Radiation-reaction force and multipolar waveforms for eccentric, spin-aligned binaries in the effective-one-body formalism

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While most binary inspirals are expected to have circularized before they enter the LIGO/Virgo frequency band, a small fraction of those binaries could have non-negligible orbital eccentricity depending on their formation channel. Hence, it is important to accurately model eccentricity effects in waveform models used to detect those binaries, infer their properties, and shed light on their astrophysical environment. We develop a multipolar effective-one-body (EOB) eccentric waveform model for compact binaries whose components have spins aligned or anti-aligned with the orbital angular momentum. The waveform model contains eccentricity effects in the radiation-reaction force and gravitational modes through second post-Newtonian (PN) order, including tail effects, and spin-orbit and spin-spin couplings. We recast the PN-expanded, eccentric radiation-reaction force and modes in factorized form so that the newly derived terms can be directly included in the state-of-the-art, quasi-circular–orbit EOB model currently used in LIGO/Virgo analyses (i.e., the SEOBNRv4HM model).

I. INTRODUCTION

The observation of gravitational waves (GWs) by the LIGO–Virgo detectors [1, 2] have corroborated the existence of binary black holes (BBHs) in our universe. But how and in which astrophysical environments these binaries form is not yet fully understood. However, the masses, spins (magnitude and orientation), and binary eccentricities inferred from GWs provide invaluable clues to determine BBH formation channels [3, 4]. So far, the observed GWs are consistent with binary coalescences of negligible eccentricity, i.e., on quasi-circular orbits [5–8].

In general, binaries are expected to circularize [9, 10] as they approach merger due to the emission of gravitational radiation. But depending on their astrophysical formation channel, a small fraction of binaries could have non-negligible orbital eccentricity, as they enter the frequency bands of current detectors. This can occur in dense stellar environments, such as globular clusters or galactic nuclei, where dynamic capture [11–16] or the Lidov-Kozai mechanism in hierarchical triples [17–19] can lead to eccentric binary inspirals at close separations.

In particular, Ref. [13] (and Ref. [14]) showed that \( \sim 5\% \) (or \( \sim 10\% \)) of all mergers in globular clusters enter the LIGO band with eccentricity \( e > 0.1 \). Binaries formed via dynamic capture in galactic nuclei are expected to have high eccentricities [16], with 92% having \( e > 0.1 \) and 50%–85% having \( e > 0.8 \) at 10 Hz. For a BBH around a supermassive BH, the Lidov-Kozai mechanism can secularly drive the BH to eccentricities near unity for some orientations [19]. Hence, inferring those eccentricities from GWs is important for understanding the origin and environment of BBHs. Interestingly, Ref. [8] pointed out that GW190521 [20] could be consistent with either an eccentric nonprecessing or a quasi-circular precessing binary, which illustrates both the difficulties and prospects of further observations in the upcoming and future LIGO, Virgo and KAGRA runs [21].

While the expected fraction of eccentric GW observations with current detectors is small, neglecting eccentricity for the parameter inference can cause significant bias [22]. This becomes more relevant for LISA where a large fraction of stellar-mass binaries is expected to be eccentric [23–27]. Hence, it is important to develop accurate waveform models for eccentric binaries to detect them, infer their properties, and shed light on their astrophysical environment and formation channels. Several studies developed post-Newtonian (PN) waveform models for eccentric orbits, such as Refs. [28–38], or hybrid models that use PN results for the inspiral and quasi-circular numerical-relativity (NR) simulations near merger [39–41]. Recently, NR simulations for eccentric binaries were reported in Refs. [41–43], and the first NR surrogate model for eccentric BBHs has been developed in Ref. [44].

The effective-one-body (EOB) formalism [45–47] improves inspiral-merger-ringdown waveforms by combining information from PN theory, NR simulations, and the strong-field test-body limit. EOB Hamiltonians have been constructed to include spin [48–56], tidal effects [57–60], information from the small mass-ratio [61–65] and the post-Minkowskian approximations [66–68], and have been refined and calibrated to NR simulations [69–77]. While the EOB Hamiltonian is valid for generic orbits, most EOB waveform models use quasi-circular orbit results for the radiation-reaction (RR) force, gravitational waveform modes, and the calibration with NR simulations.

Recent approaches to extend the EOB formalism to...
eccentric orbits include Ref. [78], which derived the RR force with eccentricity up to 2PN order, but without tail effects and for nonspinning BHs. More recently, Ref. [79] incorporated eccentricity effects in the RR force and in the (2, 2) waveform mode through 1.5PN order, including tail effects, using the Keplerian parametrization and phase variables that evolve only due to RR. References [80, 81] extended the quasi-circular SEOBNRv1 [72] model to eccentric orbits, while Ref. [82] added eccentric corrections in the SEOBNRv4 [74, 83] waveform model, notably in the (2, 2), (2, 1), (3, 3), (4, 4) modes through 2PN order, including spin-orbit (SO) and spin-spin (SS) couplings \(^1\), but not tail effects. They employed these eccentric modes to construct a RR force for eccentric orbits, which however does not include the Schott terms. As argued in Ref. [78] and Sec. II below, these Schott terms are necessary for generic orbits to satisfy the flux-balance equations. Furthermore, Refs. [84, 85] incorporated noncircular effects in the TEOBResumS_SM [76, 86] model at leading PN order in the azimuthal component of the RR force, and used a quasi-circular 2PN-expanded radial RR force without spin or tail effects. They included eccentric corrections at leading PN order to all modes \(m \neq 0\) up to \(\ell = |m| = 5\).

In this paper, we develop a multipolar EOB waveform model for eccentric binaries with the compact-objects’ spins aligned or antialigned (henceforth, for short aligned) with the orbital angular momentum. We derive the eccentric PN expressions for the RR force (including the Schott terms) and the gravitational modes up to \(\ell = |m| = 6\), including the \(m = 0\) mode, through 2PN order, including tail effects, and SO and SS couplings. We recast our results for the RR force and modes in a form that can be directly incorporated in the state-of-the-art, quasi-circular–orbbit EOB model currently used in LIGO/Virgo analyses (SEOBNRv4HM [74, 83]).

The paper is structured as follows. In Sec. II, we derive the RR force from the energy and angular momentum fluxes using the balance relations. We use the gauge freedom in the RR force to impose that it reduces to the relation used in SEOBNRv4HM in the quasi-circular–orbit limit. In Sec. III, we obtain initial conditions for eccentric orbits. In Sec. IV, we calculate all the gravitational waveform modes that contribute up to 2PN order relative to the leading order (LO) of the (2, 2) mode, i.e., up to the \(\ell = |m| = 6\) mode. These higher-order modes are even more important for eccentric orbits than for quasi-circular ones [87]. We conclude in Sec. V with a discussion of results and potential future work. Finally, Appendix A provides the coordinate transformation from harmonic to EOB coordinates, Appendix B includes a derivation of the LO spin-squared contribution to the angular momentum flux, Appendix C lists the spin contributions to the waveform modes in harmonic coordinates, Appendix D provides some relations for dynamic quantities in the Keplerian parametrization, and Appendix E includes the transformation to tortoise coordinates. We provide our results for the RR force and waveform modes as Mathematica files in the Supplemental Material [88].

\section*{Notation}

We use the metric signature \((-++,++,+\)) and use units in which \(c = G = 1\), but write \(c\) explicitly in PN expansions.

We consider an aligned-spin binary with masses \(m_1\) and \(m_2\), with \(m_1 \geq m_2\), and we define the following constants:

\[
M = m_1 + m_2, \quad \mu = \frac{m_1 m_2}{M}, \quad \nu = \frac{\mu}{M}, \quad \delta = \frac{m_1 - m_2}{M}, \quad X_1 = \frac{m_1}{M}, \quad X_2 = \frac{m_2}{M}. \tag{1}
\]

In the binary’s center of mass, we introduce the canonical phase-space variables \((R, \phi, J, P_{\phi})\), where \(R\) is the separation, \(\phi\) the azimuthal angle, \(P_R\) the radial momentum, and \(P_{\phi}\) the angular momentum. The total relative momentum \(P\) is given by \(P^2 = P_R^2 + P_{\phi}^2 / R^2\). We use the rescaled dimensionless variables

\[
r = \frac{R}{M}, \quad t = \frac{T}{M}, \quad \rho_r = \frac{P_R}{\mu}, \quad \rho_{\phi} = \frac{P_{\phi}}{\mu}, \quad \hat{H} = \frac{H}{\mu}, \quad \hat{S}_i = \frac{S_i}{M \mu}, \quad \chi_i = \frac{S_i}{m_i^2}, \tag{2}
\]

where the dimensionless quantities are denoted with either a hat or a lowercase letter.

The energy and angular momentum fluxes far away from the binary are denoted by \(\Phi_E\) and \(\Phi_J\) respectively, and scale as follows:

\[
\Phi_E = Gc^5 \tilde{\Phi}_E, \quad \Phi_J = c^5 \tilde{\Phi}_J / M, \tag{3}
\]

where quantities with a tilde are the physical dimensionful fluxes. The components of the RR force are denoted by \(F_r\) and \(F_{\phi}\), and are scaled similarly to \(\Phi_E\) and \(\Phi_J\), respectively.

\section{Radiation Reaction Force}

The RR force accounts for the energy and angular momentum losses by the system, and is added to the right-hand side of the Hamilton equations of motion (EOMs) such that

\[
\dot{r} = \frac{\partial \hat{H}}{\partial \rho_r}, \quad \dot{\rho}_r = \frac{\partial \hat{H}}{\partial r} + F_r, \quad \phi = \frac{\partial \hat{H}}{\partial \rho_{\phi}}, \quad \dot{\rho}_{\phi} = \frac{\partial \hat{H}}{\partial \phi} + F_{\phi}. \tag{4}
\]

\footnote{Our results for those modes are mostly in agreement with Ref. [82] except for the SO part, where we disagree with their findings (their expressions contain two extra SO terms).}
where the leading order of $F_{r,\phi}$ is of order $1/c^5$ (2.5 PN). From the EOMs, with $\partial H / \partial \phi = 0$, the time derivatives of energy and angular momentum are given by

$$
\dot{E}_{\text{system}} = \frac{dH}{dt} = \dot{r} F_r + \dot{\phi} F_\phi,
$$

$$
\dot{J}_{\text{system}} = \frac{dp_\phi}{dt} = F_\phi.
$$

The energy and angular momentum lost by the system are not equal to the energy and angular momentum fluxes, $\Phi_E$ and $\Phi_J$, because of additional contributions to $E$ and $J$ due to interactions with the radiation field. The balance equations are modified by Schott terms, as in electrodynamics, that appear as total time derivatives in the balance equations [78]

$$
\dot{E}_{\text{system}} + \dot{E}_{\text{Schott}} + \Phi_E = 0,
$$

$$
\dot{J}_{\text{system}} + \dot{J}_{\text{Schott}} + \Phi_J = 0.
$$

(6)

Substituting the expressions for the energy and angular momentum losses, we obtain

$$
\dot{r} F_r + \dot{\phi} F_\phi + \dot{E}_{\text{Schott}} + \Phi_E = 0,
$$

$$
\dot{F}_\phi + \dot{J}_{\text{Schott}} + \Phi_J = 0.
$$

(7)

The energy and angular momentum fluxes are gauge-independent, but the RR force and Schott terms are gauge-dependent. This coordinate gauge freedom in the RR force was discussed by Iyer and Will in Refs. [89, 90], and by Gopakumar et al. in Ref. [91]. Bini and Damour showed in Ref. [78] how the gauge freedom in $F$ is related to the freedom in defining the Schott terms.

Note that while we only consider aligned spins in this paper, an extension to precessing spins is straightforward: the RR force $F$ is added to the EOM for the total momentum $p$ and a RR contribution is added to the spin evolution equations, such that

$$
\frac{dr}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial r} + F,
$$

$$
\frac{dS_i}{dt} = \frac{\partial H}{\partial S_i} \times S_i + S_i^{RR}.
$$

(8)

The balance equations are then given by

$$
\dot{E}_{\text{system}} + \dot{E}_{\text{Schott}} + \Phi_E = 0,
$$

$$
\dot{J}_{\text{system}} + \dot{J}_{\text{Schott}} + \Phi_J = 0,
$$

(9)

with

$$
\dot{E}_{\text{system}} = \dot{r} \cdot F_r,
$$

$$
\dot{J}_{\text{system}} = r \times F + S_1^{RR} + S_2^{RR}.
$$

(10)

See, e.g., Refs. [92, 93] for more details.

A. Summary of the approach used in this paper for the RR force

The aim of this paper is to extend the quasi-circular RR force and gravitational modes employed in the SEOBNRv4HM waveform model to eccentric orbits. The Hamilton equations that describe the dynamics of the SEOBNRv4HM model use the following relations between the RR force and the energy flux for quasi-circular orbits, which are based on results from Refs. [46, 94],

$$
F_{\phi}^{qc} = -\frac{\Phi_{\phi}^{qc}}{\Omega},
$$

$$
F_{r}^{qc} = \frac{\Phi_{r}^{qc}}{\Omega} p_r - \frac{\Phi_{\phi}^{qc} p_\phi}{\Omega p_\phi},
$$

(11)

with $\Omega$ being the (angular) orbital frequency. However, these two relations are only valid for quasi-circular orbits and are not consistent for generic orbits, since they use the circular-orbit relation $\Phi_E^{qc} = \Omega \Phi_J^{qc}$ and do not include the Schott terms.

Hence, the approach we use to obtain the RR force is to write a generic ansatz with unknown coefficients for the Schott terms, and calculate the RR force from the fluxes using the balance equations

$$
\dot{F}_\phi = -\Phi_J - \dot{J}_{\text{Schott}},
$$

$$
\dot{F}_r = -\Phi_E + \dot{\phi} F_\phi - \dot{E}_{\text{Schott}} + \dot{\phi} J_{\text{Schott}}.
$$

(12)

Then, we specify the free unknown coefficients in the Schott terms such that the force reduces to the conditions in Eq. (11) in the limit of quasi-circular orbits, i.e.,

$$
\dot{F}_\phi = -\Phi_J + O(p_r) + O(p_r^2),
$$

$$
\dot{F}_r p_\phi = 1 + O(p_r^2),
$$

(13)

since both $p_r$ and $p_r$ are zero for circular orbits. Finally, we factorize the RR force into the quasi-circular part used in SEOBNRv4HM times eccentric corrections

$$
F_r = F_r^{qc} F_r^{ecc}, \quad F_\phi = F_\phi^{qc} F_\phi^{ecc},
$$

(14)

where the quasi-circular parts are given by Eq. (11), and the eccentric corrections scale as $F^{qc} \sim 1 + \dot{p}_r + p_r^2 + \ldots$. In the following subsections, we provide the details of these steps.

B. EOB Hamiltonian and angular momentum

The EOB Hamiltonian is calculated from an effective Hamiltonian $H_{\text{eff}}$ via the energy map

$$
H_{\text{EOB}} = M \sqrt{1 + 2\nu \left( \frac{H_{\text{eff}}}{\mu} - 1 \right)},
$$

(15)

with $H_{\text{eff}}$ given in Refs. [50, 51, 56]. When calculating the RR force to 2PN, we only need to work with the PN
expansion of the EOB Hamiltonian. The nonspinning part to 2PN order is given by

\[
\hat{H}_{\text{EOB}}^0 = \frac{c^2}{\nu} + \frac{p^2}{2r} - \frac{1}{r} + \frac{1}{c^2} \left[ \frac{(\nu - 1)p^2}{2r} - \frac{1 + \nu}{8} \rho^2 \right] - \frac{1 + \nu}{2r^2} + \frac{1 + \nu + \nu^2}{16} p^6 + \frac{(1 + 2\nu)p^2}{r^2} - \frac{(1 + \nu - 3\nu^2)p^2}{4r^2} + \frac{(1 + \nu)p^2\rho^2}{2r} - \frac{1 - \nu + \nu^2}{2r^3} \right],
\]

(16)

the LO (1.5PN) spin-orbit part

\[
\hat{H}_{\text{EOB}}^{\text{SO}} = \frac{p_\phi}{2c^4r^3} \left[ \chi_1 (2 + 2\delta - \nu) + \chi_2 (2 - 2\delta - \nu) \right],
\]

(17)

and the LO (2PN) spin-spin part

\[
\hat{H}_{\text{EOB}}^{\text{SS}} = \frac{1}{2c^4r^3} \left\{ \chi_1^2 \left[ X_1 \left( 1 - \frac{p_\phi^2}{r} + \frac{\rho p_\rho}{r} \right) - C_{1ES}^2X_1^2 \right] + \chi_2^2 \left[ X_2 \left( 1 - \frac{p_\phi^2}{r} + \frac{\rho p_\rho}{r} \right) - C_{2ES}^2X_2^2 \right] + 2\chi_1\chi_2 \left[ (\nu - 1)\nu - \frac{\nu^2 p_\phi^2}{r} + \nu^2\rho p_\rho^2 \right] \right\}. \quad (18)
\]

where \(C_{1ES}^2\) are the spin quadrupole constants, which equal one for BHs.

The orbital frequency expanded to 2PN is given by

\[
\Omega \equiv \dot{\phi} = \frac{\partial \hat{H}_{\text{EOB}}}{\partial \dot{p}_\phi} = \frac{p_\phi}{r^2} + \frac{p_\phi}{c^2} \left[ \frac{\nu - 1}{r^3} - \frac{(\nu + 1)p^2}{2r^2} \right] + \frac{1}{2c^2r^3} \left[ \chi_1 (2 + 2\delta - \nu) + \chi_2 (2 - 2\delta - \nu) \right] + \frac{3\nu^2 + \nu + 1}{c^4} \left[ \frac{p^2}{8r^2} + \frac{(\nu + 1)p^2}{r^3} - \frac{3\nu^2 + \nu + 1}{2r^4} \right] - \frac{p_\phi}{c^4r^3} \left[ 2\nu^2\chi_1\chi_2 + \chi_1^2 X_1^2 + \chi_2^2 X_2^2 \right].
\]

(19)

From the EOM \(\ddot{p}_r = -\nabla \Phi/\partial r\), we can obtain an expression for \(p_\phi^2/\nu\), we can obtain an expression for \(p_\phi^2/\nu\),

\[
\frac{p_\phi^2}{r} = 1 + r^2 \ddot{p}_r + \frac{1}{2c^2r} \left[ 6 + (\nu + 1)r^2 \ddot{p}_r^2 - (\nu - 5)r^2 \ddot{p}_r^2 + \frac{r^2\ddot{p}_r^3}{(\nu + 1)r^2 \ddot{p}_r + 4} \right] - \frac{3\sqrt[3]{r^3 \ddot{p}_r} + r}{2c^2r} \left[ \chi_1 (2 + 2\delta - \nu) + \chi_2 (2 - 2\delta - \nu) \right] + \frac{1}{8c^2r^2} \left[ (\nu^2 + 5\nu + 1) r^6 \ddot{p}_r^4 - (\nu^2 - \nu + 1) r^4 \ddot{p}_r^4 \ddot{p}_r^2 \right] + 2(5\nu + 8)r^4 \ddot{p}_r^2 - (\nu^2 + 7\nu - 63) r^2 \ddot{p}_r - 24(\nu - 3) + 2 \left( 8 - 24\nu + 3\nu^2 \rho p_\rho^2 \right) r^2 \ddot{p}_r^2 + 2 (\nu^2 + \nu + 3) p_\rho^2 \ddot{p}_r \nu
\]

\[
\quad + \frac{1}{2c^4r^2} \left\{ 3C_{1ES}^2X_1^2 + X_1^4 \left( 1 + 4\nu^2 \ddot{p}_r - 2r^2 \ddot{p}_r^2 \right) \right\} + \chi_2^2 \left[ 3C_{2ES}^2X_2^2 + X_2^4 \left( 1 + 4\nu^2 \ddot{p}_r - 2r^2 \ddot{p}_r^2 \right) \right] + \nu \chi_1 \chi_2 \left[ 2 (\nu + 4\nu^2 \ddot{p}_r + 3) - 4\nu r^2 \ddot{p}_r^2 \right],
\]

(20)

which we use to express the noncircular part of the RR force and modes in terms of \(p_\rho\) and \(\tilde{p}_r\). It will also be useful below, when taking the circular-orbit limit, to have an expression for \(p_\phi\) as a function of \(r\) for circular orbits. Setting \(\tilde{p}_r = 0 = p_\rho\) in the previous equation yields

\[
\frac{p_\phi^2}{r} = 1 + \frac{3}{c^2r^3 + 3(3 - \nu)}
\]

\[
- \frac{1}{2c^2r^3/2} \left[ \chi_1 (2 + 2\delta - \nu) + \chi_2 (2 - 2\delta - \nu) \right] + \frac{1}{2c^4r^2} \left[ \chi_1^2 \left( 3C_{1ES}^2X_1^2 + X_1^4 \right) + 2\nu \chi_1 \chi_2 (3 + \nu) + \chi_2^2 \left( 3C_{2ES}^2X_2^2 + X_2^4 \right) \right].
\]

(21)

C. Energy and angular momentum fluxes

The energy and angular momentum fluxes for nonspinning binaries were derived to 3PN order in harmonic and Arnowitt-Deser-Misner (ADM) coordinates in Refs. [95-97]. The 2PN instantaneous part of the fluxes for nonspinning bodies is given in EOB coordinates in Appendix A of Ref. [78]. The leading order reads

\[
\Phi_{\text{inst}}^E = \frac{8\nu^2}{15r^4} \left( 12p^2 - 11p_r^2 \right) + O \left( \frac{1}{c^2} \right), \quad \Phi_{\text{inst}}^J = \frac{8\nu^2}{5r^3} \frac{p_\phi}{r^2} \left( 2p^2 - 3p_r^2 + \frac{2}{r^2} \right) + O \left( \frac{1}{c^2} \right). \quad (22)
\]

The hereditary contributions to the fluxes can be expressed as an infinite sum over Bessel functions [79, 96] that can be evaluated numerically, or resummed analytically [31, 98]. Here, we follow the method from Ref. [79] to obtain the LO tail part (1.5 PN) of the orbit-averaged fluxes in an eccentricity expansion and we extend their derivation to \(O(e^0)\), which yields

\[
\langle \Phi_{\text{tail}}^E \rangle = \frac{128\pi a^2}{3c^3} r^{13/2} \left[ 1 + \frac{2335}{192} e^2 + \frac{42955}{768} e^4 \right]
\]

\[
\langle \Phi_{\text{tail}}^J \rangle = \frac{4}{5} j_{13/2} (t - r) \ln (\frac{r}{b}),
\]

2 Calculating the tail contribution to the fluxes is similar to that for the waveform modes (see Sec. IV B) except for using the integrals [99]

\[
\Phi_{\text{tail}}^E = \frac{4}{5} j_{13/2} \int_0^\infty dt^i j_{13} (t - r) \ln (\frac{t}{b}),
\]
\[ \langle \Phi^\text{tail}_{ij} \rangle = \frac{128\pi^2}{5e^3} x^5 \left[ 1 + \frac{209}{32} e^2 + \frac{2415}{128} e^4 + \frac{730751}{18432} e^6 + O(e^8) \right], \]

\[ \Phi^\text{tail}_{E} = \frac{128\pi^2 e_p^2}{5c^3 r^4} \left[ \frac{1}{r^3} + \frac{415p^2_r}{96r^2} + \frac{5p^4_r}{288r} - \frac{73p^6_r}{11520} + O(p^8_r) \right], \]

\[ \Phi^\text{tail}_{J} = \frac{128\pi^2}{5c^3 r^2} \left[ \frac{1}{r^3} + \frac{49p^2_r}{16r^2} - \frac{49p^6_r}{5760} + O(p^8_r) \right]. \]

For the LO (2PN) SS contributions, the LO spin1-spin2 energy and angular momentum fluxes in harmonic coordinates were derived in Refs. [93, 100], while the spin-squared energy flux was derived in Refs. [110, 111], and we obtain in Appendix B the spin-squared angular momentum flux. Transforming from harmonic to EOB coordinates, using the transformations in Appendix A, we get the following SS contributions to the fluxes for aligned-spins:

\[ \Phi^\text{SS}_{E} = \frac{\nu^2}{15c^3 r^6} \left\{ \chi_1^2 \left[ C_{1ES}^2(\delta - 2\nu + 1)(144\nu^2 - 156\nu^2) + 3(96\nu^2 - 47\delta - 96\nu^2 + 190\nu - 47)^2 \right. \right. \]
\[ \left. + \left. (149\delta - 280\nu - 280^2 - 578\nu + 149)^2 \right] \right\} + \nu\chi_1\chi_2 \left[ (10(28\nu - 33)p^2_r - 6(48\nu - 47)^2p^2) + 1 \leftrightarrow 2 \right]. \]

The total 2PN energy and angular momentum fluxes are the sum of all the above contributions, i.e.,

\[ \Phi_E = \Phi^\text{inst}_E + \Phi^\text{tail}_E + \Phi^\text{SS}_E + \Phi^\text{SO}_E, \]
\[ \Phi_J = \Phi^\text{inst}_J + \Phi^\text{tail}_J + \Phi^\text{SS}_J + \Phi^\text{SO}_J. \]

### D. Ansatz for the Schott terms

As an ansatz for the Schott terms \( E_{\text{Schott}} \) and \( J_{\text{Schott}} \), we consider

\[ J^\text{inst}_{\text{Schott}} = \frac{\nu^2 p_\phi^2}{r^2} \left[ \frac{\alpha_1}{c} + \frac{1}{c} \left( \alpha_2 p^2_r + \alpha_3 p^2 + \frac{\alpha_4}{r} \right) \right]. \]
and for the SS part,

\[ E_{\text{SS-Schott}}^{\text{inst}} = \frac{\nu^2 p_r^4}{c^3 r} \left[ \beta_0 \frac{p_r^2}{r} + \beta_7 p^2 + \frac{1}{c^2} \left( \beta_4 p_r^4 + \beta_5 p^2 p_r^2 + \beta_6 \frac{p_r^2}{r} + \beta_8 \right) \right] + \frac{1}{c^4} \left( \alpha_{10} p_r^4 + \alpha_{15} p^2 p_r^2 + \alpha_{17} p_r^2 + \alpha_8 p^4 + \alpha_9 \frac{p_r^2}{r} \right) + \frac{\alpha_{10}}{r^2}, \]

\[ J_{\text{Schott}} = J_{\text{inst-Schott}} + J_{\text{tail-Schott}} + J_{\text{SO-Schott}} + J_{\text{SS-Schott}}, \]

\[ E_{\text{Schott}} = E_{\text{inst-Schott}} + E_{\text{tail-Schott}} + E_{\text{SO-Schott}} + E_{\text{SS-Schott}}. \] 

Note that when taking the time derivative of these Schott terms using the EOMs, the LO nonspinning part contributes to the LO SO and SS parts of the RR force.

### E. Solving for the eccentric-orbits RR force

Using the fluxes and the Schott terms, the RR force can be calculated from the balance equations (12), which fix some of the unknowns in the ansatz for the Schott terms. The remaining unknowns can be determined by requiring that the RR force satisfies the conditions (13) in the circular-orbits limit.

#### 1. Leading order

At leading order, calculating the RR force with the ansatz in Eqs. (31) and expanding in \( p_r \) gives

\[ F_r = \frac{\nu^2}{5p_r r^2} \left[ p^2 \left( 5\alpha_1 - 5\beta_2 + 16 \right) - 5 \frac{\beta_3}{r} \right] + O(p_r^3). \] 

Requiring that the \( 1/p_r \) term is zero, leads to the solution

\[ \beta_3 = 0, \quad \alpha_1 = \frac{1}{5} (5\beta_2 - 16). \] 

Expanding \( F_r p_\phi / (F_\phi p_r) - 1 \) in \( p_r \) yields

\[ F_r p_\phi - 1 = \frac{9}{15} \frac{(5\beta_1 - 8) p^2 - (45\beta_1 + 30\beta_2 + 88)}{15\beta_2 p^2 + 3 (32 - 5\beta_2) / r} + O(p_r^2). \]

Requiring that the first term in that series expansion is zero gives the solution

\[ \beta_1 = \frac{8}{5}, \quad \beta_2 = -\frac{16}{3}, \quad \alpha_1 = -\frac{128}{15}. \] 

With that solution, we obtain the LO RR force

\[ F_\phi^{\text{LO}} = \frac{8\nu^2}{15p_r^2} \beta_0 \left( 10p^2 - 39p_r^2 - \frac{22}{r} \right), \]

\[ F_r^{\text{LO}} = -\frac{16\nu^2}{15p_r^2} \beta_1 \left( -5p^2 + 12p_r^2 + \frac{11}{r} \right). \]

This force satisfies the conditions in Eq. (13) for circular orbits since

\[ \frac{F_\phi^{\text{LO}} p_\phi}{F_r^{\text{LO}} p_r} = 1 + \frac{128}{15p_r^2} \frac{p_\phi^2 - 39p_r^2 - 22}{c^2}. \]
Following the same steps as above, we obtain the following solution for the unknowns at 1PN:

\[
\beta_8 = \frac{1}{15} (371 - 503 + 268 \nu),
\]
\[
\alpha_2 = \frac{1}{35} (35 \beta_5 + 108 \nu - 22),
\]
\[
\beta_6 = \frac{2}{315} (14403 - 18060 + 1743),
\]
\[
\beta_9 = \frac{32}{105} (4 \nu - 1),
\]
\[
\beta_7 = -\frac{2}{35} (4 \nu + 93),
\]
\[
\alpha_4 = -\frac{893}{21} \alpha_3 + \frac{2924}{105} \nu,
\]

with 3 arbitrary coefficients out of 9 coefficients at that order. To simplify the resulting expressions for the RR force, we choose to set all arbitrary coefficients to zero, i.e. \( \alpha_3 = \beta_4 = \beta_5 = 0 \), which yields the following 1PN contribution to the RR force:

\[
F_{\phi}^{1\text{PN}} = \frac{\nu^2 p_{\phi}}{105c^2 r^5} \left[ 18(4 \nu + 93) p_r^2 r^2 - 6(4 \nu + 93) \nu \right] + 15(196 - 9 \nu) p_r^2 r^2 + 198(6 \nu - 13) p_r^2 r^2 - (484 \nu + 3833) p_r^2 r^2 + 1684 \nu + 6213, \]

\[
F_r^{1\text{PN}} = \frac{\nu^2 p_r}{105c^2 r^5} \left[ 198(7 \nu - 5) p_r^2 - 6(4 \nu + 93) \nu \right] + 4(691 \nu + 3958) p_r^2 r^2 - 198(6 \nu - 13) p_r^2 r^2 - (484 \nu + 3833) p_r^2 r^2 + 1684 \nu + 6213. \]

3. LO tail

Solving for the unknowns at the LO tail contribution leads to the solution

\[
\lambda_2 = -\frac{334}{15}, \quad \lambda_3 = \frac{128}{5}, \quad \lambda_4 = -\frac{334}{15} + \lambda_1, \quad \lambda_5 = -\frac{334}{15}, \quad \lambda_6 = \frac{128}{5},
\]

with either \( \lambda_1 \) or \( \lambda_4 \) arbitrary. Choosing \( \lambda_1 = 0 \), the tail contribution to the RR forces becomes

\[
F_{\phi}^{\text{tail}} = \frac{\pi \nu^2}{c^3 r^2} \left( \frac{334 p_r^4}{15} - \frac{718 p_r^2}{15} \right) + \frac{49 p_r^2}{225},
\]
\[
F_r^{\text{tail}} = \frac{\nu^2 p_r p_r}{c^3 r^2} \left( \frac{334 p_r^4}{15} - \frac{p_r^4}{18} + \frac{7034 p_r^2}{45} - \frac{718}{15} \right). \]

4. LO spin-orbit

At LO SO, we obtain the solution

\[
\sigma_5 = \frac{1}{15} (-36 \delta - 32 \nu + 15 \sigma_3 + 36),
\]
\[
\sigma_6 = \frac{1}{15} (-68 \delta - 40 \nu + 15 \sigma_2 + 68),
\]
\[
\sigma_7 = \frac{1}{15} (36 \delta - 32 \nu + 15 \sigma_8 + 36),
\]
\[
\sigma_8 = \frac{1}{15} (68 \delta - 40 \nu + 15 \sigma_9 + 68),
\]
\[
\sigma_9 = \frac{8}{15} (4 \delta + 3 \nu - 4),
\]
\[
\sigma_{10} = \frac{8}{15} (4 \delta - 3 \nu + 4),
\]
\[
\sigma_{11} = \frac{1}{15} (-6 \delta + 44 \nu + 15 \sigma_{10} + 6),
\]
\[
\sigma_{12} = \frac{1}{15} (-6 \delta + 44 \nu + 15 \sigma_9 + 6),
\]
\[
\sigma_{13} = \frac{2}{15} (9 \delta - 11 \nu - 9),
\]
\[
\sigma_{14} = \frac{2}{15} (9 \delta - 11 \nu + 9).
\]

5. 2PN no spin

At 2PN, we obtain

\[
\alpha_6 = \beta_{13} - \frac{463 \nu^2}{63} - \frac{1909 \nu}{315} + \frac{922}{315},
\]
\[
\beta_{15} = -\frac{2 \beta_{18} + 1060 \nu^2}{3} - \frac{12116 \nu}{189} - \frac{347236}{2835},
\]
\[
\beta_{16} = \beta_{17} - \beta_{18} - \frac{5276 \nu^2}{315} - \frac{109609 \nu}{630} + \frac{9175}{378},
\]
\[
\beta_{14} = \alpha_7 - \frac{2 \beta_{18} + 244 \nu^2}{3} - \frac{12466 \nu}{45} - \frac{28204}{2835},
\]
\[
\beta_{19} = -\frac{640 \nu^2}{3} + \frac{416 \nu}{189} + \frac{2608}{35} - \frac{945}{189},
\]
\[
\alpha_8 = \beta_{17} - \beta_{18} - \frac{5018 \nu^2}{315} - \frac{105491 \nu}{630} + \frac{52223}{1890}. \]
\[ \alpha_{10} = \beta_{18} - \frac{92\nu^2}{9} - \frac{309\nu}{5} - \frac{10019}{315}, \]

\[ \alpha_{17} = \frac{8\nu^2}{7} + \frac{1201\nu}{90} + \frac{5711}{126}, \]

with 8 arbitrary coefficients out of 16. Choosing \( \alpha_5 = \alpha_7 = \alpha_{10} = \alpha_{11} = \alpha_{12} = \alpha_{13} = \alpha_{17} = \beta_{18} = 0 \) yields

\[ \mathcal{F}_{\phi}^{2\text{PN}} = \nu^2 p_\phi \left\{ \left( \frac{5276\nu^2}{315} + \frac{109609\nu}{630} - \frac{9175}{378} \right) p^6 + \left( \frac{1519}{315} - \frac{9964\nu^2}{315} - \frac{100847\nu}{630} \right) p^4 r + \left( \frac{3512\nu^2}{315} - \frac{4234\nu}{45} + \frac{3355}{21} \right) \frac{p^2}{r^2} \\
+ \left( \frac{9175}{126} - \frac{5276\nu^2}{105} - \frac{109609\nu}{210} \right) p^4 r + \left( \frac{104926}{945} - \frac{15121\nu^2}{105} - \frac{254732\nu}{315} \right) \frac{p^2}{r^2} p_r + \left( \frac{52}{3} - \frac{152\nu^2}{9} - \frac{310\nu}{9} \right) \frac{p^2}{r^2} \right\}, \]

\[ \mathcal{F}_{r}^{2\text{PN}} = \nu^2 p_r \left\{ \left( \frac{5276\nu^2}{315} + \frac{109609\nu}{630} - \frac{9175}{378} \right) p^6 + \left( \frac{1519}{315} - \frac{9964\nu^2}{315} - \frac{100847\nu}{630} \right) p^4 r + \left( \frac{3512\nu^2}{315} - \frac{4234\nu}{45} + \frac{3355}{21} \right) \frac{p^2}{r^2} \\
+ \left( \frac{9713}{126} - \frac{2129\nu^2}{45} - \frac{35053\nu}{630} \right) p^4 r + \left( \frac{26459}{945} - \frac{1745\nu^2}{63} - \frac{44927\nu}{315} \right) \frac{p^2}{r^2} p_r + \left( \frac{52}{3} - \frac{152\nu^2}{9} - \frac{310\nu}{9} \right) \frac{p^2}{r^2} \right\}. \]

6. LO spin-spin

At LO SS, we obtain the unique solution

\[ \zeta_1 = \frac{1}{30} \left[ \delta(73 - 128\nu) + 128\nu^2 - 274\nu + 73 \right], \]

\[ \zeta_2 = -\frac{42}{5} (\delta - 2\nu + 1), \quad \zeta_3 = \frac{2}{15} \nu(64\nu - 261), \]

\[ \zeta_4 = \frac{1}{30} \left[ \delta(128\nu - 73) + 128\nu^2 - 274\nu + 73 \right], \]

\[ \zeta_5 = \frac{42}{5} (\delta - 2\nu - 1), \]

\[ \zeta_6 = -\frac{74}{15} (-2\nu + \delta + 2\nu^2 - 4\nu + 1), \]

\[ \zeta_7 = \frac{1}{10} \left[ \delta(43 - 80\nu) + 80\nu^2 - 166\nu + 43 \right], \]

\[ \zeta_8 = 0, \quad \zeta_9 = 2 (\delta - 2\nu + 1), \]

\[ \zeta_{10} = -6 (\delta - 2\nu + 1), \quad \zeta_{11} = 0, \]

\[ \zeta_{12} = \frac{8}{15} (15 - 37\nu)\nu, \quad \zeta_{13} = \frac{2}{5} \nu(40\nu - 63), \]

\[ \zeta_{14} = 0, \quad \zeta_{15} = -\frac{74}{15} (\delta(2\nu - 1) + 2\nu^2 - 4\nu + 1), \]

\[ \zeta_{16} = \frac{1}{10} \left[ \delta(80\nu - 43) + 80\nu^2 - 166\nu + 43 \right], \]

\[ \zeta_{17} = 0, \quad \zeta_{18} = -2 (\delta + 2\nu - 1), \]

\[ \zeta_{19} = 6 (\delta + 2\nu - 1), \quad \zeta_{20} = 0. \]

With that solution, we get

\[ \mathcal{F}_{\phi}^{SS} = \frac{\nu^2 p_\phi}{30c^4 r^3} \left\{ \chi_1^2 \left[ 24X^2 C_{1ES} \left( 15p^2 - 90r_p^2 - \frac{49}{r} \right) \right. \right. \]

\[ + \left( 361\delta - 632\nu + 632\nu^2 - 1354\nu + 361 \right) \frac{p^2}{r} \]

\[ + \left( 400\nu\delta - 209\delta + 400\nu^2 + 818\nu - 209 \right) \frac{p^2}{r^2} \]

\[ + \left( 355\delta - 704\nu\delta + 704\nu^2 - 1414\nu + 355 \right) \frac{1}{r} \right\}, \]

\[ \mathcal{F}_{r}^{SS} = \frac{\nu^2 p_r}{30c^4 r^3} \left\{ \chi_1^2 \left[ 24X^2 C_{1ES} \left( 15p^2 - 55r_p^2 - \frac{49}{r} \right) \right. \right. \]

\[ + \left( 301\delta - 512\nu\delta + 512\nu^2 - 1114\nu + 301 \right) \frac{p^2}{r} \]

\[ + \left( 400\nu\delta - 209\delta - 400\nu^2 + 818\nu - 209 \right) \frac{p^2}{r^2} \]

\[ + \left( 355\delta - 704\nu\delta + 704\nu^2 - 1414\nu + 355 \right) \frac{1}{r} \right\}. \]
F. Factorizing the RR force into circular and noncircular parts

The total RR force is the sum of the contributions calculated in the previous section, i.e.,
\[ F_φ = F_φ^{\text{LO}} + F_φ^{\text{PN}} + F_φ^{\text{2PN}} + F_φ^{\text{tail}} + F_φ^{\text{SO}} + F_φ^{\text{SS}}. \]
\[ F_r = F_r^{\text{LO}} + F_r^{\text{PN}} + F_r^{\text{2PN}} + F_r^{\text{tail}} + F_r^{\text{SO}} + F_r^{\text{SS}}. \]  
(55)
We have checked that our gauge-dependent RR force agrees with that in Refs. [78, 92, 93] by using the balance equations. Denoting the RR force from those references by \( (\hat{F}_r, \hat{F}_φ) \) with corresponding Schott terms \( (\dot{E}_{\text{Schott}}, \dot{J}^{\text{Schott}}) \), Eqs. (7) lead to
\[ \dot{r} F_r + \dot{\phi} F_φ + \dot{E}_{\text{Schott}} = \dot{r} \hat{F}_r + \dot{\phi} \hat{F}_φ + \dot{E}_{\text{Schott}}, \]
\[ F_φ = \dot{J}^{\text{Schott}} = \dot{r} \hat{F}_r + \dot{J}^{\text{Schott}}. \]  
(56)
Then, by writing an ansatz for \( (\hat{E}^{\text{Schott}}, \dot{J}^{\text{Schott}}) \) with unknown coefficients, we solved that a solution exists, implying that \( (F_r, F_φ) \) and \( (\hat{F}_r, \hat{F}_φ) \) are related via a coordinate transformation.

To implement our results in the SEOBNRv4HM model, we factorize the RR force into a quasi-circular part times eccentric corrections as in Eqs. (14) and (11), which read
\[ F_φ = F_φ^{\text{QC}} F_φ^{\text{EC}}, \quad F_r = F_r^{\text{EC}} F_r^{\text{QC}}, \]
\[ F_φ^{\text{QC}} = -\frac{\Phi_φ^{\text{QC}}}{\Omega}, \quad F_r^{\text{QC}} = -\frac{\Phi_r^{\text{QC}} p_r}{\Omega p_φ}. \]  
(57)
and for the quasi-circular part we use the unexpanded force used in SEOBNRv4HM, in which the energy flux has the following PN expansion in terms of the orbital velocity \( v_Ω \equiv \Omega^{1/3} \):
\[ \Phi_r^{\text{QC}} = \frac{32}{5} \frac{v_Ω^2}{50} \left[ (980 r + 1247) \frac{v_Ω}{5} + \frac{128 \pi}{5} \frac{v_Ω}{5} \right] \]
\[ + \frac{4 \pi^2}{5} \chi_1 \Omega(12r - 11 \delta + 12r + 11 \delta - 11) + \chi_2(2r + 11 \delta - 11) \]
\[ + \frac{2}{28354 \pi^3} \Omega^4 (32760r^2 + 166872r - 44711) \]
\[ + \frac{2}{5 \pi^4 \pi^3} \Omega^4 \left[ (32C_{1, \Omega^4} + 1) \chi_2^2 X_2^2 + 62 \chi_1 \chi_2 \right] \]
\[ + \frac{2}{5 \pi^4 \pi^3} \Omega^4 \left[ (32C_{2, \Omega^4} + 1) \chi_2^2 X_2^2 \right]. \]  
(58)
This leads to the eccentric part
\[ F_φ^{\text{EC}} = \frac{299}{12} \frac{\phi_φ}{v_Ω} - \frac{10}{5} \phi_φ^2 + \frac{16}{5} \frac{\phi_φ}{v_Ω} \]
\[ + \frac{\phi_φ}{v_Ω} \chi_1 \Omega(12r - 11 \delta + 12r + 11 \delta - 11) + \chi_2(2r + 11 \delta - 11) \]
\[ + \frac{2}{28354 \pi^3} \Omega^4 (32760r^2 + 166872r - 44711) \]
\[ + \frac{2}{5 \pi^4 \pi^3} \Omega^4 \left[ (32C_{1, \Omega^4} + 1) \chi_2^2 X_2^2 + 62 \chi_1 \chi_2 \right] \]
\[ + \frac{2}{5 \pi^4 \pi^3} \Omega^4 \left[ (32C_{2, \Omega^4} + 1) \chi_2^2 X_2^2 \right]. \]  
(59)
The full 2PN expressions are provided in the Supplemental Material [88].

In these eccentric corrections to the RR force, we used \( \hat{p}_r \) instead of \( p_φ \) because it improves the agreement of our model with SEOBNRv4HM in the quasi-circular orbit limit, in which \( p_r = 0 = \hat{p}_r \) leading to \( F^{\text{EC}}_φ = 1 \). However, having \( \hat{p}_r \) on the right-hand side of the EOM for \( p_r \) would complicate solving the system of differential equations (4). Therefore, when evolving the EOMs, we simply replace \( p_r \) in the RR force with the derivative of the Hamiltonian with respect to \( r \) calculated numerically, i.e. \( p_r = -\frac{\partial H}{\partial r} \).

SEOBNR waveform models use \( p_{r, c} \) (the conjugate momentum to the tortoise radial coordinate \( r_\ast \)) instead of \( p_r \) since it improves stability of the EOMs near the EOB event horizon [69, 112]. The two momenta are related by Eq. (E3). In Appendix E, we also obtain Eq. (E8) for the transformation between \( p_r \) and \( p_{r, c} \).

III. INITIAL CONDITIONS

Having determined the RR force that enters the EOMs, we need to specify the initial conditions to be used in evolving the system of equations. In this section, we first review how the initial conditions are implemented in SEOBNRv4HM for quasi-circular orbits [46, 94], and then discuss a simple extension for eccentric orbits.

A. Initial conditions for quasi-circular orbits

Let us recapitulate the initial conditions for quasi-circular/spherical orbits in the SEOBNRv4HM model as derived in Refs. [46, 94]. We start by specifying an initial orbital frequency \( \Omega_0 \), with initial orbital phase \( \phi_0 = 0 \), and solve
\[ \left[ \frac{\partial H}{\partial r} \right]_0 = 0, \quad \left[ \frac{\partial H}{\partial p_φ} \right]_0 = \Omega_0 \]  
(60)
for the initial values of \( r \) and \( p_φ \), while neglecting RR, \( p_r \approx 0 \). The initial condition for \( p_r \) is then obtained by solving
\[ \left[ \frac{\partial H}{\partial r} \right]_0 = 0 \]  
(61)
for \( p_{r, c} \), after calculating \( [r]_0 \) using the result from adiabatic evolution [46]
\[ \left[ \frac{\partial H}{\partial r} \right]_0 = \frac{d p_φ}{d r} \frac{d p_φ}{d r} = \frac{\dot{E}}{d E/d r} \]  
(62)
where \( \dot{E} \) is the circular-orbits energy flux, and the derivative \( d E/d r = d H/d r \) can be determined using the following equations for circular orbits:
\[ d H = \frac{\partial H}{\partial r} d r + \frac{\partial H}{\partial p_r} d p_r + \frac{\partial H}{\partial p_φ} d p_φ, \]  
(63)
\[ p_r = 0, \quad d p_r = 0, \quad d \left( \frac{\partial H}{\partial r} \right) = 0. \]  
(64)
This leads to
\[
\frac{d}{dr} \frac{\partial H}{\partial r} = 0 = \frac{\partial^2 H}{\partial r^2} + \frac{\partial^2 H}{\partial r \partial p_\phi} \frac{dp_\phi}{dr},
\]
(65)
which can be solved for \(dp_\phi/dr\) to obtain
\[
\frac{dp_\phi}{dr} = -\frac{\partial^2 H / \partial r^2}{\partial^2 H / \partial r \partial p_\phi}.
\]
(66)
Plugging that solution into \(\frac{dH}{dr} = \frac{\partial H}{\partial p_\phi} dp_\phi / dr\) yields the result in Eq. (4.14) of Ref. [94], which reads
\[
\frac{dH}{dr} = -\left(\frac{\partial H / \partial p_\phi)(\partial^2 H / \partial r^2)}{\partial^2 H / \partial r \partial p_\phi}\right),
\]
(67)
and hence
\[
\left[\frac{\partial H}{\partial p_r}\right]_0 = [\dot{r}]_0 = \left[-\frac{dE}{dt} \frac{\partial^2 H / \partial r \partial p_\phi}{\partial^2 H / \partial r \partial p_\phi}\right]_0.
\]
(68)
The complete procedure to obtain the initial conditions for the orbital phase-space is now as follows. Given \(\Omega_0\), masses, and spins, we numerically solve the relations in Eq. (60) for the initial values \(r_0\) and \(p_{\phi 0}\), choosing \(\phi_0 = 0\) and assuming \(p_r \approx 0\). Using these values, we numerically solve Eq. (68) for the initial value \(p_\phi\).

**B. Initial conditions for eccentric orbits**

Since eccentricity is a gauge-dependent concept, we do not need to calculate accurate initial conditions for eccentric orbits in a specific gauge. Instead, we can choose a measure for eccentricity that can be adjusted to be as convenient as possible for numerical implementation. The only strict requirement is that for zero eccentricity \(e = 0\) one recovers the quasi-circular case. Hence, we can start with very accurate initial conditions for quasi-circular orbits and perturb them for eccentric orbits.

We choose to specify an initial orbital frequency \(\Omega_0\) and an initial eccentricity \(e_0\) using the Keplerian parametrization \(1/r = u_0(1 + e \cos \chi)\). We also assume that the orbit starts with \(\phi_0 = 0\) at periastron \((\chi = 0)\), where \(p_r = 0\) in absence of RR, which simplifies calculating the initial conditions for \(r\) and \(p_\phi\). An advantage of starting at periastron instead of apastron is that the specified initial frequency is then the maximum orbital frequency (over the first orbit), and can be used to estimate the frequency at which the binary enters a GW detector’s frequency band.

To obtain \(r_0\) and \(p_{\phi 0}\), we solve Eq. (60) with a nonzero \(p_r\), i.e.,
\[
\left[\frac{\partial H}{\partial r}\right]_0 = [\dot{p}_r(p_\phi, e)]_0, \quad \left[\frac{\partial H}{\partial p_\phi}\right]_0 = \Omega_0.
\]
(69)
with \(p_r \approx 0\), and \([\dot{p}_r]_0\) given as a 2PN expansion in terms of \(p_\phi\) and \(e\). For quasi-circular orbits, these equations reduce exactly to Eqs. (60) since \(\dot{p}_r \propto e\).

To obtain the PN expansion for \(p_r\) at periastron, we first invert the Hamiltonian at the turning points \(r_{\pm} = 1/(u_0(1 \pm e))\) with \(p_r = 0\) and solve for the energy and angular momentum as functions of \(e\) and \(u_0\), which are given by Eqs. (D2). Then, we invert \(p_\phi(e, u_0)\) to obtain Eq. (D3) for \(u_0(p_\phi, e)\) and insert it into the PN expansion for \(p_r = -\partial H / \partial r\) at periastron \((r = 1/(u_0(1 + e)))\). This yields
\[
[p_r]_0 = \frac{e_0(c_0 + 1)^2}{p_{\phi 0}^4} + \frac{e_0(c_0 + 1)^2}{2c_3^2 p_{\phi 0}^6} [c_0^2(5 - \nu) - 4e_0 + \nu + 7]
\]
\[
+ \frac{e_0(c_0 + 1)^2}{2c_3^3 p_{\phi 0}^8} [(\nu - 2\delta - 2)\chi_1 + (\nu + 2\delta - 2)\chi_2] + \frac{e_0(c_0 + 1)^2}{8c_3^7 p_{\phi 0}^8} [3\nu^2 - \nu + 95] + \frac{e_0(c_0 + 1)^2}{6c_3^8} (3\nu^2 - 3\nu - 3) X_1^4 + \nu \chi_1 \chi_2 + \frac{e_0(c_0 + 1)^2}{12c_3} (5\nu^2 + 3)
\]
\[
- 2e_0(3\nu + 1) - 3\nu + 3] + 1 \leftrightarrow 2.
\]
(70)

The initial condition can now be obtained in analogy to the quasi-circular case: given \(\Omega_0\), \(e_0\), masses, and spins, we obtain \(r_0\) and \(p_{\phi 0}\) from Eqs. (69) and (70) (assuming \(p_r \approx 0\), \(p_r\) then follows from Eq. (68) as before, and \(\phi_0 = 0\) by convention. We can keep using the circular-orbits energy flux in Eq. (68), instead of replacing \(E\) with \(\Omega_F\) for eccentric orbits, because the difference on the orbital dynamics is negligible since it involves \(p_r\) (which at periastron is numerically much smaller than \(r\) or \(p_\phi\)).

To assess the accuracy of the initial conditions for eccentric orbits, we compare the specified value for the eccentricity in the Keplerian parametrization with the value calculated from the orbital frequency (or the frequency of the (2, 2) mode) at periastron \(\Omega_p\) and apastron \(\Omega_a\), which is given by [113]
\[
e_\Omega = \frac{\sqrt{\Omega_p} - \sqrt{\Omega_a}}{\sqrt{\Omega_p} + \sqrt{\Omega_a}},
\]
(71)
and to calculate its, we follow the steps explained after Eq. (2.8) of Ref. [41]. Since we evaluate \(e_\Omega\) by evolving the binary over one orbit (including RR), it only holds approximately that \(\Omega_p \approx \Omega_a\) in the quasi-circular case. That is, \(e_\Omega\) does not vanish exactly for quasi-circular orbits, in contrast to our \(e_0\). Table I shows the value of the eccentricity \(e_\Omega\) calculated from the orbital frequency compared to the specified eccentricity \(e_0\), and we see good agreement between the two measures of eccentricity.
The modes $h^{\ell m}$ are the expansion of the complex polarization waveform $h = h_+ - i h_\times$ into spin-weighted $s = -2$ spherical harmonics $Y^{\ell m}_s(\Theta, \Phi)$ such that

$$h_+ - i h_\times = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} h^{\ell m} Y^{\ell m}_{-2}(\Theta, \Phi). \quad (72)$$

The modes $h^{\ell m}$ can be calculated directly from the radiative multipole moments via EOB coordinates. For the LO tail part, we extend the results of Refs. [79] to $\mathcal{O}(e^0)$ and to higher modes using the Keplerian parametrization, and then convert those results to an expansion in $p_r$ and $\dot{p}_r$. The spin contributions to the modes were derived to 2PN order for circular orbits in Ref. [116], and here we derive them for eccentric orbits.

The GW spherical harmonic modes $h^{\ell m}$ are related to the symmetrical trace-free (STF) moments $U_L$ and $V_L$ by

$$h^{\ell m} = \frac{1}{\sqrt{2D_L\ell+2}} \left[ U^{\ell m} - \frac{i}{c} V^{\ell m} \right], \quad (73)$$

where $D_L$ is the luminosity distance of the source, and the radiative multipole moments are related to the symmetric trace-free (STF) moments $U_L$ and $V_L$ by

$$U^{\ell m} = \frac{16\pi}{(2\ell + 1)!!} \sqrt{\frac{(\ell + 1)(\ell + 2)}{2(\ell - 1)}} Y_{L}^{\ell m} U_L,$$

$$V^{\ell m} = -\frac{32\pi}{(2\ell + 1)!!} \sqrt{\frac{\ell(\ell + 2)}{2(\ell + 1)(\ell - 1)}} Y_{L}^{\ell m} V_L, \quad (74)$$

where $Y_{L}^{\ell m}$ is the complex conjugate of the STF tensors relating the unit vectors $N_{(L)}$ (which point from the source to the detector) to the spherical harmonics basis $Y^{\ell m}(\Theta, \Phi)$ such that

$$Y^{\ell m}(\Theta, \Phi) = Y_{L}^{\ell m} N_{(L)}(\Theta, \Phi), \quad (75)$$

and we can express the unit vector $N$ in terms of the angles $\Theta$ and $\Phi$ as

$$N = \sin \Theta \cos \Phi \hat{e}_x + \sin \Theta \sin \Phi \hat{e}_y + \cos \Theta \hat{e}_z. \quad (76)$$

For planar binaries, nonspinning or with aligned spins, it was shown in Ref. [119] that the modes can be determined using the mass-type multipole moments for even $\ell + m$, or the current-type multipole moments for odd $\ell + m$, i.e.,

$$h^{\ell m} = \frac{1}{\sqrt{2D_L\ell+2}} Y^{\ell m}, \quad \ell + m \text{ even}$$

$$h^{\ell m} = -\frac{i}{\sqrt{2D_L\ell+3}} V^{\ell m}, \quad \ell + m \text{ odd}. \quad (77)$$

We define $H^{\ell m}$ such that

$$H^{\ell m} = -\frac{8\mu}{c^3 D_L} \sqrt{\frac{\pi}{5}} \ell^{-\ell} \delta^{m2} H^{\ell m}, \quad (80)$$

which makes the LO part of $H^{22} = x$ for circular orbits. Note that different conventions for the phase origin contribute a factor of $(-i)^m$ to the modes [120].

In this paper, we compute the modes to 2PN order beyond the leading order of the $(2,2)$ mode, which means we consider modes up to the $\ell = 6$, $m = 6$ modes. To 2PN order, the instantaneous contributions to the radiative multipole moments coincide with the source multipole moments. Including the hereditary terms that contribute to 2PN, the radiative multipole moments are given by [115, 117, 118]

$$U_{ij} = I_{ij}^{(2)} + \frac{2M}{c^3} \int_0^\infty dt \int_0^\infty dt' I_{ij}^{(4)}(t - t') \ln \left(\frac{\tau}{b_1}\right) + \mathcal{O}\left(\frac{1}{c^3}\right),$$

$$U_{ijk} = I_{ijk}^{(3)} + \frac{2M}{c^3} \int_0^\infty dt \int_0^\infty dt' I_{ijk}^{(5)}(t - t') \ln \left(\frac{\tau}{b_2}\right) + \mathcal{O}\left(\frac{1}{c^3}\right),$$

$$U_L = I_L^{(1)} + \mathcal{O}\left(\frac{1}{c^3}\right),$$

$$V_{ij} = J_{ij}^{(2)} + \frac{2M}{c^3} \int_0^\infty dt \int_0^\infty dt' J_{ij}^{(4)}(t - t') \ln \left(\frac{\tau}{b_3}\right) + \mathcal{O}\left(\frac{1}{c^3}\right),$$

$$V_L = J_L^{(1)} + \mathcal{O}\left(\frac{1}{c^3}\right), \quad (82)$$

where the constants $b_i$ are gauge parameters that will be eliminated via a phase shift as was done in Ref. [117]. The source multipole moments for nonspinning binaries are given in, e.g., Refs. [115, 117], while the spin contributions to the source moments are given in Refs. [116, 121].
A. Instantaneous nonspinning contributions

The instantaneous contributions to the modes for nonspinning binaries in eccentric orbits were derived in Ref. [14] to 2PN, and in Ref. [15] to 3PN. The results of Ref. [15] are in harmonic coordinates and in terms of the variables \((r, \phi, r, \dot{\phi})\). Hence, we can simply transform their results from harmonic to EOB coordinates using the transformations in Appendix A. For the \((2, 2)\) mode we obtain

\[
\hat{H}_{\text{inst}}^{22} = \frac{1}{2} \left( \frac{1}{r} + \frac{p^2}{r^2} - 2p_r^2 \right) + \frac{i}{c^2} \left[ \frac{\mu p_\phi p_r}{r} + \frac{1}{c^2} \left\{ \left( \frac{\nu}{28} - \frac{5}{28} \right) p^4 + \left( \frac{31\nu}{28} - \frac{157}{84} \right) \frac{p^2}{r} + \left( \frac{5}{14} - \frac{\nu}{14} \right) p_r^2 p_r^2 + \left( \frac{13}{3} - \nu \right) \frac{p_r^2}{r} \right\} \right] + \left( \frac{\mu}{2} - 2 \right) \frac{1}{r^2} + \frac{i}{c^2} \frac{\mu p_\phi p_r}{r} \left[ \left( \frac{\nu}{14} - \frac{5}{14} \right) p^2 + \left( \frac{2\nu}{7} - \frac{185}{42} \right) \frac{1}{r} \right] + \left( \frac{1}{c^2} \right) \left\{ \left( -\frac{17\nu^2}{336} - \frac{13\nu}{336} + \frac{5}{48} \right) p^6 \right\} + \left( -\frac{671\nu^2}{1008} - \frac{735\nu}{504} + \frac{481}{1008} \right) \frac{p^4}{r} + \left( \frac{127\nu^2}{54} - \frac{1355\nu}{189} - \frac{3024}{504} \right) \frac{r^2}{p^2} + \left( \frac{17\nu^2}{168} + \frac{13\nu}{168} - \frac{5}{24} \right) p_r^2 p_r^4 + \left( -\frac{67\nu}{126} + \frac{52}{9} - \frac{659}{504} \right) \frac{p^6}{r^2} \right\} \right]
\]

The expressions for the other modes that contribute to 2PN, i.e., up to \(\ell = |m| = 6\), are provided as a Mathematica file in the Supplemental Material [88]. Note that the \((\ell, 0)\) modes are zero for circular orbits but not for eccentric orbits. For example, the LO part of the \((2, 0)\) mode is given by

\[
\hat{H}_{\text{inst}}^{20} = \frac{1}{\sqrt{6}} \left( p^2 - \frac{1}{r} \right) + \mathcal{O} \left( \frac{1}{c^2} \right),
\]

which is zero for circular orbits since \(p^2 = 1/r + \ldots\).

B. Hereditary contributions

The hereditary contributions to the modes can be calculated analytically in an eccentricity expansion, as was done in Ref. [79] for the \((2, 2)\) mode to \(\mathcal{O}(e^2)\), and in Ref. [122] for all modes to 3PN order and to \(\mathcal{O}(e^6)\). The results of Ref. [122] use the quasi-Keplerian parametrization, while here we use the Keplerian parametrization following the method developed in Ref. [79], which is based on results from Refs. [96, 117, 123], to derive the leading order tail effects that contribute to the modes up to 2PN order and to \(\mathcal{O}(e^6)\). (See Ref. [79] for a discussion of the advantages of the Keplerian parametrization over the quasi-Keplerian parametrization.) We finally convert the eccentricity-expanded tail contributions to an expansion in \(p_r\) and \(\dot{p}_r\).

The expressions for the other modes that contribute to 2PN, i.e., up to \(\ell = |m| = 6\), are provided as a Mathematica file in the Supplemental Material [88]. Note that the \((\ell, 0)\) modes are zero for circular orbits but not for eccentric orbits. For example, the LO part of the \((2, 0)\) mode is given by

\[
\hat{H}_{\text{inst}}^{20} = \frac{1}{\sqrt{6}} \left( p^2 - \frac{1}{r} \right) + \mathcal{O} \left( \frac{1}{c^2} \right),
\]

which is zero for circular orbits since \(p^2 = 1/r + \ldots\).

1. Modes with even \(\ell + m\)

The LO mass-type multipole moments are given by [96]

\[
I^L = \mu s_\ell t^\ell n^{(L)},
\]

where \(s_\ell \equiv X_\ell^{(e-1)} + (-1)^\ell X_\ell^{(-1)}\), and the unit vectors \(n^{(L)}\) are related to spherical harmonics via Eq. (76), leading to

\[
I^L = \sum_{m=-\ell}^{\ell} Y^L_{\ell m} \alpha_{\ell m} r^\ell e^{-im\phi},
\]

with the coefficients (for equatorial orbits)

\[
\alpha_{\ell m} \equiv \frac{4\pi \mu s_\ell}{(2\ell + 1)!} Y_{\ell m} (\pi, 0).
\]

Decomposing the phase \(\phi\) into an oscillatory part and a linearly growing part \(\phi = \phi_0 + \omega_0 t + \Delta \phi_\tau\), allows expressing the oscillatory part \(\Delta \phi_\tau\) as a Fourier series expansion. Hence,

\[
I^L = \sum_{m=-\ell}^{\ell} \hat{Y}^L_{\ell m} q_{\ell m} J_{\ell m} e^{-im\psi_\phi},
\]

with the functions \(J_{\ell m}\) defined by

\[
J_{\ell m} = r^\ell e^{-im\phi_0} e^{-im\Delta \phi_\tau} = \sum_{k=-\infty}^{\infty} J_{\ell mk} e^{-ik\psi_\phi},
\]

where \(k = m - \ell\).
where \( \psi_r \) and \( \psi_\phi \) are the radial and azimuthal angle variables associated with the frequencies \( \omega_r = d\psi_r/dt \) and \( \omega_\phi = d\psi_\phi/dt \). The coefficients \( J_{\ell m k} \) are given by

\[
J_{\ell m k} = \frac{1}{2\pi} \int_0^{2\pi} d\psi_r e^{ik\psi_r} J_{\ell m}
\]

\[
= \frac{\omega_r}{2\pi} \int_0^{2\pi} d\chi e^{-im\phi_0} e^{-im\Delta\phi_\psi} e^{ik\psi_r}
\]

\[
= \frac{\omega_r}{2\pi} \int_0^{2\pi} d\chi \int_0^{\pi/2} \frac{d\chi}{1 + \epsilon \cos \chi} (\epsilon - \frac{\epsilon}{\cos \chi}) e^{-i\Delta\phi_\psi} e^{ik\psi_r},
\]

(90)

where, in the last line, we assume \( \phi_0 = 0 \). The function \( \mathcal{P} \) denotes the conservative part of \( \dot{\chi} \)

Thus, the Newtonian mass multipole moments can be expressed as

\[
I^L = \sum_{m=-\ell}^{\ell} \sum_{k=-\infty}^{\infty} Y^L_{\ell m} a_{\ell m} J_{\ell m} e^{-i(k\psi_r + m\psi_\phi)} ,
\]

(91)

where the azimuthal angle is related to the radial angle by \( \psi_\phi = \phi - \Delta\phi_\psi \), with \( \Delta\phi_\psi = \chi - \psi_\phi \) at LO. This allows us to write the LO tail contribution to the mass-type radiative moments as

\[
U^L_{\text{tail}} = \frac{2M}{c^3} \int_0^{\infty} d\tau I^L_{\ell+2}(t - \tau) \ln \left( \frac{T}{b} \right),
\]

\[
= (-1)^{\ell+2} \frac{2M}{c^3} \sum_{m=-\ell}^{\ell} \sum_{k=-\infty}^{\infty} Y^L_{\ell m} a_{\ell m} J_{\ell m} \Omega^L_{\ell m k}
\]

\[
\times e^{-i(k\psi_r + m\psi_\phi)} I(T_{\Omega_{\ell m k}}),
\]

(92)

\[
\mathcal{H}_{\text{tail}}^{22} = \frac{2\pi}{c^3} x^{5/2} \left[ 1 + e^{-i\chi} + e^{i\chi} \right] + e^2 \left[ \frac{5}{8} e^{-2i\chi} + \frac{7}{8} e^{2i\chi} + 4 \right] + e^3 \left[ \frac{121 e^{-i\chi}}{32} + \frac{143 e^{i\chi}}{32} + \frac{3}{12} e^{-3i\chi} + \frac{1}{12} e^{3i\chi} \right]
\]

\[
+ e^4 \left[ \frac{25}{16} e^{-2i\chi} + \frac{203}{96} e^{2i\chi} - \frac{5}{96} e^{4i\chi} + \frac{65}{96} \right] + e^5 \left[ \frac{55 e^{-i\chi}}{8} + 23 e^{i\chi} + \frac{15}{64} e^{-3i\chi} + \frac{281}{1536} e^{3i\chi} + \frac{53 e^{5i\chi}}{7680} \right]
\]

\[
+ e^6 \left[ \frac{175}{64} e^{-2i\chi} + \frac{1869}{512} e^{2i\chi} - \frac{449 e^{4i\chi}}{3840} + \frac{31 e^{6i\chi}}{23040} + \frac{30247}{2304} \right],
\]

(97)

while the for the (2, 0) mode

\[
\mathcal{H}_{\text{tail}}^{20} = \frac{\pi x^{5/2}}{2\sqrt{6} c^3} \left[ e^{-i\chi} + e^{i\chi} + e^2 \left( e^{-2i\chi} + e^{2i\chi} + 2 \right) + e^3 \left( 3e^{-i\chi} + 3e^{i\chi} + \frac{1}{4} e^{-3i\chi} + \frac{1}{4} e^{3i\chi} \right) \right].
\]

(98)

3 One first needs to express the leading order part in terms of the variables \((e, x, \chi)\) instead of \((r_p, p, \rho_o)\) using the relations from Appendix D. For example, for the (2, 2) mode, we obtain

\[
h_{22}^{20} = \frac{-8\mu}{c^4 DL} \sqrt{\frac{\pi}{5}} e^{-2i\phi} x \left[ 1 + \frac{e}{4} \left( e^{-i\chi} + 5e^{i\chi} \right) + \frac{e^2}{2} e^{2i\chi} \right].
\]
\[+
\] + e^4 \left( \frac{29}{12} e^{-2ix} + \frac{29}{12} e^{2ix} + \frac{9}{2} \right) + e^5 \left( \frac{179e^{-ix}}{32} + \frac{179e^{ix}}{32} + \frac{125}{192} e^{-3ix} + \frac{125}{192} e^{3ix} \right) 
\] + e^6 \left( \frac{805}{192} e^{-2ix} + \frac{805}{192} e^{2ix} - \frac{7}{960} e^{-4ix} - \frac{7}{960} e^{4ix} + \frac{121}{16} \right). \] (98)

The (3, 3) mode is given by
\[\hat{H}_{33}^{\text{tail}} = -\frac{9i\pi \delta}{4c^4} \sqrt{\frac{15}{14}} c^3 \left[ 1 + e^{i \left( \frac{47e^{-ix} + 19e^{ix}}{27} \right) + e^2 \left( \frac{61}{54} e^{-2ix} + \frac{91}{54} e^{2ix} + \frac{155}{27} \right) + e^3 \left( \frac{691e^{-ix}}{108} + \frac{841e^{ix}}{108} + \frac{35e^{-3ix}}{108} \right) 
\] + \frac{65e^{3ix}}{108} + e^4 \left( \frac{32}{9} e^{-2ix} + \frac{287}{54} e^{2ix} + \frac{5e^{-4ix}}{144} + \frac{115e^{4ix}}{1728} + \frac{3139}{216} \right) + e^5 \left( \frac{503e^{-ix}}{36} + \frac{613e^{ix}}{36} + \frac{35e^{-3ix}}{36} \right) 
\] + \frac{3095e^{3ix}}{1728} + \frac{457e^{5ix}}{25920} \right] + e^6 \left( \frac{131}{10} e^{-2ix} + \frac{150503e^{3ix}}{13824} + \frac{5}{8} e^{-4ix} + \frac{151}{810} e^{4ix} - \frac{41e^{6ix}}{20736} + \frac{219}{8} \right). \] (99)

and the (3, 1) mode
\[\hat{H}_{31}^{\text{tail}} = \frac{i\pi \delta x^3}{12\sqrt{14}c^4} \left[ 1 + e^{i \left( \frac{47e^{-ix} - 9e^{ix}}{2} \right) + e^2 \left( \frac{27}{2} e^{-2ix} - \frac{5}{4} e^{2ix} - 15 \right) + e^3 \left( \frac{177}{4} e^{-ix} - \frac{9e^{ix}}{2} - \frac{25}{4} e^{-3ix} - \frac{4}{3} e^{3ix} \right) 
\] + e^4 \left( \frac{89}{2} e^{-2ix} - \frac{15}{16} e^{2ix} - \frac{55}{192} e^{-4ix} - \frac{1703}{32} \right) + e^5 \left( \frac{101}{3} e^{-ix} - \frac{2867e^{ix}}{96} - \frac{75}{4} e^{-3ix} - \frac{629}{192} e^{3ix} \right) 
\] - \frac{7}{960} e^{5ix} + e^6 \left( \frac{-142903e^{-2ix}}{1536} - \frac{2965}{256} e^{2ix} - \frac{45}{4} e^{-4ix} - \frac{239}{240} e^{4ix} + \frac{37e^{6ix}}{23040} - \frac{16343}{144} \right). \] (100)

We checked that our results agree with those of Ref. [122] after converting between the quasi-Keplerian and Keplerian parametrization, and performing a phase shift.

To express the modes in terms of \((r, p_r, p_\phi)\) instead of \((x, e)\), we use the following leading order relations:
\[p_\phi = \frac{1}{\sqrt{u_r}}, \quad p_r = e\sqrt{u_r} \sin \chi, \quad \frac{1}{r} = u_p(1 + e \cos \chi), \quad x = u_p(1 - e^2). \] (101)

As explained above, it is advantageous to replace \(\hat{p_\phi}\) with \(\hat{p}_r\) using \(\hat{p}_r = (p_\phi^2 - r)/r^3\) and expand in both \(p_r\) and \(\hat{p}_r\) (since \(p_r\) and \(\hat{p}_r\) are both of order \(e\)) to obtain
\[\hat{H}_{32}^{\text{tail}} = \frac{2\pi}{c^3} \left[ \frac{3p_\phi}{2r^3} + \frac{i p_r}{4r^2} + \left[ \frac{7}{32} p_\phi^2 p_r^2 \hat{p}_r - \frac{7}{96} p_\phi r^3 \hat{p}_r^3 + i \left( \frac{7p_\phi}{96r} - \frac{7}{32} r^2 p_r^2 \right) \right] + \left[ \frac{3p_\phi}{32} p_r^4 \hat{p}_r^4 - \frac{p_\phi^2}{8} p_r^2 \hat{p}_r^2 + \frac{p_\phi p_r^4}{48r} \right] 
\] + i \left[ \frac{1}{12} p_\phi^3 p_r^3 \hat{p}_r \right] + \left[ \frac{31}{384} p_\phi r p_r^4 \hat{p}_r - \frac{173}{192} p_\phi r^7 \hat{p}_r^5 + \left( \frac{96p_\phi}{768} - \frac{49}{384} r^3 p_r^2 \hat{p}_r^2 + \frac{895}{3840} \right) \right] 
\] + \frac{97p_\phi r^3 p_r^3}{1152} + \frac{1}{16} p_\phi r^6 p_r^4 + \frac{47}{384} p_\phi r^3 p_r^3 p_r^2 + \frac{p_\phi p_r^6}{96} + i \left( -\frac{1}{64} r^8 p_r^5 p_r^5 + \frac{137}{1152} r^6 p_r^5 \hat{p}_r^5 + \frac{23}{640} r^2 p_r^5 \hat{p}_r^5 \right) \right] \right), \]
\[\hat{H}_{31}^{\text{tail}} = \frac{\pi}{\sqrt{6}c^3} \left[ \frac{p_\phi}{r} \hat{p}_r - p_\phi r p_r^2 + \frac{3}{4} r^3 \hat{p}_r^3 p_r - \frac{1}{4} p_r^2 \hat{p}_r^2 + \left[ \frac{7}{12} r^7 p_r^5 \hat{p}_r^4 - \frac{95}{192} r^8 \hat{p}_r^7 \right] + \frac{1192}{p_\phi r p_r^4 \hat{p}_r^4 \hat{p}_r} \right] + \left[ \frac{1393}{320} p_\phi r^6 p_r^6 + \frac{11}{16} p_\phi r^6 p_r^4 + \frac{3}{64} p_\phi r^6 p_r^2 \hat{p}_r^2 - \frac{p_\phi}{8} \right] \right] \right), \]
\[\hat{H}_{33}^{\text{tail}} = -\frac{9i\pi \delta}{4c^4} \sqrt{\frac{15}{14}} \left[ \frac{1}{r} \left[ \frac{23p_\phi}{27r} + \frac{10p_\phi}{27r^2} + \frac{2p_r^2}{27r^2} + \left( \frac{25}{432} p_\phi r^3 \hat{p}_r^3 - \frac{25}{432} r^3 p_r^3 \hat{p}_r^3 \right) \right] - \frac{25p_r^4}{1728} \right] + \left[ i \left( \frac{109p_\phi r^5 \hat{p}_r^5}{2592} + \frac{41p_\phi r^5 \hat{p}_r^5}{1296} + \frac{41p_\phi \hat{p}_r^5}{1296} \right) + \frac{293}{2592} p_\phi r^7 \hat{p}_r^7 - \frac{157}{1296} \right] \right] + \left[ \frac{2845}{4172} r^3 \hat{p}_r^3 \right] \right], \]
\[\hat{H}_{31}^{\text{tail}} = \frac{i\pi \delta}{12\sqrt{14}c^4} \left[ \frac{1}{r} \left[ \frac{101p_\phi r^5 \hat{p}_r^5}{27r} - \frac{11p_r^2}{27} \right] + \frac{131p_\phi r^5 \hat{p}_r^5}{4r^2} + \frac{3}{12} r^3 \hat{p}_r^3 \right] + \left[ i \left( \frac{6p_\phi r^5 \hat{p}_r^5}{6r^2} - \frac{35}{12} \right) \right]. \]
Then, we follow the same steps as in the previous subsection. Decomposing the phase into $\phi = \omega t + \Delta \phi$ leads to

$$J^{ij} = \mu \Omega^{ij}_{21}(e^{i(\psi + \phi)} J_{11} + \ldots, \quad (109)$$

and the current quadrupole radiative moment

$$V_{ij} = \frac{2M\mu\delta}{c^3} \sqrt{\frac{2\pi}{15}} \Omega^{ij}_{21} P_{\phi} \sum_{k=-\infty}^{\infty} J_{11k} e^{-i(k\psi + \phi)} + \ldots, \quad (113)$$

leading to the (2,1) mode

$$h_{21}^{ij} = \frac{16i}{3} \sqrt{\frac{\pi}{5}} \delta M \rho \sum_{k=-\infty}^{\infty} J_{11k} \Omega_{1k}^4 e^{i\chi}$$

with $p_{\phi} = 1/\sqrt{\nu_p} = \sqrt{(1-e^2)/x}$. Expanding in eccentricity yields

$$\psi_{21}^{ij} = \frac{i\pi\delta}{3c^2} \left[ 1 + e \left( 3e^{-i\chi} + e^{i\chi} \right) + e^2 \left( 3e^{-2i\chi} + \frac{1}{4} e^{2i\chi} + 6 \right) + e^3 \left( \frac{45e^{-i\chi}}{4} + 4e^{i\chi} + \frac{5}{4} e^{-3i\chi} + \frac{1}{6} e^{3i\chi} \right) \right]. \quad (114)$$
\[ + e^4 \left( \frac{19}{2} e^{-2ix} + \frac{25}{24} e^{2ix} + \frac{3}{16} e^{-4ix} + 17 \frac{e^{4ix}}{192} + \frac{493}{32} \right) + e^5 \left( \frac{2375 e^{-ix}}{96} + \frac{865 e^{ix}}{96} + \frac{15}{4} e^{-3ix} + \frac{91}{192} e^{3ix} \right) \]

\[ = \frac{7}{960} e^{5ix} \] + e^5 \left( \frac{29957 e^{-2ix}}{1536} + \frac{593}{256} e^{2ix} + \frac{9}{16} e^{-4ix} + \frac{241}{960} e^{4ix} + \frac{37 e^{6ix}}{23040} + \frac{8417}{288} \right), \] (115)

which is in agreement with the results of Ref. [122]. In terms of \((r, p_r, p_\phi, \dot{p}_r)\), we obtain

\[ \dot{H}_{\text{tail}}^{21} = \frac{i \pi \delta}{3 c^4} \left\{ \frac{1}{r^3} \left[ \frac{p_r}{r^2} - \frac{2 \pi p_\phi}{r^3} \right] + \left[ \frac{ip_0 p_r}{r^2}, \frac{p_r^2}{4 r^2} \right] \right\} + \frac{i}{2} \frac{2 p_r^2 \phi (38 \nu + 105)}{r^2} \delta (38 \nu + 105) + 74 \nu + 105 \right) + i \frac{(63 \nu - 38 \nu \delta - 74 \nu + 63)}{384 c^3} \] + \frac{i \chi_1}{3 r} \left\{ \chi_1 \left[ 3 C_{1 \text{ES}} X_1^2 - X_1^4 (2 p_r^2 r + 1) \right] + \chi_2 \left[ 3 C_{2 \text{ES}} X_2^2 - X_2^4 (2 p_r^2 r + 1) \right] \right\} + 2 \nu \chi_1 \chi_2 \left( \nu - 3 + 2 \nu p_r^2 r \right),

\text{C. Aligned-spin contributions}

The spin contributions to the modes were derived for circular orbits in Refs. [116, 120, 125]. To derive the spin part of the modes to 2PN for eccentric orbits, we use the source moments from Refs. [116, 121], which are in harmonic coordinates and in terms of the covariant SSC. Differentiating the source moments to obtain the radiative moments (82), and plugging them into Eq. (73), we obtain the modes listed in Appendix C. Transforming from harmonic to EOB coordinates, and from the covariant to the NW SSC using the transformations in Appendix A, we obtain the following spin contributions to the modes:

\[ \dot{H}_{\text{spin}}^{22} = \frac{1}{c^3} \left[ \frac{\chi_1}{12 r^3} \left( 6 \delta + \nu + 6 \right) p_\phi + 2 i (3 \delta - \nu + 3) r p_r \right] + \frac{\chi_2}{12 r^3} \left( 6 \delta + \nu + 6 \right) p_\phi - 2 i (3 \delta + \nu - 3) r p_r \right] + \frac{1}{4 c^4 r^3} \left\{ \frac{\chi_1}{12 r^3} \left[ 3 C_{1 \text{ES}} X_1^2 - X_1^4 (2 p_r^2 r + 1) \right] + \chi_2 \left[ 3 C_{2 \text{ES}} X_2^2 - X_2^4 (2 p_r^2 r + 1) \right] \right\} - 2 \nu \chi_1 \chi_2 \left( \nu - 3 + 2 \nu p_r^2 r \right),

\[ \dot{H}_{\text{spin}}^{21} = \frac{i}{4 c^4 r^2} \frac{\chi_1}{12 r^3} \left( 1 + \delta \right) \left( 1 - \delta \right) \chi_1 + \frac{\chi_1}{84 c^3 r^4} \left[ \frac{i p_r^2}{r^2} \right] \left[ 43 \nu - 42 \delta + 153 \nu + 42 \right] + \frac{p_\phi p_r}{r} (3 \delta (2 \nu + 49) - 104 \nu + 147) + \frac{i}{2} \frac{p_r^2 \phi (38 \nu + 105)}{r^2} (38 \nu + 105) + 74 \nu + 105 \right) + \frac{i}{63 (6 \delta - 38 \nu \delta - 74 \nu + 63)} \left[ \frac{\chi_1}{12 r^3} \left[ 3 C_{1 \text{ES}} X_1^2 - X_1^4 (2 p_r^2 r + 1) \right] + \chi_2 \left[ 3 C_{2 \text{ES}} X_2^2 - X_2^4 (2 p_r^2 r + 1) \right] \right\} - 2 \nu \chi_1 \chi_2 \left( \nu - 3 + 2 \nu p_r^2 r \right),

\[ \dot{H}_{\text{spin}}^{20} = \frac{\sqrt{3} p_\phi}{2 \sqrt{2} c^3 r^3} \left[ (2 \delta - \nu + 2) \chi_1 + (2 \delta - \nu + 2) \chi_2 \right] + \frac{\sqrt{3} \chi_1}{2 \sqrt{2} c^3 r^3} \left[ \frac{1}{3} \left[ \frac{X_1^2}{3 r^2} (2 p_r^2 r - 2 p_r^2 + r) - C_{1 \text{ES}} X_1^2 \right] \right] + \frac{2 \nu \chi_1 \chi_2}{3} \left[ \nu - 3 - 2 p_r^2 + 2 \nu p_r^2 \right],

\[ \dot{H}_{\text{spin}}^{30} = \frac{i \nu p_r}{\sqrt{2} c^3 r^3} (\chi_1 + \chi_2),

\[ \dot{H}_{\text{spin}}^{31} = \frac{1}{24 \sqrt{2} c^3 r^3} \left[ \chi_1 \left( \frac{i p_r^2}{2 r^2} (55 \nu \delta - 96 \delta + 375 \nu + 96) + i \left( \frac{p_r^2}{r} - \frac{2}{r} \right) (2 \nu \delta - 66 \nu + 23 \nu - 6) \right) \right] + \frac{p_\phi p_r}{r} (-6 \nu \delta - 127 \nu + 30) + \chi_2 \left( \frac{i p_r^2}{2 r^2} (55 \nu \delta - 96 \delta - 375 \nu + 96) + \frac{p_\phi p_r}{r} (-6 \nu \delta + 30 \delta - 127 \nu - 30) \right) + i \left( \frac{p_r^2}{r} - \frac{2}{r} \right) (2 \nu \delta - 69 \nu + 23 \nu + 6)), \right\}.

\]
\[
\hat{H}_{32}^{\text{spin}} = \frac{\nu}{6c^3H^3} \sqrt{\frac{5}{7}} (4p_\phi + i\rho_\tau) (\chi_1 + \chi_2),
\]
\[
\hat{H}_{33}^{\text{spin}} = \frac{\sqrt{5}}{8\sqrt{32\pi G^2H^4}} \left\{ \frac{-23i(\delta - 1)\nu_\phi^2}{2r^2} + \frac{2(2\nu_\delta + 6\delta - 19\nu + 6)p_\rho p_\phi}{r} + i\left( \frac{2}{r} - \rho_\tau^2 \right) \frac{(2\nu_\delta - 6\delta + 23\nu - 6)}{\chi_1} \right. 
\]
\[
+ \left. \left[ \frac{-23i(\delta - 1)\nu_\phi^2}{2r^2} + \frac{(24\nu + 6\delta + 19\nu - 6)p_\rho^2}{r} + i\left( \frac{2}{r} - \rho_\tau^2 \right) \frac{(2\nu_\delta - 6\delta - 23\nu + 6)}{\chi_1} \right] \chi_2 \right\},
\]
\[
\hat{H}_{41}^{\text{spin}} = -i\frac{5}{2\sqrt{336\pi G^2H^4}} (-10ir_\tau p_\rho + 6r^2p_\rho^2 - 12r + 11p_\phi^2) [\delta(\delta + 1)\chi_1 + (\delta + 1)\chi_2],
\]
\[
\hat{H}_{43}^{\text{spin}} = \sqrt{\frac{5}{144\sqrt{32\pi G^2H^4}}} (10ir_\tau p_\rho + 2i\rho_\tau^2p_\rho - 4ir - 23i\rho_\tau^2) [\delta(\delta + 1)\chi_1 + (\delta + 1)\chi_2].
\]

The circular-orbit limit of these modes, when expressed in terms of the orbital frequency, agrees with the results of Refs. [116, 125]. The spin contributions to the (2, 2), (2, 1), and (3, 3) modes for eccentric orbits were calculated in Ref. [82]; however, we find a small disagreement with their results for the SO part.\footnote{The difference between the modes in Ref. [82] (denoted with a bar) and the modes in Eq. (117) (with \(C_{1ES2} = C_{2ES2} = 1\)) is given by
\[
\begin{align*}
\hat{H}_2^{22}_{\text{spin}} - \hat{H}_2^{22} & = \frac{\nu p_\rho}{2cH^2} (\chi_1 + \chi_2), \\
\hat{H}_2^{21}_{\text{spin}} - \hat{H}_2^{21} & = \frac{\nu p_\rho}{6cH^2} (\rho_\tau + \rho_\tau^2) (\chi_1 + \chi_2), \\
\hat{H}_3^{23}_{\text{spin}} - \hat{H}_3^{23} & = \frac{\sqrt{5} \nu p_\rho}{8\sqrt{32\pi G^2H^4}} (17p_\rho + 5i\rho_\tau) (\chi_1 + \chi_2),
\end{align*}
\]
which is likely due to the coordinate/SSC transformations detailed in Appendix A.}

D. Factorized modes

The quasi-circular waveforms used in \texttt{SEOBNRv4HM} are factorized as follows [70, 83, 126, 127]:
\[
h_{\ell m}^{F, qc} = h_{\ell m}^{N, qc} \hat{S}_{\text{eff}}^{\text{qc}} T_{\ell m}^{\text{qc}} e^{i\delta_{\ell m}} T_{\ell m}^{\text{qc}},
\]
where \(h_{\ell m}^{N, qc}\) is the Newtonian part of the mode, \(\hat{S}_{\text{eff}}^{\text{qc}}\) is an effective source term given by
\[
\hat{S}_{\text{eff}}^{\text{qc}} = \begin{cases} 
\hat{H}_{\text{eff}}(v_{1}) & \ell + m \text{ even} \\
\hat{L}_{\text{eff}} \equiv v_{11} \rho_\tau(v_{1}) & \ell + m \text{ odd}
\end{cases},
\]
\(T_{\ell m}^{\text{qc}}\) resums the infinite number of “leading logarithms” entering the tail effects, \(\delta_{\ell m}\) contains the part of the tail not included in \(T_{\ell m}^{\text{qc}}\), and \(f_{\ell m}\) contains PN corrections such that the expansion of \(h_{\ell m}^{F, qc}\) agrees with the known PN expansion of the modes. See Refs. [70, 83] for more details and for expressions of these terms.

We include the eccentric corrections in the factorized modes as follows:
\[
\begin{align*}
h_{\ell m}^{F, e} & = \hat{S}_{\text{eff}}(1 + T_{\ell m}^{\text{ee}}) f_{\ell m}^{\text{ee}}, \\
h_{\ell m}^{F, e} & = h_{\ell m}^{N, e} \hat{S}_{\text{eff}}(T_{\ell m}^{\text{qc}} + T_{\ell m}^{\text{ee}}) e^{i\delta_{\ell m}} (f_{\ell m}^{\text{qc}} + f_{\ell m}^{\text{ee}}),
\end{align*}
\]
where the effective source term is given by
\[
\hat{S}_{\text{eff}} = \begin{cases} 
\hat{H}_{\text{eff}}(r, p_\rho, p_\phi) & \ell + m \text{ even} \\
\hat{L}_{\text{eff}} \equiv v_{11} \rho_\tau & \ell + m \text{ odd}
\end{cases},
\]
\(T_{\ell m}^{\text{ee}}\) contains the eccentric corrections to the hereditary contributions, \(\delta_{\ell m}\) is the same as in the quasi-circular case, and \(f_{\ell m}\) contains the eccentric corrections to the instantaneous contributions (both spinning and nonspinning, including the Newtonian part). For example, for the leading order of the (2, 2) mode, we obtain
\[
f_{22}^{\text{ee}} = \frac{1}{2(r^2p_\rho + 1)^{1/3}} \left[ 2 + i\rho_\tau^2p_\rho - 2(r^2p_\rho + 1)^{1/3} + 2i\rho_\tau \sqrt{r^2p_\rho + r} \right] + \ldots .
\]

For the tail part, we simplified the results of Sec. IVB and eliminated the gauge parameter by using a phase shift, which led to the circular part of the tail contribution to the (2, 2) mode simply being \(2\pi v_{10}\frac{\delta}{\Omega}\); however, this phase redefinition is not done in \texttt{SEOBNRv4HM}, and the corresponding expression reads \(v_{10}^2 (2\pi + 12i\log(2\pi \gamma_{10}) - 17i/3 + 12i\gamma_{1E}/3)\). Therefore, when including the eccentric corrections in \(T_{\ell m}^{\text{ee}}\), we assume that the phase redefinition was done only for the eccentric part and keep using the same circular part as in \texttt{SEOBNRv4HM}. In addition, since we expanded the tail part in eccentricity to \(O(\ell^0)\), when factorizing the modes as in Eq. (120) and writing the quasi-circular part in terms of frequency, we reexpand \(T_{\ell m}^{\text{ee}}\) in eccentricity (or \(p_\rho\) and \(\dot{p}_r\)). For example, for the (2, 2) mode, we obtain
\[
T_{22}^{\text{ee}} = -\frac{\pi}{4r} \left[ 4^{3/2}\dot{p}_r + i\rho_\tau (r^2\dot{p}_r + 6) + 2\sqrt{r^2p_\rho + O(p_\rho^2)} \right].
\]
The full expressions for \(T_{\ell m}^{\text{ee}}\) and \(f_{\ell m}^{\text{ee}}\) are provided in the Supplemental Material [88].
V. CONCLUSIONS

Extending the waveform models used today in GW astronomy from quasi-circular to eccentric orbits is important for future observations with LIGO, Virgo and KAGRA detectors [21], and with new facilities on the ground (Cosmic Explorer and Einstein Telescope), and in space (LISA). In fact, sources with non-negligible eccentricity might come into reach of observations soon and should routinely be included in searches and parameter inference. While this presents a challenge for waveform modeling and data analysis, it also offers the unique opportunity to unveil the formation channels of compact binaries and probe their environment (through eccentricity measurements). In this paper, we constructed an EOB waveform model for eccentric binaries. For this purpose, we obtained analytical results for the RR force and waveform modes to 2PN order, including the leading-order tail effects, and SO and SS couplings for aligned spins.

In particular, we first derived the RR force for eccentric orbits in PN expanded form, and then recast it in a form that it can be directly incorporated in the quasi-circular RR force employed in the SEOBNRv4HM [74, 83] model, currently used in LIGO/Virgo analyses [1]. We then obtained initial conditions for the binary evolution which generalize those from Ref. [94] to eccentric orbits, and which allow starting the binary’s evolution from a specified initial frequency at periastron and an initial eccentricity (in the Keplerian parametrization). We also calculated all the waveform modes that contribute up to 2PN order relative to the leading order of the (2, 2) mode. It should be noted that the (ℓ, 0) modes are proportional to the eccentricity and are hence important for eccentric orbits, especially the (2, 0) mode since it starts at the same PN order as the (2, 2) mode. Also the gravitational modes were rewritten in a factorized form to be straightforwardly implemented in the SEOBNRv4HM model.

Our results for the RR force and modes are valid for moderate to high eccentricities during the inspiral phase, since we do not use an eccentricity expansion except for the tail part, which is known analytically as an infinite series expansion. We provided expressions for the tail part numerically, or use analytical resummation methods as was done in Refs. [31, 98]. We are currently incorporating the eccentric RR force and gravitational modes of this paper in the inspiral-merger-ringdown quasi-circular–orbit SEOBNRv4HM waveform model (SEOBnRv4EHM [128]) and validating it against NR simulations with eccentricity. We leave to future work the extension of the model to higher PN orders and the inclusion of spin precession.

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Appendix A: Coordinate transformation from harmonic to EOB coordinates

The coordinate transformation from harmonic to EOB coordinates with no spin is given in Appendix A of Ref. [78]. In this appendix, we include LO SO and SS contributions to the transformation. We label harmonic, ADM, and EOB coordinates by \((x_h, v_h), (x_a, p_a),\) and \((x, p)\), respectively.

1. ADM to EOB transformation

To find the canonical transformation from the ADM Hamiltonian with LO SO and SS using the NW SSC (see e.g. Refs. [129–131]) and the 2PN expansion of the EOB Hamiltonian of Ref. [50], we write an ansatz with unknown coefficients for the generating function \(G(x, p)\), perform the following transformation on the ADM Hamiltonian [45]:

\[
x^i_a = x^i + \frac{\partial G}{\partial p_i} - \frac{\partial G}{\partial x^j} \frac{\partial^2 G}{\partial p_j \partial p_i} + \mathcal{O}\left(\epsilon^0\right),
\]

\[
p^i_a = p^i - \frac{\partial G}{\partial x^i} + \frac{\partial G}{\partial x^j} \frac{\partial^2 G}{\partial p_j \partial x^i} + \mathcal{O}\left(\epsilon^0\right),
\]

(A1)

and match it to the EOB Hamiltonian to solve for the unknowns.

The result for the generating function is given by

\[
G(x, p) = \frac{p_r}{\epsilon^2} \left[ -1 - \frac{\nu}{2} + \frac{1}{2} \nu p_r^2 r \right] + \frac{p_r}{\epsilon^4} \left[ \frac{1}{8} \nu (3\nu - 1) p_r^4 r - \frac{1}{8} \nu (\nu + 14) p_r^2 r - \frac{\nu^2 - 7\nu + 1}{4r} + \frac{1}{8} \nu^2 p_r^2 r \right]
\]

\[
+ \frac{\nu^2}{2 \epsilon^2 r} \left[ p_r (\hat{S}_1 + \hat{S}_2)^2 - (n \cdot \hat{S}_1 + n \cdot \hat{S}_2)(p \cdot \hat{S}_1 + p \cdot \hat{S}_2) \right],
\]

(A2)

which has no LO SO terms since the ADM and EOB Hamiltonian are the same at that order. This generating function
yields
\[
x_a = x + \frac{1}{c^2} \left[ x \left( \frac{\nu p^2}{2} - \frac{\nu + 2}{2r} \right) + \nu r p_r p \right] + \frac{1}{c^4} \left[ \left( \frac{3\nu(\nu + 2)}{8} \right) - \frac{1}{8} \nu(\nu + 1)p^4 - \frac{\nu(5\nu + 16)}{8r} - \frac{\nu^2 - 7\nu + 1}{4r^2} \right] \\
+ pp_r \left[ \frac{1}{2} (\nu - 1) p^2 r + \frac{\nu^2}{4r} \right] + \nu^2 \frac{\left( S_1 + S_2^2 \right)^2 x r}{r} - (S_1 + S_2)(n \cdot S_1 + n \cdot S_2) \right],
\]
\[
p_a = p + \frac{1}{c^2} \left[ p \left( \frac{\nu + 2}{2r} - \frac{p^2}{2} \right) - x \left( \frac{\nu + 2}{2r} p_r \right) \right] + \frac{1}{c^4} \left[ p \left( \frac{3\nu(3\nu + 1)}{8} \right) - \frac{\nu(7\nu + 2)p^4}{8r} + \frac{\nu(\nu + 8)p^2}{8r} + \frac{2\nu^2 - 3\nu + 5}{4r^2} \right] \\
+ x p_r \left[ \frac{3(\nu - 1)p^2}{2r^2} - \frac{3\nu - 10r + 6}{4r^3} \right] + \nu^2 \frac{x r}{r} \left[ (S_1 + S_2)p_r - (n \cdot S_1 + n \cdot S_2)(p \cdot S_1 + p \cdot S_2) \right] \\
+ \frac{\nu^2}{2r^2} \left[ -(S_1 + S_2)p_r - (n \cdot S_1 + n \cdot S_2)(p \cdot S_1 + p \cdot S_2) \right]. \tag{A3}
\]

2. Harmonic to EOB transformation

The transformation from harmonic to ADM coordinates is given by Eq. (E1) of Ref. [78], which is independent of spin since the ADM and harmonic coordinates agree at LO SO and SS. Using that equation together with Eq. (A3), we obtain the following transformation from harmonic to EOB coordinates:
\[
x_h = x + \frac{1}{c^2} \left[ x \left( \frac{\nu p^2}{2} - \frac{\nu + 2}{2r} \right) + \nu r p_r p \right] + \frac{1}{c^4} \left[ x \left( -\frac{1}{8} \nu(\nu + 1)p^4 + \frac{(3\nu - 1)p^2}{8r} - \frac{\nu(5\nu + 17)}{8r} - \frac{(\nu - 19)}{4r^2} \right) \right] \\
+ pp_r \left[ \frac{1}{4} (\nu - 19) \nu + \frac{1}{2} (\nu - 1) p^2 r \right] + \frac{\nu^2}{2r} \left[ (S_1 + S_2)^2 x r - (S_1 + S_2)(n \cdot S_1 + n \cdot S_2) \right],
\]
\[
v_h = p + \frac{1}{c^2} \left[ p \left( \frac{\nu - 1}{2} p^2 - \frac{\nu + 4}{2r} \right) - x \left( 3\nu + 2 \right) p_r \right] - \frac{1}{c^3} \left[ n \cdot S_1 (3 - 3\delta + 2\nu) + n \cdot S_2 (3 + 3\delta + 2\nu) \right] \\
+ \frac{1}{c^4} \left[ \left( \frac{3}{8} - \nu \right) \right] p^4 + \frac{\left( 7\nu^2 - 41\nu + 8 \right) p^2}{2r^2} + \frac{-15\nu^2 + 29\nu + 8}{8r^2} \right] + \frac{\nu^2}{2r^2} \left[ (S_1 + S_2)(p \cdot S_1 + p \cdot S_2) - (S_1 + S_2)^2 p \right] \\
+ \frac{\nu^2}{r^2} \left[ (S_1 + S_2)^2 p_r - (n \cdot S_1 + n \cdot S_2)(p \cdot S_1 + p \cdot S_2) \right]. \tag{A4}
\]

and for the scalars \((\phi, r, \dot{\phi}, \dot{r})\), we obtain
\[
\phi_h = \phi + \frac{p_\phi}{c^2} p_r + \frac{p_r}{c^2} - \frac{3\nu(\nu - 5\nu)}{4r^2} - \frac{\nu p^2}{2r} - \frac{\nu^2}{r^2} p^2,
\]
\[
r_h = r + \frac{1}{c^2} \left( \frac{\nu^2}{2r^2} + \frac{\nu}{2r} + 1 \right) + \frac{1}{c^2} \left( \frac{3\nu(3\nu - 1)}{8} - \frac{\nu(\nu + 1)}{r} - \frac{\nu^2}{4r} - \frac{\nu(\nu - 19)}{2r^2} \right) \\
- \frac{\nu}{8} (3\nu + 55) p_r^2 + \frac{1}{2\nu^2} r^4 + \frac{1}{2r} (X_1^2 + 2\nu 1\chi^2 + X_2^2 \chi^2),
\]
\[
\dot{\phi}_h = \frac{\dot{\phi}}{r^2} + \frac{\dot{\phi}}{r^2} \left[ \frac{\nu - 1}{2} - \frac{p^2}{2r} - \nu r p_r \right] + \frac{1}{c^2} \left[ \chi_1 \left( 2 + 2\delta - \nu \right) + \chi_2 \left( 2 - 2\delta - \nu \right) \right] + \frac{p_\phi}{c^2} \left[ 4\nu^2 p_r^4 + \frac{(3\nu - 17\nu + 2) p^2}{4r} \right] \\
- \frac{2(\nu - 1) \nu r^2 p_r^2 - \left( \nu^2 + 5\nu - 3 \right) p^2}{4r} + \frac{4 - 5\nu^2 + 65\nu}{4r^2} p_r^2 - \frac{\nu^2 - 9\nu + 2}{4r^2} + \frac{1}{c^2} \left( X_1^2 \chi^2 + 2\nu 1\chi^2 \chi^2 + X_2^2 \chi^2 \right),
\]
\[
\dot{r}_h = p_r + \frac{p_r}{c^2} \left( 2\nu - 1 \right) \frac{p^2}{2} - \frac{(2\nu + 3) p^2}{r} - \nu r p_r \right] + \frac{1}{c^2} \left[ \frac{\nu^2}{2r} + \frac{3 - \nu}{8} p^2 \right] + \frac{1}{c^2} \left( \frac{\nu^2 - 2\nu + 3}{8} p^4 + \frac{(3\nu - 17\nu + 2) p^2}{4r} + \left( \frac{5\nu^2}{2} \right) p_r^2 \right] \\
+ \frac{6 - 5\nu^2 + 39\nu}{4r^2} p_r^2 + \frac{3}{2\nu^2} p_r^4 + \frac{1}{2r} \left( X_1^2 \chi^2 + 2\nu 1\chi^2 + X_2^2 \chi^2 \right). \tag{A5}
\]

3. Transformation for the SSC

When calculating the spin contributions to the waveform modes, we used the source moments from Refs. [116], [121] which are in terms of the covariant SSC.
form the resulting modes to the NW SSC, we use the center-of-mass shift [100]

\[ x^i_A + \frac{1}{2c^2m_A} (v_A \times S_A)^i, \]  
(A6)

and the spin transformation [132]

\[ S^i_{1\text{ cov}} = (1 - \frac{m_2}{c^2r}) S^i + \frac{1}{2c^2} v_i (v \cdot S_1), \]  
(A7)

where the spin transformation is only required for the NLO SO part of the 2PN (2, 1) mode.

For the scalars \((r, \phi, \dot{r}, \phi, \chi_1, \chi_2)\), we obtain the transformations

\[
\begin{align*}
    r_{\text{cov}} &= r - \frac{\nu r^2}{2c^3} (\chi_1 + \chi_2), \\
    \phi_{\text{cov}} &= \phi + \frac{\nu r^2}{2c^3}, \\
    \dot{r}_{\text{cov}} &= \dot{r} + \frac{\nu r^2}{2c^3} (1 - \delta), \\
    \dot{\phi}_{\text{cov}} &= \dot{\phi} - \frac{\nu}{2c^3 r^2} \left( 1 + r^2 - r^3 \dot{\phi}^2 \right) (\chi_1 + \chi_2), \\
    \chi_{1\text{ cov}} &= \chi_1 - \frac{\chi_1}{2c^2 r} (1 - \delta), \\
    \chi_{2\text{ cov}} &= \chi_2 - \frac{\chi_2}{2c^2 r} (1 + \delta). \\
\end{align*}
\]  
(A8)

Appendix B: Angular momentum flux at leading-order spin-squared

In this appendix, we derive the angular momentum flux at leading spin-squared \((S_1^2)\) order. Here, we use unscaled variables in harmonic coordinates, but we drop the subscript \('h'\) to simplify the notation. We denote the orbital angular momentum \(L = \mu r \times v\), the relative position \(r = x_1 - x_2\), and relative velocity \(v = dr/dt\).

The relative acceleration \(a = a_1 - a_2\) with LO SO and SS contributions, in harmonic coordinates and the NW SSC, is given by [100]

\[
a = -M \frac{\mathbf{n}}{r^2} + \frac{1}{c^3} \left[ 3 \left(2 + \frac{3m_2}{2m_1}\right) \frac{\mathbf{n} \cdot (v \times S_1)}{r^3} - \left(4 + \frac{3m_2}{m_1}\right) \frac{\mathbf{v} \times S_1}{r^3} + 3 \left(2 + \frac{3m_2}{2m_1}\right) \frac{\dot{r} n \times S_1}{r^3} + 1 \leftrightarrow 2 \right] \\
- \frac{3}{c^4 \mu r^4} \left[ n (S_1 \cdot S_2) + S_1 (n \cdot S_2) + S_2 (n \cdot S_1) \right. \\
\left. - 5 n (n \cdot S_1) (n \cdot S_2) \right] \\
+ \frac{3}{2c^3 r^4} \left[ \frac{m_2 C_{1\text{ EST}}}{m_1 \mu} \right] \left[ -n S_1^2 + 5 n (n \cdot S_1)^2 \
- 2 S_1 (n \cdot S_1) \right] + 1 \leftrightarrow 2. \]  
(B1)

Since the spin evolution equations start at 1PN order, we can assume \(S_1 = 0 = S_2\) for the calculation of the LO fluxes.

The source multipole moments needed are the spin quadrupole \(I^{(2)}\) and the current quadrupole \(J^{(2)}\), which are given by [100, 110, 121]

\[
\begin{align*}
    I_{ij} &= m_1 x_i (x_j + 3c^3 x_i (v_1 \times S_1)^j - 4 \frac{d}{dt} x_i (v_1 \times S_1)^j \right) \\
    &= C_{1\text{ EST}} m_1 \frac{S_{ij}^1}{c^6} + 1 \leftrightarrow 2, \quad \text{and} \quad J_{ij} = m_1 x_i (x_1 + v_1)^j + \frac{3}{2c^2} x_i (v_1)^j + 1 \leftrightarrow 2, \quad \text{B3)
\end{align*}
\]

where the indices in angle brackets denote a symmetric trace-free part.

To transform from the coordinates of the two bodies \(x_i^1\) and \(x_i^2\) to the center-of-mass relative coordinates \(x_i = x_i^1 - x_i^2\), we use [133]

\[
x_i = \frac{m_2}{M} x_i^1 + \delta x^1, \quad x_i^2 = -\frac{m_1}{M} x_i + \delta x^2, \]  
(B4)

where

\[
\delta x^i = -\frac{\nu}{2c^3} \left[ (v \times S_1) \frac{n}{m_1} - (v \times S_2) \frac{n}{m_2} \right]. \]  
(B5)

The energy and angular momentum fluxes in terms of the multipole moments, to the order needed for the LO fluxes, are then calculated from [100, 134]

\[
\begin{align*}
    \Phi_E &= \frac{1}{5} f^{(3)}_i I^{(3)}_{ij} + \frac{16}{45c^2} J^{(3)}_{ij} J^{(3)}_{kj} f^{(3)}_{kl}, \\
    \Phi_J &= \frac{2}{5} \varepsilon_{ijk} J^{(2)}_{ij} J^{(2)}_{jkl} + \frac{32}{45c^2} \varepsilon_{ijk} J^{(2)}_{ij} J^{(2)}_{jkl}. \]  
(B6)
\]

This yields the LO SO and \(S_1 S_2\) fluxes derived in Refs. [92, 93, 100], in addition to the \(S_1^2\) energy flux from Ref. [110]. For the \(S_1^2\) angular momentum flux, we obtain

\[
\Phi_J = \frac{2m_2^2}{5c^4 \mu r^4} \left\{ \frac{L}{\mu r} S_1^2 - n \cdot S_1 (v \cdot S_1) + \frac{n}{S_1} (n \cdot S_1) \right\} + \frac{2m_2^2 C_{1\text{ EST}}}{5c^4 \mu r^4} \left\{ \frac{L}{\mu r} S_1^2 \left( -30 r^2 + 12c^2 + 24 \frac{M}{r} \right) \\
+ \frac{L}{\mu r} (n \cdot S_1)^2 \left( 210 r^2 - 60 c^2 - 90 \frac{M}{r} \right) \\
+ v \cdot S_1 (n \cdot S_1) \left( 30 r^2 + 18 c^2 - 12 \frac{M}{r} \right) \\
+ 6n \cdot S_1 (v \cdot S_1) - \frac{L}{\mu r} S_1^2 \frac{M}{r} \right\} - \frac{90}{\mu r} \left( n \cdot S_1 \right)^2 (v \cdot S_1) + 6 \frac{L}{\mu r} (v \cdot S_1)^2 \\
- 6 \frac{L}{\mu r} (v \cdot S_1) v \times S_1 \right\} + 1 \leftrightarrow 2. \]  
(B8)

This is in agreement with the recent results of Ref. [135], although our expression appears simpler because of using the individual spins \(S_i\) and masses \(m_i\), instead of different combinations of them.
Appendix C: Aligned-spin contributions to the modes in harmonic coordinates

The modes calculated from the source moments of Refs. [116, 121] in harmonic coordinates and using the covariant SSC have the following spin contributions:

\[
\begin{align*}
\hat{H}_{\text{spin}}^{30} &= \frac{i}{\sqrt{6}c^3r} [\chi_1(1 + \delta + \nu) + \chi_2(1 - \delta + \nu)] - \frac{\sqrt{3}}{2\sqrt{2}c^3r^3} \left[ C_{1,1,2} X_1^2 + 2\nu \chi_1 \chi_2 + C_{2,2,2} X_2^2 \right], \\
\hat{H}_{\text{spin}}^{21} &= -\frac{i}{4r^2} \left( (1 + \delta) \chi_1 + (\delta - 1) \chi_2 \right) + \frac{i}{168c^3r^3} \left\{ \chi_1 \left[ 154 + 22\delta(\nu + 7) + 34\nu + 4r^3\dot{\phi}^2(4\nu\delta - 21\delta + 66\nu - 21) \\
- 2i\dot{r}r^2\dot{\phi}(13\nu\delta + 147\delta - 83\nu + 147) + \dot{r}^2r(-60\nu + 105\delta - 52\nu + 105) \right] + \chi_2 \left[ -154 + 22\delta(\nu + 7) - 34\nu \\
+ 4r^3\dot{\phi}^2(4\nu - 21\delta - 66\nu + 21) - 2i\dot{r}r^2\dot{\phi}(13\nu\delta + 147\delta - 83\nu - 147) + \dot{r}^2r(-60\nu + 105\delta + 52\nu - 105) \right] \right\}, \\
\hat{H}_{\text{spin}}^{22} &= -\frac{1}{6c^3r^2} \left\{ \chi_1 \left[ r\dot{\phi}(3\delta - 5\nu + 3) + i\dot{r}(3\delta - 8\nu + 3) \right] + \chi_2 \left[ r\dot{\phi}(-3\delta - 5\nu + 3) - i\dot{r}(3\delta + 8\nu - 3) \right] \right\} + \frac{3}{4c^3r^3} \left[ C_{1,1,2} X_1^2 + C_{2,2,2} X_2^2 + 2\nu \chi_2 \chi_1 \right], \\
\hat{H}_{\text{spin}}^{30} &= -\frac{i}{\sqrt{42}c^3r^2} \left( \chi_1 + \chi_2 \right), \\
\hat{H}_{\text{spin}}^{31} &= \frac{i}{48\sqrt{14}c^3r^3} \left\{ \chi_1 \left[ -4 + 20\delta\nu - 4\delta + 20\nu + r^3\dot{\phi}^2(-31\delta\nu - 24\delta + 87\nu - 24) + i\dot{r}r^2\dot{\phi}(-70\delta\nu - 12\delta + 62\nu - 12) \\
+ \dot{r}^2r(-30\delta\nu + 12\delta + 50\nu + 12) \right] + \chi_2 \left[ 4 + 20\delta\nu - 4\delta - 20\nu + r^3\dot{\phi}^2(-31\delta\nu + 24\delta - 87\nu + 24) \\
+ i\dot{r}r^2\dot{\phi}(-70\delta\nu - 12\delta - 62\nu + 12) + \dot{r}^2r(-30\delta\nu + 12\delta + 50\nu - 12) \right] \right\}, \\
\hat{H}_{\text{spin}}^{32} &= \sqrt{\frac{5}{7}} \frac{\nu}{6c^3r^2} (4\dot{r}\phi + i\dot{r}) \left( \chi_1 + \chi_2 \right), \\
\hat{H}_{\text{spin}}^{33} &= \sqrt{\frac{5}{42}} \frac{i}{16c^3r^3} \left\{ \chi_1 \left[ 4 - 20\delta\nu + 4\delta - 20\nu + r^3\dot{\phi}^2(-33\delta\nu + 24\delta - 119\nu + 24) + i\dot{r}r^2\dot{\phi}(-78\delta\nu + 36\delta - 154\nu + 36) \\
+ \dot{r}^2r(30\delta\nu - 12\delta + 50\nu - 12) \right] + \chi_2 \left[ -4 - 20\delta\nu + 4\delta + 20\nu + r^3\dot{\phi}^2(-33\delta\nu + 24\delta + 119\nu - 24) \\
+ i\dot{r}r^2\dot{\phi}(-78\delta\nu + 36\delta + 154\nu - 36) + \dot{r}^2r(30\delta\nu - 12\delta - 50\nu + 12) \right] \right\}, \\
\hat{H}_{\text{spin}}^{41} &= -\frac{\nu}{2 \sqrt{3}c^3r^3} \left( 11r^3\dot{\phi}^2 - 10i\dot{r}r^2\dot{\phi} + 6i\dot{r}^2r - 12 \right) \left( (\delta - 1)\chi_1 + (\delta + 1)\chi_2 \right), \\
\hat{H}_{\text{spin}}^{42} &= \sqrt{\frac{5}{14}} \frac{\nu}{48c^3r^3} \left( -23ir^3\dot{\phi}^2 + 10i\dot{r}r^2\dot{\phi} + 2i\dot{r}^2r - 4i \right) \left( (\delta - 1)\chi_1 + (\delta + 1)\chi_2 \right). \quad \text{(C1)}
\end{align*}
\]

Appendix D: Keplerian parametrization

In the Keplerian parametrization,

\[
r = \frac{1}{u_p(1 + e \cos \chi)}, \quad \text{(D1)}
\]

This appendix provides expressions for some orbital quantities in the Keplerian parametrization that are needed for calculating the initial conditions, and the tail part of the RR force and waveform modes.
energy and angular momentum to 2PN order yields
\[ E = \frac{1}{2} \left( e^2 - 1 \right) u_p - \frac{u_p^2}{8e^3} \left( e^2 - 1 \right)^2 (\nu - 3) \]
\[ + \frac{u_p^2}{16e^3} \left( e^2 - 1 \right)^2 \left[ e^2 \left( 2\nu - 3 \nu + 5 \right) - \nu - 5\nu + 27 \right] \]
\[ + \frac{1 - e^2}{4e^4} \frac{u_p^2}{3\nu^2} \left[ \chi_1(\nu - 2\delta - 2) + \chi_2(\nu + 2\delta - 2) \right] \]
\[ + \frac{1 - e^2}{4e^4} u_p^3 \left[ \frac{1}{4} \left( C_{1ES^2} X_1^2 + X_1^4 \right) + \chi_2^2 \left( C_{2ES^2} X_2^2 + X_2^4 \right) \right], \]
\[ p_\phi = \frac{1}{\sqrt{u_p}} + \frac{u_p}{2e^2} \left( e^2 + 3 \right) + \frac{u_p^3/2}{8e^4} \left( e^2 + 3 \right) \left( 3e^2 - 4\nu + 9 \right) \]
\[ + \frac{1}{4} \left( e^2 + 3 \right) u_p \left[ \chi_1(-2\delta + \nu - 2) + \chi_2(2\delta + \nu - 2) \right] \]
\[ + \frac{1 - e^2}{4e^4} u_p^{3/2} \left[ \chi_1^2 \left( C_{1ES^2} X_1^2 + X_1^4 \right) + \chi_2^2 \left( C_{2ES^2} X_2^2 + X_2^4 \right) \right] \]
\[ + \frac{u_p^{3/2}}{2e^3} \chi_1 \chi_2 \left[ e^2(3\nu + 1) + \nu + 3 \right]. \quad (D2) \]

Inverting \( p_\phi (u_p, e) \), we obtain \( u_p(p_\phi, e) \)
\[ u_p(p_\phi, e) = \frac{1}{p_\phi} + \frac{(e^2 + 3)}{c_p^2 p_\phi} \left( e^2 + 3 \right) \left( 2e^2 - \nu + 6 \right) \]
\[ + \frac{3 + e^2}{2p_\phi^2 e^3} \left[ \nu(2\nu - 2\delta - 2) \chi_1 + (2\nu + 2\delta - 2) \chi_2 \right] \]
\[ + \frac{1}{2p_\phi^2 e^4} \left( \nu \chi_1 \chi_2 \left[ e^2(3\nu + 1) + \nu + 3 \right] \right. \]
\[ \left. + \chi_1^2 \left[ C_{1ES^2} \left( e^2 + 3 \right) X_1^2 + \left( 3e^2 + 1 \right) X_1^4 \right] \right) \]
\[ + 1 \leftrightarrow 2 \right) \}. \quad (D3) \]

Inverting the Hamiltonian to obtain \( p_r(E, p_\phi) \), and plugging \( E(e, u_p) \) and \( p_\phi(e, u_p) \), yields
\[ p_r = e \sqrt{u_p} \sin \chi + \frac{c_p^{3/2}}{2e^2} \sin \chi \left( e^2 + 2e \cos \chi + 1 \right) + \ldots \]
\[ (D4) \]

The radial and azimuthal periods are given, respectively, by
\[ T_r = \int dt = \int \left( \frac{\partial H}{\partial p_r} \right)^{-1} dr = 2 \int_0^\pi \left( \frac{\partial H}{\partial p_r} \right)^{-1} \frac{dr}{d\chi} d\chi \]
\[ = \frac{2\pi}{(u_p - e^2 u_p)^{3/2}} - \frac{c_p^2 \sqrt{u_p - e^2 u_p}}{u_p - e^2 u_p} + \ldots \]
\[ T_\phi = \int \frac{d\phi}{dt} = \int \frac{\partial H}{\partial p_\phi} dt = 2\pi + \frac{6\pi u_p}{c_p^2} + \ldots \]
\[ (D5) \]

The associated frequencies are
\[ \omega_r = \frac{2\pi}{T_r}, \quad \omega_\phi = \frac{T_\phi}{T_r}. \quad (D6) \]

The dimensionless frequency variable \( x \equiv \frac{\omega_{3/2}}{\phi} \) to 2PN order is given by
\[ x = u_p - e^2 u_p + \frac{u_p^2}{3e^2} \left( e^2 - 1 \right) \left( e^2(\nu - 6) - \nu \right) \]
\[ - \frac{u_p^3}{6e^2} \left( e^2 - 1 \right) \left\{ e^4 \left( 8\nu - 33\nu + 180 \right) + 8v^2 \right\} \]
\[ - 2e^2 \left( 3 \left( 12\nu - 1 \right)^2 + 5 \nu - 90 \sqrt{1 - e^2} + 8v^2 + 27 \right) \]
\[ + 9 \left( \sqrt{1 - e^2} - 13 \right) \nu - 180 \left( \sqrt{1 - e^2} - 1 \right) \}
\[ + \frac{u_p^{3/2}}{6e^4} \left( e^2 - 1 \right) \left( 3e^2 + 1 \right) \left[ \chi_1(2\delta + \nu) + 1 \right] \]
\[ - \frac{u_p^3}{2e^3} \left[ \chi_1^2 \left( C_{1ES^2} \left( e^4 - 1 \right) X_1^2 + \left( e^2 - 1 \right)^2 X_1^4 \right) \right. \]
\[ \left. + \nu \chi_1 \chi_2 \left[ e^4(\nu + 1) - 2e^2 \nu + \nu - 1 \right] + 1 \leftrightarrow 2 \right) \}, \quad (D7) \]

which can be inverted to obtain \( u_p(x, e) \), the 1PN part of which reads
\[ u_p = -\frac{x}{e^2 - 1} + \frac{x^2 \left[ e^2(\nu - 6) - \nu \right]}{3e^2 \left( e^2 - 1 \right)^2} + \ldots \] (D8)

**Appendix E: \( \bar{p}_r \) in tortoise coordinates**

The tortoise-coordinate \( r_\ast \) is defined by [69, 112]
\[ \frac{dr_\ast}{dr} = \frac{\sqrt{D(r)}}{A(r)} = \frac{1}{\xi(r)}. \quad (E1) \]

where \( A(r) \) and \( D(r) \) are the metric potentials
\[ ds_{\ast}^2 = -A(r)dt^2 + \frac{D(r)}{A(r)}dr^2 + r^2 d\Omega^2. \quad (E2) \]

The conjugate momentum to \( r_\ast \) is denoted \( p_{r_\ast} \), and invariance of the action gives the relation
\[ p_{r_\ast} = p_r \xi(r). \quad (E3) \]

The Hamiltonian and EOMs used in SEOBNRv4 (see Eqs. (10) of Ref. [136]) are expressed in terms of the variables \((r, p_r, \phi, p_\phi)\). However, the RR force we derived in Sec. II is expressed in terms of \((r, p_r, \bar{p}_r)\). We use Eq. (E3) to replace \( p_r \) with \( p_{r_\ast} \), and to obtain a relation between \( \bar{p}_r \) and the derivatives of \( H_{EOB}(r, p_r, p_\phi) \). We use the following relations:
\[ dH = \left( \frac{\partial H}{\partial p_r} \right)_{p_r} dt + \left( \frac{\partial H}{\partial p_r} \right)_{p_r} dp_r + \frac{\partial H}{\partial p_\phi} dp_\phi \]
\[ = \frac{\partial H}{\partial p_r} \left( \frac{d\phi}{dt} \right) \]
\[ = \frac{\partial H}{\partial p_r} \left( \frac{dp_r}{dt} \right) + \frac{\partial H}{\partial p_\phi} \frac{dp_\phi}{dt}, \quad (E4) \]
\[ dp_r = \left( \frac{dp_r}{dt} \right) \]
\[ = \left( \frac{dp_r}{dt} \right) + \left( \frac{dp_r}{dr} \right) dp_r, \quad (E5) \]
leading to
\[
\left( \frac{\partial H}{\partial r} \right)_{p_r} = \left( \frac{\partial H}{\partial r} \right)_{p_r} + \left( \frac{\partial H}{\partial p_r} \right)_{r} \left( \frac{\partial p_r}{\partial r} \right)_{p_r}, \quad (E6)
\]

where
\[
\left( \frac{\partial p_r}{\partial r} \right)_{p_r} = p_r \frac{d\xi(r)}{dr}. \quad (E7)
\]

Hence,
\[
\dot{p}_r = - \left( \frac{\partial H}{\partial r} \right)_{p_r} - \left[ \left( \frac{\partial H}{\partial p_r} \right)_{r} \left( \frac{\partial p_r}{\partial r} \right)_{p_r} \right] \frac{d\xi(r)}{dr}. \quad (E8)
\]

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