Some Properties of Generalized Foulkes Module

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Abstract

Describing the decomposition of Foulkes module $F_{a}^{b}$ into irreducible Specht modules is an open problem for $a, b > 3$. In this article we provide a new approach for the Generalized Foulkes module $F_{a}^{\nu}$ (with arbitrary partition $\nu$ of $b$) through its restriction to a maximal Young subgroup $S_{b} \times S_{ab-b}$.

1 Introduction

The modules in this paper are defined over the complex numbers. For $a, b > 1$ integers, let $n = ab$. The Foulkes module $F_{a}^{b}$ is the permutation module of $S_{n}$ acting on the set of partitions of type $(a^{b})$, that is on partitions of $\{1, 2, \ldots, n\}$ into $b$ sets of size $a$ each.

The simple $S_{b}$-modules are parametrised by the partitions of $b$, the simple module corresponding to a partition $\nu \vdash b$ is the so-called Specht module, $S_{\nu}$. In particular, $S_{(b)}$ is the trivial, while $S_{(1^{b})}$ is the sign module of $S_{b}$. The wreath product $S_{a} \wr S_{b} \leq S_{n}$ has a normal subgroup $S_{a} \times S_{a} \times \cdots \times S_{a}$ ($b$ factors), called the base group, with factor group isomorphic to $S_{b}$, hence we may consider $S_{\nu}$ as an $S_{a} \wr S_{b}$-module with kernel containing the base group. This module is the inflated Specht module, denoted by $\text{Inf} S_{a} \wr S_{b} S_{\nu}$ is a simple modele of $S_{a} \wr S_{b}$.

The $\nu$-generalized Foulkes module is the induced module of this inflation to $S_{n}$, in formula, $F_{\nu}^{a} = \text{Inf} S_{a} \wr S_{b} S_{\nu} \uparrow S_{n}$. When $\nu = (b)$, we recover the original Foulkes module, $F_{(b)}^{a} = F_{b}^{a}$. If, however, $a = 1$ then $S_{a} \wr S_{b} = S_{b}$ and $F_{\nu}^{1} = S_{\nu}$.

Thrall [1] decomposed the Foulkes module into simple components for $a = 2$ and for $b = 2$:

$$F_{b}^{2} = \bigoplus_{\lambda \vdash b} S^{2\lambda}; \quad F_{2}^{a} = \bigoplus_{\lambda \vdash a \lambda_{1}+\lambda_{2}=2a} S^{\lambda}.$$

Foulkes [2] conjectured that if $a \leq b$ then $F_{b}^{a}$ can be embedded in $F_{a}^{b}$, for $a = 2$ this is an immediate consequence of Thrall’s result. Dent [3] decomposed the Foulkes module for $F_{3}^{a}$ and $F_{a}^{3}$ into Specht modules and verified the conjecture for $a = 3$. For $a > 3$ no full decomposition of the Foulkes module is known, albeit the conjecture is proven for $a \leq 5$, see [4] [5] [6]. For an integer $0 < k < n$ we define $\Omega_{k}$ as the set of partitions of $k$ which are subpartitions of $(a^{b})$, that is all parts are of size less than or equal to $a$. Then the restriction of the generalized Foulkes module to $S_{k} \times S_{n-k}$ has a natural decomposition (see below, Definition 7) indexed by $\Omega_{k}$:

$$F_{\nu}^{a} \downarrow_{S_{k} \times S_{n-k}} = \bigoplus_{\lambda \in \Omega_{k}} V_{\nu,a}^{\lambda}. \quad (1)$$

We are concerned mainly with the $(1^{k})$-component $U_{\nu,a} = V_{\nu,a}^{(1^{k})}$. 
The main theorem of this paper is the following. Let $\mu^\perp$ denote the conjugate of the partition $\mu$. In particular, $(1^b)^\perp = (b)$. For $\mu, \lambda, \nu \vdash b$ let $c_{\mu, \lambda}^{\nu} = c_{\lambda, \mu}^{\nu}$ denote the Kronecker coefficient, that is the multiplicity of $S^\nu$ in the tensor product $S^\mu \otimes S^\lambda$. For $0 < k < n$ any pair of $S_k$-module $M$ and $S_{n-k}$-module $N$ defines an $S_k \times S_{n-k}$-module $M \times N$. The simple modules of $S_k \times S_{n-k}$ are the ones $S^\mu \times S^\lambda$ coming from pairs of Specht modules.

**Theorem 1.** Let $k = b$ and as above, $U_{\nu, a} = V^{(1^b)}_{\nu, a}$. Then

$$U_{\nu, a} \cong \bigoplus_{\mu, \lambda \vdash b} c_{\mu, \lambda}^{\nu} S^\mu \times F^a_{\lambda}.$$  

(2)

In particular, for $a = 2$

$$U_{\nu, 2} \cong \bigoplus_{\mu, \lambda \vdash b} c_{\mu, \lambda}^{\nu} S^\mu \times S^\lambda.$$  

(3)

As noted above, $S^{(b)}$ is the trivial, while $S^{(1^b)}$ is the sign module of $S_b$. So $c_{(b), \nu}^{\nu} = 1$ and $c_{(1^b), \nu^\perp}^{\nu^\perp} = 1$. Hence the multiplicity of both $S^{(b)} \times S^\nu$ and $S^{(1^b)} \times S^{\nu^\perp}$ in $U_{\nu, 2}$ are 1. Two important special cases of the main theorem are the following.

**Corollary 2.** Let $a, b \in \mathbb{N}$. The $(1^b)$-summand of the Foulkes module $F^a_{(b)}$ restricted to $S_b \times S_{n-b}$ is

$$U_{(b), a} \cong \bigoplus_{\lambda \vdash b} S^\lambda \times F^a_{\lambda}.$$  

In particular, for $a = 2$

$$U_{(b), 2} \cong \bigoplus_{\lambda \vdash b} S^\lambda \times S^\lambda.$$  

**Corollary 3.** Let $a, b \in \mathbb{N}$. The $(1^b)$-summand of the generalized Foulkes module $F^a_{(1^b)}$ restricted to $S_b \times S_{n-b}$ is

$$U_{(1^b), a} \cong \bigoplus_{\lambda \vdash b} S^{\lambda^\perp} \times F^a_{\lambda}.$$  

In particular, for $a = 2$

$$U_{(1^b), 2} \cong \bigoplus_{\lambda \vdash b} S^{\lambda^\perp} \times S^\lambda.$$  

With the help of the so-called semistandard homomorphism de Boeck [7] proved that the multiplicity of $S^{\lambda + (b)}$ in $F^a_{(b)}$ is at least the multiplicity of $S^\lambda$ in $F^a_{(b)}$ and the multiplicity of $S^{\lambda^\perp + (b^\perp)}$ in $F^a_{(1^b)}$ is equal to the multiplicity of $S^\lambda$ in $F^a_{(1^b)}$. This we establish as a corollary of our results. See Corollary [13].

## 2 Preliminaries

Our main reference for the representations of the symmetric group is [8], recall especially the notions of the $\nu$-tableau $t$, the tabloid $\{t\}$ and the polytabloid $e_t$. We call SYT($\nu$) the set of standard $\nu$-tableaux.

Let $a, b \geq 2$ fixed integers and $n = ab$. Denote by $H = H^a_b$ the set of ordered partitions of $\{1, 2, \ldots, n\}$ into $b$ sets of size $a$ each. Clearly, $S_n$ acts on $H$ by permuting the letters. But $S_b$ also acts on $H$ by permuting the indices, that is, for $X = (X_1, \ldots, X_b) \in H^a_b$ we have $\sigma X = (X_{\sigma(1)}, \ldots, X_{\sigma(b)})$. Let $I = I^a_b$ be a set of representatives of $S_b$ orbits. Therefore $|H| = n!/((a!)^b$ and $|I| = n!/((a!)^b b!$. 

2
Lemma 5. The inflation $\text{Inf}^{S_a \wr S_b}_{S_a} S^\nu$ has basis $B_{\nu,X} = \{ e_{t_X} \mid t \in \text{SYT}(\nu) \}$ in the vector space $V$ of $\nu$-"polytabloids."

Before the proof we remark that the role of $X$ in the definition is to fix the wreath product, which acts on the parts of $X$.

Proof. For $g \in S_a \wr S_b$ the image in $S_b \cong S_a \wr S_b / S^b_a$ is denoted by $\tau_g$. Now $\tau_g \in S_b$ acts on the entries of $t$ and on the indices of $X$ and we have $(\tau_g t)_X = t_{\tau_g X}$ and therefore $g e_{t_X} = e_{(\tau_g t)_X} = e_{\tau_g X} = e_{t_X}$. The action of $S_b$ on $S^\nu$ with respect to the basis $B_{\nu,X}$ is indeed inflated to the action of $S_a \wr S_b$ with respect to the basis $B_{\nu,X}$. The simple module $\text{Inf}^{S_a \wr S_b}_{S_a} S^\nu$ is generated by $e_{t_X}$ for any tableau $t$ like $S^\nu$ is generated by any $e_t$. □

Lemma 6. Fix a partition $\nu \vdash b$ and a $\nu$-tableau $t$. Let $V_0 = \langle e_{t_X} \mid X \in H \rangle \leq V$. Then $V_0$ is $S_n$-invariant and as an $S_n$-module it is isomorphic to the generalized Foulkes module

\begin{align*}
\text{Lemma 4.} \text{ For a partition } \nu \vdash b, \text{ a } \nu\text{-tableau } t, \text{ and } X \in H \text{ let } t_X \text{ be the } \nu\text{-shaped diagram with } X_l \text{ replacing } l \text{ in } t. \text{ Similarly, } \{ t_X \} \text{ is the "tableoid" where each } l \text{ in } \{ t \} \text{ is replaced by } X_l. \text{ If } g \in S_n \text{ then clearly } gt_X = t_{gX} \text{ and } g\{ t_X \} = \{ t_{gX} \}. \text{ Finally, a } \nu\text{-polytabloid, } e_t \text{ is a certain element of the vector space with the } \nu\text{-tableoids being a formal basis. The corresponding } \\
\nu\text{-"polytabloid" } e_{t_X}, \text{ is an element of the vector space } V \text{ having the } \nu\text-"tableoids" \text{ as a formal basis. Here again, the parts of } X \text{ are replacing the letters. We again have that for } g \in S_n, \\
g e_{t_X} = e_{t_{gX}}.
\end{align*}
\( F^a_\nu = \text{Inf}_{S_h} S^\nu \uparrow S_n \). Further, \( B^a_\nu = \{ e_{sZ} \mid Z \in I, s \in \text{SYT}(\nu) \} \) is a basis of \( V_0 \cong F^a_\nu \) such that if \( t \) is a \( \nu \)-tableau, \( X \in H \) and

\[ e_{tX} = \sum_{e_{sZ} \in B^a_\nu} c_{s,Z} e_{sZ} \]

then \( c_{s,Z} = 0 \) unless \( X \) is in the \( S_b \)-orbit of \( Z \).

**Proof.** As \( g e_{tX} = e_{tBX}, V_0 \) is an \( S_n \)-invariant, so an \( S_n \)-module. Also, \( V_0 \) is generated by \( e_{tX} \) for any \( X \). Let us fix a \( W = S_{\alpha} \triangleleft S_h \leq S_n \) and \( Y \in H \) the corresponding partition. As \( e_{tY} \) generates \( V_0 \) as an \( S_n \)-module, by [9] Corollary 8.3] it is enough to confirm that

\[ \dim V_0 = |S_n : W| |B_{\nu,Y}| = \frac{n!}{(a!)^b |b|!} |B_\nu| = |I| \text{ SYT}(\nu)|. \]

Pick \( Z \in I \) in the \( S_b \)-orbit of \( Y \). If \( Z = \sigma Y \) for a \( \sigma \in S_b \) then \( e_{tY} = e_{t\sigma Z} = e_{(\sigma)tZ} \). If

\[ e_{\sigma t} = \sum_{s \in \text{SYT}(\nu)} c_s e_s \]

then

\[ e_{tY} = e_{(\sigma)tZ} = \sum_{s \in \text{SYT}(\nu)} c_s e_{sZ}. \quad (4) \]

Suppose that

\[ 0 = \sum_{Z \in I} d_{s,Z} e_{sZ} = \sum_{Z \in I} \sum_{s \in \text{SYT}(\nu)} d_{s,Z} e_{sZ}. \]

If \( Z \neq Z' \) (so not in the same orbit) and \( s, t \in \text{SYT}(\nu) \) arbitrary then the “tabloids” occurring in \( e_{sZ} \) and \( e_{tZ} \) are distinct, so we must have

\[ 0 = \sum_{s \in \text{SYT}(\nu)} d_{s,Z} e_{sZ}, \forall Z \in I. \]

But \( B_{\nu,Z} \) is a basis whence \( 0 = d_{s,Z} \) for every \( s \in \text{SYT}(\nu) \) and \( Z \in I \).

Therefore \( B^a_\nu = \{ e_{sZ} \mid Z \in I, s \in \text{SYT}(\nu) \} \) is indeed a basis of \( V_0 \), \( \dim V_0 = |B^a_\nu| = |I| \text{ SYT}(\nu)| \) and thus [4] is a unique expression so the last part of the Lemma also holds.

Let \( 1 < k < ab \). We describe a decomposition of the restriction of the generalized Foulkes module to a maximal intransitive subgroup \( S_k \times S_{n-k} \). As above, \( \Omega_k \) is the set of those partitions of \( k \) that are subpartitions of \((ab)\). In the following notation the dependence on \( a \) is generally suppressed.

**Definition 7.** For \( \lambda \in \Omega_k \) let

\[ P_\lambda = P^a_\lambda = \{ X \in H^a_0 \mid \lambda \text{ is the partition type of } \{1, 2, \ldots, k\} \cap X \} \]

and let the \( \lambda \)-component, \( V^\lambda_{\nu,a} \) be the \( S_k \times S_{n-k} \)-module generated by the \( S_k \times S_{n-k} \)-invariant set \( \{ e_{tX} \mid X \in P_\lambda \} \) for any \( t \) of shape \( \nu \).
Note that \( X \in P_\lambda, \sigma \in S_b \) implies \( \sigma X \in P_\lambda \), so \( I \cap P_\lambda \) is a set of representatives of \( S_b \)-orbits of \( H \) that lie in \( P_\lambda \). The set \( \{ e_{tx} \mid X \in I \cap P_\lambda, t \in \text{SYT}(\nu) \} \) is a basis of \( V^\lambda_{\nu,a} \). Indeed, the vector space they generate is \( S_k \times S_{n-k} \)-invariant and
\[
\bigcup_{\lambda \in \Omega_k} \{ e_{tx} \mid X \in I \cap P_\lambda, t \in \text{SYT}(\nu) \} = B^a_\nu.
\]
The mentioned decomposition is thus
\[
F^a_\nu \downarrow_{S_k \times S_{n-k}} = \bigoplus_{\lambda \in \Omega_k} V^\lambda_{\nu,a}. \tag{5}
\]

Here comes an example with \( a = 3, b = k = 4 \) and \( \nu = (2^2) \). We have \( \Omega_4 = \{(3, 1), (2^2), (2, 1^2), (1^4)\} \). Then \( V = V^{(2,1^2)}_{2},a \) is generated by the set \( \{ e_{tx} \mid X \in P(2,1^2), t \in \text{SYT}((2^2)) \} \). Let \( t \) be as before and let
\[
X = \{(1,2,5), (3,7,6), (4,8,9), (10,11,12), X \cap \{1,2,3,4\} = \{(1,2), (3), (4)\}.
\]

\[
e_{tx} = \begin{cases}
\{1\ 2\ 5\} \{3\ 6\ 7\} & - \{1\ 2\ 5\} \{10\ 11\ 12\} \\
\{4\ 8\ 9\} \{10\ 11\ 12\} & - \{4\ 8\ 9\} \{3\ 6\ 7\}
\end{cases} + \begin{cases}
\{4\ 8\ 9\} \{10\ 11\ 12\} & - \{1\ 2\ 5\} \{10\ 11\ 12\} \\
\{1\ 2\ 5\} \{3\ 6\ 7\} & - \{4\ 8\ 9\} \{3\ 6\ 7\}
\end{cases}
\]

For \( g = (14)(567) \in S_4 \times S_{12} \)
\[
g e_{tx} = \begin{cases}
\{2\ 4\ 6\} \{3\ 5\ 7\} & - \{2\ 4\ 6\} \{10\ 11\ 12\} \\
\{1\ 8\ 9\} \{10\ 11\ 12\} & - \{1\ 8\ 9\} \{3\ 5\ 7\}
\end{cases} + \begin{cases}
\{1\ 8\ 9\} \{10\ 11\ 12\} & - \{2\ 4\ 6\} \{10\ 11\ 12\} \\
\{2\ 4\ 6\} \{3\ 5\ 7\} & - \{2\ 4\ 6\} \{3\ 5\ 7\}
\end{cases}
\]

### 3 Properties of \( U_{\nu,a} \)

Here we focus on \( k = b \) and especially on \( U_{\nu,a} = V^{(1^b)}_{\nu,a} \). Recall from the discussion after Definition 7 that \( U_{\nu,a} \) has basis \( \{ e_{tx} \mid X \in I^a_b \cap P(t^b), t \in \text{SYT}(\nu) \} \). To prove Theorem 1 we first deal with the \( a = 2 \) case and then connect it to the arbitrary \( a > 2 \) case.

**Definition 8.** Let \( t \) be a \( \nu \)-tableau and \( \tau \in \text{Sym}(\{b+1, \ldots, 2b\}) \). We define the ordered partition \( T(\tau) = (\{1, \tau(b+1)\}, \{2, \tau(b+2)\}, \ldots, \{b, \tau(2b)\}) \in H^2_b \cap P^2_{(1^b)} \).

It is clear that the set \( \{ T(\tau) \mid \tau \in \text{Sym}(\{b+1, \ldots, 2b\}) \} \) is a full set of representatives of the \( S_b \) orbits of \( H^2_b \cap P^2_{(1^b)} \). So \( \{ e_{t_{T(\tau)}} \mid \tau \in \text{Sym}(\{b+1, \ldots, 2b\}), t \in \text{SYT}(\nu) \} \) is a basis of \( U_{\nu,2} \).

**Lemma 9.** Let \( a, b \in \mathbb{N}, \nu \) a partition of \( b \). Then
\[
U_{\nu,a} \cong \text{Inf}_{S_b \times S_{n-b}}^S U_{\nu,2} \uparrow_{S_b \times S_{n-b}}
\]

(6)
Proof. Fix a $\nu$-tableau $t$. For $Y = \{Y_1, \ldots, Y_b\} \in H^a_{n} (\text{where the underlying set is } \{b+1, b+2, \ldots, n\})$ and $\sigma \in S_b$ let $Y_{\sigma}$ be the ordered partition $(\sigma(1)) \cup Y_1, \ldots, \{\sigma(b)\} \cup Y_b) \in H^a_{n}$. Note that $Y_{\sigma} \in H^a_{n}$ and $\sigma Y \in H^a_{n}$ are different. As is Definition $\Box$ for $\sigma = id$ put $T(Y) = Y_{id}$. Separating the smallest element of each part for $X \in P_{(1^b)}$ we see that $P_{(1^b)} = \{Y_{\sigma} \mid Y \in H^a_{n}, \sigma \in S_b\}$. Similarly, $P_{(1^b)} \cap I^n_a = \{T(Y) \mid Y \in H^a_{n-1}, \sigma \in S_b\}$. The module $U_{\nu, \alpha}$ is the $S_b \times S_{n-b}$-module generated by the $S_b \times S_{n-b}$-invariant set $\{e_{iX} \mid X \in P_{(1^b)}\} = \{e_{iY_{\sigma}} \mid Y \in H^a_{n-1}, \sigma \in S_b\}$ for any $t$ of shape $\nu$. Its basis is $\{e_{t_{(1)}} \mid Y \in H^a_{n-1}, t \in SYT(\nu)\}$.

Fix $Y \in H^a_{n-1}$ and with it a wreath product $S_{n-1} \triangleright S_b \leq S_{n-b}$. As before, for $g \in S_{n-1} \triangleright S_b$ the image in $S_b \cong S_{n-1} \triangleright S_b/S_{n-1}$ is denoted by $\tau_g$. Now for any $\sigma \in S_b$ and $g \in S_{n-1} \triangleright S_b$ we get $(gY)_{\sigma} = (\tau_g Y)_{\sigma} = \{\{\sigma(1)\} \cup Y_{\sigma(1)}, \ldots, \{\sigma(b)\} \cup Y_{\sigma(b)}\}$. Denote by $W$ the $S_b \times (S_{n-1} \triangleright S_b)$-module generated by the set $\{ge_{iY_{\sigma}} \mid \sigma \in S_b, g \in S_{n-1} \triangleright S_b\} = \{e_{t_{(1)}} \mid \tau, \sigma \in S_b\}$. Using the argument of Lemma $\Box$ we obtain that $W$ is the inflation of $U_{\nu, 2}$ to $S_b \times S_{n-1} \triangleright S_b$ and its basis is $\{e_{t_{(1)}} \mid \tau \in S_b, t \in SYT(\nu)\}$.

The argument of Lemma $\Box$ now provides

$$U_{\nu, \alpha} \cong W \uparrow^{S_b \times S_{n-b}},$$

(7)

because the induction takes place only in the second component. \qed

Thus the study of $U_{\nu, 2}$ might give some interesting information on the generalized Foulkes module.

Now we are ready to prove our main theorem.

Proof of Theorem $\Box$. Denote by $G_1 = \text{Sym}\{1, \ldots, b\}$ and $G_2 = \text{Sym}\{b+1, \ldots, 2b\}$, both isomorphic to $S_b$. Recall that a basis of $U = U_{\nu, 2}$ is $\{e_{t_{(1)}} \mid \tau \in G_2, t \in SYT(\nu)\}$.

We determine the values of the $G_1 \times G_2$-character $\chi$ of $U$ which then helps us to identify the decomposition of $U$ into irreducible modules.

Let $\chi'$ denote the irreducible character of the Specht module $S'$. We claim that for $(g_1, g_2) \in G_1 \times G_2$

$$\chi(g_1, g_2) = \begin{cases} 0, & \text{if } g_1, g_2 \text{ are of different cycle structure;} \\
|C_{S_b}(g_1)|\chi'(g_1), & \text{if } g_1, g_2 \text{ are of the same cycle structure.}
\end{cases}$$

Since the character value is the sum of the coefficients of basis element $e_{t_{(1)}}$ in $(g_1, g_2)e_{t_{(1)}}$ we need to compute them in order to prove the claim. Let $h = (1, b+1)(2, b+2) \cdots (b, 2b)$ and $g_3 = h g_1 h$ be the shifted permutation to $(b+1, \ldots, 2b)$, that is $g_3$ sends $b+l$ to $b+k$ if and only if $g_1$ sends $l$ to $k$. In particular, $g_1$ and $g_3$ have the same cycle structure. Clearly, $(g_1, g_2)e_{t_{(1)}} = e_{(g_1 t_{(1)} g_2 \cdot (g_3)^{-1})} = e_{s t_{(1)}}$ where $s = g_1 t$ and $g = g_2 \tau g_3^{-1} \in G_2$. Note that $s = g_1 t$ need not be standard so $e_{s t_{(1)}}$ might not be a basis element! However, if $g_1 e_t = e_s = \sum d_r e_r$ then

$$(g_1, g_2)e_{t_{(1)}} = e_{s t_{(1)}} = \sum d_r e_{r t_{(1)}}.$$  (8)

Therefore the coefficient of $e_{r t_{(1)}}$ in $e_{s t_{(1)}}$ is 0 unless $T(\tau) = T(g)$. In the latter case $\tau = \rho$. Hence $g_2 = \tau g_3^{-1}$ and $g_1 = h g_3 h$ must be conjugate, so of the same cycle structure.

From now on $g_1$ and $g_2$ be fixed and of the same cycle structure. For computing the character value we need to find the number of permutations $\tau$ and standard tableau $t$ such that $e_{t_{(1)}}$ is a linear summand of $(g_1, g_2)e_{t_{(1)}} = e_{s t_{(1)}}$. This can only happen if $e_t$ is a linear summand of $e_s = g_1 e_t$ and in that case $g = g_2 \tau g_3^{-1} = \tau$, in other words $\tau^{-1} g_2 = g_3$. The number of such $\tau$ is equal to the order of the centralizer $|C_{S_b}(g_3)| = |C_{S_b}(g_1)|$ which proves the claim.
Let $k_{g1} = b!/|C_{S_n}(g1)|$ denote the size of conjugacy class of $g1$. Then the multiplicity $d^\nu_{\mu,\lambda}$ of the simple module $S^\mu \times S^\lambda$ in $U$ can be expressed as the inner product of characters of $S_b \times S_b$:

$$d^\nu_{\mu,\lambda} = \langle \chi^\mu \times \chi^\lambda, \chi \rangle = \frac{1}{(b!)^2} \sum g1 \chi^\mu(g1) \cdot \chi^\lambda(g2) \cdot |C_{S_b}(g1)| \cdot \chi^\nu(g1) \cdot |C_{S_b}(g1)| \cdot \chi^\nu(g1)$$

$$= \frac{1}{b!^2} \sum g1 \chi^\mu(g1) \cdot \chi^\lambda(g1) \cdot |C_{S_b}(g1)| \cdot \chi^\nu(g1)$$

$$= \frac{1}{b!} \sum g1 \chi^\mu(g1) \cdot \chi^\lambda(g1) \cdot \chi^\nu(g1)$$

$$= c^\nu_{\mu,\lambda},$$

the so called Kronecker coefficient, the multiplicity of $S^\nu$ in the $S_b$-module $S^\mu \otimes S^\lambda$.

Thus,

$$U \cong \bigoplus_{\mu,\lambda} c^\nu_{\mu,\lambda} S^\mu \times S^\lambda,$$

which is equation 3 of Theorem 1. Equation 2 of Theorem 1 follows now from Lemma 9.

Corollaries 2 and 3 follow from the observations

$$c^{(b)}_{\lambda,\lambda} = c^{(b)}_{\lambda,\lambda} = 1$$

and

$$b! = \sum_{\lambda \vdash b} \text{deg}(S^\lambda \otimes S^\lambda) = \sum_{\lambda \vdash b} \text{deg}(S^\lambda \otimes S^\lambda) \leq \text{deg}(U_{(b),2}) = \text{deg}(U_{(b+1),2}) = b!.$$

### 4 Consequences for the Generalized Foulkes Module

**Lemma 10.** The multiplicity of $S^{(1^b)} \times S^\mu$ in $F^a_{\mu}$ is equal to the multiplicity of $S^{(1^b)} \times S^\mu$ in $U_{\mu,a}$.

**Proof.** We need prove that $V^\lambda_{\nu,a}$ of 1 has no summand $S^{(1^b)} \times S^\mu$ unless $\lambda = (1^b)$.

As above, the basis of $V^\lambda_{\nu,a}$ is the set $B = B^\lambda_{\nu,a} = \{ e_{tx} \mid X \in I \cap P_\lambda, t \in SYT(\nu) \}$. Observe that if $l_1$ and $l_2$ belong to the same part, say $X_k$, in $X = \{ X_1, X_2, \ldots, X_b \}$ then the transposition $(l_1 l_2)$ fixes $e_{tx}$, that is, $(l_1 l_2)e_{tx} = e_{tx}$.

By contradiction, suppose that $M$ is a submodule of $V^\lambda_{\nu,a}$ isomorphic to $S^{(1^b)} \times S^\mu$. Let $m \neq 0$ be an element of $M$. Then

$$m = \sum_{X \in I \cap P_\lambda, t \in SYT(\nu)} c_{tx} e_{tx}.$$

Choose $t$ and $Y$ such that $e_{ty} \neq 0$. As $\lambda \neq (1^b)$, there exist $l_1, l_2 \leq b$ such that $l_1, l_2$ belong to the same part of $Y$. Let $e_{sx} \in B \setminus \{ e_{ty} \}$ be arbitrary. If $l_1, l_2$ belong to the same part of $X$ then $(l_1 l_2)e_{sx} = e_{sx} \in B$. If $l_1, l_2$ are in different parts then $(l_1 l_2)e_{sx} = e_{s_{l_1 l_2}} = e_{s_Z}$ for some $Z \in P_\lambda$. However, this $Z$ and $Y$ cannot be in the same orbit. By Lemma 5, $e_{tx} = (l_1 l_2)e_{tx}$ is not a summand of $(l_1 l_2)e_{sx}$ in either case. Therefore the coefficient of $e_{ty}$ in $(l_1 l_2)m$ is also $c_{tx}$. However, $m \in M \cong S^{(1^b)} \times S^\mu$, so $(l_1 l_2)m = -m$, which implies that the coefficient of $e_{ty}$ is $-c_{ty} \neq c_{ty}$, a contradiction. 

□
Corollary 11. The multiplicity of \( S^{(1^b)} \times S^\mu \) in \( F^{a}_{\nu} \downarrow_{S_b \times S_a \times \nu} \) is the same as the multiplicity of \( S^\mu \) in \( F^{a-1}_{\nu} \).

Proof. By Lemma 10, the multiplicity of \( S^{(1^b)} \times S^\mu \) in \( F^{(a)}_{\nu} \downarrow_{S_b \times S_a \times \nu} \) is equal to its multiplicity in \( U_{\nu,a} \).

Let \( c_{\mu,\lambda}^{\nu} \) be the Kronecker coefficient of \( S^\nu \) in \( S^\mu \otimes S^\lambda \). By Theorem 1

\[
U_{\nu,a} = \bigoplus_{\mu,\lambda} c_{\mu,\lambda}^{\nu} S^\mu \times F^{a-1}_{\lambda}.
\]

Moreover \( c_{\mu,\lambda}^{\nu} = c_{\lambda,\mu}^{\nu} = c_{\mu,\nu}^{\lambda} \). Therefore \( c_{(1^b)\mu}^{\nu} = 0 \) unless \( \mu = \nu \), in which case \( c_{(1^b)\mu}^{\nu} = 1 \).

From the above discussion we get that \( S^{(1^b)} \times S^\mu \) can only be embedded in \( S^{(1^b)} \times F^{a-1}_{\nu} \) among the above summands of \( U_{\nu,a} \).

For a partition \( \mu \vdash n \) of length \( k \) and the Young subgroup \( S_\mu = S_{\mu_1} \times S_{\mu_2} \times \ldots \times S_{\mu_k} \), we define \( M^\mu \) as the permutation module \( 1_{S_\mu} \uparrow^{S_n} \). In particular, \( M^{(n)} \) is the regular module and \( M^{(n)} \) is the trivial module. The following lemma is specific for the regular module.

Lemma 12. Let \( a, b \in \mathbb{N} \). Then

\[
\text{Inf}_{S_b} S^a M^{(1^b)} \uparrow S_{ab} \cong M^{(a^b)}.
\]

Proof. A basis of \( M^\mu \) is the set \( \{ \{ t \} \mid t \) is a \( \mu \)-tableau.\}. Choose \( X \in I \), then a basis of \( \text{Inf}_{S_b} S^a M^{(1^b)} \) is the set \( \{ \{ tX \} \mid t \) is a \( (1^b) \)-tableau.\}. Therefore a basis for \( \text{Inf}_{S_b} S^a M^{(1^b)} \uparrow S_{ab} \) is the set \( \{ \{ tX \} \mid t \) is a \( (1^b) \)-tableau, \( X \in I \) which is also a basis for \( M^{(a^b)} \).

Now we are ready to derive the following corollary.

Corollary 13. Let \( \mu' = \mu + (1^b) \). The multiplicity of \( S^{\mu'} \) in \( F^{a+1}_{\nu} \) is the same as the multiplicity of \( S^\mu \) in \( F^{a}_{\nu} \).

Proof. From Corollary 11, we know that the multiplicity of \( S^{(1^b)} \times S^\mu \) in \( F^{a}_{\nu} \downarrow_{S_b \times S_a \times \nu} \) is the same as the multiplicity of \( S^\mu \) in \( F^{a}_{\nu} \). We also know that \( S^\nu \) embeds into the regular module \( M^{(1^b)} \). Therefore

\[
\text{Inf}_{S_b} S^\nu \text{ embeds into } \text{Inf}_{S_b} S^a M^{(1^b)}.
\]

By the definition of the generalized Foulkes module and by Lemma 12

\[
F^{a}_{\nu} = \text{Inf}_{S_b} S^\nu \uparrow S_{ab} \text{ embeds into } \text{Inf}_{S_b} S^a M^{(1^b)} \uparrow S_{ab} \cong M^{(a^b)}.
\]

Since for all simple constituents \( S^\lambda \) of \( M^{(a^b)} \), \( \lambda \) has at most \( b \) parts, this also holds for \( F^{a}_{\nu} \). By the Littlewood-Richardson Principle, [8, Theorem 16.4], all the constituents of \( S^{(1^b)} \times S^\mu \uparrow S^{(a+1)n} \) have more than \( b \) parts except for \( S^{\mu+1(1^b)} \) which occurs with multiplicity 1.

A generalized form of Corollary 13 has been proven recently by de Boeck, Paget and Wildon [10]. Their technique is much different.
5 Concluding remarks and questions

There have been many advances in the study of the Generalized Foulkes Module. One such result is given by Paget and Wildon [11]. They give a description of minimal and maximal Specht modules with respect to the dominance order on partitions.

Another result related to the Kronecker coefficients is given by Ikenmeyer, Mulmuley and Walter [12]. They proved that deciding whether a Kronecker coefficient is zero is an NP-hard problem. Bürgisser and Ikenmeyer [13] showed that computing Kronecker coefficients is $\#$P-hard. Theorem 1 gives a weak relationship between Kronecker coefficients and Foulkes modules which makes us wonder whether similar statements can be said for the multiplicity of $S^\lambda$ in $F^a_\nu$.

Question 14. What is the computational complexity of the coefficient of $S^\lambda$ in $F^a_\nu$?

There are many approaches to study the properties of $F^a_\nu$ but the problem of its decomposition into Specht modules still remains widely open. Studying the properties of the Generalized Foulkes module restricted to some small and large subgroups $G \leq S_n$, $F^a_\nu \downarrow_G$, might give some interesting information on the properties of $F^a_\nu$ and in turn help us understand its decomposition.

The generalized Foulkes module $F^a_\nu$ can be further generalized using a parameter $\lambda \vdash a$ to obtain $F^\lambda_\nu$. A description of such a generalization is given in [11].

Question 15. Find the analogue of (1) for $F^\lambda_\nu \downarrow_{S_k \times S_{n-k}}$. Ideally, there should exist a distinguished component of $F^\lambda_\nu \downarrow_{S_k \times S_{n-k}}$ with description similar to the one in Theorem 1.

In fact, the decomposition (1) of the restricted generalised Foulkes module is the Mackey decomposition, see [9, Lemma 8.7]. For more on this — for the classical Foulkes module — we refer to Definition 2.9 in [14], the proof of Theorem 6.3 and the discussion before Lemma 6.8 in [15].

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References

[1] R. M. Thrall. On symmetrized Kronecker powers and the structure of the free Lie ring. Amer. J. Math., 64:377–388, 1942.

[2] H.O. Foulkes. Concomitants of the Quintic and Sextic Up To Degree Four in the Coefficients of the Ground Form. Journal of the London Mathematical Society, s1-25:205–209, 1950.
[3] S.C. Dent. On a Conjecture of Foulkes. *Journal of Algebra*, 226:236–249, 2000.

[4] T. McKay. On Plethysm conjectures of Stanley and Foulkes. *Journal of Algebra*, 319:2050–2071, 2008.

[5] J. Müller and M. Neunhöffer. Some computations regarding Foulkes’ Conjecture. *Experiment. Math.*, 14:277–283, 2005.

[6] M.W. Cheung, C. Ikenmeyer, S. Mkrtchyan. Symmetrizing tableaux and the 5th case of the Foulkes conjecture. *Journal of Symbolic Computation*, 80:833–843, 2017.

[7] M. de Boeck. On the structure of Foulkes modules for the symmetric group. *Doctor of Philosophy (PHD) Thesis, University of Kent*, 2015.

[8] G.D. James. *Representation Theory of The Symmetry Groups*. Springer-Verlag, 1978.

[9] J.L. Alperin. *Local Representation Theory*. Cambridge University Press, 1986.

[10] M. de Boeck and R. Paget and M. Wildon. Plethysms of symmetric functions and highest weight representations. *Transactions of the American Mathematical Society*, 2021.

[11] R. Paget and M. Wildon. Generalized Foulkes modules and maximal and minimal constituents of plethysms of Schur functions. *Proc. London Math.*, 118:1153–1187, 2019.

[12] C. Ikenmeyer, K. D. Mulmuley and M. Walter. On vanishing of Kronecker coefficients. *Computational Complecity*, 26:949–992, 2017.

[13] P. Bürgisser and C. Ikenmeyer. The complexity of computing Kronecker coefficients. *FPSAC 2008*, pages 357–368, 2008.

[14] E. Giannelli. On the decomposition of the Foulkes module. *Archiv der Mathematik*, 100 (3):201–214, 2013.

[15] R. M. Adin and P. Hegedüs and Y. Roichman. Higher Lie characters and cyclic descent extension on conjugacy classes. *Algebraic Combinatorics*, 6(6):1557–1591, 2023.