GEOMETRIC REALIZATIONS OF PARA-HERMITIAN CURVATURE MODELS

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Abstract. We show that a para-Hermitian algebraic curvature model satisfies the para-Gray identity if and only if it is geometrically realizable by a para-Hermitian manifold. This requires extending the Tricerri-Vanhecke curvature decomposition to the para-Hermitian setting. Additionally, the geometric realization can be chosen to have constant scalar curvature and constant ⋆-scalar curvature.

This paper is dedicated to the memory of Professor Katsumi Nomizu

1. Introduction

1.1. Hermitian geometry. Let g be a Riemannian metric on a smooth manifold M of dimension 2n. Let J give (M, g) an almost Hermitian structure. This means that J is an almost complex structure on the tangent bundle which is compatible with g, i.e. \( J^2 = -\text{id} \) and \( J^* g = g \). We say that the almost Hermitian manifold \( M := (M, g, J) \) is Hermitian if J is integrable, i.e. if the Nijenhuis tensor vanishes or, equivalently, there exist local coordinates \( (x_1, \ldots, x_n, y_1, \ldots, y_n) \) centered at any given point of the manifold so that
\[
J \partial_{x_i} = \partial_{y_i} \quad \text{and} \quad J \partial_{y_i} = -\partial_{x_i}.
\]
We refer to [5] for further details.

The Riemann curvature tensor
\[
R(x, y) := \nabla_x \nabla_y - \nabla_y \nabla_x - \nabla_{[x,y]}
\]
of the Levi-Civita connection [6] satisfies:
\[
\begin{align*}
R(x, y, z, w) + R(y, z, x, w) + R(z, x, y, w) &= 0, \\
R(x, y, z, w) - R(y, x, z, w) &= R(z, w, x, y).
\end{align*}
\]
Gray [4] showed that there is an additional identity, which is called the Gray identity, which is satisfied by the curvature tensor of any Hermitian manifold:
\[
\begin{align*}
0 &= R(x, y, z, w) + R(Jx, Jy, Jz, Ju) - R(Jx, Jy, z, w) \\
&- R(Jx, y, Jz, w) - R(Jx, y, z, Ju) + R(Jy, Ju, Jz, w) \\
&- R(x, Jy, z, Ju) - R(x, y, Ju, Jz, w).
\end{align*}
\]

All universal curvature symmetries for Hermitian manifolds are generated by the relations of Equations (1.a) and (1.b). By contrast, there are no additional symmetries beyond those of Equation (1.a) in the almost Hermitian context. One can make this statement precise as follows. Let \( \langle \cdot, \cdot \rangle \) be a positive definite inner product on a real vector space V of dimension 2n. Let J be a Hermitian complex structure on V, and let \( \bar{J} \) be its conjugate. Then the following holds:
\[
\begin{align*}
0 &= R(x, y, z, w) + R(Jx, Jy, Jz, Ju) - R(Jx, Jy, z, w) \\
&- R(Jx, y, Jz, w) - R(Jx, y, z, Ju) + R(Jy, Ju, Jz, w) \\
&- R(x, Jy, z, Ju) - R(x, y, Ju, Jz, w).
\end{align*}
\]
structure on $V$; $J^2 = -\text{id}$ and $J^* \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle$. Let $A \in \otimes^4 V^*$ be an algebraic curvature tensor, i.e. $A$ satisfies the symmetries of Equation (1.a). Let

$$C := (V, \langle \cdot, \cdot \rangle, J, A)$$

be the associated Hermitian curvature model. We say that $C$ is geometrically realized by an almost Hermitian manifold $M = (M, g, J)$ if there is an isomorphism $\phi : V \to T_P M$ for some $P \in M$ so that $\phi^* g_P = \langle \cdot, \cdot \rangle$, $\phi^* J_P = J$, and $\phi^* R_P = A$.

We refer to [1, 2] for the proof of the following result:

**Theorem 1.1.** Let $C$ be a Hermitian curvature model.

1. $C$ is always geometrically realized by an almost Hermitian manifold.
2. $C$ is geometrically realized by a Hermitian manifold if and only if $C$ satisfies Equation (1.b).

There are analogous questions in the affine setting. For example, if $\nabla$ is both holomorphic and affine Kähler, then $R = 0$ and $\nabla$ is locally flat [7].

**1.2. Para-Hermitian geometry.** Let $(\tilde{M}, \tilde{g})$ be a pseudo-Riemannian manifold of dimension $2n$. Let $\tilde{J}$ give $(\tilde{M}, \tilde{g})$ an almost para-Hermitian structure; $\tilde{J}^2 = \text{id}$ and $\tilde{J}^* \tilde{g} = -\tilde{g}$. In this setting, necessarily $\tilde{g}$ has neutral signature $(n, n)$. The almost para-Hermitian manifold

$$\tilde{M} := (\tilde{M}, \tilde{g}, \tilde{J})$$

is said to be para-Hermitian if $\tilde{J}$ is integrable, i.e. if the Nijenhuis tensor $N_{\tilde{J}}$ vanishes (see, for instance, [3]), where

$$N_{\tilde{J}}(x, y) := [x, y] - \tilde{J}[\tilde{J}x, y] - \tilde{J}[x, \tilde{J}y] + [\tilde{J}x, \tilde{J}y].$$

Equivalently, there exist local coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ centered at any given point of $\tilde{M}$ so that $\tilde{J} \partial_{x_i} = \partial_{y_i}$ and $\tilde{J} \partial_{y_i} = \partial_{x_i}$.

In the algebraic setting, let $\langle \cdot, \cdot \rangle$ be a neutral signature inner product on a finite dimensional vector space $\tilde{V}$. Let $\tilde{J}$ be a para-Hermitian structure on $(\tilde{V}, \langle \cdot, \cdot \rangle)$, i.e. $\tilde{J}^2 = \text{id}$ and $\tilde{J}^* \langle \cdot, \cdot \rangle = -\langle \cdot, \cdot \rangle$. If $\tilde{A} \in \otimes^4 \tilde{V}^*$ is an algebraic curvature tensor, let

$$\tilde{C} := (\tilde{V}, \langle \cdot, \cdot \rangle, \tilde{J}, \tilde{A})$$

be the corresponding para-Hermitian curvature model. We change the signs in Equation (1.b) to define a corresponding para-Gray relation

$$(1.c) \quad 0 = \tilde{A}(x, y, z, w) + \tilde{A}(\tilde{J}x, \tilde{J}y, \tilde{J}z, \tilde{J}w) + \tilde{A}(\tilde{J}x, \tilde{J}y, z, w) + \tilde{A}(\tilde{J}x, \tilde{J}y, \tilde{J}z, w) + \tilde{A}(x, \tilde{J}y, \tilde{J}z, \tilde{J}w) + \tilde{A}(x, \tilde{J}y, z, \tilde{J}w) + \tilde{A}(x, \tilde{J}y, \tilde{J}z, w) + \tilde{A}(x, y, \tilde{J}z, \tilde{J}w).$$

Assertion (1) in the following Theorem was established in [1]; Assertion (2) is the main new result of this paper:

**Theorem 1.2.** Let $\tilde{C}$ be a para-Hermitian curvature model.

1. $\tilde{C}$ is always geometrically realized by an almost para-Hermitian manifold.
2. $\tilde{C}$ is geometrically realized by a para-Hermitian manifold if and only if $\tilde{C}$ satisfies Equation (1.c).

**Remark 1.3.** We make the following observations:

1. The results of [1] show that the manifolds in Theorems 1.1 and 1.2 can be chosen to have constant scalar curvature and constant $\ast$-scalar curvature.
2. The methods we will develop to establish Theorem 1.2 (2) can be used to show that Theorem 1.1 holds for pseudo-Riemannian manifolds; it is not necessary to assume that the inner product is positive definite.
We have
\[ \{ \xi_1, \ldots, \xi_n \} \]
Let indices \( \xi_1, \ldots, \xi_n \) range from 1 to 2

1.3. Outline of the paper. Here is a brief outline to the paper. In Section 2, we will show that the curvature tensor of any para-Hermitian manifold satisfies Equation (1.3) and thereby establish one implication of Theorem 1.2 (2). Rather than generalizing Gray’s proof from the Hermitian to the para-Hermitian setting, we have chosen to give a direct proof which is quite different in flavor. In Section 3, we recall the Tricerri-Vanhecke [8] decomposition of the space of algebraic curvature tensors in the Hermitian setting and extend it to the para-Hermitian setting by complexification; this result is perhaps of interest in its own right. In Section 4, we linearize the problem. We define a linear subspace \( \mathfrak{P} \) of the space of all algebraic curvature tensors which is invariant under the para-unitary structure group such that any element of \( \mathfrak{P} \) can be realized by a para-Hermitian metric with vanishing Kaehler form at the point in question. We complete the proof of Theorem 1.2 (2) in Section 5 by showing that the elements of \( \mathfrak{P} \) are precisely those algebraic curvature tensors which satisfy the para-Gray identity given in Equation (1.3).

2. The para-Gray identity for para-Hermitian manifolds

Let \( \tilde{J} \) be a para-Hermitian structure on \((\tilde{V}, \langle \cdot, \cdot \rangle)\). Let \( \{ \tilde{e}_a \} \) be a basis for \( \tilde{V} \). If \( \tilde{T} \in \otimes^4 \tilde{V}^* \), we define the para-Gray symmetrization
\[
\tilde{G}(\tilde{T})(\tilde{e}_a, \tilde{e}_b, \tilde{e}_c, \tilde{e}_d) := \tilde{T}(\tilde{e}_a, \tilde{e}_b, \tilde{e}_c, \tilde{e}_d) + \tilde{T}(\tilde{e}_a, \tilde{e}_b, \tilde{e}_c, \tilde{e}_d) + \tilde{T}(\tilde{e}_a, \tilde{e}_b, \tilde{e}_c, \tilde{e}_d).
\]

We establish one implication of Theorem 1.2 (2) by showing:

**Theorem 2.1.** If \( \tilde{M} = (\tilde{M}, \tilde{g}, \tilde{J}) \) is a para-Hermitian manifold, then \( \tilde{G}(\tilde{R}) = 0 \).

**Proof.** Introduce coordinates \((u_1, \ldots, u_{2n})\) on \( \tilde{M} \) so that \( \tilde{J} \partial_{u_1} = \partial_{u_{n+1}}, \ldots, \tilde{J} \partial_{u_n} = \partial_{u_{2n}}, \tilde{J} \partial_{u_{n+1}} = \partial_{u_1}, \ldots, \tilde{J} \partial_{u_{2n}} = \partial_{u_n} \).

We shall let indices \( a, b, c, \ldots \) range from 1 to 2n and index the coordinate frame \( \{ \xi_1, \ldots, \xi_{2n} \} := \{ \partial_{u_1}, \ldots, \partial_{u_{2n}} \} \). We also let indices \( \alpha, \beta, \gamma, \ldots \) range from 1 to 2n. Let
\[
\tilde{g}_{ab} := \tilde{g}(\xi_a, \xi_b), \quad \tilde{g}_{\alpha\beta} := \tilde{g}(\tilde{J}\xi_a, \tilde{J}\xi_b), \quad \tilde{g}_{a\beta} := \tilde{g}(\xi_a, \tilde{J}\xi_b), \quad \tilde{g}_{ab} := \tilde{g}(\tilde{J}\xi_a, \xi_b).
\]

We have \( \tilde{g}_{ab} = -\tilde{g}_{\alpha\beta} \) and \( \tilde{g}_{a\beta} = -\tilde{g}_{ab} \). Let \( \tilde{g}^{ab} \) be the inverse matrix. We adopt the Einstein convention and sum over repeated indices. Let “\( \tilde{\cdot} \) denote ordinary partial differentiation. Let \( \tilde{\Gamma} \) be the Christoffel symbols of the Levi-Civita connection. We compute:
\[
\tilde{\Gamma}_{abc} = \frac{1}{2} (\tilde{g}_{bc/a} + \tilde{g}_{ac/b} - \tilde{g}_{ab/c}), \quad \tilde{\Gamma}_{ab}^d = \tilde{g}^{cd}\tilde{\Gamma}_{abc}, \quad \tilde{R}_{abc}^d = \partial_{u_a}\tilde{\Gamma}_{bc}^d - \partial_{u_b}\tilde{\Gamma}_{ac}^d + \tilde{\Gamma}_{ac}^d \tilde{\Gamma}_{bc}^e - \tilde{\Gamma}_{bc}^d \tilde{\Gamma}_{ac}^e.
\]

This enables us to compute:
We first study the linear terms in the second derivatives of the metric:

\[ R_{abcd} = \tilde{\gamma}_{df}\partial_a\tilde{\Gamma}_{bc} + \tilde{\gamma}_{df}\partial_a\tilde{\Gamma}_{bc} + \tilde{\gamma}_{df}\partial_a\tilde{\Gamma}_{bc} - \tilde{\gamma}_{df}\partial_a\tilde{\Gamma}_{bc} - \tilde{\gamma}_{df}\partial_a\tilde{\Gamma}_{bc} \]

\[ = \tilde{\Gamma}_{bd/a} + \tilde{\gamma}_{df/a}\tilde{\Gamma}_{bc} - \tilde{\gamma}_{df/a}\tilde{\Gamma}_{bc} - \tilde{\gamma}_{df/a}\tilde{\Gamma}_{bc} - \tilde{\gamma}_{df/a}\tilde{\Gamma}_{bc} - \tilde{\gamma}_{df/a}\tilde{\Gamma}_{bc} \]

We examine the role of \( \tilde{\gamma}_{bd/\alpha} \) plays in the para-Gray identity; the remaining 3 terms play similar roles and the argument is similar after permuting the indices appropriately. We use the fact that \( \tilde{J}^*\tilde{g} = -\tilde{g} \) and apply \( \tilde{g} \) to compute

\[ \tilde{g}(\tilde{\Gamma}^1)_{abcd} = \tilde{g}_{bd/\alpha} + \tilde{g}_{bd/\alpha} + \tilde{g}_{bd/\alpha} + \tilde{g}_{bd/\alpha} + \tilde{g}_{bd/\alpha} + \tilde{g}_{bd/\alpha} \]

Next we examine the terms which are quadratic in the first derivatives of the metric; there are three different kinds of terms which must be symmetrized:

\[ \tilde{T}_{abcd} = \tilde{g}_{ad/\alpha} + \tilde{g}_{ad/\alpha} + \tilde{g}_{ad/\alpha} + \tilde{g}_{ad/\alpha} + \tilde{g}_{ad/\alpha} + \tilde{g}_{ad/\alpha} \]

The remaining quadratic terms arise by permuting the roles of \( \{a, b, c, d\} \) in these expressions. We compute:

\[ \tilde{g}(\tilde{\Gamma}^2)_{abcd} = \tilde{g}_{ad/\alpha} + \tilde{g}_{ad/\alpha} + \tilde{g}_{ad/\alpha} + \tilde{g}_{ad/\alpha} + \tilde{g}_{ad/\alpha} + \tilde{g}_{ad/\alpha} \]

The final term requires a bit more work.

\[ \tilde{g}(\tilde{\Gamma}^4)_{abcd} = \tilde{g}_{ad/\alpha} + \tilde{g}_{ad/\alpha} + \tilde{g}_{ad/\alpha} + \tilde{g}_{ad/\alpha} + \tilde{g}_{ad/\alpha} + \tilde{g}_{ad/\alpha} \]

This establishes the para-Gray identity for para-Hermitian manifolds.

3. The Tricerri-Vanhecke curvature decomposition

3.1. Hermitian models. Let \( (V, \langle \cdot, \cdot \rangle, J) \) be a Hermitian structure. Extend the inner product to tensors of all types. Let \( \Omega(x, y) := \langle x, Jy \rangle \) be the Kaehler form and let \( \mathfrak{A}(V) \subset \otimes^4 V^* \) be the space of algebraic curvature tensors on \( V \). Set
Theorem 3.1. Let $S_0^2(V^*, J) := \{ \theta \in S^2(V^*) : J^* \theta = \theta, \theta \perp \langle \cdot, \cdot \rangle \}$, 
$A_0^2(V^*, J) := \{ \theta \in A^2(V^*) : J^* \theta = \theta, \theta \perp \Omega \}$, 
$S^2_2(V^*, J) := \{ \theta \in S^2(V^*) : J^* \theta = -\theta \}$, 
$A^2_2(V^*, J) := \{ \theta \in A^2(V^*) : J^* \theta = -\theta \}$, 
$\mathcal{U} := \{ U \in \text{GL}_\mathbb{R}(V) : UJ = JU \text{ and } U^* \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle \}$. 

Pullback defines a natural orthogonal action of the unitary group $\mathcal{U}$, by the orthogonal group $O(V, \langle \cdot, \cdot \rangle)$, and by the general linear group $\text{GL}_\mathbb{R}(V)$ on $V^* \otimes V^*$ and on $\mathfrak{A}(V)$. As a $\text{GL}_\mathbb{R}(V)$ module, there is a direct sum decomposition of 

$$V^* \otimes V^* = S^2_2(V^*) \oplus A^2_2(V^*)$$

into the symmetric and the anti-symmetric 2-tensors, respectively; these modules are irreducible $\text{GL}_\mathbb{R}(V)$ modules. $A^2_2(V^*)$ is an irreducible $O(V, \langle \cdot, \cdot \rangle)$ module. Let $S^2_0(V^*, \langle \cdot, \cdot \rangle)$ be the subspace of trace free symmetric 2-tensors. There is a further irreducible orthogonal decomposition of

$$S^2_2(V^*) = \langle \cdot, \cdot \rangle \cdot \mathbb{R} \oplus S^2_0(V^*, \langle \cdot, \cdot \rangle).$$

Finally, as $\mathcal{U}$ modules, we have an orthogonal direct sum decomposition:

$$(3.3) \quad V^* \otimes V^* = \langle \cdot, \cdot \rangle \cdot \mathbb{R} \oplus S^2_0(V^*, J) \oplus S^2_2(V^*, J) \oplus \Omega \cdot \mathbb{R} \oplus A^2_0(V^*, J) \oplus A^2_2(V^*, J).$$

If $\theta \in V^* \otimes V^*$, let $\theta_{0,+}, \theta_{-,+}, \text{ and } \theta_{-,A}$ denote the components of $\theta$ in $S^2_0(V^*, J)$, $S^2_2(V^*, J)$, and $A^2_2(V^*, J)$, respectively.

Let $\{ e_i \}$ be a basis for $V$. Let $\varepsilon_{ij} := \langle e_i, e_j \rangle$ and let $\varepsilon^{ij}$ be the inverse matrix. Let $\mathbf{A} \in \mathfrak{A}(V)$. Let $\tau$, $\tau^*$, $\rho$, and $\rho^*$ be the scalar curvature, the $*$-scalar curvature, the Ricci tensor, and the $*$-Ricci tensor:

$$(3.3) \quad \rho(x, y) := \varepsilon^{ij} A(e_i, x, y, e_j), \quad \tau := \varepsilon^{ij} \rho(e_i, e_j), \quad \rho^*(x, y) := \varepsilon^{ij} A(e_i, x, Jy, Je_j), \quad \tau^* := \varepsilon^{ij} \rho^*(e_i, e_j).$$

We refer to [3] for the proof of the following result:

Theorem 3.1. Let $(V, \langle \cdot, \cdot \rangle, J)$ be a Hermitian structure.

1. We have the following orthogonal direct sum decomposition of $\mathfrak{A}(V)$ into irreducible $\mathcal{U}$ modules:
   (a) If $2n = 4$, $\mathfrak{A}(V) = W_1 \oplus W_2 \oplus W_3 \oplus W_4 \oplus W_7 \oplus W_8 \oplus W_9$.
   (b) If $2n = 6$, $\mathfrak{A}(V) = W_1 \oplus W_2 \oplus W_3 \oplus W_4 \oplus W_8 \oplus W_9 \oplus W_{10}$.
   (c) If $2n \geq 8$, $\mathfrak{A}(V) = W_1 \oplus W_2 \oplus W_4 \oplus W_7 \oplus W_8 \oplus W_9 \oplus W_{10}$.
   We have $W_1 \approx W_4$ and, if $2n \geq 6$, $W_2 \approx W_3$. The other $\mathcal{U}$ modules appear with multiplicity 1.

2. We have that:
   (a) $\tau + \tau^* : W_1 \oplus W_4 \xrightarrow{\approx} \mathbb{R} \oplus \mathbb{R}$.
   (b) If $2n = 4$, $\rho_{0,+} : W_2 \xrightarrow{\approx} S^2_0(V^*, J)$.
   (c) If $2n \geq 6$, $\rho_{0,+} : W_2 \oplus W_3 \xrightarrow{\approx} S^2_2(V^*, J) \oplus S^2_0(V^*, J)$.
   (d) $\mathcal{W}_3 = \{ A \in \mathfrak{A}(V) : A(x, y, z, w) = A(x, y, z, w) \forall x, y, z, w \} \cap \ker(\rho)$.
   (e) If $2n \geq 8$, $\mathcal{W}_5 = \ker(\rho \oplus \rho^*) \cap \{ A \in \mathfrak{A}(V) : J^* A = A \} \cap \mathcal{W}_3^2$.
   (f) $\mathcal{W}_7 = \{ A \in \mathfrak{A}(V) : A(Jx, y, z, w) = A(x, y, Jz, w) \forall x, y, z, w \}$.
   (g) $\mathcal{W}_9 \approx \mathfrak{A}(V)$.
   (h) $\rho^{\perp}_A : W_9 \xrightarrow{\approx} \mathcal{W}_9$.
   (i) If $2n \geq 6$, $\mathcal{W}_{10} = \{ A \in \mathfrak{A}(V) : J^* A = -A \} \cap \ker(\rho \oplus \rho^*)$. 
3.2. Para-Hermitian models. Let \((\tilde{V}, \langle \cdot, \cdot \rangle, \tilde{J})\) be a para-Hermitian structure; the metric is non-degenerate on the space of algebraic curvature tensors \(\mathfrak{A}(\tilde{V})\). Let \(\tilde{\Omega}(x, y) := \langle x, \tilde{J} y \rangle\) be the para-Kaehler form. We have
\[
\tilde{J}^* \tilde{\Omega} = -\tilde{\Omega} \quad \text{and} \quad \tilde{J}^* \langle \cdot, \cdot \rangle = -\langle \cdot, \cdot \rangle.
\]
Set
\[
S^2_+ (\tilde{V}^*, \tilde{J}) := \{ \theta \in S^2(\tilde{V}^*) : \tilde{J}^* \theta = \theta \},
\]
\[
A^2_1 (\tilde{V}^*, \tilde{J}) := \{ \theta \in A^2(\tilde{V}^*) : \tilde{J}^* \theta = \theta \},
\]
\[
S^2_0_-(\tilde{V}^*, \tilde{J}) := \{ \theta \in S^2(\tilde{V}^*) : \tilde{J}^* \theta = -\theta, \theta \perp \langle \cdot, \cdot \rangle \},
\]
\[
A^2_1_0_-(\tilde{V}^*, \tilde{J}) := \{ \theta \in A^2(\tilde{V}^*) : \tilde{J}^* \theta = -\theta, \theta \perp \tilde{\Omega} \},
\]
\[
\tilde{U} := \{ \tilde{U} \in \text{GL}_{\mathbb{R}}(\tilde{V}) : \tilde{U} \tilde{J} = \tilde{J} \tilde{U} \quad \text{and} \quad \tilde{U}^* \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle \}.
\]
Fix a basis \(\{ \tilde{e}_i \}\) for \(\tilde{V}\) and let \(\tilde{\varepsilon}_{ij}\) be the components of the inner product relative to this basis. If \(\tilde{A}\) is an algebraic curvature tensor, define:
\[
\rho(x, y) := \tilde{\varepsilon}^{ij} \tilde{A}(\tilde{e}_i, x, y, \tilde{e}_j), \quad \tau := \tilde{\varepsilon}^{ij} \rho(\tilde{e}_i, \tilde{e}_j),
\]
\[
\rho^*(x, y) := -\tilde{\varepsilon}^{ij} \tilde{A}(\tilde{e}_i, x, \tilde{J} y, \tilde{e}_j), \quad \tau^* := \tilde{\varepsilon}^{ij} \rho^*(\tilde{e}_i, \tilde{e}_j).
\]
The decomposition of Equation (5.a) extends to this setting to become:
\[
\tilde{V}^* \otimes \tilde{V}^* = \langle \cdot, \cdot \rangle : \mathbb{R} \oplus \tilde{S}^2_{\tilde{0}_-(\tilde{V}^*, \tilde{J})} \oplus \tilde{S}^2_{\tilde{J}}(\tilde{V}^*, \tilde{J})
\]
\[
\oplus \tilde{\Omega} : \mathbb{R} \oplus \tilde{A}^2_{\tilde{0}_-(\tilde{V}^*, \tilde{J})} \oplus \tilde{A}^2_{\tilde{J}}(\tilde{V}^*, \tilde{J})
\]
\[
\mathbb{R}.
\]
Theorem 3.2. Let \((\tilde{V}, \langle \cdot, \cdot \rangle, \tilde{J})\) be a para-Hermitian structure. 

(1) We have the following orthogonal direct sum decomposition of \(\mathfrak{A}(\tilde{V})\) into irreducible \(\tilde{U}\) modules:
(a) If \(2n = 4\), \(\mathfrak{A}(\tilde{V}) = \tilde{V}_1 \oplus \tilde{V}_2 \oplus \tilde{W}_3 \oplus \tilde{W}_4 \oplus \tilde{W}_5 \oplus \tilde{W}_6 \oplus \tilde{W}_7 \oplus \tilde{W}_8 \oplus \tilde{W}_9\),
(b) If \(2n = 6\), \(\mathfrak{A}(\tilde{V}) = \tilde{V}_1 \oplus \tilde{W}_2 \oplus \tilde{W}_3 \oplus \tilde{W}_4 \oplus \tilde{W}_5 \oplus \tilde{W}_7 \oplus \tilde{W}_8 \oplus \tilde{W}_9 \oplus \tilde{W}_{10}\),
(c) If \(2n \geq 8\), \(\mathfrak{A}(\tilde{V}) = \tilde{W}_1 \oplus \tilde{W}_2 \oplus \tilde{W}_3 \oplus \tilde{W}_4 \oplus \tilde{W}_5 \oplus \tilde{W}_7 \oplus \tilde{W}_8 \oplus \tilde{W}_9 \oplus \tilde{W}_{10}\).

We have \(\tilde{W}_1 \simeq \tilde{W}_3\) and, if \(2n \geq 6\), \(\tilde{W}_2 \simeq \tilde{W}_5\). The other \(\tilde{U}\) modules appear with multiplicity 1.

(2) We have that:
(a) \(\tau \oplus \tau^* : \tilde{V}_1 \oplus \tilde{W}_4 \overset{\approx}{\rightarrow} \tilde{S}^2_{\tilde{0}_-(\tilde{V}^*, \tilde{J})}\),
(b) If \(2n = 4\), \(\rho_{0,-.s} : \tilde{W}_2 \overset{\approx}{\rightarrow} \tilde{S}^2_{\tilde{0}_-(\tilde{V}^*, \tilde{J})}\),
(c) If \(2n \geq 6\), \(\rho_{0,-.s} \oplus \rho^*_{0,-.s} : \tilde{W}_2 \oplus \tilde{W}_3 \overset{\approx}{\rightarrow} \tilde{S}^2_{\tilde{0}_-(\tilde{V}^*, \tilde{J})} \oplus \tilde{S}^2_{\tilde{J}}(\tilde{V}^*, \tilde{J})\),
(d) \(\tilde{W}_3 = \{ \tilde{A} \in \mathfrak{A}(\tilde{V}) : \tilde{A}(x, y, z, w) = -\tilde{A}(\tilde{J} x, \tilde{J} y, z, w) \forall x, y, z, w \} \cap \ker(\rho)\),
(e) If \(2n \geq 8\), \(\tilde{W}_6 = \ker(\rho \oplus \rho^*) \cap \{ \tilde{A} \in \mathfrak{A}(\tilde{V}) : \tilde{J}^* \tilde{A} = \tilde{A} \} \cap \tilde{W}_4\),
(f) \(\tilde{W}_7 = \{ \tilde{A} \in \mathfrak{A}(\tilde{V}) : \tilde{A}(\tilde{J} x, y, z, w) = \tilde{A}(x, y, \tilde{J} z, w) \forall x, y, z, w \}\),
(g) \(\rho_{+,s} : \tilde{W}_8 \overset{\approx}{\rightarrow} \tilde{S}^2_{\tilde{J}}(\tilde{V}^*, \tilde{J})\),
(h) \(\rho^*_{+,s} : \tilde{W}_9 \overset{\approx}{\rightarrow} \tilde{A}^2_{\tilde{J}}(\tilde{V}^*, \tilde{J})\).

Proof. Let \((V, \langle \cdot, \cdot \rangle, J)\) be a Hermitian structure. We let \(V_C := V \otimes_{\mathbb{R}} \mathbb{C}\) be the complexification of \(V\). We extend \(\langle \cdot, \cdot \rangle\) to be complex bi-linear and we extend \(J\) to be complex linear. We extend an element of \(\mathfrak{A}(V)\) to be complex linear to define
\[
\mathfrak{A}(V_C) := \{ A_C \in \mathbb{C} \wedge 4 V_C^* : \text{Equation (1.a) holds} \} = \mathfrak{A}(V) \otimes_{\mathbb{R}} \mathbb{C}.
\]
Let \(A \in \mathfrak{A}(V)\). If \(\{ \xi_i \}\) is any \(\mathbb{C}\)-basis for \(V_C\), then Equation (3.1) remains valid where \(\varepsilon_{ij} := \langle \xi_i, \xi_j \rangle\). Let
\[
\mathcal{U}_C := \{ U \in \text{GL}_C(V_C) : JU = UJ \quad \text{and} \quad U^* \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle \}.
\]
Let \( \{e_1, ..., e_n, f_1, ..., f_n\} \) be an orthonormal basis for \( V \) where
\[
J e_i = f_i \quad \text{and} \quad J f_i = -e_i.
\]
We let
\[
\tilde{e}_i := \sqrt{-1}e_i, \quad \tilde{f}_i := -f_i, \quad \tilde{J} := \sqrt{-1}J, \quad \tilde{V} := \text{Span}_\mathbb{C}\{\tilde{e}_i, \tilde{f}_i\},
\]
\[
\tilde{U} := \{\tilde{U} \in \text{GL}_\mathbb{R}(\tilde{V}) : \tilde{U} \tilde{J} = \tilde{J} \tilde{U} \quad \text{and} \quad \tilde{U}^* \{\cdot, \cdot\} = \{\cdot, \cdot\}\} = \mathcal{U}_\mathbb{C} \cap \text{GL}_\mathbb{R}(\tilde{V}),
\]
\[
\mathfrak{A}(\tilde{V}) = \mathfrak{A}(V_{\mathbb{C}}) \cap \otimes^4 \tilde{V}^*.
\]

Since \( V \) has a positive definite metric, \( \tilde{V} \) inherits a metric \( \{\cdot, \cdot\} \) of neutral signature \((n, n)\); the vectors \( \tilde{e}_i \) being timelike and the vectors \( \tilde{f}_i \) being spacelike. Certain sign changes now manifest themselves:
\[
\rho^*(x, y) = -\varepsilon^{ij} A(e_i, x, \tilde{J} y, \tilde{J} e_i), \quad \tau^* = \varepsilon^{ij} \rho^*(e_i, e_j).
\]

In the decomposition of Equation (3.1), we have
\[
S_2^2(V^*, J) \otimes \mathbb{R} \mathbb{C} = \{\theta \in \otimes^2 \tilde{V}^*_\mathbb{C} : \theta(x, y) = \theta(y, x), \theta(J x, J y) = \pm \theta(x, y)\}
\]
\[
= \{\theta \in \otimes^2 \tilde{V}^*_\mathbb{C} : \theta(x, y) = \theta(y, x), \theta(J x, J y) = \mp \theta(x, y)\} = S_2^2(\tilde{V}^*, \tilde{J}) \otimes \mathbb{R} \mathbb{C},
\]
\[
\Lambda_2^1(V^*, J) \otimes \mathbb{R} \mathbb{C} = \{\theta \in \otimes^2 \tilde{V}^*_\mathbb{C} : \theta(x, y) = -\theta(y, x), \theta(J x, J y) = \pm \theta(x, y)\}
\]
\[
= \{\theta \in \otimes^2 \tilde{V}^*_\mathbb{C} : \theta(x, y) = -\theta(y, x), \theta(J x, J y) = \pm \theta(x, y)\} = \Lambda_2^1(\tilde{V}^*, \tilde{J}) \otimes \mathbb{R} \mathbb{C}.
\]
This defines a bijective correspondence which derives the decomposition of Theorem 3.2 from that of Theorem 3.1. The correspondence is reversible and hence the modules in Theorem 3.2 can not be decomposed further. \( \square \)

**Remark 3.3.** We started in the Hermitian setting to deduce a theorem in the para-Hermitian setting. Thus the Tricerri-Vanhecke decomposition works equally well in the pseudo-Hermitian setting by changing both the inner product and the operator \( J \). Suppose given integers \( p \) and \( q \) with \( p + q = n \). By setting
\[
\tilde{e}_i := \begin{cases} \sqrt{-1}e_i & \text{if } 1 \leq i \leq p \\ e_i & \text{if } p < i \leq n \end{cases}
\]
\[
\tilde{f}_i := \begin{cases} \sqrt{-1}f_i & \text{if } 1 \leq i \leq p \\ f_i & \text{if } p < i \leq n \end{cases}
\]
and by taking \( \tilde{J} := J \), we could create a pseudo-Hermitian model of signature \((2p, 2q)\). The analogous correspondence would then permit us to deduce a Tricerri-Vanhecke decomposition theorem in the pseudo-Hermitian signature as well.

4. **Linearizing the Problem**

We fix a para-Hermitian structure \((\tilde{V}, \{\cdot, \cdot\}, \tilde{J})\) hence forth. If \( \Theta \in \otimes^4 \tilde{V}^* \), set
\[
\mathcal{P}(\Theta)(x, y, z, w) := \Theta(x, z, y, w) + \Theta(y, w, x, z) - \Theta(x, w, y, z) - \Theta(y, z, x, w).
\]

**Lemma 4.1.** If \( \Theta \in S^2(\tilde{V}^*, \tilde{J}) \otimes S^2(\tilde{V}^*) \), then \( \mathcal{P}(\Theta) \) is an algebraic curvature tensor such that the complex model \((\tilde{V}, \{\cdot, \cdot\}, \tilde{J}, \mathcal{P}(\Theta))\) is geometrically realizable by a para-Hermitian manifold.

**Proof.** Let \( \{e_1, ..., e_n, f_1, ..., f_n\} \) be a basis for \( \mathbb{R}^{2n} \). Define an inner product \( \Xi \) of signature \((n, n)\) on \( \mathbb{R}^{2n} \) whose non-zero entries are
\[
\Xi(e_1, e_1) = ... = \Xi(e_n, e_n) = -1 \quad \text{and} \quad \Xi(f_1, f_1) = ... = \Xi(f_n, f_n) = +1.
\]
If \( v \in \mathbb{R}^{2n} \), expand \( v = x_1 e_1 + ... + x_n e_n + y_1 f_1 + ... + y_n f_n \) to define coordinates \((x_1, ..., x_n, y_1, ..., y_n) = (u_1, ..., u_{2n})\). Define
\[
\tilde{\mathcal{J}} \partial_{x_1} := \partial_{y_2}, \quad ..., \tilde{\mathcal{J}} \partial_{x_n} := \partial_{y_1}, \quad \tilde{\mathcal{J}} \partial_{y_1} := \partial_{x_1}, \quad ..., \tilde{\mathcal{J}} \partial_{y_n} := \partial_{x_n}.
\]
Let $\Theta \in S^2(\hat{V}^*, \hat{J}) \otimes S^2(\hat{V}^*)$. Define

\[
\tilde{g}_{ij} := \Xi_{ij} + 2\Theta_{ijkl}u^k u^l.
\]

Since $\Theta(x, y, z, w) = -\Theta(\hat{J}x, \hat{J}y, \hat{J}z, \hat{J}w)$, $\hat{J}^* \hat{g} = -\hat{g}$. Let $B_\epsilon$ be the Euclidean ball of radius $\epsilon > 0$ centered at the origin. Since $\hat{g}$ is non-singular at the origin, there exists $\epsilon > 0$ so $\hat{g}$ is non singular on $B_\epsilon$; let $M := (B_\epsilon, \hat{g}, \hat{J})$ be the resulting para-Hermitian manifold. Since the first derivatives of the metric vanish at 0,

\[
R(\partial_{u_i}, \partial_{u_j}, \partial_{u_k}, \partial_{u_l})(0) = \frac{1}{2}\{\partial_{u_i} \partial_{u_j} \tilde{g}_{kl} + \partial_{u_k} \partial_{u_l} \tilde{g}_{ij} - \partial_{u_i} \partial_{u_l} \tilde{g}_{jk} - \partial_{u_j} \partial_{u_k} \tilde{g}_{il}\} = \Theta_{ijkl} + \Theta_{jikl} - \Theta_{ijlk} - \Theta_{ikjl} = \mathcal{P}(\Theta).
\]

\section{The proof of Theorem \[1.2\] (2)}

Let $\mathcal{W}_G$ be the space of algebraic curvature tensors such that the para-Gray identity holds. Let

\[
\mathfrak{P} := \mathcal{P}\{S^2(\hat{V}^*, \hat{J}) \otimes S^2(\hat{V}^*)\}.
\]

$\mathfrak{P}$ and $\mathcal{W}_G$ are linear subspaces of $\mathfrak{A}(\hat{V})$ which are invariant under the action of the para-unitary group $\hat{U}$. The results of Section \ref{sec:proof} reduce the proof of Theorem \[1.2\] (2) to showing $\mathfrak{P} = \mathcal{W}_G$. We begin our study with the following result:

\begin{lemma}
\[5.1\] \hspace{1cm} $\mathfrak{P} \subset \mathcal{W}_G \subset \hat{W}_7^\perp$.
\end{lemma}

\begin{proof}
By Lemma \[4.1\] every element of $\mathfrak{P}$ can be geometrically realized by a para-Hermitian manifold. Theorem \[2.2\] now implies $\mathfrak{P} \subset \mathcal{W}_G$. We show $\mathcal{W}_G \subset \hat{W}_7^\perp$ by showing $\hat{W}_G \cap \hat{W}_7 = \{0\}$. Let $\hat{A} \in \hat{W}_G \cap \hat{W}_7$. Since $\hat{A} \in \hat{W}_7$, the curvature symmetries imply additionally that

\[
\hat{A}(\hat{J}x, y, z, w) = -\hat{A}(\hat{J}x, y, w, z) = -\hat{A}(x, y, \hat{J}w, z) = \hat{A}(x, y, z, \hat{J}w) = -\hat{A}(y, x, z, \hat{J}w) = -\hat{A}(\hat{J}y, x, z, w) = \hat{A}(x, \hat{J}y, z, w).
\]

Since $\hat{A} \in \hat{W}_G$, we have

\[
0 = \hat{A}(x, y, z, w) + \hat{A}(\hat{J}x, \hat{J}y, z, w)
+ \hat{A}(x, y, z, w) + \hat{A}(\hat{J}x, \hat{J}y, z, w) + \hat{A}(\hat{J}x, y, \hat{J}z, w)
+ \hat{A}(x, \hat{J}y, z, w) + \hat{A}(\hat{J}x, y, z, w) + \hat{A}(x, \hat{J}y, \hat{J}z, w)
= 8\hat{A}(x, y, z, w).
\]

We continue our study with:

\begin{lemma}
\[5.2\] \hspace{1cm}
\begin{enumerate}
\item $\tau \oplus \tau^* : \mathfrak{P} \to \mathbb{R} \oplus \mathbb{R} \to 0$. Thus $\hat{W}_1 \oplus \hat{W}_4 \subset \mathfrak{P}$.
\item If $2n = 4$, then $\rho_{0,-} : \mathfrak{P} \to S^2(\hat{V}^*, \hat{J}) \to 0$. Thus $\hat{W}_2 \subset \mathfrak{P}$.
\item $\rho_{+, S} : \mathfrak{P} \to S^2(\hat{V}^*, \hat{J}) \to 0$. Thus $\hat{W}_8 \subset \mathfrak{P}$.
\item $\rho_{+, \Lambda} : \mathfrak{P} \to \Lambda^2(\hat{V}^*, \hat{J}) \to 0$. Thus $\hat{W}_9 \subset \mathfrak{P}$.
\item If $2n \geq 6$, then $\{\rho_{0,-, 0} \otimes \rho_{0,-, 0} : \mathfrak{P} \to \{S^2(\hat{V}^*, \hat{J}) \otimes S^2(\hat{V}^*, \hat{J})\}| \to 0$.
\item $\mathfrak{P} \cap \hat{W}_3 \neq \{0\}$. Thus $\hat{W}_3 \subset \mathfrak{P}$.
\item $\mathfrak{P} \cap \hat{W}_{10} \neq \{0\}$. Thus $\hat{W}_{10} \subset \mathfrak{P}$.
\item If $2n \geq 6$, then $\mathfrak{P} \cap \hat{W}_6 \neq \{0\}$. Thus $\hat{W}_6 \subset \mathfrak{P}$.
\end{enumerate}
\end{lemma}

\begin{proof}
As in the proof of Lemma \[4.1\] we examine metrics $\hat{g} = \Xi + O(|u|^2)$; let $\hat{A} \in \mathfrak{P}$ be the curvature tensor at the origin. Set $\hat{A}^*(x, y, z, w) := \hat{A}(x, y, \hat{J}z, \hat{J}w)$. Let $\xi \otimes \eta := \frac{1}{2}(\xi \otimes \eta + \eta \otimes \xi)$
denote the symmetric product. Let \( g \) and \( \varepsilon \) be real constants. Consider the para-Hermitian metric:
\[
\hat{g} = \Xi - \varepsilon x_1^2(dx_1 \circ dx_1 - dy_1 \circ dy_1) - \rho x_2^2(dx_2 \circ dx_2 - dy_2 \circ dy_2).
\]
The non-zero entries of \( \hat{A} \) are, up to the usual \( Z_2 \) symmetries,
\[
\hat{A}(\partial_{x_1}, \partial_{y_1}, \partial_{y_1}, \partial_{x_1}) = -\varepsilon, \\
\hat{A}(\partial_{x_1}, \partial_{x_2}, \partial_{x_2}, \partial_{x_1}) = \rho, \\
\hat{A}(\partial_{x_1}, \partial_{y_2}, \partial_{y_2}, \partial_{x_1}) = -\rho.
\]
Since the \( \{\partial_{x_i}\} \) are timelike and the \( \{\partial_{y_i}\} \) are spacelike, \( \tau = 2\varepsilon + 4\rho \) and \( \tau^* = 2\varepsilon \) so \( \tau \oplus \tau^* \) is a surjective map from \( \Phi \) to \( \mathbb{R} \oplus \mathbb{R} \). Thus Assertion (1) follows from
Theorem 3.2.
\[
\mathcal{W}_1 \oplus \mathcal{W}_4 \subset \Phi.
\]
The non-zero entries in the Ricci tensor are given by:
\[
\rho(\partial_{x_1}, \partial_{x_1}) = -\varepsilon - 2\rho, \\
\rho(\partial_{y_1}, \partial_{y_1}) = \varepsilon, \\
\rho(\partial_{x_2}, \partial_{x_2}) = -\rho, \\
\rho(\partial_{y_2}, \partial_{y_2}) = 0.
\]
We take \( \rho = -1 \) and \( \varepsilon = 2 \) to ensure \( \rho \) is trace free and symmetric. We then have
\[
\rho_{+,-}(\partial_{x_1}, \partial_{x_1}) = 1, \\
\rho_{0,-}(\partial_{x_1}, \partial_{x_1}) = -1, \\
\rho_{+,-}(\partial_{y_1}, \partial_{y_1}) = 1, \\
\rho_{0,-}(\partial_{y_1}, \partial_{y_1}) = 1, \\
\rho_{+,-}(\partial_{x_2}, \partial_{x_2}) = 0, \\
\rho_{0,-}(\partial_{x_2}, \partial_{x_2}) = 1, \\
\rho_{+,-}(\partial_{y_2}, \partial_{y_2}) = 0, \\
\rho_{0,-}(\partial_{y_2}, \partial_{y_2}) = -1.
\]
This shows that \( \rho_{0,-} \) is non-zero on \( \Phi \); Assertion (2) now follows if \( 2n = 4 \) since \( \mathcal{W}_3 \) is not present:
\[
\mathcal{W}_2 \subset \Phi.
\]
It also shows \( \rho_{+,-} \) is non-trivial on \( \Phi \) and establishes Assertion (3):
\[
\mathcal{W}_8 \subset \Phi.
\]
We clear the previous notation and consider:
\[
\hat{g} = \Xi - 4\varepsilon x_1^2(-dx_1 \circ dx_2 + dy_1 \circ dy_2).
\]
There is only one non-zero curvature entry \( \hat{A}(\partial_{x_1}, \partial_{y_1}, \partial_{y_1}, \partial_{x_1}) = 2\varepsilon \). We have:
\[
\hat{A}^*(\partial_{x_1}, \partial_{y_1}, \partial_{x_2}, \partial_{y_2}) = 2\varepsilon, \\
\rho^*(\partial_{x_1}, \partial_{x_2}) = 2\varepsilon, \\
\rho^*_\lambda(\partial_{x_1}, \partial_{x_1}) = -\rho^*_\lambda(\partial_{x_2}, \partial_{x_1}) = \varepsilon, \\
\rho^*_\mu(\partial_{y_2}, \partial_{y_2}) = -\rho^*_\mu(\partial_{y_1}, \partial_{y_2}) = -\varepsilon.
\]
This shows \( \rho^*_+, \lambda \neq 0 \) so \( \hat{A} \) has a non-trivial component in \( \mathcal{W}_9 \). This completes the proof of Assertion (4):
\[
\mathcal{W}_9 \subset \Phi.
\]
Assume \( 2n \geq 6 \). We clear the previous notation and consider:
\[
\hat{g} = \Xi - 2\rho x_1^2(-dx_1 \circ dx_2 + dy_1 \circ dy_2) - 2\varepsilon x_1^2(-dx_2 \circ dx_3 + dy_2 \circ dy_3).
\]
The non-zero curvatures now become:
\[
\hat{A}(\partial_{x_1}, \partial_{y_1}, \partial_{y_2}, \partial_{x_1}) = \rho, \\
\hat{A}(\partial_{x_1}, \partial_{x_2}, \partial_{x_3}, \partial_{x_1}) = -\varepsilon, \\
\hat{A}(\partial_{x_1}, \partial_{y_2}, \partial_{y_3}, \partial_{x_1}) = \varepsilon.
\]
Note that \( \rho \) is always symmetric. We have:
\[
\rho(\partial_{y_1}, \partial_{y_2}) = -\rho, \\
\rho(\partial_{x_1}, \partial_{x_2}) = 0, \\
\rho(\partial_{x_2}, \partial_{x_2}) = \varepsilon, \\
\rho(\partial_{y_2}, \partial_{y_2}) = -\varepsilon.
\]
This leads to the decomposition:
\( \rho_{0,-,S}(\partial_{x_1}, \partial_{x_2}) = \frac{1}{2} q, \quad \rho_{+,S}(\partial_{x_1}, \partial_{x_2}) = -\frac{1}{2} q, \)
\( \rho_{0,-,S}(\partial_{y_1}, \partial_{y_2}) = -\frac{1}{2} q, \quad \rho_{+,S}(\partial_{y_1}, \partial_{y_2}) = -\frac{1}{2} q, \)
\( \rho_{0,-,S}(\partial_{x_2}, \partial_{x_3}) = \varepsilon, \quad \rho_{+,S}(\partial_{x_2}, \partial_{x_3}) = 0, \)
\( \rho_{0,-,S}(\partial_{y_2}, \partial_{y_3}) = -\varepsilon, \quad \rho_{+,S}(\partial_{y_2}, \partial_{y_3}) = 0. \)

We have:
\[
\hat{A}^*(\partial_{x_1}, \partial_{y_1}, \partial_{x_2}, \partial_{y_2}) = q, \quad \hat{A}^*(\partial_{y_2}, \partial_{x_2}, \partial_{y_1}, \partial_{x_1}) = q, \]
\[\hat{A}^*(\partial_{x_1}, \partial_{y_2}, \partial_{x_2}, \partial_{y_1}) = -\varepsilon, \quad \hat{A}^*(\partial_{x_2}, \partial_{x_1}, \partial_{y_1}, \partial_{y_2}) = -\varepsilon, \]
\[\hat{A}^*(\partial_{x_1}, \partial_{y_2}, \partial_{x_2}, \partial_{y_1}) = \varepsilon, \quad \hat{A}^*(\partial_{y_2}, \partial_{x_2}, \partial_{y_1}, \partial_{x_1}) = \varepsilon. \]

Consequently \( \rho^*(\partial_{x_1}, \partial_{x_2}) = q \) and \( \rho^*(\partial_{y_2}, \partial_{y_1}) = -q. \) This yields:
\[
\rho_{0,-,S}(\partial_{x_1}, \partial_{x_2}) = \frac{1}{2} q, \quad \rho_{+,A}(\partial_{x_1}, \partial_{x_2}) = \frac{1}{2} q, \]
\[\rho_{0,-,S}(\partial_{y_1}, \partial_{y_2}) = -\frac{1}{2} q, \quad \rho_{+,A}(\partial_{y_1}, \partial_{y_2}) = +\frac{1}{2} q. \]

If we take \( q = 0 \) and \( \varepsilon \neq 0, \) then \( \rho_{0,-,S} \neq 0 \) and \( \rho_{0,-,S}^* = 0. \) Thus
\[
\{ S_{0,-}^2(\mathring{V}^*, \mathring{J}) \oplus 0 \} \cap \{ \rho_{0,-,S} \oplus \rho_{0,-,S}^* \} \mathcal{P} \neq \{ 0 \} \quad \text{so} \quad \{ S_{0,-}^2(\mathring{V}^*, \mathring{J}) \oplus 0 \} \subset \{ \rho_{0,-,S} \oplus \rho_{0,-,S}^* \} \mathcal{P}. \]

On the other hand, if we take \( q \neq 0, \) then \( \rho_{0,-,S}^* \neq 0. \) Thus we have a non-zero component in the second factor and
\[
\{ S_{0,-}^2(\mathring{V}^*, \mathring{J}) \oplus S_{0,-}^2(\mathring{V}^*, \mathring{J}) \} \subset \{ \rho_{0,-,S} \oplus \rho_{0,-,S}^* \} \mathcal{P}. \]

This establishes Assertion (5):
\[
\mathcal{W}_2 \oplus \mathcal{W}_5 \subset \mathcal{P}. \]

To prove Assertion (6), we consider the metric
\[
\mathring{g} = \Xi - 2\{ x_1^2 + y_1^2 \}(dx_1 \circ dx_2 + dy_1 \circ dy_2). \]
The non-zero components of \( \hat{A} \) are then given, up to the usual \( Z_2 \) symmetries by:
\[
\hat{A}(\partial_{x_1}, \partial_{y_1}, \partial_{y_2}, \partial_{x_2}) = 1, \quad \hat{A}(\partial_{y_1}, \partial_{x_1}, \partial_{x_2}, \partial_{y_2}) = 1, \]
\[
\hat{A}(\partial_{x_2}, \partial_{y_1}, \partial_{y_2}, \partial_{x_1}) = -1, \quad \hat{A}(\partial_{y_2}, \partial_{x_2}, \partial_{x_1}, \partial_{y_1}) = -1. \]

We have \( \rho = 0 \) and \( \hat{A}(\mathring{J}, \mathring{J}, y, z, w) = -\hat{A}(x, y, z, w) \) for all \( x, y, z, \) and \( w. \) This shows \( \hat{A} \in \mathcal{W}_3 \) and proves Assertion (6) by showing
\[
\mathcal{W}_3 \subset \mathcal{P}. \]

Let \( 2n \geq 6. \) We consider
\[
\mathring{g} = \Xi - 2\{ x_1^2 + y_1^2 \}(dx_1 \circ dx_3 + dy_1 \circ dy_3). \]
The non-zero curvatures are then
\[
\hat{A}(\partial_{x_1}, \partial_{x_2}, \partial_{x_3}, \partial_{x_1}) = -1, \quad \hat{A}(\partial_{x_1}, \partial_{y_2}, \partial_{y_3}, \partial_{x_3}) = 1, \]
\[
\hat{A}(\partial_{y_1}, \partial_{x_2}, \partial_{x_3}, \partial_{y_1}) = -1, \quad \hat{A}(\partial_{y_1}, \partial_{y_2}, \partial_{y_3}, \partial_{x_3}) = 1. \]

We have \( \rho = \rho^* = 0. \) Since \( \mathring{J}^* \hat{A} = -\hat{A}, \) \( \hat{A} \in \mathcal{W}_{10}; \) Assertion (7) follows since
\[
\mathcal{W}_{10} \subset \mathcal{P}. \]

Let \( 2n \geq 8. \) We take
\[
\mathring{g} = \Xi - 4\{ x_1 x_2 + y_1 y_2 \}(dx_3 \circ dx_4 + dy_3 \circ dy_4). \]
The non-zero curvatures are
\[
\hat{A}(\partial_{x_1}, \partial_{x_2}, \partial_{x_3}, \partial_{x_4}) = \hat{A}(\partial_{y_1}, \partial_{x_3}, \partial_{x_4}, \partial_{y_2})
\]
\[
= \hat{A}(\partial_{x_1}, \partial_{x_4}, \partial_{x_3}, \partial_{x_2}) = \hat{A}(\partial_{y_1}, \partial_{x_4}, \partial_{x_3}, \partial_{y_2}) = -1, \]
\[
\hat{A}(\partial_{x_1}, \partial_{y_3}, \partial_{y_4}, \partial_{x_2}) = \hat{A}(\partial_{y_1}, \partial_{y_3}, \partial_{y_4}, \partial_{y_2})
\]
\[
= \hat{A}(\partial_{x_1}, \partial_{y_4}, \partial_{y_3}, \partial_{x_2}) = \hat{A}(\partial_{y_1}, \partial_{y_4}, \partial_{y_3}, \partial_{y_2}) = 1. \]
We observe that $\rho = \rho^* = 0$. Since $\tilde{A}(\tilde{J}x, \tilde{J}y, z, w) \neq -\tilde{A}(x, y, z, w)$, $\tilde{A} \notin \tilde{W}_3$. Thus $\tilde{A}$ has a non-zero component in $\tilde{W}_6 \oplus \tilde{W}_7$. As $\mathfrak{Q} \perp \tilde{W}_7$, $\tilde{A}$ has a non-zero component in $\tilde{W}_6$ and Assertion (8) follows; $\tilde{W}_6 \subset \mathfrak{Q}$.

Proof of Theorem 1.2 (2). By Lemma 5.1, we have $\mathfrak{Q} \subset \tilde{W}_G \subset \tilde{W}_7^\perp$. The assertion $\tilde{W}_7^\perp \subset \mathfrak{Q}$ follows from the Tricerri-Vanhecke decomposition described in Theorem 3.2 and from Lemma 5.2.

Proof of Remark 1.3. The construction given above yields $\tilde{M}$ with $\tilde{d}\tilde{\Omega}_P = 0$ realizing the given complex curvature model $\tilde{C}$ at $P$. Imposing the para-Kaehler identity $\tilde{d}\tilde{\Omega} \equiv 0$ globally would imply that $\tilde{R} \in \tilde{W}_1 \oplus \tilde{W}_2 \oplus \tilde{W}_3$ so this is not possible in general. In [1], we considered a further variation

$$\tilde{h} := \Xi + 2\xi(dx_1 \circ dx_1 - dy_1 \circ dy_1) + 2\eta(dx_2 \circ dx_2 - dy_2 \circ dy_2)$$

where $\{\xi, \eta\}$ are smooth functions vanishing to second order at $P$. We showed it was possible to choose $\{\xi, \eta\}$ so that the resulting metric had constant scalar curvature and constant $\star$-scalar curvature. Since $\{\xi, \eta\}$ vanish to second order, $(\tilde{M}, \tilde{h}, \tilde{J})$ realizes $\tilde{C}$ at $P$ as well and $\tilde{d}\tilde{\Omega}_{\xi,\eta} = 0$. This establishes Remark 1.3.

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