Evaluating Stability Conditions for a Rotating Droplet in a Potential Flow

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Abstract. The paper considers issues related to droplet fragmentation in air flow. We present the results of calculating the equilibrium conditions for a rotating droplet. The shape of the droplet is assumed to be ellipsoidal, and the flow is potential. The droplet equilibrium is determined by the balance of pressure due to the surface tension force, the centrifugal force and the aerodynamic effect force of the flow. We obtained an analytic expression linking the equilibrium values of the Weber number, the ellipsoid aspect ratio and the dimensionless swirl parameter. The calculation results showed that droplet swirling has a significant effect on the equilibrium conditions in a narrow range of variation in the ellipsoid aspect ratio. We obtained critical values of the Weber number and the aspect ratios for which the equilibrium state of the droplet becomes unstable.

1. Introduction

Research and development of advanced nozzle devices inevitably involves a thorough analysis of spray characteristics, such as uniformity, dispersion and flow stability [1, 2].

There are several forms of jet decay, but the last stage in all of them is the same, that is, jet instability and its fragmentation into a set of droplets in accordance with the classical Rayleigh mechanism. According to his theory, a cylindrical jet of liquid is unstable and decays into droplets if the amplitude of small perturbations symmetric about the axis of the jet reaches half its diameter. At the same time the ratio \( \frac{\lambda}{d_0} = 4.5 \), where \( \lambda \) is the perturbation wavelength, and \( d_0 \) is the initial jet diameter. This theoretical value \( \frac{\lambda}{d_0} \) corresponds to the average droplet diameter after decay of \( 1.89 \cdot d_0 \) [1].

Rayleigh believed that only the surface tension force prevents the decay, not taking the viscosity of the liquid into account. Subsequently, Weber took the effect of viscosity into account and obtained an expression for the value \( \frac{\lambda}{d_0} \) corresponding to the maximum instability of a viscous jet

\[
\frac{\lambda}{d_0} = \pi \sqrt{2 \left( 1 + 3 \frac{\mu_{liq}}{\rho_{liq}} \sqrt{\frac{d_0}{\sigma}} \right)^{0.5}},
\]

where \( \mu_{liq} \) and \( \rho_{liq} \) are the liquid viscosity and density. When \( \mu_{liq} = 0 \), the value of \( \frac{\lambda}{d_0} \) is equal to 4.44, which is close to the value of 4.5 obtained by Rayleigh. In experimental studies of decay of a viscous liquid jet [3] at \( \mu_{liq} = 0.86 \) the value \( \frac{\lambda}{d_0} \) was between 30 and 40.

In the paper [1] four jet decay modes were observed: droplet formation in response to ambient air; droplet formation in the absence of this effect; wave generation due to the friction between the liquid and air; and the final decay of the jet. In the case of jet decay in response to ambient air, the size of the...
droplets obtained is determined by the ratio of the destructive aerodynamic force $\rho_{air}U_0^2$ to the equilibrium force of the surface tension $\sigma/d_0$. It is a dimensionless ratio known as the Weber number

$$\text{We} = \frac{\rho_{air}U_0^2d_0}{\sigma},$$

(2)

where $\rho_{air}$ is the air density; $U_0$ is the relative velocity of the droplet in the air flow.

In practical applications, spraying liquids is often implemented in a swirling air flow [2]. In this case, a degree of initial jet swirl is transmitted to the droplets formed. This raises the question of how swirling affects the stability of the droplet in relation to its secondary decay in the flow. As a rule, this issue is solved by numerical methods [4] or experimentally [5]. This paper proposes an analytical solution to the problem.

2. Method of Calculation

The essence of the calculation method is that the droplet is considered to be in equilibrium if the pressure inside it at points $A$ and $B$ is equal. It is believed that the droplet has the shape of a spheroid (flattened ellipsoid), and the points $A$ and $B$ (figure 1) are in significantly different geometric and aerodynamic conditions [6].

![Figure 1. To the definition of equilibrium conditions:](image)

$U_0$ – relative velocity of the droplet in the air flow, $r_0$ – radius of the droplet, $\omega_0$ – initial angular velocity of the droplet, $\omega$ – angular velocity of the spheroid, $a$ and $c$ – semi-major and semi-minor axes, $A$ and $B$ – reference points.

At point $A$, the pressure is due to the aerodynamic effect of the incoming flow $p_{flow}^A$ and surface tension $p_{\sigma}^A$:

$$p^A = p_{flow}^A + p_{\sigma}^A.$$  

(3)

At point $B$, there is additional pressure due to the centrifugal force of the rotational motion [7]

$$p_{\omega}^B = \frac{\rho_{water}\omega^2a^2}{2}.$$  

(4)

Assuming that the volume of the droplet does not change at the transition from its original spherical shape to the spheroid, we obtain

$$a = r_0k^{1/3}, \quad c = r_0k^{-2/3}, \quad k = a/c.$$  

(5)

Since the moments of external forces do not act on the droplet, then, based on the law of conservation of angular momentum, we derive an expression linking the angular velocities of the original droplet and the spheroid

$$\omega = \omega_0k^{-2/3}.$$  

(6)
We introduce a dimensionless parameter that takes into account the effect of the droplet rotation and is equal to the ratio of the centrifugal pressure at the periphery of the spherical droplet to the dynamic head

\[ V = \frac{p_{\text{liq}}}{\rho_{\text{air}}} \left( \frac{\omega_b r_0}{U_0} \right)^2 \]  

(7)

Then, considering (5), (6) and (7), expression (4) takes the form

\[ p^b_\alpha = V \frac{p_{\text{air}} U_0 k^{2/3}}{2} \]  

(8)

The droplet equilibrium requires that the hydrostatic pressure at the point \( B \) \( p^b \) should be balanced by the aerodynamic pressure \( p^d_{\text{flow}} \) and surface tension pressure \( p^d_\sigma \). Finally, the droplet equilibrium condition takes the form

\[ p^d_{\text{flow}} + p^d_\alpha + p^b_\alpha = p^d_{\text{flow}} + p^b_\alpha \]  

(9)

Substituting (6) into (7) and using the expressions for \( p^d_\alpha \) and \( p^d_{\text{flow}} \) from [6] we obtain

\[ \text{We} = \frac{4}{k^{2/3}} \frac{k^3 + k - 2}{\eta^2 k^{2/3} + V} \]  

(10)

where \( \eta = \frac{e^3}{\sqrt{1 - e^2} \arcsin e - e(1 - e^2)} \), \( e = \sqrt{k^2 - 1} / k \).

3. Discussion of results

Figure 2 displays the calculation results.

\[ \text{Figure 2. Weber number We as a function of the aspect ratio } k. \]

Figure 2 shows that for small values of the parameter \( V \), the swirl has little effect on the droplet stability. It becomes significant at \( V > 1 \). It is also evident that when the aspect ratio \( k > 8 \), the effect of swirling is small at any value of \( V \). Thus, the calculation results show that swirling improves the droplet fragmentation conditions in the \( 1 < k < 8 \) range. At large values of \( k \) all equilibrium curves almost converge. This is due to the fact that the ratio of centrifugal pressure to the aerodynamic one decreases.

In particular for point \( B \) using (7) we obtain

\[ V^b_\alpha = \left( \frac{\Delta p_\alpha}{\Delta p_{\text{flow}}} \right)_B = \frac{V}{k^{3/2} \eta^2}. \]  

(12)

The calculation results for this expression are presented in table 1.
It should also be noted that for all equilibrium curves calculated there is a critical inflection point, after which the equilibrium states become unstable, the value of $k$ increasing with an increase in the parameter $V$. At $V = 0$, for comparison, the critical parameters are $k_{\text{crit}} = 6$, $W_{\text{crit}} = 3.75$.

Table 1. Calculation results.

| $V_B/V$ | 1     | 2     | 3     | 4     | 5     | 6     | 7     | 8     |
|---------|-------|-------|-------|-------|-------|-------|-------|-------|
|         | 0.444 | 0.089 | 0.031 | 0.014 | 0.007 | 0.004 | 0.003 | 0.002 |

| $k_{\text{crit}}$ | 6.6   | 7.1   | 7.5   | 7.9   | 8.2   | 8.5   | 8.8   | 9.1   |
|-------------------|-------|-------|-------|-------|-------|-------|-------|-------|
| $W_{\text{crit}}$ | 3.70  | 3.66  | 3.63  | 3.60  | 3.58  | 3.56  | 3.53  | 3.52  |

4. Conclusion

The results of theoretical calculation of the equilibrium conditions for a rotating droplet in a potential flow showed that swirling contributes to droplet fragmentation. It manifests as a decrease in the equilibrium value of the Weber number with an increase in swirl parameter.

The effect of swirling is significant if the droplet aspect ratio is in the range $1 < k < 8$.

For all swirl values, the equilibrium curve has an inflection point that separates the regions of stable and unstable droplet equilibrium. This means that those droplets in the air flow for which the Weber number and aspect ratio exceed the critical values cannot exist. This fact determines the practical significance of the work and can be verified experimentally.

Acknowledgments

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