HYDROSTATIC EQUILIBRIUM OF INSULAR, STATIC, SPHERICALLY SYMMETRIC, PERFECT FLUID SOLUTIONS IN GENERAL RELATIVITY

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Received (Day Month Year)
Revised (Day Month Year)

An analysis of insular solutions of Einstein’s field equations for static, spherically symmetric, source mass, on the basis of exterior Schwarzschild solution is presented. Following the analysis, we demonstrate that the regular solutions governed by a self-bound (that is, the surface density does not vanish together with pressure) equation of state (EOS) or density variation can not exist in the state of hydrostatic equilibrium, because the source mass which belongs to them, does not represent the ‘actual mass’ appears in the exterior Schwarzschild solution. The only configuration which could exist in this regard is governed by the homogeneous density distribution (that is, the interior Schwarzschild solution). Other structures which naturally fulfill the requirement of the source mass, set up by exterior Schwarzschild solution (and, therefore, can exist in hydrostatic equilibrium) are either governed by gravitationally-bound regular solutions (that is, the surface density also vanishes together with pressure), or self-bound singular solutions (that is, the pressure and density both become infinity at the centre).

Keywords: general relativity –static spherical structures; dense matter – equation of state; stars– neutron.

PACS Nos.: 04.20.Jd; 04.40.Dg; 97.60.Jd.

1. Introduction

The interior Schwarzschild solution (homogeneous density solution) of Einstein’s field equations provides two very important features towards obtaining insular (in the sense that the pressure vanishes at some finite boundary) configurations in hydrostatic equilibrium, compatible with general relativity, namely - (i) It gives an absolute upper limit on compactness ratio, \( u(\equiv M/R, \text{ mass to size ratio of the entire configuration in geometrized units}) \leq (4/9) \) for any static spherical configuration (corresponding to arbitrary density profile, provided the density decreases monotonically outwards from the centre) in hydrostatic equilibrium [1, 2], and (ii) For an

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assigned value of the compactness ratio, $u$, and the radius, $R$, the minimum central pressure, $P_0$, corresponds to the homogeneous density solution (see, e.g., ref. [2]). Recently, by using the property (ii) of homogeneous density sphere as mentioned above, we have obtained a ‘compatibility criterion for hydrostatic equilibrium’ [3]. The important feature of this criterion is that it connects the compactness ratio, $u$, of any static configuration with the corresponding ratio of central pressure to central energy-density $\sigma \equiv (P_0/E_0)$. The criterion states that in order to have compatibility with the state of hydrostatic equilibrium, for a given value of $\sigma$, the compactness ratio, $u$, of any configuration should always remain less than or equal to the compactness ratio of the homogeneous density sphere for same $\sigma$ [3].

Various insular exact solutions [4, 5; and references therein] and equations of state (EOSs) for static and spherically symmetric mass are available in the literature which can be divided, in general, into two categories:

1. The exact solutions and EOSs corresponding to the regular density variation [regular in the sense of positive finite density at the origin (i.e., the metric coefficient, $e^\lambda = 1$ at $r = 0$, defined later) which decreases monotonically outwards], such that the density at the surface vanishes together with pressure (and so called the gravitationally-bound regular structures), and

2. The exact solutions and EOSs corresponding to the density variation such that the density does not terminate together with pressure at the surface of the configuration (and so called the self-bound structures). These structures can be divided further into two sub-categories:

(a) The self-bound structures with finite central densities, generally called, the self-bound regular structures\(^a\)

(b) The self-bound structures with infinite central densities (i.e., $e^\lambda \neq 1$ at $r = 0$). We can call them, the self-bound singular structures (singular in the sense that pressure and density both become infinity at the centre).

The exact solutions in the first category include Tolman’s type VII solution with vanishing surface density [9, 10, 11; and references therein], and Buchdahl’s “gaseous” model\(^b\)[12], whereas the EOSs in this category include the well known polytropic EOSs [13, 14]. The exact solutions in the second, sub-category (a) include Tolman’s type IV solution [9], the solution independently obtained by Adler [15], Kuchowicz [16], and Adams and Cohen [17], and Durgapal and Fuloria solution [18] etc. The well known example of EOS in this category is characterized by the stiffest EOS (see, e.g., ref. [19], [20]) $(dP/dE) = 1$ (in geometrized units). Haensel and

\(^a\)There exists many self-bound exact solutions in the literature which also belong to finite central densities, but what is required to the present context is “the monotonic decrease of density outwards from the centre”, and these solutions, e.g. - Goldman I solution [6] (called Gold I in Delgaty-Lake classification [5]), Stewart’s solution [7], and Durgapal-Pande-Phuloria I solution [8] (called D-P-P I in [5]) etc. do not fulfill this criterion [5]. Apparently, such solutions are irrelevant to the context of the present study.

\(^b\)This model represents the gravitationally-bound regular structure for $u$ values $\leq 0.20$ [12].
Zdunik [20] have shown that the only EOS which can describe a sub-millisecond pulsar and the static mass of $1.442 M_\odot$ simultaneously, corresponds to the said stiffest EOS, however, they emphasized that this EOS represents an ‘abnormal’ state of matter in the sense that pressure vanishes at densities of the order of nuclear density or even higher [21]. The exact solutions in the second, sub-category (b) include Tolman’s type V and VI solutions [9], and the well known example of EOS in this category is represented by the EOS corresponding to a ‘Fermi gas’ having infinite values of pressure and density at the centre [22].

An examination of the ‘compatibility criterion’ on some well known exact solutions and EOSs indicated that this criterion, in fact, is fulfilled only by those structures which come under the category 1 and 2(b) mentioned above, that is, (i) the gravitationally-bound regular, and, (ii) the self-bound singular structures. On the other hand, it is seen that the EOSs and analytic solutions, corresponding the self-bound regular state of matter [which are mentioned above under the category 2(a)], in fact, do not fulfill this criterion [3]. We have shown this inconsistency particularly for the EOS, $(dP/dE) = 1$ (as it represents the most successful EOS to obtain the various extreme characteristics of neutron stars mentioned above), and the analytic solution put forward by Durgapal and Fuloria [18] which fulfills various properties for physically realistic structure [23].

The reason behind non-fulfillment of the ‘compatibility criterion’ by various self-bound regular EOSs and exact solutions could be resolved, if we carefully analyze the ‘specific property’ of the total mass ‘$M$’ appears in the exterior Schwarzschild solution. It immediately follows from this analysis that unlike gravitationally-bound regular structures, and self-bound singular solutions, the ‘actual’ total mass ‘$M$’, in fact, can not be attained by the configurations corresponding to a regular self-bound state of matter. This is demonstrated by considering a ‘generalized density distribution’ for the source mass ‘$M$’ and verified further on the basis of a ‘class’ of well known exact solutions, generated by what is called, ‘the algorithmic construction of all static spherically symmetric perfect fluid solutions of Einstein’s equations’ [24-26].

2. Field Equations and TOV Equations

For a spherically symmetric and static line element

$$ds^2 = e^{\nu}dt^2 - e^{\lambda}dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2,$$  

(1)

[$\nu$ and $\lambda$ are functions of $r$ alone] recalling that we are using geometrized units, the field equations yield in the following form

$$8 \pi T_0^0 = 8 \pi E = e^{-\lambda}[\nu'/(r) - (1/\nu^2)] + 1/r^2,$$  

(2)

$$-8 \pi T_1^1 = 8 \pi P = e^{-\lambda}[(\nu'/r) + (1/r^2)] - 1/r^2,$$  

(3)

$$-8 \pi T_2^2 = -8 \pi T_3^3 = 8 \pi P = e^{-\lambda}(\nu''/2) + (\nu'/4) - (\nu' \lambda'/4) + (\nu' - \lambda'/2r).$$  

(4)
where the primes represent differentiation with respect to \( r \). \( P \) and \( E \) represent, respectively, the pressure and energy-density inside the perfect fluid sphere related with the non-vanishing components of the energy-momentum tensor, \( T^i_j \), \( i = 0, 1, 2, \) and \( 3 \) respectively. Eqs. (2) - (4) represent second-order, coupled differential equations which can be written in the form of first-order, coupled differential equations, namely, TOV equations [9, 22] governing hydrostatic equilibrium in general relativity

\[
P' = -(P + E)[4\pi Pr^3 + m(r)] / r (r - 2m(r)),
\]

\[
\nu' = -2P' / (P + E),
\]

and

\[
m'(r) = 4\pi Er^2,
\]

where prime denotes differentiation with respect to \( r \), and \( m(r) \) is the mass-energy contained within the radius \( r' \), that is

\[
m(r) = \int_0^r 4\pi Er^2 dr.
\]

The equation connecting metric parameter \( \lambda \) with \( m(r) \) is given by

\[
e^{-\lambda} = 1 - [2m(r)/r] = 1 - (8\pi / r) \int_0^r Er^2 dr.
\]

The three field equations (or TOV equations) mentioned above, involve four variables, namely, \( P, E, \nu, \) and \( \lambda \). Thus, in order to obtain a solution of these equations, one more equation is needed which may be assumed as a relation between \( P \), and \( E \) (EOS), or can be regarded as an algebraic relation connecting one of the four variables with the radial coordinate \( r \) (or an algebraic relation between the parameters]). For obtaining an exact solution, the later approach is employed.

Notice that Eq. (9) yields the metric coefficient \( e^\lambda \) for the assumed energy-density, \( E \), as a function of radial distance \( 'r' \). Once the metric coefficient \( e^\lambda \) or mass \( m(r) \) is defined for assumed energy-density by using Eqs. (9) or (8), the pressure, \( P \), and the metric coefficient, \( e^\nu \), can be obtained by solving Eqs. (5) and (6) respectively which yield two constants of integration. These constants should be obtained from the following boundary conditions, in order to have a proper solution of the field equations:

3. Boundary Conditions: Hydrostatic Equilibrium for the Mass Distribution

In order to maintain hydrostatic equilibrium throughout the configuration, the pressure must vanish at the surface of the configuration, that is

\[
P = P(r = R) = P(R) = 0,
\]

where ‘\( R \)’ is the radius of the configuration.
The consequence of Eq. (10) ensures the continuity of \( \nu' \), and, therefore, that of the metric parameter \( e^{\nu} \), belonging to the interior solution with the corresponding expression for well known exterior Schwarzschild solution at the surface of the fluid configuration, that is: \( e^{\nu(r=R)} = 1 - (2M/R) \) [where \( 'M' = m(R) \) is the total mass of the configuration]. However, the exterior Schwarzschild solution guarantees that \( e^{\nu(r=R)} = e^{-\lambda(r=R)} \), at the surface of the configuration irrespective of the condition that the surface density, \( E(r = R) = E(R) \), is vanishing with pressure or not, that is

\[
E(R) = 0,
\]

together with Eq. (10), or

\[
E(R) \neq 0,
\]

So that one could assure the well known relation

\[
e^{\nu(R)} = e^{-\lambda(R)} = 1 - (2M/R) = (1 - 2u),
\]

at the surface of the configuration, for both of the cases, namely - (i) the surface density vanishes together with pressure, and (ii) the surface density does not vanish together with pressure. The total mass \( M \) which appears in Eq. (13) is defined as [Eq. (8)]

\[
M = m(R) = \int_0^R 4\pi Er^2 dr.
\]

But, the most important point which should be remembered here is the ‘specific property’ of the total mass ‘\( M = m(R) \)’ follows directly from the well known property of the exterior Schwarzschild solution, namely - it depends only upon the total mass ‘\( M' \) and not upon the ‘type’ of the density variation considered inside the radius ‘\( R' \) of this mass generating sphere. That is, the total mass ‘\( M' \) should also bear this well known property of the exterior Schwarzschild solution, what we call it, the ‘type independence’ property of the mass ‘\( M' \), and it may be defined in this manner: “The dependence of mass ‘\( M' \) upon the parameter(s) describing a particular type of the density distribution considered inside the mass generating sphere (e. g., for an assumed value of the compactness ratio, \( u \), the mass, ‘\( M' \), will depend only upon the radius, \( R \), which may either depend upon the surface density, or upon the central density, or upon both of them) should exist in such a manner that from an exterior observer’s point of view, we could not diagnose that what ‘type’ of density variation belongs to this mass”.

This point can be illustrated in the following manner: the total mass ‘\( M' \) which appears in the exterior Schwarzschild solution is called the coordinate mass - the mass measured by some external observer, and from this observer’s point of view, if we measure a sphere of mass ‘\( M' \), we can not know (by any means) that how the matter is distributed from centre to the surface of this sphere. It means, if we ‘measure’ \( M \) by the use of the non-vanishing surface density [we can calculate the
(coordinate) radius $R'$ by assigning a suitable value to the surface density in the respective expression and by using the relation, $M = uR$, the mass $M'$ can be worked out for an assumed value of $u$, we can not measure it, by any means, by the use of the central density (in order to keep the ‘type independence’ property of the mass intact), and this is possible only when there exist no relation connecting $M'$ and the central density. Or, in other words, $M'$ should be independent of the central density (meaning thereby that the surface density should be independent of the central density). On the other hand, if we measure $M'$ by using the expression for central density [by assigning a suitable value to the central density in the relevant expression, we can calculate the radius $R'$ of the configuration, and by using the relation, $M = uR$, we can work out the mass $M'$ for an assumed value of $u$, we can not be able to measure $M'$ by using the expression for surface density (in order to keep the ‘type independence’ property of the mass intact). That is, there should exist no relation connecting the total mass with surface density. It follows, therefore, that the central density should be independent of the surface density.

From the above explanation of ‘type independence’ property of mass $M'$, it is evident that the ‘actual’ total mass $M'$ which appears in the exterior Schwarzschild solution should either depend upon the surface density, or depend upon the central density of the configuration, and in any case, not upon both of them. However, the dependence of mass $M'$ upon both of the densities (surface, as well as central) is a common feature observed among all self-bound regular structures, composed of a single EOS or an analytic solution (which is the violation of the ‘type independence’ property of mass $M'$), such structures, therefore, do not correspond to the ‘actual’ total mass $M'$ required by the exterior Schwarzschild solution for the fulfillment of the boundary conditions at the surface of the structure. This also explains the reason behind non-fulfillment of the ‘compatibility criterion’ by them. It is interesting to note here that there could exist only one solution in this regard for which the mass $M'$ depends upon both, but the same value of surface and centre density, and for regular density distribution the structure would be governed by the homogeneous (constant) density throughout the configuration.

For gravitationally bound regular structures, the requirement ‘type independence’ of the mass is obviously fulfilled because the mass $M'$ depends only upon the central density (surface density is always zero for these structures). Furthermore, this demand of ‘type independence’ is also satisfied by the self-bound singular solutions, because such structures correspond to an infinite value of central density, and consequently, the mass $M'$ depends only upon surface density. Both types of these structures are also found to be consistent with the ‘compatibility criterion’ as mentioned earlier.

The above discussion regarding various types of structures is true for any single equation of state or analytic solution comprises the whole configuration. At this place, we are not claiming that the construction of a self-bound regular structure in impossible. It is quite possible, provided we consider a two-density structure, such
that the mass ‘$M$’ always turns out to be independent of the central density, and the property ‘type independence’ of the mass ‘$M$’ is satisfied. Examples of such two-density models are also available in the literature (see, e.g., ref. [27]), but in the different context. However, it should be noted here that the fulfillment of ‘type independence’ condition by the mass ‘$M’ for any two-density model will represent only a necessary condition for hydrostatic equilibrium, unless the ‘compatibility criterion’ [3] is satisfied by them, which also assure a sufficient and necessary condition for any structure in hydrostatic equilibrium (however, this issue is discussed in detail elsewhere [28]), we restrict ourself to the present context. In the following sections (4, and 5), explicit examples are given to show that a single density variation comprising a self-bound regular structure can not correspond to the actual mass ‘$M’ required for the hydrostatic equilibrium.

4. The Actual Mass $M$ and the Generalized Density Distribution

Now, consider the case of regular self-bound structures: The most smooth possible variation of density inside any regular configuration can not be other than the constant (homogeneous) density, whereas the fastest possible variation of density is well known and represented by the inverse square density variation [$E \propto (1/r^2)$]. It follows, therefore, that any possible regular self-bound configuration, characterized by an EOS or, density as a function of radial co-ordinate, can be generalized in the following form

$$ E(r) = C/(a + r)^b $$

where $C$ is the constant of proportionality, $a$ is a positive arbitrary constant to make the density positive finite at the centre, and the constant $b$ is allowed to take any value in the interval $0 \leq b \leq 2$. Eq. (15) represents a self-bound regular density distribution, the density at the centre is positive finite, decreases monotonically from centre to the outer region, and it would remain finite non-zero if one assumes that pressure vanishes at some finite radius. Thus, by using these central and surface conditions in Eq. (15), we get

$$ C = E_R(a + R)^b, $$

and

$$ C = E_0 a^b $$

One may use some other form of the equation to generalize the self-bound regular structures considered here, for example, we may consider a particular form of the ‘source function’ [see, next section] which could ‘generate’ a ‘class’ of solutions of the type considered here [category 2(a) of the present study] in the technique called ‘the algorithm for constructing all static spherically symmetric perfect fluid solutions’ [24-26]. However, the conclusions drawn on this basis will remain unaltered, because the key point is that for any self-bound regular configuration governed by a single EOS or density distribution, the surface density can not be independent of the centre density. This is what we have demonstrated in the present, and also in the next section.
where $E_0$ is the central density, $R$ is the radius of the configuration, and $E_R$ is the surface density at which pressure vanishes.

Substituting Eq. (15) into Eq. (8), we get the mass contained within the radial co-ordinate $’r’$ for the assigned density variation as

$$m(r) = 4\pi C \int_0^r \frac{r^2dr}{(a+r)^b} = 4\pi Cr_I,$$

(18)

where $I_r$ is given by

$$I_r = \frac{r}{(3-b)(a+r)^{(b-1)}} = \frac{2a}{r(3-b)} \int_0^r \frac{rdr}{(a+r)^b}.$$

(19)

The substitution of Eq. (18) into Eq. (9), yields the metric parameter, $e^\lambda$, for the assigned density variation [Eq. (15)] as

$$e^{-\lambda} = 1 - 8\pi CI_r,$$

(20)

where $C$ is defined by Eqs. (16) and (17) respectively, and $I_r$ is given by Eq. (19).

For Eq. (15), the total mass contained inside radius $’R’$ [Eq. (14)] is given by

$$M = 4\pi CRI_R.$$

(21)

Eq. (20) gives $e^\lambda$ at the surface of the configuration as

$$e^{-\lambda(R)} = 1 - 8\pi CI_R$$

(22)

where $I_R$ is given by

$$I_R = \frac{R}{(3-b)(a+R)^{(b-1)}} = \frac{2a}{R(3-b)} \int_0^R \frac{rdr}{(a+r)^b}.$$

(23)

The condition of regularity requires that

$$E_R \leq E_0.$$  (24)

Substituting the values of $E_R$ and $E_0$ from Eqs. (16) and (17) into Eq. (24), we get

$$a^b \leq (a + R)^b.$$  (25)

(A) The condition of equality in Eq (25) [that is, $a^b = (a + R)^b$] requires that

$$b = 0.$$  (26)

For this value of $b(= 0)$, Eqs. (16) and (17) give the relation

$$E_R = E_0 = C(\text{constant}) = < E > (\text{say}).$$  (27)

The substitution of Eqs. (26) and (27) into Eq. (20) gives the metric parameter $e^\lambda$ as

$$e^{-\lambda} = 1 - (8\pi/3) < E > r^2;$$

(28)

which reduces at the surface of the configuration naturally to the value prescribed by exterior Schwarzschild solution

$$e^{-\lambda(R)} = 1 - (8\pi/3) < E > R^2 = 1 - (2M/R).$$  (29)
The constant $C$ in this case is defined by both, but the same values of surface and central densities [Eq. (27)] which represents the (constant) density throughout the configuration for regular solution, given by [see, Eq. (14)]

$$< E > = \frac{3M}{4\pi R^3}. \quad (30)$$

Hence, the mass $'M'$ fulfills the property of ‘type independence’ in this case as discussed in the last section, therefore, represents the ‘actual mass’ which appears in the exterior Schwarzschild solution.

(B) Now, the condition of inequality in Eq (25) [that is, $a^b < (a + R)^b$] requires that

$$0 < b \leq 2. \quad (31)$$

For these values of $b$ [Eq. (31)], we get from Eqs. (22), the value of $e^\lambda$ at the surface of the configuration as

$$e^{-\lambda(R)} = 1 - \frac{8\pi C}{(3 - b)}\left[\frac{R}{(a + R)^{b-1}} - \frac{2aR}{R} \int_0^R \frac{rdr}{(a + r)^b}\right]. \quad (32)$$

The right-hand side of Eq. (32) can not attain the normal value, $1 - (2M/R)$, prescribed by the exterior Schwarzschild solution, because the constant $C$ in this case has been defined by both of the densities (surface, as well as central) through Eqs. (16) and (17) respectively, therefore, it can not be eliminated trivially from Eq. (32) by just using Eq. (21) [in the manner, $CI_R = (M/4\pi R)$](this common practice has been adopted by various authors in this regard) in order to keep the ‘type independent’ property of the mass $'M'$ intact. Or, in other words, the total mass of a self-bound regular configuration given by Eq. (21) does not represent the ‘actual mass’ $M$ appears in the exterior Schwarzschild solution and consequently, the boundary conditions at the surface of such configuration are not satisfied.

If, however, we set $a = 0$ in Eq. (15), we get from Eqs. (16) and (17), the relations

$$C = E_R R^b \quad (33)$$

and,

$$E_0 = \infty. \quad (34)$$

By substituting the value of $C$ from Eq. (33) into Eq. (32), we get $e^\lambda$ at the surface of the structure as

$$e^{-\lambda(R)} = 1 - \frac{8\pi E_R R^2}{(3 - b)} \quad (35)$$

or,

$$e^{-\lambda(R)} = 1 - \left(\frac{8\pi}{3}\right) < E > R^2 = 1 - \frac{2M}{R} \quad (36)$$

which is the normal value required by exterior Schwarzschild solution at the surface. In this equation, $< E >$ represents the ‘average density’ of the structure given by

$$< E > = \frac{3E_R}{(3 - b)} = \frac{3M}{4\pi R^3}. \quad (37)$$
Note that for a calculated value of $R'$, the mass $M'$ in this case depends only upon the surface density and not upon the central density (which is always infinite), thus the property ‘type independence’ of the mass is satisfied in this case and the structures will be compatible with the state of hydrostatic equilibrium.

5. The Actual Mass $M$ and a ‘Class’ of Exact Solutions Generated in the Algorithmic Construction

The results of the analysis stated under the section 3 of the present study may be discussed in the context of the recent study, generally called ‘the algorithm for the construction of all static perfect fluid solutions in general relativity’ [24-26], because it does not require any specific knowledge of the EOS (instead, it results as a byproduct of the algorithm). The basis of such type of study is the well known fact that the pressure isotropy condition in perfect fluid configurations places a single differential constraint on the metric components of the geometry of the perfect fluid. It follows, therefore, that the class of metrics representing a perfect fluid geometry should be specified by a single arbitrary monotone function called the ‘generating function’. Rahman and Visser [24] and Lake [25] have shown that an explicit closed-form (algebraic-integro-differential) solution of the pressure isotropy condition in terms of this function, in fact, exists. Rahman and Visser [24] have presented their algorithm in ‘isotropic coordinates’, because of the usefulness of this system, whereas Lake [25] has considered the ‘curvature coordinates’ (which offer a direct physical interpretation of the ‘generating function’) and also transformed it into ‘isotropic coordinates’. Martin and Visser [26] have presented a variant of the Lake’s algorithm [25] in terms of the ‘average density’ and the locally measured ‘acceleration due to gravity’, that is, the variables with a clear physical meaning.

Although, the details towards choosing the ‘generating functions’, corresponding to ‘physically reasonable’ solutions [5, 25]d, are still not known [25], nevertheless, by imposing some (but not all) of the standard ‘regularity conditions’ (which are easily possible), the ‘restricted generating function’ can generate all regular static spherically symmetric perfect fluid solutions of Einstein’s equations [24-26]. Therefore, the type of the algorithm discussed here guarantees a perfect fluid solution, but it does not guarantee a ‘physically reasonable’ perfect fluid solution [24-26]. For example, Rahman and Visser [24] have investigated the ‘general quadratic ansatz’, which yields the new form of the Goldman I solution [6], and also turns out to be

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dThe ‘physically reasonable’ solutions considered in [5, 25] would obviously fulfill the criterion of regular solutions considered in the present paper.

*The term ‘regular’ used in refs.[24-26], in fact, denotes positive finite pressure and density at the origin which do not necessarily decrease monotonically outwards (cf. the term ‘regular’ used in the present paper). Consequently, the term ‘all regular static spherically symmetric perfect fluid solutions’ used in refs.[24-26] would, therefore, represent both, (i) the ‘self-bound regular static spherically symmetric perfect fluid solutions’ which are discussed in the present paper [category 2(a)], and (ii) the ‘self-bound static spherically symmetric perfect fluid solutions’ with finite central densities which are not relevant to the context of the present study (see, footnote ‘a’ of the text).
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equivalent to Glass-Goldman solution [29] (called G-G in [5]). Furthermore, in various regions of parameter space, this general solution reduces to at least six other well known exact solutions including Stewart’s solution [7], and interior Schwarzschild solution. But, in spite of this striking feature of the ‘general quadratic ansatz’, except interior Schwarzschild solution (which we have already considered as a basis of our study [3]), both of the generated, insular, perfect fluid solutions (namely, Goldman I and Stewart’s solution) are irrelevant to the context of the present study. However, Lake [25] has investigated a particular form of the ‘generating function’ which could generate an infinite number of previously unknown (N > 5), but physically interesting exact solutions (of a particular ‘class’) [including some well known solutions already present in the literature]. Such an algorithm consider the usual metric (by using geometrized units) in ‘curvature coordinates’ as [25]

\[ ds^2 = -e^{2\Phi(r)}dt^2 + \left[1 - \left(2m(r)/r\right)\right]^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2, \]  

(38)

where \( \Phi(r) \) is also called the (gravitational) ‘red-shift’ parameter, and \( m(r) \) is the well known ‘mass inside radius \( r \)’, defined earlier. The algorithm for constructing all possible static spherically symmetric perfect fluid solutions of Einstein’s equations, in fact, connects these two parameters in the following manner [25]

\[ m(r) = \int b(r)e^{\int a(r)dr} dr + C, \]  

(39)

where

\[ a(r) \equiv \frac{2r^2[\Phi''(r) + \Phi'(r)^2] - 3r\Phi'(r) - 3}{r[\Phi'(r) + 1]}, \]  

(40)

and

\[ b(r) \equiv \frac{r[\Phi''(r) + \Phi'(r)^2] - \Phi'(r)]}{r\Phi'(r) + 1}. \]  

(41)

Here \( C \) is a constant, and a prime denotes differentiation with respect to \( r \). Eq.(39) is supplemented by the following boundary conditions [25]: \( \Phi(0) = \) finite constant (set by the scale of \( t \)); \( \Phi'(0) = 0; \Phi''(0) > 0; \) and \( \Phi'(r) \neq 0 \) for \( r > 0 \). That is, the source function \( \Phi(r) \) must be a monotone increasing function with a regular minimum at \( r = 0 \). In order to have a finite boundary \( R \), we further require that \( \Phi'(r = R) = M/R(R - 2M) \), where \( M = m(r = R) \).

For any monotonically increasing source function \( \Phi(r) \) with a regular minima at \( r = 0 \), Eq.(39) necessarily gives a regular \( \text{d} \) static spherically symmetric perfect fluid solution of Einstein’s field equations. In the present context, the term exact solution is used for those for which Eq.(39) can be evaluated without recourse to numerical methods. Lake [25] has shown that for the monotonically increasing source function (with a regular minima at the origin), given by the equation

\[ \Phi(r) = \frac{1}{2}N\ln(1 + \frac{r^2}{\alpha}), \]  

(42)
where $N$ is an integer ($\geq 1$) and $\alpha$ is a constant ($> 0$), can generate an infinite number of previously unknown ($N > 5$) but physically interesting perfect fluid solutions. The previously known exact solutions corresponding to different values of $N$ are respectively, $N = 1$ (Tolman’s type IV solution [9]), $N = 3$ (Heintzmann’s solution [30], called Heint IIa in [5]), $N = 4$ (Durgapal IV solution [31], called Durg IV in [5] which is also equivalent to Durgapal and Bannerji solution [32] called D-B in [5]), and $N = 5$ (Durgapal V solution [31], called Durg V in [5] which is also equivalent to Durgapal and Fuloria solution [18], called D-F in [5]). Since all these solutions are well known and also relevant to the context of the present study (corresponding to a positive finite density at the origin which decreases monotonically outwards [5]), we will discuss among them, the cases corresponding to the values of $N = 1$, 4, and 5 respectively.

5.1. The case $N = 1$ (Tolman’s type IV solution)

The total mass $M$ for Tolman’s type IV solution yields from Eq.(39) (which may be rearranged in terms of compactness ratio $u(\equiv M/R)$ and the surface density $E_R$ of the solution [33]) in the following form

$$M = u\left[\frac{3u(1 - 2u)}{4\pi E_R(1 - u)}\right]^{1/2}. \quad (43)$$

The relation between surface density $E_R$ and the central density $E_0$ in given by the equation

$$\frac{E_R}{E_0} = \frac{2(1 - 2u)(1 - 3u)}{(1 - u)(2 - 3u)}. \quad (44)$$

This solution is applicable for the values of $u \leq (1/3)$. It is evident from Eq.(43) that for an assigned value of the compactness ratio $u$, the mass $M$ can be ‘measured’ from the knowledge of the surface density $E_R$, which is also dependent upon the central density $E_0$ via Eq.(44). Thus, the mass given by Eq.(43) does not represent the ‘actual mass’ appears in the exterior Schwarzschild solution, because the ‘type independence’ property of the mass (which requires that surface density should be independent of the central density) is violated in this case. This is what we have discussed under section 3 of the present study.

5.2. The case $N = 4$ (Durg IV = Durgapal and Bannerji (D-B) solution)

The total mass $M$ for D-B solution yields in the following form (after rearranging the terms, similarly as in the $N = 1$ case, mentioned above)

$$M = \frac{3X}{16(1 + X)^2}\left[\frac{3X(3 + X)}{\pi E_R}\right]^{1/2}, \quad (45)$$

where $X$, in terms of $u$, is given by

$$u = \frac{3X}{4(1 + X)}, \quad (46)$$
and the surface density $E_R$ is connected with the central density $E_0$ in the following manner
\[ \frac{E_R}{E_0} = \frac{3 + X}{3(1 + X)^2}. \] 
\[ (47) \]
This solution is applicable for the values of $u \leq 0.4214$. It follows from Eqs. (45) and (46) that for an assigned value of $u$, the total mass $M$ can be worked out from the knowledge of the surface density $E_R$, which is also a function of the central density $E_0$ as indicated by Eq. (47). Evidently, the mass given by Eq. (45) does not represent the ‘actual mass’ required for the state of hydrostatic equilibrium as explained earlier.

5.3. The case $N = 5$ (Durg V = Durgapal and Fuloria (D-F) solution)

As in the case $N = 1$, and 4 above, the total mass $M$ for D-F solution yields in the following form
\[ M = \frac{4X(3 + X)}{7(1 + X)^2} \left[ \frac{X(9 + 2X + X^2)}{7\pi E_R(1 + X)^3} \right]^{1/2}, \] 
\[ (48) \]
where $X$ is connected with the compactness ratio $u$ in the following manner
\[ u = \frac{4X(3 + X)}{7(1 + X)^2}, \] 
\[ (49) \]
and the relation, connecting surface density $E_R$ and the central density $E_0$ is given by
\[ \frac{E_R}{E_0} = \frac{9 + 2X + X^2}{9(1 + X)^3}. \] 
\[ (50) \]
This solution is applicable for the values of $u \leq 0.4265$. However, Eqs. (48) and (49) clearly indicate that for an assigned value of $u$, the total mass $M$ depends upon (and not only upon) the surface density $E_R$ which is not independent of the central density $E_0$, because of the existence of relation (50). Thus, the total mass $M$ in this case also, does not represent the ‘actual mass’ which is required by the exterior Schwarzschild solution.

Let us denote the compactness ratio of homogeneous density distribution by $u_h$, and that of the exact solutions corresponding to the sub-sections 5.1 - 5.3 by $u_{T-IV}, u_{D-B}$, and $u_{D-F}$ respectively. Solving these solutions for various assigned values of the ratio of central pressure to central (energy) density, $\sigma$, we obtain the corresponding values of the compactness ratio as shown in Table 1. It is seen that for each and every assigned value of $\sigma$, the values corresponding to $u_{T-IV}, u_{D-B}$, and $u_{D-F}$ respectively, always turn out to be higher than $u_h$. Or, in other words, the configurations defined by Tolman’s type IV solution (the case $N = 1$), D-B solution (the case $N = 4$), and D-F solution (the case $N = 5$) respectively, do not show consistency with the ‘compatibility criterion’ which is also consistent with
the analysis presented under section 3, and its demonstration carried out under sections 4 and 5 respectively. However, similar results may be obtained for any other permissible value of \( N > 5 \).

### Table 1.

Table 1. Various values (round off at the fourth decimal place) of the compactness ratio \( u \equiv M/R \) as obtained for different assigned values of the ratio of the centre pressure to centre energy-density, \( \sigma \equiv (P_0/E_0) \), corresponding to the self-bound regular solutions, namely - Tolman’s type IV [9] solution [indicated by \( u_{T-IV} \)], Durg IV = Durgapal and Bannerji (D-B) [31] solution [indicated by \( u_{D-B} \)], and Durg V = Durgapal and Fuloria (D-F) [18] solution [indicated by \( u_{D-F} \)] respectively. The compactness ratio corresponding to homogeneous density distribution (interior Schwarzschild solution) is indicated by \( u_h \) for the same value of \( \sigma \). It is seen that for each and every assigned value of \( \sigma \), \( u_{T-IV} \), \( u_{D-B} \), and \( u_{D-F} \geq u_h \) which is the evidence that the self-bound regular solutions (indicated by \( u_{T-IV} \), \( u_{D-B} \), and \( u_{D-F} \) respectively) are not compatible with the state of hydrostatic equilibrium.

| \( \sigma \equiv (P_0/E_0) \) | \( u_h \) | \( u_{T-IV} \) | \( u_{D-B} \) | \( u_{D-F} \) |
|-------------------------|--------|---------|--------|--------|
| 0.1252    | 0.1654 | 0.1820  | 0.1743 | 0.1718 |
| 0.1859    | 0.2102 | 0.2387  | 0.2221 | 0.2187 |
| 0.2202    | 0.2301 | 0.2652  | 0.2429 | 0.2392 |
| 0.2800    | 0.2580 | 0.3043  | 0.2714 | 0.2676 |
| 0.3150    | 0.2714 | 0.3239  | 0.2847 | 0.2809 |
| (1/3)     | 0.2778 | (1/3)   | 0.2909 | 0.2872 |
| 0.3774    | 0.2914 | 0.3038  | 0.3003 |        |
| 0.4350    | 0.3002 | 0.3176  | 0.3145 |        |
| 0.4889    | 0.3178 | 0.3281  | 0.3253 |        |
| 0.5499    | 0.3289 | 0.3378  | 0.3354 |        |
| 0.6338    | 0.3415 | 0.3485  | 0.3465 |        |
| 0.6830    | 0.3476 | 0.3535  | 0.3519 |        |
| 0.7044    | 0.3501 | 0.3555  | 0.3541 |        |
| 0.7085    | 0.3506 | 0.3559  | 0.3545 |        |
| 0.7571    | 0.3558 | 0.3601  | 0.3589 |        |
| 0.8000    | 0.3599 | 0.3633  | 0.3624 |        |
| 0.8360    | 0.3630 | 0.3658  | 0.3650 |        |

### 6. Results and Conclusions

Thus, based upon the analysis regarding the actual mass ‘\( M' \) under section 3, and subsequently its demonstration and verification under sections 4 and 5 respectively, we conclude that

(i) The self-bound regular structures corresponding to a single EOS or exact solution can not exist, because they can not fulfill the requirement of the actual mass ‘\( M' \) set up by exterior Schwarzschild solution. This finding is consistent with the results obtained by using the ‘compatibility criterion’ [3].
(ii) The only self-bound regular structure which can exist in the state of hydrostatic equilibrium is described by the homogeneous density throughout the configuration (i.e., the homogeneous density solution).

(iii) The self-bound structures which could exist in the state of hydrostatic equilibrium (this fact is also evident from the ‘compatibility criterion’) would always correspond to singularities at the centre (because pressure and density both become infinity at \( r = 0 \)).

and,

(iv) The gravitationally-bound structures naturally fulfill the property of the actual mass ‘\( M' \) required by the exterior Schwarzschild solution, and this finding is also consistent with the ‘compatibility criterion’ mentioned above.

Acknowledgments

The author acknowledges Aryabhatta Research Institute of Observational Sciences (ARIES), Nainital for providing library and computer-centre facilities. He wishes to thank the referee for his valuable comments and in particular, for drawing his attention to the study of refs. [24-25]. S. Joshi, J. C. Pandey, and S. B. Pandey are acknowledged for their help in arranging some literature related to this work.

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