FINITE DIMENSIONAL GLOBAL ATTRACTOR FOR A DAMPED FRACTIONAL ANISOTROPIC SCHRÖDINGER TYPE EQUATION WITH HARMONIC POTENTIAL

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ABSTRACT. We study the long time behaviour of the solutions for a class of nonlinear damped fractional Schrödinger type equation with anisotropic dispersion and in presence of a quadratic potential in a two dimensional unbounded domain. We prove that this behaviour is characterized by the existence of regular compact global attractor with finite fractal dimension.

1. Introduction. Dispersive wave equations provide excellent examples of infinite dimensional dynamical systems which are either conservative or exhibit some dissipation. In the last case, one can hope to reduce the study of the flow to a bounded (or even compact) attracting set or global attractor that contains much of the relevant information about the flow. Once the global attractor is obtained, the question arises if it has special regularity properties or if it has finite-dimensional character. Hence both existence of attractors and bounds on their dimensions are of a great interest. We refer the reader to R. Temam [43], J. Robinson [40], I. Chueshov [14] and G. Raugel [39] for general frameworks of this theory.

The aims of this paper is to study the asymptotic dynamics for a dynamical system generated by a nonlinear dispersive fractional anisotropic Schrödinger type equation with harmonic trapping potential that reads

\[ \partial_t u + i(\partial_x^2 - x^2)u - i \left( \sqrt{-\partial_y^2} \right)^\alpha u + ig(|u|^2)u + \gamma u = f, \]  

where \( \partial_x^2 \) denotes for the sake of simplicity \( \frac{\partial^2}{\partial x^2} \). The unknown \( u = u(t, x, y) \) maps \( \mathbb{R}_+ \times \Omega \) into \( \mathbb{C} \) with \( \Omega = \mathbb{R} \times [0, 1] \) and \( \alpha \in (1, 2) \). The equation (1.1) is supplemented with the Dirichlet boundary conditions \( u(t, x, 0) = u(t, x, 1) = 0 \) and with initial data at \( t = 0 \)

\[ u(0) = u_0, \]  

that belongs to the anisotropic phase space \( \mathcal{H}_\alpha \) that will be specified in the sequel.

The fractional Schrödinger equations is a fundamental equation of fractional quantum mechanics which has been discovered as a result of expending the Feynman path integral from the Brownian-like to Lévy-like quantum mechanical paths.

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In some physical problems, the presence of many particles leads one to consider nonlinear terms which stimulate the interaction effect among them. The nonlinear fractional Schrödinger equations, formulated by N. Laskin [31, 32], have typically the conservative form:

$$i\partial_t \psi(t, x) = (-\Delta)^{\frac{\alpha}{2}} \psi(t, x) + V(x)\psi - F(|\psi(t, x)|^2)\psi(t, x), \quad x \in \mathbb{R}^d, \quad t \in \mathbb{R}, \quad (1.3)$$

where $\alpha \in (0, 2)$, $\psi(t, x)$ is the unknown complex-valued wave function and $V(x)$ denotes a real-valued external potential. This prototypical equation has been widely used in many branches of applied sciences arising in nonlinear optics and beam propagation, in condensed matter physics, in deep water wave dynamics, in plasma physics, in wave turbulence and in dynamics of boson stars (see for instance [18, 27, 31, 32, 35, 37, 42]).

The mathematical literature for the conservative NLS equations is so huge that we do not even try to collect here a detailed bibliography. On the contrary, to the best of our knowledge, the literature for fractional Schrödinger equations is still expending and rather young. If $\alpha = 2$, (1.3) becomes the standard Gross-Pitaevskii equation which has been extensively studied as a fundamental equation in modern mathematical physics specially for Bose-Einstein condensates (see [10, 4, 34] for instance). In the special case when $V = 0$ we refer the reader; for well-posedness results and existence of traveling waves for the resulting conservative fractional NLS; to [5, 25, 26, 21, 13, 45, 7] and the references therein.

Now going back to the case of the equation (1.3) with $F(|\psi|^2) = |\psi|^{2\sigma}$, M. Cheng prove in [12] the existence of ground state by Lagrange multiplier method as well as the standing wave with prescribed frequency by Nehari’s manifold approach. Under the same heading, in addition of proving the existence of standing waves, numerical results about the dynamics of (1.3) with harmonic potential were given in [17]. In [29], K. Kirkpatrick and Y. Zhang have also studied the following equation

$$i\partial_t \psi(t, x) = \frac{1}{2} \left( (-\Delta)^{\frac{\alpha}{2}} + |x|^2 \right) \psi(t, x) + |\psi(t, x)|^2 \psi(t, x), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^d$$

with considerable numerical analysis on soliton dynamics.

For the non conservative case, O. Goubet and E. Zahrouni have studied the following dissipative one dimensional fractional NLS that reads

$$\partial_t u(t, x) - i(-\partial_x^2)^{\frac{\alpha}{2}} u(t, x) + i|u(t, x)|^2 u(t, x) + \gamma u(t, x) = f, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}$$

where $\alpha \in (1, 2)$. They prove in [23] the existence of a regular compact global attractor in $H^{2\alpha}(\mathbb{R})$ with finite fractal dimension (under suitable assumption on $f$). Furthermore, the two dimensional Bose-Einstein equation

$$u_t + i(\Delta - x^2) u + i|u|^2 u + \gamma u = f$$

(1.4)

which is none other then (1.1) with $\alpha = 2$, was studied first of all in [1] with critical nonlinearity (cubic nonlinearity) in a thin strip. Then, with subcritical nonlinearities in [3] and [2] where the authors have prove in both cases the existence of a compact global attractor with finite fractal dimension. Finally, we point out that the case in which $\alpha = 1$ in (1.1) is very interesting and rises many questions which will be the subject of a forthcoming paper.

Now let us return to the matter at hand. The linear part of (1.1) is balanced between a skew-symmetric operator $iA_\alpha = i(-\partial_x^2 + x^2 + D_0^\alpha)$, and a zero-order dissipation term $\gamma u$ where $\gamma > 0$ is the damping parameter. $f \in L^2(\Omega)$ is a given source term that is independent of time and the nonlinearity $g$ is a $C^\infty$ mapping...
from \( \mathbb{R}^+ \) into \( \mathbb{R} \). For convenience use, we consider subcritical smooth nonlinearity, i.e. \( g \) that satisfies the following growth condition: for \( \xi \geq 0 \)

\[
\xi^2 |g''(\xi)| + \xi |g'(\xi)| + |g(\xi)| \leq c_1 \xi^\sigma,
\]

(1.5)

for a given \( \sigma \in (0, \frac{2\alpha}{2s+\alpha}) \). For later use, we also assume that the derivatives \( g^{(k)} \) are bounded for \( k \geq 3 \); moreover, we infer from (1.5) that for \( \xi \geq 0 \),

\[
\xi |g'(\xi^2)| + \xi^3 |g''(\xi^2)| \leq c_2 (1 + \xi)
\]

(1.6)

and

\[
\xi^2 |g''(\xi^2)| \leq c_3.
\]

(1.7)

Before giving the layout of this article, we recall briefly some definitions and notations. First, denoting \( D_y = \sqrt{-\partial^2_y} \), we recall (see [43, 41]) that for a given \( s \geq 0 \), the operator \( D_y^s = (\sqrt{-\partial^2_y})^s \) with Dirichlet boundary conditions is an unbounded non-negative self adjoint operator whose domain in

\[
H = \{ u \in L^2([0,1]), \text{ such that } u(0) = u(1) = 0 \}
\]

is \( \text{Dom} (D_y^s) = \{ u \in H \text{ such that } \sum_{k \geq 1} (k\pi)^{2s} |\hat{u}_k|^2 < +\infty \} \),

where \( \hat{u}_k = \sqrt{2} \int_0^1 u(y) \sin(k\pi y) dy \). Moreover, it is well known (see [16, 41]) that for a fixed fractional exponent \( s \in (0, 1) \), the fractional Sobolev space \( H_y^s([0,1]) \) defined as follows

\[
H_y^s([0,1]) = \left\{ u \in L^2([0,1]) \text{ such that } \frac{|u(x) - u(y)|}{|x - y|^{\frac{1}{2} + s}} \in L^2([0,1] \times [0,1]) \right\}
\]

(1.8)

\[
= \left\{ u \in L^2([0,1]) \text{ such that } D_y^s u \in L^2([0,1]) \right\}
\]

(1.9)

is endowed with the natural norm

\[
||u||_{H_y^s([0,1])} = \left( ||u||_{L^2([0,1])}^2 + \int_0^1 \int_0^1 \frac{|u(x) - u(y)|^2}{|x - y|^{1+2s}} \, dx \, dy \right)^\frac{1}{2}
\]

which is equivalent to \( ||D_y^s u||_{L^2([0,1])} \) thanks to the following Poincaré type inequality whose proof is rather straightforward.

Lemma 1.1. For all \( u \in H_y^s([0,1]) \),

\[
||u||_{L^2([0,1])} \leq ||D_y^s u||_{L^2([0,1])}.
\]

The Hilbert space \( \mathbb{L}^2 = L^2(\Omega) \) is equipped with the usual scalar product denoted by \( \langle u, v \rangle = \Re \int_\Omega u(x,y) \overline{v(x,y)} \, dx \, dy \). We will be regularly referring to the mixed space-time Lebesgue space denoted by \( \mathbb{L}^p_y \mathbb{L}^q_x \) equipped with the semi-norm (for a fixed \( T > 0 \))

\[
||u||_{\mathbb{L}^q_y \mathbb{L}^p_x} = \left( \int_T^{t+T} ||u(s)||_{\mathbb{L}^q_y \mathbb{L}^p_x}^q \, ds \right)^\frac{1}{q},
\]

where

\[
||u||_{\mathbb{L}^q_y \mathbb{L}^p_x} = \left( \int_\mathbb{R} \left( \int_0^1 |u(x,y)|^2 \, dy \right)^\frac{p}{2} \, dx \right)^\frac{1}{p}
\]

we extensively use the notation \( \mathbb{L}^q_y \mathbb{L}^p_x \) for anisotropic Lebesgue spaces with obvious changes if \( q \) or \( p \) is infinity.

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We use the notation \( \varrho = \sqrt{1 + x^2} \), \( x \in \mathbb{R} \) and, in a general context, for a given weight \( w = w(x) \), we set
\[
L^2_w = \left\{ u \in L^2 \text{ such that } \int_{\Omega} |u(x,y)|^2 w(x) dx dy < +\infty \right\}.
\]
From now on, \( A_0 = -\partial_x^2 + x^2 \) denotes the one dimensional harmonic oscillator. It is well known that \( A_0 \) is an unbounded non-negative self adjoint operator which has a compact inverse. Moreover, the family of eigenfunctions defined by
\[
h_n(x) = \left( 2^n \pi^\frac{1}{2} n! \right)^{-\frac{1}{2}} H_n(x) e^{-\frac{x^2}{2}}, \quad n \in \mathbb{N}
\]
constitutes an orthonormal basis in \( L^2(\mathbb{R}) \), where we denote by
\[
H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2}), \quad n \in \mathbb{N}
\]
the Hermite orthogonal polynomials (see [20] for more details), and satisfies the following spectral property:
\[
A_0 h_n = (2n + 1) h_n, \quad n \in \mathbb{N}.
\]
The phase space \( \mathcal{H}_\alpha \), \( \alpha \in (1,2) \), defined as the completion of \( C_0^\infty(\Omega) \)(the space of smooth functions with compact support in \( \Omega \) with the norm \( \| \cdot \|_{\mathcal{H}_\alpha} \) (defined below) is the Hilbert space
\[
\mathcal{H}_\alpha = \left\{ u \in L^2(\Omega) \text{ and } u(x,0) = u(x,1) = 0 \text{ such that } \partial_x u, D_y^\alpha u, \varrho u \in L^2(\Omega) \right\}
\]
that is endowed with the scalar product defined by
\[
(u, v)_{\mathcal{H}_\alpha} = (\varrho u, \varrho v) + (\partial_x u, \partial_x v) + \left( D_y^\alpha u, D_y^\alpha v \right); \quad u, v \in \mathcal{H}_\alpha
\]
and the associated norm, which is equivalent, thanks to Lemma 1.1, to
\[
\|u\|_{\mathcal{H}_\alpha} = \left( \|xu\|_{L^2}^2 + \|\partial_x u\|_{L^2}^2 + \|D_y^\alpha u\|_{L^2}^2 \right)^{\frac{1}{2}}.
\]
It is well known that \( A_\alpha \) is an unbounded non-negative self adjoint operator whose domain in \( L^2 \) is,
\[
D(A_\alpha) = \left\{ u \in \mathcal{H}_\alpha = D(A_\alpha^\frac{1}{2}) / A_\alpha u \in L^2(\Omega) \right\}
\]
The operator \( A_\alpha \) has a compact inverse. Moreover, there exists an orthonormal family in \( L^2(\Omega) \),
\[
\psi_{n,k}(x,y) = \left( 2^{n-1} \pi^\frac{1}{2} n! \right)^{-\frac{1}{2}} H_n(x) e^{-\frac{x^2}{2}} \sin(k \pi y), \quad (n, k) \in \mathbb{N} \times \mathbb{N}^*
\]
such that \( A_\alpha \psi_{n,k} = \lambda_{n,k} \psi_{n,k} \) and
\[
\lambda_{n,k} = 1 + 2n + (k \pi)^\alpha, \quad (n, k) \in \mathbb{N} \times \mathbb{N}^*. \tag{1.12}
\]
Hence, thanks to the spectral representation of \( A_\alpha \), it leads that for all \( s \geq 0 \), the operator \( A_\alpha^s \) is an unbounded non-negative self adjoint operator defined by
\[
A_\alpha^s u(x,y) = \sum_{n \geq 0, k \geq 1} \lambda_{n,k}^s (u, \psi_{n,k}) \psi_{n,k}(x,y),
\]
whose domain in \( L^2 \) is the Hilbert space
\[
D(A_\alpha^s) = \left\{ u = \sum_{n \geq 0, k \geq 1} (u, \psi_{n,k}) \psi_{n,k}(x,y) \text{ such that } \sum_{n \geq 0, k \geq 1} \lambda_{n,k}^{2s} |(u, \psi_{n,k})|^2 < +\infty \right\}
\]
endowed with the classical norm

$$||u||_{D(A_\alpha)} = \left( \sum_{n \geq 0, k \geq 1} \lambda_{n,k}^{2 \alpha} |(u, \psi_{n,k})|^2 \right)^{\frac{1}{2}}.$$  

Finally, for any positive $A$ and $B$, $A \leq B$ means that there exists $c > 0$ such that $A \leq cB$ and we recall that throughout this article the constants $C$s are numerical constants that vary from one line to another.

Our main result of this paper is stated as follows

**Theorem 1.2** (Main Theorem). Let $f \in L^2(\Omega)$ and $\alpha \in (1, 2)$. Then the equation (1.1) defines a dissipative dynamical system in $\mathcal{H}_\alpha$ that possesses a regular global attractor $\mathcal{A}_\alpha$ that is a compact subset of $D(A_\alpha)$. Moreover, the compact global attractor $\mathcal{A}_\alpha$ has a finite fractal dimension in $\mathcal{H}_\alpha$.

It should be emphasized that, to the best of our knowledge, our results are new and will open the way to consider other class of fractional Schrödinger equations.

The plan of the present paper is as follows. In section 2, we establish some anisotropic inequalities and Strichartz type estimates. The well-posedness of the Cauchy problem for (1.1) as well as the existence of the global attractor $\mathcal{A}_\alpha$ in $\mathcal{H}_\alpha$ for the associated semigroup $(S_\alpha(t))_{t \in \mathbb{R}}$ will be the subject of section 3. The issue of regularity of the global attractor $\mathcal{A}_\alpha$ will be discussed in Section 4. In the last section we establish the finite dimensionality of this global attractor.

2. Preliminary results.

Given the complexity of the current issue, it becomes immediately apparent that the first recourse must be providing tools, namely some helpful anisotropic Sobolev inequalities.

2.1. Some anisotropic inequalities.

**Lemma 2.1.** Let $p \in [1, +\infty]$. Then there exists $C_p > 0$ that depends only on $p$ such that for all $u \in \mathbb{H}^{1,2}_{\alpha,\nu} \cap \mathbb{L}^2_{\alpha,\nu}(\mathbb{R}, \mathbb{L}^2_\kappa)$,

$$||u||_{\mathbb{L}^2_{\alpha,\nu}} \leq C_p ||u||^{1-\nu(p)}_{L^2} \left||A_\alpha^\frac{\nu(p)}{2} u\right||_{L^2}$$  

(2.1)

where

$$\nu(p) = \begin{cases} \frac{1}{p} - \frac{1}{2} & \text{if } 1 \leq p \leq 2 \\ \frac{1}{2} - \frac{1}{p} & \text{if } 2 \leq p \leq +\infty \end{cases}$$

and $\mathbb{H}^{1,2}_{\alpha,\nu}$ stands for the usual Sobolev space $\mathbb{H}^1(\mathbb{R}, \mathbb{L}^2_{\nu})$.

**Proof of Lemma 2.1.** The case $p \in [2, +\infty]$ is none other than an application of the one dimensional Gagliardo-Nirenberg inequality, Minkowski’s and Hölder’s inequalities. For the case $p \in [1, 2]$, we consider $R > 0$ and then by using Hölder’s and the Cauchy-Schwarz inequalities

$$||u||_{L^p_{\alpha,\nu}}^p = \int_{|x| \leq R} ||u||_{L^p_\nu}^p dx + \int_{|x| > R} ||u||_{L^p_\nu}^p dx$$

$$\leq C(p) \left( R^{1-\frac{\nu}{2}} ||u||_{L^2}^p + \frac{1}{R^{p-\frac{\nu}{2}}} ||xu||_{L^2}^p \right).$$

Hence,

$$||u||_{L^p_{\alpha,\nu}} \leq C_p \left( R^{\frac{\nu}{2} - \frac{1}{p}} ||u||_{L^2} + \frac{1}{R^{\frac{p}{2} - \frac{\nu}{2}}} ||xu||_{L^2} \right).$$
This conclude the proof of the lemma by minimizing the last inequality with respect to $R > 0$.

**Lemma 2.2.** Let $\alpha \in (1, 2)$ and $p \in \left[\frac{2\alpha}{2\alpha - 1}, 2\alpha\right]$. Then there exists a real constant $C_{\alpha, p} > 0$ that depends only on $\alpha$ and $p$ such that for all $u \in \mathcal{H}_\alpha$,

$$
\|u\|_{L^\infty_y L^p_x} \leq C_{\alpha, p} \|u\|_{L^\infty_y L^2_x}^{1 - (\nu(p) + \frac{1}{2})} \left\| A_0^{\frac{1}{2}} u \right\|_{L^2_x} \left\| D_y^{\frac{2}{p}} u \right\|_{L^2_x} \tag{2.2}
$$

where we recall that

$$
\nu(p) = \begin{cases}
  \frac{1}{p} - \frac{1}{2} & \text{if } \frac{2\alpha}{2\alpha - 1} \leq p \leq 2 \\
  \frac{1}{p} & \text{if } 2 \leq p \leq 2\alpha
\end{cases}
$$

**Proof of Lemma 2.2.** Following the same idea used in [23], we consider $R > 0$. Thanks to the Cauchy-Schwarz inequality,

$$
\|u\|_{L^\infty_y} \leq \sum_{1 \leq k \leq R} |\hat{u}_k| + \sum_{k > R} (k\pi)^{-\frac{1}{2}} |\hat{u}_k| \frac{1}{(k\pi)^{\frac{1}{2}}} \lesssim \sqrt{R} \|u\|_{L^2_y} + R^{\frac{1-\nu(p)}{2}} \|D_y^{\frac{2}{p}} u\|_{L^2_y},
$$

where $\hat{u}_k = \frac{1}{\sqrt{2\pi}} \int_0^1 \sqrt{2} u(x, y) \sin(k\pi y) dy$. By minimizing this bound with respect to $R > 0$, yields to

$$
\|u\|_{L^\infty_y} \leq c_\alpha \|u\|_{L^2_y}^{1 - \nu(p)} \|D_y^{\frac{2}{p}} u\|_{L^2_y}. \tag{2.3}
$$

Thus, thanks to the Hölder inequality, it follows that for all $p \in (\frac{2\alpha}{2\alpha - 1}, 2\alpha]$,

$$
\|u\|_{L^\infty_y L^p_x} \leq C_{\alpha, p} \|u\|_{L^\infty_y L^2_x}^{1 - \frac{1}{2\alpha(p - 1)}} \|D_y^{\frac{2}{p}} u\|_{L^2_y}. \tag{2.4}
$$

Combining (2.4) and Lemma 2.1 achieves the proof.\[\square\]

**Lemma 2.3.** Let $\alpha \in (1, 2)$ and $q \in [2, 4]$. Then there exists a real constant $C_\alpha > 0$ that depends on $\alpha$ and $q$ such that for all $u \in \mathcal{H}_\alpha$,

$$
\|u\|_{L^\infty_y L^q_x} \leq C_\alpha \|u\|_{L^2_y} \left(\frac{1}{2} - \frac{2\alpha - 2}{2\alpha - q}\right) \left\| A_0^{\frac{1}{2}} u \right\|_{L^2_x} \left\| D_y^{\frac{2}{q}} u \right\|_{L^2_x}^{\frac{2\alpha - 2}{2\alpha - q}}. 
$$

**Proof of Lemma 2.3.** Thanks to the Agmon and Minkowski inequalities, one has

$$
\|u\|_{L^\infty_y L^q_x}^q \leq \int_0^1 \|u\|_{L^2_x}^q \|\partial_x u\|_{L^2_x}^q dy,
$$

hence, applying Hölder’s and Minkowski’s inequalities lead to

$$
\|u\|_{L^\infty_y L^q_x}^q \leq \|u\|_{L^2_y}^q \|\partial_x u\|_{L^2_x}^q. \tag{2.5}
$$

In the case $2 \leq q < 4$, we deduce from the one dimensional fractional Gagliardo Nirenberg type inequality (see [19]) that

$$
\|u\|_{L^2_y}^{\frac{2\alpha}{2\alpha - q}} \lesssim \|u\|_{L^2_y}^{\frac{2\alpha - 2}{2\alpha - q}} \left\| A_0^{\frac{1}{2}} u \right\|_{L^2_x} \left\| D_y^{\frac{2}{q}} u \right\|_{L^2_x}^{\frac{2\alpha - 2}{2\alpha - q}}, \tag{2.6}
$$

which, combined with (2.5), yields to the desired inequality. The case $q = 4$ is deduced thanks again to (2.5) and Lemma 2.2. Thus the lemma is proved.\[\square\]

**Remark 1.** By interpolation, it easily follows from Lemma 2.2 that $\mathcal{H}_\alpha$ is continuously embedded in $L^{p, q}_x$, $\forall p \in [\frac{2\alpha}{2\alpha - 1}, 2\alpha]$ and $\forall q \in [1, +\infty]$. Moreover, Lemma 2.3 implies that $\mathcal{H}_\alpha \hookrightarrow L^{p, q}_x$, $\forall p \in [1, +\infty]$ and $\forall q \in [1, 4]$.\[\square\]
Lemma 2.4. Let $\alpha \in (1, 2)$ and $p \in [1, 6]$. Then there exists $C_{\alpha, p} > 0$ that depends on $\alpha$ and $p$ such that for all $u \in H^\alpha$,  
\[ ||u||_{L^p(\Omega)} \leq \begin{cases} 
C_{\alpha, p} \left( \frac{n}{2} - \frac{1}{p} \right) \left( \frac{1}{A_0} \right)^{\frac{1}{2}} \|u\|_{L^2} + \frac{1}{(p-2)\alpha} \| A_{\alpha}^n u \|_{L^2} & \text{if } 1 \leq p \leq 2 \\
C_{\alpha, p} \left( \frac{n}{2} - \frac{1}{p} \right) \left( \frac{1}{A_0} \right)^{\frac{1}{2}} \|u\|_{L^2} + \frac{1}{(p-2)\alpha} \| D_{\alpha, y} u \|_{L^2} & \text{if } 2 \leq p \leq 6 
\end{cases} \]  
(2.7)

Proof of Lemma 2.4. As an application of the Hölder, Gagliardo-Nirenberg and Minkowski inequalities, we have for $2 \leq p \leq 6$,  
\[ ||u||_{L^p(\Omega)} \lesssim ||u||_{L^2} \left( \frac{n}{2} - \frac{1}{p} \right) \|u\|_{L^2} + \left[ \frac{p}{2} (p-2) \right] \| \partial_x u \|_{L^2}, \]  
(2.8)

which, in accordance with Lemma 2.2 and Remark 1, yields to the desired inequality. The case $p \in [1, 2]$ follows promptly from the Hölder inequality and the following inequality (see proof of Lemma 2.1)  
\[ ||u||_{L^p} \leq C_p \| u \|_{L^2} \left( \frac{n}{2} - \frac{1}{p} \right) ||ux||_{L^2}. \]

Thus the proof is achieved. \qed

In the sequel, we aims to prove that $D(A_\alpha)$ is continuously embedded in $L^\infty(\Omega)$. For this reason, we recall the following $L^\infty$–estimate for the $A_0$–eigenfunctions $(h_n)_n$ defined by (1.10) (see [30], Corollary 3.2 and [9], Lemma 2) that reads  
\[ ||h_n||_{L^\infty(\mathbb{R})} \lesssim (1 + 2n)^{-\frac{1}{\alpha}}, \quad n \in \mathbb{N}. \]  
(2.9)

Lemma 2.5. Let $\alpha \in (1, 2)$. Then the Hilbert space $D(A_\alpha)$ is continuously embedded in $L^\infty(\Omega)$.

Proof of Lemma 2.5. We write  
\[ u(x, y) = \sum_{n \geq 0, k \geq 1} (u, \psi_{n, k}) \psi_{n, k}(x, y) = \sum_{n \geq 0, k \geq 1} \lambda_{n, k}(u, \psi_{n, k}) \frac{\psi_{n, k}(x, y)}{\lambda_{n, k}}. \]

where we recall that $\psi_{n, k}(x, y) = \sqrt{2} h_n(x) \sin(k\pi y)$ and $\lambda_{n, k} = 2n + 1 + (k\pi)^\alpha$. Thanks to estimate (2.9) and the Cauchy-Schwarz inequality one has  
\[ ||u||_{L^\infty_y} \lesssim ||A_\alpha u||_{L^2} \left( \sum_{n, k \geq 1} \frac{1}{n^\frac{\alpha}{2}(n^2 + k^2)} \right)^\frac{1}{2}. \]  
(2.10)

Since $\alpha > 1$, $\sum_{n, k \geq 1} \frac{1}{n^\frac{\alpha}{2}(n^2 + k^2)} < +\infty$ and the proof of the lemma is achieved. \qed

2.2. Strichartz type estimates.

Definition 2.6. Following standard notations, we say that a pair $(q, p)$ is admissible if $2 \leq q, p \leq +\infty$ and  
\[ \frac{2}{q} = \frac{1}{2} - \frac{1}{p}. \]

In order to seek vector valued Strichartz estimates i.e. estimates in $L^q_t L^p_x(\mathbb{R}^2)$ using the real axis Strichartz estimates in $L^q_t L^p_x$ spaces, let us introduce the one dimensional Hermite operator $A_0 = -\frac{d^2}{dx^2} + x^2$, $x \in \mathbb{R}$ (the harmonic oscillator). We recall from [10], thanks to the Mehler’s formula, that for $u_0 \in L^2_x$,  
\[ e^{itA_0} u_0(x) = \int_{\mathbb{R}} G_t(x, s) u_0(s) \, ds, \quad \forall t \in \mathbb{R} \setminus \left\{ \frac{\pi}{2} \right\}, \]  
(2.11)
Moreover, for all admissible pair $(q, p)$ be an admissible pair. Then, there exists $C = C(q) > 0$ such that for all $u_0 \in \mathbb{L}^2_x$

$$\|e^{itA_0}u_0\|_{L^p(\mathbb{R}, \mathbb{L}^q_x)} \leq C \|u_0\|_{\mathbb{L}^2_x}. \quad (2.12)$$

Moreover, for every admissible pair $(\delta, \rho)$ there exists $C = C(\delta, q) > 0$ such that for all $f \in \mathbb{L}^{\delta'}_x \mathbb{L}^\rho_x'$

$$\left\| \int_0^t e^{i(t-s)A_0}f(s) \, ds \right\|_{L^q([0, T], \mathbb{L}^\rho_x)} \leq C \|f\|_{L^{\rho'}([0, T], \mathbb{L}^{\delta'}_x)} \quad (2.13)$$

where $\delta'$ and $\rho'$ denote respectively the conjugate exponent of $\delta$ and $\rho$.

**Proof of Proposition 2.7.** As a consequence of (2.11), it follows by interpolation that for $0 < t < T$, $T$ small enough (say $0 < T < \frac{\pi}{4}$) and for $p, p' \geq 2$ such that $\frac{1}{p} + \frac{1}{p'} = 1$

$$\|e^{itA_0}u_0\|_{\mathbb{L}^q_x} \leq \frac{C}{|t|^{\frac{1}{2} - \frac{1}{p}}} \|u_0\|_{\mathbb{L}^{\rho'}_x}, \quad C > 0. \quad (2.14)$$

Hence, thanks to Hardy-Littlewood-Sobolev inequality, it is standard to prove (2.12) and (2.13) (see [11, 28]). We then skip the details and the proof is completed. \( \square \)

**Proposition 2.8.** Let $T$ be fixed small enough, say $0 < T < \frac{\pi}{4}$. Then the following properties hold:

(i) For every admissible pair $(q, p)$, there exists $C_1 > 0$, depending only on $q$ such that $\forall \varphi \in \mathbb{L}^2$,

$$\|e^{itA_0}\varphi\|_{L^q(\mathbb{R}, \mathbb{L}^p_x)} \leq C_1 \|\varphi\|_{\mathbb{L}^2}. \quad (2.15)$$

(ii) Let $(\rho, \gamma)$ be an admissible pair. Then for every admissible pair $(q, p)$, there exists $C_2 > 0$ that depends only on $\rho$ and $q$ such that $\forall \varphi \in \mathbb{L}^{\rho'}([0, T], \mathbb{L}^{\gamma'}_x \mathbb{L}^2_y)$,

$$\left\| \int_0^t e^{i(t-s)A_0}\varphi(s) \, ds \right\|_{L^q([0, T], \mathbb{L}^{\rho}_x \mathbb{L}^{2}_y)} \leq C_2 \|\varphi\|_{L^{\rho'}([0, T], \mathbb{L}^{\gamma'}_x \mathbb{L}^2_y)}. \quad (2.16)$$

**Proof of Proposition 2.8.** Roughly speaking, the estimates (2.15) and (2.16) are straightforward applications of Proposition 2.7 and the Minkowski inequality. Thus it will omitted for the sake of conciseness. \( \square \)

A mere consequence of Proposition 2.8 is stated as follows

**Corollary 1.** Let $T > 0$ as in Proposition 2.8, $s \geq 0$ and $(q, p)$ an admissible pair. Then, for all $\varphi \in \mathbb{L}^2_x \mathbb{H}^k_y$,

$$\|e^{itA_0}\varphi\|_{L^q(\mathbb{R}, \mathbb{L}^p_x \mathbb{H}^k_y)} \lesssim \|\varphi\|_{\mathbb{L}^2 \mathbb{H}^k_y}. \quad (2.17)$$

Moreover, for all admissible pair $(\rho, \gamma)$ and for all $\varphi \in \mathbb{L}^{\rho'}([0, T], \mathbb{L}^{\gamma'}_x \mathbb{H}^k_y)$,

$$\left\| \int_0^t e^{i(t-s)A_0}\varphi(s) \, ds \right\|_{L^q([0, T], \mathbb{L}^p_x \mathbb{H}^k_y)} \lesssim \|\varphi\|_{L^{\rho'}([0, T], \mathbb{L}^{\gamma'}_x \mathbb{H}^k_y)}. \quad (2.18)$$
In particular, for $s > \frac{1}{2}$ the continuous embedding of $H^{s}_{p}(\{0, 1\})$ into $L^{p}_{r}(\{0, 1\})$ for all $r \in [2, +\infty]$ leads, in accordance with Corollary 1, to the following statement that reads

**Corollary 2.** Let $T > 0$ as in Proposition 2.8, $s > \frac{1}{2}$ and $(q, p)$ an admissible pair. Then, for all $\varphi \in L^{2}_{s}H^{s}_{p}$,
\[
\|e^{itA_{\alpha}}\varphi\|_{L^{q}(\mathbb{R}, L^{p}(\Omega))} \lesssim \|\varphi\|_{L^{2}_{s}H^{s}_{p}}. \tag{2.19}
\]
Moreover, for all admissible pair $(\rho, \gamma)$ and for all $\varphi \in L^{\rho'}([0, T], L^{\gamma'}H^{s}_{p})$,
\[
\left\| \int_{0}^{t} e^{i(t-s)A_{\alpha}}\varphi(s) \, ds \right\|_{L^{\rho}(\{0, T\}, L^{\rho}(\Omega))} \lesssim \|\varphi\|_{L^{\rho'}([0, T], L^{\gamma'}H^{s}_{p})}. \tag{2.20}
\]

**Remark 2.** It is important, for the sequel, to point out that the statements of Proposition 2.8 as well as Corollary 1 and Corollary 2 are still valid if one replace $A_{\alpha}$ by $A_{\alpha} = A_{\alpha} + i\gamma$ (see [28]).

3. The initial value problem and the existence of the global attractor.

3.1. Well-Posedness of the Cauchy problem. Within this framework, the current subsection aims to establish the first part of Theorem 1.2 that is

**Proposition 3.1.** Let $f \in L^{2}$ and $\alpha \in (1, 2)$. Then, under the assumption (1.5) and for every $u_{0} \in \mathcal{H}_{\alpha}$, there exists $T^{*} > 0$ and a unique solution to the problem (1.1)-(1.2) satisfying
\[
u \in \mathcal{C}([0, T^{*}), \mathcal{H}_{\alpha}) \cap L^{4}([0, T^{*}), L^{\infty}(\Omega)), \quad \text{and} \quad \sup_{t \in [0, T^{*})} \|u(t)\|_{\mathcal{H}_{\alpha}} \leq C,
\]
where $\mathcal{C}([0, T^{*}), \mathcal{H}_{\alpha})$ stands for the space of continuous functions which take values in $\mathcal{H}_{\alpha}$ and $C$ is a nonnegative real constant that only depends on $\|u_{0}\|_{\mathcal{H}_{\alpha}}$, $\|f\|_{L^{2}}$ and $\gamma$. Moreover, the maps $S(t) : u_{0} \mapsto u(t)$ are continuous on $\mathcal{H}_{\alpha}$.

**Proof of Proposition 3.1.** Let $T > 0$ small enough as in Proposition 2.8 and $R > 0$ be a fixed reels that will be chosen later. We introduce the set
\[
\mathcal{B}_{\alpha}(0, R) = \{u \in \mathcal{H}_{\alpha} \text{ such that } \|u\|_{L^{\infty}_{\alpha}} \leq R \text{ and } \|u\|_{L^{4}(\{0, T\}, L^{\infty}_{\alpha})} \leq R\},
\]
denotes the closed ball of $\mathcal{H}_{\alpha}$ with radius $R$. Equipped with the distance
\[
d(u, v) = \|u - v\|_{L^{\infty}_{\alpha}(\{0, T\}, \mathcal{H}_{\alpha})} + \|u - v\|_{L^{4}(\{0, T\}, L^{\infty}_{\alpha})},
\]
$\mathcal{B}_{\alpha}(0, R)$ is a complete metric space (see [7] for instance). We will be regularly referred to the following identity that reads
\[
g(|u|^{2})u - g(|v|^{2})v = \int_{0}^{1} [g(|\theta(\tau)|^{2})z + 2g'(|\theta(\tau)|^{2})\theta(\tau) \Re(\overline{\theta(\tau)z})] \, d\tau, \tag{3.1}
\]
where $z = u - v$ and $\theta(\tau) = \tau(u - v) + v$. We proceed in two steps

**Step 1:** A local-in-time solution. We apply Duhamel’s formula to (1.1) – (1.2),
\[
u(t) = e^{itA_{\alpha}}u_{0} + \int_{0}^{t} e^{i(t-s)A_{\alpha}} F(u(s)) \, ds,
\]
where $F(u) = f - ig(|u|^2)u$ and $\Lambda_\alpha = A_\alpha + i\gamma$ and then we introduce for a fixed $u_0 \in \mathscr{H}_\alpha$

$$
\psi : v \mapsto \psi(v) = \int_0^t e^{i(t-s)\Lambda_\alpha} F(u(s)) \, ds \quad \text{and} \quad U : v \mapsto U(v) = e^{i\Lambda_\alpha} u_0 + \psi(v).
$$

For the sake of simplicity we denote $L_r^\alpha([0, T]) = L_r^T$, $r \in [1, +\infty]$. To begin with,

**Lemma 3.2.** $\psi$ is a Lipschitz mapping on bounded subsets of $X_\alpha$.

**Proof of Lemma 3.2.** Let $z = u - v$. Thanks to Proposition 2.8, Remark 2 and (3.1),

$$
||\psi(u) - \psi(v)||_{L_r^\alpha L^2} \leq T^{\frac{2-\sigma}{\sigma}} (||u||_{L^{2\sigma}_r L^\infty_v} + ||v||_{L^{2\sigma}_r L^\infty_v}) ||z||_{L_r^\alpha L^2}.
$$

(3.2)

Independently, from Corollary 2, Remark 2 and (3.2) it follows that

$$
||\psi(u) - \psi(v)||_{L_r^\alpha L^2} \leq \sup_{\tau \in [0, 1]} \left( ||D_{\theta} \overline{g}([|\theta|^2])z||_{L^2_{\tau L^2}} + ||D_{\theta} \overline{g}([|\theta|^2] \theta \Re(z))||_{L^2_{\tau L^2}} \right).
$$

(3.3)

Firstly, under assumption (1.5) and thanks to the Kato-Ponce inequality (see [24])

$$
||D_{\theta} \overline{g}([|\theta|^2])z||_{L^2_{\tau L^2}} \leq ||\theta||_{L^\infty_v} ||z||_{L^2_{\tau L^2}} + ||z||_{L^\infty_v} ||D_{\theta} \overline{g}([|\theta|^2])||_{L^2_{\tau L^2}}.
$$

Secondly, in accordance with (1.9) we have

$$
||D_{\theta} \overline{g}([|\theta|^2])||_{L^2_{\tau L^2}} \leq ||\theta||_{L^\infty_v} ||D_{\theta} \overline{g}||_{L^2_{\tau L^2}}.
$$

Hence, thanks to the H"{o}lder inequality

$$
\sup_{\tau \in [0, 1]} \left( ||D_{\theta} \overline{g}([|\theta|^2])z||_{L^2_{\tau L^2}} \right) \leq T^{\frac{2-\sigma}{\sigma}} (||u||_{L^{2\sigma}_r L^\infty_v} + ||v||_{L^{2\sigma}_r L^\infty_v}) ||z||_{L_r^\alpha L^2}.
$$

(3.4)

Similarly, using an analogous relation to (3.1) and under assumption (1.7), we obtain that

$$
||D_{\theta} \overline{g}([|\theta|^2] \theta)||_{L^2_{\tau L^2}} \leq ||D_{\theta} \overline{g}||_{L^2_{\tau L^2}}.
$$

(3.5)

Thus, the Kato-Ponce inequality and H"{o}lder’s inequality imply that

$$
\sup_{\tau \in [0, 1]} \left( ||D_{\theta} \overline{g}([|\theta|^2] \theta \Re(z))||_{L^2_{\tau L^2}} \right) \leq T^{\frac{2-\sigma}{\sigma}} (||u||_{L^{2\sigma}_r L^\infty_v} + ||v||_{L^{2\sigma}_r L^\infty_v}) ||z||_{L_r^\alpha L^2}.
$$

(3.6)

Finally, we gather (3.5) and (3.7), the estimates (3.3) and (3.2) imply that for a given $R > 0$ and for $u, v \in \mathbb{B}_{X_\alpha}(0, R)$,

$$
||\psi(u) - \psi(v)||_{L_r^\alpha \mathscr{H}_\alpha} \lesssim \left( T^{\frac{2-\sigma}{\sigma}} R^{2\sigma} + T^{\frac{1}{2}} R^2 + T^{\frac{1}{4}} R \right) ||u - v||_{X_\alpha}.
$$
By similar computations, thanks to Corollary 2 and Proposition 2.8, we deduce that for \( u, v \in B_{X_s}(0, R) \)
\[
\| \psi(u) - \psi(v) \|_{X_s} \lesssim \left( T^{\frac{2-\alpha}{\alpha}} R^{2\sigma} + T^{\frac{1}{2}} R^2 + T^{\frac{3}{4}} R \right) \| u - v \|_{X_s},
\]
(3.8)
and the proof is completed. \( \square \)

To conclude the first step, it is enough to prove the following lemma that reads

**Lemma 3.3.** \( U \) is a contraction mapping from \( B_{X_s}(0, R) \) into itself.

**Proof of Lemma 3.3.** Observe that in accordance with Lemma 2.5,
\[
\left\| \int_0^t e^{i(t-s)\Lambda_\alpha} f \ ds \right\|_{L^{\frac{\alpha}{\alpha}}_t L^\infty_x} \lesssim \| \Lambda_\alpha^{-1} f \|_{L^\infty_x \mathcal{X}_\alpha} \lesssim \| f \|_{L^2},
\]
(3.9)
and
\[
\left\| \int_0^t e^{i(t-s)\Lambda_\alpha} f \ ds \right\|_{L^1_x L^\infty_x} \lesssim \| f \|_{L^1_x L^\infty_x} \lesssim \| f \|_{L^2},
\]
(3.10)
for a chosen \( 0 < T \leq 1 \). This implies, thanks to (3.8), that for \( \phi \in B_{X_s}(0, R) \)
\[
\| U(\phi) \|_{X_s} \leq C_1(\| u_0 \|_{\mathcal{X}_\alpha} + \| f \|_{L^2}) + C_2 \left( T^{\frac{2-\alpha}{\alpha}} R^{2\sigma} + T^{\frac{1}{2}} R^2 + T^{\frac{3}{4}} R \right) \| \phi \|_{X_s}.
\]
(3.11)
Choosing \( R > 0 \) satisfying \( R \geq 2C_1(\| u_0 \|_{\mathcal{X}_\alpha} + \| f \|_{L^2}) \) and \( 0 < T \leq 1 \) such that
\[
C_2 \left( T^{\frac{2-\alpha}{\alpha}} R^{2\sigma} + T^{\frac{1}{2}} R^2 + T^{\frac{3}{4}} R \right) \leq \frac{1}{2},
\]
where we recall that \( 0 < \sigma < \frac{2\alpha}{\alpha + 2} < 1 \), achieves the proof since
\[
\| U(\phi_1) - U(\phi_2) \|_{X_s} = \| \psi(\phi_1) - \psi(\phi_2) \|_{X_s}. \quad \square
\]

Thanks to Lemmas 3.2 and 3.3, the existence of a unique local-in-time solution for \((1.1) - (1.2)\) is obtained by a fixed point argument.

**Step 2:** Global (in time) existence in \( \mathcal{X}_\alpha \). The proof is classical and then details are omitted. Roughly speaking, the global existence is classical and easily deduced from the following energy equation that reads
\[
\frac{1}{2} \frac{d}{dt} J(u(t)) + \gamma J(u(t)) = K(u),
\]
(3.12)
where
\[
J(u) = \| A_\alpha^\frac{1}{2} u(t) \|_{L^\infty_x}^2 - \int G(|u|^2) \ dx \ dy - 2(i, u),
\]
(3.13)
\[
K(u) = \gamma \left[ (g(|u|^2), |u|^2) - (i, u) - \int G(|u|^2) \ dx \ dy \right],
\]
(3.14)
with \( G(s) = \int_0^s g(r) \ dr \). In fact, one only has to write
\[
\left\| \int_0^1 G(|u|^2) \ dx \ dy \right\| \leq \int_0^1 \int_0^1 |g(s)| \ ds \ dx \ dy
\]
Hence, under assumption (1.5) and thanks to Lemma 2.4
\[
\left\| \int_\Omega G(|u|^2) \ dx \ dy \right\| \lesssim \| u \|_{L^{2\sigma+2}_{xy}}^{2\sigma+2} \lesssim \| u \|_{L^{2\sigma+2}_{xy}} \| u \|_{\mathcal{X}_{\alpha}}^{2\sigma(\frac{1}{2} + \frac{1}{\alpha})}.
\]
This concludes the second step, knowing that the scalar product of \((1.1)\) by \( u \) leads to \( \| u \|_{L^2_{xy}} \leq C \), and then the proof of the proposition is achieved. \( \square \)
The semigroup \((S_\alpha(t))_{t \in \mathbb{R}_+}\) associated to (1.1) is well defined. At the beginning we highlight its dissipation.

**Proposition 3.4.** The semigroup \((S_\alpha(t))_{t \in \mathbb{R}_+}\) possesses a bounded absorbing ball \(B_{\mathcal{H}_\alpha}\) in \(\mathcal{H}_\alpha\), i.e.: for any bounded subset \(B \subseteq \mathcal{H}_\alpha\) there exists \(t(B) > 0\) such that
\[
S_\alpha(t)B \subseteq B_{\mathcal{H}_\alpha}, \quad \forall t \geq t(B).
\]

**Proof of Proposition 3.4.** The proof is very standard and follows from (3.12) then we omit it and we refer the reader to [33] for a complete proof. \(\square\)

### 3.2. Existence of the global attractor

The existence of an absorbing set was the first step toward the existence of the global attractor.

**Theorem 3.5.** The semigroup \((S_\alpha(t))_{t \in \mathbb{R}_+}\) associated to the dynamical system defined by (1.1) possesses a compact global attractor \(\mathcal{A}_\alpha\) in \(\mathcal{H}_\alpha\).

Thanks to Theorem 1.1 and Remark 1.4 in [43] and Theorem 5.1 in [14], one only has to prove the asymptotic compactness of the semi-group \((S_\alpha(t))_{t \in \mathbb{R}_+}\), that is

**Lemma 3.6.** The semi-group \((S_\alpha(t))_{t \in \mathbb{R}_+}\) is asymptotically compact in \(\mathcal{H}_\alpha\), i.e., for every bounded sequence \((x_k)_k\) in \(\mathcal{H}_\alpha\) and every sequence \(t_k \rightarrow +\infty\), \((S_\alpha(t_k)x_k)_k\) is relatively compact in \(\mathcal{H}_\alpha\).

**Proof of Lemma 3.6.** the proof is very standard, we sketch it for the sake of conciseness. By using the well known John Ball’s argument (see [8] and [44]), let \((u_j)_j\) be a bounded sequence in \(\mathcal{H}_\alpha\) and \(t_j \rightarrow +\infty\). Since \((S_\alpha(t_j)u_j)_j\) is bounded in \(\mathcal{H}_\alpha\) and in accordance with the compact embedding of \(H^s\) into \(L^p\), \(p \in [1, 6]\), we may assume, up to a subsequence extraction, the existence of \(a \in \mathcal{H}_\alpha\) such that
\[
S(t_j)u_j \rightharpoonup a \text{ weakly in } \mathcal{H}_\alpha \quad \text{and} \quad S(t_j)u_j \rightarrow a \text{ strongly in } L^2. \quad (3.15)
\]

Thanks to the energy equation (3.12), we deduce that
\[
J(S_\alpha(t_j)u_j) = J(S_\alpha(t_j-t)u_j) e^{-2\gamma t} + 2 \int_0^t e^{-2\gamma(t-s)} K(S_\alpha(t_j-t+s)u_j) ds \quad (3.16)
\]
and
\[
J(a) = J(S_\alpha(-t)a) e^{-2\gamma t} + \int_0^t e^{-2\gamma(t-s)} K(S_\alpha(s-t)a) ds, \quad (3.17)
\]
where we recall that \(J\) and \(K\) are defined by (3.13) and (3.14).

Thanks to the dominated convergence Theorem, (3.16) and (3.17),
\[
\limsup J(S_\alpha(t_j)u_j) \leq J(a)
\]
from which we deduce, thanks again to (3.15), that \(|S_\alpha(t_j)u_j|_{\mathcal{H}_\alpha} \rightarrow ||a||_{\mathcal{H}_\alpha}\) as \(j \rightarrow +\infty\). This concludes the proof of the current lemma as well as the proof of Theorem 3.5. \(\square\)

### 4. The regularity of the global attractor

We claim now to prove an asymptotic smoothing effect of the dynamical system defined by (1.1) thought the study of the regularity of the global attractor \(\mathcal{A}_\alpha\). This main result is stated as follows

**Theorem 4.1.** The global attractor \(\mathcal{A}_\alpha\) for the semi group \((S_\alpha(t))_{t \in \mathbb{R}_+}\) defined by the problem (1.1)-(1.2) is a compact subset of \(D(A_\alpha)\).
For that purpose, we shall briefly introduce some tools from harmonic analysis that will be used extensively in the sequel. Let $1 < p < +\infty$ and setting
\[
D_p(A_0^\frac{\alpha}{2}) = \{ u \in L^p(\mathbb{R}; A_0^\frac{\alpha}{2} u \in L^p_y) \}.
\]
Then the following statement hold true

**Lemma 4.2.** For all $u \in D_p(A_0^\frac{\alpha}{2})$, we have the following equivalence between norms
\[
\left| | A_0^\frac{\alpha}{2} u \right|_{L^p(\mathbb{R};)} \sim \left| | A_0^\frac{\alpha}{2} u \right|_{L^p_y} + \left| | D_y^\frac{\alpha}{2} u \right|_{L^p_y} + \left| | \partial_y u \right|_{L^p_y} + \left| | D_y^\alpha u \right|_{L^p_y}.
\]

**Proof of Lemma 4.2.** the proof of the first equivalence is identical to that in [3], Theorem 3.2, and then details are omitted since both of the operators $A_0 = -\partial_x^2 + x^2$ and $D_y^\alpha$ belongs to the class of bounded imaginary powers in $L^p_y$ for all $p \in (1, +\infty)$. The second one follows from Theorem 3.2 in [3] to which we refer the reader for a complete proof.

Now, for a given positive integer $N$, we denote $P_N$ and $Q_N = id - P_N$ the orthogonal projectors acting in $L^2_x(\mathbb{R}, H)$ by setting
\[
P_N(u)(x, y) = \sum_{k=1}^{N} \sum_{n=1}^{N} \tilde{u}_{n,k} \psi_{n,k}(x, y), \quad \tilde{u}_{n,k} = (u, \psi_{n,k})
\]
where we recall that $H = \{ u \in L^2([0, 1]) | u(0) = u(1) = 0 \}$ and
\[
\psi_{n,k}(x, y) = \left( 2^{n-1} \pi^{\frac{1}{4}} n! \right)^{-\frac{1}{2}} H_n(x) e^{-\frac{x^2}{4}} \sin(k\pi y), \quad (n, k) \in \mathbb{N} \times \mathbb{N}^*.
\]
Actually, $P_N$ is the projector onto the low-frequencies modes of a given function, at level $N$. Clearly, $A_0 P_N = P_N A_0$. Moreover, $P_N$ and $Q_N$ are bounded operators from $D(A_0^s)$, $s \geq 0$, into itself and satisfy the following inequalities that state as follows

**Lemma 4.3.** Let $0 \leq s_1 \leq s_2$. Then there exists $C > 0$, that does not depends on $N$, such that
\[
\left| | P_N(u) \right|_{D(A_0^{s_1})} \leq C N^{(s_2-s_1)} \left| | P_N(u) \right|_{D(A_0^{s_1})}, \quad \forall u \in D(A_0^{s_1}).
\]

**Proof of Lemma 4.3.** The proof follows merely from the very definition of $P_N$. \hfill $\square$

Besides, the following statement holds true

**Lemma 4.4.** Let $p \in (\frac{4}{3}, 4)$. Then there exists $C_p > 0$ depending only on $p$ such that for any $u \in L^p_y$,
\[
\left| | P_N(u) \right|_{L^p_x L^2_y} \leq C_p \left| | u \right|_{L^p_x L^2_y}.
\]

**Proof of Lemma 4.4.** The proof follows promptly from a well known result of R. Askey and S. Wainger (see [6] and [38]) that is,

The orthogonal projector $P_N$ defined from $L^2_x$ onto the finite dimensional subset generated by the $N$-first eigenfunctions of the operator $A_0 = -\partial_x^2 + x^2$,
\[
\left\{ h_n : x \mapsto \frac{(-1)^n}{\pi^{\frac{1}{2}} 2^{\frac{1}{2}} (n!)^{\frac{1}{2}}} H_n(x) e^{-\frac{x^2}{4}}, \quad 0 \leq n \leq N \right\},
\]
\[\]
is a bounded operator from $\mathbb{L}^2_x$ into itself if and only if $p \in \left(\frac{4}{3}, 4\right)$.

As a consequence of the previous lemma and the Minkowski inequality, we have the following space-time estimate that states as follows

**Lemma 4.5.** Let $T > 0$ small enough as in Proposition 2.8 and $(\delta, \rho)$ an admissible pair such that $\rho \in (2, 4)$. Then for every admissible pair $(q, p)$, there exists $C = C(\rho, q) > 0$ such that for all $t \in [0, T]$ and for all $f \in \mathbb{L}^{\delta'}([0, T], \mathbb{L}^p_x \mathbb{L}^q_y)$

$$
\left\| \mathbb{P}_N \int_0^t e^{i(t-s)\Lambda_\alpha} f(s) \, ds \right\|_{\mathbb{L}^{\delta'}([0, T], \mathbb{L}^p_x \mathbb{L}^q_y)} \leq C \left\| f \right\|_{\mathbb{L}^{\delta'}([0, T], \mathbb{L}^p_x \mathbb{L}^q_y)}. 
$$

(4.6)

**Proof of Lemma 4.5.** As in the proof of Lemma 4.4, let $P_N$ be the orthogonal projector from $\mathbb{L}^2_x$ onto the finite dimensional subset defined by (4.5). Then denoting $\hat{f}_k(t, x) = \sum_{n \geq 1} (f, \psi_{n,k}) h_n(x)$ and for the sake of simplicity $L^2_t = L^2([0, T])$, it follows from Lemma 4.4, Proposition 2.7 and the Minkowski inequality that

$$
\left\| \mathbb{P}_N \int_0^t e^{i(t-s)\Lambda_\alpha} f(s) \, ds \right\|_{\mathbb{L}^{\delta'}(0, T, \mathbb{L}^p_x \mathbb{L}^q_y)} \leq \left( \sum_{k \geq 1} \left\| \int_0^t e^{i(t-s)\Lambda_\alpha} P_N \hat{f}_k(s, x) \, ds \right\|_{\mathbb{L}^{\delta'}(0, T, \mathbb{L}^p_x \mathbb{L}^q_y)}^2 \right)^{\frac{1}{2}}
$$

$$
\leq \left( \sum_{k \geq 1} \left\| \hat{f}_k(s, x) \right\|_{\mathbb{L}^{\delta'}(0, T, \mathbb{L}^p_x \mathbb{L}^q_y)}^2 \right)^{\frac{1}{2}} \lesssim \left\| \left( \sum_{k \geq 1} |\hat{f}_k(s, x)|^2 \right)^{\frac{1}{2}} \right\|_{\mathbb{L}^{\delta'}(0, T, \mathbb{L}^p_x \mathbb{L}^q_y)} \lesssim \left\| f \right\|_{\mathbb{L}^{\delta'}(0, T, \mathbb{L}^p_x \mathbb{L}^q_y)}
$$

and the proof is completed.

We now turn to establish an important mixed space-time estimate

**Proposition 4.6.** Let $T > 0$ small enough as in Proposition 2.8 and $(q, p)$ an admissible pair. Consider $u(t)$, solution of (1.1)-(1.2), a complete trajectory included in $\mathcal{H}_\alpha$. We denote by $t_0$ its time entrance in the bounded absorbing set $\mathcal{B}_{\mathcal{H}_\alpha}$, the ball with radius $M > 0$ and for the sake of simplicity we suppose $t_0 = 0$. Then there exist $K_1, K_2 > 0$ that depend on $T$, $\gamma$, $\sigma$, $M$ and $f$, such that for $t \geq 0$, the following estimates hold

- $\left\| \mathcal{L}(u) \right\|_{\mathbb{L}^2_t^{\alpha} \mathbb{L}^\infty_x} \leq K_1$.  
- $\left\| \mathcal{L}(u) \right\|_{\mathbb{L}^2_t \mathbb{L}^2_x} + \left\| A_\beta^2 \mathcal{L}(u) \right\|_{\mathbb{L}^2_t^{\alpha} \mathbb{L}^2_x} \leq K_2$

(4.7)

(4.8)

where $\mathcal{L}$ denotes either $\text{id}$ or $\mathbb{P}_N$.

**Proof of Proposition 4.6.** We first prove (4.7). Using the Duhamel’s formulation for (1.1)

$$
u(t) = e^{i(t-t_0)\Lambda_\alpha} u(t) + \int_t^{t_0} e^{i(t-s)\Lambda_\alpha} F(u(s)) \, ds
$$

where $\Lambda_\alpha = A_\alpha - i\gamma$ and $F(u) = f - ig(|u|^2)u$, we deduce from Corollary 2 and Remark 1 that for all admissible pair $(\delta, \rho)$ such that the conjugate exponent $\rho' > 1$ we have

$$
\left\| u \right\|_{\mathbb{L}^{\delta'}(0, T, \mathbb{L}^p_x \mathbb{L}^q_y)} \lesssim \left\| u(t) \right\|_{\mathcal{H}_\alpha} + T^{\frac{1}{2}} \left\| f \right\|_{\mathbb{L}^2} + \left\| g(|u|^2)u \right\|_{\mathbb{L}^{\delta'}(0, T, \mathbb{L}^p_x \mathbb{L}^q_y)}. 
$$

(4.9)
\[ \lesssim C(T, M, \|f\|_{L^2}) + \left\| \left\| u \right\|_{L^{\frac{2\sigma}{2\sigma - \rho}} L_{t}^{\rho}} + \left\| D_{\sigma}^{\gamma} u \right\|_{L_{t}^{2}} \right\|_{L_{t}^{\frac{2\sigma}{2\sigma - \rho}}} \]
\[ \lesssim C(T, M, \|f\|_{L^2}) + M \left\| \left\| u \right\|_{L^{\frac{2\sigma}{2\sigma - \rho}} L_{t}^{\rho}} \right\|_{L_{t}^{\prime}}. \quad (4.10) \]

For the case \( \sigma \in (0, \frac{2}{7}) \), choosing \((\delta, \rho)\) such that
\[ \max \left(1, \frac{2}{1 + 2\sigma}\right) < \rho \leq \frac{2\alpha}{\alpha + 2\sigma} \quad (4.11) \]
ensure that \(\frac{4\sigma'}{2 - \rho'} \in [2, 2\alpha]\) and then, in accordance with Lemma 2.2
\[ \left\| \left\| u \right\|_{L^{\frac{2\sigma'}{2\sigma' - \rho'}} L_{t}^{\rho'}} \right\|_{L_{t}^{\prime}} \lesssim M^{2\sigma' T_{\frac{2 - \sigma}{2}}} \quad (4.12) \]
which, with (4.10), yields (4.7). However, in the case \( \sigma \in \left[\frac{\alpha}{7}, \frac{2\alpha}{\alpha + 2}\right) \), thanks again to Lemma 2.2
\[ \left\| \left\| u \right\|_{L^{\frac{2\sigma'}{2\sigma' - \rho'}} L_{t}^{\rho'}} \right\|_{L_{t}^{\prime}} \lesssim M^{\frac{\alpha(2 - \rho')}{\rho}} \left\| \left\| u \right\|_{L_{t}^{\infty}} \right\|^{(2\sigma + \alpha) - \frac{2\rho'}{\rho}}_{L_{t}^{\prime}} \]
\[ \lesssim M^{\frac{\alpha(2 - \rho')}{\rho}} \left( \int_{t}^{t + T} \left\| u(s) \right\|_{L_{t}^{\infty}}^{4\alpha - 2(2\alpha - \sigma)\delta'} ds \right)^{\frac{1}{\delta'}} \]
\[ \lesssim M^{\frac{\alpha(2 - \rho')}{\rho}} T \left( \frac{4\alpha}{3} + \frac{2(2\alpha - \sigma)}{3} \right) \left\| u \right\|_{L_{t}^{\infty}}^{4\left(\frac{\alpha}{3} - \frac{2\alpha - \sigma}{2}\right)}. \quad (4.13) \]

Choosing the admissible pair \((\delta, \rho)\) satisfying
\[ \frac{4\alpha}{4\alpha + 1 - 2\sigma} < \delta' < \frac{4}{3} \quad (4.14) \]
ensure that \(4\left(\frac{\alpha}{3} - \frac{2\alpha - \sigma}{2}\right) < 1\) and then (4.7) yields thanks to the Young inequality, (4.10) and (4.13).

We now proceed, similarly to \(u\) but with a slight difference, to prove (4.7) for \(P_N(u)\).

Thanks to Lemma 4.5 and Lemma 2.2 it follows that for the case \( \sigma < \frac{\alpha}{7} \), recalling that \( \alpha \in (1, 2) \), choosing an admissible pair \((\delta, \rho)\) such that
\[ \max \left(1, \frac{4}{3}, \frac{2}{1 + 2\sigma}\right) < \rho \leq \frac{2\alpha}{\alpha + 2\sigma} \quad (4.15) \]
leads to (4.12) and then (4.7) follows. For the case \( \sigma \in \left[\frac{\alpha}{7}, \frac{2\alpha}{\alpha + 2}\right) \), we choose an admissible pair \((\delta, \rho)\) such that
\[ \begin{cases} \frac{4}{3} < \rho' < 2 \\ \max(1, \frac{4\alpha}{4\alpha + 1 - 2\sigma}) < \delta' < \frac{8}{7} \end{cases} \quad (4.16) \]
leads, in accordance with Lemma 4.5 and Lemma 2.2, to (4.13) and then (4.7) yields thanks to the Young inequality. This concludes the first part of the proposition.

Now, for the second part, we consider an admissible pair \((q, p)\). Then, thanks again to (4.9), Proposition 2.8 and Remark 1 we obtain the following estimate that reads
\[ \left\| A_{\sigma}^{\frac{1}{2}} u \right\|_{L_{t}^{q} L_{x}^{p}} \lesssim \left\| u(t) \right\|_{\mathcal{X}_\omega} + \left\| A_{\sigma}^{\frac{1}{2}} [g(|u|^{2}) u] \right\|_{L_{t}^{q} L_{x}^{p}} \]
Proposition 4.7. Under the same assumptions as Proposition 4.6, there exists 

\( \rho_C > 0 \).

Lemma 4.8. Let \( \sigma \) be a function defined on \( \mathbb{R} \) and \( \gamma > 0 \), \( \tau > 0 \), \( t > 0 \), \( s > 0 \), \( x \in \mathbb{R}^d \), \( y \in \mathbb{R}^d \), \( p \) be a positive integer.

Proof of Proposition 4.7. The proof is a straightforward consequence of Proposition 4.6, then details are omitted for sake of conciseness, thanks to Proposition 2.8. Then there exist \( \rho_C > 0 \).

Proposition 4.9. Under the same assumptions as Proposition 4.6, there exists \( K > 0 \) that depend on \( T, \gamma, \sigma \) and \( f \), such that

\[
\left\| A_{\alpha}^{-\frac{3}{2}} \mathcal{L}(u_t) \right\|_{L_t^p L_x^q L_y^s} \leq K,
\]

where \( \mathcal{L} \) still denotes either \( \text{id} \) or \( \mathbb{P}_N \).

Proof of Proposition 4.7. The proof is a straightforward consequence of Proposition 4.6, then details are omitted for sake of conciseness, thanks to Proposition 2.8, Lemma 4.3 and Remark 1 (for more details see [3] for instance).

For later use, we need the following straightforward application of the previous proposition.

Lemma 4.8. Let \( T > 0 \) small enough as in Proposition 2.8. Then there exists \( C > 0 \) depending only on \( T, \gamma, \sigma \) and \( f \) such that for all admissible pairs \( (p, q) \) and for all \( t \geq 0 \) the following estimate holds

\[
\int_0^t e^{-\gamma(t-s)} \left\| A_{\alpha}^{-\frac{3}{2}} u_t \right\|_{L_t^p L_x^q L_y^s}^q \, ds \leq C.
\]

4.1. The auxiliary problem. Let \( u(t) \) be the solution of (1.1)-(1.2) which takes values in \( \mathcal{A}_\alpha \). As the previous section, we may assume that \( u(t) \) remains into the \( \mathcal{A}_\alpha \)-absorbing set, \( \mathcal{A}_{\mathcal{A}_\alpha} \), for \( t \geq 0 \).

To highlight the regularity of the global attractor \( \mathcal{A}_\alpha \), we split the solution as \( u(t) = \mathbb{P}_N(u) + \mathbb{Q}_N(u) = v(t, x, y) + w(t, x, y) \) and then, thanks to Lemma 4.3, the regularity of \( u \) depends only on the regularity of \( w \). Therefore, we shall focus on the long-time behavior of \( w(t) \) and for that purpose we approximate \( w \), solution for \( \mathbb{Q}_N(1.1) \) supplemented with initial data \( w(0) = \mathbb{Q}_N(u(0)) \), by a more regular function \( W \) which solves the following auxiliary problem that reads

\[
W_t - i A_{\alpha} W + i \mathbb{Q}_N [g(|v + W|^2)(v + W)] + \gamma W = \mathbb{Q}_N(f) \quad (4.19)
\]
\[
W(0) = 0. \quad (4.20)
\]

Proposition 4.9. Let \( f \in L^2 \) and \( T > 0 \) chosen small enough as in Proposition 2.8. Then there exist \( N_0 \); a positive integer chosen large enough depending only on \( \gamma \), \( \sigma \) and \( f \) such that (4.19)-(4.20) has a unique solution \( W \) belonging to \( \mathcal{C}_b(\mathbb{R}_+, \mathbb{Q}_N(D(A_{\alpha}))) \). Moreover, there exist two positive constants \( C = \)
Proof of Proposition 4.9. The proof is divided into three steps and sketched for the sake of conciseness. For convenience use, let us set $\Lambda = v + W$.

Step 1. Existence of a local (in-time) solution in $Q_N(D(A))$. Since, thanks to Lemma 2.5, $D(A)$ is a Banach algebra and then by applying standard fixed point argument (without recourse to use Strichartz type estimates) we show the existence of a local-in-time solution $W$ for (4.19)-(4.20) which belongs to $\mathcal{C}_b([0,T^*], Q_N(D(A)))$.

Moreover, this local-in-time solution; defined from a maximal interval $[0,T^*)$; satisfies the following alternative either $T^* = +\infty$ or $||A_n W(t)||_{L^2} \rightarrow +\infty$ as $t \rightarrow T^*, t < T^*$.

Step 2. Existence of a global (in-time) solution in $Q_N(\mathcal{H})$. The scalar product of (4.19) by $W$ then by $-i(W_\gamma + \gamma W)$ lead to the following energy equation

$$\frac{1}{2} \frac{d}{dt} \phi(W) + \gamma \phi(W) = \psi(W),$$

where we set

- $\phi(W) = ||W||^2_{\mathcal{H}} - \int_\Omega [G(|\Lambda|^2) - G(|v|^2)] dx dy - 2(\text{i} f, W)$
- $\psi(W) = 2\gamma \left(g(|\Lambda|^2) - g(|v|^2)\right) - 2\gamma \int_\Omega [G(|\Lambda|^2) - G(|v|^2)] dx dy - 2\gamma (\text{i} f, W) - 2\gamma (g(|v|^2), |v|^2) - 2 \left(g(|\Lambda|^2)|\Lambda| - g(|v|^2)v, v\right) + 2 \left(i g(|\Lambda|^2)|\Lambda|, v\right)$

with $G(s) = \int_0^s g(r) dr$. Thanks to Lemma 2.4 and Lemma 4.3, we easily obtain that for $N$ large enough there exist $C_1, C_2 > 0$ that depend only on $\gamma, \sigma$ and $f$ such that

$$\frac{1}{2} ||W||^2_{\mathcal{H}} - \frac{C_1}{N^p} ||W||^2_{\mathcal{H}} - C_2 \leq \phi(W) \leq \frac{3}{2} ||W||^2_{\mathcal{H}} + \frac{C_1}{N^p} ||W||^2_{\mathcal{H}} + C_2$$

where $\nu = 1 - \sigma (\frac{1}{\sigma} - \frac{1}{2})$.

Now we shall focus on the majorization of $\psi(W)$.

For that purpose and in accordance with Lemmas 2.4 and 4.3, it will be enough to bound $(g(|\Lambda|^2)|\Lambda| - g(|v|^2)v, v)$. Let $p \in (1, +\infty)$, then using the following identity

$$g(|\Lambda|^2)|\Lambda| - g(|v|^2)v = \int_0^1 [g(|\theta|^2)|\Lambda| + 2g(|\theta|^2)\theta \Re(\overline{\theta} W)] ds$$

with $\theta = \theta(s) = s W + v$, we have

$$\left|(g(|\Lambda|^2)|\Lambda| - g(|v|^2)v, v)\right| \leq \sup_{s \in [0,1]} \left\{\left||A_{\alpha}^\frac{\nu}{2} [g(|\theta|^2)\theta \Re(\overline{\theta} W)]\right||_{L^p_x L^q_y} + \left||A_{\alpha}^\frac{\nu}{2} [g(|\theta|^2) W]\right||_{L^p_x L^q_y} \right\} \left||A_{\alpha}^{-\frac{\nu}{2}} v\right||_{L^p_x L^q_y}$$

(4.25)
In the light of the type estimates (3.4) and (3.6), we deduce from Lemma 4.2 and the Hölder inequality that for \( p' \in (\max(1, \frac{2}{\sigma + 1}), 2) \), recalling that \( \sigma \in (0, \frac{2p}{\alpha + 2}) \),

\[
\sup_{s \in [0,1]} \left( \left| A^2 \frac{g(|\theta|^2)W}{L^p_{\infty}} + \left| A^2 \frac{g'((|\theta|^2)\theta Re(\sigma W))}{L^p_{\infty}} \right| \right) \right) \\
\lesssim \|W\|_{\mathscr{H}} \left( 1 + \|W\|_{L^p_{\infty}}^{\frac{2}{\sigma + 1} - \frac{2}{p}} + \|v\|_{L^p_{\infty}}^{\left(\frac{2}{\sigma + 1} - \frac{2}{p}\right)} \right) \\
+ \|W\|_{L^p_{\infty}}^{\left(\frac{2}{\sigma + 1} - \frac{2}{p}\right)} \|v\|_{L^p_{\infty}}^{\left(\frac{2}{\sigma + 1} - \frac{2}{p}\right)} \left( 1 + \|W\|_{\mathscr{H}} \right) \left( \|W\|_{\mathscr{H}} \|v\|_{\mathscr{H}} \right) \\
+ \|W\|_{\mathscr{H}}^{\frac{3}{2} - \frac{1}{p}} \|v\|_{\mathscr{H}}^{\frac{3}{2} - \frac{1}{p}} \left( \|W\|_{\mathscr{H}}^{\frac{1}{2}} + \|W\|_{\mathscr{H}}^{\frac{1}{2} + \frac{1}{2}} \right), \tag{4.26}
\]

which, thanks to Lemma 2.2 and Lemma 4.3, leads to

\[
\sup_{s \in [0,1]} \left( \left| A^2 \frac{g(|\theta|^2)W}{L^p_{\infty}} + \left| A^2 \frac{g'((|\theta|^2)\theta Re(\sigma W))}{L^p_{\infty}} \right| \right) \right) \\
\lesssim \left( \|W\|_{L^p_{\infty}}^{\left(\frac{2}{\sigma + 1} - \frac{2}{p}\right)} + \|v\|_{L^p_{\infty}}^{\left(\frac{2}{\sigma + 1} - \frac{2}{p}\right)} \right) \left( \|W\|_{\mathscr{H}} + \|W\|_{\mathscr{H}} \|v\|_{\mathscr{H}} \|v\|_{\mathscr{H}} \right) \|v\|_{L^p_{\infty}}^{\left(\frac{2}{\sigma + 1} - \frac{2}{p}\right)} \|v\|_{L^p_{\infty}}^{\left(\frac{2}{\sigma + 1} - \frac{2}{p}\right)} \left( \|W\|_{\mathscr{H}}^{\frac{1}{2}} + \|W\|_{\mathscr{H}}^{\frac{1}{2} + \frac{1}{2}} \right), \tag{4.26}
\]

where we denote

\[
\varsigma_1 = \left( \frac{1}{p'} - \frac{1}{2} \right) \left( 1 - \frac{1}{\alpha} \right) = \left( \frac{1}{2} - \frac{1}{p} \right) \left( 1 - \frac{1}{\alpha} \right) \quad \text{and} \quad \varsigma_2 = \frac{\varsigma_1}{2}. \tag{4.27}
\]

From the above it can be assumed, in accordance with (4.25), that

\[
\quad \text{for } p \in (2, +\infty) \quad \text{such that } p' \in \left( \max(1, \frac{2}{\sigma + 1}), 2 \right), \tag{4.28}
\]

the following estimate holds

\[
\left| (g(|A|^2)A - g(|v|^2)W, v, v) \right| \\
\leq C_0 + \frac{C_1}{N^{2\varsigma_1}} \|W\|_{\mathscr{H}}^{1 - \frac{2}{p}} + \|W\|_{\mathscr{H}}^{\frac{2}{p}} + \left( \|W\|_{\mathscr{H}} + \|W\|_{L^p_{\infty}}^{2(1 - \frac{2}{p})} \right) \left( \|W\|_{L^p_{\infty}}^{2\sigma + \frac{2}{p} - 1} + \|v\|_{L^p_{\infty}}^{2\sigma + \frac{2}{p} - 1} \right) \\
+ \left( \|W\|_{\mathscr{H}} + \|W\|_{L^p_{\infty}}^{1 - \frac{2}{p}} + \|W\|_{L^p_{\infty}}^{\frac{3}{2} - \frac{1}{p}} \right) \left( \|W\|_{L^p_{\infty}}^{\frac{1}{2} + \frac{1}{2}} + \|v\|_{L^p_{\infty}}^{\frac{1}{2} + \frac{1}{2}} \right) \left| A^{\frac{1}{2} + \frac{1}{2}} v \right|_{L^p_{\infty}}. \tag{4.29}
\]

Applying Young’s inequality to the previous estimate leads to

\[
\left| (g(|A|^2)A - g(|v|^2)W, v, v) \right| \\
\leq C_0 + \frac{C_1}{N^{2\varsigma_1}} \|W\|_{\mathscr{H}}^{1 - \frac{2}{p}} + \left( \|W\|_{\mathscr{H}} + \|W\|_{L^p_{\infty}}^{1 - \frac{2}{p}} \right) + \frac{C_2}{N^{\varsigma_1}} \|W\|_{\mathscr{H}}^{3 - \frac{2}{p}} \\
+ C_3 \left( \|W\|_{L^p_{\infty}}^{1 - \frac{2}{p}} + \|v\|_{L^p_{\infty}}^{1 - \frac{2}{p}} \right) + C_4 \left| A^{\frac{1}{2} + \frac{1}{2}} v \right|_{L^p_{\infty}}, \tag{4.30}
\]

where \((q, p)\) denotes an admissible pair satisfying the condition (4.28).
Thanks to Lemma 2.4 and Lemma 4.3, it may be concluded from (4.30) and (4.24) that for $N$ large enough

$$\psi(W) \leq \frac{\gamma}{2} \phi(W) + C_0 + \frac{C_1}{N^{\alpha}} \|W\|^3_{\mathcal{X}_\alpha} + \frac{C_2}{N^{2\alpha}} \|W\|_{\mathcal{X}_\alpha}^4$$

$$+ C_3 \left( \|W\|^4_{L^{\infty}_{t}L^{\gamma}_{y}} + \|v\|^4_{L^{\infty}_{t}L^{\gamma}_{y}} \right) + C_4 \|A_{\alpha}^{-\frac{1}{2}} v_0\|_{L^{2}_{t}L^{2}_{y}}^q ,$$

which, in accordance with (4.23), Propositions 4.6 and 4.7 and Lemma 4.8, leads to

$$\sup_{s \in [0,t]} \|W(s)\|^3_{\mathcal{X}_\alpha} \leq C_0 + \frac{C_1}{N^{\alpha}} \sup_{s \in [0,t]} \|W(s)\|^3_{\mathcal{X}_\alpha} + \frac{C_2}{N^{2\alpha}} \sup_{s \in [0,t]} \|W(s)\|_{\mathcal{X}_\alpha}^4$$

$$+ \frac{C_3}{N^{\alpha}} \sup_{s \in [0,t]} \|W(s)\|_{\mathcal{X}_\alpha}^{2\sigma+2} + C_4 \int_0^t e^{-\gamma(t-s)} \|W(s)\|^4_{L^{\infty}_{t}L^{\gamma}_{y}} ds$$

(4.32)

where $\nu = 1 - \sigma (\frac{1}{\alpha} - \frac{1}{2})$ and $\varsigma_1$ is defined by (4.27).

At this stage we need the following lemma that reads

**Lemma 4.10.** Let $T > 0$, say $(0 < T < \frac{\pi}{4})$, and $t > T$. Then there exist $N_0 > 0$ large enough and nonnegative real constants $(C_1)_{0 \leq i \leq 3}$ that depend only on $\alpha, \sigma, f$ and $T$ such that for all $N \geq N_0$ and for all $t_0 \in [0,t]$ the following estimate holds

$$\|W\|_{L^{\infty}_{t_0}L^{\gamma}_{y}} \leq C_0 + \frac{C_1}{N^{\alpha}} \|W\|_{L^{\infty}_{t}L^{\infty}_{x}} + \frac{C_2}{N^{2\alpha}} \|W\|_{L^{2\sigma}_{t}L^{\infty}_{x}}^{\sigma+1} + \frac{C_3}{N^{\alpha}} \|W\|_{L^{\infty}_{t}L^{\infty}_{x}}$$

(4.33)

where we denote

$$\|W\|_{L^{\infty}_{t}L^{\infty}_{x}} = \sup_{s \in [0,t]} \|W(s)\|_{L^{\infty}_{x}}, \quad \|W\|_{L^{2\sigma}_{t}L^{2\sigma}_{x}}^{\sigma} = \|W\|_{L^{2\sigma}([t_0,t_0+T])L^{2\sigma}_{x}}$$

and $(\nu_j)_{1 \leq j \leq 4}$ are nonnegative constants depending only on $\alpha$ and $\sigma$.

**Proof of Lemma 4.10.** By applying the Duhamel formula for (4.19) and (4.20), Corollary 2 and Lemma 4.5, it may be concluded that for every admissible pair $(\delta, \rho)$ such that $\rho' \in (\frac{4}{3}, 2)$ and for a fixed $s > \frac{1}{2}$ the following estimates hold

$$\|W\|_{L^{\infty}_{t_0}L^{\gamma}_{y}} \leq \|W(t_0)\|_{L^{\infty}_{x}} + T^{\frac{1}{4}} \|f\|_{L^2} + \|g(|A|^2)A\|_{L^{\infty}_{t_0}L^{4}_{x}}$$

$$\leq \|W(t_0)\|_{L^{\infty}_{x}} + T^{\frac{1}{4}} \|f\|_{L^2} + \|\|A\|^{2\sigma}_{L^{2\sigma}_{x}}\|_{L^{\infty}_{t_0}L^{2\sigma}_{x}} + \|\|A\|^{2\sigma}_{L^{2\sigma}_{x}}\|_{L^{\infty}_{t_0}L^{2\sigma}_{x}}$$

(4.34)

Independently, Choosing a fixed reel $\frac{1}{2} < s < \frac{q}{2}$ (say $s = \frac{\alpha+1}{2}$ for instance) then using interpolation argument and Lemma 4.3 we have

$$\|W\|_{L^{\infty}_{x}} \leq \frac{1}{N^{\frac{\alpha+2\gamma}{\alpha}}} \|W\|_{L^{\infty}_{x}}.$$

(4.35)

Following the lines of the proof of Proposition 4.6, two several cases can be distinguished:

- In case where $0 < \sigma < \frac{q}{4}$, assuming that $(\delta, \rho)$ satisfies (4.15) i.e.

$$\max \left( 4 \cdot \frac{2}{3} \cdot \frac{2}{1+2\sigma} \right) < \rho' \leq \frac{2\alpha}{\alpha + 2\sigma}$$
leads, thanks to the Hölder inequality, Lemma 2.2, Lemma 4.3, Proposition 4.6 and in accordance with (4.35) and (4.34), to the following estimate that reads
\[
||W||_{L^4_t L^\infty_x} \leq C_0 + \frac{C_1}{N^{(2 - \frac{\sigma}{4})}} ||W||_{L^\infty_t L^2_x} + C_2 \left(1 + \frac{||W||_{L^\infty_t \mathcal{H}_n}}{N^{(2 - \frac{\sigma}{4})}}\right) \left(1 + \frac{||W||^{2 \sigma}_{L^4_t \mathcal{H}_n}}{N^{(2 - \frac{\sigma}{4})}}\right)
\]
and then (4.33) follows thanks to Young’s inequality.

\[\bullet\] While in the other, where \(\frac{9}{4} \leq \sigma < \frac{2\alpha}{\alpha + 2}\), choosing an admissible pair \((\delta, \rho)\) satisfying (4.16) i.e:
\[
\begin{cases}
\frac{4}{3} < \rho' < 2 \\
\max\left(1, \frac{4\alpha}{4\alpha + 1 - 2\sigma}\right) < \delta' < \frac{8}{7}
\end{cases}
\]
allows us, thanks to (4.35), Lemma 2.2, Proposition 4.6 and the Hölder inequality, to obtain
\[
\left|||\Lambda||^{2\sigma}_{L^2_x L^\infty_y}||\right|_{L^1_t} \leq T^{\left(\frac{1 - \sigma}{4} + \frac{2\alpha - \sigma}{4}\right)} \left(1 + \frac{||W||^{\frac{\alpha(2 - \rho')}{(2 - \sigma)}}_{L^\infty_t \mathcal{H}_n}}{N^{\frac{\alpha - 1}{4} \frac{(2 - \rho')}{4\sigma - 2\rho} - \frac{(2 - \sigma)}{4\sigma - 2\rho}}}ight) \times \left(||v||^{\frac{4(\frac{2\sigma}{3} - \frac{2\alpha - \sigma}{4})}{2}}_{L^4_t L^\infty_y} + ||W||^{\frac{4(\frac{2\sigma}{3} - \frac{2\alpha - \sigma}{4})}{2}}_{L^4_t L^\infty_y}\right).
\]
Hence, in accordance with (4.34), (4.35) and Proposition 4.6 we obtain
\[
||W||_{L^4_t L^\infty_x} \leq C_0 + \frac{C_1}{N^{(2 - \frac{\sigma}{4})}} ||W||_{L^\infty_t L^2_x} + C_2 \left(1 + \frac{||W||_{L^\infty_t \mathcal{H}_n}}{N^{(2 - \frac{\sigma}{4})}}\right)^{\zeta} \left(1 + \frac{||W||^{\frac{\alpha(2 - \rho')}{(2 - \sigma)}}_{L^\infty_t \mathcal{H}_n}}{N^{\frac{\alpha - 1}{4} \frac{(2 - \rho')}{4\sigma - 2\rho} - \frac{(2 - \sigma)}{4\sigma - 2\rho}}}\right)^{\zeta}
\]
where \(\frac{1}{\zeta} = 1 - 4\left(\frac{\delta'}{8} - \frac{(2\alpha - \sigma)}{2}\right)\).

In the light of (4.37) the estimate (4.33) is deduced, thanks again to Young’s inequality and the proof of the lemma is therefore achieved. \(\square\)

Now, for \(t > T\) we write \([0, t] = \bigcup_{k=0}^{m-1} [kT, (k+1)T]\). Then, using the fact that there exists \(C = C(T) > 0\) depending only on \(T\) such that
\[
\int_0^t e^{-\gamma(t-s)} ||W(s)||^{4}_{L^\infty_y} ds \lesssim \tilde{C}(T) \sup_{0 \leq k \leq m-1} ||W||^{4}_{L^4((kT, (k+1)T), L^\infty_y)},
\]
and in accordance with Lemma 4.10, it can be assumed by classical argument that for a chosen \(N_0 > 0\) large enough \(\sup_{s \in [0, t]} ||W(s)||^{2\sigma}_{\mathcal{H}_n} \leq 2C_0\) where \(C_0 > 0\) does not depends on \(N_0\) and then \(W\) remains uniformly bounded in \(\mathcal{H}_n\). This conclude the second step as well as the proof of (4.22) thanks again to Lemma 4.10.

Step 3. The solution is bounded in \(Q_N (D(A_n))\). In order to prove that \(A_n W\) is bounded in \(L^2(\Omega)\), we will prove equivalently that \(Z = W_1\) remains bounded in \(L^2(\Omega)\) with an upper bound that may depends on \(N\).
We differentiate (4.19) with respect to \( t \), then we consider the scalar product of the resulting equation by \(-i(Z_t + \gamma Z)\). This leads to

\[
\frac{1}{2} \frac{d}{dt} \Phi(Z) + \gamma \Phi(Z) = \Psi(Z)
\]

where we set

\[
\Phi(Z) = \|Z\|^2_{\mathcal{H}_m^n}(\mathcal{H}_m^n) - 2 \big( g(|\Lambda|^2), \Re(\overline{Z}) \big) - 2 \big( g(|\Lambda|^2), \Re(\overline{Z}) \big) - 2 \big( g(|\Lambda|^2), \Re(\overline{Z}) \big)
\]

and

\[
\Psi(Z) = -\gamma \left( g(|\Lambda|^2)Z, v_t \right) - 2\gamma \left( g(|\Lambda|^2)\Lambda\Re(\overline{Z}), v_t \right)
\]

Lemma 4.11. Let \( N_0 \) be a fixed positive integer chosen large enough depending on \( \gamma, f \) and \( \sigma \). Then there exists \( C > 0 \) such that for any \( N \geq N_0 \),

\[
\frac{1}{2} \|Z\|^2_{\mathcal{H}_m^n}(\mathcal{H}_m^n) - CN^{\frac{\sigma}{(\nu+\nu+1)}} \leq \Phi(Z) \leq \frac{3}{2} \|Z\|_{\mathcal{H}_m^n}^2 + C N^{\frac{\sigma}{(\nu+\nu+1)}} .
\]

**Proof of Lemma 4.11.** Since \( W \) remains uniformly bounded in \( \mathcal{H}_m^n \), then (4.42) easily follows by application of the Hölder inequality, Lemma 2.4, Lemma 4.3 and the estimate \( \|v_t\|_{L^2} \lesssim \sqrt{N} \). We omit the details for the sake of conciseness and the proof is completed. \( \square \)

We now proceed to bound \( \Psi(Z) \).

In the one hand, thanks to the Hölder inequality, Lemma 2.4, Lemma 4.3 and the estimate \( \|v_t\|_{L^2} \lesssim \sqrt{N} \) we easily obtain

\[
\left| \left( g(|\Lambda|^2), v_t \right) \right| + \left| \left( g(|\Lambda|^2)\Lambda\Re(\overline{Z}), v_t \right) \right| \lesssim N^{\frac{\sigma}{(\nu+\nu+1)}} \|Z\|_{\mathcal{H}_m^n} . \tag{4.43}
\]

Moreover,

\[
\left| \left( g(|\Lambda|^2), \Re(\overline{Z}) \right) \right| + \left| \left( g(|\Lambda|^2)\Lambda|Z|^2, v_t \right) \right| + \left| \left( g'|(|\Lambda|^2)\Lambda\Re(\overline{Z})^2, Z \right) \right|
\]

\[
\lesssim \frac{1}{N^{\frac{\nu}{\nu+1}}} \|Z\|_{\mathcal{H}_m^n}^2 + \frac{1}{N^{\left(1-\frac{2}{\nu+1}\right)}} \|Z\|_{\mathcal{H}_m^n}^3 , \quad \nu = 1 - \sigma \left( \frac{1}{2} - \frac{1}{\alpha} \right) . \tag{4.44}
\]

In the other hand, thanks again to the Hölder inequality, Lemma 2.4 and Lemma 4.3

\[
\left| \left( g(|\Lambda|^2), \Re(\overline{Z})v_t \right) \right| + \left| \left( g(|\Lambda|^2)\Lambda|v_t|^2, Z \right) \right| + \left| \left( g''(|\Lambda|^2)\Lambda\Re(\overline{Z})^2, Z \right) \right|
\]

\[
\lesssim \frac{1}{N^{\frac{\nu}{(\nu+1)+\frac{1}{2}}}} \|Z\|_{\mathcal{H}_m^n} \|v_t\|_{L^2}^{\frac{3}{2} - \frac{\nu}{2}} \|v_t\|_{\mathcal{H}_m^n}^{\frac{1}{2} + \frac{1}{2}} \lesssim N^{\frac{\nu}{(\nu+1)+\frac{1}{2}}} \|Z\|_{\mathcal{H}_m^n} . \tag{4.45}
\]

Independently, since the following terms

\( g(|\Lambda|^2)\Re(\overline{Z})v_t \), \( g'(\Lambda|Z|^2,v_t) \) and \( g''(|\Lambda|^2)\Re(\overline{Z})^2, v_t \)
Lemma 4.12. Let $p_1 \in (2, +\infty)$ such that $1 < p'_1 \leq \alpha$. By applying Lemma 4.2 and the H"older inequality it follows, and under assumption (1.5), that
\[
\left| (g'(|\Lambda|^2) \Re(\overline{\Lambda}Z)Z, v_t) \right| \\
\leq \left\| A^\frac{1}{2}_2 [g'(|\Lambda|^2) Z \Re(\overline{\Lambda}v_t)] \right\|_{L^p_1 L^q_y} \left\| A^{-\frac{1}{2}}_2 Z \right\|_{L^p_1 L^q_y}
\lesssim \left\| \Lambda \right\|_{L^2 \otimes L^\infty} \left\| Z \right\|_{L^2 \otimes L^\infty} \left\| A^\frac{1}{2}_2 v_t \right\|_{L^p_1 L^q_y} + \left\| v_t \right\|_{L^p_1 L^q_y} \left\| Z \right\|_{L^p_1 L^q_y} \left\| \Lambda \right\|_{\mathcal{X}_n} \left\| A^{-\frac{1}{2}}_2 Z \right\|_{L^p_1 L^q_y}.
\]
Thus, for a chosen reel $p_1 \in (2, +\infty)$ such that
\[
1 < p'_1 \leq \frac{2\alpha}{\alpha + 1}, \tag{4.46}
\]
it leads from the estimate above, Lemma 2.4, Lemma 2.5 and Lemma 4.3 that there exists $C > 0$ depending only on $T$, $\gamma$, $\sigma$ and $f$ such that
\[
\left| (g'(|\Lambda|^2) \Re(\overline{\Lambda}Z)Z, v_t) \right| \leq C N^\frac{3}{2} \left\| Z \right\|_{\mathcal{X}_n} \left\| A^{-\frac{1}{2}}_2 Z \right\|_{L^p_1 L^q_y}. \tag{4.47}
\]
Now it only remains to handle the worst terms that contain $v_{tt}$.

Lemma 4.12. Let $p_2 \in (2, +\infty)$ be a fixed reel number satisfying
\[
1 < p'_2 < \min \left( \frac{2\alpha}{1 + \alpha}, \frac{1}{\sigma} \right). \tag{4.48}
\]
Then under assumption (1.5) there exists $C > 0$ depending only on $T$, $\gamma$, $\sigma$ and $f$ such that the following estimate holds
\[
\left| (g(|\Lambda|^2) Z, v_{tt}) \right| + \left| (g'(|\Lambda|^2) \Lambda \Re(\overline{\Lambda}Z), v_{tt}) \right| \\
\leq C N \left\| Z \right\|_{\mathcal{X}_n} \left( \sqrt{N} + (1 + \left\| u \right\|_{L^\infty_\gamma}) \left\| A^{-\frac{1}{2}}_2 u \right\|_{L^p_1 L^q_y} \right). \tag{4.49}
\]

Proof of Lemma 4.12. Using the Cauchy-Schwarz then the H"older inequalities, leads to
\[
\left| (g(|\Lambda|^2) Z, v_{tt}) \right| + \left| (g'(|\Lambda|^2) \Lambda \Re(\overline{\Lambda}Z), v_{tt}) \right| \lesssim \left\| \Lambda \right\|_{L^p_1 L^q_y} \left\| Z \right\|_{L^p_1 L^q_y} \left\| v_{tt} \right\|_{L^2} \tag{4.50}
\]
As can be seen from above, it remains to bound $\left\| v_{tt} \right\|_{L^2}$. Differentiating $\mathbb{P}_N (1.1)$ with respect to $t$ leads to
\[
v_{tt} = iA_{\alpha} v_t - \gamma v_t - i\mathbb{P}_N (g(|u|^2) u_t) - 2i\mathbb{P}_N (g'(|u|^2) u \Re(\overline{u} u_t)) ,
\]
and then we obtain, thanks to Lemma 4.3 and Proposition 4.7, that
\[
\left\| v_{tt} \right\|_{L^2} \lesssim N^\frac{3}{2} + \left\| \mathbb{P}_N (g(|u|^2) u_t) \right\|_{L^2} + \left\| \mathbb{P}_N (g'(|u|^2) u \Re(\overline{u} u_t)) \right\|_{L^2} \tag{4.51}
\]
Since the terms of the right hand side of (4.51) are similar, we indicate briefly how to bound one of them. Thanks to the Riesz-representation-Theorem and by a density argument it follows that for $p \in (2, +\infty)$
\[
\left\| \mathbb{P}_N (g(|u|^2) u_t) \right\|_{L^2} = \sup_{\vartheta \in L^2, \left\| \vartheta \right\|_1 = 1} \left| (\mathbb{P}_N (g(|u|^2) u_t), \vartheta) \right|
\]
Consequently, since \( u \) is a nonnegative constant, we obtain
\[
\sup_{\theta \in L^2, ||\theta||=1} \left| (u_t, g(|u|^2) P_N(\theta)) \right| \leq \sup_{\theta \in L^2, ||\theta||=1} \left\| A_\alpha^2 |g(|u|^2) P_N(\theta)| \right\|_{L^{p'}_x L^r_y} \left\| A_\alpha^{-\frac{1}{2}} u_t \right\|_{L^p_t L^q_y} (4.52)
\]
Now observe that choosing \( p \in (2, +\infty) \) such that \( p' < \min(2, \frac{1}{\sigma}) \) and applying the Hölder inequality leads to
\[
\left\| A_\alpha^2 |g(|u|^2) P_N(\theta)| \right\|_{L^{p'}_x L^r_y} \lesssim ||u||_{L^\infty} ||u||_{H_\alpha} ||P_N(\theta)||_{L^{2p'}_x L^\infty_y} + ||u||_{L^2_x L^\infty_y}^{2\sigma} \left\| A_\alpha^2 P_N(\theta) \right\|_{L^{\frac{2}{1+\sigma'}}_x L^2_y} . (4.53)
\]
Consequently, since \( u \) remains uniformly bounded in \( H_\alpha \) and in accordance with the estimate (4.52), Lemma 2.1, Lemma 2.2 and Lemma 4.3 it follows from (4.53) that for a chosen reel \( p_2 \in (2, +\infty) \) satisfying
\[
\frac{2}{3 + 2\sigma} < 1 < p'_2 < \min \left( \frac{2\alpha}{1 + \alpha' \frac{1}{\sigma}} \right),
\]
we obtain
\[
||P_N(g(|u|^2)u_t)||_{L^2} \lesssim N(1 + ||u||_{L^\infty_y}) \left\| A_\alpha^{-\frac{1}{2}} u_t \right\|_{L^{p'_2} L^2_y} . (4.54)
\]
Gathering (4.50), (4.51) and (4.54) achieves the proof of the lemma. \( \square \)

At this stage, we need the following lemma that is for \( W \) the analog of Proposition 4.6, Proposition 4.7 and Lemma 4.8 for \( u \). Since the proof of the following statement is similar to the proofs of the previous ones, we omit it.

**Lemma 4.13.** Let \( T > 0 \) small enough as in Proposition 2.8. Then there exists \( C > 0 \) depending only on \( T, \gamma, \sigma \) and \( f \) such that for all admissible pairs \( (q, p) \) the following estimate holds
\[
\left\| A_\alpha^2 W \right\|_{L^2_x L^p_y L^q_t} + \left\| A_\alpha^{-\frac{1}{2}} W_t \right\|_{L^p_t L^q_y L^2_x} \leq C.
\]
Moreover, for all \( t \geq 0 \),
\[
\int_0^t e^{-\gamma(t-s)} \left\| A_\alpha^{-\frac{1}{2}} W_t \right\|_{L^p_t L^q_y}^q ds \leq C.
\]

Thanks to (4.43), (4.44), (4.45), (4.47), Lemma 4.11 and Lemma 4.12 it may be concluded, in accordance with (4.39), that for a chosen couple of admissible pairs \( (q_1, p_1) \) and \( (q_2, p_2) \) that satisfy respectively the conditions (4.46) and (4.48) we have for \( N \) large enough
\[
\Psi(Z) \leq \frac{\gamma}{2} \phi(Z) + C_1 N^{2\kappa} + \frac{C_2}{N(1-\frac{1}{2})} \phi(Z)^{\frac{1}{2}} + C_3 ||u||_{L^\infty_y}^4
\]
\[
+ C_4 \left( \left\| A_\alpha^{-\frac{1}{2}} Z \right\|_{L^p_t L^q_y}^{q_1} + \left\| A_\alpha^{-\frac{1}{2}} u_t \right\|_{L^p_t L^q_y}^{q_2} \right) . (4.55)
\]
where \( \kappa \) denotes a nonnegative reel constant that depends only on \( \alpha \) and \( \sigma \). Hence, thanks to (4.39), (4.42), Proposition 4.6, Proposition 4.7 and Lemma 4.13, the previous estimate implies by classical arguments (as in the second step) the existence of a nonnegative constant \( \kappa \) depending only on \( \gamma \) and \( \sigma \) such that
\[
\sup_{s \in [0, t]} ||Z||_{D(A_\alpha)} \leq C N^\kappa.
\]
This concludes the third step as well as the proof of the proposition.

Let us now turn to proving Theorem 4.1,

4.2. Proof of Theorem 4.1. At first, we highlight the asymptotic smoothing
effect of the semi-group by comparing $W(t)$ to $w(t)$ when $t$ converges towards $+\infty$.

**Proposition 4.14.** Let $T > 0$ small enough as in Proposition 2.8. There exists
$C > 0$ depending only on $\gamma$, $\sigma$, $T$, $f$ and $|u_0|_{\mathcal{H}_\alpha}$ and such that
\[
\forall \ t \geq 0, \quad ||W(t) - w(t)||_{\mathcal{H}_\alpha} \leq C e^{-\gamma t}.
\]

**proof of Proposition 4.14.** Let $\chi = W - w = \Lambda - u$ where we recall that $\Lambda = W + v$.
Then $\chi$ satisfies
\[
\chi_t - iA_{\alpha} \chi + \gamma \chi + iQ_N \left( g(|u|^2)u - g(|\Lambda|^2)\Lambda \right) = 0,
\]
supplemented with initial data
\[
\chi(0) = w(0) = Q_N(u(0)).
\]
Applying the scalar product of (4.56) by $\chi$ then by $-i(\chi_t + \gamma \chi)$ we obtain that
\[
\frac{1}{2} \frac{d}{dt} \Psi(\chi) + \gamma \Psi(\chi) = \Upsilon(\chi),
\]
where we denote
\[
\Psi(\chi) = \left| \left| A_{\alpha}^{1/2} \chi \right| \right|^{2}_{L^2} - \int_{0}^{1} \left[ \left( g(|\Theta|^2), |\chi|^2 \right) + 2(g(|\Theta|^2), \Re(\Theta \chi)\Re \chi) \right] ds
\]
\[
\Upsilon(\chi) = -2 \int_{0}^{1} \left[ (g''(|\Theta|^2)\Theta \Re(\Theta \chi)\Re \chi', (\Theta \chi)' + (g'(|\Theta|^2)|\chi|^2\Theta, \Theta_i) \right] ds
\]
\[
-2 \int_{0}^{1} (g'(|\Theta|^2)\chi \Re(\Theta \chi, \Theta_i) ) ds
\]
with $\Theta = \Theta(s) = u + s \Lambda$, $s \in [0, 1]$.

On the one hand, we easily deduce from the Hölder inequality Lemma 2.2, Lemma 2.4 and Lemma 4.3 that for $N$ large enough and under assumption (1.5), we have
\[
\frac{1}{2} \left| \left| A_{\alpha}^{1/2} \chi \right| \right|^{2}_{L^2} \leq \Psi(\chi) \leq \frac{3}{2} \left| \left| A_{\alpha}^{1/2} \chi \right| \right|^{2}_{L^2}.
\]
On the other hand, in order to bound $\Upsilon(\chi)$, it is worth noticing that the majorization of the terms of the right-hand side of (4.60) involving $\Theta_i$ are similar we just handle the first one.

In the light of the type estimates (3.4) and (3.6) and in accordance with the uniform bound of $u$ and $\Lambda$ in $\mathcal{H}_\alpha$, applying Lemma 2.2, Lemma 4.2 and the Hölder inequality, it can be concluded under assumption (1.5) that
\[
\sup_{s \in [0, 1]} \left( |(g'(|\Theta|^2)|\chi|^2\Theta, \Theta_i)| \right)
\]
\[
\lesssim \sup_{s \in [0, 1]} \left( \left| \left| A_{\alpha}^{1/2} [g'(|\Theta|^2)|\chi|^2\Theta] \right| \right|^{2}_{L^2} \left( \left| \left| A_{\alpha}^{1/2} \Lambda_i \right| \right|^{2}_{L^2} + \left| \left| A_{\alpha}^{1/2} u_t \right| \right|^{2}_{L^2} \right) \right)
\]
\[
\lesssim \left( \left| \left| A_{\alpha}^{-1/2} \Lambda_i \right| \right|^{2}_{L^2} + \left| \left| A_{\alpha}^{-1/2} u_t \right| \right|^{2}_{L^2} \right) \left[ \left| \left| A_{\alpha}^{1/2} \Lambda \right| \right|^{2}_{L^2} \left| \left| \chi \right| \right|_{L^2}^{2} \right]
\]
\[
+ \left| \left| A_{\alpha}^{1/2} u \right| \right|^{2}_{L^2} \left( \left| \left| \Lambda \right| \right|_{L^2} + \left| \left| u \right| \right|_{L^2} \right) \left| \left| A_{\alpha}^{1/2} \chi \right| \right|_{L^2} \left| \left| \chi \right| \right|_{L^2}^{2} \right]
\]
Thus, thanks to Lemma 4.3 and in accordance with (4.61), applying Young’s inequality leads, for $N$ large enough, to the existence of a constant $C > 0$ that does not depend on $N$ such that the following estimate holds

$$
\sup_{s \in [0, \ell]} |\Psi(\chi(s))| \leq C e^{-\frac{\gamma}{2}\ell},
$$

(4.63)

This completes the proof of the proposition thanks again to (4.61).

Propositions 4.14 and 4.9 provide, by classical arguments, that $\mathcal{A}_a$ is a bounded subset of $D(A_a)$ (see [2, 3, 22] for instance). Moreover, let $N_0$ be a fixed positive integer chosen large enough depending on $T$, $\gamma$, $f$ and $\sigma$. Then there exist $\kappa, C > 0$ that depend on $N_0$ such that $\forall N \geq N_0,$

$$
\forall u_0 \in \mathcal{A}_a, \quad ||u_0||_{D(A_a)} \leq C N^\kappa.
$$

(4.64)

The proof of the compactness of $\mathcal{A}_a$ into $D(A_a)$ is standard and follows by using the well known J. Ball’s argument [8]. we omit it for the sake of conciseness and we refer the reader to [2], Theorem 2.10, for a similar complete proof. Hence, the proof of the theorem is achieved.

5. Finite fractal dimension of the global attractor. When studying dissipative dynamical systems, we attach a paramount importance to the fractal dimension of the global attractor. This is due to the fact that, in term of physical parameters, the fractal dimension while is finite implies that the infinite-dimensional dynamical system possesses an asymptotic behavior determined by a finite number degrees of freedom. Within this framework and for sake of completeness we recall a general result given in [15].

**Theorem 5.1.** Let $X$ be a Banach space and $M$ be a bounded closed set in $X$. Assume that there exists a mapping $V : M \to X$ such that

1. $M \subseteq V(M)$. Moreover, $V$ is Lipschitz on $M$, i.e., there exists $L > 0$ such that for all $u_1, u_2 \in M$, $||V(u_1) - V(u_2)||_X \leq L ||u_1 - u_2||_X.$

2. There exist compact semi-norms $n_1$ and $n_2$ on $X$ such that $\forall u_1, u_2 \in M,$

$$
||V(u_1) - V(u_2)||_X \leq \delta ||u_1 - u_2||_X + K [n_1(u_1 - u_2) + n_2(V(u_1) - V(u_2))].
$$

where $0 < \delta < 1$ and $K > 0$ are constants.

Then $M$ is a compact set in $X$ of a finite fractal dimension. (A semi norm $n$ on $X$ is said to be compact if and only if for any bounded set $B \subseteq X$ there exists a sequence $(x_n)_n \subseteq B$ such that $n(x_n - x_m) \to 0$ as $n, m \to +\infty$).

We now state the third part of the main result of this paper.
Theorem 5.2. Assume that the forcing term \( f \) belongs \( L^2(\Omega) \). Then the global attractor \( \mathcal{A}_\alpha \) has a finite fractal dimension in \( \mathcal{H}_\alpha \).

With the objective of proving Theorem 5.2 by checking the assumptions in Theorem 5.1, one only has, thanks to Lemma 4.3, to prove the following result

**Proposition 5.3.** There exist \( t^* > 0 \) and \( N > 0 \) depending on \( \gamma, \sigma, f \) and \( T \) such that for all \( u_0, v_0 \in \mathcal{A}_\alpha \),

\[
||Q_N(S_\alpha(t^*)u_0) - Q_N(S_\alpha(t^*)v_0)||_{\mathcal{H}_\alpha^2}^2 \leq \frac{1}{2} ||u_0 - v_0||_{\mathcal{H}_\alpha}^2 + L ||u_0 - v_0||_{L^2(\Omega)}^2,
\]

where \( L = L(t^*) > 0 \) depends on \( t^* \) and \( Q_N = \text{id} - P_N \) still denotes the orthogonal projector defined by (4.1).

**Proof of Proposition 5.3.** Let \( u \) and \( v \) be two solutions of the problem (1.1) – (1.2) issued respectively from \( u_0, v_0 \in \mathcal{A}_\alpha \). We introduce

\[
Z = Q_N(u(t)) - Q_N(v(t)) = Q_N(S_\alpha(t)u_0) - Q_N(S_\alpha(t)v_0)
\]

which solves

\[
\begin{align*}
Z_t - iA_\alpha Z + \gamma Z + iQ_N (g(|u|^2)u - g(|v|^2)v) &= 0, \\
Z(0) &= Q_N(u_0 - v_0) \in \mathcal{A}_\alpha.
\end{align*}
\]

(5.2)

Setting \( \Theta(s) = v + s(u - v) \) and using the following identity

\[
g(|u|^2)u - g(|v|^2)v = \int_0^1 g(|\Theta|^2)(u - v) + 2g'(|\Theta|^2)\Theta \text{Re}[\Theta(u - v)] ds,
\]

the scalar product of (5.2) by \(-i(Z_t + \gamma Z)\) leads to

\[
\frac{1}{2} \frac{d}{dt} \Psi(Z) + \gamma\Psi(Z) = \Phi(Z)
\]

(5.3)

where

\[
\Psi(Z) = \left|\left| A_\alpha^\frac{1}{2} Z \right|\right|_{L^2}^2 - \left\{ (g(|\Theta|^2), |Z|^2) + 2 \left( g(|\Theta|^2)P_N(u - v), Z \right) \right\}
\]

\[
- \left\{ (g'(|\Theta|^2), \text{Re}[\Theta Z]^2) + 4 \left( g'(|\Theta|^2)\Theta \text{Re}[\Theta P_N(u - v)], Z \right) \right\},
\]

(5.4)

and

\[
\Phi(Z) = -\left\{ \gamma \left( g(|\Theta|^2)P_N(u - v), Z \right) + 2\gamma \left( g'(|\Theta|^2)\Theta \text{Re}[\Theta Z], P_N(u - v) \right) \right\}
\]

\[
+ \left\{ 2 \left( g'(|\Theta|^2)\Theta \text{Re}[\Theta P_N(u - v)], \Theta_t \right) - \left( g'(|\Theta|^2) |Z|^2, \Theta_t \right) \right\}
\]

\[
- \left\{ 2 \left( g'(|\Theta|^2)Z \Theta \text{Re}[\Theta P_N(u - v)], \Theta_t \right) + 2 \left( g''(|\Theta|^2)\Theta \text{Re}[\Theta Z]^2, \Theta_t \right) \right\}
\]

\[
- \left\{ 4 \left( g''(|\Theta|^2) \Theta \text{Re}[\Theta P_N(u - v)\Theta], \Theta_t \right) \right\}
\]

\[
- \left\{ 2 \left( g'(|\Theta|^2)\Theta \text{Re}[\Theta Z], P_N(u_t - v_t) \right), \left( g(|\Theta|^2)Z, P_N(u_t - v_t) \right) \right\},
\]

(5.5)

with the notation: \( \{ \varphi \} = \int_0^1 \varphi \, ds \).

To begin with, thanks to the H"older inequality, Lemma 2.4, Lemma 2.5 and Lemma 4.3 it can be deduced that for \( N_0 > 0 \) chosen large enough, there exists \( C > 0 \) depending only on \( \gamma, \sigma, f \) and \( T \) such that for any \( N \geq N_0 \) the following estimate holds

\[
\frac{1}{2} ||Z||_{\mathcal{H}_\alpha}^2 - \frac{C}{N} ||u - v||_{L^2}^2 \leq \Psi(Z) \leq \frac{3}{2} ||Z||_{\mathcal{H}_\alpha}^2 + \frac{C}{N} ||u - v||_{L^2}^2.
\]

(5.6)

Now we handle the right hand side of (5.3)
Lemma 5.4. Let $T > 0$ as in Proposition 2.8. Then for $N_0 > 0$ chosen large enough depending on $\gamma$, $\sigma$, $f$ and $T$, there exist constants $C > 0$ that does not depends on $N_0$ such that for any $N \geq N_0$,

$$
\Phi(Z) \leq \frac{\gamma}{4} ||Z||^2_{H^s_n} + CN \|u - v\|_{L^2}^2.
$$

Proof of Lemma 5.4. Since $\mathcal{A}_\alpha$ is bounded in $D(A_\alpha)$, then by applying the Hölder inequality, Lemma 2.4, Lemma 2.5 and Lemma 4.3 we easily obtain that

$$
|\langle g'(\langle \Theta \rangle^2) \Theta \rangle \rangle[\mathbb{P}_N(u - v)\mathbb{Z}_N, \Theta_i] + |\langle g''(\langle \Theta \rangle^2) \Theta \rangle \Theta \rangle[\mathbb{P}_N(u - v)\mathbb{Z}_N, \Theta_i]| + |\langle g'(\langle \Theta \rangle^2) \Theta \rangle \rangle[\mathbb{P}_N(u - v)\mathbb{Z}_N, \Theta_i]|
$$

Similarly,

$$
|\langle g'(\langle \Theta \rangle^2) \Theta \rangle \rangle[\mathbb{P}_N(u - v)\mathbb{Z}_N, \Theta_i] + |\langle g''(\langle \Theta \rangle^2) \Theta \rangle \Theta \rangle[\mathbb{P}_N(u - v)\mathbb{Z}_N, \Theta_i]| + |\langle g'(\langle \Theta \rangle^2) \Theta \rangle \rangle[\mathbb{P}_N(u - v)\mathbb{Z}_N, \Theta_i]|
$$

This achieves the proof of the lemma thanks to (5.7) and Young’s inequality. □

According to (5.6) and Lemma 5.4 we deduce from (5.3) that

$$
\frac{d}{dt} \Phi(Z) + \gamma \Phi(Z) \leq C N \|u - v\|_{L^2}^2.
$$

Independently, since $w = u - v$ solves

$$
w_t - i A_\alpha w + \gamma w + i \left( g(|u|^2)w - g(|v|^2)v \right) = 0, \quad w(t = 0) = (u_0 - v_0) \in \mathcal{A}_\alpha,
$$

then we deduce, from the scalar product of the previous equality by $w$ and Gronwall’s Lemma, that

$$
||u(t) - v(t)||_{L^2} \leq e^{\int t} ||u_0 - v_0||_{L^2}^2,
$$

which, in accordance with (5.9) and (5.6), leads to

$$
||Q_N(S_\alpha(t)u_0) - Q_N(S_\alpha(t)v_0)||_{H^s_n} \leq C_1 e^{-\frac{\gamma}{2}t} ||u_0 - v_0||_{H^s_n} + C_2 e^{\int t} ||u_0 - v_0||_{L^2}
$$

and the proof is complete for a fixed $t = t^* > 0$ such that $C_1 e^{-\frac{\gamma}{2}t^*} = \frac{1}{2}$. □

Thanks to Proposition 5.3 and the compact embedding of $\mathcal{A}_\alpha$ into $L^2$, the proof of the Theorem 5.2 is achieved by applying Theorem 5.1 with $X = \mathcal{A}_\alpha$, $V = S_\alpha(t^*)$ and $M = \mathcal{A}_\alpha$.

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