Field Theory of the Random Flux Model

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(March 24, 2022)

The long-range properties of the random flux model (lattice fermions hopping under the influence of maximally random link disorder) are shown to be described by a supersymmetric field theory of non-linear $\sigma$ model type, where the group $\text{GL}(n|n)$ is the global invariant manifold. An extension to non-abelian generalizations of this model identifies connections to lattice QCD, Dirac fermions in a random gauge potential, and stochastic non-Hermitian operators.

Quantum disordered systems are typically realised in Hamiltonians of the general form $\hat{H} = H_0 + \hat{V}$, where $H_0$ models the underlying “clean” system, and disorder is introduced via the randomly distributed Hermitian operator $\hat{V}$. Sometimes, however, it is preferable to implement disorder in terms of unitary stochastic operators and to consider Hamiltonians of the type

$$\hat{H} = - \sum_{\langle ij \rangle} \epsilon_{ij} c_i^\dagger U_{ij} c_j,$$

(1)

where $\langle ij \rangle$ denote neighbouring sites of a $d$-dimensional hypercubic lattice, the $\epsilon$s represent $N$-component lattice fermions, and $U_{ij}$ represent $N$-dimensional unitary matrices residing on the links of the lattice. Stochasticity is introduced by drawing the $U$s from a random distribution (albeit subject to the Hermiticity requirement $U_{ij} = U_{ji}^\dagger$). Hamiltonians of the type (1) are commonly referred to as random flux (RF) Models, a denotation we will also adopt for the cases $N \neq 1$.

RF-models appear in a variety of different contexts: The $2dN = 1$ version describes the dynamics of lattice fermions subject to a random magnetic field or, more accurately, a random vector potential [4, 5]. As well as the gauge theory of high $T_c$ superconductivity [4, 5], this model has been discussed in connection with the physics of the half-filled fractional quantum Hall phase [6] as well as the spin-split Landau level [6, 7]. Identifying the two fermion components of the $N = 2$ RF-model with a spin degree of freedom, (1) describes the propagation of lattice electrons on a spin-disordered background, a situation that occurs, e.g., in connection with the physics of manganese oxides [8]. Identifying the three fermion components of the $N = 3$ model with a color degree of freedom, (1) represents a prototype [10] of the strong coupling lattice QCD-Hamiltonian.

Superficially, (1) appears to fall into the general class of (bond disordered) Anderson Hamiltonians. That conjecture indeed holds true provided one stays away from the middle of the tight-binding band, $\epsilon = 0$. Upon approaching $\epsilon = 0$, however, the phenomenology of the RF-models begins to differ drastically from a conventional disordered fermion systems. In spite of intensive numerical and analytical investigation [11], central aspects of these deviations are not yet fully understood. For example, the key question of whether or not the $2d$ RF-model possesses a band center extended metallic phase has not yet been settled; apart from the fact that the average density of states (DoS) diverges upon approaching $\epsilon = 0$, much of the structure of even that basic observable remains unknown.

The purpose of this Letter is 2-fold: Firstly we provide new information regarding the band center behavior of the RF-model. Secondly we wish to discuss a diverse network of interconnections that exist between the RF-problem and related areas of current research interest.

Both aspects of that program are based on the result that the long-range behavior of average $n$-point Green functions, $\langle G^\pm (\epsilon + \epsilon_1) \ldots G^\pm (\epsilon + \epsilon_n) \rangle$, of the RF-model can be obtained from a supersymmetric field theory defined by the action

$$S[T] = - \int \left[ c_1 \text{str} (\partial T^{-1} \partial T) + ic_2 \text{str} \left( \hat{\epsilon}(T + T^{-1}) \right) + c_3 \left( \text{str} (T^{-1} \partial T) \right)^2 \right] + S_0[T],$$

(2)

where $T \in \text{GL}(n|n)$ (the group of invertible supermatrices of dimension $2n$), ‘str’ is the standard supertrace, and the matrix $\hat{\epsilon} = \text{diag}(\epsilon_1, \ldots, \epsilon_n)$. The contribution $S_0$ [11] represents a boundary action that depends on the values of the fields $T$ at the corner points of the lattice.

Eq. (2) is derived under the assumption of maximal unitary randomness, i.e. all $U_{ij} \in U(N)$, independently distributed according to the Haar measure. In this case, the constants $c_1 = N a^{2-d}/8d$, $c_2 = N (2d - 1)^{1/2} a^{-d}/4d$, $c_3 = a^{2-d} C/16d$ where $a$ represents the lattice spacing, and $C$ denotes a geometry-dependent numerical constant $O(1)$. Below we will argue that the structure of the field theory is actually disorder independent, i.e. that RF-models are generally described by (2) [12], where the
Below we will outline how, starting from the ‘microscopic’ Hamiltonian \( H \), the effective description \( \tilde{H} \) is derived. However, before turning to that more technical part of the discussion, we first address the question of what kind of information can be gained from the field theory. Our main goal will be to demonstrate that the action \( \tilde{S} \) represents a quantitative implementation of the network of connections displayed in Fig. 1. By exploring different links, we will discuss some characteristic features of the field theory.

![Diagram](image.png)

**FIG. 1.** Connection between the RF-model and various related systems governed by the presence of chiral symmetries. The low energy limit of all models is universally described by Chiral Random Matrix theory.

**Chiral Random Matrix Theory (ChRMT):** As usual with field theories of disordered systems, the low energy regime of \( \tilde{H} \) (energies \( \epsilon < c_1/(c_2 L^2) \)) is governed by spatially constant field configurations \( T_0 = \text{const.} \),

\[
S_0[T_0] = -\frac{\pi \rho_0}{2} \text{str}(\hat{c}(T_0 + T_0^{-1})) + S_b[T_0],
\]

where \( \rho_0 \) is the bulk mean DoS of the system. Correlation functions computed with respect to the first contribution to the action \( \tilde{S} \) coincide with those otherwise obtained for the chiral unitary random matrix ensemble ChGUE [13–16], (the ensemble of symmetry AIII in the classification scheme of Ref. [17]), i.e. the ensemble of block off-diagonal matrices

\[
\begin{pmatrix}
A & \hat{A} \\
A^\dagger & \hat{A}^\dagger
\end{pmatrix},
\]

where \( \hat{A} \) is complex random. In particular, the mean DoS is found to vanish as \( \epsilon \to 0 \) on a scale set by the mean level spacing. The connection to ChRMT follows readily from the fact that the RF-Hamiltonian possesses a chiral symmetry: Partitioned into two nested sublattices \( A \) and \( B \), the bipartite lattice Hamiltonian assumes a block off-diagonal form \( \tilde{H} \) in an \( A/B \)-decomposition. To the best of our knowledge, the ramifications of the chiral structure on the physical properties of RF-models was first reported in Ref. [18]. In passing we note that, in contrast to conventional disordered systems, the low energy limit of the RF-model is not absolutely universal: The fine structure of the DoS close to \( \epsilon = 0 \) depends on the ‘parity’ of the lattice, i.e. on whether the number of sites is even or odd [19]. Without going into details, we remark that the information about this effect is encoded in the boundary term \( S_b \).

**Non-Hermitian Operators:** To investigate problems involving non-Hermitian stochastic operators \( A \neq A^\dagger \) one commonly introduces an operator like the one shown in \( \tilde{H} \), i.e. an Hermitian auxiliary operator of twice the dimension of the original problem \( \tilde{H} \). Put differently, non-Hermitian Hamiltonians possess an inherent chiral structure implying that their low-energy universal properties coincide with those of manifestly chiral problems like the RF-model. That in turn means that the basic structure of the low energy field theory \( \tilde{S} \) of the RF-model (a system with broken time-reversal invariance) should coincide with that of the (time-reversal non-invariant version of the) non-linear \( \sigma \)-model of non-Hermitian problems introduced in Ref. [21]. To make that connection explicit, we introduce the auxiliary matrix variable \( Q = \exp(\hat{W} \sigma_1/2) (\hat{s} \otimes \sigma_3) \exp(-\hat{W} \sigma_1/2) \), where \( W = \ln T \), \( \sigma_i \) are Pauli matrices operating in the block space of \( \tilde{H} \), and \( \hat{s} = \text{diag}(\text{sgn} \Im \sigma_1, \ldots, \text{sgn} \Im \sigma_n) \).

One may check by direct comparison that for \( n = 1 \) (the case there considered), the matrices \( Q \) are equivalent to the degrees of freedom employed in Ref. [21]. When represented in terms of \( Q \)'s, \( \tilde{S} \) assumes a form similar to a standard \( \sigma \)-model of novel non-linear \( \sigma \)-model, albeit one of novel symmetry [21]. That the connection is not incidental, but rather extends to the more complex variants \( n > 1 \), follows from a) the above mentioned fact that both non-Hermitian and RF-type problems possess a chiral structure, and b) that only three fundamentally different \( \sigma \)-models with chiral symmetry (corresponding to the cases of broken time-reversal invariance, broken spin rotation invariance, and invariance under both operations) exist.

For a more thorough discussion of these symmetry aspects, we refer the reader to the original Ref. [17].

**Weakly Disordered Sublattice Models:** Leaving the random matrix regime and turning to the more complicated spatially extended problem, it is important to notice that the field theory \( \tilde{S} \) has a closely related precursor: Analysis of a weakly disordered sublattice model led Gade [22] to a boson replica version of the present model, i.e. a theory over fields \( T \in \text{GL}(nR)/U(nR) \), where \( R \to 0 \) is the number of replicas. The action for these fields coincides with \( \tilde{S} \), save for the absence of the boundary term, and the important difference that, due to the weakness of the disorder, the coupling constant \( c_1 \) was parametrically larger than one.

Various conclusions concerning the physical behavior of the RF-Hamiltonian, most notably about its localization behavior, can be inferred directly from Ref. [23].
There it was shown that the conductance of the weakly disordered 2d model at the band center (which is essentially determined by the coupling constant $c_1$) did not change under one-loop perturbative renormalization. This observation suggests that a non-localized state might exist in the middle of the band. Since the stability of the perturbative RG relies merely on the smallness of the parameters $1/c_1, c_3/c_1 \ll 1$, its results can be straightforwardly carried over to the $N \gg 1$ non-abelian RF model: The one-loop renormalization indicates that, at least for $N \gg 1$, the strongly disordered RF model exhibits metallic behavior at the band center. (It is interesting to note that, according to the connections summarized above, the unusual localization properties of the zero energy states of the RF-model characterize those of all eigenstates of a stochastic non-Hermitian operator.)

It was also predicted in Ref. [28] that the 2d DoS diverges upon approaching the middle of the band. In order to understand on which energy scale that divergence sets in, and how it will eventually be cut off deep within the random matrix regime, one would have to superimpose perturbative RG techniques onto a non-perturbative treatment of the low-energy regime, a task that is well beyond the scope of the present paper.

Finally we notice that, in an RG-sense, finite energies $\epsilon_i$ represent a relevant perturbation. Renormalization of the field theory leads to a flow away from the chiral band center limit eventually leading to the standard unitary universality class. This result is consistent with the analysis of Ref. [4] where it was shown that continuum fermions (i.e. the analogue of lattice fermions close to the bottom of the band) subject to a weak random field map onto a unitary $\sigma$-model.

The tendency of sublattice models to exhibit band center delocalized behavior persists even in the (quasi) 1d case: It was shown in Refs. [5,24] that 1d sublattice models with $N$ even exhibit conventional localization behavior whilst for $N$ odd a delocalized mode remains in the band center. This parity effect is closely related to the odd/even phenomenon mentioned above in connection with the mean DoS, and indeed it is the boundary action that is responsible for the quasi 1d delocalization phenomenon within the $\sigma$-model formulation.

Lattice QCD and Random Dirac Fermions: Besides Gade's model, the field theory $[2]$, has at least two other close relatives: In QCD, $[3]$ has been suggested on phenomenological grounds as relevant for the determination of the low energy spectrum of the Dirac operator $[25]$. In that context, the base manifold is $4 + 1$-dimensional whilst the fields $T \in U(n_f + 1)$, where $n_f$ is the number of quark flavors. For a comprehensive discussion of the QCD-analogue of $[2]$, its connection with ChRMT and its relevance for lattice QCD-analyses, we refer the reader to Ref. [26]. The similarity between the theories is again a manifestation of the universality of chiral $\sigma$-models or, more physically, the universal consequences chiral symmetries have for the long-range properties of random systems. (In QCD, ‘randomness’ is represented by gauge field fluctuations in the Yang-Mills Hamiltonian.)

Secondly, an analogue of the action $[2]$ (again without the boundary operator $c_4$) with a Wess-Zumino-Novikov-Witten WZWN term

$$-c_5 \int_0^1 d\xi e^{\xi\nu} \text{str} \left( T\partial_\xi T^{-1}T\partial_\nu T^{-1}T\partial_\nu T^{-1} \right)$$

was obtained from a lattice model of random Dirac fermions $[27]$. The connection to the RF-model follows from the fact that the clean limit of the Dirac model can effectively be described in terms of a model of $\pi$-flux lattice fermions. (The Dirac-structure of the clean $\pi$-flux model is also responsible for the occurrence of a WZMN operator.)

Having reviewed some essential features of the field theory $[2]$, we finally outline how it is obtained from $[4]$. That in this Letter, the derivation is not formulated in more detail is motivated by the observation that not only the degrees of freedom but also the structure of the field theory is, to a large extent, dictated by aspects of symmetry: By analogy to the situation for the ‘conventional’ supersymmetric $\sigma$-models $[22]$, there are only a few $GL(n|n)$ invariant operators with $\leq 2$ gradients (namely the ones appearing in $[2]$ plus the WZWN operator $[28]$). Thus, the ‘only’ job that is left for a microscopic derivation is to decide whether the operators permitted by symmetry are actually realized in the field theory, and to fix their coupling constants. Here we restrict ourselves to a brief outline of that analysis. Details of the calculation will be presented in a separate publication.

(i) As usual in the construction of field theories of disordered problems, we first represent Green functions of the problem in terms of a Gaussian integral over a field $\psi$. Choosing supersymmetry as a way to normalize the resulting functional integrals to unity, the first step exactly parallels the constructions reviewed in Ref. [25].

(ii) Next we average over the set $\{U_{ij}\}$. At that stage significant deviations from the standard treatment of Hermitian disordered operators occur. A method of exactly averaging over (extended) models involving unitary stochasticity has been introduced in Ref. [29] and christened the ‘color-flavor transformation’. Following that reference we eliminate the disorder at the expense of introducing a pair of auxiliary fields $\{\xi_{ij}, \tilde{\xi}_{ij}\}$ (which play a role analogous to the Hubbard-Stratonovich field $Q$ commonly employed in $\sigma$-model constructions). (iii) Integrating out the $\psi$’s we are left with the action
\[ S[Z, \tilde{Z}] = -N \sum_{(i \in A, j \in B)} \text{str} \ln (1 - Z_{ij} \tilde{Z}_{ij}) + N \sum_{i \in A} \text{str} \ln \left( \hat{\epsilon} + \sum_{j \in B} Z_{ij} \right) + N \sum_{j \in B} \text{str} \ln \left( \hat{\epsilon} + \sum_{i \in A} \tilde{Z}_{ij} \right), \tag{5} \]

where the notation \( j \in N_i \) indicates that \( j \) is summed over all nearest neighbours of \( i \). (iv) Subjecting \( \square \) to a saddle-point analysis, the fields are conveniently parameterized as \((Z, \tilde{Z}) \equiv (ixPT, ixT^{-1}P)\), where \( x \) is a constant, and \( P, T \in GL(n|n) \) respectively have the significance of Goldstone, massive modes of the theory. (v) Integrating out \( P \) we find that, in contrast to standard \( \sigma \)-model analyses (and in accord with the construction of Gade’s action \([23]\)), a residual coupling between massive and Goldstone modes exists; it gives rise to the \( c_3 \)-term in \( \square \). (vi) The remaining, pure Goldstone action is subject to a gradient expansion which results in \( \square \).

Summarizing, we have derived an effective field theory for the maximally disordered RF-model. The theory has a status analogous to the supersymmetric non-linear \( \sigma \)-models of ‘conventional’ disordered Fermi systems, but its behavior is substantially different, a fact that is readily traced back to the presence of a chiral symmetry. It was shown that the formalism provides a platform from which interconnections to a variety of other recently investigated chiral problems can be conveniently analysed.

It is a pleasure to acknowledge valuable discussions with M. Janssen, A. Tsvelik, and J. M. Verbaarschot. We thank M. R. Zirnbauer for plenty of useful suggestions made at all stages of this work, and in particular for pointing out the significance of the operator \( O^{(1)} \).

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[11] Defining \( N_i, i = 1, \ldots, d \) as the number of sites in the \( \hat{c}_i \)-direction, and \( c_4 = N/2^d \),

\[ S_0[T] = c_4 \sum_{s_i=0}^{1} (-)^{(N_i+1)s_i} \text{str} \ln(T(s_1L_1, \ldots, s_dL_d)). \]

[12] Models where the \( U_{ij} \) are subject to further constraints may behave qualitatively differently and are not encompassed by the present analysis (e.g. the time-reversal invariant counterpart of the RF-model, \( U_{ij} \in O(N) \)).

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