TOPICS IN QUANTUM COMPUTERS

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Abstract. I provide an introduction to quantum computers, describing how they might be realized using language accessible to a solid state physicist. A listing of the minimal requirements for creating a quantum computer is given. I also discuss several recent developments in the area of quantum error correction, a subject of importance not only to quantum computation, but also to some aspects of the foundations of quantum theory.

1. What is a quantum computer?

I don’t think that I will spend many words here saying why there has been a considerable growth of interest in the last couple of years in the subject of quantum computation. There has been a spate of reviews[1, 2, 3], semi-popular articles[4], and press accounts[5] giving, on the whole, a very good overview of the subject. At some level, the recent interest simply arises from the very traditional movement of computation into ever more miniature worlds, and what could be more miniature than the world of the single quantum? At another level, though, interest has arisen because the rules of quantum dynamics changes the rules of computation itself[6], in ways which we are still working to understand. You probably can’t factor large numbers with any computer following, at the level of the logical operations, the laws of classical mechanics (which is to say, every computer ever operated up until now); with a computer, or a computation, obeying the laws of quantum dynamics, you just might be able to factor[7, 3]. This has drawn the attention of both the practical, problem-solving world, as well as that of those interested in further exploring and understanding the foundations of quantum theory itself.
So, since we are all solid state physicists here at Curacao, and all know a thing or two about quantum physics, let me immediately give a fairly sophisticated run-down of the minimal requirements for any quantum system to be a quantum computer. As you will see, the entry fee is pretty steep, which provides at least one good reason why all sorts of people aren’t already putting together their quantum processors. Anyway, here goes:

1.1. FIVE REQUIREMENTS FOR QUANTUM COMPUTING

1) The degrees of freedom required to hold data and perform computation should be available as dimensions of the Hilbert space of a quantum system. This quantum system should be more-or-less isolated from its environment. (More about this shortly.) Also, this Hilbert space should be precisely enumerable, for example, I should be able to say, “the quantum system consists of 49 spin-3/2 states on the molecule,” or something of this sort. It won’t do to be able to make only statistical statements about the number of degrees of freedom, as is often the case in solid state physics (e.g., “the quantum dot contains 100 ± 5 electrons”). No, the Hilbert space must be precisely delineated.

Now, it is furthermore very desirable for the Hilbert space to be decomposable into a direct product form. This rather formidable-sounding requirement is actually quite natural for a multi-particle quantum system. In the example mentioned a moment ago, the Hilbert space decomposes into a product of 49 parts, each of dimension \(2^{(3/2 + 1)} = 5\). Note that this means that the dimension of the Hilbert space of the entire quantum system is \(5^{49}\), a rather large number. When the Hilbert space of the individual particle is two-dimensional (e.g., for a spin-1/2 particle), we term this object a quantum bit or qubit, as its role in quantum computation is analogous to that of an individual bit in an ordinary computer. Now, it is not absolutely necessary that the Hilbert space have this direct-product form; the Hilbert spaces of collections of indistinguishable particles do not have this form because of (anti-)symmetry constraints. But it is essential that the size of the Hilbert space grow exponentially fast with the size of the system. For this, a multi-particle system is necessary; the Hilbert-space dimension of a one-particle system grows only algebraically with system size.

2) Another requirement of quantum computation is that it must be possible to place the quantum system in a fiducial starting quantum state. This state can be very simple, as in “all spins down” for a collection of spins; thus, this requirement will be satisfied if it is possible to cool the quantum system to its ground state. This may be trivial — if the qubits are embodied by a ground and excited state of an atom, room temperature...
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may be quite cool enough; this cooling requirement may also be very high
tech — cooling atoms in a trap to their motional ground state, or cooling
nuclear spins to their ground states, may require nano-Kelvins. Of course,
the experimentalists have not been remiss in achieving these sorts of condi-
tions lately. I do not think that this “initial state preparation” requirement
will be the most difficult one to achieve for quantum computation.

3) Here is a pretty tough requirement. The quantum system to be used
as a quantum computer must be to a high degree isolated from coupling to
its environment. This isolation requirement is linked up with the precision
required in quantum computation: if the state of the computer at some
instant is ideally supposed to be the state $\Psi$, then the actual state after
one clock cycle, $\rho$ (a density matrix), should differ from $\Psi$ by only a small
amount:

$$\langle \Psi | \rho | \Psi \rangle \geq 1 - \epsilon.$$

One way (not the only way) that the state of the system can depart from $\Psi$
is by evolving into a state in which the joint quantum state of the computer
and that of its environment become correlated through interaction, a state
of affairs described as entanglement by Schroedinger in 1935[8, 9]. When
the joint state is entangled, the state of the system alone must be described
as a mixed state or a density matrix; in mesoscopic physics, we would say
that the quantum system has begun to travel down the road to decoherence
or phase-breaking.

By the way, the issue of how big an $\epsilon$ is tolerable in quantum compu-
tation is probably the active question among theorists in this field today,
as it is all tied up in the question of error correction and fault tolerance
in quantum computation. There is good news and bad news on this front:
error correction is possible in quantum computation, so finite $\epsilon$ is perfectly
tolerable. The bad news is, at least at the moment, it is not known whether
a very big $\epsilon$ is tolerable. The protocols which are currently understood[10]
start to get into trouble when $\epsilon = 10^{-6}$, give or take a few orders of magni-
tude. But there may be schemes in which $\epsilon = 0.1$ will be perfectly all right;
we just don’t know, and I certainly hope that we manage to figure it out.

4) The next requirement is at the heart of quantum computation: It
must be possible to subject the quantum system to a controlled sequence
of unitary transformations. All of our quantum algorithms are expressed
in terms of such sequences. It is also required that these unitary transfor-
mations can be made to act upon specified pairs of qubits, or other small
collections of qubits. This fits hand-in-glove with the requirement 1) that
the Hilbert space be made of a direct product of the spaces of individual
small systems or particles. Note that the necessary effect of the two-bit
quantum transformations (which are referred to as quantum logic gates) is
that they produce entanglement between qubits of the quantum computer.
Notice that entanglement between different parts of the quantum computer is good; entanglement between the quantum computer and its environment is bad, since it corresponds to decoherence.

There are various ways in which such controlled unitary transformations may be achieved. In most of the ways which people are thinking about now, it is attained by subjecting the system to a particular time-dependent Hamiltonian over a fixed length of time; the resulting unitary operator is a function of the Hamiltonian according to the usual time-ordered product expression:

$$U = T \exp(i \int H(t) dt)$$

(I won’t get into the time-ordering mumbo-jumbo here.) This $H(t)$ may be imposed via a time-varying magnetic field (as in NMR), a strong (i.e. classical) time-varying optical field (as in laser spectroscopy), or by the physical motion of a massive (again, therefore, classical) object (viz., the tip in an atomic-force microscope). Such $H(t)$’s are in fact ubiquitous in experimental physics, although making these operations selective at the single-quantum level is not.

Again, there is what appears at the moment to be a pretty stringent precision requirement on these unitary operations. In the natural metric in the space of unitary operations, the distance between the specified and the actual unitary transformation should be less than $\epsilon$, and the present constraint on $\epsilon$ is about the same as in the discussion in item 3); only $\epsilon$’s less than about $10^{-5}$ can presently be considered “safe”.

5) Last, but not least, it is necessary that it be possible to subject the quantum system to a “strong” form of measurement. When I say “strong” I simply refer to the kind of quantum measurement that we learned about in our textbooks: the measurement determines which orthogonal eigenstate of some particular Hermitian operator the quantum state belongs to, while at the same time projecting the wavefunction of the system irreversibly into the corresponding eigenfunction. The standard example of such a measurement is the Stern-Gerlach experiment in which the z-component of a spin-1/2 particle is projected into one of its two eigenvalues. While this is straight out of the textbooks, it is unfortunately unlike what a lot of actual quantum measurements in the laboratory actually consist of, being more of a “weak” variety. I will not give a complete discussion of what a weak measurement is, I could point the reader to the textbook of Peres[11] for this discussion. Basically, the idea is that the individual quantum system, say a single spin-1/2 system, might interact very weakly with the measurement apparatus, such that the probability that the apparatus registers “spin up” is only very weakly correlated with the actual wavefunction amplitude for the spin to be up. To be more quantitative, there exist weak measurements
on a state $\Psi = a|\uparrow\rangle + b|\downarrow\rangle$ which register “spin up” with probability

$$p_{up} = \frac{1}{2} + \delta\left(|a|^2 - \frac{1}{2}\right)$$

(3)

for arbitrarily small $\delta$. These small-$\delta$ measurements are “weak” in the sense that after such a measurement the quantum state of the system has been disturbed hardly at all; on the other hand, hardly any information has been gained by the measurement about the state of the spin. In many areas of experimental physics a weak measurement is all that you can do, simply because it is not presently known how to make the coupling to individual quantum systems sufficiently strong; in NMR and in most solid-state spin systems, the experiments are insufficiently sensitive to detect the state of individual spins, although there is a continuing push in that direction. I would say that electrical measurements of mesoscopic quantum structures are also weak, in the sense that the course of each individual electron passing through the device is typically not determined. (This is probably not the case for certain measurements in single electronics.)

In many cases weak measurements are very satisfactory for learning a great deal about the quantum properties of systems, because they can often be done on macroscopically large ensembles, involving either many replicas of the same quantum system (very typical in NMR), or many identical runs of the same quantum measurement (as in normal electron transport). By averaging over such ensembles, a very good knowledge of $a$ in Eq. (3) can be obtained, no matter how small $\delta$ is. However, these weak measurements do not satisfy the requirements in quantum computation, at least so far as we have presently formulated them.

2. Brief survey of experimental systems

I believe that with this long-winded set of five desiderata, a reasonable evaluation of any proposed quantum computer implementation can be made. Let me illustrate this with two brief examples, both of a nano-solid state character, which show that the very act of performing this evaluation points to a lot of physics which we would like to know about these systems, about which we are presently pretty ignorant.

2.1. ATOMIC FORCE MICROSCOPE.

This is a gedanken apparatus, shown in cartoon form in Fig. 1, which I proposed some time ago[12, 2] for doing quantum computation. As will be seen, there are in fact severe problems with using it as a quantum computer; but indeed, that is what we are supposed to use the magic five criteria to reveal!
1) The idea of precisely controlling the computational Hilbert space is tied up in the precise atomic design implied by the cartoon for the tip and the surface of the instrument. Finding a spin within a solid is not a problem; indeed there are spins and other quantum states galore. The idea is to set up the system so that only the ones you want are really “available”. A qubit may be made unavailable if the energy to change its state is much larger than the energy actually available in the experiment; in solid state language, the idea is to “open a gap” for all the excitations that you don’t want to happen. In the cartoon, these means that we envision an insulating material such as intrinsic x-Si at low temperatures, so that there are no low-lying excitations available. Furthermore, the nuclei are all chosen so that no nuclear spin excitations are available (here the “unavailability” is particularly clear, since the energy to create a nuclear excitation would be somewhere in the MeV range). So there are no low-lying states, except for the proton spin of the H atoms terminating the tip and various sites on the surface. Thus, this is envisioned as a proton-NMR quantum computer. It places some requirements on atomic scale atomic engineering (the avoidance of any defects within or on the surface of the crystal, the avoidance of surface states, precise isotopic control) which are at the moment pie-in-the-sky, although conceivable as an extension of the considerable actual progress with atomic placement and manipulation in recent years.

2) The preparation of such a nuclear spin system into a pure state would require some cooling techniques considerably beyond the current state of the art. Since the energy scales for nuclear spins are so small, the temperature required to have a substantial ground-state population is below $10^{-3}\text{K}$ or so, for spins in a 1T field. Of course such temperatures can be attained in many types of experiments, but I know of no effort to achieve them in
an AFM. There are techniques which are specifically adapted to bringing nuclear spins to low temperatures[13], involving coupling by spin resonance techniques to other spins (typically electron spins). These approaches could conceivably be applied in an AFM, although no one has undertaken to do so up until now.

3) There are also many unknowns about the question of how well isolated such proton spins could be from the environment. The small energy scale of the nuclear spin is an advantage, as it reduces the phase space available for the emission of excitations (phonons, say) into the crystal. Other stray spins, from defects or thermally activated electrons, say, would be a concern. I would not want to try to predict the decoherence time attainable in such an instrument from a purely theoretical approach; ultimately, it is, I think, an experimental question. There are localized spins in solids which are known to have lifetimes in excess of 1000 sec[14].

4) We know that, almost by definition, an AFM is capable of placing the tip with respect to the surface with something approximating atomic-scale accuracy. This presumably means that one can turn on and off the spin-spin interaction Hamiltonian between a spin at the tip of the AFM and a selected spin at the surface, at least with O(1) accuracy. We normally do not think that O(1) accuracy is anywhere near good enough, as discussed above we are more comfortable with O(10\(^{-6}\)) accuracy. I think that it is completely unknown whether the AFM’s positioning accuracy could be made that good, or whether protocols for quantum computation can be devised in which a low level of positioning accuracy is acceptable. Not that even the possibly crude atomic selectivity of the AFM is not available at all in many other experimental techniques in solid state physics.

One reason why O(1) accuracy of positioning may be enough is that the principal purpose of this interaction as I have envisioned it is to shift the resonance frequency of the selected pair of spins so that their state can be manipulated independently from the other spins in the system. One can tailor pulse sequences in NMR which will accurately perform some desired unitary transformation on the pair of spins in such a way that the result is insensitive to the exact resonance frequency, so long as it falls in a given range.

Another problem which spans both the issues of isolation and controllability, which I am grateful to Prof. Jon Machta for bringing to my attention, is this: the surface will exert a force on the tip, which will be dependent upon the state of the two spins. In principle this means that the trajectory of the tip \(\vec{r}(t)\) is dependent upon the state of the spins. This can be very bad for quantum computation, since it spoils the desired isolation of the quantum state from the environment; if the quantum state of the spins becomes entangled with the state of motion of the tip, the quantum com-
putation state will be effectively decohered. Fortunately the situation is not as bad as it looks, for several reasons: first, it is reasonable to take the tip not to be in an eigenstate of position $\vec{r}$, but in a coherent state involving some reasonable degree of uncertainty in the position $\Delta \vec{r}$ and momentum $\Delta \vec{p}$ of the tip. Since the tip is rather massive and the nuclear spin forces are rather small, the deflection of the tip due to these forces will in fact quite small, and it would be reasonable to imagine that this deflection is well within the position uncertainty $\Delta \vec{r}$ of the tip. In this case, the entanglement of the motion of the tip with the quantum computation is quite small. In fact this is the same form of argument that is needed to say why the radio-frequency spectroscopy, which involves absorption and emission of phonons, does not represent a loss of coherence to the environment. The point is that the radio-frequency electric and magnetic fields are coherent states involving considerable uncertainty in the photon number, so that one photon more or less does not change the quantum state of the r.f. field appreciably.

There is one other subtlety concerning the spin forces on the tip. As noted, the trajectory is changed in no significant way by this interaction; however, the interaction does apply an impulse to the tip, dependent on the state of the spins $s_1$ and $s_2$,

$$\delta \vec{p}(s_1, s_2) = \int \vec{F}(s_1, s_2, t) dt$$

which results in a subsequent change of the phase evolution after this impulse:

$$\exp(i \delta \vec{p}(s_1, s_2) \cdot \vec{r}(t)/\hbar)$$

($\vec{r}(t)$ is the subsequent trajectory of the tip). Note that so long as knowledge of this phase evolution is retained (requiring, for example, quite accurate knowledge of the whole trajectory $\vec{r}(t)$ of the tip), this effect is not decohering; it merely appends a definite phase to each part of the quantum computer state, which must be accounted for in the bookkeeping of doing the prescribed unitary transformations properly.

5) Finally, the situation for actually performing a strong measurement of the quantum state of individual spins in the AFM is rather hopeful, in the sense that this has already been recognized as a valuable thing to do, and several researchers have been carrying out a whole experimental program to try to do such measurements[15]. They are very hard, indeed, and have not yet been accomplished — they involve, in fact, the self-same minute forces mentioned in Eq. (4), which, I have already noted, are very hard to entangle with an external variable like the tip position. However, if you really want this entanglement to occur, a few different tricks are available to you to amplify the tiny entangling tendency: the tricks involve...
keeping the tip in contact with the spin to be measured for a long time, and arranging that the force oscillate in time (flipping the spin back and forth with appropriately timed tipping pulses) so as to excite a mechanical resonance of the tip. Estimates indicate that under favorable circumstances this strategy can achieve single-spin sensitivity, although we will see if the experiments ever actually manage it.

2.2. JOSEPHSON JUNCTION DEVICE

Let me consider, in even less detail than above, the five criteria for quantum computation as applied to Josephson junction systems. I will not pretend that I seriously know how to perform this evaluation realistically, but I think that other participants at this school might, prompted by my feeble attempt, be able to do a much better job of it.

1) The Hilbert space we would have in mind would describe the quantized states of the superconducting phase. The fact that the phase is a “macroscopic quantized variable” is by no means an unalloyed advantage (see below), but at least it offers the possibility of sculpting a Hilbert space by a suitable design of an electric circuit. What would be the good way of doing this sculpting I do not really know; one possible approach, inspired in my mind by some comments by Prof. Mooij, is to use the quantized states of position of superconducting vortices[16] as the relevant degree of freedom. So, the two basis states of a qubit may be a vortex positioned in one superconducting loop or another neighboring one (with a Josephson junction in between). It would be necessary to have “macroscopic quantum coherence” in that this “macroscopic” vortex would have to be capable of existing in a quantum superposition of positions.

2) Concerning state preparation, it seems possible that, by suitable external application of supercurrents, the potential profile of the superconducting state may be biased such that, for example, the lowest energy state is the vortex in one particular position. Presumably this would result in the preparation of a pure state, although one has to worry about thermal excitations of the vortex into other states within the same ring, say. There are a lot of unknowns on this score.

3) Again, on the issue of the isolation of the system from its environment, we would quickly enter the realm of guesswork. The experiments to pin this down would correspond rather closely to those which have been attempted for years to document the occurrence of MQT and MQC (macroscopic quantum tunneling and coherence) in these systems[17]. Such experiments would have to work before we could say that the coupling to the environment is sufficiently under control to embark on quantum computation experiments.
4) The manipulation of the time evolution of the system, by applying desired time dependent Hamiltonians to the system, is probably the most achievable of the five criteria. The effective Hamiltonian of the quantized superconducting phase (or of the conjugate number operator) contains a variety of parameters which are determined by the macroscopic condition of the circuit, for instance the capacitance or inductance of various circuit elements, or the value of various externally applied supercurrents in the circuit. All of these, in principle, could be employed to implement quantum gates. It would make sense to explore protocols for doing so if the other criteria for quantum computation were closer to being sorted out.

5) Again, the fact that the system is more-or-less macroscopic makes it easy to envision various kinds of measurements, of local magnetic flux, of voltage, etc., being performed reliably on the system. The headache comes from the fact that when the means for performing these measurements are put into place, one must ask whether they form a potentially destructive part of the environment. The mere act of refraining from viewing the display of a voltmeter does not change the fact that the measurement setup is collapsing the wavefunction of the computation in an undesirable way. The point is that there must be some way of coupling and decoupling the measurement with the quantum system at any desired moment, as is routinely done by applying appropriate resonant r.f. fields in NMR. I do now know how this criterion is to be satisfied, although I believe there is some consideration of this issue in the work of Leggett[18].

Any of you who work in superconductivity who have just read this must be convinced that quantum computation is just some wild-eyed notion that bears no relationship to reality at all. But the remarkable (or perhaps depressing) fact of the matter is that in other fields of experimental physics workers have a certain degree of confidence that the five criteria which I have laid out are within the grasp of their experimental technique. The most notable example is the consideration of the linear ion trap which has been put forward in the work of Cirac and Zoller[19]. I can succinctly summarize their proposals according to my five-point plan, thus: 1) Hilbert space: very precisely understood — it is spanned by the energy levels of the isolated ions, combined with the phonon modes of the ions in the trap. 2) Cooling: done — the capability of laser cooling into the ground state was demonstrated a couple of years ago (for single ions, at least; cooling in multi-ion traps still has a ways to go). 3) Isolation: pretty good — the coherence times of the ion levels is unmeasurably long, the phonon lifetimes are adequately long to make a start of quantum computing (although these lifetimes are a few orders of magnitude shorter than are presently understood theoretically). 4) Unitary operations: done, with very high precision,
by laser spectroscopy. (There is a lot of detailed atomic physics that is being argued about now on how precisely these state transformations can actually be effected, but the arguments are at the 1-part-in-10\(^4\) level. Also, the relevant precision spectroscopy has not been demonstrated for the multi-ion experiments.) 5) Measurement: Perfect — The laser-induced fluorescence technique has 100% quantum efficiency, and can be turned on and off at will.

Despair may be the reaction of those of you reading this who had hopes that a solid-state implementation of quantum computing might be possible, or at least competitive with what can be done in atomic physics. But please don’t let the apparent tone of finality in the survey of prospects for the ion trap, nor the tone of uncertainty and pessimism in the survey of the superconducting implementation, deter you. For one thing, the ion trap has more problems than my survey revealed. First, the ion-trap scheme is not very extendible; the most ions which have ever been trapped in such an apparatus is in the neighborhood of 40; moreover, all the various cooling and spectroscopic tricks have been performed presently only on single trapped ions[20]. The potential extendibility of a solid-state system, if can be made to work at all, would seem to be much greater. Second, and most obvious, I may have stupidly missed some brilliant way of implementing quantum computation in superconducting circuits which does not suffer from any of the problems which my thoughts entail. Let not my lack of brilliance stand in the way of yours.

3. Current concepts in error correction

Having provided a grand tour in the last section of the requirements for quantum computation, I propose to get down in the trenches a little bit, and discuss how we are trying to solve some particular problems on the subject of protecting quantum computation from errors[10, 21, 22]. This is primarily a subtopic of criterion 3) of my list above, since it has to do with how well the quantum computer can survive interactions with its environment. It also bears upon criterion 4), in so far as it also provides a prescription for how to tolerate (slightly) inaccurate quantum-gate operations (that is, the unitary transformations involved in quantum computation).

The following discussion will have no pretension to be a complete description of what is going on now on the subject of error correction. It is intended as an introduction of what I consider to be an interesting subset of the current work on the subject.
3.1. DESCRIPTION OF ERRORS

The description of the interaction of a quantum system with its environment has been a subject for theorizing for many decades now, and can be as complex as you please. Let me bring up a number of elementary points about this description, which will be the only points which are needed for quantum error correction.

Suppose that we begin with a single two-level quantum system, and that it starts out at time \( t = 0 \) in a pure initial state \( \Psi \). If this system then begins to interact with its environment, the state of the system plus its environment undergoes some joint evolution. Viewing the evolution just from the point of view of the two-state system, the initial state \( \Psi \) evolves into a mixed state, which may either be thought of as an incoherent statistical mixture of an ensemble of new pure states, or it may simply described by a density matrix \( \rho \). The linear operator which specifies the time evolution of such an open system is termed a superoperator; this jargon, I believe, is meant to distinguish it from an ordinary (unitary) time-evolution operator describing the time evolution of a closed quantum system.

A very useful mathematical description of the superoperators is in terms of the so-called operator sum representation, in which this time-evolution is described as follows:

\[
\rho(t) = \sum_{i=1}^{4} A_i \rho(0) A_i^\dagger, \tag{6}
\]

\[
\sum_{i=1}^{4} A_i^\dagger A_i = 1, \tag{7}
\]

\[
\rho(0) = |\Psi\rangle \langle \Psi|. \tag{8}
\]

The action of every possible environment is completely describable by the four operators \((2\times2\) matrices) \( A_i \); these matrices may take any form consistent with the completeness condition Eq. (7). The \( A \) matrices also come up in the “ensemble” description of the effect of the quantum environment; we would say that the state \( \Psi \) evolves into the ensemble of four states

\[
|\Psi\rangle \rightarrow \{ A_1|\Psi\rangle, A_2|\Psi\rangle, A_3|\Psi\rangle, A_4|\Psi\rangle \} \tag{9}
\]

3.2. ERROR CORRECTION CONDITIONS

Now the ideal of quantum error correction is that it should be possible to subject the corrupted quantum state to some process \( R \) which combines
measurement and unitary transformation that brings any corrupted state back to the state without errors:

$$A_i |\Psi\rangle \xrightarrow{R} |\Psi\rangle.$$  \hfill (10)

Obviously it is impossible to satisfy Eq. (10) for any arbitrary $\Psi$ and error process $A_i$. The remarkable fact discovered recently is that, if the subspace in which the state $\Psi$ lies in is restricted, and the set of allowed error operators $A_i$ is also restricted (in a physically reasonable way, as it turns out), then the desire expressed by Eq. (10) can be satisfied. I will review here the derivation of the conditions to be satisfied in order that this quantum error correction process be possible, which may be found in our paper[22] and in a number of others[23, 25, 26].

The basic idea is just to formalize Eq. (10) a little more carefully. Let me specialize to the case where the error-protected states $\Psi$ are to lie in a two-dimensional subspace of the Hilbert space of $n$ spins — the jargon for this is that we will use $n$ qubits to store one qubit robustly. Introducing an orthogonal basis for this two-dimensional subspace in an obvious way, we will write the general $\Psi$ as

$$|\Psi\rangle = a|v_0\rangle + b|v_1\rangle,$$  \hfill (11)

with arbitrary complex coefficients $a$ and $b$. Now, the error correction process $R$, involving some measurements and unitary operations, can always be thought of as an entirely unitary process $R_U$ involving a larger system, the Hilbert space of $\Psi$ along with what we will term an ancilla Hilbert space. We will imagine that the ancilla, which must be under the control of the experiment in the sense of criterion 1) of the first section, is always preset to a standard state $|0\rangle$. Now, we may restate the requirement, by saying that there should exist a unitary transformation $R_U$ which performs the mapping

$$||A_i|\Psi\rangle|0\rangle|\xrightarrow{R_U} |\Psi\rangle|a_i\rangle.$$  \hfill (12)

(Here the $||\ldots||$ indicates that the state may have to be normalized first.) The operation $R_U$ is to restore the state as indicated for all error processes $A_i$ which are to be corrected. As indicated here, the ancilla will end up in some new state $a_i$ which is dependent on $A_i$; the main content of this equation is that whatever the final ancilla state, it must be in a product state with the system, and the system’s state should be restored to its original form, $\Psi$.

Now, by linearity, Eq. (12) should apply to each basis state separately:

$$N_0^i A_i |v_0\rangle|0\rangle \xrightarrow{R_U} |v_0\rangle|a_i\rangle.$$  \hfill (13)
I have now introduced explicitly the normalization constants $N_{0,1}$. Now, it is relatively easy to show that this normalization factor must be independent of $\Psi$ in order that error correction work: $N_{0} = N_{1} = N_{i}$. I will leave this as an exercise for the reader to work out (or look up), in order to move on to the more physically illuminating part of the proof.

The real point is this: in order for the map required in Eqs. (13) and (14) to be unitary, it must preserve the inner product between any pair of states. Therefore, the derivation just consists of writing down all the distinct before-and-after inner product conditions. First, the inner product of two states from Eq. (13) with different error operators $i$ and $j$ gives:

$$\langle v_{0}|A_{i}^\dagger A_{j}|v_{0}\rangle = \frac{1}{(N_{i})^{2}}\langle a_{i}|a_{j}\rangle. \quad (15)$$

Doing the same for $v_{1}$, Eq. (14), gives

$$\langle v_{1}|A_{i}^\dagger A_{j}|v_{1}\rangle = \frac{1}{(N_{i})^{2}}\langle a_{i}|a_{j}\rangle. \quad (16)$$

The right-hand sides of Eqs. (15) and (16) are equal, giving the first conditions for error correction:

$$\langle v_{0}|A_{i}^\dagger A_{j}|v_{0}\rangle = \langle v_{1}|A_{i}^\dagger A_{j}|v_{1}\rangle. \quad (17)$$

The other error correction condition is given by taking the inner product of two vectors from Eqs. (13) and (14); since $|v_{0}\rangle$ and $|v_{1}\rangle$ are orthogonal:

$$\langle v_{1}|A_{i}^\dagger A_{j}|v_{0}\rangle = 0. \quad (18)$$

### 3.3. USE OF CANONICAL ERRORS FOR GENERAL CASE

So, Eqs. (17) and (18) specify the quantum states $|v_{0}\rangle$ and $|v_{1}\rangle$ which can be error-corrected after being subject to one from among the set of errors $\{A_{i}\}$. This may seem to be not generally very useful because each quantum mechanical system will have some environment, described by some very particular set of $A_{i}$’s. Nevertheless, there are some generic error-protected spaces $|v_{0,1}\rangle$ which will be correctable against a whole class of errors. To explain this I have to introduce the idea of a canonical error set. Consider first the environment of a single qubit. Note that a complete basis for $2 \times 2$ matrices is given by the set of matrices

$$E_{0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad E_{1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_{2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad E_{3} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (19)$$
Suppose that we seek a “generic” correctable subspace of $n$ qubits for which we require that error processes which involve one of the “canonical” errors of Eq. (19) acting on only one out of the $n$ qubits. To take a specific example, if $n = 5$ the error operators for which the error-correction conditions Eq. (17) and (18) are to be satisfied should be these sixteen:

$$A_{canon.} \propto \{ E_0^{(1)} E_0^{(2)} E_0^{(3)} E_0^{(4)} E_0^{(5)}, E_1^{(1)} E_0^{(2)} E_0^{(3)} E_0^{(4)} E_0^{(5)}, E_2^{(1)} E_0^{(2)} E_0^{(3)} E_0^{(4)} E_0^{(5)}, E_3^{(1)} E_0^{(2)} E_0^{(3)} E_0^{(4)} E_0^{(5)},$$

$$E_0^{(1)} E_1^{(2)} E_0^{(3)} E_0^{(4)} E_0^{(5)}, E_1^{(1)} E_1^{(2)} E_0^{(3)} E_0^{(4)} E_0^{(5)}, E_2^{(1)} E_1^{(2)} E_0^{(3)} E_0^{(4)} E_0^{(5)}, E_3^{(1)} E_1^{(2)} E_0^{(3)} E_0^{(4)} E_0^{(5)},$$

$$E_0^{(1)} E_0^{(2)} E_1^{(3)} E_0^{(4)} E_0^{(5)}, E_1^{(1)} E_0^{(2)} E_1^{(3)} E_0^{(4)} E_0^{(5)}, E_2^{(1)} E_0^{(2)} E_1^{(3)} E_0^{(4)} E_0^{(5)}, E_3^{(1)} E_0^{(2)} E_1^{(3)} E_0^{(4)} E_0^{(5)},$$

$$E_0^{(1)} E_0^{(2)} E_0^{(3)} E_1^{(4)} E_0^{(5)}, E_1^{(1)} E_0^{(2)} E_0^{(3)} E_1^{(4)} E_0^{(5)}, E_2^{(1)} E_0^{(2)} E_0^{(3)} E_1^{(4)} E_0^{(5)}, E_3^{(1)} E_0^{(2)} E_0^{(3)} E_1^{(4)} E_0^{(5)} \} \tag{20}$$

Here $E_j^{(i)}$ refers to error $j$ on the $i$th qubit. Of course, to satisfy the completeness condition Eq. (7) these sixteen operators would have to be multiplied by some constants; but these constants have no bearing on the satisfaction of the error correction conditions (17) and (18), so we can safely ignore them.

Now of course the reason I use the example $n = 5$ is that in fact we, and others, have found vectors $v_0$ and $v_1$ that are perfectly correctable when subject to this restricted error set. Below I will present a complete description of this five-bit code after I have introduced a good general strategy for finding such codes. There is already coming to be a vast literature on these and related “codes”, group theoretic strategies for constructing them, related theorems on the capacity of noisy quantum channels, quantum gate implementations of the error correction process, applications of these codes for making quantum computation fault tolerant; all of these results are vastly important, but I will leave most of them for the reader to look up elsewhere. For I want to come back to the issue with which I started, namely, what do the existence of these quantum error-correction codes for these kinds of “canonical” error sets as in Eq. (20) have to say about the case of the “generic” environment with arbitrary error operators $A_i$?

The most general answer to this question is, “nothing.” The most generic environment, which introduces correlated errors among the various qubits, bears no resemblance whatsoever to the canonical error set in Eq. (20). However, if we consider errors which are non-generic in so far as they correspond to an independent environment acting on each qubit, then the answer to the question changes from “nothing” to “quite a bit.” For if the environments of each qubit are independent, then irrespective of the form of those environments, the error operators $A_i$ of the complete system become
a direct product of error operators on individual qubits, as in Eq. (20):

\[ A_{i_1,i_2,i_3,...} = A^{(1)}_{i_1} \otimes A^{(2)}_{i_2} \otimes A^{(3)}_{i_3} \otimes ... \]  

(21)

Now, each qubit error operator \( A^{(n)}_{i_n} \) can be expanded in terms of the matrices for the canonical error operators (since they form a complete set):

\[ A^{(n)}_{i_n} = \sum_{k=0}^{3} \alpha_{i,n,k} E_k, \]  

(22)

so that the full error operator can be expanded as a sum of products of canonical error operators:

\[ A_{i_1,i_2,i_3,...} = \sum_{k_1,k_2,k_3,...} \alpha_{\{kn\}} E^{(1)}_{k_1} \otimes E^{(2)}_{k_2} \otimes E^{(3)}_{k_3} \otimes ... \]  

(23)

Now, finally, it should be clear what good this is: if the error-correction scheme is capable of correcting all the “canonical” errors which occur in the expansion of the right-hand side of Eq. (23), then, because of the linearity of the error correcting process, the actual error operator \( A_{i_1,i_2,i_3,...} \) will be successfully corrected.

This is still not the end of the story, because the terms in Eq. (23) will generally contain all possible products of the \( E \) operators, and we have argued that there cannot exist an error correction procedure that corrects for all such errors. In fact we have to invoke one more physical requirement which we expect most sensible environments to obey: at a function of the strength of interaction between the system and the environment, or as a function of the time of interaction, we expect that for weak coupling, or short time, the dominant error operator should be very close to the identity. That is to say, the system cannot be corrupted very much after a short period of interaction. This principle allows us to argue which terms will predominate in the sum Eq. (23) for a sufficiently weak coupling to the environment. There will be one of the \( A \) operators in Eq. (23) that is close to the identity operator, which is \( 0^{th} \) order in time \( t \). Then there will be a set of \( A \)'s for which the leading terms in the sum of Eq. (23) are “single-error” terms, involving only one non-identity \( E \) operator; for a wide class of noisy environments[27], these will be of order \( O(t^{1/2}) \). The next group of \( A \)'s will be “double-error” terms, which will be \( O(t^{2/2}) \); the “triple-error” terms will be \( O(t^{3/2}) \); and so forth. Thus, the generic error correcting code which corrects some number of “canonical” errors of Eq. (20) will take care of the most important parts of the generic errors described by \( A \) (at least, at early times). The final statement (which I will make without proof) is that if one has an \( e \) error correcting code (correcting canonical errors), then
after correction the density matrix $\rho(t)$ will differ from the ideal state $\Psi(t)$ by terms of order $t^{e+1}:
\langle \Psi(t) | \rho(t) | \Psi(t) \rangle = 1 - ct^{e+1} + ...
(24)
The constant $c$ in front of the error term will be dependent on the details of the noise as expressed in the $A$’s in Eq. (23).

3.4. SIMPLE EXAMPLE OF ERROR CORRECTION

Having laid out the general principles of error correction, I want to briefly review how it actually works for the simplest “canonical” example, the one of five qubits subjected only to the one-bit standard errors of Eq. (20). In our initial work[22] we embarked on a purely numerical search for two orthogonal vectors $|v_0\rangle$ and $|v_1\rangle$ in the 32-dimensional Hilbert space of the five spins which would satisfy all the conditions Eqs. (17) and (18). We indeed succeeded by using this strategy, but in fact there is a much better strategy that was developed very rapidly by a number of other authors, which I will indicate briefly here. The good approach is a group-theoretic one[28, 29]: suppose there is a set of operators $M_i \in M$ each of which leave the code vectors invariant:
$M_i |v_0\rangle = |v_0\rangle, \quad M_i |v_1\rangle = |v_1\rangle.
(25)$
We note that if such a set of operators exists it must form a group, and that it is a reasonable guess that this group is Abelian (i.e., all the $M$’s commuting). The group structure is guaranteed because if $M_1$ and $M_2$ both satisfy Eq. (25), then obviously $M_1M_2$ does as well. The fact that these operators should also commute is not guaranteed, but is suggested by Eq. (25); an elementary fact which we learn in quantum mechanics is that commuting operators have simultaneous eigenvectors, and Eq. (25) asserts that these operators have at least two simultaneous eigenvectors, $|v_0\rangle$ and $|v_1\rangle$. (Commutivity would be assured if we could assert that the operators had all 32 eigenvectors in common, not just 2. This will turn out to be the case[28].)

Now, the next reasonable guess to be made is that if these group elements $M$ exist, they should themselves be expressible as products of the canonical $E$ operators of Eq. (19). After all, the $E$ operators are basically spin-$1/2$ angular momentum operators (i.e., the Pauli matrices), and these operators always either commute or anticommute. So, we have a chance of building a set of commuting $M$s this way.

In fact, the possibility that such operators may also anticommute is the final piece of this mathematical trickery. For suppose that we have found a set of $M$s which commute with each other, but do not commute with all
the canonical error operators $A_{i}^{\text{canon.}}$ of Eq. (20). This means that there will be one operator $A_{i}^{\text{canon.}}$ (or more) which an element $M_{\alpha}$ anticommutes with:

$$M_{\alpha}A_{i}^{\text{canon.}} = -A_{i}^{\text{canon.}}M_{\alpha}. \quad (26)$$

But watch what happens when I take the matrix elements of this equation between code vectors $v_{0,1}$:

$$\langle v_{\alpha}|M_{\alpha}A_{i}^{\text{canon.}}|v_{\beta}\rangle = -\langle v_{\alpha}|A_{i}^{\text{canon.}}M_{\alpha}|v_{\beta}\rangle \quad (27)$$

$$\langle v_{\alpha}|A_{i}^{\text{canon.}}|v_{\beta}\rangle = -\langle v_{\alpha}|A_{i}^{\text{canon.}}|v_{\beta}\rangle = 0. \quad (28)$$

So, we see that the anticommuting condition leads to the satisfaction of the error correction conditions Eq. (17) and (18), where the first condition is satisfied with the extra condition that both matrix elements are equal to zero. The argument just given is obviously not true for the case that $A_{i}^{\text{canon.}}$ is the identity operator (since this will commute with everything); but in this case $A_{i}^{\text{canon.}}$ is in the group $M$, which also leads to Eqs. (17) (18) being satisfied.

I can now finish this off by giving the full statement of the result of Gottesman[28] and of Calderbank et al.[29]: Vectors $|v_{\alpha}\rangle$ can be corrected when subjected to errors $A_{i}$ if they are the eigenvectors of an Abelian group of operators $M$ such that every operator $A_{i}^{\dagger}A_{j}$ either 1) is itself a member of the group $M$, or 2) anticommutes with at least one element of $M$.

This result has provided a very useful means of searching for and discovering a vast variety of error correcting schemes. The five-bit error correcting code can be very succinctly expressed using this language: the code vectors are eigenstates of the sixteen-element Abelian group which is generated by the four operators:

$$E_{1}^{(1)}E_{1}^{(2)}E_{3}^{(3)}E_{3}^{(5)}$$
$$E_{1}^{(2)}E_{1}^{(3)}E_{3}^{(4)}E_{3}^{(1)}$$
$$E_{1}^{(3)}E_{1}^{(4)}E_{3}^{(5)}E_{3}^{(2)}$$
$$E_{1}^{(4)}E_{1}^{(5)}E_{3}^{(1)}E_{3}^{(3)} \quad (29)$$

One choice of the two vectors which are simultaneous eigenvectors of these operators, which may be found by standard projection-operator techniques from group theory, are:

$$|v_{0}\rangle = |00000\rangle \quad (30)$$
$$+ |11000\rangle + |01100\rangle + |00110\rangle + |00011\rangle + |10001\rangle$$
$$- |10100\rangle - |01010\rangle - |00101\rangle - |10010\rangle - |01001\rangle$$
$$- |11110\rangle - |01111\rangle - |10111\rangle - |11011\rangle - |11101\rangle$$
and
\[ |c_1\rangle = |11111\rangle \tag{31} \]
\[ + |00111\rangle + |10011\rangle + |11001\rangle + |11100\rangle + |01110\rangle \]
\[ - |01011\rangle - |10101\rangle - |11010\rangle - |01101\rangle - |10110\rangle \]
\[ - |00011\rangle - |10000\rangle - |01000\rangle - |00100\rangle - |00010\rangle. \]

The entire error-correction process can also be described very succinctly and physically in this language: The prescription is to perform a measurement of the value of each of the four generators of the \( M \) group, Eq. (29). (Although I have not mentioned it previously, they are in fact Hermitian operators.) Being the product of four spin-1/2 operators, the measurement can only have two outcomes, \( \pm 1 \). If the measurement outcomes are all \( +1 \), this indicates that the state is free from error, and nothing need be done. It turns out that each other fifteen patterns of \( \pm 1 \) in this measurement indicates which of the fifteen one-error operators of Eq. (20) that the state has been subjected to. Knowing which of these (unitary) operators the state has been subjected to, simply performing the inverse unitary operation restores the state to its correct value.

Both these multi-spin measurements and the final unitary correction operations are within the capability of a quantum computer as specified in the first part of this paper. Error correction indicates a modified paradigm for quantum computation, however; instead of measurements being performed only at the final, “readout” phase of the computation, error correction indicates that measurements should be performed at fixed intervals throughout the entire course of the computation.

3.5. FINAL REMARKS ON ERROR CORRECTION, ETC.

As a summary remark on all this, I freely admit that the foregoing barely scratches the surface of the current activity in the theory of quantum quantum computation, even of the theory of error correction in quantum computation. Any such survey must be obsolete as soon as it is written, with new families of error correcting codes being discovered, their connection with classical error correcting theory being further uncovered, and proposals being advanced for how to use error correction in a full protocol for performing reliable quantum computation with faulty elements. The claim which is presently being evaluated is that if the error rate (including that induced by interaction with the environment, as well as inaccuracies in the implementation of the quantum gates) is below a certain value, then reliable quantum computation of indefinite scale and duration becomes possible. The bad news in this is that this threshold for reliable quantum computation is presently quite low, in the neighborhood of \( 10^{-5} \) per “clock cycle.”
(intervals between quantum gate operations). So, there’s clearly a lot more territory for the theory to explore at this point, and it goes without saying that the experimental situation is still in its infancy. Perhaps some fresh-minded youngster reading this will have a good idea for how to make good progress in bringing quantum computation closer to reality.

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