GENERALIZED EDGEWISE SUBDIVISIONS

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Abstract. In this paper we classify endofunctors on the simplex category, and we identify those that induce weak equivalence preserving functors on the category of simplicial sets.

1. Introduction

A generalized edgewise subdivision functor is a weak equivalence preserving endofunctor on the category of simplicial sets that is induced by an endofunctor on the simplex category. In many cases, it takes a general simplicial set and returns a nicer one. Perhaps the most famous generalized edgewise subdivision functor is Segal’s subdivision [4].

In this paper we classify all generalized edgewise subdivision functors. Segal’s edgewise subdivision functor is the endofunctor on the category of simplicial sets SSet induced by the concatenation of the opposite and the identity endofunctors on the simplex category Δ. Segal’s edgewise subdivision is used, for example, to relate Quillen’s Q-construction to Waldhausen’s S-construction. [5]

In Section 2 we recall the definitions of simplex category and the interval category and we prove that the two categories are dual.

In Section 3 we classify the endofunctors on the simplex category by studying the functors that they induce on the category of simplicial sets. The main idea of the classification is that an endofunctor $T^*$ on SSet that is induced by an endofunctor $T$ on $\Delta$ is completely determined by its value on the simplicial set $\Delta^1$. The duality between the simplex category and the interval category allows us to regard $T^*(\Delta^1)$ as a simplicial interval. A specific interpretation of the structure of this simplicial interval gives the desired classification.

The main result of the paper is proved in Section 4, where we classify the endofunctors on the simplex category that induce weak equivalence preserving functors on the category of simplicial sets. This answers a question posed to us by Clark Barwick.

2. Simplex category an simplicial sets

In this section we introduce basic definitions and notations that will be used in the paper. We show that the simplex category and the interval category are dual.
**Definition 2.1.** The *simplex category* $\Delta$ has as objects totally ordered nonempty finite sets and as morphism order preserving maps.

**Definition 2.2.** The *interval category* $I$ has as objects totally ordered finite sets with at least two elements and as morphisms order preserving and endpoint preserving maps.

Let define $[n]$ and $\{m\}$ to be the following objects:

$[n] := \{0, 1, \ldots, n\} \in \Delta$

$\{m\} := \{0, 1, \ldots, m\} \in I$.

In his paper [3] Joyal proves that the simplex category and the interval category are dual. For future use let us give a short proof here.

**Theorem 2.3.** The simplex category $\Delta$ and the interval category $I$ are dual.

**Proof.** We will prove the result by constructing explicitly the duality between the two categories.

Let $G : I \to Set$ be the functor $Hom_I(-, \{1\})$. Note that for each $n$, the set $Hom_I(\{n\}, \{1\})$ is non-empty. We will show that there is a canonical ordering of this set, such that it can be regarded as an element of $\Delta$, and we will show that with this ordering we can think of $G$ as a functor from $I$ to $\Delta$.

We can define an ordering of the set $Hom_I(\{n\}, \{1\})$ as follows: if $f_1, f_2 : \{n\} \to \{1\}$ are two different maps, we declare that $f_1 < f_2$ if and only if there exists $j$, such that $f_1(j) < f_2(j)$. Regarded as a functor from $I$ to $\Delta$, $G$ is defined on objects by $G(\{n\}) = [n-1]$.

Let $g : \{n\} \to \{m\}$ be a morphism in $I$. Given the ordering of the hom sets described above, $G(g) : [m-1] \to [n-1]$ is the following morphism:

$G(g)(i) = [g^{-1}\{0, 1, \ldots, i\}]$.

Note that $i \leq G(g)(j)$ if and only if $g(i) \leq j$. Therefore, $G(g)$ is well defined order preserving map.

In the other direction, let $F : \Delta \to Set$ be the functor $Hom_\Delta(-, [1])$. Using the same idea as before, we see that for each $[n]$, there is a canonical ordering of the set $Hom_\Delta([n], [1])$ (which must have two end points - the constant map at 0 and the constant map at 1), so that we can regard it as an element of $I$, and all of the morphisms induced by $F$ will be morphisms that preserve this ordering and the end points. This will allow us to think of $F$ as a functor from $\Delta$ to $I$.

The functors $F$ and $G$ form a quasi inverse pair. It is evident from the definitions of $F$ and $G$ that both compositions $FG$ and $GF$ are identity on objects in $I$ and $\Delta$ respectively. Let $g : \{n\} \to \{m\}$. Since $g(i) \leq g(i)$, one has $i \leq G(g)(i)$ and $FG(g)(i) \leq g(i)$. Therefore $FG(g) = g$. A similar argument shows that $GF(f) = f$. □
Notation 2.4. We write $SSet$ for the category of simplicial sets, i.e. the category of functors
\[ \Delta^{op} \to \text{Set}. \]

3. Endofunctors on the simplex category

In this section we classify the endofunctors on the simplex category $\Delta$. Note that a functor $T : \Delta \to \Delta$ induces a functor $T^* : SSet \to SSet$ that sends each simplicial set $X$ to the composition $X \circ T$.

Example 3.1. We will refer to the following functors as the basis endofunctors $\Delta \to \Delta$.

- The identity functor $Id$, that is identity on both objects and morphisms;
- The constant 0 functor $C_0$ that sends all objects of $\Delta$ to the object $[0]$, and all morphisms to the identity $[0]$ morphism.
- The opposite functor $Op$, that is identity on objects, and if $f : [n] \to [m]$ is a morphism, than $Op(f) : [n] \to [m]$ is the following morphism:
  \[ Op(f)(k) = m - f(n - k). \]

Definition 3.2. The concatenation bifunctor $* : \Delta \times \Delta \to \Delta$ is defined as follows:

- On objects $*(n, m) = [n] * [m] = [n + m + 1]$;
- On morphisms $(f, g) : ([n_1], [m_1]) \to ([n_2], [m_2])$ we have that
  \[ *(f, g) = f * g : [n_1 + m_1 + 1] \to [n_2 + m_2 + 1] \]
  is the following map:
  \[ (f * g)(k) = \begin{cases} f(k) & \text{if } 0 \leq k \leq n_1 \\ g(k) & \text{if } n_1 < k \leq n_1 + m_1 + 1. \end{cases} \]

The concatenation bifunctor can be used to define an operation ‘+’ between endofunctors on $\Delta$ in the following way:

Definition 3.3. Let $T_1$ and $T_2$ be endofunctors on the simplex category. The functor $[T_1 + T_2 : \Delta \to \Delta]$ is defined to be the following composition:

\[ \Delta^{T_1 \times T_2} \Delta \times \Delta \to \Delta. \]

Let $\circ$ denote the standard composition of functors. The strange notation ‘+’ is justified by the following lemma, whose proof is easy.

Lemma 3.4. Let $T_1$, $T_2$ and $T_3$ be endofunctors on $\Delta$, then

\[ (T_1 + T_2) \circ T_3 = (T_1 \circ T_3) + (T_2 \circ T_3). \]
In the Main Theorem of this section, Theorem 3.12, we will show that all endofunctors on $\Delta$ can be represented as a sum under ‘+’ of the basis functors defined in Example 3.1. We classify the endofunctors on $\Delta$ by studying the functors that they induce on $SSet$. Below we will prove that any functor $T^* : SSet \to SSet$ that is induced by a functor $T : \Delta \to \Delta$, is determined by its value on the simplicial set $\Delta^1$. Then, we will examine the relation between $T_1^*(\Delta^1)$, $T_2^*(\Delta^1)$ and $(T_1 + T_2)^*(\Delta^1)$.

Recall from Section 2 that the categories $\Delta$ and $I$ are dual. Let $[L : I \to Set]$ be the forgetful functor. Since $\Delta^1$ is naturally the functor $F : \Delta \to I$, as constructed in the proof of 2.3. It follows that $T^*(\Delta^1) = L \circ F \circ T$. Therefore, we can regard $T^*(\Delta^1)$ as a simplicial interval with ordering of the vertices.

Moreover, using the functors $F$ and $G$, as defined in the proof of Theorem 2.3, we see that the maps:

\[
\begin{array}{cccc}
Fun(\Delta^{op}, \Delta^{op}) & \longrightarrow & Fun(\Delta^{op}, I) & \longrightarrow & Fun(\Delta^{op}, \Delta^{op}) \\
T & \Uparrow & F \circ T & \Uparrow & G \circ F \circ T \simeq T \\
\end{array}
\]

defines an equivalence of categories. Let $ev : Fun(SSet, SSet) \to Fun(\Delta^{op}, Set)$ be the evaluation functor at $\Delta^1$. It factors through $Fun(\Delta^{op}, I)$, as explained above. Therefore we have that the following composition is injective

\[
\begin{array}{cccc}
Fun(\Delta^{op}, \Delta^{op}) & \longrightarrow & Fun(SSet, SSet) & \longrightarrow & Fun(\Delta^{op}, I) \\
T & \Uparrow & T^* & \Uparrow & T^*(\Delta^1) = F \circ T \\
\end{array}
\]

We see that an endofunctor $T^* : SSet \to SSet$ induced by a functor $T : \Delta \to \Delta$ can be recovered from its value at $\Delta^1$, by first recovering the functor $T = G(T^*(\Delta^1))$. We are thus reduced to classifying the endofunctors on the simplex category by studying the structure of the simplicial interval $T^*(\Delta^1)$.

**Observation 3.5.** Let $T_1$ and $T_2$ be endofunctors on $\Delta$. The simplicial interval induced by $(T_1 + T_2)$ is the following composition:

\[
\Delta \xrightarrow{T_1 \times T_2} \Delta \times \Delta \xrightarrow{\Delta} \Delta \xrightarrow{Hom_{\Delta}(-,[1])} Set.
\]

Let $[n]$ be an object in $\Delta$, such that $T_1([n]) = [\alpha]$ and $T_2([n]) = [\beta]$. By construction $(T_1 + T_2)([n]) = [\alpha + \beta + 1]$. Hence,

\[
[T_1^*(\Delta^1)]_n = Hom_{\Delta}([\alpha], [1]) \\
[T_2^*(\Delta^1)]_n = Hom_{\Delta}([\beta], [1])
\]
The maps in $\text{Hom}_\Delta([\alpha + \beta + 1], [1])$ are formed by concatenating the constant 0 map from $\text{Hom}_\Delta([\alpha], [1])$ with any map from $\text{Hom}_\Delta([\beta], [1])$, or by concatenating any map from $\text{Hom}_\Delta([\alpha], [1])$ with the constant 1 map from $\text{Hom}_\Delta([\beta], [1])$. Note that in this description the concatenation of the constant 0 map from $\text{Hom}_\Delta([\alpha], [1])$ with the constant 1 map from $\text{Hom}_\Delta([\beta], [1])$ appears twice. It follows that for any $n$, the $n$-th simplices of $(T_1 + T_2)^*(\Delta^1)$ are the union of the $n$-th simplices of $T_1^*(\Delta^1)$ and $T_2^*(\Delta^1)$, where the last simplex of $T_1^*(\Delta^1)$ is identified with the first simplex of $T_2^*(\Delta^1)$.

We conclude the following lemma:

**Lemma 3.6.** If $T_1$ and $T_2$ are endofunctors on $\Delta$, then

$$(T_1 + T_2)^*(\Delta^1) = T_1^*(\Delta^1) \vee T_2^*(\Delta^1).$$

**Notation 3.7.** Let $T$ be an endofunctor on $\Delta$ and $[n] \in \Delta$, then

$$T(n) := \#T([n]) - 1.$$

**Lemma 3.8.** Let $T$ be an endofunctor on $\Delta$ that is not a constant functor. For all objects $[k] \in \Delta$

$$k \leq T(k).$$

**Proof.** For the sake of contradiction, assume that $T(k) < k$ for some positive integer $k$. Without loss of generality, assume that $k$ is the smallest integer such that $T(k) < k$. Obviously, $k \neq 0$. Note that $T(k) < T(k - 1)$, since by assumption $T(k - 1) > k - 1$ and $T(k) < k$. Therefore, the maps $T(d_k) : T([k - 1]) \to T([k])$ is not injective. But this is impossible since $T(d_k) \circ T(s_k) = id$. This contradicts the assumption that $T(k) < k$. \(\square\)

**Corollary 3.9.** For any $k \geq 2$, the simplicial set $T^*(\Delta^1)$ has no non-degenerate $k$-simplices.

**Proof.** If $T$ is a constant functor, the statement of the corollary is obvious. Suppose that $T$ is not a constant functor. By 3.8 $k \leq T(k)$ for all $[k] \in \Delta$. Therefore, all maps $T^*(\Delta^1)_k \to T^*(\Delta^1)_{k-1}$ are surjective. Hence, $T^*(\Delta^1)$ has no non-degenerate $k$-simplices for $k \geq 2$. \(\square\)

**Lemma 3.10.** Let $T : \Delta \to \Delta$, such that $T([0]) = [n]$. The simplicial interval induced by $T$ is a pointed union of $n - 1$ elements among

$$\Delta^0 \sqcup \Delta^0, \Delta^1, \text{and } [\Delta^1]^{op}.$$

**Proof.** By Corollary 3.9, the simplicial set $T^*(\Delta^1)$ has no nondegenerate $k$-simplices for $k \geq 2$. Since $T^*(\Delta^1)$ has an interval structure, there is at most one directed edge between two consecutive 0-simplices.

Suppose that $n = 0$. Then $T^*(\Delta^1)$ has two 0-simplices. If there is no edge between them, $T^*(\Delta^1) = \Delta^0 \sqcup \Delta^0$. If there is an edge from the first 0-simplex to the second 0-simplex, then $T^*(\Delta^1) = \Delta^1$. If there is an edge from the second 0-simplex to the first 0-simplex, then $T^*(\Delta^1) = [\Delta^1]^{op}$.\(\square\)
Similarly, for any \( n > 0 \), \( T^*(\Delta^1) \) has \( n + 2 \) ordered 0-simplices such that there is at most one directed edge between any two consecutive simplices. Therefore, \( T^*(\Delta^1) \) is a pointed union of \( n - 1 \) elements among \( \Delta^0 \sqcup \Delta^0, \Delta^1 \) and \( [\Delta^1]^{op} \). Each of them connects two consecutive simplices. □

**Corollary 3.11.** The only endofunctors on \( \Delta \) that send the object \([0]\) to itself are the basis functors.

**Proof.** Let \( T : \Delta \to \Delta \) be a functor, such that \( T([0]) = [0] \). By the previous lemma \( T^*(\Delta^1) \) is one of the following:

\[
\Delta^0 \sqcup \Delta^0, \Delta^1, \text{ or } [\Delta^1]^{op}.
\]

If \( T^*(\Delta^1) = \Delta^0 \sqcup \Delta^0 \), then \( T = C_0 \). If \( T^*(\Delta^1) = \Delta^1 \), then \( T = Id \). If \( T^*(\Delta^1) = [\Delta^1]^{op} \), then \( T = Op \). □

**Theorem 3.12.** Let \( T \) be an endofunctor on the simplex category such that \( T([0]) = [n] \). Than \( T \) is a sum of \( n + 1 \) of the basis functors.

**Proof.** Consider the induced functor \( T^* : SSet \to SSet \). As we have shown, it is completely determined by its value on the simplicial set \( \Delta^1 \). By assumption \( T([0]) = [n] \). Hence, it is a wedge of \( n + 1 \) of the following: \( \Delta^0 \sqcup \Delta^0, \Delta^1 \) and \( [\Delta^1]^{op} \). Recall that

\[
(T_i + T_j)^*(\Delta^1) = T_i^*(\Delta^1) \vee T_j^*(\Delta^1).
\]

By the previous lemma the only functors that sends \([0]\) to \([0]\) are the basis functors. We have that \( Id(\Delta^1) = \Delta^1 \), \( Op(\Delta^1) = (\Delta^1)^{op} \) and \( C_0(\Delta^1) = \Delta^0 \sqcup \Delta^0 \). We conclude that \( T \) is the sum of \( n + 1 \) of the basis functors. □

### 4. Generalized edgewise subdivisions

In Section 3 we proved that each endofunctor on the simplex category can be expressed as a finite sum of the three basis functors. In this section we study the induced functors on the category of simplicial sets. Now let us classify the endofunctors on \( SSet \) that are induced by endofunctors on \( \Delta \) and preserve weak equivalences. Segal’s edgewise subdivision is one example of such endofuntor.

**Definition 4.1.** Let \( E \) denotes the functor \( Op + Id \). Segal’s edgewise subdivision of a simplicial set \( X \) is the simplicial set \( E(X) \) [3].

Segal’s edgewise subdivision of \( \Delta^1 \) and \( \Delta^2 \) are shown in Figure 1 and Figure 2.

![Figure 1. Segal’s edgewise subdivision of \( \Delta^1 \)](image-url)
Obviously, the constant 0 functor does not induce a functor on $SSet$ that is weak equivalence preserving, while the identity and the opposite functors do. The main result of the section is that a functor on $\Delta$ induces a weak equivalence preserving functor on $SSet$ if and only if it does not contain the constant 0 functor in its representation as a sum of the basis functors.

In his paper [1] C. Barwick proves that a functor $T^*$ on the category of simplicial sets preserves weak equivalences (and in fact is a left Quillen functor) if and only if it carries the standard $n$-simplex to a weakly contractible simplicial set.

We show that the category $[n]$ can be embedded in the category $[1]^n$ by a canonical functor $\eta : [n] \rightarrow [1]^n$, such that there exists a collapsing functor $\mu : [1]^n \rightarrow [n]$ and $\mu \circ \eta = id$.

**Construction 4.2.** Let $\eta : [n] \rightarrow [1]^n$ be define on objects as follows

$$\eta(m) = \begin{cases} 0 & \text{for } i = m+1 \\ 1 & \text{for } i \neq m+1 \end{cases}$$

If $\beta : i \rightarrow i + 1$ is the unique morphism from $i$ to $i + 1$ in $[n]$, then $\eta(\beta)$ is the unique morphism from $0\ldots01\ldots1$ to $0\ldots01\ldots1$ in $[1]^n$.

Suppose that $v$ is an object in $[1]^n$. Then $v$ is a sequence of 0-s and 1-s of length $n$. Let $v_i$ denotes the $i$-th term of this sequence. Let $\mu : [1]^n \rightarrow [n]$ be define on objects as follows

$$\mu(v) = n - \min_{v_i=1} i + 1.$$ 

Note that in both of the categories $[n]$ and $[1]^n$ if there is a morphism between two objects, it is unique. Therefore, the values of $\mu$ on morphisms in $[1]^n$ is determined by its values on objects.

It is evident that $\eta$ is a faithful functor and $\mu$ is a full functor. Moreover, it is easy to check that $\mu \circ \eta$ is the identity functor on $[n]$.

With this, we construct natural transformations $\Phi : \Delta^n \rightarrow [\Delta^1]^n$ and $\Psi : [\Delta^1]^n \rightarrow \Delta^1$ using the functors $\eta$ and $\mu$. 

![Figure 2. Segal’s edgewise subdivision of $\Delta^2$](image)
Construction 4.3. First, we construct the map $\Phi : \Delta^n \to [\Delta^1]^n$. Recall that $\Delta^n$ is the functor $\text{Hom}_\Delta(-,[n])$ and $\Delta^1$ is the functor $\text{Hom}_\Delta(-,[1])$. Note that giving $n$ maps $[m] \to [1]$ is the same as giving one map $[m] \to [1]^n$. Therefore $\text{Hom}_\Delta([m],[1])^n = \text{Hom}_{\text{Cat}}([m],[1]^n)$, where $\text{Cat}$ is the category of small categories. The components of the natural transformation $\Phi : \Delta^n \to [\Delta^1]^n$ are maps

$$\Phi_m : \text{Hom}_\Delta([m],[n]) \to \text{Hom}_\Delta([m],[1])^n = \text{Hom}_{\text{Cat}}([m],[1]^n),$$

$$f \mapsto \eta \circ f.$$ 

Similarly, the components of the natural transformation $\Psi : [\Delta^1]^n \to \Delta^n$ are maps

$$\Psi_m : \text{Hom}_\Delta([m],[1]^n) \to \text{Hom}_\Delta([m],[n]),$$

$$g \mapsto \mu \circ g.$$ 

The naturality of $\Phi$ and $\Psi$ is easy to check. Note that since $\mu \circ \eta = \text{id}$ by construction, then $\Phi \circ \Psi = \text{id}$.

Next, we prove the following stronger result:

Lemma 4.4. The induced endofunctor $T^* : SSet \to SSet$ on the category of simplicial sets preserves weak equivalences if and only if it carries the standard one-simplex $\Delta^1$ to a weakly contractible simplicial set.

Proof. Given the result in [1], it is enough to show that the map $T^*(\Phi) : T^*(\Delta^n) \to [T^*(\Delta^1)]^n$ is a weak equivalence, where $\Phi$ is the map constructed in 4.3. Note that $\Delta^n$ and $[\Delta^1]^n$ are the nerves of the categories $[n]$ and $[1]^n$ respectively. Hence, it is enough to show that the maps $\eta$ and $\mu$ form an inverse pair. By construction $\mu \circ \eta$ is the identity functor on $[n]$. Note that for all objects $\alpha \in [1]^n$ there is a unique morphism

$$\Gamma_\alpha : \underbrace{0 \ldots 0}_{n-k} \underbrace{1 \ldots 1}_{k} \to \underbrace{0 \ldots 0}_{n-k} \underbrace{1 \ldots 1}_{k}$$

The collection of the $\Gamma_\alpha$ maps defines a natural transformation $\Gamma$ from the identity functor on $[1]^n$ to the functor $\eta \circ \mu : [1]^n \to [1]^n$. We conclude that the map $\Psi$ is weak equivalence with homotopy inverse $\Phi$, as constructed in 4.3.

The maps $\Phi$ and $\Psi$ induce the following maps:

$$T^*(\Delta^1)^n \to T^*(\Delta^n) \to T^*(\Delta^1)^n$$

We will show that $T^*(\Phi) \circ T^*(\Psi)$ is hopotopic to the identity. We have proved that there exist homotopy from $\Phi \circ \Psi$ to the $\text{id}$ given by:

$$f \mapsto \eta \circ f.$$
Note that $T^*$ preserves products, therefore the above diagram becomes:

$$
\begin{array}{c}
\text{Lemma 4.5.} \\
\text{The sum of two weak equivalence preserving endofunctors on } SSet \text{ is a weak equivalence preserving functor.} \\
\text{Proof. By Lemma 4.4, it is enough to observe that if } T_1^*(\Delta^1) \text{ and } T_2^*(\Delta^1) \text{ are weakly contractible, then so is } \left( T_1 + T_2 \right)^*(\Delta^1) \simeq T_1^*(\Delta^1) \vee T_2^*(\Delta^1). \\
\text{We now conclude:} \\
\textbf{Theorem 4.6.} The functor } T^* : SSet \to SSet \text{ preserves weak equivalences if and only if the constant 0 functor does not appear in the representation of } T \text{ as a finite sum of the basis functors.} \\
\text{In the rest of section we consider some examples of the generalized edge-wise subdivisions of simplicial sets.} \\
\textbf{Example 4.7.} Segal’s edgewise subdivision can be applied more than once to a given simplicial sets. Let } E^n \text{ denote the endofunctor on } SSet \text{ that is obtained by applying the Segal’s edgewise subdivision } n \text{ times. This functor is induced by the following endofunctor on } \Delta \\
\sum_{i=1}^{n} Op + Id \\
\text{The structure of } E^2(\Delta^2) \text{ follows.}
\end{array}
$$
Example 4.8. The opposite functor $Op$ is weak equivalence preserving. Note that $Op$ does not subdivide the simplices, but it reverses the orientation of the arrows.

Example 4.9. Let denote the endofunctor on $SSet$, that is induced by the endofunctor $Id + Id$ on $\Delta$ by $ID^2$. By Theorem 4.5 it is also weak equivalence preserving. The structure of $ID^2(\Delta^2)$ is given in Figure 4.

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