On \( A \)-numerical radius inequalities for \( 2 \times 2 \) operator matrices

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Abstract

Let \((H, \langle \cdot, \cdot \rangle)\) be a complex Hilbert space and \(A\) be a positive bounded linear operator on it. Let \(w_A(T)\) be the \(A\)-numerical radius and \(\|T\|_A\) be the \(A\)-operator seminorm of an operator \(T\) acting on the semi-Hilbertian space \((H, \langle \cdot, \cdot \rangle_A)\), where \(\langle x, y \rangle_A := \langle Ax, y \rangle\) for all \(x, y \in H\). In this article, we establish several upper and lower bounds for \(B\)-numerical radius of \(2 \times 2\) operator matrices, where \(B = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}\). Further, we prove some refinements of earlier \(A\)-numerical radius inequalities for operators.

Keywords: \(A\)-numerical radius; Positive operator; Semi-inner product; Inequality; Operator matrix

1. Introduction

Let \(H\) be a complex Hilbert space with inner product \(\langle \cdot, \cdot \rangle\) and \(B(H)\) be the \(\mathbb{C}^*\)- algebra of all bounded linear operators on \(H\). For \(T \in \mathcal{L}(H)\), the numerical range of \(T\) is defined as

\[ W(T) = \{ \langle Tx, x \rangle : x \in H, \|x\| = 1 \}. \]

The numerical radius of \(T\), denoted by \(w(T)\), is defined as \(w(T) = \sup\{|z| : z \in W(T)\}\). It is well-known that \(w(\cdot)\) defines a norm on \(H\), and is equivalent to the usual operator norm \(\|T\| = \sup\{\|Tx\| : x \in H, \|x\| = 1\}\). In fact, for every \(T \in \mathcal{L}(H)\),

\[ \frac{1}{2} \|T\| \leq w(T) \leq \|T\|. \]  \hfill (1.1)

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One may refer [3, 6, 9, 10, 11, 19] for several generalizations, refinements and applications of numerical radius inequalities in different settings which appeared in the last decade. Let \( \| \cdot \| \) be the norm induced from \( \langle \cdot, \cdot \rangle \). A selfadjoint operator \( A \in \mathcal{B}(\mathcal{H}) \) is called positive if \( \langle Ax, x \rangle \geq 0 \) for all \( x \in \mathcal{H} \), and is called strictly positive if \( \langle Ax, x \rangle > 0 \) for all non-zero \( x \in \mathcal{H} \).

We denote a positive (strictly positive) operator \( A \) by \( A \geq 0 \) (\( A > 0 \)). Let \( B \) be a \( 2 \times 2 \) diagonal operator matrix, in which each of the diagonal entries is a positive operator \( A \). Through out this article, \( A \) is always assumed to a positive operator. Clearly, if \( A \) is a positive operator, it induces a positive semidefinite sesquilinear form, \( \langle \cdot, \cdot \rangle_A : \mathcal{H} \times \mathcal{H} \to \mathbb{C} \) defined by \( \langle x, y \rangle_A = \langle Ax, y \rangle \), \( x, y \in \mathcal{H} \). Let \( \| \cdot \|_A \) denote the semi-norm on \( \mathcal{H} \) induced by \( \langle \cdot, \cdot \rangle_A \), i.e., \( \| x \|_A = \sqrt{\langle x, x \rangle_A} \) for all \( x \in \mathcal{H} \). It is easy to verify that \( \| x \|_A \) is a norm if and only if \( A \) is a strictly positive operator. Also, \( (\mathcal{H}, \| \cdot \|_A) \) is complete if and only if the range of \( A (\mathcal{R}(A)) \) is closed in \( \mathcal{H} \). For \( T \in \mathcal{B}(\mathcal{H}), A \)-operator seminorm of \( T \), denoted as \( \| T \|_A \), is defined as

\[
\| T \|_A := \sup_{x \in \mathcal{R}(A), \| x \| = 0} \frac{\| T x \|_A}{\| x \|_A} = \inf \left\{ c > 0 : \| T x \|_A \leq c \| x \|_A, \, x \in \mathcal{R}(A) \right\} < \infty.
\]

We set \( \mathcal{B}^A(\mathcal{H}) := \{ T \in \mathcal{B}(\mathcal{H}) : \| T \|_A < \infty \} \). It can be seen that \( \mathcal{B}^A(\mathcal{H}) \) is not generally a subalgebra of \( \mathcal{B}(\mathcal{H}) \), and \( \| T \|_A = 0 \) if and only if \( AT A = 0 \). For \( T \in \mathcal{B}^A(\mathcal{H}) \), we also have

\[
\| T \|_A = \sup \{ \| (Tx, y)_A \| : x, y \in \mathcal{R}(A), \| x \|_A = \| y \|_A = 1 \}.
\]

If \( AT \geq 0 \), then the operator \( T \) is called \( A \)-positive. Note that if \( T \) is \( A \)-positive, then

\[
\| T \|_A = \sup \{ \| (Tx, x)_A \| : x \in \mathcal{H}, \| x \|_A = 1 \}.
\]

For \( T \in \mathcal{B}(\mathcal{H}) \), an operator \( R \in \mathcal{B}(\mathcal{H}) \) is called an \( A \)-adjoint operator of \( T \) if for every \( x, y \in \mathcal{H} \), we have \( (Tx, y)_A = \langle x, Ry \rangle_A \), i.e., \( AR = T^*A \). By Douglas Theorem [14], the existence of an \( A \)-adjoint operator is not guaranteed. In fact, an operator \( T \in \mathcal{B}(\mathcal{H}) \) may admit none, one or many \( A \)-adjoints. The set of all operators which admits \( A \)-adjoint is denoted by \( \mathcal{B}_A(\mathcal{H}) \). Note that \( \mathcal{B}_A(\mathcal{H}) \) is a subalgebra of \( \mathcal{B}(\mathcal{H}) \) which is neither closed nor dense in \( \mathcal{B}(\mathcal{H}) \). Moreover, the following inclusions \( \mathcal{B}_A(\mathcal{H}) \subseteq \mathcal{B}^A(\mathcal{H}) \subseteq \mathcal{B}(\mathcal{H}) \) hold with equality if \( A \) is injective and has a closed range.

For \( A \in \mathcal{B}(\mathcal{H}) \) and \( \mathcal{R}(A) \) is closed, the Moore-Penrose inverse of \( A \) [5] is the operator \( X \in \mathcal{B}(\mathcal{H}) \) which satisfies the following four Penrose equations:

1. \( AXA = A \),
2. \( XAX = X \),
3. \( (AX)^* = AX \),
4. \( (XA)^* = XA \).

It is unique, and is denoted by \( A^\dagger \). If \( T \in \mathcal{B}_A(\mathcal{H}) \), the reduced solution of the equation \( AX = T^*A \) is a distinguished \( A \)-adjoint operator of \( T \), which is denoted by \( T^{\#_A} \) (see [7]).
Note that $T^{\#_{A}} = A^\dagger T^* A$. If $T \in \mathcal{B}_A(\mathcal{H})$, then $AT^{\#_{A}} = T^* A$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be $A$-selfadjoint if $AT$ is selfadjoint, i.e., $AT = T^* A$. Observe that if $T$ is $A$-selfadjoint, then $T \in \mathcal{B}_A(\mathcal{H})$. However, in general, $T \neq T^{\#_{A}}$. For $T \in \mathcal{B}_A(\mathcal{H})$, $T = T^{\#_{A}}$ if and only if $T$ is $A$-selfadjoint and $\mathcal{R}(T) \subseteq \mathcal{R}(A)$. Note that if $T \in \mathcal{B}_A(\mathcal{H})$, then $T^{\#_{A}} \in \mathcal{B}_A(\mathcal{H})$, $(T^{\#_{A}})^{\#_{A}} = PTP$, where $P$ is an orthogonal projection onto $\mathcal{R}(A)$, and $((T^{\#_{A}})^{\#_{A}})^{\#_{A}} = T^{\#_{A}}$. Also $T^{\#_{A}}T$ and $TT^{\#_{A}}$ are $A$-selfadjoint and $A$-positive operators. So,

$$\|T^{\#_{A}}T\|_A = \|TT^{\#_{A}}\|_A = \|T\|^2_A = \|T^{\#_{A}}\|^2_A. \tag{1.2}$$

An operator $U \in \mathcal{B}_A(\mathcal{H})$ is said to be $A$-unitary if $\|UX\|_A = \|UX\|_A = \|X\|_A$ for all $X \in \mathcal{H}$. For $T \in \mathcal{B}_A(\mathcal{H})$ and $U \in \mathcal{B}_A(\mathcal{H})$, $w_A(U^A_T U) = w_A(T)$.

Again, for $T, S \in \mathcal{B}_A(\mathcal{H})$, $(TS)^{\#_{A}} = S^{\#_{A}}T^{\#_{A}}$, $\|TS\|_A \leq \|T\|_A \|S\|_A$ and $\|TX\|_A \leq \|T\|_A \|X\|_A$ for all $X \in \mathcal{H}$. For $T \in \mathcal{B}_A(\mathcal{H})$, we can write $Re_A(T) = \frac{T + T^{\#_{A}}}{2}$ and $Im_A(T) = \frac{T - T^{\#_{A}}}{2i}$. For further details, we refer the reader to [1, 2]. In 2012, Saddi [15] defined $A$-numerical radius of $T$, denoted as $w_A(T)$, for $T \in \mathcal{B}(\mathcal{H})$ as follows

$$w_A(T) = \sup\{|\langle Tx, x \rangle_A|: x \in \mathcal{H}, \|x\|_A = 1\}.$$  

In 2019, Zamani [12] showed that if $T \in \mathcal{B}_A(\mathcal{H})$, then

$$w_A(T) = \sup_{\theta \in \mathbb{R}} \left\|e^{\theta T} + \left(e^{i\theta T}\right)^{\#_{A}}\right\|_A. \tag{1.3}$$

The author then extended the inequality (1.1) using $A$-numerical radius of $T$, and the same is illustrated next:

$$\frac{1}{2} \|T\|_A \leq w_A(T) \leq \|T\|_A. \tag{1.4}$$

Furthermore, if $T$ is $A$-selfadjoint, then $w_A(T) = \|T\|_A$. In 2019, Moslehian et al. [8] further continued the study of $A$-numerical radius and established some inequalities for $A$-numerical radius.

For a $2 \times 2$ operator matrix $T$, $B$-numerical radius of $T$ is defined as

$$w_B(T) = \sup\{|\langle Tx, x \rangle_B|: x \in \mathcal{H}, \|x\|_B = 1\},$$

where $B = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$.

In 2019, Bhunia et al. [4] studied $B$-numerical radius inequalities of $2 \times 2$ operator matrices, where $B$ is a $2 \times 2$ diagonal operator matrix whose diagonal entries are $A$. In
this directions some authors has been studied many generalizations and refinements of $A$-numerical radius, for more details one can refer [13, 16, 17]. This motivates us to further study on this topic.

The objective of this paper is to present new $B$-numerical radius inequalities for $2 \times 2$ operator matrices. Further two refinements of the 1st inequality in (1.4) is addressed in this article. In this aspect, the article is structured as follows. In Section 2, we recall some upper and lower bounds for $B$-numerical radius inequalities for a $2 \times 2$ operator matrix. The next section contains our main results and is of two folds. First part establishes some upper and lower bounds for $2 \times 2$ operator matrices while the second part deals with certain refinements of (1.4).

2. Preliminaries

In 2020, Pintu et al. [4] proved the following lemma for $2 \times 2$ operator matrices.

**Lemma 2.1.** [Lemma 2.4, [4]]

Let $T_1, T_2 \in B_A(H)$. Then the following results hold:

(i) $w_B \left( \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \right) = \max\{w_A(T_1), w_A(T_2)\}$.

(ii) If $A > 0$, then $w_B \left( \begin{bmatrix} 0 & T_1 \\ T_2 & 0 \end{bmatrix} \right) = w_B \left( \begin{bmatrix} 0 & T_2 \\ T_1 & 0 \end{bmatrix} \right)$.

(iii) If $A > 0$, then for any $\theta \in \mathbb{R}$, $w_B \left( \begin{bmatrix} 0 & e^{i\theta}T_1 \\ T_2 & 0 \end{bmatrix} \right) = w_B \left( \begin{bmatrix} 0 & T_1 \\ T_2 & 0 \end{bmatrix} \right)$.

(iv) If $A > 0$, then $w_B \left( \begin{bmatrix} T_1 & T_2 \\ T_2 & T_1 \end{bmatrix} \right) = \max\{w_A(T_1 + T_2), w_A(T_1 - T_2)\}$.

In particular, $w_B \left( \begin{bmatrix} 0 & T_1 \\ T_1 & 0 \end{bmatrix} \right) = w_A(T_1)$.

In 2019, the authors of [13] established an upper and lower bound for a $2 \times 2$ operator matrix.

**Lemma 2.2.** [Theorem 4.3, [13]]
Let $T_1, T_2 \in B_A(\mathcal{H})$ where $A > 0$. If $T = \begin{bmatrix} 0 & T_1 \\ T_2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$ then

$$\frac{1}{2} \max\{w_A(T_1 + T_2), w_A(T_1 - T_2)\} \leq w_B(T) \leq \frac{1}{2}\{w_A(T_1 + T_2) + w_A(T_1 - T_2)\}.$$ 

In 2020, Feki [18] proved the following result.

**Lemma 2.3.** [Lemma 2.1, [18]]

Let $T = (T_{ij})_{n \times n}$ such that $T_{ij} \in B_A(\mathcal{H})$ for all $i, j$. Then

$$\|T\|_A \leq \|\tilde{T}\|,$$

where $\tilde{T} = (\|T_{ij}\|_A)_{n \times n}$.

## 3. Main Results

This section is two fold. First, we present some generalizations of $A$-numerical radius inequalities. Further we prove some upper and lower bounds for $B$-numerical radius of operator matrices. Second, we provide different refinements of $A$-numerical radius inequalities.

### 3.1. Upper and lower bounds for $B$-numerical radius of $2 \times 2$ operator matrix.

In this subsection, we establish different upper and lower bounds for $B$-numerical radius of a $2 \times 2$ block operator matrix. We start with the following lemma.

**Lemma 3.1.** Let $T_1, T_2, T_3, T_4 \in B_A(\mathcal{H})$. Then

(i) $w_B\left( \begin{bmatrix} T_1 & 0 \\ 0 & T_4 \end{bmatrix} \right) \leq w_B\left( \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \right)$.

(ii) $w_B\left( \begin{bmatrix} 0 & T_2 \\ T_3 & 0 \end{bmatrix} \right) \leq w_B\left( \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \right)$.

**Proof.** Let $T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}$ and the $B$-unitary operator $U = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$.

Here, $\begin{bmatrix} T_1 & 0 \\ 0 & T_4 \end{bmatrix} = \frac{1}{2}(T + U^*TU)$. So, we have
The following inequality generalizes (1.4).

**Theorem 3.1.** Let $T_1, T_2 \in B_A(\mathcal{H})$, where $A > 0$. If $B = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$, then

$$\max\{w_A(T_1), w_A(T_2)\} \leq w_B\left( \begin{bmatrix} T_1 & T_2 \\ -T_2 & -T_1 \end{bmatrix} \right) \leq w_A(T_1) + w_A(T_2). \tag{3.1}$$

**Proof.** By using Lemma 2.1 and Lemma 3.1

$$w_A(T_1) = w_B\left( \begin{bmatrix} T_1 & 0 \\ 0 & -T_1 \end{bmatrix} \right) \leq w_B\left( \begin{bmatrix} T_1 & T_2 \\ -T_2 & -T_1 \end{bmatrix} \right)$$

and

$$w_A(T_2) = w_B\left( \begin{bmatrix} 0 & T_2 \\ -T_2 & 0 \end{bmatrix} \right) \leq w_B\left( \begin{bmatrix} T_1 & T_2 \\ -T_2 & -T_1 \end{bmatrix} \right).$$
Replacing $T$ by $-iT$ in the identity, we have
\[
\max\{w_A(T_1), w_A(T_2)\} = \min\{w_A(T_1 + iT_2), w_A(T_1 - iT_2)\}.
\]

Therefore,
\[
\max\{w_A(T_1), w_A(T_2)\} \leq w_B \left( \begin{bmatrix} T_1 & T_2 \\ -T_2 & -T_1 \end{bmatrix} \right).
\]

On the other hand, by using Lemma 2.1 we have
\[
w_B \left( \begin{bmatrix} T_1 & T_2 \\ -T_2 & -T_1 \end{bmatrix} \right) \leq w_B \left( \begin{bmatrix} T_1 & 0 \\ -T_2 & 0 \end{bmatrix} \right) + w_B \left( \begin{bmatrix} 0 & T_2 \\ 0 & -T_1 \end{bmatrix} \right) = w_A(T_1) + w_A(T_2).
\]

A particular case of the inequality (3.1) is the following.

**Remark 3.2.** If we choose $T_2 = T_1$ in inequality (3.1), then
\[
w_A(T_1) \leq w_B \left( \begin{bmatrix} T_1 & T_1 \\ -T_1 & -T_1 \end{bmatrix} \right) \leq 2w_A(T_1).
\]

We need the following lemma to prove Theorem 3.2

**Lemma 3.3.** Let $T_1, T_2, T_3, T_4 \in \mathcal{B}(\mathcal{H})$, where $A > 0$. If $B = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$, then
\[
w_B \left( \begin{bmatrix} T_2 & -T_1 \\ T_1 & T_2 \end{bmatrix} \right) = \max\{w_A(T_1 + iT_2), w_A(T_1 - iT_2)\}.
\]

**Proof.** Let $T = \begin{bmatrix} iT_2 & -T_1 \\ T_1 & iT_2 \end{bmatrix}$ and the B-unitary operator $U = \frac{1}{\sqrt{2}} \begin{bmatrix} I & iI \\ iI & I \end{bmatrix}$. Then $U^{\#}TU = \begin{bmatrix} -i(T_1 - T_2) & 0 \\ 0 & i(T_1 + T_2) \end{bmatrix}$. Using the fact that $w_B(T) = w_B(U^{\#}TU)$, we get
\[
w_B(T) = w_B(U^{\#}TU) = w_B \left( \begin{bmatrix} -i(T_1 - T_2) & 0 \\ 0 & i(T_1 + T_2) \end{bmatrix} \right)
= \max\{w_A(-i(T_1 - T_2)), w_A(i(T_1 + T_2))\}
= \max\{w_A(T_1 - T_2), w_A(T_1 + T_2)\}.
\]

Replacing $T_2$ by $-iT_2$ in the identity, we have
\[
w_B \left( \begin{bmatrix} T_2 & -T_1 \\ T_1 & T_2 \end{bmatrix} \right) = \max\{w_A(T_1 + iT_2), w_A(T_1 - iT_2)\}.
\]
Theorem 3.2 provides an upper bound for a block operator matrix of the form \( \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \).

**Theorem 3.2.** Let \( T_1, T_2, T_3, T_4 \in \mathcal{B}_A(\mathcal{H}) \), where \( A > 0 \). If \( T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \) and \( B = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \).

Then

\[
\left| B \right| = \max \left\{ \frac{1}{2} w_A(T_1 + T_4 + i(T_2 - T_3)), \frac{1}{2} w_A(T_1 + T_4 - i(T_2 - T_3)) \right\} + \frac{1}{2} (w_A(T_4 - T_1) + w_A(T_2 + T_3)).
\]

**Proof.** Let \( U = \frac{1}{\sqrt{2}} \begin{bmatrix} I & -I \\ I & I \end{bmatrix} \) be \( B \)-unitary. Using the identity \( \left| B \right| = w_B(U^* B U) \), we have

\[
w_B \left( \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \right) = w_B \left( U^* \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} U \right) = \frac{1}{2} w_B \left( \begin{bmatrix} T_1 + T_2 + T_3 + T_4 & -T_1 + T_2 - T_3 + T_4 \\ -T_1 - T_2 + T_3 + T_4 & T_1 - T_2 - T_3 + T_4 \end{bmatrix} \right) = \frac{1}{2} w_B \left( \begin{bmatrix} T_1 + T_4 & T_2 - T_3 \\ T_3 - T_2 & T_1 + T_4 \end{bmatrix} + \begin{bmatrix} T_2 + T_3 & T_4 - T_1 \\ T_4 - T_1 & -T_3 - T_2 \end{bmatrix} \right) \leq \frac{1}{2} \left\{ \max(w_A(T_3 - T_2 + i(T_1 + T_4)), w_A(T_3 - T_2 - i(T_1 + T_4)), w_A(T_4 - T_1) + w_A(T_2 + T_3)) \right\}
\]

by Lemma 3.3 and Lemma 2.1.

The following result demonstrates an upper bound for \( B \)-numerical radius of a \( 2 \times 2 \) operator matrix.

**Theorem 3.3.** Let \( T_1, T_2, T_3, T_4 \in \mathcal{B}_A(\mathcal{H}) \), where \( A > 0 \). If \( B = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \). Then

\[
w_B \left( \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \right) \leq \max( w_A(T_1), w_A(T_3) ) + w_A(T_2 + T_3) + w_A(T_2 - T_3).\]

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Proof. Using similar argument as used in the previous theorem, we have

\[
\begin{align*}
  w_B \left( \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \right) &= \frac{1}{2} w_B \left( \begin{bmatrix} T_1 + T_4 & T_4 - T_1 \\ T_4 - T_1 & T_1 + T_4 \end{bmatrix} + \begin{bmatrix} T_2 + T_3 & T_2 - T_3 \\ T_3 - T_2 & -T_3 - T_2 \end{bmatrix} \right) \\
  &\leq \frac{1}{2} \left[ w_B \left( \begin{bmatrix} T_1 + T_4 & T_4 - T_1 \\ T_4 - T_1 & T_1 + T_4 \end{bmatrix} \right) + w_B \left( \begin{bmatrix} T_2 + T_3 & T_2 - T_3 \\ T_3 - T_2 & -T_3 - T_2 \end{bmatrix} \right) \right] \\
  &\leq \frac{1}{2} \max\{w_A(T_1 + T_4 + T_4 - T_1), w_A(T_1 + T_4 - T_4 + T_1)\} \\
  &\quad + \frac{1}{2} \{w_A(T_2 + T_3) + w_A(T_2 - T_3)\} \text{ by Lemma 2.1(iv)} \\
  &= \max\{w_A(T_1), w_A(T_4)\} + \frac{w_A(T_2 + T_3) + w_A(T_2 - T_3)}{2}.
\end{align*}
\]

The following result is an estimate of an lower bound for $B$-numerical radius of a $2 \times 2$ operator matrix.

**Theorem 3.4.** Let $T_1, T_2, T_3, T_4 \in \mathcal{B}(\mathcal{H})$, where $A > 0$. If $B = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$, then

\[
\begin{align*}
  w_B \left( \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \right) &\geq \max\left\{w_A(T_1), w_A(T_4), \frac{w_A(T_2 + T_3)}{2}, \frac{w_A(T_2 - T_3)}{2}\right\}.
\end{align*}
\]

Proof. It follows from Lemma 3.1 that

\[
\begin{align*}
  w_B \left( \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \right) &\geq \max\left\{w_B \left( \begin{bmatrix} T_1 & 0 \\ 0 & T_4 \end{bmatrix} \right), w_B \left( \begin{bmatrix} 0 & T_2 \\ T_3 & 0 \end{bmatrix} \right) \right\} \\
  &= \max\left\{w_A(T_1), w_A(T_4), \max\left\{w_A(T_2 + T_3), w_A(T_2 - T_3)\right\}\right\} \text{ by Lemma 2.2} \\
  &\geq \max\left\{w_A(T_1), w_A(T_4), \frac{w_A(T_2 + T_3) + w_A(T_2 - T_3)}{2}\right\} \\
  &= \max\left\{w_A(T_1), w_A(T_4), \frac{w_A(T_2 + T_3)}{2}, \frac{w_A(T_2 - T_3)}{2}\right\}.
\end{align*}
\]

To prove the next lemma, we need the following identities,

\[
\frac{a + b}{2} = \max(a, b) - \frac{|a - b|}{2} \tag{3.2}
\]
\[ \frac{a + b}{2} = \min(a, b) + \frac{|a - b|}{2}, \quad (3.3) \]

for any two real numbers \( a \) and \( b \).

**Lemma 3.4.** Let \( T_1, T_2 \in B_A(\mathcal{H}) \). Then

\[
\max(\|T_1 + T_2\|_A^2, \|T_1 - T_2\|_A^2) \leq \min(\|T_1^{\#A}T_1 + T_2^{\#A}T_2\|_A + \|T_1^{\#A}T_1 + T_2^{\#A}T_1\|_A, \|T_1T_1^{\#A} + T_2T_2^{\#A}\|_A + \|T_1T_2\|_A) 
\]

and

\[
\min(\|T_1 + T_2\|_A^2, \|T_1 - T_2\|_A^2) \geq \max(\|T_1^{\#A}T_1 + T_2^{\#A}T_2\|_A - \|T_1^{\#A}T_2 + T_2^{\#A}T_1\|_A, \|T_1T_1^{\#A} + T_2T_2^{\#A}\|_A - \|T_1T_2\|_A).
\]

**Lemma 3.5.** Let \( T_1, T_2 \in B_A(\mathcal{H}) \). Then

\[
\max(\|T_1 + T_2\|_A^2, \|T_1 - T_2\|_A^2) \geq \max(\|T_1^2 + T_2^2\|_A, \|T_1^{\#A}T_1 + T_2^{\#A}T_2\|_A, \|T_1T_1^{\#A} + T_2T_2^{\#A}\|_A) 
\]

\[
+ \frac{\|T_1 + T_2\|_A^2 - \|T_1 - T_2\|_A^2}{2},
\]

and

\[
\min(\|T_1 + T_2\|_A^2, \|T_1 - T_2\|_A^2) \leq \max(\|T_1^2 + T_2^2\|_A, \|T_1^{\#A}T_1 + T_2^{\#A}T_2\|_A, \|T_1T_1^{\#A} + T_2T_2^{\#A}\|_A) 
\]

\[
- \frac{\|T_1 + T_2\|_A^2 - \|T_1 - T_2\|_A^2}{2}.
\]

Following theorem demonstrates an upper bound for \( B \)-numerical radius of \( 2 \times 2 \) operator matrix using \((1.4)\) and Lemma 3.5.

**Theorem 3.5.** Let \( T_1, T_2, T_3, T_4 \in B_A(\mathcal{H}) \). Then

\[ w_B\left( \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \right) \leq \min(\alpha, \beta), \]

where,

\[ \alpha = \left( \frac{\|T_1 + T_2\|_A^2 + \|T_1 - T_2\|_A^2}{2} \right)^\frac{1}{2} + \left( \frac{\|T_4 + T_3\|_A^2 + \|T_4 - T_3\|_A^2}{2} \right)^\frac{1}{2}, \]

and

\[ \beta = \left( \frac{\|T_1 + T_3\|_A^2 + \|T_1 - T_3\|_A^2}{2} \right)^\frac{1}{2} + \left( \frac{\|T_2 + T_4\|_A^2 + \|T_2 - T_4\|_A^2}{2} \right)^\frac{1}{2}. \]
Proof. We know that
\[
w_B \left( \begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix} \right) \leq \left\| \begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix} \right\|_B
= \left\| \begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix}^{#B} \right\|_B^{1/2}
= \left\| \begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_1^{#A} & 0 \\ T_2^{#A} & 0 \end{bmatrix} \right\|_B^{1/2}
= \left\| T_1 T_1^{#A} + T_2 T_2^{#A} \right\|_A^{1/2}.
\]

By using Lemma 3.5, we get
\[
w_B \left( \begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix} \right) \leq \left( \max(\|T_1 + T_2\|_A^2, \|T_1 - T_2\|_A^2) - \frac{\|T_1 + T_2\|_A^2 - \|T_1 - T_2\|_A^2}{2} \right)^{\frac{1}{2}}
= \left( \frac{\|T_1 + T_2\|_A^2 + \|T_1 - T_2\|_A^2}{2} \right)^{\frac{1}{2}}.
\]

Let us take \( U = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \), where \( U \) is \( B \)-unitary. Now, we have
\[
w_B \left( \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \right) \leq w_B \left( \begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix} \right) + \left( \begin{bmatrix} 0 & 0 \\ T_3 & T_4 \end{bmatrix} \right)
= w_B \left( \begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix} \right) + w_B \left( U^{#B} \begin{bmatrix} T_4 & T_3 \\ 0 & 0 \end{bmatrix} U \right)
= w_B \left( \begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix} \right) + w_B \left( \begin{bmatrix} T_4 & T_3 \\ 0 & 0 \end{bmatrix} \right)
\leq \left( \frac{\|T_1 + T_2\|_A^2 + \|T_1 - T_2\|_A^2}{2} \right)^{\frac{1}{2}} + \left( \frac{\|T_4 + T_3\|_A^2 + \|T_4 - T_3\|_A^2}{2} \right)^{\frac{1}{2}} = \alpha
\]
Applying the previous calculation to \( \begin{bmatrix} T_1^#A & T_3^#A \\ T_2^#A & T_4^#A \end{bmatrix} \) in the place of \( \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \), we obtain

\[
w_B \left( \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \right) = w_B \left( \begin{bmatrix} T_1^#A & T_3^#A \\ T_2^#A & T_4^#A \end{bmatrix} \right) \\
\leq \left( \frac{\|T_1^#A + T_3^#A\|_A^2 + \|T_1^#A - T_3^#A\|_A^2}{2} \right)^{\frac{1}{2}} + \left( \frac{\|T_2^#A + T_4^#A\|_A^2 + \|T_2^#A - T_4^#A\|_A^2}{2} \right)^{\frac{1}{2}} \\
= \left( \frac{\|T_1 + T_3\|_A^2 + \|T_1 - T_3\|_A^2}{2} \right)^{\frac{1}{2}} + \left( \frac{\|T_2 + T_4\|_A^2 + \|T_2 - T_4\|_A^2}{2} \right)^{\frac{1}{2}} = \beta.
\]

Hence, we get the desired result. \( \square \)

Next result shows a lower bound for \( B \)-numerical radius of a \( 2 \times 2 \) operator matrix in which 2nd row is zero.

**Theorem 3.6.** Let \( T_1, T_2 \in B_A(H) \). Then

\[
w_B \left( \begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix} \right) \geq \frac{1}{2} \max(w_A(T_1 \pm T_2), w_A(T_1 \pm iT_2)).
\]

**Proof.** Let \( U = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \) a \( B \)-unitary operator. Then

\[
\max(w_A(T_1 + T_2), w_A(T_1 - T_2)) = w_B \left( \begin{bmatrix} T_1 & T_2 \\ T_2 & T_1 \end{bmatrix} \right) \text{ by Lemma 2.1}
= w_B \left( \begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix} + U^#B \begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix} U \right)
\leq w_B \left( \begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix} \right) + w_B \left( U^#B \begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix} U \right)
= 2w_B \left( \begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix} \right).
\]
Theorem 3.7. Let $T_1, T_2 \in B_A(H)$. 

$$w_B\left(\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}\right) \leq \min\{\alpha, \beta\},$$

where,

$$\alpha = \frac{1}{\sqrt{2}} \sqrt{\|T_1\|^2_A + \|T_2\|^2_A + \sqrt{(\|T_1\|^2_A - \|T_2\|^2_A)^2 + 4\|T_1^\# T_2\|^2_A}}$$

and

$$\beta = \frac{1}{\sqrt{2}} \sqrt{\|T_3\|^2_A + \|T_4\|^2_A + \sqrt{(\|T_3\|^2_A - \|T_4\|^2_A)^2 + 4\|T_4^\# T_3\|^2_A}}$$
We now give an special case of Theorem 3.7 in the following corollary.

**Corollary 3.1.** Let $T_1, T_2, T_3, T_4 \in \mathcal{B}(\mathcal{H})$.

(a) If $T_1^\# A = 0 = T_4 T_3^\# A$, then

$$
w_B \left( \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \right) \leq \max(\|T_1\|_A, \|T_2\|_A) + \max(\|T_3\|_A, \|T_4\|_A).
$$

(b) If $T_1^\# A T_3 = 0 = T_4^\# A T_2$, then

$$
w_B \left( \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \right) \leq \max(\|T_1\|_A, \|T_3\|_A) + \max(\|T_2\|_A, \|T_4\|_A).
$$

**Remark 3.6.** Note that equality holds in Theorem 3.5 and Theorem 3.7 by setting $T_1 = I, T_2 = T_3 = T_4 = 0$. So $A$-numerical radius inequalities for $2 \times 2$ operator matrices in Theorem 3.5 and Theorem 3.7 are sharp.

### 3.2. Refinements of $A$-numerical radius inequality for an operator

In this subsection, we present two refinements of (1.4). To do this, we need the identity (3.2).

The first refinement of inequality (1.4) is proved next.

**Theorem 3.8.** Let $T_1, T_2 \in \mathcal{B}(\mathcal{H})$. Then

$$
w_B \left( \begin{bmatrix} 0 & T_1 \\ T_2 & 0 \end{bmatrix} \right) \leq w_A(T_1) + w_A(T_2) - \frac{1}{2}|w_A(T_1 + T_2) - w_A(T_1 - T_2)|.
$$

In particular

$$
\frac{\|T_1\|_A}{2} + \frac{\|\text{Re} T_1^\# A\|_A - \|\text{Im} T_1^\# A\|_A}{2} \leq w_A(T_1).
$$

**Proof.** By Lemma 2.2 and Equality (3.2), we have

$$
w_B \left( \begin{bmatrix} 0 & T_1 \\ T_2 & 0 \end{bmatrix} \right) \leq \frac{1}{2}(w_A(T_1 + T_2) + w_A(T_1 - T_2))$$

$$
= \max\{w_A(T_1 + T_2), w_A(T_1 - T_2)\} - \frac{1}{2}|w_A(T_1 + T_2) - w_A(T_1 - T_2)|$$

$$
\leq w_A(T_1) + w_A(T_2) - \frac{1}{2}|w_A(T_1 + T_2) - w_A(T_1 - T_2)|.
$$

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Replacing $T_1$ by $T_1^{\#A}$ and $T_2$ by $(T_1^{\#A})^{\#A}$, we get

\[
\begin{align*}
&w_B\left(\begin{bmatrix} 0 & T_1^{\#A} \\ (T_1^{\#A})^{\#A} & 0 \end{bmatrix}\right) \\
&\leq w_A(T_1^{\#A}) + w_A((T_1^{\#A})^{\#A}) - \frac{1}{2} w_A(T_1^{\#A} + (T_1^{\#A})^{\#A}) - w_A(T_1^{\#A} - (T_1^{\#A})^{\#A}) \\
&= 2w_A(T_1^{\#A}) - \|ReT_1^{\#A}\|_A - \|ImT_1^{\#A}\|_A.
\end{align*}
\]

So,

\[
\frac{\|T_1\|_A}{2} + \frac{\|ReT_1^{\#A}\|_A - \|ImT_1^{\#A}\|_A}{2} \leq w_A(T_1^{\#A}) = w_A(T_1).
\]

Another refinement of the Inequality (1.4) is presented next.

**Theorem 3.9.** Let $T_1, T_2 \in B_\mathcal{A}(\mathcal{H})$. Then

\[
\begin{align*}
w_B\left(\begin{bmatrix} 0 & T_1 \\ T_2 & 0 \end{bmatrix}\right) &+ \frac{\|T_1\|_A + \|T_2\|_A}{2} + \frac{1}{2} \left| w_A(T_1 + T_2) - \frac{\|T_1\|_A + \|T_2\|_A}{2} \right| + \frac{1}{2} \left| w_A(T_1 - T_2) - \frac{\|T_1\|_A + \|T_2\|_A}{2} \right| \\
&\leq 2(w_A(T_1) + w_A(T_2)).
\end{align*}
\]

In particular,

\[
\frac{\|T_1\|_A}{2} + \frac{\|Re(T_1^{\#A})\|_A - \frac{\|T_1\|_A}{2}}{4} + \frac{\|Im(T_1^{\#A})\|_A - \frac{\|T_1\|_A}{2}}{4} \leq w_A(T_1).
\]

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4. References

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