Two-dimensional water waves in the presence of a freely floating body: conditions for the absence of trapped modes

Nikolay Kuznetsov

Laboratory for Mathematical Modelling of Wave Phenomena, Institute for Problems in Mechanical Engineering, Russian Academy of Sciences, V.O., Bol’shoy pr. 61, St. Petersburg 199178, Russian Federation
E-mail: nikolay.g.kuznetsov@gmail.com

Abstract

The coupled motion is investigated for a mechanical system consisting of water and a body freely floating in it. Water occupies either a half-space or a layer of constant depth into which an infinitely long surface-piercing cylinder is immersed, thus allowing us to study two-dimensional modes. Under the assumption that the motion is of small amplitude near equilibrium, a linear setting is applicable and for the time-harmonic oscillations it reduces to a spectral problem with the frequency of oscillations as the spectral parameter. It is essential that one of the problem’s relations is linear with respect to the parameter, whereas two others are quadratic with respect to it.

Within this framework, it is shown that the total energy of the water motion is finite and the equipartition of energy holds for the whole system. On this basis, it is proved that no wave modes can be trapped provided their frequencies exceed a bound depending on cylinder’s properties, whereas its geometry is subject to some restrictions and, in some cases, certain restrictions are imposed on the type of mode.

1 Introduction

This paper continues the rigorous study (initiated in [4]) of the coupled time-harmonic motion of the mechanical system which consists of water and a rigid body freely floating in it. The former is bounded from above by a free surface, whereas the latter is assumed to be an infinitely long cylinder which allows us to investigate two-dimensional modes orthogonal to its generators. The body is surface-piercing and no external forces acts on it (for example, due to constraints on its motion). The water domain is either infinitely deep or has a constant finite depth; the surface tension is neglected on the free surface of water whose motion is irrotational. The motion of the whole system is supposed to be small-amplitude near equilibrium which allows us to use a linear model.

In the framework of the linear theory of water waves, the time-dependent problem describing the coupled motion of water and a freely floating surface-piercing rigid body was developed by [1]. However, his formulation was rather cumbersome, and so
during the second half of the 20th century the main efforts were devoted to various
problems involving fixed bodies instead of freely floating ones (see the summarising
monograph by [5]). The cornerstone was laid by [2] himself who proved the first result
guaranteeing the absence of trapped modes at all frequencies provided an immersed
obstacle has a fixed position and is subject to a geometric restriction now usually
referred to as John’s condition. In the two-dimensional case, it includes the following
two requirements: (i) there is only one surface-piercing cylinder in the set of cylinders
forming the obstacle; (ii) the whole obstacle is confined within the strip between two
vertical lines through the points, where the surface-piercing contour intersects the free
surface of water, the part of bottom (when the depth is finite) is horizontal outside
of this strip.

[11] demonstrated that if condition (i) holds, then condition (ii) can be replaced
by a weaker one. Namely, if the depth is infinite, then the whole obstacle must be
confined to the angular domain between the lines inclined at $\pi/4$ to the vertical and
going through the two points, where the surface-piercing contour intersects the free
surface. If the depth is finite, then it is required that the whole obstacle is confined
to a smaller angular domain between the lines going through the same two points,
but inclined at a certain angle to the vertical that is a little bit less than $\pi/4$. The
results of [11] and [2] are illustrated in [5]; see pp. 125, 126 and 137, respectively.

In [3], another geometric condition alternative to (ii) was found which together
with (i) guarantees the absence of trapped modes at all frequencies for fixed bodies.
This condition does not impose any restriction on the angle between the surface-
piercing contour and the free surface (arbitrarily small angles are admissible), but
this is achieved at the expense that the wetted contour is subject to a certain point-
wise restriction (it must be transversal to curves (20) in a certain definite fashion).

On the other hand, condition (i) is essential for the absence of trapped modes.
This became clear when [8] constructed an example of such a mode for which pur-
pose she applied the so-called semi-inverse method (see, for example, [7] for its brief
description). Her example involves two fixed surface-piercing cylinders each of which
satisfies the modified condition (ii) of [11], but they are separated by a nonzero spac-
ing. Another example of a mode trapped by two fixed surface-piercing cylinders was
found by [10]. Subsequently, [4] proved that the latter cylinders can be considered as
two immersed parts of a single body which freely floats in trapped waves, but remains
motionless.

During the past decade, the problem of the coupled time-harmonic motion of water
and a freely floating rigid body has attracted much attention. Along with the just
mentioned paper [3], rigorous results were obtained in [7], where a brief review of
related papers is given. However, the substantial part of work concerns the study of
trapped modes and the corresponding trapping bodies and only the paper [6] has been
focused on conditions eliminating trapped modes in the case when a surface-piercing
or totally submerged body is present (for a surface-piercing body the original proof
of [2] was essentially simplified). In the present paper, our aim is to fill in this gap at
least partially.

In the present note, we find conditions on the frequency so that they guarantee
that no modes (or some specific modes) are trapped by a freely floating body provided
its geometry satisfies the assumptions used in [11] and [3] for establishing the absence
of modes trapped by the same body being fixed.
2 Statement of the problem

Let the Cartesian coordinate system \((x, y)\) in a plane orthogonal to the generators of a freely floating infinitely long cylinder be chosen so that the \(y\)-axis is directed upwards, whereas the mean free surface of water intersect this plane along the \(x\)-axis, and so the cross-section \(W\) of the water domain is a subset of \(\mathbb{R}^2_+ = \{ x \in \mathbb{R}, y < 0 \}\). Let \(\tilde{B}\) denote the bounded two-dimensional domain whose closure is the cross-section a floating cylinder in its equilibrium position. Let both the immersed part \(B = \tilde{B} \cap \mathbb{R}^2_+\) and the above-water part \(\tilde{B} \setminus \mathbb{R}^2_+\) be nonempty domains and \(D = \tilde{B} \cap \partial \mathbb{R}^2_+\) be a nonempty interval of the \(x\)-axis, say \(\{ x \in (-a, a), y = 0 \}\) (see figure 1). We suppose that \(W\) is either \(\mathbb{R}^2_+ \setminus B\) when water has infinite depth (see figure 1) or \(\{ x \in \mathbb{R}, -h < y < 0 \} \setminus \partial \mathbb{R}^2_+\), where \(h > b_0 = \sup_{(x,y) \in B} |y|\), when water has constant finite depth. We suppose that \(W\) is a Lipschitz domain, and so the unit normal \(n\) pointing to the exterior of \(W\) is defined almost everywhere on \(\partial W\). Finally, by \(S = \partial \tilde{B} \cap \mathbb{R}^2_+\) and \(F = \partial \mathbb{R}^2_+ \setminus D\) we denote the wetted contour and the free surface at rest, respectively; if water has finite depth, then \(H = \{ x \in \mathbb{R}, y = -h \}\) is the bottom’s cross-section.

For describing the small-amplitude coupled motion of the system it is standard to apply the linear setting in which case the following first-order unknowns are used. The velocity potential \(\Phi(x, y; t)\) and the vector-column \(q(t)\) describing the motion of body whose three components are as follows:

- \(q_1\) and \(q_2\) are the displacements of the centre of mass in the horizontal and vertical directions, respectively, from its rest position \((x^{(0)}, y^{(0)})\);
- \(q_3\) is the angle of rotation about the axis that goes through the centre of mass orthogonally to the \((x, y)\)-plane (the angle is measured from the \(x\)- to \(y\)-axis).

We omit relations governing the time-dependent behaviour (see details in [4]), and turn directly to the time-harmonic oscillations of the system for which purpose we use the ansatz

\[
(\Phi(x, y, t), q(t)) = \text{Re}\{e^{-i\omega t}(\varphi(x, y), i\chi)\},
\]

where \(\omega > 0\) is the radian frequency, \(\varphi \in H^1_{\text{loc}}(W)\) is a complex-valued function and
χ ∈ C^3. To be specific, we first assume that W is infinitely deep in which case the problem for (ϕ, χ) is as follows:

\[ \nabla^2 \varphi = 0 \quad \text{in } W, \]  \hfill (2)

\[ \partial_y \varphi - \nu \varphi = 0 \quad \text{on } F, \quad \text{where } \nu = \omega^2 / g, \]  \hfill (3)

\[ \partial_n \varphi = \omega \mathbf{N}^T \chi \left( = \omega \sum_{j=1}^{3} N_j \chi_j \right) \quad \text{on } S, \]  \hfill (4)

\[ \nabla \varphi \to 0 \quad \text{as } y \to -\infty, \]  \hfill (5)

\[ \int_{W \cap \{|x| = b\}} |\partial_x| \varphi - i \nu \varphi|^2 \, ds = o(1) \quad \text{as } b \to \infty, \]  \hfill (6)

\[ \omega^2 \mathbf{E} \chi = -\omega \int_S \varphi \mathbf{N} \, ds + g \mathbf{K} \chi. \]  \hfill (7)

Here \( \nabla = (\partial_x, \partial_y) \) is the spatial gradient, \( g > 0 \) is the acceleration due to gravity that acts in the direction opposite to the y-axis; \( \mathbf{N} = (N_1, N_2, N_3)^T \) (the operation \( T \) transforms a vector-row into a vector-column and vice versa), where \( (N_1, N_2)^T = n, \ N_3 = (x - x^{(0)}, y - y^{(0)}) \times n \) and \( \times \) stands for the vector product. In the equations of body’s motion (7), the 3 × 3 matrices are as follows:

\[ \mathbf{E} = \begin{pmatrix} I^M & 0 & 0 \\ 0 & I^M & 0 \\ 0 & 0 & I^M \end{pmatrix} \quad \text{and} \quad \mathbf{K} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I^D & I^D_x \\ 0 & I^D_x & I^D_x + I^S \end{pmatrix}. \]  \hfill (8)

The positive elements of the mass/inertia matrix \( \mathbf{E} \) are

\[ I^M = \rho_0^{-1} \int_B \rho(x, y) \, dxdy \quad \text{and} \quad I^M_2 = \rho_0^{-1} \int_B \rho(x, y) \left[ \left( x - x^{(0)} \right)^2 + \left( y - y^{(0)} \right)^2 \right] \, dxdy, \]

where \( \rho(x, y) \geq 0 \) is the density distribution within the body and \( \rho_0 > 0 \) is the constant density of water. In the right-hand side of relation (1), we have forces and their moments. In particular, the first term is due to the hydrodynamic pressure, whereas the second one is related to the buoyancy (see, for example, [1]); the non-zero elements of the matrix \( \mathbf{K} \) are

\[ I^D = \int_D \, dx > 0, \quad I^D_x = \int_D (x - x^{(0)}) \, dx, \]

\[ I^D_x = \int_D (x - x^{(0)})^2 \, dx > 0, \quad I^S_y = \int_S (y - y^{(0)}) \, dxdy. \]

Note that the matrix \( \mathbf{K} \) is symmetric.

In relations (3), (4) and (7), \( \omega \) is a spectral parameter which is sought together with the eigenvector \((\varphi, \chi)\). Since W is a Lipschitz domain and \( \varphi \in H^1_{\text{loc}}(W) \), relations (2) - (4) are, as usual, understood in the sense of the following integral identity:

\[ \int_W \nabla \varphi \nabla \psi \, dxdy = \nu \int_F \varphi \psi \, dxF + \omega \int_S \psi \mathbf{N}^T \chi \, ds, \]  \hfill (9)

which must hold for an arbitrary smooth \( \psi \) having a compact support in \( W \). Finally, relations (5) and (6) specify the behaviour of \( \varphi \) at infinity. The first of these means that the velocity field decays with depth, whereas the second one yields that the
potential given by formula (1) describes outgoing waves. This radiation condition is the same as in the water-wave problem for a fixed obstacle (see, for example, [2]).

The relations listed above must be augmented by the following conditions concerning the equilibrium position:

• The mass of the displaced liquid is equal to that of the body: \( I^M = \int_B dxdy \) (Archimedes’ law); • The centre of buoyancy lies on the same vertical line as the centre of mass: \( \int_B (x - x^{(0)}) dxdy = 0 \); • The matrix \( K \) is positive semi-definite; moreover, the \( 2 \times 2 \) matrix \( K' \) that stands in the lower right corner of \( K \) is positive definite (see [1]).

The last of these requirements yields the stability of the body’s equilibrium position, which follows from the results formulated, for example, by [1], § 2.4. The stability is understood in the classical sense that an instantaneous, infinite small disturbance causes the position changes which remain infinitesimal, except for purely horizontal drift, for all subsequent times.

In conclusion of this section, we note that relations (5) and (6) must be amended in the case when \( W \) has finite depth. Namely, the no flow condition

\[
\partial_y \varphi = 0 \quad \text{on } H
\]  

(10) replaces (3), whereas \( \nu \) must be changed to \( k_0 \) in (6), where \( k_0 \) is the unique positive root of \( k_0 \tanh(k_0 h) = \nu \).

3 Equipartition of energy, trapped modes and conditions guaranteeing their absence

3.1 Equipartition of energy

It is known (see, for example, [3], § 2.2.1), that a potential, satisfying relations (2), (3), (5) and (6), has the asymptotic representation at infinity of the same type as Green’s function. Namely, if \( W \) has infinite depth, then

\[
\varphi(x,y) = A_\pm(y) e^{i\nu|x|} + r_\pm(x,y), \quad \text{where } |r_\pm|^2, |\nabla r_\pm| = O\left(\frac{1}{x^2 + y^2}\right) \text{ as } x^2 + y^2 \to \infty,
\]  

(11)

and the following equality holds

\[
\nu \int_{-\infty}^{0} \left( |A_+(y)|^2 + |A_-(y)|^2 \right) dy = -\text{Im} \int_S \varphi \partial_n \varphi ds.
\]  

(12)

Assuming that \((\varphi, \chi)\) is a solution of problem (2)–(7), we rearrange the last formula using the coupling conditions (4) and (7). First, transposing the complex conjugate of equation (7), we get

\[
\omega^2 (E\chi)^T = -\omega \int_S \varphi N^T ds + g(K\chi)^T.
\]

This relation and condition (11) yield that the inner product of both sides with \( \chi \) can be written in the form:

\[
\omega^2 \chi^T E\chi - \chi^T K\chi = -\int_S \varphi \partial_n \varphi ds.
\]  

(13)
Second, substituting this equality into (12), we obtain
\[ \nu \int_{-\infty}^{0} \left( |A_+(y)|^2 + |A_-(y)|^2 \right) \, dy = \text{Im} \left\{ \omega^2 \chi^T E \chi - g \chi^T K \chi \right\}. \tag{14} \]

In the same way as in [7], this yields the following assertion about the kinetic and potential energy of the water motion.

**Proposition 1.** Let \((\varphi, \chi)\) be a solution of problem (2)–(7), then
\[ \int_W |\nabla \varphi|^2 \, dx \, dy < \infty \quad \text{and} \quad \nu \int_F |\varphi|^2 \, dx < \infty. \tag{15} \]

Moreover, the following equality holds:
\[ \int_W |\nabla \varphi|^2 \, dx \, dy + \omega^2 \chi^T E \chi = \nu \int_F |\varphi|^2 \, dx + g \chi^T K \chi. \tag{16} \]

Here the kinetic energy of the water/body system stands in the left-hand side, whereas we have the potential energy of this coupled motion in the right-hand side. Thus the last formula generalises the energy equipartition equality valid when a fixed body is immersed into water. Indeed, \(\chi = 0\) for such a body, and (16) turns into the well-known equality (see, for example, formula (4.99) in [5]).

Proposition 1 shows that if \((\varphi, \chi)\) is a solution of problem (2)–(7) with complex-valued components, then its real and imaginary parts separately satisfy this problem. This allows us to consider \((\varphi, \chi)\) as an element of the real product space \(H^1(W) \times \mathbb{R}^3\) in what follows (the sum of two quantities (15) defines an equivalent norm in \(H^1(W)\)).

**Definition 1.** Let the subsidiary conditions concerning the equilibrium position (see § 2) hold for the freely floating body \(\hat{B}\). A non-trivial real solution \((\varphi, \chi) \in H^1(W) \times \mathbb{R}^3\) of problem (9) and (7) is called a mode trapped by this body, whereas the corresponding value of \(\omega\) is referred to as a trapping frequency.

In order to determine when \((\varphi, \chi) \in H^1(W) \times \mathbb{R}^3\) is not trapped by \(\hat{B}\) we write (16) as follows:
\[ \chi^T (\omega^2 E \chi - g K) \chi = \nu \int_F |\varphi|^2 \, dx - \int_W |\nabla \varphi|^2 \, dx \, dy. \tag{17} \]

It is clear that the left-hand side is non-negative provided \(\omega^2\) is sufficiently large, and so we arrive at the following.

**Proposition 2.** Let \(E\) and \(K\) be given by (8) and let \(\omega^2\) be greater than or equal to the largest \(\lambda\) satisfying \(\det(\lambda E - g K) = 0\). If the domain \(W\) is such that the inequality
\[ \nu \int_F |\varphi|^2 \, dx < \int_W |\nabla \varphi|^2 \, dx \, dy \tag{18} \]
holds for every non-trivial \(\varphi \in H^1(W)\), then \(\omega\) is not a trapping frequency.

Note that if \(W\) has finite depth, then \(\nu\) must be changed to \(k_0\) in relation (11), where the behaviour of the remainder must be also replaced by the following one:
\[ |r_\pm|, \quad |\nabla r_\pm| = O(|x|^{-1}) \quad \text{as} \quad |x| \to \infty. \tag{19} \]

In relations (12) and (14) \(\nu\) must also be changed to \(k_0\). On the other hand, formula (13) remains be valid in the same form as above, and so proposition 2 is true in this case as well.
3.2 Examples of water domains for which inequality (18) holds

We begin with the case when \( W \) has infinite depth. By \( \ell_d \) and \( \ell_{-d} \) we denote the rays emanating at the angle \( \pi/4 \) to the vertical from the points \((d, 0)\) and \((-d, 0)\), respectively, and going to the right and left, respectively.

Let the whole rays \( \ell_d \) and \( \ell_{-d} \) belong to \( W \) for all \( d > a \). Thus, \( B \) is confined within the angular domain between the lines inclined at \( \pi/4 \) to the vertical and going through the points \((a, 0)\) and \((-a, 0)\) to the right and left, respectively. Under this assumption, \([11]\) proved (see also \([5]\), §§ 3.2.2.1 and 3.2.2.2) that the inequality

\[
\nu \int_F |\varphi|^2 \, dx \leq \int_{W_c} |\nabla \varphi|^2 \, dx \, dy
\]

holds provided \( \varphi \) satisfies conditions (15) and relations (2) and (3). Here \( W_c \) is the subset of \( W \) covered with rays \( \{ \ell_d : (d, 0) \in F \} \cup \{ \ell_{-d} : (-d, 0) \in F \} \). According to the last inequality, if \( \varphi \) is non-trivial, then (18) holds. Therefore, proposition 2 is applicable, thus giving a criterion which values of \( \omega \) are non-trapping frequencies for the freely floating \( \hat{B} \) whose immersed part \( B \) is confined as described above.

In order to obtain inequality (18) in the case when \( W \) has finite depth, \( \ell_d \) and \( \ell_{-d} \) must be replaced by similar segments connecting \( F \) and \( H \) and inclined at a certain angle to the vertical that is a little bit less than \( \pi/4 \). Numerical computations of \([11]\) show that the same result as for deep water is true when \( B \) is confined between the segments inclined at \( 44^1_3^\circ \).

4 Another criterion eliminating some particular trapped modes

In this section, we turn to the case when \( B \) does not satisfy the conditions of § 3.2. To be specific, we suppose that \( W \) is bounded from below by the rigid bottom \( H \). Moreover, we assume that \( \hat{B} \) is symmetric about the \( y \)-axis (see figure 1); this implies that \( N_1 = n_x \) (\( N_2 = n_y \)) attains the opposite (the same, respectively) values at every pair of points on \( \hat{B} \) which are symmetric about the \( y \)-axis. Let also \( \rho(x, y) \) be an even function of \( x \), and so \( x^{(0)} = 0 \) (the centre of mass lies on the \( y \)-axis); this implies that \( N_3 = xny - nx(y - y^{(0)}) \) has the same behaviour as \( N_1 \).

The last restriction on \( \hat{B} \) or, more precisely, on \( B \) is expressed in terms of the curves

\[
x^2 + (y - a \cot \sigma)^2 = a^2(\cot^2 \sigma + 1), \quad \pm x > 0, \quad y < 0,
\]

parametrised by \( \sigma \in (-\pi, 0) \). On curves of these two families we define directions as shown in figure 1. It is clear that all curves (20), that intersect \( H \) transversally, enter into \( W \). Let this property also hold on \( S \); that is, all transversal intersections of curves (20) with \( S \) are points of entry into \( W \) (see figure 1). In what follows, a body satisfying the listed conditions is referred to as belonging to the class \( B \) provided the conditions considered in § 3.2 are not fulfilled for it.

The following assertion generalises the criterion of \([3]\) guaranteeing the absence of trapped modes for fixed surface-piercing bodies immersed in deep water and satisfying the above transversality condition with the family of curves (20). As in proposition 2 the values of \( \omega \) that are not trapping frequencies must be sufficiently large, but what is new that some restrictions must be also imposed on the type of mode.
Proposition 3. Let $W$ have finite depth and let $\tilde{B}$ be a freely floating body belonging to the class $B$. If $\omega^2$ is strictly greater than the largest $\lambda$ such that $\text{det}(\lambda E - gK) = 0$ with $E$ and $K$ given by (25), then $\omega$ is not a trapping frequency for modes of the form:

(a) $\varphi$ is an even function of $x$ and $\chi = (d_1, 0, d_3)^T$;
(b) $\varphi$ is an odd function of $x$ and $\chi = (0, d_2, 0)^T$.

Proof. Let us write relations (2)–(4) and (10) using the bipolar coordinates $(u, v)$. The corresponding conformal mapping is usually defined as follows (see, for example, [1], §10.1):

\[
x = a \sinh u/(\cosh u - \cos v), \quad y = a \sin v/(\cosh u - \cos v).
\]

Therefore, (21) maps the strip $\{ -\infty < u < +\infty, -\pi < v < 0 \}$ onto $\mathbb{R}^2$ so that for every $\sigma \in (-\pi, 0)$ the image of the left (right) half-line $\{u > 0, v = \sigma \}$ is the circular arc (20) that lies in the left (right) half-plane (see figure 1). Moreover,

\[
\{ -\infty < u < +\infty, v = -\pi \} \quad \text{and} \quad \{ \pm u > 0, v = 0 \}
\]

are mapped onto $\{|x| < a, y = 0\}$ and $\{|x > a, y = 0\}$ respectively. Finally, we have that

\[
|z'(\zeta)| = a/(\cosh u - \cos v), \quad \text{where} \; z = x + iy \; \text{and} \; \zeta = u + iv.
\]

The inverse mapping $\zeta(z)$ has the following properties: the points $a$ and $-a$ on the $x$-axis go to infinity on the $\zeta$-plane, whereas $z = \infty$ goes to $\zeta = 0$; thus $F$ is mapped onto the whole $u$-axis.

Denoting by $W$ the image of $W$, we see that apart from the $u$-axis the boundary $\partial W$ includes the images of $S$ and $H$, say $S$ and $H$ respectively. According to properties of (21), if $\tilde{B}$ belongs to the class $B$, then $S$ is symmetric about the $v$-axis, lies within the strip $\{ -\infty < u < +\infty, -\pi < v < -\alpha \}$ and asymptotes the line $v = -\alpha$ as $u \to \pm\infty$; here $\alpha \in (0, \pi)$ is the angle between $S$ and $F$ at $(0, a)$. Moreover, the right half of $S$ is the graph of a decreasing function of $u \in (0, +\infty)$; its maximum value $v_b$ is the root of $\cos v - (a/b_0) \sin v = 1$. Finally, $H$ is a closed curve with the following properties. It is symmetric about the $v$-axis, is tangent to the $u$-axis at the origin and is the graph of a concave function of $v \in (v_b, 0)$; here $v_b \in (-\pi, 0)$ is the root of $\cos v - (a/d) \sin v = 1$. It is clear that $-\pi/4 < -\alpha < v_b < v_h < 0$.

Let $\phi(u, v) = \varphi(x(u, v), y(u, v))$, then relations (2)–(4) yield that

\[
\nabla^2 \phi = 0 \; \text{in} \; W, \quad (\cosh u - 1)\phi_v = \nu a \phi \; \text{when} \; v = 0, \quad \nabla \phi \cdot n_\zeta = \frac{\omega a N_\zeta^T \chi}{\cosh u - \cos v} \; \text{on} \; S.
\]

Here $n_\zeta$ is the unit normal to $S \cup H$ exterior with respect to $W$ and $N_\zeta = N_{z(\zeta)}$. Moreover, condition (10) implies that

\[
\nabla \phi \cdot n_\zeta = 0 \; \text{on} \; H,
\]

whereas condition (7) takes the form

\[
\omega^2 E \chi = -\omega \int_S \frac{\phi N_\zeta ds_\zeta}{\cosh u - \cos v} + g K \chi.
\]
Furthermore, conditions (15) give that
\[ \int_{W} |\nabla \phi|^2 \, du \, dv < \infty \quad \text{and} \quad \int_{-\infty}^{+\infty} \frac{\phi^2(u,0)}{\cosh u - 1} \, du < \infty, \]  
whereas equality (17) turns into the following one:
\[ \int_{W} |\nabla \phi|^2 \, du \, dv - \nu a \int_{-\infty}^{+\infty} \frac{\phi^2(u,0)}{\cosh u - 1} \, du = -\chi^T \left( \omega^2 E - g \right) \chi. \]  

Further considerations are based on the following identity (see [5], Subsection 2.2.2):
\[ (2u\phi_u + \phi) \nabla^2 \phi = \nabla \cdot (2u\phi_u + \phi) \nabla \phi - 2\phi_u^2 - (u|\nabla \phi|^2)_u. \]  
Here the left-hand side vanishes due to the Laplace equation for \( \phi \). Let us integrate this identity over \( W' = W \cap \{|u| < b\} \) and \( b \) is sufficiently large (in particular, \( H \subset \{|u| < b\} \)). Using the divergence theorem, we get
\[ 2 \int_{W'} \phi_u^2 \, du \, dv + \int_{S' \cup H} u \cdot n |\nabla \phi|^2 \, dS = \int_{-b}^{+b} [2u\phi_u(u,0) + \phi(u,0)] \varphi_v(u,0) \, du + \int_{S'} (2u\phi_u + \phi) \nabla \phi \cdot n \, dS + \sum_{\pm} \int_{C_{\pm}} (2u\phi_u + \phi) \phi_u \, du, \]  
where \( S' = S \cap \{|u| < b\} \), \( u = (u,0) \), \( \sum_{\pm} \) denotes the summation of two terms corresponding to the upper and lower signs, respectively, and \( C_{\pm} = W' \cap \{u = \pm b\} \).

All integrals on the right arise from the first term on the right in (27) and one more integral of the same type vanishes in view of the boundary condition (23) on \( H \).

Let us consider each integral standing on the right in (28). Using the free-surface boundary condition, we get that the first term is equal to
\[ \nu a \int_{-b}^{+b} [2u\phi_u(u,0) + \phi(u,0)] \frac{\phi(u,0)}{\cosh u - 1} \, du \]
\[ = \nu a \int_{-b}^{+b} \frac{u \sinh u \phi^2(u,0)}{(\cosh u - 1)^2} \, du + \nu a \int_{-b}^{+b} \frac{\phi^2(u,0)}{\cosh u - 1} \mid_{u=-b}^{u=b}, \]
where the last expression is obtained by integration by parts. It follows from (15) that \( \varphi(x,y) \) tends to constants as \( (x,y) \to (\pm a,0) \), and so \( \phi(u,v) \) has the same property as \( u \to \pm \infty \). Therefore, the integrated term in the last equality tends to zero as \( b \to \infty \), whereas the integral on the right converges in view of (25).

The second integral on the right in (28) is equal to
\[ \omega a \int_{S'} \frac{\phi_u + \phi}{\cosh u - \cos v} \, dS. \]
Since \( S \) belongs to the class \( B \), we have that \( \phi \) and \( \varphi \) are simultaneously even and odd functions of \( x \) and \( u \) respectively. Therefore, either of the assumptions (a) and (b) implies that this integral vanishes because the integrand attains opposite values at points of \( S' \) that are symmetric about the \( v \)-axis.

Finally, (25) implies that there exists a sequence \( \{b_k\}_{k=1}^{\infty} \) tending to the positive infinity and such that the last sum in (28) tends to zero as \( b_k \to \infty \). Passing to the
limit as \( k \to \infty \), we see that the transformed equation (28) with \( b = b_k \) gives the following integral identity:

\[
2 \int_{W} \phi^2 \, dudv + \int_{S \cup H} u \cdot n \left| \nabla \phi \right|^2 \, dS - \nu a \int_{-\infty}^{+\infty} \frac{u \sinh u}{(\cosh u - 1)^2} \phi^2(u, 0) \, du = 0
\]

provided either of the assumptions (a) and (b) holds.

Subtracting this from (26) multiplied by two, we get

\[
2 \int_{W} \phi^2 \, dudv - \int_{S \cup H} u \cdot n \left| \nabla \phi \right|^2 \, dS + \nu a \int_{-\infty}^{+\infty} \frac{u \sinh u - 2(cosh u - 1)}{(\cosh u - 1)^2} \phi^2(u, 0) \, du
\]

\[
= -2 \chi^T (\omega^2 \mathbf{E} - g \mathbf{K}) \chi.
\]

(29)

If \( \omega^2 \) is strictly greater than the largest \( \lambda \) such that \( \det(\lambda \mathbf{E} - g \mathbf{K}) = 0 \), then cannot hold unless \( \omega \) is not a trapping frequency for modes of the form (a) and (b). Indeed, the right-hand side is negative for such a value of \( \omega \) and a non-trivial \( \chi \), whereas the left-hand side is non-negative because \( S \) belongs to the class \( B \) and the fraction in the last integral is non-negative. The obtained contradiction proves the proposition. \( \square \)

5 Conjecture

Given the proof of a theorem guaranteeing the uniqueness of a solution to the linearised problem about time-harmonic water waves in the presence of a fixed obstacle, then this proof admits amendments transforming it into the proof of an analogous theorem for the same obstacle floating freely with additional restrictions on the non-trapping frequencies (they must be sufficiently large) and, in some cases, on body’s geometry and on the type of non-trapping modes.

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