A wave equation perturbed by viscous terms: fast and slow times diffusion effects in a Neumann problem

Monica De Angelis
Univ. di Napoli “Federico II”,
Scuola Politecnica e delle Scienze di Base,
Dip. Mat. Appl. ”R.Caccioppoli”,
Via Cintia 26, 80126, Napoli, Italia.
modeange@unina.it

A Neumann problem for a wave equation perturbed by viscous terms with small parameters is considered. The interaction of waves with the diffusion effects caused by a higher-order derivative with small coefficient $\varepsilon$, is investigated. Results obtained prove that for slow time $\varepsilon t < 1$ waves are propagated almost undisturbed, while for fast time $t > \frac{1}{\varepsilon}$ diffusion effects prevail.

1 Introduction

The paper deals with an analysis of the diffusion equation:

\[
L_\varepsilon u \equiv \partial_{xx}(\varepsilon \partial_t + 1)u - \partial_t(\partial_t + a)u = f
\]

which, as it happens for artificial viscosity methods, represents a model of wave equation perturbed by viscous terms with small parameters.

According to the meaning of source term $f$, equation (1.1) represents numerous examples of dissipative phenomena in several fields such as physics, neurobiology or engineering, and in each of these subjects an extensive bibliography exists in this regards. (see, f.i. [1]- [10] and reference therein).

In particular, for $f = \sin u + \gamma$, (1.1) expresses the perturbed Sine-Gordon equation which models the Josephson effect in Superconductivity [11]. In many other cases, extra terms can be considered to describe various Josephson junctions such as semiannular, S-shaped, window or exponentially shaped (see, f.i [12]- [15]).

Moreover, an equivalence between the third order equation (1.1) and an integro differential equation has been proved in [16]- [19] and this further allows
us to create a direct connection between biological phenomena and superconductivity. For instance, the propagation of nerve impulses described by the FitzHugh-Nagumo model can be related to the Pertubed Sine Gordon Equation. (see [8],too).

Naturally, equation (1.1) is to be complemented by initial problems and boundary conditions such as Neumann, Dirichlet, pseudo periodic or mixed problems that are meaningful in many scientific fields (see, f.i. [16,20]). Particular attention will be given to Neumann problem that is relevant in the ecological model when the exterior environment is completely hostile to the species [21], or in the study of cardiac rhythmicity, particularly for pacemakers [22]. Moreover, in superconductivity, in the case of a Josephson junction, Neumann problem can refer to the phase gradient value that is proportional to the magnetic field [23,24].

1.1 Mathematical considerations, state of the art and aim of the paper

When the behaviour of $u(x,t)$ as $\varepsilon \to 0$ is examined, the interaction of waves with the diffusion effects caused by $\varepsilon u_{xxt}$ can be estimated, and this physical aspect is meaningful to the evolution of dissipative models.

When a fixed boundary-initial problem $P_\varepsilon$ is stated and the function $f$ is linear, by means of the related Green function $G_\varepsilon$, it is possible to solve $P_\varepsilon$ explicitly. Otherwise, if the source term $f$ is non linear, the problem $P_\varepsilon$ can be reduced to an integral equation [25].

So, for $\varepsilon \equiv 0$, the parabolic equation (1.1) turns into the following equation

$$\mathcal{L}_0 U \equiv (\partial_{xx} - a \partial_t - \partial_{tt})U = f, \quad (x,t) \in \Omega,$$

and $P_\varepsilon$ changes into a problem $P_0$ for $U$, with the same initial-boundary conditions of $P_\varepsilon$.

In small time intervals, the wave behaviour is a believable approximation of $u_\varepsilon$ when $\varepsilon$ is vanishing. Conversely, when the time $t$ is large, diffusion effects should prevail and the behaviour of $u_\varepsilon$ when $\varepsilon \to 0$ and $t \to \infty$ should be analyzed.

So that, denoting by $T$ an arbitrary positive constant, let

$$\Omega = \{(x,t) : 0 \leq x \leq \pi, \quad 0 < t \leq T\},$$

and let us assume as $P_\varepsilon$ the following linear Neumann problem:

$$\begin{cases} 
\partial_{xx}(\varepsilon u_t + u) - \partial_t(u_t + a) = f(x,t), & (x,t) \in \Omega, \\
u(x,0) = f_0(x), & u_t(x,0) = f_1(x), & x \in [0,\pi], \\
u_x(0,t) = \varphi(t), & u_x(\pi,t) = \psi(t), & 0 < t \leq T.
\end{cases}$$

(1.3)
It is important to underline that problem (1.3) defined on space interval $[0, \pi]$ is equivalent to the system defined on an arbitrary interval $[0, L]$. Indeed, it is possible to consider a finite interval $[0, L]$ by rescaling $t \rightarrow \tau c$ and $x \rightarrow c \bar{x}$ with $c = \pi / L$. So that, in many cases the spatial coordinate $x$ is normalized without losing generality (see, f.i. [11], [26], [27]).

Moreover, as $f(x, t), f_0(x), f_1(x)$ are quite arbitrary, it is not restrictive to assume $\varphi(t) = 0, \psi(t) = 0$. Otherwise, it suffices to put

$$\bar{u} = u - \frac{x}{2\pi} \left[ (2\pi - x) \varphi + x \psi \right]$$

$$F = f + \left( \frac{\varepsilon}{\pi} + \frac{\alpha x^2}{2\pi} \right) \left( \dot{\varphi} - \dot{\psi} \right) + \frac{1}{\pi} \left( \varphi + \psi \right) + \frac{x^2}{2\pi} \left( \ddot{\varphi} - \ddot{\psi} \right) - x(\alpha \dot{\varphi} + \ddot{\varphi})$$

and then modify $f_0(x), f_1(x)$ accordingly. So, henceforth removing the superscripts, one obtains:

$$\partial_{xx}(\varepsilon u_t + u) - \partial_t (u_t + a) = F(x, t), \quad (x, t) \in \Omega,$$

$$u(x, 0) = F_0(x), \quad u_t(x, 0) = F_1(x), \quad x \in [0, \pi],$$

$$u_x(0, t) = 0, \quad u_x(\pi, t) = 0, \quad 0 < t \leq T$$

where

$$\begin{cases} F_0 = f_0 - \frac{x}{\pi} \left[ (2\pi - x) \varphi(0) + x \psi(0) \right]; \\ F_1 = f_1 - \frac{x}{\pi} \left[ (2\pi - x) \dot{\varphi}(0) + x \dot{\psi}(0) \right]. \end{cases}$$

Consequently, the problem $\mathcal{P}_0$ is the following:

$$\begin{cases} U_{xx} - U_{tt} - aU_t = F(x, t), \quad (x, t) \in \Omega, \\ U(x, 0) = F_0(x), \quad U_t(x, 0) = F_1(x), \quad x \in [0, \pi], \\ U_x(0, t) = 0, \quad U_x(\pi, t) = 0, \quad 0 < t \leq T. \end{cases}$$

Let us define $G_\varepsilon(x, \xi, t)$ as the Green function of the operator $\mathcal{L}_\varepsilon = \varepsilon \partial_{xxx} + \partial_{xx} - \partial_{tt} - a\partial_t$ introduced in (1.1). It is possible to determine $G_\varepsilon$ by means of Fourier series. Indeed, letting:

$$h_n = \frac{1}{2} (a + \varepsilon n^2), \quad \omega_n = \sqrt{h_n^2 - n^2}; \quad G_n^\varepsilon(t) = \frac{1}{\omega_n} e^{-h_n t} \sinh(\omega_n t),$$

it results:
\( G_\varepsilon(x,t,\xi) = \frac{1}{\pi} \frac{1 - e^{-\alpha t}}{a} + \frac{2}{\pi} \sum_{n=1}^{\infty} G_n^\varepsilon(t) \cos(n\xi) \cos(nx). \)

Moreover, assuming

\( \omega_0 = \sqrt{n^2 - (a/2)^2} \quad G_n^\varepsilon(t) = e^{-\frac{\omega_0}{\omega_0}} \frac{1}{\omega_0} \sin(\omega_0 t), \)

the Green function related to problem (1.7) is:

\( G_o(x,t,\xi) = \frac{1}{\pi} \frac{1 - e^{-\alpha t}}{a} + \frac{2}{\pi} \sum_{n=1}^{\infty} G_n^o \cos(n\xi) \cos(nx). \)

As for the state of art, the interactions between diffusion effects and wave propagation have already been studied for Dirichlet conditions. In particular, when \( \alpha = 0 \), an asymptotic approximation for the Green function has been established by means of the two characteristic times: slow time \( \tau = \varepsilon t \) and fast time \( \theta = t/\varepsilon \) [28]. Moreover, for the semilinear equation related to an exponentially shaped Josephson junction, in [29] an analytical analysis proves that the surface damping has little influence on the behaviour of oscillations, thus confirming numerical results showed in [30]. Other numerical investigations on the influence of surface losses can be found in [31], too.

As for the operator \( L_\varepsilon \), in [32] it is possible to find a short review, while for the Dirichlet and Neumann problem in [33]- [20] the Green function has already been explicitly examined by means of a Fourier series, and its properties have been proved.

Aim of this paper is to provide an estimate of the influence of the diffusion term \( \varepsilon u_{xxt} \) on wave propagation. Parameters \( 0 < \alpha < 1 \) and \( 0 < \varepsilon < 1 \) are adopted, and, according to problems (1.5) and (1.7), the following difference

\( |u(x,t,\varepsilon) - e^{-\frac{\alpha t}{2}} U(x,t)| \)

is evaluated proving, as expected, that \( u(x,t,\varepsilon) \) tends to \( U(x,t) \) as soon as \( \varepsilon \) tends to zero.

Moreover, when slow times \( \varepsilon t < 1 \) are considered, there exists a positive constant \( K \) depending on data and source term, such that

\( |u(x,t,\varepsilon) - e^{\frac{t}{\varepsilon}} U(x,t)| < K. \)

Hence, for \( t \in (0, \frac{1}{\varepsilon}) \), the wave is propagated almost undisturbed.

Conversely, for fast time \( \varepsilon t > 1 \), damped oscillations due to term \( \varepsilon u_{xxt} \) predominate.
The estimate allows us also to evaluate the behaviour when time \( t \) tends to infinity proving that, for a suitable source term, solution \( u(x,t,\varepsilon) \) is bounded.

The paper is organized as follows:

In section 2 some properties of the Green function \( G_\varepsilon(x,t,\varepsilon) \) are recalled and the solution related to problem (1.5) is determined.

In section 3 an analysis of the difference \( |G_\varepsilon(x,t,\varepsilon) - e^{\frac{\varepsilon}{a^2}} G_0(x,t)| \), and its derivative with respect to \( t \), is performed. Besides, a further inequality for Green Function is achieved.

Finally, in section 4, difference (1.12) is estimated, and behaviours due to slow and fast times are underlined together with asymptotic behaviour when time \( t \) increases.

2 On properties related to Green function \( G_\varepsilon \)

and explicit solution of the third order problem

In [14, 33] some properties of the Green function for Dirichlet-type boundary conditions have already been determined. It is possible to prove that the same properties are equally valid for the Green function related to a Neumann boundary problem.

So that, denoting by \( \beta \equiv \min \{ (\varepsilon + a)^{-1}; (a + \varepsilon)/2; a \} \), and letting

\[
G := 2 \sum_{n=1}^{\infty} G_\varepsilon^n(t) \cos(n\xi) \cos(nx),
\]

since (1.9), it results:

\[
G_\varepsilon(x,t,\xi) = \frac{1}{\pi} \frac{1 - e^{-\frac{a}{2} t}}{a} + G(x,\xi,t)
\]

and, according to Lemma 2.1 of [33] (assuming \( \ell \equiv \pi \) and \( \epsilon^2 = 1 \)), one has:

**Theorema 2.1.** For \( a, \varepsilon \in \mathbb{R}^+ \), the function \( G_\varepsilon(x,\xi,t) \) defined in (1.9) and all its time derivatives are continuous functions. Moreover, there exist positive constants \( M \), and \( D_j \) (\( j \in \mathbb{N} \)) depending on \( a \) and \( \varepsilon \), such that:

\[
|G(x,\xi,t)| \leq M e^{-\beta t} \quad \left| \frac{\partial^j G_\varepsilon}{\partial t^j} \right| \leq D_j e^{-\beta t}, \quad j \in \mathbb{N}
\]

Besides, as for \( x \)-differentiation, denoting by \( G_{\varepsilon,t} := \frac{\partial G_\varepsilon}{\partial t} \) and \( G_{\varepsilon,x} := \frac{\partial G_\varepsilon}{\partial x} \), the uniform convergence of \( G_{\varepsilon,x} \) is assured by means of standard criteria, while the absolute convergence of \( (\varepsilon G_{\varepsilon,t} + G_\varepsilon) \) and its first and second derivatives are guaranteed by means of the following inequalities already proved in Lemma 2.2 of [33]. Indeed, one has:
Teorema 2.2. For all $a, \varepsilon, \in \mathbb{R}^+$, the function $G_{\varepsilon,x}(x,\xi,t)$ is a continuous function and it converges uniformly for all $x \in [0,\pi]$. Moreover, it results

\begin{equation}
|\partial_x^{(i)}(\varepsilon G_{\varepsilon,t} + G_{\varepsilon})| \leq M_i e^{-\beta t}, \quad (i = 1, 2, 3)
\end{equation}

where $M_i$ ($i = 1, 2, 3$) are all positive constants depending on $a$ and $\varepsilon$. Furthermore, according to Theorem 2.1 of [33] it results:

\begin{equation}
L_{\varepsilon}G_{\varepsilon} = \partial_{xx}(\varepsilon G_{\varepsilon,t} + G_{\varepsilon}) - \partial_t(G_{\varepsilon,t} + aG_{\varepsilon}) = 0.
\end{equation}

Now, let $f(x)$ be a continuous function on $(0,\pi)$, and let

\begin{equation}
\tilde{f} := \frac{1}{\pi} \int_0^\pi f(\xi) \ d\xi
\end{equation}

Then, the following theorems hold:

Teorema 2.3. Let $F_1(x)$ be a $C^1$ function on $[0,\pi]$ with $\dot{F}_1(0) = \dot{F}_1(\pi) = 0$. The function

\begin{equation}
u_1(x,t) = \left(1 - e^{-a t}\right) \tilde{F}_1 + \int_0^\pi F_1(\xi) \ G(x,\xi,t) \ d\xi
\end{equation}

is a solution of the equation $L_{\varepsilon}u_1 = 0$ and it satisfies the homogeneous boundary conditions. Moreover it results:

\begin{equation}
\lim_{t \to 0} u_1(x,t) = 0, \quad \lim_{t \to 0} \partial_t u_1(x,t) = F_1(x)
\end{equation}

uniformly for all $x \in [0,\pi]$.

Proof. Theorems 2.1 and 2.2 and continuity of $F_1$ assure that function (2.20) and its partial derivatives converge absolutely for all $(x,t) \in \Omega$ and $L_{\varepsilon}u_1 = 0$. Besides, (2.21)$_1$ holds, too.

Furthermore, since:

\begin{equation}
\partial_t \int_0^\pi F_1(\xi) \ G(x,\xi,t) \ d\xi = -\frac{2}{\pi} \int_0^\pi \sum_{n=1}^\infty \hat{G}_n \sin(n\xi) \ d\xi,
\end{equation}

denoting by $\eta(x)$ the Heaviside function, it results:

\begin{equation}
\lim_{t \to 0} \partial_t u_1 = \tilde{F}_1 + \int_0^\pi \left[\frac{\xi}{\pi} - \eta(\xi - x)\right] \hat{F}_1(\xi) d\xi = F_1(x).
\end{equation}

Moreover, by means of the uniform convergence of $G_{\varepsilon}$, also boundary conditions (1.5)$_3$ hold as well.

\[\square\]
Teorema 2.4. If \( F_0(x) \in C^2[0, \pi] \) with \( \dot{F}_0(0) = \dot{F}_0(\pi) = 0 \), then the function

\[
(2.24) \quad u^*(x, t) = \tilde{F}_0 + (\partial_t + a - \varepsilon \partial_{xx}) \int_0^\pi F_0(\xi) \ G(x, \xi, t) \ d\xi
\]
is a solution of the equation \( \mathcal{L}_\varepsilon u^* = 0 \), it satisfies boundary conditions and it results:

\[
(2.25) \quad \lim_{t \to 0} u^*(x, t) = F_0(x), \quad \lim_{t \to 0} \partial_t u^*(x, t) = 0,
\]
uniformly for all \( x \in [0, l] \).

Proof. Let us define:

\[
(2.26) \quad u_{F_0} := \int_0^\pi F_0(\xi) \ G(x, \xi, t) \ d\xi; \quad u_{\dot{F}_0} := \int_0^\pi \dot{F}_0(\xi) \ G(x, \xi, t) \ d\xi
\]

Hypotheses on \( F_0(x) \) and theorem 2.1 assure that \( \partial_{xx} u_{F_0} = u_{\dot{F}_0} \) and so, since (2.18), equation \( \mathcal{L}_\varepsilon u^* = 0 \) is verified as well. Moreover, being

\[
(2.27) \quad (\partial_{tt} + a \partial_t) u_{F_0} = (\varepsilon \partial_{xxt} + \partial_{xx}) u_{\dot{F}_0}
\]
it results \( \partial_t u^* = u_{\dot{F}_0} \) and so (2.25) holds. Finally, being

\[
(2.28) \quad \lim_{t \to 0} u^* = \tilde{F}_0 + \lim_{t \to 0} \partial_t u_{\dot{F}_0}
\]
similarly to (2.22)-(2.23), (2.25) follows, too. \( \Box \)

Now, let us consider the convolution of the Green function with the source term \( F(x,t) \).

Teorema 2.5. Let the function \( F(x,t) \) be a continuous function in \( \Omega \) with continuous derivative with respect to \( x \), then the function

\[
(2.29) \quad u_F = - \int_0^t \left( \frac{1 - e^{-a(t-\tau)}}{a} \right) \dot{F}(\tau)d\tau - \int_0^t d\tau \int_0^\pi F(\xi, \tau)G(x, \xi, t-\tau)d\xi,
\]
satisfies equation \( \mathcal{L}_\varepsilon u_F = F \) and homogeneous boundary conditions are verified. Moreover, one has:

\[
(2.30) \quad \lim_{t \to 0} u_F(x, t) = 0, \quad \lim_{t \to 0} \partial_t u_F(x, t) = 0,
\]
uniformly for all \( x \in [0, \pi] \).
Proof. In the same way as (2.22) and (2.23), it results:

\[
\lim_{\tau \to t} \left( \tilde{F}(t) + \int_{0}^{\pi} F(\xi, \tau) \ G_t(x, \xi, t - \tau) \ d\xi \right) = F,
\]

so that, one has:

\[
(2.31) \quad \partial_t u_F(x, t) = -F(x, t) + a \int_{0}^{t} \tilde{F} e^{-a(t-\tau)} d\tau - \int_{0}^{\pi} F(\xi, \tau) G_{tt}(x, \xi, t-\tau) d\xi
\]

and, by means of properties (2.18), \( L_{\epsilon} u_F = F \) is verified.

Furthermore, owing to estimates (2.16), since (2.29), initial homogeneous conditions follow.

Moreover, by means of the uniform convergence of \( G_x \), boundary conditions (1.5) hold, as well.

Since uniqueness can be a consequence of the energy method (see, f.i. [33] and reference therein), the following theorem holds:

**Teorema 2.6.** When data \((F_1, F_0, F)\) satisfy respectively the hypotheses of theorems 2.3, 2.4, 2.5, then

\[
(2.32) \quad u(x, t) = \tilde{F}_0 + \left( \frac{1 - e^{-a t}}{a} \right) \tilde{F}_1 + \int_{0}^{\pi} F_1(\xi) \ G(x, \xi, t) d\xi +
\]

\[
(\partial_t + a - \varepsilon \partial_{xx}) \int_{0}^{\pi} F_0(\xi) \ G(x, \xi, t) d\xi - \int_{0}^{t} \left( \frac{1 - e^{-a(t-\tau)}}{a} \right) \tilde{F}(\tau) d\tau
\]

\[- \int_{0}^{t} d\tau \int_{0}^{\pi} G(x, \xi, t - \tau) F(\xi, \tau) d\xi.
\]

represents the unique solution of problem 1.5.

3 Estimates related to the Green Functions

Let us consider the following difference:

\[
(3.33) \quad \sum_{n=1}^{\infty} \left[ e^{-h_n} \frac{\sinh(\omega_n t)}{n^2 \omega_n} - e^{-h_1} \frac{\sin(\omega_0 t)}{n^2 \omega_0} \right] g_n = \sum_{n=1}^{\infty} H_n g_n
\]

where

\[
(3.34) \quad g_n = \frac{2}{\pi} \cos(n\xi) \cos(nx); \quad h_1 = (a + \varepsilon)/2.
\]

Assuming \( 0 < a < 1 \) and \( 0 < \varepsilon < 1 \), let us denote by \( N_1 \) the minimum natural number larger than \( \frac{1}{\varepsilon} \left( 1 - \sqrt{1 - a \varepsilon} \right) \), and by \( N_2 \) the maximum natural number smaller than \( \frac{1}{\varepsilon} \left( 1 + \sqrt{1 - a \varepsilon} \right) \).
Since $N_1 < 1$, $G_n^e(t)$ in (1.8) contains trigonometric functions for $1 \leq n \leq N_2$ and hyperbolic terms for $n \geq N_2 + 1$. Hence intervals $(1, N_2); (N_2 + 1, \infty)$ must be considered.

So, denoting

\[
(3.35) \quad \sum_{n=1}^{\infty} H_n g_n = \sum_{n=1}^{N_2} H_n(t, \varepsilon) g_n + \sum_{n=N_2+1}^{\infty} H_n(t, \varepsilon) g_n = H_1 + H_2,
\]

the following theorem holds:

**Theorem 3.7.** Whatever $1/2 < \gamma < 1$, and $0 < \delta < 2$ may be, there exists a positive constant $A$, independent from $\varepsilon$, such that the following estimate holds:

\[
(3.36) \quad \left| \sum_{n=1}^{\infty} \left[ G_n^e(t) - e^{-\frac{\varepsilon t}{n}} G_n^o(t) \right] \frac{g_n}{n^2} \right| \leq A \left[ e^{1-\gamma} r(t) e^{-\frac{\varepsilon t}{n}} + \varepsilon e^{-\frac{\varepsilon t}{4}} \right]
\]

where $\theta$ denotes the fast time $t/\varepsilon$ and $r(t) = 1 + t + t^{1-\gamma} + t^{2-\delta}$.

**Proof.** Referring to the trigonometric terms related to $H_1$, defined in (3.35), indicating by $H_n(s, \varepsilon)$ the Laplace transform of function $H_n(t, \varepsilon)$, one deduces that

\[
n^2 \hat{H}_n(s, \varepsilon) = \frac{(s + h_1)^2 + \omega_n^2 - (s + h_n)^2 - \hat{\omega}_n^2}{((s + h_1)^2 + \omega_n^2) ((s + h_n)^2 + \hat{\omega}_n^2)}
\]

where $\hat{\omega}_n^2 = n^2 - h_n^2$. So that, denoting by $\hat{g}_1 = \frac{1}{2}(a + \frac{\varepsilon}{4})$, it results

\[
(3.37) \quad \hat{H}_n(t, \varepsilon) = \frac{\hat{\omega}_n}{n^2 \omega_n} \int_0^t e^{-h_1(t-\tau)} \sin(\omega_n(t-\tau)) e^{-h_n \tau} \sin(\omega_n \tau) d\tau + \varepsilon \frac{(n-a-1)}{n^2 \omega_n} \int_0^t e^{-h_1(t-\tau)} \sin(\omega_n(t-\tau)) \left\{ e^{-h_n \tau} \left[ \frac{h_n}{\omega_n} \sin(\omega_n \tau) - \cos(\hat{\omega}_n \tau) \right] \right\}.
\]

Now, letting $g_0 = \sqrt{1 - \frac{a^2}{4}}, \quad g_1 = \frac{1}{2}(a + \frac{1}{2})$, one has

\[
(3.38) \quad \omega_0 \geq n g_0, \quad \hat{g}_1 \leq g_1.
\]

Moreover, since $\hat{\omega}_n \geq n \sqrt{1 - h_n/n} = n \Phi_n$, indicating by

\[
s = \sqrt{\frac{2(1-a^2-\varepsilon)}{2(1-a^2+1-\varepsilon)}}, \quad q = \sqrt{\frac{2-a-\varepsilon}{2}} = (\Phi_n)_{n=1}^{N_2-1}, \quad \ell = \min \{ s, q \},
\]

and by

\[
\ell = \min \{ s, q \},
\]

M. De Angelis
it is possible to choose a positive constant $g_2$ depending on parameter $a$, but independent from $\varepsilon$, such that $g_2 \leq \ell$. In such a way, for all $1 \leq n \leq N_2 - 1$, one has:

\begin{equation}
\tilde{\omega}_n \geq n \sqrt{\varepsilon} g_2.
\end{equation}

Therefore, taking into account that $h_n - h_1 = \varepsilon(n^2 - 1)/2$ and that

\begin{equation}
e^{-x} \leq \frac{\alpha^x}{(ex)^x} \quad \forall x > 0; \quad \forall \alpha > 0,
\end{equation}

considering as well that $\varepsilon^2(N_2^2 - 1) \geq 1 - a + 2\sqrt{1 - a} := \rho$, from (3.37) it results:

\begin{equation}
e^{2t} \sum_{n=1}^{N_2} H_n \leq \sum_{n=1}^{N_2 - 1} \left( \frac{g_1}{n^2g_2} + \frac{a}{2g_0g_2n^2} \right) \sqrt{\varepsilon} t + \\
\left[ \sum_{n=1}^{N_2} \frac{1}{ng_0} \frac{g_0}{g_2} \frac{g_0}{g_2} \right] \frac{(2\gamma)^\gamma}{(e)(1 - \gamma)} \frac{((e)t)^{1-\gamma}}{(n^2 - 1)^\gamma} + \\
\left[ \frac{g_1\varepsilon^4}{g_0(1 + \sqrt{1 - a\varepsilon})^3} + \varepsilon^2 a + \varepsilon(1 - a\varepsilon)^2 \right] \frac{2(\delta/e)^\delta}{\rho^\delta(2 - \delta)} e^{\delta t^2 - \delta},
\end{equation}

where it is assumed that $1/2 < \gamma < 1$, and $0 < \delta < 2$.

So it is possible to choose a positive constant $A_1$, independent from $\varepsilon$, such that:

\begin{equation}
H_1 = \sum_{n=1}^{N_2} H_ng_n \leq A_1 \left[ \sqrt{\ell} + ((e)t)^{1-\gamma} + \varepsilon^{1+\delta}t^{2-\delta} \right] e^{\frac{\delta}{2}t}.
\end{equation}

As for $H_2$ of (3.35), one has:

\begin{equation}
H_2 = \sum_{n=N_2+1}^{\infty} e^{-h_n t} \frac{\sinh(\omega_n t)}{n^2\omega_n} g_n - e^{-h_1 t} \sum_{n=1}^{\infty} \frac{\sin(\omega_0 t)}{n^2\omega_0} g_n = H'_2 - H''_2.
\end{equation}

Being

\begin{equation}
e^{-h_n t} \frac{\sinh(\omega_n t)}{\omega_n} = e^{-(h_n - \omega_n)t} \int_0^t e^{-2\omega_n \tau} d\tau \leq t e^{-(h_n - \omega_n)t}
\end{equation}

and

\begin{equation}
h_n - \omega_n = h_n - h_n \sqrt{1 - \frac{n^2}{h_n^2}} \geq h_n - h_n \left( 1 - \frac{n^2}{2h_n^2} \right),
\end{equation}
since \( n \geq N_2 + 1 > 1/\varepsilon \) and \( a\varepsilon < 1 \), one has:

\[
(3.46) \quad e^{-(h_n - \omega_n)t} \leq e^{-t \frac{a^2}{2\varepsilon n}} \leq e^{-\frac{t}{\varepsilon}}.
\]

As a consequence, denoting by \( \theta \) the fast time \( t/\varepsilon \) and by \( \zeta(z) \) the Riemann zeta function, it results:

\[
(3.47) \quad |H'_2| \leq t e^{-\frac{t}{\varepsilon}} \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2) \varepsilon \ e^{-\frac{t}{\varepsilon}} \theta.
\]

Finally, since \((3.38)_1\) one deduces that:

\[
(3.48) \quad |H''_2| \leq \frac{e^{-at/2}}{g_0} \sum_{n=N_2+1}^{\infty} \frac{e^{-at/2}}{n^3} \leq \frac{1}{g_0} \frac{e^{-at/2}}{N_2 + 1} \zeta(2) \leq A_2 \varepsilon e^{-at/2}
\]

where \( A_2 = (1 + \sqrt{1 - a\varepsilon + \varepsilon} - 1) \zeta(2)/g_0 \leq \zeta(2)/g_0. \)

Referring to \((3.47)\), by means of \((3.40)\), one has:

\[
(3.49) \quad \theta e^{-\frac{t}{\varepsilon} \theta} \leq (4/e) e^{-\frac{t}{\varepsilon} \theta} \Rightarrow |H'_2| \leq (4/e) \zeta(2) \varepsilon e^{-\frac{t}{\varepsilon} \theta}.
\]

So, by means of \((3.42),(3.48),(3.49)\), there exists a positive constant \( A \) such that inequality \((3.36)\) holds.

Now, according to solution \((2.32)\), attention must be paid to the derivative with respect to variable \( t \). So, the following theorem is proved:

**Teorema 3.8.** Whatever \( 1/2 < \gamma < 1, 0 < k < 1/2 \) and \( 0 < \eta < 1 \) may be, there exists a positive constant \( C \), independent from \( \varepsilon \), such that the following estimate holds:

\[
\left| \partial_t \left( \sum_{n=1}^{\infty} G_n^e(t) - e^{\frac{t}{\eta^2}} \sum_{n=1}^{\infty} G_n^o(t) \right) \right| \leq C \varepsilon^m \left\{ e^{-\frac{t}{\eta^2}} + p(t)e^{-\frac{t}{\eta^2}} \right\}
\]

where

\[
(3.50) \quad m = \min(1 - \gamma; 1 - \eta; 1/2 - k); \quad p(t) = 2t + t^{1-\gamma} + 3,
\]

and \( \theta \) denotes the fast time \( t/\varepsilon \).

**Proof.** As for terms of \( H_1 \) of \((3.35)\), when \( n = N_2 \), since \( \int_0^t \tau e^{-\varepsilon (N_2^2 - 1)\tau/2} d\tau \leq \varepsilon/(1 - a) \), it is possible to find a positive constant \( C_1 \), independent from \( \varepsilon \), such that, from \((3.37)\) one obtains:
(3.51)

\[ e^{\frac{\varepsilon t}{2}} |\partial_t H_1| \leq \varepsilon \sum_{n=1}^{N_2-1} \frac{\omega_0 + h_1}{\omega_0 \omega_n} \left( \frac{\bar{g}_1}{n^2} + h_n + \bar{\omega}_n \right) \int_0^t e^{-\varepsilon(n^2-1)\tau/2} d\tau + C_1 \varepsilon (1 + t) \]

where, as defined in (1.8), \( h_n = (a + \varepsilon n^2)/2 \).

Besides, letting by \( c_0 \) the Euler constant, one has \( \sum_{n=1}^{n=N_2} \frac{1}{n} \leq c_0 + \frac{1}{2N_2} + \ln N_2 \), with \( \ln N_2 \leq (\beta/\varepsilon) N_2^{1/\beta} \forall \beta > 0 \). So that, assuming \( \beta = k^{-1} (k > 0) \) it results:

(3.52)

\[ \sum_{n=N_2+1}^{N_2-1} \frac{1}{n} \leq \left[ c_0 + \frac{\varepsilon}{2(1 + \sqrt{1 - a \varepsilon})} + \frac{(1 + \sqrt{1 - a \varepsilon})^k}{ek \varepsilon^k} \right] \]

Therefore, since \( \int_0^t e^{-\varepsilon n^2\tau/2} d\tau \leq \frac{2}{\varepsilon n^2} \), there exists a positive constant \( C_2 \), independent of \( \varepsilon \), such that:

(3.53)

\[ \sum_{n=N_2}^{N_2-1} \frac{\varepsilon^2 n^2}{2\omega_n} \int_0^t e^{-\varepsilon(n^2-1)\tau/2} d\tau \leq C_2 \left( \varepsilon t^{3/2} + \varepsilon^{1/2} \right) \]

where \( 0 < k < 1/2 \).

Other terms can be evaluated taking account of

(3.54)

\[ \varepsilon \sum_{n=1}^{N_2-1} \int_0^t e^{-\varepsilon(n^2-1)\tau/2} d\tau \leq \varepsilon t + \sum_{\gamma} \frac{(2\gamma)^\gamma}{e\gamma(1 - \gamma)} \int_0^t e^{-\varepsilon n^2\tau/2} d\tau \]

with \( 1/2 < \gamma < 1 \).

As for terms of \( H_2 \), since (3.43), it results:

\[ |\partial_t H_2| \leq \sum_{n=N_2+1}^{\infty} e^{-\varepsilon n^2\tau/2} \left( |\omega_n - h_n| + (h_n + \omega_n) e^{-2\varepsilon n^2\tau} \right) + \sum_{n=N_2+1}^{\infty} e^{-\varepsilon n^2\tau/2} \left( \frac{h_n}{\omega_n} + 1 \right) = |\partial_t H_2'| + |\partial_t H_2''| \]

Since \( \forall n \geq N_2 + 1 \) function \( \frac{h_n}{\omega_n} \) decreases, one has \( \left( \frac{h_n}{\omega_n} \right)_n \leq \left( \frac{h_n}{\omega_n} \right)_{N_2+1} = \frac{1}{\sqrt{a}} \varphi \) where \( 0 < \varphi \leq 9/(2\sqrt{2}) \). So, taking also into account (3.46), it results:

(3.55)

\[ |\partial_t H_2'| \leq \sum_{n=N_2+1}^{\infty} e^{-\frac{\varphi}{n^{1+k}}} \frac{\varepsilon^{1/2 - k}}{(1 + \sqrt{1 - a \varepsilon + \varepsilon})^{1-k}} \]

Moreover, it results as well:
\[ |\partial_t H_2''| \leq \sum_{n=N_2+1}^{\infty} e^{-\frac{\pi^2}{4} t} \left( \frac{h_1}{n^{1+\eta}} + 1 \right) \frac{\varepsilon^{1-\eta}}{(1 + \sqrt{1 - a\varepsilon} + \varepsilon)^{1-\eta}} \]

with 0 < \eta < 1. So that, since (3.51) (3.53)-(3.56), it is possible to find a positive constant \( C \), independent from \( \varepsilon \), such that

\[ \left| \partial_t \left[ \sum_{n=1}^{\infty} G_n^\varepsilon(t) - e^{-\frac{\pi^2}{4} t} \sum_{n=1}^{\infty} G_n^\varepsilon(t) \right] \right| \leq C \left( e^{-\frac{\pi^2}{4} t} \varepsilon^{1/2-k} + e^{-\frac{\pi^2}{4} t} \varepsilon^{1/2-k} + \varepsilon^{1-k} + \varepsilon^k + 2\varepsilon t \right) \]

from which (3.50) follows.

Finally, it must be observed that, although the Green function \( G_\varepsilon(x, \xi, \varepsilon) \) satisfies theorems (2.1) and (2.2), a further theorem for the function \( G \), defined in (2.14), must be proved. So that, one has:

**Theorem 3.9.** Whatever 0 < \( a < 1 \) and 0 < \( \varepsilon < 1 \) may be, there exists a positive constant \( B \), independent from \( \varepsilon \), such that:

\[ \varepsilon |G(x, \xi, t)| \leq B \left[ (1 + t)^{\varepsilon/2-k} e^{-\frac{\pi^2}{4} t} + \varepsilon^{1-k} e^{-\frac{\pi^2}{4} t} \right] \]

where 0 < \( \eta < 1 \), 0 < \( k < 1/2 \), and \( \theta = t/\varepsilon \).

Indeed, for circular terms, since (3.39) and (3.52), it is possible to find a positive constant \( B_1 \), independent from \( \varepsilon \), such that:

\[ \varepsilon \sum_{n=1}^{N_3} e^{-h_n t} \frac{\sin(\omega_n t)}{\omega_n} \leq B_1 e^{-\frac{\pi^2}{4} t} \left( \varepsilon^{1/2-k} + t\varepsilon \right) \]

with 0 < \( k < 1/2 \). As for hyperbolic terms, let \( c \) be an arbitrary positive constant less than 1, and denote by \( N_c \) the integer part of \( 1/\varepsilon \sqrt{c} \) \((1 + \sqrt{1 - a\varepsilon c}) \). So that, intervals \( (N_2 + 1, N_c - 1); (N_c, \infty) \) will be considered. Since (3.44)-(3.46), letting \( B_2 = \varepsilon (N_c - N_2 - 1) \leq \frac{2}{\sqrt{c}} \), it results:

\[ \sum_{n=N_2+1}^{N_c-1} e^{-h_n t} \frac{\sin(\omega_n t)}{\omega_n} \leq \sum_{n=N_2+1}^{N_c-1} t e^{-\frac{\pi^2}{4} t} \leq B_2 \varepsilon e^{-\frac{\pi^2}{4} t} \]

Moreover, for all \( n \geq N_c \), one has \( n \geq \frac{1}{\varepsilon \sqrt{c}} (1 + \sqrt{1 - a\varepsilon c}) \Rightarrow \varepsilon \sqrt{c} n^2 + a\sqrt{c} - 2n \geq 0 \), so that \( \frac{2n}{a+\varepsilon n^2} \leq \sqrt{c} \Rightarrow \frac{n}{h_n} \leq \sqrt{c} \) and, hence, \( \omega_n = h_n \sqrt{1 - \frac{n^2}{h_n^2}} \geq h_n \sqrt{1 - c} \).

13
As a consequence, if $0 < \eta < 1$, denoting by $\zeta(x)$ the Riemann zeta function and $\theta = t/\varepsilon$, and since $\varepsilon N_c \geq 1/\sqrt{c}$, one has

\[
\sum_{n=N_c}^{\infty} e^{-h_n t} \frac{\sinh(\omega_n t)}{\omega_n} \leq \frac{2\varepsilon(1-\eta)/2}{\sqrt{1-c}} \frac{\varepsilon^{-\eta} \zeta(1+\eta)}{(1+\sqrt{1-a\varepsilon c})^{1-\eta}} e^{-\frac{t}{\varepsilon}}
\]

So, there exists a positive constant $B_3$, independent from $\varepsilon$, such that, taking also account of (3.59)-(3.60), it results:

\[
\varepsilon |G(x, \xi, t)| \leq B_1 (t\varepsilon + \varepsilon^{1/2-k}) e^{-\frac{t}{\varepsilon}} + B_3 (\varepsilon^{1-\eta} + \varepsilon) e^{-\frac{t}{\varepsilon}}
\]

from which, inequality (3.58) follows.

4 A priori estimates

By assuming that function $F_0(x), F_1(x)$ and $F(x, t)$ belong to class $C^2(\Omega)$, let

\[
||F_i^{(j)}|| = \sup_{0 \leq x \leq L} |F_i^{(j)}(x)| \text{ for } i = 0, 1 \quad j = 0, 2
\]

\[
||F|| = \sup_{\Omega} \left| \int_0^t F(x, \tau) d\tau \right| \quad ||F_{xx}|| = \sup_{\Omega} |F_{xx}(x, t)|.
\]

Besides, indicating by $\Gamma(a, z)$ the incomplete Gamma function and by $\zeta(\xi)$ the Riemann zeta function, let

\[
K_0 = ||F_0|| + \frac{1}{\pi} ||F_1|| + \frac{1}{\pi} ||F|| ; \quad K_1 = B ||\tilde{F}_0|| ;
\]

\[
(4.63) \quad K_2 = ||\tilde{F}_1|| + a ||\tilde{F}_0|| \pi A ; \quad K_3 = ||\tilde{F}_0|| \zeta(2) \pi \quad H = ||F_{xx}|| \pi A
\]

\[
H_1 = ||F_{xx}|| \pi A \frac{2/a + 4/a^2 + 4 + (2/a)^{2-\gamma} \Gamma(2-\gamma) + (2/a)^{3-\delta} \Gamma(3-\delta)}{2/a + 4/a^2 + 4 + (2/a)^{2-\gamma} \Gamma(2-\gamma) + (2/a)^{3-\delta} \Gamma(3-\delta)}
\]

\[
k(t) = (2/a)^{2-\gamma} \Gamma(2-\gamma, ta/2) + (2/a)^{3-\delta} \Gamma(3-\delta, ta/2)
\]

where constant $A$ and $B$ are referred respectively to theorem 3.7 and 3.9. Then, if we consider that

\[
\frac{d}{dt}(e^{-\frac{t}{\varepsilon}}) \sum_{n=1}^{\infty} \int_0^\pi \tilde{F}_0 G_n^0 \frac{g_0}{n^2} d\xi \leq K_3 \varepsilon t e^{-\frac{t}{\varepsilon}}
\]

where $G_n^0$ is defined in (1.10)$_2$, according to (2.32) and by means of theorems 3.7, 3.8, 3.9, the following theorem holds:
**Theorem 4.10.** When data \((F_1, F_0, F)\) satisfy respectively the hypotheses of theorem 2.3 2.4 2.5, then the following inequality holds:

\[
\left| u(x, t, \varepsilon) - e^{-\frac{1}{2}\varepsilon} U(x, t) \right| \leq K_0 \left(1 - e^{-\frac{1}{2}\varepsilon}t\right) + \varepsilon'^m[Hk(t) + H_1] + \\
+ \varepsilon'^m e^{-\frac{1}{2}\varepsilon t} \{K_1(1 + t) + K_2 r(t) + C p(t) + K_3 t\} + e^{-\frac{1}{2}\varepsilon t} \varepsilon'^m (C + K_1 + K_2)
\]

where \(m\) and \(p(t)\) are defined in (3.50), while \(r(t)\) and \(C\) refer to theorem 3.7 and 3.8, respectively.

Theorem 4.10 allows us to achieve an estimate characterized by means of fast time \(\theta = t/\varepsilon\) and slow time \(\tau = \varepsilon t\).

Indeed, for \(0 < \delta < 1/2\), since (3.40), one has \(t^{-\delta} e^{-\frac{1}{2}\varepsilon t} \leq \frac{4(1-2\delta)}{m^2} t^{1+\delta} e^{-\frac{1}{2}\varepsilon t}\). Moreover, since \(\forall z > 0\) it results \(\Gamma(a, z) \leq \Gamma(a)\), taking account of (3.42), (3.49), (3.57), (3.62) and (4.64), it is possible to introduce three positive constants \(A, B, C\), independent from \(\varepsilon\), such that it results:

\[
\left| u(x, t, \varepsilon) - e^{-\frac{1}{2}\varepsilon} U(x, t) \right| \leq K_1 \left(1 - e^{-\frac{1}{2}\varepsilon}t\right) + A \varepsilon'^m + B e^{-\frac{1}{2}\varepsilon t} \\
+ C \left[\sqrt{\varepsilon t} + (\varepsilon t)^{1-\gamma} + (\varepsilon t)^{1+\delta} + \varepsilon t\right] e^{-\frac{1}{2}\varepsilon t}
\]

(4.65)

Therefore, indicating by \(K\) a positive constant independent from \(\varepsilon\), when \(t \in (0, \frac{1}{\varepsilon})\), one has:

\[
(4.66) \quad \left| u(x, t, \varepsilon) - e^{-\frac{1}{2}\varepsilon} U(x, t) \right| \leq K
\]

and hence for slow times \(\tau = \varepsilon t < 1\), the wave is propagated nearly unperturbed. Moreover, when \(t > 1/\varepsilon\), damped oscillations prevail.

With regards to asymptotic properties, it should be remarked that obviously hypotheses on the source term have an influence on asymptotic behaviours. If for instance, we suppose that \(F_{xx}(x, t)\) is bounded also when \(t \to \infty\) and \(\int_0^\infty F(x, t) dt < \infty\) then, we obtain that the solution \(u(x, t, \varepsilon)\) is bounded when \(t\) increases to infinity.

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