Time Evolution of Harmonic Oscillator Thermal Momentum Superposition States

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The time evolution of thermal states of a mirror released from a tight harmonic trap is studied. After the release no dissipation is assumed to be present and the mirror is, after a time delay, kicked into a momentum superposition state. The thermal character of the initial state washes out the telltale interference patterns of the superposition but no loss of coherence is found. This investigation resolves a controversy about decoherence–without–dissipation and shows that entrained measurements can be surprisingly insensitive to temperature effects.

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I. INTRODUCTION

For massive objects it is hard to create a pure quantum state unentangled with the environment. Many studies of the decohering effects of the environment do however assume that the studied system is in a pure state initially, unentangled with the environment. Although realizable for atomic systems this is implausible for massive objects such as opto-mechanical mirrors [1]. Ford et al. [2-4] are therefore quite justified in insisting that for investigations of the dynamics of quantum superposition states of massive systems, weakly coupled to the environment, thermal initial states should be considered. They derived expressions for the loss of interference contrast of formally thermalized momentum superposition states of free massive objects. Their interpretations have received a somewhat mixed reception though. Notably, their interpretation of a process they dubbed “decoherence without dissipation” [3] has been either criticised [5, 6] or ignored [7, 8].

Here, following Ford et al.’s approach, a mirror is studied which initially is weakly coupled to the environment such that we can neglect dissipation other than assuming the mirror is in thermal equilibrium with the laser beams that form a tight harmonic potential holding and cooling it [9]. When the lasers are switched off the mirror becomes effectively decoupled from the environment, we model this as a ballistically expanding mirror without dissipation [2, 4]. Once the coherent width of the mirror’s spatial distribution is sufficient to imprint a photon’s momentum-recoil pattern the mirror is kicked into a momentum superposition state and then investigated using the ‘entrainment procedure’ of reference [10].

This work addresses two main questions. Can decoherence without dissipation [3] occur? How badly does the thermal nature of the initial state affect the formation and persistence of interference patterns of a mirror’s wave packet [10]?

Ford and O’Connell use the attenuation of the interference contrast as a measure for decoherence. This is not the conventional approach for the quantification of decoherence, which is based on the loss of off-diagonal elements of the reduced density matrix [5-8]. One can therefore refute Ford and O’Connell’s assertion, that they found evidence for “decoherence without dissipation” [3], on formal grounds, arguing, that no such process can occur without coupling to the environment [5-6]. I temporarily adopt Ford and O’Connell’s approach though and investigate the visibility of the interference pattern, since it is the quantity an experimentalist is most likely to measure [2-3].

Ford et al. [2, 4] consider a free particle in a Gaussian state with arbitrary width which is then formally thermalized by convolution with a Maxwell-Boltzmann velocity distribution function. One could argue, using Ford, Lewis, and O’Connell’s own logic [3], that the arbitrary spatial extension and precise form of the initial Gaussian state on the one hand, and its formal thermalization at an arbitrarily chosen temperature θ on the other, might not be independent and the details need further justification. Indeed, the environmental temperature typically determines the spatial extent of emerging position-pointer states [7] and a generic mechanism, such as photons randomly scattering off point particles with spatially extended center-of-mass wave function, can lead to wave functions that do not have Gaussian shape [11].

To avoid such ambiguities a possible physical implementation of the scenario modelled here is introduced in section [11]. Section [11] illustrates different cases of our scenario employing varied physical parameters. In section [11] we consider the impact of temperature on the expected visibility of the interference pattern and its loss (and resurgence) over time. Finally, in section [11] we consider the ‘free case’, mostly discussed by Ford et al., we will see that it can be mapped onto the case of a mirror trapped harmonically. A harmonically trapped thermal mirror shows periodic resurgence of the interference pattern, we therefore conclude that “decoherence without dissipation” is absent even when adopting Ford and O’Connell’s approach using interference contrasts instead of non-diagonal density matrix elements.

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II. STEPS OF THE SCENARIO

A. Trapped and cooled mirror

A mirror with mass \( M \), initially trapped by a stiff harmonic potential with spring constant \( K_0 \) and cooled to temperature \( \Theta \) \([9]\), is described by the thermal density matrix

\[
\hat{\rho}_0 = \sum_{n=0}^{\infty} \frac{e^{-\frac{\hbar \omega_0 n}{k_B T}}}{1 - e^{-\frac{\hbar \omega_0}{k_B T}}} |\psi_n(x, 0; K_0)\rangle \langle \psi_n(x, 0; K_0)|. \quad (1)
\]

Here \( \Theta_E = \hbar \sqrt{K_0/M}/k_B \) is the associated Einstein-temperature where \( \hbar \) is Planck’s and \( k_B \) Boltzmann’s constant. In all numerical calculations we will set \( \hbar = 1 \) and \( k_B = 1 \). For opto-mechanical oscillator masses ranging from \( M = 10^{-15} \) to \( 10^{-10} \) kg \([1]\) and recently reported large optical spring constants \( K_0 \approx 10^6 \) N m\(^{-1}\) \([9]\) the oscillator’s resonance frequency \( \Omega_0 = \sqrt{K_0/M} \) ranges from \( 3 \cdot 10^8 \) s\(^{-1}\) to \( 10^8 \) s\(^{-1}\) and the associated Einstein-temperature \( \Theta_E = \hbar \Omega_0/k_B \) from 0.24 K to 0.76 mK. Such temperatures can be reached in optical cooling setups \([9, 12, 13]\). We will see that the scheme presented here can tolerate temperatures up to several \( \Theta_E \) of the trapping potential.

The normalized harmonic oscillator eigenfunctions \( \psi_n(x, 0; K_0) = H(n, \frac{x}{\sigma_0}) e^{-\frac{x^2}{2\sigma_0^2}}/\sqrt{\pi n!} \) are parameterized by the system’s ground state position standard deviation

\[
\sigma_0 = \frac{\sqrt{\hbar}}{\sqrt{K_0 M}}. \quad (2)
\]

For the values for \( K_0 \) and \( M \) cited above, this position spread is very small \( (\sigma_0 \approx 10^{-14}) \) to \( 5 \cdot 10^{-16} \) m. This is good to protect the oscillator from decoherence but makes it hard to see interference fringes when momentum superposition states are formed \([10]\).

B. Released mirror

To be able to ‘imprint’ the interference pattern associated with the momentum superposition state \([10]\) we will therefore assume that the stiff trapping potential is suddenly switched off so that the mirror is very weakly trapped or set free. In the free case, the ensuing ballistic expansion of the stiff potential’s groundstate yields the ‘free groundstate’ wave function \([14]\)

\[
\phi_0(x, t) = \frac{1}{\sqrt{\pi \sigma(t)}} \exp \left[ -\frac{x^2}{2\sigma(t)} \right], \quad (3)
\]

where \( \sigma(t) = \sigma_0 + i \frac{\hbar}{\sigma_0 M} t \quad (4) \)

is the time-dependent position spread which gives rise to the ballistic expansion velocity of the ‘released ground-state’ (it is larger for ‘released excited states’)

\[
v = \frac{\hbar}{\sigma_0 M} = \sqrt{\frac{\hbar \sqrt{K_0}}{M^{3/4}}}. \quad (5)
\]

For \( M = 10^{-15} \) to \( 10^{-10} \) kg \([1]\) and optical spring constants \( K_0 \approx 10^6 \) N m\(^{-1}\) \([9]\) this velocity ranges from \( 60 \mu \text{m}\text{s}^{-1} \) to \( 10 \text{nm}\text{s}^{-1} \). In other words, for a mirror mass of \( 10^{-15} \) kg, the free ground state wavefunction expands to an optical wavelength size of roughly 500 nm within an expansion time of approximately 9 ms.

C. Kicked thermal mirror

In order to achieve the momentum superposition, it is suggested to use an interferometric setup featuring a half-silvered beam splitter that splits a single photon (generated in spontaneous parametric down-conversion \([15]\)) into two equally strong partial waves. These partial waves are directed onto either side of the mirror and, upon reflection, transfer their momentum recoil to the mirror in a desirable fashion: namely, after tracing over the photon wave function the mirror’s wave function becomes multiplied with an effective kick-factor \([10]\)

\[
\mathcal{K} = -i \sin(2\kappa x - \phi) \quad , \quad (6)
\]

where the relative phase \( \phi \) is fixed by the interferometric setup and the port the photon takes \([10]\). Here, it is assumed that the mirror has unit reflectivity and \( \kappa \) is the photon’s effective wavenumber, associated with its normal momentum component, for more details and the case of imperfect mirrors see \([10]\).

The ‘released eigenstates’, spawned by the eigenfunctions \( \psi_n \) of the stiff potential, are labelled \( \Psi_n(x, t; 0, \tau, k, K_0) \); here \( \tau < 0 \) denotes the time when the mirror is released from the stiff potential \( K_0 x^2/2 \) into a weak potential \( kx^2/2 \) (or set free: \( k = 0 \), in which case \( \Psi_0(x, 0; 0, -t, 0, K_0) = \phi_0(x, t) \) of Eq. \((3)\)).

Then, at time \( t = 0 \), the photon kicks the mirror into a momentum-superposition state. Through the momentum kick the wave functions become combined to form the desired momentum superposition states \( \Upsilon_n \propto \mathcal{K} \Psi_n \)

\[
\Upsilon_n(x, t; p_\gamma, \tau, k, K_0) = N_n[\Psi_n(x, t; p_\gamma, \tau, k, K_0) - e^{i\phi_0} \Psi_n(x, t; -p_\gamma, \tau, k, K_0)]. \quad (7)
\]

Here, \( N_n \) is a normalization constant that absorbs overall phase factors and primarily depends on the size of the photons’ momentum transfers \( p_\gamma = 2\hbar \kappa \), compare Eq. \((5)\), and the position spread of each state \( \Psi_n \) at the time \( (t=0) \) when the photon-kicks. The thermal momentum superposition state thus has the form

\[
\hat{\rho}(x, t; p_\gamma, \tau, k, K_0) = \sum_{n=0}^{N} e^{\frac{\omega_0 n}{k_B T}} |\Upsilon_n\rangle \langle \Upsilon_n| . \quad (8)
\]
The states $\Upsilon_n$ up to 13-th order have been determined in this work, so, the cut-off $N = 13$. Upon inclusion of the ‘free’ kicked groundstate $\Upsilon_0$, these lowest fourteen oscillator states allow us to model the stiff oscillator at temperatures of up to three times the Einstein-temperature: $\sum_{n=14}^{\infty} e^{n/3.0} < 0.01$, since this accounts for 99% of the probability distribution for temperature $\theta = 3 \cdot \Theta_E$ at which a harmonic oscillator has roughly 99.08% of the classical heat capacity $\frac{3}{2} k_B$. At such temperatures, the character of the system can be considered to be between quantum and classical.

III. DYNAMICS OF PURE STATES

We now study the dynamics of the groundstate, which is an even function in position, and the third excited state, which is odd.

Figure A shows the groundstate released from the stiff oscillator potential at $\tau = -3T/4$ where $T = 2\pi \sqrt{M/k} = 2\pi/\omega$ is the mirror’s period time. It plays out as an underlying squeezed and anti-squeezed breathing motion in evidence in all plots of Figure A. At time zero the mirror is kicked to the right and so shows a combination of breathing and oscillation. Plot B shows the same scenario except for the facts that the release from the stiff potential happens at time $\tau = -T/4$ and the momentum-kick is in superposition. The formation of the superposition is highlighted by the red line immediately before the momentum kick and the yellow line when it happens. The same yellow profile line has been moved downstream and shows that intensity patterns re-form every half period.

Plots C and D show analogous scenarios for the third excited state released from the stiff potential at times $\tau = -T/4$ and $-T/8$ respectively. Plots B and D illustrate that the interference pattern of the momentum superposition is imprinted onto the underlying spatial probability distribution of the state with full contrast.
IV. DYNAMICS OF THERMAL STATES

At the temperatures considered here, ranging from zero up to three times the Einstein temperature, the interference with full contrast can be observed for all settings of the system’s parameters such as spring constants, mass, and photon momenta, provided that the released mirror wave functions had time to ballistically expand sufficiently widely to accommodate an imprint at the effective wavelength \[10\] of the photon recoil when it is being kicked into the superposition state, compare Fig. 2 A versus B. In other words, wave packet expansion and the momentum-superposition interference imprinting operations do not commute. Each wave function \(\Upsilon_n\) is by itself fully coherent and the momentum superposition imprints occur at the same spatial positions for all of them. So, the effects add up to an interference pattern with full contrast. This implies that incoherent mixtures consisting of states spread out wide enough to carry the momentum superposition imprint can be endowed with an interference pattern when using the entrainment procedure of reference \[10\]. This observation, which also applies to states with temperatures much higher than studied here, is one of the central messages of this work.

Note that the dephasing of the imprinted interference pattern does not only depend on the temperature but also its location. Further out from the center fewer ‘released eigenstates’ \(\Upsilon_n\) contribute to the thermal wave packet and therefore dephasing is slowed down there as compared to dephasing at the center, \(x = 0\), compare Fig. 2 B with C. On the fringes the interference gets washed out more slowly.

Additionally, more highly excited ‘released eigenstates’ have a wider spread and higher ballistic expansion velocity, they therefore can carry an interference imprint, when lower lying ‘released eigenstates’ might not be able to do this, see Fig. 2 B as opposed to A. Surprisingly, higher temperatures can help in generating an interference pattern imprint because they expand the width of the mirror’s wavepacket.

Since they have different energies the various wave functions \(\Upsilon_n\) evolve at different rates and therefore mutually dephase thus washing out the interference imprint. The general scenario is varied and its detailed quantification is beyond the level of this work. We would, however, like to point out that the breathing motion underlying the dynamics in the trapped case (see breathing motion of pure states in Fig. 1) is least pronounced at every turning point or odd quarter period point: \((2n + 1)T/4 = (2n + 1)2\pi/(4\omega)\) counted from the moment of release \(\tau\). This is confirmed by Fig. 2 B and C which show slowed dephasing since the momentum-superposition kick happens at the quarter time \((\tau = -T/4, t = 0)\).

A. Benchmark for dephasing function

The dephasing times of the interference patterns in evidence in Figs. 2 and 3 can be crudely benchmarked. To do this we derive a reference approximation \(A\). In the thermal state the wave functions are incoherently added up and so the dephasing is due to the summing up of their individual dynamics, weighted with the Boltzmann-factors. To derive our benchmark approximation we form a coherent instead of an incoherent sum of wave functions using the dominant differences in their energy expressions, namely the harmonic oscillator energies of the initial stiff trapping potential, in the time evolution phase factors. This neglects variations due to the transferred kinetic energies but should be compensated for by the overestimate due to the formation of a coherent sum. Moreover, we assume that all wave functions contribute equally (weight ‘1’) rather than with their true local weight \(|\Upsilon_n|^2\); this also overestimates the dephasing effect. This yields an expression containing the ‘Boltzmann-sums’

\[
\frac{b(\theta, \tau)}{b(\theta, 0)} = \sum_{n=0}^{\infty} e^{-\omega_{\Theta}(n+1/2)\tau} = 1 - e^{-\omega_{\Theta}(1/2+\omega t)^2}. \tag{11}
\]

Since the probability distribution of the released mirror does not oscillate at frequency \(\Omega\) but at twice the frequency \(\omega\) of the weak trap, it is released into, our approximation for the interference pattern visibility adopts this and reads

\[
A(\theta, \tau) = \frac{|b(\theta, \tau \cdot \frac{n_\Omega}{\Omega})|^2}{b(\theta, 0)^2} = \frac{(1 - e^{-\omega_{\Theta}})^2}{1 + e^{-2\omega_{\Theta}} - 2 e^{-\omega_{\Theta}} \cos (2\omega t)} \tag{9}
\]

Clearly this approximation is inconsistent; for instance the probabilities instead of the amplitudes of the Boltzmann-factors are used, but then again \(A(\theta/2, \tau)\) does use the amplitudes, in short: the approximation \(A\) can only serve as a guide to the eye, compare Fig. 4 but it allows us to underline the fact that dephasing happens on time scales of the system’s evolution \(\sim 1/(2\omega)\). This answers the second question posed in the introduction: Since the approach using entrained measurements allows for very fast state preparation and analysis \[10\], we can hope that, assuming it can be experimentally implemented, thermal dephasing does not affect it too badly.

V. NO DECOHERENCE WITHOUT DISSIPATION

The state \(\Upsilon\) shows full interference contrast at the time \((t = 0)\) when it is kicked into the momentum superposition state \[10\]. The associated visibility \(\mathcal{V}\) of the mirror’s interference pattern at the origin, as a function of time, is given by the expression

\[
\mathcal{V}(t) = \frac{\rho(0, t; p_\gamma, \tau, 0, k, K_0) - \rho(0, t; p_\gamma, \tau, \pi, k, K_0)}{\rho(0, t; p_\gamma, \tau, 0, k, K_0) + \rho(0, t; p_\gamma, \tau, \pi, k, K_0).} \tag{11}
\]
The dephasing of the interference pattern is weaker for longer or minimize over the settings of the phase $C$ expansion rather than earlier (see $B$). This answers the first question posed in the introduction: there is no decoherence without dissipation. An eigenstate of the initial stiff potential has overlap with many states of a weak trapping potential it is released into, making the resulting expressions unwieldy. Instead, a trick is employed here. A mapping has been used to map between harmonically trapped and free states, for a detailed discussion see [14]. This way it became possible to generate Figs. 3 and 4.

Aside from being a mathematical trick this mapping also shows that no decoherence occurs for the case of the free particle, as was claimed in references [2–4]. Instead, the wave functions dephase but the mapping back to the trapped case and further propagation in time shows that the interference pattern reforms with full contrast. Note that, although challenging in practice for a widely spread out wave packet, in principle the mapping corresponds to the experimentalist switching on the trapping potential [14]. This answers the first question posed in the introduction: there is no decoherence without dissipation.

It raises the new question as to when to let the photon impinge on the mirror to imprint the superposition pattern. The answer to this question depends on the goals one wants to achieve experimentally. If one waits for time $T/2$ after the release the wave function has

**FIG. 3.** Probability densities $P(x,t)$ of quantum harmonic oscillator with mass $M = 1$ released from stiff oscillator potential into weak potential and kicked into momentum superposition state at time $t = 0$: Plot $A$ $\theta = 3\Theta_E$, $K = 16$, $k = 1$, $p_r = 2$ and $\tau = -T/20$; $B$ $\theta = 3\Theta_E$, $K = 64$, $k = 4$, $p_r = 2$ and $\tau = -T/4$; $C$ $\theta = 2.815\Theta_E$, $K = 32$, $k = 2$, $p_r = 1$ and $\tau = -T/10$; $D$ $\theta = 3\Theta_E$, $K = 16$, $k = 1$, $p_r = 4$ and $\tau = -T/4$. It can be seen that the interference patterns always re-form with full contrast after half a period. Moreover, the time over which the interference pattern persists is lengthened when one kicks the mirror at the point of greatest expansion rather than earlier (see $B$ as opposed to $A$). The dephasing of the interference pattern is weaker for longer wavelength imprints (smaller momenta), compare $A$ and $B$, with $C$ and $D$.

It has this definite form —without the need to maximize or minimize over the settings of the phase $\phi$— because all constituent wave functions have definite parity in the $x$-coordinate.

The visibility degrades over time because of dephasing of the components of the thermal density matrix $\rho$. But, if the mirror is released into a weak harmonic trapping potential with spring constant $k$ instead of being set free ($k = 0$), its wave function will re-form the same probability distribution (or its position-inverted mirror image) every half cycle of length $T/2$; this is illustrated in Figs. 1 and 3.

**A. Map between free and trapped states**

In theory, the ballistically expanding wave functions can be determined explicitly, using suitable propagators. In practice, this turns out to be surprisingly difficult even when using a contemporary computer algebra system [16].

**FIG. 4.** Same parameters as Fig. 3 (except for temperature $\theta$): plots of the decadic logarithm $\log_{10}(\mathcal{V})$ of the visibility $\mathcal{V}$ as defined in Eq. 11 as a function of temperature and time (colored mesh) together with the benchmark approximations $A$ of Eq. 10 (brown contour-sheets). This figure shows how with increasing temperature the visibility of the interference pattern washes out faster but always reforms fully after half a cycle. Moreover, if the mirror gets kicked into the momentum superposition state near the turning point ($\tau = -T/4$) the interference pattern persists longest, compare text and Fig. 3.
expanded furthest. This stage maybe be advantageous if the large mass of the mirror makes it impossible to achieve sufficient position spread early in its evolution. On the other hand, imprinting the interference pattern at this stage implies that the momentum kick is transferred to the mirror at a time of largest spatial extension and subsequently, the mirror will contract again, see Fig. 1B. Unless very large recoil values are employed, this precludes the formation of separate spatial humps in the probability distribution which are typically considered to be the hallmark of so-called “Schrödinger-cat” spatial superposition states.

VI. CONCLUSION

This work studied effects of thermalized initial states of a harmonically trapped quantum mirror released into a very weak potential (or set free) [17] [18]. It has shown that decoherence without dissipation, based on thermalized initial states alone, does not occur as envisaged in references [2–4]. It also studied the formation and persistence of interference patterns of the mirror’s wave packet after it is kicked into a superposition state. It was shown that dephasing and re-formation of the interference patterns happens on the time scale of the system’s time evolution and should therefore be detectable when using sufficiently fast detection schemes such as the ‘entrainment procedure’ sketched in reference [10].

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