NON-UNIQUENESS IN LAW OF THE TWO-DIMENSIONAL SURFACE QUASI-GEOSTROPHIC EQUATIONS FORCED BY RANDOM NOISE

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Abstract. Via probabilistic convex integration, we prove non-uniqueness in law of the two-dimensional surface quasi-geostrophic equations forced by random noise of additive type. In its proof we work on the equation of the momentum rather than the temperature, which is new in the study of the stochastic surface quasi-geostrophic equations. We also generalize the classical Calderón commutator estimate to the case of fractional Laplacians.

1. Introduction

1.1. Motivation from physics and mathematics. Initially investigated in 1948 by Charney [12], the two-dimensional (2-d) surface quasi-geostrophic (QG) equations can be derived as the motion of potential temperature on the boundary upon considering Laplace’s equation on $\mathbb{R}^3$ with the temperature as the partial derivative of the stream function with respect to (w.r.t.) the vertical variable (see [42 equation (2)] and [55 equation (1.2)]). Its dissipative term in the form of a fractional Laplacian appears naturally and models Ekman pumping effect in the boundary layer (see [17 equation (1)]). It has always attracted attention from physicists and engineers due to a wide breadth of applications in atmospheric sciences, meteorology, and oceanography, as well as mathematicians, especially since Constantin, Majda, and Tabak [20] demonstrated similarities between the three-dimensional (3-d) Euler equations and the non-dissipative 2-d QG equations analytically and numerically. On the other hand, the study of partial differential equations (PDEs) in general hydrodynamics forced by random noise can be traced back at least to [60] by Landau and Lifschitz in 1957 (also [2, 70]). The purpose of this manuscript is to study the QG equations forced by random noise of additive type and prove its non-uniqueness in law via probabilistic convex integration.

1.2. Previous works. We denote by $\mathbb{N} \equiv \{1, 2, \ldots\}$, $\mathbb{N}_0 \equiv \{0\} \cup \mathbb{N}$, $A_{a=b}^{(\cdot)} B$ to imply the existence of $C(a, b) \geq 0$ such that $A \leq CB$ due to an equation $(\cdot)$ and $A \approx B$ if $A \lesssim B$ and $B \lesssim A$. For any vector $x$, we denote its $j$-th component by $x_j$ for $j \in \{1, \ldots, n\}$ and $x^\perp \equiv (-x_2, x_1)$ in case $n = 2$. For any square matrix $A$, we denote its $(m, l)$-entry by $A^{ml}$ and its trace-free part by $\tilde{A}$; in particular, we denote the trace-free part of a tensor product $A \otimes B$ by $A \otimes B$. We define $\partial_\iota \equiv \frac{2}{\iota\pi}$ and $\partial_\iota \equiv \frac{2\iota}{\pi}$ for $\iota \in \{1, \ldots, n\}$. The spatial domain of our primary interest is $T^2 = (\mathbb{R}/[-\pi, \pi])^2$. For full generality, let us introduce the generalized QG equations, which was introduced by Kiselev (see [55 equation (1.3)]) and studied by many others (e.g., [66]). We denote by $\theta : \mathbb{R}_+ \times T^2 \mapsto \mathbb{R}$ the potential temperature, $\mathcal{R}$ the Riesz transform vector, $\Lambda^r \equiv (-\Delta)^r$ for any $r \in \mathbb{R}$ to be the Fourier operator with its

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Fourier symbol $|k|^\gamma$; i.e., $\Lambda^\gamma f(x) \equiv \sum_{k \in \mathbb{Z}} |k|^\gamma \hat{f}(k)e^{ikx}$ where $\hat{f}$ denotes the Fourier transform of $f$ and $\hat{f}$ will denote the Fourier inverse of $f$. Under these notations, with $\gamma_1 \in (0, 2)$, $\gamma_2 \in [1, 2]$, and $\nu \geq 0$, the generalized QG equations consist of

$$\partial_t \theta + u \cdot \nabla \theta + \nu \Lambda^{\gamma_1} \theta = 0, \quad u \equiv \Lambda^{1-\gamma_2} R_{\nu} \theta,$$  

(1)
given $\theta^0(x) \equiv \theta(0, x)$ as initial data. Hereafter, for simplicity we assume $\nu = 1$ in case $\nu > 0$. We observe that if the initial data $\theta^0$ is mean-zero, which is a standard convention in the case of a spatial domain being a torus, the solution $\theta(t)$ remains mean-zero for all $t \geq 0$.

The generalized QG equations reduce to the modified QG (MQG) equations introduced by Constantin, Iyer, and Wu [19] when $\gamma_2 = 2 - \gamma_1$ for $\gamma_1 \in (0, 1]$, the classical QG equations when $\gamma_2 = 1$ to which we refer simply as the QG equations, and the Euler equations when $\gamma_2 = 2, \nu = 0$. Due to the fact that the $L^\infty(T^2)$-norm seems to be the most useful bounded quantity in estimates that any smooth solution to (1) possesses and the rescaling property that $\Lambda^{\gamma_1+\gamma_2-2} \theta(\lambda^2 t, \lambda x)$ for any $\lambda \in \mathbb{R}_+$ solves the generalized QG equations if $\theta(t, x)$ does, we call the three cases $\gamma_1 + \gamma_2 \in (2, 4], \gamma_1 + \gamma_2 = 2$, and $\gamma_1 + \gamma_2 \in (1, 2)$, sub-critical, critical, and super-critical cases w.r.t. $L^\infty(T^2)$-norm, respectively. The issue of global existence of the unique solution to the initial value problem of the QG equations has attracted much attention. Resnick in [73] Theorem 2.1 was able to construct a global weak solution to the QG equations, even in case $\nu = 0$, starting from $\theta^0 \in L^p(T^2)$ (see [65] Theorem 1.1) for the case $\theta^0 \in L^p$ for $p > \frac{4}{3}$). Within a class of smooth solutions, the global existence was settled in the sub-critical case (e.g., [22]) and the critical case by breakthrough works [11, 58] (see also [21, 57]). While this problem remains unsolved in the super-critical case, some properties of the solution were studied in various works (e.g., [22, 24, 56]).

Many researchers have studied the following stochastic Navier-Stokes (NS) equations

$$du + [(u \cdot \nabla)u + \nabla p - \nu \Delta u] dt = G(u) dB, \quad \nabla \cdot u = 0,$$

(2)
where $u : \mathbb{R}_+ \times T^n \mapsto \mathbb{R}^n$, $p : \mathbb{R}_+ \times T^n \mapsto \mathbb{R}$, and $G(u) dB$ represent velocity field, pressure field, and random noise to be described subsequently in detail, respectively. The case $\nu = 0$ reduces (2) to the stochastic Euler equations. Flandoli and Romito [41] proved the global existence of a Leray-Hopf weak solution to the NS equation forced by an additive noise; we call their solution Leray-Hopf because it satisfies the appropriate energy inequality (see [41] MP3 on p. 421)). However, path-wise uniqueness of their solution remains unknown, and their solution was not probabilistically strong, which requires that the solution is adapted to the canonical filtration generated by the noise. In general, upon any probabilistic Galerkin approximation (e.g., [41] Appendix A), if one takes mathematical expectation to obtain uniform estimates, then the resulting solution becomes probabilistically weak. In case the noise is additive, one can consider an Ornstein-Uhlenbeck process and work on a random PDE satisfied by the difference between the solution and this Ornstein-Uhlenbeck process; nonetheless the solution obtained via this approach as a converging subsequence will depend on the fixed realization and hence not be probabilistically strong (see [40] p. 841)). By Yamada-Watanabe theorem, path-wise uniqueness will lead to the solution becoming probabilistically strong. However, the general consensus was that the proof of the path-wise uniqueness seems to be no easier than its counterpart in the deterministic case and hence a significant amount of effort has been devoted to prove uniqueness in law (e.g., [29] p. 878–879)), which is weaker than path-wise uniqueness according to Yamada-Watanabe theorem and Tanaka’s counterexample [14] Example 2.2]. Due to Cherny’s theorem, uniqueness in law and the existence of probabilistically strong
solution will imply path-wise uniqueness; nonetheless, even the uniqueness in law of the Leray-Hopf weak solution constructed in [41] has also remained open.

For the QG equations forced by random noise, Zhu in [85, Definition 4.2.1] defined its solution to require in particular the regularity of $L^\infty_T L^2_x \cap L^2_T H^1_x$ and proved the global existence of a solution in case $\gamma_1 > 0$ and the noise is additive in [85, Theorem 4.2.4]. Its proof consisted of analysis on the random PDE satisfied by the difference between the solution to the QG equation forced by a certain random noise and an Ornstein-Uhlenbeck process so that the solution that was constructed therein was probabilistically weak. In [85, Definition 4.3.1 and Theorem 4.3.2] Zhu also defined and constructed a martingale solution via the Galerkin approximation, and thus the solution is also probabilistically weak. The path-wise uniqueness of such a martingale solution in the sub-critical case with a multiplicative noise was also proven in [85, Theorem 4.4.4] under the hypothesis that its initial data $\theta$ is in $L^p_{T \times x}$ for $p$ sufficiently large. We refer to large deviation principle in the sub-critical case in [85, Sections 4.5 and 4.6], the existence of Markov selections for any $\gamma_1 > 0$, strong Feller property, support theorem, the existence of a unique invariant measure, and exponential convergence if $\gamma_1 > \frac{4}{3}$ respectively in [84, Theorems 4.2.5, 4.3.3, 4.3.8, 4.3.10, and 4.4.5] (also [83]) where the fourth result was improved to $\gamma_1 > 1$ in [74, Theorems 5.9 and 5.13], as well as regularization of multiplicative noise in [6].

Next, we review recent developments on the convex integration that was fueled in the past decade with the goal to prove Onsager’s conjecture [71], the positive direction being that every weak solution $u \in C^\alpha(T^3)$ to the 3-d Euler equations for $\alpha > \frac{1}{4}$ conserves its energy and the negative direction being the existence of a weak solution $u \in C^\alpha(T^3)$ for $\alpha < \frac{1}{4}$ that fails to conserve its energy. While Constantin, E, and Titi [18] and Eyink [38] in 1994 proved its positive direction, De Lellis and Székelyhidi Jr. [34], by partially using ideas from [69], proved the existence of a solution $u \in L^\infty_{T \times x}$ to the $n$-d Euler equations for $n \in \mathbb{N} \setminus \{1\}$ with compact support in space and time, extending previous works of Scheffer [75] and Shnirelman [77] that proved analogous results with regularity in $L^2_{T \times x}$ in the 2-d case. After further extensions (e.g., [4, 35, 36]), Isett [50] completely proved the negative direction of the Onsager’s conjecture in any dimension $n \geq 3$. Via an introduction of intermittent Beltrami waves, Buckmaster and Vicol [8] proved the non-uniqueness of weak solutions to the 3-d NS equations, and it was followed by many more: [5,16,62,64] on the NS equations; [10] on power-law model; [63] on Boussinesq system; [1,39] on magnetohydrodynamics (MHD) system; [28,67,68] on transport equation.

Concerning active scalar equations, Córdoba, Faraco, and Gancedo [26] applied the convex integration approach of [34] to the incompressible porous media equations. Subsequently, Isett and Vicol [52] proved the existence of weak solutions with compact support for general active scalar equations $\partial_t \theta + u \cdot \nabla \theta = 0$ as long as $u = T[\theta]$ is divergence-free and $T$ is a Fourier operator with a Fourier symbol that is not an odd function of frequency; consequently, their result excluded the QG equations (1). By defining $\nu = \Lambda^{-1} u$ as a potential velocity, Buckmaster, Shkoller, and Vicol in [7] worked on the QG momentum equations

$$\begin{align*}
\partial_t \nu + (u \cdot \nabla)\nu - (\nabla \nu)^T \cdot u + \nabla p + \Lambda \nu \nabla \theta &= 0, \quad \nabla \cdot \nu = 0, \\
u &= \Lambda \nu;
\end{align*}
$$

(3a)

the relationship between $\nu$ and the solution $\theta$ to (1) is

$$\theta = -\nabla \cdot \nu.$$

(4)
Considering that the solution to (5) starting from $v(0) \in H^1_0(\mathbb{T}^3)$ satisfies $\|v(t)\|_{H^1_0}^2 \leq \|v(0)\|_{H^1_0}^2$, which can be seen from the identity of

$$(u \cdot \nabla) v - (\nabla v)^T \cdot u = u^+ (\nabla^+ \cdot v)$$

(see [65, Theorem 1.2 (ii)] and [73, Section 2.2]), the authors in [7] were able to construct a solution with prescribed $\dot{v}$ (see [65, Theorem 1.2 (ii)] and [73, Section 2.2]), subsequently, Isett and Ma [51] provided a direct approach by working on the actual QG equations; moreover, Cheng, Kwon, and Li [13] proved non-uniqueness of steady-state weak solution to the QG equations.

The impact of the development of convex integration has spilled over to the community of researchers on stochastic PDEs recently. First, Breit, Feireisl, and Hofmanová [3] and Chiodaroli, Feireisl, and Flandoli [15] proved path-wise non-uniqueness of certain compressible Euler equations using some ideas of convex integration from [35] (see [46] on the incompressible Euler equations). Second, Hofmanová, Zhu, and Zhu [47] were able to prove non-uniqueness in law of the 3-d NS equations via approach similar to [9, Section 7] where non-uniqueness was proven via opposite energy inequality. This result inspired more to follow: [72, 78–82]. Furthermore, Hofmanová, Zhu, and Zhu [47] were able to prove non-uniqueness in law of the 3-d NS equations with prescribed initial data or prescribed energy by adapting the approach of [10]; we emphasize that [47, Theorem 1.1] does not imply [45, Theorem 1.2] due to the difference in the solutions’ regularity. We also refer to [43] on the 3-d NS equations forced by space-time white noise and [59] on transport equation forced by random noise of three types: additive; linear multiplicative; transport.

2. Statement of main results and new ideas to overcome difficulties

2.1. Statement of main results. Following [3] we consider

$$dv + [(u \cdot \nabla) v - (\nabla v)^T \cdot u + \nabla p + \Lambda^* v] dt = G(v) dB, \quad \nabla \cdot v = 0,$$

(6a)

$$u = \Lambda v,$$

(6b)

where $G(v)$ becomes a $G^*$-Wiener process that is independent of $v$ to be described in detail subsequently; here, we denoted an adjoint of $G$ by $G^*$. Given any deterministic initial data $v^{in} \in H^1_0(\mathbb{T}^3)$, via a Galerkin approximation under a suitable condition on $G$, one can construct a solution $v$ to (6) that satisfies

$$\mathbb{E}^P[\|v(t)\|_{H^1_0}^2] \leq \|v^{in}\|_{H^1_0}^2 + t \text{Tr}(\Delta)^{\frac{1}{2}} GG^*)$$

(7)

where $\mathbb{E}^P$ represents the mathematical expectation w.r.t. a probability measure $P$.

**Theorem 2.1.** Suppose that $\gamma_1 \in (0, \frac{1}{2})$ and $G(v)dB$ in (6) is a $G^*$-Wiener process such that

$$\text{Tr}(\Delta)^{\frac{1}{2}} + 2\gamma G^*) < \infty$$

(8)

for some $\sigma > 0$. Then, given any $T > 0, K > 1, \text{ and } \kappa \in (0, 1)$, there exist a sufficiently small $\eta \in (0, \frac{1}{2})$ and a $P$-almost surely (a.s.) strictly positive stopping time $\tau$ such that

$$P(\tau \geq T) > \kappa$$

(9)

and the following is additionally satisfied. There exists a $(\mathcal{F}_t)_{t \geq 0}$-adapted process $v$ that is a weak solution of (6) starting from a deterministic initial data $v^{in}$, satisfies

$$\text{esssup}_{\omega \in \Omega} \|v(\omega)\|_{C_t^1 C_x^{1+\eta}} + \text{esssup}_{\omega \in \Omega} \|v(\omega)\|_{C_x^{1+\eta}} < \infty \quad \forall \eta \in \left(0, \frac{1}{3}\right),$$

(10)
Hypothesis 2.1. (equations (3.3)-(3.9)) For all \( \sigma > 0 \) and handle some terms in the Reynolds stress (e.g., \( \epsilon \)).

The hypothesis (8) is needed to gain sufficiently high spatial regularity of \( z \) from (6) in Proposition 4.4 and handle some terms in the Reynolds stress (e.g., \( R_{\text{conv}} \) in (122c)).

Theorem 2.2. Suppose that \( \gamma_1 \in (0, \frac{1}{2}) \) and \( G(v)dB \) in (6) is \( GG^*-\) Wiener process such that (8) holds for some \( \sigma > 0 \). Then non-uniqueness in law for (6) holds on \( [0, \infty) \).

Moreover, for all \( T > 0 \) fixed, non-uniqueness in law holds for (6) on \( [0, T] \).

To the best of the author’s knowledge, this is the first manuscript to study the QG momentum equations forced by random noise.

2.2. Difficulties and new ideas to overcome them. Let us elaborate on the proof of non-uniqueness of the deterministic QG equations in (7). The authors fixed an arbitrary smooth function \( \mathcal{H} : [0, T] \to \mathbb{R}_+ \) with compact support, defined

\[
\lambda_q \triangleq \lambda_0^q \text{ for } \lambda_0 \in 5\mathbb{N} \text{ sufficiently large and } \delta_q \triangleq \lambda_0^2 \lambda^q, \tag{12}
\]

crafted a sequence of solutions \((v_0, p_0, \hat{R}_0) \equiv (0, 0, 0)\) and \((v_q, p_q, \hat{R}_q)_{q \in \mathbb{N}}\) to

\[
\begin{align*}
\frac{\partial v_q + (u_q \cdot \nabla) v_q - (\nabla v_q)^T \cdot u_q + \nabla p_q + \Lambda^q \cdot v_q = \nabla \cdot \hat{R}_q,}{}
\frac{\nabla \cdot v_q = 0,}{(13a)}
\end{align*}
\]

\[
\begin{align*}
u_q = \Lambda v_q, \tag{13b}
\end{align*}
\]

where \( \hat{R}_q \) is a symmetric trace-free matrix in \( \mathbb{R}^{2 \times 2} \) such that for universal constants \( C_0 > 0 \) and \( \delta_R > 0 \), they satisfy the following inductive hypothesis.

Hypothesis 2.1. (equations (3.3)-(3.9)) For all \( t \in [0, T] \),

(a) \( \text{supp } \hat{v}_q \subset B(0, 2 \lambda_q) \), a ball of radius \( 2 \lambda_q \) centered at the origin,

\[
\|v_q\|_{C^1_t C^1_x} + \|u_q\|_{C^1_x} \leq C_0 \lambda_q \delta_q, \tag{14}
\]

(b) \( \text{supp } \hat{R}_q \subset B(0, 4 \lambda_q) \),

\[
\|\hat{R}_q\|_{C^1_x} \leq \epsilon_R \lambda_{q+1} \delta_{q+1}, \tag{15}
\]

(c)

\[
\|\nabla (u_q \cdot \nabla) v_q\|_{C^1_x} \leq C_0 \lambda_q^2 \delta_q \tag{16}
\]

(d)

\[
\|\nabla (u_q \cdot \nabla) u_q\|_{C^1_x} \leq C_0 \lambda_q^3 \delta_q \tag{17}
\]

(e)

\[
\|\nabla (u_q \cdot \nabla) \hat{R}_q\|_{C^1_x} \leq \lambda_q^2 \delta_q \lambda_{q+1} \delta_{q+1} \tag{18}
\]

(f)

\[
0 \leq \mathcal{H}(t) - \|v_q(t)\|_{H^2_t}^2 \leq \lambda_{q+1} \delta_{q+1}, \tag{19}
\]

(g)

\[
\mathcal{H}(t) - \|v_q(t)\|_{H^2_t}^2 \leq \frac{\lambda_{q+1} \delta_{q+1}}{8} \Rightarrow \hat{R}_q(t) \equiv 0. \tag{20}
\]
The hypothesis \([a]\) gives the spatial regularity of the solution \(v\) to \(\mathcal{H}\) that is derived by taking limit \(q \to +\infty\) and observing the hypothesis \([b]\). The hypothesis \([c]\) gives the temporal regularity of \(v\) while the hypothesis \([d]\) guarantees that the \(H^2(T^2)\)-norm of \(v(t)\) is \(\mathcal{H}(t)\). To explain the role of other hypothesis, let us explain the construction of the solution \(v_{q+1}\) to \([7\text{a}]\) that satisfies the Hypothesis \([7\text{a}].\) The authors in \([7\] let \(q_j \in [0,1]\) be certain smooth cutoff function such that

\[
\sum_{j \in \mathbb{Z}} \chi_j^2(t-j) = 1 \quad \forall \, t \in \mathbb{R}
\]

and defined

\[
\chi_j(t) \triangleq \chi(t^{(q+1)} - j).
\]

They selected \(q_{q+1}\) in \([7\] equation (4.7)] carefully (as we will too in \([52\]), and defined \(\Phi_j\) and \(\hat{R}_{q,j}\) to be respectively the solutions to

\[
(\partial_t + u_q \cdot \nabla)\Phi_j = 0, \quad \Phi_j(\tau_{q+1}, j, x) = \chi,
\]

and

\[
(\partial_t + u_q \cdot \nabla)\hat{R}_{q,j} = 0, \quad \hat{R}_{q,j}(\tau_{q+1}, j, x) = \hat{R}_q(\tau_{q+1}, j, x).
\]

Postponing some details of notations, we mention that the authors of \([7\] defined

\[
v_{q+1} \triangleq v_q + w_{q+1}
\]

where the perturbation \(w_{q+1}\) is defined via

\[
w_{q+1}(t,x) \triangleq \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \chi_j(t) \mathbb{P}_{q+1,k}(a_{k,j}(t,x)b_{q+1}(\lambda_{q+1}\Phi_j(t,x)))
\]

with

\[
\rho(t) \triangleq \frac{1}{(2\pi)^2} \max \left\{ \mathcal{H}(t) - \|v_q(t)\|^2_{H^2}, \lambda_{q+2}^2 \delta_{q+2}^2 \right\}, \quad \rho_j \triangleq \rho(\tau_{q+1}j)
\]

(see Lemma \([32\] for \(\gamma_k\) and \(\Gamma_j\), \([37\] for \(\mathbb{P}_{q+1,k}\), and \([44\] for \(b_k\)). It is shown in \([7\] Lemma 6.1] that the hypothesis \([d]\) guarantees that if \(\rho_j = 0\), then \(\hat{R}_q(\cdot, t) \equiv 0\) for all \(t \in \text{supp} \chi_j\) which, along with \([21\], \([22\]), justified

\[
\text{div} \hat{R}_q(t) = \sum_{j \in \mathbb{Z}, \rho \neq 0} \chi_j^2(t) \text{div}(\hat{R}_q - \hat{R}_{q,j})(t) + \sum_{j \in \mathbb{Z}, \rho \neq 0} \chi_j^2(t) \text{div} \hat{R}_{q,j}(t).
\]

The decomposition \([29\) is absolutely crucial in the estimate of the difficult Reynolds oscillation term because \([24\) indicates that

\[
(\partial_t + u_q \cdot \nabla)(\hat{R}_q - \hat{R}_{q,j}) = (\partial_t + u_q \cdot \nabla)\hat{R}_q,
\]

\[
(\hat{R}_q - \hat{R}_{q,j})(\tau_{q+1}j, x) = 0,
\]

so that we can estimate

\[
|||\hat{R}_q - \hat{R}_{q,j}(t)|||_{C_1} \leq \int_{\tau_{q+1}j}^t |||\hat{R}_q(\cdot, s) - \hat{R}_{q,j}(\cdot, s)|||_{C_1} ds \leq (t - \tau_{q+1}j)\lambda_{q+1}^2 \lambda_{q+1}^2 \delta_{q+1}^2,
\]

\([31\text{b}])\]

\[
|||\hat{R}_q - \hat{R}_{q,j}(t)|||_{C_1} \leq \int_{\tau_{q+1}j}^t |||\hat{R}_q(\cdot, s) - \hat{R}_{q,j}(\cdot, s)|||_{C_1} ds \leq (t - \tau_{q+1}j)\lambda_{q+1}^2 \lambda_{q+1}^2 \delta_{q+1}^2,
\]
and thereby estimate \( \sum_{j \in \mathbb{Z}^d, j \neq 0} \chi_j^2(t)(\hat{R}_q - \hat{R}_{q_j})(t) \) in (29). Finally, having used the hypothesis (e) at level \( q \) to get the necessary bound on the Reynolds oscillation term and thereby prove the hypothesis (b) at level \( q + 1 \), one now needs to prove the hypothesis (e) at level \( q + 1 \) and the hypothesis (d) is needed for that purpose.

Let us now list some immediate difficulties in adapting this proof to the stochastic case.

(A) The primary difficulty in adapting this approach of [7] comes from the hypothesis (e) at level \( q \). We need to transform the problem of (63) to a random PDE, add a Reynolds stress as a force, and consider its solution inductively (see [80]). E.g., in the case of the 3-d NS equations (2), forced by an additive noise, authors in [45] let \( z \) be a solution to a Stokes problem forced by the same noise (cf. (66)), defined \( y = u - z \) to obtain a random PDE solved by \( y \) that has the nonlinear term of

\[
(u \cdot \nabla)u = \text{div}((y + z)\hat{\Phi}(y + z)) + \nabla \frac{|y + z|^2}{3}
\]  

(32)

so that \( \frac{|y + z|^2}{3} \) can be part of its pressure term. We observe that \( (y + z)\hat{\Phi}(y + z) \) in (32) is trace-free and symmetric and hence can become a part of the Reynolds stress. However, this implies that the Reynolds stress \( \hat{R}_q \) will be in \( C^1 \) only for \( y < \frac{1}{2} \) due to the presence of \( z \) (see [66] and Proposition [7]). Therefore, the corresponding Reynolds stress cannot possibly satisfy the hypothesis (e).

(B) With a direct application of Hypothesis 2.1 to the stochastic case out of the picture, it is tempting to try to adapt to (3) the convex integration scheme of [9, Section 7] on the 3-d NS equations that was proven to be successful in the stochastic case (e.g., [45]). However, this ends up futile, quite clearly because the nonlinear term within (5) is one derivative more singular than that of the NS equations (2), cf. \( (u \cdot \nabla)u \) in (2) and \( (\Lambda \nu \cdot \nabla)\nu - (\nabla \nu)^T \cdot \Lambda \nu \) in (3). More specifically, the convex integration in [9, Section 7] is based on intermittent jets (see [9, equation (7.17)]) that allow delicate \( L^p \)-estimates whereas the convex integration in [7] such as \( w_{q+1} \) and \( a_{q,j} \) in (26)-(27) is only fit for \( L^p \)-estimate. Similarly, adapting the convex integration on the 2-d NS equations in [62, 79] to (3) seems difficult.

In fact, upon a closer look, one realizes additional disadvantages in the case of the QG equations. The term \( (\nabla \nu)^T \cdot \Lambda \nu \) cannot be written in a divergence form and an estimate on \( B((\nabla \nu)^T \cdot \Lambda \nu) \) where \( B \) is an inverse of a divergence from Lemma 3.1 is essentially as difficult as \( (\nabla \nu)^T \cdot \Lambda \nu \) because the frequency support of \( (\nabla \nu)^T \cdot \Lambda \nu \) is not compactly supported away from the origin in general and hence \( B((\nabla \nu)^T \cdot \Lambda \nu) \) is actually two derivatives worse than the counterpart \( w_{q+1} \) in the case of the NS equations. Even for \( (\Lambda \nu \cdot \nabla)\nu \), an analogous attempt to (32) leads us to

\[
(\Lambda \nu \cdot \nabla)\nu = \text{div}(\Lambda(y + z)\hat{\Phi}(y + z)) + \nabla \frac{|\Lambda(y + z) \cdot (y + z)|^2}{2}
\]  

(33)

and we realize that while \( \Lambda(y + z)\hat{\Phi}(y + z) \) is trace-free, it is not symmetric.

(C) Wishful thinking would be to essentially just drop the hypothesis (e) and see if the proof can be modified to go through; this is not impossible if one’s goal is merely to prove non-uniqueness so that we should not have to perform a full-fledged proof of [7] that verifies high regularity of the solution with a prescribed \( H^2([0, T]) \)-norm. Unfortunately, this hope is destroyed the moment we realize that the hypothesis (e) at level \( q \) is used to prove the hypothesis (b) at level \( q + 1 \), as we described in (29)-(31), and the hypothesis (b) is indispensable as that is needed to prove that \( \hat{R}_q \) vanishes in the limit \( q \to +\infty \) and conclude that the limiting solution \( \nu \) solves (3).
(D) Besides these issues, there is another fundamental problem in adapting deterministic function $\mathcal{H} : [0, T] \mapsto \mathbb{R}_+$ with compact support such as $[a, b] \subseteq [0, T]$, constructed a solution $\nu$ to (4) such that $\|\nu(t)\|_2^2 = \mathcal{H}(t)$ which implies that $\nu$ has compact support in time, and thus conclude non-uniqueness arguing that $\nu \equiv 0$ is not the only weak solution that vanishes on the complement of $[a, b]$. In case the QG equations are forced by an additive noise, this argument breaks down because even if its solution vanishes at time $t = a$, a zero function would not be its solution for $t > a$ anyway. Considering the inequality (1) that is satisfied by the solution $\nu$ constructed via a Galerkin approximation, one may be tempted to fix $\mathcal{H}(t) \doteq 2\left(\|\nu^{\text{in}}\|_2^2 + t \text{Tr}((-\Delta)^2GG^*)\right)$ to prove non-uniqueness; unfortunately, while $\mathcal{H}(t)$ must be fixed a priori, the initial data $\nu^{\text{in}}$ cannot be prescribed in the convex integration scheme of (7).

Our new ideas to overcome these difficulties are as follows.

(A) The first idea is to replace $\rho_j$ in (28), and hence in the definition of $a_{k,j}$ in (27) and ultimately $w_{q+1}$ in (26) by $\delta_{q+1}$ (compare (27) and (29)). In particular, this implies that $\rho_j$ in (28) is always strictly positive, removing the necessity to consider distinct sets $\{j : \rho_j \neq 0\}$ and $\{j : \rho_j = 0\}$ in contrast to (7) and thus the hypothesis (g) We also do not need to include the hypothesis (f) because we are willing to prove non-uniqueness by constructing solutions with opposite $\dot{\rho}_{q+1}$.

(B) Second, we will mollify $\dot{R}_q$ in space and time; merely doing this will not make any difference because $\dot{R}_q$ remains non-differentiable in time. Specifically, we define

$$I \doteq \lambda_{q+1}^{-\alpha}$$

(34)

(see (81) for the range of $\alpha$), let $(\phi_i)_{i \geq 0}$ and $(\psi_i)_{i \geq 0}$ be families of standard mollifiers with mass one and compact support respectively on $\mathbb{R}^2$ and in $\tau_{q+1}, 2\tau_{q+1}] \subset \mathbb{R}_+$ (see (52) for a precise definition of $\tau_{q+1}$). Then we extend $\dot{R}_q$ for $q \in \mathbb{N}_0$ to $t < 0$ by its value at $t = 0$, mollify it in space and time to obtain

$$\dot{R}_{\ast} \doteq \dot{R}_q \ast_{x} \phi_t \ast_{t} \varphi_t,$$

(35)

and replace $\dot{R}_{\ast}(\tau_{q+1}, j, x)$ in (24b) by $\dot{R}_{\ast}(\tau_{q+1}, j, x)$ (see (128b)). This will allow us to obtain (29) and (30) with $\dot{R}_{\ast}$ therein replaced by $\dot{R}_{\ast}$ (see (171)) leading us to (31) in which we do not even need (18) because we can estimate within (31)

$$\|\partial_s + u_q \cdot \nabla\dot{R}_q(t)\|_{L_\infty} \leq \|\partial_s \dot{R}_q\|_{L_\infty} + \|u_q\|_{L_\infty} \|\nabla \dot{R}_q\|_{L_\infty} \leq \lambda_{q+1}^{-1} \delta_{q+1} + \lambda_{q+1}^{2} \delta_{q+1}^{2}$$

(36)

by mollifier estimates, Hypothesis (2.1 (a) and (b)) (see (212)), completely removing the necessity for the hypothesis (e) and consequently also the hypothesis (d).

(C) The two major changes we suggested above have many consequences. One important ingredient of any convex integration scheme is to cancel out the most difficult term in the Reynolds stress oscillation term. The changes we suggested, along with many others, must be made carefully to preserve the crucial cancellation (see (218)). Moreover, replacing $\text{div} \dot{R}_{\ast}$ in (29) by $\text{div} \dot{R}_{\ast}$ leads to the necessity of considering the mollified equation; i.e., we mollify not just $\dot{R}_{\ast}$ but all other functions and work on the equation satisfied by the mollified functions (see (111)). This leads to an additional Reynolds stress error that was absent in (7), informally

$$R_{\text{Com1}} \doteq \text{Nonlinear term mollified} - \text{Nonlinear term with each term therein mollified}$$
For which we can apply mollifier estimates and end up with a bound that is smaller if we take large \( \alpha > 0 \). On the other hand, \( \| v^{\alpha} - v^{\beta} \|_{H^1_x} \leq C \| v \|_{H^1_x} \) already reveals that larger \( \alpha > 0 \) implies a worse bound, and so do the bounds for other Reynolds stress errors. Thus, there is the game of whether we can find a non-empty interval for \( \alpha \) such that the addition of the new Reynolds stress error \( R_{Com1} \) still leads to the closure of the necessary estimates; in fact, we will also see another error \( R_{Com2} \) (see \( (122a) \)). In this regard, we will consider \( z_q \) (see \( (79) \)) rather than \( z \) at every step which is the only way to guarantee the frequency support of \( \tilde{R}_q \) to be contained inside \( B(0,4\lambda_q) \) and control the errors; otherwise, we get \( l^{-1} = \lambda_q^{\alpha} \) instead of \( \lambda_q \) for any spatial derivative on \( \tilde{R}_q \) and we will not be able to close our estimates as \( \alpha > 1 \) from \( (81) \). As we are going to prove non-uniqueness by constructing a solution with opposite \( H^2(T^2) \)-inequality, in contrast to the deterministic case in \( (7) \) (recall the difficulty \( (D) \)), we also need to come up with an appropriate solution to \( (14) \) at step \( q = 0 \) rather than just \( (v_0, \tilde{R}_0) \equiv (0,0) \) in \( (7) \). We also change the definition of \( \lambda_q \) and \( \delta_q \) from \( (12) \) to

\[
\lambda_q \doteq a^{\beta}\ v \quad \text{for} \ a \in \mathbb{N}, b \in \mathbb{N} \quad \text{sufficiently large and} \ \delta_q \doteq q^{2\beta}
\]

(37)

because the definition in \( (12) \) does not distinguish between \( \lambda_0 \) and \( \lambda_1 \) which makes the necessary computations at the step \( q = 0 \) very difficult. Finally, we will replace the convectons by the velocity field \( u_q \) within the transport equations \( (23)-(24) \) to the convections by a sum of \( u \) and a mollified Ornstein-Uhlenbeck process (see \( (126a) \) and \( (128a) \)); this allows us to better handle the Reynolds stress transport error, as the author discovered in \( (81) \) (see also \( (72) \)).

In Section 5 we give a minimum amount notations and previous results on convex integration, leaving the rest to the Appendix. In Section 4 we prove Theorems \( (2.1) \), \( (2.2) \). In fact, we prove the main steps of convex integration, specifically Propositions \( (4.7)-(4.10) \), for the generalized QG equations with the velocity field \( u = \Lambda^{1-\gamma_2}R^{1-\theta} \) according to \( (1) \) or equivalently the momentum equation \( (38) \)

\[
u = \Lambda^{2-\gamma_2}v.
\]

(38)

This can be derived following the same reasoning in \( (7) \) Sections 1.4 and A.1 (see also \( (72) \) Section 2.2); i.e., the self-adjoint positive-definite operator that is used to define the metric on the Lie algebra associated to the group of volume-preserving diffeomorphisms would be \( \Lambda \doteq \Lambda^{2-\gamma_2} \) so that the cases \( \gamma_2 = 1 \) and \( \gamma_2 = 2 \) give the QG and the Euler equations, respectively. The corresponding solutions emanating from initial data \( v^m \in H^{1-\frac{2\beta}{\gamma_2}}(T^2) \) constructed via Galerkin approximation would satisfy

\[
\mathbb{E}^P \left[ \| v(t) \|_{H^{1-\frac{2\beta}{\gamma_2}}}^2 \right] \leq \| v^m \|_{H^{1-\frac{2\beta}{\gamma_2}}}^2 + t \text{Tr}((\nabla^2)^{\frac{2\beta}{\gamma_2}}G' G^T),
\]

(39)

(cf. \( (7) \)). The proof of Proposition \( (4.1) \) concerning the existence of martingale solutions via a Galerkin approximation relies on Proposition \( (5,3) \) of Calderón commutator which we can also generalize (see Proposition \( (5,3) \) (2)). Unfortunately, our proof in the case of the generalized QG equations ultimately break down at the last step of the proof of Theorem \( (2.1) \) upon relying on the Calderón commutator estimate. Especially due to \( (92b) \), we need to reduce \( \beta > 1 - \frac{2\beta}{2\gamma_2} \) sufficiently close to \( 1 - \frac{2\beta}{2\gamma_2} \) and this makes \( \epsilon \in (0, \frac{1}{2} - \frac{2\beta}{2\gamma_2}) \) quite small. If the size of \( \epsilon > 0 \) was nontrivial, then we can deduce the existence of the solution to the generalized QG equations constructed via convex integration by relying on Proposition \( (5,3) \) (2). Regardless, we present in such a general way in hope that these computations may be of use in future works on the generalized QG equations.
Remark 2.1. Significant progress on this manuscript was made in early May 2022. Then, on 05/26/2022 Hofmanov, Zhu, and Zhu [49] posted a paper on ArXiV in which, among several results, they elegantly proved the non-uniqueness of the QG equations forced by random noise. Our work was completed entirely independently of [49], as can be seen clearly from the difference in results and proofs. While [49] works directly on the equation of the temperature θ, based on the convex integration technique of [13] on the steady-state weak solution to the deterministic QG equations with the probabilistic approach from [47], we work on the equation of the momentum v, based on the convex integration technique of [2] with the probabilistic strategy from [45] and some key ideas from [81]. Furthermore, the random noise tackled is white in space in [49] while white in time in this manuscript.

3. Preliminaries

3.1. Notations and assumptions. We write $C_0^\infty \equiv \{ f \in C^\infty (\mathbb{T}^2) : \int_{\mathbb{T}^2} f \, dx = 0 \}$, $C_{0, \sigma}^\infty \equiv \{ f \in C_0^\infty (\mathbb{T}^2) : \nabla \cdot f = 0 \}$, and $\dot{H}_\sigma^p \equiv \{ f \in \dot{H}^p (\mathbb{T}^2) : \int_{\mathbb{T}^2} f \, dx = 0, \nabla \cdot f = 0 \}$ where $\int_{\mathbb{T}^2} f \, dx \equiv [\mathbb{T}^2]^{-1} \int_{\mathbb{T}^2} f (x) \, dx$, and $\dot{H}_\sigma^p$ similarly for any $s \in \mathbb{R}$. We define $\mathbb{P} \equiv \text{Id} + \mathbb{R} \otimes \mathbb{R}$ to be the Leray projection onto the space of divergence-free vector fields. We define $\| \cdot \|_{C^1} \equiv \sum_{0 \leq k + |\alpha| \leq 2} \| \partial_x^k \partial^{\alpha} \mathbf{D}^\sigma \|_{L^2}$ where $k \in \mathbb{N}_0$ and $\alpha$ is a multi-index. For any Polish space $H$, we write $\mathcal{B}(H)$ to denote the $\sigma$-algebra of Borel sets in $H$. We denote by $L^2 (\mathbb{T}^2)$-inner products by $\langle \cdot, \cdot \rangle$, a duality pairing of $\dot{H}^{-1} (\mathbb{T}^2) \equiv \dot{H}^{1/2} (\mathbb{T}^2)$ by $\langle \cdot, \cdot \rangle$, a quadratic variation of $A$ and $B$ by $\langle \langle A, B \rangle \rangle$, as well as $\langle \langle A \rangle \rangle \equiv \langle \langle A, A \rangle \rangle$. We let

$$\Omega_t \equiv C ([t, \infty) ; (H_\sigma^1)^*) \cap L_{\text{loc}}^\infty ([t, \infty) ; \dot{H}_\sigma^{1/2}) \quad t \geq 0$$

(40)

where $(H_\sigma^1)^*$ denotes the dual of $H_\sigma^1$. We also denote by $\mathcal{P}(\Omega_0)$ the set of all probability measures on $(\Omega, \mathcal{B})$ where $\mathcal{B}$ is the Borel $\sigma$-field of $\Omega_0$ from the topology of locally uniform convergence on $\Omega_0$. We define the canonical process $\xi : \Omega_0 \mapsto (H_\sigma^1)^*$ by $\xi(t) (\omega) \equiv \xi(t \omega)$. For general $t \geq 0$ we equip $\Omega_t$ with Borel $\sigma$-field $\mathcal{B}_t^0 \equiv \sigma (\xi \colon s \geq t)$, and additionally define $\mathcal{B}_t^0 \equiv \sigma (\xi \colon s \leq t)$ and $\mathcal{B}_t \equiv \cap_{s \geq t} \mathcal{B}_s^0$, $t \geq 0$.

(41)

For any Hilbert space $U$, we denote by $L_2(U, \dot{H}_\sigma^p)$ with $s \in \mathbb{R}_+$ the space of all Hilbert-Schmidt operators from $U$ to $\dot{H}_\sigma^p$ with norm $\| \cdot \|_{L_2(U, \dot{H}_\sigma^p)}$. We impose on $G : \dot{H}_\sigma^{1/2} \mapsto L_2(U, \dot{H}_\sigma^{1/2})$ to be $\mathcal{B}(H_\sigma^{1/2}) - \mathcal{B}(L_2(U, \dot{H}_\sigma^{1/2}))$-measurable and satisfy

$$\| G(\phi) \|_{L_2(U, \dot{H}_\sigma^{1/2})} \leq C (1 + \| \phi \|_{H_\sigma^{1/2}}), \quad \lim_{j \to \infty} \| G(\psi^j) \phi - G(\psi^j) \phi \|_U = 0$$

(42)

for all $\phi, \psi^j, \psi \in C^\infty (\mathbb{T}^2) \cap \dot{H}_\sigma^{1/2}$ such that $\lim_{j \to \infty} \| \psi^j - \psi \|_{H_\sigma^{1/2}} = 0$.

3.2. Convex integration. Most of the following preliminaries come from [7].

Lemma 3.1. (Inverse divergence) Let $f$ be divergence-free and mean-zero. Then we define the $(i, j)$-th entry of $Bf$ for $i, j \in \{1, 2 \}$ by

$$(Bf)^{ij} \equiv - \partial_i \Lambda^{-2} f_j - \partial_j \Lambda^{-2} f_i$$

(43)

For $f$ that is not divergence-free, we define $Bf \equiv \mathcal{B}\mathcal{F} f$; for $f$ that is not mean-zero, we define $Bf \equiv \mathcal{B} \left( f - \int f \right)$. It follows that $\text{div}(Bf) = \mathcal{F} f$, and $Bf$ is a symmetric and trace-free matrix. One property that we will frequently rely on in estimates is $\| \mathcal{B} \|_{C^{r+C_1}} \leq 1$ (e.g., [8] p. 127) and [62] Lemma 7.3).
Next, for $k \in S^1$ we define
\[
 b_k(\xi) \doteq ik^\perp e^{ik \cdot \xi} \quad \text{and} \quad c_k(\xi) \doteq e^{ik \cdot \xi}
\]
which satisfies
\[
 b_k = \nabla^\perp_k c_k, \quad c_k = -\nabla^\perp_k \cdot b_k, \quad \Lambda^\perp_k b_k(\xi) = b_k(\xi) \quad \forall \alpha \geq 0, \quad b_k(\xi) = -\nabla^\perp_k c_k(\xi).
\] (45)

For any finite family of vectors $\Gamma \subset S^1$ and $|a_k| \subset \subset$ such that $\alpha_k = \tilde{\alpha}_k$, we set $W(\xi) \doteq \sum_{\alpha \in \Gamma} a_\alpha b_k(\xi)$ and $W_k(\xi) \doteq \tilde{a}_\alpha b_k(\xi)$ so that
\[
 \text{div}(W \otimes W) = \frac{1}{2} \nabla_\xi |W|^2 + (\nabla^\perp_\xi \cdot W)W^\perp \quad \text{and} \quad \sum_{\alpha \in \Gamma} W_\alpha \otimes W_{-\alpha} = \sum_{\alpha \in \Gamma} |a_\alpha|^2 k^\perp \otimes k^\perp.
\] (46)

**Lemma 3.2.** (Geometric lemma from [7, Lemma 4.2]) Let $B(\text{Id}, \varepsilon)$ denote the ball of symmetric $2 \times 2$ matrices, centered at Id of radius $\varepsilon > 0$. Then there exists $\varepsilon > 0$ with which there exist disjoint finite subsets $\Gamma_j \subset S^1$ for $j \in \{1, 2\}$ and smooth positive functions $\gamma_k \in C^\infty(B(\text{Id}, \varepsilon_j)), \quad j \in \{1, 2\}, k \in \Gamma_j,
\]

such that

(1) $5\Gamma_j \subset \subset Z^2$ for each $j$,
(2) if $k \in \Gamma_j$, then $-k \in \Gamma_j$ and $\gamma_k = \gamma_{-k},$
(3) $R = \frac{1}{2} \sum_{\alpha \in \Gamma} (\gamma_\alpha(R))^2 (k^\perp \otimes k^\perp) \quad \forall R \in B(\text{Id}, \varepsilon),$

(4) $|k + k'| \geq \frac{1}{2}$ for all $k, k' \in \Gamma_j$ such that $k + k' \neq 0$.

Next, we recall that $\mathcal{P}$ denotes the Leray projection and define for $k \in S^1$,
\[
 \mathcal{P}_{k+1, k} \doteq \mathcal{P}P_{ak,k^\perp},
\]
where $P_{ak,k^\perp}$ is a Fourier operator with a Fourier symbol $\hat{K}_{ak,k^\perp}(\xi) = \tilde{K}_{a_\perp} \left( \frac{\xi}{\lambda_{k^\perp}} - k \right)$; i.e.,
\[
 P_{ak,k^\perp} f(\xi) = \hat{K}_{ak,k^\perp}(\xi) \hat{f}(\xi) = \hat{K}_{a_\perp} \left( \frac{\xi}{\lambda_{k^\perp}} - k \right) \hat{f}(\xi)
\]
where $\tilde{K}_{a_\perp}$ is a smooth bump function such that $\text{supp} \tilde{K}_{a_\perp} \subset \{ |\xi| : |\xi| \leq \frac{1}{\lambda_{k^\perp}} \}$ and \[\tilde{K}_{a_\perp}(\xi, \xi) \equiv 1.\] We see that within the support of $\tilde{K}_{a_\perp} \left( \frac{\xi}{\lambda_{k^\perp}} - k \right)$, $\frac{7}{8} \lambda_{k^\perp} \leq |\xi| \leq \frac{9}{8} \lambda_{k^\perp}$. This implies that $\text{supp} \mathcal{P}_{k+1, k} f \subset \{ \xi : \frac{7}{8} \lambda_{k^\perp} \leq |\xi| \leq \frac{9}{8} \lambda_{k^\perp} \}$ and for any $0 \leq a, b$, for some suitable constant $C_{a,b}$ that is independent of $\lambda_{k^\perp}$,
\[
 \sup_{\xi \in \mathbb{R}^2} |\xi|^a \left| \nabla^\perp_\xi \tilde{K}_{ak,k^\perp} \right| \leq C_{a,b} \lambda_{k^\perp}^{-b},
\]
and similarly
\[
 \| x^a \nabla^b_\xi \tilde{K}_{ak,k^\perp} \|_{L^1(\mathbb{R}^2)} \leq C_{a,b} \lambda_{k^\perp}^{-b} \in \{0, 2\}.
\]
It follows that for all $f$ that is $T^2$-periodic, we can write $\mathcal{P}_{k+1, k} f(x) = \int_{T^2} K_{k+1, k}(y) f(x-y)dy$ where the kernel $K_{k+1, k}$ satisfies for all $a, b \geq 0$,
\[
 \| x^a \nabla^b_\xi K_{k+1, k}(x) \|_{L^1(\mathbb{R}^2)} \leq C_{a,b} \lambda_{k^\perp}^{-b},
\]
which allows one to deduce
\[
 \| \mathcal{P}_{k+1, k} \|_{C_{k^\perp} \rightarrow C_k} \leq C_1.
\] (51)
Finally, we define \( \tilde{P}_{q=1} \) to be the Fourier operator with a symbol supported in \( \{ \xi : \frac{3l_{q=1}}{q} \leq |\xi| \leq 4.1_{q=1} \} \) and is identically one on \( \{ \xi : \frac{1}{6.2} \leq |\xi| \leq \frac{3.1}{4.1} \} \). As the last item in this Preliminaries section, following [9, Section 5] (also [72, 81]), we split \([0, T_L] \) into finitely many subintervals of size \( \tau_{q=1} \) and let \( 0 \leq \chi \leq 1 \) be a smooth cut-off function with support in \((-1, 1)\) such that \( \chi \equiv 1 \) on \((-\frac{1}{4}, \frac{1}{4})\), and for \( l = \lambda_{q=1} \) from (53) and

\[
\tau_{q=1}^{-1} \xi = \frac{l_{q=1} - \tau_{q=1}}{q} \delta_{q=1}
\]

which is different from [7] equation (4.7),

\[
\chi_{j}(t) \equiv \chi(\tau_{q=1}^{-1} t - j) \quad \text{for } j \in \{0, 1, \ldots, \lceil \tau_{q=1}^{-1} T_L \rceil \},
\]

where we suppressed the dependence of \( \chi_{j} \) on \( q \) following [7] p. 1828, satisfies

\[
\sum_{j=0}^{\lceil \tau_{q=1}^{-1} T_L \rceil} \chi_{j}^2(t) = 1 \quad \forall t \in [0, T_L].
\]

By comparing the definitions of \( \chi_{j} \) in (53) and previous works on probabilistic convex integration such as \( \chi_{j}(t) = \chi(l^{-1} t - j) \) in [81] equation (70) (see also [9, equation (5.20)]), we see that the convex integration on the QG equations requires one to carefully select \( \tau_{q=1} \) rather than just take \( \tau_{q=1} = l \), basically optimize over the errors of \( R_T \) in (152) and \( R_{O, \text{approx}} \) in (212). On many occasions when no confusion arises, we will write for brevity

\[
\sum_{j} \sum_{k} \sum_{\Gamma_j} = \sum_{j=0}^{\lceil \tau_{q=1}^{-1} T_L \rceil} \sum_{j=0}^{\lceil \tau_{q=1}^{-1} T_L \rceil} \sum_{k \in \Gamma_j} \sum_{j=0}^{\lceil \tau_{q=1}^{-1} T_L \rceil} \sum_{k \in \Gamma_j}.
\]

4. Proofs of Theorems 2.1, 2.2

We give the general definitions of a solution to the stochastic momentum equation (6). From here to Proposition 4.6, we only consider the case \( \gamma_2 = 1 \) because Proposition 4.1 can be proven only for this special case.

**Definition 4.1.** Fix \( \epsilon \in (0, 1) \). Let \( s \geq 0 \) and \( x^m \in H^s_{\epsilon} \). Then \( P \in P(\Omega_0) \) is a martingale solution to (6) with initial data \( x^0 \) at initial time \( s \) if

(1) \( P \{ (\xi(s) = x^m \quad \forall t \in [0, s]) \} = 1 \) and for all \( l \in \mathbb{N} \),

\[
P \left\{ \xi \in \Omega_0 : \int_0^s \| G(\xi(r)) \|^2_{L_t(U^s_{\epsilon})} dr < \infty \right\} = 1,
\]

(2) for all \( \psi \in C^\infty(\mathbb{T}^2) \cap H^s_{\epsilon} \) and \( t \geq s \), the process

\[
M_{t,s} = \left\{ \langle \xi(t) - x^m, \psi \rangle \right\}
\]

\[
- \int_s^t \sum_{i,j=1}^2 \langle \Lambda \xi_j, \partial \psi \xi_j \rangle_{H^s_{\epsilon} - H^s_{\epsilon}} - \frac{1}{2} \langle \partial \psi \xi_j, [\Lambda, \psi_j \xi_j] \rangle_{H^s_{\epsilon} - H^s_{\epsilon}} - \langle \xi, \Lambda^s \psi \rangle dr
\]

is a continuous, square-integrable \( (B_t)_{t \geq 0} \)-martingale under \( P \) such that \( \langle M_{t,s} \rangle \) is a continuous, square-integrable \( (B_t)_{t \geq s} \)-martingale under \( P \) and \( \langle M_{t,s} \rangle = \int_s^t \| G(\xi(r)) \|^2_{L_t(U^s_{\epsilon})} dr \),

(3) for any \( q \in \mathbb{N} \), there exists a function \( t \mapsto C_{t,q} \in \mathbb{R} \) for all \( t \geq s \) such that

\[
\mathbb{E}^P \left[ \sup_{t \in [0, t]} \| x^m(t) \|_{H^s_{\epsilon}}^q + \int_s^t \| x^m(t) \|_{H^s_{\epsilon}}^q dr \right] \leq C_{t,q} (1 + \| x^m \|_{H^s_{\epsilon}}^q).
\]
The set of all such martingale solutions with the same constant $C_{t,q}$ in \eqref{eq:4.1} for every $q \in \mathbb{N}$ and $t \geq s$ will be denoted by $C(s, \xi^m, \{C_{t,q}\}_{q \in \mathbb{N}, t \geq s})$.

In \eqref{eq:5.3} we implicitly relied on the identity of
\[
- \int_{\mathbb{T}^2} (\nabla \xi)^T \cdot \Lambda \xi \cdot \psi^k \, dx = \sum_{i,j=1}^2 \frac{1}{2} \int_{\mathbb{T}^2} \partial_i \xi [-\Lambda, \psi^k] \xi_j \, dx
\]
and Calderón’s commutator estimate from \eqref{eq:5.2} of Proposition 5.3. If $\{\psi^k\}_{k=1}^\infty$ is a complete orthonormal basis of $\dot{H}^\frac{s}{2}$ that consists of eigenvectors of $GG^*$, then $M_{t,s} \triangleq \sum_{k=1}^\infty M_{t,s}^k \psi^k$ becomes a $GG^*$-Wiener process starting from initial time $s$ w.r.t. $(\mathcal{B}_t)_{t \geq 0}$ under $P$.

**Definition 4.2.** Fix $t \in (0, 1)$. Let $s \geq 0$, $\xi^m \in \dot{H}^\frac{s}{2}$, and $\tau : \Omega_0 \mapsto [s, \infty)$ be a stopping time of $(\mathcal{B}_t)_{t \geq 0}$. Define the space of trajectories stopped at $\tau$ by
\[
\Omega_{t,\tau} \triangleq \{\omega : \tau(\omega) \geq \omega \in \Omega_0 \} = \{\omega : \tau, (t, \omega) = \xi(t \wedge \tau(\omega), \omega) \forall t \geq 0\}.
\]
Then $P \in \mathcal{P}(\Omega_{t,\tau})$ is a martingale solution to \eqref{eq:4.2} on $[s, \tau]$ with initial data $\xi^m$ at initial time $s$ if
\[
(M1) \quad P(\xi(t) = \xi^m \forall t \in [0, s]) = 1 \quad \text{and for all } l \in \mathbb{N},
\]
\[
P \left( \int_0^{\tau \wedge T} ||G(\xi(r))||_{L_t^l(U \times H^\frac{2}{3})}^2 \, dr < \infty \right) = 1,
\]
and for all $\psi^k \in C^\infty(\mathbb{T}^2) \cap \dot{H}^\frac{s}{2}$ and $t \geq s$, the process
\[
M_{t,s}^k = \langle \xi(t \wedge \tau) - \xi^m, \psi^k \rangle
\]
\[- \int_{s}^{\tau \wedge T} \sum_{i,j=1}^2 \langle \Lambda \xi_i, \partial_j \psi^k \xi_j \rangle \cdot \dot{H}^\frac{2}{3} - \dot{H}^\frac{2}{3} - \frac{1}{2} \sum_{i,j=1}^2 \langle \partial_i \xi_j, [\Lambda, \psi^k] \xi_j \rangle \cdot \dot{H}^\frac{2}{3} - \langle \xi, \Lambda \psi^k \rangle \, dr
\]
is a continuous, square-integrable $(\mathcal{B}_t)_{t \geq 0}$-martingale under $P$ such that \langle \{M_{t,s}^k\} \rangle = \int_{s}^{\tau \wedge T} ||G(\xi(r))||_{L_t^l(U \times H^\frac{2}{3})}^2 \, dr,
\]
\[(M3) \quad \text{for any } q \in \mathbb{N}, \text{ there exists a function } t \mapsto C_{t,q} \in \mathbb{R}_+ \text{ for all } t \geq s \text{ such that}
\]
\[
\mathbb{E}^P \left[ \sup_{t \in [0, \tau \wedge T]} ||\xi(r)||_{L^q(H^\frac{2}{3})}^2 + \int_s^{\tau \wedge T} ||\xi(r)||_{L^q(H^\frac{2}{3})}^2 \, dr \right] \leq C_{t,q}(1 + \|\xi^m\|_{L^q(H^\frac{2}{3})}^2).
\]

The following result is concerned with the existence and stability of martingale solutions to \eqref{eq:4.2} in the case of an additive noise. Because the QG momentum equations forced by random noise has never been studied before, we give details of its proof in the Appendix.

**Proposition 4.1.**
\begin{enumerate}
\item For every $(s, \xi^m) \in [0, \infty) \times \dot{H}^\frac{s}{2}$, there exists a martingale solution $P \in \mathcal{P}(\Omega_0)$ to \eqref{eq:4.2} with initial data $\xi^m$ at initial time $s$ according to Definition 4.2.
\item Moreover, if there exists a family $\{(s_i, \xi_i)\}_{i \in \mathbb{N}} \subset [0, \infty) \times \dot{H}^\frac{s}{2}$ such that $\lim_{i \to \infty} ||(s_i, \xi_i) - (s, \xi^m)||_{\mathcal{E}(\mathbb{N}, \dot{H}^\frac{s}{2})} = 0$ and $P_i \in \mathcal{C}(s_i, \xi_i, \{C_{t,q}\}_{q \in \mathbb{N}, t \geq s})$ is the martingale solution corresponding to $(s_i, \xi_i)$, then there exists a subsequence $\{P_{i_k}\}_{k \in \mathbb{N}}$ and $P \in \mathcal{C}(s, \xi^m, \{C_{t,q}\}_{q \in \mathbb{N}, t \geq s})$ such that $P_{i_k}$ converges weakly to $P$.
\end{enumerate}

Proposition 4.1 leads to the following two results, which are only slight modifications of [45] Propositions 3.2 and 3.4] to which we refer interested readers for details.
Lemma 4.2. (cf. [45, Proposition 3.2]) Let $\tau$ be a bounded stopping time of $(B_t)_{t \geq 0}$. Then, for every $\omega \in \Omega_0$, there exists $Q_\omega \doteq \delta_\omega \otimes_{\mathcal{F}(\omega)} R_{\tau(\omega),\xi(\tau(\omega),\omega)} \in \mathcal{P}(\Omega_0)$ where $\delta_\omega$ is a point-mass at $\omega$ such that for $\omega \in (\xi(\tau) \in \mathcal{H}_\tau^\updownarrow)$,

$$Q_\omega((\omega' \in \Omega_0 : \xi(t,\omega') = \omega(t) \ \forall \ t \in [0, \tau(\omega)]) \equiv 1, \quad (63a)$$

$$Q_\omega(A) = R_{\tau(\omega),\xi(\tau(\omega),\omega)}(A) \ \forall \ A \in \mathcal{B}(\mathcal{H}_\tau^\updownarrow), \quad (63b)$$

where $R_{\tau(\omega),\xi(\tau(\omega),\omega)} \in \mathcal{P}(\Omega_0)$ is a martingale solution to (6) with initial data $\xi(\tau(\omega),\omega)$ at initial time $\tau(\omega)$, and the mapping $\omega \mapsto Q_\omega(B)$ is $\mathcal{B}_\tau$-measurable for every $B \in \mathcal{B}$.

Lemma 4.3. (cf. [45, Proposition 3.4]) Let $\tau$ be a bounded stopping time of $(\mathcal{B}_t)_{t \geq 0}$, $\xi^m \in H_\tau^\updownarrow$, and $P$ be a martingale solution to (6) on $[0, \tau]$ with initial data $\xi^m$ at initial time 0 that satisfies Definition 4.2. Suppose that there exists a Borel set $N \subset \Omega_0 \tau$ such that $P(N) = 0$ and $Q_\omega$ from Lemma 4.2 satisfies for every $\omega \in \Omega_0 \setminus N$,

$$Q_\omega((\omega' \in \Omega_0 : \tau(\omega') = \tau(\omega))) \equiv 1. \quad (64)$$

Then the probability measure $P \otimes_{\tau} R \in \mathcal{P}(\Omega_0)$ defined by

$$P \otimes_{\tau} R = \int_{\Omega_0} Q_\omega(\cdot) P(d\omega) \quad (65)$$

satisfies $P \otimes_{\tau} R = P$ on the $\sigma$-algebra $\sigma[\xi(t \wedge \tau), t \geq 0]$ and it is a martingale solution to (6) on $[0, \infty)$ with initial data $\xi^m$ at initial time 0.

Now we fix a GG*-Wiener process $B$ on $(\Omega, \mathcal{F}, P)$ with $(\mathcal{F}_{\tau})_{\tau \geq 0}$ as the canonical filtration of $B$ augmented by all the $P$-negligible sets. We let $\mathcal{B}_{\tau}$ represent the $\sigma$-algebra associated to the stopping time $\tau$ and consider

$$dz + [\nabla p_1 + \Lambda^T z]dt = dB, \quad \nabla \cdot z = 0, \quad (66a)$$

$$z(0, x) \equiv 0, \quad (66b)$$

so that $y \doteq v - z$, together with $p_2 \doteq p - p_1$ satisfies

$$\partial_t y + (\Lambda (y + z) \cdot \nabla)(y + z) - (\nabla (y + z))^T \cdot \Lambda (y + z) + \nabla p_2 + \Lambda^T y = 0 \quad (67)$$

due to (6). We see that

$$z(t) = \int_0^t e^{-(t-r)(-\Delta)^{1/2}} dB(r) \quad (68)$$

where $e^{-(t-r)(-\Delta)^{1/2}}$ is a semigroup generated by $-(\Delta)^{1/2}$. The following proposition informs us the regularity of $z$ from the hypothesis of Theorem 2.1.

Proposition 4.4. Suppose that $\gamma_1 \in (0, \frac{1}{2})$ and $B$ is a GG*-Wiener process that satisfies (8) for some $\sigma > 0$. Then, for all $T > 0$, $\delta \in (0, \frac{1}{2})$, and $l \in \mathbb{N}$,

$$\mathbb{E}^P \left[ \left\| z \right\|_{C_{T, H_{\frac{1}{2}}}^1}^{2^l} + \left\| z \right\|_{C_{T, H_{\frac{1}{2}}}^1}^{2} \right] < \infty. \quad (69)$$

Proposition 4.4 is a straightforward extension of previous works such as [45, Proposition 3.6] (also [78, Proposition 4.4]); thus, we sketch its proof in the Appendix for completeness. Next, we recall $\Omega_0$ from (40) and define for every $\omega \in \Omega_0$

$$M_{\omega}^0 \doteq \omega(t) - \omega(0) + \int_0^t \mathbb{P} \text{div}(\Lambda (\omega(r) \otimes \omega(r))) - \mathbb{P}((\nabla \omega)^T (r) \cdot \Lambda (\omega(r))) + \Lambda^T \omega(r) dr, \quad (70a)$$

$$Z^\omega(t) \doteq M_{\omega,0}^0 - \int_0^t \mathbb{P} \Lambda^T e^{-(t-r)(\Lambda^T)^{1/2}} M_{\omega,0}^0 dr. \quad (70b)$$
If $P$ is a martingale solution to (6), then the mapping $\omega \mapsto M_{\omega,0}^{\alpha}$ is a $GG^*$-Wiener process under $P$ and we can show using (70) and (40) that

$$Z(t) = \int_0^t e^{-(t-r)\lambda V} \mathcal{P}dM_{r,0}$$

which implies due to Proposition 4.4 that for any $\delta \in (0, \frac{1}{4})$ and $T > 0$,

$$z \in C_T H^{4+\delta} \cap C_{\delta}^{4-\delta} H^{4-\delta} \mathcal{P}-a.s.$$ 

Next, we define for $\delta \in (0, \frac{1}{4})$ and the Sobolev constant $C_{S,1} \geq 0$ such that $\|f\|_{L^p} \leq C_{S,1}\|f\|_{H^{4+\delta} \mathcal{P}}$ for all $f \in H^{4+\delta}(\mathbb{R}^2)$ that is mean-zero,

$$\tau_L(\omega) \triangleq \inf \left\{ t \geq 0 : C_{S,1}\|Z^\omega(t)\|_{H^{4+\delta}\mathcal{P}} > \left( L - \frac{1}{4} \right)^{\frac{1}{4}} \right\}$$

and $\tau_{L}(\omega) \triangleq \lim_{\lambda \to \infty} \tau_{L}(\omega)$

so that $\tau_L$ is a stopping time due to [45, Lemma 3.5]. Additionally, for $L > 1$ and $\delta \in (0, \frac{1}{4})$ we define

$$T_L \triangleq \inf \{ t \geq 0 : C_{S,1}\|z(t)\|_{H^{4+\delta}\mathcal{P}} \geq L^{\frac{1}{4}} \} \land \inf \{ t \geq 0 : C_{S,1}\|\epsilon\|_{c^{1-2\delta}H^{4+\delta}\mathcal{P}} \geq L^{\frac{1}{4}} \} \land \lambda;$$

due to Proposition 4.4 we know that $T_L > 0$ and $\lim_{L \to \infty} T_L = \infty \mathcal{P}-a.s.$ Next, we assume Theorem 2.1 on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathcal{P})$, denote by $P = \mathcal{L}(\nu)$ the law of $\nu$ constructed from Theorem 2.1 and obtain the following two results.

**Proposition 4.5.** Let $\tau_L$ be defined by (73). Then $P = \mathcal{L}(\nu)$ is a martingale solution on $[0, \tau_L]$ according to Definition 4.2.

**Proposition 4.6.** Let $\tau_L$ be defined by (73) and $P = \mathcal{L}(\nu)$. Then $P \otimes \tau_L R$ defined by (65) is a martingale solution on $[0, \infty)$ according to Definition 4.1.

For completeness, we sketch the proof of Propositions 4.5 and 4.6 in the Appendix referring to [45] Propositions 3.7-3.8 respectively for details.

**Proof of Theorem 2.2 assuming Theorem 2.1** We fix $T > 0$ arbitrarily, $K > 1$, and $\kappa \in (0, 1)$ such that $\kappa K^2 \geq 1$, rely on Theorem 2.1 and Proposition 4.6 to deduce the existence of $L > 1$ and a martingale solution $P \otimes \tau_L R$ to (6) on $[0, \infty)$ such that $P \otimes \tau_L R = P$ on $[0, \tau_L]$ where $P = \mathcal{L}(\nu)$. Hence, $P \otimes \tau_L R$ has a deterministic initial data $\nu_{in}$ from Theorem 2.1 and satisfies

$$P \otimes \tau_L R(\{\tau_L \geq T\}) \geq \mathcal{L}(\nu_{in}) \left(\{\tau_L \geq T\}\right) \mathcal{P}(\{T_L \geq T\}) > \kappa.$$

This implies

$$\mathbb{P}^{P \otimes \tau_L R}[\|\xi(T)\|^2_{H^{4+\delta}_2}] \geq \mathbb{P}^{P \otimes \tau_L R}[1_{\{\tau_L \geq T\}}]\|\xi(T)\|^2_{H^{4+\delta}_2} \geq \kappa K^2 \left[\|\nu_{in}\|^2_{H^{4+\delta}_2} + T \text{ Tr}((-\Delta)^{\frac{1}{2}} GG^*)\right].$$

where the second inequality is due to (11) and (75). On the other hand, we can construct another martingale solution $P$ starting from same initial data $\nu_{in}$ via a Galerkin approximation that satisfies (7).
Next, in addition to $\lambda_q$ and $\delta_q$ in (37) we define
\[ M_0(t) = L^t e^{ALt}. \]  
(77)

We see from (74) and Sobolev embeddings in $T^2$ that for any $\delta \in (0, \frac{1}{4})$ and $t \in [0, T_L)$,
\[ \|z(t)\|_{W^k} \leq L^\frac{1}{2} \quad \forall \, k \in \{0, 1, 2, 3\}, \quad \text{and} \quad \|\zeta\|_{C^{\gamma}_x W^{k_0}_x W^{q_0}_x} \leq L^\frac{1}{2}. \]  
(78)

Because it is not true in general that $\text{supp} \hat{z} \subset B(0, 2\lambda_q)$, via Littlewood-Paley theory we convolute $z$ in space by a smooth function $\tilde{\phi}_q$ with frequency support in $B(0, \frac{L}{2})$, $q \in \mathbb{N}_0$, such that $z_q \doteq z \ast \tilde{\phi}_q \rightarrow z$ as $q \rightarrow \infty$ in distributions so that
\[ \text{supp} \hat{z}_q \subset B \left( 0, \frac{\lambda_q}{4} \right). \]  
(79)

Here we chose $\frac{\lambda_q}{4}$ rather than $2\lambda_q$ for the convenience of estimating $R_{\text{com2}}$ in Section 4.1.6.

We will construct solutions $(v_q, \tilde{R}_q)$ for $q \in \mathbb{N}_0$ to (67) with an error iteratively. In fact, as we discussed in Subsection 2.2, we shall consider the generalized QG equations with $u = \Lambda^{\gamma_2} v$ from (38) in the following Propositions 4.7-4.10, which brings us to
\[ \partial_t y_q + (\Lambda^{2-\gamma_2}(y_q + z_q) \cdot \nabla)(y_q + z_q) - (\nabla(y_q + z_q))^T \cdot \Lambda^{2-\gamma_2}(y_q + z_q) + \nabla p_q + \Lambda^{\gamma_2} y_q = \text{div} \tilde{R}_q, \]
\[ \nabla \cdot y_q = 0, \]  
(80)

where $\tilde{R}_q$ is a symmetric trace-free matrix. The range of parameters we shall consider in the following Propositions 4.7-4.10 for the generalized QG equations described by (38) are
\[ \gamma_2 \in [1, 2), \]  
(81a)
\[ \begin{cases} \frac{1}{2} < \beta < \frac{8}{15} & \text{if } \gamma_2 = 1, \\ 1 - \frac{2}{3} < \beta < \min \left( \frac{15-7\gamma_2}{18}, \frac{8-4\gamma_2}{7}, \frac{3-2\gamma_2}{4} \right) & \text{if } \gamma_2 > 1, \end{cases} \]  
(81b)
\[ \begin{cases} -2 + 6\beta < \alpha < \frac{8-5\gamma_2}{4} & \text{if } \gamma_2 = 1, \\ 1 - \alpha < 7 - 3\gamma_2 - 5\beta & \text{if } \gamma_2 > 1, \end{cases} \]  
(81c)
\[ \gamma_1 + 2\gamma_2 < 5 - 3\beta. \]  
(81d)

We emphasize that for all $\gamma_2 \in [1, 2)$, we have $\alpha > 1$ from (81a). We also point out that $\beta < \frac{3-2\gamma_2}{4}$ for $\gamma_2 > 1$ is needed in (241) and it may seem to lead to a contradiction because $\frac{3-2\gamma_2}{4} = \frac{1}{2}$ when $\gamma_2 = 1$ and we have $\beta < \frac{1}{2}$ in (81b) in case $\gamma_2 = 1$; however, these conditions are not continuous in $\gamma_2$ due to our convenient choice of $\alpha > 1$ in (81c) (see e.g., (148) concerning its convenience). Next, we denote specific universal non-negative constants $C_{G_1}, C_{G_2}$, and $C_{S_1}$, due to Gagliardo-Nirenberg inequality and Sobolev’s inequality: for any $f$ that is sufficiently smooth in $x \in T^2$ and mean-zero,
\[ \|f\|_{L^{2+\gamma_2}_t} \leq C_{G_1} \|f\|_{L^3_t}^{\frac{2}{3}} \|\Lambda^3 f\|_{L^3_t}^{\frac{1}{3}} \|\Lambda^{2\gamma_2} f\|_{L^2_t}^{\frac{2}{3}}, \]  
\[ \|f\|_{L^2_t} \leq C_{G_2} \|f\|_{H^{\gamma_2-1}_t}. \]  
(82a)
(82b)

For a large universal constant $C_0 > 0$ from the proof of Proposition 4.8, we fix another universal constant $\bar{C} > 0$ such that
\[ \frac{2^{2+\gamma_2}}{\pi} C_0 \sqrt{1 + C_{G_1} C_{G_2} 8} < \sqrt{\bar{C}}. \]  
(83)
Similarly to \( y_t, \tilde{R}_q, \) and \( z_q \) to \( t < 0 \) by their respective values at \( t = 0 \) and mollify them to obtain
\[
y_t \triangleq y_q * \phi_t * \varphi_t, \quad \tilde{R}_t \triangleq \tilde{R}_q * \phi_t * \varphi_t, \quad \text{and} \quad z_t \triangleq z_q * \phi_t * \varphi_t. \quad (84)
\]
We informally convoluted \( z \) twice in space; this is for the convenience upon estimates that \( y_t, \tilde{R}_q, \) and \( z_q \) all have the same structure. For a subsequent purpose in (109) we assume
\[
\frac{2^j - 2^j + C_q L^2}{\pi} < a^{b(1 - \frac{1}{2} - \beta)}.
\]
Now we set a convention that \( \sum_{1 \leq j < 0} \pm 0 \) and consider the following inductive hypothesis for the universal constants \( C_0 \geq 1 \).

**Hypothesis 4.1.** For all \( t \in [0, T_L] \),
\[
(a) \quad \text{supp} \, \tilde{y}_q \subset B(0, 2 \lambda_q),
\]
\[
\|y\|_{C_{t,\lambda}} \leq C_0 \left( 1 + \sum_{1 \leq j < q} \delta_j^2 \right) L^2 M_0(t)^{\frac{1}{2}}, \quad (86)
\]
\[
\|y\|_{C_{t,\lambda_q}} + \|\Lambda^{2-\gamma_1} y\|_{C_{t,\lambda_q}} \leq C_0 L^2 M_0(t)^{\frac{1}{2}} \Lambda_{q_0}^{2-\gamma_1} \delta_q^{\frac{1}{2}}, \quad (87)
\]
\[
(b) \quad \text{supp} \, \tilde{R}_q \subset B(0, 4 \lambda_q),
\]
\[
\|\tilde{R}_q\|_{C_{t,\lambda_q}} \leq C_0 L^2 M_0(t)^{\frac{1}{2}} \Lambda_{q_0}^{2-\gamma_1} \delta_q^{\frac{1}{2}}, \quad (88)
\]
\[
(c) \quad \|(\partial_t + (\Lambda^{2-\gamma_1} y + \Lambda^{2-\gamma_2} z) \cdot \nabla) y\|_{C_{t,\lambda_q}} \leq C_0 L M_0(t)^{\frac{1}{2}} \Lambda_{q_0}^{2-\gamma_1} \delta_q^{\frac{1}{2}}. \quad (89)
\]

**Remark 4.1.** We need to consider \( z_q \) in (79) because, as we discussed in Subsection 2.2 in order to handle a Reynolds stress transport error, we will replace the convection by velocity \( u_q \) in (23) \( \Lambda^{2-\gamma_1} \) \( v_0 \) and \( \Lambda^{2-\gamma_2} \) \( z_0 \) (see (126a) and (128a)), and it will be crucial that this convection term as a whole has the frequency support in \( B(0, 2 \lambda_q) \) (see (151)), although \( v_0 \) has such a frequency support due to Hypothesis 2.7(a) \( \text{supp} \, \tilde{z} \subset \mathcal{M} \) \( \phi * \varphi \) does not have such a frequency support in general due to our choice of \( \alpha > 1 \). We also note that if we consider \( z \) in (80) rather than \( z_q \), then we cannot include the inductive hypothesis of \( \text{supp} \, \tilde{R}_q \subset B(0, 4 \lambda_q) \) in Hypothesis 2.7(b) which will make our estimates too difficult because \( \Lambda_{q+1}^{2-\gamma_1} \triangleq \Lambda_1 \) \( t^{-1} \gg \Lambda_q \) again due to \( \alpha > 1 \) in (81c).

**Proposition 4.7.** Suppose that \( \gamma_2, \beta, \alpha, \) and \( \gamma_1 \) satisfy (81). Let \( L > 1 \) and define
\[
y_0(t, x) \triangleq \frac{L^2 e^{2Lt}}{2\pi} \left( \begin{array}{c} \sin(x_2) \\ 0 \end{array} \right). \quad (90)
\]
Then, together with
\[
\hat{R}(t, x) \triangleq \frac{L^3 e^{2Lt}}{\pi} \left( \begin{array}{cc} 0 & -\cos(x_2) \\ -\cos(x_2) & 0 \end{array} \right) + \mathcal{B}(\Lambda^{2-\gamma_1} y_0)(t, x)
\]
\[
+ \mathcal{B} \left( -\nabla y_0 \cdot \Lambda^{2-\gamma_1} y_0 + \Lambda^{2-\gamma_2} z_0 \cdot \nabla \cdot z_0 + \Lambda^{2-\gamma_2} z_0 \cdot \nabla \cdot y_0 + \Lambda^{2-\gamma_2} z_0 \cdot \nabla \cdot z_0 \right)(t, x), \quad (91)
\]
the pair \((y_0, \hat{R}_0)\) solves (80) and satisfies the Hypothesis 4.7 at level \( q = 0 \) on \([0, T_L] \) provided that
\[
\frac{2LM_0(t)^{\frac{1}{2}}}{\pi} + M_0(t)^{\frac{1}{2}} C_{\gamma_1} 2^{\frac{1}{2}} + C_{G_1} 16 \sqrt{2L^2} M_0(t)^{\frac{1}{2}} + C_{G_2} 32 L^2 \leq M_0(t), \quad (92a)
\]
\[
\frac{2^{\frac{\gamma}{2}} C_0 L^\frac{\gamma}{2}}{\pi} < a^{-b(1 - \frac{\gamma}{2})} \leq \frac{\sqrt{C L^\frac{\gamma}{2}}}{\sqrt{1 + C_G^1 C_G^2 8}}.
\] (92b)

where the first inequality of (92b) guarantees (85). Finally, \( y_0(0, x) \) and \( \hat{R}_0(0, x) \) are both deterministic.

**Proof of Proposition 4.7.** First, we observe that \( y_0 \) is divergence-free, mean-zero while \( \hat{R}_0 \) is symmetric and trace-free due to Lemma 5.1. Because \( \Lambda^2 \gamma^* y_0 \cdot \nabla y_0 = 0 \), with \( p_0 = 0 \) we see to (5) that \( (y_0, \hat{R}_0) \) is a solution to (80). Next, direct computations give

\[
\int_{\mathbb{T}^2} |\sin(x_2)|^2 \, dx = \int_{\mathbb{T}^2} |\cos(x_2)|^2 \, dx = 2\pi^2,
\]

\[
\text{supp} \sin(x_2) \subset \{ (0, 1), (0, -1) \}, \quad \text{supp} \cos(x_2) \subset \{ (0, 1), (0, -1) \},
\]

\[
\sin(x_2)((0, 1)) = -i2\pi, \quad \sin(x_2)((0, -1)) = i2\pi,
\]

\[
\cos(x_2)((0, 1)) = \cos(x_2)((0, -1)) = 2\pi^2,
\] (93)

and hence \( \text{supp} \hat{y}_0 \subset B(0, 2\lambda_0) \) if we take \( \lambda_0 > \frac{\pi}{2} \). Due to the frequency support of \( y_0 \) and \( z_0 \), it follows from (91) that \( \text{supp} \hat{R}_0 \subset B(0, 4\lambda_0) \). Now as long as we choose \( C_0 \geq \frac{\pi}{2} \), we clearly have \( ||y_0||_{C^2} \leq \frac{M_0(t)}{2\pi} \leq C_0 L^\frac{\gamma}{2} M_0(t) \) as desired. Next, using the Gagliardo-Nirenberg inequality in (82a), and observing that \( \beta \) in (81) guarantees that \( 2 - \gamma_2 - \beta > 0 \), we can take \( a \in 2\mathbb{N} \) sufficiently large to deduce

\[
||y_0||_{C^2} + ||\Lambda^2 \gamma \cdot y_0||_{C^2} \leq C_G^1 2^{-\frac{\gamma}{2}} M_0(t) \leq C_0 L^\frac{\gamma}{2} M_0(t) \leq 2 L^\frac{\gamma}{2} - \gamma_2 \delta_0^\frac{\gamma}{2},
\] (94)

Thus, the hypothesis \([a]\) holds. To verify the hypothesis \([b]\), we compute

\[
||\hat{R}_0(t)||_{C^2} \leq \sum_{k=1}^{3} \Pi_k(t)
\]

(95)

where

\[
\Pi_1(t) \leq \frac{2LM_0(t)}{\pi}, \quad \Pi_2(t) \leq ||\mathcal{B}^\gamma y_0(t)||_{C^2},
\]

(96a)

\[
\Pi_3(t) \leq ||\mathcal{B}(\nabla y_0)^T \cdot \Lambda^2 \gamma y_0(t)||_{C^2} + ||\mathcal{B}(\Lambda^2 \gamma z_0 \nabla \cdot z_0(t)||_{C^2} + ||\mathcal{B}(\Lambda^2 \gamma z_0 \nabla \cdot z_0(t)||_{C^2}.
\]

(96b)

We can estimate

\[
\Pi_2(t) \leq \frac{M(t)}{\pi} C_{G^1} ||\mathcal{B}^\gamma \sin(x_2)||_{H^\gamma} = M_0(t) \frac{2}{\pi} C_{G^1} 2^{-\frac{\gamma}{2}},
\]

(97a)

\[
||\mathcal{B}(\nabla y_0)^T \cdot \Lambda^2 \gamma y_0(t)||_{C^2} \leq C_G^1 C_G^1 8 M_0(t).
\]

(97b)

\[
||\mathcal{B}(\Lambda^2 \gamma z_0 \nabla \cdot z_0(t)||_{C^2} \leq C_G^1 8 \sqrt{2} \frac{L^\frac{\gamma}{2}}{2} M_0(t),
\]

(97c)

\[
||\mathcal{B}(\Lambda^2 \gamma z_0 \nabla \cdot z_0(t)||_{C^2} \leq C_G^1 8 \sqrt{2} \frac{L^\frac{\gamma}{2}}{2} M_0(t),
\]

(97d)

\[
||\mathcal{B}(\Lambda^2 \gamma z_0 \nabla \cdot z_0(t)||_{C^2} \leq C_G^1 32 L^\frac{\gamma}{2},
\]

(97e)

where the four terms (97b) to (97e) within \( \Pi_2(t) \) were computed using the definition of \( \mathcal{B} \) from Lemma 5.1. Applying (97) to (96) and (95) allows us to compute

\[
||\hat{R}_0(t)||_{C^2} \leq [1 + C_G^1 C_G^1 8] M_0(t)
\]

(98)
thus, the Hypothesis 4.1(b) was verified. The verification of Hypothesis 4.1(c) is immediate because β in (81b) guarantees that 3 − γ_2 − 2β > 0 allowing us to estimate

\[ \| (\partial_t + (A^{2-\gamma_2} y_{\lambda t} + A^{2-\gamma_2} z_{\lambda t}) \cdot \nabla) y_0 \|_{\mathcal{C}_\alpha} \]

for \( a \in \mathbb{N} \) sufficiently large where we denoted

\[ y_{\lambda t} \overset{\Delta}{=} y_0 *_{\chi} \phi_{\lambda t} *_{\tau} \varphi_{\lambda t} \quad \text{and} \quad z_{\lambda t} \overset{\Delta}{=} z_0 *_{\chi} \phi_{\lambda t} *_{\tau} \varphi_{\lambda t}. \]

Finally, \( y_0(0, x) \) is deterministic and \( \hat{R}_0(0, x) \) is deterministic due to \( z(0, x) \equiv 0 \) from (66).

In contrast to (23), we shall define

\[ y_{q+1} \overset{\Delta}{=} y_t + w_{q+1}. \]

**Proposition 4.8.** Fix \( L > 1 \). Suppose that \( \gamma_2, \gamma_1, \alpha \), and \( y_1 \) satisfy (81) and \((y_q, \hat{R}_q)\) is a \((\mathcal{F}_t)_{t \geq 0}\)-adapted solution to (80) that satisfies Hypothesis 4.1. Then there exist \( a \in \mathbb{N} \) sufficiently large and \( \beta > 1 - \frac{2}{\gamma_2} \) in (81b) sufficiently close to \( 1 - \frac{2}{\gamma_2} \) that satisfies (92a) and (\( \mathcal{F}_t)_{t \geq 0}\)-adapted processes \((y_{q+1}, \hat{R}_{q+1})\) that solves (80) satisfies the Hypothesis 4.1 at level \( q + 1 \), and for all \( t \in [0, T_L] \),

\[ \| y_{q+1}(t) - y_q(t) \|_{\mathcal{C}_1} \leq C_0 L^\frac{1}{2} M_0(t)^{\frac{1}{2}} \delta_{q+1}^\beta. \]

Moreover, \( w_{q+1} = y_{q+1} - y_t \) satisfies

\[ \supp \hat{w}_{q+1} \subset \left\{ \xi : \frac{\Lambda_{q+1}}{2} \leq |\xi| \leq 2\Lambda_{q+1} \right\}. \]

Finally, if \( y_0(0, x) \) and \( \hat{R}_0(0, x) \) are deterministic, then so are \( y_{q+1}(0, x) \) and \( \hat{R}_{q+1}(0, x) \).

**Proof of Theorem 2.7 assuming Proposition 4.8.** Fix \( T > 0, K > 1, \) and \( \kappa \in (0, 1) \). We take \( L > 1 \) sufficiently large that satisfies (92a), \( 2C_0 L^\frac{1}{2} \geq \pi, \) and

\[ \sqrt{2\pi^2 L^2} e^{2LT} > \left( \sqrt{2\pi^2 L^2} + \sqrt{8\pi^2 L^2} + L \right) e^{LT}, \]

or

\[ e^{LT} > K \quad \text{and} \quad L e^{LT} > L^\frac{1}{2} + K \sqrt{T} \text{Tr}((-\Delta)^2 G^*). \]

Because \( \lim_{L \to \infty} T_L = \infty \) \( \mathbb{P} \)-a.s. due to (74), for the fixed \( T > 0 \) and \( \kappa > 0 \), we increase \( L \) larger if necessary to attain (9). We fix such \( L > 1 \) and rely on Proposition 4.8 to find \( a \in 5\mathbb{N} \) and \( \beta \) in (81b), allowing us to start from \((y_0, \hat{R}_0)\) in Proposition 4.7 and continue with \((y_q, \hat{R}_q)\) for all \( q \in \mathbb{N} \) that solves (80) and satisfies the Hypothesis 4.1 as well as (99).

Now, because \( z \in C_T H^{4+\kappa}_2 \) due to (69), it is immediate that \( z_q \to z \) in \( L^2_T \mathcal{H}^{1}_2 \) as \( q \to \infty \) which is all we need for our subsequent purpose. We compute for all \( \beta' \in (\frac{1}{2}, \beta) \) and \( t \in [0, T_L] \) by using the interpolation inequality (e.g., [27, p. 88]), as well as the fact that \( b^{q+1} \geq b(q + 1) \) for all \( q \geq 0 \) and \( b \geq 2 \),

\[ \sum_{q \geq 0} \| y_{q+1}(t) - y_q(t) \|_{\mathcal{C}_1}^{\beta'} \leq \sum_{q \geq 0} (L^\frac{1}{2} M_0(t)^{\frac{1}{2}} \delta_{q+1}^\frac{1}{2} 1 - \beta^\beta \| y_{q+1} \|_{\mathcal{C}_1} + \| y_q \|_{\mathcal{C}_1})^{\beta'} \]

\[ \leq L^\frac{1}{2} M_0(t)^{\frac{1}{2}} \sum_{q \geq 0} a^{b(q+1) - \beta + \beta'} < \infty. \]
Therefore, \( (y_q)_{q \in \mathbb{N}} \) is Cauchy in \( C_t C^{\alpha^p}_x \) for \( \beta' \in \left( \frac{1}{2}, \beta \right) \). This is why \( \frac{1}{2} + \epsilon < \beta' < \frac{3 - \gamma}{2} \) so that \( \tau < \frac{1}{2} - \frac{\epsilon}{2} \). We can also prove the following temporal regularity estimate; because it will be useful in the proof of Proposition 4.8 we prove this estimate for general \( \gamma_2 \in [1, 2) \) although \( \gamma_2 = 1 \) in the current proof of Theorem 2.1. Because \( 8 \) guarantees that \( \beta < 2 - \gamma_2 \), by (89), (87), and (78), we estimate
\[
\|y_q(t)\|_{C_t} \leq \|y_0\|_{C_x} + \int_0^t \|\partial^\gamma y_q(t)\|_{C_t C_x} + \|\partial^\gamma y_q(t)\|_{C_t C_x} dt \\
\leq L M_0(t) \lambda_0 \delta_q + (L^\frac{1}{2} M_0(t))^\frac{1}{2} \delta_q^\frac{1}{2} \lambda_q^\gamma + L^\frac{1}{2} M_0(t)^\frac{1}{2} \delta_q^\frac{1}{2} \lambda_q^\gamma \\
\leq L M_0(t) \lambda_0 \delta_q 
\]
(103)

Considering (103) in case \( \gamma_2 = 1 \), by interpolation inequality again, (99) and (103), this leads to, for all \( q \in \mathbb{N} \) and \( \eta \in (0, \frac{\beta}{2 - \beta}) \), where \( \beta < 2 \) due to (81),
\[
\|y_q - y_{q - 1}\|_{C_t C_x} \leq \|y_q - y_{q - 1}\|_{C_x}^\eta \|y_q - y_{q - 1}\|_{C_t}^{1 - \eta} \leq L^\frac{1}{2} M_0(t)^\frac{1}{2} \lambda_q^\gamma \delta_q^\gamma \rightarrow 0 
\]
(104)
as \( q \to \infty \). Hence, there exists a limiting solution \( \lim_{q \to \infty} y_q = y \in C([0, T_L]; C^{\alpha^p}(\mathbb{T}^2)) \cap C^\infty([0, T_L]; C(\mathbb{T}^2)) \) for which there exists deterministic constant \( C_L > 0 \) such that
\[
\|y\|_{C_t C_x} \leq C_L \forall \beta' \in \left( \frac{1}{2}, \beta \right), \eta \in \left[ 0, \frac{\beta}{2 - \beta} \right], 
\]
(105)
where \( \frac{1}{2} < \frac{\beta}{2 - \beta} \); we denote its initial data as \( y_0^m \). Because each \( y_q \) is \((\mathcal{F}_t)_{t \geq 0}\)-adapted due to Propositions 4.4, 4.8 we deduce that \( y \) is \((\mathcal{F}_t)_{t \geq 0}\)-adapted. Moreover, because the range of \( \beta \) in (81) guarantees that \( 1 - 2\beta < 0 \), we deduce that
\[
\|\tilde{R}_q\|_{C_t C_x} \leq \tilde{C} L M_0(L) \lambda_{q + 1} \delta_{q + 1} \quad \forall \beta' \in \left( \frac{1}{2}, \beta \right), \eta \in \left[ 0, \frac{\beta}{2 - \beta} \right], 
\]
(106)
as \( q \to \infty \). Using the convergence of \( y_q \) to \( y \) in \( C_{T_L} C^{\alpha^p}_x \) for \( \beta' \in \left( \frac{1}{2}, \beta \right) \), as well as (300) and (802), it follows that \( (y, z) \) solves (67) weakly; i.e. for all \( \psi \in C^\infty(\mathbb{T}^2) \cap \mathcal{H}_z^{\frac{1}{2}} \) and \( t \in [0, T_L] \)
\[
(y(t) - y(0), \psi) - \int_0^t \sum_{i,j=1}^2 \langle \Lambda(y + z)_i(s), \partial_j \psi_j(y + z)_j(s) \rangle_{H_z^{\frac{1}{2}} - H_z^{\frac{1}{2}}} - \langle y(s), \Lambda \psi \rangle \\
- \frac{1}{2} \sum_{i,j=1}^2 \langle \partial_i(y + z)_j(s), [\Lambda, \psi_j](y + z)_j(s) \rangle_{H_z^{\frac{1}{2}} - H_z^{\frac{1}{2}}} ds = 0. 
\]
(107)
It follows from (66) and definitions of \( y = v - z \) and \( p_2 = p - p_1 \) that \( v \) solves (6) weakly. As \( z(0, x) = 0 \) from (66), we denote \( v^m \). Next, we compute using Parseval theorem
\[
\|v(t)\|_{C_t C_x}^2 \leq M_0(t) 8\pi^4. 
\]
(108)
Because \( z(t) \) from (68) and \( y \) are \((\mathcal{F}_t)_{t \geq 0}\)-adapted, we see that \( v \) is \((\mathcal{F}_t)_{t \geq 0}\)-adapted. Moreover, the bounds from (105) on \( y_q \) and (78) on \( z \) give the regularity of \( v \) in (10) as follows: for almost every (a.e.) \( \omega \in \Omega, \beta' \in \left( \frac{1}{2}, \beta \right), \) and \( \eta \in \left( 0, \frac{1}{2} \right), \)
\[
\|v(\omega)\|_{C_t C_x} + \|v(\omega)\|_{C_t C_x} < \infty 
\]
(109)
Next, we compute for all \( t \in [0, T_L] \) using the fact that \( \text{supp}(y_{q + 1} - y_q) \subset B(0, 2\lambda_{q + 1}) \) due to Hypothesis 4.1(a) the fact that \( b^{q + 1} \geq b(q + 1) \) for all \( q \geq 0 \) and \( b \geq 2 \), and \( \frac{1}{2} < \beta < 2 \).
Then we can compute
\[
\|y(T)\|_{H^s_x} \geq M_0(T)^{\frac{1}{2}}\sqrt{8\pi^2} - \|y(T) - y_0(T)\|_{H^s_x} \geq \sqrt{2\pi^2}M_0(T)^{\frac{1}{2}} > (\sqrt{2\pi^2}L^2 + \sqrt{8\pi^2}L^2 + L)e^{LT}
\]
\[
\geq (\|y(0) - y_0(0)\|_{H^s_x} + \|y_0(0)\|_{H^s_x} + L)e^{LT} \geq (\|y(0)\|_{H^s_x} + L)e^{LT}.
\]
Therefore, we can deduce (111) as follows: on \(T_L \geq T\)
\[
\|v(T)\|_{H^s_x} \geq \|y(T)\|_{H^s_x} > \|z(T)\|_{H^s_x} \geq \sqrt{\|v^0\|_{H^s_x} + \sqrt{T \operatorname{Tr}((\Delta)^{\frac{1}{2}}GG^*)}}.
\]
Finally, because \(y_0(0, x)\) is deterministic due to Proposition 4.7, Proposition 4.8 implies that \(y(0, x)\) is deterministic. As \(z(0, x) = 0\) by (66b), this implies that \(v^0\) is deterministic.

4.1. Proof of Proposition 4.8

Similarly to previous works on probabilistic convex integration, there will be various functions of \(L\) in estimates and we will take \(b = b(L, \gamma_2) \in \mathbb{N}\) sufficiently large (e.g., (136), (144), (148), (212), and (232)) to bound them. Here, we emphasize that this \(b\) can be chosen independently from \(a\) and \(\beta\). Considering such \(b \in \mathbb{N}\) fixed not depending on \(a\) or \(\beta\), we notice that due to (83) we can take \(a \in 5\mathbb{N}\) sufficiently large such that \(a \geq e^{\frac{1}{\beta}}\) and \(\beta > 1 - \frac{1}{2}\) sufficiently close to 1 and achieve (92b). It follows from (80), (82), (84), and (5) that
\[
\partial_t y_i + (\Lambda^{2-\gamma_2}(y_{i} + z_i) \cdot \nabla)(y_i + z_i) - (\nabla(y_i + z_i))^T \cdot \Lambda^{2-\gamma_2}(y_i + z_i)
\]
\[
+ \nabla p_i + \Lambda^{\gamma_2}y_i = \text{div}(\hat{R}_i + R_{\text{Com}}),
\]
where
\[
p_i \equiv p_q \ast_{s} \phi_{l} \ast_{s} \varphi_{i},
\]
\[
\hat{R}_{\text{Com}} \equiv \mathcal{B}[\Lambda^{2-\gamma_2}(y_{i} + z_i) \cdot \nabla \ast \cdot (y_i + z_i) - [\Lambda^{2-\gamma_2}(y_{i} + z_i) \cdot \nabla \ast \cdot (y_q + z_q) \ast_{s} \phi_{l} \ast_{s} \varphi_{i}].
\]
Because (81b) guarantees that \(\beta < 2 - \gamma_2\), for \(a \in 5\mathbb{N}\) sufficiently large we see that
\[
\|y - y_0\|_{C_{t,s}} \leq \|y_q \ast \phi_{l} \ast_{s} \varphi_{i}\|_{C_{t,s}} + \|y_{q} \ast_{s} \phi_{l} \ast_{s} \varphi_{i}\|_{C_{t,s}} \leq ILM_0(t)^{\lambda_{q+1}^{2-\gamma_2} \delta_q}.
\]
Now we strategically decompose the Reynolds stress at level \(q + 1\); we note that especially due to \(z_{q+1}\) and \(z_q\), there will be an extra layer of complication in comparison to previous works (e.g., (45), (78), (79)). First, due to (50), (98), and (111),
\[
\text{div} \hat{R}_{q+1} - \nabla p_{q+1} = -(\Lambda^{2-\gamma_2}y_i \cdot \nabla)z_{q+1} - (\Lambda^{2-\gamma_2}z_i \cdot \nabla)y_i - (\Lambda^{2-\gamma_2}z_i \cdot \nabla)z_{q+1}
\]
\[
+ (\nabla y_i)^T \cdot \Lambda^{2-\gamma_2}z_i + (\nabla z_i)^T \cdot \Lambda^{2-\gamma_2}y_i + (\nabla z_i)^T \cdot \Lambda^{2-\gamma_2}z_{q+1} - \nabla p_i + \text{div} \hat{R}_i + \text{div} R_{\text{Com}}
\]
\[
+ \partial_t w_{q+1} + (\Lambda^{2-\gamma_2}y_i \cdot \nabla)w_{q+1} + (\Lambda^{2-\gamma_2}y_i \cdot \nabla)z_{q+1}
\]
+ (Λ^{2-γ_2}w_{q+1} \cdot \nabla) y_l + (\Lambda^{2-γ_2}w_{q+1} \cdot \nabla) w_{q+1} + (\Lambda^{2-γ_2}w_{q+1} \cdot \nabla) z_{q+1} \\
+ (\Lambda^{2-γ_2}z_{q+1} \cdot \nabla) y_l + (\Lambda^{2-γ_2}z_{q+1} \cdot \nabla) w_{q+1} + (\Lambda^{2-γ_2}z_{q+1} \cdot \nabla) z_{q+1} \\
- (\nabla y_l)^T \cdot \Lambda^{2-γ_2}y_{q+1} - (\nabla y_l)^T \cdot \Lambda^{2-γ_2}z_{q+1} \\
- (\nabla w_{q+1})^T \cdot \Lambda^{2-γ_2}y_l - (\nabla w_{q+1})^T \cdot \Lambda^{2-γ_2}z_{q+1} \\
- (\nabla z_{q+1})^T \cdot \Lambda^{2-γ_2}y_l - (\nabla z_{q+1})^T \cdot \Lambda^{2-γ_2}z_{q+1} + Λ^{γ_1}w_{q+1}.  \tag{114}

We rewrite some of these terms as follows:

$$
- (\Lambda^{2-γ_2}y_l \cdot \nabla) z_l - (\Lambda^{2-γ_2}z_l \cdot \nabla) y_l - (\Lambda^{2-γ_2}z_l \cdot \nabla) z_l 
+ (\Lambda^{2-γ_2}y_l \cdot \nabla) z_{q+1} + (\Lambda^{2-γ_2}z_l \cdot \nabla) w_{q+1} + (\Lambda^{2-γ_2}z_l \cdot \nabla) z_{q+1} \\
+ (\Lambda^{2-γ_2}z_{q+1} \cdot \nabla) w_{q+1} + (\Lambda^{2-γ_2}z_{q+1} \cdot \nabla) z_{q+1} \tag{115}
$$

$$
= \Lambda^{2-γ_2}y_{q+1} \cdot \nabla(z_{q+1} - z_q) + \Lambda^{2-γ_2}y_{q+1} \cdot \nabla(z_q - z_l) + \Lambda^{2-γ_2}y_{q+1} \cdot \nabla(y_{q+1} + \Lambda^{2-γ_2}z_l \cdot \nabla w_{q+1} + \Lambda^{2-γ_2}z_{q+1} \cdot \nabla z_{q+1} \\
+ \Lambda^{2-γ_2}z_{q+1} \cdot \nabla z_{q+1} - z_q) \cdot \nabla z_{q+1} + \Lambda^{2-γ_2}z_{q+1} \cdot \nabla z_{q+1} + \Lambda^{2-γ_2}z_{q+1} \cdot \nabla(z_q - z_l) \cdot \nabla z_{q+1} \\
+ \Lambda^{2-γ_2}z_{q+1} \cdot \nabla(z_q - z_l) \cdot \nabla z_{q+1} + \Lambda^{2-γ_2}z_{q+1} \cdot \nabla(z_q - z_l),  \tag{116}
$$

$$
(\nabla y_l)^T \cdot \Lambda^{2-γ_2}z_l + (\nabla z_l)^T \cdot \Lambda^{2-γ_2}y_l + (\nabla z_l)^T \cdot \Lambda^{2-γ_2}z_l \\
- (\nabla y_l)^T \cdot \Lambda^{2-γ_2}z_{q+1} - (\nabla z_{q+1})^T \cdot \Lambda^{2-γ_2}y_l \\
= (\nabla y_l)^T \cdot \Lambda^{2-γ_2}(z_l - z_q) + (\nabla y_l)^T \cdot \Lambda^{2-γ_2}(z_q - z_{q+1}) \\
+ (\nabla(z_l - z_q))^T \cdot \Lambda^{2-γ_2}y_l + (\nabla(z_q - z_{q+1}))^T \cdot \Lambda^{2-γ_2}y_l \\
+ (\nabla(z_l - z_q))^T \cdot \Lambda^{2-γ_2}z_l + (\nabla(z_q - z_{q+1}))^T \cdot \Lambda^{2-γ_2}z_l \\
+ (\nabla z_{q+1})^T \cdot \Lambda^{2-γ_2}(z_l - z_q) + (\nabla z_{q+1})^T \cdot \Lambda^{2-γ_2}(z_q - z_{q+1}),  \tag{117}
$$

and because $\Lambda^{2-γ_2}w_{q+1} \cdot \nabla z_l$ in (115) will still create a difficulty to handle, we couple it with $-(\nabla z_{q+1})^T \cdot \Lambda^{2-γ_2}w_{q+1}$ from (114) to write

$$
\Lambda^{2-γ_2}w_{q+1} \cdot \nabla z_l - (\nabla z_{q+1})^T \cdot \Lambda^{2-γ_2}w_{q+1} \tag{118}
$$

Finally, the products of $y_{q+1}$ with $z_q - z_l$ in (115) are difficult to treat while a product of $w_{q+1}$ with $z_q - z_l$ grants us a factor of $\Lambda^{γ_1}$, from $Ω$ considering its frequency support (see how we treat $R_T$ in Subsection 4.1.1 and $R_N$ in Subsection 4.1.2); therefore, we pair up some terms from (115) and (116) selectively to rewrite by using (28) and (5).
and furthermore for convenience by (5),

\[
\begin{align*}
(\nabla (z_l - z_q))^T \cdot \Lambda^{-2\gamma_l} \mathbf{w}_{q+1} + \Lambda^{-2\gamma_l} y_{q+1} \cdot \nabla (z_q - z_l) + (\nabla (z_l - z_q))^T \cdot \Lambda^{-2\gamma_l} y_l & \\
& = -\Lambda^{-2\gamma_l} w_{q+1}^\perp \cdot (z_l - z_q) + \Lambda^{-2\gamma_l} y_{q+1} \cdot \nabla^\perp \cdot (z_q - z_l), \\
\Lambda^{-2\gamma_l} (z_{q+1} - z_q) \cdot \nabla z_{q+1} + (\nabla z_{q+1})^T \cdot \Lambda^{-2\gamma_l} (z_q - z_{q+1}) & = \Lambda^{-2\gamma_l} (z_{q+1} - z_q)^2 \nabla^\perp \cdot (z_q - z_{q+1}), \\
\Lambda^{-2\gamma_l} (z_q - z_l) \cdot \nabla z_q + (\nabla z_l)^T \cdot \Lambda^{-2\gamma_l} (z_l - z_q) & = \Lambda^{-2\gamma_l} (z_q - z_l)^2 \nabla^\perp \cdot (z_l - z_q), \\
\Lambda^{-2\gamma_l} z_l \cdot \nabla (z_q - z_l) + (\nabla (z_l - z_q))^T \cdot \Lambda^{-2\gamma_l} z_l & = \Lambda^{-2\gamma_l} z_l^2 \nabla^\perp \cdot (z_q - z_l). 
\end{align*}
\]

Applying (115)-(120) to (114) allows us to deduce

\[
\text{div} \mathbf{R}_{q+1} = \text{div}(R_T + R_N + R_L + R_{\text{Com}1} + R_{\text{Com}2}) 
\]

where in addition to \( R_{\text{Com}1} \) in (114) we defined

\[
\begin{align*}
\text{div} \mathbf{R}_T & \equiv \partial_t w_{q+1} + (\Lambda^{-2\gamma_l} y_l + \Lambda^{-2\gamma_l} z_l) \cdot \nabla w_{q+1}, \\
\text{div} \mathbf{R}_N & \equiv \nabla (\Lambda^{-2\gamma_l} y_l + z_q)^T \cdot w_{q+1} + (\Lambda^{-2\gamma_l} w_{q+1} \cdot \nabla) y_l - (\nabla y_l)^T \cdot \Lambda^{-2\gamma_l} w_{q+1}, \\
\text{div} \mathbf{R}_L & \equiv \Lambda^{-2\gamma_l} w_{q+1} + \Lambda^{-2\gamma_l} w_{q+1}^\perp \nabla^\perp \cdot z_l, \\
\text{div} \mathbf{R}_O & \equiv \text{div} \mathbf{R}_l + (\Lambda^{-2\gamma_l} w_{q+1} \cdot \nabla) w_{q+1} - (\nabla w_{q+1})^T \cdot \Lambda^{-2\gamma_l} w_{q+1}, \\
\text{div} \mathbf{R}_{\text{Com}2} & \equiv \Lambda^{-2\gamma_l} w_{q+1} \cdot \nabla (z_{q+1} - z_q) + \Lambda^{-2\gamma_l} (z_{q+1} - z_q) \cdot \nabla y_{q+1} + (\nabla y_l)^T \cdot \Lambda^{-2\gamma_l} (z_q - z_{q+1}) \\
& + (\nabla (z_q - z_{q+1}))^T \cdot \Lambda^{-2\gamma_l} y_l + (\nabla \Lambda^{-2\gamma_l} (z_{q+1} - z_q))^T \cdot w_{q+1} \\
& - \Lambda^{-2\gamma_l} w_{q+1}^\perp \nabla^\perp \cdot (z_l - z_q) + (\nabla (z_q - z_{q+1}))^T \cdot \Lambda^{-2\gamma_l} w_{q+1} \\
& + \Lambda^{-2\gamma_l} y_l^\perp \nabla^\perp \cdot (z_q - z_l) + \Lambda^{-2\gamma_l} (z_q - z_l) \cdot \nabla w_{q+1} + \Lambda^{-2\gamma_l} (z_q - z_l)^2 \nabla^\perp \cdot y_l \\
& + \Lambda^{-2\gamma_l} (z_{q+1} - z_q)^2 \nabla^\perp \cdot z_{q+1} + \Lambda^{-2\gamma_l} \nabla^\perp \cdot z_{q+1} + \Lambda^{-2\gamma_l} z_{q+1} \nabla^\perp \cdot (z_q - z_l) + \Lambda^{-2\gamma_l} z_{q+1} \nabla^\perp \cdot (z_q - z_l), 
\end{align*}
\]

representing the Reynolds stress errors respectively of transport, Nash, linear, oscillation, and second commutator types, if we also define

\[
p_{q+1} \equiv p_l + w_{q+1} \cdot \Lambda^{-2\gamma_l} (y_l + z_{q+1}).
\]

Next, we will consider two special cases of the following transport equation

\[
\begin{align*}
\partial_t f + (u \cdot \nabla) f & = g, \\
f(t_0, x) & = f_0(x),
\end{align*}
\]

where \( u(t, x) \) is a given smooth vector field, for which if we let \( \Phi \) be the inverse of the flux \( \mathbf{X} \) of \( u \) starting at time \( t_0 \) as the identity

\[
\begin{align*}
\partial_t \mathbf{X} & = u(\mathbf{X}, t), \\
\mathbf{X}(t_0, x) & = x.
\end{align*}
\]

then one of the consequences is that \( f(t, x) = f_0(\Phi(t, x)) \) in case \( g \equiv 0 \). We defer more consequences to Lemma 5.2. Now for all \( j \in \{0, \ldots, \lceil T_{r_{q+1}} \rceil \} \) we define \( \Phi_j(t, x) \) as a special case of (124) that solves

\[
\begin{align*}
(\partial_t & + (\Lambda^{-2\gamma_l} y_l + \Lambda^{-2\gamma_l} z_l) \cdot \nabla) \Phi_j = 0, \\
\Phi_j(t_{r_{q+1}}, x) & = x,
\end{align*}
\]

similarly to (81) equation (68) (also (72) equation (43)). With \( \Gamma_1 \) and \( \Gamma_2 \) from Lemma 5.2 we define \( \Gamma_j \) to be \( \Gamma_1 \) and \( \Gamma_2 \) when \( j \) is odd and even, respectively. With \( \mathbb{P}_{q+1,k} \) from
and $a_{k,j}$ to be defined subsequently in (129), we define, recalling the notations from (85),

$$w_{q+1}(t, x) \doteq \sum_{j,k} \chi_j(t) \mathcal{E}_{q+1,k}(a_{k,j}(t, x)b_k(\lambda_{q+1} \Phi_j(t, x))),$$

(127)

where thanks to $\mathcal{E}_{q+1,k}$, we see that $w_{q+1}$ has the frequency support of (100), which also implies directly from (114) that $\text{supp} \hat{R}_{q,j} \subset B(0, 4\lambda_{q+1})$ as desired. Next, as the second special case of (124), we define $\tilde{R}_{q,j}$ to be the solution to

$$(\partial_t + (\Lambda^{2-\gamma} y_t + \Lambda^{2-\gamma} z_t) \cdot \nabla) \tilde{R}_{q,j} = 0,$$

(128a)

$$\tilde{R}_{q,j}(t, x) = \tilde{R}_j(t, x).$$

(128b)

Moreover, for $t \geq \tau_{q+1,j}$ and $\gamma_k$ for $k \in \Gamma_j \subset \mathbb{S}^1$ from Lemma 3.2, we define

$$a_{k,j}(t, x) \doteq \sqrt{C} \frac{L^2 M_0(t_{q+1,j})}{\sqrt{\gamma_j}} \delta_{q,j}^{\frac{1}{\gamma_j}} \| \tilde{R}_j(\tau_{q+1,j})\|_{C_t} \frac{\epsilon_j \| \tilde{R}_j(\tau_{q+1,j})\|_{C_t}}{C \Lambda^{2-\gamma} \delta_{q+1} \Lambda M_0(\tau_{q+1,j})},$$

(129)

where $\gamma_k \left( \text{Id} - \frac{\epsilon_j \tilde{R}_j(\tau_{q+1,j})}{C \Lambda^{2-\gamma} \delta_{q+1} \Lambda M_0(\tau_{q+1,j})} \right)$ is well-defined because

$$\left| \frac{\epsilon_j \tilde{R}_j(\tau_{q+1,j})}{C \Lambda^{2-\gamma} \delta_{q+1} \Lambda M_0(\tau_{q+1,j})} \right| \leq \frac{\epsilon_j \| \tilde{R}_j(\tau_{q+1,j})\|_{C_t}}{C \Lambda^{2-\gamma} \delta_{q+1} \Lambda M_0(\tau_{q+1,j})} \leq \epsilon_j$$

(130)

so that $\text{Id} - \frac{\epsilon_j \tilde{R}_j(\tau_{q+1,j})}{C \Lambda^{2-\gamma} \delta_{q+1} \Lambda M_0(\tau_{q+1,j})} \in B(\text{Id}, \epsilon_j)$; here we used the facts that $\varphi_t$ is supported in $(\tau_{q+1,j}, 2\tau_{q+1,j}) \subset \mathbb{R}^+$ and $M_0(t)$ from (77) is non-decreasing. We recall $\chi_j(t)$ from (33), $b_k$ and $c_k$ from (45) and define

$$\tilde{w}_{q+1,j,k}(t, x) \doteq \chi_j(t)a_{k,j}(t, x)b_k(\lambda_{q+1} \Phi_j(t, x))),$$

(131a)

$$\tilde{w}_{q+1,j,k}(t, x) \doteq \frac{c_k(\lambda_{q+1} \Phi_j(t, x)))}{c_k(\lambda_{q+1} x))} e^{i\lambda_{q+1}(\Phi_j(t, x) - x)k},$$

(131b)

so that due to (127) and (131a)

$$w_{q+1} = \sum_{j,k} \mathcal{E}_{q+1,k} \tilde{w}_{q+1,j,k} \quad \text{and} \quad b_k(\lambda_{q+1} \Phi_j(x))) = b_k(\lambda_{q+1} x) \tilde{w}_{q+1,j,k}(x).$$

(132)

In Propositions 4.9-4.10, we collect some estimates, similarly to [7] Lemmas 4.3-4.4].

**Proposition 4.9.** Define

$$D_{x,q} \doteq \partial_t + (\Lambda^{2-\gamma} y_t + \Lambda^{2-\gamma} z_t) \cdot \nabla.$$  

(133)

Then $w_{q+1}$ defined in (127) satisfies the following inequalities: for $C_1$ from (51),

$$\|w_{q+1}(t)\|_{C_t} \leq 2 \sup_{k \in \Gamma_j \cup \Gamma_k} \| \gamma_k \|_{C_t(\text{Id}, \epsilon_j)} \frac{\sqrt{C} \Lambda^{2-\gamma} M_0(t)}{\sqrt{\gamma_j}} \delta_{q,j}^{\frac{1}{\gamma_j}},$$

(134a)

$$\|D_{x,q} w_{q+1}(t)\|_{C_t} \leq L M_0(t) \delta_{q+1,j}^{\frac{1}{\gamma_j}} \tau_{q+1,j}^{-1}.$$  

(134b)

Consequently, (89), as well as all of (86), (87), and (89) at level $q+1$ and hence Hypothesis 4.1(a) and (c) at level $q+1$ hold.
Proof of Proposition 4.2 First, we can prove [134a] from (127) as
\[
\|w_{q+1}(t)\|_{C_2} \leq \sup_{k \in \mathbb{F}} \|y_k\|_{C(B(0,\ell_x,\epsilon))} \sqrt{C} \frac{1}{\sqrt{\gamma}} \sum_j \int_{\sup_{k} y_j(t) C_1 L^2 M_0(t_j + 1)^{-\frac{3}{4}} \delta_{q+1}} 1 \ (135)
\]
which implies [134a] considering the fact that for all \(t\) there exist at most two non-trivial cutoffs. Its immediate consequence is (99) because
\[
\|y_{q+1}(t) - y(t)\|_{C_2} \leq L^2 M_0(t)^{-\frac{3}{4}} \delta_{q+1} + lM_0(t) \gamma \gamma \gamma \delta_{q} \leq L^2 M_0(t)^{-\frac{3}{4}} \delta_{q+1} \ (136)
\]
where we took \(b > 2(L^2 + 2), a \in 5M\) sufficiently large and used that \(81c\) guarantees \(\alpha > \frac{3-\gamma}{2}\). Next, by Young’s inequality for convolution we can prove (86) at level \(q + 1\) as follows:
\[
\|y_{q+1}\|_{C_{q+1}} \leq \|y\|_{C_{q+1}} + \|w_{q+1}\|_{C_{q+1}} \ (137)
\]
Similarly, because \(81b\) guarantees that \(2 - C_2 \geq \beta\), we can prove (87) at level \(q + 1\) as follows: for \(C_0 \geq 2^j \sup_{k \in \mathbb{F} \setminus \{1\}} \|y_k\|_{C(B(0,\ell_x,\epsilon))} \frac{1}{\sqrt{\gamma}} \) and \(a \in 5M\) sufficiently large
\[
\|y_{q+1}\|_{C_{q+1}} \leq \|y\|_{C_{q+1}} + \|w_{q+1}\|_{C_{q+1}} \ (138)
\]
Next, \(y_{q+1} = y + w_{q+1}\) by (98) so that (100) and Hypothesis 4.1(a) at level \(q\) imply that \(\text{supp} \tilde{y}_{q+1} \subset B(0, 2^q x_{q+1})\); therefore, we conclude that the Hypothesis 4.1(a) at level \(q + 1\) holds. Next, it follows immediately from (133), (126a), and (128a) that
\[
D_{t,q} a_k(t, x) = 0, \ D_{t,q} b_k(\lambda_{q+1} \Phi(t, x)) = 0, \ D_{t,q} c_k(\lambda_{q+1} \Phi(t, x)) = 0. \ (139)
\]
By (132), (131a), and (138) we obtain
\[
D_{t,q} w_{q+1} = \sum_{j,k} [D_{t,q}, P_{q+1,k}] w_{q+1, j,k} + P_{q+1,k} [\partial k \chi_c, a_k b_k (\lambda_{q+1} \Phi_k)]. \ (140)
\]
We estimate from (139) using [7 Corollary A.8] with (50) and the facts that \(81b\) and \(81c\) respectively guarantee that \(3 - \gamma_2 - \beta > 0\) and \(\alpha > 0\) so that \(\lambda_q^{-\gamma_2} \delta_q^\alpha \leq \tau_{q+1}^{-1}\), for \(a \in 5M\) sufficiently large
\[
\|D_{t,q} w_{q+1}(t)\|_{C_2} \leq \sum_{j,k} \|\nabla (\lambda_2^{-\gamma_2} y_{j+1} + \lambda_2^{-\gamma_2} z_{j+1})(t)\|_{C_2} \|w_{q+1, j+k}(t)\|_{C_2} + \|\partial k \chi_c, a_k b_k (\lambda_{q+1} \Phi_k(t))\|_{C_2} \ (141)
\]
Next, we recall the definitions of \(y_{j+1}\) and \(z_{j+1}\) from (83) and write \(D_{t,q+1} y_{q+1}\) as
\[
D_{t,q+1} y_{q+1} = D_{t,q} w_{q+1} + (\lambda_2^{-\gamma_2} y_{j+1} \ast \lambda_{q+2} \varphi_{j+2} - \lambda_2^{-\gamma_2} y_{j+1}) \cdot \nabla w_{q+1} + \lambda_2^{-\gamma_2} w_{q+1} \ast \varphi_{j+2} \gamma \varphi_{j+2} \cdot \nabla w_{q+1} + \lambda_2^{-\gamma_2} w_{q+1} \ast \varphi_{j+2} \gamma \varphi_{j+2} \cdot \nabla y_{j+1} + \lambda_2^{-\gamma_2} w_{q+1} \ast \varphi_{j+2} \gamma \varphi_{j+2} \cdot \nabla y_{j+1} + D_{t,q} y_{j+1} \ast \varphi_{j+2} \gamma \varphi_{j+2} \cdot \nabla y_{j+1} \ (142)
\]
due to (133) and (98) so that we can prove (89) at level \(q + 1\) as follows: for \(a \in 5\mathbb{N}\) sufficiently large

\[
\|D_{t,q+1}y_{q+1}\|_{C_{t}} \lesssim \|D_{t,q}w_{q+1}\|_{C_{t}} + \|\Lambda^{2-\gamma_{q}}y_1\|_{C_{t}} + \|\Lambda^{2-\gamma_{q}}z_1\|_{C_{t}} + \|D_{t,q}w_{q+1}\|_{C_{t}} + \|\Lambda^{2-\gamma_{q}}w_{q+1}\|_{C_{t}} + \|\Lambda^{2-\gamma_{q}}y_{q+1}\|_{C_{t}} + \|\Lambda^{2-\gamma_{q}}y_{q+1}\|_{C_{t}} + \|\Lambda^{2-\gamma_{q}}z_{q+1}\|_{C_{t}} + \|\Lambda^{2-\gamma_{q}}z_{q+1}\|_{C_{t}}
\]

where we used that \(a < 3 - \gamma_{2} - \beta\) by (81c) while \(\beta < \min(\frac{3\gamma_{2}}{2}, 2 - \gamma_{2}, 1)\) by (81b).

**Proposition 4.10.** Let \(a_{t,j}\) and \(\psi_{q+1,j,k}\) be defined by (129) and (131b), respectively. Then they satisfy the following inequalities for all \(t \in \mathbb{R}\):

\[
\|D^{N}a_{t,j}(t)\|_{C_{t}} \leq L^{\frac{1}{2}}M_{0}(t)^{\frac{1}{2}}\lambda_{q}^{N+2-\gamma_{2}\delta_{q+1}} \forall N \in \mathbb{N},
\]

\[
\|\psi_{q+1,j,k}(t)\|_{C_{t}} \leq 1, \quad \|D^{N}\psi_{q+1,j,k}(t)\|_{C_{t}} \leq L^{\frac{1}{2}}M_{0}(t)^{\frac{1}{2}}\lambda_{q}^{N+2-\gamma_{2}\delta_{q+1}} \forall N \in \mathbb{N}.
\]

**Proof of Proposition 4.10.** First, immediately by definition from (129) we can estimate

\[
\|a_{t,j}(t)\|_{C_{t}} \leq L^{\frac{1}{2}}M_{0}(t)^{\frac{1}{2}}\lambda_{q}^{\delta_{q+1}}.
\]

Next, for \(N \in \mathbb{N}\), by standard chain rule estimates (e.g., (4) equation (130)),

\[
\|D^{N}a_{t,j}(t)\|_{C_{t}} \leq \frac{1}{\lambda_{q}^{N+2-\gamma_{2}\delta_{q+1}}}L^{\frac{1}{2}}M_{0}(\tau_{q+1})^{\frac{1}{2}} \times \left[ \frac{1}{\lambda_{q}^{N+\gamma_{2}+2-\gamma_{2}\delta_{q+1}}} \right]^{N} \|D_{t,q}^{N}\hat{\psi}_{q+1}(t)\|_{C_{t}}.
\]

Because (81b)–(81c) guarantee that \(3 - \gamma_{2} > \beta\), we can estimate for \(b > \frac{8}{7}(L^{2} + \frac{7}{2})\) and \(a \in 5\mathbb{N}\) sufficiently large

\[
\tau_{q+1}\|D(\Lambda^{2-\gamma_{2}}y_{1} + \Lambda^{2-\gamma_{2}}z_{1})\|_{C_{t}} \lesssim \frac{8}{7}(L^{2} + \frac{7}{2}) \leq 1,
\]

and consequently in case \(N = 1\) we can estimate by (310b) and (128)

\[
\|D_{t,q+1}y_{q+1}\|_{C_{t}} \lesssim \|D_{t,q}y_{q+1}\|_{C_{t}} \lesssim \lambda_{q}LM_{0}(\tau_{q+1})\lambda_{q}^{2-\gamma_{2}\delta_{q+1}}.
\]

and similarly in case \(N \in \mathbb{N} \setminus \{1\}\) by relying instead on (311), for \(b > 2L^{2}\)

\[
\|D^{N}y_{q+1}\|_{C_{t}} \lesssim \|D^{N}y_{q+1}\|_{C_{t}} \lesssim \lambda_{q}LM_{0}(\tau_{q+1})\lambda_{q}^{2-\gamma_{2}\delta_{q+1}}.
\]
Applying (145)-(146) to (143) gives us (141a). Next, concerning (141b), its first inequality is clear from (131b). Moreover, by \[ \text{[4 equation (130)]} \] again we have

$$
\|D\psi_{q_1,j,k}(t)\|_{C_s} \leq \lambda_{q_1} \|\nabla \Phi_j(t) - \text{Id}\|_{C_s} \leq \lambda_{q_1} \tau_{q_1} \lambda_q^{3-\gamma} \delta_q^{\frac{1}{2}} M_0(t)^{\frac{1}{2}}.
$$

(147)

In case $N \in \mathbb{N} \setminus \{1\}$, we estimate again via \([4 \text{ equation (130)}]\) from (131b) for

$$
b > \begin{cases} 
2 + 4L^2 & \text{if } \gamma_1 = 1, \\
2 + \frac{4L^2}{\gamma'_1} & \text{if } \gamma_1 > 1,
\end{cases}
$$

and $a \in 5\mathbb{N}$ sufficiently large

$$
\|D^N \psi_{q_1,j,k}(t)\|_{C_s} \leq \lambda_{q_1} \|\Phi_j(t)\|_{C_s^N} + \lambda_{q_1} \|\nabla \Phi_j(t) - \text{Id}\|_{C_s^N} \\
\leq \lambda_{q_1} \tau_{q_1} \lambda_q^{N+2-\gamma} \delta_q^{\frac{1}{2}} L^N M_0(t)^{\frac{1}{2}} + \lambda_{q_1} \tau_{q_1} \lambda_q^{3-\gamma} \delta_q^{\frac{1}{2}} L^N M_0(t)^{\frac{1}{2}} \\
\leq \lambda_{q_1} \tau_{q_1} \lambda_q^{N+2-\gamma} \delta_q^{\frac{1}{2}} L^N M_0(t)^{\frac{1}{2}}
$$

where the last inequality used the facts that $e^d \leq a$ and $\alpha > 1$ from (81c) so that

$$
\lambda_{q_1} \tau_{q_1} \lambda_q^{2-\gamma} \delta_q^{\frac{1}{2}} L^2 M_0(t)^{\frac{1}{2}} \leq \alpha^d [(b-2)(\frac{M_0(t)^{\frac{1}{2}}}{L^N})^{\alpha-1}] a^d \ll 1.
$$

(148)

Before we proceed, let us observe an immediate useful corollary of (141) and (148): for all $t \in \text{supp} \chi_j$,

$$
\|\nabla (a_{\chi_j} \psi_{q_1,j,k}(t))\|_{C_s} \leq \|\nabla a_{\chi_j}\|_{C_s} \|\psi_{q_1,j,k}\|_{C_s} + \|a_{\chi_j}\|_{C_s} \|\nabla \psi_{q_1,j,k}\|_{C_s} \leq \lambda_{q_1} \delta_q^{\frac{1}{2}} L^2 M_0(t)^{\frac{1}{2}} [1 + L^2 M_0(t)^{\frac{1}{2}} \lambda_{q_1} \tau_{q_1} \lambda_q^{2-\gamma} \delta_q^{\frac{1}{2}}] \leq \lambda_{q_1} \delta_q^{\frac{1}{2}} L^2 M_0(t)^{\frac{1}{2}}.
$$

(149)

4.1.1. Bounds on $R_T$, the transport error. Using (131a), (32)-(33), and (138), we can rewrite $R_T$ from (122a) as

$$
R_T = \sum_{j,k} 1_{\text{supp} \chi_j}(t) \mathcal{B}([\Lambda^{2-\gamma} y_j + \Lambda^{2-\gamma} z_j] \cdot \nabla, \mathcal{P}_{q_1,k} \tilde{w}_{q_1,j,k} + \mathcal{P}_{q_1,k} (\partial_a x_j a_{k,j} \tilde{b}_k(\lambda_{q_1} \Phi_j))).
$$

(150)

Next, by definitions of $\mathcal{P}_{q_1,k}$ and $\tilde{P}_{a_{k,j}}$ from Subsection 3.2, we see that for $a \in 5\mathbb{N}$ such that $a \geq \frac{4}{3\beta}$ we can rewrite furthermore as

$$
R_T = \sum_{j,k} 1_{\text{supp} \chi_j}(t) \mathcal{B} \tilde{P}_{a_{k,j}}([\Lambda^{2-\gamma} y_j + \Lambda^{2-\gamma} z_j] \cdot \nabla, \mathcal{P}_{q_1,k} \tilde{w}_{q_1,j,k} + \mathcal{P}_{q_1,k} (\partial_a x_j a_{k,j} \tilde{b}_k(\lambda_{q_1} \Phi_j))).
$$

(151)

Thus, we estimate by \([7 \text{ Lemma A.6}]\) and using the fact that (81b) and (81c) guarantee that $\beta < \frac{2-3\gamma}{1-2}$ and $\alpha < 7 - 3\gamma_2 - 5\beta$, for $a \in 5\mathbb{N}$ sufficiently large

$$
\|R_T\|_{C_s} \leq \lambda_{q_1}^{-1} \sum_{j,k} \left[ \|\nabla \Lambda^{2-\gamma} y_j\|_{C_s} + \|\nabla \Lambda^{2-\gamma} z_j\|_{C_s} \right] 1_{\text{supp} \chi_j} \|\tilde{w}_{q_1,j,k}\|_{C_s} + \tau_{q_1} \delta_q^{\frac{1}{2}} L^2 M_0(t)^{\frac{1}{2}}.
$$

(151a)
from (81b), for

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Applying (156) and (159) to (153) allows us to conclude

Bounds on \( R \)

4.1.2. Bounds on \( R_N \), the Nash error. Let us write from (122b) using (5)

\[ R_N = N_1 + N_2 \]

where

\[ N_1 = B((\nabla (\Lambda - \gamma_1 y_i + z_q))^T \cdot w_{q+1}) \quad \text{and} \quad N_2 = B((\Lambda^2 - \gamma_1 w_{q+1} (\nabla y) - y_i)). \]

(154)

Considering that \( \text{supp} \left((\nabla (\Lambda - \gamma_1 y_i + z_q))^T\right) \subset B(0, 2\lambda_q) \) and definition of \( \mathcal{P}_{q+1,k} \) within \( w_{q+1} \), we can rewrite similarly to (151)

\[ N_1 = \sum_{j,k} \mathcal{B}_{P_{q+1,k}}((\nabla (\Lambda - \gamma_1 y_i + z_q))^T \cdot \mathcal{P}_{q+1,k} (\chi \phi_{k,j} (\lambda_{q+1} \Phi_j))). \]

(155)

This allows us to immediately estimate for \( a \in 5N \) sufficiently large

\[ ||N_1||_{C_{\alpha}} \leq a^{-1} \sum_{j,k} (||\nabla (\Lambda - \gamma_1 y_i)||_{C_{\alpha}} + ||\nabla (\Lambda^2 - \gamma_1 z_q)||_{C_{\alpha}}) ||1_{\text{supp} \chi \phi_{k,j}}||_{C_{\alpha}} \]

(156)

\[ \leq a^{-1} \sum_{j,k} \mathcal{B}_{P_{q+1,k}}((\nabla (\Lambda - \gamma_1 y_i + z_q))^T \cdot \mathcal{P}_{q+1,k} (\chi \phi_{k,j} (\lambda_{q+1} \Phi_j))). \]

(157)

Applying (157) to \( N_2 \) in (154) and relying on (131b), as well as the fact that \( \mathcal{B}f = \mathcal{P}f \) for any \( f \) that is not divergence-free and \( \mathcal{P} \) eliminates any gradient gives us

\[ N_2(t,x) = \Lambda_q^{-1} \mathcal{B}((\nabla^2 \cdot \gamma_1 y_i)(t,x) \sum_{j,k} \Lambda^2 - \gamma_2 \mathcal{P}_{q+1,k} (\chi \phi_{k,j} (\lambda_{q+1} \Phi_j (t,x)))) \]

(158)

\[ + \Lambda_q^{-1} \mathcal{B}((\nabla^2 \cdot \gamma_1 y_i)(t,x) \sum_{j,k} \Lambda^2 - \gamma_2 \mathcal{P}_{q+1,k} (\chi \phi_{k,j} (\lambda_{q+1} \Phi_j (t,x)))) \]

Thus, we can compute using (17) equation (A.11) on p. 1865 and the fact that \( \beta < \frac{6-2\gamma_2}{3} \) from (81b), for \( a \in 5N \) sufficiently large

\[ ||N_2||_{C_{\alpha}} \leq a^{-2} \sum_{j,k} ||\nabla (\Lambda - \gamma_1 y_i)||_{C_{\alpha}} ||\Lambda^2 - \gamma_2 \mathcal{P}_{q+1,k} (\chi \phi_{k,j} (\lambda_{q+1} \Phi_j (t,x))))||_{C_{\alpha}} \]

(159)

\[ + ||\nabla (\Lambda - \gamma_1 y_i)||_{C_{\alpha}} ||\Lambda^2 - \gamma_2 \mathcal{P}_{q+1,k} (\chi \phi_{k,j} (\lambda_{q+1} \Phi_j (t,x))))||_{C_{\alpha}} \]

(160)
4.1.3. Bounds on $R_L$, the linear error. From (122d) let us write

$$R_L = L_1 + L_2$$

where $L_1 \doteq \mathcal{B}_1 \tilde{q}_{q+1}$ and $L_2 \doteq \mathcal{B}_2 \{ \mathcal{A}^{\gamma_1 \mathbb{Z}} \tilde{w}_{q+1} (\nabla \cdot z) \}$. We estimate from $L_1$ for $a \in 5\mathbb{N}$ sufficiently large by using $\gamma_1 + 2\gamma_2 < 5 - 3\beta$ from (81d)

$$\| L_1 \|_{L_p} \lesssim \mathcal{A}^{\gamma_1 - 1} \| w_{q+1} \|_{L_p} \quad (161)$$

$$\| L_1 \|_{L_p} \lesssim \mathcal{A}^{\gamma_1 - 1} \| w_{q+1} \|_{L_p} \quad (162)$$

For $L_2$, because $\text{supp} \tilde{w}_{q+1} \subset \{ \xi : \frac{2a+1}{a} \leq |\xi| \leq 2a \}$ and $\text{supp} \tilde{w}_{q+1} \subset B(0, \frac{a}{2})$, we can rely on (7) equation (A.11) to compute for $a \in 5\mathbb{N}$ sufficiently large

$$\| L_2 \|_{L_p} \lesssim \mathcal{A}^{\gamma_1 - 1} \| \mathcal{A}^{\gamma_2} w_{q+1} \|_{L_p} \quad (160)$$

$$\| L_2 \|_{L_p} \lesssim \mathcal{A}^{\gamma_1 - 1} \| \mathcal{A}^{\gamma_2} w_{q+1} \|_{L_p} \quad (163)$$

where we used that (81b) guarantees that $\beta < \frac{3}{2} \gamma_2$. Applying (162) and (163) in (161), we conclude that

$$\| L_1 \|_{L_p} \lesssim \mathcal{A}^{\gamma_1 - 1} \| w_{q+1} \|_{L_p} \quad (164)$$

4.1.4. Bounds on $R_o$, the oscillation error. Some of the computations in this subsection are generalizations of those in (7) Section 5.4; when the generalizations are straightforward, we only sketch them and refer to (7) Section 5.4 for details. We rewrite

$$\text{div} R_o \quad (122d) = \text{div} \left( \sum_j \mathcal{A}_j^2 (R_i - R_{q,i}) + \mathcal{A}_j^2 \tilde{R}_{q,i} \right)$$

$$+ \left( \mathcal{A}^{\gamma_2} w_{q+1} \cdot \nabla \right) w_{q+1} - \left( \nabla w_{q+1} \right)^T \cdot \mathcal{A}^{\gamma_2} w_{q+1} \quad (165)$$

By (132) we can write

$$\left( \mathcal{A}^{\gamma_2} w_{q+1} \cdot \nabla \right) w_{q+1} = \left( \mathcal{A}^{\gamma_2} \sum_{j,k} \mathbb{P}_{q+1,k} \tilde{w}_{q+1,j,k} \cdot \nabla \sum_{j,k} \mathbb{P}_{q+1,k} \tilde{w}_{q+1,j,k}^T \right) \quad (166a)$$

$$\left( \nabla w_{q+1} \right)^T \cdot \mathcal{A}^{\gamma_2} w_{q+1} = \left( \nabla \sum_{j,k} \mathbb{P}_{q+1,k} \tilde{w}_{q+1,j,k} \right)^T \cdot \mathcal{A}^{\gamma_2} \sum_{j,k} \mathbb{P}_{q+1,k} \tilde{w}_{q+1,j,k} \quad (166b)$$

We call the cases $k + k' \neq 0$ and $k + k' = 0$ respectively the high and low frequency parts. We have $\frac{1}{2} \leq |k + k'| \leq 2$ due to Lemma 5.2 in the high oscillation part that leads to

$$\mathcal{A}^{\gamma_2} \mathbb{P}_{q+1,k} \tilde{w}_{q+1,j,k} \cdot \nabla \mathbb{P}_{q+1,k} \tilde{w}_{q+1,j,k} = \tilde{P}_{n_{k+1}} \left[ \mathcal{A}^{\gamma_2} \mathbb{P}_{q+1,k} \tilde{w}_{q+1,j,k} \cdot \nabla \mathbb{P}_{q+1,k} \tilde{w}_{q+1,j,k} \right]$$

$$(\nabla \mathbb{P}_{q+1,k} \tilde{w}_{q+1,j,k})^T \cdot \mathcal{A}^{\gamma_2} \mathbb{P}_{q+1,k} \tilde{w}_{q+1,j,k} = \tilde{P}_{n_{k+1}} \left[ (\nabla \mathbb{P}_{q+1,k} \tilde{w}_{q+1,j,k})^T \cdot \mathcal{A}^{\gamma_2} \mathbb{P}_{q+1,k} \tilde{w}_{q+1,j,k} \right]$$

and thus we define formally

$$R_{o,\text{high}} \doteq \mathcal{B}_2 \tilde{P}_{n_{k+1}} \left[ \sum_{j,k, k' \neq 0} \left( \mathcal{A}^{\gamma_2} \mathbb{P}_{q+1,k} \tilde{w}_{q+1,j,k} \cdot \nabla \mathbb{P}_{q+1,k} \tilde{w}_{q+1,j,k}' \right)^T \cdot \mathcal{A}^{\gamma_2} \mathbb{P}_{q+1,k} \tilde{w}_{q+1,j,k}' \right]$$

$$- \left( \nabla \mathbb{P}_{q+1,k} \tilde{w}_{q+1,j,k} \right)^T \cdot \mathcal{A}^{\gamma_2} \mathbb{P}_{q+1,k} \tilde{w}_{q+1,j,k} \quad (167)$$

where $\sum_{j,k,k'} \doteq \sum_{j,k} \tilde{r}_{j,k} \tilde{r}_{j,k}' \sum_{k,k'} \tilde{r}_{1,k} \tilde{r}_{k,k'} \rho \tilde{r}_{j,k}$ (recall (55)). Concerning the low frequency part, Lemma 3.2 shows that $k + k' = 0$ implies that $\Gamma_j = \Gamma_{j'}$; thus, we symmetrize to define the $(j,k)$-term of low-frequency part of the nonlinear terms as

$$\mathcal{F}_{j,k} \doteq \left[ \mathcal{A}^{\gamma_2} \mathbb{P}_{q+1,k} \tilde{w}_{q+1,j,k} \cdot \nabla \mathbb{P}_{q+1,k} \tilde{w}_{q+1,j,k} \right]$$

$$- \left( \nabla \mathbb{P}_{q+1,k} \tilde{w}_{q+1,j,k} \right)^T \cdot \mathcal{A}^{\gamma_2} \mathbb{P}_{q+1,k} \tilde{w}_{q+1,j,k} \quad (168)$$
because zero and divergence-free, due to (5) we have an identity of which leads to via (132)

\[ W = \text{denote the potential vorticity associated to } \]

so that the Fourier symbol of \( \Lambda^{\gamma_2-1} \mathcal{R}_m \) is

\[ s^m(-\eta, \eta) = \int_0^1 \frac{|\eta|^{\gamma_2-1} \tilde{m} \, dr}{|\eta|^2} = |\eta|^{\gamma_2-1} \tilde{m}(\eta) \]
Denoting the $l$-th component of $\mathcal{F}(f,g)$ by $\mathcal{F}_l(f,g)$, we see that

$$2(\mathcal{F}(f,g))^\ast(\xi) \leq \int_{\mathbb{R}^2} \left( i\frac{\xi - \eta}{|\xi - \eta|^2} + \frac{i\eta}{|\eta|^2} \right) \hat{f}(\xi - \eta)\hat{g}(\eta) d\eta$$

(182)

where we can compute

$$\frac{i(\xi - \eta)}{|\xi - \eta|^2} + \frac{i\eta}{|\eta|^2} = \frac{i\xi_l}{|\xi|^2} + \frac{2}{|\eta|^2} \sum_{m=1}^2 \frac{i\xi_m\gamma_2|\xi - \eta|^2}{|\xi - \eta|^2} \gamma_2 s''(\xi - \eta, \eta).$$

(183)

Applying (183) to (182) gives

$$(\mathcal{F}(f,g))^\ast(\xi) \leq \frac{i\xi_l}{2} \int_{\mathbb{R}^2} \hat{f}(\xi - \eta)\hat{g}(\eta) d\eta + \frac{\gamma_2}{2} \sum_{m=1}^2 \int_{\mathbb{R}^2} s''(\xi - \eta, \eta) \hat{f}(\xi - \eta)\hat{g}(\eta) d\eta$$

(184)

We define $\mathcal{T}^m$ by

$$(\mathcal{T}^m(f,g))^\ast(\xi) \leq \int_{\mathbb{R}^2} s''(\xi - \eta, \eta) \hat{f}(\xi - \eta)\hat{g}(\eta) d\eta$$

(185)

so that (184) gives us

$$\mathcal{F}_l(f,g) = \frac{1}{2} \partial_l((\Lambda^{\gamma_2} f)g) + \frac{\gamma_2}{2} \sum_{m=1}^2 \partial_m(\mathcal{T}^m(\Lambda^{\gamma_2} f, \Lambda^{1-\gamma_2} R\eta g)).$$

(186)

Taking Fourier inverse on (185) and writing $K_{r\rho}$ as the Fourier inverse in $\mathbb{R}^4$ of $s''$ give us

$$\mathcal{T}^m(f,g)(x) = \int_{\mathbb{R}^2} K_{r\rho}(x - y, x - z) f(y) g(z) dy dz.$$  

(187)

(see [2] equations (5.27)-(5.28) for details). Therefore,

$$\mathcal{F}_{jk} \leq \frac{\gamma_2}{2} \partial_j(\Lambda^{\gamma_2} \theta_{jk}, \Lambda^{1-\gamma_2} R^j \theta_{1-k}) + \frac{\gamma_2}{2} \text{div}(\mathcal{T}(\Lambda^{\gamma_2} \theta_{jk}, \Lambda^{1-\gamma_2} R \theta_{1-k})).$$

(188)

Thus, we have proven our claim in (169) with

$$\mathcal{L}^m_{jk} \leq \frac{\gamma_2}{2} \partial_j(\Lambda^{\gamma_2} \theta_{jk}, \Lambda^{1-\gamma_2} R^j \theta_{1-k})$$

and

$$\mathcal{P}_{jk} \leq \frac{1}{2} (\Lambda^{\gamma_2} \partial_{jk}) \theta_{jk}.$$  

(189)

Next, we see that

$$\theta_{jk} \leq \frac{1}{k}$$

(170)

It follows due to (190), (191a), (192), and (193) that

$$\Lambda^{\gamma_2} \theta_{jk}(x) = \chi_{jk}^{\gamma_2} \cdot \Lambda^{1-\gamma_2} R^j P_{\gamma_1} \psi_{q+1,j,k}(x) c_k(\lambda_{q+1} x),$$

(191a)

$$\Lambda^{1-\gamma_2} R \theta_{1-k}(x) = -i\chi_{jk}^{\gamma_2} \cdot \Lambda^{1-\gamma_2} R^j P_{\gamma_1} \psi_{q+1,j,k}(x) c_k(\lambda_{q+1} x).$$

(191b)

It follows by using the fact that $a_{k,j} = a_{-k,j}$ due to (189) and Lemma 3.2 (2) that

$$(\Lambda^{\gamma_2} \theta_{jk})^\ast(\xi) = -\chi_{jk}^{\gamma_2} \frac{k \cdot \xi}{|k|^{\gamma_2}} \mathcal{K}_{\gamma_2} \left( \frac{\xi}{\lambda_{q+1}} - k \right) (a_{k,j} \psi_{q+1,j,k}^\ast(\xi - k\lambda_{q+1})),$$

(192a)

$$(\Lambda^{1-\gamma_2} R \theta_{1-k})^\ast(\xi) = \chi_{jk}^{\gamma_2} \frac{k \cdot \xi}{|k|^{\gamma_2}} \mathcal{K}_{\gamma_2} \left( \frac{\xi}{\lambda_{q+1}} + k \right) (a_{k,j} \psi_{q+1,j,k}^\ast(\xi + k\lambda_{q+1})).$$

(192b)

Consequently, if we define

$$M^m_{\xi}(\xi, \eta) \leq -i\delta^m(\xi + k\lambda_{q+1}, \eta - \lambda_{q+1}) \frac{k \cdot (\xi + k\lambda_{q+1})}{|\xi + k\lambda_{q+1}|^{\gamma_2}}$$

This concludes our proof.
and
\[
M_{k,r}^{ml}(ξ, η) = \left(1 - \frac{r^2 - k^2}{|1 - r^2 - k^2|} \right) M_{k,r}^{ml}(ξ, η)
\]

so that
\[
M_{k,r}^{ml}(ξ, η) = \int_0^1 M_{k,r}^{ml}(ξ, η) dr,
\]

then we obtain from (189), (185), and (192),
\[
\mathcal{L}_{jk}(x) = \gamma_2 \frac{\lambda_j^2(t)}{2 (2\pi)^2} \int_{R^2} M_{k,r}^{ml}(ξ, η) \times (a_k, η) e^{ix(ξ+η)} dξ dη.
\]

We observe that if we define for \(ξ, \xi \in R^2\)
\[
(M_{k,r}^{ml}(ξ, ξ, \xi)) = \left(1 - \frac{r^2 - k^2}{|1 - r^2 - k^2|} \right) M_{k,r}^{ml}(ξ, ξ, \xi)
\]

then we can write due to (194) and (197),
\[
M_{k,r}^{ml}(ξ, η) = \lambda_{q+1}^2 (M_{k,r}^{ml})^{1/2}(ξ, η).
\]

We note that \(M_{k,r}^*\) is independent of \(λ_{q+1}\) and due to supp \(\hat{K}_1 \subset B(0, \frac{1}{r})\) from Section 3.3, we have supp(\(M_{k,r}^*\)) \(\subset B(0, \frac{1}{r})\). This implies that \(M_{k,r}^*\) is infinitely many times differentiable with bounds that are uniform in \(r \in (0, 1)\). Next, we observe that
\[
M_{k,r}^{ml}(0, 0) \equiv -k_m k_t k_{q+1}^2 (199a)
\]

\[
\frac{1}{(2\pi)^2} \int_{R^2} \int_{R^2} (a_k, η) e^{ix(ξ+η)} dξ dη = \frac{1}{2} a_k^2 \chi_j(t, x).
\]

Thus, if we define
\[
\tilde{L}_{jk}(t, x) = \gamma_2 \frac{\lambda_j^2(t)}{2 (2\pi)^2} \int_{R^2} \int_{R^2} [M_{k,r}^{ml}(ξ, η) - M_{k,r}^{ml}(0, 0)] dr \times (a_k, η) e^{ix(ξ+η)} dξ dη,
\]

then it follows that
\[
\tilde{L}_{jk}(t, x) \equiv \gamma_2 \frac{\lambda_j^2(t)}{2 (2\pi)^2} \int_{R^2} \int_{R^2} [M_{k,r}^{ml}(ξ, η) dr \times (a_k, η) e^{ix(ξ+η)} dξ dη
\]

\[
\mathcal{L}_{jk}(t, x) \equiv \gamma_2 \frac{\lambda_j^2(t)}{2 (2\pi)^2} \int_{R^2} \int_{R^2} \lambda_{q+1}^{-2q} \nabla \bar{ξ} (M_{k,r}^*)^{1/2}(ξ, η) e^{ix(ξ+η)} dξ dη
\]

and we note that \(\text{Tr}(k^+ \otimes k^+) = 1\) because \(k \in S^1\). Next, we define
\[
\left(\tilde{L}_{jk}^{(1)}(t, x) \equiv \gamma_2 \frac{\lambda_j^2(t)}{2 (2\pi)^2} \int_{R^2} \int_{R^2} \lambda_{q+1}^{-2q} \nabla \bar{ξ} (M_{k,r}^*)^{1/2}(ξ, η) e^{ix(ξ+η)} dξ dη dξ dη
\]
so that we can compute from (200) using (198) so that we can compute from (202) using (198)
\[ t \leq T \]

\[ L^{(2)}_{jk}(t, x) = \left( L^{(1)}_{jk}(t, x) + L^{(2)}_{jk}(t, x) \right) \]

We also define for \( z \in \mathbb{R}^2 \)
\[ \mathcal{K}^{(1)}_{k,j}(z, z) = -i \left( \frac{\lambda_{q+1}}{\bar{r}} \right)^4 \left( \nabla_k (M_{k,j}^n) \right) \cdot \left( \frac{\lambda_{q+1} z}{\bar{r}}, \frac{\lambda_{q+1} \bar{z}}{\bar{r}} \right) \]

\[ \mathcal{K}^{(2)}_{k,j}(z, z) = -i \left( \frac{\lambda_{q+1}}{\bar{r}} \right)^4 \left( \nabla_\gamma (M_{k,j}^n) \right) \cdot \left( \frac{\lambda_{q+1} z}{\bar{r}}, \frac{\lambda_{q+1} \bar{z}}{\bar{r}} \right) \]

and
\[ \tilde{\mathcal{F}}^{(1)}_{k}(\nabla(a_k, \psi_{q+1,k}, a_k, \psi_{q+1,k-1})(t, x) \pm \frac{\gamma_2}{2} \lambda_{q+1}^{1-\gamma_2} \int_0^1 \int_0^1 \int_{\mathbb{R}^2 \times \mathbb{R}^2} \left( \mathcal{K}^{(1)}_{k,j}(z, z) - \mathcal{K}^{(2)}_{k,j}(z, z) \right) \nabla(a_k, \psi_{q+1,k}, a_k, \psi_{q+1,k-1})(z) \nabla(a_k, \psi_{q+1,k-1})(z) d\Sigma d\zeta \bar{r} \]
Looking at (202), we realize that an identical bound in (214) applies for so that $\tilde{R}_t$ may apply (310a) with 

$$
\text{(48)}
$$

Next, we compute from (209b)

$$(133)$$

vanishes because a trace-free part of any multiple of an identity matrix is a zero matrix. Therefore, (208) is simplified to

$$
R_{O,low} = \hat{O}_{21} + \hat{O}_{22}
$$

due to (209). We now come back to estimate $R_{O,\text{approx}}$ in (171); we recall $D_{ij}$ from (133) and realize that $D_{ij}(\tilde{R}_i - \tilde{R}_j) = D_{ij}\tilde{R}_i$ and $(\tilde{R}_i - \tilde{R}_j)(\tau_{q+1,j}, x) = 0$ due to (128) and thus we may apply (310a) with "$f_0 = 0$ and "$g" = $D_{ij}\tilde{R}_j$ to estimate for all $t \in \text{supp} \chi_t, b > L^2 + 2$, and $a \in 5\mathbb{N}$ sufficiently large, using the fact that $b < 2 - \gamma_2$ due to (81b).

$$
\| \tau_{q+1} L^1 M_{0}(t) L^2 M_{0}(t) L^2 M_{0}(t) \|_{L^1(\mathbb{R}_t^3 \times \mathbb{R}^3)}
$$

This leads to

$$
\| R_{O, \text{approx}} \|_{L^1(\mathbb{R}_t^3 \times \mathbb{R}^3)} \leq \tau_{q+1} L^1 M_{0}(t) L^2 M_{0}(t) \|_{L^1(\mathbb{R}_t^3 \times \mathbb{R}^3)} (171)
$$

Next, we compute from (209b)

$$
||O_{21}||_{L^1(\mathbb{R}_t^3 \times \mathbb{R}^3)} \leq \sum_{j,k} \chi_j(t) 1_{1_{q+1}^2} \sum_{j,k} \| \nabla a_{k,j} \|_{L^1(\mathbb{R}_t^3 \times \mathbb{R}^3)} \| \psi_{q+1,j,k} \|_{L^1(\mathbb{R}_t^3 \times \mathbb{R}^3)} \sup_{r \in [0,1]} \| (K_{ij,r}^{(1)})^{\text{w}j} \|_{L^1(\mathbb{R}_t^3 \times \mathbb{R}^3)} \| \psi_{q+1,j,k} \|_{L^1(\mathbb{R}_t^3 \times \mathbb{R}^3)} (149)
$$

Looking at (202), we realize that an identical bound in (214) applies for $O_{22}$ and thus

$$
\| R_{O, \text{approx}} \|_{L^1(\mathbb{R}_t^3 \times \mathbb{R}^3)} \leq \| O_{21} \|_{L^1(\mathbb{R}_t^3 \times \mathbb{R}^3)} + \| O_{22} \|_{L^1(\mathbb{R}_t^3 \times \mathbb{R}^3)} \leq \| O_{21} \|_{L^1(\mathbb{R}_t^3 \times \mathbb{R}^3)} + \| O_{22} \|_{L^1(\mathbb{R}_t^3 \times \mathbb{R}^3)} (214a)
$$

Next, we define

$$
O_3 \equiv \mathcal{B}_{\text{P}_{\alpha q+1}} \left( \sum_{j,k} \left( (\Lambda^2 - \gamma_1^2 \mathbb{P}_{q+1,j,k} \tilde{W}_{q+1,j,k}) \cdot \nabla \mathbb{P}_{q+1,j,k} \tilde{W}_{q+1,j,k} \right) \right)
$$

$$
O_4 \equiv \mathcal{B}_{\text{P}_{\alpha q+1}} \left( \sum_{j,k} \left( (\nabla \mathbb{P}_{q+1,j,k} \tilde{W}_{q+1,j,k})^T \cdot (\Lambda^2 - \gamma_1^2 \mathbb{P}_{q+1,j,k} \tilde{W}_{q+1,j,k}) \right) \right)
$$

so that (167) gives us

$$
R_{O, \text{high}} = O_3 - O_4.
$$

(217)
To work on $O_3$ from (216a), we use the identity of $(B \cdot \nabla) A = \text{div}(A \otimes B) - A(\nabla \cdot B)$, the fact that $\Lambda^{2-\gamma} \mathbb{P}_{\gamma+1,k} \mathbb{W}_{q+1,j,k}$ is divergence-free, (172), and (173) to split $O_3$ to

$$O_3 = \sum_{k=1}^3 O_{3k}$$

where

$$O_{31}(x) = \mathcal{B} P_{\Lambda,\gamma+1}(2-\gamma)_{q+1} \sum_{j, j', k, k' \neq 0} \text{div}(\mathbb{W}_{q+1,j,k} \otimes \mathbb{W}_{q+1,j',k'})(x),$$

$$O_{32}(x) = \mathcal{B} P_{\Lambda,\gamma+1}(2-\gamma)_{q+1} \sum_{j, j', k, k' \neq 0} \text{div}(\mathbb{W}_{q+1,j,k}(x) \otimes \chi_{j'}[\mathbb{P}_{\gamma+1,k'} \cdot a_{\gamma,j'}(x) \psi_{q+1,j',k'}(x)]b_{\gamma,j'}(\Lambda_{q+1} x)),

$$O_{33}(x) = \mathcal{B} P_{\Lambda,\gamma+1}(2-\gamma)_{q+1} \sum_{j, j', k, k' \neq 0} \text{div}(\chi_{j'}[\mathbb{P}_{\gamma+1,k'} \cdot a_{\gamma,j'}(x)]b_{\gamma,j'}(\Lambda_{q+1} x) \otimes \mathbb{P}_{\gamma+1,k} \mathbb{W}_{q+1,j,k'}(x)).$$

We now rewrite $O_{31}$ as follows; it is inspired by [81] equations (104)-(105) and different from [7] p. 1853 due to a technical reason. We first rely on (219a), (113a), (132), and symmetry to write

$$O_{31} = O_{311} + O_{312}$$

where

$$O_{311}(x) = \frac{1}{2} \mathcal{B} P_{\Lambda,\gamma+1}(2-\gamma)_{q+1} \sum_{j, j', k, k' \neq 0} \chi_{j,j'}a_{\gamma,j}(x) a_{\gamma,j'}(x) \psi_{q+1,j,j'}(x) \psi_{q+1,j,j'}(x)

\times \text{div}(b_{\gamma,j}(\Lambda_{q+1} x) \otimes b_{\gamma,j}(\Lambda_{q+1} x) + b_{\gamma,j}(\Lambda_{q+1} x) \otimes b_{\gamma,j}(\Lambda_{q+1} x)),

$$O_{312}(x) = \mathcal{B} P_{\Lambda,\gamma+1}(2-\gamma)_{q+1} \sum_{j, j', k, k' \neq 0} \chi_{j,j'}\n
\times b_{\gamma,j}(\Lambda_{q+1} x) \otimes b_{\gamma,j}(\Lambda_{q+1} x) \nabla(a_{\gamma,j}(x) a_{\gamma,j'}(x) \psi_{q+1,j,j'}(x) \psi_{q+1,j,j'}(x)).

Within $O_{311}$, we can rely on the identity

$$(A \cdot \nabla) B + (B \cdot \nabla) A = \nabla(A \cdot B) - A \times \nabla \times B - B \times \nabla \times A,$$

that was also used on [8] p. 113, (45), and that $k \in \mathbb{S}^1$ to rewrite

$$\nabla \left( b_{\gamma,j}(\Lambda_{q+1} x) \cdot b_{\gamma,j}(\Lambda_{q+1} x) + e^{(j+k') \cdot \Lambda_{q+1} x} \right).$$

(223)

Applying (223) to (221a) and using the fact that $\mathcal{B} f = \mathcal{B} \mathcal{P} f$ for any $f$ that is not divergence-free and $\mathcal{P}$ eliminates any gradient give us

$$O_{311}(x) = \frac{1}{2} \mathcal{B} P_{\Lambda,\gamma+1}(2-\gamma)_{q+1} \sum_{j, j', k, k' \neq 0} \chi_{j,j'} \nabla(a_{\gamma,j}(x) a_{\gamma,j'}(x) \psi_{q+1,j,j'}(x) \psi_{q+1,j,j'}(x))

\times \left( b_{\gamma,j}(\Lambda_{q+1} x) \cdot b_{\gamma,j}(\Lambda_{q+1} x) + e^{(j+k') \cdot \Lambda_{q+1} x} \right).$$

(224)

From (224) and (221) we are able to immediately derive

$$\|O_{31}\|_{C_{\alpha}} \lesssim \sum_{k=1}^3 \|O_{31k}\|_{C_{\alpha}}$$

(225)
\[
\sum_{j,j',k,k',k'\neq 0} \|1_{j,j',k,k'} \nabla (a_{k,j} \psi_{q+1,j,k})\|_{c_{r,s}} \|1_{j,j',k,k'} \partial_b (\Phi_{q+1,j,k})\|_{c_{r,s}}
\]

\[
+ \|1_{j,j',k,k'} \nabla (a_{k,j} \psi_{q+1,j,k})\|_{c_{r,s}} \|1_{j,j',k,k'} \partial_b (\Phi_{q+1,j,k})\|_{c_{r,s}}
\]

We can estimate \(O_{32}\) and \(O_{33}\) from \((219b), (219c)\) more immediately as follows by relying on \([7]\) equation (A.17):

\[
\|O_{32}\|_{c_{r,s}} + \|O_{33}\|_{c_{r,s}} \lesssim \sum_{j,j',k,k',k'\neq 0} \mathcal{A}_{q+1}^{2+\gamma_2} \|\chi_j a_{k,j} b_k (\Phi_{q+1,j,k})\|_{c_{r,s}}
\]

\[
\times \mathcal{A}_{q+1}^{-1} \|1_{j,j',k,k'} \nabla (a_{k,j} \psi_{q+1,j,k})\|_{c_{r,s}} \|\partial_b (\Phi_{q+1,j,k})\|_{c_{r,s}}
\]

\[
+ \mathcal{A}_{q+1}^{-1} \|1_{j,j',k,k'} \nabla (a_{k,j} \psi_{q+1,j,k})\|_{c_{r,s}} \|\partial_b (\Phi_{q+1,j,k})\|_{c_{r,s}}
\]

Thus, we conclude

\[
\|O_3\|_{c_{r,s}} \lesssim \sum_{k=1}^3 \|O_{3k}\|_{c_{r,s}} \lesssim \mathcal{A}_{q+1}^{2+\gamma_2} \lambda_q \omega_{q+1} L M_0(t).
\]

Next, we define

\[
O_{41}(x) = \sum_{j,j',k,k',k'\neq 0} \mathcal{B} \hat{P}_{\mathcal{A}_{q+1}} \chi_j a_{k,j} b_k (\Phi_{q+1,j,k}) \cdot \mathcal{A}_{q+1}^{-2+\gamma_2} \hat{w}_{q+1,j,k}(x),
\]

\[
O_{42}(x) = \sum_{j,j',k,k',k'\neq 0} \mathcal{B} \hat{P}_{\mathcal{A}_{q+1}} \left( \chi_j \nabla (\hat{w}_{q+1,j,k}) \cdot \hat{w}_{q+1,j,k} \nabla (\hat{w}_{q+1,j,k}) \right)
\]

so that \(O_4\) from \((216b)\) satisfies

\[
O_4 = O_{41} + O_{42}
\]

due to \((173), (174)\), and

\[
\mathcal{B} (\nabla \hat{w}_{q+1,j,k}) \cdot \hat{w}_{q+1,j,k} + (\nabla \hat{w}_{q+1,j,k}) \cdot \hat{w}_{q+1,j,k} = \mathcal{B} (\nabla \hat{w}_{q+1,j,k} \cdot \hat{w}_{q+1,j,k}) = 0
\]

which follows from the fact that \(\mathcal{B} f = \mathcal{B} \mathcal{P} f\) for any \(f\) that is not divergence-free and \(\mathcal{P}\) eliminates any gradient. Let us further write

\[
\nabla (\mathcal{P}_{q+1,k} a_{k,j}(x) \psi_{q+1,j,k}(x)) \partial_b (\Phi_{q+1,j,k})
\]

\[
= [\mathcal{P}_{q+1,k} \nabla (a_{k,j} \psi_{q+1,j,k})] \partial_b (\Phi_{q+1,j,k}) + [\mathcal{P}_{q+1,k} a_{k,j} \psi_{q+1,j,k}] \nabla (\Phi_{q+1,j,k}).
\]

Applying \((231)\) in \((226a)\) and then \([7]\) equation (A.17)) to \(O_{41}\), as well as using the fact that \(\alpha > 1\) due to \((81)\) give us for

\[
b > \begin{cases} 2 + 4L^2 & \text{if } \gamma_2 = 1, \\ 2 + L^2 \left( \frac{2}{1-\alpha} \right) & \text{if } \gamma_2 > 1, \end{cases}
\]

and \(\alpha \in \mathbb{N}\) sufficiently large,

\[
\|O_{41}\|_{c_{r,s}} \lesssim \sum_{j,j',k,k',k'\neq 0} \mathcal{A}_{q+1}^{-1} \|1_{j,j',k,k'} \nabla (a_{k,j} \psi_{q+1,j,k})\|_{c_{r,s}} \|\partial_b (\Phi_{q+1,j,k})\|_{c_{r,s}}
\]

\[
+ \mathcal{A}_{q+1}^{-1} \|1_{j,j',k,k'} \nabla (a_{k,j} \psi_{q+1,j,k})\|_{c_{r,s}} \|\partial_b (\Phi_{q+1,j,k})\|_{c_{r,s}}
\]
\[ \tag{149} \tag{141} \leq \lambda_{q+1}^{-1}\gamma_l^{\frac{1}{2}} L^{\frac{1}{2}} M_0(t)^{\frac{1}{2}} \cdot [\lambda_{q+1}^{-1}(\lambda_{q+1}^{\frac{1}{2}} L^{\frac{1}{2}} M_0(t)^{\frac{1}{2}} + \delta_{q+1}^{\frac{1}{2}} \lambda_{q+1} T_{q+1} + t_{q+1}^{\frac{1}{2}} \delta_{q+1}^{\frac{1}{2}} + \lambda_{q+1}^{\frac{1}{2}} \lambda_{q+1} \delta_{q+1} L M_0(t)] \] 

Similarly, we can estimate from (228b) by (equation A.17)
\[ \|O_{42}\|_{C_{\gamma}} \leq \lambda_{q+1}^{-1} \sum_{j, k, k', l' \neq 0} \|1_{\text{supp } \chi}\|_{L^2} \left\| \begin{array}{c} \nabla P_{q+1, k} \end{array} \right\|_{C_{\gamma}} \times \|1_{\text{supp } \chi}\|_{L^2} \left\| k_{q+1} \right\|_{C_{\gamma}} \]
\[ \leq \lambda_{q+1}^{-1} \sum_{j, k, k', l' \neq 0} \lambda_{q+1} \left\|1_{\text{supp } \chi}\right\|_{L^2} \left\| k_{q+1} \right\|_{C_{\gamma}} \]
\[ \times \lambda_{q+1}^{-1} \left\|1_{\text{supp } \chi}\right\|_{L^2} \left\| k_{q+1} \right\|_{C_{\gamma}} \approx \lambda_{q+1}^{-1} \lambda_{q+1} \delta_{q+1} L M_0(t). \] 

In sum, we conclude
\[ \|O_4\|_{C_{\gamma}} \leq \lambda_{q+1}^{-1} \sum_{k=1}^{\infty} \|O_{4k}\|_{C_{\gamma}} \leq \lambda_{q+1}^{-1} \lambda_{q+1} \delta_{q+1} L M_0(t). \]

Therefore,
\[ \|R_{\Omega, \text{high}}\|_{C_{\gamma}} \leq \|O_3\|_{C_{\gamma}} + \|O_4\|_{C_{\gamma}} \leq \lambda_{q+1}^{-1} \lambda_{q+1} \delta_{q+1} L M_0(t). \]

At last, we deduce by relying on the fact that \( \alpha < 7 - 3\gamma_2 - 5\beta \) and \( \beta < \frac{5 - 2\gamma_2}{2} \) due to (81b) and (81c), for \( \alpha \geq 5\) sufficiently large
\[ \|R_{\Omega}\|_{C_{\gamma}} \leq \|R_{\Omega, \text{approx}}\|_{C_{\gamma}} + \|R_{\Omega, \text{low}}\|_{C_{\gamma}} + \|R_{\Omega, \text{high}}\|_{C_{\gamma}} \leq \lambda_{q+1}^{-1} \lambda_{q+1} \delta_{q+1} L M_0(t). \]

4.1.5. Bounds on \( R_{\text{com1}}, \) the first commutator error. From (112), we compute
\[ \|R_{\text{com1}}\|_{C_{\gamma}} \leq \sum_{k=1}^{4} \|\mathcal{I}_k\| \]

where
\[ \mathcal{I}_1 \triangleq \|\mathcal{B}[(\tilde{t}^{2-\gamma_2} y_1)^{\frac{1}{2}} \cdot z^\perp \cdot y_1 - [(\tilde{t}^{2-\gamma_2} y_1)^{\frac{1}{2}} \cdot z^\perp \cdot y_1] * x, \phi_l, \phi_l]\|_{C_{\gamma}}, \]
\[ \mathcal{I}_2 \triangleq \|\mathcal{B}[(\tilde{t}^{2-\gamma_2} y_1)^{\frac{1}{2}} \cdot z^\perp \cdot z_l - [(\tilde{t}^{2-\gamma_2} y_1)^{\frac{1}{2}} \cdot z^\perp \cdot z_l] * x, \phi_l, \phi_l]\|_{C_{\gamma}}, \]
\[ \mathcal{I}_3 \triangleq \|\mathcal{B}[(\tilde{t}^{2-\gamma_2} z_l)^{\frac{1}{2}} \cdot z^\perp \cdot y_1 - [(\tilde{t}^{2-\gamma_2} z_l)^{\frac{1}{2}} \cdot z^\perp \cdot y_1] * x, \phi_l, \phi_l]\|_{C_{\gamma}}, \]
\[ \mathcal{I}_4 \triangleq \|\mathcal{B}[(\tilde{t}^{2-\gamma_2} z_l)^{\frac{1}{2}} \cdot z^\perp \cdot z_l - [(\tilde{t}^{2-\gamma_2} z_l)^{\frac{1}{2}} \cdot z^\perp \cdot z_l] * x, \phi_l, \phi_l]\|_{C_{\gamma}}, \]

Using \( \|\mathcal{B}\|_{C_{\gamma}} \leq 1 \) and the standard commutator estimate (e.g., (27) equation (5) on p. 88]) we bound them separately as follows: as \( 5 - \gamma_2 - 2\beta \leq 9 - 3\gamma_2 - 4\beta \) due to (81b),
\[ \mathcal{I}_3 \leq L^2 \|\mathcal{B}[(\tilde{t}^{2-\gamma_2} y_1)^{\frac{1}{2}} \cdot z^\perp \cdot y_1]\|_{C_{\gamma}}^2 + L^2 \|\mathcal{B}[(\tilde{t}^{2-\gamma_2} y_1)^{\frac{1}{2}} \cdot z^\perp \cdot z_l]\|_{C_{\gamma}}^2 \]
\[ \leq L^2 \|\mathcal{B}[(\tilde{t}^{2-\gamma_2} y_1)^{\frac{1}{2}} \cdot z^\perp \cdot y_1]\|_{C_{\gamma}}^2 + L^2 \|\mathcal{B}[(\tilde{t}^{2-\gamma_2} y_1)^{\frac{1}{2}} \cdot z^\perp \cdot z_l]\|_{C_{\gamma}}^2 \]
\[ \leq L^2 \|\mathcal{B}[(\tilde{t}^{2-\gamma_2} y_1)^{\frac{1}{2}} \cdot z^\perp \cdot y_1]\|_{C_{\gamma}}^2 + L^2 \|\mathcal{B}[(\tilde{t}^{2-\gamma_2} y_1)^{\frac{1}{2}} \cdot z^\perp \cdot z_l]\|_{C_{\gamma}}^2 \]
\[ \leq L^2 \|\mathcal{B}[(\tilde{t}^{2-\gamma_2} y_1)^{\frac{1}{2}} \cdot z^\perp \cdot y_1]\|_{C_{\gamma}}^2 + L^2 \|\mathcal{B}[(\tilde{t}^{2-\gamma_2} y_1)^{\frac{1}{2}} \cdot z^\perp \cdot z_l]\|_{C_{\gamma}}^2 \]
\[ \leq L^2 \|\mathcal{B}[(\tilde{t}^{2-\gamma_2} y_1)^{\frac{1}{2}} \cdot z^\perp \cdot y_1]\|_{C_{\gamma}}^2 + L^2 \|\mathcal{B}[(\tilde{t}^{2-\gamma_2} y_1)^{\frac{1}{2}} \cdot z^\perp \cdot z_l]\|_{C_{\gamma}}^2 \]
\[ \leq L^2 \|\mathcal{B}[(\tilde{t}^{2-\gamma_2} y_1)^{\frac{1}{2}} \cdot z^\perp \cdot y_1]\|_{C_{\gamma}}^2 + L^2 \|\mathcal{B}[(\tilde{t}^{2-\gamma_2} y_1)^{\frac{1}{2}} \cdot z^\perp \cdot z_l]\|_{C_{\gamma}}^2 \]
\begin{align}
\|\mathcal{R}_{\text{Com}2}\|_{C_{\alpha}} & \leq \sum_{k=1}^{14} \|\mathcal{R}_{\text{Com}2k}\|,
\end{align}
where

\begin{align}
\mathcal{R}_{\text{Com}2.1} & \triangleq \mathcal{B}(\Lambda^{2-\gamma_2} y_q \cdot \nabla z_{q+1} - z_q + \Lambda^{2-\gamma_2} (z_{q+1} - z_q) \cdot \nabla y_{q+1}), \\
\mathcal{R}_{\text{Com}2.2} & \triangleq \mathcal{B}((\nabla y) \cdot \Lambda^{2-\gamma_2} (z_q - z_{q+1})), \\
\mathcal{R}_{\text{Com}2.3} & \triangleq \mathcal{B}((\nabla (z_q - z_{q+1})) \cdot \Lambda^{2-\gamma_2} y_q), \\
\mathcal{R}_{\text{Com}2.4} & \triangleq \mathcal{B}((\nabla \Lambda^{2-\gamma_2} (z_{q+1} - z_q)) \cdot w_{q+1}), \\
\mathcal{R}_{\text{Com}2.5} & \triangleq - \mathcal{B}(\Lambda^{2-\gamma_2} w_{q+1} \cdot \nabla z_{q+1} - z_q), \\
\mathcal{R}_{\text{Com}2.6} & \triangleq \mathcal{B}((\nabla (z_q - z_{q+1})) \cdot \Lambda^{2-\gamma_2} w_{q+1}), \\
\mathcal{R}_{\text{Com}2.7} & \triangleq \mathcal{B}(\Lambda^{2-\gamma_2} y_q \cdot \nabla z_{q+1} - z_q), \\
\mathcal{R}_{\text{Com}2.8} & \triangleq \mathcal{B}(\Lambda^{2-\gamma_2} (z_q - z_{q+1}) \cdot \nabla w_{q+1}), \\
\mathcal{R}_{\text{Com}2.9} & \triangleq \mathcal{B}(\Lambda^{2-\gamma_2} (z_q - z_{q+1}) \cdot \nabla y_{q+1}), \\
\mathcal{R}_{\text{Com}2.10} & \triangleq \mathcal{B}(\Lambda^{2-\gamma_2} (z_{q+1} - z_q) \cdot \nabla z_{q+1}), \\
\mathcal{R}_{\text{Com}2.11} & \triangleq \mathcal{B}(\Lambda^{2-\gamma_2} (z_q - z_{q+1}) \cdot \nabla y_{q+1}).
\end{align}
The key observation is that for \( R_{\text{com},k} \), \( k \in \{5, 8\} \), identically to how we handled \( R_T \) (recall \( 151 \)), \( \text{supp}(z_t - z_q)^\circ \subset B(0, \frac{4}{7}) \) while \( \text{supp}(\hat{w}_{q+1,k})^\circ \subset \{ \xi : \frac{4}{7} \lambda_{q+1} \leq |\xi| \leq \frac{8}{7} \lambda_{q+1} \} \) and hence the nonlinear terms therein are equivalent to those with \( \hat{P} \approx \lambda_{q+1} \) applied which gives us a factor of \( \lambda_{q+1}^{-1} \) due to \( \mathcal{B} \). For terms with \( z_{q+1} - z_q \), we can use the fact that \( \text{supp}(z_{q+1} - z_q)^\circ \subset \{ \xi : \frac{4}{7} \leq |\xi| \leq \frac{8}{7} \} \). With these in mind, we can estimate for any \( \delta \in (0, \frac{4}{7}) \)

\[
\|R_{\text{com},1}\|_{L^2} \leq \|L^{2-\gamma}w_{q+1}\|_{L^2} + \|L^{2-\gamma}(z_{q+1} - z_q)\|_{L^2},
\]

(134a)

\[
\|R_{\text{com},2}\|_{L^2} \leq \|L^{2-\gamma}w_{q+1}\|_{L^2} + \|L^{2-\gamma}(z_{q+1} - z_q)\|_{L^2},
\]

(134b)

\[
\sum_{k=10}^{13} \|R_{\text{com},k}\|_{L^2} \leq L^2 \|\nabla(z_{q+1} - z_q)\|_{L^2} + \|\nabla(z_q - z_t)\|_{L^2},
\]

(244a)

\[
\|R_{\text{com},1}\|_{L^2} \leq \|L^{2-\gamma}w_{q+1}\|_{L^2} + \|L^{2-\gamma}(z_{q+1} - z_q)\|_{L^2},
\]

(244b)

\[
\|R_{\text{com},2}\|_{L^2} \leq \|L^{2-\gamma}w_{q+1}\|_{L^2} + \|L^{2-\gamma}(z_{q+1} - z_q)\|_{L^2},
\]

(244c)

\[
\|R_{\text{com},1}\|_{L^2} \leq \|L^{2-\gamma}w_{q+1}\|_{L^2} + \|L^{2-\gamma}(z_{q+1} - z_q)\|_{L^2},
\]

(244d)

Because

\[ \max(-7 + 4y_2 + 7\beta, -6 + 3\gamma_2 + 7\beta, 2y_2 - 6 + 6\beta, -8 + 4y_2 + 6\beta, -8 + 4y_2 + 8\beta) < \alpha \]

all due to \( 81 \), we can take \( \beta > 1 - \frac{4}{7} \) sufficiently close to \( 1 - \frac{4}{7} \), \( \delta \in (0, \frac{4}{7}) \) sufficiently small, and \( a \in 5\mathbb{N} \) sufficiently large to conclude from \( 244 \) that for \( a \in 5\mathbb{N} \) sufficiently large

\[
\|R_{\text{com},1}\|_{L^2} \leq L^2 \|\nabla(z_{q+1} - z_q)\|_{L^2} + \|\nabla(z_q - z_t)\|_{L^2},
\]

(244e)
4.1.7. Concluding the Proof of Proposition 4.1. Applying (152), (160), (164), (236), (241), and (245) to (121) allows us to conclude

\[ \| \tilde{R}_{q+1} \|_{\mathcal{C}_r} \leq \| R_T + R_N + R_L + R_{Com1} + R_{Com2} \|_{\mathcal{C}_r} \ll LM_0(t) \lambda_{q+2} \delta_{q+2} \]

and hence the Hypothesis 4.1(b) at level \( q + 1 \).

Finally, following previous works, we can readily verify that \( y_{q+1}, \tilde{R}_{q+1} \) are \((\mathcal{F}_t)_{t \geq 0}\)-adapted and \( y_{q+1}(0, x), \tilde{R}_{q+1}(0, x) \) are deterministic under the assumption that \( y_q, \tilde{R}_q \) are \((\mathcal{F}_t)_{t \geq 0}\)-adapted and \( y_q(0, x), \tilde{R}_q(0, x) \) are deterministic. First, \( z(t) \) from (66) is \((\mathcal{F}_t)_{t \geq 0}\)-adapted and consequently, so is \( z_q = z \ast \tilde{\phi}_q \) and \( z_{q+1} = z \ast \tilde{\phi}_{q+1} \) defined in (79). Due to the compact support of \( \varphi_i \) in \( (\tau_{q+1}, 2\tau_{q+1}] \), \( C_\ast \) is deterministic. It follows that \( y_q \) and \( z_q \) for all \( q \in \mathbb{N}_0 \) to \( t < 0 \) by their respective values at \( t = 0 \), it follows that

\[
y_q(t) = \int_{\tau_v}^{\tau_{v+1}} (y_q \ast \phi_i)(t-s)\varphi_i(s)ds,
\]

and \( \tilde{R}_q \) defined in (84) are all \((\mathcal{F}_t)_{t \geq 0}\)-adapted. Considering \( z_q \) and \( y_q \) in (246), we see that \( \Phi_j \) from (126) and \( \tilde{R}_q \) from (128) are both \((\mathcal{F}_t)_{t \geq 0}\)-adapted. This implies that because \( b_1 \) from (44) is deterministic, we see that \( b_1(\lambda_{q+1}\Phi_j(t, x)) \) is \((\mathcal{F}_t)_{t \geq 0}\)-adapted; similarly, because \( M_0 \) from (77), \( \chi_j \) from (83), and \( y_1 \) from Lemma 5.2 are all deterministic, \( a_{k,j} \) from (129) is \((\mathcal{F}_t)_{t \geq 0}\)-adapted. Therefore,

\[
w_{q+1}(t, x) = \sum_{j \in \mathbb{N}} \chi_j(t) \times \mathbb{P}_{q+1,k} \left( \frac{M_0(\tau_{q+1}j)}{\sqrt{2}} \delta_{q+1} \left( \frac{\tilde{R}_{q+1}(t, x)}{\lambda_{q+1}M_0(\tau_{q+1}j)} \right) b_k(\lambda_{q+1}\Phi_j(t, x)) \right)
\]

from (127) and (129), as well as \( \partial_{w_{q+1}} \) are \((\mathcal{F}_t)_{t \geq 0}\)-adapted so that \( y_{q+1} \) in (25) is \((\mathcal{F}_t)_{t \geq 0}\)-adapted. Moreover, all of \( R_T, R_O, R_N, R_L, \) and \( R_{Com1} \) in (122), as well as \( R_{Com1} \) in (112) are \((\mathcal{F}_t)_{t \geq 0}\)-adapted, implying that \( R_{q+1} \) in (121) is \((\mathcal{F}_t)_{t \geq 0}\)-adapted.

Similarly, due to the compact support of \( \varphi_i \) in \( (\tau_{q+1}, 2\tau_{q+1}] \), if \( v_q(0, x) \) and \( \tilde{R}_q(0, x) \) are deterministic, then so are \( v_q(0, x), \tilde{R}_q(0, x), \) and \( \partial_t v_q(0, x) \). Similarly, because \( z(0, x) = 0 \) by (66), \( z_q(0, x) \equiv 0 \) and hence so is \( z_q(0, x) \equiv 0 \). Moreover, within (247), because supp \( \chi_j \subset (\tau_{q+1}(j-1), \tau_{q+1}(j+1)) \), \( \chi_j(0) = 0 \) for all \( j \in \{1, \ldots, [\tau_{q+1}], T_1 \} \) leaving only one term when \( j = 0 \). As \( \Phi_0(0, x) = x \) due to (126) while \( \tilde{R}_{q+1}(0, x) = \tilde{R}_q(0, x) \) where \( \tilde{R}_q(0, x) \) was already verified to be deterministic, we conclude that \( w_{q+1}(0, x) \) is deterministic. It follows similarly that \( \partial_t w_{q+1}(0, x) \) is deterministic. It follows that \( y_{q+1}(0, x) \) from (25) is deterministic, and all of \( R_T(0, x), R_O(0, x), R_N(0, x), R_L(0, x), R_{Com2}(0, x) \) in (122), and \( R_{Com1}(0, x) \) in (112) are deterministic, allowing us to conclude that \( R_{q+1}(0, x) \) from (121) is deterministic.

5. Appendix

5.1. Proof of Proposition 4.1. The existence of a martingale solution is relatively standard (e.g., [41]); nonetheless, we sketch the proof because the QG momentum equations forced by random noise have not been studied before. In short, we rely on [43] Theorem 4.6 on p. 1739 similarly to [64] Section 4.2. We choose in accordance with notations from [43] p. 1729

\[
\mathbb{E} \triangleq \mathbb{H}_\omega^\varphi \text{ and } \mathbb{X} \triangleq (H^1_\varphi)^*.
\]

(248)
It follows that $\mathcal{Y} \subset \mathbb{H} \subset \mathcal{X}$ continuously and densely and $\mathcal{X} \hookrightarrow \mathcal{Y}$ compactly. Following \[43\] p. 152 and \[43\] equation (3.1) on p. 1730 we define an operator $\mathcal{A} : C_{0,\sigma}^\infty \rightarrow \mathcal{X}$ by
\begin{equation}
\mathcal{A}v \doteq -(u \cdot \nabla)v + (\nabla v)^T \cdot u - \Lambda_{\mathcal{Y}_1}v, \quad u = \Lambda v.
\end{equation}
(249)

The following proposition is analogous to \[43\] Lemma 4.2.3 and \[43\] Lemma 6.1 and relies on the Calderón commutator estimate from Proposition 5.3.

**Proposition 5.1.** (cf. \[43\] Lemma 4.2.3 and \[43\] Lemma 6.1) For any $v, \tilde{v} \in C_{0,\sigma}^\infty$
\begin{equation}
\|\Lambda_{\mathcal{Y}_1}v - \Lambda_{\mathcal{Y}_1}\tilde{v}\|_{\mathcal{X}} \lesssim \|v - \tilde{v}\|_{\dot{H}_{\alpha}^{\frac{1}{2}}},
\end{equation}
(250a)
\begin{equation}
\|-(u \cdot \nabla)v + (\tilde{u} \cdot \nabla)\tilde{v}\|_{\mathcal{X}} + \|(\nabla v)^T \cdot u - (\nabla \tilde{v})^T \cdot \tilde{u}\|_{\mathcal{X}} \lesssim \|v - \tilde{v}\|_{\dot{H}_{\alpha}^{\frac{1}{2}}} + \|\tilde{v}\|_{\dot{H}_{\alpha}^{\frac{1}{2}}}.
\end{equation}
(250b)

Therefore, the operator $\mathcal{A} : C_{0,\sigma}^\infty \rightarrow \mathcal{X}$ extends to an operator $\mathcal{A} : \dot{H}_{\alpha}^{\frac{1}{2}} \rightarrow \mathcal{X}$ by continuity.

**Proof.** First, because $v, \tilde{v} \in C_{0,\sigma}^\infty$ are both mean-zero, we can compute
\begin{equation}
\|\Lambda_{\mathcal{Y}_1}v - \Lambda_{\mathcal{Y}_1}\tilde{v}\|_{\mathcal{X}} \lesssim \|v - \tilde{v}\|_{\dot{H}_{\alpha}^{\frac{1}{2}}},
\end{equation}
(248)
Next, we can compute using (300) and the embedding of $H^4(\mathbb{T}^2) \hookrightarrow W^{\frac{7}{2},\infty}(\mathbb{T}^2)$
\begin{equation}
\|-(u \cdot \nabla)v + (\tilde{u} \cdot \nabla)\tilde{v}\|_{\mathcal{X}} \lesssim \sup_{w \in \dot{H}_{\alpha}^{\frac{1}{2}}, \|w\|_{\dot{H}_{\alpha}^{\frac{1}{2}}} \leq 1} \left( \frac{1}{2} \sum_{i,j=1}^{2} \int_{\mathbb{T}^2} \left| (\partial_i v_j - \partial_i \tilde{v}_j)[\Lambda, w_i]v_j + \partial_i \tilde{v}_j[\Lambda, w_i](v_j - \tilde{v}_j) \right| dx \right)
\end{equation}
(252)
Finally, we compute via (300) which are crucial as this nonlinear term is not of the divergence form:
\begin{equation}
\|\|(\nabla v)^T \cdot u - (\nabla \tilde{v})^T \cdot \tilde{u}\|_{\mathcal{X}} \lesssim \sup_{w \in \dot{H}_{\alpha}^{\frac{1}{2}}, \|w\|_{\dot{H}_{\alpha}^{\frac{1}{2}}} \leq 1} \left( \frac{1}{2} \sum_{i,j=1}^{2} \|v_j - \tilde{v}_j\|_{\dot{H}_{\alpha}^{\frac{1}{2}}} \|w_i\|_{\dot{H}_{\alpha}^{\frac{1}{2}}} \|v_j - \tilde{v}_j\|_{\dot{H}_{\alpha}^{\frac{1}{2}}} + \|\tilde{v}_j\|_{\dot{H}_{\alpha}^{\frac{1}{2}}} \|w_i\|_{\dot{H}_{\alpha}^{\frac{1}{2}}} \|v_j - \tilde{v}_j\|_{\dot{H}_{\alpha}^{\frac{1}{2}}} \right)
\lesssim \|v - \tilde{v}\|_{\dot{H}_{\alpha}^{\frac{1}{2}}} \left( \|v\|_{\dot{H}_{\alpha}^{\frac{1}{2}}} + \|\tilde{v}\|_{\dot{H}_{\alpha}^{\frac{1}{2}}} \right). \quad \Box
\end{equation}

Following \[43\] p. 1730 and 1733 and \[43\] p. 154] we define the function $\mathcal{N}_1$ and $\mathcal{N}_p$ for $p \geq 2$ on $\mathcal{Y}$ by
\begin{equation}
\mathcal{N}_1(v) \doteq \begin{cases} 
\|v\|_{\dot{H}_{\alpha}^{\frac{1}{2}}}, & \text{if } v \in \dot{H}_{\alpha}^{\frac{1}{2}+\frac{2}{p}}(\mathbb{T}^2), \\
+\infty, & \text{if } v \notin \dot{H}_{\alpha}^{\frac{1}{2}+\frac{2}{p}}(\mathbb{T}^2),
\end{cases}
\end{equation}
(253)

Under these notations, we can show that the criterion of “(C1) (Demi-Continuity)” and “(C3) (Growth Condition)” on \[43\] p. 1733 hold due to \[43\] and Proposition 5.1 while “(C2) (Coercivity Condition)” on \[43\] p. 1733 holds by definition of $\mathcal{N}_1$. Therefore, the hypothesis of \[43\] Theorem 4.6 holds, and it allows us to deduce Proposition 4.1(1).

In order to prove the Proposition 5.1(2), we need the following result first.
Proposition 5.2. ([45, Lemma A.1]) Let \( \{(s_l, \xi_l)\}_{l \in \mathbb{N}} \subset [0, \infty) \times H^\frac{1}{2}_0 \) be a family such that 
\[
\lim_{l \to \infty} \| (s_l, \xi_l) - (s, \xi^m) \|_{\mathbb{R} \times H^\frac{1}{2}_0} = 0
\]
and \( \{P_l\}_{l \in \mathbb{N}} \) be a family of probability measures on \( \Omega_0 \) satisfying 
\[
P_l(\{ (\xi(t) = \xi_l \forall t \in [0, s_l]) \}) = 1 \text{ for all } l \in \mathbb{N}, \text{ any } T > 0, \text{ and some } \varepsilon, \kappa > 0,
\]
\[
\sup_{l \in \mathbb{N}} \mathbb{E}^{P_l} \left[ \| \xi(t) - \xi_l \|_{H^\frac{1}{2}_0} \right] + \sup_{l \in \mathbb{N}} \mathbb{E}^{P_l} \left[ \| \xi(t) - \xi_l \|_{H^\frac{1}{2}_0} \right] + \| \xi_l \|_{L^2([0, T]; H^\frac{1}{2}_0)} < \infty.
\]
Then \( \{P_l\}_{l \in \mathbb{N}} \) is tight in
\[
\mathcal{S} = C_{loc}([0, \infty); (H^\frac{1}{2}_0)^{\times}) \cap L^2_{loc}([0, \infty); H^\frac{1}{2}_0).
\]

Proof of Proposition 5.2. This is an extension of [45, Lemma A.1] on the 3-d NS equations that has already been generalized to 2-d NS equations in [79, Lemma 6.4] and \( n \)-d Boussinesq system in [80, Proposition 6.6]; thus, we only sketch its proof. We recall that a set \( K \subset \mathcal{S} \) is compact if \( \{f\}_{f \in [0, T]} \subset C([0, T]; (H^\frac{1}{2}_0)^{\times}) \cap L^2(0, T; \dot{H}^\frac{1}{2}_0) \) is compact for every \( T > 0 \). Now we fix \( \varepsilon > 0 \) and \( k \in \mathbb{N} \) such that \( k \geq k_0 = \sup_{l \in \mathbb{N}} s_l \). Due to (254) we may choose \( R_k > 0 \) sufficiently large such that
\[
P_l(\{ \xi \in \Omega_0 : \sup_{r \in [0, k]} \| \xi(t) \|_{H^\frac{1}{2}_0} < \infty \}) \geq \varepsilon.
\]
and define
\[
\Omega_\varepsilon = \{ \xi \in \Omega_0 : \xi(t) = \xi_l \forall t \in [0, s_l] \}\]
and
\[
K = \bigcup_{l \in \mathbb{N}} \cap \bigcup_{k \in \mathbb{N}} \{ \xi \in \Omega_\varepsilon : \sup_{r \in [0, k]} \| \xi(t) \|_{H^\frac{1}{2}_0} < \infty \}
\]
Then we can compute \( \sup_{l \in \mathbb{N}} P_l(\Omega_0 \setminus \bar{K}) \leq \varepsilon \) using (256). By definition of tightness, if we now show that \( \bar{K} \) is compact in \( \mathcal{S} \), then it implies that \( \{P_l\}_{l \in \mathbb{N}} \) is tight in \( \mathcal{S} \) as desired. We take \( \{ \xi_w \}_{w \in \mathbb{N}} \subset K \). Suppose that for all \( N \in \mathbb{N}, \xi_w \in \Omega_N \) for only finitely many \( w \in \mathbb{N} \). Then passing to a subsequence and relabeling if necessary, we can assume that \( \xi_w \in \Omega_N \). Then, for all \( k \geq k_0 \),
\[
\sup_{r \in [0, k]} \| \xi_w(t) \|_{H^\frac{1}{2}_0} + \sup_{r \in [0, k]} \| \xi_w(t) - \xi_w(r) \|_{H^\frac{1}{2}_0} \leq R_k
\]
because \( \xi_w \in K \). Now
\[
L^\infty(0, k; H^\frac{1}{2}_0) \cap C^\ast([0, k]; (H^\frac{1}{2}_0)^{\times}) \hookrightarrow C([0, k]; (H^\frac{1}{2}_0)^{\times})
\]
is compact (e.g., [76]). Therefore, we can extract a subsequence \( \{\xi_{w_l}\}_{l} \) such that
\[
\lim_{l \to \infty} \sup_{r \in [0, k]} \| \xi_{w_l}(t) - \xi_{w_l}(r) \|_{H^\frac{1}{2}_0} = 0.
\]
It follows from (257), (258), and (261) that for all \( \delta > 0 \), there exists \( L \in \mathbb{N} \) such that \( w_l, w_q \geq L \) implies
\[
\int_0^k \| \xi_{w_l}(t) - \xi_{w_q}(t) \|_{H^\frac{1}{2}_0}^2 dt = \int_0^{t_{w_l} \wedge t_{w_q}} + \int_{t_{w_l} \wedge t_{w_q}}^{t_{w_l} \vee t_{w_q}} + \int_{t_{w_l} \vee t_{w_q}}^k \| \xi_{w_l}(t) - \xi_{w_q}(t) \|_{H^\frac{1}{2}_0}^2 dt < \delta.
\]
The case in which there exists \( N \in \mathbb{N} \) such that \( \tilde{\xi}_w \in \Omega_N \) for infinitely many \( w \) is similar and easier; thus, we conclude, along with (261), that \( \{\tilde{\xi}_w\} \) is convergent in \( C([0, k]; (\mathcal{H}^\beta_x)^*) \cap L^2(0, k; \mathcal{H}^\beta_x) \). By the arbitrariness of \( \{\tilde{\xi}_w\} \subset K \) we conclude that \( K \) is compact. \( \Box \)

With Propositions 5.1 and 5.2, we are ready to deduce Proposition 4.1 (2). Due to similarity with previous works (e.g., [35], Proof of Theorem 3.1, [78], Proof of Proposition 4.1) we only sketch its proof. We fix \( \{P_t\}_{t \in \mathbb{R}} \subset C(\mathbb{R}, \mathcal{H}_T, \{\mathcal{C}_{t,q}\}_{q \in \mathbb{R}, t \in \mathbb{R}}) \) where \( \{(\mathcal{C}_t, \mathcal{H}_T)\}_{t \in \mathbb{R}} \subset [0, \infty) \times \mathcal{H}^\beta_T \) satisfies \( \lim_{t \to \infty} \| (h_1, \mathcal{H}_T^k) - (h, \mathcal{H}_T^k) \|_{\mathcal{H}_T^k} = 0 \). To verify the hypothesis of Proposition 5.2, we first note that \( P_1 (\{\xi(t) = \xi \ \forall t \in [0, s]\}) = 1 \) for all \( l \in \mathbb{N} \) due to (M1) of Definition 4.1. Second via Proposition 5.1 we define \( F : \mathcal{H}^\beta_T \mapsto (\mathcal{H}^\beta_T)^* \) by

\[
F(\xi) \triangleq -\mathbb{E}[ (\Lambda \xi \cdot \nabla \xi) - (\nabla \xi)^T \cdot \Lambda \xi - \Lambda^\gamma \xi].
\]

(263)

By definition of \( C(\mathcal{C}_t, \mathcal{H}_T^k, \{\mathcal{C}_{t,q}\}_{q \in \mathbb{R}, t \in \mathbb{R}}) \) and (M2) of Definition 4.1 we know that for all \( l \in \mathbb{N} \) and \( t \in [s, \infty) \), the mapping \( t \mapsto M_{t,s}^{\xi,k} \) where

\[
M_{t,s}^{\xi,k} \triangleq (\xi(t) - \xi_s, \psi^k)
\]

\[
- \int \sum_{j \neq i} \langle \Lambda \xi_j, \partial_t \psi^k \xi_j \rangle - \frac{1}{2} \langle \partial_t \xi_j, [\Lambda, \psi^k] \xi_j \rangle_H^k \int_{H^\beta_T} \right)
\]

(264)

for \( \psi^k \in C^0(\mathbb{T}^2) \cap \mathcal{H}^\beta_T \) and \( \xi \in \Omega_0 \) a continuous, square-integrable \( (\mathcal{B}_t)_{t \geq 0} \)-martingale under \( P_1 \) and \( \langle \langle M_{t,s}^{\xi,k} \rangle \rangle = \int ||G(\xi(t)) \psi^k||^2_T dr \). We can thus write for all \( \kappa \in (0, \frac{1}{2}) \) due to (M2) of Definition 4.1 and (263)

\[
\sup_{t \in [0, T]} | \sup_{r \in [0, t]} \frac{\| \xi(t) - \xi(r) \|_{H^\beta_T^k}}{|t - r|^{\kappa}} |
\]

\[
= \sup_{t \in [0, T]} | \sup_{r \in [0, t]} \frac{\| \int_{r}^{t} F(\xi(\lambda))d\lambda + M_{t,s}^{\xi,k} - M_{r,s}^{\xi,k} \|_{H^\beta_T^k}}{|t - r|^{\kappa}} |
\]

(265)

where we estimate for any \( p \in (1, \infty) \) by using \( \| \int_{r}^{t} f(|d\lambda|)d\lambda \|_{H^\beta_T^k} \leq \int_{r}^{t} f \| |d\lambda| \|_{H^\beta_T} dl \) for any \( f \in L^1([r, t]; H^{-\gamma}(\mathbb{T}^2)) \), (250) with \( \tilde{\nu} \equiv 0 \), and (M3) of Definition 4.1

\[
\sup_{r \in [0, t]} \frac{\| \int_{r}^{t} F(\xi(\lambda))d\lambda \|_{H^\beta_T^k}}{|t - r|^{\kappa-1}} \leq T C_{T,\rho}(1 + \| \xi_0 \|_{H^\beta_T^k}^{2p}) \]

(266)

On the other hand, we can compute for any \( p \in (1, \infty) \)

\[
\sup_{r \in [0, t]} \| \int_{r}^{t} F(\xi(\lambda))d\lambda \|_{H^\beta_T^k} \leq T C_{T,\rho}(1 + \| \xi_0 \|_{H^\beta_T^k}^{2p})
\]

(267)

by Minkowski’s inequality, Burkholder-Davis-Gundy inequality, (M2) and (M3) of Definition 4.1. Applying Kolmogorov’s test (e.g., [31], Theorem 3.3 on p. 67) on (267) we obtain for all \( \kappa \in (0, \frac{1}{2}) \)

\[
\sup_{t \in [0, T]} | \sup_{r \in [0, t]} \frac{\| \xi(t) - \xi(r) \|_{H^\beta_T^k}}{|t - r|^{\kappa}} |
\]

(265) \( \leq (266) \)

\[
\leq C(T, \alpha, \| \xi_0 \|_{H^\beta_T^k}).
\]

(268)

Together with (M3) of Definition 4.1 we now see that (250) is satisfied so that by Proposition 5.2, we can deduce that \( \{P_t\}_{t \in \mathbb{R}} \) is tight in \( \mathbb{S} \). Then we deduce by Prokhorov’s theorem (e.g., [31], Theorem 2.3) that \( P_t \) converges weakly to some \( P \in \mathcal{P}(\Omega_0) \) and by Skorokhod’s
Next, we can compute starting from (264)

The rest of this proof consists of showing that $P \in C(\mathbb{S}, \mathbb{S} \cap \mathcal{C}_{1,t})$. First, it follows immediately from (269) and (255) that $P(|\xi(t) = \xi^*| \forall t \in [0, s]) = 1$. Next, for all $\psi^k \in C^\infty(T^2) \cap \mathcal{H}^1_T$ and $t \geq s$, due to (269) and (255)

$$\langle \hat{\xi}(t), \psi^k \rangle \rightarrow \langle \hat{\xi}(s), \psi^k \rangle$$

as $l \rightarrow \infty \tilde{P}$-a.s. Next, to prove that

$$\mathbb{E}^\tilde{P} \left[ \int_{s_t}^{t_s} \sum_{i,j=1}^{\mathcal{N}} \langle \mathcal{L}_{\xi} \tilde{\xi}_{i,j}, \partial_t \tilde{\xi}_{i,j} \rangle_{H^1_T} - \frac{1}{2} \langle \partial_t \tilde{\xi}_{i,j}, [\Lambda, \psi^k] \tilde{\xi}_{i,j} \rangle_{H^1_T} - \langle \tilde{\xi}_{i,j}, \Lambda \psi \psi^k \rangle_{H^1_T} \right] = 0$$

as $l \rightarrow \infty$, we first split the left hand side of (271) as a sum of $I_1$ and $I_2$ defined by

$$I_1 \triangleq \mathbb{E}^\tilde{P} \left[ \int_{s_t}^{t_s} \sum_{i,j=1}^{\mathcal{N}} \langle \mathcal{L}_{\xi} \tilde{\xi}_{i,j}, \partial_t \tilde{\xi}_{i,j} \rangle_{H^1_T} - \frac{1}{2} \langle \partial_t \tilde{\xi}_{i,j}, [\Lambda, \psi^k] \tilde{\xi}_{i,j} \rangle_{H^1_T} - \langle \tilde{\xi}_{i,j}, \Lambda \psi \psi^k \rangle_{H^1_T} \right],$$

$$I_2 \triangleq \mathbb{E}^\tilde{P} \left[ \int_{s_t}^{t_s} \sum_{i,j=1}^{\mathcal{N}} \langle \mathcal{L}_{\xi} \tilde{\xi}_{i,j}, \partial_t \tilde{\xi}_{i,j} \rangle_{H^1_T} - \frac{1}{2} \langle \partial_t \tilde{\xi}_{i,j}, [\Lambda, \psi^k] \tilde{\xi}_{i,j} \rangle_{H^1_T} - \langle \tilde{\xi}_{i,j}, \Lambda \psi \psi^k \rangle_{H^1_T} \right] + \frac{1}{2} \langle \partial_t \tilde{\xi}_{i,j}, [\Lambda, \psi^k] \tilde{\xi}_{i,j} \rangle_{H^1_T} - \langle \tilde{\xi}_{i,j}, \Lambda \psi \psi^k \rangle_{H^1_T} \right].$$

First, we estimate

$$I_1 \leq \mathbb{E}^\tilde{P} \left[ \int_{s_t}^{t_s} \sum_{i,j=1}^{\mathcal{N}} \langle \mathcal{L}_{\xi} \tilde{\xi}_{i,j}, \partial_t \tilde{\xi}_{i,j} \rangle_{H^1_T} - \frac{1}{2} \langle \partial_t \tilde{\xi}_{i,j}, [\Lambda, \psi^k] \tilde{\xi}_{i,j} \rangle_{H^1_T} - \langle \tilde{\xi}_{i,j}, \Lambda \psi \psi^k \rangle_{H^1_T} \right]$$

as $l \rightarrow \infty$ due to the hypothesis that $s_t \rightarrow s$ as $l \rightarrow \infty$. For $I_2$ of (272), we can estimate

$$I_2 \leq \mathbb{E}^\tilde{P} \left[ \int_{s_t}^{t_s} \sum_{i,j=1}^{\mathcal{N}} \langle \mathcal{L}_{\xi} \tilde{\xi}_{i,j}, \partial_t \tilde{\xi}_{i,j} \rangle_{H^1_T} - \frac{1}{2} \langle \partial_t \tilde{\xi}_{i,j}, [\Lambda, \psi^k] \tilde{\xi}_{i,j} \rangle_{H^1_T} - \langle \tilde{\xi}_{i,j}, \Lambda \psi \psi^k \rangle_{H^1_T} \right] + \frac{1}{2} \langle \partial_t \tilde{\xi}_{i,j}, [\Lambda, \psi^k] \tilde{\xi}_{i,j} \rangle_{H^1_T} - \langle \tilde{\xi}_{i,j}, \Lambda \psi \psi^k \rangle_{H^1_T} \right]$$

as $l \rightarrow \infty$. Applying (273) and (274) to (272) leads to (271). Next, for every $t > r \geq s$, $p \in (1, \infty)$, we estimate from (264)

$$\sup_{t \in [s, r]} \mathbb{E}^\tilde{P} \left[ ||M_{\xi, t}^k ||_{L^p_T}^2 \right] \leq \sup_{t \in [s, r]} \mathbb{E}^\tilde{P} \left[ ||\xi(t)||_{L^p_T}^2 + 1 + t^p (1 + ||\xi||_{C^0([0, t]; H^1_T)}) \right] \leq 1.$$ (275)

Next, we compute starting from (274)

$$\lim_{l \rightarrow \infty} \mathbb{E}^\tilde{P} \left[ ||M_{\xi, t}^k - M_{\xi, t}^{k, l} ||_{L^p_T} \right] = 0$$ (276)
due to (270), (271), and the hypothesis that \( \lim_{t \to \infty} \| \xi_t - \xi_0 \|_{H^2} = 0 \). Consequently, for any \( g \) that is \( \mathbb{R} \)-valued, \( \mathcal{B}_t \)-measurable, and continuous on \( \mathbb{S} \),

\[
\mathbb{E}^P[\|M_{t,s}^{k,l} - M_{r,s}^{k,l}\|^2 g(\xi_t)] \xrightarrow{269, 276} \lim_{t \to \infty} \mathbb{E}^P[\|M_{t,s}^{k,l} - M_{r,s}^{k,l}\|^2 g(\xi_t)] \xrightarrow{269} 0. \tag{277}
\]

This implies that the mapping \( t \mapsto M_{t,s}^k \) is a \( (\mathcal{B}_t)_{t \in \mathbb{R}} \)-martingale under \( P \). Next, we can estimate by Hölder’s inequality and (275–276)

\[
\lim_{t \to \infty} \mathbb{E}^P[\|M_{t,s}^{k,l} - M_{r,s}^{k,l}\|^2] \leq \lim_{t \to \infty} \left( \mathbb{E}^P[\|M_{t,s}^{k,l} - M_{r,s}^{k,l}\|^2] \right)^\frac{1}{2} \left( \mathbb{E}[\|M_{t,s}^{k,l} - M_{r,s}^{k,l}\|^4] \right)^\frac{1}{4} = 0. \tag{278}
\]

It follows that

\[
\mathbb{E}^P \left( (M_{t,s}^{k,l})^2 - (M_{r,s}^{k,l})^2 - \int_r^t \|G(\xi(\lambda))\| \|\psi\|_{L^2} d\lambda \right) \xrightarrow{269, 278, 42} 0 \tag{279}
\]

which implies that \( (M_{t,s}^{k,l})^2 - \int_r^t \|G(\xi(\lambda))\| \|\psi\|_{L^2} d\lambda \) is a \( (\mathcal{B}_t)_{t \in \mathbb{R}} \)-martingale under \( P \) so that

\[
\langle (M_{t,s}^{k,l}) \rangle = \int_s^t \|G(\xi(\lambda))\| \|\psi\|_{L^2}^2 d\lambda. \tag{280}
\]

This leads to by Burkholder-Davis-Gundy inequality (e.g., [54, p. 166])

\[
\mathbb{E}^P[\|M_{t,s}^{k,l}\|^2] \xrightarrow{280} \mathbb{E}^P \left[ \int_s^t \|G(\xi(\lambda))\| \|\psi\|_{L^2}^2 d\lambda \right] \xrightarrow{12} \mathbb{E}^P \left[ \int_s^t \|G(\xi(\lambda))\| \|\psi\|_{L^2}^2 d\lambda \right] \leq 1 \tag{281}
\]

and hence \( M_{t,s}^k \) is square-integrable. Finally, the proof of (M3) follows from defining

\[
R(t,s,\xi) \triangleq \sup_{t \in [0,1]} \| \xi(t) \|_{H^2}^2 + \int_s^t \| \xi(t) \|_{H^2}^2 d\tau \tag{282}
\]

for any fixed \( q \in \mathbb{N} \) and \( t \geq s \), and relying on the fact that the mapping \( \xi \mapsto R(t,s,\xi) \) is lower semicontinuous on \( \mathbb{S} \).

5.2. Proof of Proposition 4.4.
Following [31, p. 84] we define

\[
Y(s) = \frac{\sin(\pi \alpha)}{\pi} \int_0^s \frac{e^{-(-\Delta)^{\frac{\alpha}{2}}(s-r)^{\alpha} - \nu(x,y) d\lambda} + \frac{\alpha r}{1 + \frac{4r}{\gamma_1}} > 2\alpha, \tag{283}
\]

so that due to (272)

\[
\int_0^s (t-s)^{\alpha-1} e^{-(-\Delta)^{\frac{\alpha}{2}}(s-r)} d\lambda \approx \frac{\alpha }{1 + \frac{4r}{\gamma_1}} > 2\alpha, \tag{284}
\]

we can estimate by the Gaussian hypercontractivity theorem (e.g., [53, Theorem 3.50]), and Itô’s isometry (e.g., [53, equation (4) on p. 28]),

\[
\mathbb{E}^P[\|(-\Delta)^{\alpha} Y(s)\|_{L^2_x}^2] \lesssim \mathbb{E}^P[\|(-\Delta)^{\alpha} Y(s)\|_{L^2_x}^2] \tag{285}
\]

We integrate (285) over \([0, T]\) and use Fubini’s theorem to deduce for all \( l \in \mathbb{N} \) and \( \eta \geq 0 \) that satisfies (284).

\[
\mathbb{E}^P \left[ \int_0^T \|(-\Delta)^{\alpha} Y(s)\|_{L^2_x}^2 d\lambda \right] \lesssim 1. \tag{286}
\]

This allows us to take \( l > \frac{1}{2\alpha} \) and deduce by (283)

\[
\mathbb{E}^P[\|(-\Delta)^{\alpha} d\|^2_{L^2_x}] = \mathbb{E}^P \left[ \sup_{t \in [0, T]} \|(-\Delta)^{\alpha} \left[ \int_0^s (t-s)^{\alpha-1} e^{-(-\Delta)^{\frac{\alpha}{2}}(s-r)} Y(s) d\lambda \right] \right]_{L^2_x} \tag{287}
\]
Then, by taking \( \alpha \in (0, \frac{\sqrt{2}}{2\gamma_1} \wedge \frac{1}{2}) \), we can choose \( \eta = 2 + \frac{\gamma_1}{2} \) in \((287)\) and still satisfy \((284)\) which implies \( \mathbb{P}^\gamma[||\gamma^2_{i,t}||_{L^2} < \infty] \) which is the first claim in \((69)\). Next, in order to prove the second claim in \((69)\), we take \( \eta = \frac{8 + \sigma - \gamma_1}{4} \) in \((256)\) so that \((284)\) is satisfied for any \( \alpha \in (0, \frac{1}{2}) \), granting us
\[
\mathbb{P} \left[ \int_0^T \|(-\Delta)^{\frac{\gamma_1}{2}} Y(s)\|_{L^2}^2 ds \right] < \infty. \tag{288}
\]
We take \( I > \frac{1}{2\gamma_1} \), rely on \((30)\) Proposition A.1.1 (ii) similarly to the proof of \((33)\) Proposition 34) to deduce that the mapping \( Y \mapsto \int_0^{T-I} (t-s)^{\alpha-1} e^{-\frac{\gamma_1}{2} (t-s)} Y(s) ds \) \((283)\) is bounded linear operator from \( L^2(0, T; H_\alpha^1 T) \) to \( C^4([0, T]; H_\alpha^1 T) \) for any \( \delta \in (0, \alpha - \frac{1}{2\gamma_1}) \); this, together with \((288)\), leads to the second claim in \((69)\) identically to previous works.

5.3. **Proof of Proposition 4.5**

The stopping time \( t \) in the statement of Theorem 2.1 is \( T_L \) from \((74)\) for \( L > 0 \) sufficiently large and thus by Theorem 2.1 we know that there exists a process \( v \) which is a weak solution to \((6)\) on \([0, T_L]\) such that \((10)\) holds. Hence, we see that \( \mathcal{V}(\cdot \wedge T_L) \in \Omega_0 \). Now due to \((71)\), \((70a)\), \((6)\), and \((68)\)
\[
Z'(t) = z(t) \quad \forall \ t \in [0, T_L] \quad \mathcal{P}\text{-a.s.} \tag{289}
\]
By Proposition 4.4 we know that \( z \in C_T H_\alpha^1 T + C_T^{\frac{\gamma_1}{4} + \frac{\gamma_1}{2}} H_\alpha^1 T \) \( \mathcal{P}\text{-a.s.} \) and thus the trajectories \( t \mapsto \|z(t)\|_{H_\alpha^1 T} \) and \( t \mapsto \|z\|_{C_T^{\frac{\gamma_1}{4} + \frac{\gamma_1}{2}} H_\alpha^1 T} \) where \( \delta \in (0, \frac{1}{2}) \) are \( \mathcal{P}\text{-a.s.} \) continuous. It follows by \((73b)\), \((74)\), and \((289)\) that
\[
\tau_L(v) = T_L \quad \mathcal{P}\text{-a.s.} \tag{290}
\]
Next, we verify that \( P \) is a martingale solution to \((6)\) on \([0, T_L]\) according to Definition 4.2. The verification of (M1) follows from \((42)\) and \((10)\) while that of (M3) follows by writing \( v = y + z \), choosing \( C_{iq,q} q \in \mathbb{N} \), in the Definition 4.2 to satisfy (2\(\pi C_L + 2\pi L^2)^{-\delta} + (t \wedge T_L)(2\pi C_L + 2\pi L^2)^{\delta} \leq C_{iq,q} \), and relying on \((105)\) and \((78)\). Finally, in order to verify (M2), we let \( t \geq s \) and \( g \) be bounded, \( \mathcal{R}\)-valued, \( B_\mathcal{R}\)-measurable, and continuous on \( \Omega_0 \). By Theorem 2.1 we know that \( \mathcal{V}(\cdot \wedge T_L) \) is \((\mathcal{F}_t)_{t \geq 0}\)-adapted so that \( g(\mathcal{V}(\cdot \wedge T_L)) \) is \( \mathcal{F}_t\)-measurable. This leads to \( \mathbb{E}^\gamma[M^i_{t\wedge T_L, \psi}(\cdot) = \mathbb{E}^\gamma[M^i_{t\wedge T_L, \psi}(\cdot)] \) where \( M^i_{t\wedge T_L, 0} = \langle M^i_{t\wedge T_L, \psi}, \gamma \rangle \) and hence \( M^i_{t\wedge T_L, 0} \) is a \( (\mathcal{B}_t)_{t \geq 0}\)-martingale under \( \mathcal{P} \). Similarly, using the fact that
\[
\langle (M^i_{t\wedge T_L, \psi}(\cdot)), \mathcal{V}(\cdot \wedge T_L)(v)) \rangle = \langle (M^i_{t\wedge T_L, \psi}(\cdot)), \mathcal{V}(\cdot \wedge T_L)(\psi) \rangle \quad \forall \ \psi \in \mathcal{C}_T^{\infty}(T^2) \cap \mathcal{H}^1_{\alpha} \mathcal{F} \text{ and we can show } \mathbb{E}^\gamma[(M^i_{t\wedge T_L, 0})^2 - (t \wedge T_L)||\mathcal{G}^\psi|^2_{L^2}] \text{ which implies that } (M^i_{t\wedge T_L, 0})^2 - (t \wedge T_L)||\mathcal{G}^\psi|^2_{L^2} \text{ is a } (\mathcal{B}_t)_{t \geq 0}\text{-martingale under } \mathcal{P} \text{. It also has a consequence that } \langle (M^i_{t\wedge T_L, 0}) \rangle = \int_0^{T \wedge T_L} ||\mathcal{G}^\psi||^2_{L^2} \mathcal{P} \text{ and this also shows that } M^i_{t\wedge T_L, 0} \text{ is square-integrable, completing the proof that } \mathcal{P} \text{ satisfies (M2). Hence, } P \text{ is a martingale solution to } (6) \text{ on } [0, T_L].
\]

5.4. **Proof of Proposition 4.6**

Because \( \tau_L \) is a \( (\mathcal{B}_t)_{t \geq 0}\)-stopping time that is bounded by \( L \) due to \((73b)\) while \( P \) is a martingale solution on \([0, T_L]\) due to Proposition 4.5 we see that Lemma 4.3 completes this proof once we verify \((64)\). First, by relying on \((289)\), \((290)\), and \((69)\), we can show that there exists a \( \mathcal{P}\)-measurable set \( \mathcal{N} \subset \Omega_0, T_L \) such that \( P(\mathcal{N}) = 0 \) and for all \( \omega \in \Omega_0 \setminus \mathcal{N} \) and \( \delta \in (0, \frac{1}{2}) \), \( Z^{\omega}(\cdot \wedge T_L(\omega)) \in C_T H_\alpha^{1 + \frac{\gamma_1}{2}} \cap C_T^{\frac{\gamma_1}{4} + \frac{\gamma_1}{2} + \frac{\gamma_1}{2}} \). For every \( \omega \in \Omega_0 \) and \( \omega \in \Omega_0 \setminus \mathcal{N} \) we define
\[
Z_{\mathcal{N}}^{\omega}(t) = M^i_{t \wedge T_L, 0} - e^{-(t \wedge T_L(\omega))\Lambda} M^i_{t \wedge T_L(\omega), 0} - \int_{t \wedge T_L(\omega)}^T \mathbb{E}^\gamma[e^{-(s \wedge T_L(\omega))\Lambda}] M^i_{s \wedge T_L(\omega), 0} ds \tag{291}
\]
so that because $\nabla \cdot M_{t,0}^{\omega} = 0$ for any $\omega \in \Omega_0$ due to (10A), we see that

$$Z_{\tau_L(\omega)}^{\omega}(t) = M_{t,0}^{\omega} - M_{t_0}^{\omega} - \int_{t_0 \tau_L(\omega)}^t \mathbb{P} \Lambda^2 e^{-(t-s) \Lambda^2} \left( M_{s,0}^{\omega} - M_{s \tau_L(\omega),0}^{\omega} \right) \, ds. \quad (292)$$

Together with (10B), this leads us to

$$Z^{\omega}(t) - Z^{\omega}(t \wedge \tau_L(\omega)) = Z_{\tau_L(\omega)}^{\omega}(t) + \left( e^{-(t-t \wedge \tau_L(\omega)) \Lambda^2} - \text{Id} \right) Z^{\omega}(t \wedge \tau_L(\omega)). \quad (293)$$

It follows from (292) that $Z_{\tau_L(\omega)}^{\omega}$ is $\mathcal{B}^T(\omega)$-measurable because $M_{t,0}^{\omega} - M_{t_0}^{\omega}$ is $\mathcal{B}^T(\omega)$-measurable, and that $Q_\omega$ from Lemma 4.2 satisfies

$$Q_\omega((\omega' \in \Omega_0 : Z^{\omega'}(\cdot) \in C_T H_x^{4+\frac{\delta}{2}} \cap C^{\frac{1-\delta}{2}}_{loc} H_t^{4+\frac{\delta}{2}}))$$

$$= \delta_\omega((\omega' \in \Omega_0 : Z^{\omega'}(\cdot \wedge \tau_L(\omega)) \in C_T H_x^{4+\frac{\delta}{2}} \cap C^{\frac{1-\delta}{2}}_{loc} H_t^{4+\frac{\delta}{2}}))$$

$$\Theta_{\tau_L(\omega)} R_{\tau_L(\omega),(\xi(\tau_L(\omega)))}(\omega' \in \Omega_0 : Z_{\tau_L(\omega)}^{\omega'}(\cdot) \in C_T H_x^{4+\frac{\delta}{2}} \cap C^{\frac{1-\delta}{2}}_{loc} H_t^{4+\frac{\delta}{2}})) = 1. \quad (294)$$

where for all $\omega \in \Omega \setminus \mathcal{N}$, $\delta_\omega((\omega' \in \Omega_0 : Z^{\omega'}(\cdot) \in C_T H_x^{4+\frac{\delta}{2}} \cap C^{\frac{1-\delta}{2}}_{loc} H_t^{4+\frac{\delta}{2}})) = 1$. Next, we rewrite

$$\int_0^\infty \mathbb{P} e^{-(t-s) \Lambda^2} d(M_{s,0}^{\omega} - M_{s \tau_L(\omega),0}^{\omega}) \overset{(292)}{=} Z_{\tau_L(\omega)}^{\omega}(t) \quad (295)$$

and observe that Proposition 4.2 implies that for any $\delta \in (0, \frac{1}{4})$,

$$R_{\tau_L(\omega),(\xi(\tau_L(\omega)))}(\omega' \in \Omega_0 : Z_{\tau_L(\omega)}^{\omega'}(\cdot) \in C_T H_x^{4+\frac{\delta}{2}} \cap C^{\frac{1-\delta}{2}}_{loc} H_t^{4+\frac{\delta}{2}})) = 1. \quad (296)$$

Hence, applying this to (294) now allows us to conclude that for all $\omega \in (\Omega_0 \setminus \mathcal{N}) \cap \{ \xi(\tau) \in H_x^1 \}$, $Q_\omega((\omega' \in \Omega_0 : Z^{\omega'}(\cdot) \in C_T H_x^{4+\frac{\delta}{2}} \cap C^{\frac{1-\delta}{2}}_{loc} H_t^{4+\frac{\delta}{2}})) = 1$; i.e., for all $\omega \in (\Omega_0 \setminus \mathcal{N}) \cap \{ \xi(\tau) \in H_x^1 \}$ there exists a measurable set $N_\omega$ such that $Q_\omega(N_\omega) = 0$ and for all $\omega' \in \Omega_0 \setminus N_\omega$, the mapping $t \mapsto Z^{\omega'}(t)$ lies in $C_T H_x^{4+\frac{\delta}{2}} \cap C^{\frac{1-\delta}{2}}_{loc} H_t^{4+\frac{\delta}{2}}$. This implies by (73b) that for all $\omega \in \Omega_0 \setminus \mathcal{N}$,

$$\tau_L(\omega') = \tau_L(\omega') \quad \forall \omega' \in \Omega_0 \setminus N_\omega \quad (297)$$

if we define for $\delta \in (0, \frac{1}{4})$

$$\tau_L(\omega') \triangleq \inf\{ t \geq 0 : C_{S_1} ||Z^{\omega'}(t)||_{H_x^{4+\frac{\delta}{2}}} \geq L^\frac{\delta}{2} \} \wedge \inf\{ t \geq 0 : C_{S_1} ||Z^{\omega'}||_{C^{\frac{1-\delta}{2}}_{loc} H_t^{4+\frac{\delta}{2}}} \geq L^\frac{\delta}{4} \} \wedge L. \quad (298)$$

This leads us to, for all $\omega \in (\Omega_0 \setminus \mathcal{N}) \cap \{ \xi(\tau) \in H_x^1 \}$ with $P(\{ \xi(\tau) \in H_x^1 \}) = 1$,

$$Q_\omega((\omega' \in \Omega_0 : \tau_L(\omega') = \tau_L(\omega))) = 1 \quad (299)$$

by the identical arguments to [45].

5.5. Further preliminaries. We list various previous results which played crucial roles in the proofs of Theorems 2.1, 2.2. The following standard product estimate is convenient and can be readily proved via Fourier analysis:

$$||\Lambda^s(fg)||_{L^2} \leq ||\Lambda^s f||_{L^2} ||g||_{L^2} + ||f||_{L^2} ||\Lambda^s g||_{L^2} \quad \text{for any } s \geq 0. \quad (300)$$

**Proposition 5.3.** (Calderón commutator; e.g., [7] Lemma A.5, [65] Lemma 2.2, and [61] Theorem 10.3) Let $p \in (1, \infty)$.
If \( \phi \in W^{1,\infty}(\mathbb{T}^2) \), then for any \( f \in L^p(\mathbb{T}^2) \) that is mean-zero,
\[
\|[(\Lambda, \phi)]f\|_{L^p_T} \lesssim_p \|\phi\|_{W^{1,\infty}_{x,y}} \|f\|_{L^p_T};
\]  
(301)

moreover, if \( s \in [0,1] \), \( \phi \in W^{2,\infty}(\mathbb{T}^2) \), then for any \( f \in H^s(\mathbb{T}^2) \) that is mean-zero,
\[
\|[(\Lambda, \phi)]f\|_{H^s_T} \lesssim_s \|\phi\|_{W^{2,\infty}_{x,y}} \|f\|_{H^s_T}.
\]  
(302)

(2) Let \( \gamma_2 \in (1,2) \). If \( \phi \in C^{2-\gamma_2}(\mathbb{T}^2) \) such that \( \Lambda^{2-\gamma_2} \phi \in L^\infty(\mathbb{T}^2) \), then for all \( f \in L^p(\mathbb{T}^2) \) that is mean-zero,
\[
\|[(\Lambda^{2-\gamma_2}, \phi)]f\|_{L^p_T} \lesssim_p \|\phi\|_{C^{2-\gamma_2}} \|f\|_{L^p_T};
\]  
(303)

moreover, if \( s \in [0,1] \), \( \phi \in C^{1,2-\gamma_2}(\mathbb{T}^2) \) such that \( \Lambda^{2-\gamma_2} \phi \in W^{1,\infty}(\mathbb{T}^2) \), then for any \( f \in H^s(\mathbb{T}^2) \) that is mean-zero,
\[
\|[(\Lambda^{2-\gamma_2}, \phi)]f\|_{H^s_T} \lesssim_s \|\phi\|_{C^{1,2-\gamma_2}} \|f\|_{H^s_T}.
\]  
(304)

Proof of Proposition 5.3(1) is a statement from [7 Lemma A.5]; thus, we only sketch the proof for Proposition 5.3(2) following the argument of [7 Lemma A.5] and [65 Lemma 2.2]. First, we work on the case \( x \in \mathbb{R}^2 \) because it leads to the case \( x \in \mathbb{T}^2 \) by the same argument on [7 p. 1866]. We fix \( \phi \in C^{2-\gamma_2}(\mathbb{T}^2) \) such that \( \Lambda^{2-\gamma_2} \phi \in L^\infty(\mathbb{T}^2) \) and define \( T = [\Lambda^{2-\gamma_2}, \phi] \). We will verify the hypothesis of T(1) theorem from [32] Theorem 1 on p. 373 (see also [61] Theorem 10.2 on p. 95)), which consists of

1. \( T \) is a continuous operator from \( S(\mathbb{R}^2) \) to its dual \( S'(\mathbb{R}^2) \) where \( S(\mathbb{R}^2) \) is the Schwartz space, associated with some singular integral operator (SIO) \( K \); i.e., \( K \) is a continuous function defined on \( \mathbb{R}^2 \times \mathbb{R}^2 \setminus \Delta \) where \( \Delta = \{(x,y) \in \mathbb{R}^2 \times \mathbb{R}^2 : x = y\} \) for which there exist two constants \( \delta \in (0,1] \) and \( C_K > 0 \) such that
\[
\langle T f, g \rangle = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} K(x,y)f(y)\psi(x)dy dx \forall f, g \in C^\infty_c(\mathbb{R}^2) \text{ with disjoint supports,}
\]  
(305a)

\[
|K(x,y)| \leq C_K |x-y|^{-2} \forall (x,y) \in \mathbb{R}^2 \times \mathbb{R}^2 \setminus \Delta,
\]  
(305b)

\[
|K(x', y) - K(x, y)| + |K(y, x') - K(y, x)| \leq C_K \frac{|x' - x|^\delta}{|x - y|^{\delta + 2s}}
\]  
(305c)

\[
\forall x, x', y \text{ such that } |x' - x| \leq \frac{1}{2} |x - y|,
\]  
(305d)

2. \( T^*1 \in \text{BMO}(\mathbb{R}^2) \) where \( T^* \) is defined by \( \langle T^* f, g \rangle = \langle T g, f \rangle \) (see [32] p. 372),

3. \( T \) has the weak boundedness property (WBP) (see [32] p. 373)); i.e., for any bounded subset \( B \subset C^\infty_c(\mathbb{R}^2) \), there exists a constant \( C \) such that for all \( \psi, \tilde{\psi} \in B \), all \( x_0 \in \mathbb{R}^2 \), and \( R > 0 \), \( |\int_{B} T \psi(\frac{x-x_0}{R}) \tilde{\psi}(\frac{y-x_0}{R}) dy| \leq CR \).

First, due to Proposition 2.1 on p. 513) we may write for all \( f \in C^\infty_c(\mathbb{R}^2) \)
\[
Tf(x) = [\Lambda^{2-\gamma_2}, \phi] f(x) = C_{\gamma_2} P.V. \int_{\mathbb{R}^2} \frac{[\phi(x) - \phi(y)]}{|x-y|^{2-\gamma_2}} f(y) dy.
\]  
(306)

Therefore, off the diagonal \( \Delta \), the kernel \( K(x,y) \) associated to \( T = [\Lambda^{2-\gamma_2}, \phi] \) is
\[
K(x,y) \equiv C_{\gamma_2} \frac{\phi(x) - \phi(y)}{|x-y|^{2-\gamma_2}},
\]  
(307)

from which (305a) can be immediately verified by (306)–(307); moreover, we can immediately estimate
\[
|K(x,y)| \leq |C_{\gamma_2}| |\phi|_{C^{2-\gamma_2}} |x - y|^{-2} \text{ for all } (x,y) \in \mathbb{R}^2 \times \mathbb{R}^2 \setminus \Delta.
and conclude (305b). Next, to prove (305c), we first notice that $|K(x', y) - K(x, y)| = |K(y', x) - K(y, x)|$ by (307). Now we take $\delta \in (0, 2 - \gamma_2)$ so that $\delta < 1$ and estimate using (305d):

$$|K(x', y) - K(x, y)| \leq |C_{\gamma_2}||[\phi(x') - \phi(y)]| \frac{(4 - \gamma_2)x^2 - y^2}{|x - y|^{5-\gamma_2}} + \|\phi\|_{C^{2-\gamma_2}} |x' - x|^{2-\gamma_2} \frac{|x - y|^{5-\gamma_2}}{|x - y|^{5-\gamma_2}}. \quad (308)$$

Now using (305d), one can consider the following three cases and readily verify (305c) starting from (308):

1. $1 \leq |x' - x| < \frac{1}{4}|x - y|,$
2. $|x' - x| \leq 1 < \frac{1}{4}|x - y| \text{ or } |x' - x| < 1 \leq \frac{1}{4}|x - y|,$
3. $|x' - x| < \frac{1}{4}|x - y| \leq 1.$

This completes the proof that $T$ is a SIO and thus the first hypothesis of the T(1) theorem.

To verify the second hypothesis, we observe that $T^* = -[\Lambda^{2-\gamma_2}, \phi] = -T$ which implies $-T^*(1) = T(1) = -\sum_{k=1}^n R_k^c \Lambda^{2-\gamma_2} \phi$ where $R_k^c$ are Calderón-Zygmund operators (e.g., [61, p. 106]) and thus bounded from $L^\infty(\mathbb{R}^2)$ to $BMO(\mathbb{R}^2)$ (e.g., [61, Theorem 6.8 on p. 54]) indicating that $R_k^c \Lambda^{2-\gamma_2} \phi \in BMO(\mathbb{R}^2)$ so that $T^*(1), T(1) \in BMO(\mathbb{R}^2).$ Finally, the fact that $T$ has the WBP can be readily proven via a criteria in [32, Section IV] because $K(x, y) = -K(y, x).$ By T(1) theorem from [32], we now conclude that $T$ is bounded from $L^2(\mathbb{R}^2)$ to itself. Thus, by definition on [32, p. 372] (see also [61, Definition 6.4 on p. 53]), we deduce that it is a Calderón-Zygmund operator with a norm less than or equal to $\|\phi\|_{C^2}^{2-\gamma_2} + \|\Lambda^{2-\gamma_2} \phi\|_{L^2}$ (e.g., see [44, Theorem 8.3.3] concerning the norm). By [61, Theorem 6.8 on p. 54] this allows us to deduce that $T$ is bounded from $L^p(\mathbb{R}^2)$ to $L^p(\mathbb{R}^2)$ for all $p \in (1, \infty)$ and hence conclude (303). To prove (304), we observe that

$$\nabla [\Lambda^{2-\gamma_2} \phi] f = [\Lambda^{2-\gamma_2} \phi] \nabla f + [\Lambda^{2-\gamma_2}, \nabla \phi] f. \quad (309)$$

Because $\phi \in C^{1.2, 2-\gamma_2}(\mathbb{T}^2)$ such that $\Lambda^{2-\gamma_2} \phi \in W^{1, \infty}(\mathbb{T}^2)$ by hypothesis, we can apply (303) with $p = 2$ to $\|\Lambda^{2-\gamma_2} \phi \nabla f\|_{L^2}$ and $\|\Lambda^{2-\gamma_2}, \nabla \phi\| f\|_{L^2}$ and interpolate to deduce (304). \quad \Box

**Lemma 5.4.** (71, Lemma A.2.4, Proposition D.1) Within this lemma, we denote $D_t = \partial_t + u \cdot \nabla.$ Consider a smooth solution $f$ to (124) with a given $u, f_0,$ and $g,$ and recall that $\Phi = X^{-1} \text{ with } X$ defined by (125). Then, for $t > t_0$,

$$\|f(t)\|_{C_t} \leq \|f_0\|_{C_t} + \int_{t_0}^t \|g(\tau)\|_{C_t} d\tau, \quad (310a)$$

$$\|Df(t)\|_{C_t} \leq \|Df_0\|_{C_t} + \int_{t_0}^t e^{(t-\tau)}\|Dg(\tau)\|_{C_t} d\tau; \quad (310b)$$

more generally, for any $N \in \mathbb{N} \setminus \{1\},$ there exists a constant $C = C(N)$ such that

$$\|D^N f(t)\|_{C_t} \leq \|D^N f_0\|_{C_t} + C(t - t_0)\|D^N u\|_{C_t} \|Df_0\|_{C_t} e^{C(t-\tau)}\|Dg(\tau)\|_{C_t} \int_{t_0}^t e^{(t-\tau)}\|Dg(\tau)\|_{C_t} d\tau + \int_{t_0}^t e^{C(t-\tau)}\|Dg(\tau)\|_{C_t} \|D^N g(\tau)\|_{C_t} + C(t - \tau)\|D^N u\|_{C_t} \|Dg(\tau)\|_{C_t} d\tau. \quad (311)$$

Moreover,

$$\|D\Phi(t) - Id\|_{C_t} \leq e^{(t-\tau)}\|D\Phi_{t_0}\|_{C_t} - 1 \leq (t - t_0)\|D\|_{C_t} e^{(t-\tau)}\|D\Phi_{t_0}\|_{C_t}, \quad (312a)$$

$$\|D^N \Phi(t)\|_{C_t} \leq C(t - t_0)\|D^N u\|_{C_t} e^{C(t-\tau)}\|D\Phi_{t_0}\|_{C_t}, \quad \forall N \in \mathbb{N} \setminus \{1\}. \quad (312b)$$
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