ECONOMIC ORDER QUANTITY (EOQ) OPTIMAL CONTROL CONSIDERING SELLING PRICE AND SALESMAN INITIATIVE COST

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Abstract. Retailers usually offer several types of similar products. A larger number of available stock products in display space will lead consumer to buy more, as well as giving a negative impression on other types of less available products. However, the amount of display space is limited so capacity of carrying the products is limited. Competition among products to increase demand rate is influenced by stock levels available in display space, price and salesmen’s initiative in promoting the products. The Economic Order Quantity (EOQ) to replenish the stock of the product is dependent on the on-hand inventory. Salesman’s initiative also affects maximum profit obtained by the seller. In this paper, Potryagin’s Maximal Principle is used to determine the state of the inventory levels response to control prices of products. Sensitivity analysis of capacity allocation display space is also presented numerically.

Keyword : EOQ Model, Selling Price, Salesmen’s Initiative.

1. Introduction

Inventory has important meaning and role in a company. Without an inventory the company will be faced with the situation of the company's disruption and even discontinuity. Inventories are products that wait for further processing such as production activities on manufacturing systems and marketing activities in the distribution system [1]. Inventory usually occurs in the form of finished goods or goods to be used to fulfil a particular purpose [2]. The main function of inventory in a system is to ensure the smoothness of the fulfilment mechanism of product demand in accordance with the needs of consumers so that the system can achieve optimal performance. A well managed inventory also affects the decrease of production cost of a company. Managed systems can achieve optimal performance through production planning related to good inventory control over time [3].

Retail is a sale of small quantities of commodities to consumers. Retail usually sells some kind of similar product. The basic competition of the retail market is price competition, inventory and product promotion. Retail inventories are goods purchased for the purpose of resale. Retail inventory is closely related to the limitations of display space so that the amount of inventory is limited. The large amount of inventory in the showroom attracts consumers to buy more. Conversely supplying fewer than other similar products will lead to a negative assumption.
Differences in inventory levels affect each other at the level of consumer demand among similar products. The level of demand is the amount of goods consumers wanted at a given price level and time. The market balance can be defined as a situation where at the price level formed, the consumer can buy the desired quantity of product, and the seller can sell the desired quantity of product [4]. Generally, when the amount of inventory exceed the number of requests, the cost of storage will increase, while when the price is below the equilibrium price the amount of demand will exceed the inventory resulting in stock out.

In addition to price and quantity of product availability, product promotion to consumers also affects the level of consumer demand. Many companies involve salesmen to promote their products, to increase demand for these products. In general, many consumer goods inventory displayed in shopping centers will attract customers to buy more, which means that customers have the scope to choose better design or product quality. Customer satisfaction cannot be separated from the efforts of the sales team [5].

Inventory control is an important activity that needs special attention from retail companies. It determines the optimal inventory amount with minimum total cost. In addition to price and quantity of product availability, product promotion to consumers also affects the level of consumer demand. With optimal product and salesman prices, proper inventory control can maximize retail profit. The inventory control model that gives an important role is the Economic Order Quantity (EOQ) model. The EOQ model is used to determine the optimal order quantity of a product. This optimum order quantity leads to minimal inventory costs so as to provide maximum benefits. Technically, EOQ inventory model is a model of procurement or supply of raw materials in a company. EOQ model to determine the number of orders that meet the total cost so there is no shortage of inventory has been done by [6]. Most EOQ models consider only a single product. However, in reality a producer might be produces more than one similar products or a retail might be sells more than one similar products [7].

In modeling two similar products, the author in [7] assumes that the first product demand is positively affected by the joint venture and is negatively affected by the inventory stock of the two similar products. The same assumption is used by the author for two similar products but with different models [5]. In other paper the author discusses the dynamic analysis and optimal control on the supply control model of two different products with limited production capacity as well as joint sales team initiatives [8]. Research with inventory control model of two different products on limited product capacity as well as sales team initiatives with dynamic system approach has been done, e.g [3], other researches studies about optimal controls with sales team initiatives not only during equilibrium conditions, but optimal control with the sales team effort at any time [9]. The authors in [10] extend the model to consider more than two products and also consider different form of production growth model [11]. In this paper we discuss optimal controls for product price forecasts and salesman maximizing revenue. The Pontryagin Maximum Principle is used in solving this optimal control problem as in [12].

2. Model Formulation

A similar product is defined as a product of a uniform nature. Here a similar product is limited to two types of similar products. The establishment of EOQ models of two products in a retailer's showroom for limited inventory storage, product demand is influenced by the level of product inventory in the showroom. Salesman Initiative in EOQ model of two similar products takes form using salesmen promotion media. Some of the parameters that influence the formation of EOQ models with supply-dependent demand, pricing and salesmen initiatives from two similar products can be seen in Table 1.
Table 1. Parameters that Influence the Formation of the EOQ Model

| Notation | Information |
|----------|-------------|
| $D_x$    | Level of product demand 1 |
| $D_y$    | Level of product demand 2 |
| $E_x$    | Salesman initiative product 1 |
| $E_y$    | Salesman initiative product 2 |
| $R_x$    | Replenishment rate product 1 |
| $R_y$    | Replenishment rate product 2 |
| $r_x$    | Product replenishment rate product 1 |
| $r_y$    | Product replenishment rate product 2 |
| $L_x$    | Maximum storage capacity of the product 1 |
| $L_y$    | Maximum storage capacity of the product 2 |
| $L$      | Maximum storage capacity of both products ($L = L_x + L_y$) |
| $S_{x}^{\text{max}}$ | The upper limit of the selling price per unit of product 1 when there is no demand |
| $S_{y}^{\text{max}}$ | The upper limit of the selling price per unit of product 2 when there is no demand |
| $\delta = (r - i)$ | $r$ and $i$ represent the interest rate and inflation of currency unity |
| $l_1$    | Comparison of the level of salesmen initiative with the level of product demand 1 when the maximum salesman initiative level |
| $l_2$    | Comparison of the level of salesmen initiative with the level of product demand 2 when the maximum salesman initiative level |
| $l_3$    | Comparison of inventory levels with product demand level 1 when the maximum salesman initiative level |
| $l_4$    | Comparison of inventory levels with product demand level 2 when the maximum salesman initiative level |

\[ 0 < X(t) \leq L_x \] \hspace{1cm} (1)
and
\[ 0 < Y(t) \leq L_y \] \hspace{1cm} (2)

The general model used to describe production is the logistic growth model,

\[ D_x = \frac{C_x E_x (1 - \frac{X}{D})}{l_1 E + l_2 X} \] \hspace{1cm} (3)
and

\[ D_y = \frac{C_y E_y (1 - \frac{X}{D})}{l_3 E + l_4 Y} \] \hspace{1cm} (4)

where \( C_x = (1 - \frac{s_x}{S_{x}^{\text{max}}}) \) \hspace{1cm} (5)
and
\[ C_x = \left(1 - \frac{S_{\text{max}}}{S_y} \right), \]  

(6)
is the potential growth rate of each product. Like the production level, the demand level is also assumed to follow the logistic equation, in which the replenishment rate of product 1 and product 2 depends on the level of inventory,

\[ R_x = r_x \left(1 - \frac{X}{L_x}\right)X, \]  

(7)
and

\[ R_y = r_y \left(1 - \frac{Y}{L_y}\right)Y, \]  

(8)
where \( r_x \) and \( r_y \) are positive constants called the intrinsic growth rate.

The change in the level of product inventory at time \( t \) is the difference in the inventory level of the product at time \( t \) with the previous inventory level. The demand level is affected by the sales initiative, then for product 1, \( \frac{\partial D_x}{\partial N} = \frac{r_x X}{L_x} \) will be positive for every \( X \in (0, \infty) \) and \( \lim_{N \to \infty} D_x \to r_x X \) so that \( D_x \in [0, r_x X] \). This means the level of product demand under conditions of infinite sales initiative influence still results in limited demand. This applies equally to product 2, and hence the differential equations for supplies that available at time \( t \) for product 1 and product 2 respectively given by:

\[ \dot{X} = r_x \left(1 - \frac{X}{L_x}\right)X - \frac{C_x E_x (1-Y)}{I_1 E + I_2 X} = G_1(X, Y), \]  

(9)
and

\[ \dot{Y} = r_y \left(1 - \frac{Y}{L_y}\right)Y - \frac{C_y E_y (1-X)}{I_3 E + I_4 Y} = G_2(X, Y). \]  

(10)

3. Model Analysis

3.1. Model Stability Analysis

The fixed point or balance point is obtained when the inventory level change of each product is at the zero level, i.e., the condition in which the balance between the level of consumer demand and the level of demand for the product is met. In other words when the market equilibrium is reached. The equilibrium of the system is said to be stable if all eigenvalues of the system are negative [14]. In Equilibrium, the system of dynamic equations (9) and (10) at a fixed point \((x^*, y^*)\) meet the following conditions:

\[ r_x \left(1 - \frac{x^*}{L_x}\right)x^* = \frac{C_x E_x (1-Y)}{I_1 E + I_2 x^*}, \]  

(11)
and

\[ r_y \left(1 - \frac{y^*}{L_y}\right)y^* = \frac{C_y E_y (1-X)}{I_3 E + I_4 y^*}. \]  

(12)
The stability behaviour of the fixed points \((x^*, y^*)\) of the inventory model will be analysed locally and globally as the following.

Local Stability

From the system of dynamic equations (9) and (10) we find the Jacobian matrix at the fixed point \((x^*, y^*)\):

\[ A(x^*, y^*) = \begin{bmatrix} x^* \left( -\frac{r_x}{L_x} + \frac{C_x E_x (1-y^*)}{I_1 E + I_2 (1-x^*)} \right) & x^* \left( \frac{C_x E}{I_1 E + I_2 (1-x^*)} \right) \\ y^* \left( \frac{C_y E}{I_3 E + I_4 y^*} \right) & y^* \left( -\frac{r_y}{L_y} + \frac{C_y E_y (1-x^*)}{I_3 E + I_4 (1-y^*)} \right) \end{bmatrix}. \]  

(13)

After obtaining the Jacobian matrix from the dynamic system, we find the characteristic equation with \( P(\lambda) = \det(A(x^*, y^*) - \lambda I) = 0 \), i.e.:
The characteristic equation of \( A(x^*, y^*) \) is given by

\[
P(\lambda) = \lambda^2 - \lambda \psi_1(x^*, y^*) + \psi_2(x^*, y^*) = 0,
\]

where,

\[
\psi_1(x^*, y^*) = \frac{\partial}{\partial x} (G_1(x^*, y^*)) + \frac{\partial}{\partial y} (G_2(x^*, y^*))
\]

\[
= x^* \left( -\frac{r_x}{L_x} + \left( -\frac{r_x}{L_x} + \frac{C_x l_x E (1-x^*)^2}{(l_1 E + l_2 x^*)^2} \right) + y^* \left( -\frac{r_y}{L_y} + \frac{C_y l_y E (1-y^*)^2}{(l_1 E + l_4 y^*)^2} \right) \right)
\]

and

\[
\psi_2(x^*, y^*) = \frac{\partial}{\partial x} (G_1(x^*, y^*)) \frac{\partial}{\partial y} (G_2(x^*, y^*)) - \frac{\partial}{\partial y} (G_1(x^*, y^*)) \frac{\partial}{\partial x} (G_2(x^*, y^*))
\]

\[
= \left[ x^* \left( -\frac{r_x}{L_x} + \frac{C_x l_x E (1-x^*)^2}{(l_1 E + l_2 x^*)^2} \right) \right] \left[ y^* \left( -\frac{r_y}{L_y} + \frac{C_y l_y E (1-y^*)^2}{(l_1 E + l_4 y^*)^2} \right) \right] - x^* y^* \left( \frac{C_x E}{l_1 E + l_2 x^*} + \frac{C_y E}{l_1 E + l_4 y^*} \right)
\]

The point \((x^*, y^*)\) is stable if the eigenvalues \((\lambda)\) of equations (14) and (15) are negative. We note that two eigenvalues will be negative if the terms \(\psi_1(x^*, y^*) < 0\) and \(\psi_2(x^*, y^*) > 0\) are fulfilled.

### Global Stability

Global stability analysis is done using the Liapunov Second Method approach. This method is similar to EOQ method in investigating nonlinear system. The Liapunov method corresponding to the dynamic function of equations (4) and (5) in [9], i.e.

\[
V(X, Y) = \left( X - x^* \right)^2 + h \left( Y - y^* \right)^2
\]

where \(h\) is constant for Liapunov function. The definitive positive \(V\) function and continuous continuous derivative at \(R^2\) remain asymptotically stable \((x^*, y^*)\) where \(\dot{V} = \frac{\partial V}{\partial X} \frac{dX}{dt} + \frac{\partial V}{\partial Y} \frac{dY}{dt}\). This can be written in the form \(\dot{V} = [X - x^* \ Y - y^*]^T P \begin{bmatrix} X \\ Y \end{bmatrix}\) with \(P = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\), where

\[
a_{11} = -\frac{r_x}{L_x} - \frac{C_x l_x E (1-x^*)^2}{(l_1 E + l_2 x^*)^2}
\]

\[
a_{12} = \frac{1}{2} \left( \frac{C_x E (l_1 E + l_2 x^*)}{l_1 E + l_4 y^*} + \frac{h C_y E}{l_1 E + l_4 y^*} \right) = a_{21}
\]

\[
a_{22} = h \left( -\frac{r_y}{L_y} - \frac{C_y l_y E (1-y^*)^2}{(l_1 E + l_4 y^*)^2} \right)
\]

The characteristic equation of the matrix \(P\) with \(det(P - \lambda I) = 0\) is \(\lambda^2 - \lambda (a_{11} + a_{22}) + (a_{11} a_{22} - (a_{12})^2) = 0\). The equilibrium point \((x^*, y^*)\) is stable if the eigen value \((\lambda)\) is negative. The two eigenvalues will be negative if the conditions \(a_{11} + a_{22} < 0\) and \(a_{11} a_{22} - (a_{12})^2 > 0\) are fulfilled.

### 3.2. The Optimal Control

The main goal of a company is to get the maximum amount of profit. The profit function that can be obtained by retailers is
\[ \pi(X,Y,S_x,S_y,E,t) = (S_x - p_x) \left( \frac{C_{x} E_x(1 - \frac{Y}{L_x})}{l_x E + l_x X} \right) + (S_y - p_y) \left( \frac{C_{y} E_y(1 - \frac{Y}{L_y})}{l_y E + l_y Y} \right) - h_x X - h_y Y - \gamma E \]  

(21)

where \( \pi \) is the total profit function of product 1 and product 2 minus product storage cost and product marketing cost by the salesman. The parameters \( p_x \) and \( p_y \) are the cost to buy a unit of product 1 and product 2 while \( h_x \) and \( h_y \) are the storage costs for product 1 and product 2, and \( \gamma \) is the marketing cost incurred by the salesman.

To maximize the net income one must optimize the inventory levels of each product by taking into account the influence of the price of each product and the salesman's initiative. Using the Maximum Pontryagin principle [12], we will look for price controls for each product and sales initiative, \( S_x, S_y, E \), by optimizing the product 1 and product 2, according to the dynamic equation system (9) and (10). Here is the optimal control problem as the Maximum Pontryagin principle [15].

\[ m \text{aks PV} = \int_0^T \pi(X,Y,S_x,S_y,E,t)e^{-\delta t} \, dt. \]  

(22)

Subject to,

\[ \dot{X} = r_x \left( 1 - \frac{X}{L_x} \right) X - \frac{C_x E_x(1 - \frac{Y}{L_x})}{l_x E + l_x X} \]

\[ \dot{Y} = r_y \left( 1 - \frac{Y}{L_y} \right) Y - \frac{C_y E_y(1 - \frac{Y}{L_y})}{l_y E + l_y Y} \]

\[ 0 \leq S_x(t) \leq S_x^{\text{max}} \]

\[ 0 \leq S_y(t) \leq S_y^{\text{max}} \]

\[ 0 \leq E(t) \leq E_{\text{max}} \]

\( S_x^{\text{max}}, S_y^{\text{max}}, \) and \( E_{\text{max}} \) are the upper limit for the marketing efforts of both salesmen and the selling price of each product. To solve the maximum problem use the Hamiltonian equation

\[ H = \pi(X,Y,S_x,S_y,E,t)e^{-\delta t} + \lambda_x(t)G_1(X,Y,E) + \lambda_y(t)G_2(X,Y,E) \]  

(23)

This is equivalent to the following system:

\[ H = \left( S_x - p_x \right) \left( \frac{C_x E_x(1 - \frac{Y}{L_x})}{l_x E + l_x X} \right) + \left( S_y - p_y \right) \left( \frac{C_y E_y(1 - \frac{Y}{L_y})}{l_y E + l_y Y} \right) - h_x X - h_y Y - \gamma \]

\[ + \lambda_x(t) \left( r_x \left( 1 - \frac{X}{L_x} \right) X - \frac{C_x E_x(1 - \frac{Y}{L_x})}{l_x E + l_x X} \right) + \lambda_y(t) \left( r_y \left( 1 - \frac{Y}{L_y} \right) Y - \frac{C_y E_y(1 - \frac{Y}{L_y})}{l_y E + l_y Y} \right) \]  

(24)

where \( \lambda_x(t) \) and \( \lambda_y(t) \) are co-state variables. Optimal conditions are in accordance with the Maximum Pontryagin principle [13], if

\[ \frac{\partial H}{\partial X} = 0, \quad \frac{\partial H}{\partial S_x} = 0, \quad \frac{\partial H}{\partial S_y} = 0 \quad \text{d} m_{an} - \frac{\partial \lambda_x}{\partial t} = \frac{\partial H}{\partial X} - \frac{\partial \lambda_x}{\partial Y} \]  

(25)

The value of Lagrange multipliers \( \lambda_x(t) \) and \( \lambda_y(t) \) as \( t \to \infty \) are:

\[ \lambda_x^*(t) = \left( \frac{q_1}{\delta - \psi_1 \delta + \psi_2} \right) e^{-\delta t} \]  

(26)

and

\[ \lambda_y^*(t) = \left( \frac{q_2}{\delta - \psi_1 \delta + \psi_2} \right) e^{-\delta t} \]  

(27)

Substituting equations (26) and (27) into equation (25), we obtain the following equation,

\[ \left[ \left( S_x - p_x \right) \left( \frac{C_{x} E_x(1 - \frac{Y}{L_x})}{l_x E + l_x X} \left( \frac{Y}{L_x} \right)^2 \right) + \left( S_y - p_y \right) \left( \frac{C_{y} E_y(1 - \frac{Y}{L_y})}{l_y E + l_y Y} \left( \frac{Y}{L_y} \right)^2 \right) - \gamma \right] e^{-\delta t} = 0 \]  

(28)

\[ S_x^* = \frac{1}{2} \left[ \frac{q_1}{\delta - \psi_1 \delta + \psi_2} e^{-\delta t} \right] \]

(29)

\[ S_y^* = \frac{1}{2} \left[ \frac{q_2}{\delta - \psi_1 \delta + \psi_2} e^{-\delta t} \right] \]  

(30)
We conclude that under optimal conditions, to obtain the maximum net profit from the EOQ model for two similar products, the equations (11), (12), (28), (29) and (30) should be satisfied.

4. Numerical Analysis

The system will be simulated numerically. As illustration is given the following example with the parameter value in the numerical simulation taken from [8] where \( r_x = r_y = 10 \), \( p_x = $20 \), \( p_y = $15 \), \( s_{max}^{y} = $40 \), \( s_{max}^{x} = $35 \), \( L = 100 \) unit, \( L_x = 50 \) unit, \( L_y = 50 \) unit, \( y = $2 \), \( h_x = h_y = $0.5 \), \( l_1 = 0.4 \), \( l_2 = 0.3 \), \( l_3 = 0.5 \), \( l_4 = 0.3 \). inflation rate \( i = 11\% \) and interest rate \( r = 16\% \), so \( \delta = r - i \) = 0.05. The value of the salesman’s initiative in promoting both products is taken from [7], i.e. 37.14510186, with the selling price for product 1 is $30,000,000 and product 2 is $25,000,000. By solving the equations numerically using Maple 15, the optimal solution of the inventory levels are obtain. The best conditions of inventory levels that maximize profitability are 49 unit both for product 1 and product 2. Selling at chosen price of each product, and promoting both products simultaneously with the chosen salesman initiatives, the optimal inventory of each product will maximize profit from retailers.

The stability criterion at the equilibrium point can be illustrated by the trajectory of the product 1 and product 2 areas, in Figure 1. The figure shows that the direction of the trajectory to a certain point. Stability point at \((49.20351407; 45.191271150)\). Both eigenvalues at the equilibrium point remain negative, i.e. -9.6 and -9.9. The eigenvalues show that the stability point is in stable condition (Stable Node). The bottom line is the fact that the levels of product inventory and production capacity meet the consumer demand and also maximize earnings. Furthermore, the graphs in Figures 2 show the effect of controls performed by the sales initiatives with regard to the maximum net income over time. The figure shows that the initial inventory level of both products is 49.7 units. To achieve the optimum stable conditions it takes 0.5 units of time. To maximize retail profit at the inventory level of 149.120351407 units and 249,190,150 units, the optimum order quantity (EOQ) of product 1 is 7.837981333 and product 2 is 7.956476643.

Figure 1. Phase portrait of product 1 and product 2.
5. Conclusion
The inventory control of the EOQ model depends on inventory levels, product pricing and salesman initiatives from two similar products that use a dynamic systems approach. Using a dynamic system model, the results obtained for the inventory model of the two similar products, is the difference between the production rate limited by the production capacity, and the level of demand limited by the salesman's initiative. After obtaining a dynamic system model for fixed points \((x^*, y^*)\) inventory levels of two similar products, a condition of equilibrium between the level of consumer demand and the level of product filling when market equilibrium is reached, local stable conditions are achieved when the condition \(\psi_1(x^*, y^*) < 0\) and \(\psi_2(x^*, y^*) > 0\), and stable global conditions are reached when \(a_{11} + a_{22} < 0\) and \(a_{11}a_{22} - (a_{12})^2 > 0\). The optimal condition of the product price and the salesman's initiative in maximizing the net profit from the inventory model of two similar products will be achieved, if they satisfy (11), (12), (28), 29, and (30).

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