Is backreaction really small within concordance cosmology?

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Abstract
Smoothing over structures in general relativity leads to a renormalization of the background and potentially many other effects which are poorly understood. Observables such as the distance–redshift relation when averaged on the sky do not necessarily yield the same smooth model which arises when performing spatial averages. These issues are thought to be of technical interest only in the standard model of cosmology, giving only tiny corrections. However, when we try to calculate observable quantities such as the all-sky average of the distance–redshift relation, we find that perturbation theory delivers divergent answers in the UV and corrections to the background of order unity. There are further problems. Second-order perturbations are the same size as first-order ones, and fourth-order at least the same as second, and possibly much larger, owing to the divergences. Much hinges on a coincidental balance of two numbers: the primordial power and the ratio between the comoving Hubble scales at matter-radiation equality and today. Consequently, it is far from obvious that backreaction is irrelevant even in the concordance model, however natural it intuitively seems.

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1. Introduction

The standard Friedmann–Lemaître–Robertson–Walker (FLRW) models of cosmology can account for all observations to date with just a handful of parameters—a resounding success. These are the simplest reasonably realistic universe models possible within general relativity: homogeneous, isotropic and flat to a first approximation, with a scale-invariant spectrum of
Gaussian perturbations from inflation added on top to account for structures down to the scale of clusters of galaxies. Were there a physical motivation for the value of the cosmological constant and the associated coincidence problem, we would have a nearly complete understanding at nearly all epochs.

The coincidence problem has led some to speculate that the evidence for a cosmological constant might in fact be a misunderstanding of the standard model, and that complex relativistic ‘backreaction’ effects are present when structure forms. This might be important in a variety of ways [1–9]. The canonical mechanism might arise by altering the dynamics of spacetime when we smooth over structure to form a spatially homogeneous and isotropic ‘background’ [10–18]. In the standard picture, this would be represented by a mismatch between the background model at early versus late times—in the standard model they are assumed to be the same. Alternatively, smoothing observations over the sky might lead to a significantly different cosmology than a corresponding spatial smoothing process (sometimes known as the fitting problem [19]). Some authors have argued that it is specifically an infrared divergence which appears at second order that gives the possibility of the backreaction effect which can mimic dark energy [6, 7, 11].

That such mechanisms might lead to significant effects—let alone be the cause of apparent acceleration—has been quickly dismissed by many in the community [20–36]. We can understand the argument why backreaction must be small as follows.

The fields that are propagating on the background must alter its dynamics through the nonlinearity of the field equations. According to observers at rest with the gravitational field, the potential itself is small everywhere outside objects less dense than neutron stars, and so we write $g_{ab} = g^{(0)}_{ab} + g^{(1)}_{ab}$ where $g_{ij}^{(1)} \sim \Phi g_{ij}^{(0)}$ and where $\Phi \sim 10^{-5} \ll 1$. The affine connection or Ricci rotation coefficients determine the dynamics of the spacetime and are generically $O(\partial \Phi)$; the field equations are $O(\partial^2 \Phi)$. A perturbation of wavelength $\lambda = a/k$, where $k$ is the comoving wavenumber and $a$ is the scale factor, much less than the Hubble scale, can give rise to large fluctuations in the field equations $O(k^2 \Phi)$, even though the change from the background metric is small. Such terms describe density fluctuations, and these can be large even though the metric potentials are small. After averaging over a large domain, however, all first-order quantities vanish by assumed homogeneity and isotropy of the perturbations. This picture predicts the CMB temperature anisotropies and the matter power spectrum to all required precision. At second order, the Einstein tensor can only contain terms as large as the peculiar velocity squared $v^2 \sim (\partial \Phi)^2$. The amplitude of these terms is only the same size as $\Phi$, with contributions from higher derivative terms being contained in pure divergence terms which are zero by statistical homogeneity. Higher-order terms are suppressed by further factors of $\Phi$. Any infrared contribution must be a gauge effect and can be renormalized away. So, backreaction cannot be significant.

Much of this argument is correct. Yet it also contains elements which are startling: second-order terms are the same size as first-order terms. In particular, the second-order potential has the amplitude $\Phi^{(2)} \sim \Phi$, which is really not supposed to happen. Furthermore, if we calculate dimensionless measures of the time derivative of the Hubble rate, we find contributions from perturbation theory like $(\partial^2 \Phi)^2$, which are of order unity, and are actually divergent in the UV for a scale-invariant initial power spectrum; using any natural cutoff—from the end of inflation, say—gives colossal numbers. As we show here, these are not just a curiosity: we find the same thing if we calculate the monopole of the distance-redshift relation, a physical observable. We argue, as others have done [7, 8, 11, 37, 38], that it appears that higher-order perturbations will make things worse, not better.

This is not to say that backreaction is necessarily large or significant, let alone responsible for an apparent acceleration effect. Rather we argue that until we have a better grasp of
the consequences of these divergences, and higher-order perturbation theory in general, it is impossible to conclude that backreaction is small. Clearly, linear perturbation theory is correct in many respects. This success comes from analysing anisotropic and inhomogeneous perturbations, and observations of the CMB temperature anisotropy and the matter power spectrum, which all confirm it is basically correct. But backreaction concerns the homogeneous contribution of these perturbations at higher order when they are calculated in terms of initial perturbations. Are they small?

2. Measures of backreaction

In cosmology it is implicitly assumed that by statistical isotropy and homogeneity, we simply construct a smooth FLRW ‘spacetime’ from a lumpy inhomogeneous spacetime by averaging over structure. This might, in principle, have a metric

$$d s^2_{\text{eff}} = - d\tau^2 + a_D^2 \gamma_{ij} dy^i dy^j,$$

where $\tau$ is the average cosmic time and $a_D(\tau)$ is an averaged scale factor, the subscript $D$ indicating that it has been obtained at a certain spatial scale $D$, which is large enough so that a homogeneity scale has been reached; in this case, $\gamma_{ij}$ will be a metric of constant curvature. Unfortunately, we do not know how to construct $d s^2_{\text{eff}}$ in an invariant way—there is no unambiguous averaging scheme. We do not know what field equations this metric would obey, nor do we know how to calculate observational relations; in fact, we do not really know whether any of the standard tools of general relativity survive. There is no reason to believe any of these would be as in GR, as it has only been tested in the solar system where the averaging problem does not apply.

While different approaches to averaging exist which provide corrections to the field equations, those corrections require an underlying model with structure. In effect, it is as difficult to calculate a model with structure as the smooth average model which best corresponds to it. The scale factor itself comes from the solution to the field equations, but its first time derivative gives the Hubble expansion rate which is given algebraically from the Friedmann equation. Both of these change with averaging [1, 3]. We can estimate what the averaged scale factor in equation (1) might be by different properties of it directly within the standard model, using perturbation theory. Before we get to that, let us first consider the measures of the expansion rate locally in a spacetime; then consider averaging spatially; and finally let us consider how averages of observables look in a general spacetime—which is what we actually use to infer properties of $a_D$.

2.1. Exact spacetime dynamics

Given a 4-velocity $u^a$, the local volume expansion is defined covariantly as

$$\Theta = 3H = \nabla_au^a = 3u^a\nabla_a \ln \ell = 3\frac{\dot{\ell}}{\ell},$$

where $u^a$ is the local fluid 4-velocity with length scale $\ell$. Locally, the expansion rate obeys the usual Friedmann and Raychaudhuri equations:

$$\Theta^2 = 3\left[\rho + \Lambda + \sigma^2 - \frac{1}{3\ell^2}R\right]$$

$$\dot{\Theta} = -\frac{1}{3} \Theta^2 - \frac{1}{12} (\rho + 3p) + \Lambda - 2\sigma^2 + 2\omega^2 + A_\alpha A^\alpha + \text{div}A,$$

where the definitions of the terms are given in the appendix. For an exact solution to the field equations, the Friedmann equation is a first integral of the Raychaudhuri equation, along with
other equations such as the evolution equation for the shear—see the appendix. From this we can define a local deceleration parameter,
\[ q/\Theta_1 = -1 - \frac{3}{\Theta_1^2} \left( \frac{\rho + 3p}{\rho + \Lambda + \sigma^2 - \frac{1}{2} \rho \omega^2} - \text{div} A - A_a A^a \right), \]
(5)
\[ = \frac{3}{\rho + \Lambda + \sigma^2 - \frac{1}{2} \rho \omega^2} \left[ \frac{1}{2} (\rho + 3p) - \Lambda + 2\sigma^2 - 2\omega^2 - \text{div} A - A_a A^a \right], \]
(6)
which is commonly used as a dimensionless measure of the rate of change of the expansion rate. Over a small interval of cosmic time, both \( \Theta \) and \( q/\Theta_1 \) give important information about the dynamics of \( u^a \).

2.2. Spatial averaging formalism

Let us assume for the moment an irrotational dust spacetime, and observers and coordinates at rest with respect to the dust. The average of a scalar quantity \( S \) may be (a covariant version of the averaged equations exists [39], but they lead to almost the same form of averaged equations, because once a foliation is chosen everything else becomes 3D covariant) defined as simply its integral over a region of a spatial hypersurface \( D \) of constant proper time divided by the Riemannian volume (we follow [3]):
\[ \langle S(t, x) \rangle_D = \frac{1}{V_D} \int_D \sqrt{\det h} \, d^3x \, S(t, x). \]
(7)
Taking the time derivative of equation (7) yields the commutation relation
\[ \partial_t \langle S \rangle_D - \langle \partial_t S \rangle_D = \langle \Theta S \rangle_D - \langle \Theta \rangle_D \langle S \rangle_D , \]
(8)
where \( \Theta \) is the local expansion of the dust, and we assume that the domain is comoving with the dust. The dimensionless volume scale factor is defined as \( a_D \propto V_D^{1/3} \), which ensures \( \langle \Theta \rangle_D = 3 \partial_t \ln a_D \). Then, the equations of motion for this scale factor become
\[ 3 \left( \frac{a_D}{\dot{a}_D} \right)^2 = \langle \rho \rangle_D + \Lambda - \frac{1}{2} [Q_D + \langle R \rangle_D] \]
\[ 3 \frac{\dot{a}_D}{a_D} + \frac{1}{a_D} (\rho) = \Lambda + Q_D, \]
(9)
where \( Q_D = \frac{1}{2} \left[ (\Theta^2) - (\Theta)^2_0 \right] - 2(\sigma^2) \) is the kinematic backreaction term and \( \sigma^2 = \frac{1}{2} \sigma_{ab} \epsilon^{ab} \) is the magnitude of the shear tensor. The non-local variance of the local expansion rate can act in the same way as the cosmological constant, causing the average expansion rate to speed up, even if the local expansion rate is slowing down. For consistency, and assuming \( \langle \rho \rangle_D \propto a_D^{-3} \), these equations must lead to the integrability condition
\[ \frac{1}{a_D^3} \partial_6 \langle Q_D a_D^6 \rangle + \frac{1}{a_D^2} \partial_6 \langle (R) a_D^2 \rangle = 0. \]
(10)
A deceleration parameter \( q_D \) which describes the deceleration of the averaged hypersurface may be defined as
\[ q_D = -\frac{1}{H_D^2} \frac{\dot{a}_D}{a_D}, \]
(11)
\[ \langle \rho \rangle_D - \Lambda + (\sigma^2)_D = \frac{1}{2} \left( \langle \Theta^2 \rangle_D - \langle \Theta \rangle_D^2 \right) \]

where \( \Theta \) is given by the averaged Raychaudhuri equation. However, it should be noted that this is very different from the average of the local deceleration parameter, which is

\[ \langle q \rangle_D = -1 - \frac{3}{2} \left( \frac{\dot{\Theta}}{\Theta^2} \right) \]

As we can see, it is not easy nor unique to estimate \( \ddot{a}_D \) given \( a_D \), and we shall see that they are quite different in perturbation theory.

These quantities come with a significant drawback: even for moderately sized domains, these quantities are unobservable because they are averaged on spatial hypersurfaces, whereas we only have observational access to our past lightcone (however see [40]).

2.3. Averaging physical observables

Within the standard cosmology, cosmological parameters such as the Hubble rate and deceleration parameter are well defined based on the background metric. These cosmological parameters can be evaluated today, at \( t_0 \), by taking a Taylor series expansion of the scale factor \( a(t) \):

\[ a(t) = a_0 \left[ 1 + H_0 (t - t_0) + \frac{1}{2} q_0 H_0^2 (t - t_0)^2 + O((t - t_0)^3) \right] \]

Similarly, we can calculate the important observational quantities of interest, such as the relation between the angular-diameter distance and redshift \( d_A(z) \):

\[ d_A(z) = \frac{z}{H_0} \left( 1 - \frac{1}{2} (3 + q_0) z + O(z^2) \right) \]

Of course, these series expansions are only valid in an FLRW background spacetime, but they can be generalized to an arbitrary spacetime using the Kristain and Sachs approach [41, 42]. This gives a way to define a generalized Hubble rate and deceleration parameter in an arbitrary spacetime, which are observationally rather than dynamically motivated; that these have important differences was first shown in [43].

The corresponding equations to equations (15) are as follows:

\[ z = \left[ K^a K^b \nabla_a u_b \right] d_A + \frac{1}{2} \left[ K^a K^b K^c \nabla_a \nabla_b u_c \right] d_A^2 + \cdots , \]

\[ d_A = \frac{z}{K^a K^b \nabla_a u_b |_{t_0}} \left\{ 1 - \left[ \frac{1}{2} \left( K^a K^b K^c \nabla_a \nabla_b u_c \right) \right] |_{t_0} z + \cdots \right\} , \]

where the past pointing null vector is given by

\[ K^a = \frac{k^a}{(u_b k^b)_{t_0}} = -u^a + e^a , \]

which points in the direction \( e^a \) at the observer, located at ‘0’. Using the notation whereby \( K_{a_1} = K_{a_1} K_{a_2} \cdots K_{a_n} \) and \( \nabla_{a_1} = \nabla_{a_1} \nabla_{a_2} \cdots \nabla_{a_n} \), the \( n \)th term \( K^{a_{n+1}} \nabla_{a_1} u_{a_1} \) for a given observer is an observable, in principle, and will have a spherical harmonic expansion up to \( \ell = n + 1 \). The monopole will be given by an all sky integral, \( \int d\Omega K^{a_{n+1}} \nabla_{a_1} u_{a_1} \), and is usually what we would compare to observations in an FLRW context.
In an arbitrary spacetime, the first terms in these expansions are given by
\[
K^a K^b \nabla_a u_b = \frac{1}{3} \Theta - A_a e^a + \sigma_{ab} e^a e^b \quad (20)
\]
\[
K^a K^b K^c \nabla_a \nabla_b u_c = \frac{1}{6} (\rho + 3p) + \frac{1}{3} \Theta^2 - \frac{1}{3} \Lambda - \frac{2}{3} \omega_a \omega^a + \sigma_{ab} \sigma^{ab} + A_a A^a - \frac{2}{3} \text{div} A
- e^{[4]} e^{[4]} \left[ D_0 \Theta + \frac{3}{2} \text{div} \sigma_a + A_a - \frac{4}{3} \Theta A_a - \frac{1}{2} \sigma_{ab} A^{bc} - \epsilon_{abc} A^{bc} \right]
+ e^{[4]} e^{[4]} \left[ F_{ab} - \frac{1}{2} \sigma_{ab} + 2 \Theta \sigma_{ab} + \omega_a \omega_b + 3 \sigma_a \sigma_{bc} + 2 \epsilon_{acd} \omega^d \sigma_{bc} - 2 \text{div} A \right]
+ e^{[4]} e^{[4]} \left[ A_a \sigma_{bc} - D_0 \sigma_{bc} \right]. \quad (21)
\]

The projected symmetric trace-free (PSTF) tensors \( e^{(A)} \), for \( \ell = 0, 1, 2, 3 \), in this expression are a covariant representation of the spherical harmonics when evaluated at a given point in spacetime (see the appendix).

In the same way as we may evaluate various aspects of the spacetime by considering averages of the derivatives of \( aD_a \), we can investigate how observations might be affected by the evolution of structure. For example, the observable \( K^a K^b \nabla_a u_b \) gives us a generalized Hubble rate which now has a dipole and quadrupole in addition to the familiar monopole term \([43]\). How shall we best compare higher-order terms with the familiar FLRW expansion? Various definitions have been considered \([6, 43]\).

Note that the \( d_A(z) \) relation has powers of \( K^a K^b \nabla_a u_b \) on the denominator of each coefficient. Since this means dividing by a spherical harmonic expansion, it will be much simpler to work with the \( z(d_A) \) relation instead, and to compare that with the FLRW relation.

The definition we give below for the deceleration parameter is not unique and depends on this choice.

Let
\[
H_{0}^{\text{obs}} = \sum_{\ell=0}^{2} H_{A\ell} e^{A\ell} = [K^a K^b \nabla_a u_b]_0, \quad (22)
\]
where the \( H_{A\ell} \) are PSTF moments of the generalized Hubble rate \( H_{0}^{\text{obs}} \). Clearly the moments are simply
\[
H_{\ell} = \frac{1}{3} \Theta, \quad H_a = A_a \quad \text{and} \quad H_{ab} = \sigma_{ab}, \quad (23)
\]
evaluated at the observer. We now would like to relate the FLRW quantities \((3 + q_0) H_{0}^2\) to \( K^a K^b K^c \nabla_a \nabla_b u_c \), to define a generalized deceleration parameter. The simplest way is to define \( q_0^{\text{obs}} \) via
\[
(3 + q_0^{\text{obs}}) H_{0}^2 = [K^a K^b K^c \nabla_a \nabla_b u_c]_0. \quad (24)
\]
Then, writing \( q_0^{\text{obs}} = \sum_{\ell=0}^{3} Q_{A\ell} e^{A\ell} \), where the PSTF tensors \( Q_{A\ell} \) are the multipole moments of the observational deceleration parameter, we have the monopole
\[
Q = \frac{3}{\Theta^2} \left[ 1 + (\rho + 3p) - \Lambda + 6 \sigma^2 - 2 \omega^2 - 2 \text{div} A + 3 A^2 \right]_0. \quad (25)
\]
Note that this is not the same as the local deceleration parameter associated with \( \Theta \).

We have, then, three different ways to analyse the second derivative of the expansion rate, all of which have been written as variants of the familiar deceleration parameter. We have the local exact deceleration parameter of which we can calculate the average over a domain, the deceleration parameter associated with the deceleration of the averaged hypersurfaces, and the ‘observed’ deceleration parameter, which we have defined using the all-sky average (monopole) of the redshift–distance relation. In a general spacetime, these are quite different things.
3. Backreaction in the standard model

The standard model of cosmology ignores all the complexity of smoothing the spacetime by assuming that on 'large' scales (say larger than a few hundred Mpc) we can model the universe as homogeneous and isotropic. Linear and higher-order fluctuations describing structure then propagate as smooth fields on this background. In theory, when we sum over all these smooth fields, we should end up with a metric which describes the universe we see, with structure properly described on all relevant scales (statistically speaking).

We shall consider how backreaction comes about in the simplest cosmology which agrees with observations: a flat LCDM model with Gaussian scalar perturbations. In particular, we shall examine how backreaction affects the first and second derivatives of the scale factor, as given the Hubble rate and various deceleration parameters. Averaging FLRW perturbations has been discussed often in the literature: some authors investigate specifically the modification to the Hubble expansion rate or other variables [5, 6, 10–15, 18, 38, 44–46]; others reformulate the average of the backreaction into an effective fluid [17, 29, 35, 47–52], while one of the first attempts considered the important problem of how to calculate the averaged metric [53]. Many of these works consider only the case of backreaction in an Einstein–de Sitter model, in the hope of finding it responsible for dark energy. As it is more plausible that backreaction may lead to changes to the background at the level relevant for precision cosmology, we also need to investigate its effects in a LCDM model.

3.1. Perturbation theory

In the Poisson gauge to second order in scalar perturbations, the metric reads

\[
ds^2 = -[1 + 2\Phi + \Phi^{(2)}] dt^2 - a V_i dx^i dt + a^2 [(1 - 2\Phi - \Psi^{(2)}) \gamma_{ij} + h_{ij}] dx^i dx^j.
\]  

(26)

The background evolution of the scale factor \( a(t) \) at late times is determined by the Friedmann equation

\[
H(a)^2 = \left(\frac{\dot{a}}{a}\right)^2 = H_0^2 [\Omega_m a^{-3} + 1 - \Omega_m],
\]  

(27)

where the Hubble constant \( H_0 \) is the present day expansion rate, and \( \Omega_m \) is the normalized matter content today. The first-order scalar perturbations are given by \( \Phi, \Psi \) (and are all that is required for observations at the moment), and the second-order perturbations by \( \Phi^{(2)}, \Psi^{(2)} \) which are needed for a consistent analysis of backreaction. We also include the second-order vector modes \( V_i \) and tensor modes \( h_{ij} \). In this gauge, we have the metric in its Newtonian-like form, which we may think of as the local rest frame of the gravitational field because it is the frame in which the magnetic part of the Weyl tensor vanishes when vectors and tensors are ignored [15].

For a single fluid with zero pressure and no anisotropic stress, \( \Psi = \Phi \), and \( \Phi \) obeys the 'master' equation

\[
\ddot{\Phi} + 4H \dot{\Phi} + \Lambda \Phi = 0.
\]  

(28)

For a LCDM universe, the solution in time to this equation has \( \Phi \) constant until \( \Lambda \) becomes important, and then starts to decay as \( \Lambda \) suppresses the growth of structure on all scales by about a factor of 2. We write it as \( \Phi(t, x) = g(t)\Phi_0(x) \), where \( g(t) \) is the growing solution to equation (28) normalized to \( g = 1 \) today (we can use \( g_\infty = g(t = 0) \approx \frac{1}{2}(3 + 2\Omega_m^{-0.45}) \) as a very good approximation to its early time value). There is no scale dependence in the equation, which all comes from the initial conditions—usually a nearly scale-invariant Gaussian spectrum from frozen quantum fluctuations during inflation—and subsequent evolution during
the radiation era. Evolution during the radiation era suppresses wavelengths which enter the Hubble radius compared to those which remain larger than it until the matter era begins. Consequently, in Fourier space, assuming scale-invariant initial conditions from inflation, the power spectrum of $\frac{\Phi_1}{\Phi_1}$, $P_{\Phi_1}$, is independent of scale for modes larger than the equality scale, $k_{\mathrm{eq}} = \sqrt{2\Omega_m c_{\mathrm{eq}} H_0} \approx 0.07 \Omega_m h^2 \text{ Mpc}^{-1}$. A dimensionless transfer function describes the loss of power in the case of zero baryons (adapted from [54]):

$$T(k) = \frac{\ln(2e + 0.134\kappa)}{\ln(2e + 0.134\kappa) + \left[0.079 + \frac{4.06}{1.0466\kappa}\right]\kappa^2}$$

where $\kappa = k/k_{\mathrm{eq}}$. This is unity for $\kappa \ll 1$ and $\sim (\ln \kappa)/\kappa^2$ for $\kappa \gg 1$. The change in behaviour at the equality scale is important for backreaction because it is the modes larger than the equality scale which are primarily responsible for any backreaction at all. In essence, the equality scale determines the size of the backreaction effect.

All first-order quantities can be derived from $\Phi$; for example,

$$v^{(1)}_i = -\frac{2}{3a^2H^2\Omega_m} \partial_i (\dot{\Phi} + H\Phi)$$

is the first-order velocity perturbation, which governs the peculiar velocity between the matter flow and the rest frame of the gravitational field. Meanwhile, the gauge-invariant density perturbation is

$$\delta = \frac{\delta\rho}{\rho} = \frac{2}{3H^2\Omega_m} [a^{-2}\dot{\delta}^2\Phi - 3H(\dot{\Phi} + H\Phi)].$$

The second-order solutions for $\Psi^{(2)}$ and $\Phi^{(2)}$ are given by [55]. These are complicated expressions involving time integrals over products of $\Phi$ and its derivatives. For backreaction, however, the important thing is that if we take into account only terms which are important for backreaction, we have simply

$$\Psi^{(2)} = \Phi^{(2)} = B_3(t)\dot{\delta}^2\partial_i\partial^j(\dot{\Phi}_i\dot{\Phi}_j - \Phi_i\Phi_j) + B_4(t)\dot{\Phi}_i\dot{\Phi}_j\Phi_i\Phi_j,$$

where $B_3(t)$ and $B_4(t)$ are the time-dependent functions given by integrals over $g(t)$, with dimension $H_0^{-2}$. Their values today are given roughly by $B_3(t_0) \sim 0.85/(\Omega_{m0}^{15}H_0^2)$ and $B_4(t_0) \sim -0.25/(\Omega_{m0}^{15}H_0^2)$. $\Psi^{(2)}$ and $\Phi^{(2)}$ also contain more complicated looking inverse Laplacian terms, but these drop out after ensemble averaging [15], so we ignore them here. We shall only explicitly calculate scalar quantities at second order here, so we do not require the exact form of the vectors and tensors; for reference, generically we have [56–59]:

$$V_i \sim \Phi\partial_i\Phi, h_{ij} \sim \Phi\partial_i\partial_j\Phi,$$

which we shall return to later.

### 3.2. Backreaction of perturbations

What is the backreaction of perturbations onto the the expansion rate and its associated deceleration parameter? Similarly, what are the corrections to the monopole of the distance–redshift relation from second-order perturbations?

Expressions are quite cumbersome at second order, so let us define the Hubble normalized dimensionless derivative

$$\tilde{\dot{\delta}} = (aH)^{-1}\dot{\delta}.$$
At second order, the expansion rate of the dust is

\[
\Theta = 1 - (1 + \hat{g})\Phi - \frac{2}{9}(1 + \hat{g})\hat{g}^2 \Phi + \frac{1}{2}(3 - 2\hat{g})\Phi^2 - \frac{1}{2}\Phi^{(2)} - \frac{1}{2}H^{-1}\Psi^{(2)} + \frac{1}{6}\delta k v_k^{(2)} + \frac{2}{27\Omega_m^2}[(1 + 2\hat{g} + \hat{g}^2) + 9\Omega_m(1 + \hat{g})]\partial_k \Phi \partial^k \Phi
\]

\[
\Rightarrow H_{\text{today}} = 1 + 3(1 + \hat{g})(1 - \Omega_m)\Phi + \left[\left(1 + 2\hat{g}\right)\frac{4}{9\Omega_m} + \frac{2}{3}H\right]\Phi^2 - \frac{3}{2H}(1 - \Omega_m)(H\Phi^{(2)} + \Psi^{(2)})
\]

\[-\frac{1}{2}a\Omega_m \partial_k v_k^{(2)} - \frac{a}{3H}\partial_k v_k^{(2)} - \frac{1}{3}\delta^2 \Phi^{(2)} + \frac{1}{6}\partial^2 \Psi^{(2)}
\]

\[+ \frac{1}{9\Omega_m} \left[3\Omega_m(10 + 14\hat{g} + 6\hat{g}^2) - 8(2 + 5\hat{g} + 3\hat{g}^2)\right] \Phi \delta^2 \Phi
\]

\[+ \frac{1}{27\Omega_m^2}[(1 + 2\hat{g} + \hat{g}^2)\delta k \partial^k \partial^l \partial^j \Phi - \frac{8}{27\Omega_m^2}(1 + 2\hat{g} + \hat{g}^2)\delta^2 \Phi \delta^2 \partial_k \Phi
\]

where it is understood that the rhs is evaluated at the observer, and is not a spacetime function in the usual sense. The other deceleration parameters are given in the appendix.

### 3.3. Spatial averaging

Consider now the average of a variable over a spatial domain \(D\). The Riemannian average of a quantity \(\Upsilon\) may be defined as

\[
\langle \Upsilon \rangle_D = \frac{1}{V_D} \int_D \sqrt{\det h} d^3x \Upsilon.
\]
Here, the domain is on a hypersurface with 3-metric $h_{ij}$. If we choose that hypersurface to coincide with the spatial surfaces of the metric in the longitudinal gauge—the gravitational rest frame—then this can easily be expanded in terms of the Euclidean average defined on the background space slices, $(\Upsilon) = \int_\mathcal{D} d^3x \Upsilon/\int_\mathcal{D} d^3x$, as:

\[(\Upsilon)_{\text{grav}} = \Upsilon^{(0)} + (\Upsilon^{(1)}) + (\Upsilon^{(2)}) + 3[(\Upsilon^{(1)}) - (\Upsilon^{(1)})],\]  

(38)

where $\Upsilon^{(0)}$, $\Upsilon^{(1)}$ and $\Upsilon^{(2)}$ denote respectively the background, first-order and second-order parts of the scalar function $\Upsilon = \Upsilon^{(0)} + \Upsilon^{(1)} + \Upsilon^{(2)}$. Note the important term in square brackets, which encapsulates the relativistic part of the averaging procedure. To link with Buchert’s formulation above, we would like to average in the hypersurfaces orthogonal to $u^a$—the local rest frame of the dust—in which case we have instead (the general form of this expression was derived in [18, 60])

\[(\Upsilon)_{\text{D}} = \Upsilon^{(0)} - g_I \Upsilon^{(0)} (\Phi) + (\Upsilon^{(1)}) + (1 - 3HG_I)[(\Upsilon^{(1)}) - (\Upsilon^{(1)}) (\Phi)]
- \frac{1}{2} [\Upsilon^{(2)} - (\Phi^2) - 2\{v_i^{(1)}v_i^{(1)}\} - 2g_I a^{-2}[v_i^{(1)}\partial_i \Phi] + \frac{1}{2} \Upsilon^{(0)} + \frac{1}{2} \Upsilon^{(0)}]^2,\]

(39)

where $g_I = \frac{1}{\Omega_m} \int f g(t') dt'$. So, for example, to find $\langle q \rangle_{\text{D}}$ use $\Upsilon^{(0)} = -1 + \frac{1}{3} \Omega_m$, $\Upsilon^{(1)} = -3(1 + \frac{1}{3} \Omega_m (1 - \Omega_m) \Phi + \frac{1}{2\Omega_m}[9 + 6 \Omega_m - 4 \Omega_m (1 + \frac{1}{3} \Omega_m) \tilde{\delta}^2 \Phi$ and $\Upsilon^{(2)} = \cdots$ all the second order terms contained in $\langle q \rangle_{\text{D}}$ (cf the appendix).

With this we now have

$\langle \Theta \rangle_{\text{D}} = \text{the average of the expansion rate, averaged in the fluid rest frame;}$

$\langle q_{us} \rangle_{\text{D}} = \text{the average of the deceleration parameter, averaged in the fluid rest frame;}$

$\langle q_{D} \rangle = \text{the deceleration of the scale factor associated with} \langle \Theta \rangle_{\text{D}};\]

$\Theta$ = the monopole of the locally observed deceleration parameter.

We now turn to estimating the amplitude of these quantities given standard initial conditions from inflation.

### 3.4. Ensemble averaging

We define our Fourier transform as (suppressing any temporal quantities)

$$\Phi(x) = \frac{1}{(2\pi)^{3/2}} \int d^3k \Phi(k) e^{ikx},$$

(41)

where $\Phi^*(k) = \Phi(-k)$. Note that we assume that $\Phi(x)$ is defined over an infinite volume for this definition to be valid, as it requires the crucial identity $(2\pi)^3 \delta(x) = \int d^3k e^{ikx}$. (This can instead be reformulated in a large box with periodic boundary conditions [60]. The periodicity of the boundary conditions is instead contained in $\delta$-functions at $k = 0$ here.)

Inflationary models typically provide us with initial conditions for $\Phi$ in terms of its correlations, which appear in the form of an ensemble average or expectation value: for example, the 2-point correlation function is just

$$C(k, k') = \langle \Phi(k)\Phi(k') \rangle,$$

(42)

where an over-bar denotes an ensemble average. If $\Phi$ is Gaussian, then the distribution is given entirely by $C(k, k')$, and higher correlations are given in terms of $C$ using Wick’s theorem, and the ensemble average of odd numbers of $\Phi$ is zero. The probability distribution
function is just \( P[\Phi(k)\Phi(k')] = \exp[\Phi(k)\Phi(k')]/\Delta_1 \). Statistical homogeneity of \( \Phi(x) \) implies that different modes are uncorrelated: \( C(k,k') \propto \delta(k + k') \); statistical isotropy implies that the proportionality function cannot depend on the direction of \( k \) \([61]\). Hence, we have our power spectrum:

\[
\Phi(k)\Phi(k') = \frac{2\pi^2}{k^3} P_\Phi(k) \delta(k + k'),
\]

Assuming scale-invariant initial conditions from inflation, this is given by

\[
P_\Phi(t,k) = \left( \frac{3\Delta_R}{5\pi^2} \right)^2 g(t)^2 T(k)^2,
\]

where \( \Delta_R^2 \) is the primordial power of the curvature perturbation \([62]\), with \( \Delta_R^2 \approx 2.41 \times 10^{-9} \) at a scale \( k_{\text{CMB}} = 0.002 \text{ Mpc}^{-1} \).

What is the relation between ensemble and spatial averaging? Usually we assume ergodicity which here roughly means that, because the space is assumed infinite, the ensemble can be found within the space itself—i.e. any realization exists somewhere. Consequently, we should be able to replace ensemble averages with spatial ones and vice versa. This only works if the spatial averages are taken over an infinite domain on the background; however, which is not necessarily what we are interested in when considering the averaging problem. We would like to know how properly averaged quantities change with scale. To see this, define the Euclidean average over a domain \( D \) using a window function \( W(x-x') \):

\[
\langle X(x') \rangle(x) = \frac{1}{V} \int d^3x' W(x-x') X(x'),
\]

where \( V(x) = \int d^3x' W(x-x') \) gives the volume associated with \( W \). For two correlated Gaussian random fields \( A \) and \( B \) satisfying \( \bar{A}(k)\bar{B}(k') = A(k)B(k)\delta(k + k') \), we have \( \bar{A}(x)\bar{B}(x) = (2\pi)^{-3} \int d^3k A(k)B(k) \) directly. However, when averaged over a finite domain, we have instead

\[
\langle A(x')B(x') \rangle(x) = \frac{1}{V} \int d^3x' W(x-x') A(x')B(x')
\]

\[
= \frac{1}{(2\pi)^3} \int d^3k_1 \int d^3k_2 W(k_1 + k_2; x) A(k_1)B(k_2),
\]

where \( W(k; x) = V(x)^{-1} \int d^3x' W(x-x') e^{-ik\cdot x'} \) is the Fourier transform of \( W/V \) anchored at \( x \). In the limit we average over all space, \( W/V \to 1 \); then \( W(k) = (2\pi)^3\delta(k) \) and is independent of the anchor point \( x \). Clearly we recover the ensemble-averaged result in this case, provided the fields are isotropic.

### 3.5. The size of the backreaction

To evaluate our various averaged quantities, we could use a realization of \( \Phi \) given an inflationary model. Alternatively, we can assume a spectrum for \( \Phi \) and evaluate the statistics of the quantity in question. This allows us to calculate the expectation value of averaged variables as well as their ensemble variance, in terms of integrals over the power spectrum of \( \Phi \) multiplied by powers of \( k \). The reason we must go to second order now becomes clear when we calculate the expectation value of an average: for Gaussian perturbations from inflation, the ensemble average of \( \Phi \) is zero, which implies—assuming ergodicity—that when averaged on the background over a very large (strictly, infinite) domain they are zero too. Thus, the second-order terms provide the principal backreaction effect; the first-order terms give the statistical variance.
Let us estimate the approximate behaviour of each type of the terms which appear. The relations for determining the scaling behaviour for the backreaction terms are \((n+m)\) is even
\[
\bar{\partial}^m \Phi \bar{\partial}^n \Phi = \frac{(-1)^{(m+3n)/2}}{(aH)^{n+m}} \int_0^\infty dk \, k^{m+n-1} \mathcal{P}_\Phi(k).
\]  
(46)

The inverse Laplacian term in \(\Phi^{(2)}\) satisfies \(\{\bar{\partial}^2 \bar{\partial}^2 \mathcal{P}_\Phi \} = \frac{1}{2} \mathcal{P}_\Phi \partial_i \partial_j \Phi_0\) [15]. We also have that since \(\Phi\) is statistically homogeneous and isotropic [15],
\[
\bar{\partial}^4 \Phi^{(4)} = \bar{\partial}^2 \mathcal{P}_\Phi = 0,
\]
(47)
which means that all the potentially large terms in the second-order Hubble rate do not contribute to the expectation value (see below). For the non-connected terms, we have
\[
\langle \bar{\partial}^m \Phi \rangle \mathcal{P}_\Phi = \frac{(-1)^{(m+3n)/2}}{(aH)^{n+m}} \int_0^\infty \, dk \, k^{m+n-1} W(kR_D) \mathcal{P}_\Phi(k),
\]
(48)
where \(W\) is an appropriate window function specifying the domain. Typically this will become a delta function as the domain tends to infinity (e.g. if the window function in real space is a Gaussian of width \(R_D\), then in Fourier space it is a Gaussian of width \(1/R_D\), centred at \(k = 0\)). Note that the connected terms have no dependence on the domain size or shape at all, and that the domain dependence arises from the non-connected terms—these in turn come from using the Riemannian volume element. The integral can be written as
\[
\frac{1}{H^{n+m}} \int_0^\infty \, dk \, k^{m+n-1} \mathcal{P}_\Phi(k) = \left( \frac{3 \Delta_R}{5g_\infty} \right) ^{m+n} \left( \frac{k_{\text{eq}}}{k_H} \right) ^{m+n} \int_0^\infty \, dk \, k^{m+n-1} T(k)^2.
\]
(49)

Here, \(k_H = H_0^{-1}\) is the wavenumber of the mode entering the Hubble rate today, and \(k_{\text{eq}}/k_H = \sqrt{2\Omega_m \varepsilon_{\text{eq}}} \approx 40\). Using \(\varepsilon_{\text{eq}} \approx 2.4 \times 10^4 \Omega_m h^2\) implies the important relation
\[
\Delta_R \left( \frac{k_{\text{eq}}}{k_H} \right) ^2 \approx 2.4 \Omega_m^2 h^2.
\]
(50)

For pure CDM and a scale-invariant initial spectrum, the integral behaves as, replacing \(f_0^\infty \mapsto f_{\text{IR}}^\infty\) where necessary,
\[
\int_0^\infty \, dk \, k^{m+n-1} T(k)^2 \approx \begin{cases} 
- \ln(k_{\text{IR}}) & \text{for } m+n = 0 \\
3.9 & \text{for } m+n = 2 \\
F(k_{\text{UV}}) & \text{for } m+n = 4.
\end{cases}
\]
(51)

The function \(F\) is roughly \(F(x) \approx 0.44 x^{2.14}\) for \(1 \lesssim x \lesssim 10\), \(70 x^{-0.1} (\log_{10} x)^{4.75}\) for \(x \gg 1\), and approaches \(53 \ln^3 x\) as \(x \to \infty\). For integrals with the window function inside, \(W(k_{\text{eq}} R_D)\), we can roughly replace \(k_{\text{UV}} \mapsto 1/R_D k_{\text{eq}}\), though this depends on the details of the window function used. Combining the above equations allows us to calculate reasonably precisely the size of each term at second order.

The first type of term, \(\Phi^2\), is nominally tiny, \(\mathcal{O}(10^{-10})\). It also tells us that the IR divergence in \(\Phi^2\) must be cut-off by hand, corresponding to the first mode to leave the Hubble radius at the start of inflation. There has been speculation that this might lead to very important effects and even mimic dark energy [6, 7, 11], though this has been criticized [21–23]. Given that we only measure the primordial power spectrum to be nearly scale invariant over a comparatively narrow range of scales, the appearance of \(\langle \Phi^2 \rangle\) implies that it cannot be too tilted to the red on super-Hubble scales, or else it would give a sizeable backreaction effect, though this might be unphysical. A red spectrum would convert the logarithmic divergence into a power-law one, and constraints on the largest mode would be much stronger (e.g. \(k_{\text{IR}}/k_H \lesssim 10^{-30}\) for \(n_s = 0.95\)).
The term primarily responsible for setting the fundamental amplitude of the backreaction in the Hubble rate is \( \langle \Phi \partial^2 \Phi \rangle \propto (k_{eq}/k_H)^2 \). It is nominally quite small,

\[
\frac{\langle \Phi \partial^2 \Phi \rangle}{\Omega_m H_0^2} \sim \Delta_R^2 \frac{k_{eq}^2}{\Omega_m k_H^2} \sim \Delta_R^2 \frac{T_{eq}}{T_0} \sim 10^{-5},
\]

for the concordance model. (The overall effect is somewhat larger than this due to the contribution of several such terms.) This gives sub-per cent changes to the Hubble rate from backreaction, though non-connected terms make it significantly larger on small scales [12]. Yet, we can observe that the backreaction is small because the equality scale is large in our universe, which is because the temperature of matter-radiation equality is very low. Modes which enter the Hubble radius during the radiation era are significantly damped compared to those which remain outside until after equality; so, the longer the radiation era, the less power there is on small scales to cause a significant backreaction effect. For a scale-invariant spectrum, then, it may be considered that the long-lived radiation era is the reason that the dynamical backreaction is small. The temperature will have to drop by several orders of magnitude before backreaction in the Hubble rate becomes significant.

In the variance of the Hubble rate, the deceleration parameter \( q_{\text{obs}} \) and the ensemble average of \( Q \) divergent terms appear: \( \partial^2 \Phi \partial^2 \Phi \), which are of the order of the density fluctuation squared. The origin of such terms arises from taking the proper time derivative of the Hubble rate which contains spatial derivatives with respect to the coordinates we are using. The integral \( F \) overcomes the \( \sim 10^{-9} \times (2\Omega_m z_{eq})^2 \) pre-factor around \( k_{\text{UV}} \sim 10 k_{eq} \), so these terms are big, and it is difficult to know what to do with them: here, the UV cutoff is really a measure of our ignorance. Within linear perturbation theory it should be set by the end of inflation and the reheating temperature, as well as the small-scale physics of dark matter, both of which are sub-pc scales today. Replacing the UV cutoff with a smoothing function in \( \Phi \) implies that we might do better in perturbation theory to smooth order by order and calculate second-order terms from smoothed first-order ones, rather than calculating expectation values or spatial averages directly at a given order. Even for domains much larger than the nonlinear scale, where linear perturbation theory breaks down (somewhere around a few Mpc), \( F \) is quite sizeable, and so we have backreaction terms \( O(1) \) (note that \( F(1) = O(1) \), and the prefactor is also \( O(1) \)). From this, we also recover that the variance in the Hubble rate is \( O(1) \) on scales of Mpcs.

Observationally, can there be any signature of backreaction? When measuring the Hubble rate, perturbations are significant in the variance of the Hubble rate on sub-equality scales [12, 14]. Perturbations affect the whole distance–redshift relation which has been calculated to first order by [6, 24, 27]. Corrections to the luminosity distance include corrections \( \partial^2 \Phi \), just like for the Hubble rate. This allows us to see that the variance includes divergent terms like \( (\partial^2 \Phi)^2 \)—the lensing term—which appear in the variance of the all-sky average of the luminosity distance—see [24]. We have discussed here how the problem at second order is much more serious: the divergent terms appear in the expectation value of the distance–redshift relation itself, and not just in its variance.

Consider the ensemble average of the monopole of \( q_{\text{obs}}^0 \). This should be the value a typical observer would expect to find—from nearby supernovae for example. Considering only the

1 While they appear in the Hubble rate through the second-order velocity perturbation, the ensemble average conspires to cancel them out for an initial spectrum which is homogeneous. Because the peculiar velocity appears as a pure divergence, the spatial average is a boundary term which is small on infinite domains due to assumed homogeneity—in our analysis this is contained in assuming ergodicity of the first-order \( \Phi \).
dominant terms with four derivatives in them and using \( \Delta_{\text{QG}}^4 (k_{\text{QG}}/k_H)^4 \approx 5.7 \Omega_m^4 h^4 \), we have \( \mathcal{O}(1) \) corrections to the observed deceleration parameter:
\[
\ddot{\Omega} \approx -1 + \frac{7}{2} \Omega_m + \Omega_m^2 h^4 [1.06 + 0.03(1 - \Omega_m)^2 - 1.4(1 - \Omega_m)^{11}] F(\kappa_{\text{UV}}); \tag{53}
\]
similarly
\[
\ddot{\Omega}^* \approx -1 + \frac{7}{2} \Omega_m + \Omega_m^2 h^4 [-0.24(1 - \Omega_m)^3 + 0.66 \Omega_m^{0.37}] F(\kappa_{\text{UV}}), \tag{54}
\]
where the coefficients in square brackets are the reasonable empirical estimates for \( \Omega_m \gtrsim 0.1 \) (accurate to a per cent or so). In Buchert’s definition of \( q_D \), the divergence is nicely controlled by the domain size because the terms \( (\partial^2 \Phi)^2 \) always appear as \( (\bar{\partial}^2 \Phi)^2 \). In that case, the UV cutoff can be thought of as the smoothing scale.

It is an important open question to find out what happens to the whole distance–redshift relation to second order, where an overall ensemble-averaged offset to the luminosity distance will be present. As we have shown, this will include terms \( (\partial^2 \Phi)^2 \), which is certainly important enough to disturb the cosmic concordance.

### 3.6. Higher-order perturbation theory

Would higher-order perturbation theory affect these conclusions? On the one hand, it seems clear that higher-order perturbations should be suppressed. Provided \( \Phi \) is Gaussian, only even orders will be important, once ensemble averages are taken. We might expect the largest terms at any order \( n \) to behave like \( \Phi^{(n)} \sim (\partial \Phi)^n \) (e.g. from relativistic corrections to the peculiar velocity), the ensemble average of which goes like \( \Delta_{\text{QG}}^n (k_{\text{QG}}/k_H)^n \). Terms which appear in the Hubble rate at order \( n \) of the form \( \partial^2 \Phi^{(n)} \) do not have enough derivatives to overcome the suppression from \( (\partial \Phi)^{(n-2)} \) terms. By this argument, the second order should be as large as it gets, and backreaction from structure is irrelevant.

On the other hand, others [7, 8, 11, 37, 38] have argued that at higher-order terms such as \( (\partial \Phi)^2 (\partial^2 \Phi)^{n-2} \) are the norm. Simply squaring terms like \( \Phi \partial^2 \Phi \) gives problematic terms. In this case, from fourth order on, perturbation theory needs a UV cutoff—at least as far as calculating averages is concerned. Even if not divergent, if \( (\partial^2 \Phi)^{n-2} \sim 1 \), then higher-order terms are at least as large as at second order and must be included to evaluate backreaction properly, and so correctly identify the background. But do these terms cancel out?

Let us consider the expansion rate at fourth order, as an example. If we are interested in the expectation value, the leading terms, which contain six or more derivatives, are
\[
\frac{H^{(4)}}{H} \sim - \frac{16}{729 \Omega_m^2} (1 + 3 \hat{g} + 3 \hat{g}^2 + \hat{g}^3) \partial_k \Phi \partial^2 \Phi \partial_j \partial^2 \Phi \partial^j \Phi
\]
\[
+ \frac{8}{243 \Omega_m^4} [3(1 - \hat{g}^2 - 2 \hat{g}^3) + \Omega_m (27 - 67 \hat{g} + 30 \hat{g}^2)] \partial_k \Phi \partial^2 \Phi \partial^2 \Phi \partial^j \Phi
\]
\[
- \frac{16}{243 \Omega_m^4} [2(1 + 3 \hat{g} + 3 \hat{g}^2 + \hat{g}^3) + 7 \Omega_m (1 + \hat{g})] \partial_k \Phi \partial^2 \Phi \partial^j \Phi \partial^j \Phi
\]
\[
+ \frac{1}{81 \Omega_m^4} (\partial_k \partial^2 \Phi^{(2)} - \partial^2 \Phi^{(2)})(H \partial^2 \Phi^{(2)} + \partial^2 \Phi^{(2)})
\]
\[
+ \text{terms of the form: } \Phi^{(4)}, \partial_k v_k^{(4)}, \partial_k \Phi \partial^2 \Phi \partial^2 \Phi^{(2)},
\]
\[
\partial_k \partial^2 \Phi^{(3)} \partial^k \Phi, \partial^2 \Phi \partial^2 \Phi \partial_k \Phi^{(2)}, \partial_k \Phi^{(2)} \partial^2 \Phi^{(2)}, \ldots \tag{55}
\]
Here, we have assumed that the metric can be written in the longitudinal gauge up to \( \mathcal{O}(4) \), and we have ignored all stand-alone cubic terms as they do not appear in the ensemble average. We have also ignored coupling with vector and tensor modes which will give potentially
important contributions. We have also grouped $\Phi^{(2)}$ and $\Psi^{(2)}$ together, schematically unless explicitly written out. Note that there is the potential for eight derivative terms from terms like $(\partial_k \partial^2 \Phi^{(2)} - \partial_k \partial^2 \Psi^{(2)}) \partial^2 \Phi^{(2)}$ which appear. However, on closer inspection they cancel on small scales where $\Phi^{(2)} \simeq \Psi^{(2)}$ so should not dominate the expression. We have included the coefficients where it is obvious how to calculate the ensemble average using Wick’s theorem; where terms include $\Phi^{(2)}$ and $\Phi^{(3)}$, it is more involved and will be considered elsewhere. The full calculation of $H^{(4)}$ is non-trivial, and we give this expression only to demonstrate what might happen.

The amplitude of the six derivative terms may be estimated from equation (49). For example, in the Riemann tensor, terms such as $\langle \partial_k \Phi \partial^4 \Phi \partial^2 \Phi \partial^2 \Phi \rangle \sim \left( \frac{3 \Delta \rho}{5 \Delta g} \right)^4 \left( \frac{k_{\text{eq}}}{k_{\text{H}}} \right)^6 \times 3.9 \times F(k_{\text{UV}})$. (56)

Now, six powers of the equality scale overcome three factors of $\Delta \rho$, using equation (50), making this at least the same size as first order. If we take linear theory at face value, and assert that the UV cutoff should be on pc scales, then $k_{\text{UV}} \sim 10^5$, and $F \sim 10^5 \sim \Delta \rho^{-1}$. Plugging in the numbers, we find that these contributions to the Hubble rate—very naively—are of order unity. That is truly astonishing. This result is very sensitive to the UV cutoff, as well as the baryon fraction, which helps reduce small-scale power.

If we consider what will happen to other quantities at fourth order, it is clear that such six-derivative-with-four-Phi terms appear also from purely relativistic degrees of freedom. For example, in the Riemann tensor, terms such as $(\partial_i h_{jk})^2$ will appear. Using the fact that $h_{ij} \sim \Phi \partial_i \partial_j \Phi$, coupled with the fact that time derivatives give factors of $k$ for gravitational waves, we see that such relativistic contributions can also be large; this cannot be associated with any Newtonian effect. Do they add up to something substantial?

Perturbation theory suffers from higher-derivative terms because of the assumed scale-invariant primordial power spectrum, with mild damping of short wavelengths during the radiation era. Without this damping, even the $\Phi^2$ terms would lead to significant backreaction, as modes all the way up to the inflationary cutoff would be included. As it is, contributions for $\Phi^2$ and $\Phi \partial^2 \Phi$ are cut off by the Hubble scale at equality ($k_{\text{eq}}$), which is huge in comparison to more fundamental cutoffs. This suppresses modes shorter than the equality scale, but only in a power-law way, not exponentially. Yet it only takes four derivatives to overcome this power-law suppression at second order: $(\partial^2 \Phi)^2$. Exponential suppression of modes below around the Silk scale, or some hypothetical nonlinear scale of about that size, would keep such terms just about under control [15].

One can argue that we should insert a nonlinear cutoff scale $k_{\text{NL}}$ into our integrals over $p_g$ to represent the effective reduction of power expected on small scales which hides the UV divergence. Indeed reference [35] claims that a relativistic virial theorem provides such a scale. Of course, such a theorem cannot exist in general because energy is radiated to infinity in GR, and only the stationary part of a system virializes [63]. Nothing dynamic is truly isolated. This is realized in a non-trivial gravitational wave background from structure formation [58]. But this idea does imply that generically, at a fixed time, UV divergences represent something unphysical and can be renormalized away at that time (reference [35] deals with divergences by using a smoothing process on the background, which is effectively a UV cutoff). It is precisely this renormalization which may lead to a significant backreaction.

The integrals over the linear primordial power spectrum, however, seem to be the wrong place to insert a UV cutoff. While one might expect a reduction of power in the overall gravitational potential $\Phi^{(\text{total})}$ today (if such a thing were meaningful) compared to $\Phi$, with a change in behaviour at some scale $k_{\text{NL}}$ where linear theory becomes inaccurate, this should
come about naturally from higher-order perturbation theory. If we actually knew the late time power spectrum, everything would be fine—but an important aim is to calculate it directly from initial conditions, especially when we want to know how the background is renormalized by structure formation. In effect, we do not know what the relation of the background today is to the background at early times (if backreaction turns out to be an important effect). Higher-order perturbation theory presumably has to rely on the power at small scales in $\Phi$ to calculate $\Phi^{(\text{total})}$. There is no a priori reason to expect small scales to decouple from large order-by-order, even if they actually do decouple physically for ‘virialized’ regions after the series is summed. So, an ad hoc insertion into integrals over $\Phi$ does not seem to be the solution. It seems that higher-order perturbation theory should be considered in detail to solve this problem.

4. Discussion

We have discussed here some examples where the backreaction problem in the standard model of cosmology is non-trivial and not necessarily negligible. The two key examples we have discussed are the cases of the deceleration parameter which gives a dimensionless measure of the second time derivative of the scale factor, and the Hubble rate evaluated at fourth order. In these cases perturbation theory does not give sensible answers because of terms which have significant power in the UV.

In some respects dynamical backreaction is small by good fortune: the universe is so hot and had such a long radiation era that small-scale power is significantly reduced over its scale-invariant initial conditions (assuming those to be the case). The backreaction terms in the Hubble expansion, $\Phi \dddot{\Phi}$, are also the largest ones which appear on the lhs of the Einstein field equations (EFE), because the Einstein tensor has at most two derivatives of the metric in it. In some respects then this settles it: backreaction is small by virtue of there being a very small hierarchy of scales between the Hubble scale at equality and the Hubble scale today (they are only a factor of 50 apart in comoving terms). In this evaluation of backreaction, then, what happens on scales smaller than the equality scale is of little relevance. This is perhaps counter-intuitive given how we normally think of backreaction arising from small-scale structure—it is really power on very large scales which are responsible for the backreaction effect.

On the other hand, however, we should be able describe the universe using a tetrad version of the field equations, or the covariant formulation given in the appendix. This is entirely equivalent to the EFE but reformulates gravity as a system of first-order PDEs in the Ricci rotation coefficients and Weyl curvature tensor. In such a formulation, higher derivative terms appear which are divergent implying that backreaction may not be irrelevant. Furthermore, we should also be able to make sense of physical observables such as the fundamental distance-redshift relation or the geodesic deviation equation. In these cases, terms such as $(\dddot{\Phi})^2$ occur frequently, and as we have seen can appear in both their connected and non-connected forms giving rise to divergent backreaction terms, at least on small scales.

In Buchert’s interpretation of backreaction, where $q_D$ describes the deceleration of the average scale factor, the divergence is neatly controlled by the domain size and consequently has an elegant and straightforward interpretation, although it is not observable. This also appears to be robust against various possible gauge effects such as averaging hypersurface [18]. The fact that there is a significant difference between the ‘cosmological’ deceleration

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2 In [6, 64] it was claimed that a divergence in the IR can cause a large backreaction effect, whereas references [21–23] have strongly disagreed with this idea. To make such terms large requires fluctuations out to enormous scales beyond the horizon. The UV divergence, by contrast, arises from the use of scale-invariant initial conditions all the way down to small scales, which is usually found in many models of inflation.
and one which is smoothed on scales of a few Mpc, say, is expected since that is the scale where the Hubble flow kicks in.

On the other hand, our two other definitions of the deceleration parameter—which do not depend on Riemannian averaging—reveal significant problems. Consider \( q/\Theta_1 \) and \( q_{\text{obs}} \)—the local deceleration parameter defined relative to the dust observers and the observed one defined through the distance–redshift relation. If we calculate the expectation value of either of these, we get enormous terms. Consider cutting off at a scale suggested by either a scale associated with the end of inflation or from the dark matter suppression scale, which is around pc scales. Then we have \( \kappa_{\text{UV}} \sim 1 \, \text{pc}^{-1}/(100 \, \text{Mpc})^{-1} \sim 10^5 \). Our divergent integral then gives \( F(10^8) \sim 10^5 \). Of course, we have assumed a purely dark matter transfer function, and there is extra suppression from baryons on small scales, but this only reduces it by an order of magnitude for 20% baryons. One can reformulate the UV cut-off as a smoothing of the first-order potential to give something like the same interpretation one might give to Buchert’s \( q_D \), but this is rather ad hoc in this context—for \( \langle q/\Theta_1 \rangle_D \), smoothing has already been done. Instead it maybe tells us that smoothing is necessary order by order in perturbation theory. That is, before constructing second-order perturbation theory, one necessarily must smooth structure below a certain scale. But why? Does this imply that the very notion of ergodicity needs to be made Riemannian: should ensemble averages be replaced by Riemannian spatial averages, rather than spatial averages on the background? (Buchert also discusses this possibility [65]).

This is very unsatisfactory. It is easy to devise universes where the magic number \( \Delta_R^2(k_{\text{eq}}/k_H)^2 \sim 1 \) (though they may not be anthropically relevant). In an Einstein–de Sitter model, for example, simply one has to wait until such a time as \( H_0 \) is small enough (though that does not happen with LCDM). If there were no radiation era, things would be much worse. In such a case, perturbation theory would simply break down at second order and other formalisms for describing structure would have to be devised. The damping which happens at linear order to suppress a scale-invariant power spectrum would arise from nonlinear structure formation, as fully nonlinear GR is a well-behaved theory.

In our universe, where, apparently coincidentally, \( \Delta_R^2(k_{\text{eq}}/k_H)^2 \sim 1 \), it is \( \Delta_R^2(k_{\text{eq}}/k_H)^4 \sim 1 \) which seems to signal problems, because it is accompanied by a factor of \( F(\kappa_{\text{UV}}) \). We can ignore this in many applications of perturbation theory because it simply does not appear. Should we be worried that they do appear in the examples calculated here? Perhaps not. Maybe there are ways around them, or perhaps they are some strange gauge artefact—though this seems difficult to realize since the distance–redshift relation is a physical observable (though its ensemble average is not). But we should be worried by the fact that the (ensemble average) second-order perturbations are the same size as first, and fourth might be the same size as second—or larger if the \((\partial^2 \Phi)^2\) terms are to be taken at face value. It appears as if fourth-order perturbations could give significant changes to the Hubble rate, even if one can dismiss the divergent examples we have discussed at second order. Generically, we can see why \((\partial^2 \Phi)^2\) terms must appear at fourth order because second-order potential terms are \( \Phi \partial^2 \Phi \); second-order gradient terms (such as \( v'' \)) are then \( \partial \Phi \partial^2 \Phi \) and so have large expectation values when squared. It is tempting to assume that any terms with \((\partial^2 \Phi)^2\) in them all must cancel, or must be a gauge effect. However, as we have discussed, in models with \( \Phi \partial^2 \Phi \sim 1 \), we do not expect to have large terms cancelling just because they are large, so it need not be the case with \((\partial^2 \Phi)^2\) either. It is actually a combination of these terms together with scale-invariant initial conditions which causes the problems. If they really do cancel in all possible models, at all orders in perturbation theory, this would be a very persuasive argument that backreaction is indeed small; if not, it is hard to say either way.

Will perturbation theory even converge? Renormalization methods have been devised for Newtonian gravity [66, 67] which is comforting, but only up to a point: vector modes
which represent relativistic frame dragging have no Newtonian counterpart and are not that much smaller than $\Phi^{(2)}$ [59]; gravitational waves induced by linear scalars are larger and their time derivative is the same size as $\Phi^{(2)}$ [58]. These will easily mix together in any proper resummation scheme. However close Newtonian gravity may seem to relativity, GR is the theory of gravity we must work with to address these issues.

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Appendix A. Covariant formulation of the field equations

The 1+3 covariant Ehlers–Ellis formalism provides a physically transparent formulation of the field equations’ full nonlinear generality (see [68, 69] for reviews). The Ehlers–Ellis formalism is a covariant Lagrangian approach to gravitational dynamics, based on a decomposition relative to a chosen 4-velocity field $u^a$. The fundamental tensors are

$$h_{ab} = g_{ab} + u_a u_b, \quad \varepsilon_{abc} = \eta_{a[b|c|d]} u^d,$$

where $h_{ab}$ projects into the instantaneous rest space of comoving observers, and $\varepsilon_{abc}$ is the projection of the spacetime alternating tensor $\eta_{a[b|c|d]} = -\sqrt{-g} \delta_{[a} \delta^{b[c} \delta^{d]}$, and so

$$\eta_{abcd} = 2 u_{[a} \varepsilon_{b]cd} - 2 \varepsilon_{abc} u_{d], \quad \varepsilon_{abc} \varepsilon_{def} = 3! h_{[a}^d h_{b}^e h_{c]}^f.$$

The PSTF parts of vectors and rank-2 tensors are

$$V_{(a)} = h_{ab} V^b, \quad S_{(ab)} = \{ h_{(a}^e h_{b)}^d - \frac{1}{3} h^{cd} h_{ab} \} S_{cd}.$$

The skew part of a projected rank-2 tensor is spatially dual to the projected vector, $S_a = \frac{1}{3} \varepsilon_{abc} S^b c$, and then any projected rank-2 tensor has the decomposition $S_{ab} = \frac{1}{3} S h_{ab} + \varepsilon_{abc} S^c + S_{(ab)}$, where $S = S_{cd} h^{cd}$.

The covariant derivative $\nabla_b u_a$ defines 1+3 covariant time and spatial derivatives:

$$f_{\cdots} b = u^e \nabla_e f_{\cdots} b, \quad D_j f_{\cdots} b = h_c^d h_e^a \cdots h_b^f \nabla_d f_{\cdots} f.$$

The projected derivative $D_a$ defines a covariant PSTF divergence, $\text{div}V = D_a V_a$, $\text{div}S_a = D_d S_{da}$, and a covariant PSTF curl,

$$\text{curl}V_a = \varepsilon_{abc} D^b V^c, \quad \text{curl}S_{ab} = \varepsilon_{cd(a} D^c S_{b)}^d.$$

The relative motion of comoving observers is encoded in the PSTF kinematical quantities: the volume expansion rate, 4-acceleration, vorticity and shear, given respectively by

$$\Theta = D^a u_a, \quad A_a = \dot{u}_a, \quad \omega_a = \text{curl} u_a, \quad \sigma_{ab} = D_{(a} u_{b)}.$$

Thus

$$\nabla_b u_a = \frac{1}{3} \Theta h_{ab} + \varepsilon_{abc} \varepsilon^{cd} + \sigma_{ab} - A_b u_a.$$

The PSTF dynamical quantities describe the sources of the gravitational field: the (total) energy density $\rho = T_{ab} u^a u^b$, isotropic pressure $p = \frac{1}{3} h_{ab} T^{ab}$, momentum density $q_a = -T_{(a} h^{b)}$, and anisotropic stress $\pi_{ab} = T_{(ab)}$, where $T_{ab}$ is the total energy-momentum tensor. The locally free gravitational field, i.e. the part of the spacetime curvature not directly determined locally by dynamical sources, is given by the Weyl tensor $C_{abcd}$. This splits into the PSTF gravito-electric and gravito-magnetic fields:

$$E_{ab} = C_{abcd} u^c u^d, \quad H_{ab} = \frac{1}{3} \varepsilon_{def} C_{cde} u^c u^d,$$

which provide a covariant description of tidal forces and gravitational radiation.
The Ricci and Bianchi identities,
\[ \nabla_a \nabla_b u_c = R_{abcd} u^d, \quad \nabla^d C_{abcd} = -\nabla_a \left[ R_{bde} - \frac{1}{6} R g_{be} \right], \]
produce the fundamental evolution and constraint equations governing the covariant quantities. Einstein’s equations are incorporated via the algebraic replacement of the Ricci tensor
\[ R^{ab} = T^{ab} - \frac{1}{2} T \gamma^{ab} + \Lambda g^{ab}, \]
where \( T^{ab} \) is the total energy-momentum tensor.

The resulting equations, in fully nonlinear form and for a general source of the gravitational field, are as follows.

**Evolution:**
\[
\dot{\rho} + \frac{\rho + p}{\Theta} \Theta + \text{div} q = -2A^a q_a - \sigma^{ab} \pi_{ab},
\]
\[
\dot{\Theta} + \frac{1}{3} \Theta^2 + \frac{1}{2} (\rho + 3p) - \Lambda - \text{div} A = -\sigma_{ab} \sigma^{ab} + 2\omega_a \omega^a + A_a A^a,
\]
\[
\dot{q}_{(a)} + \frac{1}{2} \Theta q_a + (\rho + p) A_a + D_a p + \text{div} \pi_a = -\sigma_{ab} q^b + \epsilon_{abc} \omega^b q^c - A^b \pi_{ab},
\]
\[
\dot{\omega}_{(a)} + \frac{1}{2} \Theta \omega_a + \frac{1}{2} \text{curl} A_a = \sigma_{ab} \omega^b,
\]
\[
\dot{\sigma}_{(ab)} - \frac{1}{3} \Theta \sigma_{ab} + E_{ab} - \frac{1}{2} \pi_{ab} - D_{(a} A_{b)} = -\sigma_{c(a} \sigma_{b)c} - \omega_{(a} \omega_{b)} + A_{(a} A_{b)}.
\]

**Constraint:**
\[
\text{div} \omega = A^a \omega_a,
\]
\[
\text{div} \sigma_a - \text{curl} \omega_a - \frac{3}{2} D_a \Theta + q_a = -2\epsilon_{abc} \omega^b A^c,
\]
\[
\text{curl} \sigma_{ab} + D_{(a} \omega_{b)} - H_{ab} = -2A_{(a} \omega_{b)},
\]
\[
\text{div} E_a + \frac{1}{2} \text{div} \pi_a - \frac{1}{2} D_a \rho + \frac{1}{2} \Theta q_a = \epsilon_{abc} \sigma^{b} d H^{cd} - 3H_{ab} \omega^b + \frac{1}{2} \sigma_{ab} q^b - \frac{3}{2} \epsilon_{abc} \omega^b q^c,
\]
\[
\text{div} H_a + \frac{1}{3} \text{curl} q_a - (\rho + p) \omega_a = -\epsilon_{abc} \sigma^{b} d E^{cd} - \frac{1}{2} \epsilon_{abc} \sigma^{b} d \pi^{cd} + 3E_{ab} \omega^b - \frac{1}{2} \pi_{ab} \omega^b.
\]

The energy and momentum conservation equations are the evolution equations (A.11) and (A.13). The dynamical quantities \( \rho, p, q_a, \pi_{ab} \) in the evolution and constraint equations (A.11)–(A.22) are the total quantities, with contributions from all dynamically significant particle species.
A.1. Moment decomposition

Here we give a summary of the covariant spherical harmonics we use, following [70].

An observer moving with 4-velocity $u^a$ at position $x^i$, in a direction $e^a$ on the unit sphere $(e^a e_{a} = 1, e^a u_a)$ measures the luminosity of a distant supernova or galaxy. The direction $e^a$ can be given in terms of an orthonormal tetrad frame as, for example,

$$e^a(\theta, \phi) = (0, \sin \theta \sin \phi, \sin \theta \cos \phi, \cos \theta).$$  \hspace{1cm} (A.23)

There is a 1–1 mapping between all symmetric trace-free tensors of rank $l$ and the spherical harmonics of order $\ell$ [70–72]. Consider a spherical harmonic decomposition of $f(x^i; \theta, \phi)$,

$$f = \sum_{l=0}^{\infty} F_A e^A = F + F_a e^a + F_{ab} e^a e^b + F_{abc} e^a e^b e^c + F_{abcd} e^a e^b e^c e^d + \cdots,$$  \hspace{1cm} (A.25)

where the spherical harmonic coefficients $F_A$ are symmetric, trace-free tensors orthogonal to $u^a$:

$$F_A = F_{(A)}, \quad F_{A,ab} h^{ab} = 0, \quad F_{A,a} u^a.$$  \hspace{1cm} (A.26)

Round brackets ‘(··)’ denote the symmetric part of a set of indices and angle brackets ‘(··)’ the PSTF part of the indices. We use the shorthand notation using the compound index $u^a$.

Round brackets ‘(··)’ denote the symmetric part of a set of indices and angle brackets ‘(··)’ the PSTF part of the indices. We use the shorthand notation using the compound index $A_\ell = a_1 a_2 \cdots a_l$ and $e^{A_\ell} = e^{a_1} \cdots e^{a_l}$. The PSTF part of $e^{(A_\ell)}$ [71–74] is

$$e^{(A_\ell)} = \sum_{k=0}^{[\ell/2]} B_{ik} h_{a_1 a_2} \cdots h_{a_{\ell - 2k - 1} a_\ell} e^{a_1 a_2} \cdots e^{a_{\ell - 1} a_\ell},$$  \hspace{1cm} (A.27)

where $B_{ik}$ are given by [71]

$$B_{ik} = (-1)^i \ell!/(2 \ell - 2n - 1)!!/(\ell - 2n)!(2\ell - 1)!!(2n)!!.$$  \hspace{1cm} (A.28)

Here $[\ell/2]$ means the largest integer part less than or equal to $\ell/2$, and $\ell!! = \ell(\ell - 2)(\ell - 4) \cdots (2\ell + 1)$. The normalization for $e^{A_\ell}$ is given by [74], for odd and even $\ell$ respectively:

$$\frac{1}{4\pi} \int_{4\pi} e^{A_{l+1}} d\Omega = 0 \quad \text{and} \quad \frac{1}{4\pi} \int_{4\pi} e^{A_\ell} d\Omega = \frac{1}{2\ell + 1} h_{(A_\ell)}^{(A_{\ell+1})},$$  \hspace{1cm} (A.29)

which implies

$$\int_{4\pi} e^{A_\ell} e^{B_m} d\Omega = \frac{4\pi}{\ell + m + 1} h_{(A_\ell B_m)}^{(A_{\ell + 1} B_m)},$$  \hspace{1cm} (A.30)

if $\ell + m$ is even, and is zero otherwise. The orthogonality condition for $e^{(A_\ell)}$ then follows:

$$\int d\Omega e^{(A_\ell)} e_{(B_m)} = \delta_{m}^{l} \Delta_{\ell} h_{a_1 a_2 \cdots a_{\ell - 1} b_1} \cdots h_{a_{\ell - 1} a_\ell} b_1 \Delta_{\ell} = \frac{4\pi}{(2\ell + 1)} \frac{2^{\ell}(\ell)!}{(2\ell - 1)!},$$  \hspace{1cm} (A.31)

which then implies the relation between $f$ and its spherical harmonic moments $F_A$:

$$F_A(x^i) = \Delta_{\ell}^{-1} \int_{4\pi} d\Omega e^{(A_\ell)} f(x^i, e^a).$$  \hspace{1cm} (A.32)

Note that the recursion relation

$$e^{(A_{\ell+1})} = e^{(a_{\ell+1})} e^{(A_\ell)} - \frac{\ell^2}{(2\ell + 1)(2\ell - 1)} h_{a_1 a_2 a_3} e^{(A_{\ell - 1})}$$  \hspace{1cm} (A.33)

relates the $(\ell + 1)$st term to the $\ell$th term and the $(\ell - 1)$st term.
Appendix B. The deceleration parameters

We give here the full deceleration parameters for the cases we consider. First the deceleration parameter associated with the fluid expansion rate $\Theta$ is given by

$$ q_\Theta = -1 + 3 \frac{\Omega_m}{2} - 3(1 + \dot{\bar{g}})(1 - \Omega_m)\Phi + \left[ \left( 1 + \frac{2}{3}\dot{\bar{g}} \right) - \frac{4}{9\Omega_m} \left( 1 + \dot{\bar{g}} \right) \right] \bar{\delta}^2 \Phi$$

$$ - 3\dot{\bar{g}} \left( 4 + \frac{3}{2}\dot{\bar{g}} \right) (1 - \Omega_m) \Phi^2 - \frac{3}{2H} (1 - \Omega_m) (H \Phi \Phi^{(2)} + \Psi^{(2)})$$

$$ + \frac{a}{6} (1 - 3\Omega_m) \partial_k v_k^2 - \frac{a}{6H} \partial_k v_k^2 - \frac{1}{6} \left( \bar{\delta}^2 \Phi \Phi^{(2)} - \bar{\delta}^2 \Psi^{(2)} \right)$$

$$ + \frac{1}{9\Omega_m} \left[ 3\Omega_m (11 + 14\dot{\bar{g}} + 6\dot{\bar{g}}^2) - 16(1 + 2\dot{\bar{g}} + \dot{\bar{g}}^2) \right] \partial_k \Phi \partial_k \Phi^3$$

$$ - \frac{4}{27\Omega_m^2} \left[ (1 + 2\dot{\bar{g}} + \dot{\bar{g}}^2) \bar{\delta}^2 \Phi \partial_k \partial_k \Phi^3 \right]$$

$$ + \frac{2}{27\Omega_m^2} \left[ \Omega_m (5 + 8\dot{\bar{g}} + 3\dot{\bar{g}}^2) - \frac{8}{3} (1 + 2\dot{\bar{g}} + \dot{\bar{g}}^2) \right] \bar{\delta}^2 \Phi \partial_k \partial_k \Phi^3 \Phi.$$  \hspace{1cm} (B.1)

The deceleration parameter an observer would measure from the all-sky average of the redshift–distance relation is given by

$$ Q = -1 + 3 \frac{\Omega_m}{2} - 3(1 + \dot{\bar{g}})(1 - \Omega_m)\Phi + \left[ \left( 1 + \frac{2}{3}\dot{\bar{g}} \right) - \frac{4}{9\Omega_m} \left( 1 + \dot{\bar{g}} \right) \right] \bar{\delta}^2 \Phi$$

$$ - 3\dot{\bar{g}} \left( 4 + \frac{3}{2}\dot{\bar{g}} \right) (1 - \Omega_m) \Phi^2 - \frac{3}{2H} (1 - \Omega_m) (H \Phi \Phi^{(2)} + \Psi^{(2)})$$

$$ - \frac{1}{2} a \Omega_m \partial_k v_k (1 + 2\dot{\bar{g}} + \dot{\bar{g}}^2) \partial_k \Phi \partial_k \Phi^3 + \frac{1}{6} \bar{\delta}^2 \Psi^{(2)}$$

$$ + \frac{1}{9\Omega_m} \left[ 3\Omega_m (10 + 14\dot{\bar{g}} + 6\dot{\bar{g}}^2) - 8 (2 + 5\dot{\bar{g}} + 3\dot{\bar{g}}^2) \right] \Phi \partial_k \partial_k \Phi^3$$

$$ - \frac{1}{27\Omega_m^2} \left[ \frac{1}{6} \Omega_m^2 (13 + 12\dot{\bar{g}}) - \frac{4}{9} \Omega_m (2 - \dot{\bar{g}} - 12\dot{\bar{g}}^2) - \frac{4}{27} (1 + 2\dot{\bar{g}} + \dot{\bar{g}}^2) \right] \partial_k \Phi \partial_k \Phi^3$$

$$ + \frac{4}{27\Omega_m^2} (1 + 2\dot{\bar{g}} + \dot{\bar{g}}^2) \partial_k \partial_k \Phi \partial_k \partial_k \Phi$$

$$ + \frac{2}{27\Omega_m^2} \left[ \Omega_m (5 + 8\dot{\bar{g}} + 3\dot{\bar{g}}^2) - 4 (1 + 2\dot{\bar{g}} + \dot{\bar{g}}^2) \right] \bar{\delta}^2 \Phi \partial_k \partial_k \Phi^3 \Phi.$$  \hspace{1cm} (B.2)

Finally, we give the deceleration parameter defined via equation (11), averaged in the local rest frame of the dust observers:

$$ q_D = -1 + 3 \frac{\Omega_m}{2} - 3 \left[ (1 + \dot{g})(1 - \Omega_m) + \frac{3}{2} \Omega_m g_I H \right] \left( \Phi \right) + \left[ 1 + \frac{2}{3}\dot{\bar{g}} - \frac{4}{9\Omega_m} (1 + \dot{\bar{g}}) \right] \left( \bar{\delta}^2 \Phi \right)$$

$$ + \left\{ \left[ 3 \left( 4 + \dot{\bar{g}} - \frac{1}{2}\dot{\bar{g}}^2 \right) + 9 g_I H \left( (1 + \dot{g})(1 - \Omega_m) + \frac{3}{4} g_I H \right) \right] \left( \Phi \right) - 3 \left[ (4 + 5\dot{\bar{g}} + \frac{3}{2} \Omega_m (3 + 4\dot{\bar{g}} + \dot{\bar{g}}^2) \right.$$

$$ - 3 g_I H \left[ (1 + \dot{g}) \left( 1 - \frac{1}{2} \Omega_m \right) - \Omega_m (1 + 2\dot{\bar{g}} + 4 g_I H) \right] \right\} \bar{\delta}^2 \Phi \partial_k \partial_k \Phi^3 \Phi.$$
\begin{align*}
&- \frac{27}{2} \Omega_m^2 \dot{\Omega}^2 H^2 \left(1 - \frac{1}{2} \Omega_m\right) \langle \Phi \rangle^2 - \frac{1}{5 \Omega_m} \left[ \frac{1}{2} \Omega_m^3 (3 + 4 \dot{\hat{g}}) - \frac{4}{9} \Omega_m (2 + \dot{\hat{g}} - \ddot{\hat{g}}^2) \right] \\
&\quad - \frac{4}{27} \left[ 1 + 2 \dot{\hat{g}} + \ddot{\hat{g}}^2 \right] \langle \ddot{\Phi} \ddot{\Phi} \rangle + \frac{1}{3} \left[ (17 + 20 \dot{\hat{g}} + 6 \ddot{\hat{g}}^2) - \frac{2}{3 \Omega_m} (10 + 14 \dot{\hat{g}} + 4 \ddot{\hat{g}}^2) \right] \\
&\quad + g_1 H \left[ (1 - 2 \dot{\hat{g}}) - \frac{4}{3 \Omega_m} (1 + 2 \dot{\hat{g}}) - \Omega_m (2 + \dot{\hat{g}}) \right] \langle \Phi \rangle \langle \ddot{\Phi} \rangle \\
&\quad - \frac{3}{2 H} \left[ (1 - \Omega_m) (H \langle \Phi^{(2)} \rangle + \langle \Psi^{(2)} \rangle) + \frac{6}{5} \Delta (2 - 3 \Omega_m) \langle \ddot{v}_k v_k \rangle + \frac{1}{6} (\langle \ddot{\Phi} \Phi^{(2)} \rangle) \right] \\
&\quad + \frac{9}{8 \Omega_m} H \int \frac{d \tau^*}{\Omega_m} \left[ \langle \Phi^{(2)} \rangle - \frac{1}{2} \langle \Phi^2 \rangle - \langle v^k v^l \rangle - g_1 H a \langle \dddot{v}_k \rangle \right] \\
&\quad - \frac{2}{27 \Omega_m} \left[ \frac{2}{3} (1 + 2 \dot{\hat{g}} + \ddot{\hat{g}}^2) - \Omega_m (5 + 8 \dot{\hat{g}} + 3 \ddot{\hat{g}}^2) \right] \langle \ddot{\Phi} \rangle^2.
\end{align*}

Note that there are no connected $\langle \ddot{\Phi} \Phi \rangle^2$ terms in this expression.

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