An algorithm for analytical solution of basic problems featuring elastostatic bodies with cavities and surface flaws

V B Penkov\(^1\), L V Levina\(^2\), O S Novikova\(^1\) and A S Shulmin\(^3\)

\(^1\) Department of General Mechanics, Lipetsk State Technical University, Moskovskaya 30, Russia
\(^2\) Department of Applied Mathematics, Lipetsk State Technical University, Moskovskaya 30, Russia
\(^3\) Department of Mathematics, Lipetsk State Technical University, Moskovskaya 30, Russia

E-mail: vbpenkov@mail.ru

Abstract. Herein we propose a methodology for structuring a full parametric analytical solution to problems featuring elastostatic media based on state-of-the-art computing facilities that support computerized algebra. The methodology includes: direct and reverse application of P-Theorem; methods of accounting for physical properties of media; accounting for variable geometrical parameters of bodies, parameters of boundary states, independent parameters of volume forces, and remote stress factors. An efficient tool to address the task is the sustainable method of boundary states originally designed for the purposes of computerized algebra and based on the isomorphism of Hilbertian spaces of internal states and boundary states of bodies. We performed full parametric solutions of basic problems featuring a ball with a nonconcentric spherical cavity, a ball with a near-surface flaw, and an unlimited medium with two spherical cavities.

1. Introduction

The classical approach to boundary value problems in mathematical physics suggests that a solution should contain all parameters of a problem (full parametric solution, or FPS). Computer technology has made away with the traditional approach to analytical presentation of solutions. The present-day technology that makes computerized algebra possible allows to organize the presentation of solutions in the classical form. A verifiably efficient method to achieve this is the method of boundary states (MBS).

This study purports to generalize the methods for structuring full parametric analytical solutions to elastostatics problems featuring multiply connected restricted bodies and unrestricted spaces with multiple cavities and boundaries that may also contain flaws.

2. A current approach to structuring full parametric solutions

The methodology for structuring full parametric solutions is synoptically yet non-exhaustively described in study [1]. The FPS procedure includes the following stages, each of which corresponds to particular parameters to be introduced:

1) formulation of a well-defined mechanical problem;
2) non-dimensionalization of relations and parameters contained in the problem so as to come to a minimum set of parameters that allows to present the solution analytically;
3) making provisions for including the parameters of boundary states (non-dimensionalized constants);
4) if the problem features an unrestricted space, the explicit parameters must be extended to include symbols to denote remote stress states;
5) making provisions for parameters of elasticity. High-power computers of today use fixed values of medium parameters. Traditional computational approaches do not allow for their direct application in FPS;
6) accounting for geometrical parameters of the bodies involved;
7) accounting for parameters of various volume forces at work;
8) reversal to dimensional values in accordance with P-Theorem [2].

These points also apply to full parametric solutions to problems featuring multiply connected restricted bodies. A key difference of solutions addressing multiply connected bodies as opposed to simple-connected ones is that the basis of space of internal states of the medium in the former case must include a set of states describing each cavity involved [3]. The experience gained in the area so far suggests that the expansion of the basis of space by introducing sets of states to describe surface flaws significantly reduces the time needed for solution even in the case of problems featuring restricted bodies parts whose boundaries are negative curvatures.

A crucial choice to be made further is that of the solution method. As was shown in [3], the MBS is an efficient state-of-the-art method to address our needs. Its “up-to-dateness” resides in the use of computer facilities with capacity for computerized algebra. The method relies on the concept of an internal state of a body \( \xi = \{ u, \varepsilon_{ij}, \sigma_{ij} \} \) whose attributes identically match constitutive equations of the medium: Cauchy equations, generalized Hooke’s law, and balance equations [4]:

\[
\varepsilon_{ij} = \frac{1}{2} \left( u_{ij,} + u_{ji,} \right) \\
\sigma_{ij} = \lambda \theta \delta_{ij} + 2\mu \varepsilon_{ij}, \quad \theta = \varepsilon_{ii}, \\
\sigma_{ij,j} + X_i = 0,
\]

where \( \varepsilon_{ij}, \sigma_{ij} \) are components of the deformation and strain tensors, \( u_i \) is a component of the displacement vector, \( \theta \) is the volume strain, \( \delta_{ij} \) is the Kronecker delta, \( X_i \) is a component of the vector of volume forces, \( \lambda = 2\mu \ell / (1 - 2\nu) \) is the Lamé parameter, \( \mu \) is the shear modulus, and \( \nu \) is the Poisson ratio.

Simultaneous equations (1)–(3) are equivalent to three Lamé’s equations [4]

\[
\mu u_{i,j,j} + (\lambda + \mu) u_{j,j,i} + X_i = 0,
\]

general solutions to which have been separately and independently performed by H. Neuber [5] and P.F. Papkovich [6]:

\[
u_i^h = 4(1-\nu_0)B_i - (x_jB_j + B_0)_j + u_i^p,
\]

where \( B_i \) is a component of the arbitrary harmonic vector \( \text{B} \), \( B_0 \) is a harmonic scalar, and \( u_i^p \) corresponds to the displacements caused by volume forces.

To fill the basis of the Hilbertian space of internal states, it’s most convenient to use the Arzhanykh-Slobodyansky solution [4] as applied to the interior of a restricted body (volume forces being left out):
\[ u_i = 4(1 - \nu_0)B_i + x_i B_{i,j} - x_j B_{j,i}, \quad (4) \]

or as applied to the exterior of a cavity in unrestricted space:

\[ u_i = 4(1 - \nu_0)B_i - (x_j B_{j,i}), \quad (5) \]

where \( B_i \) is a component of the arbitrary harmonic vector

\[
\mathbf{B} = \begin{pmatrix} \varphi \\ 0 \\ \varphi \\ 0 \\ 0 \end{pmatrix}.
\]

For the interior of a restricted space \( \varphi \) is an element of a set of linearly independent harmonic polynomials, i.e.,

\[
\varphi \in \{x, y, z, yz, xz, xy, x^2 - z^2, y^2 - z^2, \ldots \},
\]

whereas for the exterior of a cavity it’s a harmonic of the following type [3]:

\[
\varphi \in \{y/r, x/y, y/r^3, z/r^3, yz/r^5, xz/r^5, xy/r^5, \ldots \},
\]

where the radius vector is reckoned from the center inside the cavity.

Below we provide some templates of state bases if the Poisson ratio \( \nu \) equals 0.25 in case of the interior of the restricted body:

\[
\mathbf{u}^{(1)} = \begin{pmatrix} 3x \\ -y \\ -z \end{pmatrix}, \quad \mathbf{e}^{(1)} = \begin{pmatrix} 3 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{\sigma}^{(1)} = \begin{pmatrix} 7 \\ 0 \\ 0 \\ 0 \end{pmatrix},
\]

\[
\mathbf{u}^{(2)} = \begin{pmatrix} 0 \\ 4x \\ 0 \end{pmatrix}, \quad \mathbf{e}^{(2)} = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{\sigma}^{(2)} = \begin{pmatrix} 0 \\ 4 \\ 0 \\ 0 \end{pmatrix}, \quad \ldots
\]

and now in case of the exterior of a restricted cavity:

\[
\mathbf{u}^{(1)} = r^{-3/2} \begin{pmatrix} 3x^3 + 2(\frac{y^2 + z^2}{x}) \\ xz \\ xy \end{pmatrix},
\]

\[
\mathbf{e}^{(1)} = r^{-5/2} \begin{pmatrix} -3x^3 \\ -y(7x^2 + y^2 + z^2) \\ -z(7x^2 + y^2 + z^2) \\ x(x^2 - 2y^2 + z^2) \end{pmatrix},
\]

\[
\mathbf{\sigma}^{(1)} = r^{-5/2} \begin{pmatrix} x(7x^2 + y^2 + z^2) \\ -y(7x^2 + y^2 + z^2) \\ -z(7x^2 + y^2 + z^2) \\ -3xyz \end{pmatrix}.
\]
\begin{align*}
\mathbf{u}^{(2)}(x, y, z) &= r^{-3/2} \begin{pmatrix}
x y \\
2x^2 + 3y^2 + 2z^2 \\
y z
\end{pmatrix}, \\
\mathbf{\dot{u}}^{(2)} &= r^{-5/2} \begin{pmatrix}
y(-2x^2 + y^2 + z^2) & -x(x^2 + 7y^2 + z^2) & -3xyz \\
-x(x^2 + 7y^2 + z^2) & -3y^2 & -z(x^2 + 7y^2 + z^2) \\
-3xyz & -z(x^2 + 7y^2 + z^2) & y(x^2 + y^2 - 2z^2)
\end{pmatrix}, \\
\mathbf{\ddot{u}}^{(2)} &= r^{-5/2} \begin{pmatrix}
y(-5x^2 + y^2 + z^2) & -x(x^2 + 7y^2 + z^2) & -6xyz \\
-x(x^2 + 7y^2 + z^2) & -y(x^2 + 7y^2 + z^2) & -z(x^2 + 7y^2 + z^2) \\
-6xyz & -z(x^2 + 7y^2 + z^2) & y(x^2 + y^2 - 5z^2)
\end{pmatrix}, \\
\mathbf{\dddot{u}}^{(2)} &= r^{-5/2} \begin{pmatrix}
y(-5x^2 + y^2 + z^2) & -x(x^2 + 7y^2 + z^2) & -6xyz \\
-x(x^2 + 7y^2 + z^2) & -y(x^2 + 7y^2 + z^2) & -z(x^2 + 7y^2 + z^2) \\
-6xyz & -z(x^2 + 7y^2 + z^2) & y(x^2 + y^2 - 5z^2)
\end{pmatrix}.
\end{align*}

Relations (1) and (2) enable us to formulate a k-th internal state for each displacement vector \( \mathbf{u}^{(k)} \): \( \xi^{(k)} = \{u_{ij}^{(k)}, \sigma_{ij}^{(k)}\} \). After removing linearly dependent elements from the list \( \{\xi^{(k)}\}, k \in \{1, \ldots, \infty\} \) we come to the basis of space of internal states of an isotropic elastic body \( V \). The basis of internal states \( \Xi = \{\xi^{(k)}\}_k \) engenders at boundary \( \partial V \) a basis of boundary states \( \Gamma = \{\gamma^{(k)}\}_k, \gamma^{(k)} = \{u_{ij}^{(k)}, p_{ij}^{(k)}\}, \quad p_{ij}^{(k)} = \sigma_{ij}^{(k)} n_j \) isomorphic to \( \Xi \), with \( n_j \) being a component of the unit vector of the external normal to the boundary. The isomorphism of the spaces of internal and boundary states is conditioned by the Somigliana theorem [7] and Betti’s reciprocity principle [2].

The next step is the orthogonalization of the basis in accordance with the scalar product

\[ (\xi^{(1)}, \xi^{(2)})_\Xi = \int_V \sigma_{ij}^{(2)} \epsilon_{ij}^{(1)} dV = \int_{\partial V} p_i^{(2)} u_i^{(1)} dS = (\gamma^{(1)}, \gamma^{(2)})_\Gamma, \]

followed by the formulation and solution of the infinite set of equations (ISE)

\[ Q c = q, \]

where the matrix of coefficients \( Q \) is determined only by the type of boundary conditions (BCs) and is based only on elements of the basis of space of boundary states [3], it being understood that this matrix as featured in the first and second general elasticity problems is a unit matrix and the vector \( q \) uses both basic elements and BCs. In the case of the first general problem we come to the following:

\[ \tilde{q}_k = \int_{\partial V} (p_u u_x^{(k)} + p_y u_y^{(k)} + p_w u_z^{(k)}) dS. \]

(6)

The unknown internal state \( \xi \) and boundary state \( \gamma \) are determined via the following Fourier series:

\[ u_i = \sum_k c_k u_i^{(k)}, \quad \sigma_{ij} = \sum_k c_k \sigma_{ij}^{(k)}, \quad \epsilon_{ij} = \sum_k c_k \epsilon_{ij}^{(k)}, \]

\[ u_i \big|_{\partial V} = \sum_k c_k u_i^{(k)} \big|_{\partial V}, \quad p_i = \sum_k c_k p_i^{(k)}. \]

The general advantages of the MBS reside in two facts. Firstly, the equations used to describe the medium are matched identically. Secondly, the degree of accuracy of the solution performed is easy to estimate by looking at the disparity between the resultant boundary state and the boundary conditions, after which the interval on the original basis of states may be adjusted as necessary. It removes the need for comparing test problems with ones solved by other methods (sustainability of the MBS).
3. Physical parameters of media, geometrical parameters of domains, and remote load

Study [1] describes the methodology for expanding the FPS to contain, among scale factors, the parameters of surface forces in their linear combinations as boundary conditions of the first basic problem, and the Poisson ratio. In order to obtain \( \nu \), the analytical solution was structured so as to include a Lagrange interpolation, which pre-determined the need for multiple modelling of the original basis of space \( \Xi \) and multiple orthogonalization attempts. It suffices to perform interpolation for the displacement vector only, as the deformation and strain tensors will further follow from the Cauchy formulas and the generalized Hooke’s law.

An approach that appears more rational is that based on the Poincaré perturbation method, which is quite well described in various works [8]. This approach requires to perform orthogonalization just once, which significantly reduces the energy intensity of the solution process. However, the approach brings a new difficulty: each new approximation triggers non-potential fictitious volume forces that require modelling of specific states of the elastic body. It would be easy to do if the forces were potential ones [4]. To address non-potential forces at work, we have discovered an efficient approach to modelling fields [9].

The methodology for modelling an internal state with a finite number of restricted cavities is described in study [3]. When applied to specific sets of values of the Poisson ratio \( \nu \) and specific sets of geometric parameters of the body \( h_m \), the methodology fundamentally allows us to model each of the internal states \( \xi_{m,n} \). Interpolation of the solutions taken as a whole brings us to the following description of the internal state of the medium: \( \xi(\nu, h) \). This notation explicitly accounts for both physical and geometric properties of the body. A similar approach may be applied if a problem contains multiple geometric parameters of a body. The resultant state can be safely considered to contain all non-dimensionalization parameters.

To account for a homogenous stressed state at infinity (it being enough to maintain the principal stresses \( X_\infty, Y_\infty, Z_\infty \) only, although the set can be extended to contain the shear stress), we need to have reference states \( \xi_x, \xi_y, \xi_z \) that correspond to each of \( X_\infty, Y_\infty, Z_\infty \) equaling 1 in alternative variants. The actual character representation of the cumulative state is determined by the following linear combination:

\[
\xi = X_\infty \xi_x + Y_\infty \xi_y + Z_\infty \xi_z.
\]

This equation is easy to generalize in case we wish to account for remote shear stress.

4. A ball with a non-concentric spherical cavity

In its dimensionless form (linear scale \( R \) is the radius of the ball; stress scale coincides with the shear modulus \( \mu \)), the problem reads as follows: a ball with its center at point \( O_1(0,0,1/8) \) contains a load-free spherical cavity with a radius of 1/2. The surface of the ball is loaded with hydrostatic pressure \( p \). A full parametric solution is required.

Bases of spaces of internal states were assigned alternately to each of the Poisson ratio values \( \nu \in [0.2, 0.25, 0.3] \) and were formed based on general Arzhanykh-Slobodyansky solutions [5] for the interior of a restricted body (4) and for the exterior of a restricted cavity (5).

Let’s present a fragment of the vector of the right-hand members of the ISE (rounded to the thousandth place) for a problem with a Poisson ratio \( \nu = 0.25 \). The vector fragment is calculated in accordance with (6) as follows:

\[
q = \{0, 0, 0.051, 0.458, 0, 0, 0.692, 0, -1.477, 0.493, ... \}
\]

Non-zero Fourier coefficients (rounded to the thousandth place) for a problem with a Poisson ratio \( \nu = 0.25 \) are presented in table 1.
Table 1. Non-zero Fourier coefficients in the problem featuring a ball with a cavity.

| k | $c_k$ | k | $c_k$ | k | $c_k$ | k | $c_k$ | k | $c_k$ |
|---|---|---|---|---|---|---|---|---|---|
| 3 | -0.051 | 10 | -0.493 | 24 | 0.008 | 39 | 0.002 | 52 | 0.010 |
| 4 | -0.458 | 14 | -0.359 | 28 | -0.033 | 43 | -0.005 | 56 | -0.003 |
| 7 | -0.692 | 18 | -0.284 | 32 | -0.066 | 47 | -0.004 | 59 | 0.001 |
| 9 | -1.477 | 21 | -0.015 | 36 | -0.006 | 49 | -0.002 | 63 | -0.110 |

In each instance the solution is based on the reference external pressure value of 1 unit. The real stress field differs from 1 by a factor of $p$ \[3\]. A Lagrange interpolation of the actual states in all of the problems after a reversal to dimensional values brings us to the following FPS (below we provide the initial intervals only due to the immensity of the full solution):

$$u_x = \frac{P_o x}{\mu} [ -0.554 - 0.041 z/R + 0.008 x^2 z/R^3 - 0.021 z^2/R^2 - 0.039 z^3/R^3 - 0.001 z^4/(R^4 A^{7/2}) -$$

$$-0.001/A^{5/2} - 0.036 z^2/(R^2 A^{5/2}) - 0.036 y^2/(R^2 A^{5/2}) + 0.009 z/(R A^{5/2}) - 0.037 z^2/(R^2 A^{5/2}) + \ldots +$$

$$(1.53 - 0.005 x^2/R^2 - 0.002 y^2/R^2 + 0.018 z/R - 0.063 x^2 z/R^3 + 0.042 z^2/R^2 + 0.308 z^3/R^3 -$$

$$-0.001 x^2 z/(R^3 A^{7/2}) - 0.001 (2 x^2 + y^2)/(R^2 A^{5/2}) + \ldots ) v + (-0.872 + 0.001 x^2/R^2 - 0.004 y^2/R^2 +$$

$$+ 0.143 z/R + 0.123 x^2 z/R^3 - 0.027 z^2/R^2 - 0.610 z^3/R^3 + 0.001 y^2 z^2/(R^2 A^{9/2}) + 0.001 z^4/(R^4 A^{9/2}) +$$

$$+ 0.002 x^2 z/(R^3 A^{7/2}) - 0.001 z^3/(R^3 A^{7/2}) + 0.001 (2 x^2 + y^2 - z^2)/(R^2 A^{5/2}) + \ldots ) v^2 ],$$

where $A = x^2/R^2 + y^2/R^2 + (-1/8 + z/R)^2$.

The remaining characteristics of the internal state may be obtained as usual.

Since calculations have shown that the difference between the various states corresponding to different values of $\nu$ within the limits stated above are almost imperceptible to the naked eye, we have drawn up table 3 to show charts reflecting the difference between stress strain state fields at real values of $\nu$ and $\nu_0 = 1/4$.

5. Elastic ball with a spherical surface flaw
The problem is defined in a form similar to the one above: a ball with a radius of 1 contains a spherical surface flaw (the center of the sphere being $(0,0,\sqrt{3}/2)$) with a radius of $1/2$ with the following boundary conditions: $\mathbf{p} = \begin{cases} \{ -\cos \theta \cos \varphi, -\cos \theta \sin \varphi, 0 \}, & r \in S_1 \\ \{ 0, 0, 0 \}, & r \in S_2 \end{cases}$. A full parametric solution is required.

![Figure 1. Load configuration for a ball with a surface flaw.](image)
Let’s present a fragment of the vector of the right-hand members of the ISE for a problem with a Poisson ratio of $\nu = 0.25$, rounded to a thousandth place:

$$q = \{0, 0, 0.378, -0.828, 0, 0, -1.304, 0, -0.360, -0.325, \ldots\}.$$

Non-zero Fourier coefficients (rounded to the thousandth place) for a problem with a Poisson ratio $\nu = 0.25$ are presented in table 2.

**Table 2. Non-zero Fourier coefficients in the problem featuring a ball with a surface flaw.**

| $k$ | $c_k$ | $k$ | $c_k$ | $k$ | $c_k$ | $k$ | $c_k$ | $k$ | $c_k$ |
|-----|-------|-----|-------|-----|-------|-----|-------|-----|-------|
| 3   | 0.378 | 36  | 0.082 | 73  | 0.046 | 116 | 0.017 | 154 | -0.006 |
| 4   | -0.828| 39  | -0.084| 77  | -0.051| 120 | 0.019 | 167 | -0.037 |
| 7   | -1.304| 43  | -0.021| 80  | 0.073 | 127 | -0.027| 170 | -0.006 |
| 9   | -0.360| 47  | -0.026| 84  | -0.011| 130 | 0.025 | 174 | 0.006  |
| 10  | -0.325| 49  | -0.096| 87  | 0.020 | 135 | 0.019 | 177 | 0.007  |
| 14  | -0.310| 52  | 0.035 | 93  | 0.054 | 138 | 0.012 | 178 | -0.022 |
| 18  | -0.205| 56  | -0.119| 100 | -0.014| 140 | -0.012| 182 | -0.022 |
| 21  | 0.176 | 59  | 0.084 | 103 | 0.025 | 143 | 0.029 | 186 | -0.004 |
| 24  | -0.130| 63  | -0.003| 108 | 0.070 | 145 | -0.024| 187 | -0.010 |
| 28  | -0.043| 66  | 0.005 | 111 | 0.024 | 149 | -0.024| 190 | -0.009 |
| 32  | -0.088| 70  | -0.088| 113 | -0.013| 153 | -0.007| 200 | -0.029 |

A Lagrange interpolation of the actual states in all of the problems after the reversal to dimensional values brings us to the following FPS (below we provide the initial intervals only due to the immensity of the full solution):

$$u_i = \frac{P_{i,x}}{\mu}[-0.573 + 0.047 x^2/R^2 + 0.049 y^2/R^2 - 0.020 y^4/R^4 - 0.077 z/R - 0.010/A^{1/2} +$$
\[
+ 0.015 x^2/(R^2 A^{1/2}) + 0.026 x^4/(R^4 A^{1/2}) - 0.019 y^2/(R^2 A^{1/2}) + 0.046 x^2 y^2/(R^4 A^{1/2}) +
\]
\[
+ 0.056 A^{1/2} - 0.153 A^{3/2} + 0.326 A^{7/2} - 0.554 A^{11/2} + 0.435 A^{3/2} + \ldots +
\]
\[
+ (1.19 - 0.192 x^2/R^2 + 0.047 x^4/R^4 - 0.202 y^2/R^2 - 0.019 x^2 y^2/R^4 + 0.145 y^4/R^4 -
\]
\[
- 0.072 z/R + 0.089 A^{1/2} - 0.229 x^2/(R^2 A^{1/2}) + 0.884 x^2/(R^4 A^{1/2}) + \ldots) v +
\]
\[
(1.66 + 0.557 x^2/R^2 - 0.122 x^4/R^4 + 0.577 y^2/R^2 - 0.0176 x^2 y^2/R^4 - 0.318 y^4/R^4 +
\]
\[
+ 0.099 z/R - 0.435 x^2 z/R^3 - 0.205/A^{1/2} + 0.538 x^2/(R^2 A^{1/2}) + \ldots) v^2],
\]

where $A = x^2/R^2 + y^2/R^2 + (\sqrt{3}/2 + z/R)^2$.

A careful analysis of the images (table 3) shows that:

1) qualities of the radial stresses are changeable: tensile stresses in looser media are concentrated in smaller thicknesses;
2) circumferential stresses in loose media operate in a directly opposite way: they create pulling in denser regions and compression otherwise;
3) in less loose media axial stresses concentrate compression load around the cavity, whereas in looser media the opposite is true;
4) depending on the Poisson ratio, the concentration of shear stresses operates in a directly opposite fashion with respect to the surfaces that restrict the body;
5) surface flaws concentrate all stresses in the neighborhood of the geometric singularities of the boundary.
Table 3. Variations in the internal state characteristics depending on varying Poisson ratios.

| Spherical layer | Ball with a surface flaw |
|-----------------|--------------------------|
| $\xi(0.3) - \xi(0.25)$ | $\xi(0.25) - \xi(0.2)$ | $\xi(0.3) - \xi(0.25)$ | $\xi(0.25) - \xi(0.2)$ |

| $\sigma_{xx}$ | $\sigma_{yy}$ | $\sigma_{zz}$ |
|---------------|---------------|---------------|
| ![Image](image1.png) | ![Image](image2.png) | ![Image](image3.png) |
| ![Image](image4.png) | ![Image](image5.png) | ![Image](image6.png) |
| ![Image](image7.png) | ![Image](image8.png) | ![Image](image9.png) |


Spherical layer Ball with a surface flaw

6. Single-axis loading of an unrestricted medium with two cavities

As an example, let us consider a problem about single-axis tensioning of an unrestricted medium that contains two equal-sized spherical cavities, the Poisson ratio and the distance between the cavities being understood as arbitrary values. It is possible to use the Southwell’s solution [10] combined with an iterative process of Schwartz for building the solution, but using MBS is more brief approach.

Remote stress is directed along axis $z$, which is perpendicular to the common equatorial plane of the cavities. Once non-dimensionalization is performed, the problem is found to contain three variable parameters: $\nu$, $h$, and $Z^-$. Reference solutions (at $Z^- = 1$) were performed for possible combinations of the values $\nu \in \{0.2, 0.25, 0.3\}$ and $h \in \{0.05, 0.4, 12.6\}$ (linear scale being equal to $R$ and stress scale being equal to shear modulus $\mu$). Non-zero Fourier coefficients (rounded to the thousandth place) for a problem with $\nu = 0.2$, $h = 0.05$ are presented in table 4 below:

| $k$ | $c_k$ | $c_k$ | $c_k$ | $c_k$ | $c_k$ | $c_k$ |
|-----|-------|-------|-------|-------|-------|-------|
| 1   | 0.047 | 25    | -0.019| 62    | -0.166| 90    | -0.344| 124   | -0.014| 163   | 0.008 |
| 4   | -0.085| 32    | -0.035| 65    | 0.207 | 93    | 0.210 | 131   | -0.018| 176   | 0.018 |
| 7   | 0.197 | 36    | 0.005 | 69    | -0.345| 97    | 0.015 | 135   | 0.023 | 180   | -0.001|
| 11  | 0.049 | 40    | 0.019 | 72    | 0.208 | 104   | 0.018 | 138   | -0.009| 184   | 0.010 |
| 15  | 0.555 | 47    | 0.036 | 76    | -0.279| 108   | -0.023| 139   | 0.002 | 192   | 0.020 |
| 16  | 0.197 | 51    | -0.004| 79    | 0.117 | 111   | 0.009 | 142   | -0.028| 196   | 0.008 |
| 20  | 0.049 | 55    | -0.284| 83    | -0.172| 112   | -0.002| 151   | 0.010 | 209   | 0.018 |
| 24  | 0.558 | 58    | 0.120 | 86    | 0.203 | 115   | 0.027 | 159   | 0.020 | 213   | -0.001|

Figure 2 shows stress fields $\sigma_{xz}$, $\sigma_{yy}$, $\sigma_{zz}$, $\sigma_{xx}$ at section $y = 0$ for $\nu = 0.2$, $h = 0.05$, $R = 0.7$. There is little difference in the stress fields in reference solutions at sections $y = 0$, $x > 0$, and $z > 0$.

Let’s present a fragment of the vector of the right-hand members of the ISE for a problem with a Poisson ratio of $\nu = 0.2$, rounded to a thousandth place:

$$\mathbf{q} = \{0.047, 0, 0, -0.085, 0, 0, 0.197, 0, 0, 0, \ldots\}.$$
Figure 2. Stress level lines in the problem featuring two cavities.

Interpolation of the set of reference solutions by the Lagrangian method, incorporation of the actual remote stress and further reversal to dimension values produced an explicit solution. The entire solution being cumbersome, below we provide only fragments of the results obtained for displacement values:

\[ u_x = Z^w x / \mu \left[ -0.004 + 0.029 A^{9/2} - 0.008 x/(R A^{9/2}) + 0.013 x^2 z^2/(R^4 B^{11/2}) + 
                      + 0.197 x^5/(R^5 C^{15/2}) - 0.449 v - 0.134 x^4 v/(R^4 B^{11/2}) + 
                      + 3.123 x^2 y^2/(R^5 C^{11/2}) + ... + 
                      + \left( -0.641 A^{9/2} + 0.117 x^2 y^2/(R^4 B^{9/2}) + 0.242 y^2/(R^5 C^{5/2}) + 
                      + 7.634 xv/(R B^{7/2}) + ... \right) h + 
                      + \left( 1.425 A^{9/2} + 0.015 y^2/(Rx B^{11/2}) + 0.847 v A^{9/2} + 0.082 x^2 y^2/(R^6 C^{15/2}) + ... \right) h^2, \]

\[ u_y = Z^w y / \mu \left[ -0.004 + 0.006 A^{9/2} - 0.003 x y/(R A^{9/2}) + 0.043 z^2 z^2/(R^4 B^{11/2}) - 
                      - 0.212 x^5/(R^5 C^{15/2}) - 0.449 v - 0.203 x^2 y^2/(R^5 B^{5/2}) + 0.141 x^3 y^2/(R^5 C^{13/2}) + ... + 
                      + \left( -0.128 A^{9/2} + 0.456 x^2 v/(R^2 B^{9/2}) - 1.182 x^2 z^2 v/(R^4 C^{13/2}) + ... \right) h + 
                      + \left( 0.285 A^{9/2} - 0.054 x^3 y^2/(R^5 C^{11/2}) + 0.003 x^2 y^2/(R^3 A^{9/2}) + 0.183 x^2 z^2/(R^4 C^{15/2}) + ... \right) h^2, \]

\[ u_z = Z^w z / \mu \left[ 0.496 + 0.017 A^{9/2} - 0.010 x/(R A^{9/2}) + 0.018 x^2 z^2/(R^4 B^{11/2}) + 
                      + 1.232 x^5/(R^5 C^{15/2}) - 0.449 v - 0.152 x^4 v/(R^4 B^{11/2}) + 2.924 x^4 y^2/(R^5 C^{11/2}) + ... + 
                      + \left( -0.385 A^{9/2} + 0.003 z^4/(R^4 B^{11/2}) + 2.399 x^7 y^2/(R^7 C^{15/2}) + 0.028 z^2 y^2/(R^4 A^{5/2}) + ... \right) h + 
                      + \left( 0.855 A^{9/2} - 0.013 x^2 z^2/(R^3 C^{9/2}) + 0.043 x y^2/(R^2 B^{9/2}) + 0.005 z^2 z^2/(R^2 A^{9/2}) + ... \right) h^2, \]

where \( A = (-7 + x/R)^2 + y^2/R^2 + z^2/R^2, \)
\( B = (-0.9 + x/R)^2 + y^2/R^2 + z^2/R^2, \)
\( C = (-0.725 + x/R)^2 + y^2/R^2 + z^2/R^2. \)
7. Conclusions and prospects

1. The above methodology makes it fundamentally possible to perform FPSs for restricted and unrestricted bodies with a finite number of restricted cavities.

2. The MBS proves an efficient method for formulating analytical solutions.

3. We see prospects of applying the MBS with perturbations to FPSs. This approach holds a promise of a more rational use of computer technology.

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