**Abstract.** Our aim in this paper is to introduce the relatively new concept of *-density of a fuzzy graph and *-balanced fuzzy graph. Several examples and results are also provided. In addition, many operations on fuzzy graphs that preserves *-balanced are explored.

**Keywords:** Fuzzy graph. *-density, *-balanced.

**AMS Subject Classification.** 05C72.

1. **Introduction**

Graph theory has several interesting applications in system analysis, operations research, economics and many other fields. Since most of the time the aspects of graph and graph problems are uncertain, it is a good idea to deal with these aspects via the methods of fuzzy logic. The notion of fuzzy set was first introduced by Zadeh [12] in his landmark paper “Fuzzy sets” in 1965 and the concept of fuzzy graph was first introduced by Rosenfeld [9] in his paper "Fuzzy Graphs". Since that time, several authors explored this type of graphs. As the notions of degree, complement, completeness, regularity and many others play very important role in the crisp graph case, it is a nice idea to try to see what corresponds to these notions in the case of fuzzy graphs.

Sunitha and Kumar [13] defined several new operations on fuzzy graphs and they also modified the definition of complement of a fuzzy graph so that to agree with the crisp case in graph. In 2011, AL-Hawary [1] introduced the new concept of balanced fuzzy graphs. He defined three new operations on fuzzy graphs and explored what classes of fuzzy graphs are balanced. Since then, many authors have studied the idea of balanced on distinct types of fuzzy graphs, see for example [3, 13]. Moreover, Al-Hawary and others explored the idea of balanced fuzzy graphs in [1, 2, 3, 4, 5]. We start by recalling some necessary definitions and results.

**Definition 1.** [13] A fuzzy subset of a set $V$ is mapping $\sigma : V \rightarrow [0,1]$. For any $v \in V$, $\sigma(v)$ is called the degree of membership of $v$ in $\sigma$. 
Definition 2. [13] A fuzzy relation on a set $V$ is mapping $\mu : V \times V \rightarrow [0, 1]$. A fuzzy relation $\mu$ on a fuzzy subset $\sigma$ is a fuzzy relation on $V$ such that $\mu(u, v) \leq \sigma(u) \land \sigma(v)$ for all $u, v \in V$, where $\land$ stands for minimum.

Definition 3. [13] A fuzzy graph is a pair of functions $G : (\sigma, \mu)$ where $\sigma$ is a fuzzy subset of $V$ and $\mu$ is a symmetric fuzzy relation on $\sigma$. The underlying crisp graph of $G$ is denoted by $G^* : (\sigma^*, \mu^*)$ where $\sigma^*$ is referred to as the nonempty subset $V$ of nodes and $\mu^* = E \subseteq V \times V$.

All through this paper, we only consider non-empty fuzzy graphs.

Definition 4. [8]. A fuzzy graph $G : (\sigma, \mu)$ is called complete if $\mu(u, v) = \sigma(u) \land \sigma(v)$ for all $u, v \in V$ and . A fuzzy graph $G : (\sigma, \mu)$ is called strong if $\mu(u, v) = \sigma(u) \land \sigma(v)$ for all $u, v \in E$.

Note that any complete fuzzy graph is strong, but the converse needs not be true. Let $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ be two fuzzy graphs.

Definition 5. [11]. Two fuzzy graphs $G_1$ and $G_2$ are isomorphic if there exists a bijection $h : V_1 \rightarrow V_2$ such that $\sigma_1(x) = \sigma_2(h(x))$ and $\mu_1(x, y) = \mu_2(h(x), h(y))$ for all $x, y \in V_1$.

Lemma 1. Let $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ be isomorphic fuzzy graphs. Then $\sum_{v \in V_1} \sigma_1(v) = \sum_{v \in V_2} \sigma_2(v)$ and $\sum_{u,v \in V_1} \mu_1(u, v) = \sum_{u,v \in V_2} \mu_2(u, v)$.

Several operations on fuzzy graphs were introduced in [13] such as union $G_1 \cup G_2$, the join $G_1 + G_2$, the Cartesian product $G_1 \times G_2$ and the composition $G_1 \circ G_2$. Also recently in [1], the operations of direct product $G_1 \sqcap G_2$, semi-direct product $G_1 \cdot G_2$ and strong product $G_1 \otimes G_2$. In addition, both authors studied the operations that preserves balanced notion. For more on operations on fuzzy graphs, see [1, 2, 4, 5, 13].

Definition 6. [13]. The complement of a fuzzy graph $G : (\sigma, \mu)$ is a fuzzy graph $G^* : (\sigma^*, \mu^*)$, where $\sigma^* = \sigma$ and $\mu^*(u, v) = \sigma(u) \land \sigma(v) - \mu(u, v)$, $\forall u, v \in V$.

Next we recall the following two results from [13].

Lemma 2. Let $G : (\sigma, \mu)$ be a self-complementary fuzzy graph. Then $\sum_{u,v \in V} \mu(u, v) = (1/2) \sum_{u,v \in V} (\sigma(u) \land \sigma(v))$.
Lemma 3. Let $G : (\sigma, \mu)$ be a fuzzy graph with $\mu(u, v) = (1/2)(\sigma(u) \wedge \sigma(v))$ for all $u, v \in V$. Then $G$ is self-complementary.

Definition 7. Let $G : (\sigma, \mu)$ be a fuzzy graph. The degree of a vertex $u$ is $d_G(u) = \sum_{u \neq v} \mu(u, v)$. The total degree of vertex $u$ is $td_G(u) = \sigma(u) + d_G(u)$.

Definition 8. A fuzzy graph $G : (\sigma, \mu)$ is called $k$-regular if $d_G(u) = k$ for every $u \in V$. $G : (\sigma, \mu)$ is called $k$-totally regular if $td_G(u) = k$ for every $u \in V$.

In general there does not exist any relationship between regular fuzzy graphs and totally fuzzy graphs.

Definition 9. Let $G : (\sigma, \mu)$ be a fuzzy graph. Then $\sigma$ is called a $c$-constant function if $\sigma(v) = c$ for all $v \in V$ and $\mu$ is called a $c$-constant function if $\mu(u, v) = c$ for all $u, v \in V$.

Our aim in this paper is to define the concept of $^*$-density of a fuzzy graph. In fact, it is a modification of the concept of density of fuzzy graph in which we change the denominator so as to satisfy more properties and to agree more with what is know about density of graphs. Moreover, we introduce and explore what we call $^*$-balanced fuzzy graph. Several examples and results are also provided and certain classes of $^*$-balanced fuzzy graphs are given.

2. $^*$-Balanced Fuzzy Graphs

The main idea in this section is to define the concept of $^*$-density of a fuzzy graph and $^*$-balanced fuzzy graph. We explore these notions and we get some nice results that are analogous to those in [1]. We begin by the following Definition:

Definition 10. The $^*$-density of a fuzzy graph $G : (\sigma, \mu)$ is $D^*(G) = \frac{2\sum_{u,v \in V} \mu(u, v)}{\sum_{u \in V} \sigma(u)}$. $G$ is $^*$-balanced if $D^*(H) \leq D^*(G)$ for all non-empty fuzzy subgraphs $H$ of $G$.

Theorem 4. Any complete fuzzy graph with $|V| \geq |E|$ has $^*$-density $D^*(G) \leq 2$.

Proof: Let $G$ be complete fuzzy graph. Since $\sum_{u \in V} \sigma(u) \geq \sum_{u,v \in V} \mu(u, v)$, then $\frac{\sum_{u,v \in V} \mu(u, v)}{\sum_{u \in V} \sigma(u)} \leq 1$ and hence $D^*(G) \leq 2$.

Theorem 5. Every self-complementary fuzzy graph has a density less than or equal 1.

Proof: Let $G$ be self-complementary fuzzy graph. Then as $\sum_{u,v \in V} \sigma(u) \wedge \sigma(v) \leq \sum_{u \in V} \sigma(u), D^*(G) \leq \frac{2\sum_{u,v \in V} \mu(u, v)}{\sum_{u,v \in V} \sigma(u) \wedge \sigma(v)}$. Now by Lemma 3, $D^*(G) \leq \frac{2\sum_{u,v \in V} \mu(u, v)}{2\sum_{u,v \in V} \mu(u, v)} = 1$. 

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The converse of the preceding result needs not be true.

**Theorem 6.** Let \( G : (\sigma, \mu) \) be fuzzy graph such that \( \mu(u, v) = \frac{\sigma(u) \Delta \sigma(v)}{2} \) for all \( u, v \in V \). Then \( D^*(G) \leq 1 \).

**Proof:** By Lemma \( 2 \) \( G \) is self-complementary and thus by Theorem \( 5 \) \( D^*(G) \leq 1 \).

**Lemma 7.** Let \( G_1 \) and \( G_2 \) be complete fuzzy graphs. Then \( D^*(G_i) \leq D^*(G_1 \cap G_2) \) for \( i = 1, 2 \) if and only if \( D^*(G_1) = D^*(G_2) = D^*(G_1 \cap G_2) \).

**Proof:** If \( D^*(G_i) \leq D^*(G_1 \cap G_2) \) for \( i = 1, 2 \), then
\[
D^*(G_1) = 2\left( \sum_{u_1, u_2 \in V_1} \mu_1(u_1, u_2) \right) / \sum_{u_1 \in V_1} \sigma_1(u_1) \\
\geq 2\left( \sum_{u_1, u_2 \in V_1} \mu_1(u_1, u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2) \right) / \left( \sum_{u_1, u_2 \in V_1} \sigma_1(u_1) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2) \right) \\
\geq 2\left( \sum_{u_1, u_2 \in V_1} \mu_1(u_1, u_2) \wedge \mu_2(v_1, v_2) \right) / \left( \sum_{u_1, u_2 \in V_1} \sigma_1(u_1) \wedge \sigma_2(v_1) \right) \\
= D^*(G_1 \cap G_2).
\]

Hence \( D^*(G_1) \geq D^*(G_1 \cap G_2) \) and thus \( D^*(G_1) = D^*(G_1 \cap G_2) \). Similarly, \( D^*(G_2) = D^*(G_1 \cap G_2) \). Therefore, \( D^*(G_1) = D^*(G_2) = D^*(G_1 \cap G_2) \).

The converse is trivial.

**Theorem 8.** Let \( G_1 \) and \( G_2 \) be fuzzy complete \( * \)-balanced graphs. Then \( G_1 \cap G_2 \) is \( * \)-balanced if and only if \( D^*(G_1) = D^*(G_2) = D^*(G_1 \cap G_2) \).

**Proof:** If \( G_1 \cap G_2 \) is \( * \)-balanced, then \( D^*(G_i) \leq D^*(G_1 \cap G_2) \) for \( i = 1, 2 \) and by Lemma \( 7 \) \( D^*(G_1) = D^*(G_2) = D^*(G_1 \cap G_2) \).

Conversely, if \( D^*(G_1) = D^*(G_2) = D^*(G_1 \cap G_2) \) and \( H \) is a fuzzy subgraph of \( G_1 \cap G_2 \), then there exist fuzzy subgraphs \( H_1 \) of \( G_1 \) and \( H_2 \) of \( G_2 \). As \( G_1 \) and \( G_2 \) are \( * \)-balanced and \( D^*(G_1) = D^*(G_2) = n_1/r_1 \), then \( D^*(H_1) = a_1/b_1 \leq n_1/r_1 \) and \( D^*(H_2) = a_2/b_2 \leq n_1/r_1 \). Thus \( a_1r_1 + a_2r_1 \leq b_1n_1 + b_2n_1 \) and hence \( D^*(H) \leq (a_1 + a_2)/(b_1 + b_2) \leq n_1/r_1 = D^*(G_1 \cap G_2) \). Therefore, \( G_1 \cap G_2 \) is \( * \)-balanced.

The above result needs not be true when one of the fuzzy graphs is not complete.

The preceding result need not be true if the operation \( \cap \) is replaced by \( \bullet, \odot, +, \circ, \times \). We only give an example of the case \( \circ \). We end this section by showing that isomorphism between fuzzy graphs preserve \( * \)-balanced.
Theorem 9. Let $G_1$ and $G_2$ be isomorphic fuzzy graphs. If $G_2$ is $*$-balanced, then $G_1$ is $*$-balanced.

Proof: Let $h : V_1 \rightarrow V_2$ be a bijection such that $\sigma_1(x) = \sigma_2(h(x))$ and $\mu_1(x, y) = \mu_2(h(x), h(y))$ for all $x, y \in V_1$. By Lemma 12, $\sum_{x \in V_1} \sigma_1(x) = \sum_{x \in V_2} \sigma_2(x)$ and $\sum_{x, y \in V_1} \mu_1(x, y) = \sum_{x, y \in V_2} \mu_2(x, y)$. If $H_1 = (\sigma_1, \mu_1)$ is a fuzzy subgraph of $G_1$ with underlying set $W$, then $H_2 = (\sigma_2, \mu_2)$ is a fuzzy subgraph of $G_2$ with underlying set $h(W)$ where $\sigma_2(h(x)) = \sigma_1(x)$ and $\mu_2(h(x), h(y)) = \mu_1(x, y)$ for all $x, y \in W$. Since $G_2$ is $*$-balanced, $D^*(H_2) \leq D^*(G_2)$ and so

$$2\left(\sum_{x, y \in W} \mu_2(h(x), h(y))\right)/ \sum_{x \in W} \sigma_2(x) \leq 2\left(\sum_{x, y \in W} \mu_2(x, y)\right)/ \sum_{x \in W} \sigma_2(x)$$

and so

$$2\left(\sum_{x, y \in W} \mu_1(x, y)\right)/ \sum_{x \in W} \sigma_1(x) \leq 2\left(\sum_{x, y \in W} \mu_1(x, y)\right)/ \sum_{x \in W} \sigma_1(x).$$

Thus $D^*(H_1) \leq D^*(G_1)$. Therefore, $G_1$ is $*$-balanced.

3. On regular fuzzy graphs

Theorem 10. If $G : (\sigma, \mu)$ is an $r$- regular fuzzy graph with $|V| = p$, then $G$ has a density $D^*(G) = pr/ \sum_{v \in V} \sigma(v)$.

Proof: Since $G$ is an $r$- regular fuzzy graph, then $d_G(v) = r$ for all $v \in V$. Now as $\sum_{v \in V} d_G(v) = 2 \sum_{u,v \in V} \mu(v, u), \sum_{u,v \in V} \mu(v, u) = \frac{\sum_{v \in V} r}{2} = \frac{pr}{2}$. Thus $D^*(G) = pr/ \sum_{v \in V} \sigma(v)$.

Corollary 11. If $G : (\sigma, \mu)$ is an $r$- regular and $\sigma$ is $c$- constant function, then $D^*(G) = r/c$.

Theorem 12. If $G : (\sigma, \mu)$ is an r-totally regular fuzzy graph with $|V| = p$, then $G$ has a $*$-density $D^*(G) = (pr/ \sum_{v \in V} \sigma(v)) - 1$.

Proof: Since $G$ is an r-totally regular fuzzy graph then $r = td_G(u) = d_G(u) + \sigma(u)$ for all $u \in V$. Thus $\sum_{u \in V} r = \sum_{v \in V} d_G(v) + \sum_{v \in V} \sigma(v)$. Hence $pr = 2 \sum_{u,v \in V} \mu(v, u) + \sum_{v \in V} \sigma(v)$. So $pr/ \sum_{v \in V} \sigma(v) = \frac{2 \sum_{u,v \in V} \mu(v, u)}{\sum_{v \in V} \sigma(v)} + 1$. Therefor, $D^*(G) = (pr/ \sum_{v \in V} \sigma(v)) - 1$.

Corollary 13. If $G : (\sigma, \mu)$ is an r-totally regular and $\sigma$ is $c$- constant function, then $D^*(G) = \frac{r}{c} - 1$.

Note that the operations of $\cap, \bullet, \otimes, +, \oplus, \times$ do not preserve r-totally regular property. We only give a counter example for case $\oplus$:
4. Classes of *-Balanced Fuzzy Graphs

**Theorem 14.** If the complete graph on \( n \)-vertices \( K_n \) has \( \delta \) as a c- constant function and complete, then \( K_n \) is *-balanced.

**Proof:** Now \( D^*(K_n) = \frac{2c(n-1)}{c} = n - 1 \). Any subgraph \( H \) of \( K_n \) has edges less than \( K_n \) or less edges and less vertices than \( K_n \). If \( H \) has less edges, it is clear that \( D^*(H) \leq D^*(K_n) \). Now if \( H \) has less edges and less vertices, say \( H \) has \( n - s \) vertices, then

\[
|E(H)| = \frac{n(n-1)}{2} - ((n-1) + (n-2) + ... + (n-s)) = \frac{n(n-1)}{2} - \frac{(sn - s(s+1)}{2} = \frac{n(n-1) - 2sn + s(s+1)}{2}.
\]

Thus

\[
D^*(H) = \frac{2c\left(\frac{n(n-1)-2sn+s(s+1)}{2}\right)}{c(n-s)} = \frac{n^2 - n - 2sn + s^2 + s}{n - s} = n - (s + 1).
\]

As \( 1 < s + 1 \), \( D^*(H) \leq n - 1 = D^*(K_n) \) and so \( K_n \) is *-balanced.

Even when \( \mu \) is not a constant function but \( \sigma \) is a constant function, \( K_n \) needs not be *-balanced as shown in Figure 6. Also when \( \sigma \) is not a constant function but \( \mu \) is a constant function, \( K_n \) needs not be *-balanced.

**Theorem 15.** If the cycle \( C_n \) has \( \sigma \) as a c- constant function and strong for \( n > 3 \), then \( C_n \) is *-balanced.

**Proof:** Now \( D^*(C_n) = \frac{2cn}{cn} = 2 \). Any subgraph \( H \) of \( C_n \) has edges less than \( C_n \) or less edges and less vertices than \( C_n \). If \( H \) has less edges, it is clear that \( D^*(H) \leq D^*(C_n) \). If \( H \) has less edges and less vertices than \( C_n \), say \( H \) has \( n - s \) vertices, then we have three cases:

Case 1. No two of the \( s \)-vertices are adjacent. Then \( |E(H)| = n - 2s \) and so

\[
D^*(H) = \frac{2\sigma(n-2c)}{\sigma} = 2\frac{n-2c}{n-c} \leq 2.
\]

Case 2. The subgraph consisting of these \( s \) vertices is isomorphic to a path graph.

Then \( |E(H)| = n - (2 + s - 1) = n - s - 1 \). Hence, \( D^*(H) = \frac{2\sigma(n-s-1)}{\sigma} \leq 2 \).

Case 3. The subgraph consisting of these \( s \) vertices has \( s_1 \) vertices of those in Case 1 and \( s_2 \) vertices of those in Case 2. Then \( |E(H)| = n - 2s_1 - (2 + s_2 - 1) = n - 2s_1 - s_2 - 1 \). Hence \( D^*(H) = \frac{2\sigma(n-2s_1-s_2-1)}{\sigma} \leq 2 \).

Therefore \( C_n \) is *-balanced.
**Theorem 16.** If the Petersen fuzzy graph $P_{10}$ has $\sigma$ as a c-constant function and strong, then $P_{10}$ is $^*$-balanced.

**Proof:** Now $D^*(P_{10}) = \frac{2c(15)}{c(10 - s)} = 3$. Any subgraph $H$ of $P_{10}$ has edges less than $P_{10}$ or less edges and less vertices than $P_{10}$. If $H$ has less edges, it is clear that $D^*(H) \leq D^*(P_{10})$. If $H$ has less edges and less vertices than $P_{10}$, say $H$ has $10 - s$ vertices, then we have three cases:

Case 1. No two of the s-vertices are adjacent. Then $|E(H)| = 15 - 3c$ and as $D^*(H) = \frac{2c(15 - 3s)}{c(10 - s)} \leq 3$.

Case 2. The subgraph consisting of these s vertices is isomorphic to a path graph. Then $|E(H)| = 15 - (3 + 2(s - 1)) = 14 - 2s$. Hence, $D^*(H) = \frac{2c(14 - 2s)}{c(10 - s)} \leq 3$ for all $s > 1$ and hence.

Case 3. The subgraph consisting of these s vertices has $s_1$ vertices of those in Case 1 and $s_2$ vertices of those in Case 2. Then $|E(H)| = 15 - 3s_1 - (3 + 2(s_2 - 1)) = 14 - 3s_1 - 2s_2$. Hence $D^*(H) = \frac{2c(14 - 3s_1 - 2s_2)}{c(10 - s_1 - s_2)} \leq 3$. Therefore $P_{10}$ is $^*$-balanced.

**Theorem 17.** If $K_{n,n}$ has $\sigma$ as a c-constant function and strong, then $K_{n,n}$ is $^*$-balanced.

**Proof:** Now $D^*(K_{n,n}) = \frac{2c(n^2)}{2nc} = n$. Any subgraph $H$ of $K_{n,n}$ has edges less than $K_{n,n}$ or less edges and less vertices than $K_{n,n}$. If $H$ has less edges, it is clear that $D^*(H) \leq D^*(K_{n,n})$. If $H$ has less edges and less vertices than $K_{n,n}$, say $H$ has $2n - s$ vertices, then we have three cases:

Case 1. No two of the s-vertices are adjacent. Then $D^*(K_{n-s,n}) = \frac{2c(n(n-s))}{c(2n-s)} = \frac{n(n-s)}{n-s/2} \leq n$.

Case 2. The subgraph is $K_{n-s_1,n-s_2}$ where $s_1 + s_2 = s$. Then $D^*(H) = \frac{2c((n-s_1)(n-s_2))}{c(2n-s)} = \frac{n(n-s)-s_1s_2}{n-s/2} \leq n$.

Therefore $K_{n,n}$ is $^*$-balanced.

5. **Conclusion**

In this paper, we defined the concept of $^*$-density of a fuzzy graph and we introduced and explored what we call $^*$-balanced fuzzy graph. Several examples and results were also provided and certain classes of $^*$-balanced fuzzy graphs are given. In addition, $^*$-balanced fuzzy graphs were discussed.
REFERENCES

[1] T. Al-Hawary, Certain classes of fuzzy graphs, Eur. J. Pure Appl. Math. 10(3)(2017), 552-560.
[2] T. Al-Hawary, Complete fuzzy graphs, International J. Math. Combin., 4(2011) 26-34.
[3] T. Al-Hawary, On balanced graphs and balanced matroids, Math. Sci. Res. Hot-Line 4(7)(2000), 35-45.
[4] T. Al-Hawary and Bayan Horani, On intuitionistic product fuzzy graphs, Ital. J. Pure. Appl. Math. 38(2017), 113-126.
[5] T. Al-Hawary and Bayan Horani, On product fuzzy graphs, Annals of fuzzy mathematics and Informatics 12(2)(2016), 279-294.
[6] A. Nagoor Gani and J. Malarvizhi, Isomorphism on fuzzy graphs, Int. J. Comp. and Math. Sci. 2(4)(2008), 190-196.
[7] A. Nagoor Gani and J. Malarvizhi, Isomorphism properties on strong fuzzy graphs, Int. J. Algorithms, Comp. and Math. 2(1)(2009), 39-47.
[8] A. Nagoor Gani and K. Radha, On regular fuzzy graphs, J. physical Sciences 12(2008), 33-40.
[9] A. Rosenfeld, Fuzzy graphs, in L. A. Zadeh K.S. Fu, K. Tabaka and M. Shirmura (Eds.), Fuzzy and their applications to congnitive and division processes, Academic Press, New York, 1975, 77-95.
[10] J. N. Mordeson and C. S. Peng, Operations on fuzzy graphs, Information sciences 79(1994), 381-384.
[11] K. R. Bhutani, On automorphism of fuzzy graphs, Pattern Recognition Letter 9(1989), 159-162.
[12] L. A. Zadeh, Fuzzy sets, Inform. Control. 8(1965),338-353.
[13] M. S. Sunitha and A. V. Kumar, Complements of fuzzy graphs, Indian J. pure Appl. Math, 33(9)(2002), 1451-1464.

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