') = \dim X_{k,j}(\sigma) + \langle \nu - \nu', j \rangle$. Moreover, if $j \in A_k(\sigma)$, then $f_{s,k}^{-1}$ induces a bijection $X_{k,j}(\sigma) \leftarrow X_{k,j}(\sigma')$.

**Proof.** Let $\gamma \in B_{k,j}$. Then we have the inclusions

$$\sigma'_{s,k}^{-1}(\sigma'_{s,k}(\gamma)) \subset \sigma'_{s,k}(\sigma_{s,k}(\gamma)) \simeq \mathbb{R}^e \subset p_{k, k-e}^{-1}(p_{k, k-e}(\gamma)).$$
where $e = \langle \nu, j \rangle$. Therefore, by Lemma 5.2

$$\sigma'_{s,k}^{-1}(\sigma'_{s,k}(\gamma)) \simeq R e' ,$$

where $e' = \langle \nu', \overline{j} \rangle$. This gives the first claim of corollary.

If $\overline{j} \in A'_k(\sigma)$, then $\sigma'_{s,k}^{-1}(\sigma'_{s,k}(\gamma)) = \sigma'_{s,k}(\sigma_{s,k}(\gamma))$ and the second claim of corollary follows.

The map $f_{s,k}^{-1}$ is constructible, that is its graph is an $\mathcal{AS}$ set, and therefore, by [18],

$$\beta(X_{k,j}(\sigma)) = \beta(\tilde{X}_{k,j}(\sigma')) .$$

Thus we have

$$(3) \quad \beta(L_k) - \beta(\text{Im}(\sigma'_{s,k})) = \sum_{j \in A'_{\sigma}(\sigma)} (\beta(X_{k,j}(\sigma)) - \beta(\tilde{X}_{k,j}(\sigma'))) + (\beta(Z_k(\sigma)) - \beta(\tilde{Z}_k(\sigma'))) .$$

The leading coefficient of the left-hand side is positive (if $\text{Im}(\sigma'_{s,k}) \neq L_k$), and the leading coefficient of the first summand of the right-hand side is negative. The leading coefficient of the second summand is also negative unless $\dim \tilde{Z}_k(\sigma') = \dim Z_k(\sigma)$.

Thus, necessarily, $\dim \tilde{Z}_k(\sigma') = \dim Z_k(\sigma)$ and if $j \in A'_{\sigma}(\sigma)$ then $\dim \tilde{X}_{k,j}(\sigma') \leq \dim Z_k(\sigma)$. But this is impossible by the argument of proof of Theorem 1.3. Indeed, for fixed $j$ by letting $k \to \infty$, we obtain by Lemma 4.6 the opposite inequality $\dim \tilde{X}_{k,j}(\sigma') > \dim Z_k(\sigma)$. Thus $\nu_i = \nu'_i$ for all $i \in I$ and $f$ is Lipschitz by Corollary 2.3.

Finally, $f^{-1}$ is blow-analytic by Theorem 1.5. \square

**Remark 5.4.** In the proof of Theorem 1.1 it is not necessary to use Theorem 1.5. Instead, we can argue directly as follows. By (3) we see that $\sigma'_{s,k}$ is surjective, since the codimension of $Z_k(\sigma')$ goes to infinity as $k \to \infty$. If $f^{-1}$ were not blow-analytic, there would have been a real analytic arc $\gamma(t) = \sum a_i t^i$ such that

$$(4) \quad f^{-1}(\gamma(t)) = \sum_{i=1}^m b_i t^i + bt^{p/q} + \cdots, \quad b \neq 0 ,$$

where $m < p/q < m + 1$. Changing the higher terms of $\gamma$, if necessary, we may assume that for $k \gg m$, $p_k(\gamma) \in X_{k,j}(\sigma')$, and $f_{s,k}^{-1}(\gamma) \in X_{k,j}(\sigma)$, that contradicts (4).

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**Department of Mathematics, Faculty of Science, Saitama University, 255 Shimo-Okubo, Urawa 338, Japan**

*E-mail address*: tfukui@rimath.saitama-u.ac.jp

**Laboratoire de Mathématiques, UMR 5175 du CNRS, Université de Savoie, Campus Scientifique, 73 376 Le Bourget-du-Lac Cedex, France**

*E-mail address*: kurdyka@univ-savoie.fr

**Laboratoire J. A. Dieudonné U.M.R. 6621 du CNRS, Université de Nice Sophia-Antipolis, 28, Parc Valrose 06108 Nice Cedex 02, France**

*E-mail address*: adam.parusinski@unic.fr