MABUCHI METRICS AND PROPERNESS OF THE MODIFIED DING FUNCTIONAL

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ABSTRACT. In this paper, we study Mabuchi metrics on Fano manifolds. We prove that Mabuchi metrics exist if the modified Ding functional is proper modulo a reductive subgroup of its automorphism group. On the other hand, the inverse that Mabuchi metrics implies the properness is obtained by using Darvas-Rubinstein’s properness principle. As an application, we establish a criterion for the existence of Mabuchi metrics on Fano group compactifications.

1. INTRODUCTION

The existence of canonical metrics has been a fundamental and longstanding problem in Kähler geometry. On Fano manifolds, Kähler-Einstein metrics have been studied extensively. The most remarkable progress is the resolution of Yau-Tian-Donaldson conjecture which relates the existence of Kähler-Einstein metrics to the K-stability of the Fano manifold [33, 7]. It has been known early in 1980’s that the existence of Kähler-Einstein metrics fails when the Fano manifold has nonvanishing Futaki invariant. In this case, other canonical metrics, such as extremal metrics and Kähler-Ricci solitons have attracted many attentions.

In [24, 25, 26, 27], Mabuchi studied a generalized Kähler-Einstein metric, which is neither an extremal metric nor a Kähler-Ricci soliton. Following [36], we call this metric the Mabuchi metric for simplicity. Let $M$ be a compact Fano manifold of complex dimension $n$. Let

$$\omega = \sqrt{-1}g_{ij}dz^i \wedge d\bar{z}^j \in 2\pi c_1(M)$$

be a Kähler metric and $h_\omega$ be its Ricci potential. $\omega$ is a Mabuchi metric if

$$X_\omega := -\sqrt{-1}g^{ij}\frac{\partial e^{h_\omega}}{\partial \bar{z}^j} \frac{\partial}{\partial z^i}$$

is holomorphic [24]. The uniqueness of Mabuchi metrics has been proved in [27]. Recently, Donaldson introduced a new GIT (geometric invariant theory) picture [13], in which the corresponding moment map is given by the Ricci potential. Then Yao observed that in this picture $X_\omega$ is holomorphic if and only if $\omega$ is a critical point of the norm square of the moment map, given by the following energy [36]

$$\mathcal{E}^D(\omega) = \int_M (e^{h_\omega} - 1)^2 \omega^n.$$

This brings new interests in the study of Mabuchi metrics. On toric Fano manifolds, the notion of relative Ding stability has been introduced by Yao [36]. He has also established the existence of Mabuchi metrics when the toric Fano manifold is relatively Ding stable.
The purpose of this paper is to discuss the existence of Mabuchi metrics on general Fano manifolds through properness of energy functionals.

According to [24], if \( \omega \) is a Mabuchi metric, then (1.1) coincides with the extremal vector field [14]. To state the main results, we first recall notions on extremal vector field. Denote by \( \text{Aut}_0(M) \) the identity component of its holomorphic transformation group. Its Lie algebra \( \eta(M) \) consists of all holomorphic vector fields on \( M \). \( \text{Aut}_0(M) \) admits a semi-direct decomposition

\[
\text{Aut}_0(M) = \text{Aut}_r(M) \ltimes \text{Aut}_u(M),
\]

where \( \text{Aut}_r(M) \subset \text{Aut}_0(M) \) is a reductive group and \( \text{Aut}_u(M) \) is the unipotent radical of \( \text{Aut}_0(M) \). Denote by \( \eta_r(M) \) the reductive part of \( \eta(M) \). For any \( v \in \eta(M) \), let \( K_v \) be the one-parameter group generated by the image part \( \text{Im}(v) \). For a Kähler metric \( \omega_0 \in 2\pi c_1(M) \), by Hodge theorem, there is a unique normalized potential given by

\[
(1.3) \quad i_v \omega_0 = \sqrt{-1} \partial \bar{\partial} \theta_v(\omega_0), \quad \int_M \theta_v(\omega_0) \omega_0^n = 0.
\]

Then \( \theta_v(\omega) \) is real valued if and only if \( \omega \) is \( K_v \)-invariant. For any

\[
\phi \in \mathcal{H}_v(\omega_0) := \{ \phi \in C^\infty(M) | \omega_0 + \sqrt{-1} \partial \bar{\partial} \phi > 0, \phi \text{ is } K_v \text{-invariant} \},
\]

the normalized potential \( \theta_v(\omega_0) = \theta_v(\omega_0) + v(\phi) \). Denote by \( \text{Fut}(v) \) the Futaki invariant of \( v \in \eta(M) \). The extremal vector field \( X \) is the holomorphic vector field uniquely determined by [14]

\[
(1.4) \quad \text{Fut}_X(v) := \text{Fut}(v) + \int_M \theta_v(\omega_0) \theta_X(\omega_0) \omega_0^n = 0, \forall v \in \eta(M).
\]

Moreover, \( X \in \eta_c(M) \), the centre of \( \eta_r(M) \) and \( K_X \) lies in a compact Lie group.

From now on, we assume that \( \omega_0 \) is \( K_X \)-invariant unless otherwise claimed. As pointed by Mabuchi [27], both \( \min_M \theta_X(\omega_0) \) and \( \max_M \theta_X(\omega_0) \) are independent of the choice of \( \omega_0 \in 2\pi c_1(M) \). For convenience, we denote by

\[
c_X := \min_M \{ 1 - \theta_X(\omega_0) \}, \quad C_X := \max_M \{ 1 - \theta_X(\omega_0) \}.
\]

By [24], Mabuchi metrics exist only if \( c_X > 0 \), and \( \omega_0 \in 2\pi c_1(M) \) is a Mabuchi metric if

\[
(1.5) \quad \text{Ric}(\omega_0) - \omega_0 = \sqrt{-1} \partial \bar{\partial} \log(1 - \theta_X(\omega_0)).
\]

In [30], Tian introduced the notion of properness of energy functionals as an analytic characterization of existence of Kähler-Einstein metrics. When the automorphism group of \( M \) is not discret, a notion of properness modulo a subgroup of \( \text{Aut}_0(M) \) was reformulated [9, 10, 29, 38]. In particular, Darvas-Rubinstein established a properness principle and solved Tian’s properness conjecture [10]. It is natural to ask the analogous problem for Mabuchi metrics. By [25], the Mabuchi metric is a critical point of the following modified Ding functional

\[
(1.6) \quad \mathcal{D}_X(\phi) = -\frac{1}{V} \int_0^1 \int_M \phi_*(1 - \theta_X(\omega_{h_0})) \omega^n_0 \wedge ds - \log \left( \frac{1}{V} \int_M e^{h_0 - \phi} \omega^n_0 \right),
\]

where \( V = \int_M \omega^n_0 \), \{ \phi_s \}_{s \in [0, 1]} \) is any smooth path in \( \mathcal{H}_X(\omega_0) \) joining 0 and \( \phi \), and \( h_0 \) is the Ricci potential of \( \omega_0 \), normalized by

\[
\int_M e^{h_0} \omega^n_0 = \int_M \omega^n_0.
\]

Our first main result is the following properness theorem.
Theorem 1.1. If $c_X > 0$ and the modified Ding functional is proper modulo a reductive subgroup $H^0$ of $\text{Aut}_0(M)$ which contains $K_X$, then $M$ admits Mabuchi metrics.

It is also interesting to ask the inverse of this theorem. One can show that the existence of Mabuchi metric implies the properness of $\mathcal{D}_X(\cdot)$ modulo the automorphism group of $M$ following the arguments for Kähler-Ricci solitons [9]. However, an optimal properness can be obtained by using the properness principle of Darvas-Rubinstein [10].

Theorem 1.2. Suppose $\text{Aut}_{0}^X(M)$ is the centralizer of $K^c_X$ in $\text{Aut}_0(M)$. If $M$ admits Mabuchi metrics, then there exists $C, C' > 0$, such that

$$\mathcal{D}_X(\phi) \geq C \inf_{\sigma \in \text{Aut}_{0}^X(M)} J_X(\phi_{\sigma}) - C', \quad \forall \phi \in \mathcal{H}_X(\omega_0),$$

where $J_X$ is the modified Aubin’s functional (see Section 2.1) $\phi_{\sigma}$ is defined by

$$\sigma^*(\omega_\phi) = \omega_0 + \sqrt{-1} \partial \bar{\partial} \phi_{\sigma}.$$

Remark 1.3. Suppose $\omega_0$ is a Mabuchi metric on $M$. We can define

$$\Lambda_{1,X} = \{u \in \mathcal{C}(M) | \triangle_{\omega_0} u - \frac{X}{1 - \theta_X(\omega_0)} u = -u\}.$$

Then by the similar argument as [35, Lemma 3.2], one can show that the properness modulo $\text{Aut}_{0}^X(M)$ is equivalent to the properness for Kähler potentials that are perpendicular to $\Lambda_{1,X}$ with respect to the weighted inner product

$$(\varphi, \psi) = \int_M \varphi \psi (1 - \theta_X(\omega_0)) \omega_0^n.$$

The properness condition can be verified for some special Fano manifolds. A characterization for the properness of the modified Ding functional on toric Fano manifolds has been given by [28]. We consider more general Fano group compactifications by using the ideas of [23], in which the modified K-energy is discussed. Let $G$ be a connected, complex reductive group of dimension $n$, we call $M$ a (bi-equivariant) compactification of $G$ if it admits a holomorphic $G \times G$ action on $M$ with an open and dense orbit isomorphic to $G$ as a $G \times G$-homogeneous space [4, 12]. $(M, L)$ is called a polarized compactification of $G$ if $L$ is a $G \times G$-linearized ample line bundle on $M$. In particular, when $L = -K_M$, we call $M$ a Fano group compactification. We establish the criterion for the existence of Mabuchi metrics on Fano group compactifications.

Theorem 1.4. Let $(M, -K_M)$ be a Fano compactification of $G$ and $P$ be the associated polytope. Then $M$ admits Mabuchi metrics if and only if $c_X > 0$ and

$$(1.8) \quad b_X - 4\rho \in \Xi,$$

where

$$b_X = \frac{1}{V} \int_{2P^+} y[1 - \theta_X(y)] \pi(y) dy,$$

$$\pi(y) = \prod_{\alpha \in \Phi^+_+} (\langle \alpha, y \rangle)^2, \quad V = \int_{2P^+} \pi(y) dy,$$

$\Xi$ is the relative interior of the cone generated by positive roots $\Phi^+_+, \rho = \frac{1}{V} \sum_{\alpha \in \Phi^+_+} \alpha$ and $\theta_X(y)$ is the normalized potential of $X$ viewed as a function on $2P^+$, which will be described in Lemma [4.7] below. For notations on group compactifications, see §2.3

1For the definition of properness, see Definition [2.5] below.
The paper is organized as follows: In Section 2, we first review some preliminaries on energy functionals and the definition of properness modulo an automorphism group. Then we recall basic properties of polarized compactifications. Theorem 1.1 and 1.2 will be proved in Section 3. In Section 4, we obtain Theorem 1.4. The sufficient part will be proved by the verification of properness of the modified Ding functional.

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2. Preliminaries

In this section, we first review the notions of energy functionals associated to Mabuchi metrics. Then we recall the basic knowledge for group compactifications for later use.

2.1. Reduction to the complex Monge-Ampère equations. It is clear that (1.5) is equivalent to the following equation
\[
\omega^n (1 - \theta_X (\omega_\phi)) = \omega^n_0 e^{h_0 - \phi}.
\]
We consider the following continuity path
\[
\omega^n_t (1 - \theta_X (\omega_\phi)) = \omega^n_0 e^{h_0 - t \phi}, \quad t \in [0, 1].
\]
Denote \( I := \{ t \in [0, 1] | (2.2) \) has a solution for \( t \} \). Then \( I \) is open by the implicit function theorem. For the starting point \( t = 0 \), we have

**Theorem 2.1.** When \( c_X > 0 \), (2.2) has a solution at \( t = 0 \).

Since we did not find a reference for this result, we give a proof of it for completeness in the appendix. Hence, \( 0 \in I \) and there exists an \( \varepsilon_0 > 0 \) such that (2.2) has a solution for \( t \in [0, \varepsilon_0] \). For the closedness of \( I \), it suffices to establish the \( C^0 \)-estimate of (2.2). The following lemmas will be used later.

**Lemma 2.2.** Let \( \phi_t \) be a solution of (2.2) at \( t \), then the first eigenvalue of
\[
L_t := \Delta_{\omega_\phi} - \frac{X}{1 - \theta_X (\omega_\phi)} + t
\]
is nonnegative for \( t \in [0, 1] \) and equals to 0 only if \( t = 1 \). Consequently, we have the following weighted Poincaré inequality
\[
\int_M \left| \partial \psi \right|^2_{\omega_\phi} (1 - \theta_X (\omega_\phi)) \omega_\phi^n 
\geq t \left[ \int_M \psi^2 (1 - \theta_X (\omega_\phi)) \omega_\phi^n - \frac{1}{V} \left( \int_M \psi (1 - \theta_X (\omega_\phi)) \omega_\phi^n \right)^2 \right].
\]
for any \( K_X \)-invariant \( \psi \in C^{1, \alpha} \).

**Remark 2.3.** We remark that \( L_t \) is self-dual on the space of real-valued \( K_X \)-invariant functions, equipped with the weighted inner product (cf. [27] Lemma 2.1)
\[
\langle f, g \rangle_t := \int_M f L_t(g) (1 - \theta_X (\omega_\phi)) \omega_\phi^n.
\]
Proof of Lemma 2.2] Without loss of generality, we may choose a local co-frame \( \{ \Theta^i \}_{i=1}^n \) such that \( \omega_{\phi_t} = \sqrt{-1} \sum_{i=1}^n \Theta^i \wedge \bar{\Theta}^i \). Suppose \( L_t \psi = -\lambda \psi \). Then

\[
\lambda \int_M \psi_i \psi_j (1 - \theta_X (\omega_{\phi_t})) \omega^n_{\phi_t} = - \int_M \left[ \left( \Delta \omega_{\phi_t} - \frac{X}{1- \theta_X (\omega_{\phi_t})} + t \right) \psi_i \right] \psi_j (1 - \theta_X (\omega_{\phi_t})) \omega^n_{\phi_t}
\]

\[
= - \int_M \psi_{ij} \psi_i (1 - \theta_X (\omega_{\phi_t})) \omega^n_{\phi_t} + \int_M X_i \psi_j \psi_j \omega^n_{\phi_t} + \int_M X_i \psi_j \psi_j \omega^n_{\phi_t} \]

\[
\text{(2.5)} + \int_M \frac{X(\psi) \theta_X (\omega_{\phi_t})}{1- \theta_X (\omega_{\phi_t})} \psi_j (1 - \theta_X (\omega_{\phi_t})) \omega^n_{\phi_t} - t \int M \psi_i \psi_j (1 - \theta_X (\omega_{\phi_t})) \omega^n_{\phi_t},
\]

here and below, we denote \( \phi_t \) for covariant derivatives with respect to \( \omega_{\phi_t} \), similar conventions are used for covariant derivatives of other tensors.

By Ricci identity and integration by parts, we have

\[
- \int M \psi_{ij} \psi_j (1 - \theta_X (\omega_{\phi_t})) \omega^n_{\phi_t}
\]

\[
= - \int M \psi_{ij} \psi_j (1 - \theta_X (\omega_{\phi_t})) \omega^n_{\phi_t} + \int M \psi_j \psi_j \psi_i \psi_j \psi_j \omega^n_{\phi_t}
\]

\[
= \int M \psi_{ij} \psi_j (1 - \theta_X (\omega_{\phi_t})) \omega^n_{\phi_t} - \int M X^i \psi_i \psi_j \psi_j \omega^n_{\phi_t} + \int M \psi_j \psi_j \psi_i \psi_j \psi_j \omega^n_{\phi_t}.
\]

Substituting this into (2.5) and using (2.2), it follows

\[
\lambda \int_M \psi_i \psi_j (1 - \theta_X (\omega_{\phi_t})) \omega^n_{\phi_t}
\]

\[
= \int M \psi_{ij} \psi_j (1 - \theta_X (\omega_{\phi_t})) \omega^n_{\phi_t} + (1 - t) \int M \psi_j \psi_j \psi_i g_{ij}(0) (1 - \theta_X (\omega_{\phi_t})) \omega^n_{\phi_t},
\]

where \( \omega_0 = \sqrt{-1} g_{ij}(0) \Theta^i \wedge \bar{\Theta}^j \), the lemma is proved. \( \square \)

2.2. Energy functionals. Recall that the Aubin’s functionals are given by

\[
I(\phi) = \int_M \phi (\omega^n_\phi - \omega^n_0),
\]

\[
J(\phi) = \int_0^1 \int_M \dot{\phi}_t (\omega^n_\phi - \omega^n_0) \wedge ds,
\]

where \( \{ \phi_t \}_{t \in [0,1]} \) is any smooth path in \( \mathcal{M}_X (\omega_0) \) joining \( 0 \) and \( \phi \). It is known that [32]

\[
0 \leq \frac{1}{n} J(\phi) \leq I(\phi) - J(\phi) \leq n J(\phi).
\]

To deal with Mabuchi metrics, the following modified functionals were introduced in [27]

\[
I_X (\phi) = \int_M \phi [(1 - \theta_X (\omega_0)) \omega^n_\phi - (1 - \theta_X (\omega_0)) \omega^n_0],
\]

\[
J_X (\phi) = \int_0^1 \int_M \dot{\phi}_t [(1 - \theta_X (\omega_0)) \omega^n_\phi - (1 - \theta_X (\omega_0)) \omega^n_0] \wedge ds.
\]

By [27], Remark A.1.9], when \( c_X > 0 \),

\[
0 \leq I_X (\phi) \leq (n + 2) (I_X (\phi) - J_X (\phi)) \leq (n + 1) I_X (\phi).
\]
Lemma 2.4. There are positive constants $c_1, c_2 > 0$ such that

$$c_1 I(\phi) \leq I_X(\phi) - J_X(\phi) \leq c_2 I(\phi).$$

Proof. Take a path $\phi_s = s \phi$. Then

$$\frac{d}{ds}[I_X(\phi_s) - J_X(\phi_s)] = -s \int_M \phi \cdot \left( \Delta \omega_{\phi_s} - \frac{X}{1 - \theta_X(\omega_{\phi_s})} \right) \phi \cdot (1 - \theta_X(\omega_{\phi_s})) \omega_{\phi_s}^n.

= s \int_M |\partial \phi|^2 \omega_{\phi_s} \left( 1 - \theta_X(\omega_{\phi_s}) \right) \omega_{\phi_s}^n.$$

Note that

$$\frac{d}{ds}[I(\phi_s) - J(\phi_s)] = s \int_M |\partial \phi|^2 \omega_{\phi_s} \omega_{\phi_s}^n.$$

When $c_X > 0$, it follows

$$0 \leq c_X \frac{d}{ds}[I(\phi_s) - J(\phi_s)] \leq \frac{d}{ds}[I_X(\phi_s) - J_X(\phi_s)] \leq C_X \frac{d}{ds}[I(\phi_s) - J(\phi_s)].$$

Thus the lemma follows from (2.6).

In view of [9, 10, 30, 29, 38], we have the following definition of properness:

Definition 2.5. Suppose $H^c$ is a reductive subgroup (which is the complexification of a compact Lie group $H$) of $\text{Aut}(M)$ which contains $K_X$. The modified Ding functional $\mathcal{D}_X(\cdot)$ is said to be proper modulo $H^c$ if there exists an increasing function $f(t) \geq -c$ for $t \in \mathbb{R}$ and some constant $c \geq 0$ such that $\lim_{t \to +\infty} f(t) = +\infty$ and

$$\mathcal{D}_X(\phi) \geq \inf_{\sigma \in H^c} f(I_X(\phi_\sigma) - J_X(\phi_\sigma)),$$

where $\phi_\sigma$ is defined by $\sigma^* (\omega_\phi) = \omega_0 + \sqrt{-1} \partial \bar{\partial} \phi_\sigma$.

For convenience, we write the modified Ding functional (1.6) as $\mathcal{D}_X(\phi) = \mathcal{N}(\phi) + \mathcal{D}_X^0(\phi)$, where

$$\mathcal{N}(\phi) = -\log \left( \frac{1}{V} \int_M e^{\phi - \phi_0} \omega_0^n \right),$$

(2.9)

$$\mathcal{D}_X^0(\phi) = -\frac{1}{V} \int_0^1 \int_M \phi(1 - \theta_X(\omega_{\phi_s})) \omega_{\phi_s}^n \wedge ds.$$ (2.10)

It is known that $\mathcal{N}$ is convex with respect to geodesics [6]. In the latter proof of Theorem

Lemma 2.6. The functional $\mathcal{D}_X^0(\cdot)$ satisfies:

1. When $c_X > 0$, $\mathcal{D}_X^0(\cdot)$ is monotonic, that is for any $\phi_0 \leq \phi_1$, $\mathcal{D}_X^0(\phi_0) \geq \mathcal{D}_X^0(\phi_1)$;
2. $\mathcal{D}_X^0(\cdot)$ is affine along any $C^{1,1}$-geodesic connecting two smooth potentials in $\mathcal{H}_X(\omega_\phi)$.

Proof. To see (1), by definition we have

$$\mathcal{D}_X^0(\phi_1) = \mathcal{D}_X^0(\phi_0) - \frac{1}{V} \int_0^1 \int_M \phi_s(1 - \theta_X(\omega_{\phi_s})) \omega_{\phi_s}^n \wedge ds,$$

where $\phi_s$ is any smooth path in $\mathcal{H}_X(\omega_\phi)$ joining $\phi_0$ and $\phi_1$. Take in particular $\phi_s = s(\phi_1 - \phi_0) + \phi_0$ and note that $c_X > 0$, we have

$$\mathcal{D}_X^0(\phi_1) = \mathcal{D}_X^0(\phi_0) - \frac{1}{V} \int_0^1 \int_M (\phi_1 - \phi_0)(1 - \theta_X(\omega_{\phi_s})) \omega_{\phi_s}^n \wedge ds \leq \mathcal{D}_X^0(\phi_0).$$
Next we prove (2). Let \( \{ \phi_t \} \) be the \( C^{1,1} \)-geodesic connecting \( \phi_0, \phi_1 \in H_X(\omega_0) \). By [8], \( \{ \phi_t \} \) can be approximated by a family of smooth \( \varepsilon \)-geodesic \( \{ \phi^\varepsilon_t \} \) in \( H_X(\omega_0) \) connecting \( \phi_0 \) and \( \phi_1 \), satisfying

\[
\frac{\partial^2}{\partial t \partial \tau} \phi^\varepsilon_t - |\bar{\partial} \phi^\varepsilon_t|^2_{\omega_0} (\omega_0 + \sqrt{-1} \partial \bar{\partial} \phi^\varepsilon_t)^n = \varepsilon \cdot \omega^n_0,
\]

on \( M \times \Omega \), where \( \Omega := [0, 1] \times S^1 \subset \mathbb{C} \) and \( t = \text{Re}(\tau) \). For each \( \varepsilon \), we have

\[
\frac{\partial}{\partial \tau} \mathcal{K}_0^0(\phi^\varepsilon_t) = -\frac{1}{V} \int_M \frac{\partial}{\partial \tau} \phi^\varepsilon_t (1 - \theta_X(\omega^\varepsilon_{\phi^\varepsilon_t})) \omega^n_{\phi^\varepsilon_t}.
\]

It follows

\[
\frac{\partial^2}{\partial t \partial \tau} \mathcal{K}_0^0(\phi^\varepsilon_t) = -\frac{1}{V} \int_M \frac{\partial^2}{\partial t \partial \tau} \phi^\varepsilon_t (1 - \theta_X(\omega^\varepsilon_{\phi^\varepsilon_t})) \omega^n_{\phi^\varepsilon_t} + \frac{1}{V} \int_M \frac{\partial}{\partial \tau} \phi^\varepsilon_t \frac{\partial}{\partial \tau} \theta_X(\omega^\varepsilon_{\phi^\varepsilon_t}) \omega^n_{\phi^\varepsilon_t} - \frac{\sqrt{-1}}{V} \int_M \frac{\partial}{\partial \tau} \phi^\varepsilon_t (1 - \theta_X(\omega^\varepsilon_{\phi^\varepsilon_t})) n \omega^{n-1}_{\phi^\varepsilon_t} \cdot \frac{\partial}{\partial \tau} \phi^\varepsilon_t.
\]

(2.12)

Recall that \( \theta_X(\omega^\varepsilon_{\phi^\varepsilon_t}) = \theta_X(\omega_0) + X(\phi^\varepsilon_t) \), one gets

\[
\frac{1}{V} \int_M \frac{\partial}{\partial \tau} \phi^\varepsilon_t \frac{\partial}{\partial \tau} \theta_X(\omega^\varepsilon_{\phi^\varepsilon_t}) \omega^n_{\phi^\varepsilon_t} = \frac{1}{V} \int_M \frac{\partial}{\partial \tau} \theta_X(\omega^\varepsilon_{\phi^\varepsilon_t}) \omega^n_{\phi^\varepsilon_t}.
\]

On the other hand, by integration by parts, we have

\[
\frac{\sqrt{-1}}{V} \int_M \frac{\partial}{\partial \tau} \phi^\varepsilon_t (1 - \theta_X(\omega^\varepsilon_{\phi^\varepsilon_t})) n \omega^{n-1}_{\phi^\varepsilon_t} \cdot \frac{\partial}{\partial \tau} \phi^\varepsilon_t = \frac{\sqrt{-1}}{V} \left[ \int_M \frac{\partial}{\partial \tau} \phi^\varepsilon_t (1 - \theta_X(\omega^\varepsilon_{\phi^\varepsilon_t})) n \omega^{n-1}_{\phi^\varepsilon_t} \cdot \frac{\partial}{\partial \tau} \phi^\varepsilon_t - \int_M \frac{\partial}{\partial \tau} \frac{\partial}{\partial \tau} \theta_X(\omega^\varepsilon_{\phi^\varepsilon_t}) n \omega^{n-1}_{\phi^\varepsilon_t} \cdot \frac{\partial}{\partial \tau} \phi^\varepsilon_t \right]
\]

\[
= -\frac{1}{V} \int_M \left| \frac{\partial}{\partial \tau} \phi^\varepsilon_t \right|^2_{\omega_0} (1 - \theta_X(\omega^\varepsilon_{\phi^\varepsilon_t})) \omega^n_{\phi^\varepsilon_t} + \frac{1}{V} \int_M \frac{\partial}{\partial \tau} \theta_X(\omega^\varepsilon_{\phi^\varepsilon_t}) \omega^n_{\phi^\varepsilon_t}.
\]

Plugging these into (2.12), by (2.11), we have

\[
\frac{\partial^2}{\partial t \partial \tau} \mathcal{K}_0^0(\phi^\varepsilon_t) = -\varepsilon < 0.
\]

Thus \( \mathcal{K}_0^0(\cdot) \) is concave along \( \phi^\varepsilon_t \). Sending \( \varepsilon \to 0 \), \( \sqrt{-1} \partial \bar{\partial} \mathcal{K}_0^0(\phi^\varepsilon_t) \) converges weakly to \( \sqrt{-1} \partial \bar{\partial} \mathcal{K}_0^0(\phi^\varepsilon_t) \) as Monge-Ampère measures. It follows \( \sqrt{-1} \partial \bar{\partial} \mathcal{K}_0^0(\phi^\varepsilon_t) = 0 \), thus \( \mathcal{K}_0^0(\phi^\varepsilon_t) \) is affine as desired.

\[\square\]

2.3. **Group compactifications.** As an application of Theorem 1.1, we will study the existence of Mabuchi metrics on group compactifications by testing properness of the modified Ding functional. The existence of Kähler-Einstein metrics on these manifolds has been solved by [12] by using the continuity method, while the properness of K-energy was studied in [23]. We will prove Theorem 1.4 by ideas therein later. In this subsection, we recall some facts of group compactifications from [12, 23].
2.3.1. Notations on Lie groups. Choose a maximal compact subgroup \( K \) of \( G \) such that \( G \) is its complexification. Let \( T \) be a chosen maximal torus of \( K \) and \( T^c \) be its complexification, then \( T^c \) is the maximal complex torus of \( G \). Denote their Lie algebras by the corresponding fraktur lower case letters. Assume that \( \Phi \) is the root system of \((G, T^c)\) and \( W \) is the Weyl group. Choose a set of positive roots \( \Phi_+ \). Set \( \rho = \frac{1}{2} \sum_{\alpha \in \Phi_+} \alpha \) and \( \Xi \) be the relative interior of the cone generated by \( \Phi_+ \). Let \( J \) be the complex structure of \( G \), then
\[
g = \mathfrak{t} \oplus J\mathfrak{t}.
\]
Set \( a = Jt \), it can be decomposed as a toric part and a semisimple part:
\[
a = a_t \oplus a_{ss},
\]
where \( a_t := \mathfrak{g}(g) \cap a \) and \( a_{ss} := a \cap [\mathfrak{g}, \mathfrak{g}] \). We extend the Killing form on \( a_{ss} \) to a scalar product \( \langle \cdot, \cdot \rangle \) on \( a \) such that \( a_t \) is orthogonal to \( a_{ss} \). The positive roots \( \Phi_+ \) defines a positive Weyl chamber \( a_+ \subset a \), and a positive Weyl chamber \( a^+ \) on \( a^* \), where
\[
a^+_+ := \{ y | \alpha(y) := \langle \alpha, y \rangle > 0, \forall \alpha \in \Phi_+ \},
\]
which coincides with the dual of \( a_+ \) under \( \langle \cdot, \cdot \rangle \). For later use, we fix a Lebesgue measure \( dy \) on \( a^* \) which is normalized by the lattice of the characters of \( T^c \).

2.3.2. \( K \times K \)-invariant Kähler metrics. Let \( Z \) be the closure of \( T^c \) in \( M \). It is known that \((Z, L|_Z)\) is a polarized toric manifold with a \( W \)-action, and \( L|_Z \) is a \( W \)-linearized ample toric line bundle on \( Z \). Let \( \omega_0 \in 2\pi c_1(L) \) be a \( K \times K \)-invariant Kähler form induced from \((M, L)\) and \( P \) be the polytope associated to \((Z, L|_Z)\), which is defined by the moment map associated to \( \omega_0 \). Then \( P \) is a \( W \)-invariant delzant polytope in \( a^* \). By the \( K \times K \)-invariance, for any
\[
\phi \in \mathcal{H}_{K \times K}(\omega_0) := \{ \phi \in C^\infty(M) | \omega_0 > 0, \phi \text{ is } K \times K \text{-invariant} \},
\]
the restriction of \( \omega_0 \) on \( Z \) is a toric Kähler metric. It induces a smooth strictly convex function \( \psi_0 \) on \( a \), which is \( W \)-invariant [5, 12].

By the KAK-decomposition ([20], Theorem 7.39), for any \( g \in G \), there are \( k_1, k_2 \in K \) and \( x \in a \) such that \( g = k_1 \exp(x)k_2 \). Here \( x \) is uniquely determined up to a \( W \)-action. This means that \( x \) is unique in \( \bar{a}_+ \). Thus there is a bijection between smooth \( K \times K \)-invariant functions \( \Psi \) on \( G \) and smooth \( W \)-invariant functions on \( a \) which is given by
\[
\Psi(\exp(\cdot)) = \psi(\cdot) : a \rightarrow \mathbb{R}.
\]
Clearly when a \( W \)-invariant \( \psi \) is given, \( \Psi \) is well-defined. In the following, we will not distinguish \( \psi \) and \( \Psi \). The following KAK-integral formula can be found in [19] Proposition 5.28 (see also [18]).

**Proposition 2.7.** Let \( dV_G \) be a Haar measure on \( G \) and \( dx \) the Lebesgue measure on \( a \). Then there exists a constant \( C_H > 0 \) such that for any \( K \times K \)-invariant, \( dV_G \)-integrable function \( \psi \) on \( G \),
\[
\int_G \psi(g) dV_G = C_H \int_{a_+} \psi(x) J(x) dx,
\]
where \( J(x) = \prod_{\alpha \in \Phi_+} \sinh^2 \alpha(x) \).

With out loss of generality, we can normalize \( C_H = 1 \) for simplicity. Next we recall a local holomorphic coordinates on \( G \) used in [12]. By the standard Cartan decomposition, we can decompose \( \mathfrak{g} \) as
\[
\mathfrak{g} = (t \oplus a) \oplus (\oplus_{\alpha \in \Phi} V_\alpha),
\]
where \( V_\alpha = \{ X \in g \mid ad_H(X) = \alpha(H)X, \forall H \in t \oplus \alpha \} \), the root space of complex dimension 1 with respect to \( \alpha \). By [17], one can choose \( X_\alpha \in V_\alpha \) such that \( X_\alpha = -t(X_\alpha) \) and \( [X_\alpha, X_\alpha] = \alpha^\vee \), where \( t \) is the Cartan involution and \( \alpha^\vee \) is the dual of \( \alpha \) by the Killing form. Let \( E_\alpha := X_\alpha - X_\alpha \) and \( E_{-\alpha} := J(X_\alpha + X_\alpha) \). Denote by \( \xi_\alpha, \xi_{-\alpha} \) the real line spanned by \( E_\alpha, E_{-\alpha} \), respectively. Then we have the Cartan decomposition of \( \xi \),

\[
\xi = t \oplus (\oplus_{\alpha \in \Phi_+} (\xi_\alpha \oplus \xi_{-\alpha})).
\]

Denote by \( r \) the dimension of \( T \), choose a real basis \( \{ E^0_1, \ldots, E^0_r \} \) of \( t \). Then \( \{ E^0_1, \ldots, E^0_r \} \) together with \( \{ E_\alpha, E_{-\alpha} \}_{\alpha \in \Phi_+} \) forms a real basis of \( \xi \), which is indexed by \( \{ E_1, \ldots, E_n \} \). \( \{ E_1, \ldots, E_n \} \) can also be regarded as a complex basis of \( g \). For any \( g \in G \), we define local coordinates \( \{ z^i_{(g)} \}_{i=1, \ldots, n} \) on a neighborhood of \( g \) by

\[
(z^i_{(g)}) \rightarrow \exp(z^i_{(g)} E_i) g.
\]

It is easy to see that \( \theta^i|_g = d\bar{z}^i_{(g)}|_g \), where \( \theta^i \) is the dual of \( E_i \), which is a right-invariant holomorphic 1-form. Thus \( \wedge^n \left( d\bar{z}^1_{(g)} \wedge \cdots \wedge d\bar{z}^n_{(g)} \right) \) is also a right-invariant \( (n, n) \)-form, which defines a Haar measure \( dV_G \).

The derivations of the \( K \times K \)-invariant function \( \psi \) in the above local coordinates was computed by Delcroix as follows [12, Theorem 1.2].

**Lemma 2.8.** Let \( \psi \) be a \( K \times K \)-invariant function on \( G \). Then for any \( x \in a_+ \),

\[
E^0_i(\psi)|_{\exp(x)} = d\psi(\text{Im}(E^0_i))|_x, \quad 1 \leq i \leq r,
\]

\[
E_{\pm \alpha}(\psi)|_{\exp(x)} = 0.
\]

**Lemma 2.9.** Let \( \psi \) be a \( K \times K \)-invariant function on \( G \), then for any \( x \in a_+ \), the complex Hessian matrix of \( \psi \) in the above coordinates is diagonal by blocks, and equals to

\[
\text{Hess}_C(\psi)(\exp(x)) = \\
\begin{pmatrix}
\text{Hess}_R(\psi)(x) & 0 & 0 \\
0 & M_{\alpha_1}(x) & 0 \\
0 & 0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & 0 & \cdots & 0 & M_{\alpha_{n-1}}(x)
\end{pmatrix},
\]

where \( \Phi_+ = \{ \alpha_1, \ldots, \alpha_{n-1} \} \) is the set of positive roots and

\[
M_{\alpha_1}(x) = \frac{1}{2}(\alpha_1, \nabla\psi(x)) \begin{pmatrix}
\coth \alpha_1(x) & 0 \\
0 & -\sqrt{-1}T
\end{pmatrix}
\]

By (2.13) in Lemma 2.9, we see that a \( \psi \) induced by some \( \omega_\phi \) is convex on \( a \). The complex Monge-Ampère measure is given by \( \omega^n = (\sqrt{-1} \partial \bar{\partial} \psi_\phi)^n = MA_C(\psi_\phi) dV_G \), where

\[
MA_C(\psi_\phi)(\exp(x)) = \frac{1}{2^{n+1}} MA_R(\psi_\phi)(x) \frac{1}{|J(x)|} \prod_{\alpha \in \Phi_+} \langle \alpha, \nabla \psi_\phi(x) \rangle^2.
\]

2.3.3. Legendre functions. By the convexity of \( \psi_\phi \) on \( a \), the gradient \( \nabla \psi_\phi \) defines a diffeomorphism from \( a \) to the interior of the dilated polytope \( 2P \). Let \( P' := P \cap \bar{a}_+^* \). Then

\[\text{We remark that the moment map is given by } \frac{1}{2} \nabla \psi_\phi, \text{ whose image is } P.\]
by the $W$-invariance of $\psi_\phi$ and $P$, the restriction of $\nabla \psi_\phi$ on $a_+$ is a diffeomorphism from $a_+$ to the interior of $2P_+$. Let $u_G$ be the standard Guillemin function on $2P$ \[16\]. Set 
$$
\mathcal{C}_W = \{ u | u \text{ is strictly convex, } u - u_G \in C^\infty(2P) \text{ and } u \text{ is } W\text{-invariant} \}.
$$

It is known that for any $K \times K$-invariant $\omega = \sqrt{-1} \partial \bar{\partial} \psi \in 2\pi c_1(L)$, its Legendre function $u$ is given by
\[ u(y(x)) = x^i y_i(x) - \psi(x), \quad y_i(x) = \psi_i(x) = \frac{\partial \psi}{\partial y_i} \]
is a function in $\mathcal{C}_W$ (cf. \[11\]). By a similar argument as \[15\] for toric manifolds, we have

**Lemma 2.10.** For any $\phi_0, \phi_1 \in \mathcal{H}_{K \times K}(\omega_0)$, there exists a geodesic $\{ \phi_t \}_{t \in [0, 1]}$ in $\mathcal{H}_{K \times K}(\omega_0)$ joining them, and the Legendre function of $\psi_\phi$ is given by $u_{\phi_t} = (1-t)u_{\phi_0} + tu_{\phi_1}$.

3. **Proof of the properness theorem**

Theorem \[1.1\] will be proved by steps as for Kähler-Ricci solitons \[9, 34\]. We always assume $c_X > 0$ in this section.

First, we have

**Lemma 3.1.** Let $\phi_t$ be a solution of (2.2) at $t$, if $I_X(\phi_t)$ is uniformly bounded, then there is a uniform constant $C$ such that

$$
|\phi_t| \leq C, \quad \forall t \in [0, 1].
$$

*Proof.* This estimate was essentially obtained in \[27\]. Here we will give a different proof following the arguments of \[34\]. In view of Kolodziej’s $L^\infty$-estimate \[21\] for complex Monge-Ampère equation, it suffices to obtain the $L^p$-estimate of $e^{-t\phi}$ for some $p > 1$.

By the assumption, $0 \leq I_X(\phi_t) \leq C_1$ for some uniform $C_1$. By (2.2), we have

$$
\int_M e^{h_0-t\phi} \omega_0^n = \int_M (1 - \theta_X(\omega_\phi)) \omega_\phi^n = \int_M e^{h_0} \omega_0^n,
$$

thus

$$
\inf_M \phi_t \leq 0 \leq \sup_M \phi_t.
$$

While by (2.2),

$$
-t \int_M \phi_t \omega_\phi^n = -t \int_M \phi_t \frac{e^{h_0-t\phi}}{1 - \theta_X(\omega_\phi)} \omega_0^n \geq -C_2 t \int_{\{ \phi \geq 0 \}} \phi e^{-t\phi} \omega_0^n \geq -C_3.
$$

Thus

$$
(3.1) \quad t \int_M \phi_t \omega_\phi^n \leq C_4.
$$

Let $\Gamma(\cdot, \cdot)$ be the Green function of $\omega_0$. Then by $\triangle_{\omega_0} \phi_t > -n$, $\Gamma + C_\Gamma \geq 0$ for some $C_\Gamma > 0$.

By (3.1) and Green’s formula, we have

$$
(3.2) \quad t \sup_M \phi_t \leq \frac{t}{V} \int_M \phi_t \omega_\phi^n - \frac{t}{V} \min_M \left( \int_M (\Gamma(x, \cdot) + C_\Gamma) \triangle_{\omega_0} \phi_t \omega_0^n \right) \leq C_5.
$$

By the boundness of $I_X(\phi_t)$, we have

$$
(3.3) \quad -\frac{1}{V} \int_M \phi_t \omega_\phi^n \leq C_1 - \frac{1}{V} \int_M \phi_t \omega_0^n \leq C_6.
$$
Moreover,

\[-t \int_{\{\phi \leq 0\}} \phi_t \omega^n_{\phi_t} = -t \int_M \phi_t \omega^n_{\phi_t} + t \int_{\{\phi \geq 0\}} \phi_t \omega^n_{\phi_t} \]

(3.4)

\[\leq tV C_6 + t \int_{\{\phi \geq 0\}} \phi_t \frac{e^{h_0 - t\phi_t}}{1 - \theta_X(\omega_{\phi_t})} \omega^n_{\phi_t} \leq C_7.\]

By (3.2), there is a uniform \(C > 0\) such that \(\hat{\phi}_t := \phi_t - \frac{C}{t} \leq -1\). By (3.4), it follows

\[-t \int_M \phi_t (1 - \theta_X(\omega_{\phi_t})) \omega^n_{\phi_t} \leq -t \int_{\{\phi \leq 0\}} \phi_t (1 - \theta_X(\omega_{\phi_t})) \omega^n_{\phi_t} \leq C_X C_4,\]

and consequently,

(3.5)

\[-t \int_M \hat{\phi}_t (1 - \theta_X(\omega_{\phi_t})) \omega^n_{\phi_t} \leq C_8.\]

On the other hand,

\[
\left\| \left( \frac{\hat{\phi}_t}{\phi_t} \right)^{p+1} \right\|_n^2 \omega^n_{\phi_t} = \frac{n(p+1)^2}{4p} \int_M (\hat{\phi}_t)^p (\omega^n_{\phi_t} - \omega^{n-1}_{\phi_t} \wedge \omega_0) \]

\[\leq \frac{n(p+1)^2}{4p} \int_M (\hat{\phi}_t)^p \omega^n_{\phi_t}.\]

Recall that \(0 < c_X < 1 - \theta_X(\omega_{\phi_t}) < C_X\). Combining the above inequality with (2.4),

\[
\int_M (\hat{\phi}_t)^{p+1} (1 - \theta_X(\omega_{\phi_t})) \omega^n_{\phi_t} \]

\[\leq \frac{C p}{t} \int_M (\hat{\phi}_t)^p (1 - \theta_X(\omega_{\phi_t})) \omega^n_{\phi_t} + \frac{1}{V} \left( \int_M (\hat{\phi}_t)^{p+1} (1 - \theta_X(\omega_{\phi_t})) \omega^n_{\phi_t} \right)^2 \]

\[\leq \frac{C p}{t} \int_M (\hat{\phi}_t)^p (1 - \theta_X(\omega_{\phi_t})) \omega^n_{\phi_t} \]

\[+ \frac{1}{V} \left( \int_M (\hat{\phi}_t)^p (1 - \theta_X(\omega_{\phi_t})) \omega^n_{\phi_t} \right) \left( \int_M (\hat{\phi}_t)(1 - \theta_X(\omega_{\phi_t})) \omega^n_{\phi_t} \right) \]

\[\leq \frac{C' p}{t} \int_M (\hat{\phi}_t)^p (1 - \theta_X(\omega_{\phi_t})) \omega^n_{\phi_t},\]

where we used (3.5) in the last line. By iteration and using (3.5), we have

\[
\int_M (\hat{\phi}_t)^{p+1} (1 - \theta_X(\omega_{\phi_t})) \omega^n_{\phi_t} \leq \frac{C' p (p+1)!}{t^p} \int_M (\hat{\phi}_t)(1 - \theta_X(\omega_{\phi_t})) \omega^n_{\phi_t} \leq \frac{C' p+1 (p+1)!}{t^{p+1}}.\]

Thus for \(0 < \varepsilon < 1/c\),

\[
\int_M e^{-t\varepsilon \hat{\phi}_t} (1 - \theta_X(\omega_{\phi_t})) \omega^n_{\phi_t} = \sum_{p=0}^{\infty} \frac{(t\varepsilon)^p}{p!} \int_M (\hat{\phi}_t)^p (1 - \theta_X(\omega_{\phi_t})) \omega^n_{\phi_t} \leq \frac{1}{1 - c\varepsilon}.\]

It follows

\[
\int_M e^{-t(1+\varepsilon)\phi_t} \omega^n_0 = \int_M e^{-t(1+\varepsilon)\phi_t} e^{-h_0 - t\phi_t} (1 - \theta_X(\omega_{\phi_t})) \omega^n_{\phi_t} \]

\[\leq C_9 \int_M e^{-t\varepsilon \hat{\phi}_t} (1 - \theta_X(\omega_{\phi_t})) \omega^n_{\phi_t} \leq C.\]

Then the lemma then follows from Kołodziej’s result. \(\square\)
The following lemma was proved in [22].

**Lemma 3.2.** Fix $\varepsilon_0 \in (0, 1)$. Then the modified Ding functional $\mathcal{D}_X(\phi_t)$ uniformly bounded from above for $t > \varepsilon_0$.

With the assumption of properness, Theorem 1.1 will follows from the above and the next lemma.

**Lemma 3.3.** For any solution $\phi_t$ of (2.2) with $t < 1$,
\[
\min_{\sigma \in H^t} \{I_X((\phi_t)_\sigma) - J_X((\phi_t)_\sigma)\} = I_X(\phi_t) - J_X(\phi_t).
\]

**Proof.** We will use the argument of Tian [32] to prove this lemma. For any $\phi$, set $h_{\sigma(s)} = (\phi_t)_\sigma(s)$. Note that (2.2) is equivalent to
\[
h_t + (1-t)\phi_t = \log(1 - \theta_X(\omega_{\phi})) + c_t,
\]
where $h_t$ is the normalized Ricci potential of $\omega_{\phi}$ and $c_t$ is a constant depending on $t$. Thus
\[
\frac{\partial}{\partial s} \bigg|_{s=0} (I_X - J_X)(\phi_t) = \int_M \frac{\partial}{\partial s} \bigg|_{s=0} \phi_t s \partial s \bar{s} \bar{k} (1 - \theta_X(\omega_{\phi})) \omega^n_{\phi}.
\]

Recall that $\theta_Y(\omega_{\phi})$ satisfies
\[
\triangle \omega_{\phi} \theta_Y(\omega_{\phi}) + Y(h_t) + \theta_Y(\omega_{\phi}) = \text{const.,}
\]
thus
\[
\int_M Y(h_t) \theta_X(\omega_{\phi}) \omega^n_{\phi} = - \int_M \theta_X(\omega_{\phi}) \theta_Y(\omega_{\phi}) \omega^n_{\phi} - \int_M \theta_X(\omega_{\phi}) \triangle \omega_{\phi} \theta_Y(\omega_{\phi}) \omega^n_{\phi}.
\]
Substituting this into (3.6) and by integration by parts, it holds
\[
\frac{\partial}{\partial s} \bigg|_{s=0} (I_X - J_X)(\phi_t) = - \frac{1}{1-t} \int_M Y(h_t) \theta_X(\omega_{\phi}) \omega^n_{\phi} - \frac{1}{1-t} \int_M \theta_X(\omega_{\phi}) \theta_Y(\omega_{\phi}) \omega^n_{\phi} = 0.
\]
The last equality follows from (1.4). This shows that $s = 0$ is a critical point of $(I_X - J_X)(\phi_t)$.

To prove the lemma, it suffices to show that $(I_X - J_X)(\phi_{ts})$ is convex with respect to $s$. It is direct to check that
\[
\frac{\partial^2}{\partial s^2} \phi_{ts} = \bar{s} \left( \frac{\partial}{\partial s} \phi_{ts} \right)_{\omega_{\phi s}}^2,
\]
thus $\phi_{ts}$ gives a geodesic in the space of Kähler potentials. In the following, we denote $\phi_s = \phi_{ts}$ for fixed $t$ for simplicity and $\omega_{\phi_s} = \sqrt{-1}g_{ij}(s)dz^i \wedge d\bar{z}^j$. Then
\[
\frac{\partial}{\partial s} \triangle \omega_{\phi_s} \phi = - g^{jk}g_{ij} \cdot \phi_{s,\tilde{k}j} \phi_{s,j\tilde{i}} + \triangle \omega_{\phi_s} \phi_s.
\]
Note that

\[(3.10) \frac{d}{ds} (I_X - J_X)(\phi_s) = - \int_M \phi_s \triangle_{\omega_{\phi_s}} \phi_s (1 - \theta_X(\omega_{\phi_s})) \omega_{\phi_s}^n + \int_M \phi_s X(\phi_s) \omega_{\phi_s}^n.\]

We want to differentiate the above equality. For the first term, we have by (3.9)

\[
\frac{d}{ds} \int_M \phi_s \triangle_{\omega_{\phi_s}} \phi_s (1 - \theta_X(\omega_{\phi_s})) \omega_{\phi_s}^n
= \int_M \phi_s \triangle_{\omega_{\phi_s}} \phi_s (1 - \theta_X(\omega_{\phi_s})) \omega_{\phi_s}^n - \int_M X^i \phi_s \triangle_{\omega_{\phi_s}} \phi_s \omega_{\phi_s}^n
+ \int_M \phi_s \triangle_{\omega_{\phi_s}} \phi_s (1 - \theta_X(\omega_{\phi_s})) \omega_{\phi_s}^n + \int_M \phi_s \triangle_{\omega_{\phi_s}} \phi_s (1 - \theta_X(\omega_{\phi_s})) \omega_{\phi_s}^n
- \int_M \phi_s (\phi_{s,ik} \phi_{s,ij}) g^{ik} g^{lj} (1 - \theta_X(\omega_{\phi_s})) \omega_{\phi_s}^n.
\]

Substituting (3.8) into the first term and by integration by parts, it follows

\[
\frac{d}{ds} \int_M \phi_s \triangle_{\omega_{\phi_s}} \phi_s (1 - \theta_X(\omega_{\phi_s})) \omega_{\phi_s}^n
= -\int_M \phi_s \phi_{s,ik} \phi_{s,ij} g^{ik} g^{lj} (1 - \theta_X(\omega_{\phi_s})) \omega_{\phi_s}^n - \int_M \phi_s X^i \phi_{s,ik} \phi_{s,ij} g^{ik} g^{lj} \omega_{\phi_s}^n
+ \int_M \phi_s \triangle_{\omega_{\phi_s}} \phi_s (1 - \theta_X(\omega_{\phi_s})) \omega_{\phi_s}^n
= \int_M \phi_s \phi_{s,ik} \phi_{s,ij} g^{ik} g^{lj} (1 - \theta_X(\omega_{\phi_s})) \omega_{\phi_s}^n - \int_M \phi_s X^i \phi_{s,ik} \phi_{s,ij} g^{ik} g^{lj} \omega_{\phi_s}^n
+ \int_M \phi_s \triangle_{\omega_{\phi_s}} \phi_s (1 - \theta_X(\omega_{\phi_s})) \omega_{\phi_s}^n
= \int_M \phi_s \phi_{s,ik} \phi_{s,ij} g^{ik} g^{lj} (1 - \theta_X(\omega_{\phi_s})) \omega_{\phi_s}^n - \int_M \phi_s X^i \phi_{s,ik} \phi_{s,ij} g^{ik} g^{lj} \omega_{\phi_s}^n
(3.11) + \int_M X^i \phi_{s,ik} \phi_{s,ij} \omega_{\phi_s}^n,
\]

where \(g_{ij} = g_{ij}(s)\). The second term in (3.10) gives

\[
\frac{d}{ds} \int_M \phi_s X^i \phi_{s,ij} \omega_{\phi_s}^n = \int_M \phi_s X^i \phi_{s,ij} \omega_{\phi_s}^n + \int_M \phi_s X^i \phi_{s,ij} \omega_{\phi_s}^n + \int_M \phi_s X^i \phi_{s,ij} \omega_{\phi_s}^n.
\]

Substituting (3.8) into the above equality and by integration by parts again, we have

\[
(3.12) \frac{d}{ds} \int_M \phi_s X^i \phi_{s,ij} \omega_{\phi_s}^n
= -\int_M \phi_s \phi_{s,ik} (X^i \phi_{s,ij})_k \omega_{\phi_s}^n + \int_M \phi_s X^i \phi_{s,ij} \omega_{\phi_s}^n.
\]

Combining (3.10)-(3.12), we get

\[
\frac{d^2}{ds^2} (I_X - J_X)(\phi_s) = \int_M \phi_s \phi_{s,ik} \phi_{s,ij} g^{ik} g^{lj} (1 - \theta_X(\omega_{\phi_s})) \omega_{\phi_s}^n \geq 0.
\]

Hence, the lemma is proved. \(\square\)

**Proof of Theorem 1.2.** The theorem can be proved by using the properness principle of [10]. Suppose \(\omega_0\) is the Mabuchi metric and \(I_{\omega_0}(M, \omega_0)\) is the identity component of the
corresponding isometry group. By a Calabi-Matsushima typed theorem of Mabuchi [26], we have
\[(3.13) \quad \text{aut}^X(M) = \text{iso}(M, \omega_0) + \text{Iso}(M, \omega_0),\]
where \(\text{aut}^X(M)\) and \(\text{iso}(M, \omega_0)\) are Lie algebras of \(\text{Aut}^X(M)\) and \(\text{Iso}(M, \omega_0)\), respectively. We will check that \(\mathcal{D}_X(\cdot), \text{Aut}^X(M)\) satisfy (P1)-(P2), (P4)-(P7) in the Hypothesis 3.2 of [10], which are enough for the "existence \Rightarrow properness" direction:

(P1) This is confirmed by [6, Theorem 1.1] and (2) of Lemma 2.6.
(P2) This can be shown by using (1) of Lemma 2.6 and Lemmas 5.15, 5.20, 5.29 of [10];
(P4) This is [10, Lemma 5.9];
(P5) This is shown in [27, Appendix 4];
(P6) This can be shown exactly as in [10, Theorem 8.1], by using (3.13) instead of [10, Proposition 6.10];
(P7) This follows from the co-cycle condition of \(\mathcal{D}_X(\cdot)\).

The theorem then follows from the second part of [10, Theorem 3.4].

4. Existence criterion on Fano group compactifications

In this section, we will prove Theorem 1.4. Let \(M\) be a group compactification and \(\omega_0\) be a \(K \times K\)-invariant Kähler metric in \(2\pi c_1(M)\). Assume \(\omega_0 = \sqrt{-1} \partial \bar{\partial} \psi_0\) on \(G\). For \(\phi \in \mathcal{H}_{K \times K}(\omega_0)\), we will write \(\psi_\phi\) in short for \(\psi_0 + \phi\) and \(u_\phi\) the Legendre function of \(\psi_\phi\).

4.1. Reduction of the modified Ding functional. We will give a formula of \(\mathcal{D}_X(\phi)\) in terms of \(\phi\) and \(u_\phi\). First, we compute the Futaki invariant of a vector field in \(\mathfrak{z}(\mathfrak{g})\).

**Lemma 4.1.** Let \(Y\) be a vector field of form
\[(4.1) \quad Y = \sqrt{-1} Y^i E^i_0, \quad 1 \leq i \leq r\]
for some \(Y^i \in \mathbb{C}\) such that \(\alpha_i Y^i = 0\) for any \(\alpha \in \Phi\). Then
\[(4.2) \quad \text{Fut}(Y) = -V \cdot Y^i b_i,\]
where \(b = \frac{1}{r} \int_{2P} \tau \pi(y) dy\) is the barycentre of \(2P\) with respect to the measure \(\pi(y) dy\).

**Proof.** Since \(Y \in \mathfrak{z}(\mathfrak{g})\), it is \(K \times K\)-invariant, so is its potential. Recall that
\[(4.3) \quad \text{Fut}(Y) = -\int_M \hat{\theta}_Y(\omega_0) \omega_0^n,\]
where \(\hat{\theta}_Y(\omega_0)\) is the potential of \(Y\) normalized by
\[(4.4) \quad \int_M \hat{\theta}_Y(\omega_0) e^{h_0} \omega_0^n = 0.\]
By \(\omega_0 = \sqrt{-1} \partial \bar{\partial} \psi_0\) and Lemma 2.8 it is not hard to see that
\[(4.5) \quad \hat{\theta}_Y(\omega_0) = Y^i \frac{\partial}{\partial x^i} \psi_0 + C, \quad \forall x \in \mathfrak{a}_+,\]
where \(C\) is a constant determined by (4.4). On the other hand, we have
\[(4.6) \quad \int_{\mathfrak{a}_+} Y^i \frac{\partial}{\partial x^i} \psi_0 e^{h_0} \det(\psi_{0,ij}) \prod_{\alpha \in \Phi_+} (\alpha, \nabla \psi_0)^2 dx = \int_{\mathfrak{a}_+} Y^i \frac{\partial}{\partial x^i} \psi_0 e^{-\psi_0} J(x) dx = -\int_{\mathfrak{a}_+} Y^i \frac{\partial}{\partial x^i} (e^{-\psi_0} J(x)) dx,\]
here we used the fact that
\[ Y^{i} \frac{\partial}{\partial x^i} J(x) = 2J(x) \sum_{\alpha \in \Phi^+} Y^{i} \alpha_i \cdot \coth(\alpha, x) \equiv 0. \]

Note that when $M$ is Fano, $4p \in \text{Int}(2P_+)$ (cf. [11, Remark 4.10] or [23, §3.2]), by [12, Proposition 2.10]. Hence, we have
\[ e^{-\psi_0} J(x) = e^{4p(x) - \psi_0} \prod_{\alpha \in \Phi^+} \left( \frac{1 - e^{-2\alpha(x)}}{2} \right)^2 \to 0, \quad x \to \infty \text{ in } a_+. \]

Also recall the fact that $J(x) = 0$ on $\partial (a_+)$. By integration by parts in (4.6), we see that
\[ \int_{a_+} Y^{i} \frac{\partial}{\partial x^i} \psi_0 e^{h_0} \det(\psi_{0,ij}) \prod_{\alpha \in \Phi^+} \langle \alpha, \nabla \psi_0 \rangle^2 dx = 0. \]

Thus by Proposition 2.7, we get $C = 0$ in (4.5), (4.2) then follows from (4.3).

Then we use (4.4) to determine the potential of the extremal vector field $X$.

**Lemma 4.2.** Under the coordinates chosen in §2.3, the extremal field $X$, when restricted on $Z$, can be expressed by

\[ X = \sqrt{-1}X^i E^i_0, \quad 1 \leq i \leq r \]

for some $X^i \in \mathbb{R}$ such that $\alpha(X) = 0, \forall \alpha \in \Phi$. Furthermore, $X^i$’s are determined by the condition

\[ \int_{2P_+} v^i y_i (1 - \theta_X(y)) \pi(y) dy = 0, \quad \forall v \in \mathfrak{z}(g), \]

where $\theta_X(y) = X^i y_i - X^i b_i$.

**Proof.** Since Futaki invariant is a character on $\eta_c(M)$, it suffices to consider (1.4) for all $v \in \mathfrak{z}(\eta_c(M)) \subset \mathfrak{z}(g)$. We may assume $X$ is of form (4.7). Since $K_X$ lies in a compact group, we have $X^i \in \mathbb{R}$.

For $\phi \in \mathcal{H}_{K \times K}(\omega_0)$ and $v \in \eta_c(M)$, $\theta_v(\omega_0)$ is $K \times K$-invariant, so it can be written as

\[ \theta_v(\omega_0) = v^i \frac{\partial}{\partial x^i} + c_v, \]

where $v^i$ and $c_v$ are constants with $v^i \alpha_i = 0$ for any $\alpha \in \Phi^+$.

By the second equality of (1.3), the potential is determined by

\[ \theta_v(y) = v^i y_i - v^i b_i. \]

Let $r_z = \text{dim}(\mathfrak{z}(g))$ and suppose $E^0_1, \ldots, E^0_{r_z}$ be a basis of $\mathfrak{z}(g)$. We claim that the extremal vector field $X$ is given by $X = \sum_i \sqrt{-1}X^i E^0_i \in \mathfrak{z}(g)$ such that

\[ b_i = \frac{1}{V} \left( \int_{2P_+} y_i y_j \pi(y) dy - V b_j b_j \right) X^j, \quad 1 \leq i, j \leq r_z. \]

In view of (4.9) and Lemma 4.1, it is direct to check that $X$ given by (4.10) satisfies (1.4). Hence $X$ must be extremal by the uniqueness. To see that (4.10) has a unique solution, it suffices to check that the matrix $(a_{ij})$ given by

\[ a_{ij} = \frac{1}{V} \int_{2P_+} y_i y_j \pi(y) dy - b_i b_j \]
is invertible. In fact, for any vector \( v = (v^i) \), consider the convex function \( f_v(y) = (v^i y_i)^2 \). By Jensen inequality,

\[
v^i v_i a_{ij} = \frac{1}{V} \int_{2P^+} [v(y)]^2 \pi(y) dy - [v(b)]^2 \geq 0,
\]

with equality if and only if \( f_v(y) \) is affine on \( 2P^+ \). However, this forces \( v = 0 \), thus \( (a_{ij}) > 0 \). This completes the proof. \( \square \)

**Proposition 4.3.** For \( \phi \in \mathcal{H}_{K \times K}(\omega_0) \), the modified Ding functional is given by

\[
\mathcal{D}_X(\phi) = \mathcal{L}_X(u_\phi) + \mathcal{F}(u_\phi) + \text{const.,}
\]

where

\[
\mathcal{L}_X(u_\phi) = \frac{1}{V} \int_{2P^+} u_\phi(y) \pi(y) [1 - \theta_X(y)] dy - u_\phi(4\rho),
\]

\[
\mathcal{F}(u_\phi) = -\log \left( \int_{a^+} e^{-\psi_\phi} \mathbf{J}(x) dx \right) + u_\phi(4\rho).
\]

**Proof.** By (1.6), Proposition 2.7 and (2.14), it follows\(^3\)

\[
\mathcal{D}_X^0(\phi) = -\frac{1}{V} \int_0^1 \int_{a^+} \hat{\phi}_i [1 - \theta_X(\omega_\phi)] \det(\alpha, \nabla \psi_\phi) dx \wedge ds + \text{const.,}
\]

\[
\mathcal{N}(\phi) = -\log \left( \frac{1}{V} \int_{a^+} e^{-\psi_\phi} \mathbf{J}(x) dx \right).
\]

By differentiation with Legendre transformations, we have \( \dot{u}_s(y_s(x)) = -\psi_s(x) \). Then by (4.13), \( \mathcal{D}_X^0(\phi) \) equals

\[
\int_0^1 \int_{2P^+} \dot{u}_s[1 - \theta_X(y)] \pi(y) dy \wedge ds = \int_{2P^+} u_\phi[1 - \theta_X(y)] \pi(y) dy \wedge ds + \text{const.}
\]

The proposition is proved. \( \square \)

4.2. **The linear part.** In this part, we deal with the linear part \( \mathcal{L}_X(\cdot) \). First, we introduce the spaces of normalized functions. Let \( O \) be the origin of \( \alpha^* \). Note that \( \alpha^*_s \) is the fixed point set of the \( W \)-action. Thus \( \nabla u(O) \in \alpha^*_s \) for any \( u \in \mathcal{C}_W \). We normalize \( u \in \mathcal{C}_W \) by

\[
\hat{u}(y) = u(y) - \langle \nabla u(O), y \rangle - u(O).
\]

Clearly \( \hat{u} \in \mathcal{C}_W \) and

\[
\min_{2P} \hat{u} = \hat{u}(O) = 0.
\]

The subset of normalized functions in \( \mathcal{C}_W \) will be denoted by \( \mathcal{C}_W \).

**Proposition 4.4.** Under the assumption \( c_X > 0 \) and (1.8), there exists a constant \( \lambda > 0 \) such that

\[
\mathcal{L}_X(u) \geq \lambda \int_{2P^+} u \pi(y)[1 - \theta_X(y)] dy, \forall u \in \mathcal{C}_W^0.
\]

\(^3\)Since we have assumed \( CH = 1 \), it follows \( V := \int_M \omega_0^n = \int_{2P^+} \pi(y) dy \) by Proposition 2.7. Similarly \( \int_{2P^+} [1 - \theta_X(y)] \pi(y) dy = V \).
Proof. Suppose the proposition is not true, then there’s a sequence \( \{u_k\} \subset \mathcal{C}_W \) such that

\[
(4.16) \quad \begin{cases} 
\mathcal{I}_X(u_k) \to 0, \\
\int_{2P_+} u_k \pi(y)[1 - \theta_X(y)] \, dy = 1.
\end{cases}
\]

By \( c_X > 0 \) and the argument of [23, Lemma 6.1], the second equality implies there is a subsequence (still denoted by \( \{u_k\} \)) which converges locally uniformly to some \( u_\infty \in \mathcal{C}_W \).

For any \( u \in \mathcal{C} \), by convexity, we have

\[
(4.17) \quad u - \langle \nabla u(b_X), y - b_X \rangle - u(b_X) \geq 0,
\]

thus

\[
(4.18) \quad \mathcal{I}_X(u) = \frac{1}{V} \int_{2P_+} [u - \langle \nabla u(b_X), y - b_X \rangle - u(b_X)] \pi(y)[1 - \theta_X(y)] \, dy
\]

\[
+ \frac{1}{V} \int_{2P_+} [\langle \nabla u(b_X), y - b_X \rangle + u(b_X)] \pi(y)[1 - \theta_X(y)] \, dy - u(4\rho)
\]

\[
= \frac{1}{V} \int_{2P_+} [u - \langle \nabla u(b_X), y - b_X \rangle - u(b_X)] \pi(y)[1 - \theta_X(y)] \, dy + u(b_X) - u(4\rho)
\]

\[
\geq \frac{1}{V} \int_{2P_+} [u - \langle \nabla u(b_X), y - b_X \rangle - u(b_X)] \pi(y)[1 - \theta_X(y)] \, dy + \langle \nabla u(4\rho), b_X - 4\rho \rangle \geq 0,
\]

where the last inequality follows from (4.18), (4.17) and the fact that \( \nabla u(4\rho) \in a_+ \). Applying the above inequality to \( u_k \), by (4.16), we have

\[
(4.19) \quad 0 \leq \int_{2P_+} [u_k - \langle \nabla u_k(b_X), y - b_X \rangle - u_k(b_X)] \pi(y)[1 - \theta_X(y)] \, dy \to 0,
\]

\[
(4.20) \quad 0 \leq \langle \nabla u_k(4\rho), b_X - 4\rho \rangle \to 0.
\]

By (4.19), we see that \( u_\infty \) must be affine linear. Since \( u_k(O) = 0 \), we have \( u_\infty(y) = \xi^i y_i \) for some \( (\xi^i) \in \bar{a}_+ \). Since \( u_\infty \) is normalized and \( O \) lies in the interior of \( 2P_+ \cap a_+^\circ \), it holds \( \xi \in a_{ss} \). Otherwise \( u_\infty \) is not nonnegative. Substituting \( u_\infty \) into (4.20), we see that \( \langle \xi, b_X - 4\rho \rangle = 0 \). But \( \xi \in \bar{a}_+ \) and \( b_X - 4\rho \in \mathbb{Z} \). Hence \( \xi_i = 0 \) and consequently \( u_\infty(y) \equiv 0 \).

Since \( u_k(4\rho) \to u_\infty(4\rho) = 0 \), by (4.11) and the second line of (4.16), we have \( \mathcal{I}_X(u_k) \to 1 \) by the second line of (4.16), which is a contradiction. Thus the proposition is proved.

Yao use (4.15) to define the "uniform relative Ding stability" in toric case [36]. In [36], it is shown the condition \( c_X > 0 \) is a necessary condition of (4.15). Since the arguments of [36] can be generalized to group compactifications with no difficulties, we omit the details.

**Proposition 4.5.** Inequality (4.15) cannot hold if \( c_X \leq 0 \).

4.3. **Sufficiency.** We first show the sufficient part of Theorem 1.4 by using Theorem 1.1. It suffices to prove the following theorem.

**Theorem 4.6.** If \( c_X > 0 \) and (1.8) holds, then the modified Ding functional is proper modulo \( \mathbb{Z}(G) \). Consequently, \( M \) admits Mabuchi metrics by Theorem 1.7.

First we have the following lemma on non-linear part.
Lemma 4.7. For any $\phi \in \mathcal{H}_{K \times K}(\alpha_0)$, let

$$\tilde{\psi}_\phi := \psi_\phi - 4\rho x^i, \ x \in \mathfrak{a}_+.$$  

Then

$$\mathcal{F}(u_\phi) = -\log \left( \int_{\mathfrak{a}_+} e^{-\left(\tilde{\psi}_\phi - \inf_{\mathfrak{a}_+} \tilde{\psi}_\phi \right)} \prod_{\alpha \in \Phi_+} \left( \frac{1 - e^{-2\alpha x^i}}{2} \right)^2 dx \right).$$  

Consequently, for any $c > 0$,

$$\mathcal{F}(u_\phi) \geq \mathcal{F} \left( \frac{u_\phi}{1 + c} \right) + n \cdot \log(1 + c).$$  

Proof. Since $\psi_\phi$ is convex, so is $\tilde{\psi}_\phi$. Thus if $x^* \in \mathfrak{a}_+$ satisfies $\nabla \psi_\phi(x^*) = 4\rho$, then

$$\tilde{\psi}_\phi(x) \geq \tilde{\psi}_\phi(x^*) = \inf_{\mathfrak{a}_+} \tilde{\psi}_\phi.$$  

By the definition of Legendre transformation, we have

$$\psi_\phi(x) + u_\phi(4\rho) = \psi_\phi(x) + 4x^i \rho_i - \psi_\phi(x^*) = \psi_\phi(x) - \inf_{\mathfrak{a}_+} \tilde{\psi}_\phi.$$  

Substituting this into (4.12), it follows

$$\mathcal{F}(u_\phi) = -\log \left( \int_{\mathfrak{a}_+} e^{-\left(\psi_\phi + u_\phi(4\rho)\right)} \mathcal{J}(x) dx \right) = -\log \left( \int_{\mathfrak{a}_+} e^{-\left(\tilde{\psi}_\phi - \inf_{\mathfrak{a}_+} \tilde{\psi}_\phi \right)} e^{-4\rho x^i} \mathcal{J}(x) dx \right) = -\log \left( \int_{\mathfrak{a}_+} e^{-\left(\tilde{\psi}_\phi - \inf_{\mathfrak{a}_+} \tilde{\psi}_\phi \right)} \prod_{\alpha \in \Phi_+} \left( \frac{1 - e^{-2\alpha x^i}}{2} \right)^2 dx \right).$$  

This proves (4.21).

Then we prove (4.22). For $u_c(y) = \frac{1}{1+c} u(y)$, its Legendre function $\psi_c(x) = \frac{1}{1+c} \psi((1 + c)x)$ satisfies $\tilde{\psi}_c(x) = \frac{1}{1+c} \tilde{\psi}((1 + c)x)$. In particular,

$$-\inf_{\mathfrak{a}_+} \tilde{\psi}_c(x) = -\frac{1}{1 + c} \inf_{\mathfrak{a}_+} \tilde{\psi}.$$  

By the above relations and (4.21), one gets

$$\mathcal{F} (\hat{u}) = -\log \left( \int_{\mathfrak{a}_+} e^{-\frac{1}{1+c} (\tilde{\psi}_\phi((1+c)x) - \inf_{\mathfrak{a}_+} \tilde{\psi}_\phi) \prod_{\alpha \in \Phi_+} \left( \frac{1 - e^{-2\alpha x^i}}{2} \right)^2 dx \right) = -\log \left( \int_{\mathfrak{a}_+} e^{-\frac{1}{1+c} (\psi_\phi(x) - \inf_{\mathfrak{a}_+} \psi_\phi) \prod_{\alpha \in \Phi_+} \left( \frac{1 - e^{-2\alpha x^i}}{2} \right)^2 dx \right) + r \cdot \log(1 + c).$$  

Note that $\# \Phi = n - r$. Combining the above inequality and relations

$$\log(1 + c) \geq \log(1 - e^{-t}) - \log(1 - e^{t(1+c)}) \geq 0, \ \forall t, c \geq 0$$
and
\[
\mathcal{F}(u) \geq -\log \left( \int_{a_+} e^{-\frac{1}{2} \langle \psi_\phi(x) - \inf_{a_+} \psi_\phi \rangle} \prod_{\alpha \in \Phi^+} \left( \frac{1 - e^{-\frac{1}{2} a_+ \alpha x^i}}{2} \right)^{2} \, dx \right).
\]
Hence, we have (4.22).

**Proposition 4.8.** Suppose \(c_X > 0\) and (1.8) holds. Then there are constants \(c, C > 0\) such that
\[
\mathcal{D}_X(u) \geq c \int_{2p_+} u[1 - \theta_X(y)] \pi(y) \, dy - C, \quad \forall u \in \hat{C}_W.
\]  

*Proof.* Define a function \(A\) by
\[
A(y) = \frac{V}{\int_{a_+} e^{-\psi_0 J(x)} \, dx} \, e^{\theta_0 (\nabla u_0(y))}, \quad y(x) = \nabla \psi_0(x).
\]
It is clear that
\[
\int_{a_+} e^{-\psi_0 J(x)} \, dx = \int_{M} e^{\theta_0 \omega_0^\rho} = V.
\]
Hence, \(A\) is a bounded smooth function.

Let
\[
\mathcal{D}_A(u_\phi) := \mathcal{D}_A^0(u_\phi) + \mathcal{N}(\phi), \quad \forall \phi \in \mathcal{H}_K \times K(\omega_0),
\]
where
\[
\mathcal{D}_A^0(u) := \frac{1}{V} \int_{2p_+} u A(y) \pi(y) \, dy.
\]
It is obvious that \(u_0\) is a critical point of \(\mathcal{D}_A(\cdot)\). On the other hand, along any geodesic, \(\mathcal{D}_A(\cdot)\) is affine by Lemma 2.10 and \(\mathcal{N}(\cdot)\) is convex by [6, Theorem 1.1]. Hence,
\[
\mathcal{D}_A(u) \geq \mathcal{D}_A(u_0), \quad \forall u \in \hat{C}_W.
\]  

Rewrite \(\mathcal{D}_A(\cdot) = \mathcal{L}_A(\cdot) + \mathcal{F}(\cdot)\), where
\[
\mathcal{L}_A(u) := \frac{1}{V} \int_{2p_+} u A(y) \pi(y) \, dy - u(4 \rho).
\]
By Proposition 4.4 and the boundedness of \(A\), it is clear that for any \(\delta > 0\)
\[
|\mathcal{L}_X(u) - \mathcal{L}_A(u)| = \left| \int_{2p_+} u(1 - \theta_X(y) - A(y)) \pi(y) \, dy \right|
\leq C_A \int_{2p_+} u[1 - \theta_X(y)] \pi(y) \, dy
\leq \frac{C_A(1 + \delta)}{\lambda} \mathcal{L}_X(u) - C_A \delta \int_{2p_+} u[1 - \theta_X(y)] \pi(y) \, dy, \quad \forall u \in \hat{C}_W,
\]
for some constant \(C_A > 0\). Then
\[
\mathcal{L}_X(u) \geq \frac{\lambda}{\lambda + C_A(1 + \delta)} \left[ \mathcal{L}_A(u) + C_A \delta \int_{2p_+} u[1 - \theta_X(y)] \pi(y) \, dy \right], \quad \forall u \in \hat{C}_W.
\]
Hence, taking $C = \frac{C_A(1+\delta)}{2}$, we have for any $u \in \mathcal{L}_W$,

$$\mathcal{D}_X(u) \geq \mathcal{L}_A \left( \frac{u}{1+C} \right) + \mathcal{F}(u) + \frac{C_A \delta}{1+C_A} \int_{2P_+} u[1 - \theta_X(y)] \pi(y) dy$$

$$\geq \mathcal{L}_A \left( \frac{u}{1+C} \right) + \mathcal{F} \left( \frac{u}{1+C} \right) + \frac{C_A \delta}{1+C_A} \int_{2P_+} u[1 - \theta_X(y)] \pi(y) dy - n \log (1+C)$$

$$= \mathcal{L}_A \left( \frac{u}{1+C} \right) + \frac{C_A \delta}{1+C_A} \int_{2P_+} u[1 - \theta_X(y)] \pi(y) dy - n \log (1+C)$$

$$\geq \frac{C_A \delta}{1+C_A} \int_{2P_+} u[1 - \theta_X(y)] \pi(y) dy + (\mathcal{D}_A(u_0) - n \log (1+C_A)),$$

where we used (4.21) and (4.24). This completes the proof. $\square$

To use Theorem 1.1 we introduce the following normalization: for any $\phi \in \mathcal{H}_{K \times K}(\omega_0)$, let $u_\phi$ be the Legendre function of $\psi_\phi$. Take a $v \in \eta_c(M)$ such that $\text{Re}(v) = -\nabla u_\phi(O)$. Let $\sigma_v(t)$ be the one parameter group generated by $\text{Re}(v)$. Then $\sigma_v(t) \in Z(G)$. It follows

$$(\sigma_v(1))^* \omega_\phi = \omega_0 + \sqrt{-1} \partial \bar{\partial} \hat{\phi}$$

induces a $K \times K$-invariant Kähler potential $\hat{\phi}$. Since we may also normalize $\psi_\phi$ so that $\psi_\phi(O) = 0$, thus the Legendre function $u_\hat{\phi}$ of $\psi_\phi$ is given by

$$u_\hat{\phi}(y) = u_\phi(y) - \langle \nabla u_\phi(O), y \rangle - u_\phi(O),$$

which satisfies $u_\hat{\phi} \in \mathcal{L}_W$. Then we have

**Lemma 4.9.** Under the above normalization, we have $\mathcal{D}_X(u_\phi) = \mathcal{D}_X(u_\hat{\phi})$.

**Proof.** Denote $a^i = -\text{Re}(u_\phi,i(O))$, then $(a^i) \in a_t$ and consequently $\alpha(a) = 0$ for all $\alpha \in \Phi$. On the other hand, we have

$$\psi_\hat{\phi}(x) = \psi_\phi(x - a) - u_\phi(O).$$

Taking change of variables $x \to (x - a)$ in (4.21), by the above relations, we see that $\mathcal{F}(u_\phi) = \mathcal{F}(u_\hat{\phi})$. By (4.21) and (4.8), $\mathcal{L}_X(a^i y_i - u_\phi(O)) = 0$. Hence, by (4.25) $\mathcal{L}_X(u_\phi) = \mathcal{L}_X(u_\hat{\phi})$. The lemma is proved. $\square$

The following lemma is analogous to [23, Lemma 4.14] and [35, Lemma 3.4], we omit the proof.

**Lemma 4.10.** There exists a uniform $C_F > 0$ such that

$$\left| J_X(\hat{\phi}) - \int_{2P_+} u_\hat{\phi}[1 - \theta_X(y)] \pi(y) dy \right| \leq C_F, \forall \phi \in \mathcal{H}_{K \times K}(\omega_0),$$

where $u_\hat{\phi} \in \mathcal{L}_W$ and $\psi_\hat{\phi}$ is the Legendre function of $u_\hat{\phi}$.

**Proof of Theorem 4.6** For any $\phi \in \mathcal{H}_{K \times K}(\omega_0)$, there exists $\sigma \in Z(G)$ such that

$$\sigma^* \omega_\phi = \omega_0 + \sqrt{-1} \partial \bar{\partial} \hat{\phi}$$

as above. Applying Proposition 4.8 we have

$$\mathcal{D}_X(u_\hat{\phi}) \geq \delta \int_{2P_+} u_\hat{\phi} \pi dy - C_\delta.$$
Thus by Proposition 4.3, Lemma 4.9 and Lemma 4.10,
\[ \mathcal{D}_X(\phi) = \mathcal{D}_X(\hat{\phi}) = \mathcal{D}_X(u_{\phi}) \geq \delta \cdot J_X(\hat{\phi}) - C_J - C_\delta. \]

The theorem then follows from (2.7).

4.4. **Necessity.** To complete the proof of Theorem 1.4 we will show that (1.8) is also a necessary condition of the existence of Mabuchi metrics. It is equivalent to show that

\[ \langle \xi, b_X - 4\rho \rangle > 0, \forall \xi \in a_+. \]

We will adopt the method used in [12]. By the \(K \times K\)-invariance, (2.1) can be reduced to the following Monge-Ampère equation on \(a_+\),

\[ \det(\psi_{0,ij} + \phi_{ij}) \prod_{\alpha \in \Phi_+} \langle \alpha, \nabla(\psi_0 + \phi) \rangle^2 = C \cdot \frac{e^{-(\psi_0 + \phi - \log J)}}{1 - \theta_X(\omega_0) - X(\phi)}. \]

Suppose \(\phi\) is a solution, for any \(\xi \in a_+\), we have

\[
0 = -\int_{a_+} \xi^i \frac{\partial}{\partial x^i} e^{-(\psi_0 + \phi - \log J)}
\]

\[
= \int_{a_+} \xi^i e^{-(\psi_0 + \phi - \log J)} \frac{\partial(\psi_0 + \phi - \log J)}{\partial x^i}
\]

\[
= \int_{a_+} \xi^i \det(\psi_{0,ij} + \phi_{ij}) \prod_{\alpha \in \Phi_+} \langle \alpha, \nabla(\psi_0 + \phi) \rangle^2 [1 - \theta_X(\omega_0)] \frac{\partial(\psi_0 + \phi - \log J)}{\partial x^i}
\]

\[
< \int_{a_+} \xi^i \det(\psi_{0,ij} + \phi_{ij}) \prod_{\alpha \in \Phi_+} \langle \alpha, \nabla(\psi_0 + \phi) \rangle^2 [1 - \theta_X(\omega_0)] \frac{\partial(\psi_0 + \phi - \log J)}{\partial x^i}
\]

\[
= V \int_{2P_+} \xi^i (y_i - 4\rho) \pi(y) [1 - \theta_X(y)] dy
\]

(4.28) \[ V \cdot \langle \xi, b_X - 4\rho \rangle, \]

where in the fourth line we used the fact that for any \(\xi, x \in a_+\)

\[ -\xi^i \frac{\partial}{\partial x^i} \log J = -2 \sum_{\alpha \in \Phi_+} \alpha(\xi) \cdot \coth x < -2 \sum_{\alpha \in \Phi_+} \alpha(\xi) = -4\rho(\xi). \]

Then we have (4.26).

5. **Appendix: Proof of Theorem 2.1**

In this appendix, we solve (2.2) at \(t = 0\). Following [37] for the Kähler-Ricci soliton case, we introduce the following path,

\[ (1 - \theta_X(\omega_0))^t \omega_0^n = e^{h_0} \omega_0^n, \quad t \in [0, 1]. \]

Set \(\mathcal{I} := \{ t \in [0, 1] | (5.1) \text{ has a solution for } t \}\). The Calabi-Yau theorem implies that \(0 \in \mathcal{I}\). We shall prove \(\mathcal{I}\) is both open and closed in \([0, 1]\).
5.1. Openness. Define a functional

\[ J_t(\phi) = \int_0^1 \int_M \phi_t (1 - \theta_X(\omega_{\phi}))^t \omega^n_{\phi}, \]

where \( \phi_t \) is any smooth path joining \( \phi \) and 0 in \( \mathcal{H}_X(\omega_0) \). It is standard to show that \( J_t(\cdot) \) is well-defined. Thus by taking \( \phi_t = s \phi \), we have

\[ J_t(\phi) = \int_0^1 \int_M (1 - \theta_X(\omega_{s\phi}))^t \omega^n_{s\phi}. \]

Denote an operator by

\[ L_{\lambda}(\psi) := \nabla_\omega \psi - \frac{tX(\psi)}{1 - \theta_X(\omega_0)} - \int_M \psi (1 - \theta_X(\omega_0))^t \omega^n_{\phi}, \quad \forall \psi \in \mathcal{H}_X(\omega_0). \]

Then for any \( K_X \)-invariant smooth real functions \( f \) and \( g \), it is easy to see

\[ \int_M L_{\lambda}(f)g(1 - \theta_X(\omega_0))^t \omega^n_{\phi} = \int_M fL_{\lambda}(g)(1 - \theta_X(\omega_0))^t \omega^n_{\phi}. \]

We have

**Lemma 5.1.** Suppose \( \phi_t \) is a smooth solution of (5.1) for some \( t \in [0, 1] \), then the first eigenvalue of \( L_\lambda \) is positive.

**Proof.** Suppose \( \lambda \) is the first eigenvalue and \( \psi \) is an eigenfunction. Then by \( L_\lambda \psi = -\lambda \psi \),

\[ \lambda \int_M \psi (1 - \theta_X(\omega_0))^t \omega^n_{\phi} = \int_M \psi (1 - \theta_X(\omega_0)) \cdot \int_M \psi (1 - \theta_X(\omega_0))^t \omega^n_{\phi}. \]

By the assumption \( c_X > 0 \), if \( \psi \equiv c \) for some constant \( c \neq 0 \), then \( \lambda > 0 \). Thus we may assume that \( \psi \neq \text{const} \) below.

As before, we may choose a local co-frame \( \{\Theta^i\} \) such that \( \omega_0 = \sqrt{-1} \sum_{i=1}^n \Theta^i \wedge \bar{\Theta}^i \).

By (3.2) and integration by parts, it follows

\[ \lambda \int_M \psi_i \psi_j (1 - \theta_X(\omega_0))^t \omega^n_{\phi} = -\int_M L(\psi_i) \psi_j (1 - \theta_X(\omega_0))^t \omega^n_{\phi} \]

\[ = -\int_M \psi_i \psi_j (1 - \theta_X(\omega_0))^t \omega^n_{\phi} + t \int_M \frac{X^i_j \psi_j (1 - \theta_X(\omega_0))^t \omega^n_{\phi}}{1 - \theta_X(\omega_0)} \]

\[ + t \int_M \frac{X^i_j \psi_j (1 - \theta_X(\omega_0))^t \omega^n_{\phi}}{1 - \theta_X(\omega_0)} \int_M (1 - \theta_X(\omega_0))^t \omega^n_{\phi}. \]

By Ricci identity and integration by parts, the first term on the right-hand side

\[ = -\int_M \psi_i \psi_j (1 - \theta_X(\omega_0))^t \omega^n_{\phi} \]

\[ = -\int_M (\psi_i \psi_j - R^p_{ij \phi} \psi_j (1 - \theta_X(\omega_0))^t \omega^n_{\phi}) \]

\[ = \int_M Ric_{\phi} \psi_i \psi_j (1 - \theta_X(\omega_0))^t \omega^n_{\phi} + \int_M \psi_i \psi_j (1 - \theta_X(\omega_0))^t \omega^n_{\phi}, \]

\[ - t \int_M \psi_i \psi_j X^j (1 - \theta_X(\omega_0))^{t-1} \omega^n_{\phi}. \]

By taking the limit as \( t \to 1 \), we obtain

\[ \lambda = \text{first eigenvalue of } L_\lambda \]

\[ \text{and } \psi \text{ is an eigenfunction.} \]
Plugging the above equality into (5.3), one gets

\[
\lambda \int_M \psi_i \psi_j (1 - \theta_X(\omega_\phi))^t \omega_\phi^n = \int_M \left( Ric_{i,j} + \frac{tX_{i,j}}{1 - \theta_X(\omega_\phi)} \right) \psi_i \psi_j (1 - \theta_X(\omega_\phi))^t \omega_\phi^n + \int_M \psi_i \psi_j (1 - \theta_X(\omega_\phi))^t \omega_\phi^n.
\]

(5.4)

On the other hand, by (5.1),

\[
\text{Proposition 5.2.} \quad \text{Let } \phi_t \text{ be a solution of (5.1) at } t. \text{ Then there exists a uniform constant C such that } |\phi_t| \leq C.
\]

Proof. Consider the equation

\[
\det(g_{i,j}(t))(1 - \theta_X(\omega_\phi))^t = \det(g_{i,j}(0)) e^{h_0 + J_t(\phi_t)}.
\]

By integration,

\[
\int_M (1 - \theta_X(\omega_\phi))^t \omega_\phi^n = e^{J_t(\phi_t)} V,
\]

we have

\[
J_t(\phi_t) = \log \int_M (1 - \theta_X(\omega_\phi))^t \omega_\phi^n - \log V.
\]

This implies

\[
t \log c_X \leq J_t(\phi_t) \leq t \log C_X.
\]

(5.6)

(5.5) can be rewritten as

\[
\det(g_{i,j}(t)) = \det(g_{i,j}(0)) e^{\tilde{f}_t},
\]

where \(\tilde{f}_t := h_0 + J_t(\phi_t) - t \log (1 - \theta_X(\omega_\phi)).\) Let \(\check{\phi}_t = \phi_t - c_t.\) Then \(\sup_M \check{\phi}_t = -1.\) Since

\[
|\tilde{f}_t| \leq \|h_0\|_{C^0} + 2 \max \{\log c_X, |\log C_X|\},
\]
by the argument of $C^0$-estimate in [29], we see that $|\hat{\phi}| \leq C'$ for some uniform $C' > 0$. On the other hand,

$$c_i \int_0^1 \int_M (1 - \theta_X(\omega_{s\phi}))^{i} \omega_{s\phi}^{n} \wedge ds$$

(5.7) $$= I_t(\phi_t) - \int_0^1 \int_M \hat{\phi}_t (1 - \theta_X(\omega_{\phi}))^{i} \omega_{s\phi}^{n} \wedge ds.$$ 

Combining (5.6), (5.7) and the fact that $0 < c_\epsilon \leq 1 - \theta_X(\omega_\phi) \leq C_X$, one gets a uniform constant $\hat{C}$ such that $|c_i| \leq \hat{C}$, this implies

$$|\phi_t| \leq |\hat{\phi}| + \hat{C} \leq C' + \hat{C},$$

the proposition is proved. \hfill \Box

Next we consider the $C^2$-estimate.

**Proposition 5.3.** Let $\phi_t$ be a solution of (5.7) at $t$. Then there exist two uniform constants $C, c$ such that

$$n + \triangle_{\omega_\phi} \phi \leq Ce^{c(\hat{\phi} - \inf_{t\phi} \phi)}.$$

**Proof.** By computation,

$$\triangle_{\omega_\phi} \log(1 - \theta_X(\omega_\phi)) = -\frac{\triangle_{\omega_\phi} \theta_X(\omega_\phi)}{1 - \theta_X(\omega_\phi)} - \frac{[\partial \theta_X(\omega_\phi)]^2}{1 - \theta_X(\omega_\phi)^2} \leq C_1(n + \triangle_{\omega_\phi} \phi) + C_2$$

for some constants $C_1, C_2$ independent of $\phi$. Then following the computations of [37],

$$\triangle_{\omega_\phi}((n + \triangle_{\omega_\phi} \phi)e^{-c\phi}) = e^{-c\phi} \left( \triangle_{\omega_\phi}(h_0 - t \log(1 - \theta_X(\omega_\phi))) - n^2 \inf_{l \neq k} R_{i\bar{i}l\bar{l}} \right)$$

$$+ (c + \inf_{l \neq i} R_{i\bar{i}l\bar{l}})(n + \triangle_{\omega_\phi} \phi)e^{-c\phi} \left( \sum_i \frac{1}{1 + \phi, i} \right)$$

$$-cn(n + \triangle_{\omega_\phi} \phi)e^{-c\phi}$$

(5.8) $$\geq -e^{-\phi}(C_3 + cC_4(n + \triangle_{\omega_\phi} \phi)) + C_5 e^{-c\phi} (n + \triangle_{\omega_\phi} \phi)^{\frac{2}{c-1}}$$

for sufficiently large constant $c$ and some uniform constants $C_3 \sim C_5$. The proposition then follows from (5.8) in a standard way. \hfill \Box

The higher order estimates then follow from nonlinear elliptic equation theory and we omit the details.

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