On the Riesz and Báez-Duarte criteria for the Riemann Hypothesis

Jerzy Cisło, Marek Wolf
Institute of Theoretical Physics
University of Wrocław
pl. M.Borna 9, 50-205 Wrocław, Poland
cislo@ift.uni.wroc.pl, mwolf@ift.uni.wroc.pl

July 18, 2008

Abstract

We investigate the relation between the Riesz and the Báez-Duarte criterion for the Riemann Hypothesis. In particular we present the relation between the function $R(x)$ appearing in the Riesz criterion and the sequence $c_k$ appearing in the Báez-Duarte formulation. It is shown that $R(x)$ can be expressed by $c_k$, and, vice versa, the sequence $c_k$ can be obtained from the values of $R(x)$ at integer arguments. Also, we give some relations involving $c_k$ and $R(x)$, and value of the alternating sum of $c_k$.

Dedicated to Prof. Luis Báez-Duarte
on the occasion of his 70th birthday

1 Introduction

The Riemann Hypothesis (RH) states that the nontrivial zeros of the function

$$\zeta(s) = \frac{1}{1 - 2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s},$$

(1)

where $\Re(s) > 0$ and $s \neq 1$ have the real part equal $\Re(s) = \frac{1}{2}$. Although Riemann did not request it, today it is often demanded additionally that zeros on the critical line should be simple. The function $\zeta(s)$ defined by (1) can be continued analytically to
the whole complex plane without \( s = 1 \) where \( \zeta(s) \) has the simple pole \([1]\). There are probably over 100 statements equivalent to RH, see eg. \([1], [2], [3]\). At the beginning of the 20th century M. Riesz \([4]\) considered the function

\[
R(x) = x \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k! \zeta(2k + 2)}.
\]  

(2)

We present the plot of \( R(x) \) in the Fig. 1.

In \([4]\) Riesz stated the following condition for the Riemann Hypothesis

Riesz Criterion:

\[
RH \iff R(x) = \mathcal{O}(x^{1/4+\epsilon}) \quad \text{for each } \epsilon > 0.
\]  

(3)

A few years ago L. Báez-Duarte \([5], [6]\) considered the sequence of numbers \( c_k \) defined as the forward differences of \( 1/\zeta(2j + 2) \):

\[
c_k = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{1}{\zeta(2j + 2)}.
\]  

(4)

The plot of \( c_k \) is shown on the Fig. 2.

Báez-Duarte proved

Báez-Duarte Criterion:

\[
RH \iff c_k = \mathcal{O}(k^{-\frac{3}{4}+\epsilon}) \quad \text{for each } \epsilon > 0.
\]  

(5)

Also, Baez-Duarte proved in \([6]\) that it is not possible to replace \( \frac{3}{4} \) by larger exponent, and that \( \epsilon = 0 \) implies that the zeros of \( \zeta(s) \) are simple. Next in \([7]\) Báez-Duarte has considered replacing "continuous" criteria with "sequential" criteria in more general setting.

Although the title of the Baez-Duarte paper \([6]\) was *A sequential Riesz-like criterion for the Riemann Hypothesis* he did not pursue further relation between \( c_k \) and \( R(x) \) to prove his criterion (he has used the Mellin transform).

In this paper we will present direct proof of the equivalence of the Riesz Criterion and Báez-Duarte Criterion. Besides, we will write some properties of \( R(x) \) and of \( c_k \), of two-parameter generalizations of \( R(x) \) and of \( c_k \) introduced in \([12], [16], [13]\) and \([11]\). We calculate also the alternating sum of \( c_k \) and state the conjecture about the special sum of the Möbius function.

\section{Proof of equivalence of the Riesz and Báez-Duarte Criteria}

In this section we will show that for large arguments function \( R(x)/x \) and the sequence \( c_k \) behave in a similar way.
Let $E$ denote the shift operator
\[ Ef(n) = f(n + 1). \]  
(6)

With this notation we can rewrite (4) as
\[ c_k = (1 - E)^k f(0), \quad \text{where} \quad f(j) = \frac{1}{\zeta(2j + 2)}. \]  
(7)

We have the formal identity
\[ e^{x(1-E)} = e^x e^{-xE}. \]  
(8)

Consequently we have the identity
\[ \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^j f(j) = e^x \sum_{k=0}^{\infty} \frac{(-x)^k f(k)}{k!}. \]  
(9)

After substitution $f(k) = 1/\zeta(2k + 2)$ we get
\[ \sum_{k=0}^{\infty} \frac{x^k}{k!} c_k = \frac{e^x}{x} R(x). \]  
(10)

We may also observe the above relation while comparing discrete and continuous physical models of diffusion. For both models we expect similar properties of the solutions.

Relation (10) appears also in the Exercises 67-71 in part IV of the book Polya and Szegö [15].

The formulae in the further part of this paper will involve the Möbius function
\[ \mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n \text{ is divisible by a square of a prime} \\ (-1)^k & \text{if } n \text{ is a product of } k \text{ different primes} \end{cases} \]  
(11)

Using the formula
\[ \frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}, \]  
(12)

we can rewrite $R(x)$ and $c_k$ in the suitable for us form
\[ R(x) = x \sum_{k=0}^{\infty} \frac{(-x)^k}{k! \zeta(2k + 2)} = x \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} \exp(-x/n^2), \]  
(13)

\[ c_k = \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^j}{\zeta(2j + 2)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} \left(1 - \frac{1}{n^2}\right)^k. \]  
(14)
We will also consider two two-parameter generalizations introduced in [12], [13], [16]:

\[
R_{ab}(x) = x \sum_{k=0}^{\infty} \frac{(-x)^k}{k! \zeta(ak + b)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^b} \exp(-x/n^a), \quad (15)
\]

\[
c_{ab}(k) = \sum_{j=0}^{k} \left( \binom{k}{j} \frac{(-1)^j}{\zeta(aj + b)} \right) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^b} \left(1 - \frac{1}{n^a}\right)^k, \quad (16)
\]

The original Riesz function \( R(x) \) as well as the Báez-Duarte sequence \( c_k \) correspond to the choice of parameters \( a = b = 2 \). The generalization of the original Riesz criterion to the family \( R_{ab}(x) \) was given by A. Chaudhry [13]

\[
RH \iff R_{ab}(x) = O\left(x^{1-\frac{1}{a}(b-\frac{1}{2})+\epsilon}\right) \quad \text{for each } \epsilon > 0. \quad (17)
\]

For \( a = 2, \ b = 1 \) it reproduces the Hardy–Littlewood criterion [14] for RH.

We start from following simple lemma:

**Lemma 1.** If the function \( f \) is nondecreasing for \( 1 \leq x \leq x_0 \) and nonincreasing for \( x \geq x_0 \) then

\[
\sum_{n=1}^{\infty} f(n) \leq \int_{1}^{\infty} f(x) \, dx + f(x_0). \quad (18)
\]

Proof. If \( f(1) \leq f(2) \leq \cdots \leq f(k) \) and \( f(k) \geq f(k+1) \geq \cdots \), then

\[
f(1) + f(2) + \cdots + f(k-1) \leq \int_{1}^{k} f(x) dx, \quad (19)
\]

\[
f(k + 1) + f(k+2) + f(k+3) \cdots \leq \int_{k}^{\infty} f(x) dx,
\]

\[
f(k) \leq f(x_0). \quad \Box
\]

**Corollary 1.** For \( b > 1, a > 0, \) and \( x > 0 \), we have

\[
\sum_{n=1}^{\infty} \frac{1}{n^b} \exp(-x/n^a) \leq J_{ab} x^{(1-b)/a} + \left(\frac{b}{ea}\right)^{b/a} x^{-b/a}, \quad (20)
\]

where

\[
J_{ab} = \int_{0}^{\infty} \frac{1}{t^b} \exp(-1/t^a) \, dt = \frac{1}{a} \Gamma\left(\frac{b-1}{a}\right) \quad (21)
\]

In particular we have

\[
J_{2,2} = (1/2)\sqrt{\pi}, \quad J_{2,4} = (1/4)\sqrt{\pi}, \quad J_{2,6} = (3/8)\sqrt{\pi}, \quad J_{2,8} = (15/16)\sqrt{\pi}. \quad (22)
\]
Corollary 2. We have
\[ R_{ab}(x) = O(x^{(1+a-b)/a}), \]  
\[ c_{ab}(k) = O(k^{(1-b)/a}). \]  
(23)  
(24)

In particular, for \( a = b = 2 \) we have:
\[ |R(x)| \leq (1/2)\sqrt{\pi}x^{1/2} + 1/e. \]  
(25)

The relation (24) follows from the next lemma.

Lemma 2. We have
\[ \frac{R_{ab}(k)}{k} = c_{ab}(k) + O(k^{(1-a-b)/a}). \]  
(26)

Proof. For \( x \in (0, 1) \), we have two inequalities:
\[ \exp(-x) \geq 1 - x, \quad \exp(x) \geq 1 + x + x^2/2. \]  
(27)

The first inequality implies
\[ 0 \leq \exp(-kx) - (1 - x)^k. \]  
(28)

The second inequality and Bernoulli’s inequality imply
\[ (1 - x)^k \exp(kx) \geq (1 - x^2/2 - x^3/2)^k \geq 1 - kx^2/2 - kx^3/2. \]  
(29)

After some manipulations we get
\[ \exp(-kx) - (1 - x)^k \leq (k/2)(x^2 + x^3) \exp(-kx). \]  
(30)

The inequalities (28) and (30) give us estimation
\[ \left| \frac{R_{ab}(k)}{k} - c_{ab}(k) \right| \leq \sum_{n=1}^{\infty} \frac{\exp(-k/n^a) - (1 - 1/n^a)^k}{n^b} \]  
\[ \leq \frac{k}{2} \sum_{n=1}^{\infty} \frac{1}{n^{2a+b}} \exp(-k/n^a) + \frac{k}{2} \sum_{n=1}^{\infty} \frac{1}{n^{3a+b}} \exp(-k/n^a). \]  
(31)

Here we used the triangle inequality, the inequality \(|\mu(n)| \leq 1\), and the substitution \( x = 1/n^a \). Now the thesis of Lemma 2 follows from Corollary 1. \(\square\)
The substitution $a = b = 2$ in inequality (31), and Corollary 1 give

**Lemma 3.**
\[ \left| \frac{R(k)}{k} - c_k \right| \leq \frac{3 \sqrt{\pi}}{16} k^{-3/2} + O(k^{-2}). \] (32)

More explicitly we have
\[ \left| \frac{R(k)}{k} - c_k \right| \leq \frac{3}{16} \sqrt{\pi} k^{-3/2} + \frac{27}{2} e^{-3} k^{-2} + \frac{15}{32} \sqrt{\pi} k^{-5/2} + 128 e^{-4} k^{-3}. \] (33)

Actually for $k > 16$ we have $|R(k)/k - c_k| \leq (3/16)\sqrt{\pi}k^{-3/2}$. Another proof of (33) can be found in [18], see also [17]. The fact that approximately $c_k \approx R(k)/k$ was observed previously by S. Beltraminelli and D. Merlini [16].

The Fig.3 shows depends on $k$ of $|R(k)/k - c_k|$ obtained on the computer. Here the fit was obtained by the least square method from the data with $k > 10000$ to avoid transient regime and it is given by the equation $y = 0.01175x^{-1.527}$. 

**Lemma 4.** There is a real number $A$ such that for $0 < x < y$
\[ \left| \frac{R(x)}{x} - \frac{R(y)}{y} \right| \leq A(y - x)x^{-3/2}. \] (34)

Proof. We have
\[ \left| \frac{R(x)}{x} - \frac{R(y)}{y} \right| \leq \sum_{n=1}^{\infty} \frac{\exp(-x/n^2) - \exp(-y/n^2)}{n^2}. \] (35)

From Mean-Value Theorem we conclude that there exists $z \in (x, y)$ such that
\[ \exp(-x/n^2) - \exp(-y/n^2) = \frac{y-x}{n^2} \exp(-z/n^2) < \frac{y-x}{n^2} \exp(-x/n^2). \] (36)

Finally it follows from Corollary 1 that
\[ \left| \frac{R(x)}{x} - \frac{R(y)}{y} \right| \leq (y - x) \left( \frac{\sqrt{\pi}}{4} x^{-3/2} + \frac{4}{e^2} x^{-2} \right). \] (37)

In paper [9], the following equivalence had already been anticipated:

**Theorem 1.** For any real number $\delta > -3/2$ we have
\[ R(x) = O(x^{\delta + 1}) \Leftrightarrow c_k = O(k^\delta). \] (38)
Proof.  ⇒ For integer $x$, (38) follows immediately from Lemma 3.  ⇐ For non-integer $x$, we take Lemma 4 putting $y = \lfloor x \rfloor + 1$ and use (32). □

Remark. Putting $\delta = -3/4 + \epsilon$, we see that the Riesz criterion is equivalent to the Báez-Duarte criterion.

3 The values of $c_k$ for large $k$

For large negative $x$ function $R(x)$ tends to $xe^{-x}$. For positive $x$, the behaviour of $R(x)$ is much more difficult to reveal because the series (2) is very slowly convergent. Having applied Kummer’s acceleration convergence method, we get

\[ R(x) = x \left( \frac{6}{\pi^2} + \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} \left( e^{-\frac{x}{n^2}} - 1 \right) \right). \]  \hspace{1cm} (39)

Using this formula we were able to produce the plot of $R(x)$ for $x$ up to $10^7$, see Fig.1. The first nontrivial zero of $R(x)$ is $x_0 = 1.156711643750816\ldots$. It is a reflection of the fact, that $c_0 > 0$ while $c_1 < 0$. Riesz in [4] has noticed the existence of at least one positive real zero of $R(x)$, while in [7] Báez-Duarte has proved existence of infinitely many zeros of $R(x)$. The envelopes on the Fig.2 are given by the equations

\[ y(x) = \pm Ax^{1/4}, \quad A = 0.777506\ldots \times 10^{-5}. \]  \hspace{1cm} (40)

It is very time consuming to calculate values of the sequence $c_k$ directly from the definition (4), see [8], [9]. The point is that for large $j$, $\zeta(2j)$ is practically 1, and to distinguish it from 1 high precision calculations are needed. The experience shows that to calculate $c_k$ from (4) roughly $k \log_{10}(2)$ digits of accuracy is needed [9]. However in [6] Báez-Duarte gave the explicit formula for $c_k$ valid for large $k$:

\[ c_{k-1} = \frac{1}{2k} \sum_{\rho} \frac{k^{\frac{3}{2}} \Gamma(1 - \frac{\rho}{2})}{\zeta'(\rho)} + o(1/k), \]  \hspace{1cm} (41)

where the sum runs over nontrivial zeros $\rho$ of $\zeta(s)$: $\zeta(\rho) = 0$ and $\Im(\rho) \neq 0$. Maślanka in [8] gives the similar formula which contains the term hidden in $o(1/k)$ in (41).

Let us introduce the notation

\[ \frac{\Gamma(1 - \frac{\rho}{2})}{\zeta'(\rho)} = A_i + iB_i. \]  \hspace{1cm} (42)

Assuming that $\rho_i = \frac{1}{2} + i\gamma_i$, it can be shown that $A_i$ and $B_i$ very quickly decrease to zero [12], [9]:

\[ \left| \frac{\Gamma(1 - \frac{\rho}{2})}{\zeta'(\rho)} \right| \sim e^{-\pi\gamma_i/4}. \]  \hspace{1cm} (43)
Finally, for large $k$, we obtain:

$$c_{k-1} = \frac{1}{k^3} \sum_{i=1}^{\infty} \left\{ A_i \cos \left( \frac{\gamma_i \log(k)}{2} \right) - B_i \sin \left( \frac{\gamma_i \log(k)}{2} \right) \right\}. \quad (44)$$

The above formula explains oscillations on the plots of $c_k$ published in [6] and [8], see Fig.2. Because these curves are perfect cosine-like graphs on the plots versus $\log(k)$ it means that in fact in the above formula (44) it suffices to maintain only the first zero $\gamma_1 = 14.134725 \ldots$, $A_1 = 2.0291739 \ldots \times 10^{-5}$, $B_1 = -3.315924 \ldots \times 10^{-5}$ and skip all remaining terms in the sum. It is justified by the very fast decrease of $A_i$ and $B_i$ following from (43).

4 The sums of $c_k$

Let us perform the formal calculation

$$\sum_{k=0}^{\infty} t^k (1 - E)^k = \frac{1}{1 - t(1 - E)} \quad (45)$$

$$= \frac{1}{1 - t} \frac{1}{1 + \frac{t}{1-t}E} = \frac{1}{1 - t} \sum_{k=0}^{\infty} \left( -\frac{t}{1-t} \right)^k E^k.$$

Acting with both sides on the function $j \to 1/\zeta(2j + 2)$ we get

$$\sum_{k=0}^{\infty} c_k t^k = \frac{1}{1 - t} \sum_{k=0}^{\infty} \left( -\frac{t}{1-t} \right)^k \frac{1}{\zeta(2k + 2)}. \quad (46)$$

Of course instead of $1/\zeta(2j + 1)$ we can take an arbitrary function.

The above calculation was formal and we need to know what is the domain of convergence. We will get (46) in another way. Let us consider following identity

$$\frac{1}{n^2} \sum_{k=0}^{\infty} \left( 1 - \frac{1}{n^2} \right)^k t^k = \frac{1}{t + (1-t)n^2} = \frac{1}{1 - t} \sum_{k=0}^{\infty} \left( -\frac{t}{1-t} \right)^k \frac{1}{n^{2k+2}}. \quad (47)$$

The first sum is convergent for $-1 \leq t \leq 1$ while the second one is convergent for $-\infty < t < 1/2$. Thus the common domain of convergence is the interval $(-1, 1/2)$. Hence

$$\sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{\mu(n)}{n^2} \left( 1 - \frac{1}{n^2} \right)^k t^k = \frac{1}{1 - t} \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{\mu(n)}{n^{2k+2}} \left(-\frac{t}{1-t}\right)^k. \quad (48)$$

The sums (48) are absolutely convergent and we can change the order of summation obtaining (46).
Substituting $t = -1$ in the equation (48), we get

$$\sum_{k=0}^{\infty} (-1)^k c_k = \sum_{k=1}^{\infty} \frac{1}{2^k} \zeta(2k) = 0.782527985325384234576688 \ldots$$

This number probably cannot be expressed by other known constants, because the Simon Plouffe inverter failed to find any relation [19]. Applying Abel’s summation, we can write the r.h.s. of (49) as:

$$\sum_{k=1}^{\infty} \frac{1}{2^k} \frac{1}{\zeta(2k)} = 1 + \sum_{k=1}^{\infty} \left(1 - \frac{1}{2^k}\right) \left(\frac{1}{\zeta(2k)} - \frac{1}{\zeta(2k+2)}\right)$$

$$= 1 + \int_{2}^{\infty} \left(1 - \frac{1}{2\lfloor x/2 \rfloor}\right) \frac{\zeta'(x)}{\zeta^2(x)} \, dx.$$ 

More detailed considerations gives

$$\sum_{j=0}^{k-1} (-1)^j c_j = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{1}{\zeta(2k)} - \frac{(-1)^k}{2} c_k + O(k^{-3/2}).$$

Now we turn to the sum $\sum_{i=0}^{\infty} c_i$. The partial sum can be expressed in the following way:

$$S_{k-1} = \sum_{i=0}^{k-1} c_i = \sum_{n=1}^{\infty} \mu(n) \left(1 - \left(1 - \frac{1}{n^2}\right)^k\right) = -\sum_{j=1}^{k} \left(\frac{k}{j}\right) \frac{(-1)^j}{\zeta(2j)}.$$

Computer calculations show that the partial sums initially tend from above to -2, but for $k \approx 91000$ the partial sum crosses -2 and around $k \approx 100000$ the partial sum starts to increase. These oscillations begin to repeat with growing amplitude around -2, see Fig. 4. The value -2 was informally derived in [18].

For large $k$ the oscillations are described by the integral of (44)

$$k^{1/4} \sum_{i=1}^{\infty} \frac{1}{1/4 + \gamma_i^2} \left\{(A_i + 2B_i \gamma_i) \cos \left(\frac{\gamma_i \log(k)}{2}\right) - (B_i - 2A_i \gamma_i) \sin \left(\frac{\gamma_i \log(k)}{2}\right)\right\}$$

$$= O\left(k^{3/4}\right).$$

It is interesting that the amplitude is very small, e.g. at $k \sim 10^8$ the amplitude is of the order 0.001. By combining (52) and (53) we get that

$$\sum_{n=1}^{\infty} \mu(n) \left(1 - \left(1 - \frac{1}{n^2}\right)^k\right)$$

oscillates around -2 with the amplitude growing like $k^{1/4}$. We generalize the last statement in the form of the following
Conjecture 1: Let $b \geq a > 0$. Then the sum
\[
\sum_{n=1}^{\infty} \frac{\mu(n)}{n^{b-a}} \left( 1 - \left( 1 - \frac{1}{n^a} \right)^k \right)
\] (55)
oscillate around
\[
\frac{1}{\zeta(b-a)}
\] (56)
with the amplitude given unconditionally by $k^{1-b+2a \over 2a}$ and with the amplitude growing like $k^{a-b+1/2 \over a}$ under the assumption of the Riemann Hypothesis.

It seems to be mysterious that the sum $\sum_{i=0}^{\infty} c_i$ oscillates around -2, while the alternating sum $\sum_{i=0}^{\infty} (-1)^i c_i$ gives probably transcendent number.

Acknowledgement We would like to thank Prof. L. Báez-Duarte, Prof. M. Coffey and Prof. K. Maślanka for e-mail exchange. To prepare data for some figures we used the free computer algebra system PARI/GP [20].

References

[1] E. C. Titchmarsh The Theory of the Riemann Zeta Function, 2nd ed. New York: Clarendon Press, 1987.

[2] http://www.aimath.org/WWN/rh/

[3] http://www.secamlocal.ex.ac.uk/people/staff/mrwatkin/zeta/RHreformulations.htm

[4] M. Riesz Sur l’hypothe’se de Riemann, Acta Math. 40 (1916), 185-190

[5] L. Báez-Duarte, A new necessary and sufficient condition for the Riemann Hypothesis, 2003, math.NT/0307215

[6] L. Báez-Duarte, A sequential Riesz-like criterion for the Riemann Hypothesis, International Journal of Mathematics and Mathematical Sciences (2005) pp. 3527–3537

[7] L. Báez-Duarte, Möbius convolution and the Riemann Hypothesis, International Journal of Mathematics and Mathematical Sciences (2005) pp. 3599–3608

[8] K. Maślanka, Báez-Duartes Criterion for the Riemann Hypothesis and Rices Integrals, math.NT/0603713 v2 1 Apr 2006

[9] M. Wolf Evidence in favor of the Báez-Duarte criterion for the Riemann Hypothesis math.NT/0605485 17 May 2006
Riesz and Báez-Duarte criteria for RH

[10] Ronald L. Graham, Donald E. Knuth, Oren Patashnik, *Concrete Mathematics: A Foundation for Computer Science* (2nd Edition) Reading, MA: Addison-Wesley, 1994.

[11] M.W. Coffey, “On the coefficients of the Báez-Duarte criterion for the Riemann hypothesis and their extensions”, arXiv:math-ph/0608050v2

[12] R.B. Paris, “A note on the evaluation of the Riesz function”, technical report MS (04:03), University Abertay Dundee, March 2004

[13] A. Chaudhry, “The unification of the Hardy-Littlewood and Riesz conjectures” preprint, presented at the ICM (Madryt, 2006) as a poster

[14] G.H. Hardy and J.E. Littlewood “Contributions to the theory of the Riemann zeta function and the theory of prime distribution”, Acta Mathematica 41 (1918) p.119

[15] G.Polya, G.Szegő, *Problems and Theorems in Analysis. Volume II* (Springer, Berlin, Heidelberg, New York 1976, 1998)

[16] S. Beltraminelli, D. Merlini, *The criteria of Riesz, Hardy-Littlewood et al. for the Riemann Hypothesis revisited using similar functions*, arXiv:math.NT/0601138 7 Jan 2006

[17] S. Beltraminelli, D. Merlini, “Riemann Hypothesis: a special case of the Riesz and Hardy-Littlewood wave and a numerical treatment of the Báez-Duarte coefficients up to some billions in the k-variable”, arXiv:math/0609480v1

[18] J. Cisło, M. Wolf, “Equivalence of Riesz and Báez-Duarte criterion for the Riemann Hypothesis”, math.NT/0607782

[19] [Plouffe’s inverter](http://plouffe.fr/simon/mathtables.html) and private e-mail exchange with Simon Plouffe

[20] PARI/GP, version 2.2.11, Bordeaux, 2005, [http://pari.math.u-bordeaux.fr/](http://pari.math.u-bordeaux.fr/)

[21] E. C. Titchmarsh *The Theory of Functions*, 2nd ed. Oxford, England: Oxford University Press, 1960.
Fig.1 The plot of $R(x)$ for $x \in (0, 800000)$ and for $x \in (0, 10^7)$ in the inset. The part of $R(x)$ smaller than -0.006 is skipped.

Fig.2 The plot of $c_k$ for $k \in (1, 400000)$.
Fig. 3 The log-log plot of $|R(k)/k - c_k|$ for $k \in (0, 10^6)$.

Fig. 4 The distance from -2 of the partial sums $\sum_{k=0}^{n} c_k$ for $n = 1, \ldots 500000$. 

$2 + \sum_{i=0}^{k} c_i$