Analytical Model for the Dynamical Motion of the Bulges of Two Interacting Galaxies

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Abstract

Two mathematical models of three bodies of variable masses are used to obtain a qualitative description of two interacting galaxies with mass exchange and mass loss. The reference system is centered on the largest body (the most massive galaxy), and the other two bodies are allowed to move around this one under the laws of gravity. The third body, which simulated the mass lost by the second galaxy in the form of a tail, increases its mass due to the mass lost by the second body and follows its trajectory. We are interested in knowing the time evolution of the separation of the two bulges of the interacting galaxies, and the parameters for the analytical models are obtained by running simulations with the GADGET-2 N-body code. The resulting behavior of this distance in our mathematical models is qualitatively in good agreement with that obtained by this code.

Key words: galaxies: bulges – galaxies: interactions – galaxies: kinematics and dynamics

1. Introduction

Galaxies are one of the most important structures in our universe, and studies of their formation and subsequent evolution continue to fill research journals (White & Rees 1978; Dekel & Silk 1986). All types of galaxy consist of the combination of an elliptical-like component bulge (Dwek et al. 1995; Bouwens et al. 1999; Bertin 2000; Milosavljević & Merritt 2001; Launhardt et al. 2002; Salviander et al. 2007; Xu et al. 2007) and a disk-like component (Kennicutt 1983; Kennicutt et al. 1987; Burkert et al. 1992; Bertin 2000; Giallongo et al. 2000; van der Kruit 2010), in different proportions (which can be zero), containing stars, gas, dust (Bertin 2000; Ivison et al. 2010; Kacprzak et al. 2011; Daylan et al. 2016), plus a dark matter halo, which usually contains the bulk of the mass. Visually, the bulge component (Holtzman et al. 1998; Magorrian et al. 1998) defines the trajectory of the galaxy, usually coinciding with the center of the dark matter halo.

Galaxies exist in binary systems and in small groups (less than 100 members) or large clusters of galaxies with increasingly complicated dynamics (Gallagher & Ostriker 1972; Turner 1976; Voit 2005; Paul et al. 2017). When two galaxies interact, there is an exchange of mass (Gutowski & Larson 1976), and usually one (or both) of the galaxies lose mass in the form of one (Kemp et al. 2016), or more usually, two tidal tails (Vallée & Wilson 1976; Wilson & Vallée 1977; Karachentsev & Makarov 2008). Material in tidal tails may escape the system or fall back into the galaxy eventually (Hibbard & Mihos 1995; Kemp et al. 2016), or may be incorporated in tidal dwarf galaxies, galaxies newly formed from material in tidal tails (Duc et al. 2002; Kemp et al. 2016). The distribution of material around interacting galaxies and the forms of the galaxies themselves have been successfully modeled using numerical N-body codes since the works of Toomre & Toomre (1972).

Analytically, this phenomenon corresponds to a typical mass variation problem, which is called the Gyldén-Mestschersky problem in astronomy (Gyldén 1884; Mestschersky 1893; Lovett 1902; Mestschersky 1902; Bekov 1988, 1991), which has had a long standing challenge for a complete analytical description (Sommelier 1964; Beković 1981; Bekov 1989; Spivak 2010; López & Martínez-Prieto 2014).

There have been some works (Soares 1996) with a semi-analytical description of the dynamics of two point galaxies (Sanders 1964; Bleher et al. 1988; Sanders & McGaugh 2002; Binney & Tremaine 2011; Zotos 2013), but as far as we know, there have been few attempts to consider the mass variation problem in binary galaxies (Jeans 1924, 1928). The analytical treatment of two interacting galaxies as a two-body problem cannot be considered satisfactorily since one needs to consider the dynamics of binary galaxies with mass loss or mass exchange between them.

Based on the observations made by previous studies (Spivak 2010; López & Juárez 2013), we consider that Newton’s equations of motion for mass variation problems are still valid. In addition, we consider that the trajectory of a galaxy is defined mainly by its bulge (center of the dark halo), and this is true even when two galaxies are interacting. We also consider that these bulges are treated as point bodies with variable mass, having one galaxy more massive than the other one. The less massive body loses mass (galaxy tails), which is going to be absorbed by a third body that also moves around the binary system, and it only increases its mass due to the mass loss from the second body. The parameters (mass variations) selected for our three bodies in the mass variation models will be taken from the N-body simulation program GADGET-2 (Springel & White 1999; Springel 2005; Springel et al. 2005) since the real evolution times of such systems are in Gyr. The principal point is to see if this very simple model of three bodies (galaxies) could qualitatively say something about the behavior of the very complicated real motion of two interacting galaxies, using parameters from the N-body simulations. The third body simulates a tidal tail, and if a second tail is present, it is effectively included in the mass of the first (most massive) body.
2. Analytical Models

Consider three bodies of initial masses \( m_1(0) = m_1 \), \( m_2(0) = m_2 \), and \( m_3(0) = m_3 \) under their mutual gravitational attraction. Then, the equations of motion of these bodies from an arbitrary inertial frame of reference are

\[
\begin{align*}
\frac{d(m_1v_1)}{dt} &= -G\frac{m_1m_2}{R_{12}^2}x - G\frac{m_1m_3}{R_{13}^2}x, \\
\frac{d(m_2v_2)}{dt} &= +G\frac{m_2m_1}{R_{12}^2}x - G\frac{m_2m_3}{R_{23}^2}x, \\
\frac{d(m_3v_3)}{dt} &= +G\frac{m_3m_1}{R_{13}^2}x + G\frac{m_3m_2}{R_{23}^2}x,
\end{align*}
\]

where \( G \) is the gravitational constant \((G = 6.674 \times 10^{-11} \text{m}^3 \text{kg}^{-1} \text{s}^{-2})\) (Rose et al. 1969) \( v_i = d x_i / dt \) for \( i = 1, 2, 3 \) represent the velocities of the bodies, being \( x_i, i = 1, 2, 3 \) their positions, and \( R_{ij} \) for \( i \neq j \) are the unitary vectors \( R_{ij} = R_j / |R_j| \), with \( R_{ij} = x_i - x_j \).

2.1. Model-1

Moving our reference system to the body with mass \( m_1 \) \((x_1 = 0)\), the first equation is not considered, and defining \( x_2 = x \) and \( x_3 = x' \), the following equations result,

\[
\begin{align*}
\frac{d(m_2x)}{dt} &= -G\frac{m_2m_1}{|x|^3}x + G\frac{m_2m_1}{|x'|^{3/2}}(x' - x), \\
\frac{d(m_3x')}{dt} &= -G\frac{m_3m_2}{|x - x'|^{3/2}}(x' - x) - G\frac{m_3m_1}{|x'|^{3/2}}x',
\end{align*}
\]

where the time depending mass will be taken as

\[
\begin{align*}
m_1(t) &= m_{10} + (b_1 + \beta)t, \\
m_2(t) &= m_{20} + (b_2 - \beta)t - m_3(t), \\
m_3(t) &= m_{30} + m_{3f}(1 - e^{-\gamma t}),
\end{align*}
\]

where \( \beta \) represents the mass transfer between the bodies of masses \( m_1 \) and \( m_2 \), the parameters \( b_1 \) and \( b_2 \) represent the increases in the effective masses of the bodies \( m_1 \) and \( m_2 \) due to the increases of the local particle densities of the spheres in which we define the galaxy masses when using the GADGET-2 code, \( m_{30} \) and \( m_{3f} \) are the initial and final masses of the third body, and \( \gamma \) is related to the rate of mass gain of the third body from the second one.

2.2. Model-2

In this model, we assume again that our reference system is located in the body with mass \( m_1 \), and that the trajectory of the body with mass \( m_2 \) follows the motion of the body with mass \( m_2 \) in a specified way. Thus, Equations (2a) and (1b) are not taken into account, and the motion of the system is reduced to the equation

\[
\frac{d(m_2x)}{dt} = -G\frac{m_2m_1}{|x|^3}x + G\frac{m_2m_3}{|x'|^{3/2}}(x' - x),
\]

where the motion of the body with mass \( m_3 \) will be determined by the expression

\[
x'(t) = x(t - t_c) + \alpha(t - t_c)x(t - t_c),
\]

where \( t_c \) is the retarded time which is the time delay for movement of the third body, after the second body has started to move (used to simulate the tail leaving the second galaxy). In this model, if \( t_c = 0 \) the second and third bodies would be moving tangentially on the same radial line, separated by some defined distance, on their tangential motion on the same radial line. The function \( \alpha(t - t_c) \) is a function which takes into account the separation of the third body from the second one, \( \alpha(t - t_c) = \alpha_0(2 - e^{-a(t - t_c)}) \).

Equations (2a), (2b), and (4) are solved numerically, with the functions \( m_i(t) \) for \( i = 1, 2, 3 \) determined qualitatively from the N-body GADGET-2 code, when two galaxies are interacting. The idea is to study the behavior of the separation between the body with mass \( m_1 \) and the body with mass \( m_2 \),

\[
d(t) = |x(t)|,
\]

and compare this parameter with the same parameter obtained using the GADGET-2 code.

3. Simulations of Two Interacting Galaxies with GADGET-2

The GADGET-2 code is a program that models a galaxy and galaxy interactions. Each galaxy has a bulge, a disk, central black holes, gas, and dark matter halo. The description and capabilities of this program are given in Springel (2005), Springel et al. (2005), Springel & White (1999). If \( r_{200} \) is the radius containing 75\% of the galaxy mass, where the density is 200 times the critical density and \( v_{200} \) is the rotation velocity or velocity dispersion at this radius, then the total mass of the galaxy within this radius is \( M_{200} = v_{200}^2 r_{200} / G \), where \( G \) is the gravitational constant.

\[
M_{200} = M_{\text{bulge}} + M_{\text{halos}} + M_{\text{disk}}.
\]

The halo and bulge densities, as a function of the radius \( r \) of the galaxy, follow the expression

\[
\rho_{\text{halo, bulge}}(r) = \frac{M_{\text{halo, bulge}}}{2\pi r(r + a_{b,h})},
\]

where the parameter \( a_{b,h} \) is given in term of the concentration index \( c \) as \( a_{b,h} = r_s [2\ln(1 + c) - c(1 + c)] \), and the halo scale length is defined as \( r_s = r_{200} / c \) and the proportion bulge scale length per scale radius \( f_b = a_b / R_b \), where \( R_d \) is the scale length of the disk. The angular momentum of the halo is given by

\[
J = \lambda \sqrt{2GM_{200}r_{200} / f_c},
\]

where \( \lambda \) is the twist parameter, and the parameter \( f_c \) is written in terms of the concentration index as \( f_c = c(1 - 1/(1 + c)^2 - 2 \ln (1 + c)/(1 - c)/(2\ln(1 + c) - c(1 + c)) \). The fractional bulge density of the disk is defined as \( f_b = J_{b} / JM_{\text{bulge}} / M_{\text{200}} \).

The disk (stars) density varies as

\[
\rho_d(R, z) = \frac{M_d}{4\pi z_0 R_d} \sech^2 \left( \frac{z}{2z_0} \right) \exp \left( -\frac{R}{R_d} \right),
\]

where \( R = \sqrt{R^2 + z^2} \), \( z_0 \) is the parameter determining the thickness of the disk. The angular momentum of the disk is \( J_d = J_d \), with the free parameter \( j_d \). Similarly, the gas in the
disk is modeled as

$$\Sigma_{\text{gas}} = \frac{M_{\text{gas}}}{2\pi h^2 r} \exp\left(-\frac{r}{h_r}\right),$$  \hspace{1cm} (11)

where $h_r$ is the scale length of the gas profile, and $M_{\text{gas}} + M_*= M_{\text{disk}}$. In addition, the vertical structure of gas in asymmetric galaxies is governed by the equation

$$\frac{1}{\rho_{\text{gas}}} \frac{\partial P}{\partial z} + \frac{\partial \Phi}{\partial z} = 0,$$  \hspace{1cm} (12)

where $\Phi$ is the gravitational potential due to the total mass of the gas.

Our simulation of two interacting galaxies with the GADGET-2 code were performed with the following set of parameters for the galaxies $G_1$ and $G_2$ ($N_i$ being the number of elements of each part of the galaxy):

$G_1$:

- $c = 13.0$, $v_{200} = 220$ km s$^{-1}$, $\lambda = 0.033$,
- $M_{\text{disk}}/M_{200} = 0.0$, $M_{\text{bulge}}/M_{200} = 0.007$, $a_b/R_d = 1.0$,
- $N_{\text{halo}} = 150,000$, $N_{\text{disk}} = 0.0$, $N_{\text{gas}} = 0.0$,
- $N_{\text{bulge}} = 50,000$, 

where $R_d = 3.7397h^{-1}$ kpc.

$G_2$:

- $c = 5.0$, $v_{200} = 100$ km s$^{-1}$, $\lambda = 0.033$,
- $M_{\text{disk}}/M_{200} = 0.041$, $M_{\text{bulge}}/M_{200} = 0.01367$, $M_{\text{gas}}/M_{\text{disk}} = 0.1$
- $a_b/R_d = z_0/R_d = 0.2$, $J_{\text{disk}}/J = 0.041$, $N_{\text{halo}} = 300,000$,
- $N_{\text{disk}} = 200,000$, $N_{\text{gas}} = 20,000$, $N_{\text{bulge}} = 10,000$,

where $R_d = 1.978h^{-1}$ kpc.

Figure 1 shows the initial configuration of both galaxies with the regions (circles) representing spheres used to define the total masses of the galaxies in the planes $x$-$y$ and $x$-$z$. The mass variation depends on the size of these circles. Figures 2 and 3 show the mass evolution of both galaxies during their interaction for several regions of different sizes (as the size increases, the number of elements increases and the mass increases, as shown initially in both figures) for the galaxies of mass $m_1$ and $m_2$, where the size of the regions are $50h^{-1}$ kpc and $50/2.2h^{-1}$ kpc longer than the previous one, for $G_1$ and $G_2$ ($h$ is the normalization of the Hubble constant $H_0$, $h = H_0/100$ km s$^{-1}$ Mpc$^{-1}$ and observationally, $h \sim 0.7$, and 1 kpc $\approx 206,264,806$ au $\approx 149.6 \times 10^6$ km). The maxima on the mass of the second galaxy, shown in Figure 3, are due to excess or counting the elements when the two galaxies are colliding with each other, and the minima correspond to when they are separated enough that this counting is well defined.
analytical Model-1 (without retarded time) as

\[
\begin{align*}
m_{10} &= 185.513000 \cdot 10^{10} \, h^{-1} M_\odot, \\
m_{20} &= 17.4265690 \cdot 10^{10} \, h^{-1} M_\odot, \\
m_{30} &= m_{3f} = 1.74265695 \cdot 10^{10} \, h^{-1} M_\odot, \\
\beta &= 1.30699262 \, f^{-1} M_\odot \, \text{year}^{-1}, \\
b_1 &= 74.2051983 \, f^{-1} M_\odot \, \text{year}^{-1}, \\
b_2 &= 21.7832112 \, f^{-1} M_\odot \, \text{year}^{-1}, \\
\gamma &= 1.0 \, f^{-1} \, \text{hGyear}^{-1}, \\
t_r &= 0.0 \, f^{-1} \, \text{tGyear}, \\
|x_0 - x'_0| &= 24.3476219 \, h^{-1} \, \text{kpc}, \\
x_0 &= 121.73811134032031 \, h^{-1} \, \text{kpc}, \\
y_0 &= 0.0 \, h^{-1} \, \text{kpc}, \\
v_x(0) &= 0.0 \, \text{km s}^{-1}, \\
v_y(0) &= 181.0210930177266 \, \text{km s}^{-1}, \\
v'_x(0) &= 166.4443359375 \, \text{km s}^{-1}, \\
v'_y(0) &= 221.7045793214924 \, \text{km s}^{-1}, \\
|\mathbf{v}_0| &= 0.987654328 \, hGyear , \\
0.05 \, h \, tGyear , \\
1.0 \, hGyear , \\
|\mathbf{v}_0| &= 0.98. \\
\end{align*}
\]

where \( M_\odot \) represents the Solar Mass \( (M_\odot = 1.989 \times 10^{30} \, \text{kg}) \) and the dimensionless parameter \( f = 0.98 \). For this same model, but with retarded time \( (t_r) \), the parameters are

\[
\begin{align*}
m_{10} &= 185.513000 \cdot 10^{10} \, h^{-1} M_\odot, \\
m_{20} &= 17.4265690 \cdot 10^{10} \, h^{-1} M_\odot, \\
m_{30} &= m_{3f} = 1.74265695 \cdot 10^{10} \, h^{-1} M_\odot, \\
\beta &= 0.430285633 \, f^{-1} M_\odot \, \text{year}^{-1}, \\
b_1 &= 137.417049 \, f^{-1} M_\odot \, \text{year}^{-1}, \\
b_2 &= 21.5142822 \, f^{-1} M_\odot \, \text{year}^{-1}, \\
\gamma &= 0.987654328 \, f^{-1} \, \text{hGyear}^{-1}, \\
t_r &= 0.05 \, h^{-1} \, \text{tGyear}, \\
|x_0 - x'_0| &= 24.3476219 \, h^{-1} \, \text{kpc}, \\
x_0 &= 121.73811134032031 \, h^{-1} \, \text{kpc}, \\
y_0 &= 0.0 \, h^{-1} \, \text{kpc}, \\
v_x(0) &= 0.0 \, \text{km s}^{-1}, \\
v_y(0) &= 181.0210930177266 \, \text{km s}^{-1}, \\
v'_x(0) &= 166.4443359375 \, \text{km s}^{-1}, \\
v'_y(0) &= 221.7045793214924 \, \text{km s}^{-1}, \\
|\mathbf{v}_0| &= 0.987654328 \, hGyear , \\
0.05 \, h \, tGyear , \\
1.0 \, hGyear , \\
|\mathbf{v}_0| &= 0.98. \\
\end{align*}
\]

The parameters for our analytical Model-2 can be chosen as

\[
\begin{align*}
m_{10} &= 185.513000 \cdot 10^{10} \, h^{-1} M_\odot, \\
m_{20} &= 17.4265690 \cdot 10^{10} \, h^{-1} M_\odot, \\
m_{30} &= 0, \\
m_{3f} &= 1.74265695 \cdot 10^{10} \, h^{-1} M_\odot, \\
\beta &= 0.430285633 \, f^{-1} M_\odot \, \text{year}^{-1}, \\
b_1 &= 4.58056778 \, f^{-1} M_\odot \, \text{year}^{-1}, \\
b_2 &= 21.5142822 \, f^{-1} M_\odot \, \text{year}^{-1}, \\
\gamma &= 0.987654328 \, f^{-1} \, \text{hGyear}^{-1}, \\
t_r &= 0.05 \, h^{-1} \, \text{tGyear}, \\
\alpha_0 &= 24.3476219 \, h^{-1} \, \text{kpc}, \\
\omega &= 0.987654328 \, f^{-1} \, \text{hGyear}^{-1}, \\
x_0 &= 121.73811134032031 \, h^{-1} \, \text{kpc}, \\
y_0 &= 0.0 \, h^{-1} \, \text{kpc}, \\
v_x(0) &= 0.0 \, \text{km s}^{-1}, \\
v_y(0) &= 181.0210930177266 \, \text{km s}^{-1}. \\
\end{align*}
\]

Figure 4, the green solid line, shows the evolution of the separation of the center of the bulges of the two galaxies as a function of time, where we can see the shrinking damping oscillations of the system. To make the calculation of the distance between these two galaxies, we first located the centers of both galaxies (the points of highest density) and separate these centers a distance \( d(0) = 120 \, h^{-1} \, \text{kpc} \) in the x-direction. Then, we follow the evolution of the separation. This spatial scale covers the typical separations and spatial scales of filaments seen in interacting galaxies like NGC 4038/4039 (Lahén et al. 2018) and merger remnants like NGC 7252 (Hibbard & Mihos 1995).

4. Analytical Result and Comparison

As we can see from Figures 1–3, we can consider the mass of the three bodies and the values of the parameters for our
Solving numerically Equations (2a), (2b), and (4), we obtain Figure 4, where the parameter (6) is plotted for the two mathematical models, Model-1 corresponds to the dotted green curve (no retarded time for the third body) and the dotted-dashed yellow curve (with retarded time \( t_r = 5 \, h^{-1} \text{Gyr} \) for the third body), Model-2 corresponds to the dashed blue curve (with a retarded time for the second body of \( t_r = 5 \, h^{-1} \text{Gyr} \)), and the result obtained with the GADGET-2 code (continuous red curve). As one can see, there is a qualitatively good agreement of the GADGET-2 code with both models, and Model-2 with Model-1 (with and without retarded time) have almost the same behavior, even though Model-2 is an oversimplifying approximation of Model-1.

Figures 5 and 6 show the motion of the second and third bodies (relative to the first body) on the plane of motion, and one can see that the third body tends to escape from the system (simulating the tail of the second Galaxy moving out of the binary system). Of course, one cannot expect similar behavior since in Model-2 we have forced the third body to follow the second body at some retarded time \( t_r = 5 \, h^{-1} \text{Gyr} \). The small square mark on Figures 5 and 6 indicates the position of the second body when the third body is starting to move, due to retarded time.

5. Results and Comments

We have used two mathematical models for the restricted three-body problem of variable masses to obtain a qualitative description of two interacting galaxies with mass exchange and mass loss. The main point is the use of the third body as a ghost galaxy, which simulated the mass lost by the second galaxy in the form of a tail, and this ghost galaxy increases its mass due to the mass lost by the second body. We were interested in predicting the time evolution of the distance separation of the two bulges of the interacting galaxies, using two separate models, one as two bodies with mass variation, the other with one body with mass variation, using parameters deduced by the use of the GADGET-2 \( N \)-body simulation code for two interacting galaxies.

The general result we have obtained is that the time-dependent distance between the bulges of the two galaxies, calculated with our mathematical models and deduced from the GADGET-code, agrees well qualitatively using both of our mathematical models. This agreement is better with our Model-1 (three-body problem,
between the results of our model and the interacting galaxies separations and radial velocities over a large sample of (of the GADGET-2 code could suffice to obtain the parameters (some may also be estimated more crudely by studying observed separations and radial velocities over a large sample of interacting galaxies). We also made a detailed comparison between the results of our model and the N-body code, in order to prove the reliability of the analytical model. However, once the initial parameters have been obtained, the subsequent evolution of the system using our models is obtained much more efficiently than it would be by the use of the N-body code alone.

The intention of creating the analytical model was to see if the trajectories of the bulges of galaxies could be predicted accurately and in more efficient manner than using the N-body code. Although an estimation of the mass transfer rates between galaxies can be made using our models, a detailed treatment of galaxy evolution is a many-body problem, and we are limited to using three bodies in our models. As a final comment, we expect that also including the position-depending-mass of the bodies in our models, the results can be improved appreciably to match the simulation N-body code.

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