The Alexander polynomial of a plane curve singularity via the ring of functions on it

A. Campillo  F. Delgado ∗  S. M. Gusein–Zade †

Abstract

We prove two formulae which express the Alexander polynomial $\Delta^C$ of several variables of a plane curve singularity $C$ in terms of the ring $\mathcal{O}_C$ of germs of analytic functions on the curve. One of them expresses $\Delta^C$ in terms of dimensions of some factors corresponding to a (multi-indexed) filtration on the ring $\mathcal{O}_C$. The other one gives the coefficients of the Alexander polynomial $\Delta^C$ as Euler characteristics of some explicitly described spaces (complements to arrangements of projective hyperplanes).

1 Introduction

The ring $\mathcal{O}_X$ of germs of holomorphic functions on a germ $X$ of an analytic set determines $X$ itself (up to analytic equivalence). Thus all invariants of $X$, in particular, topological ones, can “be read” from $\mathcal{O}_X$. There arises a general problem to find expressions for such invariants in terms of the ring $\mathcal{O}_X$.

Let $C$ be a germ of a reduced plane curve at the origin in $\mathbb{C}^2$ and let $C = \bigcup_{i=1}^{r} C_i$ be its representation as the union of irreducible components (with a fixed numbering). Let $\Delta^C(t_1, \ldots, t_r)$ be the Alexander polynomial of the link $C \cap S^3_\varepsilon \subset S^3_\varepsilon$ for $\varepsilon > 0$ small enough (see, e.g., [12]). The Alexander polynomial $\Delta^C(\underline{t})$ ($\underline{t} = (t_1, \ldots, t_r)$) is a complete topological invariant of a

∗First two authors were partially supported by DGICYT PB97-0471 and by Junta de Castilla y León: VA 102/01. Address: University of Valladolid, Dept. of Algebra, Geometry and Topology, 47005 Valladolid, Spain. E-mail: campillo@agt.uva.es, fdelgado@agt.uva.es

†Partially supported by grants Iberdrola, RFBR–01–01–00739, NWO 047.008.005 and INTAS–00–0259. Address: Moscow State University, Dept. of Mathematics and Mechanics, Moscow, 119899, Russia. E-mail: sabir@mccme.ru
plane curve singularity $C$ ([14]). We prove two formulae for the Alexander polynomial $\Delta^C$ in terms of the ring $O_C$ of germs of analytic functions on the curve $C$. For the case of an irreducible plane curve singularity ($r = 1$) the corresponding result was described in [14]. For the general case the result has been announced in [3]. A global analogue of the statement from [4] for plane curves with one place at infinity can be found in [3].

Let $\mathcal{C}_i$ be the complex line with the coordinate $\tau_i$ ($i = 1, \ldots, r$) and let $\varphi_i : (\mathcal{C}_i, 0) \to (\mathbb{C}^n, 0)$ be parameterizations (uniformizations) of the branches $C_i$ of the curve $C$, i.e., germs of analytic maps such that $\text{Im} \varphi_i = C_i$ and $\varphi_i$ is an isomorphism between $\mathcal{C}_i$ and $C_i$ outside of the origin. Let $O_{\mathbb{C}^2,0}$ be the ring of germs of holomorphic functions at the origin in $\mathbb{C}^2$. For a germ $g \in O_{\mathbb{C}^2,0}$, let $v_i = v_i(g)$ and $a_i = a_i(g)$ be the power of the leading term and the coefficient at it in the power series decomposition of the germ $g \circ \varphi_i : (\mathcal{C}_i, 0) \to \mathbb{C}$: $g \circ \varphi_i(\tau_i) = a_i \cdot \tau^v_i + \text{terms of higher degree } (a_i \neq 0)$. If $g \circ \varphi_i(t) \equiv 0$, $v_i(g)$ is assumed to be equal to $\infty$ and $a_i(g)$ is not defined. The numbers $v_i(g)$ and $a_i(g)$ are defined for elements $g$ of the ring $O_C$ of functions on the curve $C$ as well.

For $\underline{a} = (v_1, \ldots, v_r) \in \mathbb{Z}^r$, let $J(\underline{a}) = \{ g \in O_C : v_i(g) \geq v_i; i = 1, \ldots, r\}$ which is an ideal in $O_C$, let $c(\underline{a}) = \dim J(\underline{a})/J(\underline{a} + 1)$, where $\underline{1} = (1, \ldots, 1)$. Let

$$L_C(t_1, \ldots, t_r) = \sum_{\underline{a} \in \mathbb{Z}^r} c(\underline{a}) \cdot t^{\underline{a}},$$

$$P_C(t_1, \ldots, t_r) = \frac{L_C(1, \ldots, t_r) \cdot \prod_{i=1}^{r} (t_i - 1)}{t_1 \cdot \cdots \cdot t_r - 1}$$

($t^{\underline{a}} = t_1^{a_1} \cdots t_r^{a_r}$). We shall show that $P_C$ is a polynomial in $t_1, \ldots, t_r$. Let $F_{\underline{a}} \subset (\mathbb{C}^r)^r$ be the set of all $r$-tuples $(a_1(g), \ldots, a_r(g))$ for $g \in O_{\mathbb{C}^2,0}$ with $v_i(g) = v_i, i = 1, \ldots, r$. The subspace $F_{\underline{a}}$ is invariant with respect to multiplication by non-zero complex numbers (in fact, if $F_{\underline{a}}$ is not empty, it is the complement to an arrangement of hyperplanes in a vector space of dimension $c(\underline{a})$). Let $\mathbb{P}F_{\underline{a}}$ be the projectivization of $F_{\underline{a}}$, i.e., the factor-space $F_{\underline{a}}/\mathbb{C}^*$ with respect to this $\mathbb{C}^*$-action.

Now we formulate our main results (see more precise explanations and definitions below).

**Theorem 1** For a plane curve singularity $C = \bigcup_{i=1}^{r} C_i \subset (\mathbb{C}^2, 0)$, $r > 1$,

$$P_C(t_1, \ldots, t_r) = \Delta^C(t_1, \ldots, t_r).$$
Theorem 2 For a plane curve singularity \( C = \bigcup_{i=1}^{r} C_i \subset (\mathbb{C}^2, 0), \ r > 1, \)

\[
\Delta^C(t_1, \ldots, t_r) = \sum_{\nu \in \mathbb{Z}_{\geq 0}} \chi(P_{F_{\nu}}) \cdot t_1^{\nu}.
\]

It was found that these results can be formulated in terms of the integral with respect to Euler characteristic over the projectivization of the space of functions in two variables defined in the spirit of the motivic integration; see [7]. Recently W. Ebeling ([11]) has found that there is a relation between the Poincaré series of the natural filtration in the ring of functions on a two–dimensional quasihomogeneous hypersurface singularity and the characteristic polynomial of the monodromy operator. This permits to expect that the discussed connections are more deep and must have broader field of applications.

2 Necessary concepts and facts.

In this section we give more precise definitions of the objects used in the formulation of Theorems 1 and 2 and in the proofs and describe some of their properties.

2.1 The Alexander polynomial of an algebraic link.

The Alexander polynomial (in \( r \) variables) is an invariant of a link with \( r \) (numbered) components in the sphere \( S^3 \). The general definition can be found, e.g., in [12]. To a plane curve singularity \( C = \bigcup_{i=1}^{r} C_i \subset (\mathbb{C}^2, 0) \) there corresponds the link \( C \cap S^3_\varepsilon \) in the 3–sphere \( S^3_\varepsilon \) of radius \( \varepsilon \) centred at the origin in the complex plane \( \mathbb{C}^2 \) with \( \varepsilon \) small enough. For such a link (an algebraic one) we rather use not the general definition of the Alexander polynomial \( \Delta^C(t_1, \ldots, t_r) \), but a formula for it in terms of an embedded resolution \( \pi : (X, D) \to (\mathbb{C}^2, 0) \) of the curve singularity \( C \).

Let the curve \( C \) be given by an equation \( f = 0 \) and let \( f = \prod_{i=1}^{r} f_i \), where \( f_i = 0 \) is an equation of the curve \( C_i \). Let \( \pi : (X, D) \to (\mathbb{C}^2, 0) \) be an (embedded) resolution of the plane curve \( C = \bigcup_{i=1}^{r} C_i \). Such a resolution can be described by its dual graph \( \Gamma \). Vertices of the graph \( \Gamma \) correspond to components of the total transform \( (f \circ \pi)^{-1}(0) \) of the curve \( C \) (i.e., to components of the exceptional divisor \( D = \pi^{-1}(0) \) of the resolution and to strict transforms \( \widetilde{C}_i \) of the branches \( C_i \) of the curve \( C \); in the last case they are depicted by arrows). Two vertices of the graph \( \Gamma \) are connected by an
edge if the corresponding components intersect. The graph $\Gamma$ is a tree. The starting point of the graph (the starting divisor of the resolution) will be denoted by $1$. There is a partial order on the set of vertices of the graph $\Gamma$: $\sigma' < \sigma$ iff the geodesic in $\Gamma$ from the vertex $1$ to the vertex $\sigma$ passes through the vertex $\sigma'$. A vertex $\delta$ corresponding to a component $E_\delta$ of the exceptional divisor is said to be a dead end if it is connected with only one vertex (i.e., if $E_\delta$ intersects only one component of the total transform of the curve $C$). A vertex $\sigma$ is said to be a star point of the resolution if it is connected with at least three vertices. To each dead end $\delta$ in the graph $\Gamma$ except (possibly) the vertex $1$ there corresponds the nearest star point $st_\delta$ such that $st_\delta < \delta$. All vertices $\sigma'$ such that $st_\delta < \sigma' \leq \delta$ form the tail of the resolution graph corresponding to the dead end $\delta$. A vertex $\sigma$ is said to be a separation point of the graph $\Gamma$ if there exist two branches $C_i$ and $C_j$ of the curve $C$ such that $\sigma < \tilde{C}_i$, $\sigma < \tilde{C}_j$ and $\sigma$ is the maximal vertex with these properties (one also says that $\sigma$ is the separation point between the branches $C_i$ and $C_j$ or between $\tilde{C}_i$ and $\tilde{C}_j$). Let $st_1$ be the first (i.e., the minimal) separation point of the graph $\Gamma$. It is possible that $st_1$ is not a star point (if $st_1 = 1$). However in what follows we always include $st_1$ in the set of star vertices.

For a vertex $\sigma$ corresponding to a component $E_\sigma$ of the exceptional divisor (a complex projective line), let $\tilde{E}_\sigma$ be the “smooth part” of the component $E_\sigma$, i.e., $E_\sigma$ minus intersection points with other components of the total transform of the curve $C$. These intersection points are in one-to-one correspondence with connected components of the complement $(f \circ \pi)^{-1}(0) \setminus \tilde{E}_\sigma$. An intersection point is said to be essential if the corresponding connected component contains a component of the strict transform of the curve $C$. Let $s_\sigma$ be the number of essential points on the component $E_\sigma$, and let $\tilde{E}_\sigma$ be the complement to the set of essential points in $E_\sigma$. Let $m_j^\sigma$ ($j = 1, 2, \ldots, r$) be the multiplicity of the lifting $f_j \circ \pi$ of the function $f_j$ (the equation of the component $C_j$) to the space $X$ of the resolution along the component $E_\sigma$, $m^\sigma := (m_1^\sigma, \ldots, m_r^\sigma)$.

D.Eisenbud and W.Neumann ([12]) gave a formula for the Alexander polynomial $\Delta_C(t_1, \ldots, t_r)$ of the curve $C$ in terms of an embedded resolution of the curve $C$.

**Proposition 1** For $r > 1$,

$$\Delta_C(t_1, \ldots, t_r) = \prod_{E_\sigma \subset D} (1 - t^{m^\sigma})^{-\chi(\tilde{E}_\sigma)}. \quad (*)$$

The formula $(*)$ is an analogue of the formula of N.A’Campo for the zeta–function of the monodromy transformation of the curve $C$. 

Remarks. 1. According to the definition, the Alexander polynomial \( \Delta_C(t_1, \ldots, t_r) \) of a link is well defined only up to multiplication by monomials \( \pm t_1^{m_1} \cdot \ldots \cdot t_r^{m_r} \quad (t = (t_1, \ldots, t_r), \quad m = (m_1, \ldots, m_r) \in \mathbb{Z}^r) \). For algebraic links the formula (*) fixes the choice of the Alexander polynomial in such a way that it is really a polynomial (i.e., does not contain monomials with negative powers) and its value at the origin \( (t = 0) \) is equal to 1.

2. There is some difference in definitions (or rather in descriptions) of the Alexander polynomial for a curve with one branch \( (r = 1) \) or with many branches \( (r > 1) \) (see, e.g., [12]). In order to have all the results (Theorems 1 and 2) valid for \( r = 1 \) as well, for an irreducible plane curve singularity \( C \), \( \Delta_C(t) \) should be not the Alexander polynomial, but rather the zeta-function of the monodromy, equal to the Alexander polynomial divided by \( (1 - t) \). In this case \( \Delta_C(t) \) is not a polynomial, but an infinite power series (defined by the formula (*)). The results are valid for this case as well. However since the case of an irreducible plane curve singularity \( (r = 1) \) has been described in [4], here we shall suppose that \( r > 1 \). For \( r > 1 \), \( \Delta_C(t, \ldots, t) \) is nothing else but the zeta-function of the monodromy transformation of the curve singularity \( C \).

2.2 The extended semigroup and the Poincaré polynomial of a curve singularities.

These notions can be defined not only for plane curve singularities, but for curve singularities in spaces of any dimension. Let \( C = \bigcup_{i=1}^{r} C_i \) be a germ of a reduced curve at the origin in \( \mathbb{C}^n \) (\( C_i \) are irreducible components (branches) of the curve \( C \)). Let \( \mathbb{C}_i \) be the complex line with the coordinate \( \tau_i \) \( (i = 1, \ldots, r) \) and let \( \varphi_i : (\mathbb{C}_i, 0) \to (\mathbb{C}^n, 0) \) be parameterizations (uniformizations) of the branches \( C_i \) of the curve \( C \), i.e., germs of analytic maps such that \( \text{Im } \varphi_i = C_i \) and \( \varphi_i \) is an isomorphism between \( \mathbb{C}_i \) and \( C_i \) outside of the origin. Let \( \mathcal{O}_{\mathbb{C}^n, 0} \) be the ring of germs of holomorphic functions at the origin in \( \mathbb{C}^n \). For a germ \( g \in \mathcal{O}_{\mathbb{C}^n, 0} \), let \( v_i = v_i(g) \) and \( a_i = a_i(g) \) be the power of the leading term and the coefficient at it in the power series decomposition of the germ \( g \circ \varphi_i : (\mathbb{C}_i, 0) \to \mathbb{C} \): \( g \circ \varphi_i(\tau_i) = a_i \cdot \tau_i^{v_i} + \text{ terms of higher degree} \) \( (a_i \neq 0) \). If \( g \circ \varphi_i(t) \equiv 0 \), \( v_i(g) \) is assumed to be equal to \( \infty \) and \( a_i(g) \) is not defined. The numbers \( v_i(g) \) and \( a_i(g) \) are defined for elements \( g \) of the ring \( \mathcal{O}_C \) of functions on the curve \( C \) as well.

The semigroup \( S = S_C \) of the curve singularity \( C \) is the subsemigroup of \( \mathbb{Z}^r_{\geq 0} \) which consists of elements of the form \( \underline{v}(g) = (v_1(g), \ldots, v_r(g)) \) for all germs \( g \in \mathcal{O}_C \) with \( v_i(g) < \infty \); \( i = 1, \ldots, r \). The extended semigroup \( \hat{S} = \hat{S}_C \) of the curve singularity \( C \) was defined in [3]. It is
the subsemigroup of \( \mathbb{Z}^r_+ \times (\mathbb{C}^*)^r \) which consists of elements of the form 
\[(\mathbf{v}(g); \mathbf{a}(g)) = (v_1(g), \ldots, v_r(g); a_1(g), \ldots, a_r(g)) \text{ for all germs } g \in \mathcal{O}_C \text{ with } v_i(g) < \infty, i = 1, \ldots, r ( \text{3} ) \). The extended semigroup \( \hat{S}_C \) is well-defined (i.e., does not depend on the choice of the parameterizations \( \varphi_i \)) up to a natural equivalence relation.

It is known that both the semigroup \( S_C \) and the Alexander polynomial \( \Delta^C(t_1, \ldots, t_r) \) are complete topological invariants of a plane curve singularity \( C \), i.e., each of them determines the germ \( C \subset (\mathbb{C}^2, 0) \) up to topological equivalence \( (13), (14) \). The formulae discussed here describe a connection between them. (In fact from the Eisenbud–Neumann formula for the Alexander polynomial in terms of a resolution of a plane curve singularity (see equation (*) above) it is not difficult to understand that the Alexander polynomial \( \Delta^C(t_1, \ldots, t_r) \) may contain with non-zero coefficients only monomials \( \mathbf{t} \) for \( \mathbf{v} \) from the semigroup \( S_C \) of the curve \( C \).)

For a curve singularity \( C = \bigcup_{i=1}^r C_i \subset (\mathbb{C}^n, 0) \), let \( \pi : \hat{S}_C \rightarrow \mathbb{Z}^r \) be the natural projection: \( (\mathbf{v}, \mathbf{a}) \mapsto \mathbf{v} \). For an element \( \mathbf{v} \in \mathbb{Z}^r \), the preimage \( F_\mathbf{v} = \pi^{-1}(\mathbf{v}) \subset \{ \mathbf{v} \} \times (\mathbb{C}^*)^r \subset \{ \mathbf{v} \} \times \mathbb{C}^r \) is called the fibre of the extended semigroup. Though \( F_\mathbf{v} \) is empty for \( \mathbf{v} \notin \mathbb{Z}^r_+ \), we define it for all \( \mathbf{v} \in \mathbb{Z}^r \) in order not to meet (formal) problems with the notations in the Proof of Theorem 3. The fibre \( F_\mathbf{v} \) is not empty if and only if \( \mathbf{v} \in S_C \). For \( \mathbf{v} = (v_1, \ldots, v_r) \in \mathbb{Z}^r \), let \( J(\mathbf{v}) = \{ g \in \mathcal{O}_C : v_i(g) \geq v_i; i = 1, \ldots, r \} \) which is an ideal in \( \mathcal{O}_C \).

One has a natural linear map \( j_\mathbf{v} : J(\mathbf{v}) \rightarrow \mathbb{C}^r \), which sends \( g \in J(\mathbf{v}) \) to \( (a_1, \ldots, a_r) \), where \( a_i \) is the coefficient in the power series decomposition \( g \circ \varphi_i(\tau_i) = a_i \tau_i^{\eta_i} + \text{ terms of higher degree} \) (the number \( a_i \) may be equal to zero). Let \( C(\mathbf{v}) \subset \mathbb{C}^r \) be the image of the map \( j_\mathbf{v} \), \( c(\mathbf{v}) = \dim C(\mathbf{v}) \). It is not difficult to see that \( C(\mathbf{v}) \cong J(\mathbf{v})/J(\mathbf{v} + 1) \), where \( 1 = (1, \ldots, 1) \), and that \( F_\mathbf{v} = C(\mathbf{v}) \cap (\mathbb{C}^r)^r \) (under the natural identification of \( \{ \mathbf{v} \} \times (\mathbb{C}^*)^r \) and \( (\mathbb{C}^r)^r \)). Therefore, for \( \mathbf{v} \in S_C \), the fibre \( F_\mathbf{v} \) is the complement to an arrangement of linear hyperplanes in the linear space \( C(\mathbf{v}) \).

Remark. For a plane curve singularity \( C \), the extended semigroup \( \hat{S}_C \) contains some analytic information about the curve, however the dimensions \( c(\mathbf{v}) \) (and actually the combinatorial types of the arrangements of hyperplanes \( C(\mathbf{v}) \cap (\mathbb{C}^r \setminus (\mathbb{C}^*)^r) \subset C(\mathbf{v}) \)) depend only on the topological type of the curve \( C \) (see [3]).

Let \( \mathcal{L} = \mathbb{Z}[t_1, \ldots, t_r, t_1^{-1}, \ldots, t_r^{-1}] \) be the set of formal Laurent series in \( t_1, \ldots, t_r \). Elements of \( \mathcal{L} \) are expressions of the form \( \sum_{\mathbf{v} \in \mathbb{Z}^r} k(\mathbf{v}) \cdot \mathbf{t}^\mathbf{v} \) with \( k(\mathbf{v}) \in \mathbb{Z} \), generally speaking, infinite in all directions. \( \mathcal{L} \) is not a ring, but a \( \mathbb{Z}[t_1, \ldots, t_r] \)-module (or even a \( \mathbb{Z}[t_1, \ldots, t_r, t_1^{-1}, \ldots, t_r^{-1}] \)-module). The polynomial ring \( \mathbb{Z}[t_1, \ldots, t_r] \) can be in a natural way considered as being
Let

\[ L_C(t_1, \ldots, t_r) = \sum_{\nu \in \mathbb{Z}^r} c(\nu) \cdot \nu^\mathbf{t} \in \mathcal{L}. \]

\( L_C(\mathbf{t}) \) is not a power series, but a Laurent series infinitely long in all directions, since \( c(\nu) \) can be positive for \( \nu \) with (some) negative components \( v_i \) as well. For example, if there exists a germ \( g \in \mathcal{O}_{\mathbb{C}^n,0} \) with \( v_1(g) = v_1^* \), then for any \( v_2, \ldots, v_r \) such that \( v_i \leq v_i(g) \) (including negative ones), the germ \( g \) represents a non-trivial element in \( J(\nu)/J(\nu + 1) \) where \( \nu = (v_1^*, v_2, \ldots, v_r) \), \( J(\nu) = \{ g \in \mathcal{O}_{\mathbb{C}^n,0} : v(g) \geq \nu \} \). One can understand that along each line in the lattice \( \mathbb{Z}^r \) parallel to a coordinate one the coefficients \( c(\nu) \) stabilize in each direction, i.e., if \( v_i' \) and \( v_i'' \) are negative, or if \( v_i' \) and \( v_i'' \) are positive and large enough, then

\[
c(v_1, \ldots, v_i', \ldots, v_r) = c(v_1, \ldots, v_i'', \ldots, v_r).\]

This implies that

\[
P_C'(t_1, \ldots, t_r) = L_C(t_1, \ldots, t_r) \cdot \prod_{i=1}^{r} (t_i - 1)
\]

is a polynomial (it also follows from the proof of Theorem 3).

**Proposition 2** For a curve singularity \( C = \bigcup_{i=1}^{r} C_i \subset (\mathbb{C}^n,0), r > 1 \), the polynomial \( P'_C(t_1, \ldots, t_r) \) is divisible by \( (t_1 \cdots t_r - 1) \), i.e., the power series

\[
P_C(t_1, \ldots, t_r) = P'_C(t_1, \ldots, t_r)/(t_1 \cdots t_r - 1) \in \mathbb{Z}[[t_1, \ldots, t_r]]
\]

is, in fact, a polynomial.

The proposition follows from the proof of Theorem 3 (see below).

We call \( P_C(t_1, \ldots, t_r), r > 1 \), the (generalized) **Poincaré polynomial** of the curve singularity \( C \). For \( r = 1 \), \( P_C(t) \) is not a polynomial, but a power series and it coincides with usual Poincaré series of the filtration in \( \mathcal{O}_C \) defined by a normalization.

**Remark.** Theorem 3 implies that, for a plane curve singularity, the polynomial \( P_C(\mathbf{t}) \) determines the semigroup \( S_C \). This is not the case for non–plane curves (see [8]).

### 2.3 The semigroup of an irreducible plane curve singularity.

If the curve \( C \) is irreducible its dual graph looks like on Fig. 1. After Zariski (see e.g. [14]) it is known that the set of elements \( \{ \beta_j := m^{\alpha_j} : 0 \leq j \leq g \} \) is the minimal system of generators of the semigroup of values \( S_C \subset \mathbb{Z}_{\geq 0}; \)

[111x644]embedded into \( \mathcal{L} \).

Let

\[
L_C(t_1, \ldots, t_r) = \sum_{\nu \in \mathbb{Z}^r} c(\nu) \cdot \nu^\mathbf{t} \in \mathcal{L}.
\]
moreover $m^{\beta_j} = (n_j + 1)m^{\alpha_j}$ for some integers $n_j$, $j = 1, \ldots, g$ (integers $n_j + 1$ are in fact parts of the Puiseux pairs of the curve; the vertices $\alpha_j$ and $\beta_j$ are indicated on Fig. 1). In what follows we shall use the following properties of the minimal embedded resolution and of the semigroup $S_C$ of an irreducible curve singularity (see, e.g., [13], [10]):

1. There is only a finite number of positive integers which do not belong to the semigroup $S_C$ and the largest one $\delta - 1$ is equal to $\sum_{j=1}^g n_j \tilde{\beta}_j - \tilde{\beta}_0$ ($\delta$ is called the conductor of the semigroup $S_C$).

2. Each element $v \in S_C$ can be uniquely represented in the form $v = k_0\tilde{\beta}_0 + \sum_{j=1}^g k_j\tilde{\beta}_j$ with $k_0 \geq 0$ and $0 \leq k_j \leq n_j$ for $1 \leq j \leq g$. 

3. $(n_j + 1)\tilde{\beta}_j < \tilde{\beta}_{j+1}$ for $j = 1, \ldots, g - 1$.

4. The element $(n_j + 1)\tilde{\beta}_j$ belongs to the semigroup $\langle \tilde{\beta}_0, \ldots, \tilde{\beta}_{j-1} \rangle$ generated by $\tilde{\beta}_0, \ldots, \tilde{\beta}_{j-1}$ ($j = 1, \ldots, g$).

5. If, for a germ $\varphi \in O_{C_{2,0}}$, the strict transform of the curve $\{ \varphi = 0 \}$ intersects only divisors $\sigma$ with $\sigma < \beta_j$ (i.e., those which lie between $\alpha_0$ and $\beta_j$) then $v(\varphi) \in \langle \tilde{\beta}_0, \ldots, \tilde{\beta}_{j-1} \rangle$.

According to [1] the zeta-function $\zeta_C(t)$ of the singularity $C = \{ f = 0 \}$ is equal to $\prod_{\sigma} (1 - t^{\mu_{\sigma}})^{-\chi(E_{\sigma})}$. The property 2 (unique decomposition) permits to easily prove the equality of the Poincaré series $P_C(t)$ and the zeta-function $\zeta_C(t)$ in this case ([1]; in fact the proof is almost repeated in the proof of Proposition [1], subsection 3.2).

2.4 Graded topological spaces and the Euler characteristic.

To describe some constructions in the proof of Theorems [1] and [2] it is convenient to use the notion of a graded space. By a graded space (with
$r$–grading) we shall have in mind a disjoint union $Z$ of topological spaces $Z_v$ corresponding to elements $v$ from $\mathbb{Z}_{\geq 0}$. We shall write $Z = \sum_{v \in \mathbb{Z}_{\geq 0}} Z_v \cdot t^v$. If $Z'$ and $Z''$ are graded spaces, then their sum (disjoint union) $Z' + Z''$ and their product $Z' \times Z''$ are graded spaces as well (in the natural sense; e.g., if $Z' = \sum_{v \in \mathbb{Z}_{\geq 0}} Z'_v \cdot t^v$, $Z'' = \sum_{v \in \mathbb{Z}_{\geq 0}} Z''_v \cdot t^v$, then $Z' \times Z'' = \sum_{v \in \mathbb{Z}_{\geq 0}} \sum_{v' + v'' = v} (Z'_v \times Z''_{v'}) \cdot t^v$).

A map $Z' \to Z''$ of graded spaces is a set of maps $Z'_v \to Z''_v$. For a graded space $Z = \sum_{v \in \mathbb{Z}_{\geq 0}} Z_v \cdot t^v$ its Euler characteristic (see the remarks below)

$$\chi(Z) = \sum_{v \in \mathbb{Z}_{\geq 0}} \chi(Z_v) \cdot t^v \in \mathbb{Z}[t_1, \ldots, t_r].$$

One has $\chi(Z' + Z'') = \chi(Z') + \chi(Z'')$, $\chi(Z' \times Z'') = \chi(Z') \cdot \chi(Z'')$. A graded semigroup is a graded space with a (commutative) semigroup operation which respects the grading. The extended semigroup of a plane curve singularity is in the natural sense a graded semigroup.

There exist somewhat different definitions of the Euler characteristic which do not coincide for non-compact sets. (For compact spaces (say, for projective varieties) all definitions of the Euler characteristic are essentially the same.) The most usual definition of the Euler characteristic of a topological space (say, of a CW–complex) $X$ is

$$\chi(X) = \sum_{q \geq 0} (-1)^q \dim H_q(X; \mathbb{R}).$$

If $X = X_1 \cup X_2$ where the spaces (CW–complexes) $X$, $X_1$ and $X_2$ are compact, one has $\chi(X) = \chi(X_1) + \chi(X_2) - \chi(X_1 \cap X_2)$. Therefore for compact spaces the Euler characteristic possesses the additivity property. This permits to consider it as a generalized (nonpositive) measure on the algebra of such spaces. However spaces we are interested in (e.g., fibres of the extended semigroup of a curve) are noncompact semialgebraic sets (complex or real). The Euler characteristic defined above does not possess the additivity property for such spaces. For example, let $X$ be the circle $S^1$, let $X_1$ be a point of $X$, and let $X_2 = X \setminus X_1$ be a (real) line. Then one has $\chi(X) = 0$, $\chi(X_1) = \chi(X_2) = 1$, $\chi(X_1 \cap X_2) = \chi(\emptyset) = 0$, and $0 \neq 1 + 1 - 0$.

In order to have the desired additivity property one should define the Euler characteristic $\chi(X)$ of a semialgebraic space $X$ (the difference of two projective spaces) as

$$\sum_{q \geq 0} (-1)^q \dim H_q(X^*, *; \mathbb{R}),$$

where $X^*$ is the one-point compactification of the space $X$ (if $X$ is compact, the one-point compactification of it is the disjoint union of $X$ with a point), $*$ is the added ("infinite") point. We shall use this definition. The algebra generated by semialgebraic sets consists of constructible sets. A constructible set can be represented as the disjoint union of a finite number
of semianalytic sets. The Euler characteristic of a constructible set should be defined as the sum of Euler characteristics of the corresponding semialgebraic sets. One can show that the Euler characteristic defined this way does possess the additivity property (in the example above $\chi(X) = 0$, $\chi(X_1) = 1$, $\chi(X_2) = -1$). Moreover, a constructible set $X$ can be represented as a disjoint union of a finite number of open cells so that the boundary of a cell of some dimension (its closure in $X$ minus itself) lies in the union of cells of smaller dimensions. (This does not mean a representation of the space $X$ as a CW–complex since in general (for noncompact sets) one does not have maps of closed balls into $X$ which determine the cells. For example the real line $\mathbb{R}^1$ is simply one cell of dimension 1.) One can see that the Euler characteristic of the (constructible) set $X$ is equal to the alternating sum of numbers of cells of different dimensions. The Euler characteristic also possesses the multiplicativity property: $\chi(X_1 \times X_2) = \chi(X_1) \cdot \chi(X_2)$.

Let $X$ be a topological space and let $S^k X$ be its $k$-th symmetric power ($S^k X = \frac{X \times \ldots \times X}{S_k}$, i.e., the space of unordered $k$–tuples of points of $X$, where $S_k$ is the group of permutations of $k$ elements; $S^0 X = \bullet$ is a point). The graded space (with 1-grading) $S^0 X + S^1 X \cdot t + S^2 X \cdot t^2 + \cdots$ is in fact a graded semigroup with the semigroup operation defined by the union of $k$–tuples of points.

**Lemma 1**

$$\chi(S^0 X + S^1 X \cdot t + S^2 X \cdot t^2 + \cdots) = (1 - t)^{-\chi(X)}.$$ 

**Proof.** Let us denote $\chi(X)$ simply by $\chi$. The coefficient at $t^k$ in $(1 - t)^{-\chi}$ is equal to 

$$\binom{\chi + k - 1}{k} = (-1)^k \binom{-\chi}{k} = \frac{\chi(\chi + 1) \cdot \ldots \cdot (\chi + k - 1)}{1 \cdot \ldots \cdot k}.$$ 

It is clear that $\chi(S^k X)$ is a polynomial in $\chi$. To show that this polynomial coincides with $\binom{\chi+k-1}{k}$ one can check it for an infinite set of values of $\chi$. Let us take $\chi$ positive, and let $X$ be the (disjoint) union of $\chi$ points ($\chi(X) = \chi$). In this case $S^k X$ consists of $\binom{\chi+k-1}{k}$ points and thus $\chi(S^k X) = \#S^k X = \binom{\chi+k-1}{k}$. ~□

**Remark.** For $X = \{Z_1, \ldots, Z_\chi\}$ ($Z_i$ are points), this formula is a consequence of the equation

$$\left(1 + \sum_{k=1}^{\infty} \left( \sum_{n_1 + \ldots + n_\chi = k} Z_i^{n_i} \right) t^k \right) \prod_{i=1}^{\chi} (1 - Z_i t) = 1.$$
Corollary 1 If \( \chi(X) \leq 0 \) and \( k \geq -\chi(X) + 1 \), then \( \chi(S^kX) = 0 \). In particular, for \( X = \mathbb{CP}^1 \setminus \{ s \text{ points} \} \), \( s \geq 2 \), \( \chi(S^kX) = 0 \) for \( k \geq s - 1 \).

The group \( \mathbb{C}^* \) of non-zero complex numbers acts (freely) on \( \mathbb{Z}_{\geq 0}^r \times (\mathbb{C}^*)^r \) (by multiplication of all the coordinates \( a_i \)). The corresponding factor–space \( \mathbb{Z}_{\geq 0}^r \times (\mathbb{C}^*)^r / \mathbb{C}^* = \mathbb{Z}_{\geq 0}^r \times \mathbb{P}((\mathbb{C}^*)^r) = \sum_{\nu \in \mathbb{Z}_{\geq 0}^r} \mathbb{P}((\mathbb{C}^*)^r) \cdot t^\nu \) has a natural structure of a semigroup. The extended semigroup \( \hat{S}_C \subset \mathbb{Z}_{\geq 0}^r \times (\mathbb{C}^*)^r \) is invariant with respect to the \( \mathbb{C}^* \)-action. The factor–space \( \mathbb{P}\hat{S}_C = \hat{S}_C / \mathbb{C}^* \) will be called the projectivization of the extended semigroup of the curve \( C \) (it is also a graded semigroup in the natural sense). One has \( \mathbb{P}\hat{S}_C = \sum_{\nu \in \mathbb{Z}_{\geq 0}^r} \mathbb{P}F_{\nu} \cdot t^\nu \), where \( \mathbb{P}F_{\nu} = F_{\nu} / \mathbb{C}^* \) is the projectivization of the fibre \( F_{\nu} \). For \( \nu \in S_C \), the space \( \mathbb{P}F_{\nu} \) is the complement to an arrangement of projective hyperplanes in a \( (c(\nu) - 1) \)-dimensional complex projective space \( \mathbb{P}C(\nu) \), which can be identified with the complement to an arrangement of affine hyperplanes in a complex affine space after choosing one of the coordinates \( a_i \). Let \( \delta = (\delta_1, \ldots, \delta_r) \) be the conductor of the semigroup \( S_C \) of the curve \( C \), i.e., the minimal element for which \( \delta + \mathbb{Z}_{\geq 0}^r \subset S_C \). If \( \nu \geq \delta \) (i.e., if \( v_i \geq \delta_i \) for all \( i = 1, \ldots, r \)), then the fibre \( F_{\nu} \) of the extended semigroup coincides with \( (\mathbb{C}^*)^r \) and the Euler characteristic \( \chi(\mathbb{P}F_{\nu}) \) of its projectivization is equal to \( 0 \) (for \( r > 1 \); for \( r = 1 \) it is equal to \( 1 \)). Moreover one can show that the Euler characteristic \( \chi(\mathbb{P}\hat{S}_C) \) is a polynomial in \( t_1, \ldots, t_r \) (see Theorem 3). This polynomial participates in the formulation of Theorem 3.

3 Proofs

The following statement shows that Theorems 1 and 2 are equivalent to each other.

Theorem 3 For an arbitrary (i.e., not necessarily plane) curve singularity \( C = \bigcup_{i=1}^r C_i \subset (C^n, 0) \), \( r > 1 \), \( \chi(\mathbb{P}\hat{S}_C) \) is a polynomial and \( \chi(\mathbb{P}\hat{S}_C) \cdot (t_1 \cdots t_r - 1) = P_C(\boldsymbol{t}) \). As a consequence, \( P_C(\boldsymbol{t}) \) is a polynomial and \( \chi(\mathbb{P}\hat{S}_C) = P_C(\boldsymbol{t}) \).

Proof. Let \( \nu \) be an element of \( \mathbb{Z}_{\geq 0}^r \), and let \( b_\nu = \dim J(\nu) / J(\nu) \). For \( I \subset I_0 = \{ 1, 2, \ldots, r \} \), let \( \# I \) be the number of elements in \( I \), and let \( 1_I \) be the element of \( \mathbb{Z}_{\geq 0}^r \), the \( i \)-th component of which is equal to \( 1 \) (respectively to \( 0 \)) if \( i \in I \) (respectively if \( i \not\in I \)). One has \( 1_{\emptyset} = 1 \). Let \( L_I \subset \mathbb{C}^r \) be the subspace \( \{(a_1, \ldots, a_r) \in \mathbb{C}^r : a_i = 0 \text{ for } i \in I\} \).
One has

\[ \chi(\mathbb{P}F_v) = \chi(\mathbb{P}C(v)) - \chi(\bigcup_{i=1}^r \mathbb{P}(C(v) \cap L_{i})) = \]

\[ = \chi(\mathbb{P}C(v)) - \sum_{I \subset I_0, I \neq \emptyset} (-1)^{|I|-1} \chi(\mathbb{P}(C(v) \cap L_I)) = \]

\[ = \sum_{I \subset I_0} (-1)^{|I|} \chi(\mathbb{P}(C(v) \cap L_I)) = \sum_{I \subset I_0} (-1)^{|I|} \dim(C(v) \cap L_I). \]

If \( v \leq w - 1, \) \( \dim(C(v) \cap L_I) = b_{\underline{v}+1} - b_{\underline{v}+1} \) and therefore \( \chi(\mathbb{P}F_v) = \sum_{I \subset I_0} (-1)^{|I|}(b_{\underline{v}+1} - b_{\underline{v}+1}). \) This implies that the coefficient at \( t^v \) in the series \( \chi(\mathbb{P}\hat{S}_C) \cdot (t_1 \cdot \ldots \cdot t_r - 1) \) is equal to

\[ \sum_{I \subset I_0} (-1)^{|I|}(b_{\underline{v}+1} - b_{\underline{v}}) - \sum_{I \subset I_0} (-1)^{|I|}(b_{\underline{v}+1} - b_{\underline{v}+1}) \]

and, since \( \sum_{I \subset I_0} (-1)^{|I|} = 0, \) also to

\[ \sum_{I \subset I_0} (-1)^{|I|}(b_{\underline{v}-1} - b_{\underline{v}}) = \sum_{I \subset I_0} (-1)^{|I|} c(v - 1 + 1_I). \]

The coefficient at \( t^v \) in the polynomial \( P'_C(t) = (\sum c(v) t^v) \cdot \left( \prod_{i=1}^r (t_i - 1) \right) \) is also equal to \( \sum_{I \subset I_0} (-1)^{|I|} c(v - 1 + 1_I). \)

The facts that the series \( \chi(\mathbb{P}\hat{S}_C) \) does not contain (with non-zero coefficients) monomials \( t^v \) with \( v \geq \delta \) and \( P'_C(t) \) is a polynomial imply that \( \chi(\mathbb{P}\hat{S}_C) \) is a polynomial as well. □

**Proof of Theorem 2.** By the Eisenbud-Neumann formula, Theorem 2 follows from

**Theorem 4** \( \chi(\mathbb{P}\hat{S}_C) = \prod_\sigma(1 - t^{\alpha_\sigma})^{-\chi(E_\sigma)}. \)

The main course of the proof for Theorem 4 goes as follows. We use an embedded resolution of the curve \( C. \) In terms of the curve \( \mathbb{P}\hat{S}_C \) we construct a graded space (with \( r \)-grading; in fact a graded semigroup) \( Y \) such that its Euler characteristic is equal to the Alexander polynomial \( \Delta_C(t_1, \ldots, t_r). \) The construction of the space \( Y \) is natural after Lemma [4] above which provides the equality of the Euler characteristic of \( Y \) and the Alexander polynomial \( \Delta_C(t). \) Points of \( Y \) are represented by unordered \( k \)-tuples (with different \( k \)) of points of the exceptional divisor of the resolution without the self-intersection points of it and the intersection points of it with the
strict transform of the curve $C$. The semigroup operation is defined by the union of $k$-tuples. The grading is defined by the multiplicities $m^\sigma$ of the components $E^\sigma$ of the exceptional divisor on which the points lie.

We construct a map (a graded semigroup homomorphism) $\Pi$ from $Y$ to the projectivisation $\mathbb{P}\hat{S}_C$ of the extended semigroup $\hat{S}_C$ which is surjective up to a grading high enough, i.e., $\Pi$ maps $Y_\varphi$ onto the fibre $\mathbb{P}F_\varphi$ for $\varphi \leq V$, where $V$ is an arbitrary point of $\mathbb{Z}_{\geq 0}$ chosen in advance. This map is defined in the following way. For a point $y \in Y$, we take a function $g \in \mathcal{O}_{C^2,0}$ such that the strict transform of the curve $\{g = 0\}$ intersects the exceptional divisor of the resolution of the curve $C$ just at the points which define the element $y$ and we put $\Pi(y) = (v_1(g), \ldots, v_r(g); a_1(g) : \ldots : a_r(g))$.

If $\Pi$ would be injective this would complete the proof. However the space $Y$ is too big and $\Pi$ is very far from being injective. In 3.2 we reduce $Y$ to another space, $\tilde{Y}$, together with the corresponding map $\tilde{\Pi} : \tilde{Y} \to \mathbb{P}(\hat{S}_C)$ so that $\chi(\tilde{Y}) = \chi(Y) = \Delta^C(t_1, \ldots, t_r)$. Roughly speaking, this reduction consists of excluding all dead ends of the resolution graph. This is a way to omit some obvious repeated images by $\tilde{\Pi}$. This permits to pay the main attention only to those components $E^\sigma$, $\sigma \in \Gamma$, of the exceptional divisor $D$, for which the number $s^\sigma$ of essential points is $\geq 2$. In Lemma 3 we show that, if $\tilde{\Pi}(y_1) = \tilde{\Pi}(y_2)$ ($y_i \in \tilde{Y}$), then $y_1$ and $y_2$ have ”many” points ($\geq s^\sigma - 1 \geq 1$) on some components $E^\sigma$ of the exceptional divisor (generally speaking, different for $y_1$ and $y_2$). If an element $y \in \tilde{Y}$ is represented by a $k$-tuple with at least $s^\sigma - 1$ of them on the component $E^\sigma$ of the exceptional divisor with $s^\sigma \geq 2$, we say that $\sigma$ is a cut of $y$ (or rather of the connected component of $\tilde{Y}$ where $y$ lies).

Moreover, in 3.3 we prove more fine statements about the distribution of the cuts of $y_1$ and $y_2$ in this case. Namely, up to the numbering of $y_1$ and $y_2$ there exists a cut $\sigma$ of $y_1$ such that for each strict transform $\tilde{C}_i$ of a branch $C_i$ of the curve $C$ greater than $\sigma$ there is a cut of $y_2$ on the geodesic from $\sigma$ to $\tilde{C}_i$ on the dual graph of the resolution. This is the most complicated (combinatorial) part of the proof.

To prove that $\chi(Im \tilde{\Pi}) = \chi(\tilde{Y})$, in 3.4 we analyse places where the map $\tilde{\Pi}$ is not injective. At such places we indicate some parts of the space $\tilde{Y}$ which are fibred into complex tori $(\mathbb{C}^*)^{s-1}$, $s \geq 2$ (and thus have zero Euler characteristic). Removing these parts does not change the Euler characteristic of the source and does not change the image $\tilde{\Pi}(\tilde{Y})$. This permits to obtain a one-to-one correspondence between them. This completes the proof.

One can say that the proof consists of an explicit computation of $\chi(\mathbb{P}\hat{S}_C)$ and of $\Delta^C(t_1, \ldots, t^r)$ in terms of an embedded resolution of the curve $C$ which shows that they coincide. At the moment a direct proof which ex-
plains why the Euler characteristic $\chi(\mathbb{P}\hat{S}_C)$ coincides with the Alexander polynomial is not known.

3.1 Construction of the space $Y$ and the map $\Pi$.

One can easily see that the right–hand side of the equation of Theorem 4 does not depend on the resolution of the curve $C$. Let $V = (V_1, \ldots, V_r)$ be an arbitrary point of the lattice $\mathbb{Z}^r_{\geq 0}$. Let us take the minimal (embedded) resolution of the curve $C$ and let us make additional blow–ups of intersection points of components of the total transform of the curve $C$ so that, for each function $g \in O_{\mathbb{C}^2, 0}$ with $v(g) \leq V$, the strict transform of the curve $\{g = 0\}$ intersects the total transform $(f \circ \pi)^{-1}(0)$ of the curve $C$ only at smooth points of $(f \circ \pi)^{-1}(0)$. (These additional blow-ups do not change dead ends and star points of the dual graph $\Gamma$ of the resolution. Moreover, one can see that, if $V$ is big enough (say, $V \geq \delta$), then each strict transform $\tilde{C}$ lies on its individual "long" branch of the graph $\Gamma$.) We fix such a resolution for the rest of the paper. Let

$$Y = \prod_\sigma \left( \bullet + \sum_{k=1}^r S^1 \hat{E}_{\sigma} \cdot t_{\sum_{j=1}^r k_j m_j} + \sum_{k=1}^r S^2 \hat{E}_{\sigma} \cdot t_{2m_j} + \ldots \right)$$

where $S^k \hat{E}_{\sigma}$ is the $k$th symmetric power of $\hat{E}_{\sigma}$ and $\bullet$ is a point ($= S^0 \hat{E}_{\sigma}$). Lemma 1 implies that $\chi(Y) = \prod_\sigma (1 - \sum_{j=1}^r m_j)^{-v(\hat{E}_{\sigma})}$.

Let us define a map $\Pi : Y \to \mathbb{P}\hat{S}_C$ as follows. One has

$$Y = \sum_{\{k_\sigma\}} \left( \prod_\sigma S^{k_\sigma} \hat{E}_{\sigma} \cdot t_{\sum_{j=1}^r k_j m_j} \right).$$

A point $y$ of the space $\prod_\sigma (S^{k_\sigma} \hat{E}_{\sigma} \cdot t_{k_j m_j})$ is represented by a set of smooth points of the exceptional divisor $D$ (i.e., of $\hat{D} = \bigcup_\sigma \hat{E}_{\sigma}$) with $k_\sigma$ points $Q^\sigma_1$, $\ldots$, $Q^\sigma_{k_\sigma}$ on the component $\hat{E}_{\sigma}$. For a point $A \in \hat{D}$, let $\tilde{L}_A$ be a germ of a nonsingular (complex analytic) curve transversal to the exceptional divisor $D$ at the point $A$. Let the image $L_A = \pi(\tilde{L}_A) \subset (\mathbb{C}^2, 0)$ of the curve $\tilde{L}_A$ be given by an equation $\{g_A = 0\}$ ($g_A \in O_{\mathbb{C}^2, 0}$). By definition $\Pi(y) \in \mathbb{P}\hat{S}_C$ is represented by the element $((v(g), a(g)) \in \hat{S}_C$, where $g = \prod_\sigma \prod_{j=1}^{k_\sigma} g_{Q^\sigma_j}$.

**Lemma 2** The element $\Pi(y) \in \mathbb{P}\hat{S}_C$ does not depend on the choice of curves $\tilde{L}_A$. 

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Proof. Let \( \tilde{L}'_A \) be another germ of a nonsingular (complex) curve transversal to the exceptional divisor \( D \) at the point \( A \in \tilde{D} \), \( L'_A = \pi(\tilde{L}'_A) = \{ g'_A = 0 \} \), and let \( g' = \prod_{j=1}^{k_\sigma} g'_{Q_j} \). Let \( \bar{g} = g \circ \pi \) and \( \bar{g}' = g' \circ \pi \) be the liftings of the functions \( g \) and \( g' \) to the space \( X \) of the resolution, and let \( \psi = \bar{g}' / \bar{g} \) be their ratio. The function \( \psi \) has zeros along the curves \( \tilde{L}'_{Q_j} \) and poles along the curves \( \tilde{L}_{Q_j} \). Therefore the restriction of the function \( \psi \) to the exceptional divisor \( D \) is a regular (holomorphic) function on \( D \) and thus \( \psi \) is a constant (say, \( c \)) on \( D \). It implies that 

\[
[\nu(g'), \nu(g)] = \nu(g'_A) = \nu(g_A) = c \cdot [\nu(g), \nu(g)]
\]

and therefore the elements in \( \bar{\mathbb{P}} \hat{S}C \), represented by \( (\nu(g), \nu(g)) \) and \( (\nu(g'), \nu(g')) \), coincide. \( \Box \)

Remark. In fact one can say that \( Y \) is a graded semigroup (with respect to the operation defined by the union of sets) and \( \Pi \) is a graded semigroup homomorphism. Then it is sufficient to define \( \Pi \) only for monomials of the form \( [A] \cdot \bar{t}^{m_\sigma} \), where \( A \) is a point of \( \tilde{E}_\sigma \). We shall use this to define the map \( \bar{\Pi} \) below.

Proposition 3 For \( \nu \leq \underline{V} \) one has \( (Im \Pi)_\nu = \bar{\mathbb{P}}F_\nu \).

Proof. If \( g \in \mathcal{O}_{\mathbb{C}^2,0} \) and \( \nu(g) \leq \underline{V} \), then the strict transform \( \tilde{L} \) of the curve \( \{ g = 0 \} \) intersects the exceptional divisor \( D \) only at smooth points of \( (f \circ \pi)^{-1}(0) \). Let \( Q_j^\sigma \) (\( j = 1, \ldots, k_\sigma \)) be the points of intersection of the curve \( \tilde{L} \) with the component \( E_\sigma \) of the exceptional divisor counting with their multiplicities, i.e., each point is taken as many times as the intersection number of the curve \( \tilde{L} \) with the component \( E_\sigma \) at it, let \( K_\sigma = \{ Q_1^\sigma, \ldots, Q_{k_\sigma}^\sigma \} \subset \tilde{E}_\sigma \). For a subset \( K_\sigma \) of \( \tilde{E}_\sigma \) (or of \( \tilde{E}_\sigma \)) with \( \#K_\sigma = k_\sigma \), by \( [K_\sigma] \) we shall denote the corresponding point of the space \( \bar{S}^{k_\sigma} \tilde{E}_\sigma \) (or of \( S^{k_\sigma} \tilde{E}_\sigma \)). Then one can see that

\[
(\nu(g), \nu(g)) = \left( \nu(\prod_{\sigma} \prod_{j=1}^{k_\sigma} g_{Q_j^\sigma}), \nu(\prod_{\sigma} \prod_{j=1}^{k_\sigma} g_{Q_j^\sigma}) \right) = \Pi \left( \prod_{\sigma} ([K_\sigma] \cdot \bar{t}^{m_\sigma}) \right)
\]

(the proof repeats the one of Lemma 3). \( \Box \)

3.2 Reduction of the graded space \( Y \).

In order to reduce the space \( Y \) we will use arithmetical properties of the semigroup \( S_C \) and its relation with the dual graph of a resolution which can be found in [9], section (3.20) (see also [11], section 1).
Let $D'$ be the union of components $E_\sigma$ with at least two essential points, i.e., with $s_\sigma \geq 2$, and let $\Delta'$ be the set of the corresponding vertices. Connected components of the complement $D \setminus D'$, which do not contain the starting divisor $1$, are tails of the dual graph $\Gamma$ of the resolution and correspond to (some) dead ends $\delta$ of the graph $\Gamma$. Let $\Delta$ be the set of these dead ends. For $\delta \in \Delta$, $st_\delta$ is the vertex of $\Delta'$ such that $E_{st_\delta}$ intersects the corresponding connected component of $D \setminus D'$. Let

$$Y' = \prod_{\sigma \in \Delta'} (\bullet + S^1 \bar{E}_\sigma \cdot t^{m_\sigma} + S^2 \bar{E}_\sigma \cdot t^{2m_\sigma} + \ldots).$$

Pay attention that the spaces $\bar{E}_\sigma$ in the definition of $Y$ are substituted here by the spaces $\bar{E}_\sigma$. The possibility to deal with points of $\bar{E}_\sigma \setminus \bar{E}_\sigma$ in the same way as with other points of $\bar{E}_\sigma$ will be explained below (in the Remark after Proposition 5).

For a dead end $\delta \in \Delta$, $m^{st_\delta}$ is a multiple of $m^\delta$: $m^{st_\delta} = (n_\delta + 1) \cdot m^\delta$ (the number $n_\delta$ is the corresponding $n_j$ which appears in 2.3 for a branch $C_i$ such that the dead end $\delta$ belongs to the minimal resolution graph of the curve $C_i$). Let $Y_\delta = \sum_{k=0}^{n_\delta} \bullet \cdot t^{km_\delta}$.

Let us assume that $1 \neq st_1$. Let $\alpha_0 = 1$, $\alpha_1$, $\ldots$, $\alpha_q$ be the dead ends of the graph $\Gamma$ which do not belong to $\Delta$, and let $\beta_j$ ($j = 1, \ldots, q$) be the star point of the graph $\Gamma$ which corresponds to the dead end $\alpha_j$, $\beta_1 < \beta_2 < \ldots < \beta_q$ (see Fig 2). Let $S_1$ be the subsemigroup of the semigroup $S_C$ generated by the multiplicities $m^{\alpha_0}$, $m^{\alpha_1}$, $\ldots$, $m^{\alpha_q}$. $S_1$ coincides with the semigroup generated by all the multiplicities $m^\sigma$ with $E_\sigma$ from the connected component of $D \setminus D'$ which contains the starting divisor $1$ and is similar to a subsemigroup of $\mathbb{Z}_{\geq 0}$ because it is contained in the line $L$ in $\mathbb{R}^r \supset \mathbb{Z}_{\geq 0}$ which goes through the origin and the point $m^{\alpha_0}$. To describe it more
precisely, let \( e_i = \text{g.c.d.}(m_i^{\alpha_0}, \ldots, m_i^{\alpha_q}) \) for \( 1 \leq i \leq r \). Then the set of integers \( \{m_i^{\alpha_0}/e_i, \ m_i^{\alpha_1}/e_i, \ldots, m_i^{\alpha_q}/e_i\} \) does not depend on \( i \) and is the minimal set of generators of the semigroup, say \( \tilde{S} \), of an irreducible curve (for example, of the curve \( L_A \) with \( A \in \tilde{E}_{\beta_q} \)). Moreover \( \tilde{m} \in \tilde{S} \) if and only if \( \tilde{m} \cdot (e_1, \ldots, e_r) \in S_1 \).

As in the case of one branch one has \( \tilde{m}^{\beta_j} = (n_j + 1) \cdot \tilde{m}^{\alpha_i}, \tilde{m}^{\beta_j} \in \langle \tilde{m}^{\alpha_0}, \ldots, \tilde{m}^{\alpha_j-1} \rangle \). The multiplicity \( m^{st_1} \) belongs to the semigroup \( S_1 \) as well (see (5) in [2,3]). Let \( m^{st_1} = \sum_{j=0}^{q} \ell_j \cdot m^{\alpha_j} \), where \( \ell_j \leq n_j \) for \( j = 1, \ldots, q \) (such a representation is unique just as in the case of the semigroup of an irreducible curve). Let \( S'_1 \) be the subset of \( S_1 \) which consists of elements \( m \) such that \( m - m^{st_1} \notin S_1 \). \( S'_1 \) corresponds to the Apery base (see, e.g., [3]) of \( S \) with respect to \( m^{st_1}/e_i \) (this integer does not depends on \( i \)). Thus \( S'_1 \) is a finite set and the biggest element in it is equal to \( m^{st_1} + \sum_{j=1}^{q} n_j m^{\alpha_j} - m^{\alpha_0} \) (this follows from the expression for the conductor in [2,3]). Let \( Y_1 = \sum_{m \in S'_1} \bullet \cdot \frac{t}{m} \).

If \( 1 = st_1 \) we simply put \( Y_1 = \bullet \).

Let \( \tilde{Y} = Y' \times Y_1 \times \prod_{\delta \in \Delta} Y_\delta \).

**Proposition 4** \( \chi(\tilde{Y}) = \chi(Y) \).

**Proof.** One has \( \chi(Y) = \prod_{\sigma \in \Gamma} \left( 1 - \frac{t^{m^{\alpha_0}}}{\sigma} \right)^{-\chi(\tilde{E}_\sigma)} \). Since for all \( \sigma \) except those from \( \Delta', \Delta, \{\alpha_i\}, \) and \( \{\beta_i\} \), \( \chi(\tilde{E}_\sigma) = 0 \),

\[
\chi(Y) = \left( \prod_{\sigma \in \Delta'} \left( 1 - \frac{t^{m^{\alpha_0}}}{\sigma} \right)^{-\chi(\tilde{E}_\sigma)} \right) \times \left( \prod_{\delta \in \Delta} \left( 1 - \frac{t^{m^{\alpha_0}}}{\delta} \right)^{-1} \right) \times \frac{\prod_{i=1}^{q} (1 - t^{m^{\beta_i}})}{\prod_{i=0}^{q} (1 - t^{m^{\alpha_i}})} =
\]

\[
= \left( \prod_{\sigma \in \Delta'} \left( 1 - \frac{t^{m^{\alpha_0}}}{\sigma} \right)^{-\chi(\tilde{E}_\sigma)} \right) \times \left( \prod_{\delta \in \Delta} \frac{(1 - t^{m^{\alpha_0}/\delta})}{(1 - t^{m^{\alpha_0}})} \right) \times \frac{(1 - t^{m^{st_1}}) \prod_{i=0}^{q} (1 - t^{m^{\beta_i}})}{\prod_{i=0}^{q} (1 - t^{m^{\alpha_i}})}.
\]

The first factor coincides with \( \chi(Y') \). Now the statement follows from the facts that

\[
\chi(Y_\delta) = \frac{1 - t^{m^{st_\delta}}}{1 - t^{m^{\alpha_0}}}, \quad \chi(Y_1) = \frac{(1 - t^{m^{st_1}}) \prod_{i=1}^{q} (1 - t^{m^{\beta_i}})}{\prod_{i=0}^{q} (1 - t^{m^{\alpha_i}})}.
\]

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The first equation is obvious. To prove the second one let us notice that
\[
\sum_{\nu \in S_1} \ell^\nu = \left( \sum_{\nu \in S'_1} \ell^\nu \right) \cdot \left( 1 + t m_{st_1} + t^2 m_{st_1} + \ldots \right)
\]
(since each element \( s \in S_1 \) in a unique way is represented in the form \( \ell \cdot m_{st_1} + s' \) with \( s' \in S'_1, \ell \geq 0 \)) and
\[
\sum_{\nu \in S_1} \ell^\nu = \frac{\prod_{i=1}^q (1 - t m_{s^i})}{\prod_{i=0}^q (1 - t m_{s^i})}
\]
(it follows from the fact that \( m_{s^i} \) are multiples of \( m_{s^i} = (n_i + 1) m_{s^i} \), \( 1 \leq i \leq q \), and each element \( s \in S_1 \) in a unique way can be represented in the form \( k_0 \cdot m_{s^0} + \sum_{i=1}^q k_i \cdot m_{s^i} \) with \( k_0 \geq 0 \) and \( 0 \leq k_i \leq n_i \) for \( 1 \leq i \leq q \); see, e.g., [4]).

There exists a map \( \tilde{\Pi} : \tilde{Y} \to \mathbb{P}\tilde{S}_C \) such that \( \text{Im} \tilde{\Pi} = \text{Im} \Pi \). To define it, one can say that \( \tilde{Y} \) is a subset of the graded semigroup
\[
\tilde{Y}^* = Y'' \times \left( \sum_{m \in S_1} \bullet \cdot t^m \right) \times \prod_{\delta \in \Delta} \left( \sum_{k=0}^{\infty} \bullet \cdot t^{km^\delta} \right)
\]
(each factor of \( \tilde{Y}^* \) is a graded semigroup) and the map \( \tilde{\Pi} \) is the restriction of a graded semigroup homomorphism \( \tilde{Y}^* \to \mathbb{P}\tilde{S}_C \). Because of that it should be defined for points of \( \bigcup_{\sigma \in \Delta'} \tilde{E}_\sigma \) and also for "monomials" of the form \( \bullet \cdot t^{m^\delta} \) for \( \delta \in \Delta \) and \( \bullet \cdot t^{m_{s^i}} \) for \( i = 0, 1, \ldots, q \). For a point \( A \) of \( \tilde{E}_\sigma, \sigma \in \Delta' \) (or rather for the monomial \( [A] \cdot t^{m^\delta} \) \( \tilde{\Pi} \) coincides with \( \Pi \).

A point of \( \tilde{E}_\sigma \setminus \tilde{E}_\sigma, \sigma \in \Delta' \), corresponds either to a dead end \( \delta \in \Delta \) (and in this case \( \sigma = st_\delta \)) or to the initial divisor \( 1 \) (in this case \( \sigma = st_1 \)). In the first case one puts \( \tilde{\Pi}([A] \cdot t^{m^\delta}) = (n_\delta + 1) \cdot \Pi([A_\delta] \cdot t^{m^\delta}) \) for any point \( A_\delta \in \tilde{E}_\delta \); in the second case one puts \( \tilde{\Pi}([A] \cdot t^{m^\delta}) = \sum_{i=0}^q \ell_i \cdot \Pi([A_{a_i}] \cdot t^{m_{s^i}}) \) for any points \( A_{a_i} \in \tilde{E}_{a_i} \) (see the definitions of \( n_\delta \) and \( \ell_i \) above). One puts \( \tilde{\Pi}([A] \cdot t^{m^\delta}) = \Pi([A] \cdot t^{m^\delta}) \) for any point \( A \in \tilde{E}_\delta, \delta \in \Delta, \tilde{\Pi}([A] \cdot t^{m_{s^i}}) = \Pi([A_{a_i}] \cdot t^{m_{s^i}}) \) for any point \( A_{a_i} \in \tilde{E}_{a_i}, i = 0, 1, \ldots, q \) (one can easily see that the result does not depend on the choice of the points \( A_\delta, A_{a_i} \) in these cases).

It is not difficult to see that \( \text{Im} \tilde{\Pi} = \text{Im} \Pi \).

Now the Theorem [\[ follows from the following
Proposition 5 \( \chi(\tilde{Y}) = \chi(Im \ \tilde{\Pi}) \).

Remark. Before we prove Proposition 5, we explain why and in which sense one can deal with points of \( \tilde{E}_\sigma \setminus \tilde{E}_\sigma \), \( \sigma \in \Delta' \), just in the same way as with other points of \( \tilde{E}_\sigma \). Let \( A \in \tilde{E}_\sigma \). The point \( A \) corresponds either to a dead end \( \delta \in \Delta \) (in this case \( \sigma = st_\delta \)) or to the starting divisor \( 1 \) (in this case \( \sigma = st_1 \)). Let \( A' \in \tilde{E}_\sigma \), and let \( g \) and \( g' \) be functions \( (\mathbb{C}^2, 0) \to (\mathbb{C}, 0) \) such that \( \tilde{\Pi}(\{A\} \cdot t^m) = (\tilde{\psi}(g), \tilde{\psi}(g)) \), \( \tilde{\Pi}(\{A'\} \cdot t^m) = (\tilde{\psi}(g'), \tilde{\psi}(g')) \). Let us recall that \( g' = g_{A'} \), \( g = (g_{A_0})^{n_0} \) with \( A_0 \in \tilde{E}_\delta \) if the point \( A \) corresponds to the dead end \( \delta \in \Delta \), \( g = (g_{A_0})^{n_0} \cdot (g_{A_1})^{n_2} \cdot \ldots \cdot (g_{A_n})^{n_q} \) with \( A_\sigma \in \tilde{E}_\sigma \) if the point \( A \) corresponds to the starting divisor \( 1 \) (see the notations above). Then \( \psi(g) = \psi(g') \). To compare \( a(g) \) and \( a(g') \) one can look (and we shall regularly do it below) at the ratio \( \psi = g'/\tilde{g} \), where \( \tilde{g} = g \circ \pi \) and \( \tilde{g}' = g' \circ \pi \) are the liftings of the functions \( g \) and \( g' \) to the space \( X \) of the resolution.

The main (or rather the only) property of the function \( \psi \) which will be used below is the following one. The restriction \( \psi|_{E_\sigma} \) of the function \( \psi \) to the component \( E_\sigma \) of the exceptional divisor is a meromorphic function (in fact a ratio of two linear functions) with one pole at the point \( A \) and one zero at the point \( A' \). It has no zeroes or poles on all other components \( E_{\sigma'} \) from \( D' \). Therefore it is constant (and different from zero or infinity) on each connected component of \( D' \setminus E_\sigma \) and its value on such a component coincides with the value of the function \( \psi \) at the corresponding essential point of the component \( E_\sigma \).

Proof of Proposition 5. The proof consists in analyzing places where \( \tilde{\Pi} \) is not immersive. At each such place we explicitly show a part of \( \tilde{Y} \) which has the Euler characteristic equal to zero and which can be removed without changing the image.

Let us write \( \tilde{Y} = \sum_k Y_k \), where \( k \) is the multi-index

\[
\tilde{k} = \{(k_\sigma)_{\sigma \in \Delta'}, m, \{k_\delta\}_{\delta \in \Delta}\}
\]

with \( k_\sigma \geq 0 \) for each \( \sigma \in \Delta' \), \( m \in S_1 \), and \( 0 \leq k_\delta \leq n_\delta \) for each \( \delta \in \Delta \). One has \( Y_k = (\prod_{\sigma \in \Delta'} S^{k_\sigma} \tilde{E}_\sigma) \cdot t^{\nu(k)} \), where \( \nu(k) = \sum_{\sigma \in \Delta'} k_\sigma m_\sigma + m + \sum k_\delta m_\delta \) (\( Y_k \) are "connected components" of \( \tilde{Y} \)).

Suppose that there exist two different elements \( y_1 \) and \( y_2 \) from \( \tilde{Y} \) such that \( \tilde{\Pi}(y_1) = \tilde{\Pi}(y_2) \). Let \( y_j \in Y_{k_j} \) \( (j = 1, 2) \), and let the element \( y_j \) be represented by the sets \( K_{\sigma,j} \subset \tilde{E}_\sigma \) for \( \sigma \in \Delta' \) \( (\#K_{\sigma,j} = k_{\sigma,j}) \), i.e., \( y_j = (\prod_{\sigma \in \Delta'} ([K_{\sigma,j}] \cdot t^{k_{\sigma,j}m_\sigma}) \times x_j \), where \( x_j \in Y_1 \times \prod_{\delta \in \Delta} Y_\delta \).
Lemma 3 There exists a component $E_\sigma$ of $D'$ such that:
1) for all $\sigma^* \in \Delta'$, $\sigma^* > \sigma$, one has $K_{\sigma^*, 1} = K_{\sigma^*, 2}$;
2) for all $\delta \in \Delta$, $\delta > \sigma$, one has $k_{\delta, 1} = k_{\delta, 2}$;
3) $K_{\sigma, 1} \neq K_{\sigma, 2}$ and either $k_{\sigma, 1}$ or $k_{\sigma, 2}$ is $\geq s_\sigma - 1$.

Proof. First let us show that there exists a component $E_\sigma \subset D'$ such that $K_{\sigma, 1} \neq K_{\sigma, 2}$. If $K_{\sigma, 1} = K_{\sigma, 2}$ for all $E_\sigma \subset D'$, then $\Pi(x_1) = \Pi(x_2)$ ($x_j \in Y_1 \times \prod \ Y_\delta$; see above). Let $x_j = x^{(1)}_j \times \prod \ x^{(\delta)}_j$, where $x^{(1)}_j \in Y_1$, $x^{(\delta)}_j \in Y_\delta$, and let us suppose that $x^{(\delta)}_1 \neq x^{(\delta)}_2$ for $\delta \in \Delta$, but $x^{(\delta)}_1 = x^{(\delta)}_2$ for all dead ends $\delta$ such that $st_\delta < \delta$. Without loss of generality one can suppose that $x^{(\delta)}_j = \star$ for those $\delta$ $(j = 1, 2)$ and $x^{(\delta)}_2 = \star$, $x^{(\delta)}_1 = \star \cdot L^k m^{k_0}$ with $0 < k \leq n_{\delta_0}$.

Let $C_i$ be a component of the curve $C$ such that $st_\delta < \tilde{C}_i$. In this case $E_{\delta_0}$ and $E_{st_\delta}$ appear in the minimal embedded resolution of the (irreducible) curve $C_i$ and as a consequence $m_i^{k_0}$ (the $i$-th component of $m^{k_0}$) belongs to the minimal set of generators $\{\beta_0, \beta_1, \ldots, \beta_\epsilon\}$ of the semigroup $S_{C_i} (\subset \mathbb{Z}_{\geq 0})$ of the curve $C_i$: $m_i^{k_0} = \beta_p$ (with the notations of 2.3, in the dual graph of the minimal resolution of the curve $C_i$ one has $\delta_0 = \alpha_p$ and $st_\delta = \beta_p$).

Moreover, for any $\delta \in \Delta$ with $\delta \neq st_\delta$ (i.e., either $\delta < st_\delta$ or $\delta$ and $st_\delta$ are not comparable) the function $g_{A_i}$ satisfies the hypothesis of the point (5) in 2.3 and so the corresponding value $v_i(g_{A_i}) \in S_{C_i}$ belongs to $\langle \beta_0, \beta_1, \ldots, \beta_{p-1} \rangle$. As a consequence one has that $v_i(\Pi(x_1)) = k\beta_\epsilon + v'$ where $0 < k \leq n_{\delta_0}$, $v' \in \langle \tilde{\beta}_0, \tilde{\beta}_1, \ldots, \tilde{\beta}_{p-1} \rangle$, $v_i(\Pi(x_2)) \in \langle \tilde{\beta}_0, \tilde{\beta}_1, \ldots, \tilde{\beta}_{p-1} \rangle$ (here $v_i$ is a coordinate in $\mathbb{P} S_C \subset \mathbb{Z}_{\geq 0} \times \mathbb{P}(\mathbb{C}^r)$). This contradicts the uniqueness of the representation in the semigroup $S_{C_i}$ (see properties (2) and (4) of 2.3). This proves the statement in the discussed case. The same arguments (applied to the semigroup $S_1$) work in the case if $x^{(\delta)}_1 = x^{(\delta)}_2$ for all $\delta \in \Delta$.

Let $\sigma$ be a maximal element in the set of vertices from $\Delta'$ with $K_{\sigma, 1} \neq K_{\sigma, 2}$, i.e., $K_{\sigma', 1} = K_{\sigma', 2} = K_{\sigma'}$ for all $\sigma' > \sigma$, $\sigma' \in \Delta'$. Just the previous arguments show that, for all dead ends $\delta$ with $st_\delta \geq \sigma$, one has $k_{\delta, 1} = k_{\delta, 2}$.

Let 

\[ y_{j}' = \left( \prod_{\sigma' \in \Delta', \sigma' \neq \sigma} [K_{\sigma', j}] \cdot \tilde{t}^{k_{\sigma', j} m^{\delta}} \right) \times x_{j} ', \]

$j = 1, 2$, where $\sigma' \neq \sigma$ means that either $\sigma' \leq \sigma$ or $\sigma$ and $\sigma'$ are not comparable, $x_{j}' = \prod_{\delta \in \Delta, \delta \neq \sigma} (\bullet \cdot \tilde{t}^{k_{\delta, j} m^{\delta}} \times \bullet \cdot \tilde{m}^{\delta})$, $m_{j} \in S_1'$ (i.e., $x_{j}'$ is obtained from $x_{j}$ by dropping all factors $\tilde{t}^{k_{\delta, j} m^{\delta}}$ with $\delta > \sigma$). One has $y_{1}' \neq y_{2}'$, $\Pi(y_{1}') = \Pi(y_{2}')$ (the last equation follows from the fact that multiplication (the semigroup operation) by any element of $\mathbb{Z}^r \times (\mathbb{C}^*)^r$ is injective).
Let $k_j = k_{\sigma,j} = \#K_{\sigma,j}$, $j = 1, 2$. Without loss of generality one can suppose that $k_1 \geq k_2$. Let $Q_0, Q_1, \ldots Q_{s-1}$ $(s = s_\sigma)$ be essential points on the component $E_\sigma$. If $\sigma \neq s_1$, we suppose that the point $Q_0$ corresponds to the connected component of $D \setminus \tilde{E}_\sigma$ which contains the starting divisor $1$.

Let us fix an affine coordinate on the projective line $E_\sigma$ in such a way that the essential point $Q_0$ of $E_\sigma$ (corresponding to the starting divisor $1$ if $\sigma \neq s_1$) is the infinite one.

Let $g_1$ and $g_2$ ($g_j : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$) be functions, corresponding to $y_1'$ and $y_2'$, let $\tilde{g}_j = g_j \circ \pi$ be the lifting of the function $g_j$ to the space $X$ of the resolution, and let $\psi = \tilde{g}_1 / \tilde{g}_2$. The function $\psi$ is a meromorphic function on $E_\sigma$, and is a regular nonzero function on $\bigcup_{\sigma' > \sigma} E_{\sigma'}$. Therefore it is constant on each connected component of $\bigcup_{\sigma' > \sigma} E_{\sigma'}$. The value of $\psi$ on such a component is equal to $\psi(Q_\ell)$, where $Q_\ell$ is the essential point of $E_\sigma$ corresponding to the component.

Since $\varrho(g_1) = c \cdot \varrho(g_2)$, $\psi(Q_\ell) = c$ for a constant $c \neq 0$, $\ell = 1, \ldots, s - 1$. On $E_\sigma$ the function $\psi$ has the form $\psi = c' \cdot \frac{p_1(z)}{p_2(z)}$, where $p_j(z) = \prod_{k=1}^{k_j}(z - z_{k,j}^{(j)})$, $K_{\sigma,j} = \{z_{k,j}^{(j)}\}$, $c' \neq 0$.

The polynomial $p(z) = c' \cdot p_1(z) - c \cdot p_2(z)$ vanishes at all the points $Q_\ell$ ($\ell = 1, \ldots, s - 1$). Since $K_{\sigma,1} \neq K_{\sigma,2}$, $p(z) \neq 0$. One has $\deg p(z) \leq k_1$ and $p(z)$ has (at least) $s - 1$ zeroes. Therefore $k_1 \geq s - 1$.  

**Definition:** We shall say that a vertex $\sigma \in \Delta'$ is a **cut** of a multi-index $k = \{k_{\sigma}, \underline{m}, k_{\delta}\}$ if $k_\sigma \geq s_\sigma - 1$.

Lemma 3 implies that, if there exist $y_1 \in Y_{k_1}$, $y_2 \in Y_{k_2}$, $y_1 \neq y_2$, such that $\tilde{\Pi}(y_1) = \tilde{\Pi}(y_2)$, then either $\bar{k}_1$ or $\bar{k}_2$ has a cut.

**Remark.** Assume that a multi-index $k$ has a cut at $\sigma \in \Delta'$. Then the Euler characteristic of the component $Y_k$ of the space $\tilde{Y}$ is equal to zero and thus it makes no contribution to $\chi(\tilde{Y})$. By lemma 3, if $\tilde{\Pi}(Y_{k_1}) \cap \tilde{\Pi}(Y_{k_2}) \neq \emptyset$, then either $\bar{k}_1$ or $\bar{k}_2$ has a cut at some place. The idea is that one can remove the component with a cut (at least part of it) without changing the image. The rest of the proof explains how this can be made. The most technical part consists in showing some fine properties of cuts and their relative distribution in the dual graph (Lemmas 4 and 5. Lemma 6 is a technical tool to simplify the proof of the others).
3.3 Distribution of cuts.

Let

\[ y_j = \prod_{\sigma \in \Delta'} \left( [K_{\sigma,j}] \cdot t_{k_{\sigma,j}, m_{\sigma}} \right) \times (\cdot \cdot t_{m_j}) \times \prod_{\delta \in \Delta} (\cdot \cdot t_{k_{\delta,j}, m_{\delta}}) \]

\[(j = 1, 2, \#K_{\sigma,j} = k_{\sigma,j}, m_j \in S'_1, 0 \leq k_{\delta,j} \leq n_\delta) \text{ and let} \]

\[ g_j = \prod_{\sigma \in \Delta'} \varphi_j^{(\nu)} \cdot \varphi_j^{(1)} \cdot \prod_{\delta \in \Delta} \varphi_j^{(\delta)} \]

be the corresponding function from \(\mathcal{O}_{\mathbb{C}^2,0}: \Pi(y_j) = (v(g_j), a(g_j))\).

For a vertex \(\sigma \in \Delta'\), let \(Q_0, Q_1, \ldots, Q_{s-1}\) \((s = s_\sigma)\) be the essential points of \(E_\sigma\). Let \(\Gamma(\sigma, \ell)\) be the subgraph of \(\Gamma\) corresponding to the connected component of \(D \setminus \hat{\Delta}_E\) which contains the point \(Q_\ell\). If \(\sigma \neq st_1\), we assume that \(st_1 \in \Gamma(\sigma, 0)\). In what follows we shall use the following notations:

\[ \mathfrak{u}(g_1/g_2; \geq \sigma) = \sum_{\tau \in \Delta' \cup \Delta, \tau \geq \sigma} (\mathfrak{u}(\varphi_1^{(\tau)}) - \mathfrak{u}(\varphi_2^{(\tau)}) \mathfrak{u}(g_1/g_2; > \sigma) = \sum_{\tau \in \Delta' \cup \Delta, \tau > \sigma} (\mathfrak{u}(\varphi_1^{(\tau)}) - \mathfrak{u}(\varphi_2^{(\tau)}) \mathfrak{u}(g_1/g_2; \sigma, \ell) = \sum_{\tau \in \Gamma(\sigma, \ell)} (\mathfrak{u}(\varphi_1^{(\tau)}) - \mathfrak{u}(\varphi_2^{(\tau)}) \]

(here \(\mathfrak{u}(\varphi_1^{(\tau)}) - \mathfrak{u}(\varphi_2^{(\tau)}) = (k_{\tau,1} - k_{\tau,2}) \cdot m_{\tau}, \tau \in \Delta' \cup \Delta\)). For \(\sigma > st_1\) (respectively for \(\sigma = st_1\), \(\mathfrak{u}(g_1/g_2; > \sigma)\) is equal to \(\sum_{\ell=1}^{s-1} \mathfrak{u}(g_1/g_2; \sigma, \ell)\) (respectively to \(\sum_{\ell=1}^{s-1} \mathfrak{u}(g_1/g_2; \sigma, \ell)\) plus the contribution, \(\mathfrak{u}(\varphi_1^{(\sigma)}) - \mathfrak{u}(\varphi_2^{(\sigma)})\), of the dead end \(\delta\) such that \(st_\delta = \sigma\) if such a dead end exists.

Let \(\sigma \in \Delta'\). We define a “modified” multiplicity, \(\overline{m}_\sigma = (\overline{m}_1, \ldots, \overline{m}_r)\), in the following way. If \(\sigma\) is a star vertex and there exists a dead end \(\delta \in \Delta\) such that \(st_\delta \geq \sigma\) and \(\overline{m}_\sigma > \overline{m}_\delta\), we put \(\overline{m}_\sigma = \overline{m}_\delta\); otherwise \(\overline{m}_\sigma = \overline{m}_\sigma\) (a dead end \(\delta\) with the described properties, if it exists, is unique). The dead end \(\delta \in \Delta\) with \(st_\delta \geq \sigma\), \(\overline{m}_\sigma > \overline{m}_\delta\) (if it exists) will be denoted by \(\overline{m}(\sigma)\). Such dead end \(\delta\) exists if and only if, in the process of resolution by blow-ups of points, the component \(E_\sigma\) of the exceptional divisor is created after the component \(E_\delta\) (i.e., by blowing-up points of \(E_\delta\)) and \(\sigma \leq st_\delta\).

The reason for this construction is that the natural order in the dual graph (that is the order we use) does not coincides with the order in which the successive divisors are created. As a consequence the multiplicity map \(\sigma \mapsto m_\sigma^{\alpha}\) is not an increasing one.

Remarks. 1. If \(\sigma = st_\delta\) for \(\delta \in \Delta\), then \(\delta = \delta(\sigma)\). However \(\delta(\sigma)\) could also exist in the case when \(\sigma \neq st_\delta\) for any \(\delta \in \Delta\). Moreover, if \(\delta(\sigma)\) exists...
for some \( \sigma \) then \( \delta(\sigma) = \delta(\sigma') \) for all star vertices \( \sigma' \in \Delta' \) with \( \sigma \leq \sigma' \leq st_\delta \). Therefore one and the same dead end \( \delta \) could occur as \( \delta(\sigma) \) for several star vertices \( \sigma \in \Delta' \).

2. Let \( \sigma \) be a star vertex. Then for any \( \tau \in \Gamma \) with \( \tau > \sigma \) one has that \( m_\tau \geq \bar{m}^\sigma \) and \( m_\tau = \bar{m}^\sigma \) if and only if \( \tau = \delta(\sigma) \). Therefore \( \bar{m}^\tau \geq \bar{m}^\sigma \) for any \( \tau > \sigma \) and the equality holds if and only if either \( \tau \) is a star vertex and \( \delta(\tau) = \delta(\sigma) \) or \( \tau \) is a dead end and \( \delta = \delta(\sigma) \).

**Lemma 4** Let \( \sigma \in \Delta' \), \( \sigma > st_1 \), be a vertex such that \( v(g_1/g_2; \geq \sigma) \geq \bar{m}^\sigma \). Let \( \sigma' \) be the previous separation vertex of \( \Gamma \), i.e., either \( \sigma' = st_1 \) or \( s_{\sigma'} > 2 \), \( \sigma' < \sigma \) and \( s_{\sigma''} = 2 \) for any \( \sigma'' \in \Delta' \) with \( \sigma' < \sigma'' < \sigma \). Let \( \ell_0 \) be such that \( \sigma \in \Gamma(\sigma', \ell_0) \). Assume that there are no cuts of \( k_2 \) between \( \sigma' \) and \( \sigma \). Then:
- if there exists \( \delta(\sigma') \) and it belongs to \( \Gamma(\sigma', \ell_0) \), then \( v(g_1/g_2; \sigma', \ell_0) \geq \bar{m}^\sigma \);
- otherwise \( v(g_1/g_2; \sigma', \ell_0) \geq \bar{m}^\sigma \).

Moreover, let \( i,j \in \{1, \ldots, r\} \) be such that \( \sigma < \bar{C}_i \) and \( \sigma \not\in \bar{C}_j \). Assume that \( v_i(g_1/g_2; \geq \sigma) \cdot m_i^\sigma \leq v_i(g_1/g_2; \geq \sigma) \cdot m_j^\sigma \). Then \( v_j(g_1/g_2; \sigma', \ell_0) \cdot m_i^\sigma < v_i(g_1/g_2; \sigma', \ell_0) \cdot m_j^\sigma \).

**Proof.** Let us prove the first statement. Let \( \{\delta_1, \delta_2, \ldots, \delta_p\} \) be (all) dead ends such that \( \sigma' < st_{\delta_1} < \ldots < st_{\delta_p} < \sigma \). Suppose first that \( p = 0 \). If there exists \( \delta(\sigma') \) and it belongs to \( \Gamma(\sigma', \ell_0) \), then either \( \delta(\sigma) = \delta(\sigma') \) (if \( \sigma \) is a star vertex) and \( \bar{m}^\sigma = \bar{m}^\sigma \) or \( \bar{m}^\sigma = m^\sigma > \bar{m}^\sigma \). In both cases the statement is obvious. If \( \delta(\sigma') \) does not belong to \( \Gamma(\sigma', \ell_0) \) (or does not exist), then \( \bar{m}^\sigma < \bar{m}^\sigma \) and the statement is obvious as well.

Suppose that \( p > 0 \). Since \( \delta(st_{\delta_p}) = \delta_p \not\in \sigma, \bar{m}^\sigma > m^{st_{\delta_p}} = (n_{\delta} + 1) \cdot m^{\delta_p} \).

Taking into account that \( v(\varphi^{(\delta_p)}) \leq n_{\delta} m^{\delta_p} \), one has
\[
v(g_1/g_2; \geq st_{\delta_p}) \geq v(g_1/g_2; \geq \sigma) - n_{\delta} m^{\delta_p} > (n_{\delta} + 1) m^{\delta_p} - n_{\delta} m^{\delta_p} = m^{\delta_p} = \bar{m}^{st_{\delta_p}}.
\]
Repeating the same arguments for the star points \( st_{\delta_p-1}, \ st_{\delta_p-2}, \ldots \), one proves the statement.

Now we prove the second statement. Let \( \sigma^* \) be the separation vertex of the branches \( C_i \) and \( C_j \) (one has \( \sigma^* \leq \sigma' \)). For \( \tau \in \Gamma \), let \( h_\tau = m_\tau^\tau/m_i^\tau \). If \( S_{ij} \subseteq \mathbb{Z}_{\geq 0} \) is the semigroup of the curve \( C_i \cup C_j \) (it coincides with the projection of the semigroup \( S = S_C \) to the \((v_i, v_j)\)-plane), then \( h_\tau \) is just the slope of the line which goes through the origin and the point \((m_i^\tau, m_j^\tau) \in S_{ij} \).

It is known (see [3], proof of Theorem 2) that the slopes \( h_\tau \) are constant for \( \tau \in \Delta' \) with \( 1 \leq \tau \leq \sigma^* \); decrease strictly for \( \tau \) such that \( \sigma^* < \tau \leq \bar{C}_i \) (i.e., \( \sigma^* < \tau < \tau' \leq \bar{C}_i \) if and only if \( h_\tau > h_{\tau'} \)) and increase strictly for \( \tau \) such that \( \sigma^* \leq \tau \leq \bar{C}_j \). Moreover \( h_\tau \) is constant on each tail. If \( p = 0 \), the
Moreover proves the statement.

**Remarks**

and \( m_j^\sigma / m_i^\sigma = h_{\sigma'} > h_{\sigma} = m_j^\tau / m_i^\tau \).

If \( p > 0 \), we have the inequalities

\[
\frac{v_j (g_1 / g_2; \geq \sigma)}{v_i (g_1 / g_2; \geq \sigma)} \leq \frac{m_j^\sigma}{m_i^\sigma} < \frac{m_j^\delta_p}{m_i^\delta_p} < \cdots < \frac{m_j^\delta_1}{m_i^\delta_1} < \frac{m_j^\sigma'}{m_i^\sigma'},
\]

where \( m = (k_{\delta_p, 2} - k_{\delta_p, 1}) \cdot (m_i^\delta_p, m_j^\delta_p) \). Since \( |k_{\delta_p, 2} - k_{\delta_p, 1}| \leq n_\delta_p \), the slope of the vector \( w_{\delta_p} \) is less than \( h_{\delta_p} \) and thus less than \( h_{\delta_p - 1} \) (see Fig.3).

Moreover \( w_{\delta_p} > (m_i^\delta_p, m_j^\delta_p) > (n_{\delta_p - 1}(m_i^\delta_{p-1}, m_j^\delta_{p-1})) \). Thus one has:

\[
\frac{v_j (g_1 / g_2; \geq st_{\delta_p})}{v_i (g_1 / g_2; \geq st_{\delta_p})} < \frac{m_j^\delta_{p-1}}{m_i^\delta_{p-1}} \quad \text{and} \quad w_{\delta_p} > (n_{\delta_p - 1}(m_i^\delta_{p-1}, m_j^\delta_{p-1})).
\]

Repeating the same arguments for the star points \( st_{\delta_p-1}, st_{\delta_p-2}, \ldots \), one proves the statement. \( \square \)

**Remarks.** Let \( \tau \in \Delta' \) be such that \( \tau \) is not a cut of \( k_2 \). If there exists \( \delta(\tau) \), let \( \ell^* \in \{0, 1, \ldots, s_\tau - 1\} \) be such that \( \delta(\tau) \in \Gamma(\tau, \ell^*) \). If the conclusions of the Lemma 4 are valid for \( \tau, g_1/g_2 \) and for any \( \ell = 1, \ldots, s_\tau - 1 \) \((\ell = 0, 1, \ldots, s_\tau - 1 \) if \( \tau = st_1 \)\) then the hypothesis on the Lemma 4 are true for \( \tau \) and \( g_1/g_2 \) as well. More explicitly:
1. Suppose that $\tau \neq st_1$ and $\psi(g_1/g_2; \tau, \ell) \geq m^\tau$ for $\ell = 1, \ldots, s_\tau - 1$, $\ell \neq \ell^*$ (if it exists) and $\psi(g_1/g_2; \tau, \ell^*) \geq \tilde{m}^\tau$. Then, if $\tau \neq st_\delta$ for any dead end $\delta$, $\psi(g_1/g_2; \tau) = \sum_{\ell=1}^{s_\tau-1} \psi(g_1/g_2; \tau, \ell) \geq (s_\tau - 2) \cdot m^\tau + \tilde{m}^\tau$. If $\tau = st_\delta$ (in this case $\ell^*$ does not exist) one has $\psi(g_1/g_2; \tau) \geq (s_\tau - 2) \cdot \tilde{m}^\tau + \tilde{m}^\tau$.

Since $\tau$ is not a cut of $k_2$, one has that $\psi(\phi_{2}^{(\tau)}) \leq (s_\tau - 2) \cdot m^\tau$. Thus, in any case one has that $\psi(g_1/g_2; \geq \tau) \geq m^{st_1} + \tilde{m}^{st_1}$.

2. Let $i, j \in \{1, \ldots, r\}$ be such that $\tau < \tilde{C}_i$, $\tau \not\in \tilde{C}_j$. Suppose that $v_j(g_1/g_2; \tau, \ell) \cdot m^\tau < v_i(g_1/g_2; \tau, \ell) \cdot m^\tau$ for $\ell = 1, \ldots, s_\tau - 1$. Then $v_j(g_1/g_2; \geq \tau) \cdot m^\tau < v_i(g_1/g_2; \geq \tau) \cdot m^\tau$.

**Lemma 5** Let $k_1 \neq k_2$ be such that there exist $y_1 \in \gamma_{k_1}$ and $y_2 \in \gamma_{k_2}$ with $\bar{\Pi}(y_1) = \bar{\Pi}(y_2)$. Suppose that $\sigma_2 \in \Delta'$ is such that:

1) there are no cuts $\sigma_2'$ of $k_2$ with $\sigma_2' > \sigma_2$;
2) there exists a cut $\sigma_1$ of $k_1$ with $\sigma_1 \geq \sigma_2$.

Then on each geodesic from the vertex $\sigma_2$ to a strict transform $\tilde{C}_j$ with $\tilde{C}_j > \sigma_2$ there exists a cut of $k_1$.

**Proof.** We use the induction on the number $q$ of branches $C_j$ such that $\tilde{C}_j > \sigma_2$. The statement is obvious for $q = 1$. Let $q > 1$, and let $\tilde{C}_i, \tilde{C}_j$ ($i \neq j$) be such that $\sigma_2 < \sigma_1 < \tilde{C}_i, \sigma_2 < \tilde{C}_j$.

Suppose that there is no cut of $k_1$ on the geodesic from $\sigma_2$ to $\tilde{C}_j$, and let $\sigma^*$ be the separation vertex between $C_i$ and $C_j$. Without loss of generality one can assume that $\sigma_1$ is maximal among the cuts of $k_1$ on the geodesic from $\sigma_2$ to $C_i$ and that $\sigma_2$ is a separation vertex. If $\sigma^* > \sigma_2$, or if there is a cut $\sigma'_1$ of $k_1$ on the connected component of $D \setminus E_{\sigma^*}$ which intersects $C_j$ (this cut must be not comparable with $C_j$), the statement follows from the inductive hypothesis (in the first case the number of branches $C_j$ such that $\tilde{C}_j > \sigma^*$ is strictly smaller than $q$; in the second case one can apply the arguments to a separation point $\sigma^{**}$ such that $\sigma^* < \sigma^{**} < \tilde{C}_j$).

Thus one can assume that $\sigma^* = \sigma_2$ and that neither $k_1$ nor $k_2$ has a cut on the connected component $\Gamma(\sigma^*, \ell_i)$ of $\Gamma - \{\sigma^*\}$ which contains $\tilde{C}_j$. Let $\Gamma(\sigma^*, \ell_i)$ be the connected component of $\Gamma - \{\sigma^*\}$ which contains $\tilde{C}_i$ and let $g_u = G_u \cdot G_i^\ell \cdot G_i^\ell$ ($u = 1, 2$) where

$$G_u = \prod_{\tau \in \Gamma(\sigma^*, \ell_i)} \phi_{u}^{(\tau)}, \quad G_i^\ell = \prod_{\tau \in \Gamma(\sigma^*, \ell_i)} \phi_{u}^{(\tau)}, \quad G_u^0 = \frac{g_u}{G_i^\ell \cdot G_i^\ell}.$$
Let \( w(g) := (v_i(g), v_j(g)) \in \mathbb{Z}_{\geq 0}^2 \).

For any \( \tau \in \Delta' \cap \Gamma(\sigma^*, \ell_j) \) one has \( K_{r, 1} = K_{r, 2} \) (see Lemma 3). A simple computation (similar to the one in the proof in Lemma 3) shows that \( x_1^{(\delta)} = x_2^{(\delta)} \) for any \( \delta \in \Delta \) with \( st_\delta \in \Gamma(\sigma^*, \ell_j) \). Therefore

\[
\mathbf{w}(G_1^i) = \mathbf{w}(G_2^i).
\]

Since \( \sigma_1 \) is a maximal cut of \( k_1 \), \( \mathbf{w}(g_1/g_2; \geq \sigma_1) \geq \mathbf{w}_1 \). Since \( \Gamma(\sigma^*, \ell_i) \) does not contain cuts of \( k_2 \), by the inductive hypothesis, on each geodesic from the vertex \( \sigma_2 \) to a strict transform \( \tilde{C}_n \) with \( \tilde{C}_n > \sigma_2 \), \( \tilde{C}_n \in \Gamma(\sigma^*, \ell_i) \), there exists a cut of \( k_1 \). Thus, using repeatedly Lemma 4 and the Remark after it, one can see that

\[
\mathbf{w}(G_1^i/G_2^i) \geq (\tilde{m}_1^\sigma, \tilde{m}_j^\sigma) > (0, 0),

(v_j(G_1^i) - v_j(G_2^i)) \cdot m_i^\sigma < (v_i(G_1^i) - v_i(G_2^i)) \cdot m_j^\sigma.
\]

It is known that, for any factor \( \varphi \) of \( G_u^0 \) \( (u = 1, 2) \), one has

\[
v_j(\varphi) \cdot m_i^\sigma = v_i(\varphi) \cdot m_j^\sigma.
\]

Therefore one has

\[
\mathbf{w}(g_1) - \mathbf{w}(G_1^i) - \mathbf{w}(G_2^i) = \mathbf{w}(G_1^i/G_2^i) + \mathbf{w}(G_0^i)
\]

and this point is strictly under the line \( \mathcal{L} \) in \( \mathbb{R}^2 \supset \mathbb{Z}_{\geq 0}^2 \) which goes through the origin and the point \((m_i^\sigma, m_j^\sigma)\).

On the other hand

\[
\mathbf{w}(g_1) - \mathbf{w}(G_2^i) - \mathbf{w}(G_1^i) = \mathbf{w}(g_2) - \mathbf{w}(G_2^i) - \mathbf{w}(G_1^i) = \mathbf{w}(G_2^i).
\]

The last point lies on the line \( \mathcal{L} \). This proves the statement. \( \square \)

**Lemma 6** Let \( k_1, \ldots, k_p \) be different multi-indices such that there exist \( y_i \in Y_{k_i} \), for all \( i = 1, \ldots, p \) with \( \tilde{\Pi}(y_1) = \cdots = \tilde{\Pi}(y_p) \). Then there exist maximal cuts \( \sigma_i \) of \( k_i \), \( i = 1, \ldots, p \), which are comparable with each other, i.e., all of them lie on one and the same geodesic from the vertex \( 1 \) to a strict transform \( \tilde{C}_j \) of a branch of the curve \( C \).

**Proof.** First let us prove the existence of cuts of all the multi-indices \( k_i \), \( i = 1, \ldots, p \). It is sufficient to prove this for \( p = 2 \). By Lemma 3 there exists a cut, say of \( k_1 \), at a vertex \( \sigma_1 \in \Delta' \). If there is no cut of \( k_2 \), then Lemma 3 says that there exists a cut \( \sigma_i \) of \( k_i \) on the geodesic from \( st_1 \) to \( \tilde{C}_1 \) for all \( i = 1, \ldots, r \). Using Lemma 4 and the Remark 1 after it, one gets that

\[
\mathbf{w}(g_1/g_2; \geq st_1) \geq \mathbf{w}_{st_1} + \tilde{m}_{st_1}.
\]
One has (see, e.g., [7] or item (2) in 2.3)
\[
\bar{m}^{s_1} \geq \bar{m}^{q} = (q + 1) \cdot m^{\alpha_q} > n_q \cdot m^{\alpha_q} + (n_{q-1} + 1) \cdot m^{\alpha_{q-1}} > \cdots
\]
\[
\cdots > \sum_{p=1}^{q} n_p \cdot m^{\alpha_p} + m^{\alpha_0} > \sum_{p=1}^{q} n_p \cdot m^{\alpha_p} - m^{\alpha_0}.
\]
Since \( \bar{m} = m^{s_1} + \sum_{p=1}^{q} n_p \cdot m^{\alpha_p} - m^{\alpha_0} \) is the maximal element of (the finite set) \( S'_t \), \( m^{s_1} + \bar{m}^{s_1} \) is strictly bigger than \( m \). Therefore
\[
\psi(g_1) - \psi(g_2) = \psi(g_1/g_2; \geq 1) > 0
\]
what contradicts the supposition that \( \psi(g_1) = \psi(g_2) \).

To prove the existence of maximal cuts which are comparable with each other, we shall rather prove the following statement. Let \( \sigma \in \Delta' \) and suppose that among the vertices \( \sigma' \geq \sigma, \sigma' \in \Delta' \), there are cuts of all the multi-indices \( k_i, i = 1, 2, \ldots, p \). Then there exist maximal cuts \( \sigma_i \) of \( k_i, \sigma_i \geq \sigma \), which are comparable with each other. We shall use the simultaneous induction both on the number of multi-indices \( p \) and on the vertex \( \sigma \) (in the inverse order). If \( p = 1 \) or if \( \sigma \) is a maximal element in \( \Delta' \), the statement is trivial (in the last case all cuts coincide with \( \sigma \)). Let \( Q_0, Q_1, \ldots, Q_{s-1} \) \( (s = s_{\sigma}) \) be essential points of the component \( E_\sigma \); if \( \sigma \neq st_1 \), we assume that the component of \( D' \setminus \bar{E}_\sigma \) corresponding to the essential point \( Q_0 \) contains the divisor \( E_{st_1} \). Let \( \Gamma(\sigma, \ell) \) be the subgraph of \( \Gamma \) corresponding to the essential point \( Q_\ell, \ell = 0, 1, \ldots, s - 1 \). One can meet one of the three following situations.

1) \( \sigma \) is a maximal cut of one of the multi-indices \( k_i \), say, of \( k_p \). In this case the statement follows from one for multi-indices \( k_1, \ldots, k_{p-1} \).

2) \( \sigma \) is not a maximal cut of any of the multi-indices \( k_i \), and there exists \( \ell, 1 \leq \ell \leq s - 1 \) \( (0 \leq \ell \leq s - 1 \text{ if } \sigma = st_1) \), such that the subgraph \( \Gamma(\sigma, \ell) \) contains cuts of all the multi-indices \( k_i \). In this case the statement follows from one applied to the first (i.e., the minimal) vertex in \( \Gamma(\sigma, \ell) \).

3) \( \sigma \) is not a maximal cut of any of the multi-indices \( k_i \), and there is no \( \ell, 1 \leq \ell \leq s - 1 \) \( (0 \leq \ell \leq s - 1 \text{ if } \sigma = st_1) \), such that the subgraph \( \Gamma(\sigma, \ell) \) contains cuts of all the multi-indices \( k_i \). In this case there exist \( \ell_1, \ell_2 \) \( (1 \leq \ell_1 \leq s - 1; 0 \leq \ell_2 \leq s - 1 \text{ if } \sigma = st_1) \), \( i_1 \), and \( i_2 \) such that \( \Gamma(\sigma, \ell_1) \) contains a cut of \( k_{i_1} \), but does not contain a cut of \( k_{i_2} \), and vice versa \( \Gamma(\sigma, \ell_2) \) contains a cut of \( k_{i_2} \), but does not contain a cut of \( k_{i_1} \). Without loss of generality one can suppose that \( i_1 = 1, i_2 = 2 \). Let \( i \) (respectively \( j \)) be such that the strict transform \( \bar{C}_i \) lies in \( \Gamma(\sigma, \ell_1) \) (respectively \( \bar{C}_j \) lies in \( \Gamma(\sigma, \ell_2) \)) and the geodesic from \( \sigma \) to \( \bar{C}_i \) contains a maximal cut \( \sigma_1 \) of \( k_1 \) (respectively the geodesic from \( \sigma \) to \( \bar{C}_j \) contains a maximal cut \( \sigma_2 \) of \( k_2 \)).
Now we shall use the same arguments (and the same notations) as in the proof of Lemma 5 applied to the vertex $\sigma$, $\Gamma(\sigma, \ell_1)$ and $\Gamma(\sigma, \ell_2)$.

Let $g_u = G_u^0 \cdot G_u^i \cdot G_u^j$ where $G_u^i = \prod_{r \in \Gamma(\sigma, \ell_1)} \varphi_{\tau}^r$, $G_u^j = \prod_{r \in \Gamma(\sigma, \ell_2)} \varphi_{u}^r$,

$G_u^0 = g_u / (G_u^i \cdot G_u^j)$, $u = 1, 2$.

Since $\sigma_1$ (respectively $\sigma_2$) is a cut of $k_1$ (respectively of $k_2$) and $\Gamma(\sigma, \ell_1)$ (respectively $\Gamma(\sigma, \ell_2)$) does not contains cuts of $k_2$ (respectively of $k_1$), using repeatedly Lemma 4 and the Remark after it one can show that

$$w(G_1^i / G_2^i) \geq (\tilde{m}^i_1, \tilde{m}^i_2) > (0, 0),$$

$$v_j(G_1^i) = v_j(G_2^i) m^i_q < (v_i(G_1^i) - v_i(G_2^i)) m^i_q,$$

$$w(G_2^i / G_1^i) \geq (\tilde{m}^i_1, \tilde{m}^i_2) > (0, 0),$$

$$v_j(G_2^i) - v_j(G_1^i) m^i_q > (v_i(G_2^i) - v_i(G_1^i)) m^i_q.$$ 

One has

$$w(g_1) - w(G_2^i) - w(G_1^i) = w(G_1^i / G_2^i) + w(G_0^j),$$

and this point is strictly under the line $L$ in $\mathbb{R}^2 \supset \mathbb{Z}_{\geq 0}$ which goes through the origin and the point $(m^i_1, m^i_q)$. On the other hand,

$$w(g_1) - w(G_2^i) - w(G_1^i) = w(g_2) - w(G_2^i) - w(G_1^i) = w(G_2^i / G_1^i) + w(G_0^j),$$

where this point is strictly over the line $L$. This proves the statement. □

Remarks. 1. Under the conditions of the Lemma 5, let $\sigma_1$, $\ldots$, $\sigma_p$ be maximal cuts of multi-indices $k_1$, $\ldots$, $k_p$ such that $\sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_p$. Then, by Lemma 5, on each geodesic from the vertex $\sigma_1$ to a strict transform $\tilde{C}_j$ with $\tilde{C}_j > \sigma_1$ there exists a cut of $k_i$ for $i = 2, \ldots, p$.

2. As a consequence of Lemma 5, if a multi-index $k$ has no cuts then the image $\Pi(Y_k)$ of the corresponding component $Y_k$ of the space $\bar{Y}$ does not intersect the image $\Pi(Y_{k'})$ for $k' \neq k$.

Lemma 7 Let $Q_0, Q_1, \ldots, Q_{s-1}$ be (different) points of a projective line $E$, $\bar{E} = E - \{Q_\ell : \ell = 0, 1, \ldots, s-1\}$, let $P_0, \ldots, P_k$ be $k$ points (not necessarily different), different from $Q_0, Q_1, \ldots, Q_{s-1}$. Let $\Phi$ be the map from $S^k \bar{E}$ to $\mathbb{P}(\mathbb{C}^*)^s$ defined in the following way. For an element from $S^k \bar{E}$; i.e., for $k$ points $P_1, \ldots, P_k$, let $\psi$ be a meromorphic function on $E$ with zeroes at the points $P_1, \ldots, P_k$ and poles at the points $P_0, \ldots, P_k$; let $\Phi(P_\{j\}) := (\psi(Q_0) : \psi(Q_1) : \ldots : \psi(Q_{s-1}))$. Then, if $k \geq s - 1$, one has $\text{Im} \Phi = \mathbb{P}(\mathbb{C}^*)^s$; if $k \leq s - 1$, $\Phi$ is an embedding. Moreover in both cases $\Phi$ is a (smooth) locally trivial (in fact a trivial) fibration over its image the fibre of which is a (complex) affine space of dimension $\max(0, k - s + 1)$. 28
Proof. Without loss of generality one can suppose that $P_0^o = P_2^o = \cdots = P_k^o = P^o$. Let us choose an affine coordinate on $E$ such that $P^o = \infty$. Then $\psi$ is a polynomial of degree $\leq k$ with zeroes at those points $P_1, \ldots, P_k$ which are different from $P^o$. Let $z_\ell$ be the coordinate of the point $Q_\ell$, $\ell = 0, 1, \ldots, s - 1$.

For $k \geq s - 1$, the statement that the map $\Phi$ is onto can be reduced to the following obvious one: for an arbitrary prescribed set of values $\{\psi_0, \psi_1, \ldots, \psi_{s-1}\}$, there exists a polynomial $\psi$ of degree $\leq k$ such that $\psi(z_\ell) = \psi_\ell$, $\ell = 0, 1, \ldots, s - 1$. The statement that $\Phi$ is a locally trivial fibration over its image follows from the fact that if $\psi_1$ and $\psi_2$ are polynomials with coinciding values at the points $Q_\ell$, $\ell = 0, 1, \ldots, s - 1$, then $\psi_1 = \psi_2 + q(z)(z - z_0)(z - z_1)\ldots(z - z_{s-1})$ where $q(z)$ is an arbitrary polynomial of degree $k - s$.

For $k \leq s - 1$, the statement follows from the fact that such a polynomial of degree $\leq s - 1$ is unique. $\square$

3.4 Free action of the torus.

Let $\sigma \in \Delta'$, $s = s_\sigma$. There is defined the following free action $T = T(\sigma)$ of the group $\mathbb{P}((\mathbb{C}^*)^s) \cong (\mathbb{C}^*)^{s-1}$ on $\mathbb{P}_0^s \times \mathbb{P}((\mathbb{C}^*)^r)$. Let $Q_0, Q_1, \ldots, Q_{s-1}$ be essential points on the component $E_\sigma$ and, for $1 \leq i \leq r$, let $Q_{\ell(i)}$ be the essential point corresponding to the connected component of the complement $(f \circ \pi)^{-1}(0) \setminus \tilde{E}_\sigma$ which contains the strict transform $\tilde{C}_i$. Let $\underline{c} = (c_0 : c_1 : \cdots : c_{s-1}) \in \mathbb{P}((\mathbb{C}^*)^r)$. Then, for $(\underline{v}, \underline{a}) = (v_1, \ldots, v_r; a_1 : \cdots : a_r) \in \mathbb{P}_0^s \times \mathbb{P}((\mathbb{C}^*)^r)$, one has $T_{\underline{c}}((\underline{v}, \underline{a})) := (\underline{v}; c_{\ell(1)} \cdot a_1 : \cdots : c_{\ell(r)} \cdot a_r)$ (i.e., all coordinates $a_i$ of $\underline{a}$ such that the strict transform $\tilde{C}_i$ intersects the component of $D' \setminus \tilde{E}_\sigma$ corresponding to one essential point $Q_\ell$ are multiplied by one and the same number $c_\ell$).

Corollary 2 Suppose $\sigma$ is a cut of $k$ and $y \in Y_{\underline{k}}$, $y = [K_{\sigma}] \cdot \underline{t}^{k_{\sigma}} \cdot \underline{m}^{\sigma} \times y^*$ ($\# K_{\sigma} = k_{\sigma}$). Then $\overline{\Pi}(S^{k_{\sigma}}E_{\sigma} \cdot \underline{t}^{k_{\sigma}} \cdot \underline{m}^{\sigma} \times y^*)$ is the orbit of $\overline{\Pi}(y)$ under the described action (and thus is homeomorphic to $\mathbb{P}((\mathbb{C}^*)^s) \cong (\mathbb{C}^*)^{s-1}$).

Proof. Indeed, let $K'_{\sigma}$ be a subset of $\tilde{E}_\sigma$ with $k_{\sigma}$ elements (i.e., $[K'_{\sigma}]$ is an element of $S^{k_{\sigma}}E_{\sigma}$), let $y' = [K'_{\sigma}] \cdot \underline{t}^{k_{\sigma}} \cdot \underline{m}^{\sigma} \times y^*$, let $g$ and $g'$ be functions from $O_{C^2 \sigma}$ corresponding to the points $y$ and $y'$ of $Y_{\underline{k}}$, and let $\psi = \tilde{g}/\tilde{g}$, where $\tilde{g} = g \circ \pi$ and $\tilde{g}' = g' \circ \pi$ are the liftings of the functions $g$ and $g'$ to the space $X$ of the resolution. Then $\psi|_{E_{\sigma}}$ is a meromorphic function on the projective line $E_{\sigma}$ with $k_{\sigma}$ zeroes at the points of the set $K'_{\sigma}$ and $k_{\sigma}$ poles at the points of the set $K_{\sigma}$ (such a function is well-defined up to the multiplication by a constant). Moreover $\psi$ is constant on each connected component of $D' \setminus \tilde{E}_\sigma$ and its value on this component coincides with the value of $\psi$ at the
Corollary 3  Suppose that \( y \in Y_k \) and \( k \) has a cut on each geodesic from the vertex \( \sigma \) to a strict transform \( \widetilde{C}_i \) with \( \widetilde{C}_i > \sigma \). Then \( \overline{\Pi}(Y_k) \) contains the orbit of \( \overline{\Pi}(y) \) under the described action.

Proof. Let \( \sigma_1, \ldots, \sigma_p \) be the minimal elements in the set of cuts of \( k \) which are \( \geq \sigma \). If \( \sigma_1 = \sigma \) (and thus \( p = 1 \)), the statement is obvious (see Corollary [2]). Let \( y = \prod_{j=1}^{p} ([K_{\sigma_j}] \cdot t^{k_{\sigma_j}m_{\sigma_j}}) \times y^* \quad (K_{\sigma_j} \subset \widetilde{E}_{\sigma_j}, \#K_{\sigma_j} = k_{\sigma_j}). \)

For \( j = 1, \ldots, p \), let \( Q_0^j, Q_1^j, \ldots, Q_s^j \) be the essential points of \( E_{\sigma_j} \) numbered so that the connected component of \( D' \setminus \widetilde{E}_{\sigma_j} \), corresponding to the point \( Q_0^j \), contains the component \( E_{\sigma_j} \) of the exceptional divisor (or equivalently the first separation component \( E_{\text{std}} \)). Let \( \ell(j) \) be such that the connected component of \( D' \setminus \widetilde{E}_{\sigma_j} \), corresponding to the essential point \( Q_{\ell(j)}^j \in E_{\sigma_j} \) contains the component \( E_{\sigma_j} \) of the essential divisor. One knows (Lemma 7) that, for any set of non-zero numbers \( q_0^j, q_1^j, \ldots, q_s^j \), there exists a subset \( K'_{\sigma_j} \subset \widetilde{E}_{\sigma_j} \) with \( \#K'_{\sigma_j} = k_{\sigma_j} \), such that a meromorphic function \( \psi_j \) on \( E_{\sigma_j} \) with zeroes at the points of the set \( K'_{\sigma_j} \) and poles at the points of the set \( K_{\sigma_j} \) has values \( q_0^j, q_1^j, \ldots, q_s^j \) at the points \( Q_0^j, Q_1^j, \ldots, Q_s^j \) respectively. For \( \mathfrak{c} = (c_0 : c_1 : \ldots : c_s) \in \mathbb{P}(\mathbb{C}^s) \), let \( k_{\sigma_j} \) be such that \( \psi_j(Q_0^j) = 1, \psi_j(Q_m^j) = c_{\ell(j)}^j/c_0 \) for \( m \geq 1 \), and let \( y' = \prod_{j=1}^{p} ([K'_{\sigma_j}] \cdot t^{k_{\sigma_j}m_{\sigma_j}}) \times y^* \). Then \( \overline{\Pi}(y') = T_{\mathfrak{c}}(\overline{\Pi}(y)) \). □

Lemma 8  For a multi-index \( k = (k_0, m, k) \), the map \( \overline{\Pi}_{|Y_k} : Y_k \to \overline{\Pi}(Y_k) \) is a locally trivial fibration. (Note that \( \overline{\Pi}(Y_k) \subset \mathbb{P}F_{\mathfrak{e}} \), where \( \mathfrak{e} = \mathfrak{e}(k) \).

Proof. Let

\[
y = \prod_{\sigma \in \Delta'} ([K_{\sigma}] \cdot t^{k_{\sigma}m_{\sigma}}) \times (\bullet \cdot t^m) \times \prod_{\delta \in \Delta} (\bullet \cdot t^{k_{\sigma}m_{\delta}})
\]

be a point of \( Y_k \). For \( \sigma \in \Delta' \) let \( \Phi_{\sigma} : S^{k_{\sigma}} \widetilde{E}_{\sigma} \to \mathbb{P}(\mathbb{C}^*)^{k_{\sigma}} \) be the map (\( \Phi \)) described in Lemma [4] for \( \widetilde{E} = \widetilde{E}_{\sigma} \), \( k = k_{\sigma} \), \( \{P_{\sigma}^0, \ldots, P_{\sigma}^k\} = K_{\sigma} \), and \( \{Q_0^\sigma, Q_1^\sigma, \ldots, Q_s^\sigma\} = \{Q_0^\sigma, Q_1^\sigma, \ldots, Q_s^\sigma\} \), let \( \phi_{\sigma} = \Phi_{\sigma} \circ \pi_{\sigma} \) where \( \pi_{\sigma} : Y_k \to S^{k_{\sigma}} \widetilde{E}_{\sigma} \) is the natural projection, and let \( \Psi = \prod_{\sigma} \phi_{\sigma} : Y_k \to \prod_{\sigma} \mathbb{P}(\mathbb{C}^*)^{k_{\sigma}} \).

From Lemma [4] it follows that \( \Psi \) is a locally trivial fibration over its image. Let \( M : \prod_{\sigma} \mathbb{P}(\mathbb{C}^*)^{k_{\sigma}} \to \mathbb{P}F_{\mathfrak{e}} \) be the map defined in the following way.
For $1 \leq i \leq r$, $\sigma \in \Delta'$, let $Q_{E_{i\sigma}}$ be the essential point of the component $E_{i\sigma}$ of the exceptional divisor, corresponding to the connected component of the complement $(f \circ \pi)^{-1}(0) \setminus \overline{E_{i\sigma}}$ which contains the strict transform $C_i$. For $c^i = (c^i_0 : c^i_1 : \ldots : c^i_{s-1}) \in \mathbb{P}((\mathbb{C}^*)^{s_\sigma})$, $\sigma \in \Delta'$, $M(\Pi_{\sigma} c^i) := (\Pi_{\sigma} c^i_{e_{\sigma}(1)} : \ldots : \Pi_{\sigma} c^i_{e_{\sigma}(r)}) \cdot \Pi(y) \in \mathbb{P}F_{\omega}$. The map $M$ is a locally trivial fibration. Now the statement follows from the fact that $\Pi|_{\overline{Y}} = M \circ \Psi$. \hfill $\Box$

**Statement.** For all multi-indices $k$ with $v(k) = v$ (there is a finite number of them), one can construct subspaces $Y'_k \subset Y_k$ such that

1. $\Pi((\bigcup Y'_k)) = \Pi((\bigcup Y_k))$;
2. $\Pi(Y'_k)$ does not intersect $\Pi(Y'_j)$ for $k_1 \neq k_2$;
3. $\chi(Y'_k \cap Y'_j) = 0$;
4. either $\Pi|_{\overline{Y}}$ is one–to–one on its image, or $\chi(Y'_k) = 0$ and $\chi(\Pi(Y'_k)) = 0$.

**Proof.** Let us order multi-indices $k$ with $v(k) = v$ in an arbitrary way. For $k_1$ with $v(k_1) = v$, let $I_{k_1} = \{(\mathbf{v}, \mathbf{a}) \in \mathbb{P}F_{\omega} : \exists y_1 \in Y_{k_1}, \exists \mathbf{a} > \mathbf{k_1}, \exists y_2 \in Y_{k_2} : \Pi(y_1) = \Pi(y_2) = (\mathbf{v}, \mathbf{a}) \}, Z_{k_1} = \Pi^{-1}(I_{k_1}) \cap Y_{k_1}, Y'_k = Y_{k_1} \setminus Z_{k_1}$.

The Euler characteristic of the subspace $I_{k_1}$ is an alternative sum of Euler characteristics of the subspaces $I(k_1, k_2, \ldots, k_p) = \{(\mathbf{v}, \mathbf{a}) \in \mathbb{P}F_{\omega} : \exists y_i \in Y_{k_i}, i = 1, 2, \ldots, p : \Pi(y_1) = \Pi(y_2) = \ldots = \Pi(y_p) = (\mathbf{v}, \mathbf{a}) \}$ with $p \geq 2$.

If the set $I(k_1, k_2, \ldots, k_p)$ is empty, according to Lemma 3 there exist maximal cuts $\sigma_i$ of $k_i$, $i = 1, 2, \ldots, p$, which are comparable with each other, i.e., which lie on one and the same geodesic in the graph $F$ from $s_1$ to a strict transform of a branch of the curve $C$. Let $\sigma_{i_0}$ be (one of) the smallest of these cuts (all smallest cuts coincide with each other). The remark after Lemma 3 says that for each $i \neq i_0$ on each geodesic from the vertex $\sigma_{i_0}$ to a strict transform $C_j$ with $C_j > \sigma_{i_0}$ there exists a cut of $k_i$. By Corollary 3 the (semianalytic) subspace $I(k_1, k_2, \ldots, k_p)$ is invariant with respect to the described above free action of the group $\mathbb{P}((\mathbb{C}^*)^p) \cong (\mathbb{C}^*)^{s_{i_0} - 1}$, where $s = s_{i_0}$. Therefore $\chi(I(k_1, k_2, \ldots, k_p)) = 0$, $\chi(I_{k_1}) = 0$, and (since the map $\Pi|_{\overline{Y}}$ is a locally trivial fibration) $\chi(Z_{k_1}) = 0$.

Obviously the sets $Y'_k$ satisfy the conditions 1) – 3). If $\Pi|_{\overline{Y}}$ is not one–to–one on its image, there exists a cut $\sigma$ of $k$. In this case the space $Y_k$ is a product of a space and the symmetric power $S^k \overline{E_{i\sigma}}$ with $k_\sigma \geq k_{\sigma} - 1$. Therefore $\chi(Y_k) = 0$ and $\chi(Y'_k) = 0$. The image $\Pi(Y_k)$ is invariant with respect to the free action of the group $\mathbb{P}((\mathbb{C}^*)^s) \cong (\mathbb{C}^*)^{s_{i_0} - 1}$. Therefore $\chi(\Pi(Y_k)) = 0$, $\chi(\Pi(Y'_k)) = \chi(\Pi(Y_k)) - \chi(I_{k_1}) = 0$. \hfill $\Box$

The Statement obviously implies Proposition 3 and thus Theorem 4 has been proved.
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