A characterization of the alpha-connections on the statistical manifold of normal distributions

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Abstract
We show that the statistical manifold of normal distributions is homogeneous. In particular, it admits a 2-dimensional solvable Lie group structure. In addition, we give a geometric characterization of the Amari–Chentsov $\alpha$-connections on the Lie group.

Keywords Statistical manifolds · The Amari–Chentsov $\alpha$-connection · Lie groups

Mathematics Subject Classification Primary 53A15; Secondary 22E25

Introduction
The set of normal distributions is parametrized by $\mathbb{R} \times \mathbb{R}^+$ as

$$\mathbb{R} \times \mathbb{R}^+ \ni \theta = (\mu, \sigma) \mapsto p(t, \theta) = \frac{1}{\sqrt{2\pi} \sigma^2} \exp\left\{-\frac{(t - \mu)^2}{2\sigma^2}\right\}, \ t \in \mathbb{R},$$

where $\mu$ is the mean, and $\sigma^2$ is the variance. For tangent vectors $X$, $Y$, $Z$ of an manifold $\mathbb{R} \times \mathbb{R}^+$ at $\theta$, we define

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\begin{align*}
g_F(X, Y) &= E_\theta[(X \log p)(Y \log p)], \\
C(X, Y, Z) &= E_\theta[(X \log p)(Y \log p)(Z \log p)],
\end{align*}
where $E_\theta[f] = \int f(t) \rho(t, \theta) \, dt$ for an integrable function $f$ on $\mathbb{R}$, see [1]. For a constant $\alpha$, we define $\nabla^{(\alpha)}$ by
\begin{align*}
g_F(\nabla^{(\alpha)} X Y, Z) &= g_F(\nabla^{g_F} X Y, Z) - \frac{\alpha}{2} C(X, Y, Z),
\end{align*}
where $\nabla^{g_F}$ is the Levi–Civita connection of the Riemannian metric $g_F$. In information geometry, $g_F$ and $\nabla^{(\alpha)}$ are well known as the Fisher metric and the Amari–Chentsov $\alpha$-connection for the space of normal distributions, respectively. In the same fashion, we have the Fisher metric and the Amari–Chentsov $\alpha$-connection for spaces of certain probability densities. Abstracting an essence from these, we reach the notion of statistical manifolds.

The Fisher metric and the Amari–Chentsov $\alpha$-connection for the space of all the positive probability densities on a finite set, that is, the space of multinomial distributions, are characterized from a viewpoint of statistics, which is known as the Chentsov theorem and has been generalized for other spaces, see [4] for example and references therein. The Fisher metric and the Amari–Chentsov $\alpha$-connection for the space of normal distributions are expressed as
\begin{align*}
g_F &= \frac{dx^2 + 2dy^2}{y^2}, \\
\nabla^{(\alpha)}_x \partial_x &= \frac{1 - \alpha}{2y} \partial_y, \\
\nabla^{(\alpha)}_y \partial_y &= \frac{1 + \alpha}{y} \partial_x, \\
\nabla^{(\alpha)}_y \partial_x &= -\frac{1 + 2\alpha}{y} \partial_y,
\end{align*}
where $\partial_x = \partial/\partial x, \partial_y = \partial/\partial y$ and $(x, y) = (\mu, \sigma) \in \mathbb{R} \times \mathbb{R}^+$. The goal of this article is to characterize affine connections on the statistical manifold of normal distributions with the Fisher metric from a purely differential geometric viewpoint, especially from a viewpoint of homogeneity. For this purpose, in Sect. 3, we discuss Lie groups equipped with a left-invariant statistical structure, called statistical Lie groups, see Definition 1.4.1. In particular we give an explicit method for constructing Lie groups equipped with left-invariant statistical structure. In the final section, we first show the statistical manifold of normal distributions is a statistical Lie group. More precisely we will show that the statistical manifold of normal distributions admits a solvable Lie group structure with respect to which the statistical structure is left-invariant, Proposition 4.1.1. We finally characterize left-invariant connections on a 2-dimensional solvable Lie group with a natural left-invariant metric under the total symmetry of the covariant derivative $\nabla C$ of the cubic form $C$, Proposition 4.2.1. As a result, we have the following main theorem.

**Theorem 1** The set $G$ consisting of all normal distributions admits a Lie group structure such that the Fischer metric $g^F = (dx^2 + 2dy^2)/y^2$ is left-invariant. Furthermore,
for an affine connection $\nabla$ such that $(G, g^F, \nabla)$ is a statistical Lie group, the following statements are mutually equivalent:

1. $\nabla$ is the Amari–Chentsov $\alpha$-connection.
2. $\nabla g^F C$ is totally symmetric.
3. $\nabla C$ is totally symmetric.
4. $\nabla g^F K$ is totally symmetric.
5. $R = R^*$.

Here $K$ is the skewness operator, and $R$ and $R^*$ denote the curvature tensor fields of $\nabla$ and its dual connection $\nabla^*$, respectively. The precise definitions are given in Sects. 1.1 and 1.2.

An essential contribution of the present paper is to show that one of (2), (3), (4) and (5) implies (1) for the statistical manifold of normal distributions with the Fisher metric. It should be remarked that the equivalences of (2), (3), (4) and (5) for any statistical manifold are well known [2,10]. For the statistical manifold of normal distributions, (1) implies (2), (3), (4) and (5), see [1].

On further studies related to conjugate symmetries of constant curvature statistical manifolds, see [6]. There are several options for introducing homogeneity in information geometry. On the recent studies coming from the homogeneity of sample spaces, we refer the reader to [12,13] for example.

1 Preliminaries

In this section we recall the definition of statistical manifolds and introduce the notion of homogeneous statistical manifolds.

1.1 Statistical manifolds

Let $M$ be a manifold, $g$ a Riemannian metric and $\nabla$ a torsion free affine connection. Then $\nabla$ is said to be compatible with $g$ if the covariant derivative $\nabla g$ is symmetric. A pair $(g, \nabla)$ of a Riemannian metric and a compatible affine connection is called a statistical structure on $M$. A manifold $M$ together with a statistical structure is called a statistical manifold. In particular a statistical manifold is traditionally called a Hessian manifold if $\nabla$ is flat, see for example [11].

A Riemannian manifold $(M, g)$ together with Levi–Civita connection $\nabla^g$ of $g$ is a typical example of statistical manifold. In other words, statistical manifolds can be regarded as generalizations of Riemannian manifolds.

The conjugate connection or the dual connection $\nabla^*$ of $\nabla$ with respect to $g$ is introduced by the following formula:

$$Xg(Y, Z) = g(\nabla X Y, Z) + g(Y, \nabla^* X Z),$$  \hspace{1cm} (1.1)

where $X, Y, Z \in \mathfrak{X}(M)$. Obviously $\nabla = \nabla^*$ if and only if $\nabla$ coincides with the Levi–Civita connection $\nabla^g$. 
1.2 The associated skewness operator

On a statistical manifold \((M, g, \nabla)\), the symmetric tensor field \(C\) of type \((0, 3)\), the so-called cubic form on \(M\), is defined by

\[
C(X, Y, Z) = (\nabla_X g)(Y, Z),
\]

(1.2)

where \(X, Y, Z \in \mathfrak{X}(M)\). Then we can associate a tensor field \(K\) of type \((1, 2)\) by

\[
g(K(X, Y), Z) = C(X, Y, Z),
\]

(1.3)

where \(X, Y, Z \in \mathfrak{X}(M)\). Furthermore, for every vector field \(X\), one can introduce an endomorphism field \(K(X)\) by

\[
K(X)Y = K(X, Y).
\]

Since \(C\) is totally symmetric, \(K(X)\) is self-adjoint with respect to \(g\) and symmetric, i.e.,

\[
g(K(X)Y, Z) = g(Y, K(X)Z) \quad \text{and} \quad K(X)Y = K(Y)X.
\]

We call this tensor field \(K\) the skewness operator of \((M, g, \nabla)\), see [7]. The difference between \(\nabla\) and \(\nabla^{g}\) is given by

\[
\nabla - \nabla^{g} = -\frac{1}{2} K.
\]

Remark 1.2.1 The symbol \(K\) is sometimes used as the difference itself, for example in [9].

The Levi–Civita connection \(\nabla^{g}\) is the “mean” of \(\nabla\) and \(\nabla^{*}\), i.e., \(\nabla^{g} = \frac{1}{2} (\nabla + \nabla^{*})\). More generally for any real number \(\alpha\),

\[
\nabla^{(\alpha)}X Y = \nabla^{g}X Y - \frac{\alpha}{2} K(X)Y
\]

(1.4)

defines a torsion free affine connection \(\nabla^{(\alpha)}\). The affine connection \(\nabla^{(\alpha)}\) is called the \(\alpha\)-connection, [1]. Note that \(\nabla^{(1)} = \nabla\) and \(\nabla^{(-1)} = \nabla^{*}\). The covariant derivative of \(g\) relative to \(\nabla^{(\alpha)}\) is

\[
(\nabla^{(\alpha)}X g)(Y, Z) = \alpha C(X, Y, Z).
\]

Thus \((M, g, \nabla^{(\alpha)})\) is statistical for all \(\alpha \in \mathbb{R}\).
**Remark 1.2.2** As we have seen before, for every statistical manifold \((M, g, \nabla)\), there exists a naturally associated symmetric trilinear form \(C\). Conversely let \((M, g, C)\) be a Riemannian manifold with a symmetric trilinear form \(C\). Define a tensor field \(K\) by \(g(K(X)Y, Z) = C(X, Y, Z)\), and an affine connection \(\nabla\) by \(\nabla = \nabla^g - K/2\). Then the triplet \((M, g, \nabla)\) becomes a statistical manifold. Thus to equip a statistical structure \((g, \nabla)\) is equivalent to equip a pair \((g, C)\) consisting of a Riemannian metric \(g\) and a trilinear form \(C\). In fact, Lauritzen [7, Chapter 4] introduced the notion of statistical manifold as a Riemannian manifold \((M, g)\) together with a trilinear form \(C\).

Finally we remark that the curvature tensor field \(R\) of \(\nabla\) is related to the Riemannian curvature \(R^g = R^{(0)}\) of \(g\) by

\[
R(X, Y)Z = R^g(X, Y)Z + \frac{1}{4}[K(X), K(Y)]Z - \frac{1}{2} \left( (\nabla^g_X K)(Y, Z) - (\nabla^g_Y K)(X, Z) \right).
\]

We refer to the readers Amari and Nagaoka’s textbook [1] for general theory of statistical manifolds.

### 1.3 Covariant derivative of the cubic form \(C\)

We have many choices of affine connection \(\nabla\) compatible with a prescribed Riemannian metric \(g\), or equivalently, many choices of cubic forms \(C\) on \(M\). In this section, let us consider a condition \(\nabla C\) is totally symmetric. This condition is well known in the affine hypersurface theory. In fact, a statistical manifold with certain conditions can be realized as a Blaschke hypersurface in the equiaffine space. Then the condition \(\nabla C\) is totally symmetric means that the hypersurface is an affine hypersphere, which is the most interesting object in affine differential geometry. We refer to the readers [9] for details.

**Lemma 1.3.1** ([2,10]) The following statements are mutually equivalent:

1. \(\nabla^g C\) is totally symmetric.
2. \(\nabla C\) is totally symmetric.
3. \(\nabla^g K\) is totally symmetric.
4. \(R = R^*\).

Here \(R^*\) denotes the curvature tensor field of \(\nabla^*\). Note that statistical manifolds satisfying the condition \(R = R^*\) have been said to be *conjugate symmetric*, [7].

The proof of this lemma for statistical structures induced by affine hypersurfaces still works for our setting. We briefly review the proof for the sake of completeness. Using the relation \(\nabla = \nabla^g - \frac{1}{2} K\), we have

\[
(\nabla_X C)(U, V, W) - (\nabla_U C)(X, V, W) = (\nabla^g_X C)(U, V, W) - (\nabla^g_U C)(X, V, W).
\]

Thus the conditions (1) and (2) are equivalent. Moreover from \(C(X, Y, Z) = g(K(X)Y, Z)\), it is easy to see that

\[
(\nabla^g_X C)(U, V, W) = g((\nabla^g_X K)(U, V), W).
\]
From this, it is easy to see that the conditions (2) and (3) are equivalent. Finally, since
\[(\nabla^g_X K)(Y, Z) - (\nabla^g_Y K)(X, Z) = -R(X, Y)Z + R^*(X, Y)Z,\]
the conditions (3) and (4) are equivalent.

1.4 Homogeneous statistical manifolds

To close this section we introduce the notion of homogeneous statistical manifold.

**Definition 1.4.1** Let \((M, g, \nabla)\) be a statistical manifold and \(G\) a Lie group. Then \((M, g, \nabla)\) is said to be a **homogeneous statistical manifold** if \(G\) acts transitively on \(M\) and the action is isometric with respect to \(g\) and affine with respect to \(\nabla\). In particular, if \(G\) is equipped with a statistical structure \((g, \nabla)\) such that both \(g\) and \(\nabla\) are invariant under left translations, then \((G, g, \nabla)\) is a homogeneous statistical manifold. The resulting homogeneous statistical manifold is called a **statistical Lie group**.

From the next section we will study statistical Lie groups.

2 Left-invariant connections and left-invariant metrics on Lie groups

Let \(G\) be a connected Lie group and denote by \(\mathfrak{g}\) the Lie algebra of \(G\), that is, the tangent space \(T_eG\) of \(G\) at the unit element \(e \in G\). In this section we consider left-invariant connection, that is affine connections on \(G\) which are invariant under left translations by \(G\).

Let \(\theta\) be the Maurer-Cartan form of \(G\). By definition, for any tangent vector \(X_a\) of \(G\) at \(a \in G\), we have \(\theta_a(X_a) = (dL_a)^{-1}X_a \in \mathfrak{g}\). Here \(L_a\) denotes the left translation by \(a\) in \(G\); \(L_a : G \ni x \mapsto ax \in G\).

Hereafter, we always assume that Lie groups under consideration will be connected.

2.1 Left-invariant connections on Lie groups

Take a bilinear map \(\mu : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}\). Then we can define a left-invariant affine connection \(\nabla^\mu\) on \(G\) by its value at the unit element \(e \in G\) by
\[
\nabla^\mu_X Y = \mu(X, Y), \quad X, Y \in \mathfrak{g}.
\]

**Proposition 2.1.1** (\([8]\)) Let \(\mathcal{B}_\mathfrak{g}\) be the vector space of all \(\mathfrak{g}\)-valued bilinear maps on \(\mathfrak{g}\) and \(\mathcal{A}_G\) the affine space of all left-invariant affine connections on \(G\). Then the map
\[
\mathcal{B}_\mathfrak{g} \ni \mu \mapsto \nabla^\mu \in \mathcal{A}_G
\]
is a bijection between \(\mathcal{B}_\mathfrak{g}\) and \(\mathcal{A}_G\). The torsion \(T^\mu\) of \(\nabla^\mu\) is given by
\[
T^\mu(X, Y) = -[X, Y] + \mu(X, Y) - \mu(Y, X)
\]
for all $X, Y \in \mathfrak{g}$.

Accordingly, $\nabla^\mu$ is of torsion free if and only if

$$\mu(X, Y) - \mu(Y, X) = [X, Y], \ \text{i.e.,} \ (\text{skew } \mu)(X, Y) = \frac{1}{2}[X, Y]. \quad (2.1)$$

where $(\text{skew } \mu)$ is the skew symmetric part of $\mu$. Thus $\nabla^\mu$ has the form:

$$\nabla^\mu_X Y = \frac{1}{2}[X, Y] + (\text{sym } \mu)(X, Y), \quad (2.2)$$

where $(\text{sym } \mu)$ is the symmetric part of $\mu$.

### 2.2 Left-invariant metrics on Lie groups

We equip an inner product $\langle \cdot, \cdot \rangle$ on the Lie algebra $\mathfrak{g}$ of a real Lie group $G$ and extend it to a left-invariant Riemannian metric $g = \langle \cdot, \cdot \rangle$ on $G$. Here we define a symmetric bilinear map $U : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ by

$$2\langle U(X, Y), Z \rangle = \langle [Z, X], Y \rangle + \langle X, [Z, Y] \rangle \quad (2.3)$$

for $X, Y, Z \in \mathfrak{g}$, see [5, Chapter X.3.]. This formula implies that $g$ is bi-invariant if and only if $U = 0$.

The Levi–Civita connection $\nabla^g$ of $G$ is given, as a variant of the Koszul formula, by

$$\nabla^g_X Y = \frac{1}{2}[X, Y] + U(X, Y) \quad (2.4)$$

for $X, Y \in \mathfrak{g}$. Hence $\nabla^g$ is a left-invariant connection $\nabla^\mu$ with the bilinear map $\mu(X, Y) = \frac{1}{2}[X, Y] + U(X, Y)$. Accordingly, we have

**Proposition 2.2.1** The Levi–Civita connection $\nabla^g$ is a left-invariant connection determined by the bilinear map $\mu$ such that

$$(\text{skew } \mu)(X, Y) = \frac{1}{2}[X, Y], \quad (\text{sym } \mu)(X, Y) = U(X, Y). \quad (2.5)$$

### 3 Statistical structures on Lie groups

Let $G$ be a Lie group with a left-invariant Riemannian metric $g = \langle \cdot, \cdot \rangle$ and a left-invariant affine connection $\nabla^\mu$. The covariant derivative $C = \nabla^\mu g$ is computed as:

$$C(X, Y, Z) = -\langle \mu(X, Y), Z \rangle - \langle Y, \mu(X, Z) \rangle.$$
Since we have \[ C(Y, X, Z) = -\langle \mu(Y, X), Z \rangle - \langle X, \mu(Y, Z) \rangle \] analogously, the total symmetry condition of \( C \) is

\[
\langle \mu(X, Y) - \mu(Y, X), Z \rangle = \langle X, \mu(Y, Z) \rangle - \langle Y, \mu(X, Z) \rangle. \tag{3.1}
\]

Then by using the torsion free condition in (2.1) and the form of \( \mu \) in (2.2), the condition (3.1) is equivalent to

\[
\langle [X, Y], Z \rangle = \langle X, (\text{sym } \mu)(Y, Z) \rangle - \langle Y, (\text{sym } \mu)(X, Z) \rangle 
+ \frac{1}{2} \langle X, [Y, Z] \rangle - \frac{1}{2} \langle Y, [X, Z] \rangle. \tag{3.2}
\]

Finally using the definition of \( U \) in (2.3), the condition (3.2) is equivalent to

\[
\langle U(Y, Z), X \rangle - \langle U(X, Z), Y \rangle = \langle (\text{sym } \mu)(Y, Z), X \rangle - \langle (\text{sym } \mu)(X, Z), Y \rangle. \tag{3.3}
\]

Thus when \( g \) is bi-invariant, by (2.5) the total symmetry condition is

\[
\langle X, (\text{sym } \mu)(Y, Z) \rangle = \langle Y, (\text{sym } \mu)(X, Z) \rangle.
\]

Now let us put \( v := \text{sym } \mu \) then we obtain the following recipe for constructing statistical Lie groups.

**Proposition 3.0.1** Let \( G \) a Lie group equipped with a left-invariant metric \( g \) and the symmetric bilinear map \( U \) defined in (2.3). Then every left-invariant connection \( \nabla \) compatible with \( g \) is represented as

\[
\nabla_X Y = v(X, Y) + \frac{1}{2} [X, Y]
\]

for some symmetric bilinear map \( v : g \times g \to g \) satisfying

\[
\langle U(Y, Z), X \rangle - \langle U(X, Z), Y \rangle = \langle (\text{sym } \mu)(Y, Z), X \rangle - \langle (\text{sym } \mu)(X, Z), Y \rangle.
\]

In particular, the connection \( \nabla \) can also be represented as

\[
\nabla_X Y = \frac{1}{2} [X, Y] + U(X, Y) - \frac{1}{2} K(X)Y
\]

for a self-adjoint symmetric bilinear map \( K \), i.e., \( K : g \times g \to g \) satisfying

\[
\langle K(X)Z, Y \rangle = \langle Z, K(X)Y \rangle, \quad K(X)Y = K(Y)X.
\]

Furthermore, if the left-invariant statistical structure is bi-invariant, then the connection is expressed as \( \nabla_X Y = \frac{1}{2} [X, Y] - \frac{1}{2} K(X)Y \) for an \( \text{Ad}(G) \)-invariant self-adjoint symmetric bilinear map \( K \).
4 The statistical manifold of normal distributions

In this section we prove that the statistical manifold of normal distributions is homogeneous. More precisely, we prove that the statistical manifold of normal distributions is identified with a 2-dimensional solvable Lie group equipped with a left-invariant statistical structure, Proposition 4.1.1. Next, we characterize the Amari–Chentsov $\alpha$-connection on the statistical manifold of normal distributions in terms of the covariant derivative of the cubic form $\nabla C$.

4.1 Two dimensional Lie groups

In this subsection we give some explicit examples of 2-dimensional statistical Lie groups. It is known [3] that every 2-dimensional Lie group is either abelian or non-abelian and isomorphic to

$$g = \left\{ \begin{pmatrix} u & v \\ 0 & 0 \end{pmatrix} \right| u, v \in \mathbb{R} \right\}.$$  

The Lie algebra $g$ is generated by

$$E_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$  

with commutation relation $[E_1, E_2] = -E_1$. The simply connected and connected Lie group $G$ corresponding to $g$ is

$$G = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \right| x, y \in \mathbb{R}, \ y > 0 \right\} \quad (4.1)$$

with global coordinate system $(x, y)$. Note that $G$ is the Lie group of orientation preserving affine transformations of the real line $\mathbb{R}$. We denote by $\psi$, the inverse map of this global coordinate system. Then a direct computation shows that $\psi : \mathbb{R} \times \mathbb{R}_+ \to G$ satisfies

$$\psi(x, y)^{-1} \partial_x \psi(x, y) = y^{-1} E_1 \quad \text{and} \quad \psi(x, y)^{-1} \partial_y \psi(x, y) = y^{-1} E_2,$$

where $\partial_x = \partial / \partial x$ and $\partial_y = \partial / \partial y$ for short. Thus for a constant $\lambda > 0$, we get left-invariant vector fields on $G$ as follows:

$$e_1 = dL_{\psi(x,y)}E_1 = y\partial_x, \quad e_2 = \lambda^{-1} dL_{\psi(x,y)}E_2 = \lambda^{-1} y\partial_y.$$  

Equivalently, $\theta(e_1) = E_1, \theta(e_2) = \lambda^{-1} E_2$. Note that $[e_1, e_2] = -\lambda^{-1} e_1$.  

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We introduce a left-invariant Riemannian metric $g$ for which $\{e_1, e_2\}$ is orthonormal. Then the induced left-invariant metric $g$ is

$$g = \frac{dx^2 + \lambda^2 dy^2}{y^2}.$$  

When $\lambda = 1$, the resulting Riemannian manifold is the hyperbolic plane of curvature $-1$. In case $\lambda = \sqrt{2}$, the metric $g$ is the Fisher metric of the space of the normal distributions.

The symmetric bilinear map $U$ in (2.3) is computed as $2\langle U(e_1, e_1), e_1 \rangle = 0$ and

$$2\langle U(e_1, e_1), e_2 \rangle = \langle e_1, [e_2, e_1] \rangle + \langle e_1, [e_2, e_1] \rangle = 2\lambda^{-1}.$$  

Thus $U(e_1, e_1) = \lambda^{-1} e_2$. Similarly we have $U(e_1, e_2) = -\lambda^{-1} e_1/2$ and $U(e_2, e_2) = 0$. The Levi–Civita connection is described as

$$\nabla^g_{e_1} e_1 = \frac{1}{\lambda} e_2, \quad \nabla^g_{e_1} e_2 = -\frac{1}{\lambda} e_1, \quad \nabla^g_{e_2} e_1 = 0, \quad \nabla^g_{e_2} e_2 = 0.$$  

The Amari–Chentsov $\alpha$-connection of the statistical manifold of the normal distributions is naturally extended to the following connection on $G$:

$$\nabla^{(\alpha)}_{\partial_x} \partial_x = \frac{1-\alpha}{\lambda^2 y} \partial_y, \quad \nabla^{(\alpha)}_{\partial_x} \partial_y = \nabla^{(\alpha)}_{\partial_y} \partial_x = -\frac{1+\alpha}{y} \partial_x, \quad \nabla^{(\alpha)}_{\partial_y} \partial_y = -\frac{1+2\alpha}{y} \partial_y.$$  

It should be remarked that every connection $\nabla^{(\alpha)}$ is left-invariant. In fact the table of covariant derivatives with respect to $\nabla^{(\alpha)}$ is rephrased as

$$\nabla^{(\alpha)}_{e_1} e_1 = \frac{1-\alpha}{\lambda} e_2, \quad \nabla^{(\alpha)}_{e_1} e_2 = -\frac{1+\alpha}{\lambda} e_1, \quad \nabla^{(\alpha)}_{e_2} e_1 = -\frac{\alpha}{\lambda} e_1, \quad \nabla^{(\alpha)}_{e_2} e_2 = -\frac{2\alpha}{\lambda} e_2.$$  

(4.2)

**Proposition 4.1.1** The statistical manifold $(\{ (x, y) \in \mathbb{R}^2 \mid y > 0 \}, g, \nabla^{(\alpha)})$ of the normal distributions mentioned in Introduction is identified with the solvable Lie group $G$ in (4.1) equipped with a left-invariant metric $g = (dx^2 + 2dy^2)/y^2$ and a left-invariant connection $\nabla^{(\alpha)}$ in (4.2) with $\lambda = \sqrt{2}$. Furthermore, the skewness operator is given by

$$K(e_1, e_1) = \sqrt{2} e_2, \quad K(e_1, e_2) = \sqrt{2} e_1, \quad K(e_2, e_2) = 2\sqrt{2} e_2.$$  

**Remark 4.1.2** Under the identification of the statistical manifold of normal distributions with the solvable Lie group $G$ given by (4.1), $G$ acts on the real line $\mathbb{R}$ as affine transformations. One can see that $(\mu, \sigma)^{-1} : \mathbb{R} \ni t \mapsto (t - \mu)/\sigma \in \mathbb{R}$. This implies a transformation of random variables. Every normal distribution $(\mu, \sigma) \in G$ is translated to the standard normal distribution $(0, 1)$ by left translation $(\mu, \sigma)^{-1}$. This is nothing but the standardized form of a random variable of $(\mu, \sigma)$.  

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4.2 A characterization of the $\alpha$-connections

Finally we give a characterization of the $\alpha$-connections on the statistical manifold of normal distributions in terms of the total symmetry of $\nabla C$.

**Proposition 4.2.1** Let $G$ be a 2-dimensional solvable Lie group defined in (4.1) with a left-invariant metric $g = (dx^2 + \lambda^2 dy^2)/y^2$. If $G$ admits a left-invariant statistical structure $(G, g, \nabla)$ satisfying one of the conditions in Lemma 1.3.1. Then $\nabla$ is given by

\[
\nabla e_1 e_1 = \left( \frac{1}{\lambda} - \frac{p}{2} \right) e_2, \quad \nabla e_1 e_2 = \left( -\frac{1}{\lambda} - \frac{p}{2} \right) e_1, \\
\nabla e_2 e_1 = -\frac{p}{2} e_1, \quad \nabla e_2 e_2 = -pe_2,
\]

where $p \in \mathbb{R}$ and $\{e_1, e_2\}$ is the left-invariant orthonormal frame field as before.

**Proof** The condition (3) in Lemma 1.3.1 is equivalent to

\[
(\nabla^g e_1 K)(e_2) = (\nabla^g e_2 K)(e_1), \quad (\nabla^g e_1 K)(e_2, e_1) = (\nabla^g e_2 K)(e_1, e_1).
\]

Since $\nabla^g e_j e_j = 0 \ (j = 1, 2)$, we can compute the above equations as

\[
K^1_{22} e_2 - K^2_{22} e_1 + 2 \sum_{\ell=1}^2 K^\ell_{12} e_\ell = 0, \\
K^1_{21} e_2 - K^2_{21} e_1 + \sum_{\ell=1}^2 K^\ell_{11} e_\ell - \sum_{\ell=1}^2 K^\ell_{22} e_\ell = 0.
\]

Here we write $K(e_i, e_j) = \sum_{\ell=1}^2 K^\ell_{ij} e_\ell$. Thus the skewness operator $K$ can be explicitly given as

\[
K(e_1, e_1) = pe_2, \quad K(e_1, e_2) = K(e_2, e_1) = pe_1, \quad K(e_2, e_2) = 2pe_2. \quad (4.3)
\]

By using the relation $\nabla = \nabla^g - \frac{1}{2} K$, the claim follows. \qed

By setting $p = 2\lambda^{-1}\alpha$ with $\lambda = \sqrt{2}$ and by using Lemma 1.3.1, we arrive at Theorem 1.

**Remark 4.2.2** Differential geometric properties of the statistical manifold of multivariate normal distributions are different from those of univariate normal distributions, which will be studied in a separate publication from a viewpoint of statistical Lie groups.

On behalf of all authors, the corresponding author states that there is no conflict of interest.
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