Gauge Deformations and Embedding Theorems for Special Geometries

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Abstract
We reduce the embedding problem for hypo $SU(2)$ and $SU(3)$-structures to the embedding problem for hypo $G_2$-structures into parallel Spin(7)-manifolds. The latter will be described in terms of gauge deformations. This description involves the intrinsic torsion of the initial $G_2$-structure and allows us to prove that the evolution equations, for all of the above embedding problems, do not admit non-trivial longtime solutions.

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Introduction

In [13] N.Hitchin introduced a flow equation for cocalibrated $G_2$-structures on a manifold $M$, whose solutions yield parallel Spin(7)-structures on $I \times M$, for some interval $I \subset \mathbb{R}$. In this sense, a solution of the flow equation embeds the initial $G_2$-structure into a manifold with a parallel Spin(7)-structure and is therefore called a solution of the embedding problem for the initial structure. Similar equations are

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known for embedding $SU(2)$-structures in dimension five and $SU(3)$-structures in dimension six into manifolds with a parallel $SU(3)$ and $G_2$-structure, respectively, cf. [6, 7, 8, 9, 10]. The natural candidates for solving the embedding problem are so-called hypo structures. In the Gray-Hervella classification these are the type of structures induced on hypersurfaces of spaces with a parallel structure. Hypo $SU(3)$-structures are also called half-flat structures, whereas hypo $G_2$-structures are often called cocalibrated structures. R. Bryant shows in [3] that in the real analytic category, the embedding problem for hypo $SU(3)$ and $G_2$-structures can be solved. Bryant also provided counterexamples in the smooth category. The embedding problem for $SU(2)$-structures in dimension five was solved by D. Conti and S. Salamon in [7], cf. also [6].

The purpose of this article is to describe a unifying approach to all of the above embedding problems. We reduce the $SU(2)$ and $SU(3)$ embedding problem to the $G_2$-case, which will be studied in terms of gauge deformations, i.e. automorphism of the tangent bundle. Since the structure tensor $\varphi \in \Omega^3(M)$ of a $G_2$-structure is stable, any smooth deformation $\varphi_t$ can be described by a family of gauge deformations $A_t \in C^\infty(\text{Aut}(TM))$ via $\varphi_t = A_t \varphi$. It seems to be coincidence, that in the $G_2$-case, the intrinsic torsion $T$ takes values in the $G_2$-module $\mathfrak{gl}(7)$ and therefore can be regarded again as an (infinitesimal) gauge deformation. In Proposition 3.1 we show that the intrinsic torsion flow for $G_2$-structures

$$\dot{A}_t = T_t \circ A_t$$

can be regarded as a generalization of Hitchin’s flow equation, and hence as a generalization of the $SU(2)$, $SU(3)$ and $G_2$-embedding problem. We describe the evolution of the metric and the intrinsic torsion under the intrinsic torsion flow, cf. Theorem 3.2 As a consequence of the Cheeger-Gromoll Splitting Theorem, we prove in Theorem 3.3 and Corollary 3.4 that there are no nontrivial longtime solutions for the embedding problem.

In chapter 2 we develop a conservation law for certain integral curves in Fréchet spaces, cf. Corollary 2.5. The basic idea stems from finite dimensional geometry: If a vector field $X$ is tangent to some submanifold $N$, then any integral curve of $X$, which lies initially in $N$, stays in $N$ for all times. This does not hold for arbitrary integral curves in Fréchet spaces, but the Cauchy-Kowalevski Theorem states - beyond the existence - that the integral curves in question can be developed in a (convergent) power series. This property allows us to prove that the intrinsic torsion flow preserves certain compatibility conditions, which implies that for any real analytic hypo $SU(2)$, $SU(3)$ and $G_2$-structure on a compact manifold, the embedding problem admits a unique real analytic solution. Moreover, the solution
can be described by a family of gauge deformations
\[ A_t = \sum_{k=0}^{\infty} \frac{t^k}{k!} A_0^{(k)}, \]
where the series converges in the \( C^\infty \)-topology on \( C^\infty(\text{End}(TM)) \).

Our technique seems to be applicable to a wide class of evolution problems, where the initial structure is real analytic. For instance, instead of embedding a certain \( G \)-structure into a manifold with a parallel structure, one can ask for an embedding into a space with a nearly parallel structure, cf. [6]. The Cauchy-Kowalevski Theorem 2.12 ensures the existence of a solution for the corresponding evolution equations. This solution has to satisfy certain (non-linear) compatibility conditions. Since Corollary 2.5 can be generalized to integral curves in Fréchet manifolds, it suffices to show that the evolution equations define a vector field which is tangent to the compatibility conditions.

1. The Embedding Problem for Special Geometries

A \( G \)-structure on a manifold \( M \) is a reduction of the structure group of the frame bundle to a certain Lie subgroup \( G \subset GL(n) \). We are interested in the cases where \( M = M^n \) is a compact oriented manifold of dimension \( n \in \{5, 6, 7, 8\} \) and
\[ G \in \{SU(2), SU(3), G_2, \text{Spin}(7)\}. \]
The above groups can be realized as the isotropy group of certain model forms \( \varphi_0 \in \Lambda^k \mathbb{R}^n \), under the natural action of \( GL^+(n) \). The corresponding forms \( \varphi \in C^\infty(\Lambda^k T^*M) \) on \( M \) are called the structure tensors of the \( G \)-structure. A positive basis of \( T_p M \), for which \( \varphi \sim \varphi_0 \), is called a Cayley frame for \( \varphi \) and we say that \( \varphi \) is of type \( \varphi_0 \). Since \( G \subset SO(n) \), the structure tensors induce a metric \( g = g(\varphi) \) on \( M \) and we denote by \( \nabla^g \) the Levi-Civita connection of the metric. The structure is called parallel if \( \nabla^g \varphi = 0 \) holds. In the above cases, \( \nabla^g \varphi = 0 \) can be translated into the apparently weaker conditions \( d\varphi = d*\varphi = 0 \).

Example 1.1. A Spin(7)-structure on \( M^8 \) can be described by a single 4-form \( \Psi \) of type
\[ \Psi_0 = e^{3456} + e^{3478} + e^{5678} - e^{2358} + e^{2468} - e^{2457} - e^{2367} + e^{1357} - e^{1467} - e^{1458} - e^{1368} + e^{1234} + e^{1256} + e^{1278}, \]
where \( \text{Iso}_{GL(8)}(\Psi_0) = \text{Spin}(7) \) holds. A Spin(7)-structure is parallel if \( d\Psi = 0 \) holds, cf. [14].

**Example 1.2.** A \( G_2 \)-structure on \( M^7 \) can be described by a single 4-form \( \psi \) of type
\[
\psi_0 = e^{2345} + e^{2367} + e^{4567} - e^{1247} + e^{1357} - e^{1346} - e^{1256},
\]
where \( \text{Iso}_{GL(7)}(\psi_0) = G_2 \) holds. Given an orientation \([\varepsilon]\) for \( M^7 \), we can define a positive volume element \( \varepsilon := \varepsilon(\psi) \in \Lambda^7 T^* M^7 \) and a metric \( g = g(\psi) \), cf. [13]. Then the Hodge dual \( \varphi := *\psi \) is of model type
\[
\varphi_0 = e^{246} - e^{356} - e^{47} + e^{124} + e^{145} + e^{167}.
\]
A \( G_2 \)-structure is parallel if \( d\varphi = d\psi = 0 \) holds.

**Example 1.3.** A \( SU(3) \)-structure on \( M^6 \) can be described by a 4-form \( \sigma \) and a 3-form \( \rho \) of type
\[
\sigma_0 = e^{1234} + e^{1256} + e^{3456},
\rho_0 = e^{135} - e^{245} - e^{236} - e^{146},
\]
where \( \text{Iso}_{GL(6)}(\sigma_0, \rho_0) = SU(3) \) holds. Given an orientation for \( M^6 \), we can define positive volume elements \( \omega := \omega(\sigma, \rho) \in \Lambda^6 T^* M^6 \), cf. [13]. We consider \( \sigma \) as an element \( \sigma \in \text{Hom}(\Lambda^2 T^* M^6, \Lambda^2 T^* M^6) \) and define
\[
\omega := \frac{1}{2} \sigma(\omega^*) \in \Lambda^2 T^* M^6,
\]
where \( \omega^* \in \Lambda^2 T^* M^6 \) is defined by \( \sigma = \omega^* \otimes \varepsilon(\sigma) \in \Lambda^2 T^* M^6 \otimes \Lambda^6 T^* M^6 \). Then \( \omega \) is of type \( \omega_0 = e^{12} + e^{34} + e^{56} \) and
\[
2\alpha(X)\omega := \rho \wedge (X,\rho) \wedge \alpha,
\hat{\rho} := -I(\rho),\rho,\quad 2g(X, Y)\omega := (X,\rho) \wedge (Y,\rho) \wedge \omega,
\]
\( (X,Y) \in TM^6, \alpha \in \Lambda^1 T^* M^6 \) define tensors of type \( \mu_0 = e_1 \wedge e_2 + \ldots + e_5 \wedge e_6 \), \( \hat{\mu}_0 = e^{136} - e^{240} + e^{235} + e^{145} \) and \( \gamma_0 = \sum_{i=1}^{6} e^i \otimes e^i \), respectively. A \( SU(3) \)-structure is parallel if \( d\omega = d\rho = d\hat{\rho} = 0 \) holds.

**Example 1.4.** A \( SU(2) \)-structure on \( M^5 \) can be described by a 2-form \( \omega_1 \) and two 3-forms \( \rho_2 \) and \( \rho_3 \) of type
\[
\omega_1 = e^{23} + e^{45},
\rho_2 = e^{124} - e^{135},
\rho_3 = e^{125} + e^{134},
\]
where \( \text{Iso}_{GL(5)}(\omega_1, \rho_2, \rho_3) = SU(2) \) holds, cf. Lemma [4.1]. Given an orientation for \( M^5 \), we can define a positive volume element \( \varepsilon := \varepsilon(\omega_1, \rho_2, \rho_3) \in \Lambda^5 T^* M^5 \), see
Lemma 4.2. Then
\[2\alpha(X)\varepsilon := (X \cdot \rho_2) \wedge \rho_2,\]
\[\omega_2(X,Y)\varepsilon := -(X \cdot \omega_1) \wedge (Y \cdot \omega_1) \wedge \rho_2,\]
\[\omega_3(X,Y)\varepsilon := -(X \cdot \omega_2) \wedge (Y \cdot \omega_1) \wedge \rho_3,\]
\[g(X,Y)\varepsilon := \alpha(X)\alpha(Y)\varepsilon + \alpha \wedge \omega_1 \wedge (X \cdot \omega_2) \wedge (Y \cdot \omega_3),\]
\[(X,Y \in TM^6)\text{ define tensors of type } \alpha_0 = e_1, \omega_2 = e^{24} - e^{35}, \omega_3 = e^{25} + e^{34} \text{ and } g_0 = \sum_{i=1}^5 e^i \otimes e^i, \text{ respectively.}\]

In the previous examples, the model tensors in dimension \(n + 1\) can be constructed from the model tensors in dimension \(n\). This is due to the fact that the inclusions
\[SU(2) \subset SU(3) \subset G_2 \subset \text{Spin}(7)\]
can be realized as isotropy groups of certain unit vectors. In the following we will consider families of structures on \(M\) which depend on a parameter \(t \in I \subset \mathbb{R}\) and evolve under certain evolution equations. These equations actually guarantee that the induced structure on \(I \times M\) is parallel. For instance, consider a family of \(G_2\)-structures \(\psi_t\) on \(M^7, t \in I\). Then
\[\Psi := \psi_t + dt \wedge \varphi_t\]
defines a \(\text{Spin}(7)\)-structure on \(M^8 := I \times M^7\) and
\[d^8\Psi = d^7\psi_t + dt \wedge \dot{\psi}_t - dt \wedge d^7\varphi_t = d^7\psi_t + dt \wedge (\dot{\psi}_t - d^7\varphi_t),\]
where \(d^7, d^8\) denotes the exterior differential on \(M^7, M^8\), respectively. Hence the \(\text{Spin}(7)\)-structure is parallel if and only if \(d^7\dot{\psi}_t = 0\) and \(\dot{\psi}_t = d\varphi_t\). The second equation can be regarded as an evolution equation for the initial structure \(\varphi := \varphi_{t=0}\), whereas \(G_2\)-structures with \(d\psi_t = 0\) are called hypo structures. Note that the evolution equation preserves the hypo condition \(d\psi = 0\). In the following Proposition we list the lifting maps for the \(SU(2), SU(3)\) and \(G_2\)-case, the hypo condition for the initial structure and the evolution equations to obtain parallel structures on \(I \times M^n\).

**Proposition 1.5.** Let \(M^n\) be a manifold of dimension \(n \in \{5, 6, 7\}\), equipped with a family of
\[G_n := \begin{cases} SU(2) & , n = 5 \\ SU(3) & , n = 6 \\ G_2 & , n = 7 \\ (\text{Spin}(7) & , n = 8) \end{cases}\]
structures. Then the lift in the following table defines a \(G_{n+1}\)-structure on \(M^{n+1} := I \times M^n\):
\[
\begin{array}{|c|c|c|c|}
\hline
n & \text{Lift} & \text{Hypo Condition} & \text{Evolution} \\
\hline
5 & \omega := \omega_1 + dt \wedge \alpha & 0 = d\omega_1 & \dot{\omega}_1 = d\alpha \\
& \sigma := \frac{1}{2} \omega_1^2 + dt \wedge \alpha \wedge \omega_1 & 0 = d\rho_2 & \dot{\rho}_2 = d\omega_3 \\
& \rho := -\rho_3 + dt \wedge \omega_2 & 0 = d\rho_3 & \dot{\rho}_3 = -d\omega_2 \\
& \hat{\rho} := \rho_2 + dt \wedge \omega_3 & & \\
\hline
6 & \varphi := \rho + dt \wedge \omega & 0 = d\rho & \dot{\rho} = d\omega \\
& \psi := \sigma - dt \wedge \hat{\rho} & 0 = d\sigma & \dot{\sigma} = -d\hat{\rho} \\
\hline
7 & \Psi := \psi + dt \wedge \varphi & 0 = d\psi & \dot{\psi} = d\varphi \\
\hline
\end{array}
\]

(1) The structure on \(M^{n+1}\) is parallel if and only if the initial structure is hypo and evolves according to the evolution equations from the table.

(2) The metric of the \(G_{n+1}\)-structure on \(I \times M^n\) is given by \(g = dt^2 + g_t\), where \(g_t\) is the family of metrics induced by the \(G_n\)-structures on \(M^n\).

**Proof:** Choosing a Cayley frame \((E_1(t), ..., E_n(t))\) for the family of \(G_n\)-structures, we obtain a Cayley frame for the lift by 

\[
\left(\frac{d}{dt}, E_1(t), ..., E_n(t)\right).
\]

This proves that the lift actually defines a \(G_{n+1}\)-structure and that the metric is given by the formula in (2). The proof of (1) is similar to the \(G_2\)-case.

\[\square\]

**Definition 1.6.** Let \(M^n\) be a manifold of dimension \(n \in \{5, 6, 7\}\), equipped with a hypo \(G_n\)-structure. A family of \(G_n\)-structures which solves the evolution equations from Proposition 1.5 and equals the initial structure at \(t = 0\) is called a solution of the embedding problem for the initial \(G_n\)-structure.

The lift from Proposition 1.5 does not preserve the hypo condition. This motivates

**Definition 1.7.** Let \(M^n\) be a manifold of dimension \(n \in \{5, 6\}\), equipped with a \(G_n\)-structure. We call

\[
\begin{array}{|c|c|c|}
\hline
n = 5 & n = 6 \\
\hline
\omega := \omega_3 + d\theta \wedge \alpha & \varphi := -\hat{\rho} + d\theta \wedge \omega \\
\sigma := \frac{1}{2} \omega_3^2 + d\theta \wedge \rho_3 & \psi := \sigma - d\theta \wedge \rho \\
\rho := \rho_2 - d\theta \wedge \omega_1 & \hat{\rho} := -\alpha \wedge \omega_1 - d\theta \wedge \omega_2 \\
\hline
\end{array}
\]

the hypo lift of the \(G_n\)-structure to \(S^1 \times M^n\). Conversely, given a \(G_{n+1}\)-structure on a manifold \(M^{n+1}\), we obtain a \(G_n\)-structure on any oriented hypersurface \(i : M^n \hookrightarrow M^{n+1}\) by
\[ n = 5 \quad \begin{array}{c|c}
\omega_1 := -i^*(\frac{\partial}{\partial \rho} \rho) \\
\rho_2 := i^* \rho \\
\rho_3 := i^*(\frac{\partial}{\partial \sigma} \sigma)
\end{array} \quad n = 6 \quad \begin{array}{c|c}
\rho := -i^*(\frac{\partial}{\partial \psi} \psi) \\
\sigma := i^* \psi
\end{array} \]

where \( \frac{\partial}{\partial \theta} \) is a global vector field along \( i : M^n \hookrightarrow M^{n+1} \), which is orthonormal to \( M^n \). We call the \( G_n \)-structure the structure induced by the \( G_{n+1} \)-structure and \( \frac{\partial}{\partial \theta} \).

Note that we just applied the lifts from Proposition 1.5 to the structures

\[ (\alpha, \omega_3, -\omega_1, -\omega_2) = A(\alpha, \omega_1, \omega_2, \omega_3), \]

respectively,

\[ (\omega, -\tilde{\rho}, \rho) = I(\omega, \rho, \tilde{\rho}), \]

where \( A \in GL^+(5) \) is defined by

\[ A(e_1, \ldots, e_5) := (e_1, e_3, e_4, e_2, e_5). \]

Lemma 1.8. The hypo lift maps hypo structures to hypo structures.

Proof: In the \( SU(2) \)-case, we obtain \( d\rho = 0 \) if \( d\omega_1 = d\rho_2 = 0 \). The compatibility condition \( \omega_2^3 = \omega_1^2 \) and \( dp_3 = 0 \) imply \( d\sigma = 0 \). For a hypo \( SU(3) \)-structure we obtain immediately \( d\psi = d\sigma + d\theta \wedge d\rho = 0 \).

\[ \square \]

We will now study the compatibility of the hypo lift with the evolution equations from Proposition 1.5.

Lemma 1.9. (1) Suppose \( \psi \) is a family of \( G_2 \)-structures on \( M^7 = S^1 \times M^6 \) which is the hypo lift of some family of \( SU(3) \)-structure \( (\rho, \sigma) \) on \( M^6 \). Then

\[ \dot{\psi} = d\varphi \quad \Leftrightarrow \quad \begin{cases} 
\dot{\rho} = d\omega \\
\dot{\sigma} = -d\tilde{\rho}
\end{cases} \]

(2) Suppose \( (\rho, \sigma) \) is a family of \( SU(3) \)-structures on \( M^6 = S^1 \times M^5 \) which is the hypo lift of some family of \( SU(2) \)-structure \( (\omega_1, \rho_2, \rho_3) \) on \( M^5 \). Then

\[ \begin{cases} 
\dot{\rho} = d\omega \\
\dot{\sigma} = -d\tilde{\rho}
\end{cases} \quad \Leftrightarrow \quad \begin{cases} 
\dot{\omega}_1 = d\alpha \\
\dot{\rho}_2 = d\omega_3 \\
\dot{\rho}_3 = -d\omega_2 \\
(\frac{1}{2} \dot{\omega}_3^2) = d(\alpha \wedge \omega_1)
\end{cases} \]
Proof: By assumption we have \( \psi = \sigma - d\theta \wedge \rho \) and \( \varphi = -\hat{\rho} + d\theta \wedge \omega \). Hence

\[
\dot{\psi} = \dot{\sigma} - d\theta \wedge \dot{\rho} \quad \text{and} \quad d\varphi = -d\hat{\rho} - d\theta \wedge d\omega
\]

and part (1) follows. Similarly for part (2),

\[
\omega = \omega_3 + d\theta \wedge \alpha, \quad \sigma = \frac{1}{2}\omega_3^2 + d\theta \wedge \rho_3,
\]

\[
\rho = \rho_2 - d\theta \wedge \omega_1, \quad \hat{\rho} = -\alpha \wedge \omega_1 - d\theta \wedge \omega_2
\]

gives

\[
\dot{\rho} = \dot{\rho}_2 - d\theta \wedge \dot{\omega}_1,
\]

\[
d\omega = d\omega_3 - d\theta \wedge d\alpha,
\]

and

\[
\dot{\sigma} = \left( \frac{1}{2}\omega_3^2 \right) + d\theta \wedge \dot{\rho}_3,
\]

\[
-\dot{d}\hat{\rho} = d(\alpha \wedge \omega_1) - d\theta \wedge d\omega_2.
\]

\[\square\]

Lemma 1.10. Let \( \psi \) be a \( G_2 \)-structure on \( M^7 \) with metric \( g \).

(1) If \( M^7 = S^1 \times M^6 \), then \( \psi \) is the hypo lift of some \( SU(3) \)-structure on \( M^6 \) if and only if

\[
L_{\frac{\partial}{\partial \theta}} \psi = 0, \quad g(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta}) = 1.
\]

(2) If \( M^7 = S^1_1 \times S^1_1 \times M^5 \), then \( \psi \) is the hypo lift of some \( SU(2) \)-structure on \( M^5 \) if and only if

\[
L_{\frac{\partial}{\partial \theta^i}} \psi = 0, \quad g(\frac{\partial}{\partial \theta^i}, \frac{\partial}{\partial \theta^j}) = \delta_{ij},
\]

for \( i, j = 1, 2 \).

Proof: If \( \psi \) is the hypo lift of some \( SU(2) \) or \( SU(3) \)-structure, we get \( L_{\frac{\partial}{\partial \theta^i}} \psi = 0 \) and the orthogonality condition on the \( S^1 \)-directions. Conversely, define forms \( \sigma \) and \( \rho \) on \( M^7 \) by

\[
\psi = \frac{\partial}{\partial \theta^i} (d\theta \wedge \psi) + d\theta \wedge \left( \frac{\partial}{\partial \theta^i} \psi \right).
\]

Since \( \frac{\partial}{\partial \theta^i} \) is orthonormal to \( M^6 \) and \( G_2 \) acts transitively on \( S^6 \), we can find a Cayley frame for which \( \sigma \) and \( \rho \) are of model type. Hence \( (\sigma, \rho) \) defines a \( SU(3) \)-structure on each hypersurface \( \{ e^{i\theta} \} \times M^6 \). Since

\[
0 = L_{\frac{\partial}{\partial \theta^i}} \sigma - d\theta \wedge L_{\frac{\partial}{\partial \theta^i}} \rho
\]

implies \( L_{\frac{\partial}{\partial \theta^i}} \sigma = L_{\frac{\partial}{\partial \theta^i}} \rho = 0 \), we see that \( \sigma \) and \( \rho \) are actually constant along the flow of \( \frac{\partial}{\partial \theta^i} \). Part (2) of the Lemma follows similarly, using that \( G_2 \) acts transitively on
pairs of orthonormal vectors.

2. Integral Curves in Fréchet Spaces and the Cauchy-Kowalevski Theorem

Hamilton [12] gives an introduction to Fréchet manifolds which goes far beyond of what we require for our purposes. Although Proposition [2.4] and Corollary [2.5] can be generalized to Fréchet manifolds, we focus on Fréchet spaces to keep the technical effort at a minimum.

A locally convex topological vector space $F$ is a vector space with a collection of seminorms $\{\|\cdot\|_n\}_{n \in \mathbb{N}}$, i.e. functions $\{\|\cdot\|_n\}_n : F \to \mathbb{R}$ which satisfy

$$\|f\| \geq 0, \quad \|f + g\| \leq \|f\| + \|g\| \quad \text{and} \quad \|\lambda f\| = |\lambda|\|f\|,$$

for all $f, g \in F$ and scalars $\lambda$. Such a family defines a unique topology which is metrizable if and only if $\mathbb{N}$ is countable. In this case the topology is characterized by the property

$$\lim_{k \to \infty} f_k = f \in F \iff \lim_{k \to \infty} \|f_k - f\|_n = 0 \text{ for all } n \in \mathbb{N}.$$ 

The topology is Hausdorff if and only if $\|f\|_n = 0$ for all $n \in \mathbb{N}$, implies that $f = 0$. The space is sequentially complete if every Cauchy sequence converges, where $f_k$ is a Cauchy sequence if it is a Cauchy sequence for every seminorm $\|\cdot\|_n$.

**Definition 2.1.** A Fréchet space is a locally convex topological vector space, which is in addition metrizable, Hausdorff and complete.

**Example 2.2.** Suppose $F \to M$ is a vector bundle over a compact manifold $M$. Then the vector space

$$\mathcal{F} := C^\infty(F)$$

of smooth sections of $F$ is a Fréchet space, where the collection of seminorms

$$\|f\|_n := \sum_{j=0}^n \sup_{p \in M} |(\nabla^{(j)} f)(p)|$$

can be defined after choosing Riemannian metrics and connections on $TM$ and $F$, cf. [12] Example 1.1.5. The induced topology is the $C^\infty$ topology on $\mathcal{F}$. Given an open subset $U \subset F$, we consider the subset of all sections in $\mathcal{F}$, whose
image lies in \( U \),

\[ \mathcal{U} := \{ f \in F \mid f(M) \subset U \}. \]

For \( f \in \mathcal{U} \) we can find \( \varepsilon > 0 \) such that

\[ f \in B^0_\varepsilon(f) := \{ \tilde{f} \in F \mid \| \tilde{f} - f \|_0 < \varepsilon \} \subset \mathcal{U}. \]

Since \( B^0_\varepsilon(f) \subset F \) is open, \( \mathcal{U} \) is an open subset of the Fréchet space \( F \).

Smooth maps between Fréchet spaces can be defined as follows: Let \( U \subset F \) be an open subset of a Fréchet space \( F \) and \( P : U \to \mathcal{E} \) a continuous and nonlinear map into another Fréchet space \( \mathcal{E} \). We say that \( P \) is \( C^1 \) on \( U \) if for every \( f \in U \) and every \( v \in F \) the limit

\[ DP(f)v := \lim_{t \to 0} \frac{1}{t}(P(f + tv) - P(f)) \]

exists and the map \( DP : U \times F \to \mathcal{E} \) is continuous. Consequently, we say that \( P \) is \( C^k \) on \( U \) if \( P \) is \( C^{k-1} \) and the limit

\[ D^{(k)}P(f)v \{ v_1, \ldots, v_k \} := \lim_{t \to 0} \frac{1}{t} \left( D^{(k-1)}P(f+tv)v \{ v_1, \ldots, v_{k-1} \} - D^{(k-1)}P(f)v \{ v_1, \ldots, v_{k-1} \} \right) \]

exists for all \( f \in U \) and \( v_1, \ldots, v_k \in F \), and the map \( D^{(k)}P : U \times F \times \ldots \times F \to \mathcal{E} \) is continuous. We call \( P \) a smooth map on \( U \) if \( P \) is \( C^k \) for all \( k \in \mathbb{N} \). We summarize Corollary 3.3.5 and Theorem 3.6.2 from [12] in the following:

**Theorem 2.3.** (1) If \( P : U \subset F \to \mathcal{E} \) is \( C^1 \) and \( c(t) \in U \subset F \) is a parametrized \( C^1 \) curve, then \( P \circ c(t) \) is a parametrized \( C^1 \) curve and

\[ \frac{\partial}{\partial t}(P \circ c(t)) = DP(c(t)) \dot{c}(t). \]

(2) If \( P : U \subset F \to \mathcal{E} \) is \( C^k \), then for every \( f \in U \)

\[ D^{(k)}P(f)v \{ v_1, \ldots, v_k \} \]

is completely symmetric and linear separately in \( v_1, \ldots, v_k \in F \).

In the following we will consider curves \( c(t) \in F \) in a Fréchet space \( F \), which are integral curves of a vector field that is tangent to some subspace \( \mathcal{E} \subset F \). In finite dimension we would expect that any such integral curve with \( c(0) \in \mathcal{E} \) actually stays in the subspace for all times. This conclusion fails for Fréchet spaces, as was pointed out to us by Christian Bär: Consider \( F := C^\infty[1,2] \) and \( \mathcal{E} := \{ 0 \} \subset \mathcal{M} \).

Then

\[ c_t(x) := \begin{cases} (4\pi t)^{-\frac{1}{2}} \exp(-\frac{x^2}{4t}), & \text{for } t > 0 \\ 0, & \text{for } t \leq 0 \end{cases} \]

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solves $\dot{c}_t = \Delta c_t = \partial^2 c_t / \partial x^2$ and hence defines an integral curve of the vector field $X(c) := \Delta c$. Although $X$ is tangent to $\mathcal{E}$, i.e. $X(0) = 0$, and $c_0 = 0 \in \mathcal{E}$, the curve doesn’t stay in $\mathcal{E}$, since $c_t \neq 0$, for $t > 0$. Note also that $t \mapsto c_t(x)$ is not real analytic in $t = 0$.

**Proposition 2.4.** Suppose $\mathcal{E} \subset \mathcal{F}$ is a closed subspace of the Fréchet space $\mathcal{F}$ and that $X : U \subset \mathcal{F} \to \mathcal{F}$ is a smooth map defined on some open subset $U \subset \mathcal{F}$. Let $f \in \mathcal{F}$ and assume that $X|_{U \cap \mathcal{E}_f} : U \cap \mathcal{E}_f \to \mathcal{E}_f$, where $\mathcal{E}_f := \{f\} + \mathcal{E}$. If a smooth curve $c : (-\varepsilon, \varepsilon) \to \mathcal{F}$ satisfies $c(0) \in U \cap \mathcal{E}_f$ and $X \circ c(t) = \dot{c}(t)$, where $\dot{c} : (-\varepsilon, \varepsilon) \to \mathcal{F}$ is the derivative of $c(t)$ by $t$, then for all $k \geq 1$

$$c^{(k)}(0) \in \mathcal{E},$$

where $c^{(k)} : (-\varepsilon, \varepsilon) \to \mathcal{F}$ is the $k^{th}$ derivative of $c(t)$ by $t$.

**Proof:** First we prove by induction on $k$ that the $k^{th}$ differential $D^{(k)}X$ of $X : \mathcal{F} \to \mathcal{F}$ satisfies

$$D^{(k)}X_{|U \cap \mathcal{E}_f} : U \cap \mathcal{E}_f \times \mathcal{E} \times \ldots \times \mathcal{E} \to \mathcal{E}.$$  

For $k = 0$ this is just the assumption $X|_{U \cap \mathcal{E}_f} : U \cap \mathcal{E}_f \to \mathcal{E}_f$. For $v_0 \in U \cap \mathcal{E}_f$ and $v_1, \ldots, v_k \in \mathcal{E}$ we have by definition $D^{(k+1)}X(v_0)\{v_1, \ldots, v_k+1\}$

$$= \lim_{s \to 0} \frac{1}{s} (D^{(k)}X(v_0 + sv_{k+1})\{v_1, \ldots, v_k\} - D^{(k)}X(v_0)\{v_1, \ldots, v_k\}),$$

and since $\mathcal{E}$ is closed, we conclude that (1) holds for $k + 1$. Next we show that for $k \geq 0$ and any choice of smooth curves $t \mapsto v_0(t) \in U$ and $t \mapsto v_1(t), \ldots, v_k(t) \in \mathcal{F}$

$$\frac{\partial}{\partial t} D^{(k)}X(v_0(t))\{v_1(t), \ldots, v_k(t)\} = D^{(k+1)}X(v_0(t))\{v_1(t), \ldots, v_k(t), \dot{v}_0(t)\}$$

$$+ \sum_{j=1}^{k} D^{(k)}X(v_0(t))\{v_1(t), \ldots, \dot{v}_j(t), \ldots, v_k(t)\}$$

(2)
holds. Applying Theorem 2.3 (1) to the map $D^{(k)}X : U \times F \times \ldots \times F \to F$, we get

$$\frac{\partial}{\partial t}D^{(k)}X(v_0(t), \ldots, v_k(t))$$

$$= D(D^{(k)}X)(v_0(t), \ldots, v_k(t))\{\dot{v}_0(t), \ldots, \dot{v}_k(t)\}$$

$$= \lim_{s \to 0} \frac{1}{s} \left( D^{(k)}X(v_0(t) + s\dot{v}_0(t))\{v_1(t) + s\dot{v}_1(t), \ldots, v_k(t) + s\dot{v}_k(t)\} - D^{(k)}X(v_0(t))\{v_1(t), \ldots, v_k(t)\} \right)$$

and (2) follows, since $D^{(k)}X$ is linear in the arguments in $\{\ldots\}$, cf. Theorem 2.3 (2). We will now show by induction on $k$ that $c^{(k)}(0) \in E$ holds. For $k = 1$ we have $\dot{c}(0) = X \circ c(0) \in E$ by assumption. Since $c(t) = X \circ c(t) = D^{(0)}X(c(t))$ and $c(t) \in U$ for sufficiently small $t$, we can apply (2) to see that $c^{(k+1)}(t)$, again for sufficiently small $t$, can be expressed as a linear combination of

$$D^{(j)}X(c(t))\{v_1(t), \ldots, v_j(t)\},$$

where $j \in \{1, \ldots, k+1\}$ and $v_1(t), \ldots, v_j(t) \in \{c^{(l)}(t) | 1 \leq l \leq k\}$. Since $c(0) \in U \cap E$, we get from $c^{(1)}(0), \ldots, c^{(k)}(0) \in E$ and (1)

$$D^{(j)}X(c(0))\{v_1(0), \ldots, v_j(0)\} \in E$$

and hence $c^{(k+1)}(0) \in E$.

The following corollary can be regarded as a conservation law for certain integral curves in Fréchet spaces.

**Corollary 2.5.** If the curve $c : (-\varepsilon, \varepsilon) \to F$ from Proposition 2.4 satisfies for all $t \in (-\varepsilon, \varepsilon)$

$$c(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} c^{(k)}(0) \in F,$$

where the series converges w.r.t. the Fréchet topology in $F$, then

$$c(t) - c(0) \in E,$$

for all $t \in (-\varepsilon, \varepsilon)$.

**Proof:** From Proposition 2.4 we get $c^{(k)}(0) \in E$ for all $k \geq 1$ and hence

$$c(t) - c(0) = \sum_{k=1}^{\infty} \frac{t^k}{k!} c^{(k)}(0) \in E,$$

since $E \subset F$ is closed and the series converges in $F$. □
A formal power series in \( X = (X_1, \ldots, X_n) \) with coefficients in \( \mathbb{R} \) is an expression of the form

\[
S(X) = \sum_{p \in \mathbb{N}^n} a_p X^p,
\]
where \( a_p \in \mathbb{R} \) and \( X^p := X_1^{p_1} \cdots X_n^{p_n} \), for \( p = (p_1, \ldots, p_n) \in \mathbb{N}^n \). Given a formal power series \( S(X) \), we define

\[
\Gamma := \{ r = (r_1, \ldots, r_n) \mid r_i \geq 0 \text{ and } \sum_{p \in \mathbb{N}^n} |a_p| r^p < \infty \}
\]
and denote by \( \Delta \) the interior of \( \Gamma \), called the domain of convergence of the series.

Hence the series

\[
S(x) = \sum_{p \in \mathbb{N}^n} a_p x^p
\]
is for every \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) with \( |x| = (|x_1|, \ldots, |x_n|) \in \Gamma \) absolute convergent.

We recall the following result:

**Proposition 2.6.** Suppose \( S(X) \) is a formal power series with domain of convergence \( \Delta \). For \( \bar{x} = (\bar{x}_1, \ldots, \bar{x}_n) \in \mathbb{R}^n \) with \( |\bar{x}| \in \Delta \) and \( r_1, \ldots, r_n \) with \( 0 < r_i < |\bar{x}_i| \), define

\[
K := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid |x_i| \leq r_i \}.
\]

(1) For any subset \( P \subseteq \mathbb{N}^n \), the series

\[
S_P(x) := \sum_{p \in P} a_p x^p
\]
converges absolutely for all \( x \in K \). In particular, the series \( S(x) := \sum_{p \in \mathbb{N}^n} a_p x^p \) converges absolutely for \( x \in K \).

(2) Suppose that \( P_N \subseteq \mathbb{N}^n \) is a family of subsets, \( N \in \mathbb{N} \), such that \( \lim_{N \to \infty} P_N = \mathbb{N}^n \). Then

\[
S_N(x) := \sum_{p \in P_N} a_p x^p
\]
converges uniformly on \( K \) to the function \( S : K \to \mathbb{R}, x \mapsto S(x) \).

**Proof:** Since \( |\bar{x}| \in \Delta \) we can find \( C > 0 \) such that

\[
|a_p \bar{x}^p| \leq C, \quad \text{for all } p \in \mathbb{N}^n.
\]

Hence for \( x \in K \)

\[
|a_p x^p| = |a_p \bar{x}_1^{p_1} \cdots \bar{x}_n^{p_n} |x_1^{p_1} \cdots x_n^{p_n}| \leq C \left( \frac{r_1}{|x_1|} \right)^{p_1} \cdots \left( \frac{r_n}{|x_n|} \right)^{p_n}.
\]
Since $r_i/|\bar{x}_i| < 1$, we can apply the method of majorants to see that $S_P(x)$ converges absolutely for $x \in K$. To prove uniform convergence consider

$$\sup_{x \in K} |S(x) - S_N(x)| = \sup_{x \in K} \left| \sum_{p \in \mathbb{N}^n \setminus P_N} a_p x^P \right| \leq C \sum_{p \in \mathbb{N}^n \setminus P_N} \left( \frac{r_1}{|\bar{x}_1|} \right)^{p_1} \cdots \left( \frac{r_n}{|\bar{x}_n|} \right)^{p_n}$$

Given $\varepsilon > 0$, we can choose $M$ large, so that $\sum_{i=1}^{\infty} \sum_{p_i=M+1}^{\infty} \left( \frac{r_i}{|\bar{x}_i|} \right)^{p_i} \leq \frac{\varepsilon}{nCC_i}$, for $i = 1, \ldots, n$, where

$$C_i := \sum_{\hat{p} \in \mathbb{N}^{n-1}} \left( \frac{r_1}{|\bar{x}_1|} \right)^{p_1} \cdots \left( \frac{r_i}{|\bar{x}_i|} \right)^{p_i} \cdots \left( \frac{r_n}{|\bar{x}_n|} \right)^{p_n} < \infty \quad \text{(geometric series)}.$$ 

The notation $\hat{.}$ means that the corresponding factor is omitted. Since $\lim_{N \to \infty} P_N = \mathbb{N}^n$, we can find $N = N(M)$, such that $\{0, \ldots, M\}^n \subset P_N$. Hence

$$\sup_{x \in K} |S(x) - S_N(x)| \leq C \sum_{p \in \mathbb{N}^n \setminus \{0, \ldots, M\}^n} \left( \frac{r_1}{|\bar{x}_1|} \right)^{p_1} \cdots \left( \frac{r_n}{|\bar{x}_n|} \right)^{p_n} \leq \varepsilon.$$

\[ \square \]

**Definition 2.7.** Let $U \subset \mathbb{R}^n$ open and $x_0 \in U$.

1. A function $f : U \to \mathbb{R}$ is called real analytic in $x_0 \in U$ if there exists a formal power series $S$ with

$$f(x) = S(x - x_0),$$

for all $x$ in a neighborhood of $x_0$.

2. A function $f : U \to \mathbb{R}$ is called real analytic in $U$ if $f$ is real analytic for every $x_0 \in U$.

3. A function $F = (f_1, \ldots, f_m) : U \to \mathbb{R}^m$ is called real analytic in $U$ if each component $f_i : U \to \mathbb{R}$ is real analytic in $U$.

Note that the coefficients of $S$ can be computed in terms of partial derivatives, which shows that $S$ is uniquely determined by the condition $f(x) = S(x - x_0)$. Moreover we have the following basic properties, cf. [5] p.123:

**Lemma 2.8.**

1. If $f : U \to \mathbb{R}$ is real analytic in $x_0 \in U$, then it is differentiable in a neighborhood of $x_0$ and the derivatives are again real analytic functions...
in $x_0 \in U$.

(2) If $f$ and $g$ are real analytic in $x_0$, then the product $fg$ is real analytic in $x_0$.

(3) If $f : U \to \mathbb{R}$ is real analytic, then $1/f$ is real analytic in all points $x \in U$, where $f(x) \neq 0$.

(4) Compositions of real analytic functions are again real analytic.

A manifold $M$ is called real analytic if it admits an atlas with real analytic transition functions. Similarly to the smooth category one can define real analytic vector bundles over $M$.

In the following we will develop a global version of the Cauchy-Kowalevski Theorem, cf. [4], III. Theorem 2.1:

**Theorem 2.9.** Let $t$ be a coordinate on $\mathbb{R}$, $x = (x_i)$ be coordinates on $\mathbb{R}^n$, $y = (y_j)$ be coordinates on $\mathbb{R}^s$ and let $z = (z_j^i)$ be coordinates on $\mathbb{R}^{ns}$. Let $D \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^{ns}$ open, and let $G : D \to \mathbb{R}^s$ be a real-analytic mapping. Let $D_0 \subset \mathbb{R}^n$ be open and $f : D_0 \to \mathbb{R}^s$ be a real-analytic mapping with Jacobian $Df(x) \in \mathbb{R}^{ns}$, i.e. $z_j^i(Df(x)) = \partial f_j^i(x)/\partial x_i$, so that $\{(t_0, x, f(x), Df(x)) \mid x \in D_0\} \subset D$ for some $t_0 \in \mathbb{R}$.

Then there exists an open neighborhood $D_1 \subset \mathbb{R} \times D_0$ of $\{t_0\} \times D_0$ and a real-analytic mapping $F : D_1 \to \mathbb{R}^s$ which satisfies

$$\begin{align*}
\frac{\partial F}{\partial t}(t,x) &= G(t,x,F(t,x),\frac{\partial F}{\partial x}(t,x)) \\
F(t_0,x) &= f(x) \quad \text{for all } x \in D_0.
\end{align*}$$

$F$ is unique in the sense that any other real-analytic solution of the above initial value problem agrees with $F$ in some neighborhood of $\{t_0\} \times D_0$.

**Remark 2.10.** Since the solution $F = (f_1, \ldots, f_s) : D_1 \to \mathbb{R}^s$ from Theorem 2.9 is real analytic, we can develop each component in a convergent power series around $(t_0, x_0) = (0, 0) \in D_1$, i.e.

$$f_i(t,x) = \sum_{k=0}^{\infty} \left( \sum_{p \in \mathbb{N}^n} a_{ikp} x^p \right) t^k = \sum_{k=0}^{\infty} \left( \frac{1}{k!} f_i^{(k)}(0,x) \right) t^k.$$ 

Applying Proposition 2.6 (2) with $P_N := \{0, \ldots, N\} \times \mathbb{N}^n$ shows that

$$f_i^N(t,x) = \sum_{k=0}^{N} \left( \sum_{p \in \mathbb{N}^n} a_{ikp} x^p \right) t^k = \sum_{k=0}^{N} \frac{1}{k!} f_i^{(k)}(0,x).$$
converges locally uniformly to the function $f_i(t, x)$, for $N \to \infty$. The partial derivatives of a formal power series $S(X)$ are defined by,

$$\frac{\partial S}{\partial X_i} := \sum_{p \in \mathbb{N}^n} p_i a_p X_1^{p_1} \cdots X_i^{p_i-1} \cdots X_n^{p_n}.$$  

The formal power series $\frac{\partial S}{\partial X_i}$ has the same domain of convergence $\Delta$ as the formal power series $S$. Moreover, the function $\frac{\partial S}{\partial X_i} : \Delta \to \mathbb{R}$ is the partial derivative of the function $S : \Delta \to \mathbb{R}$ w.r.t. $x_i$, cf. Satz 3.2 in [5]. Hence we can apply again Proposition 2.6 (2) to see that all partial derivatives of the function $f_i^N(t, x)$ converge locally uniformly to the corresponding partial derivative of $f_i(t, x)$. In summary, the functions

$$F_N(t, x) := \sum_{k=0}^N \frac{t^k}{k!} F^{(k)}(0, x)$$

converge, as $N \to \infty$, locally in $C^\infty$-topology to the solution $F(t, x)$ from Theorem 2.9.

**Definition 2.11.** Suppose $M$ is a real analytic manifold and $\pi : V \to M$ is a rank $s$ real analytic vector bundle. We call a map

$$X : C^\infty(V) \to C^\infty(V)$$

a real analytic first order differential operator if every point of $M$ has a neighborhood $U \subset M$, which is the domain of a real analytic chart $u : U \to \mathbb{R}^n$, and there exists a real analytic trivialization $(\pi, v) : V|_U \cong U \times \mathbb{R}^s$, together with a real analytic function

$$G : D \subset \mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^{ns} \to \mathbb{R}^s,$$

such that for every local section $c : U \subset M \to V$

$$v(X \circ c) = G(u, v \circ c, \frac{\partial c_i}{\partial u_j})$$

holds, where $c_i$ is the $i^{th}$ component of $v \circ c : U \to \mathbb{R}^s$.

We can now prove the following global version of the Cauchy-Kowalevski Theorem,

**Theorem 2.12.** Suppose $\pi : V \to M$ is a real analytic rank $s$ vector bundle over a compact real analytic manifold $M$. Let $X : C^\infty(V) \to C^\infty(V)$ be a real analytic first order differential operator and let $c_0 \in C^\infty(V)$ be a real analytic section. Then the initial value problem

$$\begin{cases}
\dot{c}(t) = X \circ c(t) \\
c(0) = c_0
\end{cases}$$

converges locally uniformly to the function $f_i(t, x)$ for $N \to \infty$. The partial derivatives of a formal power series $S(X)$ are defined by,

$$\frac{\partial S}{\partial X_i} := \sum_{p \in \mathbb{N}^n} p_i a_p X_1^{p_1} \cdots X_i^{p_i-1} \cdots X_n^{p_n}.$$  

The formal power series $\frac{\partial S}{\partial X_i}$ has the same domain of convergence $\Delta$ as the formal power series $S$. Moreover, the function $\frac{\partial S}{\partial X_i} : \Delta \to \mathbb{R}$ is the partial derivative of the function $S : \Delta \to \mathbb{R}$ w.r.t. $x_i$, cf. Satz 3.2 in [5]. Hence we can apply again Proposition 2.6 (2) to see that all partial derivatives of the function $f_i^N(t, x)$ converge locally uniformly to the corresponding partial derivative of $f_i(t, x)$. In summary, the functions

$$F_N(t, x) := \sum_{k=0}^N \frac{t^k}{k!} F^{(k)}(0, x)$$

converge, as $N \to \infty$, locally in $C^\infty$-topology to the solution $F(t, x)$ from Theorem 2.9.

**Definition 2.11.** Suppose $M$ is a real analytic manifold and $\pi : V \to M$ is a rank $s$ real analytic vector bundle. We call a map

$$X : C^\infty(V) \to C^\infty(V)$$

a real analytic first order differential operator if every point of $M$ has a neighborhood $U \subset M$, which is the domain of a real analytic chart $u : U \to \mathbb{R}^n$, and there exists a real analytic trivialization $(\pi, v) : V|_U \cong U \times \mathbb{R}^s$, together with a real analytic function

$$G : D \subset \mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^{ns} \to \mathbb{R}^s,$$

such that for every local section $c : U \subset M \to V$

$$v(X \circ c) = G(u, v \circ c, \frac{\partial c_i}{\partial u_j})$$

holds, where $c_i$ is the $i^{th}$ component of $v \circ c : U \to \mathbb{R}^s$.

We can now prove the following global version of the Cauchy-Kowalevski Theorem,
has a unique real analytic solution $c : (-\varepsilon, \varepsilon) \to C^\infty(V)$, i.e. $c : (-\varepsilon, \varepsilon) \times M \to V$ is real analytic. Moreover, the solution $c(t)$ satisfies

$$c(t) = \sum_{k=0}^{\infty} \left( \frac{t^k}{k!} \right) c_k,$$

where the series converges in the $C^\infty$ topology on $C^\infty(V)$.

**Proof:** We will first show that we can find local sections $c_t : U \subset M \to V$, which solve the initial value problem locally. Secondly, we prove that the compactness of $M$ ensures the existence of a global solution. Eventually we will use the uniqueness part of the Cauchy-Kowalevski Theorem to prove the uniqueness statement of the Theorem.

By Definition 2.11 we can find a real analytic chart $u : U \subset M \to \mathbb{R}^n$ and a trivialization $(\pi, v) : V|_U \cong U \times \mathbb{R}^s$, such that for each local section $c : U \subset M \to V$

(1) $v(X \circ c) = G(u, v \circ c, \frac{\partial c_i}{\partial u_j})$

holds, where $G : D \subset \mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^{ns} \to \mathbb{R}^s$ is real analytic. The map

$$f : D_0 := u(U) \subset \mathbb{R}^n \to \mathbb{R}^s \text{ with } f(x) := v \circ c_0 \circ u^{-1}(x)$$

is real analytic and hence we can find by the Cauchy-Kowalevski Theorem a real analytic solution $F : (-\varepsilon, \varepsilon) \times \tilde{D}_0 \to \mathbb{R}^s$ of

$$\begin{cases}
\frac{\partial F}{\partial t}(t, x) = G(x, F(t, x), \frac{\partial F}{\partial x}(t, x)) \\
F(t_0, x) = f(x) \text{ for all } x \in D_0,
\end{cases}$$

where $\tilde{D}_0 \subset D_0$ is open. Let $\tilde{U} := u^{-1}(\tilde{D}_0) \subset U$ and define for $t \in (-\varepsilon, \varepsilon)$

(2) $c(t) : \tilde{U} \subset M \to V$ by $c(t, p) := v_p^{-1} \circ F(t, u(p))$.

where $v_p : V_p \cong \mathbb{R}^s$ is the isomorphism induced by the local trivialization $(\pi, v)$. By definition, the map $c : (-\varepsilon, \varepsilon) \times \tilde{U} \subset M \to V$ is real analytic and satisfies

(3) $c(0, p) = v_p^{-1} \circ F(0, u(p)) = v_p^{-1} \circ f(u(p)) = c_0(p)$.

Now we have for $i = 1, \ldots, s$ and $j = 1, \ldots, n$

$$\frac{\partial (v_i \circ c_j)}{\partial u_j}(p) = \frac{\partial}{\partial u_j}(v_i \circ c_j) = (u_*^{-1} \frac{\partial}{\partial x_j} \bigg|_{u(p)}) \cdot (v_i \circ c_j)$$

$$= \frac{\partial}{\partial x_j} \bigg|_{u(p)} \cdot (v_i \circ c_j \circ u^{-1}) = \left. \frac{\partial}{\partial x_j} F_i(t, \cdot) \right|_{u(p)}.$$

(4) $\frac{\partial F_i}{\partial x_j}(t, u(p))$. 

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Remark 2.10

Suppose now we apply the above construction to obtain two local sections

$$T_0 \text{ to prove uniqueness, suppose that we have two real analytic solutions } c_1, c_2 \text{ and since } M \text{ is compact, the series converges in } C^\infty \text{ topology.}

From (4) we get

$$c(t, p) = \sum_{k=0}^\infty \frac{t^k}{k!} c^{(k)}(0, p),$$

i.e. $c_i$ is the desired local solution of the initial value problem. Moreover, we get by Remark 2.10

$$c(t, p) = v_p^{-1} \circ F(t, u(p)) = v_p^{-1}(\lim_{N \to \infty} \sum_{k=0}^N \frac{t^k}{k!} F^{(k)}(0, u(p)))$$

$$= \lim_{N \to \infty} \sum_{k=0}^N \frac{t^k}{k!} F^{(k)}(0, u(p)) = \lim_{N \to \infty} \sum_{k=0}^N \frac{t^k}{k!} c^{(k)}(0, p),$$

i.e.

$$c_1 = \sum_{k=0}^\infty \frac{t^k}{k!} c^{(k)}(0),$$

where the series converges locally in $C^\infty$ topology.

Suppose now we apply the above construction to obtain two local sections

$$c_1(t) : U_1 \subset M \to V \text{ and } c_2(t) : U_2 \subset M \to V,$$

where $t \in (-\epsilon, \epsilon)$, $\epsilon := \min\{\epsilon_1, \epsilon_2\}$ and $U_1 \cap U_2 \neq \emptyset$. Since $c_1$ and $c_2$ both solve the initial value problem

$$\begin{cases} 
\dot{c}_i(t) = X \circ c_i(t) \\
c_i(0) = c_0,
\end{cases}$$

$i = 1, 2$, we see that $c_1(0) = c_2(0)$ and $\dot{c}_1(0) = \dot{c}_2(0)$ on $U_1 \cap U_2$. Differentiating the equation $\dot{c}_i(t) = X \circ c_i(t)$, shows that $c_i^{(k+1)}(t)$ can be expressed as a linear combination of

$$D^{(j)}(d) X(c_i(t))\{v_1(t), \ldots, v_j(t)\},$$

where $j \in \{1, \ldots, k+1\}$ and $v_1(t), \ldots, v_j(t) \in \{c_i^{(l)}(t) | 1 \leq l \leq k\}$, cf. the proof of Proposition 2.4. Now we obtain by induction $c_1^{(k)}(0) = c_2^{(k)}(0)$ on $U_1 \cap U_2$, for all $k \in \mathbb{N}$. Hence (5) implies $c_1(t) = c_2(t)$ on $U_1 \cap U_2$. If $M$ is compact, we can cover $M$ by finitely many domains $U_1, \ldots, U_N$ of local sections $c_i(t) : U_i \subset M \to V$, which yield a global section $c(t) : M \to V$, where $t \in (-\epsilon, \epsilon)$ and $\epsilon := \min\{\epsilon_1, \ldots, \epsilon_N\}$. From (4) we get

$$c(t) = \sum_{k=0}^\infty \frac{t^k}{k!} c^{(k)}(0),$$

and since $M$ is compact, the series converges in $C^\infty$ topology.

To prove uniqueness, suppose that we have two real analytic solutions $c_1, c_2 : (-\epsilon, \epsilon) \times M \to V$ of the initial value problem. By (1) we have for $k = 1, 2$ and
Now \( F_k(t, x) := v \circ c_k(t) \circ u^{-1}(x) \) satisfies

\[
\frac{\partial F_k}{\partial t}(t, x) = v \circ \dot{c}_k(t) \circ u^{-1}(x) = v \circ X \circ c_k(t) \circ u^{-1}(x)
\]

and by (4)

\[
\frac{\partial c_k(t)}{\partial u_j} \circ u^{-1}(x) = \frac{\partial(v \circ c_k(t))}{\partial u_j}(u^{-1}(x)) = \frac{\partial F_k}{\partial x_j}(t, x),
\]

for \( i = 1, \ldots, s \) and \( j = 1, \ldots, n \). Hence we showed

\[
\frac{\partial F_k}{\partial t}(t, x) = G(x, F_k(t, x), \frac{\partial F_k}{\partial x_j}(t, x)).
\]

Since \( F_1 \) and \( F_2 \) are both real analytic and satisfy

\[
F_1(0, x) = v \circ c_1(0) \circ u^{-1}(x) = v \circ c_0 \circ u^{-1}(x) = v \circ c_2(0) \circ u^{-1}(x) = F_2(0, x),
\]

the uniqueness part of the Cauchy-Kowalevski Theorem yields \( F_1(t, x) = F_2(t, x) \), i.e. \( c_1(t) = c_2(t) \).

\[\square\]

3. The Model Case \( G_2 \subset \text{Spin}(7) \)

Lemma [10] and [11] motivate the conjecture that the embedding problem for hypo \( SU(2) \) and \( SU(3) \)-structures might be reduced to the embedding problem for \( G_2 \)-structures. The reduction to the \( G_2 \)-case has the advantage that no compatibility conditions are involved. To solve the embedding problem for hypo structures we consequently focus on studying the evolution equation

\[
\dot{\psi}_t = d\varphi_t
\]

on a compact seven dimensional manifold \( M \). We will describe the solution \( \psi_t \) by a family of gauge deformations, i.e.

\[
\psi_t = A_t \psi,
\]

where \( A_t \in C^\infty(\text{Aut}(TM)) \). Since the orbit of the model tensor \( \psi \in \Lambda^4\mathbb{R}^7^* \) is open, it follows that any smooth deformation \( \psi_t \) of the initial structure \( \psi \) can be described in such a way. The evolution equation \( \dot{\psi}_t = d\varphi_t \) can be translated into an equation for the family of gauge deformations. This description involves the intrinsic torsion of the \( G_2 \)-structure \( \psi_t \). The intrinsic torsion \( T \in \text{End}(TM) \) of a \( G_2 \)-structure \( \varphi \) is
defined by
\[ \nabla^g_X \varphi = -T_X \varphi, \]
where we used that
\[ \nabla^g \varphi \in T^* M \otimes \Lambda^3 T^* M \]
and
\[ \Lambda^3 T^* M := \{ \alpha \in \Lambda^3 T^* M \mid \alpha = X \cdot \varphi, X \in TM \}, \]
cf. for instance [3]. From \( \psi = \ast \varphi \) it follows that \( d \psi = 2 \text{pr}_{\Lambda^2} (T) \wedge \varphi \) holds. So hypo \( G_2 \)-structures are characterized by \( T \in S^2 (TM) \) w.r.t. the metric \( g \).

**Proposition 3.1.** Suppose \( \psi_t = A_t \psi \) is a family of \( G_2 \)-structures on \( M^7 \), described by a family of gauge deformations \( A_t \in C^\infty (\text{Aut} (TM^7)) \). If \( T_t \) is the intrinsic torsion of \( \psi_t \), then
\[ \dot{\psi}_t = d\varphi_t \iff D_{\psi_t} (A_t \circ A_t^{-1}) = D_{\psi_t} (T_t), \]
where
\[ D_{\psi_t} : \text{End} (TM) \to \Lambda^2 T^* M \]
is defined by \( A \mapsto \frac{d}{ds} \bigg|_{s=0} \exp (sA) \psi_t \).

**Proof:** Since clearly \( \dot{\psi}_t = D_{\psi_t} (A_t A_t^{-1}) \), it suffices to observe that
\[ D_{\psi_t} (T_t) (X_1, ..., X_4) = - \sum_{i=1}^4 \psi_t (X_1, ..., T_t X_j, ..., X_4) \]
\[ = \sum_{i=1}^4 (-1)^i \psi_t (T_t X_j, X_1, ..., \hat{X}_j, ..., X_4) \]
\[ = \sum_{i=1}^4 (-1)^{i+1} (\nabla^g_{X_j} \varphi_t) (X_1, ..., \hat{X}_j, ..., X_4) \]
\[ = d\varphi_t (X_1, ..., X_4) \]
holds. \( \Box \)

We can now compute the evolution of the metric and the torsion endomorphism.

**Theorem 3.2.** Let \( \psi_t \) be a family of hypo \( G_2 \)-structures on \( M^7 \), which evolves under the flow \( \dot{\psi}_t = d\varphi_t \). Then the evolution of the underlying metric \( g_t \) and the torsion endomorphism \( T_t \) are given by
\[ \dot{g}_t (X, Y) = 2 g_t (T_t X, Y), \]
\[ \dot{T}_t X = \text{Ric}_t X - \text{tr}(T_t) T_t X, \]
where \( \text{Ric}_t = \text{Ric}(g_t) \) is the Ricci tensor of the metric \( g_t \).
Proof: Writing $\psi_t = A_t \psi$, Proposition 3.1 yields $D_{\psi_t}(A_t \circ A_t^{-1}) = D_{\psi_t}(T_t)$. Since the evolution $\dot{\psi}_t = d\varphi_t$ preserves the hypo condition $d\psi_t = 0$, or equivalently $T_t \in S^2_{w.r.t.} g_t$, we get
\[ \text{pr}_{S^2}(A_t \circ A_t^{-1}) = T_t, \]
since $\ker(D_{\psi_t}) = g_2$. Then we compute for $g_t = A_t g$
\[ \dot{g}_t(X,Y) = 2g_t(\text{pr}_{S^2}(A_t \circ A_t^{-1})X,Y) = 2g_t(T_tX,Y). \]
The metric $g = dt^2 + g_t$ on $I \times M^7$ has holonomy contained in Spin(7) and hence is Ricci flat. The Gauss equations and the Codazzi-Mainardi equations yield
\[ \dot{g}_t(X,Y) = 2g_t(W_t X,Y), \]
\[ g_t(W_t X,Y) = \text{ric}_t(X,Y) - \text{tr}(W_t)g_t(W_t X,Y), \]
where $W_t X := \nabla^g_{\Phi_t X} \frac{d}{dt}$ is the Weingarten map and $\Phi_t$ is the flow of the vector field $\frac{d}{dt}$, cf. for instance [1]. So $W_t = T_t$ and the Theorem follows.

We will now apply the Cheeger-Gromoll Splitting Theorem to prove that the flow $\dot{\psi} = d\varphi$ does not admit nontrivial longtime solutions.

Theorem 3.3. Suppose $\psi$ is a hypo $G_2$-structures on a compact manifold $M^7$. Then the flow $\dot{\psi}_t = d\varphi_t$ is defined for all times $t \in \mathbb{R}$ if and only if the initial structure is already parallel.

Proof: The metric on the product $M^8 := \mathbb{R} \times M^7$ has holonomy contained in Spin(7) and hence is Ricci flat. Since $g = dt^2 + g_t$, the first factor actually defines a line. Now we can apply the Cheeger-Gromoll Splitting Theorem and see that $M^8$ splits as a Riemannian product. Note that the line, i.e. the first factor of $M^8$, is actually the one dimensional factor that splits off in the decomposition as a Riemannian product, cf. Lemma 6.86 in [2]. Hence $g_t = g_0$ is constant and Theorem 3.2 yields $T_t = 0$.

In Lemma 4.9 (1) we showed that a longtime solution of the $SU(3)$ embedding problem would yield a longtime solution for the $G_2$ embedding problem. Combining part (1) and (2) of Lemma 4.9 shows that a longtime solution of the $SU(2)$ embedding problem would also yield a longtime solution for the $G_2$ embedding problem if in addition the equation $(\frac{1}{2}\omega_1^2) = d(\alpha \wedge \omega_1)$ is satisfied. If the initial $SU(2)$-structure is hypo, we have $d\omega_1 = 0$, for all times $t$. So
\[ (\frac{1}{2}\omega_1^2) = (\frac{1}{2}\omega_1^2) = \omega_1 \wedge \omega_1 = \omega_1 \wedge d\alpha = d(\alpha \wedge \omega_1) \]
and we obtain the following $SU(2)$ and $SU(3)$-analogue of Theorem 3.3.

**Corollary 3.4.** There are no nontrivial longtime solutions for the hypo $SU(2)$ and $SU(3)$ embedding problem on compact manifolds.

\[
\square
\]

In view of Proposition 5.1 the following theorem yields solutions of the $G_2$ embedding problem.

**Theorem 3.5.** Let $\psi$ be a real analytic hypo $G_2$-structure on the compact manifold $M^7$. Then the intrinsic torsion flow

\[
\begin{aligned}
\dot{A}_t &= T_t \circ A_t \\
A_0 &= \text{id}
\end{aligned}
\]

has a unique real analytic solution $A : (-\varepsilon, \varepsilon) \times M \to \text{End}(TM)$. Moreover, the solution $A_t$ is of the form

\[
A_t = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^{(k)}_0,
\]

where the series converges in the $C^\infty$-topology on $C^\infty(\text{End}(TM))$.

**Proof:** To apply Theorem 2.12 we have to show that the map

\[
X : C^\infty(\text{Aut}(TM)) \to C^\infty(\text{End}(TM)) \quad \text{with} \quad X \circ A := T(A\varphi) \circ A
\]

is a real analytic first order differential operator in the sense of Definition 2.11. For this choose local coordinates $u : U \subset M \to \mathbb{R}^7$, for which $\varphi$ is real-analytic. These coordinates induce a local trivialization $(\pi, v)$ of the bundle $\pi : \text{End}(TM) \to M$ via

\[
v(A) := \{a_{kl}\}_{k,l=1,7}, \quad \text{where} \quad A = \sum_{k,l=1}^{7} a_{kl} du_k \otimes \frac{\partial}{\partial u_l} \in \text{End}(TM).
\]

For a fixed local section $A = \sum a_{kl} du_k \otimes \frac{\partial}{\partial u_l} : U \to \text{Aut}(TM)$ write

\[
X \circ A = T(A\varphi) \circ A = \sum_{a,b=1}^{7} f_{ab} du_a \otimes \frac{\partial}{\partial u_b}.
\]

Now it suffices to find an expression

\[
f_{ab} = G_{ab}(u, a_{kl}, \frac{\partial a_{kl}}{\partial u_l})
\]

for the coefficients $f_{ab} : U \to \mathbb{R}$, where $G_{ab} : D \subset \mathbb{R}^7 \times \mathbb{R}^{49} \times \mathbb{R}^{343} \to \mathbb{R}$ is real analytic. The formula

\[
\nabla^A \varphi A\varphi = -T(A\varphi) \circ (A\varphi)
\]

shows that the intrinsic torsion is a first order invariant of the $G_2$-structure and hence we can find an expression of the form (1) that is actually polynomial in $a_{kl}$.
and $\frac{\partial a_{ij}}{\partial u^j}$, and real analytic in $u$, since the initial structure is real analytic.

\[\square\]

**Lemma 3.6.** Suppose $\psi$ is a $G_2$-structure on $M$ and $F \in \text{Diff}(M)$. Then the intrinsic torsion satisfies

$$T(F^*\psi) = F^* T(\psi) = F_*^{-1} T(\psi) F_*.$$

**Proof:** By Koszul’s formula we have $F_*(\nabla_{F_*^* X} Y) = \nabla^{F_*} X F_*^* Y$ and hence

$$(\nabla^{F_*} F_*^* \varphi) = F^* (\nabla^{F_*} \varphi).$$

Since $\nabla^g \varphi = -T \cdot \psi$, we get

$$T(F^* \psi) X \cdot F^* \varphi = -\nabla^{F_*} F_*^* \varphi = -F^* (\nabla^{F_*} \varphi)$$

$$= F^* (T(\psi) F_* X \cdot \psi) = F_*^{-1} T(\psi) F_* X \cdot F^* \psi$$

and the Lemma follows from the non-degeneracy of $F^* \psi$.

\[\square\]

**Lemma 3.7.** Suppose $\psi$ is a $G_2$-structure on $M^7 = S^1 \times \ldots \times S^1 \times M^{7-k}$, which is the hypo lift of some $SU(4-k)$-structure on $M^{7-k}$. Then the Ricci tensor $\text{Ric}$ of the metric $g = g(\psi)$ satisfies for each $S^1$-direction $\frac{\partial}{\partial \theta}$

$$L_{\frac{\partial}{\partial \theta}} \text{Ric} = \text{Ric} \frac{\partial}{\partial \theta} = d\theta \circ \text{Ric} = 0.$$

The intrinsic torsion $T$ satisfies

$$L_{\frac{\partial}{\partial \theta}} T = T \frac{\partial}{\partial \theta} = 0$$

and $d\theta \circ T = 0$ if the structure is hypo.

**Proof:** If $\psi$ is the hypo lift of some structure on $M^{7-k}$, then $g = d\theta^2 + \ldots + d\theta_k + g_{7-k}$, for some metric $g_{7-k}$ on $M^{7-k}$. Hence the Ricci tensor satisfies $\text{Ric} \frac{\partial}{\partial \theta} = 0$,

$$d\theta \circ \text{Ric} = g(\frac{\partial}{\partial \theta}, \text{Ric}) = g(\text{Ric} \frac{\partial}{\partial \theta},.) = 0$$

and

$$L_{\frac{\partial}{\partial \theta}} \text{Ric} = \frac{\partial}{\partial s} \bigg|_{s=0} \Phi_* \text{Ric}(g) = \frac{\partial}{\partial s} \bigg|_{s=0} \text{Ric}(\Phi_* g) = \frac{\partial}{\partial s} \bigg|_{s=0} \text{Ric}(g) = 0.$$

Since $0 = \nabla^g \varphi = -T \frac{\partial}{\partial \theta} \cdot \psi$, we get $T \frac{\partial}{\partial \theta} = 0$ and similarly $0 = (L_{\frac{\partial}{\partial \theta}} T) \cdot \psi$ implies $L_{\frac{\partial}{\partial \theta}} T = 0$. If the structure is hypo, i.e. $T$ is symmetric, we get in addition

$$d\theta \circ T = g(\frac{\partial}{\partial \theta}, T) = g(T \frac{\partial}{\partial \theta},.) = 0.$$

\[\square\]
Lemma 3.8. Suppose $\psi$ is a $G_2$-structure on $M^7 = S^1 \times \ldots \times S^1 \times M^{7-k}$, which is the hypo lift of some $SU(4-k)$-structure on $M^{7-k}$. If $A \in C^\infty(\text{Aut}(TM))$ satisfies
\[ A \frac{\partial}{\partial \theta_i} = \frac{\partial}{\partial \theta_i}, \quad d\theta_i \circ A = d\theta_i \quad \text{and} \quad L_{\frac{\partial}{\partial \theta_i}} A = 0, \]
then $A\psi$ is still the hypo lift of some $SU(4-k)$-structure.

PROOF: By Lemma 1.10 we have $L_{\frac{\partial}{\partial \theta_i}}(A\psi) = 0$ and
\[ (Ag)\left(\frac{\partial}{\partial \theta_i}, X\right) = g\left(\frac{\partial}{\partial \theta_i}, A^{-1}X\right) = d\theta_i(A^{-1}X) = d\theta_i(X) = g\left(\frac{\partial}{\partial \theta_i}, X\right). \]
Now the Lemma follows from Lemma 1.10.

We can now state the main result of this section,

Theorem 3.9. Suppose $\psi$ is a real analytic hypo $G_2$-structure on $M = S^1 \times \ldots \times S^1 \times M^{7-k}$, which is the hypo lift of some $SU(4-k)$-structure on $M^{7-k}$. Then the solution $A_t$ of the intrinsic torsion flow from Theorem 3.5 satisfies
\[ A_t \frac{\partial}{\partial \theta_i} = \frac{\partial}{\partial \theta_i}, \quad d\theta_i \circ A_t = d\theta_i \quad \text{and} \quad L_{\frac{\partial}{\partial \theta_i}} A_t = 0. \]
In particular, $A_t\psi$ is the hypo lift of some family of $SU(4-k)$-structures on $M^{7-k}$.

PROOF: We apply Corollary 2.5 with the following dictionary,

(1) $\mathcal{F} := C^\infty(\text{End}(TM)) \times C^\infty(\text{End}(TM))$

(2) $\mathcal{U} := C^\infty(\text{Aut}(TM)) \times C^\infty(\text{End}(TM))$

(3) $\mathcal{E} := \{(B, T) \in \mathcal{F} \mid 0 = L_{\frac{\partial}{\partial \theta_i}} B = L_{\frac{\partial}{\partial \theta_i}} T \quad \text{and} \quad 0 = B \frac{\partial}{\partial \theta_i} = T \frac{\partial}{\partial \theta_i} = d\theta_i(B) = d\theta_i(T)\}$

(4) $X : \mathcal{U} \to \mathcal{F}$ is defined w.r.t. the initial metric $g$, $X|_{(A, T)} := (T \circ A, \text{Ric}(Ag) - \text{tr}(T)T)$.

(5) $c(t) := (A_t, T_t)$.
Note that $\mathcal{U} \subset \mathcal{F}$ is open by Example 2.2 and that $X$ is smooth and $\mathcal{E} \subset \mathcal{F}$ is closed, since differential operators are smooth by Example 3.6.6. By Proposition 3.1 and the definition of $A_t$, the curve $c(t)$ is an integral curve of the vector field $X$. From Lemma 3.7 we get $c(0) = (\text{id}, T_0) \in \mathcal{E}$, where $f := (\text{id}, 0) \in \mathcal{F}$. Now it suffices to show that $X$ is tangent to $\mathcal{U} \cap \mathcal{E}$, i.e.
\[ X|_{\mathcal{U} \cap \mathcal{E}} : \mathcal{U} \cap \mathcal{E} \to \mathcal{E}. \]
For \((A = id + B, T) \in \mathcal{U} \cap \mathcal{E}_f\) we have
\[
A \frac{\partial}{\partial \theta_i} = \frac{\partial}{\partial \theta_i}, \quad d\theta_i \circ A = d\theta_i \quad \text{and} \quad L_{\frac{\partial}{\partial \theta_i}} A = 0.
\]
By Lemma 3.8 we see that \(A\psi\) is still the hypo lift of some \(SU(4-k)\)-structure and Lemma 3.7 yields
\[
L_{\frac{\partial}{\partial \theta_i}} \text{Ric}(Ag) = \text{Ric}(Ag) \frac{\partial}{\partial \theta_i} = d\theta_i \circ \text{Ric}(Ag) = 0.
\]
Now we can easily verify that \(X(A, T) \in \mathcal{E}\),
\[
\begin{align*}
&\bullet \quad L_{\frac{\partial}{\partial \theta_i}} (T \circ A) = 0 \quad \text{and} \quad L_{\frac{\partial}{\partial \theta_i}} (\text{Ric}(Ag) - \text{tr}(T)T) = 0, \\
&\bullet \quad T \circ A \frac{\partial}{\partial \theta_i} = 0 \quad \text{and} \quad (\text{Ric}(Ag) - \text{tr}(T)T) \frac{\partial}{\partial \theta_i} = 0, \\
&\bullet \quad d\theta_i (T \circ A) = 0 \quad \text{and} \quad d\theta_i (\text{Ric}(Ag) - \text{tr}(T)T) = 0
\end{align*}
\]
and the Theorem follows.

\[\square\]

**Remark 3.10.** The property \(L_{\frac{\partial}{\partial \theta_i}} A_t = 0\) from Theorem 3.5 is a consequence of the diffeomorphism invariance of the evolution equation \(A_t = T_t \circ A_t\). In fact, Lemma 3.6 shows that \(B_t := \Phi_s^* A_t\) also solves \(A_t = T_t \circ A_t\), where \(\Phi_s\) is the flow of \(\frac{\partial}{\partial \theta}\).

Since \(\Phi_s\) is real analytic, the uniqueness part of Theorem 3.5 yields \(A_t = \Phi_s^* A_t\), i.e. \(L_{\frac{\partial}{\partial \theta}} A_t = 0\).

We can now solve the embedding problem for real analytic hypo \(SU(4-k)\)-structures on \(M^{7-k}\) by reducing it to the embedding problem for real analytic hypo \(G_2\)-structures on \(M = S^1 \times ... \times S^1 \times M^{7-k}\). Namely, the hypo lift of the initial \(SU(4-k)\)-structure yields a real analytic hypo \(G_2\)-structures on \(M\). Theorem 3.5 yields a solution \(A_t\) of the intrinsic torsion flow. By Theorem 3.9 the family of \(G_2\)-structures \(\psi_t = A_t\psi\) is still the hypo lift of some family of \(SU(4-k)\)-structures.

Now Lemma 1.9 proves that the family of \(SU(4-k)\)-structures is a solution of the embedding problem.

**Corollary 3.11.** For any real analytic hypo \(SU(2), SU(3)\) and \(G_2\)-structure on a compact manifold, the embedding problem admits a unique real analytic solution. Moreover, the solution can be described by a family of gauge deformations
\[
A_t = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^{(k)}_t,
\]
where the series converges in the \(C^\infty\)-topology on \(C^\infty(\text{End}(TM))\).
Appendix: \( SU(2) \)-Structures in Dimension Five

Usually a \( SU(2) \)-structure on a five dimensional manifold is described by a quadruplet of forms \((\alpha, \omega_1, \omega_2, \omega_3)\), cf. for instance [7]. There is an alternative to the usual definition, which is justified by the last equation in the next Lemma.

Lemma 4.1.

\[
\begin{align*}
\text{Iso}_{GL(5)}(\alpha_0) &= \left\{ \begin{pmatrix} 1 & 0 \\ x & A \end{pmatrix} \mid A \in GL(4) \text{ and } x \in \mathbb{R}^4 \right\}, \\
\text{Iso}_{GL(5)}(\alpha_1) &= \left\{ \begin{pmatrix} \lambda & y^T \\ 0 & A \end{pmatrix} \mid A \in Sp(4, \mathbb{R}), y \in \mathbb{R}^4 \text{ and } \lambda \neq 0 \right\}, \\
\text{Iso}_{GL(5)}(\alpha_0, \omega_1, \omega_2, \omega_3) &= \begin{pmatrix} 1 & 0 \\ 0 & SU(2) \end{pmatrix}, \\
\text{Iso}_{GL(5)}(\omega_1, \rho_2, \rho_3) &= \begin{pmatrix} 1 & 0 \\ 0 & SU(2) \end{pmatrix}.
\end{align*}
\]

Proof: Write \( B \in GL(5) \) as

\[
B = \begin{pmatrix} \lambda & y^T \\ x & A \end{pmatrix},
\]

where \( \lambda \in \mathbb{R}, x, y \in \mathbb{R}^4 \) and \( A \in gl(4) \). Then \( \alpha(Be_1) = \lambda \) and \( \alpha(Be_j) = y^T e_j \), for \( j \in \{2, \ldots, 5\} \). Hence the stabilizer of the 1-form \( \alpha_0 := e^1 \in \Lambda^1 \mathbb{R}^{5*} \) has the above form.

For \( B \in \text{Iso}_{GL(5)}(\omega_1) \) and \( i, j \in \{2, \ldots, 5\} \) we get \( \omega_1(e_i, e_j) = \omega_1(Be_i, Be_j) = \omega_1(\lambda e_i, e_j) = \omega_1(Ae_i, e_j) \), i.e. \( A \in Sp(4, \mathbb{R}) \). This yields

\[
0 = \omega_1(Be_1, Be_j) = \omega_1(\lambda e_1 + x, (y^T e_j)e_1 + Ae_j) = \omega_1(x, Ae_j) = \omega_1(A^{-1}x, e_j)
\]

and the non-degeneracy of \( \omega_1 \), as a form on \( \mathbb{R}^4 \), implies \( x = 0 \) and proves the second equation of the lemma.

Now the third equation follows, since \( \omega_2 = \text{Re}(\Phi_0) \) and \( \omega_3 = \text{Im}(\Phi_0) \), where \( \Phi_0 = (e^2 + ie^3) \land (e^4 + ie^5) \), and \( SU(2) = \text{Sp}(4, \mathbb{R}) \cap SL(2, \mathbb{C}) \).

To obtain the last equation, we compute for \( B = \begin{pmatrix} \lambda & y^T \\ 0 & A \end{pmatrix} \in \text{Iso}_{GL(5)}(\omega_1) \cap \text{Iso}_{GL(5)}(\alpha_0 \wedge \omega_2) \) and \( i, j \in \{2, \ldots, 5\} \)

\[
\begin{align*}
\omega_2(e_i, e_j) &= (\alpha_0 \wedge \omega_2)(e_i, e_i, e_j) = (\alpha_0 \wedge \omega_2)(Be_i, Be_i, Be_j) \\
&= (\alpha_0 \wedge \omega_2)(\lambda e_1, (y^T e_i)e_1 + Ae_i, (y^T e_j)e_1 + Ae_j) \\
&= (\alpha_0 \wedge \omega_2)(\lambda e_1, Ae_i, Ae_j) \\
&= \lambda \omega_2(Ae_i, Ae_j).
\end{align*}
\]
Since the volume element $\varepsilon_0 = e^{2345}$ on $\mathbb{R}^4$ satisfies
$$\varepsilon_0 = \frac{1}{2}\omega_1^2 = \frac{1}{2}\omega_2^2 = \frac{1}{2}\omega_3^2,$$
we obtain from $A \in \text{Sp}(4, \mathbb{R}) = \text{Iso}_{\text{GL}(4)}(\omega_1)$
$$\det(A)\varepsilon_0 = A^{-1}\varepsilon_0 = A^{-1}\frac{1}{2}\omega_1^2 = \varepsilon_0,$$
i.e. $\det(A) = 1$. Now $A^{-1}\omega_2 = \lambda^{-1}\omega_2$ yields
$$\varepsilon_0 = A^{-1}\frac{1}{2}\omega_1^2 = \lambda^{-2}\varepsilon_0$$
and since $B \in \text{GL}^+(5)$, we get $\lambda = 1$. Similarly we get $A\omega_3 = \omega_3$, which yields $A \in \text{SU}(2)$. Now
$$\alpha_0 \wedge \omega_2 = B^{-1}(\alpha_0 \wedge \omega_2) = B^{-1}\alpha_0 \wedge B^{-1}\omega_2$$
$$= B^{-1}\alpha_0 \wedge A^{-1}\omega_2, \quad \text{since } \epsilon_1\omega_2 = 0$$
$$= (\alpha_0(Be_1)e^1 + \sum_{j=2}^{5} \alpha_0(Be_j)e^j) \wedge \omega_2$$
$$= (\alpha_0 + \sum_{j=2}^{5} y_j e^j) \wedge \omega_2$$
yields $\sum_{j=2}^{5} y_j e^j \wedge \omega_2 = 0$, i.e. $y = 0$.

Since the $\text{GL}^+(5)$ stabilizer of the triple $(\omega_1, \rho_2, \rho_3)$ is equal to $\{1\} \times \text{SU}(2)$, we expect that, after fixing an orientation for $\mathbb{R}^5$, we can reconstruct the forms $\alpha_0, \omega_2$ and $\omega_3$ solely from the triple $(\omega_1, \rho_2, \rho_3)$. The first step is to reconstruct the volume element $\varepsilon_0$. Then the forms $\alpha_0, \omega_2$ and $\omega_3$, as well as the metric $g_0$, can be obtained from the formulas in Example 1.4.

**Lemma 4.2.** After choosing an orientation for $V := \mathbb{R}^5$, there is a homomorphism
$$\varepsilon : \Lambda^2V^* \oplus \Lambda^3V^* \oplus \Lambda^3V^* \to \Lambda^5V^* \oplus i\Lambda^5V^*$$
of $\text{GL}^+(5)$-modules, such that for the model tensors and the canonical orientation $[\varepsilon_0]$ of $\mathbb{R}^5$
$$\varepsilon(\omega_1, \rho_2, \rho_3) = \varepsilon_0 \in \Lambda^5V^* \subset \Lambda^5V^* \oplus i\Lambda^5V^*.$$

**Proof:** Given an orientation $[\varepsilon_+]$ for $V$, represented by an element $\varepsilon_+ \in \Lambda^5V^*$, we can define a $\text{GL}^+(5)$-equivariant map
$$\sqrt{-} : \Lambda^5V^* \otimes \Lambda^5V^* \otimes \Lambda^5V^* \otimes \Lambda^5V^* \to \Lambda^5V^* \oplus i\Lambda^5V^*.$$
Now consider the $\text{GL}(5)$-equivariant map
$$K : \Lambda^2V^* \oplus \Lambda^3V^* \oplus \Lambda^3V^* \to (V^* \otimes V) \otimes (V^* \otimes V) \otimes \Lambda^5V^* \otimes \Lambda^5V^*$$
defined by

\[ K(\omega_1, \rho_2, \rho_3)(x, a, y, b) := (\rho_2 \wedge a \wedge b) \otimes (\rho_3 \wedge (x, \omega_1) \wedge (y, \omega_1)), \]

where \(x, y \in V\) and \(a, b \in V^*\). For the model tensors \(\omega_1, \rho_2, \rho_3\) let \(K_0 := K(\omega_1, \rho_2, \rho_3)\). Then we compute

\[ K_0(x, a, y, b) = (a_5 b_3 - a_3 b_5 + a_2 b_4 - a_4 b_2)(-x_3 y_4 + x_4 y_3 - x_2 y_5 + x_5 y_2) \otimes \varepsilon^2_0. \]

Taking the trace of the first factor \(V^* \otimes V\), we obtain a map

\[ L = \text{tr}(K) : \Lambda^2 V^* \oplus \Lambda^3 V^* \oplus \Lambda^3 V^* \rightarrow (V^* \otimes V) \otimes \Lambda^5 V^* \oplus \Lambda^5 V^* \]

and for the model tensors we obtain

\[ L_0(y, b) := \text{tr}(K_0)(y, b) = (-b_4 y_5 + b_5 y_4 - b_2 y_3 + b_3 y_2) \otimes \varepsilon^2_0. \]

Identifying \(V^* \otimes V = \text{Hom}(V, V)\), we define

\[ L^2 : \Lambda^2 V^* \oplus \Lambda^3 V^* \oplus \Lambda^3 V^* \rightarrow (V^* \otimes V) \otimes (\Lambda^5 V^*)^4 \]

and so

\[ L_0^2 = \begin{pmatrix} 0 & 0 \\ 0 & -\text{id}_{R^4} \end{pmatrix} \otimes \varepsilon^4_0. \]

Taking again the trace, we obtain a map

\[ \text{tr}(L^2) : \Lambda^2 V^* \oplus \Lambda^3 V^* \oplus \Lambda^3 V^* \rightarrow (\Lambda^5 V^*)^4 \]

with \(\text{tr}(L_0^2) = -4 \varepsilon^4_0\). Hence

\[ \varepsilon := \sqrt{-\frac{1}{4} \text{tr}(L^2) : \Lambda^2 V^* \oplus \Lambda^3 V^* \oplus \Lambda^3 V^* \rightarrow \Lambda^5 V^* \oplus i\Lambda^5 V^*} \]

is the desired equivariant map.

\[ \square \]
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