Does the existence of Majorana zero mode in superconducting vortices imply the superconductivity is topologically non-trivial?

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We show that the presence of Majorana zero modes (2D), and chiral-dispersing Majorana modes (3D), in the vortex cores of superconductors are neither sufficient nor necessary conditions for one to conclude the superconductivity is topologically non-trivial. We discuss the relevance of this result to the proximity-induced superconductivity, in the presence of magnetic field, on the surface of topological insulators.

Introduction

In their pioneer work\cite{1} Read and Green showed that the spin-polarized 2D superconductor with $p_x + ip_y$ pairing symmetry is topological. The signatures of it include the existence of gapless chiral Majorana edge states and Majorana zero modes in the cores of odd-vorticity vortices. Due to the relevance to topological quantum computing, Majorana zero modes has attracted strong interests in both theoretical and experimental communities\cite{2}.

In the seminal work Ref.\cite{3}, Fu and Kane proposed a way to engineer a superconductor which hosts vortex Majorana zero modes. This is achieved by coupling an s-wave superconductor to the surface states of a 3D topological insulator via the proximity effect. Following this proposal, many experimental groups carry out the search of vortex Majorana zero modes. Examples include Bi$_2$Se$_3$/NbSe$_2$\cite{4} and recently Fe(Te$_{1-x}$Se$_x$)\cite{5}. For the latter system due to the strong spin-orbit interaction of the Te atoms, a band inversion occurs at the Brillouin zone center resulting in topologically non-trivial surface states. Ref.\cite{5} reports the observation of a superconducting gap on the Fermi circle associated with the surface states, which is induced by bulk superconductivity via the proximity effect.

In the literature, the existence of vortex Majorana zero modes is commonly taken as the hallmark of topologically non-trivial superconductivity. In this work, we re-examine the validity of this association. We conclude that vortex Majorana zero mode is neither a sufficient nor a necessary condition for having topological superconductivity.

The bulk and defect classification

Our result is based on the bulk\cite{6,7} and defect\cite{8} topological classification summarized in table I of the supplementary material (SM). In the following, we briefly explain the meaning of this table, and how to apply it to our problem.

The symmetry groups given in the fourth column of table I are grouped into symmetry classes each labeled by either a mod 8 (the real class) or a mod 2 (the complex class) integer $p$ in the first column of the table. For our purpose, it is sufficient to concentrate on symmetries generated by $Q, T, C$. They denote, respectively, the U(1) charge conservation, time reversal, and charge conjugation symmetries. Moreover we shall focus on the “real class” of the classification table\cite{6,7}.

The generic Hamiltonian classified by table I has the form

$$H = \int d^dx \chi^T(x) \left( -i \sum_{j=1}^d \gamma_j \partial_j + i \sum_{\alpha=1}^m \phi_\alpha M_\alpha \right) \chi(x).$$  \hspace{1cm} (1)

Here $\chi(x)$ is a $q$-component Majorana fermion field, where $q$ is the number of spin and orbital degrees of freedom in the problem. In the following we shall refer to the internal space spanned by the spin and orbital degrees of freedom as the “mode space”\cite{6}. The $\gamma_j$ are $q \times q$ real symmetric matrices satisfying $\{ \gamma_j, \gamma_l \} = 2\delta_{jl}$. The “mass matrices” $M_\alpha$ are $q \times q$ real antisymmetric matrices satisfying $\{ \gamma_j, M_\alpha \} = 0$. The $Q, T, C$ in table I are $q \times q$ real orthogonal matrices. They are the representations of $Q, T, C$ in the mode space. Both $\gamma_j$ and $M_\alpha$ anti-commute with $T$ but commute with $Q$ and $C$.

The $\{ \phi_\alpha | \alpha = 1, \ldots, m \}$ in Eq. (1) are real scalars, their presence opens the fermion gap hence they play the role of order parameters. The “mass manifolds” are manifolds in the $R^m$ space spanned by $\{ \phi_\alpha \}$. If $\{ \phi_\alpha \}$ is a mass manifold, the fermion gap stays a constant. Thus mass manifold has the same meaning as “target space” for non-linear sigma models. Fixing the fermion gap to, say, unity, the mass manifolds are the solutions of

$$\sum_{\alpha=1}^m \phi_\alpha M_\alpha = -I_{q \times q} \quad (\text{the identity} \ q \times q \text{ matrix}).$$

The union of all mass manifolds is the “classifying space”. Because each mass matrix is required to anti-commute with all gamma matrices $\{ \gamma_j, j = 1, \ldots, d \}$, for a fixed $q$ the number of $M_\alpha$ (i.e. $m$) depends on the spatial dimension $d$. As a result, both the mass manifolds and the classifying space depend on the spatial dimension. The integer $n$ is equal to $q/q_0$ where $q_0$ is the smallest mode space dimension necessary to represent the particular symmetry class in $d = 0$. The $R_p$ listed in the second column of I are the associated classifying space.

The abelian groups given in the third column of table I are the zero$^{th}$ homotopy group of the classifying space.
$R_p$, i.e., $\pi_0(R_p)$. The number of elements in such group is equal to the number of disconnected mass manifolds, each of which corresponds to an equivalent class of topological phases. In general $\pi_0(R_p)$ depends on $n$, what listed is the “stabilized” result, namely $\pi_0(R_p)$ in the limit $n$ tends to infinity. (Note that as $n \to \infty$ so do $q$ and $m$.)

To find the bulk classification one first locates the appropriate symmetry class $p$. For example, if the problem has time reversal ($T^2 = -1$) and charge conservation ($\hat{\mathcal{Q}}$) symmetries we denote the symmetry group as $G_-(\hat{\mathcal{Q}}, T)$ (see the caption of table I for the explanation of the symbol). The symmetry class is $p = 4$. Next, we find the classifying space for a given space dimension $d$. The result (proof not given) is $R_{p-d}$ (for $d = 0$ it is $R_p$ as expected). Thus for the symmetry group $G_-(\hat{\mathcal{Q}}, T)$ and $d = 3$ the classifying space is $R_1$, for which the bulk classification is $\pi_0(R_1) = Z_2$. This means there are two inequivalent classes of fully gapped free fermion phases. Phases within the same class can go into each other without closing the energy gap or breaking the symmetries. These classes are represented by the elements of $Z_2$: 0 stands for the trivial class (phases in this class do not have gapless boundaries) and 1 stands for the non-trivial class (phases in this class possess gapless boundary excitations). The group operation of $Z_2$ corresponds to the “stacking” of different phases on top of each other. This means tensoring the mode spaces and turning on all symmetry allowed local interactions between the corresponding spin and orbital degrees of freedom. For example, $1 + 1 = 0$ means when two non-trivial phases are stacked together the result is a trivial phase, a phase with no gapless boundary.

Defects are singularities in the free fermion Hamiltonian. For example, a vortex is a point defect in the BdG Hamiltonian of a superconductor. It is important to remember that in finding out the defect classification [8] one needs to specify the Hamiltonian symmetry in the presence of defects. Each defect is characterized by its codimension, the difference between the space dimension and the defect dimension, $k$. For example a point defect has $k = d$ and a line defect has $k = d - 1$ etc. Having determined the bulk classifying space $R_{p-d}$, the defect classification is given by $\pi_{k-1}(R_{p-d})$. According to Bott’s periodicity theorem [9] $\pi_{k-1}(R_{p-d}) = \pi_0(R_{p-d+k-1})$. Hence the defect classification is given by the third column of the $(p-d+k-1)^{th}$ row in table I. For example for 2D superconductors with $T^2 = +1$ symmetry (symmetry group $G_+(T)$) the symmetry class is $p = 1$. The classification of vortices ($k = 2$) is given by the third column of the $1 - 2 + 2 - 1 = 0^{th}$ row in table I, namely, $Z$. This means a vorticity $+1$ vortex will have one Majorana zero mode and $+2$ vortex (which can be viewed as two $v = +1$ vortices stacked together) has two Majorana zero modes... etc. Of course, when a $+1$ vortex is stacked with a $-1$ vortex there exists symmetry allowed interaction that can gap out both zero modes.

Armed with table I we conclude that 2D superconductor with $T^2 = +1$ symmetry (i.e. $p = 1$) is trivial because $\pi_0(R_{1-2} = R_T) = 0$. However its vortex classification is $\pi_0(R_{1-2+2-1} = R_0) = Z$ which is non-trivial, i.e., a vorticity $+n$ vortex will harbor $n$ Majorana zero modes. Similarly 3D superconductors with no symmetry (i.e. $p = 2$) are topologically trivial because $\pi_0(R_{2-3} = R_T) = 0$. However their vortex lines are classified by $\pi_0(R_{2-3+3-2-1} = R_0) = Z$, i.e., a vorticity $n$ vortex line harbors $n$ branches of chiral-dispersing Majorana modes in the vortex core. This proves that there can exist superconductors which have topologically trivial bulk, but in the cores of their vortices there are Majorana zero modes (2D) or branches of chiral-dispersing Majorana modes (3D). However since the classification results of table I are meant for the limit where the $n$ (hence $q$) $\to \infty$, it is important to check whether the statements made in the last paragraph hold for finite $q$, i.e., for systems that might exist in experiments. To do so we need to construct explicit models with a finite number of spin and orbital degrees of freedom.

**An example in 2D** Let’s focus on the symmetry class $p = 1$. Because we are interested in superconductors (where $\hat{\mathcal{Q}}$ is not a good symmetry) there are only two symmetry groups: 1) $G_+(T)$ ($T^2 = +1$ only), and 2) $G^+_{++}(T,C)$ ($T^2 = +1$, $C^2 = +1$ $|T,C| = 0$). For symmetry group (2), $C$ commutes with the Hamiltonian and $T$, so it can be first diagonalized. After doing so the problem decouples into two sectors with ±1 eigenvalue for $C$. Therefore we can proceed to analyze each eigen sector of $C$ and the problem reduces to symmetry group (1), i.e., $T^2 = +1$ only.

For symmetry class $p = 1$ in two dimensions it can be shown that $q_0 = 4$, and the time reversal matrix is given by $T = \epsilon \sigma_z$. (Here we abbreviate the Kronecker product of two matrices each equal to $\epsilon = i\sigma_y$ as $\epsilon \sigma_z$.) Hence a $4 \times 4$ Hamiltonian can be written down. In continuum
space such a Hamiltonian read
\[
H = \int d^2 x \chi^T(x) \left(-i \mathcal{Z} \partial_1 - i X \partial_2 - i \phi_1 X \epsilon \right.
- \left. i \phi_2 Z \epsilon \right) \chi(x).
\] (2)

A lattice version of this Hamiltonian is given by Eq. (8) of the SM. In addition, we also express Eq. (2) in terms of complex fermion operators (SM Eq. (6)) so that it is easier to associate it with the BdG Hamiltonian of a superconductor.

Requiring \((\phi_1 M_1 + \phi_2 M_2)^2 = -1\) implies \(\phi_1^2 + \phi_2^2 = 1\). Therefore the mass manifold is a circle. This is consistent with the classification table since the classifying space is \(R_{12} = R_7 = U(n)/O(n)\) which is \(S^1\) for \(n = 4/\sqrt{q_0} = 4/4 = 1\). Since the classifying space consists of only one disconnected mass manifold the classification is trivial, i.e., there are only topological trivial phases.

To check whether the bulk superconductor given by Eq. (2) or Eq. (6) is indeed topologically trivial we diagonalize Eq. (8) in the SM on a cylinder with periodic (open) boundary condition in the \(x (y)\) directions. The result is shown in Fig. 1(a) which shows there is no gapless edge state, consistent with the statement that the bulk is trivial.

To check whether the vortices harbor Majorana zero modes we use the following order parameters
\[
\phi_1(x) + i \phi_2(x) = \Delta_0 \left[ (x - x_0) + i(y - y_0) \right] \left[ (x - x_1) - i(y - y_1) \right]^{n_v},
\] (3)

where \((x_0, y_0)\) and \((x_1, y_1)\) are the centers of a pair of ±1 vortices. In Fig. 2(a) and (b) we plot the energy spectrum for \(\pm 1\) (panels (a), (c)) and \(\pm 2\) (panels (b), (d)) vortices. The \(\Delta_0\) used to construct these figures is equal to .5. Panel (a) and (b) are for \(\phi_2 = 0\) superconductor, while panel (c) and (d) are for \(\phi_2 = 0.2\). The system size is \(20 \times 40\) sites \((40/20\) in the direction parallel/perpendicular to the separation vector between the vortices).

An example in 3D With a slight generalization we can write down the Hamiltonian for the topologically trivial, symmetry class \(p = 2\) (no symmetry), superconductors in 3D. The only change is to add one more gamma matrix to Eq. (2), i.e.,
\[
H = \int d^3 x \chi^T(x) \left(-i \mathcal{Z} \partial_1 - i X \partial_2 - i \epsilon \partial_3 \right.
- \left. i \phi_1 X \epsilon - i \phi_2 Z \epsilon \right) \chi(x).
\] (4)

The lattice version of this Hamiltonian is given by Eq. (16) of the SM. In Fig. 3(a) we plot the projected bandstructure for a system periodic in \(\hat{x}\) and \(\hat{y}\) and open in \(\hat{z}\). The absence of gapless boundary modes signifies the topological trivialness of the bulk superconductor.
So far we have shown that the presence of vortex Majorana modes is not a sufficient condition for concluding the parent superconductor is topologically non-trivial. However is it a necessary condition? The answer is negative. The superconductor with symmetry $G_{-\pi}(T, C)$ in the symmetry class $p = 4$ is an example. In 2D the classifying space is given by $R_{4-2} = R_2$. Its classification ($\pi_0(R_2)$) is $Z_2$ hence is non-trivial. However the vortices are classified by $\pi_0(R_{2+2-1}) = \pi_0(R_3) = 0$.

The above results lead us to conclude that the presence of vortex Majorana zero mode (2D) or chiral-dispersing Majorana modes (3D) in the vortex core of superconductors are neither sufficient nor necessary condition for concluding the superconductivity being topological. Of course, this conclusion does not affect the novelty of the Majorana modes and their possible applications.

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Classifying spaces

SUPPLEMENTARY MATERIAL

TABLE I

TABLE I. The classification tables. Here $G$ denotes the symmetry group and $\hat{Q}, \hat{T}, \hat{C}$ are generators of the U(1) charge conservation, time reversal and charge conjugation symmetries, respectively. $Q, T, C$ are orthogonal matrices representing these generators in the mode space, i.e. the spin and orbital space of Majorana fermion operators $\chi_a(x)$. More specifically, $\hat{Q} = \sum_{a,b} \frac{i}{2} \chi_a(x) Q_{ab} \chi_b(x)$, $\hat{T} i T^{-1} = -i$ and $\hat{T} \chi_a(x) T^{-1} = \sum_b T_{ab} \chi_b(x)$. $\hat{C} \chi_a(x) C^{-1} = \sum_b C_{ab} \chi_b(x)$. The meaning of the signs in the expressions, e.g., $G_{SP}(T), G_{SC}(C), G_{STC}(T, C)$ and $c_{STC}(Q, T, C)$ are the following: $T^2 = (-1)^{ST}$, $C^2 = (-1)^{SC}$, and $TC = (-1)^{STC}$. In this table it is assumed that $Q^2 = -1$ and $TQ = -QT$ hence $S_Q$ and $S_{TQ}$ are not specified.

| $p \mod 8$ | Classifying spaces $R_p$ | $\pi_0(R_p)$ | Symmetry generators |
|------------|--------------------------|-------------|---------------------|
| 0          | $O(n)$                   | $Z$         | $G_{+}(Q,T), G_{+}(T,C)$ |
| 1          | $G_{2}(n)$               | $Z$         | $G_{+}(T), G_{+}(T,C), G_{+}(Q,T,C)$ |
| 2          | $Sp(n)$                  | $Z$         | none, $G_{-}(C), G_{+}(T,C), G_{+}(Q,C)$ |
| 3          | $U(n)$                   | 0           | $G_{-}(T), G_{-}(T,C), G_{+}(Q,T,C)$ |
| 4          | $Sp(n)$                  | $Z$         | $G_{-}(Q,T), G_{-}(T,C)$ |
| 5          | $O(n)$                   | 0           | $G_{+}(Q,T,C)$ |
| 6          | $Sp(n)$                  | 0           | $G_{+}(Q,T,C), SU(2)$ |
| 7          | $S_{p}(n)$               | 0           | $G_{+}(Q,T,C)$ |

Complex class

| $p \mod 2$ | Classifying spaces $C_p$ | $\pi_0(C_p)$ | Symmetry generators |
|------------|--------------------------|-------------|---------------------|
| 0          | $U(n)$                   | $Z$         | $U(1), G_{-}(C)$ |
| 1          | $U(n)$                   | 0           | $G_{+}(T,C), G_{+}(T,C)$ |

THE COMPLEX FERMION VERSION OF EQ. (2)

Here we recast Eq. (2) into complex fermion form $\psi(x) = (\chi_1(x) - i\chi_2(x))/2$ where 1 and 2 are the indices of the first Pauli matrix in Eq. (2):

$$H = \int d^2 x \ 4\psi^\dagger (-iZ\partial_1 - iX\partial_2)\psi(x) + 2\left[(\phi_1(x) - i\phi_2(x))\right. \left.\psi^T(x) \epsilon \psi(x) + h.c.\right].$$

THE 2D LATTICE VERSION OF EQ. (2)

Here we write down a Hamiltonian on a square lattice which has the low energy effective theory described by Eq. (2). We first do so in the momentum space.
\[ H = \sum_{k \in BZ} \chi^T_k [\sin(k_x)IZ + \sin(k_y)IX - (1 - \cos(k_x))iX\epsilon - (1 - \cos(k_y))iZ\epsilon - \phi_1 iX\epsilon - \phi_2 iZ\epsilon] \chi_k \] (7)

In real space, the Hamiltonian is
\[ H = \sum_i -\chi^T_i [(1 + \phi_1) iX\epsilon + (1 + \phi_2) iZ\epsilon] \chi_i + \sum_i \left[ \chi^T_i \left( -\frac{iIZ + iX\epsilon}{2} \right) \chi_{i+\hat{x}} + \chi^T_i \left( -\frac{iIX + iZ\epsilon}{2} \right) \chi_{i+\hat{y}} + h.c. \right]. \] (8)

**The 2D Lattice Hamiltonian in the Presence of Vortices**

In the presence of vortices the lattice Hamiltonian we diagonalized to obtain Fig. 2 is given by
\[ H = \sum_i -\chi^T_i [(1 + \phi_1(x_i)) iX\epsilon + (1 + \phi_2(x_i)) iZ\epsilon + \phi_3 iI\epsilon] \chi_i + \sum_i \left[ \chi^T_i \left( -\frac{iIZ + iX\epsilon}{2} \right) \chi_{i+\hat{x}} + \chi^T_i \left( -\frac{iIX + iZ\epsilon}{2} \right) \chi_{i+\hat{y}} + h.c. \right], \] (9)

where \( \phi_1,2(x_i) \) are given by Eq. (3).

**Analytic Solution of the Majorana Zero Mode in a Vorticity +1 Vortex**

The low energy effective Hamiltonian is
\[ H = \int d^2x \chi^T(x)(-iIZ\partial_1 - iIX\partial_2 - i\phi_1(x)X\epsilon - i\phi_2(x)Z\epsilon)\chi(x) \] (10)

where
\[ \phi_1(x) + i\phi_2(x) = \Delta_0 \left( \frac{x + iy}{|x + iy|} \right)^{n_v}. \] (11)

By going to the polar coordinate the Hamiltonian becomes
\[ H = \int rd\theta \chi^T(r, \theta) \left[ -iIZ \left( \cos \theta \partial_r - \frac{\sin \theta}{r} \partial_\theta \right) - iIX \left( \sin \theta \partial_r + \frac{\cos \theta}{r} \partial_\theta \right) - i(\phi_1(x)X\epsilon + \phi_2(x)Z\epsilon) \right] \chi(r, \theta) = \int rd\theta \chi^T(r, \theta) \left[ e^{-i\theta}(IZ + iIX)/2 \left( -i\partial_r - \frac{1}{r} \partial_\theta \right) + \Delta_0 e^{-im\theta}(XY + iZY)/2 + h.c. \right] \chi(r, \theta) \] (12)

It is straightforward to check that \( H \) is invariant under rotation by angle \( \phi \), namely \( R(\phi)HR^{-1}(\phi) = H \), where \( R(\phi) = e^{iJ\phi} \) and \( J = \frac{1}{2} \partial_\theta - \frac{1}{2}(n_vYI - IY) \). The two terms are the orbital and spin angular momentum, respectively.

The eigenfunctions of \( J \) has the form \( e^\rho e^{im\theta} \) with eigenvalues \( m - \frac{1}{2}(n_v\rho - \sigma) \). Here \( e^\rho \) is a four-component spinor with eigenvalues \( \rho, \sigma = \pm 1 \) under \( YI, IY \) respectively. Their explicit forms are
\[ e^T_{++} = \frac{1}{2}(1, i, -i, 1) \]
\[ e^T_{+-} = \frac{1}{2}(1, -i, i, 1) \]
\[ e^T_{-+} = \frac{1}{2}(1, i, -i, 1) \]
\[ e^T_{--} = \frac{1}{2}(1, -i, i, 1) \]

We expand \( \chi(r, \theta) \) in eigenfunctions of \( J \) as
\[ \chi_\alpha(r, \theta) = \sum (e^\rho)^\alpha e^{im\theta} \gamma_{\sigma\tau}(r) \]
and substitute into (12), the result is

\[
H = \int 2\pi r dr \sum_{i \in \mathbb{Z}} \Gamma_j^\dagger(r) \left( \begin{array}{cccc}
0 & -i\partial_r - i/l/r & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & -i\Delta_0 & -i\partial_r + i(l - n_v - 1)/r & 0 \\
i\Delta_0 & 0 & 0 & 0 \\
\end{array} \right) \Gamma_j(r)
\]

\[
= \int 2\pi r dr \sum_j \Gamma_j^\dagger(r) D_j \Gamma_j(r)
\]

where \(l = j + \frac{1 + n_v}{2}\) and

\[
D_j = IX(-i\partial_r - i/2r) + IY(j/r) + ZY(n_v/2r) + YZ\Delta_0
\]

\[
\Gamma_j^T = (\gamma_{++(l-1)}, \gamma_{+-}, \gamma_{-+}, \gamma_{--(l-n_v)}, \gamma_{--(l-n_v)})
\]

Note that \(\gamma\)'s are complex fermions operators. Since \(\chi^\dagger = \chi\) and \(e_{\sigma\tau}^* = e_{-\sigma-\tau}\), we deduce \(\gamma_{\sigma\tau m}^\dagger = \gamma_{-\sigma-\tau-m}\). Therefore \(\Gamma_j^* = XX\Gamma_{-j}\).

Also a useful identity is

\[
\int 2\pi r dr \sum_{\alpha} f_{\alpha}(r)[D_j g_{\alpha}(r)] = \int 2\pi r dr \sum_{\alpha} [-X X D_{-j} X X f_{\alpha}(r)] g_{\alpha}(r)
\]

from which it can be shown that \(\int 2\pi r dr \sum_{\alpha} f_{\alpha}(r)[D_j g_{\alpha}(r)] = \int 2\pi r dr \sum_{\alpha} [-X X D_{-j} X X f_{\alpha}(r)] g_{\alpha}(r)\).

It turns out a normalizable zero mode solution exists\(^\text{[13]}\) when \(n_v = 1\) and \(j = 0\). The zero mode solution \(\xi = \int 2\pi r dr f_{\alpha}(r)[\Gamma_j^\dagger(r)]\alpha\) satisfies \([H, \xi] = 0\). By using (14) and \(\{[\Gamma_j^\ast(r)]_{\alpha}, [\Gamma_j^\dagger(r')]_{\beta}\} = \delta_{\alpha\beta} \delta_{j+j'} \frac{1}{2\pi r} \delta(r - r')\), this implies \(D_0 f = 0\). The zero mode solution is \(f = e^{-\Delta_0}(i, 0, 0, -i)\). The phase is chosen such that \(\xi^* = \xi\).

Note the YY in the \(\chi\) basis is diag(1, -1, -1, 1) = ZZ in the \(\Gamma\) basis. It is also readily seen that \((ZZ)f = +f\).

For \(n_v = -1\), the zero mode solution is \(f = e^{-\Delta_0}(0, 1, 1, 0)\). It is seen that \((ZZ)f = -f\).

**THE 3D LATTICE HAMILTONIAN**

The 3D version of Eq. (7) is given by

\[
H = \sum_{k \in BZ} \chi_k^T \left[ \sin(k_x)IZ + \sin(k_y)IX + \sin(k_z)\epsilon - (1 - \cos(k_x))IX\epsilon - (1 - \cos(k_y))IZ\epsilon - (1 - \cos(k_z))iZ\epsilon \right]
\]

\[
- \phi_1 IX\epsilon - \phi_2 iZ\epsilon \right] \chi_k
\]

(15)

The 3D version of Eq. (8) is given by

\[
H = -\sum_i \chi_i^T \left[ (1 + \phi_1) iX\epsilon + (2 + \phi_2) iZ\epsilon \right] \chi_i + \sum_i \chi_i^T \left( \frac{-iIZ + iX\epsilon}{2} \right) \chi_{i+\hat{z}} + \chi_i^T \left( \frac{-iIX + iZ\epsilon}{2} \right) \chi_{i+\hat{y}}
\]

\[+ \chi_i^T \left( \frac{-i\epsilon + iZ\epsilon}{2} \right) \chi_{i+\hat{z} + h.c.}\]

(16)

and the 3D version of Eq. (9) is given by

\[
H = -\sum_i \chi_i^T \left[ (1 + \phi_1(x_i)) iX\epsilon + (2 + \phi_2(x_i)) iZ\epsilon \right] \chi_i + \sum_i \chi_i^T \left( \frac{-iIZ + iX\epsilon}{2} \right) \chi_{i+\hat{z}} + \chi_i^T \left( \frac{-iIX + iZ\epsilon}{2} \right) \chi_{i+\hat{y}}
\]

\[+ \chi_i^T \left( \frac{-i\epsilon + iZ\epsilon}{2} \right) \chi_{i+\hat{z} + h.c.}\]

(17)

where \(\phi_{1,2}(x_i)\) are given by Eq. (3).
CHIRAL MAJORANA MODES AT THE INTERFACE BETWEEN SUPERCONDUCTIVITY AND FERROMAGNETISM

In the following we will show that in the interface between superconductivity and ferromagnetism, there is a chiral majorana mode. We show this by two methods; first by analytically solving the low energy effective Hamiltonian in continuum, and second by numerically diagonalize the corresponding lattice Hamiltonian.

Analytical solution

The low energy effective Hamiltonians for the 2D superconductivity and quantized anomalous Hall phases are given respectively by:

\[
H_{SC} = \int d^2x \chi^T(x)(-iIZ \partial_1 - iIX \partial_2 - i\phi_1 X \epsilon - i\phi_2 Z \epsilon) \chi(x)
\]

\[
H_{FM} = \int d^2x \chi^T(x)(-iIZ \partial_1 - iIX \partial_2 - i\phi_3 I \epsilon) \chi(x)
\]

(18)

Assume a configuration where for \( x < 0 \), the system is in the ferromagnetic phase with uniform \( \phi_3 \), whereas for \( x \geq 0 \), the system is in the superconducting phase with uniform \( \phi_1, \phi_2 \). Furthermore, we may perform a local orthogonal transformation without affecting the kinetic terms and the ferromagnetic term, such that \( \phi_1 = 0 \) and \( \phi_2 = \phi > 0 \). So

\[
H = \begin{cases}
\int d^2x \chi^T(x)(-iIZ \partial_1 - iIX \partial_2 + i\phi Z \epsilon) \chi(x) & \text{if } x \geq 0 \\
\int d^2x \chi^T(x)(-iIZ \partial_1 - iIX \partial_2 - i\phi_3 I \epsilon) \chi(x) & \text{if } x < 0
\end{cases}
\]

It is seen that \( ZI \) commutes with the Hamiltonian and the Hilbert space is decomposed into two sectors with eigenvalues \( \pm 1 \) under \( ZI \). The difference between the two sectors is that the sign of the \( \phi \) term changes. Without loss of generality we may assume \( \phi_3 > 0 \). Then the Hamiltonian in the \( ZI = -1 \) sector reads

\[
H_+ = \begin{cases}
\int d^2x \chi^T(x)(-iIZ \partial_1 - iIX \partial_2 + i\phi Z \epsilon) \chi(x) & \text{if } x \geq 0 \\
\int d^2x \chi^T(x)(-iIZ \partial_1 - iIX \partial_2 - i\phi_3 I \epsilon) \chi(x) & \text{if } x < 0
\end{cases}
\]

An eigenmode \( \xi = \int d^2x f^T(x) \chi(x) \) satisfying \( [H_+, \xi] = 2E\xi \) implies \( Dg = E \), where

\[
D = \begin{cases}
-IZ \partial_1 + X k_2 + i\phi \epsilon & \text{if } x \geq 0 \\
-IZ \partial_1 + X k_2 - i\phi_3 \epsilon & \text{if } x < 0
\end{cases}
\]

where we assumed \( f(x, y) = e^{ik_2y}g(x) \) by translation invariance along \( y \). The solution is

\[
g(x) \propto \begin{cases}
\exp(\int dx Y(k_2 I - EX) + \phi X) & \text{if } x \geq 0 \\
\exp(\int dx Y(k_2 I - EX) - \phi_3 X) & \text{if } x < 0
\end{cases}
\]

which is normalizable in the \( X = -1 \) sector and when \( E = -k_2 \). All solutions in the other sectors are not normalizable or continuous at \( x = 0 \). Thus there is only one chiral majorana mode in the interface.

Numerical solution

The corresponding lattice versions of (18) are

\[
H_{SC} = \sum_i -\chi_i^T [(1 + \phi_1)iX \epsilon + (1 + \phi_2)iZ \epsilon] \chi_i + \sum_i \chi_i^T \left(-iIZ + iX \epsilon \over 2\right) \chi_{i+\hat{z}} + \chi_i^T \left(-iIZ + iX \epsilon \over 2\right) \chi_{i+\hat{y}} + h.c.,
\]

(19)

\[
H_{QAH} = \sum_i -\chi_i^T [iX \epsilon + iZ \epsilon + \phi_3 iI \epsilon] \chi_i + \sum_i \chi_i^T \left(-iIZ + iX \epsilon \over 2\right) \chi_{i+\hat{z}} + \chi_i^T \left(-iIZ + iX \epsilon \over 2\right) \chi_{i+\hat{y}} + h.c.,
\]

(20)
FIG. 5. (a) The interface between superconducting and ferromagnetic regions. (b) The projected bandstructure where $k_x$ is the momentum around the circumference of the cylinder. (c) The wavefunction of the in-gap state at $k_x = 0.2\pi$. The left and right Majorana modes are localized on the upper and lower interface, respectively. The parameters used are $\phi_1 = 1.0 \cos(\pi/8)$, $\phi_2 = 1.0 \sin(\pi/8)$, $\phi_3 = 0.4$. The height of the cylinder is 160 lattice spacings. The widths of the three different regions are 40–80–40.

We may perform an exact diagonalization of the single body Hamiltonian defined on a cylinder geometry (Fig 5(a)) where a ferromagnetic region is sandwiched between two superconducting regions. The periodic directions is $\hat{x}$ and hence we can assume $\chi(x) = e^{ik_x} \chi(y)$, so

$$H_{SC} = \sum_{i,k_x} \chi_i^T \left[ \sin(k_x)IZ - (1-\cos(k_x) + \phi_1)iX\epsilon - (1+\phi_2)iZ\epsilon \right] \chi_i + \sum_i \left[ \chi_i^T \left( \frac{-iX + iZ\epsilon}{2} \right) \chi_{i+\hat{y}} + h.c. \right]$$

$$H_{QAH} = \sum_{i,k_x} \chi_i^T \left[ \sin(k_x)IZ - (1-\cos(k_x))iX\epsilon - iZ\epsilon - \phi_3 iI\epsilon \right] \chi_i + \sum_i \left[ \chi_i^T \left( \frac{-iX + iZ\epsilon}{2} \right) \chi_{i+\hat{y}} + h.c. \right]$$

We observe two counter propagating majorana modes in the single body spectrum. We plotted the corresponding wavefunctions and find that the left-moving/right-moving chiral majorana modes are respectively localized on the upper/lower interface between the superconducting and ferromagnetic regions. The result still holds when a chemical potential term $\Delta H = -\mu \chi^T Q \chi$ with $Q = YI$ is added to the Hamiltonians. See Fig 5.

The mass manifolds of the no symmetry class

In Fig. 6(b) we show the mass manifolds associated with the free fermion symmetry protected topological states in the no symmetry class. The superconducting mass manifold in Fig. 6(a) has $T^2 = +1$ symmetry, however since it is connected to that of Fig. 6(b) without gap closing they describe the same topological phase.