Beauty Contests and the Term Structure

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December 2020
Motivation

Can information frictions help to explain the sizeable term premia contained in Treasury yields?

Figure: Zero-coupon US Treasury yield curve (4/1/1999 - 30/6/2017)
Literature

Bond premium puzzle

- Recursive preferences—Epstein and Zin (1989), Rudebusch and Swanson (2012), van Binsbergen et al. (2012)
- Model uncertainty—Barillas et al. (2009)
- Long-run risk—Bansal and Yaron (2004), Croce (2014)
- Rare disasters—Rietz (1988), Barro (2006)
- Habit Formation—Constantinides (1990), Campbell and Cochrane (1999), Rudebusch and Swanson (2008)
- Valuation Risk—Albuquerque et al. (2016)

Information in strategic settings and volatility

- Use of public information—Morris and Shin (2002), Angeletos and Pavan (2007)
- Volatility from information frictions—Angeletos and La’O (2013), Bergemann et al. (2015), Angeletos et al. (2018)
Overview

1. Decomposing the term premium
2. Models with a representative agent
3. Models with heterogeneously informed agents
4. A beauty contest model
Decomposing the term premium

Household side of generic DSGE model

- Representative household maximises

\[
E_t \sum_{s=t}^{\infty} \beta^{s-t} u(c_s, l_s)
\]

subject to

\[
c_t + \sum_{n=1}^{N} p_t^{(n)} b_t^{(n)} = w_t l_t + d_t + \sum_{n=1}^{N} p_t^{(n-1)} b_{t-1}^{(n)}
\]

- \(b_t^{(n)}\) — non-contingent default-free zero-coupon bonds with maturity \(n = 1, 2, \ldots, N\)
- \(p_t^{(n)}\) — bond price (note \(p_t^{(0)} = 1\)
Decomposing the term premium

- **Interior solution**
  \[ p_t^{(n)} = \mathbb{E}_t m_{t+1} p_{t+1}^{(n-1)} , \quad n \in \{1, 2, \ldots, N\} \]

  with stochastic discount factor (SDF) \( m_{t+1} \equiv \beta \frac{u_c(c_{t+1}, l_{t+1})}{u_c(c_t, l_t)} \)

- **Implied yield**
  \[ i_t^{(n)} = -\frac{1}{n} \ln p_t^{(n)} \]

  where we denote \( i_t^{(1)} \equiv i_t \) for simplicity

- **Hypothetical “risk-neutral price”**
  \[ \tilde{p}_t^{(n)} = e^{-i_t} \mathbb{E}_t \tilde{p}_{t+1}^{(n-1)} , \quad n \in \{1, 2, \ldots, N\} \]

- **Term premium (in per-period terms)**
  \[ \psi_t^{(n)} = \frac{1}{n} \left( \tilde{p}_t^{(n)} - p_t^{(n)} \right) \]
Decomposing the term premium

Example – Two-period bond

• Term premium for \( n = 2 \)

\[ \psi_t^{(2)} = \frac{1}{2} \left( \tilde{p}_t^{(2)} - p_t^{(2)} \right) = -\frac{1}{2} \text{Cov}_t \left( m_{t+1}, p_{t+1}^{(1)} \right) \]

• Take unconditional expectation and apply total covariance law to obtain following result

Proposition

Assume the law of iterated expectations holds and the stochastic discount factor \( m_{t+1} \) is in the household information set \( \mathcal{I}_{t+1} \) at time \( t + 1 \). The unconditional mean real term premium is given by

\[ \mathbb{E}\psi_t^{(2)} = \frac{1}{2} \left[ -\text{Cov} \left( m_{t+1}, m_{t+2} \right) + \text{Cov} \left( \mathbb{E}_t m_{t+1}, \mathbb{E}_{t+1} m_{t+2} \right) \right] \]
Decomposing the term premium

Implications

- Mean term premium (for $n = 2$) can be decomposed into
  - covariance of successive realisations of the SDF
  - covariance of successive expectations of the SDF
- Result generalises to higher maturities ($n > 2$)
- Nominal term premium can be decomposed in analogous way
- So far theory focuses on first term (e.g. recursive preferences) ⇒ Negative autocovariance of realisations of SDF required to explain positive mean term premium
- Process of expectation formation directly affects second term ⇒ Positive autocovariance of expectations of SDF required to explain positive mean term premium

Next step

- Use decomposition to connect informational assumptions and term premia in analytical models
Models with a representative agent

Households, firms and technology

- Production function of representative firm

\[ y_t = A_t \bar{l}^{1-\alpha} \]

- Technology \( a_t \equiv \ln A_t \) follows

\[ a_t = x_t + \eta_t, \quad \eta_t \sim N(0, \sigma^2_{\eta}) \]
\[ x_t = \rho x_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim N(0, \sigma^2_{\varepsilon}) \]

- Representative household has logarithmic utility

\[ \Rightarrow \text{Coefficient of relative risk aversion tied to 1} \]

- SDF can be expressed as

\[ m_{t+1} = \beta \left( \frac{c_{t+1}}{c_t} \right)^{-1} = \beta \left( \frac{A_{t+1} \bar{l}^{1-\alpha}}{A_t \bar{l}^{1-\alpha}} \right)^{-1} \]

\[ \approx \beta (1 + a_t - a_{t+1}) \]
Models with a representative agent

**Information sets**

| Model                  | $\subset \mathcal{I}_t$ | $\not\subset \mathcal{I}_t$ |
|------------------------|--------------------------|-------------------------------|
| Full information       | $m^t, a^t, x^t, \eta^t$ |                               |
| Partial information    | $m^t, a^t$               | $x^t, \eta^t$                |
| Noisy information      | $m^t, s^t$               | $a^t, x^t, \eta^t$           |

**Table:** Information set of representative household

**Notes:** Signal given by $s_t = a_t + \xi_t$ with noise $\xi_t \sim N(0, \sigma^2_{\xi})$. 
Models with a representative agent

A. Full Information

B. Partial Information

C. Noisy Information

Figure: Components of mean real term premium \((n = 2)\)

Notes: Solid line is mean real term premium, dashed line is component in autocovariance of realisations of SDF, dotted line is component in autocovariance of expected SDF. \(\beta = 0.99, \ Var(a_t) = 0.01^2, \ \frac{Var(x_t)}{Var(a_t)} = 0.9, \ \sigma_\xi^2 = Var(a_t)/2.\)
Models with heterogeneously informed agents

Identifying conditions required to generate term premia

- Heterogeneous information on the household-side now introduced to framework described before
- Continuum of ex ante identical agents indexed \( i \in [0, 1] \)
- Each agent observes signal \( s_{i,t} = a_t + n_t + n_{i,t} \) and \( \eta_t \) allowing them to deduce

\[
x_{i,t}^n = x_t + n_t + n_{i,t}
\]

but not \( x_t \) (persistent component of technology)
- Noise persistent so that

\[
x_{i,t}^n = \rho x_{i,t-1} + \varepsilon_{i,t}^n
\]

where \( \varepsilon_{i,t}^n \equiv \varepsilon_t + \xi_t + \zeta_{i,t} \sim \mathcal{N}(0, \sigma_{\varepsilon}^2 + \sigma_{\xi}^2 + \sigma_{\zeta}^2) \)
- Forming expectation about \( m_{t+1} \) requires inferring \( x_t \) from \( x_{i,t}^n \)
Models with heterogeneously informed agents

- Focus on symmetric linear equilibrium, in which expectations are formed according to

\[ \hat{E}_{i,t}x_t = \theta x_{i,t} \quad \forall i \]

- Term premium then given by

\[ \psi_t^{(2)} = \frac{1}{2} \beta^2 \left[ \theta(1 - \rho)\sigma_{\epsilon}^2 - \sigma_{\eta}^2 \right] \]

\[ \Rightarrow \theta \uparrow \text{ implies } \psi_t^{(2)} \uparrow \]

- Rational expectations are special case with \( \theta = \theta^* = \frac{\sigma_{\epsilon}^2}{\sigma_{\epsilon}^2 + \sigma_{\xi}^2 + \sigma_{\zeta}^2} \)

- Suppose \( \hat{E}_{i,t}x_t \) formed according to general loss function – Which conditions are required to obtain expectations consistent with the mean term premium in US data?
Models with heterogeneously informed agents

- General loss function

\[
E_{i,t} \left[ \begin{pmatrix} \hat{E}_{i,t}x_t \ x_t \ \int_0^1 \hat{E}_{j,t}x_t \, dj \end{pmatrix} \begin{pmatrix} 1 & \Omega_{12} & \Omega_{13} \\ 0 & \Omega_{22} & \Omega_{23} \\ 0 & 0 & \Omega_{33} \end{pmatrix} \begin{pmatrix} \hat{E}_{i,t}x_t \\ x_t \\ \int_0^1 \hat{E}_{j,t}x_t \, dj \end{pmatrix} \right]
\]

- Optimal expectation satisfies

\[
\hat{E}_{i,t}x_t = \theta x_{i,t} = -\frac{1}{2} \left( \Omega_{12} \theta^* + \Omega_{13} \theta \frac{\sigma_{\varepsilon}^2 + \sigma_{\xi}^2}{\sigma_{\varepsilon}^2 + \sigma_{\xi}^2 + \sigma_{\zeta}^2} \right) x_{i,t}
\]

- Two degrees of freedom—If \( \Omega_{12} \) is normalised to the value consistent with MSE minimisation (and hence RE),

\[
\Omega_{13} = -2 \left( \frac{\sigma_{\varepsilon}^2 + \sigma_{\xi}^2 + \sigma_{\zeta}^2}{\sigma_{\varepsilon}^2 + \sigma_{\xi}^2} \right) \left( \frac{\theta - \theta^*}{\theta} \right)
\]

\( \Rightarrow \theta > \theta^* \) iff \( \Omega_{13} < 0 \)

\( \Rightarrow \) Sizeable term premium under strategic complementarity
A beauty contest model

Model

• More quantitative version of the model outlined just before
  • Labour supply endogenous (competitive labour market)

\[ y_t = A_t L_t^{1-\alpha} \]

• Household utility of more general form

\[ u(c_{i,t}, l_{i,t}) = \frac{1}{1 - \sigma} \left( c_{i,t} - \chi_0 \frac{l_{i,t}^{1+\chi}}{1 + \chi} \right)^{1-\sigma} \]

• Strategic complementarity through expectation formation in bond markets according to loss function with \( \Omega_{12} = -2 \) and \( \omega = \Omega_{13}/(\Omega_{13} - 1) \), i.e.

\[
(1 - \omega)E_{i,t}(\hat{E}_{i,t}x_t - x_t)^2 - \omega E_{i,t} \left( \int_0^1 \hat{E}_{j,t}x_t dj \right) \hat{E}_{i,t}x_t
\]

• Solution based on exact SDF rather than an approximation
A beauty contest model

Estimation approach

- US data, sample period 1999Q1-2017Q2
- Standard parameters calibrated \((\beta, \alpha, \chi, \chi_0)\)
- Remaining parameters estimated based on (simulated) method of moments
  - Parameters governing exogenous technology process \((\rho, \sigma_\eta, \sigma_\varepsilon)\)
    \(\Rightarrow\) Targets are the variance and first two autocovariances of detrended log consumption and variance of detrended log consumption growth
  - Parameters governing forecast formation and risk aversion \((\sigma_\xi, \sigma_\zeta, \omega, \sigma)\)
    \(\Rightarrow\) Targets are the variance and autocovariance of the median forecast of productivity growth over the next ten years and term premium at one-year maturity
A beauty contest model

Figure: Forecasts of productivity growth from the SPF.

Notes: Solid line median, dashed lines lower and upper quartiles.
A beauty contest model

| Parameter | Value | Description                    | Target (Data)                                                                 |
|-----------|-------|--------------------------------|------------------------------------------------------------------------------|
| $\beta$   | 0.9997| Discount factor                | $r^{(4)} = 0.0205 - 0.0191$  
(Treasury yields, Adrian et al. (2013), 4/1/99 - 30/6/17;  
Inflation expectations, SPF, 1999q1-2017q2) |
| $\alpha$  | 0.384 | 1 - Labour share              | $1 - \alpha = 0.6160$  
(Share of labour compensation in GDP, Penn World Table, 1999-2014) |
| $\chi$    | 0.708 | Inverse Frisch elasticity     | $\text{Var}(\ln l_t)/\text{Var}(\ln c_t) = 0.3428$  
(Consumption of nondurables and services, BEA;  
Population and hours, BLS, 1999q1-2017q2) |
| $\chi_0$  | 2.04  | Labour utility weight         | $l = 1/3$                                                                   |

**Table:** Calibrated parameters
A beauty contest model

| Parameter | Estimate | 95% Confidence Interval | Description                                           |
|-----------|----------|-------------------------|-------------------------------------------------------|
| $\rho$    | 0.90     | [0.81, 0.99]            | Shock persistence                                      |
| $\sigma_\varepsilon$ | $2.0 \times 10^{-3}$ | [9.7 $\times 10^{-4}$, 3.1 $\times 10^{-3}$] | SD innovation to persistent tech. component          |
| $\sigma_\eta$ | $8.0 \times 10^{-4}$ | [0, 2.4 $\times 10^{-3}$] | SD i.i.d. transitory tech. component                  |
| $\sigma_\xi$ | $9.9 \times 10^{-5}$ | [9.8 $\times 10^{-5}$, 1.0 $\times 10^{-4}$] | SD innovation to common noise component               |
| $\sigma_\zeta$ | $2.2 \times 10^{-3}$ | [1.9 $\times 10^{-3}$, 2.5 $\times 10^{-3}$] | SD innovation to idiosyncratic noise component       |
| $\omega$  | 0.80     | [0.78, 0.82]            | Strategic complementarity                              |
| $\sigma$  | 6.0      | [5.7, 6.3]              | Coefficient of relative risk aversion                 |

Table: Estimated parameters
A beauty contest model

| Moment                  | US data 1999Q1-2017Q2 | Estimated model | Model with full information | Model with $\omega = 0$ |
|-------------------------|-------------------------|-----------------|-----------------------------|-------------------------|
| $\text{Var} (\hat{\gamma}_t^{50})$ | $1.52 \times 10^{-5}$ | $1.42 \times 10^{-5}$ | $2.11 \times 10^{-7}$ | $4.68 \times 10^{-8}$ |
| $\text{Cov} (\hat{\gamma}_t^{50}, \hat{\gamma}_{t-4}^{50})$ | $1.25 \times 10^{-5}$ | $9.29 \times 10^{-6}$ | $1.38 \times 10^{-7}$ | $3.06 \times 10^{-8}$ |
| $\text{E} (\hat{\gamma}_t^{75} - \hat{\gamma}_t^{25})$ | $5.32 \times 10^{-3}$ | $5.40 \times 10^{-3}$ | $0$ | $3.10 \times 10^{-4}$ |
| $\mathbb{E} \psi_t^{(4)}$ | $8.2 \text{ bps}$ | $8.2 \text{ bps}$ | $2.6 \text{ bps}$ | $1.0 \text{ bps}$ |

Not targeted

| $\mathbb{E} \psi_t^{(8)}$ | $21.2 \text{ bps}$ | $16.0 \text{ bps}$ | $5.4 \text{ bps}$ | $2.0 \text{ bps}$ |
| $\mathbb{E} \psi_t^{(12)}$ | $34.5 \text{ bps}$ | $21.1 \text{ bps}$ | $7.6 \text{ bps}$ | $2.7 \text{ bps}$ |
| $\mathbb{E} \psi_t^{(16)}$ | $46.7 \text{ bps}$ | $24.4 \text{ bps}$ | $9.3 \text{ bps}$ | $3.3 \text{ bps}$ |
| $\mathbb{E} \psi_t^{(20)}$ | $57.2 \text{ bps}$ | $26.7 \text{ bps}$ | $10.7 \text{ bps}$ | $3.7 \text{ bps}$ |

**Table:** Data and model moments
A beauty contest model

Estimation results

- Estimated beauty contest model
  - matches the moments related to consumption dynamics and volatility in hours almost perfectly
  - closely matches the moments targeted from the Survey of Professional Forecasters
  - delivers sizeable term premia, between 47 and 75 per cent of the nominal term premia in US data

- Model with full information (technology observed)
  - generates autocovariance in expectations that is two orders of magnitude too small
  - gives rise to term premia that are less than half of those in the beauty contest model

- Model without strategic complementarity ($\omega = 0$)
  - yields even lower autocovariance in expectations coinciding with even lower term premia
Conclusions

- The term premia contained in bonds of any maturity depend on autocovariance terms of the realisations and expectations of the stochastic discount factor.

- Standard signal extraction problems in a representative agent framework generally do not give rise to sizeable term premia.

- In a model with heterogeneously informed households and persistent noise, strategic complementarity in expectation formation can increase term premia.

- An estimated model that allows for strategic complementarity is capable of explaining a substantial fraction of the term premia contained in the prices of US Treasuries.
Decomposing the term premium

Proposition

Assume the law of iterated expectations holds and the stochastic discount factor $m_{t+1}$ is in the household information set $I_{t+1}$ at time $t+1$. The real term premium at maturity $n \in \{2, 3, \ldots\}$ is

$$
\psi_t^{(n)} = \frac{1}{n} \sum_{k=0}^{n-2} \nu_t(k) \left[ -\text{Cov}_t \left( m_{t+k+1}, \prod_{j=k}^{n-2} m_{t+j+2} \right) 
+ \text{Cov}_t \left( E_{t+k} m_{t+k+1}, \prod_{j=k}^{n-2} E_{t+j+1} m_{t+j+2} \right) \right]
$$

where

$$
\nu_t(k) \equiv \begin{cases} 
1 & \text{for } k = 0 \\
\prod_{j=0}^{k-1} E_t e^{-i_{t+j}} & \text{otherwise}
\end{cases}
$$
Decomposing the term premium

Lemma
Assume the law of iterated expectations holds and the stochastic discount factor \( m_{t+1} \) is in the household information set \( I_{t+1} \) at time \( t + 1 \). The unconditional mean real term premium at maturity \( n \in \{2, 3, \ldots\} \) is

\[
E_{\psi}^{(n)} = \frac{1}{n} \sum_{k=0}^{n-2} \left\{ E(\iota_t(k)) \left[ -\text{Cov} \left( m_{t+k+1}, \prod_{j=k}^{n-2} m_{t+j+2} \right) + \text{Cov} \left( E_t m_{t+k+1}, E_t \prod_{j=k}^{n-2} m_{t+j+2} \right) \right] + \right.

\text{Cov} \left( E_t m_{t+k+1}, \prod_{j=k}^{n-2} E_t m_{t+j+2} \right) \left[ -\text{Cov} \left( E_t m_{t+k+1}, E_t \prod_{j=k}^{n-2} E_t m_{t+j+2} \right) \right]\right.

\text{Cov} \left( \iota_t(k), -\text{Cov}_t \left( m_{t+k+1}, \prod_{j=k}^{n-2} m_{t+j+2} \right) + \text{Cov}_t \left( E_t m_{t+k+1}, \prod_{j=k}^{n-2} E_t m_{t+j+2} \right) \right) \right\}
\]