Supersymmetry on the Surface $S_2$

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1 Abstract

By using symplectic Majorana spinors as Grassmann coordinates in a superspace associated with the supersymmetric extension of the isometry group on the spherical surface $S_2$, it proves possible to formulate supersymmetric models on $S_2$ using superspace techniques.

2 Introduction

The formulation of gauge theories on a spherical surface [1] has provided insights into their properties. The kinetic term for all fields on this surface involves the operator $L_{\mu\nu} \equiv -x_\mu \partial_\nu + x_\nu \partial_\mu$, which is the generator of the isometry group on the spherical surface.

In formulating a supersymmetric extension of the isometry group on any surface of constant curvature, one must introduce a Fermionic operator $Q$ which is the “square root” of the full angular momentum operator $J_{\mu\nu}$, whose commutation relation is the same as that of $L_{\mu\nu}$. Since $J_{\mu\nu}$ does not commute with $Q$ (in contrast to the translation operator $P_\mu$ in flat space [2]), closure of the algebra often requires introduction of further Bosonic operators that act as internal symmetry generators (which do not commute with $Q$) [3-5].

Supersymmetric models on several surfaces of constant curvature have been formulated using $L_{\mu\nu}$ directly in the kinetic term for component fields. In particular, supersymmetric models on $S_2$ and $AdS_2$ have been devised [6]. (This approach is distinct from supersymmetric models on a surface of constant curvature formulated by specializing the gravitational background field in a supergravity model; this approach has been used in [7].) The component field action for a supersymmetric model on $S_2$ is [6]

$$S = \int \frac{dA}{a^2} \left\{ \frac{1}{2} \psi^\dagger (x) (\tau \cdot L + \zeta) \psi (x) - \phi^* \left( L^2 + \zeta (1 - \zeta) \right) \phi \right. $$

$$- \frac{1}{2} F^* F \right\} + \lambda_N \left[ 2 (1 - 2 \zeta) \phi^* \phi 

$$- \left( F^* \phi + F \phi^* \right) - \psi^\dagger \psi \right\}^N \right\}. \quad (1)$$

2
In (1), \( \phi \) and \( F \) are complex scalars and \( \psi \) a two component Dirac spinor, defined on the surface of a sphere of radius \( a \) in three dimensions. The angular momentum vector is \( L = -ix \times \nabla \), the \( \tau \) are the Pauli spin matrices, and \( \zeta \) and \( \lambda_N \) are arbitrary real parameters.

The action of (1) is invariant under the supersymmetry transformations

\[
\delta \phi = \xi^\dagger \psi \tag{2a}
\]
\[
\delta \psi = [2(\tau \cdot L + 1 - \zeta)\phi - F] \xi \tag{2b}
\]
\[
\delta F = -2\xi^\dagger(\tau \cdot L + \zeta)\psi \tag{2c}
\]

and the special transformation

\[
\delta \phi = \lambda i [2(1 - \zeta)\phi - F] \tag{3a}
\]
\[
\delta \psi = \lambda i [1 + 2\tau \cdot L] \psi \tag{3b}
\]
\[
\delta F = \lambda i [-4 \left( L^2 + \zeta (1 - \zeta) \right) \phi + 2\zeta F] \tag{3c}
\]

as well as the usual rotations generated by the angular momentum operator \( \vec{J} \). These transformations are generated by \( \exp \left[ \xi^\dagger Q - Q^\dagger \xi + i\lambda Z + i\vec{\omega} \cdot \vec{J} \right] \) where \( Q, Q^\dagger, Z \) and \( J^a \) satisfy the algebra

\[
\{Q_i, Q^\dagger_j\} = Z\delta_{ij} - 2\tau^a_{ij}J^a \tag{4a}
\]
\[
[J^a, Q^\dagger_i] = -\frac{1}{2} \tau^a_{ij}Q_j \tag{4b}
\]
\[
[Z, Q_i] = -Q_i \tag{4c}
\]
\[
\left[ J^a, J^b \right] = i\epsilon^{abc}J^c. \tag{4d}
\]

(For two other superalgebras associated with \( S_2 \), see ref. [5].) This algebra has a representation in superspace

\[
Q = (\tau \cdot x + \beta) \frac{\partial}{\partial \theta^\dagger} + \left( \frac{\partial}{\partial \beta} - \tau \cdot \nabla \right) \theta \tag{5a}
\]
\[
Q^\dagger = \frac{\partial}{\partial \theta} (\tau \cdot x + \beta) - \theta^\dagger \left( \frac{\partial}{\partial \beta} - \tau \cdot \nabla \right) \tag{5b}
\]
\[
J^a = \frac{1}{2} \left[ \frac{\partial}{\partial \theta^\dagger} \tau^a \theta + \theta^\dagger \tau^a \frac{\partial}{\partial \theta} \right] + L^a \tag{5c}
\]
where $\beta$ is a Bosonic variable with no apparent physical significance and $\theta$ is a Dirac spinor that acts as a Grassmann coordinate.

A component field model similar to (1) has been formulated as a surface in 2+1 dimensions associated with the space $AdS_2$ [6,8]. For this space (as well as $AdS_3$) it has also proved possible to formulate supersymmetric models in superspace [8,9]. This has been feasible as the Grassmann coordinates in $AdS_2$ and $AdS_3$ are spinors with two independent components (Majorana spinors for $AdS_2$ and Majorana-Weyl for $AdS_3$) which limits the component fields which can contribute to a scalar superfield to being a pair of real scalars and a two component Majorana spinor. The superfield actions that are devised have viable kinetic and interaction contributions for these component fields. Curiously, the superfield actions on $AdS_2$ are distinct from the component field action that resembles the $S_2$ actions of eq. (1).

Though in eq. (5) we have a representation of the supersymmetry operators in superspace, it is not immediately clear how to construct a superfield model in this superspace. A general superfield takes the form

$$\Phi(x, \theta, \theta^\dagger) = \phi(x) + \psi^\dagger(x)\theta + \theta^\dagger\psi(x) + F(x)\theta^\dagger\theta + V^a(x)\theta^\dagger\tau^a\theta + \left(\lambda^\dagger(x)\theta + \theta^\dagger\lambda(x)\right)\theta^\dagger\theta + G(x)\left(\theta^\dagger\theta\right)^2,$$

where $\phi$, $F$ and $G$ are scalars, $\psi$ and $\lambda$ are spinors and $V^a$ is a vector. Reducing the number of independent components in $\Phi$ as is done in 3+1 dimensional space does not seem feasible, as there does not appear to be an analogue of the operators $D$ that permit one to define chiral superfields. (The possibility of having a real gauge superfield has not been pursued.)

In the next section it is shown that by replacing the Dirac spinor $\theta$ with a pair of symplectic Majorana spinors, one can write down a suitable superfield action involving just one of these two spinors. The only problem is that the action is not Hermitian; this problem is rectified by adding to this action its Hermitian conjugate which necessarily involves the second of the two symplectic Majorana spinors.
3 Superfield action on $S_2$.

We first note that as $\tau^2 \tau^a \tau^2 = -\tau^a T$, a suitable charge conjugation matrix is provided by $C = \tau^2$, and the charge conjugate of a spinor $\psi$ is $\psi_C = C \psi^{\dagger T} = (\tilde{\psi})^\dagger$. Since $(\psi_C)_C = -\psi$ one cannot have a Majorana spinor in $3 + 0$ dimensions; one can however have a pair of symplectic Majorana spinors
\[
\psi_1 = (\psi + \psi_C) / \sqrt{2} \quad (7a)
\]
\[
\psi_2 = (\psi - \psi_C) / \sqrt{2} \quad (7b)
\]
so that
\[
(\tilde{\psi}_1 \psi_1) = - (\tilde{\psi}_2 \psi_2) \quad (8a)
\]
\[
(\psi_1)C = -\psi_2 \quad (8b)
\]
\[
(\psi_2)C = +\psi_1 \quad (8c)
\]
(viz. $(\psi_\alpha)_C = -\epsilon_{\alpha\beta} \psi_\beta$). Upon decomposing the spinorial generator $Q$ of eq. (4) in this way, the anticommutator of eq. (4a) becomes
\[
\{Q_1, \tilde{Q}_1\} = -2 \tau \cdot J = - \{Q_2, \tilde{Q}_2\} \quad (9a)
\]
\[
\{Q_1, \tilde{Q}_2\} = Z = \{Q_2, \tilde{Q}_1\} \quad (9b)
\]
where $\tilde{Q}_\alpha = Q_\alpha^{\dagger T} C = (Q_\alpha)_C$. Using symplectic Majorana spinors $\theta_1$ and $\theta_2$, a superspace representation of $Q_\alpha$, $\tilde{Q}_\alpha$, by eq. (5), is given by
\[
Q_1 = \left( -\tau \cdot x \frac{\partial}{\partial \theta_1} + \beta \frac{\partial}{\partial \theta_2} \right) + \left( \tau \cdot \nabla \theta_1 - \frac{\partial}{\partial \beta} \theta_2 \right) \quad (10a)
\]
\[
Q_2 = \left( \tau \cdot x \frac{\partial}{\partial \theta_2} - \beta \frac{\partial}{\partial \theta_1} \right) + \left( \tau \cdot \nabla \theta_2 - \frac{\partial}{\partial \beta} \theta_1 \right) \quad (10b)
\]
\[
\tilde{Q}_1 = \frac{\partial}{\partial \theta_1} \tau \cdot x - \frac{\partial}{\partial \theta_2} \beta - \tilde{\theta}_1 \tau \cdot \nabla - \tilde{\theta}_2 \frac{\partial}{\partial \beta} \quad (10c)
\]
\[
\tilde{Q}_2 = \frac{\partial}{\partial \theta_2} \tau \cdot x + \frac{\partial}{\partial \theta_1} \beta - \tilde{\theta}_2 \tau \cdot \nabla - \tilde{\theta}_1 \frac{\partial}{\partial \beta} \quad (10d)
\]
\[
Z = - \left( \tilde{\theta}_1^{\dagger} \frac{\partial}{\partial \theta_1^T} + \theta_2^{\dagger} \frac{\partial}{\partial \theta_1^T} \right) \quad (10e)
\]
Excising the parts of $Q_a$ in (10) dependent on $\beta$ or $\frac{\partial}{\partial \beta}$, we define

$$q_1 = -\tau \cdot x \frac{\partial}{\partial \theta_1} + \tau \cdot \nabla \theta_1 = + (q_2)_C$$

$$q_2 = \tau \cdot x \frac{\partial}{\partial \theta_2} + \tau \cdot \nabla \theta_2 = - (q_1)_C$$

$$\tilde{q}_1 = -\frac{\partial}{\partial \theta_1} \tau \cdot x - \tilde{\theta}_1 \tau \cdot \nabla$$

$$\tilde{q}_2 = \frac{\partial}{\partial \theta_2} \tau \cdot x - \tilde{\theta}_2 \tau \cdot \nabla$$

so that

$$\{q_1, \tilde{q}_1\} = -2\tau^a J^a_1$$

$$\{q_2, \tilde{q}_2\} = +2\tau^a J^a_2$$

$$[J^a_\alpha, J^b_\alpha] = i\epsilon^{abc} J^c_\alpha \quad (\alpha = 1, 2)$$

$$[J^a_\alpha, q_\alpha] = -\frac{1}{2}\tau^a q_\alpha \quad (\alpha = 1, 2)$$

where

$$J^a_\alpha = -i(x \times \nabla)^a + \frac{1}{2} \frac{\partial}{\partial \theta_\alpha} \tau^a \theta_\alpha.$$  

We note that [8] with $Q$ given by (5a),

$$[Q, \Delta] = 0 = [Q, R^2]$$

where

$$\Delta = \theta^i \frac{\partial}{\partial \theta^i} + \theta \frac{\partial}{\partial \theta} + x \cdot \nabla + \beta \frac{\partial}{\partial \beta}$$

$$R^2 = x^2 - \beta^2 - 2\theta^1 \theta.$$  

Similarly, we find that for $\alpha = 1, 2$

$$[q_\alpha, \Delta_\alpha] = 0 = [q_\alpha, R^2_\alpha]$$

where

$$\Delta_1 = \theta_1 \frac{\partial}{\partial \theta_1} + x \cdot \nabla, \quad \Delta_2 = \theta_2 \frac{\partial}{\partial \theta_2} + x \cdot \nabla$$

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\[ R_1^2 = x^2 + \bar{\theta}_1 \theta_1 \quad R_2^2 = x^2 - \bar{\theta}_2 \theta_2 . \] (17b)

\[ = (R_2^2)^\dagger = (R_1^2)^\dagger \]

We now introduce two superfields

\[ \Phi_1 (x, \theta_1) = (\Phi_2 (x, \theta_2))^\dagger = \phi(x) + i \bar{\psi}_1(x) \theta_1 + i F(x) \bar{\theta}_1 \theta_1 \] (18a)

\[ \Phi_2 (x, \theta_2) = (\Phi_1 (x, \theta_1))^\dagger = \phi^*(x) + i \bar{\psi}_2(x) \theta_2 + i F^*(x) \bar{\theta}_2 \theta_2 \] (18b)

where \( \phi \) and \( F \) are complex scalars and \( \psi_1 \) and \( \psi_2 \) are a pair of symplectic Majorana spinors.

In addition, we define the operators

\[ e_1(\alpha, \beta) = -\alpha \tau \cdot x \frac{\partial}{\partial \theta_1} + \beta \tau \cdot \nabla \theta_1 = +(e_2)_C \] (19a)

\[ e_2(\alpha, \beta) = \alpha \tau \cdot x \frac{\partial}{\partial \theta_2} + \beta \tau \cdot \nabla \theta_2 = -(e_1)_C \] (19b)

\[ \bar{e}_1(\alpha, \beta) = -\alpha \frac{\partial}{\partial \bar{\theta}_1} \tau \cdot x - \beta \bar{\theta}_1 \tau \cdot \nabla \] (19c)

\[ \bar{e}_2(\alpha, \beta) = \alpha \frac{\partial}{\partial \bar{\theta}_2} \tau \cdot x - \beta \bar{\theta}_2 \tau \cdot \nabla, \] (19d)

where \( \alpha \) and \( \beta \) are real constants.

Under a transformation generated by \( q_1 \), we find that

\[ \delta \Phi_1 = [\bar{\epsilon}_1 q_1, \Phi_1] = [-i \bar{\epsilon}_1 \tau \cdot x \psi_1] + [-2i F \bar{\epsilon}_1 + \bar{\epsilon}_1 \tau \cdot \partial \phi] \theta_1 \]

\[ + \left[ -\frac{i}{2} \bar{\epsilon}_1 \tau \cdot \partial \psi_1 \right] \bar{\theta}_1 \theta_1 \] (20)

from which we can deduce the changes in \( \phi, \psi \) and \( F \). The change in the \( \bar{\theta}_1 \theta_1 \) contribution to \( \Phi_1 \) is a total derivative, and consequently an action invariant under the supersymmetry transformation of (20) is given by

\[ S_1 = i \int d^3 x \int d^2 \theta_2 \delta^2 (\theta_2) \int d^2 \theta_1 \delta \left( R_1^2 - a^2 \right) \Phi_1 \bar{\epsilon}_1 e_1 \Phi_1 . \] (21)

In (21) we first note that we have defined \( \theta \)-integration so that

\[ \int d^2 \theta_\alpha \bar{\theta}_\alpha \theta_\alpha = 1 \quad (\alpha = 1, 2). \] (22)
The \( \delta \)-functions are taken to be
\[
\delta^2(\theta_1) = \tilde{\theta}_1 \theta_1 = - (\tilde{\theta}_2 \theta_2)^\dagger = - [\delta(\theta_2)]^\dagger
\] (23)
\[
\delta \left( R_1^2 - a^2 \right) = \delta \left( x^2 - a^2 \right) + \tilde{\theta}_1 \theta_1 \delta' \left( x^2 - a^2 \right) \\
= \delta \left( x^2 - a^2 \right) \left[ 1 - \frac{1}{2a^2} \tilde{\theta}_1 \theta_1 (x \cdot \partial + 1) \right] \\
= \left[ \delta \left( R_2^2 - a^2 \right) \right]^\dagger.
\] (24)

The product of all of the contributions to the integrand of eq. (21) is necessarily of the form of the superfield of eq. (18a), and hence under the transformation of eq. (20), the integrand transforms as a total derivative. The action \( S_1 \) is consequently invariant under the transformations of eq. (20). However, the action is not Hermitian. Our full action is taken to be
\[
S_{(0)} = S_1 + S_2
\] (25)
where
\[
S_2 = S_1^\dagger = -i \int d^3x \int d^2\theta_1 d^2\theta_2 \tilde{\theta}_1 \theta_1 \int d^2\theta_2 \delta \left( R_2^2 - a^2 \right) \Phi_2 \tilde{c}_e c_2 \Phi_2.
\] (26)

It is evident that the action \( S_2 \) is invariant under transformations generated by \( q_2 \). Furthermore, it is possible to supplement the action \( S_{(0)} \) of eq. (25) with interactions of the form
\[
S_{(N)} = \lambda_N \int d^2\theta_1 d^2\theta_2 \left[ \delta^2(\theta_2) \delta \left( R_1^2 - a^2 \right) \Phi_1^N - \delta^2(\theta_1) \delta \left( R_2^2 - a^2 \right) \Phi_2^N \right] \quad (N = 2, 3 \ldots)
\] (27)
where \( \lambda_N \) is a coupling.

It is now feasible to determine the component field form of the action. This entails being able to define \( \Phi_\alpha \) off the spherical surface \( S_2 \). To do this, we employ the invariant conditions
\[
\Delta_\alpha \Phi_\alpha = \omega \Phi_\alpha \quad (\alpha = 1, 2)
\] (28)
with \( \Delta_\alpha \) defined in (17a) and \( \omega \) being a real constant. It is easy to establish that (28) implies that
\[
x \cdot \partial \phi = \omega \phi
\] (29a)
\[ x \cdot \partial \psi_\alpha = (\omega - 1) \psi_\alpha \quad (\alpha = 1, 2) \tag{29b} \]
\[ x \cdot \partial F = (\omega - 2) F. \tag{29c} \]

We also use \( \partial^2 = \frac{1}{x^2} (-L^2 + (x \cdot \partial)^2 + (x \cdot \partial)) \). It is now possible to show that

\[
S_{(0)} = \int d^3 x \, \delta \left( x^2 - a^2 \right) \left\{ 2 \left( -\alpha \beta + \alpha^2 \omega \right) (\phi F + \phi^* F^*) - 2i \alpha^2 a^2 \left( F^2 \right) - F^{*2} \right\} + i \beta \left[ \tilde{\psi}_1 \left( \tau \cdot L + \frac{3}{2} \right) \psi_2 + \tilde{\psi}_2 \left( \tau \cdot L + \frac{3}{2} \right) \psi_2 \right] \tag{30a} \\
+ \frac{i \beta^2}{a^2} \left[ \phi \left( L^2 - \omega (\omega + 1) \right) \phi - \phi^* \left( L^2 - \omega (\omega + 1) \phi^* \right) \right] + \frac{2i \alpha \beta \omega^2}{a^2} \left( \phi^2 - \phi^{*2} \right) \}
\]

\[
S_{(2)} = \lambda_2 \int d^3 x \, \delta \left( x^2 - a^2 \right) \left\{ 2i (\phi F - \phi^* F^*) + \frac{1}{2} \left( -\tilde{\psi}_1 \psi_1 + \tilde{\psi}_2 \psi_2 \right) - \frac{1}{2a^2} \left( 2\omega + 1 \right) \left( \phi^2 + \phi^{*2} \right) \right\} \tag{30b} \]

\[
S_{(3)} = \lambda_3 \int d^3 x \, \delta \left( x^2 - a^2 \right) \left\{ 3i \left( F \phi^2 - F^* \phi^{*2} \right) - \frac{3}{2} \left( \phi \tilde{\psi}_1 \psi_1 - \phi^* \tilde{\psi}_2 \psi_2 \right) - \frac{1}{2a^2} (3\omega + 1) \left( \phi^3 + \phi^{*3} \right) \right\}. \tag{30c} 
\]

The expressions for \( S_{(N)} \) for \( N > 3 \) can easily be generated in the same manner.

### 4 Summary

We have demonstrated in this paper how a superfield formalism can be used to construct supersymmetric models on \( S_2 \) associated with algebra of eq. (4). The resulting model, whose component field form is given in eq. (30), is quite distinct from the component field model of eq. (1).

It is quite easy to formulate superfield models associated with the superalgebra on \( S_2 \)

\[
\{ Q, Q^\dagger \} = Z + 2 \tau \cdot J \tag{31a} \]
\[
[J^a, Q] = -\frac{1}{2} \tau^a Q \tag{31b} \]
\[
[Z, Q] = Q \tag{31c} \]
\[ [J^a, J^b] = i \epsilon^{abc} J^c \]  

(31d)
as it is so akin to the algebra of eq. (4). The superalgebra [5]

\[ \{Q, \tilde{Q}\} = \tau \cdot J \quad \{Q, Q^\dagger\} = \tau \cdot Z \]  

(32a)

\[ [J^a, Q] = -\frac{1}{2} \tau^a Q \quad [Z^a, \tilde{Q}] = \frac{1}{2} Q^\dagger \tau^a \]  

(32b)

\[ [J^a, J^b] = i \epsilon^{abc} J^c \quad [Z^a, Z^b] = -i \epsilon^{abc} J^c \]  

(32c)
on \( S_2 \) is quite distinct from those of eqs. (4) and (31); a model invariant under transformations related to this algebra are more likely to be difficult to devise. It would also be interesting to discover how the superfield formalism could be used to compute radiative effects on \( S_2 \). These matters are currently under consideration.

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