Can the states of the W-class be suitable for teleportation

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Entangled states of the W-class are considered as a quantum channel for teleportation or the states to be sent. The protocols have been found by unitary transformations of the schemes, based on the multiuser GHZ channel. The main feature of the W-quantum channels is a set of non-local operators, that allow receivers recovering unknown state.

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I. INTRODUCTION

Recently a large number of new entangled states has been proposed, particulary the W-class, introduced by Cirac et al [1], the states called zero sum amplitude (ZSA) by Pati [2], a set of four-particle entanglement considered by Verstraete et al [3] and others. Although the features of these entangled states are quite various each of them can be attractive for the quantum information processing. In fact, it has been shown, that ZSA-states are useful in the quantum web for preparing an entanglement of an unknown qubit with two types of reference states [2]. Recently Lee et al [4] have considered W-states for secure communication.

It is well known, that the states of the GHZ-class and W-class are rather different with respect to loss of qubits, so that the W-channel looks more attractive because of its robustness. Now we have a lot of entangled states and one of the main questions is what informational tasks could be achieved using a given entangled state. We consider this question for the case of the W-class states. The aim of the work is to investigate teleportation of entangled states. It can be done by different ways, particularly using the GHZ (Greenberger-Horne-Zeilinger) channel, shared with a sender and receivers [5],[6], [7].

Our start point is the GHZ protocol which can be transformed to the desired case by unitary operations, involved both the channel and the state to be teleported. It is well known that the states of the GHZ and W-class can’t be converted from one to another by local operations and classical communication (LOCC) [1]. This fact results in the main feature of the W-class channel, when receivers can recover an unknown state to be teleported by non-local operations only. Because these operations don’t involve the unknown state the task is accomplished. Indeed, considering how to teleport two-qubit state by the four-particle entanglement, it was found by Lee et al [4], that the recovering operations can be non-local also. Any state from the GHZ class is not suitable as the quantum channel for teleportation. The reason is that, it needs the states, that can be converted from the GHZ triplet by two-particle unitary operation only, but it is impossible for all members of the GHZ class. Also it is true for the W-class and one finds a particular set of the states to be useful.

The work is organized as follows. First we introduce two transformations of the GHZ protocols that result in new channel and new measurement. Then a set of states from the W-class are considered as the quantum channels, next we establish new type of entangled states that can be teleported using the GHZ channel and discuss an optical implementation of the W-states.

II. TWO TYPES OF TRANSFORMATIONS

We start from the GHZ channel represented by the three-qubit entangled state of the form

\[ |GHZ⟩ = \frac{1}{\sqrt{2}}(|000⟩ + |111⟩)_{ABC}. \]

(1)

The teleportation protocol using [1] as a channel allows transmitting a two-qubit entangled state like EPR pair

\[ |A⟩ = (α|01⟩ + β|10⟩)_{12}, \]

(2)

where \(|α|^2 + |β|^2 = 1\). The scheme includes five particles in the total state, that is a product \(|A⟩_{12} \otimes |GHZ⟩_{ABC}\), where particles A, B and C of the channel are shared with Alice and two receivers Bob and Claire, spatially separated. If Alice decided sending the state \(A\) to Bob and Claire, she performs a measurement on three particles 1,2 and A in the basis \(\{Φ_x\}\), that reads

\[ \{Φ_x : π^±_1 \otimes Φ^±_{2A}, π^± \otimes Ψ^±_{2A}\}, \]

(3)

where \(Φ^± = (|00⟩ \pm |11⟩)/\sqrt{2}\), \(Ψ^± = (|01⟩ \pm |10⟩)/\sqrt{2}\) are the Bell states, \(π^±_1 = (|0⟩ \pm \exp(iθ)|1⟩)/\sqrt{2}\). The measurement projects all particles into one of the eight states \(|Φ_x⟩_{12A} \otimes |BC_x⟩_{BC}\) with equal probabilities \(Prob(Φ_x)\) to be independent from \(|A⟩_{12}\), where \(|BC_x⟩_{BC}\) is a state in the Bob and Claire hands. It results in successful teleportation because there is a set of unitary operations \(\{U_x\}\), we shall name recovering operators, that recover the unknown state. Indeed all operations can be chosen in the form of product \(U_x = B_x \otimes C_x\), where \(B_x\) and \(C_x\) acts to particle \(B\) and \(C\) [5]. Or in more detail

\[ |A⟩_{12} \otimes |GHZ⟩_{ABC} = \sum_x |Φ_x⟩_{12A} \sqrt{Prob(Φ_x)} (B_x \otimes C_x) |A⟩_{BC}. \]

(4)
where $\text{Prob}(\Phi_x) = 1/8$ and $C_x, B_x$ are the Pauli operators.

There are two main resources, such as the set of recovering operators and the measurement, that can be modified independently by some transformations of the channel or the state to be teleported.

Let $T$ be a two-particle unitary operator, that converts the GHZ state into another one

$$(1 \otimes T)(\text{GHZ})_{ABC} = |\Omega\rangle.$$  \hspace{1cm} (5)

When this transformation is applied to $4$, it results in replacing of the recovering operators

$$B_x \otimes C_x \to T(B_x \otimes C_x),$$  \hspace{1cm} (6)

In the same time equations 5 and 6 tell that to teleport entangled state 4 a new channel $\Omega$ can be used instead of the GHZ one and it needs new set of recovering operators. Generally these two-particle operators are not factorized or non-local that is a feature of the multiparticle quantum channel. In spite of the operators can be non-local, the task is accomplished, because they don’t involve any qubits of the state to be teleported. Here and later we use the simple notation, that operators are local if $U(\varphi)_{AB} = A \otimes B \varphi_{AB}$, where operator $A$ acts on the particle $A$ and don’t affect to $B$ and so on.

Consider a two-particle unitary operator $R$, that converts the state to be teleported

$$R(A) = |Z\rangle.$$  \hspace{1cm} (7)

This operator $R$ transforms 4 in such a way that instead of the basis $\Phi_x$, one finds

$$|\Phi_x\rangle \to R(|\Phi_x\rangle).$$  \hspace{1cm} (8)

It follows from 7, 8, that new state $Z$ can be teleported by the GHZ channel, when new measurement is introduced.

Two considered transformations result in a simple observation. If the GHZ channel allows teleportation of a state $A$, then any state to be unitary equivalent to $A$, can be transmitted successfully, also the unknown state $A$ can be teleported using any channel, obtained from the GHZ one by unitary operation, that involves all particles of the channel except one.

Note, the GHZ channel can be used for dense coding, when a three bit message is transmitted by manipulating two bit of the channel only 7, 9. It is possible because of there is a set of the two-particle unitary operators $\{U'_x\}$, that generate a complete basis of entangled states $\{\Phi'_x\}$ from one of them, say from the GHZ one. In other words 3 bits of classical information are encoded by the set $\Phi_x$ and three-particle measurement allows to extract the message. All operators can be chosen in the factorized form $U'_x = B'_x \otimes C'_x$, and

$$|\Phi'_x\rangle = (1 \otimes B'_x \otimes C'_x)|\text{GHZ}\rangle_{ABC}.$$  \hspace{1cm} (9)

This scheme can be modified in the same way, when a new channel or new measurement is introduced.

What kind of states from the GHZ class can be chosen as a quantum channel for teleportation or dense coding schemes? One of the subsets of the three-particle states can be obtained using equations 5 or its particular case 9. By this way one finds a collection

$$|\Psi'_x\rangle = 1/\sqrt{2}(|0\rangle \pm |1\rangle|x\rangle),$$  \hspace{1cm} (10)

where $x = 00, 01, 10, 11$, that is a complete basis. In the same time any normalized state of the form $a|000\rangle + b|111\rangle$, where $|a| \neq |b|$ is insufficient as the quantum channel. For teleportation it results from the fact, that measurement, given by 3, has outcomes, which probabilities depend from the unknown state 4 as $\text{Prob} = 1/8[1 \pm (|a|^2 - |b|^2)(|a|^2 - |b|^2)]$. For dense coding it follows from the fact, that in accordance with 9 one can generate a set of states $a(0)|x\rangle \pm b(1)|x\rangle$, but the set is not a complete basis.

Indeed the state of the GHZ class

$$|f_{\text{GHZ}}\rangle = 1/\sqrt{3}(|000\rangle + |111\rangle)$$  \hspace{1cm} (11)

can’t be used as a quantum channel. It is in agreement with equation 5, that tells, that $T$ must be a unitary operator. The reason is that any two states from the GHZ class can’t be converted from one to another by unitary transformation, if a two-particle operator is permitted only. It is true for the $|\text{GHZ}\rangle$ and $|f_{\text{GHZ}}\rangle$ state. In fact, consider the transformation

$$(1 \otimes T)(a|000\rangle + b|111\rangle) = a'|000\rangle + b'|111\rangle.$$  \hspace{1cm} (12)

Because of $T$ is the unitary operator it results in conditions $|a| = |a'|$ and $|b| = |b'|$, then there is no unitary transformation, that converts $|\text{GHZ}\rangle$ to $|f_{\text{GHZ}}\rangle$.

In other words the states of the GHZ class have different features under two-particle transformation when only a subset of the class can be used as the quantum channel for teleportation and dense coding. Indeed this observation is true for the W-class.

III. THE CHANNEL OF THE W-CLASS

Consider a particular case of transformation $T$, given by 5, that converts the GHZ state into a state of the W-class.

It follows from the result of Cirac et al 1, that the operator $T$ is non-local. Now we are interested in two states from the W-class only, say of the form

$$|W\rangle = 1/\sqrt{3}(|100\rangle + |010\rangle + |001\rangle)$$ \hspace{1cm} (13)

$$|\tilde{W}\rangle = 1/\sqrt{2}(|100\rangle + |\Psi^+\rangle).$$ \hspace{1cm} (14)

Introduce the unitary non-local operator $V$

$$V = |\Psi^+\rangle(00) + |11\rangle(01) + |\Psi^-\rangle(10) + |00\rangle(11).$$ \hspace{1cm} (15)

It can be represented by network, including the conditional gates $V = C_{12}(C - H)_{21} \sigma_{x2}C_{21}$, where $C_{21}$ is $C$ - NOT operation, $c$ is controlled bit, $t$ is target one, $(C - H)_{21}$ is the controlled Hadamard gate, that transforms $|00\rangle \to |b0\rangle$, $|b1\rangle \to (H \otimes 1)|b1\rangle$, $b = 0, 1$. The operator $V$ converts $00 \to \Psi^+$, $01 \to 11$, $10 \to \Psi^-$, $11 \to 00$ and generates a complete set of states $\Psi^+, |00\rangle, |11\rangle$ that was used by
Basharov to consider the problem of atomic relaxation under the entangled thermostat.[10]

The introduced unitary operator $V$ converts the states from the GHZ class into the W-class

$$(1 \otimes V) |\text{GHZ}\rangle_{ABC} = |\bar{W}\rangle$$

$$(1 \otimes V) |\text{f}_{\text{GHZ}}\rangle_{ABC} = |\bar{W}\rangle.$$  (17)

Because of two states $|\text{GHZ}\rangle$ and $|\text{f}_{\text{GHZ}}\rangle$ are inequivalent under two-particle transformation, the obtained states of the W-class have also their properties to be different. Clear, that in contrast $|W\rangle$, the $|\bar{W}\rangle$ state is sufficient as a quantum channel for teleportation of entangled states and dense coding. As well a collection $(1 \otimes V) |\Phi'_x\rangle$, including $|\bar{W}\rangle$, where $\Phi'_x$ is given by (19), represents a set of the quantum channels of the W-class. For teleportation it results in new feature of recovering operators which become non-local ones.

The original state of the W-class in the standard unique form reads $[3]$  

$$\varphi_W = \sqrt{a}|001\rangle + \sqrt{b}|010\rangle + \sqrt{c}|100\rangle + \sqrt{d}|000\rangle,$$  (18)

where $a, b, c > 0$ and $d = 1 - a - b - c \geq 0$. Considering $\varphi_W$ as a quantum channel, there is a question whether the two qubits in the general state is successful. The answer is not because of the following reasons. Let a total state be a product $|A'\rangle_{12} \otimes |\varphi_W\rangle_{ABC}$, where $|A'\rangle = (\gamma|00\rangle + \alpha|01\rangle + \beta|10\rangle + \delta|11\rangle)_{12}$ is a general two-qubit state. When the measurement in the basis, given by $[3]$, performs, all probabilities of outcomes depend on the unknown state $A'$

$$\text{Prob}(\pi^\pm \Phi^\pm) = |(\gamma \pm \exp(i\theta)\beta)|^2 (1 - a) + a|\alpha \pm \exp(i\theta)\delta|^2/4$$

$$\text{Prob}(\pi^\pm \Psi^\pm) = |(\gamma \pm \exp(i\theta)\beta)|^2 (1 - a) + (1 - a)|\alpha \pm \exp(i\theta)\delta|^2/4.$$  (19)

However the task can be accomplished for the considered particular cases. Indeed, it is clear without any calculations from the following reasons. From the physical point of view any two-particle entangled state looks more as one piece of reality. To teleport a qubit or one piece, it needs an EPR channel of two entangled particles, where one of them is involved in measurement and another particle is a blank. For teleportation of two entangled qubits it needs a channel, including two particles as blanks and one particle for measurement, then one finds a three-particle channel at last. It is not true, when two qubits are not entangled.

IV. THE STATES TELEPORTED BY THE GHZ-CHANNEL

In accordance with [7] and [8] a unitary transformation $R$ allows to investigate, what kind of states could be teleported using the GHZ-channel.

Let $R = V$, for example. Then

$$V|A\rangle = \alpha|11\rangle + \beta|\Psi^-\rangle.$$  (20)

To teleport this entangled state, it needs a new three-particle measurement in the basis, obtained from (3), when $|\pi^\pm \Phi^\pm\rangle \leftrightarrow 1/\sqrt{2}(|101\rangle \pm |\pi^\pm 1\rangle)$ and so on. The found set is represented by the states of the W-class.

Consider the GHZ channel of four qubits

$$|\text{GHZ}\rangle = 1/\sqrt{2}(|0000\rangle + |1111\rangle)_{ABCD}.$$  (21)

Let particles $A, B, C$ and $D$ share a sender Alice and three receivers $B, C, D$. It has been found, that a three-particle state of the form $|A\rangle_{123} = \alpha|000\rangle + \beta|111\rangle$ can be teleported, when Alice performs a joint measurement, described by the set $\{\Phi_x \otimes \varphi_2 \otimes \Phi_3^\pm \otimes \Phi_4^\pm\}$. A simple observation shows, that the channel is useful for transmitting a state of the W-class. In fact, it follows from (19), that

$$(1 \otimes V) (\alpha|000\rangle + \beta|111\rangle) = \alpha|100\rangle + \beta|0\Psi^+\rangle.$$  (22)

Then the useful teleportation of entangled state $\alpha|100\rangle + \beta|\Psi^+\rangle$ is obtained.

Consider one of the way how to prepare the states of the W-class for particular case of optical implementation. Let a collection of two-level identical atoms interacts with a set of modes, which frequencies are close to frequency of an atomic transition for simplicity. The density matrix of atoms and field $F$ obeys equation of the form

$$\partial F/\partial t = -i\hbar^{-1}[V, F],$$  (23)

where $V = SB^\dagger + S^\dagger B$ is Hamiltonian, $S, S^\dagger$ are atomic operators, $B = g_1 b_1 + g_2 b_2 + g_3 b_3; b_k, b_k^\dagger$ $k = 1, 2, 3$ are the photon operators of modes. Let atoms have coherence, in the sense, that their polarization $\langle S \rangle \neq 0$. It can be done, if atoms are illuminated by a strong coherent wave of amplitude $\alpha$, then $\langle S \rangle = q\alpha$. One can describe behavior of the field by a master equation for density matrix of modes $\rho = SP_{\text{atoms}}F$. In the first non-vanishing order the master equation for $\rho$ takes the form

$$\partial \rho/\partial t = -i\hbar^{-1}[V', \rho],$$  (24)

where an effective Hamiltonian $V' = q(\alpha^* B + \alpha B^\dagger)$ is obtained. In fact, it describes a parametric scattering of the strong wave into three weak modes and can be rewritten as

$$\partial |\Psi\rangle/\partial t = -i\hbar^{-1}V'|\Psi\rangle.$$  (25)

Let the state of the week modes be vacuum, then in the first order over interaction one finds

$$|\Psi(t)\rangle = |000\rangle - i\hbar^{-1}at|w\rangle,$$  (26)

where $|w\rangle = g_1|100\rangle + g_2|010\rangle + g_3|001\rangle$, is a state of the W-class.

V. CONCLUSION

Being attractive because of its robustness with respect to losses of particle the W-states can be used for the teleportation and dense coding tasks. In the same time the states
from the W-class are different under the unitary two-particle transformations and only some of them can be converted from one to another by this way. It results in a subset of the states to be suitable only. We find the desired collection of states using unitary transformation of the GHZ protocols. The W-channel has new feature, that is non-local operators recovering unknown state teleported.

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