A note on sign of a self-dual representation

Manish Mishra

Department of Mathematics, Indian Institute of Science Education and Research Pune, Pune, Maharashtra, India

ABSTRACT
D. Prasad showed that the sign of a self-dual representation of a finite or $p$-adic reductive group is often detected by a central element. We study the extension of his results to some more general situations and make some observations about the consequences of his results.

1. Introduction

Let $G$ be a group and let $(\tau, V)$ be an irreducible complex representation of $G$. If $\tau$ is self-dual, that is, it is isomorphic to its contragradient $\tau'$, then there exists a non-degenerate $G$-invariant bilinear form $B : V \times V \to \mathbb{C}$ which is unique up to scalars. It is thus either symmetric or skew symmetric. The sign or the Frobenius–Schur indicator $\text{sgn}(\tau)$ of $\tau$ is defined to be $+1$ (resp. $-1$) according as $B$ is symmetric (resp. skew-symmetric).

Let $G$ be a connected reductive group over a field $F$ where $F$ is either finite or non-archimedean local. In [6, 7], D. Prasad showed that the sign of a self-dual representation $\pi$ of $G(F)$ is often detected by a certain order element $\epsilon$ of the center $Z(F)$ of $G(F)$. This element $\epsilon$ can be described in terms of half the sum of positive roots (see §3.2, see also [1, §5] for real groups). When $F$ is finite, $Z$ is connected or finite with odd cardinality and $\pi$ is a generic (i.e., a representation admitting a Whittaker model) self-dual representation of $G(F)$, he showed [6] that $\text{sgn}(\pi)$ is given by the central character $\omega_n$ evaluated at $\epsilon$. For an arbitrary group, it can be shown that $\epsilon$ fails to detect the sign in general.

In this note, we make some observations based on the results of Prasad. For $F$ finite, in Theorem 1′ we make a small improvement to the Theorem of Prasad (Theorem 1), namely that it works with a weaker hypothesis on $Z$. More specifically, if the Frobenius co-invariants of the component group of $Z$ is of odd cardinality, then $\epsilon$ detects the sign of an irreducible generic self-dual representation. In Section §3.2, Theorem 1′ is further refined to cover more groups. For an arbitrary finite reductive group $G$, we show in Theorem 6 that the method of Prasad of detecting $\text{sgn}(\pi)$ can be made to work by embedding $G$ into a reductive group $G'$ with connected center and having the same derived group. This is illustrated more concretely in Corollary 7 in the case of an irreducible generic, self-dual Deligne–Lusztig character $\pm R^{\mathfrak{g}}(\theta)$.
Corollaries 2 and 9 observe another consequence of Prasad’s theorems, namely that for split reductive - finite or p-adic groups - with connected center, the sign of an irreducible generic self-dual representation is always 1.

2. Notations

In §3, \( \mathbb{F}_q \) denotes a finite field of order \( q \) with absolute Galois group \( \Gamma \) and \( \mathbb{F} \) denotes an algebraic closure of \( \mathbb{F}_q \). For an algebraic group \( \mathcal{G} \) defined over a field \( \mathbb{F} \), we denote its identity component by \( \mathcal{G}^o \) and its derived group by \( \mathcal{G}_{\text{der}} \). If \( \tau \) is a representation of \( \mathcal{G}(F) \), we denote its dual (or contragradient) by \( \tau' \). If \( \tau \) is self-dual, we denote its Frobenius-Schur indicator by \( \text{sgn}(\tau) \).

The central character of \( \tau \) will be denoted by \( \chi_{\tau} \).

3. Finite reductive group

3.1. Groups with connected center

Let \( \mathcal{G} \) be a connected reductive group defined over \( \mathbb{F}_q \) and let \( \mathbb{Z} \) denote the center of \( \mathcal{G} \).

Let \( \mathcal{B} = TU \) be an \( \mathbb{F}_q \)-Borel subgroup of \( \mathcal{G} \), where \( U \) is the unipotent radical of \( \mathcal{B} \) and \( T \) is an \( \mathbb{F}_q \)-maximal torus of \( \mathcal{G} \) contained in \( \mathcal{B} \).

Denote by \( X(T) \) (resp. \( X(T)/\mathbb{Z} \)) the character (resp. co-character) lattice of \( T \) and by \( \Phi \), the set of roots of \( T \) in \( \mathcal{G} \). Let \( \Delta \) denote the set of simple roots in \( \mathcal{G} \).

Recall that an irreducible representation \( \pi \) of \( \mathcal{G}(\mathbb{F}_q) \) is called generic if

\[
\text{Hom}_{U(\mathbb{F}_q)}(\pi, \psi) \neq 0
\]

for some non-degenerate character \( \psi : U(\mathbb{F}_q) \rightarrow \mathbb{C}^\times \). Here non-degenerate means that the stabilizer of \( \psi \) in \( (T/\mathbb{Z})(\mathbb{F}_q) \) is trivial.

We now state a Theorem of D. Prasad on signs [6, Thm. 3].

**Theorem 1** (Prasad). If \( \mathbb{Z} \) is connected or of odd order, then there exists an element \( s_0 \in T(\mathbb{F}_q) \) such that it operates by \(-1\) on all the simple root spaces of \( U \). Further, \( s_0^2 \) belongs to \( \mathbb{Z}(\mathbb{F}_q) \) and \( \text{sgn}(\pi) = \omega_{\tau}(s_0^2) \) for any irreducible, generic, self-dual representation \( \pi \) of \( \mathcal{G}(\mathbb{F}_q) \).

**Proof.** Let \( T_{\text{ad}} \) denote the adjoint torus of \( T \). From the short exact sequence

\[
1 \rightarrow \mathbb{Z} \rightarrow T \rightarrow T_{\text{ad}} \rightarrow 1,
\]

we get the long exact sequence

\[
1 \rightarrow \mathbb{Z}(\mathbb{F}_q) \rightarrow T(\mathbb{F}_q) \rightarrow T_{\text{ad}}(\mathbb{F}_q) \rightarrow H^1(\Gamma, \mathbb{Z}) \rightarrow \cdots .
\]

There is an element \( t_- \in T_{\text{ad}}(\mathbb{F}_q) \) which acts by \(-1\) on all simple root spaces of \( U \). We claim that \( t_- \) admits a pull back in \( T(\mathbb{F}_q) \). To show that, it suffices to show that \( H^1(\Gamma, \mathbb{Z}) \) is of odd cardinality. Now using the exact sequence

\[
1 \rightarrow \mathbb{Z}^o \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}/\mathbb{Z}^o \rightarrow 1,
\]

and then taking the long exact sequence, we get that

\[
H^1(\Gamma, \mathbb{Z}^o) \rightarrow H^1(\Gamma, \mathbb{Z}) \rightarrow H^1(\Gamma, \mathbb{Z}/\mathbb{Z}^o)
\]
is exact. By Lang’s Theorem, $H^1(\Gamma; Z^\circ) = 0$. Since $H^1(\Gamma; Z/Z^\circ) \cong (Z/Z^\circ)_\rho$, the claim follows from our hypothesis. Now the result, as in the proof of Theorem 1, follows as a corollary of [6, Lemma 1].

We record an easy corollary of Theorem 1.

**Corollary 2.** If $G$ is split, $Z$ is connected and $\pi$ is an irreducible, generic, self-dual representation, then $\text{sgn}(\pi) = 1$.

**Proof.** Since $Z$ is connected and $T$ is split, there is an $\mathbb{F}_q$-subtorus $Z^c$ of $T$ such that $T = Z \times Z^c$. The character $\omega_\pi$ of $Z(\mathbb{F}_q)$ then extends trivially to a character $\theta$ of $T(\mathbb{F}_q)$. Note that since $\pi$ is self-dual, $\omega_\pi^2 = 1$. Let $s_0 = (z, zc) \in T(\mathbb{F}_q)$ be as in Theorem 1. Since $s_0^2 \in Z(\mathbb{F}_q)$, we have $\omega_\pi(s_0^2) = \theta(s_0)^2 = \omega_\pi(z)^2 = 1$.

### 3.2. Further refinements of Theorem 1’

In this subsection, we discuss cases where the sign of an irreducible, generic, self-dual representation can be detected by a central element even though $(Z/Z^\circ)_\rho$ is not necessarily of odd order. All of these observations are due to the anonymous referee.

Let $\lambda$ denote the sum of positive co-roots in $\Phi^+$ determined by $B$. Choose $\zeta \in F$ such that $\zeta^2 = -1$ and put $s = \lambda(\zeta) \in T(F)$. In other words, $s$ is the element $\chi \in X(T) \mapsto \zeta^{(\chi, \lambda)} \in F$.

Then for all $x \in \Delta$, $x(s) = \zeta^{(x, \lambda)} = \zeta^2 = -1$. Thus $s^2 \in \ker(x)$ for all $x \in \Delta$. Consequently $\epsilon := s^2 \in Z(\mathbb{F}_q)$ which is of order at most 2. Now note that $\sigma(\lambda(\zeta)) = \lambda(\zeta^q)$. So $s^{-1}\sigma(s) = \lambda(\zeta^{q-1}) \in Z(\mathbb{F}_q)$ since $\zeta^{q-1} = \pm 1$. Also note that if $\frac{1}{2} \lambda \in X^\vee(T)$, then $\zeta^{(\chi, \lambda)} = -1^{(\chi, \frac{1}{2} \lambda)} \in \mathbb{F}_q$ and thus $s = \sigma(s)$.

We can thus further refine Theorem 1’.

**Theorem 3.** For any irreducible, generic, self-dual representation $\pi$ of $G(\mathbb{F}_q)$, $\text{sgn}(\pi) = \omega_\pi(\epsilon)$ provided $s \in T(\mathbb{F}_q)$ and this happens if any of the following conditions hold:

(a) $(Z/Z^\circ)_\rho$ is of odd order.
(b) $q$ is even.
(c) $(c) q \equiv 1 \pmod{4}$.
(d) $(d) \frac{1}{2} \lambda \in X^\vee(T)$.

**Example 4.** Condition (d) holds when

- $G = \text{SO}(n, \mathbb{F}_q)$ is the special orthogonal group.
- $G = \text{Spin}(n, \mathbb{F}_q)$ is a spin group and $n \equiv 0, \pm 1 \pmod{8}$.

Put $t = s^{-1}\sigma(s)$ and $\bar{t}$ the image of $t$ in $(Z/Z^\circ)_\rho$. If $\bar{t}$ is trivial, then by Lang-Steinberg’s Theorem, $t = z^{-1}\sigma(z)$ for some $z \in Z(F)$. Put $r = sz^{-1}$. Then $r = \sigma(r)$, so $r \in T(\mathbb{F}_q)$ and satisfies $\sigma(r) = -1$ for all $r \in \Delta$. We conclude that if $\bar{t}$ is trivial, then $r^2 \in Z(\mathbb{F}_q)$ detects the sign of an irreducible, generic self-dual representation of $G(\mathbb{F}_q)$.

Note that the image of $t$ in $Z/Z^\circ$ has order at most 2. If $Z/Z^\circ$ is cyclic of even order, then it has a unique element of order 2. Therefore in this case, $t$ is trivial if and only if $(\sigma - 1)(Z/Z^\circ)$ has even order. Using this criterion, we obtain:
Lemma 5. When $G$ is simple and simply connected, the sign of an irreducible, generic self-dual representation of $G(F_q)$ is detected by $\varepsilon$ if any of the conditions (i) to (vii) in [8, Theorem 1.7] are satisfied.

3.3. Groups with disconnected center

Let $G$ be as before but without any connectedness assumption on $Z$. Let $i : Z \to Z'$ be an embedding of $Z$ into a torus $Z'$ defined over $F_q$. Let $G'$ denote the pushout of $i$ and the natural inclusion $Z \to G$. Explicitly, let $G'$ be the quotient of $G \times Z'$ by the closed normal subgroup \{(z, z^{-1}) \mid z \in Z\}. Then the inclusions $G \to G'$ and $Z' \to G'$ induced by $g \in G \to (g, 1) \in G \times Z'$ and $z \in Z' \mapsto (1, z) \in G \times Z'$ respectively are $\sigma$-equivariant monomorphisms. The inclusion $G, G' \to G'$ is a regular embedding in the sense of [3, Def. 1.7.1] by [3, Lem. 1.7.3]. We identify $Z'$ with its image in $G'$. Then $G'$ is a reductive group over $F_q$ with the following properties (see [3, Remark 1.7.6, Lemma 1.7.7] or [5, §1]):

(a) The center of $G'$ is $Z'$ and $G'_\text{der} = G\text{der}$. Also, $Z = Z' \cap G$ and $Z(F_q) = Z'(F_q) \cap G(F_q)$.

(b) We have a canonical exact sequence:

$$1 \to G(F_q) \cdot Z'(F_q) \to G'(F_q) \to (Z/Z')_\sigma \to 1.$$  

(c) Let $S$ be a maximal $F_q$-torus in $G$. Then $S' = S$. $Z'$ is a maximal $F_q$-torus in $G'$ and every maximal $F_q$-torus of $G'$ is of this form.

Now let $(\pi, V)$ be an irreducible self-dual representation of $G(F_q)$. By (a) and (b) above the quotient $G'(F_q)/G(F_q)$ is finite abelian. Therefore, there exists an irreducible representation $\pi'$ of $G'(F_q)$ whose restriction to $G(F_q)$ contains $\pi$ as a constituent. The representation $\pi'$ is unique up to a linear character of $G'(F_q)$ which is trivial on $G(F_q)$. Since $\pi$ is self-dual, there is an isomorphism

$$f : \pi' \cong \pi'^* \nu^{-1}$$  

for some $\nu \in \text{Hom}(G'(F_q), \mathbb{C}^\times)$ trivial on $G(F_q)$. This determines a non-degenerate form

$$B_f : V \times V \to \mathbb{C},$$

on the space $V$ realizing $\pi'$ satisfying

$$B_f(\pi'(g)u, \pi'(g)v) = \nu^{-1}(g)B_f(u, v)$$

for all $u, v \in V$ and $g \in G'(F_q)$. The form $B_f$ is either symmetric or skew-symmetric and we write $\text{sgn}(\pi') \in \{\pm 1\}$ to be the sign of this form. The restriction of $\pi'$ to $G(F_q)$ is multiplicity free [3, Theorem 1.7.15] and from this it easily follows [2, Lemma 4.15] that $\text{sgn}(\pi) = \text{sgn}(\pi')$. If $\pi$ is generic, it immediately follows that $\pi'$ is also generic [2, Lemma 4.10].

Now let $T' = T \times Z'$. Then by (c) above, $T'$ is a maximal $F_q$-torus contained in $G'$. Since $Z'$ is connected, there exists $t'_0 \in T'(F_q)$ such that it operates by $-1$ on each of the simple root spaces of $U$. Then the argument in the proof of [6, Lemma 1] shows that

$$\text{sgn}(\pi') = \omega_{\pi'}(t'_0) \nu(t'_0).$$

We have proved,

Theorem 6. Let $\pi$ be an irreducible, generic, self-dual representation of $G(F_q)$. Then $\text{sgn}(\pi) = \omega_{\pi'}(t'_0) \nu(t'_0)$.

Now assume that $\pi$ corresponds to an irreducible generic self-dual Deligne–Lusztig character $\pm R_{\pi}^G(\theta)$. Let $\pi'$ to be an extension of $\pi$ corresponding to the irreducible Deligne–Lusztig character
\[ \pm R^G_{S'}(\theta'), \text{ where } S' = S. \] Z' and \( \theta' \) is an extension of \( \theta \) to \( S'(\mathbb{F}_q) \). Let \( W(G, S) \) (resp. \( W(G', S') \)) denote the Weyl group of \( G \) (resp \( G' \)). Note that there is a \( \sigma \)-equivariant natural isomorphism \( W(G, S) \cong W(G', S') \) \textcolor{red}{[3, Lemma 1.7.7].} Since \( \pi \) is self-dual, there is an element \( w \in W(G, S)(\mathbb{F}_q) \) which conjugates \( \theta \) to \( \theta^{-1} \). Write

\[
\mu = \theta' - 1 \cdot (w \theta)^{-1}.
\]

Then \( \mu \) is trivial on \( S(\mathbb{F}_q) \). From \textcolor{red}{[3, Lemma 1.7.7, 1.7.8]} it follows that \( G'(\mathbb{F}_q)/G(\mathbb{F}_q) \cong S'(\mathbb{F}_q)/S(\mathbb{F}_q) \). Therefore \( \mu \) extends to a linear character \( \nu \) of \( G'(\mathbb{F}_q) \) which is trivial on \( G(\mathbb{F}_q) \).

Corollary 7. \( \text{sgn}(\pi) = \theta'(s_0/2)\nu(s'_0) \).

**4. Remarks on sign for \( p \)-adic groups**

Let \( G \) be a quasi-split connected reductive group defined over a non-archimedean local field \( F \) and let \( Z \) denote the center of \( G \). Let \( B \) denote an \( F \)-Borel subgroup of \( G \) with Levi factor \( T \) and unipotent radical \( U \).

**Lemma 8.** Assume \( Z \) is an induced torus. Then there exists an element \( t_0 \) of \( T(F) \) such that it acts by \( -1 \) on all simple root spaces of \( U \) and satisfying \( t_0^2 \in Z(F) \). Consequently for any irreducible generic self-dual representation \( \pi \) of \( G(F) \), \( \text{sgn}(\pi) = \omega_{\pi}(t_0^2) \).

**Proof.** We have an exact sequence \( 1 \to Z \to T \to T_{\text{ad}} \to 1 \), where \( T_{\text{ad}} \) denotes the adjoint torus. Since \( Z \) is induced, by Hilbert’s Theorem 90 and Shapiro’s Lemma, \( H^1(F, Z) = 1 \). Therefore we get an exact sequence

\[ 1 \to Z(F) \to T(F) \to T_{\text{ad}}(F) \to 1. \]

Now let \( t_{-1} \) denote the element of \( T_{\text{ad}}(F) \) which acts by \( -1 \) on all simple root spaces of \( U \) and choose \( t_0 \) to be a pullback of \( t_{-1} \) in \( T(F) \). The result then follows from \textcolor{red}{[7, Prop. 2]}. \( \square \)

**Corollary 9.** Assume \( G \) is split and \( Z \) is connected. Let \( \pi \) be an irreducible self-dual generic representation of \( G(F) \). Then \( \text{sgn}(\pi) = 1 \).

**Proof.** By **Lemma 8**, there is an element \( t_0 \in T(F) \) such that \( \text{sgn}(\pi) = \omega_{\pi}(t_0^2) \). Now the rest of the argument is the same as in the proof of **Corollary 2**. \( \square \)

**Remark 10.** Let \( \rho^\vee \) denote half the sum of positive roots determined by \( B \) and let \( \epsilon = 2\rho^\vee(-1) \). Then \( \epsilon \in Z(F) \). If \( Z^\vee \) is anisotropic and \( \pi \) is an irreducible discrete series representation of \( G(F) \), then Conjecture 8.3 in \textcolor{red}{[4]} asserts that the sign \( \mu(\pi) \) associated to the Deligne–Langlands root number of \( \pi \) is \( \omega_{\pi}(\epsilon) \). Consequently if \( \pi \) is also self-dual and generic, then \( \text{sgn}(\pi) \) often matches \( \mu(\pi) \).

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ORCID

Manish Mishra http://orcid.org/0000-0002-1471-0682

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