FEIGIN-ODESSKII BRACKETS, SYZYGIES, AND CREMONA TRANSFORMATIONS

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Abstract. We identify Feigin-Odesskii brackets \( q_{n,1}(C) \), associated with a normal elliptic curve of degree \( n \), \( C \subset \mathbb{P}^{n-1} \), with the skew-symmetric \( n \times n \) matrix of quadratic forms introduced by Fisher in [7] in connection with some minimal free resolutions related to the secant varieties of \( C \). On the other hand, we show that for odd \( n \), the generators of the ideal of the secant variety of \( C \) of codimension 3 give a Cremona transformation of \( \mathbb{P}^{n-1} \), generalizing the quadro-cubic Cremona transformation of \( \mathbb{P}^4 \). We identify this transformation with the one considered in [13] and find explicit formulas for the inverse transformation.

1. Introduction

Let \( C \subset \mathbb{P}^{n-1} = \mathbb{P}(V) \) be an elliptic normal curve of degree \( n \geq 3 \) over a field \( k \) of characteristic 0. With \( C \) one can associate canonically (up to rescaling) a Poisson bracket \( q_{n,1}(C) \) on the ambient projective space \( \mathbb{P}(V) \). Namely, if \( L = \mathcal{O}(1)|_C \), a line bundle of degree \( n \) on \( C \), then we can identify \( \mathbb{P}(V) \) with \( \mathbb{P}H^0(C, L)^* \cong \mathbb{P}\text{Ext}^1(L, \mathcal{O}) \) (where we used Serre duality on \( C \)). Now there is a natural Poisson bracket on the projective space \( \mathbb{P}\text{Ext}^1(L, \mathcal{O}) \) described in [5] and [14].

The Poisson brackets \( q_{n,1}(C) \) correspond to quadratic brackets on \( V \) arising as semi-classical limit of Feigin-Odesskii elliptic algebras (see [4]). The Poisson geometry of \( q_{n,1}(C) \) has been well studied (see e.g., [5], [13], [14], [10], [12], [15], [8], [9]). In particular, it is known that for odd \( n \) the stratification of \( \mathbb{P}^{n-1} \) by the rank of this Poisson bracket is related to the secant varieties of \( C \).

The first goal of the present work is to establish a direct relation of \( q_{n,1}(C) \) with a certain skew-symmetric matrix of quadratic forms \( \Omega \) related to the secant varieties of \( C \), introduced and studied by Fisher in [7].

Assume first that \( n \) is odd and let \( r = (n-1)/2 \). Then as is well known the secant variety \( \text{Sec}^r C \) is a hypersurface of degree \( n \) in \( \mathbb{P}^{n-1} \). Let \( F(x_1, \ldots, x_n) = 0 \) be its equation (of degree \( n \)). Then Fisher showed in [7] that the module of syzygies between the partial

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derivatives \( (\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}) \) is generated by the relations

\[
\sum_i \frac{\partial F}{\partial x_i} \Omega_{ij} = 0,
\]

for a unique (up to rescaling) skew-symmetric \( n \times n \) matrix of quadratic forms \( \Omega \) (see Sec. 2.3 for details).

In the case when \( n \) is even, we set \( r = (n - 2)/2 \). Then the secant variety \( \text{Sec}^r C \) is a complete intersection of codimension 2 given by \( F_1 = F_2 = 0 \), where \( \text{deg}(F_1) = \text{deg}(F_2) = r + 1 \). In this case the \( n \times n \) skew-symmetric matrix of quadratic forms \( \Omega \) is characterized by the condition that the relations

\[
\sum_i \frac{\partial F_a}{\partial x_i} \Omega_{ij} = 0,
\]

where \( a = 1, 2 \),

generate the module of syzygies between the columns of the \( 2 \times n \) matrix \( (\frac{\partial F_a}{\partial x_j}) \).

Theorem A. (i) The formula \( \{x_i, x_j\} = \Omega_{ij} \) defines a Poisson bracket on \( V \) such that the induced Poisson bracket on \( \mathbb{P}(V) \) is \( q_{n,1}(C) \) (up to rescaling).

(ii) If \( n \) is odd then \( \{x_i, F\} = 0 \) for every \( i \). If \( n \) is even then \( \{x_i, F_1\} = \{x_i, F_2\} = 0 \) for every \( i \).

Note that part (ii) is stated in [11, Sec. 5] (see also [10]).

Assume now that \( n \) is odd and \( n \geq 5 \). Our second result gives an explicit formula for the Poisson birational transformation from \( (\mathbb{P}^{n-1}, q_{n,1}(C)) \) to \( (\mathbb{P}^{n-1}, q_{n,(n-1)/2}(C)) \) constructed in [14]. Recall that for any relatively prime \( (n, k) \), the Feigin-Odesskii bracket \( q_{n,k}(C) \) is a natural Poisson bracket on \( \mathbb{P}^{n-1} = \mathbb{P} \text{Ext}^1(V, \mathcal{O}) \), where \( V \) is a stable bundle of rank \( k \) and degree \( n \) on \( C \) (see [5], [14]).

Let us start with a line bundle \( L = \mathcal{O}(1)|_C \) of degree \( n \), and let \( V_0 \) be the unique stable bundle of rank 2 with \( \det(V_0) = L \) (it exists since \( n \) is odd). In [14] we constructed a natural birational map (a Cremona transformation)

\[
\phi : \mathbb{P}^{n-1} = \mathbb{P} \text{Ext}^1(\mathcal{O}, L) \dashrightarrow \mathbb{P} H^0(C, V_0) = \mathbb{P}^{n-1}
\]

by observing that for a generic extension

\[
0 \to \mathcal{O} \to E \to L \to 0
\]

the bundle \( E \) is stable, so there is an isomorphism \( E \simeq V_0 \), unique up to rescaling. Hence, from the above extension we get a nonzero section of \( V_0 \), well defined up to rescaling. Furthermore, we showed in [14] that \( \phi \) is compatible with Poisson structures, where \( \mathbb{P} H^0(C, V_0) \) is equipped with a natural Poisson bracket \( q \), such that there exists a Poisson isomorphism

\[
(\mathbb{P} H^0(C, V_0), q) \simeq (\mathbb{P} \text{Ext}^1(V, \mathcal{O}), q_{n,r}(C)),
\]

where \( V \) is a stable bundle of degree \( n \) and rank \( r = (n - 1)/2 \).

Now let us look at the variety \( \text{Sec}^{r-1} C \) of codimension 3 in \( \mathbb{P}^{n-1} \). It is known that its homogeneous ideal is generated by \( n \) forms \( (p_1, \ldots, p_n) \) of degree \( r \).
Theorem B. The above birational map \( \phi : \mathbb{P}^{n-1} \to \mathbb{P}^{n-1} \) is given by
\[
(x_1 : \ldots : x_n) \mapsto (p_1(x) : \ldots : p_n(x)).
\]
The inverse is given by
\[
(y_1 : \ldots : y_n) \mapsto (f_1(y) : \ldots : f_n(y)),
\]
where \( f_i \) are certain homogeneous polynomials of degree \( n - 2 \).

For example, when \( n = 5 \), we have \( r = 2 \) and \( (p_1, \ldots, p_5) \) are the quadrics generating the ideal of \( C \) in \( \mathbb{P}^4 \). The corresponding Cremona transformation of \( \mathbb{P}^4 \) is well known classically as quadro-cubic Cremona transformation (see [17]). The polynomials \( (p_1, \ldots, p_n) \) can be computed as (signed) submaximal pfaffians of a skew-symmetric \( n \times n \) matrix of linear forms \( \Phi \), called the Klein matrix in [6]. For a Heisenberg invariant curve \( C \), there is a simple formula for \( \Phi \) (see [6, Prop. 3.7]). We also give a recipe for computing the polynomials \( f_i \) in terms of \( \Phi \).

Corollary C. For \( r \geq 2 \), any hypersurface of degree \( r \) in \( \mathbb{P}^{2r} \) containing \( \text{Sec}^{r-1} C \), where \( C \subset \mathbb{P}^{2r} \) is an elliptic normal curve, is rational.

Indeed, the corresponding birational map \( \phi \) induces a birational map from such a hypersurface to a hyperplane in \( \mathbb{P}^{2r} \).

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2. Preliminaries

2.1. Quadratic Poisson brackets. Let \( V \) be a vector space of dimension \( n \). By a quadratic Poisson bracket on \( V \) we mean a Poisson structure on the algebra \( S(V^*) \) compatible with the grading, i.e., such that the Poisson brackets of linear forms are given by homogeneous quadratic forms. Equivalently, the corresponding bivector on \( V \) has to be homogeneous (i.e., preserved by the natural \( \mathbb{G}_m \)-action). It is well known that every such bracket induces a Poisson bracket on the projective space \( \mathbb{P}(V) \) and that every Poisson bracket on \( \mathbb{P}(V) \) comes from a quadratic Poisson bracket on \( V \), not necessarily unique (see [1], [13]).

Different liftings of a bivector on \( \mathbb{P}(V) \) to a homogeneous bivector on \( V \) differ by bivectors on \( V \) of the form \( \langle x, y \rangle = A(x)y - A(y)x \), where \( x, y \in V^* \), for some linear operator \( A : V^* \to V^* \). This easily implies that different liftings of a Poisson bracket \( \Pi \) on \( \mathbb{P}(V) \) to quadratic Poisson brackets on \( V \) are numbered by Poisson vector fields for \( \Pi \) on \( \mathbb{P}(V) \), i.e., global vector fields on \( \mathbb{P}(V) \) preserving \( \Pi \).

The following simple criterion is helpful in identifying a lifting of a Poisson bracket on \( \mathbb{P}(V) \) to a quadratic Poisson bracket on \( V \).

Lemma 2.1.1. Let \( \{ \cdot, \cdot \} \) be a homogeneous bivector on \( V \), inducing a Poisson bracket on \( \mathbb{P}(V) \). Assume that there exists a nonzero homogeneous polynomial \( F \in S^d V^* \) of positive
degree \(d\) such that \(\{x, F\} = 0\) for every \(x \in V^*\). Then \(\{\cdot, \cdot\}\) is a Poisson bracket, i.e., it satisfies the Jacobi identity.

**Proof.** Set \(J(x, y, z) := \{(x, y), z\} + \{(y, z), x\} + \{(z, x), y\}\). First, let us check that \(J(x, y, z) = 0\) for all \(x, y, z \in S^d V^*\). Indeed, \(x/F, y/F\) and \(z/F\) are local functions on \(\mathbb{P}(V)\), so we know that

\[
J\left(\frac{x}{F}, \frac{y}{F}, \frac{z}{F}\right) = 0.
\]

Furthermore, since \(\{x, F\} = \{y, F\} = \{z, F\} = 0\), we get \(\frac{x}{F}, \frac{y}{F}, \frac{z}{F}\), etc., hence,

\[
J\left(\frac{x}{F}, \frac{y}{F}, \frac{z}{F}\right) = \frac{J(x, y, z)}{F^3}.
\]

Thus, we get that \(J(x, y, z) = 0\) for all \(x, y, z \in S^d V^*\).

Hence, if \(x, y, z\) are in \(V^*\) then we have \(J(x^d, x^{d-1}y, x^{d-1}z) = 0\). Now the vanishing of \(J(x, y, z)\) follows from the formal identity

\[
J(x^d, x^{d-1}y, x^{d-1}z) = dx^{3d-3}J(x, y, z).
\]

\[\square\]

2.2. **Formula for Feigin-Odesskii bracket.** Let us recall the formula for the Poisson bivector \(\Pi = q_{n,1}(C)\) on \(\mathbb{P}\text{Ext}^1(L, \mathcal{O}) \simeq \mathbb{P}H^0(C, L)^*\) in terms of Szegö kernel, established in [9] (equivalent formulas are also stated in [10] and in [12]).

Let \(p \in C\) be a point, and let us fix a trivialization of \(\omega_C\). The Szegö kernel is the unique section \(S \in H^0(C \times C, \mathcal{O}(p) \boxtimes \mathcal{O}(p)(\Delta))\) such that \(\text{Res}_\Delta(S) = 1\) and \(S(y, x) = -S(x, y)\). If \(C\) is identified with a plane cubic \(y^2 = P(x)\) and the trivializing global differential is \(dx/2y\), then one has

\[
S = \frac{y_1 + y_2}{x_2 - x_1}
\]

(see [9] Sec. 5.1.2]).

To give a Poisson bivector \(\Pi\) we need to specify for each \(\phi \in H^0(L)^*\) a skew-symmetric form \(\Pi_\phi\) on the cotangent space

\[
T^*_\phi \mathbb{P}H^0(L)^* \simeq \ker(\phi) \subset H^0(L).
\]

This form is given by the formula (see [9] Lem. 2.1, Prop. 5.8])

\[
\Pi_\phi(s_1 \wedge s_2) = \pm \langle \tilde{\phi} \otimes \tilde{\phi}, S \cdot (s_1 \boxtimes s_2 - s_2 \boxtimes s_1) \rangle,
\]

(2.1)

for \(s_1, s_2 \in \ker(\phi)\). Here we consider \(H^0(L)\) as a subspace in \(H^0(L(p))\) and denote by \(\tilde{\phi}\) any extension of \(\phi\) to a functional on \(H^0(L(p))\). Further, we view \(s_1 \boxtimes s_2 - s_2 \boxtimes s_1\) as a section of \(L \boxtimes L\) on \(C \times C\), vanishing on the diagonal. Hence, the product \(S \cdot (s_1 \boxtimes s_2 - s_2 \boxtimes s_1)\) is a global section of \(L(p) \boxtimes L(p)\).

The fact that the right-hand side of (2.1) does not depend on a choice of \(\tilde{\phi}\) follows from the existence of a linear operator \(D : H^0(L) \rightarrow H^0(L(p))\) such that for any \(s_1, s_2 \in H^0(L)\) one has

\[
S \cdot (s_1 \boxtimes s_2 - s_2 \boxtimes s_1) + D\{s_1, s_2\} \in H^0(L) \otimes H^0(L),
\]

where
\[ D\{s_1, s_2\} := s_1 \otimes D(s_2) + D(s_2) \otimes s_1 - s_2 \otimes D(s_1) - D(s_1) \otimes s_2 \]

(so \( D \) cancels the poles at \( p \)). The existence of such \( D \) was first observed in [10] (see also [12] and [9] Sec. 5.3). For \( s_1, s_2 \in \ker(\phi) \) this allows to rewrite the formula (2.1) as

\[ \Pi_\phi(s_1 \wedge s_2) = \pm \langle \phi \otimes \phi, S \cdot (s_1 \otimes s_2 - s_2 \otimes s_1 + D\{s_1, s_2\}) \rangle. \]

(2.2)

2.3. Secant varieties and syzygies. Let us recall some results of Fisher in [6] and [7] on secant varieties of a normal elliptic curve \( C \subset \mathbb{P}^{n-1} \).

First, assume that \( n \) is odd, and set \( r = (n - 1)/2 \). The variety \( \text{Sec}^r C \) is a hypersurface in \( \mathbb{P}^{n-1} \) given by \( F = 0 \), where \( \deg(F) = n \). The first result, [7, Thm. 1.1(i)] gives a form of the minimal free resolution of the ideal \( (\partial F, \ldots, \partial F) \). Namely, there is a unique skew-symmetric \( n \times n \) matrix \( \Omega \) of quadratic forms, such that the following complex is exact, where \( R = k[x_1, \ldots, x_n] \):

\[ 0 \to R(-2n) \xrightarrow{\nabla^T} R(-n - 1)^n \xrightarrow{\Omega} R(-n + 1)^n \xrightarrow{\nabla} R, \]

(2.3)

where \( \nabla = (\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}) \). Note that the fact that this is a complex corresponds to the identity (1.1).

In the case when \( n \) is even, let us set \( r = (n - 2)/2 \). Then the variety \( \text{Sec}^r C \) is a complete intersection of codimension 2 given by \( F_1 = F_2 = 0 \), where \( \deg(F_i) = r + 1 \) (see [2, Sec. 8], [7, Thm. 9.1]). Now setting

\[ \nabla = \left( \begin{array}{ccc} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_2}{\partial x_1} & \cdots & \frac{\partial F_2}{\partial x_n} \end{array} \right), \]

the statement of [7, Thm. 1.1(ii)] is that there is a unique skew-symmetric \( n \times n \) matrix \( \Omega \) of quadratic forms, such that the following complex is exact:

\[ 0 \to R(-n)^2 \xrightarrow{\nabla^T} R(-r - 1)^n \xrightarrow{\Omega} R(-r)^n \xrightarrow{\nabla} R^2. \]

(2.4)

Again, the fact that this is a complex corresponds to the identities (1.2).

Next, in the case when \( n \) is odd, we will use the following description of the minimal free resolution of the ideal of \( \text{Sec}^{r-1} C \) in \( R[x_1, \ldots, x_n] \), where \( r = (n - 1)/2 \) (see [6, Sec. 4]). This ideal is generated by \( n \) linearly independent forms of degree \( r \), \( (p_1, \ldots, p_n) \), and there exists a unique skew-symmetric \( n \times n \) matrix of linear forms \( \Phi \) such that the following complex is exact:

\[ 0 \to R(-n) \xrightarrow{\nabla^T} R(-r - 1)^n \xrightarrow{\Phi} R(-r)^n \xrightarrow{\Phi^T} R, \]

(2.5)

where \( p = (p_1, \ldots, p_n) \).

Furthermore, we will use the following description of \( \Phi \), which is called Klein matrix in [6]. Let \( V_0 \) be the unique stable bundle on \( C \) with \( \det(V_0) \simeq L = \mathcal{O}(1)|_C \). Then identifying the space of linear forms on \( \mathbb{P}^{n-1} \) with \( H^0(C, L) \), the skew-symmetric matrix \( \Phi \) corresponds to the natural \( H^0(C, L) \)-valued skew-symmetric pairing on \( H^0(C, V_0) \),

\[ \Phi : \bigwedge^2 H^0(C, V_0) \to H^0(C, \det(V_0)) \simeq H^0(C, L). \]

(2.6)
Remark 2.3.1. Both complexes (2.3) and (2.5) for odd \( n \) are examples of Buchsbaum-Eisenbud pfaffian presentations of the corresponding Gorenstein ideals of height 3 (see [3]). In particular, the generators of the ideal \( (\frac{\partial F}{\partial x_i}) \) in the former case and \( p_i \) in the latter case, are equal to the (signed) submaximal pfaffians of the corresponding skew-symmetric matrix (\( \Omega \) in the former case and \( \Phi \) in the latter case).

Remark 2.3.2. The fact that the submaximal pfaffians \( (p_i) \) vanish on \( \text{Sec}^{r-1} C \subset \mathbb{P}H^0(C,L)^* \) means that for \( \xi \in \text{Sec}^{r-1} C \) the skew-symmetric form \( \Phi_\xi \) on \( H^0(C,V_0) \) has rank \( \leq n-3 \). This can be explained geometrically as follows. A point \( \xi \in \text{Sec}^{r-1} C \) corresponds to a functional that factors as \( H^0(C,L) \to H^0(C,L|_D) \to k \), where \( D \) is an effective divisor of degree \( r-1 \) on \( C \). But then the 3-dimensional subspace \( H^0(C,V_0(-D)) \subset H^0(C,V_0) \) is in the kernel of \( \Phi_\xi \).

2.4. Skew-symmetric matrices of linear forms. Let \( A \) and \( B \) be a pair of \( n \)-dimensional vector spaces, and let \( \Phi \in \wedge^2 A^* \otimes B \). Typically one views \( \Phi \) as a skew-symmetric matrix of linear forms on \( B^* \). However, \( \Phi \) also gives a linear map \( \nu_\Phi : A \to B \otimes A^* \), so we can view it as an \( n \times n \) matrix of linear forms on \( A \).

More precisely, consider the polynomial algebra \( R = S^*(A^*) \). Then we can view \( \nu_\Phi \) as an \( R \)-valued linear map \( A \to B \otimes R \), and consider the corresponding matrix of minors,

\[
\bigwedge^{n-1}(\nu_\Phi) : \bigwedge^{n-1} A \to \bigwedge^{n-1} B \otimes R
\]

with entries in \( S^{n-1}(A^*) \subset R \).

Lemma 2.4.1. (i) Consider the composed map

\[
\sigma(\Phi) : A^* \xrightarrow{\sim} \det(A)^* \otimes \bigwedge^{n-1}(A) \xrightarrow{\text{id} \otimes \bigwedge^{n-1}(\nu_\Phi)} \det(A)^* \otimes \bigwedge^{n-1}(B) \otimes R \\
\simeq \det(A)^* \otimes \det(B) \otimes B^* \otimes R.
\]

Then for every \( \xi \in A^* \), the element \( \sigma(\Phi)(\xi) \) (of the target \( R \)-module) is divisible by \( \xi \).

(ii) There exists a unique element

\[
\sigma_{n-2}(\Phi) \in \det(A)^* \otimes \det(B) \otimes B^* \otimes S^{n-2}(A^*),
\]

such that

\[
\sigma(\Phi)(\xi) = \sigma_{n-2}(\Phi) \cdot \xi
\]

(where the product is in \( R \)).

(iii) Viewing \( \sigma_{n-2}(\Phi) \) as a \( \det(A)^* \otimes \det(B) \otimes R \)-valued linear form on \( B \), we have

\[
\sigma_{n-2}(\Phi) \circ \nu_\Phi = 0
\]

as a \( \det(A)^* \otimes \det(B) \otimes R \)-valued linear form on \( A \).

(iv) Assume that \( \Phi(a,?) : A \to B \) has rank \( \geq n-1 \) for some \( a \in A \). Then \( \sigma_{n-2}(\Phi)(a) \neq 0 \).

Proof. (i) Let us think of \( \nu_\Phi \) as a \( \text{Hom}(A,B) \)-valued function on \( A \). Then \( \sigma(\Phi)(\xi) \) is given by the \((n-1) \times (n-1)\)-minors of the corresponding \( \text{Hom}(\langle \xi \rangle^+,B) \)-valued function on \( A \),

\[
\nu_{\Phi,\xi} : \langle \xi \rangle^+ \to B \otimes A^*,
\]
where we use the restriction from $A$ to the hyperplane $\langle \xi \rangle^\perp \subset A$. The condition that all $n \times n$-minors of $\nu_{\Phi, \xi}$ are divisible by $\xi$ is equivalent to the condition that for every $a \in \langle \xi \rangle^\perp$, the induced linear map

$$\nu_{\Phi, \xi, a} : \langle \xi \rangle^\perp \to B$$

has rank $< n - 1$. It is enough to check that for $a \neq 0$, the map $\nu_{\Phi, \xi, a}$ has nonzero kernel. But this follows from the identity

$$\nu_{\Phi, \xi, a}(a) = 0$$

which holds by the skew-symmetry of $\Phi$.

(ii) Note that the target of $\sigma(\Phi)$ is a free $R$-module. Thus, it is enough to prove that if a linear map $\sigma : A^* \to R = S(A^*)$ has the property that $\sigma(\xi)$ is divisible by $\xi$ for every $\xi \in A^*$, then $\sigma(\xi) = f \cdot \xi$ for a fixed polynomial $f \in R$.

Indeed, assume that for every $\xi \neq 0$, we have

$$\sigma(\xi) = f_\xi \cdot \xi$$

for some $f_\xi \in R$. We need to check that $f_{\xi_1} = f_{\xi_2}$ for a linearly independent pair $\xi_1, \xi_2 \in A^*$. For any $c \in k^*$ we have

$$\sigma(\xi_1 + c\xi_2) = f_{\xi_1} \xi_1 + f_{\xi_2} \xi_2 = f_c(\xi_1 + c\xi_2),$$

where $f_c := f_{\xi_1 + c\xi_2}$. This implies that $(f_c - f_{\xi_1})\xi_1$ is divisible by $\xi_2$, so we can write

$$f_c = f_{\xi_1} + f\xi_2.$$ 

From this we get

$$f_{\xi_2} - f_{\xi_1} = (c^{-1}x_1 + x_2) \cdot f,$$

so $f_{\xi_2} - f_{\xi_1}$ is divisible by $c^{-1}x_1 + x_2$. Since $k$ is infinite, this implies that $f_{\xi_2} - f_{\xi_1} = 0$.

(iii) We use again the fact that for a nonzero $\xi \in A^*$, $\sigma(\Phi)(\xi)$, which is an $R$-valued functional on $B$, is given by $\wedge^{n-1}(\Phi|_{\langle \xi \rangle^\perp})$, where we consider the restriction

$$\Phi|_{\langle \xi \rangle^\perp} : \langle \xi \rangle^\perp \to B \otimes R,$$

where $\dim(\langle \xi \rangle^\perp) = n - 1$. It follows that the composition

$$\langle \xi \rangle^\perp \xrightarrow{\Phi} B \otimes R \xrightarrow{\sigma(\Phi)(\xi)} \det(A)^* \otimes \det(B) \otimes R$$

is zero. By part (ii) this implies that the composition

$$\sigma_{n-2}(\Phi) \circ \Phi : A \to \det(A)^* \otimes \det(B) \otimes R$$

has zero restriction to $\langle \xi \rangle^\perp$ for every $\xi \neq 0$. Hence, this composition is zero.

(iv) The assumption implies that $\sigma(\Phi)(a) \neq 0$. Now the assertion follows from (ii). $\square$
3. Proofs

3.1. Proof of Theorem A. Working over an algebraic closure of $k$, we can assume that $L = \mathcal{O}(np)$ for a point $p \in C$. In this case Fisher gives the following explicit description of the matrix

$$\Omega : \bigwedge^2 H^0(C, L) \to S^2 H^0(C, L)$$

in [7 Sec. 5]. For $s_1, s_2 \in H^0(C, L)$ and $\phi \in H^0(C, L)^*$, one has

$$\frac{1}{n} \Omega(s_1, s_2)(\phi) = [S \cdot (s_1 \boxtimes s_2 - s_2 \boxtimes s_1)](\phi, \phi) + \mathcal{D}\{s_1, s_2\}(\phi, \phi),$$

for some operator $\mathcal{D} : H^0(C, \mathcal{O}(np)) \to H^0(C, \mathcal{O}((n + 1)p))$ (which is given up to rescaling by $f \mapsto df/\omega$, where $\omega$ is a global differential on $C$).

Comparing the above formula for $\Omega$ with (2.2) we immediately deduce that the bivector on $\mathbb{P}^{n-1}$ induced by $\Omega$ is $q_{n,1}(C)$, up to rescaling.

In the case $n$ is odd we have

$$\{F, x_j\} = \sum_i \frac{\partial F}{\partial x_i} \{x_i, x_j\} = \sum_i \frac{\partial F}{\partial x_i} \Omega_{ij} = 0$$

by (1.1). Hence, the assertion of the theorem follows in this case from Lemma 2.1.1.

Similarly, in the case $n$ is even, we deduce from (1.2) that

$$\{F_1, x_j\} = \{F_2, x_j\} = 0.$$

3.2. Proof of Theorem B. We start with the following description of

$$\phi^{-1} : \mathbb{P} H^0(C, V_0) \longrightarrow \mathbb{P} \text{Ext}^1(\mathcal{O}, L) \simeq \mathbb{P} H^0(C, L)^*.$$

Given a generic global section $s \in H^0(C, V_0)$, the corresponding map $s : \mathcal{O} \to V_0$ is an embedding of a subbundle, and we have a canonical identification of $V_0/s(\mathcal{O})$ with $\det(V_0) = L$. Hence, we get an extension of $L$ by $\mathcal{O}$. Note that the Serre duality isomorphism $\text{Ext}^1(\mathcal{O}, L) \simeq H^0(C, L)^*$ associates with an extension

$$0 \to \mathcal{O} \to E \to L \to 0$$

the corresponding coboundary homomorphism $H^0(C, L) \to H^1(C, \mathcal{O}) \simeq k$. The exact sequence of cohomology shows that the kernel of this homomorphism coincides with the image of the map $H^0(C, E) \to H^0(C, L)$.

Thus, $\phi^{-1}$ associates with a generic $s \in H^0(C, V_0)$ the unique functional $\xi \in H^0(C, L)^*$ (up to rescaling), such that $\ker(\xi)$ is equal to the image of the map

$$d_s : H^0(C, V_0) \to H^0(C, V_0/s(\mathcal{O})) \simeq H^0(C, L).$$

Recall that an isomorphism $\alpha_s : V_0/s(\mathcal{O}) \overset{\sim}{\longrightarrow} L$ induces an isomorphism

$$\beta_s : \det(V_0) \overset{\sim}{\longrightarrow} L : s \wedge x \mapsto \alpha(x).$$

Note that $\beta_s$ does not depend on $s$ up to rescaling.

Thus, up to rescaling, the map $d_s$ can be identified with the map

$$\Phi(s, ?) : H^0(C, V_0) \to H^0(C, L),$$
where $\Phi$ is the Klein matrix (2.6). In other words, $\xi = \phi^{-1}(s)$ is characterized by the condition

$$\ker(\xi) = \operatorname{im} \Phi(s, ?).$$

Let us set $A := H^0(C, V_0)$ and $B := H^0(C, L)$ for brevity (note that $B$ is the space of linear forms on our projective space $\mathbb{P}^{n-1} = \mathbb{P}H^0(C, L)^*$), so that $\Phi$ can be viewed as a skew-symmetric linear map

$$\Phi : A \rightarrow A^* \otimes B.$$

We showed above that the birational map

$$\phi^{-1} : \mathbb{P}A \dashrightarrow \mathbb{P}B^*$$

sends a generic $a$ to the unique functional $b^*$ (up to rescaling) such that $b^*$ vanishes on the image of $\Phi(a, ?) : A \rightarrow B$. Now we recall that with $\Phi$ we can associate canonically (up to rescaling) an element $\sigma_{n-2}(\Phi) \in B^* \otimes S^{n-2}(A^*)$, such that the composition

$$A \xrightarrow{\Phi(a, ?)} B \xrightarrow{\sigma_{n-2}(\Phi)(a)} k$$

is zero (see Lemma 2.4.1). Furthermore, since $\Phi(a, ?)$ has rank $n - 1$ for generic $a$, we have $\sigma_{n-2}(\Phi)(a) \neq 0$ for such $a$ (see Lemma 2.4.1(iv)). This implies that

$$\phi^{-1}(a) = \sigma_{n-2}(\Phi)(a)$$

in $\mathbb{P}B^*$. If we choose a basis in $B$, then $\sigma_{n-2}(\Phi)$ is given by $n$ polynomials $(f_1, \ldots, f_n)$, where $f_i \in S^{n-2}(A^*)$, and we have

$$\phi^{-1}(a) = (f_1(a) : \ldots : f_n(a)).$$

It remains to prove the formula for $\phi : \mathbb{P}B^* \rightarrow \mathbb{P}A$. We can rewrite the complex (2.5) as

$$0 \rightarrow R(-n) \xrightarrow{p^T} A \otimes R(-r - 1) \xrightarrow{\Phi} A^* \otimes R(-r) \xrightarrow{p} R$$

where $R = S^*(B)$. This complex shows that for each $b^* \in B^*$ the specialization $\Phi_{b^*} : A \rightarrow A^*$ satisfies

$$\Phi_{b^*}(p(b^*)^T) = 0,$$

where $p(b^*)^T \in A$.

We need to check that $\phi(b^*) = p(b^*)^T$. Equivalently, setting

$$a := p(b^*)^T,$$

we need to prove that

$$\phi^{-1}(a) = b^*.$$

By the above description of $\phi^{-1}$, it is enough to check that $b^*$ annihilates the image of $\Phi(a, ?) : A \rightarrow B$ (since the latter image is a hyperplane in $B$). In other words, we need to check that $b^* \circ \Phi(a, ?) = 0$. But we have

$$b^* \circ \Phi(a, ?) = \Phi_{b^*}(a) = 0$$

by (3.1). This ends the proof.
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