On Gluing CFT Correlators to Higher-Spin Amplitudes in AdS

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Abstract

We demonstrate how three-point correlation functions of the free scalar U(N) model involving two scalar operators and one spin-$s$ conserved current organize themselves into corresponding AdS amplitudes involving two scalar and one spin-$s$ bulk to boundary propagators, coupled via the bulk gauge invariant interaction vertex. Our analysis relies on the general program advocated in hep-th/0308184 and some features of the embedding space formalism also play an important role.

1 Introduction

Higher-Spin/CFT dualities [1, 2, 3], see [4, 5] for reviews, have been of great interest since their discovery for a variety of reasons. Firstly, since unitary representations of the Poincare group exist for arbitrarily high spins, it is certainly of interest to enquire if consistent, interacting theories involving such fields may be constructed, especially since a wide variety of no go theorems seem to preclude the existence of such theories at first sight. While we refer the reader to the reviews [6, 7] for a fuller account of the history of the field, it is important to especially note here that consistent, interacting theories involving higher-spin fields do exist in Anti-de Sitter space, and turn out to be of foundational importance to AdS/CFT duality.

1 Indeed, while the construction of the first such theories [11], reviewed in [12], significantly predates the proposal of AdS/CFT duality, the duality also almost seems to require the existence of higher-spin theories in AdS [13, 14, 15]. We briefly recollect the elements of this argument.

For definiteness, let us consider the original duality between Type IIB string theory on $\text{AdS}_5 \times S^5$ and $\mathcal{N} = 4$ SYM defined on the boundary of the $\text{AdS}_5$ 2 [21]. The boundary parameters $\lambda$ and $N$ are related to the bulk parameters string tension $\alpha'$, Planck length $\ell_P$, and string length $\ell_s$ via

$$N^2 \sim \left( \frac{\ell_{\text{AdS}}}{\ell_P} \right)^8, \lambda \sim \frac{\ell_{\text{AdS}}^4}{(\alpha')^2}. \quad (1)$$

From the above, we see that the large-$N$ limit on the boundary maps to a weakly coupled (semiclassical) limit on the bulk, while the $\lambda = 0$ limit on the boundary sets the string tension to zero in the bulk. Hence, a free planar limit of the CFT maps to a classical tensionless string in AdS. It has long been expected

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1 We also note recent work on the construction of consistent interacting higher-spin theories in flat space in [8, 9, 10], where the no-go theorems are evaded by working in light-cone coordinates.

2 Here and throughout, we shall be concerned with the Euclidean version of the duality, with the $\text{AdS}_{d+1}$ taken to be the Poincare patch, with boundary $\mathbb{R}^d$. A significant role is played by the embedding space formalism, in which these spaces are taken to lie in $\mathbb{M}^{1,d+1}$. Details of this formalism are briefly presented in Appendix C and reviewed more extensively in [16, 17, 18, 19, 20].
that many hidden symmetries of string theory might be manifest in this limit, much as the massless limit of QFTs makes their hidden symmetries manifest. Carrying over intuition from tensile string theory in flat space, the tensionless limit is also precisely where we expect massless higher-spins to appear in the string spectrum. We shall shortly review how AdS/CFT furnishes much sharper reasons to expect the same. In fact, a careful analysis, see e.g. [22, 23] indicates that higher-spin symmetry would indeed play an important role in this limit of the duality. Finally, though the above expressions were for one specific instance of AdS/CFT, the arguments presented here are expectedly generic to all known cases of the duality. We particularly note the recent developments [24] where these expectations have been very explicitly realized for the tensionless string in AdS3.

Moreover, in this limit both the bulk and the boundary theories admit a weak coupling expansion, so perturbative tests of the duality are possible, and indeed have been carried out for the case of ‘pure’ higher-spin/CFT dualities [25, 26, 27], see also [28, 29, 30]. Further, holographic reconstructions of the interaction vertices of the AdS higher-spin theory have been explored [31, 32, 33, 34], as reviewed for instance in [18, 19].

Given this situation, it is natural to enquire if one may use this window to gain some insight into the underlying mechanics of AdS/CFT duality. In particular, since both the bulk and boundary expressions are perturbative, and presumably admit at least some degree of simultaneous control, one may wish to systematically rewrite expressions on one side of the duality as their counterparts on the other side. Specifically, we ask if we may take a CFT correlation function, computable by Wick contractions, and systematically rewrite it as the sum of Witten diagrams in AdS? It would clearly be very interesting to identify a precise mechanism by which this would happen. For one, the original proposal of [21] seems to rely deeply on dynamics of D-branes in string theory, supersymmetry, and open-closed string duality. However, the actual computational prescription for comparing bulk and boundary theories in the classical bulk limit makes little to no reference to a string theory, using instead the symmetries of the bulk and boundary [35, 36] as a guiding principle. Identifying the precise mechanism by which bulk and boundary theories organize into each other could shed new light on the interplay between these two seemingly disparate features that underpin the duality.

A general prescription for carrying this out, inspired by open-closed string duality, was proposed in [37, 38, 39, 40]. Schematically, one starts with the CFT correlator written in a slight variation of Schwinger parametrization and carries out a change of variables on these parameters. For the all-scalar three-point function 3, the resulting expression quite naturally became a product of three scalar bulk to boundary propagators meeting at a bulk point, which was integrated over [37]. The boundary coordinates of the bulk point arose from momentum conservation in the correlator, while the extra holographic coordinate was obtained from the Schwinger parameters. These facts may be quickly reviewed by following our equations (9) through to (38) all the while setting $s = 0$ there. In what follows, we shall follow the nomenclature of [37] and refer to their proposed prescription as ‘gluing’. In this paper, we shall extend this analysis to the three point function of two scalar operators with one conserved current of arbitrary spin. Even at the outset one may notice several potential subtleties. On the AdS side, the explicit form of the interaction vertex may be changed on integration by parts, and the form of the spinning propagator is also gauge dependent. Additionally, there is the freedom of doing field redefinitions in the bulk. Having glued the CFT correlation function as per [37], how may one recognize it as having the form of the appropriate AdS amplitude? In this regard, the embedding space formalism [43] turns out to be especially useful, which should perhaps not surprise us, given its great utility in studying higher-spin fields and their interactions, see e.g. [17, 44, 45, 46, 47, 48, 49, 50, 51], as well as correlators in conformal field theory, see e.g. [52, 53, 54] and [20] for an introduction to the formalism for CFTs.

3The three-point function is a natural starting point for this program, as there is only one dual Witten diagram to organize the CFT correlator into.

4Gluing for some ‘low spin’ three-point correlators was previously studied in [41, 42].
We shall now turn to a brief account of the free CFT and its spectrum and holographic dual, then in Section 2 obtain the form of CFT correlator in embedding space after gluing. We then show in Section 3 that this indeed has the structure we expect from the AdS amplitude. We then conclude.

A final note on conventions. We work Euclidean signature, using coordinates \( \vec{x} \) on the \( d \) dimensional boundary, and coordinates \( x \equiv (t, \vec{w}) \) on the Poincare patch of AdS\(_{d+1} \). The usual holographic coordinate \( z \) on the Poincare patch is related to \( t \) via \( z = \sqrt{t} \). The AdS\(_{d+1} \) metric in our coordinates is

\[
d s^2 = \frac{d t^2}{t^2} + \frac{1}{t} d \vec{w}^2, \quad \sqrt{g} d^{d+1} x = \frac{d t}{t^{d/2+1}} d^d w.
\]

We shall often drop the overhead arrows. Further, since we work extensively with symmetric tensors, we shall repackaged indices in terms of a polarization vector. For example, a spin-\( s \) operator in the boundary will be written as \( J_s(x, z) = J_{\mu_1 \ldots \mu_s}(x) z^{\mu_1} \ldots z^{\mu_s} \). The polarization vector in the bulk will be denoted by \( u \). Small letters denote intrinsic quantities, and capital letters denote quantities in embedding space.

1.1 The Free CFT on the Boundary and the Interacting Theory in the Bulk

We shall most concretely be working with the free \( U(N) \) vector model with fundamental scalars in \( \mathbb{R}^d \), i.e.

\[
S = \int d^d x \varphi^{i*}(x) \partial^2 \varphi_i(x),
\]

where we shall consider the set of single trace conformal primary operators involving two insertions of the field \( \varphi_i \). This is given by [55] \(^5\)

\[
J_s(x, z) = \varphi^*(x) f^{(s)} \left( z \cdot \frac{\delta}{\delta z}, z \cdot \frac{\delta}{\delta \bar{z}} \right) \varphi(x),
\]

\[
f^{(s)}(u, v) = \left[ z^{s(\Delta-1)} z^{(\Delta-1)2u} \right]^{-1/2} (u + v)^s C_s^{(\Delta/2-1)} \left( \frac{u - v}{u + v} \right),
\]

where \( z^2 = 0 \) encodes tracelessness of the primary, \( s \) is an arbitrary positive integer, corresponding to the spin of the primary, and \( C_s^{(\Delta/2-1)} \) is a Gegenbauer polynomial. Here, and throughout, \( \Delta = d - 2 \), which is the dimension of the scalar primary \( \varphi^* \varphi_i \). The normalization constant is from [31], with a relative \( \sqrt{2} \) since we have a complex scalar instead of a real one, and is chosen such that

\[
\langle J_s(x_1, z_1) J_s(x_2, z_2) \rangle = \frac{1}{x_{12}^{\Delta+2s}} \left( z_1 \cdot z_2 - 2 \frac{z_1 \cdot x_{12} z_2 \cdot x_{12}}{x_{12}^2} \right)^s.
\]

For most computations done in this paper it is usually more convenient to utilize the series expansion of \( f^{(s)} \) and instead write

\[
f^{(s)}(u, v) = n_s^{-1/2} \sum_{k=0}^{s} \frac{(-1)^k}{k! (s-k)! (k + \frac{d-4}{2})!} \frac{1}{(s-k + \frac{d-4}{2})!} u^k v^{s-k},
\]

where

\[
n_s = \frac{(-1)^s 2^{\Delta+3s-2} \Gamma \left( s + \frac{\Delta-1}{2} \right)}{\sqrt{\pi} s! \Gamma \left( \frac{\Delta}{2} \right)^2 \Gamma \left( s + \frac{\Delta}{2} \right) \Gamma (s + \Delta - 1)}.
\]

In particular, note that \( n_0 = \Gamma \left( \frac{\Delta}{2} \right)^{-4} \), so that \( f^{(0)}(u, v) = 1 \). One may check by explicit series expansion that (4) and (6) define the same currents, as we have done in Appendix B. We also note that with the

\(^5\)Higher-spin conserved currents in Minkowski space were also constructed in [56, 57, 58, 59].
exception of the \( s = 0 \) operator, these primaries all saturate the unitarity bound \([60, 61]\) and hence are conserved currents.

As should be apparent, for the vector model this completely exhausts the set of single trace conformal primaries. For a matrix valued scalar, this would be a subset, the full spectrum comprising of long multiplets of the conformal algebra as well as representations of mixed symmetry. Explicit enumerations of the spectrum are (partially) available in \([62, 63]\). However, even in the matrix case, our analysis would continue to apply for that subset and should also extend to scalar bilinear operators carrying additional internal symmetry indices, as in \( \mathcal{N} = 4 \) SYM. A few examples of such primaries in \( \mathcal{N} = 4 \) SYM are available in Table 1 of \([15]\), see \([22, 23]\) for more details regarding this spectrum.

As a final observation regarding the CFT, we note that though the theory is free, the correlation functions of the operators \( J_s \) contain non-zero connected components, as is well known and is widely available in the literature. The connected component of \( \langle J_0 J_0 J_s \rangle \), first evaluated in \([64]\), has also been explicitly evaluated below and is proportional to \( \frac{1}{\sqrt{N}} \). Standard AdS/CFT arguments, based on the dictionary \([35, 36]\) therefore lead us to expect a dual theory \([11, 65]\) with an AdS\(_{d+1}\) vacuum with an infinite tower of higher spins weakly coupled to each other. As mentioned before, it is remarkable that though the CFT \((3)\) is not embedded in string theory, a well defined AdS dual has been explicitly proposed in many cases \([1, 2]\), note also the low-dimensional instance of \([3]\), and the dualities seem to obey all the standard features of a stringy AdS/CFT duality.

## 2 The CFT Three-Point Functions

The three-point function \( \langle J_0 (x_1) J_0 (x_2) J_s (x_3, z) \rangle \) may readily be computed in the CFT by Wick contractions and determined to be \(^6\)

\[
\langle J_0 (x_1) J_0 (x_2) J_s (x_3, z) \rangle = g \frac{1}{(x_{12})^{\Delta} (x_{23})^{\Delta} (x_{31})^{\Delta}} \left[ \frac{z \cdot x_{13}}{x_{13}^2} - \frac{z \cdot x_{23}}{x_{23}^2} \right]^s .
\]

This is the form of the expression as expected from conformal symmetry. For the purposes of gluing into the AdS amplitude, we note that there is an alternate form of this correlator.

\[
\langle J_0 (x_1) J_0 (x_2) J_s (x_3, z) \rangle = (1 + (-1)^s) \int d^d w \int_0^{\infty} \frac{d \tau}{\tau^{d/2+1}} \int_0^1 \int d^3 \alpha \times \\
\times f^s \left( \frac{z \cdot x_{13}}{2 \tau \alpha_2 \alpha_1}, \frac{z \cdot x_{23}}{2 \tau \alpha_1} \right) e^{-\sum \frac{(\alpha_i - \omega)^2}{4 \tau \alpha_i \alpha_j}} \frac{1}{\tau^{3s/2 (\alpha_1 \alpha_2 \alpha_3)}} .
\]

Here, and in the rest of the paper, the measure

\[
d^3 \alpha \equiv d \alpha_1 \, d \alpha_2 \, d \alpha_3 \, \delta (\Sigma_i \alpha_i - 1 ) .
\]

While \((8)\) follows straightforwardly by computing the given correlator in the theory \((3)\) by Wick contractions in position space, it is less obvious that \((9)\) is also an expression for the correlator. In practice \((9)\) may most readily be arrived at by starting with the corresponding expression in momentum space, and Fourier transforming to position space. Carrying out this analysis to get a form suitable for gluing requires some care, and we provide details below.

\(^6\) We refer the reader to \([66, 52, 67, 68]\) where two and three-point functions of arbitrary-spin conserved currents have been extensively studied. We also note previous work \([69, 28, 29, 70]\) in \( d = 3 \). Explicit expressions for these current correlators in the \( O(N) \) vector model were obtained in \([64, 31]\).
2.1 The Three-Point Function in Momentum space

We now Equation (9), which is the essential starting-point for gluing into the corresponding AdS amplitude. We closely follow the approach of [37], starting with the momentum space representation of the amplitude and then Fourier transforming to position space. However, we work directly with the second quantized version of the theory rather than attempting to develop a worldline description incorporating vertex operators for higher-spin currents. It would be interesting to develop such a formalism, expectedly using [71, 72] as a starting point. The spin-s conserved current in position space is

\[ J(x, z) = \varphi^* (x) f^{(s)} \left( z \cdot \frac{\partial}{\partial z} \right) \varphi(x), \]  

(11)

where \( z^2 = 0 \) and \( f^{(s)} \) was defined in (6). We now take the Fourier transform of this equation to momentum space. We have the following conventions.

\[ f(x) = \int \frac{d^d k}{(2\pi)^{d/2}} e^{i k \cdot x} f(k), \quad f(k) = \int \frac{d^d x}{(2\pi)^{d/2}} e^{-i k \cdot x} f(x), \]  

(12)

\[ \delta^{(d)}(p) = \int \frac{d^d w}{(2\pi)^{d}} e^{-i w \cdot p}. \]

Then

\[ J_s(p, z) = i^{s+2d} \int \frac{d^d k}{(2\pi)^{d/2}} \varphi^*(k) f^{(s)}(z \cdot k, z \cdot (p - k)) \varphi(p - k), \]

(13)

and the 3-point function in momentum space is then given by

\[ \langle J_0(p_1) J_0(p_2) J_s(p_3, z) \rangle = i^{s+2d} \int \prod_{i=1}^3 \frac{d^d k_i}{(2\pi)^{d/2}} f^{(s)}(z \cdot k_i, z \cdot (p_3 - k_i)) \times \]

(14)

\[ \langle \varphi^*(k_1) \varphi(p_1 - k_1) \varphi^*(k_2) \varphi(p_2 - k_2) \varphi^*(k_3) \varphi(p_3 - k_3) \rangle. \]

Two sets of Wick contractions contribute to the connected part of this correlator. These are will be denoted by \( \langle \ldots \rangle_{(1,2)} \) respectively. We find that

\[ \langle J_0(p_1) J_0(p_2) J_s(p_3, z) \rangle_{(i)} = \frac{(-1)^d}{(2\pi)^{3d/2}} \left[ \frac{4\pi^{d/2}}{\Gamma \left( \frac{d}{2} \right)} \right]^3 \langle \langle J_0(p_1) J_0(p_2) J_s(p_3, z) \rangle \rangle_{(i)}, \]

(17)

where

\[ \langle \langle J_0(p_1) J_0(p_2) J_s(p_3, z) \rangle \rangle_{(1)} = i^s \delta(\Sigma_k p_i) \int d^d k \frac{f^{(s)}(z \cdot k, z \cdot (p_3 - k))}{(k + p_1)^2 (k - p_3)^2 k^2}, \]

(18)

and

\[ \langle \langle J_0(p_1) J_0(p_2) J_s(p_3, z) \rangle \rangle_{(2)} = i^s \delta(\Sigma_k p_i) \int d^d k \frac{f^{(s)}(z \cdot k, z \cdot (p_3 - k))}{(k - p_3)^2 (k + p_2)^2 k^2}. \]

(19)

We use the basic Wick contraction

\[ \varphi^*(k_1) \varphi(k_2) = \frac{4\pi^{d/2}}{\Gamma \left( \frac{d}{2} \right)} \delta^d(k_1 + k_2) \frac{1}{k_1^d}, \]

(15)

obtained by Fourier transforming the position space expression

\[ \varphi^*(x_1) \varphi(x_2) = \frac{1}{\Gamma \left( \frac{d}{2} \right)} \int_0^\infty dt t^d e^{-tx_1 x_2}. \]

(16)
We simplify the denominator using Schwinger parametrization. In the first Wick contraction,
\[
\frac{1}{(k + p_1)^2 (k - p_3)^2 k^2} = \int_0^\infty d^3 \tau e^{-\tau_1 (k - p_3)^2 - \tau_2 k^2 - \tau_3 (k + p_1)^2} .
\] (20)

We now change integration variables to $(\tau, \alpha_i)$ where $\tau_i = \tau \alpha_i$ and $d^3 \tau = \tau^2 d\tau d^3 \alpha$ where $d^3 \alpha = d\alpha_1 d\alpha_2 d\alpha_3 \delta \left( \sum_i (\alpha_i - 1) \right)$ & we therefore find
\[
\frac{1}{(k + p_1)^2 (k - p_3)^2 k^2} = \int \frac{d\tau}{\tau} \tau^3 \int d^3 \alpha e^{-\tau (\ell^2 + \alpha_1 p_3^2 + \alpha_2 p_3^2 + \alpha_3 p_3^2)} ,
\] (21)

where $\ell = k + \alpha_3 p_1 - \alpha_1 p_3$, and we used $\alpha_1 + \alpha_2 + \alpha_3 = 1$. As a result,
\[
\langle \langle J_0 (p_1) J_0 (p_2) J_s (p_3, z) \rangle \rangle \rangle (1) = i^s \pi^{d/2} \delta \left( \sum_k p_k \right) \int \frac{d\tau}{\tau^{d/2+1}} \tau^3 \int d^3 \alpha e^{-\tau \sum_k \alpha_k^2} \times \frac{\partial}{\partial \alpha_1} f^{(s)} (z \cdot (\alpha_1 p_3 - \alpha_3 p_1), z \cdot (\alpha_1 p_3 - \alpha_3 p_1)) .
\] (22)

$f^{(s)}$ is polynomial in its arguments. Terms odd in $\ell$ integrate to zero because they are odd functions. Terms even in $\ell$ integrate to zero because the $\ell$ integral is proportional to appropriately symmetrized combinations of $\eta^\mu\nu$, and $z^2 = 0$. Hence only the $\ell^0$ term contributes non-vanishingly to the integral, and may be evaluated using the standard Gaussian integral formula $\int d^d \ell e^{-\ell^2} = \left( \frac{\pi}{\tau} \right)^{d/2}$. We also have the identities
\[
\alpha_3 p_1 + (1 - \alpha_1) p_3 = -\alpha_3 (p_2 + p_3) + (1 - \alpha_1) p_3 = -\alpha_3 p_2 + (1 - \alpha_1 - \alpha_3) p_3 = -\alpha_3 p_2 + \alpha_2 p_3 .
\] (23)

Hence
\[
\langle \langle J_0 (p_1) J_0 (p_2) J_s (p_3, z) \rangle \rangle \rangle (1) = i^s \pi^{d/2} \delta \left( \sum_k p_k \right) \int \frac{d\tau}{\tau^{d/2+1}} \tau^3 \int d^3 \alpha e^{-\tau \sum_k \alpha_k^2} \times f^{(s)} (z \cdot (\alpha_1 p_3 - \alpha_3 p_1), z \cdot (\alpha_2 p_3 - \alpha_3 p_2)) .
\] (24)

The sum in the exponent above runs over values $i \neq j \neq k$. The second set of Wick contractions may be evaluated exactly analogously and we obtain
\[
\langle \langle J_0 (p_1) J_0 (p_2) J_s (p_3, z) \rangle \rangle \rangle (2) = i^s \pi^{d/2} \delta \left( \sum_k p_k \right) \int \frac{d\tau}{\tau^{d/2+1}} \tau^3 \int d^3 \alpha e^{-\tau \sum_k \alpha_k^2} \times f^{(s)} (z \cdot (\alpha_2 p_3 - \alpha_3 p_2), z \cdot (\alpha_1 p_3 - \alpha_3 p_1)) .
\] (25)

Since $f^{(s)} (y, x) = (-1)^s f^{(s)} (x, y)$, we have
\[
\langle \langle J_0 (p_1) J_0 (p_2) J_s (p_3, z) \rangle \rangle \rangle (2) = (1 + (-1)^s) i^s \pi^{d/2} \delta \left( \sum_k p_k \right) \int \frac{d\tau}{\tau^{d/2+1}} \tau^3 \int d^3 \alpha e^{-\tau \sum_k \alpha_k^2} f^{(s)} (z \cdot (\alpha_2 p_3 - \alpha_3 p_2), z \cdot (\alpha_1 p_3 - \alpha_3 p_1)) .
\] (26)

Restoring the normalization, we have
\[
\langle \langle J_0 (p_1) J_0 (p_2) J_s (p_3, z) \rangle \rangle \rangle = \frac{1 + (-1)^s}{[2^{d/2-2} \Gamma \left( \frac{d}{2} \right)]^2} \delta \left( \sum_k p_k \right) \int \frac{d\tau}{\tau^{d/2+1}} \tau^3 \int d^3 \alpha e^{-\tau \sum_k \alpha_k^2} f^{(s)} (z \cdot (\alpha_2 p_3 - \alpha_3 p_2), z \cdot (\alpha_1 p_3 - \alpha_3 p_1)) .
\] (27)

The reader may set $s = 0$ and compare the resulting expression with Equation (3.2) of [37]. We explicitly see that the parameter $\tau$ constructed from the Schwinger parameters as above plays the role of the worldline modulus in a first quantized formulation of the CFT. This connection was further developed on in [38].
2.2 Fourier Transforming to Position Space

Having obtained the correlator in momentum space we now Fourier transform it to position space via

\[
\langle \langle J_0 (x_1) J_0 (x_2) J_s (x_3, z) \rangle \rangle = \int \frac{d^d p_1 d^d p_2 d^d p_3}{(2\pi)^{3d/2}} e^{i \hat{p}_1 \cdot x_1} \langle \langle J_0 (p_1) J_0 (p_2) J_s (p_3, z) \rangle \rangle .
\]

(28)

Using the Fourier representation of the Dirac delta function,

\[
\langle \langle J_0 (p_1) J_0 (p_2) J_s (p_3, z) \rangle \rangle = i^s \frac{\pi^{d/2}}{2^{2d/2}} (1 + (-1)^s) \int \frac{d^d w}{(2\pi)^d} e^{-i (p_1 + p_2 + p_3) \cdot w} \times
\]

\[
\times \int \frac{d \tau}{\tau^{d/2 + 1}} \tau^3 d^3 \alpha f^{(s)} (z \cdot (\alpha_1 p_3 - \alpha_3 p_1), z \cdot (\alpha_2 p_3 - \alpha_3 p_2)) e^{-\tau \sum \alpha_i \alpha_j p_i^2} .
\]

(29)

The basic momentum integral is

\[
\int \frac{d^d p_1 d^d p_2 d^d p_3}{(2\pi)^{3d/2}} e^{i \hat{p}_1 \cdot (x_1 - w)} e^{-\tau \sum \alpha_i \alpha_j p_i^2} = \frac{1}{(2\pi)^{3d/2} (\alpha_1 \alpha_2 \alpha_3)^d} e^{-\sum \frac{(\alpha_i - w_i)^2}{4 \alpha_i \alpha_j}} .
\]

(30)

As a result, since \( p = -i \partial \),

\[
\langle \langle J_0 (x_1) J_0 (x_2) J_s (x_3, z) \rangle \rangle = (1 + (-1)^s) \frac{\pi^{d/2}}{2^{2d/2}} \int \frac{d^d w}{(2\pi)^d} \int \frac{d \tau}{\tau^{d/2 + 1}} \tau^3 d^3 \alpha \times
\]

\[
\times f^{(s)} (z \cdot (\alpha_1 \partial_3 - \alpha_3 \partial_1), z \cdot (\alpha_2 \partial_3 - \alpha_3 \partial_2)) e^{-\sum \frac{(\alpha_i - w_i)^2}{4 \alpha_i \alpha_j}} \tau^{d/2} .
\]

(31)

We finally obtain, also using the fact that \( z^2 = 0 \),

\[
f^{(s)} (z \cdot (\alpha_1 \partial_3 - \alpha_3 \partial_1), z \cdot (\alpha_2 \partial_3 - \alpha_3 \partial_2)) e^{-\sum \frac{(\alpha_i - w_i)^2}{4 \alpha_i \alpha_j}} \tau^{d/2} .
\]

(32)

As a result,

\[
\langle \langle J_0 (x_1) J_0 (x_2) J_s (x_3, z) \rangle \rangle = (1 + (-1)^s) \frac{\pi^{d/2}}{2^{2d/2}} \int \frac{d^d w}{(2\pi)^d} \int \frac{d \tau}{\tau^{d/2 + 1}} \tau^3 d^3 \alpha \times
\]

\[
\times f^{(s)} \left( \frac{z \cdot x_1}{2 \tau \alpha_2}, \frac{z \cdot x_2}{2 \tau \alpha_1} \right) e^{-\sum \frac{(\alpha_i - w_i)^2}{4 \alpha_i \alpha_j}} \tau^{d/2} .
\]

(33)

Comparing with (17) we see that this implies that the correlator is given by

\[
\langle J_0 (x_1) J_0 (x_2) J_s (x_3, z) \rangle = (1 + (-1)^s) \frac{\pi^{d/2}}{2^{3d} \Gamma \left( \frac{d}{2} \right)} \int \frac{d^d w}{(2\pi)^d} \int \frac{d \tau}{\tau^{d/2 + 1}} \tau^3 d^3 \alpha \times
\]

\[
\times f^{(s)} \left( \frac{z \cdot x_1}{2 \tau \alpha_2}, \frac{z \cdot x_2}{2 \tau \alpha_1} \right) e^{-\sum \frac{(\alpha_i - w_i)^2}{4 \alpha_i \alpha_j}} \tau^{d/2} .
\]

(34)
2.3 Writing in Embedding Space

We now multiply the right hand side with “\(1 = \frac{\Gamma(1\!+\!d\!-\!3)}{1\!+\!d\!-\!3} \int_0^\infty \frac{d\rho}{\rho} \rho^{s+d-3} e^{-\rho}\)” and carry out the change of variables proposed in [37], namely,

\[ t = 4 \rho \alpha_1 \alpha_2 \alpha_3. \]  

It is quite straightforward to find the resulting expression

\[
\langle J_0 (x_1) J_0 (x_2) J_0 (x_3, z) \rangle = \int d^4 w \int \frac{dt}{t^{d/2+1}} \int \frac{d^3 \rho \rho^2 d^3 \alpha e^{-\rho}}{t^{3/2}} \times f(s) \left( \frac{2 \rho_1 \rho_3}{t} z \cdot x_{13}, \frac{2 \rho_2 \rho_3}{t} z \cdot x_{23} \right) e^{-\sum \rho_i (x_i - w)^2} \left( \rho_1 \rho_2 \rho_3 \right)^{d-3}. \]

Next, change variables to \( \rho_i = \rho_\alpha i \) to obtain

\[
\langle J_0 (x_1) J_0 (x_2) J_0 (x_3, z) \rangle = \int d^4 w \int \frac{dt}{t^{d/2+1}} \int d^3 \rho e^{-\sum \rho_i (x_i - w)^2} \times f(s) \left( \frac{2 \rho_1 \rho_3}{t} z \cdot x_{13}, \frac{2 \rho_2 \rho_3}{t} z \cdot x_{23} \right) \left( \rho_1 \rho_2 \rho_3 \right)^{d-3}. \]

We next rescale \( \rho_i \) to \( \sqrt{t} \rho_i \) to obtain

\[
\langle J_0 (x_1) J_0 (x_2) J_0 (x_3, z) \rangle = 2^3 \int d^4 w \int \frac{dt}{t^{d/2+1}} \int d^3 \rho f(s) \left( \rho_1 \rho_3 z \cdot x_{13}, \rho_2 \rho_3 z \cdot x_{23} \right) \times \prod_{i=1}^3 \rho_i^{d-3} e^{-\rho_i \left( \sqrt{t} (x_i - w)^2 \right)}. \]

We would now like to argue that this expression represents the spin 00s amplitude in AdS_{d+1}. As remarked previously, while this match was relatively clean for the \( s = 0 \) case, for spinning fields there are potential subtleties. It is far from obvious how the expression (39) may be organized into an expectedly complicated bulk expression involving AdS covariant derivatives and a gauge dependent bulk to boundary propagator.

As remarked previously, it is convenient to re-express (39) in \( d + 2 \) dimensional embedding space \( \mathbb{M}^{1,d+1} \) of signature \((- + \ldots +)\) in order to carry out the match with AdS. Define the points

\[ P_i = \left( \frac{x_i^2}{2}, \bar{x}_i, \frac{1 - x_i^2}{2} \right), \quad W = \left( \frac{1 + w^2}{2}, \bar{w}, \frac{1 - w^2}{2} \right). \]

As a result,

\[ (x_i - w)^2 = -2 P_i \cdot W. \]

These are the embedding space representatives of the boundary points \( x_i \) and \( w \) respectively. Also, we define

\[ X = \left( \frac{t + w^2 + 1}{2t^{1/2}}, \frac{1 - w^2}{2t^{1/2}}, \frac{t - w^2}{2t^{1/2}} \right) = t^{-1/2} W + \frac{t^{1/2}}{2} \left( 1, \bar{0}, -1 \right). \]

\[ \text{The reader would readily recognize the } s = 0 \text{ case, namely,} \]

\[
\langle J_0 (x_1) J_0 (x_2) J_0 (x_3) \rangle = \int d^4 w \int \frac{dt}{t^{d/2+1}} \int d^3 \rho \prod_{i=1}^3 \rho_i^{d-3} e^{-\rho_i \left( \sqrt{t} (x_i - w)^2 \right)} \]

\[
= \int d^4 w \int \frac{dt}{t^{d/2+1}} \int d^3 \rho \prod_{i=1}^3 K_{d-2} (x_i; t, z),
\]

which is the integrated product of three bulk to boundary propagators of \( \Delta = d - 2 \) scalars in AdS_{d+1}. This is of course the essential computation of [37].
and hence
\[ -2 P_i \cdot X = -2t^{-1/2} P_i \cdot W + t^{1/2} = \sqrt{t} + \frac{(x_i - w)^2}{\sqrt{t}}. \] (43)

The reader would recognize $X$ as a point in $\text{AdS}_{d+1}$ embedded as the locus $X^2 = -1$ in $\mathbb{M}^{1,d+1}$. The embedding space representative of the intrinsic polarization vector $z$ is denoted by $Z$ and is defined to obey the constraints via $Z^2 = 0$ and $Z \cdot P_3 = 0$. A canonical choice is
\[ Z_c = \frac{\partial P_3}{\partial x_3} \cdot z = (\vec{x}_3 \cdot \vec{z}, \vec{z}, -\vec{x}_3 \cdot \vec{z}), \] (44)

As a result, for $i = 1, 2$,
\[ Z_c \cdot P_i = \vec{z} \cdot \vec{x}_{i3}. \] (45)

Hence, for a generic choice of polarization vector $Z = Z_c + \alpha P_3$ we would have
\[ z \cdot x_{i3} = Z \cdot P_i - \alpha P_1 \cdot P_3. \] (46)

Choosing $\alpha = \frac{(Z \cdot X)}{(P_3 \cdot X)}$, we have
\[ z \cdot x_{i3} = Z \cdot P_i - \frac{(Z \cdot X)}{(P_3 \cdot X)} P_1 \cdot P_3. \] (47)

With these inputs, and defining $\Delta = d - 2$, (39) reduces to
\[ \langle J_0 (P_1) J_0 (P_2) J_s (P_3, Z) \rangle = \sum_{k=0}^{s} \frac{(-1)^k}{k! (s-k)! (k+\frac{d+1}{2})! (s-k+d+1)!} \int_{\text{AdS}} d^{d+1}x \]
\[ \times \int \frac{dp_1 dp_2 dp_3}{\rho_1 \rho_2 \rho_3} \rho_1^{\Delta+k} \rho_2^{\Delta+s-k} \rho_3^{\Delta+s} (Z \cdot P_1 P_3 X - Z \cdot X P_3 \cdot P_1)^k \]
\[ \times (Z \cdot P_2 P_3 \cdot X - Z \cdot X P_3 \cdot P_2)^{s-k} \frac{1}{(P_3 \cdot X)^s} \prod_{i=1}^{3} e^{2 \rho_i P_i \cdot X}. \] (48)

The reader familiar with embedding space expressions of AdS propagators [16] might already recognize a structure reminiscent of two scalar bulk to boundary propagators multiplied to a spinning bulk to boundary propagator, i.e. an AdS amplitude.

## 3 The AdS Amplitude in Embedding Space

Having organized the CFT expression into a form comparable to an AdS amplitude, we now turn to the AdS computation. The on-shell coupling of a spin-$s$ field to two scalars is given in intrinsic AdS coordinates $x$ as
\[ \hat{V}_{0,0,s} = g \varphi_{\mu_1...\mu_s} \varphi^{(a)} \nabla^{\mu_1} \cdots \nabla^{\mu_s} \varphi^{(b)} = g s! \varphi^{(a)} (x) (\partial_u \cdot \nabla)^s \varphi^{(b)} (x) \varphi_s (x, u), \] (49)

which in embedding space may be re-expressed as
\[ \hat{V}_{0,0,s} = g s! \varphi^{(a)} (X) (\partial_U \cdot \partial_X)^s \varphi^{(b)} (X) \varphi_s (X, U), \] (50)

The coupling of two scalars to a spin-$s$ field, invariant under the gauge transformation $\varphi (x) \rightarrow \varphi (x) + \nabla \xi (x)$, is given by
\[ V_{00s} = \int d^{d+1}x J^{\mu_1...\mu_s} (x) \cdot \varphi_{\mu_1...\mu_s} (x). \] (51)
where $J^{(s)}$ is a conserved current constructed from the scalars. A specific example of such a current may be constructed in embedding space, leading to the interaction [17]

$$\mathcal{V}_{00s} = g_s \frac{s!}{k!} \sum_{k=0}^{s} (-1)^k \frac{s!}{k!(s-k)!} (\partial_U \cdot \partial_X)^k \varphi^+(X) (\partial_U \cdot \partial_X)^{s-k} \varphi(X) \Phi(X,U).$$

(52)

Since the spin-$s$ field is transverse traceless, we may integrate (52) by parts to obtain (50) where the respective couplings are related by $g = g_s i^s 2^s$.

To match with the CFT expression (48) we shall consider instead the following current in embedding space $^{10}$

$$\tilde{J}_s = \sum_{k=0}^{s} \frac{(-1)^k}{k!(s-k)!(k+\frac{d-1}{2})!(s-k+\frac{d-1}{2})!} (U \cdot \partial_X)^k \varphi^+(X) (U \cdot \partial_X)^{s-k} \varphi(X),$$

(54)

which we couple to the spin-$s$ field via $^{11}$

$$\mathcal{V}_{00s} = g_s \Phi(X,\partial_U) \sum_{k=0}^{s} \frac{(-1)^k}{k!(s-k)!} (k+\frac{d-1}{2})!(s-k+\frac{d-1}{2})! (U \cdot \partial_X)^k \varphi^+(X) (U \cdot \partial_X)^{s-k} \varphi(X).$$

(55)

Computing the corresponding amplitude,

$$\mathcal{A}_{00s} = g_s \int d^{d+1}X \Pi_s (P_3, Z; X, \partial_U) \sum_{k=0}^{s} \frac{(-1)^k}{k!(s-k)!} (k+\frac{d-1}{2})!(s-k+\frac{d-1}{2})! \times$$

$$\times (U \cdot \partial_X)^k \Pi (P_1, X) (U \cdot \partial_X)^{s-k} \Pi (P_2, X).$$

(56)

To proceed further, we need explicit expressions of bulk to boundary propagators in embedding space. These were obtained in [16], to which we refer the reader for details.

$$\Pi_{\Delta_s,s} (P_3, Z; X, U) = C_{\Delta_s,s} \frac{[2(U \cdot P_3)(X \cdot Z) - 2(U \cdot Z)(X \cdot P_3)]^s}{(-2P_3 \cdot X)^{\Delta_s + s}},$$

(57)

here $\Delta_s$ is the dimension of the spin-$s$ field, i.e. $\Delta_s = \Delta + s = d - 2 + s$. Also

$$C_{\Delta,s} = \frac{(s + \Delta_s - 1) \Gamma(\Delta_s)}{2\pi^{d/2} \Gamma(\Delta + 1 - \frac{d}{2})}.$$

(58)

One may explicitly convert this expression to intrinsic coordinates and recognize that this is the spin-$s$ bulk to boundary propagator in the transverse-traceless gauge [78]. We refer the reader to [18] for a review

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$^9$Conserved currents in various AdS dimensions were also constructed in [73, 74, 75, 76] and for a conformally coupled scalar in general dimensions in [77]. The embedding space formalism allows us to work with any scalar field in arbitrary dimensions, as pointed out in [17].

$^{10}$As outlined in Appendix C, any field of the form (90)

$$J^{(s)}(X, U) = \sum_{k=0}^{s} t_{s,k} (U \cdot \partial_X)^k \varphi^+(X) (U \cdot \partial_X)^{s-k} \varphi$$

(53)

obeys the embedding space as well as AdS covariant conservation equation, provided the homogeneties of $\varphi(X)$ and $\varphi^+(X)$ are chosen appropriately. All such conserved currents of a given spin in embedding space would differ from each other by longitudinal and trace terms. Since the spin-$s$ field they are coupling to is transverse traceless, this distinction should be irrelevant.

$^{11}$We may again integrate this vertex by parts, and use the transversality condition to obtain (50) where the respective couplings are now related by $g = g_s \frac{2^s \Gamma(\frac{1}{2} (2s + \Delta - 1))}{\sqrt{\pi} \Gamma(s+1) \Gamma(s+\Delta-1) \Gamma(\frac{1}{2} (2s + \Delta))}$. 

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10
of propagators in embedding space for different gauge choices. Further, using a Schwinger parametrization for \( \frac{1}{(-2P_3 \cdot X)^2} \), and identifying \( \Delta_s = \Delta + s \), we may write

\[
\Pi_{\Delta, s} (P_3, Z; X, U) = \frac{C_{\Delta, s+3s}}{\Gamma (\Delta + s)} \left[ (2P_3 \cdot X) - 2Z \cdot (X \cdot P_3) \right] \cdot U \int_0^\infty \frac{d \rho}{\rho} \rho^{\Delta + s} e^{-2 \rho P_3 \cdot X}. \tag{59}\]

The amplitude then evaluates to

\[
A_{00s} = \frac{2^s g_s}{\Gamma (\Delta)^2} \int d^{d+1}X \Pi_s (P_3, Z; X, \partial U) \sum_{k=0}^s \frac{(-1)^k}{k! (s-k)! (k+\frac{d-4}{2})! (s+k+\frac{d-4}{2})!} \times \int \frac{d \rho_1 d \rho_2}{\rho_1 \rho_2} \rho_1^{\Delta + k} \rho_2^{\Delta + s - k} (U \cdot P_1)^k (U \cdot P_2)^{s-k} e^{2 \rho_1 P_1 \cdot X} e^{2 \rho_2 P_2 \cdot X}. \tag{60}\]

As a result,

\[
A_{00s} = g_s \frac{2^s s!}{\Gamma (\Delta)^2} \frac{C_{\Delta, s+3s}}{\Gamma (\Delta + s)} \sum_{k=0}^s \frac{(-1)^k}{k! (s-k)! (k+\frac{d-4}{2})! (s+k+\frac{d-4}{2})!} \int d^{d+1}X \frac{1}{(-2P_3 \cdot X)^s} \times \int \frac{d \rho_1 d \rho_2 d \rho_3}{\rho_1 \rho_2 \rho_3} \rho_1^{\Delta + k} \rho_2^{\Delta + s - k} \rho_3^{\Delta + s} [(2P_3 \cdot X) - 2Z \cdot (X \cdot P_3)] \cdot P_1^k \left[ (2P_3 \cdot X) - 2Z \cdot (X \cdot P_3) \right] \cdot P_2^{s-k} e^{2 \rho_1 P_1 \cdot X} e^{2 \rho_2 P_2 \cdot X} e^{2 \rho_3 P_3 \cdot X}. \tag{61}\]

This further evaluates to

\[
A_{00s} = g_s \frac{2^s s!}{\Gamma (\Delta)^2} \frac{C_{\Delta, s+3s}}{\Gamma (\Delta + s)} \sum_{k=0}^s \frac{(-1)^k}{k! (s-k)! (k+\frac{d-4}{2})! (s+k+\frac{d-4}{2})!} \int d^{d+1}X \frac{1}{(-2P_3 \cdot X)^s} \times \int \frac{d \rho_1 d \rho_2 d \rho_3}{\rho_1 \rho_2 \rho_3} \rho_1^{\Delta + k} \rho_2^{\Delta + s - k} \rho_3^{\Delta + s} [P_3 \cdot P_1 X \cdot Z - Z \cdot P_1 X \cdot P_3]^k \times \times \left[ P_3 \cdot P_2 X \cdot Z - Z \cdot P_2 X \cdot P_3 \right]^{s-k} \prod_{i=1}^3 e^{2 \rho_i P_i \cdot X}. \tag{62}\]

which is the same as the CFT answer (48), up to an overall proportionality. We therefore see that the CFT expression naturally organizes itself into an AdS amplitude determined by an interaction vertex of the form (51), i.e.

\[
S_{\text{int}} = \int d^{d+1}x J^{\mu_1 \ldots \mu_s} (x) \varphi_{\mu_1 \ldots \mu_s} (x), \tag{63}\]

which is the gauge invariant three-point interaction of a spin-s field with two scalars. We now conclude with some final comments.

### 4 Conclusions

In this paper we showed how the approach of [37] may be used to organize correlators of spinning currents into dual AdS amplitudes. The computation presented here may be taken as a hint that a skeletal version of open-closed string duality continues to operate even for the higher-spin/CFT dualities considered here, which are non-supersymmetric and have no obvious string embedding. In fact, applying the prescription of [37] leads us quite naturally to the putative AdS amplitude one would evaluate to match with the CFT three-point function as per [35, 36].
At this juncture we would like to point to the intriguing role played by the embedding space formalism in our analysis. In the simplest instance of three scalar operators, we have seen

$$\langle J_0(P_1) J_0(P_2) J_0(P_3) \rangle = \int d^d w \ dt \frac{dt}{t^{d/2+1}} \int_0^\infty \prod_{i=1}^3 d\rho_i \ (\rho_1 \rho_2 \rho_3)^{d-2} e^{-\sum \rho_i \left(2 \frac{P_i \cdot \vec{w}}{t^{d/2}} - t^{1/2}\right)}$$

$$= \int d^d w \ dt \frac{dt}{t^{d/2+1}} \prod_{i=1}^3 \int_0^\infty \frac{d\rho_i}{\rho_i} \rho_i^{d-2} e^{-2\rho_i \cdot P_i \cdot X} = \int d^{d+1} X \prod_{i=1}^3 K_{d-2}(P_i, X). \tag{64}$$

The above equations make completely manifest a central feature of the AdS/CFT duality. Namely, the fact that both the CFT and the AdS expressions are in fact the same. The identification of the quantities $t$ and $\vec{w}$ as Schwinger parameter and auxiliary point or the coordinates of an interior AdS point is a matter of book-keeping and interpretation. If one wishes to interpret (64) as a CFT correlator then the first choice is natural, while interpreting it as an AdS amplitude requires the second choice. The analysis for the spinning correlators illustrates the same phenomenon, with the additional feature that the choice of boundary polarization vector representative $Z$ in embedding space seems to translate to gauge choice for the spin-$s$ field in the AdS bulk. We would like to better understand this point.

Further, it is interesting that the CFT amplitude when organized into AdS, doesn’t just produce the on-shell interaction vertex, but seems to yield the interaction of a higher-spin field with a conserved current, as required by bulk gauge invariance. In a sense therefore, the CFT amplitude does contain information about the full gauge invariant coupling of spinning fields to scalars. This is another remarkable, and somewhat gratifying, feature of the above analysis. Hopefully, these results are indicative of a promising starting point to unravel the mechanics of AdS/CFT further. We also note here the results of [79] where again embedding space played a crucial role in uncovering a correspondence between the CFT two-point function and the AdS bulk to boundary propagator, albeit through somewhat different means than those adopted here. It would be interesting to explore connections between the two approaches.

Note: While this paper was being readied for submission we learned of [80] which makes use of the analogy between Feynman graphs and electrical circuits to study conformal field theories in momentum space. It would also be of interest to develop our methods further with a view to explicating the AdS/CFT duality directly in momentum space.

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Appendix

A The Standard Form of the CFT Three-Point Function

In this section we reduce the expression (34) to the standard form, of the conformal structure times a prefactor. In order to do so, we carry out the $w$ integral using

$$\pi^{d/2} \int \frac{d^d w}{(2\pi)^d} e^{-\sum \frac{(x_j - w)^2}{4\alpha_j \alpha_k}} = (\tau \alpha_1 \alpha_2 \alpha_3)^{d/2} e^{-\sum \frac{(x_j - x_k)^2}{4\alpha_j \alpha_i}}, \tag{65}$$
and as a result

\[ \langle J_0(x_1) J_0(x_2) J_s(x_3, z) \rangle = \frac{1}{2^{3\Delta}} \frac{1 + (-1)^s}{\Gamma \left( \frac{s}{2} \right)^3} \int d\tau \tau^2 d^3 \alpha f(s) \left( \frac{x_{13}}{2\tau \alpha_2}, \frac{x_{23}}{2\tau \alpha_1} \right) \prod_{i=1}^{3} e^{-\frac{x_{ik}^2}{4\tau \alpha_i}}. \]  

(66)

Now defining \( \tau_i = \tau \alpha_i \) we have

\[ \langle J_0(x_1) J_0(x_2) J_s(x_3, z) \rangle = \frac{1}{2^{s+3\Delta}} \frac{1 + (-1)^s}{n_s^{1/2} \Gamma \left( \frac{s}{2} \right)^3} \int d^3 \tau \tau^s f(s) \left( \frac{x_{13}}{\tau_2}, \frac{x_{23}}{\tau_1} \right) e^{-\sum \frac{x_{ik}^2}{\tau_i \tau_j}}. \]  

(67)

The \( \tau \) integrals can be done to obtain the expected conformal structure. To see this, expand

\[ \langle J_0(x_1) J_0(x_2) J_s(x_3, z) \rangle = \frac{1}{2^{s+3\Delta}} \frac{1 + (-1)^s}{n_s^{1/2} \Gamma \left( \frac{s}{2} \right)^3} \int d^3 \tau \sum_{k=0}^{s} \frac{n!}{(s-k)!(k+\frac{\Delta+3}{2})!(s-k+\frac{\Delta+1}{2})!} \times \]

\[ \times \left( \frac{x_{13}}{\tau_2} \right)^k \left( \frac{x_{23}}{\tau_1} \right)^{s-k} e^{-\sum \frac{x_{ik}^2}{\tau_i \tau_j}}. \]

(68)

We see that the \( 4\tau_i \) are the inverses of the usual Schwinger parameters in terms of which which one would typically write the the three-point amplitude in position space. This is somewhat unsurprising since the \( \tau_i \) were the usual Schwinger parameters in momentum space. Finally, we carry out the \( \tau_i \) integrals using the identity

\[ \int d\tau \frac{1}{\tau^n} e^{-x^2/4\tau} = \frac{(2\pi)^{n/2}}{\Gamma(n-1)} \]  

(69)

and subsequently summing over \( k \), we obtain

\[ \langle J_0(x_1) J_0(x_2) J_s(x_3, z) \rangle = \frac{1 + (-1)^s}{n_s^{1/2} \Gamma \left( \frac{s}{2} \right)^3} \frac{2^{s-1/2}}{(r_2)^{s/2} (r_3)^{s/2}} \left[ \frac{x_{13}}{r_2} - \frac{x_{23}}{r_3} \right]^s, \]  

(70)

which contains the expected conformal structure. The prefactor may be evaluated using (7), which states

\[ n_s = \frac{(-1)^s 2^{s+3\Delta-2}}{\sqrt{\pi} s! \Gamma \left( \frac{s}{2} \right)^2} \left[ \frac{\Gamma \left( s + \frac{\Delta-1}{2} \right)}{\Gamma \left( s + \frac{\Delta}{2} \right) \Gamma \left( s + \Delta - 1 \right)} \right], \]  

(71)

and hence the prefactor is

\[ -i^s (1 + (-1)^s) 2^{s+2-s+\Delta} \pi^{1/4} \left[ \frac{\Gamma \left( s + \frac{\Delta}{2} \right) \Gamma \left( s + \Delta - 1 \right)}{s! \Gamma \left( s + \frac{\Delta-1}{2} \right) \Gamma \left( \frac{s}{2} \right)^2} \right]^{1/2}. \]  

(72)

## B Normalized Traceless Currents

In this section we provide some details of how the spin-\( s \) currents (4) are normalized via (7). We consider the two-point function, where conformal invariance demands that the result is of the form

\[ \langle J_s(x_1, z_1) J_s(x_2, z_2) \rangle = \frac{g}{(x_{12}^2)^{\Delta+s}} \left[ z_1 \cdot z_2 - 2 \frac{z_1 \cdot x_{12} z_2 \cdot x_{12}}{x_{12}^2} \right]^s, \]  

(73)
where we have to normalize the currents so that \( g = 1 \). If we adopt the series expansion form (6) for (4), then the simplest way to determine this normalization is to compute the coefficient of the term \( \frac{(z_1 \cdot z_2)^s}{(x_{12})^{\Delta+2s}} \).

Start with the two-point function in the Schwinger parametrization

\[
\langle J_s(x_1, z_1) J_s(x_2, z_2) \rangle = \sum_{k_1, k_2=0}^{s} c_{k_1}^s c_{k_2}^s \frac{1}{(\Delta)^2} \int_0^\infty \frac{du_1}{u_1} \frac{du_2}{u_2} (u_1 u_2)^{\Delta/2} \times
\]

\[
\times (z_1 \cdot \partial_{y_1})^{k_1} (z_1 \cdot \partial_{y_2})^{s-k_1} (z_2 \cdot \partial_{y_2})^{k_2} (z_2 \cdot \partial_{y_2})^{k_2} e^{-u_1(y_1-y_2)^2-u_2(y_1-y_2)^2}|_{(y, \bar{y}) \rightarrow x}.
\]

Since \( z_1^2 = 0 \), one readily sees that the only terms that contribute to \((z_1 \cdot z_2)^s\) come with \( k_1 = k \), \( k_2 = s-k \).

The relevant term evaluates to

\[
\langle J_s(x_1, z_1) J_s(x_2, z_2) \rangle = \frac{2^s (-1)^s (z_1 \cdot z_2)^s}{\Gamma(\Delta)^2} \frac{1}{(x_{12})^{\Delta+2s}} \sum_{k=0}^{s} \frac{1}{k!(s-k)!((s-k)\Delta+\frac{k}{2}-1)!} + \ldots,
\]

\[
= n_s \frac{(z_1 \cdot z_2)^s}{(x_{12})^{\Delta+2s}} + \ldots
\]

with \( n_s \) as defined in (7). As an example, the explicit form of the normalized traceless spin two current in \( d \) dimensions is

\[
J_2(x, z) = \varphi^*(x, z) \left\{ \left( \frac{\partial \cdot \bar{\varphi}}{2 \sqrt{2}} \right)^2 + \frac{2d-2}{d-2} \frac{z \cdot \partial \varphi}{2} - \frac{4d-2}{d-2} \frac{\bar{\varphi} \cdot \partial \varphi}{2} + \left( \frac{z \cdot \bar{\varphi}}{2} \right)^2 \right\} \varphi(x, z),
\]

which may be reproduced by expanding (4).

C Embedding Space for AdS and CFT

In this section we provide an overview of the embedding space formalism, which dates back to Dirac, and was further developed by Fronsdal \[43\]. The essential idea is to consider a \( d + 2 \) dimensional Minkowski space \( \mathbb{M}^{d+1} \) of metric signature \((- , + , \ldots , +)\). The \( \text{so}(d+1,1) \) symmetry acts linearly on this space, which is the great advantage of this formalism. Vectors in \( \mathbb{M}^{d+1} \) carry capital latin indices, and the \( \text{so}(d+1,1) \) invariant inner product of two such vectors \( X^A \) and \( Y^B \) is denoted by

\[
X \cdot Y = \eta_{AB} X^A Y^B, \quad X^2 = X \cdot X.
\]

AdS\(_{d+1}\), with intrinsic coordinates \( x^\mu \), may be embedded into this space as the locus \( X^2 = -R \). In the main text we shall set \( R = 1 \). We consider the embedding \(^{12}\)

\[
x^\mu \mapsto X^A(x^\mu).
\]

\(^{12}\) A concrete choice, for \( R = 1 \), that we use in this paper is

\[
X = \left( \frac{t + w^2 + 1}{2t^{1/2}}, \frac{w}{t^{1/2}}, \frac{1 - t - w^2}{2t^{1/2}} \right).
\]

This is Euclidean AdS\(_{d+1}\). Its boundary \( \mathbb{R}^d \) is the locus \( P^2 = 0 \) parametrized by

\[
P_t = \left( \frac{1 + x^2}{2}, \frac{1 - x^2}{2} \right).
\]

The parametrization is somewhat different if one chooses light-cone coordinates in embedding space. These choices are of course trivially related to each other.
Tensors in AdS\(_{d+1}\) are mapped to their embedding space representatives in \(M^{1, d+1}\) via
\[
 t_{\mu_1 \ldots \mu_s} (x) = \frac{\partial X^{A_1}}{\partial x^\mu_1} \ldots \frac{\partial X^{A_s}}{\partial x^\mu_s} T_{A_1 \ldots A_s} (X),
\]
where the tensor \(T_{A_1 \ldots A_r}\) is taken to satisfy the additional constraints [43] of

1. **homogeneity** of fixed non-zero degree \(k \in \mathbb{C}\),

   \[
   T_{A_1 \ldots A_s} (\lambda X) = \lambda^k T_{A_1 \ldots A_s} (X),
   \]

   ensured by \((X \cdot \partial X - k) T_{A_1 \ldots A_s} (X) = 0\),

   and,

2. **tangentiality** to the AdS\(_{d+1}\) loci of any given \(R\), i.e.

   \[
   X^{A_1} T_{A_1 \ldots A_s} (X) = 0.
   \]

The second condition is implemented by the projection operator
\[
P_B^A = \delta_B^A - \frac{X_A X^B}{X^2}.
\]

It is straightforward to check that for a given vector \(Y^A\), \(X \cdot P (Y) = 0\). Further, the embedding space representative of the AdS\(_{d+1}\) covariant derivative defined with respect to the Levi-Civita connection is
\[
\nabla \mapsto D = \nabla \circ P \circ \partial \circ \partial P.
\]

An important example of these statements is the scalar field itself. Using the above statements, one may show that a scalar field \(\varphi (x)\) in AdS\(_{d+1}\) may be represented by \(\varphi (X)\) of homogeneity \(\mu - d/2\) where

\[
\partial^2 \varphi (X) = 0 \iff \left[ \nabla^2_{\text{AdS}_{d+1}} + \frac{1}{R^2} \left( \frac{d^2}{4} - \mu^2 \right) \right] \varphi (X) = 0.
\]

If we identify this with the AdS\(_{d+1}\) Klein Gordon equation \(\left[ \nabla^2_{\text{AdS}_{d+1}} - m^2 \right] \varphi (x) = 0\) and note the dictionary that \((m R)^2 = \Delta (\Delta - d)\) then

\[
\mu = \frac{d}{2} - \Delta, \Delta - \frac{d}{2} \implies k = -\Delta, -(d - \Delta).
\]

In the following, we shall take the embedding representative of \(\varphi (x)\) to have homogeneity \(k = -\Delta\) while the embedding representative of the complex conjugate \(\varphi^* (x)\) will be a field \(\varphi^i (X)\) of homogeneity \(k^i = \Delta - d\).

Finally, since we shall always work with completely symmetric tensors, we find it convenient to introduce polarization vectors \(u^i \mapsto U^A\) with which we shall contract all indices. Therefore, a tensor \(T_{A_1 \ldots A_s} (X)\) will be written as
\[
T_s (X, U) = T_{A_1 \ldots A_s} (X) U^{A_1} \ldots U^{A_s}.
\]

The subscript will be dropped when there is no danger of ambiguity. We now turn to fields in AdS\(_{d+1}\) which obey the conservation law
\[
\partial_u \cdot \nabla J_s (x, u) \approx 0.
\]

\[\text{We refer the reader to [17] for a complete account of the results mentioned here, with the reminder that in their conventions,}\]
\[\text{\(d\) is the AdS dimension, which for us is \(d + 1\).}\]
To this end, we consider fields in embedding space of the form

$$J_s(X,U) = \sum_{m=0}^{s} t_{s,m} \left[ (U \cdot \partial_X)^m \varphi^\dagger (X) \right] \left[ (U \cdot \partial_X)^{s-m} \varphi (X) \right], \quad (90)$$

where $t_{s,m}$ are arbitrary coefficients. This clearly obeys the conservation equation

$$\partial_U \cdot \partial_X J_s(X,U) \approx 0,$$

provided $\partial^2_X \varphi = 0 = \partial^2_X \varphi^\dagger$. This also obeys the covariant conservation law

$$D_{A_i} J_{A_1 \ldots A_s} \approx 0,$$

since $[17]$, as one may readily check, 14

$$(X \cdot \partial_X + U \cdot \partial_U + d) J_s(X,U) = 0. \quad (95)$$

As a result, (90) also defines a conserved current in AdS$_{d+1}$.

The $\mathbb{R}^d$ boundary, with intrinsic coordinates $\vec{x}$, is parametrized in embedding space by points $P$ obtained from the AdS$_{d+1}$ locus $X$ in the scaling limit

$$P^A = \epsilon X^A \quad \epsilon \to 0,$$

while keeping $P^+ = P^0 + P^{d+1}$ fixed. Since $X^2$ is fixed, this limit sets $P^2 = 0$. The embedding space representative of a traceless boundary tensor $f_{i_1 \ldots i_s}$, denoted by $F_{A_1 \ldots A_s}$ obeys

$$\eta^{A_1 A_2} F_{A_1 \ldots A_s} = 0,$$

along with the usual criteria of homogeneity and tangentiality as in the AdS$_{d+1}$ case. However, since $P^2 = 0$, the tangentiality criterion

$$P^{A_1} F_{A_1 \ldots A_s} = 0,$$

leaves a residual ambiguity

$$F_{A_1 \ldots A_s} \sim F_{A_1 \ldots A_s} + P_{(A_1} G_{A_2 \ldots A_s)}. \quad (99)$$

Again, as we are working with completely symmetric traceless tensors it is convenient to introduce polarization vectors $z$ and $\tilde{Z}$ in intrinsic and embedding spaces respectively. It is relatively straightforward to show that the above properties of the embedding space representative are satisfied if $Z$ obeys the conditions

$$Z^2 = 0, \quad Z \cdot P = 0. \quad (100)$$

Then a traceless spin-$s$ field $j_s(x,z)$ maps to the embedding space representative $J_s(P,Z)$. Also, we may shift the polarization vector $Z$ to $\tilde{Z} = Z + \alpha P$, which also obeys the properties

$$\tilde{Z}^2 = 0, \quad \tilde{Z} \cdot P = 0. \quad (101)$$

Hence $J_s(P,Z + \alpha P)$, is an equally good representative of $j_s(x,z)$. This fact is useful in the computations presented in the main text. Further details about this formalism for CFTs may be found in [52].

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14It is useful to note the identity

$$X \cdot \partial_X (U \cdot \partial_X)^n = (U \cdot \partial_X)^n (X \cdot \partial_X - n). \quad (93)$$

As a result,

$$X \cdot \partial_X (U \cdot \partial_X)^m \varphi^\dagger (X) (U \cdot \partial_X)^{s-m} \varphi (X)
= (k + k^\dagger - s) (U \cdot \partial_X)^n \varphi^\dagger (X) (U \cdot \partial_X)^{s-m} \varphi (X)
= -(d + s) (U \cdot \partial_X)^m \varphi^\dagger (X) (U \cdot \partial_X)^{s-m} \varphi (X), \quad (94)$$

and (95) readily follows.
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