Abstract

In a general measure space \((X, \mathcal{L}, \lambda)\), a characterization of weakly null sequences in \(L_\infty(X, \mathcal{L}, \lambda)\) \((u_k \rightharpoonup 0)\) in terms of their pointwise behaviour almost everywhere is derived from the Yosida-Hewitt identification of \(L_\infty(X, \mathcal{L}, \lambda)^*\) with finitely additive measures, and extreme points of the unit ball in \(L_\infty(X, \mathcal{L}, \lambda)^*\) with \(\pm \mathcal{G}\), where \(\mathcal{G}\) denotes the set of finitely additive measures that take only values 0 or 1. When \((X, \tau)\) is a locally compact Hausdorff space with Borel \(\sigma\)-algebra \(\mathcal{B}\), the well-known identification of \(\mathcal{G}\) with ultrafilters means that this criterion for nullity is equivalent to localized behaviour on open neighbourhoods of points \(x_0\) in the one-point compactification of \(X\). Notions of weak convergence at \(x_0\) and the essential range of \(u\) at \(x_0\) are natural consequences. When a finitely additive measure \(\nu\) represents \(f \in L_\infty(X, \mathcal{B}, \lambda)^*\) and \(\hat{\nu}\) is the Borel measure representing \(f\) restricted to \(C_0(X, \tau)\), a minimax formula for \(\hat{\nu}\) in terms of \(\nu\) is derived and those \(\nu\) for which \(\hat{\nu}\) is singular with respect to \(\lambda\) are characterized.

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1 Introduction

In the usual Banach space \(C(Z)\) of real-valued continuous functions on a compact metric space \(Z\) with the maximum norm, it is well-known [3] that \(v_k\) converges weakly to \(v\) (\(v_k \rightharpoonup v\)) if and only if \(\|v_k\|\) is bounded and \(v_k(z) \rightarrow v(z)\) for all \(z \in Z\). This observation amounts to a simple test for weak convergence in \(C(Z)\) from which follows, for example, the weak sequential continuity [2] of composition maps \(u \mapsto f \circ u, \ u \in C(Z)\), when \(f : \mathbb{R} \rightarrow \mathbb{R}\) is continuous. However \(u_k \rightharpoonup u\) in \(L_\infty(X, \mathcal{L}, \lambda)\) implies that \(\{\|u_k\|_\infty\}\) is bounded and often that \(u_k(x) \rightarrow u(x)\) almost everywhere (Lemma 3.3), but the converse is false (Remark 3.5) and, despite the identification of \(L_\infty(X, \mathcal{L}, \lambda)\) with \(C(Z)\) for some compact \(Z\) [5, VIII 2.1], it can be difficult to decide whether or not a given sequence is weakly convergent in \(L_\infty(X, \mathcal{L}, \lambda)\). To address this issue Theorem 3.6 characterises sequences that are weakly convergent to 0 in \(L_\infty(X, \mathcal{L}, \lambda)\) (hereafter referred to as weakly null) purely in terms of their pointwise behaviour almost everywhere, and a practical test for weak nullity ensues (Corollary 3.7 and Section 3.1). When \((X, \tau)\) is a locally compact Hausdorff topological space, localization in terms of opens sets, as opposed to pointwise, follows from the identification of ultrafilters in the corresponding Borel measure space \((X, \mathcal{B}, \lambda)\) with extreme points in the unit ball of \(L_\infty(X, \mathcal{B}, \lambda)^*\). When \(\nu\) is the finitely additive measure corresponding to \(f \in L_\infty(X, \mathcal{B}, \lambda)^*\) we give a formula for the Borel measure \(\hat{\nu}\) that represent the restriction \(\hat{f}\) of \(f\) to
Notation summarised in Section 2.1, the analogue of the Riesz Representation Theorem [12, Thm. 6.19] for functionals in $L_\infty(X,\mathcal{L},\lambda)^*$ is the following.

**Theorem 2.1.** (Yosida & Hewitt [15], see also [6, Theorem IV.8.16]). For every bounded linear func-
tional on \( L_∞(X, \mathcal{L}, \lambda) \) there exists a finitely additive measure (Definition 2.2) \( \nu \) on \( \mathcal{L} \) such that

\[
f(w) = \int_X w \, d\nu \text{ for all } w \in L_∞(X, \mathcal{L}, \lambda), \tag{2.1}
\]

\[
\nu(N) = 0 \text{ for all } N \in \mathcal{N} \text{ and } |\nu|(X) = \|f\|_∞ < \infty.
\]

Conversely if \( \nu \) is a finitely additive measure on \( X \) with \( \nu(N) = 0 \) for all \( N \in \mathcal{N} \), then \( f \) defined by (2.1) is in \( L_∞(X, \mathcal{L}, \lambda) \). We write \( \nu \in L_∞^*(X, \mathcal{L}, \lambda) \) if (2.1) holds for some \( f \in L_∞(X, \mathcal{L}, \lambda) \).

Because \( \nu \) is finitely additive, but not necessarily \( \sigma \)-additive, integrals in (2.1) should be treated with care. For example, the Monotone Convergence Theorem and Fatou’s Lemma do not hold, and the Dominated Convergence Theorem holds only in a restricted form. The next section is a review of notation and standard theory; for a comprehensive account see [15], [6, Ch. III] or [4, Ch. 4]. When combined with the Hahn-Banach theorem, Theorem 2.1 yields the existence of a variety of finitely additive measures.

### 2.1 Finitely Additive Measures: Notation and Definitions

Although finitely additive measures are defined on algebras (closed under complementation and finite unions), here they are considered only on \( \sigma \)-algebras, where their theory is somewhat more satisfactory, because \( \mathcal{L} \) in Theorem 2.1 is a \( \sigma \)-algebra.

**Definition 2.2.** [15, §1.2-§1.7] A finitely additive measure \( \nu \) on \( \mathcal{L} \) is a mapping from \( \mathcal{L} \) into \( \mathbb{R} \) with

\[
\nu(\emptyset) = 0 \text{ and } \sup_{A \in \mathcal{L}} |\nu(A)| < \infty;
\]

\[
\nu(A \cup B) = \nu(A) + \nu(B) \text{ for all } A, B \in \mathcal{L} \text{ with } A \cap B = \emptyset.
\]

A finitely additive measure is \( \sigma \)-additive if and only if

\[
\nu(\bigcup_{k \in \mathbb{N}} E_k) = \sum_{k \in \mathbb{N}} \nu(E_k) \text{ for all } \{E_k\} \subset \mathcal{L} \text{ with } E_j \cap E_k = \emptyset, j \neq k.
\]

Let \( \mathcal{Y}(\mathcal{L}) \) and \( \Sigma(\mathcal{L}) \) denote, respectively, the families of finitely additive and \( \sigma \)-additive measures on \( \mathcal{L} \).

Since finitely-additive measures are not one-signed, the hypothesis that \( \sup_{A \in \mathcal{L}} |\nu(A)| < \infty \) does not follow from the fact that \( \nu(X) < \infty \). The following results are from [15, §1.9-§1.12].

For \( \nu_1, \nu_2 \in \mathcal{Y}(\mathcal{L}), E \in \mathcal{L}, \) let

\[
(\nu_1 \vee \nu_2)(E) = \sup_{F \supseteq E \in \mathcal{L}} \{\nu_1(F) + \nu_2(E \setminus F)\},
\]

\[
(\nu_1 \wedge \nu_2)(E) = -((\neg \nu_1) \vee (\neg \nu_2))(E).
\]

Then \( \nu_1 \vee \nu_2, \nu_1 \wedge \nu_2 \in \mathcal{Y}(\mathcal{L}) \), which is a lattice, and \( \nu \in \mathcal{Y}(\mathcal{L}) \) can be written

\[
\nu = \nu^+ - \nu^- \text{ where } \nu^+ = \nu \vee 0, \quad \nu^- = (\neg \nu) \vee 0 \text{ and } \nu^+ \wedge \nu^- = 0. \tag{2.2b}
\]

\( \nu^+ \) are the positive and negative parts of \( \nu \) and \( |\nu| := \nu^+ + \nu^- \) is its total variation (see Theorem 2.1).

For \( \nu_1, \nu_2 \in \mathcal{Y}(\mathcal{L}) \) write \( \nu_1 \ll \nu_2 \) (\( \nu_1 \) is absolutely continuous with respect to \( \nu_2 \)), if for all \( \epsilon > 0 \) there exists \( \delta > 0 \) such that \( |\nu_1(E)| < \epsilon \text{ when } |\nu_2|(E) < \delta \), and write \( \nu_1 \perp \nu_2 \) if for every \( \epsilon > 0 \) there exists \( E \in \mathcal{L} \) such that \( |\nu_1|(E) + |\nu_2|(X \setminus E) < \epsilon \).

**Remark 2.3.** When \( \nu_1, \nu_2 \in \Sigma(\mathcal{L}) \subset \mathcal{Y}(\mathcal{L}) \) the above definitions imply:

\[
\nu_1 \ll \nu_2 \text{ if and only if } |\nu_2|(E) = 0 \text{ implies } \nu_1(E) = 0 \text{ for all } E \in \mathcal{L};
\]

\[
\nu_1 \perp \nu_2 \text{ if and only if } |\nu_1|(E) + |\nu_2|(X \setminus E) = 0 \text{ for some } E \in \mathcal{L}.
\]

However it is important that a non-negative finitely additive measure \( \nu \) which vanishes on \( \mathcal{N} \) (see Theorems 2.1 and 2.9) need not satisfy \( \nu \ll \lambda \) if \( \nu \notin \Sigma(\mathcal{L}) \).
Definition 2.4. [15, §1.13] A non-negative \( \nu \in \mathcal{Y}(\mathcal{L}) \) is purely finitely additive (written \( \nu \in \Pi(\mathcal{L}) \)) if

\[
\{ \gamma \in \Sigma(\mathcal{L}) : 0 \leq \gamma \leq \nu \} = \{ 0 \}.
\]

Equivalently, \( 0 \leq \nu \in \Pi(\mathcal{L}) \) if and only if \( \nu \wedge \gamma = 0 \) for all \( 0 \leq \gamma \in \Sigma(\mathcal{L}) \). In general, \( \nu \in \mathcal{Y}(\mathcal{L}) \) is purely finitely additive if \( \nu^+ \) and \( \nu^- \) are purely finitely additive.

Note that \( \Pi(\mathcal{L}) \cap \Sigma(\mathcal{L}) = \{ 0 \} \) and if \( \alpha \in \mathbb{R} \) and \( \nu \in \Pi(\mathcal{L}) \) then \( \alpha \nu \in \Pi(\mathcal{L}) \). Moreover \( \Pi(\mathcal{L}) \) is a lattice [15, Thm. 1.17]: if \( \nu_i \in \Pi(\mathcal{L}) \), \( i = 1, 2 \), then \( \nu_1 + \nu_2, \nu_1 \wedge \nu_2, \nu_1 \vee \nu_2 \in \Pi(\mathcal{L}) \). The sense in which a purely finitely additive measure on a \( \sigma \)-algebra is singular with respect to any \( \sigma \)-additive measure is captured by the following observation which is not true if \( \mathcal{L} \) is only an algebra.

Theorem 2.5. [15, Thm 1.22] For \( 0 \leq \gamma \in \Sigma(\mathcal{L}) \) and \( 0 \leq \mu \in \Pi(\mathcal{L}) \) there exists \( \{ E_k \} \subset \mathcal{L} \) with

\[
E_{k+1} \subset E_k, \quad \mu(E_k) = \mu(X) \text{ for all } k \text{ and } \gamma(E_k) \to 0 \text{ as } k \to \infty.
\]

Conversely if \( 0 \leq \mu \in \mathcal{Y}(\mathcal{L}) \) and for all \( 0 \leq \gamma \in \Sigma(\mathcal{L}) \) a sequence \( \{ E_k \} \) with these properties exists, then \( \mu \in \Pi(\mathcal{L}) \).

The significance of purely finitely additive measures is evident from the following.

Theorem 2.6. [15, Thms 1.23 & 1.24] Any \( \nu \in \mathcal{Y}(\mathcal{L}) \) can be written uniquely as \( \nu = \mu + \gamma \) where \( \mu \in \Pi(\mathcal{L}) \) and \( \gamma \in \Sigma(\mathcal{L}) \). Any \( \nu \in L^*_\infty(X, \mathcal{L}, \lambda) \) can be written uniquely as

\[
\nu = \mu + \gamma \in (L^*_\infty(X, \mathcal{L}, \lambda) \cap \Pi(\mathcal{L})) \oplus (L^*_\infty(X, \mathcal{L}, \lambda) \cap \Sigma(\mathcal{L})). \tag{2.3}
\]

If \( \nu \geq 0 \) the elements of the decomposition are non-negative. This is the Yosida-Hewitt Decomposition of finitely additive measures.

By (2.3), \( \nu = \mu + \gamma \) where \( \mu \in (L^*_\infty(X, \mathcal{L}, \lambda) \cap \Pi(\mathcal{L})) \) and \( \lambda \gg \gamma \in \Sigma(\mathcal{L}) \). If \( (X, \mathcal{L}, \lambda) \) is \( \sigma \)-finite, by the Lebesgue-Radon-Nikodym Theorem [7, Ch. 3.8] there exists \( g \in L_1(X, \mathcal{L}, \lambda) \) with

\[
\int_X u \, d\gamma = \int_X u g \, d\lambda \text{ for all } u \in L_\infty(X, \mathcal{L}, \lambda). \tag{2.4}
\]

In this case (2.3) can be re-written

\[
\nu = \mu + g \lambda, \quad \mu \in \Pi(\mathcal{L}) \cap L^*_\infty(X, \mathcal{L}, \lambda), \quad g \in L_1(X, \mathcal{L}, \lambda). \tag{2.5}
\]

The relation between this and the Lebesgue decomposition of Borel measures is the topic of Section 5.

2.2 \( \mathcal{G} : 0 \text{-1 Measures} \)

Recall that \( L^*_\infty(X, \mathcal{L}, \lambda) \) is the set of finitely additive measures on \( \mathcal{L} \) that are zero on \( \mathcal{N} \). Let

\[
\mathcal{G} = \{ \omega \in L^*_\infty(X, \mathcal{L}, \lambda) : \omega(X) = 1 \text{ and } \omega(A) \in \{ 0, 1 \} \text{ for all } A \in \mathcal{L} \}. \tag{2.6}
\]

\( A \in \mathcal{L} \) is called a \( \lambda \)-atom if \( \lambda(A) > 0 \) and if \( A \supset E \in \mathcal{L} \) implies \( \lambda(E) \in \{ 0, \lambda(A) \} \).

Theorem 2.7. Suppose \( \omega \in \mathcal{G} \). (a) Either \( \omega \in \Pi(\mathcal{L}) \) or \( \omega \in \Sigma(\mathcal{L}) \). (b) If \( (X, \mathcal{L}, \lambda) \) is \( \sigma \)-finite and \( \omega \in \Sigma(\mathcal{L}) \), there exists a \( \lambda \)-atom \( E_\omega \) such that \( \omega(E) = \lambda(E \cap E_\omega) / \gamma(E_\omega) \) for all \( E \in \mathcal{L} \).

Remark. Hence \( \mathcal{G} \subset \Pi(\mathcal{L}) \) if \( (X, \mathcal{L}, \lambda) \) is \( \sigma \)-finite and \( \mathcal{L} \) has no \( \lambda \)-atoms. A stronger statement, Lemma 4.1, can be made when \( \mathcal{L} \) is the Borel \( \sigma \)-algebra of a locally compact Hausdorff space. \( \square \)
Proof. (a) For $\omega \in \mathcal{G}$, by Theorem 2.6, $\omega = \mu + \gamma$ where $\mu \in \Pi(\mathcal{L})$ and $\gamma \ll \lambda, \gamma \in \Sigma(\mathcal{L})$ are non-negative. By Theorem 2.5 there exists $\{E_k\} \subset \mathcal{L}$ with $\mu(E_k) = \mu(X)$ for all $k$ and $\gamma(E_k) \to 0$ as $k \to \infty$. If $\omega(E_k) = 0$ for some $k$ then $0 = \omega(E_k) \geq \mu(E_k) = \mu(X)$ and so $\omega = \gamma \in \Sigma(\mathcal{L})$. If $\omega(E_k) = 1$ for all $k$, then

$$1 = \omega(E_k) = \mu(E_k) + \gamma(E_k) = \mu(X) + \gamma(E_k) \to \mu(X) \text{ as } k \to \infty.$$ 

Hence $\omega(X) = 1 = \mu(X)$ and consequently $\gamma(X) = 0$. Thus $\omega = \mu \in \Pi(\mathcal{L})$.

(b) Since $\omega \ll \lambda$ where $\omega \in \Sigma(\mathcal{L})$ is finite and $\lambda$ is $\sigma$-additive, by (2.4) there exists $g \in L_1(X, \mathcal{L}, \lambda)$ with $\omega(E) = \int_E g \, d\lambda$ for all $E \in \mathcal{L}$. So $g \geq 0$ $\lambda$-almost everywhere on $X$. Since $g \in L_1(X, \mathcal{L}, \lambda)$, $\lambda(\{x \in X : g(x) \geq n\}) \to 0$ as $n \to \infty$, and hence, by [7, Cor. 3.6],

$$\omega(\{x \in X : g(x) \geq n\}) = \int_{\{x \in X : g(x) \geq n\}} g \, d\lambda \to 0 \text{ as } n \to \infty.$$ 

Since $\omega \in \mathcal{G}$ it follows that $\omega(\{x \in X : g(x) \geq N\}) = 0$ for some $N \in \mathbb{N}$. Now, by finite additivity, $\omega(X) = 1$ and $\omega(E) \in \{0, 1\}$ implies that for every $K \in \mathbb{N}$ there exists a unique $k_K \in \{1, \cdots, N2^K\}$ such that

$$1 = \omega(X) = \sum_{k=1}^{N2^K} \omega(E_k) = \omega(E_{k_K}) \text{ where } E_k = \left\{x \in X : \frac{k-1}{2^K} \leq g(x) < \frac{k}{2^K}\right\}.$$ 

Hence $E_{k_{K+1}} \subset E_{k_K}$ and since $\omega$ is $\sigma$-additive it follows that $\omega(E_{\omega}) = 1$ where $E_{\omega} = \{x \in X : g(x) = \alpha\}$ for some $\alpha \in [0, N]$. Then $\lambda(E_{\omega}) > 0$ because $\omega(E_{\omega}) = 1$ and, for all $E \in \mathcal{L}$,

$$\omega(E) = \omega(E \cap E_{\omega}) = \int_{E \cap E_{\omega}} \alpha \, d\lambda = \alpha \lambda(E \cap E_{\omega}).$$ 

Hence $\alpha = 1/\lambda(E_{\omega})$, and $E_{\omega}$ is a $\lambda$-atom with the required properties because $\omega \in \mathcal{G}$. \hfill \Box

Theorem 2.8. For $u \in L_\infty(X, \mathcal{L}, \lambda)$ and $\omega \in \mathcal{G}$ there is a unique $\alpha \in I := [-\|u\|_\infty, \|u\|_\infty]$ such that

$$\omega(\{x \in X : |u(x) - \alpha| < \epsilon\}) = 1 \text{ for all } \epsilon > 0, \quad (2.7a)$$

$$\int_X u \, d\omega = \alpha \text{ and } \int_X |u| \, d\omega = |\alpha|, \quad (2.7b)$$

Remark. Thus, on $L_\infty(X, \mathcal{L}, \lambda)$ elements of $\mathcal{G}$ are analogous to Dirac measures $\mathcal{D}$ in the theory of continuous functions on topological spaces. When (2.7a) holds we say that $u = \alpha$ on $X \omega$-almost everywhere even though it does not imply that $\omega(\{x \in X : u(x) = \alpha\}) = \omega(X)$ if $\omega \notin \Sigma(\mathcal{L})$. \hfill \Box

Proof. Since $\omega$ is zero on $N$, it is clear that $\alpha \in I$ if (2.7a) holds. Now (2.7a) cannot hold for distinct $\alpha_1 < \alpha_2$ because, with $\epsilon = (\alpha_2 - \alpha_1)/4$ the sets $\omega(\{x \in X : |u(x) - \alpha| < \epsilon\})$, $i = 1, 2$, are disjoint and by finite additivity the $\omega$-measure of their union would be 2. Since $\omega \in \mathcal{G}$, there is at most one $\alpha$ for which (2.7a) holds.

Now suppose that there is no $\alpha$ for which (2.7a) holds. Then for each $\alpha \in I$ there is an $\epsilon_\alpha > 0$ such that $\omega(\{x \in X : |u(x) - \alpha| < \epsilon_\alpha\}) = 0$. By compactness there exists $\{\alpha_1, \cdots, \alpha_K\} \subset I$ such that $I \subset \cup_{k=1}^{K}(\alpha_k - \epsilon_\alpha_k, \alpha_k + \epsilon_\alpha_k)$ and consequently

$$1 = \omega(X) = \omega(\{x : u(x) \in \cup_{k=1}^{K}(\alpha_k - \epsilon_\alpha_k, \alpha_k + \epsilon_\alpha_k)\}) \leq \sum_{k=1}^{K} \omega(\{x : u(x) \in (\alpha_k - \epsilon_\alpha_k, \alpha_k + \epsilon_\alpha_k)\}) = 0.$$
Hence (2.7a) holds for a unique $\alpha$. The first part of (2.7b) follows because, by (2.7a), $u = \alpha \omega$-almost everywhere on $X$ and $\omega(X) = 1$. Finally, $\omega \{ x \in X : |u(x)| - |\alpha| < \epsilon \} = 1$ for all $\epsilon > 0$, and the second part of (2.7b) follows.

The next result gives the existence elements of $\mathfrak{G}$.

**Theorem 2.9.** [15, Thm. 4.1] Let $E \subset \mathcal{L} \setminus \mathcal{N}$ have the property that $E_\ell \in \mathcal{E}$, $1 \leq \ell \leq L$ implies that $\cap_{\ell=1}^L E_\ell \notin \mathcal{N}$. Then there exists $\omega \in \mathfrak{G}$ with $\omega(E) = 1$ for all $E \in \mathcal{E}$.

The proof is by Zorn's lemma and for given $E$ there can be uncountably many $\omega$. The same argument underlies the correspondence between elements of $\mathfrak{G}$ and ultrafilters.

**Definition 2.10.** Given $(X, \mathcal{L}, \lambda)$, a filter is a family $\mathcal{F}$ of subsets of $X$ satisfying: (i) $X \in \mathcal{F}$ and $\mathcal{N} \cap \mathcal{F} = \emptyset$; (ii) $E_1, E_2 \in \mathcal{F}$, implies that $E_1 \cap E_2 \in \mathcal{F}$; (iii) $E_2 \supset E_1 \in \mathcal{F}$ implies that $E_2 \in \mathcal{F}$. A maximal filter $\mathcal{F}$, one which satisfies (iv) $\mathcal{F} \subset \hat{\mathcal{F}}$ implies $\mathcal{F} = \hat{\mathcal{F}}$, is called an ultrafilter. Let $\mathfrak{F}$ denote the family of ultrafilters.

It is obvious that when $\omega \in \mathfrak{G}$

$$\mathcal{F}(\omega) := \{ E \in \mathcal{L} : \omega(E) = 1 \} \in \mathfrak{F}. \quad (2.8a)$$

Conversely, when $\mathcal{F} \in \mathfrak{F}$,

$$\omega(E) := \begin{cases} 1 & \text{if } E \in \mathcal{F} \\ 0 & \text{otherwise} \end{cases} \in \mathfrak{G}. \quad (2.8b)$$

This holds because, exactly as in the proof of [15, Thm. 4.1], the maximality of $\mathcal{F} \in \mathfrak{F}$ implies that for $E \in \mathcal{L}$ precisely one of $E$ and $X \setminus E$ is in $\mathcal{F}$. Thus (2.8b) defines $\omega \in \mathfrak{G}$ with $\mathcal{F} = \mathcal{F}(\omega)$ and hence $\omega \leftrightarrow \mathcal{F}(\omega)$ is a one-to-one correspondence between $\mathfrak{G}$ and $\mathfrak{F}$.

By the essential range of $u$ is meant the set

$$\mathcal{R}(u) := \{ \alpha \in \mathbb{R} : \lambda(\{ x : |u(x) - \alpha| < \epsilon \}) > 0 \text{ for all } \epsilon > 0 \}. \quad (2.9)$$

**Corollary 2.11.** For $u \in L_\infty(X, \mathcal{L}, \lambda)$,

$$\left\{ \int_X u \, d\omega : \omega \in \mathfrak{G} \right\} = \mathcal{R}(u).$$

**Proof.** It follows from Theorems 2.8 and 2.9 that the right side is a subset of the left. Since $\omega(E) = 1$, $E \in \mathcal{L}$, implies $\lambda(E) > 0$, it is immediate from Theorem 2.8 that the right side contains the left. \qed

In a topological space (2.8), (2.9) and Corollary 2.11 can be localized to points, (4.1), (4.2) and (4.3).

For $A \in \mathcal{L}$, let $\Delta_A = \{ \omega \in \mathfrak{G} : \omega(A) = 1 \}$ and let $\{ \Delta_A : A \in \mathcal{L} \}$ be a base for the topology $t$ on $\mathfrak{G}$. Note from Theorem 2.9 that $\Delta_A$ is empty if and only if $A \in \mathcal{N}$ and $\Delta_A$ is both open and closed because $\mathfrak{G} \setminus \Delta_A = \Delta_{X \setminus A}$. For $u \in L_\infty(X, \mathcal{L}, \lambda)$ let $L[u] : \mathfrak{G} \to \mathbb{R}$ be defined by

$$L[u](\omega) = \int_X u \, d\omega \text{ for all } \omega \in \mathfrak{G}. \quad (2.10)$$

**Theorem 2.12.** [15, Thms. 4.2 & 4.3] (a) $(\mathfrak{G}, t)$ is a compact Hausdorff topological space.

(b) For $u \in L_\infty(X, \mathcal{L}, \lambda)$, $L[u]$ is continuous on $(\mathfrak{G}, t)$ with

$$\| u \|_\infty = \| L[u] \|_{C(\mathfrak{G}, t)} := \sup_{\omega \in \mathfrak{G}} | L[u](\omega) |,$$
and $u \mapsto L[u]$ is linear from $L_\infty(X, \mathcal{L}, \lambda)$ to $C(\mathfrak{U}, t)$. Moreover, for $u, v \in L_\infty(X, \mathcal{L}, \lambda)$,

$$L[u](\omega)L[v](\omega) = L[uv](\omega) \text{ for all } \omega \in \mathfrak{U}. \quad (2.11)$$

Conversely, for every real-valued continuous function $U$ on $(\mathfrak{U}, t)$ there exists $u \in L_\infty(X, \mathcal{L}, \lambda)$ with $U = L[u]$. So $L$ is an isometric isomorphism between Banach algebras $L_\infty(X, \mathcal{L}, \lambda)$ and $C(\mathfrak{U}, t)$.

Since $L_\infty(X, \mathcal{L}, \lambda)$ and $C(\mathfrak{U}, t)$ are isometrically isomorphic, $u_k \rightharpoonup u_0$ in $L_\infty(X, \mathcal{L}, \lambda)$ if and only if $L[u_k] \rightarrow L[u_0]$ in $C(\mathfrak{U}, t)$. Since $(\mathfrak{U}, t)$ is a compact Hausdorff topological space, it follows from the opening remarks of the Introduction that $L[u_k] \rightarrow L[u_0]$ in $C(\mathfrak{U}, t)$ if and only if $\{\|L[u_k]\|_{C(\mathfrak{U}, t)}\}$ is bounded and $L[u_k] \rightarrow L[u_0]$ pointwise on $\mathfrak{U}$. Hence $u_k \rightharpoonup u_0$ in $L_\infty(X, \mathcal{L}, \lambda)$ if and only if

$$\|u_k\|_\infty \leq M \text{ and } \int_X u_k \, d\omega \rightarrow \int_X u_0 \, d\omega \text{ as } k \rightarrow \infty \text{ for all } \omega \in \mathfrak{U}. \quad (2.12)$$

Sequential weak continuity of composition operators is an obvious consequence.

**Theorem 2.13.** If $u_{k_n}^n \rightharpoonup u_0^n$ in $L_\infty(X, \mathcal{L}, \lambda)$ as $k \rightarrow \infty$, $n \in \{1, \ldots, N\}$, and $F : \mathbb{R}^N \rightarrow \mathbb{R}$ is continuous, then $F(u_{k_1}^1, \ldots, u_{k_N}^N) \rightarrow F(u_0^1, \ldots, u_0^N)$ in $L_\infty(X, \mathcal{L}, \lambda)$.

**Proof.** When $u_{k_1}^1 \rightharpoonup u_0^1$ in $L_\infty(X, \mathcal{L}, \lambda)$, $L[u_{k_1}^1] \rightarrow L[u_0^1]$ in $C(\mathfrak{U}, t)$ and consequently $L[u_{k_1}^1](\omega) \rightarrow L[u_0^1](\omega)$ pointwise in $\mathfrak{U}$ as $k \rightarrow \infty$. Therefore, for continuous $F$,

$$F(L[u_{k_1}^1](\omega), \ldots, L[u_{k_N}^N](\omega)) \rightarrow F(L[u_0^1](\omega), \ldots, L[u_0^N](\omega)), \ \omega \in \mathfrak{U}.\]$$

If $F$ is a polynomial it follows from (2.11) that

$$L[F(u_{k_1}^1, \ldots, u_{k_N}^N)](\omega) \rightarrow L[F(u_0^1, \ldots, u_0^N)](\omega), \ \omega \in \mathfrak{U},$$

and this holds for continuous $F$, by approximation. Consequently, for continuous $F$,

$$L[F(u_{k_1}^1, \ldots, u_{k_N}^N)] \rightarrow L[F(u_0^1, \ldots, u_0^N)] \text{ in } C(\mathfrak{U}, t)$$

and so $F(u_{k_1}^1, \ldots, u_{k_N}^N) \rightarrow F(u_0^1, \ldots, u_0^N)$ in $L_\infty(X, \mathcal{L}, \lambda)$. \hfill \Box

### 3 Pointwise and Weak Convergence in $L_\infty(X, \mathcal{L}, \lambda)$

The goal is to characterise weakly null sequences in $L_\infty(X, \mathcal{L}, \lambda)$ in terms of their pointwise behaviour, but we begin with some observations on the pointwise behaviour of weakly convergent sequences.

**Lemma 3.1.** In $L_\infty(X, \mathcal{L}, \lambda)$, $u_k \rightharpoonup 0$ if and only if $|u_k| \rightharpoonup 0$.

**Proof.** ‘Only if’ follows from Theorem 2.13 and ‘if’ is a consequence of (2.12) since $u_k = u_k^+ - u_k^-$, $0 \leq u_k^\pm \leq |u_k|$ and $\omega \geq 0$. \hfill \Box

**Lemma 3.2.** If $(X, \mathcal{L}, \lambda)$ is $\sigma$-finite and $\{u_k\}$ is weakly null, there is a subsequence $\{u_{k_j}\}$ with $u_{k_j}(x) \rightarrow 0$ $\lambda$-almost everywhere on $X$.

**Proof.** Since $(X, \mathcal{L}, \lambda)$ is $\sigma$-finite there exists $f \in L_1(X, \mathcal{L}, \lambda)$ which is positive almost everywhere. Since $|u_k|f \rightarrow 0$ in $L_1(X, \mathcal{L}, \lambda)$, there is a subsequence with $|u_{k_j}(x)| \rightarrow 0$ for $\lambda$-almost all $x \in X$. \hfill \Box
Lemma 3.3. Suppose that \((X, \rho)\) is a metric space on which \(\lambda\) is a regular Borel measure with the property that for all locally integrable functions \(f\) and balls \(B(x, r)\) centred at \(x\) and radius \(r\),
\[
\lim_{0<r \to 0} \int_{B(x, r)} f \, d\lambda = f(x) \quad \text{for } \lambda\text{-almost all } x \in X \quad \text{where} \quad \int_{B(x, r)} f \, d\lambda := \frac{1}{\lambda(B(x, r))} \int f \, d\lambda. \quad (3.1)
\]
Then \(u_k \to u_0\) in \(L_\infty(X, \mathcal{L}, \lambda)\) implies that \(u_k(x) \to u_0(x)\) pointwise \(\lambda\)-almost everywhere.

Remark 3.4. From [9, Ch. 1], (3.1) holds in particular when \(\lambda\) is a doubling measure on \((X, \rho)\) (i.e. there exists a constant \(C\) such that \(\lambda(B(x, 2r)) \leq C \lambda(B(x, r))\)), or on \(\mathbb{R}^n\) with the standard metric when \(\lambda\) is any Radon measure (i.e. \(\lambda\) is finite on compact sets).

Proof. By hypothesis, for \(u \in L_\infty(X, \mathcal{L}, \lambda)\) there exists a set \(E(u) \in \mathcal{B}\) with \(\lambda(X \setminus E(u)) = 0\) and
\[
u(x) := \lim_{0<r \to 0} \int_{B(x, r)} u \, d\lambda \quad \text{for all } x \in E(u). \quad (3.2)
\]
Now for \(u_k \to u_0\) the set \(E = \bigcap_0^\infty E(u_k)\) has full measure. Let \(V\) denote the subspace of \(L_\infty(X, \mathcal{L}, \lambda)\) spanned by \(\{u_k: k \geq 0\}\) and for fixed \(x \in E\) define a linear functional \(\ell_x\) on \(V\) by \(\ell_x(u) = u(x)\). Then
\[
|\ell_x(u)| = |u(x)| = \lim_{0<r \to 0} \int_{B(x, r)} u \, d\lambda \leq \|u\|_\infty, \quad u \in V,
\]
and, by the Hahn-Banach Theorem, there exists \(L_x \in L_\infty(X, \mathcal{L}, \lambda)^*\) with \(L_x(u) = \ell_x(u)\) for all \(u \in V\). Therefore since \(u_k \to u_0\),
\[
u_k(x) = \ell_x(u_k) = L_x(u_k) \to L_x(u_0) = \ell(u_0) = u_0(x) \quad \text{for all } x \in E.
\]
Hence \(u_k \to u_0\) in \(L_\infty(X, \mathcal{L}, \lambda)\) implies \(u_k(x) \to u_0(x)\) for almost all \(x \in X\). \(\Box\)

Remark 3.5. By contrast there follows an example where \(\{\|u_k\|_\infty\}\) is bounded, \(u_k\) is continuous except at one point and \(u_k(x) \to 0\) everywhere as \(k \to \infty\), but \(u_k \not\to 0\) in \(L_\infty(X, \mathcal{L}, \lambda)\). Let \(X = (-1, 1)\), for each \(k \geq 2\) let \(u_k(0) = 0\), \(u_k(x) = 0\) when \(|x| \geq 2/k\), \(u_k(x) = 1\) if \(0 < |x| < 1/k\), and linear elsewhere. Now in Theorem 2.9 let \(E_\ell = (-1/2\ell, 0) \cup (0, 1/2\ell)\) for each \(\ell\) and let \(\omega\) be a finitely additive measure that takes the value 1 on \(E_\ell\) for all \(\ell\). Then \(\omega \in \mathcal{G}\) and, by Theorem 2.8, \(\int_X u_k \, d\omega = 1\) for all \(k\). So \(u_k \not\to 0\), yet it is clear that \(u_k(x) \to 0\) for all \(x \in X\). \(\Box\)

By a well-known result of Mazur, \(y_k \to y\) in a normed linear space implies, for any strictly increasing sequence \(\{k_j\} \in \mathbb{N}\), that some \(\{y_{k_j}\}\) in the convex hull of \(\{y_{k_j}: j \in \mathbb{N}\}\) converges strongly to \(y\). Hence if \(u_k \to 0\) in \(L_\infty(X, \mathcal{L}, \lambda)\), by Lemma 3.1 there exists \(\{\pi_i\}\) in the convex hull of \(\{u_{k_j}: j \in \mathbb{N}\}\) with
\[
\pi_i \to 0 \quad \text{as } i \to \infty \quad \text{and, for all } i, \quad \pi_i = \sum_{j=1}^{m_i} \gamma_j^i |u_{k_j}|, \quad \gamma_j^i \in [0, 1] \quad \text{and} \quad \sum_{j=1}^{m_i} \gamma_j^i = 1, \quad \text{for some } m_i \in \mathbb{N}.
\]
Since \(\gamma_j^i\) may be zero there is no loss in assuming that \(\{m_i\}\) is increasing. Therefore, for a strictly increasing sequence \(\{k_j\} \in \mathbb{N}\),
\[
0 \leq w_i(x) := \inf \{ |u_{k_j}(x)| : j \in \{1, \cdots, m_i\} \} \leq \pi_i(x), \quad x \in X,
\]
defines a non-increasing sequence in \(L_\infty(X, \mathcal{L}, \lambda)\) with \(\|w_i\|_\infty \to 0\). It follows that if \(u_k \to 0\)
\[
v_J(x) = \inf \{ |u_{k_j}(x)| : j \in \{1, \cdots, J\} \}
\]
is a non-increasing sequence in \(L_\infty(X, \mathcal{L}, \lambda)\) with \(\|v_J\|_\infty \to 0\) as \(J \to \infty\). We now show that a sequence is weakly null in \(L_\infty(X, \mathcal{L}, \lambda)\) if and only if every sequence \(\{v_J\}\), defined as above in terms of a strictly increasing \(\{k_j\}\), converges strongly to 0 in \(L_\infty(X, \mathcal{L}, \lambda)\). To do so, for \(u \in L_\infty(X, \mathcal{L}, \lambda)\) and \(\alpha > 0\), let
\[
A_\alpha(u) = \{ x \in X : |u(x)| > \alpha \}.
\]
Theorem 3.6. A bounded sequence \( \{u_k\} \) in \( L_\infty(X, \mathcal{L}, \lambda) \) converges weakly to zero if and only if for every \( \alpha > 0 \) and every strictly increasing sequence \( \{k_j\} \) in \( \mathbb{N} \) there exists \( J \in \mathbb{N} \) with the property that

\[
\lambda \{ \cap_{j=1}^J A_\alpha(u_{k_j}) \} = 0. \tag{3.4}
\]

This criterion is equivalent to saying that for every strictly increasing sequence \( \{k_j\} \) in \( \mathbb{N} \) the corresponding sequence \( \{v_J\} \) in (3.3) converges strongly to zero in \( L_\infty(X, \mathcal{L}, \lambda) \).

Proof. Suppose, for a strictly increasing sequence \( \{k_j\} \) and \( \alpha > 0 \), that (3.4) is false for all \( J \in \mathbb{N} \). Then \( \mathcal{E} = \{ A_\alpha(u_{k_j}) : j \in \mathbb{N} \} \) satisfies the hypothesis of Theorem 2.9. Hence there exists \( \omega \in \mathfrak{G} \) such that \( \omega(A_\alpha(u_{k_j})) = 1 \) for all \( j \). It follows that

\[
\int_X |u_{k_j}|d\omega \geq \int_{A_\alpha(u_{k_j})} |u_{k_j}|d\omega \geq \alpha > 0 \text{ for all } j.
\]

Hence \( |u_k| \neq 0 \) by (2.12) and so, by Lemma 3.1, \( u_k \neq 0 \).

Conversely suppose \( u_k \neq 0 \). Then by Lemma 3.1 and (2.12), there exists \( \alpha > 0 \), a strictly increasing sequence \( \{k_j\} \subset \mathbb{N} \) and \( \omega \in \mathfrak{G} \) such that

\[
\int_X |u_{k_j}|d\omega =: \alpha_j > \alpha > 0 \text{ for all } j \in \mathbb{N}.
\]

Since \( \alpha_j - \alpha > 0 \), by Theorem 2.8,

\[
\omega(\{x : |u_{k_j}| - \alpha_j < \alpha_j - \alpha\}) = 1 \text{ for all } j.
\]

Therefore, since \( |u_{k_j}| - \alpha = |u_{k_j}| - \alpha_j + \alpha_j - \alpha \), it follows that \( \omega(A_\alpha(u_{k_j})) = 1 \) for all \( j \). Hence, since \( \omega \) is a 0-1 measure, by finite additivity \( \omega \left( \cap_{j=1}^J A_\alpha(u_{k_j}) \right) = 1 \) for all \( J \). Since \( \omega \in \mathfrak{G} \subset L^*_\infty(X, \mathcal{L}, \lambda) \), it follows that (3.4) is false for all \( J \). Finally note that for a strictly increasing sequence \( \{k_j\} \) and \( \alpha > 0 \),

\[
\lambda \{ x : v_J(x) > \alpha \} = \lambda \{ x : |u_{k_j}(x)| > \alpha \text{ for all } j \in \{1, \cdots, J\} \} = \lambda \{ \cap_{j=1}^J A_\alpha(u_{k_j}) \}.
\]

Since \( v_J(x) \geq v_{J+1}(x) \geq 0 \) it follows that \( v_J \to 0 \) in \( L_\infty(X, \mathcal{L}, \lambda) \) if and only if (3.4) holds for every \( \alpha > 0 \). This completes the proof. \( \square \)

There follows an analogue of Dini’s theorem that on compact topological spaces monotone, pointwise convergence of sequences of continuous functions to a continuous function is uniform; equivalently, for bounded monotone sequences weak and strong convergence coincide.

Corollary 3.7. Suppose \( \{u_k\} \) is bounded in \( L_\infty(X, \mathcal{L}, \lambda) \) and \( |u_k(x)| \geq |u_{k+1}(x)|, k \in \mathbb{N}, \) for \( \lambda \)-almost all \( x \in X \). Then \( u_k \to 0 \) if and only if \( u_k \to 0 \) in \( L_\infty(X, \mathcal{L}, \lambda) \).

Proof. The monotonicity of \( \{|u_k|\} \) implies that \( v_J \) coincides with \( |u_J| \) in Theorem 3.6 and so that \( |u_J| \to 0 \) in \( L_\infty(X, \mathcal{L}, \lambda) \) as \( J \to \infty \) when \( u_k \to 0 \) if in \( L_\infty(X, \mathcal{L}, \lambda) \). The converse is obvious. \( \square \)

3.1 Illustrations of Theorem 3.6

(1) In this example \( X = (-1, 1) \) with Lebesgue measure, \( u_k \) is supported in \([-1/2, 1/2], \|u_k\|_\infty = 1 \) and \( u_k \) weakly converges to zero, but \( u_k \neq 0 \) where \( u_k^\pm(x) = u_k(x \pm 1/2^{k+1}) \). To see this, let \( A_k = [1/2^{k+1}, 1/2^k], \) \( A_k^\pm = A_k + 1/2^k, \) \( u_k = \chi_{A_k} \) and \( u_k^\pm = \chi_{A_k^\pm} \). Clearly \( u_k^\pm(x) = u_k(x \pm 1/2^{k+1}) \) and \( u_k \neq 0 \) because \( v_J \), defined in (3.3) by \( u_k^\pm \), is 1 on \((0, 1/2^{j+1})\). But since \( \{A_k\} \) and \( \{A_k^\pm\} \) are two mutually disjoint families,
in (3.3) \(v_J\), defined for any \(\{k_j\} \subset \mathbb{N}\) by \(u_{k_j}\) or \(u_{-k_j}\), is zero for \(J \geq 2\). Hence \(u_k \to 0\) and \(u_k^- \to 0\). That \(\chi_{A_k} \to 0\) for a disjoint family of sets is used in Remark 4.5.

(2) In \(L_\infty(X, \mathcal{L}, \lambda)\) let \(u_k(x) = \sum_{i=1}^{\infty} \alpha_i \chi_{A_k^j}, x \in X\), where \(\sum_{i=1}^{\infty} |\alpha_i| < \infty\) and, for each \(i \in \mathbb{N}\), \(\{A_k^j\}_{k \in \mathbb{N}}\) is a family of mutually disjoint non-null measurable sets. Then \(u_k \to 0\) in \(L_\infty(X, \mathcal{L}, \lambda)\).

To see this, note that for each \(x \in X\) and \(i \in \mathbb{N}\) there exists at most one \(k \in \mathbb{N}\), denoted, if it exists, by \(\kappa(x, i)\), such that \(x \in A_k^j\) if and only if \(k = \kappa(x, i)\). Note also that for \(\epsilon > 0\) there exists \(I_\epsilon \in \mathbb{N}\) such that \(\sum_{i \in I_\epsilon} |\alpha_i| < \epsilon\). Hence, for any given \(k \in \mathbb{N}\) and \(x \in X\),

\[
|u_k(x)| \leq \sum_{i \in I_\epsilon} |\alpha_i| \chi_{A_k^j}(x) + \epsilon = \sum_{i \in \{1, \ldots, I_\epsilon\}} |\alpha_i| + \epsilon.
\]

Since \(\{\kappa(x, i) : i \in \{1, \ldots, I_\epsilon\}\}\) has at most \(I_\epsilon\) elements, there exists \(k \in \{1, \ldots, I_\epsilon + 1\}\) such that \(k \neq \kappa(x, i)\) for any \(i \in \{1, \ldots, I_\epsilon\}\). Consequently \(\inf \{|u_k(x)| : 1 \leq k \leq I_\epsilon + 1\} \leq \epsilon\), independent of \(x \in X\). Since this argument can be repeated with \(k \in \mathbb{N}\) replaced by any strictly increasing subsequence \(\{k_j\}\), it follows that \(\{v_J\}\) defined in terms of any subsequence in (3.3) has \(\|v_J\|_\infty \to 0\) in \(L_\infty(X, \mathcal{L}, \lambda)\). The weak convergence of \(\{u_k\}\) follows. For the special case, take \(\alpha_1 = 1\) and \(\alpha_i = 0, i \geq 2\).

(3) Let \(u : \mathbb{R} \to \mathbb{R}\) be essentially bounded and measurable with \(|u(x)| \to 0\) as \(|x| \to \infty\) and let \(u_k(x) = u(x + k)\). Then \(u_k \to 0\) in \(L_\infty(X, \mathcal{L}, \lambda)\) where \(\lambda\) is Lebesgue measure on \(\mathbb{R}\). To see this, for \(\epsilon > 0\) suppose that \(|u(x)| < \epsilon\) if \(|x| > K_\epsilon\). Then for any \(\{k_j\} \subset \mathbb{N}\), \(\|v_J\|_\infty < \epsilon\) for all \(J \geq K_\epsilon\) where \(\{v_J\}\) is defined in terms of \(\{u_k\}\) by (3.3), and the result follows.

(4) Let \(u : \mathbb{R} \to \mathbb{R}\) be essentially bounded and measurable with \(|u(x)| \to 0\) as \(x \to \infty\) and \(u(x) \to 1\) as \(x \to -\infty\). Let \(u_k(x) = u(x + k)\). Then \(u_k(x) \to 0\) as \(k \to \infty\) for all \(x \in \mathbb{R}\), but \(u_k\) is not weakly convergent to 0 because of Theorem 3.6. However, in the notation of Definition 4.3, \(u_k \to 0\) at every point of \(\mathbb{R}\), but not at the point at infinity in its one-point compactification.

(5) Define \(\{u_k\}_{k \in \mathbb{N}} \subset L_\infty(X, \mathcal{L}, \lambda)\) by \(u_k(x) = \sin(1/(kx))\), \(x \in X = (0, 2\pi)\), with the standard measure \(\lambda\) on the Lebesgue \(\sigma\)-algebra on \(X\). Clearly \(|u_k(x)| \to 0\) as \(k \to \infty\) uniformly on \((\epsilon, 2\pi)\) for any \(\epsilon \in (0, 2\pi)\). Therefore if a subsequence \(\{u_{k_j}\}\) is weakly convergent, its weak limit must be zero.

To see that no subsequence of \(\{u_k\}\) is weakly convergent to 0, consider first a strictly increasing sequence \(\{k_j\}\) of natural numbers for which there exists a prime power \(p^{m_j}\) which does not divide \(k_j\) for all \(j\). Then, for \(J \in \mathbb{N}\) sufficiently large, let

\[
x_J = \left\{ \frac{\pi}{p^m} \text{lcm} \{k_1, \ldots, k_j\} \right\}^{-1} \in (0, 2\pi),
\]

where \(\text{lcm}\) denotes the least common multiple. Then, since \(p^{m_j} \nmid k_j\) and \(p\) is prime,

\[
\frac{1}{k_j x_j} = \frac{\text{lcm} \{k_1, \ldots, k_j\}}{p^m k_j} \pi \text{ where } \frac{\text{lcm} \{k_1, \ldots, k_j\}}{k_j} = r \mod p^m, \quad r \in \{1, \ldots, p^m - 1\},
\]

from which it follows that \(|u_{k_j}(x_J)| \geq |\sin(\pi/p^{m_j})| > 0\), independent of \(J\). Since, for all \(j \in \{1, \ldots, J\}\), \(u_{k_j}\) is continuous at \(x_J\), it follows that \(|v_J|_{L_\infty(X, \mathcal{L}, \lambda)} \geq |\sin(\pi/p^{m_j})| > 0\) for all \(J\) sufficiently large. By Theorem 3.6 this shows that \(u_{k_j} \not\to 0\) if \(\{k_j\}\) has a subsequence \(\{k_{j}^{'}\}\) for which \(p^{m_j} \nmid k_{j}^{'}\) for all \(j \in \mathbb{N}\). Note that if this hypothesis is not satisfied by \(\{k_j\}\) for any prime \(p\) and \(m \in \mathbb{N}\), then every \(K \in \mathbb{N}\) is a divisor of \(k_j\) for all \(j\) sufficiently large, how large depending on \(K\). Consequently, if \(u_{k_j} \to 0\), \(\{k_j\}\) has subsequence \(\{k_{j}^{'}\}\) with the property that \(2^{j+2}k_{j}^{'}\) divides \(k_{j+1}^{'}\) for all \(j\). In other words \(2^{j+2}n_jk_{j}^{'} = k_{j+1}^{'}\), \(n_j \in \mathbb{N}\), and \(0, k_{j+1}^{'}\) is a union of \(2^{j+2}n_j\) disjoint intervals of length \(k_{j}^{'}\).

Now fixed \(J \in \mathbb{N}\), let \(m_J\) denote the mid-point of \(I_J := [0, k_{j}^{'}]\), and let \(I_{j-1} := [m_j - k_{j-1}^{'}, m_j]\), which is a half open interval of length \(k_{j-1}^{'}\) to the left of \(m_j\). Then

\[
x = r \mod k_{j}^{'} \text{ where } r \in \left[ \frac{k_{j}^{'}}{2} \left( 1 - \frac{1}{2^{n_{j-1}}} \right), \frac{k_{j}^{'}}{2} \right] \text{ for all } x \in I_{j-1},
\]
This section deals with only values 0 or 1 is a Dirac measure concentrated at a point (the "point at infinity"), and a subset such that \( \omega \) with \( \| \omega \| = 1 \) for all open sets \( \mathcal{G} \) with \( \mathcal{G} \) is Hausdorff, if there is another \( x_1 \in X \) with this property there are open sets with \( x_0 \in G_{x_0}, x_1 \in G_{x_1} \) and \( G_{x_0} \cap G_{x_1} = \emptyset \). But this is impossible because by finite additivity \( \omega(G) = 0 \) for all compact sets \( K \). By local compactness, for \( x \in X \) there is an open set \( G_x \) with \( x \in G_x \) and its closure \( \overline{G_x} \) is compact. Since \( \omega(G_x) \leq \omega(\overline{G_x}) = 0 \), there is no \( x \in X \) with the required property. Finally, the existence of \( x_0 \) when \( X \) is compact follows because \( \omega(X) = 1 \). This completes the proof.

Let \( (X, \tau) \) be a locally compact Hausdorff space and \( \omega \in \mathcal{G} \). Then there exists a unique \( x_0 \in X_\infty \) such that \( \omega_\infty(G) = 1 \) for all open sets \( G \) in \( X_\infty \) with \( x_0 \in G \), \( x_0 = x_\infty \) if and only if \( \omega(K) = 0 \) for all compact \( K \subset X \) and \( x_0 \in X \) if \( (X, \tau) \) is compact.
4.1 Localization of Weak Convergence in $L_\infty(X, \mathcal{B}, \lambda)$

By (2.8) there is a one-to-one correspondence between $\mathcal{G}$ and $\mathcal{F}$. For $x_0 \in X_\infty$, let $\mathcal{G}(x_0) \subset \mathcal{G}$ denote the set of $\omega \in \mathcal{G}$ for which the conclusions of Lemma 4.1 holds, and let $\mathcal{F}(x_0) \subset \mathcal{F}$ be the corresponding family of ultrafilters. Then, by Lemma 4.1,

$$\mathcal{G} = \bigcup_{x_0 \in X_\infty} \mathcal{G}(x_0), \quad \mathcal{F} = \bigcup_{x_0 \in X_\infty} \mathcal{F}(x_0),$$

(4.1)

which leads to the following definition of weak pointwise convergence.

**Definition 4.3.** $u_k$ converges weakly to $u$ at $x_0 \in X_\infty$ if

$$\int_X u_k \, d\omega \to \int_X u \, d\omega \text{ for all } \omega \in \mathcal{G}(x_0).$$

The localized version of Theorem 3.6 is immediate. For $u \in L_\infty(X, \mathcal{L}, \lambda)$, $\alpha > 0$ and $E \in \mathcal{L}$ let

$$A_\alpha(u|_E) = \{x \in E : |u(x)| > \alpha\}.$$

**Theorem 4.4.** A bounded sequence $\{u_k\}$ in $L_\infty(X, \mathcal{B}, \lambda)$ converges weakly to zero at $x_0 \in X_\infty$ if and only if for every $\alpha > 0$, every strictly increasing sequence $\{k_j\}$ in $\mathbb{N}$ and every open $G \subset X_\infty$ with $x_0 \in G$ there exists $\lambda$ with $\lambda\{ \cap_{j=1}^{\infty} A_\alpha(u_{k_j}|_G) \} = 0$. Equivalently, in (3.3), $v_j \to 0$ in $L_\infty(G, \mathcal{B}, \lambda)$.

By analogy with (2.9), for $x_0 \in X_\infty$ the essential range of $u$ at $x_0 \in X_\infty$ is defined by

$$\mathcal{R}(u)(x_0) = \left\{ \int_X u \, d\omega : \omega \in \mathcal{G}(x_0) \right\}.$$  

(4.2)

As in Corollary 2.11, for open $G$ with $x_0 \in G$,

$$\mathcal{R}(u)(x_0) = \left\{ \int_G u \, d\omega : \omega \in \mathcal{G}(x_0) \right\} = \{ \alpha : \lambda\{ x \in G : |\alpha - u(x)| < \epsilon \} > 0 \text{ for all } \epsilon > 0 \}.$$  

(4.3)

Note that $\mathcal{R}(u)(x_0)$ is closed in $\mathbb{R}$ because, by (4.3), for any $x_0 \in X$ its complement is open. It is immediate from (2.12), Lemmas 4.1 and 4.2 that $u_k \rightharpoonup u$ in $L_\infty(X, \mathcal{B}, \lambda)$ if and only if for all $x_0 \in X_\infty$

$$\alpha_k := \int_X u_k \, d\omega \to \int_X u \, d\omega =: \alpha \text{ as } k \to \infty \text{ for all } \omega \in \mathcal{G}(x_0),$$

which is not equivalent to $\alpha_k \to \alpha$ when $\alpha_k \in \mathcal{R}(u_k)(x_0)$ and $\alpha \in \mathcal{R}(u)(x_0)$ because, possibly,

$$\alpha_k = \int_X u_k \, d\omega_k \text{ and } \alpha = \int_X u \, d\omega, \text{ but } \omega_k \neq \omega.$$

However, $\alpha = \int_X u \, d\omega \in \mathcal{R}(u)(x_0)$, $\omega \in \mathcal{G}(x_0)$, may be thought of as a directional limit of $u$ at $x_0$, the “direction” being determined by $\mathcal{F}(\omega)$. Then weak convergence in $L_\infty(X, \mathcal{B}, \lambda)$ is equivalent to convergence, for each $\mathcal{F} \in \mathcal{F}(x_0)$, of the directional limits of $u_k$ at $x_0$ to corresponding directional limits of $u$ at $x_0$, for each $x_0 \in X_\infty$. Therefore, by Theorem 2.8, $u_k \rightharpoonup u$ in $L_\infty(X, \mathcal{B}, \lambda)$ if and only if for all $x_0 \in X_\infty$ and all $\omega \in \mathcal{G}(x_0)$

$$|\alpha_k - \alpha| \to 0 \quad \text{and} \quad \omega \{ x \in G : |u_k(x) - \alpha_k| + |u(x) - \alpha| < \epsilon \} = 1$$

for all $\epsilon > 0$ and all open sets $G \subset X_\infty$ with $x_0 \in G$,  

(4.4a)

equivalently $u_k \rightharpoonup u$ in $L_\infty(X, \mathcal{B}, \lambda)$ if and only if for all $x_0 \in X_\infty$ and all $\mathcal{F} \in \mathcal{F}(x_0)$,

$$|\alpha_k - \alpha| \to 0 \quad \text{and} \quad \omega \{ x \in G : |u_k(x) - \alpha_k| + |u(x) - \alpha| < \epsilon \} \in \mathcal{F}$$

for all $\epsilon > 0$ and all open sets $G \subset X_\infty$ with $x_0 \in G$.  

(4.4b)
Remark 4.5. It follows that for \( u_k \to u \) it is necessary that for every \( x_0 \in X_\infty \) and every \( \alpha \in \mathcal{R}(u)(x_0) \) there exist \( \alpha_k \in \mathcal{R}(u_k)(x_0) \) such that \( \alpha_k \to \alpha \) as \( k \to \infty \) and sufficient that for every \( x_0 \in X_\infty \)

\[
\sup \{ |\gamma| : \gamma \in \mathcal{R}(u_k - u)(x_0) \} \to 0 \quad \text{as} \quad k \to \infty.
\]

As noted earlier, the necessary condition is not sufficient. To see that the sufficient condition is not necessary, let \( u_k = \chi_{A_k} \) where \( \{A_k\} \) is a sequence of disjoint segments centred on the origin 0 of the unit disc \( X \) in \( \mathbb{R}^2 \). Then \( \mathcal{R}(u_k)(0) = \{0, 1\} \) but \( u_k \to 0 \) by the last remark in Section 3.1 (1) or, equivalently, by Section 3.1 (2) with \( \alpha_1 = 1 \) and \( \alpha_i = 0, \ i \geq 2 \). In this example \( \int_X u_k d\omega \to 0 \), but not uniformly, for every \( \omega \in \mathcal{G}(0) \).

5 Restriction to \( C_0(X, \tau) \) of Elements of \( L^*_\infty (X, \mathcal{B}, \lambda) \)

Throughout this section \((X, \tau)\) is a locally compact Hausdorff topological space and \( C_0(X, \tau) \) is the space of real-valued continuous functions \( v \) on \( X \) with the property that for all \( \epsilon > 0 \) there exists a compact set \( K \subset X \) such that \( |v(x)| < \epsilon \) for all \( x \in X \setminus K \). When endowed with the maximum norm

\[
\|v\|_\infty = \max_{x \in X} |v(x)|, \quad v \in C_0(X, \tau),
\]

\( C_0(X, \tau) \) is a Banach space which if \( X \) is compact consists of all real-valued continuous functions on \( X \). Let \( \nu \in L^*_\infty (X, \mathcal{B}, \lambda) \), as in Theorem 2.1 define \( f \in L_\infty (X, \mathcal{B}, \lambda)^* \) by

\[
f(u) = \int_X u d\nu, \quad u \in L_\infty (X, \mathcal{B}, \lambda),
\]

and let \( \hat{f} \) denote the restriction of \( f \) to \( C_0(X, \tau) \). By the Riesz Representation theorem [12, Thm. 6.19] there is a unique bounded regular Borel measure \( \hat{\nu} \in \Sigma(\mathcal{B}) \) corresponding to \( \hat{f} \), and consequently

\[
\int_X v d\nu = \int_X v d\hat{\nu} \quad \text{for all} \quad v \in C_0(X, \tau).
\]

The goal is to understand how \( \hat{\nu} \) depends on \( \nu \) and, since \( \hat{\nu}^\pm = \hat{\nu}^\mp \) (see (2.2b)), there is no loss of generality in restricting attention to non-negative \( \nu \in L^*_\infty (X, \mathcal{B}, \lambda) \). Recall

(i) from the Yosida-Hewitt decomposition (2.5), \( \nu = \mu + g\lambda \) where \( \mu \in L^*_\infty (X, \mathcal{B}, \lambda) \) is purely finitely additive and \( g\lambda, g \in L_1(X, \mathcal{B}, \lambda) \), is \( \sigma \)-additive.

(ii) from the Lebesgue-Radon-Nikodym Theorem [7, Thm. 3.8], [12, Thm. 6.10], \( \hat{\nu} = \rho + k\lambda \) where \( \rho \) and \( k\lambda \) are \( \sigma \)-additive, \( k \in L_1(X, \mathcal{B}, \lambda) \), and \( \rho \) is singular with respect to \( \lambda \). Thus \( \hat{\nu} \) has a singularity with respect to \( \lambda \) if \( \hat{\nu}(E) \neq 0 \) (equivalently \( \rho(E) \neq 0 \)) for some \( E \in \mathcal{N} \), and \( \hat{\nu} \) is singular if in addition \( k = 0 \), where

\[
\int_X v d\mu + \int_X vg d\lambda = \int_X v d\nu = \int_X v d\hat{\nu} = \int_X v d\rho + \int_X vk d\lambda \quad \text{for all} \quad v \in C_0(X, \tau),
\]

where \( \rho \perp \lambda \) in \( \Sigma(\mathcal{B}) \), \( \mu \perp \lambda \) in \( \gamma(\mathcal{B}) \) (see Remark 2.3 for the distinction), and \( g, k \in L_1(X, \mathcal{B}, \lambda) \). Valadier was first to note that the relation between \( \mu \) and \( \rho \), and \( g \) and \( k \) is not straightforward.

Theorem (Valadier [13]). When \( \lambda \) is Lebesgue measure on \([0, 1]\) there is a non-negative \( \nu \in \Pi(\mathcal{B}) \) with

\[
\int_0^1 v d\nu = \int_0^1 v d\lambda \quad \text{for all} \quad v \in C^*[0, 1].
\]

Thus in (i), (ii) and (5.3), \( 0 \neq \mu \in \Pi(\mathcal{B}) \) and \( g = 0 \) but \( \rho = 0 \) and \( k \equiv 1 \), and \( \hat{\nu} \) has no singularity.
Hensgen independently observed that the last claim in [15, Theorem 3.4] is false.

**Theorem** (Hensgen [8]). With $X = (0, 1)$ there exists $\nu \in L^\infty_\infty(X, \mathcal{B}, \lambda)$ which is non-zero and not purely finitely additive but $\int_0^1 v \, d\nu = 0$ for all $v \in C(0, 1)$.

Subsequently Abramovich & Wickstead [1] provided wide ranging generalizations and recently Wrobel [14] gave a sufficient condition on $\nu$ for $\hat{\nu}$ to be singular with respect to Lebesgue measure on $[0, 1]$. To find a formula for $\hat{\nu}$ satisfying (5.2) for a given non-negative $\nu \in L^\infty_\infty(X, \mathcal{B}, \lambda)$, and to characterise those $\nu$ for which $\hat{\nu}$ has a singularity, recall the following version of Urysohn’s Lemma.

**Lemma 5.1.** [12, §2.12] If $K \subset G \subset X$ where $K$ is compact and $G$ is open, there exists a continuous function $f : X \to [0, 1]$ such that $f(K) = 1$, $\{x : f(x) > 0\} \subset G$ is compact, and hence $f \in C_0(X, \tau)$.

**Lemma 5.2.** Suppose $0 \leq \nu \in L^\infty_\infty(X, \mathcal{B}, \lambda)$ and $B \subset \mathcal{B}$. Then $\nu(K) \leq \hat{\nu}(B) \leq \nu(G)$ for compact $K$ and open $G$ with $K \subset B \subset G$. Moreover

$$\nu(K) \leq \hat{\nu}(K) \text{ and } \hat{\nu}(G) \leq \nu(G) \text{ for all compact } K \text{ and open } G \text{ in } X.$$  

and $\nu(F) \leq \hat{\nu}(F) + \nu(X) - \hat{\nu}(X)$ if $F$ is closed. Thus $\hat{\nu}(X) = \nu(X)$ implies that $\nu(F) \leq \hat{\nu}(F)$ for all closed sets $F \subset X$. (That $\nu(X) = \hat{\nu}(X)$ when $(X, \tau)$ is compact was noted following (5.2).)

**Proof.** For a given Borel set $B$ and $K \subset B \subset G$ as in the statement, let $f$ be the continuous function determined in Lemma 5.1 by $K$ and $G$. Then

$$\nu(K) \leq \int_X f \, d\nu \leq \nu(G) \text{ and } \hat{\nu}(K) \leq \int_X f \, d\hat{\nu} \leq \hat{\nu}(G).$$

It follows from (5.2) that $\nu(K) \leq \hat{\nu}(G)$ and $\hat{\nu}(K) \leq \nu(G)$ whence, since $\hat{\nu}$ is regular [12, Thm. 6.19], $\nu(K) \leq \hat{\nu}(B) \leq \nu(G)$. In particular, if $B = K$ is compact, $\nu(K) \leq \hat{\nu}(K)$, and if $B = G$ is open, $\hat{\nu}(G) \leq \nu(G)$. That $\nu(F) \leq \hat{\nu}(F) + \nu(X) - \hat{\nu}(X)$ when $F$ is closed follows by finite additivity since $0 \leq \nu(X)$, $\hat{\nu}(X) < \infty$. \hfill \Box

**Remark 5.3.** A non-negative finitely additive set function $\nu$ on $\mathcal{B}$ is said to be regular [6, III.5.11] if for all $E \in \mathcal{B}$ and $\epsilon > 0$ there are sets $F \subset E \subset G$ with $F$ closed, $G$ open and $\nu(G \setminus F) < \epsilon$. If $X$ is compact and $\nu$ is regular, by a theorem of Alexandroff [6, III.5.13] $\nu$ is $\sigma$-additive and hence $\hat{\nu} = \nu$. By Lemma 5.2, if $\nu(X) = \hat{\nu}(X)$ and $F \subset E \subset G$, where $F$ is closed and $G$ is open,

$$\nu(F) \leq \hat{\nu}(F) \leq \hat{\nu}(E) \leq \hat{\nu}(G) \leq \nu(G).$$

Hence if $\nu(X) = \hat{\nu}(X)$ and $\nu \geq 0$ regular implies that $\nu = \hat{\nu}$ is $\sigma$-additive on $\mathcal{B}$. \hfill \Box

**Theorem 5.4.** Suppose $K$ is compact, $G$ is open, $K \subset G$ and $0 \leq \nu \in L^\infty_\infty(X, \mathcal{B}, \lambda)$. Then for $n \in \mathbb{N}$ there exist compact $K_n$ and open $G_n$ with

$$K \subset G_n \subset K_n \subset G, \quad G_n \subset G_{n-1}, \quad K_n \subset K_{n-1},$$

$$\hat{\nu}(K) \leq \nu(K_n), \quad \hat{\nu}(G) \geq \nu(G_n) \quad \text{and} \quad \lambda(K_n) < \lambda(K) + 1/n.$$

**Proof.** Since $\lambda$ is a regular Borel measure that is finite on compact sets there exist open sets $G^k$ with $K \subset G^k \subset G$ and $\lambda(G^k) < \lambda(K) + 1/k$ for $k \in \mathbb{N}$. By Lemma 5.1 there exists a continuous function $f_k : X \to [0, 1]$ such that $f_k(K) = 1$ and $\{x : f_k(x) > 0\}$ is a compact subset of $G^k$. For $x \in X$, let $g_n(x) = \min\{f_k(x) : k \leq n\}$ so that $g_n \leq g_{n-1}$, $g_n$ is continuous on $X$, $g_n(K) = 1$ and $\{x : g_n(x) > 0\} \subset G^n$ is compact.
Let $G_n = \{ x : g_n(x) > 0 \}$ and $K_n = \{ x : g_n(x) > 0 \}$. Then $K \subset G_n \subset K_n \subset G^n \subset G$ and, by Lemma 5.2,

$$\hat{\nu}(K) \leq \hat{\nu}(G_n) \leq \nu(G_n) \leq \nu(K_n), \quad \hat{\nu}(G) \geq \hat{\nu}(K_n) \geq \nu(K_n) \geq \nu(G_n),$$

and $\lambda(K_n) < \lambda(K) + 1/n$ because $K_n \subset G^n$. Now $\{G_n\}$ and $\{K_n\}$ are nested sequences of open and compact sets, respectively, because $g_n(x)$ is decreasing in $n$, with the required properties.

**Corollary 5.5.** For $G$ open, $K$ compact and $\nu \in L^*_\infty(X, \mathcal{B}, \lambda)$ non-negative,

$$\hat{\nu}(G) = \sup\{ \nu(K) : K \subset G, \ K \text{ compact} \}, \quad \hat{\nu}(K) = \inf\{ \nu(G) : K \subset G, \ G \text{ open} \}.$$

**Proof.** Let $G$ be open. Then for any $\epsilon > 0$ there exists compact $K_\epsilon \subset G$ with $\nu(K_\epsilon) > \hat{\nu}(G) - \epsilon$, since $\hat{\nu}$ is regular, and $\nu(K_\epsilon) \leq \nu(K) \leq \hat{\nu}(G)$ by Lemma 5.2. Now by Theorem 5.4 there exists compact $K_1$ with $K_\epsilon \subset K_1 \subset G$ and $\hat{\nu}(K_1) \geq \nu(K_1) \geq \nu(K_\epsilon) > \hat{\nu}(G) - \epsilon$. This establishes the first identity. Similarly for compact $K$ and $\epsilon > 0$ there exists open $G_\epsilon$ with $K \subset G_\epsilon$ and $\hat{\nu}(G_\epsilon) < \nu(K) + \epsilon$, and an open $G_1$ with $\nu(G_1) \geq \nu(G_1)$, $K \subset G_1 \subset G_\epsilon$, whence $\hat{\nu}(K) + \epsilon > \nu(G_\epsilon) \geq \nu(G_1)$.

**Corollary 5.6.** For $0 \leq \nu \in L^*_\infty(X, \mathcal{B}, \lambda)$, $\hat{\nu} \in \Sigma(\mathcal{B})$ has a singularity if and only if there exists $\alpha > 0$ and a sequence of compact sets with $\nu(K_n) \geq \alpha$ for all $n$, and $\lambda(K_n) \rightarrow 0$ as $n \rightarrow \infty$.

**Proof.** If $\alpha > 0$ and such a sequence exists, by Lemma 5.2, $\hat{\nu}(K_n) \geq \alpha$ for all $n$. Since $\{K_n\}$ is nested and $\hat{\nu}$ is $\sigma$-additive it follows that $\hat{\nu}(K) \geq \alpha$ where $K = \bigcap_n K_n$. Since $K \in \mathcal{N}$, because $\lim_{n \rightarrow \infty} \lambda(K_n) = 0$ and $\lambda$ is $\sigma$-additive, $\nu$ has a singularity. Conversely if $\hat{\nu}(K) \geq \alpha$ there exists $E \in \mathcal{N}$ and $\alpha > 0$ with $\hat{\nu}(E) = 2\alpha$. Since $\hat{\nu}$ is regular, there exists a compact $K \subset E$ with $\hat{\nu}(K) \geq \alpha > 0$. Now since $\lambda(K) = 0$ because $K \subset E \in \mathcal{N}$, the existence of compact sets with $\nu(K_n) \geq \alpha$ for all $n$, and $\lambda(K_n) \rightarrow 0$ as $n \rightarrow \infty$ follows from Theorem 5.4.

**Theorem 5.7.** For $B \in \mathcal{B}$ and $0 \leq \nu \in L^*_\infty(X, \mathcal{B}, \lambda)$,

$$\hat{\nu}(B) = \inf_{G \text{ open}} \left\{ \sup_{K \text{ compact}} \nu(K) \right\} = \sup_{K \text{ compact}} \left\{ \inf_{G \text{ open}} \nu(G) \right\}.$$

**Proof.** This follows from Corollary 5.5 since $\hat{\nu}$ is a regular Borel measure.

**Corollary 5.8.** (a) For $\omega \in \mathcal{G}$, either $\hat{\omega}$ is zero or $\hat{\omega}$ is a Dirac measure. (b) Both possibilities in (a) may occur when $(X, \tau)$ is not compact. (c) If $\hat{\omega} = \delta_{x_0} \in \mathcal{D}$, then $\omega \in \mathcal{G}(x_0)$.

**Proof.** (a) If $\omega(K) = 0$ for all compact $K$, the first formula of (5.4) implies that $\omega = 0$. If $\omega(K) = 1$ for some compact $K$, by Lemma 4.1 there is a unique $x_0 \in X$ for which $\omega(G) = 1$ if $x_0 \in G$ and $G$ is open. From the second part of (5.4) it is immediate that $\hat{\omega}(B) = 1$ if and only if $x_0 \in B$. Hence $\hat{\omega} \in \mathcal{D}$.

(b) For an example of both possibilities let $X = (0, 1)$ with the standard locally compact topology and Lebesgue measure. Let $\omega \in \mathcal{G}$ be defined by Theorem 2.9 with $E_\ell = (0, 1/\ell)$, $\ell \in \mathbb{N}$. Then $\omega(K) = 0$ for all compact $K \subset (0, 1)$ and hence $\omega = 0$. On the other hand if $E_\ell = (1/2 + 1/\ell, 1/2)$ in Theorem 2.9, $\omega \in \mathcal{G}$ with $\omega([1/2 + 1/\ell, 1/2]) = 1$ for all $\ell$ and hence $\omega = \delta_{1/2} \in \mathcal{D}$.

(c) When $\omega = \delta_{x_0}$, let $x_0 \in G$ open. Since $\{x_0\}$ is compact by Lemma 5.1 there exists $v \in C_0(X, \tau)$ with $v(X) \subset [0, 1]$, $v(x_0) = 1$, $v(X \setminus G) = 0$. Now $\omega(G) = 1$ for every open set with $x_0 \in G$ since

$$1 \geq \omega(G) \geq \int_G v \ d\omega = \int_X v \ d\omega = \int_X v \ d\omega = v(x_0) = 1.$$  

\[\square\]
A Appendix: $\mathcal{G}$ and Extreme Points of the Unit Ball in $L^*_\infty(X,\mathcal{L},\lambda)$

**Theorem A.1** (Rainwater [11]). *In a Banach space $B$, $x_k \to x$ as $k \to \infty$ if and only if $f(x_k) \to f(x)$ in $\mathbb{R}$ for all extreme points $f$ of the closed unit ball in $B^*$.*

Recall that $L^*_\infty(X,\mathcal{L},\lambda)$ is the set of finitely additive measures on $\mathcal{L}$ that are zero on $\mathcal{N}$. Let $U^*_\infty = \{\nu \in L^*_\infty(X,\mathcal{L},\lambda) : |\nu|(X) \leqslant 1\}$, the closed unit ball in $L^*_\infty(X,\mathcal{L},\lambda)$. Then $\nu \in U^*_\infty$ is an extreme point of $U^*_\infty$ if for $\nu_1, \nu_2 \in U^*_\infty$ and $\alpha \in (0, 1)$

$$
\nu(E) = \alpha \nu_1(E) + (1 - \alpha) \nu_2(E) \text{ for all } E \in \mathcal{L} \text{ implies that } \nu = \nu_1 = \nu_2.
$$

Clearly extreme points have $|\nu|(X) = 1$ and, by Theorem A.1, $u_k \to u_0$ in $L^*_\infty(X,\mathcal{L},\lambda)$ if and only if for some $M$

$$
\|u_k\|_\infty \leqslant M \text{ and } \int_X u_k \, d\nu \to \int_X u_0 \, d\nu \text{ as } k \to \infty \text{ for all extreme points } \nu \text{ of } U^*_\infty.
$$

Thus (2.12) is a consequence of the following result.

**Lemma A.2.** $\nu$ is an extreme point of $U^*_\infty$ if and only if either $\nu$ or $-\nu \in \mathcal{G}$, see (2.6).

*Proof.* If $|\nu|(X) = 1$ but $\nu$ is not one signed, then $|\nu| = \nu^+ + \nu^-$ where $\nu^+ \wedge \nu^- = 0$ and $\nu^+(X) \in (0, 1)$. Let $0 < \epsilon_0 = \frac{1}{2} \min\{|\nu^+(X)|, 1 - \nu^+(X)|\}$ and, by (2.2b), choose $A \in \mathcal{L}$ such that $\nu^+(X \setminus A) + \nu^-(A) = \epsilon < \epsilon_0$. If $\nu(A) = 0$ then $\nu^+(X) = \nu^+(X \setminus A) + \nu^+(A) = \nu^+(X \setminus A) + \nu^-(A) = \epsilon < \epsilon_0$, which is false. So $\nu(A) \neq 0$ and hence $|\nu|(A) > 0$. If $|\nu|(A) = 1$ then $\nu^+(X) = 1 + \epsilon - 2\nu^-(A) \geqslant 1 - \epsilon$, and hence $1 - \nu^+(X) < \epsilon < \epsilon_0$, which is false. So $|\nu|(A) \in (0, 1)$. Let

$$
\nu_1(E) = \frac{\nu(A \cap E)}{|\nu|(A)}, \quad \nu_2(E) = \frac{\nu((X \setminus A) \cap E)}{|\nu|(X \setminus A)} \quad \text{for all } E \in \mathcal{L}.
$$

Then $\nu_1, \nu_2 \in U^*_\infty$ and, for all $E \in \mathcal{L}$,

$$
\nu(E) = \alpha \nu_1(E) + (1 - \alpha) \nu_2(E), \text{ where } \alpha = |\nu|(A), \quad (1 - \alpha) = |\nu|(X \setminus A).
$$

Since $\alpha \in (0, 1)$, $\nu_1(A) = \nu(A)/|\nu|(A) \neq 0$ and $\nu_2(A) = 0$, this shows that $\nu$ is not an extreme element of $U^*_\infty$ if $\nu$ is not one-signed.

Now suppose $0 \leqslant \nu \in U^*_\infty$ (for $\nu \leqslant 0$ replace $\nu$ with $-\nu$). If $\nu \notin \mathcal{G}$ there exists $A \in \mathcal{L}$ with $\nu(A) \in (0, 1)$. Let

$$
\nu_1(E) = \frac{\nu(A \cap E)}{\nu(A)}, \quad \nu_2(E) = \frac{\nu((X \setminus A) \cap E)}{\nu(X \setminus A)} \quad \text{for all } E \in \mathcal{L}.
$$

Then $\nu_1, \nu_2 \in U^*_\infty$,

$$
\nu(E) = \alpha \nu_1(E) + (1 - \alpha) \nu_2(E) \text{ for all } E \in \mathcal{L}, \text{ where } \alpha = \nu(A), \quad (1 - \alpha) = \nu(X \setminus A).
$$

Since $\nu_1(A) = 1 \neq 0 = \nu_2(A)$, $\nu$ is not extreme. Hence $\nu$ extreme implies that $\pm \nu \in \mathcal{G}$.

Now suppose that $\nu \in \mathcal{G}$ and for all $E \in \mathcal{L}$,

$$
\nu(E) = \alpha \nu_1(E) + (1 - \alpha) \nu_2(E), \quad \alpha \in (0, 1), \quad \nu_1, \nu_2 \in U^*_\infty.
$$

Then $\nu \geqslant 0$ and if $\nu(E) = 1$,

$$
1 = \nu(E) = \alpha \nu_1(E) + (1 - \alpha) \nu_2(E) \leqslant \alpha |\nu_1|(X) + (1 - \alpha) |\nu_2|(X) \leqslant 1
$$

which implies that $\nu_1(E) = \nu_2(E) = \nu(E) = 1$. In particular $\nu_1(X) = \nu_2(X) = 1$. If $\nu(E) = 0$ then $\nu(X \setminus E) = 1$ and so $\nu_1(X \setminus E) = \nu_2(X \setminus E) = 1$, whence $\nu_1(E) = \nu_2(E) = \nu(E) = 0$. Thus $\nu = \nu_1 = \nu_2$ and $\nu$ is extreme if $\nu \in \mathcal{G}$.  \qed
Closing Remark. Although the main result, Theorem 3.6, is derived above from Yosida-Hewitt theory [15] without reference to other sources, Theorem A.1 and Lemma A.2 lead to (2.12), and Lemma 2.8 yields Corollary 3.1, thus dispensing with any need for Theorems 2.12 and 2.13.

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