Cosmological application of the Maxwell gravity

Salih Kibaroğlu*

Maltepe University, Faculty of Engineering and Natural Sciences, 34857, Istanbul, Turkey and
Institute of Space Sciences (IEEC-CSIC) C. Can Magrans s/n, 08193 Barcelona, Spain
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In this study, we consider a cosmological model for the Maxwell gravity which is constructed by gauging the semi-simple extended Poincaré algebra. Inspired by the Einstein-Yang-Mills theory, we describe the Maxwell gauge field in terms of two additional time-dependent scalar fields. Within the context of a homogeneous and isotropic Friedmann-Lemaître-Robertson-Walker universe, we derive the Friedmann equations together with new contributions. Additionally, we find the exact expressions for the additional scalar fields, considering the exponential evolution of the scale factor. Moreover, we investigate the gauge theory of gravity based on the Maxwell algebra and demonstrate that this model leads to the (anti) de Sitter universe as well as a non-accelerated universe model.

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I. INTRODUCTION

Einstein’s general theory of relativity is widely regarded as the most effective theoretical framework for describing gravitational phenomena in a large range of scales. It has been tested through many experiments and observations. However, despite its remarkable observational accomplishments, several unresolved puzzles remain. One such example is the nature of dark energy, which is thought to be responsible for the accelerated expansion of the universe still remains unknown. Additionally, the theory is not useful at very high energies where quantum effects are expected to become important. These considerations constitute the driving force for examining generalized theories of gravity.

There exists an interesting extended gravitation theory that comes from the gauge theory of the Maxwell algebra. The Maxwell algebra can be interpreted as a modification of the Poincaré algebra by six additional tensorial Abelian generators that make the four-momenta non-commutative \([P_a, P_b] = iZ_{ab}\) [1–3]. If one constructs a gauge theory of gravity based on this algebra, it leads to a generalized theory of gravity that includes the cosmological constant and an additional term to the energy-momentum tensor and this gravitational model is called the Maxwell gravity or the Maxwell gauge theory of gravity [4–15]. Up to now, this energy-momentum term has not been extensively analyzed yet, but it is known that such an additional term may be related to dark energy [16, 17]. In the cosmological framework, a minimal cosmological model related to this symmetry can be found in [18] with a small discussion. Besides, it is also discussed that the gauge fields of the Maxwell symmetry may provide a geometric background to describe vector inflatons in cosmological models [19]. For the non-gravitational case, this symmetry group is used to describe a particle moving in a Minkowski spacetime filled with a constant electromagnetic background field which is formed by the additional degrees of freedom related to \(Z_{ab}\). From this idea, Maxwell symmetry is considered as the symmetry group of a particle moving in a constant electromagnetic field [20, 21]. It is also used to describe planar dynamics of the Landau problem [22], higher spin fields [23, 24], and applied to the string theory as an internal symmetry of the matter gauge fields [25].

The gauge field configurations in the context of the Friedmann-Lemaître-Robertson-Walker (FLRW) cosmology have been discussed in the literature before [26–33]. For example, in the Yang-Mills (YM) theories in which a non-Abelian YM field couples to the scalar curvature, one examines the cosmological consequences of the nonminimal gravitational coupling of the YM field. YM theory is also widely used in inflationary cosmology [32, 34–36] (for a complete review see [37]) and used in the models for dark energy [30, 31, 38–43] and dark matter [44–46]. Our aim in this paper is to find the potential effects of the additional energy-momentum tensor produced by the Maxwell gravity on the evolution of the universe. For this purpose, inspired by the YM theory and gauge field construction method proposed by [19], we study to construct a cosmological model for a class of the Maxwell gravity which is obtained gauging the semi-simple extended Poincaré algebra [6, 20, 47].

* salihkibaroglu@maltepe.edu.tr
This paper is organized as follows. In Section II, we provide a concise overview of gauging the semi-simple extended Poincaré algebra. In Section III, we investigate two scenarios within a homogeneous and isotropic FLRW universe to derive the Friedman equations for the Maxwell gravity. Finally, the last section concludes the paper by giving some discussion.

II. GAUGE THEORY OF THE SEMI-SIMPLE EXTENDED POINCARÉ GROUP

The Maxwell algebra [3] is a non-central extension of the Poincaré algebra which has non-commutative momentum generators as $[P_a, P_b] = i\lambda Z_{ab}$. Here $Z_{ab}$ is the additional antisymmetric tensorial generator which has an Abelian characteristic. One can derive the Maxwell algebra by applying the Lie algebra expansion method outlined in [48, 49]. from the (A)dS algebra. This technique makes it possible to generate two sets of algebras known as generalized Poincaré algebras (also referred to as $\mathfrak{B}_n$ algebra, where the Maxwell algebra is a specific instance called $\mathfrak{B}_4$ algebra) and generalized AdS algebras (which is known as $\mathfrak{AdS}_n$ algebras).

In this section, we shortly review the gauge theory of gravity based on the semi-simple extended Poincaré algebra [6, 47] which corresponds to $\mathfrak{AdS}_4$ algebra [50] and it can be established by direct sum of the anti-de Sitter AdS algebra $so(3, 2)$ and the Lorentz algebra $so(3, 1)$ [47]. This algebra extends the Poincaré algebra by using a non-Abelian tensorial generator $Z_{ab}$ and can be seen as the modification of the Maxwell algebra [20]. The gauge theory of gravity based on this algebra lead to generalizing the Maxwell gravity which was discussed in [6] in the context of the cosmological term problem.

The Lie algebra of this extended symmetry can be given as follows,

$$
[M_{ab}, M_{cd}] = i (\eta_{ad}M_{bc} + \eta_{bc}M_{ad} - \eta_{ac}M_{bd} - \eta_{bd}M_{ac}),
$$
$$
[M_{ab}, Z_{cd}] = i (\eta_{ad}Z_{bc} + \eta_{bc}Z_{ad} - \eta_{ac}Z_{bd} - \eta_{bd}Z_{ac}),
$$
$$
[Z_{ab}, Z_{cd}] = i \alpha (\eta_{ad}Z_{bc} + \eta_{bc}Z_{ad} - \eta_{ac}Z_{bd} - \eta_{bd}Z_{ac}),
$$
$$
[P_a, P_b] = i\lambda Z_{ab},
$$
$$
[M_{ab}, P_c] = i (\eta_{bc}P_a - \eta_{ac}P_b),
$$
$$
[Z_{ab}, P_c] = i \alpha (\eta_{bc}P_a - \eta_{ac}P_b),
$$

(1)

where $M_{ab}$ is the generator of rotation, $P_a$ is the generator of translation, $Z_{ab}$ is the tensor generator, $\eta_{ab}$ is the Minkowski metric which have $\text{diag}(\eta_{ab}) = (+, -, -, -)$ and the group indices $a, b, ... = 0, ..., 3$. Here, the constant $\lambda$ has the unit of $L^{-2}$ which will be related to the cosmological constant where $L$ is considered as the unit of length and $\alpha$ is a dimensionless constant. In addition to the Poincaré algebra, this algebra contains six new additional non-Abelian tensorial generators $Z_{ab}$ which behave as an antisymmetric second-rank Lorentz tensor.

To construct gauge theory of gravity based on the semi-simple Poincaré algebra, we firstly introduce the following one-form gauge field,

$$
\mathcal{A}(x) = e^a P_a + \frac{1}{2} B^{ab} Z_{ab} - \frac{1}{2} \omega^{ab} M_{ab},
$$

(2)

where $e^a(x) = e^a dx^a$, $B^{ab}(x) = B^a_{\mu} dx^\mu$, and $\omega^{ab}(x) = \omega^{\mu}_{ab} dx^\mu$ are the one form gauge fields of corresponding generators. Also, the unit dimension of all gauge fields have zero other than $[e^a] = L$. Using the structure equation $F = dA + \frac{1}{2} [A, A]$ and defining the curvatures as $F(x) = F^a P_a + \frac{1}{2} F^{ab} Z_{ab} - \frac{1}{2} R^{ab} M_{ab}$, we find the associated two-form curvatures as,

$$
F^a = D e^b - \alpha B^{a}_{c} \wedge e^c,
$$
$$
F^{ab} = DB^{ab} - \alpha B^{a}_{c} \wedge B^{b}_{c} - \lambda e^{a} \wedge e^{b},
$$
$$
R^{ab} = D\omega^{ab},
$$

(3)

where $D\Phi = [d + \omega]\Phi$ is the Lorentz covariant derivative. Then, taking the covariant derivative of the curvatures, we can obtain the Bianchi identities as follows;

$$
DF^a = R^a_{c} \wedge e^c - \alpha DB^a_{c} \wedge e^c + \alpha B^{a}_{c} \wedge D e^c,
$$
$$
DF^{ab} = R^{a}_{c} \wedge B^{b}_{c} - \alpha DB^{a}_{c} \wedge B^{b}_{c} - \lambda de^{a} \wedge e^{b},
$$
$$
DR^{ab} = 0.
$$

(4)
At this point, it is worth noting that a shifted connection can be established through the utilization of the one-forms $B^{ab}(x)$ to extend the Riemannian connection $\omega^{ab}(x)$ in the subsequent manner \cite{4, 51},

$$\tilde{\omega}^{ab} = \omega^{ab} - \alpha B^{ab}.$$ \hspace{1cm} (5)

This modified connection can be construed as a generalization of the Riemannian connection $\omega^{ab}$ to a non-Riemannian connection. In this particular context, the antisymmetry of $B^{ab}$ indicates that the system under consideration can be described by an Einstein–Cartan geometry. Furthermore, the non-metricity tensor associated with this geometry is identically zero due to the vanishing of the symmetric component of the shifted connection, i.e., $\tilde{\omega}^{(ab)} = 0$. In this sense, it is possible to regard this model as a particular extension to a non-Riemannian framework that is determined by the structure of the semi-simple Poincaré algebra.

A. Gravitational action

From this background, one can construct the following action by introducing a shifted curvature defined as $J^{ab} = R^{ab} - \mu F^{ab}$,

$$S_g = \frac{1}{4\kappa \mu \lambda} \int J \land * J$$

$$= \frac{1}{4\kappa} \int \left( \epsilon_{abcd} R^{ab} \land e^c \land e^d + \frac{\mu \lambda}{2} \epsilon_{abcd} e^a \land e^b \land e^c \land e^d \right)$$

$$+ \epsilon_{abcd} \left( \frac{\alpha}{\lambda} R^c \land B^{e} \land B^{ed} - \mu \Delta B^{ab} \land e^c \land e^d + \frac{\alpha}{2} B^{e}_{...} \land B^{eb} \land e^c \land e^d \right)$$

$$+ \epsilon_{abcd} \left( \frac{\mu}{2\lambda} \Delta B^{ab} \land \Delta B^{cd} + \frac{\mu \alpha}{\lambda} \Delta B^{ab} \land B^{e} \land B^{ed} + \frac{\mu \alpha^2}{2\lambda} B^{a}_{...} \land B^{eb} \land B^{e}_{f} \land B^{fd} \right),$$ \hspace{1cm} (6)

where we have neglected the total derivative terms. The first part of the above action contains the Einstein-Hilbert term together with the cosmological term and the other parts consist of mixed terms. Note that the asterisk represents Hodge duality operation, $\kappa = 8\pi G$ and $\mu$ is a constant.

In order to obtain the corresponding field equations, the action can be varied with respect to the gauge fields $\omega^{ab}(x)$, $B^{ab}(x)$, and $e^a(x)$, respectively. Thus one gets the following equations of motion,

$$\epsilon_{abcd} D J^{ab} + \mu \epsilon_{abcd} J^c \land B^{[b]} = 0,$$ \hspace{1cm} (7)

$$\epsilon_{abcd} D J^{ab} \land e^c = 0,$$ \hspace{1cm} (8)

Here, we want to note that the variation $\omega^{ab}(x)$ and $B^{ab}(x)$ satisfy the same equation (7).

Using Eq.(8) and transforming the tangent indices to world space-time indices, one can show that the generalized Einstein field equations can be written as follows,

$$R^\mu_{\alpha} - \frac{1}{2} \delta^\mu_\alpha R = T^\mu_{\alpha},$$ \hspace{1cm} (9)

where the Greek indices $\mu, \nu, ... = 0, 1, 2$ and $T^\mu_{\alpha}$ represents the Maxwell symmetry contributions which can be shown as,

$$T^\mu_{\alpha} = \mu \left[ e^\mu_\alpha e^\beta_\beta \left( D_{[\dot{\alpha}} B^{ab}_{\beta]} - \alpha B^{a}_{\dot{\alpha} c} B^{b}_{\beta c} \right) - \frac{1}{2} \delta^\mu_\alpha \epsilon^\beta_\beta \left( D_{\mu B^{ab}} - \alpha B^{a}_{\mu c} B^{b}_{c \beta} \right) + 3 \delta^\mu_\alpha \lambda \right].$$ \hspace{1cm} (10)

We also assume that $T^\mu_{\alpha}$ has a perfect fluid characteristics as $T^\mu_{\alpha} = \text{diag} (\rho, -P, -P, -P)$ with the energy density $\rho(t)$ and pressure $P(t)$. So the all matter-energy content of the universe is described by the Maxwell curvature contributions in this formulation.

On the other hand, the term $3 \delta^\mu_\alpha \lambda$ can be considered as a cosmological constant when we multiply with the parameter $\mu$. So we can divide the Eq.(10) into two parts such as

$$T^\mu_{\alpha} = T^\mu_{B,\alpha} + T^\mu_{\Lambda,\alpha},$$ \hspace{1cm} (11)
where $T_{B\mu}^{\alpha}$ and $T_{\Lambda \mu}^{\alpha}$ represents the energy-momentum tensors of the Maxwell gauge field $B^{ab}$ and the cosmological constant $\Lambda = \lambda \mu$. Thus we can write the field equation in Eq. (9) as the following form,

$$R^\mu_\alpha - \frac{1}{2} \delta^\mu_\alpha R - 3\Lambda \delta^\mu_\alpha = T_{B\mu}^{\alpha},$$

(12)

where we have used the explicit expression $T_{\Lambda \mu}^{\alpha} = 3\Lambda \delta^\mu_\alpha$ to indicate the cosmological constant in a more clear formulation. We assume again that $T_{B\mu}^{\alpha}$ and $T_{\Lambda \mu}^{\alpha}$ represent perfect fluids and thus the total energy density and total pressure take the following forms,

$$\rho = \rho_{B} + \rho_{\Lambda},$$

(13)

and

$$P = P_{B} + P_{\Lambda},$$

(14)

where we assume that all energy densities are positive definite. In the light of this result, we can say that the standard gravitational equation is extended to include a cosmological constant and additional energy-momentum tensors in terms of the Maxwell symmetry. These are the main characteristics of the Maxwell extended (super)-gravitational theories. In the next section, we will analyze the cosmological behavior of this gravitational equation.

### III. COSMOLOGICAL SETUP

Motivated by the cosmological implications of the Einstein-Yang-Mills framework, where a non-Abelian gauge field was utilized to explore inflationary or late-time accelerated expansion, we consider finding potential cosmological consequences of the Maxwell gauge fields $B^{ab}$ which present in gravitational field equation in Eq. (12). In general, for most cosmological models, the metric is assumed to be a flat FLRW metric because the universe seems to be nearly flat so we begin our analysis with the following line element,

$$ds^2 = dt^2 - a(t)^2 (dx^2 + dy^2 + dz^2).$$

(15)

In this space, the space-time metric can be defined in terms of the vierbein fields $e^a_\mu (x)$,

$$g_{\mu \nu} (x) = \eta_{ab} e^a_\mu e^b_\nu,$$

(16)

where $\eta_{ab}$ is the tangent space metric which is

$$\eta_{ab} = g_{\mu \nu} (x) e^a_\mu e^b_\nu.$$  

(17)

Here the vierbein field can be defined as,

$$e^a_\mu (x) = (e^a_0, e^a_i) = (\delta^a_0, -\delta^a_i a(t)),$$

(18)

and satisfy the following relations

$$e^a_\mu (x) e^b_\mu (x) = \delta^a_b, \quad e^a_\mu (x) e^a_\nu (x) = \delta^\mu_\nu,$$

(19)

with

$$e = det (e^a_\mu) = \sqrt{-det (g_{\mu \nu})}.$$  

(20)

In the conventional non-Abelian gauge theory which is minimally coupled to Einstein’s gravity in four dimensions, one can consider a simple gauge invariant Lagrangian such as

$$\mathcal{L} = \frac{1}{2} R + \mathcal{L}_G (F^A_{\mu \nu}),$$

(21)
where \( \mathcal{L}_G \) (\( F^{\mu \nu}_{A} \)) is a generic gauge invariant action which may include the Yang-Mills term made out of powers of the field strength \( F^{\mu \nu}_{A} \) and \( A, B, ... = 1, 2, ... \text{dim} \mathcal{G} \). In general, we note that the gauge group \( \mathcal{G} \) can be any non-Abelian compact group. Here the field strength tensor is defined as

\[
F^{\mu \nu}_{A} = \partial_{\mu} A^{A}_{\nu} - \partial_{\nu} A^{A}_{\mu} - g f^{A}_{BC} A^{B}_{\mu} A^{C}_{\nu},
\]

where \( A^{A}_{\mu} \) is the gauge field, \( g \) is the gauge coupling and \( f^{A}_{BC} \) are the structure constants of the gauge group. If we choose the gauge group \( \mathcal{G} \) to be \( SU(2) \), the gauge field can be introduced as follows,

\[
A^{\alpha}_{\mu} = \{ 0, \psi(t) e^{\alpha}_{t} \},
\]

where \( \psi(t) \) is a scalar field under rotations and \( e^{\alpha}_{t} = -a(t) \delta^{\alpha}_{t} \) are the triads of the spatial hyper-surface (for more details, see the report [37]).

In our scenario, the formulation of the Maxwell gauge field slightly differs from the expression given in Eq.(23). According to [19], we define the Maxwell gauge fields \( B^{ab}_{\mu} \) using one-dimensional fields \( \psi(t) \) and \( \zeta(t) \),

\[
B^{bx}_{\mu}(x) \rightarrow B^{bs}_{\mu}(t) = (0, \delta^{e}_{s} \psi(t)),
\]

\[
B^{tx}_{\mu}(x) \rightarrow B^{ts}_{\mu}(t) = (0, \epsilon^{e}_{s} \zeta(t)),
\]

where \( r,s = 1,2,3 \) are the tangent indices and \( i,j = 1,2,3 \) are the space-time indices. Here, \( \delta^{e}_{s} \) and \( \epsilon^{e}_{s} \) are three dimensional \( so(3) \) tensors.

\section{A. \( \alpha = \mu \) case}

In this case, we interested in the torsion-free condition which requires the condition \( e^{a} \wedge e^{e} = \frac{\mu}{2} B^{e}_{j} \wedge B^{t}_{c} \) to satisfy the equations of motion in Eq.(7). Therefore, the spin connection \( \omega^{ab}_{\mu}(x) \) can be expressed in terms of vierbein and the Levi Civita connection \( \Gamma^{\nu}_{\mu \sigma} \) as

\[
\omega^{ab}_{\mu} = e^{a}_{\nu} \partial_{\mu} e^{b \nu} + e^{a}_{\nu} \Gamma^{\nu}_{\mu \sigma} e^{b \sigma}.
\]

Then applying Eqs.(24), (25) and (26) to the field equation in Eq.(12), we are ready to find the Friedmann equations. The first equation can be derived from the \( (0,0) \) component of Eq.(12),

\[
\left( \frac{\dot{a}}{a} \right)^{2} = \frac{\mu}{a^{2}} \left[ 2 \dot{\psi} \dot{\psi} - \mu \left( \psi^{2} - \zeta^{2} \right) \right] + \Lambda,
\]

where the dot denotes derivative with respect to the cosmic time \( t \). The \( (i,i) \) component of Eq.(12) leads to the following equation

\[
\frac{2 \dot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^{2} = \frac{\mu}{a^{2}} \left[ 2 \dot{\psi} \dot{\psi} + 2 a \ddot{\psi} - \mu \left( \psi^{2} - \zeta^{2} \right) \right] + 3 \Lambda,
\]

Then by making use of Eqs.(27) and (28), the acceleration equation can be obtained as,

\[
\frac{\ddot{a}}{a} = \frac{\mu \ddot{\psi}}{a} + \Lambda,
\]

Here according to this equation, we see that the acceleration of the universe depends on \( \psi(t) \) and \( \Lambda \). If \( \psi(t) \) is constant or zero the corresponding universe exhibits behavior like the ordinary de Sitter universe in which the acceleration arises due to the cosmological constant. On the other hand, when \( \dot{\psi} = -\alpha \Lambda / \mu \), the acceleration vanishes.

Moreover, by using the Eqs.(27) and (28) one can find the energy density and the pressure expressions for the Maxwell gauge field and the cosmological constant as follows,
\[ \rho_B = \frac{3\mu}{\kappa a^2} \left[ 2\dot{a}\psi - \mu \left( \psi^2 - \zeta^2 \right) \right], \]  

(30)

\[ P_B = -\frac{\mu}{\kappa a^2} \left( 2\dot{a}\psi + 2a\dot{\psi} - \mu \left( \psi^2 - \zeta^2 \right) \right) = -\frac{2\mu\dot{\psi}}{\kappa a} - \frac{1}{3} \rho_B, \]  

(31)

\[ \rho_\Lambda = \frac{3\Lambda}{\kappa}, \quad P_\Lambda = -\frac{3\Lambda}{\kappa}. \]  

(32)

For completeness, if we combine Eqs. (27) and (29), we also derive the time derivative of the Hubble parameter \( H(t) = \frac{\dot{a}(t)}{a(t)} \) as

\[ \dot{H} = \frac{\mu}{a^2} \left( a\dot{\psi} - 2\dot{a}\psi + \mu \left( \psi^2 - \zeta^2 \right) \right) = -\frac{\kappa}{2} (\rho_B + P_B), \]  

(33)

and taking the time derivation of Eq. (27) and using Eq. (33), we get

\[ \dot{\rho}_B + 3H(\rho_B + P_B) = 0, \]  

(34)

which is the energy conservation equation for the Maxwell gauge field contribution.

For the Friedmann equations

In the context of the exponential evolution of the scale factor, we proceed to analyze the Friedmann equations of this model. Assuming the scale factor takes the form \( a(t) = a_0 e^{ht} \) (\( a_0 \) and \( h \) are constants) for the spatially-flat FLRW universe (15) which is a solution of Eq. (12). Building upon this background, we begin by solving the acceleration equation in Eq. (29) with respect to \( \psi(t) \), we get

\[ \psi(t) = \frac{a_0 e^{ht} \left( h^2 - \Lambda \right)}{h\mu} + C, \]  

(35)

and putting this equation into Eq. (27) or Eq. (28), one can get the following solution for \( \zeta(t) \),

\[ \zeta(t) = \pm \frac{1}{\mu h} \sqrt{a_0^2 \Lambda e^{2ht} (\Lambda - h^2) - 2\Lambda h a_0 C e^{ht} + h^2 \mu^2 C^2}. \]  

(36)

On the other hand, if we solve Eq. (27) with respect to \( \psi \), we get;

\[ \psi(t) = \frac{1}{\mu} \left( a_0 he^{ht} \pm \sqrt{a_0^2 \Lambda e^{2ht} + \mu^2 \zeta(t)^2} \right), \]  

(37)

then equate with Eq. (35) and solve with respect to \( \zeta(t) \), we reach the same solution as given in Eq. (36). Thus these results show the consistency of the Friedmann equations in Eqs. (27), (28) and (29) under the exponential evolution of the scale factor. So Eqs. (35) and (36) together with the definition of the scale factor turn out to be a solution to the Friedmann equations.

Furthermore, by setting the parameter \( h = \sqrt{\Lambda} \) and fixing the integration constant \( C \) to zero in Eq. (35), the acceleration equations (29) can be transformed into those describing the pure de Sitter universe. We also note that utilizing a similar approach, exact formulations for \( \psi(t) \) and \( \zeta(t) \) can be derived in a more comprehensive scenario where \( a(t) = a_0 e^{f(t)} \) with \( f(t) \) representing an arbitrary function of the cosmic time.
B. $\alpha = 0$ case

If we apply this condition to Eq. (1), this means that we are working in the Maxwell algebra background with the following commutation relations [3, 4],

\[
\begin{align*}
[M_{ab}, M_{cd}] &= i (\eta_{ad} M_{bc} + \eta_{bc} M_{ad} - \eta_{ac} M_{bd} - \eta_{bd} M_{ac}) , \\
[M_{ab}, Z_{cd}] &= i (\eta_{ad} Z_{bc} + \eta_{bc} Z_{ad} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac}) , \\
[P_a, P_b] &= i \lambda Z_{ab} , \\
[M_{ab}, P_c] &= i (\eta_{bc} P_a - \eta_{ac} P_b) .
\end{align*}
\]

(38)

Now the additional tensor generator $Z_{ab}$ exhibits Abelian characteristics. Gauging this algebra lead to a generalized theory of gravity known as Maxwell gravity which was suggested as a potential alternative approach to addressing the cosmological constant problem [4].

After following the same gauging procedure given in Section (II), one can derive that the curvature two-forms reduces to,

\[
\begin{align*}
F^a &= D e^b , \\
F^{ab} &= D B^{ab} - \lambda e^a \wedge e^b , \\
R^{ab} &= D \omega^{ab} ,
\end{align*}
\]

(39)

Then taking account of the action in Eq. (6) by using Eq. (39), the equations of motion with respect to the gauge fields $\omega^{ab}(x), B^{ab}(x)$, and $e^a(x)$, takes the following form

\[
\begin{align*}
\epsilon_{abcd} D J^{ab} + \mu \epsilon_{abcd} J_a \wedge B^{|b|} &= 0 , \\
\epsilon_{abcd} D J^{ab} &= 0 , \\
\epsilon_{abcd} J^{ab} \wedge e^c &= 0 .
\end{align*}
\]

(40)

Finally, using the $e^a(x)$, one can find the field equation similar to the formulation of Eq. (8) with the following energy-momentum tensor,

\[
T_B^{\mu \alpha} = \mu \left( \epsilon_{a}^{\mu} \epsilon_{b}^{\beta} D_{[a} B_{b]}^{\alpha \beta} - \frac{1}{2} \delta_{a}^{\mu} \epsilon_{a}^{\alpha} \epsilon_{b}^{\beta} D_{[\mu} B_{\nu]}^{ab} \right) .
\]

(41)

In this background, we again interested in the torsion-free condition with the spin connection in Eq. (26). Therefore the Friedmann equations in Eqs. (27), (28) and (29) reduce to the following forms,

\[
\left( \frac{\dot{a}}{a} \right)^2 = 2 \mu \dot{\psi} a^2 + \Lambda ,
\]

(42)

\[
2 \ddot{a} + \left( \frac{\dot{a}}{a} \right)^2 = 2 \mu \left( \dot{\psi} + a \ddot{\psi} \right) + 3 \Lambda ,
\]

(43)

\[
\frac{\dot{a}}{a} = \frac{\mu \ddot{\psi}}{a} + \Lambda ,
\]

(44)

and the energy density and the pressure expressions related to the $B^{ab}_\mu(x)$ gauge field reduce to

\[
\rho_B = \frac{6 \mu \dot{\psi}}{\kappa a^2} , \quad P_B = -\frac{2 \mu \ddot{\psi}}{\kappa a^2} - \frac{1}{3} \rho_B .
\]

(45)

At this point, it should be noted that the second scalar field, denoted as $\zeta(t)$, exhibits no influence on the Friedmann equations under these particular circumstances.
If we solve Eq. (42) with respect to $\psi(t)$, we get
\[ \psi = \frac{\dot{a}^2 - \Lambda a^2}{2\mu\dot{a}}, \] (46)
then substituting this result to Eq. (43) or Eq. (44), we find the following solutions for the scale factor;
\[ a = e^{\pm\sqrt{\Lambda}(t-C_1)}, \quad C_1 t + C_2, \] (47)
where $C_1$ and $C_2$ are the integration constants. For consistency, one can use the solution of Eq. (44)
\[ \psi = \int \frac{\dot{a} - \Lambda a}{\mu} dt + C_1, \] (48)
and equate with Eq. (46), this calculation leads to the same solution as Eq. (47). The first solution in Eq. (47) correspond to (anti) de Sitter universe and this solution forces the function $\psi$ to be zero. On the other hand, the second solution results in a non-accelerating universe due to $\ddot{a} = 0$. These findings suggest that the Maxwell gravity formulation under consideration offers a limited range of solutions and therefore lacks a diverse foundation for investigating the universe’s evolution.

IV. CONCLUSION

In this paper, we analyzed the Maxwell gravity model which is based on gauging the semi-simple extended Poincaré algebra [6] in the context of cosmology. In analogy with the Einstein-Yang-Mills theory, we described the Maxwell gauge field as a function of time in terms of two time-dependent functions $\psi(t)$ and $\zeta(t)$ in Eqs. (24) and (25). By using this definition, we found the Friedmann equations involving contributions of the Maxwell gauge field together with the cosmological constant. Then we analyzed the solution of Friedman equations under the exponential evolution of the scale factor and derived the explicit expression of $\psi(t)$ and $\zeta(t)$. Thus we show that this model provides a useful background to study one of the most common cosmological models. In addition to this approach, it is known that the power series evolution of the scale factor such as $a(t) \approx t^{\alpha}$ produces well-known cosmological models such as the radiation dominated ($\alpha = 1/2$) or the matter dominated ($\alpha = 2/3$) universe. So one can study this model and derive the exact expression of $\psi(t)$ and $\zeta(t)$.

It is important to note that if one specifies $\psi(t)$ or $\zeta(t)$ at first instead of defining the scale factor, this approach may lead to interesting cosmological models. On the other hand, one can add a matter term to the action (6) instead of trying to describe the all matter-energy content of the universe by the Maxwell curvature contributions to find more general solutions.

We also analyzed $\alpha = 0$ condition which corresponds to the minimal model of the Maxwell algebra given in [3, 4] and we find that this model only allows the (anti) de Sitter universe models and a non-accelerated universe model (see Eq. (47)). We also see that the function $\zeta(t)$ does not make any contribution to the Friedmann equations for this condition. Despite these limitations, we can easily say that by using different gravitational actions (see [4]) one may derive different cosmological models.

Note that similar to the Maxwell algebra, the non-commutativity of the momentum generators would lead to important experimental consequences and a modification of the uncertainty principle (for the cosmological effects, see [52] and references therein). If one establishes a connection between these studies and the Maxwell algebra, this idea may produce interesting results or gets some constraints or limits for the parameter $\lambda$.

Finally, we can say that the Maxwell symmetry offers a potential alternative approach that may help the explanation of the evolution of our universe. Furthermore, the additional gauge fields to the gravitation theory presents a useful foundation to examine unresolved phenomena in General Relativity, such as the inflationary universe (for more information [37]) and dark energy models [38]. Therefore, the model derived from this study will be explored elsewhere in the context of these fields.

CONFLICTS OF INTEREST

The authors declare no conflicts of interest.
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