TWISTED DIAGRAMS AND HOMOTOPY SHEAVES

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Abstract. Twisted diagrams are generalised diagrams: The vertices are allowed to live in different categories, and the structure maps act through specified “twisting” functors between these categories. Examples include spectra (in the sense of homotopy theory) and quasi-coherent sheaves of modules on an algebraic variety. We construct a twisted version of Kan extensions and establish various model category structures (with pointwise weak equivalences). Using these, we propose a definition of “homotopy sheaves” and show that a twisted diagram is a homotopy sheaf if and only if it gives rise to a “sheaf in the homotopy category”. 

1. Introduction

One often encounters constructions which pretend to be diagrams in some category but cannot quite be described with that formalism. An important example is the notion of (naive) spectra, a sequence of pointed spaces \( X_0, X_1, \ldots \) and structure maps \( \Sigma X_n \longrightarrow X_{n+1} \). This almost determines a diagram indexed over \( \mathbb{N} \) (regarded as a category), and in fact can be described by a “twisted diagram” with “twists” given by iterated suspension functors. Another example (and the origin of the present paper) is the category of quasi-coherent sheaves on projective spaces as defined by the first author in [Hüt02]: a “sheaf” is a collection of equivariant spaces, each equipped with an action of a different monoid, together with structure maps which are equivariant with respect to the “smaller” monoid. A detailed description is contained in the examples in this paper.—The new formalism also applies, as a special case, to diagram categories in the usual sense (i.e., functor categories).

To illustrate the general idea, suppose we have two categories \( \mathcal{C} \) and \( \mathcal{D} \) and a functor \( F: \mathcal{C} \longrightarrow \mathcal{D} \) which has a right adjoint \( U \). A twisted diagram (with respect to this data) is a morphism (in \( \mathcal{D} \))

\[
F(Y) \xrightarrow{f} Z
\]

where \( Y \) is an object of \( \mathcal{C} \) and \( Z \) is an object of \( \mathcal{D} \). This gadget should be thought of as a generalised diagram of the form \( Y \xrightarrow{f} Z \). Since \( Y \) and \( Z \) live in different categories, the “structure map” \( f \) has to act by a “twist” given by \( F \).
The twisted diagram \( Y \to Z \) is a \textit{strict sheaf} if its structure map \( F(Y) \to Z \) is an isomorphism. The terminology is motivated by the description of quasi-coherent sheaves on a quasi-compact scheme by twisted diagrams of modules: The category of quasi-coherent sheaves is equivalent to the full subcategory of those twisted diagrams which are strict sheaves; cf. Example 4.3.2.—Provided the categories \( C \) and \( D \) are equipped with compatible model structure, we can define the twisted diagram to be a \textit{homotopy sheaf} if the structure map \( F(Y) \to Z \) is a weak equivalence (the technical definition given in 4.3.3 involves a cofibrant replacement of the source). The rough idea is that, by passing to the homotopy class of the structure map, we obtain a twisted diagram involving the homotopy categories of \( C \) and \( D \) only, and that \( Y \to Z \) is a homotopy sheaf if and only if it is mapped to a strict sheaf in the homotopy categories. However, there are technical issues which make this process of “passing to the homotopy categories” slightly more complicated than one would wish; these issues are addressed in §4.

Homotopy sheaves of the kind described here appeared in the context of the algebraic \( K \)-theory of spaces \[HKV^+01, \text{H"ut}02, \text{H"ut}04, \text{H"ut}a\]. However, none of these papers addressed the question of the exact relationship between sheaves and homotopy sheaves, and the present paper is intended to provide clarification of this point (see the remarks at the very end of this paper). In a forthcoming paper, it will be shown that homotopy sheaves can be used to describe the derived category, in the usual sense, of toric varieties, and that homotopy sheaves can be characterised as colocal objects in a twisted diagram category \[H"utb\].

To make this rather abstract paper accessible to a broad audience, we include a quite detailed discussion of elementary topics, in particular in the foundational material. This includes completeness of the category of twisted diagrams and the construction of Kan extensions. Apart from some elementary category theory \[Mac71\] and and basic model category theory \[DS95, Hov99\] no prerequisites are required. While the model structures are well-known among experts in homotopy theory, it seems that there has been no accessible published account so far. We hope to fill the gap with the present paper.

\textbf{Organisation of the paper.} The paper is divided into three parts. §2 is devoted to the definition of twisted diagrams and the development of basic machinery. The fundamental notion is that of an \textit{adjunction bundle}, consisting of a collection of categories and adjoint functor pairs. It encodes the shape of the diagrams and carries all the necessary information about twists. We discuss the behaviour of twisted diagrams with respect to morphisms of adjunction bundles and prove a convenient criterion for completeness. In §2.4 we construct a twisted version of Kan extensions. Section 2.5 includes a different description of twisted diagrams and shows how to construct important examples of adjunction bundles.
In \[3\] we prove the existence of several Quillen closed model category structures on categories of twisted diagrams. This part is based on model category structures for diagram categories as in \[Hov99\]. In more detail, we consider “pointwise” weak equivalences. Depending on properties of the adjunction bundle (the index category is required to be a “direct” or “inverse” category), we establish Reedy-type model structures using (generalised) latching or matching spaces. If the adjunction bundle consists of cofibrantly generated model categories, we construct (for arbitrary index categories) a cofibrantly generated model structure.

Finally, in \[4\] we propose definitions of sheaves and homotopy sheaves. Starting from an adjunction bundle of model categories we construct an associated bundle of homotopy categories. A twisted diagram over the original adjunction bundle gives rise to a twisted diagram over the homotopy bundle, and the former is a homotopy sheaf if and only if the latter is a sheaf.

A special case of the results on model structures has been used by the first author to study the algebraic $K$-theory of projective toric varieties \[H"{u}t02, H"{u}ta\]. Homotopy sheaves are important to control finiteness of homotopy colimits of infinite CW complexes \[H"{u}t04\]. Twisted diagrams and their model structures also appear implicitly in \[HKV^+01\].

2. Foundations

2.1. Adjunction Bundles. Let $\mathcal{I}$ be a small category. It will serve as the index category for our diagrams.

**Definition 2.1.1.** An adjunction bundle $\mathfrak{B} = (\mathcal{C}, F, U)$ over $\mathcal{I}$, or $\mathcal{I}$-bundle, consists of the following data:

- for each object $i \in \mathcal{I}$ a category $\mathcal{C}_i$,
- for each morphism $\sigma : i \to j$ in $\mathcal{I}$ a pair of adjoint functors $F_\sigma : \mathcal{C}_i \to \mathcal{C}_j$ and $U_\sigma : \mathcal{C}_j \to \mathcal{C}_i$ (with $F_\sigma$ being the left adjoint),

such that $U$ determines a functor $\mathcal{I}^{\text{op}} \to \text{Cat}$, i.e., $U_{id_i} = \text{id}_{\mathcal{C}_i}$, and for each pair of composable arrows $i \xymatrix{\sigma \ar[r] & j \ar[r]^{\tau} & k}$, the equality $U_{\tau \sigma} = U_\tau \circ U_\sigma$ holds. In addition, we require $F_{id_i} = \text{id}_{\mathcal{C}_i}$. The properties of adjunctions guarantee that there is a canonical isomorphism $F_{\tau \sigma} \cong F_\tau \circ F_\sigma$ (which will be referred to as uniqueness isomorphism), since both functors are left adjoint to $U_{\tau \sigma} = U_\tau \circ U_\sigma$ (\[Mac71\], IV.1, Corollary 1, p. 83).

**Example 2.1.2.** Any category $\mathcal{C}$ gives rise to a trivial $\mathcal{I}$-bundle with $\mathcal{C}_i = \mathcal{C}$ for all $i$, and all adjunctions being the identity adjunction.

**Example 2.1.3** (The non-linear projective line). Let $M\text{-}\text{Top}$ be the category of pointed topological spaces with a basepoint-preserving action of the monoid $M$. A monoid homomorphism $f : M \to M'$ determines an adjunction

$$\cdot \wedge M' : M\text{-}\text{Top} \rightleftharpoons M'\text{-}\text{Top} : R_f$$


with \( R_f \) being restriction along \( f \), and \( \cdot \wedge M' M' \) being its left adjoint (inducing up). The integers \( \mathbb{Z} \) form a monoid under addition, and we have submonoids \( \mathbb{N}_+ \) (non-negative integers) and \( \mathbb{N}_- \) (non-positive integers). Hence we can form the adjunction bundle \( \mathcal{P}^1 \) over \( \mathcal{I} = (\alpha \beta \leftarrow 0 \rightarrow \beta \alpha) \), consisting of the categories \( \mathbb{N}_+\text{-}\text{Top}, \mathbb{Z}\text{-}\text{Top} \) and \( \mathbb{N}_-\text{-}\text{Top} \), and the adjoint pairs “inducing up” and “restriction” along the inclusions \( \mathbb{N}_+ \subseteq \mathbb{Z} \) and \( \mathbb{N}_- \subseteq \mathbb{Z} \).

2.2. Twisted Diagrams.

**Definition 2.2.1 (Twisted diagrams).** Let \( \mathcal{B} \) be an adjunction bundle over \( \mathcal{I} \). A twisted diagram \( Y \) with coefficients in \( \mathcal{B} \) consists of the following data:

- for each object \( i \in \mathcal{I} \) an object \( Y_i \in C_i \),
- for each morphism \( \sigma: i \to j \) in \( \mathcal{I} \) a map \( y_\sigma^b: Y_i \to U_\sigma(Y_j) \) in \( C_i \) such that \( Y \) behaves like a functor, i.e., \( y_{id_i}^b = id_{Y_i} \) and \( y_{\tau \circ \sigma}^b = U_\sigma(y_{\tau}^b) \circ y_\sigma^b \) for each pair \( i \to \sigma \to j \to \tau \) of composable arrows in \( \mathcal{I} \). (A reformulation using the left adjoints will be given below.)

A map \( f: Y \to Z \) of twisted diagrams is a collection of maps \( f_i: Y_i \to Z_i \) in \( C_i \), one for each object \( i \in \mathcal{I} \), such that for each morphism \( \sigma: i \to j \) in \( \mathcal{I} \) the equality \( U_\sigma(f_j) \circ y_\sigma^b = z_\sigma^b \circ f_i \) holds. (A reformulation using the left adjoints will be given below.) The category of twisted diagrams and their maps is denoted \( \text{Tw}(\mathcal{I}, \mathcal{B}) \).

For each of the structure maps \( y_\sigma^b: Y_i \to U_\sigma(Y_j) \) there is a corresponding adjoint map \( y_\sigma^g: F_\sigma(Y_i) \to Y_j \). The idea is to think of the (meaningless) symbol \( y_\sigma: Y_i \to Y_j \) as a kind of “structure map” having two incarnations as a \( b \)-type map (a morphism in \( C_i \)) and a \( g \)-type map (a morphism in \( C_j \)).

The definition of twisted diagrams does not make explicit use of the left adjoints provided by the adjunction bundle. However, the properties of adjunctions will play a crucial rôle for the discussion of limits and colimits in \( \text{Tw}(\mathcal{I}, \mathcal{B}) \).

**Example 2.2.2 (Spectra).** Let \( \mathbb{N} \) denote the ordered set of natural numbers, considered as a category. For each \( n \in \mathbb{N} \), define \( C_n \) to be the category \( S \) of pointed simplicial sets. If \( n \leq m \), we have an adjunction \( \Sigma^{m-n}: S \to S; \Omega^{m-n} \) of iterated loop space and suspension functors. It is clear that this defines an adjunction bundle \( Sp \) over \( \mathbb{N} \). A twisted diagram \( X \) with coefficients in \( Sp \), graphically represented by the “diagram”

\[
X_0 \to X_1 \to X_2 \to \ldots,
\]

is nothing but a spectrum in the sense of Bousfield and Friedlander, cf. [BF78].

**Remark 2.2.3.**

1. If \( \mathcal{B} \) is a trivial \( \mathcal{I} \)-bundle (Example 2.1.2), we recover the functor category: \( \text{Tw}(\mathcal{I}, \mathcal{B}) = \text{Fun}(\mathcal{I}, C) \).
(2) If \( I \) is a discrete category (i.e., contains no non-identity morphisms), an adjunction bundle over \( I \) is nothing but a collection of categories \( \{ C_i \}_{i \in I} \), and the category of twisted diagrams is the product category \( \prod_{i \in I} C_i \).

(3) Suppose \( \mathcal{B}_\nu = (C_\nu, F_\nu, U_\nu) \) is a family of adjunction bundles indexed by \( I_\nu \). Then we can form the following adjunction bundle

\[ \prod_\nu \mathcal{B}_\nu =: \mathcal{B} = (C, F, U) \]

indexed by the disjoint union \( I := \coprod_\nu I_\nu \); for each \( i \in I \) there is exactly one \( \nu \) with \( i \in I_\nu \), and we define \( C_i = C_\nu \) (and similarly for the \( F \) and \( U \)). It is easy to see that

\[ \text{Tw}(I, \mathcal{B}) = \prod_\nu \text{Tw}(I_\nu, \mathcal{B}_\nu) \]

in this case.

Given twisted diagrams \( Y, Z \in \text{Tw}(I, \mathcal{B}) \) and maps \( f_i : Y_i \to Z_i \) in \( C_i \), we can form two squares for each morphism \( \sigma : i \to j \) in \( I \)

\[
\begin{array}{ccc}
Y_i & \xrightarrow{f_i} & Z_i \\
\downarrow y_<^\sigma & & \downarrow z_>^\sigma \\
U_\sigma(Y_j) & \xrightarrow{U_\sigma(f_j)} & U_\sigma(Z_j)
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
F_\sigma(Y_i) & \xrightarrow{F_\sigma(f_i)} & F_\sigma(Z_i) \\
\downarrow y_>^\sigma & & \downarrow z_<^\sigma \\
Y_j & \xrightarrow{f_j} & Z_j
\end{array}
\]

and the definition of adjunctions imply that the left square commutes if and only if the right square commutes. Thus the family \( (f_i)_{i \in I} \) determines a map of twisted diagrams if and only if \( z_<^\sigma \circ F_\sigma(f_i) = f_j \circ y_>^\sigma \).

For later use, we record the following fundamental fact:

**Lemma 2.2.4.** Suppose we have a map \( y_<^\sigma : Y_i \to U_\sigma(Y_j) \) in \( C_i \) for each morphism \( \sigma : i \to j \) in \( I \) satisfying \( y_<^\sigma \circ \text{id}_i = \text{id} \), and denote by \( y_>^\sigma \) the adjoint map \( F_\sigma(Y_i) \to Y_j \). Let \( \tau : j \to k \) be another morphism in \( I \). Then if one of the squares

\[
\begin{array}{ccc}
F_\tau \circ F_\sigma(Y_i) & \xrightarrow{\cong} & F_{\tau \circ \sigma}(Y_i) \\
\downarrow F_\tau(y_<^\sigma) & & \downarrow y_>^{\tau \circ \sigma} \\
F_\tau(Y_j) & \xrightarrow{y_<^\tau} & Y_k
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
Y_i & \xrightarrow{=} & Y_i \\
\downarrow y_<^\sigma & & \downarrow y_>^\tau \\
U_\sigma(Y_j) & \xrightarrow{U_\sigma(y_<^\tau)} & U_{\tau \circ \sigma}(Y_k)
\end{array}
\]

commutes so does the other (the upper horizontal map in the left square is the uniqueness isomorphism). In other words, if for all composable morphisms \( \sigma \) and \( \tau \) one of the squares commutes, the objects \( Y_i \) together with the maps \( y_<^\sigma \) form a twisted diagram.

**Proof.** This follows from naturality of and composition rules for units and counits of adjunctions, cf. [Mac71 §§IV.1+8]. We omit the details.  \( \square \)
2.3. Limits, Colimits, Direct and Inverse Image. The next proposition says that $\text{Tw}(\mathcal{I}, \mathcal{B})$ is as complete and cocomplete as all the $\mathcal{C}_i$, and that limits resp., colimits can be computed “pointwise” in the categories $\mathcal{C}_i$. For $i \in \mathcal{I}$, let $E_{v_i}: \text{Tw}(\mathcal{I}, \mathcal{B}) \rightarrow \mathcal{C}_i$ denote the $i$th evaluation functor which maps a twisted diagram $Y$ to its $i$th term $Y_i$.

**Proposition 2.3.1** (Limits and colimits of diagrams of twisted diagrams). Let $G: \mathcal{D} \rightarrow \text{Tw}(\mathcal{I}, \mathcal{B})$ be a functor, and suppose that for all $i$ the limit of $E_{v_i} \circ G$ exists. Then $\lim \leftarrow G$ exists and the canonical map

$$E_{v_i}(\lim \leftarrow G) \rightarrow \lim (E_{v_i} \circ G)$$

is an isomorphism. A similar assertion holds for colimits.

*Proof.* The proof relies on the compatibility of left (resp., right) adjoint functors with colimits (resp., limits): if $F$ is a left adjoint, and $D$ is a functor, then there is a unique natural isomorphism $\lim \leftarrow (F \circ D) \rightarrow F(\lim \leftarrow D)$, and similarly for right adjoints and limits ([Mac71, §V.5, Theorem 1, p. 114]).

To prove the lemma, we treat the case of colimits only. (For limits one has to use similar techniques. Since $U$ is supposed to be functorial, this is slightly easier.) Let $G_i := E_{v_i} \circ G$, and define $C_i := \lim \leftarrow G_i$. We claim that the objects $C_i$ assemble to a twisted diagram $C$, and it is almost obvious that $C$ is “the” colimit of $G$.

Let $\sigma: i \rightarrow j$ denote a morphism in $\mathcal{I}$. The $\sharp$-type structure maps of the twisted diagrams $G(d)$ (for objects $d \in \mathcal{D}$) assemble to a natural transformation

$$G^\sharp_\sigma: F_{\sigma} \circ G_i \rightarrow G_j$$

of functors $\mathcal{D} \rightarrow \mathcal{C}_j$. Hence we can define the $\sharp$-type structure map $c^\sharp_\sigma$ as the composite

$$F_{\sigma}(C_i) = F_{\sigma}(\lim \leftarrow G_i) \cong \lim \leftarrow (F_{\sigma} \circ G_i) \xrightarrow{f} \lim \leftarrow G_j = C_j$$

with $f$ induced by $G^\sharp_\sigma$.

By Lemma [2.2.4] we are left to show that the following square commutes for all composable morphisms $\sigma$ and $\tau$ in $\mathcal{I}$:

$$\begin{array}{ccc}
F_{\tau} \circ F_{\sigma}(C_i) & \cong & F_{\tau \circ \sigma}(C_i) \\
F_{\tau}(c^\sharp_\sigma) & & c^\sharp_{\tau \circ \sigma} \\
F_{\tau}(C_j) & \rightarrow & C_k
\end{array}$$
We replace the symbols $C_\ell$ and the structure maps by their definition and obtain the following bigger diagram:

\[
\begin{array}{cccccc}
F_\tau \circ F_\sigma (\lim G_i) & \cong & F_\tau \circ F_\sigma (\lim G_i) & \cong & F_\tau \circ F_\sigma (\lim G_i) \\
1 & \cong & 2 & \cong & 3 & \cong & 4
\end{array}
\]

All the small squares commute: for 1 this is true by uniqueness of the isomorphisms for commuting left adjoints with colimits. The horizontal maps of 2 are induced by the uniqueness isomorphism, the vertical maps are induced by the isomorphism for commuting left adjoints with colimits. By uniqueness, 2 commutes. Both horizontal maps of 3 are induced by the isomorphism for commuting colimits with $F_\tau$, and both vertical maps are induced by the natural transformation $G^\#_\sigma : F_\sigma \circ G_i \to G_j$. Hence 3 commutes. Finally, square 4 commutes by Lemma 2.2.4, applied componentwise to the diagrams $G_\ell$, and by functoriality of $\lim$.

Hence the diagram (**) commutes. But the square (*) is contained in there as the outer square, thus is commutative as claimed. □

If $\mathcal{I}$ is a small category and $\mathcal{C}$ is an arbitrary category, the category of diagrams Fun$(\mathcal{I}, \mathcal{C})$ is the value of an internal hom functor on the category of categories. Hence it is functorial in both variables (provided the entries in the first variable are small). The case of twisted diagrams is more complicated since it involves “maps” of adjunction bundles as well as of the actual diagrams.

**Definition 2.3.2** (Inverse image of bundles). Given a functor $\Phi : \mathcal{I} \to \mathcal{J}$ and a $\mathcal{J}$-bundle $\mathfrak{B} = (\mathcal{D}, G, V)$, we define the inverse image of $\mathfrak{B}$ under $\Phi$, denoted $\Phi^* \mathfrak{B}$, to be the $\mathcal{I}$-bundle $(\mathcal{C}, F, U)$ given by $C_i := D_{\Phi(i)}$, $U_i := V_{\Phi(i)}$ and $F_i := G_{\Phi(i)}$.

If $\Phi : \mathcal{I} \to \mathcal{J}$ is the inclusion of a subcategory, we write $\mathfrak{B}|_{\mathcal{I}}$ instead of $\Phi^* \mathfrak{B}$ and call the resulting $\mathcal{I}$-bundle the restriction of $\mathfrak{B}$ to $\mathcal{I}$.

Forming inverse images is functorial, i.e., $id^*_\mathcal{I} \mathfrak{B} = \mathfrak{B}$ and $(\Phi \circ \Theta)^* \mathfrak{B} = \Theta^* \Phi^* \mathfrak{B}$. The inverse image of a trivial bundle is a trivial bundle.

**Definition 2.3.3** (Morphisms of bundles). Suppose $\mathfrak{A} = (\mathcal{C}, F, U)$ and $\mathfrak{B} = (\mathcal{D}, G, V)$ are $\mathcal{I}$-bundles. An $\mathcal{I}$-morphism $\Psi : \mathfrak{A} \to \mathfrak{B}$ consists of two families of functors $\rho_i : C_i \to D_i$ and $\lambda_i : D_i \to C_i$ where $i$ ranges
over the objects of \( I \) such that \( \lambda_i \) is left adjoint to \( \rho_i \), and such that for each morphism \( \sigma: i \to j \) in \( I \) we have \( V_\sigma \circ \rho_j = \rho_i \circ U_\sigma \).

Given an \( I \)-bundle \( \mathcal{A} \) and a \( J \)-bundle \( \mathcal{B} \), a morphism of bundles

\[
\Xi: \mathcal{A} \to \mathcal{B}
\]

is a pair \( \Xi = (\Phi, \Psi) \) where \( \Phi: I \to J \) is a functor and \( \Psi: \mathcal{A} \to \Phi^* \mathcal{B} \) is an \( I \)-morphism of \( I \)-bundles.

**Definition 2.3.4** (Inverse image of twisted diagrams). Suppose we have a functor \( \Phi: I \to J \), a \( J \)-bundle \( \mathcal{B} \), and a twisted diagram \( Y \in \text{Tw}(J, \mathcal{B}) \). We define the *inverse image of \( Y \) under \( \Phi \)*, denoted \( \Phi^* Y \), as the twisted diagram over \( I \) with coefficients in \( \Phi^* \mathcal{B} \) given by \( (\Phi^* Y)_i := Y_{\Phi(i)} \) and \( (\Phi^* y)_\sigma := y_{\Phi(\sigma)} \) for all objects \( i \in I \) and all morphisms \( \sigma \in I \). We obtain a functor \( \Phi^*: \text{Tw}(J, \mathcal{B}) \to \text{Tw}(I, \Phi^* \mathcal{B}) \).

Now suppose we have \( I \)-bundles \( \mathcal{A} = (C, F, U) \) and \( \mathcal{B} = (D, G, V) \), and an \( I \)-morphism \( \Psi = (\rho, \lambda): \mathcal{A} \to \mathcal{B} \). The functor invertible image under \( \Psi \), denoted \( \Psi^*: \text{Tw}(I, \mathcal{B}) \to \text{Tw}(I, \mathcal{A}) \), assigns to a twisted diagram \( Y \in \text{Tw}(I, \mathcal{A}) \) given by \( (\Psi^* Y)_i := \lambda_i(Y_i) \) with \( \sharp \)-type structure maps \( (\Psi^* y)_\sigma \) given by the composition

\[
F_\sigma((\Psi^* Y)_i) = F_\sigma(\lambda_i(Y_i)) \equiv \lambda_j(G_\sigma(Y_i)) \xrightarrow{\lambda_i(y^\sharp_\sigma)} \lambda_j(Y_j) = (\Psi^* Y)_j
\]

for all objects \( i \in I \) and morphisms \( \sigma: i \to j \). (We will prove in the next lemma that \( \Psi^* \) is well-defined, i.e., that \( \Psi^* Y \) is a twisted diagram.)

More generally, a morphism \( \Xi = (\Phi, \Psi): \mathcal{A} \to \mathcal{B} \) of bundles induces an inverse image functor \( \Xi^* = \Psi^* \circ \Phi^*: \text{Tw}(J, \mathcal{B}) \to \text{Tw}(I, \mathcal{A}) \).

If \( \Phi: I \to J \) is the inclusion of a subcategory, we write \( Y|_I \) instead of \( \Phi^* Y \) and call the resulting twisted diagram with coefficients in \( \mathcal{B}|_I \) the *restriction of \( Y \) to \( I \)*. This defines the restriction functor

\[
\text{Tw}(J, \mathcal{B}) \to \text{Tw}(I, \mathcal{B}|_I)
\]

As a special case of restriction (if \( I = \{i\} \) is the trivial subcategory consisting of \( i \)), we obtain the evaluation functors \( Ev_i \) as defined above.

**Lemma 2.3.5.** Given \( I \)-bundles \( \mathcal{A} = (C, F, U) \) and \( \mathcal{B} = (D, G, V) \), an \( I \)-morphism \( \Psi = (\lambda, \rho): \mathcal{A} \to \mathcal{B} \), and a twisted diagram \( Y \in \text{Tw}(I, \mathcal{B}) \), the object \( \Psi^* Y \) defined in Definition 2.3.4 is a twisted diagram with coefficients in \( \mathcal{A} \).
Proof. Let \( \sigma: i \rightarrow j \) and \( \tau: j \rightarrow k \) be morphisms in \( I \) and consider the diagram

\[
\begin{array}{ccc}
F_\tau \circ F_\sigma \circ \lambda_i(Y_i) & \cong & F_{\tau \circ \sigma} \circ \lambda_i(Y_i) \\
\Downarrow & & \Downarrow \\
F_\tau \circ \lambda_j \circ G_\sigma(Y_i) & \cong & \lambda_k \circ G_\tau \circ G_\sigma(Y_i) \\
\Downarrow & & \Downarrow \\
F_\tau \circ \lambda_j(y_\delta^i) & \cong & \lambda_k \circ G_\tau(y_\delta^j) \\
\Downarrow & & \Downarrow \\
F_\tau \circ \lambda_j(Y_j) & \cong & \lambda_k \circ G_\tau(Y_j) \\
\end{array}
\]

in which all arrows labelled with \( \cong \) denote uniqueness isomorphisms. Recall that the compositions of functors appearing in the upper rectangle are left adjoints to the functor \( U_\sigma \circ U_\tau \circ \rho_k \). Thus the upper rectangle commutes by uniqueness. The lower left square commutes by naturality. The lower right square commutes since \( Y \) is a twisted diagram (Lemma 2.2.4) and \( \lambda_k \) is a functor. Hence the whole diagram commutes and \( \Psi^*Y \) is a twisted diagram by another application of Lemma 2.2.4. \( \square \)

Definition 2.3.6 (Direct image of twisted diagrams). Suppose we have a bundle morphism \( \Xi = (\Phi, \Psi): \mathcal{A} \rightarrow \mathcal{B} \), where \( \mathcal{A} = (C, F, U) \) is an \( I \)-bundle, \( \mathcal{B} = (D, G, V) \) is a \( J \)-bundle, \( \Phi \) is a functor \( I \rightarrow J \), and \( \Psi = \{ (\lambda_i, \rho_i) \}_{i \in I} \) is an \( I \)-morphism \( \mathcal{A} \rightarrow \Phi^*\mathcal{B} \). Let \( Y \) be a twisted diagram with coefficients in \( \mathcal{A} \). It is straightforward to check that the definition \( \Psi^*(Y)_i := \rho_i(Y_i) \) yields a twisted diagram with coefficients in \( \Phi^*\mathcal{B} \) having the structure maps

\[
\Psi^*(y_\alpha^i): \rho_i(Y_i) \xrightarrow{\rho_i(y_\alpha^i)} \rho_j(U_\alpha(Y_j)) = V_{\Phi(\alpha)} \circ \rho_j(Y_j)
\]
for \( \alpha: i \rightarrow j \). In this way we obtain a functor

\[
\Psi^*: \text{Tw}(I, \mathcal{A}) \rightarrow \text{Tw}(I, \Phi^*\mathcal{B})
\]
Suppose the right adjoint \( R\Phi \) of \( \Phi^* \) exists. The composition

\[
\Xi^* := R\Phi \circ \Psi^*: \text{Tw}(I, \mathcal{A}) \rightarrow \text{Tw}(J, \mathcal{B})
\]

is called the direct image functor.

We will see below that if the bundle \( \mathcal{B} \) consists of complete categories, the functor \( R\Phi \) exists and can be constructed by twisted Kan extension. Using this, we can prove:

Corollary 2.3.7. Let \( \Xi = (\Phi, \Psi): \mathcal{A} \rightarrow \mathcal{B} \) be a bundle morphism, with \( \mathcal{B} \) consisting of complete categories. Then the functor \( \Xi^* \) (inverse image under \( \Xi \)) has a right adjoint \( \Xi^* \) (direct image under \( \Xi \)).
Proof. Since $R \Phi$ is right adjoint to $\Phi^\ast$ by assumption, it remains to show that $\Psi_\ast$ is right adjoint to $\Psi^\ast$. However, this is true because $\Psi_\ast$ is pointwise right adjoint to $\Psi^\ast$, and it can be checked that adjoining pointwise respects maps of twisted diagrams. We omit the details. \hfill \Box

2.4. Twisted Kan Extensions. Assume that $\mathcal{B}$ is a trivial bundle over $\mathcal{J}$, consisting of the category $\mathcal{C}$ (and identity functors), and $\Phi: \mathcal{I} \longrightarrow \mathcal{J}$ is a functor. In this case, the inverse image of $\mathcal{B}$ under $\Phi$ is the trivial bundle over $\mathcal{I}$ (consisting of $\mathcal{C}$ and identity functors), and $\Phi^\ast$ is the functor $\text{Fun} (\mathcal{J}, \mathcal{C}) \longrightarrow \text{Fun} (\mathcal{I}, \mathcal{C})$ mapping $Y$ to $Y \circ \Phi$. If $\mathcal{C}$ is complete, the functor $\Phi^\ast$ has a right adjoint given by right Kan extension along $\Phi$ ([Mac71, §X.3, Corollary 2]).

It is possible to construct Kan extensions in our framework. We consider only left Kan extensions, the other case being similar (and easier).

Let $\Phi: \mathcal{I} \longrightarrow \mathcal{J}$ be a functor, $\mathcal{B} = (\mathcal{C}, F, U)$ a $\mathcal{J}$-bundle, and $Y$ a twisted diagram over $\mathcal{I}$ with coefficients in $\Phi^\ast \mathcal{B} = (\mathcal{D}, G, U)$. First, we have to define a twisted diagram $L(Y)$ over $\mathcal{J}$ with coefficients in $\mathcal{B}$. (Later, we will convince ourselves that the assignment $Y \longmapsto L(Y)$ is a functor which is left adjoint to $\Phi^\ast$.) Let $j \in \mathcal{J}$ be given, and let $\Phi \downarrow j$ denote the category of objects $\Phi$-over $j$. Its objects are maps of the form $\sigma: \Phi(i) \longrightarrow j \in \mathcal{J}$ (for $i$ an object of $\mathcal{I}$). The morphisms from $\sigma: \Phi(i) \longrightarrow j$ to $\tau: \Phi(i') \longrightarrow j$ are morphisms $\alpha: i \longrightarrow i' \in \mathcal{I}$ (satisfying $\tau \circ \Phi(\alpha) = \sigma$). Consider the assignment

$$D^Y_j: \Phi \downarrow j \longrightarrow \mathcal{C}_j, \quad (\Phi(i) \overset{\sigma}{\longrightarrow} j) \mapsto F_\sigma(Y_i)$$

This is well-defined because $Y_i$ is an object of $\mathcal{D}_i = \mathcal{C}_{\Phi(i)}$ by definition of $\Phi^\ast \mathcal{B}$, so $F_\sigma(Y_i)$ is an object of $\mathcal{C}_j$.

The assignment $D^Y_j$ is in fact a functor, as one can deduce as follows. Let $\text{pr}_\mathcal{I}$ denote the obvious projection functor $\Phi \downarrow j \longrightarrow \mathcal{I}$ mapping the object $\Phi(i) \longrightarrow j$ to $i$, and define $\text{pr}_\mathcal{J} := \Phi \circ \text{pr}_\mathcal{I}$. Using the equality $\text{pr}_\mathcal{J}^\ast \mathcal{B} = \text{pr}_\mathcal{I}^\ast (\Phi^\ast \mathcal{B})$, we get a functor $\text{pr}_\mathcal{J}^\ast: \mathcal{Tw}(\mathcal{I}, \Phi^\ast \mathcal{B}) \longrightarrow \mathcal{Tw}(\Phi \downarrow j, \text{pr}_\mathcal{J}^\ast \mathcal{B})$. Let $\{j\}$ denote the subcategory of $\mathcal{J}$ given by the object $j$ (and no non-identity morphism) and consider the category $\mathcal{C}_j$ as a (trivial) bundle over $\{j\}$. Then we have a morphism of bundles $\Xi: \mathcal{C}_j \longrightarrow \text{pr}_\mathcal{J}^\ast \mathcal{B}$ consisting of the functor $\Phi \downarrow j \longrightarrow \{j\}$ and the $(\Phi \downarrow j)$-morphism $\Psi$ from $\text{pr}_\mathcal{J}^\ast \mathcal{B}$ to the trivial bundle with $\sigma$-component the adjunction $F_\sigma: \mathcal{C}_{\Phi(i)} \longrightarrow \mathcal{C}_j: U_\sigma$ (for $\sigma: \Phi(i) \longrightarrow j$). The inverse image under $\Psi$ is a functor

$$\Psi^\ast: \mathcal{Tw}(\Phi \downarrow j, \text{pr}_\mathcal{J}^\ast \mathcal{B}) \longrightarrow \text{Fun} (\Phi \downarrow j, \mathcal{C}_j).$$

Tracing the definitions shows $D^Y_j = \Psi^\ast \text{pr}_\mathcal{I}^\ast(Y)$.

Now assume that the bundle $\mathcal{B}$ consists of cocomplete categories. Define $L(Y)_j$ as the colimit of $D^Y_j$. To prove that the $L(Y)_j$ assemble to a twisted diagram, we construct for each $\alpha: j \longrightarrow k$ a structure map $l^Y_\alpha: F_\alpha(L(Y)_{jj}) \longrightarrow L(Y)_k$ and apply Lemma 2.2.4.
Since $F_\alpha$ is a left adjoint, we have a unique isomorphism
\[ u_\alpha : F_\alpha(\lim D^Y_j) \cong \lim (F_\alpha \circ D^Y_j). \]

Let $\Phi(i) \xrightarrow{\sigma} j$ be an object of $\Phi \downarrow j$. Then $\alpha \circ \sigma$ is an object of $\Phi \downarrow k$, and there is a canonical map $F_{\alpha\circ\sigma}(Y_i) \longrightarrow \lim D^Y_k = L(Y)_k$ (since $F_{\alpha\circ\sigma}(Y_i)$ appears in the diagram $D^Y_k$). The composition with a uniqueness isomorphism yields a map
\[ t_\sigma : F_\sigma \circ F_\sigma(Y_i) \longrightarrow L(Y)_k. \]

The $t_\sigma$ assemble to a natural transformation from $F_\alpha \circ D^Y_j$ to the constant diagram with value $L(Y)_k$ (a proof involves the uniqueness of the uniqueness isomorphisms and the naturality of the canonical maps mentioned above; we omit the details). By taking colimits, this determines a map
\[ v_\alpha : \lim (F_\alpha \circ D^Y_j) \longrightarrow L(Y)_k, \]
and we set $i^\sharp_\alpha := v_\alpha \circ u_\alpha$.

Now we have to check that, for $j \xrightarrow{\alpha} k \xrightarrow{\beta} l \in J$, the square
\[
\begin{array}{ccc}
F_\beta \circ F_\alpha(L(Y)_j) & \longrightarrow & F_{\beta\circ\alpha}(L(Y)_j) \\
\downarrow F_\beta(i^\sharp_\alpha) & & \downarrow F_{\beta\circ\alpha}(i^\sharp_{\beta\circ\alpha}) \\
F_\beta(L(Y)_k) & \longrightarrow & L(Y)_l
\end{array}
\]
commutes. First of all, the diagram
\[
\begin{array}{ccc}
F_\beta \circ F_\alpha(L(Y)_j) & \cong & F_{\beta\circ\alpha}(L(Y)_j) \\
\downarrow \cong & & \downarrow \cong \\
\lim (F_\beta \circ F_\alpha \circ D^Y_j) & \cong & \lim (F_{\beta\circ\alpha} \circ D^Y_j)
\end{array}
\]
consisting of uniqueness isomorphisms commutes because of their uniqueness. By the universal property of the colimit and the definition of the
structure maps, we are left to show that for every \( \sigma : \Phi(i) \rightarrow j \) the diagram

\[
\begin{array}{ccc}
F_\beta \circ F_\alpha \circ F_\sigma(Y_i) & \cong & F_\beta \circ F_\sigma(Y_i) \\
\downarrow \cong & & \downarrow \cong \\
F_\beta \circ F_{\alpha \circ \sigma}(Y_i) & \cong & F_{\beta \circ \alpha \circ \sigma}(Y_i) \\
\downarrow c_{\alpha \sigma} & & \downarrow c_{\beta \circ \alpha \circ \sigma} \\
\text{lim} (F_\beta \circ D^Y_k) & \cong & \text{lim} (F_\beta \circ D^Z_k) \\
\downarrow \cong & & \downarrow \cong \\
F_\beta(L(Y)_k) & \underset{l^\beta_j}{\longrightarrow} & L(Y)_l
\end{array}
\]

commutes, where the maps \( c_{\alpha \sigma} \) and \( c_{\beta \circ \alpha \circ \sigma} \) are canonical maps to the colimit, and all maps labelled with \( \cong \) are uniqueness isomorphisms. The upper square commutes by uniqueness, and the lower square commutes by definition of \( l^\beta_j \). This implies that the square (\( \ast \)) commutes, and Lemma 2.2.4 shows that \( L(Y) \) is a twisted diagram as claimed.

**Theorem 2.4.1** (Left Kan extensions). Let \( \mathcal{B} \) be a \( \mathcal{J} \)-bundle consisting of cocomplete categories, \( \Phi : \mathcal{I} \rightarrow \mathcal{J} \) a functor and \( Y \) a twisted diagram with coefficients in \( \Phi^* \mathcal{B} \). The assignment \( Y \mapsto L(Y) \) described above is the object function of a functor \( L\Phi : \text{Tw}(\mathcal{I}, \Phi^* \mathcal{B}) \rightarrow \text{Tw}(\mathcal{J}, \mathcal{B}) \) which is left adjoint to \( \Phi^* \).

**Proof.** Abbreviate \( L\Phi \) by \( L \) and keep the notation used in the construction of \( L(Y) \).

We start by describing the effect of \( L \) on morphisms. Let \( f : Y \longrightarrow Z \) be a map of twisted diagrams with coefficients in \( \Phi^* \mathcal{B} \), and fix an object \( j \in \mathcal{J} \). For each \( \sigma : \Phi(i) \rightarrow j \), the maps \( F_\sigma(f_i) \) form a natural transformation from \( D^Y_j \) to \( D^Z_j \), because the uniqueness isomorphisms are natural, \( f \) is a map of twisted diagrams and \( F_\sigma \) is a functor. This defines a map on the colimits \( L(f)_j : L(Y)_j \longrightarrow L(Z)_j \).
We claim that the maps $L(f)_j$ assemble to a map $L(f)$ of twisted diagrams. For $\alpha : j \longrightarrow k$ in $\mathcal{J}$, consider the diagram

$$
\begin{array}{c}
F_\alpha(L(Y)_j) \xrightarrow{F_\alpha(L(f)_j)} F_\alpha(Z_j) \\
\downarrow l_\alpha \downarrow \quad \quad \quad \downarrow m_\alpha \\
L(Y)_k \xrightarrow{L(f)_k} L(Z)_k
\end{array}
$$

where $l$ and $m$ denote the structure maps of $L(Y)$ and $L(Z)$. It commutes if and only if for each object $\sigma : \Phi(i) \longrightarrow j$ of $\Phi \downarrow j$, the diagram

$$
\begin{array}{c}
F_\alpha \circ F_\sigma(Y_i) \xrightarrow{F_\alpha \circ F_\sigma(f_i)} F_\alpha \circ F_\sigma(Z_i) \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \\
F_{\alpha \circ \sigma}(Y_i) \xrightarrow{F_{\alpha \circ \sigma}(f_i)} F_{\alpha \circ \sigma}(Z_i) \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \\
L(Y)_k \xrightarrow{L(f)_k} L(Z)_k
\end{array}
$$

commutes. The isomorphisms are uniqueness isomorphisms, which are natural, hence the upper square commutes. The lower vertical arrows denote the canonical map to the colimit, and the naturality of these make the lower square commute.

Having checked that $L(f)$ is indeed a map of twisted diagrams, it is clear that $L$ is a functor, because maps of twisted diagrams are defined pointwise, and $L_j$ is defined as the composition of functors $\lim \circ \Psi^* \circ \text{pr}_I^*$ (with $\Psi$ and $\text{pr}_I$ being explained below the definition of $D(Y)$). To prove that $L$ is left adjoint to $\Phi^*$, we construct natural transformations $\eta : \text{id} \longrightarrow \Phi^* \circ L$ and $\epsilon : L \circ \Phi^* \longrightarrow \text{id}$ satisfying the triangular identities [Mac71 §IV.1, Theorem 2 (v)].

For $Y \in \text{Tw}(\mathcal{I}, \Phi^*\mathcal{B})$, the $Y$-component $\eta_Y$ is given (pointwise) as the canonical map to the colimit $Y_i \longrightarrow \Phi^*(L(Y))_i = L(Y)_{\Phi(i)}$ which corresponds to the identity id : $\Phi(i) \longrightarrow \Phi(i)$ (an object of $\Phi \downarrow \Phi(i)$). We check that $\eta_Y$ is a map of twisted diagrams. Let $\alpha : i \longrightarrow j \in \mathcal{I}$ be given and
consider the diagram

\[
\begin{array}{ccc}
F_{\Phi(\alpha)}(Y_i) & \xrightarrow{F_{\Phi(\alpha)}((\eta Y)_i)} & F_{\Phi(\alpha)}(L(Y)_{\Phi(i)}) \\
\downarrow y^\sharp_\alpha & & \downarrow \iota^\sharp_{\Phi(\alpha)} \\
Y_j & \xrightarrow{(\eta Y)_j} & L(Y)_{\Phi(j)}
\end{array}
\]

with the structure map \(y^\sharp_\alpha\) starting from \(G_\alpha(Y_i) = F_{\Phi(\alpha)}(Y_i)\) by definition of \(\Phi^*\mathfrak{B}\). Since the structure map \(\iota^\sharp_{\Phi(\alpha)}\) is defined via the canonical maps to the colimit

\[
F_{\Phi(\alpha)} \circ F_{\sigma}(Y_k) \cong F_{\Phi(\alpha) \circ \sigma}(Y_k) \xrightarrow{} L(Y)_{\Phi(j)}
\]

(for \(\sigma: \Phi(k) \rightarrow \Phi(i)\) an object of \(\Phi \downarrow \Phi(i)\)), the composition

\[
\iota^\sharp_{\Phi(\alpha)} \circ F_{\Phi(\alpha)}((\eta Y)_i)
\]

coincides with the canonical map to the colimit

\[
c: F_{\Phi(\alpha)}(Y_i) \rightarrow L(Y)_{\Phi(j)}
\]

(the special case \(\sigma = id_{\Phi(i)}\)). Hence we have to show that the triangle

\[
\begin{array}{ccc}
F_{\Phi(\alpha)}(Y_i) & \xrightarrow{c} & L(Y)_{\Phi(j)} \\
\downarrow y^\sharp_\alpha & & \downarrow \eta(Y)_i \\
Y_j & \xrightarrow{(\eta Y)_j} & L(Y)_{\Phi(j)}
\end{array}
\]

commutes. But this is true by the definition of \(L(Y)_{\Phi(j)}\) as the colimit of \(D_{\Phi(j)}^Y\). The naturality of \(\eta\) can be explained as follows. For \(i \in I\), the canonical maps to the colimit \(Y_j \rightarrow L(Y)_{\Phi(i)}\) for varying \(\sigma: j \rightarrow \Phi(i)\) are a natural transformation of diagrams (with shape \(\Phi \downarrow \Phi(i)\)). In particular, the \(\Phi(i)\)-component, being the map \((\eta_Y)_i\), is natural. We turn to the definition of \(\epsilon: L \circ \Phi^* \rightarrow id\). For \(Z \in \text{Tw}(J, \mathfrak{B})\), the map \(\epsilon_Z\) is given pointwise as follows: for every \(j \in J\) and every \(\sigma: \Phi(i) \rightarrow j\) in \(\Phi \downarrow j\), the structure maps \(F_{\sigma}(\Phi^*(Z)_i) = F_{\sigma}(Z_{\Phi(i)}) \xrightarrow{\iota^\sharp_\alpha} Z_j\) assemble to a natural transformation from \(D_j^Y\) to the constant diagram with value \(Z_j\) (this follows from Lemma 2.2.4 and the fact that \(Z\) is a twisted diagram). By the universal property of the colimit, this natural transformation defines a unique map

\[
(\epsilon_Z)_j: L(\Phi^*(Z))_j \rightarrow Z_j.
\]
To prove that $\epsilon_Z$ is a map of twisted diagrams, let $\alpha : j \longrightarrow k \in \mathcal{J}$ and consider the following diagram:

$$
\begin{array}{ccc}
F_\alpha(L(\Phi^*(Z))_j) & \xrightarrow{F_\alpha((\epsilon_Z)_j)} & F_\alpha(Z_j) \\
\downarrow m^Z_\alpha & & \downarrow \epsilon^Z_k \\
L(\Phi^*(Z))_k & \xrightarrow{(\epsilon_Z)_k} & Z_k
\end{array}
$$

Using the universal property of the colimit, the definition of $\epsilon_Z$ and the definition of the structure map $m^Z_\alpha$, we are left to show that, for each $\sigma : \Phi(i) \longrightarrow j$ in $\Phi \downarrow j$, the diagram

$$
\begin{array}{ccc}
F_\alpha(F_\sigma(Z_i)) & \xrightarrow{F_\alpha(\epsilon^Z_\sigma)} & F_\alpha(Z_j) \\
\downarrow \epsilon^Z_k & & \downarrow \epsilon^Z_k \\
L(\Phi^*(Z))_k & \xrightarrow{(\epsilon_Z)_k} & Z_k
\end{array}
$$

commutes, where the left vertical map is the composition of the uniqueness isomorphism and the canonical map to the colimit $F_{\alpha \circ \sigma}(Z_i) \longrightarrow L(\Phi^*(Z))_k$. However, the definition of $\epsilon_Z$ implies that the diagram above commutes since $Z$ is a twisted diagram. To prove the naturality of $\epsilon$, let $f : Y \longrightarrow Z$ be a map in $\mathbf{Tw}(\mathcal{J}, \mathcal{B})$. For $j \in \mathcal{J}$ and every $\sigma : \Phi(i) \longrightarrow j$ in $\Phi \downarrow j$, the maps $F_\sigma(f_{\phi(i)}) : F_\sigma(Y_{\phi(i)}) \longrightarrow F_\sigma(Z_{\phi(i)})$ assemble to a natural transformation $D^\Phi f$ of functors on $\Phi \downarrow j$ making the diagram

$$
\begin{array}{ccc}
D^\Phi f_Y & \xrightarrow{D^\Phi f_Y} & Y_j \\
\downarrow D^\Phi f & & \downarrow f_j \\
D^\Phi f_Z & \xrightarrow{D^\Phi f_Z} & Z_j
\end{array}
$$

commute. The horizontal maps are the ones appearing in the definition of $\epsilon$. Since the colimit functor is left adjoint to the “constant diagram” functor, the square

$$
\begin{array}{ccc}
L(\Phi^*(Y))_j & \xrightarrow{(\epsilon_Y)_j} & Y_j \\
\downarrow L(\Phi^*f) & & \downarrow f_j \\
L(\Phi^*(Z))_j & \xrightarrow{(\epsilon_Z)_j} & Z_j
\end{array}
$$
commutes, proving the naturality of \( \epsilon \).

It remains to prove that the composites

\[
L \xrightarrow{L\eta} L \circ \Phi^* \circ L \xrightarrow{\epsilon L} L \quad \text{and} \quad \Phi^* \xrightarrow{\eta \Phi^*} \Phi^* \circ L \circ \Phi^* \xrightarrow{\Phi^* \epsilon} \Phi^*
\]

are identity natural transformations. The verification is straightforward; we omit the details. \(\square\)

The right adjoint of \( \Phi^* \), obtained by the corresponding twisted version of right Kan extension along \( \Phi \), will be denoted \( R \Phi \). By the dual of Theorem 2.4.1 it exists if \( \mathcal{B} \) consists of complete categories.

Recall the functor \( \text{Ev}_i \) defined as the restriction along \( \{i\} \rightarrow \mathcal{J} \). If the bundle \( \mathcal{B} \) consists of cocomplete categories, its left adjoint \( \text{Fr}_i : \mathcal{C}_i \rightarrow \text{Tw}(\mathcal{J}, \mathcal{B}) \) exists by Theorem 2.4.1. It is the analogue of the free diagram at \( i \) and will be needed later in the construction of a cofibrantly generated model structure. We call \( \text{Fr}_i(K) \) the free twisted diagram generated by \( K \in \mathcal{C}_i \).

**Example 2.4.2 (Spectra, continued).** Let \( \text{Sp} \) be the bundle defined in Example 2.2.2 which leads to ordinary spectra. The \( n \)th evaluation functor maps a spectrum to its \( n \)th term, and the corresponding \( n \)th free twisted diagram of a pointed simplicial set \( K \) is the spectrum

\[
\ast \rightarrow \ast \rightarrow \ldots \rightarrow \ast \rightarrow K \rightarrow \Sigma K \rightarrow \Sigma^2 K \rightarrow \ldots
\]

with \( K \) appearing at the \( n \)th spot and all \( \sharp \)-type structure maps being identities except for the map \( \Sigma(\ast) = \ast \rightarrow K \).

### 2.5. Construction of Adjunction Bundles

We think of twisted diagrams as generalised diagrams. However, there is an alternative approach using fibred and cofibred categories in the sense of Grothendieck. For definitions and notation the reader may wish to consult [Qui73].

Let us recall the GROTHENDIECK construction \( \widehat{\text{Gr}}(U) \) of a contravariant functor \( U \) defined on \( \mathcal{I} \) with values in the category of (small) categories. The objects of \( \widehat{\text{Gr}}(U) \) are the pairs \((i, Y)\) with \( i \) an object of \( \mathcal{I} \) and \( Y \) an object of \( U(i) \). A morphism \((i, Y) \rightarrow (j, Z)\) consists of a morphism \( i \rightarrow j \) in \( \mathcal{I} \) and a morphism \( Y \xrightarrow{A} U(\sigma)(Z) \) in \( U(i) \). Composition is given by the rule

\[
(\tau, B) \circ (\sigma, A) := (\tau \circ \sigma, U(\sigma)(B) \circ A) .
\]

This construction comes equipped with a functor \( \widehat{\text{Gr}}(U) \rightarrow \mathcal{I} \).

**Remark 2.5.1.** An adjunction bundle determines, by definition, a functor \( U : \mathcal{I}^{\text{op}} \rightarrow \text{Cat} \), hence a functor \( \widehat{\text{Gr}}(U) \rightarrow \mathcal{I} \). The existence of the left adjoints \( F_\sigma \) make \( \widehat{\text{Gr}}(U) \) a cofibred category over \( \mathcal{I}^{\text{op}} \), even a bifibred bundle in the sense of the next definition.

**Definition 2.5.2.** Given a functor \( \pi : \mathcal{E} \rightarrow \mathcal{A} \), we call \( \mathcal{E} \) a bifibred bundle over \( \mathcal{A} \) if the following conditions are satisfied (using notation from [Qui73]):
The functor $\pi$ is fibred, and for all composable morphisms $\alpha$ and $\beta$ in $A$, the natural isomorphism $\alpha^* \circ \beta^* \sim (\beta \circ \alpha)^*$ is the identity.

(2) The functor $\pi$ is cofibred, and for all morphisms $\alpha \in A$ the functor $\alpha^*$ is right adjoint to $\alpha^*$.

In this situation, a functor $f : I \rightarrow A$ determines an $I$-indexed adjunction bundle $f \bowtie \pi = I \bowtie E$ which sends the object $i \in I$ to the category $\pi^{-1}(f(i))$ and the morphism $\mu \in I$ to the adjoint pair $f(\mu)^*$ and $f(\mu)^*$.

**Remark 2.5.3** (M. Brun’s reformulation of twisted diagrams). Recall from Remark 2.5.1 the functor $\pi : \tilde{G}(U) \rightarrow I$ associated to an adjunction bundle. A straightforward calculation which we omit shows that $\text{Tw}(I, \mathcal{B})$ is the category of sections of $\pi$.

More generally, given a bifibred bundle $\pi$ and an adjunction bundle $f \bowtie \pi$ as in 2.5.2, the category of twisted diagrams $\text{Tw}(I, f \bowtie \pi)$ is the category of lifts of $f$ to $E$, i.e., the category of functors $g : I \rightarrow E$ satisfying $\pi \circ g = f$.

**Example 2.5.4.** Let $\text{Mod} \rightarrow \text{Rng}$ denote the canonical functor from the category of all modules over all rings to the category of rings. (The objects of Mod are pairs $(R, M)$ with $R$ a ring and $M$ an $R$-module. A morphism $(R, M) \rightarrow (S, N)$ consists of a ring map $f : R \rightarrow S$ and an $f$-semi-linear additive map $M \rightarrow N$.) This defines a bifibred bundle.

A toric variety determines a functor into $\text{Rng}$, hence an adjunction bundle (cf. Definition 2.5.2). In fact, the fan $\Sigma$ of a toric variety can be regarded as a poset (ordered by inclusion of cones), hence as a category, and we obtain a functor

$$\Sigma^{op} \rightarrow \text{Rng}, \sigma \mapsto \mathbb{C}[\sigma \cap M]$$

where $\sigma$ is the dual cone of $\sigma$ and $M$ is the dual lattice (see Oda [Oda88] for details). Thus the toric variety $X(\Sigma)$ determines the adjunction bundle $\Sigma^{op} \bowtie_{\text{Rng}} \text{Mod}$. As a more explicit example, the $n$-dimensional projective space is a toric variety. Its fan is isomorphic, as a poset, to the set of non-empty subsets of $[n] = \{0, 1, \ldots, n\}$ (ordered by reverse inclusion). The monoids $\tilde{\sigma} \cap M = M^\Sigma$ are described in Example 2.5.6 below.

This example can be generalised to obtain an adjunction bundle from a diagram of monoids and a cocomplete category $\mathcal{D}$. We proceed with a construction.

It is well known that we can consider any monoid $M$ as a category with one object and morphisms corresponding to the elements of $M$. A morphism of monoids then is a functor between such categories. Suppose that $\mathcal{D}$ is a cocomplete category. We define the category of $M$-equivariant objects in $\mathcal{D}$, denoted $M^{\mathcal{D}}$, as the category of functors $M \rightarrow \mathcal{D}$. A monoid homomorphism $f : M \rightarrow M'$ induces the “restriction” functor $f^* = R_f : M'^{\mathcal{D}} \rightarrow M^{\mathcal{D}}$ (given by pre-composing with $f$). Since $\mathcal{D}$ is cocomplete, this functor has a left adjoint $f_*=\cdot\wedge_M M' : M^{\mathcal{D}} \rightarrow M'^{\mathcal{D}}$. For composable monoid homomorphisms we have the relations $(g \circ f)^* = f^* \circ g^*$ and $(g \circ f)_* \cong g_* \circ f_*$. Moreover $\text{id}^* = \text{id}$, and we choose $\text{id}_* = \text{id}$. 

Let $\text{Eq}\mathcal{D}$ denote the category of equivariant objects in $\mathcal{D}$. Objects are the pairs $(M, D)$ where $M$ is a monoid and $D$ is a functor $M \rightarrow \mathcal{D}$. A morphism from $(M, D)$ to $(M', D')$ is a pair $(\alpha, \nu)$ where $\alpha: M \rightarrow M'$ is a monoid homomorphism and $\nu$ is a natural transformation of functors $D \rightarrow D' \circ \alpha$. The forgetful functor $\pi: \text{Eq}\mathcal{D} \rightarrow \text{Mon}$ on into the category of monoids make $\text{Eq}\mathcal{D}$ into a bifibred bundle in the sense of 2.5.2. The fibre over the monoid $M$ is the category $M\cdot\text{Eq}\mathcal{D}$ of $M$-equivariant objects in $\mathcal{D}$.

**Definition 2.5.5.** Suppose we have a (small) category $\mathcal{I}$ and an $\mathcal{I}$-indexed diagram $G$ of monoids, i.e., a functor $G: \mathcal{I} \rightarrow \text{Mon}$. For a cocomplete category $\mathcal{D}$ we define the $\mathcal{I}$-indexed adjunction bundle $\text{Ad}_{\mathcal{D}}G = (\mathcal{C}, F, U)$ by

$$\text{Ad}_{\mathcal{D}}G := \mathcal{I} \bowtie \Downarrow_{\text{Mon}} \text{Eq}\mathcal{D}.$$ 

Explicitly, for an object $i \in \mathcal{I}$ we let $\mathcal{C}_i := G(i)\cdot\mathcal{D}$, the category of $G(i)$-equivariant objects in $\mathcal{D}$, and for a morphism $\sigma \in \mathcal{I}$ we define $F_\sigma := G(\sigma)_*$ and $U_\sigma := G(\sigma)^*$.

This definition is clearly natural in $G$, i.e., given a natural transformation of diagrams of monoids $G \rightarrow G'$ we obtain an $\mathcal{I}$-morphism of adjunction bundles $\text{Ad}_{\mathcal{D}}G' \rightarrow \text{Ad}_{\mathcal{D}}G$.

**Example 2.5.6** (Non-linear projective spaces). This generalises the non-linear projective line (2.1.3). Let $[n]$ denote the set $\{0, 1, \ldots, n\}$, and write $\langle n \rangle$ for the category of non-empty subsets of $[n]$; morphisms are given by inclusion of sets. For $A \subseteq [n]$, define the (additive) monoid

$$M^A := \left\{(a_0, \ldots, a_n) \in \mathbb{Z}^{n+1} \mid \sum_{0}^{n} a_i = 0 \text{ and } \forall i \notin A: a_i \geq 0 \right\}.$$ 

These monoids assemble to a functor $D: \langle n \rangle \rightarrow \text{Mon}$. Let $\text{Eq}\mathcal{Top}$ denote the category of equivariant spaces as constructed above. (Objects are pairs $(M, T)$ where $M$ is a monoid and $T$ is a pointed topological space with a base-point preserving $M$-action. Maps are semi-equivariant continuous maps of pointed topological spaces.) This category is a bifibred bundle over the category of monoids. Thus we are in the situation of Definition 2.5.5 (with $\mathcal{I} = \langle n \rangle$); denote the resulting adjunction bundle $\text{Ad}_{\text{top}}D = \langle n \rangle \bowtie_{\text{Mon}} \text{Eq}\mathcal{Top}$ by $\mathcal{P}^n$. The category of twisted diagrams $\mathcal{Tw}(\langle n \rangle, \mathcal{P}^n)$ is nothing but the category $\mathcal{P}^n$ of presheaves as defined in [Hüt02, 6.1].

3. Model Structures

3.1. Bundles of Model Categories. A model category is a category $\mathcal{C}$ equipped with three classes of distinguished morphisms, called weak equivalences, cofibrations and fibrations. All these classes have to be closed under composition, and they are required to contain the identity morphisms. This set of data is subject to the following axioms:

**(MC1)** All finite limits and colimits exist in $\mathcal{C}$.
(MC2) If \( f \) and \( g \) are composable morphisms, and if two of the three morphisms \( f \), \( g \) and \( g \circ f \) are weak equivalences, so is the third.

(MC3) The classes of weak equivalences, cofibrations and fibrations are closed under retracts.

(MC4) Given a commutative square diagram in \( C \)

\[
\begin{array}{ccc}
A & \xrightarrow{f} & X \\
\downarrow i & & \downarrow p \\
B & \xrightarrow{g} & Y
\end{array}
\]

where \( i \) is a cofibration, \( p \) is a fibration, and at least one of \( i \) and \( p \) is a weak equivalence, there is a lift in the diagram, i.e., a morphism \( \ell : B \rightarrow X \) with \( \ell \circ i = f \) and \( p \circ \ell = g \).

(MC5) Given any morphism \( f \) there is a factorisation \( f = q \circ i \) where \( i \) is a cofibration, and \( q \) is a fibration and a weak equivalence. Given any morphism \( f \) there is a factorisation \( f = p \circ j \) where \( j \) is a cofibration and a weak equivalence, and \( p \) is a fibration.

The term “model category” is always to be understood in the above sense which is the definition given by Dwyer and Spalinski [DS95]. It is slightly more general than the definition given by Hovey [Hov99], the differences being the following: In [Hov99], it is required that a model category has all small limits and colimits (instead of just finite ones), and the factorisations have to be functorial and are part of the structure (instead of assuming that they simply exist).

**Definition 3.1.1.** Let \( \mathcal{B} = (C, F, U) \) be an adjunction bundle over \( \mathcal{I} \). We call \( \mathcal{B} \) an adjunction bundle of model categories if all the \( C_i \) are model categories, and all the \( F_\sigma \) preserve cofibrations and acyclic cofibrations. In other words, we require the pair \((F_\sigma, U_\sigma)\) to form a Quillen adjoint pair.—If in addition all the \( C_i \) are left proper model categories, \( \mathcal{B} \) is called left proper, and similarly for “right proper” and “proper”. Note that the inverse image of an adjunction bundle of model categories \( \mathcal{B} \) is again an adjunction bundle of model categories, which is as proper as \( \mathcal{B} \).

**Example 3.1.2.** The projective space bundles \( \mathbb{P} \) [2.5.6] and spectra \( Sp \) [2.2.2] are examples of proper adjunction bundles of model categories. The model structure defined on \( M\text{-}\mathbf{Top} \) (for \( M \) a monoid) has weak equivalences and fibrations on underlying spaces, the model structure on the category of pointed simplicial sets is the usual one.

Before defining the model structures on twisted diagrams, we make a technical observation.
Remark 3.1.3. Suppose $\mathcal{C} = \prod_\nu \mathcal{C}_\nu$ is the product of model categories $\mathcal{C}_\nu$. Then there is a product model structure on $\mathcal{C}$ where a map is a weak equivalence (resp., fibration, resp., cofibration) if its image under the canonical projection is a weak equivalence (resp., fibration, resp., cofibration) in $\mathcal{C}_\nu$ for all $\nu$ (see [Hov99, 1.1.6]). If all the $\mathcal{C}_\nu$ are left proper, $\mathcal{C}$ is a left proper model category, and similarly for “right proper”.

3.2. The $\alpha$-Structure. The first model structure on $\text{Tw}(\mathcal{I}, \mathcal{B})$ we want to consider has pointwise weak equivalences and pointwise fibrations. The price one has to pay for the simple definition of fibrations is that the description of cofibrations is rather involved. Moreover, we have to restrict to “nice” indexing categories.

Definition 3.2.1 (Direct categories). A category with degree function is a (small) category $\mathcal{I}$ together with a $\mathbb{Z}$-valued function $d$, defined on the objects, such that whenever there is a non-identity morphism $i \to j$ we have $d(i) \neq d(j)$. (We say that all non-identity arrows change the degree. In particular, objects have no non-trivial endomorphisms.) The category is called bounded if $d$ is bounded below, and it is called locally bounded if each connected component is bounded. Without restriction, the degree of a bounded category has values in an honest ordinal, namely $\mathbb{N}$. If non-identity arrows always increase the degree and the category is (locally) bounded, we say that $\mathcal{I}$ is a (locally) direct category.

All finite dimensional categories (i.e., categories with finite dimensional nerve) admit degree functions and can be made into direct categories. A disjoint union of locally direct categories is locally direct. If $\mathcal{I}$ is (locally) direct, so are subcategories, under and over categories formed with $\mathcal{I}$. In particular, the full subcategory $\mathcal{I}_n$ of objects of degree less than or equal to $n$ is (locally) direct. A finite product of direct categories is direct (with degree given by sum of partial degrees).

In what follows, $\mathcal{B} = (\mathcal{C}, F, U)$ is an adjunction bundle of cocomplete model categories over $\mathcal{I}$. Let $Y$ be a twisted diagram with coefficients in $\mathcal{B}$ and $i$ an object of $\mathcal{I}$. To describe the cofibrations in the model structure we are going to construct, we have to introduce the latching object of $Y$ at $i$. Recall that for a diagram $Z$ (untwisted case) the latching object at $i$ is defined as the colimit over all components $Z_j$ which map to $Z_i$. For a twisted diagram $Y$, we mimic this construction, using the “twisting” functors $F_\sigma$ to push everything into the category $\mathcal{C}_i$. The colimit is to be taken with respect to the $\sharp$-type structure maps of $Y$.

Technically, we can describe the latching spaces as follows. For each object $i \in \mathcal{I}$, let $\mathcal{I} \downarrow i$ denote the category of objects over $i$. Let $\mathcal{I} \downarrow i$ denote the full subcategory of $\mathcal{I} \downarrow i$ which consists of all objects $\sigma: j \to i$ with $\sigma \neq \text{id}_i$. There are canonical functors $i: \mathcal{I} \downarrow i \to \mathcal{I} \downarrow i$ (the inclusion) and $\text{pr}: \mathcal{I} \downarrow i \to \mathcal{I}$ (the projection $(\sigma: j \to i) \mapsto j$). Set $\mathcal{P}_{\mathcal{I} \downarrow i} := \text{pr} \circ i$ and denote the trivial bundle over $\mathcal{I} \downarrow i$ with value $\mathcal{C}_i$ by $\mathcal{C}_i$ again. We define an
I ⇓ i-morphism of bundles Ψ: C_i → (P_{I; i})^*B as follows: For σ: j → i, the adjoint pair

F_σ: C_j ≅ C_i: U_σ

is the σ-component of Ψ, and it is obvious from the definitions that Ψ is in fact a bundle morphism. Hence we have a functor

Ψ*: Tw(I ⇓ i, (P_{I; i})^*B) → Fun(I ⇓ i, C_i).

Define G_i: Tw(I ⇓ i, (P_{I; i})^*B) → C_i as the composition G_i := lim_→ Ψ^*

Definition 3.2.2. The latching object L_iY of Y at i is defined as L_iY := G_i ∘ (P_{I; i})^*(Y). It is an object of C_i.

Remark 3.2.3. Note that L_i is a composition of functors, hence itself a functor. The structure maps y^♯σ: F_σ(Y_j) → Y_i for σ: j → i define a natural transformation L_i → Ev_i. If a map L_iY → Y_i is mentioned, it is always this natural map.

Example 3.2.4. If X is a spectrum and n > 0, the latching object of X at n is the pointed simplicial set ΣX_{n-1}, and the natural map ΣX_{n-1} → X_n of [3.2.3] is the (♯-type) structure map of the spectrum.

Example 3.2.5. Let Y = (Y_+ y_α → Y_0 ← y_β Y_-) be a twisted diagram with coefficients in the projective line bundle B_1 (cf. [2.1.3]). The latching objects of Y at + and at - are the initial objects in N_+·Top and N_-·Top, respectively. The latching object at 0 is the Z-equivariant pointed space (Y_+ ∧_{N_+} Z) ∨ (Y_- ∧_{N_-} Z). The ♯-type structure maps induce a map to Y_0.

Definition 3.2.6 (The c-structure). Let f: Y → Z be a map in of twisted diagrams in Tw(I, B). We call f a weak equivalence if f_i is a weak equivalence in C_i for every object i ∈ I. We call f a c-cofibration if for all objects i of I, the induced map Y_i ∪_{L_i Y} L_i Z → Z_i is a cofibration. We call f a c-fibration if all f_i are fibrations in C_i.

To prove that the c-structure is a model structure, we concentrate on the lifting axiom first. Call a map f ∈ Tw(I, B) a good acyclic c-cofibration if for all objects i of I, the induced map Y_i ∪_{L_i Y} L_i Z → Z_i is an acyclic cofibration. Later, we will prove that the class of good acyclic c-cofibrations coincides with the class of acyclic c-cofibrations.

Lemma 3.2.7. Let I be a direct category, and let B be an adjunction bundle of cocomplete model categories over I. Good acyclic c-cofibrations have the left lifting property with respect to c-fibrations. Similarly, c-cofibrations have the left lifting property with respect to acyclic c-fibrations.
Proof. We treat the first case only, the other is similar. Let

\[
\begin{array}{ccc}
A & \xrightarrow{g} & X \\
\downarrow{f} & & \downarrow{p} \\
B & \xrightarrow{h} & Y
\end{array}
\]

be a commutative diagram in $\mathbf{Tw}(I, \mathcal{B})$ such that $f$ is a good acyclic $c$-cofibration and $p$ is a $c$-fibration. We will construct the desired lift by induction on the degree of objects of $I$.

Since $I$ is direct, the degree function $d$ has a minimum $k$. If $i$ is an object in $I$ of degree $k$, then $L_i$ is the constant functor with the initial object as value. By definition of a good acyclic cofibration, the map $f_i$ is an acyclic cofibration in $C_i$. Hence we can find a lift $l_i$ in the following diagram:

\[
\begin{array}{ccc}
A_i & \xrightarrow{g_i} & X_i \\
\downarrow{f_i} & & \downarrow{p_i} \\
B_i & \xrightarrow{h_i} & Y_i
\end{array}
\]

Since the full subcategory $I_k$ of objects of degree $k$ is discrete, the lifts $l_i$ for the various $i \in I_k$ assemble to a map $l|_{I_k}: B|_{I_k} \longrightarrow X|_{I_k}$ in $\mathbf{Tw}(I_k, \mathcal{B}|_{I_k})$.

Now let $n > k$, and assume that we have constructed a lift in the diagram

\[
\begin{array}{ccc}
A|_{I_{n-1}} & \xrightarrow{g|_{I_{n-1}}} & X|_{I_{n-1}} \\
\downarrow{f|_{I_{n-1}}} & & \downarrow{p|_{I_{n-1}}} \\
B|_{I_{n-1}} & \xrightarrow{h|_{I_{n-1}}} & Y|_{I_{n-1}}
\end{array}
\]

making it a commutative diagram in $\mathbf{Tw}(I|_{n-1}, \mathcal{B}|_{I_{n-1}})$. If $i$ is an object of degree $n$ and $\sigma: j \longrightarrow i$ an object of $I \downarrow i$, the map

\[
F(\sigma)(B_j) \xrightarrow{F(\sigma)(l_j)} F(\sigma)(X_j) \xrightarrow{x^\sigma_i} X_i
\]
is part of a natural transformation \( \phi: L_iB \to X_i \) such that the following square diagram commutes:

\[
\begin{array}{ccc}
L_iA & \longrightarrow & A_i \\
\downarrow & & \downarrow \\
L_if & \longrightarrow & g_i \\
\downarrow & & \downarrow \\
L_iB & \phi & X_i \\
\end{array}
\]

Hence we get another diagram

\[
\begin{array}{ccc}
A_i \cup_{L_iA} L_iB & \longrightarrow & X_i \\
\downarrow & & \downarrow \\
B_i & \sim & p_i \\
\downarrow & & \downarrow \\
Y_i & \downarrow h_i & Y_i \\
\end{array}
\]

in which, by hypothesis, the left vertical map is an acyclic cofibration and the right vertical map is a fibration. Thus a lift \( l_i: B_i \to X_i \) exists, and it is straightforward to check that these maps \( l_i \), together with the morphism \( l|_{I_{n-1}} \), define a map of twisted diagrams \( l|_{I_n}: B|_{I_n} \to X|_{I_n} \) such that the diagram

\[
\begin{array}{ccc}
A|_{I_n} & \longrightarrow & X|_{I_n} \\
\downarrow & & \downarrow \\
f|_{I_n} & \searrow & p|_{I_n} \\
\downarrow & & \downarrow \\
B|_{I_n} & \longrightarrow & Y|_{I_n} \\
\end{array}
\]

commutes. This completes the induction. \( \square \)

Let \( \Phi: I \longrightarrow J \) be a functor and \( \mathcal{A} \) an adjunction bundle of cocomplete model categories over \( J \). Obviously, the functor

\[
\Phi^*: \text{Tw}(J, \mathcal{A}) \longrightarrow \text{Tw}(I, \Phi^*\mathcal{A})
\]

preserves weak equivalences and \( c \)-fibrations. The question is whether \( \Phi^* \) also preserves \( c \)-cofibrations. Under certain conditions (which are satisfied in the case of interest) we can give a positive answer.

Suppose the functor \( \Phi: I \longrightarrow J \) is injective at identities, i.e., whenever \( \Phi(\sigma) \) is an identity morphism, so is \( \sigma \). (For example, a faithful functor is injective at identities.) Then \( \Phi \) induces a functor

\[
\Phi \downarrow i: I \downarrow i \longrightarrow J \downarrow \Phi(i)
\]
which sends $\sigma: k \rightarrow i$ to $\Phi(\sigma): \Phi(k) \rightarrow \Phi(i)$. This construction is compatible with the projection functors:

$$\Phi \circ P_{I \downarrow i} = P_{I \downarrow \Phi(i)} \circ \Phi \downarrow i.$$ 

Recall that a functor $F: C \rightarrow D$ is called final if for each $A \in D$ the category $A \downarrow F$ of objects $F$-under $A$ is non-empty and connected.

We say that the functor $\Phi$ satisfies the finality condition if it is injective at identities, and the functor $\Phi \downarrow i$ is final for all objects $i \in I$.

**Lemma 3.2.8.** Let $\Phi: I \rightarrow J$ be a functor, $\mathcal{B}$ an adjunction bundle of cocomplete model categories over $I$ and $i$ an object of $I$. Denote by $L_i$ the $i$-th latching object functor of $\mathcal{T}w(I, \Phi^*B)$, and by $L_{\Phi(i)}'$ the $\Phi(i)$-th latching object functor of $\mathcal{T}w(J, \mathcal{B})$. If $\Phi$ satisfies the finality condition, then there is a natural isomorphism $L_i \circ \Phi^* \cong L'_{\Phi(i)}.$

**Proof.** The functor $L_i$ is defined as the composition $\lim \circ \Psi^* \circ P_{I \downarrow i}^*$, with $\Psi$ being an $I \downarrow i$-morphism with $\sigma$-component given by the adjunction

$$F_{\Phi(\sigma)}: C_{\Phi(j)} \rightarrow C_{\Phi(i)}: U_{\Phi(\sigma)}$$

where $\sigma: j \rightarrow i$ is an object of $I \downarrow i$. On the other hand, $L'_{\Phi(i)}$ is the composition $L'_{\Phi(i)} = \lim \circ \Theta^* \circ P_{J \downarrow \Phi(i)}^*$, with $\Theta$ having the $\tau$-component given by the adjunction

$$F_{\tau}: C_{\Phi(j)} \rightarrow C_{\Phi(i)}: U_{\tau}$$

where $\tau: j \rightarrow \Phi(i)$ is an object of $J \downarrow \Phi(i)$. It is straightforward to check that the equality $L_i \circ \Phi^* = \lim \circ ( \Phi \downarrow i)^* \circ \Theta^* \circ P_{J \downarrow \Phi(i)}^*$ holds. Hence the $i$-th latching object of $\Phi^*(A)$ is given by

$$L_i(\Phi^*(A)) = \lim \circ ( \Phi \downarrow i)^* \circ P_{J \downarrow \Phi(i)}^*(A) \circ (\Phi \downarrow i).$$

The functor $\Phi \downarrow i$ induces a map $L_i(\Phi^*(A)) \rightarrow L'_{\Phi(i)}(A)$ which is an isomorphism by [Mac71, IX.3.1] since $\Phi \downarrow i$ is final.

**Corollary 3.2.9.** If $\Phi$ satisfies the finality condition, then $\Phi^*$ preserves c-cofibrations and good acyclic c-cofibrations.

**Proof.** This follows from 3.2.8 since the maps $L_i(\Phi^*A) \rightarrow A_{\Phi(i)}$ and $L'_{\Phi(i)}A \rightarrow A_{\Phi(i)}$ correspond under the isomorphism.

**Remark 3.2.10.** The functor $P_{I \downarrow i}$ satisfies the finality condition because $(P_{I \downarrow i}) \downarrow x$ is an isomorphism of categories for each object $x \in I \downarrow i$.

**Lemma 3.2.11.** Let $I$ be direct. For each $i \in I$, the latching object functor $L_i$ maps c-cofibrations to cofibrations and good acyclic c-cofibrations to acyclic cofibrations.

**Proof.** Recall that $L_i$ was defined as the composite $G_i \circ (P_{I \downarrow i})^*$. By Remark 3.2.10 and Corollary 3.2.9 we are left to show that $G_i$ maps c-cofibrations to cofibrations and good acyclic c-cofibrations to acyclic cofibrations.
However, $G_i$ has a right adjoint $V_i := \Psi_* \circ \delta$, where $\delta: C_i \rightarrow \text{Fun}(I \downarrow i, C_i)$ denotes the constant diagram functor and $\Psi_*$ is the direct image under the $I \downarrow i$-morphism $\Psi$ having $\sigma$-component

$$F_\sigma: C_j \longrightarrow C_i: U_\sigma$$

where $\sigma: j \rightarrow i$ is an object of $I \downarrow i$. It is easy to see that $V_i$ maps (acyclic) fibrations to (acyclic) c-fibrations. Hence the statement follows from Lemma 3.2.7 and the fact that $C_i$ is a model category. □

**Corollary 3.2.12.** If $f$ is a (good acyclic) c-cofibration, all its components are (acyclic) cofibrations in their respective categories. In particular, a good acyclic c-cofibration is an acyclic c-cofibration.

**Proof.** Let $f: A \longrightarrow B$ be a c-cofibration. By Lemma 3.2.11 the map $L_i f: L_i A \longrightarrow L_i B$ is a cofibration in $C_i$, hence $A_i \longrightarrow A_i \cup_{L_i A} L_i B$ is a cofibration. Observe that $f_i$ factors as this last map followed by the map $A_i \cup_{L_i A} L_i B \longrightarrow B_i$. Since the latter is a cofibration by hypothesis, we conclude that $f_i$ is a cofibration.—The other case is similar. □

**Theorem 3.2.13.** Suppose $I$ is a locally direct category, and $B$ is an adjunction bundle of cocomplete model categories over $I$.

1. The c-structure is a model structure.
2. A map $f$ of twisted diagrams is an acyclic c-cofibration if and only if for all objects $i \in I$, the induced map $Y_i \cup_{L_i Y} L_i Z \longrightarrow Z_i$ is an acyclic cofibration in $C_i$.
3. If $B$ is a left (resp., right) proper bundle, the c-structure is left (resp., right) proper.

**Proof.** Let $(I_\nu)$ denote the family of path components of $I$. Then $I = \coprod I_\nu$, and each of the $I_\nu$ is a direct category. Since $\text{Tw}(I, B) = \coprod_\nu \text{Tw}(I_\nu, B|_{I_\nu})$, it is enough to show that the c-structure is a model structure for each of the categories $\text{Tw}(I_\nu, B|_{I_\nu})$; by 3.1.3 we can equip $\text{Tw}(I, B)$ with the product model structure. Consequently, we can assume that $I$ is direct.

We use the axioms for model categories as given in [DS95]. First we note that the class of weak equivalences is closed under composition since weak equivalences are defined pointwise. Similarly, the composition of two c-fibrations is a c-fibration again.

Now assume we have two composable c-cofibrations $A \xrightarrow{f} B \xrightarrow{g} C$. To show that $g \circ f$ is a c-cofibration, we have to prove that for all objects $i \in I$ the induced map

$$A_i \cup_{L_i A} L_i C \longrightarrow C_i$$

is a cofibration in $C_i$. But we can factor this map as

$$A_i \cup_{L_i A} L_i C \xrightarrow{x} A_i \cup_{L_i A} L_i B \cup_{L_i B} L_i C \xrightarrow{y} B_i \cup_{L_i B} L_i C \longrightarrow C_i$$
where \( x \) is induced by \( f \), and \( y \) is induced by \( g \). But both of these maps are cofibrations (since they are cobase changes of cofibrations), hence so is their composite.

It is obvious that each of the classes above contains all identities.

**Axiom MC1:** existence of finite limits and colimits is guaranteed by \( \text{(2.3.1)} \) since they exist in all \( \mathcal{C}_i \).

**Axiom MC2:** the "2-of-3" property for weak equivalences is satisfied since weak equivalences are defined pointwise and \( \text{MC2} \) holds in all the categories \( \mathcal{C}_i \).

**Axiom MC3:** the class of weak equivalences is closed under retracts since weak equivalences are defined pointwise, and in each category \( \mathcal{C}_i \) a retract of a weak equivalence is a weak equivalence. Similarly, the class of fibrations is closed under retracts.

Suppose \( g: Y \rightarrow Z \) is a retract of \( f: A \rightarrow B \) and \( f \) is a \( c \)-cofibration. We have to show that for all objects \( i \in \mathcal{I}_n \), the map \( L_i Z \cup_{L_i Y} Y_i \rightarrow Z_i \) induced by \( g \) is a cofibration in \( \mathcal{C}_i \). But by functoriality of pushouts and latching objects, this map is a retract of the map \( L_i B \cup_{L_i A} A_i \rightarrow B_i \) induced by \( f \), which is a cofibration by hypothesis. Since \( \text{MC3} \) is valid in \( \mathcal{C}_i \), the former map is a cofibration. Hence \( g \) is a \( c \)-cofibration as claimed.

This argument also shows that the class of good acyclic \( c \)-cofibrations is closed under retracts.

**Axiom MC5:** let \( f: A \rightarrow X \) be a map in \( \text{Tw}(\mathcal{I}, \mathfrak{B}) \). We will construct inductively a factorisation of \( f \) as a good acyclic \( c \)-cofibration followed by a \( c \)-fibration. (The other factorisation axiom is proved in a similar manner). Let \( k \) be the minimum of the degree function on \( \mathcal{I} \), and let \( i \) be of degree \( k \). Then \( f_i \) factors in \( \mathcal{C}_i \) as \( A_i \xrightarrow{g_i} T_i \xrightarrow{p_i} X_i \), with \( g_i \) being an acyclic cofibration and \( p_i \) being a fibration. The collection of these factorisations (where \( i \) ranges through all objects of degree \( k \)) yields a factorisation of \( f|_{\mathcal{I}_k} \) in \( \text{Tw}(\mathcal{I}_k, \mathfrak{B}|_{\mathcal{I}_k}) \) as \( g|_{\mathcal{I}_k}: A|_{\mathcal{I}_k} \rightarrow T|_{\mathcal{I}_k} \) followed by \( p|_{\mathcal{I}_k}: T|_{\mathcal{I}_k} \rightarrow X|_{\mathcal{I}_k} \).

Let \( n > k \), and assume we have already constructed a factorisation of \( f|_{\mathcal{I}_{n-1}} \) in \( \text{Tw}(\mathcal{I}_{n-1}, \mathfrak{B}|_{\mathcal{I}_{n-1}}) \) as the composite

\[
A|_{\mathcal{I}_{n-1}} \xrightarrow{g|_{\mathcal{I}_{n-1}}} T|_{\mathcal{I}_{n-1}} \xrightarrow{p|_{\mathcal{I}_{n-1}}} X|_{\mathcal{I}_{n-1}}.
\]

Let \( i \) be of degree \( n \). The canonical functor \( P_{\mathcal{I} \downarrow i}: \mathcal{I} \downarrow i \rightarrow \mathcal{I} \) factors through the inclusion \( \Phi: \mathcal{I}_{n-1} \leftarrow \mathcal{I} \) as \( \Theta: \mathcal{I} \downarrow i \rightarrow \mathcal{I}_{n-1} \) since \( \mathcal{I} \) is direct. Recall the functor

\[
G_i: \text{Tw}(\mathcal{I} \downarrow i, (P_{\mathcal{I} \downarrow i})^* \mathfrak{B}) \rightarrow \mathcal{C}_i
\]

appearing in the definition of the \( i \)-th latching object functor \( L_i \) \( \text{(3.2.2)} \). By definition, \( L_i = G_i \circ P_{\mathcal{I} \downarrow i} = G_i \circ \Theta^* \circ \Phi^* \), hence \( G_i \circ \Theta^*(A|_{\mathcal{I}_{n-1}}) = L_i A \). The maps \( F_\sigma(T_j) \xrightarrow{F_\sigma(p_j)} F_\sigma(X_j) \xrightarrow{x_j} X_i \) for the different objects \( \sigma: j \rightarrow i \) of
$T \downarrow i$ induce a map $G_i \circ \Theta^* (T|_{I_n-1}) \rightarrow X_i$ which makes the diagram

$$G_i \circ \Theta^* (A|_{I_{n-1}}) = L_i A \rightarrow A_i$$

 commute. Now factor the induced map $A_i \cup_{L_i(A)} (G_i \circ \Theta^*) (T|_{I_{n-1}}) \rightarrow X_i$ as an acyclic cofibration $h_i: A_i \cup_{L_i(A)} (G_i \circ \Theta^*) (T|_{I_{n-1}}) \rightarrow T_i$ followed by a fibration $p_i: T_i \rightarrow X_i$ in $\mathcal{C}$. The collection of the $T_i$ for the different objects $i$ of degree $n$, together with $T|_{I_{n-1}}$ define a twisted diagram in $\text{Tw}(I_n, \mathcal{B}|_{I_n})$. The new structure maps for $\sigma: j \rightarrow i$ are the compositions

$$F_\sigma(T_j) \rightarrow G_i \circ \Theta^* (T|_{I_{n-1}}) \rightarrow A_i \cup_{L_i(A)} (G_i \circ \Theta^*) (T|_{I_{n-1}}) \xrightarrow{h_i} T_i$$

 where the first two maps are the canonical ones. If we define $g_i$ as the composition of the canonical map $A_i \rightarrow A_i \cup_{L_i(A)} (G_i \circ \Theta^*) (T|_{I_{n-1}})$ with $h_i$, it is straightforward to check that we get a factorisation $f|_{I_n} = p|_{I_n} \circ g|_{I_n}$ in $\text{Tw}(I_n, \mathcal{B}|_{I_n})$. This completes the induction.

We end up with a factorisation of $f$ as $A \xrightarrow{g} T \xrightarrow{p} X$. The object $T|_{I_n}$ we constructed in the induction step coincides with the restriction of $T$, and similarly for the maps $g$ and $p$. It is clear that $p$ is a c-fibration in $\text{Tw}(I, \mathcal{B})$. To complete the proof of axiom $\textbf{MC5}$, it remains to show that the map $g$ is a good acyclic c-cofibration. However, if $i$ is of degree $k = \text{min} d$, the map $A_i \cup_{L_i(A)} L_i T = A_i \xrightarrow{g_i} T_i$ is an acyclic cofibration in $\mathcal{C}$, and if $i$ is of degree $n > k$, the map $A_i \cup_{L_i(A)} L_i T \rightarrow T_i$ coincides with the map $h_i: A_i \cup_{L_i(A)} (G_i \circ \Theta^*)(T|_{I_{n-1}}) \rightarrow T_i$ which is an acyclic cofibration in $\mathcal{C}$. Hence $g$ is a good acyclic c-cofibration.

We prove part (2) of the theorem. We have already seen that every good acyclic c-cofibration is an acyclic c-cofibration \textbf{(3.2.12)}. To prove the converse, let $f: A \rightarrow X$ be an acyclic c-cofibration. Factor $f$ as a good acyclic c-cofibration $g: A \rightarrow T$ followed by a c-fibration $p: T \rightarrow X$, and note that $p$ is an acyclic c-fibration by axiom $\textbf{MC2}$. The map $f$ is in particular a $c$-cofibration, so we can find a lift in the diagram

$$\begin{array}{ccc}
A & \xrightarrow{g} & T \\
| & & | \\
F & \downarrow & p \\
X & \xrightarrow{id_X} & X
\end{array}$$

which expresses $f$ as a retract of $g$. Since good acyclic c-cofibrations are closed under retracts, we are done.
Knowing (2), we see that axiom MC4 is an immediate consequence of Lemma 3.2.7. This finishes the proof of (1).

Finally, recall from Proposition 2.3.1 that pushouts and pullbacks are calculated pointwise. Since the components of a weak equivalence (c-fibration, c-cofibration) are weak equivalences (fibrations, cofibrations) in the respective categories (use Corollary 3.2.12 for the c-cofibrations), assertion (3) follows. □

Remark 3.2.14. The definition of a direct category can be extended to more general degree functions having ordinals as values, cf. [Hov99]. The two inductive proofs of 3.2.7 and 3.2.13 can be completed with a discussion of the “limit ordinal case”, thus giving the c-structure for a larger class of indexing categories.

3.3. The f-Structure. The construction of the c-structure can be dualised. There is a notion of a (locally) inverse category, and matching objects allow us to define an f-structure with pointwise cofibrations and weak equivalences. In the following, let \( B = (C, F, U) \) be an adjunction bundle of complete model categories over \( I \). Denote by \( i \downarrow I \) the full subcategory of the under category \( i \downarrow I \) consisting of objects \( \sigma: i \rightarrow j \) with \( \sigma \neq id_i \).

Again we have a canonical functor \( \Phi: i \downarrow I \rightarrow I \). Consider \( C_i \) as a trivial bundle over \( i \downarrow I \), and let \( \Psi: \Phi^*B \rightarrow C_i \) be the \( i \downarrow I \)-morphism of bundles with \( \sigma \)-component given by the adjunction

\[ F_\sigma: C_i \rightarrow C_j: U_\sigma \]

for \( \sigma: i \rightarrow j \). Define \( H_i: Tw(i \downarrow I, \Phi^*B) \rightarrow C_i \) as the composition \( \lim \circ \Psi \). In fact, \( H_i \) coincides with the direct image functor \( \Xi \) where \( \Xi \) is the bundle morphism given by the pair \( (\Psi, i \downarrow I \rightarrow \{i\}) \) (here \( \{i\} \) is the trivial category).

Definition 3.3.1. Let \( Y \) be a twisted diagram with coefficients in \( B \). The matching object of \( Y \) at \( i \) is defined as \( M_iY := H_i \circ \Phi^*(Y) \).

Remark 3.3.2. The structure maps \( y^\#_\sigma: Y_i \rightarrow U_\sigma(Y_j) \) for \( \sigma: i \rightarrow j \) define a natural transformation \( Ev_i \rightarrow M_i \). If a map \( Y_i \rightarrow M_iY \) is mentioned, it is always this natural map.

Definition 3.3.3 (The f-structure). Let \( f: Y \rightarrow Z \) be a map of twisted diagrams in \( Tw(I, B) \). We call \( f \) a weak equivalence if \( f_i \) is a weak equivalence in \( C_i \) for every object \( i \in I \). We call \( f \) an f-fibration if for all objects \( i \in I \), the induced map \( Y_i \rightarrow Z_i \times_{M_iZ} M_iY \) is a fibration. We call \( f \) an f-cofibration if all \( f_i \) are cofibrations in \( C_i \).

Definition 3.3.4. A category with degree function is called a (locally) inverse category if its opposite category (with the same degree function) is (locally) direct (3.2.1).

Theorem 3.3.5. Suppose \( I \) is a locally inverse category, and \( B \) is an adjunction bundle of complete model categories over \( I \).
(1) The $f$-structure is a model structure.
(2) If $f$ is an $f$-fibration, all its components are fibrations in their respective categories.
(3) A map $f: Y \to Z$ of twisted diagrams is an acyclic $f$-fibration if and only if for all objects $i \in I$, the induced map

$$Y_i \to Z_i \times_{M_i Z} M_i Y$$

is an acyclic fibration in $C_i$.
(4) If $\mathcal{B}$ is a left resp.right proper bundle, the $f$-structure is left resp.right proper.

\[\Box\]

Remark 3.3.6. In fact, it is possible to construct a model structure on the category $\mathbf{Tw}(I, \mathcal{B})$ if $I$ is a Reedy category and $\mathcal{B}$ consists of complete and cocomplete model categories. One has to combine the construction of the $c$-structure and the $f$-structure. The weak equivalences are pointwise weak equivalences, the fibrations and cofibrations are more complicated to define. In the case of diagram categories, this is done in section 5.2 of [Hov99], and the proof given there applies to our situation as well.

3.4. The $g$-Structure. In this section we consider a cofibrantly generated model structure with pointwise weak equivalences and pointwise fibrations. (In particular, the $g$-structure coincides with the $c$-structure provided both are defined.) Terminology is taken from [Hov99].

Definition 3.4.1. An $I$-bundle $\mathcal{B}$ of cocomplete model categories is called a cofibrantly generated adjunction bundle if for all objects $i \in I$ the model category $C_i$ is cofibrantly generated.

Examples of cofibrantly generated adjunction bundles include the spectrum bundle $Sp$ of Example [2.2.2] and the projective space bundle $\mathbb{P}^n$ of [2.5.6]. The inverse image of a cofibrantly generated adjunction bundle is cofibrantly generated.

Since $C_i$ is cocomplete, the $i$-th evaluation functor $Ev_i: \mathbf{Tw}(I, \mathcal{B}) \to C_i$ has a left adjoint $Fr_i: C_i \to \mathbf{Tw}(I, \mathcal{B})$, the $i$-th free twisted diagram functor obtained by twisted left Kan extension (Theorem 2.4.1). Explicitly, for an object $A$ of $C_i$ the $j$-component of $Fr_i(A)$ is given by the coproduct

$$\coprod_{\alpha \in \text{hom}_I(i,j)} F_\alpha(A)$$
and the structure maps are given in the following way: if $\beta: j \to k$ is a morphism in $I$, the map $Fr_i(A)^j_\beta$ is the composition

$$F_\beta(Fr_i(A)^j_\beta) = \bigoplus_{\alpha \in \text{hom}_I(i,j)} F_\alpha(A)$$

$$\cong \bigoplus_{\alpha \in \text{hom}_I(i,j)} F_\beta \circ F_\alpha(A)$$

$$\cong \bigoplus_{\alpha \in \text{hom}_I(i,j)} F_{\beta \circ \alpha}(A)$$

$$\quad \bigoplus_{\gamma \in \text{hom}_I(i,k)} F_\gamma(A)$$

where the last map is the canonical map induced by the identity on each summand, mapping the $\alpha$-summand of the source into the $\beta \circ \alpha$-summand of the target.

Define $M$ to be the set of maps in $\text{Tw}(I, B)$ of the form $Fr_i(f)$ with $i$ some object of $I$ and $f$ a generating cofibration in $C_i$. Define $N$ to be the set of maps in $\text{Tw}(I, B)$ of the form $Fr_i(f)$, with $i$ some object of $I$ and $f$ a generating acyclic cofibration in $C_i$. Note that $M$ and $N$ are sets because $I$ is small.

**Definition 3.4.2 (The $g$-structure).** Let $f: Y \to Z$ be a map of twisted diagrams in $\text{Tw}(I, B)$. We call $f$ a weak equivalence if $f_i$ is a weak equivalence in $C_i$ for every object $i \in I$. We call $f$ a $g$-fibration if $f$ has the right lifting property with respect to the set $N$. We call $f$ a $g$-cofibration if $f$ has the left lifting property with respect to every $g$-fibration which is also a weak equivalence.

**Lemma 3.4.3.** A map has the right lifting property with respect to the set $N$ (resp. $M$) if and only if all its components are fibrations (resp. acyclic fibrations).

**Proof.** This follows from the adjointness of $Fr_i$ and $Ev_i$, and the fact that $B$ is cofibrantly generated. □

**Lemma 3.4.4.** The domains of the maps of $M$ are small relative to $M$-cell. The domains of the maps of $N$ are small relative to $N$-cell.

**Proof.** This follows from the adjointness of $Fr_i$ and $Ev_i$, and the fact that $B$ is cofibrantly generated. We give a detailed argument for the case of $M$. Let $A$ be the domain of a map in $M$, so $A$ is of the form $Fr_i(X)$ for some $i \in I$, with $X$ being the domain of a generating cofibration in $C_i$. Denote the set of generating cofibrations in $C_i$ by $J$ and recall that $X$ is $\kappa$-small relative to the class $J$-cell for some cardinal $\kappa$, because $C_i$ is cofibrantly generated. We will prove that $A = Fr_i(X)$ is $\kappa$-small relative to the class $M$-cell.

Let $\lambda$ be a $\kappa$-filtered ordinal and $B: \lambda \to \text{Tw}(I, B)$ be a functor such that the map $B_\beta \to B_{\beta + 1}$ is in $M$-cell for all $\beta$ with $\beta + 1 < \lambda$. We have
to prove that the canonical map
\[
\lim \rightarrow \mathbf{Tw}(\mathcal{I}, \mathcal{B})(A, B) \to \mathbf{Tw}(\mathcal{I}, \mathcal{B})(A, \lim B)
\]
is an isomorphism. The adjointness of \(Fr_i\) and \(Ev_i\) provides that this map is isomorphic to the composite
\[
\lim C_i(X, Ev_i(B_\beta)) \to C_i(X, Ev_i \circ \lim B) \cong C_i(X, \lim Ev_i \circ B)
\]
(where the isomorphism is the one from Proposition 2.3.1). This composite is the canonical map, and \(X\) is \(\kappa\)-small relative to \(J\)-cell. By [Hov99, 2.1.16], \(X\) is then even \(\kappa\)-small relative to the class of cofibrations in \(C_i\). Hence we are done if for all \(\beta\) with \(\beta + 1 < \lambda\) the map \(Ev_i(b) : Ev_i(B_\beta) \to Ev_i(B_{\beta + 1})\) is a cofibration. However, since the maps in \(M\) are in particular pointwise cofibrations, and the class of pointwise cofibrations is closed under cobase changes and transfinite compositions, every map in \(M\)-cell is a pointwise cofibration. This finishes the proof.

\[\square\]

**Theorem 3.4.5.** Let \(\mathcal{B}\) be a cofibrantly generated bundle over \(\mathcal{I}\). The \(g\)-structure is a model structure on \(\mathbf{Tw}(\mathcal{I}, \mathcal{B})\) which is cofibrantly generated by the sets \(M\) and \(N\).

**Proof.** We use Theorem 2.1.19 of [Hov99], which applies also for model categories in the sense of [DS95]. The weak equivalences clearly define a subcategory which is closed under retracts and satisfies \(MC2\), so condition 1 holds. Lemma 3.4.4 implies conditions 2 and 3, and Lemma 3.4.3 implies conditions 5 and 6, and one half of condition 4. It remains to prove that every map in \(N\)-cell is a weak equivalence. Since every map in \(N\) is pointwise an acyclic cofibration, and the class of pointwise acyclic cofibrations is closed under pushouts and transfinite compositions, every map in \(N\)-cell is pointwise an acyclic cofibration, so in particular a weak equivalence. \(\square\)

**Remark 3.4.6.** From the general theory of cofibrantly generated model structures, we know that a morphism \(f\) of twisted diagrams is a \(g\)-cofibration if and only if it is a retract of a transfinite composition of cobase changes of maps in \(M\). Similarly, acyclic \(g\)-cofibrations can be characterised using the set \(N\).

### 4. Sheaves and Homotopy Sheaves

Let \(\mathcal{C}\) be a model category, and suppose the diagram category \(\mathbf{Fun}(\mathcal{I}, \mathcal{C})\) carries a model structure with pointwise weak equivalences as described in one of the previous sections. There is a canonical functor
\[
h : \mathbf{Fun}(\mathcal{I}, \mathcal{C}) \longrightarrow \mathbf{Fun}(\mathcal{I}, \mathcal{Ho}\mathcal{C})
\]
which replaces each structure map of a diagram by its homotopy class (or, more precisely, by its image under the localisation functor \(\mathcal{C} \longrightarrow \mathcal{Ho}\mathcal{C}\)). In particular, the structure maps of a diagram \(Y\) are weak equivalences if and only if the structure maps of \(h(Y)\) are isomorphisms.
The functor $h$ factors through a functor

$$h: \text{HoFun} (I, \ C) \rightarrow \text{Fun} (I, \ \text{HoC})$$

which is, in general, not an equivalence of categories.

In this section, we construct such functors $h$ and $\mathbb{h}$ for twisted diagrams. Unfortunately, this is not as straightforward as one could expect since formation of total derived functors is not functorial. One way to explain this is the following: if the functor $U$ preserves weak equivalences between fibrant objects, its total right derived $RU$ exists, and $RU(Y)$ is given by evaluating $U$ on a fibrant replacement of $Y$. Thus for $U = \text{id}_C$ we see that if $C$ contains objects which are not fibrant, the total right derived of the identity functor $\text{id}_C$ is isomorphic to, but different from, the functor $\text{id}_{\text{HoC}}$.

We remedy this by focusing on the full subcategory $C^f$ of fibrant objects in $C$. This is possible since by a theorem of Quillen, the localisation of $C^f$ with respect to weak equivalences is equivalent to the homotopy category of $C$.

4.1. Associated Homotopy Bundle. Let $C$ denote a model category, and denote by $C^f$ the full subcategory of fibrant objects. The homotopy category $\text{HoC}$ is the localisation of $C$ with respect to the class of weak equivalences. We will use the rather explicit model described in [DS95]: the objects of $\text{HoC}$ are the objects of $C$, and morphisms are homotopy classes of maps between cofibrant-fibrant replacements. Let $\text{Ho}_f C$ denote the full subcategory of $\text{HoC}$ generated by the fibrant objects.

Lemma 4.1.1. The following diagram commutes:

$$\begin{array}{ccc}
C^f & \rightarrow & C \\
\downarrow \gamma^f & & \downarrow \gamma \\
\text{Ho}_f C & \rightarrow & \text{HoC}
\end{array}$$

The vertical arrows are localisations with respect to the class of weak equivalences, the horizontal arrows are full embeddings. The lower horizontal arrow is an equivalence of categories.

Proof. This follows from Quillen’s theorem on existence of homotopy categories [Qui67, §1.1, Theorem 1]. We omit the details. \hfill \Box

Now choose, for each object $X \in C$, a cofibrant replacement

$$p_X: X^c \sim X.$$

If $X$ is cofibrant itself, we choose $p_X = \text{id}_X$. Similarly, we choose fibrant replacements

$$q_X: X \sim X^f.$$
with $q_X = \text{id}_X$ for fibrant $X$.—The following Proposition is a standard exercise in model category theory:

**Proposition 4.1.2.** Suppose $U: \mathcal{C} \rightarrow \mathcal{D}$ is right Quillen with left adjoint $F$.

1. The total right derived $RU: \text{Ho}\mathcal{C} \rightarrow \text{Ho}\mathcal{D}$ exists and is given by $RU(X) := U(X^f)$ on objects. Moreover, the functor $RU$ has a left adjoint $LF$.

2. The image of the functor $RU$ lies inside $\text{Ho}_f\mathcal{D}$, hence $RU$ induces (by restriction) a functor $R_fU: \text{Ho}_f\mathcal{C} \rightarrow \text{Ho}_f\mathcal{D}$.

3. Every map $\alpha: X \rightarrow Y$ in $\text{Ho}_f\mathcal{C}$ is represented by a diagram of the form $X \xymatrix{\sim \ar[r]^f & Y}$ in $\mathcal{C}$.

4. The functor $R_fU$ is given by the identity on objects and, using the description of (3), by $R_fU(\alpha) = \gamma_f(U(f)) \circ \gamma_f(U(p_X))^{-1}$ on morphisms.

5. We have $R_fU \circ \gamma_f^\mathcal{C} = \gamma_f^\mathcal{D} \circ U$. Moreover, the functor $R_fU$ is a left Kan extension of $U$ along $\gamma_f$.

6. The equalities $R_fU \circ \gamma_f^\mathcal{C} = \text{id}_{\text{Ho}_f\mathcal{C}}$ and $R_f(V \circ U) = R_fV \circ R_fU$ hold.

7. The functor $R_fU$ has a left adjoint, denoted $L_fF$, given (on objects) by the formula $L_fF(X) := F(X^c)^f$.

\[\square\]

In view of the previous lemma, parts (2) and (5) mean that $R_fU$ is a “good” substitute for $RU$. Moreover, by parts (6) and (7), the following definition makes sense:

**Definition 4.1.3 (Associated homotopy bundle).** If $\mathfrak{B} = (\mathcal{C}, F, U)$ is an $\mathcal{I}$-indexed adjunction bundle of model categories, we define its associated homotopy bundle of fibrant objects $\text{Ho}_f\mathfrak{B} = (\text{Ho}_f\mathcal{C}, L_fF, R_fU)$ as the $\mathcal{I}$-indexed adjunction bundle given by $i \mapsto \text{Ho}_f\mathcal{C}_i$ for objects $i \in \mathcal{I}$ and $\sigma \mapsto L_fF_\sigma$ and $\sigma \mapsto R_fU_\sigma$ for morphisms $\sigma \in \mathcal{I}$.

### 4.2. Construction of $h$ and $h'$

Suppose $\mathfrak{B} = (\mathcal{C}, F, U)$ is an $\mathcal{I}$-indexed adjunction bundle of model categories. We assume that we can equip $\text{Tw}(\mathcal{I}, \mathfrak{B})$ with a model structure with pointwise weak equivalences (this is certainly possible if $\mathcal{I}$ is locally direct or locally inverse, or if $\mathfrak{B}$ is a cofibrantly generated bundle). We want to associate to each twisted diagram $Y \in \text{Tw}(\mathcal{I}, \mathfrak{B})$ a corresponding twisted diagram $h(Y) \in \text{Tw}(\mathcal{I}, \text{Ho}_f\mathfrak{B})$.

Assume for the moment that $Y$ is a twisted diagram with fibrant components. Let $Z$ denote the following twisted diagram:

\[
i \mapsto \gamma_f(Y_i) = Y_i \quad \sigma \mapsto \gamma_f(y^\sigma): Y_i \xrightarrow{U_\sigma(Y_j)} R_fU_\sigma(\gamma_f(Y_j))
\]
We need to check the commutativity condition: if \( i \xrightarrow{\sigma} j \xrightarrow{\tau} k \) are composable morphisms in \( \mathcal{I} \), the following diagram is supposed to commute:

\[
\begin{array}{ccc}
\gamma_f(Y_i) & \xrightarrow{\gamma_f(y^\flat_{\sigma})} & R_f U_{\sigma}(\gamma_f(Y_j)) \\
\downarrow & & \downarrow \\
R_f U_{\tau \sigma}(\gamma_f(Y_k)) & \xrightarrow{\gamma_f(y^\flat_{\tau \sigma})} & R_f U_{\sigma}(\gamma_f(y^\flat_\tau)) \\
\end{array}
\]

Using \( \text{4.1.2 (5)} \) we see that this is just the corresponding diagram for \( Y \) after application of \( \gamma_f \), hence commutes as desired.

A morphism \( f: Y \rightarrow \tilde{Y} \) between pointwise fibrant twisted diagrams induces a map \( g: Z \rightarrow \tilde{Z} \) with components \( g_i = \gamma_f(f_i) \) as can be shown using \( \text{4.1.2 (5)} \) and functoriality of \( \gamma_f \).

Now we use this construction to define the actual functor \( h \) (or \( \hbar \)). We discuss three cases in order of increasing difficulty.

**Case 1:** All the model categories \( \mathcal{C}_i \) used in the bundle \( \mathcal{B} \) consist of fibrant objects only. Then \( \mathcal{C}_i^f = \mathcal{C}_i \) and \( R_f U_{\sigma} = R U_{\sigma} \). The assignment \( Y \mapsto \tilde{Y} \) defines (the object function of) a functor \( h: \text{Ho Tw}(\mathcal{I}, \mathcal{B}) \rightarrow \text{Tw}(\mathcal{I}, \text{Ho f } \mathcal{B}) \).

**Case 2:** Suppose that fibrant objects of \( \text{Tw}(\mathcal{I}, \mathcal{B}) \) are pointwise fibrant. Suppose moreover that \( \text{Tw}(\mathcal{I}, \mathcal{B}) \) has a fibrant replacement functor \( Y \mapsto \tilde{Y} \) by \( \text{4.1.7} \). Then we can apply the above construction to \( Y^f \) instead of \( Y \), and the composite \( Y \mapsto Y^f \mapsto Z \) defines (the object function of) a functor \( h \). By construction it maps weak equivalences to isomorphisms, hence descends to a functor \( \hbar: \text{Ho Tw}(\mathcal{I}, \mathcal{B}) \rightarrow \text{Tw}(\mathcal{I}, \text{Ho f } \mathcal{B}) \).

**Case 3:** Suppose that fibrant objects of \( \text{Tw}(\mathcal{I}, \mathcal{B}) \) are pointwise fibrant. Let \( \mathcal{K} \) denote the category with objects the fibrant and cofibrant twisted diagrams in \( \text{Tw}(\mathcal{I}, \mathcal{B}) \), and morphisms the homotopy classes of maps between such objects. By \( \text{DS95 5.6} \) the inclusion \( \nu: \mathcal{K} \rightarrow \text{Ho Tw}(\mathcal{I}, \mathcal{B}) \) is an equivalence of categories. Thus it suffices to construct a functor \( \phi: \mathcal{K} \rightarrow \text{Tw}(\mathcal{I}, \text{Ho f } \mathcal{B}) \); then we can define \( h \) by the composition of an inverse of \( \nu \) with \( \phi \).

An object \( Y \in \mathcal{K} \) is in particular a pointwise fibrant twisted diagram. Hence the construction preceding case 1 applies, and we can define \( \phi(Y) := Z \).

A morphism \( f: Y \rightarrow \tilde{Y} \) in \( \mathcal{K} \) can be represented by a map \( \tilde{f}: Y \rightarrow \tilde{Y} \) in \( \text{Tw}(\mathcal{I}, \mathcal{B}) \) by \( \text{DS95 5.7} \), and \( \tilde{f} \) induces \( \phi(f): \phi(Y) \rightarrow \phi(\tilde{Y}) \) with components \( \phi(f)_i = \gamma_f f_i \). To show that \( \phi(f) \) does not depend on the choice of \( \tilde{f} \), recall that homotopy is an equivalence relation for maps \( Y \rightarrow \tilde{Y} \) by \( \text{DS95 4.22} \). Moreover, the evaluation functors \( \text{Ev}_i \) (given by \( Y \mapsto Y_i \)) commute with products and preserve weak equivalences. Hence they preserve path
objects and right homotopies. Thus if \( f \) and \( g \) are homotopic, so are \( f_i \) and \( g_i \). Since the localisation functor identifies homotopic maps, this proves that \( \phi(f) \) is well defined.

Since homotopy is compatible with composition \([DS95, 4.11 \text{ and } 4.19]\), and since the identity morphisms in \( \mathfrak{R} \) are represented by identity maps, \( \phi \) is a functor as required.

### 4.3. Comparison of Sheaves and Homotopy Sheaves.

**Definition 4.3.1** (Left strict sheaves). Given an \( \mathcal{I} \)-indexed adjunction bundle \( \mathfrak{B} \), we call an object \( Y \in \text{Tw}(\mathcal{I}, \mathfrak{B}) \) a left strict sheaf if the \( \# \)-type structure map \( y^\#: F_\sigma(Y_i) \to Y_j \) is an isomorphism for all morphisms \( \sigma: i \to j \) of \( \mathcal{I} \). We write \( \text{Shv}(\mathcal{I}, \mathfrak{B}) \) for the full subcategory of \( \text{Tw}(\mathcal{I}, \mathfrak{B}) \) generated by left strict sheaves.

There is also a dual notion of a right strict sheaf requiring that all \( \flat \)-type structure maps are isomorphisms.

**Example 4.3.2** (Quasi-coherent sheaves on toric varieties). Recall the adjunction bundle \( \Sigma^{op} \bowtie_{\text{Rng Mod}} \) associated to a toric variety \( X \) with fan \( \Sigma \), cf. 2.5.4. We claim that the category \( \text{Shv}(\Sigma^{op}, \Sigma^{op} \bowtie_{\text{Rng Mod}}) \) is equivalent to the category of quasi-coherent sheaves on \( X \). To see this, recall that a cone \( \sigma \in \Sigma \) corresponds to an open affine sub-scheme \( U_\sigma \) of \( X \). Given a quasi-coherent sheaf \( F \), the associated twisted diagram is given by \( \sigma \mapsto F(U_\sigma) \) with \( \flat \)-type structure maps given by restriction maps. Conversely, a left strict sheaf \( Y \) defines quasi-coherent sheaves \( \tilde{Y}_\sigma \) on the sub-schemes \( U_\sigma \) which can be glued via the \( \# \)-type structure maps to give a quasi-coherent sheaf on \( X \). The details are left to the reader.

**Definition 4.3.3** (Left homotopy sheaves). Suppose that \( \mathfrak{B} \) is an adjacency bundle of model categories. We call an object \( Y \in \text{Tw}(\mathcal{I}, \mathfrak{B}) \) a left homotopy sheaf if for all morphisms \( \sigma: i \to j \) of \( \mathcal{I} \) there is an acyclic fibration \( \bar{Y}_i \sim \to Y_i \) in \( \mathcal{C}_i \) with \( \bar{Y}_i \) cofibrant such that the adjoint to the composite \( \bar{Y}_i \sim \to Y_i \xrightarrow{y^\flat} U_\sigma(Y_j) \) is a weak equivalence in \( \mathcal{C}_j \). We write \( \text{hShv}(\mathcal{I}, \mathfrak{B}) \) for the full subcategory of \( \text{Tw}(\mathcal{I}, \mathfrak{B}) \) generated by left homotopy sheaves.

**Theorem 4.3.4** (Comparison of strict sheaves and homotopy sheaves). Let \( \mathfrak{B} \) denote an \( \mathcal{I} \)-indexed adjunction bundles of model categories. Assume that we have a map \( h \) as given by one of the cases of \( [1,2] \). An object \( Y \in \text{Tw}(\mathcal{I}, \mathfrak{B}) \) is a left homotopy sheaf if and only if \( h(Y) \in \text{Tw}(\mathcal{I}, \text{Ho}_\mathfrak{B}) \) is a left strict sheaf. In particular, if \( Y \sim \to Z \) is a weak equivalence of twisted diagrams, \( Y \) is a left homotopy sheaf if and only if \( Z \) is.

**Proof.** Fix a morphism \( \sigma: i \to j \) of \( \mathcal{I} \), and define \( Z := h(Y) \). By construction, \( z^\sigma_\# \) is a morphism in \( \text{Ho}_\mathfrak{C}_i \) which is isomorphic, in \( \text{Ho}_\mathfrak{C}_i \), to a morphism \( k^\#: Y_i \to \mathbf{R}U_\sigma(Y_j) \). The isomorphism is given by the fibrant
replacement used in the construction of \( h \). If \( Y_i \) and \( Y_j \) happen to be fibrant, the maps \( z^\flat_\sigma \) and \( k^\flat \) agree.

There is a commutative diagram of categories and functors

\[
\begin{array}{ccc}
\text{Ho} C_i & \xleftarrow{R_f U_\sigma} & \text{Ho} C_j \\
\downarrow & & \downarrow \\
\text{Ho} C_i & \xrightarrow{U_\sigma} & \text{Ho} C_j
\end{array}
\]

where both vertical arrows are equivalences. Hence \( z^\sharp_\sigma \) is isomorphic, in the category \( \text{Ho} C_j \), to the adjoint \( k^\sharp : LF_\sigma(Y_i) \rightarrow Y_j \) of \( k^\flat \). In particular, the morphism \( z^\sharp_\sigma \) is an isomorphism if and only if \( k^\sharp \) is.

Choose a cofibrant replacement \( q_i : Y^c_i \sim Y_i \) of \( Y_i \) and a fibrant replacement \( p_j : Y_j \sim Y^f_j \) of \( Y_j \). Let \( \ell^b \) denote the composite map

\[
Y^c_i \xrightarrow{q_i} Y_i \xrightarrow{y^\flat_\sigma} U_\sigma(Y_j) \xrightarrow{U_\sigma(p_j)} U_\sigma(Y^f_j).
\]

By the proof of [DS95, 9.7] we know that \( k^\flat \) is isomorphic to \( \gamma_i(\ell^b) \) where \( \gamma_i : C_i \rightarrow \text{Ho} C_i \) denotes the localisation functor. Similarly, \( k^\sharp \) is isomorphic to \( \gamma_j(\ell^\sharp) \), where \( \gamma_j \) denotes the localisation functor for \( C_j \), and \( \ell^\sharp \) is adjoint to \( \ell^\flat \). In particular, \( k^\sharp \) is an isomorphism if and only if \( \ell^\sharp \) is a weak equivalence. But \( \ell^\sharp \) factors as \( F_\sigma(Y^c_i) \rightarrow Y_j \xrightarrow{p_j} Y^f_j \) which shows that \( \ell^\sharp \) is a weak equivalence if and only if the homotopy sheaf condition (“at \( \sigma \)”)
holds for \( Y \).

The second assertion follows immediately since \( h \) maps weak equivalences to isomorphisms and the property of being a left strict sheaf is clearly invariant under isomorphism. \( \square \)

The theorem applies, for example, to the category of non-linear sheaves on projective \( n \)-space. Recall the adjunction bundle \( \mathfrak{P}^n \) from [2.5.6]. This is an adjunction bundle of model categories. The resulting category of homotopy sheaves \( \text{hShv}(\langle n \rangle, \mathfrak{P}^n) \) is the category \( \mathfrak{P}^n \) of sheaves as defined in [Hut02, 6.3]. The index category \( \langle n \rangle \) is direct with degree function \( d(A) := \#A \). Hence the c-structure exists. Moreover, all objects of \( \text{Tw}(\langle n \rangle, \mathfrak{P}^n) \) are c-fibrant. Thus we can use the construction of Case 1 in [4.2] and Theorem 4.3.4 applies.—More generally, we can consider the category of non-linear sheaves over any projective toric variety [Hut04], and Theorem 4.3.4 identifies non-linear sheaves (or rather, homotopy classes of such) with “sheaves in the homotopy category”.

Theorem
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