Bour’s spacelike maximal and timelike minimal surfaces in the 3-dimensional Minkowski space

Erhan Güler

Abstract. Bour’s minimal surface has remarkable properties in three dimensional Minkowski space. We reveal the definite and indefinite cases of the Bour’s surface using Weierstrass representations, and give some differential geometric properties of the astonishing maximal and minimal surfaces.

1. Introduction

The origins of minimal surface theory can be traced back to 1744 with the Swedish mathematician Leonhard Euler’s paper, and to the 1760 French mathematician Joseph Louis Lagrange’s paper. A minimal surface in three dimensional Euclidean space \( \mathbb{E}^3 \) is a regular surface for which the mean curvature vanishes identically, firstly defined by Lagrange in 1760.

Classical minimal surfaces in Riemannian geometry are known by nearly all the mathematicians, especially the geometers. However, there is a little knowledge about the Bour’s minimal surface. In 1862, the French mathematician Jacques Edmond Émile Bour used semigeodesic coordinates and found a number of new cases of deformations of surfaces. He gave a well known theorem about the helicoidal and rotational surfaces. And also the Bour-Enneper equation (today called the sine-Gordon wave equation) used in soliton theory and quantum field theories in Physics was first set down by Bour.

Minimal surfaces applicable onto a rotational surface were first determined by Bour \cite{bour} in 1862. These surfaces have been called \( \mathcal{B}_m \) (following Haag) to emphasize the value of \( m \). We see the papers dealing with the \( \mathcal{B}_m \) in the literature:

Bour, E. Theorie de la deformation des surfaces. Journal de l’École Imperiale Polytechnique, tome 22, cahier 39 (1862), pp. 99-109.

Schwarz, H. A. Miscellen aus dem Gebiete der Minimalflächen. Journal de Crelle, vol. 80 (1875), p. 295, published also in Gesammelte Mathematische Abhandlungen.

2000 Mathematics Subject Classification. Primary 53A35; Secondary 53C42.

Key words and phrases. Bour’s surface, spacelike surface, timelike surface, Weierstrass representation.

This paper is in final form and no version of it will be submitted for publication elsewhere.
Ribaucour, A. Etude sur les classoides ou surfaces a courbure moyenne nulle. Memoires Couronnes de l’Academie Royale de Belgique, vol. XLIV (1882), chapter XX, pp. 215-224.

Demoulin, A. Bulletin des Sciences Mathematiques (2), vol. XXI (1897), pp. 244-252.

Haag, J. Bulletin des Sciences Mathematiques (2), vol. XXX (1906), pp. 75-94, also pp. 293-296.

St¨ ubler, E. Mathematische Annalen, vol. 75 (1914), pp. 148-176.

Whittemore, J. K. Minimal surfaces applicable to surfaces of revolution. Ann. of Math. (2) 19 (1917), no. 1, 1–20.

All real minimal surfaces applicable to rotational surfaces setting $F(s) = Cs^{m-2}$ in the Weierstrass representation equations, where $s, C \in \mathbb{C}$, $m \in \mathbb{R}$, and $F(s)$ is an analytic function. For $C = 1$, $m = 0$ we obtain the Catenoid, $C = i$, $m = 0$, the right Helicoid, $C = 1$, $m = 2$, the Enneper’s surface (see also [3, 8]).

The Bour’s surface has not been studied up till now in three dimensional Minkowski space $\mathbb{L}^3$. In this paper, we reveal the Bour’s surface in $\mathbb{L}^3$.

Next, we focus on the definite (resp. indefinite) case of the Bour’s maximal (resp. minimal) surface in three dimensional Minkowski space $\mathbb{L}^3$.

2. Bour’s spacelike maximal surface $\mathcal{B}_m$

Throughout this work, we shall identify a vector $\mathbf{v} = (u, v, w)$ with its transpose, the surfaces will be smooth, and simply connected.

2.1. Definite case. Let $\mathbb{L}^3$ be a 3-dimensional Minkowski space with natural Lorentzian metric

$$\langle \cdot, \cdot \rangle = dx^2 + dy^2 - dz^2.$$ A vector $\mathbf{v}$ in $\mathbb{L}^3$ is called spacelike if $\langle \mathbf{v}, \mathbf{v} \rangle > 0$ or $\mathbf{v} = 0$, timelike if $\langle \mathbf{v}, \mathbf{v} \rangle < 0$, lightlike if $\mathbf{v} \neq 0$ satisfies $\langle \mathbf{v}, \mathbf{v} \rangle = 0$.

A surface in $\mathbb{L}^3$ is called a spacelike (resp. timelike, degenerate (lightlike)) if the induced metric on the surface is a positive definite Riemannian (resp. Lorentzian, degenerate) metric. A spacelike surface with vanishing mean curvature is called a maximal surface in three dimensional Minkowski space.

The Weierstrass representation theorem for minimal surfaces in $\mathbb{E}^3$ is discovered by K. Weierstrass in 1866 [9]. Next, we give it for the maximal surfaces in $\mathbb{L}^3$.

**Theorem 1.** (Weierstrass representation for maximal surfaces in $\mathbb{L}^3$). Let $\mathcal{F}$ and $\mathcal{G}$ be two holomorphic functions defined on a simply connected open subset $U$ of $\mathbb{C}$ such that $\mathcal{F}$ does not vanish and $|\mathcal{G}| \neq 1$ on $U$. Then the map

$$x(\zeta) = \text{Re} \int_{\zeta}^{\bar{\zeta}} \left( \frac{\mathcal{F}(1 + \mathcal{G}^2)}{2\mathcal{F}\mathcal{G}} \right) d\zeta$$

is a conformal immersion of $U$ into $\mathbb{L}^3$ whose image is a maximal surface.

See also details for the maximal surfaces in literature [1, 5, 7].

**Lemma 1.** The Weierstrass patch determined by the functions

$$\langle \mathcal{F}(\zeta), \mathcal{G}(\zeta) \rangle = (\zeta^{m-2}, \zeta)$$
is a representation of the Bour’s surface of value $\zeta \in \mathbb{C}$, $m \in \mathbb{R}$ in $\mathbb{L}^3$.

**Theorem 2.** Bour’s surface of value $m$

\[
\mathcal{B}_m(r, \theta) = \left( \frac{r^{m-1} \cos ((m-1) \theta) + r^{m+1} \cos ((m+1) \theta)}{m}, \frac{r^{m-1} \sin ((m-1) \theta) + r^{m+1} \sin ((m+1) \theta)}{m}, \frac{2r^m \cos (m\theta)}{m} \right)
\]

is a maximal surface in $\mathbb{L}^3$, where $m \in \mathbb{R} - \{-1, 0, 1\}$.

**Proof.** The coefficients of the first fundamental form of the surface $\mathcal{B}_m$ are

\[
E = r^{2m-4} (1 - r^2)^2,
\]

\[
F = 0,
\]

\[
G = r^{2m-2} (1 - r^2)^2.
\]

We have

\[
\det I = \left[ r^{2m-3} (1 - r^2)^2 \right]^2.
\]

So, $\mathcal{B}_m$ is a spacelike surface. The Gauss map of the surface is

\[
e = \frac{1}{r^2 - 1} \left( \begin{array}{c} 2r \cos (\theta) \\ 2r \sin (\theta) \\ r^2 + 1 \end{array} \right).
\]

The coefficients of the second fundamental form of the surface are

\[
L = 2r^{m-2} \cos (m\theta),
\]

\[
M = -2r^{m-1} \sin (m\theta),
\]

\[
N = -2r^m \cos (m\theta).
\]

Then, we have

\[
\det II = -4r^{2m-2}.
\]

In spacelike case, the Gaussian curvature is defined by

\[
K = \epsilon \frac{\det II}{\det I},
\]

where $\epsilon := \langle e, e \rangle = -1$ in $\mathbb{L}^3$. Hence, the Gaussian curvature and the mean curvature of the Bour’s surface of value $m$, respectively, are

\[
K = \left( \frac{2r^{2-m}}{(1 - r^2)^2} \right)^2, \quad H = 0.
\]

Therefore, the $\mathcal{B}_m$ is a maximal surface in $\mathbb{L}^3$.

**2.2. Bour’s spacelike surface $\mathcal{B}_3$.** If take $m = 3$ in $\mathcal{B}_m(r, \theta)$, we have Bour’s maximal surface $\mathcal{B}_3$ (see Fig. 1 and Fig. 2)

\[
\mathcal{B}_3(r, \theta) = \left( \frac{r^2 \cos (2\theta) + r^4 \cos (4\theta)}{2}, \frac{-r^2 \sin (2\theta) + r^4 \sin (4\theta)}{2}, \frac{r^3 \cos (3\theta)}{2} \right)
\]

in Minkowski 3-space, where $r \in [-1, 1]$, $\theta \in [0, \pi]$. 
The coefficients of the first fundamental form of the Bour’s maximal surface of value 3 are
\[ E = r^2 \left(1 - r^2\right)^2, \quad F = 0, \quad G = r^4 \left(1 - r^2\right)^2. \]

So,
\[ \det I = r^6 \left(1 - r^2\right)^4. \]

The Gauss map of the surface is
\[ e = \frac{1}{r^2 - 1} \left(2r \cos(\theta), 2r \sin(\theta), 1 + r^2\right). \]
The coefficients of the second fundamental form of the surface are

\[ L = 2r \cos (3\theta), \quad M = -2r^2 \sin (3\theta), \quad N = -2r^3 \cos (3\theta). \]

Then,

\[ \det II = -4r^4. \]

The mean and the Gaussian curvatures of the Bour’s maximal surface of value 3 are, respectively,

\[ H = 0, \quad K = \frac{4}{r^2 (1-r^2)^4}. \]

The Weierstrass patch determined by the functions

\[ (\tilde{F}, \tilde{G}) = (\zeta, \zeta) \]

is a representation of the Bour’s maximal surface of value 3 in \( \mathbb{L}^3 \). The parametric form of the maximal surface \( \mathfrak{B}_3 \) (see Fig. 3, and Fig. 4) is

\[ \mathfrak{B}_3 (u, v) = \begin{pmatrix} u^4 + \frac{3}{4}u^2v^2 + \frac{u^2}{2} - \frac{v^2}{2} \\ \frac{3}{2}uv^3 - uv - uv^3 \\ \frac{2}{3}u^3 - 2uv^2 \end{pmatrix}, \]

where \( u, v \in \mathbb{R} \).

\[ (a) \quad \text{Figure 3} \quad \text{Maximal surface } \mathfrak{B}_3 (u, v), \quad (b) \quad u, v \in [-1, 1] \]
Figure 4 Maximal surface $\mathcal{B}_3(u,v)$ with its shadows

The coefficients of the first fundamental form of the Bour’s surface of value 3 are

$$E = (u^2 + v^2)(u^2 + v^2 - 1)^2 = G, \quad F = 0,$$

So,

$$\det I = (u^2 + v^2)^2 (u^2 + v^2 - 1)^4.$$ 

The Gauss map of the surface $\mathcal{B}_3$ is

$$e = \frac{1}{u^2 + v^2 - 1} (2u, 2v, u^2 + v^2 + 1).$$

The coefficients of the second fundamental form of the surface are

$$L = 2u, \quad M = -2v, \quad N = -2u.$$ 

Then,

$$\det II = -4 (u^2 + v^2).$$

The mean and the Gaussian curvatures of the Bour’s minimal surface of value 3 are

$$H = 0, \quad K = \frac{4}{(u^2 + v^2)(1 + u^2 + v^2)^3}.$$
2.3. Applications of the definite case in $L^3$.

**Example 1.** If take $m = 2$, we have Enneper’s maximal surface without self-intersections (see Fig. 5)

$$
\mathcal{B}_2 (r, \theta) = \begin{pmatrix}
    r \cos (\theta) + \frac{r^3}{3} \cos (3\theta) \\
    -r \sin (\theta) + \frac{r^3}{3} \sin (3\theta) \\
    r^2 \cos (2\theta)
\end{pmatrix},
$$

and $(\mathcal{F}, \mathcal{G}) = (1, \zeta)$ in Minkowski 3-space, where $r \in [-1, 1], \theta \in [0, \pi]$.

\begin{figure}[h]
\centering
(a) \hspace{1cm} (b)
\caption{Maximal surface $\mathcal{B}_2$ without self-intersections}
\end{figure}

**Example 2.** If take $m = 2$, we have Enneper’s maximal surface with self-intersections (see Fig. 6) in Minkowski 3-space, where $r \in [-3, 3], \theta \in [0, \pi]$.

\begin{figure}[h]
\centering
(a) \hspace{1cm} (b)
\caption{Maximal surface $\mathcal{B}_2$ with self-intersections}
\end{figure}

**Example 3.** If take $m = \frac{1}{2}$, we have (see Fig. 7)

$$
\mathcal{B}_{1/2} (r, \theta) = \begin{pmatrix}
    -2r^{-1/2} \cos \left( \frac{\theta}{2} \right) + \frac{2}{3} r^{3/2} \cos \left( \frac{3\theta}{2} \right) \\
    -2r^{-1/2} \sin \left( \frac{\theta}{2} \right) + \frac{2}{3} r^{3/2} \sin \left( \frac{3\theta}{2} \right) \\
    4r^{-1/2} \cos \left( \frac{\theta}{2} \right)
\end{pmatrix},
$$

and $(\mathcal{F}, \mathcal{G}) = \left( \zeta^{-3/2}, \zeta \right)$ in Minkowski 3-space, where $r \in [-1, 1], \theta \in [-2\pi, 2\pi]$.
Example 4. If $m = \frac{3}{2}$, we have (see Fig. 8)

$$\mathfrak{B}_{3/2}(r, \theta) = \begin{pmatrix}
2r^{-1/2}\cos\left(\frac{\theta}{2}\right) + \frac{2}{3}r^{5/2}\cos\left(\frac{5\theta}{2}\right) \\
-2r^{-1/2}\sin\left(\frac{\theta}{2}\right) + \frac{2}{3}r^{5/2}\sin\left(\frac{5\theta}{2}\right) \\
\frac{4}{3}r^{3/2}\cos\left(\frac{3\theta}{2}\right)
\end{pmatrix},$$

with self-intersections, and $(\mathfrak{F}, \mathcal{G}) = \left(\zeta^{-1/2}, \zeta\right)$ in Minkowski 3-space, where $r \in [-3, 3], \theta \in [-2\pi, 2\pi]$.

Example 5. If $m = \frac{3}{2}$, we have $\mathfrak{B}_{3/2}(r, \theta)$ without self-intersections (see Fig. 9), and $(\mathfrak{F}, \mathcal{G}) = (z^{-1/2}, z)$ in Minkowski 3-space, where $r \in [-1, 1], \theta \in [-2\pi, 2\pi]$.
Example 6. If \( m = \frac{2}{3} \), we have (see Fig. 10)

\[
\mathfrak{B}_{2/3} (r, \theta) = \begin{pmatrix}
-3r^{-1/3} \cos \left( \frac{\theta}{3} \right) + \frac{1}{3}r^{5/3} \cos \left( \frac{5\theta}{3} \right) \\
-3r^{-1/3} \sin \left( \frac{\theta}{3} \right) + \frac{1}{3}r^{5/3} \sin \left( \frac{5\theta}{3} \right) \\
3r^{2/3} \cos \left( \frac{2\theta}{3} \right)
\end{pmatrix},
\]

and \((\mathcal{F}, \mathcal{G}) = \left( \zeta^{-4/3}, \zeta \right)\) in Minkowski 3-space, where \( r \in [-1, 1] \), \( \theta \in [-3\pi, 3\pi] \).

Example 7. If \( m = \frac{4}{3} \), then we have (see Fig. 11)

\[
\mathfrak{B}_{4/3} (r, \theta) = \begin{pmatrix}
3r^{1/3} \cos \left( \frac{\theta}{3} \right) + \frac{1}{3}r^{7/3} \cos \left( \frac{7\theta}{3} \right) \\
-3r^{1/3} \sin \left( \frac{\theta}{3} \right) + \frac{1}{3}r^{7/3} \sin \left( \frac{7\theta}{3} \right) \\
\frac{1}{3}r^{2/3} \cos \left( \frac{2\theta}{3} \right)
\end{pmatrix},
\]

and \((\mathcal{F}, \mathcal{G}) = \left( \zeta^{-2/3}, \zeta \right)\) in Minkowski 3-space, where \( r \in [-2, 2] \), \( \theta \in [-3\pi, 3\pi] \).
Example 8. If \( m = \frac{5}{2} \), then we have (see Fig. 12)

\[
\mathcal{B}_{5/2} (r, \theta) = \left( \frac{2}{3} r^{3/2} \cos \left( \frac{3\theta}{2} \right) + \frac{2}{7} r^{7/2} \cos \left( \frac{7\theta}{2} \right) , \frac{2}{3} r^{3/2} \sin \left( \frac{3\theta}{2} \right) + \frac{2}{7} r^{7/2} \sin \left( \frac{7\theta}{2} \right) , \frac{1}{4} r^{5/2} \cos \left( \frac{5\theta}{2} \right) \right),
\]

and \((\mathcal{F}, \mathcal{G}) = \left( \zeta^{1/2}, \zeta \right)\) in Minkowski 3-space, where \( r \in [-1, 1], \theta \in [-2\pi, 2\pi] \).

Example 9. If \( m = 4 \), then we have (see Fig. 13)

\[
\mathcal{B}_4 (r, \theta) = \left( \frac{1}{4} r^3 \cos (3\theta) + \frac{1}{4} r^5 \cos (5\theta) , \frac{1}{3} r^3 \sin (3\theta) + \frac{1}{3} r^5 \sin (5\theta) , \frac{1}{4} r^4 \cos (4\theta) \right),
\]

and \((\mathcal{F}, \mathcal{G}) = (\zeta^2, \zeta)\) in Minkowski 3-space, where \( r \in [-1, 1], \theta \in [0, 2\pi] \).
3. Bour’s timelike minimal surface $\mathcal{B}_m$

3.1. Indefinite case. Let $\mathbb{L}^2 = (\mathbb{R}^2, -dx^2 + dy^2)$ be Minkowski plane, and $\mathbb{L}^3$ be a 3-dimensional Minkowski space with natural Lorentzian metric

$$\langle \cdot, \cdot \rangle = -dx^2 + dy^2 + dz^2.$$ 

THEOREM 3. (Weierstrass representation for timelike minimal surfaces in $\mathbb{L}^3$)

Let $\mathbf{x} : \mathbb{M} \rightarrow \mathbb{L}^3$ be a timelike surface parametrized by null coordinates $(u, v)$, where $u := -x + y, v := x + y$. Timelike minimal surface is represented by

$$\mathbf{x}(u, v) = \int^u f(u) du + \int^v g(v) dv,$$

The functions $f(u), g(u), f(v)$ and $g(v)$ are defined by

$$f = \frac{-\phi_1 + \phi_2}{2}, \quad g = \frac{\phi_3}{-\phi_1 + \phi_2},$$
$$\bar{f} = \frac{\mu_1 + \mu_2}{2}, \quad \bar{g} = \frac{\mu_3}{\mu_1 + \mu_2},$$

and $\phi = (\phi_1, \phi_2, \phi_3), \mu = (\mu_1, \mu_2, \mu_3)$ vector valued functions, $\phi(u) := x_u, \mu(v) := x_v$ satisfy

$$\phi^2 = 0, \quad \mu^2 = 0.$$ 

Hence, the timelike minimal surface has the form

$$\mathbf{x}(u, v) = \int^u \phi(u) du + \int^v \mu(v) dv, \quad \Omega(u) + \Psi(v),$$

and its conjugate

$$\mathbf{x}^\ast(u, v) = \Omega(u) - \Psi(v),$$

where $\phi(u)$ and $\mu(v)$ are linearly independent, $\Omega(u)$ and $\Psi(v)$ are null curves in $\mathbb{L}^3$. Weierstrass formula for the timelike minimal surfaces obtained by M. Magid [6] (see also [4], for details).
Lemma 2. The Weierstrass patch determined by the functions

\[(f(u), g(u)) = (u^{m-2}, u) \quad \text{and} \quad (f(v), g(v)) = (v^{m-2}, v)\]

is a representation of the Bour’s timelike minimal surface of value \(m\) in \(\mathbb{L}^3\), where \(m \in \mathbb{R}\).

Bour’s timelike minimal surface of value \(m\) is

\[
\int^u \left( \frac{-u^{m-2} (1 + u^2)}{2u^{m-1}} \right) du + \int^v \left( \frac{v^{m-2} (1 + v^2)}{2v^{m-1}} \right) dv,
\]

and it has the form

\[
(3.2) \quad \mathfrak{M}_m(u, v) = \left( \frac{- \frac{1}{m-1} (u^{m-1} - v^{m-1}) - \frac{1}{m+1} (u^{m+1} - v^{m+1})}{2\frac{1}{m} (u^m + v^m)} \right).
\]

Therefore, \(\mathfrak{M}_m(r, \theta)\) is

\[
x = - \frac{r^{m-1}}{m-1} \left( \cos^{(m-1)}(\theta) - \sin^{(m-1)}(\theta) \right),
\]

\[
y = - \frac{r^{m+1}}{m+1} \left( \cos^{(m+1)}(\theta) - \sin^{(m+1)}(\theta) \right),
\]

\[
z = 2 \frac{r^m}{m} (\cos^{(m)}(\theta) + \sin^{(m)}(\theta)).
\]

Theorem 4. Bour’s surface \(\mathfrak{M}_m(r, \theta)\) is a timelike minimal surface in \(\mathbb{L}^3\), where \(m \in \mathbb{R} - \{1, 0, -1\}\).

Proof. The coefficients of the first fundamental form of the \(\mathfrak{M}_m\) are

\[
E = 4r^{2m-4} (\sin \theta \cos \theta)^{m-1} (1 + r^2 \sin \theta \cos \theta)^2,
\]

\[
F = 2r^{2m-3} (\sin \theta \cos \theta)^{m-2} (1 + r^2 \sin \theta \cos \theta)^2 \cos (2\theta),
\]

\[
G = -4r^{2m-2} (\sin \theta \cos \theta)^{m-1} (1 + r^2 \sin \theta \cos \theta)^2.
\]

Then we have

\[
\det I = \left[ 2r^{2m-3} (\sin \theta \cos \theta)^{m-2} (1 + r^2 \sin \theta \cos \theta)^2 \right]^2.
\]

So, \(\mathfrak{M}_m\) is a timelike surface. The Gauss map is

\[
e = \frac{1}{1 + r^2 \sin \theta \cos \theta} \left( \begin{array}{c} r (\cos \theta - \sin \theta) \\ r (\cos \theta + \sin \theta) \\ r^2 \cos \theta \sin \theta - 1 \end{array} \right).
\]

The coefficients of the second fundamental form of the surface are

\[
L = -2r^{m-2} (\sin^m(\theta) + \cos^m(\theta)),
\]

\[
M = 2r^{m-1} (\sin(\theta) \cos^{m-1}(\theta) - \cos(\theta) \sin^{m-1}(\theta)),
\]

\[
N = -2r^m (\sin^2(\theta) \cos^{m-2}(\theta) + \cos^2(\theta) \sin^{m-2}(\theta)).
\]
We have
\[
\det II = -4r^{2m-2} (\sin \theta \cos \theta)^{m-2}.
\]
Hence, the Gaussian curvature and the mean curvature, respectively, are
\[
K = (\sin \theta \cos \theta)^{2-m} \left( \frac{r^{2-m}}{1 + r^2 \sin \theta \cos \theta} \right)^2,
\]
and
\[
H = 0.
\]
So, the $B_m$ is a timelike minimal surface in $L^3$. □

3.2. Bour's timelike surface $B_3$. If take $m = 3$ in $B_m (r, \theta)$, we have Bour's timelike minimal surface (see Fig. 14)
\[
B_3 (r, \theta) = \left( \begin{array}{c}
\left( -\frac{r^2}{2} - \frac{r^4}{4} \right) \cos (2\theta) \\
\frac{r^2}{2} - \frac{r^4}{4} (\cos^4 (\theta) + \sin^4 (\theta)) \\
\frac{r^3}{3} (\cos^3 (\theta) + \sin^3 (\theta))
\end{array} \right)
\]
in Minkowski 3-space, where $r \in [-1, 1]$, $\theta \in [0, \pi]$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{bourfig14.png}
\caption{(a) Bour's timelike minimal surface $B_3 (r, \theta)$}
\end{figure}

The coefficients of the first fundamental form of the Bour's timelike minimal surface of value 3 are
\[
E = 4r^2 (\sin \theta \cos \theta)^2 (1 + r^2 \sin \theta \cos \theta)^2,
\]
\[
F = r^3 \sin 2\theta (1 + r^2 \sin \theta \cos \theta)^2 \cos (2\theta),
\]
\[
G = -4r^4 (\sin \theta \cos \theta)^2 (1 + r^2 \sin \theta \cos \theta)^2.
\]
Then
\[
\det I = -4r^6 (\sin \theta \cos \theta)^2 (1 + r^2 \sin \theta \cos \theta)^4.
\]
The coefficients of the second fundamental form of the surface are
\[
L = -2r(\sin^3 (\theta) + \cos^3 (\theta)),
\]
\[
M = 2r^2 (\sin (\theta) \cos^2 (\theta) - \cos (\theta) \sin^2 (\theta)),
\]
\[
N = -2r^3 (\sin^2 (\theta) \cos (\theta) + \cos^2 (\theta) \sin (\theta)).
\]
So, \[ \det I = -4r^4 \sin \theta \cos \theta. \]
The mean and the Gaussian curvatures of the Bour’s minimal surface of value 3 are, respectively,
\[
H = 0, \quad K = \frac{1}{r^2 \sin \theta \cos \theta (1 + r^2 \sin \theta \cos \theta)^2}.
\]
The Weierstrass patch determined by the functions
\[(f, g) = (u, u) \quad \text{and} \quad (f, g) = (v, v)\]
in \(\mathbb{L}^3\).

The parametric form of the surface (see Fig. 15) is
\[
(3.4) \quad \mathcal{B}_3(u, v) = \begin{pmatrix} -\frac{1}{2} \left( u^2 - v^2 \right) - \frac{1}{4} \left( u^4 - v^4 \right) \\ \frac{1}{2} \left( u^2 + v^2 \right) - \frac{1}{4} \left( u^4 + v^4 \right) \\ \frac{2}{3} (u^3 + v^3) \end{pmatrix},
\]
where \(u, v \in I \subset \mathbb{R}\).

\(\begin{array}{cc}
(a) & (b) \\
\text{Figure 15} \quad \text{Timelike minimal surface } \mathcal{B}_3(u, v), u, v \in [-1, 1] & \\
\end{array}\)

The coefficients of the first fundamental form of the timelike Bour’s surface of value 3 are
\[
E = 0 = G, \quad F = 2uv (1 + uv)^2,
\]
So, 
\[ \det I = -4u^2v^2 (1 + uv)^4. \]
The Gauss map of the surface \(\mathcal{B}_3\) is
\[
e = \frac{1}{1 + uv} (u - v, u + v, uv - 1).
\]
The coefficients of the second fundamental form of the surface are
\[
L = -2u, \quad M = 0, \quad N = -2v.
\]
Then,
\[ \det II = 4uv. \]
The mean and the Gaussian curvatures of the timelike Bour’s minimal surface \(\mathcal{B}_3\) are
\[
H = 0, \quad K = -\frac{1}{uv (1 + uv)}.
\]
3.3. Applications of the indefinite case in $\mathbb{L}^3$.

**Example 10.** If take $m = 2$, we have $\mathcal{B}_2(r, \theta)$ (see Fig. 16)

\[
\begin{pmatrix}
-r(\cos{\theta} - \sin{\theta}) - \frac{r^3}{3}(\cos^3{\theta} - \sin^3{\theta}) \\
\frac{r^5}{5}(\cos^5{\theta} + \sin^5{\theta})
\end{pmatrix}
\]

in Minkowski 3-space, where $r \in [-2, 2]$, $\theta \in [-\pi/2, \pi/2]$.

---

**Figure 16** Bour’s timelike minimal surface $\mathcal{B}_2(r, \theta)$

**Example 11.** If take $m = 2$, we have $\mathcal{B}_2(r, \theta)$ (see Fig. 17) in Minkowski 3-space, where $r \in [-3, 3]$, $\theta \in [-\pi/2, \pi/2]$.

---

**Figure 17** Bour’s timelike minimal surface $\mathcal{B}_2(r, \theta)$

**Example 12.** If take $m = 4$, we have $\mathcal{B}_4(r, \theta)$ (see Fig. 18)

\[
\begin{pmatrix}
-\frac{r^3}{3}(\cos^3{\theta} - \sin^3{\theta}) \\
\frac{r^5}{5}(\cos^5{\theta} + \sin^5{\theta})
\end{pmatrix}
\]

in Minkowski 3-space, where $r \in [-1, 1]$, $\theta \in [0, \pi]$. 

---
Example 13. If take $m = 4$, we have $B_4(r, \theta)$ (see Fig. 19) in Minkowski 3-space, where $r \in [-2, 2]$, $\theta \in [0, \pi/2]$.

$$$
\begin{pmatrix}
-\frac{r^4}{4} (\cos^4 (\theta) - \sin^4 (\theta)) - \frac{r^6}{6} (\cos^6 (\theta) - \sin^6 (\theta)) \\
\frac{r^4}{4} (\cos^4 (\theta) + \sin^4 (\theta)) - \frac{r^6}{6} (\cos^6 (\theta) + \sin^6 (\theta)) \\
\frac{r^5}{5} (\cos^5 (\theta) + \sin^5 (\theta))
\end{pmatrix}
$$$
in Minkowski 3-space, where $r \in [-0.003, 0.003]$, $\theta \in [0, \pi]$. 

Example 14. If take $m = 5$, we have $B_5(r, \theta)$ (see Fig. 20)
Acknowledgement. A large part of this work had been completed by the author, when he visited as a post-doctoral researcher at the Katholieke Universiteit Leuven, Belgium in 2011-2012 academic year. The author would like to thanks to the hospitality of the members of the geometry section of the K.U. Leuven, especially to the Professor Franki Dillen.

References

[1] Anciaux, H. Minimal Submanifolds in Pseudo-Riemannian Geometry. World Sci. Pub., USA, 2011.
[2] Bour, E. Théorie de la déformation des surfaces. Journal de l’École Imperiale Polytechnique, 22, Cahier 39, 1–148, 1862.
[3] Gray, A. Modern Differential Geometry. CRC Press, Florida, 1998.
[4] Inoguchi, J., Lee, S. Null curves in Minkowski 3-space. Int. Electron. J. Geom., 1 (2), 40–83, 2008.
[5] Kobayashi, O. Maximal surfaces in the 3-dimensional Minkowski space $\mathbb{L}^3$. Tokyo J. Math. 6 (2), 297–309, 1983.
[6] Magid, M. Timelike surfaces in Lorentz 3-space with prescribed mean curvature and Gauss map. Hokkaido Math. J. 20 (3), 447-464, 1991.
[7] McNerney, L.V. One-parameter families of surfaces with constant curvature in Lorentz 3-space. Ph.D. Thesis, Brown Un., 1980.
[8] Nitsche, J.C.C. Lectures on Minimal Surfaces. Vol. 1. Introduction, fundamentals, geometry and basic boundary value problems. Cambridge University Press, Cambridge, 1989
[9] Weierstrass, K. Untersuchungen über die Flächen, deren mittlere Krümmung überall gleich Null ist, Monatsber. d. Berliner Akad. 612-625, 1866.

BARTIN UNIVERSITY, FACULTY OF SCIENCE, DEPARTMENT OF MATHEMATICS, BARTIN, TURKEY, E-mail address, E. Güler: ergler@gmail.com