A SINGULAR INTEGRAL APPROACH TO THE MAXIMAL $L^p$ REGULARITY OF PARABOLIC EQUATIONS

BUYANG LI

1 Introduction

Consider a parabolic problem
\begin{equation}
\partial_t u + Au = f
\end{equation}
on a Banach space $X$. The maximal $L^p$ regularity of the parabolic problem \ref{1.1} refers to such estimates as
\begin{equation}
\|\partial_t u\|_{L^p(\mathbb{R}^+;X)} + \|Au\|_{L^p(\mathbb{R}^+;X)} \leq C\|f\|_{L^p(\mathbb{R}^+;X)}, \quad \text{for } 1 < p < \infty.
\end{equation}

Traditionally, this estimate was derived for parabolic equations by using parabolic singular integrals \cite{1, 6} for $X = L^p$. For example, when $Au = -\Delta u$, we have
\begin{equation}
\partial_t u(t, x) = \int_0^t \int_{\mathbb{R}^d} k(t - s, x - y)f(s, y) \, dy \, ds,
\end{equation}
where $k(t, x) = \left(\frac{|x|^2}{t^2} - \frac{d}{2}\right) e^{-|x|^2/(2t)}$ is a standard singular kernel of parabolic type:
\begin{equation}
|\partial_t^m \partial_x^\alpha k(t, x)| \leq C_{\alpha, m}/(t + |x|^2)^{m+|\alpha|/2}
\end{equation}
Therefore, by the theory of singular integrals, \ref{1.2} holds for $X = L^p$.

Unfortunately, this argument has not been extended to the settings of a general Banach space $X$, or more specifically, $X = L^r$ with $r \neq p$. Most existing methods in establishing \ref{1.2} for a general Banach space $X$ rely on the concepts of analytic semigroups \cite{2, 5}, operator-valued Fourier multiplier theory and $R$-boundedness \cite{9, 10} or $H^\infty$-functional calculus \cite{7}.

In this note, we present a simple and fundamental approach to the maximal $L^p$ regularity of parabolic problems, which only uses the concept of singular integrals of Volterra type (which are straightforward modification of the standard singular integrals). Knowledge of analytic semigroups, $R$-boundedness or $H^\infty$-functional calculus are not required.

2 Singular integrals of Volterra type

Frequently we meet the following type of operators
\begin{equation}
Tf(t) = \int_0^t K(t, s)f(s) \, ds,
\end{equation}
where \( f \in L^r(\mathbb{R}_+ \mapsto X) \), \( K(t, s) \) is a map from the Banach space \( X \) to the Banach space \( Y \) for any fixed \( 0 < s < t < \infty \), and the kernel \( K(t, s) \) exhibits singularity such as

\[
\|K(t, s)\|_{\mathcal{L}(X,Y)} \leq \frac{M}{t - s},
\]

\[
\|K(t, s) - K(t, s_0)\|_{\mathcal{L}(X,Y)} \leq \frac{M|s - s_0|^\sigma}{(t - s_0)^{1+\sigma}} \quad \text{if } t - s_0 \geq 2|s - s_0|,
\]

\[
\|K(t, s) - K(t_0, s)\|_{\mathcal{L}(X,Y)} \leq \frac{M|t - t_0|^\sigma}{(t_0 - s)^{1+\sigma}} \quad \text{if } t_0 - s \geq 2|t - t_0|,
\]

for some \( \sigma \in (0, 1] \), where \( \| \cdot \|_{\mathcal{L}(X,Y)} \) denotes the operator norm. A kernel \( K(t, s) \) which satisfies the above conditions is called a standard singular kernel of Volterra type. The above properties imply Hörmander’s condition: \( K(t, s) \) is integrable in any bounded domain \( U \times V \) such that \( U \cap V = \emptyset \) and \( (t, s) \in U \times V \) implies \( t > s > 0 \), and

\[
\sup_{s, s_0 \in \mathbb{R}_+} \int_{t - s_0 \geq 2|s - s_0|} \|K(t, s) - K(t, s_0)\|_{\mathcal{L}(X,Y)} \, dt \leq M,
\]

\[
\sup_{t, t_0 \in \mathbb{R}_+} \int_{t_0 - s \geq 2|t - t_0|} \|K(t, s) - K(t_0, s)\|_{\mathcal{L}(X,Y)} \, ds \leq M.
\]

A singular integral operator of Volterra type is a bounded linear operator from \( L^r(\mathbb{R}_+ \mapsto X) \) to \( L^r(\mathbb{R}_+ \mapsto Y) \) for some \( r \in (1, \infty] \), given by \( (2.3) \) when \( f \) has compact support and \( t \) does not lie in the support of \( f \), where the kernel \( K(t, s) \) satisfies the conditions \( (2.4)-(2.6) \). Of course, those conditions also imply Hörmander’s conditions \( (2.7)-(2.8) \).

**Remark 2.1** The above definition mimic the definition of the usual singular integral operators \( 3, 4, 8 \). We should be cautious that a singular kernel of Volterra type does not define the singular integral operator directly. For example, the kernel \( K(t, s) = 1/(t - s) \) satisfies \( (2.4)-(2.6) \) but the integral operator \( T f(t) = \int_0^t K(t, s)f(s) \, ds \) is not defined even in the Cauchy principal sense.

The following theorem indicates that singular integral operators of Volterra type preserve the essential properties of standard singular integral operators.

**Theorem 2.1** Let \( X \) and \( Y \) be reflexive Banach spaces. Then a singular integral operator of Volterra type is of weak-type \((1, 1) \) and strong-type \((p, p) \) for \( 1 < p < \infty \). In particular, if \( \|T\|_{L^r(\mathbb{R}_+ \mapsto X) \to L^r(\mathbb{R}_+ \mapsto Y)} = B \), then

\[
\|Tf\|_{L^{1,\infty}(\mathbb{R}_+ \mapsto Y)} \leq C(M + B)\|f\|_{L^1(\mathbb{R}_+ \mapsto X)},
\]

\[
\|Tf\|_{L^p(\mathbb{R}_+ \mapsto Y)} \leq C_p(M + B)\|f\|_{L^p(\mathbb{R}_+ \mapsto X)}, \quad 1 < p < \infty.
\]

**Proof** The theorem is proved based on the Calderón-Zygmund decomposition (see Appendix). We only prove the case \( 1 < r < \infty \), as the case \( r = \infty \) can be proved similarly.

Without loss of generality, we can first consider \( f \) as a smooth function with compact support and then extend the result to \( L^p(\mathbb{R}_+ \mapsto X) \) for \( 1 \leq p < \infty \). Let \( f = g + \sum_{Q_j \in \mathcal{Q}} b_j \) be the Calderón–Zygmund decomposition so that both \( g \) and \( b_j \) are in
$L^1(\mathbb{R}_+ \mapsto X) \cap L^\infty(\mathbb{R}_+ \mapsto X)$ and the sum $\sum_{Q_j \in \mathcal{Q}} b_j$ converges in $L^r(\mathbb{R}_+ \mapsto X)$. Since the operator $T$ is bounded on $L^r(\mathbb{R}_+ \mapsto X)$, it follows that

$$Tf(t) = Tg(t) + \sum_{Q_j \in \mathcal{Q}} Tb_j(t)$$

for almost all $t \in \mathbb{R}_+$. The idea of such decomposition is that, if we let $Q^*_j$ be the unique cube with the same center as $Q_j$ (denoted by $s_j$), with sides parallel to the sides of $Q_j$ and have side length $l(Q^*_j) = 2l(Q_j)$, then

$$Tb_j(t) = \int_0^t K(t, s)b_j(s) \, ds = \begin{cases} \int_{Q_j} (K(t, s) - K(t, s_j))b_j(s) \, ds & \text{for } t \in (Q^*_j)^c \cap \{t > s_j\}, \\ 0 & \text{for } t \in (Q^*_j)^c \cap \{t < s_j\}. \end{cases}$$

Let $b = \sum_{Q_j \in \mathcal{Q}} b_j$ and we note that

$$|\{t \in \mathbb{R}_+ : \|Tf(t)\|_Y > 1\}| \leq |\{t \in \mathbb{R}_+ : \|Tg(t)\|_Y > 1\}| + |\{t \in \mathbb{R}_+ : \|Tb(t)\|_Y > 1/2\}| \leq 2^r B'^r\|g\|_{L^r(\mathbb{R}_+ \mapsto X)} + \{t \notin \cup_j Q^*_j : \|Tb(t)\|_Y > 1/2\}| + \cup_j Q^*_j\]

$$\leq (2^{r-1}B'^r\alpha^{r-1} + C\alpha^{-1})\|f\|_{L^1(\mathbb{R}_+ \mapsto X)} + 2\sum_j \int_{(Q^*_j)^c} \|Tb_j(t)\|_Y \, dt,$$

where $\alpha$ is the parameter in the Calderón–Zygmund decomposition. We choose $\alpha$ to satisfy $2^{r-1}B'^r\alpha^{r-1} = 2B$ so that

$$|\{t \in \mathbb{R}_+ : \|Tf(t)\|_Y > 1\}| \leq CB\|f\|_{L^1(\mathbb{R}_+ \mapsto X)} + 2\sum_j \int_{(Q^*_j)^c} \|Tb_j(t)\|_Y \, dt.$$

For the second term, we have

$$\sum_j \int_{(Q^*_j)^c} \|Tb_j(t)\|_Y \, dt$$

$$= \sum_j \int_{(Q^*_j)^c \cap \{t > s_j\}} \|Tb_j(t)\|_Y \, dt$$

$$\leq \sum_j \int_{(Q^*_j)^c \cap \{t > s_j\}} \int_{Q_j} \|K(t, s) - K(t, s_j)\|_{L(X,Y)} \|b_j(s)\|_X \, ds \, dt$$

$$\leq \sum_j \sup_{s \in Q_j} \int_{t-s_j \geq 2|s-s_j|} \|K(t, s) - K(t, s_j)\|_{L(X,Y)} \, dt \int_{Q_j} \|b_j(s)\|_X \, ds$$

$$\leq \sum_j M \int_{Q_j} \|b_j(s)\|_X \, ds \leq CM\|f\|_{L^1(\mathbb{R}_+ \mapsto X)}.$$

Therefore, we have proved the weak-type $(1, 1)$ estimate. The strong-type $(p, p)$ estimates for $1 < p < r$ follows from real interpolation.

For $r < p < \infty$, we consider the transpose operator $T'$ defined by $(Tg, f) = (g, T'f)$ for any given $g \in L^r(\mathbb{R}_+ \mapsto X)$ and $f \in L^r(\mathbb{R}_+ \mapsto Y')$. Clearly, the operator $T'$ is
given by
\[ T' f(t) = \int_t^\infty K'(t, s) f(s) \, ds \]
with the kernel \( K'(t, s) = K(s, t)' \in \mathcal{L}(Y', X') \). Then \( T' \) is bounded from \( L^r(\mathbb{R}_+ \to Y') \) to \( L'^r(\mathbb{R}_+ \to X') \). For \( f \in L^\infty(\mathbb{R}_+ \to Y') \) with compact support, we let \( f = g + \sum_{Q_j \in Q} b_j \) be the Calderón–Zygmund decomposition so that both \( g \) and \( b_j \) are in \( L^1(\mathbb{R}_+ \to Y') \cap L^\infty(\mathbb{R}_+ \to Y') \) and the sum \( \sum_{Q_j \in Q} b_j \) converges in \( L^r(\mathbb{R}_+ \to Y') \), and
\[ T' f(t) = T' g(t) + \sum_{Q_j \in Q} T' b_j(t) \]
for almost all \( t \in \mathbb{R}_+ \), and
\[ T' b_j(t) = \int_t^\infty K(s, t)' b_j(s) \, ds = \begin{cases} 0 & \text{for } t \in (Q_j')^c \cap \{ t > s_j \}, \\ \int_{Q_j} (K(s, t)' - K(s_j, t)') b_j(s) \, ds & \text{for } t \in (Q_j')^c \cap \{ t < s_j \}. \end{cases} \]
Then
\[ \left| \{ t \in \mathbb{R}_+ : \| T' f(t) \|_{X'} > 1 \} \right| \]
\[ \leq \left| \{ t \in \mathbb{R}_+ : \| T' g(t) \|_{X'} > 1/2 \} \right| + \left| \{ t \in \mathbb{R}_+ : \| T' b(t) \|_{X'} > 1/2 \} \right| \]
\[ \leq 2^r B \| g \|_{L^r(\mathbb{R}_+ \to Y')} + \left| \{ t \notin \cup_j Q_j' : \| T' b(t) \|_{X'} > 1/2 \} \right| + \left| \cup_j Q_j' \right| \]
\[ \leq (2^{2r-1} B^r \alpha^{r-1} + C \alpha^{-1}) \| f \|_{L^1(\mathbb{R}_+ \to Y')} + 2 \sum_j \int_{(\cup Q_j')^c} \| T' b_j(t) \|_{X'} \, dt, \]
where \( \alpha \) is the parameter in the Calderón–Zygmund decomposition. We choose \( \alpha \) to satisfy \( 2^{2r-1} B^r \alpha^{r-1} = 2B \) so that
\[ \left| \{ t \in \mathbb{R}_+ : \| T' f(t) \|_{X'} > 1 \} \right| \leq C B \| f \|_{L^1(\mathbb{R}_+ \to Y')} + 2 \sum_j \int_{(Q_j')^c} \| T' b_j(t) \|_{X'} \, dt. \]
For the second term, we have
\[ \sum_j \int_{(Q_j')^c} \| T' b_j(t) \|_{X'} \, dt \]
\[ = \sum_j \int_{(Q_j')^c \cap \{ t > s_j \}} \| T' b_j(t) \|_{X'} \, dt \]
\[ \leq \sum_j \int_{(Q_j')^c \cap \{ t > s_j \}} \int_{Q_j} \| K(s, t)' - K(s_j, t)' \|_{\mathcal{L}(Y', X')} \| b_j(s) \|_{Y'} \, ds \, dt \]
\[ \leq \sum_j \sup_{s \in Q_j} \int_{s_j - t \geq 2|s - s_j|} \| K(s, t) - K(s_j, t) \|_{\mathcal{L}(X,Y)} \, dt \int_{Q_j} \| b_j(s) \|_{Y'} \, ds \]
\[ \leq \sum_j M \int_{Q_j} \| b_j(s) \|_{Y'} \, ds \leq CM \| f \|_{L^1(\mathbb{R}_+ \to Y')} \]
Therefore, we have proved the weak-type \((1, 1)\) estimate. The strong-type \((p', p')\) estimates for \( 1 < p' < r' \) follows from real interpolation. In other words, \( T' \) is bounded
from $L^{p'}(\mathbb{R}_+ \mapsto Y)$ to $L^{p''}(\mathbb{R}_+ \mapsto X')$ for $1 < p' < r'$. By a simple duality argument, this implies that $T$ is bounded from $L^p(\mathbb{R}_+ \mapsto X)$ to $L^{p'}(\mathbb{R}_+ \mapsto Y)$ for $r < p < \infty$.

Overall, $T$ is bounded from $L^p(\mathbb{R}_+ \mapsto X)$ to $L^{p'}(\mathbb{R}_+ \mapsto Y)$ for $1 < p < \infty$. □

3 Maximal $L^p$ regularity of parabolic equations

Consider the parabolic problem

\[(3.1) \quad \partial_t u - \sum_{i,j=1}^{d} \partial_j (a_{ij}(x) \partial_i u) + \sum_{j=1}^{d} b_j(x) \partial_j u + c(x) u = f\]

with the initial condition $u(0, x) \equiv 0$, where the coefficients $a_{ij}, b_j$ and $c$ are bounded, measurable and satisfying the strongly ellipticity condition:

$$\Lambda^{-1} |\xi|^2 \leq \sum_{i,j=1}^{d} a_{ij}(x) \xi_i \xi_j \leq \Lambda |\xi|^2, \quad \forall \, x, \xi \in \mathbb{R}^d.$$ 

This corresponds to (3.1) with $Au = - \sum_{i,j=1}^{d} \partial_j (a_{ij}(x) \partial_i u) + \sum_{j=1}^{d} b_j(x) \partial_j u + c(x) u$.

Let $G(t, x, y)$ denote the Green function of the parabolic equation so that the solution of (3.1) is given by

$$u(t, x) = \int_0^t \int_{\mathbb{R}^d} G(t - s, x, y) f(s, y) \, dy \, ds.$$ 

Thus

$$\partial_t u(t, x) = f(t, x) + \int_0^t \int_{\mathbb{R}^d} \partial_t G(t - s, x, y) f(s, y) \, dy \, ds.$$ 

Let $X = L^r$ and let the mapping from $f \in L^r(\mathbb{R}_+; X)$ to $\partial_t u - f \in L^r(\mathbb{R}_+; X)$ be denoted by $T$. Then $T f(t) = \int_0^t K(t - s) f(s) \, ds$ when $f$ has compact support and $t$ is not in the support of $f$, where the operator-valued kernel $K(t) : L^r \to L^r$ can be expressed as

$$[K(t)g](x) = \int_{\mathbb{R}^d} \partial_t G(t, x, y) g(y) \, dy.$$ 

The kernel $K(t)$ obeys the standard estimates:

\[(3.2) \quad ||K(t - s)||_{L^r(X; L^r)} \leq \frac{C}{t - s}, \quad \text{for } 0 < s < t < \infty,\]

\[(3.3) \quad ||\partial_t K(t - s)||_{L^r(X; L^r)} \leq \frac{C}{(t - s)^2}, \quad \text{for } 0 < s < t < \infty,\]

which indicate that $T : L^r(\mathbb{R}_+; X) \to L^r(\mathbb{R}_+; X)$ is a singular integral operator of Volterra type. By Theorem 2.1 this operator must be bounded on $L^p(\mathbb{R}_+; X)$ for all $1 < p < \infty$.

Remark 3.1 By this approach, the maximal regularity

$$||\partial_t u||_{L^p(\mathbb{R}_+; L^r')} + ||Au||_{L^p(\mathbb{R}_+; L^r')} \leq C ||f||_{L^p(\mathbb{R}_+; L^r')}, \quad \text{for } 1 < p, r < \infty.$$
reduces to the homogeneous estimate
\[ \| \partial_t u \|_{L^r(\mathbb{R}^+; L^r)} + \| Au \|_{L^r(\mathbb{R}^+; L^r)} \leq C \| f \|_{L^r(\mathbb{R}^+; L^r)}, \quad 1 < r < \infty. \]
This approach can also be applied to problems defined on a finite time interval as well as problems defined on the bounded domain.

**Appendix: Calderón–Zygmund decomposition on \( \mathbb{R}_+ \)**

Let \( Z = \{0, \pm 1, \pm 2 \cdots \} \) denote the set of all integers and let \( N = \{0, 1, 2, \cdots \} \) denote the set of all natural numbers. For any integers \( n \in Z \) and \( k \in N \) we define \( Q_{n,k} = (2^n k, 2^n (k + 1)] \), called a dyadic cube. Then \( \mathcal{Q}_+ := \{Q_{n,k} : n \in Z \text{ and } k \in N\} \) is called the set of all dyadic cubes on the half-line \( \mathbb{R}_+ \). For any two dyadic cubes \( Q_1, Q_2 \in \mathcal{Q}_+ \), either \( Q_1 \subset Q_2 \) or \( Q_2 \subset Q_1 \) or \( Q_1 \cap Q_2 = \emptyset \). This is often referred to as the nesting property of dyadic cubes.

**Proposition** Let \( X \) be a Banach space, \( f \in L^1(\mathbb{R}_+ \mapsto X) \) and \( \alpha > 0 \). Then there is a decomposition
\[ f = g + \sum_{j=1}^{\infty} b_j \]
which satisfies that
(1) \( \|g\|_{L^1(\mathbb{R}_+ \mapsto X)} \leq \|f\|_{L^1(\mathbb{R}_+ \mapsto X)} \) and \( \|g\|_{L^\infty(\mathbb{R}_+ \mapsto X)} \leq 2\alpha \),
(2) the functions \( b_j \) are supported in disjoint dyadic cubes \( Q_j \in \mathcal{Q}_+ \), respectively,
(3) \( \int_{Q_j} b_j(t) \, dt = 0, \quad \int_{Q_j} |b_j| \|x\| \, dt \leq 4\alpha |Q_j| \),
(4) \( \sum_j |Q_j| \leq \|f\|_{L^1(\mathbb{R}_+ \mapsto X)}/\alpha \),
(5) if \( f \in L^\infty(\mathbb{R}_+ \mapsto X) \) with compact support in \( \mathbb{R}_+ \), then \( g \in L^\infty(\mathbb{R}_+ \mapsto X) \) with compact support in \( \mathbb{R}_+ \), \( \int_{\mathbb{R}_+} \| \sum_{j=k}^{l} b_j \|_X \, dt \leq 2r \int_{Q_{j=k}} \|f\|_X \, dt \) and so the series \( \sum_{j=1}^{\infty} b_j \) converges in \( L^r(\mathbb{R}_+ \mapsto X) \) for any \( 1 \leq r < \infty \).

**Proof** Let us say that a dyadic cube \( Q \) is bad if \( \frac{1}{|Q|} \int_Q \|f(t)\|_X \, dt > \alpha \), and good otherwise. A maximal bad dyadic cube is a bad dyadic cube such that any dyadic cube strictly containing it is good. Since \( f \in L^1(\mathbb{R}_+ \mapsto X) \), any bad dyadic cube is contained in a maximal bad dyadic cube. Let \( \mathcal{Q} \) be the collection of all maximal bad dyadic cubes. By the nesting property of dyadic cubes, cubes in \( \mathcal{Q} \) are disjoint. For any \( Q \in \mathcal{Q} \), \( \frac{1}{|Q|} \int_Q \|f(t)\|_X \, dt \geq \alpha \) and \( \frac{1}{|Q'|} \int_{Q'} \|f(t)\|_X \, dt \leq \alpha \) for any dyadic cube \( Q' \) strictly containing \( Q \). Therefore,
\[ \alpha < \frac{1}{|Q|} \int_Q \|f(t)\|_X \, dt \leq 2\alpha. \]
For any dyadic cube \( Q \) outside \( \bigcup \mathcal{Q} \), \( \frac{1}{|Q|} \int_Q \|f(t)\|_X \, dt \leq \alpha \). By the Lebesgue differentiation theorem,
\[ \|f(t)\|_X = \lim_{\text{diam}(Q) \to 0} \frac{1}{|Q|} \int_Q \|f(s)\|_X \, ds \leq 2\alpha \]
for almost all \( t \) outside \( \bigcup \mathcal{Q} \), where \( Q \) extends over all sequence of dyadic cubes disjoint from \( \bigcup \mathcal{Q} \) and containing \( t \).
Let
\[ b_j(t) = \left( f(t) - \frac{1}{|Q_j|} \int_{Q_j} f(s) \, ds \right) 1_{Q_j}(t), \quad g(t) = \frac{1}{|Q_j|} \int_{Q_j} f(s) \, ds \]
for \( t \in Q_j \subset \mathcal{Q} \). Let \( g = f \) outside \( \cup \mathcal{Q} \). Then \( f = g + \sum_{Q_j \in \mathcal{Q}} b_j \) satisfies the requirements. \( \square \)

References

[1] Calderón, *Singular integrals*, Colloquium Lectures given in Aug 31–Sep 3, 1965 at the Seventieth Summer Meeting of the American Mathematical Society held in Ithaca, New York.

[2] P. Cannarsa and V. Vespri, *On maximal \( L^p \) regularity for the abstract Cauchy problem*, Boll. Un. Mat. Ital. B, 5 (1986), pp. 165-175.

[3] L. Grafakos, *Classical Fourier analysis*, Springer Science+Business Media, LLC, 2008.

[4] L. Grafakos, *Modern Fourier analysis*, Springer Science+Business Media, LLC, 2009.

[5] M. Hieber and J. Prüss, *Heat kernels and maximal \( L^p-L^q \) estimates for parabolic evolution equations*, Comm. Partial Differential Equations, 22 (1997), pp. 1647-1669.

[6] B.F. Jones, *A Class of Singular Integrals*, American J. Math., 86 (1964), pp. 441-462.

[7] C.L. Merdy, *\( H^\infty \)-functional calculus and applications to maximal regularity*, Publ. Math. UFR Sci. Tech. Besancon. 16 (1998), pp. 41-77.

[8] E.M. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, 1970.

[9] L. Weis, *A new approach to maximal \( L^p \)-regularity*, Evolution equations and their applications in physical and life sciences (Bad Herrenalb, 1998), Dekker, New York, 2001, pp. 195-214.

[10] L. Weis, *Operator-valued Fourier multiplier theorems and maximal \( L^p \)-regularity*, Math.Ann., 319 (2001), pp.735-758.