SOME CHARACTERIZATIONS OF RELATIVE SEQUENTIALLY COHEN-MACAULAY AND RELATIVE COHEN-MACAULAY MODULES

MAJID RAHRO ZARGAR

ABSTRACT. Let $M$ be an $R$-module over a Noetherian ring $R$ and $\mathfrak{a}$ be an ideal of $R$ with $c = \mathfrak{cd}(\mathfrak{a}, M)$. First, we prove that $M$ is finite $\mathfrak{a}$-relative Cohen-Macaulay if and only if $H_i(\Lambda \mathfrak{a}(H^c_0(M))) = 0$ for all $i \neq c$ and $H_i(\Lambda \mathfrak{a}(H^c_0(M))) \cong \hat{M}$

Throughout this paper, $R$ is a commutative Noetherian ring and $\mathfrak{a}$ is a proper ideal of $R$. In the case where $R$ is local with the maximal ideal $\mathfrak{m}$, $\hat{R}$ denotes the $\mathfrak{m}$-adic completion of $R$, $E_R(R/\mathfrak{m})$ denotes the injective hull of the residue field $R/\mathfrak{m}$ and $\hat{R}^\mathfrak{a}$ denotes the $\mathfrak{a}$-adic completion of $R$. Also, for an $R$-module $M$, the $R$-modules $H^\mathfrak{a}_i(M)$ for all $i$ denote the local cohomology modules, and $H_i(\Lambda \mathfrak{a}(M))$ for all $i$ denote the left derived functor of the $\mathfrak{a}$-adic completion $\Lambda \mathfrak{a}(-)$. The concept of sequentially Cohen-Macaulay was defined by combinatorial commutative algebraists [22, 3.9] to answer a basic question to find a non-pure generalization of the concept of a Cohen-Macaulay module so that the face ring of a shellable (non-pure) simplicial complex has this property. This concept was then applied by commutative algebraists to study some algebraic invariants or special algebras that come from graphs. Recall that a finitely generated $R$-module $M$ over a local ring $R$ is called sequentially Cohen-Macaulay if there exists a finite filtration

$0 = M_0 \subset M_1 \subset \cdots \subset M_{r-1} \subset M_r = M$ of $M$ by submodules of $M$ such that each quotient module $M_i/M_{i-1}$ is a Cohen-Macaulay $R$-module and that $\dim(M_1/M_0) < \dim(M_2/M_1) < \cdots < \dim(M_r/M_{r-1})$ (For more details see [19]). There are some nice characterizations of the concept of sequentially Cohen-Macaulay modules which one of them is the following basic theorem of J. Herzog and D. Popescu in [8, Theorem 2.4]:

**Theorem 1.1.** Let $(R, \mathfrak{m})$ be a $d$-dimensional Cohen-Macaulay local ring with the canonical $R$-module $\omega_R$ and $M$ be a finitely generated $R$-module. Then, the following statements are equivalent:

(i) $M$ is sequentially Cohen-Macaulay.

(ii) $\operatorname{Ext}^i_R(M, \omega_R)$ is zero or Cohen-Macaulay of dimension $i$ for all $i \geq 0$.

2010 Mathematics Subject Classification. 13D45, 13C14, 13D02.

Key words and phrases. Local Cohomology, Completion, Relative sequentially Cohen-Macaulay module, Relative Cohen-Macaulay module.
The author, in [14], introduced the notion of relative Cohen-Macaulay modules with respect to an ideal \( a \) as a natural generalization of the concept of Cohen-Macaulay modules over a local ring. Indeed, we say that an \( R \)-module \( M \) is \( a \)-relative Cohen-Macaulay if there is precisely one non-vanishing local cohomology module of \( M \) with respect to \( a \). Also, in [15], we studied the \( R \)-module \( D_a = \text{Hom}_R(H_a^c(R), E(R/\mathfrak{m})) \) in the case where \( R \) is a relative Cohen-Macaulay local ring with respect to \( a \) and \( c = \text{ht}_R a \). Indeed, we showed that these modules behave like the canonical modules over Cohen-Macaulay local rings. On the other hand, as a generalization of the notion of sequentially Cohen-Macaulay, the concept of relative sequentially Cohen-Macaulay was defined and studied by some authors (see [1], [13], [12] and for definition see 2.3). Therefore, in section 3, among other things we were able to provide the following result which can be considered a suitable generalization of the above-mentioned theorem of J. Herzog and D. Popescu.

**Theorem 1.2.** Let \((R, \mathfrak{m})\) be an \( a \)-relative Cohen-Macaulay local ring with \( \text{cd}(a, R) = c \). Then, the following statements hold:

(i) If \( M \) is finite \( a \)-relative sequentially Cohen-Macaulay, then \( \text{Ext}_R^{c-i}(M, D_a) \) is zero or is \( a \)-relative Cohen-Macaulay of cohomological dimension \( i \) for all \( i \geq 0 \).

(ii) If for all \( i \geq 0 \), \( \text{Ext}_R^{c-i}(M, D_a) \) is zero or is \( a \)-relative Cohen-Macaulay of cohomological dimension \( i \) and \( H_a^j(\text{Ext}_R^{c-i}(M, D_a)) = 0 \) for all \( j > i \), then \( M \otimes_R \hat{R} \) is \( a \)-relative sequentially Cohen-Macaulay.

R. N. Roberts, in [17], as a dual of the notion of Krull-dimension, introduced the Noetherian dimension, \( \text{Ndim} \ M \), for an Artinian \( R \)-module \( M \). On the other hand, the concept of co-regular sequence for Artinian \( R \)-module \( M \) was introduced by E. Matlis in [11] as a natural dual of the concept of regular sequence. The width of \( M \), denoted by \( \text{width}(M) \), is the length of any maximal \( M \)-co-regular sequence in \( \mathfrak{m} \). In general, for any Artinian \( R \)-module \( M \), \( \text{width} M \leq \text{Ndim} M < \infty \). Z. Tang and H. Zakeri, in [24], introduced the concept of co-Cohen-Macaulay for Artinian \( R \)-module \( M \). Indeed, they defined that \( M \) is co-Cohen-Macaulay if and only if \( \text{width} M = \text{Ndim} M \). Also, they proved that \( \text{width} M = \inf \{ i \mid \text{Tor}_i^R(R/\mathfrak{m}, M) \neq 0 \} \). Furthermore, in view of [21, Corollary 3.4.2] one has \( \text{width} M = \inf \{ i \mid H_i^R(M) \neq 0 \} \). Also, in [4, Propositions 4.8, 4.10], it was proved that \( \text{Ndim} M = \sup \{ i \mid H_i^R(M) \neq 0 \} \) whenever \( M \) is an Artinian module over a local ring \((R, \mathfrak{m})\). Therefore, an Artinian \( R \)-module \( M \) is co-Cohen-Macaulay if and only if \( H_i^R(M) = 0 \) for all \( i \neq \text{Ndim} M \). Hence, in section 4, as a generalization of this concept our main aim is to study the \( R \)-modules \( X \) such that its local homology modules with respect to an arbitrary ideal \( a \), \( H_i(\mathcal{L}A_a(X)) \), has just one non-vanishing point. Indeed, we are going to provide some characterizations for relative Cohen-Macaulay and relative sequentially Cohen-Macaulay modules in terms of local homology modules. In this direction, first we provide the two lemmas 4.1 and 4.2, to prove the first main theorem 4.3 as follows:

Let \( a \) be a proper ideal of \( R \) and \( M \) be an \( R \)-module with \( \text{cd}(a, M) = c \). Then, we have \( M \) is finite \( a \)-relative Cohen-Macaulay if and only if

\[
H_i(\mathcal{L}A_a(H_a^c(M))) \cong \begin{cases} 
\hat{M}^a & \text{if } i = c \\
0 & \text{otherwise,}
\end{cases}
\]
Z. Tang, in [23], proved that over a local ring \((R, m)\) if \(M\) is a \(d\)-dimensional Cohen-Macaulay \(R\)-module, then the \(R\)-module \(H^d_m(M)\) is co-Cohen-Macaulay, but the converse is not true, in general. Therefore, the above our result is a generalization of this result of Z. Tang. Also, it shows that the converse of [20, Theorem 1.2] is also true. Next, in Theorem 4.7 which is another main result in this section we provide a characterization of a relative sequentially Cohen-Macaulay as follows:

**Theorem 1.3.** Let \(a\) be an ideal of \(R\) and \(M\) be a finitely generated \(R\)-module and set \(T^i(-) := H^i_a(-)\) or \(T^i(-) := H^i_a(-)\). Then, consider the following statements:

(i) If \(M\) is a finite \(a\)-relative sequentially Cohen-Macaulay, then \(T^j(H^i_a(M)) = 0\) for all \(0 \leq i \leq \text{cd}(a, M)\) and for all \(j \neq i\) and \(T^j(H^i_a(M))\) is zero or \(a\)-relative Cohen-Macaulay of cohomological dimension \(i\).

(ii) If \(H_j(L\Lambda_a(H^i_a(M))) = 0\) for all \(0 \leq i \leq \text{cd}(a, M)\) and for all \(j \neq i\), then \(\tilde{M}^a\) is a finite \(a\)-relative sequentially Cohen-Macaulay as an \(\tilde{R}^a\)-module.

In [21, Theorem 10.5.9], it is shown that the local ring \((R, m)\) is a \(d\)-dimensional Gorenstein ring if and only if \(H_1(L\Lambda_m(E_R(k))) = 0\) for all \(i \neq d\) and \(H_d(L\Lambda_m(E_R(k))) \cong \hat{R}^m\). So, in this direction, we provide the following result which is an improvement of the above result.

**Theorem 1.4.** Let \((R, m)\) be a local ring of dimension \(d\) and let \(a\) be an ideal of \(R\). Then, the following statements are equivalent:

(i) \(R\) is Gorenstein

(ii) \(H_i(L\Lambda_m(E_R(k))) = 0\) for all \(i \neq d\) and \(H_d(L\Lambda_m(E_R(k))) \cong \hat{R}^m\).

(iii) There exists an ideal \(a\) of \(R\) with \(\text{cd}(a, R) = c\) such that \(H_i(L\Lambda_a(E_R(k))) = 0\) for all \(i \neq c\) and \(\text{fd}_R(H_c(L\Lambda_a(E_R(k)))) < \infty\).

### 2. Prerequisites

An \(R\)-complex \(X\) is a sequence of \(R\)-modules \((X_v)_{v \in \mathbb{Z}}\) together with \(R\)-linear maps \((\partial^X_v : X_v \rightarrow X_{v-1})_{v \in \mathbb{Z}}\),

\[
X = \cdots \rightarrow X_{v+1} \xrightarrow{\partial^X_v} X_v \xrightarrow{\partial^X_v} X_{v-1} \rightarrow \cdots,
\]
such that \(\partial^X_v \partial^X_{v+1} = 0\) for all \(v \in \mathbb{Z}\). The derived category of \(R\)-modules are denoted by \(\text{D}(R)\) and we use the symbol \(\simeq\) for denoting isomorphisms in \(\text{D}(R)\). Any \(R\)-module \(M\) can be considered as a complex having \(M\) in its 0-th spot and 0 in its other spots. We denote the full subcategory of homologically left (resp. right) bounded complexes by \(\text{D}_<(R)\) (resp. \(\text{D}_>(R)\)). Let \(X \in \text{D}(R)\) and/or \(Y \in \text{D}(R)\). The left-derived tensor product complex of \(X\) and \(Y\) in \(\text{D}(R)\) is denoted by \(X \otimes^L_R Y\) and is defined by

\[
X \otimes^L_R Y \simeq F \otimes_R Y \simeq X \otimes_R F' \simeq F \otimes_R F',
\]

where \(F\) and \(F'\) are flat resolutions of \(X\) and \(Y\), respectively. Also, let \(X \in \text{D}(R)\) and/or \(Y \in \text{D}(R)\). The right derived homomorphism complex of \(X\) and \(Y\) in \(\text{D}(R)\) is denoted by \(\text{RHom}_R(X, Y)\) and is defined by

\[
\text{RHom}_R(X, Y) \simeq \text{Hom}_R(P, Y) \simeq \text{Hom}_R(X, I) \simeq \text{Hom}_R(P, I),
\]
where $P$ and $I$ are the projective resolution of $X$ and injective resolution of $Y$, respectively. For any two complexes $X$ and $Y$, we set $\operatorname{Ext}^i_R(X,Y) = H_{-i}(R \operatorname{Hom}_R(X,Y))$ and $\operatorname{Tor}^i_R(X,Y) = H_i(X \otimes^R_R Y)$. For any integer $n$, the $n$-fold shift of a complex $(X, \xi^X)$ is the complex $\Sigma^n X$ given by $(\Sigma^n X)_v = X_{v-n}$ and $\xi^\Sigma^n X = (-1)^n \xi^X_v$. Also, we have $H_i(\Sigma^n X) = H_{i-n}(X)$. Next, for any contravariant, additive and exact functor $T : \mathcal{D}(R) \to \mathcal{D}(R)$ and $X \in \mathcal{D}(R)$, we have $H_i(T(X)) \cong T(H_{-i}(X))$ for all $i \in \mathbb{Z}$. For any $R$-module $M$ and ideal $a$ of $R$, set $\Gamma_a(M) := \{ x \in M \mid \operatorname{Supp}_R Rx \subseteq V(a) \}$. Now, for any $R$-complex $X$ in $\mathcal{D}(R)$, the right derived functor of the functor $\Gamma_a(-)$ in $\mathcal{D}(R)$, $R \Gamma_a(X)$, exists and is defined by $R \Gamma_a(X) := \Gamma_a(I)$, where $I$ is any injective resolution of $X$. Also, for any integer $i$, the $i$-th local cohomology module of $X$ with respect to $a$ is defined by $H^a_i(X) := H_{-i}(R \Gamma_a(X))$.

Let $a$ be an ideal of $R$ and $M$ an $R$-module. The $i$-th local homology module $H^a_i(M)$ of $M$ with respect to $a$ is defined by

$$H^a_i(M) = \lim_{\longrightarrow} \operatorname{Tor}^i(\mathcal{D}(R/a^n, M) = \lim_{\longrightarrow} H_i(R/a^n \otimes_R F_\bullet),$$

where $F_\bullet$ is a projective resolution of $M$. On the other hand, we use $\Lambda_a(M) := \lim_{\longrightarrow} (R/a^n \otimes_R M)$ to denote the $a$-adic completion of $M$. It is known that the functor of the $a$-adic completion, $\Lambda_a(-)$, is an additive covariant functor from the category of $R$-modules and $R$-homomorphisms to itself. We denote by $H_i(\mathcal{L}A_a(M))$ the $i$-th left derived module of $\Lambda_a(M)$. Since the tensor functor is not left exact and the inverse limit is not right exact on the category of $R$-modules, the functor $\Lambda_a(-)$ is neither left nor right exact. Therefore $H_0(\mathcal{L}A_a(M)) \neq \Lambda_a(M)$, in general. Notice that $H_i(\mathcal{L}A_a(M)) = H_i((\lim_{\longrightarrow} (R/a^n \otimes_R F_\bullet))$.

Thus, there are natural maps $\phi_i : H_i(\mathcal{L}A_a(M)) \to H^a_i(M)$, which are epimorphisms by [5, (1.1)]. Also, in view of [4, Proposition 4.1], these maps are isomorphisms provided $M$ is Artinian. Clearly, $H^a_0(M) \cong \Lambda_a(M)$. Moreover, if $M$ is a finitely generated module, then by [4, Remark 2.1(iii)] for all $i > 0$, $H_i(\mathcal{L}A_a(M)) = 0$ and so $H^a_i(M) = 0$. Let $\underline{x} = x_1, \ldots, x_t$ denote a system of elements in $R$ and $a = (\underline{x})R$. Then, we recall the definition of the Čech, $C_\underline{x}$, and its free resolution $\hat{L}_{\underline{x}}$ as done in [21, 6.2.2 and 6.2.3]. For any $R$-module $M$, one has $H_i(\mathcal{L}A_a(M)) = H_i(\mathcal{H}om_R(\hat{L}_{\underline{x}}, M))$, (see [21] for more details and generalizations).

**Definition 2.1.** We say that an $R$-module $M$ is relative Cohen-Macaulay with respect to $a$ if there is precisely one non-vanishing local cohomology module of $M$ with respect to $a$. In the case where $M$ is finitely generated, clearly this is the case if and only if $\text{grade}(a, M) = \text{cd}(a, M)$, where $\text{cd}(a, M)$ is the largest integer $i$ for which $H^a_i(M) \neq 0$. For convenience, we use the notation $a$-relative Cohen-Macaulay, for an $R$-module which is relative Cohen-Macaulay with respect to $a$. Also, in the case where $(R, m)$ is a relative Cohen-Macaulay local ring with respect to $a$ and $c = \text{ht}_R a$, we define the $R$-module $D_a := \operatorname{Hom}_R(H^a_0(R), E(R/m))$ as a relative canonical $R$-module.

Observe that the above definition provides a generalization of the concept of Cohen-Macaulay modules. Also, notice that the notion of relative Cohen-Macaulay modules is connected with the notion of cohomologically complete intersection ideals which has been studied in [6] and has led to some interesting results. Furthermore, recently, such modules have been studied in [7], [14] and [16]. In [15], we showed
the $R$-module $D_a$ treat like canonical modules over Cohen-Macaulay local rings. Indeed, we provided the following result which will be used in section 3 of the present paper.

**Theorem 2.2.** Let $(R, m)$ be an $a$-relative Cohen-Macaulay local ring with $\text{ht}_R a = c$. Then, the following statements hold:

(i) For all ideals $b$ of $R$ such that $a \subseteq b$, $H^i_b(D_a) = 0$ if and only if $i \neq c$.

(ii) $\text{id}_R(D_a) = c$.

(iii) $\hat{R} \cong \text{Hom}_R(D_a, D_a)$ and $\text{Ext}^i_R(D_a, D_a) = 0$ for all $i > 0$.

(iv) For all $t = 0, 1, \ldots, c$ and for all finitely generated $a$-relative Cohen-Macaulay $R$-modules $M$ with $\text{cd}(a, M) = t$, one has:

(a) $\text{Ext}^i_{R, a}(M, D_a) = 0$ if and only if $i \neq t$.

(b) $H^0_a(\text{Ext}^{c-t}_{R, a}(M, D_a)) = 0$ if and only if $i \neq t$

**Definition 2.3.** Let $M$ be an $R$-module and $a$ be an ideal of $R$. Then we say that $M$ is *sequentially Cohen-Macaulay* with respect to $a$ (Abb. $a$-sequentially Cohen-Macaulay) if there exists a finite filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_{r-1} \subset M_r = M$$

of $M$ by submodules of $M$ such that each quotient module $M_i/M_{i-1}$ is an $a$-relative Cohen-Macaulay $R$-module and that $\text{cd}(a, M_1/M_0) < \text{cd}(a, M_2/M_1) < \cdots < \text{cd}(a, M_{r-1}/M_{r-2}) < \text{cd}(a, M/M_{r-1})$. In the case where, $M$ is finitely generated a sequentially Cohen-Macaulay with respect to $a$ is called *finite $a$-sequentially Cohen-Macaulay*.

Here notice that the concept of relative sequentially Cohen-Macaulay is a generalization of the notion of sequentially Cohen-Macaulay which is defined in the introduction. Indeed, for a finitely generated $R$-module $M$ over a local ring $(R, m)$, one has $\text{cd}(m, M) = \dim_R(M)$ and also we can see that the concept of $m$-relative Cohen-Macaulay modules coincide with the ordinary concept of Cohen-Macaulay modules, and so $m$-sequentially Cohen-Macaulay modules are precisely the sequentially Cohen-Macaulay modules.

Also, it is clear that every $a$-relative Cohen-Macaulay $R$-modules are also $a$-relative sequentially Cohen-Macaulay, but in general, the converse is no longer true. For this purpose, let $R = k[[x, y]]$ where $k$ is a field, $M = k[[x, y]]/(xy)$ and $a = (x)$. Notice that $\text{cd}(a, M) = 1$ and $\Gamma_a(M) \neq 0$; and so $M$ is not $a$-relative Cohen-Macaulay. But, $0 \subset \Gamma_a(M) \subset M$ is a finite filtration such that $\Gamma_a(M)$ and $M/\Gamma_a(M)$ are $a$-relative Cohen-Macaulay $R$-modules with cohomological dimension 0 and 1, respectively. Therefore $M$ is a finite $a$-relative sequentially Cohen-Macaulay $R$-module which is not finite $a$-relative Cohen-Macaulay.

3. Relative Sequentially Cohen-Macaulay and Relative Canonical modules

The starting point of this section is the next two lemmas, which will play an essential role in proving our main result 3.3.

**Lemma 3.1.** Let $(R, m)$ be an $a$-relative Cohen-Macaulay local ring with $\text{cd}(a, R) = c$ and suppose that $M$ is a finite $a$-relative sequentially Cohen-Macaulay module with the filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_{r-1} \subset M_r = M$ and set $\text{cd}(a, M_i/M_{i-1}) := c_i$. Then $\text{Ext}^{c_i}_{R, a}(M, D_a) = 0$ for all $j \notin \{c_1, \ldots, c_r\}$ and $\text{Ext}^{c_i}_{R, a}(M, D_a)$ is $a$-relative Cohen-Macaulay of cohomological dimension $c_i$ for all $i = 1, \ldots, r$. 

Proof. We use induction on \( r \) to prove the result. To do this, let \( r = 1 \), then by our assumption \( M \) is finite \( \alpha \)-relative Cohen-Macaulay with \( \text{cd}(\alpha, M) = c_1 \). So, by Theorem 2.2, one has \( \text{Ext}^{c-i}_R(M, D_a) = 0 \) for all \( i \neq c_1 \) and \( \text{Ext}^{c-i}_R(M, D_a) \) is an \( \alpha \)-relative Cohen-Macaulay module with \( \text{cd}(\alpha, \text{Ext}^{c-i}_R(M, D_a)) = c_1 \). Now, assume that \( r > 1 \) and the result has been proved for all finite \( \alpha \)-relative sequentially Cohen-Macaulay modules with a finite filtration of length \( r - 1 \). Notice that \( M/M_1 \) is a finite \( \alpha \)-relative sequentially Cohen-Macaulay \( R \)-module with the following finite filtration of length \( r - 1 \):

\[
0 = M_1/M_1 \subset M_2/M_1 \subset \cdots \subset M_r/M_1 = M/M_1.
\]

Therefore, by induction hypothesis one has \( \text{Ext}^{c-j}_R(M/M_1, D_a) = 0 \) for all \( j \notin \{c_2, \ldots, c_r\} \) and \( \text{Ext}^{c-j}_R(M/M_1, D_a) \) is an \( \alpha \)-relative Cohen-Macaulay of cohomological dimension \( c_i \) for all \( i = 2, \ldots, r \).

Now consider the short exact sequence \( 0 \rightarrow M_1 \rightarrow M \rightarrow M/M_1 \rightarrow 0 \) and the induced long exact sequence

\[
\cdots \rightarrow \text{Ext}^{c-i}_R(M/M_1, D_a) \rightarrow \text{Ext}^{c-i}_R(M, D_a) \rightarrow \text{Ext}^{c-i}_R(M_1, D_a)
\]

\[
\cdots \rightarrow \text{Ext}^{c-i}_R(M/M_1, D_a) \rightarrow \text{Ext}^{c-j}_R(M, D_a) \rightarrow \text{Ext}^{c-j}_R(M_1, D_a)
\]

\[
\cdots \rightarrow \text{Ext}^{c-j}_R(M/M_1, D_a) \rightarrow \text{Ext}^{c-j}_R(M, D_a) \rightarrow \text{Ext}^{c-j}_R(M_1, D_a)
\]

\[
\cdots \rightarrow \text{Ext}^{c-j}_R(M/M_1, D_a) \rightarrow \text{Ext}^{c-j}_R(M_1, D_a) \rightarrow \text{Ext}^{c-j+1}_R(M_1, D_a) \rightarrow \cdots.
\]

Notice that, by Theorem 2.2, \( \text{Ext}^{c-j}_R(M/M_1, D_a) = 0 \) for all \( j \neq c_1 \) and \( \text{Ext}^{c-j}_R(M/M_1, D_a) \) is an \( \alpha \)-relative Cohen-Macaulay with \( \text{cd}(\alpha, \text{Ext}^{c-j}_R(M/M_1, D_a)) = c_1 \). Therefore, by the above induced sequence one has the isomorphisms \( \text{Ext}^{c-j}_R(M, D_a) \cong \text{Ext}^{c-j}_R(M/M_1, D_a) \) for \( j \neq c_1, c_1 - 1 \) and the following exact sequence:

\[
0 \rightarrow \text{Ext}^{c-j}_R(M/M_1, D_a) \rightarrow \text{Ext}^{c-j}_R(M, D_a) \rightarrow \text{Ext}^{c-j}_R(M_1, D_a)
\]

\[
\rightarrow \text{Ext}^{c-j-1}_R(M/M_1, D_a) \rightarrow \text{Ext}^{c-j+1}_R(M_1, D_a) \rightarrow 0.
\]

Since \( \text{Ext}^{c-j}_R(M/M_1, D_a) = 0 \) for \( j = c_1, c_1 - 1 \), one has \( \text{Ext}^{c-j}_R(M, D_a) \cong \text{Ext}^{c-j}_R(M_1, D_a) \) and \( \text{Ext}^{c-j+1}_R(M_1, D_a) = 0 \), as required.

\[\square\]

Lemma 3.2. Let \( \alpha \) be an ideal of \( R \) with \( \text{cd}(\alpha, R) = c_1 \), \( M \) be a finitely generated \( R \)-module and \( D \) be an \( R \)-module which the following conditions are established:

(i) \( \text{Hom}_R(D, D) \) is flat and \( \text{Ext}^i_R(D, D) = 0 \) for all \( i > 0 \).

(ii) For all \( i \), we have \( \text{Ext}^{c-i}_R(M, D) \) is zero, or

(a) \( \text{Ext}^{c-j}_R(\text{Ext}^{c-i}_R(M, D), D) = 0 \) if and only if \( j \neq i \), and

(b) \( \mathbb{H}^j(\text{Ext}^{c-i}_R(\text{Ext}^{c-j}_R(M, D), D)) = 0 \) if and only if \( j \neq i \).

Then \( M \otimes_R \text{Hom}_R(D, D) \) is \( \alpha \)-sequentially Cohen-Macaulay.

Proof. Let \( P_{\bullet} : \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow 0 \) be a deleted free resolution with finitely generated terms for \( M \) and \( I_{\bullet} : 0 \rightarrow I^0 \rightarrow I^1 \rightarrow \cdots \rightarrow I^c \rightarrow \cdots \) be a deleted injective resolution for \( D \). Now, consider the first quadrant double complex \( M_{p,q} := \text{Hom}_R(\text{Hom}_R(P_p, D), I^{-q}) \). Let \( I^1 E \) (resp. \( I^1 \mathbb{H}^j \)) denote the vertical (resp. horizontal) spectral sequence associated to the double complex \( M = \{M_{p,q}\} \). Now, with the notation of [18], \( E^1 \) is the bigraded module whose \( (p, q) \) term is \( H_q^p(M_{p, \ast}) \), the \( q \)-th homology of the \( p \)-th column. Therefore, \( H_q^p(M_{p, \ast}) = \text{Ext}^{c-q}_R(\text{Hom}_R(P_p, D), D) \), and so in view of (i) one has

\[
I^1 E^1_{p,q} = H_q^p(M_{p, \ast}) = \begin{cases} 0 & \text{if } q \neq c \\ \text{Hom}_R(\text{Hom}_R(P_p, D), D) & \text{if } q = c, \end{cases}
\]
As \( P_p \) is finite free module for all \( p \), then in view of \([3, \text{Theorem 2.5.6}]\), one has \( \text{Hom}_R(\text{Hom}_R(P_\bullet, D), D) \simeq P_\bullet \otimes_R \text{Hom}_R(D, D) \). So, we have

\[
I E^2_{p,q} = H^i_p H^\infty_q(M) = \begin{cases} 
0 & \text{if } q \neq c \\
\text{Tor}_p^0(M, \text{Hom}_R(D, D)) & \text{if } q = c.
\end{cases}
\]

Hence, in view of (i), \( I E^2_{p,q} = 0 \) for all \( p > 0 \) and so the spectral sequence collapses on the \( q \)-axis. Therefore, by \([18, \text{Proposition 10.21}]\), one has \( I E^3_{p,c} = M \otimes_R \text{Hom}_R(D, D) \cong H_c(\text{Tot}(M)) \).

A similar argument applies to the second iterated homology, using the fact that each \( I_{c-q} \) is injective, yields \( I E^2_{p,q} = H^i_p H^q_q(M) = \text{Ext}^{p+1}_p(\text{Ext}^q_p(M, D), D) \) and that \( I E^2_{p,q} \Rightarrow H_{p+q}(\text{Tot}(M)) \). Thus, for all \( p, q \) such that \( c = p + q \), there is the following filtration

\[
0 = \Phi^{-1} H_c \subseteq \Phi^0 H_c \subseteq \ldots \subseteq \Phi^{c-1} H_c \subseteq \Phi^c H_c = H_c(\text{Tot}(M)) = M \otimes_R \text{Hom}_R(D, D)
\]

such that \( \text{Ext}^\infty_{p,c-p} := I E^\infty_{p,c-p} = \Phi^p H_c/\Phi^{p-1} H_c. \)

On the other hand, by our assumption \((ii)\), one has \( \text{Ext}^{p+1}_p(\text{Ext}^q_p(M, D), D) = 0 \) for all \( p \neq c - q \) and so \( E^3_{p,q} = 0 \) for all \( p \neq c - q \). Next, for each \( r \geq 2 \) and \( p \geq 0 \), let

\[
Z^r_{p,c-p} := \text{Ker}(E^r_{p,c-p} \longrightarrow E^r_{p-r,c-p+r-1})
\]

and

\[
B^r_{p,c-p} := \text{Im}(E^r_{p+r,c-p-r+1} \longrightarrow E^r_{p,c-p}).
\]

By using the assumption, \( E^r_{p-r,c-p+r-1} = E^r_{p+r,c-p-r+1} = 0 \) for all \( r \geq 2 \) and \( p \geq 0 \), because for all \( p \), \( E^r_{p,c-p} \) is subquotients of \( E^2_{p,c-p} \) and so \( E^r_{p,c-p} = Z^r_{p,c-p} \) and \( B^r_{p,c-p} = 0 \). On the other hand, one has \( E^{r+1}_{p,c-p} = Z^r_{p,c-p} \), and thus \( E^3_{p,c-p} \cong \text{Ext}^3_{p,c-p} \cong \ldots \cong \text{Ext}^\infty_{p,c-p} \). Therefore, by our assumption \((iii)\) for all \( p \), \( \text{Ext}^{p+1}_p(\text{Ext}^q_p(M, D), D) = 0 \) is a-relative Cohen-Macaulay of cohomological dimension \( p \). Therefore, \( \Phi^p H_c/\Phi^{p-1} H_c \) is a-relative Cohen-Macaulay of cohomological dimension \( p \) for all \( 0 \leq p \leq c \), and so the above filtration is a a-relative Cohen-Macaulay filtration for \( M \otimes_R \text{Hom}_R(D, D) \), as required.

\[\square\]

**Theorem 3.3.** Let \((R, \mathfrak{m})\) be an a-relative Cohen-Macaulay local ring with \( \text{cd}(a, R) = c \). Then the following statements hold:

(i) If \( M \) is finite a-relative sequentially Cohen-Macaulay, then \( \text{Ext}^{c-i}_R(M, D_a) \) is zero or is a-relative Cohen-Macaulay of cohomological dimension \( i \) for all \( i \geq 0 \).

(ii) If for all \( i \geq 0 \), \( \text{Ext}^{c-i}_R(M, D_a) \) is zero or is a-relative Cohen-Macaulay of cohomological dimension \( i \) and \( H^i_\bullet(\text{Ext}^{c-i}_R(M, D_a), D_a) = 0 \) for all \( j > i \), then \( M \otimes_R \tilde{R} \) is a-relative sequentially Cohen-Macaulay.

**Proof.** The statement (i) is an immediate consequence of Lemma 3.1. For the statement (ii), first in view of Theorem 2.2(iii) one has \( \tilde{R} \cong \text{Hom}_R(D_a, D_a) \) and \( \text{Ext}^i_R(D_a, D_a) = 0 \) for all \( i > 0 \). Also, by our assumption and \([15, \text{Corollary 3.2}]\), we have \( \text{Ext}^{c-j}_R(\text{Ext}^{c-j}_R(M, D_a), D_a) = 0 \) if and only if \( j \neq i \).

Now, we show that for all \( i \), \( H^i_\bullet(\text{Ext}^{c-j}_R(\text{Ext}^{c-j}_R(M, D_a), D_a)) = 0 \) for all \( j < i \), which to do this it is enough to show that \( \text{Ext}^j_R(R/a, \text{Ext}^{c-j}_R(\text{Ext}^{c-j}_R(M, D_a), D_a)) = 0 \) for all \( j < i \). To do this, notice that
since $\text{Ext}^{i-1}_R(M, D_a)$ is $a$-relative Cohen-Macaulay of cohomological dimension $i$, in view of [15, Theorem 3.1(iii)], we have $\text{Tor}^R_{i-1}(R/a, \text{Ext}^{i-1}_R(M, D_a)) \cong \text{Tor}^R_i(R/a, H^i_a(\text{Ext}^{i-1}_R(M, D_a)))$ for all $t$. Therefore, one can use [15, Corollary 3.2] to get the following isomorphisms

$$\text{Ext}^i_j(R/a, \text{Ext}^{i-1}_R(M, D_a), D_a)) \cong (\text{Tor}^R_j(R/a, H^i_a(\text{Ext}^{i-1}_R(M, D_a))))^\vee$$

$$\cong (\text{Tor}^R_j(R/a, \text{Ext}^{i-1}_R(M, D_a)))^\vee,$$

which implies that $\text{Ext}^i_j(R/a, \text{Ext}^{i-1}_R(M, D_a), D_a)) = 0$ for all $j < i$, as required. Therefore, by our assumption we can see that $\text{Ext}^{c-1}_R(\text{Ext}^{c-1}_R(M, D_a), D_a)$ is a-relative Cohen-Macaulay of cohomological dimension $i$, for all $i$. Therefore, the assertion is done by Lemma 3.2.

The following result is a consequence of Lemma 3.2 which has already been proved by J. Herzog and D. Popescu in [8, Theorem 2.4].

**Corollary 3.4.** Let $(R, m)$ be a $d$-dimensional Cohen-Macaulay local ring with the canonical $R$-module $\omega_R$ and $M$ be a finitely generated $R$-module. Then, the following statements are equivalent:

(i) $M$ is sequentially Cohen-Macaulay.

(ii) $\text{Ext}^{c-1}_R(M, \omega_R)$ is zero or Cohen-Macaulay of dimension $i$ for all $i \geq 0$.

**Proof.** First notice that for all finitely generated $R$-modules $X$ we have $\dim_R(X) = \dim_\hat{R}(X \otimes R \hat{R})$, $\text{depth}_R(X) = \text{depth}_R(X \otimes R \hat{R})$ and $\text{id}_R(X) = \text{id}_R(X \otimes R \hat{R}) = \text{depth} R$. On the other hand, , it is clear to see that

$$\omega_\hat{R} \cong \omega_R \otimes R \hat{R}$$

$$\cong \text{Hom}_R(H^0_\omega(R), E_R(R/m))$$

$$\cong \text{Hom}_\hat{R}(H^0_\omega(R) \otimes R \hat{R}, E_R(R/m))$$

$$\cong \text{Hom}_\hat{R}(H^0_\omega(\hat{R}), E_R(\hat{R}/m\hat{R}))$$

$$= D_m\hat{R}.$$ 

Hence, for all integers $i$, one has

$$\text{Ext}^{d-i}_\hat{R}(M \otimes R \hat{R}, D_m\hat{R}) \cong \text{Ext}^{d-i}_\hat{R}(M \otimes R \hat{R}, \omega_\hat{R})$$

$$\cong \text{Ext}^{d-i}_\hat{R}(M, \omega_R) \otimes R \hat{R}.$$ 

Therefore, for the implication (ii) $\implies$ (i), in view of Theorem 2.2 and the fact that $\hat{R}$ is faithfully flat, one can deduce that $\text{id}_R(\omega_R) = \dim R$, $R \cong \text{Hom}_R(\omega_R, \omega_R)$ and $\text{Ext}^i_R(\omega_R, \omega_R) = 0$ for all $i > 0$. Also, by our assumption and Theorem 3.3(iv), for all $i = 0, 1, \ldots, \dim R$, the $i$-dimensional finitely generated Cohen-Macaulay $R$-modules $\text{Ext}^{d-i}_R(M, \omega_R)$, have the following conditions:

(a) $\text{Ext}^{d-i}_R(\text{Ext}^{d-i}_R(M, \omega_R), \omega_R) = 0$ if and only if $j \neq i$.

(b) $H^2_\omega(\text{Ext}^{d-i}_R(M, \omega_R), \omega_R) = 0$ if and only if $j \neq i$.

Therefore, one can use Lemma 3.2 to complete the proof. Note that, by use of definition, one can check that $M \otimes R \hat{R}$ is a sequentially Cohen-Macaulay $\hat{R}$-module whenever $M$ is a sequentially Cohen-Macaulay $R$-module. Hence, the implication (i) $\implies$ (ii) follows from Theorem 3.3(i) and the above two isomorphisms (1) and (2).
4. Relative sequentially Cohen-Macaulay and local homology

Based on the discussion made in the introduction, an Artinian $R$-module $M$ is co-Cohen-Macaulay if and only if $H^m_i(M) = 0$ for all $i \neq \text{Ndim } M$. Hence, in this section, as a generalization of this concept, our main aim is to study the $R$-modules $X$ such that its local homology modules concerning an arbitrary ideal $a$, $H_i(\Lambda a(X))$, has just one non-vanishing point. Indeed, in this section, we are going to provide some characterizations for relative Cohen-Macaulay and relative sequentially Cohen-Macaulay modules.

The starting point of this section is the next two lemmas which play an essential role in the proof of the next main theorem.

**Lemma 4.1.** Let $X_\bullet := \cdots \to X_{i+1} \overset{d_{i+1}}\to X_i \overset{d_i}\to X_{i-1} \to \cdots$ be an $R$-complex such that $H_i(X_\bullet) = 0$ for all $i \neq c$, then there exists the isomorphism $\Sigma^c H_c(X_\bullet) \simeq X_\bullet$ in derived category $D(R)$.

**Proof.** Consider the following quasi-isomorphisms:

\[
\begin{array}{ccccccccccccccccccc}
\Sigma^c H_c(X_\bullet) & : & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & H_c(X_\bullet) & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
\downarrow & & & & & \downarrow & & & & i & \downarrow & & & & 0 & \downarrow & \downarrow \\
\subset_c X_\bullet & : & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \frac{X_c}{\ker d_c} & \overset{\pi}\longrightarrow & X_{c-1} & \overset{d_{c-1}}\longrightarrow & X_{c-2} & \longrightarrow & \cdots \\
\uparrow & & & & & & & & & & & & & & & \uparrow \\
X_\bullet & : & \cdots & \longrightarrow & X_{c+2} & \overset{d_{c+2}}\longrightarrow & X_{c+1} & \overset{d_{c+1}}\longrightarrow & X_c & \overset{d_c}\longrightarrow & X_{c-1} & \longrightarrow & X_{c-2} & \longrightarrow & \cdots,
\end{array}
\]

and

\[
\begin{array}{ccccccccccccccccccc}
\Sigma^c H_c(X_\bullet) & : & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & H_c(X_\bullet) & \longrightarrow & 0 & \longrightarrow & \cdots \\
\downarrow & & & & & \downarrow & & & & i & \uparrow & & & & 0 & \downarrow & \downarrow \\
X_{c, \subseteq} & : & \cdots & \longrightarrow & X_{c+2} & \overset{d_{c+2}}\longrightarrow & X_{c+1} & \overset{\text{ker } d_c}{\longrightarrow} & 0 & \longrightarrow & \cdots \\
\downarrow & & & & & & & & & & & & & & \downarrow \\
\Sigma^c H_c(X_\bullet) & : & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & H_c(X_\bullet) & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}
\]

Therefore, there exists the quasi-isomorphisms

\[
\begin{align*}
\Sigma^c H_c(X_\bullet) & \xrightarrow{\sim} \subset_c X_\bullet \xleftarrow{\sim} X_\bullet \quad \text{and} \\
\Sigma^c H_c(X_\bullet) & \xleftarrow{\sim} X_{c, \subseteq} \xrightarrow{\sim} X_\bullet,
\end{align*}
\]

where $\subset_c X_\bullet$ and $X_{c, \subseteq}$ are the soft left truncation and the soft right truncation of the complex $X_\bullet$, respectively, for more detail see [2, A.1.14]. Therefore, by the definition of isomorphism in derived category $D(R)$ one has $\Sigma^{-i} H^c(M) = 0$ for all $i \neq c$. Then, $H_c(\Lambda a(H^c(M))$ is a nonzero $a$-relative Cohen-Macaulay $R$-module of cohomological dimension $c$.

**Lemma 4.2.** Let $M$ be an $R$-module, $a$ be an ideal of $R$ such that for a fix integer $c$, $H^c(M) \neq 0$ and $H_i(\Lambda a(H^c(M))) = 0$ for all $i \neq c$. Then, $H_c(\Lambda a(H^c(M)))$ is a nonzero $a$-relative Cohen-Macaulay $R$-module of cohomological dimension $c$. 

\[\square\]
Proof. First notice that by our assumption and Lemma 4.1 we have the isomorphism $\Sigma^c H_a(\Lambda_a(H^c_a(M))) \cong \Lambda_a(H^c_a(M))$, and so $H_c(\Lambda_a(H^c_a(M))) \cong \Sigma^{-c} \Lambda_a(H^c_a(M))$. Next, consider the following isomorphisms in derived category $D(R)$:

\[
\text{R} \Gamma_a(H_c(\Lambda_a(H^c_a(M)))) \cong \text{R} \Gamma_a(\Sigma^{-c} \Lambda_a(H^c_a(M))) \\
\cong \hat{C} \otimes_R \Sigma^{-c} \Lambda_a(H^c_a(M)) \\
\cong \Sigma^{-c} (\hat{C} \otimes_R \Lambda_a(H^c_a(M))) \\
\cong \Sigma^{-c} \text{R} \Gamma_a(\Lambda_a(H^c_a(M))) \\
\cong \Sigma^{-c} \text{R} \Gamma_a(H^c_a(M)),
\]

where the second and last isomorphisms follow from [21, Theorem 7.4.4] and [21, Theorem 9.1.3], respectively. Therefore, for all $i$, one has the following isomorphisms:

\[
H^i_a(H_c(\Lambda_a(H^c_a(M)))) \cong H_{-i}(\text{R} \Gamma_a(H_c(\Lambda_a(H^c_a(M)))) \\
\cong H_{-i}(\Sigma^{-c} \text{R} \Gamma_a(H^c_a(M))) \\
\cong H_{-i+c}(\text{R} \Gamma_a(H^c_a(M))) \\
\cong H^{-i-c}_a(H^c_a(M)),
\]

which show that $H^i_a(H_c(\Lambda_a(H^c_a(M)))) = 0$ if and only if $i \neq c$, as required. \qed

The following theorem, which is one of our main results, provides a characterization of a finite $a$-relative Cohen-Macaulay $R$-module $M$ in terms of vanishing of the local homology modules of the local cohomology module $H^{cd(a,M)}_a(M)$. Furthermore, it shows that the converse of [20, Theorem 1.2] is also true.

**Theorem 4.3.** Let $a$ be an ideal of $R$ and $M$ be an $R$-module with $cd(a,M) = c$. Then, the following statements hold:

(i) If $M$ is $a$-relative Cohen-Macaulay, then there exists the following isomorphism:

\[
\lim_{n \in \mathbb{N}} \text{Tor}_i^R(R/a^n, H^c_a(M)) \cong \lim_{n \in \mathbb{N}} \text{Tor}_i^R(R/a^n, M),
\]

for all integers $i$. In particular, we have $H^i_{-i+c}(H^c_a(M)) \cong H^i_a(M)$ for all $j$.

(ii) If $M$ is finite $a$-relative Cohen-Macaulay, then

\[
H^i_a(H^c_a(M)) \cong \begin{cases} 
\hat{M}^a & \text{if } i = c \\
0 & \text{otherwise},
\end{cases}
\]

(iii) $M$ is finite $a$-relative Cohen-Macaulay if and only if

\[
H_i(\Lambda_a(H^c_a(M)) \cong \begin{cases} 
\hat{M}^a & \text{if } i = c \\
0 & \text{otherwise},
\end{cases}
\]

**Proof.** (i). Let

\[
I^* : 0 \rightarrow I_0 \xrightarrow{d_0} I_{-1} \xrightarrow{d_{-1}} \cdots \rightarrow I_{-c+1} \xrightarrow{d_{-c+1}} I_{-c} \xrightarrow{d_{-c}} I_{-c-1} \rightarrow \cdots
\]
be a deleted injective resolution for $M$. By applying the functor $\Gamma_a(-)$ on this complex, we obtain the complex $\Gamma_a(I^*)$. As, $H^c_a(M) = H_{-c}(\Gamma_a(I^*))$, by our assumption and Lemma 4.1 we have $\Sigma^{-c}H^c_a(M)$ and $\Gamma_a(I^*)$ are isomorphic in derived category modules $D(R)$. Now, as $\Gamma_a(I^*)$ is a bounded above complex of injective modules, by \cite[Lemma 3.3.8]{3} one has $\Gamma_a(I^*)$ is a semi-injective complex. So, in view of \cite[Proposition 4.1.11]{3} there exists a quasi-isomorphism $f : \Sigma^{-c}H^c_a(M) \xrightarrow{\sim} \Gamma_a(I^*)$. Then, we have the following functorial morphisms in the derived category $D(R)$:

$$(1) \quad - \otimes_R \Sigma^{-c}H^c_a(M) \xrightarrow{id(-) \otimes_R f} - \otimes_R \Gamma_a(I^*)$$

$$(\psi(-, I^*)) \quad \xrightarrow{\psi(-, I^*)} R\Gamma_a(- \otimes_R I^*)$$

$$(\iota(- \otimes_R I^*)) \quad \xrightarrow{\iota(- \otimes_R I^*)} - \otimes_R I^*$$

Here, we should notice that for any semi-projective complex $P_\bullet$, the functorial morphisms $i_{P_\bullet} \otimes_R f$, by \cite[Theorem 3.4.5]{3}, is quasi-isomorphism and $\psi(-, I^*)$ follows from \cite[Corollary 3.3.1]{10} which is always an isomorphism. Also, the morphism $\iota(- \otimes_R I^*)$ follows from \cite[Corollary 3.2.1]{10} which is isomorphism for all $R$-complex $C$ such that $\text{Supp}_R(C) \subseteq V(a)$. Therefore, there is a functorial isomorphism $f_{P_\bullet} : P_\bullet \otimes_R \Sigma^{-c}H^c_a(M) \rightarrow P_\bullet \otimes_R I^*$ in Derived category $D(R)$ for each deleted projective resolution $P_\bullet$ of an $\alpha$-torsion $R$-module $N$. Now, let $n < m$ be natural numbers and $P^m_\bullet$ and $P^n_\bullet$ be deleted projective resolutions for $R/a^m$ and $R/a^n$, respectively. Consider the natural map $\gamma^{m}_n : R/a^m \rightarrow R/a^n$ which induces the $R$-complex morphism $(\gamma^{m}_n)_{i \in N} : P^m_\bullet \rightarrow P^n_\bullet$. Therefore, one can obtain the following commutative diagrams

$$
\begin{array}{c}
P^m_\bullet \otimes_R \Sigma^{-c}H^c_a(M) \xrightarrow{f_{P^m_\bullet}} P^m_\bullet \otimes_R I^*
\\
(\gamma^{m}_n)_{i \in N} \otimes_R id_{\Sigma^{-c}H^c_a(M)} \downarrow
\\
P^n_\bullet \otimes_R \Sigma^{-c}H^c_a(M) \xrightarrow{f_{P^n_\bullet}} P^n_\bullet \otimes_R I^*
\end{array}
$$

Now, taking homology of the above commutative diagram yields the following commutative diagram:

$$
\begin{array}{c}
\text{Tor}^R_{i+c}(R/a^m, H^c_a(M)) \xrightarrow{\dagger} \text{Tor}^R_i(R/a^m, M)
\\
\downarrow
\\
\text{Tor}^R_{i+c}(R/a^n, H^c_a(M)) \xrightarrow{\dagger} \text{Tor}^R_i(R/a^n, M)
\end{array}
$$

for all integers $i$, where $\dagger$ and $\ddagger$ are isomorphisms. Here, one should notice that $H_i(P^m_\bullet \otimes_R \Sigma^{-c}H^c_a(M)) = H_i(\Sigma^{-c}(P^m_\bullet \otimes_R H^c_a(M))) = H_{i+c}(P^n_\bullet \otimes_R H^c_a(M)) = \text{Tor}^R_{i+c}(R/a^m, H^c_a(M))$, and also $H_i(P^n_\bullet \otimes_R I^*) = \text{Tor}^R_{i}(R/a^m, M)$. Therefore, one can deduce that $\lim_{n \in \mathbb{N}} \text{Tor}^R_{i+c}(R/a^n, H^c_a(M)) \simeq \lim_{n \in \mathbb{N}} \text{Tor}^R_{i}(R/a^n, M)$ and thus $H^c_{i+c}(H^c_a(M)) \simeq H^c_i(M)$, as required.

(ii). It follows from the isomorphism in (i) and \cite[Remark 3.2(ii)]{4}.

(iii). For the implication $\Rightarrow$, first in view of Lemma 4.1 we have the isomorphism $\Sigma^{-c}H^c_a(M) \simeq R\Gamma_a(M)$. On the other hand, by \cite[Theorem 9.1.3]{21} there is the isomorphism $L\Lambda_\alpha(R\Gamma_a(M) \simeq L\Lambda_\alpha(M)$ in derived category $D(R)$, and also in view of \cite[Theorem 7.5.12]{21} one has $L\lambda_\alpha(-) \simeq R\text{Hom}_R(\check{C}_\alpha, -)$.
Therefore, one can get the following isomorphisms in the derived category \( D(R) \):
\[
H_i(\mathcal{L}\Lambda_a(M)) \cong H_i(\mathcal{L}\Lambda_a(\mathcal{R}\Gamma_a(M))) \\
\cong H_i(\mathcal{L}\Lambda_a(\Sigma^{-c}\mathcal{H}_a^c(M))) \\
\cong H_i(\mathcal{R}\text{Hom}_R(\mathcal{C}_a^\text{c}, \Sigma^{-c}\mathcal{H}_a^c(M)))) \\
\cong H_i(\Sigma^{-c}\mathcal{R}\text{Hom}_R(\mathcal{C}_a^\text{c}, \mathcal{H}_a^c(M)))) \\
\cong H_{i+c}(\mathcal{R}\text{Hom}_R(\mathcal{C}_a^\text{c}, \mathcal{H}_a^c(M)))) \\
\cong H_{i+c}(\mathcal{L}\Lambda_a(H_a^c(M)))).
\]
Hence, by [4, Remark 2.1](iii) the assertion is done. For the implication \( \Leftarrow \), by Lemma 4.2 and our assumption one has \( H_a^c(\mathcal{M}^\alpha) = 0 \) for all \( i \neq c \). On the other hand, by the Flat Base Change Theorem and Independent Theorem we have the following isomorphisms for all integers \( i \):
\[
H_a^i(\mathcal{M}^\alpha) \cong H_{aR^\alpha}(M \otimes_R \mathcal{R}^\alpha) \\
\cong H_a^i(M) \otimes_R \mathcal{R}^\alpha \\
\cong H_a^i(M).
\]
Here we should notice the last isomorphism holds because the \( R \)-module \( H_a^i(M) \) is \( a \)-torsion, and so has \( \mathcal{R}^\alpha \)-module structure and also, by [9, Lemma 1.4], there exists the isomorphism \( H_a^i(M) \otimes_R \mathcal{R}^\alpha \cong H_a^i(M) \) both as \( R \)-modules and \( \mathcal{R}^\alpha \)-modules. Therefore, we can deduce that \( H_a^i(M) = 0 \) for all \( i \neq c \) which says that \( M \) is \( a \)-relative Cohen-Macaulay.

In [4, Proposition 4.1] it is shown that for any Artinian \( R \)-module \( M \) there is an isomorphism \( H_a^i(M) \cong H_i(\mathcal{L}\Lambda_a(M)) \) for all \( i \). Also we know that \( H_a^i(M) \) is Artinian for all \( i \) and all finitely generated \( R \)-modules \( M \). So, the following result is an immediate consequence of the previous theorem.

**Corollary 4.4.** Let \((R, m)\) be a local ring and \( M \) be a finitely generated \( R \)-module of dimension \( d \). Then the following statements are equivalent:

(i) \( M \) is Cohen-Macaulay.

(ii)
\[
H_i^m(H_a^i(M)) \cong \begin{cases} 
\overline{M}^m & \text{if } i = d \\
0 & \text{otherwise.}
\end{cases}
\]

Notice that the local homology and cohomology functors are \( R \)-linear. So, the following result follows from Theorem 4.3.

**Corollary 4.5.** Let \( a \) be an ideal of \( R \) and \( M \) be a finitely generated \( a \)-relative Cohen-Macaulay \( R \)-module with \( \text{cd}(a, M) = c \). Then \( \text{Ann}_R(H_a^c(M)) = \text{Ann}_R(M) \).

The following lemma has an essential role in the proof of the next theorem, which is one of the main results in the present paper.

**Lemma 4.6.** Let \( M \) be a finite \( a \)-relative sequentially Cohen-Macaulay module with the filtration \( 0 = M_0 \subset M_1 \subset \cdots \subset M_{r-1} \subset M_r = M \) and set \( \text{cd}(a, M_i/M_{i-1}) := c_i \). Set \( T^i(-) := H_a^i(-) \) or \( T^i(-) := H_i(\mathcal{L}\Lambda_a(-)) \). Then
(i) $H^i_a(M) = 0$ for all $i \notin \{c_1, \ldots, c_r\}$.
(ii) $\mathcal{T}^j(H^i_a(M)) = 0$ for all $j \neq i$ where $i \in \{c_1, \ldots, c_r\}$.
(iii) $\mathcal{T}^{c_i}(H^i_a(M))$ is $a$-relative Cohen-Macaulay of cohomological dimension $c_i$ for all $i = 1, \ldots, r$.

Proof. We use induction on $r$ to prove the result. To do this, let $r = 1$, then by our assumption $M$ is finite $a$-relative Cohen-Macaulay module with $\text{cd}(a, M) = c_1$. So, by Theorem 4.3, one has $\mathcal{T}^j(H^1_a(M)) = 0$ for all $j \neq c_1$ and $\mathcal{T}^{c_1}(H^1_a(M)) \cong \widehat{M}^a$ which implies that is an $a$-relative Cohen-Macaulay module with cohomological dimension $c_1$. Now, assume that $r > 1$ and the result has been proved for all finite $a$-relative sequentially Cohen-Macaulay modules with a finite filtration of length $r - 1$. Notice that $M/M_1$ is a sequentially $R$-module with the following finite filtration of length $r - 1$:

$$0 = M_1/M_1 \subset M_2/M_1 \subset \cdots \subset M_r/M_1 = M/M_1.$$ 

Therefore, by induction hypothesis we have $H^i_a(M/M_1) = 0$ for all $i \notin \{c_2, \ldots, c_r\}$, $\mathcal{T}^j(H^i_a(M/M_1)) = 0$ for all $j \neq i$ where $i \in \{c_2, \ldots, c_r\}$ and $\mathcal{T}^{c_i}(H^i_a(M/M_1))$ is $a$-relative Cohen-Macaulay of cohomological dimension $c_i$ for all $i = 2, \ldots, r$. Now, consider the short exact sequence $0 \rightarrow M_1 \rightarrow M \rightarrow M/M_1 \rightarrow 0$ and the induced long exact sequence

$$\cdots \rightarrow H^i_a(M_1) \rightarrow H^i_a(M) \rightarrow H^i_a(M/M_1) \rightarrow H^{i+1}_a(M_1) \rightarrow H^{i+1}_a(M) \rightarrow \cdots$$

Therefore, by the above induced sequence one has the isomorphism $H^i_a(M) \cong H^i_a(M/M_1)$ for $i \neq c_1, c_1 - 1$ and the following exact sequence

$$0 \rightarrow H^{c_i-1}_a(M) \rightarrow H^{c_i-1}_a(M/M_1) \rightarrow H^{c_i}_a(M_1) \rightarrow H^{c_i}_a(M) \rightarrow H^{c_i}_a(M/M_1) \rightarrow 0.$$

As $H^i_a(M/M_1) = 0$ for $i = c_1, c_1 - 1$, one has $H^{c_i}_a(M_1) \cong H^{c_i}_a(M)$ and $H^{c_i-1}_a(M) = 0$. Therefore, using the hypothesis assumption and Theorem 4.3 for the $R$-module $M_1$ completes the proof. \qed

**Theorem 4.7.** Let $a$ be an ideal of $R$ and $M$ be a finitely generated $R$-module and set $\mathcal{T}^i(-) := H^i_a(-)$ or $\mathcal{T}^i(-) := H_i(L\Lambda_a(-))$. Then, the following statements hold:

(i) If $M$ is a finite $a$-relative sequentially Cohen-Macaulay, then $\mathcal{T}^j(H^i_a(M)) = 0$ for all $0 \leq i \leq \text{cd}(a, M)$ and for all $j \neq i$ and $\mathcal{T}^j(H^i_a(M))$ is zero or $a$-relative Cohen-Macaulay of cohomological dimension $i$.

(ii) If $H_j(L\Lambda_a(H_a^1(M))) = 0$ for all $0 \leq i \leq \text{cd}(a, M)$ and for all $j \neq i$, then $\widehat{M}^a$ is a finite $a$-relative sequentially Cohen-Macaulay as an $\widehat{R}^a$-module.

Proof. The statement (i) immediately follows from Lemma 4.6. For the statement (ii), we first prove that it will be enough to show the claim for $M = \widehat{M}^a$ the $a$-adic completion of $M$ over the completed ring $\widehat{R}^a$. This follows because, in view of [21, Corollary 9.8.2], one has $H_j(L\Lambda_a(H_a^1(M))) \cong H_j(L\Lambda_a(H_a^1(M) \otimes_R \widehat{R}^a)) \cong H_j(L\Lambda_{\widehat{a}}(\widehat{H}_a^1(\widehat{M}^a)))$. Now, consider $\hat{L}_x$ and $E_x^\bullet(M)$ as a bounded free resolution of the Cech complex $C_x$ and a deleted injective resolution for $M$, respectively. Let

$$\hat{L}_x := 0 \rightarrow L^0_x \rightarrow L^1_x \rightarrow \cdots \rightarrow L^{k-1}_x \rightarrow \hat{L}^k_x \rightarrow 0,$$

and

$$\Gamma_a(E_x^\bullet(M)) := \cdots \leftarrow \Gamma_a(E_{-1}) \leftarrow \cdots \Gamma_a(E_{-1}) \leftarrow \Gamma_a(E_0) \leftarrow 0,$$
where \( k \) is the number of generators of \( \mathfrak{a} \) and for all \( i \geq 0 \), \( \Gamma_\mathfrak{a}(E_{-i}) = \Gamma_\mathfrak{a}(E^i) \). Now, consider the forth quadrant double complex \( \mathcal{M} = (M_{p,-q} = \text{Hom}_R(L^p,\Gamma_\mathfrak{a}(E_{-q}))) \) for all \( q \geq 0 \) and \( 0 \leq p \leq k \) with the total complex \( \text{Tot}(M)_n = \bigoplus_{p+q=n} M_{p,-q} \) for \( n \leq k \). Next, consider the first filtration defined by \( I^p \text{Tot}(\mathcal{M}) := \bigoplus_{i\geq p} M_{i,i-n} \) and note that since over line \( p+q = n \) there are finitely non zero terms of double complex, then this filtration is bounded for all \( n \). Now, with the notation of [18], \( E^1 \) is the bigraded module whose \((p,-q)\) term is \( H^p_{m}(M_{p,-q}) \), the -q-th homology of the p-th column. Therefore, \( H^p_\mathfrak{a}(M, \mathcal{M}) = \text{Hom}_R(L^p, H^p_\mathfrak{a}(M)) \), and so \( E^2_{p,-q} = H^p_\mathfrak{a}(H^p_\mathfrak{a}(M)) = H^p_\mathfrak{a}(\Lambda_\mathfrak{a}(H^p_\mathfrak{a}(M))) \). Now, as only a finite number of rows \( M_{p,-q} \) are non-zero, then the first spectral sequence converges, that is, \( I^p E^2_{p,-q} = H^p_\mathfrak{a}(\Lambda_\mathfrak{a}(H^p_\mathfrak{a}(M))) \implies H_{p-q}(\text{Tot}(\mathcal{M})) \)

Therefore, for all \( q \geq 0 \) and \( 0 \leq p \leq k \) there is the following filtration in the stage \( p-q = 0 \)

\[
0 = \Phi_{-1}H^0 \subseteq \Phi_0H^0 \subseteq \ldots \subseteq \Phi_kH^0 \subseteq \Phi_kH^0 = H^0(\text{Tot}(\mathcal{M}))
\]

such that \( I^p E^\infty_{p,-q} \cong \Phi_pH^0/\Phi_{p-1}H^0 \).

Also, by our assumption we have \( \Phi_pH^0/\Phi_{p-1}H^0 = 0 \) for all \( p \neq q \) and \( I^p E^\infty_{p,-p} \cong \Phi_pH^0/\Phi_{p-1}H^0 \) for all \( p \). Therefore, by our assumption \( \Phi_pH^0/\Phi_{p-1}H^0 \) is zero or \( \mathfrak{a} \)-relative Cohen-Macaulay \( R \)-module of cohomological dimension \( p \) and so \( H^0(\text{Tot}(\mathcal{M})) \) is \( \mathfrak{a} \)-relative sequentially Cohen-Macaulay. On the other hand, first notice that \( \text{Supp}_R(L_\mathfrak{a}) \subseteq \mathcal{V}(\mathfrak{a}) \) and so by [20, Lemma 2.3] one has the \( R \)-complex isomorphism \( \text{Hom}_R(L_\mathfrak{a}, \Gamma_\mathfrak{a}(E^\bullet(M))) \cong \text{Hom}_R(L_\mathfrak{a}, E^\bullet(M)) \). Also, as \( E^\bullet(M) \) is an injective resolution for \( M \), there exists the quasi isomorphism \( M \xrightarrow{\sim} E^\bullet(M) \), and hence we have the quasi isomorphism \( \text{Hom}_R(L_\mathfrak{a}, M) \xrightarrow{\sim} \text{Hom}_R(L_\mathfrak{a}, E^\bullet(M)) \). Therefore, one can use the following isomorphisms

\[
\text{H}_0(\text{Tot}(\mathcal{M})) \cong H^0(\text{Hom}_R(L_\mathfrak{a}, \Gamma_\mathfrak{a}(E^\bullet(M))))
\]

\[
\cong \text{H}_0(\text{Hom}_R(L_\mathfrak{a}, M))
\]

\[
\cong \text{H}_0(\Lambda_\mathfrak{a}(M))
\]

\[
\cong \mathcal{M}^\mathfrak{a},
\]

to deduce that \( \mathcal{M}^\mathfrak{a} \) is \( \mathfrak{a} \)-relative sequentially Cohen-Macaulay, as required.

\[\square\]

Here we should notice that if \( c \) is a fix integer such that \( \text{H}_j(\Lambda_\mathfrak{a}(H^c_\mathfrak{a}(M))) = 0 \) for all \( i \), then in view of [21, Corollary 3.4.2] one has \( \text{Tor}^R_0(R/\mathfrak{a}, H^c_\mathfrak{a}(M)) = 0 \) for all \( i \). Therefore, by [21, Theorem 3.2.6] we have that \( \text{Ext}^R_1(R/\mathfrak{a}, H^c_\mathfrak{a}(M)) = 0 \) for all \( i \), which implies that \( \text{H}^c_\mathfrak{a}(H^c_\mathfrak{a}(M)) = 0 \) for all \( i \). Hence \( H^c_\mathfrak{a}(M) = 0 \).

So, in the statement (i) of the previous theorem, there exists at least one integer \( i \) such that \( \text{T}^i(H^c_\mathfrak{a}(M)) \) is nonzero.

The following result is an immediate consequence of Theorem 4.7 for the case where \( \mathfrak{a} = \mathfrak{m} \).

**Corollary 4.8.** Let \((R, \mathfrak{m})\) be a local ring and \( M \) be a finitely generated \( R \)-module. Then, the following statements hold:

(i) If \( M \) is sequentially Cohen-Macaulay, then \( \text{H}_j(\Lambda_{\mathfrak{m}}(H^c_{\mathfrak{m}}(M))) = 0 \) for all \( 0 \leq t \leq \dim M \) and for all \( j \neq t \).
(ii) If $H_j(\Lambda_m(H^t_m(M))) = 0$ for all $0 \leq t \leq \dim M$ and for all $j \neq t$, then $\hat{M}$ is a sequentially Cohen-Macaulay $\hat{R}$-module.

**Definition and Remarks 4.9.** Let $(R, \mathfrak{m})$ be a local ring of dimension $d$ and $M$ be a finitely generated $R$-module with $n = \dim R M$. For an integer $i \in \mathbb{Z}$, define $K^j_M := H^j(\operatorname{Hom}_R(M, D^+_R))$ where $D^+_R$ is a normalized dualizing complex for $R$. The module $K^j_M := K^j_M$ is called the canonical module of $M$. For $i \neq n$ the modules $K^j_M$ are called the modules of deficiency of $M$. By the local duality theorem, for all $i$ there are the following canonical isomorphisms:

$$H^i_m(M) \cong \operatorname{Hom}_R(K^i_M, E_R(R/m)).$$

Recall that all of the $K^i_M$, $i \in \mathbb{Z}$, are finitely generated $R$-modules. Moreover $M$ is a Cohen-Macaulay module if and only if $K^i_M = 0$ for all $i \neq n$. Whence the modules of deficiencies of $M$ measure the deviation of $M$ from being a Cohen-Macaulay module. In the case of $(R, \mathfrak{m})$ the quotient of a local Gorenstein ring $(S, \mathfrak{n})$ there are the following isomorphisms $K^i_M = \operatorname{Ext}_S^{s-i}(M, S)$, $s = \dim S$, for all $i \in \mathbb{Z}$. Therefore, one has $K^i_M \cong \operatorname{Hom}_R(H^i_m(M), E_R(R/m)) \cong \operatorname{Hom}_{\hat{R}}(\hat{H}^i_m(\hat{M}), E_{\hat{R}}(\hat{R}/\mathfrak{m}\hat{R}))$ and is finitely generated $\hat{R}$-module for all $i$.

**Proposition 4.10.** Let $(R, \mathfrak{m})$ be a local ring with a dualizing complex and $M$ be a finitely generated $R$-module. Then, for any integer $t$, the following statements are equivalent:

(i) $K^t_M$ is zero or a Cohen-Macaulay $R$-module of dimension $t$.

(ii) $H_j(\Lambda_m(H^t_m(M))) = 0$ for all $j \neq t$.

**Proof.** First, consider the following isomorphisms:

$$H_j(\Lambda_m(H^t_m(M))) \cong H_j(\operatorname{Hom}_R(L_m, H^t_m(M))$$

$$\cong H_j(\operatorname{Hom}_R(L_m, \operatorname{Hom}_R(K^t_M, E_R(R/m))))$$

$$\cong H_j(\operatorname{Hom}_R(L_m \otimes_R K^t_M, E_R(R/m)))$$

$$\cong \operatorname{Hom}_R(H^j(L_m \otimes_R K^t_M), E_R(R/m))$$

$$\cong \operatorname{Hom}_R(H^i_m(K^t_M), E_R(R/m)).$$

Here, the last isomorphism follows from [21, Theorem 7.4.4]. Therefore, $K^t_M$ is a Cohen-Macaulay if and only if $H_j(\Lambda_m(H^t_m(M))) = 0$ for all $j \neq t$, as required.

It is well-known that a complete local ring $(R, \mathfrak{m})$ with respect to $\mathfrak{m}$-adic completion has a dualizing complex. Therefore, we provide the following result as an immediate consequence of Corollary 4.8 and Proposition 4.10.

**Corollary 4.11.** Let $(R, \mathfrak{m})$ be a complete local ring with respect to $\mathfrak{m}$-adic completion and $M$ be a finitely generated $R$-module. Then, the following statements are equivalent:

(i) $M$ is a sequentially Cohen-Macaulay $R$-module.

(ii) $K^t_M$ is zero or a Cohen-Macaulay $R$-module of dimension $t$ for all $0 \leq t \leq \dim M$.

In [20, Corollary 5.4], for a $d$-dimensional finitely generated $R$-module $M$ over the local ring $(R, \mathfrak{m})$ it is proven that $K^d_M$ is a Cohen-Macaulay $\hat{R}$-module if and only if $H_j(\Lambda_m(H^d_m(M))) = 0$ for all $j \neq d$. So,
the following corollary, which is a consequence of Proposition 4.10, is a generalization of [20, Corollary 5.4].

**Corollary 4.12.** Let \((R, \mathfrak{m})\) be a local ring and \(M\) be a finitely generated \(R\)-module. Then, for any \(0 \leq t \leq \dim M\) the following statements are equivalent:

(i) \(K^t_M\) is zero or a Cohen-Macaulay \(\hat{R}\)-module of dimension \(t\).

(ii) \(H_j(L\Lambda_m(H^t_m(M))) = 0\) for all \(j \neq t\).

**Proof.** First notice that, by [21, Corollary 2.2.5], \(H_j(L\Lambda_m(H^t_m(M))) \cong H_j(L\Lambda_{\hat{R}}(H^t_m(\hat{M})))\) for all \(j\). Hence we may and do assume that \(R\) is a complete local ring, and so it has a dualizing complex. Therefore, the proof is complete by Proposition 4.10. \(\square\)

In [21, Theorem 10.5.9], it is shown that \(R\) is a \(d\)-dimensional Gorenstein ring if and only if \(H_0(L\Lambda_m(E_R(k))) = 0\) for all \(i \neq d\) and \(H_d(L\Lambda_m(E_R(k))) \cong \hat{R}^m\). So, in this direction we provide the following result:

**Theorem 4.13.** Let \((R, \mathfrak{m})\) be a local ring of dimension \(d\). and let \(\mathfrak{a}\) be an ideal of \(R\). Then, the following statements are equivalent:

(i) \(R\) is Gorenstein

(ii) \(H_i(L\Lambda_{\mathfrak{a}}(E_R(k))) = 0\) for all \(i \neq d\) and \(H_d(L\Lambda_{\mathfrak{a}}(E_R(k))) \cong \hat{R}^m\).

(iii) There exist an ideal \(\mathfrak{a}\) of \(R\) with \(\mathfrak{a}d(\mathfrak{a}, R) = c\) such that \(H_i(L\Lambda_{\mathfrak{a}}(E_R(k))) = 0\) for all \(i \neq c\) and \(\text{fd}_R(H_c(L\Lambda_{\mathfrak{a}}(E_R(k)))) < \infty\).

**Proof.** The implication \((i) \implies (ii)\) follows from Corollary 4.4 and the fact that \(R\) is Cohen-Macaulay and \(H^d_\mathfrak{a}(R) \cong E_R(k)\). The implication \((ii) \implies (iii)\) is obvious. For the implication \((iii) \implies (i)\), in view of [21, Corollary 9.2.5] one has \(H_i(L\Lambda_{\mathfrak{a}}(E_R(k))) \cong \text{Hom}_R(H^c_\mathfrak{a}(R), E_R(k))\) for all \(i\). Therefore, \(H^c_\mathfrak{a}(R) = 0\) for all \(i \neq c\), and also \(H_c(L\Lambda_{\mathfrak{a}}(E_R(k))) \cong \text{Hom}_R(H^c_\mathfrak{a}(R), E_R(k))\). Hence, by our assumption \(\text{fd}_R(\text{Hom}_R(H^c_\mathfrak{a}(R), E_R(k)))\) is finite. On the other hand, by [15, Theorem 3.4], we have \(H^c_\mathfrak{a}(\text{Hom}_R(H^c_\mathfrak{a}(R), E_R(k))) = 0\) for all \(i \neq c\) and also by [15, Lemma 3.5] one has \(H^c_\mathfrak{a}(\text{Hom}_R(H^c_\mathfrak{a}(R), E_R(k))) \cong E_R(k)\). Therefore, by [16, Theorem 3.2], \(\text{fd}_R(E_R(k))\) is finite which it follows that \(R\) is Gorenstein, as required. \(\square\)

**Acknowledgments.** The author would like to thank Dr. Mohsen Gheibi and Prof. Hossein Zakeri for their valuable and profound comments during the preparation of the manuscript, and Prof. Kamaran Divaani-Aazar for having a helpful conversation with him.

**References**

[1] A. Atazadeh, M. Sedghi and R. Naghipour, *Cohomological dimension filtration and annihilators of top local cohomology modules*, Colloquium Mathematicum, 139, (2015), 25–35.

[2] L.W. Christensen, *Gorenstein Dimensions*, Lecture notes in Mathematics, Springer-Verlag, Berlin, 2000.

[3] L.W. Christensen and H-B. Foxby, *Hyperhomological Algebra with Applications to Commutative Rings*, in preparation.

[4] N.T. Cuong and T.T. Nam, *The i-adic completion and local homology for Artinian modules*, Math. Proc. Cambridge Philos. Soc., (131)(1), (2001), 61–72.
[5] J. P. C. Greenlees and J. P. May, Derived functors of $I$-adic completion and local homology, Journal of Algebra (149), (1992), 438–453.
[6] M. Hellus and P. Schenzel, On cohomologically complete intersections, J. Algebra, 320, (2008), 3733–3748.
[7] M. Hellus and P. Schenzel, Notes on local cohomology and duality, J. Algebra 401, (2014), 48–61.
[8] J. Herzog, D. Popescu, Finite filtrations of modules and shellable multisimplices, Manuscripta Math. 121, (2006), 385–410.
[9] B. Kubik, M. J. Leamer, and S. Sather-Wagstaff, Homology ofartinian and mini-max modules, I, J. Pure Appl. Algebra 215, no. 10, (2011), 2486–2503.
[10] J. Lipman, Lectures on local cohomology and duality, Local cohomology and its applications (Guanajuato, 1999), Lecture Notes in Pure and Appl. Math., 226, Dekker, New York, (2002), 39–89.
[11] E. Matlis, The Koszul complex and duality, Communications in algebra, (1)(2), (1974), 87–144.
[12] P. L. Majd and A. Rahimi, On the structure of sequentially Cohen-Macaulay bigraded modules. Czech Math J, 65, (2015), 1011–1022.
[13] A. Rahimi, Sequentially Cohen-Macaulayness of bigraded modules, Rocky Mountain J. Math., 65(2), 2017, 621–635.
[14] M. Rahro Zargar and H. Zakeri, On injective and Gorenstein injective dimensions of local cohomology modules, Algebra Colloq. 22, (2015), 935–946.
[15] M. Rahro Zargar, Relative canonical module and some duality results, Algebra Colloq. 26(02), (2019), 351–360.
[16] M. Rahro Zargar and H. Zakeri, On flat and Gorenstein flat dimensions of local cohomology modules, Canad. Math. Bull. 59, (2016), 403–416.
[17] R.N. Roberts, Krull-dimension for Artinian modules over quasi local commutative Rings, Quart. J. Math. Oxford Ser. 26(2), no. 103, (1975), 269–273.
[18] J.J. Rotman, An introduction to homological algebra, Second ed., Springer, New York, 2009.
[19] P. Schenzel, On the dimension filtration and Cohen-Macaulay filtered modules, in: F. Van Oystaeyen (Ed.), Commutative Algebra and Algebraic Geometry, in: Lecture Notes Pure Appl. Math., vol. 206, Dekker, New York, 1999, pp. 245–264.
[20] P. Schenzel, Notes on endomorphisms, local cohomology and completion, arXiv preprint arXiv:2105.00664, 2021 - arxiv.org.
[21] P. Schenzel and A. M. Simon, Completion, Čech and local homology and cohomology, Interactions between them., Cham: Springer, 2018.
[22] R.P. Stanley, Combinatorics and Commutative Algebra, Birkhäuser, Basel, 1983.
[23] Z. Tang, Local Homology and Local Cohomology, Algebra Colloq. 11(4), (2004), 467–476.
[24] Z. Tang and H. Zakeri, Co-Cohen-Macaulay modules and modules of generalized fractions, Communications in Algebra, 22, no. 6, (1994), 2173–2204.

Majid Rahro Zargar, Department of Engineering Sciences, Faculty of Advanced Technologies, University of Mohaghegh Ardabili, Namin, Ardabil, Iran,

Email address: zargar9077@gmail.com
Email address: m.zargar@uma.ac.ir