COMPLETE GRAPH MINORS AND
THE GRAPH MINOR STRUCTURE THEOREM

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ABSTRACT. The graph minor structure theorem by Robertson and Seymour shows that every graph that excludes a fixed minor can be constructed by a combination of four ingredients: graphs embedded in a surface of bounded genus, a bounded number of vortices of bounded width, a bounded number of apex vertices, and the clique-sum operation. This paper studies the converse question: What is the maximum order of a complete graph minor in a graph constructed using these four ingredients? Our main result answers this question up to a constant factor.

1. INTRODUCTION

Robertson and Seymour [8] proved a rough structural characterization of graphs that exclude a fixed minor. It says that such a graph can be constructed by a combination of four ingredients: graphs embedded in a surface of bounded genus, a bounded number of vortices of bounded width, a bounded number of apex vertices, and the clique-sum operation. Moreover, each of these ingredients is essential.

In this paper, we consider the converse question: What is the maximum order of a complete graph minor in a graph constructed using these four ingredients? Our main result answers this question up to a constant factor.

To state this theorem, we now introduce some notation; see Section 2 for precise definitions. For a graph $G$, let $\eta(G)$ denote the maximum integer $n$ such that the complete graph $K_n$ is a minor of $G$, sometimes called the Hadwiger number of $G$. For integers $g, p, k \geq 0$, let $\mathcal{G}(g, p, k)$ be the set of graphs obtained by adding at most $p$ vortices, each with width at most $k$, to a graph embedded in a surface of Euler genus at most $g$. For an integer $a \geq 0$, let $\mathcal{G}(g, p, k, a)$ be the set of graphs $G$ such that $G \setminus A \in \mathcal{G}(g, p, k)$ for some set $A \subseteq V(G)$ with $|A| \leq a$. The vertices in $A$ are called apex vertices. Let $\mathcal{G}(g, p, k, a)^+$ be the set of graphs obtained from clique-sums of graphs in $\mathcal{G}(g, p, k, a)$.

The graph minor structure theorem of Robertson and Seymour [8] says that for every integer $t \geq 1$, there exist integers $g, p, k, a \geq 0$, such that every graph $G$ with $\eta(G) \leq t$ is in $\mathcal{G}(g, p, k, a)^+$. We prove the following converse result.

**Theorem 1.1.** For some constant $c > 0$, for all integers $g, p, k, a \geq 0$, for every graph $G$ in $\mathcal{G}(g, p, k, a)^+$,

$$\eta(G) \leq a + c(k + 1)\sqrt{g + p} + c.$$
Moreover, for some constant $c' > 0$, for all integers $g, a \geq 0$ and $p \geq 1$ and $k \geq 2$, there is a graph $G$ in $\mathcal{G}(g, p, k, a)$ such that

$$\eta(G) \geq a + c' k \sqrt{g + p}.$$  

Let $RS(G)$ be the minimum integer $k$ such that $G$ is a subgraph of a graph in $\mathcal{G}(k, k, k, k)^+$. The graph minor structure theorem [8] says that $RS(G) \leq f(\eta(G))$ for some function $f$ independent of $G$. Conversely, Theorem 1.1 implies that $\eta(G) \leq f'(RS(G))$ for some (much smaller) function $f'$. In this sense, $\eta$ and $RS$ are “tied”. Note that such a function $f'$ is widely understood to exist (see for instance Diestel [2, p. 340] and Lovász [5]). However, the authors are not aware of any proof. In addition to proving the existence of $f'$, this paper determines the best possible function $f'$ (up to a constant factor).

Following the presentation of definitions and other preliminary results in Section 2, the proof of the upper and lower bounds in Theorem 1.1 are respectively presented in Sections 3 and 4.

2. Definitions and Preliminaries

All graphs in this paper are finite and simple, unless otherwise stated. Let $V(G)$ and $E(G)$ denote the vertex and edge sets of a graph $G$. For background graph theory see [2].

A graph $H$ is a minor of a graph $G$ if $H$ can be obtained from a subgraph of $G$ by contracting edges. (Note that, since we only consider simple graphs, loops and parallel edges created during an edge contraction are deleted.) An $H$-model in $G$ is a collection $\{S_x : x \in V(H)\}$ of pairwise vertex-disjoint connected subgraphs of $G$ (called branch sets) such that, for every edge $xy \in E(H)$, some edge in $G$ joins a vertex in $S_x$ to a vertex in $S_y$. Clearly, $H$ is a minor of $G$ if and only if $G$ contains an $H$-model. For a recent survey on graph minors see [4].

Let $G[k]$ denote the lexicographic product of $G$ with $K_k$, namely the graph obtained by replacing each vertex $v$ of $G$ with a clique $C_v$ of size $k$, where for each edge $vw \in E(G)$, each vertex in $C_v$ is adjacent to each vertex in $C_w$. Let $tw(G)$ be the treewidth of a graph $G$; see [2] for background on treewidth.

Lemma 2.1. For every graph $G$ and integer $k \geq 1$, every minor of $G[k]$ has minimum degree at most $k \cdot tw(G) + k - 1$.

Proof. A tree decomposition of $G$ can be turned into a tree decomposition of $G[k]$ in the obvious way: in each bag, replace each vertex by its $k$ copies in $G[k]$. The size of each bag is multiplied by $k$; hence the new tree decomposition has width at most $k(w + 1) - 1 = kw + k - 1$, where $w$ denotes the width of the original decomposition. Let $H$ be a minor of $G[k]$. Since treewidth is minor-monotone,

$$tw(H) \leq tw(G[k]) \leq k \cdot tw(G) + k - 1.$$  

The claim follows since the minimum degree of a graph is at most its treewidth. \qed

Note that Lemma 2.1 can be written in terms of contraction degeneracy; see [1, 3].

Let $G$ be a graph and let $\Omega = (v_1, v_2, \ldots, v_l)$ be a circular ordering of a subset of the vertices of $G$. We write $V(\Omega)$ for the set $\{v_1, v_2, \ldots, v_l\}$. A circular decomposition of $G$ with perimeter $\Omega$ is a multiset $\{C\langle w \rangle \subseteq V(G) : w \in V(\Omega)\}$ of subsets of vertices of $G$, called bags, that satisfy the following properties:
• every vertex \( w \in V(\Omega) \) is contained in its corresponding bag \( C\langle w \rangle \);
• for every vertex \( u \in V(G) \setminus V(\Omega) \), there exists \( w \in V(\Omega) \) such that \( u \) is in \( C\langle w \rangle \);
• for every edge \( e \in E(G) \), there exists \( w \in V(\Omega) \) such that both endpoints of \( e \) are in \( C\langle w \rangle \), and
• for each vertex \( u \in V(G) \), if \( u \in C\langle v_i \rangle, C\langle v_j \rangle \) with \( i < j \) then \( u \in C\langle v_i+1 \rangle, \ldots, C\langle v_j-1 \rangle \) or \( u \in C\langle v_j+1 \rangle, \ldots, C\langle v_k \rangle, C\langle v_1 \rangle, \ldots, C\langle v_{i-1} \rangle \).

(The last condition says that the bags in which \( u \) appears correspond to consecutive vertices of \( \Omega \).) The width of the decomposition is the maximum cardinality of a bag minus 1. The ordered pair \((G, \Omega)\) is called a vertex; its width is the minimum width of a circular decomposition of \( G \) with perimeter \( \Omega \).

A surface is a non-null compact connected 2-manifold without boundary. Recall that the Euler genus of a surface \( \Sigma \) is \( 2 - \chi(\Sigma) \), where \( \chi(\Sigma) \) denotes the Euler characteristic of \( \Sigma \). Thus the orientable surface with \( h \) handles has Euler genus \( 2h \), and the non-orientable surface with \( c \) cross-caps has Euler genus \( c \). The boundary of an open disc \( D \subset \Sigma \) is denoted by \( \partial D \).

See [6] for basic terminology and results about graphs embedded in surfaces. When considering a graph \( G \) embedded in a surface \( \Sigma \), we use \( G \) both for the corresponding abstract graph and for the subset of \( \Sigma \) corresponding to the drawing of \( G \). An embedding of \( G \) in \( \Sigma \) is 2-cell if every face is homeomorphic to an open disc.

Recall Euler’s formula: if an \( n \)-vertex \( m \)-edge graph is 2-cell embedded with \( f \) faces in a surface of Euler genus \( g \), then \( n - m + f = 2 - g \). Since \( 2m \geq 3f \),

\[
1 \leq 3n + 3g - 6
\]

which in turn implies the following well-known upper bound on the Hadwiger number.

**Lemma 2.2.** If a graph \( G \) has an embedding in a surface \( \Sigma \) with Euler genus \( g \), then

\[
\eta(G) \leq \sqrt{6g + 4}.
\]

**Proof.** Let \( t := \eta(G) \). Then \( K_t \) has an embedding in \( \Sigma \). It is well-known that this implies that \( K_t \) has a 2-cell embedding in a surface of Euler genus at most \( g \) (see [6]). Hence \( \binom{t}{2} \leq 3t + 3g - 6 \) by (1). In particular, \( t \leq \sqrt{6g + 4} \). \( \square \)

Let \( G \) be an embedded multigraph, and let \( F \) be a facial walk of \( G \). Let \( v \) be a vertex of \( F \) with degree more than 3. Let \( e_1, \ldots, e_d \) be the edges incident to \( v \) in clockwise order around \( v \), such that \( e_1 \) and \( e_d \) are in \( F \). Let \( G' \) be the embedded multigraph obtained from \( G \) as follows. First, introduce a path \( x_1, \ldots, x_d \) of new vertices. Then for each \( i \in [1, d] \), replace \( v \) as the endpoint of \( e_i \) by \( x_i \). The clockwise ordering around \( x_i \) is as described in Figure 1. Finally delete \( v \). We say that \( G' \) is obtained from \( G \) by splitting \( v \) at \( F \). Each vertex \( x_i \) is said to belong to \( v \). By construction, \( x_i \) has degree at most 3. Observe that there is a one-to-one correspondence between facial walks of \( G \) and \( G' \). This process can be repeated at each vertex of \( F \). The embedded graph that is obtained is called the splitting of \( G \) at \( F \). And more generally, if \( F_1, \ldots, F_p \) are pairwise vertex-disjoint facial walks of \( G \), then the embedded graph that is obtained by splitting each \( F_i \) is called the splitting of \( G \) at \( F_1, \ldots, F_p \). (Clearly, the splitting of \( G \) at \( F_1, \ldots, F_p \) is unique.)

For \( g, p, k \geq 0 \), a graph \( G \) is \((g, p, k)\)-almost embeddable if there exists a graph \( G_0 \) embedded in a surface \( \Sigma \) of Euler genus at most \( g \), and there exist \( q \leq p \) vortices \((G_1, \Omega_1), \ldots, (G_q, \Omega_q)\), each of width at most \( k \), such that
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**Figure 1.** Splitting a vertex $v$ at a face $F$.

- $G = G_0 \cup G_1 \cup \cdots \cup G_q$;
- the graphs $G_1, \ldots, G_q$ are pairwise vertex-disjoint;
- $V(G_i) \cap V(G_0) = V(\Omega_i)$ for all $i \in [1, q]$, and
- there exist $q$ disjoint closed discs in $\Sigma$ whose interiors $D_1, \ldots, D_q$ are disjoint from $G_0$, whose boundaries meet $G_0$ only in vertices, and such that $\text{bd}(D_i) \cap V(G_0) = V(\Omega_i)$ and the cyclic ordering $\Omega_i$ is compatible with the natural cyclic ordering of $V(\Omega_i)$ induced by $\text{bd}(D_i)$, for all $i \in [1, q]$.

Let $\mathcal{G}(g, p, k)$ be the set of $(g, p, k)$-almost embeddable graphs. Note that $\mathcal{G}(g, 0, 0)$ is exactly the class of graphs with Euler genus at most $g$. Also note that the literature defines a graph to be $h$-almost embeddable if it is $(h, h, h)$-almost embeddable. To enable more accurate results we distinguish the three parameters.

Let $G_1$ and $G_2$ be disjoint graphs. Let $\{v_1, \ldots, v_k\} \cup \{w_1, \ldots, w_k\}$ be cliques of the same cardinality in $G_1$ and $G_2$ respectively. A **clique-sum** of $G_1$ and $G_2$ is any graph obtained from $G_1 \cup G_2$ by identifying $v_i$ with $w_i$ for each $i \in [1, k]$, and possibly deleting some of the edges $v_iv_j$.

The above definitions make precise the definition of $\mathcal{G}(g, p, k, a)^+$ given in the introduction. We conclude this section with an easy lemma on clique-sums.

**Lemma 2.3.** If a graph $G$ is a clique-sum of graphs $G_1$ and $G_2$, then

$$\eta(G) \leq \max\{\eta(G_1), \eta(G_2)\}.$$  

**Proof.** Let $t := \eta(G)$ and let $S_1, \ldots, S_t$ be the branch sets of a $K_t$-model in $G$. If some branch set $S_i$ were contained in $G_1 \setminus V(G_2)$, and some branch set $S_j$ were contained in $G_2 \setminus V(G_1)$, then there would be no edge between $S_i$ and $S_j$ in $G$, which is a contradiction. Thus every branch set intersects $V(G_1)$, or every branch set intersects $V(G_2)$. Suppose that every branch set intersects $V(G_1)$. For each branch set $S_i$ that intersects $G_1 \cap G_2$ remove from $S_i$ all vertices in $V(G_2) \setminus V(G_1)$. Since $V(G_1) \cap V(G_2)$ is a clique in $G_1$, the modified branch sets yield a $K_t$-model in $G_1$. Hence $t \leq \eta(G_1)$. By symmetry, $t \leq \eta(G_2)$ in the case that every branch set intersects $G_2$. Therefore $\eta(G) \leq \max\{\eta(G_1), \eta(G_2)\}$. \hfill $\square$

### 3. Proof of Upper Bound

The aim of this section is to prove the following theorem.

**Theorem 3.1.** For all integers $g, p, k \geq 0$, every graph $G$ in $\mathcal{G}(g, p, k)$ satisfies

$$\eta(G) \leq 48(k + 1)\sqrt{g + p} + \sqrt{6g} + 5.$$
Combining this theorem with Lemma 2.3 gives the following quantitative version of the first part of Theorem 1.1.

**Corollary 3.2.** For every graph \( G \in \mathcal{G}(g,p,k,a)^+ \),
\[
\eta(G) \leq a + 48(k+1)\sqrt{g+p} + \sqrt{6g} + 5.
\]

**Proof.** Let \( G \in \mathcal{G}(g,p,k,a)^+ \). Lemma 2.3 implies that \( \eta(G) \leq \eta(G') \) for some graph \( G' \in \mathcal{G}(g,p,k,a) \). Clearly, \( \eta(G') \leq \eta(G' \setminus A) + a \), where \( A \) denotes the (possibly empty) apex set of \( G' \). Since \( G' \setminus A \in \mathcal{G}(g,p,k) \), the claim follows from Theorem 3.1.

The proof of Theorem 3.1 uses the following definitions. Two subgraphs \( A \) and \( B \) of a graph \( G \) touch if \( A \) and \( B \) have at least one vertex in common or if there is an edge in \( G \) between a vertex in \( A \) and another vertex in \( B \). We generalize the notion of minors and models as follows. For an integer \( k \geq 1 \), a graph \( H \) is said to be an \((H,k)\)-minor of a graph \( G \) if there exists a collection \( \{S_x : x \in V(H)\} \) of connected subgraphs of \( G \) (called branch sets), such that \( S_x \) and \( S_y \) touch in \( G \) for every edge \( xy \in E(H) \), and every vertex of \( G \) is included in at most \( k \) branch sets in the collection. The collection \( \{S_x : x \in V(H)\} \) is called an \((H,k)\)-model in \( G \). Note that for \( k = 1 \) this definition corresponds to the usual notions of \( H \)-minor and \( H \)-model. As shown in the next lemma, this generalization provides another way of considering \( H \)-minors in \( G[k] \), the lexicographic product of \( G \) with \( K_k \). (The easy proof is left to the reader.)

**Lemma 3.3.** Let \( k \geq 1 \). A graph \( H \) is an \((H,k)\)-minor of a graph \( G \) if and only if \( H \) is a minor of \( G[k] \).

For a surface \( \Sigma \), let \( \Sigma_c \) be \( \Sigma \) with \( c \) cuffs added; that is, \( \Sigma_c \) is obtained from \( \Sigma \) by removing the interior of \( c \) pairwise disjoint closed discs. (It is well-known that the locations of the discs are irrelevant.) When considering graphs embedded in \( \Sigma_c \) we require the embedding to be 2-cell. We emphasize that this is a non-standard and relatively strong requirement; in particular, it implies that the graph is connected, and the boundary of each cuff intersects the graph in a cycle. Such cycles are called cuff-cycles.

For \( g \geq 0 \) and \( c \geq 1 \), a graph \( G \) is \((g,c)\)-embedded if \( G \) has maximum degree \( \Delta(G) \leq 3 \) and \( G \) is embedded in a surface of Euler genus at most \( g \) with at most \( c \) cuffs added, such that every vertex of \( G \) lies on the boundary of the surface. (Thus the cuff-cycles induce a partition of the whole vertex set.) The graph \( G \) is \((g,c)\)-embeddable if there exists such an embedding. Note that if \( C \) is a contractible cycle in a \((g,c)\)-embedded graph, then the closed disc bounded by \( C \) is uniquely determined even if the underlying surface is the sphere (because there is at least one cuff).

**Lemma 3.4.** For every graph \( G \in \mathcal{G}(g,p,k) \) there exists a \((g,p)\)-embeddable graph \( H \) with \( \eta(G) \leq \eta(H[k+1]) + \sqrt{6g} + 4 \).

**Proof.** Let \( t := \eta(G) \). Let \( S_1, \ldots, S_t \) be the branch sets of a \( K_t \)-model in \( G \). Since \( \eta(G) \) equals the Hadwiger number of some connected component of \( G \), we may assume that \( G \) is connected. Thus we may ‘grow’ the branch sets until \( V(S_1) \cup \cdots \cup V(S_t) = V(G) \).

Write \( G = G_0 \cup G_1 \cup \cdots \cup G_q \) as in the definition of \((g,p,k)\)-almost embeddable graphs. Thus \( G_0 \) is embedded in a surface \( \Sigma \) of Euler genus at most \( g \), and \((G_1, \Omega_1), \ldots, (G_q, \Omega_q)\) are pairwise vertex-disjoint vortices of width at most \( k \), for some \( q \leq p \). Let \( D_1, \ldots, D_q \) be the proper interiors of the closed discs of \( \Sigma \) appearing in the definition.
Define $r$ and reorder the branch sets, so that each $S_i$ contains a vertex of some vortex if and only if $i \leq r$. If $t > r$, then $S_{t+1}, \ldots, S_t$ is a $K_{t-r}$-model in the embedded graph $G_0$, and hence $t - r \leq \sqrt{6g} + 4$ by Lemma 2.2. Therefore, it suffices to show that $r \leq \eta(H[k+1])$ for some $(g,p)$-embeddable graph $H$.

Modify $G$, $G_0$, and the branch sets $S_1, \ldots, S_r$ as follows. First, remove from $G$ and $G_0$ every vertex of $S_i$ for all $i \in [r+1,t]$. Next, while some branch set $S_i$ ($i \in [1,r]$) contains an edge $uv$ in $G_0$ where $u$ is in some vertex, but $v$ is in no vertex, contract the edge $uv$ into $u$ (this operation is done in $S_i$, $G$, and $G_0$). The above operations on $G_0$ are carried out in its embedding in the natural way. Now apply a final operation on $G$ and $G_0$: for each $j \in [1,q]$ and each pair of consecutive vertices $a$ and $b$ in $\Omega_j$, remove the edge $ab$ if it exists, and embed the edge $ab$ as a curve on the boundary of $D_j$.

When the above procedure is finished, every vertex of the modified $G_0$ belongs to some vertex. It should be clear that the modified branch sets $S_1, \ldots, S_r$ still provide a model of $K_r$ in $G$. Also observe that $G_0$ is connected; this is because $V(\Omega_j)$ induces a connected subgraph for each $j \in [1,q]$, and each vortex intersects at least one branch set $S_i$ with $i \in [1,r]$. By the final operation, the boundary of the disc $D_j$ of $\Sigma$ intersects $G_0$ in a cycle $C_j$ of $G_0$ with $V(C_j) = V(\Omega_j)$ and such that $C_j$ (with the right orientation) defines the same cyclic ordering as $\Omega_j$ for every $j \in [1,q]$.

We claim that $G_0$ can be a 2-cell embedded in a surface $\Sigma'$ with Euler genus at most that of $\Sigma$, such that each $C_j$ ($j \in [1,q]$) is a facial cycle of the embedding. This follows by considering the combinatorial embedding (that is, circular ordering of edges incident to each vertex, and edge signatures) determined by the embedding in $\Sigma$ (see [6]), and observing that under the above operations, the Euler genus of the combinatorial embedding does not increase, and facial walks remain facial walks (so that each $C_j$ is a facial cycle). Now, removing the $q$ open discs corresponding to these facial cycles gives a 2-cell embedding of $G_0$ in $\Sigma_q'$.

We now prove that $\eta(G_0[k+1]) \geq r$. For every $i \in [1,q]$, let \{ $C(w) \subseteq V(G_i) : w \in V(\Omega_i)$ \} denote a circular decomposition of width at most $k$ of the $i$-th vortex. For each $i \in [1,r]$, mark the vertices $w$ of $G_0$ for which $S_i$ contains at least one vertex in the bag $C(w)$ (recall that every vertex of $G_0$ is in the perimeter of some vertex), and define $S'_i$ as the subgraph of $G_0$ induced by the marked vertices. It is easily checked that $S'_i$ is a connected subgraph of $G_0$. Also, $S'_i$ and $S'_j$ touch in $G_0$ for all $i \neq j$. Finally, a vertex of $G_0$ will be marked at most $k+1$ times, since each bag has size at most $k+1$. It follows that $\{S'_1, \ldots, S'_r\}$ is a $(K_r, k+1)$-model in $G_0$, which implies by Lemma 3.3 that $K_r$ is minor of $G_0\{k+1\}$, as claimed.

Finally, let $H$ be obtained from $G_0$ by splitting each vertex $v$ of degree more than 3 along the cuff boundary that contains $v$. (Clearly the notion of splitting along a face extends to splitting along a cuff.) By construction, $\Delta(H) \leq 3$ and $H$ is $(g,q)$-embedded. The $(K_r, k+1)$-model of $G_0$ constructed above can be turned into a $(K_r, k+1)$-model of $H$ by replacing each branch set $S'_i$ by the union, taken over the vertices $v \in V(S'_i)$, of the set of vertices in $H$ that belong to $v$. Hence $r \leq \eta(G_0[k+1]) \leq \eta(H[k+1])$. $\square$

We need to introduce a few definitions. Consider a $(g,c)$-embedded graph $G$. An edge $e$ of $G$ is said to be a cuff or a non-cuff edge, depending on whether $e$ is included in a cuff-cycle. Every non-cuff edge has its two endpoints in either the same cuff-cycle or in two distinct cuff-cycles. Since $\Delta(G) \leq 3$, the set of non-cuff edges is a matching.
Lemma 3.5. Let $G$ be a $(g,c)$-embedded graph, and let $e_1, e_2, e_3$ be distinct noncontractible non-cuff edges of $G$, such that $e_1$ is homotopic to $e_2$ and to $e_3$. Then $e_2$ and $e_3$ are also homotopic. Moreover, given a contractible $\{e_1, e_2\}$-cycle $C_{12}$ bounding a closed disc $D_{12}$, for some distinct $i, j \in \{1, 2, 3\}$, there is a contractible $\{e_i, e_j\}$-cycle bounding a closed disc containing $e_1, e_2, e_3$ and all noncontractible non-cuff edges of $G$ contained in $D_{12}$.

Proof. Let $C_{13}$ be a contractible $\{e_1, e_3\}$-cycle. Let $P_{12}, Q_{12}$ be the two paths in the graph $C_{12} \setminus \{e_1, e_2\}$. Let $P_{13}, Q_{13}$ be the two paths in the graph $C_{13} \setminus \{e_1, e_3\}$. Exchanging $P_{13}$ and $Q_{13}$ if necessary, we may denote the endpoints of $e_i$ ($i = 1, 2, 3$) by $u_i, v_i$ so that the endpoints of $P_{12}$ and $P_{13}$ are $u_1, u_2$ and $u_3, u_4$, respectively, and similarly, the endpoints of $Q_{12}$ and $Q_{13}$ are $v_1, v_2$ and $v_3, v_4$, respectively.

Let $D_{13}$ be the closed disc bounded by $C_{13}$. Each edge of $P_{1i}$ and $Q_{1i}$ ($i = 2, 3$) is on the boundaries of both $D_{1i}$ and a cuff; it follows that every non-cuff edge of $G$ incident to an internal vertex of $P_{1i}$ or $Q_{1i}$ is entirely contained in the disc $D_{1i}$. The paths $P_{12}$ and $P_{13}$ are subgraphs of a common cuff-cycle $C_P$, and $Q_{12}$ and $Q_{13}$ are subgraphs of a common cuff-cycle $C_Q$. Note that these two cuff-cycles could be the same.

Recall that non-cuff edges of $G$ are independent (that is, have no endpoint in common). This will be used in the arguments below. We claim that

\[ \text{every noncontractible non-cuff edge } f \text{ contained in } D_{1i} \text{ has one endpoint in } P_{1i} \text{ and the other in } Q_{1i}, \text{ for each } i \in \{2, 3\}. \]

The claim is immediate if $f \in \{e_1, e_i\}$. Now assume that $f \not\in \{e_1, e_i\}$. The edge $f$ is incident to at least one of $P_{1i}$ and $Q_{1i}$ since there is no vertex in the proper interior of $D_{1i}$. Without loss of generality, $f$ is incident to $P_{1i}$. The edge $f$ can only be incident to internal vertices of $P_{1i}$, since $f$ is independent of $e_1$ and $e_i$. Say $f = xy$. If $x, y \in V(P_{1i})$ then the $\{f\}$-cycle obtained by combining the $x-y$ subpath of $P_{1i}$ with the edge $f$ is contained in $D_{1i}$ and thus is contractible. Hence $f$ is a contractible non-cuff edge, a contradiction. This proves (2).
First we prove the lemma in the case where $e_3$ is incident to $P_{12}$. Since $e_3$ is incident to an internal vertex of $P_{12}$, it follows that $e_3$ is contained in $D_{12}$. This shows the second part of the lemma. To show that $e_2$ and $e_3$ are homotopic, consider the endpoint $v_3$ of $e_3$. Since $e_3$ is in $D_{12}$ and $u_3 \in V(P_{12})$, we have $v_3 \in V(Q_{12})$ by (2). Now, combining the $u_2$–$u_3$ subpath of $P_{12}$ and the $v_2$–$v_3$ subpath of $Q_{12}$ with $e_2$ and $e_3$, we obtain an $\{e_2, e_3\}$-cycle contained in $D_{12}$, which is thus contractible. This shows that $e_2$ and $e_3$ are homotopic.

By symmetry, the above argument also handles the case where $e_3$ is incident to $Q_{12}$. Thus we may assume that $e_3$ is incident to neither $P_{12}$ nor $Q_{12}$.

Suppose $P_{12} \subseteq P_{13}$. Then, by (2), all noncontractible non-cuff edges contained in $D_{12}$ are incident to $P_{12}$, and thus also to $P_{13}$. Hence they are all contained in the disc $D_{13}$. Moreover, a contractible $\{e_2, e_3\}$-cycle can be found in the obvious way. Therefore the lemma holds in this case. Using symmetry, the same argument can be used if $P_{12} \subseteq Q_{13}$, $Q_{12} \subseteq P_{13}$, or $Q_{12} \subseteq Q_{13}$. Thus we may assume

$$P_{12} \not\subseteq P_{13}; \quad P_{12} \not\subseteq Q_{13}; \quad Q_{12} \not\subseteq P_{13}; \quad Q_{12} \not\subseteq Q_{13}. \quad (3)$$

Next consider $P_{12}$ and $P_{13}$. If we orient these paths starting at $u_1$, then they either go in the same direction around $C_P$, or in opposite directions. Suppose the former. Then one path is a subpath of the other. Since by our assumption $u_3$ is not in $P_{12}$, we have $P_{12} \subseteq P_{13}$, which contradicts (3). Hence the paths $P_{12}$ and $P_{13}$ go in opposite directions around $C_P$. If $V(P_{12}) \cap V(P_{13}) \neq \{u_1\}$, then $u_3$ is an internal vertex of $P_{12}$, which contradicts our assumption on $e_3$. Hence

$$V(P_{12}) \cap V(P_{13}) = \{u_1\}. \quad (4)$$

By symmetry, the above argument show that $Q_{12}$ and $Q_{13}$ go in opposite directions around $C_Q$ (starting from $v_1$), which similarly implies

$$V(Q_{12}) \cap V(Q_{13}) = \{v_1\}. \quad (5)$$

Now consider $P_{12}$ and $Q_{13}$. These two paths do not share any endpoint. If $C_P \neq C_Q$ then obviously the two paths are vertex-disjoint. If $C_P = C_Q$ and $V(P_{12}) \cap V(Q_{13}) \neq \emptyset$, then at least one of $v_1$ and $v_3$ is an internal vertex of $P_{12}$, because otherwise $P_{12} \subseteq Q_{13}$, which contradicts (3). However $v_1 \notin V(P_{12})$ since $v_1 \in V(Q_{12})$, and $v_3 \notin V(P_{12})$ by our assumption that $e_3$ is not incident to $P_{12}$. Hence, in all cases,

$$V(P_{12}) \cap V(Q_{13}) = \emptyset. \quad (6)$$

By symmetry,

$$V(Q_{12}) \cap V(P_{13}) = \emptyset. \quad (7)$$

It follows from (4)–(7) that $C_{12}$ and $C_{13}$ only have $e_1$ in common. This implies in turn that $D_{12}$ and $D_{13}$ have disjoint proper interiors. Thus the cycle $C_{23} := (C_{12} \cup C_{13}) - e_1$ bounds the disc obtained by gluing $D_{12}$ and $D_{13}$ along $e_1$. Hence $C_{23}$ is an $\{e_2, e_3\}$-cycle of $G$ bounding a disc containing $e_3$ and all edges contained in $D_{12}$. This concludes the proof. \( \square \)

The next lemma is a direct consequence of Lemma 3.5. An equivalence class $Q$ for the homotopy relation on the noncontractible non-cuff edges of $G$ is trivial if $|Q| = 1$, and non-trivial otherwise.
Lemma 3.6. Let \( G \) be a \((g, c)\)-embedded graph and let \( Q \) be a non-trivial equivalence class of the noncontractible non-cuff edges of \( G \). Then there are distinct edges \( e, f \in Q \) and a contractible \( \{e, f\} \)-cycle \( C \) of \( G \), such that the closed disc bounded by \( C \) contains every edge in \( Q \).

Our main tool in proving Theorem 3.1 is the following lemma, whose inductive proof is enabled by the following definition. Let \( G \) be a \((g, c)\)-embedded graph and let \( k \geq 1 \). A graph \( H \) is a \( k \)-minor of \( G \) if there exists an \((H, 4k)\)-model \( \{S_x : x \in V(H)\} \) in \( G \) such that, for every vertex \( u \in V(G) \) incident to a noncontractible non-cuff edge in a non-trivial equivalence class, the number of subgraphs in the model including \( u \) is at most \( k \). Such a collection \( \{S_x : x \in V(H)\} \) is said to be a \( k \)-model of \( H \) in \( G \). This provides a relaxation of the notion of \((H, k)\)-minor since some vertices of \( G \) could appear in up to \( 4k \) branch sets (instead of \( k \)). We emphasize that this definition depends heavily on the embedding of \( G \).

Lemma 3.7. Let \( G \) be a \((g, c)\)-embedded graph and let \( k \geq 1 \). Then every \( k \)-minor \( H \) of \( G \) has minimum degree at most \( 48k\sqrt{c + g} \).

Proof. Let \( q(G) \) be the number of non-trivial equivalence classes of noncontractible non-cuff edges in \( G \). We proceed by induction, firstly on \( g + c \), then on \( q(G) \), and then on \( |V(G)| \). Now \( G \) is embedded in a surface of Euler genus \( g' \leq g \) with \( c' \leq c \) cuffs added. If \( g' < g \) or \( c' < c \) then we are done by induction. Now assume that \( g' = g \) and \( c' = c \).

We repeatedly use the following observation: If \( C \) is a contractible cycle of \( G \), then the subgraph of \( G \) consisting of the vertices and edges contained in the closed disc \( D \) bounded by \( C \) is contractible, and thus has treewidth at most 2. This is because the proper interior of \( D \) contains no vertex of \( G \) (since all the vertices in \( G \) are on the cuff boundaries).

Let \( \{S_x : x \in V(H)\} \) be a \( k \)-model of \( H \) in \( G \). Let \( d \) be the minimum degree of \( H \). We may assume that \( d \geq 20k \), as otherwise \( d \leq 48k\sqrt{c + g} \) (since \( c \geq 1 \)) and we are done. Also, it is enough to prove the lemma when \( H \) is connected, so assume this is the case.

Case 1: Some non-cuff edge \( e \) of \( G \) is contractible. Let \( C \) be a contractible \( \{e\} \)-cycle. Let \( u, v \) be the endpoints of \( e \). Remove from \( G \) every vertex in \( V(G) \setminus \{u, v\} \) and modify the embedding of \( G \) by redrawing the edge \( e \) where the path \( C - e \) was. Thus \( e \) becomes a cuff-edge in the resulting graph \( G' \), and \( u \) and \( v \) both have degree 2. Also observe that \( G' \) is connected and remains simple (that is, this operation does not create loops or parallel edges). Since the embedding of \( G' \) is 2-cell, \( G' \) is \((g, c)\)-embedded also.

If \( e_1 \) and \( e_2 \) are noncontractible non-cuff edges of \( G' \) that are homotopic in \( G' \), then \( e_1 \) and \( e_2 \) were also noncontractible and homotopic in \( G \). Hence, \( q(G') \leq q(G) \). Also, \( |V(G')| < |V(G)| \) since we removed at least one vertex from \( G \). Thus, by induction, every \( k \)-minor of \( G' \) has minimum degree at most \( 48k\sqrt{c + g} \). Therefore, it is enough to show that \( H \) is a \( k \)-minor of \( G' \).

Let \( G_1 \) be the subgraph of \( G \) lying in the closed disc bounded by \( C \); observe that \( G_1 \) is outerplanar. Moreover, \((G_1, G')\) is a separation of \( G \) with \( V(G_1) \cap V(G') = \{u, v\} \). (That is, \( G_1 \cup G' = G \) and \( V(G_1) \setminus V(G') \neq \emptyset \) and \( V(G') \setminus V(G_1) \neq \emptyset \).)

First suppose that \( S_x \subseteq G_1 \setminus \{u, v\} \) for some vertex \( x \in V(H) \). Let \( H' \) be the subgraph of \( H \) induced by the set of such vertices \( x \). In \( H \), the only neighbors of a vertex \( x \in V(H') \) that are not in \( H' \) are vertices \( y \) such that \( S_y \) includes at least one of
$u, v$. There are at most $2 \cdot 4k = 8k$ such branch sets $S_i$. Hence, $H'$ has minimum degree at least $d - 8k \geq 12k$. However, $H'$ is a minor of $G_1[4k]$ and hence has minimum degree at most $4k \cdot \text{tw}(G_1) + 4k - 1 \leq 12k - 1$ by Lemma 2.1, a contradiction.

It follows that every branch set $S_x (x \in V(H))$ contains at least one vertex in $V(G')$. Let $S'_x := S_i \cap G'$. Using the fact that $uv \in E(G')$, it is easily seen that the collection $\{S'_x : x \in V(H)\}$ is a $k$-model of $H$ in $G'$.

**Case 2: Some equivalence class $Q$ is non-trivial.** By Lemma 3.6, there are two edges $e, f \in Q$ and a contractible $\{e, f\}$-cycle $C$ such that every edge in $Q$ is contained in the disc bounded by $C$. Let $P_1, P_2$ be the two components of $C \setminus \{e, f\}$. These two paths either belong to the same cuff-cycle or to two distinct cuff-cycles of $G$.

Our aim is to eventually contract each of $P_1, P_2$ into a single vertex. However, before doing so we slightly modify $G$ as follows. For each cuff-cycle $C'$ intersecting $C$, select an arbitrary edge in $E(C') \setminus E(C)$ and subdivide it twice. Let $G'$ be the resulting $(g, c)$-embedded graph. Clearly $q(G') = q(G)$, and there is an obvious $k$-model $\{S'_x : x \in V(H)\}$ of $H$ in $G'$: simply apply the same subdivision operation on the branch sets $S_x$.

Let $G'_1$ be the subgraph of $G'$ lying in the closed disc $D$ bounded by $C$. Observe that $G'_1$ is outerplanar with outercycle $C$. Suppose that some edge $xy \in E(G'_1) \setminus E(C)$ has both its endpoints in the same path $P_i$, for some $i \in \{1, 2\}$. Then the cycle obtained by combining $xy$ and the $x$-$y$ path in $P_i$ is a contractible cycle of $G'_1$, and its only non-cuff edge is $xy$. The edge $xy$ is thus a contractible edge of $G'$, and hence also of $G$, a contradiction.

It follows that every non-cuff edge included in $G'_1$ has one endpoint in $P_1$ and the other in $P_2$. Hence, every such edge is homotopic to $e$ and therefore belongs to $Q$.

Consider the $k$-model $\{S'_x : x \in V(H)\}$ of $H$ in $G'$ mentioned above. Let $e = uv$ and $f = u'v'$, with $u, v, u', v' \in V(P_1)$ and $v, v' \in V(P_2)$. Let $X := \{u, u', v, v]\}$. For each $w \in X$, the number of branch sets $S'_x$ that include $w$ is at most $k$, since $e$ and $f$ are homotopic noncontractible non-cuff edges.

Let $J := G'_1 \setminus X$. Note that $\text{tw}(J) \leq 2$ since $G'_1$ is outerplanar. Let $Z := \{x \in V(H) : S'_x \subseteq J\}$. First, suppose that $Z \neq \emptyset$. Every vertex of $J$ is in at most $4k$ branch sets $S'_x (x \in Z)$. It follows that the induced subgraph $H[Z]$ is a minor of $J[4k]$. Thus, by Lemma 2.1, $H[Z]$ has a vertex $y$ with degree at most $4k \cdot \text{tw}(J) + 4k - 1 \leq 4k \cdot 2 + 4k - 1 = 12k - 1$. Consider the neighbors of $y$ in $H$. Since $X$ is a cutset of $G'$ separating $V(J)$ from $V(G_1')$, the only neighbors of $y$ in $H$ that are not in $H[Z]$ are vertices $x$ such that $V(S'_x) \cap X \neq \emptyset$. As mentioned before, there are at most $4k$ such vertices; hence, $y$ has degree at most $12k - 1 + 4k = 16k - 1$. However this contradicts the assumption that $H$ has minimum degree $d \geq 20k$. Therefore, we may assume that $Z = \emptyset$; that is, every branch set $S'_x (x \in V(H))$ intersecting $V(G_1')$ contains some vertex in $X$.

Now, remove from $G'$ every edge in $Q$ except $e$, and contract each of $P_1$ and $P_2$ into a single vertex. Ensuring that the contractions are done along the boundary of the relevant cuffs in the embedding. This results in a graph $G''$ which is again $(g, c)$-embedded. Note that $G''$ is guaranteed to be simple, thanks to the edge subdivision operation that was applied to $G$ when defining $G'$.

If a non-cuff edge is contractible in $G''$ then it is also contractible in $G'$, implying all non-cuff edges in $G''$ are noncontractible. Two non-cuff edges of $G''$ are homotopic in $G''$ if and only if they are in $G'$. It follows $q(G'') = q(G'') - 1 = q(G) - 1$, since $e$ is not homotopic to another non-cuff edge in $G''$. By induction, every $k$-minor of $G''$ has
minimum degree at most \(48k\sqrt{c + g}\). Thus, it suffices to show that \(H\) is also a \(k\)-minor of \(G''\).

For \(x \in V(H)\), let \(S''_x\) be obtained from \(S'_x\) by performing the same contraction operation as when defining \(G''\) from \(G'\): every edge in \(Q \setminus \{e\}\) is removed and every edge in \(E(P_1) \cup E(P_2)\) is contracted. Using that every subgraph \(S'_y\) either is disjoint from \(V(G'_1)\) or contains some vertex in \(X\), it can be checked that \(S''_x\) is connected.

Consider an edge \(xy \in E(H)\). We now show that the two subgraphs \(S''_x\) and \(S''_y\) touch in \(G''\). Suppose \(S'_x\) and \(S'_y\) share a common vertex \(w\). If \(w \notin V(G'_1)\), then \(w\) is trivially included in both \(S''_x\) and \(S''_y\). If \(w \in V(G'_1)\), then each of \(S''_x\) and \(S''_y\) contain a vertex from \(X\), and hence either \(u\) or \(v\) is included in both \(S''_x\) and \(S''_y\), or \(w\) is included in one and \(v\) in the other. In the latter case \(uv\) is an edge of \(G''\) joining \(S''_x\) and \(S''_y\). Now assume \(S'_x\) and \(S'_y\) are vertex-disjoint. Thus there is an edge \(ww' \in E(G')\) joining these two subgraphs in \(G'\). Again, if neither \(w\) nor \(w'\) belong to \(V(G'_1)\), then obviously \(ww'\) joins \(S'_x\) and \(S'_y\) in \(G''\). If \(w, w' \in V(G'_1)\), then each of \(S'_x\) and \(S'_y\) contain a vertex from \(X\), and we are done exactly as previously. If exactly one of \(w\) and \(w'\) belongs to \(V(G'_1)\), say \(w\), then \(w \in X\) and \(w'\) is the unique neighbor of \(w\) in \(G'\) outside \(V(G'_1)\). The contraction operation naturally maps \(w\) to a vertex \(m(w) \in \{u, v\}\). The edge \(w'm(w)\) is included in \(G''\) and thus joins \(S''_x\) and \(S''_y\).

In order to conclude that \(\{S''_x : x \in V(H)\}\) is a \(k\)-model of \(H\) in \(G''\), it remains to show that, for every vertex \(w \in V(G'')\), the number of branch sets including \(w\) is at most \(4k\), and is at most \(k\) if \(w\) is incident to a non-cuff edge homotopic to another non-cuff edge. This condition is satisfied if \(w \notin \{u, v\}\), because two non-cuff edges of \(G''\) are homotopic in \(G''\) if and only if they are in \(G'\). Thus assume \(w \in \{u, v\}\). By the definition of \(G''\), the edge \(e = uv\) is not homotopic to another non-cuff edge of \(G''\). Moreover, for each \(z \in X\), there are at most \(k\) branch sets \(S'_z (x \in V(H))\) containing \(z\). Since \(|X| = 4\), it follows that there are at most \(4k\) branch sets \(S''_z (x \in V(H))\) containing \(w\). Therefore, the condition holds also for \(w\), and \(H\) is a \(k\)-minor of \(G''\).

**Case 3:** There is at most one non-cuff edge. Because \(G\) is connected, this implies that \(G\) consists either of a unique cuff-cycle, or of two cuff-cycles joined by a non-cuff edge. In both cases, \(G\) has treewidth exactly 2. Since \(H\) is a minor of \(G'[4k]\), Lemma 2.1 implies that \(H\) has minimum degree at most \(4k \cdot \text{tw}(G) + 4k - 1 = 12k - 1 \leq 48k\sqrt{c + g}\), as desired.

**Case 4:** Some cuff-cycle \(C\) contains three consecutive degree-2 vertices. Let \(u, v, w\) be three such vertices (in order). Note that \(C\) has at least four vertices, as otherwise \(C = G\) and the previous case would apply. It follows \(uw \notin E(G)\). Let \(G'\) be obtained from \(G\) by contracting the edge \(uv\) into the vertex \(u\). In the embedding of \(G'\), the edge \(uv\) is drawn where the path \(uw\) was; thus \(uw\) is a cuff-edge, and \(G'\) is \((g, c)\)-embedded. We have \(q(G') = q(G)\) and \(|V(G')| < |V(G)|\), hence by induction, \(G'\) satisfies the lemma, and it is enough to show that \(H\) is a \(k\)-minor of \(G'\).

Consider the \(k\)-model \(\{S_x : x \in V(H)\}\) of \(H\) in \(G\). If \(V(S_x) = \{v\}\) for some \(x \in V(H)\), then \(x\) has degree at most \(3 \cdot 4k - 1 = 12k - 1\) in \(H\), because \(xy \in E(H)\) implies that \(S_y\) contains at least one of \(u, v, w\). However this contradicts the assumption that \(H\) has minimum degree \(d \geq 20k\). Thus every branch set \(S_x\) that includes \(v\) also contains at least one of \(u, w\) (since \(S_x\) is connected).

For \(x \in V(H)\), let \(S''_x\) be obtained from \(S_x\) as expected: contract the edge \(uv\) if \(uv \in E(S_x)\). Clearly \(S''_x\) is connected. Consider an edge \(xy \in E(H)\). If \(S_x\) and \(S_y\) had a
common vertex then so do \(S'_x\) and \(S'_y\). If \(S_x\) and \(S_y\) were joined by an edge \(e\), then either \(e\) is still in \(G'\) and joins \(S'_x\) and \(S'_y\), or \(e = uv\) and \(u \in V(S'_x), V(S'_y)\). Hence in each case \(S'_x\) and \(S'_y\) touch in \(G'\). Finally, it is clear that \(\{S'_x : x \in V(H)\}\) meets remaining requirements to be a \(k\)-model of \(H\) in \(G'\), since \(V(S'_x) \subseteq V(S_x)\) for every \(x \in V(H)\) and the homotopy properties of the non-cuff edges have not changed. Therefore, \(H\) is a \(k\)-minor of \(G'\).

**Case 5: None of the previous cases apply.** Let \(t\) be the number of non-cuff edges in \(G\) (thus \(t \geq 2\)). Since there are no three consecutive degree-2 vertices, every cuff edge is at distance at most 1 from a non-cuff edge. It follows that

\[
|E(G)| \leq 9t.
\]

(This inequality can be improved but is good enough for our purposes.)

For a facial walk \(F\) of the embedded graph \(G\), let \(nc(F)\) denote the number of occurrences of non-cuff edges in \(F\). (A non-cuff edge that appears twice in \(F\) is counted twice.) We claim that \(nc(F) \geq 3\). Suppose on the contrary that \(nc(F) \leq 2\).

First suppose that \(F\) has no repeated vertex. Thus \(F\) is a cycle. If \(nc(F) = 0\), then \(F\) is a cuff-cycle, every vertex of which is not incident to a non-cuff edge, contradicting the fact that \(G\) is connected with at least two non-cuff edges. If \(nc(F) = 1\) then \(F\) is a contractible cycle that contains exactly one non-cuff edge \(e\). Thus \(e\) is contractible, and Case 1 applies. If \(nc(F) = 2\) then \(F\) is a contractible cycle containing exactly two non-cuff edges \(e\) and \(f\). Thus \(e\) and \(f\) are homotopic. Hence there is a non-trivial equivalence class, and Case 2 applies.

Now assume that \(F\) contains a repeated vertex \(v\). Let

\[
F = (x_1, x_2, \ldots, x_i = v, x_{i+1}, x_{i+2}, \ldots, x_j = v).
\]

All of \(x_1, x_{i-1}, x_{i+1}, x_{j-1}\) are adjacent to \(v\). Since \(x_1 \neq x_{j-1}\) and \(x_{i-1} \neq x_{i+1}\) and \(\deg(v) \leq 3\), we have \(x_{i+1} = x_{j-1} = x_1 = x_{i-1}\). Without loss of generality, \(x_{i+1} = x_{j-1}\). Thus the path \(x_{i-1}vx_{i+1}\) is in the boundary of the cuff-cycle \(C\) that contains \(v\). Moreover, the edge \(vx_{i+1} = vx_{j-1}\) counts twice in \(nc(F)\). Since \(nc(F) \leq 2\), every edge on \(F\) except \(vx_{i+1}\) and \(vx_{j-1}\) is a cuff-edge. Thus every edge in the walk \(v, x_1, x_2, \ldots, x_{i-1}, x_i = v\) is in \(C\), and hence \(v, x_1, x_2, \ldots, x_{i-1}, x_i = v\) is the cycle \(C\). Similarly, \(x_{i+1}, x_{i+2}, \ldots, x_{j-2}, x_{j-1} = x_{i+1}\) is a cycle \(C'\) bounding some other cuff. Hence \(vx_{i+1}\) is the only non-cuff edge incident to \(C\), and the same for \(C'\). Therefore \(G\) consists of two cuff-cycles joined by a non-cuff edge, and Case 3 applies.

Therefore, \(nc(F) \geq 3\), as claimed.

Let \(n := |V(G)|\), \(m := |E(G)|\), and \(f\) be the number of faces of \(G\). It follows from Euler’s formula that

\[
n - m + f + c = 2 - g.
\]

Every non-cuff edge appears exactly twice in faces of \(G\) (either twice in the same face, or once in two distinct faces). Thus

\[
2t = \sum_{F, \text{ face of } G} nc(F) \geq 3f.
\]

Since \(n = m - t\), we deduce from (9) and (10) that

\[
t = f + c + g - 2 \leq \frac{2}{3}t + c + g - 2.
\]
Thus $t \leq 3(c + g)$, and $m \leq 9t \leq 27(c + g)$ by (8). This allows us, in turn, to bound the number of edges in $G[4k]$:

$$|E(G[4k])| = \left(\frac{4k}{3}\right)n + (4k)^2m \leq (4k)^2 \cdot 2m \leq 54(4k)^2(c + g) \leq 2(24k)^2(c + g).$$

Since $H$ is a minor of $G[4k]$, we have $|E(H)| \leq |E(G[4k])|$. Thus the minimum degree $d$ of $H$ can be upper bounded as follows:

$$2|E(H)| \geq d|V(H)| \geq d^2,$$

and hence

$$d \leq \sqrt{2|E(H)|} \leq \sqrt{2|E(G[4k])|} \leq \sqrt{2 \cdot 2(24k)^2(c + g)} = 48k\sqrt{c + g},$$

as desired.

Now we put everything together and prove Theorem 3.1.

**Proof of Theorem 3.1.** Let $G \in \mathcal{G}(g, p, k)$. By Lemma 3.4, there exists a $(g, p)$-embedded graph $G'$ with

$$\eta(G) \leq \eta(G'[k + 1]) + \sqrt{6g + 4}.$$

Let $t := \eta(G'[k + 1])$. Thus $K_t$ is a $(k + 1)$-minor of $G'$ by Lemma 3.3. Lemma 3.7 with $H = K_t$ implies that

$$\eta(G'[k + 1]) - 1 = t - 1 \leq 48(k + 1)\sqrt{g + p}.$$

Hence $\eta(G) \leq 48(k + 1)\sqrt{g + p} + \sqrt{6g + 5}$, as desired. \hfill \Box

4. **CONSTRUCTIONS**

This section describes constructions of graphs in $\mathcal{G}(g, p, k, a)$ that contain large complete graph minors. The following lemma, which in some sense, is converse to Lemma 3.4 will be useful.

**Lemma 4.1.** Let $G$ be a graph embedded in a surface with Euler genus at most $g$. Let $F_1, \ldots, F_p$ be pairwise vertex-disjoint facial cycles of $G$, such that $V(F_1) \cup \cdots \cup V(F_p) = V(G)$. Then for all $k \geq 1$, some graph in $\mathcal{G}(g, p, k)$ contains $G[k]$ as a minor.

**Proof.** Let $G'$ be the embedded multigraph obtained from $G$ by replacing each edge $vw$ of $G$ by $k^2$ edges between $v$ and $w$ bijectively labeled from $\{(i, j) : i, j \in [1, k]\}$. Embed these new edges 'parallel' to the original edge $vw$. Let $H_0$ be the splitting of $G'$ at $F_1, \ldots, F_p$. Edges in $H_0$ inherit their label in $G'$. For each $\ell \in [1, p]$, let $J_\ell$ be the face of $H_0$ that corresponds to $F_\ell$.

Let $H_\ell$ be the graph with vertex set $V(J_\ell) \cup \{(v, i) : v \in V(F_\ell), i \in [1, k]\}$, where:

(a) each vertex $x$ in $J_\ell$ that belongs to a vertex $v$ in $F_\ell$ is adjacent to each vertex $(v, i)$ in $H_\ell$, and

(b) vertices $(v, i)$ and $(w, j)$ in $H_\ell$ are adjacent if and only if $v = w$ and $i \neq j$.

We now construct a circular decomposition $\{B(x) : x \in V(J_\ell)\}$ of $H_\ell$ with perimeter $J_\ell$. For each vertex $x$ in $J_\ell$ that belongs to a vertex $v$ in $F_\ell$, let $B(x)$ be the set $\{x\} \cup \{(v, i) : i \in [1, k]\}$ of vertices in $H_\ell$. Thus $|B(x)| \leq k + 1$. For each type-(a) edge between $x$ and $(v, i)$, the endpoints are both in bag $B(x)$. For each type-(b) edge between $(v, i)$ and $(v, j)$ in $H_\ell$, the endpoints are in every bag $B(x)$ where $x$ belongs to $v$. Thus the endpoints of every edge in $H_\ell$ are in some bag $B(x)$. Thus $\{B(x) : x \in V(J_\ell)\}$ is a circular decomposition of $H_\ell$ with perimeter $J_\ell$ and width at most $k$. 

Let $H$ be the graph $H_0 \cup H_1 \cup \cdots \cup H_p$. Thus $V(H_0) \cap V(H_\ell) = V(J_\ell)$ for each $\ell \in [1,p]$. Since $J_1, \ldots, J_p$ are pairwise vertex-disjoint facial cycles of $H_0$, the subgraphs $H_1, \ldots, H_p$ are pairwise vertex-disjoint. Hence $H$ is $(g, p, k)$-almost embeddable.

To complete the proof, we now construct a model $\{D_{v,i} : v^{(i)} \in V(G[k])\}$ of $G[k]$ in $H$, where $v^{(i)}$ is the $i$-th vertex in the $k$-clique of $G[k]$ corresponding to $v$. Fix an arbitrary total order $\preceq$ on $V(G)$. Consider a vertex $v^{(i)}$ of $G[k]$. Say $v$ is in face $F_\ell$. Add the vertex $(v,i)$ of $H_\ell$ to $D_{v,i}$. For each edge $v^{(i)}w^{(j)}$ of $G[k]$ with $v \prec w$, by construction, there is an edge $xy$ of $H_0$ labeled $(i,j)$, such that $x$ belongs to $v$ and $y$ belongs to $w$. Add the vertex $x$ to $D_{v,i}$. Thus $D_{v,i}$ induces a connected star subgraph of $H$ consisting of type-(a) edges in $H_\ell$. Since every vertex in $J_\ell$ is incident to at most one labeled edge, $D_{v,i} \cap D_{w,j} = \emptyset$ for distinct vertices $v^{(i)}$ and $w^{(j)}$ of $G[k]$.

Consider an edge $v^{(i)}w^{(j)}$ of $G[k]$. If $v = w$ then $i \neq j$ and $v$ is in some face $F_\ell$, in which case a type-(b) edge in $H_\ell$ joins the vertex $(v,i)$ in $D_{v,i}$ with the vertex $(w,j)$ in $D_{w,j}$. Otherwise, without loss of generality, $v \prec w$ and by construction, there is an edge $xy$ of $H_0$ labeled $(i,j)$, such that $x$ belongs to $v$ and $y$ belongs to $w$. By construction, $x$ is in $D_{v,i}$ and $y$ is in $D_{w,j}$. In both cases there is an edge of $H$ between $D_{v,i}$ and $D_{w,j}$. Hence the $D_{v,i}$ are the branch sets of a $G[k]$-model in $H$.\hfill $\square$

Our first construction employs just one vortex and is based on an embedding of a complete graph.

**Lemma 4.2.** For all integers $g \geq 0$ and $k \geq 1$, there is an integer $n \geq k\sqrt{6g}$ such that $K_n$ is a minor of some $(g,1,k)$-almost embeddable graph.

**Proof.** The claim is vacuous if $g = 0$. Assume that $g \geq 1$. The map color theorem [7] implies that $K_m$ triangulates some surface if and only if $m \mod 6 \in \{0,1,3,4\}$, in which case the surface has Euler genus $\frac{1}{2}(m-3)(m-4)$. It follows that for every real number $m_0 \geq 2$ there is an integer $m$ such that $m_0 \leq m \leq m_0 + 2$ and $K_m$ triangulates some surface of Euler genus $\frac{1}{6}(m-3)(m-4)$. Apply this result with $m_0 = \sqrt{6g} + 1$ for the given value of $g$. We obtain an integer $m$ such that $\sqrt{6g} + 1 \leq m \leq \sqrt{6g} + 3$ and $K_m$ triangulates a surface $\Sigma$ of Euler genus $g' := \frac{1}{6}(m-3)(m-4)$. Since $m-4 < m-3 \leq \sqrt{6g}$,
we have \( g' \leq g \). Every triangulation has facewidth at least 3. Thus, deleting one vertex from the embedding of \( K_n \) in \( \Sigma \) gives an embedding of \( K_{n-1} \) in \( \Sigma \), such that some face contains every vertex. Let \( n := (m - 1)k \geq k\sqrt{6g} \). Lemma 4.1 implies that \( K_{m-1}[k] \cong K_n \) is a minor of some \((g', 1, k)\)-almost embeddable graph. \( \square \)

Now we give a construction based on grids. Let \( L_n \) be the \( n \times n \) planar grid graph. This graph has vertex set \([1, n] \times [1, n]\) and edge set \{\((x, y)(x', y') : |x - x'| + |y - y'| = 1\}\). The following lemma is well known; see [9].

**Lemma 4.3.** \( K_{nk} \) is a minor of \( L_n[2k] \) for all \( k \geq 1 \).

**Proof.** For \( x, y \in [1, n] \) and \( z \in [1, 2k] \), let \((x, y, z)\) be the \( z \)-th vertex in the \( 2k \)-clique corresponding to the vertex \((x, y)\) in \( L_n[2k] \). For \( x \in [1, n] \) and \( z \in [1, k] \), let \( B_{x,z} \) be the subgraph of \( L_n[2k] \) induced by \{\((x, y, 2z - 1), (y, x, 2z) : y \in [1, n]\)\}. Clearly \( B_{x,z} \) is connected. For all \( x, x' \in [1, n] \) and \( z, z' \in [1, k] \) with \((x, z) \neq (x', z')\), the subgraphs \( B_{x,z} \) and \( B_{x',z'} \) are disjoint, and the vertex \((x, x', 2z - 1)\) in \( B_{x,z} \) is adjacent to the vertex \((x, x', 2z')\) in \( B_{x',z'} \). Thus the \( B_{x,z} \) are the branch sets of a \( K_{nk}\)-minor in \( L_n[2k] \). \( \square \)

**Lemma 4.4.** For all integers \( k \geq 2 \) and \( p \geq 1 \), there is an integer \( n \geq \frac{2}{3}\sqrt{3} k\sqrt{p}, \) such that \( K_n \) is a minor of some \((0, p, k)\)-almost embeddable graph.

**Proof.** Let \( m := \lceil \sqrt{p} \rceil \) and \( \ell := \lceil \frac{k}{2} \rceil \). Let \( n := 2m\ell \geq 2 \cdot \sqrt{\frac{p}{3}} \cdot \frac{k}{3} = \frac{2}{3}\sqrt{3} k\sqrt{p} \). For \( x, y \in [1, m] \), let \( F_{x,y} \) be the face of \( L_{2m} \) with vertex set \{\((2x - 1, 2y - 1), (2x, 2y), (2x - 1, 2y)\)\}. There are \( m^2 \) such faces, and every vertex of \( L_{2m} \) is in exactly one such face. By Lemma 4.3, \( K_n \) is a minor of \( L_{2m}[2\ell] \). Since \( L_{2m} \) is planar, by Lemma 4.1, \( K_n \) is a minor of some \((0, m^2, 2\ell)\)-almost embeddable graph. The result follows since \( p \geq m^2 \) and \( k \geq 2\ell \). \( \square \)

The following theorem summarizes our constructions of almost embeddable graphs containing large complete graph minors.

**Theorem 4.5.** For all integers \( g \geq 0 \) and \( p \geq 1 \) and \( k \geq 2 \), there is an integer \( n \geq \frac{1}{k^2} k\sqrt{p + \bar{g}}, \) such that \( K_n \) is a minor of some \((g, p, k)\)-almost embeddable graph.

**Proof.** First suppose that \( g \geq p \). By Lemma 4.2, there is an integer \( n \geq k\sqrt{6g} \), such that \( K_n \) is a minor of some \((g, 1, k)\)-almost embeddable graph, which is also \((g, p, k)\)-embeddable (since \( p \geq 1 \)). Since \( n \geq k\sqrt{5p + 3g} > \frac{1}{k} k\sqrt{p + \bar{g}} \), we are done.

Now assume that \( p > g \). By Lemma 4.4, there is an integer \( n \geq \frac{2}{3}\sqrt{3} k\sqrt{p} \), such that \( K_n \) is a minor of some \((0, p, k)\)-almost embeddable graph, which is also \((g, p, k)\)-embeddable (since \( g \geq 0 \)). Since \( n \geq \frac{2}{3}\sqrt{3} k\sqrt{p} > \frac{1}{k} k\sqrt{g + \bar{p}} \), we are done. \( \square \)

Adding a dominant vertices to a graph increases its Hadwiger number by \( a \). Thus Theorem 4.5 implies:

**Theorem 4.6.** For all integers \( g, a \geq 0 \) and \( p \geq 1 \) and \( k \geq 2 \), there is an integer \( n \geq a + \frac{1}{k} k\sqrt{p + \bar{g}}, \) such that \( K_n \) is a minor of some graph in \( \mathcal{G}(g, p, k, a) \).

Corollary 3.2 and Theorem 4.6 together prove Theorem 1.1.

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