A Simplified Method for Dynamic Equation of Robot in Generalized Coordinate System

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Abstract. This paper presents a simplified method of dynamic equations in a generalized coordinate system, which decouples the relative motion of the front and back links of the robot joint, and maps generalized angular variables to the same angular datum. Because of the decoupling between absolute angle variables, the complexity of the coefficient matrix of the dynamic equation of the system is reduced, which facilitates the application of the actual series robot system. The simplified process is derived in detail, and the equivalent relation between the generalized relative coordinates and the generalized absolute coordinate dynamics is demonstrated. The relation can be extended to the dynamic equations of different generalized coordinate variables. Aimed at the problem that the quasi-input moment vectors obtained in generalized absolute coordinates cannot be directly used without considering the dynamic coupling of the whole system, the linear relationship between the input moment vectors in absolute and relative coordinates is found. In the end, a simplified example of the Euler-Lagrange dynamic equation for multi-degree-of-freedom manipulator is given. The relationship between the two input torque vectors is simulated by MATLAB and verifies that the input torque can be indirectly calculated by finding the quasi input torque under the absolute coordinates.

1. Introduction

The analysis and control strategy design of robots generally includes kinematics analysis and dynamics analysis based on dynamic models [1]. The dynamics model reflects the mathematical relationship between the motion of the robot, the driving torque and load. This embodies the robot's athletic ability and carrying capacity from the side. Correct, simple, and rapid establishment of system dynamics models is important in robot control, such as, the study of passive gait without energy input in biped robot research [2-3], control strategy and system limit cycle stability study trying to achieve certain goals in the production of gait required speed or energy consumption [4-5], and the research on rigid-flexible coupled nonlinear arm trajectory tracking in the study of flexible manipulators [6-7]. They are all inseparable from the system dynamics model.

On the other hand, the complexity of the model also needs to be taken seriously. The predecessors summarized several methods for establishing system dynamics equations, such as Newton-Euler dynamic equilibrium method [8-9] and Euler-Lagrange functional balance method [10-11]. At present, most of the robot dynamics equations are established by the Euler-Lagrange method in the generalized coordinate system. The coordinate variable that is customarily selected is the joint angle measured based on the extension of the previous link (generalized relative coordinates) [12-13]. However, most of the established models are complex and inconvenient to solve. And the complexity of the model
will increase greatly with the increase of the degree of freedom [14]. So, there is a need for a way to simplify the building of the model.

Because the selection of the generalized coordinate system variables is different, the obtained dynamic equations are not the same, but these equations are equivalent. However, what kind of relationship is there between them, and whether there is a relatively simple dynamic equation in mathematical form is also unknown. In this paper, a method of simplifying system dynamics equation modeling is found by variable substitution. This method uses joint angle vectors measured in generalized absolute coordinates (the same reference) to establish system dynamics equation. It can effectively decouple the mutual coupling between the joint angle vector elements, simplify the equation coefficient matrix and save the control system’s calculate ability. The simplified process of the model will also help to understand the analysis process of the robot control method [15-16].

2. Euler-Lagrange dynamic equation

According to the structure of the robot, the Euler-Lagrange dynamic equation of the robot can be obtained under the generalized coordinate system. In engineering practice, the D-H rule [17] is generally used to establish the forward and inverse kinematics model of the system, and then the dynamic model of the system is established according to the selected joint variables. Two-degree-of-freedom (2-DOF) manipulators select joint variables based on general relative coordinates \( q \). As shown in figure 1. It is also possible to choose to build a dynamic model of the system based on the same reference coordinates (generalized absolute coordinates). At this point, the joint variables are shown in figure 2.

![Figure 1. Relative coordinate model of 2 DOF manipulator](image1)

![Figure 2. Absolute coordinate model of 2 DOF manipulator](image2)

The established dynamic model is written in a compact form according to figure 1:

\[
M(q)\ddot{q} + d(q, \dot{q}) = \tau
\]

(1)

Where \( q, \dot{q}, \ddot{q} \in \mathbb{R}^n \) are joint angle vector, joint angular velocity vector and joint angular acceleration vector. \( M(q) = M(q)^T > 0 \) is inertia matrix, \( d(q, \dot{q}) \) is the remaining dynamic component, and \( \tau \in \mathbb{R}^n \) is joint angular moment input vector. Similarly, if you choose a generalized absolute coordinate variable \( \theta \), you can build a dynamic model to get the compact Euler-Lagrange equation:

\[
\tilde{M}(\theta)\ddot{\theta} + \tilde{d}(\theta, \dot{\theta}) = \tilde{\tau}
\]

(2)

Where \( \theta, \dot{\theta}, \ddot{\theta} \in \mathbb{R}^n \), they are absolute joint angle, absolute joint angular velocity and absolute joint angular acceleration. \( \tilde{M}(\theta) = \tilde{M}(\theta)^T > 0 \) is inertia matrix, \( \tilde{d}(\theta, \dot{\theta}) \) is the remaining dynamic component, \( \tilde{\tau} \in \mathbb{R}^n \) is quasi-input torque vector. The obtained \( \tilde{\tau} \) cannot directly used as the robot’s actual torque input. There is no direct hinge relationship between vector elements \( \theta \). And equation (2) does not reflect the moment that the upper joint acts on the joint.
There is an equivalence relation between equation (1) and equation (2). By applying generalized absolute coordinates, the dynamic equations under generalized relative coordinates can be obtained. Because there is no coupling between vector elements of $\theta$ in generalized absolute coordinates, the complexity of $\tilde{M}(\theta)$ and $\tilde{d}(\theta, \dot{\theta})$ is smaller than the complexity of the $M(\theta)$ and $d(\theta, \dot{\theta})$. And it can be obtained $\tau$ indirectly by finding $\tilde{\tau}$.

3. Simplification of dynamic equations in generalized coordinate system
The Lagrange function under generalized relative coordinate measurement can be written as the difference between kinetic energy and potential energy,

$$L(q, \dot{q}) = K(q, \dot{q}) - P(q)$$  \hspace{1cm} (3)

Where

$$K(q, \dot{q}) = \frac{1}{2} \dot{q}^T M(q) \dot{q}$$  \hspace{1cm} (4)

Kinetic equations

$$\frac{d}{dt} \left( \frac{\partial}{\partial q} L(q, \dot{q}) \right) - \frac{\partial}{\partial q} L(q, \dot{q}) = \tau$$  \hspace{1cm} (5)

can be written as

$$\frac{d}{dt} \left( \frac{\partial}{\partial q} K(q, \dot{q}) \right) - \frac{\partial}{\partial q} K(q, \dot{q}) + \frac{\partial}{\partial q} P(q) = \tau$$  \hspace{1cm} (6)

In addition

$$\frac{\partial}{\partial q} \left( \frac{1}{2} \dot{q}^T M(q) \dot{q} \right) = M(q) \dot{q}$$  \hspace{1cm} (7)

$$\frac{d}{dt} \left( \frac{\partial}{\partial q} \left( \frac{1}{2} \dot{q}^T M(q) \dot{q} \right) \right) = \dot{M}(q) \dot{q} + M(q) \ddot{q}$$  \hspace{1cm} (8)

Let

$$\frac{\partial}{\partial q} P(q) = f(q)$$  \hspace{1cm} (9)

$$\dot{M}(q) \dot{q} - \frac{\partial}{\partial q} K(q, \dot{q}) = b(q, \dot{q}) \ddot{q}$$  \hspace{1cm} (10)

Where $f(q)$ is gravity moment component, $b(q, \dot{q}) \in \mathbb{R}^{n_x}$ is Coriolis matrix. So, equation (1) can converted into:

$$M(q) \ddot{q} + b(q, \dot{q}) \dddot{q} + f(q) = \tau$$  \hspace{1cm} (11)

In the generalized absolute coordinate system, $\mathbf{\theta}$ is based on the horizontal x-axis. There is a linear relationship between $q$ and $\mathbf{\theta}$: $q = q(\mathbf{\theta}) = k \theta + b$, and $\dot{q} = q'(\mathbf{\theta}) = k \dot{\theta}$, $\ddot{q} = k \ddot{\theta}$, and $k$ is reversible, then,

$$L(q(\mathbf{\theta}), \dot{q}(\mathbf{\theta})) = K(q(\mathbf{\theta}), \dot{q}(\mathbf{\theta})) - P(q(\mathbf{\theta}))$$  \hspace{1cm} (12)

Where
\[ K\left( q(\theta).q(\dot{\theta}) \right) = \frac{1}{2} \dot{q}(\dot{\theta})^T M(q(\theta)) \dot{q}(\dot{\theta}) \]  

(13)  

Kinetic equations:

\[ \frac{d}{dt} \left( \frac{\partial}{\partial q} L\left( q(\theta).q(\dot{\theta}) \right) \right) - \frac{\partial}{\partial q} L\left( q(\theta).q(\dot{\theta}) \right) = \tau \]  

(14)  

Therefore, the dynamic equation under \( \theta \) can be written as:

\[ \frac{d}{dt} \left( \frac{\partial}{\partial \theta} L\left( q(\theta).q(\dot{\theta}) \right) \right) - \frac{\partial}{\partial \theta} L\left( q(\theta).q(\dot{\theta}) \right) = \tau \]  

(15)  

In addition

\[ \frac{\partial}{\partial \theta} L\left( q(\theta).q(\dot{\theta}) \right) = \frac{\partial}{\partial \theta} K\left( q(\theta).q(\dot{\theta}) \right) \]  

(16)  

\[ \frac{\partial}{\partial \theta} L\left( q(\theta).q(\dot{\theta}) \right) = \frac{\partial}{\partial \theta} K\left( q(\theta).q(\dot{\theta}) \right) - \frac{\partial}{\partial \theta} P(q(\theta)) \]  

(17)  

\[ \frac{\partial}{\partial \theta} L\left( q(\theta).q(\dot{\theta}) \right) = k^T M\left( q(\theta) \right) \dot{q}(\dot{\theta}) \]  

(18)  

\[ \frac{\partial}{\partial \theta} K\left( q(\theta).q(\dot{\theta}) \right) = k^T M\left( q(\theta) \right) \dot{q}(\dot{\theta}) \]  

(19)  

\[ \frac{\partial}{\partial \theta} P(q(\theta)) = k^T \frac{\partial P(q(\theta))}{\partial q} \]  

(20)  

So, equation (15) can be rewritten as:

\[ k^T \left[ M(q) \ddot{q} + M(q) \ddot{q} - \frac{\partial}{\partial q} \left( K(q, \dot{q}) \right) + \frac{\partial P(q)}{\partial q} \right] = \tau \]  

(21)  

Bringing equation (9) and equation (10) into equation (21), get,

\[ k^T \left[ M(q) \ddot{q} + b(q, \dot{q}) \ddot{q} + f(q) \right] = \tau \]  

(22)  

Bringing equation (11) into equation (22),

\[ k^T \tau = \tau \]  

(23)  

What’s more

\[ \vec{M}(\theta) \ddot{\theta} + \vec{b}(\theta, \dot{\theta}) \dot{\theta} + \vec{f}(\theta) = \tau \]  

(24)  

Comparing it with equation (22) get,

\[ \vec{M}(\theta) = k^T M(q) k \]
\[ \vec{b}(\theta, \dot{\theta}) = k^T b(q, \dot{q}) k \]
\[ \vec{f}(\theta) = k^T f(q) \]  

(25)
Inverse calculation of equation (23) and equation (25) can be used to get $\tau$, $M(q)$, $b(q, \dot{q})$ and $f(q)$.

In summary, the above derivation process proves the equivalence of the dynamic equations under different coordinate measurements. The equivalence can be used to map the dynamic equations which under generalized relative coordinate measurement $q$ to the generalized absolute coordinate measurement $\theta$. The kinetic equation under $q$ can be indirectly solved by measuring the dynamic equation under $\theta$. This process is also a simplified process for the dynamic equation under $q$.

4. Dynamic Analysis of Planar Two-DOF Manipulator

In order to verify the above algorithm, this section uses a two-link manipulator to illustrate the equivalence of the model in two coordinate systems, and the simplified process of the dynamic equation in the absolute coordinate system.

4.1. Dynamic Model of Planar Two-Link in Generalized Relative Coordinate System

The two-link manipulator applies the D-H modeling method and selects the general relative joint angle variable. As shown in figure 3, in this way, the dynamic model of the system is established. $m_1$ and $m_2$ are the masses of the link 1 and the link 2. It is expressed by the point mass of the central connecting rod, $l_1$ and $l_2$ represent half the length of the link 1 and link 2, $q_1$ and $q_2$ represent generalized coordinate variables. The gravitational acceleration is $g$. Respectively solve the potential energy and kinetic energy of link 1 and link 2.

Potential energy of system:

$$P_1 = -m_1 g l_1 \cos q_1$$

$$P_2 = -2m_2 g l_1 \cos q_1 - m_2 g l_2 \cos (q_1 + q_2)$$

kinetic energy of system:

$$K_1 = \frac{1}{2} m_1 l_1^2 \dot{q}_1^2$$

$$K_2 = 2m_2 l_1^2 \dot{q}_1^2 + \frac{1}{2} m_2 l_2^2 (\dot{q}_1 + \dot{q}_2)^2 + 2m_2 l_1 l_2 \cos q_2 (\dot{q}_1^2 + \dot{q}_1 \dot{q}_2)$$

So

$$K = \left( \frac{1}{2} m_1 + 2m_2 \right) l_1^2 \dot{q}_1^2 + \frac{1}{2} m_2 l_2^2 (\dot{q}_1 + \dot{q}_2)^2 + 2m_2 l_1 l_2 \cos q_2 (\dot{q}_1^2 + \dot{q}_1 \dot{q}_2)$$
\[ P = -(m_1 + 2m_2)gl_1 \cos q_1 - m_2 gl_2 \cos(q_1 + q_2) \]  

Lagrange function is
\[ L = K - P \]

Write
\[ \frac{d}{dt} \left( \frac{\partial}{\partial \dot{q}} L(q, \dot{q}) \right) - \frac{\partial}{\partial q} L(q, \dot{q}) = \tau \]

as a compact:
\[ M(q)\ddot{q} + b(q, \dot{q})\dot{q} + f(q) = \tau \]

Where
\[ \begin{align*} 
M(q) &= \begin{bmatrix} m_1 l_1^2 + m_2 l_2^2 + m_2 l_1 l_2 \cos q_2 & m_2 l_2^2 + 2m_2 l_1 l_2 \cos q_2 \\
& m_2 l_2^2 \end{bmatrix} \\
b(q, \dot{q}) &= \begin{bmatrix} -4m_2 l_1 l_2 \sin q_1 \dot{q}_2 - 2m_2 l_1 \sin q_1 \dot{q}_2 \\
2m_2 l_1 l_2 \sin q_1 \dot{q}_1 & 0 \end{bmatrix}, \\
f(q) &= \begin{bmatrix} (m_1 + 2m_2) gl_1 \sin q_1 + m_2 gl \sin(q_1 + q_2) \\
m_2 gl_2 \sin(q_1 + q_2) \end{bmatrix} 
\end{align*} \]

4.2. Dynamic Model of Planar Two-Link in Generalized Absolute Coordinate System

Because the dynamic of the system is different when different generalized coordinate variables are chosen. Here we can separate the movement of the second degree of freedom from the movement of the first degree of freedom. Therefore, take the generalized absolute coordinate variable shown in figure 4 to establish the dynamic model of the system. In the figure, \( \theta_i \) and \( \dot{\theta}_i \) represent generalized coordinates, and other parameters are the same as in Section 4.1. Find system kinetic energy:
\[
K_i = \frac{1}{2} m_i l_i^2 \dot{\theta}_i^2 
\]

potential energy of system:
\[
P_i = -2m_i gl_1 \cos\left(\theta_i - \frac{3\pi}{2}\right) = m_i gl_1 \sin \theta_i 
\]

Lagrange function is:
\[ L = \left( \frac{1}{2} m_1 + 2m_2 \right) l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_1^2 \dot{\theta}_1^2 + 2m_2 l_1 l_2 \cos(\theta_2 - \theta_1) \dot{\theta}_1 \dot{\theta}_2 - (m_1 + 2m_2) gl_1 \sin \theta_1 - m_2 gl_2 \sin \theta_2 
\]

Kinetic equation is:
\[ \frac{d}{dt} \left( \frac{\partial}{\partial \dot{\theta}} L(\theta, \dot{\theta}) \right) - \frac{\partial}{\partial \theta} L(\theta, \dot{\theta}) = \tau \]

Written in compact
\[ \ddot{\theta} + b(\theta, \dot{\theta}) + f(\theta) = \tau \]

Where
\[
\begin{bmatrix}
(m_1 + 4m_2)l_1^2 & 2m_2l_2l_2 \cos (\theta_2 - \theta_1) \\
2m_2l_2l_2 \cos (\theta_2 - \theta_1) & m_2l_2^2
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & -2m_2l_2l_2 \sin (\theta_2 - \theta_1) \dot{\theta}_2 \\
-2m_2l_2l_2 \sin (\theta_2 - \theta_1) \dot{\theta}_2 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
(m_1 + 2m_2)gl_1 \cos \theta_1 \\
m_2gl_2 \cos \theta_2
\end{bmatrix}
\]

4.3. Uniformity Analysis and Verification of Planar Two-Link Dynamics Model

In the above, under different generalized coordinate systems, different system dynamics models are obtained. By comparing we can find that equation (41) has a more concise mathematical expression than equation (34). Let’s look for the linear relationship between \( q \) and \( \theta \). According to the planar geometry principle, it can be seen that:

\[
 q = k\theta + b = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \theta + \begin{bmatrix} -3\pi \\ 0 \end{bmatrix}
\]

And can verify

\[
\begin{align*}
\mathbf{\bar{M}}(\theta) &= k^T \mathbf{M}(q)k, \\
\mathbf{\bar{b}}(\theta, \dot{\theta}) &= k^T b(q, \dot{q})k, \\
\mathbf{\bar{f}}(\theta) &= k^T f(q)
\end{align*}
\]

Through the above derivation, we can see the simplified process of the model under absolute coordinates. However, the obtained \( \mathbf{\bar{F}} \) cannot be directly applied to the actual control of the robot. The reason is that \( \mathbf{\bar{F}} \) does not consider the reaction torque of the successor link of the joint. In the following, the two-link manipulator model is used to calculate the joint torque in the same motion state to verify the correctness of equation (23). Select the same motion state:

\[
q_1 = \frac{\pi}{6} + \frac{\pi}{4} \sin(2\pi t), \quad q_2 = \frac{\pi}{4} + \frac{\pi}{3} \cos(\pi t)
\]

The angle under the \( \theta \) is:

\[
\theta_1 = \frac{5\pi}{3} + \frac{\pi}{4} \sin(2\pi t), \quad \theta_2 = \frac{23\pi}{12} + \frac{\pi}{3} \cos(2\pi t) + \frac{\pi}{3} \cos(\pi t)
\]

![Figure 5. Wobble state of two degree of freedom manipulator](image)

![Figure 6. Input torque comparison of two degree of freedom manipulator swing state](image)

Other system parameters are shown in Table 1.

| Variable | Value | Variable | Value |
|----------|-------|----------|-------|
| \( m_1 \) (Kg) | 10 | \( l_1 \) (m) | 0.6 |
| \( m_2 \) (Kg) | 5 | \( l_2 \) (m) | 0.5 |
By calculating the applied torques, we can see that \( \tau_1 > \tau_2 \), if the torque \( \tau_2 \) required to move the link 2 is added to \( \tau_1 \), \( \tau_1 = \tau_1 + \tau_2 \) will be found, as shown in figure 6.

It is proved by simulation calculation that

\[
\tau = (k^{-1} \bar{\tau})
\]

In this way, the torque vector that can be applied to the joint of the robot is indirectly found only by the calculation of the simplified absolute coordinate system dynamic equation.

5. Dynamic Analysis of Multi-degree-of-freedom Manipulator

Through the analysis in the previous chapter, we can find that the \( \theta \) and the \( \theta \) have a linear relationship in the planar robot, and then deduce the linear relationship of the driving torque. This section will use the PUMA560 to study the advantages of spatial multi-DOF robots using absolute coordinate variables to establish dynamic equations.

Because the PUMA560 (shown in figure 7) has the spatial configuration and multiple degrees of freedom, its dynamic equation is more complicated. Therefore, if the dynamic equations established by using absolute coordinate variables can reduce the computational complexity, it will be helpful for the research of robots. The first three degrees of freedom of PUMA560 already have a typical human arm structure. The achievable area covers most of the surrounding space. In order to explain the problem simply and clearly, the dynamic equations of the generalized relative and absolute coordinate systems are studied with PUMA560 as the object. Appropriate simplification of the model, focusing the mass on one point of the respective links, the boom and the arm are coplanar.

5.1. Dynamic Equation of Space Manipulator

The simplified PUMA560 is shown in figure 8, \( m_1, m_2, m_3 \) are the mass of the turntable, the boom and the arm, respectively. In this section, the mass of the turntable can be ignored, \( l_2, l_3 \) are the length of the boom and the length of the arm, respectively, \( r_2, r_3 \) are the distance from the center of mass of the boom to the axis of rotation and the distance from the center of mass of the arm to the axis of rotation. \( I_1, I_2, I_3 \) are the moment of inertia about the rotation of each axis. Let

\[
a_1 = m_1 r_2^2 + m_2 r_2^2, \quad a_2 = m_2 r_3^2, \quad a_3 = m_3 r_3^2, \quad h_1 = (m_2 r_2^2 + m_3 r_3^2) g, \quad h_2 = m_3 r_3 g
\]

Establishing the dynamic equation based on the D-H rule

\[
M(q)\ddot{q} + b(q, \dot{q})\dot{q} + f(q) = \tau
\]

Where
Select generalized absolute coordinate variables as shown in figure 9. Geometrically, the two coordinate variables still have a linear relationship, and the dynamic equations are established,

$$\ddot{\overline{M}}(\theta)\ddot{\theta} + \overline{F}(\theta, \dot{\theta})\dot{\theta} + \overline{f}(\theta) = \overline{\tau}$$

(44)

| \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} | \begin{bmatrix} h_{11} \\ h_{21} \\ h_{31} \end{bmatrix} | \begin{bmatrix} \cos(q_2 + q_3) \\ \cos(q_2 + q_3) \\ \cos(q_2 + q_3) \end{bmatrix} = 0

where,

$$M(q) = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{21} \\ h_{31} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$f(q) = \begin{bmatrix} b_1 \cos(q_2 + q_3) + b_2 \cos(q_2 + q_3) \\ b_2 \cos(q_2 + q_3) \end{bmatrix}$$

$$M(q) = a_1 \cos^2(q_2 + q_3) + 2a_1 \cos(q_2 + q_3) + I_1, \quad m_{23} = m_{31} = m_{33} = 0$$

$$h_{11} = -\frac{1}{2}a_1 \dot{q}_2 \sin(2q_2) - \frac{1}{2}a_1 \dot{q}_3 \sin(2q_2 + 2q_3) - a_1 \dot{q}_2 \sin(q_2 + q_3)$$

$$h_{21} = -\frac{1}{2}a_1 \dot{q}_3 \sin(2q_2 + q_3) - a_1 \dot{q}_2 \sin(q_2 + q_3)$$

$$h_{31} = -\frac{1}{2}a_1 \dot{q}_3 \sin(q_2 + q_3)$$

$$b_1 = -b_2, \quad b_{22} = -a_1 \dot{q}_3 \sin(q_2 + q_3)$$

$$b_{23} = -a_1 \dot{q}_2 \sin(q_2 + q_3)$$

5.2. Manipulator Dynamics Uniformity Verification

The visual model of PUMA560 is established by using the robot toolbox V9.10 in MATLAB. The structural parameters of the robot are shown in Table 2. There is a linear relationship between the variables:
\[ q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \theta \]

Table 2 Parameter value of puma560

| Variable | Value | Variable | Value |
|----------|-------|----------|-------|
| \( m_0 \) (Kg) | 0 | \( l_2 \) (m) | 0.4318 |
| \( m_1 \) (Kg) | 17.4 | \( l_1 \) (m) | 0.4318 |
| \( m_2 \) (Kg) | 4.8 | \( r_2 \) (m) | 0.215 |
| \( I_2 \) (Kg.m\(^2\)) | 0.35 | \( r_2 \) (m) | 0.215 |
| \( l_3 \) (Kg.m\(^2\)) | 0 | \( l_3 \) (Kg.m\(^2\)) | 0 |

Figure 10. Visual model of PUMA560 in MATLAB

Figure 11. Angle, angular velocity and angular acceleration

Draw the trajectory of the manipulator from the point \( q_\varepsilon = [0 \ 0 \ 0]^T \) to the other point \( q_n = [-\pi/3 \ \pi/2 \ -\pi/4]^T \) in the joint space (shown in figure 10). Angle, angular velocity and angular acceleration obtained by interpolated are shown in figure 11, the arm smoothly move to the target point, using the toolbox’s own torque solution function to draw the three load moment of the joints are shown in figure 12. The torque \( \tau \) in the relative coordinate and the torque \( \bar{\tau} \) in the absolute coordinate obtained by mathematical modelling are shown in figure 13, and we can see that:

1. The torque obtained by mathematical modelling is the same as the torque solved by the toolbox function.

2. Torque in generalized relative coordinate system,

\[ \tau = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bar{\tau} \]

a linear relationship between the two moments is satisfied.

3. Comparing the two dynamic models, the spatial multi-DOF robot model in absolute coordinates also has a relatively simple form.
6. Discussion on the Selection of Coordinate Variables of Multi-DOF Robot

The choice of reference coordinate system is very important for the modeling of robots. In many robot systems, the generalized coordinate variables selected by the kinetic model are relative coordinate variables that facilitate sensor measurements and the physical meaning of each kinetic parameter is clear. The calculated torque directly reflects the interaction between the two links. However, the dynamic equations of the multi-DOF space configuration robot established by this method are complex. The dynamic equation established by the absolute reference coordinate system splits the coupling relationship between the front and rear links. The absolute coordinate variable can be measured by absolute sensor, the control process is not affected. It also facilitates better motion analysis of the robot. For example, analysing the limit cycles of each individual joint.

For the 3DOF mechanical arm in Section 5, the front and rear links’ torque coupling relationship is mainly reflected in the second and third joints, and for the spatial multi-DOF robot shown in figure 14, In terms of geometric relationship, depending on the selected angle reference, a linear relationship can be found for $q = k\theta + b$, and then the absolute coordinate variable $\theta$ can be used to simplify the dynamic equation to a large extent. Finally, according to equation (23), linearly combine $\tau^r$ to find $\tau$ .

In terms of energy, the total energy of the robot is the sum of kinetic energy and potential energy, which can be written as:

$$E(q,\dot{q}) = \frac{1}{2} q^r M(q) \dot{q} + P(q)$$  \hspace{1cm} (45)

Get the power of the system in different coordinate systems is equal:

$$\dot{E}(q,\dot{q}) = \dot{\tau}^r = \dot{\theta}^r \tau$$  \hspace{1cm} (46)
7. Conclusion

1) In the derivation process, the relationship between dynamic equations under different generalized coordinates is given. Based on no loss of generality, the equivalence of generalized absolute and relative coordinates of 2 DOF and spatial multi-DOF manipulators is verified. It is found that the matrix elements of the dynamic model based on the absolute coordinates $\theta$ is simpler than that obtained by using $q$, which reduces the complexity of the model.

2) The degree of simplification of the system with more degrees of freedom will be more obvious. There is only a simple linear relationship between the input moments, which is beneficial to the popularization of the method and helps to save the calculation time of limited systems, such as embedded systems. It is convenient for model’s dynamic optimization analysis and design of control law.

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