IMPACT OF REORDER OPTION IN SUPPLY CHAIN COORDINATION

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Abstract. This paper studies the impacts of some reorder options on the performance as well as the coordination issues in a supply chain. A large category of products requires a long procurement lead time yet only has a relatively short selling season. Hence the purchase decisions usually have to be made well in advance of the opening of the sales. However, when uncertainty exists, the actual market demand may turn out to severely deviate from the initial order amount. To make up for the deficiency arising from this situation, a reorder option is introduced which renders a second manufacturing chance available shortly before the selling season. This reorder option facilitates an adjustment of the inventory level according to the realization of market demand. Since the market under investigation is facing a downward sloping demand curve, the effect of implementing this option is multi-fold. Moreover, the launch of the reorder option may also affect the decision makings at other levels of operations, such as altering the size of the initial order. Therefore, the overall impact of such option is not immediately clear. In this paper, it is shown that a properly designed reorder option is able to bring in profit growth and stabilize the fluctuations in the market retail price. Besides, quantity discount contracts are constructed to coordinate decisions on the initial inventory amount within the supply chain, so as to achieve higher economic efficiency. Finally, numerical examples are given to demonstrate the conclusions obtained in this paper.

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1. Introduction. A large class of consumer merchandise such as toys, fashion apparels and computer hardware, etc, is characterized by having a short selling season. While on the other hand, their production lead time is comparatively long, especially under the popular trend of outsourcing manufacturing to lower-cost countries for the sake of cost reduction. As a result of the long production period, retailers are often required by the suppliers to place initial orders well in advance of the actual sales, see for example, [10]. However, an important problem arises here is that since the initial commitments are made before the actual market demand is revealed, which may severely deviate from the original plan, but the risk from the uncertain market demand is mainly borne by the retailers. This problem is of particular importance for innovative or fashionable products, because these markets may turn out to be either very positive or the opposite while the trend is difficult to accurately predict long ahead of the selling season. As a consequence of the risk arising from the mismatch between the realized market demand and the order amount, the retailers’ incentives to participate in the business may be suppressed and economic deficiency in the supply chain occurs. In this paper, we study one possible solution which requires the supplier to share the risk with the retailer, and hence gives an incentive for the latter to purchase early.

Two relevant approaches are related to the study in this paper, which include options and the utilization of contracting tools to assist in coordinating the supply chains.

The launch of options in an operational context provides flexibilities at different levels of decision making, and these flexibilities are proved to be highly valuable in the presence of uncertainty. A large body of research investigates such function of options. For example, Dasu and Li [8] and Kogut and Kulatilaka [19] both study multinational cooperations who operate plants in several locations, and hence possess the flexibilities to shift production among these spots. Li and Kouvelis [21] investigate how to efficiently reduce the sourcing cost by properly designed supply chain contracts that provide retailer with flexibility either in terms of time, quantity or both when making their orders. Huang et al. [12] propose a real option approach to study an optimal inventory level of retail products. They apply an actuarial approach to value the profit structure of the retailer which can be represented in terms of the payoff of the real option. Wagner et al. [35] introduce real option valuation to quantify risks and uncertainties, which improves the efficiency and the performance of the supply chain departments. Liang et al. [22] apply the option contract mechanism into relief material supply chain management, and give a feasible price range of option contract such that members of relief material supply chain are profitable.

In this paper, we consider a reorder option contract offered by the supplier which gives the retailer a second chance to adjust the inventory size. With this contract, the retailer has the option to choose whether or not to acquire more products shortly before the sales when more information about the demand is available and most of the market uncertainty is resolved. Since this urgent additional order has to be filled within a short period, the supplier incurs a higher production cost. However, The reorder option is shown to be able to increase supply chain profit. In addition, it also has the effect of rendering the retail price less volatile. Such characteristics are realized in the following way. On one hand, the opportunity to place an additional order when unanticipated demand emerges decreases the risk of lost sales and helps the retailer capture the benefit from a high demand. On the other hand, because of
the provision of reordering, the retailer is able to protect profit from encountering a low market demand by keeping a lower amount of products at the beginning when the initial inventory decision is being made. Apart from securing profits, since reorder options have actually adjusted the inventory level according to the actual market demand and the demand curve is downward sloping, the retail price is now more stable.

In addition to investigating the influence of introducing reorder options, we also explore the coordination problem for the supply chain when such options are implemented. The motivation comes from the following facts. When decisions are made at an individual level with a view to maximizing private profits, the aggregate profit may be compromised due to possible conflicts of interests within the supply chain. However, since the total profit is to be shared by all agents, only when operational decisions are well coordinated and the overall supply chain performance is optimized, Pareto optimality can be achieved and every participant can be better off. Here we identify that when the initial decision on the inventory level is determined by retailers, which is usually the case in reality, they are going to order at a level that is lower than which is optimal for the supply chain as whole. Hence, the existence of the reorder option does not eliminate the “double marginalization” problem, (see, for example, [31]).

To deal with the above problem and hence further increase profit for the supply chain, contracting tools are often applied. There are extensive research works on various kinds of contracts that provide incentive to induce the retailers to make inventory decisions that are in alignment with those optimal ones from the viewpoint of the supply chain. Such contracts include: buyback contracts, as explored in [23] which are commitments made by the supplier to buy the leftover inventories at the end of the selling season at a pre-determined price; Krishnan et al. [18] and Taylor [32] both look into the sales rebate contracts which require the supplier to award a fixed payment to the retailers for every unit sold; Huang et al. [14] define an efficiency measure of the wholesale price contract. Based on this measure, supply chain participants can decide whether extra effort is worth or not. Revenue sharing contracts with which retailers pay supplier a certain portion of the revenue apart from the wholesale price that are analyzed in [7]. Furthermore, Palsule-Desai [24] conducts a study on revenue-dependent revenue sharing contracts and finds that there exist situations in which revenue-dependent contracts outperform revenue-independent contracts. To solve the problems induced by the leftovers from the initial sales and the consumer returns, Huang et al. [13] introduce a secondary market and show numerically that the total wholesale volume increases in the presence of a secondary market.

In this paper, we explore the application of a quantity discount contract to encourage a larger initial order from the retailer so as to improve supply chain profit. Specifically, this contract has a simple linear form, and hence the associated administration cost is low; better still, it is also shown to be able to achieve a large range of desired allocation of profit within the supply chain. The wholesale price specified in this quantity discount contract is decreasing in the order amount. Jeuland and Shugan [16] examine the function of such kind of contracts; Kolay et al. [20] discuss different types of quantity discounts. Huang et al. [11] apply the quantity discount contract to supply chain coordination for false failure returns. A general literature review of quantity discounts can be found in [9] and Wilson [34] offers an encompassing discussion of other non-linear pricing schemes. Donohue [5] focus on contracts
of the wholesale prices in two stages and the return price on the condition that the retail price and demand distribution are given. In their conclusions, they find that if new market information is highly predictable, then the supplier is able to get more profit. In particular, Burnetas and Ritchken [3] assume demand distributions depend on pricing decisions. They adopt an uncertain downward sloping demand curve to describe the relation between demand and price, which also changes with respect to the state of the market.

The study here contributes to the literature in several aspects. First, it investigates the application as well as the optimal design of reorder options to alleviate the risk arising from the unanticipated market demand so as to increase profit. Second, incentive contracts are formulated in this new context with reorders to coordinate the decisions on the initial order amount and hence further enhance the overall performance of the entire supply chain. The production system may experience sudden disruptions caused by some unexpected events such as machine break-downs, transportation failures and natural disasters. Some studies have performed for inventory models with disruptions, including Hishamuddin et al. [15] and Paul et al. [28], [29], [30]. Furthermore, a dynamic solution approach is provided by Paul et al. [27] to deal with multiple disruptions which may occur one after another as a series. Readers who are interested in the research domain of risk and disruption management in production inventory and supply chain systems can refer a review by Paul et al. [26]. In this paper, supply disruptions may happen at the initial order stage and (or) the reorder stage. Then it will be an interesting topic for our future research to investigate the impacts of supply disruptions on the profit of the supply chain and design a recovery strategy to decrease the additional costs to recover from the negative impacts of disruption.

This work is based on the model in Burnetas and Ritchken [3], however, the focuses are different. Since the call option conducted in [3] allows the retailer to choose the size of the reorder, attention is paid to the decision of the retailer with the call option. Although we consider the offering of reorder options with the function to assist the supplier to share demand risk with the retailer and hence improve profit, we focus on the optimal design of the option such that the overall supply chain profit can be optimized. More importantly, although Burnetas and Ritchken [3] point out that the performance in the decentralized case is worse than the one where decisions are coordinated, it provides no solution to articulate this problem. However, in our study, a quantity discount contract is also presented to reconcile profit conflicts and hence facilitates the achievement of better overall supply chain performance. This result also constitutes a significant contribution of this paper to the existing literature by bridging the research gap.

The rest of this paper is structured as follows. In Section 2 the model of the reorder option is constructed. Comparison is carried out to illustrate the impacts of this option. In addition, a detailed discussion concerning the design of the reorder is also presented. In Section 3 the coordination of the supply chain with the reorder contract is under investigation. Specifically, attention is focused on the derivation of a quantity discount contract to induce the retailer to make the initial order at the supply chain optimal level, which helps to further improves the chain-wide economic efficiency. Section 4 gives some numerical examples. Finally, concluding remarks and some future research issues are exhibited in Section 5.
2. The model. We consider a supply chain consisting of a single supplier and a single retailer in a one-period setting. In reality, there may be multiple suppliers and retailers in a dynamic environment. However, as in [3], the simple one-period paradigm with a single supplier and a single retailer may provide us with key insights into understanding the impacts of the reorder option on the supply chain. The retailer is facing a market with a downward sloping demand curve embedded with uncertainty which is only resolved at the end of the period. In this section, we restrict our attention to the decision-making in the centralized case. That is to find an optimal solution with a view to maximizing the total profit for the entire supply chain. The decentralized case will be discussed in the next section. To facilitate our discussion, in Table 1, we give the notations which will be used in the model.

| Notation | Description |
|----------|-------------|
| $P$ | market clearing price of the product |
| $Q$ | amount of products at time 0 with the reorder option |
| $Q^*$ | optimal amount of products at time 0 with the reorder option |
| $Q_0$ | amount of products at time 0 without the reorder option |
| $Q^*_0$ | optimal amount of products at time 0 without the reorder option |
| $\delta$ | the slope of the demand curve |
| $a_H$ ($a_L$) | the indicator of the market condition in the high (low) state |
| $e_H$ ($e_L$) | Arrow-Debreu state price for the high (low) state |
| $\rho$ | $\rho = e_H / e_L$ |
| $B$ | present value of the risk-free coupon that pays 1 dollar regardless of the state |
| $A$ | present value of the security that pays $a_H$ ($a_L$) in the high (low) state |
| $\mu$ | risk-neutral probability of the occurrence of the high (low) state |
| $\sigma^2$ | the variance of the uncertain factor $a$ |
| $c$ | the unit production cost of a product at time 0 |
| $c_1$ | the unit production cost of a product at time $T$ |
| $R$ | a pre-determined amount of products in the reorder option |

Let $Q$ be the amount of all products in stock available for sale. Then the market clearing price is given by

$$P = a - \delta Q$$  \hspace{1cm} (1)

where $\delta$ is a positive known constant. It is assumed that $a$ is a random variable whose value is realized at the end of the period. Such formulation for downward sloping demand curve has been widely used (see, for instance, [1, 3, 17]). The value of $a$ can be interpreted as an indicator of the market condition and it also reflects the uncertainty of the market demand, $(0 \leq p \leq 1)$:

$$a = \begin{cases} 
  a_H & \text{with probability } p \\
  a_L & \text{with probability } 1 - p.
\end{cases}$$  \hspace{1cm} (2)

The market is said to be in the high demand state if $a = a_H$; and it is in the low demand state if $a = a_L$. In this paper, for simplicity of discussion, the market demand is considered to be randomly distributed with two possible states. Similar assumption can be found in Burnetas and Ritchken [3]. In Paul et al. [25], they consider the problem of an imperfect production system under fuzzy demand. We remark that the market demand can be assumed to follow an uniform distribution or a normal distribution for our future research. Use $e_H$ and $e_L$ to denote the Arrow-Debreu state prices [2] for the high demand state and the low demand state, respectively:
• $e_H$ stands for the present value of the high-state security that pays 1 dollar if the high state occurs and 0 dollar otherwise;

• $e_L$ stands for the present value of the low-state security that pays 1 dollar if the low state occurs and 0 dollar otherwise;

Let $B$ be the present value of the risk-free bond that pays 1 dollar regardless of the state. Indeed, the value of $B$ can also be interpreted as the present value of a risk-free zero-coupon bond, which plays the role as a discount factor in the one-period model. Besides, we also assume there is a tradeable security, whose current value is denoted by $A$, to be traded in the market. This security pays $a_H$ in the high state and $a_L$ in the low state. The payments in all the securities described above are made at the end of the period.

Under the risk neutral probability concept, the expected growth rate of all traded securities equals the risk-free rate. Here $1/B$ can be considered as the risk-free rate. The current value of the high-state security is $e_H$ and its expected value at time $T$ under the risk neutral probability is $p_H$. Therefore with these values, we have $p_H/e_H = 1/B$ according to the concept of risk neutral probability. Then the risk-neutral probability of the occurrence of the high state can be calculated by

$$p_H = e_H/B.$$  \hfill (3)

Similarly,

$$p_L = e_L/B.$$  \hfill (4)

For ease of exposition, we denote

$$\rho = e_L/e_H = p_L/p_H.$$  \hfill (5)

Without taxes and transaction costs, the market of these securities is complete. Consequently, we can use the state securities to replicate the payoffs of the risk-free bond $B$ as well as the security $A$. Then refer to Baxter et al. [4], under the no-arbitrage principle, the law of one price implies the followings:

$$A = e_Ha_H + e_La_L \quad \text{and} \quad B = e_H + e_L.$$  \hfill (6)

This framework has been employed in Burnetas et al. [3]. The mean and variance of the uncertain factor $a$ in the market demand can be determined by $\mu = a_Hp_H + a_Lp_L$ and $\sigma^2 = (a_H - \mu)^2p_H + (a_L - \mu)^2p_L$. Under the risk-neutral probability given by Eq. (3) and Eq. (4), we simplify the expressions and get

$$\mu = \frac{A}{B} \quad \text{and} \quad \sigma^2 = \frac{e_He_L}{B^2}(a_H - a_L)^2.$$  \hfill (7)

We remark that with these settings, $A > c$, where $c$ is the unit production cost. Since the total amount of products $Q > 0$ and the factor $\delta > 0$, $a_H$ ($a_L$) is the highest possible unit market price of the product according to Eq. (1) with respect to the high (low) demand state. Therefore, $A$ is the present value of a unit of product sold at the highest possible market price, which should be above the unit production cost $c$ to assure a positive profit for the supply chain. We further assume that $a_HB > a_LB > c$. This means that even in the low demand state, the highest possible market price is still higher than the production cost.

2.1. The base case without the reorder option. In this section, we study the performance of the supply chain when the reorder option is absent. In this case, the following events happen in sequence. At the beginning of the period, which we refer to as time 0 afterwards, the supply chain makes the only decision on the inventory level and then starts the manufacturing immediately at a unit cost $c$. At
the end of the period, which we refer to time $T$ hereafter, the actual market demand is revealed and the supply chain releases all the available inventory to the market and sells the products out at the market clearing price. The time line of this case is shown in Figure 1. The result here serves as a benchmark for one to observe and study the effective impact brought by the reorder option in the later discussion.

Let $Q_0$ be the supply chain’s inventory level. Then the present value of the profit for the supply chain as a whole in this case without any reorder option available at time $T$, is given by

$$\Pi_0(Q_0) = e_H(a_H - \delta Q_0)Q_0 + e_L(a_L - \delta Q_0)Q_0 - cQ_0$$

$$= -\delta BQ_0^2 + (A - c)Q_0.$$  \hfill (8)

Note that in the first expression, the sum of the first two terms is the expected present value of the profit from the sales of the product. When the total production cost is subtracted, it gives the net profit. Since the Arrow-Debreu state prices are used in Eq. (8), the present value of the net profit is interpreted in the risk-neutral sense. This may not be unreasonable since an equilibrium framework for the market demand and market clearing price are considered here. The net profit of the supply chain here may be interpreted in a market equilibrium sense. The goal of optimizing the supply chain is to choose an order amount $Q_0$ such that the net profit is maximized.

Since $\Pi_0(Q_0)$ is a concave quadratic function of $Q_0$, we can find an optimal value of $Q$ by setting

$$\frac{\partial \Pi_0}{\partial Q_0} = -2\delta BQ_0 + (A - c) = 0.$$  \hfill (9)

Solving the above equation for $Q_0$, we have

$$Q_0^* = \frac{A - c}{2\delta B}.$$  \hfill (9)

and

$$\Pi_0^* = \Pi_0(Q_0^*) = \frac{(A - c)^2}{4\delta B}.$$
2.2. The case with the reorder option. In this section, we assume that the supply chain allows a reorder provision at the end of the period when the actual market demand is revealed. The option works in the following way. The first order is made at time 0, and the initial production starts straightaway at a unit cost $c$. At time $T$, the uncertainty in market demand is resolved and there is an option available for the supply chain to decide whether to place a reorder or not. If this reorder option is exercised, a pre-determined amount $R$ ($R > 0$) of products are then manufactured. However, since now only a very short manufacturing time is allowed, the unit production cost $c_1$ for fulfilling this reorder must be higher than $c$. We further assume that $c_1 B > c$, which reflects the fact that, generally, it is more beneficial to commit an early order and allow a longer procurement time. The time line of this case is given in Figure 2.

It should be optimal for the reorder option to be exercised only in the high demand state. This can be explained as follows. If the option is not exercised in either the high demand state or the low demand state, then this option needs not to be offered at all; On the other hand, if the option is exercised in both the high and the low demand state, more products are required to increase profit from the sales at time $T$. However, since the production cost is higher at time $T$ than at time 0, the supply chain should have manufactured more at time 0 instead of using the reorder option to satisfy the market demand. The last possibility is that the reorder option is exercised in the low demand state but not in the high demand state, but this is obviously sub-optimal because the market demands more in the high state than the low one.

With these assumptions, and we use $Q$ to denote the amount of the production at time 0, the profit of the supply chain in the presence of reorder option is then given by

$$
\Pi(Q) = e_H\{(a_H - \delta(Q + R))(Q + R) - Rc_1\} + e_L\{(a_L - \delta Q)Q - cQ\}
= -\delta BQ^2 + (A - c - 2R\delta e_H)Q + R(e_H(a_H - \delta R - c_1)).
$$

Note that the first term in the first expression represents the expected value of profit in the high state which is the proceeds from sales subtracted by the cost of the second production.
Since $\Pi(Q)$ is a concave quadratic function in $Q$, maximization gives the optimal value of $Q^*$:

$$Q^* = \frac{A - c - 2R\delta e_H}{2\delta B}. \quad (11)$$

Substituting $Q^*$ into the profit function Eq. (10) yields

$$\Pi^* = \Pi(Q^*) = \frac{(A - c - 2R\delta e_H)^2}{4B\delta} + Re_H(a_H - R\delta - c_1). \quad (12)$$

To ensure that $Q^* > 0$, we need the following condition:

$$R < \frac{(A - c)}{2\delta e_H}. \quad (13)$$

This restriction on $R$ can be explained as follows. By comparing Eq. (11) and Eq. (11), $Q_0^* > Q^*$. That is, in the presence of the reorder option, the supply chain has reduced the amount of the initial production. The reason is that the availability of a second chance of production has partially mitigated the shortage risk attributed to high sales. Hence the supply chain has a stronger incentive to decrease the inventory level at time 0 to help protecting profit in the case of low sales. Therefore, $Q^* < Q_0^*$. This can be regarded as the feedback effect of offering the reorder option. Furthermore, note that the option only allows the supply chain to choose between whether or not to reorder a fixed pre-determined amount of products. So if the size of the reorder $R$ is too large, the supply chain will cut back substantially on the initial order. Hence we have to impose a floor for the value of $R$ to make sure that the optimal initial order remains positive.

Furthermore, since it is easy to prove that $Q^* + R > Q_0^*$, the net effect of offering the reorder is the following. The supply chain has more products available for sale in the high state but less inventory released to the market in the low state, when compared to the case without the reorder. This makes intuitive sense. But note that since the market has a downward sloping demand curve, even in the high state, if too many products are on sale, the market price will decline. Consequently, it may decrease the total profit. Hence we also need $R$ to be not exceeding a certain amount to guarantee the growth in the net profit when the reorder is possible. We have the following lemma to address this problem and its proof can be found in Appendix.

**Lemma 2.1.** The supply chain profit is higher with the reorder option only when $R$ satisfies the following condition:

$$0 < R < \frac{-(Bc_1 - c) + B\sigma\sqrt{\rho}}{e_L\delta}. \quad (14)$$

To ensure the existence of such $R$, it is required that

$$\sigma > \frac{Bc_1 - c}{B\sqrt{\rho}}. \quad (15)$$

An interpretation is given for this result. From Eq. (15), we know that only when the market fluctuation is large enough, it is possible for the optimally designed reorder option to truly contribute to the growth of the supply chain total profit. Indeed, when the volatility of the market is high, there may be a need for the reorder option. Put it into more details, when the supply chain is aware of a second chance for order, it cuts back on the initial order amount as has been exhibited before, and supplement the high market demand by a reorder. But if the market is fairly stable,
the supply chain may then be keeping too low an inventory level in the low state while acting too aggressive in the other case. Eventually, the launch of the reorder, along with its feedback effect, actually has a negative impact on the supply chain total profit.

The reorder option is able to effectively alleviate fluctuations in retail market prices caused by the volatility of the market. To see this more clearly, recall that in the absence of the reorder, the amount of inventory available for sale is fixed to be $Q^*_0$ as in Eq. (9), regardless of market condition. Hence the gap of the market price is

$$\left( a_H - \delta Q^*_0 \right) - \left( a_L - \delta Q^*_0 \right) = a_H - a_L.$$ 

On the other hand, when the reorder option has been introduced, the level of inventory to meet the market demand becomes $Q^*$ in the low state and $Q^* + R$ in the high state. Hence, the difference in market prices is now narrowed down to

$$\left( a_H - \delta (Q^* + R) \right) - \left( a_L - \delta Q^* \right) = a_H - a_L - \delta R.$$ 

This means that the reorder option stabilizes prices while adjusting the availability of the commodity for sale.

### 2.2.1. The design of the reorder option.

As previously explained, the reorder option should be designed such that it is only optimal to be exercised in the case of high sales. This section aims to deal with this problem. Note that at time $T$, the only factor that matters when deciding whether to reorder or not is the comparative relationship between the proceeds from the sales and the possible cost of the second production. Besides, since the decision on the initial production size has already been made at time 0, the value of $Q$ and the cost of the initial production are both irrelevant here. Moreover, as proved above, $Q^*$ maximizes the supply chain total profit, so we take $Q = Q^*$ here as given. Firstly, we try to derive the condition for the reorder to be used in the case of high sales. By comparing Eq. (8) with Eq. (10), we require the following to be true:

$$\left( a_H - \delta Q^* \right)Q^* < \left( a_H - \delta (Q^* + R) \right)(Q^* + R) - Rc_1,$$

and

$$\left( a_L - \delta Q^* \right)Q^* > \left( a_L - \delta (Q^* + R) \right)(Q^* + R) - Rc_1.$$

We summarize the results into the following lemma and its proof is given in Appendix.

**Lemma 2.2.** Assume that Eq. (15) is satisfied. Then for the option to be only optimal for exercising in the high state, the size of reorder should satisfy:

- If $e_H > e_L$,
  $$0 < R < \frac{Bc_1 - c + B\sigma \frac{1}{\sqrt{\rho}}}{(\epsilon_H - \epsilon_L)\delta};$$

- If $e_H < e_L$,
  $$0 < R < \frac{Bc_1 - c - B\sigma \frac{1}{\sqrt{\rho}}}{(\epsilon_H - \epsilon_L)\delta};$$

- If $e_H = e_L$,
  $$0 < R.$$

Based on Lemma 2.1 and Lemma 2.2, we summarize the feasible range of $R$ in the following proposition.
Proposition 1. Suppose there is a reorder option which has the following three properties: 1) Its existence leads to an increase in supply chain total profit; 2) It ensures a positive amount of initial order; 3) It is optimal to be exercised when the market demand is in the high demand state at time $T$. This kind of reorder option exists if and only if the size of the reorder amount $R$ satisfies the following conditions.

Case I:

$$0 < R < \frac{-(Bc_1 - c) + B\sigma\sqrt{\rho}}{eL}\delta,$$

when

$$\frac{Bc_1 - c}{B\sqrt{\rho}} \sigma < \frac{(Bc_1 - c) + \frac{1}{2}\rho(A - c)}{B\sqrt{\rho}}.$$  

Case II:

$$0 < R < \frac{A - c}{2eH}\delta,$$

when

$$\sigma > \frac{(Bc_1 - c) + \frac{1}{2}\rho(A - c)}{B\sqrt{\rho}}.$$  

2.2.2. The optimal reorder size. Recall that with the reorder size being $R$, it is optimal for the supply chain to keep the initial inventory amount to be $Q^* = \left(\frac{A - c - 2R\delta eH}{2B}\right)/2B\delta$.

We substitute this into the supply chain profit function Eq. (10) and obtain

$$\Pi^* = \Pi(Q^*) = -\frac{\epsilon_HeL\delta}{B}R^2 + \frac{\epsilon_H}{B}\{(a_H - a_L)eL - (Bc_1 - c)\}R + \frac{(A - c)^2}{4B\delta}.$$  

This is a concave function of $R$. Maximization gives the optimal value of $R$ to be

$$(- (Bc_1 - c) + B\sigma\sqrt{\rho})/2eL\delta.$$  

This, combined with the results in Proposition 1, we have

$$R^* = \min \left\{ \frac{-(Bc_1 - c) + B\sigma\sqrt{\rho}}{2eL\delta}, \frac{A - c}{2eH\delta} \right\}.$$  

Direct computation gives the corresponding optimized supply chain profit to be either

$$\Pi \left( Q^* | R^* \right) = \frac{-(Bc_1 - c) + B\sigma\sqrt{\rho}}{2eL\delta} = \frac{[-(Bc_1 - c) + B\sigma\sqrt{\rho}]^2}{4B\delta\rho} + \frac{(A - c)^2}{4B\delta},$$

or

$$\Pi \left( Q^* | R^* \right) = \frac{A - c}{2eH\delta} = \frac{(A - c)[-(Bc_1 - c) + B\sigma\sqrt{\rho}]}{2B\delta} + \frac{(1 - \rho)(A - c)^2}{4B\delta}.$$  

Note that the constraint-free optimal solution is only achieved when the volatility of the market is high enough. This is because the reorder option is designed to provide the supply chain with a flexible inventory level to manage the risk arising from the uncertainty in the market demand. Hence when the market demand is more volatile, the benefit from the reorder option becomes more obvious. On the other hand, in a more stable market, the effect of the option is limited.
3. Coordination of the supply chain with reorder option. In this section, we focus on the decentralized case where the chronology of events is as follows. At time 0, the supplier acts first as a Stackelberg leader by setting the wholesale price scheme of the product, the retailer then follows by placing the initial order without knowing the actual market demand with a view to maximizing her own profit. The supplier starts the manufacturing right after the order. At time $T$, the actual market demand is revealed. The retailer decides whether or not to exercise the reorder option to buy a fixed amount $R$ of products. Here the value of $R$ is pre-determined by the suppliers. If the reorder option is to be exercised, the supplier begins the second manufacturing at time $T$. We assume the second manufacturing can be finished within a very short time and have the merchandise ready right before the market opens, but at the price of a higher unit production cost $c_1$. Finally, the market opens and the retailer sells out all the inventory in stock at the market clearing price.

We will show that when the decisions on the initial order level as well as the exercise of the reorder are both made by the retailer, then at time 0, the retailer is going to order an amount of products which is lower than the supply chain optimal level, and hence the decentralized case is sub-optimal. Then we proceed to construct one kind of linear quantity-discount contract for coordination and hence induce the retailer to make a supply chain optimal decision on the inventory level. To facilitate our discussion, a list of notations adopted in the following section is given in Table 2.

### Table 2. A List of Notations

| Symbol | Description |
|--------|-------------|
| $w$    | a fixed wholesale price which is higher than $c$ |
| $X$    | strike price which is pre-determined by the supplier |
| $Q_R$  | amount of products ordered by the retailer at time 0 with the reorder option |
| $Q^*_R$| optimal amount of products ordered by the retailer at time 0 with the reorder option |
| $Q_{max}$ | maximum size of products ordered by the retailer at time 0 with the reorder option |
| $a, b$ | parameters in the function of wholesale price $w$ |
| $\eta (\eta)$ | the portion of the maximized supply chain total profit earned by the retailer (supplier) |

3.1. The decentralized case. In this case, the retailer is charged a fixed wholesale price $w$ which is higher than the initial unit production cost $c$ for every unit initially ordered and the retailer also has to pay a strike price $X$ ($X > c_1$) pre-determined by the supplier, for every unit reordered at time $T$. We suppose now that the supplier sets the values of $R$ and $X$ such that it is optimal to only exercise reorder when high sales occurs as should be the more profitable case for the supply chain due to the reason explained in the previous section. A more detailed discussion about the value of $X$ will be presented in a later section.

Denote retailer’s order at time 0 as $Q_R$. The present value of the retailer’s profit will be
\[ \Pi_R(Q_R) = \epsilon_H ((a_H - \delta(Q_R + R))(Q_R + R) - RX) + \epsilon_L (a_L - \delta Q_R)Q_R - wQ_R \]

\[ = -\delta BQ_R^2 + (A - w - 2R\delta e_H)Q_R + Re_H (a_H - \delta R - X). \quad (23) \]

Maximization gives the optimal value of \( Q_R \) for the retailer to be

\[ Q_R^* = \frac{A - w - 2R\delta e_H}{2B\delta}. \quad (24) \]

Comparing Eq. (11) and Eq. (24), we conclude that \( Q_R^* < Q^* \) since \( w > c \). This shows the sub-optimality of this decentralized case because the retailer’s decision fails to maximize the supply chain total profit.

3.2. The linear quantity-discount contract. The sub-optimality exhibited in the last section arises because for the supply chain as a whole, the cost of per initial order is just the production cost \( c \), but the retailer is paying the unit wholesale price \( w \), and \( w > c \). Hence this higher cost deters the retailer from keeping a high enough inventory level at time 0 if a wholesale-price-only contract is offered by the supplier. In this subsection, our goal is to derive a price scheme that charges the retailer a unit wholesale price \( w(Q_R) \) which is linearly decreasing in the retailer’s order amount \( Q_R \) to provide an incentive for the retailer to act in the globally optimal way. We present Proposition 2 below and its proof can be found in Appendix.

**Proposition 2.** A linear quantity-discount wholesale price contract is able to induce the retailer to order at \( Q^* \) at time 0, and hence coordinates the supply chain and maximizes supply chain total profit if the unit wholesale price \( w(Q_R) \) which is linearly decreasing in retailer’s order amount \( Q_R \) satisfies

\[ w(Q_R) = -aQ_R + b \quad \text{for} \quad 0 \leq Q_R \leq Q_{\text{max}} < \frac{b}{a}, \quad (25) \]

with

\[ \frac{0}{a} < \frac{B\delta}{a} \quad \text{and} \quad b = \frac{a(A - c - 2R\delta e_H)}{B\delta} + c. \]

Here \( Q_{\text{max}} \) is the maximum possible order size.

Moreover, denote \( D = (A - c - 2R\delta e_H)^2 \), \( E = 4BR\delta e_H (a_H - R\delta - c) \) and \( F = 4BR\delta e_H (a_H - R\delta - X) \). If

\[ \frac{F}{D + E} < \eta < \frac{D + F}{D + E}, \quad (26) \]

set \( \eta = 1 - \eta \),

\[ a = \eta B\delta - \frac{4B^2\delta^2 e_H R (-a_H + \eta R\delta - \eta c + X)}{(A - c - 2Re_H\delta)^2} \quad \text{and} \quad b = \frac{a(A - c - 2R\delta e_H)}{B\delta} + c, \quad (27) \]

then with this contract, when the supply chain is coordinated, the retailer is actually earning a portion \( \eta \) of the maximized supply chain total profit and the remaining portion \( \eta \) of the maximized supply chain total profit will be allocated to the supplier.

Proposition 2 does not only specify the explicit form of a coordination linear quantity-discount contract but it also illustrates that the contract is powerful in assisting the allocation of total profit. Note that \( \eta \), which stands for the portion of total wealth directed to the retailer, is required to be not exceeding a ceiling value (see Eq. (26)). The only way to relax this restriction is for the supplier to set the wholesale price to be lower than the unit production cost \( c \), which is
obviously infeasible. The cause of it comes from the fact that when the reorder option is exercised, the retailer has to purchase an additional amount of products at the unit price $X$, which is higher than the cost $c_1$, that means another double marginalization exists. However, the quantity-discount contract only adjusts the wholesale price to address the double marginalization problem arising in the initial order. Hence it fails to achieve perfect coordination in the sense that retailer’s share of profit has to be kept within a certain level, although the contract is capable of realizing a large range of desired profit allocation.

Next, we give some conditions that should be satisfied by the strike price $X$ which is pre-determined by the supplier. When these restrictions of $X$ are satisfied, the retailer will be orchestrated to exercise the reorder option when the market demand turns out to be high, but give up the reorder in the low sales case, which helps to maximize profit due to the reasons explained before in the decentralized case.

We note that if it is optimal to use the option with high sales, then the following should be true:

$$(a_H - \delta(Q + R))(Q + R) - RX > (a_H - \delta Q)Q,$$

which is solved to be

$$X < a_H - \delta R - 2Q\delta. \quad (28)$$

Similarly, in order to deter the reorder option from being used in the low states, we have

$$(a_L - \delta(Q + R))(Q + R) - RX < (a_L - \delta Q)Q$$

which is then solved to be

$$X > a_L - \delta R - 2Q\delta. \quad (29)$$

Suppose that the coordination contract is already implemented and hence the retailer is induced to order exactly at $Q^*$, whose value is independent of $X$. And note that when the retailer is making the decision on whether to exercise the option, the initial order amount has already been made and hence it is no longer relevant here, so we can take $Q = Q^*$ as given and fixed. Substituting this into the above two inequalities and simplifying yield:

$$\frac{1}{B} \left( - B\sigma \frac{1}{\sqrt{\rho}} + (e_H - e_L)R\delta + c \right) < X < \frac{1}{B} \left( B\sigma \sqrt{\rho} + (e_H - e_L)R\delta + c \right). \quad (30)$$

Furthermore, $X > c_1$ should also be true, otherwise, the supplier has no incentive to provide any reorder opportunity for the retailer since it only brings the supplier non-positive income. But since $R$ has to satisfy Lemma 2.2, it shows that the left (right) part in Eq. (30) is always greater (less) than $c_1$. Therefore, we arrive at the following conclusions. The proof of the following proposition can be found in Appendix.

**Proposition 3.** In order for the retailer to only exercise the reorder option on condition that high sales occurs when the coordination contract is in force, the reorder size $R$ and the unit strike price $X$ at which the retailer can reorder more inventory at time $T$, both set by the supplier at time 0, should satisfy the following condition:

$$c_1 < X < \frac{1}{B} \left( B\sigma \sqrt{\rho} + (e_H - e_L)R\delta + c \right). \quad (31)$$
4. **Numerical examples.** This section aims to give some numerical examples and gain some insights from the examples. We adopt the following values of parameters: the security price, $A = 16$ dollars, the risk-free coupon price, $B = 0.8$ dollar, the ratio of state prices, $\rho = 4$, and the slope of the demand curve, $\delta = 1$. Similar assumptions can be found in Burnetas et al. [3]. Furthermore, the unit production prices are: $c = 4$ dollars at time 0 and $c_1 = 7$ dollars at time $T$. Then according to Eq. (5), Eq. (6) and Eq. (7), we have $e_L = 0.64$ dollar, $e_H = 0.16$ dollar and $\mu = 20$ dollars. Table 3 summarizes the values of the parameters.

| $A$     | $B$     | $c$     | $c_1$    | $\rho$ | $\delta$ |
|---------|---------|---------|----------|--------|---------|
| 16 dollars | 0.8 dollar | 4 dollars | 7 dollars | 4      | 1       |
| $e_L$   | $e_H$   | $\mu$  |
| 0.64 dollar | 0.16 dollar | 20 dollars |

**Example 1.** To illustrate the results in Proposition 1, we set $\sigma_1 = 10$ which satisfies Eq. (20) and $\sigma_2 = 25$ which satisfies Eq. (22). The right-hand-side in Eq. (19) is $(Bc_1 - c + \frac{1}{2} \rho (A - c)) = 22.5$ and the right-hand-side of Eq. (21) is $\frac{A - c}{2c_1 \delta} = 37.5$.

Case I: $\sigma_1 = 10$, then $a_L = 15$ and $a_H = 40$ according to Eq. (6) and Eq. (7). The results from Figures 3, 4 and 5 suggest that if the supplier sets $R$ in $(0, 22.5)$, then the total supply chain profit would increase, which agrees with Proposition 1.

Figure 3 shows that the total profit of the supply chain will increase by setting the reorder option with $R < 22.5$ units. Otherwise, if we choose the amount of products $R > 22.5$ in the reorder option, it implies that the supply chain will not benefit from the existence of the reorder option. Figure 4 shows that if $R$ ranges from 0 to 22.5, the optimal initial order $Q^*$ is always greater than 0. The value of $Q^*$ may be positive when $R > 22.5$, however, according to Figure 3, the supply chain will gain less profit with a reorder option than that without a reorder option. Thus, it is useless to design a reorder option for $R > 22.5$. Figure 5 shows that if $R$ taken from $(0, 22.5)$, it is optimal to exercise the option when the market demand is in the high demand state at time $T$. This is because if the market turns to be in high demand, exercising the reorder option is able to increase the incomes of the supply chain, while if it turns to be in low demand, exercising the reorder option would reduce the incomes of the supply chain.

Case II: $\sigma_2 = 25$, then $a_L = 7.5$ and $a_H = 70$ according to Eq. (6) and Eq. (7). Figures 6, 7 and 8 imply that $R$ determined by the supplier should be less than 37.5 units in this case. The results agree with Proposition 1.

Figure 6 shows that the total profit without a reorder option $\Pi_0(Q^*_0)$ is less than that with a reorder option $\Pi(Q^*)$ if the value of $R$ is set to be in the range of $(0, 37.5)$. If the reorder number $R$ is set to be above 37.5 units, the optimal initial order $Q^*$ would be negative, which is shown in Figure 6. That is impossible in reality, thus $R$ is required to be less than 37.5, which ensures that $Q^*$ is positive. When the supplier allows a reorder provision with the amount of products $R \in (0, 37.5)$ at time $T$, Figure 6 indicates that it is beneficial for the whole supply chain if the retailer exercises (drops) the option when the market demand is in high (low) demand. In Figure 6 it shows that if the market demand is in the high state, the retailer
exercising the reorder option is able to increase the incomes of the supply chain. If the market demand is in the low state, the retailer by dropping the reorder option, is able to reduce the loss of the supply chain.

\[ \Pi(Q^*) \]

\[ \Pi(Q^*) \]

**Figure 3.** Total profits with and without reorder option under $\sigma_1 = 10$.

\[ Q^* \]

\[ Q^* \]

**Figure 4.** Optimal initial order $Q^*$ with a reorder option under $\sigma_1 = 10$.

**Example 2.** To illustrate the results in Proposition 2, let $\sigma = 10$, the predetermined amount of products $R$ in the reorder option to be 15 units and the strike price $X$ to be 9 dollars, which satisfies $X > c_1$. Set the parameter $a$ in Eq. (25) to be 0.4, which satisfies $0 < a < B\delta$, then $b = 7.6$ according to the function of $b$ in Proposition 2.

Figure 4 gives the supply chain’s total profit $\Pi(Q)$ defined in Eq. (10) and the retailer’s profit $\Pi_R(Q_R)$ defined in Eq. (24), with respect to the amount of products
ordered at time 0. It implies that both $\Pi(Q)$ and $\Pi_R(Q_R)$ are concave functions of the initial order amount of products. The two curves reach their highest values at the same point of $Q = Q_R = 4.5$ units. This result suggests that in order to maximize the profit of the whole supply chain with a reorder option, the initial amount of products would be chosen as $Q^* = 4.5$ units. The retailer, who has the linear quantity-discount wholesale price contract with the supplier, would order 4.5 units products at time 0 to maximize his profit. Thus the linear quantity-discount wholesale price contract is able to introduce the retailer to order the same amount of products $Q_R^*$ as $Q^*$ at time 0. This result shows that the contract promotes the coordination of the supply chain successfully.
Furthermore, according to Eq. (26) in Proposition 2, we set the parameter $\eta$ to range from 0.65 to 0.92, and the parameters $a$ and $b$ are determined by Eq. (27). In Figure 10, $\Pi^*$ is the maximized supply chain total profit defined in Eq. (12), $\Pi_R^*$ is the maximized profit of the retailer given in Eq. (23), and $\Pi_S^* = \Pi^* - \Pi_R^*$ is the profit of the supplier. It shows that $\frac{\Pi^*_S}{\Pi^*_R} = \eta$ and $\frac{\Pi^*_R}{\Pi^*_R} = 1 - \eta$, which implies that the allocation of the maximized supply chain total profit depends on the value of $\eta$ under the linear quantity-discount contract. The retailer earns a portion $\eta$ of the maximized supply chain total profit and the supplier gets the rest of the profit.

**Example 3.** Set $\sigma = 10$ and $R = 15$ units, which are same to Example 2. We keep other parameters constant and change the strike price $X$. Since the strike price $X$
should be higher than the unit production cost $c_1$, $X$ is assumed to be greater than 7 dollars.

Figure 11 shows that it is optimal for the retailer to exercise the reorder option at the high demand rate when the strike price $X$ belongs to $(7, 16)$. If $X > 16$ and the demand rate is $a_H$, exercising the option would decrease the incomes of the retailer. Thus, in this case, in order to design a reorder option which is able to introduce the retailer to exercise the option only at the high demand rate, the supplier is suggested to set the value of $X$ in $(7, 16)$, which can also be found by using Eq. (31). This result agrees with Proposition 3.

![Figure 9](image-url)  
**Figure 9.** Supply chain’s total profit and the retailer’s profit with respect to $Q$ and $Q_R$, respectively.

![Figure 10](image-url)  
**Figure 10.** Distribution of the maximised supply chain total profit between the retailer and the supplier with respect to $\eta$. 
5. Conclusions. This paper considers the performance of a supply chain consisting of a single retailer and a single supplier in a market facing downward sloping demand curve with uncertainty. Since the initial order of inventory has to be placed well in advance of the start of the selling season because the production takes a long lead time, it is possible that the actual market demand level turns out to substantially deviate from the level of the procured inventory and causes damage to the profit. To deal with this problem, we consider opening up a second manufacturing chance with a fixed size shortly before the sales when most of the uncertainty in market demand is revealed. However, such second production is costing more because only a short lead time is allowed, and hence the satisfaction of such additional order comes at the price of higher-paid extra working hours, more expensive raw materials and more severely depreciation of the machines, etc. It is demonstrated that such reorder option benefits the supply chain in multiple aspects. First of all, the options is able to bring in profit growth. Moreover, it assists in stabilizing the fluctuation of market retail price. Next, our research efforts are devoted to the optimal design of such reorder options to best increase supply chain wealth.

Another contribution of our study is the derivation of a linear quantity discount contract to align interests among the supply chain in making the decision on the initial inventory level so as to induce the retailer to order exactly at such an amount that maximizes the supply chain profit. Such quantity discount contract also has other merits apart from coordination. Firstly, it specifies a wholesale price which is linearly decreasing in the order amount, such simple form of the contract term incurs a low administration cost. More importantly, it is proved that this coordination contract is capable of achieving a large range of desired allocation of total profit within the supply chain.

The analysis in this paper may be enriched in several directions. Firstly, the impacts of reorder options and the corresponding coordination issues are explored with respect to a supply chain consisting of only one supplier and one retailer. However, if the analysis can be generalized to supply chains with multiple retailers, then the competition and other possible strategic interactions can be properly taken into account and therefore provide more in-depth insights of the problem. Secondly,
the reorder options here can be further modified to deal with more complicated problems. For example, we have assumed the reorder time point to be fixed with a higher production cost, yet in the reality, the production cost may actually evolve with time in a more sophisticated manner. Therefore, it may be possible to integrate more time flexibility into the contract terms to balance the benefit from reducing the cost and the benefit from getting more information of the uncertain market demand by delaying the decision. Another possibility for future research is to investigate the design of similar reorder options while relaxing the assumption that the actual market demand can only turn out to be either one of the two states. One may allow the market to have continuously fluctuating demand, and then how to design reorder options for multi-period model will be an interesting topic for our future research.

6. Appendix.

6.1. Proof of Lemma 2.1.

Proof. The condition for the supply chain to be better off with the reorder is

\[ 0 < \Delta \Pi = \Pi^* - \Pi_0^* = (A - c - R\delta e_H) - \Pi_0(Q_0^*) - \Pi_0(Q_0^*)^2, \]

where

\[ \Pi_0(Q_0^*) = aH B - R\delta B - Bc_1 - A + c + R\delta e_H \]

\[ = \frac{eH R}{B}(aH + aL + c) \]

\[ = \frac{eH R}{B}(-R\delta e_L + (aH - aL)e_L - (Bc_1 - c)), \]

Since \( e_H, R \) and \( B \) are all positive, the above inequality is equivalent to

\[ -R\delta e_L + (aH - aL)e_L - (Bc_1 - c) > 0, \]

by making use of Eq. (7), the above is then equivalent to

\[ R < \frac{(Bc_1 - c) + B\sigma}{e_L \delta}. \]

To ensure the existence of \( R > 0 \), it is required that \( -(Bc_1 - c) + B\sigma \sqrt{\rho} > 0 \), that is

\[ \sigma > \frac{Bc_1 - c}{B\sqrt{\rho}}. \]

This completes the proof. \( \square \)

6.2. Proof of Lemma 2.2.

Proof. As proved in Section 3.2, \( Q^* \) in Eq. (11) maximizes the supply chain total profit, so we take \( Q = Q^* \) here as given. Firstly, we try to derive the condition for the reorder to be used in the case of high sales. By comparing Eq. (8) with Eq. (10), we require the following to be true:

\[ (a_H - \delta Q^*)Q^* < (a_H - \delta(Q^* + R))(Q^* + R) - Rc_1. \]
It is simplified as follows.

\[
\begin{align*}
\delta R & < a_H - 2\delta \left(\frac{A - c - 2R\delta e_H}{2B\delta}\right) - c_1 \\
R\delta e_H + R\delta e_L & < a_H e_H + a_H e_L - a_He_H - a_le_L + c + 2R\delta e_H - Bc_1 \\
Bc_1 - c - (a_H - a_L)e_L & < (e_H - e_L)R\delta \\
Bc_1 - c - B\sigma \sqrt{\rho} & < (e_H - e_L)R\delta \\
\end{align*}
\]

In the above, we have made use of Eq. (6), Eq. (7) and Eq. (11).

Similarly, for the reorder to be optimally not exercised in the low state, from Eq. (8) and Eq. (10) we have

\[
(a_L - \delta Q^*)Q > (a_L - \delta(Q^* + R))(Q^* + R) - Rc_1.
\]

Simplifications with Eq. (6), Eq. (7) and Eq. (11) give

\[
\begin{align*}
\delta R & > a_L - 2\delta \left(\frac{A - c - 2R\delta e_H}{2B\delta}\right) - c_1 \\
Bc_1 - c + (a_H - a_L)e_H & > (e_H - e_L)R\delta, \\
Bc_1 - c + B\sigma \frac{1}{\sqrt{\rho}} & > (e_H - e_L)R\delta.
\end{align*}
\]

Assume that Eq. (15) is satisfied, that is

\[
\sigma > \frac{Bc_1 - c}{B\sqrt{\rho}}.
\]

Therefore, if \(e_H > e_L\),

\[
\frac{Bc_1 - c - B\sigma \sqrt{\rho}}{(e_H - e_L)\delta} < R < \frac{Bc_1 - c + B\sigma \frac{1}{\sqrt{\rho}}}{(e_H - e_L)\delta}.
\]

Since \(\frac{Bc_1 - c - B\sigma \sqrt{\rho}}{(e_H - e_L)\delta} < 0\), thus

\[
0 < R < \frac{Bc_1 - c + B\sigma \frac{1}{\sqrt{\rho}}}{(e_H - e_L)\delta}.
\]

If \(e_H < e_L\),

\[
\frac{Bc_1 - c + B\sigma \frac{1}{\sqrt{\rho}}}{(e_H - e_L)\delta} < R < \frac{Bc_1 - c - B\sigma \sqrt{\rho}}{(e_H - e_L)\delta}.
\]

Since \(\frac{Bc_1 - c + B\sigma \frac{1}{\sqrt{\rho}}}{(e_H - e_L)\delta} < 0\), thus

\[
0 < R < \frac{Bc_1 - c - B\sigma \sqrt{\rho}}{(e_H - e_L)\delta}.
\]

If \(e_H = e_L\),

\[
0 < R.
\]

This completes the proof.\(\Box\)
6.3. Proof of Proposition 1

Proof. If \( e_H > e_L \),

\[
\begin{align*}
\frac{Bc_1 - c + B\sigma \sqrt{\rho} - (Bc_1 - c) + B\sigma \sqrt{\rho}}{(e_H - e_L) \delta} &= \frac{e_L \delta}{(Bc_1 - c)e_L + (a_H - a_L)e_L + (Bc_1 - c)e_H - (Bc_1 - c)e_L - (a_H - a_L)e_L + (a_H - a_L)e_L^2}{(e_H - e_L) \delta} \\
&= \frac{1}{(e_H - e_L) \delta } \left( (a_H - a_L)e_L^2 + (Bc_1 - c)e_H \right) > 0.
\end{align*}
\]

Hence Eq. (16) can be replaced by Eq. (14).

If \( e_H < e_L \),

\[
\begin{align*}
\frac{Bc_1 - c - B\sigma \sqrt{\rho} - (Bc_1 - c) + B\sigma \sqrt{\rho}}{(e_H - e_L) \delta} &= \frac{e_L \delta}{(Bc_1 - c)e_L + (a_H - a_L)e_L + (Bc_1 - c)e_H - (Bc_1 - c)e_L - (a_H - a_L)e_L + (a_H - a_L)e_L^2}{(e_H - e_L) \delta} \\
&= \frac{1}{(e_H - e_L) \delta } \left( (a_H - a_L)e_L^2 + (Bc_1 - c)e_H \right) > 0.
\end{align*}
\]

Hence Eq. (17) can be replaced by Eq. (14).

If \( e_H = e_L \), Eq. (18) can be replaced by Eq. (14).

In addition, we have

\[
\frac{-(Bc_1 - c) + B\sigma \sqrt{\rho}}{e_L \delta} \cdot \frac{A - c}{2e_H \delta} = \frac{1}{2e_H \delta} e_L \delta \left( -2e_H (Bc_1 - c) + 2e_H B\sigma \sqrt{\rho} - e_L (A - c) \right) > 0
\]

when

\[
\sigma > \frac{(Bc_1 - c) + \frac{1}{2} \rho (A - c)}{B \sqrt{\rho}}.
\]

And the above difference in Eq. (32) is negative otherwise, combine this with Eq. (15), we arrive at the conclusion in Case II. \( \square \)

6.4. Proof of Proposition 2

Proof. With this wholesale price scheme, the retailer’s profit becomes

\[
\begin{align*}
\Pi_R(Q_R) &= e_H \left\{ (a_H - \delta(Q_R + R))(Q_R + R) - RX \right\} + e_L \left\{ a_L - \delta Q_R \right\} Q_R \\
&\quad - (-aQ_R + b)Q_R \\
&= -\delta BQ_R^2 + (A - 2R\delta e_H)Q_R + Re_H(a_H - \delta R - X) + aQ_R^2 \\
&\quad - \left( \frac{a(A - c - 2R\delta e_H)}{B\delta} + c \right) Q_R \\
&= -(B\delta - a)Q_R^2 + (B\delta - a) \left\{ \frac{A - c - 2R\delta e_H}{B\delta} Q_R + Re_H(a_H - \delta R - X) \right\}.
\end{align*}
\]

Since \( 0 < a < B\delta \), the above profit function is a concave quadratic function of \( Q_R \), hence the maximizing solution is

\[
Q^*_R = \frac{A - c - 2R\delta e_H}{2B\delta} = Q^*.\]

This means that the linearly discounted quantity-discount contract succeeds in coordinating the supply chain.
In addition, when the contract with Eq. (27) is in force, the retailer, after optimizing its order by choosing $Q^*_R(Q_{RC})$, earns a profit of 

$$
\Pi^*_R = \Pi_R(Q_{RC}) = -(B\delta - a) \left( \frac{A - c - 2R \delta e_H}{2B\delta} \right)^2 + (B\delta - a) \left( \frac{A - c - 2R \delta e_H}{2B^2\delta^2} \right)^2 
+ Re_H(a_H - \delta R - X) 
+ e_H R(a_H - R\delta - X) 
\eta B\delta - 4B^2\delta^2 e_H R(-\eta a_H + \eta R\delta - \eta c_1 + X) 
\left( A - c - 2R \delta e_H \right)^2
\right) 
\eta \Pi^*. 
$$

Similarly, it can be proved that the supplier’s profit

$$
\Pi^*_S = \Pi_S(Q^*_R) = e_H R(X - c_1) + Q^*_R(w(Q^*_R) - c) = \eta \Pi^*. 
$$

Moreover, $a > 0$ should be guaranteed such that the wholesale price is indeed decreasing in the ordered amount, which then makes economic sense. By Eq. (27), set

$$
\frac{\eta B\delta - 4B^2\delta^2 e_H R(-\eta a_H + \eta R\delta - \eta c_1 + X)}{\left( A - c - 2R \delta e_H \right)^2} > 0,
$$

solving it yields

$$
\eta < \frac{(A - c - 2R \delta e_H)^2 + 4BR \delta e_H(a_H - R\delta - X)}{(A - c - 2R \delta e_H)^2 + 4BR \delta e_H(a_H - R\delta - c_1)},
$$

with the right hand side less than 1 because $X > c_1$. In addition,

$$
\frac{4BR \delta e_H(a_H - R\delta - R - c_1)}{(A - c - 2R \delta e_H)^2 + 4BR \delta e_H(a_H - R\delta - c_1)}
$$

assures that $a < B\delta$. This completes the proof. $\square$

### 6.5. Proof of Proposition 3

**Proof.** If $e_H > e_L$, then

$$
\frac{1}{B} \left( Bc_1 - c + B\sigma \sqrt{\rho} + (e_H - e_L) \right) R\delta + c > c_1,
$$

since

$$
\sigma > \frac{Bc_1 - c}{B\sqrt{\rho}} \quad \text{and} \quad R > 0.
$$

$$
\frac{1}{B} \left( -B\sigma \frac{1}{\sqrt{\rho}} + (e_H - e_L) \right) R\delta + c < c_1,
$$

since

$$
0 < R < \frac{Bc_1 - c + B\sigma}{(e_H - e_L)\delta}.
$$

If $e_H < e_L$, then

$$
\frac{1}{B} \left( B\sigma \sqrt{\rho} + (e_H - e_L) \right) R\delta + c > c_1,
$$

since

$$
0 < R < \frac{Bc_1 - c - B\sigma \sqrt{\rho}}{(e_H - e_L)\delta}.
$$
\[
\frac{1}{B} \left( -B \sigma \frac{1}{\sqrt{\rho}} + (e_H - e_L) R \delta + c \right) < c_1,
\]

since
\[
c < Bc_1 \quad \text{and} \quad R > 0.
\]

If \( e_H = e_L \), then
\[
\frac{1}{B} \left( B \sigma \sqrt{\rho} + (e_H - e_L) R \delta + c \right) > c_1,
\]

since
\[
\sigma > \frac{Bc_1 - c}{B \sqrt{\rho}}.
\]

\[
\frac{1}{B} \left( -B \sigma \frac{1}{\sqrt{\rho}} + (e_H - e_L) R \delta + c \right) < c_1,
\]

since
\[
c < Bc_1.
\]

Therefore, by summarising these discussions, the conclusion in Proposition is obtained,
\[
c_1 < X < \frac{1}{B} \left( B \sigma \sqrt{\rho} + (e_H - e_L) R \delta + c \right).
\]

This completes the proof. \( \square \)

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