N-point Correlation Functions of the Spin-1 XXZ Model

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Abstract

We extend the recent approach of M. Jimbo, K. Miki, T. Miwa, and A. Nakayashiki to derive an integral formula for the N-point correlation functions of arbitrary local operators of the antiferromagnetic spin-1 XXZ model. For this, we realize the quantum affine symmetry algebra $U_q(su(2)_2)$ of level 2 and its corresponding type I vertex operators in terms of a deformed bosonic field free of a background charge, and a deformed fermionic field. Up to GSO type projections, the Fock space is already irreducible and therefore no BRST projections are involved. This means that no screening charges with their Jackson integrals are required. Consequently, our N-point correlation functions are given in terms of usual classical integrals only, just as those derived by Jimbo et al in the case of the spin-1/2 XXZ model through the Frenkel-Jing bosonization of $U_q(su(2)_1)$.

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1 Introduction

Solving many-body interacting quantum (statistical-mechanical or field-theoretic) systems is still a challenging problem even in low space-time dimensions. Recently, in a remarkable paper, Davies et al. [1] presented a new method for solving the antiferromagnetic spin-1/2 XXZ model directly in the thermodynamic limit. This contrasts with the more conventional Bethe Ansatz approach, in which the theory is formulated on a finite chain before taking the thermodynamic limit in order to make the Bethe equations more tractable. Davis et al.’s approach is appealing for two obvious reasons: firstly, because in addition to being able to diagonalise the transfer matrix, it is also possible to calculate the correlation functions, form factors, eigenstates and S-matrix of the model; and secondly because it highlights the role and makes extensive use of the quantum affine symmetry of the model which is present only in the thermodynamic limit. The role of this symmetry is analogous to that of infinite-dimensional symmetries in conformal field theory.

The Hamiltonian of the spin 1/2 XXZ Heisenberg quantum spin chain is [2, 3, 4, 5]

\[ H_{\text{XXZ}} = -\frac{1}{2} \sum_{i=-\infty}^{\infty} (\sigma^x_i \sigma^x_{i+1} + \sigma^y_i \sigma^y_{i+1} + \Delta \sigma^z_i \sigma^z_{i+1}), \] (1.1)

where \( \Delta = (q + q^{-1})/2 \) is an isotropy parameter. Davis et al. consider this model in the \( \Delta < -1 \) antiferromagnetic regime. The Hamiltonian (1.1) acts formally on the infinite tensor product,

\[ \cdots V \otimes V \otimes V \otimes V \cdots \] (1.2)

where \( V \) is a two dimensional representation of \( U'_q(su(2)_1) \) (in what follows the subscript \( k \) in \( U_q(su(2)_k) \) or \( U'_q(su(2)_k) \) stands for the level of these algebras). The latter is the symmetry algebra of (1.1) and is a subalgebra (by dropping the grading operator) of the full quantum affine algebra \( U_q(su(2)_1) \). The central idea of the approach of Davies et al. is to replace this infinite tensor product by the \( U'_q(su(2)_0) \) module

\[ \mathcal{F}_{\lambda,\mu} = \text{Hom}(V(\mu), V(\lambda)), \] (1.3)

where \( V(\lambda) \) and \( \lambda = \Lambda_0, \Lambda_1 \) are the \( U_q(su(2)_1) \) highest weight modules and highest weights respectively. Following Frenkel and Reshetikhin [3], Davies et al. realised this homomorphism by introducing the following vertex operators (VOs) that intertwine the \( U_q(su(2)_1) \)
modules
\[ \tilde{\Phi}_\lambda^\mu V(z) : V(\lambda) \to V(\mu) \otimes V(z). \] (1.4)

Here \( z \) is a spectral parameter and \( V^j(z) \) is the ‘evaluation representation’ of \( U_q(su(2)_0) \), and is isomorphic to \( V \otimes \mathbb{C}[z, z^{-1}] \). These VOs 1.4 are the type I VOs of ref. [1]. By introducing a variant of these type I VOs, i.e.,
\[ \tilde{\Phi}_\lambda^\mu V(z) : V(\lambda) \otimes V(z) \to V(\mu), \] (1.5)

Jimbo et al. [7] were able to re-formulate the action of local operators of the vertex model (such as the spin local operator \( \sigma^z_i \)) on the infinite product 1.2 in terms of their action on \( F_{\lambda, \mu} \). In this manner, and by bosonizing the VOs, they were able to write down an integral formula for N-point correlation functions of local operators in the 6 vertex model. Evaluating this integral for \( N=1 \), enabled them to triumphantly reproduce Baxter’s result for the staggered polarization of the 6-vertex model [3].

Generalisations of the 6-vertex model, corresponding to spin-\( S \) quantum chains, were introduced by Zamolodchikov and Fateev [9] and Kulish and Reshitikhin [10]. The theory associated with \( S = 1 \) is a 19-vertex model [3, 10]. The Hamiltonian is that of the antiferromagnetic XXZ model [9, 3, 11]
\[ H = J \sum_{i=\infty}^{\infty} \left\{ S_i \cdot S_{i+1} - (S_i \cdot S_{i+1})^2 + \frac{1}{2}(q - q^{-1})^2[S_i^z \cdot S_{i+1}^z - (S_i^z \cdot S_{i+1}^z)^2 + 2(S_i^z)^2] - (q + q^{-1})^2 \left[ (S_i^x \cdot S_{i+1}^x + S_i^y \cdot S_{i+1}^y)S_i^z \cdot S_{i+1}^z + S_i^z \cdot S_{i+1}^z (S_i^x \cdot S_{i+1}^x + S_i^y \cdot S_{i+1}^y) \right] \right\}, \] (1.6)

where \( S^x, S^y \) and \( S^z \) are (3 \times 3) spin-1 matrices, and \( J > 0 \). This Hamiltonian is symmetric under \( U'_q(su(2)_2) \) [12]. The theory is massless for \( q = \exp(i\theta) \), \( 0 \leq \theta \leq \pi \) [13, 14], and for \( q = 1 \) it reduces to the XXX model discussed for example in [3, 15]. We consider it instead in the massive region with \( q \) real and \( -1 < q < 1 \). The general spin-\( S \) Hamiltonian is a complicated polynomial in the \( 2S + 1 \) dimensional spin matrices, and is given in the paper by Sogo [13].

Idzumi et al. [12] have extended the work of Davis et al. to general level \( k \). The associated spin/vertex models are the generalised XXZ models with \( S = k/2 \). The step of bosonizing the currents and vertex operators is however more complicated for \( k > 1 \). Firstly, one conventionally uses a deformation of the Wakimoto bosonization [16, 17, 18, 19].
which requires three bosonic fields unlike the Frenkel-Jing bosonization for the case $k = 1$ \cite{23}, which requires only one. Secondly, and more seriously when calculating correlation functions it is necessary to take a trace over a $U_q(su(2)_k)$ irreducible highest weight module. For $k > 1$, the bosonic Fock space can no longer be equated with any $U_q(su(2)_k)$ irreducible highest weight module, and one must therefore take its non-trivial BRST cohomology structure into account. This BRST structure can be studied by means of screening charges \cite{24}, but the latter are given in terms of Jackson integrals, and a formula for the N-point correlation function derived using this approach would be highly complicated.

In this paper, we realise the $U_q(su(2)_2)$ algebra and its type I vertex operators in terms of one boson and one fermion. The major advantage of this realisation is that now, up to a GSO-like projection (though not exactly the familiar one), the Fock space is irreducible as for $k = 1$. Exploiting this simplicity, we are able to produce an integral formula for the N-point correlation function of local operators of the spin-1 XXZ chain.

The paper is organized as follows. In section 2, we fix the notation and make this paper self contained by reviewing the basic properties of the $U_q(su(2)_k)$ quantum affine algebra. For the purpose of free field realization we write this algebra in an operator product expansion form that we refer to as the $U_q(su(2)_k)$ quantum current algebra. In section 3, we use this quantum current algebra to realize $U_q(su(2)_2)$ and its associated type I vertex operators in terms of one boson and one fermion. Then we define the Fock space built from the modes of these two fields, and describe how, with the aid of a GSO-like projector, it is related to the highest weight representations of $U_q(su(2)_2)$. In section 4, we describe how the local operators of the spin-1 XXZ model act in $\mathcal{F}$, the space of excitation over the ground state. Here we introduce the variants of type I vertex operators. In section 5, we use the realization of section 3 to write an integral formula for the N-point correlation functions of arbitrary local operators. All integrals involved here are the usual classical ones and not Jackson integrals. Finally, section 6 is devoted to our conclusions.
2 The $U_q(su(2)_k)$ Quantum Current Algebra

In this section we review some basic properties of the $U_q(su(2)_k)$ quantum affine algebra at level $k$, which will make the subsequent sections more transparent. This algebra reads in the Chevalley basis as \[25, 26, 27\]

\[
\begin{align*}
t_i t_j &= t_j t_i, \\
t_i e_i t_i^{-1} &= q^{2} e_i, & t_i e_j t_i^{-1} &= q^{-2} e_j, & i \neq j, \\
t_i f_i t_i^{-1} &= q^{-2} f_i, & t_i f_j t_i^{-1} &= q^{2} f_j, & i \neq j, \\
[e_i, f_j] &= \delta_{i,j} \frac{t_i - t_i^{-1}}{q - q^{-1}}, \\
q^d e_i q^{-d} &= q^{\delta_{i,0}} e_i, & q^d f_i q^{-d} &= q^{-\delta_{i,0}} f_i & q^d t_i q^{-d} &= t_i,
\end{align*}
\]

where \{e_i, f_i, t_i, i, j = 0, 1\} are the usual Chevalley generators and $d$ is the grading operator. Its main feature is that it is an associative Hopf algebra with comultiplication

\[
\begin{align*}
\Delta(e_i) &= e_i \otimes 1 + t_i \otimes e_i, \\
\Delta(f_i) &= f_i \otimes t_i^{-1} + 1 \otimes f_i, \\
\Delta(t_i) &= t_i \otimes t_i, \\
\Delta(q^d) &= q^d \otimes q^d,
\end{align*}
\]

and antipode

\[
\begin{align*}
& a(e_i) = -t_i^{-1} e_i, & a(f_i) = -f_i t_i, & a(t_i) = t_i^{-1}, & a(q^d) = q^{-d}, & i = 0, 1. \quad (2.9)
\end{align*}
\]

When $d$ is omitted, the above algebra is referred to as $U'_q(su(2)_k)$. For many practical purposes, it is convenient to use the Drinfeld realization of \[2.7\] \[26\]. This is constructed from the following redefinitions of the Chevalley generators \{e_i, f_i, t_i\}:

\[
\begin{align*}
t_0 &= \gamma^k q^{-\sqrt{2} H_0}, & t_1 &= q^{\sqrt{2} H_0}, \\
e_0 &= E_1^- q^{-\sqrt{2} H_0}, & e_1 &= E_0^+, \\
f_0 &= q^{\sqrt{2} H_0} E_{-1}^+, & f_1 &= E_0^-.
\end{align*}
\]

(2.10)
Using (2.10) and (2.11) one recursively generates the following algebra, which is known as the Drinfeld realization of $U_q(su(2)_k)$:

\[
[H_n, H_m] = \frac{[2n]}{2n} \gamma^{nk} - \gamma^{-nk} \delta_{n+m,0}, \quad n \neq 0,
\]

\[
[q^{\pm \sqrt{2} H_0}, H_m] = 0,
\]

\[
[H_n, E_m^+] = \pm \sqrt{2} \gamma^{n+k/2} [2n] E_{n+m}^+, \quad n \neq 0,
\]

\[
q^{\sqrt{2} H_0} E_{m}^+ q^{-\sqrt{2} H_0} = q^{\pm 2} E_{m}^+,
\]

\[
[E_m^+, E_m^-] = \frac{\gamma^k (n-m)/2}{q - q^{-1}} (\psi_{n+m} - \gamma^{k(m-n)/2} \varphi_{n+m}),
\]

\[
E_{n+1}^+ E_m^+ - q^{\pm 2} E_m^+ E_{n+1}^+ = q^{\pm 2} E_{n+1}^+ E_m^+ - E_{m+1}^+ E_n^+,
\]

\[
q^d E_n^+ q^{-d} = q^n E_n^+, \quad q^d H_n q^{-d} = q^n H_n.
\]

This is an algebra generated by the Drinfeld operators $\{E_n^+ (n \in \mathbb{Z}), H_m (m \in \mathbb{Z} \neq 0), q^{\pm \sqrt{2} H_0}, q^\pm, \gamma^{\pm 1/2}\}$, where $\gamma^{\pm 1/2}$ commute with all the generators, and as usual \([x] = (q^x - q^{-x})/(q - q^{-1})\). $\psi_n$ and $\varphi_n$ are the modes of fields $\psi(z)$ and $\varphi(z)$ defined by

\[
\psi(z) = \sum_{n \geq 0} \psi_n z^{-\frac{n}{2}} q^{\sqrt{2} H_0} \exp\{\sqrt{2} (q - q^{-1}) \sum_{n > 0} H_n z^{-\frac{n}{2}}\},
\]

\[
\varphi(z) = \sum_{n \leq 0} \varphi_n z^{-\frac{n}{2}} q^{-\sqrt{2} H_0} \exp\{-\sqrt{2} (q - q^{-1}) \sum_{n < 0} H_n z^{-\frac{n}{2}}\}.
\]

Using the basic identifications (2.10) one can re-express the comultiplication (2.8) in terms of the Drinfeld generators (2.11) as

\[
\Delta(E_n^+) = E_n^+ \otimes \gamma^{kn} + \gamma^{2kn} q^{\sqrt{2} H_0} \otimes E_n^+ + \sum_{i=0}^{n-1} \gamma^{k(n+3i)/2} \psi_{n-i} \otimes \gamma^{k(n-i)} E_i^+ \mod N_- \otimes N_+^2,
\]

\[
\Delta(E_{-n}^-) = E_{-n}^- \otimes \gamma^{-kn} - \gamma^{2kn} q^{-\sqrt{2} H_0} \otimes E_{-n}^- - \sum_{i=0}^{n-1} \gamma^{k(n-3i)/2} \varphi_{n-i} \otimes \gamma^{k(n-i)} E_i^- \mod N_+ \otimes N_-^2,
\]

\[
\Delta(E_{-m}^-) = E_{-m}^- \otimes \gamma^{-kn} - \gamma^{2kn} q^{\sqrt{2} H_0} \otimes E_{-m}^- - \sum_{i=0}^{m-1} \gamma^{k(n-3i)/2} \varphi_{m-i} \otimes \gamma^{k(n-i)} E_i^- \mod N_+ \otimes N_-^2,
\]

\[
\Delta(H_m) = H_m \otimes \gamma^{kn} + \gamma^{3km} \otimes H_m \mod N_+ \otimes N_+,
\]

\[
\Delta(H_{-m}) = H_{-m} \otimes \gamma^{-3km} + \gamma^{-kn} \otimes H_{-m} \mod N_- \otimes N_-,
\]

\[
\Delta(q^{\pm \sqrt{2} H_0}) = q^{\pm \sqrt{2} H_0} \otimes q^{\pm \sqrt{2} H_0},
\]

\[
\Delta(q^{\pm d}) = q^{\pm d} \otimes q^{\pm d},
\]

(2.13)

where $m > 0$, $n \geq 0$, and $N_\pm$ and $N_\pm^2$ are left $Q(q)[\gamma^{\pm}, \psi_m, \varphi_{-n}; m, n \in \mathbb{Z}_{\geq 0}]$ modules generated by $\{E_m^+, m \in \mathbb{Z}\}$ and $\{E_m^\pm, E_n^\pm; m, n \in \mathbb{Z}\}$ respectively [4, 28]. The main virtue of this comultiplication is that it will make the derivation of the intertwining properties of the
vertex operators explicit. These intertwining relations in turn allow the free field realizations (in terms of either free bosons, fermions, or ghosts) of the vertex operators. However for the purpose of the free field realization, it is convenient to rewrite the algebra \[ \text{2.11} \] in terms of operator product expansions (OPEs), that is, as a quantum current algebra (QCA). The \( U_{q}(su(2)_{k}) \) QCA then reads \[ \text{2.12} \]

\[
\begin{align*}
\psi(z) \cdot \varphi(w) &= \frac{(z-wq^{2+k})(z-wq^{-2-k})}{(z-wq^{2-k})(z-wq^{-2+k})} \varphi(w) \cdot \psi(z), \\
\psi(z) \cdot E^{\pm}(w) &= q^{\pm 2} \frac{(z-wq^{(2+k/2)})}{z-wq^{(2-k/2)}} E^{\pm}(w) \cdot \psi(z), \\
\varphi(z) \cdot E^{\pm}(w) &= q^{\pm 2} \frac{(z-wq^{(2-k/2)})}{z-wq^{(2+k/2)}} E^{\pm}(w) \cdot \varphi(z), \\
E^{+}(z) \cdot E^{-}(w) &\sim \frac{1}{w(q-q^{-1})} \left\{ \frac{\psi(wq^{k/2})}{z-wq^{k}} - \frac{\varphi(wq^{-k/2})}{z-wq^{-k}} \right\}, \quad |z| > |wq^{\pm k}|, \\
E^{\pm}(z) \cdot E^{\pm}(w) &= \frac{(z^{\pm 2} - w)}{z-wq^{\pm k}} E^{\pm}(w) \cdot E^{\pm}(z). 
\end{align*}
\]

Here the quantum currents \( \psi(z) \) and \( \varphi(z) \) are given by \[ \text{2.12} \], whereas \( E^{\pm}(z) \) are the following generating functions in terms of the Drinfeld generators:

\[
E^{\pm}(z) = \sum_{n=-\infty}^{+\infty} E^{\pm}_{n} z^{-n-1}.
\]

The free field realization of the \( U_{q}(su(2)_{k}) \) QCA therefore translates into solving the relations \[ \text{2.14} \] in terms of free fields. For example, the bosonization (i.e., the free field realization in terms of bosons) of \( U_{q}(su(2)_{k}) \) has recently been studied intensively. For \( k = 1 \) this is known as the Frenkel-Jing bosonization and requires only a single deformed boson field \[ \text{23} \]. For general \( k \), there are many bosonizations available in the literature. We shall refer to these as q-deformations of the Wakimoto bosonization \[ \text{16, 17, 18, 19, 20, 21, 22, 30} \]. (See \[ \text{30} \] for a detailed discussion of the equivalence of these different bosonizations.) Very recently, the bosonization of \( U_{q}(su(n)_{k}) \) has been constructed in Ref. \[ \text{31} \], and the difference realizations of \( U_{q}(su(n)) \) and general quantum Lie algebras have been achieved in Refs. \[ \text{32, 33} \] respectively.

## 2.1 Intertwining relations of the vertex operators

The vertex operators relevant to this discussion are the type I intertwiners of Refs \[ \text{1, 12} \]. They are defined as maps between \( U_{q}(su(2)_{k}) \) modules in the following way:

\[
\tilde{\Phi}^{\mu,V_{j}}(z) : V(\lambda) \to V(\mu) \otimes V^{j}(z).
\]
Here $V(\lambda)$ are $U_q(su(2)k)$ left highest weight modules, with $\{\lambda = \lambda_i = (k-i)\Lambda_0 + i\Lambda_1, \ i = 0, \ldots, k\}$ and $\{\Lambda_0, \Lambda_1\}$ denoting the sets of $U_q(su(2)k)$ dominant highest weights and fundamental weights respectively. $V^j(z) (0 \leq j \leq k/2)$ is the spin $j$ ‘evaluation representation’ of $U_q(su(2)0)$. It is isomorphic to $V^j \otimes C[z, z^{-1}]$, where $V^j$ is the $U'_q(su(2)0) 2j + 1$ dimensional representation, with the basis $\{v^j_m, \ -j \leq m \leq j\}$ such that:

$$
e_1v^j_m = [j + m]v^j_{m-1}, \quad f_1v^j_m = [j - m]v^j_{m+1}, \quad t_1v^j_m = q^{-2m}v^j_m, \quad (2.17)$$

$$e_0v^j_m = [j - m]v^j_{m+1}, \quad f_0v^j_m = [j + m]v^j_{m-1}, \quad t_0v^j_m = q^{2m}v^j_m,$$

where it is understood that $v^j_m = 0$ if $|m| > j$. $V^j(z)$ is equipped with the following $U'_q(su(2)0)$ module structure \[12\] :

$$e_1v^j_m \otimes z^n = [j + m]v^j_{m-1} \otimes z^n, \quad e_0v^j_m \otimes z^n = [j - m]v^j_{m+1} \otimes z^{n+1},$$

$$f_1v^j_m \otimes z^n = [j - m]v^j_{m+1} \otimes z^n, \quad f_0v^j_m \otimes z^n = [j + m]v^j_{m-1} \otimes z^{n-1}, \quad (2.18)$$

$$t_1v^j_m \otimes z^n = q^{-2m}v^j_m \otimes z^n, \quad t_0v^j_m \otimes z^n = q^{2m}v^j_m \otimes z^n.$$

In terms of the Drinfeld realization this becomes

$$\gamma^{\pm1/2}v^j_m \otimes z^\ell = v^j_m \otimes z^\ell, \quad q^{\sqrt{2}H_0}v^j_m \otimes z^\ell = q^{-2m}v^j_m \otimes z^\ell,$$

$$E^+_m v^j_m \otimes z^\ell = q^{2n(1-m)}[j + m]v^j_{m-1} \otimes z^{\ell+n}, \quad (2.19)$$

$$E^-_m v^j_m \otimes z^\ell = q^{-2nm}[j - m]v^j_{m+1} \otimes z^{\ell+n},$$

$$H_n v^j_m \otimes z^\ell = \frac{1}{\sqrt{2n}} \{(2nj) - q^{n(j-m+1)}(q^n + q^{-n})[n(j + m)]\}v^j_m \otimes z^{\ell+n}.$$

Let $V^j_z$ be the evaluation representation dual to $V^j_z$ and endowed with the left $U'_q(su(2)0)$ module structure through the action of the antipode \[2.9\], which is an anti-automorphism of this algebra, i.e.,

$$< xu, v > = < u, a(x)v >, \quad x \in U_q(su(2)0), \quad u \in V^j_z, \quad v \in V^j_z. \quad (2.20)$$

Then it can easily be shown that the following two evaluation representations are isomorphic to each other \[12\] :

$$C : V^j_{zq^{-1/2}} \sim V^j_z, \quad (2.21)$$

7
where
\[ C^j_m \otimes (zq^{-2})^n = C^j_m v^j_m \otimes z^n, \]
\[ C^j_m = (-1)^{j+m} q^{-(j+m)(j-m-1)} \left( \frac{2j}{j+m} \right)^{-1}, \quad -j \leq m \leq j. \]  

(2.22)

Here \( \{v^j_m, -j \leq m \leq j\} \) is the basis of \( V^{j*} \), which is dual to \( V^j \), and the notation \( \left[ \frac{x}{y} \right] \) defines the q-analogue of the binomial coefficient as
\[ \left[ \frac{x}{y} \right] = \frac{[x]!}{[y]![x-y]!}, \]
\[ [x]^! = [x][x-1] \ldots [1]. \]  

(2.23)

By definition the vertex operators \( \check{\Phi}^\mu_{\lambda} V^j(z) \) obey the intertwining condition [6, 1, 12]
\[ \check{\Phi}^\mu_{\lambda} V^j(z) \circ x = \Delta(x) \circ \check{\Phi}^\mu_{\lambda} V^j(z) \quad \forall x \in U_q(su(2)_k). \]  

(2.24)

Here \( x \) denotes a generator in the Drinfeld realization, which is appropriate for explicitly constructing the vertex operators in terms of free fields. It is also convenient to define components of these vertex operators through
\[ \check{\Phi}^\mu_{\lambda} V^j(z) = s^\mu_{\lambda} V^j(z) \sum_{m=-j}^j \phi^\mu_m(z) \otimes v^j_m, \]  

(2.25)

where the normalisation function \( s^\mu_{\lambda} V^j(z) \) is given by [34]
\[ s^\mu_{\lambda} V^j(z) = (-zq^{k+2})^{\Delta(\lambda_2j)+\Delta(\lambda)-\Delta(\mu)} \]  

(2.26)

with
\[ \Delta(\lambda_2j) = \frac{j(j+1)}{k+2}. \]  

(2.27)

\( \phi^\mu_m(z) \) are referred to as the bare vertex operators. Using the above relation [2.24], the comultiplication [2.13], and the fact that \( N_+ v^j_{-j} = N_- v^j_j = 0, N_+ v^j_m \in F[z, z^{-1}] v^j_{m+1} \), we arrive at the following commutation relations:
\[ [E^+(w), \phi^j_m(z)] = 0, \]
\[ [H_n, \phi^j_m(z)] = j \sqrt{2} \left\{ q^{(n(k+2)+|n|k/2)[2m]} z^n \right\} \phi^j_m(z), \]  

(2.28)

\[ \phi^j_m(z) = \frac{1}{[m]!} [\ldots [\phi^j_m(z), E^0_{-2}] \ldots, E^0_{-2}]_{q^2(m+1)}. \]  

In [2.28] there are \( (j-m) \) quantum commutators, where the quantum commutator \([A, B]_{q^x}\) is defined by
\[ [A, B]_{q^x} = AB - q^x BA. \]  

(2.29)
3 Realization of $U_q(su(2)_2)$ in Terms of One Boson and One Fermion

When $k = 2$, one needs one boson, deformed in two different ways as $\chi^\pm(z)$, and one deformed fermion $\Psi(z)$ in order to realize the quantum current algebra $2.14$. The quantum currents are given in terms of these fields as

\[
\begin{align*}
\psi(z) & = \exp(i\chi^+(zq) - i\chi^-(zq^{-1})) \\
& = q^{2a_0} \exp \left( 2(q - q^{-1}) \sum_{n>0} a_n z^{-n} \right), \\
\varphi(z) & = \exp(i\chi^+(zq^{-1}) - i\chi^-(zq)) \\
& = q^{-2a_0} \exp \left( -2(q - q^{-1}) \sum_{n<0} a_n z^{-n} \right), \\
E^\pm(z) & = \sqrt{2}\Psi(z) \exp(\pm i\chi^\pm(z)),
\end{align*}
\]

where

\[
\begin{align*}
\chi^\pm(z) & = a - ia_0 \ln z + 2i \sum_{n>0} \frac{q^{n}a_n z^{-n}}{[2n]} + 2i \sum_{n<0} \frac{q^{-n}a_n z^{-n}}{[2n]}, \\
\Psi(z) & = \sum_{r \in \mathbb{Z}+1/2} b_r z^{-r-1/2}.
\end{align*}
\]

The bosonic modes $\{a, a_n\}$ of $\chi^\pm(z)$ and the fermionic modes $b_r$ in the Neuveu Schwartz sector of $\Psi(z)$ satisfy the following Heisenberg algebra and anticommutation relations respectively:

\[
\begin{align*}
[a_n, a_m] & = \frac{|2n|^2}{4n} \delta_{n+m,0}, \\
[a, a_0] & = i, \\
\{b_r, b_s\} & = \frac{|4r|}{2[2r]} \delta_{r+s,0}.
\end{align*}
\]

All the other commutation and anticommutation relations are trivial. $\chi^\pm(z)$ and $\Psi(z)$ have the following simple OPEs:

\[
\begin{align*}
\chi^\pm(z) \cdot \chi^\mp(w) & = -\ln(z-w) + :\chi^\pm(z) \cdot \chi^\mp(w):, \quad |z| > |w|, \\
\chi^\pm(z) \cdot \chi^\pm(w) & = -\ln(z - \sqrt{q^2} w) + :\chi^\pm(z) \cdot \chi^\pm(w):, \quad |z| > |q^2 w|, \\
\Psi(z) \cdot \Psi(w) & = \frac{|2|}{2} \frac{\bar{z} - w}{(z-q^2)(z-q^2^{-1})} + :\Psi(z) \cdot \Psi(w):, \quad |z| > |q^2w|.
\end{align*}
\]

It is clear from the above expressions that $\chi^\pm(z)$ ($\Psi(z)$) are two different deformations of a usual real bosonic field (real fermionic field) and reduce to it in the limit $q \to 1$. The normal ordering symbol $:\cdots:$ introduced in $3.33$ means that the annihilation modes $\{a_{n \geq 0}, b_{r \geq 1/2}\}$
where the basic OPE's needed to derive (3.36) are given by

\[
\xi(z) = a - i a_0 \ln (-q^4) + 2i \sum_{n>0} q^{-3n} a_n z^{-n} + 2i \sum_{n<0} q^{-5n} a_n z^{-n},
\]

(3.34)

with

\[
\xi(z) \xi(w) = - \ln(q^4(wq^2 - z)) + : \xi(z) \xi(w) :, \quad |z| > |q^2 w|,
\]

(3.35)

then the bare vertex operators are realized as follows:

\[
\begin{align*}
\phi_1(z) &= \exp(i \xi(z)), \\
\phi_0(z) &= [\phi_1(z), E_0] q^2 = \sqrt{2} \oint_{2\pi i} : \phi_1(z) \bar{E}^- (\omega) : I_0(\omega, z) \psi(\omega), \\
\phi_{-1}(z) &= \frac{[\phi_0(z), E_0] \slash 2}{\sqrt{2} \oint_{2\pi i}} : \phi_1(z) \bar{E}^- (\omega) \bar{E}^- (\eta) : I_0(\omega, z) I_1(\eta, \omega, z) \psi(\omega) + I_1^-(\eta, \omega, z) \psi(\eta). \psi(\omega). \\
I_0(\omega, z) &= \frac{(q^2-q^{-2})}{zq^2 (1-q^2 z/\omega)(1-q^2 w/\omega)}, \quad |zq^6| < |w| < |q^2 z|, \\
I_1^1(\eta, \omega, z) &= \frac{(-\omega^{-2} q^6)}{2 |q^4 z (1-q^{-2} \eta/\omega)|}, \quad |q^{-2} \eta/|z| < 1, \\
I_1^-(\eta, \omega, z) &= \frac{(-q^2 - z^{-2})}{2 |q^6 \eta (1-q^{-2} \eta/\omega)|}, \quad |q^6 \eta/|\eta| < 1.
\end{align*}
\]

(3.36)

(3.37)

The denominators are left in the form \((1 - a)\) in order that the domain of convergence of the OPE sums \(|a| < 1\) is apparent. The \(w\) and \(\eta\) integration contours must be chosen to lie within the domains indicated. This point will be discussed more in Section 5. Here the basic OPE's needed to derive (3.36) are given by

\[
\begin{align*}
\phi_1(z) \bar{E}^- (\omega) &= : \phi_1(z) \bar{E}^- (\omega) : \frac{1}{zq^4 (1-q^{-2} \omega/z)}, \\
\bar{E}^- (\omega_1) \phi_1(z) &= \bar{E}^- (\omega_1) \phi_1(z) : \frac{1}{\omega (1-q^2 \omega/z)} \\
\bar{E}^- (\omega_1) \bar{E}^- (\omega_2) &= \bar{E}^- (\omega_1) \bar{E}^- (\omega_2) : (\omega_1 - q^2 \omega_2), \\
\end{align*}
\]

(3.38)

where \(\bar{E}^- (\omega)\) indicates the purely bosonic part of \(\bar{E}^- (\omega)\). In order to check our vertex operators [3.31], we have computed two matrix elements involving the vertex operators [3.4]

\[
\Phi_{\lambda_i}^{\lambda_2-i,v_1} (z) = (-q^4)^{i/2} (\phi_1(z) \otimes v_{1}^{\dagger} + \phi_0(z) \otimes v_{0}^{\dagger} + \phi_{-1}(z) \otimes v_{-1}^{\dagger}), \quad i = 0, 2.
\]

(3.39)
After performing the various double integrals, we find
\[
<\lambda_0|\tilde{\Phi}_{\lambda_2}^{V_1} (z) \tilde{\Phi}_{\lambda_0}^{V_1} (w)|\lambda_0> = \frac{1}{1 - q^{w/z}} (q^2 v_{+1}^1 \otimes v_{-1}^1 + v_{-1}^1 \otimes v_{+1}^1 - q^2 [2] v_0^1 \otimes v_0^1),
\]
\[
<\lambda_2|\tilde{\Phi}_{\lambda_2}^{V_1} (z) \tilde{\Phi}_{\lambda_0}^{V_1} (w)|\lambda_2> = \frac{1}{1 - q^{w/z}} (v_{+1}^1 \otimes v_{-1}^1 + q^2 (\frac{w}{z})^2 v_{-1}^1 \otimes v_{+1}^1 - q^2 [2] (\frac{w}{z}) v_0^1 \otimes v_0^1).
\]

The above results coincide exactly with those given in formula 3.27 of Ref. [12] which were obtained by solving the quantum Knizhnik-Zamolodchikov equation.

### 3.1 Fock spaces

From the realization [3.30] and [3.36] it is clear that the representation of the $U_q(su(2)_2)$ are the Fock modules
\[
F(n) = F_- |n>, \quad F_+ |n> = 0,
\]
where $F_\mp$ are free $\mathbb{Q}(q)$ modules generated by $\{a_{\mp m}, b_{\mp r}, m \in \mathbb{Z}_{>0}, r \in \mathbb{Z}_{\geq 0} + 1/2\}$. The states $|n>$, labelled by the integers $n$, are defined by
\[
|n> = \exp(ina)|0>,
\]
where $|0>$ is the ‘in’ vacuum and it is annihilated by $\{a_m, m \geq 0; b_r, r \geq 1/2\}$. The dual Fock spaces are defined through $a_{\mp}^\dagger = a_{\pm}, a_{\mp}^\dagger = a$, and $b_{\mp}^\dagger = b_{\mp}$. $F(n)$ is not irreducible but splits into two irreducible Fock modules $F^0(n)$ and $F^2(n)$, which are isomorphic to the representations $V(\lambda_0)$ and $V(\lambda_2)$ of $U_q(su(2)_2)$ respectively. The latter can be obtained from $F(n)$ through the following GSO (though not exactly the usual GSO) projectors:
\[
P_\pm = (1 \pm \exp(-2\pi i d))/2
\]
as
\[
F^0(n) = P_+ F(n),
F^2(n) = P_- F(n),
\]
and where
\[
d = 4 \sum_{n>0} \frac{n^2}{[2n]^2} a_{-n} a_n + 2 \sum_{r \geq 1/2} \frac{r[2r]}{[4r]} b_{-r} b_r + (a_0)^2 / 2.
\]
Moreover, $F^0(n)$ and $F^2(n)$ are highest weight representations of $U_q(su(2)_2)$, with highest weights $|0>$ and $|1>$ respectively. The construction of the Fock module $F^1(n)$, which
is isomorphic to the representation $V(\lambda_1)$, is slightly more complicated but as it is unnecessary for our purposes we will not discuss it here.

## 4 Correlation Functions of Local Operators

A ‘local operator’ $L_N$ is one that acts naturally on the $N$ tensor product $V \otimes V \otimes \ldots \otimes V$ ($V = V^{k/2}$) corresponding to $N$ adjacent sites of a quantum spin chain. In order to interpret its action instead in $\mathcal{F} = \text{End}(V(\lambda_i))$, which is the space of excitations above the ground state labelled by $\lambda_i$, it is necessary to introduce another set of type I vertex operators denoted by $\Phi^\lambda_{\sigma(\lambda_i),V}(z)$ [33, 12]. These vertex operators intertwine $U_q(su(2)_k)$ modules in the following way:

$$\Phi^\lambda_{\sigma(\lambda_i),V}(z) : V(\sigma(\lambda_i)) \otimes V(z) \to V(\lambda_i).$$

(4.46)

Here $\lambda_i = (k - i)\Lambda_0 + i\Lambda_1$ and $\sigma(\lambda_i) = i\Lambda_0 + (k - i)\Lambda_1$. This VO has $k + 1$ components $\Phi^\lambda_{\sigma(\lambda_i),V,m}(z)$ defined by

$$\Phi^\lambda_{\sigma(\lambda_i),V,m}(z)|u> = \Phi^\lambda_{\sigma(\lambda_i),V}(z)(|u > \otimes v_m) = (id_{V(\lambda_i)} \otimes < v_m,..>)\Phi^\lambda_{\sigma(\lambda_i)}(z)|u >,$$

(4.47)

where $|u > \in V(\sigma(\lambda_i))$ and

$$\Phi^\lambda_{\sigma(\lambda_i)}(z) = \alpha^\lambda_{\sigma(\lambda_i)}(id_{V(\lambda_i)} \otimes C)\Phi^\lambda_{\sigma(\lambda_i)}(zq^{-2}).$$

(4.48)

This implies that

$$\Phi^\lambda_{\sigma(\lambda_i),V,m}(z) = \alpha^\lambda_{\sigma(\lambda_i)}C_{-m}^{k/2}\Phi^\lambda_{\sigma(\lambda_i),m}(zq^{-2}),$$

(4.49)

where $\Phi^\lambda_{\sigma(\lambda_i)}(z) = \sum_{m=-k/2}^{k/2} \Phi^\lambda_{\sigma(\lambda_i),m}(z) \otimes v_m$ and $\alpha^\lambda_{\sigma(\lambda_i)}$ is a normalization constant, which we fix by [12]

$$\Phi^\lambda_{\sigma(\lambda_i),V,k/2-i}(z)|\sigma(\lambda_i)> = \Phi^\lambda_{\sigma(\lambda_i),V}(z)|\sigma(\lambda_i) > \otimes v_{k/2-i}) = |\lambda_i > + \ldots$$

(4.50)

Using in addition the normalization

$$\Phi^\lambda_{\sigma(\lambda_i),V}(z)|\sigma(\lambda_i)> = |\lambda_i > + \ldots$$

(4.51)

we arrive then at

$$\alpha^\lambda_{\sigma(\lambda_i)} = \frac{1}{C_{i-k/2}}.$$  

(4.52)
This in turn leads to

\[ \tilde{\Phi}_{\sigma(\lambda_i),V,m}(z) = \frac{C^{k/2}_{-m}}{C^{k/2}_{k-m}} \Phi_{\sigma(\lambda_i),-m}(zq^{-2}), \quad -k/2 \leq m \leq k/2. \] (4.53)

If we set \( k = 2 \) then \( \sigma(\lambda_i) = \lambda_{2-i} \) and we get explicitly

\[ \tilde{\Phi}_{\lambda_{2-i},V}(z) = f_i(\epsilon)\tilde{\Phi}_{\lambda_{i-\epsilon},V}(zq^{-2}), \] (4.54)

where \( f_i(+1) = q^{i-2}, \quad f_i(0) = -q^{i-2}/[2] \) and \( f_i(-1) = q^i \). Here \( i = 0, 2 \) because only the ground states labelled by \( \lambda_0 \) and \( \lambda_2 \) will be considered in this paper. These ground states correspond to the two antiferromagnetic spin configurations, \((\cdots + 1, -1, +1, -1 \cdots)\) and \((\cdots - 1, +1, -1, +1 \cdots)\) respectively. These are the ground states for which the staggered polarisation for the spin-1 XXZ model would be non-zero [8, 7].

From the above definitions one can show [12] that

\[ \tilde{\Phi}_{\lambda_{2-i},V}(z) \circ \tilde{\Phi}_{\lambda_{i-\epsilon},V}(z) = g_\lambda \epsilon V(\lambda_i), \] (4.55)

where \( g_\lambda \) are scalar functions which can be obtained from the two-point matrix elements [3, 4] and relation [4.54]. We find then

\[ g_\lambda = \frac{1}{1 - q^2}, \quad \epsilon = 0, 2. \] (4.56)

The action of \( L_N \) on \( \mathcal{F} \) can be specified by using the VOs \( \tilde{\Phi}_{\lambda}^\mu(z) \) and \( \tilde{\Phi}_{\lambda}^\mu(z) \) [8, 12], in the following way:

\[ L_N^\lambda = (g_\lambda)^{-N} \]

\[ \tilde{\Phi}_{\lambda_{i+2},V}(z_1)\tilde{\Phi}_{\lambda_{i+4},V}(z_2) \cdots \tilde{\Phi}_{\lambda_{i+2(N-1)},V}(z_N)(id_{V_{\lambda_{i+2N}}}) \times L_N \tilde{\Phi}_{\lambda_{i+2},V}(z_N) \cdots \tilde{\Phi}_{\lambda_{i+4},V}(z_2)\tilde{\Phi}_{\lambda_{i+2},V}(z_1) \]

Here \( i = 0 \) or \( 2 \) labels the choice of ground state (or equivalently the \( U_q(su(2))_2 \) module over which the trace is to be taken), \( z_1, \cdots, z_N \) are local spectral parameters, and the subscripts on the \( \lambda \)'s are understood as modulo 4. Then the correlation function of this operator is given by:

\[ < L > = L_{\epsilon_1,\cdots,\epsilon_N}^{\epsilon'_{1},\cdots,\epsilon'_{N}} P_{\epsilon_1,\cdots,\epsilon_N}^{\epsilon'_{1},\cdots,\epsilon'_{N}}(z_1, z_2, \cdots, z_N|\epsilon), \] (4.58)

where

\[ P_{\epsilon_1,\cdots,\epsilon_N}^{\epsilon'_{1},\cdots,\epsilon'_{N}}(z_1, z_2, \cdots, z_N|\epsilon) = (1 - q^2)^N \times \]

\[ Tr_{V(\lambda)}(q^{-2}\tilde{\Phi}_{\lambda_{i+2},V,\epsilon_1}(z_1) \cdots \tilde{\Phi}_{\lambda_{i+2(N-1)},V,\epsilon_N}(z_N)\tilde{\Phi}_{\lambda_{i+2},V}(z_N) \cdots \tilde{\Phi}_{\lambda_{i+2},V}(z_1))/Tr_{V(\lambda)}(q^{-2}), \] (4.59)
and $\rho = 2d + a_0$. This trace may be rewritten purely in terms of the bare vertex operators of 3.36 as

$$P'_{N\cdots e_1} (z_1, z_2, \ldots, z_N | i) = (1 - q^2)^N \prod_{l=1}^{N} f_{(i+2l) \mod 4} (\epsilon'_l) (-z_l q^{4-(i+2l) \mod 4}) \times$$

$$\left( T^{-e'_1 \cdots e'_N} (z_1 q^{-2}, \ldots, z_N q^{-2}, z_N, \ldots, z_1 | x, y) + \right)$$

$$(-1)^{i/2} T^{-e'_1 \cdots e'_N} (z_1 q^{-2}, \ldots, z_N q^{-2}, z_N, \ldots, z_1 | \bar{x}, y) \right) \times$$

$$\left( Tr (x^{-d} y^{2a_0}) + (-1)^{i/2} Tr (\bar{x}^{-d} y^{2a_0}) \right)^{-1}$$

(4.60)

where $x = q^4$, $\bar{x}^{1/2} = -x^{1/2}$, $y = q^{-1}$ and

$$T^{e_1, \cdots, e_n} (z_1, \ldots, z_n | x, y) = Tr (x^{-d} y^{2a_0} \phi_{e_1} (z_1) \cdots \phi_{e_n} (z_n)).$$

(4.61)

The traces in 4.60 and 4.61 are taken over the complete Fock space of Section 3.1.

5 Evaluation of the Trace

The trace 4.61 splits naturally into a product of contributions from the bosonic non-zero modes, fermionic modes and bosonic zero modes 36. The evaluation of these three terms will be dealt with separately.

5.1 The trace over bosonic non-zero modes

From the expressions 3.36 for $\phi_1 (z), \phi_0 (z)$ and $\phi_{-1} (z)$, it is apparent that the complete contribution of bosonic non-zero modes to 4.61 is

$$Tr (x^{-d_b} \prod_{i=1}^{n} F_{e_i}),$$

(5.62)

where $-d_b = 4 \sum_{n>0} \frac{a^2}{(2\eta)^2} a_n a_n$, and

$$F_{e_i=1} = \hat{\phi}_1 (z_i),$$

$$F_{e_i=0} = : \hat{E}^- (\omega_i) \hat{\phi}_1 (z_i) : ;$$

$$F_{e_i=-1} = : \hat{E}^- (\eta_i) \hat{E}^- (\omega_i) \hat{\phi}_1 (z_i) : ;$$

(5.63)
The hat indicates the bosonic non-zero mode part, and $\omega_i$ and $\eta_i$ are integration variables. To evaluate $\overline{5.62}$ we use the trace reduction technique of Clavelli and Shapiro [30]. The trace of an operator $O$ which is a function of the bosonic non-zero modes $a_n$, is given by

$$Tr(x^{-d_b}O) = \frac{<0|\tilde{O}|0>}{(x; x)},$$  \hspace{1cm} (5.64)$$

where $(s; x) \equiv \prod_{n=0}^{\infty} (1 - sx^n)$, and $\tilde{O}$ is the same operator expressed in terms of oscillators $\tilde{a}_n$ defined by

$$\tilde{a}_n = \frac{a_n}{1 - x^n} + c_n \hspace{0.2cm} (n > 0), \hspace{0.3cm} \tilde{a}_n = a_n + \frac{c_n}{x^n - 1} \hspace{0.2cm} (n < 0).$$  \hspace{1cm} (5.65)$$

Here an extra set of oscillators $c_n$ have been introduced which have the same commutation relations as the $a_n$, and commute with them. The technique for calculating $<0|\tilde{O}|0>$ is to completely normal order the operator $\tilde{O}$ with respect to $a_n$ and $c_n$, and then to use Wick’s theorem (for exponentials) in order to express it in terms of contractions between pairs of operators. When carrying out the integrations associated with $\phi_0(z)$ and $\phi_{-1}(z)$, the contour must always be chosen to be consistent with the domain of convergence of the series that occur in the normal ordering calculation. Specifically this means that whenever $1/(1 - a\omega)$ or $(aw; x)$ appears on the right-hand side of an OPE the contour for integration over $\omega$ must be chosen such that $|aw| < 1$.

After normal ordering with respect to $a_n$ and $c_n$, $\tilde{\phi}_1(z)$ and $\tilde{E}^-(\omega)$ become

$$\tilde{\phi}_1(z) = (q^2x; x) \exp \left( -2 \sum_{n<0} \frac{(a_n q^{-5n} z^{-n} - c_n q^{3n} z^n)}{[2n]} \right) \exp \left( 2 \sum_{n>0} \frac{(a_n q^{-3n} z^{-n} - c_n q^{5n} (xz)^n)}{(1 - x^n)[2n]} \right),$$

$$\tilde{E}^-(\omega) = (q^2x; x) \exp \left( -2 \sum_{n<0} \frac{(a_n \omega^{-n} - c_n \omega^n q^{-n})}{[2n]} \right) \exp \left( 2 \sum_{n>0} \frac{(a_n \omega^{-n} - c_n (\omega x)^n q^n)}{(1 - x^n)[2n]} \right).$$  \hspace{1cm} (5.66)$$

Re-normal ordering with respect to $a_n$ and $c_n$ (indicated by $\circ \cdots \circ$), gives

$$: \tilde{E}^-(\omega)\tilde{\phi}_1(z) : = \circ \cdots \circ \frac{(q^2x; x)^2}{g(\omega/z)} ,$$

$$: \tilde{E}^-(\omega_1)\tilde{E}^-(\omega_2)\tilde{\phi}_1(z) : = \circ \cdots \circ \frac{(q^2x; x)^3 h(\omega_1/\omega_2)}{g(\omega_1/z)g(\omega_2/z)},$$  \hspace{1cm} (5.67)$$

where

$$g(\omega/z) = (q^{-2}(\omega/z); x)(q^{6}(z/\omega); x),$$

$$h(\omega_1/\omega_2) = (q^2(\omega_1/\omega_2); x)(q^2(\omega_2/\omega_1); x).$$  \hspace{1cm} (5.68)$$
Normal ordering in pairs gives

\[
\begin{align*}
\circ \tilde{\phi}_1(z_1) \circ \circ \tilde{\phi}_1(z_2) \circ &= \circ \cdots \circ G_1(z_1, z_2), \\
\circ \tilde{\phi}_1(z) \circ \circ \tilde{E}^-(\omega) \circ &= \circ \cdots \circ G_2(z, \omega), \\
\circ \tilde{E}^-(\omega) \circ \circ \tilde{\phi}_1(z) \circ &= \circ \cdots \circ G_3(\omega, z), \\
\circ \tilde{E}^-(\omega_1) \circ \circ \tilde{E}^-(\omega_2) \circ &= \circ \cdots \circ G_4(\omega_1, \omega_2),
\end{align*}
\]

(5.69)

where

\[
\begin{align*}
G_1(z_1, z_2) &= (q^2(z_2/z_1); x)(q^2(z_1/z_2)x; x), \\
G_2(z, \omega) &= \frac{1}{(q^{-2}(\omega/z); x)(q^{\omega}(z/\omega)x;x)}, \\
G_3(\omega, z) &= \frac{1}{(q^{\omega}(z/\omega); x)(q^{-2}(\omega/z)x; x)}, \\
G_4(\omega_1, \omega_2) &= (q^2(\omega_2/\omega_1); x)(q^2(\omega_1/\omega_2)x; x).
\end{align*}
\]

To label the fields we use the indices

\[
\begin{align*}
a, a' &\in A = \{l|\epsilon_l = 1, 0, -1\}, \quad \text{dim} \ A = n, \\
b, b' &\in B = \{l|\epsilon_l = 0, -1\}, \quad \text{dim} \ B = n_B, \\
c, c' &\in C = \{l|\epsilon_l = -1\}, \quad \text{dim} \ C = n_C.
\end{align*}
\]

(5.71)

Then in terms of these functions the non-zero bosonic mode contribution \[4.61\] to the trace \[4.61\] is

\[
\begin{align*}
\text{Tr}(x^{-d_b} \prod_{i=1}^n F_{c_i}) &= \frac{(q^2 x; x)^{n+n_B+n_C}}{(x; x)} \prod_b \frac{1}{g(\omega_b/z_b)} \prod_c \frac{1}{g(\eta_c/\omega_c)} \\
&\quad \prod_{a<a'} G_1(z_a, z_{a'}) \prod_{a<b} G_2(z_a, \omega_b) \prod_{a<c} G_2(z_a, \eta_c) \\
&\quad \prod_{b<a} G_3(\omega_b, z_a) \prod_{b<b'} G_4(\omega_b, \omega_{b'}) \prod_{b<c} G_4(\omega_b, \eta_c) \\
&\quad \prod_{c<a} G_3(\eta_c, z_a) \prod_{c<b} G_4(\eta_c, \omega_b) \prod_{c<c'} G_4(\eta_c, \eta_{c'}). \\
\end{align*}
\]

(5.72)

where \(\prod_b \equiv \prod_{b \in B}\) etc.

**5.2 The fermionic trace**

Introducing the set of integers \(\{l_c\}\) with \(l_c = 1\) or \(-1\) for each \(c \in C\), which shall label both the order of the fermions and the associated \(I_{-1}^c(\eta_c, \omega_c, z_c)\) functions of \(3.30\), the fermionic contribution to \(4.61\) becomes

\[
\text{Tr}(x^{-d_f} \prod_b H_b),
\]

(5.73)
where $-d_f = 2 \sum_{r \geq 1/2} \frac{r [2r]}{[4r]} b_r b_r$, and

$$H_b = \begin{cases} 
\Psi(\omega_b)\Psi(\eta_b), & b \in C, \\
\Psi(\eta_b)\Psi(\omega_b), & b \in C, \\
\Psi(\omega_b), & b \not\in C.
\end{cases}$$

Using Clavelli and Shapiro's formula for fermionic trace reduction [36] we get

$$Tr(x^{-d_f}O) = \langle 0|\tilde{O}|0 \rangle \prod_{r \geq \frac{1}{2}} (1 + x^r).$$

(5.74)

Now $\tilde{O}$ implies $b_r \rightarrow \bar{b}_r$, where,

$$\bar{b}_r = \frac{b_r}{1 + x^r} \quad (r \geq \frac{1}{2}), \quad \bar{b}_r = b_r + \frac{d_r}{1 + x^r} \quad (r \leq -\frac{1}{2}).$$

(5.75)

Here $d_r$ are anticommuting copies of the $b_r$. Normal ordering with respect to $b_r$ and $d_r$ gives

$$\tilde{\Psi}(\omega_1)\tilde{\Psi}(\omega_2) = \circ \cdots \circ + \Delta(\omega_1, \omega_2),$$

(5.76)

where,

$$\Delta(\omega_1, \omega_2) = \frac{(\omega_1\omega_2)^{-\frac{1}{2}}}{2} \sum_{r \geq \frac{1}{2}} (q^{2r} + q^{-2r}) (x^{\omega_1/\omega_2} + (\omega_2/\omega_1)^r) (1 + x^r).$$

(5.77)

This expression can be split into two parts, $\Delta_1(\omega_1, \omega_2)$ and $\Delta_{-1}(\omega_1, \omega_2)$, where,

$$\Delta_l(\omega_1, \omega_2) = \frac{(\omega_1\omega_2)^{-\frac{1}{2}}}{2} \sum_{r \geq \frac{1}{2}} (q^{2l}x^{\omega_1/\omega_2})^r + (q^{2l}\omega_2/\omega_1)^r (1 + x^r), \quad l = \pm 1.$$  

(5.78)

$\Delta_1(\omega_1, \omega_2)$ and $\Delta_{-1}(\omega_1, \omega_2)$ are convergent in the strips $|x| < |q^2\omega_1/\omega_2| < 1$ and $|x| < |q^{-2}\omega_2/\omega_1| < 1$ respectively. For the purpose of integration $\Delta_l(\omega_1, \omega_2)$ may be re-expressed in terms of the usual $\theta$ functions through the identity of Goddard and Waltz [37]:

$$\Delta_l(\omega_1, \omega_2) = \frac{i(\omega_1\omega_2)^{-\frac{1}{4}}}{4} \theta_2(0|\tau)\theta_4(0|\tau)\theta_3(\nu|\tau)/\theta_1(\nu|\tau),$$

(5.79)

where $\nu_l = \ln(q^{2l}\omega_2/\omega_1)/(2\pi i)$ and $\tau = \ln(x)/(2\pi i)$. Writing the theta functions in terms of infinite products [38] gives

$$\Delta_l(\omega_1, \omega_2) = \frac{2(q^{2l}\omega_2/\omega_1)}{2(q^{-l}\omega_1 - q^l\omega_2)} \frac{(x^{\omega_1/\omega_2})(-x^{\omega_1/\omega_2})}{(x(q^{2l}\omega_2/\omega_1); x)(-x^{\omega_1/\omega_2}(q^{2l}\omega_2/\omega_1)^{-1}; x)}. \quad (5.80)$$
Then (5.73) is given by

\[ Tr(x^{-d_f} \prod_b H_b) = (x^{\frac{1}{2}}; x) < 0| \prod_b \tilde{H}_b|0> \]  

(5.81)

where the vacuum-vacuum correlation function is given by Wick’s theorem as a sum over all possible products of pair wise contractions of the fermionic fields. Each of these contractions is given in terms the fermionic “propagator”

\[ < \bar{\Psi}(\omega_1)\bar{\Psi}(\omega_2) > = \Delta_1(\omega_1, \omega_2) + \Delta_{-1}(\omega_1, \omega_2). \]  

(5.82)

5.3 The trace over zero modes

The contribution of the bosonic zero modes to (5.61) is the factor

\[ Tr(x^{-d_0} y^{2a_0} \prod_{i=1}^n F^0_{\epsilon_i}), \]  

(5.83)

where \(-d_0 = a_0^2/2\), and in analogy to equations (5.63) we have

\[ F^0_{\epsilon_i=1} = \phi^0(z_i), \]
\[ F^0_{\epsilon_i=0} = : E_0^-(\omega_i)\phi^0(z_i) :, \]
\[ F^0_{\epsilon_i=-1} = : E_0^-(\eta_i)E_0^-(\omega_i)\phi^0(z_i) :, \]  

(5.84)

with the zero sub/superscript indicating zero modes. We define the functions

\[ \phi^0(z_1)\phi^0(z_2) = : \phi^0(z_1)\phi^0(z_2) : G^0_1(z_1, z_2), \]  

\[ G^0_1(z_1, z_2) = -zq^4; \]
\[ \phi^0(z)E_0^-(\omega) = : \phi^0(z)E_0^-(\omega) : G^0_2(z, \omega), \]  

\[ G^0_2(z, \omega) = (-zq^4)^{-1}; \]
\[ E_0^-(\omega)\phi^0(z) = : E_0^-(\omega)\phi^0(z) : G^0_3(\omega, z), \]  

\[ G^0_3(\omega, z) = \omega^{-1}; \]
\[ E_0^-(\omega_1)E_0^-(\omega_2) = : E_0^-(\omega_1)E_0^-(\omega_2) : G^0_4(\omega_1, \omega_2), \]  

\[ G^0_4(\omega_1, \omega_2) = \omega. \]  

(5.85)

After completely normal ordering all operators in (5.83) and provided that \( \sum_{i=1}^n \epsilon_i = 0 \), without which the trace vanishes, the factor left over is

\[ Tr\left( x^{-d_0} y^{2a_0} \prod_a \exp(a_0 \ln(-z_a q^4)) \prod_b \exp(-a_0 \ln(\omega_b)) \prod_c \exp(-a_0 \ln(\eta_c)) \right). \]  

(5.86)
Thus (5.83) is given by

\[ Tr(x^{-d_0}y^{2a_0} \prod_{i=1}^{n} F_{i}^{0}) = \sum_{m \in \mathbb{Z}} (x^{m^2/2} \left( \frac{y^2 z (-q^4)^n}{\omega \eta} \right)^m) \times \]

\[ \prod_{a < a'} G_{1}^{0}(z_{a}, z_{a'}) \prod_{a < b} G_{2}^{0}(z_{a}, \omega_{b}) \prod_{a < c} G_{2}^{0}(z_{a}, \eta_{c}) \]

\[ \prod_{b < a} G_{3}^{0}(\omega_{b}, z_{a}) \prod_{b < \nu} G_{4}^{0}(\omega_{b}, \omega_{\nu}) \prod_{b < c} G_{4}^{0}(\omega_{b}, \eta_{c}) \]

\[ \prod_{c < a} G_{3}^{0}(\eta_{c}, z_{a}) \prod_{c < b} G_{4}^{0}(\eta_{c}, \omega_{b}) \prod_{c < c'} G_{4}^{0}(\eta_{c}, \eta_{c'}) , \]

where \( \bar{z} = \prod_{a} z_{a} \), \( \bar{\omega} = \prod_{b} \omega_{b} \) and \( \bar{\eta} = \prod_{c} \eta_{c} \).

### 5.4 Collecting terms

Defining \( \tilde{G}_{l} = G_{l} G_{l}^{0} \), \( (l = 1, \ldots, 4) \), the complete expression for (4.6) is given by

\[ T^{x_1, \ldots, x_n}(z_1, \ldots, z_n|x, y) = (\sqrt{2})^{n_B+n_C} (q^2 x; x)^{n+n_B+n_C} \times \]

\[ \prod_{b} \int \frac{d\omega_{b}}{2\pi i} I_{0}(\omega_{b}, z_{b}) \sum_{\{l_c\}} \left\{ \prod_{c} \int \frac{d\eta_{c}}{2\pi i} I_{-1}(\eta_{c}, \omega_{c}, z_{c}) \right\} < 0 \prod_{b} \tilde{H}_{b} |0> \]

\[ \prod_{b} \frac{1}{g(\omega_{b}/z_{b})} \prod_{c} \frac{h(\eta_{c}/\omega_{c})}{g(\eta_{c}/z_{c})} \]

\[ \prod_{a < a'} \tilde{G}_{1}(z_{a}, z_{a'}) \prod_{a < b} \tilde{G}_{2}(z_{a}, \omega_{b}) \prod_{a < c} \tilde{G}_{2}(z_{a}, \eta_{c}) \]

\[ \prod_{b < a} \tilde{G}_{3}(\omega_{b}, z_{a}) \prod_{b < \nu} \tilde{G}_{4}(\omega_{b}, \omega_{\nu}) \prod_{b < c} \tilde{G}_{4}(\omega_{b}, \eta_{c}) \]

\[ \prod_{c < a} \tilde{G}_{3}(\eta_{c}, z_{a}) \prod_{c < b} \tilde{G}_{4}(\eta_{c}, \omega_{b}) \prod_{c < c'} \tilde{G}_{4}(\eta_{c}, \eta_{c'}) \]

\[ \left[ \frac{x^{1/2}}{(x; x)} \right] \sum_{m \in \mathbb{Z}} \left( x^{m^2/2} \left( \frac{y^2 \bar{z} (-q^4)^n}{\bar{\omega} \bar{\eta}} \right)^m \right) , \]

(5.88)

where the first sum is over all sets of \( \{l_{c}\} \) with each \( l_{c} = \pm 1 \). The contour for each integral in this sum is chosen in the way described in section 5.1. Each contributing integral will correspond to one choice of \( \{l_{c}\} \), and a particular fermionic contraction. Using the Jacobi triple product identity 38,

\[ \sum_{m \in \mathbb{Z}} x^{m^2/2} z^{2m} = (x; x)(-x^{1/2} z^2; x)(-x^{1/2} z^{-2}; x), \]

(5.89)

the last factor in (5.88) within \([\cdots]\) can be written as

\[ (x^{1/2}; x)(-x^{1/2} \left( \frac{y^2 \bar{z} (-q^4)^n}{\bar{\omega} \bar{\eta}} \right); x)(-x^{1/2} \left( \frac{y^2 \bar{z} (-q^4)^n}{\bar{\omega} \bar{\eta}} \right)^{-1}; x) . \]

(5.90)
Similarly,

\[
Tr(x^{-d} y^{2a_0}) = \frac{(x^{\frac{1}{2}}; x)}{(x; x)} \sum_{m \in \mathbb{Z}} (x^{m^2/2} y^{2m})
= (x^{\frac{1}{2}}; x)(-x^{\frac{1}{2}} y^2; x)(-x^{\frac{1}{2}} y^{-2}; x).
\]

(5.91)

N-point correlation functions of local operators are then given in terms of 4.58, 4.60 and 5.88.

6 Conclusions

In this paper, we have derived an integral formula for the N-point correlation functions of arbitrary local operators of the antiferromagnetic spin-1 XXZ model in the thermodynamic limit. In doing this, we have derived a one boson one fermion realization of \( U_q(su(2)_2) \). This realization is more convenient than the q-deformation of the Wakimoto realization of this algebra, which is given in terms of three deformed bosons. This is because the latter realization leads to infinitely many Fock spaces that are highly reducible. To project out the irreducible subspaces, which are isomorphic to the highest weight representations of \( U_q(su(2)_2) \), one has to study the cohomology structure of the Fock spaces by means of the screening charges, which act as BRST operators among them. These screening charges are non-local and given as Jackson integrals in terms of the screening currents. This means that had we used the q-deformation of the Wakimoto construction, the N-point correlation functions would have involved, in addition to the usual integrals, both Jackson integrals and infinite summations arising from the BRST projections. These two would be shortcomings are avoided in our one boson one fermion realization of \( U_q(su(2)_2) \) because the Fock spaces are already irreducible (up to simple GSO-like projections as explained in section 3).

It is still a challenging problem to try to explicitly integrate, if not the N-point correlations functions, at least the 1-point correlation functions. Choosing the local operator as \( S^z \) would lead, as for \( k = 1 \) [7], to the staggered polarisation of the spin-1 XXZ model. Other quantities of physical interest, which might be extracted with this technique, are the local energy density, and form factors of the model.
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