Two distance-regular graphs

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Abstract

We construct two families of distance-regular graphs, namely the subgraph of the dual polar graph of type $B_3(q)$ induced on the vertices far from a fixed point, and the subgraph of the dual polar graph of type $D_4(q)$ induced on the vertices far from a fixed edge. The latter is the extended bipartite double of the former.

1 The extended bipartite double

We shall use $\sim$ to indicate adjacency in a graph. For notation and definitions of concepts related to distance-regular graphs, see [BCN]. We repeat the definition of extended bipartite double.

The bipartite double of a graph $\Gamma$ with vertex set $X$ is the graph with vertex set $\{x^+, x^- \mid x \in X\}$ and adjacencies $x^+ \sim y^-$ iff $\delta e = -1$ and $x \sim y$. The bipartite double of a graph $\Gamma$ is bipartite, and it is connected iff $\Gamma$ is connected and not bipartite. If $\Gamma$ has spectrum $\Phi$, then its bipartite double has spectrum $(-\Phi) \cup \Phi$. See also [BCN], Theorem 1.11.1.

The extended bipartite double of a graph $\Gamma$ with vertex set $X$ is the graph with vertex set $\{x^+, x^- \mid x \in X\}$, and the same adjacencies as the bipartite double, except that also $x^- \sim x^+$ for all $x \in X$. The extended bipartite double of a graph $\Gamma$ is bipartite, and it is connected iff $\Gamma$ is connected. If $\Gamma$ has spectrum $\Phi$, then its extended bipartite double has spectrum $(-\Phi - 1) \cup (\Phi + 1)$. See also [BCN], Theorem 1.11.2.

2 Far from an edge in the dual polar graph of type $D_4(q)$

Let $V$ be a vector space of dimension 8 over a field $F$, provided with a nondegenerate quadratic form of maximal Witt index. The maximal totally isotropic subspaces of $V$ (of dimension 4) fall into two families $\mathcal{F}_1$ and $\mathcal{F}_2$, where the dimension of the intersection of two elements of the same family is even (4 or 2 or 0) and the dimension of the intersection of two elements of different families is odd (3 or 1).

The geometry of the totally isotropic subspaces of $V$, where $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$ are incident when $\dim A \cap B = 3$ and otherwise incidence is symmetrized inclusion, is known as the geometry $D_4(F)$. The bipartite incidence graph on
the maximal totally isotropic subspaces is known as the dual polar graph of type $D_4(F)$.

Below we take $F = \mathbb{F}_q$, the finite field with $q$ elements, so that graph and geometry are finite. We shall use projective terminology, so that 1-spaces, 2-spaces and 3-spaces are called points, lines and planes. Two subspaces are called disjoint when they have no point in common, i.e., when the intersection has dimension 0.

**Proposition 2.1** Let $\Gamma$ be the dual polar graph of type $D_4(\mathbb{F}_q)$. Fix elements $A_0 \in \mathcal{F}_1$ and $B_0 \in \mathcal{F}_2$ with $A_0 \sim B_0$. Let $\Delta$ be the subgraph of $\Gamma$ induced on the set of vertices disjoint from $A_0$ or $B_0$. Then $\Delta$ is distance-regular with intersection array \{(\text{q}^3, \text{q}^3 - 1, \text{q}^1 - \text{q}, \text{q}^1 - \text{q}^2 + 1; 1, \text{q}, \text{q}^3 - 1, \text{q}^3)\}.

The distance distribution diagram is

\[
\begin{array}{cccccccc}
1 & q^3 & \text{q}^2 & q^2 - 1 & q^3 - \text{q} + 1 & q^3 - \text{q}^2 + 1 & q^3 - \text{q}^2 + 1 & \text{q}^3 - 1 \\
\end{array}
\]

**Proof:** There are $q^6$ elements $A \in \mathcal{F}_1$ disjoint from $A_0$ and the same number of $B \in \mathcal{F}_2$ disjoint from $B_0$, so that $\Delta$ has $2q^6$ vertices.

Given $A \in \mathcal{F}_1$, there are $q^3 + q^2 + q + 1$ elements $B \in \mathcal{F}_2$ incident to it. Of these, $q^2 + q + 1$ contain the point $A \cap B_0$ and hence are not vertices of $\Delta$. So, $\Delta$ has valency $q^3$.

Two vertices $A, A' \in \mathcal{F}_1$ have distance 2 in $\Delta$ if and only if they meet in a line, and the line $L = A \cap A'$ is disjoint from $B_0$. If this is the case, then $L$ is in $q + 1$ elements $B \in \mathcal{F}_2$, one of which meets $B_0$, so that $A$ and $A'$ have $c_2 = q$ common neighbours in $\Delta$.

Given vertices $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$ that are nonadjacent, i.e., that meet in a single point $P$, the neighbours $A'$ of $B$ at distance 2 to $A$ in $\Delta$ correspond to the lines $L$ on $P$ in $A$ disjoint from $B_0$ and nonorthogonal to the point $A_0 \cap B$. There are $q^2 + q + 1$ lines $L$ on $P$ in $A$, $q + 1$ of which are orthogonal to the point $A_0 \cap B$, and one further of which meets $B_0$. (Note that the points $A_0 \cap B$ and $A \cap B_0$ are nonorthogonal since neither point is in the plane $A_0 \cap B_0$ and $V$ does not contain totally isotropic 5-spaces.) It follows that $c_3 = q^2 - 1$, and also that $\Delta$ has diameter 4, and is distance-regular.

The geometry induced by the incidence relation of $D_4(F)$ on the vertices of $\Delta$, together with the points and lines contained in the planes disjoint from $A_0 \cup B_0$, has Buekenhout-Tits diagram (cf. [P])

that is, the residue of an object $A \in \mathcal{F}_1$ is an affine 3-space, where the objects incident to $A$ in $\mathcal{F}_2$ play the rôle of points. Similar things hold more generally for $D_n(F)$ with arbitrary $n$, and even more generally for all diagrams of spherical type. See also [BB], Theorem 6.1.

Let $P$ be a nonsingular point, and let $\phi$ be the reflection in the hyperplane $H = P^\perp$. Then $\phi$ is an element of order two of the orthogonal group that fixes $H$ pointwise, and consequently interchanges $\mathcal{F}_1$ and $\mathcal{F}_2$. For each $A \in \mathcal{F}_1$ we
have \( \phi(A) \sim A \). The quotient \( \Gamma/\phi \) is the dual polar graph of type \( B_3(q) \), and we see that more generally the dual polar graph of type \( D_{m+1}(q) \) is the extended bipartite double of the dual polar graph of type \( B_m(q) \). The quotient \( \Delta/\phi \) is a new distance-regular graph discussed in the next section. It is the subgraph consisting of the vertices at maximal distance from a given point in the dual polar graph of type \( B_3(q) \). For even \( q \) we have \( B_3(q) = C_3(q) \), and it follows that the symmetric bilinear forms graph on \( F_q^3 \) is distance-regular, see [BCN]. Proposition 9.5.10 and the diagram there on p. 286.

3 Far from a point in the dual polar graph of type \( B_3(q) \)

First a very explicit version of the graph of this section.

**Proposition 3.1** (i) Let \( W \) be a vector space of dimension 3 over the field \( F_q \), provided with an outer product \( \times \). Let \( Z \) be the graph with vertex set \( W \times W \) where \( (u, u') \sim (v, v') \) if and only if \( (u, u') \neq (v, v') \) and \( u \times v + u' \times v' = 0 \). Then \( Z \) is distance-regular of diameter 3 on \( q^3 \) vertices. It has intersection array \( \{ q^3-1, q^3-q, q^3-q^2+1; 1, q, q^2-1 \} \) and eigenvalues \( q^3-1, q^2-1, -1, -q^2-1 \) with multiplicities 1, \( \frac{1}{q}(q+1)(q^3-1) \), \( (q^3-q^2+1)(q^3-1) \), \( \frac{1}{q^2}(q-1)(q^3-1) \), respectively.

(ii) The extended bipartite double \( \hat{Z} \) of \( Z \) is distance-regular with intersection array \( \{ q^3, q^3-1, q^3-q, q^3-q^2+1; 1, q, q^2-1, q^3 \} \) and eigenvalues \( \pm q^3, \pm q^2, 0 \) with multiplicities 1, \( q^2(q^3-1) \), \( 2(q^3-q^2+1)(q^3-1) \), respectively.

(iii) The distance-1-or-2 graph \( Z_1 \cup Z_2 \) of \( Z \), which is the halved graph of \( \hat{Z} \), is strongly regular with parameters \( (v, k, \lambda, \mu) = (q^6, q^2(q^3-1), q^2(q^2+q-3), q^2(q^3-1)) \).

The distance distribution diagram of \( Z \) is

\[
\begin{array}{cccc}
1 & q^3-1 & q^3-q & q^3-q^2+1 \\
q^3-1 & 1 & q^3-q & q^3-q^2+1 \\
q^3-q & q^3-q^2+1 & \frac{(q^3-1)(q^3-1)}{q} & \frac{(q^3-q^2+1)(q^3-1)}{q^3-q^2} \\
q^3-q^2+1 & q^3-q^2+1 & \frac{(q^3-1)(q^3-1)}{q} & \frac{(q^3-q^2+1)(q^3-1)}{q^3-q^2} \\
\end{array}
\]

**Proof:** Note that the adjacency relation is symmetric, so that \( Z \) is an undirected graph. The computation of the parameters is completely straightforward. Clearly, \( Z \) has \( q^3 \) vertices. For \( a, b \in W \) the maps \( (u, u') \mapsto (u+a, u'+(a \times u)+b) \) are automorphisms of \( Z \), so \( \text{Aut}(Z) \) is vertex-transitive.

The \( q^3-1 \) neighbours of \((0,0)\) are the vertices \((v,0)\) with \( v \neq 0 \). The common neighbours of \((0,0)\) and \((v,0)\) are the vertices \((cv,0)\) for \( c \in F_q, c \neq 0,1 \). Hence \( a_1 = q - 2 \).

The \( (q^3-1)(q^3-1) \) vertices at distance 2 from \((0,0)\) are the vertices \((u,u')\) with \( u, u' \neq 0 \) and \( u' \perp u \). The common neighbours of \((0,0)\) and \((u,u')\) are \((v,0)\) with \( v \times u = u' \), and together with \((v,0)\) also \((v + cu,0)\) is a common neighbour, so \( c_2 = q \). Vertices \((u,u')\) and \((v,v')\), both at distance 2 from \((0,0)\) are adjacent when \( 0 \neq v \perp u' \) and \( v \neq 0 \) and \( v = u \neq u' \) and \( v' = u \times v + u' \), so that \( a_2 = q^3 - q - 2 \).

The remaining \( (q^3-1)(q^3-q^2+1) \) vertices have distance 3 to \((0,0)\). They are the \((u,w')\) with \( w \not\perp u' \) or \( w = 0 \neq w' \). The neighbours \((u,u')\) of \((w,w')\) that lie at distance 2 to \((0,0)\) satisfy \( 0 \neq u \perp w' \) and \((0 \neq u)' = w \times u + w' \),
so that \( c_3 = q^2 - 1 \). This shows that \( Z \) is distance-regular with the claimed parameters. The spectrum follows.

The fact that the extended bipartite double is distance-regular, and has the stated intersection array, follows from \([BCN]\), Theorem 1.11.2(vi).

The fact that \( Z_2 \) is strongly regular follows from \([BCN]\), Proposition 4.2.17(ii) (which says that this happens when \( Z \) has eigenvalue \(-1\)). \( \square \)

For \( q = 2 \), the graphs here are (i) the folded 7-cube, (ii) the folded 8-cube, (iii) the halved folded 8-cube. All are distance-transitive. For \( q > 2 \) these graphs are not distance-transitive.

When \( q \) is a power of two, the graphs \( \hat{Z} \) have the same parameters as certain Kasami graphs, but for \( q > 2 \) these are nonisomorphic.

Next, a more geometric description of this graph.

Let \( H \) be a vector space of dimension 7 over the field \( \mathbb{F}_q \), provided with a nondegenerate quadratic form. Let \( \Gamma \) be the graph of which the vertices are the maximal totally isotropic subspaces of \( H \) (of dimension 3), where two vertices are adjacent when their intersection has dimension 2. This graph is known as the dual polar graph of type \( B_3(q) \). It is distance-regular with intersection array \((q(q^2 + q + 1), q^3(q + 1), q^2 + 1, q + 1, q^2 + q + 1)\). (See \([BCN]\), §9.4.)

**Proposition 3.2** Let \( \Gamma \) be the dual polar graph of type \( B_3(q) \). Fix a vertex \( \pi_0 \) of \( \Gamma \), and let \( \Delta \) be the subgraph of \( \Gamma \) induced on the collection of vertices disjoint from \( \pi_0 \). Then \( \Delta \) is isomorphic to the graph \( Z \) of Proposition 3.1. Its extended bipartite double \( \hat{\Delta} \) (or \( \hat{Z} \)) is isomorphic to the graph of Proposition 2.1.

**Proof:** Let \( V \) be a vector space of dimension 8 over \( \mathbb{F}_q \) (with basis \( \{e_1, \ldots, e_8\} \)), provided with the nondegenerate quadratic form \( Q(x) = x_1x_5 + x_2x_6 + x_3x_7 + x_4x_8 \). The point \( P = (0, 0, 0, 1, 0, 0, 0, -1) \) is nonisotropic, and \( P^\perp \) is the hyperplane \( H \) defined by \( x_4 = x_8 \). Restricted to \( H \) the quadratic form becomes \( Q(x) = x_1x_5 + x_2x_6 + x_3x_7 + x_4^2 \).

The \( D_4 \)-geometry on \( V \) has disjoint maximal totally isotropic subspaces \( E = \langle e_1, e_2, e_3, e_4 \rangle \) and \( F = \langle e_5, e_6, e_7, e_8 \rangle \). Fix \( E \) and consider the collection of all maximal totally isotropic subspaces disjoint from \( E \). This is precisely the collection of images \( F_A \) of \( F \) under matrices \( \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} \), where \( A \) is alternating with zero diagonal (cf. \([BCN]\), Proposition 9.5.1(i)). Hence, we can label the \( q^6 \) vertices \( F_A \cap H \) of \( \Delta \) with the \( q^6 \) matrices \( A \).

Two vertices are adjacent when they have a line in common, that is, when they are the intersections with \( H \) of maximal totally isotropic subspaces in \( V \), disjoint from \( E \), that meet in a line contained in \( H \). Let

\[
A = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix}.
\]

Then \( \det A = (af - be + cd)^2 \), and if \( \det A = 0 \) but \( A \neq 0 \), then \( \ker A \) has dimension 2, and is spanned by the four vectors \( (0, f, -e, d)^\top, (-f, 0, c, -b)^\top, (e, -c, 0, a)^\top, (-d, b, -a, 0)^\top \). Writing the condition that matrices \( A \) and \( A' \) belong to adjacent vertices we find the description of Proposition 3.1 if we take \( u = (c, e, f) \) and \( u' = (-d, b, -a) \). \( \square \)
4 History

In 1991 the second author constructed the graphs from Section 2 and the first author those from Section 3. Both were mentioned on the web page [ac], but not published thus far. These graphs have been called the Pasechnik graphs and the Brouwer-Pasechnik graphs, respectively, by on-line servers.

References

[BB] R. J. Blok & A. E. Brouwer, The geometry far from a residue, pp. 29–38 in: Groups and Geometries, L. di Martino, W. M. Kantor, G. Lunardon, A. Pasini, M. C. Tamburini (eds.), Birkhäuser Verlag, Basel, 1998.

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