Topology-Induced Symmetry Breaking for Vortex with Artificial Monopole

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We construct an artificial $U(1)$ gauge field in the cold atom system to form a monopole along with vortices. It is supposed that the cold atoms are confined on a spherical surface, and two sets of identical laser beams in the opposite propagating directions shine on two sides of the sphere. Arbitrary Chern number $CN$, proportional to the quantized magnetic flux, can be obtained by selecting proper laser modes. This construction meets the condition of Chern’s theorem, so that the vortices of the atom wave function will emerge on the sphere, whose winding number equals $CN$. It is found that a geometric symmetry is broken spontaneously for odd $CN$, which corresponds to a topology-induced quantum phase transition. In particular for $CN = 1$, the ground state of the cold atoms are double-degenerate and can be applied to make a stable qubit. Since the ground-state degeneracy is protected (led) by topology-induced symmetry breaking against dissipation, the proposed topological structure has vast potential in quantum storage.

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Gauge field plays an elementary role in various fields of physics. The gauge intensity reflects a local geometric property, the curvature, of intrinsic space (vector bundle) \[ \text{property}, \text{the curvature}, \text{of intrinsic space (vector bundle)} \]

Cold atom scheme resorts to a working about the geometric phase [8]. The idea of the work [2–4], and optical lattice [5–7], based on Berry’s pioneering and computation. Thus, it is interesting and significant to see their realization in the artificial gauge system.

Topology is closely related to the gauge field, which reflects the global twisting of the intrinsic space. Nevertheless, there are few studies of investigating the topological structure in the cold atom scheme. Although the artificial magnetic field with $1/e^2$ behavior has been constructed and called monopole [3], it is the $U(2)$ gauge of 3 dimension that always has a vanishing Chern number [1] and does not generate a vortex, in other words, a trivial topology. The Chern number as well as the winding number of vortices are important topological invariants and extremely useful in the topological physics such as instaton [18], integer quantum Hall effect [19], and Kosterlitz-Thouless transition [21]. These topological invariants are robust against local disturbances, therefore, they have vast potential for topological quantum storage and computation. Thus, it is interesting and significant to see their realization in the artificial gauge system.

In this letter, we will construct an artificial gauge field with an adjustable Chern number by the scheme of cold atom, in which the monopole and vortex will be generated naturally. We recall Chern’s theorem [1], also known as the Dirac quantization condition [1, 20], that the (first) Chern number $\int_{\Sigma} B/2\pi \times iB$ is the curvature 2-form of a $U(1)$ gauge field equals to the winding number of vortices for the charged particle on a 2 dimensional (2D) closed surface $\Sigma$. Then we follow this condition to propose the cold atoms being confined on a 2D spherical surface $S^2$, and couple them to a $U(1)$ gauge potential. The $U(1)$ gauge can be realized by the laser-atom coupling introduced above, while the laser beam can only shine on a semi sphere. Thus we resort to an identical set of laser beams with the opposite propagating direction to cover the other part. These gauge potentials for two sides should be equivalent along the equator, namely, differ only by a gauge transformation. This construction is similar to the well known Dirac monopole [22], and leads to a non-zero Chern number $CN$ adjusted by the choice of laser modes.

Chern’s theorem guarantees the emergence of vortices for the cold atoms on $S^2$. We solve the Schrödinger equation for the atoms and obtain the wave functions of the vortices. A geometric symmetry is found spontaneously broken when $CN$ adjusted from even to odd, which indicates the existence of a topology-induced quantum phase transition. The ground states of the cold atoms are degenerate in the symmetry breaking phase, in which the winding number of vortex keeps invariant in their linear superposition. In particular, for the case of $CN = 1$, the degenerate ground states can be used to make a stable qubit, with the location of its single vortex representing the qubit state. This type of qubit, utilizing the ground-state degeneracy led by topology-induced symmetry breaking, avoids the dissipation of the traditional qubit [23]. Therefore, the topological structure proposed in this letter has great capability in quantum storage.

We start to construct an artificial $U(1)$ gauge field on
$S^2 \mathbb{R}$. The origin Hamiltonian consists of the kinetic energy of the mass center characterized by the Laplacian on sphere, and a potential acting on the internal space. It takes the form of $H_0 = -\nabla^2 + V (\mathbf{r})$ in which

$$V (\mathbf{r}) = \begin{pmatrix} g_1 (\mathbf{r}) & g_2 (\mathbf{r}) \\ \frac{g_1 (\mathbf{r})}{g_2 (\mathbf{r})} \end{pmatrix},$$

with the exterior differentiation $d$, Hodge star $*$, $3 \times 3$ identity matrix $I$, atomic mass $M$, and $\hbar = 1$. This model describes a group of non-interacting cold atoms in Bose-Einstein condensation coupled to laser light. Metric elements $g_{1,2} (\mathbf{r})$ represent the laser fields that induce the Rabi oscillations between the internal energy levels of atoms, which satisfy the Helmholtz equations. They are usually parameterized as $g_1 = g \cos \beta \cdot e^{i \gamma_1}$ and $g_2 = g \sin \beta \cdot e^{i \gamma_2}$ by real functions. The common treatment is to find the dressed state $|D (\mathbf{r}) \rangle$ that satisfies $V (\mathbf{r}) |D (\mathbf{r}) \rangle = 0$ and to assume a heavy atomic mass $M \gg g (\mathbf{r})$ to apply the Born-Oppenheimer approximation [10][24]. As a result, the mass-center motion of the dressed state $|D (\mathbf{r}) \rangle$ is dominated by the reduced Hamiltonian $H = \langle D | H_0 | D \rangle$ with a U (1) gauge potential, given by

$$H = -\frac{1}{2M} (d - iA) \cdot (d - iA) + \frac{W (\mathbf{r})}{2M}.$$  

The gauge and scalar potentials are respectively given by [8][4]

$$A = -\cos^2 \beta d \gamma_1 - \sin^2 \beta d \gamma_2 + d \eta, \quad (3)$$

$$W = \cos^2 \beta \sin^2 \beta \|d \gamma_1 - d \gamma_2\|^2 + \|d \beta\|^2.$$  

As our scheme requires, potentials $A$ and $W$ are confined on the sphere $r = r_0$, so that parameters $\beta$ and $\gamma$ are the functions of only spherical coordinates $\theta$ and $\varphi$. Furthermore, $A$ is undetermined with an unfixed gauge, an arbitrary exact 1-form $d \eta$ on the sphere. One can design the fields of laser to construct the required U (1) gauge.

Next we select the solutions of the Helmholtz equation to realize the gauge with non-trivial topological structures. It has been mentioned that we need two sets of laser beams with the opposite propagation directions to cover the entire sphere, with the sketch presented in Fig. 1. The propagating directions are chosen as $\pm z$ axis, respectively. The two sets of laser beams are made symmetric under the $180^\circ$ rotation along the $x$ axis, i.e. the 2D parity for the $y - z$ coordinate $P : (\theta, \varphi) \rightarrow (\pi - \theta, -\varphi)$. For the north semi-spherical part, the laser beams $g_{1,2} (\mathbf{r})$ have wave vector $k$ along the $-z$ axis, whose solution in the spherical coordinate are given by the Gauss-Laguerre beam [22]

$$g_j (r, \theta, \varphi) = G c_j \left( \frac{\sqrt{2} r \sin \theta}{w (r \cos \theta)} \right)^{|l_j|} e^{-w r^2 / 2}, \quad (5)$$

in which $w (z) = w_0 \sqrt{1 + z^2 / z_R^2}$ with $z_R = k w_0^2 / 2$ the Rayleigh range and $w_0$ the waist size. We now assume that the sphere locates at the origin with its radius far less than the Rayleigh range $r_0 \ll z_R$, so that we obtain $w (z) \approx w_0$ on the sphere. Furthermore, the common factor $G (r, \theta, \varphi)$ is a Gaussian beam but has nothing to do with the potentials, for Eqs. (3)-(4) contain only the angular parameters. Besides, $c_j$ are constants representing the laser intensities, which are adjusted to $c_1 / c_2 = (\sqrt{2} r_0 / w_0) |l_2| - |l_1|$ for simplicity. Then the parameters $\beta$ and $\gamma$ are calculated and given by $\gamma_j (\varphi) = -l_j \varphi$ and $\tan \beta (\theta) = \sin |l_2| - |l_1| \theta$. Without losing generality, we choose $0 \leq l_1 < l_2$ and obtain the potentials from Eqs. (3)-(4) on the north semi sphere as follows,

$$A_N = f (\theta) \, d \varphi + d \eta, \quad (6)$$

$$W_N = \frac{1}{r_0^2} \left[ 1 + \cos^2 \theta \frac{d f (\theta)}{d \theta} \right], \quad (7)$$

with a continuous function defined by $f (\theta) = \left[ l_1 + l_2 (\sin \theta)^{2l_2 - 2l_1} \right] / \left[ 1 + (\sin \theta)^{2l_2 - 2l_1} \right]$. Since $d \varphi$ has coordinate singularities at the north and south poles, we fix the gauge as $\eta = -f (0) \varphi$ to make $A_N = [f (\theta) - f (0)] d \varphi$ analytical.

The potentials on the south semi sphere is the 2D parity transformation $P$ of the north one, that the gauge potential reads $A_S = [f (\theta) - f (0)] d (-\varphi)$ and the scalar potential $W_S$ shares the same expression of Eq. (7). Similar to the well known Dirac monopole, the gauge potential on the equator $(A_N - A_S)_{\theta = \pi / 2} = 2 [f (\varphi) - f (0)] d \varphi$ should be a periodic 1-form, which requires $2 [f (\varphi) - f (0)]$ to be an integer [22][26]. Actually this number is the proper (first) Chern number of

![Figure 1](image-url)
the $U(1)$ Berry curvature,

$$CN = \frac{1}{2\pi} \int_{S^2} dA = 2 \left[ f\left(\frac{\pi}{2}\right) - f\left(0\right) \right].$$ \hspace{1cm} (8)

Using \( f(\theta) \), we find \( CN = l_2 - l_1 \) is indeed an integer. We also see that arbitrary Chern number can be realized by choosing a proper set of laser modes in this scheme. The physical interpretation of \( CN \) is given as follows: the integrand \( B = dA \) is a differential 2-form known as the Berry curvature like a magnetic field, then \( 2\pi \cdot CN \) represents the magnetic flux \( \int_{S^2} B \). The magnetic flux always equals to zero in the trivial case, and the non-zero flux indicates the existence of a monopole, a non-trivial topology. As revealed by differential geometry, a non-trivial topology requires \( A \) being defined in different areas, and joined together via a gauge transformation. Our designation satisfies this construction, therefore it forms an artificial monopole.

The topology of the monopole will influence the “charged” particle field. Denoting the dressed state by \(|D\rangle = \int_{S^2} d\psi(r) |r\rangle \), we see that the phase of the wave function \( \psi(r) \), known as the Berry phase, is adaptive to the \( U(1) \) Berry connection \( A \). By Chern’s theorem, we immediately know the index of zeros (total winding number of the Berry phase around zero points) of \( \psi \) on the sphere equals to \( CN \). The zero point with a non-zero winding number is a vortex. Therefore, in our well-constructed system, the artificial monopole induces vortices of the cold atoms.

Then we investigate the eigenstates of the cold atoms to check the properties of vortex, by solving the time-independent Schrödinger equation \( H\psi = (\lambda/2M\gamma^2)\psi \). Since both the equation and the Chern number are invariant under the 2D parity transformation \( P \), the solution space must be \( P \) symmetric. Besides this discrete symmetry, there is also the z-axial rotation symmetry \( R \) with generator \(-i\partial_R \), so that the spherical variables can be separated. The basis of the solution to the Schrödinger equation takes the form [20]

$$\psi(\theta, \varphi) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{i m_2 \varphi \Theta(\theta)}, & 0 \leq \theta < \frac{\pi}{2}, \\ \frac{1}{\sqrt{2\pi}} e^{-i m_2 \varphi \Theta(\theta)}, & \frac{\pi}{2} \leq \theta \leq \pi. \end{cases} \hspace{1cm} (9)$$

Since the gauge potential on the equator satisfies \( A_{\lambda} = A_{\lambda} + CN\partial_{\phi} \), the rule of gauge transformation requires \( \psi_{|\theta=\pi/2} = e^{iCN\varphi} \psi_{|\theta=-\pi/2} \), which demands \( m_N = CN - m_S \) according to Eq. (10). Actually, this is the result of Chern’s theorem as revealed later. To obtain the equation for the latitude wave function \( \Theta \), we insert Eq. (9) to the Schrödinger equation and make substitution \( z = \cos \theta \), arriving at

$$\left[ -\frac{d}{dz} \left( 1 - z^2 \right) \frac{d}{dz} + \frac{F^2}{1 - z^2} + r_0^2 W - \lambda \right] \Theta = 0, \hspace{1cm} (10)$$

in which \( F(z) \) is defined by \( m_s = f(\pi - \theta) + f(0) \) at \( z \in (-1, 0) \) and \( f(\theta) - f(0) - m_N \) at \( z \in (0, 1) \) with

\[ F'(z) \] existing at \( z = 0 \). Since \( F(z) \) is continuous, the wave function is second-order differentiable. Without the artificial gauge, i.e. setting \( f = 0 \), Eq. (10) reduces to the usual Legendre equation.

Figure 2 presents the behaviors of the wave functions for various \( CN \) by numerical solution, in which \( l_1 = 0 \) is chosen for simplicity so that \( CN = l_2 \) from Eq. (8). The most important conclusion shown in Fig. 2 is that the non-zero \( m_NS \) induces zero of the wave function at the corresponding pole. This happens even for the ground states, unlike the normal quantum mechanical system whose ground state usually contains no zero. And considering the phase around the zero reads \( m_N\varphi \) or \( m_S(\varphi) \) according to Eq. (9), we find the winding number of the phase around the north and south pole is just \( m_N \) and \( m_S \) (notice the right-handed direction of \( -\varphi \) corresponds to the outside normal direction at the south pole), respectively. Thus, that the total winding number reads \( m_N + m_S = CN \) is the result of Chern’s theorem, which is consistent with the condition of gauge transformation.

The another interesting effect presented in Fig. 2 is the \( RP \) symmetry \( SO(2) \otimes \mathbb{Z}_2 \) of the Hamiltonian Eq. (4) is spontaneously broken for the odd Chern number. It is seen that the ground states for the even Chern number \( CN = 0, 2 \) (Fig. 2(a) and (c)) are non-degenerate, which certainly contains the complete symmetry; while those for the odd Chern number \( CN = 1, 3 \) (Fig. 2(b) and (d)) are double-degenerate, which means the \( RP \) symmetry is broken, for the \( RP \) symmetric solution must have an even winding number. Therefore, there is a topology-induced quantum phase transition: when the discrete variable \( CN \) varies from even to odd such as from 0 to 1, the \( RP \) symmetric phase vanishes and the asymmetric phase emerges. At the same time, the Landau order...
The degenerate ground states for \( CN \) keeps correct for the case of higher odd Chern number. The superposition state of \( \psi \) making a qubit. The superposition state of \( R \) symmetry; or it locates at the equator to maintain the \( P \) symmetry. All in all, the \( RP \) symmetry must be broken for \( CN = 1 \), which still keeps correct for the case of higher odd Chern number.

What is the application of this topological structure? The degenerate ground states for \( CN = 1 \) are suitable for making a qubit. The superposition state of \( \psi_N \) and \( \psi_S \), locating on the Bloch sphere, represents a bit of quantum information. The qubit state can be read by measuring the location of the single vortex formed by a group of cold atoms in condensation. We know that the intensity of linear response is proportional to the density of matter, thus the vortex as the zero point can be detected by the response signal to a weak external field \([27]\). Next, we point out that this qubit is protected against dissipation. From the beginning if the dissipation rate of the internal levels for potential Eq. (1) is much less than the light intensity \( g \), the dissipation will be negligible and the dressed state will have an appreciable lifetime. Then suppose the \( RP \) symmetry and topological structures proposed in this letter are realized, then the degenerate ground states, resulted from topology-induced symmetry breaking, evidently have no dissipation. Thus, the qubit is stable against dissipation. Besides, the required topology is determined by the lasers which can be fined tuned and keep stable. And suppose there is small deficiency of the required symmetry, one can still adjust the light field to compensate the asymmetry and restrict the degeneracy splitting. In short, our topological scheme provides an ideal realization of quantum storage device.

In summary, we construct an artificial \( U(1) \) gauge potential in the cold atom system on a spherical surface to form a monopole and vortex. The monopole with arbitrary Chern number can be obtained by selecting proper coupling laser beams, and it induces the vortex of the atoms appearing on the sphere via Chern’s theorem. Then the Schrödinger equation is solved to check the total winding number of vortices equal \( CN \). For the odd \( CN \), it is found that the geometric \( RP \) symmetry is spontaneously broken, for the ground states of atoms are double-degenerate. This topology-induced phase transition has a discrete dependent variable \( CN \), without a transition order. In particular for \( CN = 1 \), it is shown that the linear coefficient of superposition ground state is one-to-one mapped to the location of single vortex. These degenerate ground states are suitable for making a qubit whose state can be read by detecting the vortex location. Besides, the ground-state degeneracy led by topology-induced symmetry breaking is stable against dissipation. It is interesting to see the realization of the proposed topological structure in the cold atom experiments, which possesses an application significance in quantum storage.

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