Color dipole cross section in the DGLAP improved saturation model

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Abstract We show that the geometric scaling of the dipole cross section can be explained using standard DGLAP perturbative evolution. The DGLAP-improved saturation model due to the Laplace transform method is considered at LO and NNLO approximations from the experimental data by relying on a Froissart-bounded parametrization of $F_2(x, Q^2)$. These results are comparable with the Golec-Biernat–Wüsthoff (GBW) model in a wide kinematic region $r Q_s$ which takes into account charm mass. The successful description of $\sigma_{\text{dip}}(x, r)/\sigma_0$ and $\sigma_{\text{dip}}(r Q_s)/\sigma_0$ are presented.

1 Introduction

An update [1] on the saturation model of deep inelastic scattering (DIS) was recently presented by Golec-Biernat et al. by introducing the results of new fits [2] to the extracted Hadron-Electron Ring Accelerator (HERA) data [3,4] on the proton structure function at small $x$ with the Golec-Biernat–Wüsthoff (GBW) saturation model and its modification to cover high values of $Q^2$. When $x \ll 1$, the Dokshitzer–Gribov–Lipatov–Altarelli–Parisi (DGLAP) [5–7] or the Balitsky–Fadin–Kuraev–Lipatov (BFKL) [8–10] evolution equations predict that the small $x$ structure of the proton is dominated by a strongly increasing gluon density, which drives a similar increase in the sea quark densities. In this region, gluons in the proton form a dense system with mutual interaction and recombination which leads to the saturation of the total cross section [8–10]. For $x \approx Q^2/W^2 \ll 1$, the virtual spacelike photon on the proton fluctuations are defined into an on-shell quark-antiquark, $q\bar{q}$, vector state. Here, $Q^2$ refers to the photon virtuality, and $W$ to the photon–proton center-of-mass energy. In this process, a photon interacts with the proton via coupling of two gluons to the $q\bar{q}$ color dipole, which is called the color dipole model (CDM).

The mass of the $q\bar{q}$ dipole in terms of the transverse momentum $k_\perp$ is given by $M_{q\bar{q}}^2 = \frac{k_\perp^2}{(1-z)}$, where $k_\perp$ is defined with respect to the photon direction, and the variable $z$ characterizes the distribution of the momenta between quark and antiquark [11–13]. The lifetime of the $q\bar{q}$ dipole is defined by $\tau = \frac{W^2}{Q^2 + M_{q\bar{q}}^2} \gg \frac{1}{M_p}$, which is much longer than its typical interaction time with the target at small $x$. This condition not only restricts the kinematic range of the color dipole model to $x \ll 1$, but also saturates the $\gamma^*\text{-proton}$ cross section with $x < 0.1$ [14].

Some years ago [15,16] the saturation model was shown by Golec-Biernat and Wüsthoff, which gives an elegant and accurate account of DIS at small $x$ and has been formulated to new models in recent years [17,23]. This type of saturation occurs when the photon wavelength $1/Q$ reaches the size of the proton. It is well known that the dipole picture is a factorization scheme for DIS, which is particularly convenient for the inclusion of unitarity corrections at small $x$. In the mixed representation, the scattering between the virtual photon $\gamma^*$ and the proton is seen as the color dipole where the transverse dipole size $r$ and the longitudinal momentum fraction $z$ are defined with respect to the photon momentum. The amplitude for the complete process is simply the production of these subprocess amplitudes, as the DIS cross section is factorized into a light-cone wave function and a dipole cross section. Using the optical theorem, this leads to the following expression for the $\gamma^*\text{-proton}$ cross sections

$$\sigma^{\gamma^*p}_{L,T}(x, Q^2) = \int d^2r |\Psi_{L,T}(r, z, Q^2)|^2 \sigma_{\text{dipole}}(x, r),$$

(1)

and the $F_2$ structure function is defined as

$$F_2(x, Q^2) = \frac{Q^2}{4\pi^2} \left[ \sigma^{\gamma^*p}_L(x, Q^2) + \sigma^{\gamma^*p}_T(x, Q^2) \right].$$

(2)

The subscript $L$ and $T$ refer to the transverse and longitudinal polarization state of the exchanged boson. Here, $\Psi_{L,T}$ are the
appropriate spin-averaged light-cone wave functions of the photon, \( \sigma_{\text{dip}}(x_f, r) \) is the dipole cross section which is related to the imaginary part of the \((q\bar{q})\) forward scattering amplitude, and \( \tilde{X} \equiv x(1 + 4m_f^2/Q^2) \) is equivalent to the Bjorken variable and provides an interpolation for the \( Q^2 \rightarrow 0 \) limit; \( m_f \) is the mass of the quark of flavor \( f \). The variable \( z \), with \( 0 \leq z \leq 1 \), characterizes the distribution of the momenta between the quark and antiquark. The square of the photon wave function describes the probability for the occurrence of a \((q\bar{q})\) fluctuation of transverse size with respect to the photon polarization [1, 15–27].

The dipole hadron cross section \( \sigma_{\text{dip}} \) contains all information about the target and the strong interaction physics. There are several phenomenological implementations for this quantity, and the main feature is to be able to match the soft (low \( Q^2 \)) and hard (large \( Q^2 \)) regimes in a unified way. In Ref. [15, 16], the dipole cross section was proposed to have the eikonal-like form

\[
\sigma_{\text{dip}}(x_f, r) = \sigma_0 \left( 1 - e^{-r^2Q_s^2/4} \right),
\]

where \( Q_s(x) \) plays the role of the saturation momentum, parametrized as \( Q_s^2(x) = Q_0^2(x/x_0)^{-\lambda} \). Parameters \( Q_0 \) and \( x_0 \) set the dimension and absolute value of the saturation scale and exponent \( \lambda \) governs \( x \) behavior of \( Q_s^2 \). The saturation [nonlinear quantum chromodynamics (QCD)] is energy-dependent and marks the transition between the linear (leading twist) perturbative QCD regime and saturation domain. The resulting dipole cross section presents the color transparency property, i.e., \( \sigma_{\text{dip}} \sim r^2 \) when \( r \rightarrow 0 \), which is a purely perturbative QCD (pQCD) phenomenon, and the saturation property, i.e., \( \sigma_{\text{dip}} \sim \sigma_0 \) at large \( r \), which imposes the unitarity condition. The GBW model was updated in [18, 19, 28] to improve the large \( Q^2 \) description of \( F_2 \) by a modification of the small \( r \) behavior of the dipole cross section to include the DGLAP evolved gluon distribution. A parameterization similar in spirit to the dipole scattering amplitude, based on the Balitsky–Kovchegov (BK) equation solution, was proposed in [17]. The BK equation [29–31] for a dipole scattering amplitude was proposed in terms of the hierarchy of equations for Wilson line operators in the limit of large number of colors \( N_c \). The geometrical scaling (GS) [32] at the high-energy limit of pQCD is obtained from the BK equation [29–31] and the color glass condensate (CGC) formalism [33]. Geometrical scaling is connected to the existence of the saturation scale and is defined as the dependence of the dipole cross section on only one dimensionless variable.

In the limit of large \( Q^2 \) values, the structure function (2) does not exactly match with the DGLAP formula for \( F_2 \), i.e., the saturation model does not include logarithmic scaling violations. Since the energy dependence in a large \( Q^2 \) region is mainly due to the behavior of the dipole cross section at small dipole size \( r \), the authors in Refs. [18, 19] investigated the DGLAP evolution for small dipoles. Bartels–Golec-Bienat–Kowalski (BGBK) improved the dipole cross section by adding the collinear DGLAP effects. Indeed, the BGBK model is the implementation of QCD evolution in the dipole cross section which depends on the gluon distribution. The following modification of the DGLAP improved saturation model [1] proposed for the dipole cross section is

\[
\sigma_{\text{dip}} = \sigma_0 \left\{ 1 - \exp \left( -\frac{\pi^2r^2}{3\sigma_0}G(\tilde{x}, \mu_s^2) \right) \right\},
\]

where the hard scale is assumed to have the form

\[
\mu_s^2 = C/r^2 + \mu_0^2,
\]

and the parameters \( C \) and \( \mu_0^2 \) are obtained from the fit to the DIS data [1]. The gluon distribution \( g(x, \mu^2) \) obeys the DGLAP evolution equation truncated to the gluonic sector, as reported in Refs. [1, 15–33], by the form

\[
\frac{\partial g(x, \mu^2)}{\partial \ln\mu^2} = \frac{\alpha_s(\mu^2)}{2\pi} \int_0^1 \frac{dz}{z} P_{gg}(z) g \left( \frac{x}{z}, \mu^2 \right),
\]

where \( g(x, \mu^2) \) is the gluon density, and \( G(x, \mu^2) = xg(x, \mu^2) \). The splitting function \( P_{gg} \) at the leading-order (LO) approximation reads

\[
P_{gg}^{\text{LO}}(z) = 2C_A \left( \frac{z}{1-z} + \frac{1-z}{z} + z(1-z) \right) + \delta(1-z) \left( 11C_A - 4n_fT_R \right) \frac{6}{6},
\]

with \( C_A = N_c = 3, C_F = \frac{N_c^2-1}{2N_c} = \frac{4}{3}, \) and \( T_f = \frac{1}{2}n_f \) where \( n_f \) is the active quark flavor. The convolution integrals in (6) which contain a plus prescription, \( (),_+ \), can be easily calculated by

\[
\int_0^1 \frac{dy}{y} f \left( \frac{x}{y} \right)_+ g(y) = \int_0^1 \frac{dy}{y} f \left( \frac{x}{y} \right)_+ \left[ g(y) - \frac{x}{y} g(x) \right] - g(x) \int_0^x f(y)dy.
\]

The initial gluon distribution is defined at the scale \( \mu_0^2 \) in the form

\[
xg(x, \mu_0^2) = A_g x^{-\lambda_g}(1-x)^{5.6}.
\]

The choice of the power 5.6, which regulates the large-x behavior, and other parameters (i.e., \( A_g \) and \( \lambda_g \)) is motivated by global fits to DIS data with the LO DGLAP equation in the literature.

Although the BGBK model is successful in describing the dipole cross section at large values of \( r \), as the two models (GBW and BGBK) overlap in this region, they differ in the small \( r \) region where the running of the gluon distribution starts to play a significant role. Indeed, the DGLAP improved model of \( \sigma_{\text{dip}} \) significantly improves agreement at large values of \( Q^2 \) without affecting the physics of saturation.
responsible for the transition to small $Q^2$. As expected, GS is true for the DGLAP improved model curve for the scaling variable $rQ_s \geq 1$ and for the GBW model curve for the whole region [1].

It is well known that the color dipole cross sections are determined from the original structure functions with a parametrization of the deep inelastic structure function for electromagnetic scattering with protons in Ref. [34]. The authors in Ref. [34] presented the dipole cross section from an approximate form of the presumed dipole cross section convoluted with the perturbative photon wave function for virtual photon splitting into a color dipole with massless quarks. Some approximated analytical solutions in the color dipole model have been reported in recent years [35–42] with considerable phenomenological success. The analytical methods of the unpolarized DGLAP evolution equations have been discussed extensively in Mellin and Laplace transformation [43, 44].

We present a modification of the DGLAP improved saturation model with respect to the Laplace transform technique by employing the parametrization of the proton structure function at LO up to next-to-next-to-leading order (NNLO) approximations, which preserves its behavior success in the low- and high-$Q^2$ regions. We show that GS holds for the DGLAP improved model in a wide kinematic region $rQ_s$.

In the next section, we introduce the theoretical details of the model with regard to the Laplace transform technique and discuss its qualitative features. We then derive the dipole cross section with respect to the parametrization of $F_2$ at LO up to NNLO approximations. In Sect. 3, we describe our results and discuss their physical implications in comparison with the GBW model. Section 4 contains conclusions.

2 The model

An analytical expression for $F_2(x, Q^2)$ was suggested by the authors in Ref. [45] which describes fairly well the available experimental data on the reduced cross section in full accordance with the Froissart predictions [46]. This parameterization provides a reliable structure function $F_2(x, Q^2)$ according to a combined fit of the H1 and ZEUS Collaborations' data [47] in a range of the kinematic variables $x$ and $Q^2$, $x \leq 0.1$ and 0.15 GeV$^2 < Q^2 < 3000$ GeV$^2$ as

$$F_2(x, Q^2) = D(Q^2) (1-x)^n \sum_{m=0}^{2} A_m(Q^2) L^m$$  \hspace{1cm} (10)

and can be applied as well in analyses of ultrahigh-energy processes with cosmic neutrinos. The effective parameters are defined by the following forms

$$D(Q^2) = \frac{Q^2 (Q^2 + \lambda M^2)}{(Q^2 + M^2)^2}, \quad A_0(Q^2) = a_{00} + a_{01} L_2,$$

$$A_i(Q^2) = \sum_{k=0}^{2} a_{ik} L_2^k, \quad i = (1, 2),$$  \hspace{1cm} (11)

with the logarithmic terms $L$ as

$$L = \ln \frac{1}{x} + L_1, \quad L_1 = \ln \left( \frac{Q^2}{Q^2 + \mu^2} \right)$$

$$L_2 = \ln \left( \frac{Q^2 + \mu^2}{Q^2} \right),$$  \hspace{1cm} (12)

where the effective parameters $M$ and $\mu^2$ are the effective mass and a scale factor, respectively. The additional parameters with their statistical errors are given in Table 1. According to the DGLAP $Q^2$-evolution equation, the singlet and gluon distribution functions are related by the following form [43] (for further discussion, please refer to Appendix A)

$$\frac{\partial F_2(x, Q^2)}{\partial \ln Q^2} = -\frac{a_s(Q^2)}{2} \left[ P_{qq}(x) \otimes F_2(x, Q^2) + <e^2> P_{gq}(x) \otimes x g(x, Q^2) \right],$$  \hspace{1cm} (13)

where

$$P_{a,b}(x) = P_{a,b}^{(0)}(x) + a_s(Q^2) \tilde{P}_{a,b}^{(1)}(x) + a_s^2(Q^2) \tilde{P}_{a,b}^{(2)}(x)$$  \hspace{1cm} (14)

and

$$\tilde{P}_{a,b}^{(n)}(x) = P_{a,b}^{(n)}(x) + \left[ C_{2,s} + C_{2,g} + \cdots \right] \otimes P_{a,b}^{(0)}(x) + \cdots .$$

The quantities $\tilde{P}_{a,b}$ are expressed via the known splitting and Wilson coefficient functions in the literature [43, 48–58], and $a_s(Q^2) = \alpha_s(Q^2)/4\pi$. In Eq. (13), $<e^k>$ is the average of the charge $e^k$ for the active quark flavors, $<e^k> = n_f^{-1} \sum_{f=1}^{n_f} e^k_f$.

One can substantially simplify the calculations by considering Eq. (13) in the space of Laplace transform techniques and taking advantage of the fact that the convolution form $f_1(x) \otimes f_2(x)$ in $x$ space becomes merely a product of individual Laplace transforms of the corresponding functions in the Laplace space. By considering the variable definitions $v \equiv \ln(1/x)$ and $w \equiv \ln(1/z)$, Eq. (13) reads as

$$\frac{\tilde{F}_2(v, Q^2)}{\partial \ln Q^2} = \int_0^v \left[ \tilde{F}_2(v, Q^2) \tilde{F}_2^{(\psi)}_{2,s}(a_s(Q^2), v-w) + <e^2> \tilde{g}(v, Q^2) \tilde{F}_2^{(\psi)}_{2,g}(a_s(Q^2), v-w) \right] dw$$  \hspace{1cm} (15)

where

$$\frac{\partial \tilde{F}_2(v, Q^2)}{\partial \ln Q^2} = \frac{\partial F_2(e^{-v}, Q^2)}{\partial \ln Q^2},$$

$$\tilde{g}(v, Q^2) = G(e^{-v}, Q^2),$$

$$\tilde{P}^{(\psi)}_{a,b}(a_s(Q^2), v) = e^{-v} \tilde{P}^{(\psi)}_{a,b}(a_s(Q^2), v).$$
Here, $\phi$ denotes the order in running coupling $\alpha_s(Q^2)$, and
\[
P^{(\psi)}_{a,b}(a_s, x) = \sum_{\phi=0}^{\psi} \alpha_s^{\phi+1}(Q^2) P^{(\phi)}_{a,b}(x).
\]

The Laplace transform of $\mathcal{H}(a_s(Q^2), \nu) \cdot s$ are given by the following forms
\[
\Phi^{(\psi)}(a_s(Q^2), s) \equiv \mathcal{L}[\mathcal{H}^{(\psi)}(a_s(Q^2), \nu); s]
= \int_0^\infty \mathcal{H}^{(\psi)}(a_s(Q^2), \nu) e^{-sv} dv,
\]
\[
\Theta^{(\psi)}(a_s(Q^2), s) \equiv \mathcal{L}[\mathcal{H}^{(\psi)}(a_s(Q^2), \nu); s]
= \int_0^\infty \mathcal{H}^{(\psi)}(a_s(Q^2), \nu) e^{-sv} dv.
\]

We know that the Laplace transforms of the convolution factors are simply the ordinary products of the Laplace transforms of the factors. Therefore, Eq. (15) in the Laplace space $s$ reads as
\[
\frac{\partial f_2(s, Q^2)}{\partial \ln Q^2} = \Phi^{(\psi)}(a_s(Q^2), s) f_2(s, Q^2)
+ \left< e^2 > \Theta^{(\psi)}(a_s(Q^2), s) g(s, Q^2), \right.
\]
where
\[
\mathcal{L}[\mathcal{F}(v, Q^2); s] = f_2(s, Q^2),
\mathcal{L}[\mathcal{G}(v, Q^2); s] = g(s, Q^2).
\]

The gluon insertion into the parametrization of the proton structure function and its derivative with respect to $\ln Q^2$ in $s$-space in Eq. (16) is given by the following form
\[
g^{(\psi)}(s, Q^2) = k^{(\psi)}(a_s(Q^2), s) Df_2(s, Q^2)
- hs^{(\psi)}(a_s(Q^2), s) f_2(s, Q^2),
\]
where
\[
Df_2(s, Q^2) = \frac{\partial f_2(s, Q^2)}{\partial \ln Q^2},
k^{(\psi)}(a_s(Q^2), s) = \frac{1}{(e^2 > \Theta^{(\psi)}(a_s(Q^2), s))},
h^{(\psi)}(a_s(Q^2), s) = \Phi^{(\psi)}(a_s(Q^2), s) k^{(\psi)}(a_s(Q^2), s).
\]

The coefficient functions $\Phi_f$ and $\Theta_f$ in the Laplace space $s$ are given by

- at LO approximation
\[
\Theta_f^{(0)}(a_s, s) = 2n_f a_s(Q^2) \left[\frac{1}{1+s} - \frac{2}{2+s} + \frac{2}{3+s}\right],
\]
\[
\Phi_f^{(0)}(a_s, s) = a_s(Q^2) \left[4 - \frac{8}{3} \left(\frac{1}{1+s} + \frac{1}{2+s} + 2S_1(s)\right)\right].
\]
Here, $S_1(s) = \psi(s+1) + \gamma_E$, where $\psi(x)$ is the digamma function, and $\gamma_E = 0.5772156 \ldots$ is the Euler constant.

The explicit expressions for the NLO and NNLO kernels in $s$ space are rather cumbersome; therefore, we recall that we are interested in investigation of the kernels in small $x$ [59,60] as

- at NLO approximation
\[
\Theta_f^{(1)}(a_s, s) \simeq \Theta_f^{(0)}(s) + \alpha_s^2(Q^2) C_A T_f \left[\frac{40}{9s}\right],
\]
\[
\Phi_f^{(1)}(a_s, s) \simeq \Phi_f^{(0)}(s) + \alpha_s^2(Q^2) C_F T_f \left[\frac{40}{9s}\right].
\]

- at NNLO approximation
\[
\Theta_f^{(2)}(a_s, s) \simeq \Theta_f^{(1)}(s) + \alpha_s^2(Q^2) \left[n_f \left[ -\frac{1268.300}{s} + \frac{896}{3s^2} \right] + \frac{n_f^2}{243s} \right],
\]
\[
\Phi_f^{(2)}(a_s, s) \simeq \Phi_f^{(1)}(s) + \alpha_s^2(Q^2) \left[n_f \left[ -\frac{206}{s} + \frac{3584}{27s^2} \right] + \frac{n_f^2}{256s} \right].
\]

The standard representation for QCD couplings in LO up to NNLO (within the $\overline{MS}$ scheme) approximations are defined by
\[
\alpha_s^{LO}(t) = \frac{4\pi}{\beta_0 t},
\]
\[
\alpha_s^{NLO}(t) = \frac{4\pi}{\beta_0 t} \left[ 1 - \frac{\beta_1}{\beta_0} \ln \frac{t}{\Lambda^2} \right],
\]
\[
\alpha_s^{NNLO}(t) = \frac{4\pi}{\beta_0 t} \left[ 1 - \frac{\beta_1}{\beta_0} \ln \frac{t}{\Lambda^2} \right] + \frac{1}{\beta_0^3 t^2} \left[ \frac{\beta_1^2}{\beta_0^3} \left( \ln^2 t - \ln t - 1 \right) + \beta_2 \right].
\]
where $\beta_0, \beta_1$, and $\beta_2$ are the one-, two-, and three-loop corrections to the QCD $\beta$ function, respectively, and $t = \ln \frac{Q^2}{\Lambda^2}$ is the QCD cutoff parameter.

Now the inverse Laplace transforms of Eq. (17) can be easily performed by the following form as
\[
\mathcal{G}^{(\psi)}(v, Q^2) \equiv \mathcal{L}^{-1}[g^{(\psi)}(s, Q^2)(s, Q^2); v]
= \mathcal{L}^{-1}[k^{(\psi)}(a_s(Q^2), s) Df_2(s, Q^2)
- hs^{(\psi)}(a_s(Q^2), s) f_2(s, Q^2); v].
\]
where the inverse transform of a product to the convolution of the original functions gives
\[
\mathcal{L}^{-1}[f(s) \times h(s); v] = \int_0^v \mathcal{F}(w) \mathcal{H}(v - w) dw.
\]
The inverse Laplace transforms of the functions $k$ and $h$ in Eq. (21) are defined by $\hat{J}(\upsilon) = \mathcal{L}^{-1}[k(s); \upsilon]$ and $\hat{M}(\upsilon) = \mathcal{L}^{-1}[h(s); \upsilon]$, respectively. The result for the color dipole cross section at scale $\mu^2$ is

$$
\sigma_d^{(\psi)} = \sigma_0 \left\{ 1 - \exp \left( -\frac{\pi^2 r^2 \alpha_s(\mu^2) G(\psi)(\bar{x}, \mu^2)}{3 \sigma_0} \right) \right\},
$$

where

$$
G(\psi)(x, \mu^2) = \int_x^1 \left[ D F_2 \left( \ln \frac{1}{x}, \mu^2 \right) J(\psi) \left( \alpha_s(\mu^2), \ln \frac{y}{x} \right) 
- F_2 \left( \ln \frac{1}{y}, \mu^2 \right) M(\psi) \left( \alpha_s(\mu^2), \ln \frac{y}{x} \right) \right] dy \frac{dy}{y}.
$$

The explicit expressions for the functions $J(\psi)$ and $M(\psi)$ are defined in Appendix B. We therefore obtained an explicit solution for the color dipole cross section $\sigma_{dip}$ of $\sigma_0$ in terms of the parametrization of $F_2(x, \mu^2)$ and its derivative with respect to $\ln \mu^2$ at LO up to NNLO approximations due to the form of kernels.

### 3 Numerical results

The effective parameters in the GBW model have been extracted from a fit of the HERA data according to Ref. [1]:

- $\sigma_0 = 23$ mb, $\lambda = 0.288$, $x_0/10^{-4} = 3.04$,
- $C = 0.38$, $\mu_0^2 = 1.73 \text{ GeV}^2$

We have calculated the $r$ dependence, at low $x$, of the ratio $\sigma_{dip}/\sigma_0$ [i.e., Eq. (22)] in the LO up to NNLO approximations. Results of calculations and comparison with the GBW model [1] are presented in Figs. 1, 2, and 3, where the cricel-dot lines correspond to the extracted $\sigma_{dip}/\sigma_0$ at LO up to NNLO approximations, respectively.

Calculations were performed at the Bjorken variable $x$ to vary in the interval $x = 10^{-6} \ldots 10^{-2}$. The DGLAP improved model due to the parameterization of $F_2(x, Q^2)$ gives a good description of the ratio $\sigma_{dip}/\sigma_0$ in comparison with the GBW saturation model at low $x$ in a wide range of the momentum transfer $Q^2$. Figures 1, 2, and 3 clearly demonstrate that the extraction procedure provides correct behaviors of the extracted $\sigma_{dip}/\sigma_0$ within the LO up to NNLO approximations. At low and high $Q^2$, the extracted values of $\sigma_{dip}/\sigma_0$ are in good agreement with the GBW saturation model. We observe that the higher-order corrections are in a very good agreement with the GBW model in comparison with the LO approximation in a wide range of $r$. We see that the two results (the GBW and DGLAP improved models) overlap in small and large values of $r$, where the gluon distribution obtained from the parametrization of the proton structure function plays a significant role in the evolution of the
gluon distribution. To emphasize the size of the higher-order corrections, we show the ratio “order/GBW” for the ratio $\sigma_{\text{dip}}/\sigma_0$ at $x = 10^{-6}$ in a wide range of $r$ at LO up to NNLO approximations. As can be seen, these corrections are determined in the interval $10^{-3} \text{ fm} < r < 5 \text{ fm}$ for $x = 10^{-6}$. In Fig. 4, the results for the NLO and NNLO approximations are very similar. It is seen that the NLO corrections are smaller than the NNLO corrections in the interval $0.1 \text{ fm} < r < 1 \text{ fm}$ and are larger in the interval $10^{-3} \text{ fm} < r < 0.1 \text{ fm}$. The LO up to NNLO corrections are completely equivalent for $r > 1 \text{ fm}$. Indeed, these results due to the NLO and NNLO corrections are comparable to the GBW model in a wide range of domains.

Of particular interest is the ratio $\sigma_{\text{dip}}/\sigma_0$ defined by the scaling variable $r Q_s(x)$, where all the curves in the GBW model merge into one solid line.

In Figs. 5, 6, and 7, we show that the ratio $\sigma_{\text{dip}}(x, r)/\sigma_0$ has a property of geometric scaling as $\sigma_{\text{dip}}(x, r) = \sigma_{\text{dip}}(r Q_s(x))$. The results of the DGLAP improved saturation model due to the parametrization of the proton structure function have become a function of a single variable, $r Q_s$, for all values of $r$ and $x$ at LO up to NNLO approximations in Figs. 1, 2, and 3, respectively. From Fig. 7, one can infer that the NNLO results essentially improve the agreement with the geometric scaling in the GBW model in comparison with the LO and NLO calculations. The geometric scaling in the dipole cross sections in these calculations is visible in a wide range of $r Q_s$ at LO up to NNLO approximations. In these figures we observe that the violation between the geometric scaling of our results and the GBW model for low $r Q_s$ is clearly visible. The violations in this region are rather small and can be covered by the statistical errors in the parametrization of the proton structure function and its derivative. In Fig.
with the statistical errors in the parametrization of the distribution function is derived from the parametrization of $F_2$ and its derivative, compared with the GBW model (solid curve).  

Fig. 8 The extracted ratio $\sigma_{dip}(r Q_s(x))/\sigma_0$ as a function $r Q_s$ for $x = 10^{-4}$ within the NNLO approximation (dashed curves), accompanied with the statistical errors in the parametrization of $F_2$ and its derivative, compared with the GBW model (solid curve).

8, the $r Q_s$ dependence of the ratio $\sigma_{dip}(r Q_s(x))$ at $x = 10^{-4}$ is compared with the GBW saturation model. The error bands illustrated in this figure are the statistical errors in the parametrization of $F_2$ and its derivative, where the fit parameter errors are shown in Table 1. As can be seen from the related figures, the ratio results with respect to the Laplace transform method are consistent with the geometric scaling at low and large values of $r Q_s$.

To summarize, the essential elements of the GBW model, the saturation scale and geometric scaling, are preserved in the DGLAP improved dipole cross section when the gluon distribution function is derived from the parametrization of the proton structure function and its derivative due to the Laplace transform method in a wide range of the variables $r$ and $r Q_s$, respectively.

### Table 1

| Parameters | Value                  |
|------------|------------------------|
| $a_{00}$   | $2.550 \times 10^{-1}$ $+ 1.600 \times 10^{-2}$ |
| $a_{01}$   | $1.475 \times 10^{-1}$ $+ 3.025 \times 10^{-2}$ |
| $a_{10}$   | $8.205 \times 10^{-4}$ $+ 4.62 \times 10^{-4}$ |
| $a_{11}$   | $-5.148 \times 10^{-2}$ $+ 8.19 \times 10^{-3}$ |
| $a_{12}$   | $-4.725 \times 10^{-3}$ $+ 1.01 \times 10^{-3}$ |
| $a_{20}$   | $2.217 \times 10^{-3}$ $+ 1.42 \times 10^{-4}$ |
| $a_{21}$   | $1.244 \times 10^{-2}$ $+ 8.56 \times 10^{-4}$ |
| $a_{22}$   | $5.958 \times 10^{-4}$ $+ 2.32 \times 10^{-4}$ |
| $n$        | $11.49 \pm 0.99$       |
| $\lambda$  | $2.430 \pm 0.153$      |
| $\chi^2$ (goodness of fit) | 0.95 |

The results according to the saturation scale and geometric scaling are consistent with the GBW saturation model in a wide range of $r$ and $r Q_s$, respectively. With regard to the statistical errors due to the effective parameters, the NNLO results give a reasonable data description in comparison with the other models. Indeed, the small size of the dipole cross section is improved in the DGLAP improved model, which is based on the evolution of gluon density in this region. In summary, we have analyzed the dipole cross section at low values of $x$ and shown that the geometric scaling holds for the DGLAP improved model if the gluon distribution is defined by the parameterization of the proton structure function, and this is comparable to the GBW model curve in the whole region $r Q_s$.

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**Data Availability Statement** This manuscript has no associated data or the data will not be deposited. [Authors’ comment: We did not use experimental data directly. Rather, we have used models corresponding to the dipole cross sections].

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Appendix A

The flavor-singlet quark density of a hadron is given by

\[ f_\text{s} = \sum_{i=1}^{n_f} [f_i + \overline{f}_i], \]

where \( f_i \) and \( \overline{f}_i \) represent the number distribution of quarks and antiquarks, respectively, in the fractional hadron momentum \( x \). The subscripts \( i \) indicate the flavor of the quarks and antiquarks. In the common \( \overline{\text{MS}} \) renormalization scheme, the proton structure function, extracted from the DIS \( ep \) process, can be written as the sum of a flavor singlet, using the fact that the non-singlet contribution \( F_2^{\text{NS}} \) can be ignored safely at low values of \( x \). So we will have

\[
\frac{F_2(x, Q^2)}{x} = \frac{1}{x} \left( F_2^s(x, Q^2) + F_2^q(x, Q^2) \right).
\]

Indeed,

\[
F_2(x, Q^2) = \langle e^2 \rangle \left( C_{2,s}(x, Q^2) \otimes x f_s(x, Q^2) \right.
\]

\[
+ C_{2,g}(x, Q^2) \otimes x f_g(x, Q^2) \right),
\]

(24)

where the corresponding gluon density is denoted by \( f_g(x, Q^2) \) (or \( g(x, Q^2) \)), and \( C_{2,i}(i = s, g) \) are the common Wilson coefficient functions. In the space-like region, the proton structure function can be expressed in terms of quarks and gluon densities as the following convolution of the proton coefficient functions

\[
F_2(x, Q^2) = x < e^2 > \left[ C_{2,s}(x, Q^2) \otimes f_s(x, \mu_f^2) \right.
\]

\[
+ C_{2,g}(x, Q^2) \otimes f_g(x, \mu_f^2) \right].
\]

(25)

The scale \( \mu_f \) denotes the factorization scale. Considering coupling renormalization, the proton structure function is expressed as follows \[48–54\]

\[
F_2(x, Q^2) = \left[ C_{2,s}(x, \mu_f^2, Q^2, \mu_f^2, \mu_f^2) \otimes f_s(x, \mu_f^2, \mu_f^2) \right.
\]

\[
+ C_{2,g}(x, \mu_f^2, Q^2, \mu_f^2, \mu_f^2) \otimes f_g(x, \mu_f^2, \mu_f^2) \right] \times (x < e^2 >),
\]

(26)

where the renormalized parton density is defined by the following form

\[
f_\text{a} \left( z, \alpha_s(\mu_f^2), \frac{\mu_f^2}{\mu_r^2}, \frac{\mu_f^2}{\mu_r^2} \right)
\]

\[
= \sum_{k=q,g} \left( \Gamma_k(\alpha_s(\mu_f^2), \frac{\mu_f^2}{\mu_r^2}, \frac{\mu_f^2}{\mu_r^2}, \epsilon) \otimes \hat{f}_k \right)(z),
\]

for \( a = s, g \).

Here, \( \mu_r \) is the renormalization scale, \( \epsilon = n - 4 \) represents the collinear singularities, and \( \hat{f}_k \) is defined as the bare parton density. Due to changes in the scales \( \mu_r \) and \( \mu_f \), the changes in the parton densities and the coefficient functions can be expressed in terms of the renormalization group equation \[50–54\]. The renormalization group equation of the parton densities and the scale dependence of the coefficient functions causes the parton distribution functions to be expressed in terms of the splitting functions \( P_{ab}(x, \alpha_s) \). For the splitting functions \( P \) and coefficient functions \( C \), we employ the convention

\[
P(x, \alpha_s) = \sum_{n=0}^{n+1} \left( \frac{\alpha_s}{4\pi} \right)^n P^{(n)}(x),
\]

\[
C(x, \alpha_s) = \sum_{n=0}^{n+1} \left( \frac{\alpha_s}{4\pi} \right)^n C^{(n)}(x)
\]

for the expansion in the running coupling, where \( C \)'s appears in proton structure functions and \( P \)'s in the evolution equations \[43\].

The evolution equations of distribution functions in the singlet sector at LO analysis are given by

\[
\frac{\partial (x f_a(x, Q^2))}{\partial \ln Q^2} = - \frac{1}{2} \sum_{a,b = s,g} P_{ab}^{(0)} \otimes x f_b(x, Q^2).
\]

(27)

For the high-order contribution to scaling violation of distribution functions, we extend the discussion to the NNLO level. Within the pQCD, and up to the NNLO corrections, the splitting function \( P_{ab}(x) \) reads as

\[
P_{a,b}(x) = a_s(Q^2) P_{a,b}^{(0)}(x) + a_s(Q^2) P_{a,b}^{(1)}(x)
\]

\[
+ a_s^3(Q^2) P_{a,b}^{(2)}(x).
\]

Thus, the high-order correction to the evolution of the singlet structure function can be written as

\[
\frac{\partial F_2^s(x, Q^2)}{\partial \ln Q^2} = - \frac{a_s(Q^2)}{2} \left[ (P_{ss}^{(0)}(x) + a_s(Q^2) P_{ss}^{(1)}(x))
\]

\[
+ a_s^2(Q^2) P_{sg}^{(2)}(x)) \otimes F_2^s(x, Q^2) + < e^2 > (P_{sg}^{(0)}(x) + a_s(Q^2) P_{sg}^{(1)}(x))
\]

\[
+ a_s^2(Q^2) P_{sg}^{(2)}(x)) \otimes x f_g(x, Q^2),
\]

(28)
where
\[ F_{ab}^{(n)}(x) = P_{ab}^{(n)}(x) + [C_{2,s} + C_{2,g} + \cdots] \otimes P_{ab}^{(0)}(x) + \cdots. \]

For brevity, the leading-order contribution to the evolution of the singlet structure function can be written as
\[
\frac{4\pi}{\alpha_s(Q^2)} \frac{\partial F_1^s(x, Q^2)}{\partial \ln Q^2} = 4F_2^s(x, Q^2) + \frac{16}{3} F_2^s(x, Q^2) \ln \frac{1-x}{x}
\]
\[
+ \frac{16}{3} \int_x^1 \frac{dz}{z} \left( \frac{F_2^s(z, Q^2)}{z} - \frac{F_2^s(x, Q^2)}{x} \right)
\]
\[
- \frac{8}{3} x \int_x^1 F_2^s(z, Q^2) \left( 1 + \frac{x}{z} \right) \frac{dz}{z^2}
\]
\[
+ 2n_f x \int_x^1 G(z, Q^2) \left( 1 - 2 \frac{x}{z} + 2 \frac{x^2}{z^2} \right) \frac{dz}{z}.
\] (29)

Here, \( \alpha_s(Q^2) \) is the running strong coupling at an LO approximation. Therefore, we write the evolution of the singlet structure function at higher-order approximation as
\[
\frac{4\pi}{\alpha_s(Q^2)} \frac{\partial F_1^s(x, Q^2)}{\partial \ln Q^2} = F_2^s \otimes (P_{gs}^{(0)} + \alpha_s(Q^2) P_{ss}^{(1)})
\]
\[
+ a_s^2(Q^2) P_{ss}^{(2)}(x, Q^2)
\]
\[
+ \alpha_s^2(Q^2) P_{gs}^{(1)}(x, Q^2)
\]
\[
+ a_s^2(Q^2) P_{gs}^{(2)}(x, Q^2),
\] (30)

with \( \alpha_s(Q^2) \) the higher-order running strong coupling.

Appendix B

The functions \( k \) and \( h \) in \( s \)-space at LO approximation are given by
\[
k(s, Q^2) = \frac{2\pi}{< e^2 > n_f \alpha_s(Q^2)} \left[ \frac{1}{1+s} - \frac{2}{2+s} + \frac{2}{3+s} \right].
\]
\[
h(s, Q^2) = \frac{1}{2n_f < e^2 >} \left[ 4 - \frac{8}{3} \left( \frac{1}{1+s} + \frac{1}{2+s} + 2S_1(s) \right) \right]
\]
\[
\times \left[ \frac{1}{1+s} - \frac{2}{2+s} + \frac{2}{3+s} \right].
\] (31)

The inverse Laplace transform of functions \( k \) and \( h \) in Eq. (31) are defined by the following forms
\[
\hat{f}(s, Q^2) \equiv \mathcal{L}^{-1}[k(s, Q^2); s] \]
\[
= \frac{\pi}{2 < e^2 >} \alpha_s \left[ \delta'(s) + 3\delta(s) \right]
\]
\[
- \exp \left( -\frac{3}{2} s \right) \left[ 2 \cos \left( \frac{1}{2} \sqrt{7} s \right) + \frac{6}{7} \sqrt{7} \sin \left( \frac{1}{2} \sqrt{7} s \right) \right].
\]

The evaluation of the higher-order coefficients is straightforward but is too lengthy to be included in this note, and will be given in the future when we numerically evaluate \( G(x, Q^2) \) in NLO and NNLO approximations [61,62]. Finally, the gluon distribution function directly is obtained from the parameterization of the structure function \( F_2(x, Q^2) \) and its derivatives by the following form at LO approximation
\[
G(x, Q^2) = \frac{\pi}{2 < e^2 >} \alpha_s \left[ \frac{\partial D F_2(x, Q^2)}{\partial \ln x} + 3D F_2(x, Q^2)
\]
\[
- \int_x^1 \frac{dy}{y} D F_2(x, Q^2) \left( \frac{y}{x} \right) \left[ 2 \cos \left( \frac{1}{2} \sqrt{7} \ln \left( \frac{y}{x} \right) \right) + \frac{6}{7} \sqrt{7} \sin \left( \frac{1}{2} \sqrt{7} \ln \left( \frac{y}{x} \right) \right) \right]
\]
\[
- \frac{1}{e^2 >} \left( \frac{1}{2} + \frac{2}{3} \ln 2 \right) \frac{\partial D F_2(x, Q^2)}{\partial \ln x}
\]
\[
+ \left( \frac{1}{6} + 2\ln 2 \right) \frac{\partial D F_2(x, Q^2)}{\partial \ln x}
\]
\[
\times \left( \frac{y}{x} \right)^2 \left[ 3.606 \cos \left( \frac{1}{2} \sqrt{7} \ln \left( \frac{y}{x} \right) \right) + 1.371 \sin \left( \frac{1}{2} \sqrt{7} \ln \left( \frac{y}{x} \right) \right) \right] - \int_x^1 \frac{dy}{y} D F_2(x, Q^2)
\]
\[
\times \left( \frac{1}{2} \sqrt{7} \ln \left( \frac{y}{x} \right) - \frac{8}{7} \frac{y}{x} \right) + \cdots - \frac{120}{23} \left( \frac{x}{y} \right)^{11} \right].
\] (33)

where \( D F_2(x, Q^2) \equiv \frac{\partial D F_2(x, Q^2)}{\partial \ln Q^2} \).

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