On modifications of the exponential integral with the Mittag-Leffler function

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Abstract

In this paper we survey the properties of the Schelkunoff modification of the Exponential integral and we generalize it with the Mittag-Leffler function. So doing we get a new special function (as far as we know) that may be relevant in linear viscoelasticity because of its complete monotonicity properties in the time domain. We also consider the generalized sine and cosine integral functions.

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1 Introduction

One purpose of this paper is to point out the relevance of a modification due to Schelkunoff on 1944 of the Exponential integral that turns out to be not sufficiently pointed out in the majority of textbooks on special functions. The merit of this modification was later recognized by the great Italian mathematician F.G Tricomi because it is proved to provide an entire function. Here we propose to generalize this modification by introducing in the kernel of the integral the Mittag-Leffler function which is known to be the most direct generalization of the exponential function like it is the gamma function for the factorial.
The plane of the paper is the following. In Section 2 we recall the known expressions of the Exponential integral and introduce the Schelkunoff modification. This modification leads to an entire function in the complex plane $\text{Ein}(z)$ that in the time positive domain $t > 0$ is of Bernstein type, namely its derivative is completely monotone. This property is equivalent to have a non-negative spectral distribution that indeed was adopted by Becker in 1926 for an interesting model in linear viscoelasticity even if independently of the later note by Schelkunoff.

In Section 3 we introduce the Mittag-Leffler function depending on a positive parameter $\nu$ in the kernel in such a way that for $\nu = 1$ the transcendental function $\text{Ein}_\nu(z)$ reduces to the Schelkunoff function and for $0 < \nu \leq 1$ and $t > 0$ keeps the Bernstein character thanks the well-known complete monotonicity of the Mittag-Leffler function. We consider also the limiting case $\nu = 0$ and recall for $0 \leq \nu \leq 1$ the relevance of our function in describing the creep features of a viscoelastic model already discussed recently by Mainardi et al [15]. For our function we exhibit the spectral distributions for selected values of $\nu \in (0, 1)$ derived from its Laplace transform by using the Mathematica tool box.

In section 4, after recalling the Schelkunoff modification of the functions related to the exponential integrals: sine and cosine exponential integrals, we generalize them by introducing in the kernel of the integral the Mittag-Leffler function. On this respect, we adopt for fractional circular functions the definitions by Herrmann [11] noting that there exist different definitions for them in the literature.

Finally, in Section 5 we draw our conclusions.

2 The Exponential integral and its Schelkunoff modification

2.1 The classical Exponential integral $\text{Ei}(z)$

A classical definition of the Exponential integral is

$$\text{Ei}(z) := -\int_{-z}^{\infty} \frac{e^{-u}}{u} \, du, \quad z \in \mathbb{C}^-, \quad (2.1)$$

where the Cauchy principal value of the integral is understood if $z = x > 0$. Above and from now on, with $\mathbb{C}^-$ we denote the complex plane $\mathbb{C}$ cut along
the negative real negative axis, that is $|\arg z| < \pi$.

Some authors such as Jahnke and Emde \[12\] adopt the following definition for $\text{Ei}(z)$,

$$\text{Ei}(z) := \int_{-\infty}^{z} \frac{e^u}{u} du, \quad z \in \mathbb{C}^{-}. \quad (2.2)$$

which is equivalent to (2.1). For this we note that

$$\int_{-z}^{\infty} \frac{e^{-u}}{u} du = -\int_{-\infty}^{z} \frac{e^u}{u} du,$$

where the Cauchy principal value is understood for $z = x > 0$.

Recalling the incomplete Gamma functions

$$\Gamma(\nu, z) := \int_{z}^{\infty} u^{\nu-1} e^{-u} du, \quad \nu \in \mathbb{R}, \quad z \in \mathbb{C}^{-},$$

we note the identity

$$-\text{Ei}(-z) = \Gamma(0, z) = \int_{z}^{\infty} \frac{e^{-u}}{u} du, \quad z \in \mathbb{C}^{-}. \quad (2.3)$$

## 2.2 The function $\mathcal{E}(z)$

In many texts on special functions the function $\Gamma(0, z)$ is usually taken as definition of Exponential integral and denoted by $\mathcal{E}_1(z)$. Then we can account for the following equivalent expressions for $z \in \mathbb{C}^{-}$:

$$\mathcal{E}_1(z) = \Gamma(0, z) = -\text{Ei}(-z) = \int_{z}^{\infty} \frac{e^{-u}}{u} du = \int_{1}^{\infty} \frac{e^{-zt}}{t} dt. \quad (2.4)$$

This definition is then generalized to yield, see e.g. \[16\]

$$\mathcal{E}_\nu(z) = \int_{1}^{\infty} \frac{e^{-zt}}{t^\nu} dt = z^{\nu-1} \Gamma(1-\nu, z), \quad \nu \in \mathbb{R}, \quad z \in \mathbb{C}^{-}. \quad (2.5)$$

We note that, in contrast with the standard literature where the Exponential integrals are denoted by the letter $E$, we have used for them the letter $\mathcal{E}$: this choice is to avoid confusion with the standard notation for the Mittag-Leffler function $E_\nu(z)$ ($\nu > 0$) later used in this paper to generalize the Exponential integral. We recall that the Mittag-Leffler function plays a relevant role in fractional calculus; for more details see any treatise on fractional calculus and in particular the 2014 monograph by Gorenflo et al. \[9\].
2.3 The modified Exponential integral \( \text{Ein}(z) \)

The whole subject matter can be greatly simplified if we agree to follow F.G. Tricomi [21], see also his former assistant L. Gatteschi [5], who has proposed to consider the following \textit{entire} function, formerly introduced in 1944 by Schelkunoff [18]:

\[
\text{Ein}(z) := \int_0^z \frac{1-e^{-u}}{u} \, du, \quad z \in \mathbb{C}. \tag{2.6}
\]

Indeed, such a function, that we refer to as the \textit{modified exponential integral}, turns out to be entire, being the primitive of an entire function. We found an account of this function in Ch. 6 of the handbook NIST edited by Nico Temme [16], where it is referred to as the \textit{complementary exponential integral}.

The \textit{power series expansion} of \( \text{Ein}(z) \), valid in all of \( \mathbb{C} \), can be easily obtained by term-by-term integration and reads

\[
\text{Ein}(z) := z - \frac{z^2}{2 \cdot 2!} + \frac{z^3}{3 \cdot 3!} - \frac{z^4}{4 \cdot 4!} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n \, n!}. \tag{2.7}
\]

The relation between \( \text{Ein}(z) \) and \( \mathcal{E}_1(z) = -\text{Ei}(-z) \) can be obtained from the series expansion of the incomplete gamma function \( \Gamma(\alpha, z) \) in the limit as \( \alpha \to 0 \), as shown \textit{e.g.} in the appendix D of the book by Mainardi [14]. We get

\[
\mathcal{E}_1(z) = -\text{Ei}(-z) = \Gamma(0, z) = -C - \log z + \text{Ein}(z), \quad \tag{2.8}
\]

with \( |\arg z| < \pi \), where \( C = -\Gamma'(1) = 0.577215\ldots \), denotes the Euler-Mascheroni constant.

This relation is important for understanding the analytic properties of the classical exponential integral functions in that it isolates the multi-valued part represented by the logarithmic function from the regular part represented by the entire function \( \text{Ein}(z) \) given by the power series in (2.7), absolutely convergent in all of \( \mathbb{C} \).

2.4 The Exponential integrals in the time domain

Let us consider in the time domain \( t > 0 \) the following two causal functions \( \phi(t), \psi(t) \) related to Exponential integrals:

\[
\phi(t) := \mathcal{E}_1(t), \quad t > 0, \quad \tag{2.9}
\]
\[ \psi(t) := \Ein(t) = C + \log t + \mathcal{E}_1(t), \quad t > 0, \quad (2.10) \]

The corresponding Laplace transforms, analytically continued in the complex \( s \) plane cut along the negative real axis, turn out to be:

\[ \mathcal{L}\{\phi(t)\}(s) = \frac{1}{s} \log (s + 1), \quad s \in \mathbb{C}^-, \quad (2.11) \]

\[ \mathcal{L}\{\psi(t)\}(s) = \frac{1}{s} \log \left( \frac{1}{s} + 1 \right), \quad s \in \mathbb{C}^-. \quad (2.12) \]

The proof of Eq. (2.11) can be found \textit{e.g.} in the book by Ghizzetti and Ossicini \footnote{6}, see Eq. [4.6.16], pp 104–105. The proof of (2.12) is hereafter provided in two ways. The first proof is obtained as a consequence of the identity (2.10), \textit{i.e.} \( \Ein(t) = \mathcal{E}_1(t) + C + \log t \), and the Laplace transform pair

\[ \mathcal{L}\{\log t\}(s) = -\frac{1}{s} [C + \log s], \quad s \in \mathbb{C}^- \]

whose proof is found \textit{i.e.} in \footnote{6}, see Eq. [4.6.15] and p. 104. The second proof is direct and instructive. For this it is sufficient to compute the Laplace transform of the elementary function provided by the derivative of \( \Ein(t) \), that is, according to a standard exercise in the theory of Laplace transforms,

\[ \mathcal{L}\left\{ \frac{1 - e^{-t}}{t} \right\}(s) = \log \left( \frac{1}{s} + 1 \right), \quad s \in \mathbb{C}^- \]

so that, with \( f(t) = (1 - e^{-t})/t \) and \( \tilde{f}(s) = \mathcal{L}\{f(t)\}(s) \),

\[ \psi(t) := \Ein(t) = \int_0^t f(t') \, dt' \div \frac{\tilde{f}(s)}{s} = \frac{1}{s} \log \left( \frac{1}{s} + 1 \right), \quad s \in \mathbb{C}^- \]

in agreement with (2.12), where we have denoted with \( \div \) the juxtaposition of a function \( f(t) \) with its Laplace transform \( \tilde{f}(s) \).

We outline the different asymptotic behaviours of the two functions \( \phi(t) \), \( \psi(t) \) for small argument \( t \to 0^+ \) and large argument \( t \to +\infty \) that can be easily obtained by the known asymptotical expressions for \( \mathcal{E}_1(z) \) and Ein \( z \) available \textit{e.g.} in \footnote{10}. However, it is instructive to derive the required asymptotic representations by using the Karamata Tauberian theory for Laplace transforms, see Feller \footnote{4}, Chapter XIII.5, as pointed out in \footnote{14}. We have

\[ \phi(t) \sim \begin{cases} \log (1/t), & t \to 0^+, \\ e^{-t}/t, & t \to +\infty, \end{cases} \quad (2.13) \]
\[
\psi(t) \sim \begin{cases} 
t, & t \to 0^+, \\
C + \log t, & t \to +\infty. 
\end{cases}
\quad (2.14)
\]

We note that the modified exponential integral (2.10) was adopted by Becker for describing the creep law in his 1926 viscoelastic model [1] and more later in 1982 by Strick and Mainardi [20] and recently by Mainardi et al. [15], even if the modified expression of the exponential integral was not known to Becker himself. The Becker model is also discussed in [10], in [14] and references therein. The main reason for adopting this model in linear viscoelasticity is that the derivative of the function (2.10) in the time domain \( t > 0 \) is interpreted (unless a suitable normalization factor) as the rate of creep:

\[
\psi'(t) := \frac{d}{dt} \operatorname{Ein}(t) = \frac{1 - e^{-t}}{t} = \frac{1 - e^{-t}}{t}, \quad t > 0,
\quad (2.15)
\]

It is straightforward to prove that the function (2.15) is completely monotonic (CM) (that is a non-negative function for \( t > 0 \) with infinitely many derivatives with alternating sign) as shown hereafter. Indeed the proof is carried out by the following steps. At first, based on excellent book by Schilling et al. [19] that is the most recent treatise on CM functions, we recognize that \( \psi'(t) \) is CM being a non-negative linear combination of two CM functions, \( 1/t \) and \( e^{-t}/t \). Then, in view of the Bernstein theorem the function \( \psi'(t) \) can be interpreted as the Laplace transform of a positive measure, that is represented by the following integrals

\[
\frac{d\psi}{dt}(t) = \int_0^\infty e^{-rt} K(r) \, dr = \int_0^\infty e^{-t/\tau} H(\tau) \, d\tau,
\quad (2.16)
\]

where \( K(r) \geq 0 \) and \( H(\tau) \geq 0 \) are referred to as the spectra in frequency \( (r) \) and in time \( (\tau = 1/r) \), respectively and read

\[
K(r) = \begin{cases} 
1 & 0 \leq r < 1, \\
0 & 1 \leq r < \infty;
\end{cases} \quad H(\tau) = \begin{cases} 
0 & 0 \leq \tau < 1, \\
1/\tau^2 & 1 \leq \tau < \infty.
\end{cases}
\quad (2.17)
\]

We note that the frequency spectrum \( K(r) \) can be determined from the Laplace transform of \( \psi'(t) \) by the Titchmarsh formula that reads in our notation noting \( \psi(0^+) = 0 \)

\[
K(r) = \pm \frac{1}{\pi} \Im[\tilde{s} \tilde{\psi}(s)]\bigg|_{s = re^{\pm i\pi}}, \quad \tilde{s} \tilde{\psi}(s) = \mathcal{L}\{\psi(t)\}(s) \quad s \in \mathbb{C}.
\quad (2.18)
\]
This a consequence of the fact that the Laplace transform of $\psi'(t)$ is the iterated Laplace transform of the frequency spectrum, that is the Stieltjes transform of $K(r)$ and henceforth the Titchmarsh formula provides the inversion of the Stieltjes transform, see e.g. the treatise by Widder [23]. Indeed, the Titchmarsh formula, intended in the limit of $s$ tending from above and below to the negative real axis $s = -r$ with $r > 0$, reads from (2.12)

$$K(r) = \pm \frac{1}{\pi} \Im[s\tilde{\psi}(s)]_{s = re^{\mp\pi}} = \frac{1}{\pi} \Im \left[ \log \left( -\frac{1}{r} + 1 \right) \right].$$

Then the function $\psi(t)$, being non negative with a CM derivative is of Bernstein type following the usual terminology in [19].

3 The generalized modified Exponential integral via the Mittag-Leffler function

3.1 The new function $E_{\nu}(z)$, $\nu > 0$

Let us now generalize the modified exponential integral as depending on a real parameter $\nu > 0$ by considering the Mittag-Leffler function

$$E_{\nu}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(k\nu + 1)}, \quad z \in \mathbb{C}, \quad \nu > 0, \quad (3.1)$$

that is known to be an entire function for $\nu > 0$ and generalize the exponential function $\exp(z)$ to which it reduces just for $\nu = 1$. For details on this transcendental function the reader is referred to the 2014 treatise by Gorenflo et al. [9]. This entire function can be seen as the particular case of the Mittag-Leffler in two parameters $\nu, \mu$

$$E_{\nu,\mu}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(k\nu + \mu)}, \quad z \in \mathbb{C}, \quad \nu, \mu > 0, \quad (3.2)$$

for $\mu = 1$. Then we propose to define for any $\nu > 0$, in the cut plane $\mathbb{C}^-$,

$$E_{\nu}(z) = \int_{0}^{z} \frac{1 - E_{\nu}(-u^\nu)}{u^\nu} \, du = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1} z^{\nu n - \nu + 1}}{\Gamma(n \nu + \nu + 1)}. \quad (3.3)$$

For $\nu = 1$ the function (2.6) is recovered, that is we have $E_{1}(z) = E(z)$. 7
3.2 The limiting regularized case $\nu = 0$ for $t > 0$

We note that the limiting case $\nu = 0$ requires special attention because in this case the Mittag-Leffler is no longer an entire function. Indeed, limiting to the Mittag-Leffler function in the non-negative time domain, we see from the corresponding plots that for $\nu \to 0$ the function becomes discontinuous around $t = 0$ assuming the value 1 at $t = 0$ and $\frac{1}{2}$ at $t > 0$, so we get

$$Ein_0(t) = \frac{t}{2}, \quad t > 0. \quad (3.4)$$

However, in this limiting case, we recover formally this result by summing according to Cesàro the undefined series of the corresponding limit of the Mittag-Leffler function, known as Grandi’s series

$$E_0(-t^0) = \sum_{k=0}^{\infty} (-1)^k = 1 - 1 + 1 - 1 + \cdots = \frac{1}{2}, \quad t > 0. \quad (3.5)$$

This series is a particular realization of the so called Dirichlet $\eta$ function [16]. The latter is part of a broad class of function series, known as Dirichlet series, more known in rheology as Prony series, that have recently found new physical applications in the so-called Bessel models, see e.g. [8, 2, 7].

3.3 The generalized Becker model in linear viscoelasticity

Following the analysis in the paper by Mainardi et al. [15] it is straightforward to introduce the generalized Becker model. The corresponding creep function and the rate of creep read with $0 < \nu \leq 1$, $t \geq 0$,

$$\psi_\nu(t) = \left\{ \begin{array}{ll} \Gamma(1 + \nu) Ein_\nu(t) \\ \Gamma(1 + \nu) \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{t^{\nu n-\nu+1}}{(\nu n - \nu + 1) \Gamma(1+n\nu)}, \end{array} \right. \quad (3.6)$$

and

$$\psi'_\nu(t) := \frac{d\psi_\nu}{dt} = \Gamma(1 + \nu) \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{t^{\nu n-\nu}}{\Gamma(1+n\nu)}, \quad (3.7)$$

where the factor $\Gamma(1 + \nu)$ is settled to get $\psi'(0) = 1$. The regularized limiting result for $\nu = 0$ corresponds in linear viscoelasticity to the linear creep law for
a Maxwell body. As a consequence, our generalized Becker model is ranging from the Maxwell body at $\nu = 0$ to the Becker body at $\nu = 1$.

As in [15], in the following figure (Fig. 1) we show versus time the creep function $\psi_\nu(t)$ and its derivative (the rate of creep) $\psi'_\nu(t)$ in a linear scale $0 \leq t \leq 10$ for the particular values of $\nu = 0, 0.25, 0.50, 0.75, 1$, from where we can note the tendency to the linear creep law for the Maxwell model as $\nu \to 0^+$.  

![Figure 1: The creep function $\psi_\nu(t)$ (left) and the rate of creep $\psi'_\nu(t)$ (right) for selected values of $\nu \in [0, 1]$.](image)

We point out that also for $0 < \nu < 1$ the functions characterizing the generalized Becker model $\psi_\nu(t)$ and $\psi'_\nu(t)$ keep the property to be Bernstein and CM functions as it is for the Maxwell and Becker bodies. The proof is straightforward because it is obtained by following the same steps of the Becker model illustrated in the previous section. This is due to the property of complete monotonicity of the Mittag-Leffler function even if divided for a power law function of Bernstein type. Indeed it is known that a CM function divided by a Bernstein function is still CM, see e.g. [19]. Furthermore, the function $\psi_\nu(t)$, being non negative with a CM derivative is of Bernstein type following the usual terminology in [19].

Indeed, following the same arguments of the previous section we can compute the spectral distributions $K_\nu(r)$, $H_\nu(\tau)$ related to $\psi'_\nu(t)$ applying the Titchmarsh formula to the Laplace transform of $\psi'_\nu(t)$. However, for $0 < \nu < 1$ the Laplace transform of the rate of creep is not known in analytic form so that it can be obtained integrating term by term the series representation of the original function. The series in the Laplace domain of the functions $\psi_\nu(t)$ and $\psi'_\nu(t)$ turn out to be, with $0 < \nu \leq 1$, $s \in \mathbb{C}^-$:

$$
\mathcal{L}\{\psi_\nu(t)\}(s) = \Gamma(1 + \nu) \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{s^{\nu-n\nu-2} \Gamma(n\nu - \nu + 2)}{(\nu n - \nu + 1) \Gamma(1 + n\nu)}. 
$$

$(3.8)$
\[ \mathcal{L}\{\psi'_\nu(t)\}(s) = \Gamma(1 + \nu) \sum_{n=1}^{\infty} (-1)^{n-1} s^{\nu-n-1} \Gamma(n\nu - \nu + 1) \Gamma(1 + n\nu). \] (3.9)

We show in Fig. 3 the spectra in frequency and in time of the rate of the creep for selected values of \( \nu \) in the range \( 0 < \nu \leq 1 \) obtained by using the MATHEMATICA\textsuperscript{\textregistered} tool box on the series of \( \mathcal{L}\{\psi'_\nu(t)\}(s) \) entering the Titchmarsh formula (2.18) with \( \tilde{\psi}_\nu(s) \). These spectra turn out to be non-negative (with a semi-infinite support \([0, +\infty)\) except in the Becker case \( \nu = 1 \)) that, as a matter of fact, are consistent with the CM property of \( \psi'_\nu(t) \).

![Figure 2: The spectra for the generalized Becker model for \( \nu = 0.25, 0.50, 0.75 \) compared with those of the Becker model \( \nu = 1 \): left in frequency \( K_\nu(r) \); right in time \( H_\nu(\tau) \).](image)

We can note the power-law behaviour for the spectra \( K_\nu(r) \) and \( H_\nu(\tau) \), due to power law asymptotics of the rate of creep \( \psi'_\nu(t) \) for \( t \to 0 \) and \( t \to \infty \) for \( 0 < \nu < 1 \), in contrast with the exponential decay of this function occurred for \( \nu = 1 \). Indeed the spectrum \( K_\nu(r) \) decays like \( r^{-(1+\nu)} \) at \( r \to 0 \) and \( r \to \infty \) as we can see considering the major contribution related to the series of \( \tilde{\psi}(s) \) entering the Titchmarsh formula for \( s = -r \).

4 The generalized Sine and Cosine integrals

4.1 The classical Sine and Cosine integrals

As usual in books on special functions, see e.g. [16] we define the sine integral and the cosine integral by the following integrals in the complex plane \( \mathbb{C} \),

\[ \text{Si}(z) := \int_0^z \frac{\sin(t)}{t} \, dt, \quad \text{Ci}(z) := -\int_{+\infty}^z \frac{\cos(t)}{t} \, dt, \] (4.10)
where the path does not cross the negative real axis or pass through the origin. It is easy to recognize that \( \text{Si}(z) \) is an (odd) entire function whereas \( \text{Ci}(z) \) is a polydrome function with a branch cut on the negative real axis. We also recognize that following limits on the positive real axis

\[
\lim_{x \to \infty} \text{Si}(x) = \frac{\pi}{2}, \quad \lim_{x \to \infty} \text{Ci}(x) = 0. \tag{4.11}
\]

In some books we find another function related to the sine integral

\[
\text{si}(z) := -\int_{z}^{\infty} \frac{\sin(t)}{t} \, dt = \text{Si}(z) - \frac{\pi}{2}. \tag{4.12}
\]

To remove the polydromy of \( \text{Ci}(z) \) we follow Shelkunoff and introduce the (even) entire function \( \text{Cin}(z) \), referred to as the modified cosine integral as

\[
\text{Cin}(z) = \int_{0}^{z} \frac{1 - \cos(t)}{t} \, dt = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}z^{2n}}{\Gamma(2n+1)2n}, \quad z \in \mathbb{C}. \tag{4.13}
\]

Because of a known relation concerning the exponential integrals, we easily get the relation between the standard cosine integral and the modified cosine integral, that is

\[
\text{Ci}(z) = -\text{Cin}(z) + \log z + C, \quad z \in \mathbb{C}^- . \tag{4.14}
\]

For completeness we define

\[
\text{Sin}(z) = \text{Si}(z) = \int_{0}^{z} \frac{\sin(t)}{t} \, dt = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}z^{2n-1}}{\Gamma(2n)(2n-1)}, \quad z \in \mathbb{C}. \tag{4.15}
\]

Now we like to generalize the entire functions \( \text{Sin}(z) \) and \( \text{Cin}(z) \) in the cut-plane \( \mathbb{C}^- \) by introducing a real (positive) parameter \( \nu \) and using the Mittag-Leffler functions. We restrict our analysis to \( \nu \in (0, 1) \) with the request to recover the above classical functions for \( \nu = 1 \). We first note that in the literature there exist some different definitions of the fractional circular functions depending on the parameter \( \nu \in (0, 1) \) adopted to generalize the classical circular function for \( \nu = 1 \), see the 1999 textbook by Podlubny [17], p.19, Eqs. (1.69)-(170), (1.71)-(1.72), formerly provided by Tsytlin in 1984 [22] and Luchko and Srivastava in 1995 [13].
4.2 The generalized Sine and Cosine integrals

For generalizing Sine and Cosine integral functions in the cut plane \( C^- \), we agree to start with the fractional circular functions defined more recently in the 2014 textbook by Herrmann [11], Ch. 6. They are related to the Mittag-Leffler functions defined in (3.2) and read

\[
\begin{align*}
\sin_\nu(z) &= z^\nu E_{2\nu,1+\nu} (-z^{2\nu}) = \sum_{k=0}^{+\infty} (-1)^k \frac{z^{(2k+1)\nu}}{\Gamma((2k+1)\nu+1)}, \\
\cos_\nu(z) &= E_{2\nu,1} (-z^{2\nu}) = \sum_{k=0}^{+\infty} (-1)^k \frac{z^{2k\nu}}{\Gamma(2k\nu+1)}.
\end{align*}
\]

(4.16)

For the fraction sine and fractional cosine we get the following plots in the range \( 0 \leq x \leq 10 \), for \( 0 < \nu \leq 1 \): We easily recognize because of the power law asymptotics of the Mittag-Leffler functions that both the fractional circular functions are decaying for \( \nu \in (0, 1) \) as a power law after a finite number of oscillations. Furthermore for \( \nu = 0 \) the fractional sine and cosine functions are expressed by Grandi’s series so the corresponding plots tend as \( \nu \to 0 \) to the constant value 0.5 for \( x > 0 \) with the value 0 and 1 at \( x = 0 \), respectively.

Then we define the functions \( \text{Sin}_\nu(z) \) and \( \text{Cin}_\nu(z) \) by the following integrals

\[
\begin{align*}
\text{Sin}_\nu(z) &= \int_{0}^{z} \frac{\sin_\nu(t)}{t^\nu} \, dt, \\
\text{Cin}_\nu(z) &= \int_{0}^{z} \frac{1 - \cos_\nu(t)}{t^\nu} \, dt.
\end{align*}
\]

(4.17)

The plots for \( z = x \) of these functions for selected values of \( \nu \), that is \( \nu = 0.25, 0.50, 0.75, 1 \), in the range \( 0 \leq x \leq 10 \). We note that for \( \nu = 1 \)
The plots by Gatteschi [5] of $\sin(x)$, and $\cos(x)$ are reproduced in our range $x \in [0, 10]$. As a consequence we derive the following power series for the desired functions

$$
\begin{align*}
\sin_\nu(z) &= \sum_{k=0}^{+\infty} (-1)^k \frac{z^{2k+1}}{\Gamma((2k+1)\nu+1)(2k\nu+1)}, \\
\cos_\nu(z) &= \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{z^{2n+1}}{\Gamma(2n\nu+1)((2n-1)\nu+1)}.
\end{align*}
$$

(4.18)

5 Conclusions

In this paper, after having recalled the properties of the Schelkunoff modification of the Exponential integral, we have generalized it with the Mittag-Leffler function. So doing we got a new special function (as far as we know) that was shown to be relevant in linear viscoelasticity because of its complete monotonicity properties in the time domain. Indeed this new model depending on the parameter $\nu \in [0, 1]$ allows a transition from from the standard Maxwell model for $\nu = 0$ to the Becker model for $\nu = 1$. We have also considered an approach the generalized sine and cosine integral functions based on a particular definition of the fractional sine and cosine functions, that appear reasonable in absence of a unique definition for them. We believe that this point is still open because also the fractional circular functions have not found a precise application up to now, as far as we know.
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