Strong Ill-Posedness in $L^\infty$ for the Riesz Transform Problem

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Abstract

We prove strong ill-posedness in $L^\infty$ for linear perturbations of the 2d Euler equations of the form:

$$\partial_t \omega + u \cdot \nabla \omega = R(\omega),$$

where $R$ is any non-trivial second order Riesz transform. Namely, we prove that there exist smooth solutions that are initially small in $L^\infty$ but become arbitrarily large in short time. Previous works in this direction relied on the strong ill-posedness of the linear problem, viewing the transport term perturbatively, which only led to mild growth. In this work we derive a nonlinear model taking all of the leading order effects into account to determine the precise pointwise growth of solutions for short time. Interestingly, the Euler transport term does counteract the linear growth so that the full nonlinear equation grows an order of magnitude less than the linear one. In particular, the (sharp) growth rate we establish is consistent with the global regularity of smooth solutions.

1 Introduction

The Euler equations for incompressible flow are a fundamental model in fluid dynamics that describe the motion of ideal fluids:

$$\partial_t u + u \cdot \nabla u + \nabla p = 0$$
$$\nabla \cdot u = 0.$$ (1.1)

In this equation, $u$ is the velocity field and $p$ is the pressure of an ideal fluid flowing in $\mathbb{R}^2$. A key difficulty in understanding the dynamics of 2d Euler flows is the non-locality of the system due to the presence of the pressure term.

Defining the vorticity $\omega := \nabla \perp \cdot u$, it is insightful to study the Euler equations in vorticity form:

$$\partial_t \omega + u \cdot \nabla \omega = 0,$$
$$\nabla \cdot u = 0$$
$$u = \nabla \perp \Delta^{-1} \omega.$$ (1.2)

Because the $L^\infty$ norm of vorticity is conserved in the Euler equations in two dimensions, Yudovich proved that there is a unique global-in-time solution to the Euler equation corresponding to every initial bounded and decaying vorticity. See also ([23], [1], [17], [25], [18], [22], [21]). This bound on the $L^\infty$ norm is unfortunately unstable even to very mild perturbations of the equation [3, 10, 13]. To understand this phenomenon, we are interested in studying linear perturbations of the Euler equations in two dimensions as follows:

$$\partial_t u + u \cdot \nabla u + \nabla p = \begin{pmatrix} 0 \\ u_1 \end{pmatrix}$$
$$\nabla \cdot u = 0.$$ (1.3)

(1.3) is a model for many problems in fluids dynamics that have a coupling with the Euler equations. For instance, similar types of equations appear in viscoelastic fluids see [3, 7, 14, 20].
and in magnetohydrodynamics see [2, 4, 16, 24]. Further, they also appear when studying stochastic Euler equation, see [15].

Writing (1.3) in vorticity form, we get

\[
\begin{align*}
\partial_t \omega + u \cdot \nabla \omega &= \partial_x u_1 \\
\nabla \cdot u &= 0 \\
u &= \nabla^\bot \Delta^{-1} \omega,
\end{align*}
\]  

where \( u = \nabla^\bot \Delta^{-1} \omega \). 

we observe that the challenge of studying these equations is that the right hand side of (1.4) can be written as the Riesz transform of vorticity \( \partial_x u_1 = R(\omega) \), which is unbounded on \( L^\infty \). P. Constantin and V. Vicol considered these equations with weak dissipation in [6], and they proved global well-posedness. However, without dissipation it is an open question whether these equations are globally well-posed. In this work, we are interested in the question of \( L^\infty \) ill/well-posedness of the Euler equations with Riesz forcing and the local rate of \( L^\infty \) growth. The first author and N. Masmoudi studied the Euler equations with Riesz forcing in [13], where they proved that it is mildly ill-posed. This means that there is a universal constant \( c > 0 \) such that for all \( \epsilon > 0 \), there is \( \omega_0 \in C^\infty \) for which the unique local solution to (1.4) satisfies:

\[
|\omega_0|_{L^\infty} \leq \epsilon, \quad \text{but} \quad \sup_{t \in [0, \epsilon]} |\omega(t)|_{L^\infty} \geq c
\]  

The authors in [13] conjectured that the Euler equations with Riesz forcing is actually strongly ill-posed in \( L^\infty \). Namely, that we can take \( c \) in (1.5) to be arbitrarily large. The goal of our work here is to show that indeed this is possible. To show this, we use the first author’s Biot-Savart law decomposition [9] to derive a leading order system for the Euler equations with Riesz forcing. We then show that the leading order system is strongly ill-posed in \( L^\infty \). Using this, we can show that the Euler equations with Riesz forcing is strongly ill-posed by estimating the error between the leading order system and the Euler with Riesz forcing system on a specific time interval.

We should remark that the main application of the approach of the first author and N. Masmoudi in [13] was to prove ill-posedness of the Euler equation in the integer \( C^k \) spaces, which was also proved independently by J. Bourgain and D. Li in [3]. Regarding the notion of mild ill-posedness in \( L^\infty \) for models related to the Euler with Riesz forcing system, see the work of J. Wu and J. Zhao in [24] about the 2D resistive MHD equations.

1.1 Statement of the main result

**Theorem 1.** For any \( \alpha, \delta > 0 \), there exists an initial data \( \omega_0^{\alpha, \delta} \in C^\infty(\mathbb{R}^2) \) and \( T(\alpha) \) such that the corresponding unique global solution, \( \omega^{\alpha, \delta} \), to (1.4) is such that at \( t = 0 \) we have

\[
|\omega_0^{\alpha, \delta}|_{L^\infty} = \delta,
\]

but for any \( 0 < t \leq T(\alpha) \) we have

\[
|\omega^{\alpha, \delta}(t)|_{L^\infty} \geq |\omega_0|_{L^\infty} + c \log(1 + \frac{c}{\alpha} t),
\]

where \( T(\alpha) = c|\log(\alpha)| \) and \( c > 0 \) is a constant independent of \( \alpha \).

**Remark 1.1.** Note that at time \( t = T(\alpha) \), we have that

\[
|\omega^{\alpha, \delta}|_{L^\infty} \geq c \log(c |\log \alpha|),
\]

which can be made arbitrarily large as \( \alpha \to 0 \). Fixing \( \delta > 0 \) small and then taking \( \alpha \) sufficiently small thus gives strong ill-posedness for (1.4) in \( L^\infty \).

**Remark 1.2.** As we will discuss below, we in fact establish upper and lower bounds on the solutions we construct so that on the same time-interval we have:

\[
|\omega^{\alpha, \delta}(t)|_{L^\infty} \approx |\omega_0|_{L^\infty} + c \log(1 + \frac{c}{\alpha} t).
\]
This should be contrasted with the linear problem where the upper and lower bounds for the same data come without the log:

\[ |\omega_{\text{linear}}^{\alpha,\delta}(t)|_{L^\infty} \approx |\omega_0|_{L^\infty} + c(1 + \frac{t}{\alpha}). \]

**Remark 1.3.** Our ill-posedness result applies to the equation:

\[ \partial_t \omega + u \cdot \nabla \omega = R(\omega), \]

where \( R = R_{12} = \partial_{12} \Delta^{-1} \). Note that a direct consequence of the result gives strong ill-posedness when \( R = R_{11} \) or \( R = R_{22} \) even though these are dissipative on \( L^2 \). This can be seen just by noting that a linear change of coordinates can transform \( R_{12} \) to a constant multiple of \( R_{11} - R_{22} = R_{11} - \text{Id} \).

The strong ill-posedness for the Euler equation with forcing by any second order Riesz transform (other than the identity) follows. We further remark that the same strategy can be used to study the case of general Riesz transforms though we do not undertake this here since the case of forcing by second order Riesz transforms is the most relevant for applications we are aware of (such as the 3d Euler equations, the Boussinesq system, visco-elastic models, MHD, etc.).

### 1.2 Comparison with the linear equation and the effect of transport

We now move to compare the result of this paper with the corresponding linear results and emphasize the regularizing effect of the non-linearity in this problem. The ill-posedness result of [13] relies on viewing (1.4) as a perturbation of

\[ \partial_t f = R(f). \]  

For this simple linear equation, it is easy to show that \( L^\infty \) data can immediately develop a logarithmic singularity. Let us mention two ways to quantify this logarithmic singularity. One way is to study the growth of \( L^p \) norms as \( p \to \infty \). For the linear equation (1.6), it is easy to show that the upper bound:

\[ |f(t)|_{L^p} \leq \exp(Ct)p|f_0|_{L^p} \]

is sharp in the sense that we can find localized \( L^\infty \) data for which the solution satisfies

\[ |f(t)|_{L^p} \geq c(t) \cdot p. \]

This can be viewed as approximating \( L^\infty \) “from below.” Similarly, the \( C^\alpha \) bound for (1.6),

\[ |f(t)|_{C^\alpha} \leq \frac{\exp(\alpha C t)}{\alpha}|f_0|_{C^\alpha} \]

can also be shown to be sharp for short time in that we can find for each \( \alpha > 0 \) smooth and localized data with \( |f_0|_{C^\alpha} = 1 \) for which

\[ |f(t)|_{L^\infty} \geq \frac{c(t)}{\alpha}. \]

The main result of [13] was that these upper and lower bounds remain unchanged in the presence of a transport term by a Lipschitz continuous velocity field. This is not directly applicable to our setting since the coupling between \( \omega \) and \( u \) is such that \( u \) may not be Lipschitz even if \( \omega \) is bounded. Interestingly, in [10], it was shown that this growth could be significantly stronger in the presence of a merely bounded velocity field.

All of the above discussion leads us to understand that the nature of the well/ill-posedness of (1.4) will depend on the precise relationship between the velocity field and the linear forcing term in (1.4). In particular, for a natural class of data, we construct solutions to (1.4) satisfying

\[ |\omega|_{L^\infty} \approx 1 + \log(1 + \frac{t}{\alpha}). \]
for short time, which is the best growth rate possible in this setting. This should be contrasted with the corresponding growth rate for the linear problem

$$|\omega_{\text{lin}}|_{L^\infty} \approx 1 + \frac{t}{\alpha}.$$  

In particular, the nonlinear term in (1.4) actually tries to prevent $L^\infty$ growth. Let us finally remark that the weak growth rate we found is consistent with the vorticity trying to develop a log log singularity. It is curious that, in the Euler equation, vorticity with nearly log log data are perfectly well-behaved and consistent with global regularity but with a triple exponential upper bound on gradients. Though establishing the global regularity rigorously remains a major open problem, this appears to be a sign that perhaps smooth solutions to (1.3) are globally regular.

### 1.3 A short discussion of the proof

The first step of the proof is to use the Biot-Savart law decomposition by the first author [9] to derive a leading order model:

$$\partial_t \Omega + \frac{1}{2\alpha}(L_s(\Omega) \sin(2\theta) + L_c(\Omega) \cos(2\theta)) \partial_\theta \Omega = \frac{1}{2\alpha} L_s(\Omega),$$

where the operators $L_s$ and $L_c$ are bounded linear operators on $L^2$ defined by

$$L_s(f)(R) = \frac{1}{\pi} \int_R^\infty \int_0^{2\pi} \frac{f(s,\theta)}{s} \sin(2\theta) \, d\theta \, ds$$

and

$$L_c(f)(R) = \frac{1}{\pi} \int_R^\infty \int_0^{2\pi} \frac{f(s,\theta)}{s} \cos(2\theta) \, d\theta \, ds.$$

Essentially all we do here is replace the velocity field by its most singular part. Upon inspecting this model, we observe that the forcing term on the right hand side is purely radial while the direction of transport is angular. Upon choosing a suitable unknown, we thus reduce the problem to solving a transport equation for some unknown $f$:

$$\partial_t f + \frac{1}{2\alpha} L_s(f) \sin(2\theta) \partial_\theta f = 0.$$

Surprisingly, this reduced equation propagates the usual “odd-odd” symmetry even though the original system does not. The leading order model will then be strongly ill-posed if we can ensure that the solution of this transport equation satisfies that \(\int_0^t L_s(f)\) can be arbitrarily large. One subtlety is that the growth of $L_s(f)$ enhances the transport effect, which in turn depletes the growth of $L_s(f)$. In fact, were the transport term to be stronger even by a log, the problem would not be strongly ill-posed. By a careful study of the characteristics of this equation, we obtain a closed non-linear integro-differential equation governing the evolution of $L_s(f)$ (see equation (3.4)). We study this non-linear integro-differential equation and establish upper and lower bounds on $L_s(f)$ proving strong ill-posedness for the leading order equation; see section 3 for more details. Finally, we close the argument by estimating the error incurred by approximating the dynamics with the leading order model. An important idea here is to work on a time scale long enough to see the growth from the leading order model but short enough to suppress any potential stronger non-linear growth; see section 6 for more details.

### 1.4 Organization

This paper is organized as follows: In section 2 we derive a leading order model for the Euler equations with Riesz forcing (1.3) based on the first author’s Biot-Savart law approximation [9]. Then, in section 3 we obtain a pointwise estimate on the leading order model which is the main ingredient in obtaining the strong ill-posedness result for the Euler with Riesz forcing system. In addition, in section 3 we also obtain some estimates on the leading order model in suitable norms which will be then used in estimating the reminder term in section 6. After that, in section 4 we will recall the first author’s Biot-Savart law decomposition obtained in [9], and we will include a short sketch of the proof. In section 5 we will obtain some embedding estimates which will also be used in section 6 for the reminder term estimates. Then, in section 6 we show that the reminder term remains small which will then allow us to prove the main result in section 7.
1.5 Notation

In this paper, we will be working in a form polar coordinates introduced in $\mathbb{R}^2$. Let $r$ be the radial variable:

$$r = \sqrt{x^2 + y^2}$$

and since we will be working with functions of the variable $r^\alpha$, where $0 < \alpha < 1$, we will use $R$ to denote it:

$$R = r^\alpha$$

We will use $\theta$ to denote the angle variable:

$$\theta = \arctan \frac{y}{x}$$

We will use $|f|_{L^\infty}$ and $|f|_{L^2}$ to denote the usual $L^\infty$ and $L^2$ norms, respectively. In addition, we will use $f_t$ or $f_r$ to denote the time variable. Further, in this paper, following [9], we will be working on $(R, \theta) \in [0, \infty) \times [0, \frac{\pi}{2}]$ where the $L^2$ norm will be with measure $dR d\theta$ and not $R dR d\theta$.

We define the weighted $H^k([0, \infty) \times [0, \frac{\pi}{2}])$ norm as follows:

$$|f|_{H^k} = \sum_{m=0}^{k} \sum_{i=0}^{m} \left| \partial_R^i \partial_\theta^{m-i} f \right|_{L^2} + \sum_{m=0}^{k} \sum_{i=1}^{m} \left| R^i \partial_R^i \partial_\theta^{m-i} f \right|_{L^2} + \sum_{m=0}^{k} \sum_{i=0}^{m} \left| \partial_R^i \partial_\theta^{m-i} f \right|_{L^\infty}$$

We also define $W^k,\infty$ norm as follows:

$$|f|_{W^k,\infty} = \sum_{m=0}^{k} \sum_{i=0}^{m} \left| \partial_R^i \partial_\theta^{m-i} f \right|_{L^\infty} + \sum_{m=0}^{k} \sum_{i=1}^{m} \left| R^i \partial_R^i \partial_\theta^{m-i} f \right|_{L^\infty}$$

Throughout this paper, we will use the following notation to define the following operators:

$$L(f)(R) = \int_{R}^{\infty} \frac{f(s)}{s} ds$$

and by adding a subscript $L_s$ or $L_c$, we denote the project onto $\sin(2\theta)$ and $\cos(2\theta)$ respectively. Namely,

$$L_s(f)(R) = \frac{1}{\pi} \int_{R}^{\infty} \int_{0}^{2\pi} \frac{f(s, \theta)}{s} \sin(2\theta) d\theta ds \quad \text{and} \quad L_c(f)(R) = \frac{1}{\pi} \int_{R}^{\infty} \int_{0}^{2\pi} \frac{f(s, \theta)}{s} \cos(2\theta) d\theta ds$$

2 Leading Order Model

In this section, we will derive a leading order model for the Euler equation with Riesz forcing:

$$\begin{align*}
\partial_t \omega + u \cdot \nabla \omega &= \partial_s u_1 \\
\nabla \cdot u &= 0 \\
u &= \nabla^\perp \Delta^{-1} \omega
\end{align*}$$

(2.1)

To do this, we follow [9] and we write the equation in a form of polar coordinates. Namely, we set $r = \sqrt{x^2 + y^2}$, $R = r^\alpha$, and $\theta = \arctan \frac{y}{x}$. We will the rewrite the equation (2.1) in the new functions $\omega(x, y) = \Omega(R, \theta)$ and $\psi(x, y) = r^2 \Psi(R, \theta)$ with $u = \nabla^\perp \psi$, where $u_1 = -\partial_\theta \psi$, and $u_2 = \partial_s \psi$.

Equations of $u$ in terms of $\Psi$

$$u_1 = -r(2 \sin(\theta) \Psi + \alpha \sin(\theta) R \partial_R \Psi + \cos(\theta) \partial_\theta \Psi)$$
\[ u_2 = r (2 \cos(\theta) \Psi + \alpha \cos(\theta) R \partial R \Psi - \sin(\theta) \partial \theta \Psi) \]

**Evolution Equation for** \( \Omega \)

\[
\partial t \Omega + \left( - \alpha R \partial \theta \Psi \right) \partial R \Omega + \left( 2 \Psi + \alpha R \partial R \Psi \right) \partial \theta \Omega = \left( - 2 \alpha R \sin(\theta) \cos(\theta) - \alpha^2 R \sin(\theta) \cos(\theta) \right) \partial R \Psi
+ \left( - 1 + 2 \sin^2(\theta) \right) \partial \theta \Psi + \left( - \alpha R \cos^2(\theta) + \alpha R \sin^2(\theta) \right) \partial R \theta \Psi
- \left( \alpha^2 R^2 \sin(\theta) \cos(\theta) \right) \partial R R \Psi + \left( \sin(\theta) \cos(\theta) \right) \partial \theta \theta \Psi
\]

The elliptic equation for \( \Delta (r^2 \Psi(R, \theta)) = \Omega(R, \theta) \)

\[
4 \Psi + \alpha^2 R^2 \partial R R \Psi + \partial \theta \theta \Psi + (4 \alpha + \alpha^2) R \partial R \Psi = \Omega(R, \theta)
\]

Now using the first author’s Biot-Savart decomposition [9], see section [11] for more details, by defining the operators

\[
L_s(\Omega)(R) = \frac{1}{\pi} \int_R^\infty \int_0^{2\pi} \frac{\Omega(s, \theta)}{s} \sin(2\theta) \, d\theta \, ds \quad \text{and} \quad L_c(\Omega)(R) = \frac{1}{\pi} \int_R^\infty \int_0^{2\pi} \frac{\Omega(s, \theta)}{s} \cos(2\theta) \, d\theta \, ds
\]

we have

\[
\Psi(R, \theta) = - \frac{1}{4\alpha} L_s(\Omega) \sin(2\theta) - \frac{1}{4\alpha} L_c(\Omega) \cos(2\theta) + \text{lower order terms}
\]

Thus, if we ignore the \( \alpha \) terms in the evolution equation, we obtain

\[
\partial t \Omega + \left( 2 \Psi \right) \partial t \Omega = \left( - 1 + 2 \sin^2(\theta) \right) \partial \theta \Psi + \left( \sin(\theta) \cos(\theta) \right) \partial \theta \theta \Psi
\]

(2.2)

Now we consider \( \Psi \) of the form

\[
\Psi = - \frac{1}{4\alpha} L_s(\Omega) \sin(2\theta) - \frac{1}{4\alpha} L_c(\Omega) \cos(2\theta)
\]

and plug it into the evolution equation, we have

\[
\partial t \Omega = \left( \frac{1}{2\alpha} L_s(\Omega) \sin(2\theta) + \frac{1}{2\alpha} L_c(\Omega) \cos(2\theta) \right) \partial \theta \Omega = - \left( \cos(2\theta) \right) \left( - \frac{1}{2\alpha} L_s(\Omega) \cos(2\theta) + \frac{1}{2\alpha} L_c(\Omega) \sin(2\theta) \right)
+ \left( \frac{1}{2} \sin(2\theta) \right) \left( \frac{1}{\alpha} L_s(\Omega) \sin(2\theta) + \frac{1}{\alpha} L_c(\Omega) \cos(2\theta) \right)
\]

which simplifies to

\[
\partial t \Omega = \left( \frac{1}{2\alpha} L_s(\Omega) \sin(2\theta) + \frac{1}{2\alpha} L_c(\Omega) \cos(2\theta) \right) \partial \theta \Omega = \frac{1}{2\alpha} L_s(\Omega)
\]

In order to work with positive solutions and have the angular trajectories moving to the right, we make the change \( \Omega \rightarrow -\Omega \) and get the final model:

\[
\partial t \Omega + \left( \frac{1}{2\alpha} L_s(\Omega) \sin(2\theta) + \frac{1}{2\alpha} L_c(\Omega) \cos(2\theta) \right) \partial \theta \Omega = \frac{1}{2\alpha} L_s(\Omega).
\]

(2.3)

We now move to study the dynamics of solutions to (2.3).

**Proposition 2.1.** Let \( \Omega \) be a solution to the leading order model

\[
\partial t \Omega + \left( \frac{1}{2\alpha} L_s(\Omega) \sin(2\theta) + \frac{1}{2\alpha} L_c(\Omega) \cos(2\theta) \right) \partial \theta \Omega = \frac{1}{2\alpha} L_s(\Omega)
\]

(2.4)

with initial data of the form \( \Omega_{|t=0} = f_0(R) \sin(2\theta) \) then we can write \( \Omega \) as follow:

\[
\Omega = f + \frac{1}{2\alpha} \int_0^t L_s(f_r) \, d\tau
\]

(2.5)

where \( f \) satisfies the following transport equation:

\[
\partial_t f + \frac{1}{2\alpha} \sin(2\theta) L_s(f) \partial \theta f = 0
\]

(2.6)
Proof. The righthand side term of (2.4) is radial, and hence if we take the inner product with \( \sin(2\theta) \) it will be zero. Now if we write \( \Omega \) as:

\[
\Omega_t(R, \theta) = f_t(R, \theta) + \frac{1}{2\alpha} \int_0^t L_s(\Omega_{\tau})(R) d\tau
\]

and consider it to be a solution to (2.4), we obtain that \( f \) satisfies the following:

\[
\partial_t f_t + \left( \frac{1}{2\alpha} L_s(f_t) \sin(2\theta) + \frac{1}{2\alpha} L_s(f_t) \cos(2\theta) \right) \partial_\theta f_t = 0 \tag{2.7}
\]

Here we used that \( L_s(\Omega_{\tau})(R) \) is a radial function. Notice that (2.7) is a transport equation that preserves odd symmetry. Now if we set:

\[
f_t^s = \int_0^{2\pi} f_t(R, \theta) \sin(2\theta) d\theta \quad \text{and} \quad \Omega_t^s = \int_0^{2\pi} \Omega_t(R, \theta) \sin(2\theta) d\theta,
\]

we notice that \( f_t^s \) and \( \Omega_t^s \) will satisfy the same equation. Thus, if we start with the same initial conditions \( f_0 = \Omega_0 \), then

\[
f_t^s = \Omega_t^s \quad \text{for all } t
\]

Thus, we have \( L_s(\Omega_t) = L_s(f_t) \), and hence

\[
\Omega_t = f_t + \frac{1}{2\alpha} \int_0^t L_s(f_{\tau}) d\tau
\]

Now since the initial data which we are considering have odd symmetry, it suffices to consider the following transport equation:

\[
\partial_t f_t + \frac{1}{2\alpha} \sin(2\theta) L_s(f_t) \partial_\theta f_t = 0 \tag{2.8}
\]

\[\Box\]

3 Leading Order Model Estimate

The purpose of this section is to obtain \( L^\infty \) estimates for the leading order model which is the main ingredient in obtaining the ill-posedness result for the Euler with Riesz forcing system. This will be done in subsection 3.1 in three steps: Lemma 3.1, Lemma 3.2, and Proposition 3.3. Then in subsection 3.2, we will obtain some estimate for the leading order model which will be useful in reminder estimates in section 6.

3.1 Pointwise Leading Order Model Estimate

Lemma 3.1. Let \( f \) be a solution to the following transport equation:

\[
\partial_t f + \frac{1}{2\alpha} \sin(2\theta) L_s(f) \partial_\theta f = 0 \tag{3.1}
\]

with initial data \( f_{|t=0} = f_0(R) \sin(2\theta) \), then we have the following estimate on the operator \( L_s(f) \):

\[
c_1 \int_R^\infty \frac{f_0(s)}{s} \exp\left( -\frac{1}{\alpha} \int_0^t L_s(f_{\tau})(s) d\tau \right) ds \leq L_s(f_t)(R) \leq c_2 \int_R^\infty \frac{f_0(s)}{s} \exp\left( -\frac{1}{\alpha} \int_0^t L_s(f_{\tau})(s) d\tau \right) ds \tag{3.2}
\]

where \( c_1 \) and \( c_2 \) are independent of \( \alpha \)

Proof. To prove this, we consider the following variable change. For \( \theta \in [0, \frac{\pi}{2}) \), let \( \gamma \) be defined as follows

\[
\gamma := \tan(\theta) \implies \frac{d\gamma}{d\theta} = \sec^2(\theta), \quad \text{and} \quad \sin(2\theta) = \frac{2\gamma}{1 + \gamma^2}
\]

Applying chain rule, we rewrite \ref{2.8} in the \((R, \gamma)\) variables:
\[ \partial_t f_t + \frac{1}{\alpha} \gamma L_\alpha(f_t)(R) \partial_t f = 0 \]  

(3.3)

with initial date

\[ f|_{t=0} = f_0(R) \sin(2\theta) = f_0(R) \frac{2\gamma}{1 + \gamma^2} \]

Let \( \phi_t(\gamma) \) be the flow map associated with (3.3), so we have

\[ \frac{d\phi_t(\gamma)}{dt} = \frac{1}{\alpha} \phi_t(\gamma) L_\alpha(f_t) \implies \phi_t(\gamma) = \gamma \exp\left( \frac{1}{\alpha} \int_0^t L_\alpha(f_r) \, dr \right) \]

Thus,

\[ \phi_t^{-1}(\gamma) = \gamma \exp\left( -\frac{1}{\alpha} \int_0^t L_\alpha(f_r) \, dr \right) \]

Hence, we now write the solution to (3.3) as follows:

\[ f_t(R, \gamma) = f_0(R, \phi_t^{-1}(\gamma)) = f_0(R) \frac{2\phi_t^{-1}(\gamma)}{1 + \phi_t^{-1}(\gamma)^2} = f_0(R) \frac{2\gamma}{1 + \gamma^2} \exp\left( -\frac{1}{\alpha} \int_0^t L_\alpha(f_r) \, dr \right) \]

Now we consider the operator \( L_\alpha \) in the \((R, \gamma) \in [0, \infty) \times [0, \frac{\pi}{2}]\) variables:

\[ L_\alpha(f_t)(R) = \frac{1}{\pi} \int_R^\infty \frac{1}{s} \int_s^\infty f_t(s, \gamma) \frac{2\gamma}{(1 + \gamma^2)^2} \, d\gamma \, ds \]

Plugging the expression for \( f_t \), we have

\[ L_\alpha(f_t)(R) = \frac{1}{\pi} \int_R^\infty \frac{1}{s} \int_s^\infty f_0(s) \frac{\exp\left( -\frac{1}{\alpha} \int_0^s L_\alpha(f_r)(s) \, dr \right)}{1 + \gamma^2} \exp\left( -\frac{2}{\alpha} \int_0^s L_\alpha(f_r)(s) \, dr \right) \frac{4\gamma^2}{(1 + \gamma^2)^2} \, d\gamma \, ds \]  

(3.4)

Now since \( 0 \leq \exp\left( -\frac{2}{\alpha} \int_0^s L_\alpha(f_r)(s) \, dr \right) \leq 1 \), we have a upper and lower bound on the operator on \( L_\alpha(f_t)(R) \) with constants \( c_1, c_2 \) independent of \( \alpha \) (In fact, these constants can be explicitly computed). Namely,

\[ c_1 \int_R^\infty \frac{f_0(s)}{s} \exp\left( -\frac{1}{\alpha} \int_0^s L_\alpha(f_r)(s) \, dr \right) ds \leq L_\alpha(f_t)(R) \leq c_2 \int_R^\infty \frac{f_0(s)}{s} \exp\left( -\frac{1}{\alpha} \int_0^s L_\alpha(f_r)(s) \, dr \right) ds \]

Thus, we have our desired inequalities.

\[ \square \]

**Lemma 3.2.** Define the operator

\[ \hat{L}(f_t)(R) := \int_R^\infty \frac{f_0(s)}{s} \exp\left( -\frac{1}{\alpha} \int_0^t \hat{L}(f_r)(s) \, ds \right) \, ds \]  

(3.5)

Then we have

\[ \int_0^t \hat{L}(f_r)(R) \, dr = 2\alpha \log(1 + \frac{t}{2\alpha} L(f_0)(R)) \]

where \( L(f_0)(R) = \int_R^\infty \frac{f_0(s)}{s} \, ds \)

**Proof.** We introduce \( g_t(R) := \exp\left( -\frac{1}{\alpha} \int_0^t \hat{L}(f_r)(R) \, dr \right) \) and \( K(R) := \frac{f_0(R)}{R} \), then the operator \( \hat{L} \) can be rewritten as:

\[ \hat{L}(f_t)(R) = \int_R^\infty K(s)g_t(s) \, ds \]  

(3.6)

Now taking time derivative of (3.6), and using that \( \partial_t g_t(R) = -2g_t(R) \int_R^\infty K(s)g_t(s) \, ds \), we can obtain:

\[ \partial_t \hat{L}(f_t) = -\frac{1}{2\alpha} (\hat{L}(f_t))^2 \]
which can be solved explicitly:

\[ \hat{L}(f_t)(R) = \frac{L(f_0)(R)}{1 + \frac{t}{2\alpha} L(f_0)(R)} \]  \hfill (3.7)

and then it follows that

\[ \int_0^t \hat{L}(f_t)(R) d\tau = 2\alpha \log(1 + \frac{t}{2\alpha} L(f_0)(R)) \]

\[ \square \]

**Proposition 3.3.** Let \( f \) be a solution to the following transport equation:

\[ \partial_t f + \frac{1}{2\alpha} \sin(2\theta)L_s(f) \partial_\theta f = 0 \]  \hfill (3.8)

with initial data \( f|_{t=0} = f_0(R) \sin(2\theta) \), then we have the following estimate on the operator \( L_s(f) \):

\[ \frac{2\alpha}{c_1} \log(1 + \frac{c_1}{2\alpha} t L(f_0)(R))) \geq \int_0^t L_s(f_t)(R) \geq \frac{2\alpha}{c_2} \log(1 + \frac{c_2}{2\alpha} t L(f_0)(R)) \]  \hfill (3.9)

where \( c_1 \) and \( c_2 \) are independent of \( \alpha \)

**Proof.** In the section, we will use the bounds in (3.2), Namely

\[ c_1 \int_R \frac{f_0(s)}{s} \exp(-\frac{1}{\alpha} \int_0^t L_s(f_t)(s) d\tau) ds \leq L_s(f_t)(R) \leq c_2 \int_R \frac{f_0(s)}{s} \exp(-\frac{1}{\alpha} \int_0^t L_s(f_t)(s) d\tau) ds, \]  \hfill (3.10)

to obtain and upper and lower estimate on \( \int_0^t L_s(f) \). As before we set:

\[ g_t(R) = \exp(-\frac{1}{\alpha} \int_0^t L_s(f_t)(R) d\tau) \quad \text{and} \quad K(R) = \frac{f_0(R)}{R} \]

Using (3.10), we can obtain that

\[ -\frac{c_1}{2\alpha} \left( \int_R g_t(K(s) ds \right)^2 \geq \partial_t \int_R g_t(s) K(s) ds \geq -\frac{c_2}{2\alpha} \left( \int_R g_t(s) K(s) ds \right)^2 \]  \hfill (3.11)

Similar to Lemma 3.2, we define

\[ L_s(f_t)(R) := \int_R g_t(s) K(s) ds \]

Now from (3.11), we have

\[ -\frac{c_1}{2\alpha} (L_s(f_t)(R))^2 \geq \partial_t L_s(f_t)(R) \geq -\frac{c_2}{2\alpha} (L_s(f_t)(R))^2 \]

Thus,

\[ \frac{L(f_0)(R)}{1 + \frac{t}{2\alpha} L(f_0)(R)} \geq L_s(f_t)(R) \geq \frac{L(f_0)(R)}{1 + \frac{t}{2\alpha} L(f_0)(R)} \]  \hfill (3.12)

which will give us that:

\[ \frac{2\alpha}{c_1} \log(1 + \frac{c_1}{2\alpha} t L(f_0)(R))) \geq \int_0^t L_s(f_t)(R) \geq \frac{2\alpha}{c_2} \log(1 + \frac{c_2}{2\alpha} t L(f_0)(R)) \]

and this completes the proof \[ \square \]
3.2 Estimate for the leading order model in $W^{k,\infty}$ and $H^k$ norms

The purpose of this subsection is to obtain some estimate on the leading order model in $W^{k,\infty}$ and $H^k$ norms. These will be used to estimate the size of the remainder term in section 6. First we will obtain estimates on $\Psi_2$ in Lemma 3.4. Then in Lemma 3.5 we will obtain estimates on $\Omega_2$.

**Lemma 3.4.** Let $\Omega_2$ to solution to the leading order model:

$$\partial_t \Omega_2 + \left( \frac{1}{2\alpha} L_s(\Omega_2) \sin(2\theta) + \frac{1}{2\alpha} L_c(\Omega_2) \cos(2\theta) \right) \partial_\theta \Omega_2 = \frac{1}{2\alpha} L_s(\Omega_2)$$

with initial data $\Omega_2|_{t=0} = f_0(R) \sin(2\theta)$, where $f_0(R)$ is smooth with compactly support. Consider

$$\Psi_2 = \frac{1}{4\alpha} L_s(\Omega_2) \sin(2\theta) + \frac{1}{4\alpha} L_c(\Omega_2) \cos(2\theta)$$

Then, we have the following estimates on $\Psi_2$:

$$|\Psi_2|_{W^{k+1,\infty}} \leq c_k \frac{c}{\alpha}, \quad |\Psi_2|_{H^{k+1}} \leq c_k \frac{c}{\alpha}$$

(3.13)

where $c_k$ depends on the initial conditions and is independent of $\alpha$

**Proof.** Recall that from Proposition 2.1, we can write $\Omega_2$ as follows:

$$\Omega_2 = f + \frac{1}{2\alpha} \int_0^t L_s(f_\tau) \, d\tau,$$

and since the initial data is odd in $\theta$, we have

$$\Psi_2 = \frac{1}{4\alpha} L_s(\Omega_2) \sin(2\theta) = \frac{1}{4\alpha} L_s(f_t) \sin(2\theta)$$

To estimate the size of $\Psi_2$, from (3.4), we have

$$L_s(f_t)(R) = \int_R^\infty \frac{1}{s} \int_0^\infty f_0(s) \frac{\exp(-\frac{1}{\alpha} \int_0^t L_s(f_\tau)(s) \, d\tau)}{1 + \gamma^2 \exp(-\frac{1}{\alpha} \int_0^t L_s(f_\tau)(s) \, d\tau)} \frac{4\gamma^2}{(1 + \gamma^2)^2} \, d\gamma \, ds$$

Using (3.2), we have

$$|\Psi_2|_{L^\infty} \leq \frac{c}{\alpha} \int_R^\infty \frac{f_0(s)}{s} \, ds \leq \frac{c_0}{\alpha}$$

For $\partial_\theta \Psi_2$, it is clear that we have

$$|\partial_\theta \Psi_2|_{L^\infty} \leq \frac{c_0}{\alpha}$$

where, similarly, $c_0$ depends on the initial condition.

Now for $\partial_R \Psi_2$, we have

$$\partial_R \Psi_2 = \frac{1}{4\alpha} \partial_R L_s(f_t) \sin(2\theta)$$

Thus,

$$\partial_R L_s(f_t)(R) = -\frac{1}{R} \int_R^\infty f_0(R) \frac{\exp(-\frac{1}{\alpha} \int_0^t L_s(f_\tau)(R) \, d\tau)}{1 + \gamma^2 \exp(-\frac{1}{\alpha} \int_0^t L_s(f_\tau)(R) \, d\tau)} \frac{4\gamma^2}{(1 + \gamma^2)^2} \, d\gamma$$

and similarly, we have

$$|\partial_R \Psi_2|_{L^\infty} \leq \frac{c}{\alpha}$$

Now the estimate on $R \partial_R \Psi_2$ follows from the estimate on $\partial_R \Psi_2$ and the fact that the initial data have compact support. Thus,

$$|R \partial_R \Psi_2|_{L^\infty} \leq \frac{c}{\alpha}$$

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For higher order derivative, we can obtain the estimate following the same steps. Hence, we have

\[ |\Psi|_{W^{k+1,\infty}} \leq \frac{c_k}{\alpha} \]

The \( \mathcal{H}^k \) estimates also follows using the same steps.

\[ |\Psi|_{H^{k+1}} \leq \frac{c_k}{\alpha} \]

In the following Lemma, we will obtain the \( \mathcal{H}^k \) estimates on \( \Omega_2 \). Here we will use Lemma 3.4 and transport estimates.

**Lemma 3.5.** Let \( \Omega_2 \) to solution to the leading order model:

\[ \partial_t \Omega_2 + \left( \frac{1}{2\alpha} L_s(\Omega_2) \sin(2\theta) + \frac{1}{2\alpha} L_c(\Omega_2) \cos(2\theta) \right) \partial_\theta \Omega_2 = \frac{1}{2\alpha} L_s(\Omega_2) \]

with initial data \( \Omega_2|_{t=0} = f_0(R) \sin(2\theta) \), where \( f_0(R) \) is smooth with compactly support. Then, we have the following estimates on \( \Omega_2 \):

\[ |\Omega_2|_{H^k} \leq c_k e^{\frac{c_0}{\alpha} t} \]  \hspace{1cm} (3.14)

where \( c_k \) depends on the initial conditions and is independent of \( \alpha \)

**Proof.** Recall that from Proposition 2.1 we can write \( \Omega_2 \) as follows:

\[ \Omega_2 = f + \frac{1}{2\alpha} \int_0^t L_s(f_\tau) \, d\tau, \]

where \( f \) satisfies the following transport equation:

\[ \partial_t f + 2\Psi_2 \partial_\theta f = 0 \]

When we consider the derivatives of \( \Omega_2 \), the transport term \( f \) will dominates the radial term \( \frac{1}{2\alpha} \int_0^t L_s(f) \, d\tau \). Thus, it suffices to consider the \( \mathcal{H}^k \) estimates on \( f \) which will follow from the standard \( L^2 \) estimate for the transport equation. Thus, Since we have

\[ \partial_t f + 2\Psi_2 \partial_\theta f = 0 \implies \partial_t \partial_\theta f + 2\partial_\theta \Psi_2 \partial_\theta f + 2\Psi_2 \partial_\theta^2 f = 0 \]

Hence,

\[ |\partial_\theta f_1|_{L^2} \leq |\partial_\theta f_0|_{L^2} e^{\frac{c_0}{\alpha} t} |\partial_\theta \Psi_2|_{L^\infty} \]

From (3.13) we have \( |\partial_\theta \Psi_2|_{L^\infty} \leq \frac{c_0}{\alpha} \). Thus, applying Gronwall inequality, we have

\[ |\partial_\theta f_1|_{L^2} \leq |\partial_\theta f_0|_{L^2} e^{\frac{c_0}{\alpha} t} \]  \hspace{1cm} (3.15)

To obtain \( \mathcal{H}^k \) estimates, we need to estimate terms of the form \( R^k \partial_R^k \). We will show how to obtain the \( R \partial_R \) estimate, and for general \( k \), it will follow similarly. Thus, similar to \( L^2 \) estimate for \( \partial_\theta f \) case, since

\[ \partial_t f + 2\Psi_2 \partial_\theta f = 0 \]

we have

\[ \partial_t \partial_R f + 2\partial_\theta \Psi_2 \partial_\theta f + 2\Psi_2 \partial_\theta^2 f = 0 \]

and thus,

\[ \partial_t |R \partial_R f_1|_{L^2} \leq 2 |R \partial_R \Psi_2|_{L^\infty} |\partial_\theta f_1|_{L^2} + |\partial_\theta \Psi_2|_{L^\infty} |R \partial_R f_1|_{L^2} \]

Now from (3.13), (3.15), and applying Gronwall inequality we have

\[ |R \partial_R f_1|_{L^2} \leq (|R \partial_R f_0|_{L^2} + |\partial_\theta f_0|_{L^2} e^{\frac{c_0}{\alpha} t}) e^{\frac{c_0}{\alpha} t} \]

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Hence,
\[ |f(t)|_{\mathcal{H}^1} \leq |f_0|_{\mathcal{H}^1} e^{\frac{c_1}{t}} \]
which implies that
\[ |\Omega_2(t)|_{\mathcal{H}^1} \leq |\Omega_2(0)|_{\mathcal{H}^1} e^{\frac{c_1}{t}} \]

Similarly, using (3.13), the transport estimate, and following the same steps as above, we can obtain for the general \( H^k \) estimates. Hence
\[ |\Omega_2|_{\mathcal{H}^k} \leq |\Omega_2(0)|_{\mathcal{H}^k} e^{\frac{c_k}{t}} \]

\[ \square \]

## 4 Elliptic Estimate

The purpose of this section is to recall the first author’s Biot-Savart law decomposition [9] which is used here to derive the leading order model. In this section, we highlight the main ideas in the proof, and for more details, see [9] and [8]. We remark that this is also related to the Key Lemma of A. Kiselev and V. Šverák, see also the work of the first author [11], and the first author and I. Jeong [12] for generalization.

**Proposition 4.1.** ([9]) Given \( \Omega \in H^k \) such that for every \( R \) we have
\[ \int_0^{2\pi} \Omega(R, \theta) \sin(n\theta) d\theta = \int_0^{2\pi} \Omega(R, \theta) \cos(n\theta) d\theta = 0 \]
for \( n = 0, 1, 2 \) then the unique solution to
\[ 4\Psi + \partial_\theta \Psi + \alpha^2 R^2 \partial_{RR} \Psi + (4\alpha + \alpha^2) R \partial_R \Psi = \Omega(R, \theta) \]
satisfies
\[ |\partial_\theta \Psi|_{H^k} + \alpha |R \partial_R \Psi|_{H^k} + \alpha^2 |R^2 \partial_{RR} \Psi|_{H^k} \leq C_k |\Omega|_{H^k} \]  
(4.1)
where \( C_k \) is independent of \( \alpha \). In addition, we have the weights estimate
\[ |\partial_\theta D_R^k(\Psi)|_{L^2} + \alpha |R \partial_R D_R^k(\Psi)|_{L^2} + \alpha^2 |R^2 \partial_{RR} D_R^k(\Psi)|_{L^2} \leq C_k |D_R^k(\Omega)|_{L^2} \]  
(4.2)
where \( C_k \) is independent of \( \alpha \). Recall that \( D_R = R \partial_R \)

**Proof.** First, we will show how to obtain (4.1). Since \( \Omega \) is orthogonal to \( \sin(n\theta) \) and \( \cos(n\theta) \) for \( n = 0, 1, 2 \). Then, \( \Psi \) must also be orthogonal to \( \sin(n\theta) \) and \( \cos(n\theta) \) for \( n = 0, 1, 2 \). Consider the elliptic equation, and we consider \( L^2 \) estimate.
\[ 4\Psi + \partial_\theta \Psi + \alpha^2 R^2 \partial_{RR} \Psi + (4\alpha + \alpha^2) R \partial_R \Psi = \Omega(R, \theta) \]

Taking the inner product with \( \partial_\theta \Psi \) and integrating by parts, we have obtain
\[ -4|\partial_\theta \Psi|_{L^2}^2 + |\partial_\theta \Psi|^2_{L^2} - \alpha^2 |\partial_\theta \Psi|^2_{L^2} + \alpha^2 |R \partial_R \Psi|^2_{L^2} + \frac{(4\alpha + \alpha^2)}{2} |\partial_\theta \Psi|^2_{L^2} \leq |\Omega|_{L^2} |\partial_\theta \Psi|_{L^2} \]

Now by assumption, we have
\[ \Psi(R, \theta) = \sum_{n \geq 3} \Psi_n(R) e^{in\theta} \]
and hence
\[ |\partial_\theta \Psi|^2_{L^2} \leq \frac{1}{9} |\partial_\theta \Psi|^2_{L^2} \]
Using the above inequality, we can show that
\[
\frac{5}{9} |\partial_\theta \Psi|_{L^2}^2 + \alpha^2 |\partial_\partial \Psi|_{L^2}^2 + \frac{(4\alpha - \alpha^2)}{2} |\partial_\theta \Psi|_{L^2}^2 \leq |\Omega|_{L^2} |\partial_\theta \Psi|_{L^2}
\]
and thus we have that
\[
|\partial_\theta \Psi|_{L^2} \leq C_0 |\Omega|_{L^2}
\]
where \(C_0\) is independent of \(\alpha\). The estimate for the \(R^2 \partial_\partial \Psi\) term will follow similarly. We can also obtain the \(H^k\) estimates by following the same strategy. To obtain (4.2) estimates, recall that \(D_\partial = R \partial_\partial\) and we notice that we can write the elliptic equation in the following form:
\[
4\Psi + \partial_\theta \partial_\theta \Psi + \alpha^2 D_\partial^2 R(\Psi) + 4\alpha D_\partial R(\Psi) = \Omega(R, \theta)
\]
From this, we observe that the \(D_\partial\) operator commutes with the elliptic equation, and hence (4.2) estimates will follow from (4.1).

**Theorem 2.** (11) Given \(\Omega \in H^k\) where \(\Omega\) has the form of
\[
\Omega(R, \theta) = f(R) \sin(2\theta) \quad \Omega(R, \theta) = f(R) \cos(2\theta)
\]
then the unique solution to
\[
4\Psi + \partial_\theta \Psi + \alpha^2 R^2 \partial_\partial \Psi + (4\alpha + \alpha^2) R \partial_\partial \Psi = \Omega(R, \theta)
\]
is
\[
\Psi = -\frac{1}{4\alpha} L(f)(R) \sin(2\theta) + R(f) \quad \Psi = -\frac{1}{4\alpha} L(f)(R) \cos(2\theta) + R(f)
\]
where
\[
L(f)(R) = \int_R^\infty \frac{f(s)}{s} ds
\]
and
\[
|R(f)|_{H^k} \leq c |f|_{H^k}
\]
where \(c\) is independent of \(\alpha\)

**Proof.** Consider the case where \(\Omega(R, \theta) = f(R) \sin(2\theta)\), the case where \(\Omega(R, \theta) = f(R) \cos(2\theta)\) can be handled similarly. In this case \(\Psi(R, \theta)\) will be of the form \(\Psi(R, \theta) = \Psi_2(R) \sin(2\theta)\). Where \(\Psi_2(R)\) will satisfy the following ODE:
\[
\alpha^2 R^2 \partial_\partial \partial_\theta \Psi_2 + (4\alpha + \alpha^2) R \partial_\partial \Psi_2 = f(R)
\]
we can solve the ODE and obtain
\[
\partial_\partial \Psi_2(R) = \frac{1}{\frac{1}{\alpha^2} R^{2\alpha + 1}} \int_R^\infty \frac{f(s)}{s^{\alpha + \frac{1}{2}}} ds
\]
Now using that \(\Psi_2(R) \rightarrow 0\) as \(R \rightarrow \infty\), we obtain
\[
\Psi_2(R) = -\frac{1}{\alpha^2} \int_R^\infty \frac{f(s)}{s^{\alpha + \frac{1}{2}}} ds
\]
By integrating by parts, it follows that
\[
\Psi_2(R) = -\frac{1}{4\alpha} \int_R^\infty \frac{f(s)}{s} ds - \frac{1}{4\alpha} R^{2\alpha} \int_0^R \frac{f(s)}{s^{1-\frac{1}{2}}} ds := -\frac{1}{4\alpha} L(f)(R) + R(f)
\]
Using Hardy-type inequality one can show that:
\[
|R(f)|_{L^2} \leq c |f|_{L^2}
\]
where \(c\) is independent of \(\alpha\)
5 Embedding estimate in terms of $H^k$ norm

In this section we consider some embedding estimate in the $H^k$ norm which will be used in section 6. These estimates will be used various times as we estimate the reminder term. Recall that the $H^k$ norm is defined as follows:

$$|f|_{H^m} = \sum_{i=0}^{m} |\partial_i^m f|_{L^2} + \sum_{i=1}^{m} |R^i \partial_i^m f|_{L^2}$$

$$|f|_{H^k} = \sum_{m=0}^{k} |f|_{H^m}$$

**Lemma 5.1.** Let $\Omega \in H^N$, where $N \in \mathbb{N}$, then we have

$$|R^k \partial_k \partial^m \Omega|_{L^\infty} \leq c_{k,m} |\Omega|_{H^{k+m+2}} \quad \text{for any} \quad k + m + 2 \leq N \quad (5.1)$$

**Proof.** Using Sobolev embedding, we have

$$|R^k \partial_k \partial^m \Omega|_{L^\infty} \leq c_{k,m} |R^k \partial_k \partial^m \Omega|_{H^2_{R,\theta}}$$

where $H^2_{R,\theta}$ is the standard $H^2$ norm in $R$ and $\theta$. When considering the second derivative terms of $R^k \partial_k \partial^m \Omega$, for the angular derivatives term, we have $|R^k \partial_k \partial^m \Omega|_{L^2} \leq |\Omega|_{H^{k+m+2}}$. Now for the radial derivatives, we have three cases. Considering the case when the two radial derivatives land on $\partial_k \partial^m \Omega$, we have

$$|R^k \partial_k \partial^m \Omega|_{L^2} \leq |R^{k+2} \partial_k \partial^m \Omega|_{L^2} + |\partial_k^2 \partial^m \Omega| \leq |\Omega|_{H^{k+m+2}}$$

where the last inequality follows from the definition of the $H^N$ norm. The other two cases follow in a similar way.

We will also need some embedding estimates for the stream function $\Psi$ in terms of $\Omega$.

**Lemma 5.2.** Let $\Omega \in H^N$, where $N \in \mathbb{N}$, satisfying the same conditions as in Proposition 4.1, then for the solution $\Psi$ of

$$4\Psi + \partial_{\theta\theta} \Psi + \alpha^2 R^2 \partial_R \Psi + (4\alpha + \alpha^2) R \partial_R \Psi = \Omega(R, \theta),$$

we have

$$|\partial_R \partial^m \Psi|_{L^\infty} \leq c_{k,m} |\Omega|_{H^{k+m+1}} \quad (5.2)$$

for $k, m \in \mathbb{N}$ with $k + m + 1 \leq N$.

**Proof.** As in Lemma 5.1 applying the Sobolev embedding, we have

$$|\partial_R \partial^m \Psi|_{L^\infty} \leq c_{k,m} |\partial_R \partial^m \Psi|_{H^2_{R,\theta}}$$

From the elliptic estimates in Proposition 4.1 for any $i, n \in \mathbb{N}$, we have

$$|\partial_R \partial^m \Psi|_{L^2} \leq c_{i,n} |\Omega|_{H^{i+n-1}} \quad (5.3)$$

Thus, to bound $|\partial_R \partial^m \Psi|_{H^2_{R,\theta}}$, we take $\Omega$ to be in $H^{k+m+1}$. Hence, we have

$$|\partial_R \partial^m \Psi|_{L^\infty} \leq c_{k,m} |\Omega|_{H^{k+m+1}} \quad (5.4)$$

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Lemma 5.3. Let $\Omega \in \mathcal{H}^N$, where $N \in \mathbb{N}$, satisfying the same conditions as in Proposition 4.1, then for the solution $\Psi$ of

$$4\Psi + \partial_{\theta \theta} \Psi + \alpha^2 R^2 \partial_{RR} \Psi + (4\alpha + \alpha^2) R \partial_\theta \Psi = \Omega(R, \theta),$$

we have

$$|R^k \partial^k_R \partial^m_\theta \Psi|_{L^\infty} \leq c_{k,m} |\Omega|_{\mathcal{H}^{k+m+1}}$$

(5.5)

for $k, m \in \mathbb{N}$ with $k + m + 1 \leq N$.

Proof. As in Lemma 5.1 applying the Sobolev embedding, we have

$$|R^k \partial^k_R \partial^m_\theta \Psi|_{L^\infty} \leq c_{k,m} |R^k \partial^k_R \partial^{m-1}_\theta \Psi|_{H^2_{R,\theta}}$$

From the elliptic estimates in Proposition 4.1 for any $i, n \in \mathbb{N}$, we have

$$|\partial^i_R \partial^m_\theta \Psi|_{L^2} \leq c_{i,n} |\partial^i_R \partial^{m-1}_\theta \Omega|_{L^2} \leq c_{i,n} |\Omega|_{\mathcal{H}^{i+n-1}}$$

(5.6)

and

$$|R^k \partial^k_R \partial^m_\theta \Psi|_{L^2} \leq c_{i,n} |\Omega|_{\mathcal{H}^{i+n-1}}$$

(5.7)

Thus, if we look at the second derivative terms of $R^k \partial_R^k \partial_\theta^m \Psi$, we can use the above inequalities to obtain the desired estimate. For the angular derivative term, we have $|R^k \partial^k_R \partial^{n+2}_\theta \Psi|_{L^2} \leq c_{k,n} |\Omega|_{\mathcal{H}^{k+n+1}}$. When considering the radial derivative terms, we have three terms. For $R^k \partial^k_R \partial_\theta^m \Psi$ term, applying (5.6) and (5.7), we have

$$|R^k \partial^k_R \partial^m_\theta \Psi|_{L^2} \leq |R^k \partial^k_R \partial^{n+2}_\theta \Psi|_{L^2} + |R^k \partial^k_R \partial^m_\theta \Psi| \leq c_{k,m} |\Omega|_{\mathcal{H}^{k+m+1}}$$

The other terms can be handled in similar way. Hence, we have our desired result.

\[\square\]

6 Reminder estimate

In this section, we obtain an error estimate on the remaining terms in the Euler with Riesz forcing. Recall that $\Omega$ satisfies the following evolution equation:

$$\partial_t \Omega + \left( - \alpha R \partial_\theta \Psi \right) \partial_R \Omega + \left( 2\Psi + \alpha \partial_R \Psi \right) \partial_\theta \Omega = \left( 2\alpha R \sin(\theta) \cos(\theta) + \alpha^2 R \sin^2(\theta) \cos(\theta) \right) \partial_R \Psi$$

$$+ \left( 1 - 2 \sin^2(\theta) \right) \partial_\theta \Psi + \left( \alpha R \sin^2(\theta) + R \sin(\theta) \cos(\theta) \right) \partial_\theta \Psi$$

$$+ \left( \alpha^2 R \sin(\theta) \cos(\theta) \right) \partial_{RR} \Psi - \left( \sin(\theta) \cos(\theta) \right) \partial_{\theta \theta} \Psi$$

and the elliptic equation is the following:

$$4\Psi + \alpha^2 R^2 \partial_{RR} \Psi + \partial_{\theta \theta} \Psi + (4\alpha + \alpha^2) R \partial_\theta \Psi = \Omega(R, \theta)$$

From section 2 the leading order model for the Euler with Riesz forcing equation satisfies the following:

$$\partial_t \Omega_2 + (2\Psi_2) \partial_\theta \Omega_2 = \left( - 1 + 2 \sin^2(\theta) \right) \partial_\theta \Psi_2 + \left( \sin(\theta) \cos(\theta) \right) \partial_{\theta \theta} \Psi_2$$

(6.1)

Where

$$\Psi_2(R, \theta) = \frac{1}{4\alpha} L_s(\Omega) \sin(2\theta) + \frac{1}{4\alpha} L_c(\Omega) \cos(2\theta)$$
Now set \( \Omega_r = \Omega - \Omega_2 \) to be the reminder term for the vorticity, and similarly set \( \Psi_r = \Psi - \Psi_2 \) to be the reminder term for the stream function. Thus, we have that reminder, \( \Omega_r \), satisfies the following evolution equation:

\[
\partial_t \Omega_r + \left( -\alpha R(\partial_\theta \Psi_2 + \partial_\theta \Psi_r) \right) (\partial_\theta \Omega_2 + \partial_\theta \Omega_r) + \left( 2 \Psi_2 \partial_\theta \Omega_r + 2 \Psi_r \partial_\theta \Omega_2 + 2 \Psi_r \partial_\theta \Omega_r \right) \\
+ \left( \alpha R(\partial_\theta \Psi_2 + \partial_\theta \Psi_r) \right) (\partial_\theta \Omega_2 + \partial_\theta \Omega_r) = \left( 2 \alpha R \sin(\theta) \cos(\theta) + \alpha^2 R \sin(\theta) \cos(\theta) \right) (\partial_\theta \Psi_2 + \partial_\theta \Psi_r) \\
+ (1 - 2 \sin^2(\theta)) \partial_\theta \Psi_r + \alpha (R \cos^2(\theta) - R \sin^2(\theta)) (\partial_\theta \Psi_2 + \partial_\theta \Psi_r) \\
+ \alpha^2 (R^2 \sin(\theta) \cos(\theta)) (\partial_{RR} \Psi_2 + \partial_{RR} \Psi_r) - (\sin(\theta) \cos(\theta)) \partial_\theta \Psi_r.
\]

(6.2)

The goal of this section is to show that \( \Omega_r \) remains small. Namely, using energy methods, for some time \( T \), we show that

\[
\sup_{t \leq T} |\Omega_r(t)|_{L^\infty} \leq C \alpha^{\frac{1}{2}}
\]

for some constant \( C \) independent of \( \alpha \). We define the following term to shorten the notation:

\[
I_1 = -\alpha R(\partial_\theta \Psi_2 + \partial_\theta \Psi_r) (\partial_\theta \Omega_2 + \partial_\theta \Omega_r), \quad I_2 = (2 \Psi_2 \partial_\theta \Omega_r + 2 \Psi_r \partial_\theta \Omega_2 + 2 \Psi_r \partial_\theta \Omega_r) \quad I_3 = \alpha R(\partial_\theta \Psi_2 + \partial_\theta \Psi_r) (\partial_\theta \Omega_2 + \partial_\theta \Omega_r)
\]

\[
I_4 = 2 \alpha (1 - \alpha) R \sin(\theta) \cos(\theta) (\partial_\theta \Psi_2 + \partial_\theta \Psi_r), \quad I_5 = (1 - 2 \sin^2(\theta)) \partial_\theta \Psi_r, \quad I_6 = \alpha (R \cos^2(\theta) - R \sin^2(\theta)) (\partial_\theta \Psi_2 + \partial_\theta \Psi_r)
\]

\[
I_7 = \alpha^2 (R^2 \sin(\theta) \cos(\theta)) (\partial_{RR} \Psi_2 + \partial_{RR} \Psi_r), \quad I_8 = - (\sin(\theta) \cos(\theta)) \partial_\theta \Psi_r
\]

and now we have the error estimate proposition.

**Proposition 6.1.** Let \( \Omega_r = \Omega - \Omega_2 \) satisfying \( \Omega_r|_{t=0} = 0 \) then

\[
\sup_{0 \leq t \leq T} |\Omega_r(t)|_{L^\infty} \leq c_N \alpha^{\frac{1}{2}}
\]

where \( T = c_\alpha |\log \alpha| \) and \( c \) is a small constant independent of \( \alpha \).

**Proof.** We will use \( \partial^N \) to refer to any mixed derivatives in \( R \) and \( \theta \) of order \( N \) (not excluding pure \( R \) and \( \theta \) derivatives). From the definition of \( \mathcal{H}^N \) norm, to obtain \( \mathcal{H}^N \) estimate we will take the following inner product with each \( I_i \) term:

\[
\langle \partial^N I_i, \partial^N \Omega_r \rangle \quad \text{and} \quad \langle R^k \partial_R^k \partial_\theta^{N-k} I_i, R^k \partial_R^k \partial_\theta^{N-k} \Omega_r \rangle
\]

for \( 0 \leq k \leq N \) and \( 1 \leq i \leq 8 \).

**Estimate on \( I_1 \) and \( I_3 \)**

Here we will estimate \( I_1 \) and \( I_3 \). The estimate of \( I_3 \) is very similar to \( I_1 \), and so we will just show how to obtain the estimate on \( I_1 \).

**Estimate on \( I_1 \)**

We can write \( I_1 \) as

\[
I_1 = -\alpha R (\partial_\theta \Psi_2 + \partial_\theta \Psi_r) (\partial_\theta \Omega_2 + \partial_\theta \Omega_r) = -\alpha (\partial_\theta \Psi_2) R (\partial_\theta \Omega_2) - \alpha (\partial_\theta \Psi_2) R (\partial_\theta \Omega_r)
\]

\[
- \alpha (\partial_\theta \Psi_r) R (\partial_\theta \Omega_2) - \alpha (\partial_\theta \Psi_r) R (\partial_\theta \Omega_r)
\]

\[
= I_{1,1} + I_{1,2} + I_{1,3} + I_{1,4}
\]

and we will estimate each term separately.

- \( I_{1,1} = -\alpha \partial_\theta \Psi_2 R \partial_\theta \Omega_2 \)

Here we have

\[
\langle \partial^N (\alpha \partial_\theta \Psi_2 R \partial_\theta \Omega_2), \partial^N \Omega_r \rangle = \sum_{i=0}^N c_i N \int \partial^i (\alpha \partial_\theta \Psi_2) \partial^{N-i} (R \partial_\theta \Omega_2) \partial^N \Omega_r
\]

Now from Lemma 3.4 and Lemma 3.5 we know that

\[
|\Psi_2|_{W^{k+1, \infty}} \leq \frac{c_k}{\alpha} \quad \text{and} \quad |\Omega_2|_{H^k} \leq |\Omega_2(0)|_{H^k} e^{-\frac{c}{\alpha}}
\]

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Thus, we have
\[
\sum_{i=0}^{N} \int \alpha \partial_i^j (\partial_\theta \Psi_2) \partial^{N-i}(R \partial_\Omega \Omega_2) \partial^N \Omega_r \leq c_N \sum_{i=0}^{N} \alpha |\partial_i^j \partial_\theta \Psi_2|_{L^\infty} |\partial^{N-i}(R \partial_\Omega \Omega_2)|_{L^2} |\partial^N \Omega_r|_{L^2}
\]
\[
\leq c_N \alpha |\Psi_2|_{W^{N+1, \infty}} |\Omega_2|_{H^{N+1}} |\Omega_r|_{H^N} \leq \alpha \frac{c_N}{\alpha} \frac{c_N}{\alpha} |\Omega_r|_{H^N}
\]

and similarly we have
\[
\langle \partial_R^k \partial_\theta^{N-k} (\alpha \partial_\theta \Psi_2 R \partial_R \Omega_2), R^{2k} \partial_R^k \partial_\theta^{N-k} \Omega_r \rangle =
\sum_{i+m=0}^{N} \int R_i \partial_R^m (\alpha \partial_\theta \Psi_2) \partial_R^{k-i} \partial_\theta^{N-k-m} (R \partial_\Omega \Omega_2) R^{2k} \partial_R^k \partial_\theta^{N-k} \Omega_r
\]
\[
\leq c_N \sum_{i+m=0}^{N} \alpha |R_i \partial_R^m \partial_\theta^{N+1} \Psi_2|_{L^\infty} |R^{k-i} \partial_R^{k-i} \partial_\theta^{N-k-m} (R \partial_\Omega \Omega_2)|_{L^2} |R^{2k} \partial_R^k \partial_\theta^{N-k} \Omega_r|_{L^2}
\]
\[
\leq c_N \alpha |\Psi_2|_{W^{N+1, \infty}} |\Omega_2|_{H^{N+1}} |\Omega_r|_{H^N} \leq \alpha \frac{c_N}{\alpha} \frac{c_N}{\alpha} |\Omega_r|_{H^N}
\]

From the definition of $W^{N+1, \infty}$ norm, we have for $i + m \leq N$
\[
|R_i \partial_R^m \partial_\theta^{N+1} \Psi_2|_{L^\infty} \leq |\Psi_2|_{W^{N+1, \infty}}
\]

Again, applying Lemma 3.4 and Lemma 3.5 we obtain
\[
\sum_{i+m=0}^{N} \int R_i \partial_R^m (\alpha \partial_\theta \Psi_2) \partial_R^{k-i} \partial_\theta^{N-k-m} (R \partial_\Omega \Omega_2) R^{2k} \partial_R^k \partial_\theta^{N-k} \Omega_r
\]
\[
\leq c_N \sum_{i+m=0}^{N} \alpha |R_i \partial_R^m \partial_\theta^{N+1} \Psi_2|_{L^\infty} |R^{k-i} \partial_R^{k-i} \partial_\theta^{N-k-m} (R \partial_\Omega \Omega_2)|_{L^2} |R^{2k} \partial_R^k \partial_\theta^{N-k} \Omega_r|_{L^2}
\]
\[
\leq c_N \alpha |\Psi_2|_{W^{N+1, \infty}} |\Omega_2|_{H^{N+1}} |\Omega_r|_{H^N} \leq \alpha \frac{c_N}{\alpha} \frac{c_N}{\alpha} |\Omega_r|_{H^N}
\]

Thus, we have
\[
\langle I_{1, 1}, \Omega_r \rangle_{H^N} \leq c_N e^{\frac{c_N}{\alpha}} |\Omega_r|_{H^N} \tag{6.3}
\]

\* $I_{1, 2} = -\alpha \partial_\theta \Psi_2 R \partial_R \Omega_r$

Here we have
\[
\langle \partial^N (\alpha \partial_\theta \Psi_2 R \partial_R \Omega_r), \partial^N \Omega_r \rangle = \sum_{i=0}^{N} \int \partial^i (\alpha \partial_\theta \Psi_2) \partial^{N-i}(R \partial_\Omega \Omega_r) \partial^N \Omega_r
\]

To obtain this estimate, we again apply Lemma 3.4. Namely, that $|\Psi_2|_{W^{k+1, \infty}} \leq \frac{c_N}{\alpha}$. When $i = 0$, we integrate by parts and obtain
\[
\int (\alpha \partial_\theta \Psi_2) \partial^N (R \partial_\Omega \Omega_r) \partial^N \Omega_r \leq c |\Psi_2|_{W^{2, \infty}} |\Omega_r|_{H^N}^2 \leq \frac{c_N}{\alpha} |\Omega_r|_{H^N}^2
\]

For $1 \leq i \leq N$ we have,
\[
\sum_{i=1}^{N} \int \alpha \partial_i^j (\partial_\theta \Psi_2) \partial^{N-i}(R \partial_\Omega \Omega_r) \partial^N \Omega_r \leq c_N \sum_{i=1}^{N} \alpha |\partial_i^j \partial_\theta \Psi_2|_{L^\infty} |\partial^{N-i}(R \partial_\Omega \Omega_r)|_{L^2} |\partial^N \Omega_r|_{L^2}
\]
\[
\leq c_N \alpha |\Psi_2|_{W^{N+1, \infty}} |\Omega_r|_{H^N} |\Omega_r|_{H^N} \leq \alpha \frac{c_N}{\alpha} |\Omega_r|_{H^N}^2 \leq c_N |\Omega_r|_{H^N}^2
\]

Similarly, Now for the $R^{2k} \partial_R^k \partial_\theta^{N-k}$ terms we have
Thus, we have

\[
\langle R^i \partial^N \Omega \partial_\theta R^{N-k}(\alpha \partial_\theta \Psi_2 R^1 \partial_\theta \Omega_r), R^{i} \partial^N \Omega \partial_\theta R^{N-k}\partial_\theta \Omega_r \rangle = c_{i,m,N} \int \sum_{i+m=0}^{N} R^{i} \partial^N \Omega \partial_\theta R^{N-k}(\alpha \partial_\theta \Psi_2) \partial^N \Omega \partial_\theta R^{N-k-m}(\partial_\theta \Omega_r) R^{i} \partial^N \Omega \partial_\theta R^{N-k}\partial_\theta \Omega_r.
\]

We again use \(|\Psi_2|_{W^k+1,\infty} \leq \frac{\alpha}{c} \). Hence, we have

\[
\sum_{i+m=0}^{N} R^{i} \partial^N \Omega \partial_\theta R^{N-k}(\alpha \partial_\theta \Psi_2) R^{i} \partial^N \Omega \partial_\theta R^{N-k-m}(\partial_\theta \Omega_r) R^{i} \partial^N \Omega \partial_\theta R^{N-k}\partial_\theta \Omega_r \leq c_N \sum_{i+m=0}^{N} \alpha |R^{i} \partial^N \Omega \partial_\theta R^{N-k}(\alpha \partial_\theta \Psi_2)|_{L^\infty} |R^{i} \partial^N \Omega \partial_\theta R^{N-k-m}(\partial_\theta \Omega_r)|_{L^2} |R^{i} \partial^N \Omega \partial_\theta R^{N-k}\partial_\theta \Omega_r|_{L^2}
\]

\[
\leq c_N |\Psi_2|_{W^{k+1,\infty}} |\Omega_r|_{H^N} |\Omega_r|_{H^N} \leq \alpha \frac{c_N}{\alpha} |\Omega_r|^2_{H^N} \leq c_N |\Omega_r|^2_{H^N}
\]

Thus, we have

\[
\langle I_{1,2}, \Omega_r \rangle_{H^N} \leq c_N |\Omega_r|^2_{H^N} \tag{6.4}
\]

\( I_{1,3} = -\alpha (\partial_\theta \Psi_r) R^1 \partial_\theta \Omega_r \)

To obtain the estimate on \( I_{1,3} \), we will use Lemma 3.2 which will give us the estimate on \(\Omega_r\). In addition, to bound the \( \partial_\theta \Psi_r \) term, we will use the elliptic from Proposition 4.1 and embedding estimates from Lemma 5.2. Now we have

\[
\langle \partial^N (\alpha \partial_\theta \Psi_r R^1 \partial_\theta \Omega_r), \partial^N \Omega_r \rangle = \sum_{i=0}^{N} c_{i,N} \int \partial^N (\alpha \partial_\theta \Psi_r) \partial^N \Omega_r = \sum_{i=0}^{N} c_{i,m,N} \int \partial^N (\alpha \partial_\theta \Psi_r) \partial^N \Omega_r \]

When \( 0 \leq i \leq \frac{N}{2} \), we will use the embedding from Lemma 5.2. Namely that

\[
|\partial^N \Omega_r|_{L^\infty} \leq c_{k,m,N} |\Omega_r|_{H^{k+m+1}}
\]

Thus, we have

\[
\sum_{i=0}^{\frac{N}{2}} \int \partial^N (\alpha \partial_\theta \Psi_r) \partial^N \Omega_r \leq \sum_{i=0}^{\frac{N}{2}} \alpha |\partial^N \Omega_r|_{L^\infty} \leq c_{k,m,N} |\Omega_r|_{H^{k+m+1}} \Omega_r \leq c_{N} \alpha \frac{c_N}{\alpha} |\Omega_r|^2_{H^N}
\]

Here we used Lemma 3.2 for \(|\Omega_r|_{H^{k+1}}\) term.

When \( \frac{N}{2} \leq i \leq N \) we will use the elliptic estimate from Proposition 4.1. Namely,

\[
|\partial^N \Omega_r|_{L^2} \leq c_{k} |\Omega_r|_{H^k}
\]

Thus we have

\[
\int \partial^N (\alpha \partial_\theta \Psi_r) \partial^N \Omega_r \leq \sum_{i=0}^{\frac{N}{2}} \alpha |\partial^N \Omega_r|_{L^\infty} \leq \sum_{i=0}^{\frac{N}{2}} \alpha |\partial^N \Omega_r|_{L^\infty} \leq c_{N} \alpha \frac{c_N}{\alpha} |\Omega_r|^2_{H^N}
\]
Similarly, to estimate the following inner product

\[
\langle \partial_R^k \partial_\theta^{N-k} (\alpha (\partial_\theta \Psi) R \partial_R \Omega_r), R^{2k} \partial_R^k \partial_\theta^{N-k} \Omega_r \rangle \leq c_N \alpha e c_N |\Omega_r|^2 |H^{N}|
\]

we will use (4.2) in Proposition 4.1 and embedding estimates from Lemma 5.3. Following the same steps as we did in the previous inner product, we obtain that

\[
\langle I_{1,3}, \Omega_r \rangle_{\mathcal{H}^N} \leq c_N \alpha e c_N |\Omega_r|^2 |H^{N}|
\]

(6.5)

- **Estimate on** \(I_{1,4} = -\alpha (\partial_\theta \Psi) R \partial_R \Omega_r\)

To obtain the estimate on \(I_{1,4}\), we will use the elliptic estimates from Proposition 4.1. Namely, (4.1) and (4.2), then we will also use the embedding estimates from Lemma 5.2 and Lemma 5.3. We will only show how to obtain the estimate on the following term:

\[
\langle \partial_R^k \partial_\theta^{N-k} (\alpha (\partial_\theta \Psi) R \partial_R \Omega_r), R^{2k} \partial_R^k \partial_\theta^{N-k} \Omega_r \rangle =
\]

\[
c_i, m, N \sum_{i+m=0}^{N} \partial_\theta^i \partial_\theta^m (\alpha (\partial_\theta \Psi)) \partial_R^{k} \partial_\theta^{N-k-m} (R \partial_R \Omega_r) \]

For the other inner product, the idea is the same. To start the estimate, first we consider the case when \(i = m = 0\), we integrate by parts and use the embedding estimates in Lemma 5.2 and Lemma 5.3, we have

\[
\int \alpha (\partial_\theta \Psi) \left( R^{k+1} \partial_\theta^{k+1} \partial_\theta^{N-k} \Omega_r + R^{k} \partial_\theta^{k} \partial_\theta^{N-k} \Omega_r \right) R^{k} \partial_\theta^{k} \partial_\theta^{N-k} \Omega_r
\]

\[
\leq \alpha |R \partial_\theta \Psi|_{L^\infty} |R^{k} \partial_\theta^{k} \partial_\theta^{N-k} \Omega_r|_{L^2}^2 + \alpha |\partial_\theta \Psi|_{L^\infty} |R^{k} \partial_\theta^{k} \partial_\theta^{N-k} \Omega_r|_{L^2}^2
\]

\[
\leq c_N (|\Omega_r|_{\mathcal{H}^2}^2 |\Omega_r|^2_{\mathcal{H}^N} + |\Omega_r|_{\mathcal{H}^2}^2 |\Omega_r|^2_{\mathcal{H}^N}) \leq c_N |\Omega_r|^3_{\mathcal{H}^N}
\]

Now when \(1 \leq i + m \leq N\), we use Lemma 6.3 and the definition of the \(H^k\) norm to obtain:

\[
\sum_{i+m \geq 1}^{N} R^{k} \partial_\theta^{k} \partial_\theta^{N-k} \Omega_r
\]

\[
\leq \sum_{i+m \geq 1}^{N} \alpha |R^i \partial_\theta^{i} \partial_\theta^{m+1} \Psi|_{L^\infty} |R^{k+1-i} \partial_\theta^{k+1-i} \partial_\theta^{N-k-m} \Omega_r|_{L^2}^2 + |R^{k-i} \partial_\theta^{k-i} \partial_\theta^{N-k-m} \Omega_r|_{L^2}^2
\]

\[
\sum_{i+m \geq 1}^{N} \alpha |R^i \partial_\theta^{i} \partial_\theta^{m+1} \Psi|_{L^\infty} |R^{k+1-i} \partial_\theta^{k+1-i} \partial_\theta^{N-k-m} \Omega_r|_{L^2}^2 + |R^{k-i} \partial_\theta^{k-i} \partial_\theta^{N-k-m} \Omega_r|_{L^2}^2
\]

\[
\leq c_N \sum_{i+m \geq 1}^{N} |\Omega_r|_{\mathcal{H}^{i+m+2}}^2 (|\Omega_r|_{\mathcal{H}^N} + |\Omega_r|_{\mathcal{H}^{N-1}}) |\Omega_r|_{\mathcal{H}^N} \leq c_N |\Omega_r|^3_{\mathcal{H}^N}
\]

Now for the case when \(\frac{N}{2} \leq i + m \leq N\), we will use Lemma 5.1 and the elliptic estimates from Proposition 4.1 to obtain
\[
\sum_{i+m \geq \frac{N}{2}}^N R^i \partial_R^i \partial_\theta^m (\alpha \partial_\theta \Psi) \left( R^{k+i-1} \partial_R^{k+i-1} \partial_\theta^{N-k-m} \Omega_r + R^{k-i} \partial_R^{k-i} \partial_\theta^{N-k-m} \Omega_r \right) R^k \partial_R^k \partial_\theta^N \partial_\theta \Omega_r
\]
\[
\leq \sum_{i+m \geq \frac{N}{2}}^N \alpha |R^i \partial_R^i \partial_\theta^{m+1} \psi|_{L^2} \left( |R^{k+i-1} \partial_R^{k+i-1} \partial_\theta^{N-k-m} \Omega_r|_{L^\infty} \right) |R^k \partial_R^k \partial_\theta^N \partial_\theta \Omega_r|_{L^2} +
\]
\[
\sum_{i+m \geq \frac{N}{2}}^N \alpha |R^i \partial_R^i \partial_\theta^{m+1} \psi|_{L^2} \left( |R^{k-i} \partial_R^{k-i} \partial_\theta^{N-k-m} \Omega_r|_{L^\infty} \right) |R^k \partial_R^k \partial_\theta^N \partial_\theta \Omega_r|_{L^2}
\]
\[
\leq \sum_{i+m \geq \frac{N}{2}}^N |\Omega_r|_{H^{i+m-1}} \left( |\Omega|_{H^{i+m-1}} + |\Omega|_{H^{i+m-1}} \right) |\Omega_r|_{H^N} \leq c_N |\Omega_r|_{H^N} \leq c_N |\Omega_r|_{H^N}
\]

and thus, we have the following:
\[
\left\langle I_{1,4}, \Omega_r \right\rangle_{\Omega^N} \leq c_N |\Omega_r|_{H^N}
\] (6.6)

Thus, we have the following estimate on \( I_1 \) term
\[
\left\langle I_1, \Omega_r \right\rangle_{\Omega^N} \leq c_N e^{\frac{c_N}{\alpha} |\Omega_r|_{H^N}} + c_N e^{\frac{c_N}{\alpha} |\Omega_r|_{H^N}} + c_N |\Omega_r|_{H^N}^3
\] (6.7)

**Estimate on \( I_3 \)**

The estimate on \( I_3 \) follows similarly to \( I_1 \), so we skip the details for this case. One can obtain the following:
\[
\left\langle I_3, \Omega_r \right\rangle_{\Omega^N} \leq c_N e^{\frac{c_N}{\alpha} |\Omega_r|_{H^N}} + c_N e^{\frac{c_N}{\alpha} |\Omega_r|_{H^N}} + c_N |\Omega_r|_{H^N}^3
\] (6.8)

**Estimate on \( I_2 \)**

Here we have
\[
I_2 = (2 \Psi_2 \partial_\theta \Omega_r + 2 \Psi_2 \partial_\theta \Omega_r + 2 \Psi_2 \partial_\theta \Omega_r) = I_{2,1} + I_{2,2} + I_{2,3}
\]

- \( I_{2,1} = 2 \Psi_2 \partial_\theta \Omega_r \)

To estimate \( I_{2,1} \), we follow the same steps as in the \( I_1 \) term. Using Lemma 3.3, Namely, that \( |\Psi_2|_{W^{N,\infty}} \leq \frac{c_N}{\alpha} \), we have:
\[
\left\langle I_{2,1}, \Omega_r \right\rangle_{\Omega^N} \leq c_N |\Omega_r|_{H^N}^2
\] (6.9)

- \( I_{2,2} = 2 \Psi_2 \partial_\theta \Omega_r \)

Similarly, To estimate \( I_{2,2} \), we also follow the same steps as we did in \( I_1 \). Using Lemma 3.5, that \( |\Omega_2|_{H^N} \leq |\Omega_2(0)|_{H^N} e^{\frac{c_N}{\alpha}} \), we obtain:
\[
\left\langle I_{2,2}, \Omega_r \right\rangle_{\Omega^N} \leq c_N e^{\frac{c_N}{\alpha} |\Omega_r|_{H^N}^2}
\] (6.10)

- \( I_{2,3} = 2 \Psi_2 \partial_\theta \Omega_r \)

This terms \( I_{2,3} \) can be estimated in a similar way as in the \( I_{1,4} \) term, by using embedding and elliptic estimate, we have
\[
\left\langle I_{2,3}, \Omega_r \right\rangle_{\Omega^N} \leq c_N |\Omega_r|_{H^N}^3
\] (6.11)
Hence, we obtain

\[
\left\langle I_2, \Omega_r \right\rangle_{\mathcal{H}^N} \leq \frac{c_N}{\alpha} |\Omega_r|_{\mathcal{H}^N}^2 + c_N e^{\frac{c_N}{\alpha t}} |\Omega_r|_{\mathcal{H}^N}^2 + c_N |\Omega_r|_{\mathcal{H}^N}^3 \leq c_N e^{\frac{c_N}{\alpha t}} |\Omega_r|_{\mathcal{H}^N}^2 + c_N |\Omega_r|_{\mathcal{H}^N}^3
\]  

(6.12)

Estimate on \(I_4, I_5, I_6, I_7, \) and \(I_8\)

We can write \(I_4\) as:

\[
I_4 = 2\alpha R \sin(\theta) \cos(\theta) + \alpha^2 R \sin(\theta) \cos(\theta) \left( \partial_R \Psi_2 + \partial_R \Psi_r \right)
\]

\[
= \alpha(2 + \alpha) \sin(\theta) \cos(\theta) R \partial_R \Psi_2 + \alpha(2 + \alpha) \sin(\theta) \cos(\theta) R \partial_R \Psi_r
\]

\[
= I_{4,1} + I_{4,2}
\]

Recall that

\[
I_5 = (1 - 2 \sin^2(\theta))\partial_\theta \Psi_r
\]

We can also rewrite and \(I_6\) and \(I_7\) as follows:

\[
I_6 = \alpha(\cos^2(\theta) - \sin^2(\theta)) R(\partial_{R\theta} \Psi_2 + \partial_{R\theta} \Psi_r)
\]

\[
= \alpha(\cos^2(\theta) - \sin^2(\theta)) R \partial_{R\theta} \Psi_2 + \alpha(\cos^2(\theta) - \sin^2(\theta)) R \partial_{R\theta} \Psi_r
\]

\[
= I_{6,1} + I_{6,2}
\]

and

\[
I_7 = \alpha^2(\sin(\theta) \cos(\theta)) R^2(\partial_{RR} \Psi_2 + \partial_{RR} \Psi_r)
\]

\[
= \alpha^2(\sin(\theta) \cos(\theta)) R^2 \partial_{RR} \Psi_2 + \alpha^2(\sin(\theta) \cos(\theta)) R^2 \partial_{RR} \Psi_r
\]

\[
= I_{7,1} + I_{7,2}
\]

Recall that

\[
I_8 = -\sin(\theta) \cos(\theta) \partial_{\theta \theta} \Psi_r
\]

Now for \(i = 4, 6, \) and \(7,\) using Lemma 3.4, Namely, that \(|\Psi|_{\mathcal{H}^{k+1}} \leq \frac{c_N}{\alpha},\) we have the following estimate:

\[
\left\langle I_{i,1}, \Omega_r \right\rangle_{\mathcal{H}^N} \leq c_N |\Omega_r|_{\mathcal{H}^N} \quad \text{for} \quad i = 4, 6, 7
\]  

(6.13)

Using the elliptic estimates in Proposition 4.1 we obtain that:

\[
\left\langle I_{i,2}, \Omega_r \right\rangle_{\mathcal{H}^N} \leq c_N |\Omega_r|_{\mathcal{H}^N}^2 \quad \text{for} \quad i = 4, 6, 7
\]  

(6.14)

and

\[
\left\langle I_i, \Omega_r \right\rangle_{\mathcal{H}^N} \leq c_N |\Omega_r|_{\mathcal{H}^N}^3 \quad \text{for} \quad i = 5, 8
\]  

(6.15)

Hence, from (6.13), (6.14), (6.15), we have that

\[
\left\langle I_i, \Omega_r \right\rangle_{\mathcal{H}^N} \leq c_N |\Omega_r|_{\mathcal{H}^N} + c_N |\Omega_r|_{\mathcal{H}^N}^2 \quad \text{for} \quad i = 4, 5, \ldots 8
\]  

(6.16)

Totally reminder estimate:

Here we obtain the totally error estimate. From our previous work we have,

\[
\frac{d}{dt} |\Omega_r|_{\mathcal{H}^N}^2 = \langle \partial_t \Omega_r, \Omega_r \rangle_{\mathcal{H}^N} \leq \sum_{i=1}^8 \left| \left\langle I_i, \Omega_r \right\rangle_{\mathcal{H}^N} \right|
\]

and thus from (6.7), (6.8), (6.12), and (6.16), we have

\[
\frac{d}{dt} |\Omega_r|_{\mathcal{H}^N}^2 \leq c_N e^{\frac{c_N}{\alpha t}} |\Omega_r|_{\mathcal{H}^N} + \left( \frac{c_N}{\alpha} + c_N e^{\frac{c_N}{\alpha t}} \right) |\Omega_r|_{\mathcal{H}^N} + c_N |\Omega_r|_{\mathcal{H}^N}^3
\]

and hence

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\[
\frac{d}{dt}|\Omega_r|_{H^N} \leq c_N e^{\frac{c_N}{\alpha}t} + \left(\frac{c_N}{\alpha} + c_N e^{\frac{c_N}{\alpha}t}\right) |\Omega_r|_{H^N} + c_N |\Omega_r|^2_{H^N}
\]

Now since we have \(\Omega_r|_{t=0} = 0\), then by bootstrap, it follows that

\[
|\Omega_r|_{H^N} \leq \left(\int_0^t c_N e^{\frac{c_N}{\alpha}\tau} d\tau\right) \exp\left(\int_0^t \frac{c_N}{\alpha} + c_N e^{\frac{c_N}{\alpha}\tau} d\tau\right) \leq \alpha c_N (e^{\frac{c_N}{\alpha}t} - 1) \exp\left(\frac{c_N}{\alpha} t + \alpha c_N e^{\frac{c_N}{\alpha}t}\right)
\]

Thus, if we choose \(t < T = c\alpha|\log\alpha|\) for \(c\) small, say \(c = \frac{1}{\epsilon c_N}\), then we have

\[
|\Omega_r|_{H^N} \leq \alpha c_N^{\frac{1}{2}}
\]

and this completes the proof of Proposition 6.1.

\[\square\]

7 Main result

We now recall and prove the main theorem of this work.

**Theorem 3.** For any \(\alpha, \delta > 0\), there exists an initial data \(\omega^{\alpha, \delta}_0 \in C_0^\infty(\mathbb{R}^2)\) and \(T(\alpha)\) such that the corresponding unique global solution, \(\omega^{\alpha, \delta}\), to (1.4) is such that at \(t = 0\) we have

\[
|\omega^{\alpha, \delta}_0|_{L^\infty} = \delta,
\]

but for any \(0 < t \leq T(\alpha)\) we have

\[
|\omega^{\alpha, \delta}(t)|_{L^\infty} \geq |\omega_0|_{L^\infty} + c \log(1 + \frac{c_0}{\alpha t}),
\]

where \(T(\alpha) = c\alpha|\log(\alpha)|\) and \(c > 0\) is a constant independent of \(\alpha\).

**Proof.** Consider the initial data of the form

\[
\omega_0 = \Omega|_{t=0} = f_0(R) \sin(2\theta)
\]

where \(f(R)\) is non-negative compactly support smooth function which is zero on \([0, 1]\) and positive outside. We know that we can write \(\Omega = \Omega_2 + \Omega_r\), and from the form of initial data, we have \(\Omega_r|_{t=0} = 0\) and thus from Proposition 6.1, we have

\[
|\Omega_r(t)|_{L^\infty} \leq c_N^{\frac{0.5}{2}}
\]

for \(0 \leq t \leq c|\alpha|\log\alpha|\), where recall that \(c\) is a small constant independent of \(\alpha\). Recall also that we can write \(\Omega_2\) as:

\[
\Omega_2 = f + \frac{1}{2\alpha} \int_0^t L_s(f_r) d\tau
\]

and thus from Proposition 3.3, we obtain that

\[
\Omega_2 = f + \frac{1}{2\alpha} \int_0^t L_s(f_r) d\tau \geq f + c_0 \log(1 + \frac{c_0}{\alpha t})
\]

for some \(c_0\) independent of \(\alpha\) and thus we have our desired result.

\[\square\]

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References

[1] J. T. Beale, T. Kato, A. Majda. Remarks on the breakdown of smooth solutions for the 3-D Euler equations, Comm. Math. Phys, 94(1):61-66, 1984.

[2] N. Boardman, H. Lin, J. Wu. Stabilization of a background magnetic field on a 2 dimensional magnetohydrodynamic flow. SIAM J. Math. Anal. 52 (2020): 5001-35.

[3] J. Bourgain, D. Li. Strong illposedness of the incompressible Euler equation in integer $C^m$ spaces. Geom. Funct. Anal. 25 (2015), no. 1, 1786

[4] C. Cao, J. Wu. Global regularity for the 2D MHD equations with mixed partial dissipation and magnetic diffusion. Adv. Math. 226 (2011): 1803-22.

[5] J. Y. Chemin and N. Masmoudi. About lifespan of regular solutions of equations related to viscoelastic fluids. SIAM J. Math. Anal. 33(1), 84?112, 2001

[6] P. Constantin, V. Vicol, Nonlinear maximum principles for dissipative linear nonlocal operators and applications, Geom. Funct. Anal. 22(5), 1289-1321 (2012)

[7] P. Constantin, M. Kliegl, on Global Regularity for Two-Dimensional Oldroyd-B Fluids with Diffusive Stress, Arch. Ration. Mech. Anal, 206, no. 3, 725-74

[8] T. D. Drivas, T. M. Elgindi, Singularity formation in the incompressible Euler equation in finite and infinite time. arXiv:2203.17221, (2022).

[9] T. M. Elgindi . Finite-Time Singularity Formation for $C^{1,\alpha}$ Solutions to the Incompressible Euler Equations on $\mathbb{R}^3$, Annals of Mathematics 194.3 (2021): 647-727.

[10] T. M. Elgindi, Sharp $L^p$ estimates for singular transport equations, Adv. Math. 329 (2018), 1285-1306

[11] T. M. Elgindi, Remarks on functions with bounded Laplacian. arXiv:1605.05266, 2016

[12] T. M. Elgindi, and I-J. Jeong, On singular vortex patches, I: Well-posedness issues. to appear in Memoirs of the American Mathematical Society

[13] T. M. Elgindi and N. Masmoudi. $L^\infty$ ill-posedness for a class of equations arising in hydrodynamics Arch. Ration. Mech. Anal. 235 (2020), no. 3, 1979-2025, arXiv:1405.2478

[14] T. M. Elgindi and F. Rousset Global regularity for some Oldroyd type models Commun. Pure.Appl. Math. 68 2005-2021, 2015

[15] N. Glatt-Holtz, and V. Vicol, Local and global existence of smooth solutions for the stochastic Euler equations on a bounded domain. Ann. Probab. 42(1), 80?145, (2014)

[16] T. Hmidi, On the Yudovich solutions for the ideal MHD equations, Nonlinearity. 27 (2014), 3117-3158.

[17] E. Holder, Über die unbeschränkte Fortsetzbarkeit einer stetigen ebenen Bewegung in einer unbegrenzten inkompressiblen Flüssigkeit. Math. Z. 37 (1933), 727-738

[18] T. Kato, On classical solutions of the two-dimensional non-stationary Euler equation. Arch. Rat. Mech. 27 (1968), 188?200

[19] A. Kiselev and V. Šverák. Small scale creation for solutions of the incompressible two-dimensional Euler equation. Ann. of Math. (2), 180(3):1205?1220, 2014.

[20] P.-L. Lions, N. Masmoudi, Global solutions for some Oldroyd models of non-Newtonian flows, Chinese Ann. Math. Ser. B 21 (2000) 131146.
[21] A. Majda, A. Bertozzi. *Vorticity and incompressible flow*. Cambridge Texts in Applied Mathematics, 27, 2002

[22] C. Marchioro and M. Pulvirenti, *Mathematical Theory of Incompressible Nonviscous Fluids*. Applied Mathematical Sciences, 96, Springer-Verlag, New York, 1994

[23] W. Wolibner, *Un théorème sur l’existence du mouvement plan d’unuide parfait, homogène, incompressible, pendant un temps infiniment long* (French), Mat. Z., 37 (1933), 698–726

[24] J. Wu, J. Zhao, *Mild ill-posedness in $L^\infty$ for 2D resistive MHD equations near a background magnetic field*. International Mathematics Research Notices, 2022, rnac007, https://doi.org/10.1093/imrn/rnac007

[25] Y. Yudovich. *Nonstationary flow of an ideal incompressible liquid*. Zh. Vych. Mat., 3 (1963), 1032–1066.

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