COMPARISON OF FUNDAMENTAL GROUP SCHEMES OF A PROJECTIVE VARIETY AND AN AMPLE HYPERSURFACE

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Abstract. Let $X$ be a smooth projective variety defined over an algebraically closed field, and let $L$ be an ample line bundle over $X$. We prove that for any smooth hypersurface $D$ on $X$ in the complete linear system $|L^\otimes d|$, the inclusion map $D \hookrightarrow X$ induces an isomorphism of fundamental group schemes, provided $d$ is sufficiently large and $\dim X \geq 3$. If $\dim X = 2$, and $d$ is sufficiently large, then the induced homomorphism of fundamental group schemes remains surjective. We give an example to show that the homomorphism of fundamental group schemes induced by the inclusion map of a reduced ample curve in a smooth projective surface is not surjective in general.

1. Introduction

Let $X$ be a projective variety defined over an algebraically closed field $k$. Let $D$ be a reduced ample hypersurface in $X$. If $X$ is normal, and $\dim X \geq 2$, then it is known that the induced homomorphism of étale fundamental groups

$$\pi_1(D, x_0) \rightarrow \pi_1(X, x_0)$$

is surjective; here $x_0$ is any $k$-rational point of $D$. If $X$ is smooth with $\dim X \geq 3$, then from Grothendieck’s Lefschetz theory it follows that the above homomorphism of étale fundamental groups is an isomorphism (see [Gr, Exposé X]).

When the characteristic of $k$ is positive, Nori constructed an invariant of $X$ which is finer than the étale fundamental group $\pi_1(X, x_0)$ [No1]. He calls this invariant the fundamental group scheme of $X$.

To describe the fundamental group scheme, we first recall that the étale fundamental group $\pi_1(X, x_0)$ coincides with group scheme associated to the Tannakian category defined by all vector bundles $E$ over $X$ with the following property: there is a finite étale Galois cover $Y$ of $X$ such that the pull back of $E$ to $Y$ is trivializable. The fiber functor for this Tannakian category sends $E$ to its fiber $E_{x_0}$. The fundamental group scheme is the group scheme associated to the Tannakian category defined by all vector bundles $E$ over $X$ with the following property: there is a principal bundle $Y \rightarrow X$ over $X$ with a finite group scheme as the structure group scheme such that the pull back of $E$ to $Y$ is trivializable. Note that if the characteristic of the base field $k$ is zero, then the fundamental group scheme coincides with the étale fundamental group.

The fundamental group scheme of $X$ with $x_0$ as the base point is denoted by $\pi(X, x_0)$. Comparing the above definitions of $\pi_1(X, x_0)$ and $\pi(X, x_0)$ it follows that there is a canonical surjective (faithfully flat) homomorphism from $\pi(X, x_0)$ to $\pi_1(X, x_0)$. The kernel of this surjective homomorphism $\pi(X, x_0) \rightarrow \pi_1(X, x_0)$ is a local group scheme.
Although in the case of positive characteristics the group scheme \( \pi(X, x_0) \) can be larger than \( \pi_1(X, x_0) \), the fundamental group scheme continues to share some of the basic properties of the étale fundamental group. Our aim here is to investigate the relationship between \( \pi(D, x_0) \) and \( \pi(X, x_0) \), where \( D \subset X \) is an ample hypersurface.

We first give an example of a pair \((X, D)\), where \( X \) is a smooth projective surface and \( D \) a reduced ample curve in \( X \), such that the natural homomorphism

\[
\pi(D, x_0) \longrightarrow \pi(X, x_0)
\]

is not surjective. This example is based on the fact that the Kodaira vanishing theorem may fail in positive characteristics. More precisely, given an ample hypersurface \( D \hookrightarrow X \) with \( \dim X \geq 2 \), the induced homomorphism

\[
\pi(D, x_0) \longrightarrow \pi(X, x_0)
\]

fails to be surjective whenever \( H^1(X, \mathcal{O}_X(-D)) \neq 0 \).

On the other hand, the following theorem shows that the above homomorphism between fundamental group schemes behaves like the homomorphism between étale fundamental groups if \( D \) is sufficiently positive.

**Theorem 1.1.** Let \( X \) be a smooth projective variety defined over an algebraically closed field and \( L \) an ample line bundle over \( X \).

1. Assume that \( \dim X \geq 2 \). There is an integer \( d_1(X, L) \) such that for any smooth divisor \( D \in |L^{\otimes d}| \), where \( d > d_1(X, L) \), the natural homomorphism between fundamental group schemes

\[
\rho : \pi(D, x_0) \longrightarrow \pi(X, x_0)
\]

is surjective (faithfully flat), where \( x_0 \) is any \( k \)-rational point of \( D \).

2. Assume that \( \dim X \geq 3 \). There is an integer \( d_2(X, L) \) such that for any smooth divisor \( D \in |L^{\otimes d}| \), where \( d > d_2(X, L) \), the homomorphism \( \rho \) in Eqn. (1.1) is a closed immersion. (So combining with statement (1) it follows that \( \rho \) is an isomorphism if \( d > d_1(X, L), d_2(X, L) \).)

An effective estimate of the constant \( d_1(X, L) \) in Theorem 1.1(1) is given in Theorem 3.5. An effective estimate of the constant \( d_2(X, L) \) in the second part of the theorem is described in Remark 5.3.

The first part of the above theorem is proved in Theorem 3.5, and the second part is proved in Theorem 5.1.

In Section 2, the earlier mentioned example is constructed. In Section 4, some results needed in the proof of Theorem 5.1 are established.

## 2. An example

Let \( k \) be an algebraically closed field. Let \( F_k \) be the Frobenius homomorphism of the field \( k \). Let \( X \) be a variety defined over \( k \). We then have the geometric Frobenius
morphism
\[ F_X : X \to F_k^*X. \]
Note that since the field \( k \) is perfect, the variety \( F_k^*X \) is isomorphic to \( X \). For notational convenience, we will denote the iterated pull back \( (F_k^n)^*X \) by \( X \) itself, where \( n \) is any positive integer. With this notation, the \( n \)-fold iteration of the geometric Frobenius morphism
\[ F_X^n : X \to (F_k^n)^*X \]
will be considered as a morphism from \( X \) to \( X \). We will use this convention throughout the manuscript.

**Definition 2.1.** A vector bundle \( E \) over \( X \) will be called an \( F \)-trivial vector bundle if there is a nonnegative integer \( m \) such that the vector bundle \( (F_k^m)^*E \) is isomorphic to a trivial vector bundle over \( X \).

Let \( X \) be a projective variety defined over \( k \). For any \( k \)-rational point \( x_0 \in X \), the étale fundamental group of \( X \) with \( x_0 \) as the base point will be denoted by \( \pi_1(X, x_0) \).

Let \( D \subset X \) be a reduced ample hypersurface in \( X \). If we have \( \dim X \geq 2 \) with \( X \) normal, then from Grothendieck’s Lefschetz theory, \( [Gr, \text{Exposé X}] \), it follows that for any \( k \)-rational point \( x_0 \in D \), the natural homomorphism
\[ (2.1) \quad \rho_1 : \pi_1(D, x_0) \to \pi_1(X, x_0) \]
is surjective. To prove that \( \rho_1 \) is surjective, take any connected étale Galois cover
\[ \phi : \tilde{X} \to X. \]
Since \( X \) is normal, we conclude that \( \tilde{X} \) is irreducible. As the divisor is reduced and ample, the inverse image \( \phi^{-1}(D) \) is also a reduced and ample hypersurface in \( \tilde{X} \). Therefore, we know that \( \phi^{-1}(D) \) is connected (see \( [Ha, \text{page 79, Corollary 6.2}] \) and \( [Ha, \text{page 64, Proposition 2.1}] \)). This immediately implies that the homomorphism \( \rho \) in Eqn. \( (2.1) \) is surjective.

For any \( k \)-rational point \( x_0 \) of \( D \), the fundamental group scheme of \( X \) (respectively, \( D \)) with \( x_0 \) as the base point will be denoted by \( \pi(X, x_0) \) (respectively, \( \pi(D, x_0) \)); the fundamental group scheme was introduced in \( [No1], [No2] \).

Let
\[ (2.2) \quad \rho : \pi(D, x_0) \to \pi(X, x_0) \]
be the homomorphism of fundamental group schemes induced by the inclusion map \( D \hookrightarrow X \). Our aim in this section is to give an example of a pair \((X, D)\), where \( D \) is an ample reduced curve on a smooth projective surface \( X \), for which the homomorphism \( \rho \) in Eqn. \( (2.2) \) is not surjective. This is in contrast with the situation for étale fundamental groups.

Assume that the characteristic of the field \( k \) is positive. In \( [Ra] \), Raynaud constructed a pair \((S, C)\), where \( S \) is a smooth projective surface and \( C \) a reduced ample hypersurface on \( S \), such that
\[ (2.3) \quad H^1(S, \mathcal{O}_S(-C)) \neq 0. \]
It may be noted that the Kodaira vanishing theorem says that $H^1(X',\mathcal{O}_{X'}(-D')) = 0$ for any ample divisor $D'$ on a smooth projective surface $X'$ defined over an algebraically closed field of characteristic zero.

For our example, set $X = S$, and take $D$ to be the divisor $C$ on $S$ in the above mentioned example $(S, C)$ of [Ra].

We consider the natural short exact sequence of sheaves

$$0 \rightarrow \mathcal{O}_X(-D) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_D \rightarrow 0$$

on $X$. Let

(2.4) $H^0(X, \mathcal{O}_X) = H^0(D, \mathcal{O}_D) \rightarrow H^1(X, \mathcal{O}_X(-D)) \xrightarrow{\gamma} H^1(X, \mathcal{O}_X) \xrightarrow{\delta} H^1(D, \mathcal{O}_D)$

be a segment of the long exact sequence of cohomologies corresponding to this exact sequence of sheaves. Take any nonzero element

(2.5) $c \in H^1(X, \mathcal{O}_X(-D)) \setminus \{0\}$

which exists by Eqn. (2.3). Therefore, the cohomology class $c$ gives a non–split exact sequence of vector bundles

(2.6) $0 \rightarrow \mathcal{O}_X \rightarrow W \xrightarrow{\psi} \mathcal{O}_X(D) \rightarrow 0$

over $X$.

Set

(2.7) $\alpha := \gamma(c) \in H^1(X, \mathcal{O}_X) \setminus \{0\}$,

where $\gamma$ is the homomorphism in Eqn. (2.4) and $c$ is the element in Eqn. (2.5). Let

(2.8) $0 \rightarrow \mathcal{O}_X \rightarrow V \rightarrow \mathcal{O}_X \rightarrow 0$

be the nontrivial extension corresponding to the cohomology class $\alpha$ defined in Eqn. (2.7). So we have $V = \psi^{-1}(\mathcal{O}_X)$, where $\psi$ is the projection in Eqn. (2.6).

Since

$$\delta(\alpha) = \delta(\gamma(c)) = 0,$$

where $\delta$ is the homomorphism in Eqn. (2.4), we conclude that the restriction of the vector bundle $V$ to $D$ splits as

(2.9) $V|_D = \mathcal{O}_D \oplus \mathcal{O}_D$.

Let $F_X : X \rightarrow X$ be the Frobenius morphism. For any positive integer $n$, let

$$F^n_X := \underbrace{F_X \circ \cdots \circ F_X}_{\text{n-times}} : X \rightarrow X$$

be the $n$–fold iteration of the self–morphism $F_X$. By $F^0_X$ we will mean the identity morphism of $X$.

**Lemma 2.2.** There is a positive integer $n$ such that the vector bundle $(F^n_X)^*V$ over $X$ is isomorphic to $\mathcal{O}_X \oplus \mathcal{O}_X$, where $V$ is the vector bundle constructed in Eqn. (2.8).
Proof. For any morphism \( f : Y \to X \), consider the short exact sequence of vector bundles over \( Y \)
\[
0 \to f^* \mathcal{O}_X = \mathcal{O}_Y \to f^* \mathcal{V} \to f^* \mathcal{O}_X = \mathcal{O}_Y \to 0
\]
obtained by pulling back the exact sequence Eqn. (2.8). The cohomology class in \( H^1(Y, \mathcal{O}_Y) \) corresponding to it coincides with
\[
f^* \alpha \in H^1(Y, f^* \mathcal{O}_X) = H^1(Y, \mathcal{O}_Y),
\]
where \( \alpha \) is defined in Eqn. (2.7). Let
\[
h : H^1(Y, \mathcal{O}_Y(-f^{-1}(D))) \to H^1(Y, \mathcal{O}_Y)
\]
by the homomorphism obtained from the exact sequence
\[
0 \to \mathcal{O}_Y(-f^{-1}(D)) \to \mathcal{O}_Y \to \mathcal{O}_{f^{-1}(D)} \to 0.
\]
From the definition of \( \alpha \) it follows that
\[
f^* \alpha \in h(H^1(Y, \mathcal{O}_Y(-f^{-1}(D))))
\]
where \( h \) is the homomorphism in Eqn. (2.10). Indeed, the identity \( f^* \alpha = h(f^* c) \) holds, where \( c \) is the cohomology class in Eqn. (2.5).

We set \( f := F^m_X : X \to X \). Then
\[
f^{-1}(D) = p^m D,
\]
where \( p \) is the characteristic of the field \( k \) (which is positive by our assumption).

Let \( K_X \) denote the canonical line bundle of \( X \). Since \( D \) is an ample divisor on \( X \), and \( \dim X = 2 \), we have
\[
H^1(X, \mathcal{O}_X(-jD)) = H^1(X, K_X \otimes \mathcal{O}_X(jD))^* = 0
\]
for all \( j \) sufficiently large [EGA3 page 111, Proposition 6.2.1], where the first isomorphism is the Serre duality. Therefore, using Eqn. (2.12) we conclude that
\[
H^1(X, \mathcal{O}_X(-f^{-1}(D))) = 0
\]
for all \( m \) sufficiently large, where \( f = F^m_X \). Consequently, from Eqn. (2.11) it follows that \( (F^m_X)^* \alpha = 0 \) for all \( m \) sufficiently large. Hence the exact sequence
\[
0 \to \mathcal{O}_X \to (F^m_X)^* \mathcal{V} \to \mathcal{O}_X \to 0
\]
obtained by pulling back the exact sequence in Eqn. (2.8) using \( F^m_X \) splits for all \( m \) sufficiently large. This completes the proof of the lemma. \( \square \)

We will now prove a proposition from which it will follow that the vector bundle \( \mathcal{V} \) in Eqn. (2.8) is essentially finite; see [No1 page 38] for the definition of essentially finite vector bundles.

**Proposition 2.3.** Let \( M \) be a projective variety defined over an algebraically closed field \( k \). Let \( E \) be a vector bundle over \( M \) with the following property. There is an étale Galois covering
\[
\beta : \tilde{M} \to M
\]
such that the vector bundle $\beta^*E$ over $\tilde{M}$ is $F$–trivial (see Definition 2.1). Then $E$ is an essentially finite vector bundles.

**Proof.** Let $r$ denote the rank of $E$. Let $n$ be a positive integer such that the vector bundle $(F^*_n\beta^*E)$ is trivializable, where

$$F^*_{\tilde{M}} : \tilde{M} \to \tilde{M}$$

is the Frobenius morphism.

Consider the exact sequence of group schemes

$$e \to \ker(F^*_nGL(r,k)) \to GL(r,k) \xrightarrow{F^*_nGL(r,k)} GL(r,k) \to e,$$

where $F^*_nGL(r,k) : GL(r,k) \to GL(r,k)$ is the Frobenius morphism. Let

$$(2.13) \quad H^1(\tilde{M}, \ker(F^*_nGL(r,k))) \to H^1(\tilde{M}, GL(r,k)) \xrightarrow{\phi} H^1(\tilde{M}, GL(r,k))$$

be the exact sequence of pointed sets obtained from this short exact sequence.

For any principal $G$–bundle $E_G$ over $\tilde{M}$, where $G$ is an algebraic group defined over $k$, the pull back $F^*_{M}E_G$ is identified with the principal $G$–bundle over $\tilde{M}$ obtained by extending the structure group of $E_G$ using the Frobenius morphism $F\_G : G \to G$ [RR 287, Remark 3.22]. Using this, together with the fact that $(F^*_n\beta^*E)$ trivializable, we conclude that the element $\alpha \in H^1(\tilde{M}, GL(r,k))$ corresponding to $(F^*_n\beta^*E)$ has the property that $\phi(\alpha)$ is the base point in $H^1(\tilde{M}, GL(r,k))$ (the point corresponding to the trivial $GL(r,k)$–bundle), where $\phi$ is the map in Eqn. (2.13).

This implies that the vector bundle $\beta^*E$ is associated to a principal bundle over $\tilde{M}$ with the finite group scheme $\ker(F^*_nGL(r,k))$ as the structure group scheme. Since $\beta$ is an étale Galois covering, from this it follows that the vector bundle $E$ is associated to a principal bundle over $M$ with the finite group scheme as the structure group scheme. Consequently, the vector bundle $E$ is essentially finite [No1 page 38, Proposition 3.8]. This completes the proof of the proposition. $\square$

Using Proposition 2.3 from Lemma 2.2 it follows that the vector bundle $V$ in Eqn. (2.8) is essentially finite. The vector bundle $V$, being essentially finite, corresponds to a representation of the fundamental group scheme $\pi(X, x_0)$, where $x_0$ is any $k$–rational point of $X$. The fundamental group scheme is defined in [No1 page 40].

Consider the two essentially finite vector bundles over $X$, namely $V$ and $O_X \oplus O_X$. Since $V$ is a nontrivial extension of $O_X$ by $O_X$, these two vector bundles are not isomorphic. On the other hand, Eqn. (2.9) says that their restrictions to $D$ are isomorphic.

Consequently, we have two non-isomorphic representations of $\pi(X, x_0)$, namely $V$ and $O_X \oplus O_X$, with the following property: when these two are considered as representation of $\pi(D, x_0)$ using $\rho$ in Eqn. (2.2), then they become isomorphic representations of $\pi(D, x_0)$.

From this it follows immediately that the homomorphism $\rho$ (defined in Eqn. (2.2)) for this pair $(X, D)$ is not surjective.
Remark 2.4. Let $X$ be a projective variety of dimension at least two. Let $D \subset X$ be an ample hypersurface such that $H^1(X, \mathcal{O}_X(-D)) \neq 0$. Then the above arguments show that the induced homomorphism $\pi(D, x_0) \to \pi(X, x_0)$ is not surjective. See [Ek, page 120, Proposition 2.14] for further examples of such pairs $(X, D)$.

Remark 2.5. Consider the vector bundle $W$ in Eqn. (2.6). We note that $c_1(W) = D$ and $c_2(W) = 0$. Since $D$ is an ample divisor, we know that $D^2 > 0$. Therefore, $c_1(W)^2 > 4c_2(W) = 0$. In other words, the vector bundle $W$ violates the Bogomolov inequality condition. However the vector bundle $W$ is semistable [Mu, pages 251–252]. More precisely, in [Mu, pages 251–252] Mumford shows that if $W$ is not semistable, then the exact sequence in Eqn. (2.6) splits. Hence $W$ provides a counter-example to the Bogomolov inequality for positive characteristic.

3. Fundamental group scheme and hyperplane section

Let $k$ be an algebraically closed field of characteristic $p$, with $p > 0$.

Let $Y$ be a smooth projective variety defined over $k$. Fix an ample line bundle $L$ over $Y$ to define degree of coherent sheaves on $Y$. Let $F_Y : Y \to Y$ be the Frobenius morphism of $Y$.

Take any vector bundle $E$ over $Y$. We will briefly recall the definition of $L_{\text{max}}(E)$ (see [La, page 257]). For any positive integer $m$, let

$$0 = E_{m,0} \subset E_{m,1} \subset \cdots \subset E_{m,a(m)} \subset E_{m,a(m)-1} = (F^m_Y)^*E$$

be the Harder–Narasimhan filtration of the vector bundle $(F^m_Y)^*E$ for the polarization $L$ on $Y$. It is known that for any sufficiently large $m$, we have

$$E_{m+n,1} = (F^m_Y)^*E_{m,1}$$

for all $n \geq 0$ [La, page 259, Claim 2.7.1]. Consequently,

$$\frac{\text{degree}(E_{m,1})}{p^m \cdot \text{rank}(E_{m,1})} \in \mathbb{Q}$$

is independent of $m$ as long as $m$ is sufficiently large. This well-defined rational number $\frac{\text{degree}(E_{m,1})}{p^m \cdot \text{rank}(E_{m,1})}$, where $m$ is sufficiently large, is denoted by $L_{\text{max}}(E)$.

A torsionfree coherent sheaf $E$ on $Y$ is called strongly semistable if $(F^m_Y)^*E$ is semistable for all $m \geq 0$.

Lemma 3.1. Let $E$ and $E'$ be vector bundles over $Y$ such that $L_{\text{max}}(E) + L_{\text{max}}(E') < 0$. Then

$$H^0(Y, E \otimes E') = 0.$$ 

Proof. From Eqn. (3.1) it follows immediately that $E_{m,1}$ is strongly semistable for all $m$ sufficiently large. Since the torsionfree part of the tensor product of any two strongly
semistable sheaves is again strongly semistable [RR page 288, Theorem 3.23], we conclude that
\[ \mu_{\text{max}}((F_Y^m)^*(E \otimes E')) = \mu_{\text{max}}((F_Y^m)^*E) + \mu_{\text{max}}((F_Y^m)^*E') = p^m(L_{\text{max}}(E) + L_{\text{max}}(E')) < 0 \]
for all \( m \) sufficiently large. Consequently,
\[ H^0(Y, (F_Y^m)^*(E \otimes E')) = 0 \]
for all \( m \) sufficiently large. This immediately implies that
\[ H^0(Y, E \otimes E') = 0, \]
and the proof of the lemma is complete. \( \square \)

The cotangent bundle of \( Y \) will be denoted by \( \Omega_Y \).

**Lemma 3.2.** Let \( r \) be any integer satisfying the condition \( r \cdot \text{degree}(L) > L_{\text{max}}(\Omega_Y) \). Let \( E \) be any essentially finite vector bundle over \( Y \). Then
\[ H^0(Y, E \otimes \Omega_Y \otimes (L^*)^{\otimes r}) = 0, \]
where \( L^* \) is the dual of \( L \). Moreover, if \( r \cdot \text{degree}(L) > p \cdot L_{\text{max}}(\Omega_Y) \) then we have
\[ H^0(Y, E \otimes F_Y^*\Omega_Y \otimes (L^*)^{\otimes r}) = 0. \]

**Proof.** Since any essentially finite vector bundle is strongly semistable of degree zero [No1 page 37, Corollary 3.5], we conclude that \( L_{\text{max}}(E) = 0 \). Therefore, we have
\[ L_{\text{max}}(E) + L_{\text{max}}(\Omega_Y \otimes (L^*)^{\otimes r}) = L_{\text{max}}(\Omega_Y \otimes (L^*)^{\otimes r}) = L_{\text{max}}(\Omega_Y) - r \cdot \text{degree}(L) < 0. \]
Hence from Lemma 3.1 it follows that \( H^0(Y, E \otimes \Omega_Y \otimes (L^*)^{\otimes r}) = 0. \)

The second part of the lemma follows similarly using the fact that \( L_{\text{max}}(F_Y^*\Omega_Y) = p \cdot L_{\text{max}}(\Omega_Y) \). This completes the proof of the lemma. \( \square \)

**Lemma 3.3.** Let \( D \subset Y \) be a smooth divisor in the complete linear system \( |L^{\otimes r}| := \mathbb{P}H^0(Y, L^{\otimes r})^* \). Let \( I \subset \mathcal{O}_Y \) be the ideal sheaf defining \( D \). Then
\[ H^0(D, (E \otimes (I/I^2))|_D) = 0 \]
and
\[ H^0(D, (E \otimes F_Y^*(I/I^2))|_D) = 0 \]
for any essentially finite vector bundle \( E \) over \( Y \).

**Proof.** Since \( E \) is essentially finite, the restriction \( E|_D \) is an essentially finite vector bundle over \( D \).

Since \( D \in |L^{\otimes r}| \), the Poincaré adjunction formula says that the line bundle over \( D \) defined by \( I/I^2 \) coincides with the restriction of the line bundle \( L^{-r} := (L^*)^{\otimes r} \) to \( D \). As \( L \) is ample, the line bundle \( L^{-r}|_D \) is of negative degree. Therefore, it now follows from Lemma 3.1 that \( H^0(D, (E \otimes (I/I^2))|_D) = 0. \)

Since \( F_Y^*L = L^{\otimes p} \), the restriction of the line bundle \( F_Y^*L^{-r} \) to \( D \) is of negative degree. Using this the second vanishing result follows similarly. This completes the proof of the lemma. \( \square \)
Let $G$ and $H$ be group schemes defined over the algebraically closed field $k$. We will denote by $G$–rep (respectively, $H$–rep) the category of all finite dimensional left $G$–modules (respectively, $H$–modules). Given a homomorphism of group schemes
\begin{equation}
\rho_0 : G \longrightarrow H
\end{equation}
we have a contravariant functor
\begin{equation}
\tilde{\rho}_0 : H\text{-rep} \longrightarrow G\text{-rep}
\end{equation}
that considers a $H$–module as a $G$–module through $\rho_0$.

We now reproduce Proposition 2.21 of [DM].

**Proposition 3.4** (DM, page 139, Proposition 2.21). Let $\rho_0$ be the homomorphism in Eqn. (3.2) and $\tilde{\rho}_0$ the corresponding functor in Eqn. (3.3).

(1) The homomorphism $\rho_0$ is surjective (faithfully flat) if and only if the following two conditions hold: the functor $\tilde{\rho}_0$ is fully faithful and for each exact sequence
\begin{equation}
0 \longrightarrow W' \longrightarrow \tilde{\rho}_0(V) \longrightarrow W'' \longrightarrow 0
\end{equation}
of $G$–representations, where $V$ is a $H$–representation, there is an exact sequence
\begin{equation}
0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0
\end{equation}
of $H$–representations and a commutative diagram
\begin{equation}
\begin{array}{ccc}
0 & \longrightarrow & W' \\
\downarrow \cong & & \downarrow \text{Id}
\end{array}
\begin{array}{ccc}
\longrightarrow & \tilde{\rho}_0(V) & \longrightarrow W'' \\
\downarrow \cong & & \downarrow \cong
\end{array}
\begin{array}{ccc}
0 & \longrightarrow & \tilde{\rho}_0(V') \\
\longrightarrow & \tilde{\rho}_0(V) & \longrightarrow \tilde{\rho}_0(V'')
\end{array}
\end{equation}
of $G$–representations.

(2) The homomorphism $\rho_0$ in Eqn. (3.2) is a closed immersion if and only if for each $G$–representation $V$ there exists some $H$–representation $W$ such that $V$ is a subquotient of the $G$–representation $\tilde{\rho}_0(W)$.

Let $X$ be a smooth projective variety defined over $k$. Fix an ample line bundle $L$ over $X$. So $L$ gives a polarization on $X$. For any nonnegative integer $d$, we will denote by $|L^\otimes d|$ the complete linear system $\mathbb{P}H^0(X, L^\otimes d)^*$. Our aim in this section is to prove the following theorem:

**Theorem 3.5.** Let $X$ be a smooth projective variety with $\dim X \geq 2$ and $L$ an ample line bundle over $X$. Let $D \in |L^\otimes d|$ be a smooth divisor on $X$, where $d$ is any positive integer satisfying the inequality
\begin{equation}
d > L_{\text{max}}(\Omega_X)/\text{degree}(L),
\end{equation}
and let $x_0$ be a $k$–rational point of $D$. Then the homomorphism between fundamental group schemes
\begin{equation}
\pi(D, x_0) \longrightarrow \pi(X, x_0)
\end{equation}
induced by the inclusion map $D \hookrightarrow X$ is surjective.
Proof. We will use the criterion in Proposition 3.4(1) for surjectivity.

Let $E$ be an essentially finite vector bundle over $X$. From Proposition 3.10 in [No1, page 40] we know that $E$ is trivializable over a principal bundle over $X$ whose structure group scheme is finite. Taking the local group scheme corresponding to the closure of the identity element (see the proof of Proposition 7 in [No2, Chapter II]) we conclude that there is a connected étale Galois covering

$$f_1 : X_1 \longrightarrow X$$

such that $f_1^*E$ is an $F$–trivial vector bundle over $X_1$. See Definition 2.1 for $F$–trivial vector bundles.

Let $V$ be another essentially finite vector bundle over $X$. Let

$$f_2 : X_2 \longrightarrow X$$

be a connected étale Galois covering such that $f_2^*V$ is an $F$–trivial vector bundle over $X_2$. Let

$$Z \subset X_1 \times_X X_2$$

be any connected component of the fiber product. Set

$$f := f_1 \times f_2 : Z \longrightarrow X.$$ 

Therefore, both $f^*E$ and $f^*V$ are $F$–trivial vector bundles over the connected smooth projective variety $Z$.

Let $D \in |L^{\otimes d}|$ be a smooth divisor on $X$, where $d > L_{\text{max}}(\Omega_X)/\text{degree}(L)$. Set $D' := f^{-1}(D)$. Since $D \subset X$ is an ample smooth divisor, and dim $X \geq 2$, the divisor $D'$ is irreducible and smooth. Thus, the restriction

$$f_D := f|_{D'} : D' \longrightarrow D$$

is also a connected étale Galois covering.

In our set–up, the first condition in Proposition 3.4(1) says that the restriction homomorphism

$$H^0(X, \mathcal{H}om(E, V)) \longrightarrow H^0(D, \mathcal{H}om(E, V)|_D)$$

is an isomorphism.

To prove this, set $E_1 := f^*E$ and $V_1 := f^*V$, where $f$ is the morphism defined in Eqn. (3.5). Since $f$ and $f_D$ (defined Eqn. (3.6)) are connected étale Galois coverings, we have

$$H^0(X, \mathcal{H}om(E, V)) = H^0(Z, \mathcal{H}om(E_1, V_1))^{\Gamma}$$

and

$$H^0(D, \mathcal{H}om(E|_D, V|_D)) = H^0(D', \mathcal{H}om(E_1|_{D'}, V_1|_{D'})^{\Gamma},$$

where $\Gamma := \text{Gal}(f)$ is the Galois group for the covering $f$, which is also the Galois group for $f_D$. Therefore, to prove that Eqn. (3.7) is an isomorphism it suffices to show that the restriction homomorphism $\mathcal{H}om(E_1, V_1) \longrightarrow \mathcal{H}om(E_1|_{D'}, V_1|_{D'})$ gives an isomorphism

$$H^0(Z, \mathcal{H}om(E_1, V_1)) = H^0(D', \mathcal{H}om(E_1|_{D'}, V_1|_{D'})).$$
Note that the restriction homomorphism
\[ \mathcal{H}om(E_1, V_1) \rightarrow \mathcal{H}om(E_1|_{D'}, V_1|_{D'}) \]
is \(\Gamma\)-equivariant. If Eqn. (3.8) holds, then taking the \(\Gamma\)-invariants of both sides of Eqn. (3.8) it follows that the homomorphism in Eqn. (3.7) is an isomorphism.

Let
\[ F_Z : Z \rightarrow Z \]
be the Frobenius morphism of the variety \(Z\) in Eqn. (3.4). Since both \(E_1\) and \(V_1\) are \(F\)-trivial vector bundles over \(Z\), there is a nonnegative integer \(m\) such that both \((F_Z^m)^* E_1\) and \((F_Z^m)^* V_1\) are trivializable vector bundles.

We proceed by induction on \(m\). If \(m = 0\), then both \(E_1\) and \(V_1\) are trivializable vector bundles over \(Z\). In that case,
\[ H^0(Z, \mathcal{H}om(E_1, V_1)) = \text{Hom}_k((E_1)_{x_0}, (V_1)_{x_0}) = H^0(D', \mathcal{H}om(E_1|_{D'}, V_1|_{D'})), \]
where \(x_0\) is any \(k\)-rational point of \(D'\). Therefore, Eqn. (3.8) is valid if \(m = 0\).

Assume that Eqn. (3.8) is valid for all \(E, V\) and \(f\) as above for which \(m \leq n_0 - 1\). We will show that Eqn. (3.8) is valid for all \(E, V\) and \(f\) for which \(m = n_0\). It should be emphasized that we are not fixing \(f\) in the induction hypothesis. The induction hypothesis says that for any triple \((E, V, f)\) as above such that \((F_Z^m)^* E_1\) and \((F_Z^m)^* V_1\) are trivializable vector bundles, Eqn. (3.8) is valid provided \(m \leq n_0 - 1\).

Take any triple \(E, V\) and \(f\) as above for which \(m = n_0\). Set
\[ E' := F_Z^* E_1 \]
and
\[ V' := F_Z^* V_1. \]
Since \(E'\) and \(V'\) are pull backs under the Frobenius morphism \(F_Z\), by Cartier, there are canonical connections \(\nabla^{E'}\) and \(\nabla^{V'}\) on \(E'\) and \(V'\) respectively, with the property that the \(p\)-curvature of the connections \(\nabla^{E'}\) and \(\nabla^{V'}\) vanish (see [Ka, Section 5]). Furthermore, there is a natural isomorphism
\[ (3.9) \quad \mathcal{H}om_{\mathcal{O}_Z}(E_1, V_1) = \mathcal{H}om_{\mathcal{O}_Z}((E', \nabla^{E'}), (V', \nabla^{V'})) \]
between the sheaf of \(\mathcal{O}_Z\)-linear homomorphisms from \(E_1\) to \(V_1\) and the sheaf of connection preserving \(\mathcal{O}_Z\)-linear homomorphisms from \(E'\) to \(V'\) [Ka, page 190, Theorem 5.1.1].

We will use the polarization \(f^* L\) on \(Z\) to define degree of coherent sheaves on \(Z\). For notational convenience, given any vector bundle \(W\) on \(Z\), the vector bundle \(W \otimes_{\mathcal{O}_Z} \mathcal{O}_Z(-D')\) will be denoted by \(W(-D')\). We have
\[ (3.10) \quad L_{\text{max}}(\mathcal{H}om(E_1, V_1(-D'))) = L_{\text{max}}(V_1) - L_{\text{max}}(E_1) + \text{degree}(\mathcal{O}_Z(-D')). \]
We note that as both \(E_1\) and \(V_1\) are \(F\)-trivial,
\[ (3.11) \quad L_{\text{max}}(V_1) = 0 = L_{\text{max}}(E_1). \]
Combining Eqn. (3.10) and Eqn. (3.11),
\[ L_{\text{max}}(\mathcal{H}om(E_1, V_1(-D'))) = \text{degree}(\mathcal{O}_Z(-D')) < 0, \]
where $D' = f^{-1}(D)$. Hence from Lemma 3.1 we conclude that
\begin{equation}
H^0(Z, \mathcal{H}om(E_1, V_1(-D'))) = 0.
\end{equation}

Using Eqn. (3.12) in the left exact sequence of global sections for the exact sequence of sheaves
\begin{equation}
0 \longrightarrow \mathcal{H}om(E_1, V_1(-D')) \longrightarrow \mathcal{H}om(E_1, V_1) \longrightarrow \mathcal{H}om(E_1|_{D'}, V_1|_{D'}) \longrightarrow 0
\end{equation}

it follows that the restriction homomorphism
\begin{equation}
H^0(Z, \mathcal{H}om(E_1, V_1)) \longrightarrow H^0(D', \mathcal{H}om(E_1|_{D'}, V_1|_{D'}))
\end{equation}
is injective. Hence Eqn. (3.8) holds if the homomorphism in Eqn. (3.13) is surjective.

Take any homomorphism of vector bundles
\begin{equation}
\phi \in H^0(D', \mathcal{H}om_{\mathcal{O}_{D'}}(E_1|_{D'}, V_1|_{D'})).
\end{equation}
over $D'$. This homomorphism $\phi$ defines a homomorphism
\begin{equation}
\tilde{\phi} := F^*_{D'} \phi : E'|_{D'} \longrightarrow V'|_{D'},
\end{equation}
where $F_{D'} : D' \longrightarrow D'$ is the Frobenius morphism of $D'$; note that $E'|_{D'}$ (respectively, $V'|_{D'}$) is identified with $F^*_{D'}(E_1|_{D'})$ (respectively, $F^*_{D'}(V_1|_{D'})$).

Let $\nabla'$ (respectively, $\nabla''$) be the restriction to $D'$ of the Cartier connection $\nabla^E'$ (respectively, $\nabla^{V'}$) on $E'$ (respectively, $V'$). Since Cartier connection is compatible with pull back operation, the connection $\nabla'$ (respectively, $\nabla''$) coincides with the Cartier connection on the Frobenius pull back $F^*_{D'}(E_1|_{D'}) = E'|_{D'}$ (respectively, $F^*_{D'}(V_1|_{D'}) = V'|_{D'}$). The homomorphism $\tilde{\phi}$ in Eqn. (3.14), being a pulled back one, intertwines the two connections $\nabla'$ and $\nabla''$. In other words, the following identity holds:
\begin{equation}
(\tilde{\phi} \otimes \text{Id}_{\Omega_{D'}}) \circ \nabla' = \nabla'' \circ \tilde{\phi},
\end{equation}
where $\Omega_{D'}$ is the cotangent bundle of $D'$. Both sides of Eqn. (3.15) are $k$–linear homomorphisms from $E'|_{D'}$ to $V'|_{D'} \otimes \Omega_{D'}$.

Using the induction hypothesis, which says that Eqn. (3.8) is valid for all $E$, $V$ and $f$ with $m \leq n_0 - 1$, the restriction homomorphism gives an isomorphism
\begin{equation}
H^0(Z, \mathcal{H}om(E', V')) \cong H^0(D', \mathcal{H}om(E'|_{D'}, V'|_{D'})).
\end{equation}
Note that as $F^*_X E_1 = f^* F^*_X E$ and $F^*_X V_1 = f^* F^*_X V$, and $m = n_0$ for the triple $(E, V, f)$, the induction hypothesis indeed applies for the triple $(F^*_X E, F^*_X V, f)$ giving the isomorphism in Eqn. (3.16).

Let
\begin{equation}
\tilde{\phi}' : E' \longrightarrow V'
\end{equation}
be the homomorphism that corresponds to $\tilde{\phi}$ (defined in Eqn. (3.14)) by the isomorphism in Eqn. (3.16). Therefore, $\tilde{\phi}$ is the restriction of $\tilde{\phi}'$ to $D'$.

To prove that Eqn. (3.8) is valid for all $E$, $V$ and $f$ for which $m \leq n_0$, it suffices to show that the homomorphism $\tilde{\phi}'$ in Eqn. (3.17) intertwines the connections $\nabla^{E'}$ and $\nabla^{V'}$. Indeed, if $\tilde{\phi}'$ intertwines $\nabla^{E'}$ and $\nabla^{V'}$, then using Eqn. (3.9) the homomorphism $\tilde{\phi}'$ descends
to a homomorphism from $E_1$ to $V_1$. Sending any $\phi$ to the descend of the homomorphism $\tilde{\phi}'$ constructed from $\phi$ in Eqn. (3.17), the inverse of the restriction homomorphism in Eqn. (3.13) is obtained.

To prove that $\tilde{\phi}'$ intertwines $\nabla^{E'}$ and $\nabla^{V'}$, consider
\[(3.18)\quad \gamma := (\tilde{\phi}' \otimes \text{Id}_{\Omega_Z})\nabla^{E'} - \nabla^{V'}\tilde{\phi}' \in H^0(Z, \mathcal{H}om(E', V' \otimes \Omega_Z)).\]
(This homomorphism $\gamma : E' \to V' \otimes \Omega_Z$ is clearly $\Omega_Z$–linear.) The homomorphism $\tilde{\phi}'$ intertwines $\nabla^{E'}$ and $\nabla^{V'}$ if and only if $\gamma = 0$. Let
\[(3.19)\quad \gamma' \in H^0(D', (\mathcal{H}om(E', V') \otimes \Omega_Z)|_{D'})\]
be the restriction to $D'$ of $\gamma$ defined in Eqn. (3.18).

As mentioned earlier, the degree of a coherent sheaf on $Z$ will be defined using $f^*L$. Therefore, we have $\text{deg}(f^*W) = \text{deg}(f^*\text{dim}_X \text{deg}(W))$ for any coherent sheaf $W$ on $X$. Also, $f^*\Omega_X = \Omega_Z$. Therefore, the given condition in the theorem that $d > L_{\text{max}}(\Omega_X)/\text{deg}(L)$, which is equivalent to the condition $\text{deg}(\mathcal{O}_X(D)) > L_{\text{max}}(\Omega_X)$, implies that
\[(3.20)\quad \deg(\mathcal{O}_Z(D')) > L_{\text{max}}(\Omega_Z).\]
In view of Eqn. (3.20), from Lemma 3.2 it follows that
\[H^0(Z, \mathcal{H}om(E', V' \otimes \Omega_Z)(-D')) = 0\]
(the vector bundle $\mathcal{H}om(E', V')$ is $F$–trivial as both $E'$ and $V'$ are so). Therefore, considering the left exact sequence of global sections for the exact sequence of sheaves
\[0 \to \mathcal{H}om(E', V' \otimes \Omega_Z)(-D') \to \mathcal{H}om(E', V' \otimes \Omega_Z) \to (\mathcal{H}om(E', V' \otimes \Omega_Z)|_{D'}) \to 0\]
we conclude that the restriction homomorphism
\[(3.21)\quad H^0(Z, \mathcal{H}om(E', V' \otimes \Omega_Z)) \to H^0(D', (\mathcal{H}om(E', V') \otimes \Omega_Z)|_{D'})\]
is injective.

Consequently, the homomorphism $\gamma$ constructed in Eqn. (3.18) vanishes if its restriction $\gamma'$ (defined in Eqn. (3.19)) vanishes.

Let $\mathcal{I} \subset \mathcal{O}_Z$ be the ideal sheaf defining the smooth divisor $D'$. We have an exact sequence of vector bundles over $D'$
\[(3.22)\quad 0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega_Z|_{D'} \to \Omega_{D'} \to 0.\]
Tensoring the exact sequence in Eqn. (3.22) with $\mathcal{H}om(E'|_{D'}, V'|_{D'})$, and then taking global sections, we obtain the following exact sequence:
\[(3.23)\quad 0 \to H^0(D', \mathcal{H}om(E'|_{D'}, V'|_{D'}) \otimes (\mathcal{I}/\mathcal{I}^2)) \to H^0(D', \mathcal{H}om(E'|_{D'}, V'|_{D'}) \otimes \Omega_Z|_{D'}) \xrightarrow{\beta} H^0(D', \mathcal{H}om(E'|_{D'}, V'|_{D'}) \otimes \Omega_{D'}).\]
We note that the identity in Eqn. (3.15) is equivalent to the assertion that $\beta(\gamma') = 0$, where $\gamma'$ is the section defined in Eqn. (3.19) and $\beta$ is the homomorphism in Eqn. (3.23). On the other hand, as both $E'$ and $V'$ are $F$–trivial, using the first part of Lemma 3.3 we
conclude that the term $H^0(D', \mathcal{H}om(E'|_{D'}, V'|_{D'}) \otimes (I/I^2))$ in Eqn. (3.23) vanishes. In other words, $\beta$ is injective. Since $\beta$ is injective with $\beta(\gamma') = 0$, we conclude that $\gamma' = 0$.

We noted earlier that $\gamma$ constructed in Eqn. (3.18) vanishes if $\gamma' = 0$. Therefore, we have proved that $\gamma = 0$. Since $\gamma = 0$, the homomorphism $\tilde{\rho}'$ in Eqn. (3.17) intertwines the connections $\nabla^{E'}$ and $\nabla^{V'}$. We noted earlier that this implies that Eqn. (3.8) is valid for all $E, V$ and $f$ for which $m \leq n_0$.

Using induction on $m$ it now follows that Eqn. (3.8) is valid. It was also shown earlier that Eqn. (3.8) implies that the homomorphism in Eqn. (3.7) is an isomorphism. This completes the proof of the assertion that the homomorphism $\pi(D, x_0) \rightarrow \pi(X, x_0)$ in the statement of the theorem satisfies the first condition in Proposition 3.4(1).

We will now prove that the homomorphism in Eqn. (3.24) also satisfies the second condition in Proposition 3.4(1). For that we will use the above set-up, and we will again employ induction on $m$.

Let $V$ be an essentially finite vector bundle over $X$ such that $(F^m_Z)^*f^*V$ is a trivial vector bundle over $Z$, where $f$ and $Z$ are as above. Let

\begin{equation}
E_0 \subset V|_D
\end{equation}

be an essentially finite subbundle over $D$. Therefore, the vector bundle $(F^m_{D'})^*f_D^*E_0$ over $D' := f^{-1}(D)$, where $f_D$ is the restriction morphism in Eqn. (3.6) and $F_{D'}$ is the Frobenius morphism of $D'$ (as in Eqn. (3.14)), is an essentially finite subbundle of the trivializable vector bundle $((F^m_Z)^*f^*V)|_{D'}$. Since any subbundle of degree zero of a trivializable vector bundle is also trivializable, and any essentially finite vector bundle is of degree zero, we conclude that $(F^m_{D'})^*f_D^*E_0$ is a trivializable vector bundle over $D'$.

If $m = 0$, then it is easy to see that $E_0$ is the restriction to $D$ of an essentially finite subbundle of $V$ over $X$. Indeed, this follows immediately from the fact that the natural homomorphism of étale fundamental groups

\[
\pi_1(D, x_0) \twoheadrightarrow \pi_1(X, x_0)
\]

is surjective (a proof of this is given in Section 2).

Assume that for all triples of the form $(V, E_0, f)$ as above with $m < n_0$, the vector bundle $E_0$ is the restriction to $D$ of an essentially finite subbundle of $V$. We will show that for all triples $(V, E_0, f)$ as above with $m = n_0$, the vector bundle $E_0$ is the restriction to $D$ of an essentially finite subbundle of $V$.

Take any triple $(V, E_0, f)$ as above for which $m = n_0$. Let

\begin{equation}
V' := F^*_Zf^*V
\end{equation}

be the vector bundle over $Z$ and

\begin{equation}
E_0' := F^*_Df_D^*E_0
\end{equation}

the vector bundle over $D'$. Since $f \circ F_Z = F_X \circ f$, we have

\[
(F^m_Z)^*f^*V = (F^m_Z)^*f^*F_X^*V.
\]
Also, since $F_X \circ \iota_D = \iota_D \circ F_D$, where
\[ \iota_D : D \hookrightarrow X \]
is the inclusion morphism and $F_D : D \rightarrow D$ is the Frobenius morphism, the inclusion $E_0 \subset V_D$
in Eqn. (3.25) induces an injective homomorphism of vector bundles
\[ F_0^*E_0 \subset (F_X^*V)|_D. \]
Consequently, the induction hypothesis applies for the triple $(F_X^*V, F_D^*E_0, f)$. (Note that $F_0^*E_0$ is essentially finite as $E_0$ is so.) Therefore, there is an essentially finite subbundle $E_1 \subset F_X^*V$
such that $E_1|_D = F_0^*E_0 \subset (F_X^*V)|_D$.

Set
\[ E' := f^*E_1 \subset V'. \]
Therefore, the two subbundles of $V'|_{D'}$, namely $E'|_{D'}$ and $E'_0$, coincide, where $E'_0$ is defined in Eqn. (3.27). Note that
\[ E'_0 = F^*\delta_D E_0 \subset F^*\delta_D f^*(V|_D) = F^*\delta_D ((f^*V)|_{D'}) = (F^*_Z f^*V)|_{D'} = V'|_{D'}. \]

As before, let $\nabla^V$ denote the Cartier connection on the vector bundle $V'$ defined in Eqn. (3.26). Let
\[ \delta : E' \hookrightarrow V' \xrightarrow{\nabla^V} V' \otimes \Omega_Z \longrightarrow (V'/E') \otimes \Omega_Z \]
be the composition homomorphism giving the second fundamental form of the subbundle $E'$ (defined in Eqn. (3.28)) for the Cartier connection $\nabla^V$. From Leibniz identity, this composition homomorphism is $O_Z$–linear. In other words,
\[ \delta \in H^0(Z, \mathcal{H}om(E', (V'/E') \otimes \Omega_Z)). \]
This homomorphism $\delta$ vanishes if and only if the subbundle $E'$ is the pull back, by $F_Z$, of a subbundle of $f^*V$ [Ka, page 190, Theorem 5.1].

Since the vector bundle $V'$ is $F$–trivial and $E'$ is an essentially finite subbundle of $V'$, it follows immediately that both $E'$ and $V'/E'$ are $F$–trivial vector bundles (a subbundle of degree zero of a trivial vector bundle is trivializable and the quotient is also trivializable). As we noted earlier, the inequality in Eqn. (3.20) follows from the given condition on $d$. Since $E'$ and $V'/E'$ are $F$–trivial, using Lemma 3.2 it follows that
\[ H^0(Z, \mathcal{H}om(E', (V'/E') \otimes \Omega_Z)(-D')) = 0. \]
In view of this, considering the left exact sequence of global sections for the exact sequence
\[ 0 \longrightarrow \mathcal{H}om(E', (V'/E') \otimes \Omega_Z)(-D') \longrightarrow \mathcal{H}om(E', (V'/E') \otimes \Omega_Z) \]
\[ \dashrightarrow \mathcal{H}om(E', (V'/E') \otimes \Omega_Z)|_{D'} \longrightarrow 0 \]
we conclude that the restriction homomorphism
\[ \psi : H^0(Z, \mathcal{H}om(E', (V'/E') \otimes \Omega_Z)) \longrightarrow H^0(D', \mathcal{H}om(E', (V'/E') \otimes \Omega_Z)|_{D'}) \]
is injective.

Set

\[ \delta' := \psi(\delta) \in H^0(D', \mathcal{H}om(E', (V'/E') \otimes \Omega_Z)|_{D'}) , \]

where \( \psi \) is the homomorphism in Eqn. (3.32) and \( \delta \) is the section in Eqn. (3.31). In other words, \( \delta' \) is the restriction of \( \delta \) to \( D' \).

Since the homomorphism \( \psi \) in Eqn. (3.32) is injective, the section \( \delta \) vanishes if \( \delta' \), defined in Eqn. (3.33), vanishes.

Tensoring the short exact sequence in Eqn. (3.22) with \( \mathcal{H}om(E'|_{D'}, (V'/E'|_{D'}) \otimes (\mathcal{I}/\mathcal{I}^2)) \), and then taking global sections, we get a left exact sequence

\[ H^0(D', \mathcal{H}om(E'|_{D'}, (V'/E'|_{D'}) \otimes (\mathcal{I}/\mathcal{I}^2)) \rightarrow H^0(D', \mathcal{H}om(E'|_{D'}, (V'/E'|_{D'}) \otimes \Omega_Z|_{D'}) \rightarrow H^0(D', \mathcal{H}om(E'|_{D'}, (V'/E'|_{D'}) \otimes \Omega_{D'}) . \]

As \( E' \) and \( V'/E' \) are \( F \)-trivial vector bundles, the vector bundles \( E'|_{D'} \) and \( (V'/E'|_{D'}) \) are also \( F \)-trivial. Hence the first part of Lemma 3.3 gives that

\[ H^0(D', \mathcal{H}om(E'|_{D'}, (V'/E'|_{D'}) \otimes (\mathcal{I}/\mathcal{I}^2)) = 0 . \]

Therefore, the homomorphism \( \alpha \) in Eqn. (3.34) is injective.

Recall that \( E'|_{D'} = E_0' \), or in other words, the restriction \( E'|_{D'} \) is the pull back, by the Frobenius morphism \( F_{D'} \), of the subbundle \( f_D^*E_0 \subset (f^*V)|_{D'} \). Therefore, the second fundamental form of the subbundle

\[ E'|_{D'} \subset V'|_{D'} = F_{D'}((f^*V)|_{D'}) \]

(see Eqn. (3.29)) for the Cartier connection on \( F_{D'}f_D(V|_D) \) vanishes identically. This immediately implies that \( \alpha(\delta') = 0 \), where \( \delta' \) is defined in Eqn. (3.33) and \( \alpha \) is the homomorphism in Eqn. (3.34).

Since \( \alpha \) is injective (this was proved above), and \( \alpha(\delta') = 0 \), we conclude that \( \delta' = 0 \).

We saw earlier that if \( \delta' = 0 \), then the homomorphism \( \delta \) constructed in Eqn. (3.30) vanishes. Therefore, we have proved that \( \delta = 0 \). Since \( \delta = 0 \), the subbundle \( E' \subset V' \) is the pull back, by \( F_Z \), of a subbundle of \( f^*V \). Let

\[ E'' \subset f^*V \]

be the subbundle such that \( F_Z^*E'' = E' \subset V' \). Since \( E' \) is essentially finite, it follows that the vector bundle \( E'' \) is also essentially finite. Indeed, as \( E' \) is essentially finite, there is an étale Galois cover \( Z' \) of \( Z \) such that the pull back of \( E' \) to \( Z' \) is \( F \)-trivial, hence the pull back \( E'' \) to \( Z' \) is \( F \)-trivial, which, using Proposition 2.3, implies that \( E'' \) is essentially finite.

We will show that the natural action of the Galois group \( \Gamma := \text{Gal}(f) \) on \( f^*V \) leaves the subbundle \( E'' \) invariant. For this, take any element \( g \in \Gamma \) of the Galois group. Let

\[ H_g : E'' \oplus (g^{-1})^*E'' \rightarrow f^*V \]
be the homomorphism defined by \( H_g(v, w) = \iota(v) + g((g^{-1})^*\iota(w)) \), where \( \iota : E'' \hookrightarrow f^*V \) is the inclusion map in Eqn. (3.35),

\[
(g^{-1})^*\iota : (g^{-1})^*E'' \hookrightarrow (g^{-1})^*f^*V = f^*V
\]

is the pull back of \( \iota \) and \( g(((g^{-1})^*\iota)(w)) \) is the image of \(((g^{-1})^*\iota)(w) \in f^*V \) for the action of \( g \) on \( f^*V \). The vector bundle \( E'' \oplus (g^{-1})^*E'' \) is essentially finite as \( E'' \) is so. Therefore, the image \( H_g(E'' \oplus (g^{-1})^*E'') \) is a subbundle of the essentially finite vector bundle \( f^*V \) \( \text{[No1, page 38, Proposition 3.7(b)]} \).

Since

\[
F^s_D(E''|_{D'}) = (F^s_2E'')|_{D'} = E'|_{D'} = E'_0 = F^s_Df^s_0E_0,
\]

we know that \( f^*_DE_0 = E''|_{D'} \). Therefore, the action of \( \Gamma \) on \((f^*V)|_{D'}\) leaves the subbundle

\[
f^*_DE_0 = E''|_{D'} \subset (f^*V)|_{D'}
\]

invariant. Hence we have

(3.37) \[
\text{rank}(H_g(E'' \oplus E'')) = \text{rank}(E'')
\]

for the homomorphism \( H_g \) in Eqn. (3.36). From Eqn. (3.37) it follows immediately that the action of \( g \in \Gamma \) on \( f^*V \) leaves the subbundle \( E'' \) invariant.

Since the subbundle \( E'' \subset f^*V \) is left invariant by \( \Gamma \), it descends as a subbundle of \( V \), and furthermore, the descended subbundle extends to \( X \) the subbundle \( E_0 \subset V|_D \) over \( D \). Since \( E'' \) is essentially finite, it follows that the descend of \( E'' \) is an essentially finite vector bundle over \( X \). Indeed, as \( f \) is étale Galois, and \( E'' \) is essentially finite, there is an étale Galois cover \( X' \) of \( X \) such that the pull back to \( X' \) of the descend of \( E'' \) to \( X \) is \( F \)-trivial, which, using Proposition (2.3) implies that the descend of \( E'' \) to \( X \) is essentially finite.

Thus, if \( V \) is an essentially finite vector bundle over \( X \) and \( E_0 \subset V|_D \) an essentially finite subbundle over \( D \), where \( D \in |L \otimes d| \) is a smooth divisor on \( X \) with \( d > L_{\text{max}}(\Omega_X)/\text{degree}(L) \), then \( E_0 \) extends to \( X \) as an essentially finite subbundle of \( V \). Therefore, the homomorphism in Eqn. (3.24) satisfies the second condition in Proposition (3.4(1)). Hence this homomorphism is surjective. This completes the proof of the theorem. \( \square \)

Combining Remark (2.4) and Theorem (3.5) we have the following corollary:

**Corollary 3.6.** Let \( X \) be a smooth projective variety of dimension at least two and \( L \) an ample line bundle over \( X \). Let \( D \in |L \otimes d| \) be a smooth divisor on \( X \), where \( d \) is any positive integer satisfying the inequality

\[
d > L_{\text{max}}(\Omega_X)/\text{degree}(L).
\]

Then \( H^1(X, \mathcal{O}_X(-D)) = 0 \).

4. Some vanishing results

In this section we will prove some vanishing results which will be used in Section 5.

Let \( k \) be an algebraically closed field of characteristic \( p \), with \( p > 0 \). Let \( X \) be a smooth projective variety defined over \( k \) and \( F_X \) the Frobenius morphism of \( X \). Given
any vector bundle $E$ over $X$, we will construct a natural filtration of coherent subsheaves of $F_X^*F_{X*}E$.

Let $W$ be a vector bundle over $X$ equipped with a connection $\nabla$. Let $W_1 \subset W$ be a coherent subsheaf. Set $W_2$ to be the kernel of the second fundamental form $S(W_1) : W_1 \rightarrow (W/W_1) \otimes \Omega_X$ of $W_1$ for the connection $\nabla$, where $\Omega_X$ is the cotangent bundle of $X$. We recall that 

$$S(W_1) = (q_{W_1} \otimes \text{Id}_{\Omega_X}) \circ \nabla \circ \iota_{W_1},$$

where $\iota_{W_1} : W_1 \hookrightarrow W$ is the inclusion map and $q_{W_1} : W \rightarrow W/W_1$ is the quotient map. Next set $W_3$ to be the kernel of the second fundamental form $S(W_2) : W_2 \rightarrow (W/W_2) \otimes \Omega_X$ of $W_2$ for the connection $\nabla$. This way, for $i \geq 1$, define $W_{i+1}$ inductively to be the kernel of the second fundamental form $S(W_i)$ of $W_i$ for the connection $\nabla$. For notational convenience, set $W_0 := W$. From the construction of this filtration

$$W = W_0 \supset W_1 \supset W_2 \supset \cdots,$$

it follows immediately that

$$\text{Image}(S(W_i)) \subset (W_{i-1}/W_i) \otimes \Omega_X \subset (W/W_i) \otimes \Omega_X$$

for each $i \geq 1$. Consequently, the second fundamental form $S(W_i)$ induces a homomorphism

$$\tilde{S}(W_i) : W_i/W_{i+1} \rightarrow (W_{i-1}/W_i) \otimes \Omega_X$$

for each $i \geq 1$.

Iterating the homomorphisms in Eqn. (4.2) we have a homomorphism

$$\tilde{S}(W_1) \circ \cdots \circ \tilde{S}(W_{i-1}) \circ \tilde{S}(W_i) : W_i/W_{i+1} \rightarrow (W/W_i) \otimes \Omega_X^{\otimes i}$$

for all $i \geq 1$. Let

$$S_i : W_i/W_{i+1} \rightarrow (W/W_i) \otimes \Omega_X^{\otimes i}$$

be the composition homomorphism.

Take any vector bundle $E$ over $X$. The vector bundle $F_X^*F_{X*}E$ is equipped with the Cartier connection. First set $E_0 := F_X^*F_{X*}E$. There is a natural surjective homomorphism

$$E_0 := F_X^*F_{X*}E \rightarrow E.$$

Let $E_1 \subset E_0$ denote the kernel of the homomorphism in Eqn. (4.4). Using the construction of the filtration in Eqn. (4.1), a filtration of $F_X^*F_{X*}E$ is obtained from the subbundle $E_1$ and the Cartier connection on $F_X^*F_{X*}E$. Thus, we get a filtration of coherent subsheaves

$$F_X^*F_{X*}E =: E_0 \supset E_1 \supset E_2 \supset \cdots.$$

Since the homomorphism in Eqn. (4.4) is surjective, we have $E_0/E_1 = E$. 
Proposition 4.1. Let $p$ denote the characteristic of the field $k$, and $d = \dim X$. With the above notation we have the following:

1. Each $E_i$ in Eqn. (4.5) is a vector bundle over $X$, and $E_i = 0$ if $i > (p-1)d$.
2. For each $i \geq 1$, the homomorphism

\[
S_i : E_i/E_{i+1} \rightarrow E \otimes \Omega_X^{\otimes i}
\]

constructed as in Eqn. (4.3) is an injective homomorphism of vector bundles.
3. Set $E = \mathcal{O}_X$. Then for each $i \geq 1$, consider the subbundle

\[G_i \subset \Omega_X^{\otimes i}\]

given by the image of the homomorphism $S_i$ in Eqn. (4.6) (for $E = \mathcal{O}_X$). Let $E$ be an arbitrary vector bundle over $X$. Then the image $S_i(E_i/E_{i+1})$ coincides with the subbundle

\[E \otimes G_i \subset E \otimes \Omega_X^{\otimes i},\]

where $G_i$ is the above subbundle of $\Omega_X^{\otimes i}$.

Proof. If $f : X_1 \rightarrow X_2$ is a morphism, then $f \circ F_{X_1} = F_{X_2} \circ f$, where $F_{X_j}, j = 1, 2,$ is the Frobenius morphism of $X_j$. Furthermore, if $V$ is a vector bundle over $X_2$, then the isomorphism $F_{X_1}^*f^*V = f^*F_{X_2}^*V$ takes the Cartier connection on $F_{X_1}^*f^*V$ to the pullback, by $f$, of the Cartier connection on $F_{X_2}^*V$. Using these and the fact that the proposition is local in nature we conclude that it is enough to prove the proposition assuming that $X = \text{Spec}(A)$, where $A := k[[x_1, \ldots, x_d]]$. We will assume that $X = \text{Spec}(A)$ unless specified otherwise. Therefore, we may also assume that the vector bundle $E$ over $X$ is trivial. We will identify $E$ with $A^{\oplus r}$ equipped with the standard basis $\{e_1, \ldots, e_r\}$.

Therefore, the cotangent bundle $\Omega_X$ is a free $A$–module with a basis $\{dx_1, \ldots, dx_d\}$.

Set $B := k[[x_1^p, \ldots, x_d^p]] \subset A$, which is isomorphic to $A$. The Frobenius morphism $F_X$ is seen as the inclusion map

\[B \hookrightarrow A.\]

Using this inclusion we see that $A$ is a free $B$–module with a basis given by monomials of the form $x_1^{m_1} \cdots x_d^{m_d}$, where $0 \leq m_j \leq p-1$. The total degree of such a monomial $x_1^{m_1} \cdots x_d^{m_d}$ is defined to be $\sum_{j=1}^d m_j$.

We will identify $F_X^*F_XE$ with $\bigoplus_{i=1}^r (A \otimes_B A)e_i$ using the trivialization of $E$ and the identification of $F_X$ as the inclusion of $B$ in $A$. The $B$–module structure of

\[F_X^*E = \bigoplus_{i=1}^r Ae_i\]

is defined by the following rule:

\[x_j^p(\sum_{i=1}^r a_ie_i) = \sum_{i=1}^r x_j^p a_ie_i,\]

where $j \in [1,d]$ and $a_i \in A$. We note that $F_X^*E$ is a free $B$–module with a basis given by $\{x_1^{m_1}x_2^{m_2} \cdots x_d^{m_d}e_i\}$, where $1 \leq i \leq r$ and $0 \leq m_j < p$ for each $j \in [1,d]$. 
The natural projection $F_X^* F_{X*} E \rightarrow E$ in Eqn. (4.4) is now identified with a homomorphism

$$
\bigoplus_{i=1}^{r} (A \otimes_B A)e_i \rightarrow \bigoplus_{i=1}^{r} A e_i
$$

which is defined as follows:

$$
(4.7) \quad c \otimes x_1^{m_1} \cdots x_d^{m_d} e_i \mapsto c a_1^{m_1} \cdots x_d^{m_d} \otimes 1 e_i,
$$

where $C_{i_{m_1, \ldots, m_d}} \in A$.

For $i \in [1, d]$, let $y_i$ be the variable defined by $y_i := x_i \otimes 1 - 1 \otimes x_i \in A \otimes_B A$.

The homomorphism defined in Eqn. (4.7) is surjective and its kernel is generated by linear combinations of all $e_i$ with coefficients that are homogeneous polynomials in the variables $y_i$ of total degree at least one. This already shows that the subsheaf $E_1$ in Eqn. (4.5) is a subbundle of $E_0$.

The Cartier connection operator on $F_X^* F_{X*} E$

$$
F_X^* F_{X*} E \rightarrow (F_X^* F_{X*} E) \otimes \Omega_X
$$

coincides with the map

$$
\bigoplus_{i=1}^{r} (A \otimes_B A)e_i \rightarrow \bigoplus_{i=1}^{r} (A \otimes_B A) \otimes_A \Omega_X e_i
$$

defined by

$$
\sum_{i=1}^{r} C^i(y_1, \ldots, y_d) e_i \mapsto \sum_{i=1}^{r} d(C^i(y_1, \ldots, y_d)) \otimes e_i,
$$

where

$$
d(C^i(y_1, \ldots, y_d)) := \sum_{j=1}^{d} \frac{\partial C^i(y_1, \ldots, y_d)}{\partial x_j} \otimes dx_j
$$

with $C^i(y_1, \ldots, y_d) \in A \otimes_B A$, and

$$
\frac{\partial}{\partial x_j} : A \otimes_B A \rightarrow A \otimes_B A
$$

being the pullback of the $B$–derivation

$$
\frac{\partial}{\partial x_j} : A \rightarrow A.
$$

The second fundamental form of $E_1$ for the Cartier connection

$$
S(E_1) : E_1 \rightarrow E \otimes \Omega_X
$$

is identified with the $A$–linear homomorphism defined as follows:

$$
\sum_{i=1}^{r} \sum_{\{m_j\}_{j=1}^{d} \in [0, p-1] \setminus \{0\}^d} C_{i_{m_1, \ldots, m_d}} \otimes x_1^{m_1} \cdots x_d^{m_d} e_i \mapsto \sum_{i=1}^{r} \sum_{j=1}^{d} C_{i} e_i \otimes dx_j,
$$
where \( j = (0, \cdots, 1, \cdots, 0) \in \mathbb{Z}^d \), i.e., 0 everywhere except at the \( j \)-th position where it is 1. From this expression of the second fundamental form \( S(E_1) \) it follows immediately that the kernel of \( S(E_1) \) is a free \( \mathbb{A} \)-module.

For the general case, using induction we see that for any \( \ell \geq 1 \), the subsheaf \( E_\ell \subseteq F_X^* \mathcal{O}_X \) is a free \( \mathbb{A} \)-module generated by monomials of the form \( y_1^{m_1} \cdots y_d^{m_d} e_i \), where \( 1 \leq i \leq r \), \( \sum_{j=1}^d m_j = \ell + 1 \) and \( 0 \leq m_j < p \) for each \( j \in [1, d] \). The second fundamental form

\[
S(E_\ell) : E_\ell \rightarrow (E_{\ell-1}/E_\ell) \otimes \Omega_X
\]

coincides with the homomorphism that sends any element

\[
\sum_{i=1}^r d(C_i(y_1, \ldots, y_d)) e_i \in E_\ell
\]

to the image of \( \sum_{i=1}^r d(C_i(y_1, \ldots, y_d)) e_i \) in \( (E_{\ell-1}/E_\ell) \otimes \Omega_X \). Note that the kernel of this homomorphism is a free \( \mathbb{A} \)-module generated by the monomials of the form \( y_1^{m_1} \cdots y_d^{m_d} e_i \), where \( 1 \leq i \leq r \), \( \sum_{j=1}^d m_j = \ell + 1 \) and \( 0 \leq m_j < p \) for all \( j \in [1, d] \).

From this description it follows that each \( E_\ell \) is a vector bundle. It also follows that \( E_\ell = 0 \) if \( \ell > (p-1)d \). Therefore, the proofs of statement (1) and statement (2) in the proposition are complete.

To complete the proof of the proposition we need to show that the two subbundles of \( E \otimes \Omega_X^{\otimes \ell} \), namely \( \text{image}(S_\ell) \) and \( E \otimes G_\ell \), coincide.

To prove this we may assume that \( X = \text{Spec}(A) \), where \( A := k[[x_1, \ldots, x_d]] \), and we may also assume that \( E = \mathcal{O}_X^{\oplus r} = A^{\oplus r} \). If \( E = \mathcal{O}_X = A \), then

\[
\text{image}(S_\ell) = G_\ell = E \otimes G_\ell \subset E \otimes \Omega_X^{\otimes \ell} = \Omega_X^{\otimes \ell}
\]

for all \( \ell \geq p \). Therefore,

\[
\text{image}(S_\ell) = E \otimes G_\ell \subset E \otimes \Omega_X^{\otimes \ell}
\]

if \( E = \mathcal{O}_X^{\oplus r} \) and \( \ell \geq p \). This completes the proof of the proposition. \( \square \)

The following proposition is proved using Proposition 4.1(1).

**Proposition 4.2.** Let \( X \) be a smooth projective variety of dimension \( d \) defined over \( k \). Let \( M \) be a nonnegative rational number such that

\[
M \geq L_{\text{max}}(\Omega_X^{\otimes i})
\]

for all \( i \in [1, (p-1)d] \), where \( L_{\text{max}} \) is defined in Section 3 and \( p \) is the characteristic of \( k \). Then for each positive integer \( n \) the following inequality holds:

\[
L_{\text{max}}((F_X^n)_* \mathcal{O}_X) \leq M.
\]

**Proof.** For notational convenience, the vector bundle \( (F_X^{n-1})_* \mathcal{O}_X \) over \( X \) will be denoted by \( W \). Consider the filtration of quotients of \( F_X^* F_X_* W \)

\[
F_X^* F_X_* W =: W_0 \supset W_1 \supset W_2 \supset \cdots
\]
constructed as in Eqn. (4.5). Pulling it back by the morphism $(F_{X}^{n-1})^{*}$ we obtain a filtration of subbundles

\begin{equation}
(F_{X}^{n-1})^{*}F_{X}F_{X}^{*}W =: W_{0}' \supset W_{1}' \supset W_{2}' \supset \cdots ,
\end{equation}

where $W_{i}' := (F_{X}^{n-1})^{*}W_{i}$ for all $i \geq 0$. Since

\begin{equation}
(F_{X}^{n-1})^{*}F_{X}F_{X}^{*}W = (F_{X}^{n})^{*}(F_{X}^{n})\mathcal{O}_{X} ,
\end{equation}

the filtration in Eqn. (4.8) gives a filtration of subbundles of $(F_{X}^{n})^{*}(F_{X}^{n})\mathcal{O}_{X}$.

Using Proposition 4.1(1) we know that

\begin{equation}
W_{i}'/W_{i+1}' = (F_{X}^{n-1})^{*}(W_{i}/W_{i+1}) \subset (F_{X}^{n-1})^{*}(W \otimes \Omega_{X}^{\otimes i})
\end{equation}

and $W_{i}'/W_{i+1}' = 0$ if $i > (p-1)d$, where $d = \dim X$. On the other hand, given any two vector bundles $V_{1}$ and $V_{2}$ over $X$, we have

\begin{equation}
L_{\max}(V_{1} \otimes V_{2}) = L_{\max}(V_{1}) + L_{\max}(V_{2})
\end{equation}

(this follows from [RR, page 288, Theorem 3.23]). Combining these we conclude that there exists a nonnegative integer $i \leq (p-1)d$ such that

\begin{equation}
L_{\max}((F_{X}^{n})^{*}(F_{X}^{n})^{*}\mathcal{O}_{X}) \leq L_{\max}((F_{X}^{n-1})^{*}W) + L_{\max}((F_{X}^{n-1})^{*}\Omega_{X}^{\otimes i}) .
\end{equation}

(If $0 = V_{0} \subset V_{1} \subset \cdots \subset V_{m-1} \subset V_{m}$ is a filtration of subbundles and $V' \subset V_{m}$ is a coherent subsheaf, then there is a nonzero homomorphism of $V'$ to some quotient $V_{i}/V_{i-1}$.)

Let $M$ be any nonnegative rational number such that

\[ M \geq L_{\max}(\Omega_{X}^{\otimes j}) \]

for all $j \in [1, (p-1)d]$. From the definition of $L_{\max}$ we have

\begin{equation}
L_{\max}((F_{X}^{n-1})^{*}\Omega_{X}^{\otimes j}) = p^{n-1}L_{\max}(\Omega_{X}^{\otimes j}) \leq p^{n-1}M
\end{equation}

for all $j \in [1, (p-1)d]$.

As $W = (F_{X}^{n-1})^{*}\mathcal{O}_{X}$, from the inequalities in Eqn. (4.9) and Eqn. (4.10) we have

\[ L_{\max}((F_{X}^{n})^{*}(F_{X}^{n})^{*}\mathcal{O}_{X}) \leq \sum_{j=0}^{n-1} p^{j}M = M(p^{n} - 1)/(p - 1) .\]

Since $L_{\max}((F_{X}^{n})^{*}(F_{X}^{n})^{*}\mathcal{O}_{X}) = p^{n}L_{\max}((F_{X}^{n})^{*}\mathcal{O}_{X})$, this inequality implies that

\[ L_{\max}((F_{X}^{n})^{*}\mathcal{O}_{X}) \leq M(p^{n} - 1)/p^{n}(p - 1) \leq M .\]

This completes the proof of the proposition. \hfill \square

Let

\begin{equation}
f : Z \longrightarrow X
\end{equation}

be a connected Galois étale cover of the smooth projective variety $X$. As before, the Frobenius morphism of $Z$ will be denoted by $F_{Z}$. 

Lemma 4.3. The following diagram is Cartesian

\[
\begin{array}{ccc}
Z & \xrightarrow{F_Z} & Z \\
\downarrow f & & \downarrow f \\
X & \xrightarrow{F_X} & X \\
\end{array}
\]

Proof. As the above diagram is commutative, we obtain a morphism

\[ g : Z \to X \times_X Z \]

to the fiber product. The composition

\[ Z \xrightarrow{g} X \times_X Z \xrightarrow{pr_X} X \]

coinsides with \( f \), and hence it is an étale morphism. On the other hand, the projection \( pr_X \) is étale as \( f \) is so. Therefore, the morphism \( g \) is étale [Mi, page 24, Corollary 3.6]. Since the étale morphism \( g \) is bijective on the set of closed points, it must be an isomorphism. This completes the proof of the lemma. \( \square \)

Note that as a consequence of Lemma 4.3 and the fact that the morphism \( f \) is flat, the following holds: For any vector bundle \( E \) over \( X \), the base change morphism

\[
(4.12) \quad f^* F_X_* E \to F_Z^* f^* E
\]

is an isomorphism, where \( f \) is as in Eqn. (4.11).

The following corollary is proved using the fact that the homomorphism in Eqn. (4.12) is an isomorphism.

Corollary 4.4. For any \( f \) as in Eqn. (4.11) and any integer \( n \geq 1 \), there is a canonical isomorphism

\[
f^*(F^n_X)_* E \to (F^n_Z)_* f^* E.
\]

Proof. We will use induction on \( n \). For \( n = 1 \), this is the isomorphism in Eqn. (4.12). Assume that we have constructed the isomorphism

\[
(4.13) \quad f^*(F^{n-1}_X)_* E \to (F^{n-1}_Z)_* f^* E.
\]

Taking direct image by \( F_Z \), the isomorphism in Eqn. (4.13) gives an isomorphism

\[
F_Z^* f^*(F^{n-1}_X)_* E \to (F^n_Z)_* f^* E.
\]

On the other hand, substituting \( E \) by \((F^{n-1}_X)_* E \) in Eqn. (4.12) we get an isomorphism

\[
f^*(F^n_X)_* E = f^* F_X (F^{n-1}_X)_* E \to F_Z^* f^*(F^{n-1}_X)_* E.
\]

Combining the above two isomorphisms, an isomorphism

\[
f^*(F^n_X)_* E \to (F^n_Z)_* f^* E
\]

is obtained. This completes the proof of the corollary. \( \square \)
For any integer \( n \geq 1 \), there is a natural homomorphism of \( \mathcal{O}_X \) into the direct image \((F^n_X)_*\mathcal{O}_X\). Let \( B_n \) be the quotient bundle. So we have a short exact sequence
\[
0 \longrightarrow \mathcal{O}_X \longrightarrow (F^n_X)_*\mathcal{O}_X \longrightarrow B_n \longrightarrow 0
\]
(4.14)
of vector bundles over \( X \). We note that the pullback of this exact sequence by \( F^n_X \) splits using the adjunction homomorphism \((F^n_X)^*(F^n_X)_*\mathcal{O}_X\longrightarrow \mathcal{O}_X\). Consequently,
\[
(F^n_X)^*(F^n_X)_*\mathcal{O}_X = \mathcal{O}_X \oplus (F^n_X)^*B_n.
\]
(4.15)

Let \((B_n)_Z\) denote the quotient bundle \((F^n_Z)_*\mathcal{O}_Z/O_Z\) over \( Z \) which is constructed by replacing \( X \) by \( Z \) in Eqn. (4.14).

Corollary 4.4 gives the following:

**Corollary 4.5.** For any \( f \) as in Eqn. (1.11), the vector bundle \((B_n)_Z\) over \( Z \) is canonically identified with \( f^*B_n \), where \( n \) is any positive integer.

**Proof.** Set \( E = \mathcal{O}_X \) in Corollary 4.4. The isomorphism in Corollary 4.4 evidently takes the line subbundle
\[
\mathcal{O}_Z = f^*\mathcal{O}_X \subset f^*(F^n_X)_*\mathcal{O}_X
\]
to the line subbundle \( \mathcal{O}_Z \subset (F^n_Z)_*\mathcal{O}_Z \). Hence the isomorphism in Corollary 4.4 for \( E = \mathcal{O}_X \) induces an isomorphism of \( f^*B_n \) with \((B_n)_Z\). This completes the proof of the corollary. \(\square\)

Let
\[
F^n_Z f^*F_X, E \longrightarrow F^n_Z f^*E
\]
be the pull back of the isomorphism in Eqn. (4.12) by the morphism \( F_Z \). On the other hand, since \( F_X \circ f = f \circ F_Z \), the vector bundle \( F^n_Z f^*F_X, E \) is identified with \( f^*F^n_X F_X, E \). Combining these two isomorphisms we have an isomorphism
\[
f^*F^n_X F_X, E \longrightarrow F^n_Z f^*E.
\]
(4.16)

The following lemma shows that the isomorphism in Eqn. (4.16) is compatible with the filtration of subbundles constructed in Eqn. (4.5).

**Lemma 4.6.** Let \( \{E_i\}_{i \geq 0} \) (respectively, \( \{E'_i\}_{i \geq 0} \)) be the filtration of subbundles of the vector bundle \( F^n_X F_X, E \) (respectively, \( F^n_Z F_Z, f^*E \)) constructed as in Eqn. (4.3). For each \( i \geq 0 \), the isomorphism in Eqn. (4.16) takes the subbundle \( f^*E_i \subset f^*F^n_X F_X, E \) to the subbundle \( E'_i \subset F^n_Z F_Z, f^*E \).

**Proof.** The following diagram is commutative
\[
\begin{array}{ccc}
f^*F^n_X F_X, E & \longrightarrow & F^n_Z f^*E \\
\downarrow & & \downarrow \\
f^*E & = & f^*E
\end{array}
\]
where the vertical homomorphisms are defined as in Eqn. (4.4) and the top horizontal isomorphism is constructed in Eqn. (4.16). Consequently, the isomorphism in Eqn. (4.16) takes the subbundle \( f^*E_1 \subset f^*F^n_X F_X, E \) to \( E'_1 \subset F^n_Z F_Z, f^*E \).
The Cartier connection on $F_X^* F_X^* E$ induces a connection on $f^* F_X^* F_X^* E$. This induced connection is taken to the Cartier connection on $F_Z^* F_Z^* f^* E$ by the isomorphism in Eqn. (4.16).

Since the filtration in Eqn. (4.5) is constructed from the subbundle $E_1$ and the Cartier connection on $F_X^* X$, we conclude that for each $i \geq 0$, the isomorphism in Eqn. (4.16) takes the subbundle $f^* E_i$ to $E'_i$. This completes the proof of the lemma. □

Lemma 4.7. Let $X$ be an irreducible smooth projective variety of dimension at least three. Fix an ample line bundle $L$ over $X$. There exist integers $r_1$ and $r_2$ such that for each connected étale Galois covering $f : Z \to X$,

\[ H^i(Z, f^* L^{-r}) = 0 \]

provided $r > r_i$, where $i = 1, 2$. Furthermore, $r_1$ can be chosen to be $M/\text{degree}(L)$, where $M$ is given in Proposition 4.2.

Proof. Take any connected étale Galois covering $f : Z \to X$. Since $f$ is a finite morphism,

\[ H^i(Z, f^* L^{-r}) = H^i(X, f_* f^* L^{-r}) \]

for all $i \geq 0$. Combining this with the projection formula

\[ f_* f^* L^{-r} = f_*(O_Z) \otimes L^{-r} \]

we have

\[ H^i(Z, f^* L^{-r}) = H^i(X, f_*(O_Z) \otimes L^{-r}) \]

for all $i \geq 0$.

For notational convenience, the vector bundle $f_* O_Z$ over $X$ will be denoted by $W$.

Tensoring the exact sequence in Eqn. (4.14) with $W \otimes L^{-r}$ we have the short exact sequence

\[ 0 \to W \otimes L^{-r} \to ((F^n_X)_* O_X) \otimes W \otimes L^{-r} \to W \otimes B_n \otimes L^{-r} \to 0. \]

The long exact sequence of cohomologies for this exact sequence gives the following:

If

\[ H^{i-1}(X, W \otimes B_n \otimes L^{-r}) = 0 \]

and

\[ H^i(X, ((F^n_X)_* O_X) \otimes W \otimes L^{-r}) = 0, \]

then $H^i(X, W \otimes L^{-r}) = 0$.

The lemma will be proved by showing that for suitable values of $r$, Eqn. (4.18) holds for all $n$ and Eqn. (4.19) holds for all sufficiently large $n$.

To prove that Eqn. (4.19) holds for $i = 1, 2$, first note by the projection formula

\[ H^i(X, ((F^n_X)_* O_X) \otimes W \otimes L^{-r}) = H^i(X, (F^n_X)_*(F_X^n)^*(W \otimes L^{-r})) \]

for all $i \geq 0$. Now, since $F^n_X$ is a finite morphism, we have

\[ H^i(X, (F_X^n)^*(W \otimes L^{-r})) = H^i(X, (F^n_X)^*(W \otimes L^{-r})) \]
for all \( i \geq 0 \). Therefore, we have
\[
H^i(X, ((F^n_X)_*O_X) \otimes W \otimes L^{-r}) = H^i(X, (F^n_X)^*(W \otimes L^{-r}))
\]
for all \( i \geq 0 \).

As the next step, we will show that there is a natural isomorphism \((F^n_X)^*W = W\) for all \( n \geq 1 \). To construct this isomorphism, first note that since the vector bundle \( f^*W \) over \( Z \) is trivializable, there is a natural isomorphism
\[
f^*W \longrightarrow F^*_Z f^*W.
\]
On the other hand, since \( F_X \circ f = f \circ F_Z \), there is a natural isomorphism of \( F^*_Z f^*W \) with \( f^*F^*_X W \). This isomorphism and the isomorphism in Eqn. (4.21) together give an isomorphism
\[
f^*W \longrightarrow f^*F^*_X W
\]
for all \( n \geq 1 \).

The isomorphism in Eqn. (4.22) intertwines the actions of the Galois group \( \text{Gal}(f) \) on \( f^*W \) and \( f^*F^*_X W \). Therefore, the isomorphism in Eqn. (4.22) descends to an isomorphism of \( F^*_X W \) with \( W \). Since \( F^*_X W = W \), we have
\[
(F^n_X)^*W = W.
\]

The isomorphism in Eqn. (4.20) and the isomorphism in Eqn. (4.23) together give
\[
H^i(X, ((F^n_X)_*O_X) \otimes W \otimes L^{-r}) = H^i(X, W \otimes (F^n_X)^*L^{-r})
\]
(note that \((F^n_X)^*(W \otimes L^{-r}) = ((F^n_X)^*W) \otimes (F^n_X)^*L^{-r} = W \otimes (F^n_X)^*L^{-r})\). Since
\[
(F^n_X)^*L^{-r} = L^{-p^n r},
\]
where \( p \) is the characteristic of the field \( k \), and \( \dim X \geq 3 \), using the Serre vanishing it follows that
\[
H^i(X, W \otimes (F^n_X)^*L^{-r}) = H^i(X, W \otimes L^{-p^n r}) = 0
\]
if \( i = 1, 2 \) and \( n \) is sufficiently large [EGA3 page 111, Proposition 6.2.1]. Therefore, using Eqn. (4.24) we conclude that Eqn. (4.19) holds provided \( i = 1, 2 \) and \( n \) is sufficiently large.

We will now prove that Eqn. (4.18) holds for \( i = 1, 2 \) under appropriate conditions.

From the short exact sequence in Eqn. (4.15) it follows that \( L_{\max}((F^n_X)_*O_X) \geq 0 \) and
\[
L_{\max}(B_n) \leq L_{\max}((F^n_X)_*O_X).
\]
We will use the ample line bundle \( L \) to define degree of a coherent sheaf on \( X \).

Take any integer \( r \) such that \( r \cdot \text{degree}(L) > M \), where \( M \) is given in Proposition 4.2. From Proposition 4.2 it follows that
\[
L_{\max}((F^n_X)_*O_X) < r \cdot \text{degree}(L) = \text{degree}(L^r).
\]
Combining Eqn. (4.25) and Eqn. (4.26) we conclude that
\[
L_{\max}(B_n \otimes L^{-r}) = L_{\max}(B_n) - \text{degree}(L^r) \leq L_{\max}((F^n_X)_*O_X) - \text{degree}(L^r) < 0.
\]
Since $f$ is an étale Galois covering, the vector bundle $f^*W$ over $Z$ is trivializable. Therefore, from Eqn. (4.27) it follows that

$$L_{\text{max}}(f^*(W \otimes B_n \otimes L^{-r})) = L_{\text{max}}(f^*(B_n \otimes L^{-r})) < 0$$

with respect to the polarization $f^*L$ on $Z$. This inequality implies that

$$H^0(Z, f^*(W \otimes B_n \otimes L^{-r})) = 0.$$ 

Hence Eqn. (4.18) holds for all $n$ if $i = 1$ and $r > M/\text{degree}(L)$.

Since both Eqn. (4.18) and Eqn. (4.19) hold for $i = 1$ and $r > M/\text{degree}(L)$, we know that

(4.28) $$H^1(X, W \otimes L^{-r}) = 0$$

provided $r > M/\text{degree}(L)$. In view of Eqn. (4.17), this implies that

$$H^1(Z, f^*L^{-r}) = 0$$

if $r > r_1 := M/\text{degree}(L)$.

To prove Eqn. (4.18) for $i = 2$, consider the exact sequence of sheaves

$$0 \to F_X^* \mathcal{O}_X \to F_X^* \mathcal{O}_X \to F_X^* B_{n-1} \to 0$$

obtained by taking the direct image of Eqn. (4.14) by the morphism $F_X$ after substituting $n - 1$ for $n$ in Eqn. (4.14). It gives an exact sequence

(4.29) $$0 \to B_1 \to B_n \to F_X^* B_{n-1} \to 0$$

of vector bundles over $X$. Replacing $n$ by $n - 1$ in Eqn. (4.29), and then taking direct image by $F_X$, we have

$$0 \to F_X^* B_1 \to F_X^* B_{n-1} \to F_X^* F_X^* B_{n-2} \to 0.$$ 

More generally, replacing $n$ by $n - j$ in Eqn. (4.29), and then taking direct image by $F_X^j$, we have

$$0 \to F_X^j B_1 \to F_X^j B_{n-j} \to F_X^{j+1} B_{n-j-1} \to 0,$$

where $0 \leq j \leq n - 1$.

For $j \in [0, n - 1]$, set $U_j := F_X^j B_{n-j}$. Also, set $U_n = 0$. We have a filtration of quotients

(4.30) $$B_n := U_0 \to U_1 \to U_2 \to \cdots \to U_{n-1} \to U_n = 0$$

of the vector bundle $B_n$, where the homomorphisms $h_j, j \in [0, n - 1]$, are constructed above. Note that kernel($h_j$) = $F_X^j B_1, j \in [0, n - 1]$, with the convention that $(F_X^0)_* B_1 = B_1$ (recall that $F_X^0$ denotes the identity morphism of $X$).

Using the filtration in Eqn. (4.30) we conclude that Eqn. (4.18) holds for $i = 2$ if

(4.31) $$H^1(X, W \otimes L^{-r} \otimes (F_X^j)_* B_1) = 0$$

for all $j \in [0, n - 1]$. 
The projection formula says that $W \otimes L^{-r} \otimes (F_X^j)_* B_1 = (F_X^j)_* ((F_X^j)^*(W \otimes L^{-r}) \otimes B_1)$. We have $(F_X^j)^* W = W$ (see Eqn. (4.23)) and $(F_X^j)^* L^{-r} = L^{-p^r}$. Using these and the fact that $F_X^j$ is a finite morphism we have

\begin{equation}
(4.32) \quad H^1(X, W \otimes L^{-r} \otimes (F_X^j)_* B_1) = H^1(X, (F_X^j)_* ((F_X^j)^*(W \otimes L^{-r}) \otimes B_1))
\end{equation}

\begin{equation}
= H^1(X, (F_X^j)^*(W \otimes L^{-r}) \otimes B_1) = H^1(X, W \otimes L^{-p^r} \otimes B_1).
\end{equation}

There is a positive integer $m$, such that there exists a surjective homomorphism

$$\psi : (L^{-m})^\oplus m \longrightarrow B_1^\ast.$$ Given such a homomorphism $\psi$, consider the exact sequence of vector bundles

\begin{equation}
(4.33) \quad 0 \longrightarrow B_1 \xrightarrow{\psi^\ast} (L^m)^\oplus m \longrightarrow \mathcal{H} \longrightarrow 0
\end{equation}

over $X$.

Fix an integer $r_2$ satisfying the two conditions:

1. $r_2 \cdot \text{degree}(L) \geq \text{L}_{\text{max}}(\mathcal{H})$, and
2. $r_2 \geq m + r_1$, where $r_1$ is defined in the statement of the lemma.

We will show that Eqn. (4.18) with $i = 2$ holds for all $n$ and all $r > r_2$. For that purpose we will show that

\begin{equation}
(4.34) \quad H^1(X, W \otimes B_1 \otimes L^{-r}) = 0
\end{equation}

for all $r > r_2$.

Take any $r > r_2$, where $r_2$ is the above integer. Let

\begin{equation}
(4.35) \quad \longrightarrow H^0(X, W \otimes \mathcal{H} \otimes L^{-r}) \longrightarrow H^1(X, W \otimes B_1 \otimes L^{-r}) \longrightarrow H^1(X, W \otimes L^{m-r})^\oplus m
\end{equation}

be the long exact sequence of cohomologies for the short exact sequence of sheaves obtained by tensoring the exact sequence in Eqn. (4.33) with $W \otimes L^{-r}$.

We already proved that $H^1(X, W \otimes L^{-r'}) = 0$ if $r' > r_1$; see Eqn. (4.28). Therefore,

\begin{equation}
(4.36) \quad H^1(X, W \otimes L^{m-r}) = 0
\end{equation}

as $r - m > r_2 - m \geq r_1$ (see the definition of $r_2$ given above).

On the other hand, as $L_{\text{max}}(W) = 0$ (the pull back $f^*W$ is trivializable), we have

$$L_{\text{max}}(W \otimes \mathcal{H} \otimes L^{-r}) = L_{\text{max}}(\mathcal{H} \otimes L^{-r}) = L_{\text{max}}(\mathcal{H}) - \text{degree}(L^r) < 0$$

(the last inequality follows from the conditions that $r > r_2$ and $r_2 \cdot \text{degree}(L) \geq L_{\text{max}}(\mathcal{H})$). Therefore, we have

\begin{equation}
(4.37) \quad H^0(X, W \otimes \mathcal{H} \otimes L^{-r}) = 0.
\end{equation}

Combining Eqn. (4.36) and Eqn. (4.37) with Eqn. (4.35) we conclude that Eqn. (4.34) holds. In view of Eqn. (4.31) and Eqn. (4.32), this implies that Eqn. (4.18) with $i = 2$ holds for all $n$. We showed earlier that Eqn. (4.19) $i = 2$ holds for all $n$ sufficiently large.

Therefore, if $r \geq r_2$, then $H^2(X, W \otimes L^{-r}) = 0$. In view of Eqn. (4.17), this completes the proof of the lemma. □
We will need the following lemma for our purpose.

**Lemma 4.8.** Let $X$ and $L$ be as in Lemma 4.7. Let $V$ be a vector bundle over $X$. Then there is an integer $N(V)$ such for any $r > N(V)$, and for each connected étale Galois cover $f : Z \to X$,

$$H^1(Z, (F^n_Z)^* f^*(V \otimes L^{-r})) = 0$$

for all $n \geq 0$.

**Proof.** There is an integer $m$ such that there is a surjective homomorphism

$$\psi : (L^{-m})^\oplus m \to V^*$$

of vector bundles. The homomorphism $\psi$ gives an exact sequence of vector bundles

$$0 \to V \xrightarrow{\psi^*} (L^m)^\oplus m \to A \to 0$$

over $X$.

Fix an integer $N(V)$ satisfying the two conditions:

1. $N(V) \cdot \text{degree}(L) \geq \text{L}_{\text{max}}(A)$, and
2. $N(V) \geq m + r_1$, where $r_1$ is prescribed in Lemma 4.7.

These conditions ensure that if $r > N(V)$, then

$$L_{\text{max}}(A \otimes L^{-r}) < 0$$

and

$$H^1(Z, f^* L^p(n-m-r)) = 0$$

for all $n \geq 0$ (see Lemma 4.7). Since

$$(F^n_Z)^* f^* L^m = f^* L^p(n-m-r),$$

from Eqn. (4.40) we conclude that

$$H^1(Z, (F^n_Z)^* f^* L^m) = 0$$

for all $n \geq 0$.

Take any $r > N(V)$. Take any $f$ as in the statement of the lemma. From Eqn. (4.39) we have

$$L_{\text{max}}(f^*(A \otimes L^{-r})) < 0$$

with respect to the polarization on $Z$ obtained by pulling back the polarization on $X$. This implies that

$$L_{\text{max}}((F^n_Z)^* f^*(A \otimes L^{-r})) = p^n L_{\text{max}}(f^*(A \otimes L^{-r})) < 0$$

for all $n \geq 0$. Consequently,

$$H^0(Z, (F^n_Z)^* f^*(A \otimes L^{-r})) = 0$$

and for all $n \geq 0$.

Tensor the exact sequence in Eqn. (4.38) with $L^{-r}$ and then pull it back by the morphism $f \circ F^n_Z$. This gives the exact sequence of vector bundles

$$0 \to (F^n_Z)^* f^*(V \otimes L^{-r}) \to ((F^n_Z)^* f^* L^m)^{\oplus m} \to (F^n_Z)^* f^*(A \otimes L^{-r}) \to 0$$
over $Z$. Consider the corresponding long exact sequence of cohomologies

$$H^0(Z, (F^n_Z)^* f^*(A \otimes L^{-r})) \to H^1(Z, (F^n_Z)^* f^*(V \otimes L^{-r})) \to H^1(Z, (F^n_Z)^* f^*L^{m-r}) \oplus m.$$ 

In view of Eqn. (4.42) and Eqn. (4.41) it implies that

$$H^1(Z, (F^n_Z)^* f^*(V \otimes L^{-r})) = 0.$$ 

This completes the proof of the lemma.

The following lemma is very similar to Lemma 4.8.

**Lemma 4.9.** Let $X$ and $L$ be as in Lemma 4.7. Fix vector bundles $V_1, V_2, \cdots, V_\ell$ over $X$. Then there is an integer $N = N(V_1, \cdots, V_\ell)$ such that

$$H^1(Z, (\bigotimes_{j=0}^{n-1} (F^j_Z)^* f^*V_{n_j}) \otimes f^*L^{-p^n r}) = 0$$

provided $r > N$, where $n$ is any positive integer and $n_j \in [1, \ell]$ for all $j \in [0, n-1]$, and $f : Z \to X$ is any connected étale Galois cover.

**Proof.** As in Eqn. (4.38), there is an integer $m$ and exact sequences

$$0 \to V_i \xrightarrow{\psi_i} (L^m)^\oplus m \to A_i \to 0$$

for all $i \in [1, \ell]$. For any positive integer $n$ and integers $n_j \in [1, \ell]$, where $j \in [0, n-1]$, the homomorphisms $\psi_i$ in Eqn. (4.43) together give an injective homomorphism of vector bundles

$$\bigotimes_{j=0}^{n-1} (F^j_Z)^* f^*V_{n_j} \to (f^*L^m(p^n - 1)/(p-1))^{\oplus m^n}.$$ 

The quotient bundle for the homomorphism in Eqn. (4.44) is filtered by subbundles such that each successive quotient is of the form

$$(\bigotimes_{j \in I} (F^j_Z)^* f^*V_{n_j}) \otimes (\bigotimes_{j \in \overline{I}} (F^j_Z)^* f^*A_{n_j}),$$

where $I$ is some subset of $[0, n-1]$. Therefore, using this filtration, from the long exact sequence of cohomologies for the short exact sequence of vector bundles obtain from Eqn. (4.44) tensored with $L^{-p^n r}$ we conclude that if

$$H^0(Z, (\bigotimes_{j \in I} (F^j_Z)^* f^*V_{n_j}) \otimes (\bigotimes_{j \in \overline{I}} (F^j_Z)^* f^*A_{n_j}) \otimes f^*L^{-p^n r}) = 0$$

for all $I \subset [0, n-1]$, and

$$H^1(Z, f^*L^m(p^n - 1)/(p-1) \otimes f^*L^{-p^n r}) = 0,$$

then

$$H^1(Z, (\bigotimes_{j=0}^{n-1} (F^j_Z)^* f^*V_{n_j}) \otimes f^*L^{-p^n r}) = 0.$$ 

Take a nonnegative integer $N := N(V_1, \cdots, V_\ell)$ satisfying the following conditions:

(1) $N \cdot \text{degree}(L) \geq L_{\max}(A_i)$ and $N \cdot \text{degree}(L) \geq L_{\max}(V_i)$ for all $i \in [1, \ell]$;
(2) $N \geq r_1 + m$, where $r_1$ is defined in Lemma 4.7 and $m$ is as in Eqn. (4.43).

If $r > N$, then from the given condition $N \geq r_1 + m$ it follows that

$$p^n r - m(p^n - 1)/(p - 1) > r_1.$$  

Therefore, Eqn. (4.46) follows from Lemma 4.7 provided $r > N$.

To prove Eqn. (4.46), we note that

$$L_{\max}((\bigotimes_{j \in I} (F^j \mathbb{Z})^* f^* V_{n_j}) \otimes (\bigotimes_{j \in I} (F^j \mathbb{Z})^* f^* \mathcal{A}_{n_j}) \otimes f^* L^{-p^n r}) = \bigoplus_{j \in I} L_{\max}((F^j \mathbb{Z})^* f^* (V_{n_j} \otimes L^{-r}))$$

$$+ \sum_{j \in I} L_{\max}((F^j \mathbb{Z})^* f^* (\mathcal{A}_{n_j} \otimes L^{-r})) + \frac{r(p^n - 1)}{p - 1} - rp^n \deg(f) \leq \frac{r(p^n - 1)}{p - 1} - rp^n \deg(f)$$

as $L_{\max}(V_i \otimes L^{-r}), L_{\max}(\mathcal{A}_i \otimes L^{-r}) \leq 0$ for all $i \in [1, \ell]$ (this is the first of the two conditions that define $N$). Therefore, as $r(p^n - 1)/(p - 1) < rp^n$, we conclude that Eqn. (4.45) holds if $r > N$. This completes the proof of the lemma.

**Proposition 4.10.** Let $X$ and $L$ be as in Lemma 4.7. There is an integer $N$ such that for all $r > N$ the following holds:

$$H^1(Z, (F^n Z)^* (F^n Z)^* f^* L^{-p^n r}) = 0$$

for all $n \geq 0$ and all connected étale Galois cover $f : Z \to X$.

**Proof.** Set $E := (F^n Z)^* f^* \mathcal{O}_Z$. Consider the filtration of subbundles of $F^n Z f^* E$ constructed as in Eqn. (4.5). Pulling this filtration back by $F^{-1}_Z$ we get a filtration of subbundles of

$$(F^n Z)^* F^n Z f^* E = (F^n Z)^* f^* \mathcal{O}_Z$$

whose each successive quotient is the subbundle

$$(F^n Z)^* (E \otimes \mathcal{G}_i) \subset (F^n Z)^* (E \otimes \Omega_X^{p^n i}),$$

where $0 \leq i \leq (p - 1) \dim(X)$ and $\mathcal{G}_i$ is the vector bundle on $Z$ defined in Proposition 4.1(3).

In view of Lemma 4.6 and the fact that the connection on $f^* F_X f^* \mathcal{O}_X$ induced by the Cartier connection on $F_X f^* \mathcal{O}_X$ is taken to the Cartier connection on $F^n Z f^* \mathcal{O}_Z$ by the isomorphism in Eqn. (4.16) (see the proof of Lemma 4.6), we conclude that $\mathcal{G}_i = f^* \mathcal{G}_i^X$, where $\mathcal{G}_i^X \subset \Omega_X^{p^n i}$ is the subbundle defined in Proposition 4.1(3). Therefore,

$$(F^n Z)^* (E \otimes \mathcal{G}_i) = (((F^n Z)^* E) \otimes (F^n Z)^* f^* \mathcal{G}_i^X) = ((F^n Z)^* (F^n Z)^* \mathcal{O}_Z) \otimes (F^n Z)^* f^* \mathcal{G}_i^X.$$  

Repeating the above argument after replacing $n$ by $n - 1$, for all $i \in [0, (p - 1) \dim(X)]$ we get a filtration of subbundles of the vector bundle in Eqn. (4.47) whose successive quotients are of the form

$$(F^n Z)^* (F^n Z)^* \mathcal{O}_Z) \otimes (F^n Z)^* f^* \mathcal{G}_j^X \otimes (F^n Z)^* f^* \mathcal{G}_j^X,$$

where $0 \leq j \leq (p - 1) \dim(X)$.

In fact, iterating the above argument we conclude the following:
To prove the proposition it is enough to show that there is an integer $N$ such that for any $r > N$,

$$
H^1(Z, \left( \bigotimes_{i=0}^{n-1} (F^i_Z)^* f^* G^X_i \right) \otimes f^* L^{-p^n r}) = 0
$$

for all nonnegative integer $n$ and all $n_i \in [0, (p-1) \dim(X)]$.

Setting $\ell = (p-1) \dim(X) + 1$ and $V_j = G^X_{j-1}$ in Lemma 4.9, we get an integer $N$ such that Eqn. (4.48) holds for all $r > N$. This completes the proof of the proposition. \qed

**Proposition 4.11.** Let $X$ be an irreducible smooth projective variety of dimension $d$, with $d \geq 3$. Fix an ample line bundle $L$ over $X$. There is an integer $N = N(X, L)$ with the following property. Let $f : Z \to X$ be a connected étale Galois cover and $E$ an $F$-trivial vector bundle over $Z$. Then

(1) $H^1(Z, E \otimes f^* L^{-r}) = 0$ and

(2) $H^1(Z, E \otimes \Omega_Z \otimes f^* L^{-r}) = 0$

for all $r > N$.

**Proof.** Take $f$ and $E$ as in the statement of the proposition. Let $n$ be a positive integer such that the vector bundle $(F^n_Z)^* E$ is trivializable. To prove statement (1), let

$$
0 \to \mathcal{O}_Z \to (F^n_Z)_* \mathcal{O}_Z \to (B_n)_Z \to 0
$$

be the exact sequence of vector bundles over $Z$ constructed as in Eqn. (4.14). Consider the long exact sequence of cohomologies for the short exact sequence obtained by tensoring Eqn. (4.49) with $E \otimes f^* L^{-r}$. From it we conclude that if

$$
H^0(Z, (B_n)_Z \otimes E \otimes f^* L^{-r}) = 0
$$

and

$$
H^1(Z, E \otimes f^* L^{-r} \otimes (F^n_Z)_* \mathcal{O}_Z) = 0,
$$

then $H^1(Z, E \otimes f^* L^{-r}) = 0$.

Using the projection formula and the fact that the Frobenius morphism $F_Z$ is flat we have

$$
H^1(Z, E \otimes f^* L^{-r} \otimes (F^n_Z)_* \mathcal{O}_Z) = H^1(Z, (F^n_Z)_* (F^n_Z)^* (E \otimes f^* L^{-r}))
$$

$$
= H^1(Z, (F^n_Z)^* (E \otimes f^* L^{-r})) = H^1(Z, ((F^n_Z)^* E) \otimes f^* L^{-p^n r}) \cong H^1(Z, f^* L^{-p^n r}) \otimes \mathcal{O}_E,
$$

where $r_E = \text{rank}(E)$; recall that the vector bundle $(F^n_Z)^* E$ is trivializable.

Lemma 4.7 says that $H^1(Z, f^* L^{-p^n r}) = 0$ if $p^n r > r_1$. Therefore, Eqn. (4.51) holds if $r > r_1$, where $r_1$ is given by Lemma 4.7.

To prove Eqn. (4.50) we first recall Eqn. (4.26) which says that

$$
L_{\text{max}}((B_n)_Z) \leq L_{\text{max}}((F^n_Z)_* \mathcal{O}_Z).
$$

Since $L_{\text{max}}(E) = 0$, this inequality gives

$$
L_{\text{max}}((B_n)_Z \otimes E \otimes f^* L^{-r}) = L_{\text{max}}((B_n)_Z) - \text{degree}(f^* L^r) \leq L_{\text{max}}((F^n_Z)_* \mathcal{O}_Z) - \text{degree}(f^* L^r).
$$
Corollary 4.4 implies that
\[ L_{\max}((F^n_Z)_* \mathcal{O}_Z) = L_{\max}(f^*(F^n_X)_* \mathcal{O}_X). \]
Hence from Proposition 4.2, we conclude that the right-hand side of Eqn. (4.52) is negative if \( r > r_1 \), where \( r_1 \) is given in Lemma 4.7.

Consequently, Eqn. (4.50) holds for all \( r > r_1 \). This completes the proof of statement (1) of the proposition.

Statement (2) is proved in a similar way.

Tensoring Eqn. (4.49) with \( E \otimes \Omega_Z \otimes f^*L^{-r} \) and considering the corresponding long exact sequence of cohomologies, we conclude that if
\begin{equation}
H^0(Z, (B_n)_Z \otimes E \otimes \Omega_Z \otimes f^*L^{-r}) = 0
\end{equation}
and
\begin{equation}
H^1(Z, E \otimes \Omega_Z \otimes f^*L^{-r} \otimes (F^n_Z)_* \mathcal{O}_Z) = 0,
\end{equation}
then \( H^1(Z, E \otimes \Omega_Z \otimes f^*L^{-r}) = 0. \)

We have
\[ L_{\max}((B_n)_Z \otimes E \otimes \Omega_Z \otimes f^*L^{-r}) = L_{\max}((B_n)_Z \otimes E \otimes f^*L^{-r}) + L_{\max}(\Omega_Z) \]
\[ = L_{\max}((B_n)_Z \otimes E \otimes f^*L^{-r}) + L_{\max}(f^*\Omega_X). \]
We saw that \( L_{\max}((B_n)_Z \otimes E \otimes f^*L^{-r}) < 0 \) if \( r > r_1 \). Therefore,
\[ L_{\max}((B_n)_Z \otimes E \otimes \Omega_Z \otimes f^*L^{-r}) < 0 \]
for all \( r > r_1 + L_{\max}(\Omega_X)/\text{degree}(L) \). Hence Eqn. (4.53) holds if
\[ r > r_1 + L_{\max}(\Omega_X)/\text{degree}(L). \]

As before, using the projection formula and the fact that the Frobenius morphism \( F_Z \) is flat we have
\[ H^1(Z, E \otimes \Omega_Z \otimes f^*L^{-r} \otimes (F^n_Z)_* \mathcal{O}_Z) = H^1(Z, (F^n_Z)^* (E \otimes \Omega_Z \otimes f^*L^{-r})) \]
\[ \cong H^1(Z, (F^n_Z)^* (\Omega_Z \otimes f^*L^{-r}))^{\text{Gr}} = H^1(Z, (F^n_Z)^* f^*(\Omega_X \otimes L^{-r}))^{\text{Gr}}. \]

Now, Lemma 4.8 says that there is an integer \( N' \) such that
\[ H^1(Z, (F^n_Z)^* f^*(\Omega_X \otimes L^{-r})) = 0 \]
for all \( r > N' \). Therefore, Eqn. (4.54) holds if \( r > N' \).

Thus, statement (2) of the proposition holds if \( r > \text{Max}\{N', r_1 + L_{\max}(\Omega_X/\text{degree}(L))\} \). This completes the proof of the proposition. \( \square \)

Proposition 4.11(2) will be very useful in Section 5

**Lemma 4.12.** Let \( X \) and \( L \) be as in Proposition 4.11. There is an integer \( N = N(X, L) \) with the following property. Let \( f : Z \longrightarrow X \) be any connected étale Galois cover and \( E \) an \( F \)-trivial vector bundle over \( Z \). Then
\[ H^2(Z, E \otimes f^*L^{-r}) = 0 \]
for all \( r > N \).

Proof. Take any \( f \) and \( E \) as in the statement of the lemma. Let \( n \) be an integer such that the vector bundle \((F^r_Z)^*E\) is trivializable.

Let
\[
H^1(Z, (B_n)_Z \otimes E \otimes f^*L^{-r}) \longrightarrow H^2(Z, E \otimes f^*L^{-r}) \longrightarrow H^2(Z, E \otimes ((F^r_Z)_n, \mathcal{O}_Z) \otimes f^*L^{-r})
\]
be the exact sequence of cohomologies for the short exact sequence of vector bundles obtained by tensoring the exact sequence in Eqn. (4.49) with \( \mathcal{O}_Z \). Using this exact sequence it follows that if
\[
H^2(Z, E \otimes ((F^r_Z)_n, \mathcal{O}_Z) \otimes f^*L^{-r}) = 0
\]
and
\[
H^1(Z, (B_n)_Z \otimes E \otimes f^*L^{-r}) = 0,
\]
then \( H^2(Z, E \otimes f^*L^{-r}) = 0 \).

Using the projection formula and the fact that the Frobenius morphism \( F_Z \) is flat we have
\[
H^2(Z, E \otimes ((F^r_Z)_n, \mathcal{O}_Z) \otimes f^*L^{-r}) = H^2(Z, (F^n_Z)_n(E \otimes f^*L^{-r})) \cong H^2(Z, f^*L^{-p^r}) \otimes E,
\]
where \( r_E = \text{rank}(E) \). Now, Lemma 4.7 says that \( H^2(Z, f^*L^{-p^r}) = 0 \) if \( r > r_2 \), where \( r_2 \) is given by Lemma 4.7.

Therefore, Eqn. (4.55) holds if \( r > r_2 \).

To prove Eqn. (4.56), tensor the exact sequence in Eqn. (4.49) with \( (B_n)_Z \otimes E \otimes f^*L^{-r} \) and consider the corresponding long exact sequence of cohomologies. From it we conclude that Eqn. (4.56) holds if
\[
H^0(Z, (B_n)_Z \otimes (B_n)_Z \otimes E \otimes f^*L^{-r}) = 0
\]
and
\[
H^1(Z, ((F^r_Z)_n, \mathcal{O}_Z) \otimes (B_n)_Z \otimes E \otimes f^*L^{-r}) = 0.
\]

Note that
\[
L_{\max}((B_n)_Z \otimes (B_n)_Z \otimes E \otimes f^*L^{-r}) = 2L_{\max}((B_n)_Z) - r \cdot \deg(f^*L)
\]
as \( L_{\max}(E) = 0 \) and \( L_{\max}((B_n)_Z \otimes (B_n)_Z) = 2L_{\max}((B_n)_Z) \).

Corollary 4.5 says that \((B_n)_Z = f^*B_n\). Using Eqn. (4.25) and Proposition 4.2 we know that
\[
L_{\max}(B_n) \leq M,
\]
where \( M \) is given in Proposition 4.2. In other words,
\[
2L_{\max}(B_n) - r \cdot \deg(L) < 0
\]
if \( r > 2M/\deg(L) \). Since \((B_n)_Z = f^*B_n\), this implies that
\[
2L_{\max}((B_n)_Z) - r \cdot \deg(f^*L) < 0
\]
if \( r > 2M/\deg(L) \).
Therefore, using Eqn. (4.59) we conclude that Eqn. (4.57) holds if \( r > 2M/\text{degree}(L) \).

To prove Eqn. (4.58), first note that using projection formula and the fact that \((F^n_Z)_*E\) is trivializable we have

\[
H^1(Z, ((F^n_Z)_*\mathcal{O}_Z) \otimes (B_n)_Z \otimes E \otimes f^*L^{-r}) = H^1(Z, (F^n_Z)_*(F^m_Z)*((B_n)_Z \otimes E \otimes f^*L^{-r}))
\]

\[
= H^1(Z, (F^n_Z)*((B_n)_Z \otimes E \otimes f^*L^{-r})) \cong H^1(Z, (F^n_Z)*((B_n)_Z \otimes f^*L^{-r}))^{r_E},
\]

where \( r_E = \text{rank}(E) \). Therefore, Eqn. (4.58) holds if

\[
(4.60) \quad H^1(Z, (F^n_Z)*((B_n)_Z \otimes f^*L^{-r})) = 0.
\]

Tensor the exact sequence in Eqn. (4.49) with \( f^*L^{-r} \) and then pull it back by \( F^n_Z \). Consider the long exact sequence of cohomologies

\[
(4.61) \quad \rightarrow H^1(Z, (F^n_Z)*(((F^n_Z)_*\mathcal{O}_Z) \otimes f^*L^{-r})) \rightarrow H^1(Z, (F^n_Z)*((B_n)_Z \otimes f^*L^{-r}))
\]

\[
\rightarrow H^2(Z, (F^n_Z)*f^*L^{-r}) \rightarrow
\]

corresponding to the resulting exact sequence of vector bundles.

Using projection formula we have

\[
(F^n_Z)*(((F^n_Z)_*\mathcal{O}_Z) \otimes f^*L^{-r}) = (F^n_Z)*((F^n_Z)*f^*L^{-r} - (F^n_Z)*f^*L^{-r}).
\]

Therefore, from Proposition 4.10 we conclude that there is an integer \( N' \) such that

\[
H^1(Z, (F^n_Z)*(((F^n_Z)_*\mathcal{O}_Z) \otimes f^*L^{-r})) = H^1(Z, (F^n_Z)*f^*L^{-r}) = 0
\]

if \( r > N' \). In view of this, using Lemma 4.7 from Eqn. (4.61) we conclude that there is an integer \( N'' \) such that Eqn. (4.60) holds if \( r > N'' \). This completes the proof of the proposition.

**Proposition 4.13.** Let \( X \) be an irreducible smooth projective variety of dimension \( d \), with \( d \geq 3 \). Fix an ample line bundle \( L \) over \( X \). There is an integer \( N = N(X, L) \) with the following property. Let \( f : Z \rightarrow X \) be a connected étale Galois cover and \( E \) an \( F \)-trivial vector bundle over \( Z \). Then for any smooth divisor \( D \in |L'| \) with \( r > N \),

\[
H^1(f^{-1}(D), (E|_{f^{-1}(D)}) \otimes N_{f^{-1}(D)}) = 0,
\]

where \( N_{f^{-1}(D)} \) is the normal bundle of the hypersurface \( f^{-1}(D) \).

**Proof.** Consider the exact sequence of sheaves

\[
0 \rightarrow E \otimes \mathcal{O}_Z(-2f^{-1}(D)) \rightarrow E \otimes \mathcal{O}_Z(-f^{-1}(D)) \rightarrow (E|_{f^{-1}(D)}) \otimes N_{f^{-1}(D)} \rightarrow 0
\]

over \( Z \). Let

\[
(4.62) \quad \rightarrow H^1(Z, E \otimes \mathcal{O}_Z(-f^{-1}(D))) \rightarrow H^1(f^{-1}(D), (E|_{f^{-1}(D)}) \otimes N_{f^{-1}(D)})
\]

\[
\rightarrow H^2(Z, E \otimes \mathcal{O}_Z(-2f^{-1}(D)))
\]

be the long exact sequence of cohomologies corresponding to it.

Proposition 4.11 says that there is an integer \( N' \) such that

\[
H^1(Z, E \otimes \mathcal{O}_Z(-f^{-1}(D))) = 0
\]
for all \( r > N' \). Therefore, using Lemma 4.12 from Eqn. (4.62) we conclude that there is an integer \( N \) such that

\[
H^1(f^{-1}(D), (E|_{f^{-1}(D)}) \otimes N^*_{f^{-1}(D)}) = 0
\]

for all \( r > N \). This completes the proof of the proposition.

\[\square\]

5. Injectivity of homomorphism of fundamental group schemes

As before, let \( X \) be a smooth projective variety defined over an algebraically closed field \( k \) of characteristic \( p \), with \( p > 0 \). Let \( L \) be an ample line bundle over \( X \).

**Theorem 5.1.** Assume that \( \dim X \geq 3 \). There is an integer \( d_0 = d(X, L) \) with the following property. Let \( D \in |L^\otimes d| \) be a smooth divisor, where \( d > d_0 \). Let \( x_0 \) be any \( k \)-rational point of \( D \). Then the homomorphism between fundamental group schemes

\[
\pi(D, x_0) \to \pi(X, x_0)
\]

induced by the inclusion map \( D \hookrightarrow X \) is a closed immersion.

**Proof.** We will use the criterion given in Proposition 3.4(2) for a homomorphism to be a closed immersion.

Let \( D \in |L^\otimes d| \) be a smooth divisor. Let \( E \) be an essentially finite vector bundle over \( D \). Using Proposition 3.4(2), to prove the theorem it suffices to show the following:

If \( d \geq d_0 \), then there is an essentially finite vector bundle \( V \) over \( X \) whose restriction \( V|_D \) to \( D \) is isomorphic to \( E \).

Take any essentially finite vector bundle \( E \) over \( D \). Let

\[
f_D : \tilde{D} \to D
\]

be a connected étale Galois covering such that \( f_\tilde{D}E \) is an \( F \)-trivial vector bundle over \( \tilde{D} \) (see Definition 2.1 for \( F \)-trivial vector bundles).

As \( \dim X \geq 3 \) with \( D \) ample, from Grothendieck’s Lefschetz theory we know that the inclusion of \( D \) in \( X \) induces an isomorphism of étale fundamental groups [Gr page 123, Théorème 3.10], [Ha page 177, Corollary 2.2]. Therefore, there is a unique (up to an isomorphism) connected étale Galois covering

\[
f : Z \to X
\]

and an isomorphism

\[
\alpha : \tilde{D} \to f^{-1}(D)
\]

such that \( f_D = f \circ \alpha \), where \( f_D \) is the covering morphism in Eqn. (5.2).

Let

\[
F_\tilde{D} : \tilde{D} \to \tilde{D}
\]
be the Frobenius morphism of the variety $\tilde{D}$. Since $f_D^*E$ is an $F$–trivial vector bundle, there is a nonnegative integer $m$ such that the vector bundle $(F^m_D)^* f_D^* E$ over $\tilde{D}$ is trivializable. As done in the proof of Theorem 3.5, we will employ induction on $m$.

First assume that $m = 0$, in other words, the vector bundle $f_D^* E$ itself is trivializable. As done in the proof of Theorem 3.5, we will consult the inductive hypothesis. There is a nonnegative integer $m$ such that the restriction $V|_D$ is isomorphic to $E$, and furthermore, the vector bundle $f^* V$ is trivializable, where $f$ is the morphism in Eqn. (5.7). In particular, the criterion in Proposition 3.4 is valid if $m = 0$.

We assume that $E$ extends to an essentially finite vector bundle over $X$ provided $m \leq n_0 - 1$. We will show that $E$ extends to an essentially finite vector bundle over $X$ if $m = n_0$.

Let $E$ be an essentially finite vector bundle over $D$ with $m = n_0$. Consider the vector bundle $F_D^* E$ over $D$, where $F_D : D \to D$ is the Frobenius morphism. Since

$$f_D^* f_D^* E = F_D^* f_D^* E$$

over $\tilde{D}$, where $f_D$ is the morphism in Eqn. (5.2) and $F_D$ is defined in Eqn. (5.5), we conclude that the vector bundle $(F^m_D)^* f_D^* E$ is isomorphic to $(F^m_D)^* f_D^* E$. Hence $(F^m_D)^* f_D^* F_D^* E$ is trivializable. Therefore, by the induction hypothesis, the vector bundle $F_D^* E$ extends to $X$ as an essentially finite vector bundle.

Let $V'$ be an essentially finite vector bundle over $X$ such that $V'|_D \cong F_D^* E$. Fix an isomorphism of $V'|_D$ with $F_D^* E$. Let

$$V_1 := f^* V'$$

be the essentially finite vector bundle over $Z$, where $f$ is the morphism in Eqn. (5.3).

Henceforth, we will assume that $d > p\max(\Omega_X)/\text{degree}(L)$.

We will identify $\tilde{D}$ with $f^{-1}(D)$ using $\alpha$ in Eqn. (5.4). In view of the above assumption on $d$ and Eqn. (3.20), from Theorem 3.5 it follows that the homomorphism of fundamental group schemes

$$\pi(\tilde{D}, x_0) \to \pi(Z, x_0)$$

induced by the inclusion map $\tilde{D} \to Z$ is surjective, where $x_0$ is any $k$–rational point of $\tilde{D}$. Note that using Eqn. (5.6), Eqn. (5.7) and the identity $\iota \circ F_D = F_Z \circ \iota$, where $\iota : \tilde{D} \to Z$ is the inclusion map and $F_Z$ is the Frobenius morphism of $Z$, we have

$$((F^m_Z)^* V_1)|_{\tilde{D}} = (F^m_D)^* ((f^* V')|_{\tilde{D}}) = (F^m_D)^* f_D^* F_D^* E = (F^m_D)^* f_D^* E.$$

Hence $((F^m_Z)^* V_1)|_{\tilde{D}}$ is trivializable as $(F^m_D)^* f_D^* E$ is so (by assumption on $E$). Therefore, from the fact that the homomorphism in Eqn. (5.8) is surjective it follows immediately that the vector bundle $(F^m_Z)^* V_1$ over $Z$ is trivializable.

Using the isomorphism in Eqn. (5.6), the Cartier connection on $F^*_D f_D^* E$ induces a connection on the vector bundle $f^*_D F^*_D E$. This induced connection on $f^*_D F^*_D E$ will also be called the Cartier connection. Let $\nabla^{\tilde{D}}$ denote the Cartier connection on $f^*_D F^*_D E$. This
connection $\nabla^{\tilde{D}}$ coincides with the pull back, by $f_{\tilde{D}}$, of the Cartier connection on $F_{\tilde{D}}^*E$. Indeed, this follows from the fact that the Cartier connection is compatible with the pull back operation.

We will show that $\nabla^{\tilde{D}}$ extends to a connection on the vector bundle $V_1$ defined in Eqn. (5.7). For that we will first show that $V_1$ admits a connection.

We recall from [At] that a connection on $V_1$ is a splitting of the Atiyah exact sequence

$$0 \rightarrow \text{End}(V_1) \rightarrow \text{At}(V_1) \rightarrow TZ \rightarrow 0.$$ 

Let

$$\theta(V_1) \in H^1(Z, \text{End}(V_1) \otimes \Omega_Z)$$

be the obstruction class for the existence of a splitting of the Atiyah exact sequence. This obstruction class is also called the Atiyah class of $V_1$.

Let

$$\tau : H^1(Z, \text{End}(V_1) \otimes \Omega_Z) \rightarrow H^1(\tilde{D}, (\text{End}(V_1) \otimes \Omega_Z)|_{\tilde{D}})$$

be the homomorphism induced by the inclusion map $\tilde{D} \hookrightarrow Z$. For notational convenience, the vector bundle $f_{\tilde{D}}^*F_{\tilde{D}}^*E = V_1|_{\tilde{D}}$ over $\tilde{D}$ will be denoted by $E_1$. Let

$$\tau' : H^1(\tilde{D}, (\text{End}(V_1) \otimes \Omega_Z)|_{\tilde{D}}) \rightarrow H^1(\tilde{D}, \text{End}(E_1) \otimes \Omega_{\tilde{D}})$$

be the homomorphism obtained from the natural projection

$$\Omega_Z|_{\tilde{D}} \rightarrow \Omega_{\tilde{D}}.$$

From general properties of the Atiyah class it follows that

$$c(E_1) := \tau' \circ \tau(\theta(V_1)) \in H^1(\tilde{D}, \text{End}(E_1) \otimes \Omega_{\tilde{D}})$$

is the Atiyah class for $E_1$ where $\theta(V_1)$, $\tau$ and $\tau'$ are defined in Eqn. (5.9), Eqn. (5.10) and Eqn. (5.11) respectively. Since $E_1$ admits the Cartier connection $\nabla^{\tilde{D}}$, the Atiyah class $c(E_1)$ vanishes.

Since $\tau' \circ \tau(\theta(V_1)) = c(E_1) = 0$, we conclude that $\theta(V_1) = 0$ if both the homomorphisms $\tau$ and $\tau'$ are injective.

Consider the long exact sequence of cohomologies

$$H^1(Z, \text{End}(V_1) \otimes \Omega_Z(-\tilde{D})) \rightarrow H^1(Z, \text{End}(V_1) \otimes \Omega_Z) \xrightarrow{\tau} H^1(\tilde{D}, \text{End}(E_1) \otimes (\Omega_Z|_{\tilde{D}}))$$

corresponding to the short exact sequence of sheaves

$$0 \rightarrow \text{End}(V_1) \otimes \Omega_Z(-\tilde{D}) \rightarrow \text{End}(V_1) \otimes \Omega_Z \rightarrow \text{End}(E_1) \otimes (\Omega_Z|_{\tilde{D}}) \rightarrow 0.$$ 

Using it together with Proposition 4.11(2), which says that there exists an integer $N(X, L)$ such that

$$H^1(Z, \text{End}(V_1) \otimes \Omega_Z(-\tilde{D})) = 0$$

if $r > N(X, L)$, we conclude that the homomorphism $\tau$ in Eqn. (5.10) is injective if $r > N(X, L)$. 
To prove that \( \tau' \) is injective, consider the long exact sequence of cohomologies
\[
H^1(\tilde{D}, \mathcal{E}nd(E_1) \otimes N^*_B) \to H^1(\tilde{D}, \mathcal{E}nd(E_1) \otimes (\Omega_Z|_B)) \xrightarrow{\tau'} H^1(\tilde{D}, \mathcal{E}nd(E_1) \otimes \Omega_\tilde{D})
\]
corresponding to the short exact sequence of sheaves
\[
0 \to \mathcal{E}nd(E_1) \otimes N^*_D \to \mathcal{E}nd(E_1) \otimes (\Omega_Z|_D) \to \mathcal{E}nd(E_1) \otimes \Omega_\tilde{D} \to 0,
\]
where \( N_D \) as before is the normal bundle to the divisor \( \tilde{D} \). Using it together with Proposition 4.13 which says that there exists some integer \( N'(X, L) \) such that
\[
H^1(\tilde{D}, \mathcal{E}nd(E_1) \otimes N^*_D) = 0
\]
if \( r > N'(X, L) \), we conclude that \( \tau' \) defined in Eqn. (5.11) is also injective.

Henceforth, we will assume that \( r > N(X, L), N'(X, L) \).

Since both \( \tau \) and \( \tau' \) are injective and \( \tau' \circ \tau(V_1) = c(E_1) = 0 \), the Atiyah class \( \theta(V_1) \) defined in Eqn. (5.9) vanishes. In other words, the vector bundle \( V_1 \) admits a connection.

We will now show that \( V_1 \) admits a connection that restricts to the Cartier connection \( \nabla^\tilde{D} \) on \( V_1|_B = E_1 \).

Let \( \nabla^{V_1} \) be a connection on \( V_1 \). Let \( \nabla^1 \) denote the connection on \( E_1 = V_1|_B \) induced by this connection \( \nabla^{V_1} \). On the other hand, the vector bundle \( E_1 \) has the Cartier connection \( \nabla^E \). Therefore, we have
\[
\gamma(\nabla^1) := \nabla^\tilde{D} - \nabla^1 \in H^0(\tilde{D}, \mathcal{E}nd(E_1) \otimes \Omega_\tilde{D}).
\]

Consider the long exact sequence of cohomologies
\[
H^0(Z, \mathcal{E}nd(V_1) \otimes \Omega_Z) \xrightarrow{\tau_0} H^0(\tilde{D}, \mathcal{E}nd(E_1) \otimes (\Omega_Z|_D)) \to H^1(Z, \mathcal{E}nd(V_1) \otimes \Omega_Z|_D)
\]
corresponding to the exact sequence in Eqn. (5.12). In view of Eqn. (5.13), it follows from this long exact sequence that the restriction homomorphism
\[
\tau_0 : H^0(Z, \mathcal{E}nd(V_1) \otimes \Omega_Z) \to H^0(\tilde{D}, \mathcal{E}nd(E_1) \otimes (\Omega_Z|_D))
\]
is surjective.

Similarly, using Eqn. (5.15) and the long exact sequence of cohomologies
\[
H^0(\tilde{D}, \mathcal{E}nd(E_1) \otimes (\Omega_Z|_D)) \xrightarrow{\tau'_0} H^0(\tilde{D}, \mathcal{E}nd(E_1) \otimes \Omega_\tilde{D}) \to H^1(\tilde{D}, \mathcal{E}nd(E_1) \otimes N^*_\tilde{D})
\]
corresponding to the exact sequence in Eqn. (5.14) we conclude that the above homomorphism \( \tau'_0 \) is surjective. Therefore, the composition homomorphism
\[
\tau'_0 \circ \tau_0 : H^0(Z, \mathcal{E}nd(V_1) \otimes \Omega_Z) \to H^0(\tilde{D}, \mathcal{E}nd(E_1) \otimes \Omega_\tilde{D})
\]
is surjective, where \( \tau_0 \) and \( \tau'_0 \) are defined in Eqn. (5.17) and Eqn. (5.18) respectively.

Fix any
\[
\beta \in H^0(Z, \mathcal{E}nd(V_1) \otimes \Omega_Z)
\]
such that
\[
\tau'_0 \circ \tau_0(\beta) = \gamma(\nabla^1),
\]
where the section $\gamma(\nabla^1)$ is constructed in Eqn. (5.16) and $\tau'_0 \circ \tau_0$ is the surjective homomorphism constructed in Eqn. (5.19).

Let
\begin{equation}
\nabla' := \nabla^{V_1} + \beta
\end{equation}
be the connection on $V_1$, where $\beta$ is defined in Eqn. (5.20) and $\nabla^{V_1}$ is the earlier mentioned connection on $V_1$ (recall that $\nabla^{V_1}$ restricts to $\nabla^1$ on $\tilde{D}$). Since $\nabla^1$ is the restriction of $\nabla^{V_1}$ to $\tilde{D}$, from Eqn. (5.21) it follows immediately that the restriction of the connection $\nabla'$ (defined in Eqn. (5.22)) to $\tilde{D}$ coincides with the Cartier connection $\nabla^{\tilde{D}}$ on $E_1$.

Let $\Gamma = \text{Gal}(f)$ be the Galois group for the covering $f$ defined in Eqn. (5.3). The vector bundle $V_1$ in Eqn. (5.7), being a pull back by $f$, is equipped with an action of $\Gamma$. For any $g \in \text{Gal}(f)$ the connection $g^*\nabla'$ is another connection on $V_1$. Since the space of all connections on $V_1$ is an affine space for the vector space $H^0(Z, \text{End}(V_1) \otimes \Omega_Z)$, we conclude that $g^*\nabla' - \nabla'$ is an element of $H^0(Z, \text{End}(V_1) \otimes \Omega_Z)$.

We showed that the restriction of the connection $\nabla'$ to $\tilde{D}$ coincides with the Cartier connection $\nabla^{\tilde{D}}$. On the other hand, the Cartier connection $\nabla^{\tilde{D}}$ is invariant under the action of $\text{Gal}(f_D) = \text{Gal}(f) = \Gamma$ on $E_1$. Therefore, the element $g^*\nabla' - \nabla' \in H^0(Z, \text{End}(V_1) \otimes \Omega_Z)$ is in the kernel of the homomorphism
\[
\tau'_0 \circ \tau_0 : H^0(Z, \text{End}(V_1) \otimes \Omega_Z) \rightarrow H^0(\tilde{D}, \text{End}(E_1) \otimes \Omega_\tilde{D}),
\]
where $\tau'_0 \circ \tau_0$ is the homomorphism in Eqn. (5.19). This homomorphism $\tau'_0 \circ \tau_0$ coincides with the composition of the homomorphism in Eqn. (3.21) with the homomorphism $\beta$ in Eqn. (3.23) with the substitution $E' = V_1 = V'$. In the proof of Theorem 3.5 we saw that the homomorphism in Eqn. (3.21) and the homomorphism $\beta$ in Eqn. (3.23) are both injective. Therefore, the homomorphism $\tau'_0 \circ \tau_0$ is also injective. We showed above that
\[
\tau'_0 \circ \tau_0(g^*\nabla' - \nabla') = 0.
\]
Consequently, $g^*\nabla' = \nabla'$.

Hence we have proved that the connection $\nabla'$ on $V_1$ is preserved by the action of $\Gamma$ on $V_1$. Therefore, it descends to a connection on the vector bundle $V'$ over $X$ (see Eqn. (5.7)). Let $\nabla$ denote the connection on $V'$ given by $\nabla'$. Therefore, we have
\begin{equation}
\nabla' = f^*\nabla.
\end{equation}
Moreover, it follows that the restriction to the divisor $D$ of the connection $\nabla$ on $V'$ coincides with the Cartier connection on $F^*_DE$ (recall that $V'|_D = F^*_DE$). Indeed, this is a consequence of the fact that the restriction of the connection $\nabla'$ to $\tilde{D}$ coincides with the Cartier connection $\nabla^{\tilde{D}}$.

We will now show that the $p$–curvature of $\nabla$ vanishes (see [Ka, page 190, (5.0.4)] for the definition of $p$–curvature of a connection).
Let
\[(5.24) \quad \psi(\nabla) \in H^0(X, \mathcal{E}nd(V') \otimes F_X^*\Omega_X)\]
be the $p$–curvature of the connection $\nabla$ on $V'$ in Eqn. (5.23). Let
\[\tilde{\alpha} : H^0(X, \mathcal{E}nd(V') \otimes F_X^*\Omega_X) \to H^0(D, (\mathcal{E}nd(V') \otimes F^*_D\Omega_X)|_D)\]
be the restriction homomorphism. In view of our assumption that
\[d > p \cdot L_{\text{max}}(\Omega_X)/\text{degree}(L),\]
from the second part of the Lemma 3.2 we know that
\[H^0(X, \mathcal{E}nd(V') \otimes (F_X^*\Omega_X)(-D)) = 0.\]
Hence the above homomorphism $\tilde{\alpha}$ is injective. Let
\[(5.25) \quad \tilde{\beta} : H^0(D, (\mathcal{E}nd(V') \otimes F_X^*\Omega_X)|_D) \to H^0(D, (\mathcal{E}nd(V'|_D) \otimes F^*_D\Omega_D)|_D)\]
be the homomorphism induced by the natural projection $\Omega_X|_D \to \Omega_D$. Since
\[H^0(D, (\mathcal{E}nd(V'|_D) \otimes F^*_D\Omega_D)|_D) = 0\]
(see the second part of Lemma 3.3), from the left exact sequence of global sections for the short exact sequence of sheaves
\[0 \to \mathcal{E}nd(V'|_D) \otimes F^*_D\mathcal{N}_D \to (\mathcal{E}nd(V') \otimes F_X^*\Omega_X)|_D \to \mathcal{E}nd(V'|_D) \otimes F^*_D\Omega_D \to 0\]
it follows that the homomorphism $\tilde{\beta}$ defined in Eqn. (5.25) is injective.

Since the $p$–curvature of any Cartier connection vanishes, and the restriction to $D$ of the connection $\nabla$ coincides with the Cartier connection on $F^*_DE$, we have
\[\tilde{\beta} \circ \tilde{\alpha}(\psi(\nabla)) = 0,\]
where $\psi(\nabla)$ is the $p$–curvature in Eqn. (5.24). As both $\tilde{\alpha}$ and $\tilde{\beta}$ are injective homomorphisms, this implies that $\psi(\nabla) = 0$.

Since the $p$–curvature of $\nabla$ vanishes, there is a vector bundle $W$ over $X$ such that $F_X^*W$ equipped with the Cartier connection is identified with the vector bundle $V'$ equipped with the connection $\nabla$ [Ka, page 190, Theorem 5.1.1]. Since $F_X^*W = V'$ is an essentially finite vector bundle, it follows that the vector bundle $W$ is also essentially finite (this was noted in the proof of Theorem 3.5).

Therefore, using induction, any essentially finite vector bundle over $D$ extends to $X$ as an essentially finite vector bundle. Using Proposition 3.4(2) this completes the proof of the theorem. $\square$

Using the proof of Theorem 5.1 we have the following corollary:

**Corollary 5.2.** Let $D$ be a smooth hypersurface in the projective space $\mathbb{P}_k^n$, with $n \geq 3$. Then the fundamental group scheme $\pi(D, x_0)$ is trivial.
Proof. We will follow the steps of the proof of Theorem 5.1.

Take any smooth hypersurface $D$ of $\mathbb{P}^n_k$. Since the étale fundamental group of $D$ is trivial, we have $Z = \mathbb{P}^n_k$ and $f$ to be the identity map of $\mathbb{P}^n_k$. Next we note that the fundamental group scheme of $\mathbb{P}^n_k$ is trivial (see the corollary in [No2, page 93] following the proof of Proposition 9). In other words, any essentially finite vector bundle over $\mathbb{P}^n_k$ is a trivializable vector bundle. Next we note that the cotangent bundle $\Omega_{\mathbb{P}^n_k}$ is strongly semistable. This can be proved using the facts that the tangent bundle $T_{\mathbb{P}^n_k}$ is globally generated and $\text{Aut}(\mathbb{P}^n_k)$ acts irreducibly on $H^0(\mathbb{P}^n_k, T_{\mathbb{P}^n_k})$.

In particular,

$$L_{\text{max}}(\Omega_{\mathbb{P}^n_k}) = -\frac{n+1}{n}.$$ 

Using the above observations in the proof of Theorem 5.1 we conclude that the proof of the corollary will be complete once we establish the following two assertions:

(5.26) 

$$H^1(\mathbb{P}^n_k, \Omega_{\mathbb{P}^n_k}(-D)) = 0$$

and

(5.27) 

$$H^1(D, N^*_D) = 0.$$ 

Note that these two correspond to Eqn. (5.13) and Eqn. (5.15) respectively.

Since $n \geq 3$, we have

$$H^1(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(\delta)) = 0 = H^2(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(\delta))$$

for all $\delta$. Therefore, Eqn. (5.26) follows from the long exact sequence of cohomologies for the short exact sequence of vector bundles

$$0 \rightarrow \Omega_{\mathbb{P}^n_k}(-D) \rightarrow \mathcal{O}_{\mathbb{P}^n_k}(-1)(-D)^{\otimes(n+1)} \rightarrow \mathcal{O}_{\mathbb{P}^n_k}(-D) \rightarrow 0$$

obtained from the Euler sequence.

Similarly, Eqn. (5.27) follows from the long exact sequence of cohomologies for the short exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n_k}(-2D) \rightarrow \mathcal{O}_{\mathbb{P}^n_k}(-D) \rightarrow N^*_D \rightarrow 0.$$ 

This completes the proof of the corollary. \qed

Remark 5.3. The integer $d_0$ in Theorem 5.1 can be taken to be any integer satisfying the following conditions.

1. $d_0 \geq p \cdot r_1$, where $r_1$ is defined in Lemma 4.7.
2. $d_0 \geq N(B_1)$, where $B_1$ is defined in Eqn. (4.14), and $N(V)$ of a vector bundle $V$ is defined in the proof of Lemma 4.8.
3. $d_0 \geq N(G_1, \cdots, G_{(p-1)\dim(X)})$, where $G_i$ are defined in Proposition 4.1(3) and $N(G_1, \cdots, G_{(p-1)\dim(X)})$ is defined in the proof of Lemma 4.9.
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