Remarks on the energy of regular graphs

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Abstract

The energy of a graph is the sum of the absolute values of the eigenvalues of its adjacency matrix. This note is about the energy of regular graphs. It is shown that graphs that are close to regular can be made regular with a negligible change of the energy. Also a $k$-regular graph can be extended to a $k$-regular graph of a slightly larger order with almost the same energy. As an application, it is shown that for every sufficiently large $n$, there exists a regular graph $G$ of order $n$ whose energy $\|G\|_*$ satisfies

$$\|G\|_* > \frac{1}{2} n^{3/2} - n^{13/10}.$$ 

Several infinite families of graphs with maximal or submaximal energy are given, and the energy of almost all regular graphs is determined.

**Keywords:** graph energy; regular graph; random regular graph; degree deviation; maximal energy graphs.

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1 Introduction

In [8] Gutman introduced the energy of a graph as the sum of the absolute values of the eigenvalues of its adjacency matrix. Since the energy of a graph $G$ is the trace norm of its adjacency matrix, we write $\|G\|_*$ for the energy of $G$.

In this note we discuss the energy of regular graphs, an area that has been studied (see, e.g., [10] and [9]), but which is still vastly unexplored.

In the groundbreaking paper [12], Koolen and Moulton showed that

$$\|G\|_* \leq k + \sqrt{k(n-k)(n-1)}$$

(1)

for every $k$-regular graph of order $n$. Equality holds in (1) if and only if $G = K_n$, or $G = (n/2) K_2$, or $G$ is a strongly regular graph with parameters $(n,k,a,a)$, i.e., a $k$-regular design graph (see, e.g., [4], p. 144). Since bound (1) is precise for an amazing variety of graphs, in [3] Balakrishnan proposed to study how tight this bound is in general, and asked the following relevant question, which is quoted here verbatim:

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Question 1 (Balakrishnan [3]) For any two positive integers \( n \) and \( k \), \( n - 1 > k \geq 2 \) and \( \varepsilon > 0 \), does there exist a \( k \)-regular graph \( G \) with \( \|G\|_*/B_2 > 1 - \varepsilon \), where \( B_2 = k + \sqrt{k(n-1)(n-k)} \).

Unfortunately, despite its sound idea, Question 1 is incoherent in the above form, because the quantifier of \( \varepsilon \) is unclear, and if \( n \) and \( k \) do not depend on \( \varepsilon \), the answer is almost always negative. These weaknesses have been exploited in the literature to trivialize Balakrishnan’s question, e.g., in [13]. Moreover, important recent results of van Dam, Haemers, and Koolen [6] imply that the original question of Balakrishnan’s cannot be preserved without excising many combinations of \( n \) and \( k \). To clarify this point, we state an essential corollary of the paper [6], which, however, has eluded its authors:

**Theorem 2** Let \( k \geq 2 \) and \( n \geq k^2 - k + 1 \). If \( G \) is a \( k \)-regular graph of order \( n \), then

\[
\|G\|_* \leq \left( \sqrt{k-1} + \frac{1}{k + \sqrt{k-1}} \right) n.
\]

Equality holds in (2) if and only if \( G \) is a disjoint union of incidence graphs of projective planes of order \( k - 1 \) or \( k = 2 \) and \( G \) is disjoint union of triangles and hexagons.

Note that for \( n > k^2 - k + 1 \) the right side of (1) is greater than the right side of (2), so Theorem 2 is a clear improvement over (1). Therefore, if we want to investigate \( k \)-regular graphs for which (1) is almost tight, we must suppose that \( k \geq \sqrt{n} \).

With this hindsight, it seems natural to split Question 1 into two conjectures: First, study Question 1 for \( k \)-regular graphs whenever \( k \) grows not too fast with \( n \), say, \( k = o(n) \). For this case we venture the following conjecture:

**Conjecture 3** For every \( \varepsilon > 0 \), there exist \( \delta > 0 \) and \( k_0(\varepsilon) \) such that if \( n\delta > k > k_0(\varepsilon) \) and \( kn \) is even, there exists a \( k \)-regular graph \( G \) of order \( n \) with

\[
\|G\|_* \geq (1 - \varepsilon) \sqrt{kn}.
\]

Second, study Question 1 for dense regular graphs, for which the following conjecture might hold:

**Conjecture 4** For every \( \varepsilon > 0 \) and \( n \) sufficiently large, if \( n > k > \sqrt{n} \) and \( kn \) is even, there exists a \( k \)-regular graph \( G \) of order \( n \) with

\[
\|G\|_* \geq (1 - \varepsilon) \sqrt{k(n-k)n}.
\]

As it turns out that the important factor in Conjecture 4 is not the degree of \( G \), but its density. Thus, it turns out that Conjecture 4 follows from the following simpler one:

**Conjecture 5** Let \( 0 < c \leq 1/2 \). For every \( \varepsilon > 0 \), if \( n \) is sufficiently large, there exists a graph \( G \) of order \( n \) with at most \( cn^2/2 \) edges such that

\[
\|G\|_* \geq (1 - \varepsilon) \sqrt{c(1-c)n^{3/2}}.
\]
First, note that in Conjecture 5 the requirement $0 < c \leq 1/2$ is not restrictive and is just a convenience, for the energy of a graph is roughly the same as the energy of its complement, as shown in Section 3 below.

However, the crucial implication here is that if a graph $G$ of order $n$ has at most $cn^2/2$ edges and satisfies (4), then it must be close to regular, and such graphs can be made regular with a negligible loss of energy, easily concealed by the $(1 - \epsilon)$ coefficient in (3).

As another illustration of these ideas, we shall improve a result on graphs of maximal energy: Recall that in [17], a question of Koolen and Moulton [12] was answered by showing that if $n$ is sufficiently large, there exists a graph $G$ of order $n$ with

$$\|G\|_* > \frac{1}{2} n^{3/2} - n^{11/10}.$$  

In Section, 2 we show that $G$ can be chosen regular at a negligible loss of energy:

**Theorem 6** If $n$ is sufficiently large, there exists a regular graph $G$ of order $n$ with

$$\|G\|_* > \frac{1}{2} n^{3/2} - n^{13/10}.$$  

The remaining part of this note covers the following topics: In Section 2, we establish relations between the energy and the degree deviation of a graph, from which we deduce Theorem 6. Section 3 is dedicated to inequalities between the energy of a graph and the energy of its complement. In Section 4, we use some design graphs to show the tightness of bound (1) for infinite classes of graphs. Finally, in Section 5, we determine the energy of almost all regular graphs, which so far seems to have gone unnoticed.

## 2 The energy of graphs and degree deviation

The principal goal of this section is to show that graphs that are close to regular can be made regular with just a minor change of the energy. Another goal is to show that a $k$-regular graph can be extended to a $k$-regular graph of a slightly larger order with almost the same energy. These two results can be used in various graph constructions related to the energy of regular graphs.

Let $G$ be a graph of order $n$ and size $m$. In [15], the author suggested to use the function

$$s\left(G\right) = \sum_{u \in V(G)} \left|d\left(u\right) - \frac{2m}{n}\right|$$

as a measure of irregularity of $G$. Here $d\left(u\right)$ stands for the degree of the vertex $u$. Clearly, $G$ is regular if and only if $s\left(G\right) = 0$, so we say that $G$ is close to regular if $s\left(G\right) = o\left(n^2\right)$.

The following theorem was proved in [15]:

**Theorem R** For every graph $G$ of order $n$ and size $m$, there exists a graph $R$ of order $n$ and size $m$ such that $\Delta\left(R\right) \leq \delta\left(R\right) + 1$ and $R$ differs from $G$ in at most $s\left(G\right)$ edges. In particular, if $2m/n$ is an integer then $R$ is $(2m/n)$-regular.

Applications of Theorem R often use the following simple bound:
Proposition 7 Let $H$ and $G$ be two graphs on the same vertex set. If $H$ differs from $G$ in at most $m$ edges, then
$$\|\| H \|_* - \| G \|_* \| \leq \sqrt{2mn}.$$ 

Proof Indeed, write $A(G)$ and $A(H)$ for the adjacency matrices of $G$ and $H$. Clearly the matrix $A(G) - A(H)$ has at most $2m$ entries of modulus one. Hence,
$$\|A(G) - A(H)\|_* \leq \sqrt{n \sum \lambda_i^2 (A(G) - A(H))} = \sqrt{2mn},$$
as claimed. □

Theorem R, Proposition 7, and the triangle inequality for the trace norm help to construct regular graphs from irregular ones keeping control on the energy change:

Proposition 8 Let $n > k \geq 1$ and $nk$ be even. If $G$ is a graph of order $n$ with $kn/2$ edges, then there exists a $k$-regular graph $R$ of order $n$ such that
$$\|\| R \|_* - \| G \|_* \| \leq \sqrt{2s(G)n}.$$ 

Here we outline a concrete application of this proposition, which is one of the main results of this section:

Theorem 9 Let $n > t > k \geq 2$ and suppose that $nk$ is even. If $H$ is a $k$-regular graph of order $t$, there exists a $k$-regular graph $G$ of order $n$ such that
$$\|\| H \|_* - \| G \|_* \| < 3\sqrt{(n-t)kn}.$$ 

Proof Write $H_0$ for the graph of order $n$ obtained by adding $n-t$ isolated vertices to $H$ and note that $\|H_0\|_* = \|H\|_*$. Let $G_0$ be the graph of order $n$ obtained by joining every new vertex of $H_0$ to $\lceil k/2 \rceil$ or $\lfloor k/2 \rfloor$ of the vertices of $H$ so that $G_0$ has $nk/2$ edges. It is not hard to see that
$$s(G_0) = (n-t)k.$$ (5)

Since $G_0$ has $(n-t)k/2$ edges in addition to those of $H$, Proposition 7 implies that
$$\|\| G_0 \|_* - \| H_0 \| \| \leq \sqrt{(n-t)kn}.$$ (6)

On the other hand, Theorem R implies that there exists a $k$-regular graph $G$ of order $n$ such that $G$ differs from $G_0$ in at most $s(G_0)$ edges. Hence, Proposition 8, together with (5), implies that
$$\|\| G \|_* - \| G_0 \|_* \| \leq \sqrt{2s(G_0)n} \leq \sqrt{2(n-t)kn}.$$ 

Therefore, in view of (6), we find that
$$\|\| G \|_* - \| H \|_* \| \leq \|\| G_0 \|_* - \| H_0 \| \| + \|\| G \|_* - \| G_0 \|_* \| \leq \sqrt{2(n-t)kn} + \sqrt{(n-t)kn} < 3\sqrt{(n-t)kn},$$
Armed with Theorem 9, we shall encounter no difficulty in proving Theorem 6:

**Proof of Theorem 6** Recall that for \( n \) sufficiently large, there exists a prime \( p \) such that \( p \equiv 1 \mod 4 \) and \( p \leq n + n^{11/20 + \epsilon} \) (see, e.g., [2], Theorem 3). Suppose that \( n \) is large enough and fix some prime \( p \leq n + n^{3/5} \) such that \( p \equiv 1 \mod 4 \). Write \( G_p \) for the Paley graph of order \( n \). Recall that \( V(G_p) = \{1, \ldots, p\} \) and \( \{i,j\} \) is an edge of \( G_p \) if and only if \( i - j \) is a quadratic residue mod \( p \). Paley graphs are conference graphs, and their spectra are well known. A simple bound on the energy of \( G_p \) gives \( \|G_p\|_* > p^{3/2}/2 \) (see, e.g., [17]).

Let \( k := (p - 1)/2 \), and note that \( k \) is even and \( G_p \) is \( k \)-regular. Theorem 9 implies that there exists a \( k \)-regular graph \( G \) of order \( n \) with

\[
\|G\|_* > \left|\|G_p\|_* - 3\sqrt{(n - p) kn} > \frac{1}{2} p^{3/2} - 3\sqrt{n^{8/5} k/8}\right|
\[
> \frac{1}{2} \left(n - n^{3/5}/8\right)^{3/2} - \frac{3}{4} n^{13/10}.
\]

On the other hand, using Bernoulli’s inequality, we find that

\[
\frac{1}{2} \left(n - n^{3/5}/8\right)^{3/2} - \frac{3}{4} n^{13/10} = \frac{1}{2} n^{3/2} \left(1 - n^{-2/5}/8\right)^{3/2} > \frac{1}{2} n^{3/2} - \frac{3}{32} n^{13/10}.
\]

Hence, \( \|G\|_* > n^{3/2}/2 - n^{13/10} \), completing the proof of Theorem 6. \( \square \)

### 3 The energy of the complement of a graph

It seems not widely known that the energy of a graph \( G \) and the energy of its complement \( \overline{G} \) are quite close. Indeed, let \( G \) be a graph of order \( n \), let \( J_n \) be the all-ones square matrix of order \( n \), and let \( I_n \) be the identity matrix of order \( n \). If \( A \) and \( \overline{A} \) are the adjacency matrices of \( G \) and \( \overline{G} \), then \( A + \overline{A} = J_n - I_n \), and using the triangle inequality for the trace norm, we find that

\[
\|\overline{A}\|_* = \|J_n - I_n - A\|_* \leq \|G\|_* + \|J_n - I_n\|_* = \|G\|_* + 2n - 2.
\]

By symmetry, we get the following proposition:

**Proposition 10** If \( G \) is a graph of order \( n \) and \( \overline{G} \) is the complement of \( G \), then

\[
\|\overline{G}\|_* - \|G\|_* \leq 2n - 2.
\]

Equality in (7) holds if and only if \( G \) or \( \overline{G} \) is a complete graph.

Instead of tackling the characterization for equality in (7) directly, we shall prove a more elaborate bound, which implies this characterization right away. Hereafter, we write \( \lambda_1(G), \ldots, \lambda_n(G) \) for the eigenvalues of the adjacency matrix of \( G \) arranged in descending order.
Theorem 11  If $G$ is a graph of order $n$ and $\overline{G}$ is the complement of $G$, then

$$\|G\|_* - \|\overline{G}\|_* \leq 2\lambda_1(G)$$

and

$$\|\overline{G}\|_* - \|G\|_* \leq 2\lambda_1(\overline{G}).$$

Proof By definition we have

$$\|\overline{G}\|_* = \lambda_1(\overline{G}) + |\lambda_2(\overline{G})| + \cdots + |\lambda_n(\overline{G})|,$$

$$\|G\|_* = \lambda_1(G) + |\lambda_n(G)| + \cdots + |\lambda_2(G)|.$$

Thus, the triangle inequality for the absolute value implies that

$$\|\overline{G}\|_* - \lambda_1(\overline{G}) - \|G\|_* + \lambda_1(G) = \sum_{k=2}^{n} |\lambda_k(\overline{G})| - |\lambda_{n-k+2}(G)| \leq \sum_{k=2}^{n} |\lambda_k(\overline{G}) + \lambda_{n-k+2}(G)|.$$

On the other hand, Weyl's inequalities for the eigenvalues of Hermitian matrices (see, e.g., [5], p. 181) imply that

$$\lambda_k(G) + \lambda_{n-k+2}(\overline{G}) \leq \lambda_2(J_n - I_n) = -1$$

for any $k \in \{2, \ldots, n\}$. Thus, we get

$$\sum_{k=2}^{n} |\lambda_k(\overline{G}) + \lambda_{n-k+2}(G)| = - \sum_{k=2}^{n} \lambda_k(\overline{G}) + \lambda_{n-k+2}(G) = \lambda_1(\overline{G}) + \lambda_1(G),$$

and the required inequalities follow. $\square$

It seems that the bounds in Theorem 11 can be improved, so we raise the following problem.

Problem 12  Find the best possible upper bounds for $\|G\|_* - \|\overline{G}\|_*$ for general and for regular graphs.

4  The energy of some strongly regular graphs

The goal of this section is to exhibit infinite classes of graphs for which the bound (1) is exact or almost exact. Our first example uses the rich class of symplectic graphs $Sp(2m,q)$ in the general form given by Tang and Wan in [18] (see the references of [18] for previous work on symplectic graphs). The graph $Sp(2m,q)$ is a design graph and therefore forces equality in (1). Its complement performs just slightly worse as seen in the next statement:
Proposition 13 Let $q$ be a prime power.

(a) For every $n_0$ there exists a $k$-regular graph $G$ of order $n > n_0$ such that

$$k = \frac{q-1}{q} n + \frac{1}{q},$$

and

$$\|G\|_* = k + \sqrt{k(n-k)(n-1)}.$$

(b) For every $n_0$, there exists a $k$-regular graph $G$ of order $n > n_0$ such that

$$k = \frac{n}{q} - \frac{q+1}{q},$$

and

$$\|G\|_* > \sqrt{k(n-k)(n-n+k+1)}.$$

Proof Recall that in [18] Tang and Wan defined a class of strongly regular graphs $Sp(2m,q)$ with parameters

$$\left(\frac{q^{2m} - 1}{q-1}, q^{m-1}, q^{2m-2} (q-1), q^{2m-2} (q-1)\right),$$

where $q$ is any prime power and $m$ is any positive integer.

Note that the eigenvalues of $Sq(2m,q)$ are $q^{2m-1}$, $q^{m-1}$, and $-q^{m-1}$. Therefore, letting $G := Sq(2m,q),

$$n := \frac{q^{2m} - 1}{q-1} \text{ and } k := q^{m-1},$$

we see that

$$k - \frac{q-1}{q} n = \frac{q^{2m} - 1}{q-1} - \frac{q-1}{q} q^{m-1} = 1,$$

and

$$\|G\|_* = q^{2m-1} + (n-1) q^{m-1} = k + \sqrt{k(n-k)(n-1)}.$$

This observation proves (a). To prove (b) take $G$ to be the complement of $Sp(2m,q)$. Now $G$ is $k$-regular with

$$k = n - \frac{q-1}{q} n - \frac{1}{q} - 1 = \frac{n}{q} - \frac{q+1}{q}.$$

The eigenvalues of $G$ are:
- $k$ with multiplicity 1
- $q^{m-1} - 1$ with multiplicity $((n-1) + q^m) / 2$
- $-q^{m-1} - 1$ with multiplicity $((n-1) - q^m) / 2$

Hence, the energy of $G$ satisfies

$$\|G\|_* = \frac{(n-1) + q^m}{2} (q^{m-1} - 1) + \frac{(n-1) - q^m}{2} (q^{m-1} + 1) + n - q^{2m-1} - 1$$

$$= \|Sp(2m,q)\|_* - q^{2m-1} - q^m + n - q^{2m-1} - 1$$

$$= \sqrt{(k+1)(n-k-1)(n-1) + k - \frac{q+1}{q}(n-k-1)}$$
Two simple calculations show that
\[
\sqrt{(k+1)(n-k-1)} n > \sqrt{k(n-k)} n,
\]
and if \( m \geq 2 \), then
\[
k - \frac{q+1}{q} (n-k-1) > -(n-k-1).
\]
Hence, for \( m \geq 2 \) we see that
\[
\|G\|_* > \sqrt{k(n-k)} n - n + k + 1,
\]
as claimed.

The main implication of Proposition 13 is the fact that the bound (1) is exact or asymptotically exact for an infinite set of edge densities of graphs. Among these densities are the numbers \( 1/5, 1/4, 1/3, 1/2, 2/3, 3/4, 4/5, \) etc. The first unknown case is \( 1/6 \), so we ask the following question:

**Question 14** Is it true that for every \( \varepsilon > 0 \) and \( n_0 > 0 \), there exists a graph \( G \) of order \( n > n_0 \) such that
\[
\left(\frac{1}{6} - \varepsilon\right) \binom{n}{2} < e(G) < \left(\frac{1}{6} + \varepsilon\right) \binom{n}{2}
\]
and
\[
\|G\|_* > \left(\sqrt{5/6} - \varepsilon\right) n^{3/2}.
\]

We conclude this section with a family of sparse graphs implying equality in (1). In [1] Ahrens and Szekeres constructed a family of strongly regular graphs with parameters
\[
\left(q^2 (q+2), q (q+1), q, q\right),
\]
where \( q \) is a prime power. These graphs give some credibility to Conjecture 3:

**Proposition 15** For every \( n_0 \) there exists a \( k \)-regular graph \( G \) of order \( n > n_0 \) such that
\[
n^{2/3} - \frac{1}{3} n^{1/3} < k < n^{2/3},
\]
with
\[
\|G\|_* = k + \sqrt{k(n-k) (n-1)}.
\]

Using the results of Section 2, one can deduce the following extension:

**Proposition 16** For every \( \varepsilon > 0 \), and sufficiently large \( n \), if \( k \) satisfies
\[
(1 - \varepsilon) n^{2/3} < k < (1 + \varepsilon) n^{2/3},
\]
and \( kn \) is even, there exists a \( k \)-regular graph \( G \) of order \( n \) with
\[
\|G\|_* \geq (1 - \varepsilon) \sqrt{kn}.
\]
5 The energy of random regular graphs

In [5], Chen, Li and Lin studied the skew-energy of a random \( k \)-regular oriented graph. Somewhat surprisingly, these authors missed the fact that the similar methods apply to the energy of \( k \)-regular graphs. Thus, in this section we fill in this void. In what follows, we use “almost any \( k \)-regular graph” as a synonym of “randomly chosen \( k \)-regular graph”.

**Theorem 17** Let \( k \geq 2 \) be a fixed integer. The energy of almost any \( k \)-regular graph \( G \) of order \( n \) satisfies

\[
\|G\|_* = \frac{n}{\pi} \left( 2k\sqrt{k-1} - k(k-2) \arctan \frac{2\sqrt{k-1}}{k-2} \right) + o(n).
\]

**Proof** Let \( G \) be a randomly chosen \( k \)-regular graph of order \( n \). In [14] McKay showed that, as \( n \) increases, the empirical spectral distribution of \( G \) converges to the density function

\[
f(x) = \begin{cases} \frac{k\sqrt{4(k-1) - x^2}}{2\pi(k^2 - x^2)}, & \text{if } |x| \leq 2\sqrt{k-1}, \\ 0, & \text{otherwise}. \end{cases}
\]

This implies that the energy of \( G \) almost surely satisfies

\[
\|G\|_* = n \int_{-2\sqrt{k-1}}^{2\sqrt{k-1}} |x| \frac{k\sqrt{4(k-1) - x^2}}{2\pi(k^2 - x^2)} dx + o(n).
\]

After a change of variable, the indefinite integral can be calculated, and we find that, almost surely, \( \|G\|_* \) satisfies

\[
\|G\|_* = \frac{n}{\pi} \left( 2k\sqrt{k-1} - k(k-2) \arctan \frac{2\sqrt{k-1}}{k-2} \right) + o(n),
\]

which completes the proof.

Note that the energy of a randomly chosen \( k \)-regular graph of order \( n \) is almost equal to the skew energy of a randomly chosen \( k \)-regular oriented graph of order \( n \) (Theorem 4.3 of [5]). This is not very surprising, as \( k \)-regular graphs of large order locally are trees, and the skew energy of oriented trees is equal to the energy of the underlying unoriented tree.

By some involved calculations one can show that

\[
\frac{8}{3}\sqrt{k} < 2k\sqrt{k-1} - k(k-2) \arctan \frac{2\sqrt{k-1}}{k-2} < \frac{8}{3}\sqrt{k-1} \left( 1 + \frac{1}{k} \right).
\]

Hence, as \( n \) increases, the energy of almost any \( k \)-regular graph \( G \) of order \( n \) satisfies

\[
\frac{8}{3\pi}\sqrt{kn} < \|G\|_* < \frac{8}{3\pi}\sqrt{k-1} \left( 1 + \frac{1}{k} \right) n.
\]
Let us reiterate that in the above discussion $k$ is fixed, and $n$ tends to $\infty$. On the other hand, the distribution of the eigenvalues of random $k$-regular graphs whenever $k \to \infty$ with $n$ has been found only recently, by Dumitriu and Pal [7], and Tran, Vu, and Wang [19]. Based on the latter work, we prove the following theorem.

**Theorem 18** Let $k \to \infty$ with $n$. The energy of almost any $k$-regular graph $G$ of order $n$ satisfies

$$\| G \|_* = \left( \frac{8}{3\pi} + o(1) \right) \sqrt{k(n-k)n}.$$

**Proof** Let $G_{n,k}$ be a randomly chosen $k$-regular graph of order $n$, let $A_n$ be its adjacency matrix, and let

$$M_n := \frac{1}{\sqrt{k_n / (1 - k_n)}} \left( A - k_n J_n \right).$$

A recent result of Tran, Vu, and Wang [19] states that if $k \to \infty$ with $n$, then the empirical spectral distribution of the matrix $n^{-1/2}M_n$, converges to the standard semicircle distribution, i.e., to the density function

$$f(x) = \begin{cases} \frac{1}{2\pi} \sqrt{4 - x^2}, & \text{if } |x| \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

First, let us note that

$$\| A \|_* = \sqrt{\frac{k_n}{n} \left( 1 - \frac{k_n}{n} \right)} \| M \|_* + k.$$

For $\| M \|_*$ we get

$$\| M \|_* = \frac{n^{3/2}}{2\pi} \int_{-2}^{2} |x| \sqrt{4 - x^2} + o \left( n^{3/2} \right) = \frac{n^{3/2}}{\pi} \int_{0}^{2} x \sqrt{4 - x^2} + o \left( n^{3/2} \right)$$

$$= \frac{8}{3\pi} n^{3/2} + o \left( n^{3/2} \right).$$

Hence,

$$\| A \|_* = \left( \frac{8}{3\pi} + o(1) \right) \sqrt{k(n-k)n},$$

as claimed. $\square$

Comparing Theorem 17 with (2) and Theorem 18 with bound (1), we see that the energy of almost all $k$-regular graphs is more than 84% of the maximum one. This high value adds some extra credibility to Conjecture 4.
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