Stochastic Impulse Control of Non-Markovian Processes

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Abstract

We consider a class of stochastic impulse control problems of general stochastic processes i.e. not necessarily Markovian. Under fairly general conditions we establish existence of an optimal impulse control. We also prove existence of combined optimal stochastic and impulse control of a fairly general class of diffusions with random coefficients. Unlike, in the Markovian framework, we cannot apply quasi-variational inequalities techniques. We rather derive the main results using techniques involving reflected BSDEs and the Snell envelope.

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1 Introduction

Finding a stochastic impulse control policy amounts to determining the sequence of random dates at which the policy is exercised and the sequence of impulses describing the magnitude of the applied policies, which maximizes a given reward function. Given the general applicability

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of stochastic impulse control models in various fields such as finance, e.g. cash management (see Korn (1999) for an excellent survey and the textbook by Jeanblanc et al. (2005) and the references therein), and management of renewable resources (see e.g. Alvarez (2004), Alvarez and Koskel (2007) and the references therein), it is not surprising that the mathematical framework of such problems is well established (see Lepeltier-Marchal (1984), Øksendal and Sulem (2006) and the references therein and the seminal textbook by Bensoussan and Lions (1984) on quasi-variational inequalities and impulse control). Indeed, in most cases, the impulse control problem is studied relying on quasi-variational inequalities, which is possible only through tacitly assuming that the underlying dynamics of the controlled system is Markovian and the instantaneous part of the reward function a deterministic function of the value of the process at a certain instant. These assumptions are obviously not realistic in most applications, such as in certain models in commodities trading. Even if the underlying process is Markov, the instantaneous part of the reward function may depend on the whole path of the process or is simply random.

In this study we consider a class of stochastic impulse control problems where the underlying dynamics of the controlled system is typically not Markov and where the instantaneous reward functional is random, in which case, we cannot rely on the well established quasi-variational inequalities technique to solve it. Instead, we solve the problem using techniques involving reflected BSDEs and the Snell envelope that seem suit well this general situation. The main idea is to express the value-process of the control problem as a Snell envelope and show that it solves a reflected BSDE, whose existence and uniqueness are guaranteed provided some mild integrability conditions of the involved coefficients. This is done through an appropriate approximation scheme of the system of reflected BSDEs that is shown to converge to our value process. The underlying approximating sequence is shown to be the value process of an impulse control over strategies which have only a bounded number of impulses, for which an optimal policy is also shown to exist. Finally, passing to the limit, letting the number of impulses become large, we prove existence of an optimal policy of our stochastic impulse control problem.

The paper is organized as follows. In Section 2 we recall the main tools on reflected BSDEs and Snell envelope we will use to establish the main results. In Section 3, we formulate the considered stochastic impulse control. In Section 4, we consider an appropriate approximation scheme of the system of reflected BSDEs that is shown to converge to our value process. In
Section 5, we establish existence of an optimal impulse control over strategies with a bounded number of impulses, in Section 6, we prove existence of an optimal impulse control over all admissible strategies. Moreover, the corresponding value process is the limit of the sequence of value processes associated with the optimal impulse control over finite strategies, as their number becomes large. Finally, in Section 7, we consider a mixed stochastic control and impulse control problem of a fairly large class of diffusion processes that are not necessarily Markovian. Using a Beneš-type selection theorem, we derive an optimal policy using similar tools.

2 Preliminaries and notation

Throughout this paper \((\Omega, F, \mathcal{IP})\) is a fixed probability space on which is defined a standard \(d\)-dimensional Brownian motion \(B = (B_t)_{0\leq t \leq T}\) whose natural filtration is \((\mathcal{F}_t^0 := \sigma\{B_s, s \leq t\})_{0\leq t \leq T}\); \((\mathcal{F}_t)_{0\leq t \leq T}\) is the completed filtration of \((\mathcal{F}_t^0)_{0\leq t \leq T}\) with the \(\mathcal{IP}\)-null sets of \(\mathcal{F}\), hence \((\mathcal{F}_t)_{0\leq t \leq T}\) satisfies the usual conditions, i.e., it is right continuous and complete. Let

- \(\mathcal{P}\) be the \(\sigma\)-algebra on \([0,T] \times \Omega\) of \(\mathcal{F}_t\)-progressively measurable processes.
- for any \(p \leq 2\), \(\mathcal{H}^{p,k}\) be the set of \(\mathcal{P}\)-measurable processes \(v = (v_t)_{0\leq t \leq T}\) with values in \(\mathbb{R}^k\) such that \(\mathbb{E}\left[\int_0^T |v_s|^p ds\right] < \infty\).
- \(\mathcal{S}^2\) (resp. \(\mathcal{S}^2_c\)) be the set of \(\mathcal{P}\)-measurable and càdlàg (abbreviation of right continuous and left limited) (resp. continuous) processes \(Y = (Y_t)_{0\leq t \leq T}\) such that \(\mathbb{E}[\sup_{0\leq t \leq T} |Y_t|^2] < \infty\).
- \(\mathcal{S}_t^2\) (resp. \(\mathcal{S}_{t,c}^2\)) the set of non-decreasing processes \(k = (k_t)_{0\leq t \leq T}\) of \(\mathcal{S}^2\) (resp. \(\mathcal{S}^2_c\)) which satisfy \(k_0 = 0\).
- for \(t \leq T\), \(\mathcal{T}_t\) the set of \(\mathcal{F}_t\)-stopping times \(\nu\) such that \(\mathcal{IP} - a.s., t \leq \nu \leq T\). Finally for any stopping time \(\nu\), \(\mathcal{F}_\nu\) is the \(\sigma\)-algebra on \(\Omega\) which contains the sets \(A\) of \(\mathcal{F}\) such that \(A \cap \{\nu \leq t\} \in \mathcal{F}_t\). \(\square\)

Consider now an \(\mathcal{S}^2\)-process \(X = (X_t)_{0\leq t \leq T}\). The Snell envelope of \(X\), which we denote by \(N(X) = (N(X)_t)_{0\leq t \leq T}\), is defined as

\[
\mathcal{IP} - a.s. \quad N(X)_t = \text{ess sup}_{\nu \in \mathcal{T}_t} \mathbb{E}[X_{\nu}|\mathcal{F}_t], \quad 0 \leq t \leq T.
\]
It is the smallest càdlàg \((\mathcal{F}_t, \mathbb{P})\)-supermartingale of class \([D]\) (see the appendix for the definition) which dominates \(X\), i.e., \(\mathbb{P} - \text{a.s.}, N(X)_t \geq X_t\), for all \(0 \leq t \leq T\).

For the sequel, we need the following result related to the continuity of the Snell envelope with respect to increasing sequences whose proof can be found in Cvitanic and Karatzas (1996) or Hamadène and Hdiri (2007).

**Proposition 2.1.** Let \((U_n)_{n \geq 1}\) be a sequence of càdlàg and uniformly square integrable processes which converges increasingly and pointwisely to a càdlàg and uniformly square integrable process \(U\), then \((N(U_n))_{n \geq 1}\) converges increasingly and pointwisely to \(N(U)\).

In the Appendix at the end of the paper, we collect further results on the Snell envelope we will refer to in the rest of the paper.

Let us underline that in the Markovian case, the problem under consideration is solved using PDEs techniques. However, in our framework, we can no longer apply these techniques. Instead, we use backward stochastic differential equations (BSDEs in short) which we will introduce with others properties.

Let \(X = (X_t)_{0 \leq t \leq T}\) be a barrier process of \(S^2\) and \(f : [0, T] \times \Omega \times \mathbb{R}^{1+d} \rightarrow \mathbb{R}\) a drift coefficient such that \((f(t, \omega, 0, 0))_{0 \leq t \leq T} \in \mathcal{H}^{2,1}\) and uniformly Lipschitz in \((y, z)\), i.e. there exists a constant \(C > 0\) such that

\[|f(t, y, z) - f(t, y', z')| \leq C(|y - y'| + |z - z'|)\]

for any \(t, y, z, y'\) and \(z'\).

Then we have the following

**Theorem 2.1.** (Hamadène (2002)). There exists a unique \(\mathcal{F}\)-measurable triple of processes \((Y, Z, K) = (Y_t, Z_t, K_t)_{0 \leq t \leq T}\) with values in \(\mathbb{R}^{1+d+1}\) solution of the reflected BSDE associated with \((f, X)\), i.e.,

\[
\begin{cases}
Y \in S^2, \ Z \in \mathcal{H}^{2,d} \text{ and } K \in S^2, \\
Y_t = X_T + \int_t^T f(s, Y_s, Z_s) ds + K_T - K_t - \int_t^T Z_s dB_s, \ 0 \leq t \leq T, \\
Y_t \geq X_t, \ \text{ for all } 0 \leq t \leq T, \\
\int_0^T (Y_t - X_t) dK_t^c = 0, \ \text{and} \ \Delta_t Y := Y_t - Y_{t-} = -(X_{t-} - Y_t)^+ \ [Y_{t-} - Y_t < 0],
\end{cases}
\]

where \(K^c\) is the continuous part of \(K\). Moreover, \(Y\) admits the following representation.

\[
\mathbb{P} - \text{a.s.}, \ Y_t = \text{ess sup}_{\tau \in T_t} \mathbb{E}\left[\int_\tau^T f(s, Y_s, Z_s) ds + X_\tau | \mathcal{F}_\tau\right], \ t \leq T.
\]
In addition, if $X$ is left upper semi-continuous, i.e., it has only positive jumps, then the process $Y$ is continuous.

From (2.1) we note that $(Y_t + \int_0^t f(s, Y_s, Z_s)ds)_{0 \leq t \leq T}$ is the Snell envelope of the process $(\int_0^t f(s, Y_s, Z_s)ds + X_t)_{0 \leq t \leq T}$.

In view of the results in El-Karoui et al. (1995), solutions of BSDEs with one reflecting barrier can be compared when we can compare the generators, the terminal values and the barriers. This remains true in this framework of discontinuous processes. Indeed, the following result holds.

**Proposition 2.2. (Hamadène (2002))** Let $\tilde{f}$ (resp. $\tilde{X}$) be another map from $[0, T] \times \Omega \times \mathbb{R}^{1+d}$ into $\mathbb{R}$ (resp. another process of $\mathcal{S}^2$) such that:

1. there exists a process $(\tilde{Y}, \tilde{Z}, \tilde{K}) = (\tilde{Y}_t, \tilde{Z}_t, \tilde{K}_t)_{t \leq T}$ solution of the reflected BSDE associated with $(\tilde{f}, \tilde{X})$

2. $\mathbb{P}$ -- a.s. $\forall t \leq T$, $f(t, \tilde{Y}_t, \tilde{Z}_t) \leq \tilde{f}(t, \tilde{Y}_t, \tilde{Z}_t)$

3. $\mathbb{P}$ -- a.s., for all $t \leq T$, $X_t \leq \tilde{X}_t$

Then, we have $\mathbb{P}$ -- a.s., for all $t \leq T$, $Y_t \leq \tilde{Y}_t$. □

Now, let us consider a sequence $(y^n, z^n, k^n)_{n \geq 1}$ of processes defined as follows:

\[
\begin{cases}
(y^n, z^n, k^n) \in \mathcal{S}^{2}_{c} \times \mathcal{H}^{2,d} \times \mathcal{S}^{2}_{c,i}, \\
y^n_t = y^n_T + \int_t^T f(s, y^n_s, z^n_s)ds + k^n_T - k^n_t - \int_t^T z^n_s dB_s, \quad t \leq T, \\
y^n_t \geq X_t, \quad \text{for all } t \leq T, \quad \text{and } \int_0^T (y^n_t - X_t)dk^n_t = 0.
\end{cases}
\]

We now recall the following result by S. Peng (1999) which generalizes a well known property of supermartingales which tells that an increasing limit of càdlàg supermartingales is a also a càdlàg supermartingale.

**Proposition 2.3. (Peng (1999, pp.485))** Assume the sequence $(y^n)_{n \geq 1}$ converges increasingly to a process $(y_t)_{0 \leq t \leq T}$ such that $\mathbb{E}[\sup_{0 \leq t \leq T} |y_t|^2] < \infty$, then there exist two processes $(z, k) \in \mathcal{H}^{2,d} \times \mathcal{S}^{2}_{i}$ such that

\[
y_t = y_T + \int_t^T f(s, y_s, z_s)ds + k_T - k_t - \int_t^T z_s dB_s.
\]

In addition, $z$ is the weak (resp. strong) limit of $z^n$ in $\mathcal{H}^{2,d}$ (resp. in $\mathcal{H}^{0,d}$, for $p < 2$) and for any stopping time $\tau$, the sequence $(k^n_\tau)_{n \geq 1}$ converges to $k_\tau$ in $L^p(\mathbb{P})$. 

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In this result, the assumption $E[\sup_{0 \leq t \leq T} |y_t|^2] < \infty$ can be replaced by $E[\sup_{n \geq 1} \sup_{t \leq T} |y^n_t|^2] < \infty$.

### 3 Formulation of the impulse control problem

Let $L = (L_t)_{0 \leq t \leq T}$ be a stochastic process that describes the evolution of a system. We assume it $\mathcal{F}$-measurable, with values in $\mathbb{R}^l$ and is such that $E[\int_0^T |L_s|^2 ds] < \infty$. An impulse control is a sequence of pairs $\delta = (\tau_n, \xi_n)_{n \geq 0}$ in which $(\tau_n)_{n \geq 0}$ is a sequence of $\mathcal{F}_t$-stopping times such that $0 \leq \tau_0 \leq \tau_1 \leq \ldots \leq T \text{ P.a.s.}$ and $(\xi_n)_{n \geq 0}$ a sequence of random variables with values in a finite subset $U$ of $\mathbb{R}^l$ such that $\xi_n$ is $\mathcal{F}_{\tau_n}$-measurable. Considering the subset $U$ finite is in line with the fact that, in practice, the controller has only access to limited resources which allow him to exercise impulses of finite size.

The sequence $\delta = (\tau_n, \xi_n)_{n \geq 0}$ is said to be an admissible strategy of the control, and the set of admissible strategies will be denoted by $\mathcal{A}$. The controlled process $L^\delta = (L^\delta_t)_{0 \leq t \leq T}$ is described as follows:

$$L^\delta_t = \begin{cases} L_t, & \text{if } 0 \leq t \leq \tau_0, \\ L_t + \xi_n, & \text{if } \tau_n \leq t < \tau_{n+1}, n \geq 0, \end{cases}$$

or, in compact form,

$$L^\delta_t = L_t + \sum_{n \geq 0} \xi_n 1_{[\tau_n \leq t]} , \quad 0 \leq t \leq T.$$

The associated reward of controlling the system is

$$J(\delta) = E[\int_0^T h(s, \omega, L^\delta_s) ds - \sum_{n \geq 0} \psi(\xi_n) 1_{[\tau_n < T]}],$$

where $h$, represents the instantaneous reward and $\psi$ the costs due to the impulses.

This formulation of impulse control also falls within the class of singular stochastic control problems, since the bounded variation part of the process, which controls the dynamic of the system, is allowed to be only purely discontinuous- See Øksendal and Sulem (2006) for further details. Finally, note that if for example the process $L$ satisfies

$$L_t = L_0 + \int_0^t b(s, \omega) ds + \int_0^t \sigma(s, \omega) dB_s, \quad t \leq T,$$

where, $(b(s))_{0 \leq s \leq T}$ and $(\sigma(s))_{0 \leq s \leq T}$ are adapted stochastic processes, the existing theory on impulse control cannot be applied to the associated problem, since the processes $b$ and $\sigma$ are random.
We make the following assumptions on $h$ and $\psi$.

**Assumption (A)**

(A1) $h : [0, T] \times \Omega \times \mathbb{R}^l \rightarrow [0, +\infty)$ is uniformly bounded by a constant $\gamma$ in all its arguments i.e. for any $(t, \omega, x) \in [0, T] \times \Omega \times \mathbb{R}^l$, $0 \leq h(t, \omega, x) \leq \gamma$.

(A2) $\psi : U \rightarrow [0, +\infty)$ is bounded from below, i.e. there exists a constant $c > 0$ such that $\inf_{\beta \in U} \psi(\beta) \geq c$.

Assumption (A2) is motivated by the following form of proportional and fixed transaction costs (see Korn (1999) or Baccarin and Sanfelici (2006) for further examples).

$$\psi(\xi) = \phi(\xi) + c,$$

where $\phi \geq 0$, $\phi(0) = 0$ and $c$ is a positive constant.

**Definition 3.1.** A strategy $\delta^* \in \mathcal{A}$ such that

$$J(\delta^*) = \sup_{\delta \in \mathcal{A}} J(\delta)$$

is called optimal.

The properties of $h$ and $\psi$ make the supremum of the reward function $J$ over the set $\mathcal{A}$ coincides with the one over the set of finite strategies, $\mathcal{D}$ defined as

$$\mathcal{D} = \{\delta = (\tau_n; \beta_n)_{n \geq 0} \in \mathcal{A}; \ \mathbb{P}(\tau_n(\omega) < T, n \geq 0) = 0\}.$$ 

That is,

$$\sup_{\delta \in \mathcal{A}} J(\delta) = \sup_{\delta \in \mathcal{D}} J(\delta).$$

Indeed, consider a strategy $\delta = (\tau_n; \beta_n)_{n \geq 0}$ of $\mathcal{A}$ which does not belong to $\mathcal{D}$ and let $B = \{\omega \in \Omega; \ \tau_n(\omega) < T, n \geq 0\}$. Since $\delta$ is not finite, $\mathbb{P}(B) > 0$. But, since $h$ is bounded, we have

$$J(\delta) = \mathbb{E}[\int_0^T h(s, L^\delta_s)ds - \sum_{n \geq 0} \psi(\beta_n)\mathbb{1}_{[\tau_n < T]}]$$

$$\leq \gamma T - \mathbb{E}[(\sum_{n \geq 0} \psi(\beta_n)\mathbb{1}_{[\tau_n < T]})\mathbb{1}_B - (\sum_{n \geq 0} \psi(\beta_n))\mathbb{1}_{[\tau_n < T]}\mathbb{1}_{B^c}]$$

$$= -\infty,$$

whence the desired result.
4 An approximation scheme

For any stopping time \( \nu \) and an \( \mathcal{F}_\nu \)-measurable random variable \( \xi \), let \( (Y^0_t(\nu, \xi), Z^0_t(\nu, \xi))_{0 \leq t \leq T} \) be the solution in \( \mathcal{S}^2_c \times \mathcal{H}^{2,d} \) of the following standard BSDE:

\[
Y^0_t(\nu, \xi) = \int_t^T h(s, L_s + \xi) \mathbb{1}_{[s \geq \nu]} ds - \int_t^T Z^0_s(\nu, \xi) dB_s, \quad 0 \leq t \leq T. \tag{4.4}
\]

The solution of this BSDE exists and is unique by the well known Pardoux-Peng’s Theorem (see Pardoux and Peng (1990)) since the terminal value is null and the function \( h \) is bounded.

Next, for any \( n \geq 1 \), let \( (Y^n_t(\nu, \xi), K^n_t(\nu, \xi), Z^n_t(\nu, \xi))_{0 \leq t \leq T} \) be the sequence of processes defined recursively as solutions of reflected BSDEs in the following way:

\[
\begin{align*}
(Y^n(\nu, \xi), Z^n(\nu, \xi), K^n(\nu, \xi)) &\in \mathcal{S}^2_c \times \mathcal{H}^{2,d} \times \mathcal{S}^2_{c,i}, \\
Y^n_t(\nu, \xi) &= \int_t^T h(s, L_s + \xi) \mathbb{1}_{[s \geq \nu]} ds + K^n_t(\nu, \xi) - \int_t^T Z^n_s(\nu, \xi) dB_s, \quad 0 \leq t \leq T, \\
Y^n_t(\nu, \xi) &\geq O^n_t(\nu, \xi) := \max_{\beta \in U} \{-\psi(\beta) + Y^{n-1}_t(\nu, \xi + \beta)\}, \quad 0 \leq t \leq T, \\
\int_0^T (Y^n_t(\nu, \xi) - O^n_t(\nu, \xi)) dK^n_t(\nu, \xi) &= 0. \tag{4.5}
\end{align*}
\]

**Proposition 4.1.** For any \( n \geq 0, \; \nu \in \mathcal{T}_0 \) and any \( \mathcal{F}_\nu \)-measurable r.v. \( \xi \), the triple \((Y^n(\nu, \xi), K^n(\nu, \xi), Z^n(\nu, \xi))\) of (4.5) is well posed. Moreover, it satisfies the following properties.

(i) \( \mathbb{P} \)-a.s. \( 0 \leq Y^n_t(\nu, \xi) \leq Y^{n+1}_t(\nu, \xi), \quad 0 \leq t \leq T. \)

(ii) \( \mathbb{P} \)-a.s. \( Y^n_t(\nu, \xi) \leq \gamma(T-t), \quad 0 \leq t \leq T. \)

**Proof:** We prove the result by induction on \( n \). We first begin to show the well-posedness of \((Y^n(\nu, \xi), K^n(\nu, \xi), Z^n(\nu, \xi))\) for any \( n \geq 0 \). As pointed out previously for \( n = 0 \), for any stopping time \( \nu \) and any \( \mathcal{F}_\nu \)-measurable r.v. \( \xi \), the pair \((Y^0(\nu, \xi), Z^0(\nu, \xi))\) exists and belongs to \( \mathcal{S}^2_c \times \mathcal{H}^{2,d} \). Suppose now for some \( n \geq 1 \), for any stopping time \( \nu \) and any \( \mathcal{F}_\nu \)-measurable r.v. \( \xi \), the triplet \((Y^n(\nu, \xi), K^n(\nu, \xi), Z^n(\nu, \xi))\) exists and belongs to \( \mathcal{S}^2_c \times \mathcal{S}^2_{c,i} \times \mathcal{H}^{2,d} \). Hence, thanks to the finiteness of \( U \), \((O^{n+1}_t(\nu, \xi))_{0 \leq t \leq T}\) is a continuous process and satisfies \( O^{n+1}_T(\nu, \xi) \leq 0 \).

In view of Theorem [2.1], the triplet \((Y^{n+1}_t(\nu, \xi), K^{n+1}_t(\nu, \xi), Z^{n+1}_t(\nu, \xi))\) exists and belongs to \( \mathcal{S}^2_c \times \mathcal{S}^2_{c,i} \times \mathcal{H}^{2,d} \). Thus, for any \( n \geq 0 \), any stopping time \( \nu \) and any \( \mathcal{F}_\nu \)-measurable r.v. \( \xi \), the triplet \((Y^n(\nu, \xi), K^n(\nu, \xi), Z^n(\nu, \xi))\) exists and belongs to \( \mathcal{S}^2_c \times \mathcal{S}^2_{c,i} \times \mathcal{H}^{2,d} \).

Let us now show (i) and (ii). Once more we will use an induction argument. First writing \( Y^0_t(\nu, \xi) \) as a conditional expectation w.r.t. \( \mathcal{F}_t \) and taking into account of \( 0 \leq h \leq \gamma \) we
obtain that \(0 \leq Y^0_t(\nu, \xi) \leq \gamma(T - t)\), for any stopping time \(\nu\) and any \(\mathcal{F}_\nu\)-measurable r.v. \(\xi\).

Next, as \(K^1(\nu, \xi)\) is an increasing process then using standard comparison result of solutions of BSDEs (see e.g. El-Karoui et al. (1995)), we obtain \(Y^0(\nu, \xi) \leq Y^1(\nu, \xi)\). Therefore, Properties (i) and (ii) hold for \(n = 0\). Suppose now that for some \(n\), for any stopping time \(\nu\) and any \(\mathcal{F}_\nu\)-measurable r.v. \(\xi\), (i) and (ii) hold. Then, \(O^{n+1}(\nu, \xi) \leq O^{n+2}(\nu, \xi)\) and then the characterization (2.1) implies that \(Y^{n+1}(\nu, \xi) \leq Y^{n+2}(\nu, \xi)\). On the other hand, since, for any \(\zeta \in \mathcal{F}_\nu\), \(Y^n(\nu, \zeta) \leq \gamma(T - t)\), it holds that \(O^{n+1}_t(\nu, \xi) = \max_{\beta \in U} (\psi(\beta) + Y^n_t(\nu, \xi + \beta)) \leq \max_{\beta \in U} (\psi(\beta) + \gamma(T - t)) \leq \gamma(T - t)\), \(0 \leq t \leq T\).

Now, once more by (2.1), we have, for any \(n \geq 1\),

\[
Y^{n+1}_t(\nu, \xi) = \text{ess sup}_{t \in T_t} \mathbb{E}[\int_t^T h(s, L_s + \xi) ds + O^{n+1}_t(\nu, \xi) \mathbf{1}_{\tau < T} |\mathcal{F}_t], t \leq T. \tag{4.6}
\]

Therefore,

\[
Y^{n+1}_t \leq \text{ess sup}_{t \in T_t} \mathbb{E}[\gamma(\tau - t) + \gamma(T - \tau) |\mathcal{F}_t] = \gamma(T - t)
\]

and this completes the proof of the claim. \(\square\)

In the next proposition we identify the limit process \(Y_t(\nu, \xi) := \lim_{n \to \infty} Y^n_t(\nu, \xi)\) (which exists according to the last proposition) as a Snell envelope. Note that, as a limit of a non-decreasing sequence of continuous processes, \(Y(\nu, \xi)\) is upper semi-continuous. Moreover, it holds that

\[
0 \leq Y_t(\nu, \xi) \leq \gamma(T - t), \text{ for all } t \leq T, \text{ and } Y_T(\nu, \xi) = 0. \tag{4.7}
\]

Finally, once more thanks to the finiteness of \(U\), the sequence of processes \((O^n(\nu, \xi))_{n \geq 0}\) converges to \(O(\nu, \xi)\) as \(n \to \infty\), where, \(O_t(\nu, \xi) := \max_{\beta \in U} [-\psi(\beta) + Y_t(\nu, \xi + \beta)]\), \(0 \leq t \leq T\).

**Proposition 4.2.** (i) Let \(\nu\) and \(\nu'\) be two stopping times such that \(\nu \leq \nu'\) and \(\xi\) an \(\mathcal{F}_\nu\)-measurable random variable, then it holds that \(\mathbb{P}\)-a.s., \(Y_t(\nu, \xi) = Y_t(\nu', \xi)\) for all \(t \geq \nu'\).

(ii) For any stopping time \(\nu\) and \(\mathcal{F}_\nu\)-measurable random variable \(\xi\), the process \(Y(\nu, \xi)\) is càdlàg and satisfies:

\[
Y_t(\nu, \xi) = \text{ess sup}_{t \in T_t} \mathbb{E}[\int_t^T h(s, L_s + \xi) \mathbf{1}_{s \geq \nu} ds + \mathbf{1}_{\tau < T} O_{\tau}(\nu, \xi) |\mathcal{F}_t], t \leq T. \tag{4.8}
\]

**Proof:** (i) We proceed by induction on \(n\). We note that the solution of the BSDE

\[
Y^0_t(\nu, \xi) = \int_t^T h(s, L_s + \xi) \mathbf{1}_{s \geq \nu} ds - \int_t^T Z_s(\nu, \xi) d\mathbb{B}_s
\]
is unique. It follows that, for any $\xi \in \mathcal{F}_\nu$, \( Y^0_t(\nu, \xi) = Y^0_t(\nu', \xi) \) for any $t \geq \nu'$. Assume now that the property holds true for some fixed $n$. Then $O^{n+1}_t(\nu, \xi) = O^{n+1}_t(\nu', \xi), \forall t \geq \nu'$. Once more the uniqueness of the solution of (4.5) yields $Y^{n+1}_t(\nu, \xi) = Y^{n+1}_t(\nu', \xi), \forall t \geq \nu'$. Hence the property holds true for any $n \geq 0$ and the desired result is obtained by taking the limit as $n \to \infty$.

(ii) The sequence of processes \( \left( Y^n_t(\nu, \xi) + \int_0^t h(s, L_s + \xi) ds \right)_{0 \leq t \leq T} \) is of càdlàg supermartingales which converges increasingly and pointwisely to the process \( \left( Y_t(\nu, \xi) + \int_0^t h(s, L_s + \xi) ds \right)_{0 \leq t \leq T} \). Therefore, according to Dellacherie and Meyer (1980, p. 86) and taking into account (4.7), the limit is also a càdlàg supermartingale. It follows that the process $Y(\nu, \xi)$ is also càdlàg. Next, the processes $O^n(\nu, \xi), n \geq 1$, are càdlàg and converge increasingly to $O(\nu, \xi)$. The rest of the proof is a direct consequence of Proposition 2.1.

Remark 4.1. Propositions 4.1 and 4.2 are generalizations of Corollaries 7.6 and 7.7 in Øksendal and Sulem (2006).

5 Optimal impulse control over bounded strategies

In this section we establish existence of an optimal impulse control over the set of strategies which have only a bounded number of impulses. Indeed, for fixed $n \geq 0$, let $\mathcal{A}_n$ be the following set of bounded strategies:

\[ \mathcal{A}_n = \{ (\tau_m, \xi_m)_{m \geq 1} \in \mathcal{D}, \text{ such that } \tau_n = T, IP - a.s. \}. \]

Then, the following result, which is a generalizations of Theorem 7.2 in Øksendal and Sulem (2006), holds.

Proposition 5.1. For $n \geq 1$, we have

\[ Y^n_0(0, 0) = \sup_{\delta \in \mathcal{A}_n} J(\delta). \]  

(5.9)

In addition, there exists a strategy $\delta^*_n \in \mathcal{A}_n$ which is optimal, i.e.,

\[ J(\delta^*_n) = \sup_{\delta \in \mathcal{A}_n} J(\delta). \]  

(5.10)

Proof. Let $\delta^*_n = (\tau^n_k, \beta^n_k)_{k \geq 0}$ be the strategy defined as follows.

\[ \tau^n_0 = \inf\{ s \geq 0; O^n_s(0, 0) = Y^n_s(0, 0) \} \land T, \]

and...
and
\[
O^\alpha_{\tau_0}(0, 0) := \max_{\beta \in U} (\mathcal{L}(\bar{S}) + Y_{\tau_0}^{n-1}(0, \beta)) = \max_{\beta \in U} (\mathcal{L}(\bar{S}) + Y_{\tau_0}^{n-1}(\tau_0^n, \beta))
\]
(5.11)

\[
= -\psi(\beta_0^n) + Y_{\tau_0}^{n-1}(\tau_0^n, \beta_0^n),
\]

and, for any \( k \in \{1, \ldots, n - 1\}, \)
\[
\tau_k^n = \inf\{ s \geq \tau_{k-1}^n; O_s^n(\tau_{k-1}^n, \beta_0^n + \ldots + \beta_{k-1}^n) = Y_s^{n-k}(\tau_{k-1}^n, \beta_0^n + \ldots + \beta_{k-1}^n) \} \wedge T,
\]

and \( O_s^n(\tau_{k-1}^n, \beta_0^n + \ldots + \beta_{k-1}^n) = -\psi(\beta_k^n) + Y_s^{n-k-1}(\tau_{k-1}^n, \beta_0^n + \ldots + \beta_{k-1}^n + \beta_k^n). \)

Note that in (5.11) we have taken into account the fact that \( Y_{\tau_0}^{n-1}(0, \beta) = Y_{\tau_0}^{n-1}(\tau_0^n, \beta) \).

This equality is valid since \( \beta \) is deterministic and thanks to the uniqueness of the solutions of BSDEs (4.5) which define \( Y^{n-1}(0, \beta) \) and \( Y^{n-1}(\tau_0^n, \beta) \) for \( t \geq \tau_0^n \). Finally, \( \tau_0^n = T \) and \( \beta_0^n \in U \) arbitrary. The choice of \( \beta_n \) is not very significant since there are no impulses at \( T \). We will show that \( \delta_n^* \) is an optimal strategy.

For any \( k \leq n \), the random variables \( \beta_k^n \) are \( \mathcal{F}_T \) measurable. Thanks to (2.1) and (4.5) we obtain
\[
Y_0^n(0, 0) = \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[ \int_0^\tau h(s, L_s)ds + \mathbb{1}_{[\tau < T]} O_\tau^n(0, 0) \right].
\]

Moreover, since the process \( O^n(0, 0) \) is continuous and \( O_T^n(0, 0) \leq 0 \), then the stopping time \( \tau_0^n \) is optimal after 0. Therefore,
\[
Y_0^n(0, 0) = \mathbb{E}\left[ \int_0^{\tau_0^n} h(s, L_s)ds + \mathbb{1}_{[\tau_0^n < T]} O_{\tau_0^n}(0, 0) \right].
\]
(5.12)

Now, since for any \( n \geq 1 \),
\[
O_{\tau_0^n}(0, 0) = \max_{\beta \in U} \{-\psi(\beta) + Y_{\tau_0^n-1}(0, \beta)\} = \max_{\beta \in U} \{-\psi(\beta) + Y_{\tau_0^n-1}(\tau_0^n, \beta)\}
\]
(5.13)

The second equality is valid since for any \( \beta \in U \) we have \( Y_{\tau_0^n-1}(0, \beta) = Y_{\tau_0^n-1}(\tau_0^n, \beta) \).

Then, it holds that
\[
Y_0^n(0, 0) = \mathbb{E}\left[ \int_0^{\tau_0^n} h(s, L_s)ds + \mathbb{1}_{[\tau_0^n < T]} (-\psi(\beta_0^n) + Y_{\tau_0^n-1}(\tau_0^n, \beta_0^n)) \right].
\]

But, once again using (2.1) and (4.5), we have
\[
Y_{\tau_0^n-1}(\tau_0^n, \beta_0^n) = \text{ess sup}_{\tau \in \mathcal{T}_0^n} \mathbb{E}\left[ \int_0^\tau h(s, L_s + \beta_0^n)ds + \mathbb{1}_{[\tau < T]} O_{\tau}^{n-1}(\tau_0^n, \beta_0^n) | \mathcal{F}_\tau \right].
\]

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and $\tau^n_1$ is an optimal stopping time after $\tau^n_0$. It yields that
\[
Y^n_{\tau^n_0} (\tau^n_0, \beta^n_0) = \mathbb{E} \left[ \int_{\tau^n_0}^{\tau^n_1} h(s, L_s + \beta^n_0) ds + \mathbb{1}_{[\tau^n_1 < T]} O^n_{\tau^n_0} (\tau^n_0, \beta^n_0) \mid \mathcal{F}_{\tau^n_0} \right]
\]
\[
= \mathbb{E} \left[ \int_{\tau^n_0}^{\tau^n_1} h(s, L_s + \beta^n_0) ds \right. + \mathbb{1}_{[\tau^n_1 < T]} ( - \psi (\beta^n_0) + Y^n_{\tau^n_1} (\tau^n_1, \beta^n_0 + \beta^n_1)) \mid \mathcal{F}_{\tau^n_0} \right].
\]
By combining the last equality and (5.12) we get
\[
Y^n_0 (0, 0) = \mathbb{E} \left[ \int_{0}^{\tau^n_0} h(s, L_s) ds \right. + \int_{\tau^n_0}^{\tau^n_1} h(s, L_s + \beta^n_0) ds \left. + \mathbb{1}_{[\tau^n_1 < T]} ( - \psi (\beta^n_0)) \right. + \mathbb{1}_{[\tau^n_1 < T]} Y^n_{\tau^n_1} (\tau^n_1, \beta^n_0 + \beta^n_1) \right],
\]
since $[\tau^n < T] \subset [\tau^n_0 < T]$ and $\mathbb{1}_{[\tau^n_1 < T]} \int_{\tau^n_0}^{\tau^n_1} h(s, L_s + \beta^n_0) ds = \int_{\tau^n_0}^{\tau^n_1} h(s, L_s + \beta^n_0) ds$.
Repeating this argument as many times as necessary yields
\[
Y^n_0 (0, 0) = \mathbb{E} \left[ \int_{0}^{\tau^n_0} h(s, L_s) ds \right. + \sum_{1 \leq k \leq n-1} \int_{\tau^n_k}^{\tau^n_{k+1}} h(s, L_s + \beta^n_0 + \ldots + \beta^n_{k-1}) ds \left. + \sum_{0 \leq k \leq n-1} \mathbb{1}_{[\tau^n_k < T]} ( - \psi (\beta^n_k)) \right] + \mathbb{1}_{[\tau^n_{n-1} < T]} Y^n_{\tau^n_{n-1}} (\tau^n_{n-1}, \beta^n_0 + \ldots + \beta^n_{n-1}) \right].
\]
But, according to (4.4) we have
\[
Y^n_{\tau^n_{n-1}} (\tau^n_{n-1}, \beta^n_0 + \ldots + \beta^n_{n-1}) = \mathbb{E} \left[ \int_{\tau^n_{n-1}}^{T} h(s, L_s + \beta^n_0 + \ldots + \beta^n_{n-1}) ds \right. \mid \mathcal{F}_{\tau^n_{n-1}} \right].
\]
Therefore,
\[
Y^n_0 (0, 0) = \mathbb{E} \left[ \int_{0}^{\tau^n_0} h(s, L_s) ds \right. + \sum_{1 \leq k \leq n} \int_{\tau^n_k}^{\tau^n_{k+1}} h(s, L_s + \beta^n_0 + \ldots + \beta^n_{k-1}) ds \left. \right. + \sum_{0 \leq k \leq n} \mathbb{1}_{[\tau^n_k < T]} ( - \psi (\beta^n_k)) \right] \right] + \sum_{0 \leq k \leq n} \mathbb{1}_{[\tau^n_k < T]} ( - \psi (\beta^n_k)) \right] \right] + \mathbb{1}_{[\tau^n_{n-1} < T]} Y^n_{\tau^n_{n-1}} (\tau^n_{n-1}, \beta^n_0 + \ldots + \beta^n_{n-1}) \right]
\]
\[
= J(\delta^n_0).
\]
It remains to show that $J(\delta^n_0) \geq J(\beta^n)$ for any strategy $\delta^n$ of $A_n$.

Indeed, let $\delta^n = (\tau^n_0, \ldots, \tau^n_{n-1}, T, T, \ldots; \beta^n_0, \ldots, \beta^n_{n-1}, \beta^n_n, \beta^n_{n}, \ldots)$ be a strategy of $A_n$.

Since $\tau^n_0$ is optimal after 0, we have
\[
Y^n_0 (0, 0) \geq \mathbb{E} \left[ \int_{0}^{\tau^n_0} h(s, L_s) ds + \mathbb{1}_{[\tau^n_0 < T]} O^n_{\tau^n_0} (0, 0) \right].
\]
But,
\[
O^n_{\tau^n_0} (0, 0) = \max_{\beta \in U} \{- \psi (\beta) + Y^n_{\tau^n_0} (0, \beta)\} = \max_{\beta \in U} \{- \psi (\beta) + Y^n_{\tau^n_0} (\tau^n_0, \beta)\} \geq - \psi (\beta^n_0) + Y^n_{\tau^n_0} (\tau^n_0, \beta^n_0).
\]
Therefore, we have
\[
Y_0^n(0, 0) \geq \mathbb{E}\left[ \int_0^{\tau_0^n} h(s, L_s) ds + \mathbb{I}_{[\tau_0^n < T]}(-\psi(\beta_0^n) + Y_{\tau_0^n}(\tau_0^n, \beta_0^n)) \right]
\]
\[
\geq \mathbb{E}\left[ \int_0^{\tau_0^n} h(s, L_s) ds + \int_{\tau_0^n}^{\tau_1^n} h(s, L_s + \beta_0^n) ds + \mathbb{I}_{[\tau_1^n < T]}(-\psi(\beta_0^n)) \right]
\]
\[
+ \mathbb{I}_{[\tau_1^n < T]} Y_{\tau_1^n}(\tau_1^n, \beta_0^n + \beta_1^n) \right].
\]

Finally, iterating as many times as necessary we obtain
\[
Y_0^n(0, 0) \geq \mathbb{E}\left[ \int_0^{\tau_0^n} h(s, L_s) ds + \sum_{1 \leq k \leq n} \int_{\tau_k^n}^{\tau_{k+1}^n} h(s, L_s + \beta_0^n + \ldots + \beta_k^n) ds \right]
\]
\[
+ \sum_{0 \leq k \leq n} \left\{ \mathbb{I}_{[\tau_k^n < T]}(-\psi(\beta_k^n)) \right\} = J(\delta^n).
\]

Hence, \(J(\delta^*_n) \geq J(\delta^n)\), for any \(\delta^n \in \mathcal{A}_n\). The proof is now complete. □

6 An optimal impulse control result.

We now give the main result of this paper.

**Theorem 6.1.** Under Assumption (A), the strategy \(\delta^* = (\tau_n^*, \beta_n^*)_{n \geq 0}\) defined by
\[
\tau_0^* = \inf\{ s \geq 0; O_s(0, 0) = Y_s(0, 0) \} \land T,
\]
\[
\max_{\beta \in U}(-\psi(\beta) + Y_{\tau_0^*}(0, \beta)) = -\psi(\beta_0^*) + Y_{\tau_0^*}(\tau_0^*, \beta_0^*),
\]
for \(n \geq 1,
\[
\tau_n^* = \inf\{ s \geq \tau_{n-1}^*; Y_s(\tau_{n-1}^*, \beta_0^* + \ldots + \beta_{n-1}^*) = O_s(\tau_{n-1}^*, \beta_0^* + \ldots + \beta_{n-1}^*) \} \land T,
\]
and
\[
\max_{\beta \in U}(-c - \psi(\beta) + Y_{\tau_n^*}(\tau_{n-1}^*, \beta_0^* + \ldots + \beta_{n-1}^* + \beta)) = -c - \psi(\beta_0^*) + Y_{\tau_n^*}(\tau_n^*, \beta_0^* + \ldots + \beta_{n-1}^* + \beta_n^*).
\]
is optimal for the impulse control problem.

Furthermore, we have
\[
Y_0(0, 0) = J(\delta^*).
\]

**Proof.** The proof is performed in three steps.
Step 1. Continuity of the value process \((Y_t(\nu, \xi))_{0 \leq t \leq T}\). We note that, by (1.8), we have, for any \(0 \leq t \leq T\),

\[
Y_t(\nu, \xi) + \int_0^t h(s, L_s + \xi) \mathbb{1}_{[s \geq \nu]} ds = \operatorname{ess} \sup_{\tau \in T} \mathbb{E} \left[ \int_0^\tau h(s, L_s + \xi) \mathbb{1}_{[s \geq \nu]} ds + \mathbb{1}_{[\tau < T]} O_{\tau}(\nu, \xi) | \mathcal{F}_t \right],
\]

meaning that the process \(\left( Y_t(\nu, \xi) + \int_0^t h(s, L_s + \xi) \mathbb{1}_{[s \geq \nu]} ds \right)_{0 \leq t \leq T}\) is the Snell envelope of \(\left( \int_0^t h(s, L_s + \xi) \mathbb{1}_{[s \geq \nu]} ds + \mathbb{1}_{[t < T]} O_t(\nu, \xi) \right)_{0 \leq t \leq T}\). Therefore, using Proposition 7.4 in the appendix below, there exist a continuous martingale \(M(\nu, \xi)\) and two increasing processes \(A(\nu, \xi)\) and \(B(\nu, \xi)\) belonging to \(\mathcal{S}_t^2\) such that \(B_0(\nu, \xi) = 0\) and, for \(0 \leq t \leq T\),

\[
\int_0^t h(s, L_s + \xi) \mathbb{1}_{[s \geq \nu]} ds + Y_t(\nu, \xi) = M_t(\nu, \xi) - A_t(\nu, \xi) - B_t(\nu, \xi).
\]

In addition, the process \(A(\nu, \xi)\) is optional and continuous, and \(B(\nu, \xi)\) is predictable and purely discontinuous. The continuity of the value process \(Y_t(\nu, \xi)\), will follow once we show that, for any stopping time \(\nu\) and \(\mathcal{F}_\nu\)-measurable random variable \(\xi\), \(B(\nu, \xi) \equiv 0\). Indeed, let us assume that \(B(\nu, \xi)\) is different to zero. Since the process is non-decreasing and purely discontinuous, there exists \(\tau \in T_\nu\) such that \(B_\tau(\nu, \xi) - B_{\tau^-}(\nu, \xi) > 0\). Thanks to (7.32), in the appendix, we have \(Y_{\tau^-}(\nu, \xi) = O_{\tau^-}(\nu, \xi)\). Hence,

\[
Y_{\tau^-}(\nu, \xi) = \max_{\beta \in U} (-\psi(\beta) + Y_{\tau^-}(\nu, \xi + \beta)) > Y_\tau(\nu, \xi) \geq O_\tau(\nu, \xi) = \max_{\beta \in U} (-\psi(\beta) + Y_\tau(\tau, \xi + \beta)).
\]

Therefore, since \(U\) is finite, there exists \(\beta_1 \in U\) such that the set

\[
\Lambda_1 = \{Y_{\tau^-}(\nu, \xi) = -\psi(\beta_1) + Y_{\tau^-}(\nu, \xi + \beta_1) \text{ and } \Delta Y_{\tau^-}(\nu, \xi + \beta_1) < 0\}
\]

satisfies \(P(\Lambda_1) > 0\). But, the same holds for \(\Delta Y_{\tau^-}(\nu, \xi + \beta_1)\). Therefore, there exists \(\beta_2 \in U\) such that the set

\[
\Lambda_2 = \{Y_{\tau^-}(\nu, \xi + \beta_1) = -\psi(\beta_2) + Y_{\tau^-}(\nu, \xi + \beta_1 + \beta_2) \text{ and } \Delta Y_{\tau^-}(\nu, \xi + \beta_1 + \beta_2) < 0\}
\]

satisfies \(P[\Lambda_1 \cap \Lambda_2] > 0\). It follows that, on the set \(\Lambda_1 \cap \Lambda_2\), we have

\[
Y_{\tau^-}(\nu, \xi) = -\psi(\beta_1) - \psi(\beta_2) + Y_{\tau^-}(\nu, \xi + \beta_1 + \beta_2).
\]

Making this reasoning as many times as necessary we obtain the existence of \(\beta_1, \ldots, \beta_n\) elements of \(U\) and a subset \(\Lambda_n\) of positive probability such that, on \(\Lambda_n\), we have

\[
Y_{\tau^-}(\nu, \xi) = -\sum_{i=1}^n \psi(\beta_i) + Y_{\tau^-}(\nu, \xi + \beta_1 + \ldots + \beta_n) \leq -nc + \gamma T.
\]
But, this is impossible for \( n \) large enough since the process \( Y(\tau, \xi) \) is non-negative. Therefore, the purely discontinuous process \( B(\nu, \xi) \) has no jumps and then it is null. Thus, the process \( Y(\nu, \xi) \) is continuous.

**Step 2.** The strategy \( \delta^* = (\tau_{n}^*, \beta_{n}^*)_{n \geq 0} \in \mathcal{D} \) and is such that \( Y_0(0, 0) = J(\delta^*) \).

Using Proposition 4.2 we get

\[
Y_0(0, 0) = \text{ess sup}_{\tau \in \mathcal{T}_0} \mathbb{E}\left[ \int_0^\tau h(s, L_s) ds + \mathbb{1}_{[\tau < T]} O_\tau(0, 0) \right].
\]

(6.14)

Now, since \( Y(\nu, \xi) \) is continuous for any \( \nu \in \mathcal{T} \) and any \( \mathcal{F}_\nu \)-measurable random variable \( \xi \) and \( O_T(0, 0) \leq 0 \), then the stopping time \( \tau_0^* \) is optimal for the problem (6.14). This yields

\[
Y_0(0, 0) = \mathbb{E}\left[ \int_0^{\tau_0^*} h(s, L_s) ds + \mathbb{1}_{[\tau_0^* < T]} O_{\tau_0^*}(0, 0) \right].
\]

But,

\[
O_{\tau_0^*}(0, 0) = \max_{\beta \in \mathcal{U}} \{-\psi(\beta) + Y_{\tau_0^*}(0, \beta) = \max_{\beta \in \mathcal{U}} \{-\psi(\beta) + Y_{\tau_0^*}(\tau_0^*, \beta)\} = -\psi(\beta^*) + Y_{\tau_0^*}(\tau_0^*, \beta^*)\}
\]

where \( \beta^* \in \mathcal{F}_{\tau_0^*} \). Note that the second equality is valid thanks to Proposition 4.2(i). Therefore,

\[
Y_0(0, 0) = \mathbb{E}\left[ \int_0^{\tau_0^*} h(s, L_s) ds + \mathbb{1}_{[\tau_0^* < T]}(-\psi(\beta^*) + Y_{\tau_0^*}(\tau_0^*, \beta^*)) \right].
\]

Next,

\[
Y_{\tau_0^*}(\tau_0^*, \beta_0^*) = \mathbb{E}\left[ \int_{\tau_0^*}^{\tau_1^*} h(s, L_s + \beta_0^*) ds + \mathbb{1}_{[\tau_1^* < T]} O_{\tau_1^*}(\tau_0^*, \beta_0^*) | \mathcal{F}_{\tau_0^*} \right]
\]

\[
= \mathbb{E}\left[ \int_{\tau_0^*}^{\tau_1^*} h(s, L_s + \beta_0^*) ds + \mathbb{1}_{[\tau_1^* < T]}(-\psi(\beta_1^*) + Y_{\tau_1^*}(\tau_1^*, \beta_1^* + \beta_0^*)) | \mathcal{F}_{\tau_0^*} \right].
\]

Replacing \( Y_{\tau_0^*}(\tau_0^*, \beta_0^*) \) by its expression in (6.15), we obtain

\[
Y_0(0, 0) = \mathbb{E}\left[ \int_0^{\tau_0^*} h(s, L_s) ds + \int_{\tau_0^*}^{\tau_1^*} h(s, L_s + \beta_0^*) ds + (-\psi(\beta_0^*)) \mathbb{1}_{[\tau_0^* < T]} \right]
\]

\[
+ \left( (-\psi(\beta_1^*)) \mathbb{1}_{[\tau_1^* < T]} + Y_{\tau_1^*}(\tau_1^*, \beta_1^* + \beta_0^*) \right) \mathbb{1}_{[\tau_0^* < T]}
\]

since \( [\tau_1^* < T] \subset [\tau_0^* < T] \) and \( [\tau_0^* < T] \in \mathcal{F}_{\tau_0^*} \). Proceeding in the same way as many times as necessary we get

\[
Y_0(0, 0) = \mathbb{E}\left[ \int_0^{\tau_0^*} h(s, L_s) ds + \ldots + \int_{\tau_{n-1}^*}^{\tau_n^*} h(s, L_s + \beta_{n-1}^*) ds + (-\psi(\beta_{n-1}^*)) \mathbb{1}_{[\tau_{n-1}^* < T]} + \ldots 
\]

\[
+ \left( (-\psi(\beta_n^*)) \mathbb{1}_{[\tau_n^* < T]} + Y_{\tau_n^*}(\tau_n^*, \beta_n^* + \ldots + \beta_{n-1}^* + \beta_0^*) \right) \mathbb{1}_{[\tau_n^* < T]} \right].
\]

(6.15)
Let us now show that $\delta^* \in D$. Assume that $P\{\tau^*_n < T; \ n \geq 0 \} > 0$. Then we have

$$
\begin{align*}
Y_0(0,0) & \leq E[\int_0^{T_0} h(s,L_s) ds + \sum_{t \leq T_0} Y_0(\tau^*_n, \beta^*_n + \ldots + \beta^*_{n-1}) + \max_{s \leq T_0} |Y_0(\tau^*_n, \beta^*_n + \ldots + \beta^*_{n-1})|]
+ \sum_{0 \leq k \leq n} (-\psi(\beta^*_k)) I_{[\tau^*_k < T]} + \sum_{0 \leq k \leq n} (-\psi(\beta^*_k)) I_{[\tau^*_k < T]} \leq \gamma T + E[\max_{s \leq T} |Y_0(\tau^*_n, \beta^*_n + \ldots + \beta^*_{n-1})| - nc \ P\{\tau^*_n < T; \ n \geq 0 \}.
\end{align*}
$$

The last quantity tends to $-\infty$ as $n \to \infty$, then $Y_0(0,0) = -\infty$ which contradicts the fact that $Y(0,0) \in S^2$. Therefore, $P\{\tau^*_n < T; \ n \geq 0 \} = 0$ i.e. $\delta^* \in D$. Finally, by taking limit as $n \to \infty$ in (8.13) we obtain $Y_0(0,0) = J(\delta^*)$.

**Step 3.** $J(\delta^*) \geq J(\delta)$ for any strategy $\delta \in A$. Let $\delta = (\tau_n, \beta_n)_{n \geq 0}$ be a finite strategy. Since $\tau^*_0$ is optimal after 0, we have

$$
Y_0(0,0) \geq E[\int_0^{T_0} h(s,L_s) ds + I_{[\tau_0 < T]} O_{\tau_0}(0,0)]
\geq E[\int_0^{T_0} h(s,L_s) ds + I_{[\tau_0 < T]} \{ -\psi(\beta_0) + Y_{\tau_0}(\tau_0, \beta_0) \}].
$$

But,

$$
O_{\tau_0}(0,0) = \max_{\beta \in U} \{ -\psi(\beta) + Y_{\tau_0}(0, \beta) \} = \max_{\beta \in U} \{ -\psi(\beta) + Y_{\tau_0}(\tau_0, \beta) \} \geq -\psi(\beta_0) + Y_{\tau_0}(\tau_0, \beta_0).
$$

It follows that

$$
Y_0(0,0) \geq E[\int_0^{T_0} h(s,L_s) ds + I_{[\tau_0 < T]} \{ -\psi(\beta_0) + Y_{\tau_0}(\tau_0, \beta_0) \}].
$$

Next,

$$
Y_{\tau_0}(\tau_0, \beta_0) = \text{ess sup}_{\tau \in T_0} E[\int_0^{T_0} h(s,L_s + \beta_0) ds + I_{[\tau < T]} O_{\tau}(\tau_0, \beta_0) | \mathcal{F}_{\tau_0}]
\geq E[\int_{\tau_0}^{T_0} h(s,L_s + \beta_0) ds + I_{[\tau_0 < T]} \{ -\psi(\beta_1) + Y_{\tau_1}(\tau_1, \beta_0 + \beta_1) \} | \mathcal{F}_{\tau_0}].
$$

Therefore,

$$
Y_0(0,0) \geq E[\int_0^{T_0} h(s,L_s) ds + \sum_{t \leq T_0} h(s,L_s + \beta_0) ds + (-\psi(\beta_0)) I_{[\tau_0 < T]} + (-\psi(\beta_1)) I_{[\tau_1 < T]} + I_{[\tau_1 < T]} Y_{\tau_1}(\tau_1, \beta_0 + \beta_1)].
$$

Now, by following this reasoning as many times as necessary we obtain,

$$
Y_0(0,0) \geq E[\int_0^{T_0} h(s,L_s) ds + \sum_{1 \leq k \leq n} \int_{\tau_{k-1}}^{\tau_k} h(s,L_s + \beta_0 + \ldots + \beta_{k-1}) ds
+ \sum_{0 \leq k \leq n} (-\psi(\beta_k)) I_{[\tau_k < T]} + Y_{\tau_k}(\tau_k, \beta_0 + \ldots + \beta_k)].
$$
and since the strategy $\delta$ is finite, by taking the limit as $n \to \infty$, we obtain $Y_0(0,0) \geq J(\delta)$ since $|Y_{n}(\tau_n, \beta_0+\ldots+\beta_n)| \leq \gamma_1[\tau_n<T]$. As $\delta \in \mathcal{A}$ is arbitrary, then $Y_0(0,0) = J(\delta^*) = \sup_{\delta \in \mathcal{D}} J(\delta) = \sup_{\delta \in \mathcal{A}} J(\delta)$. □

**Corollary 6.1.** Under Assumptions (A) and (B) it holds that

$$
\sup_{\delta \in \mathcal{A}} J(\delta) = Y_0(0,0) = \lim_{n \to \infty} Y_0^n(0,0) = \lim_{n \to \infty} \sup_{\delta \in \mathcal{A}_n} J(\delta).
$$

(6.16)

### 7 Combined stochastic and impulse controls

In this section we study a mixed stochastic and impulse control problem, where, we allow the process $L$, that describes the evolution of the system and subject to impulses, to also depend on a control $u$ from some appropriate set $\mathcal{V}$. Therefore, the dynamics of the system is subject to a combination of control and impulses. To begin with, we describe this dynamics.

Let $C$ be the set of continuous functions $w$ from $[0,T]$ into $\mathbb{R}^d$ endowed with the uniform norm. For $t \leq T$, let $\mathcal{G}_t$ be the $\sigma$-field of $C$ generated by $\{\pi_s : w \mapsto w_s, s \leq t\}$. By $\mathcal{G}$ we denote the $\sigma$-field on $[0,T] \times C$ consisting of all the subsets $G$, which have the property that the section of $G$ at time $t$ is in $\mathcal{G}_t$ and the section of $G$ at $w$ is Lebesgue measurable (see Elliott (1976) for more details on this subject). Finally if $w \in C$ and $a$ is a deterministic function then $w + a$ is the function which with $t \in [0,T]$ associates $(w + a)_t = w_t + a$.

Let us now consider a function from $[0,T] \times C \to \mathbb{R}^d$ which satisfies the following

**Assumption (H).**

(i) $\sigma$ is $\mathcal{G}$-measurable and there exists a constant $k$ such that

$$
\text{for every } t \in [0,T] \text{ and every } w \text{ and } w' \in C, \ |\sigma(t,w) - \sigma(t,w')| \leq k\|w - w'\|_t \text{ where } \|w\|_t = \sup_{s \leq t} |w_s|, \ t \leq T;
$$

(ii) for every $t \in [0,T]$, $|\sigma(t,0)| \leq k$, $\sigma$ is invertible and its inverse $\sigma^{-1}$ is bounded.

Let $\mathcal{V}$ be a compact metric space and $\mathcal{V}$ the set of $\mathcal{P}$-measurable processes $v = (v_t)_{t \leq T}$ with values in $\mathcal{V}$. Hereafter, $\mathcal{V}$ is called the set of admissible controls.

We consider now the process $(L_t)_{0 \leq t \leq T}$ which is the unique solution for the following stochastic differential equation:

$$
\begin{cases}
   dL_t = \sigma(t,L_t) dB_t, & 0 < t \leq T, \\
   L_0 = x, & x \in \mathbb{R}^d,
\end{cases}
$$
whose existence is guaranteed by Assumption (H1). The process $L$ stands for the state of the system when non-controlled.

Let $f$ and (resp. $h$) be a measurable and uniformly bounded function from $[0, T] \times \mathcal{C} \times \mathcal{V}$ into $\mathbb{R}^d$ (resp. $\mathbb{R}^+$) such that

(H2) $f$ and $h$ are $\mathcal{G} \otimes \mathcal{B}(\mathcal{V})$-measurable

(H3) for every $t \in [0, T]$, $w \in \mathcal{C}$, the function which with $u \in \mathcal{V}$ associates $f(t, w, u)$ (resp. $h(t, w, u)$) is continuous.

Now, given a control $u \in \mathcal{V}$, let $\mathbb{P}^u$ be the probability measure on $(\Omega, \mathcal{F})$ defined by

$$
\frac{d\mathbb{P}^u}{d\mathbb{P}} = \exp\left\{\int_0^T \sigma^{-1}(s, L.) f(s, L., u_s) dB_s - \frac{1}{2} \int_0^T |\sigma^{-1}(s, L.) f(s, L., u_s)|^2 ds\right\}.
$$

Thanks to Girsanov’s Theorem (see e.g. Revuz and Yor (1991)), for every $u \in \mathcal{V}$ the process $B^u := \left(B_t - \int_0^t \sigma^{-1}(s, L.) f(s, L., u_s) ds\right)_{0 \leq t \leq T}$ is a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P}^u)$, and $L$ is a weak solution for the following functional differential equation.

$$
\begin{cases}
    dL_t = f(t, L., u_t) dt + \sigma(t, L.) dB^u_t, & 0 < t \leq T, \\
    L_0 = x.
\end{cases}
$$

Under $\mathbb{P}^u$, the process $L$ represents the evolution of the system when controlled by $(u_t)_{0 \leq t \leq T}$ but not subject to impulses. Next, for a strategy $\delta = (\tau_n, \xi_n)_{n \geq 1} \in \mathcal{A}$, we denote by $(L^\delta_t)_{0 \leq t \leq T}$ the process defined by

$$
L^\delta_t = L_t + \sum_{n \geq 1} \xi_n \mathbb{1}_{[\tau_n < t]} = x + \int_0^t f(s, L., u_s) ds + \int_0^t \sigma(s, L.) dB^u_s + \sum_{n \geq 1} \xi_n \mathbb{1}_{[\tau_n < t]}.
$$

Under $\mathbb{P}^u$, the process $L^\delta$ stands for the evolution of the system when controlled by $(u_t)_{0 \leq t \leq T}$ and subject to the impulse strategy $\delta$. Note that the control and impulses are interconnected. The reward function associated with the pair $(\delta, u)$ is

$$
J(\delta, u) = \mathbb{E}^u\left[\int_0^T h(s, L^\delta, u_s) ds - \sum_{n \geq 1} \psi(\xi_n) \mathbb{1}_{[\tau_n < T]}\right],
$$

where, $\mathbb{E}^u$ is the expectation with respect to the probability measure $\mathbb{P}^u$. With, $\xi_0 = 0$ and $\tau_0 = 0$, we have

$$
\int_0^T h(s, L^\delta, u_s) ds = \sum_{n \geq 0} \int_{\tau_n}^{\tau_{n+1}} h(s, L + \xi_1 + \ldots + \xi_n, u_s) ds.
$$
The objective is to find a pair \((\delta^*, u^*)\) such that
\[
J(\delta^*, u^*) = \sup_{(\delta, u) \in A \times V} J(\delta, u).
\]

Next let \(H\) be the Hamiltonian associated with the control problem, i.e., the function which with \((t, w, z, u) \in [0, T] \times C \times \mathbb{R}^d \times V\) associates \(H(t, w, z, u) = z\sigma^{-1}(t, w) f(t, w, u) + h(t, w, u)\). The function \(H\) is Lipschitz \(w.r.t.\) \(z\) uniformly in \((t, w, u)\) and through Beneš Selection Lemma (cf. Beneš (1970), Lemma 1), there exists a \(\beta\) such that for any \((t, w, z)\)
\[
H^*(t, w, z) := \sup_{u \in V} H(t, w, z, u) = H(t, w, z, u^*(t, w, z)). \tag{7.19}
\]

Moreover, the function \(H^*\) is Lipschitz in \(z\) uniformly \(w.r.t.\) \((t, w)\) as a supremum over \(u \in V\) of functions uniformly Lipschitz \(w.r.t.\) \((t, w, u)\).

For any stopping time \(\nu \in T\), and any \(\mathcal{F}_\nu\)-measurable random variable \(\xi\), let \((Y^n(\xi, \nu), Z^n(\xi, \nu), K^n(\xi, \nu))_{n \geq 0}\) be the sequence of processes defined as follows.
\[
Y^n_t(\nu, \xi) = \int_t^T H^*(s, L(\omega) + \xi, Z^n_s(\xi, \nu)) \mathbb{1}_{[s \geq \nu]} ds - \int_t^T Z^n_s(\xi, \nu) dB_s, \quad 0 \leq t \leq T, \tag{7.20}
\]
and, for any \(n \geq 1\),
\[
\begin{cases}
(Y^n(\nu, \xi), Z^n(\nu, \xi), K^n(\nu, \xi)) \in S^2_c \times \mathcal{H}^{2, d} \times S^2_c, \\
Y^n_t(\nu, \xi) = \int_t^T H^*(s, L_s + \xi, Z^n_s(\xi, \nu)) \mathbb{1}_{[s \geq \nu]} ds + K^n_t(\nu, \xi) - \int_t^T Z^n_s(\nu, \xi) dB_s, \quad t \leq T, \\
Y^n_t(\nu, \xi) \geq O^n_t(\nu, \xi) := \max_{\beta \in U} (-\psi(\beta) + Y^{n-1}_t(\nu, \xi + \beta)), \quad t \leq T, \\
\int_0^T (Y^n_t(\nu, \xi) - O^n_t(\nu, \xi)) dK^n_t(\nu, \xi) = 0. \tag{7.21}
\end{cases}
\]

We can easily see by induction that for any \(n \geq 0\), the processes \(Y^n(\xi, \nu), Z^n(\xi, \nu)\) and \(K^n(\xi, \nu)\) are well defined, since \(H^*\) is Lipschitz in \(z\) and \(U\) is finite. In addition, the process \(Y^n(\xi, \nu)\) is continuous, since \(\max_{\beta \in U} (-\psi(\beta) + Y^{n-1}_T(\nu, \xi + \beta)) < 0\). Next, in view of Proposition 2.2 it holds that, for any \(n \geq 0\), for any \(\nu\) and \(\xi\), \(Y^n(\xi, \nu) \leq Y^{n+1}(\xi, \nu)\) since \(Y^0(\xi, \nu) \leq Y^1(\xi, \nu)\).

Now, according to (7.20) and (7.21), there are controls \(u^n \in V\) such that:
\[
Y^n_t(\nu, \xi) = \mathbb{E}^\nu \left[ \int_t^T h(s, L_s + \xi, u^n_s) \mathbb{1}_{[s \geq \nu]} ds | F_t \right], \quad 0 \leq t \leq T, \tag{7.22}
\]
and, for any \(n \geq 1\),
\[
Y^n_t(\nu, \xi) = \text{esssup}_{\tau \in T} \mathbb{E}^{\nu^n} \left[ \int_t^\tau h(s, L_s + \xi, u^n_s) \mathbb{1}_{[s \geq \nu]} ds + \mathbb{1}_{[\tau < T]} O^n_t(\nu, \xi) | F_{\tau} \right], \quad t \leq T. \tag{7.23}
\]
The last inequality is valid since $K^n(\nu, \xi)$ is non-decreasing and $Y^n_t(\nu, \xi) \geq \mathbb{1}_{\{\tau < T\}}O^n_T(\nu, \xi)$. Therefore, $Y^n_t(\nu, \xi)$ is greater than the expression inside the ess sup. On the other hand, there is equality when $\tau = \inf\{s \geq t, K^n_s(\nu, \xi) - K^n_t(\nu, \xi) > 0\} \land T$.

Now, by induction, as in the proof of Proposition 4.1, we obtain that, for any $n \geq 0$, $\tau$ a stopping time and any $\mathcal{F}_\tau$-measurable r.v. $\xi$, the process $Y^n(\nu, \xi)$ satisfies the following property:

$$0 \leq Y^n_t(\nu, \xi) \leq \gamma(T - t), \quad t \leq T,$$

where, $\gamma$ is the constant of boundedness of $h$. Therefore, using Proposition 2.3, there exists a càdlàg process $(Y^*_t(\nu, \xi))_{t \leq T}$ limit of the increasing sequence $(Y^n(\nu, \xi))_{n \geq 0}$ as $n \to \infty$. Moreover we have

$$0 \leq Y^*_t(\nu, \xi) \leq \gamma(T - t), \quad t \leq T.$$

In the next proposition, we give a characterization of $Y^*(\nu, \xi)$.

**Proposition 7.1.** The process $Y^*(\nu, \xi)$ is continuous. Moreover, there exist processes $Z^*(\nu, \xi) \in \mathcal{H}^{2,d}$ and $K^*(\nu, \xi) \in \mathcal{S}^2_{\text{ci}}$ such that, for all $t \leq T$,

$$
\begin{cases}
Y^*_t(\nu, \xi) = \int_t^T H^*(s, L + \xi, Z^*_s(\xi, \nu))\mathbb{1}_{\{s \geq t\}}ds + K^*_t(\nu, \xi) - \int_t^T Z^*_s(\nu, \xi)d\mathbb{B}_s, \\
Y^*_t(\nu, \xi) \geq O_t(\nu, \xi) := \max_{\beta \in U} (-\psi(\beta) + Y^*_t(\nu, \xi + \beta)) \\
\int_0^T (Y^*_t(\nu, \xi) - O_t(\nu, \xi))dK^*_t(\nu, \xi) = 0.
\end{cases}
$$

(7.24)

Furthermore, for any pair $(\nu, \xi)$ and any stopping time $\nu' \geq \nu$, we have $Y^*_\nu(\nu, \xi) = Y^*_\nu(\nu, \xi)$.

**Proof.** Thanks to Proposition 2.3 there exists a process $Z^*(\nu, \xi) \in \mathcal{H}^{2,d}$ such that, for any $p \in [1, 2)$, the sequence $(Z^n(\nu, \xi))_{n \geq 0}$ converges to $Z^*(\nu, \xi)$ in $\mathcal{H}^{p,d}$. This convergence holds also weakly in $\mathcal{H}^{2,d}$. Additionally, there exists an increasing process $K^*(\nu, \xi) \in \mathcal{S}^2_{\text{ci}}$ such that for any stopping time $\tau$ the sequence $(K^n_T(\nu, \xi))_{n \geq 0}$ converges to $K^*_T(\nu, \xi)$ in $L^p(dP)$. Therefore, we have

$$
\begin{cases}
Y^*_t(\nu, \xi) = \int_t^T H^*(s, L + \xi, Z^*_s(\xi, \nu))\mathbb{1}_{\{s \geq t\}}ds + K^*_t(\nu, \xi) - \int_t^T Z^*_s(\nu, \xi)d\mathbb{B}_s, \\
Y^*_t(\nu, \xi) \geq O_t(\nu, \xi) := \max_{\beta \in U} (-\psi(\beta) + Y^*_t(\nu, \xi + \beta)), \quad 0 \leq t \leq T.
\end{cases}
$$

(7.25)

The last inequality is valid, since $U$ is finite.

Next, for $t \leq T$, let us set

$$R_t = \text{ess sup}_{\tau \in T_t} \mathbb{E}\left[\int_t^\tau H^*(s, L + \xi, Z^*_s(\xi, \nu))\mathbb{1}_{\{s \geq t\}}ds + \mathbb{1}_{\{\tau < \tau\}}O^*_T(\nu, \xi)|\mathcal{F}_t\right].$$

(7.26)
Using Characterization (2.1) of $(R_t)_{0 \leq t \leq T}$ as a solution of a BSDE yields that, in using the comparison result (Proposition 2.2), for any $t \leq T$, $R_t \geq Y^n_t(\nu, \xi)$ and then $R_t \geq Y^*_t(\nu, \xi)$. On the other hand, a result by Peng and Xu (2005) implies that $(R_t)_{0 \leq t \leq T}$ is the smallest $H^*(s, L, + \xi, z)\mathbb{I}_{[s \geq v]}$-supermartingale which dominates $O_t(\nu, \xi) := \max_{\beta \in U} (-\psi(\beta) + Y^*_t(\nu, \xi + \beta))$. But, by (7.25), the process $Y^*(\nu, \xi)$ is a $H^*(s, L, + \xi, z)\mathbb{I}_{[s \geq v]}$-supermartingale such that $Y^*_t(\nu, \xi) \geq O_t(\nu, \xi) := \max_{\beta \in U} (-\psi(\beta) + Y^*_t(\nu, \xi + \beta))$. Thus, $Y^*_t(\nu, \xi) \geq R_t$, for any $t \leq T$. Finally, since both processes are càdlàg, then P-a.s., $R = Y^*(\nu, \xi)$. This means that $Y^*(\nu, \xi)$ is equal to the second term in (7.26). Now, using the characterization of Theorem 2.1 it holds that $Y^*(\nu, \xi)$ and, $Z^*(\nu, \xi)$ and $K^*(\nu, \xi)$ satisfy (7.24). The continuity of $Y^*(\nu, \xi)$ is obtained in a similar fashion as in Theorem 6.1 since $U$ is finite.

Now, if $\nu' \geq \nu$ then thanks to uniqueness result we have, for any $n \geq 0$, $Y^n_{\nu'}(\nu, \xi) = Y^n_{\nu'}(\nu, \xi)$, and then it is enough to take the limit as $n \rightarrow \infty$. \(\square\)

In the same way as previously, for any admissible control $u \in \mathcal{U}$, a stopping time $\nu$, an $\mathcal{F}_\nu$-measurable r.v. $\xi$ and $n \geq 0$, let us consider the sequence of processes defined recursively by

$$Y^u_{t,0}(\nu, \xi) = \int_t^T H(s, L, \omega + \xi, Z^u_{s,0}(\xi, \nu), u_s)\mathbb{I}_{[s \geq v]}ds - \int_t^T Z^u_{s,0}(\xi, \nu)d\mathbb{B}_s, \quad t \leq T$$

(7.27)

and, for any $n \geq 1,$

$$
\begin{cases}
(Y^u_{t,n}(\nu, \xi), Z^u_{t,n}(\nu, \xi), K^u_{t,n}(\nu, \xi)) \in \mathcal{S}_c^2 \times \mathcal{H}_c^2 \times \mathcal{S}_c^2

Y^u_{t,n}(\nu, \xi) = \int_t^T H(s, L, + \xi, Z^u_{s,n}(\xi, \nu), u_s)\mathbb{I}_{[s \geq v]}ds + K^u_{t,n}(\nu, \xi) - K^u_{t,0}(\nu, \xi) - \int_t^T Z^u_{s,n}(\nu, \xi)d\mathbb{B}_s,

Y^u_{t,n}(\nu, \xi) \geq O_{t,n}(\nu, \xi) := \max_{\beta \in U} (-\psi(\beta) + Y^u_{t,n-1}(\nu, \xi + \beta))

\text{and} \quad \int_0^T (Y^u_{t,n}(\nu, \xi) - O_{t,n}(\nu, \xi))dK^u_{t,n}(\nu, \xi) = 0.
\end{cases}
$$

(7.28)

As above, the sequence of processes $(Y^u_{t,n}(\nu, \xi))_{n \geq 0}$ is increasing and converges to a càdlàg process $Y^u(\nu, \xi)$ which satisfies $0 \leq Y^u_t(\nu, \xi) \leq \gamma(T-t)$, for any $t \leq T$. We also have the following

**Proposition 7.2.** The process $Y^u(\nu, \xi)$ is continuous. Furthermore, there exist two processes $(Z^u(\nu, \xi), K^u(\nu, \xi)) \in \mathcal{H}_c^2 \times \mathcal{S}_c^2$ such that, for all $t \leq T,$

$$
\begin{cases}
Y^u_t(\nu, \xi) = \int_t^T H(s, L, + \xi, Z^u_s(\xi, \nu), u_s)\mathbb{I}_{[s \geq v]}ds + K^u_t(\nu, \xi) - K^u_0(\nu, \xi) - \int_t^T Z^u_s(\nu, \xi)d\mathbb{B}_s,

Y^u_t(\nu, \xi) \geq O_t(\nu, \xi) := \max_{\beta \in U} (-\psi(\beta) + Y^u_t(\nu, \xi + \beta)),

\text{and} \quad \int_0^T (Y^u_t(\nu, \xi) - O_t(\nu, \xi))dK^u_t(\nu, \xi) = 0.
\end{cases}
$$

(7.29)
Moreover, we have
\[ Y_0^u(0,0) = \sup_{\delta \in \mathcal{A}} J(u, \delta). \]

**Proof.** The proof of the two first claims is the same as the one of Proposition \ref{prop:existence_control}. It remains to show the last one. Indeed, since the triple \((Y^u(\nu, \xi), (Z^u(\nu, \xi), K^u(\nu, \xi))\) satisfies
\[
\begin{align*}
Y^u_t(\nu, \xi) &= \int_t^T h(s, L. + \xi, u_s) \mathbb{I}_{[s \geq \nu]} ds + K^u_T(\nu, \xi) - K^u_T(\nu, \xi) - \int_t^T Z^u_s(\nu, \xi) d\mathbb{B}^u_s, \quad t \leq T \\
Y^u_T(\nu, \xi) &\geq O_t(\nu, \xi) := \max_{\beta \in U} (-\psi(\beta) + Y^u_t(\nu, \xi + \beta)) \quad \text{and} \\
\int_0^T (Y^u_t(\nu, \xi) - O_t(\nu, \xi)) dK^u_t(\nu, \xi) &= 0.
\end{align*}
\]

it follows, as in Theorem \ref{thm:existence_control} that \(Y_0^u(0,0) = \sup_{\delta \in \mathcal{A}} J(u, \delta). \)

We give now the main result of this section.

**Theorem 7.1.** There exist a control \(u^* \in \mathcal{V}\) and a strategy \(\delta^* = (\tau^*_n, \beta^*_n)_{n \geq 0} \in \mathcal{A}\) such that
\[ J(\delta^*, u^*) = \sup_{(\delta, u) \in \mathcal{A} \times \mathcal{V}} J(\delta, u). \]

In addition,
\[ Y_0^{u^*}(0,0) = J(\delta^*, u^*). \]

**Proof:** Let \(u \in \mathcal{V}\), then through the definitions of \(Y^u(\nu, \xi)\) and \(Y^*(\nu, \xi)\) it holds true that \(Y^*(\nu, \xi) \geq Y^u(\nu, \xi)\) since, in using the comparison result of Proposition \ref{prop:comparison} and an induction argument, we have \(Y^{*,n}(\nu, \xi) \geq Y^{u,n}(\nu, \xi)\), for any \(n \geq 0\). Hence, we have
\[ Y_0^*(0,0) \geq Y_0^u(0,0) = \sup_{\delta \in \mathcal{A}} J(u, \delta), \]
and then
\[ Y_0^*(0,0) \geq \sup_{(\delta, u) \in \mathcal{A} \times \mathcal{V}} J(\delta, u) \geq \sup_{u \in \mathcal{V}} \sup_{\delta \in \mathcal{A}} J(u, \delta). \]

Now, let \(u^*\) and \(\delta^*\) be defined as follows.
\[
\begin{align*}
\tau^*_1 &= \inf \{s \geq 0; \ O_s(0,0) = Y^*_s(0,0) \} \land T, \\
-\psi(\beta^*_1) + Y^*_{t_1^*}(\tau^*_1, \beta^*_1) &= \max_{\beta \in U} \{-\psi(\beta) + Y^*_{t_1^*}(0, \beta)\} = O_{t_1^*}(0,0), \\
u^*_1 \mathbb{I}_{[t \leq \tau^*_1]}(t, L., Z^*_1(t, 0, 0))
\end{align*}
\]
and, for \(n \geq 2,
\[
\begin{align*}
\tau^*_n &= \inf \{s \geq \tau^*_{n-1}; \ Y^*_{s}(\tau^*_{n-1}, \beta^*_1 + \ldots + \beta^*_n) = O_s(\tau^*_{n-1}, \beta^*_1 + \ldots + \beta^*_{n-1}) \} \land T, \\
-\psi(\beta^*_n) + Y^*_{t_n^*}(\tau^*_n, \beta^*_1 + \ldots + \beta^*_n) &= \max_{\beta \in U} \{-\psi(\beta) + Y^*_{t_n^*}(\tau^*_{n-1}, \beta^*_1 + \ldots + \beta^*_{n-1} + \beta)\} \\
&= O_{t_n^*}(\tau^*_{n-1}, \beta^*_1 + \ldots + \beta^*_{n-1})
\end{align*}
\]
and \(u^*_n \mathbb{I}_{[\tau^*_{n-1}, \tau^*_n]}(t) = u^*(t, L. + \beta^*_1 + \ldots + \beta^*_n, Z^*_1(\tau^*_n, \beta^*_1 + \ldots + \beta^*_{n-1})). \)
Therefore,

\[ Y^*_0(0, 0) = \mathbb{E}^u \left[ \int_0^{\tau^*_1} h(s, L, u^*_s)ds + O_{\tau^*_1}(0, 0) \mathds{1}_{[\tau^*_1 < T]} \right] \]

and as \( O_{\tau^*_1}(0, 0) = -\psi(\beta^*_1) + Y^*_0(\tau^*_1, \beta^*_1) \) then

\[ Y^*_0(0, 0) = \mathbb{E}^u \left[ \int_0^{\tau^*_1} h(s, L, u^*_s)ds + (-\psi(\beta^*_1) + Y^*_0(\tau^*_1, \beta^*_1)) \mathds{1}_{[\tau^*_1 < T]} \right]. \]

But,

\[ Y^*_0(\tau^*_1, \beta^*_1) = Y^*_0(\tau^*_1, \beta^*_1) + \int_{\tau^*_1}^{\tau^*_2} h(s, L + \beta^*_1, u^*_s)ds - \int_{\tau^*_1}^{\tau^*_2} \mathds{1}_{[\tau^*_1 < T]}. \]

Plugging the last quantity in the previous equality to obtain

\[ Y^*_0(0, 0) = \mathbb{E}^u \left[ \int_0^{\tau^*_2} h(s, L, u^*_s)ds + \int_{\tau^*_1}^{\tau^*_2} h(s, L + \beta^*_1, u^*_s)ds - \psi(\beta^*_1) \mathds{1}_{[\tau^*_1 < T]} + Y^*_0(\tau^*_1, \beta^*_1) \mathds{1}_{[\tau^*_2 < T]} \right] \]

\[ = \mathbb{E}^u \left[ \int_0^{\tau^*_2} h(s, L^*, u^*_s)ds - \psi(\beta^*_1) \mathds{1}_{[\tau^*_1 < T]} + Y^*_0(\tau^*_1, \beta^*_1) \right] \]

\[ = \mathbb{E}^u \left[ \int_0^{\tau^*_2} h(s, L^*, u^*_s)ds - \psi(\beta^*_1) \mathds{1}_{[\tau^*_1 < T]} + O_{\tau^*_2}(\tau^*_1, \beta^*_1) \mathds{1}_{[\tau^*_2 < T]} \right], \]

since \( Y^*_0(\tau^*_1, \beta^*_1) = Y^*_0(\tau^*_1, \beta^*_1) \mathds{1}_{[\tau^*_2 < T]} \subset [\tau^*_1 < T] \) and finally

\[ Y^*_0(\tau^*_1, \beta^*_1) = O_{\tau^*_2}(\tau^*_1, \beta^*_1) \mathds{1}_{[\tau^*_2 < T]} \]

Repeating now this reasoning as many times as necessary to obtain, for all \( n \geq 1 \),

\[ Y^*_0(0, 0) = \mathbb{E}^u \left[ \int_0^{\tau^*_n} h(s, L^*, u^*_s)ds - \sum_{k=1}^n \psi(\beta^*_k) \mathds{1}_{[\tau^*_k < T]} + O_{\tau^*_n+1}(\tau^*_n, \beta^*_n + ... + \beta^*_n) \mathds{1}_{[\tau^*_n+1 < T]} \right]. \]

This property implies first that the strategy \( \delta^* \) is finite since \( Y^*(0, 0) \) is a real constant. On the other hand taking the limit as \( n \to \infty \) to obtain:

\[ Y^*(0, 0) = J(u^*, \delta^*). \]

Thus,

\[ Y^*(0, 0) = J(\delta^*, u^*) = \sup_{u \in \mathcal{F}} \sup_{\delta \in \mathcal{A}} J(\delta, u), \]

and the proof is complete. \( \square \)
Appendix

Let \( \theta \) (resp. \( \pi \)) be the optional (resp. predictable) tribe on \( \Omega \times [0,T] \), i.e., the tribe generated by the càdlàg and \( \mathcal{F}_t \)-adapted processes \( X = (X_t)_{0 \leq t \leq T} \) (resp. the left continuous and \( \mathcal{F}_t \)-adapted processes \( Y = (Y_t)_{t \leq T} \)).

**Definition 7.1.** A measurable process \( U = (U_t)_{t \leq T} \) is said to be of class \([D]\) if the set of random variables \( \{U_\tau, \tau \in T\} \) is uniformly integrable.

**Proposition 7.3.** Let \( U = (U_t)_{t \leq T} \) be an optional process which is of class \([D]\) and \( N = (N_t)_{t \leq T} \) the Snell envelope of \( U \) defined by:

\[
N_t = \text{ess sup}_{\tau \in T} \mathbb{E}[U_\tau | \mathcal{F}_t], \quad t \leq T.
\]

If \( U \) is right upper semi-continuous, then the process \( N \) is continuous.

**Proposition 7.4.** Let \( (U_t)_{t \leq T} \) be an optional process of class \([D]\) and \( N \) its Snell envelope. Then

(i) there exist a martingale \( M \) and two increasing, integrable and right continuous processes \( A \) and \( B \) such that,

\[
N_t = M_t - A_t - B_t, \quad 0 \leq t \leq T. \tag{7.31}
\]

The process \( A \) is optional and continuous, and \( B \) is predictable, i.e., \( \pi \)-measurable and purely discontinuous. This decomposition is unique. In addition for any \( t \leq T \) we have:

\[
\{\Delta_t B > 0\} \subset \{U_t - N_t = 0\} \tag{7.32}
\]

and

\[
\Delta_t B = (U_t - N_t)^+ \mathbb{1}_{\Delta_t U < 0}. \tag{7.33}
\]

(ii) If \( Y \in \mathcal{S}^2 \) and \( M \) is a continuous martingale with respect to \( \mathcal{F} \), then the processes \( A \) and \( B \) are also in \( \mathcal{S}^2 \).

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