Novel phases in strongly coupled four-fermion theories

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ABSTRACT: We study a lattice model comprising four flavors of reduced staggered fermion in four dimensions interacting via a specific four-fermion interaction. We present both theoretical arguments and numerical evidence that support the idea that the system develops a mass gap for sufficiently strong four-fermi coupling via the formation of a symmetric four-fermion condensate. In contrast to other lattice four-fermion models studied previously our results do not favor the formation of a symmetry-breaking bilinear condensate for any value of the four-fermi coupling and we find evidence for one or more continuous phase transitions separating the weak and strong coupling regimes.

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1. Introduction

This paper is devoted to the study of a model consisting of four flavors of massless reduced staggered fermions interacting via a strong four-fermion interaction. The same model was studied previously in three dimensions utilizing three different numerical algorithms: fermion bags, rational hybrid Monte Carlo (RHMC) and quantum Monte Carlo [1, 2, 3, 4]. These studies revealed an interesting two-phase structure for the model; a massless phase at weak coupling is separated by a continuous phase transition from a massive phase at strong coupling. Most importantly, and in contrast to all earlier work with four-fermion models by the lattice gauge theory community [5, 6, 7, 8, 9, 10, 11], no intermediate phase characterized by symmetry-breaking bilinear condensates was found.

Recently results were reported for the same model in four dimensions [12]. The conclusion of that work was that a narrow broken phase reappears in four dimensions. Unlike earlier models this phase appeared to be separated from the weak- and strong-coupling phases by continuous rather than first-order phase transitions. In our work we have augmented the action used in that recent study with source terms to directly address the question of whether spontaneous symmetry breaking associated with the formation of bilinear condensates takes place. As in [12] we see no evidence for first-order phase transitions but in contrast to that work our results do not favor the presence of a symmetry-broken phase.
The plan of the paper is as follows: in the next section we describe the lattice model and its symmetries and in section 3 we describe the phases expected at strong and weak four-fermi coupling. In section 4 we show how to replace the four-fermi interaction by appropriate Yukawa terms and prove that the resulting Pfaffian is real, positive definite. This fact allows us to simulate the model using the RHMC algorithm and we show results for the phase diagram from those simulations in section 5. To examine the question of whether spontaneous symmetry breaking occurs we have conducted the bulk of our simulations with an action that includes explicit symmetry-breaking source terms and we include a detailed study of the volume and source dependence of possible bilinear condensates in section 6. In section 7 we compute the Coleman–Weinberg potential associated with a single-site condensate that breaks the $SU(4)$ symmetry of the model and show that the unbroken state remains a minimum of the potential for all values of the four-fermi coupling in agreement with our numerical study. Finally we summarize our findings and outline future work in section 8.

2. Lattice action and symmetries

Consider a theory of four reduced staggered fermions in four dimensions whose action contains a single-site $SU(4)$-invariant four-fermion term.\(^1\) The action is given by

\[
S = \sum_x \sum_{\mu} \eta_\mu(x) \Delta^a_b(x) \psi^a(x) \psi^b(x) - \frac{1}{4} G^2 \left( \sum_x \epsilon_{abcd}(x) \psi^a(x) \psi^b(x) \psi^c(x) \psi^d(x) \right)
\]  

(2.1)

where $\Delta^a_b(x) = \frac{1}{2} \delta^a_b (\psi^b(x + \hat{\mu}) - \psi^b(x - \hat{\mu}))$ with $\hat{\mu}$ representing unit displacement in the lattice in the $\mu$ direction and $\eta_\mu(x)$ is the usual staggered fermion phase $\eta_\mu(x) = (-1)^{\sum_{i=0}^{d-1} x_i}$. The reduced staggered fermions are taken to transform according to

\[
\psi(x) \rightarrow e^{i\epsilon(x)\alpha} \psi(x)
\]  

(2.2)

with $\alpha$ an arbitrary element of the algebra of $SU(4)$ and $\epsilon(x) = (-1)^{\sum_{i=0}^{d-1} x_i}$ denoting the lattice parity. The presence of the four-fermion interaction breaks the usual global $U(1)$ symmetry down to $Z_4$ whose action is given explicitly by

\[
\psi \rightarrow \Gamma \psi
\]  

(2.3)

where $\Gamma = [1, -1, i\epsilon(x), -i\epsilon(x)]$. The action is also invariant under the shift symmetry

\[
\psi(x) \rightarrow \xi_\mu(x) \psi(x + \hat{\rho})
\]  

(2.4)

where the flavor phase $\xi_\mu(x) = (-1)^{\sum_{i=0}^{d-1} x_i}$. These shift symmetries can be thought of as a discrete remnant of continuum chiral symmetry [13].

\(^1\)The $SO(4)$ symmetry discussed in \cite{3} naturally enhances to $SU(4)$ if the fermions are allowed to be complex. Such an enlargement of the symmetry group does not invalidate the arguments needed to construct an auxiliary field representation or to show the Pfaffian is real, positive definite.
These symmetries strongly constrain the possible bilinear terms that can arise in the lattice effective action as a result of quantum corrections. For example, a single-site mass term of the form $\psi^a(x)\psi^b(x)$ breaks the $SU(4)$ invariance and the $Z_4$ symmetry but maintains the shift symmetry, while $SU(4)$-invariant bilinear terms constructed from products of staggered fields within the unit hypercube generically break the shift symmetries [14, 15] and, depending on the relative site parities, also the $Z_4$ symmetry. The possible $SU(4)$-invariant bilinear operators for a reduced staggered fermion that correspond to a Dirac mass in the continuum limit are

$$O_1 = \sum_{x,\mu} m_{\mu} \epsilon(x) \xi_{\mu}(x) S_{\mu} \psi(a)(x)$$

$$O_3 = \sum_{x,\mu,\nu,\lambda} m_{\mu\nu\lambda} \xi_{\mu}(x + \hat{\mu}) \xi_{\nu}(x + \hat{\nu}) \psi(a)(x) S_{\mu} S_{\nu} S_{\lambda} \psi(a)(x)$$

where $m_{\mu\nu\lambda}$ is totally antisymmetric in its indices. The symmetric translation operator $S_{\mu}$ acts on a lattice field according to

$$S_{\mu} \psi(x) = \psi(x + \hat{\mu}) + \psi(x - \hat{\mu}).$$

It is straightforward to show that these mass terms are all antisymmetric operators, as required for reduced staggered fermions, and break the shift symmetries. They are hence prohibited from appearing in the effective action. Nevertheless condensates of such operators can appear if the vacuum state spontaneously breaks the shift symmetry.

3. Strong-coupling behavior

Before turning to the auxiliary field representation of the four-fermi term and our numerical simulations we can first attempt to understand the behavior of the theory in the limits of both weak and strong coupling. At weak coupling one expects that the fermions are massless and there should be no bilinear condensate since the four-fermi term is an irrelevant operator by power counting.

In contrast the behavior of the system for large coupling can be deduced from a strong-coupling expansion. The leading term corresponds to the static limit $G \to \infty$ in which the kinetic operator is dropped and the exponential of the four-fermi term is expanded in powers of $G$. In this limit the partition function for lattice volume $V$ is saturated by terms of the form

$$Z \sim \left[ 6G^2 \int d\psi^1(x) d\psi^2(x) d\psi^3(x) d\psi^4(x) \psi^1(x) \psi^2(x) \psi^3(x) \psi^4(x) \right]^V$$

corresponding to a single-site four-fermi condensate. To leading order in this expansion it should also be clear that the vev of any bilinear operator will be zero since one cannot then saturate all the Grassmann integrals using just the four-fermion operator.

\[2\text{The usual single-site mass term } \bar{\psi}(x) \psi^a(x) \text{ that is possible for a full staggered field is invariant under all symmetries but this term is absent for a reduced staggered field since in this case there is no independent } \bar{\psi} \text{ field.}\]
To compute the fermion propagator at strong coupling it is convenient to rescale the fermion fields by $\sqrt{\alpha}$ where $\alpha = \frac{1}{\sqrt{6G}} \ll 1$ which removes the coupling from the interaction term and instead places a factor of $\alpha$ in front of the kinetic term. To leading order in $\alpha$ the partition function is now unity. The strong-coupling expansion then corresponds to an expansion in $\alpha$. We follow the procedure described in [16] and consider the fermion propagator $F(x) = \langle \psi^1(x)\psi^1(0) \rangle$. To integrate out the fields at site $x$ one needs to bring down $\psi^2(x)$, $\psi^3(x)$, $\psi^4(x)$ from the kinetic term. This yields a leading contribution

$$F(x) = \left(\frac{\alpha}{2}\right)^3 \int_x D\psi \sum_\mu \eta_\mu(x) (\Psi^1(x + \hat{\mu}) - \Psi^1(x - \hat{\mu})) \psi^1(0) e^{-S}$$

(3.2)

where $\Psi^1 = \psi^2\psi^3\psi^4$ and $\int_x$ means we no longer include an integration over the fields at $x$. We then repeat this procedure at $x \pm \hat{\mu}$ leading to

$$F(x) = \left(\frac{\alpha}{2}\right)^3 \sum_\mu \eta_\mu(x) (\delta_{x+\hat{\mu},0} - \delta_{x-\hat{\mu},0})$$

$$+ \left(\frac{\alpha}{2}\right)^4 \int_{x,x\pm\hat{\mu}} D\psi \sum_\mu (\psi^1(x + 2\hat{\mu}) + \psi^1(x - 2\hat{\mu})) \psi^1(0) e^{-S}.$$  

(3.3)

Notice that to this order in $\alpha$ we can restore the integrations over $x, x \pm \hat{\mu}$ and we now recognize that the right-hand side of this expression contains the propagator at the displaced points $F(x \pm 2\hat{\mu})$. A closed-form expression for the latter can hence be found by going to momentum space where one finds

$$F(p) = \frac{(i/\alpha) \sum_\mu \sin p_\mu}{\sum_\mu \sin^2 p_\mu + m_F^2}$$

(3.4)

with $m_F^2 = -2 + \frac{4}{\alpha^2}$. Thus the strong-coupling calculation indicates that for sufficiently large $G$ the system should realize a phase in which the fermions acquire a mass without breaking the $SU(4)$ symmetry.

An analogous calculation can be performed for the bosonic propagator $B(x) = \langle b(x)b(0) \rangle$ corresponding to the single-site fermion bilinear $b = \psi^1\psi^2 + \psi^3\psi^4$:

$$B(x) = 2\delta_{x,0} + \left(\frac{\alpha}{2}\right)^2 \sum_\mu (B(x + \hat{\mu}) + B(x - \hat{\mu}))$$

(3.5)

or in momentum space

$$B(p) = \frac{8/\alpha^2}{4\sum_\mu \sin^2 p_\mu / 2 + m_B^2}$$

(3.6)

yielding a corresponding boson mass $m_B^2 = -8 + \frac{4}{\alpha^2}$. Thus one expects both bosonic and fermionic excitations to be gapped at strong coupling. Furthermore, this strong-coupling

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$^3$One might have imagined that there are additional contributions arising from sites $x \pm \hat{\mu} \pm \hat{\nu}$ but these in fact cancel due to the staggered fermion phases.
expansion suggests that the mechanism of dynamical mass generation in this model corresponds to the condensation of a bilinear formed from the original elementary fermions $\psi^a$ and a composite three-fermion bound state $\Psi_a = \epsilon_{abcd} \psi^b \psi^c \psi^d$ – the latter transforming in the complex conjugate representation of the $SU(4)$ symmetry. Clearly this is a non-perturbative phenomena invisible in weak-coupling perturbation theory.

The weak- and strong-coupling phases must be separated by at least one phase transition. Previous work with similar lattice Higgs–Yukawa models employing staggered or naive fermions had revealed such a paramagnetic strong-coupling (PMS) phase in a variety of models. However such studies also typically revealed the presence of a third, intermediate phase in which the symmetries of the system were spontaneously broken by the formation of a bilinear fermion condensate [5, 6, 7]. In all these earlier studies this intermediate phase was separated from the weak- and strong-coupling regimes by first-order phase transitions. One of the goals of the current work is to ascertain whether such bilinear condensates appear at intermediate coupling in the current model.

4. Auxiliary field representation

We follow the standard strategy and rewrite the original action (eqn. 2.1) in a new form quadratic in the fermions but including an auxiliary real scalar field. In our case this auxiliary field $\sigma^+_{ab}$ is an antisymmetric matrix in the internal space and possesses an important self-dual property as described below. This transformation preserves the free energy up to a constant:

$$S = \sum_{x, \mu} \psi^a \left[ \eta \cdot \Delta \delta_{ab} + G \sigma^+_{ab} \right] \psi^b + \frac{1}{4} (\sigma^+_{ab})^2$$

where

$$\sigma^+_{ab} = P^+_{abcd} \sigma_{cd} = \frac{1}{2} \left( \sigma_{ab} + \frac{1}{2} \epsilon_{abcd} \sigma_{cd} \right)$$

and we have introduced the projectors

$$P^\pm_{abcd} = \frac{1}{2} \left( \delta_{ac} \delta_{bd} \pm \frac{1}{2} \epsilon_{abcd} \right).$$

In principle one can now integrate over the fermions to produce a Pfaffian Pf$(M)$ where the fermion operator $M$ is given by

$$M = \eta \cdot \Delta + G \sigma^+.$$  

Rather remarkably one can show that the Pfaffian of this operator is in fact positive semi-definite. To see this consider the associated eigenvalue equation

$$(\eta \cdot \Delta + G \sigma^+) \psi = \lambda \psi.$$  

Since the operator is real and antisymmetric the eigenvalues of $M$ are pure imaginary and come in pairs $i\lambda$ and $-i\lambda$. Sign changes in the Pfaffian would then correspond to an odd
number of eigenvalues passing through the origin as the field $\sigma^+$ varies. But in our case we can show that all eigenvalues are doubly degenerate – so no sign change is possible.

This degeneracy stems from the fact that $M$ is invariant under a set of $SU(2)$ transformations that form a subgroup of the original $SU(4)$ symmetry. Specifically $SU(4)$ contains a subgroup $SO(4) \simeq SU(2) \times SU(2)$. While the fermion transforms as a doublet under each of these $SU(2)$s the auxiliary $\sigma^+$ is a singlet under one of them.\(^4\) Since the fermion operator is invariant under this $SU(2)$ its eigenvalues are doubly degenerate. This conclusion has been checked numerically and guarantees positivity of the Pfaffian. It is of crucial importance for our later numerical work since it is equivalent to the statement that the system does not suffer from a sign problem – we can replace $\text{Pf}(M) \rightarrow \det \frac{1}{V} (MM^T)$.

5. Phase diagram

One interesting observable is $\frac{1}{4} \sigma_+^2 = \frac{1}{2} \sum_{a < b} (\sigma_{ab}^+)^2$, which serves as a proxy for the four-fermion condensate and can be computed analytically in the limits $G \to 0$ and $G \to \infty$. Consider the modified action

$$S(G, \beta) = \sum \frac{\beta}{4} \sigma_+^2 + \sum \psi (\eta \Delta + G \sigma_+) \psi.$$  (5.1)

Clearly

$$\left\langle \frac{1}{4} \sigma_+^2 \right\rangle = - \frac{\partial \ln Z(G, \beta)}{\partial \beta}. \quad (5.2)$$

Rescaling $\sigma_+$ by $1/\sqrt{\beta}$ allows us to write the partition function $Z(G, \beta)$ as

$$Z(G, \beta) = \int D\sigma_+ \int D\psi e^{-S} = (\beta)^{-3V/2} Z \left( \frac{G}{\sqrt{\beta}}, 1 \right) \quad (5.3)$$

where we have exploited the antisymmetric self-dual character of $\sigma_+$ by allowing for just 3 independent $\sigma$ integrations at each lattice site. Thus

$$\frac{1}{V} \sum \frac{1}{4} \sigma_+^2 = \frac{3}{2\beta} - \frac{\partial \ln Z \left( \frac{G}{\sqrt{\beta}}, 1 \right)}{\partial \beta}. \quad (5.4)$$

Integrating over the fermions yields

$$Z \left( \frac{G}{\sqrt{\beta}}, 1 \right) = \int D\sigma_+ \text{Pf} \left( \eta \Delta + \frac{G}{\sqrt{\beta}} \sigma_+ \right) e^{-\frac{1}{4} \sigma_+^2}. \quad (5.5)$$

For $G = 0$ the partition function is $\beta$ independent, while its $\beta$ dependence is simply $\beta^{-V}$ in the strong-coupling limit (eqn. 3.1). Using these results and setting $\beta = 1$ one finds

$$\left\langle \frac{1}{4} \sigma_+^2 \right\rangle = \begin{cases} 
3/2 & \text{as } G \to 0 \\
5/2 & \text{as } G \to \infty.
\end{cases} \quad (5.6)$$

\(^4\) $\sigma^-$ is a singlet under the other $SU(2)$ – this is just the standard representation theory of $SO(4)$. 

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In practice we simulate the full antisymmetric $\sigma$ field which allows us to monitor the vev of the anti-selfdual component $\sigma_-$ also. Since this component does not couple to the fermions we expect $\langle \frac{1}{4} \sigma_-^2 \rangle = \frac{3}{2}$ independent of $G$.

Our numerical results for $\langle \frac{1}{4} \sigma_+^2 \rangle - \frac{3}{2}$ shown in fig. 1 are consistent with these predictions. The observed behavior of $\sigma_+^2$ appears to interpolate smoothly between the weak- and strong-coupling limits of eqn. 5.6, while $\sigma_-^2$ shows no dependence on $G$. There are no signs of first-order phase transitions and indeed on $L^4$ lattices with $L > 4$ the observed finite-volume effects are small. In our simulations we have employed a thermal boundary condition; namely the fermions wrapping the temporal direction pick up a minus sign. This has the merit of removing an exact fermion zero mode arising at $G = 0$ and preserves all symmetries of the system.\(^5\)

In addition, to probe the question of spontaneous symmetry breaking, we have also augmented the action shown in eqn. 2.1 by adding source terms which couple to both $SU(4)$-breaking fermion bilinear terms and the shift-symmetry breaking one-link terms described in eqn. 2.5:

$$\Delta S = m \sum_x \left[ \psi^a(x)\psi^b(x) \right]_+ + m \sum_{x,\mu} \epsilon(x)\xi_{\mu}(x)\psi(x)S_{\mu}\psi(x).$$

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\(^5\)This corrects a comment in our earlier paper \[^3\], which stated that the use of antiperiodic boundary conditions causes a breaking of the shift symmetries. We thank Shailesh Chandrasekharan for pointing this out.
In the latter case we assume a rotationally invariant form of the link coupling. The results for the link and site bilinear vevs from runs at $m = 0.1$ with varying $G$ are shown in fig. 2.

While the presence of the source terms clearly leads to non-zero vevs for the bilinears at any coupling $G$, these plots make it clear that these vevs are monotonically suppressed as one enters the strongly coupled regime. We have checked that the vevs of both operators vanish within statistical errors at all values of $G$ when the source terms are set to zero ($m = 0$), due to the exact symmetries.

The transition from weakly coupled free fields to strongly coupled four-fermion condensates is most clearly seen by plotting a susceptibility defined by

$$\chi = \frac{1}{V} \sum_{x,y,a,b} \left\langle \bar{\psi}^a(x) \psi^b(x) \bar{\psi}^a(y) \psi^b(y) \right\rangle. \quad (5.8)$$

Using Wick’s theorem this can be written as sums of products of fermion propagators. We group these into connected $\chi_{\text{conn}}$ and disconnected $\chi_{\text{dis}}$ contributions corresponding to

$$\chi_{\text{conn}} = \frac{1}{V} \sum_{x,y} \left\langle \bar{\psi}^a(x) \psi^a(y) \right\rangle \left\langle \bar{\psi}^b(x) \psi^b(y) \right\rangle - \left\langle \bar{\psi}^a(x) \psi^b(y) \right\rangle \left\langle \bar{\psi}^a(x) \psi^b(y) \right\rangle \quad (5.9)$$

and

$$\chi_{\text{dis}} = \frac{1}{V} \left[ \sum_x \left\langle \bar{\psi}^a(x) \psi^b(x) \right\rangle \right]^2. \quad (5.10)$$

In fig. 3 we plot the logarithm of the connected contribution to $\chi$ for both $m = 0.1$ (left panel) and $m = 0.0$ (right panel). The fermion propagators used in this measurement were obtained by inverting the fermion operator on sixteen point sources located at $(p_1, p_2, p_3, p_4)$ with $p_i \in \{0, L/2\}$ on each configuration and subsequently averaging the results over the
Figure 3: $\ln \chi_{\text{conn}}$ vs $G$ for $L = 4, 6, 8, 12$, considering $m = 0.1$ (left) and $m = 0$ (right).

| $m$ | $\gamma$ | $\chi^2$/dof |
|-----|----------|--------------|
| 0.0 | 3.82(10) | 0.4          |
| 0.1 | 3.33(8)  | 0.8          |

Table 1: Susceptibility exponents from power-law fits at peak $G_c = 1.05$.

Monte Carlo ensemble. A well defined peak that scales rapidly with increasing volume is seen centered around $G_c \approx 1.05$. The position, width and (for $m = 0$) height of this peak agree well with those reported in [12], using the mapping $G^2 = \frac{2}{3} U$ to relate our coupling $G$ to the coupling $U$ appearing in that work. This mapping requires rescaling the fermions by a factor of $\sqrt{2}$ to fix the coefficient of the kinetic term. If we assume that the height of the connected susceptibility peak scales as $\chi_{\text{max}} \sim L^\gamma$ we can try to estimate $\gamma$ from a log–log plot of the susceptibility versus the lattice size. Such a plot is shown in fig. 4 for both $m = 0$ and $m = 0.1$. The exponents extracted from least-squares fits are shown in table 1.

The value $\gamma = 3.82(10)$ for $m = 0$ is in approximate agreement with the volume scaling reported in [12], where an exponent $\gamma = 4$ is attributed to the formation of an $SU(4)$-breaking fermion bilinear condensate. However, such a condensate would manifest itself through the disconnected $\chi_{\text{dis}}$ which is not included in fig. 3. We have monitored $\chi_{\text{dis}}$ separately, computing it using $Z_2$ stochastic volume sources. The logarithm of the ratio $\chi_{\text{dis}}/\chi_{\text{conn}}$ is shown in fig. 5. It is clear that the disconnected contribution arising from any bilinear condensate is relatively small and is not driving the growth of the peak in the susceptibility. Indeed fig. 5 shows that the disconnected contribution is in fact suppressed in the critical region. Thus the divergence in the susceptibility does not appear to have its origin in the formation of a bilinear condensate. In the next section we will report that we could find no evidence of spontaneous symmetry breaking in this model.
To explain the divergence of the connected susceptibility there must necessarily be long-range correlations in the fermion bilinear operator and hence also the fermion propagator. One piece of evidence for this can be seen in fig. 6 where we plot the logarithm of the smallest eigenvalue of the fermion operator vs the coupling. It can be seen that the smallest eigenvalue falls rapidly in a region between $G \approx 1.0-1.1$ consistent with the peak seen in the connected susceptibility.\footnote{This dramatic drop in the smallest eigenvalue is paired with a corresponding rapid increase in the number of conjugate gradient (CG) iterations needed to invert the fermion operator. It is this fact that has limited the largest lattice that we can easily simulate; at the critical point the $L = 12$ lattice requires approximately 20,000 CG iterations per solve.}

We can gain further insight into this issue by computing a bosonic two-point function whose integral yields

$$\chi = \frac{1}{V} \sum_t G(t)$$

where

$$G(t) = \sum_{x,y,a,b} \langle B^{ab}(x)B^{ab}(y) \rangle \delta(x_t - y_t - t)$$

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure4.png}
\caption{ln $\chi_{\text{conn}}$ vs ln $L$ at $G = 1.05$ for $m = 0.1$ (lower points) and $m = 0$ (upper points). The straight lines guide the eye by illustrating power laws $\chi_{\text{conn}} \propto L^\gamma$ with $\gamma = 3.4$ for $m = 0.1$ and $\gamma = 3.8$ for $m = 0$.}
\end{figure}

with $B^{ab}(x) = \psi^a(x)\psi^b(x)$ and the delta function picks out points separated by $t$ units in the time direction. This connected correlation function $G(t)$ is shown in fig. 7 for $8^3 \times 16$ lattices at $m = 0$. The solid lines correspond to cosh fits and allow us to read off the
mass of the bosonic state created by operating on the vacuum with the bilinear operator $\psi^a(x)\psi^b(x)$. Fig. 8 shows this mass as a function of the coupling $G$. Notice that in the critical region $1.0 \leq G \leq 1.1$ corresponding to the peak in the susceptibility the mass is very small and independent of $G$. This structure together with the observed rather broad peak in the susceptibility prompts one to conjecture that the system may indeed possess a narrow intermediate phase as reported in [12]. Where we differ from [12] is in the question of whether such a phase is characterized by a bilinear condensate. While we see no evidence for a bilinear condensate there are clear indications of the presence of gapless excitations in this region of the phase diagram. Indeed, it is quite possible that this bosonic correlator develops a power-law behavior in this regime although the accuracy of our current data is insufficient to determine any power-law exponent. It should also be noted that the amplitude of the correlator is quite sensitive to both the coupling and the lattice volume in this critical region. The latter fact is likely partly responsible for the rapid scaling seen in the susceptibility in this region.

6. Bilinear condensates and spontaneous symmetry breaking

Consider first a possible mass operator of the form

$$O_0 = \sum_{a,b} \sum_x \left[ \psi^a(x)\psi^b(x) \right] \Sigma^a_{\downarrow}$$  \hspace{1cm} (6.1)
Figure 6: $2 \ln \lambda_{\text{min}}$ vs $G$ for $L = 8, 12$ and $m = 0.1$.

where the external self-dual source takes the symmetry-breaking form

$$\Sigma_+ = \begin{pmatrix} i \sigma_2 & 0 \\ 0 & i \sigma_2 \end{pmatrix}. \quad (6.2)$$

As described earlier we have added such a term to the action with coupling $m$ and computed its vev at both $G = 1.0$ and $G = 1.05$ for a range of $m$ and several lattice volumes. This operator serves as an order parameter for the spontaneous breaking of the $SU(4)$ symmetry. As can be seen from the scan in $G$ given earlier in fig. 2 the coupling $G = 1.0$ represents one edge of the critical region and a point where a small, brief increase is visible in the one-link condensate as $G$ increases. This marks it out as a candidate location for possible spontaneous breaking of the shift symmetries. The value $G = 1.05$ corresponds to the peak in the susceptibility and center of the critical region and hence marks another likely point to search for spontaneous symmetry breaking. Fig. 9 shows the behavior of the site vev for lattices with $L = 4, 6, 8, 12$ for both $G = 1.0$ and $G = 1.05$. As expected the vev of $O_0$ vanishes for $m \to 0$ for any fixed volume. What is more significant is the lack of any evidence for a strong volume dependence – a prerequisite for spontaneous symmetry breaking. Indeed, one expects that the vev must increase with $V$ at small $m$ in order that a finite value be obtained as one takes the limits $V \to \infty$ followed by $m \to 0$. While the statistical errors are relatively large in the $G = 1.05$ data there is no sign of the necessary volume dependence in the plot. At best any condensate must be small, $O(0.005)$ in lattice units, to be compatible with our data. And it is quite consistent to conclude that in fact the condensate vanishes.
The results for the one-link operator $O_1$ defined in eqn. 2.5 are shown in fig. 10. We have chosen to monitor the vacuum expectation value of this operator as an order parameter for the spontaneous breaking of the lattice shift symmetries. Again it is clear that $\langle O_1 \rangle$ approaches zero as $m \to 0$ for any fixed volume as expected on the basis of the shift symmetries. The question of whether these symmetries break spontaneously can be answered by examining the volume dependence of the vev for small $m$. As can be seen in fig. 10 the volume dependence is very small and there is no hint of a condensate growing with volume for small $m$ as would be expected if such a condensate were to arise from spontaneous breaking of the shift symmetries.

To summarize our numerical results provide evidence that the symmetry-breaking condensates vanish for all values of the four-fermi coupling.

7. Coleman–Weinberg effective potential

The question of whether spontaneous symmetry breaking occurs can be examined by computing the effective potential for the auxiliary field $\sigma$.\footnote{We thank Jan Smit for pointing out this possibility.} We assume the latter takes the form

$$\sigma_+ = \mu \begin{pmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}.$$  \hfill (7.1)
Figure 8: Mass of the bilinear state $B^{ab} = \psi^a \psi^b$ versus $G$, for $8^3 \times 16$ lattices at $m = 0$. Most error bars are smaller than the symbols.

Figure 9: Site bilinear vs $G$ for $L = 4, 6, 8, 12$ with $G = 1.0$ (left) or $G = 1.05$ (right).

After integrating over the fermions the effective action takes the form

$$S_{\text{eff}}(\sigma_+) = -\frac{1}{2} \text{Tr} \ln(\eta, \Delta + G\sigma_+).$$  \hspace{1cm} (7.2)
In a constant $\sigma$ background the kinetic term can be diagonalized and written

$$V_{\text{eff}} (\sigma_+) = -\frac{1}{4} \text{tr} \sum_k [\ln (i\lambda_k + G\sigma_+) + \ln (-i\lambda_k + G\sigma_+)]$$  \hspace{1cm} (7.3)

where we have exploited the fact that the eigenvalues of $\eta_D$ come in pairs $(i\lambda, -i\lambda)$ and $\text{tr}$ denotes the remaining trace over $SU(4)$ indices. Combining these terms, diagonalizing in the internal space and recalling that the Pfaffian is real positive semidefinite yields

$$V_{\text{eff}} = -\sum_k \ln |\lambda_k^2 - G^2\mu^2|.$$  \hspace{1cm} (7.4)

Clearly the effective potential is extremized at $\mu = 0$ and it is trivial to further show that $\frac{\partial^2 V_{\text{eff}}}{\partial \mu^2}|_{\mu=0} > 0$ independent of $G$. Thus the symmetric state $\mu = 0$ remains a local minimum of the effective potential for all $G$ – there can be no spontaneous symmetry breaking. It should be clear that the anti-hermitian nature of the Yukawa coupling arising from this four-fermion interaction combined with the positivity of the Pfaffian of the fermion operator play a key role in ensuring this result. On reflection this should not be too surprising; at least at weak coupling and in the continuum limit the $SU(4)$ symmetry should be interpreted as a vector-like flavor symmetry in which case the Vafa–Witten theorem would generally prohibit spontaneous symmetry breaking [17].

8. Conclusions

In this paper we have studied a four-dimensional lattice theory comprising four flavors of reduced staggered fermions coupled through an $SU(4)$-invariant four-fermion interaction. Strong-coupling arguments allow us to infer that the system develops a massive phase for sufficiently large four-fermi coupling without breaking symmetries. Such a (paramagnetic
strong-coupling or PMS) phase has been seen before in other lattice fermion models and is generically separated from a massless paramagnetic weak-coupling (PMW) phase by an intermediate phase characterized by a symmetry-breaking bilinear fermion condensate – see the numerical results in [5, 6, 7] and a large-$N$ argument given in [18]. Furthermore, in all previous studies, this intermediate phase was bordered by first-order phase transitions precluding any new continuum limits.

The novel aspect of the current model is that we find no evidence for a bilinear condensate for any value of the coupling $G$ and no evidence for first-order phase transitions. Instead a broad critical region is observed in which the lowest eigenvalue of the fermion operator becomes very small and long-range correlations are observed in a bilinear fermion operator. These long-range correlations lead to a corresponding susceptibility which diverges with increasing lattice size. The width of the peak in the susceptibility does not diminish quickly with increasing lattice volume and fits to the mass of the bosonic state in this region yield small $G$-independent values. This observation leads to the possibility that the system possesses a narrow intermediate phase as argued for in ref [12]. However, since we do not find evidence for bilinear condensates in our work we cannot exclude the possibility that a single phase transition separates the weak- and strong-coupling phases. In either scenario any phase transitions(s) appear continuous.

This situation is reminiscent of the mechanism by which the two-dimensional Thirring model develops a mass gap without breaking chiral symmetry [19]. In the two-dimensional case the corresponding correlator of fermion bilinears falls off with distance as a power law with a small power $\Delta \sim 1/N_f$, where $N_f$ is the number of continuum flavors, and the spectral density develops a branch cut rather than a simple pole.

There has been considerable interest in recent years within the condensed matter community in the construction of models in which fermions can be gapped without breaking symmetries using carefully chosen quartic interactions [21, 22]. Although the condensed matter models are constructed using Hamiltonian language and describe non-relativistic fermions it is nevertheless intriguing that the sixteen Majorana fermions they require match the sixteen Majorana fermions that are expected at weak coupling in this lattice theory. It has been proposed that such quartic interactions can be used in the context of domain wall fermion theories to provide a path to achieve chiral lattice gauge theories [23, 24]. If indeed the current model avoids symmetry-breaking phases it may be possible to revisit the original Eichten–Preskill proposal for the construction of chiral lattice gauge theories using strong four-fermion terms in the bulk to lift fermion doubler modes [16, 25]. However, it is not clear to the authors how such constructions can work in detail; the model described here uses reduced staggered rather than Wilson or naive fermions which negates a simple transcription of the four-fermion interaction appearing in this model to those earlier constructions.

Independently of these speculations one can wonder whether the phase transition(s) in the model described here are evidence of new continuum limit(s) for strongly interacting

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8We thank Simon Hands for bringing this and related papers to our attention [20].
fermions in four dimensions. One must be careful in drawing too strong a conclusion at this stage; even if a new fixed point exists it might not be Lorentz invariant. Indeed, given the connection between staggered fermions and Kähler–Dirac fermions such a scenario is possible given that the latter are invariant only under a twisted group comprising both Lorentz and flavor symmetries [26]. In staggered approaches to QCD one can show that the theory becomes invariant under both symmetries in the continuum limit. However this may not be true when taking the continuum limit in the vicinity of a strongly coupled fixed point.

Clearly, further work, both theoretical and computational, will be required to understand these issues. On the numerical front one will need to simulate larger systems to improve control over finite-volume effects and allow for a more precise determination of critical exponents. It is possible that higher-resolution studies will reveal small but non-zero bilinear condensates on larger volumes or that the continuous transitions we observe will become first order. Such future studies will likely require significant improvements to the simulation algorithm, for example by using deflation techniques and/or carefully chosen preconditioners to handle the small fermion eigenvalues.

Acknowledgments

This work is supported in part by the U.S. Department of Energy (DOE), Office of Science, Office of High Energy Physics, under Award Number DE-SC0009998. We thank Venkt Ayyar, Shailesh Chandrasekharan, Poul Damgaard, Simon Hands, Jan Smit and Erich Poppitz for useful discussions at various stages of this work. Numerical simulations were performed at Fermilab using USQCD resources.

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