Polynomial identities with involution for the algebra of $3 \times 3$ upper triangular matrices

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Abstract

Let $\mathbb{F}$ be a field of characteristic $p$, and let $UT_n(\mathbb{F})$ be the algebra of $n \times n$ upper triangular matrices over $\mathbb{F}$ with an involution of the first kind. In this paper we describe: the set of all $*$-central polynomials for $UT_n(\mathbb{F})$ when $n \geq 3$ and $p \neq 2$; the set of all $*$-polynomial identities for $UT_3(\mathbb{F})$ when $\mathbb{F}$ is infinite and $p > 2$.

1 Introduction

Let $\mathbb{F}$ be a field of characteristic $p \neq 2$. In this paper, every algebra is unitary associative over $\mathbb{F}$ and every involution is of the first kind.

We will talk a little about the involutions of the matrix algebra $M_n(\mathbb{F})$ and its subalgebra $UT_n(\mathbb{F})$. There are two important involutions on $M_n(\mathbb{F})$: the transpose and symplectic. When $\mathbb{F}$ is algebraically closed, these are the only involutions up to isomorphism. With respect to algebra $UT_n(\mathbb{F})$, there exist two classes of inequivalent involutions when $n$ is even and a single class otherwise (see [12, Proposition 2.5]) for all $\mathbb{F}$ (finite or infinite).
Given two disjoint infinite sets \( Y = \{y_1, y_2, \ldots\} \) and \( Z = \{z_1, z_2, \ldots\} \), denote by \( \mathbb{F}(Y \cup Z) \) the free unitary associative algebra, freely generated by \( Y \cup Z \), with the involution \( * \) where

\[
y_i^* = y_i \quad \text{and} \quad z_i^* = -z_i,
\]

for all \( i \geq 1 \). Given an algebra with involution \( (A, \otimes) \), denote by \( \text{Id}(A, \otimes) \) the set of its \(*\)-polynomial identities, that is, the set of all \( f(y_1, \ldots, y_m, z_1, \ldots, z_n) \in \mathbb{F}(Y \cup Z) \) such that

\[
f(a_1, \ldots, a_m, b_1, \ldots, b_n) = 0
\]

for all \( a_1, \ldots, a_m \in A^+ \) and \( b_1, \ldots, b_n \in A^- \). Here, \( A^+ \) (\( A^- \)) is the set of all symmetric (skew-symmetric) elements of \( A \).

When we study \( \text{Id}(M_n(\mathbb{F}), \otimes) \) and \( F \) is infinite, it is sufficient to consider the transpose and symplectic involutions (see \[6\] Theorem 3.6.8). The case \( n = 2 \) was described as follows: Levchenko \[7, 8\] for \( p = 0 \) or \( \mathbb{F} \) finite; Colombo and Koshlukov \[4\] for \( \mathbb{F} \) infinite with \( p > 2 \).

With respect to \( \text{Id}(UT_n(\mathbb{F}), \otimes) \), the case \( n = 2 \) was described as follows: Di Vincenzo, Koshlukov and La Scala \[12\] when \( \mathbb{F} \) is infinite; Urure and Gonçalves \[10\] when \( \mathbb{F} \) is finite. The case \( n = 3 \) also was described in \[12\] when \( p = 0 \).

The main result of this paper is the description of \( \text{Id}(UT_3(\mathbb{F}), \otimes) \) for all involutions of the first kind \( \otimes \) when \( \mathbb{F} \) is infinite and \( p > 2 \) (see Theorem 3.43).

Recently, Aljadeff, Giambruno, Karasik \[1\] and Sviridova \[9\] proved that if \( A \) is an algebra with involution \( \otimes \) and \( p = 0 \), then \( \text{Id}(A, \otimes) \) is finitely generated as a \( T(\otimes) \)-ideal. We find a finite generating set of \( \text{Id}(UT_3(\mathbb{F}), \otimes) \) as a \( T(\otimes) \)-ideal when \( \mathbb{F} \) is infinite and \( p > 2 \). It is the same of the case \( p = 0 \) (see Theorem 3.43 and \[12\] Theorem 6.6).

Given an algebra with involution \( (A, \otimes) \), denote by \( C(A, \otimes) \) the set of its \(*\)-central polynomials, that is, the set of all \( f(y_1, \ldots, y_m, z_1, \ldots, z_n) \in \mathbb{F}(Y \cup Z) \) such that

\[
f(a_1, \ldots, a_m, b_1, \ldots, b_n) \in Z(A)
\]

for all \( a_1, \ldots, a_m \in A^+ \) and \( b_1, \ldots, b_n \in A^- \). Here, \( Z(A) \) is the center of \( A \).

If \( \mathbb{F} \) is infinite, then Brandão and Koshlukov \[3\] described \( C(M_2(\mathbb{F}), \otimes) \). For every \( \mathbb{F} \) (finite and infinite), Urure and Gonçalves \[11\] described \( C(UT_2(\mathbb{F}), \otimes) \). Differently of central polynomials, there exists non trivial \(*\)-central polynomial for \( UT_2(\mathbb{F}) \). But this is not true in general. In this paper, we prove that if \( n \geq 3 \) then

\[
C(UT_n(\mathbb{F}), \otimes) = \text{Id}(UT_n(\mathbb{F}), \otimes) + \mathbb{F}
\]

for all \( \mathbb{F} \) and \( \otimes \).

## 2 Involution

We suggest to the reader to see Section 2, page 546 and Section 5 of \[12\]. We will use several results from there.
Given \( n \geq 1 \), let \( J \in M_n(F) \) and \( D \in M_{2m}(F) \) (if \( n = 2m \)) be the following matrices:

\[
J = \begin{bmatrix}
0 & \cdots & 0 & 1 \\
0 & \cdots & 1 & 0 \\
\vdots & \ddots & \vdots & \vdots \\
1 & \cdots & 0 & 0
\end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix}
I_m & 0 \\
0 & -I_m
\end{bmatrix},
\]

where \( I_m \) is the identity matrix. Define the maps \( \ast : UT_n(F) \to UT_n(F) \) and \( s : UT_n(F) \to UT_n(F) \) (if \( n \) is even) by

\[
A^\ast = JA^tJ \quad \text{and} \quad A^s = DA^\ast D,
\]

where \( A^t \) is the transpose matrix of \( A \). We known that \( \ast \) and \( s \) are involutions on \( UT_n(F) \). Moreover:

a) The involution \( \ast \) is not equivalent to \( s \).

b) Every involution on \( UT_n(F) \) is equivalent either to \( \ast \) or to \( s \).

See [12, Propositions 2.5 and 2.6] for details. In particular, we have the following corollary:

**Corollary 2.1.** If \( \otimes \) is an involution on \( UT_3(F) \) then \( \otimes \) is equivalent to \( \ast \), where

\[
\begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
0 & a_{22} & a_{23} \\
0 & 0 & a_{33}
\end{bmatrix}^\ast = \begin{bmatrix}
a_{33} & a_{23} & a_{13} \\
0 & a_{22} & a_{12} \\
0 & 0 & a_{11}
\end{bmatrix}.
\]

Moreover, \( \text{Id}(UT_3(F), \otimes) = \text{Id}(UT_3(F), \ast) \).

### 3 \( \ast \)-Polynomial Identities for \( UT_3(F) \)

Let \( \ast \) be the involution on \( UT_3(F) \) defined in (1). From now on \( F \) is an infinite field of characteristic \( p > 2 \). We denote

\[
UT_3(F) = UT_3 \quad \text{and} \quad \text{Id}(UT_3(F), \ast) = \text{Id}.
\]

In this section we will describe \( \text{Id} \).

The vector spaces of symmetric and skew-symmetric elements of \( UT_3 \) are respectively

\[
UT_3^+ = \text{span} \{ e_{11} + e_{33}, e_{22}, e_{12} + e_{23}, e_{13} \} \quad \text{and} \quad UT_3^- = \text{span} \{ e_{11} - e_{33}, e_{12} - e_{23} \}.
\]

Thus, we have the following lemma.

**Lemma 3.1.** If \( f(y_1, \ldots, y_n, z_1, \ldots, z_m) \in F(Y \cup Z) \) and

\[
f(a_1, \ldots, a_n, b_1, \ldots, b_m) \in \text{span} \{ e_{11}, \ldots, e_{13} \}
\]

for all \( a_1, \ldots, a_n \in UT_3^+ \), \( b_1, \ldots, b_m \in UT_3^- \), then

\[
(f - f^\ast) \in \text{Id}.
\]
Proof. Since \( f - f^* \) is skew-symmetric we have that \( (f - f^*)(a_1, \ldots, a_n, b_1, \ldots, b_m) \) is skew-symmetric. But \( (f - f^*)(a_1, \ldots, a_n, b_1, \ldots, b_m) = \alpha e_{13} \) is symmetric, where \( \alpha \in \mathbb{F} \). Thus \( \alpha = 0 \) and \( (f - f^*) \in Id \).

If \( f \in \mathbb{F}(Y \cup Z)^+ \) and \( g \in \mathbb{F}(Y \cup Z)^- \), we denote \( |f| = 1 \) and \( |g| = 0 \). Thus, if \( h \in \mathbb{F}(Y \cup Z)^+ \cup \mathbb{F}(Y \cup Z)^- \) then

\[
h^* = -(-1)^{|h|} h.
\]
From now on, we denote by \( x_i \) any element of \( \{ y_i, z_i \} \) and write \( |[x_i, x_j]| = |x_i x_j| \). Here,

\[
[x_i, x_j] = x_i x_j - x_j x_i \quad \text{and} \quad [x_1, \ldots, x_n] = [[[x_1, \ldots, x_{n-1}], x_n]
\]

are the commutators.

**Proposition 3.2.** The following polynomials belong to \( Id \):

1. \( s_3(z_1, z_2, z_3) = z_1 [z_2, z_3] - z_2 [z_1, z_3] + z_3 [z_1, z_2] \)
2. \( (1)^{x_1 x_2}[x_1, x_2][x_3, x_4] - (1)^{x_1 x_4}[x_3, x_4][x_1, x_2] \)
3. \( (1)^{x_1 x_3}[x_1, x_2][x_3, x_4] - (1)^{x_1 x_3}[x_1, x_3][x_2, x_4] + (1)^{x_3 x_4}[x_1, x_4][x_2, x_3] \)
4. \( z_1 [x_3, x_4] z_2 + (1)^{x_3 x_4} z_2 [x_3, x_4] z_1 \)
5. \( [x_1, x_2] [x_3, x_4] \)
6. \( z_1 [x_4, x_5] z_2 [x_3, x_4] \)

Proof. Since \( s_3 \) is the standard polynomial and \( \dim UT^+_3 = 2 \) it follows that \( s_3 \in Id \).

By [2], the polynomial (ii) has the form \( f - f^* \) where \( f = (1)^{x_1 x_2}[x_1, x_2][x_3, x_4] \). Thus, by Lemma 3.1, it is a \(*\)-identity for \( UT^+_3 \).

Defining \( f = z_1 [x_3, x_4] z_2 \), we can use the same argument as in (ii) to prove that (iv) belongs to \( Id \).

Defining \( f = z_1 [x_4, x_5] z_2 x_3 \), we can use [2], Lemma 3.1 and (iv) to prove that (vi) belongs to \( Id \).

The proof that (iii) and (v) are \(*\)-identities for \( UT^+_3 \) consists of a straightforward verification.

**Notation 3.3.** From now on, we denote by \( I \) the \( T(\ast) \)-ideal generated by the polynomials of Proposition 3.2. We will deduce some consequences of these identities.

**Lemma 3.4.** The following polynomials belong to \( I \):

1. \( [x_1, x_2][x_3, x_4][x_5, x_6] \)
2. \( [x_1, x_2][x_3, x_4] x_5 + (1)^{x_3 x_5}[x_1, x_2][x_3, x_4] \)
3. \( [x_1, x_2, x_5][x_3, x_4] - (1)^{x_3 x_5}[x_1, x_2][x_3, x_4, x_5] \)

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Proof. The proof of this lemma is similar to the proof of Lemma 5.2, Lemma 5.3 and Lemma 5.5 in [12]. □

Lemma 3.5. Consider the quotient algebra $F(Y,Z)/I$. If $\sigma \in \text{Sym}(n)$ and $\rho \in \text{Sym}(m)$ then:

a) $[z_a, z_b, z_{\sigma(1)}, \ldots, z_{\sigma(n)}] + I = [z_a, z_b, z_1, \ldots, z_n] + I$,
b) $z_\rho(1) \ldots z_\rho(m) [x_a, x_b, x_{\sigma(1)}, \ldots, x_{\sigma(n)}] [x_c, x_d] + I = z_1 \ldots z_m [x_a, x_b, x_1, \ldots, x_n] [x_c, x_d] + I$,
c) $[x_a, x_b, x_{\sigma(1)}, \ldots, x_{\sigma(n)}, y_1] + I = [x_a, x_b, x_1, \ldots, x_n, y_1] + I$,
d) $x_{\sigma(1)} \ldots x_{\sigma(n)} z_j [x_a, x_b] + I = x_1 \ldots x_n z_j [x_a, x_b] + I$,
e) $[x_a, x_b] z_j x_{\sigma(1)} \ldots x_{\sigma(n)} + I = [x_a, x_b] z_j x_1 \ldots x_n + I$.

Proof. Here, Sym$\,(n)$ is the symmetric group of $\{1, \ldots, n\}$. The proof of this lemma is similar to the proof of Remark 5.4 and Remark 5.19 in [12]. □

The next result is the Lemma 5.6 of [12]. In the page 554 line 6 there is a little error and we correct it bellow.

Lemma 3.6. For all $n \geq 0$, the following polynomial belongs to $I$:

\begin{align*}
(-1)^{|x_4 x_5|} [x_4, x_3, x_{i_1}, \ldots, x_{i_n}] [x_2, x_1] \\
-(-1)^{|x_4 x_2|} [x_4, x_2, x_{i_1}, \ldots, x_{i_n}] [x_3, x_1] \\
+(-1)^{|x_3 x_2|} [x_3, x_2, x_{i_1}, \ldots, x_{i_n}] [x_4, x_1].
\end{align*}

Proof. The proof is by induction on $n$. Note that it suffice to prove that the following polynomial is in $I$:

$g = (-1)^{|x_4 x_3|} [x_4, x_3] x_5 [x_2, x_1] \\
-(-1)^{|x_4 x_2|} [x_4, x_2] x_5 [x_3, x_1] \\
+(-1)^{|x_3 x_2|} [x_3, x_2] x_5 [x_4, x_1].$

If $x_5$ is skew-symmetric then $g \in I$ by Proposition 3.2 - (v).
Suppose that $x_5$ is symmetric and denote $x_5 = y_5$. Write

$\begin{align*}
f(x_1, x_2, x_3, x_4) &= (-1)^{|x_4 x_3|} [x_4, x_3] [x_2, x_1] \\
&\quad -(-1)^{|x_4 x_2|} [x_4, x_2] [x_3, x_1] \\
&\quad +(-1)^{|x_3 x_2|} [x_3, x_2] [x_4, x_1].
\end{align*}$

By the identities (ii) and (iii) of Proposition 3.2 we have that $f \in I$ and therefore $f(y_5 x_1, x_2, x_3, x_4) \in I$. Now we use the equality

$[x_1, y_5 x_1] = y_5 [x_1, x_1] + [x_1, y_5] x_1$

to finish the proof. □
**Lemma 3.7.** The following polynomials belong to $I$:

i) $[z_1, z_2][x_3, x_4] - z_1[x_3, x_4]z_2$, when $|x_3| = |x_4|$.

ii) $[z_1, z_2][z_3, y_4] + z_1[z_2, y_4]z_3 = z_2[z_1, y_4]z_3$.

*Proof.* The proof of this lemma is similar to the proof of Lemma 5.10 in [12].

**Lemma 3.8.** For all $n \geq 3$, the following polynomial belongs to $I$:

$$f_n = \sum_{i=1}^{n} z_i[z_3, z_2, x_4, \ldots, x_n] - \sum_{i=1}^{n} z_i[z_1, x_4, \ldots, x_n] + \sum_{i=1}^{n} z_i[z_2, z_1, x_4, \ldots, x_n].$$

*Proof.* The proof of this lemma is similar to the proof of Lemma 5.11 in [12].

**Lemma 3.9.** For all $m \geq 2$, the following polynomials are elements of $I$:

a) $z_1z_2[y_1, y_2, \ldots, y_m] - z_2[y_2, z_1, y_1, y_3, \ldots, y_m] + z_2[y_1, z_1, y_2, \ldots, y_m]$.

b) $z_1z_2[y_1, z_3, y_2, \ldots, y_m] + z_2[y_1, z_3, z_1, y_2, \ldots, y_m]$.

c) $z_1z_2[z_3, z_4, y_1, \ldots, y_m] + z_2[z_3, z_4, z_1, y_1, \ldots, y_m]$.

*Proof.* The proof of this lemma is similar to the proof of Lemma 5.13, Lemma 5.14 and Lemma 5.15 in [12].

Let $B$ be the subspace of $\mathbb{F}(Y, Z)$ formed by all $Y$-proper polynomials. Since $F$ is an infinite field, it is known that $Id$ and $I$ are generated, as a $T(*)$-ideals, by its multihomogeneous elements in $B$. See [5] Lemma 2.1 and [12] Page 546 for details.

If $M = (m_1, \ldots, m_k)$ and $N = (n_1, \ldots, n_s)$, denote by $B_{MN}$ the following multihomogeneous subspace of $B$:

$$B_{MN} = \{ f(y_1, \ldots, y_k, z_1, \ldots, z_s) \in B : \deg_y f = m_i, \deg_z f = n_j, 1 \leq i \leq k, 1 \leq j \leq s \}.$$ 

We shall prove that $I = Id$, that is, $I \cap B_{MN} = Id \cap B_{MN}$ for all $M, N$.

**Observation 3.10.** Let $\sigma \in \text{Sym}(k)$ and $\rho \in \text{Sym}(s)$. Since the $T(*)$-ideals generated by

$$f(y_1, \ldots, y_k, z_1, \ldots, z_s) \quad \text{and} \quad f(y_{\sigma(1)}, \ldots, y_{\sigma(k)}, z_{\rho(1)}, \ldots, z_{\rho(s)})$$

are equal it is sufficient to prove $I \cap B_{MN} = Id \cap B_{MN}$ for

$$1 \leq m_1 \leq \ldots \leq m_k \quad \text{and} \quad 1 \leq n_1 \leq \ldots \leq n_s.$$  

From now on we assume (3).

Denote $B_{MN}(I) = B_{MN}/I \cap B_{MN}$ and $B_{MN}(Id) = B_{MN}/Id \cap B_{MN}$.

When $m_1 = \cdots = m_k = n_1 = \cdots = n_s = 1$, we write $B_{MN} = \Gamma_{ks}$. 

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Suppose \( m_1, \ldots, m_k \geq 1 \) and \( n_1, \ldots, n_s \geq 1 \). Write \( m_1 + \cdots + m_k = m \) and \( n_1 + \cdots + n_s = n \). Let \( \varphi_{MN} : \Gamma_{mn}(I) \to B_{MN}(I) \) be the linear map defined by

\[
\varphi_{MN}(f(y_1, \ldots, y_m, z_1, \ldots, z_n) + I \cap \Gamma_{mn}) = f(y_1, \ldots, y_{m_k}, y_k, \ldots, y_{m_k}, z_1, \ldots, z_{n_1}, \ldots, z_{n_s}) + I \cap B_{MN}.
\]

Since \( \varphi_{MN} \) is onto, we have the following proposition:

**Proposition 3.11.** Consider the above notations. If the vector space \( \Gamma_{mn}(I) \) is spanned by a subset \( S \), then \( B_{MN}(I) \) is spanned by \( \varphi_{MN}(S) \).

Fix the following order on \( Y \cup Z \):

\[
z_1 < z_2 < \ldots < y_1 < y_2 < \ldots
\]

**Definition 3.12.** Let \( S_1 \) be the set of all polynomials

\[
f = z_{i_1} \cdots z_{i_t} [x_{j_1}, \ldots, x_{j_l}] [x_{k_1}, \ldots, x_{k_q}]
\]

where \( t, l, q \geq 0, l \neq 1, q \neq 1 \), \( z_i \leq \ldots \leq z_i, x_{j_1} > x_{j_2} \leq \ldots \leq x_{j_l} \) and \( x_{k_1} > x_{k_2} \leq \ldots \leq x_{k_q} \). We say that \( f \) is an \( S_1 \)-standard polynomial.

**Definition 3.13.** Let \( S_2 \subset S_1 \) be the set of all polynomials

\[
f = z_{i_1} \cdots z_{i_t} [x_{j_1}, \ldots, x_{j_l}] [x_{k_1}, \ldots, x_{k_q}] \in S_1
\]

such that: if \( l \geq 2 \) then \( q = 0 \) or \( q = 2 \), and when \( q = 2 \) we have that \( x_{j_1} \geq x_{k_1} \) and \( x_{j_2} \geq x_{k_2} \). If \( f \in S_2 \) we say that \( f \) is an \( S_2 \)-standard polynomial.

**Proposition 3.14.** The vector space \( B_{MN}(I) \) is spanned by the set of all elements \( f + I \cap B_{MN} \) where \( f \in B_{MN} \) is \( S_2 \)-standard.

**Proof.** This proposition is true for \( \Gamma_{mn}(I) \). In fact, we can use the same proof as in [12] Proposition 5.8]. Now, if \( x_i < x_j \) then \( \varphi_{MN}(x_i) \leq \varphi_{MN}(x_j) \). Thus, by Proposition 6.11 the general case is proved.

**Observation 3.15.** Let \( F[y_{ij}, z_{ij}] = F[y_{ij}^k, z_{ij}^k : i, j, k \geq 1] \) be the free commutative algebra freely generated by the set of variables \( L = \{ y_{ij}^k, z_{ij}^k : i, j, k \geq 1 \} \).

Given an order \( > \) on \( L \), consider the order on the monomials of \( F[y_{ij}, z_{ij}] \) induced by \( > \) as follows: if \( w_1 \geq w_2 \geq \ldots \geq w_n \), \( w'_1 \geq w'_2 \geq \ldots \geq w'_m \) are in \( L \) then

\[
w_1 w_2 \ldots w_n > w'_1 w'_2 \ldots w'_m
\]

if and only if

- either \( w_1 = w'_1, \ldots, w_l = w'_l, w_{l+1} > w'_{l+1} \) for some \( l \),
- or \( w_1 = w'_1, \ldots, w_m = w'_m \) and \( n > m \).
Given \( f \in F[y_{ij}^k, z_{ij}^k] \), we denote by \( m(f) \) its leading monomial.

In \( UT_3(F[y_{ij}^k, z_{ij}^k]) \) consider the \( q \)generic matrices

\[
Z_k = \begin{bmatrix}
y_{11}^k & y_{12}^k & 0 \\
0 & 0 & -y_{12}^k \\
0 & 0 & -y_{11}^k
\end{bmatrix} \quad \text{and} \quad Y_k = \begin{bmatrix}
y_{11}^k & y_{12}^k & y_{13}^k \\
0 & 0 & y_{12}^k \\
0 & 0 & y_{11}^k
\end{bmatrix}.
\]

Note, the \((2, 2)\)-entry of \( Y_k \) is 0.

By using an analogous argument to the \([12, \text{Lemma 6.1}]\) we obtain the next lemma.

**Lemma 3.16.** If \( f(y_1, \ldots, y_k, z_1, \ldots, z_s) \in Id \), then \( f(Y_1, \ldots, Y_k, Z_1, \ldots, Z_s) = 0 \).

### 3.1 Subspaces \( B_{MN} \) where \( N = 0 \)

If \( M = (m_1, \ldots, m_k) \) and \( N = (0) \), then denote \( B_{MN} = B_{M0} \), that is

\[
B_{M0} = \{ f(y_1, \ldots, y_k) \in B : \deg_y f = m_i, \ 1 \leq i \leq k \}.
\]

A polynomial in \( S_2 \cap B_{M0} \) has the form

\[
f^{(i_1)} = [y_{i_1}, y_{i_2}, \ldots, y_{i_m}] \quad \text{or} \quad f^{(i_1, j_1)} = [y_{i_1}, y_{i_2}, \ldots, y_{i_{m-2}}][y_{i_1}, y_{j_2}]
\]

where \( i_1 > i_2 \leq i_3 \leq \ldots \leq i_m \) for \( f^{(i_1)} \); \( i_1 > i_2 \leq i_3 \leq \ldots \leq i_{m-2} \leq i_1 \geq j_1 > j_2 \) and \( i_2 \geq j_2 \) for \( f^{(i_1, j_1)} \).

**Proposition 3.17.** The set \( \{ f + Id \cap B_{M0} : f \in S_2 \cap B_{M0} \} \) is a basis for the vector space \( B_{M0}(Id) \). In particular, \( Id \cap B_{M0} = I \cap B_{M0} \).

**Proof.** Since \( I \subseteq Id \) we have, by Proposition 3.14

\[
B_{M0}(Id) = \text{span}\{ f + Id \cap B_{M0} : f \in S_2 \cap B_{M0} \}.
\]

We will use the notations of Observation 3.15. Consider some order \( > \) on \( L \) such that

\[
y_1^{i+1} > y_1^i > y_1^{i+1} > y_1^i
\]

for all \( i \geq 1 \). By using the \( q \)generic matrices we have the following equalities:

(a) \([Y_1, Y_2, \ldots, Y_{2l}] = (y_{11}^{1} y_{12}^{2} - y_{11}^{2} y_{12}^{1}) \left( \prod_{s=3}^{2l} y_{11}^{s} \right) (e_{12} - e_{23}),\]

(b) \([Y_1, Y_2, \ldots, Y_{2l+1}] = -(y_{11}^{1} y_{12}^{2} - y_{11}^{2} y_{12}^{1}) \left( \prod_{s=3}^{2l} y_{11}^{s} \right) (y_{11}^{2l+1} (e_{12} + e_{23}) - 2y_{12}^{2l+1} e_{13}),\]

(c) \([Y_1, Y_2, \ldots, Y_{2l-2}] [Y_{2l-1}, Y_{2l}] = (y_{11}^{1} y_{12}^{2} - y_{11}^{2} y_{12}^{1}) \left( \prod_{s=3}^{2l-2} y_{11}^{s} \right) (y_{11}^{2l-1} y_{12}^{2l+1} - y_{12}^{2l-1} y_{11}^{2l-1}) e_{13} ,\]

8
(d) \([Y_1, Y_2, \ldots, Y_{2l-1}, Y_{2l}, Y_{2l+1}] = -(y_{11} y_{12}, y_{21} y_{22}) \left(\prod_{s=3}^{2l-1} y_{s1}\right) (y_{11} y_{12}, y_{21} y_{22}) e_{13}\). 

Let \(f^{(i)}, f^{(i, j)}\) as in (4). Write

\(f^{(i)}(Y_1, \ldots, Y_k) = \sum f^{(i)} e_{ij}\) and \(f^{(i, j)}(Y_1, \ldots, Y_k) = \sum f^{(i, j)} e_{ij}\).

We shall prove that \(\{f + \text{Id} \cap B_{M0} : f \in S_2 \cap B_{M0}\}\) is a linearly independent set of \(B_{M0}(\text{Id})\). Suppose

\[\sum \alpha_{i1} f^{(i)} + \sum \alpha_{i, j} f^{(i, j)} \in \text{Id},\]

where \(\alpha_{i1}, \alpha_{i, j} \in \mathbb{F}\). By Lemma 3.16 we have

\[\sum \alpha_{i1} f^{(i)}(Y_1, \ldots, Y_k) + \sum \alpha_{i, j} f^{(i, j)}(Y_1, \ldots, Y_k) = 0.\]

The leading monomials of \(f^{(i)}\) and \(f^{(i, j)}\) are

\[m(f_{12}^{(i)}) = y_{12}^{j} \left(\prod_{s=2}^{m} y_{s1}\right)\] and \[m(f_{13}^{(i, j)}) = y_{12}^{j} y_{11}^{j} \left(\prod_{s=2}^{m} y_{s1}\right) y_{11}^{j}.\]

Moreover, the coefficients of \(m(f_{12}^{(i)})\) and \(m(f_{13}^{(i, j)})\) in \(f_{12}^{(i)}\) and \(f_{13}^{(i, j)}\) are \(\alpha_{i1}\) and \(\alpha_{i, j}\), respectively. 

Since \(f_{12}^{(i, j)} = 0\), we have \(\sum \alpha_{i1} f_{12}^{(i)} = 0\). Thus, the coefficient of the maximal monomial of the set \(\{m(f_{12}^{(i)}) : i_1 \geq 1\}\) is 0. By induction, every coefficient \(\alpha_{i1}\) is 0. This implies \(\sum \alpha_{i1} f_{13}^{(i, j)} = 0\) and we can use similar argument to prove that every \(\alpha_{i, j}\) is 0.

\[\square\]

### 3.2 Subspaces \(B_{MN}\) where \(M = (0)\)

If \(M = (0)\) and \(N = (n_1, \ldots, n_s)\), then denote \(B_{MN} = B_{0N}\), that is

\[B_{0N} = \{f(z_1, \ldots, z_s) \in B : \deg_{z_i} f = n_i, 1 \leq i \leq s\}.\]

**Definition 3.18.** Let \(S_3 \subset S_2\) be the set of all polynomials \(f, f^{(j)}; f^{(i, j)} \in B_{0N}\) such that:

- \(f = z_1^{n_1} \ldots z_s^{n_s} \in S_2,\)
- \(f^{(j)} = [z_j, z_{j+1}, \ldots, z_s] \in S_2,\)
- \(f^{(i, j)} = z_i [z_j, \ldots, z_{j-1}] \in S_2\) where \(i \leq j_1.\)

If \(f \in S_3\) we say that \(f\) is an \(S_3\)-standard polynomial.

**Proposition 3.19.** The vector space \(B_{0N}(I)\) is spanned by the set of all elements \(f + I \cap B_{0N}\) where \(f \in B_{0N}\) is \(S_3\)-standard.
Proof. This proposition is true for $\Gamma_{0n}(I)$. In fact, we can use the same proof as in [12 Proposition 5.12]. Now, if $z_i < z_j$ then $\varphi_{0N}(z_i) \leq \varphi_{0N}(z_j)$. Thus, by Proposition 3.11 the general case is proved. □

**Proposition 3.20.** The set $\{ f + Id \cap B_{0N} : f \in S_3 \}$ is a basis for the vector space $B_{0N}(Id)$. In particular, $Id \cap B_{0N} = I \cap B_{0N}$.

Proof. Since $I \subseteq Id$ we have, by Proposition 3.19

\[ B_{0N}(Id) = \text{span}\{ f + Id \cap B_{0N} : f \in S_3 \}. \]

We will use the notations of Observation 3.15. Consider some order $> \in D$ such that

\[ z_i^{1+} > z_i^{1-} > z_i^{1+} > z_i^{1-} \]

for all $i \geq 1$. By using the generic matrices we have the following equalities:

(a) $[Z_1, Z_2, \ldots, Z_n] = (-1)^n \left( z_{11}^1 z_{12}^2 - z_{12}^1 z_{11}^2, \prod_{i=3}^{n} z_{ii}^i \right)(\epsilon_{12} - \epsilon_{23}),$

(b) $Z_1[Z_2, Z_3, \ldots, Z_n] = (-1)^{n-1} \left( z_{11}^2 z_{12}^3 - z_{12}^2 z_{11}^3, \prod_{i=4}^{n} z_{ii}^i \right)(\epsilon_{11}^2 - \epsilon_{12}^1, \epsilon_{13}^1, \epsilon_{23}^1).$

Let $f, f^{(j_1)}$ and $f^{(i_1, j_1)}$ as in (5). Write $f(Z_1, \ldots, Z_s) = \sum f_{ij} e_{ij}$, $f^{(j_1)}(Z_1, \ldots, Z_s) = \sum f_{ij}^{(j_1)} e_{ij}$ and $f^{(i_1, j_1)}(Z_1, \ldots, Z_s) = \sum f_{ij}^{(i_1, j_1)} e_{ij}$.

Suppose

\[ \alpha f + \sum \alpha_{j_1} f^{(j_1)} + \sum \alpha_{i_1, j_1} f^{(i_1, j_1)} \in Id, \]

where $\alpha, \alpha_{j_1}, \alpha_{i_1, j_1} \in \mathbb{F}$. By Lemma 3.16 we have

\[ \alpha f(Z_1, \ldots, Z_s) + \sum \alpha_{j_1} f^{(j_1)}(Z_1, \ldots, Z_s) + \sum \alpha_{i_1, j_1} f^{(i_1, j_1)}(Z_1, \ldots, Z_s) = 0. \]

Since $f^{(j_1)}_{11} = f^{(i_1, j_1)}_{11} = 0$, we obtain $\alpha f_{11} = 0$ and so $\alpha = 0$. Note that

\[ m(f^{(j_1)}_{23}) = z_{12}^1 \prod_{i=2}^{n} z_{ii}^i \] and \[ m(f^{(i_1, j_1)}_{13}) = z_{12}^1 z_{1i}^1 \prod_{i=2}^{n} z_{ii}^i. \]

Moreover, the coefficients of $m(f^{(j_1)}_{23})$ and $m(f^{(i_1, j_1)}_{13})$ in $f^{(j_1)}_{23}$ and $f^{(i_1, j_1)}_{13}$ are $\pm \alpha_{j_1}$ and $\pm \alpha_{i_1, j_1}$, respectively.

Since $f^{(i_1, j_1)}_{23} = 0$, we have $\sum \alpha_{j_1} f^{(j_1)}_{23} = 0$. Thus, the coefficient of the maximal monomial of the set $\{ m(f_{23}^{(j_1)} : j_1 \geq 1 \}$ is 0. By induction, every coefficient $\alpha_{j_i}$ is 0. We can use similar argument to prove that every $\alpha_{i_1, j_1}$ is 0. □
3.3 Subspaces $B_{MN}$ where $M \neq (0), (1)$ and $N = (1)$

If $M = (m_1, \ldots, m_k) \neq (0), (1)$ and $N = (1)$, then denote $B_{MN} = B_{M1}$, that is, \[ B_{M1} = \{ f(y_1, \ldots, y_k, z_1) \in B : \deg y_i f = m_i \text{ and } \deg z_1 f = 1, 1 \leq i \leq k \} \]
and $m = m_1 + \cdots + m_k > 1$.

Definition 3.21. Let $S_3 \subset S_2$ be the set of all polynomials $f^{(i_1)}, g^{(i_1)}, f^{(i_1, i_j)} \in B_{M1}$ such that:

\begin{itemize}
  \item $f^{(i_1)} = [y_{i_1}, z_1, y_{i_2}, \ldots, y_{i_m}] \in S_2$;
  \item $g^{(i_1)} = z_1[y_{i_1}, y_{i_2}, \ldots, y_{i_m}] \in S_2$,
  \item $f^{(i_1, i_j)} = [y_{i_1}, y_{i_2}, \ldots, y_{i_m-1}][y_{i_j}, z_1] \in S_2$.
\end{itemize}

If $f \in S_3$, we say that $f$ is an $S_3$-standard polynomial.

Proposition 3.22. The vector space $B_{M1}(I)$ is spanned by the set of all elements $f + I \cap B_{M1}$ where $f \in B_{M1}$ is $S_3$-standard.

Proof. This proposition is true for $\Gamma_m(I)$. In fact, we can use the same proof as in [12, Proposition 5.17]. Now, if $y_i < y_j$ then $\varphi_{M1}(y_i) \leq \varphi_{M1}(y_j)$. Thus, by Proposition 3.11, the general case is proved.

Proposition 3.23. The set $\{ f + Id \cap B_{M1} : f \in S_3 \}$ is a basis for the vector space $B_{M1}(Id)$. In particular, $Id \cap B_{M1} = I \cap B_{M1}$.

Proof. Since $I \subset Id$, we have, by Proposition 3.22
\[ B_{M1}(Id) = \text{span}\{ f + Id \cap B_{M1} : f \in S_3 \}. \]

We will use the notations of Observation 3.14. Consider some order $>$ on $L$ such that \[ y_{i_1}^{z_{i_2}^2} > y_{i_2}^{z_{i_1}^2} > y_{i_1}^{z_{i_2}^2} > y_{i_2}^{z_{i_1}^2} > z_{i_2}^{z_{i_1}^2} > z_{i_1}^{z_{i_2}^2} \]
for all $i \geq 1$. By using the generic matrices we have the following equalities:

\begin{align*}
[Y_1, Z_{1}, Y_{2}, \ldots, Y_{2l}] &= - (y_{i_1}^{z_{i_2}^2} - y_{i_2}^{z_{i_1}^2}) \left( \prod_{s=2}^{2l} y_{i_1}^{s} \right) (e_{12} - e_{23}), \\
[Y_1, Z_{1}, Y_{2}, \ldots, Y_{2l+1}] &= (y_{i_1}^{z_{i_2}^2} - y_{i_2}^{z_{i_1}^2}) \left( \prod_{s=2}^{2l+1} y_{i_1}^{s} \right) (y_{i_1}^{2l+1}(e_{12} + e_{23}) - 2y_{i_2}^{2l+1}e_{13}), \\
Z_{1}[Y_1, Y_{2}, \ldots, Y_{2l}] &= (y_{i_1}^{z_{i_2}^2} - y_{i_2}^{z_{i_1}^2}) \left( \prod_{s=3}^{2l} y_{i_1}^{s} \right) (z_{i_1}^{z_{i_2}^2}e_{12} - z_{i_2}^{z_{i_1}^2}e_{13}), \\
Z_{1}[Y_1, Y_{2}, \ldots, Y_{2l+1}] &= - (y_{i_1}^{z_{i_2}^2} - y_{i_2}^{z_{i_1}^2}) \left( \prod_{s=3}^{2l+1} y_{i_1}^{s} \right) (z_{i_1}^{z_{i_2}^2}y_{i_1}^{2l+1}e_{12} + (-2z_{i_1}^{z_{i_2}^2}y_{i_1}^{2l+1} + z_{i_2}^{z_{i_1}^2}y_{i_1}^{2l+1})e_{13}).
\end{align*}
Let \( B \) is 0. By induction, every coefficient \( \alpha \) is 0. Thus, the coefficient of the maximal monomial of the set \( f \) where \( g \) is 3.4. 

\[ Y_1, \ldots, Y_{2l-1}, \alpha(G) = \frac{1}{2} \left( \prod_{i=3}^{2l-1} y_i \right) \left( y_1^2 - y_1 y_2 \right) \left( y_1^2 - y_1 y_{2l-1} \right) \epsilon_{l1}. \]

\[ Y_{2l}, \ldots, Y_{2l+1}, \alpha(G) = \frac{1}{2} \left( \prod_{i=3}^{2l} y_i \right) \left( y_1^2 - y_1 y_{2l+1} \right) \epsilon_{l1}. \]

Let \( f^{(i_1)}, g^{(i_1)}, f^{(i_1,i_2)} \) as in (b). Write \( f^{(i_1)}(Y_1, \ldots, Y_k, Z_1) = \sum f^{(i_1)} e_{ij}, g^{(i_1)}(Y_1, \ldots, Y_k, Z_1) = \sum g^{(i_1)} e_{ij} \) and \( f^{(i_1,i_2)}(Y_1, \ldots, Y_k, Z_1) = \sum f^{(i_1,i_2)} e_{ij}. \)

Suppose \( \sum \alpha_i f^{(i_1)} + \sum \beta_i g^{(i_1)} + \sum \alpha_{i_1,i_2} f^{(i_1,i_2)} \in I_1, \) where \( \alpha_i, \beta_i, \alpha_{i_1,i_2} \in \mathbb{F}. \)

Note that

\[ m(f^{(i_1)}) = y_1^{i_1} \left( \prod_{i=2}^{m} y_i \right) z_1^{i_1}, \]

and its coefficient in \( f^{(i_1)} \) is \(-\alpha_i\). Since \( g^{(i_1)} = f^{(i_1,i_2)} = 0 \), we have \( \sum \alpha_i f^{(i_1)} = 0. \) Thus, the coefficient of the maximal monomial of the set \( \{ m(f^{(i_1)}): i_1 \geq 1 \} \) is 0. By induction, every coefficient \( \alpha_i \) is 0. Now,

\[ m(g^{(i_1)}) = y_1^{i_1} \left( \prod_{i=2}^{m} y_i \right) z_1^{i_1}, \]

and its coefficient in \( g^{(i_1)} \) is \( \pm \alpha_i \). Since \( f^{(i_1,i_2)} = 0 \), we have \( \sum \beta_i g^{(i_1)} = 0. \) Thus, the coefficient of the maximal monomial of the set \( \{ m(g^{(i_1)}): i_1 > 1 \} \) is 0. By induction, every coefficient \( \beta_i \) is 0. This implies \( \sum \alpha_{i_1,i_2} f^{(i_1,i_2)} = 0. \) Since

\[ m(f^{(i_1,i_2)}) = y_1^{i_1} y_2^{i_2} \left( \prod_{i=3}^{m} y_i \right) z_1^{i_1}, \]

and its coefficient in \( f^{(i_1,i_2)} \) is \( \pm \alpha_{i_1,i_2} \), we can use similar argument to prove that every \( \alpha_{i_1,i_2} \) is 0. 

3.4 Subspaces \( B_{MN} \) where \( M = (1) \) and \( N \neq (0) \)

If \( M = (1) \) and \( N = (n_1, \ldots, n_s) \neq (0) \), then denote \( B_{MN} = B_{1N} \), that is

\[ B_{1N} = \{ f(y_1, z_1, \ldots, z_s) \in B : \deg y_i f = 1 \text{ and } \deg z_i f = n_i, 1 \leq i \leq s \}. \]

Definition 3.24. Let \( S_3 \) be the set of all polynomials \( f^{(j)}, g^{(j)}, h^{(j)}, f^{(i,j)} \in B_{1N} \) such that

(a) \( f^{(j)} = z_1^{n_1} \ldots z_j^{n_j-1} \ldots z_s^{n_s} [y_1, z_j] \), where \( 1 \leq j \leq s. \)

(b) \( g^{(j)} = [y_1, z_j] z_1^{n_1} \ldots z_j^{n_j-1} \ldots z_s^{n_s} \), where \( 1 \leq j \leq s. \)

(c) \( h^{(j)} = z_1^{n_1} \ldots z_j^{n_j-1} \ldots z_s^{n_s-1} [y_1, z_j] z_s \), where \( 1 \leq j \leq s. \)
Proposition 3.11. The general case is proved.

If $f$ as in [12, Proposition 5.20]. Now, if $z$ this proposition is true for $\Gamma$

Proof. The vector space $B_{1N}(I)$ is spanned by the set of all elements $f + I \cap B_{1N}$ where $f$ is $S_3$-standard.

**Proposition 3.25.** The vector space $B_{1N}(I)$ is spanned by the set of all elements $f + I \cap B_{1N}$ where $f$ is $S_3$-standard.

Proof. This proposition is true for $\Gamma_{1n}(I)$. In fact, we can use the same proof as in [12 Proposition 5.20]. Now, if $z_i < z_j$ then $\varphi_{1N}(z_i) \leq \varphi_{1N}(z_j)$. Thus, by Proposition 3.11 the general case is proved.

3.4.1 Subspace $B_{1N}$ where $n_s > 1$

We start this subsection with the next proposition. By using similar arguments as the ones used in [2] Theorem 6 in Chapter 4 we obtain:

**Proposition 3.26.** Let $F$ be an infinite field of $\text{char}(F) = p > 2$. If $H$ is a $T(\ast)$-ideal then $H$ is generated, as a $T(\ast)$-ideal, by its multihomogeneous elements $f(y_1, \ldots, y_k, z_1, \ldots, z_s) \in H$ with multidegree $(p^{a_1}, \ldots, p^{a_k}, p^{b_1}, \ldots, p^{b_v})$ where $a_1, \ldots, a_k, b_1, \ldots, b_s \geq 0$.

We want to show that $I = Id$ by proving $Id \cap B_{MN} = I \cap B_{MN}$. By the last proposition, in this subsection is sufficient to consider the case $1 < n_s = p^{b_v}$. Since $\text{char}(F) = p \geq 3$ we have $n_s \geq 3$. Thus, from now on, we assume $n_s \geq 3$ in $B_{1N}$.

**Definition 3.27.** Let $S_4$ be the set of all polynomials $f^{(j)}, g^{(j)}, h^{(s)}, p^{(i,j)} \in B_{1N}$ such that:

- $f^{(j)}, g^{(j)}, h^{(s)} \in S_3$ as in Definition 3.24
- $p^{(i,j)} = z_1^{n_1} \cdots z_i^{n_{i-1}} \cdots z_j^{n_{j-1}} \cdots z_s^{n_{s-1}}[z_1, \ldots, z_s][y_1, z_j], 1 \leq i \leq j \leq s$ and $i < s$.

If $f \in S_4$, we say that $f$ is an $S_4$-standard polynomial.

**Proposition 3.28.** If $n_s \geq 3$, then the vector space $B_{1N}(I)$ is spanned by the set of all elements $f + I \cap B_{1N}$ where $f$ is $S_4$-standard.

Proof. Let $\Lambda = \text{span}\{f + I \cap B_{1N} : f \in S_4\}$. By Proposition 3.26 it is enough to prove that

- $h^{(j)} + I \cap B_{1N} \in \Lambda, j < s$;
- $f^{(i,j)} + I \cap B_{1N} \in \Lambda, 1 \leq i \leq j \leq s$ and $i < s$.

By Lemma 3.3 (4) it follows that $f^{(i,j)} + I = p^{(i,j)} + f^{(j)} + I$. Thus $f^{(i,j)} + I \cap B_{1N} \in \Lambda$. 

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Write \( h^{(j)} = wz_s z_{[y_1, z_j]} z_k \) where \( w = z_1^{n_1} \ldots z_j^{n_j-1} \ldots z_s^{n_s-3} \). By Lemma 3.7 ii), Lemma 3.5 d) and Lemma 3.4 ii) we obtain:

\[
\begin{align*}
&h^{(j)} + I = wz_s z_{[y_1, z_s]} z_k + wz_s z_{[y_1, z_s]} z_k + I \\
&= wz_s z_{[y_1, z_s]} z_k + w[z_s, z_j][y_1, z_s] z_k - wz_s z_{[y_1, z_s]} z_k + I \\
&= h(s) - wz_s z_{[y_1, z_s]} z_k - wz_s z_{[y_1, z_s]} z_k + I \\
&= h(s) - 2p^{(j,s)} + I.
\end{align*}
\]

Therefore, \( h^{(j)} + I \cap B_{1N} \in \Lambda \).

\[\square\]

**Proposition 3.29.** If \( n_s \geq 3 \), then \( \{f + Id \cap B_{1N} : f \in S_4\} \) is a basis for the vector space \( B_{1N}[Id] \). In particular, \( Id \cap B_{1N} = I \cap B_{1N} \).

**Proof.** Since \( I \subset Id \), we have, by Proposition 3.28

\[
B_{1N}(Id) = \text{span}\{f + Id \cap B_{1N} : f \in S_4\}.
\]

We will use the notations of Observation 3.15. Consider some order \( > \) on \( L \) such that

\[
z_{i_1}^{j_1} > z_{i_2}^{j_2} > z_{i_3}^{j_3} > z_{i_4}^{j_4} > z_{i_5}^{j_5} > z_{i_6}^{j_6} > z_{i_7}^{j_7} > z_{i_8}^{j_8} > z_{i_9}^{j_9} > z_{i_10}^{j_10} \]

for all \( 1 \leq l \leq s - 2 \) and \( i \geq 1 \). By using the \( q \)-generic matrices we have the following equalities:

\[
\begin{align*}
&\bullet Z_1 \ldots Z_j \ldots Z_m[Y_1, Z_j] = \left[ \prod_{i=1}^{m} \left( z_{i_1}^{j_1} \right) \right] (y_{11}^{j_1} z_{i_2}^{j_2} - y_{12}^{j_2} z_{i_1}^{j_1}) e_{12} + ue_{13}, \\
&\bullet [Y_1, Z_j]Z_1 \ldots Z_j \ldots Z_m = (-1)^{m-1} (y_{11}^{j_1} z_{i_2}^{j_2} - y_{12}^{j_2} z_{i_1}^{j_1}) \left[ \prod_{i=1}^{m} \left( z_{i_1}^{j_1} \right) \right] e_{23} + ve_{13}, \\
&\bullet Z_1 \ldots Z_{s-1} Z_s[Y_1, Z_s] = -\left[ \prod_{i=1}^{s} \left( z_{i_1}^{j_1} \right) \right] (y_{11}^{j_1} z_{i_2}^{j_2} - y_{12}^{j_2} z_{i_1}^{j_1}) z_{i_1}^{j_1} e_{13}, \\
&\quad + \left( \sum_{i=1}^{s} \left( z_{i_1}^{j_1} \right) (y_{12}^{j_2} z_{i_3}^{j_3} + y_{13}^{j_3} z_{i_2}^{j_2}) z_{i_1}^{j_1} - \left[ \prod_{i=1}^{s} \left( z_{i_1}^{j_1} \right) \right] z_{i_2}^{j_2} (y_{11}^{j_1} z_{i_2}^{j_2} - y_{12}^{j_2} z_{i_1}^{j_1}) z_{i_1}^{j_1} \right) e_{13}, \\
&\bullet Z_1 \ldots Z_i \ldots Z_s[Z_s, Z_i][Y_1, Z_j] = \left[ \prod_{i=1}^{s} \left( z_{i_1}^{j_1} \right) \right] (z_{i_1}^{j_1} z_{i_2}^{j_2} - z_{i_2}^{j_2} z_{i_1}^{j_1}) (y_{11}^{j_1} z_{i_2}^{j_2} - y_{12}^{j_2} z_{i_1}^{j_1}) e_{13}.
\end{align*}
\]

for some polynomials \( u, v \in F[L] \).

Let \( f^{(j)}, g^{(j)}, h^{(s)}, p^{(j,s)} \) as in [4]. Write

\[
\begin{align*}
f^{(j)}(Y_1, Z_1, \ldots, Z_s) &= \sum f_{ab}^{(j)} e_{ab}, \quad g^{(j)}(Y_1, Z_1, \ldots, Z_s) = \sum g_{ab}^{(j)} e_{ab}, \\
h^{(s)}(Y_1, Z_1, \ldots, Z_s) &= \sum h_{ab}^{(s)} e_{ab}, \quad p^{(j,s)}(Y_1, Z_1, \ldots, Z_s) = \sum p_{ab}^{(j,s)} e_{ab}.
\end{align*}
\]
and suppose
\[ \sum \alpha_j f^{(j)} + \sum \beta_j g^{(j)} + \gamma h^{(s)} + \sum \beta_{i,j} p^{(i,j)} \in \text{Id}, \]
where \( \alpha_j, \beta_j, \gamma, \beta_{i,j} \in \mathbb{F} \). Now we use the same arguments as Propositions 3.17, 3.20 and 3.23. In short, by the following table

| Entry   | Information             | Monomial           | Its coefficient |
|---------|-------------------------|--------------------|-----------------|
| (1, 2)  | \( g^{(1,2)} = h^{(s)}_{12} = p^{(i,j)}_{12} = 0 \) | \( m(f^{(1,2)}_{12}) \) | \( \alpha_j \) |
| (2, 3)  | \( h^{(s)}_{23} = p^{(i,j)}_{23} = 0 \) | \( m(g^{(2,3)}_{23}) \) | \( \pm \beta_j \) |
| (1, 3)  | | \( w \) | \( 2 \gamma \) |
| (1, 3)  | | \( m(p^{(i,j)}_{13}) \) | \( \beta_{i,j} \) |

where
\[
\begin{align*}
    m(f^{(1,2)}_{12}) &= (z_{11}^{n_1} \cdots z_{1j}^{n_{j-1}} \cdots z_{11}^{n_j})^{n_s} y_{11}^{12}, \\
    m(g^{(2,3)}_{23}) &= y_{11}^{12} z_{2j}^{n_1} \cdots (z_{11}^{n_j})^{n_s}, \\
    w &= y_1^{13} (z_{11}^{n_1})^{n_s}, \\
    m(p^{(i,j)}_{13}) &= (z_{11}^{n_1} \cdots z_{1j}^{n_{j-1}} \cdots z_{11}^{n_j})^{n_s} y_{11}^{12} z_{12}^{ij},
\end{align*}
\]
we have \( \alpha_j = 0, \beta_j = 0, \gamma = 0, \beta_{i,j} = 0 \) respectively. \( \square \)

### 3.4.2 Subspace \( B_{1N} \) where \( n_s = 1 \)

By Observation 3.10 if \( n_s = 1 \) then \( n_1 = \ldots = n_s = 1 \). In this case, \( B_{1N} = \Gamma_{1s} \).

**Proposition 3.30.** If \( n_s = 1 \), then \( \{ f + \text{Id} \cap \Gamma_{1s} : \ f \in S_3 \} \) is a basis for the vector space \( \Gamma_{1s}(\text{Id}) \). In particular, \( \text{Id} \cap \Gamma_{1s} = \Gamma \cap \Gamma_{1s} \).

**Proof.** If \( f \) is \( S_3 \)-standard then \( f \) is \( T_2 \)-standard in [12, Definition 5.18]. Now we can use the same proof of [12, Lemma 6.5]. \( \square \)

### 3.5 Subspaces \( B_{MN} \) where \( M \neq (0), (1) \) and \( N \neq (0), (1) \)

Let \( M = (m_1, \ldots, m_k) \) and \( N = (n_1, \ldots, n_s) \). In this section,
\[
m = m_1 + \ldots + m_k \geq 2 \quad \text{and} \quad n = n_1 + \ldots + n_s \geq 2.
\]

**Definition 3.31.** Let \( S_3 \subset S_2 \) be the set of all polynomials \( f^{(i)}, g^{(i)}, f^{(i,i)}, g^{(i,i)}, h^{(j,ji)} \in B_{MN} \) such that:

- \( f^{(i)} = [z_{i_1}, z_{i_2}, \ldots, x_{i-t_i}] \in S_2 \),
- \( g^{(i)} = [y_{i_1}, z_{i_2}, \ldots, x_{i-t_i}] \in S_2 \),
- \( f^{(i,i)} = z_i [z_{i_1}, x_{i_2}, \ldots, x_{i-t_i}] \in S_2 \) and \( z_i \leq z_{i_1} \),
- \( g^{(i,i)} = z_i [y_{i_1}, x_{i_2}, \ldots, x_{i-t_i}] \in S_2 \),
- \( h^{(j,ji)} = [y_{j_1}, x_{j_2}, \ldots, x_{j_{t_j}}][y_{p_1}, z_1] \in S_2 \).
where $t = m + n$. If $f \in S_3$, we say that $f$ is an $S_3$-standard polynomial.

**Proposition 3.32.** The vector space $B_{MN}(I)$ is spanned by the set of all elements $f + I \cap B_{MN}$ where $f$ is $S_3$-standard.

**Proof.** This proposition is true for $\Gamma_{mn}(I)$. In fact, we can use the same proof as in [12] Proposition 5.17. Now, if $x_i < x_j$, then $\varphi_{MN}(x_i) \leq \varphi_{MN}(x_j)$. Thus, by Proposition 3.11 the general case is proved. \hfill \Box

In $UT_3(F[y_{ij}^k, z_{ij}^k])$ consider the *generic* matrices

$$Z_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad Z_l = \begin{bmatrix} 1 & z_{i2}^l & 0 \\ 0 & 0 & -z_{i2}^l \\ 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad Y_j = \begin{bmatrix} 1 & y_{i1}^j & y_{i3}^j \\ 0 & 0 & y_{i2}^j \\ 0 & 0 & 1 \end{bmatrix}$$

for all $l \geq 2$ and $j \geq 1$. If $w(y_1, \ldots, y_k, z_1, \ldots, z_s)$ is $S_3$-standard then we write

$$w(Y_1, \ldots, Y_k, Z_1, \ldots, Z_s) = \sum_{a,b=1}^3 w_{ab} e_{ab}.$$

Since $F$ is an infinite field we have the following lemma:

**Lemma 3.33.** Let $Y_1, \ldots, Y_k, Z_1, \ldots, Z_s$ be generic matrices. If $f(y_1, \ldots, y_k, z_1, \ldots, z_s) \in Id$, then $f(Y_1, \ldots, Y_k, Z_1, \ldots, Z_s) = 0$.

**Lemma 3.34.** Let $Z_i$ and $Y_i$ be the generic matrices, where $l \geq 1$.

a) If $m \geq 2$ is even then:

$$[Z_{i1}, Z_{i2}, \ldots, Z_{in}, Y_{j1}, \ldots, Y_{jm}] = (-1)^n(z_{i2}^{j1} - z_{i2}^{j1})(e_{12} - e_{23}), \quad \text{where} \quad n \geq 2;$$

$$[Y_{j1}, Z_{i2}, \ldots, Z_{in}, Y_{j2}, \ldots, Y_{jm}] = (-1)^n(z_{i2}^{j1} - y_{i1}^{j1})(e_{12} - e_{23}), \quad \text{where} \quad n \geq 2;$$

$$Z_i[Z_{i1}, Z_{i2}, \ldots, Z_{in-1}, Y_{j1}, \ldots, Y_{jm}] = (-1)^{n-1}(z_{i2}^{j1} - z_{i2}^{j1})e_{12} + (-1)^n z_{i2}^{j1}(z_{i2}^{j1} - z_{i2}^{j1})e_{13}, \quad \text{where} \quad n \geq 3;$$

$$Z_i[Y_{j1}, Z_{i2}, \ldots, Z_{in-1}, Y_{j2}, \ldots, Y_{jm}] = (-1)^{n-1}(z_{i2}^{j1} - y_{i1}^{j1})e_{12} + (-1)^n z_{i2}^{j1}(z_{i2}^{j1} - y_{i1}^{j1})e_{13}, \quad \text{where} \quad n \geq 2;$$

$$[Y_{j1}, Z_{i1}, \ldots, Z_{in-1}, Y_{j2}, \ldots, Y_{jm-1}][Y_{p1}, Z_{p2}] = (-1)^n(z_{i1}^{j1} - y_{i1}^{j1})(z_{i2}^{p1} - y_{i1}^{p1})e_{13},$$

where $n \geq 2$.

b) If $m \geq 2$ is odd then:

$$[Z_{i1}, Z_{i2}, \ldots, Z_{in}, Y_{j1}, \ldots, Y_{jm}] = (-1)^{n-1}(z_{i2}^{j1} - z_{i2}^{j1})(e_{12} + e_{23}) + 2(-1)^n(z_{i2}^{j1} - z_{i2}^{j1})y_{i2}^m e_{13}, \quad \text{where} \quad n \geq 2;$$

$$[Y_{j1}, Z_{i1}, \ldots, Z_{in}, Y_{j2}, \ldots, Y_{jm}] = (-1)^{n-1}(z_{i2}^{j1} - y_{i1}^{j1})(e_{12} + e_{23}) + 2(-1)^n(z_{i2}^{j1} - y_{i1}^{j1})y_{i2}^m e_{13}, \quad \text{where} \quad n \geq 2;$$

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\[ Z_i[Z_{i1}, Z_{i2}, \ldots, Z_{in-1}, Y_{j1}, \ldots, Y_{jm}] = (-1)^n(z_{12}^i - z_{12}^l)e_{12} + \\
- (-1)^n(z_{12}^i - z_{12}^l)(-2y_{12}^i + z_{12}^i)e_{13}, \text{ where } n \geq 3; \]
\[ Z_i[Y_{j1}, Z_{i1}, \ldots, Z_{in-1}, Y_{j2}, \ldots, Y_{jm}] = (-1)^n(z_{12}^i - y_{12}^j)e_{12} + \\
- (-1)^n(z_{12}^i - y_{12}^j)(-2y_{12}^i + z_{12}^i)e_{13}, \text{ where } n \geq 2; \]
\[ [Y_{j1}, Z_{i1}, \ldots, Z_{in-1}, Y_{j2}, \ldots, Y_{jm-1}][Y_{p1}, Z_{p2}] = (-1)^{n-1}(z_{12}^i - y_{12}^j)(z_{12}^i - y_{12}^j)e_{13}, \]
\[ \text{where } n \geq 2. \]

**Proof.** We leave the proof to the reader. \(\square\)

### 3.5.1 Case \(m\) even and \(n_1 > 1\)

Let \(M = (m_1, \ldots, m_k), N = (n_1, \ldots, n_s), m = m_1 + \ldots + m_k \geq 2\) and \(n = n_1 + \ldots + n_s \geq 2\). In this subsection, we consider the case where \(m\) is even and \(n_1 > 1\).

**Proposition 3.35.** If \(m\) is even and \(n_1 > 1\), then \((f + \text{Id} \cap B_{MN} : f \in S_3)\) is a basis for the vector space \(B_{MN}(\text{Id})\). In particular, \(\text{Id} \cap B_{MN} = I \cap B_{MN}\).

**Proof.** Since \(I \subset \text{Id}\), we have, by Proposition 3.32

\[ B_{MN}(\text{Id}) = \text{span}\{f + \text{Id} \cap B_{MN} : f \in S_3\}. \]

Consider some order \(>\) on \(L\) such that

\[ z_{12}^{i+1} > z_{12}^l > y_{12}^{i+1} > y_{12}^i \]

for all \(i \geq 1\). Let \(Z_i\) and \(Y_i\) be the generic matrices, where \(l \geq 1\). By Lemma 3.34 we have

\[ [Z_{i1}, Z_{i2}, \ldots, Z_{in-1}, Y_{j1}, \ldots, Y_{jm}] = (-1)^{n+1}z_{12}^i(e_{12} - e_{23}), \]
\[ [Y_{j1}, Z_{i1}, \ldots, Z_{in-1}, Y_{j2}, \ldots, Y_{jm}] = (-1)^{n+1}y_{12}^j(e_{12} - e_{23}), \]
\[ Z_i[Y_{j1}, Z_{i1}, \ldots, Z_{in-1}, Y_{j2}, \ldots, Y_{jm}] = (-1)^n z_{12}^i y_{12}^j e_{12} - (-1)^n z_{12}^i z_{12}^j e_{13}, \]
\[ Z_i[Y_{j1}, Z_{i1}, \ldots, Z_{in-1}, Y_{j2}, \ldots, Y_{jm-1}][Y_{p1}, Z_{p2}] = (-1)^ny_{12}^j y_{12}^p e_{13}. \]

Let \(f^{(i)}(j_1), f^{(i,j_1)}, g^{(i,j_1)}, h^{(j_1,p_1)}\) as in (12) and suppose

\[ \sum \alpha_i f^{(i)} + \sum \beta_j g^{(j_1)} + \sum \alpha_{i,j_1} f^{(i,j_1)} + \sum \beta_{i,j_1} g^{(i,j_1)} + \sum \gamma_{j_1,p_1} h^{(j_1,p_1)} \in \text{Id}, \]

where \(\alpha_i, \beta_j, \alpha_{i,j_1}, \beta_{i,j_1}, \gamma_{j_1,p_1} \in \mathbb{F}\). Now we use the same arguments as in the previous propositions. In short, by the following table
| Entry | Information | Monomial | Its coefficient |
|-------|-------------|----------|-----------------|
| (2,3) | $f_{(i,1)}^{(i,1)} = g_{(i,1)}^{(i,1)} = h_{(i,1)}^{(i,1)} = 0$ | $m(f_{(i,1)}^{(i,1)})$ | $\pm a_{i,1}$ |
| (2,3) | $f_{23}^{(i,1)} = g_{23}^{(i,1)} = h_{23}^{(i,1)} = 0$ | $m(g_{23}^{(i,1)})$ | $\pm b_{i,1}$ |
| (1,3) | $i > 1$ | $m(f_{13}^{(i,1)})$ | $\pm a_{1,1}$ |
| (1,3) | $i > 1$ | $m(g_{13}^{(i,1)})$ | $\pm b_{1,1}$ |
| (1,2) | $i = 1$ | $m(f_{12}^{(1,1)})$ | $\pm a_{1,1}$ |
| (1,2) | $i = 1$ | $m(g_{12}^{(1,1)})$ | $\pm b_{1,1}$ |
| (1,3) | | $m(h_{13}^{(1,1)})$ | $\pm g_{1,1,1}$ |

where

$$m(f_{23}^{(i,1)}) = z_{12}^{i}, \quad m(g_{23}^{(i,1)}) = y_{12}^{i},$$

$$m(f_{13}^{(1,1)}) = z_{12}^{1}y_{12}^{1}, \quad m(g_{13}^{(1,1)}) = y_{12}^{1},$$

$$m(f_{12}^{(1,1)}) = z_{12}^{1}, \quad m(g_{12}^{(1,1)}) = y_{12}^{1},$$

$$m(h_{13}^{(1,1)}) = y_{12}^{1}y_{12}^{1},$$

we have $a_{1,1} = 0$, $b_{1,1} = 0$, $a_{1,1} = 0$, $b_{1,1} = 0$, $a_{1,1} = 0$, $b_{1,1} = 0$, $g_{1,1,1} = 0$, respectively.

### 3.5.2 Case $m$ even and $n_1 = 1$

Let $M = (m_1, \ldots, m_k)$, $N = (n_1, \ldots, n_s)$, $m = m_1 + \ldots + m_k \geq 2$ and $n = n_1 + \ldots + n_s \geq 2$. In this subsection, we consider the case where $m$ is even and $n_1 = 1$.

**Proposition 3.36.** If $m$ is even and $n_1 = 1$, then $\{f + Id \cap B_{MN} : f \in S_3\}$ is a basis for the vector space $B_{MN}(Id)$. In particular, $Id \cap B_{MN} = I \cap B_{MN}$.

**Proof.** Since $I \subset Id$, we have, by Proposition 3.32,

$$B_{MN}(Id) = \text{span}\{f + Id \cap B_{MN} : f \in S_3\}.$$

Consider some order $>$ on $L$ such that

$$y_{12}^{i+1} > y_{12}^{i} > z_{12}^{i+1} > z_{12}^{i}$$

for all $i \geq 1$. Let $Z_l$ and $Y_l$ be the sgeneric matrices, where $l \geq 1$. By Lemma

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we have

\[
\begin{align*}
[Z_{i_1}, Z_{i_2}, \ldots, Y_{j_m}] &= (-1)^{n+1} z_{12}^i (e_{12} - e_{23}), \\
[Y_{j_1}, Z_{i_1}, \ldots, Y_{j_m}] &= (-1)^{n+1} y_{12}^i (e_{12} - e_{23}), \\
Z_i [Z_{i_1}, Z_{i_2}, \ldots, Y_{j_m}] &= (-1)^n z_{12}^i e_{12} - (-1)^n z_{12}^i z_{12}^i e_{13}, \\
Z_i [Z_{i_1}, Z_{i_2}, \ldots, Y_{j_m}] &= (-1)^{n-1} (z_{12}^i - z_{12}^i) e_{12}, \\
Z_i [Z_{j_1}, Z_{j_2}, \ldots, Y_{j_m}] &= (-1)^n y_{12}^i e_{12} - (-1)^n y_{12}^i y_{12}^i e_{13}, \\
Z_i [Y_{j_1}, Z_{j_2}, \ldots, Y_{j_m}] &= (-1)^{n-1} (z_{12}^i - y_{12}^i) y_{12}^i e_{13}. \\
\end{align*}
\]

Let \( f^{(i_1)}, g^{(j_1)}, f^{(i_1)}, g^{(i_2, p_1)}, h^{(j_1, p_1)} \) as in (12) and suppose

\[
\sum \alpha_i f^{(i_1)} + \sum \beta_j g^{(j_1)} + \sum \alpha_{i,i} f^{(i_1)} + \sum \beta_{i,j} g^{(i_1, j_1)} + \sum \gamma_{j_1, p_1} h^{(j_1, p_1)} \in Id,
\]

where \( \alpha_i, \beta_j, \alpha_{i,i}, \beta_{i,j}, \gamma_{j_1, p_1} \in F \). Now we use the same arguments as in the previous propositions. In short, by the following table

| Entry | Information | Monomial | Its coefficient |
|-------|-------------|----------|----------------|
| (2,3) | \( f^{(i_1)} \) | \( g^{(i_1, j_1)} = h^{(j_1, p_1)} = 0 \) | \( \pm \beta_{j_1} \) |
| (2,3) | \( f^{(i_1)} \) | \( g^{(i_1, j_1)} = h^{(j_1, p_1)} = 0 \) | \( \pm \alpha_{i,i} \) |
| (1,3) | \( i > 1 \) | \( g^{(i_1, j_1)} \) | \( \pm \beta_{i,j_1} \) |
| (1,3) | \( i > 1 \) | \( f^{(i_1)} \) | \( \pm \alpha_{i,i} \) |
| (1,2) | \( i = 1 \) | \( g^{(i_1, j_1)} \) | \( \pm \beta_{1,j_1} \) |
| (1,2) | \( i = 1 \) | \( f^{(i_1)} \) | \( \pm \alpha_{1,i} \) |

where

\[
\begin{align*}
m(g^{(i_1, j_1)}) &= g_{12}^i, \\
m(h^{(j_1, p_1)}) &= g_{12}^i y_{12}^i, \\
m(g^{(i_1, j_1)}) &= z_{12}^i e_{12}, \\
m(g^{(i_1, j_1)}) &= y_{12}^i, \\
m(f^{(i_1)}) &= z_{12}^i, \\
m(f^{(i_1)}) &= z_{12}^i, \\
\end{align*}
\]

we have \( \beta_{i_1} = 0, \alpha_{i,i} = 0, \gamma_{j_1, p_1} = 0, \beta_{i,j_1} = 0, \alpha_{i,i} = 0, \beta_{1,j_1} = 0, \alpha_{1,i} = 0 \), respectively.

\( \square \)

### 3.5.3 Case \( m \) odd and \( n_1 > 1 \)

Let \( M = (m_1, \ldots, m_k), N = (n_1, \ldots, n_s), \) \( m = m_1 + \ldots + m_k \geq 2 \) and \( n = n_1 + \ldots + n_s \geq 2. \) In this subsection, we consider the case where \( m \) is odd and \( n_1 > 1. \)

**Proposition 3.37.** If \( m \) is odd and \( n_1 > 1, \) then \( \{ f + Id \cap B_{MN} : f \in S_3 \} \) is a basis for the vector space \( B_{MN}(Id) \). In particular, \( Id \cap B_{MN} = I \cap B_{MN}. \)
Proof. Let $Z_l$ and $Y_l$ be the generic matrices, where $l \geq 1$. By Lemma 3.34 we have

\[ [Z_1, Z_1, \ldots, Z_m] = (-1)^n z_{12}^1 (e_{12} + e_{23}) - 2(-1)^n z_{12}^1 y_{12} e_{13}, \]
\[ [Y_1, Z_1, \ldots, Y_m] = (-1)^n y_{12}^1 (e_{12} + e_{23}) - 2(-1)^n y_{12}^1 y_{12}^1 e_{13}, \]
\[ Z_i[Z_1, Z_1, \ldots, Z_m] = (-1)^{n+1} z_{12}^1 e_{12} + (-1)^{n+1} z_{12}^1 (2y_{12}^1 + z_{12}^1) e_{13}, \]
\[ Z_i[Y_1, Z_1, \ldots, Y_m] = (-1)^{n+1} y_{12}^1 e_{12} + (-1)^{n+1} y_{12}^1 (2y_{12}^1 + z_{12}^1) e_{13}, \]
\[ [Y_1, Z_1, \ldots, Y_{m-1}][Y_{p_1}, Z_1] = (-1)^{n+1} y_{12}^1 y_{12}^1 e_{13}. \]

Now we use the same order $>$, table and leading monomials in Proposition 3.35.

3.5.4 Case $m$ odd where $n_1 = 1$ and $m_k > 1$

Let $M = (m_1, \ldots, m_k)$, $N = (n_1, \ldots, n_s)$, $m = m_1 + \ldots + m_k \geq 2$ and $n = n_1 + \ldots + n_s \geq 2$. In this subsection, we consider the case $m$ odd where $n_1 = 1$ and $m_k > 1$.

**Proposition 3.38.** If $m$ is odd, $n_1 = 1$ and $m_k > 1$, then $\{f + Id \cap B_{MN} : f \in S_3\}$ is a basis for the vector space $B_{MN}(Id)$. In particular, $Id \cap B_{MN} = I \cap B_{MN}$.

**Proof.** Since $I \subset Id$, we have, by Proposition 3.32

\[ B_{MN}(Id) = \text{span}\{f + Id \cap B_{MN} : f \in S_3\}. \]

Consider some order $>$ on $L$ such that

\[ z_{12}^{i+1} > z_{12}^i > y_{12}^{i+1} > y_{12}^i \]

for all $i \geq 1$. Let $Z_l$ and $Y_l$ be the generic matrices, where $l \geq 1$. By Lemma 3.34 we have

\[ [Z_1, Z_1, \ldots, Z_k] = (-1)^n z_{12}^1 (e_{12} + e_{23}) - 2(-1)^n z_{12}^1 y_{12} e_{13}, \]
\[ [Y_1, Z_1, \ldots, Y_k] = (-1)^n y_{12}^1 (e_{12} + e_{23}) - 2(-1)^n y_{12}^1 y_{12}^1 e_{13}, \]
\[ Z_i[Z_1, Z_1, \ldots, Z_k] = (-1)^{n+1} z_{12}^1 e_{12} + (-1)^{n+1} z_{12}^1 (2y_{12}^1 + z_{12}^1) e_{13}, \]
\[ Z_i[Y_1, Z_1, \ldots, Y_k] = (-1)^{n+1} y_{12}^1 e_{12} + (-1)^{n+1} y_{12}^1 (2y_{12}^1 + z_{12}^1) e_{13}, \]
\[ Z_i[Y_1, Z_1, \ldots, Y_{k-1}][Y_{p_1}, Z_1] = (-1)^n (z_{12}^1 - y_{12}^1) y_{12}^1 e_{13}. \]

Let $f^{(i)}, g^{(j)}, f^{(i; i)}, g^{(i; j)}, h^{(i; p_1)}$ as in (12), and suppose

\[ \sum \alpha_i f^{(i)} + \sum \beta_j g^{(j)} + \sum \alpha_{i; i} f^{(i; i)} + \sum \beta_{i; j} g^{(i; j)} + \sum \gamma_{j; p_1} h^{(j; p_1)} \in Id, \]

where $\alpha_i, \beta_j, \alpha_{i; i}, \beta_{i; j}, \gamma_{j; p_1} \in \mathbb{F}$. By the following table
For the remaining coefficients, by the following table

| Entry | Information | Monomial | Its coefficient |
|-------|-------------|----------|-----------------|
| (1, 3) | $j_1 < k$ | $m(g_{13}^{(2, j_1)})$ | $\pm \beta_{2, j_1}$ |
| (1, 3) | $j_1 < k$ | $u$ | $\pm \beta_{3, j_1}$ |
| (1, 2) | $j_1 = k$ | $m(g_{12}^{(1, k)})$ | $\pm \beta_{1, k}$ |
| (1, 2) | $j_1 = k$ | $m(g_{12}^{(2, k)})$ | $\pm \beta_{2, k}$ |

where

$m(g_{13}^{(2, j_1)}) = y_{i_2}^{j_1} z_{j_2}, \quad u = y_{i_2}^{j_1} y_{k_1}^{j_2}$

we have $\beta_{2, j_1} = 0, \beta_{1, j_1} = 0, \beta_{1, k} = 0$ and $\beta_{2, k} = 0$, respectively.
3.5.5 Case \( m \) odd where \( n_1 = m_k = 1 \) and \( \text{char}(F) > 3 \)

Let \( M = (m_1, \ldots, m_k) \), \( N = (n_1, \ldots, n_s) \), \( m = m_1 + \ldots + m_k \geq 2 \) and \( n = n_1 + \ldots + n_s \geq 2 \). In this subsection, we consider the case \( m \) odd where \( n_1 = m_k = 1 \) and \( \text{char}(F) > 3 \).

**Proposition 3.39.** If \( m \) is odd, \( n_1 = m_k = 1 \) and \( \text{char}(F) > 3 \), then \( \text{Id} \cap B_{MN} = I \cap B_{MN} \).

**Proof.** By Observation 3.10 we have \( m_1 = \ldots = m_{k-1} = m_k = 1 \).

If \( n_s = 1 \) then \( n_1 = n_2 = \ldots = n_s = 1 \) and we can use the same proof of [12 Lemma 6.4].

Suppose \( n_s > 1 \). By a change of variables \( z_1 \leftrightarrow z_s \) we can suppose \( n_1 > 1 \).

Note that

\[
\begin{align*}
n_s &\leq n_2 \leq n_3 \leq \ldots \leq n_{s-1} \leq n_1.
\end{align*}
\]

But the Proposition 3.37 is also true in this case, the proof is the same. \( \square \)

3.5.6 Case \( m \) odd where \( n_1 = m_k = 1 \) and \( \text{char}(F) = 3 \)

Let \( M = (m_1, \ldots, m_k) \), \( N = (n_1, \ldots, n_s) \), \( m = m_1 + \ldots + m_k \geq 2 \) and \( n = n_1 + \ldots + n_s \geq 2 \). In this subsection, we consider the case \( m \) odd where \( n_1 = m_k = 1 \) and \( \text{char}(F) = 3 \).

We remember that \( S_3 \) is the set of all polynomials defined in [12].

**Definition 3.40.** Denote by \( S_4 \) the set

\[
S_4 = S_3 - \{g^{(1,k)}\}.
\]

We say that the polynomials in \( S_4 \) are \( S_4 \)-standard.

**Proposition 3.41.** The vector space \( B_{MN}(I) \) is spanned by the set of all elements \( f + I \cap B_{MN} \) where \( f \in B_{MN} \) is \( S_4 \)-standard.

**Proof.** We work modulo \( I \). By Proposition 3.32 it is sufficient to prove that \( g^{(1,k)} \) is a linear combination of \( S_4 \)-standard polynomials. In fact, we will prove that

\[
g^{(1,k)} = g^{(1,k-1)} - g^{(2,k-1)} + g^{(2,k)}. \tag{13}
\]

By Lemma 3.39 (b,c), we have

\[
z_{i_n} \ldots z_{i_3} z_1[y_k, z_l, y_1, \ldots, y_{k-1}] = (-1)^n z_1[y_k, z_l, z_{i_3}, \ldots, z_{i_n}, y_1, \ldots, y_{k-1}].
\]

Thus it is sufficient to prove (13) when \( n = 2 \) that is

\[
z_1[y_k, z_2, y_1, \ldots, y_{k-1}] = z_1[y_{k-1}, z_2, y_1, \ldots, y_k] - z_2[y_{k-1}, z_1, y_1, \ldots, y_k] + z_2[y_k, z_1, y_1, \ldots, y_{k-1}].
\]
Claim: If \( i \neq j \) and \( a \neq b \), then:

\[
2i[y_1, z_j, y_0, \ldots, y_k] = 2i[y_1, z_j, y_1, \ldots, y_0] - 2i[y_0, y_1, z_j, \ldots, y_0] + 2z_i[y_1, z_j, \ldots, [y_0, y_b]].
\]

In fact, by Lemma 3.4-iii), equality \([a, b], c, d] = [[a, b], [c, d]],\) Jacobi identity and Proposition 3.8-b), we obtain

\[
2i[y_1, z_j, y_0, \ldots, y_k] = 2i[y_1, z_j, \ldots, y_0, y_k] + 2z_i[y_1, z_j, \ldots, [y_0, y_b]]
\]

and the claim is proved.

Now, by the Jacobi identity, we have

\[
g^{(1,k)} = 2i[y_1, z_2, y_k, \ldots, y_{k-1}] - 2i[y_1, y_k, z_2, \ldots, y_{k-1}]
\]

and applying Lemma 3.8 and Jacobi identity in the second summand,

\[
g^{(1,k)} = 2i[y_1, z_2, y_k, \ldots, y_{k-1}] - 2i[y_1, y_k, z_2, \ldots, y_{k-1}]
+ [y_1, y_k][z_2, z_1, \ldots, y_{k-1}]
\]

\[
= z_1[y_1, z_2, y_k, \ldots, y_{k-1}] + g^{(2,k)}
- 2z_2[y_1, z_1, y_k, \ldots, y_{k-1}] + [y_1, y_k][z_2, z_1, \ldots, y_{k-1}].
\]

By applying the Claim in the summands \(z_1[y_1, z_2, y_k, \ldots, y_{k-1}]\) and \(z_2[y_1, z_1, y_k, \ldots, y_{k-1}]\), we have

\[
g^{(1,k)} = g^{(1,k-1)} - g^{(2,k-1)} + g^{(2,k)} + f
\]

where

\[
f = - z_1[y_k, y_k-1, y_1, z_2, \ldots, y_{k-1}] + z_2[y_k, y_1, z_2, \ldots, y_{k-1}]
- 2z_2[y_1, z_1, \ldots, y_{k-1}] + z_3[y_1, z_2, \ldots, y_{k-1}]
+ [y_1, y_k][z_2, z_1, \ldots, y_{k-1}].
\]

We shall prove that \( f = 0 \). By Lemma 3.8 and Lemma 3.4-iii),

\[
- z_1[y_k-1, y_1, z_2, \ldots, y_k] + z_2[y_k-1, y_1, z_1, \ldots, y_k] = [y_k-1, y_1][z_2, z_1, \ldots, y_{k-1}]
\]

By Lemma 3.4-iii),

\[
[y_1, y_k][z_2, z_1, \ldots, y_{k-1}] = -[y_1, y_k, y_k-1][z_2, z_1, \ldots, y_{k-1}]
\]
and then, applying the Jacobi identity, Lemma 3.4-iii), Lemma 3.5-b) and Proposition 3.4-ii) we have:

\[-[y_{k-1}, y_1, y_k][z_2, z_1, \ldots] - [y_1, y_k, y_{k-1}][z_2, z_1, \ldots] = [y_k, y_{k-1}, y_1][z_2, z_1, \ldots] = -[y_k, y_{k-1}][z_2, z_1, y_1, \ldots] = [z_2, z_1, y_1, \ldots][y_k, y_{k-1}] = [y_1, z_1, z_2, \ldots][y_k, y_{k-1}] - [y_1, z_2, z_1, \ldots][y_k, y_{k-1}].\]

By Proposition 3.2-v) and Lemma 3.5-b),

\[-2z_2[y_1, z_1, \ldots][y_k, y_{k-1}] + 2z_1[y_1, z_2, \ldots][y_k, y_{k-1}] = 2[y_1, z_1, z_2, \ldots][y_k, y_{k-1}] - 2[y_1, z_2, z_1, \ldots][y_k, y_{k-1}].\]

Therefore, since \(\text{char}(F) = 3\), we have

\[f = [y_1, z_1, z_2, \ldots][y_k, y_{k-1}] - [y_1, z_2, z_1, \ldots][y_k, y_{k-1}] + 2[y_1, z_1, z_2, \ldots][y_k, y_{k-1}] - 2[y_1, z_2, z_1, \ldots][y_k, y_{k-1}] = 3[y_1, z_1, z_2, \ldots][y_k, y_{k-1}] - 3[y_1, z_2, z_1, \ldots][y_k, y_{k-1}] = 0.\]

We finished the proof.

**Proposition 3.42.** If \(m \) is odd, \(n_1 = m_k = 1 \) and \(\text{char}(F) = 3\), then \(\{f + \text{Id} \cap B_{MN} : f \in S_4\} \) is a basis for the vector space \(B_{MN}(\text{Id})\). In particular, \(\text{Id} \cap B_{MN} = I \cap B_{MN}\).

**Proof.** Since \(I \subset \text{Id}\), we have, by Proposition 3.4-ii)

\[B_{MN}(\text{Id}) = \text{span}\{f + \text{Id} \cap B_{MN} : f \in S_4\}.\]

Consider some order \(\succ\) on \(L\) such that

\[z_{12}^{i+1} > z_{12}^i > y_{12}^{i+1} > y_{12}^i\]

for all \(i \geq 1\). Let \(Z_1\) and \(Y_1\) be the generic matrices, where \(l \geq 1\). By Lemma 3.34 we have

\[
\begin{align*}
[Z_{11}, Z_{12}, \ldots, Y_k] &= (-1)^n z_{12}^{i1}(e_{12} + e_{23}) - 2(-1)^n z_{12}^{i1}y_{12}^{i}e_{13}, \\
[Y_{j1}, Z_{12}, \ldots, Y_{jm}] &= (-1)^n y_{12}^{j1}(e_{12} + e_{23}) - 2(-1)^n y_{12}^{j1}y_{12}^{m}e_{13}, \\
Z_{1}[Z_{11}, Z_{12}, \ldots, Y_k] &= (-1)^{n+1} z_{12}^{i1}e_{12} + (-1)^{n+1} z_{12}^{i1}(-2y_{12}^{k} + z_{12}^{k})e_{13}, \\
Z_{1}[Z_{12}, \ldots, Y_{k-1}] &= (-1)^n (z_{12}^2 - z_{12}^{i1})e_{12} - 2(-1)^n (z_{12}^2 - z_{12}^{i1})y_{12}^{i}e_{13}, \\
Z_{1}[Y_{j1}, Z_{12}, \ldots, Y_{kj}] &= (-1)^{n+1} y_{12}^{j1}e_{12} + (-1)^{n+1} y_{12}^{j1}(-2y_{12}^{k} + z_{12}^{k})e_{13}, \\
Z_{1}[Y_{j1}, \ldots, Y_{jm}] &= (-1)^n (z_{12}^2 - y_{12}^{j1})e_{12} - 2(-1)^n (z_{12}^2 - y_{12}^{j1})y_{12}^{j}e_{13}, \\
Y_{j1}, Z_{12}, \ldots, Y_{jm-1}][Y_{pi}, Z_1] &= (-1)^n (z_{12}^2 - y_{12}^{p1})y_{12}^{i}e_{13}. \\
\end{align*}
\]
Let $f^{(i_1)}, g^{(j_1)}, f^{(i,i_1)}, g^{(i,j_1)}, h^{(j_1,p_1)}$ be $S_4$-standard polynomials, and suppose
\[
\sum \alpha_{i_1} f^{(i_1)} + \sum \beta_{j_1} g^{(j_1)} + \sum \alpha_{i,i_1} f^{(i,i_1)} + \sum \beta_{i,j_1} g^{(i,j_1)} + \sum \gamma_{j_1,p_1} h^{(j_1,p_1)} \in Id,
\]
where $\alpha_{i_1}, \beta_{j_1}, \alpha_{i,i_1}, \beta_{i,j_1}, \gamma_{j_1,p_1} \in \mathbb{F}$. Now we use the same arguments as in the previous propositions. In short, by the following table

| Entry | Information | Monomial | Its coefficient |
|-------|-------------|----------|-----------------|
| (2,3) | $f^{(i,i_1)}_{23} = g^{(j_1,i_1)}_{23} = h^{(j_1,p_1)}_{23} = 0$ | $m(f^{(i,i_1)}_{23})$ | $\pm \alpha_{i_1}$ |
| (2,3) | $f^{(i,i_1)}_{33} = g^{(j_1,i_1)}_{33} = h^{(j_1,p_1)}_{33} = 0$ | $m(g^{(j_1,i_1)}_{33})$ | $\pm \beta_{j_1}$ |
| (1,3) | $i > 1$ | $m(f^{(1,i)}_{i,j})$ | $\pm \alpha_{i,j}$ |
| (1,2) | $i = 1$ | $m(f^{(1,1)}_{1,j})$ | $\pm \alpha_{1,j}$ |
| (1,3) | $i > 2$ | $m(g^{(1,i)}_{i,j})$ | $\pm \beta_{i,j}$ |
| (1,2) | $j_1 < k$ | $m(g^{(2,k)}_{j_1})$ | $\pm \beta_{2,k}$ |

where
\[
\begin{align*}
m(f^{(i,i_1)}_{23}) &= z_{12}^{i}, \\
m(g^{(j_1,i_1)}_{23}) &= y_{12}^{j_1}, \\
m(f^{(1,i)}_{13}) &= z_{12}^{1}, \\
m(g^{(1,i)}_{13}) &= y_{12}^{1}, \\
m(g^{(1,i)}_{12}) &= y_{12}^{1}, \\
m(g^{(2,k)}_{j_1}) &= y_{12}^{k},
\end{align*}
\]
we have $\alpha_{i_1} = 0$, $\beta_{j_1} = 0$, $\alpha_{i,i_1} = 0$ for $i > 1$, $\alpha_{1,i_1} = 0$, $\beta_{i,j_1} = 0$ for $i > 2$, $\gamma_{j_1,p_1} = 0$ for $j_1 < k$ and $\beta_{2,k} = 0$, respectively.

Thus, now we have
\[
\sum_{j_1=1}^{k-1} \beta_{2,j_1} g^{(2,j_1)} + \sum_{j_1=1}^{k-1} \beta_{1,j_1} g^{(1,j_1)} + \sum_{p_1=1}^{k-1} \gamma_{k,p_1} h^{(k,p_1)} \in Id.
\]

By the monomial $y_{12}^{j_1}$ in the (1,2)-entry, we have
\[
\beta_{1,j_1} + \beta_{2,j_1} = 0
\]
for all $j_1 = 1, \ldots, k-1$, and by the monomial $y_{12}^{j_1}y_{12}^{k}$ in the (1,3)-entry we have
\[
-2\beta_{1,l} - 2\beta_{2,l} + \gamma_{k,l} = 0
\]
for all $l = 1, \ldots, k-1$. Therefore, $\gamma_{k,l} = 0$ for all $l = 1, \ldots, k-1$.

For the remaining coefficients, by the following table

| Entry | Information | Monomial | Its coefficient |
|-------|-------------|----------|-----------------|
| (1,3) | $m(g^{(2,j_1)}_{13})$ | $\pm \beta_{2,j_1}$ |
| (1,3) | $m(g^{(1,j_1)}_{13})$ | $\pm 2\beta_{1,j_1}$ |
where
\[ m(g(z_{13}^{(2,j_1)})) = y_{12}^{-1} y_{12}^{2} \quad u = y_{12}^{j_1} y_{12}^{k}, \]
we have \( \beta_{2,j_1} = 0 \) and \( \beta_{1,j_1} = 0 \), respectively.

### 3.6 Conclusion

Since \( \mathbb{F} \) is an infinite field and \( B_{MN} \cap Id = B_{MN} \cap I \) for all \( M, N \), we have the first main result of this paper.

**Theorem 3.43.** Let \( \mathbb{F} \) be an infinite field with \( \text{char}(\mathbb{F}) > 2 \). If \( \ast \) is an involution of the first kind on \( UT_3(\mathbb{F}) \) then \( \text{Id}(UT_3(\mathbb{F}), \ast) \) is the \( T(\ast) \)-ideal generated by the polynomials of Proposition \( 3.2 \).

Note that this theorem is also true when \( \text{char}(\mathbb{F}) = 0 \). See [12, Theorem 6.6].

### 4 \( \ast \)-Central Polynomials for \( UT_n(\mathbb{F}) \)

Let \( \mathbb{F} \) be a field (finite or infinite) of characteristic \( \neq 2 \). In this section we study the \( \ast \)-central polynomials for \( UT_n(\mathbb{F}) \), where \( n \geq 3 \).

Consider the involutions \( \ast \) and \( s \) in Section [2]. If \( \circ \) is an involution on \( UT_n(\mathbb{F}) \) then \( \circ \) is equivalent either to \( \ast \) or to \( s \), see Section [2]. Thus

\[ C(UT_n(\mathbb{F}), \circ) = C(UT_n(\mathbb{F}), \ast) \quad \text{or} \quad C(UT_n(\mathbb{F}), \circ) = C(UT_n(\mathbb{F}), s). \quad (14) \]

**Theorem 4.1.** If \( \circ \) is an involution on \( UT_n(\mathbb{F}) \) and \( n \geq 3 \) then

\[ C(UT_n(\mathbb{F}), \circ) = \text{Id}(UT_n(\mathbb{F}), \circ) + \mathbb{F}. \]

**Proof.** By [14] we can suppose \( \circ = \ast \) or \( \circ = s \). In this case we have that \( e_{11}^\circ = e_{nn} \).

In particular, \( A = e_{11} + e_{nn} \) and \( B = e_{11} - e_{nn} \) are symmetric and skew-symmetric elements respectively.

Since

\[ C(UT_n(\mathbb{F}), \circ) \supseteq \text{Id}(UT_n(\mathbb{F}), \circ) + \mathbb{F} \]

we shall prove the inclusion \( \subseteq \). Let \( g(y_1, \ldots, y_k, z_1, \ldots, z_s) \in C(UT_n(\mathbb{F}), \circ) \).

Write

\[ g(y_1, \ldots, y_k, z_1, \ldots, z_s) = f(y_1, \ldots, y_k, z_1, \ldots, z_s) + \lambda \]

where \( f(0,0, \ldots, 0) = 0 \) \( (f \) without constant term) and \( \lambda \in \mathbb{F} \).

**Claim 1:** \( f(y_1, \ldots, y_k, z_1, \ldots, z_s) \) is a polynomial identity for \( \mathbb{F} \).

In fact, let \( a_1, \ldots, a_k, b_1, \ldots, b_s \in \mathbb{F} \). Write

\[ f(a_1 A, \ldots, a_k A, b_1 B, \ldots, b_s B) = \sum \alpha_{ij} e_{ij}. \]

Since \( \alpha_{11} = f(a_1, \ldots, a_k, b_1, \ldots, b_s), \ \alpha_{22} = 0 \) and \( f(y_1, \ldots, y_k, z_1, \ldots, z_s) \in C(UT_n(\mathbb{F}), \circ) \) it follows that \( \alpha_{11} = \alpha_{22} = 0 \) as desired.
Claim 2: \( f(y_1, \ldots, y_k, z_1, \ldots, z_s) \in Id(UT_n(F), \circ). \)

Let \( A_1, \ldots, A_k \in UT_n(F)^+ \) and \( B_1, \ldots, B_s \in UT_n(F)^- \) where
\[
A_l = \sum a_{ij}^l e_{ij} \quad \text{and} \quad B_l = \sum b_{ij}^l e_{ij}.
\]

Write
\[
f(A_1, \ldots, A_k, B_1, \ldots, B_s) = \sum \alpha_{ij} e_{ij}.
\]

Since \( f(y_1, \ldots, y_k, z_1, \ldots, z_s) \in C(UT_n(F), \circ) \) it follows that
\[
f(A_1, \ldots, A_k, B_1, \ldots, B_s) = \sum_{i=1}^{n} \alpha e_{ii},
\]
where \( \alpha = \alpha_{11} = \ldots = \alpha_{nn} \). Since \( \alpha_{11} = f(a_{11}^1, \ldots, a_{11}^k, b_{11}^1, \ldots, b_{11}^s) \), by Claim 1 we have \( \alpha = 0 \) as desired.

By Claim 2 we have \( g(y_1, \ldots, y_k, z_1, \ldots, z_s) \in Id(UT_n(F), \circ) + F. \) \( \square \)

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