On Stability of a Class of Filters for Non-Linear Stochastic Systems

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Abstract—This article considers stability properties of a broad class of commonly used filters, including the extended and unscented Kalman filters, for discrete and continuous-time stochastic dynamic systems with non-linear state dynamics and linear measurements. We show that contractivity of the filtering error process and boundedness of the error covariance matrix induce stability of the filter. The results are in the form of time-uniform mean square bounds and exponential concentration inequalities for the filtering error. As is typical of stability analysis of filters for non-linear systems, the assumptions required are stringent. However, in contrast to much of the previous analysis, we provide a number of example of model classes for which these assumptions can be verified a priori. Typical requirements are a contractive drift or sufficient inflation of the error covariance matrix and fully observed state. Numerical experiments using synthetic data are used to validate the derived error bounds.

I. INTRODUCTION

The extended Kalman filter (EKF) and other extensions of the classical Kalman filter, such as the unscented Kalman filter (UKF), are fundamental tools widely used in automatic control, robotics, and signal processing [1]. For linear systems, these filters reduce to the optimal linear Kalman filter whose stability properties are well understood, having been extensively studied since the 1960s in continuous [2]–[4] and discrete-time [5]–[7] settings. However, most systems of interest are non-linear, and in this case the aforementioned extensions of the Kalman filter inherit no global optimality properties. Even though these filters typically provide useful estimates in practice, analysing their stability or convergence is far from trivial.

This article analyses stochastic stability of a broad class of extensions of the Kalman filter for systems with non-linear dynamics and linear measurements. The focus is on the continuous-time setting, but we also briefly cover the discrete case. Our main results, Theorems III.1 and IV.3, are inspired by recent work by Del Moral et al. [8] on the EKF and provide time-uniform bounds on the mean square filtering error and related exponential concentration inequalities. Two principles distinguish our analysis and results from much of the previous work on the topic: (i) the analysis is unified in that the class of (generic) filters we consider contains most commonly used non-linear Kalman filters and (ii) we emphasise that error bounds should be a priori computable (i.e., before the filter is run) and accordingly review three model classes for which this is possible in Section V. A more detailed presentation of the contributions of this article is given in Section I-B after a review of literature and technical problems posed by the stability analysis.

A. Literature Review and Technical Aspects

A Kalman filter\(^1\) or its non-linear extension provides, at time \(t \geq 0\), a real-time estimate \(\hat{X}_t\) of the true latent state \(X_t\) of a dynamic system based on partial and noisy measurement sequence \(\{Y_t\}_{t=0}^T\). The state estimate is typically accompanied with a positive-semidefinite matrix \(P_t\) that is an estimate of covariance of the estimation error \(E_t = X_t - \hat{X}_t\). This error covariance matrix and the associated gain matrix \(K_t\) are computed from a Riccati-type differential equation.

Stability of extensions of the Kalman filter for non-linear systems can be analysed either in the deterministic or stochastic setting. In the former case, both the state dynamics and measurements are noiseless and the positive-semidefinite matrices \(Q\) and \(R\), that in the stochastic case would be covariances of Gaussian state and measurement noise terms, respectively, are viewed as tuning parameters. The goal is then to prove that the estimation error converges to zero as \(t \to \infty\). In the stochastic setting it cannot be expected that the error vanishes, and one instead (for example) attempts to find time-uniform upper bounds on mean square estimation error \(E[\|E_t\|^2]\). In both settings, the main difficulties one encounters when trying to carry out stability analysis are very similar:

\(a)\) Uniform boundedness of \(P_t\): In the linear setting, asymptotic stability of the Kalman filter is obtained in [5], [6] by employing the Lyapunov function \(V_t = E_t^T P_t^{-1} E_t\). Once stability results have been obtained for \(V_t\), time-uniform bounds on the error covariance \(P_t\) are needed for concluding that the filter itself is stable. While in the linear case \(P_t\) does not depend on the estimates \(\hat{X}_t\), bounds on this matrix following from standard results on Riccati equations under certain observability and controllability conditions, in the non-linear case the local structure, arising from linearisations of some sort around the estimated trajectory, of most Kalman filters introduces dependencies—and, consequently, stochasticity—of \(P_t\) on the estimates whose behaviour is difficult, if not impossible, to anticipate and control. As a result, most articles dealing with stability of non-linear Kalman filters rely on assumptions about uniform boundedness of \(P_t\); see [9]–[16] for the EKF as a non-linear observer, [17]–[20] for the stochastic EKF, and [21]–[27] for the UKF and other related filters. Such assumptions are highly non-trivial, and are in practice hardly guaranteed

\(1\) Later on, when we want to refer specifically to continuous-time Kalman filters, we use the term Kalman–Bucy filter.

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to hold, since the estimate $\hat{X}_t$ depends on the Kalman gain $K_t$, which is a function of the error covariance $P_t$, which in turn again depends on the estimate, so that in principle some positive feedback phenomenon may trigger error amplification. In the recent article [28] this assumption is relaxed, but the class of systems considered is such that the dependency of the Riccati equation on the estimated trajectory is in fact much reduced.

b) Non-linearity: If the system is non-linear, stability analysis of a Kalman filter necessarily involves analysing non-linear (stochastic) differential equations. This is obviously much more involved than analysis of linear differential equations. As such, the approach taken in many articles is to assume that the error associated to the linearisation method used in a particular non-linear Kalman filter is “small”. This allows for deriving a linear differential inequality for the Lyapunov function that can be easily analysed. In [17], [18] this is via certain Lipschitz-type bounds on the remainder and in [21], [25], [26] via assumptions on boundedness of some residual-correcting random matrices. However, these assumptions are sometimes difficult to verify beforehand and potentially very restrictive.

The explicit assumption on boundedness of $P_t$ has been addressed essentially in two ways. If the system is fully observed, that is, $dY_t = X_t \, dt + R^{1/2} \, dV_t$, there is hope for the Riccati equation to be well behaved despite the fact it depends on $\hat{X}_t$, since, essentially, the quadratic correction term in the Riccati equation will prevent $\hat{X}_t$ and $P_t$ from drifting indefinitely; see [20], Section IV and [26], Section 4 for the discrete and [29] for the continuous-time case. Alternatively, one can consider certain difficult-to-verify non-linear extensions of the standard observability and controllability conditions [14], [17], [18], [30]. Another situation of interest is when the estimates are explicitly known to remain in a bounded region of the state space, so that one recovers some control over the estimate-dependent terms in the Riccati equation, which in turn limits the possible values of $P_t$. For example, both of these conditions are leveraged in [31] to prove stochastic stability of the EKF for a robotics application.

B. Contributions

Recently, Del Moral et al. [8] studied stochastic stability of the extended Kalman–Bucy filter using a slightly different approach to the ones cited above. Namely, they directly consider the squared error $\|E_t\|^2$ instead of a Lyapunov function involving inverse of the error covariance and based their analysis on logarithmic norm inequalities. The class of systems they consider is rather restrictive as they assume the state to be fully observed and, moreover, require that the dynamical model specified by the drift function $f$ itself is exponentially stable (i.e., the homogeneous system $\partial_t x_t = f(x_t)$ is exponentially stable).

In this article we pursue stability along the lines of [8], but the results are generalized in various directions:

1) We consider a broad class, defined in Section II-B, of generic Kalman-type filters for continuous-time non-linear systems when the measurements are linear; see (5) for the model. As discussed in Section II-C, this class includes most commonly used filters, including the extended Kalman–Bucy filter and the more recent Gaussian integration filters such as the extended Kalman–Bucy filter [32], [33] and the Gauss–Hermite filter [34], [35]. The unified framework developed in this article is exceedingly convenient as every filter needs not be analysed individually and has been the case previously.

2) The system is not explicitly required to be exponentially stable nor fully observed for the stability analysis in Section III. However, in practice validation of our assumptions often requires exponential stability or full observability. Example classes of models are presented in Section V.

3) Although our main focus is on continuous-time systems, Section IV contains analogous results for the discrete-time case.

The stability results that appear in Theorems III.1 and IV.3 come in the form of a time-uniform upper bounds on the mean square filtering error and an associated exponential concentration inequalities. These results are validated numerically for two example models in Section VI.

Our focus is on finding conditions that do not involve boundedness of the state and from which stability of the filter can be concluded before the filter is run. This is in some contrast to earlier work where it is occasionally suggested that (a) having values of certain parameters, as computed when the filter is run, satisfy the bounds or conditions required for stability allows for concluding that a stochastic filter is stable (see, e.g., [17, p. 716] and [25, p. 244]), which is problematic if one is considering stability in mean square sense since the conditions are validated only for this particular trajectory, though more acceptable in the deterministic observer setting [16, p. 566] or (b) the true state can be assumed to remain in a compact set (see [17, Theorem 4.1] and [31]). A particular consequence of this is that we end up working only with linear measurement models. However, it should be noted that, out of necessity, many models that have been previously used in demonstrating stability results indeed have linear measurements; see for example the model examined in [17], [21].

C. Some Preliminaries

This section introduces some notation and basic results that are used throughout the article.

1) Logarithmic Norm and Matrix Inequalities: The smallest and largest eigenvalue of a symmetric real matrix $A$ are denoted $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$, respectively. The logarithmic norm $\mu(A)$ of a square matrix $A \in \mathbb{R}^{d \times d}$ is

$$\mu(A) := \frac{1}{2} \lambda_{\max}(A + A^T),$$

coinciding with $\lambda_{\max}(A)$ when $A$ is symmetric. We also define the quantity

$$\nu(A) := \frac{1}{2} \lambda_{\min}(A + A^T) = - \mu(-A).$$

Basic results that will be repeatedly used are

$$\nu(A) \|x\|^2 \leq \langle x, Ax \rangle = x^T Ax \leq \mu(A) \|x\|^2$$
for any \(x \in \mathbb{R}^d\) and the “triangle inequalities”
\[
\lambda_{\text{max}}(A + B) \leq \lambda_{\text{max}}(A) + \lambda_{\text{max}}(B),
\]
\[
\lambda_{\text{min}}(A + B) \geq \lambda_{\text{min}}(A) + \lambda_{\text{min}}(B)
\]
and
\[
\mu(A + B) \leq \mu(A) + \mu(B),
\]
\[
\nu(A + B) \geq \nu(A) + \nu(B).
\]

For a positive-semidefinite \(B\), recall also the trace inequality [36, Chapter 8]
\[
\nu(A) \text{ tr}(B) \leq \text{tr}(AB) \leq \mu(A) \text{ tr}(B)
\]
for any square matrix \(A\) and its special case
\[
\lambda_{\text{min}}(A) \text{ tr}(B) \leq \text{tr}(AB) \leq \lambda_{\text{max}}(A) \text{ tr}(B)
\]
for a symmetric \(A\). See [37, 38] for more detailed expositions of properties of the logarithmic norm.

2) **Lipschitz Constants:** The Lipschitz constant of a differentiable function \(g: \mathbb{R}^d \to \mathbb{R}^d\), having the Jacobian matrix
\[
[J_g]_{ij} := \partial g_i / \partial z_j,
\]
is
\[
\|J_g\| := \sup_{x \in \mathbb{R}^d} \|J_g(x)\|
\]
where the matrix norm is the norm induced by the Euclidean norm (i.e., the spectral norm). This constant satisfies
\[
\|g(x) - g(x')\| \leq \|J_g\| \|x - x'\|
\]
for any \(x, x' \in \mathbb{R}^d\). If \(\|J_g\| < \infty\), the function \(g\) is said to be Lipschitz. The logarithmic Lipschitz constants of \(g\) are
\[
N(g) := \inf_{x \in \mathbb{R}^d} \nu[J_g(x)] \quad \text{and} \quad M(g) := \sup_{x \in \mathbb{R}^d} \mu[J_g(x)],
\]
and they satisfy
\[
N(g) \|x - x'\|^2 \leq \langle x - x', g(x) - g(x') \rangle \leq M(g) \|x - x'\|^2,
\]
for any \(x, x' \in \mathbb{R}^d\). Note that \(M(g) \leq \|J_g\|\) [38, Proposition 3.1].

3) **Grönwall’s Inequality:** The classical Grönwall inequality is a basic ingredient in most of our proofs. Its discrete counterpart is introduced in conjunction with other discrete-time results in Section IV.

**Theorem I.1** (Continuous-time Grönwall’s inequality). Suppose that \(\beta_t\) is a continuous real-valued function of \(t \in \mathbb{R}\) and \(x_t\) is continuously differentiable on \(\mathbb{R}_+\) and satisfies the differential inequality
\[
\dot{x}_t \leq \alpha x_t + \beta_t, \quad t \geq 0,
\]
for some constant \(\alpha\). Then
\[
x_t \leq x_0 e^{\alpha t} + \int_0^t e^{\alpha(t-s)} \beta_s \, ds
\]
for every \(t \geq 0\). If \(\beta_t \equiv \beta\), (3) reduces to
\[
x_t \leq x_0 e^{\alpha t} + \frac{1 - e^{\alpha t}}{\alpha} \beta.
\]

The form of (4) that we need the most is the one where \(\beta_t \equiv \beta \geq 0\) and \(\alpha = -\gamma\) for \(\gamma > 0\). Then the inequality takes the form
\[
x_t \leq x_0 e^{-\gamma t} + \beta / \gamma.
\]

**II. NON-LINEAR SYSTEMS AND FILTERING**

This section introduces the continuous-time stochastic dynamic system with non-linear state dynamics and linear measurements this article is concerned with, as well as the class of stochastic Kalman–Bucy filters our results apply to. A number of prominent members of this class are also given. Discrete-time systems and filters are introduced in Section IV.

**A. The Non-Linear System**

The models that we consider are in the form of a system of stochastic differential equations
\[
dX_t = f(X_t) \, dt + Q^{1/2} \, dW_t, \quad (5a)
\]
\[
dY_t = HX_t \, dt + R^{1/2} \, dV_t, \quad (5b)
\]
where \(X_t \in \mathbb{R}^{d_x}\) is the latent state that evolves according to the continuously differentiable and potentially non-linear drift function \(f: \mathbb{R}^{d_x} \to \mathbb{R}^{d_x}\). Throughout the article, it is assumed that the drift is Lipschitz (i.e., \(\|J_f\| < \infty\)) and that its Jacobian is bounded in logarithmic norm:
\[
-\infty < N(f) = \inf_{x \in \mathbb{R}^{d_x}} \nu[J_f(x)], \quad (6a)
\]
\[
M(f) = \sup_{x \in \mathbb{R}^{d_x}} \mu[J_f(x)] < \infty. \quad (6b)
\]

These conditions ensure that the state and the filters defined in Section II-B remain almost surely bounded in finite time. The measurements \(Y_t \in \mathbb{R}^{d_y}\) are obtained linearly through the measurement model matrix \(H \in \mathbb{R}^{d_y \times d_x}\). Both the state and measurements are disturbed by independent Brownian motions \(W_t\) and \(V_t\) with positive-definite covariance matrices \(Q \in \mathbb{R}^{d_x \times d_x}\) and \(R \in \mathbb{R}^{d_y \times d_y}\). The state is initialised from \(X_0 \sim N(x_0, P_0)\) for some mean \(x_0 \in \mathbb{R}^{d_x}\) and a positive-definite covariance \(P_0 \in \mathbb{R}^{d_x \times d_x}\).

The results of this article remain valid if the time-invariant function \(f\) and matrices \(H, Q,\) and \(R\) in (5) are replaced with time-varying versions that satisfy appropriate regularity and uniform boundedness conditions. As nothing fundamentally different would occur in this case, we work with the time-invariant case so as to avoid cumbersome notation.

**B. A Class of Generic Filters for Non-Linear Systems**

Consider a positive linear functional \(L_{x,P}\) with parameters \(x \in \mathbb{R}^{d_x}\) and \(P \in \mathbb{R}^{d_x \times d_x}\) that

(i) is applied elementwise to vector and matrix valued functions;

(ii) is Lipschitz (and hence continuous) in the parameters \(x\) and \(P\) in the sense that \(L_{x,P}(g)\) is a Lipschitz function from \(\mathbb{R}^{d_x \times d_x}\) to \(\mathbb{R}\) for any fixed Lipschitz function \(g: \mathbb{R}^{d_x} \to \mathbb{R}^{d_x}\);

(iii) satisfies \(L_{x,P}(A) = Ax\) and \(L_{x,P}(J_A) = A\) for any matrix \(A \in \mathbb{R}^{d_x \times d_x}\) when interpreted as a linear function \(A: \mathbb{R}^{d_x} \to \mathbb{R}^{d_x}\).

All these conditions are usually easily verifiable and non-restrictive. The following assumption on the linear functional will be crucial to the stability analysis in Section III.
Assumption II.1. Assume that for every differentiable $g: \mathbb{R}^{d_z} \to \mathbb{R}^{d_z}$ with finite $N(g)$ and $M(g)$ there is a constant $C_g \geq 0$, varying continuously with $M(g)$ and $N(g)$, such that

$$\langle x - \hat{x}, g(x) - L_{\hat{x}, P}(g) \rangle \leq M(g) \|x - \hat{x}\|^2 + C_g \text{tr}(P)$$

for any $x, \hat{x} \in \mathbb{R}^{d_z}$ and $P \in \mathbb{R}^{d_z \times d_z}$.

Since $\langle x - \hat{x}, g(x) - g(\hat{x}) \rangle \leq M(g) \|x - \hat{x}\|^2$ by (2), what the above assumption essentially entails is that $L_{\hat{x}, P}(g)$ cannot deviate too much from $g(\hat{x})$ and that magnitude of their difference is controlled by the size of $P$.

A filter computes a quantity $\tilde{X}_t \in \mathbb{R}^{d_z}$ that is used as an estimate of the latent state $X_t$. We consider generic filters defined as

$$d\tilde{X}_t = L_{\tilde{X}_t, P_t}(f) \, dt + P_tH^T R^{-1} (dY_t - H \tilde{X}_t \, dt),$$

(7)

where the time-varying matrices $P_t \in \mathbb{R}^{d_z \times d_z}$ are user-specified and allowed to depend on all the system parameters as well as all preceding measurements and state estimates. Lipschitzianity in $x$ and $P$ of $L_{\hat{x}, P}(f)$ guarantees that there exists a unique solution to (7). The matrices

$$K_t := P_t H^T R^{-1} \quad \text{and} \quad S := H^T R^{-1} H,$$

(8)

the former of which are known as Kalman gain matrices, will be useful later. In this article, the filter (7) is initialised with $\tilde{X}_0 = x_0 = E(X_0)$ and $P_0 = \text{Var}(X_0)$. That is, we do not consider potentially erroneous initial conditions as is done in, for example, [18]. In Section III we will see that, as long as they remain uniformly bounded, the construction of the matrices $P_t$ does not substantially affect our analysis. This is a vestige of the rather crude proof technique that we need to employ.

The class of filters of the form (7) that use a linear functional $\langle \cdot, \cdot \rangle$ does not substantially affect our analysis. This is because (2), what the above assumption essentially entails is that $L_{\hat{x}, P}(g)$ cannot deviate too much from $g(\hat{x})$ and that magnitude of their difference is controlled by the size of $P$.

3) Kalman–Bucy Filters for Continuous-Time Non-Linear Systems

A Kalman–Bucy filter for the model (5) computes approximations $\tilde{X}_t$ and $P_t$, latter of which goes by the name of error covariance in this setting, to the conditional filtering means and covariances $E(X_t \mid F_t)$ and $\text{Var}(X_t \mid F_t)$, respectively, where $F_t = \sigma(Y_s, s \leq t)$ is the $\sigma$-algebra generated by the measurements. These quantities are in general intractable unless $f$ is an affine function. A generic Kalman–Bucy filter for the non-linear model (5) is

$$d\tilde{X}_t = L_{\tilde{X}_t, P_t}(f) \, dt + P_tH^T R^{-1} (dY_t - H \tilde{X}_t \, dt),$$

(9a)

$$\partial_t P_t = L_{\tilde{X}_t, P_t}(f) + L_{\tilde{X}_t, P_t}(f)^T + Q_u - P_tSP_t, \quad \text{(9b)}$$

where $L_{\tilde{X}_t, P_t}(f)$ maps functions to $d_z \times d_z$ matrices.\footnote{The existence of a unique solution to (9b) is guaranteed if $f$ is Lipschitz in $x$ and $P$. This holds typically when the derivatives of $f$ satisfy a Lipschitz condition: $\|f'(x) - f'(x')\| \leq L \|x - x'\|$ for some $L < \infty$ and all $x, x' \in \mathbb{R}^{d_x}$.}

Equation (9b) governing evolution of $P_t$ is known as the (non-linear) Riccati equation. The matrix $Q_u$ is some positive-definite matrix that does not have to equal to $Q$, the state noise covariance, in which case we can speak of tuning this matrix [39]. As we will see later, proper tuning (in practice, inflation) is often necessary to induce (provable) stability of a Kalman–Bucy filter. The above formulation in terms of the linear functionals $L_{\tilde{X}, P}$ and $L_{\tilde{X}, P}$ is still somewhat unconventional. Next we provide a few examples of classical Kalman–Bucy filters of the above form that satisfy the assumptions in Section II-B.

1) The Extended Kalman–Bucy Filter: The classical extended Kalman–Bucy filter (EKF) is based on local first-order linearisations around the estimated state $\tilde{X}_t$. For this purpose, select $L_{\tilde{X}, P}(g) = g(x)$ and $L_{\tilde{X}, P}(f) = J_f(x)g(x)$ so that the generic Kalman–Bucy filtering equations (9) become

$$d\tilde{X}_t = f(\tilde{X}_t) \, dt + P_tH^T R^{-1} (dY_t - H \tilde{X}_t \, dt), \quad (10a)$$

$$\partial_t P_t = J_f(\tilde{X}_t)P_t + P_tJ_f(\tilde{X}_t)^T + Q_u - P_tSP_t, \quad (10b)$$

Assumption II.1 holds with $C_g = 0$ since, by (2),

$$\langle x - \tilde{x}, g(x) - L_{\tilde{x}, P}(g) \rangle = \langle x - \tilde{x}, g(x) - g(\tilde{x}) \rangle \leq M(g) \|x - \tilde{x}\|^2$$

for any $x, \tilde{x} \in \mathbb{R}^{d_z}$.

2) Gaussian Assumed Density Filters: In Gaussian assumed density filters [40], the point evaluations of the model functions and their Jacobians in the EKF are replaced with Gaussian expectations with mean $\tilde{X}_t$ and variance $P_t$. That is,

$$L_{\tilde{X}, P}(g) = \mathbb{E}_{N(\tilde{X}_t, P_t)}(g) := \int_{\mathbb{R}^{d_z}} g(z) N(z \mid x, P) \, dz,$n

$$L_{\tilde{X}, P}(f) = \mathbb{E}_{N(\tilde{X}_t, P_t)}(J_f)\mathbb{E}_{N(\tilde{X}_t, P_t)}(f).$$

The filtering equations (9) are

$$d\tilde{X}_t = \mathbb{E}_{N(\tilde{X}_t, P_t)}(f) \, dt$$

$$+ P_tH^T R^{-1} (dY_t - H \tilde{X}_t \, dt), \quad (11a)$$

$$\partial_t P_t = \mathbb{E}_{N(\tilde{X}_t, P_t)}(J_f)P_t + P_tJ_f(\tilde{X}_t)^T + Q_u - P_tSP_t. \quad (11b)$$

All the properties required of $L_{\tilde{X}, P}$ in Section II-B hold. That Assumption II.1 holds is slightly more complicated. For any differentiable $g: \mathbb{R}^{d_x} \to \mathbb{R}^{d_x}$, we have

$$\langle x - \tilde{x}, g(x) - L_{\tilde{x}, P}(g) \rangle = \langle x - \tilde{x}, g(x) - \mathbb{E}_{N(\tilde{X}_t, P_t)}(g) \rangle$$

$$= \int_{\mathbb{R}^{d_x}} \langle x - z, g(x) - g(z) \rangle N(z \mid \tilde{x}, P) \, dz$$

$$= \int_{\mathbb{R}^{d_x}} \langle x - z, g(x) - g(z) \rangle N(z \mid \tilde{x}, P) \, dz$$

$$- \int_{\mathbb{R}^{d_x}} \langle z - \tilde{x}, g(x) \rangle N(z \mid \tilde{x}, P) \, dz.$$
The first term can be bounded as
\[
\int_{\mathbb{R}^d_e} \langle x - z, g(x) - g(z) \rangle N(z \mid \hat{x}, P) \, dz \\
\leq M(g) \int_{\mathbb{R}^d_e} \|x - z\|^2 N(z \mid \hat{x}, P) \, dz \\
= M(g) \left( \int_{\mathbb{R}^d_e} (\|z - \tilde{x}\|^2 + \|x - \tilde{x}\|^2) \, N(z \mid \hat{x}, P) \, dz \right) \\
= M(g) [\|x - \tilde{x}\|^2 + \text{tr}(P)],
\]
whereas the second has the bound
\[
- \int_{\mathbb{R}^d_e} \langle z - \tilde{x}, g(z) \rangle N(z \mid \hat{x}, P) \, dz \\
= - \int_{\mathbb{R}^d_e} \langle z - \tilde{x}, g(z) - g(\tilde{x}) \rangle N(z \mid \hat{x}, P) \, dz \\
\leq - N(g) \int_{\mathbb{R}^d_e} \|z - \tilde{x}\|^2 N(z \mid \hat{x}, P) \, dz \\
= - N(g) \text{tr}(P).
\]
Combination of these estimates shows that Assumption II.1 holds with \( C_g = M(g) - N(g) \geq 0. \)

3) Gaussian Integration Filters and the Unscented Kalman–Bucy Filter: In practice, the Gaussian expectations required in implementation of the Gaussian assumed density filter (11) are typically unavailable in closed form, necessitating the use of numerical integration formulas. We call such filters Gaussian integration filters. Popular alternatives include fully symmetric formulas, such as the ubiquitous unscented transform [33], [41], and Gaussian tensor-product rules [34], [35].

A Gaussian integration filter replaces the Gaussian expectations occurring in the Gaussian assumed density filter (11) with the numerical cubature approximations
\[
L_{x,P}(g) = \sum_{i=1}^{n} w_i g(x + \sqrt{P} \xi_i) \approx \mathbb{E}_{N(x,P)}(g), \tag{12}
\]
where \( \xi_1, \ldots, \xi_n \in \mathbb{R}^d_s \) and \( w_1, \ldots, w_n \in \mathbb{R} \) be user-specified unit sigma-points and weights, respectively, and \( \sqrt{P} \) is some form of symmetric matrix square root. Usually the Jacobian in the Riccati equation (11b) is not integrated directly. Instead, one computes an approximation to the right-hand side of Stein’s identity
\[
\mathbb{E}_{N(x,P)}(J_g) P = \int_{\mathbb{R}^d_e} g(z) (z - x)^T N(z \mid x, P) \, dz.
\]
That is,
\[
L_{x,P}^{\text{Ric}}(g) = \sum_{i=1}^{n} w_i g(x + \sqrt{P} \xi_i) \xi_i^T \sqrt{P}
\]
is used. Obviously, it is not necessary to use the same numerical integration scheme in \( L_{x,P} \) and \( L_{x,P}^{\text{Ric}} \). The resulting filter takes the form
\[
d\hat{X}_t = \sum_{i=1}^{n} w_i f(\hat{X}_t + \sqrt{P} \xi_i) \, dt \\
+ P_t H^T R^{-1} [dY_t - H \hat{X}_t] dt, \\
\partial_t P_t = \sum_{i=1}^{n} w_i \left( \sqrt{P} \xi_i f(\hat{X}_t + \sqrt{P} \xi_i)^T + Q_w - P_t S P_t \right) \\
+ \sum_{i=1}^{n} w_i \sqrt{P} \xi_i f(\hat{X}_t + \sqrt{P} \xi_i)^T \\
\text{To show that Assumption II.1 holds, assume that the weights are non-negative and}
\[
L_{x,P}(p) = L_{N(x,P)}(p) \tag{13}
\]
whenever \( p : \mathbb{R}^d \to \mathbb{R} \) is a \( d_x \)-variate polynomial of total degree at most two. In particular, this implies that \( \sum_{i=1}^{n} w_i = 1 \) and \( \sum_{i=1}^{n} w_i \sqrt{P} \xi_i = 0 \) since \( L_{x,P} \) is exact for the constant function and the polynomials \( p(x) = x_i \) for every \( i = 1, \ldots, d_x \).

Among many others, (13) is satisfied by the aforementioned Kalman–Bucy filters based on the unscented transform and Gaussian tensor-product rules. Under the above assumptions we can proceed analogously to Section II-C2:
\[
\langle x - \tilde{x}, g(x) - L_{\tilde{x},P}(g) \rangle \\
= \left\langle x - \tilde{x}, g(x) - \sum_{i=1}^{n} w_i g(\tilde{x} + \sqrt{P} \xi_i) \right\rangle \\
= \sum_{i=1}^{n} w_i \left( \langle x - (\tilde{x} + \sqrt{P} \xi_i), g(x) - g(\tilde{x} + \sqrt{P} \xi_i) \rangle \\
+ \langle \sqrt{P} \xi_i, g(x) - g(\tilde{x} + \sqrt{P} \xi_i) \rangle \right)
\]
Hence
\[
\langle x - \tilde{x}, g(x) - L_{\tilde{x},P}(g) \rangle \\
\leq M(g) \sum_{i=1}^{n} w_i \|x - (\tilde{x} + \sqrt{P} \xi_i)\|^2 \\
+ \sum_{i=1}^{n} w_i \left( \sqrt{P} \xi_i, g(x) - g(\tilde{x} + \sqrt{P} \xi_i) \right).
\]
The first term is a sigma-point approximation of a quadratic function. Using (13) and proceeding as in Section II-C2,
\[
\sum_{i=1}^{n} w_i \|x - (\tilde{x} + \sqrt{P} \xi_i)\|^2 = \int_{\mathbb{R}^d_e} \|x - z\|^2 N(z \mid \hat{x}, P) \, dz \\
= \|x - \hat{x}\|^2 + \text{tr}(P).
To bound the second term, notice that
\[
\sum_{i=1}^{n} u_i \left( \sqrt{n} \xi_i, g(x) - g(x) \right) = - \sum_{i=1}^{n} u_i \left( \hat{x} - (x + \sqrt{n} \xi_i), g(x) - g(x) \right) \leq -N(n) \sum_{i=1}^{n} \left\| \sqrt{n} \xi_i \right\|^2 = -N(n) \text{tr}(P)
\]
by exactness of $L_{x,p}$ for quadratic polynomials. Assumption II.1 thus holds with the same constant $C_g = M(g) - N(g)$ as in Section II-C2.

As remarked before, the above analysis covers most Gaussian numerical integration filters in use due to them being designed to integrate exactly polynomials of low degree. However, the analysis fails for some numerical integration filters. These include, for example, kernel-based Gaussian process cubature filters [42], [43].

III. STABILITY OF KALMAN–BUCY FILTERS

This section contains the main theoretical result of this article. This result, Theorem III.1, consists of an upper bound on the mean square filtering error and an exponential concentration for the error. Discrete-time filters and their error behaviour are discussed in Section IV and examples of models that satisfy our assumptions are given in Section V. Exponential concentration inequalities similar to those of Theorem III.1 have appeared in [8] for the extended Kalman–Bucy filter and in [44] for the ensemble Kalman–Bucy filter. See also [4], [45] for work regarding the linear case and [46] for analysis, somewhat similar to ours, for ensemble Kalman–Bucy filters.

A. Main Result

Theorem III.1 below is based on the evolution equation
\[
dE_t = \left[ f(x_t) - L_{\hat{x}_t, P_t} f - P_t S(x_t - \hat{x}_t) \right] dt + Q^{1/2} dW_t - P_t H^T R^{-1/2} dV_t
\]
for the filtering error $E_t = x_t - \hat{x}_t$ of the generic filter (7). This equation is easily derived by differentiating $E_t$, inserting the expressions for $dX_t$ and $dY_t$ from (5) and for $d\hat{X}_t$ from (7) into the resulting stochastic differential equation, and recalling the definition $S = H^T R^{-1/2} H$.

**Theorem III.1.** Consider the generic filter (7) for the continuous-time model (5) and let $L_{x,p}$ satisfy Assumption I.I.1. Suppose that there are positive constants $\lambda_P$ and $\lambda$ and time $T \geq 0$ such that $\sup_{t \geq 0} \text{tr}(P_t) \leq \lambda_P$ and
\[
M(f - P_t S) = \sup_{x \in \mathbb{R}^d} \mu(J_f(x) - P_t S) \leq -\lambda < 0
\]
holds for every $t \geq T$ almost surely. Denote
\[
\beta(\delta) = e^{\sqrt{2\delta + \delta}}.
\]
Then there are non-negative constants $C_\lambda$ (continuously dependent on $\lambda$, $M(f)$, $N(f)$, tr($S$), and $\lambda_P$) and $C_T$ such that, for any $t \geq T$ and $\delta > 0$, we have the exponential concentration inequality
\[
\|E_t\|^2 \geq \left( C_T e^{-2\lambda(t-T)} + \frac{\text{tr}(Q) + 2C\lambda\lambda_P + \text{tr}(S)\lambda_P^2}{2\lambda} \right) \beta(\delta)
\]
with probability smaller than $e^{-\delta}$ and the mean square filtering error bound
\[
E(\|E_t\|^2) \leq E(\|E_T\|^2) e^{-2\lambda(t-T)} + \frac{\text{tr}(Q) + 2C\lambda\lambda_P + \text{tr}(S)\lambda_P^2}{2\lambda}
\]
Proof. Only the proof of (16) is presented here; that of the concentration inequality (15) is contained in Appendix B. The filtering error $E_t = x_t - \hat{x}_t$ evolves according to
\[
dE_t = \left[ f(x_t) - L_{\hat{x}_t, P_t} f - P_t S(x_t - \hat{x}_t) \right] dt + Q^{1/2} dW_t - P_t H^T R^{-1/2} dV_t.
\]
Itô’s lemma yields
\[
d\|E_t\|^2 = dM_t + \left[ \text{tr}(Q) + \text{tr}(S P_t^2) \right] dt + 2(\langle f(x_t) - L_{\hat{x}_t, P_t} f - P_t S(x_t - \hat{x}_t), x_t - \hat{x}_t \rangle dt,\]
where
\[
dM_t = 2(\langle Q^{1/2} dW_t - P_t H^T R^{-1/2} dV_t, x_t - \hat{x}_t \rangle
\]
is a zero-mean (local) martingale. Keeping in mind that $L_{\hat{x}_t, P_t} (A) = A \hat{x}_t$ for any $A \in \mathbb{R}^{d_t \times d_x}$, we write
\[
f(x_t) - L_{\hat{x}_t, P_t} f - P_t S(x_t - \hat{x}_t) = f(x_t) - P_t S x_t - L_{\hat{x}_t, P_t} (f - P_t S)
\]
and can apply Assumption II.1 to the function $g := f - P_t S$ with $x = X_t$ and $\hat{x} = \hat{X}_t$, obtaining
\[
\langle f(x_t) - L_{\hat{x}_t, P_t} f - P_t S(x_t - \hat{x}_t), x_t - \hat{x}_t \rangle dt \leq -\lambda \|x_t - \hat{x}_t\|^2 + C_\lambda \text{tr}(P_t),
\]
where $C_\lambda \geq 0$ is finite because $M(f - P_t S) \leq -\lambda$ and
\[
N(f - P_t S) \geq N(f) + \nu(-P_t S) = N(f) - \mu(P_t S) \geq N(f) - \|P_t \| \|S\| \geq N(f) - \text{tr}(S) \lambda_P,
\]
which is finite by (6) and the assumption $\sup_{t \geq 0} \text{tr}(P_t) \leq \lambda_P$. For $t \geq T$, the assumption (14), together with (2), produces the almost sure bound
\[
d\|E_t\|^2 \leq -2\lambda \|E_t\|^2 dt + u dt + dM_t,
\]
where $u = \text{tr}(Q) + 2C\lambda\lambda_P + \text{tr}(S)\lambda_P^2$. Taking expectations and using Grönewall’s inequality then yield the claimed mean square bound
\[
E(\|E_t\|^2) \leq E(\|E_T\|^2) e^{-2\lambda(t-T)} + \frac{u}{2\lambda},
\]
with $E(\|E_T\|^2)$ being finite due to finiteness of $N(f)$ and $M(f)$. Taking the analysis further, we obtain the exponential concentration inequality (15) by deriving bounds on $E(\|E_t\|^2)$ for every $n \geq 1$ and using the Bernstein inequality (Theorem A.1). Details can be found in Appendices A and B.
Note that the above proof (or the one in Appendix B) makes no use of properties of a Riccati-type equation (9b) that is usually used to determine $P_t$. Any $P_t$ that has a uniformly bounded trace and that satisfies (14) corresponds to a generic filter to which Theorem III.1 is applicable. Section V contains a number of examples on how such a sequence of matrices can be constructed.

**Remark III.2.** In the generic filter (7), the true noise covariance matrix $R$ could be replaced with a tuned version $R_u$ as $Q$ is replaced with $Q_u$ in the Riccati equation for the Kalman–Bucy filter (9). The only modification needed in Theorem III.1 and its proof would be replacement of $S$ with $S_u := H^T R_u^{-1} H$.

**Remark III.3.** For the EKF, the constant $C_\lambda$ is zero. For Gaussian assumed density and integration filters it was shown in Sections II-C2 and II-C3 that $C_\lambda = M(g) - N(g)$. Because $M(f - P_t S) \leq -\lambda$ and $N(f - P_t S) \geq N(f) - \text{tr}(S) \lambda_P$, these filters have $C_\lambda = -\lambda - N(f) + \text{tr}(S) \lambda_P$.

### B. On the Stability Condition

The assumption
\[
\sup_{t \geq 0} M(f - P_t S) = \sup_{t \geq 0} \sup_{x \in \mathbb{R}^d} \mu[J_f(x) - P_t S] < 0
\]
is a time-uniform condition on contractivity of the filtering error process $E_t$. Indeed, it is the uniformity of this condition that enabled the relatively straightforward analysis in the preceding section. We are essentially ignoring any non-linear couplings between elements of $X_t$ that would need to be exploited were the analysis to be extended for systems that are not fully observed. See Section V for examples demonstrating full observability is (almost always) necessary and [45, Section 4] for more relevant discussion.

Even if one were to ignore issues with uniformity, the condition is still an extremely stringent one as it does not necessarily hold even for stable Kalman–Bucy filters for linear time-invariant systems. The Kalman–Bucy filter for the time-invariant linear system
\[
\begin{align*}
\dot{d}X_t &= AX_t dt + Q^{1/2} dW_t, \\
\dot{d}Y_t &= HX_t dt + R^{1/2} dV_t
\end{align*}
\]
is
\[
\begin{align*}
\dot{d}\hat{X}_t &= A\hat{X}_t dt + P_t H^T R^{-1} (dY_t - H \hat{X}_t dt), \\
\partial_t P_t &= AP_t + P_t A^T + Q - P_t S P_t.
\end{align*}
\]
Under certain observability and stabilizability conditions [47–49] the error covariance has a limiting steady state: $P_t \to P$ as $t \to \infty$ for the solution $P$ of the algebraic Riccati equation
\[
AP + PA^T + Q - PSP = 0.
\]
Furthermore, the system $\partial_t x_t = (A - PS)x_t$ (i.e., homogeneous part of the linear filtering error equation) is exponentially stable in the usual sense that the eigenvalues of the system matrix are located in the left half-plane:
\[
\alpha(A - PS) := \max_{i = 1, \ldots, d_x} \text{Re} \left[ \lambda_i(A - PS) \right] < 0.
\]
However, the general inequality linking $\alpha(A - PS)$ and $M(A - PS) = \mu(A - PS)$ is in the “wrong” direction [38, Equation (1.3)]:
\[
\alpha(A - PS) \leq \mu(A - PS).
\]
That is, assumption (14) need not be satisfied even though the linear filter is stable.

**IV. Discrete-Time Models and Filters**

We now turn our attention to discrete-time systems, introducing first a class of generic discrete-time filters analogous to that of the continuous-time filters discussed in Section II-B. Then we provide discrete-time analog of the stability result of Section III. When necessary, we differentiate between the continuous and discrete cases by reserving $k$ for discrete time-indices and using an additional subscript $d$ for parameters related to the discrete case.

### A. A Class of Discrete-Time Filters for Non-Linear Systems

In discrete-time, we consider systems of the form
\[
\begin{align*}
X_k &= f(X_{k-1}) + Q^{1/2} W_k, \quad (18a) \\
Y_k &= HX_k + R^{1/2} V_k, \quad (18b)
\end{align*}
\]
where $W_k$ and $V_k$ are now independent standard Gaussian random vectors. In this discrete case, the drift $f$ is assumed to be Lipschitz (i.e., $\|f(x) - f(y)\| < \infty$). We again consider a linear functional $L_{x,P}$ satisfying the three basic properties listed in Section II-B. However, Assumption II.1 will need to be replaced with a slightly modified version.

**Assumption IV.1.** Assume that for every differentiable $g: \mathbb{R}^{d_x} \to \mathbb{R}^{d_x}$ with finite $\|J_g\|$ there is a constant $C_g \geq 0$, varying continuously with $\|J_g\|$, such that
\[
\|g(x) - L_{x,g}(P)\|^2 \leq \|J_g\|^2 \|x - \tilde{x}\|^2 + C_g \text{tr}(P)
\]
for any points $x, \tilde{x} \in \mathbb{R}^{d_x}$ and any $P \in \mathbb{R}^{d_x \times d_x}$.

Again, this assumption says that $L_{x,g}(P)$ cannot deviate too much from $g(\tilde{x})$ since the standard Lipschitz bound is
\[
\|g(x) - g(\tilde{x})\| \leq \|J_g\| \|x - \tilde{x}\|.
\]

A generic discrete-time filter for the non-linear system (18) produces the state estimates
\[
\begin{align*}
\hat{X}_k &= L_{\hat{X}_{k-1},P_{k-1}}(f) + P_k H^T (H P_{k|k-1} H^T + R)^{-1} \left[ Y_k - H L_{\hat{X}_{k-1},P_{k-1}}(f) \right], \quad (19)
\end{align*}
\]
where $P_k$ and $P_{k|k-1}$ are some user-specified positive-definite $d_x \times d_x$ matrices that are allowed to depend on the state estimates and measurements up to time $k - 1$. The matrices
\[
K_k := P_{k|k-1} H^T (H P_{k|k-1} H^T + R)^{-1}
\]
are the discrete-time Kalman gain matrices.
B. Kalman Filters for Discrete-Time Non-Linear Systems

Analogously to Kalman–Bucy filters introduced in Section II-C, a Kalman filter for the discrete-time model (18) computes the approximations \( \hat{x}_{k|k-1} \) and \( P_{k|k-1} \) to the conditional filtering means and covariances \( \mathbb{E}(X_k \mid Y_1, \ldots, Y_k) \) and \( \text{Var}(X_k \mid Y_1, \ldots, Y_k) \). Such a filter consists of the prediction step

\[
\hat{x}_{k|k-1} = L_{X_k|k-1} p_{k-1}(f), \tag{20a}
\]

\[
P_{k|k-1} = P_{X_k|k-1}^\text{Rec} p_{k-1}(f) + Q_u, \tag{20b}
\]

where \( L_{X_k|k-1} p_{k-1}(g) \) maps functions to positive-semidefinite matrices and \( Q_u \) is again a potentially tuned version of \( Q \), and the update step

\[
K_k = P_{k|k-1} H^T (HP_{k|k-1} H^T + R)^{-1}, \tag{21a}
\]

\[
\hat{x}_k = \hat{x}_{k|k-1} + K_k (y_k - H \hat{x}_{k|k-1}), \tag{21b}
\]

\[
P_k = (I - K_k H) P_{k|k-1}. \tag{21c}
\]

All standard extensions of the Kalman filter for non-linear systems fit this framework. For example, \( L_{x,p}(g) = g(x) \) and \( L_{x,p}^\text{Rec}(g) = J_g(x) P J_g(x)^T \) yield the extended Kalman filter, while

\[
L_{x,p}(g) = \mathbb{E}_{N(x,p)}(g) = \int_{\mathbb{R}^d_x} g(z) N(z \mid x, P) \, dz,
\]

\[
L_{x,p}^\text{Rec}(g) = \text{Var}_{N(x,p)}(g) = \int_{\mathbb{R}^d_x} \left[ g(z) - L_{x,p}(g) \right] \left[ g(z) - L_{x,p}(g) \right]^T N(z \mid x, P) \, dz
\]
correspond to discrete-time Gaussian assumed density filters. Obviously, by replacing the exact integrals with their numerical approximations we obtain different discrete-time Gaussian integration filters. For the EKF, Assumption IV.1 holds again with \( C_g = 0 \), whereas similar arguments as those appearing in Section II-C show that \( C_g = \| J_g \| \) for the Gaussian assumed density filter and Gaussian integration filters whose associated numerical integration rule satisfies the second-degree exactness condition (13).

C. Stability of Discrete-Time Filters

A discrete version of Grönwall inequality of Theorem I.1 is needed in stability analysis of discrete-time filters.

**Theorem IV.2** (Discrete-time Grönwall’s inequality). Let \( 0 \leq \alpha < 1 \) and \( \beta \geq 0 \) and suppose that \( x_k \geq 0 \) satisfy the difference inequality

\[
x_k \leq \alpha x_{k-1} + \beta
\]

for \( k \geq 1 \). Then

\[
x_k \leq \alpha^k x_0 + \beta \sum_{n=0}^{k-1} \alpha^n \leq \alpha^k x_0 + \frac{\beta}{1 - \alpha}.
\]

Discrete-time stability analysis is based on the non-linear difference equation

\[
E_k = f(X_{k-1}) + Q^{1/2} W_k - \hat{X}_{k|k-1} - K_k (Y_k - H \hat{X}_{k|k-1})
\]

\[
= f(X_{k-1}) - L_{\hat{x}_{k-1}, p_{k-1}}(f) - K_k H (X_k - \hat{X}_{k|k-1})
\]

\[
+ Q^{1/2} W_k - K_k R^{1/2} V_k
\]

\[
= (I - K_k H) [f(X_{k-1}) - L_{\hat{x}_{k-1}, p_{k-1}}(f)]
\]

\[
+ (I - K_k H) Q^{1/2} W_k - K_k R^{1/2} V_k.
\]

for the filtering error \( E_k := X_k - \hat{X}_k \).

**Theorem IV.3.** Consider the generic discrete-time filter (19) for the model (18) and let \( L_{x,p} \) satisfy Assumption IV.1. Suppose that there are positive constants \( \lambda_0^p \), \( \lambda_1^p \), and \( \lambda_d \) such that \( \sup_{k \geq 0} \text{tr}(P_{k|k-1}) \leq \lambda_0^p \), \( \sup_{k \geq 0} \text{tr}(P_k) \leq \lambda_1^p \),

\[
\sup_{k \geq 1} \| I - K_k H \| \leq \lambda_d < \infty, \text{ and } \lambda_d^f := \| J_f \| < 1 \tag{22}
\]

hold almost surely. Denote

\[
\beta(\delta) = e(\sqrt{2\delta} + \delta) \quad \text{and} \quad \kappa = \sup_{k \geq 1} \| K_k \| \leq \lambda_0^p \| H \| \| R^{-1} \|.
\]

Then there is a non-negative constant \( C_f \) such that, for any \( \delta > 0 \), we have the exponential concentration inequality

\[
\| E_k \|^2 \geq 3(d_x + 2) \beta(\delta) \times \left( \lambda_2^f \| P_0 \| + \frac{\lambda_2^f [\text{tr}(Q) + C_f \lambda_1^p] + 4\kappa^2 \text{tr}(R)}{(1 - \lambda_d^f)^2} \right)
\]

with probability smaller than \( e^{-\delta} \) and the mean square filtering error bound

\[
\mathbb{E}(\| E_k \|^2) \leq \lambda_2^f \| P_0 \| \text{tr}(Q) + C_f \lambda_1^p \| + \kappa^2 \text{tr}(R) \frac{1}{1 - \lambda_d^f^2}.
\]

**Proof.** As in the proof of Theorem III.1, we only provide the argument for the mean square bound (24) and prove the exponential concentration inequality (23) in Appendix C. Norm of the filtering error is

\[
\| E_k \|^2 = \left[ f(X_{k-1}) - L_{\hat{x}_{k-1}, p_{k-1}}(f) \right]^T (I - K_k H)^T
\]

\[
\times [I - K_k H] \left[ f(X_{k-1}) - L_{\hat{x}_{k-1}, p_{k-1}}(f) \right]
\]

\[
+ 2 \left[ f(X_{k-1}) - L_{\hat{x}_{k-1}, p_{k-1}}(f) \right]^T (I - K_k H)^T
\]

\[
\times [I - K_k H] Q^{1/2} W_k - K_k R^{1/2} V_k
\]

\[
+ [I - K_k H] Q^{1/2} W_k - K_k R^{1/2} V_k \times [I - K_k H] Q^{1/2} W_k - K_k R^{1/2} V_k.
\]

We immediately obtain

\[
\kappa = \sup_{k \geq 1} \| K_k \|
\]

\[
= \| P_{k|k-1} H^T (HP_{k|k-1} H^T + R) \|^{-1} \leq \lambda_1^p \| H \| \| R^{-1} \|.
\]

Using Assumption IV.1 and (22), we get the recursive filtering error bound

\[
\| E_k \|^2 \leq \lambda_2^f \| E_{k-1} \|^2 + C_f \lambda_1^p \lambda_2^p \| U_k \|^2 + 2M_k,
\]
where \( C_f \) is the constant appearing in Assumption IV.1 for the function \( f \),
\[
U_k := (I - K_k H)Q^{1/2}W_k - K_k R^{1/2}V_k
\]
admits the bound
\[
E(\|U_k\|^2) \leq u_d := \lambda_d^2 \text{tr}(Q) + \kappa^2 \text{tr}(R),
\]
and the random variable
\[
M_k := \left[ f(X_{k-1}) - L \widehat{X}_{k-1}, P_{k-1}(f) \right]^T (I - K_k H)U_k
\]
\[
\leq \lambda_{df} \|U_k\| \|E_{k-1}\| + \lambda_d(C_f \lambda_p^2)^{1/2} \|U_k\|
\]
is zero-mean because \( W_k \) and \( V_k \) are independent of \( X_{k-1}, \widehat{X}_{k-1}, P_{k-1} \), and \( K_k \). Therefore
\[
E(\|E_k\|^2) \leq \lambda_{df}^2 E(\|E_{k-1}\|^2) + u_d + C_f \lambda_d^2 \lambda_p^2.
\]
The discrete Grönwall’s inequality then produces
\[
E(\|E_k\|^2) \leq \lambda_{df}^2 E(\|E_0\|^2) + u_d + C_f \lambda_d^2 \lambda_p^2 \frac{1}{1 - \lambda_{df}^2}
\]
\[
= \lambda_{df}^2 \text{tr}(P_0) + u_d + C_f \lambda_d^2 \lambda_p^2 \frac{1}{1 - \lambda_{df}^2},
\]
which is (24). The exponential concentration inequality (23) is derived in Appendices A and C by bounding \( E(\|E_k\|^2) \) for every \( n \geq 1 \) and using Bernstein’s inequality.

The stability condition (22) is merely a discrete-time version of the continuous-time version (14) and is equally stringent, requiring in practice that the system be fully observed.

V. EXAMPLE MODELS

This section examines three model classes for which certain Kalman filters satisfy Theorem III.1 or IV.3, possibly under sufficient covariance inflation. All these models are essentially fully observed, by which we mean that \( S = H^T R^{-1} H = sI \) for some \( s > 0 \). Such an assumption is very strong, but it appears that the techniques used in this article are simply inadequate for considering more general measurement models. This assumption is widely used in most related works on stability of Kalman filters for non-linear models and ensemble Kalman filters that fit our framework. In particular, we provide explicit values for the various constants appearing in Theorems III.1 and IV.3.

1) The Continuous-Time Case: First, consider a generic Kalman–Bucy filter of the form (9) and suppose that there is \( \ell_c \) such that \( M(f) \leq -\ell_c < 0 \). This means that the homogeneous system \( \partial x_t = f(x_t) \) is exponentially stable: \( x_t \to 0 \) with an exponential rate as \( t \to \infty \). Assume also that the matrix-valued operator \( \text{tr} L_{x,P}^\text{Ric} \) in the Riccati equation (9b) satisfies
\[
\text{tr} \left[ L_{x,P}^\text{Ric} \right] \leq M(f) \text{ tr}(P).
\]
As shown in [29], this assumption is natural and satisfied by all Kalman–Bucy filters discussed in Section II-C. From this assumption it follows that
\[
\partial_t \text{ tr}(P_t) = \text{ tr} \left[ L_{x,P}^\text{Ric} \right] + \text{tr}(Q_u) - \text{tr}(P_t S P_t) \leq -2\ell_c \text{ tr}(P_t) + \text{tr}(Q_u).
\]
Consequently, by Grönwall’s inequality,
\[
\text{tr}(P_t) \leq \lambda P_t \leq \lambda P,
\]
which is a zero-mean Gaussian random vector with variance \( r/h^2 \). That is,
\[
E(\|X_k - \widehat{X}_{Y,k}\|^2) = \frac{d_y r}{h^2}.
\]
If the assumptions of Theorem IV.3 hold, the mean square bound (24) is
\[
E(\|E_k\|^2) \leq \lambda_{df}^2 \text{tr}(Q) + C_f \lambda_d^2 \lambda_p^2 \frac{1}{1 - \lambda_{df}^2},
\]
where \( \lambda_{df} < 1 \) and
\[
\kappa = \sup_{k \geq 1} \|K_k\| = \sup_{k \geq 1} \|hP_{k|k-1}(h^2 P_{k|k-1} + rI)^{-1}\|
\leq \frac{h}{h^2 + r/\lambda_p^2} = O(r^{-1})
\]
as \( r \to \infty \). It can be now observed that the bound (26) is smaller than (25) if \( \text{tr}(Q) \) and \( C_f \) are sufficiently small and \( r \) is sufficiently large. From Section IV-B we recall that \( C_f = 0 \) for the EKF and \( C_f = \|J_f\| \) for the UKF and its relatives. This result is entirely intuitive: if there is little process noise but the measurement noise level is high, the filter is able to produce accurate estimates by following the dynamics. This also demonstrates that in the somewhat limited setting where Theorem IV.3 is applicable the bounds it yields are sensible.

B. Contractive Dynamics

Stability analysis in [8] was restricted to the extended Kalman–Bucy filter for fully observed models with a contractive (or uniformly monotone) drift: \( M(f) < 0 \). This section revisits stability analysis for such models and for the classes of Kalman filters that fit our framework. In particular, we provide explicit values for the various constants appearing in Theorems III.1 and IV.3.

A. Accuracy of Measurements

If a discrete-time model is fully observed, one can simply use the measurements themselves as state estimates. For certain regimes of the system parameters it can be shown that the mean square error bound of Theorem IV.3 is an improvement over that for such naive state estimators. Consider the discrete-time system (18) and suppose the measurement model is
\[
Y_k = hX_k + \sqrt{\nu} V_k
\]
for some positive scalars \( h \) and \( r \). Error of the naive estimate \( \widehat{X}_{Y,k} := Y_k/h \) is
\[
E_{Y,k} := X_k - \widehat{X}_{Y,k} = X_k - Y_k/h = (\sqrt{r}/h)V_k,
\]
where
\[
\lambda_{P,t} = e^{-2\ell_c t} \text{tr}(P_0) + \frac{\text{tr}(Q_u)}{2\ell_c} \leq \lambda_P := \text{tr}(P_0) + \frac{\text{tr}(Q_u)}{2\ell_c}.
\]

where
Furthermore, if the model is in addition fully observed,\[ M(f - P_t S) \leq M(f) + s\mu(-P_t) \leq -\ell_c.\]

That is, the assumptions of Theorem III.1 are satisfied for this class of exponentially stable and fully observed models for any positive-definite \(Q_{tu}\).

**Proposition V.1.** Consider a generic Kalman–Bucy filter (9), defined by \(L_x, P\) satisfying Assumption II.1, for the continuous-time model (5). Suppose that there is a positive \(\ell_c\) such that \(M(f) \leq -\ell_c < 0, S = H^TR^{-1}H = sI\) for some \(s > 0\), and that (27) holds. Then Theorem III.1 holds with \(T = 0, C_T = 0, \lambda = \ell_c, \) and \(\lambda_P = \text{tr}(P_0) + \frac{\text{tr}(Q_{tu})}{2\ell_c}\).

In particular, under the assumptions of the above proposition and when using the time-dependent bound \(\lambda_{P,t}\), the mean square bound (16) for the EKF takes the form
\[
\mathbb{E}(\|E_t\|^2) \leq \text{tr}(P_0) e^{-2\ell_c t} + \text{tr}(Q) + d_x s \frac{\text{tr}(P_0) + \text{tr}(Q_{tu})/(2\ell_c)^2}{2\ell_c} \lim_{t \to \infty} \left( \text{tr}(Q) + d_x s \frac{\text{tr}(Q_{tu})}{2\ell_c} \right) 1 - \frac{2\ell_c}{2\ell_c}.
\]

2) The Discrete-Time Case: For discrete-time systems (18), the exponential stability condition takes the form
\[
\|J_f\| = \sup_{x \in \mathbb{R}^d} \|J_f(x)\| \leq \ell_d < 1.
\]

For simplicity, consider then the discrete-time extended Kalman filter defined by (20) and (21) with \(L_{x,t}, P(g) = g(x)\) and \(L_{R_x,t}(g) = J_f(x)P J_f(x)^T\). Let \(Q_{tu} = q_{tu}I\) for \(q_{tu} > 0\). In addition, assume that \(H = hI\) and \(R = rI\) for positive scalars \(h\) and \(r\). Under these assumptions it is easy to show that
\[
\lambda_d = \sup_{k \geq 1} \|I - K_t H\| \leq \frac{r/h^2}{q_{tu} + r/h^2} < 1.
\]

Consequently, \(\lambda_{df} = \lambda_d \|J_f\| = \lambda_d \ell_d < 1\). The error covariances satisfy \(\text{tr}(P_k) \leq \text{tr}(P_{k-1})\) by the standard equivalence\[ P_k = (I - K_t H) P_{k-1} = (I - K_t H) (H P_{k-1} H^T + R)^{-1} H P_{k-1}.\]

Furthermore,
\[
\text{tr}(P_{k-1}) = \text{tr} \left[ J_f (\tilde{X}_{k-1}) P_{k-1} J_f (\tilde{X}_{k-1})^T \right] + d_x q_u \leq \|J_f\|^2 \text{tr}(P_{k-1}) + d_x q_u.
\]

so that the discrete Gronwall inequality produces the bound \(\text{tr}(P_k) \leq \text{tr}(P_{k-1}) \leq \ell_d \text{tr}(P_0) + d_x q_u \frac{1}{1 - \ell_d} \leq \text{tr}(P_0) + d_x q_u \frac{1}{1 - \ell_d} \).

See [20, Section IV] and [26, Section 4] for other bounds. We thus obtain a discrete-time counterpart of Proposition V.1.

**Proposition V.2.** Consider the extended Kalman filter for the discrete-time model (18). Suppose that there is \(0 < \ell_d < 1\) such that \(\|J_f\| \leq \ell_d \) and \(H = hI, R = rI, \) and \(Q_{tu} = q_{tu}I\) for some \(h, r, q_u > 0\). Then Theorem IV.3 holds with
\[
\lambda_d = \frac{r/h^2}{q_{tu} + r/h^2} \quad \text{and} \quad \lambda_P = \lambda_P^d = \text{tr}(P_0) + d_x q_u \frac{1}{1 - \ell_d}.
\]

C. Covariance Inflation

Intuitively, if the state is observed linearly and “well enough”, artificial inflation of the error covariance matrix \(P_t\) makes the filter more stable (or robust) since this results in less emphasis being placed on the state dynamics, mitigating instability potentially caused by non-linearity of the drift. Covariance inflation is an important topic in the study of ensemble Kalman filters [50], [51] and has been suggested also for stabilising the discrete-time UKF [21], [25].

Covariance inflation allows for considering models whose drift is not necessarily contractive. Suppose that \(S = sI\) for some positive \(s\). Then
\[
\sup_{x \in \mathbb{R}^d} \mu(J_f(x) - P_t S) \leq M(f) + s\mu(-P_t) = M(f) - s\lambda_{\min}(P_t),
\]

and becomes evident that for large enough \(\lambda_{\min}(P_t)\) this quantity becomes negative as required in Theorem III.1. Specifically, \(\lambda_{\min}(P_t) \geq (M(f) + \lambda)/s\) is sufficient to ensure that \(\sup_{x \in \mathbb{R}^d} \mu(J_f(x) - P_t S) \leq -\lambda\). As next demonstrated, this can be achieved using covariance inflation in Kalman–Bucy filters by choosing a large enough tuned dynamic noise covariance matrix \(Q_{tu}\). For simplicity, consider the extended Kalman–Bucy filter. The inversion formula \(\partial_t P_t^{-1} = -P_t^{-1} (\partial_t P_t) P_t^{-1}\) yields the Riccati equation
\[
\partial_t P_t^{-1} = -P_t^{-1} J_f(\tilde{X}_t) - J_f(\tilde{X}_t)^T P_t^{-1} + S - P_t^{-1} Q P_t^{-1}
\]
for the inverse error covariance. The result below follows by arguments similar to those appearing in [29].

**Proposition V.3.** Consider the extended Kalman–Bucy filter (10) for the continuous-time model (5). Then
\[
\text{tr}(P_t^{-1}) \leq \frac{\lambda_{\min}(Q_{tu}) \lambda_{\max}(S)/d_x + N(f)^2 - N(f)}{\lambda_{\min}(Q_{tu})/d_x} + \alpha_1 e^{-\beta_1 t}
\]

for some positive constants \(\alpha_1\) and \(\beta_1\) that depend on the system parameters.

Since \(\text{tr}(P_t^{-1}) = \sum_{i=1}^{d_x} \lambda_i(P_t)^{-1}\), (28) implies the eigenvalue bound
\[
\lambda_{\min}(P_t) \geq \frac{1}{\text{tr}(P_t^{-1})} \geq \frac{\lambda_{\min}(Q_{tu})/d_x}{\sqrt{\lambda_{\min}(Q_{tu}) \lambda_{\max}(S)/d_x + N(f)^2 - N(f)}} + \alpha_2 e^{-\beta_2 t}
\]
for some positive constants \(\alpha_2\) and \(\beta_2\). As this eigenvalue bound grows as square root of \(\lambda_{\min}(Q_{tu})\), the inequality \(\lambda_{\min}(P_t) \geq (M(f) + \lambda)/s\) that induces the stability condition (14) is satisfied when \(\lambda_{\min}(Q_{tu})\) and \(t\) are large enough.

D. Integrated Velocity Models

Let \(h \neq 0, \alpha_2, q_2, r_2 > 0\, \text{and}\, \alpha_1\) be constants and \(g: \mathbb{R} \to \mathbb{R}\) a continuously differentiable function such that
\[
N(g) = \inf_{x \in \mathbb{R}} g'(x) \geq \ell_g > 0
\]
(29)
for a constant $\ell_g$. Consider the integrated velocity model
\begin{equation}
\begin{aligned}
dX_{t,1} &= \left[ a_1 X_{t,1} + a_2 X_{t,2} \right] dt + \left[ \begin{array}{c} q_{1/2} \\ 0 \end{array} \right] \sqrt{q_{1/2}} dV_t, \\
dY_t &= \left[ \begin{array}{c} h \\ 0 \end{array} \right] X_t dt + r/\sqrt{d} dw_t,
\end{aligned}
\end{equation}
for a two-dimensional state $X_t = (X_{t,1}, X_{t,2}) \in \mathbb{R}^2$ of which one-dimensional measurements $Y_t$ are obtained. When $a_1 = 0$, the first state component $X_{t,1}$ can be interpreted as the position of a target, obtained by integrating the velocity $X_{t,2}$ that evolves independently. By only using measurements of the position one then wants to estimate both the position and the velocity.

We now show the extended Kalman–Bucy filter (10) for this model satisfies Theorem III.1 if enough covariance inflation is employed (see [52, Section 6.1] for related computations). Because
\[ J_f(x) = \begin{bmatrix} a_1 & a_2 \\ 0 & -g'(x_2) \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} h^2/r & 0 \\ 0 & 0 \end{bmatrix}, \]
we will denote $s_1 = h^2/r$, the EKF for the integrated velocity model (30) takes the form
\begin{equation}
\begin{aligned}
d\hat{X}_t &= \begin{bmatrix} a_1 \hat{X}_{t,1} + a_2 \hat{X}_{t,2} \\ -g'(\hat{X}_{t,2}) \end{bmatrix} dt + \begin{bmatrix} P_{t,11} & P_{t,12} \\ P_{t,12} & P_{t,22} \end{bmatrix} \frac{h}{r} \sqrt{P_{t,22}} dY_t, \\
\partial_t P_t &= \begin{bmatrix} a_1 & a_2 \\ 0 & -g'(\hat{X}_{t,2}) \end{bmatrix} \left[ \begin{array}{cc} P_{t,11} & P_{t,12} \\ P_{t,12} & P_{t,22} \end{array} \right] \\
&+ \left[ \begin{array}{c} P_{t,11} \\ P_{t,12} \end{array} \right] \begin{bmatrix} a_1 & 0 \\ a_2 & -g'(\hat{X}_{t,2}) \end{bmatrix} \\
&+ \left[ \begin{array}{c} q_{u,1} \\ 0 \end{array} \right] \begin{bmatrix} 0 & q_{u,2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P_{t,11} & P_{t,12} \\ P_{t,12} & P_{t,22} \end{bmatrix} + \left[ \begin{array}{c} h^2/r \\ 0 \end{array} \right] \left[ \begin{array}{cc} P_{t,11} & P_{t,12} \\ P_{t,12} & P_{t,22} \end{array} \right], \end{aligned}
\end{equation}
where $q_{u,1}$ and $q_{u,2}$ are elements of the diagonal tuned noise covariance $Q_u$. Differential equations for the three distinct elements of $P_{t,11}$ are
\[ \partial_t P_{t,11} = 2a_1 P_{t,11} + q_{u,1} - sP_{t,11}^2 + 2a_2 P_{t,12}, \]
\[ \partial_t P_{t,12} = \left[ a_1 - g'(\hat{X}_{t,2}) - sP_{t,11} \right] P_{t,12} + a_2 P_{t,22}, \]
\[ \partial_t P_{t,22} = -2g'(\hat{X}_{t,2}) P_{t,22} + q_{u,2} - sP_{t,12}^2. \]
From (29) it follows that
\[ \partial_t P_{t,22} \leq -2\ell_g P_{t,22} + q_{u,2}, \]
which yields the upper bound
\[ P_{t,22} \leq e^{-2\ell_g t} P_{0,22} + \frac{q_{u,2}}{2\ell_g} =: C_{22}(t). \]
Suppose that $P_{0,12} \geq 0$. Since $a_2 P_{t,22} > 0$, this implies that $P_{t,12} \geq 0$ for every $t \geq 0$. Consequently,
\[ \partial_t P_{t,11} \geq 2a_1 P_{t,11} + q_{u,1} - sP_{t,11}^2, \]
and from this it can be established that [29, Lemma 3]
\[ P_{t,11} \geq \frac{a_1 + (s q_{u,1} + a_1^2)^{1/2}}{s} - \alpha_1 e^{-\beta_1 t} \]
for some positive constants $\alpha_1$ and $\beta_1$. It follows that
\[ a_1 - g'(x) - sP_{t,11} \leq a_1 - \ell_g - sP_{t,11} \leq a_1 - \ell_g - sP_{t,11} \leq -\lambda_1 < 0 \]
when $t \geq T_{\lambda_1}$. Thus
\[ \partial_t P_{t,12} \leq -\lambda_1 P_{t,12} + a_2 P_{t,22} \leq -\lambda_1 P_{t,12} + a_2 C_{22}(t) \]
for $t \geq T_{\lambda_1}$, implying that there is a time-uniform upper bound $C_{12}$ on $P_{t,12}$. From this we obtain an upper bound for $P_{t,11}$:
\[ \partial_t P_{t,11} = 2a_1 P_{t,11} + q_{u,1} - sP_{t,11}^2 + 2a_2 P_{t,12} \leq 2a_1 P_{t,11} - sP_{t,11}^2 + q_{u,1} + 2a_2 C_{12} \]
implies that
\[ P_{t,11} \leq \frac{a_1 + (s q_{u,1} + 2a_2 C_{12}) + a_1^2)^{1/2}}{s} + \alpha_2 e^{-\beta_2 t} \]
for some positive constants $\alpha_2$ and $\beta_2$. Since the both diagonal elements $P_{t,11}$ and $P_{t,22}$ are bounded, we have thus obtained an upper bound on $\text{tr}(P_t)$.

Finally, to show that Theorem III.1 is applicable, we need to prove that the matrix
\[ A := (J_f(x) - P_t S)^{\text{sym}} = \begin{bmatrix} a_1 - sP_{t,11} & a_2 \\ -sP_{t,12} & -g'(x) \end{bmatrix} \]
\[ = \begin{bmatrix} 2(a_1 - sP_{t,11}) & a_2 - sP_{t,12} \\ a_2 - sP_{t,12} & 2g'(x) \end{bmatrix} \]
is negative-definite for every $x \in \mathbb{R}$ and large enough $t$. The eigenvalues of this matrix are
\[ \frac{1}{2} \left( \text{tr}(A) \pm \sqrt{\text{tr}(A)^2 - 4 \det(A)} \right). \]
Having previously selected $q_{u,1}$ and $T_{\lambda_1}$ such that
\[ \frac{1}{2} \text{tr}(A) = a_1 - g'(x) - s P_{t,11} \leq -\lambda_1 < 0, \]
we see that the larger of the eigenvalues is negative because
\[ \sqrt{\text{tr}(A)^2 - 4 \det(A)} < |\text{tr}(A)|. \]
To summarise, we have proved that error covariance inflation can be used to induce provable stability of the extended Kalman–Bucy filter for the two-dimensional integrated velocity model (30).

VI. NUMERICAL EXAMPLES

This section contains numerical examples that validate the mean square error bound of Theorem III.1 for the extended and unscented Kalman–Bucy filters applied to two toy models.
A. Contractive Dynamics

In this example we consider the EKF and the UKF for the fully observed model

\[
\begin{align*}
\text{d}X_t &= f(X_t)\,\text{d}t + \text{d}W_t, \\
\text{d}Y_t &= X_t\,\text{d}t + \sqrt{2}\,\text{d}V_t,
\end{align*}
\]

initialised from \(X_0 \sim N(0,1)\), with the drift

\[
f(x) = \begin{bmatrix}
-x_1(1 + \frac{1}{1+x_1}) - 3x_1 \\
-x_1 - x_2 - x_3 \\
x_1^2 e^{-x_1^2 - x_3^2} - x_1 - 2x_3
\end{bmatrix}.
\]

We can compute

\[
N(f) \approx -3.617 \quad \text{and} \quad M(f) \approx -0.672.
\]

This means that the model is exponentially stable and the assumptions of Proposition V.1 are satisfied with \(\ell_c = -M(f)\).

For any generic Kalman–Bucy filter, this proposition yields the error covariance bound (when \(Q_m = Q\))

\[
\text{tr}(P_t) \leq \text{tr}(P_0) + \frac{\text{tr}(Q)}{2\ell_c} =: \lambda_P \approx 5.231.
\]

The mean square bound of Theorem III.1 is

\[
\mathbb{E}(\|X_t - \hat{X}_t\|^2) \leq \text{tr}(P_0) e^{-2\lambda t} + \frac{\text{tr}(Q) + 2C_\lambda \lambda_P + \text{tr}(S)\lambda_P^2}{2\ell_c},
\]

where \(C_\lambda = 0\) for the EKF and

\[
C_\lambda = M(f) - N(f) + \text{tr}(S)\lambda_P \approx 10.792
\]

for the UKF (see Remark III.3). Note that this is merely a shortcoming of the proof technique we have used rather than a manifestation of greater accuracy of the EKF.

Figure 1 depicts the theoretical upper bounds on \(\mathbb{E}(\|X_t - \hat{X}_t\|^2)\) for the EKF and the UKF and the empirical mean square error based on 1,000 state and measurement trajectory realisations. The results were obtained using Euler–Maruyama discretisation with step-size 0.01. It is evident that the theoretical bounds are valid and extremely conservative, which is quite typical in stability theory of non-linear Kalman filters (see, e.g., numerical examples in [17], [18]).

B. Integrated Velocity Model

We now validate the theoretical bounds obtained in Section III on the integrated velocity model discussed in Section V-D. This example is interesting because the state is not fully observed, only the first state component being measured.

Consider the EKF for the integrated velocity model

\[
\begin{align*}
\text{d} \begin{bmatrix} X_{t,1} \\ X_{t,2} \end{bmatrix} &= \begin{bmatrix} a_1X_{t,1} + a_2X_{t,2} \\ -g(X_{t,2}) \end{bmatrix} \text{d}t + \begin{bmatrix} q_1^{1/2} & 0 \\ 0 & q_2^{1/2} \end{bmatrix} \text{d}W_t, \\
\text{d}Y_t &= [h & 0] X_t \,\text{d}t + r^{1/2} \,\text{d}V_t,
\end{align*}
\]

with the parameters \(a_1 = 0, a_2 = 1, q_1 = q_2 = 0.05, h = 1, r = 0.05, \hat{X}_0 = 0, P_0 = 0.3I\), and

\[
g(x) = x \left(1 + \frac{\sin x}{1 + x^2}\right).
\]

The derivative is

\[
g'(x) = 1 + \frac{x^3 + x}{(1 + x^2)^2} \cos x - (x^2 - 1) \sin x
\]

and its maximum and minimum are

\[
\sup_{x \in \mathbb{R}} g'(x) \approx 1.581 \quad \text{and} \quad \inf_{x \in \mathbb{R}} g'(x) \approx 0.419.
\]

That is, \(g\) satisfies (29) with \(\ell_g = 0.419\). Based on the derivations in Section V-D we are able to compute that \(\text{tr}(P_t) \leq \lambda_P \approx 0.217\) for all sufficiently large \(t\). Because \(a_1 = 0\), no covariance inflation is needed for (31) to hold. In this particular case, the value \(\lambda = 0.5478\) can be used in Theorem III.1.

Figure 2 depicts the limiting (i.e., all exponentially decaying terms are disregarded) theoretical mean square filtering error bound for the EKF thus obtained and the empirical mean square error based on 1,000 state and measurement trajectory realisations. Again, Euler–Maruyama discretisation with step-size 0.01 was used.
VII. CONCLUSIONS AND DISCUSSION

In this article we have shown that large classes of generic filters for both continuous and discrete-time systems with non-linear state dynamics and linear measurements are stable, in the sense of time-uniformly bounded mean square filtering error, if certain stringent conditions on boundedness of error covariance matrices and exponential stability of filtering error equations are met. Similar analysis and results have appeared previously for the extended Kalman–Bucy filter and exponentially stable state processes in [8] and, as such, our main contribution has been in providing the first comprehensive framework that allows for carrying out such analysis for most commonly used extensions of the Kalman–Bucy or Kalman filter to non-linear systems, such as Gaussian assumed density filters and their numerical approximation, including the uncented Kalman filter. In Section V, we have also presented three different classes of models that satisfy the stringent stability assumptions. This is in stark contrast to earlier work for, for example, the UKF that has relied on unverifiable assumptions on certain auxiliary random matrices [21].

As discussed in Section III-B, the uniform bounding technique used in this article appears too rough for obtaining significantly stronger results. This stems from the fact that the analysis essentially discards potential non-linear couplings between different state components (and non-linear stability analysis) in favor of simple bounds that transform the problem into a linear one and allow for applying Grönwall’s inequality. As a consequence, the state has to be typically fully observed. The only meaningful extensions that we believe are possible are for fully detected systems, in adaptation of terminology of [7]. These systems are essentially generalisations of the integrated velocity model we considered in Section V-D: not all state components need to be (fully) observed, but those that are not must be exponentially stable so that their effect on observed components is small.

APPENDIX A

USING BERNSTEIN’S CONCENTRATION INEQUALITY

In contrast to Del Moral et al. [8] who base their exponential concentration inequality for the EKF on the exponential inequality appearing in Proposition 11.6.6 of [53], we use the classical Bernstein inequality.

Theorem A.1 (Bernstein’s inequality). Let $X$ be a non-negative random variable. Suppose that there exists $\alpha > 0$ such that

$$\mathbb{E}(X^n) \leq n^n \alpha^n$$

for every integer $n \geq 2$. Then

$$\Pr[X \geq \alpha e (\sqrt{2\delta} + \delta)] \leq e^{-\delta}$$

(35)

for any $\delta > 0$.

Proof. By the standard Stirling bound,

$$\mathbb{E}(X^n) \leq n^n \alpha^n \leq \frac{n!}{\sqrt{2\pi}} e^n \alpha^n \leq \frac{n!}{2} (e \alpha)^n$$

for every $n \geq 2$. The “standard” version of Bernstein’s inequality (see, e.g., [54, Theorem 2.10]) posits that $\mathbb{E}(X^2) \leq \sigma < \infty$ and

$$\mathbb{E}(X^n) \leq \frac{n!}{2} \sigma \gamma^{n-2}$$

for some $\sigma > 0$ and $\gamma > 0$ and every $n \geq 3$ imply

$$\Pr[X \geq \sqrt{2\sigma \delta} + \gamma \delta] \leq e^{-\delta}$$

for any $\delta > 0$. Thus, setting $\gamma = e \alpha$ and $\sigma = \gamma^2$ produces the claim.

The concentration inequality used in [8] is based on the same moment assumption but states instead that

$$\Pr[X \geq \alpha e (\sqrt{2\delta} + \delta)] = \mathbb{P} \left[ X \geq \alpha \sqrt{2} \left( \sqrt{\delta} + \frac{1}{\sqrt{2} \delta} \right) \right]$$

and $\sqrt{2} e < e^2 / \sqrt{2}$, the concentration inequality (35) is the tighter of the two for every $\delta > 0$.

Lemma A.2. Let $X \in \mathbb{R}^d$ be a zero-mean Gaussian random vector with a positive-semidefinite covariance $P$. Then

$$\mathbb{E}((\|X\|^2)^n) \leq \|P\|^n (d + 2)^n n^n$$

for every $n \geq 1$.

Proof. We know that $X = P^{1/2} U$ for a standard normal $U \in \mathbb{R}^d$. Therefore

$$\mathbb{E}((\|X\|^2)^n) = \mathbb{E}((U^T P U)^n) \leq \|P\|^n \mathbb{E}(\|U\|^2n^n),$$

where $\mathbb{E}(\|U\|^2n^n)$ is the $n$th moment around zero of the chi-squared distribution with degrees of freedom $d$. That is,

$$\mathbb{E}(\|U\|^2n^n) = d \times \cdots \times (d + 2(n - 1)) \leq (d + 2(n - 1))^n \leq (d + 2)^n n^n.$$

APPENDIX B

COMPLETE PROOF OF THEOREM III.1

This appendix contains the complete proof for the exponential concentration inequality (15) of Theorem III.1. We begin with a proposition providing bounds for functions satisfying certain differential inequalities.

Proposition B.1. Let $\alpha \neq 0$ and $\beta \geq 0$ be constants and $n$ a positive integer. Suppose that a non-negative and differentiable function $x_t$ satisfies the differential inequality

$$\partial_t x_t \leq \alpha x_t + \beta n^2 x_t^{1-1/n}$$

for $t \geq t_0$. Then

$$x_t^{1/n} \leq x_{t_0}^{1/n} e^{\alpha(t-t_0)} + \frac{\beta n}{\alpha} \left( e^{\alpha(t-t_0)} - 1 \right).$$
Proof. The proof is similar to that of Grönwall’s inequality. For \( t \geq t_0 \), the function \( z_t := e^{-\alpha n(t-t_0)} x_t \geq 0 \) satisfies the differential inequality
\[
\partial_t z_t = -\alpha n e^{-\alpha n(t-t_0)} x_t + e^{-\alpha n(t-t_0)} \partial_t x_t \\
\leq n^2 \beta e^{-\alpha n(t-t_0)} Z_t^{1-1/n} \\
= \beta n^2 e^{-\alpha n(t-t_0)} Z_t^{1-1/n}.
\]
Consequently, for \( t \geq t_0 \),
\[
\partial_t z_t^{1/n} = \frac{1}{n} z_t^{1/n-1} \partial_t z_t \leq \beta n e^{-\alpha (t-t_0)}
\]
and direct integration yields
\[
z_t^{1/n} \leq z_{t_0}^{1/n} + \beta n \int_{t_0}^{t} e^{-\alpha (s-t_0)} ds \\
= z_{t_0}^{1/n} + \frac{\beta n}{\alpha} (1 - e^{-\alpha (t-t_0)})
\]
The claim is obtained by observing that \( x_t^{1/n} = e^{\alpha (t-t_0)} z_t^{1/n} \).

We are now in a position to provide a complete proof of Theorem III.1.

Proof of Theorem III.1. As shown in Section III-A, squared norm of the filter error process \( E_t = X_t - \hat{X}_t \) obeys the stochastic differential inequality [55], [56]
\[
d\|E_t\|^2 \leq -2\lambda \|E_t\|^2 dt + u dt + dM_t
\]
for \( t \geq T \) with \( \lambda > 0 \), \( u = \text{tr}(Q) + 2C_\lambda \rho_p + \text{tr}(S) \lambda_p^2 > 0 \), and
\[
dM_t = 2\left( Q^{1/2} dW_t - P_t H^T R^{-1/2} dV_t, X_t - \hat{X}_t \right)
\]
a zero-mean (local) martingale. When \( 0 \leq t \leq T \), we instead have the inequality
\[
d\|E_t\|^2 \leq 2\rho \|E_t\|^2 dt + u dt + dM_t
\]
where
\[
\rho = M(f) + \|S\| \text{tr}(P_t) \geq M(f) + \mu(-P_t S) \geq M(f - P_t S).
\]
We can assume that \( \rho \) is positive for if it were negative, we could set \( -\lambda = \rho \) and \( T = 0 \).

Let \( \gamma \) stand for either \( -\lambda \) or \( \rho \). We compute upper bounds on \( \mathbb{E}(\|E_t\|^{2n}) \) for every \( n \geq 1 \) in order to use Bernstein’s inequality. First, observe that \( \langle M \rangle_t \), the quadratic variation of \( M_t \) (i.e., the increasing process such that \( M_t^2 - \langle M \rangle_t \) is a martingale), satisfies
\[
d\langle M \rangle_t \leq 4 \|E_t\|^2 \left[ \text{tr}(Q) + \text{tr}(SP_t^2) \right] dt \leq 4 \|E_t\|^2 u dt.
\]
For \( n \geq 2 \), the above inequality, the identity
\[
d\|E_t\|^{2n} = n \|E_t\|^{2(n-1)} d\|E_t\|^2 + \frac{n(n-1)}{2} \|E_t\|^{2(n-2)} d\|E_t\|^2
\]
and the general form of Itô’s lemma then produce
\[
\mathbb{E}(\|E_t\|^{2n}) \leq 2\gamma n \mathbb{E}(\|E_t\|^{2n-2}) + 2\gamma n \mathbb{E}(\|E_t\|^{2n-2}) d\|E_t\|^2
\]
\[
\leq 2\gamma n \|E_t\|^{2n} dt + 2\gamma n \|E_t\|^{2n-2} d\|M_t\|^2.
\]
Simple induction establishes that \( \mathbb{E}(\|E_t\|^{2n}) \) does not explode in finite time. Therefore the term \( \|E_t\|^{2(n-1)} d\|M_t\|^2 \) vanishes when expectations are taken. See for example [57, Section 4.5] for similar arguments. Using Hölder’s inequality with \( p = n/(n-1) \), we get
\[
\partial_t \mathbb{E}(\|E_t\|^{2n}) \leq 2\gamma n \mathbb{E}(\|E_t\|^{2n}) + 2\gamma n \mathbb{E}(\|E_t\|^{2(n-1)})
\]
\[
\leq 2\gamma n \mathbb{E}(\|E_t\|^{2n}) + 2\gamma n \mathbb{E}(\|E_t\|^{2(n-1)} - \|E_t\|^{2n-1/n}).
\]
We can now apply Proposition B.1 with \( x_t = \|E_t\|^{2n} \) and \( \beta = \gamma n \). Setting \( \alpha = 2\rho \) and \( t_0 = 0 \) and considering \( t \leq T \), we obtain
\[
\mathbb{E}(\|E_T\|^{2n})^{1/n} \leq \left( \mathbb{E}(\|E_0\|^{2n})^{1/n} + \frac{\gamma n \|E_0\|^{2n}}{2\rho} \right) e^{2\rho T}.
\]
Noting that \( E_0 \sim \mathcal{N}(0, P_0) \), Lemma A.2 gives
\[
\mathbb{E}(\|E_T\|^{2n})^{1/n} \leq \left[ \|P_0\| (d_2 + 2) + \frac{\gamma n \|E_0\|^{2n}}{2\rho} \right] e^{2\rho T}.
\]
Denote \( C_T = \|P_0\| (d_2 + 2) + \rho/2 \). This provides a bound on the initial value for the case \( \alpha = -2\lambda \), \( t_0 = T \), and \( t \geq T \) in Proposition B.1:
\[
\mathbb{E}(\|E_T\|^{2n})^{1/n} \leq \left( C_T e^{-2\lambda(t-T)} + \frac{\gamma n \|E_0\|^{2n}}{2\lambda} \right) n^n.
\]
That is,
\[
\mathbb{E}(\|E_t\|^{2n}) \leq \left( C_T e^{-2\lambda(t-T)} + \frac{\gamma n \|E_0\|^{2n}}{2\lambda} \right) n^n.
\]
The claimed exponential concentration inequality follows by applying Bernstein’s inequality of Theorem A.1 to \( X = \|E_t\|^2 \) with \( \alpha = C_T e^{-2\lambda(t-T)} + u/(2\lambda) \).

Appendix C

Complete Proof of Theorem IV.3

This appendix contains the complete proof for the exponential concentration inequality (23) of Theorem IV.3.

Proof of Theorem IV.3. Recall from Section IV-C that the filtering error \( E_k = X_k - \hat{X}_k \) satisfies
\[
\|E_k\|^2 \leq \lambda_d^2 \|E_{k-1}\|^2 + \|U_k\|^2 + 2M_k + \eta^2
\]
We now bound all even moments \( E \) with Minkowski's inequality yields Bernstein's inequality. We have

\[ \text{Var}(U_k) = (I - K_k H) Q (I - K_k H)^T + K_k R K_k^T, \quad (36) \]

having the bound

\[ \| \text{Var}(U_k) \| \leq u_d = \lambda_d^2 \| Q \| + \kappa^2 \| R \|. \]

The latter admits the bound

\[ M_k \leq \lambda_d f \| U_k \| \| E_k - 1 \| + \eta \| U_k \|. \]

We now bound all even moments \( E(\| E_k \|^{2n}) \) in order to apply Bernstein's inequality. We have

\[ E(\| E_k \|^{2n}) \leq E\left( \left( \lambda_d^2 \| E_k - 1 \|^{2n} + \| U_k \|^{2} + 2M_k + \eta^2 \right)^n \right). \]

Minkowski's inequality yields

\[ E(\| E_k \|^{2n})^{1/n} \leq E\left( \left( \lambda_d^2 \| E_k - 1 \|^{2n} + \| U_k \|^{2} + 2M_k + \eta^2 \right)^n \right)^{1/n} \]

\[ \leq \lambda_d^2 E\left( \| E_k - 1 \|^{2n} \right)^{1/n} + E(\| U_k \|^{2n})^{1/n} + 2E(M_k) + \eta^2. \]

By Lemma A.2 and (36),

\[ E(\| U_k \|^{2n})^{1/n} \leq (d_x + 2) \eta u_d, \]

and by Minkowski's and Hölder's inequalities,

\[ E(M_k)^{1/n} \leq E\left( \| U_k \|^{n} \right)^{1/n} \left[ \left( \lambda_d^2 \| E_k - 1 \| + \eta \right)^n \right]^{1/n} \]

\[ \leq E\left( \| U_k \|^{n} \right)^{1/n} \left[ \lambda_d E\left( \| E_k - 1 \|^{n} \right) \left( \| E_k - 1 \| + \eta \right) \right] \]

\[ \leq \sqrt{(d_x + 2) \eta u_d E\left( \| E_k - 1 \|^{2n} \right)^{1/(2n)}} + \eta. \]

Inserting these bounds into (37) and recognising that the result can be bounded by a sum of two quadratic terms yields

\[ E(\| E_k \|^{2n})^{1/(2n)} \leq \lambda_d E\left( \| E_k - 1 \|^{2n} \right)^{1/(2n)} + 2\sqrt{(d_x + 2) \eta u_d}. \]

Then the discrete Grönwall's inequality and Lemma A.2 yield

\[ E(\| E_k \|^{2n})^{1/(2n)} \leq \lambda_d E\left( \| E_0 \|^{2n} \right)^{1/(2n)} + \frac{2\sqrt{(d_x + 2) \eta u_d}}{1 - \lambda_d} \]

\[ \leq \left( \lambda_d^k \| P_0 \|^{1/2} + \frac{2\sqrt{\eta u_d}}{1 - \lambda_d} \right) \sqrt{(d_x + 2) n}. \]

Because

\[ E(\| E_k \|^{2n}) \leq \left( \lambda_d^k \| P_0 \|^{1/2} + \frac{2\sqrt{\eta u_d}}{1 - \lambda_d} \right)^{2n} \]

\[ \leq 3(d_x + 2) \left( \lambda_d^{2k} \| P_0 \|^{1/2} + \frac{4\sqrt{\eta u_d}}{(1 - \lambda_d)^2} \right) n^n, \]

the claim follows from Bernstein's inequality with \( X = \| E_k \|^2 \) and

\[ \alpha = 3(d_x + 2) \left( \lambda_d^{2k} \| P_0 \|^{1/2} + \frac{4\sqrt{\eta u_d}}{(1 - \lambda_d)^2} \right). \]
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