Topological T-duality via Lie algebroids and $Q$-flux in Poisson-generalized geometry

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Abstract

It is known that the topological T-duality exchanges $H$ and $F$-fluxes. In this paper, we reformulate the topological T-duality as an exchange of two Lie algebroids in the generalized tangent bundle. Then, we apply the same formulation to the Poisson-generalized geometry, which is introduced in [1] to define $R$-fluxes as field strength associated with $\beta$-transformations. We propose a definition of $Q$-flux associated with $\beta$-diffeomorphisms, and show that the topological T-duality exchanges $R$ and $Q$-fluxes.

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1 Introduction

In the effective theory of string, there are various types of fluxes. Some of the fluxes are known to appear in the analysis on the string spectrum of the worldsheet theory. In particular, $H$-fluxes and geometric $F$-fluxes fit well in the framework of the generalized geometry [2, 3, 4]. Whereas some others of them, referred to as “non-geometric fluxes,” are expected to appear after performing duality transformations [5, 6, 7, 8, 9, 10], but the geometrical characterizations of them are still missing.

Our main motivation for introducing a new geometric structure in the previous paper [1] was to reveal the geometrical origin of such non-geometric fluxes. We proposed a variant of the generalized geometry as a candidate of a geometrical structure, which describes the one of the non-geometric fluxes.

The structure is based on a Courant algebroid, $(TM)_0 \oplus (T^*M)_\theta$, defined on a Poisson manifold $M$ equipped with a Poisson tensor $\theta$ [1]. In the standard generalized geometry [2, 3, 4], the Courant algebroid $TM \oplus T^*M$ is a basic object and can be considered as an extension of the Lie algebroid $TM$. Similarly, in the variant of the generalized geometry, which we call Poisson-generalized geometry, the Courant algebroid $(TM)_0 \oplus (T^*M)_\theta$ can be regarded as an extension of the Lie algebroid $(T^*M)_\theta$ of the Poisson manifold. They are dual with each other in the sense that the roles of the tangent and the cotangent bundles are exchanged. Apart from some differences, they are indeed equipped with analogous mathematical structures [1]. Hence, various concepts known in the standard generalized geometry such as Dirac structures, generalized Riemannian structures can be established also in the Poisson-generalized geometry.

One of the major differences is in their symmetries. The symmetry of the Courant algebroid $TM \oplus T^*M$ is given by the semidirect product of the diffeomorphism and $B$-transformation, whereas that of the the Courant algebroid $(TM)_0 \oplus (T^*M)_\theta$ is given by the semidirect product of $\beta$-diffeomorphism and $\beta$-transformation. As a result, in a similar way as an $H$-flux is associated with the twist of $(TM)_0 \oplus (T^*M)_\theta$ by the local $B$-gauge transformations [2, 11], the twist of $(TM)_0 \oplus (T^*M)_\theta$ by the local $\beta$-gauge transformations indicates the proper definition of the so-called $R$-flux, which is a 3-vector $R \in \Gamma(\Lambda^3TM)$, as a gauge field strength of local bivector gauge potentials [1].

In this paper, we investigate the Poisson-generalized geometry further. The aim of this paper is to propose a proper definition of another kind of non-geometric flux, the $Q$-flux, in the framework of Poisson-generalized geometry. The strategy is to require consistency with the topological T-duality, and to define a $Q$-flux as the Poisson analogue of a geometric $F$-flux.

The notion of topological T-duality [12, 13] is well understood in the framework of the generalized geometry [2, 14, 15, 16]. A remarkable feature of the topological T-duality is that it exchanges $H$-flux and geometric $F$-flux. Since the $F$-flux is defined as the curvature 2-form of a principal circle bundle, the topological T-duality provides a relation between two generalized geometries defined on two different circle bundles. This formulation is not suitable for our purpose, since we do not know the Poisson analogue of circle bundles.
Thus, in the former part of this paper, we reformulate the topological T-duality in the standard generalized geometry. We demonstrate the $S^1$-dimensional reduction of the generalized tangent bundle $TM \oplus T^*M$ and show that two Lie algebroids appear in two different ways in this setting. Both of them are isomorphic to the Lie algebroid $TN \oplus \mathbb{R}$ over the reduced base space $N$ with $M = N \times S^1$. Using this fact, we show that the topological T-duality can be reformulated as an exchange of these two Lie algebroids. Then we see that both the 2-form part of an $H$-flux and of a geometric $F$-flux are associated with the twisting of the same Lie algebroid $TN \oplus \mathbb{R}$. Thus, the topological T-duality results in the exchange of these fluxes, when fluxes are present.

An advantage of this reformulation of the topological T-duality from the viewpoint of Lie algebroid is to keep the base space $M$ unchanged under the topological T-duality. Thus, it is easy to apply the same procedure to the case of the Poisson-generalized geometry. In the latter part of this paper, we show that we can find an analogue of the topological T-duality in the $S^1$-reduction of the new Courant algebroid $(TM)_0 \oplus (T^*M)_\theta$, where the Lie algebroid $(T^*N)_\theta \oplus \mathbb{R}$ appears in two different ways. By considering the twisting of this Lie algebroid, $Q$-flux can be naturally defined. This $Q$-flux is defined as a gauge field strength bivector associated with $\beta$-diffeomorphisms, as a counterpart of the geometric $F$-flux associated with ordinary diffeomorphisms in the standard generalized geometry. We then show the consistency of our $R$ and $Q$-flux with the topological T-duality, which is summarized as the exchange of the bivector parts of $R$-flux and $Q$-flux.

The organization of this paper is as follows: In §2, we recall the basic setting of both the standard generalized geometry and the Poisson-generalized geometry. We also give characterizations of $H$ and $R$-fluxes in terms of bundle maps $\varphi$, which is useful to define twisted brackets and to give a simple proof of topological T-duality in the subsequent sections. In §3, we demonstrate the $S^1$-dimensional reduction of the generalized tangent bundle $TM \oplus T^*M$ and reformulate the topological T-duality using the Lie algebroid $TN \oplus \mathbb{R}$. By considering $H$ and $F$-fluxes as twistings of $TN \oplus \mathbb{R}$, we recover the exchange of $H$ and $F$-fluxes in this formulation. Then in §4, we apply the topological T-duality in the new formulation in §3 to the case of the Poisson-generalized geometry $(TM)_0 \oplus (T^*M)_\theta$. After formulating the topological T-duality using the Lie algebroid $(T^*N)_\theta \oplus \mathbb{R}$, we propose a geometrical definition of $Q$-fluxes. Then, we show that $R$ and $Q$-fluxes are also exchanged by the topological T-duality. §5 is devoted to the conclusion and discussion.

2 Generalized geometry and Poisson-generalized geometry

We first recall the basic setting of the generalized and the Poisson-generalized geometries. See [1] in more detail.
2.1 Generalized geometry

The generalized geometry is formulated in terms of the Courant algebroid $\mathcal{T} M \oplus \mathcal{T}^* M$, the generalized tangent bundle, with the inner product, the anchor map and the Courant bracket being defined as, for the generalized tangent vectors $e_1 = X + \xi$ and $e_2 = Y + \eta,$

\[
\langle e_1, e_2 \rangle = \frac{1}{\mathcal{Z}}(i_X \eta + i_Y \xi), \quad \rho(e_1) = X,
\]

\[
[e_1, e_2]_C = [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2} d(i_X \eta - i_Y \xi),
\]

respectively. The symmetry of the Courant algebroid consists of diffeomorphisms and $B$-transformations. These transformations can be represented by using a vector $Z \in \Gamma(\mathcal{T} M)$ and a 2-form $b \in \Gamma(\wedge^2 \mathcal{T}^* M)$ with $\mathcal{Z}b = 0$, as matrices

\[
\begin{pmatrix}
e^{\mathcal{L}Z} & 0 \\
0 & e^{\mathcal{L}Z}
\end{pmatrix}, \quad \begin{pmatrix}1 & 0 \\
b & 1
\end{pmatrix},
\]

where each matrix element is a bundle map and the matrices are acting on the column vector $(\Gamma(\mathcal{T} M), \Gamma(\mathcal{T}^* M))^t$ in the representation space.

An $H$-flux is specified by the data $(H, B_i, A_{ij}) \in \Gamma(\wedge^3 \mathcal{T}^* M) \times \Gamma(\wedge^2 \mathcal{T}^* U_i) \times \Gamma(\mathcal{T}^* U_{ij})$ such that

\[
H|_{U_i} = dB_i, \quad B_j - B_i|_{U_{ij}} = dA_{ij},
\]

where $\{U_i\}$ is a good open covering of $M$, and $U_{ij} = U_i \cap U_j$. An $H$-twisting of $\mathcal{T} M \oplus \mathcal{T}^* M$ is a construction of a new Courant algebroid $E$ from the data, satisfying the exact sequence

\[
0 \to \mathcal{T}^* M \xrightarrow{s^*} E \xrightarrow{\rho} \mathcal{T} M \to 0,
\]

with a splitting $s : \mathcal{T} M \to E$, where $\rho$ denotes the anchor map $E \to \mathcal{T} M$. The splitting $s$ is given locally by the $B$-transformation of $X \in \Gamma(\mathcal{T} M)$ as

\[
s(X) = X + B_i(X).
\]

The Courant algebroid $E$ is written as $E = s(\mathcal{T} M) \oplus \mathcal{T}^* M$. In this paper, we also denote the splitting as a bundle map $\varphi_H = s \oplus \text{id} : \mathcal{T} M \oplus \mathcal{T}^* M \to E$. The map $\varphi_H$ is globally-defined, and can be locally represented by a matrix

\[
\varphi_H = \begin{pmatrix}1 & 0 \\
B_i & 1
\end{pmatrix}.
\]

Then, the $H$-twisted bracket on $\mathcal{T} M \oplus \mathcal{T}^* M$ can be defined by

\[
[e_1, e_2]_H := \varphi_H^{-1} [\varphi_H(e_1), \varphi_H(e_2)]_C.
\]

Substituting $e_1 = X + \xi$ and $e_2 = Y + \eta$, the above $H$-twisted bracket gives

\[
[X + \xi, Y + \eta]_H = [X + \xi, Y + \eta]_C - i_X i_Y H.
\]

It is shown that the Courant algebroid $E$ with the Courant bracket and $\mathcal{T} M \oplus \mathcal{T}^* M$ with the $H$-twisted bracket are isomorphic to each other.

\[\text{We omit the symbol } \rho^* \text{ of the inclusion for notational simplicity.}\]
2.2 Poisson-generalized geometry

The Poisson-generalized geometry, introduced in [1], is formulated in terms of the Courant algebroid $(TM)_0 \oplus (T^*M)_\theta$, with the inner product and the anchor map

$$\langle e_1, e_2 \rangle = \frac{1}{2}(i_\xi Y + i_\eta X), \quad \rho(e_1) = \theta(\xi),$$

(2.10)

for generalized tangent vectors $e_1 = X + \xi$ and $e_2 = Y + \eta$, and the bracket

$$[e_1, e_2] = [\xi, \eta]_\theta + \mathcal{L}_\xi Y - \mathcal{L}_\eta X - \frac{1}{2}d_\theta(i_\xi Y - i_\eta X).$$

(2.11)

Here, $\mathcal{L}_\xi$, $d_\theta$ and $i_\xi$ are the $A$-Lie derivative, the $A$-differential and the $A$-interior product of the Lie algebroid $A = (T^*M)_\theta$ of the Poisson manifold, respectively. The bracket $[\cdot, \cdot]_\theta$ is the Lie bracket of the Lie algebroid $(T^*M)_\theta$, which is so-called the Koszul bracket. The symmetry of this Courant algebroid consists of $\beta$-diffeomorphisms and $\beta$-transformations. For a 1-from $\zeta \in \Gamma(T^*M)$ with $\mathcal{L}_\zeta \theta = 0$, and a bi-vector $\beta \in \Gamma(\wedge^2 T^*M)$ with $d_\theta \beta = 0$, they are represented as

$$\begin{pmatrix} e^\mathcal{L}_\zeta & 0 \\ 0 & e^\mathcal{L}_\zeta \end{pmatrix}, \quad \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix},$$

(2.12)

as a bundle map acting on $(\Gamma(TM), \Gamma(T^*M))^\sharp$.

An $R$-flux is specified by the data $(R, \beta_i, \alpha_{ij}) \in \Gamma(\wedge^3 TM) \times \Gamma(\wedge^2 TU_i) \times \Gamma(TU_{ij})$ such that

$$R|_{U_i} = d_\theta \beta_i, \quad \beta_j - \beta_i|_{U_{ij}} = d_\theta \alpha_{ij},$$

(2.13)

where $\{U_i\}$ is a good open covering of $M$, and $U_{ij} = U_i \cap U_j$. An $R$-twisting of $(TM)_0 \oplus (T^*M)_\theta$ is a construction of a new Courant algebroid $E$ from the data, satisfying the exact sequence

$$0 \to (TM)_0 \xrightarrow{\pi} E \xrightarrow{s} (T^*M)_\theta \to 0,$$

(2.14)

with a splitting $s : (T^*M)_\theta \to E$, where $\pi : E \to (T^*M)_\theta$ denotes the canonical projection (not the anchor map). Here the splitting $s$ is given locally

$$s(\xi) = \xi + \beta_i(\xi),$$

(2.15)

and the Courant algebroid $E$ is written as $E = (TM)_0 \oplus s((T^*M)_\theta)$. We denote this splitting as a bundle map $\varphi_R = \text{id} \oplus s : (TM)_0 \oplus (T^*M)_\theta \to E$, which is locally given by

$$\varphi_R = \begin{pmatrix} 1 & \beta_i \\ 0 & 1 \end{pmatrix}. $$

(2.16)

By using this, the $R$-twisted bracket on $(TM)_0 \oplus (T^*M)_\theta$ is defined by

$$[e_1, e_2]_R := \varphi_R^{-1} [\varphi_R(e_1), \varphi_R(e_2)],$$

(2.17)

and it is calculated for $e_1 = X + \xi$ and $e_2 = Y + \eta$ as

$$[e_1, e_2]_R = [e_1, e_2] - i_\xi i_\eta R.$$

(2.18)

It is shown that the Courant algebroid $E$ with the bracket (2.11) and $(TM)_0 \oplus (T^*M)_\theta$ with the $R$-twisted bracket are isomorphic to each other.
3 Topological T-duality of Generalized Geometry

In this section, we discuss a property of the topological T-duality in the standard generalized geometry, by using $S^1$-dimensional reduction of a target manifold $M = N \times S^1$. Although the concepts of the topological T-duality are well-known and explained already in the framework of generalized geometry in references [2],[14],[15],[16], we here present a new formulation of the topological T-duality in terms of the Lie algebroid. In this formulation, the following aspects of the topological T-duality become manifest:

1) A Lie algebroid $TN \oplus \mathbb{R}$ appears in the generalized tangent bundle $TM \oplus T^*M$ in two different ways, namely as $TN \oplus \langle \partial_y \rangle$ and $TN \oplus \langle dy \rangle$. The topological T-duality in the absence of fluxes is formulated as an exchange of these two Lie algebroids.

2) An $F$-flux is introduced as a field strength associated with a twisting of the Lie algebroid $TN \oplus \mathbb{R}$.

3) The twist of 2) is included in $TM \oplus T^*M$ as two different twistings, $H_2$- and $F$-twisting. The former twist is caused by the 2-form part of the $H$-flux, while the latter by the geometric $F$-flux.

4) The topological T-duality exchanges $H_2$- and $F$-fluxes.

The notations above and the details will be explained in the following subsections.

3.1 Topological T-duality without flux

Let us assume that a target manifold $M$ is a direct product $M = N \times S^1$, with local coordinates $(x^m, y)$, and regard $M$ as a trivial $S^1$-bundle over the base space $N$. Then, the $S^1$-dimensional reduction of the tangent bundle $TM$ corresponds to restricting the space of vector fields $\Gamma(TM)$ to the $S^1$-invariant vector fields, which we call basic throughout this paper. The basic vector field has the form

$$X = X_1 + f \partial_y,$$  \hspace{1cm} (3.1)

where $X_1 = X^m(x) \partial_m \in \Gamma(TN)$ and $f(x) \in C^\infty(N)$ are a vector field and a function on $N$, respectively. Note that they are independent of local $S^1$-coordinate $y$. We denote the space of basic vector fields as $\Gamma(TM)_{\text{basic}}$. The tangent bundle $TM$ is a Lie algebroid $(TM, \rho = \text{id.}, [\cdot, \cdot])$, with the anchor map $\rho$ being the identity map, and with the Lie bracket of vector fields. The space of the basic vector fields $\Gamma(TM)_{\text{basic}}$ closes under the Lie bracket

$$[X, Y] = [X_1 + f \partial_y, Y_1 + g \partial_y] = [X_1, Y_1]_{TN} + (\mathcal{L}_{X_1}g - \mathcal{L}_{Y_1}f) \partial_y.$$  \hspace{1cm} (3.2)

This bracket is the same as the Lie bracket of the Lie algebroid $A = TN \oplus \mathbb{R}$ over $N$, where $\mathbb{R}$ denotes the trivial line bundle over $N$. In the Lie algebroid $A = TN \oplus \mathbb{R}$, the elements have

\footnote{Our presentation is valid also for $M = N \times \mathbb{R}$.}
the form $X_1 + f \in TN \oplus \mathbb{R}$, the anchor map $\rho_A : A \to TN$ is defined by $\rho_A(X_1 + f) = X_1$, and its Lie bracket is given by

$$[X_1 + f, Y_1 + g]_A = [X_1, Y_1]_{TN} + (\mathcal{L}_{X_1} g - \mathcal{L}_{Y_1} f).$$ (3.3)

Thus the dimensional reduction reduces the Lie algebroid $TM$ over $M$ to $TN \oplus \mathbb{R}$ over $N$, with the identification of the anchor map $\rho_A = \rho|_{TN}$, the restriction of the anchor map of $TM$ to $TN$. To distinguish with another Lie algebroid given below, we denote the above Lie algebroid as $TN \oplus \langle \partial_y \rangle$, and thus $\Gamma(TM)_{\text{basic}} \simeq \Gamma(TN \oplus \langle \partial_y \rangle)$.

The same Lie algebroid $A = TN \oplus \mathbb{R}$ can appear differently in the generalized geometry $TM \oplus T^*M$. Consider a subbundle $L = \text{span}\{\partial_m, dy\}$ over $M$, which is a Dirac structure in the generalized tangent bundle $TM \oplus T^*M$. In general, a Dirac structure is a Lie algebroid with respect to the Courant bracket $\mathcal{L}_X g - \mathcal{L}_Y f$ dy, (3.4)

where $X_1 \in \Gamma(TN)$ and $f(x) \in C^\infty(N)$ are basic, i.e., independent of the local $S^1$-coordinate $y$. The Courant bracket of the basic sections of $L$ gives

$$[X_1 + fdy, Y_1 + gdy]_C = [X_1, Y_1]_{TN} + (\mathcal{L}_{X_1} g - \mathcal{L}_{Y_1} f) dy,$$ (3.5)

which is identical to (3.3), the Lie bracket of $A = TN \oplus \mathbb{R}$. We denote this Lie algebroid as $TN \oplus \langle dy \rangle$, and thus $\Gamma(L)_{\text{basic}} \simeq \Gamma(TN \oplus \langle dy \rangle)$.

In summary, the Lie algebroid $A = TN \oplus \mathbb{R}$ appears in two different ways in the framework of $TM \oplus T^*M$, and the bundle map $\mathcal{T} : TN \oplus \langle \partial_y \rangle \to TN \oplus \langle dy \rangle$ defines a Lie algebroid isomorphism, which induces the map between the sections as

$$\mathcal{T} : X_1 + f\partial_y \mapsto X_1 + fdy.$$ (3.6)

The situation can be summarized schematically as follows:

$$\begin{array}{c}
TM \oplus T^*M \\
\xymatrix{ & TN \oplus \langle \partial_y \rangle \ar[ld] \ar[rd] \ar@{~}[r] & TN \oplus \langle dy \rangle,} \\
TN \oplus \langle \partial_y \rangle & \\
\end{array}$$ (3.7)

where each diagonal arrow represents the dimensional reduction to the corresponding bundle over $N$. Note that the left diagonal arrow is accompanied by the restriction of the anchor map. The horizontal arrow represents the map $\mathcal{T}$.

This isomorphism is the key of the topological T-duality. In fact, the map $\mathcal{T}$ can be extended to the automorphism of the $S^1$-reduced generalized tangent bundle $TN \oplus \langle \partial_y \rangle \oplus T^*N \oplus \langle dy \rangle$. For the basic sections $\Gamma(TM \oplus T^*M)_{\text{basic}}$ of the form

$$e = X_1 + f\partial_y + \xi_1 + hdy,$$ (3.8)
where $X_1 \in \Gamma(TN)$, $\xi_1 \in \Gamma(T^*N)$ and $f, h \in C^\infty(N)$, the extension of the map $T$ is given by

$$T : X_1 + f \partial_y + \xi_1 + hdy \mapsto X_1 + h \partial_y + \xi_1 + f dy,$$

which just yields the interchange of $f$ and $h$. Then, it can be shown (see appendix A) that the map $T$ preserves the inner product, anchor map (restricted to $TN$), and the Courant bracket

$$\langle Te_1, Te_2 \rangle = \langle e_1, e_2 \rangle, \quad \rho|_{TN}(Te) = \rho|_{TN}(e),$$

$$[Te_1, Te_2]_C = T[e_1, e_2]_C .$$

Thus, the map $T$ defines an automorphism. In other words, $T$ defines an extra symmetry valid only for the basic sections.

### 3.2 $F$-twisting of the Lie algebroid $TN \oplus \mathbb{R}$

It is known\(^6\) that the twisting of $TN \oplus \mathbb{R}$ gives another Lie algebroid $A$ over $N$, satisfying the exact sequence

$$0 \to \mathbb{R} \to A \to TN \to 0,$$

and which is classified by $[F] \in H^2_{dR}(N)$ in the de Rham cohomology. The corresponding bracket in $TN \oplus \mathbb{R}$ becomes an $F$-twisted one:

$$[X_1 + f, Y_1 + g]_F = [X_1, Y_1]_{TN} + (L_{X_1}g - L_{Y_1}f) - F(X_1, Y_1).$$

The procedure to obtain the $F$-twisted bracket is analogous to the case of the $H$-twist. Given a good cover $\{U_i\}$ of $N$ and a trivialization $(F, a_i, \lambda_{ij})$ of a closed 2-form $F$ such that

$$F |_{U_i} = da_i, \quad a_j - a_i |_{U_{ij}} = d\lambda_{ij},$$

we define a transition function $G_{ij} : U_{ij} \to GL(d)$ ($d = \dim N + 1 = \dim M$) by

$$G_{ij} = \begin{pmatrix} 1 & 0 \\ d\lambda_{ij} & 1 \end{pmatrix} .$$

Then, since $G_{ij}$ satisfies the cocycle condition, we obtain a vector bundle over $N$

$$A = \coprod_{i \in N} TU_i \oplus \mathbb{R}_i / \sim .$$

Moreover, $A$ is a Lie algebroid, since each $TU_i \oplus \mathbb{R}_i \to U_i$ is a Lie algebroid and the gluing condition preserves the anchor map and the Lie bracket. The set $\{a_i\}$ of local 1-forms gives a splitting $\tilde{s} : TN \to A$, locally defined by

$$\tilde{s}(X_1) = X_1 - a_i(X_1),$$

---

\(^6\)The content of this section is a well-known material as Atiyah algebroids. See for example [2, 17, 18].
for $X_1 \in TU_i$. Since $\tilde{s}(TN)$ is globally well-defined, any section of $A$ is uniquely specified by
\[ \tilde{s}(X_1) + f, \] (3.17)
for $X_1 \in \Gamma(TN)$ and $f \in C^\infty(N)$. The Lie bracket of these sections can be calculated straightforwardly as
\[ [\tilde{s}(X_1) + f, \tilde{s}(Y_1) + g]_A = \tilde{s}([X_1, Y_1]) + \mathcal{L}_{X_1}g - \mathcal{L}_{Y_1}f - F(X_1, Y_1), \] (3.18)
which is identified with the $F$-twisted Lie bracket (3.12) under the identification of $A = \tilde{s}(TN) \oplus \mathbb{R}$ and $TN \oplus \mathbb{R}$.

It is obvious from the construction that the 2-form flux $F$ is a field strength associated with an abelian gauge symmetry defined by a set of local gauge parameters $f_i \in C^\infty(U_i)$ where the gauge transformation of the data $(F, a, \lambda)$ is given by
\[ F \mapsto F, \ a_i \mapsto a_i + df_i, \ \lambda_{ij} \mapsto \lambda_{ij} + f_j - f_i. \] (3.19)

If $[F/2\pi]$ is the image of the map of cohomologies $H^2(N; \mathbb{Z}) \rightarrow H^2(N; \mathbb{R}) \simeq H^2_{dR}(N)$, i.e., the 1st Chern class, then $A$ is identified with the Atiyah algebroid $TP/U(1)$, where $P$ is a principal $S^1$-bundle over $N$ with connection.

### 3.3 $H$ and $F$-fluxes in generalized geometry

The $F$-twisting of Lie algebroid $TN \oplus \mathbb{R}$ above can also appear in two different twistings of the dimensional reduction of the generalized tangent bundle $TM \oplus T^*M$:

1. By the $F$-twisting of $TN \oplus \langle \partial_y \rangle$, the $F$-twisted bracket (3.12) corresponds to
\[ [X_1 + f \partial_y, Y_1 + g \partial_y]_F = [X_1, Y_1] + ((\mathcal{L}_{X_1}g - \mathcal{L}_{Y_1}f) - F(X_1, Y_1)) \partial_y. \] (3.20)

   In this case $F$ is called a geometric flux, since this Lie algebroid is equivalent to a principal $S^1$-bundle $P$ over $N$ with curvature $F$.

2. By the $F$-twisting of $TN \oplus \langle dy \rangle$, the $F$-twisted bracket (3.12) corresponds to
\[ [X_1 + f dy, Y_1 + g dy]_F = [X_1, Y_1] + ((\mathcal{L}_{X_1}g - \mathcal{L}_{Y_1}f) - F(X_1, Y_1)) dy. \] (3.21)

   This twisting is necessarily a part of an $H$-twisting, since the twisting of $\partial_m$ and $dy$ is achieved by a $B$-transformation. We denote such 2-form $F$ as $H_2 \in \wedge^2 T^*N$. Then, the corresponding 3-form $H$-flux is $H = -H_2 \wedge dy$.

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7It is valid for both gauge group $G = \mathbb{R}$ and $G = U(1)$. For $G = U(1)$, it can be written as more familiar form $g_{ij} \mapsto e^{i\lambda_{ij}}g_{ij}e^{-i\lambda_{ij}}$ for $g_{ij} = e^{i\lambda_{ij}}$. 

8
We may consider more generally twistings of basic sections of the generalized tangent bundle by both an $H$-flux and an $F$-flux, represented schematically as

\[ \begin{align*}
& TM \oplus T^*M & TN \oplus \langle \partial_y \rangle & TN \oplus T^*N & TN \oplus \langle dy \rangle \\
\uparrow & & & & \downarrow
\end{align*} \]

Here 2-forms $F$ and $H_2$ on $N$ correspond to the $F$-twistings described above, and the 3-form $H_3$ on $N$ corresponds to the $H$-twisting of $TN \oplus T^*N$.

In the following, we formulate this schematic picture more precisely in the framework of the Courant algebroid. To this end, first we twist $TN \oplus \langle \partial_y \rangle$ by $F$ to obtain a Courant algebroid $A \oplus A^*$ over $N$ with $A = \tilde{s}(TN) \oplus \langle \partial_y \rangle$. Then, we twist $A \oplus A^*$ by $H$ to obtain a Courant algebroid $E = s(A) \oplus A^*$, which is equivalent to the $S^1$-reduced generalized tangent bundle with $(H, F)$-twisted Courant bracket.

### 3.3.1 Geometric $F$-flux

By the $F$-twisting of the Lie algebroid $TN \oplus \langle \partial_y \rangle$ for given data $(F, a_i, \lambda_{ij})$, we obtain a Lie algebroid $A \simeq \tilde{s}(TN) \oplus \langle \partial_y \rangle$, whose section has the form

\[ X = \tilde{s}(X_1) + f \partial_y, \quad \tilde{s}(X_1) = X_1 - a_i(X_1) \partial_y. \] (3.23)

If we regard $A = TP/U(1)$, where $P$ is a principal $S^1$-bundle over $N$ with curvature $F$, $\tilde{s}(X_1)$ defines the horizontal lift of $X_1$ in $TP$, while $\partial_y$ is the vertical direction. Considering its dual $A^* = T^*P/U(1)$, the corresponding connection 1-form $a$ on $P$ is defined locally by

\[ a|_{U_i} = dy + a_i, \] (3.24)

and an arbitrary basic 1-form $\xi$ on $M$ is decomposed with respect to $a$ as

\[ \xi = \xi_1 + ha, \] (3.25)

where $\xi_1 \in \Gamma(T^*N)$ and $h \in C^\infty(N)$. As usual, the following relations hold:

\[ i_{\tilde{s}(X_1)} a = 0, \quad i_{\partial_y} a = 1, \quad i_{\tilde{s}(X_1)} \xi_1 = i_X \xi_1, \quad i_{\partial_y} \xi_1 = 0. \] (3.26)

With this in mind, any section of the $F$-twisted Courant algebroid $A \oplus A^*$ is written of the form

\[ \tilde{s}(X_1) + f \partial_y + \xi_1 + ha. \] (3.27)

Therefore, corresponding to the representation of the twisting in §2, the $F$-twisting is specified by a bundle map $\varphi_F : TN \oplus \langle \partial_y \rangle \oplus T^*N \oplus \langle dy \rangle \rightarrow A \oplus A^*$,

\[ \varphi_F = \begin{pmatrix} e^{-a_i \cdot \partial_y} & 0 \\ 0 & e^{-a_i \cdot \partial_y} \end{pmatrix}, \] (3.28)
locally defined on \( U_i \). Indeed, it is easily seen that
\[
\varphi_F(X_1 + f\partial_y + \xi_1 + hd\gamma) = \tilde{s}(X_1) + f\partial_y + \xi_1 + ha, \tag{3.29}
\]
because
\[
e^{-a^i\wedge\partial_y}(X_1 + f\partial_y) = X_1 + f\partial_y - a_i(X_1)\partial_y = \tilde{s}(X_1) + f\partial_y,
\]
\[
e^{-a^i\wedge\partial_y}(\xi_1 + hd\gamma) = \xi_1 + hd\gamma + ha_i = \xi_1 + ha. \tag{3.30}
\]

We emphasize that the \( F \)-twisting in the previous subsection affects both on \( TN \oplus \langle\partial_y\rangle \) and \( T^*N \oplus \langle dy\rangle \), diagonally.

Corresponding to (3.28), the transition function (3.14) is also embedded diagonally as
\[
G^F_{ij} = \begin{pmatrix} e^{-d\lambda_{ij}\wedge\partial_y} & 0 \\ 0 & e^{-d\lambda_{ij}\wedge\partial_y} \end{pmatrix}. \tag{3.31}
\]

It is a diffeomorphism \( e^{\mathcal{L}_{Z_{ij}}} \) generated by the vector field \( Z_{ij} = \lambda_{ij}\partial_y \in \langle \partial_y \rangle \) on \( U_{ij} \).

**Proof.** We see that the action of \( \mathcal{L}_{Z_{ij}} \) is equivalent to the gluing condition.
\[
\mathcal{L}_{Z_{ij}}(X_1 + f\partial_y) = -(\mathcal{L}_{X_1}\lambda_{ij})\partial_y = -(i_{X_1}d\lambda_{ij})\partial_y = -d\lambda_{ij}(X_1)\partial_y,
\]
\[
\mathcal{L}_{Z_{ij}}(\xi_1 + hd\gamma) = h(\mathcal{L}_{Z_{ij}}dy) = h(d\lambda_{ij}) = h\lambda_{ij}. \tag{3.32}
\]

(End of the proof.)

This shows that \( F \)-fluxes are associated with diffeomorphisms, and thus called geometric fluxes.\(^8\)

The \( F \)-twisted bracket on \( TN \oplus \langle\partial_y\rangle \oplus T^*N \oplus \langle dy\rangle \) is defined by using (3.28) as
\[
[e_1, e_2]_F := \varphi_F^{-1}[\varphi_F(e_2), \varphi_F(e_1)]_C. \tag{3.33}
\]

By direct calculation, we see that the vector part of this bracket is indeed (3.20).

### 3.3.2 \((H, F)\)-flux

By twisting \( A \oplus A^* \) with \( H_2 \) and \( H_3 \) further, we obtain a Courant algebroid \( E \), specified by a bundle map \( \varphi_H : A \oplus A^* \to E \). Hence, the twisting of \( TN \oplus \langle\partial_y\rangle \oplus T^*N \oplus \langle dy\rangle \) by both \( H \) and \( F \)-flux is specified by the bundle map
\[
\varphi_{H,F} := \varphi_H \varphi_F = \begin{pmatrix} 1 & 0 \\ B_i & 1 \end{pmatrix} \begin{pmatrix} e^{-a_i\wedge\partial_y} & 0 \\ 0 & e^{-a_i\wedge\partial_y} \end{pmatrix}. \tag{3.34}
\]

Here the set of local 2-forms \( \{B_i\} \) defines a splitting \( s : A \to E \), and \( A \) already carries the information of the \( F \)-flux. Correspondingly, any basic \( k \)-form on \( M \), say \( \omega \), is decomposed with respect to \( a \) in (3.24) as
\[
\omega = \omega_k + \omega_{k-1} \wedge a, \tag{3.35}
\]

\(^8\)When considering a generalized metric, this diffeomorphism is seen as the Kaluza-Klein \( U(1) \)-gauge symmetry.
where $k$ and $k - 1$ are degrees as forms on $N$. In particular, we decompose the $H$-flux into

$$H = H_3 - H_2 \wedge a.$$  \hfill (3.36)

The decomposition is also applied to local differential forms defined on each open set $U_i$ of $N$. In particular, we decompose $B_i$ in (3.31) into

$$B_i = B_{2i} - B_{1i} \wedge a, \quad B_{2i} \in \Gamma(\wedge^2 T^* U_i), \quad B_{1i} \in \Gamma(T^* U_i).$$  \hfill (3.37)

Then, by noting

$$H|_{U_i} = dB_i = (dB_{2i} + B_{1i} \wedge F) - dB_{1i} \wedge a,$$

we should identify

$$H_3|_{U_i} = dB_{2i} + B_{1i} \wedge F, \quad H_2|_{U_i} = dB_{1i}.$$  \hfill (3.39)

Thus, local 1-form $B_{1i}$ is the gauge potential for the $H_2$-flux.

With this decomposition, (3.34) means that the section of $E$ is locally given by

$$\varphi_{H,F}(e) = \varphi_H(s(X_1) + f \partial_y + \xi_1 + ha)$$

$$= s(X_1) + f \partial_y + \xi_1 + ha + i\langle s(X_1) + f \partial_y \rangle B_i$$

$$= s(X_1) + f \partial_y + \xi_1 + ha + i\langle s(X_1) + f \partial_y \rangle (B_{2i} - B_{1i} \wedge a)$$

$$= s(X_1) + f \partial_y + (\xi_1 + iX_1 B_{2i} + f B_{1i}) + (h - B_{1i}(X_1)) a,$$  \hfill (3.40)

for $e = X_1 + f \partial_y + \xi_1 + hdy$.

The $(H,F)$-twisted bracket on $TN \oplus \langle \partial_y \rangle \oplus T^*N \oplus \langle dy \rangle$ is defined by

$$[e_1, e_2]|_{H,F} := \varphi^{-1}_{H,F}[\varphi_{H,F}(e_1), \varphi_{H,F}(e_2)]_C.$$  \hfill (3.41)

By a direct calculation using (3.40), we have explicitly

$$[X_1 + f \partial_y + \xi_1 + hdy, Y_1 + g \partial_y + \eta_1 + kdy]|_{H,F}$$

$$= [X_1 + f \partial_y + \xi_1 + hdy, Y_1 + g \partial_y + \eta_1 + kdy]_C$$

$$- i_{X_1} i_{Y_1} H_3 + (i_{X_1} i_{Y_1} F) \partial_y + (i_{X_1} i_{Y_1} H_2) a + ki_{X_1} F - hi_{Y_1} F - fi_{Y_1} H_2 + gi_{X_1} H_2.$$  \hfill (3.42)

### 3.4 Topological T-duality with $H$ and $F$-fluxes

We will see that the topological T-duality yields following relations:

$$\begin{array}{c}
\text{T-duality diagram}
\end{array}$$

$$\begin{array}{c}
\text{TM} \oplus T^* M \\
\text{TN} \oplus \langle \partial_y \rangle
\end{array}$$

$$\begin{array}{c}
\text{TN} \oplus T^* N \\
\text{TN} \oplus \langle dy \rangle
\end{array}$$

$$\begin{array}{c}
\text{F} \\
\text{H}_5
\end{array}$$

$$\begin{array}{c}
\text{H}_2
\end{array}$$

$$11$$
Here the T-dual fluxes \((\hat{H}, \hat{F})\) define the T-dual Courant algebroid \(\hat{E}\), or equivalently the \(S^1\)-reduced generalized tangent bundle \(TN \oplus \langle \partial_y \rangle \oplus T^*N \oplus \langle dy \rangle\) with the \((\hat{H}, \hat{F})\)-twisted bracket. This twisting is governed by a map \(\varphi_{\hat{H}, \hat{F}}\), the analogue of \((3.34)\). 3-form \(\hat{H}\) on \(M\) is decomposed like \((3.40)\) into
\[
\hat{H} = \hat{H}_3 - \hat{H}_2 \wedge \hat{a},
\]  
(3.44)
where \(\hat{a}\) with \(\hat{F} = d\hat{a}\) is the connection 1-form on the T-dual \(S^1\)-principal bundle \(\hat{P}\) over \(N\).

Since the T-duality interchanges \(\partial_y\) and \(dy\), the T-dual fluxes \((\hat{H}, \hat{F})\) are related to the original fluxes by
\[
(\hat{H}_3, \hat{H}_2, \hat{F}) = (H_3, F, H_2).
\]  
(3.45)
From this we can observe the standard result, the T-duality exchanges \(H_2\) and \(F\) [12] [13].

Note that in this presentation, the total space \(M = N \times S^1\) as a manifold is unchanged by the T-duality. In another view, we may also consider the space \(M\) with an \(F\)-flux as a principal \(S^1\)-bundle \(P\). In this case the topological T-duality relates different principal \(S^1\)-bundles with 3-form fluxes \(\mathcal{T} : (P, H) \to (\hat{P}, \hat{H})\) as [12]. Now we elaborate on the derivation of this result.

### 3.4.1 Details on topological T-duality with \((H,F)\)-flux

The topological T-duality is defined still as a bundle map of \(S^1\)-reduced generalized tangent bundle \(TN \oplus \langle \partial_y \rangle \oplus T^*N \oplus \langle dy \rangle\). It acts on basic sections as
\[
\mathcal{T} : X_1 + f\partial_y + \xi_1 + hdy \mapsto X_1 + h\partial_y + \xi_1 + fdy.
\]  
(3.46)
After twisting, it is also regarded as a map \(\mathcal{T} : E \to \hat{E}\) of twisted Courant algebroids. The key relation between sections of \(E\) and \(\hat{E}\) is
\[
\mathcal{T}\varphi_{H,F}(e) = \varphi_{\hat{H},\hat{F}}(\mathcal{T}e).
\]  
(3.47)

**Proof.** The section of \(E\) has the form \((3.40)\). Thus, applying the rule \((3.46)\), the T-dual of this section becomes
\[
\mathcal{T}\varphi_{H,F}(e) = (X_1 - a_i(X_1)dy) + fdy + (\xi_1 + iX_1 B_{2i} + fB_{1i}) + (h - B_{1i}(X_1)) (\partial_y + a_i)
\]  
(3.48)
\[
= (X_1 - B_{1i}(X_1)\partial_y) + h\partial_y + (\xi_1 + iX_1 (B_{2i} - B_{1i} \wedge a_i) + ha_i) + (f - a_i(X_1)) (dy + B_{1i}).
\]

On the other hand, the section of \(\hat{E}\) has the form \((3.40)\) with replacing \((H,F)\) to \((\hat{H}, \hat{F})\). Thus, we have for \(\mathcal{T}e = X_1 + h\partial_y + \xi_1 + fdy\),
\[
\varphi_{\hat{H},\hat{F}}(\mathcal{T}e) = (X_1 - \hat{a}_i(X_1)\partial_y) + h\partial_y + (\xi_1 + iX_1 \hat{B}_{2i} + h\hat{B}_{1i}) + (f - \hat{B}_{1i}(X_1)) (dy + \hat{a}_i).
\]  
(3.49)
By comparing them, \((3.47)\) holds by the identification
\[
\hat{B}_{2i} = B_{2i} - B_{1i} \wedge a_i, \quad \hat{B}_{1i} = a_i, \quad \hat{a}_i = B_{1i}.
\]  
(3.50)
As a result, we have on one hand,
\[ \hat{a} = dy + \hat{a}_i = dy + B_{1i} \]  
(3.51)
so that the dual $F$-flux is written as
\[ \hat{F} = d\hat{a} = dB_{1i} = H_2. \]  
(3.52)

On the other hand, the dual local 2-form becomes
\[
\hat{B}_i = \hat{B}_{2i} - B_{1i} \wedge \hat{a} = (B_{2i} - B_{1i} \wedge a_i) - a_i \wedge (dy + B_{1i}) = B_{2i} - a_i \wedge dy,
\]  
(3.53)
and correspondingly, it yields the dual $H$-flux as
\[
\hat{H} = dB_{2i} - da_i \wedge dy = (dB_{2i} + da_i \wedge B_{1i}) - da_i \wedge (dy + B_{1i}) = (dB_{2i} + B_{1i} \wedge F) - F \wedge \hat{a} = H_3 - F \wedge \hat{a}.
\]  
(3.54)

These results are summarized as
\[
(H_3, H_2, F) \rightarrow (H_3, F, H_2).
\]  
(3.55)

(End of the proof.)

It is now straightforward to show that $\mathcal{T}$ is a morphism of Courant algebroids $\mathcal{T} : E \rightarrow \hat{E}$, or equivalently, a morphism of $TN \oplus \langle \partial_y \rangle \oplus T^*N \oplus \langle dy \rangle$ with the change of bracket from $[\cdot, \cdot]_{H,F}$ to $[\cdot, \cdot]_{\hat{H},\hat{F}}$.

\[
\langle \mathcal{T} e_1, \mathcal{T} e_2 \rangle = \langle e_1, e_2 \rangle, \quad \rho|_{TN}(\mathcal{T} e) = \rho|_{TN}(e),
\]
\[
[\mathcal{T} e_1, \mathcal{T} e_2]_{\hat{H},\hat{F}} = \mathcal{T}[e_1, e_2]_{H,F}.
\]  
(3.56)

Proof. Since the first two equations are obvious to hold, we show the third equation. We know already that $[\mathcal{T} e_1, \mathcal{T} e_2]_C = \mathcal{T}[e_1, e_2]_C$ in (3.10) for the Courant bracket. By using (3.10), the definition of the twisted brackets (3.41) and the key relation (3.47), we have
\[
[\mathcal{T} e_1, \mathcal{T} e_2]_{\hat{H},\hat{F}} = \varphi_{\hat{H},\hat{F}}^{-1}([\varphi_{\hat{H},\hat{F}}(\mathcal{T} e_1), \varphi_{\hat{H},\hat{F}}(\mathcal{T} e_2)]_C
\]  
\[
= \varphi_{\hat{H},\hat{F}}^{-1}([\mathcal{T} \varphi_{H,F}(e_1), \mathcal{T} \varphi_{H,F}(e_2)]_C
\]  
\[
= \varphi_{\hat{H},\hat{F}}^{-1}([\varphi_{H,F}(e_1), \varphi_{H,F}(e_2)]_C
\]  
\[
= \mathcal{T} \varphi_{\hat{H},\hat{F}}^{-1}([\varphi_{H,F}(e_1), \varphi_{H,F}(e_2)]_C
\]  
\[
= \mathcal{T}[e_1, e_2]_{H,F}.
\]  
(3.57)
where we used $\varphi_{\hat{H},\hat{F}}^{-1} \mathcal{T} = \mathcal{T} \varphi_{H,F}^{-1}$. (End of the proof.)

This shows the advantages of representing various twists in terms of bundle maps $\varphi$. 

13
4 Topological T-duality of Poisson-Generalized geometry

Along the line of the above reformulation of the topological T-duality in the generalized geometry, here we will formulate the topological T-duality in the Poisson-generalized geometry, and show the following aspects:

1) A Lie algebroid \((T^*N)_\theta \oplus \mathbb{R}\) appears in the Courant algebroid \((TM)_0 \oplus (T^*M)_\theta\) in two different ways as \((T^*N)_\theta \oplus \langle dy \rangle\) and \((T^*N)_\theta \oplus \langle \partial y \rangle\). The topological T-duality without flux is formulated as an exchange of these two Lie algebroids.

2) A \(Q\)-flux is introduced associating with a twisting of the Lie algebroid \((T^*N)_\theta \oplus \mathbb{R}\).

3) In \((TM)_0 \oplus (T^*M)_\theta\), the twist of 2) corresponds to two different twistings, \(R_2\)- and \(Q\)-twisting. The former is the bivector part of the \(R\)-flux, while the latter is the \(Q\)-flux.

4) The topological T-duality exchanges \(R_2\) and \(Q\).

To this end, let us first formulate the dimensional reduction of the Lie algebroid \((T^*M)_\theta\).

4.1 Topological T-duality without flux

Consider a Poisson manifold \((M, \theta)\) with a Poisson structure \(\theta\) and the Lie algebroid \((T^*M)_\theta\) of this Poisson manifold. As in the previous section, we consider that \(M\) is a trivial \(S^1\) bundle, \(M = N \times S^1\) with local coordinates \((x^m, y)\). Then a basic section \(\xi\) of \((T^*M)_\theta\) has the form

\[
\xi = \xi_1 + fdy, \tag{4.1}
\]

where \(\xi_1 = \xi_m(x)dx^m \in \Gamma(T^*N)\) is a 1-form on \(N\), and \(f(x) \in C^\infty(N)\).

The Poisson bivector \(\theta\) on \(M\) is also assumed to be basic, decomposed in general as

\[
\theta = \theta_2 + \theta_1 \wedge \partial y, \quad \theta_2 = \frac{1}{2} \theta^mn(x)\partial_m \wedge \partial_n, \quad \theta_1 = \theta^m(x)\partial_m. \tag{4.2}
\]

where \(\theta_2 \in \Gamma(\wedge^2TN)\) is a bivector and \(\theta_1 \in \Gamma(TN)\) is a vector field on \(N\). The condition \([\theta, \theta]_S = 0\) for \(\theta\) to be a Poisson bivector is equivalent to

\[
[\theta_2, \theta_2]_S = 0, \quad [\theta_2, \theta_1]_S = 0, \tag{4.3}
\]

with respect to the Schouten bracket for \(\Gamma(\wedge^*TN)\). That is, \(\theta_2\) is a Poisson structure on \(N\), while \(\theta_1\) is a Poisson vector field which preserves \(\theta_2\).

We further assume that \(\theta_1 = 0\) (equivalently \(\theta(dy) = 0\)) in (4.2), that is, \(\theta = \theta_2\) is a Poisson structure on the base space \(N\). In this case, we may identify \((T^*M)_{\theta, \text{basic}} = (T^*N)_\theta \oplus \langle dy \rangle\) as vector bundles over \(N\). Then, the Koszul bracket between basic 1-forms reduces to

\[
[\xi_1 + fdy, \eta_1 + gdy]_\theta = [\xi_1, \eta_1]_\theta + (\mathcal{L}_{\xi_1}g - \mathcal{L}_{\eta_1}f) dy, \tag{4.4}
\]

which is a counterpart of (3.2) See appendix B for a proof. The image of the anchor map \(\theta : (T^*M)_\theta \rightarrow TM\) is also restricted to \(TN\), since \(\theta(\xi_1 + fdy) = \theta(\xi_1)\) for \(\theta_1 = 0\).

By definition, \(\theta_1 = -\theta(dy)\) is a Hamiltonian vector field of the function \(y\).
It is apparent that there is the same kind of structure as seen in §3.1 in the dimensional reduction of $TM$. First, the $S^1$-reduced Koszul bracket \([4.4]\) is the same as that of the Lie algebroid $A = (T^*N)_\theta \oplus \mathbb{R}$ over $N$,

$$[\xi_1 + f, \eta_1 + g]_A = [\xi_1, \eta_1]_\theta + (\mathcal{L}_{\xi_1} g - \mathcal{L}_{\eta_1} f), \quad (4.5)$$

which is an analogue of \([3.3]\). The anchor maps are also identical because $\rho_A(\xi_1 + f) = \theta(\xi_1) = \rho_{(T^*M)_\theta}(\xi_1 + f \, dy)$. Second, the Dirac structure $L = \text{span}\{dx^m, \partial_y\}$ of the Courant algebroid $(TM)_0 \oplus (T^*M)_\theta$ has the same bracket, corresponding to a counterpart of \([3.5]\),

$$[\xi_1 + f \partial_y, \eta_1 + g \partial_y] = [\xi_1, \eta_1]_\theta + (\mathcal{L}_{\xi_1} g - \mathcal{L}_{\eta_1} f) \partial_y, \quad (4.6)$$

when restricting to basic sections (see appendix \([3]\) for a proof.). That is, $(L)_{\text{basic}} = (T^*N)_\theta \oplus (\partial_y)$.

Then, there is an isomorphism $\mathcal{T} : (T^*N)_\theta \oplus (dy) \rightarrow (T^*N)_\theta \oplus (\partial_y)$ of these Lie algebroids, and it is extended to the automorphism of $(TN)_0 \oplus (\partial_y) \oplus (T^*N)_\theta \oplus (dy)$ defined by the exchange of $\partial_y$ and $dy$ as

$$\mathcal{T} : X_1 + f \partial_y + \xi_1 + hdy \mapsto X_1 + h \partial_y + \xi_1 + fdy. \quad (4.7)$$

This is the analogue of the topological T-duality for the Poisson-generalized geometry, in the case of vanishing fluxes. In fact, we have

$$\langle \mathcal{T} e_1, \mathcal{T} e_2 \rangle = \langle e_1, e_2 \rangle, \quad \rho(\mathcal{T} e) = \rho(e), \quad [\mathcal{T} e_1, \mathcal{T} e_2] = \mathcal{T}[e_1, e_2], \quad (4.8)$$

where $[,]$ denotes the bracket of $(TN)_0 \oplus (\partial_y) \oplus (T^*N)_\theta \oplus (dy)$, which is also the $S^1$-reduced bracket of $(TM)_0 \oplus (T^*M)_\theta$, given by

$$\begin{align*}
[X_1 + f \partial_y + \xi_1 + hdy, \ Y_1 + g \partial_y + \eta_1 + kdy] \\
= & [\xi_1, \eta_1]_\theta + (\mathcal{L}_{\xi_1} k - \mathcal{L}_{\eta_1} h) \, dy \\
+ & \mathcal{L}_{\xi_1} Y_1 - \mathcal{L}_{\eta_1} X_1 - \frac{1}{2} d\theta (i_{\xi_1} Y_1 - i_{\eta_1} X_1) + \frac{1}{2} (d_\theta h g - h d_\theta g - d_\theta k f + k d_\theta f) \\
+ & (\mathcal{L}_{\xi_1} g - \mathcal{L}_{\eta_1} f) \partial_y. \quad (4.9)
\end{align*}$$

**Proof.** The first two equations in \((4.8)\) are obvious to hold. For the last equation, we need to show that \((4.9)\) is valid. Note first that

$$\begin{align*}
[\xi_1 + hdy, \ Y_1 + g \partial_y] \\
= & \mathcal{L}_{\xi_1 + hdy} (Y_1 + g \partial_y) - \frac{1}{2} d\theta (i_{\xi_1 + hdy} (Y_1 + g \partial_y)) \\
= & \mathcal{L}_{\xi_1} Y_1 - \frac{1}{2} d\theta (i_{\xi_1} Y_1) + (\mathcal{L}_{\xi_1} g) \partial_y + \frac{1}{2} (d_\theta h g - h d_\theta g), \quad (4.10)
\end{align*}$$

where

$$\begin{align*}
\mathcal{L}_{\xi_1 + hdy} (Y_1 + g \partial_y) \\
= & (i_{\xi_1 + hdy} d\theta + d\theta i_{\xi_1 + hdy})(Y_1 + g \partial_y) \\
= & i_{\xi_1 + hdy} (d\theta Y_1 + (d_\theta g) \partial_y) + d\theta (i_{\xi_1} Y_1 + h g) \\
= & \mathcal{L}_{\xi_1} Y_1 + (\mathcal{L}_{\xi_1} g) \partial_y + (d_\theta h) g, \quad (4.11)
\end{align*}$$
is used. From this and (4.4), we have the reduced bracket (4.9). Then, it is straightforward to show the last equation in (4.8). (End of the proof.)

The situation is summarized schematically as the following diagram:

\[
\begin{array}{ccc}
(TM)_0 \oplus (T^*M)_\theta & \cong & (T^*N)_\theta \oplus (dy) \\
(T^*N)_\theta \oplus (dy) & \mapsto & (T^*N)_\theta \oplus (\partial_y).
\end{array}
\] (4.12)

We close this part with a remark on our two assumptions. We have assumed that sections are basic, and that \(\theta\) is basic and \(\theta_1 = 0\) in (4.2). In the standard generalized geometry \(TM \oplus T^*M\), the dimensional reduction is directly related to the \(S_1\)-invariance: A basic vector field \(X_1 + f \partial_y\) and a basic 1-form \(\xi_1 + hdy\) is \(S_1\)-invariant, that is, invariant under the shift along the fiber direction generated by the vector field \(\partial_y\):

\[
\mathcal{L}_{\partial_y}(X_1 + f \partial_y) = 0, \quad \mathcal{L}_{\partial_y}(\xi_1 + hdy) = 0.
\] (4.13)

In the Poisson-generalized geometry \((TM)_0 \oplus (T^*M)_\theta\), the reasoning by using the vector field \(\partial_y\) seems to be subtle, since in general a shift is generated by a 1-form. Nevertheless, this assumption is natural in the dimensional reduction scheme. For example, for a surjection \(p : M \to N\), a bundle \(T^*N \oplus \mathbb{R}\) over \(N\) has the pull-back \(p^*T^*N\) over \(M\), whose sections are identified with basic sections, i.e., \(\Gamma(T^*M)_{\text{basic}} = \Gamma(p^*(TN \oplus \mathbb{R}))\). Thus, the former assumption corresponds is needed if the T-duality is formulated on the base space \(N\).

On the other hand, the latter assumption on the Poisson tensor is used to reduce the bracket and the anchor map to that of the Lie algebroid \(TN \oplus \mathbb{R}\). We may think it also as the invariance under the shift generated by the 1-form \(dy\). For generic sections of \((TM)_0 \oplus (T^*M)_\theta\) and for a Poisson tensor of the form (4.2) which is not necessary basic, we have

\[
\mathcal{L}_{dy}X = \mathcal{L}_{\theta(dy)}X + \theta(i_X d^2y) = -\mathcal{L}_{\theta_1}X,
\]

\[
\mathcal{L}_{dy}\xi = \mathcal{L}_{\theta(dy)}\xi - i_{\theta(\xi)}d^2y = -\mathcal{L}_{\theta_1}\xi.
\] (4.14)

They vanish if \(\theta_1 = 0\). In other words, our assumptions mean the invariance by both \(\mathcal{L}_{\partial_y}\) and \(\mathcal{L}_{dy}\). As shown in appendix [13] however, these are rather strong assumptions in order to obtain two isomorphic Lie algebroids and satisfy the T-duality property. In particular, the T-duality can also be formulated using Lie algebroids over \(M\), without assuming the dimensional reduction (see the case i) in appendix [13]. This is peculiar to the Poisson geometry where the exterior derivative \(dy\) depends on \(\theta\). Although we do not consider this possibility in this paper, it is worth to investigate this case further.

### 4.2 Q-twisting of the Lie algebroid \((T^*N)_\theta \oplus \mathbb{R}\)

In this subsection, we give a definition of Q-flux given in a parallel manner of F-flux as discussed in §3.2. Here, we investigate the twisting of the Lie algebroid \((T^*N)_\theta \oplus \mathbb{R}\), which we call Q-twisting. The strategy is the same as the case of F-twisting in §3.2.
Given a good cover \( \{ U_i \} \) of \( N \) and a trivialization \((Q, \alpha_i, \gamma_{ij})\) of a \( d_\theta \)-closed bivector \( Q \) such that

\[
Q_{|U_i} = d_\theta \alpha_i, \quad \alpha_j - \alpha_i|_{U_{ij}} = d_\theta \gamma_{ij}, \tag{4.15}
\]

we define a transition function \( G_{ij} : U_{ij} \to GL(d) \) \((d = \dim N + 1 = \dim M)\) by

\[
G_{ij} = \begin{pmatrix}
1 & d_\theta \gamma_{ij} \\
0 & 1
\end{pmatrix}, \tag{4.16}
\]

Then, since \( G_{ij} \) satisfies the cocycle condition, we obtain a vector bundle over \( N \)

\[
A = \coprod_{x \in N} (T^*U_i)_\theta \oplus \mathbb{R}_i/\sim. \tag{4.17}
\]

Since each \((T^*U_i)_\theta \oplus \mathbb{R}_i \to U_i\) is a Lie algebroid and the gluing condition preserves the anchor map and the Lie bracket, \( A \) is in fact a Lie algebroid, which satisfies the exact sequence

\[
0 \to \mathbb{R} \to A \to (T^*N)_\theta \to 0. \tag{4.18}
\]

The set \( \{ \alpha_i \} \) of local vector fields gives a splitting \( \tilde{s} : (T^*N)_\theta \to A \), locally defined by

\[
\tilde{s}(\xi_1) = \xi_1 - \alpha_i(\xi_1) \tag{4.19}
\]

for \( \xi_1 \in (T^*U_i)_\theta \). Since \( \tilde{s}((T^*N)_\theta) \) is globally well-defined, we may identify \( A \simeq \tilde{s}((T^*N)_\theta) \oplus \mathbb{R} \), and any section of \( A \) is uniquely specified by

\[
\tilde{s}(\xi_1) + f \tag{4.20}
\]

for \( \xi_1 \in \Gamma((T^*N)_\theta) \) and \( f \in C^\infty(N) \). The Lie bracket of these sections, which is the counterpart of \((3.18)\), is given by

\[
[\tilde{s}(\xi_1) + f, \tilde{s}(\eta_1) + g]_A = \tilde{s}([\xi_1, \eta_1]_\theta) + \mathcal{L}_{\xi_1} g - \mathcal{L}_{\eta_1} f + i_{\xi_1} i_{\eta_1} Q, \tag{4.21}
\]

which corresponds to the \( Q \)-twisted Lie bracket on \((T^*N)_\theta \oplus \mathbb{R} \):

\[
[\xi_1 + f, \eta_1 + g]_Q = [\xi_1, \eta_1]_\theta + \mathcal{L}_{\xi_1} g - \mathcal{L}_{\eta_1} f + i_{\xi_1} i_{\eta_1} Q. \tag{4.22}
\]

Proof.

\[
[\tilde{s}(\xi_1) + f, \tilde{s}(\eta_1) + g]_A = [\xi_1 + (f - \alpha_i(\xi_1)), \eta_1 + (g - \alpha_i(\eta_1))]_{(T^*N)_\theta \oplus \mathbb{R}}
\]

\[
= [\xi_1, \eta_1]_\theta + \mathcal{L}_{\xi_1} (g - \alpha_i(\eta_1)) - \mathcal{L}_{\eta_1} (f - \alpha_i(\xi_1))
\]

\[
= [\xi_1, \eta_1]_\theta + \mathcal{L}_{\xi_1} g - \mathcal{L}_{\eta_1} f - \mathcal{L}_{\xi_1} (\alpha_i(\eta_1)) + \mathcal{L}_{\eta_1} (\alpha_i(\xi_1))
\]

\[
= [\xi_1, \eta_1]_\theta - \alpha_i([\xi_1, \eta_1]_\theta) + \mathcal{L}_{\xi_1} g - \mathcal{L}_{\eta_1} f + i_{\xi_1} i_{\eta_1} d_\theta \alpha_i
\]

\[
= \tilde{s}([\xi_1, \eta_1]_\theta) + \mathcal{L}_{\xi_1} g - \mathcal{L}_{\eta_1} f + i_{\xi_1} i_{\eta_1} d_\theta \alpha_i, \tag{4.23}
\]

17
where
\[-\mathcal{L}_{\xi_1}(\alpha_i(\eta_1)) + \mathcal{L}_{\eta_1}(\alpha_i(\xi_1)) = -d_\theta i_{\xi_1} i_{\eta_1} \alpha_i - i_{\xi_1} d_\theta i_{\eta_1} \alpha_i + \mathcal{L}_{\eta_1} i_{\xi_1} \alpha_i \]
\[= 0 - i_{\xi_1}(\mathcal{L}_{\eta_1} - i_{\eta_1} d_\theta) \alpha_i + \mathcal{L}_{\eta_1} i_{\xi_1} \alpha_i \]
\[= [\mathcal{L}_{\eta_1}, i_{\xi_1}] \alpha_i + i_{\xi_1} i_{\eta_1} d_\theta \alpha_i \]
\[= -i_{[\xi_1, \eta_1]} \alpha_i + i_{\xi_1} i_{\eta_1} d_\theta \alpha_i, \quad (4.24)\]
is used. (End of the proof)

By construction, \(Q\) is a \(d_\theta\)-closed bivector \(Q \in \Gamma(\wedge^2 TN)\) on \(N\), and is regarded as the field strength of the gauge potential \(\{\alpha_i\}\). In local coordinates, \(Q\) is written as
\[Q|_{U_i} = d_\theta \alpha_i = [\theta, \alpha_i]|_S \]
\[= \frac{1}{2} \left( \theta^m \partial_i \alpha_i^m - \theta^n \partial_i \alpha_i^n - \alpha_i^l \partial_i \theta^m \partial_j \right) \partial_m \wedge \partial_n. \quad (4.25)\]
Although it has a rather complicated form, \(Q\) is an abelian-type field strength. In fact, there is a gauge symmetry of the form
\[Q \mapsto Q, \quad \alpha_i \mapsto \alpha_i + d_\theta f_i, \quad \gamma_{ij} \mapsto \gamma_{ij} + f_j - f_i, \quad (4.26)\]
for a set of local gauge parameters \(f_i \in C^\infty(U_i)\), and this transformation preserves \(Q\). On the other hand, a shift of \(\{\alpha_i\}\)
\[\alpha_i \mapsto \alpha_i + \alpha \quad (4.27)\]
by a global vector field \(\alpha \in \Gamma(TN)\), corresponds to a change of splitting. This changes \(Q \mapsto Q + d_\theta \alpha\) but preserves its cohomology class \([Q] \in H^2_\theta(N)\) in the Poisson cohomology.

Recall that the \(F\)-twisting of the Lie algebroid \(TN \oplus \mathbb{R}\), leads to \(A = \tilde{s}(TN) \oplus \mathbb{R}\), but when considering it as a tangent bundle of some space, it is identified with \(A = TP/U(1)\). That is, the underlying topological space has been changed from \(M = N \times S^1\) to \(P\). In this sense, the Lie algebroid \(A = \tilde{s}((T^*N)_\theta) \oplus \mathbb{R}\) here would be identified with some Poisson-version of a principal bundle, but we do not know what this is. It would be related to a question what a non-geometric space is, and in general it is a challenging issue. This is why we need a reformulation of the topological T-duality so far, where a manifold \(M\) is unchanged and without recourse to the use of “principal bundles”.

### 4.3 \(R\) and \(Q\)-fluxes in Poisson-generalized geometry

Similar to the case of \(F\)-twisting in the standard Courant algebroid, the \(Q\)-twisted Lie algebroid described in §4.2 can appear in two different places in \((TM)_0 \oplus (T^*M)_\theta\). The structure is
completely analogous to the case of $(H, F)$-fluxes:

\[
\begin{array}{ccc}
(TM)_0 \oplus (T^* M)_\theta & \xrightarrow{Q} & (T^* N)_\theta \oplus (dy) \\
(TM)_0 \oplus (T^* M)_\theta & \xrightarrow{R_3} & (T N)_0 \oplus (T^* N)_\theta \oplus \langle \partial y \rangle \\
(TM)_0 \oplus (T^* M)_\theta & \xrightarrow{R_2} & (T^* N)_\theta \oplus (dy) \\
\end{array}
\]

(4.28)

Here $Q$ denotes the $Q$-twisting of $(T^* N)_\theta \oplus (dy)$, while $R_2$ denotes the $Q$-twisting of $(T^* N)_\theta \oplus \langle \partial y \rangle$. As we will elaborate on, we call the former as a $Q$-flux, the counterpart of a $F$-flux, because it is glued by $\beta$-diffeomorphism. On the other hand, the latter is a bivector part of the $R$-flux, since it is a result of local $\beta$-transformation. By combining with $R_3$, appearing in the $R$-twisting of $(T^* N)_\theta \oplus (TN)_0$, we have an $R$-flux in the total space $M$ as

\[
R = R_3 - R_2 \wedge \alpha,
\]

(4.29)

where $\alpha$ is defined locally by

\[
\alpha|_{U_i} = \partial_y + \alpha_i
\]

(4.30)

such that $Q = d_\theta \alpha$.

In the following, we describe this structure in more detail. We first twist $(T^* N)_\theta \oplus (dy)$ by $Q$ to obtain a Courant algebroid $A \oplus A^*$ with $A = \tilde{s}((T^* N)_\theta) \oplus (dy)$, and then we twist $A \oplus A^*$ by $R$ to obtain a Courant algebroid $E = s(A) \oplus A^*$.

**4.3.1 $Q$-flux**

Given data $(Q, \alpha_i, \gamma_{ij})$ in §4.2, we obtain a $Q$-twisted Lie algebroid $A = \tilde{s}((T^* N)_\theta) \oplus (dy)$, whose section has the form, analogue of (3.23),

\[
\tilde{s}(\xi_1) + hdy, \quad \tilde{s}(\xi_1) = \xi_1 - \alpha_i(\xi_1)dy.
\]

(4.31)

As in §3.3, this $Q$-twisting also affects its dual $A^*$, reflecting in the decomposition

\[
X + f\alpha,
\]

(4.32)

where $\alpha$ is given in (4.30). The following relations hold:

\[
i_{\tilde{s}(\xi_1)} \alpha = 0, \quad id_y \alpha = 1, \quad i_{\tilde{s}(\xi_1)} X_1 = i_{\xi_1} X_1, \quad id_y X_1 = 0,
\]

(4.33)

to be compared with (3.26). By combining them, the $Q$-twisting of the Courant algebroid $(TM)_0 \oplus (T^* M)_\theta$ is specified by a bundle map $\varphi_Q : (TN)_0 \oplus \langle \partial y \rangle \oplus (T^* N)_\theta \oplus (dy) \to A \oplus A^*$, locally given by

\[
\varphi_Q = \begin{pmatrix}
e^{-\alpha_i \wedge dy} & 0 \\
0 & e^{-\alpha_i \wedge dy}
\end{pmatrix}.
\]

(4.34)

\[\text{[10]}\text{It is an $A$-1 form on $A$ and is regarded as a global vector field on the “principal bundle”}.\]
Indeed, it is shown that
\[ \varphi_Q(X_1 + f \partial_y + \xi_1 + hdy) = X_1 + f \alpha + \tilde{s}(\xi_1) + hdy, \quad (4.35) \]

because of
\[ e^{-\alpha_i \wedge dy}(X_1 + f \partial_y) = X_1 + f \partial_y + f \alpha_i = X_1 + f \alpha, \]
\[ e^{-\alpha_i \wedge dy}(\xi_1 + hdy) = \xi_1 + hdy - \alpha_i(\xi_1)dy = \tilde{s}(\xi_1) + hdy. \quad (4.36) \]

Correspondingly, the transition function is given by
\[ G^Q_{ij} = \begin{pmatrix} e^{-d\theta \lambda_{ij} \wedge dy} & 0 \\ 0 & e^{-d\theta \lambda_{ij} \wedge dy} \end{pmatrix}. \quad (4.37) \]

It is a \( \beta \)-diffeomorphism \( e^{\mathcal{L}_{\xi_{ij}}} \) generated by the 1-form \( \zeta_{ij} = \lambda_{ij} dy \in \langle dy \rangle \) on \( U_{ij} \).

**Proof.** We see that the action of \( \mathcal{L}_{\xi_{ij}} \) is equivalent to the gluing condition.
\[
\mathcal{L}_{\xi_{ij}}(X_1 + f \partial_y) = \mathcal{L}_{\theta(\zeta_{ij})}(X_1 + f \partial_y) + \theta(i_{X_1} + f \partial_y \theta \zeta_{ij}) = \theta(i_{X_1} + f \partial_y \theta \lambda_{ij} \wedge dy) = \theta(d\lambda_{ij}(X_1)dy - f \lambda_{ij}) = -f d\theta \lambda_{ij},
\]
\[
\mathcal{L}_{\xi_{ij}}(\xi_1 + hdy) = \mathcal{L}_{\theta(\zeta_{ij})}(\xi_1 + hdy) - i_{\theta(\xi_1 + hdy)}\theta \zeta_{ij} = -i_{\theta(\xi_1)}(d\lambda_{ij} \wedge dy) = -\theta(\xi_1, d\lambda_{ij})dy = -(d\theta \lambda_{ij})(\xi_1)dy,
\]
where \( \theta(dy) = 0 \) is used. *(End of the proof.)*

Because \( \mathcal{L}_{\zeta_{ij}} \theta = d\theta i_{\zeta_{ij}} \theta = 0 \), this \( \beta \)-diffeomorphism satisfies the condition for the symmetry \([\Pi] \).

This shows that \( Q \)-fluxes are associated with \( \beta \)-diffeomorphisms.

The \( Q \)-twisted bracket on \((TN)_0 \oplus \langle \partial_y \rangle \oplus (T^*N)_0 \oplus \langle dy \rangle\) is defined through \((4.32)\) as
\[
[e_1, e_2]_Q := \varphi_Q^{-1}[\varphi_Q(e_1), \varphi_Q(e_2)]. \quad (4.39)
\]

The explicit form will be given by
\[
[e_1, e_2]_Q = [e_1, e_2] + (i_{\xi_1} i_{\eta_1} Q)dy + g i_{\xi_1} Q - f i_{\eta_1} Q, \quad (4.40)
\]
for \( e_1 = X_1 + f \partial_y + \xi_1 + hdy \) and \( e_2 = Y_1 + g \partial_y + \eta_1 + kdy \).

**Proof.** We first calculate each term in
\[
[\varphi_Q(e_1), \varphi_Q(e_2)] = [X_1 + f \alpha + \tilde{s}(\xi_1) + hdy, Y_1 + g \alpha + \tilde{s}(\eta_1) + kdy]
\]
\[
= [\tilde{s}(\xi_1) + hdy, \tilde{s}(\eta_1) + kdy]_\theta + [\tilde{s}(\xi_1) + hdy, Y_1 + g \alpha] + [X_1 + f \alpha, \tilde{s}(\eta_1) + kdy]. \quad (4.41)
\]

Here the first term
\[
[\tilde{s}(\xi_1) + hdy, \tilde{s}(\eta_1) + kdy]_\theta = \tilde{s}([\xi_1, \eta_1]_\theta) + (\mathcal{L}_{\xi_1} k - \mathcal{L}_{\eta_1} h)dy + (i_{\xi_1} i_{\eta_1} Q)dy
\]
\[
= \varphi_Q([\xi_1 + hdy, \eta_1 + kdy] + (i_{\xi_1} i_{\eta_1} Q)dy), \quad (4.42)
\]
is a consequence of [1,22]. The second term is written as

\[
[\tilde{s}(\xi_1) + hdy, Y_1 + g\alpha] = \mathcal{L}_{\tilde{s}(\xi_1) + hdy}(Y_1 + g\alpha) - \frac{1}{2} d\theta i_{\tilde{s}(\xi_1) + hdy}(Y_1 + g\alpha)
\]

\[
= i_{\tilde{s}(\xi_1) + hdy} d\theta(Y_1 + g\alpha) + \frac{1}{2} d\theta i_{\tilde{s}(\xi_1) + hdy}(Y_1 + g\alpha)
\]

\[
= i_{\xi_1} d\theta Y_1 + (i_{\xi_1} d\theta g)\alpha - h d\theta g + gi_{\xi_1} Q + \frac{1}{2} d\theta (i_{\xi_1} Y_1 + hg)
\]

\[
= \mathcal{L}_{\xi_1} Y_1 + (\mathcal{L}_{\xi_1} g)\alpha - \frac{1}{2} d\theta i_{\xi_1} Y_1 + \frac{1}{2} (d\theta hg - h d\theta g) + gi_{\xi_1} Q
\]

\[
= \varphi_Q(\xi_1 + hdy, Y_1 + g\alpha) + gi_{\xi_1} Q),
\]

(4.43)

and similar for the third term. Therefore, it leads to

\[
[\varphi_Q(e_1), \varphi_Q(e_2)] = \varphi_Q([e_1, e_2] + (i_{\xi_1} i_{\eta} Q)dy + gi_{\xi_1} Q - f_{i\eta} Q).
\]

(End of the proof.)

By construction, our \(Q\)-flux proposed here in the Poisson-generalized geometry is completely analogous to the geometric \(F\)-flux in the standard generalized geometry. However, there are of course some differences, since the underlying Lie algebroid \((T^*N)_{\theta}\) is different from \(TN\).

Recall that an \(F\)-flux can be in general characterized either by the Lie bracket of frame vector fields \(e_a\) \((a = 1, \cdots, \text{dim}M)\) or the Maurer-Cartan equation of 1-forms \(e^a\) as\(^{11}\)

\[
[e_a, e_b] = f^c_{ab} e_c, \quad de^c = -\frac{1}{2} f^c_{ab} e^a \wedge e^b.
\]

(4.45)

In the case of the \(F\)-flux in §3.3.1, if we define locally

\[
e_m = \tilde{s}(\partial_m) = \partial_m - A_i(\partial_m)\partial_y, \quad e_y = \partial_y,
\]

(4.46)

\[
e^m = dx^m, \quad e^y = A = dy + A_i,
\]

(4.47)

then it is easy to show that

\[
[e_m, e_n] = -F_{mn} e_y, \quad de^y = \frac{1}{2} F_{mn} e^m \wedge e^n,
\]

(4.48)

with \(F_{mn} = F(\partial_m, \partial_n)\). It says that \(f^y_{mn} = -F_{mn}\) and zero for others.

On the other hand, for the \(Q\)-flux here, if we define frame fields as

\[
e_m = \partial_m, \quad e_y = \alpha = \partial_y + \alpha_i,
\]

(4.49)

\[
e^m = \tilde{s}(dx^m) = dx^m - \alpha_i(dx^m)dy, \quad e^y = dy,
\]

(4.50)

then it can be shown that

\[
d_\theta e_m = [\theta, \partial_m]_S = -\partial_m \theta^n e_n \wedge e_l,
\]

\[
d_\theta e_y = d_\theta \alpha_i = \frac{1}{2} Q^n l e_n \wedge e_l,
\]

\[
[e^m, e^n] = \tilde{s}([dx^m, dx^n]_\theta) - Q(dx^m, dx^n)dy = \partial_l \theta^n m e^l - Q^n m e^y,
\]

(4.51)

\(^{11}\) It is also consistent with the relation \([e_a, e_b]_C = f^c_{ab} e_c\) and \([e^c, e_a]_C = -f^c_{ab} e^b\) of the Courant bracket.
with $Q^{mn} = Q(dx^m, dx^n)$. It says that they satisfy the relations\(^\text{12}\)

$$[e^a, e^b] = q^{ab}_c e^c, \ d_\theta e_c = -\frac{1}{2} q^{ab}_c e_a \wedge e_b,$$

(4.52)

with $q^{mn}_{ij} = \partial_i \theta^{mn}$, $q^{mn}_{ij} = -Q^{mn}$ and zero for others. We observe that $q^{mn}_{ij}$ coming from the structure of $(T^*N)_\theta$ does not vanish even when $Q = 0$.

### 4.3.2 \((R, Q)\)-flux

By twisting $A \oplus A^*$ with $R$ further, we obtain a Courant algebroid $E$, specified by a bundle map $\varphi_R : A \oplus A^* \to E$. Hence, the twisting of $(TN)_0 \oplus \langle \partial_\theta \rangle \oplus (T^*N)_\theta \oplus \langle dy \rangle$ by both $R$ and $Q$-flux is specified by the bundle map

$$\varphi_{R,Q} := \varphi_R \varphi_Q = \begin{pmatrix} 1 & \beta_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-\alpha_i \wedge dy} & 0 \\ 0 & e^{-\alpha_i \wedge dy} \end{pmatrix}.$$  

(4.53)

Here the set of local bivectors $\{\beta_i\}$ defines a splitting $s : A \to E$, and $A$ carries already the information of the $Q$-flux. Correspondingly, any basic $k$-vector on $M$ is decomposed with respect to $\alpha$ in (4.30) into

$$V = V_k + V_{k-1} \wedge \alpha,$$

(4.54)

where $k$ and $k - 1$ are degrees as polyvectors on $N$. In particular, we decompose the global 3-vector $R$ as in (4.29), and local bivector $B_i$ in (4.53) into

$$\beta_i = \beta_{2i} - \beta_{1i} \wedge \alpha, \quad \beta_{2i} \in \Gamma(\wedge^2 TU_i), \quad \beta_{1i} \in \Gamma(TU_i).$$

(4.55)

Then, by noting

$$R|_{U_i} = d_\theta \beta_i = (d_\theta \beta_{2i} + \beta_{1i} \wedge Q) - d_\theta \beta_{1i} \wedge \alpha,$$

(4.56)

we should identify

$$R_3|_{U_i} = d_\theta \beta_{2i} + \beta_{1i} \wedge Q, \quad R_2|_{U_i} = d_\theta \beta_{1i}.$$  

(4.57)

Thus, the local vector field $\beta_{1i}$ is the gauge potential for the $R_2$-flux.

With this decomposition, (4.53) says that the section of $E$ is locally given by

$$\varphi_{R,Q}(e) = \varphi_R(X_1 + f \alpha + \tilde{s}(\xi_1) + hdy)$$

$$= X_1 + f \alpha + \tilde{s}(\xi_1) + hdy + i_{\tilde{s}(\xi_1) + hdy} \beta_i$$

$$= X_1 + f \alpha + \tilde{s}(\xi_1) + hdy + i_{\tilde{s}(\xi_1) + hdy} (\beta_{2i} - \beta_{1i} \wedge \alpha)$$

$$= (X_1 + i\xi_1 \beta_{2i} + h\beta_{1i}) + (f - \beta_{1i}(\xi_1)) \alpha + \tilde{s}(\xi_1) + hdy,$$

(4.58)

for $e = X_1 + f \partial_\theta + \xi_1 + hdy$.

\(^{12}\) It is consistent with the relations $[e^a, e^b] = q^{ab}_c e^c$ and $[e^a, e_c] = -q^{ab}_c e_b$ of the bracket in the Poisson-generalized geometry.
The $(R, Q)$-twisted bracket on $(TN)_0 \oplus \langle \partial_y \rangle \oplus (T^* N)_\theta \oplus \langle dy \rangle$ is defined by

$$[e_1, e_2]_{R,Q} := \varphi_{R,Q}^{-1} [\varphi_{R,Q}(e_1), \varphi_{R,Q}(e_2)].$$

By a direct calculation using (4.58), we have explicitly

$$[e_1, e_2]_{R,Q} = [e_1, e_2] - i_{\xi_1} i_{\eta_1} R_3 + (i_{\xi_1} i_{\eta_1} Q) dy + (i_{\xi_1} i_{\eta_1} R_2) \alpha + gi_{\xi_1} Q - fi_{\eta_1} Q - hi_{\eta_1} R_2 + ki_{\xi_1} R_2,$$

for $e_1 = X_1 + f \partial_y + \xi_1 + hdy$ and $e_2 = Y_1 + g \partial_y + \eta_1 + kdy$.

**Proof.** We first show the validity of the $R$-twisted bracket. For $e_1 = X + \xi$ and $e_2 = Y + \eta$, we calculate

$$[\varphi_R(e_1), \varphi_R(e_2)] = [X + i\xi \beta_i + \xi, Y + i\eta \beta_i + \eta]$$

$$= [X + \xi, Y + \eta] + [\xi, i \eta \beta_i] + [i \xi \beta_i, \eta]$$

$$= [X + \xi, Y + \eta] + \beta_i ([\xi, \eta]_\theta) - i \xi i \eta R$$

$$= \varphi_R([X + \xi, Y + \eta] - i \xi i \eta R),$$

(4.61)

where

$$[\xi, i \eta \beta_i] + [i \xi \beta_i, \eta] = L \xi i \eta \beta_i - L \eta i \xi \beta_i - \frac{1}{2} d \theta (i \xi i \eta \beta_i - i \eta i \xi \beta_i) = i \xi d \theta i \eta \beta_i - L \eta i \xi \beta_i = i \xi (L \eta - i \eta d \theta) \beta_i - L \eta i \xi \beta_i = -L \eta, i \xi \beta_i - i \xi i \eta R = i [\xi, \eta]_\theta \beta_i - i \xi i \eta R$$

(4.62)

is used. This shows $[X + \xi, Y + \eta]_R = [X + \xi, Y + \eta] - i \xi i \eta R$, which is also valid for the decomposition like $e_1 = X_1 + f \alpha + \tilde{s}(\xi_1) + hdy$. Thus,

$$i \xi i \eta R = i \tilde{s}(\xi_1) + hdy i \tilde{s}(\eta_1) + kdy (R_3 - R_2 \wedge \alpha) = i \tilde{s}(\xi_1) + hdy (i \eta_1 R_3 - i \eta_1 R_2 \wedge \alpha - k R_2) = i \xi_1 i \eta_1 R_3 - (i \xi_1 i \eta_1 R_2) \alpha + hi_{\eta_1} R_2 - ki_{\xi_1} R_2.$$

(4.63)

By combining with the explicit form of the $Q$-twisted bracket (4.40), we obtain the result.

(End of the proof.)

**4.4 Topological T-duality with $R$ and $Q$-fluxes**

The topological T-duality is a bundle map of $S^1$-reduced Courant algebroid $(TN)_0 \oplus \langle \partial_y \rangle \oplus (T^* N)_\theta \oplus \langle dy \rangle$ acting as

$$T : X_1 + f \partial_y + \xi_1 + hdy \mapsto X_1 + h \partial_y + \xi_1 + f dy.$$  

(4.64)
After \((R,Q)\)-twisting, it is also regarded as a map \(\mathcal{T} : E \to \hat{E}\) of twisted Courant algebroids. The structure is exactly the same as the case of the \((H,F)\)-twisting in §3.4. The key relation between sections of \(E\) and \(\hat{E}\) in this case, similar to (3.47), is

\[
\mathcal{T}\varphi_{R,Q}(e) = \varphi_{\hat{R},\hat{Q}}(\mathcal{T}e),
\]

provided that

\[
(R = R_3 - R_2 \wedge \alpha, Q) \mapsto (\hat{R} = R_3 - Q \wedge \hat{\alpha}, R_2).
\]

**Proof.** The section of \(E\) has the form (4.58). Thus, applying the rule (4.64), the T-dual of this section becomes

\[
\mathcal{T}\varphi_{R,Q}(e) = (X_1 + i\xi_1\beta_{2i} + h\beta_{1i}) + (f - \beta_{1i}(\xi_1))(dy + \alpha_i) + (\xi_1 - \alpha_i(\xi_1)\partial_y) + h\partial_y
\]

\[
= (X_1 + i\xi_1(\beta_{2i} - \beta_{1i} \wedge \alpha_i) + f\beta_{1i}) + (h - \alpha_i(\xi_1))(\partial_y + \beta_{1i}) + (\xi_1 - \beta_{1i}(\xi_1)dy) + fdy.
\]

On the other hand, the section of \(\hat{E}\) has the form (4.58) with replacing \((R,Q)\) to \((\hat{R},\hat{Q})\). Thus, we have for \(\mathcal{T}e = X_1 + h\partial_y + \xi_1 + fdy\),

\[
\varphi_{\hat{R},\hat{Q}}(\mathcal{T}e) = (X_1 + i\xi_1\hat{\beta}_{2i} + f\beta_{1i}) + (h - \beta_{1i}(\xi_1))(\partial_y + \hat{\alpha}_i) + (\xi_1 - \hat{\alpha}_i(\xi_1)dy) + fdy.
\]

By comparing them, (4.65) holds by the identification

\[
\hat{\beta}_{2i} = \beta_{2i} - \beta_{1i} \wedge \alpha_i, \quad \hat{\beta}_{1i} = \alpha_i, \quad \hat{\alpha}_i = \beta_{1i}.
\]

As a result, we have on one hand,

\[
\hat{\alpha} = \partial_y + \hat{\alpha}_i = \partial_y + \beta_{1i},
\]

so that the dual \(Q\)-flux is written as

\[
\hat{Q} = d_\theta \hat{\alpha} = d_\theta \beta_{1i} = R_2.
\]

On the other hand, the dual local bivector becomes

\[
\hat{\beta}_i = \beta_{2i} - \beta_{1i} \wedge \hat{\alpha}
\]

\[
= (\beta_{2i} - \beta_{1i} \wedge \alpha_i) - \alpha_i \wedge (\partial_y + \beta_{1i})
\]

\[
= \beta_{2i} - \alpha_i \wedge \partial_y,
\]

and correspondingly, it yields the dual \(R\)-flux as

\[
\hat{R} = d_\theta \beta_{2i} - d_\theta \alpha_i \wedge \partial_y
\]

\[
= (d_\theta \beta_{2i} + d_\theta \alpha_i \wedge \beta_{1i}) - d_\theta \alpha_i \wedge (\partial_y + \beta_{1i})
\]

\[
= (d_\theta \beta_{2i} + \beta_{1i} \wedge Q) - Q \wedge \hat{\alpha}
\]

\[
= R_3 - Q \wedge \hat{\alpha}.
\]
These results are summarized as

$$(R_3, R_2, Q) \to (R_3, Q, R_2).$$

(End of the proof.)

It is now straightforward to show that $\mathcal{T}$ is a morphism of Courant algebroids.

$$(\mathcal{T} e_1, \mathcal{T} e_2) = (e_1, e_2), \quad \rho(\mathcal{T} e) = \rho(e),$$

$$[\mathcal{T} e_1, \mathcal{T} e_2]_{R,Q} = \mathcal{T}[e_1, e_2]_{R,Q}. \quad (4.75)$$

Proof. By using (4.7), (4.59) and (4.65), we have

$$[\mathcal{T} e_1, \mathcal{T} e_2]_{R,Q} = \varphi^{-1}_{R,Q} \mathcal{T} \varphi_{R,Q}(e_1), \varphi_{R,Q}(e_2)$$

$$= \varphi^{-1}_{R,Q} \mathcal{T} \varphi_{R,Q}(e_1), \varphi_{R,Q}(e_2)$$

$$= \varphi^{-1}_{R,Q} \mathcal{T} \varphi_{R,Q}(e_1), \varphi_{R,Q}(e_2)$$

$$= \mathcal{T} \varphi^{-1}_{R,Q} \varphi_{R,Q}(e_1), \varphi_{R,Q}(e_2)$$

$$= \mathcal{T}[e_1, e_2]_{R,Q}. \quad (4.76)$$

where we used $\varphi^{-1}_{R,Q} \mathcal{T} = \mathcal{T} \varphi^{-1}_{R,Q}$. (End of the proof.)

5 Conclusion and Discussion

In this paper we have studied the topological T-duality in the standard generalized geometry as well as the Poisson-generalized geometry, and examined the property of the fluxes related to the twisting of the underlying Courant algebroids.

By the $S^1$-dimensional reduction of the generalized tangent bundle $TM \oplus T^*M$, we reformulated the topological T-duality as an exchange of two isomorphic Lie algebroids. By using this reformulation, we proved the topological T-duality in the Poisson-generalized geometry based on the Courant algebroid $(TM)_0 \oplus (T^*M)_{\theta}$.

In the standard generalized geometry, when $TM \oplus T^*M$ is twisted, $H$-fluxes and geometric $F$-fluxes are exchanged under the topological T-duality. In particular, a $F$-flux associated with diffeomorphism is obtained form the 2-form part $H_2$ of a $H$-flux associated with $B$-gauge transformations. Similarly, in the Poisson generalized geometry, as proposed in the previous paper [1], an $R$-flux appears associated with the $\beta$-gauge transformation. Then applying the topological T-duality, as an natural analogue of the geometric $F$-flux, we obtain a bivector flux associated with $\beta$-diffeomorphisms. We proposed to identify it with a $Q$-flux, which corresponds to the definition of $Q$-fluxes as $Q$-twisting of the Lie algebroid $(T^*N)_{\theta} \oplus \mathbb{R}$. As a result, we obtained a clear classification of four kinds of fluxes in terms of symmetries as

- $H$: $B$-gauge transformation,
- $F$: diffeomorphism,
- $R$: $\beta$-gauge transformation,
- $Q$: $\beta$-diffeomorphism.
The twistings corresponding to these fluxes are expressed in an unified way by introducing the bundle maps, which can be used also to formulate the twisted brackets.

Using this formalism, we showed that \((R_2, Q)-\)fluxes are exchanged by the topological T-duality of the Poisson-generalized geometry, just as \((H_2, F)-\)fluxes are exchanged by the topological T-duality of the standard generalized geometry. However, it is apparent that the topological T-duality never relate \((H, F)-\)fluxes and \((R, Q)-\)fluxes in this way, because two underlying Courant algebroids are different. In order to relate all the fluxes and complete the duality chain, we need another kind of map than the topological T-duality.

The paper includes a reformulation of the standard topological T-duality for the circle bundles. Recently, an extension to \(SU(2)-\)bundle, called the spherical T-duality, is proposed \cite{19}. Along the same line, our viewpoint using Lie algebroids would help to understand such cases further.

We also pointed out the possibility to formulate the T-duality without dimensional reduction. As already stated, it is worth to investigate this case further. In particular, it is interesting to apply it to field theories with keeping all the Kaluza-Klein modes. To this end, we need to formulate a supergravity theory with \(R\) and \(Q\)-fluxes based on the Poisson-generalized geometry. As stated already in the previous paper \cite{1}, the relevance of this new geometry in physics should be further studied in the context of string theory. We will come to this point in the future publication.

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A Proof of (3.10)

The invariance of the inner product and anchor is apparent. To see the compatibility with the Courant bracket, we need the formula of the Courant bracket of the basic sections. From (3.12) and

\[
[X_1 + f\partial_y, \eta_1 + kdy]_{C} = \mathcal{L}_{X_1 + f\partial_y} (\eta_1 + kdy) - \frac{1}{2} d \left( i_{X_1 + f\partial_y} (\eta_1 + kdy) \right) \\
= \mathcal{L}_{X_1} \eta_1 - \frac{1}{2} d (i_{X_1} \eta_1) + \mathcal{L}_{X_1 + f\partial_y} (kdy) - \frac{1}{2} d (fk) \\
= \mathcal{L}_{X_1} \eta_1 - \frac{1}{2} d (i_{X_1} \eta_1) + (\mathcal{L}_{X_1} k) dy + k \mathcal{L}_{\partial_y} (dy) - \frac{1}{2} d (fk) \\
= \mathcal{L}_{X_1} \eta_1 - \frac{1}{2} d (i_{X_1} \eta_1) + (\mathcal{L}_{X_1} k) dy + \frac{1}{2} (dfk - fdk),
\]

(A.1)
we have
\[ [X_1 + f\partial_y + \xi_1 + hdy, Y_1 + g\partial_y + \eta_1 + kdy]_C \]
\[ = [X_1, Y_1] + (L_{X_1}g - L_{Y_1}f)\partial_y \]
\[ + L_{X_1}\eta_1 - L_{Y_1}\xi_1 - \frac{1}{2}d(i_{X_1}\eta_1 - i_{Y_1}\xi_1) + \frac{1}{2}(dfk - fdk - dhg + gdh) \]
\[ + (L_{X_1}k - L_{Y_1}h)dy. \] (A.2)

By using the above formula (A.2), the map $T$ of the Courant bracket of the two basic vectors is given by exchanging $\partial_y$ and $dy$ components as
\[ [X_1 + f\partial_y + \xi_1 + hdy, Y_1 + g\partial_y + \eta_1 + kdy]_C \]
\[ = [X_1, Y_1] + (L_{X_1}k - L_{Y_1}h)\partial_y \]
\[ + L_{X_1}\eta_1 - L_{Y_1}\xi_1 - \frac{1}{2}d(i_{X_1}\eta_1 - i_{Y_1}\xi_1) + \frac{1}{2}(dfk - fdk - dhg + gdh) \]
\[ + (L_{X_1}g - L_{Y_1}f)dy. \] (A.3)

On the other hand, by exchanging $(f, h)$ and $(g, k)$ first in the same formula (A.2), we have
\[ [T(X_1 + f\partial_y + \xi_1 + hdy), T(Y_1 + g\partial_y + \eta_1 + kdy)]_C \]
\[ = [X_1 + h\partial_y + \xi_1 + fdy, Y_1 + k\partial_y + \eta_1 + gdy]_C \]
\[ = [X_1, Y_1] + (L_{X_1}k - L_{Y_1}h)\partial_y \]
\[ + L_{X_1}\eta_1 - L_{Y_1}\xi_1 - \frac{1}{2}d(i_{X_1}\eta_1 - i_{Y_1}\xi_1) + \frac{1}{2}(dhg - hdg - dkf + kdf) \]
\[ + (L_{X_1}g - L_{Y_1}f)dy. \] (A.4)

Thus, they are equivalent. *(End of the proof.)*

**B Reduction of the Koszul bracket**

We here calculate that the Koszul bracket
\[ [\xi, \eta]_\theta = i_{\theta(\xi)}d\eta - i_{\theta(\eta)}d\xi + d(\theta(\xi, \eta)), \] (B.1)

for general sections $\xi = \xi_1 + fdy$ and $\eta = \eta_1 + gdy$, that are not necessarily basic, and for a general Poisson tensor $\theta$. We show that the Koszul bracket reduces to the form (4.4) if either of the following conditions i) or ii) are satisfied:

i) $\theta$ is basic and $\theta_1 = 0$.

ii) $\theta$ is basic. The sections $(\xi$ and $\eta)$ are basic and $\theta_1$-invariant.

Our assumption in this paper is a particular case of i).

Note that
\[ \theta(\xi) = \theta_2(\xi_1) - f\theta_1 + \theta_1(\xi_1)\partial_y, \]
\[ \theta(\xi, \eta) = \theta_2(\xi_1, \eta_1) - f\theta_1(\eta_1) + g\theta_1(\xi_1). \] (B.2)
We divide the exterior differential of $M$ into $d = d_N + dy\partial_y$ so that

$$d\xi = d_N\xi_1 + dy \wedge \mathcal{L}_{\partial_y}\xi_1 + d_N f \wedge dy.$$  \hfill (B.3)

Similarly, we have

$$d(\theta(\xi, \eta)) = d_N(\theta_2(\xi_1, \eta_1) - f\theta_1(\eta_1) + g\theta_1(\xi_1)) + dy\mathcal{L}_{\partial_y}(\theta_2(\xi_1, \eta_1) - f\theta_1(\eta_1) + g\theta_1(\xi_1)), \hfill (B.4)$$

and

$$i_{\theta(\xi)}d\eta = i_{\theta_2(\xi_1)}d_N\eta_1 + (i_{\theta_2(\xi_1)}d_Ng - i_{\theta_2(\xi_1)}\mathcal{L}_{\partial_y}\eta_1) dy$$

$$= i_{\theta_2(\xi_1)}d_N\eta_1 + \left( i_{\theta_2(\xi_1)}d_N g - i_{\theta_2(\xi_1)}\mathcal{L}_{\partial_y}\eta_1 \right) dy$$

$$- f i_{\theta_1} d_N \eta_1 - f \left( i_{\theta_1} d_N g - i_{\theta_1} \mathcal{L}_{\partial_y} \eta_1 \right) dy + \theta_1(\xi_1) \left( \mathcal{L}_{\partial_y} \eta_1 - d_N g \right). \hfill (B.5)$$

By using these, we compute

$$[\xi, \eta]_\theta = i_{\theta_2(\xi_1)}d_N\eta_1 - i_{\theta_2(\eta_1)}d_N\xi_1 + d_N(\theta_2(\xi_1, \eta_1))$$

$$- f i_{\theta_1} d_N \eta_1 + g i_{\theta_1} d_N \xi_1 + d_N(-f\theta_1(\eta_1) + g\theta_1(\xi_1))$$

$$+ \theta_1(\xi_1) \left( \mathcal{L}_{\partial_y} \eta_1 - d_N g \right) - \theta_1(\eta_1) \left( \mathcal{L}_{\partial_y} \xi_1 - d_N f \right)$$

$$+ \left( i_{\theta_2(\xi_1)}d_N g + \theta_1(\xi_1) \mathcal{L}_{\partial_y} g - i_{\theta_2(\eta_1)}d_N f - \theta_1(\eta_1) \mathcal{L}_{\partial_y} f \right) dy$$

$$- \left( i_{\theta_2(\xi_1)}\mathcal{L}_{\partial_y} \eta_1 - i_{\theta_2(\eta_1)}\mathcal{L}_{\partial_y} \xi_1 + \mathcal{L}_{\partial_y}(\theta_2(\xi_1, \eta_1)) \right) dy$$

$$- f \left( i_{\theta_1} d_N g - i_{\theta_1} \mathcal{L}_{\partial_y} \eta_1 + \mathcal{L}_{\partial_y}(\theta_1(\eta_1)) \right) dy + g \left( i_{\theta_1} d_N f - i_{\theta_1} \mathcal{L}_{\partial_y} \xi_1 + \mathcal{L}_{\partial_y}(\theta_1(\xi_1)) \right) dy$$

$$= [\xi_1, \eta_1]_\theta^N$$

$$- f \mathcal{L}_{\partial_1}^N \eta_1 + g \mathcal{L}_{\partial_1}^N \xi_1 + \theta_1(\xi_1) \left( \mathcal{L}_{\partial_1}^N \eta_1 \right) - \theta_1(\eta_1) \left( \mathcal{L}_{\partial_1}^N \xi_1 \right)$$

$$+ \left( \mathcal{L}_{\partial_1}^N g - \mathcal{L}_{\partial_1}^N f \right) dy - f \left( \mathcal{L}_{\partial_1}^N g \right) dy + g \left( \mathcal{L}_{\partial_1}^N f \right) dy. \hfill (B.6)$$

Here in the second line, we defined $[\xi_1, \eta_1]_\theta^N = i_{\theta_2(\xi_1)}d_N\eta_1 - i_{\theta_2(\eta_1)}d_N\xi_1 + d_N(\theta_2(\xi_1, \eta_1))$ and $\mathcal{L}_{\partial_1}^N = d_Ni_{\partial_1} + i_{\partial_1}d_N$. We also used

$$\mathcal{L}_{\partial_1}^N g = i_{\theta(\xi_1)}d_N g = i_{\theta_2(\xi_1)+\theta_1(\xi_1)}d_N g + dy\mathcal{L}_{\partial_1}^N g$$

$$= i_{\theta_2(\xi_1)}d_N g + \theta_1(\xi_1) \mathcal{L}_{\partial_1}^N g$$

$$= \mathcal{L}_{\partial_2(\xi_1)}^N g + \theta_1(\xi_1) \mathcal{L}_{\partial_1}^N g. \hfill (B.7)$$

From this result, we show the claim. We first assume that only $\theta$ is basic. Then we may drop the terms involving $\mathcal{L}_{\partial_1}$ acting on $\theta_2$ and $\theta_1$, and we obtain

$$[\xi, \eta]_\theta = [\xi_1, \eta_1]_\theta^N$$

$$- f \mathcal{L}_{\partial_1}^N \eta_1 + g \mathcal{L}_{\partial_1}^N \xi_1 + \theta_1(\xi_1) \left( \mathcal{L}_{\partial_1}^N \eta_1 \right) - \theta_1(\eta_1) \left( \mathcal{L}_{\partial_1}^N \xi_1 \right)$$

$$+ \left( \mathcal{L}_{\partial_1}^N g - \mathcal{L}_{\partial_1}^N f \right) dy - f \left( \mathcal{L}_{\partial_1}^N g \right) dy + g \left( \mathcal{L}_{\partial_1}^N f \right) dy. \hfill (B.8)$$
It reduces to the desired form

\[ [\xi, \eta]_\theta = [\xi_1, \eta_1]_{\partial_2}^N + (\mathcal{L}_{\xi_1} g - \mathcal{L}_{\eta_1} f) \, dy, \quad (B.9) \]

if \( \theta_1 = 0 \) (note that \( \mathcal{L}_{\xi_1} g = \mathcal{L}_{\partial_2(\xi_1)} g \)). This is the condition i).

Next, let us assume that \( \xi, \eta \) and \( \theta \) are all basic. Then we have

\[
[\xi, \eta]_\theta = [\xi_1, \eta_1]_{\partial_2}^N - f \mathcal{L}_{\partial_1}^N \eta_1 + g \mathcal{L}_{\partial_1}^N \xi_1 + \left( \mathcal{L}_{\partial_1}^N g - \mathcal{L}_{\partial_2(\eta_1)}^N f \right) \, dy - f \left( \mathcal{L}_{\partial_1}^N g \right) \, dy + g \left( \mathcal{L}_{\partial_1}^N f \right) \, dy. \quad (B.10)
\]

if it reduces to the desired form iff

\[
\mathcal{L}_{\partial_1}^N \xi_1 = 0, \quad \mathcal{L}_{\partial_1}^N \eta_1 = 0, \quad \mathcal{L}_{\partial_1}^N f = 0, \quad \mathcal{L}_{\partial_1}^N g = 0. \quad (B.11)
\]

These are satisfied either \( \theta_1 = 0 \) (a particular case of i)), or ii) sections are \( \theta_1 \)-invariant.

Next, we consider the case of the Dirac structure \( \mathcal{L} = \text{span}\{dx^m, \partial_y\} \ni \xi_1 + f \partial_y \). The bracket of the Courant algebroid \((TM)_0 \oplus (T^*M)_\theta\) for sections of \( \mathcal{L} \) is given by

\[
[\xi_1 + f \partial_y, \eta_1 + g \partial_y] = [\xi_1, \eta_1]_\theta + \mathcal{L}_{\xi_1}(g \partial_y) - \mathcal{L}_{\eta_1}(f \partial_y) + \frac{1}{2} d_\theta (i_{f \partial_y} \eta_1 - i_{g \partial_y} \xi_1). \quad (B.12)
\]

We assume that only \( \theta \) is basic. Then, the first term is written by using \((B.8)\) as

\[
[\xi_1, \eta_1]_\theta = [\xi_1, \eta_1]_{\partial_2}^N + \theta_1(\xi_1) (\mathcal{L}_{\partial_y} \eta_1) - \theta_1(\eta_1) (\mathcal{L}_{\partial_y} \xi_1) \quad (B.13)
\]

The second term is written as (see \((B.7)\))

\[
\mathcal{L}_{\xi_1}(g \partial_y) = (\mathcal{L}_{\xi_1} g) \partial_y - g i_{\xi_1} \mathcal{L}_{\partial_y} \theta = (\mathcal{L}_{\xi_1} g) \partial_y. \quad (B.14)
\]

Thus, we obtain

\[
[\xi_1 + f \partial_y, \eta_1 + g \partial_y] = [\xi_1, \eta_1]_{\partial_2}^N + \theta_1(\xi_1) (\mathcal{L}_{\partial_y} \eta_1) - \theta_1(\eta_1) (\mathcal{L}_{\partial_y} \xi_1) + (\mathcal{L}_{\xi_1} g - \mathcal{L}_{\eta_1} f) \partial_y. \quad (B.15)
\]

It is again an element of \( \Gamma(\mathcal{L}) \) either if \( \theta_1 = 0 \) (condition i)), or if \( \xi_1, \eta_1, f \) and \( g \) are all basic (the condition ii)). In both cases, the bracket reduces to the desired form. Therefore, the condition i) or ii) is sufficient to prove the T-duality.

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