Quantum derivation of Manley Rowe type relations

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Abstract

The Ermakov Lewis quantum invariant for the time dependent harmonic oscillator is expressed in terms of number and phase operators. The identification of these variables is made in accordance with the correspondence principle and the amplitude and phase representation of the classical orthogonal functions invariant. The relationship between the number and phase operators is established through this invariant as the system evolves from one frequency to another. In the specific case where the excitations represent the photon number, these relations are equivalent to the power density transport equations derived in nonlinear optical processes.

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I. INTRODUCTION

Different exact invariants have been used either to solve the one dimensional time dependent Shrödinger equation [1] or to obtain the general differential equation solution from a particular one [2]. The Ermakov-Lewis invariant and the orthogonal functions invariants are two schemes that have been used to solve the quantum mechanical time dependent harmonic oscillator equation in one dimension (QM-TDHO) [3]. These invariants are usually expressed in terms of coordinate and momentum operators. Nonetheless, an amplitude and phase operators representation of these invariants is also possible. However, such a representation is not unique and furthermore, it is not straightforward to associate a number operator representing the number of excitations to these variables [4].

On the other hand, despite the mathematical success of these formalisms there is scarcely any literature regarding the physical interpretation of these quantities. Two exceptions, to the best of our knowledge, are the proposal made by Eliezer [5] regarding the Ermakov-Lewis classical invariant and the interpretation of the orthogonal functions invariant given by us [6].

The conservation of electromagnetic field power density in non-linear processes is described by the Manley-Rowe relations [7]. These equations have been obtained in semi-classical theory using the nonlinear wave equation derived from Maxwell’s equations as a starting point. The participating waves are considered harmonic in time, colinear and the slowly varying spatial envelope approximation is imposed in order to obtain a set of coupled first order linear differential equations [8]. The nonlinear susceptibility tensor for the specific wave mixing process is then evaluated in a lossless medium and its symmetries are established through the Kleinmann’s conditions [9]. The relations thus obtained are interpreted in terms of the conservation of the photon flux as the fields propagate in space.

In this presentation, we first recreate the procedure used to obtain the amplitude and phase representation in the classical and quantum cases. To this end, the orthogonal functions and the Ermakov-Lewis invariants are described trying to stress the physical meaning of the variables involved. The phase and number operators are then associated according with the role that the variables play. The Ermakov-Lewis quantum invariant is shown to represent the energy conservation of the closed oscillator ensemble.

The formalism is applied to a photon field in an analogous fashion as the black body
radiation problem was tackled by Planck [10]. Namely, the field properties are obtained from the oscillator ensemble with which the radiation interacts. A quantum derivation of the conservation equations is then obtained for arbitrary non-linear frequency conversion processes.

II. CLASSICAL AND QUANTUM INVARIANTS

A. Orthogonal functions invariant

Consider the time dependent Schrödinger equation with $\hbar = 1$

$$i \frac{\partial |\psi(t)\rangle}{\partial t} = \hat{H}|\psi(t)\rangle,$$  
(1)

for a time dependent harmonic oscillator Hamiltonian $\hat{H}(t) = 1/2 (\hat{p}^2 + \Omega^2(t)\hat{q}^2)$. The quantum orthogonal functions invariants of this system are

$$\hat{G}_1 = u_1 \hat{p} - \dot{u}_1 \hat{q}, \quad \hat{G}_2 = -u_2 \hat{p} + \dot{u}_2 \hat{q}.$$  
(2)

These invariants obey the commutation relation

$$[\hat{G}_1, \hat{G}_2] = -iG;$$  
(3)

$G$ is a constant given by the classical orthogonal functions invariant [11]

$$G = u_1 \dot{u}_2 - u_2 \dot{u}_1,$$  
(4)

where $u_1$ and $u_2$ are real linearly independent solutions of the TDHO equation $\ddot{u} + \Omega^2(t)u = 0$. The amplitude $\rho$ and phase $s_\rho$ representation of this $c$-number is straight forward from the substitution of $u_1 = -\rho \sin s_\rho$, $u_2 = \rho \cos s_\rho$

$$G = \rho^2 \dot{s}_\rho.$$  
(5)

Let the constant $G$ at an initial time be given by

$$G = \rho_0^2 \omega_0,$$  
(6)

where the derivative of the phase is defined as the frequency $\omega(t) \equiv \dot{s}_\rho$. The squared amplitude times the frequency at any subsequent time obey the relationship

$$\rho^2(t) \omega(t) = \rho_0^2 \omega_0,$$  
(7)

$\rho$ satisfies the differential equation $\ddot{\rho} + \Omega^2(t)\rho = 1/\rho^3$. 

3
B. Ermakov Lewis Invariant

The Ermakov Lewis invariant may be written as [12]

\begin{align*}
\hat{I} &= \frac{1}{2} \left( \hat{G}_1^2 + \hat{G}_2^2 \right) \\
&= \frac{1}{2} \left[ \left( \frac{\hat{G} \hat{q}}{\rho} \right)^2 + \left( \rho \hat{p} - \hat{\rho} \hat{q} \right)^2 \right],
\end{align*}

where the amplitude function is \( \rho = \sqrt{u_1^2 + u_2^2} \). The classical Ermakov Lewis constant of motion in the amplitude and phase representation follows from the substitution \( \hat{q} \to \rho \cos s_{\rho}, \hat{p} \to \dot{\rho}q/dt : \)

\begin{align*}
I &= \frac{1}{2} \rho^4 \dot{s}_{\rho}^2.
\end{align*}

This expression, according to Eliezer and Gray [5] may be interpreted as the square of an angular momentum through the introduction of an auxiliary imaginary axis perpendicular to the actual direction of the oscillator motion.

III. CREATION AND ANNIHILATION OPERATORS

An operator that can be written as the sum of two squares may be expressed in terms of two adjoint complex quantities. The invariant operator \( \hat{I} \) may be written as the sum of two squares in two different ways, namely (8) and (9). The former expression leads to annihilation and creation operators of the form

\begin{align*}
\hat{A} &= \frac{1}{\sqrt{2}} \left( \hat{G}_1 - i\hat{G}_2 \right), \\
\hat{A}^\dagger &= \frac{1}{\sqrt{2}} \left( \hat{G}_1 + i\hat{G}_2 \right).
\end{align*}

These operators may also be obtained from the non Hermitian linear invariant which arises from the complex solution of the TDHO equation [13]. These annihilation and creation operators are also invariant since they are composed by invariant operators. On the other hand, the operators arising from (8) yield

\begin{align*}
\hat{a} (t) &= \frac{1}{\sqrt{2}} \left( \frac{\hat{G} \hat{q}}{\rho} + i(\rho \hat{p} - \hat{\rho} \hat{q}) \right), \\
\hat{a}^\dagger (t) &= \frac{1}{\sqrt{2}} \left( \frac{\hat{G} \hat{q}}{\rho} - i(\rho \hat{p} - \hat{\rho} \hat{q}) \right).
\end{align*}
These time dependent annihilation and creation operators were originally introduced by Lewis [12]. The Ermakov invariant in terms of these operators is

\[ \hat{I} = \hat{a}^\dagger(t) \hat{a}(t) + \frac{G}{2} = \hat{A}^\dagger \hat{A} + \frac{G}{2}, \]  

where the second equality follows from the definition of this invariant in terms of the orthogonal functions quantum invariants (8). The time dependent annihilation (creation) operators may be written as the product of the time independent annihilation (creation) operators times a phase that only involves a \( c \)-number function. This expression may be written as a unitary transformation of a phase shift

\[ \hat{a} = \exp \left( is_\rho \hat{I} \right) \hat{A} \exp \left( -is_\rho \hat{I} \right) \]  

such that the equation of motion of this operator is then

\[ \hat{a} = i\omega(t) [\hat{I}, \hat{a}]. \]  

It is thus seen that the operator \( \omega(t) \hat{I} \) in the QM-TDHO plays the role that the Hamiltonian does in the time independent harmonic oscillator case. This assertion is consistent with two previous results. On the one hand, the transformation that relates the invariant and the time dependent Hamiltonian [3]

\[ \omega(t) \hat{I} = \hat{H}(t) - i\frac{\partial \hat{T}^\dagger}{\partial t} \hat{T}, \]  

where the \textit{squeeze} transformation [14] is given by

\[ \hat{T} = \exp \left( i\frac{\ln \rho \sqrt{\omega_0}}{2} (\hat{q}\hat{p} + \hat{p}\hat{q}) \right) \exp \left( -i\frac{\hat{p}^2}{2\rho \omega_0} \right). \]  

On the other hand, the propagator that describes the time evolution of the transformed wave function \( |\psi(t)\rangle = \hat{U}_I \hat{T}^\dagger \hat{T}(0) |\psi(0)\rangle \) in the time dependent case is \( \hat{U}_I = \exp \left( -is_\rho \hat{I} \right) \). In the time independent case of course \( \hat{U}_I \) becomes the unity operator. It is thus clear that the invariant in the time dependent case enters the propagator expression in an analogous fashion as the Hamiltonian does in the time independent case.

The energy of a time dependent classical oscillator in the adiabatic approximation is proportional to \( E \propto \rho^2(t) \omega^2(t) \). Therefore from (5) it is seen that the energy is proportional to the frequency \( E \propto \omega(t) \) [6] whereas the Lewis constant has a quadratic of energy over frequency dependence. However, the quantum versions of the orthogonal functions invariants
produce a linear form in the coordinate and momentum operators as seen in Eqs. (2). It has been necessary to evaluate the square of these operators in order to obtain expressions proportional to the Hamiltonian of the system. Therefore, a quantum invariant with a quadratic dependence on the coordinate and momentum variables should be in correspondence with the classical orthogonal functions invariant.

IV. PHASE OPERATOR FOR TIME DEPENDENT COHERENT STATES

Coherent states are states that follow classical trajectories and are a standard to define non-classical features of quantum states. Maybe because of their classicality, they have been useful to describe phase in quantum optics [15, 16]. The invariant formalism will be applied here to the phase operator given by Turski. This operator will allow an appropriate translation of the classical amplitude-phase invariant into the quantum one.

By using the annihilation operator (14) the displacement operator can be written as
\[ \hat{D}(\alpha) = \exp(\alpha \hat{a}^\dagger - \alpha^* \hat{a}) \], \[ \alpha = r \exp(i\theta) \]. The vacuum state may then be displaced to obtain a coherent state |\alpha\rangle = \hat{D}(\alpha)|0\rangle and the phase operator introduced by Turski [16] is then generalized to the time dependent case
\[ \hat{\Phi} = \int \theta|\alpha\rangle \langle \alpha| d^2\alpha. \] (20)

This operator obeys the commutation relation [\hat{\Phi}, \hat{I}] = -i. In order to evaluate the time evolution of \( \hat{\Phi} \), this operator can be written in terms of the invariant annihilation and creation operators using (16)
\[ \hat{\Phi} = e^{is_p\hat{I}} \left( \int \theta \hat{D}_A(\alpha)e^{-is_p\hat{I}}|0\rangle \langle 0|e^{is_p\hat{I}} \hat{D}_A^\dagger(\alpha)d^2\alpha \right) e^{-is_p\hat{I}}, \] (21)
where \( \hat{D}_A(\alpha) = \exp(\alpha \hat{A}^\dagger - \alpha^* \hat{A}) \). The invariant acting over the vacuum state is \( \hat{I}|0\rangle = \frac{1}{2}|0\rangle \) and the phase is then
\[ \hat{\Phi} = e^{is_p\hat{I}} \left( \int \theta \hat{D}_A(\alpha)|0\rangle \langle 0|\hat{D}_A^\dagger(\alpha)d^2\alpha \right) e^{-is_p\hat{I}}, \] (22)
the time derivative of this expression yields the equation of motion for \( \hat{\Phi} \):
\[ \dot{\hat{\Phi}} = i\omega(t)[\hat{I}, \hat{\Phi}] = -\omega(t). \] (23)

The operator \( \omega(t)\hat{I} \) once again takes the role of the Hamiltonian.
V. NUMBER AND PHASE OPERATORS

The coordinate operator from (14) is
\[ \hat{q} = \sqrt{\frac{1}{2G\omega(t)}}(\hat{a} + \hat{a}^\dagger), \] (24)
and following Dirac [20] the creation and annihilation operators may be written as
\[ \hat{a} = \sqrt{I}e^{-i\hat{\Phi}}, \quad \hat{a}^\dagger = e^{i\hat{\Phi}}\sqrt{I}. \] (25)
The coordinate operator (24) in the form of amplitude and phase variables is then
\[ \hat{q} = \sqrt{\frac{I}{2G\omega(t)}}e^{-i\hat{\Phi}} + e^{i\hat{\Phi}}\sqrt{\frac{I}{2G\omega(t)}}, \] (26)
where the amplitude \( \rho \) and phase \( s_\rho \) are identified as
\[ \rho \to \sqrt{\frac{I}{G\omega(t)}}, \quad s_\rho \to \hat{\Phi}. \] (27)

The invariant with the aid of (23) is given in amplitude and phase operators as
\[ \hat{I} = -\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\frac{1}{G\omega(t)}\hat{\Phi}\right), \] (28)
which has the same structure of the orthogonal functions classical invariant written in am-
plitude and phase variables (5). The number operator is then identified with
\[ \hat{n}(t) = \frac{1}{G\omega(t)}\hat{a}^\dagger\hat{a}. \] (29)

It should be recalled that the number operator is identified with the product of the
annihilation and creation pair when these operators arise from the Hamiltonian operator.
However, the identification of the number operator in terms of the annihilation - creation
pair arising from the invariant is obtained from the correspondence with the orthogonal
functions classical invariant. The invariant in terms of the number and phase operators is then
\[ \hat{I} = -\left(\hat{n}(t) + \frac{1}{2}\frac{1}{\omega(t)}\right)\hat{\Phi}(t). \] (30)
Since \( \hat{a}^\dagger\hat{a} \) is invariant from (15), if the frequency is constant the number of excitations is
then also constant. Nonetheless, in the time dependent case, the number of excitations is
inversely proportional to the time dependent frequency in correspondence with the intensity dependence obtained in the classical limit.

The energy of the excitation at a given time \( t_s \) is given by \( E = \hat{n}(t_s) \omega(t_s) \) (with \( \hbar = 1 \)) but this is precisely the quantum invariant value above the vacuum state \( \hat{I} = -\hat{n}(t_s) \dot{\Phi}(t_s) \). Therefore, the invariant represents the energy conservation of the closed system. In contrast, the time dependent Hamiltonian is no longer a constant of motion whose eigenvalue is necessarily related to an open system.

VI. CONSERVATION EQUATIONS

Consider, as an example of this formalism, the number of excitations to represent the photon number. Let an ensemble of oscillators be followed along their propagating path so that the properties of the oscillators are time dependent quantities. This scheme corresponds to a Lagrangian hydrodynamic framework.

Allow for a nonlinear process where the wave mixing at an initial time consists of \( \hat{n}_1(t_i), \hat{n}_2(t_i), \ldots, \hat{n}_n(t_i) \) oscillators with frequencies \( \omega_1(t_i), \omega_2(t_i), \ldots, \omega_n(t_i) \). The system then evolves to frequencies \( \omega_1(t_f), \omega_2(t_f), \ldots, \omega_n(t_f) \) with \( \hat{n}_1(t_f), \hat{n}_2(t_f), \ldots, \hat{n}_n(t_f) \) photons at a time \( t_f \). For any pair it is possible to establish the corresponding invariant relationship (30), thus

\[
\omega_k(t_i) \hat{n}_k(t_i) = \omega_k(t_f) \hat{n}_k(t_f) \quad k \in 1, 2, \ldots, n. \tag{31}
\]

Each of these equations is stating that the energy of an ensemble of \( \hat{n}_k(t_i) \) oscillators with frequency \( \omega_k(t_i) \) is equal to the energy of \( \hat{n}_k(t_f) \) oscillators with frequency \( \omega_k(t_f) \).

The particular nonlinear process taking place then imposes a relationship between the input and output frequencies. For example, sum frequency generation used to produce VUV tunable radiation [17] implies that

\[
2\omega_1 + \omega_2 = \omega_f. \tag{32}
\]

In the above notation this expression corresponds to

\[
2\omega_1(t_i) + \omega_2(t_i) = \omega_1(t_f) = \omega_2(t_f). \tag{33}
\]

If we multiply this equation by the number of photons in the final state \( \hat{n}_1(t_f) \), the equation reads

\[
2\omega_1(t_i) \hat{n}_1(t_f) + \omega_2(t_i) \hat{n}_1(t_f) = \omega_1(t_f) \hat{n}_1(t_f), \tag{34}
\]
but since \( \omega_1 (t_i) \hat{n}_1 (t_i) = \omega_1 (t_f) \hat{n}_1 (t_f) \) and \( \omega_2 (t_i) \hat{n}_2 (t_i) = \omega_2 (t_f) \hat{n}_2 (t_f) \), the above expression may be casted as

\[
2\omega^2_1 (t_i) \frac{\hat{n}_1 (t_i)}{\omega_1 (t_f)} + \omega^2_2 (t_i) \frac{\hat{n}_2 (t_i)}{\omega_1 (t_f)} = \hat{n}_1 (t_f) \omega_1 (t_f) .
\] (35)

This result is often cited in the literature in terms of the power density, defined as the energy per unit time \( W_k = E_k / \tau_k = \omega^2_k \hat{n}_k \), at these two times is then given by

\[
W_1 (t_i) + W_2 (t_i) = W_1 (t_f) .
\] (36)

This reasoning may be applied to any other nonlinear processes such as parametric amplification or frequency difference.

VII. CONCLUSIONS

The Ermakov Lewis invariant may be used in an equivalent fashion as the Hamiltonian is used in the time independent case. Namely, to obtain evolution operators, to cast the equations of motion of different operators in commutative expressions, and to produce a phase shift with its exponential form. The number and phase representation of this invariant corresponds to the classical orthogonal functions invariant, which in turn is proportional to the ratio of energy over frequency in the adiabatic approximation. These properties of the Ermakov Lewis invariant suggest that this quantity is proportional to the total energy of the closed system.

A quantum derivation of the conservation equations has been obtained for arbitrary nonlinear frequency conversion processes. These results in a Lagrangian frame of reference are analogous to the flux conservation in the semiclassical Eulerian framework. An important difference between the two descriptions is that in the semiclassical case, the fields at different fixed frequencies change their amplitudes as a function of position. In contrast, this quantum treatment considers the time evolution of the frequency with its corresponding amplitude variation.

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