GENERALIZATION OF MODULAR LOWERING OPERATORS FOR $\text{GL}_n$

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Abstract. We consider the generalization of Kleshchev’s lowering operators obtained by raising all the Carter-Lusztig operators in their definition to a power less than the characteristic of the ground field. If we apply such an operator to a nonzero $\text{GL}_{n-1}$-high weight vector of an irreducible representation of $\text{GL}_n$, shall we get a nonzero $\text{GL}_{n-1}$-high weight vector again? The present paper gives the explicit answer to this question. In this way we obtain a new algorithm for generating some normal weights.

1. Introduction

Informally speaking, the aim of this paper is to suggest a possible generalization of the operators introduced by A. S. Kleshchev in [K, Definition 2.5] that would be suitable for removing several nodes instead of one. Our main sources of inspiration are therefore [K] and [CL]. In the latter paper, Carter and Lusztig developed useful formulae to work with powers of their lowering operators. We have made trivial reformulations of their results (see Propositions 2.4 and 5.1) and use them as our principal tool.

In what follows, $L_n(\lambda)$ denotes the irreducible rational $\text{GL}_n$-module with highest weight $\lambda$. As is well known, Kleshchev’s lowering operators are enough for constructing:

- all $\text{GL}_{n-1}$-high weight vectors belonging to the first level of $L_n(\lambda)$ ([K, Theorem 4.2]);
- all $\text{GL}_{n-1}$-high weight vectors if $\lambda$ is a generalized Jantzen-Seitz weight ([BKS, Main Theorem]).

The second result was proved by successive application of Kleshchev’s operators to the $\text{GL}_{n-1}$-high weight vectors of $L_n(\lambda)$ already obtained. However, it follows from [S] and the tables at the end of [B] that some $\text{GL}_{n-1}$-high weight vectors (even belonging to levels with number less than the characteristic of the base field) can not be reached in this way. This fact forces us to consider new lowering operators $T_{i,j}^{(d)}(M, 1)$ (see Definition 3.2 where $R = 1$), whose effectiveness can be demonstrated by the number of $\text{GL}_{n-1}$-high weight vectors (up to proportionality) that can be reached by them (see Table 1). In this paper, we exploit the simplest approach to constructing such operators: we raise all the Carter-Lusztig operators in the definition of Kleshchev’s operators to a fixed power $d$. Thus all Kleshchev’s operators correspond to $d = 1$.

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It would be natural to expect these new operators to behave similarly to original Kleshchev’s operators, which they in most cases actually do. The major obstacle to overcome, however, is the impossibility of direct generalization of Lemmas 2.13 and 2.15 from \cite{K}, which say how the lowering operators behave under multiplication by strictly positive elements of the hyperalgebra. We have achieved the required modifications in Lemmas 3.3 and 3.4, but at the price of introducing the extra parameter \( R \) in Definition 3.2.

We now pass to strict formulations. Let \( \mathbf{K} \) be an algebraically closed field of characteristic \( p > 0 \). We assume that \( \mathbb{Z}/p\mathbb{Z} \subset \mathbf{K} \). We denote by \( \text{GL}_n \) the general linear group of size \( n \) over \( \mathbf{K} \). Let \( D_n \) and \( T_n \) denote the subgroups of \( \text{GL}_n \) consisting of all diagonal matrices and all upper triangular matrices respectively. We put \( X(n) := \mathbb{Z}^n \) and call the elements of this set \textit{weights}. Any weight \((\lambda_1, \ldots, \lambda_n)\) will be identified with the character of \( D_n \) that takes \( \text{diag}(t_1, \ldots, t_n) \) to \( t_1^{\lambda_1} \cdots t_n^{\lambda_n} \). A vector \( v \) of a rational \( \text{GL}_n \)-module is called a \textit{\( \text{GL}_n \)-high weight vector} if the line \( \mathbf{K} \cdot v \) is fixed by \( T_n \). We denote by \( X^+(n) \) the subset of \( X(n) \) consisting of all weakly decreasing sequences and call the elements of \( X^+(n) \) \textit{dominant weights}.

For \( n > 1 \), the group \( \text{GL}_{n-1} \) will be identified with a subgroup of \( \text{GL}_n \) consisting of matrices having zeros in the last row and the last column except the position of their intersection, where they have 1.

The main results of this paper are as follows. For a \( \text{GL}_{n-1} \)-high weight vector \( v \) of the irreducible module \( L_n(\lambda) \),

1. Theorems 6.4 and 6.5 show if the vector \( T^{(d)}_{i,j}(M,1)v \) is a nonzero \( \text{GL}_{n-1} \)-high weight vector for any \( 1 \leq d < p \), \( 1 \leq i < j \leq n \) and \( M \subset (i..j) \);
2. Theorems 6.6 and 6.8(ii) show if for fixed \( 1 \leq d < p \) and \( 1 \leq i < j \leq n \), there exists \( M \) such that \( T^{(d)}_{i,j}(M,1)v \) is a nonzero \( \text{GL}_{n-1} \)-high weight vector.

Of main interest for us, however, are the weights \( \mu \in X^+(n-1) \) such that there exists a nonzero \( \text{GL}_{n-1} \)-high weight vector of \( L_n(\lambda) \) having weight \( \mu \). These weights are called \textit{normal}. It is a major problem to find a direct combinatorial description of all normal weights, from the solution of which the structure of the socle of the restriction \( L_n(\lambda) \downarrow_{\text{GL}_{n-1}} \) would follow (see \cite{BK} Theorem D). Now we are going to formulate a property of the normal weight pattern following from result \cite{2} above. This property implies an algorithm to construct normal weights that generalizes any such algorithm of \cite{K}, \cite{BKS} or \cite{S}.

**Definition 1.1.** Let \( A \subset \mathbb{Z}^n \), \( B \subset \mathbb{Z}^m \) and \( \varphi : A \to B \). We say that \( \varphi \) is \textit{weakly increasing} (decreasing) w.r.t. the \( k \)th coordinate, where \( 1 \leq k \leq \min\{n,m\} \) if \( y_k \geq x_k \) (resp. \( y_k \leq x_k \)), whenever \( \varphi(x_1, \ldots, x_n) = (y_1, \ldots, y_m) \).

For integers \( 1 \leq i < j \leq n \), \( 1 \leq d < p \) and weights \( \mu \in X^+(n-1), \lambda \in X^+(n) \), we put (assuming \( j < n \) in the second line)

\[
\mathcal{C}_d^\mu(i,j) := \{ t \in (i..j) : t - i + \mu_t - \mu_i \equiv 0 \pmod{p} \},
\]

\[
\mathcal{X}_d^\mu(i,j) := \{(t, s) \in [i..j] \times [1..d] : t - i + \mu_t - \mu_{t+1} \equiv d - s \pmod{p} \},
\]

\[
\mathcal{X}_d^{\mu,\lambda}(i,j) := \{(t, s) \in [i..j] \times [1..d] : t - i + \mu_t - \lambda_{t+1} \equiv d - s \pmod{p} \}.
\]
In these definitions and in what follows, by \([i..j], [i..j], (i..j), (i..j)\) we denote the sets \(\{x \in \mathbb{Z} : i \leq x \leq j\}\), \(\{x \in \mathbb{Z} : i \leq x < j\}\), \(\{x \in \mathbb{Z} : i < x \leq j\}\), \(\{x \in \mathbb{Z} : i < x < j\}\) respectively. Moreover, for any sequence \(a\), we denote its length by \(|a|\) and its \(i\)th entry by \(a_i\).

**Property 1.2.** Let \(\mu \in X^+(n-1)\) be normal for \(\lambda \in X^+(n)\) and \(1 \leq d < p\).

(i) If \(1 \leq i < j < n\) and there exists an injection \(\varepsilon : \mathcal{X}^{\mu, \lambda}_{j}(i, n) \to \mathcal{C}^{\mu}(i, n)\) weakly decreasing w.r.t the first coordinate, then the weight \((\mu_1, \ldots, \mu_{i-1}, \mu_i - d, \mu_{i+1}, \ldots, \mu_{n-1})\) is also normal for \(\lambda\).

(ii) If \(1 \leq i < j < n\), \((j-1, 1) \in \mathcal{X}^{\mu}(i, j), (j-1, 1) \notin \mathcal{X}^{\mu, \lambda}_{d}(i, j)\), there exists an injection \(\varepsilon : \mathcal{X}^{\mu, \lambda}_{d}(i, j) \to \mathcal{C}^{\mu}(i, j)\) weakly decreasing w.r.t the first coordinate and an injection \(\tau : \mathcal{X}^{\mu, \lambda}_{d}(i, j) \to \mathcal{X}^{\mu, \lambda}_{d}(i, j) \setminus \{(j-1, 1)\}\) weakly increasing w.r.t. the first coordinate and weakly decreasing w.r.t. the second coordinate, then the weight \((\mu_1, \ldots, \mu_{i-1}, \mu_i - d, \mu_{i+1}, \ldots, \mu_{j-1}, \mu_j + d, \mu_{j+1}, \ldots, \mu_{n-1})\) is also normal for \(\lambda\).

The table below shows the maximum over all \(p\)-restricted weights \(\lambda \in X^+(n)\) of the number of the \(GL_{n-1}\)-high weight vectors in \(L_n(\lambda)\) (up to proportionality) that can be reached by the operators \(T^{(d)}_{i,j}(M, 1)\) for \(1 \leq d < p\), but can not be reached by Kleshchev’s lowering operators (i.e. \(T^{(1)}_{i,j}(M, 1)\)).

| \(p\) | \(3\) | \(5\) | \(7\) |
|-----|-----|-----|-----|
| \(n\) | 2   | 3   | 5   | 6   | 7   | 8   | 2   | 3   | 4   | 5   | 6   | 7   |
| \(\max\) | 0   | 1   | 3   | 4   | 5   | 14  | 43  | 153 | 465 | 0   | 4   | 26  | 153 | 917 | 0   | 9   | 87  | 713 |

**Table 1**

We fix the following sequences of \(X(n)\): \(\varepsilon_i\) having 1 at position \(i\), where \(1 \leq i \leq n\), and zeros elsewhere; \(\alpha(i, j) := \varepsilon_i - \varepsilon_j\), where \(1 \leq i, j \leq n\); \(\alpha_i = \alpha(i, i+1)\), where \(1 \leq i < n\). The lengths of these sequences will always be clear from the context and the following stipulation: an elementwise linear combination of two sequences \(a\) and \(b\) is well defined only if \(|a| = |b|\). Elements of \(X(n)\) are ordered as follows: \(\lambda \geq \mu\) if \(\lambda - \mu\) is a sum of \(\alpha_i\) with nonnegative coefficients. The descending factorial power \(x^\alpha\) equals \(x(x-1)\cdots(x-n+1)\) if \(n \geq 0\) and equals \(1/(x+1)\cdots(x-n)\) if \(n < 0\). The principal relation for this power is \(x^\alpha x^\mu = x^\mu(x-m)^\alpha\). Any product of the form \(\prod_{i=m}^b X_i\), with not necessarily commuting factors \(X_i\), will mean \(X_a \cdots X_b\). Following the standard agreement, we shall interpret any expression \(a^n\) as the sequence of length \(n\) whose every entry is \(a\), if this notation does not cause confusion. For example, \((a^3, 1^2) = (a, a, a, 1, 1)\). For two finite sequences \(a = (a_1, \ldots, a_n)\) and \(b = (b_1, \ldots, b_m)\), we define their product by \(a \ast b := (a_1, \ldots, a_n, b_1, \ldots, b_m)\). A formula \(A \cup B = C\) will mean \(A \cup B = C\) and \(A \cap B = \emptyset\). For any condition \(\mathcal{P}\), let \(\delta\) be 1 if \(\mathcal{P}\) is true and 0 if it is false.

The layout of the paper is as follows. Section 2 is devoted to various basic constructions in the hyperalgebra over integers \(\mathbb{U}(n)\) and its extension \(\bar{\mathbb{U}}(n)\). The central topics here are Lemma 2.3 on intersections and Lemmas 2.8 and 2.12 on block products (see also Definition 2.4). In Section 3 we introduce the operators \(T^{(d)}_{i,j}(M, R)\) (Definition 3.2) and prove the multiplication
formulae for them (Lemmas 3.3 and 3.4). In Section 4, we introduce the rational expressions similar to $\xi_{r,s}(M)$ of [K, Definition 2.9] virtually for the same purpose (see 4.1, 4.2 and Lemma 4.3). While up to now we have not been interested whether our elements belong to $U(n)$, Section 5 considers this question. Finally, in Section 6 the main results are proved.

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2. Hyperalgebra over $\mathbb{Z}$

In Sections 2–5, we fix an integer $n > 0$. Let $U_Q(n)$ be the universal enveloping algebra of the Lie algebra $gl_Q(n)$ of all $n \times n$-matrices over rationals. As usual, $gl_Q(n)$ is embedded into $U_Q(n)$. We denote by $X_{i,j}$ the matrix of $gl_Q(n)$ having 1 at $ij$-entry and 0 elsewhere. The ring $U_Q(n)$ is graded by the subgroup of $\mathbb{Z}^n$ consisting of sequences with sum zero so that $X_{i,j}$ has weight $\alpha(i,j)$. In this paper, weights and homogeneity of elements are always understood with respect to this grading.

The hyperalgebra over $\mathbb{Z}$ is the subring $U(n)$ of $U_Q(n)$ generated by

$$X_{i,j}^{(r)} := \frac{(X_{i,j})^r}{r!} \quad \text{for integers } 1 \leq i \neq j \leq n \text{ and } r \geq 0;$$

$$X_{i,i}^{(r)} := \frac{X_{i,i}^{(r-1)} - (X_{i,i} - r + 1)}{r!} \quad \text{for integers } 1 \leq i \leq n \text{ and } r \geq 0.$$

In this definition, empty product means the identity of $U_Q(n)$. It is convenient to define the above elements as zero if $r$ is a negative integer. For mutually distinct $i, j, k$, we have

$$X_{i,j}^{(a)}X_{j,k}^{(b)} = \sum_t X_{j,i}^{(b-t)}\left(\mathbb{H}_i - \mathbb{H}_j - (a + b) + 2t\right)X_{i,j}^{(a-t)};$$

$$X_{i,j}^{(a)}X_{j,k}^{(b)}X_{k,i}^{(c)} = \sum_t (-1)^t X_{i,j}^{(a-t)}X_{i,k}^{(b-t)}X_{j,i}^{(c-t)}. \quad (2.1)$$

We shall use the notation $E_{i,j}^{(r)} := X_{i,j}^{(r)}$, $F_{i,j}^{(r)} := X_{i,j}^{(r)}$, where $1 \leq i < j \leq n$, and $\mathbb{H}_i := X_{i,i}$. We also put $E_{i,i}^{(r)} := X_{i,i}^{(r+1)}$ and omit the superscript $^{(1)}$.

Let $UT(n)$ denote the set of integer $n \times n$-matrices $N$ with nonnegative entries such that $N_{a,b} = 0$ unless $a < b$ and $e_{i,j}$, where $i < j$, denote the element of $UT(n)$ with 1 at $ij$-entry and 0 elsewhere. For any integer $n \times n$-matrix $N$, we define

$$F(N) := \prod_{1 \leq a < b \leq n} E_{a,b}^{(N_{a,b})}, \quad E(N) := \prod_{1 \leq a < b \leq n} E_{a,b}^{(N_{a,b})},$$

where $F_{a,b}^{(N_{a,b})}$ precedes $F_{c,d}^{(N_{c,d})}$ if $b < d$ or $b = d$ and $a < c$ in the first product and $E_{a,b}^{(N_{a,b})}$ precedes $E_{c,d}^{(N_{c,d})}$ if $a < c$ or $a = c$ and $b < d$ in the second product. Obviously, $F(N) = E(N) = 0$ if $N$ contains a negative entry.

**Proposition 2.1 ([CL 2.1]).** Elements $F(N)\mathbb{H}_1 \cdots \mathbb{H}_n E(M)$, where $N, M \in UT(n)$ and $r_1, \ldots, r_n$ are nonnegative integers form a $Q$-basis of $U_Q(n)$. These elements generate $U(n)$ as a $\mathbb{Z}$-module.

In particular, the $Q$-subalgebra $U^0_Q(n)$ of $U_Q(n)$ generated by $\mathbb{H}_1, \ldots, \mathbb{H}_n$ is generated by them freely as a commutative $Q$-algebra.
For any $N \in UT(n)$, we define the ring automorphism $\tau_N$ of $U^0_Q(n)$ by

$$\tau_N(\mathbb{H}):= \mathbb{H} + \sum_{1 \leq a < i} N_{a,i} - \sum_{i < b \leq n} N_{i,b} \quad \text{for} \quad i = 1, \ldots, n.$$  

This definition is made to ensure

$$f\mathbb{E}(N) = \mathbb{E}(N)\tau_N(f), \quad \mathbb{E}(N)f = \tau_N(f)\mathbb{E}(N)$$

(2.2)

for every $f \in U^0_Q(n)$ and $N \in UT(n)$. It follows from these formulae and Proposition 2.1 that every element $f \in U^0_Q(n) \setminus \{0\}$ is not a zero divisor in $U_Q(n)$. Let $\mathbb{U}(n)$ be the right ring of quotients of $U_Q(n)$ with respect to $U^0_Q(n) \setminus \{0\}$. Its existence follows from the right Ore condition, which can be easily checked. It is easy to prove that $\mathbb{U}(n)$ is also the left ring of quotients for the same pair. In Sections 2-5 we use miscellaneous rings, all of which are subrings of $\mathbb{U}(n)$. Below we give the table that defines them.

| $R$ | $\mathbb{U}$ | $\mathbb{U}$ | $\mathbb{U}$ |
|-----|---------------|---------------|---------------|
| $R^+(a,b)$ | $\{\mathbb{E}_{i,j}^{(r)} : a \leq i < j \leq b\}$ subring | $\mathbb{H}_k : a \leq k \leq b$ subfield | $\mathbb{H}_k : a \leq k \leq b$ subring |
| $R^0(a,b)$ | $\{\mathbb{H}_r : a \leq k \leq b\}$ subring | $\mathbb{H}_k : a \leq k \leq b$ subring | $\mathbb{H}_k : a \leq k \leq b$ subring |
| $R^-(a,b)$ | $\{\mathbb{F}_{i,j}^{(r)} : a \leq i < j \leq b\}$ subring | $\{\mathbb{F}_{i,j}^{(r)} : a \leq i < j \leq b\}$ subring | $\mathbb{H}_k : a \leq k \leq b$ subring |
| $R^{0,-}(a,b)$ | $\{\mathbb{F}_{i,j}^{(r)} : a \leq i < j \leq b\}$ subring | $\{\mathbb{F}_{i,j}^{(r)} : a \leq i < j \leq b\}$ subring | $\mathbb{H}_k : a \leq k \leq b$ subring |

The last three columns correspond to values of $R$, the last four cells in each of these columns contain generating sets and the word showing if we generate a subring or a subfield. We abbreviate $R^i(1,n)$ to $R^i(n)$.

If we extend the automorphism $\tau_N$ of $U^0_Q(n)$ to the automorphism of the field $\overline{U^0}(n)$ by $\tau_N(fg^{-1}) = \tau_N(f)\tau_N(g)^{-1}$, where $f, g \in U^0_Q(n)$ and $g \neq 0$, then formulae (2.2) will hold for any $f \in \overline{U^0}(n)$.

**Lemma 2.2.** Any element $x \in \overline{U}(n)$ is uniquely represented in the form $x = \sum_{N,M \in UT(n)} \mathbb{E}(N)H_{N,M}\mathbb{E}(M)$, where $H_{N,M} \in \overline{U^0}(n)$.

**Proof.** It suffices to apply (2.2) and Proposition 2.1.

We shall use two types of ideals in either ring $\mathbb{U}(n)$ and $\overline{U}(n)$. For $S \subset [1..n)$, let $I_S$ and $\overline{I}_S$ denote the left ideals of $\mathbb{U}(n)$ and $\overline{U}(n)$ respectively generated by $\mathbb{E}(r)$, where $r \geq 1$ and $i \in S$. Clearly $\overline{I}_S$ is the left ideal of $\overline{U}(n)$ generated by $\mathbb{E}_i$, where $i \in S$. Let $C = (c_1, \ldots, c_{n-1})$ be the sequence with entries belonging to $Z \cup \{+\infty\}$. Let $J^{(C)}$ and $\overline{J}^{(C)}$ denote the left ideals of $\mathbb{U}(n)$ and $\overline{U}(n)$ respectively generated by homogeneous elements having weight with $\alpha_i$-coefficient strictly greater than $c_i$ for some $i = 1, \ldots, n - 1$. Obviously $\overline{I}_S = \overline{I}_S = \mathbb{J}^{(c_1, \ldots, +\infty)} = \mathbb{J}^{(+\infty, \ldots, +\infty)} = 0$.

**Lemma 2.3.** $(\overline{I}_S + \overline{J}^{(C)}) \cap \mathbb{U}(n) = I_S + J^{(C)}$.

**Proof.** Applying the automorphism of $\overline{U}(n)$ permuting the indexes, it suffices to consider the case $S = [k..n)$. We only need to prove that the left-hand side is contained in right-hand side.
Let \( f_1 \in \mathbb{I}_S \). The definition of \( \mathbb{I}_S \) and Lemma 2.2 yield
\[
\sum_{i=k}^{n-1} \left( \sum_{N,M \in UT(n)} \mathbb{F}^{(N)} H_{N,M}^{(i)} \mathbb{E}^{(M)} \right) E_i,
\]
where \( H_{N,M}^{(i)} \in \mathbb{U}^0(n). \) The order of factors in \( \mathbb{E}^{(M)} \), we agreed upon, and formulae (2.1) show that \( \mathbb{E}^{(M)} E_i \) is equal to the \(\mathbb{Z}\)-linear combination of elements \( \mathbb{E}^{(M')} \), where \( M' \in UT(n) \) such that \( M'_{s,t} \neq 0 \) for some \( s \) and \( t \) with \( i < s < t \leq n \).

Now let \( f_2 \in \mathbb{U}(C) \). It is easy to see that \( f_2 = \sum_{N,M \in UT(n)} \mathbb{F}^{(N)} H_{N,M} \mathbb{E}^{(M)}, \) where \( H_{N,M} \in \mathbb{U}^0(n) \) and if \( H_{N,M} \neq 0 \) then there is some \( i = 1, \ldots, n-1 \) such that the \( \alpha_i \)-coefficient of the weight of \( \mathbb{E}^{(M)} \) is strictly greater than \( \alpha_i \).

Now assume \( f_1 + f_2 \in \mathbb{U}(n) \). The above representations of \( f_1 \) and \( f_2 \) and Lemma 2.2 yield \( f_1 + f_2 = \sum_{N,M \in UT(n)} \mathbb{F}^{(N)} H_{N,M} \mathbb{E}^{(M)} \), where \( H_{N,M} \in \mathbb{U}^0(n) \) and if \( H_{N,M} \neq 0 \) then either the \( \alpha_i \)-coefficient of the weight of \( \mathbb{E}^{(M)} \) is greater than \( \alpha_i \) for some \( i = 1, \ldots, n-1 \) or \( M_{s,t} \neq 0 \) for some \( s \) and \( t \) with \( k \leq s < t \leq n \). In the first case \( \mathbb{E}^{(M)} \in \mathbb{J}(C) \).

Therefore, consider the second case. Let \((i,j)\) be the lexicographically greatest pair such that \( 1 \leq i < j \leq n \) and \( M_{i,j} > 0 \). Hence \((i,j)\) is greater than or equal to \((s,t)\) and thus \( k \leq s < i < j \leq n \). Hence by (2.1) (inductively) we get \( \mathbb{E}^{(M_{i,j})} \in \mathbb{I}_{[k,n]} \). Now recall that \( \mathbb{E}^{(M)} = \mathbb{E}^{(M-M_{i,j}e_{i,j})} \mathbb{E}^{(M_{i,j})} \) and thus \( \mathbb{E}^{(M)} \in \mathbb{I}_{[k,n]} \).

In what follows, we shall use the abbreviations
\[
\mathbb{C}(i,j) := j - i + \mathbb{H}_i - \mathbb{H}_j, \quad \mathbb{B}(i,j) := j - i + \mathbb{H}_i - \mathbb{H}_{i+1}.
\]

The main tool of our investigation is the following elements of the hyperalgebra
\[
\mathbb{S}_{i,j} := \sum_{A \subseteq \{i,j\}} \left( \mathbb{F}^A_{i,j} \prod_{t \in \{i,j\} \setminus A} \mathbb{C}(i,t) \right),
\]
where \( 1 \leq i < j \leq n \) and \( \mathbb{F}^A_{i,j} = \mathbb{F}_{a_0,a_1} \mathbb{F}_{a_1,a_2} \cdots \mathbb{F}_{a_m,a_{m+1}} \) for \( A \cup \{i,j\} = \{a_0 < \cdots < a_{m+1}\} \). In particular, \( \mathbb{S}_{i,j} = 1 \). Elements \( \mathbb{S}_{i,j} \) were first introduced in [CL]. In this connection, we call them the Carter–Lusztig lowering operators.

The operators \( \mathbb{S}_{i,j} \) possess the property
\[
\mathbb{E}_{l-1} \mathbb{S}_{i,j} \equiv 0 \pmod{\mathbb{U}(n) \cdot \mathbb{E}_{l-1}}, \quad (\text{2.3})
\]
for \( l \neq j \) (see [CL] Lemma 2.4 or [B] Lemma 3.6]). If \( j = l \) then it follows from [CL] Lemma 2.6] that the following equivalences hold.

**Proposition 2.4.** Let \( 1 \leq i < j \leq n \) and \( l, d \geq 0 \). Then modulo the left ideal of \( \mathbb{U}(n) \) generated by \( \mathbb{E}_{l-1}, \ldots, \mathbb{E}_{j-1}^{(l)} \), we have
\[
\mathbb{E}_{j-1}^{(l)} \mathbb{S}_{i,j}^{d} \equiv \left( \mathbb{S}_{i,j}^{l} \right)^{l-d+1} (\mathbb{C}(i,j) - d + l - 1) \cdots (\mathbb{C}(i,j) - d) \text{ if } l \leq d;
\]
\[
\mathbb{E}_{j-1}^{(l)} \mathbb{S}_{i,j}^{d} \equiv 0 \text{ if } l > d.
\]
Corollary 2.5. Let $1 \leq i < j_1 \leq \cdots \leq j_d \leq n$ and $l = 1, \ldots, d$. Then modulo the left ideal of $\mathcal{U}(n)$ generated by $E_{j_1-1}$, we have

$$E_{j_1-1}S_{i,j_1} \cdots S_{i,j_d} = S_{i,j_1} \cdots S_{i,j_d}(d_{j_1} - d_{j_1+1})(C(i, j_1) - d_{j_1}),$$

where $d_s := |\{t \in [1..d] : s \leq j_t\}|$, $j'_t = j_t - 1$ if $t$ equals the smallest of the numbers of $[1..d]$ such that $j_t = j_i$, and $j'_t = j_t$ otherwise.

We shall abbreviate $S_{i,j} := S_{i,j_1} \cdots S_{i,j_d}$, where $J = (j_1, \ldots, j_d)$ is a sequence of integers. For $N \in UT(n)$ and $1 < m \leq n$, one can easily check the formula

$$[E_{m-1}, F(N)] = \sum_{1 \leq s < m-1} (N_{s,m-1} + 1)F(N-e_s,m-1)$$

$$+F(N-e_{m-1,m}) \left( H_{m-1} - H_m + 1 - \sum_{m-1 < b \leq n} N_{m-1,b} + \sum_{m < b \leq n} N_{m,b} \right) \quad (2.4)$$

$$- \sum_{m < t \leq n} (N_{m,t} + 1)F(N-e_{m-1,t+e_m,t}).$$

For $i = 1, \ldots, n-1$, let $\theta_i$ be the abelian group automorphism of $\mathcal{U}(n)$ defined by the rules:

- $\theta_i(H_j) = h_{j,i} + \delta_{j=i+1} - \delta_{j=i}$ for any $j \in [1..n]$ and the restriction of $\theta_i$ to $\mathcal{U}^0(n)$ is a field automorphism;
- $\theta_i(FHE) = F\theta_i(H)E$, where $F \in \mathcal{U}^-(n)$, $H \in \mathcal{U}^0(n)$ and $E \in \mathcal{U}^+(n)$.

Proposition 2.6.

1. Let $1 \leq a \leq b \leq n$, $1 < m \leq n$ and $N \in UT(n)$ be such that $F(N) \in \mathcal{U}^-(a,b)$. Then $[E_{m-1}, F(N)] \in \mathcal{U}^{-,0}(a,b).

2. Let $R \in \mathcal{U}^{-,0}(a,b)$. We put $R_0 := R$. Suppose that every $R_{k+1}$ is obtained from $R_k$ either by left multiplication by $E_i$, where $i \in M$, or by application of some $\theta_j$. Then all $R_m$ belong to $\mathcal{U}^{-,0}(a,b) \oplus \bar{I}_M$.

Proof. This result follows immediately from (2.4).

For $1 \leq a \leq b \leq c \leq d \leq n$, we have

$$[\mathcal{U}^+(a,b), \mathcal{U}^+(c,d)] = 0. \quad (2.5)$$

Proposition 2.7.

1. If $1 \leq a \leq b \leq i < n$ and $Y \in \mathcal{U}^-(a,b)\mathcal{U}^0(n)$, then $E_iY = \theta_i(Y)E_i$.

2. If $1 \leq i < n$, $m \in \mathcal{Z}$, $X \in \mathcal{U}(n)$ and $H \in \mathcal{U}^0(n)$, then $\theta_i^m(XH) = \theta_i^m(X)\theta_i^m(H)$.

3. If $1 \leq i < m \in \mathcal{Z}$ and $Y_1, Y_2 \in \mathcal{U}^{-,0}(n)$, then $\theta_i^m(Y_1Y_2) = \theta_i^m(Y_1)\theta_i^m(Y_2)$.

4. If $1 \leq i < j < n$ and $m \in \mathcal{Z}$, then $\theta_i^m(C(i,j)) = C(i,j) + m$.

5. The ideals $I_M$ and $\bar{I}_M$ are stable under $\theta_i$.

Proof. It suffices to notice that the restriction of $\theta_i$ to $\mathcal{U}^0(n)$ commutes with $\tau_N$ and apply formulae (2.2) and (2.3).

Lemma 2.8. Suppose $1 \leq a_1 < b_1 \leq \cdots \leq a_k < b_k \leq n$ and for each $i = 1, \ldots, k$, we are given an element $X_i \in \mathcal{U}^{-,0}(a_i,b_i) \oplus \bar{I}_{M_i}$, where $M_i \subset [a_i..b_i]$.

1. If for each $i = 1, \ldots, k$, we are given another $X'_i \in \mathcal{U}(n)$ such that $X_i \equiv X'_i (mod \bar{I}_{M_i})$, then $X_1 \cdots X_k \equiv X'_1 \cdots X'_k (mod \bar{I}_{M_1 \cup \cdots \cup M_k})$. 

Proof. This result follows immediately from (2.4).
(ii) If \( s \in [1,n) \) then modulo \( \mathbb{I}_{M_1 \cup \cdots \cup M_k \cup \{s\}} \) the element \( E_sX_1 \cdots X_k \) equals

(a) 0 if \( s \notin \{a_1, b_1\} \cup \cdots \cup \{a_k, b_k\}; \)
(b) \( X_1 \cdots X_{m-1}E_sX_m \cdots X_k \) if \( s \in [a_m, b_m) \) and \( m = 1 \) or \( s > b_{m-1}; \)
(c) \( X_1 \cdots X_{m-2}E_s(x_{m-1})E_sX_m \cdots X_k \) if \( m > 1 \) and \( s = a_m = b_{m-1}. \)

Proof. Let \( Y_i \) be the element of \( \mathbb{U}^{-\theta}(a_i, b_i) \) such that \( X_i = Y_i \) (mod \( \mathbb{I}_{M_i} \)). We use induction on \( k. \) The case \( k = 0 \) is obvious. Assume that \( k > 0 \) and that the assertion is true for smaller number of factors. By the inductive hypothesis, our \( E_sX_1 \cdots X_k \equiv \mathbb{E}_sX_1 \cdots X_k \) (mod \( \mathbb{I}_{M_1 \cup \cdots \cup M_k} \)). Therefore, it remains to prove that \( (X_1 - X')_sX_2 \cdots X_k \in \mathbb{I}_{M_1 \cup \cdots \cup M_k} \). Applying the inductive hypothesis one more time, we get \( X_2 \cdots X_k \equiv Y_2 \cdots Y_k \) (mod \( \mathbb{I}_{M_1 \cup \cdots \cup M_k} \)). Hence \( (X_1 - X')_sX_2 \cdots X_k \equiv (X_1 - X')_sY_2 \cdots Y_k \) (mod \( \mathbb{I}_{M_1 \cup \cdots \cup M_k} \)). Moreover, (2.5) implies \( (X_1 - X')_sY_2 \cdots Y_k \in \mathbb{I}_{M_1}. \)

Part (i) yields \( E_sX_1 \cdots X_k \equiv E_sY_1 \cdots Y_k \) (mod \( \mathbb{I}_{M_1 \cup \cdots \cup M_k} \)). In case (a) the result immediately follows from (2.5).

Now we consider the case where \( s \in [a_m, b_m) \) for some \( m = 1, \ldots, k. \) Then Proposition 2.7(i) implies \( E_sX_1 \cdots X_k \equiv \mathbb{E}_sX_1 \cdots X_k \) (mod \( \mathbb{I}_{M_1 \cup \cdots \cup M_k} \)). Note that \( E_sX_1 \cdots X_k \in \mathbb{U}^{-\theta}(a_m, b_m) \oplus \mathbb{I}_{\{s\}} \), which easily follows from Proposition 2.7(i). Let us prove inductively on \( j = 0, \ldots, m-1 \) that

\[
\theta_s(Y_1) \cdots \theta_s(Y_{m-1}) \mathbb{E}_sY_m \cdots Y_k \equiv \theta_s(X_1) \cdots \theta_s(X_j) \times \\
\times \theta_s(Y_{j+1}) \cdots \theta_s(Y_{m-1}) \mathbb{E}_sY_m \cdots Y_k \quad \text{(mod \( \mathbb{I}_{M_1 \cup \cdots \cup M_{j+1}} \)).}
\]

Actually the proof follows from \( \theta_s(Y_{j+1}) \equiv \theta_s(X_{j+1}) \) (mod \( \mathbb{I}_{M_{j+1}} \)), which holds by Proposition 2.7(v). The equivalence we just proved applied for \( j = m - 1 \) and part (i) of the current lemma yield

\[
\mathbb{E}_sX_1 \cdots X_k \equiv \theta_s(X_1) \cdots \theta_s(X_{m-1}) \mathbb{E}_sY_m \cdots Y_k \equiv \\
\theta_s(X_1) \cdots \theta_s(X_{m-1}) \mathbb{E}_sX_m \cdots X_k \quad \text{(mod \( \mathbb{I}_{M_1 \cup \cdots \cup M_k \cup \{s\}} \)).}
\]

Cases (b) and (c) now follow from the definition of \( \theta_s. \)

We shall use the notation \( E(i, j) := E_i \cdots E_{j-1}, \) where \( 1 \leq i \leq j \leq n. \)

Definition 2.9. Let \( 1 \leq j \leq n \) and \( K = (k_1, \ldots, k_m), \) \( L = (l_1, \ldots, l_q) \) be sequences of integers such that \( 1 \leq k_1 \leq \cdots \leq k_m \leq j \) and \( q = m \) or \( q = m + 1. \) For \( f \in \mathbb{U}(n) \), we define \( \mathbb{E}_j(K, L)(f) \) to equal

- \( f \) if \( K = L = \emptyset; \)
- \( \mathbb{E}(k_m, j) \cdot \mathbb{E}_j(K', L)(f) \) if \( m = q > 0, \) where \( K' = (k_1, \ldots, k_{m-1}); \)
- \( \theta_j^{m+1}(\mathbb{E}_j(K, L')(f)) \) if \( q = m + 1, \) where \( L' = (l_1, \ldots, l_m). \)

The maps \( \mathbb{E}_j(K, L) \) we have defined are abelian group automorphisms of \( \mathbb{U}(n). \) We also see that \( \mathbb{E}_j(K, L) = \mathbb{E}_j(K, L \ast (0)) \) if the sequences \( K \) and \( L \) have the same length.

Proposition 2.10. We have \( \mathbb{E}_j(K, L)(fH) = \mathbb{E}_j(K, L)(f) \cdot \theta_j^E,L(H) \) for any \( f \in \mathbb{U}(n) \) and \( H \in \mathbb{U}(n)^0. \)

Proof. The result follows by induction on \( |K| + |L| \) from Definition 2.9 and Proposition 2.7(ii).
For integers $i \leq j$ and $k$, we define $k^{(i,j)} := \min\{j, \max\{i, k\}\} = \max\{i, \min\{j, k\}\}$. For a sequence of integers $K = (k_1, \ldots, k_m)$ and integers $i \leq j$, we put $K^{(i,j)} := (k_1^{(i,j)}, \ldots, k_m^{(i,j)})$ and $K^{(j)} := (\delta_{k_1 \leq j}, \ldots, \delta_{k_m \leq j})$.

Below we collect the main properties of the above operations that we shall use throughout this paper.

Proposition 2.11. Let $K = (k_1, \ldots, k_d)$ be a sequence of integers and $i, j, s, t$ be integers such that $i \leq j$ and $s \leq t$.

(i) If $[i..j] \cap [s..t] \neq \emptyset$ then $(K^{(i,j)})^{(s,t)} = K^{(\max\{i, s\}, \min\{j, t\})}$.

(ii) If $i \leq s < j$ then $(K^{(i,j)})^{(s)} = K^{(s)}$.

(iii) $(K - \varepsilon_x)^{(i,j)} = K^{(i,j)} - \delta_{1 \leq k_x \leq \varepsilon_x}$ for $1 \leq x \leq d$.

(iv) $(K - \varepsilon_x)^{(j)} = K^{(j)} + \delta_{k_x - 1 \leq j - \varepsilon_x}$ for $1 \leq x \leq d$.

Now we are going to prove a lemma that will explain the role of sequences $K^{(i,j)}$ and $K^{(j)}$.

Lemma 2.12. Suppose $1 \leq a_1 < \cdots < a_q \leq n$, where $q \geq 2$, and for each $i = 1, \ldots, q - 1$, we are given an element $X_i \in \bigcup M_i$ with entries from $\{a_i, a_{i+1}\}$, where $M_i \subset \{a_i, a_{i+1}\}$. Let $K$ be a weakly decreasing sequence of integers with entries from $\{a_1..a_q\}$ and $L$ be a sequence of integers of length $|K|$ or $|K| + 1$. We have

\[
E_{a_q}(K, L)(X_1 \cdots X_{q-1}) \equiv E_{a_q}(K^{(a_1, a_2)}, K^{(a_2)})(X_1) \cdots E_{a_{q-1}}(K^{(a_{q-2}, a_{q-1})}, K^{(a_{q-1})})(X_{q-2}) \times
\]

\[
\times E_{a_q}(K^{(a_{q-1}, a_q)}, L)(X_{q-1}) \quad \text{(mod } \bar{I}_{M_1 \cup \cdots \cup M_{q-1} \cup [\min K..a_q]}\). \]

Proof. Let $K = (k_1, \ldots, k_m)$ and $L = (l_1, \ldots, l_t)$. We apply induction on $m + t$. The case $m + t = 0$ is obvious. Therefore assume $m + t > 0$.

First we consider the case $m = t = 0$. Let $\tilde{K} := (k_1, \ldots, k_{m-1})$ and

\[
\hat{X}_1 := E_{a_2}(\hat{K}^{(a_1, a_2)}, \hat{K}^{(a_2)})(X_1),
\]

\[
\vdots
\]

\[
\hat{X}_{q-2} := E_{a_{q-1}}(\hat{K}^{(a_{q-2}, a_{q-1})}, \hat{K}^{(a_{q-1})})(X_{q-2}),
\]

\[
\hat{X}_{q-1} := E_{a_q}(\hat{K}^{(a_{q-1}, a_q)}, L)(X_{q-1}).
\]

We put $\hat{c} := \min \tilde{K}$ for brevity. Clearly, $k_1 = \min K \leq \hat{c}$. By Definition 2.9 and the inductive hypothesis, we get

\[
E_{a_q}(K, L)(X_1 \cdots X_{q-1}) \equiv E(k_m, a_q)\hat{X}_1 \cdots \hat{X}_{q-1} \quad \text{(mod } \bar{I}_{M_1 \cup \cdots \cup M_{q-1} \cup [\hat{c}..a_q]}\).
\]

The required result follows from this formula if $k_m = a_q$. Therefore, we consider the case $k_m < a_q$. Let $r = 1, \ldots, q - 1$ be a number such that $a_1 < \cdots < a_r < k_m < a_{r+1} < \cdots < a_q$. Hence

\[
E(k_m, a_q) = E(k_m, a_{r+1})E(a_{r+1}, a_{r+2}) \cdots E(a_q, a_q),
\]

where all factors in the right-hand side are nonunitary. By Proposition 2.10 for every $i = 1, \ldots, q - 1$, we have $X_i \in \bigcup M_i \cup [a_i, a_{i+1}] \cup \bar{I}_{M_1 \cup \cdots \cup M_{q-1} \cup [\hat{c}..a_q]}$. Lemma 2.6 now implies

\[
E(k_m, a_q)\hat{X}_1 \cdots \hat{X}_{q-1} \equiv \hat{X}_1 \cdots \hat{X}_{r-1}E(k_m, a_{r+1}) \times
\]

\[
\times E_{a_{r+1}}(\hat{X}_r)E(a_{r+1}, a_{r+2}) \cdots E_{a_{q-1}}(\hat{X}_{q-2})E(a_q, a_q) \times
\]

\[
\times \hat{X}_{q-1} \quad \text{(mod } \bar{I}_{M_1 \cup \cdots \cup M_{q-1} \cup [k_1..a_q]}\).
\]
if \( r = 1 \) or \( a_r < k_m \) and
\[
\mathbb{E}(k_m, a_q) \hat{X}_1 \cdots \hat{X}_{q-1} = \hat{X}_1 \cdots \hat{X}_{r-2} \theta_{a_r} (\hat{X}_{r-1}) \mathbb{E}(k_m, a_{r+1}) \times \\
\times \theta_{a_{r+1}} (\hat{X}_r) \mathbb{E}(a_{r+1}, a_{r+2}) \cdots \theta_{a_{q-1}} (\hat{X}_{q-2}) \mathbb{E}(a_{q-1}, a_q) \times \\
\times \hat{X}_{q-1} \pmod{\mathbb{I}_{M_1 \cup \cdots \cup M_{q-1} \cup [k_1, a_q)}}
\]
if \( r > 1 \) and \( a_r = k_m \). The result now follows from Definition 2.9.

Now let \( t = m + 1 \). By the inductive hypothesis and Proposition 2.7(iv), we get
\[
\mathbb{E}_{a_q} (K, L) (X_1 \cdots X_{q-1}) \equiv \theta_{a_q} \left( \mathbb{E}_{a_2} (K^{(a_1, a_2)}, K^{(a_2)}) (X_1) \cdots \\
\mathbb{E}_{a_{q-1}} (K^{(a_{q-2}, a_{q-1})}, K^{(a_{q-1})}) (X_{q-2}) \mathbb{E}_{a_q} (K^{(a_{q-1}, a_q)}, \hat{L}) (X_{q-1}) \right) \pmod{\mathbb{I}_{M_1 \cup \cdots \cup M_{q-1} \cup [\min K, a_q)}}.
\]
where \( \hat{L} = (l_1, \ldots, l_m) \). It follows from Proposition 2.7(iii) that for every \( i = 1, \ldots, q - 1 \), there exists \( \hat{Y}_i \in \mathbb{U}^{-0}(a_i, a_{i+1}) \) such that
\[
\mathbb{E}_{a_{i+1}} (K^{(a_i, a_{i+1})}, N_i) (X_i) \equiv \hat{Y}_i \pmod{\mathbb{I}_{M_1 \cup \cdots \cup M_{q-1} \cup [\min K, a_q)}}.
\]
where \( N_i = K^{(a_{i+1})} \) if \( i < q - 1 \) and \( N_{q-1} = \hat{L} \). It follows from Lemma 2.8(i), Proposition 2.7(iv) and formula (2.6) that
\[
\mathbb{E}_{a_q} (K, L) (X_1 \cdots X_{q-1}) \equiv \hat{Y}_1 \cdots \hat{Y}_{q-2} \theta_{a_q} (\hat{Y}_{q-1}) \\
\equiv \mathbb{E}_{a_2} (K^{(a_1, a_2)}, K^{(a_2)}) (X_1) \cdots \mathbb{E}_{a_{q-1}} (K^{(a_{q-2}, a_{q-1})}, K^{(a_{q-1})}) (X_{q-2}) \times \\
\times \theta_{a_q} \left( \mathbb{E}_{a_q} (K^{(a_{q-1}, a_q)}, \hat{L}) (X_{q-1}) \right) \pmod{\mathbb{I}_{M_1 \cup \cdots \cup M_{q-1} \cup [\min K, a_q)}}.
\]
It remains to apply Definition 2.9 to the last factor. \(\square\)

3. Multiplication Formulae

In this section, we introduce certain elements \( T^{(d)}_{r, j} (M, R) \) of \( \mathbb{U}(n) \). We are interested in the behavior of \( \mathbb{E}_j (K, L) (T^{(d)}_{r, j} (M, R)) \) under the left multiplication by \( \mathbb{E}_{l-1} \) modulo a suitable ideal. This is done in Lemmas 3.3 and 3.4.

In what follows, \( \mathbb{Z} (\zeta) \) denotes the field of fractions of the ring of polynomials \( \mathbb{Z} [\zeta] \).

Lemma 3.1. Let \( 1 \leq i < j \leq n, d \geq 1, R \in \mathbb{Z} (\zeta) \) and \( K = (k_1, \ldots, k_m) \), \( L = (l_1, \ldots, l_q) \) be sequences of integers such that \( m \leq d, i \leq k_1 \leq \cdots \leq k_m \leq j \) and \( q = m \) or \( q = m + 1 \). Modulo \( \mathbb{I}_{[\min K', j]} \), we have
\[
\mathbb{E}_j (K, L) \left( \mathbb{S}_{i,j} R (\mathbb{C}(i, j)) \right) \equiv \mathbb{S}_{i, K'} d! \times \\
\times \prod_{s=1}^{r} \left( \mathbb{C}(i, j) - d + s - 1 + \sum_{s<h \leq q} l_h \prod_{k_s < t \leq j} (\mathbb{C}(i, t) - d + s - 1) \right) \times \\
\times R (\mathbb{C}(i, j) + \Sigma L),
\]
where \( K' = (k_1, \ldots, k_m, j^{d-m}) \) and \( r \) is the number of entries of \( K \) less than \( j \).

Proof. We apply induction on \( m + q \). If \( q = m + 1 \) then we obtain the required formula from the inductive hypothesis by applying Definition 2.9 and Proposition 2.7(iv).
Now let \( m = q \). Since the case \( K = \emptyset \) is obvious, we assume \( K \neq \emptyset \). By definition \(^{22,3}\) we have
\[
E_j(K, L)\left( S_{i,j}^d R(C(i,j)) \right) = E(k_m, j)L E_j(\hat{K}, L)\left( S_{i,j}^d R(C(i,j)) \right),
\]
where \( \hat{K} = (k_1, \ldots, k_{m-1}) \). The inductive hypothesis implies
\[
E_j(\hat{K}, L)\left( S_{i,j}^d R(C(i,j)) \right) \equiv S_{i,K''}d'' \times \]
\[ \times \prod_{s=1}^\hat{r} \left( \left( C(i,j) - d + s - 1 + \sum_{s<h \leq m} l_h \right) \prod_{k_s < t < j} (C(i,t) - d + s - 1) \right) (3.1) \]
\[ \times R(C(i,j) + \Sigma L) \quad (\text{mod } \mathbb{I}_{\min K''} \ldots j) \]
where \( K'' = (k_1, \ldots, k_{m-1}, j^{d-m+1}) \) and \( \hat{r} \) is the number of entries of \( \hat{K} \) less than \( j \). Clearly, \( \min K'' \geq \min K' \). If \( k_m = j \) then \( E(k_m, j) = 1, \hat{r} = r \) and \( K'' = K' \). Hence the required result.

Now let \( k_m < j \). We have \( r = m \) and \( \hat{r} = m - 1 \). By Corollary \(^{2,3}\) we obtain
\[
E(k_m, j)S_{i,K''} \equiv S_{i,K'}(d - m + 1) \prod_{k_m < t \leq j} (C(i,t) - (d - m + 1)) \quad (\text{mod } \mathbb{I}_{[k_m \ldots j]}). \]
Using this formula and multiplying \(^{24,1}\) by \( E(k_m, j) \) on the left, we obtain the required formula. \( \square \)

Now we introduce the central object of our study.

**Definition 3.2.** Let \( 1 \leq i < j \leq n, d \geq 1 \) and \( R \in \mathbb{Z}(\zeta) \) We define
- \( T_{i,j}^{(d)}(\emptyset, R) = S_{i,j}^d R(C(i,j)) \);
- \( T_{i,j}^{(d)}(M, R) = \left( T_{i,j}^{(d)}(M', R) - S_{i,m}^d T_{m,j}^{(d)}(M', R) \right) C(i,m)^{-1} \), where \( m = \min M \) and \( M' = M \setminus \{m\} \), if \( M \) is a nonempty subset of \((i..j)\).

The elements \( T_{i,j}^{(d)}(M, R) \) belong to \( \overline{U}^{-0}(i, j) \).

**Lemma 3.3.** Let \( 1 \leq i < j \leq n, d \geq 1 \), \( M \subset (i..j), R \in \mathbb{Z}(\zeta), K = (k_1, \ldots, k_d) \) and \( L \) be sequences of integers such that \( i \leq k_1 \leq \cdots \leq k_d \leq j \) and \( |L| = d \) or \( |L| = d + 1 \). Take \( l \in (1..n) \setminus K \). Then modulo the ideal \( \mathbb{I}_{[i..j] \cup \{l-1\}} \) and even the ideal \( \mathbb{I}_{(l-1)} \) if \( K = (j^d) \), we have
\[
E_{l-1} E_j(K, L)\left( T_{i,j}^{(d)}(M, R) \right) \equiv 0
\]
if \( l \notin M \) or \( k_d < l \) and
\[
E_{l-1} E_j(K, L)\left( T_{i,j}^{(d)}(M, R) \right) \equiv
\]
\[ - E_l(K^{(l)}, \varepsilon_{a+1}, K^{(l)}) \left( T_{i,l}^{(d)} \left( M \cap (i..l), \frac{1}{\zeta - a} \right) \right) \times \]
\[ \times E_j(K^{(l)}, L) \left( T_{i,j}^{(d)}(M \cap (l..j), R) \right) \]
otherwise, where \( a \) is the number of entries of \( K \) less than \( l \).

**Proof.** Let \( P := \emptyset \) if \( K = (j^d) \) and \( P := [i..j] \) otherwise. We apply induction on \(|M|\). Let \( M = \emptyset \). By Lemma \(^{3,3}\) we have \( E_j(K, L)\left( T_{i,j}^{(d)}(\emptyset, R) \right) \equiv S_{i,K}H \quad (\text{mod } \mathbb{I}_P) \) for some \( H \in \overline{U}^0(n) \). Now the result follows from \(^{2,3}\) and \( l \notin K \).
Now let $M \neq \emptyset$ and the result holds for smaller sets. We put $m := \min M$ and $M' := M \setminus \{m\}$. By Definition 3.2 Proposition 2.10 and Lemma 2.12 we have

$$\mathbb{E}_j(K, L)(T_{i,j}^{(d)}(M, R)) \equiv \left( \mathbb{E}_j(K, L)(T_{i,j}^{(d)}(M', R)) - \right.$$

$$\left. \mathbb{E}_m(K^{(i,m)}, K^{(m)})(S_{i,m}^d) \cdot \mathbb{E}_j(K^{(m,j)}, L)(T_{m,j}^{(d)}(M', R)) \right) \cdot \mathbb{C}(i, m)^{-1} \quad (3.2)$$

We are going to multiply this equivalence by $\mathbb{E}_{l-1}$ on the left and apply the inductive hypothesis.

Case $l \not\in M$ or $k_d < l$. The inductive hypothesis implies

$$\mathbb{E}_{l-1} \mathbb{E}_j(K, L)(T_{i,j}^{(d)}(M', R)) \equiv 0 \pmod {\mathbb{P}_{l-1}}. \quad (3.3)$$

If $l - 1 \not\in [i..j]$ then by Proposition 2.11 and Lemma 2.14 we have

$$\mathbb{E}_{l-1} \mathbb{E}_m(K^{(i,m)}, K^{(m)}) (S_{i,m}^d) \times \mathbb{E}_j(K^{(m,j)}, L)(T_{m,j}^{(d)}(M', R)) \equiv 0 \pmod {\mathbb{P}_{l-1}}. \quad (3.4)$$

If $l - 1 \in [i..m]$ then $l \not\in K^{(i,m)}$. Therefore, the inductive hypothesis implies $\mathbb{E}_{l-1} \mathbb{E}_m(K^{(i,m)}, K^{(m)}) (S_{i,m}^d) \equiv 0$ modulo $\mathbb{P}_{(P \cap [i..m]) \cup (l-1)}$. If we multiply the last ideal by $\mathbb{E}_1(K^{(m,j)}, L)(T_{m,j}^{(d)}(M', R))$ on the right, then we obtain a subset of $\mathbb{P}_{l-1}$. Hence we get (3.4) again.

Finally assume $l - 1 \in [m..j]$. By proposition 2.11 and Lemma 2.14 we get

$$\mathbb{E}_{l-1} \mathbb{E}_m(K^{(i,m)}, K^{(m)}) (S_{i,m}^d) \cdot \mathbb{E}_j(K^{(m,j)}, L)(T_{m,j}^{(d)}(M', R)) \equiv$$

$$\equiv \mathbb{E}_m(K^{(i,m)}, K^{(m)})(S_{i,m}^d) \cdot \mathbb{E}_{l-1} \mathbb{E}_j(K^{(m,j)}, L)(T_{m,j}^{(d)}(M', R)) \cdot \mathbb{P}_{l-1}. \quad (3.5)$$

Clearly $l \not\in K^{(m,j)}$ and all entries of $K^{(m,j)}$ are less than $l$ if $k_d < l$. Therefore, by the inductive hypothesis $\mathbb{E}_{l-1} \mathbb{E}_j(K^{(m,j)}, L)(T_{m,j}^{(d)}(M', R)) \equiv 0$ modulo $\mathbb{P}_{l-1}$, whence we have (3.3) one more time.

Case $l = m$ and $k_d \geq l$. By the inductive hypothesis formula (3.3) holds. From Definition 2.9 it follows that

$$\mathbb{E}_{l-1} \mathbb{E}_m(K^{(i,m)}, K^{(m)})(S_{i,m}^d) = \mathbb{E}_m(K^{(i,m)}, K^{(m+1)})(S_{i,m}^d).$$

It remains to deal with $\mathbb{C}(i, m)^{-1}$. Firstly, we have

$$\mathbb{E}_j(K^{(m,j)}, L)(T_{m,j}^{(d)}(M', R)) \cdot \mathbb{C}(i, m)^{-1} =$$

$$(\mathbb{C}(i, m) - d + a)^{-1} \cdot \mathbb{E}_j(K^{(m,j)}, L)(T_{m,j}^{(d)}(M', R)).$$

Secondly, applying Proposition 2.10 and Proposition 2.14 we get

$$\mathbb{E}_m(K^{(i,m)}, K^{(m)}) (S_{i,m}^d) \cdot (\mathbb{C}(i, m) - d + a)^{-1}$$

$$= \mathbb{E}_m(K^{(i,m)}, K^{(m)}) (S_{i,m}^d)(\mathbb{C}(i, m) - d)^{-1}.$$
Case $l \in M'$ and $k_d \geq l$. Multiplying (3.2) by $E_{l-1}$ on the left and applying Proposition 2.11 and Lemma 2.8 we obtain

$$E_{l-1}E_j(K, L)\left( T^{(d)}_{i,j} (M, R) \right) \equiv$$

$$- \left( E_l(K^{(i,l)} - \varepsilon_{a+1, K^{(l)}}) \left( T^{(d)}_{i,l} \left( M' \cap (i..l), \frac{1}{\xi_d} \right) \right) \right) \times$$

$$- \left[ \theta_{m-l-m} \left( E_m(K^{(i,m)}, K^{(m)}) \left( T^{d}_{i,m} \left( M' \cap (m..l), \frac{1}{\xi_d} \right) \right) \right) \right] \times$$

$$\times E_j(K^{(l,j)}, L)\left( T^{(d)}_{l,j} (M \cap (l..j), R) \right) C(i, m)^{-1} \quad \text{(mod } \bar{P}_{l-j} l_{l-1} \text{)}.$$  

(3.6)

Applying Lemma 2.12 and Proposition 2.11 we get

$$E_l(K^{(i,l)} - \varepsilon_{a+1, K^{(l)}}) \left( S^{d}_{i,m} T^{(d)}_{m,l} \left( M' \cap (m..l), \frac{1}{\xi_d} \right) \right) \equiv$$

$$E_m(K^{(i,m)}, K^{(m)}) + \delta_{l-1=m} \varepsilon_{a+1} \left( S^{d}_{i,m} \right) \times$$

$$\times E_j(K^{(m,l)} - \varepsilon_{a+1, K^{(l)}}) \left( T^{(d)}_{m,l} \left( M' \cap (m..l), \frac{1}{\xi_d} \right) \right) \quad \text{(mod } \bar{P}_{l-j} l_{l-1} \text{)}.$$  

(3.7)

Moreover, Proposition 2.9 [ii] shows that both sides of (3.7) belong to $\bar{U}^{-d}(i, l) + \bar{S}_{l-j}(l-1)$. Finally, we observe

$$E_m(K^{(i,m)}, K^{(m)}) + \delta_{l-1=m} \varepsilon_{a+1} = \theta_{m-l-m} \circ E_m(K^{(i,m)}, K^{(m)}).$$  

(3.8)

Indeed, in the only nontrivial case $l-1 = m$, this formula follows from Definition 2.9 and the equalities $K^{(i,m)} = (k_1, \ldots, k_a, m^{l-a})$ and $K^{(m)} = (1^a, 0^{d-a})$. Therefore, the right-hand side of (3.7) is exactly the product in the square brackets of (3.6). Applying Lemma 2.8 [ii] we get

$$E_{l-1}E_j(K, L)\left( T^{(d)}_{i,j} (M, R) \right) \equiv$$

$$- E_l(K^{(i,l)} - \varepsilon_{a+1, K^{(l)}}) \left( T^{(d)}_{i,l} \left( M' \cap (i..l), \frac{1}{\xi_d} \right) \right) -$$

$$S^{d}_{i,m} T^{(d)}_{m,l} \left( M' \cap (m..l), \frac{1}{\xi_d} \right) \times$$

$$\times E_j(K^{(l,j)}, L)\left( T^{(d)}_{l,j} (M \cap (l..j), R) \right) \quad \text{(mod } \bar{P}_{l-j} l_{l-1} \text{)}.$$  

Applying Proposition 2.10 we carry $C(i, m)^{-1}$ (unchanged) under the sign of $E_l(K^{(i,l)} - \varepsilon_{a+1, K^{(l)}})$ and use Definition 3.2. □

Lemma 3.4. Let $1 \leq i < j \leq n$, $d \geq 1$, $M \subset (i..j)$, $R \in \mathbb{Z}(\zeta)$ and $K = (k_1, \ldots, k_d)$, $L = (l_1, \ldots, l_q)$ be sequences of integers such that $i \leq k_1 \leq \cdots \leq k_d \leq j$ and $q = d$ or $q = d+1$. Take $l \in (i..j) \cap K$. Then modulo $\bar{P}_{i,j}$, or even modulo the zero ideal if $l = j$, we have

$$E_{l-1}E_j(K, L)\left( T^{(d)}_{i,j} (M, R) \right) \equiv c E_j(K - \varepsilon_{a+1, K^{(l)}}) \left( T^{(d)}_{i,j} (M, R) \right).$$
if \( l \notin M \) and
\[
E_{l-1}E_j(K, L)\left( T^{(d)}_{i,j} (M, R) \right) \equiv cE_j(K - \varepsilon_{a+1}, L)\left( T^{(d)}_{i,l} (M, R) \right) + \\
\left[ (c-1)E_j(K^{(i,j)} - \varepsilon_{a+1}, K^{(l)}) \left( T^{(d)}_{i,l} \left( M \cap (i..l), \frac{1}{\zeta} \right) \right) \right] \times \\
\times E_j(K^{(l,j)}, L)\left( T^{(d)}_{i,j} (M \cap (l..j), R) \right)
\]
if \( l \in M \), where \( a \) is the number of entries of \( K \) less than \( l \); \( L' = L \) if \( l < j \) and \( L' = (l_1, \ldots, l_a, \sum_{a < h \leq q} l_h, 0^{q-a-1}) \) if \( l = j \); \( c \) is the number of entries of \( K \) equal to \( l \) if \( l < j \) and \( c = 1 \) if \( l = j \).

**Proof.** If \( l = j \) then Definition 2.9 implies
\[
E_j(K, L) = \theta_j \sum_{a<h<q} \circ E_j((k_1, \ldots, k_a), (l_1, \ldots, l_a)),
\]
whence the required result easily follows. Therefore we assume \( l < j \) for the rest of the proof. Let \( b \) and \( r \) denote the number of entries of \( K \) not greater than \( l \) and less than \( r \) respectively. We have \( c = b - a \) and \( L' = L \).

Now let \( M \neq \emptyset \) and suppose that the lemma is true for sets of lesser cardinality. We put \( m := \min M \) and \( M' := M \setminus \{ m \} \). Clearly, (3.2) holds in the present case, where \( P = [i..j] \).

**Case \( l \notin M \).** By the inductive hypothesis, we have
\[
E_{l-1}E_j(K, L)\left( T^{(d)}_{i,j} (M', R) \right) \equiv (b - a)E_j(K - \varepsilon_{a+1}, L)\left( T^{(d)}_{i,l} (M', R) \right) \quad (\text{mod } \mathbb{I}_{[i..j]}). \tag{3.9}
\]
We first consider the case \( i < l < m \). The inductive hypothesis and Proposition 2.11 yield
\[
E_{l-1}E_m(K^{(i,m)}, K^{(m)})\left( S^d_{i,m} \right) \equiv \\
(b - a)E_m((K - \varepsilon_{a+1})^{(i,m)}, (K - \varepsilon_{a+1})^{(m)})\left( S^d_{i,m} \right) \quad (\text{mod } \mathbb{I}_{[i..m]}).
\]
Applying Proposition 2.6(iii), Lemma 2.8(i) and Proposition 2.11(iii) we get
\[
E_{l-1}E_m(K^{(i,m)}, K^{(m)})\left( S^d_{i,m} \right) \circ E_j(K^{(m,j)}, L)\left( T^{(d)}_{m,j} (M', R) \right) \equiv \\
(b - a)E_m((K - \varepsilon_{a+1})^{(i,m)}, (K - \varepsilon_{a+1})^{(m)})\left( S^d_{i,m} \right) \times \\
\times E_j((K - \varepsilon_{a+1})^{(m,j)}, L)\left( T^{(d)}_{m,j} (M', R) \right) \quad (\text{mod } \mathbb{I}_{[i..j]}). \tag{3.10}
\]
Now we consider the case \( m < l < j \). In that case, (3.5) also holds, where \( P = [i..j] \) (i.e., the equivalence holds modulo \( \mathbb{I}_{[i..j]} \)). To rearrange the second factor of the right-hand side of (3.5), we apply the inductive hypothesis and Proposition 2.11. Thus we get
\[
E_{l-1}E_j(K^{(m,j)}, L)\left( T^{(d)}_{m,j} (M', R) \right) \equiv \\
(b - a)E_j((K - \varepsilon_{a+1})^{(m,j)}, L)\left( T^{(d)}_{m,j} (M', R) \right) \quad (\text{mod } \mathbb{I}_{[m..j]}).
\]
Since in the present case (3.8) also holds, we can apply this formula to the first factor of the right-hand side of (3.5). Finally, Proposition 2.11 shows that (3.11) is again true.
In both cases, (3.10) and Lemma 2.12 yield

\[
\begin{align*}
\mathbb{E}_{l-1} & \mathbb{E}_m(K^{(i,m)}, K^{(m)}) (S_{l,m}^d, \mathbb{E}_j(K^{(m,j)}), L) (T_{m,j}^{(d)} (M', R)) \\
& \equiv (b - a) \mathbb{E}_j(K - \varepsilon_{a+1}, L) (S_{l,m}^d, T_{m,j}^{(d)} (M', R)) \quad \text{(mod } \bar{I}_{[i-j]}). \numberthis (3.11)
\end{align*}
\]

Therefore, multiplying (3.2) on the left by \( \mathbb{E}_{l-1} \), using (3.9) and (3.11), Proposition 2.11(ii) to carry \( C(i, m)^{-1} \) under the sign of \( \mathbb{E}_j(K - \varepsilon_{a+1}, L) \), and Definition 3.2 we obtain the required result.

Case \( l = m \). The inductive hypothesis implies that (3.9) holds. We introduce the auxiliary notation \( K' := (1^a, b - a, 0^{d-a-1}) \) and

\[
X := (b - a) \mathbb{E}_m((K - \varepsilon_{a+1})^{(i,m)}, (K - \varepsilon_{a+1})^{(m)}) (S_{l,m}^d) \\
- \mathbb{E}_m(K^{(i,m)} - \varepsilon_{a+1}, K') (S_{l,m}^d).
\]

It follows from Proposition 2.11(ii) that \( X \in \bar{U}^{-d}(i, m) \oplus \bar{I}_{[i,m]} \). With regard to Proposition 2.11(iii) the inductive hypothesis implies

\[
\mathbb{E}_{l-1} \mathbb{E}_m(K^{(i,m)}, K^{(m)}) (S_{l,m}^d) = \mathbb{E}_m(K^{(i,m)} - \varepsilon_{a+1}, K') (S_{l,m}^d). \numberthis (3.12)
\]

Therefore, multiplying (3.2) on the left by \( \mathbb{E}_{l-1} \), using (3.9) and (3.12), and applying Proposition 2.11(iii) we get

\[
\begin{align*}
\mathbb{E}_{l-1} & \mathbb{E}_j(K, L) (T_{i,j}^{(d)} (M, R)) \equiv (b - a) \mathbb{E}_j(K - \varepsilon_{a+1}, L) (T_{i,j}^{(d)} (M', R)) - \\
& \mathbb{E}_m(K^{(i,m)} - \varepsilon_{a+1}, K') (S_{l,m}^d) \cdot \mathbb{E}_j(K^{(m,j)}, L) (T_{m,j}^{(d)} (M', R)) C(i, m)^{-1} = \\
& (b - a) \left( \mathbb{E}_j(K - \varepsilon_{a+1}, L) (T_{i,j}^{(d)} (M', R)) - \\
& \left[ \mathbb{E}_m(K^{(i,m)} - \varepsilon_{a+1})^{(i,m)}, (K - \varepsilon_{a+1})^{(m)} (S_{l,m}^d) \right] \times \\
& \times \mathbb{E}_j((K - \varepsilon_{a+1})^{(m,j)}, L) (T_{m,j}^{(d)} (M', R)) \right) C(i, m)^{-1} \\
& + X \cdot \mathbb{E}_j(K^{(m,j)}, L) (T_{m,j}^{(d)} (M', R)) C(i, m)^{-1} \quad \text{(mod } \bar{I}_{[i,j]}).
\end{align*}
\]

By Lemma 2.12 the product in the square brackets of the above formula equals \( \mathbb{E}_j(K - \varepsilon_{a+1}, L) (S_{l,m}^d, T_{m,j}^{(d)} (M', R)) \) modulo \( \bar{I}_{[i-j]} \). Thus we obtain

\[
\begin{align*}
\mathbb{E}_{l-1} & \mathbb{E}_j(K, L) (T_{i,j}^{(d)} (M, R)) \equiv (b - a) \mathbb{E}_j(K - \varepsilon_{a+1}, L) (T_{i,j}^{(d)} (M, R)) \\
& + X \cdot \mathbb{E}_j(K^{(m,j)}, L) (T_{m,j}^{(d)} (M', R)) C(i, m)^{-1} \quad \text{(mod } \bar{I}_{[i,j]}). \numberthis (3.13)
\end{align*}
\]

Let us calculate \( X \) modulo \( \bar{I}_{[i,m]} \). By Lemma 3.1 we have

\[
\begin{align*}
X & \equiv S_{K^{(i,m)} - \varepsilon_{a+1}}^{a+1} \prod_{s=1, k_s < t < m} (C(i, t) - d + s - 1) \times \\
& \times (C(i, m) - d + b - 1)^a ((b - a)(C(i, m) - d + b - 1) - (C(i, m) - d + a)) \equiv \numberthis (3.14)
\end{align*}
\]

\[
(b - a - 1) \mathbb{E}_m(K^{(i,m)} - \varepsilon_{a+1}, K^{(m)}) (S_{l,m}^d) \cdot \frac{C(i,m) - d + b}{C(i,m) - d + b - 1} \quad \text{(mod } \bar{I}_{[i,m]}).\]
To rearrange the second summand of the right-hand side of (3.13), we apply Proposition 2.1(ii), equivalence (3.14), Lemma 2.8(iv) and Propositions 2.10 and 2.7(iv). Thus we obtain

\[
X \cdot E_j(K^{(m,j)}, L)\left(\mathbb{T}^{(d)}_{m,j}(M', R)\right) \cap (i, m)^{-1}
\]

\[
= X(\mathbb{C}(i, m) - d + b)^{-1}E_j(K^{(m,j)}, L)(\mathbb{T}^{(d)}_{m,j}(M', R))
\]

\[
\equiv (b - a - 1)E_m(K^{(i,m)} - \varepsilon_{a+1}, K^{(m)})\left(\mathbb{S}^d_{i,m}\right)(\mathbb{C}(i, m) - d + b - 1)^{-1} \times
\]

\[
\times E_j(K^{(m,j)}, L)(\mathbb{T}^{(d)}_{m,j}(M', R))
\]

\[
= (b - a - 1)E_m(K^{(i,m)} - \varepsilon_{a+1}, K^{(m)})\left(\mathbb{T}^{(d)}_{i,m}\left(\varnothing, \frac{1}{\zeta - d - 1}\right)\right) \times
\]

\[
\times E_j(K^{(m,j)}, L)(\mathbb{T}^{(d)}_{m,j}(M', R)) \pmod {\mathbb{I}_{(i,j)}}.
\]

This formula together with (3.13) yield the required result.

Case \(l \in M'\). By the inductive hypothesis, we have

\[
E_{l-1}E_j(K, L)(\mathbb{T}^{(d)}_{i,j}(M', R)) \equiv (b - a)E_j(K - \varepsilon_{a+1}, L)(\mathbb{T}^{(d)}_{i,j}(M', R)) +
\]

\[
(b - a - 1)E_l(K^{(i,l)} - \varepsilon_{a+1}, K^{(l)})\left(\mathbb{T}^{(d)}_{i,l}\left(M' \cap (i..l), \frac{1}{\zeta - d - 1}\right)\right) \times
\]

\[
\times E_j(K^{(l,j)}, L)(\mathbb{T}^{(d)}_{m,j}(M \cap (l..j), R)) \pmod {\mathbb{I}_{(m,j)}}.
\]

Multiplying (3.2) by \(E_{l-1}\) on the left and taking into account (3.6) together with the above equivalences, we obtain

\[
E_{l-1}E_j(K, L)(\mathbb{T}^{(d)}_{i,j}(M, R)) \equiv (b - a)\left(E_j(K - \varepsilon_{a+1}, L)(\mathbb{T}^{(d)}_{i,j}(M', R)) -
\right.
\]

\[
\left.\left[\theta^{l-1-m}_m\left(E_m(K^{(i,m)}, K^{(m)})\mathbb{S}^d_{i,m}\right)E_j(K^{(m,j)} - \varepsilon_{a+1}, L)(\mathbb{T}^{(d)}_{m,j}(M', R))\right]\right) \times
\]

\[
\times \mathbb{C}(i, m)^{-1} + (b - a - 1)E_l(K^{(i,l)} - \varepsilon_{a+1}, K^{(l)})\left(\mathbb{T}^{(d)}_{i,l}\left(M' \cap (i..l), \frac{1}{\zeta - d - 1}\right)\right) \times
\]

\[
\times E_j(K^{(l,j)}, L)(\mathbb{T}^{(d)}_{m,j}(M \cap (l..j), R)) \pmod {\mathbb{I}_{(i,j)}}.
\]

Applying (3.5), Proposition 2.11 and Lemma 2.12, we obtain that the product in the first pair of the square brackets of the above formula equals \(E_j(K - \varepsilon_{a+1}, L)(\mathbb{S}^d_{i,m} \mathbb{T}^{(d)}_{m,j}(M', R)) \pmod {\mathbb{I}_{(i,j)}}\) and the product in the second pair of the square brackets equals \(E_l(K^{(i,l)} - \varepsilon_{a+1}, K^{(l)})\left(\mathbb{S}^d_{i,m} \mathbb{T}^{(d)}_{m,i}(M' \cap (i..l), \frac{1}{\zeta - d - 1}\right)\right) \times
\]

\[
\times E_j(K^{(l,j)}, L)(\mathbb{T}^{(d)}_{m,j}(M \cap (l..j), R)) \pmod {\mathbb{I}_{(i,j)}}.
\]
\[
\cap (m,i), \frac{1}{m-i-1} \right) \text{ modulo } \mathbb{I}(n,i). \text{ Now application of Lemma } \text{2.10} \text{ and Definition } \text{3.2} \text{ concludes the proof.} \]

\section{Coefficients}

\textbf{Proposition 4.1.} Let \(1 \leq i \leq j \leq n, c_1, \ldots, c_{n-1}\) be nonnegative integers and \(I\) denote the left ideal of \(\mathbb{U}(n)\), generated by \(E_t^{(c_t+1)} U^+(n)\) for \(t \in [i..j]\). Assuming additionally \(c_0 = 0\), we have
\[
\left( \prod_{t=1}^{n-1} E_t^{(c_t+\delta \in [i..j])} \right) S_{i,j} = \left( \prod_{t=[i..j]} \left( B(i,t) + c_{t-1} - c_t \right) \right) E_1^{(c_1)} \cdots E_n^{(c_{n-1})} \pmod{I}.
\]

\textbf{Proof.} This equivalence was actually proved in the course of the proof of [BKS] Proposition 4.5. \(\Box\)

\textbf{Proposition 4.2.} Let \(1 \leq i_1 \leq j_1 \leq n, \ldots, 1 \leq i_m \leq j_m \leq n\) and \(c_1, \ldots, c_{n-1}\) be nonnegative integers. We put \(c_0 := 0, c_t := c_t + \delta \in [i_{m+1}, j_{m+1}] + \cdots + \delta \in [i_{m+j}, j_m]\) for \(t = 0, \ldots, n-1\) and \(C := (c_1, \ldots, c_{n-1})\). We have
\[
\left( \prod_{t=1}^{n-1} E_t^{(c_t)} \right) S_{i_1,j_1} \cdots S_{i_m,j_m}
\]
\[
= \left( \prod_{s=1}^{m} \left( B(i_s,t) + c_{i_s-1}^{(s)} - c_{i_s}^{(s)} \right) \right) \left( \prod_{t=1}^{n-1} E_t^{(c_t)} \right) \pmod{\mathbb{J}(C)}.
\]

\textbf{Proof.} We apply induction on \(m\). Since the case \(m = 0\) is obvious, we assume that \(m > 0\). The inductive hypothesis implies
\[
\left( \prod_{t=1}^{n-1} E_t^{(c_t)} \right) S_{i_1,j_1} \cdots S_{i_m,j_m}
\]
\[
= \left( \prod_{s=1}^{m} \left( B(i_s,t) + c_{i_s-1}^{(s)} - c_{i_s}^{(s)} \right) \right) \left( \prod_{t=1}^{n-1} E_t^{(c_t)} \right) \pmod{\mathbb{J}(C)}.
\]

On the other hand, Proposition 1.1 implies
\[
\left( \prod_{t=1}^{n-1} E_t^{(c_t)} \right) S_{i_1,j_1} = \left( \prod_{t=[i_1..j_1]} \left( B(i_1,t) + c_{i_1-1}^{(1)} - c_{i_1}^{(1)} \right) \right) \left( \prod_{t=1}^{n-1} E_t^{(c_t)} \right) \pmod{I},
\]
where \(I\) is the left ideal of \(\mathbb{U}(n)\) generated by \(E_t^{(c_0)} U^+(n)\) for \(t \in [i_1..j_1]\). Finally, it remains to notice the inclusion \(\mathbb{I}_{[i_2,j_2]} \cdots \mathbb{I}_{[i_m,j_m]} \subset \mathbb{J}(C)\). \(\Box\)

For a sequence \(C = (c_1, \ldots, c_{n-1})\) of integers and integers \(k, i, t\) such that \(1 \leq i, t < n\), we put for brevity \(B^C(k,i,t) := B(i,t) + c_{i-t} - c_i + \delta_{i \geq k}(c_{i+1} - c_t)\), where \(c_0 = c_n = 0\).

To formulate the next lemma, we need to introduce rational expressions similar to \(E_{r,s}(M)\) of [K, Definition 2.9]. For any integers \(i, j\), sequences of integers \(C = (c_1, \ldots, c_{n-1})\), \(K = (k_1, \ldots, k_d)\) \(L = (l_1, \ldots, l_q)\), a subset \(M \subset (i..j)\) and a rational expression \(R \in \mathbb{Z}(\zeta)\) such that \(1 \leq i < j \leq n, i \leq k_1 \leq \cdots \leq k_d \leq j\) and \(q = d\) or \(q = d + 1\), we define \(\rho^{(C)}(i, j, K, L, M, R) \in \mathbb{U}^0(n)\) inductively on \(|M|\) as follows, additionally assuming \(c_0 = 0\) and \(c_n = 0\).
Case $M = \emptyset$. We put

$$\rho^{(C)}(i, j, K, L, \emptyset, R) := \prod_{s=1}^{d} \left( \prod_{t \in [i..j]} (d-s+1) \delta_{t=m-1 \geq k_s} \left( \mathbb{B}^{C,k_s}(i, t) - d + s + \delta_{t=j-1 \geq k_s} \cdot \sum_{h \leq d} b_h \right) \right) R \left( C(i, j) + c_{i-1} - c_i - c_{j-1} + c_j + \Sigma L \right).$$  \hspace{1cm} (4.1)

Case $M \neq \emptyset$. Let $m := \min M$ and $M' := M \setminus \{m\}$. We put

$$\rho^{(C)}(i, j, K, L, M, R) := \left( \rho^{(C)}(i, j, K, L, M', R) - \zeta^{(C)}(i, m, K) \right) \times \rho^{(C)}(m, j, K^{(m,j)}, L, M', R) \left( \mathbb{C}(i, m) + c_{i-1} - c_i - c_{m-1} + c_m \right)^{-1},$$

where $\zeta^{(C)}(i, m, K) := \prod_{s=1}^{d} \prod_{t \in [i..m)} (d-s+1) \delta_{t=m-1 \geq k_s} \times \left( \mathbb{B}^{C,k_s}(i, t) + (\delta_{t<m-1} + \delta_{t=m-1 < k_s})(s-d) \right) \zeta^{(C)}_{\min K, j)}$.

The reason for introducing these expressions is explained as follows.

**Lemma 4.3.** Let $1 \leq i < j \leq n$, $d \geq 1$, $M \subseteq (i..j)$ and $R \in \mathbb{Z}(\zeta)$. Let $K = (k_1, \ldots, k_d)$, $L = (l_1, \ldots, l_q)$ and $C = (c_1, \ldots, c_{n-1})$ be sequences of integers such that all entries of $C$ are nonnegative, $i \leq k_1 \leq \ldots \leq k_d \leq j$ and $q = d$ or $q = d + 1$. We put $c_t := c_t + \sum_{s=1}^{d} \delta_{t \in [i..k_s)}$. Modulo $\mathbb{J}(C)^{+} + \mathbb{I}_{\min K, j)}$, we have

$$\left( \prod_{t=1}^{n-1} \mathbb{E}_t^{(c_t)} \right) \mathbb{E}_j(K, L) \left( \mathbb{T}_{i,j}^{(d)}(M, R) \right) \equiv \rho^{(C)}(i, j, K, L, M, R) \left( \prod_{t=1}^{n-1} \mathbb{E}_t^{(c_t)} \right).$$

**Proof.** We apply induction on $|M|$. By Lemma 3.1, we have

$$\mathbb{E}_j(K, L) \left( \mathbb{T}_{i,j}^{(d)}(\emptyset, R) \right) \equiv \mathbb{S}_{i,K} \mathbb{E}^{d} \times$$

$$\times \prod_{s=1}^{r} \left( \mathbb{B}(i, j-1) - d + s + \sum_{h \leq d} b_h \prod_{t \in [k_s, j-1)} \mathbb{B}(i, t) - d + s \right) \times \times R(\mathbb{C}(i, j) + \Sigma L) \pmod{\mathbb{I}_{\min K, j)},$$

where $r$ is the number of entries of $K$ less than $j$. Now Proposition 4.2 implies

$$\left( \prod_{t=1}^{n-1} \mathbb{E}_t^{(c_t)} \right) \mathbb{S}_{i,K} \equiv \prod_{s=1}^{d} \prod_{t \in [i..k_s)} \left( \mathbb{B}(i, t) + c_{i-1} - (c_i + q_s) \right) \times$$

$$\times \left( \prod_{t=1}^{n-1} \mathbb{E}_t^{(c_t)} \right) \pmod{\mathbb{J}(C)},$$

where $q_s$ is the number of elements of the sequence $k_{s+1}, \ldots, k_d$ that are greater than $i$. Let $a$ denote the number of entries of $K$ equal to $i$. If $s \leq a$
then the product in the square brackets of \((13)\) is empty. However, if \(s > a\) then \(q_s = d - s\). Thus \((13)\) allows the following reformulation
\[
\left( \prod_{t=1}^{n-1} E_t^{(c_t)} \right) S_{i,K} \equiv \left( \prod_{s=1}^{d} \prod_{t \in [i..k_s]} (B(i,t) + c_{i-1} - c_i - d + s) \right) \times \\
\left( \prod_{t=1}^{n-1} E_t^{(c_t)} \right) \pmod{\mathbb{J}(C)},
\]
Now throwing the corresponding elements of \(\bar{U}^0(n)\) over \(\prod_{t=1}^{n-1} E_t^{(c_t)}\) and applying \((11)\), we obtain the required formula.

Now suppose that \(M \neq \emptyset\) and that the lemma is true for sets of smaller cardinality. We put \(m := \min M\) and \(M' := M \setminus \{m\}\). In the present case, equivalence \((5.2)\) also holds, where \(P = \bar{I}_{[\min K..j]}\). The inductive hypothesis implies
\[
\left( \prod_{t=1}^{n-1} E_t^{(c_t)} \right) E_j(K, L) (T_{i,j}^{(d)} (M', R)) \equiv \\
\rho(C) (i, j, K, L, M', R) \left( \prod_{t=1}^{n-1} E_t^{(c_t)} \right) \pmod{\mathbb{J}(C) + \bar{I}_{[\min K..j]}}.
\]
We put \(\hat{c}_t := c_t + \sum_{s=1}^{d} \delta_{t \in [m..k_s]}\) and \(\hat{C} := (\hat{c}_1, \ldots, \hat{c}_{n-1})\). Clearly, \(\hat{c}_t = \hat{c}_t + \sum_{s=1}^{d} \delta_{t \in [i..k_s]}\). Now the inductive hypothesis implies
\[
\left( \prod_{t=1}^{n-1} E_t^{(c_t)} \right) E_{m}(K^{(i,m)}, K^{(m)}) (S_{i,m}) \equiv \\
\rho(\hat{C}) (i, m, K^{(i,m)}, K^{(m)}, \emptyset, 1) \left( \prod_{t=1}^{n-1} E_t^{(c_t)} \right) \pmod{\mathbb{J}(\hat{C}) + \bar{I}_{[\min K..m]}}.
\]
One can easily observe that \(\rho(\hat{C})(i, m, K^{(i,m)}, K^{(m)}, \emptyset, 1) = \zeta(C)(i, m, K)\).
(the last element was introduced exactly for this equality). Since by the inductive hypothesis we have
\[
\left( \prod_{t=1}^{n-1} E_t^{(c_t)} \right) E_{j}(K^{(m,j)}, L) (T_{m,j}^{(d)} (M', R)) \equiv \\
\rho(C) (m, j, K^{(m,j)}, L, M', R) \left( \prod_{t=1}^{n-1} E_t^{(c_t)} \right) \pmod{\mathbb{J}(C) + \bar{I}_{[\min K..j]}},
\]
it remains to notice the following quite obvious formulae
\[
\bar{I}_{[\min K..m]} E_{j}(K^{(m,j)}, L) (T_{m,j}^{(d)} (M', R)) \in \bar{I}_{[\min K..j]} \\
\mathbb{J}(\hat{C}) \mathbb{J}(\hat{C}) (i, m, K^{(i,m)}, K^{(m)}, \emptyset, 1) \in \mathbb{J}(C). \quad \Box
\]

Lemma 4.4. Let \(X \in \mathbb{U}^{-0}(n)\) be an element of weight \(\sigma = -m_1 a_1 - \cdots - m_{n-1} a_{n-1}\) and \(C = (c_1, \ldots, c_{n-1})\) be a sequence of nonnegative integers. Then there exists some \(\rho \in \mathbb{U}^0(n)\) such that
\[
\left( \prod_{t=1}^{n-1} E_t^{(c_t + m_1)} \right) X \equiv \rho \left( \prod_{t=1}^{n-1} E_t^{(c_t)} \right) \pmod{\mathbb{J}(C)}.
\]
Proof. Take any integers \( t = 1, \ldots, n - 1 \) and \( i, j, r, c \) such that \( 1 \leq i < j \leq n \) and \( r, c \geq 0 \). Using (2.2), we see that \( E_r^{(c)\epsilon_{i,j}} = \sum_{s \geq 0} F_s E^{(c-s)}_t \), where \( F_s \) is an element of \( \mathbb{U}^{-0}(n) \) having weight \(-r\alpha(i,j) + s\alpha_t\).

Now notice that it suffices to prove the lemma for \( X = \mathbb{P}(N) \), where \( N \in UT(n) \). Applying the formula we have just proved, we obtain
\[
\left( \prod_{t=1}^{n-1} E_t^{(\alpha+mu_t)} \right) \mathbb{P}(N) = \sum_{s_1, \ldots, s_{n-1} \geq 0} F_{s_1, \ldots, s_{n-1}} \left( \prod_{t=1}^{n-1} E_t^{(\alpha+mu_t-s_t)} \right),
\]
where \( F_{s_1, \ldots, s_{n-1}} \) is an element of \( \mathbb{U}^{-0}(n) \) having weight \( \sigma + s_1\alpha_1 + \cdots + \alpha_{n-1} \). In particular, \( F_{s_1, \ldots, s_{n-1}} = 0 \) if some \( s_t > m_t \) and \( F_{m_1, \ldots, m_{n-1}} \) belongs to \( \mathbb{U}^{0}(n) \), since this element has weight zero. The proof concludes with noticing that \( \prod_{t=1}^{n-1} E_t^{(\alpha+mu_t-s_t)} \in \mathbb{P}(C) \) if some \( s_t < m_t \).

The next lemma follows directly from the definition of \( \rho^{(C)}(i, j, K, L, M, R) \).

Lemma 4.5. If the entries at positions \( i-1, \ldots, j \) of \( C \) and \( C' \) coincide, then \( \rho^{(C)}(i, j, K, L, M, R) = \rho^{(C')}(i, j, K, L, M, R) \). Let \( a \in \mathbb{Z}, j < n, L' = L + a \mathbb{C} \) if \( |L| = d \) and \( L' = L + a \mathbb{E}_{d+1} \) if \( |L| = d + 1 \). We have
\[
\rho^{(C+a\mathbb{E})}(i, j, K, L, M, R) = \rho^{(C)}(i, j, K, L', M, R).
\]

5. Integral elements

In this section, for \( N \in UT(n) \) we denote by \( N_t \) the sum \( \sum_{1 \leq a \leq n} N_{a, t} \) (the sum of elements in column \( t \) of \( N \)).

Proposition 5.1 ([CL 2.9]). Let \( 1 \leq i \leq j \leq n \) and \( d \geq 1 \). Then we have
\[
S_{i,j}^d = \sum_N \mathbb{P}^{(N)} \prod_{i < t < j} N_t! \mathbb{C}(i, t)^{d-N_t},
\]
where the summation runs over all \( N \in UT(n) \) such that \( \mathbb{P}^{(N)} \) has weight \(-d\alpha(i, j)\).

In other words, the summation runs over all \( N \in UT(n) \) such that \( \sum_{1 \leq a \leq t < b \leq n} N_{a, b} = \delta_{i < t < j} \) for any \( t = 1, \ldots, n - 1 \). Clearly, if \( i < j \) then \( N_j = d \) for any such \( N \).

Let \( X \in \mathbb{U}^{-0}(n) \). By Lemma 2.2 we have a unique representation \( X = \sum_{N \in UT(n)} \mathbb{P}^{(N)} H_N \), where \( H_N \in \mathbb{U}^{0}(n) \). In that case, \( H_N \) is called the \( \mathbb{P}^{(N)} \)-coefficient of \( X \). The proof of the following lemma is similar to that of [K] Lemma 2.4.

Lemma 5.2. For any integers \( 1 \leq i < j \leq n \) and \( d \geq 1 \) and set \( M \subset (i, j) \), we have \( T_{i,j}^{(d)}(M, 1) \in \mathbb{U}^{-0}(i, j)[\mathbb{U}^{0}(i, j-1)] \).

Proof. We fix \( d \geq 1 \). For any matrix \( N \in UT(n) \) and any nonempty set \( M \) such that \( \mathbb{P}^{(N)} \) has weight \(-d\alpha(i, j)\) and \( M \subset (i, j) \), where \( i \) and \( j \) are some (uniquely determined) integers, we define the polynomial \( P_{N,M} \in \mathbb{U}^{0}(\min M, j-1)[x] \). We require the two conditions
\[
(1) P_{N,M} = P_{K,M} \text{ if } N_{a,b} = K_{a,b} \text{ for all } 1 \leq a \leq n \text{ and } \min M \leq b \leq n;
\]
(2) for any $1 \leq i < j \leq n$, nonempty $M \subset (i..j)$ and $N \in UT(n)$ such that $F(N)$ has weight $-d\alpha(i, j)$, the $F(N)$-coefficient of $T_{i,j}^{(d)}(M, 1)$ is

$$d! \left( \prod_{i<t<m\in M} N! [C(i, t) \frac{d-N_t}{d}] \right) P_{N, M}(C(i, \min M)).$$

In view of Proposition 5.1, constructing such polynomials will automatically prove the lemma. Indeed, if $M = \emptyset$ then $T_{i,j}^{(d)}(M, 1) = S_{i,j}^d \in U^-(i, j)U^0(i, j - 1)$ by Definition 3.2 and Proposition 5.1. If $M \neq \emptyset$ then by condition (2) the $F(N)$-coefficient of $T_{i,j}^{(d)}(M, 1)$ is an integer polynomial in $H_i, \ldots, H_{j-1}$, that is an element of $U^0(i, j - 1)$.

We apply induction on $|M| > 0$. Put $m := \min M, M' := M \setminus \{m\}$ and $m' := \min M'$ if $M'$ is not empty. We take a matrix $N \in UT(n)$ such that $F(N)$ has weight $-d\alpha(i, j)$ for some $i$ and $j$ with $M \subset (i..j)$. We are going to define $P_{N, M}$ so that condition (2) holds.

Case $N_m < d$. Since the $F(N)$-coefficient of $S_{i,m}^d T_{m,j}^{(d)}(M', 1)$ equals zero in this case, we can define

$$P_{N, M} := N_m! x^{d-N_m} \left( \prod_{m < t < m'} N_t!(x + C(m, t))^{d-N_t} \right) P_{N, M'}(x + C(m, m'))$$

if $M' \neq \emptyset$ and

$$P_{N, M} := N_m! x^{d-N_m} \left( \prod_{m < t < m'} N_t!(x + C(m, t))^{d-N_t} \right)$$

if $M' = \emptyset$.

Case $N_m = d$. We have the decomposition $N = N' + N''$, where $N', N'' \in UT(n), N'_{a,b} = \delta_{a\leq m} N_{a,b}$ and $N''_{a,b} = \delta_{a>m} N_{a,b}$. This decomposition follows from $\sum_{a<m-1<b} N_{a,b} = d$ and the fact that all entries of $N$ are nonnegative. Thus we have $F(N) = F(N')F(N'')$. If $M' \neq \emptyset$ then by condition (1) and the inductive hypothesis we get $P_{N, M'} = P_{N', M'}$. Therefore, we can define

$$P_{N, M} := \frac{d!}{x} \left( \prod_{m < t < m'} N_t!(x + C(m, t))^{d-N_t} \right) P_{N, M'}(x + C(m, m'))$$

if $M' \neq \emptyset$ and

$$P_{N, M} := \frac{d!}{x} \left( \prod_{m < t < m'} N_t!(x + C(m, t))^{d-N_t} \right) - \left( \prod_{m < t < m'} N_t!(x + C(m, t))^{d-N_t} \right)$$

if $M' = \emptyset$. One can easily see that $P_{N, M} \in U^0(\min M, j - 1)[x]$. Moreover, condition (1) inductively follows from the formulae defining $P_{N, M}$. □

Each commutative ring $U^0(a, b)$ is generated by elements $H_i, \ldots, H_b$ freely over $Z$. Therefore, $U^0(a, b)$ is isomorphic to $Z[x_a, \ldots, x_b]$ and is a UFD. This fact is used below.
Corollary 5.3. In Lemmas 5.3 and 3.4, the expressions in square brackets belong to $U^{-0}(i, l) \oplus \mathbb{F}_{\min K, l} \cup (l-1)$, whenever the corresponding case occurs.

Proof. Denote this expression by $\bar{X}$ and let $X$ be the element of $\bar{U}^{-0}(i, l)$ such that $X \equiv X \pmod{\mathbb{F}_{\min K, l} \cup (l-1)}$. Without loss of generality we can assume that $R = 1$ and $M \cap (l, j) = \emptyset$. Let $\bar{Y} := \mathbb{E}(K^{(l, j)}, L)((T^{(d)}_{i,j}) (\emptyset, 1))$. Thus $\bar{Y}$ is the second factor in the product where $\bar{X}$ occurs. Let $Y$ be the element of $\bar{U}^{-0}(i, j)$ such that $\bar{Y} \equiv Y \pmod{\mathbb{F}_{[l, j]}}$. By Lemma 4.3, we have $Y \neq 0$, since $\rho^{(n-1)}(l, j, K^{(l, j)}, L, \emptyset, 1) \neq 0$. On the other hand, $Y \in U^{0, \ast}(l, j)$. By Lemma 5.2, applying Lemma 3.3 or Lemma 3.4, respectively, we get $XY \in U^{0, \ast}(i, j)$. Let $D(i, l)$ denote the set of all products (including empty) of elements of the form $C(s, t) + N$, where $i \leq s < t \leq j$ and $N \in \mathbb{Z}$.

Choose some $N \in UT(n)$ such that the $\mathbb{F}^{(N)}$-coefficient $H_N$ of $Y$ is not equal to zero. Let $M \in UT(n)$ be a matrix such that the $\mathbb{F}^{(M)}$-coefficient $H_M$ of $X$ is not equal to zero. We have $\mathbb{F}^{(M)} \mathbb{F}^{(N)} = \mathbb{F}^{(M+N)}$. Thus the $\mathbb{F}^{(M+N)}$-coefficient of $XY$ is $\tau_N(H_M) H_N$ by the first formula of (2.2) and the remark before Lemma 2.2. We have $H_N \in U^0(l, j)$, $\tau_N(H_M) H_N \in U^0(i, j)$, $H_M = h/f$ for some $h \in U^0(i, l)$ and $f \in D(i, l)$. The above representation for $H_M$ is derived from Definition 2.2 and Proposition 2.4. We have $\tau_N(H_M) = \tau_N(h)/\tau_N(f)$, $\tau_N(h) \in U^0(i, l)$ and $\tau_N(f) \in D(i, l)$. Therefore, $\tau_N(f)$ divides $\tau_N(h) H_N$ in the ring $U^0(i, j)$. Since $H_N \in U^0(l, j)$, the prime decomposition of $H_N$ can not contain factors of the form $C(s, t) + N$, where $i \leq s < t \leq l$. Therefore, $\tau_N(H_M) = \tau_N(h)/\tau_N(f) \in U^0(i, l)$, whence $H_M \in U^0(i, l)$. To prove the latter, one needs to apply the inverse map to $\tau_N$.

In the next lemma, we use the following notation: $C = (c_1, \ldots, c_{n-1})$, $c_0 = c_n = 0$, $K = (k_1, \ldots, k_d)$, $L = (l_1, \ldots, l_q)$, where $d \geq 1$ and $q = d$ or $q = d + 1$.

Lemma 5.4. We have $\rho^{(C)}(i, j, K, L, M, R) \in U^0(i, j)$ if $R = 1$ or if $R = 1/((\zeta - d)$, $K \neq (j^d)$ and $l_1 = 0$.

Proof. We shall only sketch the proof, leaving technical details to the reader. For $m \in (i, j)$, let $\sigma_m$ be the endomorphism of the ring $U^0(i, j)$ defined by $\sigma_m(\mathbb{H}_t) := i - m + \mathbb{H}_m - c_{i-1} + c_i + c_{m-1} - c_m$ and $\sigma_m(\mathbb{H}_t) := \mathbb{H}_t$ for $t \neq i$. We shall prove inductively on $|M|$ the two facts

(i) $\rho^{(C)}(i, j, K, L, M, R) \in U^0(i, j)$;
(ii) $\sigma_m(\rho^{(C)}(i, j, K, L, M, R) - \zeta^{(C)}(i, m, K) \rho^{(C)}(m, j, K^{(m, j)}, L, M, R)) = 0$ for any $m \in (i, \min M \cup \{j\})$.

The case $M = \emptyset$ follows from the definition of $\rho^{(C)}(i, j, K, L, \emptyset, R)$ and direct calculations.

Now let $M \neq \emptyset$, $m' = \min M$ and $M' = M \setminus \{m'\}$. To prove condition (i), we first note that by the inductive hypothesis the element

$$\rho^{(C)}(i, j, K, L, M', R) - \zeta^{(C)}(i, m, K) \rho^{(C)}(m, j, K^{(m, j)}, L, M', R)$$

belongs to $U^0(i, j)$. Thus it can be considered as a polynomial in $\mathbb{H}_t$ over $U^0(i+1, j)$. Applying $\sigma_{m'}$ to (5.1) and using the inductive hypothesis, we obtain that $i - m' + \mathbb{H}_{m'} - c_{i-1} + c_i + c_{m'-1} - c_{m'}$ is a root of this polynomial. Therefore, it is divisible by $\mathbb{C}(i, m') + c_{i-1} - c_i - c_{m'-1} + c_{m'}$.
and the coefficients of the quotient belong to \( U^0(i+1, j) \). The required result follows now from the definition of \( \rho^{(C)}(i, j, K, L, M, R) \). Condition \( [ii] \) can be checked by direct calculations.

**Proposition 5.5.** Let \( f_1, \ldots, f_a \) be first degree polynomials of \( U^0(i, j) \) having the form \( f_h = m_m + g_h \), where \( g_h \) is a \( \mathbb{Z} \)-linear combination of the unit and the variables \( \mathbb{H}_t \) for \( t > m_h \) and \( i < m_1 < \cdots < m_a < j \). Let \( I \) denote the ideal of \( U^0(i, j) \) generated by \( f_1, \ldots, f_a \). Then

(i) \( I \) is a prime ideal;
(ii) a first degree polynomial belongs to \( I \) if and only if it is a \( \mathbb{Z} \)-linear combination of \( f_1, \ldots, f_a \).

The next result states the property of \( \rho^{(C)}(i, j, K, (0^{d+1}), M, 1) \) similar to that of \( \xi_{r,s}(M) \) proved in [K, Lemma 2.11].

**Lemma 5.6.** Let \( 1 \leq i < j \leq n, \ d \geq 1, \ C = (c_1, \ldots, c_{n-1}) \) and \( K = (k_1, \ldots, k_d) \) be sequences of integers and \( M = \{m_1, \ldots, m_a\} \) be a set such that \( i \leq k_1 \leq \cdots \leq k_d \leq j \) and \( i < m_1 < \cdots < m_a < j \). Suppose that \( h \mapsto (t_h, s_h) \) is an injective map from \([1..a]\) to \([i..j] \times [1..d]\) such that \( t_h \geq m_h \) for all \( h \). Modulo the ideal of \( U^0(i, j) \) generated by \( \mathbb{B}^{C,k_s}(m_h, t_h) - d + s_h \), where \( h = 1, \ldots, a \), we have

\[
\rho^{(C)}(i, j, K, (0^{d+1}), M, 1) \equiv d^a \prod_{t,s} \left\{ \mathbb{B}^{C,k_s}(i, t) - d + s : (t, s) \in [i..j] \times [1..d] \setminus \{(t_h, s_h) : h = 1, \ldots, a\} \right\},
\]

where \( r \) is the number of entries of \( K \) less than \( j \).

**Proof.** We apply induction on \( |M| \). The case \( M = \emptyset \) follows immediately from the definition. Now let \( M \neq \emptyset \). We put \( M' := \{m_2, \ldots, m_a\}, \ X := [i..j] \times [1..d], \ Y := [m_1..j] \times [1..d], \ \Phi := \{(t_h, s_h) : h = 1, \ldots, a\} \) and \( \Psi := \{(t_h, s_h) : h = 2, \ldots, a\} \). The inductive hypothesis implies

\[
\rho^{(C)}(i, j, K, (0^{d+1}), M', 1) = d^a \left( \prod_{(t,s) \in X \setminus \Psi} \mathbb{B}^{C,k_s}(i, t) - d + s \right) + f,
\]

\[
\rho^{(C)}(m_1, j, K(m_1..j), (0^{d+1}), M', 1) = d^a \left( \prod_{(t,s) \in Y \setminus \Psi} \mathbb{B}^{C,k_s}(m_1, t) - d + s \right) + g,
\]

where \( f \) and \( g \) belong to the ideal \( I \) of \( U^0(i, j) \) generated by \( \mathbb{B}^{C,k_s}(m_h, t_h) - d + s_h \) for \( h = 2, \ldots, a \). We put for brevity \( x := C(i, m_1) + c_{i-1} - c_i - c_{m_1-1} + c_{m_1} \).
Therefore \( q \) in the square brackets to zero, and therefore, this expression is divisible by \( g \) where \( \delta_{m-1 \geq k_s} \). Lemma 5.7.

The substitution \( x \rightarrow 0 \) takes the expression in the square brackets to zero, and therefore, this expression is divisible by \( x \). Hence \( f + g' = xq \) in \( I \) for some \( q \in \mathcal{U}^0(i,j) \). By Proposition 5.5(ii) we have \( x \in I \) or \( q \in I \). The former case is impossible by Proposition 5.5(ii).

Therefore \( q \in I \). Now dividing (5.2) by \( x \), we prove the required result. \( \square \)

For \( d \geq 1 \), \( i < j \) and \( M \subset (i..j) \), we define the following polynomials

\[
\begin{align*}
&f_{i,j}^{(d)}(\emptyset) := \left( \prod_{t \in [i..j]} (y_t - x_i)^d \right), \quad g_{i,j}^{(d)}(\emptyset) := (y_j - x_i)^d \left( \prod_{t \in [i..j-1]} (y_t - x_i)^d \right), \\
&f_{i,j}^{(d)}(M) := f_{i,j}^{(d)}(M') - f_{i,m}^{(d)}(\emptyset) \frac{f_{i,j}^{(d)}(M')}{x_m - x_i}, \quad g_{i,j}^{(d)}(M) := g_{i,j}^{(d)}(M') - g_{i,m}^{(d)}(\emptyset) \frac{g_{i,j}^{(d)}(M')}{x_m - x_i},
\end{align*}
\]

where \( M \neq \emptyset \), \( m = \min M \) and \( M' = M \setminus \{m\} \).

Similarly to Lemma 5.3, one can prove that \( f_{i,j}^{(d)}(M) \) and \( g_{i,j}^{(d)}(M) \) are elements of \( \mathbb{Z}[x_i, x_j-1, y_i+1, \ldots, y_j] \). For \( M \subset (i..j) \), \( l \in M \) and a strictly increasing sequence \( N = (i_1, \ldots, i_k) \subset M \cap (i..l) \), we put

\[
G_{i,l}^{(d)}(M, N) := \prod_{r=0}^{k} g_{i_r, i_{r+1}}^{(d)}(M \cap (i_r..i_{r+1})) , \quad \text{where } i_0 = i \text{ and } i_{k+1} = l .
\]

**Lemma 5.7.** Let \( M \subset (i..j) \) and \( l \in M \). Then \( f_{i,j}^{(d)}(M) \) belongs to the ideal of \( \mathbb{Z}[x_i, x_j-1, y_i+1, \ldots, y_j] \) generated by \( G_{i,l}^{(d)}(M, N) \), where \( N \subset M \cap (i..l) \).

**Proof.** We apply induction on \( |M| \). For \( M = \{l\} \), we have

\[
\begin{align*}
f_{i,l}^{(d)}(M) := (y_l - x_i - d + 1) g_{i,l}^{(d)}(\emptyset) \frac{(\prod_{t \in [i..l]} (y_l - x_i)^d) - (\prod_{t \in [i..l]} (y_l - x_i)^d)}{x_l - x_i},
\end{align*}
\]

The substitution \( x_i \rightarrow x_l \) shows that the fraction in the right-hand side of this expression is a polynomial of \( \mathbb{Z}[x_i, x_j-1, y_i+1, \ldots, y_j] \).

Now let \( |M| > 1 \). We put \( m := \min M \) and \( M' := M \setminus \{m\} \). First we consider the case \( l = m \). We put \( m' := \min M' \). By the inductive hypothesis, we have \( f_{i,j}^{(d)}(M') = g_{i,m'}^{(d)}(\emptyset) h \), where \( h \in \mathbb{Z}[x_i, x_j-1, y_i+1, \ldots, y_j] \).

Hence \( f_{i,j}^{(d)}(M') = g_{i,l}^{(d)}(\emptyset) h' \), where \( h' = h(y_l - x_i - d + 1)(y_m' - x_i)^{d-1} \times \)
\[ \times \prod_{i \in [l, m']}(y_{i+1} - x_i)^d. \]

We have \( f_{i,j}^{(d)}(M) = g_{i,j}^{(d)}(0)(h' - (y_i - x_i - d + 1) f_{i,j}^{(d)}(M'))/(x_i - x_l). \]

Since \( x_i - x_l \) and \( g_{i,j}^{(d)}(0) \) are relatively prime, we have \( h' - (y_i - x_i - d + 1) f_{i,j}^{(d)}(M')/(x_i - x_l) \in \mathbb{Z}[x_i, \ldots, x_{j-1}, y_i+1, \ldots, y_j]. \)

Now we consider the case \( m < l \). The inductive hypothesis implies that \( f_{i,j}^{(d)}(M') \) belongs to the ideal of \( \mathbb{Z}[x_i, \ldots, x_{j-1}, y_i+1, \ldots, y_j] \) generated by the polynomials \( G_{i,l}^{(d)}(M', N) \), where \( N \subset M' \cap (i..l) \). Since for \( i_1 \in M' \) one has

\[
\begin{align*}
g_{i_1}^{(d)}(M' \cap (i..i_1)) &= (x_m - x_i)g_{i_1}^{(d)}(M \cap (i..i_1)) \quad \text{and} \quad \rho^{(d)}(i, j, (d), (0^{d+1}), (M, 1), M) \\
&= (x_m - x_i - d + 1)g_{i,m}^{(d)}(0)g_{i_1}^{(d)}(M \cap (m..i_1)),
\end{align*}
\]

\( f_{i,j}^{(d)}(M') \) belongs to the ideal \( I \) of \( \mathbb{Z}[x_i, \ldots, x_{j-1}, y_i+1, \ldots, y_j] \), generated by the polynomials \( (x_m - x_i)^{\rho_{z,n} G_{i,l}^{(d)}(M, N)} \), where \( N \subset M \cap (i..l) \). By the inductive hypothesis \( f_{i, m}^{(d)}(0)f_{j, m}^{(d)}(M') \in I \). Hence \( f_{i,j}^{(d)}(M)(x_m - x_i) \in I \).

We have \( f_{i,j}^{(d)}(M)(x_m - x_i) = \sum_{N \subset M \cap (i..l)}(x_m - x_i)^{\rho_{z,n} G_{i,l}^{(d)}(M, N)}h_N \) for some \( h_N \in \mathbb{Z}[x_i, \ldots, x_{j-1}, y_i+1, \ldots, y_j] \).

For every subset \( N \subset M \cap (i..l) \) such that \( m \in N \), we consider the representation \( h_N = h''_N(x_m - x_i) + h''_N \), where \( h''_N \in \mathbb{Z}[x_i, \ldots, x_{j-1}, y_i+1, \ldots, y_j] \) and \( h''_N \in \mathbb{Z}[x_i+1, \ldots, x_{j-1}, y_i+1, \ldots, y_j] \). Hence the product

\[
g_{i,m}^{(d)}(0) \cdot \sum_{N \subset M \cap (m..l)} G_{i,m}^{(d)}(M', N')h''_N
\]

is divisible by \( (x_m - x_i) \). Since \( (x_m - x_i) \) does not divide the first factor of (5.3), it divides the second factor, which however does not depend on \( x_i \).

Therefore, the whole polynomial (5.3) equals zero. Thus we have obtained

\[
f_{i,j}^{(d)}(M) = \sum_{N \subset M \cap (i..l)} G_{i,l}^{(d)}(M, N)h_N + \sum_{N \subset M \cap (i..l)} G_{i,l}^{(d)}(M, N)h''_N. \quad \Box
\]

**Corollary 5.8.** Let \( C = (c_1, \ldots, c_{n-1}) \) be a sequence of integers, \( d \geq 1 \), \( 1 \leq i < j \leq n \), \( M \subset (i..j) \) and \( l \leq M \). Then \( \rho^{(d)}(i, j, (d), (0^{d+1}), (M, 1)) \) belongs to the ideal of \( \mathbb{U}^{0}(i, j) \) generated by

\[
\prod_{r=0}^{k} \left( \frac{1}{\rho^{(d)}(i_r, i_{r+1}, (i_{r+1} - 1, i_{r+1} - d), (0^{d}, d), M \cap (i_r..i_{r+1}), \frac{1}{\mathbb{Z}}}) \right),
\]

where \( i = i_0 < i_1 < \cdots < i_k < i_{k+1} = l \), \( k \geq 0 \) and \( i_1, \ldots, i_k \in M \).

**Proof.** The result follows from the previous lemma and formulae (4.1) and (1.2) by the substitution \( y_l \mapsto t - 1 \in \mathbb{U}l \), \( x_l \mapsto t - \mathbb{U}l - c_{l-1} + c_l \). \( \Box \)

6. **Proof of the main results**

We define the hyperalgebra \( U(n) \) over \( K \) to be \( \mathbb{U}(n) \otimes_{\mathbb{Z}} K \). It is well known that every rational \( GL_m \)-module can be considered as a \( U(n) \)-module (see [1, I.7.11–I.7.16]). We shall use the following notation: \( E_i^{(r)} := E_i^{(r)} \otimes 1_K \), \( (H_i)^r := (H_i)^r \otimes 1_K \), \( U^{j}(n) := U^{j}(n) \otimes K \), \( U_{i,j}^{j}(n) := U_{i,j}^{j}(n) \otimes K \), \( U_{i,j}^{j}(i, j) := U_{i,j}^{j}(i, j) \otimes K \). Any weight \( \lambda = (\lambda_1, \ldots, \lambda_n) \) will also be identified with the \( K \)-algebra homomorphism \( \pi_\lambda : U^{0}(n) \rightarrow K \) that takes \( (H_i)^r \) to \( \lambda_i \) and \( p\mathbb{Z} \). A vector \( v \) of a \( U(n) \)-module is called a \( U(n) \)-high weight vector if it is a \( U^{0}(n) \)-weight
vector and \( E^{(r)}_i v = 0 \) for all \( i = 1, \ldots, n - 1 \) and \( r > 0 \). One can observe that a vector of a rational \( \text{GL}_n \) module is a \( \text{GL}_n \) high weight vector if and only if it is a \( U(n) \) high weight vector.

For the rest of the paper, we fix \( n > 1 \). For \( \lambda \in X^+(n) \), we denote by \( \nabla_n(\lambda) \) the \( \text{GL}_n \) module contravariantly dual to the Weyl module with highest weight \( \lambda \). It is well known that \( \text{soc} \nabla_n(\lambda) \cong L_n(\lambda) \). We need the module \( \nabla_n(\lambda) \) because we know all its \( U(n - 1) \) high weight vectors. Explicitly, for any \( \mu \in X^+(n - 1) \) such that \( \lambda_i + 1 \leq \mu_i \leq \lambda_i \) for \( i = 1, \ldots, n - 1 \), there is a nonzero \( U(n - 1) \) high weight vector \( f_{\mu, \lambda} \in \nabla_n(\lambda) \) of weight \( \mu \). We shall express the above relation between \( \mu \) and \( \lambda \) by \( \mu \leftarrow \lambda \). These vectors have the property that for any nonzero \( U(n - 1) \) high weight vector \( v \in \nabla_n(\lambda) \) of weight \( \mu \in X^+(n - 1) \), there holds \( \mu \leftarrow \lambda \) and \( v \) is a scalar multiple of \( f_{\mu, \lambda} \) (see [BKS, Corollary 3.3]). We shall abbreviate \( f_\lambda := f_{\lambda, \lambda} \) where \( \lambda = (\lambda_1, \ldots, \lambda_{n - 1}) \). Thus \( \text{soc} \nabla_n(\lambda) \) is generated by \( f_{\lambda} \) as a \( \text{GL}_n \) module.

**Proposition 6.1.** We can choose the vectors \( f_{\mu, \lambda} \) so that \( E^{(a_1)}_1 \cdots E^{(a_{n - 1})}_{n - 1} f_{\mu, \lambda} = f_{\lambda} \), where \( a_i = \sum_{s = 1}^i (\lambda_s - \nu_s) \).

**Proof.** If all entries of \( \lambda \) are nonnegative, then this can be done by [BKS, Lemma 2.6]. To this case, the arbitrary case can be reduced by tensoring \( \nabla_n(\lambda) \) with a sufficiently high power of the determinant representation. \( \square \)

Now we are going to associate with every homogeneous vector \( v \in \nabla_n(\lambda) \) the element \( \text{cf}(v) \in K \) as follows. Let \( v \) have weight \( \nu \in X(n - 1) \) and \( a_i = \sum_{s = 1}^i (\lambda_s - \nu_s) \). We choose \( \text{cf}(v) \) so that \( E^{(a_1)}_1 \cdots E^{(a_{n - 1})}_{n - 1} v = \text{cf}(v) f_{\lambda} \).

We define the relation \( \rightarrow \rightarrow \) between vectors of \( \nabla_n(\lambda) \) by the rules: \( v \rightarrow \rightarrow v \); \( v \rightarrow \rightarrow E^{(r)}_l v \) for \( 1 \leq l < n - 1 \) and \( r > 0 \); if \( v \rightarrow u \) and \( u \rightarrow w \) then \( v \rightarrow w \).

The next theorem is our main tool to find out whether a vector of \( \nabla_n(\lambda) \) is a nonzero \( U(n - 1) \) high weight vector.

**Theorem 6.2.** Let a vector \( v \) of \( \nabla_n(\lambda) \) have weight \( \nu \in X(n - 1) \).

(i) \( v = 0 \) if and only if \( \text{cf}(u) = 0 \) for any \( u \) such that \( v \rightarrow u \),

(ii) \( v \) is a nonzero \( U(n - 1) \) high weight vector if and only if \( \text{cf}(v) \neq 0 \) and \( \text{cf}(u) = 0 \) for any \( u \) such that \( u \neq v \) and \( v \rightarrow u \).

**Proof.** (i) Clearly, it suffices to prove that \( v = 0 \) if \( \text{cf}(u) = 0 \) for any \( u \) such that \( v \rightarrow u \). Assume this is wrong. Then there is \( \nu \in X(n - 1) \) and a vector \( v \in \nabla_n(\lambda) \) of weight \( \nu \) such that \( v \neq 0 \) but \( \text{cf}(u) = 0 \) for any \( u \) such that \( v \rightarrow u \). We can assume that \( \nu \) is the maximal weight with this property.

Take arbitrary \( 1 \leq l < n - 1 \) and \( r > 0 \). We have \( v \rightarrow \rightarrow E^{(r)}_l v \). If \( E^{(r)}_l v \rightarrow u \) then \( v \rightarrow u \) and thus \( \text{cf}(u) = 0 \). Since \( E^{(r)}_l v \) has \( U(n - 1) \) weight strictly greater than \( v \), we have \( E^{(r)}_l v = 0 \).

We have proved that \( v \) is a nonzero \( U(n - 1) \) high weight vector of \( \nabla_n(\lambda) \). By [BKS, Corollary 3.3], we have \( \nu \leftarrow \lambda = \beta f_{\nu, \lambda} \) for some \( \beta \in K \setminus \{0\} \).

Multiplying this equality by \( E^{(a_1)}_1 \cdots E^{(a_{n - 1})}_{n - 1} \), where \( a_i = \sum_{s = 1}^i (\lambda_s - \nu_s) \), and taking into account Proposition 6.1, we obtain \( 0 = \text{cf}(v) f_\lambda = \beta f_\lambda \). Hence \( \beta = 0 \) contrary to assumption.

(ii) Let \( v \) be a nonzero \( U(n - 1) \) high weight vector. For any vector \( u \) such that \( u \neq v \) and \( v \rightarrow u \), there exist \( 1 \leq l < n - 1 \) and \( r > 0 \) such that...
Proposition 6.3. Let $X$ be an element of $U^{-,0}(n)$ of $U(n)$-weight $\sigma$, $1 \leq t < n$ and $r \geq 0$. Then we have $E_t^{(r)}X = \sum_{s=0}^{b} X_s E_t^{(r-s)}$, where $X_s$ is an element of $U^{-,0}(n)$ of weight $\sigma + s\alpha_t$ and $b$ is the $\alpha_t$-coefficient of $-\sigma$.

In the sequel, we shall use the notation

\[ B^{\mu,\lambda}(i, t) := t - i + \mu_i - \lambda_{t+1}, \quad B^{\mu,\lambda,k}(i, t) := \begin{cases} t - i + \mu_i - \lambda_{t+1} & \text{if } k \leq t; \\ t - i + \mu_i - \lambda_{t+1} + k & \text{if } k > t. \end{cases} \]

Theorem 6.4. Let $\lambda \in X^+(n)$, $\mu \in X^+(n-1)$, $\mu \prec \lambda$, $1 \leq i < n$, $1 \leq d < p$ and $M \subseteq (i..n)$. Then $T_{i,n}^{(d)}(M, 1)f_{\mu, \lambda}$ is a nonzero $U(n-1)$-high weight vector if and only if there is an injection $\gamma: M \to [i..n] \times [1..d]$ such that

(i) $\gamma_1(m) \geq m$ and $B^{\mu,\lambda}(m, \gamma_1(m)) \equiv d - \gamma_2(m) \pmod{p}$ for any $m \in M$;

(ii) $B^{\mu,\lambda}(i, t) \not\equiv d - s \pmod{p}$ for any $(t, s) \in [i..n] \times [1..d] \setminus \Im \gamma$,

where $\gamma(m) = (\gamma_1(m), \gamma_2(m))$.

Proof. We put $v := T_{i,n}^{(d)}(M, 1)f_{\mu, \lambda}$, $a_t := \sum_{s=1}^{t} (\lambda_s - \mu_s)$, $A := (a_1, \ldots, a_{n-1})$ and $A_t := [t..n] \times [1..d]$. By Lemma 3.3 we have

\[
\prod_{t=1}^{n-1} E_t^{(a_t + \delta_t \in [i..n])}(M, 1) \equiv \rho^{(A)}(i, n, (n^d), (0^d), M, 1) \prod_{t=1}^{n-1} E_t^{(a_t)} \pmod{\mathbb{Z}}.
\]

modulo $\mathbb{Z}$. By Lemmas 5.2 and 5.4 both sides of this equivalence belong to $U(n)$. Therefore, applying Lemma 2.6 we obtain that this equivalence holds modulo $J^{(A)}$. Tensoring the above equivalence with $1_K$ gives

\[
\prod_{t=1}^{n-1} E_t^{(a_t + \delta_t \in [i..n])}(M, 1) \equiv \xi^{(A)}(i, n, (n^d), (0^d), M, 1) \prod_{t=1}^{n-1} E_t^{(a_t)} \pmod{J^{(A)}}.
\]

modulo $J^{(A)}$. Since $J^{(A)}f_{\mu, \lambda} = 0$, we can multiply this equivalence by $f_{\mu, \lambda}$ on the right and apply Proposition 6.1 to the right-hand side. Hence

\[ \text{cf}(v) = \pi_\lambda(\xi^{(A)}(i, n, (n^d), (0^d), M, 1)). \tag{6.1} \]

"If part". Lemma 5.6 and condition (ii) give $\pi_\lambda(\xi^{(A)}(i, n, (n^d), (0^d), M, 1)) = \prod_{(t,s) \in A_t \setminus \Im \gamma} B^{\mu,\lambda}(i, t) - d + s + p\mathbb{Z}$. This element is not equal to zero by virtue of condition (ii). Thus by (6.1), we have proved $\text{cf}(v) \neq 0$. 

Let \( V \) be the subspace of \( \nabla_n(\lambda) \) spanned by all vectors of the form
\[
X T_{l,n}^{(d)}(M \cap (l..n), 1) f_{\mu,\lambda},
\]
where \( X \in U^{-0}(i, l) \) and \( l \in M \). We claim that
(a) \( \text{cf}(u) = 0 \) for any \( u \) of a \( U(n) \)-weight space of \( V \);
(b) \( u \in V \) for any \( u \) such that \( u \neq v \) and \( v \rightarrow u \).

These properties, once proved, will immediately imply that \( v \) is a nonzero \( U(n-1) \)-high weight vector by virtue of Theorem 6.2.

To prove (a) we can restrict ourselves to the case \( u = X T_{l,n}^{(d)}(M \cap (l..n), 1) f_{\mu,\lambda} \), where \( X \) is an element of \( U^{-0}(i, l) \) having \( U(n) \)-weight \( \sigma = -m_1 \alpha_1 - \cdots - m_{n-1} \alpha_{n-1} \) and \( l \in M \). We put \( \bar{a}_t := a_t + d \delta_t \in [l..n] \) and \( A := (\bar{a}_1, \ldots, \bar{a}_{n-1}) \). By Lemma 2.3 we get
\[
\left( \prod_{t=1}^{n-1} E_t^{(\bar{a}_t+m_t)} \right) X \equiv (\prod_{t=1}^{n-1} E_t^{(\bar{a}_t)}) (\text{mod } J^{(A)})
\]
for some \( \xi \in U^0(n) \). Since \( \lambda \) is the highest weight of \( \nabla_n(\lambda) \), we have \( J^{(A)} T_{l,n}^{(d)}(M \cap (l..n), 1) f_{\mu,\lambda} = 0 \). Hence
\[
\left( \prod_{t=1}^{n-1} E_t^{(\bar{a}_t+m_t)} \right) X T_{l,n}^{(d)}(M \cap (l..n), 1) f_{\mu,\lambda} = \xi \left( \prod_{t=1}^{n-1} E_t^{(\bar{a}_t)} \right) T_{l,n}^{(d)}(M \cap (l..n), 1) f_{\mu,\lambda}.
\]

Lemma 2.3 yields
\[
\left( \prod_{t=1}^{n-1} E_t^{(\bar{a}_t)} \right) T_{l,n}^{(d)}(M \cap (l..n), 1)
\equiv \rho^{(A)}(n, (n^d), (0^d), M \cap (l..n), 1) \left( \prod_{t=1}^{n-1} E_t^{(a_t)} \right) (\text{mod } J^{(A)}).
\]
Again applying Lemmas 5.2 5.3 and 2.3, we obtain that this equivalence holds modulo \( J^{(A)} \). Therefore, tensoring this equivalence with \( 1_X \), multiplying both sides by \( f_{\mu,\lambda} \) on the right and applying Proposition 6.1 we get
\[
\left( \prod_{t=1}^{n-1} E_t^{(\bar{a}_t)} \right) T_{l,n}^{(d)}(M \cap (l..n), 1) f_{\mu,\lambda} = \pi_\lambda(\xi^{(A)}(n, (n^d), (0^d), M \cap (l..n), 1)) f_\lambda.
\]
By Lemma 5.6 and condition (ii) the right-hand side of the last equality equals
\[
\prod_{(t, s) \in A_\gamma(M \cap (l..n))} \left( B_{\mu,\lambda}(l, t) - d + s \right) f_\lambda = 0.
\]
Hence \( \text{cf}(u) = 0 \).

To prove (b) we shall prove at first that \( V \) is closed under multiplication by elements \( E_t^{(r)} \), where \( 1 \leq t < n - 1 \) and \( r \geq 0 \). We use induction on \( r \). Since the case \( r = 0 \) is obvious, we assume that \( r > 0 \) and that \( V \) is closed under multiplication by \( E_t^{(r-1)} \). It suffices to prove \( E_t^{(r)} u \in V \) for \( u = X T_{l,n}^{(d)}(M \cap (l..n), 1) f_{\mu,\lambda} \), where \( l \in M \) and \( X \in U^{-0}(i, l) \). Clearly, this is true if \( t \notin [i..n] \) as in that case \( E_t^{(r)} u = 0 \). In the case \( t \in [i..l] \), the result follows from Lemma 6.3. Finally, let \( t \in [l..n] \). Then we have \( E_t^{(r)} u = Y E_t^{(r)} T_{l,n}^{(d)}(M \cap (l..n), 1) f_{\mu,\lambda} \) for some \( Y \in U^{-0}(i, l) \). It follows from Lemma 6.3 that \( E_t^{(r)} T_{l,n}^{(d)}(M \cap (l..n), 1) f_{\mu,\lambda} = 0 \) if \( r > d \). Thus we assume that \( r \leq d < p \). Taking into account the formula \( E_t^{(r)} = r^{-1} E_t^{(r-1)} E_t \) and the inductive hypothesis, we can assume that \( r = 1 \). In that case, the required result follows from Lemma 3.3 2.3 Corollary 5.3 and Lemma 2.3.

Now we are going to prove that \( E_t^{(r)} v \in V \) for \( 1 \leq t < n - 1 \) and \( r > 0 \). If \( r > d \) then \( E_t^{(r)} v = 0 \) by Lemma 6.3. Therefore we assume that \( r \leq d < p \).
Since $E_t^{(r)} = r^{-1} E_t^{(r-1)} E_t$ and $E_t^{(r-1)} V \subset V$, it remains to show that $E_t v \in V$.

This follows from Lemma 3.3 Lemma 5.2 Corollary 5.3 and Lemma 2.3

“Only if part”. We shall prove by downward induction on $q \in M \cup \{n\}$ that

for any $m \in M \cap \{q..n\}$ there are $t_m \in \{m..n\}$ and $s_m \in \{1..d\}$
such that the pairs $(t_m, s_m)$ are mutually distinct and

$$B^\mu(m, t_m) \equiv d - s_m \pmod p$$

for all $m \in M \cap \{q..n\}$.

Suppose that $M \neq \emptyset$, $q \in (M \cup \{n\}) \setminus \{\min M\}$ and for all $m \in M \cap \{q..n\}$
the numbers $t_m$ and $s_m$ have already been defined. Let $l$ denote the element of $M$
directly preceding $q$. We must define $t_l$ and $s_l$.

By Corollary 5.8 the polynomial $\rho^{(A)}(i, n, (n^d), (0^d+1), M, 1)$ belongs to the ideal of $U^0(i, n)$ generated by the polynomials

$$\prod_{r=0}^{k} \frac{1}{d} \rho^{(A)} \left( i_r, i_{r+1}, (i_{r+1} - 1, i_{r+1}^d - 1), (0^d, d), M \cap (i_r..i_{r+1}) \right),$$

where $i = i_0 < i_1 < \ldots < i_k < i_{k+1} = l$, $k \geq 0$ and $i_1, \ldots, i_k \in M$. By (6.1)
and Theorem 6.2(ii) we get $\pi_{\lambda}(\xi^{(A)}(i, n, (n^d), (0^d), M, 1)) \neq 0$. Therefore,
there exist integers $t_0, i_1, \ldots, i_k, i_{k+1}$ satisfying the above conditions such that

$$\prod_{r=0}^{k} \pi_{\lambda} \left( \xi^{(A)}(i_r, i_{r+1}, (i_{r+1} - 1, i_{r+1}^d - 1), (0^d, d), M \cap (i_r..i_{r+1}) \right) \neq 0. (6.3)$$

We put $K^{(s)} := (i_s - 1, i_s^d - 1)$ for $1 \leq s \leq k + 1$, $L^{(s)} := (0^d, \delta_{i,s=\{i_s+1\}-1})$ for
$1 \leq s \leq k$ and $L^{(k+1)} := (0^d)$. Lemma 3.3, Lemma 2.3(ii) and Corollary 5.3 yield

$$E_{i_{k+1}} \cdots E_{i_1} \bar{p}^{(d)}_{l,n}(M, 1) \equiv X_1 \cdots X_{k+1} \bar{p}^{(d)}_{l,n}(M \cap (l..n), 1) (6.4)$$

modulo $\bar{I}_{\{i_1, \ldots, i_{k+1}\}}$, where each $X_s \in U^0(\{i_{s-1}, i_s\})$ and

$$X_s \equiv -E_{i_s}(K^{(s)}, L^{(s)}) \left( \bar{p}^{(d)}_{i_{s-1}, i_s}(M \cap (i_{s-1}..i_s), \frac{1}{i_s}) \right) \pmod {\bar{I}_{\{i_{s-1}\}}}.$$

Lemma 5.2 and Lemma 2.3 show that equivalence (6.4) holds modulo $\bar{I}_{\{i_1, \ldots, i_{k+1}\}}$. Let $X_1, \ldots, X_{k+1}$ denote the images in $U(n)$ of $X_1, \ldots, X_{k+1}$ respectively. Then it follows from (6.4) that

$$0 = E_{i_{k+1}} \cdots E_{i_1} v = X_1 \cdots X_{k+1} T^{(d)}_{l,n}(M \cap (l..n), 1) f_{\mu,\lambda}. (6.5)$$

Let $a^{(s)} := a + d\delta_{i,s=\{i_1\}-1} - \delta_{i,s=\{i_{s+1}\}-1}$ and $A^{(s)} := (a^{(s)}, \ldots, a^{(s)})$.

By Lemmas 1.3 3.4 and 2.3 for any $s = 1, \ldots, k + 1$, we get

$$\left( \prod_{l=1}^{n-1} E_{t}^{(a^{(s-1)}_{l-1})} \right) X_s \equiv -\rho^{(A^{(s)})} \left( i_{s-1}, i_s, K^{(s)}, L^{(s)}, M \cap (i_{s-1}..i_s), \frac{1}{i_s} \right) \times$$

$$\times \left( \prod_{l=1}^{n-1} E_{t}^{(a^{(s)}_{l})} \right) \pmod {\bar{I}^{(A^{(s)})} + \bar{I}_{\{i_{s-1}\}}}.$$
It is elementary to see that \((\mathbb{J}^{(A)}_{(s)}) + \mathbb{I}_{(i_{s-1})})X_{s+1} \cdots X_{k+1}T_{l,n}^{(d)}(M \cap (l..n), 1) \subseteq \mathbb{J}^{(A)} + \mathbb{I}_{(i_{s-1})}\). Therefore, applying Lemma \ref{lem:elementary}, we have

\[
\left(\prod_{i=1}^{n-1} E_{a_i}^{(a_i)}(0)\right) X_1 \cdots X_{k+1} T_{l,n}^{(d)}(M \cap (l..n), 1) \equiv (-1)^{k+1} \times \left(\prod_{s=1}^{k+1} \rho^{(A)}(i_{s-1}, i_s, K^{(s)}, (0^d, d), M \cap (i_{s-1}..i_s), \frac{1}{\xi^d})\right) \times \left(\prod_{t=1}^{n-1} E_{a_t + d\delta_{t \not\in [1..n]}}(a_t)\right) T_{l,n}^{(d)}(M \cap (l..n), 1) \pmod{\mathbb{J}^{(A)} + \mathbb{I}_{(i_{s-1},..,i_{k+1} - 1)}}.
\]

Tensoring this with \(1_K\), multiplying by \(f_{\mu, \lambda}\) and applying \ref{lem:elementary}, \ref{prop:main} and Lemma \ref{prop:factorization} and Proposition \ref{prop:factorization2} we get

\[
0 = \left(\prod_{i=1}^{n-1} E_{a_i + d\delta_{i \not\in [1..n]}}(a_i)\right) T_{l,n}^{(d)}(M \cap (l..n), 1) f_{\mu, \lambda} = \pi_{\lambda}(\xi^{(A)}(l, n, (n^d), (0^d), M \cap (l..n), 1)) f_{\lambda} = \prod_{(t,s) \in \Lambda_i \setminus \{(t_m, s_m) : m \in M \cap [q..n]\}} f_{\lambda}.
\]

This formula allows us to choose \(t_i\) and \(s_i\) as required.

Applying \ref{prop:factorization2} for \(q = \min M \cup \{n\}\), we define \(\gamma(m) := (t_m, s_m)\) for \(m \in M\). For this \(\gamma\) condition \ref{cond:gamma} is clearly satisfied. Lemma \ref{prop:factorization} now implies

\[
0 \neq \pi_{\lambda}(\xi^{(A)}(i, n, (n^d), (0^d), M, 1)) = \prod_{(t,s) \in \Lambda_i \setminus \Im \gamma} (B^{\mu, \lambda}(i, t) - d + s + pZ),
\]

which gives condition \ref{cond:gamma}.

\begin{flushright}
\Box
\end{flushright}

**Theorem 6.5.** Let \(\lambda \in X^{+}(n), \mu \in X^{+}(n-1), \mu \leftarrow \lambda, 1 \leq i < j < n\), \(1 \leq d < p\) and \(M \subset (i..j)\). Then \(T_{i,j}^{(d)}(M, 1) f_{\mu, \lambda}\) is a nonzero \(U(n-1)\)-high weight vector if and only if for any sequence of integers \(K = (k_1, \ldots, k_d)\) such that \(i \leq k_1 \leq \cdots \leq k_d \leq j\), there exists an injection \(\gamma_K : M \rightarrow [i..j] \times [1..d]\) such that

\begin{enumerate}[(i)]
\item \(\gamma_{K, 1}(m) \geq m\) and \(B^{\mu, \lambda, \gamma_K, 2}(m, \gamma_{K, 1}(m)) \equiv d - \gamma_{K, 2}(m) \pmod{p}\) for any \(m \in M\);
\item if \(K \neq (j^d)\) then the exits a pair \((t, s) \in [i..j] \times [1..d] \setminus \Im \gamma_K\) such that \(B^{\mu, \lambda, \gamma_K}(i, t) \equiv d - s \pmod{p}\);
\item \(B^{\mu, \lambda, \gamma_K}(i, t) \neq d - s \pmod{p}\) for any pair \((t, s) \in [i..j] \times [1..d] \setminus \Im \gamma_{j^d}\) such that \(\gamma_K(m) = (\gamma_{K, 1}(m), \gamma_{K, 2}(m))\).
\end{enumerate}

**Proof.** We put \(v := T_{i,j}^{(d)}(M, 1) f_{\mu, \lambda}, a_t := \sum_{s=1}^{t} (\lambda_s - \mu_s), A := (a_1, \ldots, a_{n-1})\) and \(\Lambda_i := \{t..j\} \times [1..d]\).

"If part". Using condition \ref{cond:gamma} for \(K = (j^d)\) and condition \ref{cond:gamma3}, we prove that \(\cf(v) \neq 0\) similarly to how it was done in Theorem 6.4.

Let \(V\) be the subspace of \(\nabla_n(\lambda)\) spanned by vectors of the form

\begin{itemize}
\item \(E_j(K, (0^d)) (T_{i,j}^{(d)}(M, 1)) f_{\mu, \lambda}\), where \(K\) is a weakly increasing sequence distinct from \((j^d)\) with entries from \([i..j]\);
\item \(X E_j(K, (0^d)) (T_{i,j}^{(d)}(M \cap (l..j), 1)) f_{\mu, \lambda}\), where \(K\) is a weakly increasing sequence with entries from \([l..j]\), \(X \in U^{-\Delta}(i, l)\) and \(l \in M\).
\end{itemize}
Properties $[a]$ and $[b]$ from the "if-part" of Theorem 5.4 also hold in the present situation. The only alteration that must be made in the proofs is that we now additionally apply Lemma 3.4.

"Only if part". We fix a sequence of integers $K = (k_1, \ldots, k_d)$ such that $i \leq k_1 \leq \cdots \leq k_d \leq j$. We shall prove by downward induction on $q \in M \cup \{j\}$ that for any $m \in M \cap [q..j]$ there are $t_m \in [m..j]$ and $s_m \in [1..d]$ such that the pairs $(t_m, s_m)$ are mutually distinct and

$$B^{\mu, \lambda, k_m}(m, t_m) \equiv d - s_m \pmod{p}$$

for any $m \in M \cap [q..j]$.

Suppose that $M \neq \emptyset$, $q \in (M \cup \{j\}) \setminus \{\min M\}$ and for all $m \in M \cap [q..j]$ the numbers $t_m$ and $s_m$ have already been defined. Let $l$ denote the element of $M$ directly preceding $q$. We must define $t_l$ and $s_l$.

Similarly to (6.3) we prove that there are integers $i_1, \ldots, i_k$ belonging to $M$ such that $k \geq 0$, $i = i_0 < i_1 < \cdots < i_k < i_{k+1} = l$ and

$$\prod_{r=0}^{k} \pi_{r}(a^{(4)}(i_{r}, i_{r+1}, (i_{r+1} - 1, i_{r+1}^d), (0^d), M \cap (i_{r+1} - 1, i_{r+1}^d), \frac{1}{\zeta - \delta})) \neq 0. \quad (6.7)$$

We put for brevity $K^{(s)} := (i_s - 1, i_s^d)$ for $s = 0, 1, \ldots, k + 1$, $L^{(s)} := (0^d, \delta_s, i_{s+1} - 1)$ for $s = 1, \ldots, k$ and $L^{(k+1)} := (0^d, b)$, where $b$ is the number of entries of $K$ not greater than $l$. Lemmas 3.3, 2.8(ii) and Corollary 5.3 yield

$$\mathbb{E}(k^{(l)}_d, l) \mathbb{E}(k^{(l)}_1, l) \mathbb{E}I_{l+j}^{(d)}(M, 1) \equiv X_1 \cdots X_{k+1} \mathbb{E}I_{l+j}(M \cap (l, j), 1) \quad (6.8)$$

modulo $\mathbb{I}_{i_1, \ldots, i_k, i_{k+1}}$, where each $X_s \in U^{-\theta}(i_s, i_s)$ and

$$X_s \equiv -\mathbb{E}(K^{(s)}, L^{(s)})(\mathbb{T}_{i_s+1, i_s}^{(d)}(M \cap (i_s, i_s), \frac{1}{\zeta - \delta})) \pmod{\mathbb{I}_{i_s-1}}.$$

Lemmas 5.2 and 2.3 show that (6.8) holds modulo $\mathbb{I}_{i_1, \ldots, i_k, i_{k+1}}$. Let $X_1, \ldots, X_{k+1}$ denote the images in $U(n)$ of $X_1, \ldots, X_{k+1}$ respectively.

Let $a^{(s)}_t := a_t + \sum_{h=1}^{d} \delta_{t \in [i_s, i_s^d]} - \delta_{t \in [i_{s+1}, i_{s+1}]}$ and $A^{(s)} := (a^{(1)}_1, \ldots, a^{(s)}_{n-1})$. By Lemmas 4.3, 5.4 and 2.3 for any $s = 1, \ldots, k + 1$, we get

$$\prod_{t=1}^{n-1} \mathbb{E}(a^{(s-1)}_t) X_s \equiv -\rho(A^{(s)})(i_{s-1}, i_s, K^{(s)}, L^{(s)})(M \cap (i_{s-1}, i_s), \frac{1}{\zeta - \delta}) \times$$

$$\times \prod_{t=1}^{n-1} \mathbb{E}(a^{(s-1)}_t) \pmod{\mathbb{I}_{i_s-1}}.$$

It is elementary to verify that $(\mathbb{J}(A^{(s)} + \mathbb{I}_{i_s-1})) X_{s+1} \cdots X_{k+1} \times \mathbb{E}(K^{(l)}, L^{(l)}, L^{(s)})(\mathbb{T}_{i_s+1}^{(d)}(M \cap (l, j), 1) \subset \mathbb{J}(A^{(s)} + \mathbb{I}_{i_s-1})$. Hence Lemma 4.5 yields

$$\prod_{t=1}^{k+1} \mathbb{E}(a^{(l+1)}_t) X_1 \cdots X_{k+1} \mathbb{E}(K^{(l)}, L^{(s)})(\mathbb{T}_{i_s+1}^{(d)}(M \cap (l, j), 1)) \equiv (-1)^{k+1} \times$$

$$\times \prod_{s=1}^{k+1} \rho(A^{(s)})(i_{s-1}, i_s, K^{(s)}, L^{(s)})(M \cap (i_{s-1}, i_s), \frac{1}{\zeta - \delta}) \times$$

$$\times \prod_{t=1}^{n-1} \mathbb{E}(a^{(l+1)}_t) \mathbb{E}(K^{(l)}, L^{(s)})(\mathbb{T}_{i_s+1}^{(d)}(M \cap (l, j), 1)) \pmod{\mathbb{J}(A^{(s)} + \mathbb{I}_{i_s-1})}.$$
By Lemmas 6.3 and 6.4 and 2.3 the last product of this equivalence equals
\( \rho^{(A)}(l, j, K^{(d)}, (0^d), M \cap (l..j), 1) \prod_{i=1}^{n-1} E_t(a_i) \) modulo (mod \( J^{(A)} + \Gamma_{[l..j]} \)). We note that \( B^{\mu,\lambda,k_s^{(d)}}(q, t) = B^{\mu,\lambda,k_s}(q, t) \) for \( t \geq l \). Applying this remark, the above equivalences, equivalence (6.8), inequality (6.7), Theorem 6.2(ii) and Lemma 6.6 we get
\[
0 = \pi_\lambda(\xi^{(A)}(l, j, K^{(d)}, (0^d), M \cap (l..j), 1)) f_\lambda = \\
= \prod \{ B^{\mu,\lambda,k_s}(l, t) - d + s : (t, s) \in \Lambda_1 \setminus \{(t_m, s_m) : m \in M \cap [g..j]\} \} f_\lambda.
\]
This formula allows us to choose \( t_i \) and \( s_i \) as required.

Applying (6.6) for \( q = \min M \cup \{ j \} \), we define \( \gamma_K(m) := (t_m, s_m) \) for \( m \in M \). For this \( \gamma_K \) condition (iii) is clearly satisfied. Lemma 5.6 now implies
\[
\left( \prod_{i=1}^{n-1} E_t(a_i) \right) (E_{k_{d}} \cdots E_{j-1}) \cdots (E_{k_{1}} \cdots E_{j-1}) T_{i,j}^{(d)}(M, 1)f_{\mu,\lambda} = \\
\pi_\lambda(\xi^{(A)}(i, j, K, (0^d), M, 1)) f_\lambda = \left( \prod_{(t,s) \in \Lambda_1 \setminus \text{Im} \gamma_K} (B^{\mu,\lambda,k_s}(i, t) - d + s) \right) f_\lambda,
\]
where \( a_t = a_t + \sum_{s=1}^{d} \delta_{l \leq i..k_s} \). By Theorem 6.2(ii), the left-hand side of the above formula equals zero if and only if \( K \neq (j^d) \). Hence conditions (ii) and (iii) easily follow.

Simpler criterions will appear if we consider the problem of existence of \( M \) such that \( T_{i,j}^{(d)}(M, 1)f_{\mu,\lambda} \) is a nonzero \( U(n-1) \)-high weight vector in Theorem 6.3 or in Theorem 6.5 in the cases \( j = n \) or \( j < n \), respectively.

For what follows, let us recall the definition of the sets \( \mathcal{E}^{\mu}(i, j) \), \( \mathcal{X}^{\mu,\lambda}_d(i, j) \), \( \mathcal{X}^{\mu,\lambda}_d(i, j) \) and Definition 1.1 (see the introduction). We also use the notation \( C^{\mu}(i, t) = i + t - \mu_i - \mu_t \).

**Theorem 6.6.** Let \( 1 \leq i < n, 1 \leq d < p, \mu \in X^+(n-1), \lambda \in X^+(n) \) and \( \mu \leftarrow \lambda \). There exists \( M \subset \{i..n\} \) such that \( T_{i,j}^{(d)}(M, 1)f_{\mu,\lambda} \) is a nonzero \( U(n-1) \)-high weight vector if and only if there exists an injection \( \varepsilon : \mathcal{X}^{\mu,\lambda}_d(i, n) \rightarrow \mathcal{E}^{\mu}(i, n) \) weakly decreasing w.r.t. the first coordinate.

**Proof.** At first, we suppose that such \( M \) exists. Let \( \gamma : M \rightarrow [i..n] \times [1..d] \) be an injection satisfying conditions (i) and (ii) of Theorem 6.3. Condition (ii) immediately implies \( \mathcal{X}^{\mu,\lambda}_d(i, n) \subset \text{Im} \gamma \). Now let \( (m) \in \mathcal{X}^{\mu,\lambda}_d(i, n) \) for some \( m \in M \). Applying condition (i), we easily get \( C^{\mu}(i, m) \equiv 0 \pmod{p} \) and \( m \in \mathcal{E}^{\mu}(i, n) \). Now we can take for \( \varepsilon \) the map with domain \( \mathcal{X}^{\mu,\lambda}_d(i, n) \) partially inverting \( \gamma \).

Now let \( \varepsilon : \mathcal{X}^{\mu,\lambda}_d(i, n) \rightarrow \mathcal{E}^{\mu}(i, n) \) be an injection as in the formulation of the theorem. We put \( M := \text{Im} \varepsilon \) and denote by \( \gamma \) the inverse map of \( \varepsilon \). Condition (ii) of Theorem 6.3 follows from \( \text{Im} \gamma = \mathcal{X}^{\mu,\lambda}_d(i, n) \). Let \( m \in M \).

We have \( m = \varepsilon(t, s) \) for some \( (t, s) \in \mathcal{X}^{\mu,\lambda}_d(i, n) \) and \( C^{\mu}(i, m) \equiv 0 \pmod{p} \). Since \( B^{\mu,\lambda}(i, t) \equiv d - s \pmod{p} \), we have \( B^{\mu,\lambda}(m, t) \equiv d - s \pmod{p} \), which can be reformulated as \( B^{\mu,\lambda}(m, \gamma_1(m)) \equiv d - \gamma_2(m) \pmod{p} \) to conform to condition (iii) of Theorem 6.3.

Before proceeding further, we make a remark on partially ordered sets. Let \( X \) be a finite set with nonstrict partial order \( \preceq \). We put \( \text{cone}(x) := \{ y \in X : y \preceq x \} \).
X : x ≍ y) for any x ∈ X and \( \text{cone}(S) = \bigcup_{x \in S} \text{cone}(x) \) for any \( S \subset X \). A map \( \alpha : A \to B \), where \( A, B \subset X \), is called weakly increasing if \( x ≍ \alpha(x) \) for any \( x \in A \).

**Theorem 6.8.**

There exist \( \gamma \) such that \( \gamma \) gives condition (i). To see that condition (ii) holds, it suffices to use condition (ii). Finally condition (iii) holds, since \( \text{Im} \gamma \subseteq X \). For the rest of the paper, we fix integers \( i, j, d \) and weights \( \mu \in X^+(n-1) \). Let \( K \) be a sequence of integers \( K = (k_1, \ldots, k_d) \) such that \( i \leq k_1 \leq \cdots \leq k_d \leq j \). We put \( \mathbf{X}^{\mu, \lambda, K}(i, j) := \{ (t, s) \in X : B_{\mu, \lambda, K}(i, t) \equiv d-s \pmod p \} \). Using \( K \), we define the following subsets of \( X \) : \( Y_K := \{(t, s) \in X : t \leq k_s \} \) and \( Z_K := \{(t, s) \in X : t \geq k_s \} \). Clearly \( X = Y_K \cup Z_K \). We have

\[
\mathbf{X}_d^{\mu, \lambda, K}(i, j) = \left( \mathbf{X}_d^{\mu, \lambda}(i, j) \cap Y_K \right) \cup \left( \mathbf{X}_d^{\mu, \lambda}(i, j) \cap Z_K \right).
\]

**Theorem 6.8.** Let \( 1 \leq i < j \leq n, 1 \leq d < p, \mu \in X^+(n-1), \lambda \in X^+(n) \) and \( \mu \leftarrow \lambda \).

(i) There exists \( M \subset (i, j) \) such that \( T_{i,j}(d)(M, 1)_{f_{\mu, \lambda}} \) is a nonzero \( U(n-1) \)-high weight vector if and only if there exists an injection \( \varepsilon : \mathbf{X}_d^{\mu, \lambda}(i, j) \to \mathcal{C}(i, j) \) weakly decreasing w.r.t. the first coordinate and for any sequence of integers \( K = (k_1, \ldots, k_d) \) such that \( i \leq k_1 \leq \cdots \leq k_d \leq j \) and \( K \neq (j^d) \) there exists an injection \( \theta_K : \{ i \} \cup \text{Im} \varepsilon \to X_d^{\mu, \lambda, K}(i, j) \) weakly increasing w.r.t. the first coordinate.

(ii) There exists \( M \subset (i, j) \) such that \( T_{i,j}(d)(M, 1)_{f_{\mu, \lambda}} \) is a nonzero \( U(n-1) \)-high weight vector if and only if \( (j-1, 1) \in X_d^{\mu, \lambda}(i, j), (j-1, 1) \notin X_d^{\mu, \lambda}(i, j), \) there exists an injection \( \varepsilon : \mathbf{X}_d^{\mu, \lambda}(i, j) \to \mathcal{C}(i, j) \) weakly decreasing w.r.t. the first coordinate and an injection \( \tau : \mathbf{X}_d^{\mu, \lambda}(i, j) \to \mathbf{X}_d^{\mu, \lambda}(i, j) \setminus \{(j-1, 1)\} \) weakly increasing w.r.t. the first coordinate and weakly decreasing w.r.t. the second coordinate.

**Proof.** (i) Suppose that the required set \( M \) exists and let \( \gamma_K : M \to X \) be injections as in Theorem 6.5. It follows from condition (iii) of Theorem 6.5 that \( X_d^{\mu, \lambda}(i, j) \subset \text{Im} \gamma \). Now let \( \gamma_{(j^d)}(m) \in X_d^{\mu, \lambda}(i, j) \) for some \( m \in M \).

Condition (i) of Theorem 6.5 and the definition of \( X_d^{\mu, \lambda}(i, j) \) imply \( m \in \mathcal{C}(i, j) \). It is clear now that \( \varepsilon \) can be chosen so that \( \gamma_{(j^d)}(m) \otimes \varepsilon = \text{id}_{X_d^{\mu, \lambda}(i, j)} \).

For a sequence of integers \( K = (k_1, \ldots, k_d) \) such that \( i \leq k_1 \leq \cdots \leq k_d \leq j \) and \( K \neq (j^d) \), we put \( \theta_K(m) = \gamma_K(m) \) for \( m \in \text{Im} \varepsilon \) and put \( \theta_K(i) \) equal to the pair \((t, s)\) that is mentioned in condition (ii) of Theorem 6.5.

Let \( m \in \text{Im} \varepsilon \). Applying the congruence \( C_{\mu}(i, m) \equiv 0 \pmod p \) and condition (i) of Theorem 6.5 we get \( \theta_K(m) = \gamma_K(m) \in X_d^{\mu, \lambda, K}(i, j) \).

Now suppose, on the contrary, that the required maps \( \varepsilon \) and \( \theta_K \) exist. We put \( M := \text{Im} \varepsilon, \gamma_{(j^d)} := \varepsilon^{-1} \) and \( \gamma_K := \theta_K|_M \) for \( K \neq (j^d) \). We claim that \( M \) and \( \gamma_K \) so defined satisfy conditions (i) (ii) of Theorem 6.5.

Let \( m \in M \) and \( K = (k_1, \ldots, k_d) \) be a sequence of integers such that \( i \leq k_1 \leq \cdots \leq k_d \leq j \). Since \( \gamma_K(m) \in X_d^{\mu, \lambda, K}(i, j) \) and \( C_{\mu}(i, m) \equiv 0 \pmod p \), we have \( B_{\mu, \lambda, K}(m, \gamma_K, 1(m)) \equiv d - \gamma_K, 2(m) \pmod p \), which gives condition (i). To see that condition (ii) holds, it suffices to \( \theta_K(i) \) for \((t, s)\). Finally condition (iii) holds, since \( \text{Im} \gamma_{(j^d)} = X_d^{\mu, \lambda}(i, j) \).
\(\text{(ii)}\) Let \(\varepsilon\) and \(\tau\) be injections as in the formulation of the theorem. For a sequence of integers \(K = (k_1, \ldots, k_d)\) such that \(i \leq k_1 \leq \cdots \leq k_d \leq j\) and \(K \neq (j^d)\), we define \(\theta_K : \{i\} \cup \text{Im} \varepsilon \to \mathcal{X}_d^{\mu,\lambda,K}(i, j)\) by

\[
\theta_K(x) := \begin{cases} 
(j - 1, 1) & \text{if } x = i; \\
\varepsilon^{-1}(x) & \text{if } i < x \text{ and } \varepsilon^{-1}(x) \in Y_K; \\
\tau(\varepsilon^{-1}(x)) & \text{if } i < x \text{ and } \varepsilon^{-1}(x) \in Z_K.
\end{cases}
\]

Clearly, these maps \(\varepsilon\) and \(\theta_K\) satisfy conditions of part \(\text{(i)}\) of the current theorem (see \(6.9\)).

Now, on the contrary, let \(\varepsilon\) and \(\theta_K\) be injections as in part \(\text{(i)}\). For any sequence of integers \(K = (k_1, \ldots, k_d)\) such that \(i \leq k_1 \leq \cdots \leq k_d \leq j\) and \(K \neq (j^d)\), we have

\[
|\mathcal{X}_d^{\mu,\lambda,K}(i, j)| \geq |\text{Im} \theta_K| = |\{i\} \cup \text{Im} \varepsilon| = 1 + |\mathcal{X}_d^{\mu,\lambda}(i, j)|.
\] (6.10)

Putting \(K = (j - 1, j^{d-1})\) and applying \(6.9\), we get

\[
|\mathcal{X}_d^{\mu,\lambda}(i, j) \cap Y_{(j-1,j^{d-1})}| + |\mathcal{X}_d^{\mu}(i, j) \cap Z_{(j-1,j^{d-1})}| = |\mathcal{X}_d^{\mu,\lambda,(j-1,j^{d-1})}(i, j)| \\
\geq 1 + |\mathcal{X}_d^{\mu,\lambda}(i, j) \cap Y_{(j-1,j^{d-1})}| + |\mathcal{X}_d^{\mu,\lambda}(i, j) \cap Z_{(j-1,j^{d-1})}|.
\]

Hence \(|\mathcal{X}_d^{\mu}(i, j) \cap Z_{(j-1,j^{d-1})}| \geq 1 + |\mathcal{X}_d^{\mu,\lambda}(i, j) \cap Z_{(j-1,j^{d-1})}|\). Since \(Z_{(j-1,j^{d-1})} = \{(j - 1, 1)\}\), we have \((j - 1, 1) \in \mathcal{X}_d^{\mu}(i, j)\) and \((j - 1, 1) \notin \mathcal{X}_d^{\mu,\lambda}(i, j)\).

To prove the existence of \(\tau\), we apply Proposition \(6.7\). We introduce the partial order \(\preceq\) on \(X\) by \((a, b) \preceq (x, y) \iff a \leq x \& b \geq y\). Take a nonempty subset \(S \subset X\). Then we have cone(\(S\)) = \(Z_K\) for \(K = (k_1, \ldots, k_d)\), where \(k_s = \min(\text{cone}(S) \cap (Z \times \{s\}))\) \(\cup \{j\}\). Notice that \(K \neq (j^d)\) and \((j - 1, 1) \in Z_K\), since \(S \neq \emptyset\). By \(6.9\) and \(6.10\), we have

\[
1 + |\mathcal{X}_d^{\mu,\lambda}(i, j) \cap Y_K| + |\mathcal{X}_d^{\mu,\lambda}(i, j) \cap Z_K| = 1 + |\mathcal{X}_d^{\mu,\lambda}(i, j)| \\
\leq |\mathcal{X}_d^{\mu,\lambda,K}(i, j)| = |\mathcal{X}_d^{\mu,\lambda}(i, j) \cap Y_K| + |\mathcal{X}_d^{\mu}(i, j) \cap Z_K| \\
= |\mathcal{X}_d^{\mu,\lambda}(i, j) \cap Y_K| + |\mathcal{X}_d^{\mu}(i, j) \cap \{j - 1, 1\}) \cap Z_K| + 1.
\]

Hence

\[
|\mathcal{X}_d^{\mu,\lambda}(i, j) \cap Z_K| \leq |(\mathcal{X}_d^{\mu}(i, j) \setminus \{(j - 1, 1)\}) \cap Z_K|,
\]

which by Proposition \(6.7\) yields the required map \(\tau\). \(\square\)
GENERALIZATION OF LOWERING OPERATORS FOR GLₙ  

7. APPENDIX: LIST OF NOTATIONS

\( \mathbb{Z}, \mathbb{Q}, \mathbb{K} \) sets of integers and rationals, algebraically closed field of characteristic \( p > 0 \);

\( L_n(\lambda) \) irreducible \( GL_n \)-module with highest weight \( \lambda \), p. 11

\( X(n) \) \( \mathbb{Z}^n \), weights, p. 2

\( X^+(n) \) \( \{ \lambda \in \mathbb{Z}^n : \lambda_1 \geq \cdots \geq \lambda_n \} \), dominant weights, p. 2

\( C^\mu(i,j) \) \{ \( t \in (i..j) : t - i + \mu_t - \mu_i \equiv 0 \mod p \} \}, p. 2

\( \mathcal{X}_d^\mu(i,j) \) \{ \( (t, s) \in [i..j] \times [1..d] : t - i + \mu_t - \mu_i \equiv d - s \mod p \} \}, p. 2

\( \mathcal{X}_d^\mu(i,j) \) \{ \( (t, s) \in [i..j] \times [1..d] : t - i + \mu_t - \mu_i \equiv d - s \mod p \} \}, p. 2

\( [i..j] \), etc. \( \{ x \in \mathbb{Z} : i \leq x \leq j \} \), etc., p. 8

\( \varepsilon_i \) (0, 0, 0, 0, 0, 0) having 1 at position \( i \), p. 8

\( \alpha(i,j), \alpha_i \) \( \varepsilon_i - \varepsilon_j, \alpha(i,i+1) \) resp., p. 8

\( x^\alpha \) \( x \cdots (x - n + 1) \) if \( n \geq 0 \) and \( 1/(x + 1) \cdots (x - n) \) if \( n < 0 \), p. 8

\( \delta_P \) 1 or 0 if \( P \) is true or false respectively, p. 8

\( U_{\mathbb{Q}}(n) \) universal enveloping algebra of \( \mathfrak{g} \mathfrak{l}_{\mathbb{Q}}(n) \), p. 11

\( \mathcal{U}(n) \) hyperalgebra over \( \mathbb{Z} \), p. 11

\( UT(n) \) set of integer strictly upper triangular \( n \times n \)-matrices with nonnegative entries, p. 11

\( \mathbb{F}(N), \mathbb{E}(N), \mathbb{H}(N) \) \( \prod_{1 \leq a < b \leq n} \mathbb{E}_{a,b}^{(N_{a,b})}, \prod_{1 \leq a < b \leq n} \mathbb{E}_{a,b}^{(N_{a,b})} \) resp., p. 11

\( U^0_{\mathbb{Q}}(n) \) \( \mathbb{Q} \)-subalgebra of \( U_{\mathbb{Q}}(n) \) generated by \( \mathcal{H}_1, \ldots, \mathcal{H}_n \), p. 11

\( \tau_N \) automorphism of \( U^0_{\mathbb{Q}}(n) \) satisfying (2.2), p. 11

\( \bar{U}(n) \) right ring of quotients of \( U_{\mathbb{Q}}(n) \) with respect to \( U^0_{\mathbb{Q}}(n) \setminus \{0\} \), p. 11

\( \mathbb{I}_S, \bar{I}_S \) left ideals generated by \( \mathbb{E}_i^{(r)} \) for \( r \geq 1, i \in \mathbb{S} \) in \( \mathcal{U}(n) \) and \( \bar{U}(n) \) resp., p. 11

\( \mathbb{J}^{(C)}, \bar{J}^{(C)} \) left ideals generated by elements of weight with \( \alpha_i \)-coefficient \( > c_i \) for some \( i \) in \( \mathbb{S} \) and \( \bar{U}(n) \) resp., p. 11

\( \mathbb{C}(i,j), \mathbb{B}(i,j) \) \( j - i + \mathbb{H}_i - \mathbb{H}_j \) and \( j - i + \mathbb{H}_i - \mathbb{H}_{j+1} \) resp. p. 11

\( \mathbb{S}_{i,j} \) Carter–Lusztig lowering operator, p. 6

\( \theta_i \) p. 7

\( E_j(K,L) \) Definition 24.4 p. 8

\( K^{(i,j)}, K^{(j)} \) p. 9

\( T_{i,j}^{(d)}(M,R) \) Definition 5.4 p. 11

\( B^{C,i}(k,t) \) \( \mathbb{B}(i,t) + c_i - c_i + \delta_{i<k}(c_{i+1} - c_i) \), p. 17

\( \rho^{(C)}(i,j,K,L,M,R) \) equations (4.1) and (4.2), p. 18

\( f_{i,j}^{(d)}(M), g_{i,j}^{(d)}(M) \) p. 23

\( G_{i,j}^{(d)}(M,N) \) p. 23

\( U(n) \otimes_{\mathbb{Z}} \mathbb{K} \), hyperalgebra over \( \mathbb{K} \), p. 25
\[ \pi_\lambda \ \text{K-algebra homomorphism from } U^0(n) \text{ to } K \ \text{taking} \ \left( \frac{b_i}{r} \right) \ \text{to} \ \left( \frac{a_i}{r} \right) + p \mathbb{Z}, \ \text{p. } 26 \]

\[ \nabla_n(\lambda) \ \text{co-Weyl module with highest weight } \lambda, \ \text{p. } 26 \]

\[ f_{\mu,\lambda} \ \text{U}(n-1)\text{-high weight vector of } \nabla_n(\lambda) \text{ of weight } \mu, \ \text{p. } 26 \]

\[ f_\lambda \ \text{image of } f_{\lambda_1,\ldots,\lambda_{n-1},\lambda}, \ \text{p. } 26 \]

\[ \text{cf}(v) \ \text{element of } K \ \text{such that} \ E_1^{(a_1)} \cdots E_{n-1}^{(a_{n-1})} v = \text{cf}(v)f_\lambda \ \text{for } v \in \nabla_n(\lambda) \ \text{of weight} \ \lambda - \alpha_1 - \cdots - \alpha_{n-1} \alpha_{n-1}, \ \text{p. } 26 \]

\[ S_{i,j} \ \text{image of } S_{i,j} \ \text{in } U(n), \ \text{p. } 27 \]

\[ T^{(d)}_{i,j}(M, R) \ \text{image of } T^{(d)}_{i,j}(M, R) \ \text{in } U(n), \ \text{p. } 27 \]

\[ \xi^{(C)}(i, j, K, L, M, R) \ \text{image of } \rho^{(C)}(i, j, K, L, M, R) \ \text{in } U(n), \ \text{p. } 27 \]

\[ I_S, J^{(C)} \ \text{images of } I_S \ \text{and } J^{(C)} \ \text{in } U(n) \ \text{resp., p. } 27 \]

\[ B^{\mu,\lambda}(i, t) \ \text{t} - i + \mu_i - \lambda_{t+1}, \ \text{p. } 27 \]

\[ B^{\mu,\lambda,k}(i, t) \ \text{t} - i + \mu_i - \mu_{t+1} \ \text{if } k \leq t \ \text{or} \ \text{t} - i + \mu_i - \lambda_{t+1} \ \text{if } k > t, \ \text{p. } 27 \]

\[ C^{\mu}(i, t) \ \text{t} - i + \mu_i - \mu_t, \ \text{p. } 32 \]

\[ \text{cone}(x) \ \{ y \in X : x \leq y \}, \ \text{p. } 33 \]

\[ \text{cone}(S) \ \bigcup_{x \in S} \text{cone}(x), \ \text{p. } 33 \]

\[ X_{d,\mu,K}(i, j) \ \{(t, s) \in [i..j] \times [1..d] : B^{\mu,\lambda,K}(i, t) \equiv d - s \ \text{(mod } p)\}, \ \text{p. } 33 \]

\[ Y_K \ \{(t, s) \in [i..j] \times [1..d] : t < k_s\}, \ \text{p. } 33 \]

\[ Z_K \ \{(t, s) \in [i..j] \times [1..d] : t \geq k_s\}, \ \text{p. } 33 \]

**Remark.** See the table on page 5 for the definitions of \( \mathbb{U}^+(a, b), \mathbb{U}^0(a, b), \mathbb{U}^-(a, b), \mathbb{U}^{-0}(a, b), \mathbb{U}^0(a, b), \mathbb{U}^{-0}(a, b), \mathbb{U}^0(a, b), \mathbb{U}^{-0}(a, b) \).

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