New analytical approach for geometrically nonlinear buckling analysis of an inclined rod using the arc length method

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Abstract. This paper is concerned with nonlinear buckling problem of inclined rod subjected to concentrated loads and moments at the ends. Rigorous analysis of geometrically nonlinear structures demands creating mathematical models that accurately include loading and support conditions. This work introduces a new analytical approach to construct the governing equations considering large displacement. The mathematical formulation based on geometrical compatibility, equilibrium of forces and moments and constitutive relations considering large displacements. The geometrical compatibility relationship is getting from integrating along the elastic curve of the deformed rod. A system of nonlinear and integral equations with boundary conditions prescribed at both end is constructed. Using the arc length technique the paper developed incremental-iterative algorithm for solving the system of nonlinear equation. Based on the proposed algorithm, the paper established the calculation procedure and the programs for determining the equilibrium path for generally supported inclined rod subjected to concentrated loads and moments at the end.

1. Introduction

The stability of slender rods is a phenomenon associated with buckling and post buckling. Buckling and post buckling behavior for slender rods is very important since post buckling means loss the stability of structure associated with large displacement and may be lead to destruction the structure. Since the early classical contributions from Bernoulli, Euler and Lagrange in the 18th century, the subject of buckling, post buckling and large deflection analyses of slender rods has experienced significant evolution. In recent years, a large number of research work addressed the buckling and post-buckling behavior of rods [1-6]. Most of preceding work deals with numerical approach of problem formulation. This paper proposes a new analytical approach to formulating nonlinear buckling problem of inclined rods subjected to concentrated loads and moments at the ends. The mathematical model is built from geometrical compatibility, equilibrium of forces and moments and constitutive relations considering large displacement. The geometrical relationship is implemented by integrating along the elastic curve of the deformed rod. For solving the system of integral equations with boundary conditions prescribed at both end, the paper presents a new algorithm based on arc length technique [7-9]. The calculation programs for solving the nonlinear buckling problem of inclined rods sub which generally supported at two ends. For investigating the nonlinear equilibrium path and critical point presented numerical test for commonly type of inclined rods.

2. Problem formulation
Let us consider the geometrical compatibility, equilibrium of forces and moments and constitutive relations for the flexural bar. In formulating nonlinear problem, assuming that mechanical behaviour of materials is ideally elastic. Axial displacement due to stretching is negligible in comparison with the normal displacement.

\[ \text{Figure 1. Large displacement model of flexural bar} \]

Based on the Euler Bernoulli beam theory [10], from moment-curvature relationship and elementary calculus, we get

\[ \frac{1}{\rho} = \frac{d\phi}{ds} = \frac{y'(z)}{\left[1 + (y'(z))^2\right]^{3/2}} = \frac{M}{EI} \]  
(1)

Where: \( \rho \) is radius of curvature; \( d\phi \) is differential angle; \( ds \) is differential curve length; \( y \) is displacement function; \( M \) is bending moment and \( EI \) is flexural stiffness.

The equation (2) is differential equation of elastic curve considering large displacement. It can be compactly written as follows

\[ \frac{d\phi}{ds} = -\frac{M}{EI} \]  
(2)

The bending moment is defined by the expression

\[ M = P_{A'} \cdot y - P_{yA'} \cdot z + M_{A'} \]  
(3)

From (2) and (3), we get

\[ \frac{d\phi}{ds} = -\left( \frac{P_{A'} \cdot y - P_{yA'} \cdot z + M_{A'}}{EI} \right) \]  
(4)

Taking a derivative of both sides of equation (4) with respect to \( s \), getting

\[ \frac{d^2\phi}{ds^2} = -\left( \frac{P_{A'} \cdot \sin(\phi) - P_{yA'} \cdot \cos(\phi)}{EI} \right) \]  
(5)

Multiply both sides of an equation (5) by \( d\phi \), then taking integral of both sides, creating
\[
\int \frac{d^2 \varphi}{ds^2} \frac{d\varphi}{ds} ds = \int \left[ -\frac{P_{sA} \cdot \sin(\varphi) - P_{sA} \cdot \cos(\varphi)}{EI} \right] d\varphi
\]

From (6) we get

\[
\frac{1}{2} \left[ \frac{d\varphi}{ds} \right]^2 = -\frac{\left[ -P_{sA} \cdot \cos(\varphi_A) - P_{sA} \cdot \sin(\varphi_A) \right]}{EI} + C
\]

The constants of integration can be determined from the prescribed constraints (the boundary conditions) at the end point \( A' \), having

\[
\frac{1}{2} \left[ \frac{M_A}{EI} \right]^2 = -\frac{\left[ -P_{sA} \cdot \cos(\varphi_A) - P_{sA} \cdot \sin(\varphi_A) \right]}{EI} + C
\]

Incorporating constants of integration \( C \) to (7), we get

\[
\frac{1}{2} \left[ \frac{d\varphi}{ds} \right]^2 = \frac{1}{2} \left[ \frac{M_A}{EI} \right]^2 + \frac{\left[ P_{sA} \cdot \left( \cos \varphi - \cos \varphi_A \right) + P_{sA} \cdot \left( \sin \varphi - \sin \varphi_A \right) \right]}{EI}
\]

It can be written in compact form as follows

\[
\left[ \frac{d\varphi}{ds} \right]^2 = \left[ \frac{M_A}{EI} \right]^2 + \frac{2P_{sA} \cdot \left( \cos \varphi - \cos \varphi_A \right) + 2P_{sA} \cdot \left( \sin \varphi - \sin \varphi_A \right)}{EI}
\]

From (8) and (2), getting the function of bending moment

\[
M = \sqrt{\left[ \frac{M_A}{EI} \right]^2 + \frac{2P_{sA} \cdot \left( \cos \varphi - \cos \varphi_A \right) + 2P_{sA} \cdot \left( \sin \varphi - \sin \varphi_A \right)}{EI}}
\]

From (8) we get

\[
ds = \pm \frac{d\varphi}{\sqrt{\left[ \frac{M_A}{EI} \right]^2 + \frac{2P_{sA} \cdot \left( \cos \varphi - \cos \varphi_A \right) + 2P_{sA} \cdot \left( \sin \varphi - \sin \varphi_A \right)}{EI}}}
\]

The sign in (10) depends on convexo-concave type of elastic curve.

The axial displacement due to stretching is negligible in comparison with the normal displacement and the length of the bar remains unchanged. Integrating along the elastic curve of the deformed rod and from (8), getting

\[
L = \int_{\varphi_A}^{\varphi} ds = \int_{\varphi_A}^{\varphi} \frac{d\varphi}{\sqrt{\left[ \frac{M_A}{EI} \right]^2 + \frac{2P_{sA} \cdot \left( \cos \varphi - \cos \varphi_A \right) + 2P_{sA} \cdot \left( \sin \varphi - \sin \varphi_A \right)}{EI}}}
\]

Using geometrical differential relation \( dy = \sin(\varphi) ds; \ dz = \cos(\varphi) ds \), incorporating to (10), we get
\[
\begin{align*}
  y_B &= \int_0^{y_B} dy = \int_0^L \sin(\varphi) \, ds \\
  z_B &= \int_0^{z_B} dz = \int_0^L \cos(\varphi) \, ds
\end{align*}
\]  

(11’)

Where: \( y_B = Y_A \); \( Z_A = L - z_B \), incorporating \( ds \) from (10) to (11’) we get

\[
Y_A = \int_{\varphi_A}^{\varphi_B} \sin(\varphi) \, d\varphi \\
Z_A = L - \int_{\varphi_A}^{\varphi_B} \cos(\varphi) \, d\varphi
\]

The equilibrium equations are as follows:

\[
\sum z = 0 \rightarrow P_{zB} = P_{zA}
\]

(14)

\[
\sum y = 0 \rightarrow P_{yB} = P_{yA}
\]

(15)

\[
\sum m_B = 0 \rightarrow M_B = -M_A + P_{yA} \cdot Y_A + P_{zA} \cdot (L - Z_A)
\]

(16)

The system of equations from (11) to (16) is governing system for solving the nonlinear buckling problem.

3. Algorithm for solving the nonlinear buckling problem based on the arc length method

Choosing three displacement \( Y_A, \varphi_A, \varphi_B \) as primary unknowns. The equations (11), (12) and (2.13) are the solving equations, expressing as

\[
\int_{\varphi_A}^{\varphi_B} \sqrt{\frac{M_A^2}{EI} + \frac{2P_{yA}}{EI} \cdot (\cos \varphi - \cos \varphi_A) + \frac{2P_{zA}}{EI} \cdot (\sin \varphi - \sin \varphi_A)} \, d\varphi = -L = 0
\]

(17)

\[
\int_{\varphi_A}^{\varphi_B} \frac{\sin(\varphi) \, d\varphi}{\sqrt{\frac{M_A^2}{EI} + \frac{2P_{yA}}{EI} \cdot (\cos \varphi - \cos \varphi_A) + \frac{2P_{zA}}{EI} \cdot (\sin \varphi - \sin \varphi_A)}} = -Y_A
\]

(18)

\[
L - \int_{\varphi_A}^{\varphi_B} \sqrt{\frac{M_A^2}{EI} + \frac{2P_{yA}}{EI} \cdot (\cos \varphi - \cos \varphi_A) + \frac{2P_{zA}}{EI} \cdot (\sin \varphi - \sin \varphi_A)} \, d\varphi = Z_A
\]

(19)

Displacement vector of unknowns \( \mathbf{u} \) is expressed as
$\{ u, u_2, u_3 \}$

(20)

Components of left side of equations (17-19) is expressed as vector $q(u)$ following

$$q(u) = \{ q_1(u), q_2(u), q_3(u) \}^T$$

(21)

Components of right side of equations (17-19) is expressed as vector $P$ following

$$P = \{ P_1, P_2, P_3 \}^T \equiv \{ 0, 0, Z_A \}^T$$

(22)

The unknown $P_3 \equiv Z_A$ is used as a regulator-unknown in arc length method [7]

The system of equations (17-18) can be written in matrix form as

$$q(u) = P$$

(23)

Using arc length method incremental equation of equilibrium (23) is described following

$$K(u) \delta u = P + \Delta P - q(u)$$

(24)

Where:

$$K(u) = \frac{\partial q(u)}{\partial u} = \begin{bmatrix}
\frac{\partial q_1(u)}{\partial u_1} & \frac{\partial q_1(u)}{\partial u_2} & \frac{\partial q_1(u)}{\partial u_3} \\
\frac{\partial q_2(u)}{\partial u_1} & \frac{\partial q_2(u)}{\partial u_2} & \frac{\partial q_2(u)}{\partial u_3} \\
\frac{\partial q_3(u)}{\partial u_1} & \frac{\partial q_3(u)}{\partial u_2} & \frac{\partial q_3(u)}{\partial u_3}
\end{bmatrix}, \delta u = \{ \delta u_1, \delta u_2, \delta u_3 \}^T$$

The block diagram of algorithm for solving the nonlinear buckling problems is established (shown in Figure 2).
Figure 2. Incremental-iterative procedure for solving nonlinear buckling problem based on arc length technique

4. Numerical analysis
Based on proposed above incremental-iterative algorithm, the calculation program for solving nonlinear buckling problem is written in MathCAD software.

4.1. Example formulation
The inclined rod subjected to concentrated load and moment at the end $A$. The geometric parameters, material parameters and loading parameters are given:

$$M_A = 1500kN \cdot cm; L = 400cm; D = 10cm; d = 8.8cm; E = 2 \cdot 10^4 \frac{kN}{cm^2}$$

4.2. Numerical results
The calculating results are equilibrium path, limit points, relationship load-displacement at any point within the rod (shown in Fig. 4-6)

![Equilibrium path P-Δ](image)

The limit load is determined from equilibrium path, we get

$$P_{L_{limit}}^+ = 284.447kN$$ and $$P_{L_{limit}}^- = -80.278kN.$$
5. Summary
The proposed method for solving the geometrically nonlinear buckling problem of inclined rod is effective way in analytical approach.

The solution of the geometrically nonlinear buckling problem of inclined rod using the proposed analytical formulation has important advantage in possibility of getting the load-displacement relations at any point within element.

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