QCD Sum-Rule Interpretation of X(3872) with $J^{PC} = 1^{++}$ Mixtures of Hybrid Charmonium and $\bar{D}D^*$ Molecular Currents

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Abstract

QCD sum-rules are employed to determine whether the X(3872) can be described as a mixed state that couples to $J^{PC} = 1^{++}$ charmonium hybrid and $\bar{D}D^*$ molecular currents. After calculating the mixed correlator of hybrid and molecular currents, we formulate the sum-rule in terms of a mixing parameter that interpolates between the pure molecular and hybrid scenarios. As the mixing parameter is increased from the pure molecular case, the predicted mass increases until it reaches a maximum value in good agreement with the X(3872) and the resulting sum-rule analysis appears more robust than the pure molecular case.

1 Introduction

The X(3872) state was first detected in August 2003 by Belle$^1$ in $B^\pm \rightarrow \pi^+\pi^-J/\psi K^\pm$, and its existence was quickly confirmed by the CDF$^2$, D0$^3$ and BaBar$^1$ collaborations. The observation of the X(3872) and other new states with masses in the range 3.8–4.7 GeV has led to a resurgence of interest in exotic meson spectroscopy$^5$. Recently the LHCb collaboration has measured the X(3872) mass as $m_{X(3872)} = 3871.95 \pm 0.48 \text{(stat)} \pm 0.12 \text{(syst)} \text{MeV}$.$^6$. The PDG summary table has listed the X(3872) with an average mass $m_{X(3872)} = 3871.68 \pm 0.17 \text{MeV}$ and a narrow width $\Gamma_{X(3872)} \leq 1.2 \text{MeV}$.$^7$. The radiative decay $X(3872) \rightarrow \gamma J/\psi \pi^0 \pi^0$ implies the C-parity of the X(3872) is even. The LHCb collaboration has recently provided evidence to preclude $J^{PC} = 2^{-+}$ from the 1$^{++}$ and 2$^{++}$ assignments$^1$ (Ref. $13$ is another recent work supporting $J^P = 1^{--}$).

With strong evidence supporting the 1$^{++}$ quantum numbers, the X(3872) does not appear to be a pure conventional charmonium state because the mass of 1$^{++}$ states in the quark model calculation is too large. The 1$^{++}$ assignment of the X(3872), due to its exotic interpretation, has attracted a lot of attention. To date, many structures for the X(3872) have been proposed, such as a tetraquark state$^{[14, 15, 16, 17, 18]}$, a charmonium molecule$^{[19]}$, and a $D^0 \bar{D}$ molecule$^{[20, 21]}$. More recently, a mixing scenario has been proposed$^{[22, 23, 24, 25, 26, 27, 28, 29, 30]}$ (see Refs.$^{[5, 31]}$ for comprehensive reviews).

The $DD^*$ molecule picture was developed before the experimental observation, so it attracted considerable attention, but the molecular scenario is hard to reconcile with the large radiative $J/\psi \pi \pi$ decay rate because the loosely bound $DD^*$ is unlikely to annihilate into $J/\psi \pi \pi$.

The tetraquark or hybrid scenarios may explain the decay rate puzzle. But the expected mass of the pure axial-vector charmonium hybrid is about 5 GeV$^{[32]}$, well above the X(3872). Furthermore, the diquark-antidiquark (tetraquark) scenario should have partners of X(3872), with the charged partner possibly identified as the $Z_c(3895)$ seen by BES and Belle$^{[33, 34]}$ (see also Ref. $35$). More recently, a mixing scenario has been proposed$^{[36, 37, 38, 39, 40]}$, where it is suggested that the $c \bar{c}$ 1$^{++}$ $\chi_{c1}(2P)$ strongly couples to $DD^*$ so that its mass is dynamically shifted about 100 MeV downwards from the bare $\chi_{c1}(2P)$ state. Thus this scenario may both solve both the mass and decay width puzzles.

QCD sum-rules have been used to study a number of 1$^{++}$ scenarios for the X(3872)$^{[16, 41, 42, 43, 44]}$. It is difficult to assess whether the tetraquark or molecular currents provide an adequate description of X(3872). For example, Refs.$^{[41, 42]}$ find a tetraquark mass of about 4.2 GeV and a molecular mass of about 4.1 GeV, while Refs.$^{[16, 43]}$ find nearly degenerate molecular and tetraquark masses of about 3.9 GeV. However, in these analyses it is not possible to find a sum-rule window of validity unless a 50% continuum contribution is permitted. Mass predictions for 1$^{++}$ states seem to be insensitive to the choice of tetraquark or molecular currents$^{[45]}$, but lead to different consequences for branching ratios$^{[40]}$. However, the tetraquark and molecular currents are related through
Fierz transformations, leading to ambiguities in interpreting the underlying quark configuration from the structure of the currents \[47\].

QCD sum-rules have also been used to study mixed interpretations of the X(3872). If mixed molecular and charmonium currents are used, the mass prediction decreases to about 3.8 GeV \[44\]. From the QCD sum-rules perspective, the four-quark state (or $\overline{DD}^*$ molecular state) may couple more strongly to the charmonium hybrid than the $c\bar{c}$ charmonium, because the latter is $\alpha_s$ suppressed in perturbation theory (i.e., the leading-order correction is chirally suppressed). The purpose of this paper is to explore whether the X(3872) can be described through a mixture of hybrid and molecular currents. Although naive considerations seem to preclude such a mixing scenario because the expected masses of both the four-quark state and hybrid from QCD sum-rules are widely separated, Fierz transformations, leading to ambiguities in interpreting the underlying quark configuration from the structure \[47\].

Our paper is organized as follows: Section 2 contains the field-theoretical calculation of the mixed correlator of molecular and hybrid charmonium currents, Section 3 presents the Laplace QCD sum-rule analysis, and concluding remarks are in Section 4.

## 2 Correlation Functions

The fundamental two-point correlation functions of interest in the QCD sum-rule mixing analysis are given by

\[
\Pi_{\mu \nu}^{ij} (q) = i \int d^4 x \, e^{i q \cdot x} \langle 0 | J_{\mu}^i (x) J_{\nu}^j (0) | 0 \rangle,
\]

where the indices $\{i,j\} \in \{h, m\}$ denote the either the hybrid or molecular currents. The hybrid diagonal correlation function $(i = j = h)$ was studied in Ref. \[32\] using the current

\[
J_{\mu}^h = \frac{1}{2} g \bar{c} \gamma^\mu \lambda^a \tilde{G}_{\mu \nu}^a c, \quad \tilde{G}_{\mu \nu}^a = \frac{1}{2} \epsilon_{\mu \alpha \beta \nu} \tilde{G}_{\alpha \beta}^a,
\]

where $c$ denotes a charm quark field. The molecular diagonal correlation function $(i = j = m)$ was investigated in Ref. \[42\], using the current

\[
J_{\mu}^m = \frac{1}{\sqrt{2}} \left( \bar{q} a_5 c a \bar{b} \gamma^\nu q b - \bar{q} a_5 \gamma^\nu c a \bar{b} \gamma^5 q b \right),
\]

where $q$ denotes a light quark field and $a, b$ are spinor indices. As mentioned earlier, the mass predictions emerging from tetraquark and molecular currents are very similar \[44\]; we have focussed on the molecular currents because the analysis can be contrasted with the scenario of mixed molecular and charmonium currents \[44\]. In this work we consider mixed molecular/hybrid currents

\[
J_{\mu}^\xi = \sqrt{1 - \xi^2} J_{\mu}^m + \xi \sigma J_{\mu}^h.
\]

where $\xi$ is a dimensionless parameter that will be varied in the Section 3 analysis and $\sigma$ is a mass scale that accounts for the different mass dimensions of the hybrid and molecular currents. With no loss of generality, we set $\sigma = 1 \text{GeV}$ comparable to the $\overline{MS}$ charm mass; changes in $\sigma$ are compensated by a change in $\xi$. The correlation function associated with this current

\[
\Pi_{\mu \nu}^\xi (q) = i \int d^4 x \, e^{i q \cdot x} \langle 0 | J_{\mu}^\xi (x) J_{\nu}^\xi (0) | 0 \rangle
\]

is simply a linear combination of the Eq. 11 correlators. Thus the remaining quantity needed for the mixing analysis is the non-diagonal correlation function given by

\[
\Pi_{\mu \nu}^{hm} (q) = i \int d^4 x \, e^{i q \cdot x} \langle 0 | J_{\mu}^h (x) J_{\nu}^m (0) | 0 \rangle.
\]

Only the transverse part of the correlation function \[11\] couples to hadronic states with $J^{PC} = 1^{++}$ that are mixtures of hybrid and molecular states, and hence we calculate

\[
\Pi_{hm} (q) = \frac{1}{d - 1} \Pi_{hm}^{\mu \nu} (q) \left( g_{\mu \nu} - \frac{q_\mu q_\nu}{q^2} \right),
\]

2
where $d$ denotes the number of spacetime dimensions.

We begin by calculating the leading-order perturbative contribution to the non-diagonal correlation function \(\Pi\), which is depicted in Fig. 1. As in Ref. \[52\], we have chosen to calculate the entire correlation function, rather than just the imaginary part. This approach provides greater clarity in the renormalization issues detailed below, and as argued in Ref. \[52\], justifies any limiting procedures that may be needed in forming the Laplace sum-rule. We first note that Fig. 1 would appear to require the calculation of non-trivial massive three-loop momentum integrals. However, the light quark loop in Fig. 1 can be integrated immediately and the remaining two-loop integrals are tabulated in Ref. \[49\], rendering the calculation tractable. Integrals with irreducible tensor structures can be calculated using the method described in Refs. \[50, 51\] whereby tensor integrals in $d$ dimensions are related to scalar integrals in $d + N$ dimensions. In order to use this approach, the integrals given in Ref. \[49\] must be expressed in an arbitrary number of dimensions. The relevant result is

$$
\frac{1}{\mu^{2(d-4)}} \int \frac{dk_1}{(2\pi)^d} \frac{dk_2}{(2\pi)^d} \frac{1}{(k^2 - m^2 + i0^+)^\alpha (k^2 - q^2 - m^2 + i0^+)^\beta (k_1 - k_2)^2 + i0^+},
$$

which is depicted in Fig. 1.

where $\mu$ is the renormalization scale, $d$ is the number of spacetime dimensions, $m$ is the charm quark mass, $q^2$ is the external momentum and \(\alpha, \beta, \gamma\) denote the number of spacetime dimensions.

$$
\Pi_{\text{pert}}(z) = -\frac{m^7 \alpha}{24 \sqrt{2\pi}^5} \left[ 3L_3 \left( 1 - 2z - 2i\sqrt{z(1-z)} \right) + 6iL_2 \left( 1 - 2z - 2i\sqrt{z(1-z)} \right) \right] \sin^{-1}(\sqrt{z})
$$

$$
+ \frac{1}{192z^3} \left( 896z^4 - 1280z^3 + 96z^2 (12 \log 2 - 1) + 80z + 7 \right) \sin^{-1}(\sqrt{z})^2
$$

$$
- \frac{1}{1440z} \left( 2340z^5 + 7568z^4 - 2656z^3 + 15426z^2 + 116z + 105 \right) \frac{\sin^{-1}(\sqrt{z})}{\sqrt{z(1-z)}}
$$

$$
+ 2i \left[ \sin^{-1}(\sqrt{z})^3 + 6 \log \left( z + i\sqrt{z(1-z)} \right) \right] \sin^{-1}(\sqrt{z})^2 + \frac{7}{192z}, \quad z = \frac{q^2}{4m^2},
$$

where $L_3$ and $L_2$ denote the trilogarithm and dilogarithm functions, respectively \[54\]. Terms corresponding to dispersion relation subtraction constants have been omitted, and the coupling $\alpha$ and charm quark mass $m$ are

\[\text{A related perturbative diagram where the gluon connects to the heavy quark line is trivially zero because of the massless (light-quark) tadpole.}\]

\[\text{Ref. \[55\] also gives a result for the integral \[9\] which is consistent with our conventions.}\]
implicitly functions of the renormalization scale \( \mu \). Although (10) superficially appears to be singular at \( z = 0 \), it is in fact well-defined at this point. The imaginary part can be determined through analytic continuation of the functions in (10), yielding

\[
\text{Im}\Pi_{\text{hm}}^{\text{pert}}(z) = -\frac{m^7 \alpha}{48 \sqrt{2} \pi^4} \left[ 6 \left( \text{Li}_2 \left[ -\left( \sqrt{z} + \sqrt{z-1} \right)^2 \right] + \log^2 \left[ \sqrt{z} + \sqrt{z-1} \right] + \log \left[ \sqrt{z} + \sqrt{z-1} \right] \log |z| \right) \\
+ \frac{1}{96z^2} \left( 896z^4 - 1280z^3 + 96z^2 (12 \log |z| - 1) + 80z + 7 \right) \log \left[ \sqrt{z} + \sqrt{z-1} \right] \\
- \frac{1}{1440z} \left( 2340z^5 + 7568z^4 - 26568z^3 + 15426z^2 + 1165z + 105 \right) \frac{1}{\sqrt{z(z-1)}} \\
+ \frac{\pi^2}{2} \right] , \quad z > 1.
\]

Figure 2: Feynman diagrams for the leading order contributions of the (light) quark condensate to the non-diagonal correlation function. The diagram on the left corresponds to \( \Pi_{\text{hm}}^{\text{q1}} \) and the diagram on the right corresponds to \( \Pi_{\text{hm}}^{\text{q2}} \). Diagrams related by symmetry are not shown. All notations are identical to those in Fig. 1.

We first consider the quark condensate, for which there are two distinct contributions as depicted in Fig. 2. For convenience in what follows we define

\[
w = \sqrt{\frac{z}{1-z}},
\]

where \( z \) is as defined in (10). For the left diagram in Fig. 2 we find

\[
\Pi_{\text{hm}}^{\text{q1}}(z) = -\frac{m^4 \alpha \langle \bar{q}q \rangle}{1152 \sqrt{2} \pi^3 z^2} \left[ 4i \left( 16z^4 - 104z^3 + 46z^2 + 51z - 9 \right) w \log \left[ \frac{i-w}{i+w} \right] \\
+ 3 \left( 3 - 16z + 48z^2 \right) \log^2 \left[ \frac{i-w}{i+w} \right] \right].
\]

The imaginary part of (13) is

\[
\text{Im}\Pi_{\text{hm}}^{\text{q1}}(z) = \frac{m^4 \alpha \langle \bar{q}q \rangle}{96 \sqrt{2} \pi^3} \left[ \frac{1}{z^2} \left( 48z^2 - 16z + 3 \right) \log \left[ \sqrt{z} + \sqrt{z-1} \right] \\
+ \frac{1}{3z} \left( 16z^4 - 104z^3 + 46z^2 + 51z - 9 \right) \frac{1}{\sqrt{z(z-1)}} \right] , \quad z > 1.
\]
For the right diagram in Fig. 2 we find

\[
\Pi_{\text{hm,bare}}^{qq^2}(z) = \frac{m^4 \alpha \langle \bar{q} q \rangle}{162 \sqrt{2} \pi^3} \left[ \frac{30i(z-1)}{w} \log \left[ \frac{i-w}{i+w} \right] + \frac{30i(z-1)}{w} \left( L_{22} \left[ \frac{1-iw}{2} \right] - L_{22} \left[ \frac{1+iw}{2} \right] \right) + \frac{i}{2} \log^2 \left[ 1-iw \right] - \frac{i}{2} \log^2 \left[ 1+iw \right] \right] 
\]

\[
\ldots \ldots + \frac{i}{4zw} \left( 96z^3 + 140z^2 - 16z - 15 - 240z(z-1) \log \left[ \frac{m^2}{\mu^2} \right] \right) \log \left[ \frac{i-w}{i+w} \right] 
\]

\[
- \frac{3}{16z^2} \left( 80z^3 - 24z^2 - 6z + 5 \right) \log^2 \left[ \frac{i-w}{i+w} \right]. 
\]

Notice that the first term in (15) contains a non-local divergence. This term is problematic since it cannot be removed through dispersion relation subtraction constants or application of the Borel transform when the sum rules are formulated, nor can it be removed through a multiplicative renormalization. Similar to the mixed scalar gluonic and quark currents \([60]\), the origin of this divergence is the renormalization-induced mixing of the composite operator \(J^{hc}\), Ref. \([61]\). The renormalized \(1^{++}\) hybrid is non-exotic, it can and will mix with conventional charmonium states with the same quantum numbers. This is in contrast to hybrids with exotic \(J^{PC}\), which do not mix with conventional charmonium states (see e.g., Ref. \([61]\)). The renormalized \(1^{++}\) hybrid current can be expressed as

\[
\left[ J_{\mu}^h \right]_R = Z_1 \left[ J_{\mu}^h \right]_B + Z_2 m^2 \left[ O_\mu \right]_B + \ldots, \quad Z_1 = 1 + \frac{\alpha Z_{h1}}{\pi \epsilon}, \quad Z_2 = \frac{\alpha Z_{h2}}{\pi \epsilon}, 
\]

(16)

where \(m\) denotes the charm quark mass, the subscripts \(R\) and \(B\) represent renormalized and bare quantities, respectively, and the ellipses in (16) are to indicate that additional lower dimensional operators may be present that would only contribute the the mixed-correlator \([60]\) at higher-loop level. The first term in (16) corresponds to the multiplicative renormalization of the hybrid current, which could be relevant for higher-order studies of hybrid-molecular state mixing, but is irrelevant to us at present because it represents a higher-loop effect. The composite operator in (16) is given by

\[
O_\mu = c \Gamma_\mu c, \quad \Gamma_\mu = \epsilon_{\mu\nu\alpha\beta} \left( \gamma^\nu \sigma^{\alpha\beta} + \gamma^\alpha \sigma^{\beta\nu} - \gamma^\beta \sigma^{\alpha\nu} \right), 
\]

(17)

where \(\sigma^{\alpha\beta} = \frac{i}{2} \left[ \gamma^{\alpha}, \gamma^{\beta} \right]\) and \(c\) denotes a charm quark field. Therefore, there is a renormalization induced contribution to the quark condensate, arising from the composite operator \(O_\mu\) in (16). This contribution is denoted as \(\Pi_{\text{hm,opmix}}^{qq^2}\) and is represented in Fig. 3:

\[
\text{Figure 3: Feynman diagram representing the contribution to the non-diagonal correlation function due to the renormalization of the hybrid current. The black box represents an insertion of the lower dimensional operator } O_\mu \text{ that mixes with the hybrid current. All other notations are identical to those in Fig. 1.}
\]

To our knowledge, the renormalization properties of the \(1^{++}\) hybrid current have never been studied. As such a full study is beyond the scope of the present work, we simply tune the renormalization factor \(Z_{h2}\) such that the non-local divergence in (15) is cancelled. That is, we require that the sum

\[
\Pi_{\text{hm,renorm}}^{qq^2} = \Pi_{\text{hm,bare}}^{qq^2} + \Pi_{\text{hm,opmix}}^{qq^2} 
\]

(18)

\[\text{The renormalization-induced perturbative diagram is trivially zero because of the massless (light-quark) tadpole.}\]
has no non-local divergences. Doing so, we find
\[ Z_{h2} = -\frac{10}{243} \frac{\alpha \gamma_1}{\pi \epsilon}, \] (19)
and the renormalized induced contribution to the quark condensate is
\[ \Pi_{\text{qm,opmix}}^{q2}(z) = -\frac{m^4 \alpha}{162 \sqrt{2\pi^3}} \left[ \frac{30i(z-1)}{w} \log \left( \frac{i-w}{i+w} \right) + \frac{30i(z-1)}{w} \left\{ \text{Li}_2 \left[ \frac{1-iw}{2} \right] - \text{Li}_2 \left[ \frac{1+iw}{2} \right] \right\} - \frac{1}{2} \log^2 \left[ 1+iw \right] + \frac{1}{2} \log \left[ \frac{1-iw}{4} \right] \log \left[ \frac{1-iw}{1+iw} \right] + \frac{1}{2} \log \left[ 1-iw \right] \log \left[ 1+iw \right] + \frac{i(z-1)}{3w} \left( 5 + 3 \log \left( \frac{m^2}{\mu^2} \right) \right) \log \left[ \frac{i-w}{i+w} \right] \right]. \] (20)
Comparing (15) and (20), it is clear that the non-local divergence will be eliminated. The renormalized quark condensate contribution is then
\[ \Pi_{\text{hm,renorm}}^{q2}(z) = -\frac{m^4 \alpha}{162 \sqrt{2\pi^3}} \left[ \frac{30i(z-1)}{w} \log \left( \frac{i-w}{i+w} \right) + \frac{30i(z-1)}{w} \left\{ \text{Li}_2 \left[ \frac{1-iw}{2} \right] - \text{Li}_2 \left[ \frac{1+iw}{2} \right] \right\} - \frac{1}{2} \log^2 \left[ 1+iw \right] + \frac{1}{2} \log \left( \frac{1-iw}{1+iw} \right) \right] \log \left[ \frac{i-w}{i+w} \right] \left( \frac{z}{96} \frac{80z^3 - 24z^2 - 6z + 5}{z^2 - z - 1} \log \left( \frac{\sqrt{z} + \sqrt{z-1}}{\sqrt{z}} \right) + \frac{3}{16z^2} (80z^3 - 24z^2 - 6z + 5) \log^2 \left[ \frac{i-w}{i+w} \right] \right). \] (21)
The imaginary part of (21) can now be easily extracted, yielding
\[ \text{Im} \Pi_{\text{hm,renorm}}^{q2}(z) = -\frac{m^2 \alpha}{216 \sqrt{2\pi^2}} \left( \frac{1}{z^2} (80z^3 - 24z^2 - 6z + 5) \log \left[ \sqrt{z} + \sqrt{z-1} \right] - \frac{1}{3z} \left( 96z^3 + 340z^2 - 184z - 15 - 120z(z-1) \log \left( \frac{m^2}{\mu^2} \right) \right) \sqrt{z - 1} \right), \quad z > 1. \] (22)

Figure 4: Feynman diagram for the leading order contribution of the gluon condensate to the non-diagonal correlation function. All notations are identical to those in Fig. [1].

Note that the gluon condensate, as would be obtained from Fig. [4] is chirally suppressed by the light quark loop and hence has a negligible effect on the analysis. This same chiral suppression of the gluon condensate occurs in the mixed correlator of scalar (light) quark and gluonic currents [60].

Finally we consider contributions from the mixed condensate, which is depicted in Fig. [5]. For this contribution we find
\[ \Pi_{\text{hm}}^{Gq}(z) = \frac{im^2 \langle \bar{q} \sigma G q \rangle}{72 \sqrt{2\pi^2}} \frac{w}{z} \left( 2z^2 - z - 1 \right) \log \left[ \frac{i-w}{i+w} \right] \right), \] (23)
where we define \( \langle \bar{q} \sigma G q \rangle = \langle g \bar{q} \frac{\gamma_5}{2} \sigma_{\mu \nu} G_{\mu \nu} q \rangle \). The corresponding imaginary part is
\[ \text{Im} \Pi_{\text{hm}}^{Gq}(z) = -\frac{m^2 \langle \bar{q} \sigma G q \rangle}{72 \sqrt{2\pi}} \left( 1 + 2z \right) \frac{\sqrt{z - 1}}{\sqrt{z}}, \quad z > 1. \] (24)

\textsuperscript{4}A related gluon condensate diagram where the gluon connects to the heavy quark line is trivially zero because of the massless (light-quark) tadpole.
3 QCD Laplace Sum-Rule Analysis

Utilizing the results given above for the non-diagonal hybrid-molecular correlation function $\Pi_{hm}$, along with the results from Refs. [32, 43] for the diagonal hybrid correlation function $\Pi_{hh}$ and molecular correlation function $\Pi_{mm}$, we now perform the QCD Laplace sum-rules analysis of mixing between $J^{PC} = 1^{++}$ molecular and hybrid charmonium. We do not review the QCD Laplace sum-rules methodology here, but the reader is directed to the original papers [62, 63] and, for example, reviews given in Refs. [64, 65]. Invoking the standard resonance plus continuum model for the hadronic spectral function, the Laplace sum-rules take the form

$$R_k(\tau, s_0) = \frac{1}{\pi} \int_{t_0}^{\infty} t^k \exp[-t\tau] \rho_{\text{had}}(t) \, dt,$$

(25)

where $t_0$ is the hadronic threshold. The left hand side of (25) is given by

$$R_k(\tau, s_0) = \frac{1}{\tau} \hat{B} \left[ (-1)^k Q^2 \Pi (Q^2) \right] - \frac{1}{\pi} \int_{s_0}^{\infty} t^k \exp[-t\tau] \text{Im}\Pi(t) \, dt,$$

(26)

where $s_0$ is the continuum threshold for the hadronic spectral function $\rho_{\text{had}}(t)$, $Q^2 = -q^2$ is the Euclidean momentum, and $\hat{B}$ is the Borel transform operator. For our purposes $\Pi(Q^2)$ in (26) corresponds to either the non-diagonal correlator $\Pi_{hm}$, or the diagonal correlators $\Pi_{hh}$ and $\Pi_{mm}$.

We now construct the non-diagonal hybrid-molecular sum-rules. Using the results obtained above for the perturbative (11), quark condensate (14), (22), and mixed condensate (24) contributions to the non-diagonal correlation function, the QCD Laplace sum-rules take the form

$$R_{0}^{hm}(\tau, s_0) = \frac{4m^2}{\pi} \int_{1/s_0/4m^2}^{1} \left[ \text{Im}\Pi_{\text{pert}}^{hm}(4m^2x) + \text{Im}\Pi_{\text{pert}}^{q1}(4m^2x) + \text{Im}\Pi_{\text{pert}}^{q2}(4m^2x) + \text{Im}\Pi_{\text{pert}}^{g}(4m^2x) \right] \exp(-4m^2\tau x) \, dx,$$

(27)

$$R_{1}^{hm}(\tau, s_0) = -\frac{\partial}{\partial \tau} R_{0}^{hm}(\tau, s_0).$$

(28)

The mass and coupling in (27), (28) are implicitly functions of the renormalization scale $\mu$ in the $\overline{\text{MS}}$-scheme, and renormalization group improvement may be implemented by setting $\mu = 1/\sqrt{\tau}$ after evaluating the derivative with respect to $\tau$ [66] or by setting $\mu$ to the charm quark mass scale. The sum-rules for the diagonal hybrid and molecular correlation functions are given in Refs. [32, 43], respectively. As these are both somewhat lengthy expressions, we do not repeat them here. In terms of the correlation function (4) of the mixed current (3), the Laplace sum-rule is then a linear combination of the diagonal and non-diagonal expressions

$$R_{k}^{\xi}(\tau, s_0) = \xi^2 \sigma^2 R_{k}^{\text{hh}}(\tau, s_0) + (1 - \xi^2) R_{k}^{\text{mm}}(\tau, s_0) + 2\xi \sqrt{1 - \xi^2} \sigma R_{k}^{\text{hm}}(\tau, s_0).$$

(29)

If the parameter $\xi$ is chosen appropriately, then a single narrow resonance model can be used:

$$\frac{1}{\pi}\rho_{\text{had}}(t) = f^2 \delta(t - M_X^2).$$

(30)

Eq. (25) then yields

$$R_{k}^{\xi}(\tau, s_0) = f^2 M_X^{2k} \exp(-M_X^2 \tau),$$

(31)
The QCD input parameters within the sum-rules are chosen to maintain consistency with the diagonal charmonium hybrid analysis [32] and the molecular analysis [43] (see [7, 67, 68, 69] for the original sources of the parameter values):

\begin{align}
\langle q\sigma Gq \rangle &= M_0^2 \langle \bar{q}q \rangle, \quad M_0^2 = (0.8 \pm 0.1) \text{ GeV}^2, \\
\langle \bar{q}q \rangle &= -(0.23 \pm 0.03) \text{ GeV}^3, \\
\langle \alpha G^2 \rangle &= (7.5 \pm 2.0) \times 10^{-2} \text{ GeV}^4, \\
\langle g^3 G^3 \rangle &= (8.2 \pm 1.0) \text{ GeV}^2 \langle \alpha G^2 \rangle.
\end{align}

Our analysis methodology is to optimize the parameter \( \xi \) to find the best agreement between \( M_X \) and the \( X(3872) \). For a given value of \( \xi \) we find a sum-rule window of validity where the continuum contribution is less than 30\% (providing an upper bound on the Borel scale \( M^2 = 1/\tau \)) and where the condensate contributions are less than 10\% (providing a lower bound on the Borel scale \( M \)). In this sum-rule window of validity, we require that \( M_X \) has a critical point (typically a minimum) as a function of the Borel scale. Within the range of continuum 4.1 GeV \( \lesssim \sqrt{s_0} \lesssim 5.5 \text{ GeV} \) previously found for the pure charmonium hybrid [32] and molecular [43] sum-rules, \( s_0 \) is optimized to minimize the dependence of \( M_X \) on the Borel scale.

As \( \xi \) is increased from zero (i.e. the pure molecular case), \( M_X \) increases until it reaches a maximum value near \( \xi = 0.002 \), and then decreases until the sum-rule becomes unstable. For this value \( \xi = 0.002 \), the optimization of \( s_0 \) shown in Fig. 6 results in \( \sqrt{s_0} \approx 4.3 \text{ GeV} \), and Fig. 7 shows the prediction of \( M_X \) corresponding to the largest mass prediction. The critical point in Fig. 7 provides the mass prediction \( M_X = 3.88 \text{ GeV} \) in excellent correspondence with the \( X(3872) \). Although the value of \( \xi \) appears to represent a small mixing between the hybrid and molecular currents, it in fact represents a significant mixing effect because the ratio of the perturbative contributions to the sum-rules is quite large: \( 10^3 \lesssim \sigma^2 \mathcal{R}_{S_0}^{\bar{h} h}/\mathcal{R}_{S_0}^{m m} \lesssim 10^5 \). As mentioned earlier, the choice of \( \sigma \) affects \( \xi \); if we rescale the hybrid current by setting \( 1/\sigma \sim \sqrt{\mathcal{R}_{S_0}^{h h}/\mathcal{R}_{S_0}^{m m}} \), then \( 0.06 \lesssim \xi \lesssim 0.6 \). The mixing of hybrid and molecular currents seems to have an important stabilizing effect on the sum-rule analysis. For the pure molecular case, it is difficult to control the continuum contributions and it is necessary to accept a sum-rule analysis where the continuum reaches 50\% [10, 11, 42, 43]. However, in the mixed case we are able to find a working region where the continuum contribution is less than 30\%.

We have chosen to use the \( \overline{\text{MS}} \) charm mass to align with the pure molecular and hybrid analyses [32, 43]. However, since these analyses and our mixing correlator calculation are leading order, there is no field-theoretical distinction between the pole and \( \overline{\text{MS}} \) scheme masses (see e.g., Ref. [70] for a next-to-leading order sum-rule analysis that employs both the pole and \( \overline{\text{MS}} \) schemes). Nevertheless, we explore the effect of using the pole-scheme charm mass \( M_c = 1.6 \text{ GeV} \) as a source of theoretical uncertainty. In this case the mass in the pure molecular limit (\( \xi = 0 \)) is significantly above the \( X(3872) \), and decreases with increasing \( \xi \) until reaching the optimized mixing parameter \( \xi = 0.007 \) (see Figs. 8 and 9). There is minimal effect on the sum-rule window and the continuum. Thus in the pole scheme, mixing effects seem to be necessary for consistency with the \( X(3872) \).
Figure 6: The sum-rule prediction of $M_X$ as a function of the continuum $s_0$ for $\xi = 0.002$ for Borel scales $M$ within the sum-rule region of validity: $M^2 = \{1.6, 1.75, 1.9, 2.05, 2.2\}$ GeV$^2$. The minimum value of $s_0$ corresponds to the pure molecular analysis [43] and the maximum value corresponds to the pure hybrid case [32]. The MS scheme has been used for the charm quark mass.

Figure 7: The sum-rule prediction of $M_X$ as a function of the Borel scale $M$ for $\xi = 0.002$ and the optimized continuum $s_0 = 19$ GeV$^2$ (red curve) and a slightly smaller value $s_0 = 18$ GeV$^2$ (blue curve). The range of $M$ corresponds to the sum-rule region of validity $1.6$ GeV$^2 < M^2 < 2.2$ GeV$^2$ for the central $s_0$ values. The MS scheme has been used for the charm quark mass.
Figure 8: The sum-rule prediction of $M_X$ as a function of the continuum $s_0$ for $\xi = 0.007$ for Borel scales $M$ within the sum-rule region of validity: $M^2 = \{1.65, 1.75, 1.9, 2.05, 2.2\}$ GeV$^2$. The pole scheme has been used for the charm quark mass.

Figure 9: The sum-rule prediction of $M_X$ as a function of the Borel scale $M$ for $\xi = 0.007$ and the optimized continuum $s_0 = 21$ GeV$^2$ (blue curve) and a slightly smaller value $s_0 = 19$ GeV$^2$ (red curve). The range of $M$ corresponds to the sum-rule region of validity $1.6$ GeV$^2 < M^2 < 2.1$ GeV$^2$ for the central $s_0$ values. The pole scheme has been used for the charm quark mass.
4 Conclusions

In summary, we have calculated the mixed correlation function of \(1^{++}\) hybrid charmonium and molecular currents, enabling a QCD sum-rule analysis of the X(3872) as a mixed state coupling to both hybrid and molecular currents. In the range of mixing parameters near the pure molecular limit, the largest mass prediction is in good agreement with X(3872) and represents a significant mixing of hybrid and molecular currents. In general, the mixed current increases the stability of the sum-rule analysis compared to the pure molecular case \([10, 41, 42, 43]\). A mixed scenario where the X(3872) couples to a mixture of hybrid and molecular currents is thus viable. However, as mentioned earlier, the sum-rule molecular and tetraquark currents are related by Fierz transformations and hence the interpretation of the underlying quark configuration is ambiguous. We anticipate that use of tetraquark currents would not lead to a substantial change in our conclusions because the molecular and tetraquark QCD sum-rule analysis mass predictions are virtually indistinguishable \([45]\).

Since only the four-quark state with \(I = 0\) can mix with the hybrid, the mixing pattern may have a role in explaining the X(3872) isospin-violating decay puzzle because the hybrid component \(D^*D\) decay channel is highly suppressed \([73]\) and will mainly decay into \(J/\psi\omega\) \([21]\). The standard explanation of isospin violation in X(3872) decay is that, for some non-perturbative reasons, quarks like to be in diquark (or meson) pairs, so that \([cu]\) and \([cd]\) diquarks (or meson clusters) in the multi-quark system can be considered as a quasiparticle. For instance, in \([14]\), \([cu][\bar{c}u]\) and \([cd][\bar{c}d]\) are independently bound. This pattern predicts two neutral “X(3872)” states with a small mass difference, and charged partners of the X(3872) \([14, 74]\). Similarly, the molecular scenario explains isospin violation in the decays \([24]\) and predicts a second neutral state with a larger mass mass splitting between the charged and neutral states (see e.g., Ref. [74] for a recent discussion). Previously, the charged partners of X(3872) had not been observed \([9, 72]\), but there is now evidence for their existence \([33, 34, 35]\). Note that the experimental mass difference from the X(3872) is about 20 MeV, which does not seem to fit either the tetraquark or molecular expectation.

The discovery of the neutral partner of the X(3872) is thus crucial for four-quark models of the X(3872). In both models, the neutral partners are composed of \(I = 0, 1\) states, so a hybrid mixture in the \(I = 0\) component provides a new mechanism for a mass splitting between the neutral states. However, future work is needed to determine how a mixed charmonium hybrid and \(\bar{D}D^*\) molecular scenario would affect the mass splitting of the neutral states and whether it would accomodate the observed isospin violation in X(3872) decays \([8, 9, 75]\).

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References

[1] S.K. Choi et al., Phys. Rev. Lett. 91 (2003) 262001.
[2] D. Acosta et al., Phys. Rev. Lett. 93 (2004) 072001.
[3] V.M. Abazov et al., Phys. Rev. Lett. 93 (2004) 162002.
[4] Bernard Aubert et al., Phys. Rev. Lett. 93 (2004) 041801.
[5] Eric S. Swanson. Phys. Rept. 429 (2006) 243.
[6] R. Aaij et al., Eur. Phys. J. C72 (2012) 1972.
[7] J. Beringer et al., Phys. Rev. D86 (2012) 010001.
[8] K. Abe et al. [Belle Collaboration], [hep-ex/0505037].
[9] S. -K. Choi, S. L. Olsen, K. Trabelsi, I. Adachi, H. Aihara, K. Arinstein, D. M. Asner and T. Aushev et al., Phys. Rev. D 84 (2011) 052004 [arXiv:1107.0163 [hep-ex]].
[10] Bernard Aubert et al., Phys. Rev. D74 (2006) 071101.
[11] Bernard Aubert et al., Phys. Rev. Lett. 102 (2009) 132001.
[12] R. Aaij et al. [LHCb Collaboration], [arXiv:1302.6269 [hep-ex]].
[13] C. Hanhart, Yu.S. Kalashnikova, A.E. Kudryavtsev, and A.V. Nefediev, Phys. Rev. D85 (2012) 011501.
[14] L. Maiani, F. Piccinini, A.D. Polosa, and V. Riquer, Phys. Rev. D71 (2005) 014028.
[15] D. Ebert, R.N. Faustov, and V.O. Galkin, Phys. Lett. B634 (2006) 214.
[16] Ricardo D’Elia Matheus, S. Narison, M. Nielsen, and J.M. Richard, Phys. Rev. D75 (2007) 014005.
[17] Kunihiko Terasaki, Prog. Theor. Phys. 118 (2007) 821.
[18] Stanislav Dubnicka, Anna Z. Dubnickova, Mikhail A. Ivanov, and Juergen G. Korner, Phys. Rev. D81 (2010) 114007.
[19] Bing An Li, Phys. Lett. B605 (2005) 306.
[20] Ted Barnes and Stephen Godfrey, Phys. Rev. D69 (2004) 054008.
[21] Mahiko Suzuki, Phys.Rev. D72 (2005) 114013.
[22] Frank E. Close and Philip R. Page, Phys. Lett. B578 (2004) 119.
[23] M.B. Voloshin, Phys. Lett. B579 (2004) 316.
[24] Eric S. Swanson, Phys. Lett. B588 (2004) 189.
[25] Nils A. Tornqvist, Phys. Lett. B590 (2004) 209.
[26] Mohammad T. AlFiky, Fabrizio Gabbiani, and Alexey A. Petrov, Phys. Lett. B640 (2006) 238.
[27] C.E. Thomas and F.E. Close, Phys. Rev. D78 (2008) 034007.
[28] Xiang Liu, Zhi-Gang Luo, Yan-Rui Liu, and Shi-Lin Zhu, Eur. Phys. J. C61 (2009) 411.
[29] Ian Woo Lee, Amand Faessler, Thomas Gutsche, and Valery E. Lyubovitskij, Phys. Rev. D80 (2009) 094005.
[30] D. Gamermann, J. Nieves, E. Oset, and E. Ruiz Arriola, Phys. Rev. D81 (2010) 014029.
[31] N. Brambilla, S. Eidelman, B.K. Heltsley, R. Vogt, G.T. Bodwin, et al., Eur. Phys. J. C71 (2011) 1534.
[32] D. Harnett, R. T. Kleiv, T. G. Steele and H. -Y. Jin, J. Phys. G39 (2012) 125003 [arXiv:1206.6776 [hep-ph]].
[33] M. Ablikim et al. [ BESIII Collaboration], [arXiv:1303.5949 [hep-ex]].
[34] Z. Q. Liu et al. [Belle Collaboration], Phys. Rev. Lett. 110 (2013) 252002 [arXiv:1304.0121 [hep-ex]].
[35] T. Xiao, S. Dobbs, A. Tomaradze and K. K. Seth, [arXiv:1304.3036 [hep-ex]].
[36] S. Coito, G. Rupp and E. van Beveren, Eur. Phys. J. C 71 (2011) 1762 [arXiv:1008.5100 [hep-ph]].
[37] S. Coito, G. Rupp and E. van Beveren, Eur. Phys. J. C 73 (2013) 2351 [arXiv:1212.0648 [hep-ph]].
[38] S. Coito, G. Rupp and E. van Beveren, Acta Phys. Polon. Supp. 3 (2010) 983 [arXiv:1005.2480 [hep-ph]].
[39] S. Coito, G. Rupp and E. van Beveren, Acta Phys. Polon. Supp. 5 (2012) 1015 [arXiv:1209.1313 [hep-ph]].
[40] M. Takizawa and S. Takeuchi, [arXiv:1206.4877 [hep-ph]].
[41] Wei Chen and Shi-Lin Zhu, EPJ Web Conf. 20 (2012) 01003 [arXiv:1209.4748 [hep-ph]].
[42] W. Chen, S. -L. Zhu and , Phys. Rev. D83 (2011) 034010 [arXiv:1010.3397 [hep-ph]].
[43] S. H. Lee, M. Nielsen and U. Wiedner, Jour. Korean Phys. Soc. 55 (2009) 424 [arXiv:0803.1168 [hep-ph]].
[44] R. D’E. Matheus, F. S. Navarra, M. Nielsen, C. M. Zanetti and , Phys. Rev. D 80 (2009) 056002 [arXiv:0907.2683 [hep-ph]].
[45] S. Narison, F. S. Navarra, M. Nielsen and , Phys. Rev. D 83 (2011) 016004 [arXiv:1006.4802 [hep-ph]].
[46] M. Karliner, H. J. Lipkin and [arXiv:1008.0203 [hep-ph]].
[47] A. Zhang, T. Huang and T. G. Steele, Phys. Rev. D 76 (2007) 036004 [hep-ph/0612146].
