CALCULATION OF TOPOLOGICAL CHARGE OF REAL FINITE-GAP SINE-GORDON SOLUTIONS USING THETA-FUNCTIONAL FORMULAE

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The most basic characteristic of real solutions \( u(x,t) \) of the sine-Gordon equation \( u_{tt} - u_{xx} + \sin u = 0 \) which are quasiperiodic in the \( x \)-variable is the density of topological charge defined as:

\[
\bar{n} = \lim_{T \to \infty} \frac{u(x + T,t) - u(x,t)}{2\pi T}
\]

The real finite-gap solutions \( u(x,t) \) are expressed in terms of the Riemann theta function of non-singular hyperelliptic curves \( \Gamma : \mu^2 = \lambda \prod_{i=1}^{2g} (\lambda - E_i) \) and a generic positive divisor \( D \) of degree \( g \) on \( \Gamma \), where the spectral data \((\Gamma, D)\) must satisfy some reality conditions. The problem of calculating \( \bar{n} \) in terms of the spectral data was first studied in [3] and [6]. A solution to this problem was obtained in [4] avoiding the use of the \( \theta \)-functional formula for \( u(x,t) \). As pointed out by S.P. Novikov, if the \( \theta \)-functional form of solutions is to be considered as an effective one, then it should be possible to calculate \( \bar{n} \) directly from the \( \theta \)-functional formulæ. We achieve this goal in the present note, using a new multiscale or elliptic limit of real finite-gap sine-Gordon solutions. The authors would like to express their gratitude to Professor Novikov for attracting their attention to this problem and stimulating discussions.

The reality condition on \( \Gamma \) is that \( \{E_1, \cdots, E_{2g}\} = \{\overline{E_1}, \cdots, \overline{E_{2g}}\} \) for all \( k \), and \( E_i < 0 \) if \( E_i \in \mathbb{R} \). The reality condition on the divisor found by Cherednik in [1] is: \( D + \tau D - 0 - \infty = \mathcal{K} \) where \( \tau(\lambda, \mu) = (\overline{\lambda}, \overline{\mu}) \) and \( \mathcal{K} \) is the canonical class. Such a divisor will be called admissible. In order to write the \( \theta \)-functional formula for \( u(x,t) \) we need some notation. Let \( a_i, b_j \) for \( 1 \leq i, j \leq g \) be a symplectic basis of cycles on \( \Gamma \) and \( \omega = (\omega_1, \cdots, \omega_g) \) holomorphic differentials satisfying \( \int_{a_j} \omega_i = \delta_{ij} \). The Riemann matrix of \( \Gamma \) is the matrix defined by \( B_{ij} = \int_{b_j} \omega_i \). The Abel-Jacobi map \( A : \Gamma \to J(\Gamma) \) is defined by \( P \mapsto \int_{P}^\infty \omega \), where \( J(\Gamma) = \mathbb{C}^g/\{\mathbb{Z}^g + B\mathbb{Z}^g\} \) is the Jacobian variety of \( \Gamma \). Let \( K \in J(\Gamma) \) be the associated vector of Riemann constants. Let \( A(0) \equiv \epsilon'/2 + B \epsilon/2 \), for some \( \epsilon, \epsilon' \in \mathbb{Z}^g \). We define \( U = (\frac{d}{d\sqrt{A}})(\infty) \) and \( V = (\frac{d}{d\sqrt{B}})(0) \). The \( \theta \)-functional formula for \( e^{iu(x,t)} \) can be written as follows (see [3], [2]):

\[
e^{iu(x,t)} = C_1 \frac{\theta(A(0) + z(x,t)) \theta(-A(0) + z(x,t))}{\theta^2(z(x,t))}
\]

where \( z(x,t) = -A(D) + ix(V - U)/4 - it(U + V)/4 - K, C_1^2 = \exp(\pi i \epsilon' B \epsilon/2), \) and

\[
\theta(z|B) = \sum_{n \in \mathbb{Z}^g} \exp(\pi i n^i B n) \exp(2\pi i n^l z) \quad \text{for} \quad z \in \mathbb{C}^g
\]

In order to compute \( \bar{n} \) for real solutions, we choose a special basis of cycles \( a_i, b_j \) (figure 1) with parameter \( k = 1 \) as suggested in [1]. Here \( E_1, \cdots, E_{2m} \) are the non-zero real branch points and \( E_{2j} = E_{2j-1} \) for \( m + 1 \leq j \leq g \). This basis satisfies \( \tau a_i = -a_i \) for all \( i, \tau b_i = b_i \).
for $1 \leq i \leq m$ and $\tau b_i = b_i + a_i$ for $m + 1 \leq i \leq g$, therefore $\text{Re}(B) = -1/2 \begin{pmatrix} 0 & 0 \\ 0 & I_{g-m} \end{pmatrix}$ where $I_{g-m}$ is the identity matrix of size $g - m$. (From here on, all vectors $v$ written as $(v_1, v_2)^t$ are understood to be split into blocks of length $m$ and $g - m$). We have $A(0) = \epsilon'/2 + B\epsilon/2$ with $\epsilon' = (0,1)^t$ and $\epsilon = (1,0)^t$. The reality condition on the divisor becomes

$$A(D) = x + B \begin{pmatrix} 1/2 - s/4 \\ 1/2 \end{pmatrix}, \quad s = \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_m \end{pmatrix}$$

for some $x \in \mathbb{R}^g/\mathbb{Z}^g$ and for some $s_k \in \{-1,1\}$. Thus the set $\{z \in J(\Gamma) \mid z = -A(D) - K \}$ with $D$ admissible has $2^m$ components characterized by the symbol $\tilde{s} = (s_1, s_2, \ldots, s_m)$, each of which is a real $g$-dimensional torus denoted by $T_{\tilde{s}}$. The vectors $iU, iV$ are purely real and hence $z(x,t) \in T_{\tilde{s}}$. It is a consequence of the Cherednik reality condition that the $T_{\tilde{s}}$ are disjoint from $\Theta \cup (\Theta - A(0))$. Therefore the real solutions $u(x,t)$ given by equation (1) are non-singular. We also define functions $u_j(T)$ for $T \in [0,1]$ by the formula:

$$e^{iu_j(T)} = C_1 \frac{\theta(A(0) + z_j(T))\theta(-A(0) + z_j(T))}{\theta^2(z_j(T))}, \quad z_j(T) = -A(D) - Te_j - K$$

where $e_j$ is the $j$-th standard basis vector of $\mathbb{C}^g$, and therefore $z_j(T)$ represents the $j$-th basic cycle of $T_{\tilde{s}}$. The topological charge density $n$ is given by the formula (see [4]):

$$n = \sum_{j=1}^g (iU_j - iV_j) n_j/4, \quad \text{where} \quad n_j = \frac{1}{2\pi i} \int_{T=0}^{T=1} d\log e^{iu_j(T)}$$

It is convenient to replace $\epsilon = (1,0)^t$ in the expression $A(0) = \epsilon'/2 + B\epsilon/2$ with $\tilde{\epsilon}$ defined by $\tilde{\epsilon}_j = (-1)^j s_j$ for $1 \leq j \leq m$ and $\tilde{\epsilon}_j = 0$ for $j > m$. Using the transformation rule $\theta(z + N + BM) = \theta(z) \exp(-2\pi i M^t z - \pi i M^t BM)$ we get

$$n_j = -\tilde{\epsilon}_j + \frac{2}{2\pi i} \int_{T=0}^1 d\log \left( \frac{\theta(B\tilde{\epsilon}/2 + z_j(T))}{\theta(z_j(T))} \right)$$

where we have also used the fact that the cycles $\epsilon'/2 + B\tilde{\epsilon}/2 + z_j(T)$ and $B\tilde{\epsilon}/2 + z_j(T)$ are homologous in the real torus $T_{\tilde{s}}$. The charges $n_j$ are easily calculated using formula (3) in two special cases:

**Lemma 1.** In the case when $m = 0$ (no real branch points) we have $n_j = 0$ for all $j$.

In the case $g = m = 1$, we have $n_1 = s_1$. For later use, we also calculate $n_1$ from (3) if $K$ is taken to be $1/2$ instead of the correct value $(1 + \tau)/2$, and $s_1$ is replaced by $-s_1$. In this case $n_1 = -s_1$.

We reduce the calculation of the charges $n_j$ in the general case to the two special cases of Lemma 1. We consider the following family of real nonsingular hyperelliptic curves $\Gamma(k)$ depending on the real parameter $k \in [1, \infty)$:

$$\Gamma(k) : \mu^2 = \lambda \prod_{i=1}^m (\lambda - k^{i-1} E_{2i-1})(\lambda - k^{i-1} E_{2i}) \prod_{i=2m+1}^{2g} (\lambda - k^m E_i)$$

The basic cycles $a_i(k), b_j(k)$ on $\Gamma(k)$ are chosen as shown in figure (1). The original curve $\Gamma$ coincides with $\Gamma(1)$. Let $C_i$ for $1 \leq i \leq m$ be the real elliptic curves defined by $C_i : y^2 = x(x - E_{2i-1})(x - E_{2i})$ and let $C_{m+1}$ be the real hyperelliptic curve $C_{m+1} : y^2 = x \prod_{i=2m+1}^{2g} (\lambda - E_i)$. 

Let $B(k)$ denote the Riemann matrix of the curve $\Gamma(k)$ with respect to the basic cycles $a_i(k), b_j(k)$.

**Figure 1.** Basic cycles and cuts on $\Gamma(k)$ for $g = 4, m = 2$

**Theorem 1.** The limiting curve $\Gamma(\infty) = \lim_{k \rightarrow \infty} \Gamma(k)$ is a nodal curve with $m$ nodes when $m < g$, and $m - 1$ nodes when $m = g$. The irreducible components of the normalization of $\Gamma(\infty)$ are $\{C_i | 1 \leq i \leq m + 1\}$ when $m < g$, and $\{C_i | 1 \leq i \leq g\}$ when $m = g$. The limiting Riemann matrix $B(\infty) = \lim_{k \rightarrow \infty} B(k)$ is block diagonal $B(\infty) = \text{diag}(B_1, B_2)$ where $B_1 = \text{diag}(\tau_1, \cdots, \tau_m)$ for some purely imaginary $\tau_j$ in the complex upper half plane, and $\text{Re}(B_2) = -1/2 I_{g-m}$.

The charges $n_j$ being integers are constant during the deformation $k : 1 \rightarrow \infty$. Therefore $n_j$ can be calculated from formula (3) using the Riemann matrix $B(\infty)$.

Since $\theta((z_1, z_2)^t | \text{diag}(B_1, B_2)) = \theta(z_1 | B_1) \theta(z_2 | B_2)$, using Lemma 1 together with the formula

$$K = \sum_{i=1}^g A(E_{2i-1}) = \frac{1}{2} \binom{g}{1} + \frac{1}{2} B(\nu_1, \nu_2)$$

where $(\nu_1, \nu_2)^t = (1, 2, \cdots, g)^t$, we obtain the result:

$$n_j = \begin{cases} (-1)^{j-1}s_j & \text{if } 1 \leq j \leq m \\ 0 & \text{if } j > m \end{cases}$$

(4)

The details will appear in a future article.

**Remarks**

(1) In [4], the admissible divisors were characterized by certain symbols $\{s'_1, \cdots, s'_m\} \in \{-1\}^m$ defined as follows. Given an admissible divisor $D = \{(\lambda_i, \mu_i) | 1 \leq j \leq g\}$ let $P(\lambda)$ be the unique polynomial of degree $g - 1$ interpolating the $g$ points $(\lambda_i, \mu_i/\lambda_i)$. Then $P(\lambda)$ is real and $s'_j$ is defined to be the sign of $P(\lambda)$ over $[E_{2j}, E_{2j-1}]$. It was shown in [4] that the charges $n_j$ are equal to $(-1)^{j-1}s'_j$ for $j \leq m$ and $n_j = 0$ for $j > m$. Comparing with formula (4), it follows that the symbols $s'_j$ and $s_j$ coincide for all $j$. 

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The multiscale limit of the spectral curve constructed above was used only for a topological argument. The sine-Gordon solutions \( u(x, t, k) \) associated with the spectral curve \( \Gamma(k) \) (and admissible divisors \( D(k) \)) depend on the vectors \( U(k) \) and \( V(k) \) mentioned in the introduction. As \( k \to \infty \), some component of \( U(k) \) will diverge to \( \infty \). Thus there is no limiting solution. However asymptotic expansion in the parameter \( k \) of \( u(x, t, k) \) involving elliptic (genus 1) solutions can be written. This will be investigated in a future work.

**References**

[1] I.V. Cherednik, *Reality conditions in “finite-zone” integration*, Sov. Phys. Dokl. **25** (1980), 450–452.
[2] B.A. Dubrovin, S.M. Natanzon, *Real two-zone solutions of the sine-Gordon equation*, Funct. Anal. Appl. **16** (1982), 21–33.
[3] B.A. Dubrovin, S.P. Novikov, *Algebrao-geometrical Poisson brackets for real finite-zone solutions of the Sine-Gordon equation and the nonlinear Schrödinger equation*, Sov. Math. Dokl. **26** (1982), no. 3, 760–765.
[4] P.G. Grinevich, S.P. Novikov, *Topological charge of the real periodic finite-gap sine-Gordon solutions*, Comm. Pure Appl. Math. **56** (2003), no. 7 956–978.
[5] V.A. Kozel, V.P. Kotlyarov, *Almost periodic solutions of the equation \( u_{tt} - u_{xx} + \sin u = 0 \).* (Russian), Dokl. Akad. Nauk Ukrain. SSR Ser. A (1976), no. 10 878–881.
[6] S.P. Novikov, *Algebrotopological approach to the reality problems. Real action variables in the theory of finite-zone solutions of the Sine-Gordon equation*, Zap. Nauchn. Sem. LOMI: Differential geometry, Lie groups and mechanics. VI, **133** (1984), 177–196.

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