BOUNDED WEIGHTED COMPOSITION OPERATORS ON FUNCTIONAL QUASI-BANACH SPACES AND STABILITY OF DYNAMICAL SYSTEMS

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ABSTRACT. In this paper, we investigate the boundedness of weighted composition operators defined on a quasi-Banach space continuously included in the space of smooth functions on a manifold. We show that the boundedness of a weighted composition operator strongly limits the dynamics of the original map, and it provides us an effective method to investigate properties of weighted composition operators via the theory of dynamical system. As a result, we prove that only an affine map can induce a bounded composition operator on an arbitrary quasi-Banach space continuously included in the space of entire functions of one-variable. We also obtain the same result for bounded weighted composition operators on infinite dimensional quasi-Banach space under certain condition for weights. We also prove that any polynomial automorphism except an affine transform cannot induce a bounded weighted composition operator with non-vanishing weight on a quasi-Banach space composed of entire functions in the two-dimensional complex affine space under certain conditions.

1. INTRODUCTION

In this paper, we investigate the boundedness of weighted composition operators defined on a quasi-Banach space continuously included in the space of smooth functions on a smooth manifold. We provide a connection between the boundedness of weighted composition operators and the dynamics of the original maps. As a result, we prove that only affine maps can induce a bounded weighted composition operator on any quasi-Banach space continuously included in the space of entire functions on \( \mathbb{C} \), in particular, on any reproducing kernel Hilbert space (RKHS) composed of entire functions on \( \mathbb{C} \), using utilize a powerful theory developed in holomorphic dynamics, in contrast to existing studies, which heavily depend on the explicit structure of their function spaces. To be precise, let us introduce several notions and state our main results. We include other basic notations at the end. Let \( X \) be a smooth manifold of dimension \( d \). Let \( \mathcal{E}(X) \) be the space of smooth functions on \( X \). We equip the weak Whitney topology with \( \mathcal{E}(X) \), which is the topology defined by uniform convergence of any derivatives on compact sets. The quasi-Banach space is a complete Hausdorff topological vector space \( V \) which has a bounded neighborhood of 0. We always assume \( V \) is a subset of \( \mathcal{E}(X) \) and the

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inclusion \( t : V \hookrightarrow \mathcal{E}(X) \) is continuous. We say a quasi-Banach space is functional if it is a subspace of \( \mathbb{C}^X \) (we equip \( \mathbb{C}^X \) with the pointwise convergence topology). Thus, \( V \) is functional in this sense. A typical example of \( V \) is an RKHS (see Section 3, or [27] for more detail), Hardy spaces, and Bergman spaces (see, for example, [11]).

Let \( f : X \to X \) be a smooth map and let \( u \in \mathcal{E}(X) \). The weighted composition operator with weight \( u \) is a linear operator \( uC_f \) on \( V \) defined on \( \{ h \in V : u \cdot (h \circ f) \in V \} \) such that \( uC_fh := u \cdot (h \circ f) \). There have been numerous studies on the properties of the weighted composition operators, especially in the case where \( X \) is a complex manifold. [34] and [10] are standard references for the classical function spaces of analytic functions. In the case of \( X = \mathbb{C}^d \), there exist many works on characterization of the boundedness of weighted composition operators on quasi-Banach spaces, and many results show that boundedness implies the affineness of the original holomorphic map [6, 29, 26, 7, 1, 25, 17, 8, 18, 23, 22, 32, 33]. These results suggest that quite a wide class of function spaces of entire functions have the same property. As in Theorem 1.2 below, we solve this problem in the one-dimensional case in quite a general situation. The composition operator is also important in applied mathematics, for example, in signal processing [9, 2, 3] as the time-warping and recently attracts researchers in machine learning [16, 19, 21] as the Koopman operator.

Let

\[
\mathcal{D}(\mathbb{R}^d) := \bigoplus_{n_1, \ldots, n_d = 0}^\infty \mathbb{C} \partial_{x_1}^{n_1} \cdots \partial_{x_d}^{n_d}
\]

be the space of differential operators on \( \mathcal{E}(\mathbb{R}^d) \) and for each \( n \geq 0 \), we define a finite-dimensional subspace of \( \mathcal{D}(\mathbb{R}^d) \) by

\[
\mathcal{D}_n(\mathbb{R}^d) := \bigoplus_{n_1 + \cdots + n_d = n} \mathbb{C} \partial_{x_1}^{n_1} \cdots \partial_{x_d}^{n_d}.
\]

Let \( p \in X \) and fix a local coordinate \( \phi \) from an open neighborhood of \( p \) onto an open subset \( \mathbb{R}^d \). Then, we define an injective linear map \( \delta_{p, \phi} : \mathcal{D}(\mathbb{R}^d) \to \mathcal{E}(X)' \) by

\[
\delta_{p, \phi}(D)(h) := D(h \circ \phi^{-1})(\phi(x))
\]

for \( D \in \mathcal{D}(\mathbb{R}^d) \) and \( h \in \mathcal{E}(X) \), where \((\cdot)'\) stands for the dual space with the strong topology. The set \( \delta_{p, \phi}(\mathcal{D}(\mathbb{R}^d)) \) independent of the choice of the local coordinate \( \phi \), and we define subspaces of \( \mathcal{E}(X)' \) by

\[
\mathcal{D}(X)_p := \delta_{p, \phi}(\mathcal{D}(\mathbb{R}^d)),
\]

\[
\mathcal{D}_n(X)_p := \delta_{p, \phi}(\mathcal{D}_n(\mathbb{R}^d)).
\]

As we below show (Lemma 2.2), the dual operator \( f^* : \mathcal{E}(X)' \to \mathcal{E}(X)' \) of \( f^* : h \mapsto h \circ f \) preserves the finite dimensional subspace \( \mathcal{D}_n(X)_p \) for any \( n \geq 0 \) if \( p \) is a fixed point of \( f \). Thus, \( f^* \) naturally induces the linear map on \( \mathcal{D}_n(X)_p / \mathcal{D}_{n-1}(X)_p \) and we denote it by \( \text{gr}_{f^*, n} \). For \( n \geq 0 \), we define a surjective linear map induced by
the dual of the inclusion $\iota^\prime$:
$$\kappa^a_p : \mathcal{D}_n(X)_p / \mathcal{D}_{n-1}(X)_p \longrightarrow \iota^\prime(\mathcal{D}_n(X)_p) / \iota^\prime(\mathcal{D}_{n-1}(X)_p).$$

Then, we have the following theorem:

**Theorem 1.1.** Let $p \in X$ be a fixed point of $f$, namely, $f(p) = p$. Assume the following conditions:

1. $u C_f$ is bounded on $V$ for some $u \in \mathcal{E}(X)$ with $u(p) \neq 0$.
2. For infinitely many $n \geq 0$, $\text{Ker}(\kappa^n_p) \subset \text{Ker}(\text{gr}^a_{f^r})$.

Then, any eigenvalue $\alpha$ of $df_p : T_p(X) \to T_p(X)$ satisfies the inequality $|\alpha| \leq 1$. Here, $T_p(X)$ is the tangent space of $X$ at $p$.

We note that for any holomorphic $f$ and all but finitely many $p \in X$, the condition (2) holds when, for example, the space $V$ is the Fock type space $\mathcal{F}$ (see Remark 4.2). As an immediate corollary, we have
Corollary 1.3. Let $V \subset \mathcal{A}(\mathbb{C})$ be an infinite dimensional quasi-Banach space and the inclusion map is continuous. Then, for any holomorphic map $f : \mathbb{C} \to \mathbb{C}$, if $e^{w}C_{f}$ is bounded on $V$ for an entire function $w \in \mathcal{A}(\mathbb{C})$, we have $f(z) = az + b$ for some $a, b \in \mathbb{C}$ with $|a| \leq 1$.

As $\mathcal{H}(f)$ is compact when $f$ is a polynomial (see, for example, [12, (1)]), we also have the interesting result as follows:

Corollary 1.4. Let $V \subset \mathcal{A}(\mathbb{C})$ be an infinite dimensional quasi-Banach space and the inclusion map is continuous. Then, for any polynomial map $f : \mathbb{C} \to \mathbb{C}$, if $uC_{f}$ is bounded on $V$ for a nonzero entire function $u \in \mathcal{A}(\mathbb{C})$, we have $f(z) = az + b$ for some $a, b \in \mathbb{C}$ with $|a| \leq 1$.

As we mentioned above, these types of results are known in various existing works and what we prove is that it is always true in the one-dimensional case in a quite general situation. Theorem 1.1 does not hold if $V$ is finite-dimensional (consider, for example, $V = \mathbb{C}e^{z}$, $f(z) = (z + 1)^{2}/2$, and $u(z) = e^{-z^{2}/2}$). However, if we take a constant weight, the similar result to Corollary 1.3 still holds and thus we have the following theorem:

Theorem 1.5. Let $V \subset \mathcal{A}(\mathbb{C})$ be a quasi-Banach space and the inclusion map is continuous. For any holomorphic map $f : \mathbb{C} \to \mathbb{C}$, if the composition operator $C_{f}$ is bounded on $V$, then $f(z) = az + b$ for some $a, b \in \mathbb{C}$. Moreover, if $V$ is infinite dimensional we have $|a| \leq 1$.

For higher dimensional cases, the problem gets more complicated in contrast to the one-dimensional case. Let $\mathcal{G}_{d}(V)$ which is a set composed of regular matrices $A \in \text{GL}_{d}(\mathbb{C})$ satisfying there exist $b \in \mathbb{C}^{d}$ and non-vanishing $v \in \mathcal{A}(\mathbb{C}^{d})$ such that $vC_{A(\cdot)+b}$ is bounded on $V$. Then, we have the following result in the two-dimensional case:

Theorem 1.6. Let $V$ be a quasi-Banach space continuously included in $\mathcal{A}(\mathbb{C}^{2})$. Assume the following conditions:

(1) $\kappa_{p}^{\text{hol}}$ is injective for all but finitely many $p \in \mathbb{C}^{2}$ and infinitely many $n \geq 0$,

(2) $\langle \mathcal{G}_{2}(V) \rangle_{\mathbb{C}} = \mathcal{M}_{2}(\mathbb{C})$.

Then, for any polynomial automorphism $f : \mathbb{C}^{2} \to \mathbb{C}^{2}$, if $e^{w}C_{f}$ is bounded for some $w \in \mathcal{A}(\mathbb{C}^{2})$, we have $f(z) = Az + b$ for some $A \in \text{GL}_{2}(\mathbb{C})$ and $b \in \mathbb{C}^{2}$.

Here, $\langle S \rangle_{\mathbb{C}}$ means the vector space generated by $S$ over $\mathbb{C}$ and $\kappa_{p}^{\text{hol}}$ is defined in the same way as $\kappa_{p}^{s}$ in the complex situation (see Section 2.2 for the definition). The conditions (1) and (2) hold in a certain large class of function spaces, for example, the RKHS treated in [29, 26, 18] (see Section 3). The theory of holomorphic dynamics is a powerful tool to tackle this problem and enables us to reduce it to infinitely many existence of repelling or saddle periodic points for holomorphic dynamics, which is an important problem in the theory of dynamical system (see [14, Question 2.16]). For the details, see Section 4.

The structure of this paper is as follows: In Section 2, we introduce notions, several preliminary lemmas and provide the proof of Theorem 1.1 (Theorem 2.4).
We also obtain parallel results in complex manifolds and briefly summarize their statements. In Section 3, we provide several explicit examples of quasi-Banach spaces continuously included in $\mathcal{E}'(X)$ which satisfies the condition (2) in Theorem 1.1. We here illustrate that our framework covers many previous results. In Section 4, we prove Theorems 1.2 and 1.6 and the corollaries.

**BASIC NOTATION**

We denote the set of the real (resp. complex) values by $\mathbb{R}$ (resp. $\mathbb{C}$). For any subset of $S \subset \mathbb{R}$, we denote by $S_{>0}$ (resp. $S_{\geq 0}$) the set of positive (resp. non-negative) elements of $S$. We denote by $\partial_{x_j}$ (resp. $\partial_{x_j}$) the partial derivative with respect to the $j$-th variable of a differentiable (resp. holomorphic) function on an open set of $\mathbb{R}^d$ (resp. $\mathbb{C}^d$). We denote by $\text{GL}_d(\mathbb{C})$ (resp. $\text{M}_d(\mathbb{C})$) the set of regular matrices (resp. matrices) of size $d$. For any subset $S \subset V$ of a complex linear space $V$, we denote by $\langle S \rangle_C$ the linear subspace generated by $S$.

We sometimes use the multi-index notation. For example, for $(\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}_{\geq 0}^d$, $(z_1, \ldots, z_d)$ and $(\partial_{x_1}, \ldots, \partial_{x_d})$, we write $z^{\alpha} := z_1^{\alpha_1} \cdots z_d^{\alpha_d}$ and $\partial^{\alpha} := \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}$.

For a smooth (complex) manifold $X$, we denote by $\mathcal{E}'(X)$ (resp. $\mathcal{A}(X)$) the space of $\mathbb{C}$-valued $C^\infty$-class (resp. holomorphic) functions on $X$.

For a topological linear space $V$, we denote the strong dual by $V'$. For a bounded linear map $L : V_1 \rightarrow V_2$ between topological linear spaces $V_1$ and $V_2$, we denote by $L'$ the dual linear operator $L' : V'_2 \rightarrow V'_1$.

For a vector space $V$ with an ascending filtration, which is an ascending family of subspaces $\{V_n\}_{n \geq -1}$, we define $\text{gr}^n[V] := V_n/V_{n-1}$ and $\text{gr}[V] := \bigoplus_{n \geq 0} \text{gr}^n[V]$. For a linear map $A : V \rightarrow W$ such that there exists families of ascending subspaces $\{V_n\}_{n \geq -1}$ (resp. $\{W_n\}_{n \geq -1}$) of $V$ (resp. $W$) satisfying $A(V_n) \subset W_n$, we denote by $\text{gr}^n : V_n/V_{n-1} \rightarrow W_n/W_{n-1}$ the induced linear map. We define $\text{gr} := \bigoplus_{n \geq 0} \text{gr}^n : \bigoplus_{n \geq 0} V_n/V_{n-1} \rightarrow \bigoplus_{n \geq 0} W_n/W_{n-1}$

**2. BOUNDEDNESS OF COMPOSITION OPERATORS AND A STABILITY OF DYNAMICAL SYSTEMS**

In this section, we prove that the boundedness of a weighted composition operator strongly limits behavior of a dynamical system on a smooth manifold. In Subsection 2.1, we describe the main result in a general smooth manifold, and in Subsection 2.2, we summarize corresponding results in the complex case.

**2.1. A STABILITY OF DYNAMICAL SYSTEMS ON A SMOOTH MANIFOLD WITH BOUNDED COMPOSITION OPERATORS.** Let $X$ be a smooth manifold of dimension $d$, and let $f : X \rightarrow X$ be a smooth map. Let $V$ be a quasi-Banach space, which is a complete Hausdorff topological vector space with respect to a quasi-norm $\| \cdot \|_V$. Here, the quasi-norm is a map $\| \cdot \|_V : V \rightarrow \mathbb{R}_{\geq 0}$ satisfying the following three conditions: (1) $\|av\|_V = |a| \cdot \|v\|_V$, (2) there exists $K \geq 1$ such that $\|v + w\|_V \leq K(\|v\|_V + \|w\|_V)$, and (3) $v = 0$ if $\|v\|_V = 0$, where $a \in \mathbb{C}$ and $v, w \in V$. We note that although a quasi-Banach space is not necessarily locally convex, bounded linear operators on a quasi-Banach space share several same properties as those well-known in a Banach space. A quasi-Banach space is also characterized as a complete Hausdorff...
where we identify $\prod_{u \in C}$.

Here, we regard $(2.2)$ $D_u C \subset X$ be the space of differential operators on $\mathbb{R}$ such that for all $p \in \mathbb{R}$, namely, for any $i, j \in \mathbb{R}$, there exists $i_j, j_i \in I$ such that $V_j \subset U_{i_j} \cap U_{j_i}$. Then, we regard $\mathcal{E}(X)$ as a closed subspace of the locally convex Hausdorff space $\prod_{i \in I} \mathcal{E}(U_i)$ defined by the kernel of the following continuous linear map:

$$
\prod_{i \in I} \mathcal{E}(U_i) \longrightarrow \prod_{j \in J} \mathcal{E}(V_j); \ (h_i)_{i \in I} \mapsto (h_j|_{V_j} - h_{j_i}|_{V_j})_{i, j \in I},
$$

where we identify $\mathcal{E}(U_i)$ and $\mathcal{E}(V_j)$ with $\mathcal{E}(\phi_i(U_i))$ and $\mathcal{E}(\psi_j(V_j))$, respectively, and their topologies are defined by uniformly convergence of any derivatives on compact sets (see [31, p.26-27]). As in the following lemma, this topology is independent of the choice of local coordinate systems:

**Lemma 2.1.** Let $p \in \mathbb{R}^d$ and let $U \subset \mathbb{R}^d$ be an open neighborhood of $p$. Let $F : U \rightarrow \mathbb{R}^r$ be a smooth map and let $h, u : U \rightarrow \mathbb{C}$ be a smooth function. Then, for any $\partial_{x_{i_1}} \cdots \partial_{x_{i_n}} \in \mathcal{D}_n(\mathbb{R}^d)$ $(i_j \in \{1, \ldots, d\})$, there exists a smooth map $D_n$ from $U$ to $\mathcal{D}_{n+1}(\mathbb{R}^d)$ such that for all $p \in U$,

$$(2.2) \quad (\partial_{x_{i_1}} \cdots \partial_{x_{i_n}})(u \cdot (h \circ F))(p) - u(p) \left[ \left( \prod_{j=1}^n \partial_{x_{i_j}} F_p \cdot \partial \right) h \right](F(p)) + (D_n(p)h)(F(p)).$$

Here, we regard $\mathcal{D}_{n+1}(\mathbb{R}^d)$ as a finite product of $\mathbb{C}$’s as it is finite-dimensional, and

$$
\partial_{x_{i_1}} F_p \cdot \partial := \sum_{m=1}^r \frac{\partial F^m}{\partial x_j}(p) \partial_{x_{i_m}} \in \mathcal{D}_1(\mathbb{R}^d),
$$

where we write $F = (F^1, \ldots, F^r)$ for some $F^j \in \mathcal{E}(U)$ $(j = 1, \ldots, r)$. 

Proof. First we assume \( u \equiv 1 \). The general case follows from the Leibniz rule. We prove by induction. In the case of \( n = 1 \), it immediately follows from the chain rule and we have \( \mathbf{D}_1 = 0 \). In the case of \( n = k > 1 \). Put \( D = \partial_{x_2} \cdots \partial_{x_k} \). Fix \( p \in U \) and let \( p_t := p + te_{i_t} \), where \( t \in \mathbb{R} \) and \( e_{i_t} \) is the vector whose \( i_t \)-th component is 1 and other ones are 0. Then, by induction hypothesis, there exists \( \mathbf{D}_{k-1} : U \to \mathcal{D}_{k-2}(\mathbb{R}^d) \) such that for sufficiently small \( t \),

\[
(2.3) \quad D(h \circ F)(p_t) = \left[ \left( \prod_{j=2}^{k} \partial_{x_j} F_{p_t} \cdot \partial \right) h \right] (F(p_t)) + (\mathbf{D}_{k-1}(p_t)h)(F(p_t)).
\]

We define a smooth function from \( U \) to \( \mathcal{D}_{k-1}(\mathbb{R}^d) \) by

\[
\mathbf{D}_k(p) := \frac{d}{dt} \left( \mathbf{D}_{k-1}(p_t) - \prod_{j=1}^{k} \left( \partial_{x_j} F_{p_t} \cdot \partial \right) \right) \bigg|_{t=0} + (\partial_{x_i} F_{p_t} \cdot \partial) \cdot \mathbf{D}_{k-1}(p).
\]

Then, we apply \( d/dt \big|_{t=0} \) to (2.3), and by direct computation, we have the formula (2.2) in the case of \( n = k \).

We define the pull-back \( f^* : \mathcal{E}(X) \to \mathcal{E}(X) \) by allocating \( h \circ f \) to \( h \in \mathcal{E}(X) \). For \( u \in \mathcal{E}(X) \), we define \( uf^* \) by \( uf^*(h) := u \cdot (h \circ f) \). Then, \( uf^* \) induces a continuous linear map on \( \mathcal{E}(X) \) by Lemma 2.1. The dual operator \( (uf^*)' : \mathcal{E}(X)' \to \mathcal{E}(X)' \) is also a continuous linear operator. We note that on the domain of \( C_f \), we have \( (uf^*)t = t(uC_f) \).

Let \( p \in X \) and fix a local coordinate \( \phi \) from an open neighborhood of \( p \) onto an open subset \( \mathbb{R}^d \). Then, we define an injective linear map \( \delta_{p, \phi} : \mathcal{D}(\mathbb{R}^d) \to \mathcal{E}(X)' \) by

\[
(2.4) \quad \delta_{p, \phi}(D)(h) := D(h \circ \phi^{-1})(\phi(x))
\]

for \( D \in \mathcal{D}(\mathbb{R}^d) \) and \( h \in \mathcal{E}(X) \). By Lemma 2.1, the set \( \delta_{p, \phi}(\mathcal{D}(\mathbb{R}^d)) \) independent of the choice of the local coordinate \( \phi \), and we define subspaces of \( \mathcal{E}(X)' \) by

\[
\mathcal{D}(X)_{p} := \delta_{p, \phi}(\mathcal{D}(\mathbb{R}^d)), \quad \mathcal{D}_{n}(X)_{p} := \delta_{p, \phi}(\mathcal{D}_{n}(\mathbb{R}^d)).
\]

We equip an ascending filtration \( \{ \mathcal{D}_{n}(X)_{p} \}_{n \geq 0} \) with \( \mathcal{E}(X)' \). In the case where \( X \) is an open subset of \( \mathbb{R}^d \), we always take the local coordinate \( \phi \) as a canonical inclusion, and we denote by \( \delta_p \) instead \( \delta_{p, \phi} \).

As an immediate consequence of Lemma 2.1 with the Leibniz formula, we have the following statement:

**Lemma 2.2.** Let \( p \in X \) be a fixed point of \( f \), namely, \( f(p) = p \). Fix \( u \in \mathcal{E}(X) \). Let \( \phi \) be a diffeomorphism from an open neighborhood of \( p \) into an open subset of \( \mathbb{R}^d \). Then, for any \( \partial_{x_i} \cdots \partial_{x_u} \in \mathcal{D}_u(\mathbb{R}^d) \) \( (i_j \in \{1, \ldots, d\}) \), we have

\[
(2.5) \quad (uf^*)' \left[ \delta_{p, \phi}(\partial_{x_i} \cdots \partial_{x_u}) \right] - \delta_{p, \phi} \left( u(p) \prod_{j=1}^{d} (\partial_{x_j} f^\phi_{(p)} \cdot \partial) \right) \in \mathcal{D}_{n-1}(X)_p.
\]
Moreover, we have the following commutative diagram:

\[
\begin{array}{ccc}
\text{gr}[\mathcal{D}(X)_p] & \xrightarrow[\text{gr}(u^*)'] & \text{gr}[\mathcal{D}(X)_p] \\
\cong & \uparrow & \cong \\
\mathbb{C}[t_1, \ldots, t_d] & \xrightarrow{u(p)S(d_{\phi(p)}^\phi)} & \mathbb{C}[t_1, \ldots, t_d]
\end{array}
\]

where the vertical maps are defined by the correspondence \( t^\alpha \mapsto \delta_{p,\phi}(\partial^\alpha) \) and \( S(d_{\phi(p)}^\phi) \) is the linear map defined by the correspondence \( t_i \mapsto \sum_{m=1}^d \frac{\partial f^\phi_j}{\partial x_i}(\phi(p))t_m \).

Proof. By Lemma 2.2, \( (uf^*)'(\mathcal{D}(X)_p) \subset \mathcal{D}(X)_p \) is obvious. By (2.5), \( \text{gr}(u^*)' \) allocates the element \( u(p)(\partial_{\alpha}f^\phi_{\phi(p)} \cdot \partial)^\alpha \) to \( \partial^\alpha \), and thus we have the second statement. \( \square \)

As a result, we have the following corollaries:

**Corollary 2.3.** We use the same notation as in Lemma 2.2. Then, we have

\[
(uf^*)'(\mathcal{D}_n(X)_p) \subset \mathcal{D}_n(X)_p.
\]

Moreover, we have the following commutative diagram:

\[
\begin{array}{ccc}
\text{gr}[\mathcal{D}(X)_p] & \xrightarrow[\text{gr}(u^*)'] & \text{gr}[\mathcal{D}(X)_p] \\
\cong & \uparrow & \cong \\
\mathbb{C}[t_1, \ldots, t_d] & \xrightarrow{u(p)S(d_{\phi(p)}^\phi)} & \mathbb{C}[t_1, \ldots, t_d]
\end{array}
\]

where the vertical maps are defined by the correspondence \( t^\alpha \mapsto \delta_{p,\phi}(\partial^\alpha) \) and \( S(d_{\phi(p)}^\phi) \) is the linear map defined by the correspondence \( t_i \mapsto \sum_{m=1}^d \frac{\partial f^\phi_j}{\partial x_i}(\phi(p))t_m \).

Proof. By Lemma 2.2, \( (uf^*)'(\mathcal{D}_n(X)_p) \subset \mathcal{D}_n(X)_p \) is obvious. By (2.5), \( \text{gr}(u^*)' \) allocates the element \( u(p)(\partial_{\alpha}f^\phi_{\phi(p)} \cdot \partial)^\alpha \) to \( \partial^\alpha \), and thus we have the second statement. \( \square \)

For \( n \geq 0 \), we define a surjective linear map induced by \( t^\alpha \):

\[
(2.6) \quad \kappa^\alpha_p : \mathcal{D}_n(X)_p / \mathcal{D}_{n-1}(X)_p \rightarrow t^\alpha(\mathcal{D}_n(X)_p) / t^\alpha(\mathcal{D}_{n-1}(X)_p)
\]

Here, we define \( \mathcal{D}_{n}(X)_p := \{0\} \). Then, we have the following theorem:

**Theorem 2.4.** Let \( X \) be a smooth manifold, and let \( f : X \rightarrow X \) be a smooth map. Let \( V \subset \mathcal{E}(X) \) be a quasi-Banach space continuously included in \( \mathcal{E}(X) \). Let \( p \in X \) be a fixed point of \( f \), namely, \( f(p) = p \). Assume the following conditions:

1. \( uC_f \) is bounded on \( V \) for some \( u \in \mathcal{E}(X) \) with \( u(p) \neq 0 \),
2. for infinitely many \( n \geq 0 \), \( \text{Ker}(\kappa^\alpha_p) \subset \text{Ker}(\text{gr}^\alpha(u^*)') \).

Then, any eigenvalue \( \alpha \) of \( df_p : T_p(X) \rightarrow T_p(X) \) satisfies the inequality \( |\alpha| \leq 1 \). Here, \( T_p(X) \) is the tangent space of \( X \) at \( p \).

Proof. We may assume \( \alpha \neq 0 \). By Corollary 2.3, for any \( n \geq 0 \), there exists nonzero \( v_n \in \text{gr}^\alpha[\mathcal{D}(X)_p] \) such that

\[
\text{gr}^\alpha_{u^*}(v_n) = \alpha^n v_n.
\]

Let \( w_n := \kappa^\alpha_p(v_n) \). Then, we have

\[
\text{gr}^\alpha_{u^*}(w_n) = \kappa^\alpha_p(\text{gr}^\alpha_{u^*})(v_n) = u(p)\alpha^n w_n.
\]
Since \( v_n \notin \text{Ker}(\text{gr}^n_{f^*}) \), we see that \( w_n \neq 0 \) for infinitely many \( n \geq 0 \). We note that 
\[
\| (uC_j)^i' \| \geq \| \text{gr}^n_{(uC_j)^i'} \| \| u \| ,
\]
where the norm \( \| \cdot \| \) (resp. \( \| \cdot \|_0 \)) is the operator norm on \( V \) (resp. the image of \( \kappa^n_p \)). Thus, for any \( n \geq 0 \) with \( w_n \neq 0 \), we have 
\[
(2.7) \quad |u(p)|^{1/n} \cdot |\alpha| \leq \| (uC_j)^i' \|^{1/n}.
\]
Since \( u(p) \neq 0 \) and (2.7) holds infinitely many \( n \), we have \( |\alpha| \leq 1 \). \( \Box \)

This theorem implies that if a dynamical system induces a bounded weighted composition operator on a certain functional quasi-Banach space, it does not have periodic points with a repelling direction.

2.2. Boundedness of composition operators and stabilities of holomorphic dynamics. Let \( X \) be a complex manifold of complex dimension \( d \). We equip the topology of uniform convergence on compact sets with \( \mathcal{A}(X) \). We here provide several corresponding results in the complex case. Let
\[
\mathcal{D}^\text{hol}(\mathbb{C}^d) := \bigoplus_{n_1,\ldots,n_d=0}^\infty \mathbb{C} \partial_{z_1}^{n_1} \cdots \partial_{z_d}^{n_d}
\]
be the space of holomorphic differential operators on \( \mathcal{A}(\mathbb{C}^d) \) and for each \( n \geq 0 \), we define
\[
\mathcal{D}^\text{hol}_n(\mathbb{C}^d) := \bigoplus_{n_1+\cdots+n_d \leq n} \mathbb{C} \partial_{z_1}^{n_1} \cdots \partial_{z_d}^{n_d}.
\]
Let \( p \in X \) and fix a local complex coordinate \( \phi \) from an open neighborhood of \( p \) onto an open subset \( \mathbb{C}^d \). Then, we define an injective linear map 
\[
\delta^\text{hol}_{p,\phi} : \mathcal{D}^\text{hol}(\mathbb{C}^d) \longrightarrow \mathcal{A}(X)
\]
in the same way as (2.4). We also define \( \mathcal{D}^\text{hol}(X)_p \) and \( \mathcal{D}^\text{hol}_n(X)_p \) by the images of \( \mathcal{D}^\text{hol}(\mathbb{C}^d) \) and \( \mathcal{D}^\text{hol}_n(\mathbb{C}^d) \) under \( \delta^\text{hol}_{p,\phi} \), respectively. In terms of Lemma 2.2 and Corollary 2.3, for \( n \geq 0 \), we define
\[
\text{gr}^n_{f^*} : \mathcal{D}^\text{hol}_n(X)_p / \mathcal{D}^\text{hol}_{n-1}(X)_p \longrightarrow \mathcal{D}^\text{hol}_n(X)_p / \mathcal{D}^\text{hol}_{n-1}(X)_p ;
\]
\[
\kappa^n_p : \mathcal{D}^\text{hol}_n(X)_p / \mathcal{D}^\text{hol}_{n-1}(X)_p \longrightarrow \tau' \left( \mathcal{D}^\text{hol}_n(X)_p / \mathcal{D}^\text{hol}_{n-1}(X)_p \right),
\]
where \( p \) is a fixed point of \( f \).

We also obtain an analogous result of Theorem 2.4 with a similar proof:

**Theorem 2.5.** Let \( X \) be a complex manifold, and let \( f : X \to X \) be a holomorphic map. Let \( V \subset \mathcal{A}(X) \) be a quasi-Banach space continuously included in \( \mathcal{A}(X) \). Let \( p \in X \) be a fixed point of \( f \). Assume the following conditions:

1. \( uC_j \) is a bounded linear operator on \( V \) for some \( u \in \mathcal{A}(X) \) with \( u(p) \neq 0 \).
2. For infinitely many \( n \geq 0 \), \( \text{Ker}(\kappa^n_p) = \text{Ker}((\text{gr}^n_{f^*})^* \kappa^n_p) \).

Then, any eigenvalue \( \alpha \) of \( df_p : T^{1,0}_p(X) \to T^{1,0}_p(X) \) satisfies the inequality \( |\alpha| \leq 1 \). Here, \( T^{1,0}_p(X) \) is the \((1,0)\) part of the tangent vector over \( \mathbb{C} \) of \( X \) at \( p \).

We prove a simple but important lemma for the complex case:
Lemma 2.6. Let $X$ be a connected complex manifold. Let $V \subset \mathscr{A}(X)$ be a quasi-Banach space with a continuous inclusion $\iota : V \hookrightarrow \mathscr{A}(X)$. Then, $V$ is infinite-dimensional if and only if
\[
t'\left(\mathscr{D}_n(X)_{p}\right)/t'\left(\mathscr{D}_{n-1}(X)_{p}\right) \neq 0
\]
for infinitely many $n \geq 0$.

Proof. We prove the “only if” part as the “if” part is obvious. Suppose there exists $k \geq 0$ such that for any $n \geq k+1$,
\[
t'\left(\mathscr{D}_n(X)_{p}\right)/t'\left(\mathscr{D}_{n-1}(X)_{p}\right) = 0.
\]
Then, we see that
\[
t'\left(\mathscr{D}_k(X)_{p}\right) = t'\left(\mathscr{D}_k(X)_{p}\right).
\]

Let $\ell : V \rightarrow \mathbb{C}^{k+1}$ be a linear map defined by $\ell(h) := (h(p), h^{(1)}(p), \cdots, h^{(k)}(p))$, where $h^{(n)}(p) := \delta_{p,\phi}(\partial^n_p)(h)$ with a local coordinate $\phi$ at $p$. Then, by (2.8), $\ell(h) = 0$ implies $h^{(n)}(p) = 0$ for all $n \geq 0$. Thus, we see that $h \equiv 0$ and $F$ is injective as $X$ is connected. Therefore, we conclude $V$ is finite-dimensional. \hfill \Box

As a corollary, if $X$ is 1-dimensional, we have the following statement:

Corollary 2.7. Assume $X$ is an 1-dimensional connected complex manifold. Let $V \subset \mathscr{A}(X)$ be a quasi-Banach space with continuous inclusion $\iota : V \hookrightarrow \mathscr{A}(X)$. Unless $V$ is finite-dimensional, for infinitely many $n \geq 0$, we have $\text{Ker}(\kappa_p^{n,\text{hol}}) = \{0\}$, namely, $\kappa_p^{n,\text{hol}}$ is an isomorphism. In particular, if $V$ is infinite dimensional, the condition (2) of Theorem 2.5 is always true for any holomorphic map on $X$.

Proof. Since $\mathscr{D}_n(X)_{p}/\mathscr{D}_{n-1}(X)_{p}$ is always one dimensional and $\kappa_p^{n,\text{hol}}$ is a surjective linear map onto $t'\left(\mathscr{D}_n(X)_{p}\right)/t'\left(\mathscr{D}_{n-1}(X)_{p}\right)$, the homomorphism $\kappa_p^{n,\text{hol}}$ is isomorphism if and only if
\[
t'\left(\mathscr{D}_n(X)_{p}\right)/t'\left(\mathscr{D}_{n-1}(X)_{p}\right) \neq 0.
\]
Thus, the corollary follows from Lemma 2.6. \hfill \Box

3. Function spaces continuously included in the space of smooth or holomorphic functions

Let $X$ be a smooth manifold of dimension $d$. In this section, we provide several important and typical examples of function spaces on $X$ continuously included in $\mathscr{E}(X)$.

3.1. Reproducing kernel Hilbert space (RKHS). The theory of reproducing kernel Hilbert space is among the most typical frameworks to construct a Hilbert space continuously included in $\mathscr{E}(X)$ or $\mathscr{A}(X)$. We first review a general notion of reproducing kernel Hilbert spaces.

Definition 3.1. Let $X$ be an arbitrary abstract non-empty set and let $k : X \times X \rightarrow \mathbb{C}$ be a map. We call $k$ a positive definite kernel if for any finite $p_1, \ldots, p_n \in X$, the matrix $(k(p_i, p_j))_{i,j=1,\ldots,n}$ is a Hermitian positive semi-definite matrix.
Then, there is a classical result, known as the Moore-Aronszajn theorem, that states for any positive definite kernel $k$, there uniquely exists a Hilbert space $H_k$ composed of $\mathbb{C}$-valued maps on $X$ such that the following conditions hold:

1. for any $p \in X$, the map $k_p := k(p, \cdot)$ is an element of $H_k$,
2. for any $p \in X$ and $h \in H_k$, we have $\langle h, k_p \rangle_{H_k} = h(p)$.

We call $H_k$ a reproducing kernel Hilbert space, or RKHS for short (see [27] for more details).

**Remark 3.2.** Let $H$ be an abstract Hilbert space and let $H_k$ be a RKHS associated to a positive definite kernel $k$ on $X$. If there exists a map $\varphi : X \to H$ such that $\langle \varphi(p), \varphi(p') \rangle_H = k(p, p')$ and a linear subspace generated by $\varphi(X)$ is dense in $H$, then the correspondence $\varphi(p) \mapsto k_p$ define a well-defined isomorphism from $H$ to $H_k$. Thus, as a Hilber space, RKHS is characterized as a pair of a Hilbert space $H$ and a map $\varphi : X \to H$ satisfying the above properties.

We say a positive definite kernel $k$ is smooth (resp. holomorphic) if $k_p \in \mathcal{E}(X)$ (resp. $\mathcal{A}(X)$) for any $p \in X$. Note that by the symmetric property of $k$ if $k$ is a smooth (resp. holomorphic) positive definite kernel, then $k$ is smooth (resp. anti-holomorphic) with respect to the first variable as a function on $X \times X$. As in the following proposition, $H_k$ inherits regularity of $k$:

**Proposition 3.3.** Let $X$ be a smooth (resp. complex) manifold, and let $k : X \times X \to \mathbb{C}$ be a smooth (resp. holomorphic) positive definite kernel. Then, $H_k$ is contained in $\mathcal{E}(X)$ (resp. $\mathcal{A}(X)$). Moreover, the natural inclusion $H_k \hookrightarrow \mathcal{E}(X)$ (resp. $\mathcal{A}(X)$) is continuous.

**Proof.** It follows from [27, Theorem 2.6, 2.7].

We remark that RKHS associated with a smooth (resp. holomorphic) positive definite kernel is characterized as a Hilbert space realized as a subspace of $\mathcal{E}(X)$ (resp. $\mathcal{A}(X)$), namely, we have the following proposition:

**Proposition 3.4.** Let $X$ be a smooth (resp. complex) manifold. Then, the correspondence $k \mapsto H_k$ provides a one-to-one correspondence between smooth (holomorphic) positive definite kernels on $X$ and Hilbert spaces continuously included in $\mathcal{E}(X)$ (resp. $\mathcal{A}(X)$).

**Proof.** We construct an inverse correspondence of $k \mapsto H_k$. Let $H \subset \mathcal{E}(X)$ be a Hilbert space such that the inclusion $H \hookrightarrow \mathcal{E}(X)$ is continuous. For $p \in X$, the linear functional $h \mapsto h(p)$ on $H$ is continuous. Thus, by the Riesz representation theorem, there exists $k_p \in H$ such that $\langle h, k_p \rangle_H = h(p)$. Then, we $k(p, p') := \langle k_p, k_{p'} \rangle_H$ define a positive definite kernel on $X$. Since $H \subset \mathcal{E}(X)$, $k$ is obviously a smooth function on $M \times M$, and we have $H = H_k$.

Since $H_k$ is a Hilbert space, we may identify $H_k'$ with $H_k$ via the Riesz representation theorem (note that this identification is antilinear). We provide an explicit formula of an element of $t'(\mathcal{E}(X)_p)$ as follows:
Proposition 3.5. Let $X$ be a smooth manifold and let $H_k$ be an RKHS associated to a smooth positive definite kernel $k$ on $X$. We identify $H_k'$ with $H_k$ via the Riesz representation theorem. Let $1 : H_k \hookrightarrow \mathcal{E}(X)$ be the continuous inclusion. Then, for any $p \in X$, $v \in \mathcal{D}(X)_p$ and $p_0 \in X$, we have 

$$t'(v)(p_0) = v(k p_0).$$

Proof. In fact, by definition, we have 

$$t'(v)(p_0) = \langle t'(v), k_{p_0} \rangle_{H_k},$$

$$= v(k p_0).$$

□

3.2. Positive definite kernels defined by a formal power series. We here consider a positive definite kernel defined by a formal power series. It covers various typical positive definite kernels and provides a natural generalization of those in [29, 26]. In addition, in some particular cases, it coincides with a Bergman kernel on an open domain of $\mathbb{C}^d$, and thus provides quasi-Banach spaces. We here explain the Fock spaces treated in numerous works.

3.2.1. RKHS in a general setting. Let $\Phi(z) = \sum c_{\alpha} z^{\alpha}$ be a holomorphic function and let $\Omega \subset \mathbb{C}^d$ be an open neighborhood of 0. Assume that for $z \in \Omega$, the series 

$$\Phi(|z_1|^2, \ldots, |z_d|^2) = \sum_{n \in \mathbb{Z}_d^d} c_{\alpha} |z|^2 \alpha,$$

absolutely converges and 

(3.1) 

$$c_{\alpha} \geq 0 \text{ for all } \alpha \in \mathbb{Z}_d^d.$$ 

Then, we can define a holomorphic positive definite kernel on $\Omega$ by 

(3.2) 

$$k(z, w) := \Phi(z_1 w_1, \ldots, z_d w_d).$$

Remark 3.6. The condition (3.1) actually a necessary and sufficient condition for the positive definiteness of $k(z, w)$ defined as in (3.2), namely, for any holomorphic function on $\Omega$, if $k(z, w) := \Phi(z_1 w_1, \ldots, z_d w_d)$ determines a positive definite kernel, then the coefficients $c_{\alpha_1, \ldots, \alpha_d}$ are nonnegative. In fact, let $\varepsilon > 0$ be a sufficiently small positive number and put $e(\theta) := (\varepsilon e^{i n_1 \theta}, \ldots, \varepsilon e^{i n_d \theta})$. Since $k$ is positive definite, we see that 

$$c_{n_1, \ldots, n_d} = \frac{1}{4\pi^2 \varepsilon^2 \Sigma \alpha_j} \int_0^{2\pi} \int_0^{2\pi} e^{\Sigma \alpha_j (\theta - \phi)} k(e(\theta), e(\phi)) d\theta d\phi \geq 0.$$ 

Let 

(3.3) 

$$\mathcal{I}_\Phi := \{n \in \mathbb{Z}_d^d : c_n \neq 0\}.$$ 

Then, we consider the following assumption on $\mathcal{I}_\Phi$: 

Assumption 3.7. For any proper subset $I \subsetneq \{1, \ldots, d\}$ and element $a \in \mathbb{Z}_d^I$, the set $p_I^{-1}(a) \cap \mathcal{I}_\Phi$ is a non-empty infinite set. Here, $p_I : \mathbb{Z}^d \rightarrow \mathbb{Z}^I; (a_j)_{j=1}^d \mapsto (a_j)_{j \in I}$ is a natural projection ($\mathbb{Z}_I := \{0\}$).
For example, \( \Phi(z) = \phi(z_1 + \cdots + z_d) \) for an entire function \( \phi \) satisfies this assumption. It provides a positive definite kernel treated in \([29, 26]\). Bergman kernels on the ball or polydisc also satisfies this assumption.

Then, we have the following proposition:

**Proposition 3.8.** The notation as above. Then, for any \( p \in \Omega \) and \( \alpha \in \mathbb{Z}_+^d \), we have

\[
t'(\delta_p^{\text{hol}}(\partial_z^\alpha)) (w) = w^\alpha (\partial_z^\alpha \Phi)(\overline{p_1}w_1, \ldots, \overline{p_d}w_d).
\]

Here, we regard \( H'_k = H_k \) via the Riesz representation theorem (via an antilinear isomorphism). Moreover, if the set \( \mathcal{O} \) satisfies Assumption 3.7, then for any \( p \neq 0 \), \( t'|_{\Omega(X)} \) is injective (and thus \( \kappa^n \) is injective for \( n \geq 0 \)). In particular, the condition (2) in Theorem 2.5 holds for any holomorphic map \( f \) on \( \Omega \).

**Proof.** The first formula follows from Proposition 3.5. We prove the second statement. Let \( p \in \mathbb{C}^{\setminus \{0\}} \). Since for any \( Q \in \mathbb{C}[t_1, \ldots, t_d] \),

\[
t'(\delta_p^{\text{hol}}(\partial_z)) = \sum_{\alpha} b_{\alpha} w^\alpha (\partial_z^\alpha \Phi)(\overline{p_1}w_1, \ldots, \overline{p_d}w_d),
\]

it suffices to show that for any \( \Phi \) satisfying Assumption 3.7 and \( Q(t) = \sum_{\alpha} b_{\alpha} t^\alpha \in \mathbb{C}[t_1, \ldots, t_d] \),

\[
\sum_{\alpha} b_{\alpha} w^\alpha (\partial_z^\alpha \Phi)(\overline{p_1}w_1, \ldots, \overline{p_d}w_d) = 0
\]

(3.4) implies \( b_{\alpha} = 0 \). We prove it by induction of \( d \). In the case of \( d = 1 \), (3.4) is equivalent to

\[
c_m p_1^m \sum_{\alpha} b_{\alpha} m(m-1) \cdots (m-\alpha+1) = 0
\]

for all \( m \geq \alpha \). Since \( p_1 \neq 0 \) and \( c_m \neq 0 \) for infinitely many \( m \) by Assumption 3.7, we have \( b_{\alpha} = 0 \). We next consider the case of \( d > 1 \). We only treat the case of \( p_1 \neq 0 \), and other cases are proved in a similar way. Suppose \( Q \neq 0 \). Then, there exists \( Q_0 \in \mathbb{C}[t_1, \ldots, t_d] \) such that \( Q_0(t_1, \ldots, t_{d-1}, 0) \neq 0 \) and \( Q = t_d^k Q_0 \). Write \( Q_0 := \sum_{\alpha} b'_{\alpha} t^\alpha \). Then, we have

\[
w_d \sum_{\alpha} b'_{\alpha} w^\alpha (\partial_z^\alpha \Phi)(\overline{p_1}w_1, \ldots, \overline{p_d}w_d) = \sum_{\alpha} b_{\alpha} w^\alpha (\partial_z^\alpha \Phi)(\overline{p_1}w_1, \ldots, \overline{p_d}w_d)
\]

\[
= 0.
\]

Thus, we see that

\[
\sum_{\alpha} b'_{\alpha} w^\alpha (\partial_z^\alpha \Phi)(\overline{p_1}w_1, \ldots, \overline{p_d}w_d) = 0.
\]

Let \( \Psi(z_1, \ldots, z_{d-1}) := \partial_{z_d} \Phi(z_1, \ldots, z_{d-1}, 0) \). We note that \( \Psi \) also satisfies the Assumption 3.7. Then, by substituting 0 for \( w_d \) in the above formula, we have

\[
\sum_{\alpha_{d}=0} b'_{\alpha} w^\alpha (\partial_z^\alpha \Psi)(\overline{p_1}w_1, \ldots, \overline{p_{d-1}}w_{d-1}) = 0.
\]

This equality corresponds to (3.4) in the case \( \Phi = \Psi \) and \( Q = Q_0(t_1, \ldots, t_{d-1}, 0) \). Therefore, by induction hypothesis, \( b'_{\alpha} = 0 \) for any \( \alpha = 0 \) with \( \alpha_d \neq 0 \), which contradicts the fact that \( Q_0(t_1, \ldots, t_{d-1}, 0) \neq 0 \). \( \square \)
3.2.2. **Fock spaces.** Let $A$ be a Lebesgue measure on $\mathbb{C}^d$ via the isomorphism $\mathbb{R}^d \times \mathbb{R}^d \cong \mathbb{C}^d$; $(x,y) \mapsto x + iy$. For $\alpha > 0$, let 

$$\Phi(z) := \left(\frac{\alpha}{\pi}\right)^d e^{\alpha z},$$

and define $k$ as in (3.2). For $0 < q \leq \infty$ and $h \in \mathcal{A}(\mathbb{C}^d)$, we define 

$$\|h\|_{q,\alpha} := \begin{cases} \int_{\mathbb{C}^d} |h(z)|^q e^{-\alpha \|z\|^2/2} dA(z) < \infty & \text{if } q < \infty, \\ \sup_{z \in \mathbb{C}^d} |h(z)|e^{-\alpha \|z\|^2/2} & \text{if } q = \infty. \end{cases}$$

Then, for $0 < q \leq \infty$, we define 

$$F^q_{\alpha} := \left\{ h \in \mathcal{A}(\mathbb{C}^d) : \|h\|_{q,\alpha} < \infty \right\}.$$

Then, for any $h \in F^q_{\alpha}$ and $p \in \mathbb{C}^d$, we have the following formula (see, for example, [24, Theorem 8]):

$$h(p) = \int_{\mathbb{C}^d} h(z)k_p(z)e^{-\alpha \|z\|^2} dA(z).$$

In particular, $F^q_{\alpha}$ is a continuously included subspace of $\mathcal{A}(X)$ and for any $Q \in \mathbb{C}[t_1, \ldots, t_d]$,

$$Q(\partial_z)h(p) = \int_{\mathbb{C}^d} h(z)\overline{Q(\alpha z)k_p(z)}e^{-\alpha \|z\|^2} dA(z).$$

(3.5)

In particular, $t'(\delta_p(Q(\partial_z))) \in (F^q_{\alpha})'$ is represented by the function $\overline{Q(\alpha z)k_p(z)}$ via the integral on the right hand side of (3.5), and we have the following proposition:

**Proposition 3.9.** Let $\alpha > 0$ and $0 < q \leq \infty$. The linear map $\kappa^q_{\alpha}$ is injective for the Fock space $F^q_{\alpha}$ for any $p \in \mathbb{C}^d$ and $n \geq 0$

**Proof.** It suffices to show that $t'_{|_{\mathcal{A}(X)p}}$ is injective. Since $t'(\delta_p(Q(\partial_z))) = 0$ if and only if $\overline{Q(\alpha z)k_p(z)} = 0$. Since $k_p$ has zero nowhere, we see that $\overline{Q(\alpha z)k_p(z)} = 0$ if and only if $Q = 0$. \qed

3.3. **Shift invariant kernels on the Euclidean spaces.** In this subsection, we introduce several RKHSs and Banach spaces defined by continuous positive definite functions, which is a fourier transform of finite Borel measure on the Euclidean space. We first provide a general example of RKHS and then give a Banach space in the case where the base measure is absolutely continuous.

3.3.1. **RKHS for continuous positive definite functions.** Let $\mu$ be a Borel measure on $\mathbb{R}^d$ such that for any non-negative integer $n \geq 0$,

$$\int_{\mathbb{R}^d} |\xi|^n d\mu(\xi) < \infty.$$ 

We employ $\xi = (\xi_1, \ldots, \xi_d)$ to describe the variables of functions in $L^2(\mu)$. We define a smooth positive definite kernel on $\mathbb{R}^d$ by

$$k(x,y) := \tilde{\mu}(x-y) := \int e^{i(x-y) \cdot \xi} d\mu(\xi).$$
Here for $x, y \in \mathbb{C}^d$, we denote $\langle x, y \rangle$ by $x \cdot y$.

We note that the correspondence $k_p \mapsto e^{ip \cdot \xi}$ induces an isomorphism (see Remark 3.2).

$$\rho : H_k \cong L^2(\mu); k_p \mapsto e^{ip \cdot \xi}.$$ 

Then, we have the following proposition:

**Proposition 3.10.** The notation as above, we regard $H'_k$ as $H_k$ via the Riesz representation theorem (via an antilinear isomorphism). Then, for any $Q \in \mathbb{C}[t_1, \ldots, t_d]$ and $p \in \mathbb{R}^d$, we have

$$(3.6) \quad \rho \iota'(\delta_p(Q(\partial_x))) = \overline{Q}(i\xi)e^{ip \cdot \xi},$$

where $\tilde{\xi}_j$ is the equivalent class of $\xi_j$ in $L^2(\mu)$. In particular, if $\mathbb{C}[\xi_1, \ldots, \xi_d] \hookrightarrow L^2(\mu)$, $\iota'_{\overline{\rho}(X)_p}$ is injective for all $p \in \mathbb{R}^d$ (and $\kappa_n^p$ is injective for any $n \geq 0$), and the condition (2) in Theorem 2.4 holds for any smooth function on $\mathbb{R}^d$.

**Proof.** We first remark that for any $h \in H_k$, $\rho(h)$ is characterized by

$$h(x) = \int_{\mathbb{R}^d} \rho(h)(\xi)e^{-ix \cdot \xi}d\mu(\xi)$$

for all $x \in \mathbb{R}^d$. Since $k(x, y) = \int e^{i(x-y) \cdot \xi}d\mu(\xi)$, by Lemma 3.5, we have

$$\iota'(\delta_p(Q(\partial_x)))(x) = \int_{\mathbb{R}^d} \overline{Q}(i\xi)e^{ip \cdot \xi}e^{-ix \cdot \xi}d\mu(\xi).$$

According to the first remark, we have (3.6). \qed

If $\mu$ has a Zariski dense support (for example, $\mu$ is absolutely continuous), the natural map $\mathbb{C}[\xi_1, \ldots, \xi_d] \to L^2(\mu)$ is injective.

If we additionally impose $\int \prod_j e^{a_j|\xi_j|}d\mu(\xi) < \infty$ for some $a = (a_1, \ldots, a_d) \in \mathbb{R}_{>0}^d$, $k$ also define a positive definite kernel on a complex manifold $\mathbb{I}_a := \{z = (z_j)_{j=1}^d \in \mathbb{C}^d : |\text{Im}(z_j)| < a_j\}$ of complex dimension $d$, where we set $k(z, w) := \hat{\mu}(z - \overline{w})$ for $z, w \in \mathbb{I}_a$. Then, we have a similar proposition as follows:

**Proposition 3.11.** Assume $\int \prod_j e^{a_j|\xi_j|}d\mu(\xi) < \infty$ for some $a = (a_1, \ldots, a_d) \in \mathbb{R}_{>0}^d$. We regard $H'_k$ as $H_k$ via the Riesz representation theorem (via an antilinear isomorphism). Then, for any $Q \in \mathbb{C}[t_1, \ldots, t_d]$ and $p \in \mathbb{I}_a$,

$$\rho \iota'(\delta_p(Q(\partial_x))) = \overline{Q}(i\xi)e^{\overline{\xi} \cdot \xi}.$$

where $\tilde{\xi} = (\tilde{\xi}_j)_j$ and each $\tilde{\xi}_j$ is the equivalent class of $\xi_j$ in $L^2(\mu)$. In particular, if $\mathbb{C}[\xi_1, \ldots, \xi_d] \to L^2(\mu)$, $\iota'_{\overline{\rho}(\mathbb{I}_a)_p}$ is injective for all $p \in \mathbb{I}_a$ (and $\kappa_n^p$ is injective for any $n \geq 0$), and the condition (2) in Theorem 2.4 holds for any holomorphic maps on $\mathbb{I}_a$.

There are several works regarding composition operators on this type of RKHSs [7, 9, 18, 25, 17].

**Remark 3.12.** The function space treated in [25, 17] is isomorphic to the above RKHS with respect to an atomic measure via the change of variable $z \mapsto iz$. 

3.3.2. Banach spaces defined by positive definite functions. Assume $\mu$ is absolutely continuous, namely $\mu = w(\xi)\,d\xi$ for some $w \in L^1(\mathbb{R}^d)$. We further assume $w \in L^q(\mathbb{R}^d)$ for all $q > 0$. We define

$$\mathcal{B}_w^q := \left\{ h \in \mathcal{S}(\mathbb{R}^d) \cap \mathcal{H}' : \mathcal{F} h \in L^q(\mathbb{R}^d, \omega^{-1}) \right\},$$

where the $\mathcal{S}'$ is the space of tempered distributions on $\mathbb{R}^d$, the operator $\mathcal{F}$ is the Fourier transform on $\mathcal{S}'$ such that $\mathcal{F}[h](\xi) = \int h(x) e^{-i\xi x} \, dx$ for any Schwartz function $h$, and we denote by $L^q(\mathbb{R}^d, \omega^{-1})$ the space composed of the measurable functions $h$ on $\mathbb{R}^d$ such that $h(\xi) = 0$ on $\{w(\xi) = 0\}$ and $\int |h|^q w^{-1}(\xi) \, d\xi < \infty$.

Then we have the following proposition [13, Theorem 4.1]:

**Proposition 3.13.** Assume $\mu = w(\xi)\,d\xi$ for some $w \in \cap_{r>0} L^r(\mathbb{R}^d)$. Then, we have $\mathcal{B}_w^q$ is a Banach space isometrically isomorphic to $L^q(\mu)$ and the inclusion $\iota : \mathcal{B}_w^q \hookrightarrow \mathcal{S}(\mu)$ is continuous. In addition, if we further assume $\int \prod_j e^{a_j|\xi_j|} \, d\mu(\xi) < \infty$ for some $a = (a_1, \ldots, a_d) \in \mathbb{R}_{\geq 0}^d$, we have $\mathcal{B}_w^q \subset \mathcal{A}(\mathbb{I}_d)$ and its inclusion map is continuous as well.

**Proof.** The first statement follows from the fact that the linear map $\theta : L^q(\mu) \to \mathcal{B}_w^q$ defined by

$$\theta(g) := \mathcal{F}^{-1}(gw^{2/q})$$

is actually an isometric isomorphism (see the proof of [13, Theorem 4.1]). The second and third statements follows from the Fourier inversion formula: for any $Q \in \mathbb{C}[t_1, \ldots, t_d]$ and $h \in \mathcal{B}_w^q$ and $p \in \mathbb{R}^d$, we have

$$Q(\partial_d)h(p) = \int_{\mathbb{R}} \theta^{-1}(h)(\xi)Q(-i\xi)w(\xi)^{2/q-1}e^{-ip\xi} \, d\mu(\xi) \leq C\|h\|_{\mathcal{B}_w^q}$$

for some $C > 0$ by the Hölder inequality. \hfill \square

A measurable function $g$ on $\mathbb{R}^d$ with suitable decay can be embedded in $(\mathcal{B}_w^q)'$ when we regard $g$ as $h \mapsto \int \theta^{-1}(h)g \, d\mu$. Thus, by (3.7), the $\iota'(\delta_0(Q(\partial_d))) \in (\mathcal{B}_w^q)'$ is described by the function $Q(i\xi)w(\xi)^{2/q-1}e^{ip\xi}$. Then, we obtain that similar statements to Propositions 3.10 and 3.11 holds. In particular, we have the following proposition:

**Proposition 3.14.** Assume $\mu = w(\xi)\,d\xi$ for some $w \in \cap_{r>0} L^r(\mathbb{R}^d)$. Then, $\mathcal{B}_w^q$ satisfies the condition (2) in Theorem 2.4 for any smooth map on $\mathbb{R}^d$. If we further assume $\int \prod_j e^{a_j|\xi_j|} \, d\mu(\xi) < \infty$ for some $a = (a_1, \ldots, a_d) \in \mathbb{R}_{\geq 0}^d$, the space $\mathcal{B}_w^q$ satisfies the condition (2) in Theorem 2.5 for any holomorphic map on $\mathbb{I}_d$.

4. BOUNDED COMPOSITION OPERATORS FOR THE COMPLEX AFFINE SPACE

In this section, we discuss bounded weighted composition operators on functional quasi-Banach spaces composed of entire functions on $\mathbb{C}^d$. We first treat the one-dimensional case and prove that any map except affine map cannot induce a bounded composition operator (Theorem 4.1). We next discuss higher dimensional cases. In this case, the situation is much more complicated. We prove that any
polynomial automorphism except affine maps cannot induce bounded composition operators under mild conditions in the two dimensional case (Theorem 4.6). We end this paper with remarks for general dimensional cases.

4.1. One-dimensional case. By combining several classical results of holomorphic dynamics in one variable, we provides affineness of holomorphic dynamics on $\mathbb{C}$ which induces bounded weighted composition operators on a quasi-Banach subspace of $A(\mathbb{C})$: For $f: \mathbb{C} \to \mathbb{C}$ is a holomorphic map, we define the set of repelling periodic point of period $r$:

$$R_r(f) := \{p \in \mathbb{C} : f^r(p) = p, |(f^r)'(p)| > 1, f^i(p) \neq p \text{ for } i = 1, \ldots, r-1\}.$$ 

Let $R(f) := \cup_{r \geq 0} R_r(f)$ and define $J_f := \overline{R(f)}$.

**Theorem 4.1.** Let $f: \mathbb{C} \to \mathbb{C}$ be a holomorphic map. Let $V \subset A(\mathbb{C})$ be an infinite dimensional quasi-Banach space and the inclusion map is continuous. We assume that, there exists $p \in R_r(f)$ and $u \in A(\mathbb{C})$ such that $u(p) \neq 0$ and $uC_f$ is bounded on $V$ for an $r \geq 1$. Then, $f(z) = az + b$ for some $a, b \in \mathbb{C}$ with $|a| \leq 1$.

**Proof.** Suppose $f$ is not affine but $C_f$ is a bounded linear operator on $V$. It is well-known that $J_f$ coincides with the Julia set of $f$ and $J_f$ is a nonempty closed subset of $\mathbb{C}$ without any isolated point ([4, Theorem 6.9.2] and [28, Theorem 1.20]) unless $f$. In particular $R(f) \neq \emptyset$. Therefore, Theorem 2.5 with Corollary 2.7 implies that $uC_f$ cannot be bounded for $u \in A(\mathbb{C})$ with $u(p) \neq 0$ for some $p \in R_r(f)$. \[\Box\]

**Remark 4.2.** We here illustrate that Theorem 4.1 and [30, Theorem 1] provide a different proof of the affineness of $f$ from the boundedness of $uC_f$ on the Fock space $F_{1/2}^2$, which is proved in [22]. Let $V$ be the Fock space $F_{1/2}^2$. Then, Ueki proves that $uC_f$ is bounded on $V$ if and only if

$$B_f(u) := \sup_{w \in \mathbb{C}, \lambda \in \mathbb{C}} \int |u(z)|^2 |e^{\lambda w/2}|^2 e^{-|w|^2/2} e^{-|z|^2/2} dA(z) < \infty,$$

where $dA$ is the Lebesgue measure on $\mathbb{C}$. Suppose $f$ is not affine map and let $p \in R_r(f)$. Then, there exists an entire function $v$ such that $v(f^j(p)) \neq 0$ for $j = 0, \ldots, r-1$ and

$$u(z) = (z - p)^{m_1} (z - f(p))^{m_2} \cdots (z - f^{r-1}(p))^{m_r} v(z)$$

for some nonnegative integers $m_1, \ldots, m_r$. Put $v_r(z) := v(z)v(f(z)) \cdots v(f^{r-1}(z))$. Then, we easily see that $B_f(v) < \infty$ and we have $vC_f$ is bounded, thus $(vC_f)' = v_rC_f'$ is bounded. Since the assumption of Theorem 4.1 holds as $v_r(p) \neq 0$, we conclude that $f(z) = az + b$ with $|a| \leq 1$.

As an immediate corollary, we have

**Corollary 4.3.** Let $V \subset A(\mathbb{C})$ be an infinite dimensional quasi-Banach space and the inclusion map is continuous. Then, for any holomorphic map $f: \mathbb{C} \to \mathbb{C}$, if $e^wC_f$ is bounded on $V$ for an entire function $w \in A(\mathbb{C})$, we have $f(z) = az + b$ for some $a, b \in \mathbb{C}$ with $|a| \leq 1$.

When $f$ is polynomial, we have a stronger result:
Corollary 4.4. Let \( V \subset \mathscr{A}(\mathbb{C}) \) be an infinite dimensional quasi-Banach space and the inclusion map is continuous. Then, for any polynomial map \( f : \mathbb{C} \to \mathbb{C} \), if \( u C_f \) is bounded on \( V \) for a nonzero entire function \( u \in \mathscr{A}(\mathbb{C}) \), we have \( f(z) = az + b \) for some \( a, b \in \mathbb{C} \) with \( |a| \leq 1 \).

Proof. Let \( u_r(z) := u(z)(f(z))^r \cdots u(f^{-r-1}(p)) \). We prove that \( u_r(p) \neq 0 \) for some \( r \geq 1 \) and \( p \in \mathbb{R}_r(f) \). If not, \( u \) has infinitely many zeros in \( J_f \). On the other hand, since \( J_f \) is compact when \( f \) is a polynomial (see, for example, [12, (1)]), we conclude \( u \equiv 0 \), which is a contradiction. Thus, by the formula \( (u C_f)' = u C_{f'} \) and Theorem 4.1, we see that \( f(z) = az + b \) for some \( a, b \in \mathbb{C} \) with \( |a| \leq 1 \).

We note that Theorem 4.1 is not valid if \( V \) is finite-dimensional. In fact, let \( V = \mathbb{C} e^z, f(z) := (z + 1)^2/2 \) and let \( u(z) = e^{-z^2/2} \). Then, since

\[
(u C_f)e^z = e^{-z^2/2}e^{(z+1)^2/2} = e^{1/2}e^z,
\]

we see that \( u C_f \) induces a bounded linear map on \( V \). However, if we further assume that \( u \equiv 1 \), we have the similar result, and thus we have the following general result for composition operators:

Theorem 4.5. Let \( V \subset \mathscr{A}(\mathbb{C}) \) be a quasi-Banach space and the inclusion map is continuous. For any holomorphic map \( f : \mathbb{C} \to \mathbb{C} \), if the composition operator \( C_f \) is bounded on \( V \), then \( f(z) = az + b \) for some \( a, b \in \mathbb{C} \). Moreover, if \( V \) is infinite dimensional we have \( |a| \leq 1 \).

Proof. As the statement is true if \( V \) is infinite dimensional by Corollary 4.3, we assume \( V \) is finite-dimensional. We first assume there exists a nonconstant entire function \( h \in V \) and \( \lambda \in \mathbb{C} \) such that \( C_f[h] = \lambda h \). Let \( J_f \) be the Julia set of \( f \), which is the same as the closure of the set of repelling periodic points of \( f \) ([4, Theorem 6.9.2] and [28, Theorem 1.20]). Thus, there exists \( p \in \mathbb{C} \) such that \( f^n(p) = p \) and \( h(p) \neq 0 \) for some \( n \geq 1 \). Then, we have \( h(p) = h(f^n(p)) = \lambda^n h(p) \), and thus \( \lambda^n = 1 \). As \( f^n \) is also not an affine map, we may assume \( \lambda = 1 \), namely \( h \circ f = h \). Since the Julia set \( J_f \) is the same as the closure of \( \bigcup_{n \geq 0} f^{-n}(S) \) for a finite set \( S \subset \mathbb{C} \) of cardinality greater than 2 ([4, p.68] and [28, Theorem 1.7]), we see that \( h(J_f) \subset S \), and thus \( h \) is constant, which is contradiction. We next assume there is no eigenfunction of \( C_f \) other than constant functions. Then, there exists a nonconstant function \( h \in V \) such that \( C_f[h] = h + 1 \). Let \( p \in \mathbb{C} \) be a periodic point of \( f \). Let \( r \) be the period, namely, \( f^r(p) = p \). Then, we see that \( h(p) = C_f[h](p) = h(p) + r \), and thus \( r = 0 \). We again have a contradiction. Therefore, \( f(z) \) is in the form of \( az + b \) for some \( a, b \in \mathbb{C} \).

4.2. Higher dimensional cases. Here, we discuss a higher dimensional case, especially the two dimensional case, under the condition that \( \kappa_p^{n,\text{hol}} \) is injective for all but finitely many \( p \in \mathbb{C}^d \) and infinitely many \( n \geq 0 \). In contrast to one dimensional case, the relation between behavior of dynamics and boundedness of weighted composition operators gets much more complicated.

Let \( V \subset \mathscr{A}(\mathbb{C}^d) \) be a quasi-Banach space and its inclusion is continuous. Recall \( \mathcal{G}_d(V) \) which is a set composed of regular matrices \( A \in \text{GL}_d(\mathbb{C}) \) satisfying there
exist \( b \in \mathbb{C}^d \) and \( v \in \mathscr{A}(\mathbb{C}^d) \) vanishing nowhere such that \( vC_{A()}+b \) is bounded on \( V \). We note that \( \mathscr{A}(\mathbb{C}^d) \) is a sub semigroup of \( \text{GL}_2(\mathbb{C}) \).

We first consider a two dimensional case. We call a holomorphic map \( f = (f_1, f_2) : \mathbb{C}^2 \to \mathbb{C}^2 \) polynomial map if \( f_1 \) and \( f_2 \) are polynomial. If there exists another polynomial map \( g : \mathbb{C}^2 \to \mathbb{C}^2 \) such that \( f \circ g = g \circ f = \text{id} \), we call \( f \) a polynomial automorphism.

**Theorem 4.6.** Let \( V \) be a quasi-Banach space continuously included in \( \mathscr{A}(\mathbb{C}^2) \).

Assume the following conditions:

1. \( \kappa_{p, \text{hol}}^n \) is injective for all but finitely many \( p \in \mathbb{C}^2 \) and infinitely many \( n \geq 0 \).
2. \( (\mathscr{A}_2(V))_C = M_2(\mathbb{C}) \).

Then, for any polynomial automorphism \( f : \mathbb{C}^2 \to \mathbb{C}^2 \), if \( e^nC_f \) is bounded for some \( w \in \mathscr{A}(\mathbb{C}^2) \), we have \( f(z) = Az + b \) for some \( A \in \text{GL}_2(\mathbb{C}) \) and \( b \in \mathbb{C}^2 \).

**Proof.** Let \( f \) be a non-affine polynomial automorphism on \( \mathbb{C}^2 \). Put \( u = e^w \). As in [15], we may describe \( f = g_n \circ \cdots \circ g_1 \) with a reduced word \( (g_1, \ldots, g_n) \) (see [15, Definition, p.69]), where we may assume each \( g_i \) is either a non-upper triangular regular matrix or an elementary transform. Here, the elementary transform is a polynomial automorphism is in the form of

\[
e_{Q(a,b,c)}(x,y) := (ax + Q(y), by + c),
\]

Here, \( a, b, c \in \mathbb{C} \) with \( ab \neq 0 \) and \( Q \) is a polynomial of degree greater than 1.

If \( n \) is even, then \( f \) is conjugate to a cyclically reduced element ([15, p.70]). Thus, by [15, Theorem 2.6], the map \( f \) is a finite composition of generalized Hénon maps \( h_{Q,b} = \circ \cdots \circ h_{Q_1,b_1} \), for some \( r \geq 0 \), where \( h_{Q,b} \) is defined by

\[
h_{Q,b}(x,y) = (Q(x) - by,x).
\]

where, \( b \in \mathbb{C} \) with \( ab \neq 0 \) and \( Q \) is a polynomial of degree greater than 1. Then, by [5, Theorem 3.4], there exist infinitely many periodic points \( p \in \mathbb{C}^2 \) of period \( r \) such that the absolute value of an eigenvalue of \( d f_p^r \) is greater than 1. Thus, Theorem 2.5 implies that \( f \) cannot induce a bounded weighted composition operator \( uC_f \) on \( V \) as \( u \) is nonvanishing.

Suppose \( n \) is odd. We separately treat the two cases where \( g_n \) is a regular matrix or an elementary transform. We first assume \( g_n \) is a regular matrix. If \( g_1g_n \) is not an upper triangular matrix, then \( g_n^{-1}fg_n \) is conjugate to cyclically reduced and thus, as in the above argument, \( uC_f \) cannot become bounded. We now assume \( g_1g_n \) is an upper triangular matrix. We may assume \( g_n \) is in the form of

\[
g_n = \begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \beta & 0 \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

Since for any \( A = (a_{ij}) \),

\[
\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} * & -a_{21}\alpha^2 - \alpha(a_{11} - a_{22}) + s_{12} \\ * & \end{pmatrix},
\]

\[
\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} * & -a_{21}\alpha^2 - \alpha(a_{11} - a_{22}) + s_{12} \\ * & \end{pmatrix},
\]

\[
\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} * & -a_{21}\alpha^2 - \alpha(a_{11} - a_{22}) + s_{12} \\ * & \end{pmatrix},
\]

\[
\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} * & -a_{21}\alpha^2 - \alpha(a_{11} - a_{22}) + s_{12} \\ * & \end{pmatrix},
\]
there exists $A \in G_2(V)$ such that $g_n^{-1}Ag_n$ is not an upper triangular matrix by Lemma 4.7 below. Let $h := A(\cdot) + b$ such that $vC_h$ is bounded for some non-vanishing $v \in \mathscr{A}(\mathbb{C}^2)$. Then, we see that
\[
g_n^{-1}h \circ f \circ g_n = (g_n^{-1} \circ h \circ g_n) \circ g_{n-1} \circ \cdots \circ (g_2 \circ g_1 g_n)
\]
is conjugate to a finite composition of generalized Hénon maps, by [15, Theorem 2.6]. We note that $(vC_h)(uC_f) = (vh^*(u))C_{ho f}$. Since $(vh^*(u))C_{ho f}$ cannot induce a bounded linear operator by Theorem 2.5, $uC_f$ cannot be bounded as well. Next, we assume $g_n$ is an elementary transform. By Lemma 4.7 below, there exists $A \in G_2(V)$ such that $A = (a_{ij})$ with $a_{21} \neq 0$ and $h := A(\cdot) + b$ induces a bounded composition operator. Then, $h \circ f$ is conjugate to a finite composition of generalized Hénon maps by [15, Theorem 2.6]. Thus, $uC_f$ cannot be bounded as above. Therefore, $f$ must be an affine map if $uC_f$ is bounded on $V$.

In other words, this theorem says that if sufficiently many affine maps induce bounded composition operators on the function spaces introduced in Section 3, no polynomial automorphism except affine maps can induce a bounded composition operator.

The following lemma is used in the above proof of Theorem 4.6.

**Lemma 4.7.** Let $\mathcal{G} \subset GL_2(\mathbb{C})$ be a sub semigroup. For $S = (s_{ij}) \in \mathcal{G}$, let
\[
\mathfrak{A}_S := \{ \alpha \in \mathbb{C} : s_{21} \alpha^2 + (s_{11} - s_{22}) \alpha - s_{12} = 0 \}.
\]
Then $(\mathcal{G})_\mathbb{C} = M_2(\mathbb{C})$ if and only if $\cap_{S \in \mathcal{G}} \mathfrak{A}_S = \emptyset$ and there exists $B = (b_{ij}) \in \mathcal{G}$ such that $b_{21} \neq 0$.

**Proof.** We note that for $S = (s_{ij})$, $\alpha$ satisfies the equation $s_{21} \alpha^2 + (s_{11} - s_{22}) \alpha - s_{12} = 0$ if and only if there exists $\lambda \neq 0$ such that $(1, \alpha)(S - \lambda) = 0$. In fact, since
\[
s_{21} \alpha^2 + (s_{11} - s_{22}) \alpha - s_{12} = \det \begin{pmatrix} s_{11} + s_{21} \alpha & s_{12} + s_{22} \alpha \\ 1 & \alpha \end{pmatrix} = 0,
\]
there exists $\lambda \in \mathbb{C}$ such that
\[
\lambda (1, \alpha) = (1, \alpha) \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}.
\]
We here regard any element in $\mathbb{C}^2$ as a horizontal vector. We first prove the “only if” part. Assume $\cap_{S \in \mathcal{G}} \mathfrak{A}_S \neq \emptyset$. Let $\alpha \in \cap_{S \in \mathcal{G}} \mathfrak{A}_S$. Then, for any $B \in (\mathcal{G})_\mathbb{C}$,
\[
(1, \alpha)B = \lambda_B (1, \alpha)
\]
for some $\lambda_B \in \mathbb{C}$, thus $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ cannot contains in $(\mathcal{G})_\mathbb{C}$. We next prove the “if” part. Assume $\cap_{S \in \mathcal{G}} \mathfrak{A}_S = \emptyset$. Fix $B = (b_{ij}) \in \mathcal{G}$ such that $b_{21} \neq 0$. Since $\mathfrak{A}_B \neq \emptyset$, take $\alpha \in \mathfrak{A}_B$ and let $v := (1, \alpha)$. Let $\lambda \in \mathbb{C}$ such that $vB = \lambda v$. We take another element $C \in \mathcal{G}$ such that $C \notin \mathbb{C} + CB$. It suffices to show that there exists $D \in \mathcal{G}$
such that $D \notin \mathbb{C} + \mathbb{C}B + \mathbb{C}C$. If $BC \notin \mathbb{C} + \mathbb{C}B + \mathbb{C}C$, the element $BC$ is the desired one. Suppose $BC = a + bB + cC$ for some $a, b, c \in \mathbb{C}$. Then, we have

$$vBC = \lambda vC = av + b\lambda v + cvC.$$  

Thus, we have

$$(\lambda - c)vC = (a + b\lambda)v.$$  

In the case of $\lambda \neq c$, since $vC = (\lambda - c)^{-1}(a + b\lambda)v$, we have $\alpha \in \mathfrak{A}_C$, and thus $\alpha \in \mathfrak{A}_B \cap \mathfrak{A}_C$. In the case of $\lambda = c$, we see that $a = -b\lambda$, and we have

$$(\lambda - B)(b - C) = 0.$$  

Let $w = b_{21}^{-1}(0, 1)(\lambda - B) = (1, \beta)$. Then $w(b - C) = 0$, namely $\beta \in \mathfrak{A}_C$ and we have either $w(\lambda - B) \in \mathbb{C}w$ or $C = b$. Thus, we see that $\beta$ or $\alpha$ must be contained in $\mathfrak{A}_B \cap \mathfrak{A}_C$. Therefore, we have $\mathfrak{A}_B \cap \mathfrak{A}_C \neq \emptyset$. Since $\mathfrak{A}_B \cap \mathfrak{A}_C \cap \mathfrak{A}_C \neq \emptyset$ for any $C' \in \mathbb{C} + \mathbb{C}B + \mathbb{C}C$, there exists $D \in \mathfrak{A}$ such that $D \notin \mathbb{C} + \mathbb{C}B + \mathbb{C}C$ as we are now assuming $\cap_{\lambda \in \mathfrak{A}} \mathfrak{A}_S = \emptyset$. \hfill $\square$

We expect that the higher dimensional version of Theorem 4.6 is valid not only for polynomial automorphisms but also any holomorphic maps, namely, the following conjecture would be true:

**Conjecture 4.8.** Let $V \subset \mathfrak{A}(\mathbb{C}^d)$ be a quasi-Banach space and the inclusion map is continuous. Assume the following conditions:

1. $\kappa^\text{hol}_p$ is injective for all but finitely many $p \in \mathbb{C}^d$ and infinitely many $n \geq 0$.
2. $\mathsf{M}_d(\mathbb{C}) = \langle \mathfrak{A}_d(V) \rangle_{\mathbb{C}}$.

Then, for any holomorphic map $f : \mathbb{C}^d \rightarrow \mathbb{C}^d$, if $e^wC_f$ is bounded for some $w \in \mathfrak{A}(\mathbb{C}^d)$, we have $f(z) = Az + b$ for some $A \in \mathsf{GL}_d(\mathbb{C})$ and $b \in \mathbb{C}^d$.

The theory of holomorphic dynamics is quite an effective approach and the problem is reduced to infinitely many existence of repelling or saddle periodic points of the composition of a holomorphic dynamical system and a suitable regular matrix. However, existence of those types of points seems to be still open in general and existence of such points is an important problem in holomorphic dynamics (see [14, Question 2.16]).

We note that this conjecture is actually proved in specific quasi-Banach spaces. When $V$ is the Fock space, it is proved in [32, 33] under assumption that weighted composition operators are invertible or unitary. If we impose $\mu \equiv 1$, there are many results as well. For example, in [6, 8], they prove it for the Fock and the Fock-Sobolev spaces. In [26, 29], they prove it if the kernel is in the form of $k(z, w) = \Phi(zw)$ for an entire function $\Phi$, which is a special form of a kernel introduced in Section 3.2. In [18], they investigate the RKHS associated with a shift invariant kernel (Section 3.3). They treat certain general class of shift invariant kernels, and the condition $\langle \mathfrak{A}_d(V) \rangle_{\mathbb{C}} = \mathsf{M}_d(\mathbb{C})$ appears as well. However, they do not use any results developed in the theory of holomorphic dynamics, and they introduce this condition in a different context. We expect that their results ([18, Theorem 1]) hold without the technical condition (called Assumption (B) there).
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