INVERSION OF SEISMIC-TYPE RADON TRANSFORMS ON THE PLANE

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ABSTRACT. We study integral transforms mapping a function on the Euclidean plane to the family of its integration on plane curves, that is, a function of plane curves. The plane curves are given by the graphs of functions with a fixed axis of the independent variable, and are imposed some symmetry with respect to the axes. These transforms contain the parabolic Radon transform and the hyperbolic Radon transforms arising from seismology. We prove the inversion formulas for these transforms under some vanishing conditions of functions.

1. INTRODUCTION

Fix arbitrary $c \in \mathbb{R}$ and $\alpha, \beta > 1$. Let $(x, y), (s, u) \in \mathbb{R}^2$ be variables of functions. We study three types of integral transforms $\mathcal{P}_\alpha f(s, u)$, $\mathcal{Q}_\alpha f(s, u)$ and $\mathcal{R}_{\alpha, \beta} f(s, u)$ of a function $f(x, y)$. These are the integration of $f(x, y)$ on the graph of functions of the fixed axis $x$. Their precise definitions are the following.

Firstly, $\mathcal{P}_\alpha f(s, u)$ is defined by

$$\mathcal{P}_\alpha f(s, u) = \int_{-\infty}^{\infty} f(x, s|x-c|^\alpha + u) dx = \int_{-\infty}^{\infty} f(x + c, s|x|^\alpha + u) dx.$$  

$\mathcal{P}_\alpha f(s, u)$ is the integration of $f$ on a curve

$$\Gamma_\mathcal{P}(\alpha; s, u) = \{ (x, s|x-c|^\alpha + u) : x \in \mathbb{R} \},$$

and slightly different from the standard contour integral on $\Gamma_\mathcal{P}(\alpha; s, u)$ because the integration does not contain the line element. In particular $\Gamma_\mathcal{P}(2l; s, u)$ with $l = 1, 2, 3, \ldots$ and $s \neq 0$ is the translation of the graph of monomial $x^{2l}$. Moreover $\mathcal{P}_2 f(s, u)$ is called the parabolic Radon transform of $f$ in seismology. Note that

$$f(-x + c, s|x|^\alpha + u) = f(-x + c, s|x|^\alpha + u)$$

and $f$ can be splitted into

$$f(x + c, y) = \frac{f(x + c, y) + f(-x + c, y)}{2} + \frac{f(x + c, y) - f(-x + c, y)}{2}.$$  

The second term of the right hand side of the above does not contribute to $\mathcal{P}_\alpha f(s, u)$. So we have only to consider functions satisfying $f(-x + c, y) = f(x + c, y)$ for $\mathcal{P}_\alpha$.

Secondly, $\mathcal{Q}_\alpha f(s, u)$ is defined by

$$\mathcal{Q}_\alpha f(s, u) = \int_{-\infty}^{\infty} f(x, s|x-c|^\alpha - 1 + u) dx = \int_{-\infty}^{\infty} f(x + c, sx|x|^\alpha - 1 + u) dx.$$  

$\mathcal{Q}_\alpha f(s, u)$ is the integration of $f$ on a curve

$$\Gamma_\mathcal{Q}(\alpha; s, u) = \{ (x, s|x-c|^\alpha - 1 + u) : x \in \mathbb{R} \},$$

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and slightly different from the standard contour integral on $\Gamma_Q(\alpha; s, u)$. In particular $\Gamma_Q(2l + 1; s, u)$ with $l = 1, 2, 3, \ldots$ and $s \neq 0$ is a translation of the graph of $x^{2l+1}$. Unfortunately, however, $Q_0 f(s, u)$ does not seem to be used in science and technology.

Finally, $R_{\alpha, \beta} f(s, u)$ is defined by

$$R_{\alpha, \beta} f(s, u) = \int_{x \in \mathbb{R}} \frac{f(x, (s|x-c|^\alpha + u)^{1/\beta})}{(s|x-c|^\alpha + u)^{1/\beta}} \, dx$$

$$= \int_{x \in \mathbb{R}} \frac{f(x + c, (s|x|^\alpha + u)^{1/\beta})}{(s|x|^\alpha + u)^{1/\beta}} \, dx.$$  

It is natural to impose $f(x, 0) = 0$ on $f(x, y)$ in order to resolve the singularities at $s|x|^\alpha + u = 0$. Note that

$$\frac{f(-x + c, (s|x|^\alpha + u)^{1/\beta})}{(s|-x|^\alpha + u)^{1/\beta}} = f(x + c, (s|x|^\alpha + u)^{1/\beta})$$

Moreover, if $f(x, y)$ satisfies the symmetry

$$f(x + c, y) = f(-x + c, y) = f(x + c, -y),$$

$R_{\alpha, \alpha} f(s, u)$ can be regarded as the integration of $f$ on a curve

$$\Gamma_R(\alpha, \beta; s, u) = \{(x, y) \in \mathbb{R}^2 : |y|^\beta = s|x - c|^\alpha + u\},$$

and slightly different from the standard contour integral on $\Gamma_R(\alpha, \beta; s, u)$. In particular $\Gamma_R(2, 2; s, u)$ with $s > 0$ is a hyperbola, and $R_{2,2} f(s, u)$ is called the hyperbolic Radon transform of $f$ in seismology.

Here we recall the mathematical background of our transforms. In the early 1980s, Cormack introduced the Radon transform of a family of plane curves and studied the basic properties in his pioneering works [2] and [3]. More than a decade later, Denecker, van Overloop and Sommen in [4] studied the parabolic Radon transform without fixed axis, in particular, the support theorem, higher dimensional generalization and etc. More than a decade later, Jollivet, Nguyen and Truong in [7] studied some properties of the parabolic Radon transform with fixed axis, which is the exact contour integration. Recently, Moon established the inversion of the parabolic Radon transform $P_2$ and the inversion of the hyperbolic Radon transform $R_{2,2}$ respectively in his interesting paper [9]. He introduced some change of variables in $(x, y) \in \mathbb{R}^2$ so that the Radon transform of a family of plane curves became the X-ray transform, that is, the Radon transform of a family of lines. More recently, replacing $x^2$ by some function $\varphi(x)$ in the parabolic Radon transform, Ustaoglu developed Moon’s idea to study the inversion of more general Radon transforms on the plane in [10]. The author believes that this is an very interesting challenge. Unfortunately, however, his inversion formulas are not complete and contain some values of $f$ due to some difficulties coming from the generalization. More precisely, the reduction to the X-ray transforms does not work well due to the lack symmetry of plane curves. Here we say that the inversion formula is complete if $f$ can be reconstructed only from the transformation of $f$ corresponding to the data of observation in case of CT-scan.

Now we also recall the scientific background of the parabolic Radon transform $P_2$ and the hyperbolic transform $R_{2,2}$. These are used for processing of the data of observation of seismic waves. In our notation $x$ means the distance of the source and $y$ means the travel time of the wave. In an early stage, the hyperbolic Radon transform seemed to have been used in seismology. In 1986, Hampton proposed to replace hyperbolas by parabolas and introduced the parabolic Radon transform in [5]. See also expository papers due to seismologists [8] for the parabolic Radon transform and [11] for the hyperbolic Radon transform respectively.
Theorem 2. Let \( x \rightarrow m \) prepare some lemmas in Section 2, and prove Theorem 2 in Section 3.

The X-ray transform of \( f \) is defined by

\[
\partial_x^k f(x, y) = 0 \quad \text{for} \quad k = 0, 1, \ldots, m.
\]

Definition 1. Fix \( c \in \mathbb{R} \). Let \( m \) be a nonnegative integer. We define functions spaces \( \mathcal{S}_{c,m}(\mathbb{R}^2) \), \( \mathcal{S}_{c,m}^p(\mathbb{R}^2) \) and \( \mathcal{S}_{c,m}^r(\mathbb{R}^2) \) by

\[
\mathcal{S}_{c,m}(\mathbb{R}^2) = \left\{ f \in \mathcal{S}(\mathbb{R}^2) : \frac{\partial^k f}{\partial x^k}(x, y) = 0 \quad \text{for} \quad k = 0, 1, \ldots, m \right\},
\]

\[
\mathcal{S}_{c,m}^p(\mathbb{R}^2) = \left\{ f \in \mathcal{S}_{c,m}(\mathbb{R}^2) : f(x + c, y) = f(-x + c, y) \right\},
\]

\[
\mathcal{S}_{c,m}^r(\mathbb{R}^2) = \left\{ f \in \mathcal{S}_{c,m}(\mathbb{R}^2) : f(x + c, y) = f(x + c, y - c), \ f(x + c, 0) = 0 \right\}.
\]

Recall \( \alpha > 1 \) and \( \beta > 1 \). Throughout the present paper, we here assume that the vanishing order \( m \) at \( x = c \) satisfies \( m \geq \alpha - 2 \). This condition guarantees the existence of finite boundary value at \( x \rightarrow c \pm 0 \) after the reduction to X-ray transform. Our main results are the following.

Theorem 2. Let \( c \in \mathbb{R} \) and let \( \alpha, \beta > 1 \). Suppose that \( m \) is a nonnegative integer satisfying \( m \geq \alpha - 1 \).

(i) For any \( f \in \mathcal{S}_{c,m}(\mathbb{R}^2) \),

\[
f(x, y) = \alpha|x - c|^\alpha \frac{\partial \mathcal{Q}_\alpha f(s, u)}{2\pi^2} \int_{-\infty}^{\infty} \left( \mathrm{vp} \int_{-\infty}^{\infty} \frac{\partial \mathcal{Q}_\alpha f(s, u)}{y - s(x - c)} dx \right) ds.
\]

(ii) For any \( f \in \mathcal{S}_{c,m}^p(\mathbb{R}^2) \),

\[
f(x, y) = \alpha|x - c|^\alpha \frac{\partial \mathcal{P}_\alpha f(s, u)}{3\pi} \int_{-\infty}^{\infty} \left( \mathrm{vp} \int_{-\infty}^{\infty} \frac{\partial \mathcal{P}_\alpha f(s, u)}{y - s(x - c)} dx \right) ds.
\]

(iii) For any \( f \in \mathcal{S}_{c,m}^r(\mathbb{R}^2) \),

\[
f(x, y) = \alpha|x - c|^\alpha |y| \frac{\partial \mathcal{R}_{\alpha,\beta} f(s, u)}{4\pi^2} \int_{-\infty}^{\infty} \left( \mathrm{vp} \int_{-\infty}^{\infty} \frac{\partial \mathcal{R}_{\alpha,\beta} f(s, u)}{|y|\beta - s(x - c)} dx \right) ds.
\]

Here \( \partial_u = \partial / \partial u \) and \( \mathrm{vp} \) denotes the Cauchy principal value for improper integrals.

The proof of Theorem 2 depends on Moon’s idea of the reduction to X-ray transform in [9]. We prepare some lemmas in Section 2 and prove Theorem 2 in Section 3.

2. Preliminaries

We begin with the X-ray transform on \( \mathbb{R}^2 \).

Definition 3. The X-ray transform of \( f \in \mathcal{S}(\mathbb{R}^2) \) is defined by

\[
\mathcal{X} f(\theta, t) = \int_{-\infty}^{\infty} f(t \cos \theta - \sigma \sin \theta, t \sin \theta + \sigma \cos \theta) d\sigma
\]

for \( (\theta, t) \in \mathbb{R}^2 \). Note that \( \mathcal{X} f(\theta \pm \pi, -t) = \mathcal{X} f(\theta, t) \).

The inversion formula of the X-ray transform is well-known as follows.

Theorem 4. For \( f \in \mathcal{S}(\mathbb{R}^2) \),

\[
f(x, y) = \frac{1}{2\pi^2} \int_0^{\pi} \left( \mathrm{vp} \int_{-\infty}^{\infty} \frac{\partial \mathcal{X} f(\theta, t)}{x \cos \theta + y \sin \theta - t} dt \right) d\theta.
\]
It is very important in the present paper that the X-ray transform and its inversion formula are valid also for a smooth function \( f(x, y) \) satisfying \( f(x, y) = O((1 + |x| + |y|)^{-d}) \) with some \( d > 1 \), compactly supported distributions, rapidly decaying Lebesgue measurable functions and etc. See, e.g., Chapter 1 of Helgason’s textbook [4] for the detail.

Secondly, we give a lemma to make full use of the vanishing conditions.

**Lemma 5.**

(i) For \( f \in \mathcal{S}_{c,m}(\mathbb{R}^2) \),

\[
    f(x + c, y) = \frac{x^{m+1}}{m!} \int_0^1 (1-t)^m \frac{\partial^{m+1} f}{\partial x^{m+1}}(tx + c, y) dt.
\]

(ii) For \( f \in \mathcal{S}^R_{c,m}(\mathbb{R}^2) \),

\[
    \frac{\partial^l f}{\partial x^l}(x + c, y) = y \int_0^1 \frac{\partial^{l+1} f}{\partial x^{l+1}}(x + c, \tau y) d\tau, \quad l = 0, 1, 2, \ldots,
\]

\[
    f(x + c, y) = \frac{x^{m+1}}{m!} \int_0^1 \int_0^1 (1-t)^m \frac{\partial^{m+2} f}{\partial x^{m+1} \partial y}(tx + c, \tau y) dtd\tau.
\]

**Proof.** Let \( f \in \mathcal{S}_{c,m}(\mathbb{R}^2) \). Then \( \partial^k f/\partial x^k(c, y) = 0 \) for \( k = 0, 1, \ldots, m \). Taylor’s formula in \( x \) gives

\[
    f(x + c, y) = \sum_{k=0}^m \frac{m!}{k!} \frac{\partial^k f}{\partial x^k}(c, y) + \frac{x^{m+1}}{m!} \int_0^1 (1-t)^m \frac{\partial^{m+1} f}{\partial x^{m+1}}(tx + c, y) dt
\]

which is (4).

Let \( f \in \mathcal{S}^R_{c,m}(\mathbb{R}^2) \). Then (4) holds since \( \mathcal{S}^R_{c,m}(\mathbb{R}^2) \subset \mathcal{S}_{c,m}(\mathbb{R}^2) \). Recall \( f(x + c, 0) = 0 \). Then \( \partial^l f/\partial x^l(x + c, 0) = 0 \) for \( l = 0, 1, 2, \ldots \). Applying the mean value theorem to \( \partial^l f/\partial x^l(x + c, y) \) in \( y \), we have

\[
    \frac{\partial^l f}{\partial x^l}(x + c, y) = \frac{\partial^l f}{\partial x^l}(x + c, 0) - \frac{\partial^l f}{\partial x^l}(x + c, 0) = y \int_0^1 \frac{\partial^{l+1} f}{\partial x^{l+1}}(x + c, \tau y) d\tau,
\]

which is (5). Applying this with \( l = m + 1 \) to (4), we obtain (6). This completes the proof. \( \square \)

We make use of the change of variable \( x = \xi/|\xi|^{1+1/\alpha} = sgn(\xi)|\xi|^{1/\alpha} \) for \( \xi \neq 0 \) in order to reduce our transforms to the X-ray transform. Since \( |x| = |\xi|^{1/\alpha} \), we have \( s|\xi|^{\alpha-1} + u = s\xi + u \) for \( \xi \neq 0 \) and in particular \( s|x|^{\alpha} + u = s\xi + u \) for \( \xi > 0 \). For this purpose, we introduce new functions of \((\xi, \eta) \in \mathbb{R}^2\) defined by \( f(x, y) \) as follows:

\[
    F_\alpha(\xi, \eta) = \frac{f(\xi|\xi|^{-1+1/\alpha+\varepsilon}, \eta)}{\alpha|\xi|^{(\alpha-1)/\alpha}} \quad (\xi \neq 0),
\]

\[
    F_\alpha^P(\xi, \eta) = \begin{cases} 
    2F_\alpha(\xi, \eta) & (\xi > 0), \\
    0 & (\text{otherwise}), 
    \end{cases} \quad \frac{2f(\xi^{1/\alpha} + c, \eta)}{\alpha\xi^{(\alpha-1)/\alpha}} \quad (\xi > 0),
\]

\[
    F_\alpha^{R, \beta}(\xi, \eta) = \begin{cases} 
    2f(\xi^{1/\alpha} + c, \eta^{1/\beta}) & (\xi > 0, \eta > 0), \\
    0 & (\text{otherwise}). 
    \end{cases} \quad \frac{\alpha\xi^{(\alpha-1)/\alpha}\eta^{1/\beta}}{\alpha\xi^{(\alpha-1)/\alpha}\eta^{1/\beta}}
\]

We consider the properties of these new functions. To avoid the confusion, we split our statements into three lemmas.

Firstly, the properties \( F_\alpha \) for \( f \in \mathcal{S}_{c,m}(\mathbb{R}^2) \) are the following.
Lemma 6. For $f \in \mathcal{F}_{c,m}(\mathbb{R}^2)$, $F_{\alpha}(\xi, \eta)$ satisfies
\[
f(x, y) = \alpha |x - c|^{\alpha - 1} F_{\alpha}((x - c)|x - c|^{\alpha - 1}, y),
\]
and for any $N > 0$, there exists a constant $C_N > 0$ such that
\[
|F_{\alpha}(\xi, \eta)| \leq C_N (1 + |\xi| + |\eta|)^{-N}.
\]
Moreover, when $\xi \to \pm 0$,
\[
F_{\alpha}(\xi, s\xi + u) \to \begin{cases} 0 & (m > \alpha - 2), \\ \frac{(\pm 1)^{m+1}}{\alpha(m+1)!} \frac{\partial^{m+1} f}{\partial x^{m+1}}(c, u) & (m = \alpha - 2). \end{cases}
\]

Proof. Suppose that $f \in \mathcal{F}_{c,m}(\mathbb{R}^2)$. We can check (7) since $x|x|^{\alpha - 1} = \xi$ and $|x|^\alpha = |\xi|$. The decay estimate (8) is obvious for $|\xi| > 1$ since $f \in \mathcal{F}(\mathbb{R}^2)$. By using (4), we have for $\xi \neq 0$
\[
F_{\alpha}(\xi, \eta) = \frac{f(\xi|\xi|^{\alpha - 1} + c, \eta)}{\alpha|\xi|^{(\alpha-1)/\alpha}} = \frac{(\text{sgn}(\xi))^{m+1} |\xi|^{(m+2-\alpha)/\alpha}}{\alpha m!} \times \int_0^1 (1-t)^m \frac{\partial^{m+1} f}{\partial x^{m+1}}(t\xi|\xi|^{-1+\alpha} + c, \eta) dt.
\]
By using this, we have $F_{\alpha}(\xi, \eta) = O((1 + |\eta|^{-N}))$ uniformly in $0 < |\xi| \leq 1$ for any $N > 0$. Thus we proved (8). Using the above again, we have
\[
F_{\alpha}(\xi, s\xi + u) = \frac{(\text{sgn}(\xi))^{m+1} |\xi|^{(m+2-\alpha)/\alpha}}{\alpha m!} \times \int_0^1 (1-t)^m \frac{\partial^{m+1} f}{\partial x^{m+1}}(t\xi|\xi|^{-1+\alpha} + c, s\xi + u) dt.
\]
This implies (9) immediately. \(\square\)

Secondly, the properties $F_{\alpha}^P$ for $f \in \mathcal{F}_{c,m}(\mathbb{R}^2)$ are the following.

Lemma 7. For $f \in \mathcal{F}_{c,m}^P(\mathbb{R}^2)$, $F_{\alpha}(\xi, \eta)$ satisfies
\[
f(x, y) = \frac{\alpha}{2} |x - c|^{\alpha - 1} F_{\alpha}^P(|x - c|^\alpha, y),
\]
and for any $N > 0$, there exists a constant $C_N > 0$ such that
\[
|F_{\alpha}(\xi, \eta)| \leq C_N (1 + |\xi| + |\eta|)^{-N}.
\]
Moreover, when $\xi \to -0$, $F_{\alpha}^P(\xi, s\xi + u) \to 0$, and when $\xi \to +0$,
\[
F_{\alpha}^P(\xi, s\xi + u) \to \begin{cases} 0 & (m > \alpha - 2), \\ \frac{2}{\alpha(m+1)!} \frac{\partial^{m+1} f}{\partial x^{m+1}}(c, u) & (m = \alpha - 2). \end{cases}
\]

Proof. We can prove Lemma 7 in the same way as the proof of Lemma 6. Here we omit the detail. \(\square\)

Finally, the properties $F_{\alpha,\beta}^R$ for $f \in \mathcal{F}_{c,m}^R(\mathbb{R}^2)$ are the following.

Lemma 8. For $f \in \mathcal{F}_{c,m}^R(\mathbb{R}^2)$, $F_{\alpha}(\xi, \eta)$ satisfies
\[
f(x, y) = \frac{\alpha}{2} |x - c|^{\alpha - 1} |y| F_{\alpha,\beta}^R(|x - c|^\alpha, |y|^\beta),
\]
for any $N > 0$.
and for any $N > 0$, there exists a constant $C_N > 0$ such that
\[
|F^{R}_{\alpha}(\xi, \eta)| \leq C_N (1 + |\xi| + |\eta|)^{-N}.
\] (14)

Moreover, when $\xi \to -0$, $F^{R}_{\alpha}(\xi, s\xi + u) \to 0$, and when $\xi \to +0$,
\[
F^{R}_{\alpha}(\xi, s\xi + u) \to \begin{cases} 
0 & (m > \alpha - 2 \text{ or } u \leq 0), \\
\frac{1}{\alpha(m + 1)!} \int_{0}^{1} \frac{\partial^{m+2} f}{\partial x^{m+1} \partial y} (c, \tau u^{1/\beta}) d\tau & (m = \alpha - 2 \text{ and } u > 0).
\end{cases}
\] (15)

**Proof.** Suppose $f \in \mathcal{S}_{c,m}(\mathbb{R}^2)$. By using the symmetries, we have
\[
f(x, y) = f((x - c) + c, y) = f(|x - c| + c, |y|) = \frac{\alpha}{2} |x - c|^{\alpha-1} |y| F^{R}_{\alpha,\beta}(|x - c|^\alpha, |y|^{\beta}),
\]
which proves (13). To complete the proof of Lemma 8, it suffices to show (14) for $\xi > 0$ and $\eta > 0$, and (15) for $\xi \to +0$.

It is easy to see (14) for $\xi > 1$ and $\eta > 1$. So we show (14) for $0 < \xi \leq 1$ or $0 < \eta \leq 1$. When $\xi > 1$ and $0 < \eta \leq 1$, (5) with $l = 0$ implies
\[
F^{R}_{\alpha,\beta}(\xi, \eta) = \frac{1}{\alpha \xi^{(\alpha-1)/\alpha}} \int_{0}^{1} \frac{\partial f}{\partial y} (\xi^{1/\alpha} + c, \tau \eta^{1/\beta}) d\tau.
\]
By using this, we obtain $F^{R}_{\alpha,\beta}(\xi, \eta) = O((1 + \xi)^{-N})$ for any $N > 0$ uniformly in $0 < \eta \leq 1$.

When $0 < \eta \leq 1$ and $\eta > 1$, (3) implies
\[
F^{R}_{\alpha,\beta}(\xi, \eta) = \frac{\xi^{(m+2-\alpha)/\alpha}}{\alpha m!} \int_{0}^{1} \int_{0}^{1} (1 - t)^m \frac{\partial^{m+2} f}{\partial x^{m+1} \partial y} (t \xi^{1/\alpha} + c, \tau \eta^{1/\beta}) dt d\tau.
\]
By using this, we obtain $F^{R}_{\alpha,\beta}(\xi, \eta) = O((1 + \eta)^{-N})$ for any $N > 0$ uniformly in $0 < \xi \leq 1$.

When $0 < \xi \leq 1$ and $0 < \eta \leq 1$, (6) implies
\[
F^{R}_{\alpha,\beta}(\xi, \eta) = \frac{\xi^{(m+2-\alpha)/\alpha}}{\alpha m!} \int_{0}^{1} \int_{0}^{1} (1 - t)^m \frac{\partial^{m+2} f}{\partial x^{m+1} \partial y} (t \xi^{1/\alpha} + c, \tau (s\xi + u)^{1/\beta}) dt d\tau = O(1).
\]

Combining the above estimates, we obtain (14).

If $u \leq 0$, then $s\xi + u \to 0$ and $F^{R}_{\alpha,\beta}(\xi, s\xi + u) \to 0$ as $\xi \to +0$. Note that if $u > 0$, then $s\xi + u > 0$ near $\xi = 0$. By using (6) again, we have for $0 < \xi \ll 1$
\[
F^{R}_{\alpha,\beta}(\xi, s\xi + u) = \frac{\xi^{(m+2-\alpha)/\alpha}}{\alpha m!} \times \int_{0}^{1} \int_{0}^{1} (1 - t)^m \frac{\partial^{m+2} f}{\partial x^{m+1} \partial y} (t \xi^{1/\alpha} + c, \tau (s\xi + u)^{1/\beta}) dt d\tau.
\]
By using this, we obtain (15) for $u > 0$. This completes the proof. \hfill \square

3. PROOF OF MAIN THEOREMS

We begin with computing $Q_{\alpha} f(s,u)$, $P_{\alpha} f(s,u)$ and $R_{\alpha,\beta} f(s,u)$.

**Lemma 9.**

(i) For $f \in \mathcal{S}_{c,m}(\mathbb{R}^2)$,
\[
Q_{\alpha} f(s,u) = \frac{1}{\sqrt{1 + s^2}} \cdot X F_{\alpha} \left( - \text{Arccot} s, \frac{u}{\sqrt{1 + s^2}} \right),
\] (16)
\[
\partial_u Q_{\alpha} f(s,u) = \frac{1}{1 + s^2} \cdot (\partial_t X F_{\alpha}) \left( - \text{Arccot} s, \frac{u}{\sqrt{1 + s^2}} \right).
\] (17)
(i) For \( f \in \mathcal{S}_{c,m}^P(\mathbb{R}^2) \),
\[
\mathcal{P}_\alpha f(s, u) = \frac{1}{\sqrt{1 + s^2}} \cdot \mathcal{X} F^P_{\alpha} \left( -\text{Arccot} s, \frac{u}{\sqrt{1 + s^2}} \right), \tag{18}
\]
\[
\partial_u \mathcal{P}_\alpha f(s, u) = \frac{1}{1 + s^2} \cdot (\partial_u F^P_{\alpha}) \left( -\text{Arccot} s, \frac{u}{\sqrt{1 + s^2}} \right). \tag{19}
\]

(ii) For \( f \in \mathcal{S}_{c,m}^R(\mathbb{R}^2) \),
\[
\mathcal{R}_{\alpha,\beta} f(s, u) = \frac{1}{\sqrt{1 + s^2}} \cdot \mathcal{X} F^R_{\alpha,\beta} \left( -\text{Arccot} s, \frac{u}{\sqrt{1 + s^2}} \right), \tag{20}
\]
\[
\partial_u \mathcal{R}_{\alpha,\beta} f(s, u) = \frac{1}{1 + s^2} \cdot (\partial_u F^R_{\alpha,\beta}) \left( -\text{Arccot} s, \frac{u}{\sqrt{1 + s^2}} \right). \tag{21}
\]

Proof. Firstly, we show (i). Suppose that \( f \in \mathcal{S}_{c,m}^P(\mathbb{R}^2) \). Recall the definition of \( \mathcal{Q}_\alpha f(s, u) \):
\[
\mathcal{Q}_\alpha f(s, u) = \int_{-\infty}^\infty f(x + c, x|x|^{\alpha-1} + u) \, dx.
\]
We want to make use of the change of variable \( x = \xi|\xi|^{-1+1/\alpha} \) for \( \xi \neq 0 \). For this reason, we regard the above integration as the sum of improper integrals on intervals \((-\infty, 0)\) and \((0, \infty)\). Since \( dx/d\xi = 1/\alpha|\xi|^{(\alpha-1)/\alpha} \) and \( x|x|^{\alpha-1} = \xi \) for \( \xi \neq 0 \), we have
\[
\mathcal{Q}_\alpha f(s, u) = \int_{-\infty}^\infty \frac{f(\xi|\xi|^{-1+1/\alpha}, \xi s + u)}{\alpha|\xi|^{(\alpha-1)/\alpha}} \, d\xi = \int_{-\infty}^\infty F_\alpha(\xi, \xi s + u) \, d\xi. \tag{22}
\]
(8) implies \( F_\alpha(\xi, \xi s + u) = O((1 + |\xi|)^{-2}) \), and (9) shows that \( F_\alpha(\xi, \xi s + u) \) has finite limits at \( \xi = \pm 0 \). Then the computation in the above integration can be justified.

Now we apply another change of variable \( \mathbb{R} \ni \xi \mapsto \sigma \in \mathbb{R} \) defined by
\[
\xi = -\frac{s}{\sqrt{1 + s^2}} - \frac{su}{1 + s^2}.
\]
We can see
\[
-\frac{s}{\sqrt{1 + s^2}} = \cos \theta, \quad \frac{1}{\sqrt{1 + s^2}} = \sin \theta
\]
for \( s \in \mathbb{R} \). When \( s \) moves from \(-\infty\) to \( \infty \), \( \theta \) moves from \( 0 \) to \( \pi \). Hence \( s = -\cot \theta \), that is, \( \theta = -\text{Arccot} s \). In this case,
\[
(\xi, \xi s + u) = \left( \frac{u}{1 + s^2} \cos \theta - \sigma \sin \theta, \frac{u}{1 + s^2} \sin \theta + \sigma \cos \theta \right),
\]
and \( d\xi/\sigma = -1/\sqrt{1 + s^2} \). Hence (22) becomes
\[
\mathcal{Q}_\alpha f(s, u) = \frac{1}{\sqrt{1 + s^2}} \int_{-\infty}^\infty F_\alpha \left( \frac{u}{1 + s^2} \cos \theta - \sigma \sin \theta, \frac{u}{1 + s^2} \sin \theta + \sigma \cos \theta \right) \, d\sigma
\]
\[
= \frac{1}{\sqrt{1 + s^2}} \cdot \mathcal{X} F_{\alpha} \left( -\text{Arccot} s, \frac{u}{\sqrt{1 + s^2}} \right),
\]
which is (16). Differentiating this with respect to \( u \), we can get (17).

Secondly, we prove (ii). Differentiating (13) with respect to \( u \), one can get (19). Here we show only (18). Suppose that \( f \in \mathcal{S}_{c,m}^P(\mathbb{R}^2) \). By using the symmetry \( f(x + c, y) = f(-x + c, y) \) and the same change of variable \( x \mapsto \xi \) in the proof of (i), we deduce that
\[
\mathcal{P}_\alpha f(s, u) = \int_{-\infty}^\infty f(x + c, s|x|^{\alpha} + u) \, dx
\]
Using the same change of variable \( \xi \mapsto \sigma \) as in the proof of (16), we obtain (18).

Finally, we prove (iii). Differentiating (20) with respect to \( u \), one can get (21). Here we show only (20). Suppose that \( f \in \mathcal{K}_{c,m}(\mathbb{R}^2) \). By using the symmetry \( f(x + c, y) = f(-x + c, y) \) and the same change of variable \( x \mapsto \xi \) in the proof of (i), we deduce that

\[
\mathcal{R}_{\alpha, \beta} f(s, u) = \int_{x \in \mathbb{R}} f(x + c, \{s|x|^\alpha + u\}^{1/\beta}) dx.
\]

Now we shall prove Theorem 2.

**Proof of Theorem 2** Firstly, we prove (11). Suppose that \( f \in \mathcal{K}_{c,m}(\mathbb{R}^2) \). By using (7) in Lemma 6 and Theorem 4, we deduce that

\[
f(x, y) = \alpha|x - c|^{\alpha - 1} F_\alpha((x - c)|x - c|^{\alpha - 1}, y)
= \frac{\alpha|x - c|^{\alpha - 1}}{2\pi^2} \int_0^\pi \left( \text{vp} \int_{-\infty}^\infty \frac{(\partial_\beta X F_\alpha)(\theta, t)}{(x - c)|x - c|^{\alpha - 1} \cos \theta + y \sin \theta - t} dt \right) d\theta.
\]

Here we use the change of variables \((\theta, t) = (-\text{Arccot } s, u / \sqrt{1 + s^2})\), whose Jacobian is

\[
\frac{\partial(\theta, t)}{\partial(s, u)} = \det \begin{bmatrix} 1/(1 + s^2) & 0 \\ s & 1/\sqrt{1 + s^2} \end{bmatrix} = \frac{1}{(1 + s^2)^{3/2}}.
\]

Using this and (17) in Lemma 6, in order, we deduce that

\[
f(x, y) = \frac{\alpha|x - c|^{\alpha - 1}}{2\pi^2} \int_{-\infty}^\infty \left( \text{vp} \int_{-\infty}^\infty \frac{1}{-(x - c)|x - c|^{\alpha - 1} \sqrt{1 + s^2} + y \frac{1}{\sqrt{1 + s^2}} - u}{(1 + s^2)^{3/2}} \cdot (\partial_\beta X F_\alpha) \left(-\text{Arccot } s, \frac{u}{\sqrt{1 + s^2}}\right) du \right) ds
\]

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f(x, y) = \frac{\alpha|x - c|^{\alpha - 1}}{2\pi^2} \int_{-\infty}^\infty \left( \text{vp} \int_{-\infty}^\infty \frac{1}{-(x - c)|x - c|^{\alpha - 1} \sqrt{1 + s^2} + y \frac{1}{\sqrt{1 + s^2}} - u}{(1 + s^2)^{3/2}} \cdot (\partial_\beta X F_\alpha) \left(-\text{Arccot } s, \frac{u}{\sqrt{1 + s^2}}\right) du \right) ds
\]
By using the change of variables

\[
\frac{\alpha |x - c|^{\alpha - 1}}{2\pi^2} \int_{-\infty}^{\infty} \left( \text{vp} \int_{-\infty}^{\infty} \frac{1}{y - s(x - c)|x - c|^{\alpha - 1} - u} \right) ds \times \frac{1}{1 + s^2} : (\partial_t \mathcal{X} F_{\alpha})(-\text{Arccot} s, \frac{u}{\sqrt{1 + s^2}}) du \right) ds
\]

\[
= \frac{\alpha |x - c|^{\alpha - 1}}{2\pi^2} \int_{-\infty}^{\infty} \left( \text{vp} \int_{-\infty}^{\infty} \frac{\partial_u Q_{\alpha} f(s, u)}{y - s(x - c)|x - c|^{\alpha - 1} - u} \right) du \right) ds,
\]

which is (1).

Secondly, we prove (2). Suppose that \( f \in \mathcal{S}^P_{c,m}(\mathbb{R}^2) \). By using (10) in Lemma 7 and Theorem 4 we deduce that

\[
f(x, y) = \frac{\alpha}{2} |x - c|^{\alpha - 1} F^P_{\alpha, \beta}(|x - c|^{\alpha}, y)
\]

\[
= \frac{\alpha |x - c|^{\alpha - 1}}{4\pi^2} \int_{0}^{\pi} \left( \text{vp} \int_{-\infty}^{\infty} \frac{(\partial_t \mathcal{X} F^P_{\alpha})(\theta, t)}{|x - c|^{\alpha} \cos \theta + y \sin \theta - t} dt \right) d\theta.
\]

By using the change of variables \((\theta, t) = (-\text{Arccot} s, u/\sqrt{1 + s^2})\) and \((19)\) in Lemma 6 in order, we can obtain (2) in the same way as \(1\). Here we omit the detail.

Finally, we prove (3). Suppose that \( f \in \mathcal{S}^R_{c,m}(\mathbb{R}^2) \). By using (13) in Lemma 8 and Theorem 4 we deduce that

\[
f(x, y) = \frac{\alpha}{2} |x - c|^{\alpha - 1} F^R_{\alpha, \beta}(|x - c|^{\alpha}, y)
\]

\[
= \frac{\alpha |x - c|^{\alpha - 1}}{4\pi^2} \int_{0}^{\pi} \left( \text{vp} \int_{-\infty}^{\infty} \frac{(\partial_t \mathcal{X} F^R_{\alpha, \beta})(\theta, t)}{|x - c|^{\alpha} \cos \theta + y \sin \theta - t} dt \right) d\theta.
\]

By using the change of variables \((\theta, t) = (-\text{Arccot} s, u/\sqrt{1 + s^2})\) and \((21)\) in Lemma 6 in order, we can obtain (3) in the same way as \(1\). Here we omit the detail. This completes the proof. 

\[\square\]

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