Modelling spontaneous four-wave mixing in periodically-tapered waveguides

Mohammed F. Saleh$^{1, 2}$

$^1$Scottish Universities Physics Alliance (SUPA), Institute of Photonics and Quantum Sciences, Heriot-Watt University, EH14 4AS Edinburgh, UK
$^2$Department of Mathematics and Engineering Physics, Alexandria University, Alexandria, Egypt

Periodically-tapered-waveguides technique is an emerging potential route to establish quasi-phase-matching schemes for efficient on-demand parametric interactions in third-order nonlinear materials. In this paper, I investigate this method in enhancing spontaneous photon-pairs emission in fibres and planar waveguides with sinusoidally-varying cross sections. I have developed a general robust quantum model to study this process under continuous or pulsed-pump excitations. The model shows a great enhancement in photon-pairs generation in waveguides with a small number of tapering periods that are feasible via the current fabrication technologies. I envisage that this work will open a new area of research to investigate how the tapering patterns can be fully optimised to tailor the spectral properties of the output photons in third-order nonlinear guided structures.
I. INTRODUCTION

Spontaneous photon-pair emission in nonlinear media has become the standard approach for preparing quantum states of light in labs due to its efficiency, ease of use, and operation at practical conditions [1]. In second-order nonlinear media, a mother photon can be spontaneously downconverted into two daughter photons in three-wave mixing process. Whereas, two pump photons will combine to generate two other photons in a spontaneous four-wave mixing (SFWM) parametric process in third-order nonlinear media.

Efficient parametric wave-mixing processes are inherently constrained by energy and momentum conservation (the latter is known as the phase-matching condition). Quasi-phase-matching schemes have been successfully employed in second-order nonlinear media to enable on-demand parametric processes, using periodically-poled ferroelectric crystals [2, 3]. This technique has had a remarkable impact in the field of nonlinear and quantum optics; it eliminates the stringent constraints posed on the directions and polarisations of the interacting waves, and allows for on-chip collinear co-polarised strong nonlinear interactions to take place. Using this technique, highly-efficient and pure single-photon sources have been widely demonstrated in bulk and integrated configurations [4, 5].

Silica fibres and silicon-photonic waveguides are the practical and convenient platforms for generating single photons for quantum communications and integrated-quantum applications, respectively [6, 7]. Enabling SFWM parametric processes in these third-order nonlinear platforms can be achieved via exploiting the nearly-phase-matching regime, or balancing waveguide and material dispersions [8]. Similar to second-order nonlinear media, these conventional methods pose restrictions, for instance, the frequency-separation between the interacting photons and the operating dispersion regime. Other alternative techniques such as photonic crystal fibres [10, 11], birefringent silica waveguides [1, 5], microresonators [12], symmetric and asymmetric directional couplers [13, 14] have shown the ability to circumvent these limitations to obtain highly-efficient and pure single photons in third-order nonlinear media.

In analogue to periodically-poled structures, periodically-tapered waveguides (PTWs) have been also introduced to correct phase mismatching of third-order nonlinear interactions via sinusoidally modulating the width of planar waveguides or the core of microstructured fibres (coined as dispersion-oscillating fibres) [15, 16]. The tapering period was initially restricted to the millimetre range in planar structures. Very recently, PTWs with tapering periods in the micrometer range have been successfully realised in planar waveguides in two different configurations with either modulating the core or the cladding using the Photonic Damascene process [17]. These structures have been mainly exploited in tuning dispersive-wave emission, controlling supercontinuum generation, and manipulating modulation instability. On the theoretical side, I have presented a new analysis based on the Fourier-series to explain how PTWs can be employed as a complete analogue to periodically-poled structures [18]. In PTWs both the linear and nonlinear coefficients are longitudinally varying resulting in multiple Fourier components of the phase-mismatching and nonlinear terms. Opposite Fourier components of the two terms can simultaneously cancel each other, allowing the conversion efficiency of a parametric process to grow along the structure, if right combinations of the tapering period and modulation amplitude of the waveguide are used.

In this paper, I have applied the PTW-technique to generate photon pairs via SFWM processes at on-demand frequencies and with high spectral purity. The paper is organised as follows. In Sec. II a robust quantum model have been developed to study this process in tapered waveguides for continuous and pulsed pump sources. This model has then been applied to study SFWM processes in dispersion-oscillating fibres and width-modulated planar silicon-nitride waveguides in Sec. III! Finally, my conclusions are summarised in Sec. IV.

II. MODELLING SFWMS PROCESSES IN PTWS

Consider a SFWM parametric process in a single-mode third-order nonlinear waveguide. I have developed a model follows an approach that determines the energy flux across a transverse plane over a quantisation time $T$ in the Heisenberg picture [19]. Initially, it is assumed that the pump is a monochromatic wave and the waveguide is uniform. Later, the model will be generalised to take into account propagation in tapered waveguides, as well as having pulsed pump sources. Raman nonlinearity is neglected, for simplicity, which is also the case in silicon-nitride waveguides [20] and hollow-core fibres filled by noble gases [21].

a. Continuous pump wave. Assuming that the pump ($p$) is a strong undepleted monochromatic classical wave, its electric field can be described as

$$\mathcal{E}_p(x, y, z, t) = \text{Re} \{E_p(x, y, z, t)\},$$  \hspace{1cm} (1)

where $x, y$ are the transverse coordinates, $z$ is the longitudinal direction, $t$ is the time, $\text{Re}$ is the real part, $E_p = A_p F_p(x, y) e^{-j(\omega_p t - k_p z)}$, $A_p$, $F_p$, $\omega_p$, $k_p$ are the amplitude, transverse profile, angular frequency, and propagation constant of the pump wave, respectively, $k_p = n_p \omega_p/c$, $n_p$ is the refractive index at the pump frequency, and $c$ is
the speed of light in vacuum. Using Maxwell equations and applying slowly-varying approximation \(^9\), self-phase modulation effect that influences the pump propagation can be included by replacing \(k_p\) with

\[
\kappa_p = k_p \left[ 1 + \frac{3\chi^3 p^2}{8\pi^2 S_p} \int \left| F_p \right|^4 dxdy \right],
\]

(2)

where \(\chi^3\) is the nonlinear coefficient, and \(S_p = \int \left| F_p \right|^2 dxdy\) is the pump beam area.

d. Signal and idler photons. The electric field operators of the converted single photons (either signal or idler) can be written as

\[
\hat{\mathcal{E}}(x, y, z, t) = \hat{\mathcal{E}}^+(x, y, z, t) + \hat{\mathcal{E}}^-(x, y, z, t),
\]

(3)

where \(\hat{\mathcal{E}}^+\) and \(\hat{\mathcal{E}}^-\) are the positive and negative frequency-components of the field, \(\hat{\mathcal{E}}^- = (\hat{\mathcal{E}}^+)\dagger\), and \(\hat{\mathcal{E}}^+\) can be expressed in terms of a superposition of frequency-dependent mode operators \(^{19}\),

\[
\hat{\mathcal{E}}^+(x, y, z, t) = \sum_s \sqrt{\frac{\hbar\omega_s}{2\epsilon_o T n_s S_s}} F_s(x, y) \hat{a}(z, \omega_s) e^{-j\omega_s t},
\]

(4)

\(F, S, \omega,\) and \(n\) have the same definitions as for the pump wave, \(\hbar\) is the reduced Planck’s constant, \(\epsilon_o\) is the dielectric permittivity, and \(\hat{a}\) is annihilation operator. The discrete spectral modes are separated by a frequency spacing 2\(\pi/T\), and \(T\) tends to \(\infty\) in the continuous case. The photon-number operator of mode \(s\) during a time window \(T\) is \(N(z, \omega_s) = \hat{a}\dagger(z, \omega_s) \hat{a}(z, \omega_s)\), with \(\hat{a}\dagger\) the creation operator. The space evolution of the operators are governed by

\[
-j\hbar \frac{\partial}{\partial z} \hat{\mathcal{E}} = \left[ \hat{\mathcal{E}}, \hat{G} \right],
\]

(5)

where \(\hat{G}(z) = \int \int \int \hat{g}(x, y, z, t) dt dxdy\) is the momentum operator, \(\hat{g} = \hat{D}^\dagger \hat{\mathcal{E}}^- + H.c.\) is the momentum flux, \(\hat{D} = \epsilon_o n^2 \hat{\mathcal{E}} + \hat{P}_{NL}\) is the electric displacement field, \(\hat{P}_{NL}\) is the nonlinear polarisation, and \(H.c.\) is the Hermitian conjugate. In the linear regime (L) and substituting the electric field and momentum operators in Eq. \(5\), it can be easily shown that

\[
\frac{\partial \hat{a}^{(L)}_s}{\partial z} = jk_s \hat{a}_s(z),
\]

(6)

where the commutation relation \([\hat{a}(z, \omega_s), \hat{a}^\dagger(z, \omega_{s'})]\) = \(\delta_{ss'}\) has been applied, with \(\delta\) a Kroenecker delta, \(k_s = n_s \omega_s/c\), and \(\hat{a}_s(z) = \hat{a}(z, \omega_s)\) for brevity.

e. Spontaneous four-wave mixing process. To study a SFWM interaction in this picture, the positive frequency-part of the nonlinear polarisation associated with this process should be first determined, \(\hat{P}_{SFWM}^+ = 3\epsilon_o \chi^3 p^2 \hat{E}_p \hat{E}_s^* \hat{E}_i\). Then, the momentum flux operator is \(\hat{G}_{SFWM} = \hat{P}_{SFWM}^+ \hat{E}^- + H.c.\). Applying the definition of the delta function \(\delta(\Delta \omega) = \frac{1}{T} \int_0^T \exp(-j\Delta \omega t) dt\), the momentum operator in this case is determined as

\[
\hat{G}_{SFWM} = \frac{3\hbar \chi^3 p^2 \epsilon_o^2 n_s S_s}{8c} \sum_s \frac{\omega_s \omega_i}{n_s n_i S_s S_i} \int \int F_p^2 F_s^* F_i dx dy \hat{a}_i^\dagger(z) \hat{a}_s(z) + H.c.,
\]

(7)

with \(\omega_i = 2\omega_p - \omega_s\). The subscript \(i\) refers to another mode that is coupled to the mode \(s\) within the spectrum. Substituting \(\hat{G}_{SFWM}\) in Eq. \(4\), the space evolution of the mode operator \(s\) due to this process is

\[
\frac{\partial \hat{a}^{(SFWM)}_s}{\partial z} = j\frac{3\chi^3 p^2 \epsilon_o^2 n_s S_s}{4c} \sum_s \frac{\omega_s \omega_i}{n_s n_i S_s S_i} \int \int F_p^2 F_s^* F_i dx dy \hat{a}_i^\dagger(z).
\]

(8)

d. Cross-phase modulation (XPM) effect. Following the aforementioned procedure, the nonlinear part of the momentum flux operator due to the XPM effect is \(\hat{G}_{XPM} = 6\epsilon_o \chi^3 p^2 \hat{E}_p \hat{E}_s^* \hat{E}_s + H.c.\). Hence, the momentum operator of this effect is

\[
\hat{G}_{XPM} = \frac{3\hbar \chi^3 p^2 \epsilon_o^2}{4c} \sum_s \frac{\omega_s}{n_s S_s} \int \left| F_p \right|^2 dx dy \hat{a}_s(z) \hat{a}_s^\dagger(z) + H.c.,
\]

(9)

and the annihilation operator is spatially evolved as

\[
\frac{\partial \hat{a}^{(XPM)}_s}{\partial z} = j\frac{3\chi^3 p^2 \epsilon_o^2}{2c} \frac{\omega_s}{n_s S_s} \int \left| F_p \right|^2 \left| F_s \right|^2 dx dy \hat{a}_s(z).
\]

(10)
e. Total effects. Therefore, the evolution of the mode operator $s$ due to linear and nonlinear contributions can be written as

$$\frac{\partial \hat{a}_s}{\partial z} = \frac{\partial \hat{a}_s^{(L)}}{\partial z} + \frac{\partial \hat{a}_s^{(XPM)}}{\partial z} + \frac{\partial \hat{a}_s^{(FWM)}}{\partial z} = j\kappa_s \hat{a}_s(z) + j\gamma_{s,i} e^{j2\kappa_p z} \hat{a}_s^\dagger(z),$$

with

$$\kappa_s = k_s \left(1 + \frac{3\chi^{(3)} A_p^2}{2n_s^2 S_s} \int |F_p|^2 |F_s|^2 \, dx \, dy\right),$$

and

$$\gamma_{s,i} = \frac{3\chi^{(3)} A_p^2}{4c} \sqrt{\frac{\omega_s \omega_i}{n_s n_i S_s}} \int F_p^2 F_s^2 F_i^2 \, dx \, dy.$$

A similar equation can be written for the mode operator $i$,

$$\frac{\partial \hat{a}_i^\dagger}{\partial z} = -j\kappa_i^\ast \hat{a}_i^\dagger(z) - j\gamma_{s,i}^\ast e^{-j2\kappa_p z} \hat{a}_s(z).$$

The two coupled Eqs. (11,14) can be solved exactly in the case of uniform waveguides, similar to the spontaneous parametric downconversion process [19].

f. Transfer matrix method. Introducing the phase transformation $\hat{a}_q(z) = \hat{b}_q(z) e^{ip_{q} z}$, $q = s, i$, Eqs. (11,14) can be written in the common compact form,

$$\frac{\partial \hat{b}_s}{\partial z} = j\gamma_{s,i} e^{j\Delta \kappa z} \hat{b}_i^\dagger(z), \quad \frac{\partial \hat{b}_i^\dagger}{\partial z} = -j\gamma_{s,i}^\ast e^{-j\Delta \kappa z} \hat{b}_s(z),$$

with $\Delta \kappa = 2\kappa_p - \kappa_s - \kappa_i$. If the waveguide is tapered, the coefficients in these equations become spatial dependent. Also, the term $e^{j\Delta \kappa z}$ should be replaced by $e^{j\Delta \phi}$, with $\Delta \phi = \int_0^z \Delta \kappa(z') \, dz'$ the proper accumulated phase-mismatching [3,22]. To solve the coupled equations in this case, the waveguide can be truncated into discrete infinitesimal elements with constant cross sections, then the equations can be solved within each element. The outcome of an element $m$ can be written in a transfer-matrix form,

$$\begin{bmatrix} \hat{b}_s \hat{b}_i^\dagger \end{bmatrix}_{z=z_m+\Delta z} = \mathcal{T}_m \begin{bmatrix} \hat{b}_s \hat{b}_i^\dagger \end{bmatrix}_{z=z_m}, \quad \mathcal{T}_m = \begin{bmatrix} 1 & f_{s,i}(\omega_p) \\ f_{s,i}^\ast(\omega_p) & 1 \end{bmatrix}_{z=z_m},$$

with $f_{s,i}(\omega_p) = j\gamma_{s,i} \Delta \kappa e^{j\Delta \phi}$ and $\Delta \kappa$ the element thickness. The explicit dependence of the parameter $f_{s,i}$ on $\omega_p$ has been shown here, to remind the reader that all these matrix elements depend on the pump frequency.

The transfer matrix that describes the whole structure and relates the output operators to the input ones is given by the multiplication of the transfer matrices of all elements in a descending order, $\mathcal{T}_{s,i} = \mathcal{T}_M \mathcal{T}_{M-1} \ldots \mathcal{T}_2 \mathcal{T}_1$, with $M$ the elements number. It is worth to note that each element has a different transfer matrix even in periodic structures, because of the parameter $\Delta \phi$ that counts the accumulated phase from the beginning of the structure.

g. Expected number of photons. The expected number of photons of a certain mode $s$ at the end of the waveguide is $\langle \psi | \hat{N} (L, \omega_s) | \psi \rangle = \langle \psi | \hat{b}_i^\dagger (L) \hat{b}_s (L) | \psi \rangle$, with $| \psi \rangle = | 0 \rangle$, the input quantum state and $L$ the waveguide length. Using the transfer matrix of the whole structure, $\hat{b}_s (L) = \mathcal{T}_{s,i} (1,1) \hat{b}_s (0) + \mathcal{T}_{s,i} (1,2) \hat{b}_i^\dagger (0)$, and $| \psi \rangle | \hat{N} (L, \omega_s) | \psi \rangle = | \mathcal{T}_{s,i} (1,2) |^2$. It is important to mention that this procedure is only for a single mode $s$ and it should be repeated for all other modes that compose the signal spectrum.

h. Pulsed pump source. The model that I have developed so far is based on having a monochromatic pump wave. If the pump is assumed to be a Gaussian pulse with a characteristic temporal width $\tau$ and a central frequency $\omega_{p0}$, then $E_p(t) = A(t) F_p(x, y) e^{j\kappa_p z}$, where $A(t) = A_0 e^{-t^2/\tau^2} e^{-j\omega_{p0} t}$ is the pulse envelope and the full-width-half-maximum of this pulse equals $2\tau \sqrt{\ln(2)}$. Using the Fourier Transform, the pump electric field can be decomposed into a superposition of multiple monochromatic waves with frequencies $\omega_p$,

$$E_p(x, y, z, t) = \text{Re} \left\{ \sum_{\omega_p} A_p (\omega_p) e^{-j(\omega_p t - \kappa_p z)} \right\},$$

(17)
where \( A_p(\omega_p) = \frac{A_{0p}e^{-i\omega_p^2/2\Delta\omega}}{\sqrt{2\pi}} \) is the amplitude of each component, and \( \Delta\omega \) is the sample frequency of the pump pulse.

To study how the evolution of two certain coupled modes \( s \) and \( i \) is influenced by having a pulsed pump source, each possible combination of two monochromatic pump waves \( \omega_{p1} \) and \( \omega_{p2} \) in the spectrum should be counted towards the evolution of the mode operators. Because of the energy conservation or the delta function used in determining the momentum operator \( G_{\text{FWM}} \), the double summation over \( \omega_{p1} \) and \( \omega_{p2} \) is reduced to a single summation over \( \omega_{p1} \) with \( \omega_{p2} = \omega_s + \omega_i - \omega_{p1} \). The above procedure to determine the expected number of photons can still be followed, however, the transfer matrix of an element \( m \) becomes

\[
T_m = \begin{bmatrix}
1 & \sum_{\omega_{p1}} f_{s,i}^*(\omega_{p1}, \omega_{p2}) \\
\sum_{\omega_{p1}} f_{s,i}^{*}(\omega_{p1}, \omega_{p2}) & 1
\end{bmatrix}_{z=z_u}.
\]

For non-degenerate combinations of \( \omega_{p1} \) and \( \omega_{p2} \),

\[
\gamma_{s,i} = \frac{3\chi^3(3)A_{p1}A_{p2}}{2c} \int \frac{\omega_s\omega_i}{n_s n_i S_s S_i} \int F_{p1} F_{p2} F_{s}^* F_{i}^* dxdy,
\]

and \( \Delta\kappa = \kappa_{p1} + \kappa_{p2} - \kappa_s - \kappa_i \). Also, each monochromatic pump wave in this case will induce a XPM effect that will influence the other pump wave as well as the converted single photons. Therefore,

\[
\kappa_{p1} = \kappa_p \left[ 1 + \frac{3\chi^3(3)}{8n_{p1}^2 S_{p1}} \int \left( A_{p1}^2 |F_{p1}|^2 + 2A_{p1}^2 |F_{p2}|^2 \right) |F_{p1}|^2 dxdy \right], \quad \left\{ u, v = 1, 2, \quad u \neq v \right\},
\]

and

\[
\kappa_p = \kappa_q \left[ 1 + \frac{3\chi^3(3)}{2n_q^2 S_q} \int \left( A_{p1}^2 |F_{p1}|^2 + 2A_{p2}^2 |F_{p2}|^2 \right) |F_{q}|^2 dxdy \right]. \quad q = s, i.
\]

Finally, it is more common in third-order nonlinear media to use the nonlinear refractive \( n_2 \) in units \( m^2/W \) rather than \( \chi^{(3)} \) in units \( m^2/V^2 \). Using the definitions of \( n_2 \) and \( \chi^{(3)} \), one can show that \( 3\chi^{(3)}A_{p1}^2 = 8n_2I_p \), where \( I_p = \frac{E_{p1}^2 \sigma \Delta\omega^2}{2\pi \sqrt{\pi}} e^{-2(\omega_p - \omega_{p0})^2} \) is the intensity of a Fourier pump component \( p \), and \( E_{p1} \) is the input pulse energy. Similarly, the quantity \( 3\chi^{(3)}A_{p1}A_{p2} \) can be written as \( 8n_2\sqrt{I_{p1}I_{p2}} \).

III. RESULTS

In this section, the developed model will be exploited in investigating photon-pair generation via SFWM parametric processes in periodically-tapered waveguides (PTWs), in particular, dispersion-oscillating fibres and width-modulated silicon-nitride waveguides. The platforms are designed to operate in the normal-dispersion regime, where satisfying the phase-matching condition is usually hard to achieve in uniform single-mode waveguides. Operating in this regime will suppress unwanted nonlinear phenomena such as supercontinuum generation, soliton formation, or dispersive-wave emission that could ruin the aimed SFWM process under strong pumping. To ensure adiabatic propagation, only small modulation amplitude and relatively-large tapering periods have been explored. The tapering period in the simulated structures has been discretised into 200 steps to increase the accuracy of the computational results.

A. Dispersion-oscillating fibres

These are silica solid-core sinusoidally-tapered microstructured fibres made of a stack of hollow capillary tubes with a cross-sectional pitch \( \sigma \), and a hole-diameter \( d \). The pitch varies as \( \sigma(z) = \sigma_0 [1 - \Delta\sigma \cos(2\pi z/\Lambda_T)] \), where \( \sigma_{av} \) is the average pitch, \( \Delta\sigma \) is the modulation amplitude, and \( \Lambda_T \) is the tapering period. The group index \( n_g = c\beta_1 \) (where \( \beta_1 \) is the first-order dispersion), and the second-order dispersion coefficient \( \beta_2 \) of a fibre with \( \sigma_{av} = 1\mu m, \Delta\sigma = 0.1, d = 0.5\mu m, \) and an output diameter \( 40\sigma \) are displayed in Fig. (a,b). The fibre effective refractive indices in these calculations have been computed via the empirical equations, including the material dispersion of silica. Over the shown spectrum from 0.6 to 1 \( \mu m \), the fibre dispersion is completely normal and the group index is gradually decrease with varying slopes.
FIG. 1. (a,b) Wavelength-dependence of the group-index and second-order dispersion of a sinusoidally-tapered microstructured fibres with a pitch $\sigma$ varies between 0.9 and 1.1 $\mu$m, and a hole-diameter $d = 0.5\sigma$. (c) Dependence of the mismatching in the propagation constants $\Delta\kappa$ of a SFWM process and the tapering period $\Lambda_T$ on the signal wavelength $\lambda_s$ with a pump at 780 nm and an input power 1 W. The assumed fibre nonlinear refractive index is $n_2 = 2.25 \times 10^{-20}$ m$^2$/W. These parameters will be used in subsequent simulations, unless stated otherwise.

Assuming a monochromatic pump source with a wavelength 780 nm and an input power 1 W, Fig. 1(c) shows the dependence of the mismatching in the propagation constants $\Delta\kappa$ of SFWM processes and the corresponding tapering period $\Lambda_T = 2\pi/\Delta\kappa$ on the signal wavelength $\lambda_s$ for the average pitch. These presented values of $\Lambda_T$ are regarded as good estimates for the right ones needed to correct the phase-mismatching via the PTW-technique at a particular wavelength $\lambda_s$ [18]. Close to the pump, $\Lambda_T$ is in the few-centimetres range, which can be achievable using advanced post-treatment processes [16]. The PTW-technique can be used to generate photon-pairs, in principle, at any on-demand frequency. However, due to the current fabrication limitations, I will exploit this technique to demonstrate the ability to generate signal and idler photons at 750 and 812.5 nm. All interacting photons are in the fibre fundamental mode, and their transverse profiles are approximated as Gaussian distributions [9].

The right combinations between the modulation amplitude and tapering period that lead to an enhanced conversion efficiency of the expected number of photons $\langle \hat{N} \rangle_{L,\omega_s}$ of a SFWM process at the targeted wavelengths are portrayed in Fig. 2(a) for the same number of periods $M$. The bright trajectories from left to right represent the 1st, 2nd, 3rd, ... -order tapering periods. The values of $\langle \hat{N} \rangle$ are normalised to the case when $\Delta\sigma = 0$, to quantify the enhancement using the PTW-technique. The expected number of signal photons are enhanced by 35 dB for only $M = 50$. As shown Fig. 2(b), the number of single photons spatially grows as an amplified oscillating function with single and double peaks within each cycle for the 1st- and 2nd-order tapering periods. Similar to periodically-poled structures, the growth rate is higher for shorter periods [3]. Also, the value of the 2nd-order period slightly deviates from the exact one, which results in accumulation of small phase mismatching during propagation that subsequently affects the growth rate.

![Output spectrum of the photon pairs around the targeted wavelengths at the end of the fibre](fig3a.png)

The output spectrum of the photon pairs around the targeted wavelengths at the end of the fibre is depicted in Fig. 3(a) for a continuous pump wave at 780 nm. The spectrum is featured as a narrow sinc-function with very weak
FIG. 3. (a,b) 1D and 2D spectral dependence of the expected number of photons $\langle \hat{N} \rangle$ generated in a modulated fibre with $\Delta \sigma = 0.1$ around the targeted wavelengths. (c) Dependence of $\langle \hat{N} \rangle$ on the signal wavelength over the entire spectrum. (d) Dependence of $\langle \hat{N} \rangle$ on the signal and idler wavelengths for a Gaussian pump with $E_i = 1$ nJ and $\tau = 2.47$ ps. $M = 50$ and $\Lambda_T = 4.5$ cm in (b–d).

sidelobes that are significantly diminished for large number of periods. Panel (b) in Fig. 3 shows the 2D representation of the spectrum as a function of the signal $\lambda_s$ and idler $\lambda_i$ wavelengths with $\omega_s + \omega_i = 2\omega_p$ satisfied at each point. This plot is equivalent to the phase-matching function determined in the interaction Hamiltonian picture via tracking the evolution of the quantum two-photon state \[26\].

The whole spectra in the absence and presence of the waveguide modulation are displayed in Fig. 3(c). This plot validates the developed quantum model, it resembles the output spectrum in the normal dispersion regime of fibre due to the classical nonlinear modulation instability effect. In a uniform waveguide with a normal dispersion, the propagation of a CW is stable against small perturbations \[4\]. However, this scenario is completely changed in a nonuniform waveguide, where modulation instability amplifies the background noise regardless the dispersion type. The frequencies of the amplified sidebands can be approximated with $\approx 3$ nm difference from the simulations using the relation $\beta_2 \Omega^2 + 2\gamma_{s,i} P = 2\pi l / \Lambda_T$, where $\Omega = \omega_s - \omega_p$ is the frequency shift from the pump source, $\beta_2$, and $\gamma_{s,i}$ are the average second-order dispersion and nonlinear coefficient, $P$ is the input power, and $l$ is an integer \[16\]. The slight deviation is due to the negligence of the effect of the modulation amplitude in this equation.

Considering a Gaussian-pulse pump source with an input energy 1 nJ and a full-width-half-maximum 4 ps, the corresponding expected number of photons is portrayed in Fig. 3(d). This plot is equivalent to the joint-spectral intensity distribution, which is usually obtained in the Hamiltonian picture and regarded as the magnitude square of the product of the phase-matching function and the pump-spectral envelope \[27\]. Therefore, the joint-spectral amplitude function, used in calculating the spectral-purity or discorrelation between the output single photons, can be constructed from the elements $T_{s,i}(1, 2)$ in the introduced model. The group-velocity-matching condition, required for high spectral-purity \[11\], is satisfied in the simulated case via setting the group-velocity of the pump in between those of the signal and idler, as shown in Fig. 1(a). Using the Schmidt decomposition of the matrix constructed from the elements $T_{s,i}(1, 2)$ results in a spectral purity 0.74 \[20\]. For a shorter Gaussian pump source with width 1 ps the spectral purity increases to 0.93, which shows the ability of the PTW-technique in producing high-pure single photons at any sought frequencies.
A planar silicon-nitride PTW with $w_{av} = 1650$ nm, $\Delta w = 0.05$, $h = 450$ nm, $\Lambda_T = 370$ $\mu$m, $M = 50$, and $\tau_2 = 1.5 \times 10^{-18}$ m$^2$/W: (a) Fundamental TE mode at $\lambda = 1064$ nm and $w = w_{av}$. (b) Spectral-dependence of $\beta_2$ of the waveguide at different widths. (c) Dependence of $\Delta \kappa$ and the corresponding $\Lambda_T$ on the signal wavelength $\lambda_s$ at the average width, and a pump wavelength at 1064 nm with an input power 1 W. (d) Dependence of $\langle \hat{N} \rangle$ on the signal and idler wavelengths for a Gaussian-pulse centred at 1064 nm with $E_i = 1$ nJ and $\tau = 0.29$ ps.

B. Width-modulated silicon-nitride waveguides.

A current key challenge in developing integrated single photons sources is isolating the pump photons from the converted single photons that are only within few nanometers spectral range $[28]$. The PTW-technique would provide a direct solution for this problem by enabling photon-pairs generation at on-demand frequencies spectrally far from the pump. For this purpose, planar silicon-nitride waveguides, characterised by CMOS-compatibility, negligible two-photon absorption and Raman nonlinearity $[20]$, have been also examined in this work. A sketch of the cross-sectional view of the planar waveguide used in the simulations is shown in Fig. 4(a). In this case, the core thickness $h$ is fixed, while its width varies sinusoidally as $w(z) = w_{av} \left[1 - \Delta w \cos \left(2\pi z/\Lambda_T \right)\right]$ with $w_{av}$ the average width and $\Delta w$ the modulation. The fundamental TE mode of a wave at 1.064 $\mu$m in a waveguide with a core width $w = 1650$ nm and a thickness $h = 450$ nm is shown in Fig. 4(a). Simulations are performed using ‘COMSOL’, a commercial finite-element method, including the material dispersion of silica and silicon nitride. The dispersion of the waveguide is normal over the entire spectrum, as depicted in Fig. 4(b). The guiding loss is less than 0.1 dB/m for $\lambda < 1.4$ $\mu$m, then it increases rapidly to 3.8 dB/m at 1.6 $\mu$m.

Assuming a pump source with a central wavelength 1.064 $\mu$m and an input pump power 1 W, Fig. 4(c) shows the spectral dependence of the propagation-constant mismatching $\Delta \kappa$ of SFWM processes and the corresponding tapering periods for the average waveguide width. $\Lambda_T$ in the range of few-hundreds micrometers is required to generate photon-pairs with one photon in the telecommunication regime using this pump source. The expected number of photons in a PTW with 50 periods under a Gaussian-pulse excitation is depicted in Fig. 4(d). The emission of photon-pairs at 810 and 1550 nm has been enhanced by 25 dB via the proper combination of the tapering period and modulation amplitude. The presented data has been smoothed to amend small irregularities, which might be due to the fact that an extremely-fine mesh grid is required to precisely resolve the modes of this planar configuration especially at longer wavelengths. The spectral purity of the output photons is 0.82 (0.9 with smoothing) using Schmidt decomposition,
which demonstrates the potential of the proposed structure as a pure highly-efficient single-photon source for integrated quantum applications.

IV. CONCLUSIONS

In conclusion, I have developed a rigorous quantum model to study SFWM parametric processes in tapered waveguides for continuous and pulsed pump sources. The model calculates the expected number photons of a certain spectral mode in the Heisenberg picture. The transverse profiles of the interacting modes, material and waveguide dispersions, self and cross-phase modulations have been accurately treated in this analysis. Both dispersion-oscillating fibres and width-modulated silicon-nitride waveguides, examples of PTWs, have been examined for enhancing photon-pairs generation at on-demand frequencies and with high spectral purities. The model results have been verified via retrieving the dynamics of the classical nonlinear modulation instability process in dispersion-oscillating fibres. In comparison to uniform waveguides, a 35 (25) dB enhancement in the expected number of photons can be achieved via a proper choice of the tapering period and modulation amplitude in fibres (planar waveguides) with only 50 periods. The output photons are characterised by having very narrow bandwidths, and high spectral purities. The presented simulations theoretically demonstrate the concept of the PTW-technique as a new efficient quasi-phase-matching scheme for photon-pairs generation in third-order nonlinear materials, which would have a great leverage in the field of quantum optics. Finally, I anticipate that exploring non-periodic tapering patterns could be a fruitful future-research direction for optimising the spectral properties of the output photons.

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