GENERALIZED GEOMETRICAL STRUCTURES
OF ODD DIMENSIONAL MANIFOLDS

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Abstract. We define an almost–cosymplectic–contact structure which generalizes cosymplectic and contact structures of an odd dimensional manifold. Analogously, we define an almost–coPoisson–Jacobi structure which generalizes a Jacobi structure. Moreover, we study relations between these structures and analyze the associated algebras of functions.

As examples of the above structures, we present geometrical dynamical structures of the phase space of a general relativistic particle, regarded as the 1st jet space of motions in a spacetime. We describe geometric conditions by which a metric and a connection of the phase space yield cosymplectic and dual coPoisson structures, in case of a spacetime with absolute time (a Galilei spacetime), or almost–cosymplectic–contact and dual almost–coPoisson–Jacobi structures, in case of a spacetime without absolute time (an Einstein spacetime).

Introduction

In [2, 3, 5, 6, 7] we studied geometrical structures on the phase space of a spacetime naturally induced (in the sense of [10]) by a metric and a phase connection. Some of these structures are well known and some are less standard. In the present paper, we generalize these structures on odd dimensional manifolds and study general properties of such structures.

First, in Section 1, we recall some standard structures and introduce new structures, namely almost–cosymplectic–contact, coPoisson and almost–coPoisson–Jacobi structures. In Section 2 we study algebras of functions which are associated with the new geometrical structures.

As examples of the above new structures, we study the geometrical structures on the phase space of a spacetime. Actually, the geometric objects arising in Section 3.1 in the framework of the Galilei’s phase space [2, 5, 6], involve mainly the concepts of cosymplectic and (regular) coPoisson structures. On the other hand, the analogous geometric
objects arising in Section 3.2 in the framework of the Einstein’s phase space \[\text{[3, 7]},\] involve mainly the concepts of almost–cosymplectic–contact and almost–coPoisson–Jacobi structures (eventually contact and Jacobi structures).

1. Geometrical structures

We use the inner product \(i\) of \(k\)–vectors with \(r\)–forms defined by \(i_{X_1\wedge\ldots\wedge X_k}\beta = i_{X_k}\ldots i_{X_1}\beta\), for each \(r\)–form \(\beta\) and \(k\) vector fields \(X_1,\ldots, X_k\), with \(k \leq r\). We use the same symbol for the dual inner product of \(k\)–forms with \(r\)–vectors.

For the Schouten bracket we use the identity, \[\text{[11, 12, 16]},\]
\[
i_{[P,Q]}\beta = (-1)^{q(p+1)}i_Pdi_Q\beta + (-1)^pi_Qdi_P\beta - i_{P\wedge Q}d\beta,
\]
for each \(p\)–vector \(P\), \(q\)–vector \(Q\) and \((p+q-1)\)–form \(\beta\). In particular, for each vector field \(E\) and 2–vector \(\Lambda\), we have \(i_{[E,\Lambda]}\beta = i_Edi_{\Lambda}\beta - i_{\Lambda}di_E\beta\), for each closed 2–form \(\beta\), and \(i_{[\Lambda,\Lambda]}\beta = 2i_{\Lambda}di_{\Lambda}\beta\), for each closed 3–form \(\beta\).

In what follows, \(M\) is a \((2n+1)\)–dimensional smooth manifold.

1.1. Covariant and contravariant pairs.

1.1. Definition. We define a covariant pair to be a pair \((\omega, \Omega)\) consisting of a 1–form \(\omega\) and a 2–form \(\Omega\) of constant rank \(2r\), with \(0 \leq r \leq n\), such that \(\omega \wedge \Omega^r \neq 0\), and a contravariant pair to be a pair \((E, \Lambda)\) consisting of a vector field \(E\) and a 2–vector \(\Lambda\) of constant rank \(2s\), with \(0 \leq s \leq n\), such that \(E \wedge \Lambda^s \neq 0\). Thus, by definition, we have \(\Omega^r \neq 0\), \(\Omega^{r+1} \equiv 0\) and \(\Lambda^s \neq 0\), \(\Lambda^{s+1} \equiv 0\).

We say that the pairs \((\omega, \Omega)\) and \((E, \Lambda)\) are regular if, respectively,
\[
\omega \wedge \Omega^n \neq 0 \quad \text{and} \quad E \wedge \Lambda^n \neq 0.
\]

Let us consider a covariant pair \((\omega, \Omega)\) and a contravariant pair \((E, \Lambda)\).

We define the following linear maps and subspaces
\[
\Omega^\flat : TM \to T^*M : X \mapsto X^\flat =: i_X\Omega, \quad \Lambda^\sharp : T^*M \to TM : \alpha \mapsto \alpha^\sharp =: i_\alpha\Lambda,
\]
\[
\langle \omega \rangle =: \{\lambda\omega \mid \lambda \in \mathbb{R}\} \subset T^*M, \quad \langle E \rangle =: \{\lambda E \mid \lambda \in \mathbb{R}\} \subset TM,
\]
\[
\ker E =: \{\alpha \in T^*M \mid \alpha(E) = 0\}, \quad \ker \omega =: \{X \in TM \mid \omega(X) = 0\}.
\]

We have \(\dim(\text{im } \Omega^\flat) = 2r\) and \(\dim(\text{im } \Lambda^\sharp) = 2s\).

If \((\omega, \Omega)\) is regular, then \(r = n\), \(\dim(\text{im } \Omega^\flat) = 2n\), \(\dim(\ker \Omega^\flat) = 1\), \(\dim(\ker \omega) = 2n\).

If \((E, \Lambda)\) is regular, then \(s = n\), \(\dim(\text{im } \Lambda^\sharp) = 2n\), \(\dim(\ker \Lambda^\sharp) = 1\), \(\dim(\ker E) = 2n\).

1.2. Structures given by covariant pairs. According to \[\text{[12]},\] a pre cosymplectic structure on \(M\) is defined by a regular covariant pair \((\omega, \Omega)\).

Two distinguished types of pre cosymplectic structures appear in the literature. Namely, we recall that a cosymplectic structure \[\text{[1]},\] and a contact structure \[\text{[11]}\] are defined by a covariant pair \((\omega, \Omega)\) such that, respectively,
\[
d\omega = 0, \quad d\Omega = 0, \quad \omega \wedge \Omega^n \neq 0, \quad (1.1)
\]
\[
\Omega = d\omega, \quad \omega \wedge \Omega^n \neq 0. \quad (1.2)
\]
Thus, a contact structure is characterised just by a 1–form $\omega$ such that

$$\omega \wedge (d\omega)^n \neq 0.$$  

We can easily generalize the above structures in the following way.

1.2. **Definition.** We define an *almost–cosymplectic–contact structure* to be a covariant pair $(\omega, \Omega)$ such that

$$d\Omega = 0, \quad \omega \wedge \Omega^n \neq 0. \quad \square$$

Clearly, for $d\omega = 0$ we obtain a cosymplectic structure and for $\Omega = d\omega$ a contact structure. So, almost–cosymplectic–contact structures are regular structures which generalize both cosymplectic and contact structures.

1.3. **Structures given by contravariant pairs.** Two distinguished types of contravariant pairs appear in the literature.

Namely, we recall that a *Jacobi structure* is defined by a contravariant pair $(E, \Lambda)$ such that

$$[E, \Lambda] = 0, \quad [\Lambda, \Lambda] = -2E \wedge \Lambda,$$

where $[,]$ denotes the Schouten bracket.

In the particular case when $E = 0$, we obtain

$$[\Lambda, \Lambda] = 0$$

and the pair $(E, \Lambda) =: (0, \Lambda)$ is called *Poisson structure*.

On the other hand, in the particular case when $\Lambda = 0$, we obtain $[E, \Lambda] = 0$ and $[\Lambda, \Lambda] = 0$ and the pair $(E, \Lambda) =: (E, 0)$ is called *trivial structure*.

In the following we assume $E \neq 0$ and $\Lambda \neq 0$.

1.3. **Remark.** In the literature (see for instance [12]) the condition $E \wedge \Lambda^s \neq 0$ is considered just as a possible non necessary property of the Jacobi pair $(E, \Lambda)$. So, our definition is a little more restrictive; however, the assumption $E \wedge \Lambda^s \neq 0$ is quite reasonable and it is needed for our subsequent developments.

In the literature (see for instance [11, 12, 16]) the Jacobi structure is usually defined by the identities $[E, \Lambda] = 0, \quad [\Lambda, \Lambda] = 2E \wedge \Lambda$. The difference in the sign in the second identity, with respect to our definition, is caused by the different convention on the inner product, hence by the different sign in definition of $\Lambda^s$. \square

In order to exhibit a certain symmetry between geometric structures given by covariant and contravariant pairs, we introduce the following notions.

1.4. **Definition.** We define a *pre coPoisson structure* to be a contravariant pair $(E, \Lambda)$.

In particular, a *coPoisson structure* is defined by a contravariant pair $(E, \Lambda)$ such that

$$[E, \Lambda] = 0, \quad [\Lambda, \Lambda] = 0. \quad \square$$

1.5. **Definition.** We define an *almost–coPoisson–Jacobi structure* to be a 3–plet $(E, \Lambda, \omega)$, where $(E, \Lambda)$ is a contravariant pair and $\omega$ a 1–form, such that

$$[E, \Lambda] = -E \wedge \Lambda^s(L_E \omega), \quad [\Lambda, \Lambda] = 2E \wedge (\Lambda^s \otimes \Lambda^s)(d\omega), \quad i_E \omega = 1, \quad i_\omega \Lambda = 0.$$
The 1-form \( \omega \) is said to be the \textit{fundamental 1-form} of the almost–coPoisson–Jacobi structure. \( \square \)

1.6. \textbf{Remark.} Almost–coPoisson–Jacobi structures generalize both coPoisson and Jacobi structures.

Indeed, if \( d\omega = 0 \), then we have \( L_E \omega = i_E d\omega = 0 \), hence from Definition 1.5 we obtain \( [E, \Lambda] = 0 \) and \( [\Lambda, \Lambda] = 0 \), i.e. \((E, \Lambda)\) turns out to be a coPoisson structure.

Moreover, if \( L_E \omega = 0 \) and \( (\Lambda^2 \otimes \Lambda^2)(d\omega) = -\Lambda \), then we obtain \( [E, \Lambda] = 0 \) and \( [\Lambda, \Lambda] = -2E \wedge \Lambda \), i.e. \((E, \Lambda)\) turns out to be a Jacobi structure. \( \square \)

1.7. \textbf{Proposition.} Let \((E, \Lambda)\) be a regular contravariant pair. Then, there exists a unique 1–form \( \omega \), such that \( i_\omega (E \wedge \Lambda^n) = \Lambda^n \).

Indeed, such an \( \omega \) satisfies the equalities \( i_E \omega = 1 \) and \( i_\omega \Lambda = 0 \).

Thus, the 3–plet \((E, \Lambda, \omega)\) turns out to be an almost–coPoisson–Jacobi structure if and only if \([E, \Lambda] = -E \wedge \Lambda^2(L_E \omega)\) and \([\Lambda, \Lambda] = 2E \wedge (\Lambda^2 \otimes \Lambda^2)(d\omega)\). \( \square \)

Thus, a regular almost–coPoisson–Jacobi structure can be defined just as a suitable contravariant pair \((E, \Lambda)\), as the additional 1–form \( \omega \) is naturally determined by the above pair itself.

1.4. \textbf{Dual structures.} Let us consider a covariant pair \((\omega, \Omega)\) and a contravariant pair \((E, \Lambda)\).

1.8. \textbf{Definition.} The pairs \((\omega, \Omega)\) and \((E, \Lambda)\) are said to be \textit{mutually dual} if they are regular, the maps \(\Omega^b_{|\text{im}(\Lambda^2)} : \text{im}(\Lambda^2) \to \text{im}(\Omega^b) \subset T^*M\) and \(\Lambda^2_{|\text{im}(\Omega^b)} : \text{im}(\Omega^b) \to \text{im}(\Lambda^2) \subset TM\) are isomorphisms and

\[
(\Omega^b_{|\text{im}(\Lambda^2)})^{-1} = \Lambda^2_{|\text{im}(\Omega^b)}, \quad (\Lambda^2_{|\text{im}(\Omega^b)})^{-1} = \Omega^b_{|\text{im}(\Lambda^2)}, \quad i_E \Omega = 0, \quad i_\omega \Lambda = 0, \quad i_E \omega = 1. \]

\( \square \)

1.9. \textbf{Theorem.} [12] The relation of duality yields a bijection between regular covariant pairs \((\omega, \Omega)\) and regular contravariant pairs \((E, \Lambda)\).

Thus, the geometric structures given by dual covariant and contravariant pairs are essentially the same.

In the literature \(E\) is called the \textit{fundamental vector field} [12], or the \textit{Reeb vector field} [13], and \(\Lambda\) the \textit{fundamental 2-tensor} of \((\omega, \Omega)\).

Now, let us consider dual pairs \((\omega, \Omega)\) and \((E, \Lambda)\) and state some results.

1.10. \textbf{Lemma.} We have

\[
\langle E \rangle = \ker \Omega^b, \quad \text{im}(\Lambda^2) = \ker \omega \quad \text{and} \quad \langle \omega \rangle = \ker \Lambda^2, \quad \text{im}(\Omega^b) = \ker E.
\]

\( Proof. \) 1) We have \(\langle E \rangle \subset \ker \Omega^b\); hence, \(\dim(\ker \Omega^b) = 1 = \dim \langle E \rangle\) implies \(\langle E \rangle = \ker \Omega^b\).

If \(X \in \text{sec}(M, \text{im}(\Lambda^2))\), then there exists \(\alpha \in \text{sec}(M, T^*M)\), such that \(i_\alpha \Lambda = X\); hence,

\[
\omega(X) = \omega(i_\alpha \Lambda) = \Lambda(\alpha, \omega) = -i_\alpha \Lambda^2(\omega) = 0, \quad \text{hence} \quad X \in \text{sec}(M, \ker \omega).
\]
Then, \( \dim(\ker \omega) = 2n = \dim(\ker \omega) \) implies \( \ker \omega = 2n \).

2) In the same way we prove the other two identities. \( \square \)

1.11. Proposition. We have the splittings
\[
TM = \langle E \rangle \oplus \text{im}(\Lambda^2) \quad \text{and} \quad T^*M = \langle \omega \rangle \oplus \text{im}(\Omega^\flat).
\]
Accordingly, for each \( X \in \text{sec}(M, TM) \) and \( \alpha \in \text{sec}(M, T^*M) \), we have the splittings
\[
X = \omega(X) E + (X - \omega(X)) E \quad \text{and} \quad \alpha = \alpha(E) \omega + (\alpha - \alpha(E)) \omega,
\]
Thus, the maps
\[
\Lambda^2 \circ \Omega^\flat : TM \rightarrow \text{im}(\Lambda^2) \quad \text{and} \quad \Omega^\flat \circ \Lambda^2 : T^*M \rightarrow \text{im}(\Omega^\flat)
\]
are the “orthogonal” projections of the splittings of \( TM \) and \( T^*M \).

Proof. The equalities \( \dim(\langle E \rangle) + \dim(\text{im}(\Lambda^2)) = 1 + 2n \) and \( \langle E \rangle \cap \text{im}(\Lambda^2) = \langle E \rangle \cap \ker \omega \) yield \( TM = \langle E \rangle \oplus \text{im}(\Lambda^2) \).
Clearly, we have
\[
\omega(X) E \in \text{sec}(M, \langle E \rangle), \quad X - \omega(X) E \in \text{sec}(M, \text{im}(\Lambda^2)) = \text{sec}(M, \ker \omega).
\]
Then, we obtain
\[
X - \omega(X) E = (\Lambda^2 \circ \Omega^\flat)(X - \omega(X)) = (\Lambda^2 \circ \Omega^\flat)(X).
\]
The dual result can be obtained in the same way. \( \square \)

1.12. Proposition. For each \( X, Y \in \text{sec}(M, TM) \) and \( \alpha, \beta \in \text{sec}(M, T^*M) \), we have
\[
(\Lambda^2 \otimes \Lambda^2)(\Omega) = -\Lambda \quad \text{and} \quad (\Omega^\flat \otimes \Omega^\flat)(\Lambda) = -\Omega.
\]
Proof. We have
\[
\Omega(\Lambda^2(\alpha), \Lambda^2(\beta)) = i_{\Lambda^2(\beta)} \Omega^\flat(\Lambda^2(\alpha)) = i_{\Lambda^2(\beta)}(\alpha - \alpha(E)) = \Lambda(\beta, \alpha - \alpha(E)) = -\Lambda(\alpha, \beta).
\]
The second identity can be proved in the same way. \( \square \)

1.13. Lemma. Let us consider the functions \( f, g, h \in \text{map}(M, \mathbb{R}) \), the closed forms \( \alpha, \beta, \gamma \in \text{sec}(M, T^*M) \), and the induced vector fields \( X, Y, Z \in \text{sec}(M, TM) \), given by
\[
X = : \alpha^\sharp + fE, \quad Y = : \beta^\sharp + gE, \quad Z = : \gamma^\sharp + hE,
\]
where \( f = \omega(X) \), \( g = \omega(Y) \), \( h = \omega(Z) \).
Then, the following equality holds
\[
d\Omega(X, Y, Z) = (i_{E^\Lambda(\Lambda^\sharp \Lambda^\sharp)}(d\omega) - \frac{1}{2} i_{[\Lambda, \Lambda]})(\alpha \wedge \beta \wedge \gamma) + f (i_{[E, \Lambda]} + i_{E \wedge (LE\omega)^\sharp})(\beta \wedge \gamma) + g (i_{[E, \Lambda]} + i_{E \wedge (LE\omega)^\sharp})(\gamma \wedge \alpha) + h (i_{[E, \Lambda]} + i_{E \wedge (LE\omega)^\sharp})(\alpha \wedge \beta).
\]
Proof. Let $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$ be the projections of $\alpha, \beta, \gamma$ on $\text{sec}(M, \text{im}(\Omega^\flat)) \subset \text{sec}(M, T^*M)$. We have

$$d\Omega(X, Y, Z) =$$

$$= d\Omega(\alpha^\flat + \omega(X) E, \beta^\flat + \omega(Y) E, \gamma^\flat + \omega(Z) E)$$

$$= d\Omega(\alpha^\flat, \beta^\flat, \gamma^\flat) + \omega(X) d\Omega(E, \beta^\flat, \gamma^\flat) + \omega(Y) d\Omega(\alpha^\flat, E, \gamma^\flat) + \omega(Z) d\Omega(\alpha^\flat, \beta^\flat, E).$$

Then, we obtain

$$d\Omega(\alpha^\flat, \beta^\flat, \gamma^\flat) =$$

$$= \alpha^\flat.\Omega(\beta^\flat, \gamma^\flat) + \beta^\flat.\Omega(\gamma^\flat, \alpha^\flat) + \gamma^\flat.\Omega(\alpha^\flat, \beta^\flat)$$

$$- \Omega([\alpha^\flat, \beta^\flat], \gamma^\flat) - \Omega([\beta^\flat, \gamma^\flat], \alpha^\flat) - \Omega([\gamma^\flat, \alpha^\flat], \beta^\flat)$$

$$= -\alpha^\flat.\Lambda(\beta, \gamma) - \beta^\flat.\Lambda(\alpha, \beta) - \gamma^\flat.\Lambda(\alpha, \beta) + i_{[\alpha^\flat, \beta^\flat]}i_{\gamma^\flat}\Omega + i_{[\beta^\flat, \gamma^\flat]}i_{\alpha^\flat}\Omega + i_{[\gamma^\flat, \alpha^\flat]}i_{\beta^\flat}\Omega$$

$$= -i_{\alpha^\flat}d(\Lambda(\beta, \gamma)) - i_{\beta^\flat}d(\Lambda(\alpha, \beta)) - i_{\gamma^\flat}d(\Lambda(\alpha, \beta)) + (i_{\alpha^\flat}d\gamma - i_{\gamma^\flat}d\alpha)i_{\beta^\flat}\Omega + (i_{\beta^\flat}d\gamma - i_{\gamma^\flat}d\beta)i_{\alpha^\flat}\Omega + (i_{\gamma^\flat}d\gamma - i_{\alpha^\flat}d\gamma)i_{\beta^\flat}\Omega$$

$$- i_{\alpha^\flat}d\gamma - i_{\beta^\flat}d\gamma - i_{\gamma^\flat}d\gamma$$

$$= \Lambda(\alpha, d(\Lambda(\beta, \gamma))) + \Lambda(\beta, d(\Lambda(\alpha, \beta))) + \Lambda(\gamma, d(\Lambda(\alpha, \beta)))$$

$$- d\tilde{\alpha}(\beta^\flat, \gamma^\flat) - d\tilde{\beta}(\gamma^\flat, \alpha^\flat) - d\tilde{\gamma}(\alpha^\flat, \beta^\flat)$$

$$= -i_{\Lambda}d\alpha(\alpha \wedge \beta \wedge \gamma) + \alpha(E)d\omega(\beta^\flat, \gamma^\flat) + \beta(E)d\omega(\gamma^\flat, \alpha^\flat) + \gamma(E)d\omega(\alpha^\flat, \beta^\flat)$$

Similarly, we obtain

$$d\Omega(\alpha^\flat, \beta^\flat, E) =$$

$$= E.\Omega(\alpha^\flat, \beta^\flat) - \Omega([\beta^\flat, E], \alpha^\flat) - \Omega([E, \alpha^\flat], \beta^\flat)$$

$$= -E.\Lambda(\alpha, \beta) + i_{[\beta^\flat, E]}i_{\alpha^\flat}\Omega + i_{[E, \alpha^\flat]}i_{\beta^\flat}\Omega$$

$$= -E.\Lambda(\alpha, \beta) + (i_{\beta^\flat}dE - i_{E}d\beta)\alpha + (i_{E}d\alpha - i_{\alpha^\flat}dE - i_{E\wedge\alpha^\flat}d)\beta$$

$$= E.\Lambda(\alpha, \beta) + i_{[\beta^\flat, E]}i_{\alpha^\flat}\Omega + i_{E\wedge\alpha^\flat}d(\beta(E)\omega)$$

$$= i_{E}d\alpha(\alpha \wedge \beta) - \Lambda(d(\alpha(E)), \beta) + \Lambda(d(\beta(E)), \alpha) - \alpha(E)d\omega(E, \beta^\flat) + (\beta(E)d\omega(E, \alpha^\flat)$$

Then, the above equalities imply (1.6). \square
It is well known [9, 12] that if \((\omega, \Omega)\) is contact, then \((E, \Lambda)\) is Jacobi. Thus, the geometric structures given by dual contact and regular Jacobi pairs are essentially the same. But we obtain the equivalence of structures also for other types of dual covariant and contravariant pairs.

1.14. **Theorem.** The following facts hold:

1. \((\omega, \Omega)\) is an almost–cosymplectic–contact pair if and only if \((E, \Lambda, \omega)\) is an almost–coPoisson–Jacobi 3–plet;

2. \((\omega, \Omega)\) is a symplectic pair if and only if \((E, \Lambda)\) is a coPoisson pair;

3. \((\omega, \Omega)\) is a contact pair if and only if \((E, \Lambda)\) is a Jacobi pair.

**Proof.** Let us consider a point \(x \in M\). All 1–forms on \(M\) can be obtained, pointwisely, from closed 1–forms. Then, according to the splitting (1.5), all vectors \(X, Y, Z \in T_x M\) can be obtained, pointwisely, by means of closed forms; conversely, all closed forms \(\alpha, \beta, \gamma \in \sec(M, T^* M)\) can be obtained, pointwisely, from all vectors above.

Therefore, from Lemma 1.13 we deduce the following facts, by means of a pointwise reasoning, by taking into account the fact that the equality (1.6) involves the vectors \(X, Y, Z\) and the forms \(\alpha, \beta, \gamma\) only pointwisely and by considering their arbitrariness at \(x \in M\).

1) \(d\Omega = 0\) if and only if \([\Lambda, \Lambda] = 2E \wedge (\Lambda^2 \otimes \Lambda^2)(d\omega)\) and \([E, \Lambda] = -E \wedge (L_E\omega)^2\), i.e. the pair \((\omega, \Omega)\) is almost–cosymplectic–contact if and only if the 3–plet \((E, \Lambda, \omega)\) is almost–coPoisson–Jacobi.

2) Moreover, if \(d\Omega = 0\) and \(d\omega = 0\) then \([E, \Lambda] = 0\) and \([\Lambda, \Lambda] = 0\), i.e. \((E, \Lambda)\) is coPoisson.

On the other hand, if \(d\Omega = 0\) and \((E, \Lambda)\) is coPoisson, then \((\Lambda^2 \otimes \Lambda^2)(d\omega) = 0\) and \((L_E\omega)^2 = 0\), i.e. \(d\omega(\alpha^2, \beta^2) = 0\) and \(d\omega(E, \alpha^2) = 0\), for all 1–forms \(\alpha, \beta \in \sec(M, T^* M)\).

Then, from the splitting \(TM = \langle E \rangle \oplus \im(\Lambda^2)\), we have \(d\omega = 0\) and the pair \((\omega, \Omega)\) is cosymplectic.

Hence the pair \((\omega, \Omega)\) is cosymplectic if and only if the pair \((E, \Lambda)\) is coPoisson.

3) Finally, if \(d\omega = \Omega\), hence \(d\Omega = 0\), we have \([E, \Lambda] = -E \wedge (L_E\omega)^2 = 0\) and \([\Lambda, \Lambda] = 2E \wedge (\Lambda^2 \otimes \Lambda^2)(\Omega) = -2E \wedge \Lambda\), hence the pair \((E, \Lambda)\) is Jacobi.

On the other hand, if \(d\Omega = 0\) and the pair \((E, \Lambda)\) is Jacobi, then \((\Lambda^2 \otimes \Lambda^2)(d\omega) = -\Lambda\) and \((L_E\omega)^2 = 0\), i.e. \(d\omega(\alpha^2, \beta^2) = -\Lambda(\alpha, \beta)\) and \(d\omega(E, \alpha^2) = 0\), hence \(d\omega = \Omega\), i.e. the pair \((\omega, \Omega)\) is contact.

Thus, the pair \((\omega, \Omega)\) is contact if and only if the pair \((E, \Lambda)\) is Jacobi. \(\square\)

1.5. **Darboux’s charts.** First, let us consider an almost–cosymplectic–contact structure \((\omega, \Omega)\).

1.15. **Note.** [11] In a neighborhood of each \(x \in M\) there exists a local chart (a Darboux’s chart) \((t, x^i, x^{i+n})\), with \(i = 1, \ldots, n\), adapted to an almost–cosymplectic–contact structure \((\omega, \Omega)\), i.e. such that

\[
(1.7) \quad \omega = dt + \sum_{1 \leq i \leq n} (\omega^i dt^i + \omega^{i+n} dx^{i+n}), \quad \Omega = \sum_{1 \leq i \leq n} dx^i \wedge dx^{i+n},
\]

where \(\omega^i, \omega^{i+n} \in \text{map}(M, \mathbb{R})\).
Indeed, the above almost–cosymplectic–contact pair is cosymplectic if, for instance, \( \omega^i = \omega^{i+n} = 0 \) \( \square \) and contact if, for instance, \( \omega^i = -x^{i+n} \) and \( \omega^{i+n} = 0 \).

Then, let us consider an almost–coPoisson–Jacobi structure \( (E, \Lambda, \omega) \). We can find Darboux’s charts adapted to this structure, analogously to the case of almost–cosymplectic–contact structures.

1.16. Lemma. Let \( \alpha, \beta \in \text{sec}(M, T^*M) \). Then, we have

\[
[E, \alpha^2] = (i_E d\alpha - \alpha(E)(L_E \omega))^2 + \Lambda(L_E \omega, \alpha) E,
\]

\[
[\alpha^2, \beta^2] = (d\Lambda(\alpha, \beta) + \frac{1}{2} i_{\beta} d\alpha + \frac{1}{2} \alpha(E)(i_{\beta} d\omega) - \frac{1}{2} i_o d\beta - \frac{1}{2} \beta(E)(i_o d\omega))^2
\]

\[
+ \frac{1}{2} d\omega(\alpha^2, \beta^2) E.
\]

Proof. For each \( h \in \text{map}(M, \mathbb{R}) \), we have

\[
[E, \alpha^2].h = E.(\alpha^2.h) - \alpha^2.(E.h) = E.\Lambda(\alpha, dh) - \Lambda(\alpha, d(E.h))
\]

\[
= i_{E, \Lambda}(\alpha \wedge dh) + (i_E d\alpha)^2.h
\]

\[
= -i_{E\wedge(L_E \omega)\beta}(\alpha \wedge dh) + (i_E d\alpha)^2.h
\]

\[
= -i_E \alpha((L_E \omega)^2.h) + \Lambda(L_E \omega, \alpha)(E.h) + (i_E d\alpha)^2.h
\]

\[
= (i_E d\alpha - \alpha(E)(L_E \omega))^2.h + \Lambda(L_E \omega, \alpha) E.h
\]

and

\[
[\alpha^2, \beta^2].h = \alpha^2.(\beta^2.h) - \beta^2.(\alpha^2.h)
\]

\[
= \Lambda(\alpha, d\lambda(\beta, dh)) - \Lambda(\beta, d\lambda(\alpha, dh))
\]

\[
= -\frac{1}{2} i_{[\Lambda, \lambda]}(\alpha \wedge \beta \wedge dh) - \Lambda(dh, d\lambda(\alpha, \beta)) + \frac{1}{2} (i_{\beta} d\alpha)^2.h - \frac{1}{2} (i_o d\beta)^2.h
\]

\[
= (d\lambda(\alpha, \beta) + \frac{1}{2} i_{\beta} d\alpha - \frac{1}{2} i_o d\beta)^2.h + \frac{1}{2} i_{E\wedge(L^2 \otimes M)\omega}(\alpha \wedge \beta \wedge dh)
\]

\[
= (d\lambda(\alpha, \beta) + \frac{1}{2} i_{\beta} d\alpha + \frac{1}{2} \alpha(E)(i_{\beta} d\omega) - \frac{1}{2} i_o d\beta - \frac{1}{2} \beta(E)(i_o d\omega))^2.h
\]

\[
+ \frac{1}{2} d\omega(\alpha^2, \beta^2) E.h. \quad \square
\]

1.17. Proposition. If \( f, g \in \text{map}(M, \mathbb{R}) \), then

\[
[E, df^2] = -\alpha(E)(L_E \omega)^2 + \Lambda(L_E \omega, df) E,
\]

\[
[df^2, dg^2] = \Lambda(df, dg) + \frac{1}{2} E.f (i_{dg} d\omega) - \frac{1}{2} E.g (i_{dg} d\omega) + \frac{1}{2} d\omega(df^2, dg^2) E.
\]

Proof. It follows from the above Lemma \( \square \) by putting \( \alpha = df \) and \( \beta = dg \).

1.18. Theorem. The \( (2s + 1) \)-dimensional distribution \( \langle E \rangle \oplus \text{im} \Lambda^2 \) is completely integrable and \( (E, \Lambda, \omega) \) induces a regular almost–coPoisson–Jacobi structure on the integral submanifolds of \( \langle E \rangle \oplus \text{im} \Lambda^2 \).

Proof. By the above Lemma \( \square \) the distribution \( \langle E \rangle \oplus \text{im} \Lambda^2 \) is involutive and of constant rank, so it is completely integrable.
Let us consider a $(2s+1)$-dimensional integral submanifold $\iota : \mathcal{N} \hookrightarrow \mathcal{M}$ passing through $x \in \mathcal{M}$.

If $\tilde{f}, \tilde{g} \in \text{map}(\mathcal{N}, \mathbb{R})$, then we can extend them (locally) to $f, g \in \text{map}(\mathcal{M}, \mathbb{R})$, such that $\tilde{f} = f \circ \iota$, $\tilde{g} = g \circ \iota$. Then, we define $E_N \in \text{sec}(\mathcal{N}, T\mathcal{N})$ and $\Lambda_N \in \text{sec}(\mathcal{N}, \Lambda^2 T\mathcal{N})$ by

$$E_N(\tilde{f}) = E.f, \quad \Lambda_N(d\tilde{f}, d\tilde{g}) = \Lambda(df, dg) = (df^\sharp).g = -(dg)^\sharp.f.$$ 

Indeed, the above $E_N$ and $\Lambda_N$ depend only on $\tilde{f}, \tilde{g}$, since they are computed along the integral curves of $E$, $(df)^\sharp, (dg)^\sharp$ through $x$ and these curves belong to $\mathcal{N}$.

Clearly, $(E_N, \Lambda_N)$ satisfy the equalities

$$E_N(\iota^* \alpha) = E(\alpha) \circ \iota, \quad \Lambda_N(\iota^* \alpha, \iota^* \beta) = \Lambda(\alpha, \beta) \circ \iota, \quad \forall \alpha, \beta \in \text{sec}(\mathcal{M}, T^* \mathcal{M}).$$

Then, from the naturality of the Schouten bracket [10], we have

$$[E_N, \Lambda_N](\iota^* \alpha, \iota^* \beta) = [E, \Lambda](\alpha, \beta) \circ \iota,$$

$$[\Lambda_N, \Lambda_N](\iota^* \alpha, \iota^* \beta, \iota^* \gamma) = [\Lambda, \Lambda](\alpha, \beta, \gamma) \circ \iota.$$ 

Let us set $\omega_N = \iota^* \omega$ and $\Lambda^2 : T^* \mathcal{N} \rightarrow T\mathcal{N} : \iota^* \alpha \mapsto (\iota^* \alpha)^2_N =: i_{\omega_N} \Lambda_N$.

Then, $i_{E_N} \omega = 1$ implies $i_{E_N} \omega_N = 1$ and $i_{\omega_N} \Lambda = 0$ implies $i_{\omega_N} \Lambda_N = 0$.

Moreover, for each $\alpha, \beta \in \text{sec}(\mathcal{M}, T^* \mathcal{M})$, we have

$$\iota^*(L_{E\omega}) = L_{E_N} \omega_N \quad \text{and} \quad d\omega_N((\iota^* \alpha)^2_N, (\iota^* \beta)^2_N) = d\omega(\alpha^2_N, \beta^2_N) \circ \iota.$$ 

Then, we have

$$[E_N, \Lambda_N](\iota^* \alpha, \iota^* \beta) =$$

$$= [E, \Lambda](\alpha, \beta) \circ \iota$$

$$= -(E \wedge (L_{E\omega})^\sharp)(\alpha, \beta) \circ \iota = -E(\alpha) \Lambda(L_{E\omega}, \beta) \circ \iota + E(\beta) \Lambda(L_{E\omega}, \alpha) \circ \iota$$

$$= -E_N(\iota^* \alpha) \Lambda_N(\iota^*(L_{E\omega}), \iota^* \beta) + E_N(\iota^* \beta) \Lambda_N(\iota^*(L_{E\omega}), \iota^* \alpha)$$

$$= -(E_N \wedge (L_{E_N} \omega_N)^2_N)(\iota^* \alpha, \iota^* \beta).$$

Similarly, we have

$$[\Lambda_N, \Lambda_N](\iota^* \alpha, \iota^* \beta, \iota^* \gamma) =$$

$$= [\Lambda, \Lambda](\alpha, \beta, \gamma) \circ \iota$$

$$= 2(E \wedge (\Lambda^2 \otimes \Lambda^2) d\omega)(\alpha, \beta, \gamma) \circ \iota$$

$$= 2(E(\alpha) d\omega(\beta^2_N, \gamma^2_N) + E(\beta) d\omega(\alpha^2_N, \gamma^2_N) + E(\gamma) d\omega(\alpha^2_N, \beta^2_N)) \circ \iota$$

$$= 2(E_N(\iota^* \alpha) d\omega_N(\iota^* \beta)^2_N, (\iota^* \gamma)^2_N) - E_N(\iota^* \beta) d\omega_N((\iota^* \alpha)^2_N, (\iota^* \gamma)^2_N)$$

$$+ E_N(\iota^* \gamma) d\omega_N((\iota^* \alpha)^2_N, (\iota^* \beta)^2_N))$$

$$= 2(E_N \wedge (\Lambda_N^2 \otimes \Lambda_N^2) d\omega_N)(\iota^* \alpha, \iota^* \beta, \iota^* \gamma).$$

Hence, $(E_N, \Lambda_N, \omega_N)$ is a regular almost–coPoisson–Jacobi 3–plet on $\mathcal{N}$. □
1.19. Proposition. In a neighborhood of each \( x \in M \) there exists a local chart (a Darboux’s chart) \((W; t, x^i, x^{i+n})\), with \( i = 1, \ldots, n \), adapted to the almost–coPoisson–Jacobi 3–plet \((E, \Lambda, \omega)\) i.e. such that

\[
E = \frac{\partial}{\partial t}, \quad \Lambda = \sum_{1 \leq i \leq s} \frac{\partial}{\partial x^{i+n}} \wedge \frac{\partial}{\partial x^i} - \sum_{1 \leq i \leq s} \left( \omega^{i+n} \frac{\partial}{\partial t} \wedge \frac{\partial}{\partial x^i} - \omega^i \frac{\partial}{\partial t} \wedge \frac{\partial}{\partial x^{i+n}} \right),
\]

\[
\omega = dt + \sum_{1 \leq i \leq n} \left( \omega^i dx^i + \omega^{i+n} dx^{i+n} \right),
\]

where \( \omega^i, \omega^{i+n} \in \text{map}(M, \mathbb{R}) \).

Proof. First, let us suppose that \( \Lambda \) be of rank \( 2s = 2n \) and let us consider a Darboux’s chart adapted to the dual almost–cosymplectic–contact pair \((\omega, \Omega)\). Then, from (1.7) we can easily see that (1.8) is satisfied.

Next, let us suppose that \( 2s < 2n \).

Let \( s = 0 \). Then, \( i_E \omega = 1 \) implies that there exists a chart \((t, x^i, x^{i+n})\) such that

\[
E = \frac{\partial}{\partial t}, \quad \Lambda = 0, \quad \omega = dt + \sum_{1 \leq i \leq n} \left( \omega^i dx^i + \omega^{i+n} dx^{i+n} \right).
\]

Let \( s > 0 \). Then, let us consider an integral submanifold \( N \) of the distribution \((E) \oplus \text{im} \Lambda^i \). There exists a coordinate neighborhood \((W; t, x^i, x^{i+n})\) of each \( x \in N \), with \( i = 1, \ldots, n \), such that \( N \) is given by \( x^i = 0, x^{i+n} = 0 \), with \( j = s + 1, \ldots, n \), and such that the coordinate neighborhood \((W \cap N; t, x^i, x^{i+j})\), with \( i = 1, \ldots, s \), is the Darboux’s chart on \( N \) adapted to \((E_N, \Lambda_N, \omega_N)\).

1.20. Remark. It is easy to see that \((E, \Lambda, \omega)\) given by (1.8) satisfies the conditions for almost–coPoisson–Jacobi 3–plets. Indeed, we have

\[
[E, \Lambda] = \frac{\partial}{\partial t} \wedge \sum_{1 \leq i \leq s} \left( - \frac{\partial \omega^{i+n}}{\partial t} \frac{\partial}{\partial x^i} + \frac{\partial \omega^i}{\partial x^{i+n}} \frac{\partial}{\partial t} \right),
\]

\[
[\Lambda, \Lambda] = 2 \frac{\partial}{\partial t} \wedge \left\{ \sum_{i,j=1}^s \left( \omega^{i+n} \frac{\partial \omega^{j+n}}{\partial t} + \frac{\partial \omega^j}{\partial x^{i+n}} \right) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} \right\}
+ \sum_{i,j=1}^s \left( \omega^i \frac{\partial \omega^{i+n}}{\partial t} - \omega^{i+n} \frac{\partial \omega^i}{\partial t} + \frac{\partial \omega^j}{\partial x^{i+n}} - \frac{\partial \omega^{j+n}}{\partial x^i} \right) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^{j+n}}
+ \sum_{i,j=1}^s \left( \omega^i \frac{\partial \omega^j}{\partial t} + \frac{\partial \omega^i}{\partial x^{j+n}} \right) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^{j+n}} \right];
\]

\[
\Lambda^2(L_{E\omega}) = \sum_{1 \leq i \leq s} \left( \frac{\partial \omega^{i+n}}{\partial t} \frac{\partial}{\partial x^i} - \frac{\partial \omega^i}{\partial t} \frac{\partial}{\partial x^{i+n}} + \left( \frac{\partial \omega^i}{\partial t} \omega^{i+n} - \frac{\partial \omega^{i+n}}{\partial t} \omega^i \right) \frac{\partial}{\partial t} \right),
\]
coPoisson if, for instance, \( \omega \in \Lambda^2 \). by a pre coPoisson pair. \( \omega \) given by (1.8), then the functions \( \{ \omega^i, \omega^j \} \) form is not unique. \( E \), \( \Lambda \), \( \omega \) are given uniquely by \( \Lambda \) and \( \Lambda \) are 

\[
(1.12) \quad (\Lambda^s \otimes \Lambda^t)(d\omega) = \\
= \sum_{1 \leq i,j \leq s} (\omega^i \omega^j + \frac{\partial \omega^i}{\partial t} + \omega^i \omega^j + \frac{\partial \omega^i}{\partial x^j}) \frac{\partial}{\partial t} \wedge \frac{\partial}{\partial x^j} \\
+ \omega^i \omega^j + \frac{\partial \omega^i}{\partial x^j} \frac{\partial}{\partial x^j} + \omega^i \omega^j + \frac{\partial \omega^i}{\partial x^j} \frac{\partial}{\partial x^j} \\
+ \sum_{1 \leq i,j \leq s} (\omega^i \omega^j + \frac{\partial \omega^i}{\partial x^j} \frac{\partial}{\partial x^j} + \omega^i \omega^j + \frac{\partial \omega^i}{\partial x^j} \frac{\partial}{\partial x^j}) \frac{\partial}{\partial x^j} \wedge \frac{\partial}{\partial x^j}.
\]

1.21. Remark. Let \((E, \Lambda)\) be a contravariant pair with \( s < n \). Then, there exists a 1–form \( \omega \) which satisfies \( i_E \omega = 1 \) and \( i_\omega \Lambda = 0 \), (hence also \( i_\omega(E \wedge \Lambda^s) = \Lambda^s \)). But such a form is not unique.

Moreover, the 3–plet \((E, \Lambda, \omega)\) turns out to be almost–coPoisson–Jacobi if and only if the equalities \([E, \Lambda] = -E \wedge (L_E \omega)^s \) and \([\Lambda, \Lambda] = E \wedge (\Lambda^s \otimes \Lambda^t)(d\omega)\) are satisfied. We can see it in adapted Darboux’s charts; in fact, if the coordinate expressions of \( E \) and \( \Lambda \) are given by (1.8), then the functions \( \omega^i, \omega^j \), with \( i = 1, \ldots, s \), are given uniquely by \( \Lambda \), but \( \omega^i, \omega^j \), with \( i = s + 1, \ldots, n \), are arbitrary, so \( \omega \) is not unique. \( \square \)

1.22. Note. The almost–coPoisson–Jacobi 3–plet given in Darboux’s charts by (1.8) is coPoisson if, for instance, \( \omega^i = \omega^i + 0 = 0 \), with \( i = 1, \ldots, s \), and is Jacobi if, for instance, \( \omega^i = -x^i + n \), \( \omega^j + n = 0 \), with \( i = 1, \ldots, s \). \( \square \)

2. Lie algebra of functions

Next, we study the algebras of functions associated with the geometrical structure given by a pre coPoisson pair.

2.1. Poisson algebra of functions. First, let us start by considering just a 2–vector \( \Lambda \in \sec(M, \Lambda^2 TM) \).

2.1. Definition. The Poisson bracket of functions \( f, g \in \text{map}(M, \mathbb{R}) \) is defined as

\[
(2.1) \quad \{ f, g \} = i_{df \wedge dg} \Lambda = i_\Lambda(df \wedge dg) = \Lambda(df, dg).
\]
2.2. Lemma. For each \( f, g, h \in \text{map}(\mathcal{M}, \mathbb{R}) \), we have

\[
\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = \frac{1}{2} i_{[\Lambda, \Lambda]}(df \wedge dg \wedge dh).
\]

Proof. We have

\[
\{\{f, g\}, h\} = \Lambda(df(dg, dh) - \Lambda(df, dh) dg + \Lambda(dg, dh) df).
\]

Then,

\[
i_{[\Lambda, \Lambda]}(df \wedge dg \wedge dh) = 2i_{\Lambda}i_{\Lambda}(df \wedge dg \wedge dh)
= 2i_{\Lambda}d(\Lambda(df, dg) dh - \Lambda(df, dh) dg + \Lambda(dg, dh) df)
= 2\Lambda(d\Lambda(df, dg), dh) + \Lambda(d\Lambda(dg, df), dg) + \Lambda(d\Lambda(dg, dh), df)
= 2\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\}.
\]

2.3. Proposition. The following facts are equivalent:

1. \([\Lambda, \Lambda] = 0\);
2. \(\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0, \quad \forall f, g, h \in \text{map}(\mathcal{M}, \mathbb{R})\);
3. the bracket \(\{,\}\) is a Lie bracket.

Thus, a Poisson structure yields a Lie algebra of functions.

2.4. Lemma. The following facts are equivalent:

1. \([df^z, dg^z] = d\{f, g\}^z, \quad \forall f, g \in \text{map}(\mathcal{M}, \mathbb{R})\);
2. \(\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0, \quad \forall f, g, h \in \text{map}(\mathcal{M}, \mathbb{R})\).

Proof. We have

\[
d\{f, g\}^z.h = \Lambda(\{f, g\}, dh) = \{\{f, g\}, h\},
\]

\[
[df^z, dg^z].h = df^z.dg^z.h - dg^z.df^z.h = \{f, \{g, h\}\} - \{g, \{f, h\}\}.
\]

Then,

\[
(d\{f, g\}^z - [df^z, dg^z]).h = \{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\}.
\]

2.5. Proposition. The following facts are equivalent:

1. \([df^z, dg^z] = d\{f, g\}^z, \quad \forall f, g \in \text{map}(\mathcal{M}, \mathbb{R})\);
2. the map

\[
\Lambda^z \circ d : \text{map}(\mathcal{M}, \mathbb{R}) \to \text{sec}(\mathcal{M}, TM)
\]

is a Lie algebra homomorphism with respect to the Poisson bracket of functions and the Lie bracket of vector fields;
3. \((\Lambda)\) is a Poisson structure, i.e. \([\Lambda, \Lambda] = 0\).

2.6. Corollary. Let \((E, \Lambda)\) be a coPoisson pair then \(\Lambda\) defines a Poisson algebra of functions.
2.2. Jacobi algebra of functions. Then, let us consider a contravariant pair \((E, \Lambda)\).

2.7. Remark. If \((E, \Lambda)\) is a Jacobi pair with \(s > 0\), then the Poisson bracket does not satisfy the Jacobi identity. In fact, the Jacobi identity turns out to be equivalent to \(E \wedge \Lambda = 0\). But this condition conflicts with our hypothesis \(E \wedge \Lambda^s \neq 0\). □

2.8. Definition. The Hamiltonian lift of a function \(f \in \text{map}(M, \mathbb{R})\) is defined to be the vector field
\[
X_f =: i_{df} \Lambda - fE = df^2 - fE.
\]

2.9. Definition. The Jacobi bracket of two functions \(f, g \in \text{map}(M, \mathbb{R})\) is defined as
\[
[f, g] =: \{f, g\} - fE.g + gE.f = \Lambda(df, dg) - fE.g + gE.f.
\]

2.10. Lemma. For each \(f, g \in \text{map}(M, \mathbb{R})\), we have
\[
E.\{f, g\} = \{E.f, g\} + \{f, E.g\} + i_{[E, \Lambda]}(df \wedge dg).
\]

Proof. We have
\[
i_{[E, \Lambda]}(df \wedge dg) = i_E di_{\Lambda}(df \wedge dg) - i_{\Lambda} di_E(df \wedge dg)
\]
\[
= i_E di_{\Lambda}(df \wedge dg) - i_{\Lambda} d(E.f dg - E.g df)
\]
\[
= i_E di_{\Lambda}(df \wedge dg) - i_{\Lambda} (d(E.f) \wedge dg) - i_{\Lambda} (df \wedge d(E.g))
\]
\[
= E.\{f, g\} - \{E.f, g\} - \{f, E.g\}. □
\]

2.11. Lemma. For each \(f, g, h \in \text{map}(M, \mathbb{R})\), we have
\[
[[f, g], h] + [[g, h], f] + [[h, f], g] =
\]
\[
= (\frac{1}{2} i_{[\Lambda, \Lambda]} + i_{E \wedge \Lambda})(df \wedge dg \wedge dh) + i_{[E, \Lambda]}(f df \wedge dh + g dh \wedge df + h df \wedge dg).
\]

Proof. We have
\[
[[f, g], h] = \{(f, g), h\} - \{(f, g), E.h\} - \{f, h\}(E.g) + \{g, h\}(E.f)
\]
\[
+ h\{E.f, g\} + h\{f, E.g\} - f\{E.g, h\} + g\{E.f, h\} + hi_{[E, \Lambda]}(df \wedge dg)
\]
\[
+ f(E.g)(E.h) - g(E.f)(E.h) - hf E^2.g + hg E^2.f.
\]

Then,
\[
[[f, g], h] + [[g, h], f] + [[h, f], g] =
\]
\[
= \{(f, g), h\} + \{(g, h), f\} + \{(h, f), g\} + \{f, g\}(E.h) + \{g, h\}(E.f) + \{h, f\}(E.g)
\]
\[
+ f i_{[E, \Lambda]}(dh \wedge df) + gi_{[E, \Lambda]}(df \wedge dh) + h i_{[E, \Lambda]}(df \wedge dg)
\]
\[
= (\frac{1}{2} i_{[\Lambda, \Lambda]} + i_{E \wedge \Lambda})(df \wedge dg \wedge dh) + i_{[E, \Lambda]}(f df \wedge dh + g dh \wedge df + h df \wedge dg). □
\]

2.12. Proposition. \[\] The Jacobi bracket defines a Lie algebra of functions if and only if \([E, \Lambda] = 0\) and \([\Lambda, \Lambda] = -2E \wedge \Lambda\).

So, a Jacobi pair \((E, \Lambda)\) defines a Lie algebra of functions with respect to the Jacobi bracket (the Jacobi algebra of functions). □
2.13. **Remark.** A coPoisson pair does not define a Lie algebra of functions with respect to the Jacobi bracket. Indeed, for a coPoisson pair, we have

\[
[[f, g], h] + [[g, h], f] + [[h, f], g] = i_{E \wedge \Lambda}(df \wedge dg \wedge dh),
\]
so, in general, the Jacobi identity is not satisfied. Indeed, it is satisfied in the particular case when \( E \wedge \Lambda = 0 \), but this condition conflicts with our hypothesis \( E \wedge \Lambda^* \neq 0 \). □

2.14. **Lemma.** For each \( f, g, h \in \text{map}(M, \mathbb{R}) \), we have

\[
([X_f, X_g] - X_{[f,g]}).h = -(\frac{1}{2} i_{[\Lambda, \Lambda]} + i_{E \wedge \Lambda})(df \wedge dg \wedge dh) - f_i_{[E, \Lambda]}(dg \wedge dh) - g_i_{[E, \Lambda]}(df \wedge dh).
\]

**Proof.** We have

\[
[X_f, X_g].h = [df^2, dg^2].h - [df^2, gE].h - [f E, dg^2].h + [f E, gE].h
\]

\[
= d\{f, g\}^2.h - \{\{f, g\}, h\} - \{\{g, h\}, f\} - \{\{h, f\}, g\}
\]

\[
- df^2(gE.h) + gE.(df^2.h) - f E.(dg^2.h) + dg^2.(f E.h) + f E.(gE.h) - gE.(f E.h)
\]

\[
= d\{f, g\}^2.h - \frac{1}{2} i_{[\Lambda, \Lambda]}(df \wedge dg \wedge dh) - 2\{f, g\}(E.h)
\]

\[
- g\{f, E.h\} + gE.\{f, h\} + f\{g, E.h\} - f E.\{g, h\} + f(E.g)(E.h) - g(E.f)(E.h)
\]

\[
= d\{f, g\}^2.h - \frac{1}{2} i_{[\Lambda, \Lambda]}(df \wedge dg \wedge dh) - 2\{f, g\}(E.h) - f \{E.g, h\} + g \{E.f, h\}
\]

\[
- f i_{[E, \Lambda]}(dg \wedge dh) + g i_{[E, \Lambda]}(df \wedge dh) + f(E.g)(E.h) - g(E.f)(E.h)
\]

On the other hand,

\[
X_{[f,g]}h = d\{f, g\}^2.h - \{f, h\}(E.g) + \{g, h\}(E.f) - f E.\{g, h\} + g \{E.f, h\}
\]

\[
- \{f, g\}(E.h) + f(E.g)(E.h) - g(E.f)(E.h).
\]

Hence,

\[
([X_f, X_g] - X_{[f,g]}).h = -(\frac{1}{2} i_{[\Lambda, \Lambda]} + i_{E \wedge \Lambda})(df \wedge dg \wedge dh) - f i_{[E, \Lambda]}(dg \wedge dh) - g i_{[E, \Lambda]}(df \wedge dh).\]

□

2.15. **Proposition.** [12]. The following facts are equivalent:

(1) \( [X_f, X_g] = X_{[f,g]} \), \( \forall f, g \in \text{map}(M, \mathbb{R}) \);

(2) the Hamiltonian lift of functions with respect to a pair \((E, \Lambda)\) is a Lie algebra homomorphism with respect to the Jacobi bracket and the Lie bracket of vector fields;

(3) the pair \((E, \Lambda)\) is a Jacobi structure, i.e. \( [E, \Lambda] = 0 \) and \([\Lambda, \Lambda] = -2 E \wedge \Lambda\).

**Proof.** The equivalence follows from Lemma 2.13 and the arbitrariness of the functions \( f, g, h \). □
2.3. **Uniqueness of the Jacobi structure.** Now, we revisit the well known Proposition 2.12 [9] in the context of our almost–coPoisson–Jacobi structures. Actually, we prove that an almost–coPoisson–Jacobi 3–plet \((E, \Lambda, \omega)\) defines a Lie algebra of functions with respect to the Jacobi bracket if and only if the pair \((E, \Lambda)\) is Jacobi.

Let us consider an almost–coPoisson–Jacobi 3–plet \((E, \Lambda, \omega)\).

2.16. **Lemma.** The following facts are equivalent:

(1) for each \(f, g, h \in \text{map}(M, \mathbb{R})\),

\[
\left( \frac{1}{2} i_{[\Lambda, \Lambda]} + i_{E \wedge \Lambda} \right)(df \wedge dg \wedge dh) + i_{E, \Lambda}(f \, dg \wedge dh + g \, dh \wedge df + h \, df \wedge dg) = 0,
\]

(2) for each \(f, g \in \text{map}(M, \mathbb{R})\),

\[
\{f, g\} = -d \omega(X_f, X_g).
\]

**Proof.** We have

\[
i_{\Lambda^*(L_E \omega)} df = -d \omega(E, df^2).
\]

Then,

\[
\left( \frac{1}{2} i_{[\Lambda, \Lambda]} + i_{E \wedge \Lambda} \right)(df \wedge dg \wedge dh) + i_{E, \Lambda}(f \, dg \wedge dh + g \, dh \wedge df + h \, df \wedge dg) =
\]

\[
= (E.f) (\Lambda(dg, dh) + (\Lambda^2 \otimes \Lambda^2)(d\omega)(dg, dh))
\]

\[
+ (E.g) (\Lambda(dh, df) + (\Lambda^2 \otimes \Lambda^2)(d\omega)(dh, df))
\]

\[
+ (E.h) (\Lambda(df, dg) + (\Lambda^2 \otimes \Lambda^2)(d\omega)(df, dg))
\]

\[
+ f(E.g) d\omega(E, dh^2) - f(E.h) d\omega(E, df^2)
\]

\[
+ g(E.h) d\omega(E, df^2) - g(E.f) d\omega(E, dh^2)
\]

\[
+ h(E.f) d\omega(E, df^2) - h(E.g) d\omega(E, dg^2)
\]

\[
= (E.f) \{g, h\} + d \omega(df^2, dg^2) - gd \omega(E, dh^2) + hd \omega(E, dg^2)
\]

\[
+ (E.g) \{h, f\} + d \omega(dh^2, df^2) - hd \omega(E, df^2) + f d \omega(E, dh^2)
\]

\[
+ (E.h) \{f, g\} + d \omega(df^2, dg^2) - f d \omega(E, dg^2) + gd \omega(E, df^2)
\]

\[
= (E.f) \{g, h\} + d \omega(df^2 - g E, dh^2 - h E)
\]

\[
+ (E.g) \{h, f\} + d \omega(dh^2 - h E, df^2 - f E)
\]

\[
+ (E.h) \{f, g\} + d \omega(df^2 - f E, dg^2 - g E).
\]

\[\square\]

2.17. **Proposition.** The almost–coPoisson–Jacobi structure \((E, \Lambda, \omega)\) yields a Lie algebra of functions with respect to the Jacobi bracket if and only if the Poisson bracket satisfies \((2.7)\).

**Proof.** It follows from the above Lemma 2.16 end from Lemma 2.11.\[\square\]
2.18. **Corollary.** A Jacobi pair \((E, \Lambda)\) yields a Lie algebra with respect to the Jacobi bracket.

A coPoisson pair \((E, \Lambda)\) yields a Lie algebra with respect to the Jacobi bracket if and only if \(\Lambda = 0\).

**Proof.** Let \((E, \Lambda)\) be a Jacobi pair. Then, for each \(\alpha, \beta \in \text{sec}(M, T^*M)\), we have

\[
d\omega(\alpha^\sharp, \beta^\sharp) = -\Lambda(\alpha, \beta) \quad \text{and} \quad d\omega(E, \alpha^\sharp) = 0,
\]

hence, for each \(f, g \in \text{map}(M, \mathbb{R})\), we obtain

\[
\{f, g\} =: \Lambda(df, dg) = -d\omega(df^\sharp, dg^\sharp) = -d\omega(df^\sharp, dg^\sharp) + f d\omega(E, dg^\sharp) - g d\omega(df^\sharp, E)
\]

hence condition (2.7) is satisfied.

Let \((E, \Lambda)\) be a coPoisson pair. Then, we have

\[
d\omega = 0
\]

hence condition (2.7) is satisfied if and only if \(\{f, g\} = 0\), i.e. if and only if \(\Lambda = 0\).\]

2.19. **Theorem.** An almost–coPoisson–Jacobi 3–plet \((E, \Lambda, \omega)\) yields a Lie algebra of functions with respect to the Jacobi bracket if and only if the pair \((E, \Lambda)\) is Jacobi.

**Proof.** It is sufficient to prove that (2.7) implies that the pair \((E, \Lambda)\) is Jacobi.

We can prove it in a local chart.

In a Darboux's chart adapted to an almost–coPoisson–Jacobi 3–plet \((E, \Lambda, \omega)\) according to (1.8) we have

\[
(2.8) \quad X_f = \left( -f + \sum_{1 \leq i \leq s} \left( \omega^{i+n} \frac{\partial f}{\partial x^i} - \omega^i \frac{\partial f}{\partial x^{i+n}} \right) \frac{\partial}{\partial t} \right)
\]

\[
+ \sum_{1 \leq i \leq s} \left( \omega^{i+n} \frac{\partial f}{\partial x^i} - \omega^i \frac{\partial f}{\partial x^{i+n}} \right) \frac{\partial}{\partial x^i} + \sum_{1 \leq i \leq s} \left( \omega^i \frac{\partial f}{\partial x^i} - \omega^i \frac{\partial f}{\partial x^{i+n}} \right) \frac{\partial}{\partial x^{i+n}}.
\]

Then,

\[
d\omega(X_f, X_g) =
\]

\[
= \left( f \frac{\partial g}{\partial t} - g \frac{\partial f}{\partial t} \right) \cdot \sum_{i=1}^{s} \left( \frac{\partial \omega^i}{\partial t} \omega^{i+n} - \frac{\partial \omega^{i+n}}{\partial t} \omega^i \right)
\]

\[
+ \sum_{1 \leq i \leq s} \left( f \frac{\partial g}{\partial x^i} - g \frac{\partial f}{\partial x^i} \right) \frac{\partial \omega^{i+n}}{\partial t} - \sum_{i=1}^{s} \left( f \frac{\partial g}{\partial x^{i+n}} - g \frac{\partial f}{\partial x^{i+n}} \right) \frac{\partial \omega^i}{\partial t}
\]

\[
+ \sum_{1 \leq i, j \leq s} \left( \frac{\partial f}{\partial t} \frac{\partial g}{\partial x^i} - \frac{\partial g}{\partial t} \frac{\partial f}{\partial x^i} \right) \left( \omega^{j+n} \omega^{i+n} \frac{\partial \omega^j}{\partial t} - \omega^j \omega^{i+n} \frac{\partial \omega^j}{\partial t} \right.
\]

\[
+ \omega^j \omega^{i+n} \frac{\partial \omega^{i+n}}{\partial x^j} - \omega^{i+n} \omega^j \frac{\partial \omega^{i+n}}{\partial x^j} + \omega^j \omega^{i+n} \frac{\partial \omega^{i+n}}{\partial x^j} - \omega^j \omega^{i+n} \frac{\partial \omega^{i+n}}{\partial x^j} \right)
\]

\[
+ \sum_{1 \leq i, j \leq s} \left( \frac{\partial f}{\partial t} \frac{\partial g}{\partial x^{i+n}} - \frac{\partial g}{\partial t} \frac{\partial f}{\partial x^{i+n}} \right) \left( \omega^i \omega^{j+n} \frac{\partial \omega^j}{\partial t} - \omega^j \omega^{i+n} \frac{\partial \omega^j}{\partial t} \right)
\]

\[
+ \sum_{1 \leq i, j \leq s} \left( \frac{\partial f}{\partial t} \frac{\partial g}{\partial x^{i+n}} - \frac{\partial g}{\partial t} \frac{\partial f}{\partial x^{i+n}} \right) \left( \omega^i \omega^{j+n} \frac{\partial \omega^j}{\partial t} - \omega^j \omega^{i+n} \frac{\partial \omega^j}{\partial t} \right).
\]
Now, if we use the above identities, then we obtain

\[
\{f, g\} = \sum_{1 \leq i, j \leq s} \left( \frac{\partial f}{\partial x^{i+n}} \frac{\partial g}{\partial x^{i}} - \frac{\partial g}{\partial x^{i+n}} \frac{\partial f}{\partial x^{i}} \right) - \omega^{i+n} \left( \frac{\partial f}{\partial t} \frac{\partial g}{\partial x^{i}} - \frac{\partial g}{\partial t} \frac{\partial f}{\partial x^{i+n}} \right) + \omega^{i} \left( \frac{\partial f}{\partial t} \frac{\partial g}{\partial x^{i+n}} - \frac{\partial g}{\partial t} \frac{\partial f}{\partial x^{i+n}} \right)
\]

Now, if we assume \(\{f, g\} = -\omega (X_f, X_g)\), then we obtain the following system of partial differential equations, for all \(i, j = 1, \ldots, s\),

\[
0 = \frac{\partial \omega^{i+n}}{\partial t}, \quad 0 = \frac{\partial \omega^{i}}{\partial t}, \quad 0 = \sum_{1 \leq j \leq s} \left( \omega^{j} \frac{\partial \omega^{j+n}}{\partial x^{i+n}} - \omega^{j+n} \frac{\partial \omega^{j}}{\partial x^{i+n}} \right),
\]

\[
0 = \left( \frac{\partial \omega^{i}}{\partial x^{i}} - \frac{\partial \omega^{i}}{\partial x^{j+n}} \right), \quad 0 = \left( \frac{\partial \omega^{i+n}}{\partial x^{j+n}} - \frac{\partial \omega^{j+n}}{\partial x^{i+n}} \right), \quad \delta_j^i = \left( \frac{\partial \omega^{i+n}}{\partial x^{j}} - \frac{\partial \omega^{j}}{\partial x^{i+n}} \right).
\]

Now, if we use the above identities, then we obtain

\[
[E, \Lambda] = 0
\]

\[
[\Lambda, \Lambda] = 2 \frac{\partial}{\partial t} \wedge \left( \sum_{1 \leq i, j \leq s} \frac{\partial \omega^{j+n}}{\partial x^{i+n}} \frac{\partial}{\partial x^{i}} \wedge \frac{\partial}{\partial x^{j}} + \sum_{1 \leq i \leq s} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^{i+n}} \right.
\]

\[
\left. + \sum_{1 \leq i, j \leq s} \frac{\partial \omega^{j}}{\partial x^{i}} \frac{\partial}{\partial x^{i+n}} \wedge \frac{\partial}{\partial x^{j+n}} \right) = -2E \wedge \Lambda.
\]

So, \((E, \Lambda)\) is a Jacobi pair. \(\square\)

3. Examples: Dynamical structures

As examples of the geometric structures analysed above, now we discuss the dynamical structures arising on the phase space of a spacetime in classical relativistic theories. We consider the relativistic Galilei and the Einstein spacetimes, emphasizing the analogies and the differences between the two cases.

In order to make our theory explicitly independent from units of measurement, we introduce the “spaces of scales” \(\mathbb{S}\). Roughly speaking, a space of scales \(\mathbb{S}\) has the algebraic structure of \(\mathbb{R}^+\) but has no distinguished “basis”. We can naturally define the tensor
product of scales of spaces and the tensor product of spaces of scales and vector spaces. We can also naturally define rational tensor powers $S^{m/n}$ of a space of scales $S$. Moreover, we can make a natural identification $S^* \simeq S^{-1}$.

The basic objects of our theory (the metric field, the phase 2–form, the phase 2–vector, etc.) will be valued into scaled vector bundles, that is into vector bundles multiplied tensorially with spaces of scales. In this way, each tensor field carries explicit information on its “scale dimension”. Actually, we assume the following basic spaces of scales: the space of time intervals $\mathbb{T}$, the space of lengths $\mathbb{L}$ and the space of masses $\mathbb{M}$. Moreover, we consider the following “universal scales”: the speed of light $c \in \mathbb{T}^{-1} \otimes \mathbb{L}$ and the Planck constant $\hbar \in \mathbb{T}^* \otimes \mathbb{L}^2 \otimes \mathbb{M}$.

A time unit is defined to be an element $u_0 \in \mathbb{T}$, or, equivalently, its dual $u^0 \in \mathbb{T}^*$.

3.1. **Galilei spacetime.** First, we study the geometrical structures arising on the phase space of a Galilei spacetime $[2, 3, 5, 14]$.

3.1.1. **Spacetime.** We assume absolute time to be an affine 1–dimensional space $T$ associated with the vector space $\mathbb{T} = \mathbb{T} \otimes \mathbb{R}$.

We assume spacetime to be an oriented $(3+1)$–dimensional fibred manifold $E$ equipped with a time fibring $t : E \to T$.

A spacetime chart is defined to be a chart $(x^\lambda) \equiv (x^0, x^i)$ of $E$, adapted to the orientation, to the fibring, to the affine structure of $T$ and to a time unit $u_0$. Greek indices will span all spacetime coordinates and Latin indices will span the fibre coordinates. In the following, we shall always refer to spacetime charts. The induced local bases of $TE$ and $T^*E$ are denoted, respectively, by $(\partial_\lambda)$ and $(d^\lambda)$.

The vertical restriction of forms will be denoted by the “check” symbol $\check{\cdot}$. The differential of the time fibring is the scaled 1–form $dt : E \to \mathbb{T} \otimes T^*E$, with coordinate expression $dt = u_0 \otimes d^0$.

We assume spacetime to be equipped with a scaled spacelike Riemannian metric $g : E \to \mathbb{L}^2 \otimes (V^*E \otimes V^*E)$. The contravariant metric is denoted by $\bar{g} : E \to \mathbb{L}^{-2} \otimes (VE \otimes VE)$.

We have the coordinate expressions

$$g = g_{ij} \check{d}^i \otimes \check{d}^j,$$

$$\bar{g} = g^{ij} \partial_i \otimes \partial_j,$$

with $g_{ij} \in \text{map}(E, \mathbb{L}^2 \otimes \mathbb{R})$, $g^{ij} \in \text{map}(E, \mathbb{L}^{-2} \otimes \mathbb{R})$.

3.1.2. **Phase space.** A motion is defined to be a section $s : T \to E$. The 1st differential of a motion $s$ is defined to be the map $ds : T \to T^* \otimes TE$. We have $dt(ds) = 1$.

We assume as phase space the 1st jet space $J_1E$ of motions.

A space time chart $(x^\lambda)$ induces naturally a chart $(x^0, x^i_0)$ on $J_1E$.

The velocity of a motion $s$ is defined to be its 1st jet $j_1s : T \to J_1E$.

We define the contact map to be the unique fibred morphism $\lambda : J_1E \to T^* \otimes TE$ over $E$ such that $\lambda \circ j_1s = ds$, for each motion $s$. We have $\lambda \circ dt = 1$. The coordinate expression of $\lambda$ is

$$\lambda = u^0 \otimes \partial_0 \equiv u^0 \otimes (\partial_0 + x^i_0 \partial_i).$$

The map $\lambda$ is injective. Accordingly, the 1st jet space can be naturally identified with the subbundle $J_1E \subset T^* \otimes TE$, of scaled vectors which project on $1 : T \to T^* \otimes \mathbb{T}$. 


Thus, the bundle $J_1E \to E$ turns out to be affine and associated with the vector bundle $T^* \otimes V E$. Indeed, $J_1E \subset T^* \otimes TE$ is the fibred submanifold over $E$ characterised by the constraint $\dot{x}_0 = 1$.

We define also the complementary contact map $\theta := 1 - \delta \circ dt : J_1E \to T^* E \otimes V E$. The coordinate expression of $\theta$ is

$$\theta = \theta^i \otimes \partial_i \equiv (d^i - x^i_0 d^0) \otimes \partial_i .$$

3.1.3. Vertical bundle of the phase space. Let $V_0J_1E \subset V J_1E \subset T J_1E$ be the vertical tangent subbundle over $E$ and the vertical tangent subbundle over $T$, respectively. The affine structure of the phase space yields the equality $V_0J_1E = J_1E \times (T^* \otimes V E)$, hence the natural map $\nu : J_1E \to T \otimes (V^* E \otimes V_0J_1E)$, with coordinate expression $\nu = u_0 \otimes \dot{d}^i \otimes \partial_i^0$.

3.1.4. Spacetime connections. We define a spacetime connection to be a torsion free linear connection $K : TE \to T^* E \otimes TTE$ of the bundle $TE \to E$. Its coordinate expression is of the type

$$K = d^\lambda \otimes (\partial_\lambda + K_\lambda^\mu \nu \dot{x}^\nu \dot{\nu}_\mu) , \quad \text{with} \quad K_\lambda^\mu \nu = K_\mu^\nu \lambda \in \text{map}(E, \mathbb{R}) .$$

A spacetime connection $K$ is said to be time preserving if it preserves the time fibering, i.e. if $\nabla dt = 0$. In coordinates, this reads $K_0^0 = 0$.

A time preserving spacetime connection $K$ is said to be metric if it preserves the metric $g$, i.e. if $\nabla g = 0$. In coordinates, it reads

$$K_0^i = \frac{1}{2} g^{ij} \phi_{0j} ,$$

$$K_0^i = -\frac{1}{2} g^{ij} (2 \phi_{0hj} + \partial_0 g_{hj}) ,$$

$$K_0^i = -\frac{1}{2} g^{ij} (\partial_h g_{jk} + \partial_k g_{jh} - \partial_j g_{hk}) ,$$

where $\phi \in \text{sec}(E, T^* \otimes \mathbb{L}^2 \otimes \Lambda^2 T^* E)$ is a scaled spacetime 2–form (which depends on $K$ and on the chosen chart).

The vertical restriction of of a metric spacetime connection $K$ is just the Levi Civita connection of the spacetime fibres.

A spacetime connection $K$ is said to be a Galilei connection if it is time preserving, metric and such that its curvature tensor $R$ fulfills a symmetry condition which in coordinates reads $R_\lambda^i \mu^j = R_\mu^j \lambda^i$, where $R_\lambda^i \mu^j := g^{jp} R_\lambda^j \mu p$.

3.1.5. Phase connections. We define a phase connection to be a connection of the bundle $J_1E \to E$.

A phase connection can be represented, equivalently, by a tangent valued form $\Gamma : J_1E \to T^* E \otimes T J_1E$, which is projectable over $1 : E \to T^* E \otimes TE$, or by the complementary vertical valued form $\nu[\Gamma] : J_1E \to T^* J_1E \otimes V J_1E$, respectively, with coordinate expressions

$$\Gamma = d^\lambda \otimes (\partial_\lambda + \Gamma_\lambda^i \partial_i^0) , \quad \nu[\Gamma] = (d^i_\lambda - \Gamma_\lambda^i d^b^0) \otimes \partial_i^0 , \quad \text{with} \quad \Gamma_\lambda^i \in \text{map}(J_1E, \mathbb{R}) .$$

The coordinate expression of an affine phase connection $\Gamma$ is $\Gamma_0^i = \Gamma_0^i \dot{x}_0 = \Gamma_0^i \partial_i^0 + \Gamma_0^i \lambda_0 p$. We can prove \([3]\) that there is a natural bijective map $\chi : K \mapsto \Gamma$ between time preserving linear spacetime connections $K$ and affine phase connections $\Gamma$, with coordinate expression $\chi_{\lambda 0 \mu}^i \Gamma = K_\lambda^i \mu$. 
3.1.6. Dynamical phase connection. The space of 2–jets of motions $J_2E$ can be naturally regarded as the affine subbundle $J_2E \subset T^* \otimes T^*J_1E$, which projects on $\pi : J_1E \rightarrow T^* \otimes TE$.

A dynamical phase connection is defined to be a 2nd–order connection, i.e. a section $\gamma : J_1E \rightarrow J_2E$, or, equivalently, a section $\gamma : J_1E \rightarrow T^* \otimes T_1J_1E$, which projects on $\pi$.

The coordinate expression of a dynamical phase connection is of the type

$$\gamma = c\alpha^0(\partial_0 + x^i_0 \partial_i + \gamma_0^0 \partial_0^0), \quad \text{with} \quad \gamma_0^0 \in \text{map}(J_1E, \mathbb{R}).$$

If $\gamma$ is a dynamical phase connection, then we have $\gamma \circ d\tau = 1$.

The contact map $\pi$ and a phase connection $\Gamma$ yield the section $\gamma = \gamma[\pi, \Gamma] =: \pi \circ \Gamma : J_1E \rightarrow T^* \otimes T_1J_1E$, which turns out to be a dynamical phase connection, with coordinate expression

$$\gamma_0^0 = \Gamma_0^0 + \Gamma_{0j}^j x_0^j.$$ 

In particular, a time preserving spacetime connection $K$ yields the dynamical phase connection $\gamma =: \gamma[\pi, K] =: \pi \circ \chi(K)$, with coordinate expression

$$\gamma_0^0 = K_{i0}^i x_0^i + 2 K_{i0}^j x_0^j + K_0^0.$$ 

3.1.7. Phase 2–form and 2–vector. The metric $g$ and a phase connection $\Gamma$ yield the scaled 2–form $\Omega$, called (scaled) phase 2–form, and the scaled vertical 2–vector $\Lambda$, called (scaled) phase 2–vector,

$$\Omega = \Omega[g, \Gamma] =: g \circ (\nu[\Gamma] \wedge \theta) : J_1E \rightarrow T^* \otimes \mathbb{L}^2 \otimes \Lambda^2 T^*J_1E,$$

$$\Lambda = \Lambda[g, \Gamma] =: \tilde{g} \circ (\Gamma \wedge \nu) : J_1E \rightarrow T \otimes \mathbb{L}^{-2} \otimes \Lambda^2 V J_1E,$$

with coordinate expressions

$$\Omega[g, \Gamma] = g_{ij} u^0 \otimes (d^i_0 - \Gamma_i^0 d^0) \wedge (d^j_0 - \Gamma_j^0 d^0),$$

$$\Lambda[g, \Gamma] = g^{ij} u_0 \otimes (\partial_i + \Gamma_i^h \partial_0^h) \wedge \partial_j^0.$$

We can easily see that $dt \wedge \Omega^3 \neq 0$ and $\gamma \wedge \Lambda^3 \neq 0$.

There is a unique dynamical phase connection $\gamma$, such that $\gamma \circ \Omega[g, \Gamma] = 0$. Namely, $\gamma = \gamma[\pi, \Gamma]$.

In particular, a metric spacetime connection $K$ yields the (scaled) phase 2–form $\Omega = \Omega[g, K] =: \Omega[g, \chi(K)]$ and the (scaled) phase 2–vector $\Lambda = \Lambda[g, K] =: \Lambda[g, \chi(K)]$ with coordinate expressions

$$\Omega = -g_{ij} (d^i - x^i_0 d^0) \wedge d^j_0 + \left(\frac{1}{2} \partial_j g_{hk} x^h_0 x^k_0 + \partial_0 g_{jk} x^k_0 + \phi_{0,0j}\right) d^0 \wedge d^j +$$

$$+ \left(\frac{1}{2} \left(\partial_i g_{hk} - \partial_j g_{hi}\right) x^h_0 + \frac{1}{2} \phi_{0,ij}\right) d^i \wedge d^j,$$

$$\Lambda = g^{ij} \partial_i \wedge \partial_j - \frac{1}{2} g^{ij} g^{ik} \left((\partial_k g_{lr} - \partial_r g_{lk}) x^l_0 + \phi_{0,kr}\right) \partial_j^0 \wedge \partial_j^0.$$

3.1.8. Dynamical structures of the phase space. We have the following result [2, 5].

**Theorem.** Let us consider a spacetime connection $K$ and the induced objects $\Gamma =: \chi(K)$, $\gamma =: \gamma[\pi, \Gamma]$, $\Omega =: \Omega[g, \Gamma]$ and $\Lambda =: \Lambda[g, \Gamma]$. Then, the following assertions are equivalent.

(1) $K$ is a Galilei connection.

(2) $\Omega$ is closed, i.e. $(-dt, \Omega)$ is a scaled cosymplectic pair.
(3) \([\gamma, \Lambda] = 0\) and \([\Lambda, \Lambda] = 0\), i.e. \((-\gamma, \Lambda)\) is a scaled (regular) coPoisson pair.

Moreover, the cosymplectic pair \((-dt, \Omega)\) and the coPoisson pair \((-\gamma, \Lambda)\) are mutually dual. \[\square\]

3.2. **Remark.** If \(K\) is a time preserving spacetime connection, then the induced pairs \((-dt, \Omega[K])\) and \((-\gamma[K], \Lambda[K]\) are scaled.

On the other hand, some results of the general theory of geometrical structures developed in the first two sections requires unscaled pairs.

Indeed, if we refer to a particle of mass \(m \in M\) and consider the universal scales \(\hbar \in T^{-1} \otimes L^2 \otimes M\) and \(c \in T^{-1} \otimes L\), then we obtain unscaled pairs in the following natural way.

We have the unscaled spacetime 1–form
\[
\frac{m c^2}{\hbar} dt : E \to T^* E.
\]

Moreover, the rescaled contact map \(\overline{D} =: \frac{\hbar}{m c^2} \gamma[K]: J_1 E \to TE\) yields the unscaled phase vector field
\[
\gamma \equiv \gamma[K, K] = \frac{\hbar}{m c^2} \gamma[K, K] : E \to T J_1 E.
\]

Furthermore, the rescaled metric \(G =: \frac{m}{\hbar} g : E \to T \otimes V^* E \otimes V^* E\) yields the unscaled phase 2–form and phase 2–vector
\[
\Omega \equiv \Omega[K, K] = \frac{m}{\hbar} \Omega[K, K] : J_1 E \to \Lambda^2 T^* J_1 E,
\]
\[
\Lambda \equiv \Lambda[K, K] = \frac{\hbar}{m} \Lambda[K, K] : J_1 E \to \Lambda^2 T J_1 E.
\]

Thus, if \(K\) is a Galilei spacetime connection, then \((-\frac{m c^2}{\hbar} dt, \Omega)\) and \((-\frac{\hbar}{m c^2} \gamma, \Lambda)\) turn out to be mutually dual unscaled cosymplectic and coPoisson pairs of the phase space.

Indeed, the Plank constant does not play any direct role in classical mechanics; nevertheless, such a scale is necessary for getting unscaled objects as above. \[\square\]

3.2. **Einstein spacetime.** Then, we study the geometrical structures arising on the phase space of an Einstein spacetime \([3, 7]\).

3.2.1. **Spacetime.** We assume *spacetime* to be an oriented 4–dimensional manifold \(E\) equipped with a scaled Lorentzian metric \(g : E \to \mathbb{L}^2 \otimes (T^* E \otimes T^* E)\), with signature \((-+++)\); we suppose spacetime to be time oriented. The contravariant metric is denoted by \(\bar{g} : E \to \mathbb{L}^{-2} \otimes (TE \otimes TE)\).

A *spacetime chart* is defined to be a chart \((x^\lambda) \equiv (x^0, x^i) \in \text{map}(E, \mathbb{R} \times \mathbb{R}^3)\) of \(E\), which fits the orientation of spacetime and such that the vector field \(\partial_0\) is timelike and time oriented and the vector fields \(\partial_1, \partial_2, \partial_3\) are spacelike. Greek indices \(\lambda, \mu, \ldots\) will span spacetime coordinates, while Latin indices \(i, j, \ldots\) will span spacelike coordinates. In the following, we shall always refer to spacetime charts. The induced local bases of \(TE\) and \(T^* E\) are denoted, respectively, by \((\partial_\lambda)\) and \((d^\lambda)\). We have the coordinate expressions
\[
g = g_{\lambda\mu} d^\lambda \otimes d^\mu, \quad \text{with} \quad g_{\lambda\mu} \in \text{map}(E, \mathbb{L}^2 \otimes \mathbb{R}),
\]
\[
\bar{g} = g^{\lambda\mu} \partial_\lambda \otimes \partial_\mu, \quad \text{with} \quad g^{\lambda\mu} \in \text{map}(E, \mathbb{L}^{-2} \otimes \mathbb{R}).
\]
3.2.2. Jets of submanifolds. In view of the definition of the phase space, let us consider a manifold \( M \) of dimension \( n \) and recall a few basic facts concerning jets of submanifolds \([15]\).

Let \( k \geq 0 \) be an integer. A \( k \)-jet of 1–dimensional submanifolds of \( M \) at \( x \in M \) is defined to be an equivalence class of 1–dimensional submanifolds touching each other at \( x \) with a contact of order \( k \). The \( k \)-jet of a 1–dimensional submanifold \( s : N \hookrightarrow M \) at \( x \in N \) is denoted by \( j_k s(x) \). The set of all \( k \)-jets of all 1–dimensional submanifolds at \( x \in M \) is denoted by \( J_k(M, 1) \). The set \( J_k(M, 1) =: \bigsqcup_{x \in M} J_k(x, 1) \) is said to be the \( k \)-jet space of 1–dimensional submanifolds of \( M \). In particular, for \( k = 0 \), we have the natural identification \( J_0(M, 1) = M \), given by \( j_0 s(x) = x \), for each 1–dimensional submanifold \( s : N \hookrightarrow M \). For each integers \( k \geq h \geq 0 \), we have the natural projection \( \pi^k_h : J_k(M, 1) \to J_h(M, 1) : j_k s(x) \mapsto j_h s(x) \).

A chart of \( M \) is said to be divided if the set of its coordinate functions is divided into two subsets of 1 and \( n - 1 \) elements. Our typical notation for a divided chart will be \( (x^0, x^i) \), with \( 1 \leq i \leq n - 1 \). A divided chart and a 1–dimensional submanifold \( s : N \hookrightarrow M \) are said to be related if the map \( \bar{x}^0 =: x^0|_N \in \text{map}(N, \mathbb{R}) \) is a chart of \( N \). In such a case, the submanifold \( N \) is locally characterised by \( s^i \circ (\bar{x}^0)^{-1} =: (x^i \circ s) \circ (\bar{x}^0)^{-1} \in \text{map}(\mathbb{R}, \mathbb{R}) \).

In particular, if the divided chart is adapted to the submanifold, then the chart and the submanifold are related.

Let us consider a divided chart \( (x^0, x^i) \) of \( M \).

Then, for each submanifold \( s : N \hookrightarrow M \) which is related to this chart, the chart yields naturally the local fibred chart \( (x^0, x^i; x^i_{\underline{\alpha}})_{1 \leq |\underline{\alpha}| \leq k} \in \text{map}(J_k(M, 1), \mathbb{R}^n \times \mathbb{R}^{k(n-1)}) \) of \( J_k(M, 1) \), where \( \underline{\alpha} =: (h) \) is a multi–index of "range" 1 and "length" \( |\underline{\alpha}| = h \) and the functions \( x^i_{\underline{\alpha}} \) are defined by \( x^i_{\underline{\alpha}} = j_1 N : = \partial_{0 \cdots 0} s^i \), with \( 1 \leq |\underline{\alpha}| \leq k \).

We can prove the following facts:

1) the above charts \( (x^0, x^i; x^i_{\underline{\alpha}}) \) yield a smooth structure of \( J_k(M, 1) \);
2) for each 1–dimensional submanifold \( s : N \subset M \) and for each integer \( k \geq 0 \), the subset \( j_k s(N) \subset J_k(M, 1) \) turns out to be a smooth 1–dimensional submanifold;
3) for each integers \( k \geq h \geq 1 \), the maps \( \pi^{k}_h : J_k(M, 1) \to J_h(M, 1) \) turn out to be smooth bundles.

We shall always refer to such divided charts \( (x^0, x^i) \) of \( M \) and to the induced fibred charts \( (x^0, x^i; x^i_{\underline{\alpha}}) \) of \( J_k(M, 1) \).

Let \( m_1 \in J_1(M, 1) \), with \( m_0 = \pi^1_0(m_1) \in M \). Then, the tangent spaces at \( m_0 \) of all 1–dimensional submanifolds \( s : N \hookrightarrow M \), such that \( j_1 s(m_0) = m_1 \), coincide. Accordingly, we denote by \( T[m_1] \subset T_{m_0} M \) the tangent space at \( m_0 \) of the above 1–dimensional submanifolds \( N \) generating \( m_1 \). We have the natural fibred isomorphism \( J_1(M, 1) \to \text{Grass}(M, 1) : m_1 \mapsto T[m_1] \subset T_{m_0} M \) over \( M \) of the 1st jet bundle with the Grassmannian bundle of dimension 1. If \( s : N \hookrightarrow M \) is a 1–dimensional submanifold, then we obtain \( T[j_1 s] = \text{span}(\partial_{0} + \partial_{0} s^i \partial_i) \), with reference to a related chart.

3.2.3. Phase space. A motion is defined to be a 1–dimensional timelike submanifold \( s : T \hookrightarrow E \).

For every arbitrary choice of a "proper time origin" \( t_0 \in T \), we obtain the "proper time scaled function" given by the equality \( \sigma : T \to \mathbb{R} : t \mapsto \frac{1}{c} \int_{[t_0, t]} \| \frac{ds}{d\sigma} \| d\bar{x}^0 \).
This map yields, at least locally, a bijection $T \to \mathbb{T}$, hence a (local) affine structure of $T$ associated with the vector space $\mathbb{T}$. Indeed, this (local) affine structure does not depend on the choice of the proper time origin and of the spacetime chart.

Let us choose a time origin $t_0 \in T$ and consider the associated proper time scaled function $\sigma : T \to \mathbb{T}$ and the induced linear isomorphism $TT \to T \times \mathbb{T}$. Moreover, let us consider a spacetime chart $(x^\lambda)$ and the induced chart $(\dot{x}^0)$ in map($T$, $\mathbb{R}$). Let us set $\partial_0 s^\lambda := \frac{ds^\lambda}{dt} \ |

The 1st differential of the motion $s$ is defined to be the map $ds := \frac{ds}{dt} : T \to T^* \otimes TE$. We have $g(ds, ds) = -c^2$.

We assume as phase space the subspace $\mathcal{J}_1E \subset J_1(E, 1)$ consisting of all 1–jets of motions.

For each 1–dimensional submanifold $s : T \subset E$ and for each $x \in T$, we have $j_1s(x) \in \mathcal{J}_1E$ if and only if $T[j_1s(x)] = T_xE$ is timelike.

Any spacetime chart $(x^0, x^i)$ is related to each motion $s$. Hence, the fibred chart $(x^0, x^i, \dot{x}^i)$ is defined on tubelike open subsets of $\mathcal{J}_1E$.

We shall always refer to the above fibred charts.

The velocity of a motion $s$ is defined to be its 1–jet $j_1s : T \to \mathcal{J}_1(E, 1)$.

We define the contact map to be the unique fibred morphism $\xi : \mathcal{J}_1E \to \mathbb{T}^* \otimes TE$ over $E$, such that $\xi \circ j_1s = ds$, for each motion $s$. We have $g(\xi, \xi) = -c^2$. The coordinate expression of $\xi$ is

$$\xi = c \alpha^0 (\partial_0 + x^i \partial_i), \quad \text{where} \quad \alpha^0 = 1/\sqrt{|g_{00} + 2g_{0j}x^0_j + g_{ij}x^0_i x^0_j|}.$$

The map $\xi : \mathcal{J}_1E \to \mathbb{T}^* \otimes TE$ is injective. Indeed, it makes $\mathcal{J}_1E \subset \mathbb{T}^* \otimes TE$ the fibred submanifold over $E$ characterised by the constraint $g_{\lambda i} x^\lambda_0 x^i_0 = -(c_0)^2$.

We define the time form to be the map $\tau = -\frac{1}{c} g^t(\xi) : \mathcal{J}_1E \to \mathbb{T} \otimes T^*E$. We have $\tau(\xi) = 1$ and $\bar{g}(\tau, \tau) = -\frac{1}{c^2}$. The coordinate expression of $\tau$ is

$$\tau = \tau_\lambda d^\lambda, \quad \text{where} \quad \tau_\lambda = -\frac{\alpha^0}{c} (g_{0\lambda} + g_{i\lambda} x^i_0).$$

We define also the complementary contact map $\theta = 1 - \xi \otimes \tau : \mathcal{J}_1E \to T^*E \otimes TE$. The coordinate expression of $\theta$ is

$$\theta = d^\lambda \otimes \partial_\lambda + (\alpha^0)^2 (g_{0\lambda} + g_{i\lambda} x^i_0) d^\lambda \otimes (\partial_0 + x^i_0 \partial_i).$$

### 3.2.4. Vertical bundle of the phase space.

Let $V\mathcal{J}_1E \subset T\mathcal{J}_1E$ be the vertical tangent subbundle over $E$. The vertical prolongation of the contact map yields the mutually inverse linear fibred isomorphisms

$$\nu_\tau : \mathcal{J}_1E \to T \otimes V^*_\tau E \otimes V\mathcal{J}_1E \quad \text{and} \quad \nu_\tau^{-1} : \mathcal{J}_1E \to V^*\mathcal{J}_1E \otimes \mathbb{T}^* \otimes V^\tau E,$$

with coordinate expressions

$$\nu_\tau = \frac{1}{c \alpha^0} (d^i - x^i_0 \partial^0) \otimes \partial^0, \quad \nu_\tau^{-1} = c \alpha^0 d^i_0 \otimes (\partial_i - c \alpha^0 \tau_i (\partial_0 + x^0_0 \partial_0)).$$

Thus, for each $Y \in \sec(\mathcal{J}_1E, V\mathcal{J}_1E)$ and $X \in \sec(E, TE)$, we obtain

$$\nu_\tau^{-1}(Y) \in \text{fib}(\mathcal{J}_1E, T^* \otimes V^\tau E) \quad \text{and} \quad \nu_\tau(X) \in \sec(\mathcal{J}_1E, T \otimes V\mathcal{J}_1E),$$
with coordinate expressions
\[ \nu^{-1}_\tau(Y) = c\alpha^0 Y^i_0 \left( \partial_t - c\alpha^0 \tau_i(\partial_0 + x_0^i \partial_\rho) \right) \quad \text{and} \quad \nu_\tau(X) = \frac{1}{c\alpha^0} \tilde{X}^i \partial_i^0, \]
where \( \tilde{X}^i = X^i - x_0^i X^0. \)

3.2.5. **Spacetime connections.** We define a spacetime connection to be a torsion free linear connection \( K : \mathcal{T} \mathcal{E} \rightarrow T^* \mathcal{E} \otimes \mathcal{T}^2 \mathcal{E} \) of the bundle \( \mathcal{T} \mathcal{E} \rightarrow \mathcal{E}. \) Its coordinate expression is of the type
\[ K = d^\lambda \otimes (\partial_\lambda + K^\nu_\mu \dot{i}^\mu \dot{\partial}_\nu), \quad \text{with} \quad K^\nu_\mu = K^\nu_\mu \in \text{map}(\mathcal{E}, \mathbb{R}). \]

We denote by \( K[g] \) the Levi Civita connection, i.e. the torsion free linear spacetime connection such that \( \nabla g = 0. \)

3.2.6. **Phase connections.** We define a phase connection to be a connection of the bundle \( \mathcal{J}_1 \mathcal{E} \rightarrow \mathcal{E}. \)

A phase connection can be represented, equivalently, by a tangent valued form \( \Gamma : \mathcal{J}_1 \mathcal{E} \rightarrow T^* \mathcal{E} \otimes \mathcal{T} \mathcal{E}, \) which is projectable over \( 1 : \mathcal{E} \rightarrow T^* \mathcal{E} \otimes \mathcal{T} \mathcal{E}, \) or by the complementary vertical valued form \( \nu[\Gamma] : \mathcal{J}_1 \mathcal{E} \rightarrow T^* \mathcal{J}_1 \mathcal{E} \otimes V \mathcal{J}_1 \mathcal{E}, \) or by the vector valued form \( \nu_\tau[\Gamma] : \mathcal{J}_1 \mathcal{E} \rightarrow T^* \mathcal{J}_1 \mathcal{E} \otimes (T^* \otimes V \tau \mathcal{E}). \) Their coordinate expressions are
\[ \Gamma = d^\lambda \otimes (\partial_\lambda + \Gamma^i_0 \partial_i^0), \quad \nu[\Gamma] = (d^\lambda_0 - \Gamma^i_0 d^\lambda_0) \otimes \partial_i^0, \]
\[ \nu_\tau[\Gamma] = c\alpha^0 (d^\lambda_0 - \Gamma^i_0 d^\lambda_0) \otimes (\partial_t - c\alpha^0 \tau_i(\partial_0 + x_0^i \partial_\rho)), \quad \text{with} \quad \Gamma^i_0 \in \text{map}(\mathcal{J}_1 \mathcal{E}, \mathbb{R}). \]

We define the curvature of a phase connection \( \Gamma \) to be the vertical valued 2–form
\[ R = R[\Gamma] := -[\Gamma, \Gamma] : \mathcal{J}_1 \mathcal{E} \rightarrow \Lambda^2 T^* \mathcal{E} \otimes V \mathcal{J}_1 \mathcal{E}, \]
where \([,] \) is the Frölicher–Nijenhuis bracket.

We can prove that there is a natural map \( \chi : K \rightarrow \Gamma \) between linear spacetime connections \( K \) and phase connections \( \Gamma, \) with coordinate expression
\[ \Gamma^i_0 = K^i_0 + K^i_0 p x_0^0 - x_0^i (K^0_0 + K^0_0 p x_0^0). \]

3.2.7. **Dynamical phase connection.** The space of 2–jets of motions \( \mathcal{J}_2 \mathcal{E} \) can be naturally regarded as the affine subbundle \( \mathcal{J}_2 \mathcal{E} \subset \mathcal{T} \mathcal{E} \otimes \mathcal{T} \mathcal{J}_1 \mathcal{E}, \) which projects on \( \mathcal{J} : \mathcal{J}_1 \mathcal{E} \rightarrow \mathcal{T} \mathcal{E} \otimes \mathcal{T} \mathcal{E}. \)

A dynamical phase connection is defined to be a 2nd–order connection, i.e. a section \( \gamma : \mathcal{J}_1 \mathcal{E} \rightarrow \mathcal{J}_2 \mathcal{E}, \) or, equivalently, a section \( \gamma : \mathcal{J}_1 \mathcal{E} \rightarrow \mathcal{T} \mathcal{E} \otimes \mathcal{T} \mathcal{J}_1 \mathcal{E}, \) which projects on \( \mathcal{J}. \)

The coordinate expression of a dynamical phase connection is of the type
\[ \gamma = c\alpha^0 (\partial_0 + x_0^i \partial_i + \gamma^i_0 \partial_i^0), \quad \text{with} \quad \gamma^i_0 \in \text{map}(\mathcal{J}_1 \mathcal{E}, \mathbb{R}). \]

If \( \gamma \) is a dynamical phase connection, then we have \( \gamma \mathcal{J} \tau = 1. \)

The contact map \( \mathcal{J} \) and a phase connection \( \Gamma \) yield the section \( \gamma = \gamma[\mathcal{J} \Gamma] =: \mathcal{J} \Gamma : \mathcal{J}_1 \mathcal{E} \rightarrow \mathcal{T} \mathcal{E} \otimes \mathcal{T} \mathcal{J}_1 \mathcal{E}, \) which turns out to be a dynamical phase connection, with coordinate expression
\[ \gamma^i_0 = \Gamma^i_0 + \gamma^{j_0}_0 x_0^j. \]
In particular, a linear spacetime connection \( K \) yields the dynamical phase connection \( \gamma =: \gamma[\alpha, K] =: \alpha \cdot \chi(K) \), with coordinate expression

\[
\gamma^0_0 = K^0_0 + K^0_h x^h_0 + K^i_h x^i_0 + K^i_h K^i_j x^i_0 x^j_0 - x^i_0 (K^0_0 + K^0_h x^h_0 + K^i_0 x^i_0 + K^0_h x^h_0 + K^i_0 x^i_0 x^j_0).
\]

3.2.8. Phase 2–form and 2–vector. The metric \( g \) and a phase connection \( \Gamma \) yield the scaled 2–form \( \Omega \), called (scaled) phase 2–form, and the scaled vertical 2–vector \( \Lambda \), called (scaled) phase 2–vector,

\[
\Omega =: \Omega[g, \Gamma] =: g \cdot (\nu[\Gamma] \land \theta) : \mathcal{J}_1 E \rightarrow (T^* \otimes \Omega^2) \otimes \Lambda^2 T^* \mathcal{J}_1 E,
\]

\[
\Lambda =: \Lambda[g, \Gamma] =: g \cdot (\Gamma \land \nu[\Gamma]) : \mathcal{J}_1 E \rightarrow (T \otimes \Omega^{-2}) \otimes \Lambda^2 T \mathcal{J}_1 E,
\]

with coordinate expressions

\[
\Omega = c \alpha^0 (g^\mu_i + c^2 \tau_i \tau^i) (d^\mu_i - \Gamma^i_0 d^\lambda_i) \land d^\mu_i,
\]

\[
\Lambda = \frac{1}{c \alpha^0} (g^{i\lambda} - x^i_0 g^{\lambda 0}) (\partial_\lambda + \Gamma^i_0 \partial_i) \land \partial_0.
\]

We can easily see that \(-c^2 \tau \land \Omega^3 \neq 0 \) and \(-\frac{1}{c^2} \gamma \land \Lambda^3 \neq 0 \).

There is a unique dynamical phase connection \( \gamma \), such that \( \gamma \cdot \Omega[g, \Gamma] = 0 \). Namely, \( \gamma = \gamma[\alpha, \Gamma] \).

In particular, a metric and time preserving spacetime connection \( K \) yields the (scaled) phase 2–form \( \Omega[g, K] =: \Omega[g, \chi(K)] \) and the (scaled) phase 2–vector \( \Lambda[g, K] =: \Lambda[g, \chi(K)] \) with coordinate expressions

\[
\Omega = -c (g^\mu_i + c^2 \tau_i \tau^i) (d^\mu_i - (K^i_0 + K^{ij} x^j_0 - K^0_0 x^i_0 - K^i_0 x^j_0 x^0_0) d^\lambda_i) \land d^\mu_i,
\]

\[
\Lambda = \frac{1}{c \alpha^0} (g^{i\lambda} - g^{\lambda 0} x^h_0) (\partial_\lambda + (K^i_0 + K^{ij} x^j_0 - K^0_0 x^i_0 - K^i_0 x^j_0 x^0_0) \partial_i) \land \partial_0.
\]

3.2.9. Dynamical structures of the phase space. Let us consider a phase connection \( \Gamma \) and the induced phase objects \( \gamma =: \gamma[\alpha, \Gamma] \), \( \Omega =: \Omega[g, \Gamma] \), and \( \Lambda =: \Lambda[g, \Gamma] \).

We define the Lie derivatives

\[
L_\Gamma \tau = (i_\Gamma d - di_\Gamma) \tau \quad \text{and} \quad L_R \tau = (i_R d + di_R) \tau.
\]

Then, the following results holds \cite{7}.

3.3. Theorem. The following assertions are equivalent.

1. \( L_{\nu[\chi]} L_\Gamma \tau = 0 \), \( \forall X \in \text{sec}(E, TE) \), and \( L_R \tau = 0 \).

2. \( d\Omega = 0 \), i.e. \( (-c^2 \tau, \Omega) \) is a (scaled) almost–cosymplectic–contact pair.

3. \( [-\frac{1}{c^2}, \gamma, \Lambda] = \frac{1}{c^2} \gamma \land (\Lambda^2 \land \Lambda^2) (d\tau) \) and \( [\Lambda, \Lambda] = 2 \gamma \land (\Lambda^2 \land \Lambda^2) (d\tau) \), i.e. \( (-\frac{1}{c^2}, \gamma, \Lambda, -c^2 \tau) \) is a (scaled regular) almost–coPoisson–Jacobi 3–plet.

Moreover, the almost–cosymplectic–contact pair \( (-c^2, \Omega) \) and the (regular) almost–coPoisson–Jacobi 3–plet \( (-\frac{1}{c^2}, \gamma, \Lambda, -c^2 \tau) \) are mutually dual. \hfill \Box

3.4. Lemma. We have

\[
\Omega - c^2 L_\Gamma \tau = -c^2 d\tau.
\]
3.5. **Theorem.** The following assertions are equivalent.

1. \( L_T \tau = 0 \).
2. \( \Omega = -c^2 d\tau \), i.e. \((-c^2 \tau, \Omega)\) is a (scaled) contact pair.
3. \( [-\frac{1}{c^2} \gamma, \Lambda] = 0 \) and \( [\Lambda, \Lambda] = \frac{2}{c^2} \gamma \wedge \Lambda \), i.e. \((-\frac{1}{c^2} \gamma, \Lambda)\) is a (scaled regular) Jacobi pair.

Moreover, the contact pair \((-c^2 \tau, \Omega)\) and the (regular) Jacobi pair \((-\frac{1}{c^2} \gamma, \Lambda)\) are mutually dual.  

Next, let us consider a linear spacetime connection \( K \) and the induced phase objects \( \Gamma = : \chi(K) \), \( \gamma = : \gamma[d, \Gamma] \), \( \Omega = : \Omega[g, \Gamma] \), and \( \Lambda = : \Lambda[g, \Gamma] \).

3.6. **Theorem.** The following assertions are equivalent.

1. \( L_{\chi(K)} \tau = 0 \).
2. \( g(Z, Z) \left( (\nabla_X g)(Y, Z) - (\nabla_Y g)(X, Z) + g(T(X, Y), Z) \right) \)
   \[ + \frac{1}{2} g(Z, X)(\nabla_Y g)(Z, Z) - \frac{1}{2} g(Z, Y)(\nabla_X g)(Z, Z) = 0, \]
   for each \( X, Y, Z \in \text{sec}(E, T E) \), where \( T \) is the torsion of \( K \).
3. \( \Omega = -c^2 d\tau \), i.e. \((-c^2 \tau, \Omega)\) is a (scaled) contact pair.
4. \( [-\frac{1}{c^2} \gamma, \Lambda] = 0 \) and \( [\Lambda, \Lambda] = \frac{2}{c^2} \gamma \wedge \Lambda \), i.e. \((-\frac{1}{c^2} \gamma, \Lambda)\) is a (scaled regular) Jacobi pair.

Moreover, if the above conditions are fulfilled, then the contact pair \((-c^2 \tau, \Omega)\) and the (regular) Jacobi pair \((-\frac{1}{c^2} \gamma, \Lambda)\) are mutually dual.  

3.7. **Corollary.** Let \( K \) be a torsion free spacetime connection. If \( \nabla g \) and \( g \otimes \nabla g \) are symmetric (0,3) and (0,5) tensor fields, respectively, then \((-c^2 \tau, \Omega)\) and \((-\frac{1}{c^2} \gamma, \Lambda)\) are mutually dual contact and Jacobi pairs, respectively.

3.8. **Theorem.** We have:

1. \( \Omega = -c^2 d\tau \), i.e. \((-c^2 \tau, \Omega)\) is a (scaled) contact pair.
2. \( [-\frac{1}{c^2} \gamma, \Lambda] = 0 \) and \( [\Lambda, \Lambda] = \frac{2}{c^2} \gamma \wedge \Lambda \), i.e. \((-\frac{1}{c^2} \gamma, \Lambda)\) is a (scaled regular) Jacobi pair.

Moreover, the contact pair \((-c^2 \tau, \Omega)\) and the (regular) Jacobi pair \((-\frac{1}{c^2} \gamma, \Lambda)\) are mutually dual.  

3.9. **Remark.** If \( K \) is a spacetime connection, then the induced pairs \((-c^2 \tau, \Omega)\) and \((-\frac{1}{c^2} \gamma, \Lambda)\) are scaled.

On the other hand, some results of the general theory of geometrical structures developed in the first two sections requires unscaled pairs.
Indeed, if we refer to a particle of mass $m \in \mathbb{M}$ and consider the universal scales $\hbar \in \mathbb{T}^{-1} \otimes \mathbb{L}^2 \otimes \mathbb{M}$ and $c \in \mathbb{T}^{-1} \otimes \mathbb{L}$, then we obtain unscaled pairs in the following natural way.

We have the unscaled spacetime 1–form

$$-\frac{m c^2}{\hbar} \tau : E \to T^*E.$$  

Moreover, the rescaled contact map $\mathcal{J} = \frac{h}{m c^2} \mathcal{J} : \mathcal{J}_1 E \to T \mathcal{J}_1 E$ yields the unscaled phase vector field

$$-\gamma[\mathcal{J}, K] = -\frac{h}{m c^2} \gamma[\mathcal{J}, K] : E \to T \mathcal{J}_1 E.$$  

Furthermore, the rescaled metric $G = \frac{m}{\hbar} g : E \to T \otimes T^*E \otimes T^*E$ yields the unscaled phase 2–form and phase 2–vector

$$\Omega \equiv \Omega[G, K] = \frac{m}{\hbar} \Omega[g, K] : \mathcal{J}_1 E \to \Lambda^2 T^* \mathcal{J}_1 E,$$

$$\Lambda \equiv \Lambda[G, K] = \frac{h}{m} \Lambda[g, K] : \mathcal{J}_1 E \to \Lambda^2 T \mathcal{J}_1 E.$$  

Thus, if $K$ is the Levi Civita spacetime connection, then $\left(-\frac{m c^2}{\hbar} \tau, \frac{m}{\hbar} \Omega\right)$ and $\left(-\frac{h}{m c^2} \gamma, \frac{h}{m} \Lambda\right)$ turn out to be mutually dual unscaled contact and Jacobi pairs of the phase space.

Indeed, the Plank constant does not play any direct role in classical mechanics; nevertheless, such a scale is necessary for getting unscaled objects as above. □

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