DEPTHS AND CORES IN THE LIGHT OF DS-FUNCTORS

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Abstract. The Dulfo-Serganova functors DS are tensor functors relating representations of different Lie superalgebras. In this paper we study the behaviour of various invariants, such as the defect, the dual Coxeter number, the atypicality and the cores, under the DS-functor. We introduce a notion of depth playing the role of defect for algebras and atypicality for modules. We mainly concentrate on examples of symmetrizable Kac-Moody and $Q$-type superalgebras.

0. Introduction

0.1. Let $g$ be a Lie superalgebra over the field $\mathbb{C}$. We set

$$X(g) := \{x \in g_1 | [x, x] = 0\}.$$ 

For $x \in X(g)$ we consider the functor $DS_x$ introduced by Dulfo and Serganova in [DS]:

$$M \mapsto DS_x(M) := M^x / xM.$$ 

This is a functor from the category of $g$-modules to the category of $g^x$-modules, where $g^x := g^x / [x, g]$.

Let $g$ be an indecomposable finite-dimensional Kac-Moody superalgebra with a fixed non-degenerate invariant bilinear form. Consider the following notions:

(i) the defect of $g$;
(ii) the dual Coxeter number;
(iii) the atypicality of a central character;
(iv) for a “non-exceptional” $g$: the type of the root system ($A$, $B$, $C$ or $D$) and the “core” of a central character (see 4.11, 4.12).

By [DS], $g^x$ is a finite-dimensional Kac-Moody superalgebra. The functor $DS_x$ reduces the defect of $g$ and the atypicality of each central character by the same non-negative integer; $DS_x$ preserves the dual Coxeter number, the type of the root system and the core of a central character. In this paper we establish the similar results for the queer superalgebra $q_n$.

Let $g$ be an indecomposable affine Kac-Moody superalgebra with a fixed non-degenerate invariant bilinear form. For some “bad” values of $x$, $DS_x(g)$ is not a Kac-Moody superalgebra. We show that, for “nice” values of $x$, $DS_x$ preserves the dual Coxeter number and the type of the root system, reducing the defect of $g$ by a non-negative integer which
we denote by rank $x$. In the absence of central characters, we define the atypicality and the core for each block in the BGG-category $\mathcal{O}(\mathfrak{g})$. We show that, for certain values of $x$, $DS_x$ reduces the atypicality by rank $x$ and preserves the cores.

Let $\mathfrak{g}$ be an arbitrary Lie superalgebra. We suggest a notion of depth, which plays the role of defect for algebras and atypicality for modules; this notion is defined in terms of $DS$-functors. For a finite-dimensional Kac-Moody superalgebra $\mathfrak{g}$ the formulae

\begin{align}
(i) \quad & \text{depth}(\mathfrak{g}) = \text{defect } \mathfrak{g} \\
(ii) \quad & \text{depth}(\mathcal{B}) = \text{atyp } \mathcal{B} \quad \text{for a block } \mathcal{B} \text{ in } \mathcal{O}(\mathfrak{g})
\end{align}

can be easily obtained from the results of [DS]. We will check (i) for $\mathfrak{p}_n, \mathfrak{q}_n$ and the symmetrizable affine Kac-Moody superalgebras. We will prove (ii) for $\mathfrak{q}_n$ and show that depth $\mathcal{B} = n$ for each block in $\mathcal{F}\text{in}(\mathfrak{p}_n)$.

0.2. Content of the paper. In Section [1] we recall some properties of $DS$-functor and a construction of the map $\theta_x : Z(\mathfrak{g}) \rightarrow Z(\mathfrak{g}_x)$.

In Section [2] we introduce the notion of iso-set, which generalizes the notion of isotropic sets introduced in [KW1]. We denote by defect $\mathfrak{g}$ the maximal cardinality of an iso-set for $\mathfrak{g}$ (for the finite-dimensional Kac-Moody superalgebras this agrees with the standard definition). We introduce the set $X_{iso}(\mathfrak{g})$; for the finite-dimensional Kac-Moody superalgebras $X_{iso}(\mathfrak{g}) = X(\mathfrak{g})$.

In Section [3] we introduce the notion of depth for a superalgebra and its modules. This notion is defined recursively using the functors $DS_x$ for $x \in X_{iso}$. We show that depth $\mathfrak{g} \geq \text{defect } \mathfrak{g}$. We briefly consider several examples: the "relatives" of $\mathfrak{gl}(m|n)$ and of $\mathfrak{p}_n$. In [3,5] we prove that depth $\mathcal{B} = n$ for each block in $\mathcal{F}\text{in}(\mathfrak{p}_n)$ and give an example when $DS_1(\mathfrak{p}_1(L)) \not\cong DS_2(L)$.

In Section [4] we consider the case when $\mathfrak{g}$ is a symmetrizable Kac-Moody superalgebra or one of $Q$-type superalgebras. The classical results in [BGG], [J] and [KK] determine the list of simple modules for each block in the BGG-category $\mathcal{O}(\mathfrak{g})$ (for the $Q$-type this description is up to parity shift). In Theorem [4,8] we show that, up to the parity shift, the non-critical modules $L(\lambda - \rho)$ and $L(\nu - \rho)$ lie in the same block if and only if $\nu \in W(\lambda)(\lambda + S_\lambda)$, where $S_\lambda$ is the maximal iso-set orthogonal to $\lambda$ and $W(\lambda)$ is a certain subgroup of the Weyl group $W$ (for some cases this description was established in [ChW] and in [CCL]). In Corollary [4,9,2] we give a similar formula for the critical blocks of several affine superalgebras. Using these results we introduce the following block invariants: the atypicality and the Core (as in the finite-dimensional case, Core is defined when $\mathfrak{g}$ is not exceptional).

In Section [5] we obtain several results for $DS$-functors in the case when $\mathfrak{g}$ is a $Q$-type superalgebra ($\mathfrak{q}_n$ or its relatives). One has

$$\text{depth } \mathfrak{q}_n = \left[\frac{n}{2}\right], \quad DS_x(\mathfrak{q}_n) \cong \mathfrak{q}_{n-2 \text{rank } x}.$$
We show that the map $\theta_x$ is surjective and that the dual map $\theta^*_x$ preserves the core of a central character and increases the atypicality of a central character by rank $x$. In particular, $\text{DS}_x$ commutes with translation functors.

In Section 6 we recall some results of [DS] for the finite-dimensional Kac-Moody superalgebras and prove Theorem 6.4 describing the image of $\theta_x$ in this case.

In Section 7 we describe $g_x$ for the case when $g$ is a symmetrizable Kac-Moody superalgebra and $x \in X_{\text{iso}}(g)$. As in the finite-dimensional case, $g_x$ has the same type of the root system as $g$ (for instance, $\text{DS}_x(A(2m|2n)^{(4)}) = A(2m - 2r|2n - 2r)^{(4)}$). We check (i) and explain why $\text{DS}_x$ preserves the dual Coxeter number.

In Sections 8 and 9 we consider a symmetrizable Kac-Moody superalgebra $g$ and the case when $x \in X(g)_{\text{iso}}$ has a special form. The main result of this section is Theorem 9.1 stating that $\text{DS}_x$ reduces the atypicality by rank $x$ and "preserves the core". In addition, we obtain the formula (33) which can be useful for studying the DS-functor on the category $\mathcal{O}(g)$ (note that it is not clear whether $\text{DS}_x(\mathcal{O}(g)) \subset \mathcal{O}(g_x)$).

0.3. Index of frequently used notation. Throughout the paper the ground field is $\mathbb{C}$; $\mathbb{N}$ stands for the set of non-negative integers. We always assume that the dimension of $g$ is at most countable. We will frequently used the following notation.

\[
\begin{align*}
\text{Mod, } & \mathcal{Z}(g), \mathcal{C}(\chi), \text{mspec}_C, \theta_x, \theta^*_x, \\
\text{supp, } & \text{iso-set, defect, } X_{\text{iso}}, \\
\Omega(N), & \text{Irr}(\mathcal{F}), \mathcal{F}\text{in}(g), \\
\text{depth, } & \text{rank, } X(g)_{\text{iso}}, \hat{X}(N), \\
W, & (-|-), \Delta_{\text{red}}, \Delta_{\text{nis}}, \Delta_{\text{iso}}, \alpha^\vee, \hat{h}, \delta, \Delta(\lambda), W(\lambda), \\
M(\lambda), & L(\lambda), \mathcal{O}_{\text{inf}}, \mathcal{O}, \mathcal{O}_{KK}, \sim, \\
\text{atyp, } & \\
\text{Core}(\lambda), & \\
\chi_{\lambda}, \text{Core}(\chi), \\
\Omega_{h}, & \mathcal{O}_{h}^{\text{inf}}, \\
\Omega_{h}, & \mathcal{O}_{h}^{\text{inf}}. \\
\end{align*}
\]

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1. DS-functor

The DS-functor was introduced in [DS]. We recall definitions and some results of [DS] below. In 3.2 we list some properties of depth. We retain notation of Section 0.

1.1. Construction. We set $X(g) := \{x \in g_1 \mid [x, x] = 0\}$. For a $g$-module $M$ and $g \in g$ we set $M^g := \text{Ker}_M g$. For $x \in X(g)$ we introduce

\[\text{DS}_x(M) := M^g / xM.\]
Notice that \( g^x \) and \( g_x := DS_x(g) = g^x/[x,g] \) are Lie superalgebras. Since \( M^x, xM \) are \( g^x \)-invariant and \([x,g]M^x \subset xM, DS_x(M) \) is a \( g^x \)-module and \( g_x \)-module. Thus
\[
DS_x : M \mapsto DS_x(M)
\]
is a functor from the category of \( g \)-modules to the category of \( DS_x(g) \)-modules. One has
\[
DS_x(\Pi(N)) = \Pi(\Pi(DS_x(N)))
\]
and
\[
DS_x(M) \otimes DS_x(N) = DS_x(M \otimes N), \quad \text{sdim}(N) = \text{sdim}(DS_x(N)).
\]

The following result is called Hinich’s Lemma: each exact sequence of \( g \)-modules
\[
0 \to M_1 \to N \to M_2 \to 0
\]
duces a long exact sequence of \( g_x \)-modules
\[
0 \to Y \to DS_x(M_1) \to DS_x(N) \to DS_x(M_2) \to \Pi(Y) \to 0,
\]
where \( Y \) is some \( g_x \)-module. Lemma 2.1 in [HW] gives a similar result.

### 1.2. Action on the centre.
Denote by \( \mathcal{Mod} \) (resp., by \( \mathcal{Mod}_x \)) the category of \( g \) (resp., \( g_x \))-modules. We denote by \( U(g) \) the universal enveloping algebra of \( g \) and by \( Z(g) \) its centre. We denote by \( \text{mspec} Z(g) \) the set of central characters \( \text{Hom}(Z(g), \mathbb{C}) \). For each central character \( \chi \) we denote by \( C(\chi) \) the full subcategory of \( C \) consisting of the modules annihilated by \( \text{Ker} \chi \) and set
\[
\text{mspec}_C Z(g) := \{ \chi \in \text{mspec} Z(g) | C(\chi) \neq 0 \}.
\]

#### 1.2.1. Consider the following algebra homomorphisms
\[
Z(g) \hookrightarrow U(g)^{\text{ad}x} \to DS_x(U(g)).
\]
By [DS], \( DS_x(U(g)) = U(g_x) \). Since \( Z(g) = U(g)^{\text{ad}g} \) we obtain the algebra homomorphism
\[
\theta_x : Z(g) \to Z(g_x)
\]
and the dual map \( \text{Hom}(Z(g_x), \mathbb{C}) \to \text{Hom}(Z(g), \mathbb{C}) \); the map \( \theta_x^* \) is the restriction of the latter to \( \text{mspec}_{\mathcal{Mod}_x} Z(g_x) \). One has \( DS_x(\mathcal{Mod}(\chi)) = 0 \) if \( \chi \notin \text{Im} \theta_x^* \). Note that a surjectivity of \( \theta_x^* \) implies the injectivity of \( \theta_x^* \).

#### 1.2.2. Proposition
Take \( N \in \mathcal{Mod}(\chi) \).

(i) If \( \theta_x \) is surjective, then \( DS_x(N) \) lies in \( \mathcal{Mod}_x((\theta_x^*)^{-1}(\chi)) \).

(ii) For each simple subquotient \( L' \) of \( DS_x(N) \) there exists \( \chi' \in (\theta_x^*)^{-1}(\chi) \) such that \( L' \in \mathcal{Mod}_x(\chi') \). In particular, \( DS_x(N) = 0 \) if \( \chi \notin \text{Im} \theta_x^* \).

**Proof.** Set \( m := \text{Ker} \chi = \text{Ann}_{Z(g)} N \). One has \( \theta_x(m) \subset \text{Ann}_{Z(g_x)} DS_x(N) \).

If \( \theta_x \) is surjective, then \( \theta_x(m) \) is a maximal ideal in \( Z(g_x) \); this gives (i).

For (ii) recall that the dimension of \( g_x \) is at most countable. By Dixmier generalization of Schur’s Lemma (see [DS]), each simple \( g_x \)-module admits a central character, so
\[(\text{Ker } \chi') L' = 0 \text{ for some } \chi' \in \text{mspec}_{\text{Mod}} Z(g_x). \] By above, \(\theta_x(m) \subset \text{Ker } \chi'\) which implies \(\theta_x(\chi') = \chi\) as required. \[\square\]

1.2.3. Remark. It is not clear when \(\text{mspec}_{\text{Mod}} Z(g) = \text{mspec } Z(g)\), see Conjecture 13.5.1 in [M2]. In [M1] I. M. Musson proved the following generalization of Duflo’s Theorem: for a basic classical Lie superalgebra \(g\) any primitive ideal in \(\mathcal{U}(g)\) is the annihilator of a simple highest weight module. The proof of [M1] works for any quasi-reductive superalgebra\(^1\). As a result, for a quasi-reductive superalgebra \(g\) one has

\[\text{mspec}_{\text{Mod}} Z(g) = \text{mspec } \mathcal{O}(g) Z(g),\]

where \(\mathcal{O}(g)\) is the BGG-category for \(g\).

2. ISO-SETS

In this section we introduce the iso-sets and prove Lemmata 2.3, 2.4.1 which seem to be useful for computations. For a finite-dimensional Kac-Moody superalgebra \(g\) the iso-sets are the same as isotropic sets introduced in [KW1] (see 2.1.5).

2.1. Definitions and first properties. Let \(h\) be a commutative subalgebra of \(g_0\) which acts diagonally in the adjoint representation of \(g\). We introduce the multisets of even and odd roots in the usual way \((\Delta_0, \Delta_1 \subset h^* \setminus \{0\})\). We write each \(a \in g_i\) (for \(i = 0, 1\)) in the form

\[a = \sum_{\alpha \in \text{supp}(a)} a_\alpha, \text{ where } a_\alpha \in g_\alpha \setminus \{0\}, \text{ supp}(a) \subset \Delta_i \cup \{0\}.\]

2.1.1. Definition. We say that \(S \subset \Delta_1\) is an iso-set if the elements of \(S\) are linearly independent and for each \(\alpha, \beta \in \Delta_1 \cap (S \cup (-S))\) one has \(\alpha + \beta \not\in \Delta_0\). We denote by \(\text{defect } g\) the maximal cardinality of an iso-set for all possible choices of \(h\).

2.1.2. Lemma. Let \(x \in g\) be such that \(S := \text{supp}(x)\) is an iso-set. Then \(x \in X(g)\). Moreover, for each \(y \in g_1\) with \(\text{supp}(y) \subset (-S)\) one has \([x, y] \in h\) and \(x, y, [x, y]\) span a subalgebra which is a quotient of \(\mathfrak{sl}(1|1)\).

Proof. Set \(S + S = \{\alpha + \beta | \alpha, \beta \in S\}\). One has

\[\text{supp}([x, x]) \subset (S + S) \cap (\Delta_0 \cup \{0\}).\]

By definition, \((S + S) \cap \Delta_0 = \emptyset\). Since the elements of \(S\) are linearly independent, \(0 \not\in S + S\). Hence \(\text{supp}([x, x]) = \emptyset\), so \([x, x] = 0\). Since \(\text{supp}(x) \neq 0\), the algebra \(h\) contains \(h\) satisfying \([h, x] \neq 0\). Hence \(x \in X\).

\(^1\)a finite-dimensional Lie superalgebra is called quasi-reductive if \(g_0\) is reductive and \(g_1\) is a semisimple \(g_0\)-module, see [SI].
2.1.3. **Definition.** We denote by $X_{\text{iso}}(g)$ the set of all $x \in g$ with the following property: there exists a subalgebra $h$ as above such that supp$(x)$ is an iso-set. By above, $X_{\text{iso}}(g) \subset X(g)$.

We say that $x, x' \in X_{\text{iso}}(g)$ are equivalent if there exists $h$ as above and an inner automorphism $\phi \in \text{Aut}(g)$ such that $S := \text{supp}(x), S' := \text{supp}(\phi(x'))$ are iso-sets and

$$-S' \cup S' = -S \cup S.$$ 

We will often use $X$ instead of $X(g)$ and $X_{\text{iso}}$ instead of $X_{\text{iso}}(g)$.

2.1.4. **Notation.** In this paper $h$ is always as in [2.1] For a semisimple $h$-module $N$ we denote by $\Omega(N)$ the set of weights of $N$. For each category $C$ we denote by $\text{Irr}(C)$ the set of simple modules in $C$. If $g$ is finite-dimensional, we denote by $\text{Fin}(g)$ the full category of finite-dimensional modules which are semisimple over $g_0$.

2.1.5. **Remark.** Consider the case when $h^0 \cap g_0 = h$ and $g$ admits an even non-degenerate symmetric invariant bilinear form. Then this form induces a non-degenerate symmetric form on $h^*$ which we denote by $(-|-)$.

Fix $\alpha \in \Delta_1$ with $2\alpha \not\in \Delta_0$ and a pair $x_{\alpha}, x_{-\alpha} \in g_\alpha, x_{-\alpha} \in g_{-\alpha}$ with $(x_{\alpha}, x_{-\alpha}) = 1$. Arguing as in [K3], Thm.2.2 we see that $x_{\alpha}, x_{-\alpha}, [x_{\alpha}, x_{-\alpha}]$ form an $\mathfrak{sl}(1|1)$-triple and $(\nu | \alpha) = \nu([x_{\alpha}, x_{-\alpha}])$ for each $\nu \in h^*$. This implies $(\alpha | \alpha) = 0$. As a result, each iso-set $S \subset \Delta_1$ is a basis of an isotropic subspace of $h^*$.

2.2. **Iso-sets and DS-functor.** Let $S = \{ \beta_i \}_{i=1}^r$ be an iso-set and $y_i \in X$ be such that supp$(y_i) = \{ \beta_i \}$. Fix $s$ with $1 \leq s < r$ and set

$$x := \sum_{i=1}^s y_i, \quad h_x := h^*/([x,g] \cap h) = h^*/(\sum_{i=1}^s [y_i,g_{-\beta_i}]).$$

We view $h_x$ as a subalgebra of $g_x$; clearly, $h_x$ is a finite-dimensional commutative subalgebra of $(g_x)_0$ which acts diagonally in the adjoint representation of $g_x$. We introduce the multisets $(\Delta_x)_0, (\Delta_x)_1 \subset h_x^* \setminus \{0\}$ of even and odd roots in the usual way.

The space $\sum_{i=s+1}^r g_{\beta_i}$ lies in $g^x$ and has zero intersection with $[x,g]$. This gives an embedding of $\sum_{i=s+1}^r g_{\beta_i} \to g_x$ for each $i = s+1, \ldots, r$ the image of $g_{\beta_i}$ lies in $(g_x)_{\beta_i'}$ for some $\beta_i' \in (\Delta_x)_1 \cup \{0\}$. 

Similar arguments give $[y, y] = 0$ and supp$([x, y]) \subset \{0\}$, that is $[x, y] \in h$. One has

$$2[x, y] = [[x, y]] = 0, \quad 2[y, x] = [[y, x]] = 0,$$

so $x, y, [x, y]$ span a quotient of $\mathfrak{sl}(1|1)$.  \hfill \Box
2.2.1. **Lemma.** The set \( \{ \beta_i \}_{i=s+1}^r \) is an iso-set in \( \Delta_x \).

**Proof.** Assume that \( \sum_{i=s+1}^r c_i \beta_i' = 0 \) for some scalars \( c_i \in \mathbb{C} \) which are not all equal to zero. Since \( \{ \beta_i \}_{i=1}^r \) are linearly independent there exists \( h \in \mathfrak{h} \) such that
\[
\sum_{i=s+1}^r c_i \beta_i(h) = 1, \quad \beta_1(h) = \beta_2(h) = \ldots = \beta_s(h) = 0.
\]
Then \( h \in \mathfrak{h}^r \); denoting by \( h \) the image of \( h \) in \( \mathfrak{h}_x \) we obtain \( \sum_{i=s+1}^r c_i \beta_i'(h) = 1 \), a contradiction. \( \square \)

2.3. **Lemma.** Let \( x = \sum_{\alpha \in \text{supp}(x)} x_\alpha \) be such that \( \text{supp}(x) \) is an iso-set. For each \( \alpha \in \text{supp}(x) \) fix any element \( h_\alpha \in [g - \alpha, x_\alpha] \).

If \( N \) is a \( g \)-module with a diagonal action of \( \mathfrak{h} \), then
\[
M := \bigcap_{\alpha \in \text{supp}(x)} N^{h_\alpha}
\]
is a \( C \)-submodule of \( N \) and the natural map \( DS_x(M) \to DS_x(N) \) is bijective.

**Proof.** The proof follows the idea of the proof of Lemma 4.2 in [S2]. Set
\[
S := \{ \beta \in \text{supp}(x) | h_\beta \neq 0 \}.
\]
If \( S \) is empty, then \( M = N \) and the assertion is tautological. Assume that \( S \neq \emptyset \) and that \( N \) is indecomposable. Let \( F \) be the minimal subfield of \( \mathbb{C} \) which contains
\[
\{ \lambda(h_\beta) \}_{\beta \in S, \lambda \in \Omega(N)}.
\]
Since \( \Delta \) is at most countable and \( N \) is indecomposable, the set \( \Omega(N) \) is at most countable, so the degree \([\mathbb{C} : F]\) is infinite. We fix a \( F \)-linearly independent set \( \{ q_\beta \}_{\beta \in S} \subset \mathbb{C} \). For each \( \beta \in S \) take \( x_\beta \in g - \beta \) such that \( h_\beta = [x_\beta, x_\beta] \). We introduce
\[
y := \sum_{\beta \in S} q_\beta x_\beta, \quad h := [x, y] = \sum_{\beta \in S} q_\beta h_\beta.
\]
By 2.1.2 \( x, y, h \) form an \( \mathfrak{sl}(1|1) \)-triple (since \( h \neq 0 \)). As an \( \mathfrak{sl}(1|1) \)-module \( N \) can be written as \( N = N^h \oplus N^\perp \), where
\[
N^h = \sum_{\nu \in \Omega(N): \mu(h)=0} N_\nu, \quad N^\perp := \sum_{\mu \in \Omega(N): \mu(h)\neq 0} N_\mu.
\]
Since \( \{ q_\beta \}_{\beta \in S} \) are linearly independent over \( F \) one has \( N^h = M \). Notice that \( N^\perp \) is a sum of typical \( \mathfrak{sl}(1|1) \)-modules, so \( DS_x(N^\perp) = 0 \). The assertion follows. \( \square \)

2.3.1. **Remark.** For \( g \) as in 2.1.3 one has \( M = \sum_{\nu \in \mathfrak{h}^r: (\nu|S)=0} N_\nu. \)
2.4. Case of disjoint isosets. Let \( x, y \in X \) be such that
\[
\text{supp}(x) \cap \text{supp}(y) = \emptyset, \quad \text{supp}(x) \cup \text{supp}(y) \text{ is an iso-set.}
\]
By [2.1.4] \( y \) has a non-zero image in \( \mathfrak{g}_x \); we denote this image by \( \overline{y} \).

2.4.1. Lemma. Let \( x, y \in X \) be as in [2.4]. Let \( N \) be a finite-dimensional \( \mathfrak{g} \)-module with a diagonal action of \( \mathfrak{h} \). If \( \dim DS_{\mathfrak{g}} DS_x(N) =: (p|q) \), then
\[
\dim DS_{x+y}(N) = (p - j|q - j)
\]
for some \( j \geq 0 \).

Proof. Take \( h \in \mathfrak{h} \) such that \( \alpha(h) = 1 \), \( \beta(h) = -1 \) for each \( \alpha \in \text{supp}(x) \) and \( \beta \in \text{supp}(y) \). Then
\[
[x, y] = 0, \quad [h, x] = x, \quad [h, y] = -y,
\]
so the algebra spanned by \( x, y, h \) is isomorphic to \( \mathfrak{pgl}(1|1) \). We identify \( y \) with \( \overline{y} \).

It is enough to check the inequality on each finite-dimensional indecomposable \( \mathfrak{pgl}(1|1) \)-module. These modules were classified in [Ger]. For each \( c \) there are one-dimensional modules \( L(c) \), \( \Pi(L(c)) \), where \( h \) acts by \( cId \) and \( x, y \) by zero. Up to the multiplication on such modules, we have the following classes of non-isomorphic indecomposable modules: a 4-dimensional projective modules \( M_4 \) satisfying \( DS_x(M_4) = DS_{x+y}(M_4) = 0 \) and the “zigzag” modules \( V_1^{\pm} \). Each zigzag module has a basis \( \{ v_i \}_{i=1}^s \) with \( p(v_{i+1}) = \overline{1} \) and \( hv_i = iv_i \); we depict each module by the diagram, where \( xv_i = v_{i+1} \) is depicted as \( v_i \rightarrow v_{i+1} \) and \( yv_i = v_{i-1} \) is depicted as \( v_i \rightarrow v_{i-1} \). We have
\[
\begin{align*}
V_1^+ & \quad v_1 \rightarrow v_2 \leftarrow v_3 \rightarrow v_4 \leftarrow \ldots \rightarrow v_{2n} \\
V_2^+ & \quad v_1 \leftarrow v_2 \rightarrow v_3 \leftarrow v_4 \rightarrow \ldots \leftarrow v_{2n} \\
V_{2n-1}^+ & \quad v_1 \leftarrow v_2 \rightarrow v_3 \leftarrow v_4 \rightarrow \ldots \leftarrow v_{2n-1} \\
V_{2n-1}^- & \quad v_1 \rightarrow v_2 \leftarrow v_3 \rightarrow v_4 \leftarrow \ldots \rightarrow v_{2n-1}
\end{align*}
\]
(The modules \( V_1^{\pm} \) are trivial). One sees that
\[
\begin{align*}
\dim DS_x(V_1^{2n-1}) &= \dim DS_y(V_2^{2n+1}) = \dim DS_{x+y}(V_2^{2n+1}) = (1|0), \\
DS_x(V_2^{2n}) &= DS_x(V_2^-) = 0, \quad DS_{x+y}(V_2^{2n}) = 0
\end{align*}
\]
and \( \dim DS_{\mathfrak{g}} DS_x(V_2^{2n}) = \dim DS_x(V_2^n) = (1|1) \) for \( n > 1 \). The assertion follows. \( \square \)

2.4.2. Remark. Note that \( DS_{\mathfrak{g}} DS_x(V_2^2) = 0 \) and for \( n > 1 \) one has
\[
\begin{align*}
\dim DS_y(V_2^-) = 0, \quad & \dim DS_x(V_2^-) = \dim DS_{\mathfrak{g}} DS_x(V_2^n) = (1|1); \\
\dim DS_x(V_2^n) = 0, \quad & \dim DS_y(V_2^n) = \dim DS_{\mathfrak{g}} DS_x(V_2^n) = (1|1).
\end{align*}
\]
In many examples \( \dim DS_{x+y}(N) = \dim DS_{\mathfrak{g}} DS_x(N) \) for any \( x, y \) as in [2.4]. By above, this property is equivalent to the fact that \( N \) does not contain \( \mathfrak{pgl}(1|1) \)-modules \( V_2^{2n} \) with \( n > 1 \) as direct summands; by [GQS], the modules with this property “form a tensor subalgebra” in \( \mathcal{F}\text{in}(\mathfrak{pgl}(1|1)) \). This property holds if \( N \) is a simple finite-dimensional
module over a Kac-Moody superalgebra $g$ (see [HW], [GH]); this does not hold for $g = p_2$: see 3.5.3 below.

3. DS-depth

In this section we introduce a notion of depth for a superalgebra and its modules. In 3.3 and 3.4 we briefly consider several examples: we find depth $g$ for the case when $g$ is a "relative" of $gl(n|n)$ or a "relative" of $p_n$.

3.1. Definitions. We define $\text{depth}(g) \in \mathbb{N} \cup \{\infty\}$ by the formula

$$\text{depth}(g) = \begin{cases} 0 & \text{if } X_{iso} = 0 \\ 1 + \max_{x \in X_{iso} \setminus \{0\}} \text{depth}(g_x) & \text{if } X_{iso} \neq 0. \end{cases}$$

For $x \in X$ we define $\text{rank } x := \text{depth}(g) - \text{depth}(g_x)$. We set

$$X(g)_r := \{ x \in X \mid \text{rank } x = r \}.$$

For a $g$-module $N$ we set

$$\hat{X}(N) := \{ x \in X_{iso} \setminus \{0\} \mid \text{DS}_x(N) \neq 0 \}$$

and introduce $\text{depth}(N) \in \mathbb{N} \cup \{\infty\}$ recursively by

$$\text{depth}(N) := \begin{cases} 0 & \text{if } \hat{X}(N) = \emptyset \\ \max_{x \in \hat{X}(N)} \left( \text{depth}(\text{DS}_x(N)) + \text{rank } x \right) & \text{if } \hat{X}(N) \neq \emptyset. \end{cases}$$

For a full subcategory of $g$-modules $C$ we define $\text{depth}(C)$ as the maximum of $\text{depth}(N)$ for $N \in C$. For $\chi \in \text{mspec } Z(g)$ we set

$$\text{depth } \chi := \text{depth } \text{Mod}(\chi).$$

3.1.1. Remark. In 3.5.3 we give an example of a simple module $L$ satisfying $\max_{x \in \hat{X}(L)} \text{rank } x \leq \text{depth}(L)$.

3.2. Properties of depth. Clearly, $\text{depth}(g' \times g'') = \text{depth}(g') + \text{depth}(g'')$.

One has $\text{depth}(N) = 0$ if and only if $\hat{X}(N) = \emptyset$. Moreover, $\text{depth}(N) \leq \text{depth}(g)$ and $\text{depth}(N) = \text{depth } g$ if $\text{sdim } N \neq 0$. Note that depth $g$ may be greater than the depth of the adjoint module (for $g = gl(1|1), osp(2|2), osp(3|2), psq_3$ we have $\text{depth}(g) = 1$ and $\text{depth}(Ad) = 0$).

3.2.1. By 3.1.1 for any $N', N'' \in \text{Mod}$ one has

$$\text{depth}(N' \oplus N'') = \max(\text{depth}(N'), \text{depth}(N'')),$$

$$\text{depth}(N' \otimes N'') = \min(\text{depth}(N'), \text{depth}(N'')).$$
3.2.2. In the light of Proposition 1.2.2 (ii) for \( \chi \in \text{mspec } \mathcal{Z}(\mathfrak{g}) \) we have
\[
\text{depth } \chi \leq \max \{ \text{rank } x \mid \chi \in \text{Im} \theta_x^* \}.
\]

3.2.3. Using Lemma 2.2.1 and the induction on \( r \) we obtain:

- if \( \text{supp}(x) \) is an iso-set of cardinality \( r \), then \( \text{rank } x \geq r \);
- \( \text{depth } \mathfrak{g} \geq \text{defect } \mathfrak{g} \) (where defect as in 2.1).

For an example when \( \text{rank } x \) is greater than the cardinality of \( \text{supp}(x) \), see 3.4.2.

3.2.4. Example. Let \( \mathfrak{g} \) be a finite-dimensional Kac-Moody superalgebra. From the results of [DS], it follows that \( X(\mathfrak{g}) = X(\mathfrak{g})_{\text{iso}} \) and \( \text{depth}(\mathfrak{g}) \) is equal to the defect of \( \mathfrak{g} \). For each block \( \mathcal{B} \) in \( \mathcal{F}\text{in}(\mathfrak{g}) \) (or in \( \mathcal{O}(\mathfrak{g}) \)) \( \text{depth}(\mathcal{B}) \) is equal to the atypicality of \( \mathcal{B} \). Moreover, by [S2], the atypicality of a simple finite-dimensional module \( L \) is equal to \( \text{depth}(L) \) and
\[
\text{depth}(L) = \text{depth } \text{DS}_x(L) + \text{rank } x.
\]

3.2.5. Let \( \mathcal{B} \) be a Serre subcategory of \( \mathcal{M}\text{od} \), i.e. the full subcategory consisting of the modules of finite length whose all simple subquotients lie in a given set \( \text{Irr}(\mathcal{B}) \). Recall that \( \text{depth}(\mathcal{B}) \) is the maximal \( \text{depth}(N) \) for \( N \in \mathcal{B} \). By Hinich’s Lemma one has
\[
\text{depth}(\mathcal{B}) = \max_{L \in \text{Irr}(\mathcal{B})} \text{depth}(L).
\]
We say that a block \( \mathcal{B} \) satisfies Serganova property if
\[
\text{depth}(\mathcal{B}) = \text{depth}(L) \quad \text{for each } L \in \text{Irr}(\mathcal{B}).
\]

By [S2], for a finite-dimensional Kac-Moody superalgebra \( \mathfrak{g} \) this property holds for each block in \( \mathcal{F}\text{in}(\mathfrak{g}) \). This property does not hold for strange superalgebras, see 3.5.3,5.1 below.

3.2.6. Remark. One might expect that in ”good cases” depth has the following properties:

- if \( \mathfrak{m} \) be an ideal in \( \mathfrak{g} \) and \( N \) is a \( \mathfrak{g}/\mathfrak{m} \)-module, then \( \text{depth}(N) = \text{depth } \text{Res}_{\mathfrak{g}/\mathfrak{m}}^\mathfrak{g}(N) \);
- \( \text{depth}(N) = \text{depth } \text{DS}_x(N) + \text{rank } x \).

3.2.7. In 3.3 we will see that ”the relatives of \( \mathfrak{gl}(m|n) \)” satisfy the following properties:

(i) \( \text{depth}(\mathfrak{g}) = \text{defect } \mathfrak{g} \);
(ii) all elements of a given rank in \( X_{\text{iso}} \) are equivalent with respect to the equivalence relation introduced in 2.1.3
(iii) if \( \text{supp}(x) \) is an iso-set, then \( \text{rank } x \) is equal to the cardinality of \( \text{supp}(x) \);
(iv) the elements \( x \in (X \setminus X_{\text{iso}}) \) have the maximal possible rank;
(v) for \( x \in X_{\text{iso}} \) the rank of \( x \) is equal to the rank of the corresponding matrix (i.e. the dimension of the image of \( x \) in the natural representation).
In §3.3.2 we will see that $p_n$ satisfy (i), (ii), (iv) and (v). Later we will show that the symmetrizable Kac-Moody superalgebras and the $Q$-type superalgebras satisfy the properties (i)—(iv).

3.3. Examples: $\mathfrak{gl}(n|n)$ and its relatives. Recall that $\mathfrak{gl}(m|n)$ consists of the block matrices

$$T_{A,B,C,D} := \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

We consider Lie superalgebras $\mathfrak{g} = \mathfrak{gl}(n|n), \mathfrak{sl}(n|n), \mathfrak{pgl}(n|n), \mathfrak{psl}(n|n)$ and identify $\mathfrak{g}_1$ for all these superalgebras.

3.3.1. Case $n = 1$. The algebra $\mathfrak{gl}(1|1)$ is spanned by even elements $h, z$ and odd elements $E, F$ with the relations

$$[h, E] = 2E, \quad [h, F] = -2F, \quad [E, F] = z, \quad [z, h] = [z, E] = [z, F] = 0$$

and $\mathfrak{pgl}(1|1) = \mathfrak{gl}(1|1)/\mathbb{C}z$. One has $\text{defect}(\mathfrak{gl}(1|1)) = \text{defect}(\mathfrak{pgl}(1|1)) = 1$ and

$$X(\mathfrak{gl}(1|1)) = X(\mathfrak{gl}(1|1))_{iso} = CE \cup CF = X(\mathfrak{pgl}(1|1)) = X(\mathfrak{pgl}(1|1))_{iso}.$$

For $x \in X(\mathfrak{gl}(1|1)) \setminus \{0\}$ we have

$$\text{DS}_x(\mathfrak{gl}(1|1)) = \mathfrak{gl}(0|0) = 0, \quad \text{DS}_x(\mathfrak{pgl}(1|1)) \simeq \mathbb{C}.$$ 

Hence depth $\mathfrak{gl}(1|1) = \text{depth}(\mathfrak{pgl}(1|1)) = 1$ and

$$X((\mathfrak{gl}(1|1))_1 = X((\mathfrak{pgl}(1|1))_1 = X(\mathfrak{gl}(1|1)) \setminus \{0\}.$$

The algebra $\mathfrak{sl}(1|1)$ is spanned by $z, E, F$. One has defect $\mathfrak{sl}(1|1) = 0$

$$X(\mathfrak{sl}(1|1)) = X(\mathfrak{gl}(1|1)) = CE \cup CF, \quad X(\mathfrak{sl}(1|1))_{iso} = 0,$$

so depth($\mathfrak{sl}(1|1)$) = 0 (in particular, all $x \in X(\mathfrak{sl}(1|1))$ have zero rank). Note that for non-zero $x$ the algebra $\text{DS}_x(\mathfrak{sl}(1|1))$ can be identified with $\mathbb{C}x$.

The algebra $\mathfrak{psl}(1|1)$ is a commutative superalgebra of dimension $(0|2)$. One has $X(\mathfrak{psl}(1|1)) = \mathfrak{psl}(1|1)$ and $X(\mathfrak{psl}(1|1))_{iso} = 0$, so depth($\mathfrak{psl}(1|1)$) = defect $\mathfrak{psl}(1|1) = 0$. Note that $\text{DS}_x(\mathfrak{psl}(1|1)) = \mathfrak{psl}(1|1)$ for all $x$.

3.3.2. Case $\mathfrak{gl}(m|n)$. One has defect $\mathfrak{gl}(m|n) = \min(m,n)$. By [DS],

$$X(\mathfrak{gl}(m|n)) = X(\mathfrak{gl}(m|n))_{iso} = \{T_{0,B,C,0} \mid BC = 0, \quad CB = 0\}$$

and depth($\mathfrak{gl}(m|n)$) = min($m,n$), $\text{DS}_x(\mathfrak{gl}(m|n)) = \mathfrak{gl}(m-r|n-r)$, where $r := \text{rank} x$. Moreover, rank $x$ is equal to the rank of the matrix of $x$ in (5). The same formulae hold for $\mathfrak{sl}(m|n)$ with $m \neq n$. 
3.3.3. *Cases* $\mathfrak{sl}(n|n)$. The iso-sets for $\mathfrak{sl}(n|n)$ coincide with the iso-sets of cardinality $r < n$ for $\mathfrak{gl}(n|n)$ (the iso-sets of cardinality $n$ becomes linearly dependent for $\mathfrak{sl}(n|n)$). This gives $\text{defect}(\mathfrak{sl}(n|n)) = n - 1$. One has

$$X(\mathfrak{sl}(n|n)) = X(\mathfrak{gl}(n|n)) = X(\mathfrak{sl}(n|n))_{iso} \coprod X(\mathfrak{gl}(n|n)),$$

and for $x \in X(\mathfrak{gl}(n|n))_r$ we have

$$\text{DS}_x(\mathfrak{sl}(n|n)) = \mathfrak{sl}(n - r|n - r) \quad \text{if } r < n, \quad \text{DS}_x(\mathfrak{sl}(n|n)) = \Pi(\mathbb{C}) \quad \text{if } r = n.$$

Thus $\text{depth}(\mathfrak{sl}(n|n)) = n - 1$ and $X(\mathfrak{sl}(n|n))_r = X(\mathfrak{gl}(n|n))_r$ for $r < n - 1$ with

$$X(\mathfrak{sl}(n|n))_{n-1} = X(\mathfrak{gl}(n|n))_{n-1} \coprod X(\mathfrak{gl}(n|n)).$$

3.3.4. *Cases* $\mathfrak{pgl}(n|n), \mathfrak{psl}(n|n)$. One has

$$X(\mathfrak{pgl}(n|n)) = X(\mathfrak{psl}(n|n)) = X(\mathfrak{gl}(n|n)) \coprod X', \quad \text{where } X' := \{T_{0,B,C,0} \mid BC \in \mathbb{C}^*Id\}.$$

For $x \in X(\mathfrak{gl}(n|n))_r$ with $r < n$ one has

$$\text{DS}_x(\mathfrak{pgl}(n|n)) = \mathfrak{pgl}(n - r|n - r), \quad \text{DS}_x(\mathfrak{psl}(n|n)) = \mathfrak{psl}(n - r|n - r) \quad \text{and} \quad \text{DS}_x(\mathfrak{pgl}(n|n)) = \Pi(\mathbb{C}), \quad \text{DS}_x(\mathfrak{psl}(n|n)) = \mathfrak{psl}(1|1) \quad \text{for } x \in X(\mathfrak{gl}(n|n))_n \cup X'.$$

We have $X(\mathfrak{pgl}(n|n))_{iso} = X(\mathfrak{gl}(n|n))$ and

$$\text{depth}(\mathfrak{pgl}(n|n)) = \text{defect} \mathfrak{pgl}(n|n) = n.$$

This gives $X(\mathfrak{pgl}(n|n))_r = X(\mathfrak{gl}(n|n))_r$ for $r < n$,

$$X(\mathfrak{pgl}(n|n))_n = X(\mathfrak{gl}(n|n))_n \coprod X', \quad X' = X(\mathfrak{pgl}(n|n)) \setminus X(\mathfrak{pgl}(n|n))_{iso}.$$

We have $X(\mathfrak{psl}(n|n))_{iso} = X(\mathfrak{sl}(n|n))_{iso}$ and

$$\text{depth}(\mathfrak{psl}(n|n)) = \text{defect} \mathfrak{psl}(n|n) = n - 1.$$

This gives $X(\mathfrak{psl}(n|n))_r = X(\mathfrak{sl}(n|n))_r$ for $r < n - 1$ and

$$X(\mathfrak{psl}(n|n))_n = X(\mathfrak{sl}(n|n))_{n-1} \coprod X'.$$

3.4. *P-type*. A strange Lie superalgebra $\mathfrak{p}_n$ is a subalgebra of $\mathfrak{gl}(n|n)$ consisting of the matrices with the block form

$$T_{A,B,C} := \begin{pmatrix} A & B \\ C & -A' \end{pmatrix}$$

where $B' = B$ and $C' = -C$. The commutant $\mathfrak{p}_n' := [\mathfrak{p}_n, \mathfrak{p}_n]$ is simple for $n \geq 3$; one has $\mathfrak{p}_n' = \{T_{A,B,C} \mid A \in \mathfrak{sl}_n\}$. 
3.4.1. One has $p_0 = 0$, $sp_1 = \Pi(\mathbb{C})$ and
\[
X(p_n) = X(p_n)_iso = X(gl(n|n)) \cap p_n, \quad X(p_n)_r = X(gl(n|n))_r \cap p_n, \\
X(p'_{n}) = X(sp_n)_iso \bigcup X(p_n)_n, \quad X(p'_{n})_r = X(p_n)_r \quad \text{for } r < n - 1
\]
and $X(p'_{n})_{n-1} = X(p_n)_{n-1} \bigcup X(p_n)_n$. One has
\[
\text{depth}(p_n) = \text{defect } p_n = n, \quad DS_x(p_n) = p_{n-rank x}, \\
\text{depth}(p'_n) = n - 1, \quad DS_x(p'_n) = p'_{n-rank x}.
\]

Notices that $GL_n$ acts transitively on $X(p_n)_1$; we denote by $DS_1$ the functor $DS_x$ for some $x \in X(p_n)_1$ and set $DS_1^{n+1} := DS_1 \circ DS_1^n$ (these functors are not uniquely defined).

3.4.2. Remark. Take $x$ with $\text{supp}(x) = \{\varepsilon_1 + \varepsilon_2\}$ (using the standard notation of [K1]). Notice that $\text{supp}(x)$ is an iso-set of cardinality 1 and rank $x = 2$. It is not hard to see that $p_n$ contains a Cartan subalgebra $h'$ such that the support of $x$ computed with respect to $h'$ is an iso-set of cardinality two (i.e., $\{2\varepsilon_1, 2\varepsilon_2\}$).

3.5. Example: $\mathcal{F}\text{in}(p_n)$. Consider $p_n$ with the simple roots
\[
\varepsilon_2 - \varepsilon_1, \ldots, \varepsilon_n - \varepsilon_{n-1}, -\varepsilon_n - \varepsilon_{n-1}.
\]
Denote by $\rho$ the Weyl vector: $\rho := \frac{1}{2} \sum_{\alpha \in \Delta^+} (-1)^{\bar{\alpha}} \alpha$.

We denote by $L_{p_n}(\mu)$ the simple $p_n$-module of the highest weight $\mu$ with the even highest weight space. The module $L_{p_n}(\sum_{i=1}^n a_i \varepsilon_i - \rho)$ is finite-dimensional if and only if $a_i + 1 - a_i \in \mathbb{N}_{>0}$ for $i = 1, \ldots, n - 1$.

3.5.1. Lemma. Take $n \geq 2$ and a weight $\lambda = \sum_{i=1}^n a_i \varepsilon_i$ with $a_{i+1} - a_i \in \mathbb{N}_{>2n+2}$ for $i = 1, \ldots, n - 1$. Then $\dim DS_1^n(L_{p_n}(\lambda - \rho)) = \left(\frac{n}{2}, \frac{n}{2}\right)$.

Proof. The composition factors of $DS_1(L)$ for $L \in \text{Irr}(\mathcal{F}\text{in}(p_n))$ are described in [ES]. From this description one sees that the composition factors of $DS_1(L_{p_n}(\lambda - \rho_n))$ are
\[
\{\Pi_s L_{p_{n-1}}(\nu_s - \rho')\}_{s=0}^{n-1},
\]
where $\rho'$ is the Weyl vector for $p_{n-1}$ and $\nu_s = \sum_{i=1}^{n-1} b_i \varepsilon_i$ with
\[
b_1 \leq b_2 \leq b_{n-1}, \quad \{b_1\}_{i=1}^{n-1} = \{a_1\}_{i=1}^n \setminus \{a_{n-s}\}.
\]
In particular, $|(\nu_s - \nu_j)\varepsilon_{n-s-1}| > 2n + 2$ for all $0 < s < j < n - 1$. Using [BD+], Prop. 3.7.1 we conclude that $DS_1(L(\lambda))$ is completely reducible. For $n = 2$ this gives
\[
DS_1(L_{p_2}(\lambda - \rho)) = \Pi(L_{p_1}(a_1 \varepsilon_1 - \rho')) \oplus L_{p_1}(a_2 \varepsilon_1 - \rho')
\]
which implies the required formula (since $\dim L_{p_1}(a \varepsilon_1) = (1|0)$ for each $a \in \mathbb{C}$). The general case follows by induction on $n$, since, by above, $DS_1(L(\lambda))$ is a completely reducible module of length $n$ and the highest weight of each composition factor satisfies the condition of the lemma (i.e., $b_{i+1} - b_i > 2n$). \qed
3.5.2. By [BD+], each block in the category $\mathcal{F}\text{in}(\mathfrak{p}_n)$ contains a simple module $L_{p_n}(\lambda - \rho)$ with $\lambda$ satisfying the conditions of Lemma 3.5.1. Hence

$$\text{depth } \mathcal{B} = n$$
for each block $\mathcal{B}$ in $\mathcal{F}\text{in}(\mathfrak{p}_n)$.

3.5.3. Example: $\mathcal{F}\text{in}(\mathfrak{p}_2)$. Fix the root vectors $x_i \in \mathfrak{g}_2, i = 1, 2, \quad x_\pm \in \mathfrak{g}_\pm(\epsilon_1 + \epsilon_2)$.

Note that $\text{rank } x_i = 1$ and $\text{rank } x_\pm = 2$.

By [BD+], the category $\mathcal{F}\text{in}(\mathfrak{p}_2)$ contains the following blocks: $\mathcal{B}_0$, containing the trivial module, and $\mathcal{B}_1$ containing the natural $(2|2)$-dimensional module; any block in $\mathcal{F}\text{in}(\mathfrak{p}_2)$ is equivalent to one of these blocks via an equivalence given by the tensoring on a one-dimensional module.

One has $\text{Irr}(\mathcal{B}_0) = \{L^{(2j+1)}\}_{j=0}^\infty$, where $\dim L^{(1)} = 1$ and $\dim L^{(j)} = (2j + 1|2j + 1)$ for $j > 1$. For example, $L^{(3)}$ is a simple submodule of the adjoint representation and

$$\text{DS}_{x_-}(L^{(3)}) = 0, \quad \text{dim } \text{DS}_{x_+}(L^{(3)}) = (1|1).$$

One has $\text{Irr}(\mathcal{B}_1) = \{L^{(2j)}\}_{j=1}^\infty$, where $\dim L^{(2j)} = (2j|2j)$. The modules $L^{(2j)}$ provide interesting examples for Remark 2.4.2; consider $\mathfrak{pgl}(1|1)$ spanned by $x_1, x_2$; as a $\mathfrak{pgl}(1|1)$-module, $L^{(2j)} = V_{2j}^+ \oplus V_{2j}^-$ (see Lemma 2.4.1 for notation) and so

$$\text{DS}_{x_1 + x_2}(L^{(2j)}) = 0, \quad \text{DS}_{x_1}(\text{DS}_{x_2}(L^{(2j)})) = \mathbb{C} \oplus \Pi(\mathbb{C}) \quad \text{for } j > 1.$$  

The $\mathfrak{p}_1$-module $\text{DS}_1(L^{(2j)})$ is an indecomposable (resp., semisimple) for $j = 1$ (resp., $j > 1$) and $\dim \text{DS}_1(L^{(2j)}) = (1|1)$. This gives

$$\text{depth } L^{(2)} = 1, \quad \text{depth } L^{(2j)} = 2 \quad \text{for } j > 1.$$  

Note that $X(\mathfrak{p}_2)_2 = GL_2 x_- \prod GL_2(x_1 + x_2)$. It is easy to check that $\text{DS}_{x_-}(L^{(2j)}) = 0$; using $\text{DS}_{x_1 + x_2}(L^{(2j)}) = 0$ we obtain $\text{DS}_x(L^{(2j)}) = 0$ for each $x \in X(\mathfrak{p}_2)_2$. This gives

$$\max_{x \in X(L^{(2j)})} \text{rank } x = 1 \neq \text{depth}(L^{(2j)}) \quad \text{for } j > 1.$$

4. Iso-sets and blocks in the category $\mathcal{O}$

In this section $\mathfrak{g}$ is either an indecomposable symmetrizable Kac-Moody superalgebra with an isotropic root or one of the $Q$-type superalgebras $\mathfrak{psq}_m, \mathfrak{pq}_m$ for $m \geq 3$, $\mathfrak{sq}_m, \mathfrak{q}_m$ for $m \geq 2$. We will refer to the former case as the "KM-case" and to the latter case as the "$Q$-type case". By [H], in the KM-case $\mathfrak{g}$ is either finite-dimensional (classified in [K1]) or affine (classified in [vdL]).
4.1. **Notation.** We denote by $W$ the Weyl group of $g_0$. The algebra $g_0$ admits a non-degenerate invariant bilinear form and we denote by $(-|-)$ the corresponding form on $\mathfrak{h}^*$. We say that two triangular decompositions of $g$ are compatible if they induce the same triangular decomposition of $g_0$; similarly, we say that two bases (the sets of simple roots) are compatible if the corresponding triangular decompositions are compatible.

We denote by $\Delta_{re}$ the set of real roots. If $g$ is finite-dimensional, then $\Delta_{re} = \Delta$ and we set $\delta := 0$. If $g$ is affine we denote by $\delta$ the minimal imaginary root. In both cases $\Delta \setminus \Delta_{re} = \mathbb{Z}\delta \setminus \{0\}$, $(\delta|\Delta) = 0$.

We introduce $\mathfrak{h}^* := \{\mathfrak{h}^* \text{ if } \dim g < \infty \} \setminus \{\lambda \in \mathfrak{h}^* \mid (\lambda|\delta) \neq 0\} \text{ if } \dim g = \infty$.

We denote by $\Sigma$ a base (the set of simple roots). For the Kac-Moody case we fix a Weyl vector $\rho \in \mathfrak{h}^*$ satisfying $2(\rho|\alpha) = (\alpha|\alpha)$ for each $\alpha \in \Sigma$; for the $Q$-type superalgebras we set $\rho := 0$. We introduce $\Delta_{iso} := \{\alpha \in \Delta_{re} \cap \Delta_0 \mid \alpha \notin \Delta\}$, $\Delta_{iso} := \{\alpha \in \Delta_{re} \cap \Delta_1 \mid 2\alpha \notin \Delta\}$, $\Delta_{nis} := \{\alpha \in \Delta_{re} \cap \Delta_1 \mid 2\alpha \in \Delta\}$, and $\Delta_{iso} := \Delta_{iso} \cap \Delta_{iso}$ and so on. For $\alpha \in \Delta_{iso}$ the subalgebra generated by $g_\alpha$ and $g_{-\alpha}$ is isomorphic to $\mathfrak{sl}(1|1)$. One has $\Delta_{iso} = \{\alpha \in \Delta_{iso} \cap \Delta_1 \mid (\alpha|\alpha) = 0\}$ for the $Q$-type case for the KM-case.

4.1. For $\alpha \in \Delta_{iso}$ one has $(\alpha|\alpha) \neq 0$ and we set $\alpha^\vee := \frac{2\alpha}{(\alpha|\alpha)}$. For the KM-case we set $\alpha^\vee := \alpha$ for each $\alpha \in \Delta_{iso}$. In the $Q$-type case $\Delta_{iso} = \{\tau \in \Sigma \mid i \neq j\}$ (we use the standard $\mathfrak{gl}$-notation) and we set $\alpha^\vee := \gamma_i + \gamma_j$ for $\alpha := \gamma_i - \gamma_j \in \Delta_1$.

In this way $\alpha^\vee$ is defined for each $\alpha \in \Delta_{iso}$ (if we identify $\mathfrak{h}^*$ with $\mathfrak{h}$ via the form $(-|-)$), then the image of $\alpha^\vee$ in the usual coroot lying in $[\mathfrak{g}_\alpha,\mathfrak{g}_{-\alpha}]$. Notice that in all cases $wa^\vee = (wa)^\vee$ for every $w \in W$.

4.1.2. **Affine root systems.** Let $g$ be affine. We call $\hat{\Sigma} \subset \Sigma$ a finite part of $\Sigma$ if $\Sigma \setminus \hat{\Sigma}$ contains exactly one root and $\Sigma$ has a connected Dynkin diagram. The finite parts of $\Sigma$ are described in [GK], 13.2. We fix $d \in \mathfrak{h}$ with $\delta(d) = 1$ and $\alpha(d) = 0$ for $\alpha \in \Sigma$. Then $\hat{\Delta} := \{\alpha \in \Delta \mid \alpha(d) = 0\}$ is a finite root system with a base $\hat{\Sigma}$. One has $\Delta_{iso} = \hat{\Delta}_{iso} + \mathbb{Z}\delta$ except for the cases $A(2m|2n)^{(4)}$, $D(m|n)^{(2)}$, where $\Delta_{iso} = \hat{\Delta}_{iso} + 2\mathbb{Z}\delta$. 
4.1.3. The group $W(\lambda)$. We set

$$\Delta(\lambda) := \{ \alpha \in \Delta_{re} \cap \Delta_0 | (\lambda|\alpha^\vee) \in \mathbb{Z} \text{ if } \frac{\alpha}{2} \not\in \Delta, \ (\lambda|\alpha^\vee) \in \mathbb{Z} + \frac{1}{2} \text{ if } \frac{\alpha}{2} \in \Delta \}$$

and denote by $W(\lambda)$ the subgroup of $W$ generated by the reflections $\{r_\alpha\}_{\alpha \in \Delta(\lambda)}$. Note that for $w \in W(\lambda)$ one has $w\lambda \in \lambda + \mathbb{Z}\Delta$. Since $\text{ad} g_a$ acts locally nilpotently, one has $(\beta|\alpha^\vee) \in \mathbb{Z}$ for each $\beta \in \Delta$ and $\alpha \in \Delta_{re} \cap \Delta_0$. This gives $\Delta(\lambda) = \Delta(\lambda + \mu)$ for $\mu \in \mathbb{Z}\Delta$ and, in particular, $\Delta(w\lambda) = \Delta(\lambda)$ for $w \in W(\lambda)$.

4.2. Category $\mathcal{O}$ and the equivalence relation $\sim$. We denote by $M^g(\lambda)$ (resp., $L^g(\lambda)$) a Verma (resp., a simple) module of the highest weight $\lambda$ (usually we won’t distinguish between the modules which differ by $\Pi$); if $g$ is fixed, we will denote these modules by $M(\lambda), L(\lambda)$ respectively.

We denote by $\mathcal{O}^{inf}(g)$ the full category of $g$-modules $N$ with the following properties:

(C1) $N$ is a semisimple $\mathfrak{h}$-module;
(C2) $\mathfrak{n}_0^+$ acts locally nilpotently on $N$.

The BGG-category $\mathcal{O}(g)$ is the full subcategory of $\mathcal{O}^{inf}(g)$ consisting of finitely generated modules. The Verma modules lie in $\mathcal{O}(g)$. In [KK] the authors consider another version of $\mathcal{O}$-category, which we denote by $\mathcal{O}_{KK}(g)$. One has

$$\mathcal{O}(g) \subset \mathcal{O}_{KK}(g) \subset \mathcal{O}^{inf}(g).$$

Note that $\mathcal{O}(g), \mathcal{O}^{inf}(g)$ do not depend on the choice of triangular decomposition in the following sense: two compatible triangular decompositions define the same categories (the category $\mathcal{O}_{KK}(g)$ does not possess this property).

4.2.1. Let $\mathcal{O}$ be one of the categories $\mathcal{O}(g), \mathcal{O}_{KK}(g), \mathcal{O}^{inf}(g)$. One has

$$\text{Irr}(\mathcal{O}) = \{ L(\lambda) \}_{\lambda \in \mathfrak{h}^*}.$$  

Consider the graph, where the set of vertices is $\mathfrak{h}^*$ and the number of edges $\lambda_1 \rightarrow \lambda_2$ is equal to $\dim \text{Ext}^1(L(\lambda_1 - \rho), L(\lambda_2 - \rho))$. It is easy to see that this graph is the same for all three categories. We write $\lambda_1 \sim \lambda_2$ if $\lambda_1, \lambda_2$ lie in the same connected component of this graph.

By [DGK], the modules in $\mathcal{O}_{KK}(g)$ admit "local composition series", which play a role of Jordan-Hölder series for the modules of infinite length. As a result, the following conditions are equivalent:

- $\lambda_1 \sim \lambda_2$;
- $L(\lambda_1 - \rho), L(\lambda_2 - \rho)$ lie in the same block in $\mathcal{O}_{KK}(g)$;
- $L(\lambda_1 - \rho), L(\lambda_2 - \rho)$ lie in the same block in $\mathcal{O}(g)$.

4.3. Definition of atypicality. The following definition extends the usual notion of atypicality to affine case.
4.3.1. **Definition.** We say that an iso-set $S \subset \Delta_1$ is **orthogonal to** $\lambda$ if $(\lambda|\alpha^\vee) = 0$ for each $\alpha \in S$. For $\lambda \in \mathfrak{h}^*$ we denote by $\text{atyp}(\lambda)$ the maximal cardinality of an iso-set orthogonal to $\lambda$. Clearly,

$$0 \leq \text{atyp} \lambda \leq \text{defect} \mathfrak{g}, \quad \text{atyp} 0 = \text{defect} \mathfrak{g}.$$  

We say that $\lambda$ is **typical** if $\text{atyp}(\lambda) = 0$.

4.3.2. Using the formulae for $\Delta_{\text{iso}}$ given in 4.11 it is easy to see that all maximal iso-sets orthogonal to $\lambda$ have the same cardinality. Moreover, $\nu \sim \lambda$ implies $\text{atyp} \nu = \text{atyp} \lambda$ (see Corollary 4.10). In the light of 4.2 this allows to introduce the atypicality of of an indecomposable module $N$ in $\mathcal{O}_{KK}(\mathfrak{g})$ via the formula $\text{atyp}(N) := \text{atyp}(\lambda)$ where $[N : L(\lambda - \rho)] \neq 0$.

By 4.6, the atypicality of $N \in \mathcal{O}(\mathfrak{g})$ does not depend on the triangular decomposition. Using this fact it is easy to show that the atypicality of a one-dimensional module is equal to the defect of $\mathfrak{g}$.

4.4. **Main results and content of the section.** The classical results in [BGG], [J] and [KK] determine the list of irreducible modules for each block in the category $\mathcal{O}$ of a symmetrizable Kac-Moody superalgebra (see [GS2], 1.13.2 for details). For the $Q$-type superalgebras the same arguments determine these modules up to a parity shift. This give a description of the relation $\sim$, which we will call the "Kac-Kazhdan description" (the reasoning in [KK] works equally well for supercase, while some of the arguments [BGG] and [J] should be modified in the infinite-dimensional setting). We recall this description in 4.5 below. In Theorem 4.8 we show that for $\lambda \in \mathfrak{h}^*$ one has

$$\nu \sim \lambda \iff \nu \in W(\lambda)(\lambda + ZS_\lambda),$$  

where $S_\lambda$ is a maximal iso-set orthogonal to $\lambda$. (For $\dim \mathfrak{g} < \infty$, this formula was known in many cases, see, for example, [CCL].) Remark that, by [DS], Lemma 6.1 and [ChW], Chapter II, for a finite-dimensional $\mathfrak{g}$ one has

$$\text{Ann}_Z(g) L(\lambda - \rho) = \text{Ann}_Z(g) L(\nu - \rho) \iff \nu \in W(\lambda + C S_\lambda).$$  

In 4.9 we study the equivalence classes of $\sim$ in the set $\mathfrak{h}^* \setminus \hat{\mathfrak{h}}^*$. In Corollary 4.10 we show that $\nu \sim \lambda$ implies $\text{atyp} \nu = \text{atyp} \lambda$. In 4.11 we introduce $\text{Core}(\lambda)$; in 4.12 we discuss the connection between $\text{Core}(\lambda)$ and $\text{Ann}_Z(g) L(\lambda - \rho)$. By [DS], the functor $\text{DS}_x$ preserves $\text{Core}(\lambda)$ for $\mathfrak{g} = \mathfrak{gl}(m|n), \mathfrak{osp}(m|2n)$ (in Theorem 9.1 we will check this statement for certain values of $x$ in the affine case).

4.4.1. **Conjecture.** The formula (8) can be generalized to the infinite-dimensional case for $Z(\mathfrak{g})$ substituted by the algebra of Laplace operators introduced in [K2].

These operators might be useful for generalization of Theorem 9.1 to other $x \in X_{\text{iso}}$. 
4.5. Kac-Kazhdan description of the relation $\sim$. We introduce
\[K_{\text{red}} := \{ (\lambda, \lambda - m\alpha) \in h^* \times h^* | \alpha \in \Delta_{\text{red}}, m \in \mathbb{N}_{>0} \text{ s.t. } (\lambda|\alpha^\vee) = m \}\]
\[K_{\text{nis}} := \{ (\lambda, \lambda - m\alpha) \in h^* \times h^* | \alpha \in \Delta_{\text{nis}}, m \in 2\mathbb{N} + 1 \text{ s.t. } (\lambda|\alpha^\vee) = m \}\]
\[K_{\text{iso}} := \{ (\lambda, \lambda - \alpha) \in h^* \times h^* | \alpha \in \Delta_{\text{iso}}, (\lambda|\alpha^\vee) = 0 \}\]
\[K_{\text{im}} := \{ (\lambda, \lambda - \delta) \in h^* \times h^* | (\lambda|\delta) = 0 \}.\]

We set
\[\hat{K} := K_{\text{red}} \coprod K_{\text{iso}} \coprod K_{\text{nis}}, \quad K := \hat{K} \coprod K_{\text{im}}.\]

Take $\lambda \in h^*$. Let $M'(\lambda)$ be the maximal submodule of $M(\lambda)$ which satisfies the property $M'(\lambda)_\lambda = 0$. The factorization of Shapovalov determinants, obtained in [KK], [K3] ([G1] for the $Q$-type) gives:

— $M'(\lambda)_\nu \neq 0$ if $(\lambda, \nu) \in \mathcal{K},$
— if $M'(\lambda)_\mu \neq 0$, then $(\lambda, \nu) \in \mathcal{K}$ for some $\nu \in \mu + \mathbb{N}\Delta^+.$

By the density arguments of [BGC], Lemma 10 (see also [KK] and [G1], 11.4.4 for the $Q$-type) we have $\text{Hom}(M(\nu), M(\lambda)) \neq 0$ if $(\lambda, \nu) \in \mathcal{K}$. Finally, the classical arguments of [J] (see also [KK], Thm. 2) give

\[(9) \text{ the equivalence relation $\sim$ is generated by the set } \mathcal{K}.\]

Note that the restriction of the equivalence relation $\sim$ on $\hat{h}^*$ is generated by the set $\hat{K}$.

4.6. Change of the base. For an odd simple root $\beta \in \Delta^+$ with $(\beta|\beta) = 0$, we can construct a new subset of positive roots $r_\beta(\Delta^+)$ using so-called odd reflection:

\[(10) r_\beta(\Delta^+) = (\Delta^+ \setminus \{\beta\}) \cup \{-\beta\};\]

taking $\rho' := \rho + \beta$ we obtain a Weyl vector for $r_\beta(\Delta^+)$. Note that there are no odd reflections in the $Q$-type case. By [S3] any two compatible triangular decompositions are connected by a chain of odd reflections. Each module $L(\nu)$ is a highest weight module with respect to $r_\beta(\Delta^+)$ and the highest weight $\nu'$ is given by the formula

\[\nu' + \rho' = \begin{cases} 
\nu + \rho & \text{if } (\nu|\beta) \neq 0, \\
\nu + \rho + \beta & \text{if } (\nu|\beta) = 0.
\end{cases}\]

Notice that $(\nu' + \rho') \sim (\nu + \rho)$.

4.6.1. Corollary. If $\lambda' - \rho'$ is the highest weight of $L(\lambda - \rho)$ with respect to a base $\Sigma'$ which is compatible with $\Sigma$, then $\lambda \sim \lambda'$.

4.7. Iso-sets. Recall that for each $\beta \in \Delta_{\text{iso}}$ the root spaces $g_{\beta}, g_{-\beta}$ generate a subalgebra isomorphic to $\mathfrak{s}(1|1)$ and $[g_{\beta}, g_{-\beta}] = h_{\beta}$, where $\mu(h_{\beta}) = (\mu|\beta^\vee)$ for each $\mu \in h^*$. 
4.7.1. **Lemma.** For $\beta_1, \beta_2 \in \Delta_{iso}$ one has
\[(\beta_1 | \beta_2^\vee) \neq 0 \iff \beta_1 + \beta_2 \in \Delta_{re} \text{ or } \beta_1 - \beta_2 \in \Delta_{re}.\]

**Proof.** The implication $\implies$ follows from the facts that $g_{\beta}, g_{-\beta}, h_\beta$ span $\mathfrak{sl}(1|1)$. Assume that $\gamma := \beta_1 - \beta_2 \in \Delta_{re}$. For the KM-case $\beta \in \Delta_{iso}$ implies $(\beta | \beta) = 0$, so
\[0 \neq (\gamma | \gamma) = -2(\beta_1 | \beta_2) = -2(\beta_1 | \beta_2^\vee).\]
For the Q-type case we have $\beta_1 = \epsilon_i - \epsilon_j$ and $\beta_2 = \epsilon_i - \epsilon_k$ or $\beta_2 = \epsilon_k - \epsilon_j$ for some $i \neq j \neq k$, so $(\beta_1 | \beta_2^\vee) \neq 0$. $\square$

4.7.2. **Corollary.**
(i) A set $S \subset \Delta_1$ is an iso-set if and only if for each $\alpha, \beta \in S$ one has $(\alpha | \beta^\vee) = 0$ and $\alpha \pm \beta \not\in \mathbb{Z}\delta$ for $\alpha \neq \beta$.
(ii) For two iso-sets $S', S''$ of the same cardinality there exists $w \in W$ such that $w(S' \cup (-S')) = S'' \cup (-S'')$.
(iii) Let $S$ be an iso-set of the maximal cardinality. For each $x \in X_{iso}$ (see 2.1.3 for notation) there exists an automorphism $\phi \in \text{Aut}(\mathfrak{g})$ such that $\text{supp} \phi(x)$ lies in $-S \cup S$.

**Proof.** Lemma 4.7.1 implies “only if” in (i). Fix $S \subset \Delta_1$ such that for each $\alpha, \beta \in S$ one has $(\alpha | \beta^\vee) = 0$ and $\alpha \pm \beta \not\in \mathbb{Z}\delta$ for $\alpha \neq \beta$.

For the Q-type case these conditions give $S = w\{\varepsilon_{2i} - \varepsilon_{2i-1}\}_{i=1}^r$ for some $r < n$ and $w \in W$; thus $S$ is an iso-set; this also establishes (ii) for this case.

In the remaining KM-case one has $\beta^\vee = \beta$. Let $\overline{S} \subset S$ be the maximal linearly independent subset of $S$. By Lemma 4.7.1, $\overline{S}$ is an iso-set. Using the description in [vdL] it is easy to check that $C\overline{S} \cap \Delta = -\overline{S} \cup \overline{S}$; this gives $S = \overline{S}$, so $S$ is an iso-set. This establishes (i). For dim $\mathfrak{g} < \infty$ (ii) was verified in [DS]; using [R], Table I one can easily check (ii) case-by-case for affine $\mathfrak{g}$.

For (iii) let $\mathfrak{h}'$ be as in Section 2. By [KP], $\phi(\mathfrak{h}') \subset \mathfrak{h}$ for some (inner) automorphism $\phi \in \text{Aut}(\mathfrak{g})$. Now (iii) follows from (ii). $\square$

4.7.3. **Remark.** If $\mathfrak{g}$ is finite-dimensional, then $X = X_{iso}$, see [DS] and 5.6.1. This does not hold for the affine case: taking $x$ with $\text{supp}(x) = \{\alpha, \alpha + \delta\}$ for $\alpha \in \Delta_{iso}$ we obtain $x \in X \setminus X_{iso}$.

4.8. **Theorem.** Fix $\lambda \in \mathfrak{h}^\vee$. Let $S \subset \Delta^+$ be a maximal iso-set orthogonal to $\lambda$. One has
\[\nu \sim \lambda \iff \nu \in W(\lambda)(\lambda + \mathbb{Z}S).\]
Proof. The implication "\(\Longleftarrow\)" easily follows from the formula \(W(\lambda + ZS) = W(\lambda)\). In order to verify the implication "\(\Longrightarrow\)" it is enough to show that

\[
\nu \in W(\lambda)(\lambda + ZS), \quad (\nu, \nu - s\alpha) \in K \implies (\nu - s\alpha) \in W(\lambda)(\lambda + ZS).
\]

Take \(\nu, \alpha\) as above, that is

\[
\nu = w\lambda' \quad \text{for} \quad w \in W(\lambda), \quad \lambda' \in \lambda + ZS \quad \text{and} \quad (\nu, \nu - s\alpha) \in K.
\]

By [4.1.3] one has \(W(\lambda) = W(\nu)\). If \(\alpha \in \Delta^+_{red}\), then \(\alpha \in \Delta(\nu)\); if \(\alpha \in \Delta^+_{n_{is}}\), then \(2\alpha \in \Delta(\nu)\).

In both cases one has \(r_\alpha \in W(\lambda) = W(\nu)\) and so \(\nu - s\alpha = r_\alpha \nu\) lies in \(W(\lambda)(\lambda + ZS)\) as required. Consider the remaining case \(\alpha \in \Delta^+_{iso}\). Then \(s = 1, (\nu|\alpha^\vee) = 0\), so

\[
\nu - s\alpha = w(\lambda' - \beta), \quad \text{where} \quad \beta := w^{-1}\alpha \in \Delta_{iso}.
\]

Since \(w \in W(\lambda)\), it suffices to verify that

\[
(11) \quad \lambda' - \beta \in W(\lambda)(\lambda + ZS).
\]

If \(\beta \in (S \cup (-S))\), then \(\lambda' - \beta\) lies in \(\lambda + ZS\) as required. Assume that \(\beta \notin (-S \cup S)\).

Observe that

\[
(12) \quad 0 = (\nu|\alpha^\vee) = (w\lambda'|(w\beta)^\vee) = (\lambda'|\beta^\vee).
\]

If \(S \cup \{\beta\}\) is an iso-set, then \((S|\beta^\vee) = 0\) and \((12)\) implies \((\lambda|\beta^\vee) = 0\), so \((S \cup \{\beta\}\) is an iso-set orthogonal to \(\lambda\), which contradicts to the maximality of \(S\). Hence \(S \cup \{\beta\}\) is not an iso-set. In the light of [4.7.2] \(S\) contains \(\beta_1\) such that either \(\beta_1 \in \mathbb{Z}\delta \pm \beta\) or \((\beta|\beta_1^\vee) \neq 0\).

Note that \(\beta_1 \in S\) implies

\[
(13) \quad (\lambda'|\beta_1^\vee) = 0.
\]

If \(\beta_1 \in \mathbb{Z}\delta \pm \beta\), then \(g\) is affine and \((\lambda'|\beta) = (\lambda'|\beta_1) = 0\); this gives \((\lambda'|\delta) = 0\) in contradiction to \(\lambda \in \mathfrak{h}^\vee\). Hence \((\beta|\beta_1^\vee) \neq 0\). We claim that there exists \(\gamma \in \Delta_{red}\) satisfying

\[
(14) \quad r_\gamma \lambda' = \lambda', \quad \beta_1 = \{r_\gamma \beta, -r_\gamma \beta\}.
\]

Indeed, if \(g\) is a \(Q\)-type, then \((\beta|\beta_1^\vee) \neq 0\) gives \(\beta = \pm(\varepsilon_i - \varepsilon_j), \beta_1 = \pm(\varepsilon_i - \varepsilon_k)\) for some \(i \neq j \neq k\). Combining \((12)\) with \((13)\) we obtain

\[
(15) \quad (\lambda'|\varepsilon_i + \varepsilon_j) = (\lambda'|\varepsilon_i + \varepsilon_k) = 0,
\]

so \((14)\) holds for \(\gamma := \varepsilon_j - \varepsilon_k\) (since \((\lambda'|\gamma) = 0\)).

Consider the KM-case. Notice that \((\mathbb{C} \beta + \mathbb{C} \beta_1) \cap \Delta\) is a generalized root system in the sense of [S1] which is spanned by two non-orthogonal isotropic roots \((\beta \text{ and } \beta_1)\). By [S1] this root system is of the type \(A(1|0), C(2)\) or \(B(1|1)\); in each case it is easy to check the existence of \(\gamma \in \Delta_{red}\) satisfying \(\beta_1 = \pm r_\gamma \beta\). Since \(r_\gamma \lambda' = \lambda' - c\gamma\) for some \(c \in \mathbb{C}\), combining \((12)\) and \((13)\) we obtain

\[
0 = (\lambda'|\beta_1) = (\lambda'|r_\gamma \beta) = (\lambda' - c\gamma|\beta),
\]

so \(c(\gamma|\beta) = 0\). By above, \(r_\gamma \beta \neq \beta\) so \((\gamma|\beta) \neq 0\) and thus \(c = 0\). This establishes \((14)\).
By (14), \( r_\gamma \in W(\lambda') = W(\lambda) \). Using \( \lambda' \in \lambda + \mathbb{Z}S \) and \( \beta_1 \in S \) we get
\[
\lambda' - \beta = r_\gamma (\lambda' + \beta_1) \in W(\lambda)(\lambda + \mathbb{Z}S)
\]
as required. \qed

4.9. The relation \( \sim \) on \( \mathfrak{h}^* \setminus \mathfrak{b}^* \). Consider the case when \( \mathfrak{g} \) is affine. One has
\[
\mathfrak{h}^* \setminus \mathfrak{b}^* = \{ \lambda \in \mathfrak{h}^* \mid (\lambda|\delta) = 0 \} = \mathfrak{b}^* + \mathbb{C}\delta
\]
(see [4.1.2] for the notation). We will write \( \lambda \in \mathfrak{h}^* \setminus \mathfrak{b}^* \) in the form \( \lambda + a_\lambda \delta \) for \( \lambda \in \mathfrak{h}^* \) and \( a_\lambda \in \mathbb{C} \). We denote by \( \hat{W} \) the Weyl group of \( \hat{\mathfrak{g}} \) and define \( \hat{W}(\lambda) \) as in [4.1.3].

4.9.1. Take \( \lambda \in \mathfrak{h}^* \setminus \mathfrak{b}^* \). By [4.1.3], one has \( \lambda \sim \lambda - \delta \). Using Theorem [4.8] for \( \hat{\lambda} \) we get
\[
\dot{\nu} \in \hat{W}(\hat{\lambda})(\hat{\lambda} + S_\lambda), \quad a_\nu \in a_\lambda + \mathbb{Z} \quad \implies \quad \nu \sim \lambda
\]
if \( S_\lambda \subset \hat{\Delta}^+ \) is a maximal iso-set orthogonal to \( \hat{\lambda} \).

4.9.2. Corollary. For the cases \( \mathfrak{g} = \hat{\mathfrak{g}}^{(1)} \) with \( \hat{\mathfrak{g}} = \mathfrak{gl}(m|n), \mathfrak{osp}(2m|2n), D(2,1|a) \) or \( F(4) \) one has
\[
\dot{\nu} \in \hat{W}(\hat{\lambda})(\hat{\lambda} + S_\lambda), \quad a_\nu \in a_\lambda + \mathbb{Z} \quad \iff \quad \nu \sim \lambda.
\]

Proof. For the implication \( \leftarrow \rightarrow \) note that
\[
\Delta_{\text{red}} = \hat{\Delta}_{\text{red}} + \mathbb{Z}\delta, \quad \Delta_{\text{iso}} = \hat{\Delta}_{\text{iso}} + \mathbb{Z}\delta, \quad \Delta_{\text{ais}} = \emptyset.
\]
Take \( \dot{\nu} \in \hat{W}(\hat{\lambda})(\hat{\lambda} + \mathbb{Z}S_\lambda) \) and \( (\nu, \mu) \in \mathcal{K} \). Using the notation of [4.5], we have \( \mu = \nu - s\alpha \) for \( \alpha \in \Delta \). Then \( (\nu|\alpha) = (\dot{\nu}|\hat{\alpha}) \) and \( \dot{\mu} = \dot{\nu} - s\dot{\alpha} \). Using Theorem [4.8] for \( \hat{\lambda} \) we obtain \( \dot{\mu} \in \hat{W}(\hat{\lambda})(\hat{\lambda} + \mathbb{Z}S_\lambda) \) as required. \qed

4.9.3. Take \( \hat{W} \subset GL(\mathfrak{h}^*) \) generated by \( r_\alpha \) for all even real roots \( \alpha \) (one has \( \hat{W} = \hat{W} \) if \( \mathfrak{g} = \hat{\mathfrak{g}}^{(1)} \)). From [4.1.4] we see that \( \hat{\Delta}_{\text{iso}} \) is \( \hat{W} \)-invariant. Arguing as in the proof above, it is easy to show that
\[
\nu \sim \lambda \quad \implies \quad \dot{\nu} \in \hat{W}(\hat{\lambda} + \mathbb{Z}S_\lambda), \quad a_\nu \in a_\lambda + \mathbb{Z}.
\]

4.10. Corollary. \( \nu \sim \lambda \quad \implies \quad \text{atyp} \nu \geq \text{atyp} \lambda \).

Proof. Since \( \sim \) is an equivalence relation, it is enough to show that for \( \nu \sim \lambda \) one has \( \text{atyp} \nu \geq \text{atyp} \lambda \). Let \( S \) be a maximal iso-set orthogonal to \( \lambda \). If \( \lambda \in \mathfrak{h}^* \), then, by Theorem [4.8], \( \nu = w(\lambda + \mu) \), where \( \mu \in \mathbb{Z}S \) and \( w \in W(\lambda) \). Since \( wS \) is an iso-set orthogonal to \( \nu \), we obtain \( \text{atyp} \nu \geq \text{atyp} \lambda \) as required. Similarly, for \( \lambda \notin \mathfrak{h}^* \), the required inequality follows from [10] and the fact that \( \Delta_{\text{iso}} \) is \( \hat{W} \)-invariant. \qed
4.11. **Cores.** We introduce $\text{Core}(\lambda)$ in the cases when $g$ is not exceptional. For $g = \mathfrak{gl}(m|n), \mathfrak{osp}(M|2n)$, our definition is similar to one used in [GrS1].

We will consider the root systems of the types

- **Finite**: $\mathfrak{gl}(m|n), B(m|n) = \mathfrak{osp}(2m + 1|2n), D(m|n) = \mathfrak{osp}(2m|2n),$ $q_m$;
- **Affine**: $\mathfrak{gl}(m|n)^{(1)}, \mathfrak{osp}(M|2n)^{(1)}, A(M|2n)^{(2)}, A(2m|2n)^{(4)}, D(m|n)^{(2)}.$

For the finite root systems we will use the standard notation of [K1]. For affine case we retain notation of [4.1.2] note that the finite root system $\Delta$ is of the $\mathfrak{gl}(m|n)$-type for $\mathfrak{gl}(m|n)^{(1)}$ and of the $\mathfrak{osp}$-type for the rest of the cases. In all cases we set

$$a_i := (\lambda|\varepsilon_i), \quad b_j := (\lambda|\delta_j) \quad \text{for} \quad i = 1, \ldots, m, \quad j = 1, \ldots, n.$$

4.11.1. **Case $\mathfrak{gl}(m|n)$**. In this case $\Delta_{\text{iso}} = \{\pm(\varepsilon_i - \delta_j)\}_{i=1}^{m}.$

**Definition.** For $\lambda \in \mathfrak{h}^*$ let $\text{Core}(\lambda)$ be the multiset obtained from $\{a_i\}_{i=1}^{m} \bigcup \{b_j\}_{j=1}^{n}$ by deleting the maximal number of pairs satisfying $a_i = b_j$.

For instance, for $\lambda = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \delta_1 - 2\delta_2$ we have $a_1 = a_2 = a_3 = 1, b_1 = 1, b_2 = 2$ and $\text{Core}(\lambda) = \{1, 1\} \bigcup \{2\}$.

4.11.2. **Cases $B(m|n), D(m|n)$**. In this case $\Delta_{\text{iso}} = \{\pm(\varepsilon_i \pm \delta_j)\}_{i=1}^{m}.$

**Definition.** For $\lambda \in \mathfrak{h}^*$ let $\text{Core}(\lambda)$ be the multiset obtained from $\{a_i^2\}_{i=1}^{m} \bigcup \{b_j^2\}_{j=1}^{n}$ by deleting the maximal number of pairs satisfying $a_i^2 = b_j^2$.

4.11.3. **Case $q_m$**. In this case $\Delta_{\text{iso}} = \{\varepsilon_i - \varepsilon_j\}_{1 \leq i \neq j \leq m}.$

**Definition.** For $\lambda \in \mathfrak{h}^*$ we let $\text{Core}(\lambda)$ be the multiset obtained from $\{a_i\}_{i=1}^{n}$ by deleting the maximal number of pairs satisfying $a_i + a_j = 0$ for $i \neq j$.

For example, for $\lambda = \varepsilon_1 + \varepsilon_2 - \varepsilon_m$ we have $\text{Core}(\lambda) = \{0, 1\}$ if $m$ is even, $\text{Core}(\lambda) = \{1\}$ if $m$ is odd.

4.11.4. **Case $\mathfrak{gl}(m|n)^{(1)}$**. In this case $\Delta_{\text{iso}} = \{Z\delta \pm (\varepsilon_i - \delta_j)\}_{i=1}^{m}.$

**Definition.** Take $\lambda \in \mathfrak{h}^*$ and set $k := (\lambda|\delta)$. We let $\text{Core}(\lambda)$ be the multiset obtained from $\{a_i\}_{i=1}^{m} \bigcup \{b_j\}_{j=1}^{n}$ by deleting the maximal number of pairs satisfying $a_i - b_j \in \mathbb{Z}k$. We view the elements of the multiset $\text{Core}(\lambda)$ as elements in $\mathbb{C}/\mathbb{Z}k$.

4.11.5. **Cases $B(m|n)^{(1)}, D(m|n)^{(1)}, A(2m|2n - 1)^{(2)}, A(2m - 1|2n - 1)^{(2)}$**. In this case

$$\Delta_{\text{iso}} = \{Z\delta \pm \varepsilon_i \pm \delta_j\}_{i=1}^{m}.$$

**Definition.** Take $\lambda \in \mathfrak{h}^*$ and set $k := (\lambda|\delta)$. Let $\text{Core}(\lambda)$ be the multiset obtained from $\{a_i\}_{i=1}^{m} \bigcup \{b_j\}_{j=1}^{n}$ by deleting the maximal number of pairs satisfying $a_i \pm b_j \in \mathbb{Z}k$. 
We view the elements of the multiset $\text{Core}(\lambda)$ as elements in the set $\mathbb{C}$ modulo the action of the group $\mathbb{Z} \times \mathbb{Z}_2$ generated by $z \mapsto z + k$ and $z \mapsto -z$.

4.11.6. *Case $A(2m|2n)(4), D(m|n)(2)$.* In this case $\Delta_{\text{iso}} = \{2\mathbb{Z}\delta \pm \varepsilon_i \pm \delta_j\}_{i=1, \ldots, m}$ and the definition can be obtained from above by substituting $k$ by $2k$.

4.11.7. *Remark: degenerate cases.* In the case when $mn = 0$ the set $\Delta_{\text{iso}}$ is empty, but we use the same definition for $\text{Core}(\lambda)$. For instance, $\mathfrak{osp}(2|0) = \mathbb{C}$; for this algebra $\mathfrak{h}^* = \mathbb{C}\varepsilon_1$ and we set $\text{Core}(a\varepsilon_1) = a^2$. Note that $\text{gl}(0|0) = \mathfrak{osp}(0|0) = \mathfrak{osp}(0|0)(1) = \mathbb{C}K \times \mathbb{C}d$; in all these cases we set $\text{Core}(\lambda) = \emptyset$ for each weight $\lambda \in \mathfrak{h}^*$.

4.11.8. Take $\lambda \in \mathfrak{h}^*$. Let $S_\lambda$ be a maximal iso-set orthogonal to $\lambda$. It is easy to see that

\begin{align}
W(\lambda + \mathbb{C}S_\lambda) \subset \{\nu\mid \text{Core}(\nu) = \text{Core}(\lambda)\},
W(\lambda + \mathbb{C}S_\lambda) = \{\nu\mid \text{Core}(\nu) = \text{Core}(\lambda)\} & \text{ for } A(m|n), B(m|n), q_m.
\end{align}

For $D(m|n)$ the equality holds if atyp $\lambda \neq 0$.

4.11.9. *Corollary.*

(i) The cardinality of $\text{Core}(\lambda)$ is equal to $m + n - 2$ atyp $\lambda$ (resp., $m - 2$ atyp $\lambda$) for the KM-case (resp., for the Q-type case).

(ii) For $\lambda, \nu \in \mathfrak{h}^*$ one has $\text{Core}(\lambda) = \text{Core}(\nu)$ if $\nu \sim \lambda$.

(iii) Let $\mathfrak{g}$ be finite-dimensional. If $\lambda' - \rho$ is the highest weight of the module $L(\lambda - \rho)$ with respect to a base $\Sigma'$, which is compatible to $\Sigma$, then $\text{Core}(\lambda) = \text{Core}(\lambda')$. For affine case this holds if $\Delta = \Delta'$.

*Proof.* For (i) note that in all cases the number of deleted pairs is equal to atyp $\lambda$. Combining Theorem 4.8 with (17) we get (ii); (iii) follows from 4.6. \qed

4.11.10. *Remark.* For affine case the construction of $\text{Core}(\lambda)$ depends on the choice of $\hat{\Sigma}$. Using the tables in [GK], 13.2, it is not hard to check that, apart from the case $\mathfrak{gl}(m|n)(1)$ $\text{Core}(\lambda)$ does not depend on the choice of $\hat{\Sigma}$.

4.12. **Central characters.** Consider the case when $\dim \mathfrak{g} < \infty$.

For $z \in \mathfrak{Z}(\mathfrak{g})$ let $\mathcal{HC}(z) \in \mathcal{S}(\mathfrak{h})$ be such that $z$ acts on $M(\lambda - \rho)$ by $\mathcal{HC}(z)\text{Id}$. Then

$$\mathcal{HC} : \mathfrak{Z}(\mathfrak{g}) \to \mathcal{S}(\mathfrak{h})$$

is an algebra monomorphism ($\mathcal{HC}$ is the composition of the Harish-Chandra projection HC with $\rho$-twisting; by contrast with HC, the map $\mathcal{HC}$ does not depend on the triangular
decomposition of $\Delta$). The arguments of [K2] (see also [G1]) imply that the image of $HC$ is given by

$$Z(g) := \{ f \in S(h) \mid f(\lambda) = f(\nu) \text{ if } (\nu, \lambda) \in K \}.$$ 

Take any $\beta \in \Delta_{iso}$. It is easy to see that $\Delta_{iso} = W\beta \cup W(-\beta)$; this allows to rewrite the above formula as

$$(18) \quad Z(g) = \{ f \in S(h) \mid \forall \lambda \in h^* \text{ with } (\lambda|\beta') = 0 \text{ one has } f(\lambda) = f(\lambda - c\beta) \text{ for } c \in \mathbb{C} \}.$$ 

(The above formulae were established earlier by A. Sergeev in [Ser2], [Ser1] by other methods; see also [GN] for another approach in the $Q$-type case).

4.12.1. Let $\chi_{\lambda} : Z(g) \to \mathbb{C}$ be the central character of $L(\lambda - \rho)$, i.e.

$$\chi_{\lambda}(z) := HC(z)(\lambda).$$

By [1.2.3] each central character $\chi \in \text{mspec}_{\text{Mod}} Z(g)$ is of the form $\chi = \chi_{\lambda}$ for some $\lambda \in h^*$. We set atyp $\chi_{\lambda} := \text{atyp } \chi$. If $g$ is not exceptional we set Core($\chi_{\lambda}$) := Core($\lambda$). These notions are well-defined; moreover, using [4.11.8] it is not hard to see that the assignment $\chi \mapsto \text{Core(}\chi)$ is bijective for $g \neq \mathfrak{osp}(2m|2n)$. We give a proof for the $Q$-type cases in Proposition 4.12.2 below (the proof is similar to one for $\mathfrak{g}_n$ in [ChW], Ch. II). For $g = \mathfrak{osp}(2m|2n)$ the fiber is of the form $\{\chi, \sigma(\chi)\}$, where $\sigma$ is as in 6.1; one has $\sigma(\chi) = \chi$ if atyp $\chi \neq 0$ (see [ChW], Ch. II for the details).

4.12.2. Proposition. Let $g$ be a $Q$-type algebra. For $\lambda, \nu \in h^*$ one has $\chi_{\lambda} = \chi_{\nu}$ if and only if $\text{Core}(\lambda) = \text{Core}(\nu)$.

Proof. Our goal is to show that

$$\text{Core}(\lambda) = \text{Core}(\nu) \iff \forall f \in Z(g) \ f(\lambda) = f(\nu).$$

The implication $\Rightarrow$ follows from (17). For the implication $\Leftarrow$ we introduce the following notation. For $\mathfrak{q}_n$ and $\mathfrak{sq}_n$ we denote by $\{h_i\}_{i=1}^n$ the basis of $h$ which is dual to the basis of $\{\varepsilon_i\}_{i=1}^n$ in $h^*$; for $\mathfrak{pd}_n$ and $\mathfrak{psq}_n$ we denote by the same symbol the image of $h_i$ in $h$. Consider the symmetric polynomials

$$p_{2r+1} := \sum_{i=1}^n h_i^{2r+1}$$

(one has $p_1 = 0$ for $\mathfrak{pq}_n$, $\mathfrak{psq}_n$). From (18) one has $p_{2r+1} \in Z(g)$. We set

$$\Phi := \sum_{r=1}^\infty p_{2r+1} z^r = \sum_{i=1}^n \frac{h_i}{1 - h_i^2 z}.$$ 

For each $\mu = \sum_{i=1}^n a_i \varepsilon_i \in h^*$ we introduce the function

$$\phi(z) := \Phi(\mu) = \sum_{i=1}^n \frac{a_i}{1 - a_i^2 z}.$$
Clearly, \( \phi(z) \) is a meromorphic function in \( z \) with the set of poles equal to
\[
\{ a^{-2} | a \in \text{Core}(\mu) \setminus \{0\} \}.
\]
Moreover, the residue of \( \phi(z) \) at the point \( a^{-2} \) is equal to \( \text{mult}(a)a^3 \), where \( \text{mult}(a) \) stands for the multiplicity of \( a \) in the multiset \( \text{Core}(\mu) \).

Let \( \nu \) be such that \( p_{2r+1}(\lambda) = p_{2r+1}(\nu) \) for each \( r \). Then \( \Phi(\lambda) = \Phi(\nu) \). By above, this gives \( \text{Core}(\lambda) \setminus \{0\} = \text{Core}(\nu) \setminus \{0\} \). Since the multiplicity of 0 in the multisets \( \text{Core}(\lambda), \text{Core}(\nu) \) is at most 1 and
\[
\# \text{Core}(\lambda) \equiv n \equiv \# \text{Core}(\nu) \mod 2
\]
we obtain \( \text{Core}(\lambda) = \text{Core}(\nu) \) as required. \( \square \)

4.13. Useful fact. The following fact will be used later. Let \( g \) be a Kac-Moody superalgebra or \( q_n \). Fix \( \Sigma' \subset \Sigma \) and choose \( h \in \mathfrak{h} \) such that \( \alpha(h) = 0 \) for \( \alpha \in \Sigma' \) and \( \alpha(h) = 1 \) for \( \alpha \in \Sigma \setminus \Sigma' \). The algebra \( g^h \) has the triangular decomposition:
\[
g^h = g^h \oplus (n \cap g^h) \oplus (n^- \cap g^h)
\]
( the corresponding set of simple roots is \( \Sigma' \)).

Lemma. Take \( \lambda \in \mathfrak{h}^* \) and denote by \( L \) the module \( L(\lambda) \) viewed as a \( g^h \)-module. Then
\[
L' := \sum_{\nu \in \mathbb{N} \Sigma'} L_{\lambda - \nu}
\]
is a simple \( g^h \)-module of the highest weight \( \lambda \) and \( L' \) is a direct summand of \( L \).

Proof. Each \( g^h \)-module \( N \) with a diagonal action of \( \mathfrak{h} \) can be decomposed as
\[
N = \bigoplus_{\mu \in \mathfrak{h}^*/\mathbb{Z} \Sigma'} N_{\mu},
\]
where each \( N_{\mu} \) is a \( g^h \)-submodule. This shows that \( L' \) is a direct summand of \( L \). Let \( v \in L_{\lambda - \nu} \) be a \( g^h \)-primitive vector, i.e. \( (g^h \cap n)v = 0 \). Take \( \alpha \in \Delta^+ \setminus \Delta(g_h) \). Since \( \nu \in \mathbb{N} \Sigma' \) one has
\[
\lambda - \nu + \alpha \notin \lambda - \mathbb{N} \Sigma',
\]
so \( g_{\alpha} L(\lambda)_{\lambda - \nu} = 0 \). Hence \( v \) is a \( g \)-primitive, that is \( v \in L'_{\lambda} \). However, \( L'_{\lambda} = L(\lambda)_{\lambda} \) is a simple \( g^h \)-module and \( g^h \subset g^h \). Hence \( L' \) is simple. \( \square \)

5. DS-functor for \( Q \)-type algebras

Queer (\( Q \)-type) Lie superalgebras are the superalgebras \( q_n, sq_n, pq_n \) and \( psq_n \). Recall that \( q_n \) is a subalgebra of \( \mathfrak{gl}(n|n) \) consisting of the matrices with the block form
\[
T_{A,B} := \begin{pmatrix} A & B \\ B & A \end{pmatrix}
\]
The centre of $\mathfrak{q}_n$ is spanned by the identity matrix, which we denote by $z$. The commutant $\mathfrak{sq}_n := [\mathfrak{q}_n, \mathfrak{q}_n]$ is a subalgebra of $\mathfrak{q}_n$ consisting of the matrices $T_{A,B}$ with $TrB = 0$. One has $\mathfrak{pq}_n := \mathfrak{q}_n/\mathbb{C}z$ and $\mathfrak{psq}_n$ is the image of $\mathfrak{sq}_n$ in $\mathfrak{pq}_n$. The algebra $\mathfrak{psq}_n$ is simple for $n \geq 3$.

The group $GL_n$ acts on these algebras by the inner automorphisms $g.T := T_g^{-1}g.Tg^{-1}$.

We retain notation of 2.1.4 and Section 4. We denote by $B$ the trivial module in $\mathcal{F}in(\mathfrak{g})$, which is the block containing the trivial module.

5.1. Main results. For $\mathfrak{g} = \mathfrak{q}_n, \mathfrak{sq}_n, \mathfrak{pq}_n, \mathfrak{psq}_n$ we have

$$\text{defect } \mathfrak{g} = \text{depth}(\mathfrak{g}) = \left[\frac{n}{2}\right]$$

For $x \in X(\mathfrak{q}_n)_r$ we have $DS_x(\mathfrak{q}_n) \cong q_{n-2r}$. The similar formulae hold for $\mathfrak{sq}_n, \mathfrak{pq}_n, \mathfrak{psq}_n$ if $r \neq \left[\frac{n}{2}\right]$; for $r = \left[\frac{n}{2}\right]$ we have

$$\begin{align*}
DS_x(\mathfrak{sq}_n) &\cong \mathfrak{e}_1 = \mathbb{C}, & DS_x(\mathfrak{pq}_n) &\cong \mathfrak{p}_1 = \Pi(\mathbb{C}) \\
DS_x(\mathfrak{psq}_n) &\equiv \mathfrak{p}_1 = 0 & \text{if } n \text{ is odd}, & DS_x(\mathfrak{psq}_n) &\equiv \mathbb{C} \times \Pi(\mathbb{C}) & \text{if } n \text{ is even}.
\end{align*}$$

One has $X(\mathfrak{q}_n)_iso = X(\mathfrak{sq}_n)_iso$ and $X(\mathfrak{pq}_n)_iso = X(\mathfrak{psq}_n)_iso$ is the image of $X(\mathfrak{q}_n)_iso$. Moreover, $X(\mathfrak{g})_iso = X(\mathfrak{g})$ for $\mathfrak{g} = \mathfrak{q}_n, \mathfrak{sq}_n$ and $X(\mathfrak{g}) \setminus X(\mathfrak{g})_iso \subset X(\mathfrak{g})_? = X(\mathfrak{g})$ for $\mathfrak{g} = \mathfrak{pq}_n, \mathfrak{psq}_n$.

In Corollary 5.8.1 we prove that the map $\theta_x^r$ preserves the core of a central character, increases the atypicality by $r$ and that the image of $\theta_x^r$ consists of the central characters of atypicality $\geq r$. This implies that $DS$ commutes with translation functors (see 5.9). We also show that depth $\chi$ coincides with the atypicality of $\chi$. In Proposition 5.8.3 we prove that $\theta_x$ is surjective.

View $\mathfrak{psq}_3$ as a quotient of the adjoint representation for $\mathfrak{q}_3$. This is a simple module which lie in the principal block $B_0(\mathfrak{q}_3)$. Since $DS_x(\mathfrak{psq}_3) = 0$ for all non-zero $x \in X(\mathfrak{q}_3)$, this module has zero depth; this gives an example of a of zero depth module which is not projective (by [DS], for finite-dimensional Kac-Moody superalgebras, $N \in \mathcal{F}in(\mathfrak{g})$ has zero depth if and only if $N$ is projective). Note that depth($B_0(\mathfrak{q}_3)$) = 1, since depth of the trivial module is equal to depth($\mathfrak{g}$).

5.2. Questions and content of the section. Let $L$ be a simple finite-dimensional $\mathfrak{g}$-module.

5.2.1. Question. Is it true that $DS_x(L) \cong DS_y(L)$ if $\text{rank } x = \text{rank } y$? More general, is it true that $DS_x(L) \cong DS_y(DS_y(L))$ if $\text{rank } x = \text{rank } y + \text{rank } y'$?

If $\text{dim } L = \infty$, this does not hold, see [10].

5.2.2. Question. Is it true that $DS_x(L)$ is semisimple for $\mathfrak{g} = \mathfrak{q}_n$?

This property holds for finite-dimensional Kac-Moody superalgebras (see [HW] and [GH]) and does not hold for $\mathfrak{sq}_2$ and for $\mathfrak{pq}_2$. 
5.2.3. In 5.3 we introduce additional notation. In 5.4, 5.5 we consider the case \( q \). Finally, in 5.6–5.8 we study the case \( g \) subalgebra of \( C \) representation of a Clifford algebra.

5.3. Notation. Let \( g \) be one of the \( Q \)-type algebras. We denote by \( h \) the standard Cartan subalgebra of \( g_0 \) (consisting of \( T_{A,0} \) with diagonal \( A \)). The sets \( \Delta_0 = \Delta_1 \subset h^* \) coincide with the root system of \( g_1 \). All triangular decompositions are \( GL_n \)-conjugated. We fix a triangular decomposition: \( g = g^b \oplus n + n^- \), where \( g^b \) is the Cartan subalgebra (one has \( g^b \cap g_0 = h \)). We use the standard \( g_1 \)-notation for the root system \( \Delta_0 \) and take \( \Sigma := \{ \varepsilon_i - \varepsilon_{i+1} \}^n_{i=1} \). We identify the Weyl group \( W \) with the permutations of \( \{ \varepsilon_i \}^n_{i=1} \).

For \( g = q_n, sq_n \) we set

\[
h_i := \varepsilon^*_i \in h;
\]

for \( pq_n, psq_n \) we denote by \( h_i \) the image of the corresponding element in \( h \). One has

\[
\sum^n_{i=1} h_i = \begin{cases} z & \text{for } q_n, sq_n, \\
0 & \text{for } pq_n, psq_n. \end{cases}
\]

We denote by \( \iota \) the canonical map \( q_n \to pq_n \) and its restriction \( sq_n \to psq_n \).

5.3.1. Remark. Let \( L(\lambda) \) be a typical finite-dimensional module and \( \mathcal{F}\text{in}(\chi_\lambda) \) be the full subcategory of \( \mathcal{F}\text{in}(g) \) which corresponds to the central character \( \chi_\lambda \) (one has \( L(\lambda) \in \mathcal{F}\text{in}(\chi_\lambda) \)). By [P1] the category \( \mathcal{F}\text{in}(\chi_\lambda) \) is equivalent to the finite-dimensional representation of a Clifford algebra \( \mathcal{C}\text{l}(\lambda) := S(g^b)/S(g^b) \text{Ker} \lambda \). For instance, for \( q_n \)-case, the category \( \mathcal{F}\text{in}(\chi_\lambda) \) is equivalent to the category of the finite-dimensional modules over the algebra \( \mathbb{C}[\xi]/(\xi^2) \) if \( 0 \in \text{Core}(\lambda) \) and \( \mathcal{F}\text{in}(\chi_\lambda) \) is semisimple if \( 0 \not\in \text{Core}(\lambda) \).

5.4. Case \( n = 1 \). We have \( q_1 = \mathbb{C}z + \mathbb{C}H \), where \( H \) is odd and \( [H, H] = z \). One has \( sq_1 = \mathbb{C}z \cong \mathbb{C} \) and \( psq_1 = 0 \); the algebra \( pq_1 \cong \Pi(\mathbb{C}) \) is spanned \( \iota(H) \). In all these cases \( X_{nso} = 0 \), so depth \( g = 0 \). One has \( X(q_1) = X(sq_1) = 0 \) and \( X(pq_1) = pq_1 \).

The action of \( z \) decomposes the category \( \mathcal{F}\text{in}(q_1) \) into blocks \( B_c \) with \( c \in \mathbb{C} \). All these blocks are typical. For \( c \neq 0 \) the block \( B_c \) is semisimple with \( \text{Irr}(B_c) = \{ L_{q_1}(c) \} \), where \( L_{q_1}(c) \cong \Pi(L_{q_1}(c)) \) and \( \dim L_{q_1}(c) = (1|1) \). For the principal block \( B_0 \) we have

\[
\text{Irr}(B_0) = \{ L_{q_1}(0), \Pi(L_{q_1}(0)) \}, \quad \text{Ext}^1(L_{q_1}(0), \Pi(L_{q_1}(0))) = \mathbb{C}.
\]

5.5. Case \( n = 2 \). The modules \( L(\lambda) \) are explicitly described in [P1]. The module \( L(\lambda) \) is finite-dimensional if and only if \( \lambda(h_1 - h_2) \in \mathbb{Z}_{>0} \) or \( \lambda = 0 \). The module \( L(\lambda) \) is atypical if and only if \( \lambda(z) = 0 \) (for \( pq_2, psq_2 \) all modules are atypical). In 5.6 5.11.2 we will see that for \( x \neq 0 \) one has \( DS_x(N) = 0 \) for a typical module \( N \).

Set \( g := q_2 \). Recall that \( g_0 = \mathbb{C}z \times sl_2 \); we fix the standard basis \( f, h, e \in sl_2 \). Note that \( g_1 \) has a basis \( H_1 := T_{0,1d}, H_2, E, F \), where \( H_2 \in sq_2^b \) is such that \( [H_1, H_2] = 2h \) and...
We will use the same notation for the images of these elements in $\mathfrak{pq}$ and in $\mathfrak{psq}$. 

5.5.1. Typical case. Let $\lambda$ be such that $\lambda(z) \neq 0$. Then $L_{q_2}(\lambda), L_{sq_2}(\lambda)$ are typical and $L_{q_2}(\lambda) = L_{sq_2}(\lambda)$ as $\mathfrak{sq}_2$-module. One has $\Pi(L_{sq_2}(\lambda)) \cong L_{sq_2}(\lambda)$. The typical blocks in $\mathcal{F}\text{in}(\mathfrak{sq}_2)$ are semisimple. By 5.6 below, for each typical $q_2$-module we have $DS_x(N) = 0$ for $x \neq 0$. This gives $DS_x(L_{sq_2}(\lambda)) = 0$ and thus $DS_x(N) = 0$ if $N$ is a typical module of a finite length.

5.5.2. Cases $q_2, \mathfrak{sq}_2$. One has

$$X(\mathfrak{sq}_2) = X(\mathfrak{sq}_2)_{iso} = X(q_2) = X(q_2)_{iso} = GL_2E \cup \{0\}.$$ 

One has $DS_E(q_2) = 0$; the algebra $DS_E(\mathfrak{sq}_2)$ can be identified with $C_\lambda$.

Let $L := L_{sq_2}(\lambda)$ be atypical. Then $L$ is a simple $\mathfrak{sl}_2$-module (so $g_1L = 0$). In particular, $EL = 0$, so $DS_E(L) = L$ as a $DS_E(\mathfrak{sq}_2)$-module (the action of $e \in DS_E(\mathfrak{sq}_2)$ on $DS_E(L)$ coincides with the action of $e$ on $L = L_{q_2}(\lambda)$).

Let $L := L_{q_2}(\lambda)$ be an atypical module for $\lambda \neq 0$. As a $\mathfrak{sq}_2$-module, $L$ is a non-splitting extension of $\Pi(L_{sq_2}(\lambda))$ by $L_{q_2}(\lambda)$. If $L$ is finite-dimensional, then

$$DS_x(L) \cong \mathbb{C} \oplus \Pi(\mathbb{C}) \quad \text{for } x \in X_{iso} \setminus \{0\}.$$ 

If $L$ is infinite-dimensional, then

$$(19) \quad DS_E(L_{q_2}(\lambda)) \cong \mathbb{C}, \quad DS_F(L_{q_2}(\lambda)) \cong \Pi(\mathbb{C}).$$

5.5.3. Case $\mathfrak{pq}_2$. One has

$$X(\mathfrak{pq}_2)_{iso} = X(q_2), \quad X(\mathfrak{pq}_2) \setminus X(\mathfrak{pq}_2)_{iso} = \mathbb{C}^*t(H_1) \coprod \mathbb{C}^*t(H_2);$$

for each non-zero $x \in X(\mathfrak{pq}_2)$ one has rank $x = 1$ and $DS_x(\mathfrak{pq}_2) \cong \Pi(\mathbb{C}) \cong \mathfrak{pq}_1$.

The $\mathfrak{pq}_2$-modules are $q_2$-modules annihilated by $z$ (in particular, $O(\mathfrak{pq}_2)$ corresponds to the subcategory of atypical modules in $O(q_2)$). Take $L := L(\lambda)$ for $\lambda \neq 0$.

The algebra $DS_E(\mathfrak{pq}_2)$ can be identified with $\mathbb{C}F$. If $L$ is finite-dimensional and $\lambda \neq 0$, then $\dim DS_x(L) = (1|1)$ for each $x \in X(\mathfrak{pq}_2)_{iso}$; it is easy to check that $DS_E(L)$ is an indecomposable $\mathfrak{pq}_1$-module if and only if $\dim L = (2|2)$. If $L$ is infinite-dimensional, then $DS_E(L), DS_F(L)$ are given by (19) (with $\mathfrak{pq}_1 \cong \Pi(\mathbb{C})$ acting by zero).

For $x \in \mathbb{C}^*t(H_1) \coprod \mathbb{C}^*t(H_2)$ the algebra $DS_x(\mathfrak{pq}_2)$ can be identified with $\mathbb{C}x$. It is easy to see that $DS_x(L) = 0$ for $\lambda \neq 0$. 

$$E = \frac{1}{2}[H_2, c], \quad F = \frac{1}{2}[f, H_2].$$ One has $[g_0, H_1] = 0$ and

$$[H_1, H_1] = [H_2, H_2] = 2c, \quad [h, E] = 2E, \quad [h, F] = -2F, \quad [E, F] = z, \quad [E, E] = [F, F] = 0.$$
5.5.4. Case $\mathfrak{psq}_2$. One has
\[ X(\mathfrak{psq}_2)_{iso} = \iota(X(q_2)), \quad X(\mathfrak{psq}_2) \setminus X(\mathfrak{psq}_2)_{iso} = \mathbb{C}^*\iota(H_2); \]
for each non-zero $x \in X(\mathfrak{psq}_2)$ one has rank $x = 1$ and $\text{DS}_x(\mathfrak{psq}_2) \cong \mathbb{C} \times \Pi(\mathbb{C})$.

The $\mathfrak{psq}_2$-modules are $\mathfrak{sq}_2$-modules annihilated by $z$. In particular, $L_{\mathfrak{psq}_2}(\lambda)$ is a simple $\mathfrak{sl}_2$-module.

The algebra $\text{DS}_E(\mathfrak{psq}_2)$ can be identified with $\mathbb{C}e \oplus \mathbb{C}F$. By above, $\text{DS}_E(L) = L$ as a $\mathbb{C}e$-module and $F(\text{DS}_E(L)) = 0$.

For $x \in \mathbb{C}^*H_2$ the algebra $\text{DS}_x(\mathfrak{psq}_2)$ can be identified with $\mathbb{C}h \oplus \mathbb{C}x$. This gives $\text{DS}_x(L) = L$ as a $\mathbb{C}h$-module and $x(\text{DS}_x(L)) = 0$.

5.5.5. Note that for $g = q_2, \mathfrak{sq}_2, \mathfrak{pq}_2, \mathfrak{psq}_2$ we have depth $g = \text{defect } g = 1$.

5.6. The set $X(q_n)$. In 5.6–5.8 we consider the case $g := q_n$ with $n \geq 2$.

For $i = 1, \ldots, n-1$ we set $\alpha_i := \varepsilon_i - \varepsilon_{i+1}$. For each $s \leq n$ we identify $q_s$ with the subalgebra of $q_n$ with the set of simple roots $\{\alpha_1, \ldots, \alpha_s\}$. For $r = 0, 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor$ we set
\[ S_r := \{\alpha_{n-2i} \}_{i=0}^{r-1} \]
and fix $x_r \in X$ such that $\text{supp}(x_r) = S_r \quad (x_0 := 0)$.

5.6.1. Lemma. The elements $x_0, \ldots, x_{\left\lfloor \frac{n}{2} \right\rfloor}$ form a set of representatives of $GL_n$-orbits in $X$.

Proof. One has $X = \{T_{0,B} | B^2 = 0\}$. The elements of $GL_n$ act by conjugation on $B$, so the $GL_n$-orbits correspond to the Jordan forms. □

5.6.2. Recall that $g \in GL_n$ acts on $\mathfrak{g}$ by an inner automorphism. If $N$ is a finite-dimensional $\mathfrak{g}$-module, then $g$ acts on $N$; as a result, $g$ induces a bijection between $\text{DS}_x(N)$ and $\text{DS}_{gx}(N)$, which is compatible with the algebra isomorphism $g \xrightarrow{\sim} g_{gx}$.

5.6.3. Proposition. The algebra $\text{DS}_{x_r}(q_n)$ can be identified with $q_{n-2r}$.

Proof. Set $x := x_r = \sum_{\alpha \in S_r} x_\alpha$, where $x_\alpha \in \mathfrak{g}_\alpha$. By Lemma 2.3 one has
\[ \text{DS}_x(\mathfrak{g}) = \text{DS}_x(\sum_{\alpha \in Y} \mathfrak{g}_\alpha + \mathfrak{g}^b), \quad \text{where } (\alpha|\beta^b) = 0 \quad \text{for } \beta \in \text{supp}(x) \}
\]

Denote by $\mathfrak{q}_2(\alpha)$ the copy of $\mathfrak{q}_2$ corresponding to the root $\alpha$. One has
\[ \sum_{\alpha \in Y} \mathfrak{g}_\alpha + \mathfrak{h} = \bigoplus_{\alpha \in S_r} \mathfrak{q}_2(\alpha) \bigoplus q_{n-2r} \]
as a $\mathbb{C}x$-module. One has $xq_{n-2r} = 0$. For each $\alpha \in S_r$, the action of $x$ on $q_2(\alpha)$ coincides with the adjoint action of $x_\alpha \in q_2(\alpha)$, so $DS_x(q_2(\alpha)) = 0$ (by \[5.5.2\]). This implies the statement. \hfill \Box

5.6.4. **Corollary.** \(\text{depth}(q_n) = \left\lfloor \frac{n}{2} \right\rfloor\) and $X_r = GL_n x_r$.

5.6.5. **Remark.** One has $X(q_n)_r = X(gl(n|2r)) \cap q_n$.

5.7. **Case when supp\((x)\) is an iso-set.** The maximal cardinality of an iso-set is $\left\lfloor \frac{n}{2} \right\rfloor$ and each iso-sets of the cardinality $r$ is $W$-conjugated to $S_r$.

Take $x := x_r$. Using Proposition \[5.6.3\] we identify $g_x$ with $q_{n-2r}$. The algebra $h_x := g_x \cap h$ is spanned by $h_1, \ldots, h_{n-2r}$. We identify $h_x^*$ with the span of $\varepsilon_1, \ldots, \varepsilon_{n-2r}$.

5.7.1. Now take an arbitrary $x \in X_r$ such that supp\((x)\) is an iso-set, i.e. supp\((x)\) = $wS_r$ for some $w \in W$. Take $g \in GL_n$ such that $g\mathfrak{h} = h$ and that $g\mathfrak{g}_\alpha = \mathfrak{g}_{w\alpha}$. We identify $DS_x(g)$ with $gDS_x(g)$. Then $h_x := DS_x(g) \cap h$ is a Cartan subalgebra of $(DS_x(g))_0 \cong gl(n-2k)$. We fix the following triangular decomposition of $g_x$:

$$g_x = n_x^- \oplus g_x^{h^*} \oplus n_x,$$

where $n_x^- := n^- \cap g_x$ and $n_x := n \cap g_x$. Notice that $h_x$ is spanned by $\left\{ h_{w(i)} \right\}_{i=1}^{n-2r}$; we identify $h_x^*$ with the span of $\{ \varepsilon_{w(i)} \}_{i=1}^{n-2r}$. Observe that

\[
\{ \nu \in h^* | \nu|_{h_x} = 0, \ \nu \text{ orthogonal to } S \} = CS
\]

(these spaces have the same dimension and the inclusion $\supset$ is straightforward).

Let $N$ be a $\mathfrak{g}$-module with a diagonal action of $h$. For each $\mu \in h_x^*$ we set

$$N_\mu := \sum_{\gamma \in CS} N_{\mu + \gamma}.$$

Note that $N_\mu$ is $(h + \mathbb{C}x)$-submodule of $N$. Combining (20) and Lemma 2.3 we get

\[
DS(N) = \sum_{\mu \in h_x^*} DS_x(N)_\mu, \quad DS_x(N)_\mu = DS_x(N_\mu).
\]

5.7.2. Recall that $g_x^{h^*}$ is the Cartan subalgebra of $g_x$ and $h_x = (g_x^{h^*})_0$. The following statement demonstrates a peculiarity of $q_n$.

**Proposition.** Let $x \in X$ be such that $S := \text{supp}(x)$ is an iso-set and $S \subset \Pi$. Assume $\lambda \in h^*$ is orthogonal to $S$. We set $\lambda' := \lambda|_{h_x}$ and view $M' := DS_x(L(\lambda))_{\lambda'}$ as a $g_x^{h^*}$-module.

(i) One has $n_x M' = 0$.
(ii) For $\lambda \in h_x^*$, $M'$ can be identified with $L(\lambda)_\lambda$.
(iii) One has $M' \cong L_{g_x^{h^*}}(\lambda') \otimes (\mathcal{O} \oplus \Pi C)^{\otimes s}$, where $s$ is the cardinality of the set

$$\{ \beta \in S | (\lambda|\beta) \in \mathbb{N}_{>0} \}.$$


Proof. Set $N := L(\lambda)$ and $M := \text{DS}_x(L)$. Identify $\mathfrak{h}_x^*$ with a subspace in $\mathfrak{h}^*$ as in 5.7.1. The formula (20) gives $\lambda - \lambda' \in \mathbb{C}S$. Using (21) we get

$$M' = M_{\lambda'} = \text{DS}_x(N_{\lambda'}) = \sum_{\nu \in \mathbb{C}S} N_{\lambda - \nu}.$$

For (i) assume that $(\mathfrak{n}_x)^{\lambda'} M' \neq 0$ for some $\alpha \in \Delta^+(\mathfrak{g}_x)$. Since $(\mathfrak{n}_x)^{\lambda'} M_{\lambda'} \subset M_{\lambda' + \alpha}$, the formula (21) implies the existence of $\gamma \in \mathbb{N} \Delta^+$ satisfying

$$\lambda - \gamma \in (\lambda' + \alpha) + \mathbb{C}S.$$

Then $\gamma + \alpha \in \mathbb{C}S$, which is impossible since $\alpha \in \Delta^+$ and $\alpha \notin \mathbb{C}S$. This establishes (i).

For (ii) recall that $(\mathfrak{g}_x^{\lambda'})_0 = \mathfrak{h}_x$ is spanned by $\{h_j\}_{j \in J}$ where $J \subset \{1, \ldots, n\}$ has cardinality $n - 2r$. Therefore

$$\mathfrak{g}_x^{\lambda'} \cong q_1 \times \ldots \times q_1\bigg|_{n-2r \text{ times}}.$$

Retain notation of 4.13 and set $\Sigma' := \mathbb{S} := \{\beta_1, \ldots, \beta_r\}$. We have

$$\mathfrak{g}^h = \mathfrak{g}_x^{\lambda'} \times I, \quad \text{where } I := q_2(\beta_1) \times \ldots \times q_2(\beta_r).$$

By 4.13 $N_{\lambda'}$ is a simple $\mathfrak{g}^h$-module of the highest weight $\lambda$ and is a direct summand of $N$ viewed as a $\mathfrak{g}^h$-module. Since $x \in \mathfrak{g}^h$, the latter property allows to identify $M'$ with $\text{DS}_x(N_{\lambda'})$. By above, $N_{\lambda'} \cong L_{\mathfrak{g}^h}(\lambda)$. In the light of (22), $L_{\mathfrak{g}^h}(\lambda)$ can be decomposed as

$$L_{\mathfrak{g}^h}(\lambda) \cong L_{\mathfrak{g}_x^{\lambda'}}(\lambda') \otimes L_1(\nu),$$

where $\nu$ is the restriction of $\lambda$ to $\mathfrak{h} \cap I$ and $I$ (resp., $\mathfrak{g}_x^{\lambda'}$) acts by zero on the first (resp., on the second) factor in the above decomposition. Since $x \in I$ one has $x L_{\mathfrak{g}_x^{\lambda'}}(\lambda') = 0$ and

$$\text{DS}_x(L_{\mathfrak{g}^h}(\lambda)) \cong L_{\mathfrak{g}_x^{\lambda'}}(\lambda') \otimes \text{DS}_x(L_1(\nu)).$$

For $\lambda \in \mathfrak{h}_x^*$ we have $\nu = 0$, so $L_{\mathfrak{g}^h}(\lambda) = L_{\mathfrak{g}_x^{\lambda'}}(\lambda')$; this establishes (ii).

For (iii) we denote by $\nu_{i}$ the restriction of $\lambda$ to $\mathfrak{h} \cap q_2(\beta_i)$ and write

$$L_{\nu}(\nu) \cong \bigotimes_{i=1}^r L_{q_2}(\nu_i).$$

The action of $x$ on each $L_{q_2}(\nu_i)$ coincides with the action of a non-zero $x' \in X(q_2)$ and using 5.5.2 we obtain (iii). \hfill $\Box$

5.8. The map $\theta_x$ for $q_n$. By 1.2 the functor $\text{DS}_x$ induces the algebra homomorphism $\theta_x : Z(\mathfrak{g}) \rightarrow Z(\mathfrak{g}_x)$.

By 1.6.1 each $y \in X(q_n)_r$ is of the form $y = \phi(x)$, where $\phi$ is an inner automorphism. Note that $\phi$ induces an isomorphism $\iota : \mathfrak{g}_x \sim \mathfrak{g}_y$ and that $\theta_y = \iota \circ \theta_x$. Hence we can (and will) take $x := x_r$ and use the identification of 5.7.

For $\nu \in \mathfrak{h}_x^*$ we denote by $\chi'_{\nu}$ the corresponding central character $\chi' : Z(\mathfrak{g}_x) \rightarrow \mathbb{C}$. 

5.8.1. **Corollary.** Take $x := x_r$.

(i) For $\lambda \in h_x^*$ the $\mathfrak{g}_x$-module $DS_x(L(\lambda))$ has a non-zero primitive vector of weight $\lambda$.
(ii) $(HC_x \circ \theta_x)(z) = HC(z)_{h^*_x}$, where $HC_x : Z(\mathfrak{g}_x) \to S(h_x)$ is the Harish-Chandra homomorphism for $\mathfrak{g}_x \cong \mathfrak{q}_{n-2r}$.
(iii) One has $\theta_x^*(\chi_\lambda) = \chi_\lambda$ for each $\lambda \in h_x^*$.
(iv) $\text{Core}(\theta_x^*(\chi)) = \text{Core}(\chi)$ and $\theta_x^*$ increases atypicality by rank $x$;
(v) $\theta_x^*$ is injective and $\text{Im}\theta_x^*$ consists of the central characters of atypicality at least rank $x$.

**Proof.** The assertion (i) follows from Proposition 5.7.2 and implies (ii). In its turn, (ii) implies (iii) and (iv) follows from (iii). Finally, (v) follows from Proposition 4.12.2 and (iv). □

5.8.2. **Corollary.** For each $\chi \in \text{mspec}_{\text{Mod}} Z(q_n)$ one has \[\text{atyp} \chi = \text{depth} \chi = \text{depth} O(\chi).\]

**Proof.** Set $k := \text{atyp} \chi$ and take $x := x_k$. Using (2) and Corollary 5.8.1 (v) we obtain $\text{depth} \chi \leq k$ and thus $\text{depth} O(\chi) \leq k$. Let $\text{Core}(\chi) = \{a_i\}_{i=1}^{n-2k}$. Note that $\chi = \chi_\nu$ for $\nu := \sum_{i=1}^{n-2k} a_i e_i \in h_x^*$. By Lemma 5.7.2, $DS_x(L(\nu)) \neq 0$, so $\text{depth}(L(\nu)) \geq k$. This gives $\text{depth} \chi, \text{depth} O(\chi) \geq k$ and completes the proof. □

5.8.3. **Proposition.** Take $x \in X_r$.

(i) The map $\theta_x$ is surjective;
(ii) $DS_x(\text{Mod}(\chi)) \subset \text{Mod}_x((\theta_x^*)^{-1}\chi)$, where $\text{Mod}_x$ stands for the category of $\mathfrak{g}_x$-modules.

**Proof.** By above, it is enough to consider $x := x_r$. Using notation of 5.8.1 (ii) we set $Z_n := HC(Z(q_n)) \subset S(h), \quad Z_{n-2r} := HC_x(Z(q_{n-2r})) \subset S(h_x) \subset S(h)$ and identify $Z(q_n)$ with $Z_n$ and $Z(q_{n-2r})$ with $Z_{2n-2r}$.

Consider the projection $\psi : S(h) \to S(h_x)$ given by $h_i \mapsto h_i, \quad h_j \mapsto 0$ for $1 \leq i \leq n - 2r < j \leq n$.

Using the above identification the formula in 5.8.1 (ii) takes the form $\theta_x(z) = z|_{h^*_x}$, that is $\theta_x$ coincides with the restriction of $\psi$ to $Z_n$.

By [Ser1], the algebra $Z_n$ is generated by the set $\{p_{2k+1}^{(n)}\}_{k=0}^\infty$, where
\[p_{2k+1}^{(n)} := \sum_{i=1}^{n} h_i^{2k+1}.\]
5.8.4. **Remark.** Note that, as in [5.4] below, for injectivity of $\theta_x^*$ it is enough to see that $p_{2k+1}^{(n)}$ lies in $Z_n$, whereas the proof of surjectivity is based on the fact that $Z_n$ is generated by these elements.

5.9. **DS and translation functors.** Let $\chi_1, \chi_2$ be central characters of $g$. Consider the translation functor $T_{\chi_1,\chi_2} : O(\chi_1) \to O(\chi_2)$ given by

$$T_{\chi_1,\chi_2}(M) := (M \otimes V_{st})^{\chi_2},$$

where $V_{st} = L(\varepsilon_1)$ is the standard $q_n$-module and $M \mapsto M^{\chi_2}$ stands for the projection $O \to O(\chi_2)$. Similarly to Corollary 4.4 in [S2], combining Proposition 5.8.1 (iv) and the formula $DS_x(L_g(\varepsilon_1)) = L_{g_x}(\varepsilon_1)$, we obtain the following corollary.

5.9.1. **Corollary.** Let $\chi_1, \chi_2$ (resp., $\chi_1', \chi_2'$) be the central characters of $g$ (resp., $g_x$) satisfying $\text{Core}(\chi_i) = \text{Core}(\chi_i')$ for $i = 1, 2$. Then

$$DS_x \circ T_{\chi_1,\chi_2} = T_{\chi_1',\chi_2'} \circ DS_x.$$ 

5.10. **KW-conditions.** Take $g := q_n$. We denote by $P^+(g)$ the set of dominant weights, i.e.

$$P^+(g) := \{ \lambda \in h^* | \dim L(\lambda) < \infty \}.$$

Character formulae for finite-dimensional simple modules were obtained in [PS1, PS2, Br, ChK].

By [P1], $\lambda \in P^+(g)$ if and only if $\lambda = \sum a_i \varepsilon_i$ with $a_i - a_{i+1} \in \mathbb{Z}_{\geq 0}$ and $a_i = a_{i+1}$ implies $a_i = 0$. Recall that $L(\lambda)$ is typical if $a_i + a_j \neq 0$ for any $1 \leq i < j \leq n$. For a typical central character $\chi$ there exists at most one weight $\lambda \in P^+(g)$ with $\chi_{\lambda} = \chi$: in this case $a_i - a_{i+1} \in \mathbb{Z}_{>0}$ for each $i$.

Arguing as in Corollary 5.8.2 we obtain $\text{depth} \, F(\chi) = \text{depth} \chi$ if $F(\chi) \neq 0$.

5.10.1. **Definition.** We say that $\lambda \in P^+(g)$ satisfies the **KW-conditions** if there exists an iso-set $S \subset \Pi$ which is a maximal iso-set orthogonal to $\lambda$.

5.10.2. **Proposition.** Let $\lambda = \sum a_i \varepsilon_i \in P^+(g)$ has atypicality $k$ and satisfies the KW-condition. Take $x$ of atypicality $k$.

(i) There exists a unique $\lambda' \in P^+(g_x)$ with $\text{Core} \lambda' = \text{Core} \lambda$.

(ii) If for some $s$ one has

$$a_s = a_{s+1} = \ldots = a_{s+2k-1} = 0,$$

then $DS_x(L(\lambda)) = L_{g_x}(\lambda')$. 

Since $\psi(p_{2k+1}^{(n)}) = p_{2k+1}^{(n-2r)}$, the map $\theta_x$ is surjective. Now (ii) follows from Proposition 1.2.2. $\square$
(iii) If (23) does not hold, then \( k = 1 \) and
\[
\text{DS}_x(L(\lambda)) = L_{\theta_x}(\lambda') \oplus \Pi(L_{\theta_x}(\lambda')).
\]

Proof. Take \( S \) as in [5.10.1]. By [6.6.2] we can (and will) assume that \( \text{supp}(x) = S \). Define \( g_x \) and \( h_x \) as in [5.7]. Since \( \text{rank} x = k \), \( \text{Core}(\lambda) \) has the cardinality \( n - 2k = \text{dim} h_x \); this gives (i). By Corollary 5.8.2
\[
(\theta_x^\ast)^{-1}(\lambda_x) = \chi_{\lambda_x'},
\]
where \( \chi_{\lambda_x'} \in \text{mspec} Z(g_x) \). Since \( M := \text{DS}_x(L(\lambda)) \) is finite-dimensional, all simple sub-quotients of \( M \) are of the form \( L_{\theta_x}(\lambda') \) or \( \Pi(L_{\theta_x}(\lambda')) \). Using Proposition 5.7.2 we obtain \( \lambda' = \lambda|_{h_x} \) and deduce (ii).

It is easy to see that if \( \lambda \in P^+(g) \) satisfies the KW-conditions and (23) does not hold, then for some index \( i \) one has \( a_{i+1} = -a_i \neq 0 \) and \( S = \{ \varepsilon_i - \varepsilon_{i+1} \} \). Since \( \lambda \) is dominant, \( a_i - a_{i+1} \in \mathbb{N}_{>0} \) and (iii) follows from from Proposition 5.7.2 (iii). \( \Box \)

5.11. \( \text{DS}_x \) for \( \text{sq}_n, \text{pq}_n \) and \( \text{psq}_n \). We retain notation of 5.6. Recall that \( z := T_{Id,0} \). Recall that \( \iota \) stands for the canonical maps \( q_n \to \text{pq}_n \) and \( \text{sq}_n \to \text{psq}_n \).

5.11.1. Proposition. Take \( x \in g \) with \( \text{supp}(x) = S_r \).

1. For \( r = \frac{n}{2} \) one has \( \text{DS}_x(\text{sq}_n) \cong \mathbb{C} \), \( \text{DS}_x(\text{pq}_n) \cong \Pi(\mathbb{C}) \) and \( \text{DS}_x(\text{psq}_n) \cong \mathbb{C} \times \Pi(\mathbb{C}) \).

2. For \( r > \frac{n}{2} \) the algebra \( \text{DS}_x(\text{sq}_n) \) (resp., \( \text{DS}_x(\text{pq}_n) \), \( \text{DS}_x(\text{psq}_n) \)) can be identified with \( \text{sq}_{n-2r} \) (resp., with \( \text{pq}_{n-2r}, \text{psq}_{n-2r} \)).

Proof. For (i) recall that \( \text{DS}_x(q_n) = 0 \) if \( r = \frac{n}{2} \). Since \( q_n/\text{sq}_n \cong \Pi(\mathbb{C}) \) as \( \text{sq}_n \)-modules, Hinich's Lemma gives \( \text{DS}_x(\text{sq}_n) \cong \mathbb{C} \); the case \( \text{pq}_n \) is similar. Take \( \tilde{g} := \text{psq}_2 \). Since \( g = \text{sq}_2/\mathbb{C} \), Hinich's Lemma implies that \( \text{DS}_x(g) \) is either 0 or \( \mathbb{C} \times \Pi(\mathbb{C}) \). It is easy to see that \( q_n \) contains \( y \) with \( \text{supp}(y) = -\text{supp}(x) \) and \( [y, x] = z \). Therefore \( [\iota(x), \iota(y)] = 0 \). One readily sees that \( \iota(y) \not\in [x, g] \), so \( \iota(y) \) has a non-zero image in \( \text{DS}_x(g) \). This gives (i).

For (ii) take \( r < \frac{n}{2} \) and denote by \( \tilde{q}_2 \) the copy of \( q_2 \) corresponding to the root system \( \{ \varepsilon_i - \varepsilon_j \}_{i < j, i,j \leq n} \).

Take \( g := \text{sq}_n \) and set \( \tilde{\text{sq}}_2 := \text{sq}_n \cap \tilde{q}_2 \). It is easy to see that
\[
p := \text{sq}_{n-2r} + \tilde{\text{sq}}_2 + g^b
\]
is a subalgebra of \( \text{sq}_n \). By Lemma 2.3
\[
\text{DS}_x(p) = \text{DS}_x(\sum_{\alpha \in Y} p_\alpha + p^b), \quad \text{DS}_x(g) = \text{DS}_x(\sum_{\alpha \in Y} g_\alpha + g^b),
\]
where \( Y := \{ \alpha \in \Delta | (\alpha|\beta^\vee) = 0 \} \) for \( \beta \in \text{supp}(x) \}. \) Since \( g^b = p^b \) and \( g_\alpha = p_\alpha \) for \( \alpha \in Y \), we obtain \( \text{DS}_x(p) = \text{DS}_x(g) \).

As \( \text{sq}_2 \)-module \( p \) can be decomposed as
\[
p = \text{sq}_{n-2r} \oplus (\tilde{\text{sq}}_2 + \mathbb{C}H),
\]
(24)
where \( H \) spans \( p_1^{\text{ps}} \). Note that \( x \in \sl_2 \), and that \([\sl_2, \s_{n-2}] = 0.\) Therefore
\[
\DS_x(p) = \s_{n-2} \oplus \DS_x(\sl_2 + CH).
\]

Define a linear bijection \( \sl_2 + CH \xrightarrow{\sim} \sl_2 + CT_0,Id = \sl_2 \) via the identification \( t = \s_2 \), and \( H \to T_0,Id. \) One readily sees that this map is a \( \sl_2 \)-isomorphism. One has \( \DS_x(\sl_2 + CH) = \DS_x(\sl_2) = 0 \) and so \( \DS_x(\s_n) = \DS_x(p) = \s_{n-2r} \) as required.

For the case \( pq \), we identify \( \sl_2 \) with \( \iota(\sl_2) \subset \s_{pq} \). Then \( x \in \sl_2 \) and \( \s_{pq} \oplus \s_{n-2} \) is a subalgebra of \( \s_{pq} \). Arguing as above, we obtain
\[
\DS_x(\s_{pq}) = \DS_x(\s_{pq} \oplus \s_{n-2}) = \s_{pq}.
\]

For the remaining case \( g = \ps_{pq} \), we identify \( \sl_2 \) with \( \iota(\sl_2) \subset g \) and substitute \( p \) by
\[
\ps_{pq} + \sl_2 + g^0 = \ps_{pq} \oplus (\sl_2 + CH).
\]

By above, \( \DS_x(\sl_2 + CH) = 0 \), so \( \DS_x(\ps_{pq}) = \ps_{pq} \) as required. \( \square \)

5.11.2. \textbf{Case} \( \s_{pq} \). One has \( X(\s_{pq}) = X(\s_{pq}) = X(q_n). \) By Proposition 5.11.4 we have
\[
\text{depth}(\s_{pq}) = \lfloor \frac{n}{2} \rfloor \quad \text{and} \quad X(\s_{pq}) = X(q_n). \quad \text{For } \s_{pq}, \text{ the analogue of Proposition 5.8.3 holds for } r \neq \frac{n}{2} \text{ (the proof is the same).}
\]

5.11.3. \textbf{Case} \( \ps_{pq} \). One has \( X(\ps_{pq}) = \iota(X(q_n)) \) and
\[
X(\ps_{pq}) = \iota(X(q_n)) \cup X', \quad \text{where } X' = \{\iota(T_0,B) \mid B^2 \in \mathbb{C}^*Id\}.
\]

Each element in \( X' \) is \( GL_n \)-conjugated to \( x' \) := \( T_0B \), where \( B \) is a diagonal matrix with \( B^2 \in \mathbb{C}^*Id \). It is easy to see that the algebra \( \DS_x'(\ps_{pq}) \) can identified with \( \mathbb{C}x' \cong \Pi(\mathbb{C}) \). We obtain
\[
\text{depth}(\ps_{pq}) = \lfloor \frac{n}{2} \rfloor, \quad \iota(X(q_n)) \subset X(\ps_{pq}), \quad X' \subset X(\ps_{pq})[\frac{n}{2}].
\]

Proposition 5.8.3 holds for \( r < \frac{n}{2} - 1 \) (the proof is the same).

5.11.4. \textbf{Case} \( \ps_{pq} \). One has \( X(\ps_{pq}) = X(\ps_{pq}) \cap \ps_{pq} \) and \( X(\ps_{pq}) = X(\ps_{pq}). \) This gives \( X(\ps_{2n+1}) = \iota(X(\ps_{2n+1})) \) and
\[
X(\ps_{2n}) = \iota(X(\ps_{2n})) \cup X', \quad \text{where } X' = \{\iota(T_0,B) \mid B^2 \in \mathbb{C}^*Id, TrB = 0\}.
\]

Note that \( X' \) is \( GL_n \)-conjugated to \( x' \) := \( T_0B \), where \( B \in \sl_2n \) is a diagonal matrix satisfying \( B^2 \in \mathbb{C}^*Id \). It is easy to see that the algebra \( \DS_x'(\ps_{2n}) \) can identified with \( \mathbb{C}x' \cong \mathbb{C} \times \Pi(\mathbb{C}) \). Summarizing we have
\[
\text{depth}(\ps_{pq}) = \lfloor \frac{n}{2} \rfloor, \quad \iota(X(q_n)) \subset X(\ps_{pq}), \quad X' \subset X(\ps_{pq})[\frac{n}{2}].
\]

Proposition 5.8.3 holds for \( r < \frac{n}{2} - 1 \) (the proof is the same).
6. THE CASE OF FINITE-DIMENSIONAL KAC-MOODY SUPERALGEBRAS

Let \( g \) be a finite-dimensional Kac-Moody superalgebra. We retain notation of Section 4. From [DS] it follows that

\[
X(g) = X(g)_{iso}, \quad \text{depth}(g) = \text{defect } g
\]

if \( \text{supp}(x) \) is an iso-set, then rank(\( x \)) is equal to the cardinality of \( \text{supp}(x) \).

We recall some other results of [DS] in 6.2, 6.3 below.

6.1. Choice of base. The algebra \( D(m|n) \) with \( m \neq 0 \) admits an involutive automorphism \( \sigma \) which acts on \( g_0 = \mathfrak{o}_{2m} \times \mathfrak{o}_{2m} \) as follows: \( \sigma|_{\mathfrak{sp}_{2m}} = \text{Id} \) and \( \sigma|_{\mathfrak{o}_{2m}} \) is a Dynkin diagram involution if \( m > 1 \) and \( -\text{Id} \) for \( m = 1 \) (one has \( \mathfrak{o}_2 = \mathbb{C} \)). Note that \( \sigma(\mathfrak{h}) = \mathfrak{h} \).

We denote by \( \sigma \) also the induced map on \( \mathfrak{h}^* \) (then \( \sigma = r_{\varepsilon_m} \)).

We fix a base \( \Sigma \subset \Delta \) with the following properties:

- \( \Sigma \) contains a maximal possible number of odd roots;
- \( \sigma(\Sigma) = \Sigma \) for \( D(m|n) \) if \( m > 1 \).

Then \( \Sigma \) contains an iso-set of the maximal possible cardinality (equal to \( \text{defect } g \)).

For instance, for \( D(2|2) = \mathfrak{osp}(4|4) \) we may take \( \Sigma = \{ \delta_1 - \varepsilon_1, \varepsilon_1 - \delta_2, \delta_2 \pm \varepsilon_2 \} \).

6.2. The algebra \( g_x \). Let \( S \) be an iso-set of cardinality \( r > 0 \) satisfying

\[
S \subset \Sigma \cup (-\Sigma).
\]

We fix \( x \in X \) such that \( \text{supp}(x) = S \).

6.2.1. One has \( h^x = \{ h \in \mathfrak{h} \mid S(h) = 0 \} \). We introduce

\[
\Delta_x := (S^+ \cap \Delta) \setminus (-S \cup S).
\]

By [DS], \( g_x \) can be identified with a subalgebra of \( g \) generated by the root spaces \( g_\alpha \) with \( \alpha \in \Delta_x \) and a subalgebra \( \mathfrak{h}_x \subset \mathfrak{h}^x \) with the following properties

\[
\mathfrak{h}_x \oplus \left( \sum_{\beta \in S} \mathbb{C}h_\beta \right) = \mathfrak{h}^x; \quad \forall \alpha \in \Delta_x \quad [g_\alpha, g_\alpha] \subset \mathfrak{h}_x.
\]

Moreover, \( \mathfrak{h}_x \) is a Cartan subalgebra of \( g_x \).

If \( \Delta_x \) is not empty, then \( \Delta_x \) is the root system of the Lie superalgebra \( g_x \) and one can choose a base \( \Sigma_x \) in \( \Delta_x \) such that

\[
\Delta^+(\Sigma_x) = \Delta^+ \cap \Delta_x.
\]

(If \( \Delta_x = \emptyset \) we take \( \Sigma_x = \emptyset \).) If \( g = \mathfrak{gl}(m|n) \) (resp., \( \mathfrak{osp}(m|n) \)), then \( g_x = \mathfrak{gl}(m-r|n-r) \) (resp., \( \mathfrak{osp}(m-2r|n-2r) \)). For \( g = D(2|1; a), G_3, F_4 \) with \( x \neq 0 \), one has \( r = 1 \) and \( g_x = \mathbb{C}, \mathfrak{sl}_2, \mathfrak{sl}_3 \) respectively.
6.2.2. Dual Coxeter number. The restriction of the non-degenerate invariant bilinear form on \( g \) gives a non-degenerate invariant bilinear form on \( g_x \).

Recall that the dual Coxeter number \( h^\vee(g) \) is the eigenvalue of the Casimir operator on the adjoint representation. By [HW], \( \theta \) maps the Casimir operator of \( g \) to the Casimir operator of \( g_x \). Therefore

\[ h^\vee(g) = h^\vee(g_x). \]

6.2.3. We identify \( h^*_x \) with a subspace in \( h^*_x \): we take \( h^*_x \) spanned by \( \Delta_x \) if \( g_x \neq \mathbb{C} \) and \( g \neq \mathfrak{gl}(m|n) \); for \( g = \mathfrak{gl}(m|n) \) we take the minimal span of \( \epsilon_i \)s and \( \delta_j \)s which contains \( \Delta_x \); for \( g_x = \mathbb{C} \) we choose an arbitrary \( h^*_x \) with the property \( S^\perp = CS \oplus h^*_x \). We set

\[ W'' := \{ w \in W \mid w(-S \cup S) = (-S \cup S) \}. \]

Then \( W''h^*_x = h^*_x \) and \( W''\Delta_x = \Delta_x \). Note that \( W'' \) contains \( W(g_x) \) (the Weyl group of \( g_x \)) which is generated by \( r_\alpha \) with \( \alpha \in (\Delta_x \cap \Delta_0) \). Viewing \( W'' \) as a subgroup of \( GL(h^*_x) \) we obtain \( W'' = W(g_x) \cup W(g_x)\sigma_x \), where \( \sigma_x \) is as follows:

- for \( g = D(m|n) \) with \( m > r \) we have \( g_x = D(m-r|n-r) \) and \( \sigma_x \) is as above;
- for \( g = F(4) \) \( \sigma_x \) is the involution of the Dynkin diagram of \( g_x = \mathfrak{sl}_3 \);
- for \( g = D(2|1;a) \) we have \( g_x = \mathbb{C} \) and \( \sigma_x := -Id; \)
- \( \sigma_x := Id \) for all other cases.

In all cases \( \sigma_x(\Sigma_x) = \Sigma_x \).

6.3. The map \( \theta^*_x \). Consider the usual \( \rho \)-twisted action of \( W \) on \( h^* \):

\[ w\lambda := w(\lambda + \rho) - \rho. \]

The restriction of the Harish-Chandra map gives an algebra monomorphism

\[ HC : Z(g) \to \mathcal{S}(h)^W. \]

Using Lemma 2.3 and 4.13 it is easy to see that

\[ (27) \quad HC_x(\theta_x(z)) = HC(z)|_{h^*_x}. \]

6.3.1. Take \( \lambda \in h^*_x \subset h^* \). Let \( \chi'_x \) be the corresponding central character of \( g_x \). By above, \( \theta_x^*(\chi'_x) = \chi_x \). As in Corollary 5.8.1 this implies that \( \theta_x^* \) preserves the cores for non-exceptional \( g \) and that \( Im\theta^*_x \) consists of the central characters of atypicality at least rank \( x \). Recall that \( DS_x(\mathcal{M}od(\chi)) = 0 \) if \( \chi \not\in Im \theta^*_x \), see Proposition 1.2.2. Arguing as in Corollary 5.8.2 we obtain

\[ \text{atyp } \chi = \text{depth } \mathcal{O}(\chi) = \text{depth } \chi \]

and \( \text{depth } \mathcal{F}in(g)(\chi) = \text{depth } \chi \) if \( \mathcal{F}in(g)(\chi) \neq 0 \).
6.3.2. Since $\HC(Z(g)) \subset S(h)^W$, and $W''h_x^* = h_x^*$ we have

$$\HC_x(\theta_x(Z(g)) \subset S(h_x)^{W''}.$$ 

Take $\sigma_x$ as in 6.2.3. Since $\sigma_x(\Sigma_x) = \Sigma_x$, the projection $\HC_x$ commutes with $\sigma_x$. It is not hard to see that the Weyl vector of $g_x$ is equal to $\rho|_{h_x}$, so the dot action of $W(g_x)$ on $h_x^*$ is the restriction of the dot action of $W$. Moreover, $\sigma_x \rho_x = \rho_x$, so $\sigma_x \lambda = \sigma_x \lambda$ for $\lambda \in h_x^*$. This gives

$$\text{(28)} \quad \HC_x(\theta_x(Z(g)) \subset S(h_x)^{\sigma_x}, \quad \theta_x(Z(g)) \subset Z(g_x)^{\sigma_x}.$$ 

6.4. **Theorem.** Take $x \in X(g)_r$ such that $\text{supp}(x) \subset \Sigma \cup (-\Sigma)$ is an iso-set.

(i) For each $y \in X(g)_r$ there exists $\iota_{x,y} : g_x \xrightarrow{\sim} g_y$ such that $\theta_y = \theta_x \circ \iota_{x,y}$.

(ii) For $r \neq 0$ we have $\theta_x(Z(g)) = Z(g_x)^{\sigma_x}$.

Proof. For (i) note that each $\phi \in \text{Aut}(g)$ induces the required isomorphism for $y := \phi(x)$. By Corollary 4.7.2 we can assume that $\text{supp}(y) \subset (-\Sigma \cup S)$. Then, by 6.2 we can identify $g_x$ with $g_y$ and $h_x^*$ with $h_y^*$. Now the assertion (i) follows from (27).

For (ii) observe that, by (28), $\theta_x(Z(g)) \subset Z(g_x)^{\sigma_x}$. For the opposite inclusion $\supset$ we retain notation of 1.12 and identify $Z(g)$ with $Z(g) \subset S(h)^W$. We denote by $\theta'$ the corresponding map $Z(g) \to Z(g_x)$ and denote by $\psi : S(h) \to S(h_x)$ the map

$$\psi(f) := f|_{h_x^*}.$$ 

Since the Weyl vector of $g_x$ is equal to $\rho|_{h_x}$, the formula (27) implies that $\theta'$ coincides with the restriction of $\psi$ to $Z(g)$. Thus $\supset$ can be rewritten as

$$\text{(29)} \quad \psi(Z(g)) \supset Z(g_x)^{\sigma_x}.$$ 

For the cases $\mathfrak{gl}(m|n)$, $\mathfrak{osp}(2m|2n)$, $\mathfrak{osp}(2m+1|2n)$ let $\{e_i\}_{i=1}^m \cup \{\delta_i\}_{i=1}^n$ be the standard basis of $h^*$ and let $B_{m|n} := \{e_i\}_{i=1}^m \cup \{d_i\}_{i=1}^n$ be the dual basis of $h$. By (i) we can assume that $\text{supp}(x) = \{\delta_{m+1-i} - \varepsilon_{m+1-i}\}_{i=1}^r$. Then $B_{m-r'n-r}$ is a basis of $h_x$ and $\psi$ is the projection $h \to h_x$ given by

$$\psi(a) = a \text{ if } a \in B_{m'|n'}, \quad \psi(a) = 0 \text{ if } a \in (B_{m|n} \setminus B_{m'|n'})$$

where $m' := m - r$ and $n' := n - r$. Consider the polynomials

$$p_k^{(m|n)} := \sum_{i=1}^m e_i^k - \sum_{i=1}^n d_i^k, \quad k \in \mathbb{N}.$$ 

By [Ser2], the algebra $Z(\mathfrak{gl}(m|n))$ is generated by $\{p_k^{(m|n)}\}_{k=1}^\infty$ and $Z(\mathfrak{osp}(2m+1|2n))$ is generated by $\{p_k^{(m|n)}\}_{k=1}^\infty$. Since $\psi(p_k^{(m|n)}) = p_k^{(m'|n')}$, this gives $\psi(Z(g)) = Z(g_x)$ for the cases $\mathfrak{gl}(m|n)$, $\mathfrak{osp}(2m+1|2n)$ and establishes (ii) for these cases. For the case $\mathfrak{osp}(2m|2n)$ one has

$$W(\mathfrak{osp}(2m+1|2n)) = W(\mathfrak{osp}(2m|2n)) \prod W(\mathfrak{osp}(2m|2))\sigma.$$
Taking $\beta = \varepsilon_1 - \delta_1$ in the formula (18), we obtain

$$Z(\mathfrak{osp}(2m|2n))^\sigma = Z(\mathfrak{osp}(2m + 1|2n)).$$

In particular, $\psi(Z(\mathfrak{osp}(2m|2n)))$ contains

$$\psi(Z(\mathfrak{osp}(2m + 1|2n))) = Z(\mathfrak{osp}(2m' + 1|2n')) = Z(\mathfrak{osp}(2m'|2n'))^{\sigma*},$$

that is $\psi(Z(\mathfrak{osp}(2m|2n))) \supset Z(\mathfrak{osp}(2m'|2n'))^{\sigma*}$. This establishes (29) for $\mathfrak{osp}(2m|2n)$.

The remaining cases are exceptional Lie superalgebras. In this case rank $x = 1$. We view $S(\mathfrak{h})$ as a polynomial algebra. The algebra $Z(\mathfrak{g})$ contains a homogeneous polynomial $L_2$ of degree 2 (the Casimir element) and $\psi(L_2)$ is a polynomial of degree 2 in $S(\mathfrak{h}_x)$.

For $\mathfrak{g} = G(3)$ one has $\mathfrak{g}_x \cong \mathfrak{sl}_2$, $\sigma_x = \text{Id}$ and for $\mathfrak{g} = D(2,1|a)$ one has $\mathfrak{g}_x = \mathbb{C}z$, $\sigma_x = -\text{Id}$. In both cases the algebra $Z(\mathfrak{g}_x)^{\sigma_x}$ is generated by a non-zero polynomial of degree 2 (the Casimir element and $z^2$ respectively); this gives (29).

Consider the remaining case $\mathfrak{g} = F(4)$. In this case $\{\varepsilon_i\}_{i=1}^3 \cup \{d_i\}$ form a basis of $\mathfrak{h}^*$ and we denote by $\{e_i\}_{i=1}^3 \cup \{d_i\}$ the dual basis of $\mathfrak{h}$. We choose

$$\Sigma := \{\beta, \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_3\}, \quad \beta := \frac{1}{2}(\delta_1 - \sum_{i=1}^3 \varepsilon_i).$$

Take $x$ such that $\text{supp}(x) = \{\beta\}$. Then $\mathfrak{g}_x \cong \mathfrak{sl}_3$ with $\mathfrak{h}_x$ spanned by $e_1 - e_2, e_2 - e_3$ and $\Sigma_x = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3\}$. The map $\psi$ is given by

$$\psi(d_1) = 0, \quad \psi(e_1 + e_2 + e_3) = 0, \quad \psi(e_1 - e_2) = e_1 - e_2, \quad \psi(e_2 - e_3) = e_2 - e_3.$$

The involution $\sigma_x$ permutes $e_1 - e_2$ with $e_2 - e_3$ and can be extended to the span of $e_1, e_2, e_3$ by $\sigma_x(e_1) = -e_3, \sigma_x(e_2) = -e_2$. Recall that $Z(\mathfrak{g}_3)$ is generated by the symmetric polynomials $p_s := \sum_{i=1}^3 \varepsilon_i^s$. One has $\sigma_x(p_s) = (-1)^s p_s$. Identifying $Z(\mathfrak{sl}_3)$ with $\mathbb{C}[p_2, p_3]$ we obtain

$$Z(\mathfrak{sl}_3)^{\sigma_x} = \mathbb{C}[p_2, p_3^2] = \mathbb{C}[p_2, p_6].$$

By [Ser2], $Z(\mathfrak{g})$ contains $L_2$ (as above) and $L_6$ satisfying $\psi(L_2) = p_2, \psi(L_6) = p_6$. This establishes (29) and completes the proof. \qed

6.5. **Corollary.** Fix $x \in X(\mathfrak{g})_r$ and denote by $\mathcal{Mod}_x$ the full category of $\mathfrak{g}_x$-modules. Take $\chi \in \text{Im} \theta_x^*$ and $\chi' \in (\theta_x^*)^{-1}(\chi)$.

(i) If $\sigma_x = \text{Id}$, then $DS_x(\mathcal{Mod}(\chi)) \subset \mathcal{Mod}_x(\chi')$.
(ii) If $\sigma_x(\chi') \neq \chi'$, then $DS_x(\mathcal{Mod}(\chi)) \subset \mathcal{Mod}_x(\chi') + \mathcal{Mod}_x(\sigma_x(\chi'))$.
(iii) If $\sigma_x \neq \text{Id}$ and $\sigma_x(\chi') = \chi'$, then $(\ker \chi')^2 DS_x(N) = 0$ for each $N \in \mathcal{Mod}(\chi)$.

**Proof.** Take $N \in \mathcal{Mod}(\chi)$ and set $N' := DS_x(N)$. Set $m := \ker \chi$ and $m' := \ker \chi'$. Then $\theta_x(m) \subset \text{Ann}_{\mathcal{Z}(\mathfrak{g}_x)} N'$.
If \( \theta_x \) is surjective, then \( \theta_x(m) = m' \). This gives (i). Consider the remaining case when \( \theta_x \) is not surjective. By Theorem 6.4, \( \theta_x(m) \) is a maximal ideal in \( \mathbb{Z}(g_x)^{\sigma_x} \). Set \( A := \mathbb{Z}(g_x)/\theta_x(m) \). By above, \( N' \) is an \( A \)-module and \( A \neq \mathbb{C} \).

The algebra \( A \) inherits the action of \( \sigma_x \) and \( A^{\sigma_x} = \mathbb{Z}(g_x)^{\sigma_x}/\theta_x(m) \cong \mathbb{C} \), so
\[
A = \mathbb{C} \oplus A_- \quad \text{where} \quad A_- := \{ a \in A \mid \sigma_x(a) = -a \} \neq 0.
\]

If \( A_- = \mathbb{C}a \) with \( a^2 = 1 \), then \( N' = N'_+ + N'_- \), where \( N'_+ = \{ v \in N' \mid av = \pm v \} \) and \( m', \sigma(m') \) are the preimages of the ideals \( \mathbb{C}(a - 1), \mathbb{C}(a + 1) \) in \( \mathbb{Z}(g_x) \). This gives (ii).

Assume that \( A_- \neq \mathbb{C}a \) with \( a^2 = 1 \). For any \( a_1, a_2 \in A_- \) one has \( a_1a_2 \in A^{\sigma_x} = \mathbb{C} \), so \( a_1a_2 = 0 \). Then \( A_- \) is the unique maximal ideal in \( A \) and thus \( m' = \sigma_x(m') \) is the preimage of \( A_- \). This gives (iii). \( \Box \)

7. The algebra \( g_x \) in the affine case

Let \( g \) be an indecomposable symmetrizable affine superalgebra or \( \mathfrak{gl}(m|n)^{(1)} \). We retain notation of 4.1.2. The dual Coxeter number for \( g \) is given by
\[
h^\vee(g) := (\rho|\delta).
\]
Note that \( h^\vee(g) \) depends on the choice of \( (-|-) \). It is easy to see that \( h^\vee(g) \) does not depend on the choice of \( \rho \); in the light of 4.6, \( h^\vee(g) \) does not depend on the triangular decomposition of \( g \).

Recall that \( \hat{\Sigma} \) is a finite part of \( \Sigma \) and \( d \in \mathfrak{h} \) satisfies \( \delta(d) = 1 \) and \( \alpha(d) = 0 \) for \( \alpha \in \hat{\Sigma} \). One has
\[
\hat{g}^d = \hat{g} \times \mathbb{C}K \times \mathbb{C}d, \quad \mathfrak{h} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}d
\]
where \( K \) is a central element of \( g \), \( \hat{g} \) is a finite-dimensional Kac-Moody superalgebra or \( \mathfrak{gl}(m|n) \), and \( \mathfrak{h} \) is the Cartan subalgebra of \( \hat{g} \).

We consider the triangular decomposition of \( \hat{g} \) which is induced by the triangular decomposition of \( g \) (then \( \hat{\Sigma} \) is the base of \( \Delta^+ \)). We set
\[
\hat{\rho} = \frac{1}{2} \sum_{\alpha \in \Delta^+} (-1)^{p(\alpha)} \alpha
\]
and fix \( \rho \in \mathfrak{h}^* \) such that \( \rho_{\mathfrak{h}} = \hat{\rho}|_{\mathfrak{h}} \) (this condition holds for any choice of \( \rho \) if \( g \neq \mathfrak{gl}(m|n) \)).

7.1. The algebra \( g_x \) for \( x \in X_{iso} \). By contrast with the finite-dimensional case, \( g_x \) is not always Kac-Moody, see the example 7.2 below. The following proposition shows that \( g_x \) is Kac-Moody for \( x \in X_{iso} \).
7.1.1. Proposition. Take \( x \in X_{\text{iso}} \) (see 2.1.3 for the notation) and let \( r \) be the cardinality of \( \text{supp}(x) \).

(i) One has \( \text{DS}_x(\hat{\mathfrak{g}}^{(1)}) \cong \hat{\mathfrak{g}}^{(1)}_x \) and \( \text{DS}_x(H(M|N)^{(i)}) = H(M - 2r|N - 2r)^{(i)} \) for \( H(M|N)^{(i)} = A(M|2n - 1)^{(2)} \), \( A(2m|2n)^{(4)} \), \( D(M + 1|N)^{(2)} \).

(ii) One has depth \( \mathfrak{g} = \text{defect } \mathfrak{g} \) and rank \( x = r \).

Proof. By Corollary 4.7.2 we can assume that \( x \in X(\hat{\mathfrak{g}})_r \). Now (i) follows from the fact that \( \text{DS}_x \) maps the adjoint (resp., the standard) \( \hat{\mathfrak{g}} \)-module to the adjoint (resp., standard) \( \hat{\mathfrak{g}}_x \)-module; (ii) follows from (i). \( \square \)

7.1.2. Take \( x \in X(\hat{\mathfrak{g}})_r \). Using 6.2 we view \( \mathfrak{g}_x \) as a subalgebra of \( \mathfrak{g} \) which contains \( \hat{\mathfrak{g}}_x \), \( K \) and \( d \); one has \( \mathfrak{g}_x^d = \hat{\mathfrak{g}}_x \times \mathbb{C}K \times \mathbb{C}d \). The restriction of the non-degenerate invariant bilinear form \((-|-)\) on \( \mathfrak{g} \) gives a non-degenerate invariant bilinear form on \( \mathfrak{g}_x \).

7.1.3. A \( \mathfrak{g} \)-module \( N \) is called restricted if for each \( v \in N \) \( \mathfrak{g}_a v = 0 \) for almost all positive roots \( \alpha \). By [K3], Ch. 2 the bilinear form \((-|-)\) gives rise to the Casimir operator which acts on restricted \( \mathfrak{g} \)-modules by \( \mathfrak{g} \)-endomorphisms; in particular, this operator acts on \( M(\lambda) \) by \( (\lambda + 2\rho|\lambda) \text{Id} \).

The above definition of a restricted module can be reformulated as follows: for each \( v \in N \) there exists \( j \in \mathbb{N} \) such that \( \mathfrak{g}_a v = 0 \) if \( a(d) > j \). From this reformulation it follows that \( \text{DS}_x(N) \) is a restricted \( \mathfrak{g}_x \)-module if \( N \) is a restricted \( \mathfrak{g} \)-module.

7.1.4. Proposition. Take \( x \in X(\hat{\mathfrak{g}}) \).

(i) If the Casimir element of \( \mathfrak{g} \) acts on a restricted \( \mathfrak{g} \)-module \( N \) by \( c \text{Id} \) (where \( c \in \mathbb{C} \)), then the Casimir element of \( \mathfrak{g}_x \) acts on \( \text{DS}_x(N) \) by \( c \text{Id} \).

(ii) If \( \mathfrak{g}_x \neq 0, \mathbb{C} \), then \( h^\wedge(\mathfrak{g}) = h^\wedge(\mathfrak{g}_x) \).

Proof. In the light of Corollary 4.7.2 it is enough to consider the case \( \text{supp}(x) \subset (\Sigma \cup \check{\Sigma}) \). For such \( x \) (i) is established in [GS1]. For (ii) consider the module \( L(\delta) \). This module is one-dimensional, so as the vector space \( \text{DS}_x(L(\delta)) \) is equal to \( L(\delta) \). Since \( d \) acts on \( L(\delta) \) by \( \text{Id} \), we obtain \( \text{DS}_x(L(\delta)) = L_{\mathfrak{g}_x}(\delta_x) \), where \( \delta_x \) is the minimal imaginary root of \( \mathfrak{g}_x \). Using (i) we get \( (\delta + 2\rho|\delta) = (\delta_x + 2\rho_x|\delta_x) \) which gives \( (\rho|\delta) = (\rho_x|\delta_x) \) as required. \( \square \)

7.1.5. Remark. Using the notation of 4.14 and normalizing the form by the condition \( (\varepsilon_1|\varepsilon_1) = 1 \) we have

| \( \text{osp}(M|N)^{(1)} \) | \( A(2m|2n)^{(4)} \) | \( D(2|1;\alpha)^{(1)} \) | \( G(3)^{(1)} \) | \( F(4)^{(1)} \) |
|----------------|-----------------|-----------------|-----------------|-----------------|
| \( M - N - 2 \) | \( m - n \) | 0 | 2 | 3 |

and \( h^\wedge(\mathfrak{g}) = M - N \) for \( \mathfrak{g} = A(M|N)^{(1)} \), \( A(M|2n - 1)^{(2)} \), \( D(M + 1|N)^{(2)} \).
7.2. Examples. Consider the case when $g = \mathfrak{sl}(2|1)^{(1)}$. For each real root $\alpha$ fix a root vector $e_\alpha \in g_\alpha$. Set $y := e_{\varepsilon_1 - \varepsilon_2}$ and $h := [\varepsilon_{\delta_1 - \varepsilon_1}, e_{\varepsilon_1 - \delta_1}]$.

One has $\text{DS}_y(g) \cong \mathbb{C}K \times \mathbb{C}d = \mathfrak{sl}^{(1)}_1$.

One has $\text{DS}_{y + pt}(g) \cong \mathbb{C}K \times t$, where $t$ is $(2|1)$-dimensional superalgebra with a basis $h, e := e_{\varepsilon_1 - \varepsilon_2}, F := e_{\varepsilon_2 - \delta_1}$. Thus $t$ has the relations
\[ [e, F] = 0, \quad [h, e] = e, \quad [h, F] = -F. \]

The algebra $\text{DS}_{y + pt + y^2}(g)$ is spanned by the images of $K, y, h, ht^{-1}, e, et, F, Ft$.

7.3. Induced modules. We consider the functor $\text{Ind} : g^d - \text{Mod} \to g - \text{Mod}$ which is given by the following construction: we set $g_{>0} := \{ g \in g | [d, g] = ig, \ i > 0 \}$, $\text{Ind}(V) := \text{Ind}_{g^d + g_{>0}} V$,

where a $g^d$-module $V$ is viewed as a $(g^d + g_{>0})$-module with the zero action of $g_{>0}$.

For $x \in X(\mathfrak{g})$ we introduce similarly the functor $\text{Ind}_x : g^d_x - \text{Mod} \to g_x - \text{Mod}$. We will use the following result.

7.3.1. Lemma. Take $x \in X(\mathfrak{g})$. Let $V$ be a $g^d$-module, where $d$ acts diagonally. Then
\[ \text{DS}_x(\text{Ind}(V)) = \text{Ind}_x(\text{DS}_x(V)). \]

Proof. Set $m := \{ g \in g | [d, g] = ig, \ i < 0 \}$ and note that $m$ is ad $g^d$-invariant. One has $m_x = \{ g \in g_x | [d, g] = ig, \ i < 0 \} = \text{DS}_x(m)$.

The embedding $m \to \mathcal{U}(m)$ induces the map $m_x \to \text{DS}_x(\mathcal{U}(m))$ which gives the canonical map
\[ \phi : \mathcal{U}(m_x) \to \text{DS}_x(\mathcal{U}(m)). \]

Let us show that $\phi$ is bijective. As in [DS], this follows from the existence of the following commutative diagram
\[
\begin{array}{ccc}
\mathcal{U}(m_x) & \xrightarrow{\phi} & \text{DS}_x(\mathcal{U}(m)) \\
\downarrow^{\text{sym}_x} & & \downarrow^{\text{sym}'} \\
S(m_x) & \xrightarrow{\phi'} & \text{DS}_x(S(m))
\end{array}
\]

The map $\text{sym}_x$ is the usual symmetrization map. The map $\text{sym'}$ is induced by the symmetrization map $\text{sym} : \mathcal{U}(m) \xrightarrow{\phi} S(m)$, which is a bijection of $g^d$-modules. The map $\phi'$ is the natural map. By [DS], $\text{DS}_x$ is a tensor functor and $\phi'$ is bijective. Since $\text{sym}, \text{sym}'$ are also bijective, $\phi$ is bijective.
Now let us prove the formula (30). We can assume that \( d \) acts on \( V \) by \( a \text{Id}_V \) \((a \in \mathbb{C})\). Then the \( d \)-eigenvalues of \( \text{Ind}(V) \) are of the form \( a - N \) and \( V \) coincides with the \( a \)th eigenspace. Therefore the \( d \)-eigenvalues of \( \text{DS}_x(\text{Ind}(V)) \) are of the form \( a - N \) and the \( a \)th eigenspace coincides with \( \text{DS}_x(V) \). This gives a \( \mathfrak{g}_x \)-homomorphism
\[
\iota : \text{Ind}_x(\text{DS}_x(V)) \to \text{DS}_x(\text{Ind}(V)).
\]
Since \( \text{DS}_x \) is a tensor functor we have
\[
\text{DS}_x(\text{Ind}(V)) = \text{DS}_x(\mathcal{U}(\mathfrak{m}) \otimes V) = \text{DS}_x(\mathcal{U}(\mathfrak{m})) \otimes \text{DS}_x(V) = \mathcal{U}(\mathfrak{m}) \otimes \text{DS}_x(V).
\]
Since \( \text{Ind}_x(\text{DS}_x(V)) = \mathcal{U}(\mathfrak{m}_x) \otimes \text{DS}_x(V) \), the map \( \iota \) is bijective. \( \square \)

8. The category \( \mathcal{O}^{inf}_h(\mathfrak{g}) \)

Let \( \mathfrak{g} \) be a symmetrizable indecomposable Kac-Moody superalgebra with a base \( \Sigma \). We retain notation of 4.2. In this section we consider a certain subcategory of \( \mathcal{O}^{inf}(\mathfrak{g}) \) which contains the BGG category \( \mathcal{O}(\mathfrak{g}) \). We will use this category in the proof of Theorem 9.1.

8.1. Definition. We fix \( h \in \mathfrak{h} \) with the following properties
\[
(31) \quad \alpha(h) \in \mathbb{N} \text{ for all } \alpha \in \Sigma; \quad \{ \alpha \in \Sigma \mid \alpha(h) = 0 \} \text{ is an iso-set.}
\]

For each \( \mathfrak{g} \)-module \( N \) we denote by \( \Omega_h(N) \) the set of \( h \)-eigenvalues in \( N \):
\[
\Omega_h(N) := \{ b \in \mathbb{C} \mid \exists v \in N \setminus \{ 0 \} \text{ } hv = bv \}.
\]
We denote by \( \mathcal{O}^{inf}_h(\mathfrak{g}) \) the full subcategory of \( \mathcal{O}^{inf}(\mathfrak{g}) \) with the modules \( N \) satisfying the following properties: all \( h \)-eigenspaces are finite-dimensional and there exists a finite set \( \{ c_i \}_{i=1}^s \subset \mathbb{C} \) such that
\[
N = \bigoplus_{i=1}^s N_i, \quad \Omega_h(N_i) \subset c_i - \mathbb{N}.
\]
Note that \( N \) has finite-dimensional weight spaces.

8.2. Properties of \( \mathcal{O}^{inf}_h(\mathfrak{g}) \). Clearly, \( \mathcal{O}^{inf}_h(\mathfrak{g}) \) is a “locally small subcategory”, i.e. for each exact sequence
\[
0 \to A \to B \to C \to 0
\]
one has \( B \in \mathcal{O}^{inf}_h(\mathfrak{g}) \) if and only if \( A, C \in \mathcal{O}^{inf}_h(\mathfrak{g}) \).

8.2.1. We will show that the category \( \mathcal{O}^{inf}_h(\mathfrak{g}) \) has the following properties:

— The BGG category \( \mathcal{O}(\mathfrak{g}) \) lies in \( \mathcal{O}^{inf}_h(\mathfrak{g}) \).
— The modules in \( \mathcal{O}^{inf}_h(\mathfrak{g}) \) have “local composition series” introduced in Proposition 8.4.3 (this statement is a modification of Prop. 3.2 in [DGK]).
8.2.2. The existence of the "local composition series" for \( N \in O^{inf}(g) \) allows to define the multiplicities \([N : L(\lambda)]\) using the formalism of Section 3 in [DGK]. One has
\[
\text{ch } N = \sum_{\lambda} [N : L(\lambda)] \text{ch } L(\lambda), \quad [N : L(\lambda)] \in \mathbb{N}.
\]

8.3. Applications to DS-functor. Let \( x \in X(g) \) be such that \( \text{supp}(x) \subset \Sigma \cup (-\Sigma) \). It is easy to see that \( S := \text{supp}(x) \) is an iso-set. By [6.27.31] \( g_x := DS_x(g) \) is a Kac-Moody superalgebra which can be viewed as a subalgebra of \( g \). The Cartan subalgebra \( h_x \) lie in \( h^x \); the root system \( \Delta_x \subset \Delta \) is given by the same formula as in the finite-dimensional case described in [6.2].

We consider the triangular decomposition of \( g_x \) which is induced by the triangular decomposition of \( g \). Clearly, \( DS_x(O^{inf}(g)) \subset O^{inf}(g_x) \).

8.3.1. Fix \( h \) satisfying (31) and such that \([h, x] = 0\). (For instance, take \( h \) with \( \alpha(h) = 0 \) for \( \alpha \in S \) and \( \alpha(h) = 1 \) for \( \alpha \in \Sigma \setminus (-S \cup S) \)). We denote by \( h_x \) the image of \( h \) in \( h_x \).

Clearly, \( \alpha(h_x) \in \mathbb{N}_{\geq 0} \) for each \( \alpha \in \Delta_x^+ \) and
\[
\{ \alpha \in \Delta_x^+ \mid \alpha(h_x) = 0 \} \subset \{ \alpha \in \Delta^+ \mid \alpha(h) = 0 \}
\]
is an iso-set. Hence \( h_x \in h_x \) satisfies (31).

8.3.2. Take \( N \in O^{inf}_h(g) \). Since the \( h_x \)-eigenspaces of \( DS_x(N) \) are the images of the \( h \)-eigenspaces of \( N \) (with the same eigenvalues), we have
\[
(32) \quad DS_x(O^{inf}_h(g)) \subset O^{inf}_{h_x}(g_x)
\]
and, in particular, \( DS_x(O(g)) \subset O^{inf}_{h_x}(g_x) \).

8.3.3. Take \( N \in O^{inf}_h(g) \). By above, the multiplicities \([N : L(\lambda)]\) and \([DS_x(N) : L_{g_x}(\nu)]\) are well-defined (where \( L_{g_x}(\nu) \) stands for the corresponding simple \( g_x \)-module). In 8.5 we will prove the following useful formula
\[
(33) \quad \sum_{\lambda \in h^*} [N : L(\lambda)] \cdot [DS_x(L(\lambda)) : L_{g_x}(\nu)] \in [DS_x(N) : L_{g_x}(\nu)] + 2\mathbb{N}.
\]
(In particular, we will show that the left-hand side is finite).

8.4. Proof of the properties We start from the following lemma.
8.4.1. Lemma. For each \( j \in \mathbb{N} \) the set
\[
\Delta(j) := \{ \alpha \in \Delta^+| \alpha(h) = j \}
\]
is finite.

Proof. Recall that \( S := \{ \beta \in \Delta^+| \beta(h) = 0 \} = \{ \beta \in \Sigma| \beta(h) = 0 \} \) is an iso-set and write \( S =: \{ \beta_i \}_{i=1}^r \). By [S3], for each \( i = 1, \ldots, r - 1 \) the root \( \beta_{i+1} \) is a simple root for \( r_{\beta_i} \ldots r_{\beta_1}(\Delta^+) \) (see 4.6 for notation), so \( r_{\beta_i} \ldots r_{\beta_1}(\Delta^+) \) is well-defined. Let \( \Sigma' \) be the base for \( r_{\beta_i} \ldots r_{\beta_1}(\Delta^+) \) and \( h' \in \mathfrak{h} \) be such that \( \alpha(h') = 1 \) for each \( \alpha \in \Sigma' \). By [S3], one has \( -S \subset \Sigma' \), so \( \beta(h') = -1 \) for \( \beta \in S \). Set
\[
a := \max\{ \alpha(h')| \alpha \in \Sigma \setminus S \}.
\]
One has \( \Delta(0) = S \). Take \( j > 0 \). By [4.6]
\[
r_{\beta_r} \ldots r_{\beta_1}(\Delta^+) = (\Delta^+ \setminus S) \cup (-S),
\]
so this set contains \( \Delta(j) \). Write \( \gamma \in \Delta(j) \) in the form \( \gamma = \sum_{\alpha \in \Sigma} m_\alpha \alpha \). One has \( m_\alpha \in \mathbb{N} \) and \( \sum_{\alpha \in \Sigma \setminus S} m_\alpha \leq j \). Since \( \gamma \in r_{\beta_r} \ldots r_{\beta_1}(\Delta^+) \) we have
\[
0 < \gamma(h') = \sum_{\alpha \in \Sigma \setminus S} m_\alpha \alpha(h') - \sum_{\beta \in S} m_\beta,
\]
so \( \sum_{\beta \in S} m_\beta < a_j \). Hence \( \Delta(j) \) lies in the set
\[
\{ \sum_{\alpha \in \Sigma} m_\alpha \alpha \text{ with } m_\alpha \in \mathbb{N} \text{ for } \alpha \in \Sigma \text{ and } \sum_{\alpha \in \Sigma} m_\alpha \leq j(a+1) \}
\]
which is finite. \( \square \)

8.4.2. Now let us prove the inclusion
\[
(34) \quad \mathcal{O}(\mathfrak{g}) \subset \mathcal{O}_{h}^{inf}(\mathfrak{g}).
\]

First, let us show that \( \mathcal{O}_{h}^{inf}(\mathfrak{g}) \) contains all Verma modules. It is enough to show that all \( h \)-eigenspaces in \( \mathcal{U}(\mathfrak{n}^-) \) are finite-dimensional. Recall that \( \mathcal{U}(\mathfrak{n}^-) \cong \mathcal{S}(\mathfrak{n}^-) \) as a \( \mathfrak{h} \)-module. Fix \( s \in \mathbb{N} \) and set
\[
\mathfrak{m} := \sum_{\alpha \in \Delta^+: 0 < \alpha(h) \leq s} \mathfrak{g}_{-\alpha}, \quad \mathfrak{s} := \sum_{\alpha \in \Delta^+: \alpha(h)=0} \mathfrak{g}_{-\alpha}.
\]
One has
\[
\{ v \in \mathcal{S}(\mathfrak{n}^-)| hv = sv \} \subset \left( \bigoplus_{i=0}^{s} \mathcal{S}^i(\mathfrak{m}) \otimes \mathcal{S}(\mathfrak{s}) \right).
\]
The assumptions [33] imply that \( \mathfrak{s} \) is an odd finite-dimensional space. By Lemma 8.4.1 \( \dim \mathfrak{m} < \infty \). Therefore each summand \( \mathcal{S}^i(\mathfrak{m}) \otimes \mathcal{S}(\mathfrak{s}) \) is a finite-dimensional space. Hence the \( h \)-eigenspaces in \( \mathcal{S}(\mathfrak{n}^-) \) are finite-dimensional.
Denote by $\text{Ver}$ the set of isomorphism classes of all quotients of Verma modules. Take $N \in \mathcal{O}(\mathfrak{g})$. Since $N$ is finitely generated, $N$ admits a finite filtration with cyclic quotients. It is easy to see that each cyclic module in $\mathcal{O}(\mathfrak{g})$ admits a finite filtration with quotients in $\text{Ver}$. Hence $N$ admits a finite filtration with quotients in $\text{Ver}$. Since $\mathcal{O}_{\text{inf}}(\mathfrak{g})$ is “locally small” and contains all Verma modules, it contains $N$. This completes the proof of (34).

8.4.3. Proposition. Take $N \in \mathcal{O}_{\text{inf}}(\mathfrak{g})$ and $a \in \mathbb{C}$. The module $N$ admits a “local composition series at $a$”, which is a finite filtration

$$0 = N^0 \subset N^1 \subset \ldots \subset N^t = N$$

with the following property: for every $i = 1, \ldots, t$

either $N^i/N^{i-1} \cong L(\lambda_i)$ with $\lambda_i(h) \in a + \mathbb{N}$

or $\Omega_h(N^i/N^{i-1}) \cap (a + \mathbb{N}) = \emptyset$.

Proof. The proof is a slight modification of the proof of Prop. 3.2 in [DGK]. For every $N \in \mathcal{O}_{\text{inf}}(\mathfrak{g})$ we set

$$N \geq a := \sum_{i=0}^{\infty} \{v \in N| hv = (a + i)v\}, \quad m(N) := \dim N \geq a.$$

From the definition of $\mathcal{O}_{\text{inf}}(\mathfrak{g})$ it follows that $m(N) < \infty$. We prove the statement by induction on $m(N)$. If $m(N) = 0$, then $0 = N^0 \subset N^1 = N$ is the required filtration. Let $m(N) > 0$. Note that $N \geq a$ is a finite-dimensional $(\mathfrak{h} + \mathfrak{n})$-module. Let $v \in N \geq a$ be a primitive vector of weight $\lambda$ (one has $\lambda(h) \in a + \mathbb{N}$). Let $M$ be a submodule generated by $v$ and $M'$ be the maximal submodule of $M$ satisfying $M'_\lambda = 0$. Then

$$0 \subset M' \subset M \subset N \quad \text{with} \quad M/M' \cong L(\lambda).$$

Since $\lambda(h) \in a + \mathbb{N}$, one has $m(L(\lambda)) > 0$ and thus

$$m(M') + m(N/M') = m(N) - m(M/M') < m(N).$$

By induction $M'$ and $N/M'$ admit suitable filtrations; combining these filtrations we get the required filtration for $N$. □

8.5. Proof of (33). Fix $\nu \in \mathfrak{h}_x^\perp$. For each $N \in \mathcal{O}_{\text{inf}}(\mathfrak{g})$ we set

$$n(N) := \sum_{\lambda} [N : L(\lambda)] \cdot [\text{DS}_x(L(\lambda)) : L_{\partial_x}(\nu)].$$

Our goal is to show that $n(N) < \infty$ and that $[\text{DS}_x(N) : L_{\partial_x}(\nu)] \in n(N) - 2\mathbb{N}$.

Consider the filtration as in Proposition 8.4.3 for $a := \nu(h_x)$. We prove the assertion by induction on the length of this filtration $t$; we denote this length by $t$. If the filtration has zero length, then $N = 0$ and the assertion holds.
Recall that for any $M \in \mathcal{O}_h^{inf}$ one has $\Omega_{h_x}(\text{DS}_x(M)) \subset \Omega_h(M)$. This implies

$$\Omega_h(M) \cap (a + N) = \emptyset \implies [\text{DS}_x(M') : L_{g_x}(\nu)] = 0 \text{ if } M' \text{ is a subquotient of } M$$

Consider the case when $\Omega_h(N/N^{t-1}) \cap (a + N) = \emptyset$. By (35),

$$[\text{DS}_x(N/N^{t-1}) : L_{g_x}(\nu)] = 0$$

and $[N/N^{t-1} : L(\lambda)] = 0$ if $[\text{DS}_x(L(\lambda)) : L_{g_x}(\nu)] \neq 0$. Using Hinich’s Lemma and the first formula we get

$$[\text{DS}_x(N) : L_{g_x}(\nu)] = [\text{DS}_x(N^{t-1}) : L_{g_x}(\nu)].$$

The second formula gives $n(N) = n(N^{t-1})$. By the induction hypothesis, the required assertions hold for $N^{t-1}$; thus the assertions hold for $N$.

Now consider the remaining case when $N/N^{t-1} \cong L(\lambda)$ with $\lambda(h) \in a + N$. One has

$$n(N) = n(N^{t-1}) + [\text{DS}_x(L(\lambda)) : L_{g_x}(\nu)].$$

By Hinich’s Lemma

$$[\text{DS}_x(N) : L_{g_x}(\nu)] \in [\text{DS}_x(N^{t-1}) : L_{g_x}(\nu)] + [\text{DS}_x(L(\lambda)) : L_{g_x}(\nu)] - 2N,$$

so

$$[\text{DS}_x(N) : L_{g_x}(\nu)] - n(N) \in [\text{DS}_x(N^{t-1}) : L_{g_x}(\nu)] - n(N^{t-1}) - 2N.$$

By induction, $n(N) < \infty$ and $[\text{DS}_x(N) : L_{g_x}(\nu)] - n(N) \in -2N$. □

9. DS-functor and cores in the affine case

In this section $\mathfrak{g}$ is an indecomposable symmetrizable affine superalgebra or $\mathfrak{gl}(m|n)^{(1)}$. We retain notation of Section 7 and of 4.11.

Take $x \in X(\mathfrak{g})$ such that $\text{supp}(x) \subset (-\Sigma \cup \hat{\Sigma})$; set $S := \text{supp}(x)$. We identify $\mathfrak{g}_x$ with a subalgebra of $\mathfrak{g}$ as in 7.1. We consider the triangular decomposition of $\mathfrak{g}_x$ which is induced by the triangular decomposition of $\mathfrak{g}$. Note that $(\mathfrak{g}^d)_x$ coincides with $(\mathfrak{g}_x)^d$; we will denote this algebra by $\mathfrak{g}^d_x$. One has

$$\mathfrak{g}^d_x = \hat{\mathfrak{g}}_x \times \mathbb{C} K \times \mathbb{C} d, \quad h_x = \hat{h}_x \oplus \mathbb{C} K \oplus \mathbb{C} d;$$

the triangular decomposition of $\hat{\mathfrak{g}}_x$ is as in 6.2 We choose the Weyl vectors $\hat{\rho}$, $\rho$, $\hat{\rho}_x$, $\rho_x$ as in Section 7. The main result of this section is the following theorem.

9.1. **Theorem.** Fix $x \in X(\mathfrak{g})$ with $\text{supp}(x) \subset (\Sigma \cup (-\Sigma))$. Let $N \in \mathcal{O}(\mathfrak{g})$ be an indecomposable $\mathfrak{g}$-module and let $L$ be a simple subquotient of $\text{DS}_x(N)$.

(i) atyp($L$) = atyp($N$) − rank $x$.

(ii) If $\hat{\mathfrak{g}}$ is not exceptional, then Core($N$) = Core($L$).
9.2. **Remark.** By contrast with $\mathcal{F}\text{in}(g)$, $DS_x$ can kill simple modules in $O(g)$ even in the case when the atypicality of the module is equal to rank $x$.

For example, take $g := sl(2|1)$ with the base $\Sigma = \{\beta_1, \beta_2\}$, where $\beta_1, \beta_2$ are odd. If $(\lambda | \beta_2) = 0$ and $L(\lambda - \rho)$ is infinite-dimensional, then

$$DS_x(L(\lambda - \rho)) = \begin{cases} \mathbb{C} & \text{if } \text{supp}(x) = \{-\beta_2\} \\ 0 & \text{if } \text{supp}(x) = \{-\beta_1\}. \end{cases}$$

9.3. **Proof of Theorem 9.1.** By Corollary 4.11.9, (i) follows from (ii) if $\dot{g}$ is non-exceptional. If $\dot{g}$ is exceptional, then $\text{defect } g = 1$ and (i) reduces to the following assertion: $DS_x(N) = 0$ if $N$ is typical. Therefore we may (and will) substitute (i) by

(i') $N$ is atypical

(meaning that the existence of a subquotient $L$ implies that $N$ is atypical).

Assume that (i') or (ii) does not hold for a certain pair $(N, L)$. Using (33), we conclude that the same assertion does not hold for the pair $(L(\lambda), L)$, where $L(\lambda)$ is a simple subquotient of $N$. Write $L = L_{g_x}(\nu)$ and let $A \subset h^*$ be the set of $\lambda$s such that the assertion does not hold for the pair $(L(\lambda), L_{g_x}(\nu))$. By above, $A$ is non-empty.

9.3.1. We fix $h \in h^*$ satisfying

$$\alpha(h) = \begin{cases} 0 & \text{for } \alpha \in \text{supp}(x), \\ 1 & \text{for } \alpha \in \Sigma \setminus (\text{supp}(x) \cup (-\text{supp}(x))). \end{cases}$$

Note that $h \in g^x$. Denote by $h_x$ the image of $h$ in $h_x$. Retain notation of Section 8 and recall that for each $M \in DS_x(O(g))$ one has

$$\Omega_{h_x}(DS_x(M)) \subset \Omega_h(M), \quad DS_x(M) \in O_{h_x}(g_x).$$

By (35), for each $\lambda \in A$ one has $\lambda(h) \in \nu(h_x) + \mathbb{N}$. We fix $\lambda \in A$ with the minimal value of $\lambda(h)$. Then $L_{g_x}(\nu)$ is a subquotient of $DS_x(L(\lambda))$ and either $\lambda + \rho$ is typical (for (i')) or $\text{Core}(\lambda + \rho) \neq \text{Core}(\nu + \rho_x)$ (for (ii)).

9.3.2. Retain notation of 7.3 and consider the induced functors

$$\text{Ind} : g^d - \text{Mod} \to g - \text{Mod}, \quad \text{Ind}_x : g^d_x - \text{Mod} \to g_x - \text{Mod}.$$

The module $L(\lambda)$ is a quotient of the induced module $\text{Ind}(L_{g^d}(\lambda))$ by its maximal proper submodule, which we denote by $N'$.

Let $L(\lambda')$ be a subquotient of $N'$. We claim that

$$(37) \quad \lambda'(h) < \lambda(h), \quad \text{atyp } \lambda = \text{atyp } \lambda', \quad \text{Core}(\lambda + \rho) = \text{Core}(\lambda' + \rho).$$

(using the notation Core we always assume that $g$ is non-exceptional). Indeed, the first formula follows from the fact that $\lambda - \lambda' \in N\Sigma \setminus N\Sigma$. The rest of the formulae follow from 4.11 and the fact that $N$ is a subquotient of $M(\lambda)$. 

Combining (37) and the minimality of \( \lambda(h) \), we get \( \lambda \not\in A \). Using (33) we obtain
\[
[N' : L_{g_x}(\nu)] = 0.
\]
Recall that \( L_{g_x}(\nu) \) is a subquotient of \( \text{DS}_x(L(\lambda)) \). Hinich’s Lemma gives
\[
[\text{DS}_x(\text{Ind}(L_{g^x}(\lambda))) : L_{g_x}(\nu)] = [\text{DS}_x(L(\lambda)) : L_{g_x}(\nu)] \neq 0.
\]
Using (30) we conclude
\[
[\text{Ind}_x(\text{DS}_x(L_{g^x}(\lambda))) : L_{g_x}(\nu)] \neq 0.
\]
Combining (37) and the minimality of \( \lambda(h) \), we get \( \lambda \not\in A \). Using (33) we obtain
\[
[N' : L_{g_x}(\nu)] = 0.
\]
Recall that \( L_{g_x}(\nu) \) is a subquotient of \( \text{DS}_x(L(\lambda)) \). Hinich’s Lemma gives
\[
[\text{DS}_x(\text{Ind}(L_{g^x}(\lambda))) : L_{g_x}(\nu)] = [\text{DS}_x(L(\lambda)) : L_{g_x}(\nu)] \neq 0.
\]
Using (30) we conclude
\[
[\text{Ind}_x(\text{DS}_x(L_{g^x}(\lambda))) : L_{g_x}(\nu)] \neq 0.
\]
In particular, \( \text{DS}_x(L_{g^x}(\lambda)) \neq 0 \). Since \( g^d = \hat{g} \times \mathbb{C}K \times \mathbb{C}d \) and \( \hat{g} \) is a finite-dimensional Kac-Moody superalgebra, \( \text{DS}_x(L_{g^x}(\lambda)) \neq 0 \) implies the atypicality of the weight \( \lambda|_h + \hat{\rho} \).

Therefore \( (\lambda|_h + \hat{\rho})|\alpha) = 0 \) for some \( \alpha \in \hat{\Delta}_{iso} \). By our choice of \( \rho \), this gives \( (\lambda + \rho)|\alpha) = 0 \), so \( \lambda + \rho \) is an atypical weight. This establishes (i).

9.3.3. By Proposition 8.4.3 the \( g^d \)-module \( \text{DS}_x(L_{g^x}(\lambda)) \) admits a finite filtration
\[
0 = M^0 \subset M^1 \subset \ldots \subset M^t = \text{DS}_x(L_{g^x}(\lambda)),
\]
where for each index \( i \) the module \( M^i/M^{i-1} \) is a simple \( g^d \)-module or
\[
\Omega_{h_x}(M^i/M^{i-1}) \subset \nu(h_x) - 1 - N.
\]
Consider the corresponding filtration of the induced module \( \text{Ind}_x(\text{DS}_x(L_{g^x}(\lambda))) \). By (38) for some index \( j \) we have
\[
[\text{Ind}_x(M^j/M^{j-1}) : L_{g_x}(\nu)] \neq 0.
\]
Since \( \Omega_{h_x}(\mathcal{U}((g_x)_{<0})) \subset -N \), for \( M^j/M^{j-1} \) satisfying (39) we have
\[
\Omega_{h_x}(\text{Ind}_x(M^j/M^{j-1})) \subset \Omega_{h_x}(M^j/M^{j-1}) - N \subset \nu(h_x) - 1 - N.
\]
Thus \( M^j/M^{j-1} \) does not satisfy (39), so \( M^j/M^{j-1} = L_{g^x}(\mu) \) for some \( \mu \in h_x^* \). Then \( L_{g_x}(\nu) \) is a subquotient of \( \text{Ind}_x(L_{g^x}(\mu)) \), which is a quotient of \( M_{g_x}(\mu) \). Hence \( L_{g_x}(\nu) \) is a subquotient of \( M_{g_x}(\mu) \). Using (4.11) we get
\[
\text{Core}(\mu + \rho_x) = \text{Core}(\nu + \rho_x).
\]
On the other hand, \( L_{g^x}(\mu) \) is a subquotient of \( \text{DS}_x(L_{g^x}(\lambda)) \). Using (30) we conclude that the module \( L_{g^x}(\mu|_{h_x}) \) is a subquotient of \( \text{DS}_x(L_{g}(\lambda|_h)) \). By [DS] (see also 6.3.1) this gives
\[
\text{Core}(\lambda|_h + \hat{\rho}) = \text{Core}(\mu|_h + \hat{\rho}_x).
\]
Clearly, \( \lambda(K) = \nu(K) = \mu(K) \). One has \( (\rho|\delta) = \rho(K) \). Using Proposition 7.1.4 (ii) we get \( \rho(K) = \rho_x(K) \). Therefore
\[
(\mu + \rho_x)(K) = (\lambda + \rho)(K).
\]
Recall that \( (\rho - \hat{\rho}|\varepsilon_i) = (\rho - \hat{\rho}|\delta_j) = 0 \) for all indices \( i, j \). Combining this fact with (40) and (41) we conclude that
\[
\text{Core}(\lambda + \rho) = \text{Core}(\mu + \rho_x).
\]
which contradicts to $\text{Core}(\lambda + \rho) \neq \text{Core}(\nu + \rho_x)$. This establishes (ii). \qed

9.4. DS$_x$ for vertex algebras. Let $\mathfrak{g} = \hat{\mathfrak{g}}^{(1)}$. Let $V_k(\mathfrak{g})$ be the affine vertex superalgebra of level $k$ and let $V_k(\mathfrak{g}_x)$ be the corresponding simple affine vertex superalgebra. A natural analogue of $\mathcal{F}\text{in}(\hat{\mathfrak{g}})$ is the category $KL_k(\mathfrak{g})$, which is the full subcategory of $\hat{\mathfrak{g}}$-locally finite $V_k(\mathfrak{g})$-modules in $\mathcal{O}(\mathfrak{g})$. This category was studied in [AKMFPP] and in other papers.

The set $\hat{X} := \{ g \in \mathfrak{g}_1 | x^2 = 0 \}$ is a subset in $X_{\text{iso}}$. Take $x \in \hat{X}$. By above, DS$_x(\mathfrak{g})$ is the affinization of DS$_x(\hat{\mathfrak{g}})$ and that DS$_x$ gives the functor from the category of $V_k(\mathfrak{g})$-modules to the category of $V^k(\mathfrak{g}_x)$-modules (see [GS3]). Take a non-zero $x \in \hat{X}$.

9.4.1. Conjecture. For “admissible” values of $k$ (introduced in [GS3]) one has

$$DS_x(V_k(\mathfrak{g})) = V_k(\mathfrak{g}_x), \quad DS_x(KL_k(\mathfrak{g})) \subset KL_k(\mathfrak{g}_x).$$

9.4.2. Conjecture. Each block in $KL_k$ satisfies the analogue of (3) and (4) for depth, where depth is defined by substituting $X_{\text{iso}}$ by $\hat{X}$ (so depth $\leq$ depth).

9.4.3. Conjecture. For $k \in \mathbb{N}_{>0}$ and $x \in \hat{X}$ the functor $DS_x : KL_k(\mathfrak{g}) \to KL_k(\mathfrak{g}_x)$ satisfies the following properties:

- $DS_x(L)$ is semisimple for each $L \in \text{Irr}(KL_k(\mathfrak{g}))$;
- the extension graphs of $KL_k(\mathfrak{g}), KL_k(\mathfrak{g}_x)$ admit bipartitions

$$\text{dex} : \text{Irr}(KL_k(\mathfrak{g})) \to \{ \pm 1 \}, \quad \text{Irr}(KL_k(\mathfrak{g}_x)) \to \{ \pm 1 \}$$

and $[DS_x(L) : L'] = 0$ if $\text{dex}(L) \neq \text{dex}(L')$.

(For $\mathcal{F}\text{in}(\hat{\mathfrak{g}})$ such bipartition is studied in [G2]).

9.4.4. Remark. Consider the case $k \in \mathbb{N}_{>0}$. By [GS3] Sect. 3, Conjecture 9.4.1 holds for $\mathfrak{g} \neq \mathfrak{osp}(2m + 2|2m), D(2|1; a)$.

Take $\hat{\mathfrak{g}} = \mathfrak{sl}(1|n)$. One has $DS_x(\hat{\mathfrak{g}}) = \mathfrak{sl}_{n-1}$ and $DS_x(\mathfrak{g}) = \mathfrak{sl}_{n-1}^{(1)}$. In this case $KL_k(\mathfrak{g})$ is the subcategory of the integrable modules in $\mathcal{O}^k$ and DS$_x$ gives a functor from $KL_k(\mathfrak{g})$ to the category of integrable $\mathfrak{sl}_{n-1}^{(1)}$-modules of level $k$. By [GS1], the conjectures 9.4.1, 9.4.3 hold in this case.

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