Thomas precession as a source of the electromagnetic angular momentum radiation

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Abstract. Using the general principles of the relativistic radiation theory two alternative definitions of the angular momentum density of the electromagnetic field by Ivanenko-Sokolov and by Teitelboim are discussed. It is shown that both definitions give identical integral characteristics of radiation for an arbitrary moving relativistic charge. It is obtained that the total power of the field spin angular momentum radiation is proportional to the Thomas precession and corresponds to the force field momentum of radiation. As an application there are considered properties of the orbital and spin moments of the synchrotron radiation.

1. Introduction
The hypothesis that electromagnetic waves have proper angular momentum was put forward by A. I. Sadowsky over 100 years ago in 1897 [1]. He proposed the method of measurement for angular momentum of light based on light transmission through anisotropic crystalline plate: «...any apparatus processing linearly polarized light into circularly polarized must rotate...».

In 1935-1936, B. A. Beth in USA [2] and A. N. S. Holborn in England [3] experimentally proved that circularly polarized light has angular momentum. These precise experiments not only confirmed Sadowsky’s hypothesis, but also allowed to estimate Planck’s constant with an accuracy of 10%.

Observation of considerably greater torque became possible with the emergence of lasers (see, for example, [4]). The existence of angular momentum of circularly polarized electromagnetic waves in the present time there is no doubt about.

However the general definition of angular momentum of the electromagnetic field (AMEF) is argued over among physicists to this day. Contentious debates about the adequacy of the theory of angular momentum and its radiation arise from time to time [5-8] et al.

2. Two alternative methods of the AMEF definition
It may be distinguished two alternative ways of the relativistically covariant methods for the description of AMEF originated by D. D. Ivanenko and A. A. Sokolov [9] and by C. Teitelboim et al. [10-12].

According to the pathway proposed by C. Teitelboim et al. the density tensor of the total AMEF

\[ \mathcal{M}^{\mu\nu\lambda} = R^\mu_{\rho} \mathcal{P}^{\rho\nu\lambda} - R^{\rho}_{\sigma\lambda} \mathcal{P}^{\mu\nu\sigma} \]

is described on basis of the symmetric density tensor of the electromagnetic field momentum

\[ \mathcal{P}^{\mu\nu} = -\frac{1}{4} \left( H^{\mu\nu} H^\nu_{\rho} + \frac{1}{4} g^{\mu\rho} H^\alpha_{\alpha3} H^{\alpha\beta} \right) = \mathcal{P}^{\mu\nu}_R, \]

which in contrast to \( \mathcal{P}^{\mu\nu}_{can} \) is gauge invariant.

Tensor \( \mathcal{P}^{\mu\nu} \) is calculated in the relativistic radiation theory of the charged particles according to that (see. [13])

\[ H^{\mu\nu} = \vec{H}^{\mu\nu} + \vec{H}^{\mu\nu}, \]

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where $\tilde{H}^{\mu\nu}$ is the strength tensor of the attached fields in the near-charge zone and $\tilde{H}^{\mu\nu}$ is in the long-distance region which is called the radiation zone.

In this representation the decomposition of the density of total AMEF $M^{\mu\nu}=A^{\mu\nu}+\Pi^{\mu\nu}$ into orbital $A^{\mu\nu}$ and spin $\Pi^{\mu\nu}$ parts arise automatically according to

$$A^{\mu\nu}=\tilde{\rho}^{\mu}\tilde{P}^{\nu}-\tilde{\rho}^{\nu}\tilde{P}^{\mu}, \quad \Pi^{\mu\nu}=\tilde{\rho}^{\mu}\tilde{P}^{\nu}-\tilde{\rho}^{\nu}\tilde{P}^{\mu}. \quad (4)$$

The dependence of the falling of field strength according to its dependence on the distance $\tilde{r}$ takes a significant role in these definitions. Tensor $\tilde{P}^{\mu\nu}$ disintegrates into three parts in accordance with general theory of relativistic radiation (see, for example [8]),

$$\tilde{P}^{\mu\nu}=\tilde{\rho}^{\mu}\tilde{P}^{\nu}+\tilde{P}^{\mu\nu}+\tilde{\rho}^{\mu\nu}, \quad (5)$$

where

$$\tilde{\rho}^{\mu\nu}=\frac{1}{4}\frac{e^2}{c^3}\left[\frac{1}{2}\tilde{g}^{\mu\nu}-\frac{\tilde{\rho}^{\mu}\tilde{\rho}^{\nu}}{(\tilde{\rho}^{\mu})^3}+\frac{\tilde{\rho}^{\mu}\tilde{\rho}^{\nu}}{(\tilde{\rho}^{\mu})^3}+\frac{1}{2}\tilde{g}^{\mu\nu}\right] \sim \frac{1}{\tilde{r}^4} \quad (5a)$$

is the momentum density tensor of the convection field in the near zone. Here $e$ is the charge of an arbitrary moving relativistic particle, $v^\mu = \frac{dr^\mu}{d\tau}$ its four-velocity, $w^\mu = \frac{dw^\mu}{d\tau}$ its four-acceleration of the charge the location of which is described by the trajectory radius-vector $r^\mu(\tau)$, $\tau$ is a proper time and $r^\mu(\tau, \tau) = R^\mu(\tau) - r^\mu(\tau)$ is the four-dimensional vector traced from the charge to the spectator point of four-dimensional vector $R^\mu(\tau)$.

$$\tilde{\rho}^{\mu\nu}=-\frac{1}{4\pi}\frac{e^2}{c^3}\left[\frac{(\tilde{\rho}^{\mu})^2}{(\tilde{\rho}^{\mu})^3}+\frac{(\tilde{\rho}^{\mu})^2}{(\tilde{\rho}^{\mu})^3}\right] \sim \frac{1}{\tilde{r}^2} \quad (5b)$$

is the momentum density tensor of the mixed fields $\tilde{H}^{\mu\nu}$ and $\tilde{H}^{\mu\nu}$,

$$\tilde{\rho}^{\mu\nu}=-\frac{1}{4\pi}\frac{e^2}{c^3}\left[\frac{1}{(\tilde{\rho}^{\mu})^3}+(\tilde{\rho}^{\mu})^2\right] \sim \frac{1}{\tilde{r}^2} \quad (5c)$$

is the momentum density tensor of the radiation field in the wave zone. In the following we will consider the AMEF only in the wave zone where in accordance with (4)-(5) it follows that

$$\tilde{A}^{\mu\nu}=\frac{1}{4\pi}\frac{e^2}{c^3}\left[\frac{1}{(\tilde{\rho}^{\mu})^3}+\frac{1}{(\tilde{\rho}^{\mu})^3}\right] \sim \frac{1}{\tilde{r}^2} \quad (6a)$$

$$\tilde{\Pi}^{\mu\nu}=-\frac{e^2}{4\pi}\left[\frac{1}{(\tilde{\rho}^{\mu})^3}+\frac{1}{(\tilde{\rho}^{\mu})^3}\right] \sim \frac{1}{\tilde{r}^2} \quad (6b)$$

In terms of the definition

$$D^\mu = \frac{\partial}{\partial R^\mu} = \tilde{\partial}^\mu + \frac{\tilde{\rho}^{\mu}}{(\tilde{\rho}^{\mu})^3} \frac{d}{d\tau} \quad (7)$$

applying the differentiation technique of the retarded in time fields [10,13] it can be shown that

$$D_\lambda\tilde{P}^{\lambda\mu} = 0 \quad (8)$$

follows from
\[ D_\lambda (\widetilde{\mathcal{P}}^{\mu\nu} + \widetilde{\mathcal{P}}^{\nu\mu}) = 0, \quad D_\lambda \widetilde{\mathcal{P}}^{\mu\nu} = 0. \]  

From this, one obtains \[ D_\lambda \widetilde{\mathcal{M}}^{\mu\nu\lambda} = 0, \] and \[ D_\lambda \widetilde{\mathcal{I}}^{\mu\nu\lambda} = D_\lambda \widetilde{\mathcal{I}}^{\mu\nu\lambda} = 0. \]

Thus, the decomposition of the density of the total AMEF onto orbital and spin parts now is gauge- and relativistically invariant and so we will assume Teitelboim’s method as a basis for our further research.

It can be seen that in the wave zone the density of the canonical momentum tensor becomes symmetrical form: \( \widetilde{\mathcal{P}}_{\text{can}}^{\mu\nu} = \widetilde{\mathcal{P}}_{\text{can}}^{\nu\mu} \). And in accordance with the above consideration the previously denoted defects of Ivanenko-Sokolov method disappear! Henceforth we will also use this method for comparison with the Teitelboim’s method.

3. Relativistic radiation theory of the orbital and spin AMEF

Then in accordance with (11) we can receive orbital and spin momenta of the radiation field on the basis of Gauss’s integral theorem in the form

\[ \tilde{L}^{\mu\nu} = \oint \widetilde{\mathcal{A}}^{\mu\nu\lambda} d\sigma_\lambda, \quad \tilde{P}^{\mu\nu\lambda} = \oint \widetilde{\mathcal{I}}^{\mu\nu\lambda\sigma} d\sigma_\lambda. \]  

Here \( d\sigma_\lambda = e_\lambda \varepsilon^\sigma c^2 \tau d\Omega_0 \) is the element of timelike hypersurface with the spacelike normal vector \( e_\lambda = \varepsilon/\varepsilon - u_\lambda/c \) and \( d\Omega_0 = (\varepsilon^2/f^2) d\Omega \) is the solid-angle element in the rest frame of particle, \( \varepsilon = -\tilde{\tau}_\lambda v^\lambda/c = \text{inv} \).

As

\[ \widetilde{\mathcal{A}}^{\mu\nu\lambda} \tilde{v}_\lambda = 0, \quad \widetilde{\mathcal{I}}^{\mu\nu\lambda} \tilde{v}_\lambda = 0, \]  

then the tensors \( \widetilde{\mathcal{A}}^{\mu\nu\lambda} \) and \( \widetilde{\mathcal{I}}^{\mu\nu\lambda} \) are lightlike and the whole of the radiation of AMEF move from the world line in the direction of generatrices of the light cone.

Due to relations (12) one can get expressions for the rate of change of the orbital and spin AMEF (of the field moments of force) in the following form

\[ \frac{d\tilde{L}^{\mu\nu}}{d\tau} = -\frac{e^2}{4\pi c^3} \oint \left\{ (e_\lambda w^\lambda)^2 - w_\lambda w^\lambda \right\} \left[ r^\mu e^\nu - r^\nu e^\mu + \frac{1}{c^2} (r^\mu u^\nu - r^\nu u^\mu) \right] d\Omega_0, \]  

\[ \frac{d\tilde{P}^{\mu\nu}}{d\tau} = \frac{e^2}{4\pi c^3} \oint \left\{ \Omega^\mu w^\nu - w^\mu \Omega^\nu + (e^\mu \Omega^\nu - e^\nu \Omega^\mu) e_\lambda w^\lambda - c \left( e^\mu \Omega^\nu - e^\nu \Omega^\mu \right) \right\} d\Omega_0. \]  

Integrating formulas (14) over the angles by well-known (see, for example, [13]) integrals we can receive expressions for the power of the orbital and spin AMEF radiation in the wave zone

\[ \frac{d\tilde{L}^{\mu\nu}}{d\tau} = \frac{2e^2}{3c^5} w_\rho \Omega^\rho (r^\mu u^\nu - r^\nu u^\mu), \quad \frac{d\tilde{P}^{\mu\nu}}{d\tau} = \frac{2e^2}{3c^5} (2e^\mu w^\rho - e^\rho w^\mu). \]  

It’s interesting that the method of D. D. Ivanenko and A. A. Sokolov yields the same results! The received formulas have clear-cut physical interpretation. If we put into operation the well-known four-dimensional vector of the radiation momentum change [13]

\[ \tilde{F}^\mu = \frac{d\tilde{P}^\mu}{d\tau} = \frac{2e^2}{3c^5} w_\rho \Omega^\rho u^\mu, \]  

then we can get the expression for the orbital momentum of radiation.
\[ dL^{\mu\nu}/d\tau = r^\mu \left( d\tilde{P}^{\nu}/d\tau \right) - r^\nu \left( d\tilde{P}^{\mu}/d\tau \right) = r^\mu \tilde{F}^{\nu} - r^\nu \tilde{F}^{\mu} = -\tilde{G} \]  

(17)

that is precisely the same relation between the torque tensor and its angular momentum tensor derivative as in the relativistic mechanics.

The power of the radiation of the spin momentum according to (15) is proportional to the Thomas’s precession frequency \( (\nu^\alpha w_{\nu} - \nu^\nu w^\alpha)/c^2 \).

4. Properties of the orbital and spin angular momenta of the synchrotron radiation

Here we will use accurate solutions of the relativistic motion equations of the charge in the homogeneous magnetic field \( \mathbf{H} = (0,0,H) \). For \( e = -e_0 < 0 \) (electron) we will have \( r^\nu = (ct,\rho \cos \omega t, \rho \sin \omega t, 0) \) and so on.

In this formula radius of the flat circular orbit of the electron \( \rho \) and the frequency of the periodic motion \( \omega \) are related by these correlations \( \rho = c\beta/\omega \), \( \omega = e_0 H/m_0 c \gamma = c\beta/\rho \).

In the sequel we will investigate only those components of tensors \( dL^{\mu\nu}/d\tau \) and \( d\tilde{P}^{\mu\nu}/d\tau \) which in accordance with (15) save their orientation in the space along the direction of the magnetic field. This is purely spatial components of tensors satisfied following equations in the wave zone of radiation

\[ \tilde{G} = \frac{dL}{dt} = \frac{2e^2a}{3c^2} \beta^3 \gamma^4 = \beta^2 \gamma^2 \frac{d\tilde{P}}{dt} = \beta^2 \gamma^2 \Sigma. \]  

(18)

This implies field orbital moment of force or that means the same the rate of losses of the synchrotron radiation (SR) in the form of the orbital angular momentum for the ultrarelativistic case. It is always much more than losses of the spin momentum.

One can also get formulas (15) in the instantaneous reference frame with \( \mathbf{b} = (0,0,\beta) \), \( \mathbf{a} = a(\sin \alpha, 0, \cos \alpha) \) and \( \mathbf{n} = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta) \).  

(19)

In these signs for the SR \( (\alpha = \pi/2) \) in accordance with (14) we will have

\[ \frac{dL}{dt} = \frac{e^2a}{4\pi c^2} \beta^3 \gamma^2 \int \rho(\theta,\varphi) d\Omega, \quad \frac{d\tilde{P}}{dt} = \frac{e^2a \beta}{4\pi c^2} \int \rho(\theta,\varphi) d\Omega, \]  

(20)

where

\[ \rho(\theta,\varphi) = \frac{1}{(1 - \beta \cos \theta)^3} - \frac{\sin^2 \theta \cos^2 \varphi}{(1 - \beta \cos \theta)^4} \left( 1 - \beta^2 \right). \]  

(20a)

Hence instantaneous radiation indicatrixes of the angular momenta of the SR have strong relativistic effect of the directivity of this radiation and the same for the angular distribution of the SR power.

It is interesting that orbital and spin angular distributions of the AMEF radiation indicatrixes are proportional to correspondingly SR indicatrixes.

It can be shown that further angular integration in the (20) of course take us again to the integral characteristic (15), and instantaneous losses of the orbital and spin angular momenta are in agreement with values averaged over the rotation period of the electron around the sense of magnetic field.

First let consider angular moment of the SR for the one electron and then for the bunch from \( N \) electrons.

Following [8] put into operation «length of free path» of electrons between two adjacent photon emissions

\[ l = cE_{ph}/W_{SR}, \]  

(21)

where \( E_{ph} = h\omega_{\max}, \omega_{\max} = c\gamma^3/2\rho \) is corresponding to the spectral maximum of the SR, and
is the power of the SR of one electron. Then according to (17) for the orbital angular moment of the electron SR one can get an expression for the module of the field angular moment in its standard «mechanical» form

\[ \tilde{\mathbf{L}} = \left( \rho I / c^2 \beta \right) W_{SR} \approx \rho \tilde{P}, \]  

where \( \tilde{P} = E_{ph} / c \) is transmitted photon pulse.

For the typical parameters of the modern electron collider with the orbital radius about 10 \( m \) and with the 10 GeV energy is

\[ l = 5.25 cm, \ E_{ph} = 1.2 \times 10^{-7} \text{ erg}, \ W_{SR} = 6.7 \times 10^{-5} \text{ erg/sec} \]  

and according to (23) for the one electron one can get \( \tilde{L} = 4 \times 10^{-15} \text{ erg} \cdot \text{sec} \) and for the bunch of \( N = 10^{11} \) electrons for noncoherent SR one can find \( \tilde{L} = NL_t = 4 \times 10^4 \text{ erg} \cdot \text{sec} \).

This value by many orders of magnitude greater than angular momentum of the laser’s light pulse with the \( 1 J \) energy for which is (see [4]) \( \tilde{L} = 2.5 \times 10^{-9} \text{ erg} \cdot \text{sec} \).

5. Conclusion

Summing up it can be stated that the work presented here discovers a new trend in the relativistic theory of radiation with regard to the investigations of angular momentum properties of the SR and connected with last one field moment of force. Therefore authors invite experimentalists to create the sensitive detectors for the investigation of this unusual phenomenon of nature. For example, it can be made by new nanostructural technologies. Now it is known that the light-driven torque is greatly enhanced by the coupling of the incident light to plasmonic waves in the nanostructural objects [14].

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