On Fixed Points of Order K of RSA

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Abstract

In this paper, we gave a preliminary dynamical analysis on the RSA cryptosystem and obtained a computational formulae of the number of the fixed points of \( k \) order of the RSA. Thus, the problem in [8, 9] has been solved.

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1 Introduction

Shortly after Diffie and Hellman [1] introduced the idea of public key cryptography, Rivest, Shamir and Adleman (RSA) [2] proposed such a cryptosystem. A simplified version of RSA is the following:

Let \( n = pq \) be the product of two large primes of the same size. Let \( e, d \) be two integers satisfying \( ed \equiv 1 \pmod{\varphi(n)} \). Call \( n \) the RSA modulus, \( e \) the encryption exponent, and \( d \) the decryption exponent. Let \( e \) and \( n \) be public keys, and let \( d \) be the corresponding secret key. A message is an integer \( m \in \mathbb{Z}_n \). To encrypt \( m \), one computes \( m^e \equiv c \pmod{n} \). To decrypt the ciphertext \( c \), the receiver computes \( m \equiv m^{ed} \equiv c^d \pmod{n} \). Denote such a cryptosystem by \( RSA(n, e) \). We call \( m \) a fixed point of \( RSA(n, e) \) if \( m^e \equiv m \pmod{n} \). And call \( m \) a fixed point of order \( k \) if \( k \) is the smallest positive integer such that \( m^{ek} \equiv m \pmod{n} \). Clearly, \( f : x \rightarrow x^e \pmod{n} \) is a dynamical system. Thus, \( k \) is exactly the period of \( m \). For more details on the arithmetic of dynamical systems, see [7].
In 1979, Blakley and Borosh [3] first pointed out that there were at least 9 fixed points in \( RSA(n, e) \). For more references on fixed points, also see [4]-[6]. Denoted the set of all fixed points of order \( k \) of \( RSA(n, e) \) by \( E_{n,e,k} \) and the cardinality of the set \( S \) by \( |S| \). In [8, 9], Yu considered the general case of fixed points of order \( k \) and gave geometric mean value of \( |T_{n,e,k}| \) and pointed out that it was difficult to give a quantitative description of \( |T_{n,e,k}| \), where \( k \) is a given positive integer and

\[
T_{n,e,k} = \{x \mid \forall m < k, m \in \mathbb{N}, x \in \mathbb{Z}_n^*, x^{e^k} \equiv x \pmod{n}, x^{e^m} \neq x \pmod{n}\}.
\]

In this essay, we preliminarily consider this question and obtain the following results:

**Theorem 1** \( |T_{n,e,k}| = \sum_{d|k} \mu(k/d)(e^d - 1, p - 1)(e^d - 1, q - 1) \), where \( \mu(\cdot) \) is the Möbius function.

Based on this result, we get Theorem 2.

**Theorem 2** \( |E_{n,e,k}| = \sum_{d|k} \mu(k/d)((e^d - 1, p - 1) + 1)((e^d - 1, q - 1) + 1) \).

## 2 Proof of Main Theorems

We denote the set of positive integers by \( \mathbb{N} \). For given positive integers \( a \) and \( b \), we write \( a|b \) if \( a \) divides \( b \). And denote the greatest common divisor of \( a \) and \( b \) by \( (a, b) \). Denote a complete set of residues modulo \( n \) by \( \mathbb{Z}_n \), where \( 1 < n \in \mathbb{N} \), and a reduced set of residues modulo \( n \) is denoted by \( \mathbb{Z}_n^* \). Let \( a \) be an integer relatively prime to \( n \). The order of \( a \) modulo \( n \), denoted by \( \text{ord}_n(a) \), which is the smallest positive integer \( d \) such that \( a^d \equiv 1 \pmod{n} \).

**Lemma 1**[8] For \( 1 < n \in \mathbb{N}, r \in \mathbb{N}, \) let the canonical factorization of \( n \) be \( \prod_{i=1}^{m} p_i^{a_i} \) and \( T_{n,r} = \{x \mid x^r \equiv 1 \pmod{n}, 1 \leq x < n\} \), then \( |T_{n,r}| = \prod_{i=1}^{m} (r, \varphi(p_i^{a_i})) \).

**Lemma 2** For \( 1 < n \in \mathbb{N}, a, m, k \in \mathbb{N}, e \in \mathbb{Z}_n^* \), if \( a^{e^k} \equiv a \pmod{n} \) and \( k|m \), then \( a^{e^m} \equiv a \pmod{n} \).

**Proof** Let \( m = tk \). When \( t = 1 \), clearly \( a^{e^k} \equiv a^{e^m} \equiv a \pmod{n} \). Suppose that \( a^{e^m} \equiv a \pmod{n} \) when \( t = l \). And when \( t = l + 1 \), we have \( a^{e^m} \equiv a^{e^{lk}e^k} \equiv a^{e^k} \equiv a \pmod{n} \). It immediately shows that Lemma 2 is true by induction.

**Lemma 3** For \( 1 < n \in \mathbb{N}, a, m, k \in \mathbb{N}, e \in \mathbb{Z}_n^* \), if \( a^{e^m} \equiv a \pmod{n} \) and \( a \in E_{n,e,k} \), then \( k|m \).
Proof Let \( m = kt + r, \ t \in \mathbb{N}, \ 0 \leq r < k \). We have \( a^e^m \equiv a^{ekt} \equiv a^{kr} \equiv a(\mod n) \) by Lemma 2. Since \( a \in E_{n,e,k} \), hence \( r = 0 \), and Lemma 3 is true.

Proof of Theorem 1 By Lemma 1 and Lemma 3, it is easy to deduce
\[
\sum_{d|k} |T_{n,e,d}| = \prod_{i=1}^{m} (e^k - 1, \varphi(p_i^{a_i})).
\]
By Möbius inversion, it immediately shows that Theorem 1 is true.

Proof of Theorem 2 By Lemma 2 and Lemma 3, analogously, using Chinese Remainder Theorem and the method of proof of Theorem 1, it is easy to deduce that Theorem 2 is true.

Corollary 1 Let \( 1 < n \in \mathbb{N}, \ r \in \mathbb{N} \), and let the canonical factorization of \( n \) be \( \prod_{i=1}^{m} p_i^{a_i} \). Then \( |\{ x | \text{ord}_n(x) = r, 1 \leq x \in \mathbb{Z}^*_n \}| = \sum_{d|r} (\mu(r/d) \prod_{i=1}^{m} (d, \varphi(p_i^{a_i}))) \).

Corollary 2 Let \( 1 < n \in \mathbb{N}, \ r \in \mathbb{N} \), let the canonical factorization of \( n \) be \( \prod_{i=1}^{m} p_i^{a_i} \), and let \( F_{n,r} = \{ x | \forall k < r, k \in \mathbb{N}, 1 \leq x \leq n, x^r \equiv x(\mod n), x^k \neq x(\mod n) \} \). Then \( |F_{n,r}| = \sum_{d|r} (\mu(r/d) \prod_{i=1}^{m} (1 + (d - 1, \varphi(p_i^{a_i}))) \).

3 Conclusion

Clearly, if the factorization of \( n \) is known, then computing the number of the fixed points of order \( k \) of the RSA cryptosystem is simple and convenient by the presented formulae. This is useful to pick the encryption exponent, which is necessary to ensure the resulting RSA safe from fixed points attack. Maybe we are not afraid of a fixed point. However, the following problem should be further considered: Is there a polynomial-time algorithm for finding a fixed point \( m \), where \( m \neq 0, \pm 1 \)? This problem and Factoring the RSA modulus perhaps are equivalent.

Remark: This paper is the revision of paper [10] in the proceedings of China Crypt’2006, whose Chinese version has been accepted by Journal of Mathematics (Wuhan, China).

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