Proposal for a Quantum Delayed-Choice Experiment with Massive Mechanical Resonators

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We present and analyze an experimentally-feasible implementation of a macroscopic quantum delayed-choice experiment to test a quantum wave-particle superposition on massive mechanical resonators. In our approach, the electronic spin of a single nitrogen-vacancy impurity is employed to control the coherent coupling between the mechanical modes of two carbon nanotubes. We demonstrate that a mechanical phonon can be in a coherent superposition of wave and particle, thus exhibiting both behaviors at the same time. Furthermore, we discuss the mechanical noise tolerable in our proposal and predict a critical temperature below which the morphing between wave and particle states can be effectively observed in the presence of environment-induced fluctuations.

PACS numbers: 03.65.Ta, 85.85.+j

Wave-particle duality lies at the heart of quantum physics. According to Bohr’s complementarity principle [1], a quantum system may behave either as a wave or as a particle depending on the measurement apparatus, and both behaviors are never observed simultaneously. This can be well demonstrated via a single photon Mach–Zehnder interferometer, as depicted in Fig. 1(a). An incident photon is split, at an input beam splitter BS1, into an equal superposition of being in the upper and lower paths. This is followed by a phase shift φ in the upper path. At the output beam splitter BS2, the paths are recombined and the detection probability in the detector D1 or D2 depends on the phase φ, heralding the wave nature of a single photon. If, however, BS2 is absent, the photon is detected with probability 1/2 in each detector, and thus, shows its particle nature. In Wheeler’s delayed-choice experiment [2, 3], the decision of whether or not to insert BS2 is randomly made after a photon is already inside the interferometer. The arrangement rules out a hidden-variable theory, which suggests that the photon may determine, in advance, which behavior, wave or particle, to exhibit through a hidden variable [4–12]. Recently, a quantum delayed-choice experiment, where BS2 is engineered to be in a quantum superposition of being present and absent, has been proposed [13]. Such a version allows a single system to be in a quantum superposition of a wave and a particle, so that both behaviors can be observed in a single measurement apparatus at the same time [14, 15]. This extends the conventional boundary of Bohr’s complementarity principle. The quantum delayed-choice experiment has already been implemented in nuclear magnetic resonance [16–18], optics [19–24], and superconducting circuits [25, 26]. However, all these experiments were performed essentially at the microscopic scale.

Here, as a first step in the macroscopic test for a coherent wave-particle superposition on massive objects, we propose and analyze a novel approach for a mechanical quantum delayed-choice experiment. Mechanical systems are not only being explored now for potential quantum technologies [27–38], but they also have been considered as a promising candidate to test fundamental principles in quantum theory [39], including, e.g., quantum superposition [40–43], wave-function collapse [44, 45], quantum entanglement [46–48], and Bell’s nonlocality [49–52]. In this manuscript, we demonstrate that, similar to a single photon, the mechanical phonon can be prepared in a quantum superposition of both a wave and a particle. The basic idea is to use a single nitrogen-vacancy (NV) center in diamond to control the coherent coupling between two separated carbon nanotubes (CNTs) [53]. We focus on the electronic ground state of the NV center, which is a spin $S = 1$ triplet with a zero-field splitting $D \simeq 2\pi \times 2.87$ GHz between spin states $|0\rangle$ and $|\pm 1\rangle$ [see Fig. 1(b)]. If the spin is in $|0\rangle$, the mechanical modes are decoupled, and otherwise are coupled. Moreover, the mechanical noise tolerated by our proposal is evaluated and we show a critical temperature, below which the coherent signal is resolved.

Physical model.—We consider a hybrid system [54, 55] consisting of two (labelled as $k = 1, 2$) parallel CNTs and an NV electronic spin, as illustrated in Fig. 1(c). The CNTs, both suspended along the $\hat{x}$-direction, carry dc currents $I_1$ and $I_2$, respectively, while the spin is placed between them, at a distance $d_1$ from the first CNT and at a distance $d_2$ from the second CNT. When vi-
brating along the $\hat{y}$-direction, the CNTs can parametrically modulate the Zeeman splitting of the intermediate spin through the magnetic field, yielding a magnetic coupling to the spin [56–60]. For simplicity, below we assume that the CNTs are identical such that they have the same vibrational frequency $\omega_m$ and the same vibrational mass $m$. The mechanical vibrations are modelled by quantized harmonic oscillators with a Hamiltonian $H_{mv} = \sum_{k=1,2} \hbar \omega_m b_k^{\dagger} b_k$, where $b_k$ ($b_k^{\dagger}$) denotes the phonon annihilation (creation) operator. The Hamiltonian characterizing the coupling of the mechanical modes to the spin is $H_{int} = \sum_{k=1,2} \hbar g_k S_z q_k$, where $S_z = |+1\rangle\langle+1| - |-1\rangle\langle-1|$ is the $z$-component of the spin, $q_k = b_k + b_k^{\dagger}$ represents the canonical phonon position operator, and $g_k = \mu_B g \gamma_m G_k / \hbar$ refers to the Zeeman shift corresponding to the zero-point motion $y_{zp} = \sqrt{\hbar / (2m\omega_m)}$. Here, $\mu_B$ is the Bohr magneton, $g_s \simeq 2$ is the Landé factor, and $G_k = \mu_0 I_k / (2\pi d_k^2)$ is the magnetic-field gradient, where $\mu_0$ is the vacuum permeability. In order to mediate the coherent coupling of the CNT mechanical modes through the spin, we apply a time-dependent magnetic field $B_x(t) = B_0 \cos(\omega_0 t)$ (with amplitude $B_0$ and frequency $\omega_0$) along the $\hat{x}$-direction to drive the $|0\rangle \rightarrow |\pm1\rangle$ transitions with Rabi frequency $\Omega = \mu_B g \omega_0 / (2\sqrt{2}\hbar)$. We apply a static magnetic field $B_z = \sum_{k=1,2} (-1)^k d_k G_k$ along the $\hat{z}$-direction to eliminate the Zeeman splitting between the spin states $|\pm1\rangle$ [59]. This causes the same Zeeman shift, $\Delta = \Delta_+ + 3\Omega^2 / \Delta_+$, where $\Delta_+ = D \pm \omega_0$, to be imprinted on $|\pm1\rangle$, and a coherent coupling, of strength $\Omega^2 / \Delta_+$, between them, as shown in Fig. 1(b). We can thus, introduce a dark state $|\tilde{D}\rangle = (\pm 1) / \sqrt{2}$ and a bright state $|\tilde{B}\rangle = (|1\rangle + |0\rangle) / \sqrt{2}$, with an energy splitting $\simeq 2\Omega^2 / \Delta_+$. In this case, the spin state $|0\rangle$ is decoupled from the dark state, and is dressed by the bright state.

Under the assumption of $\Omega / \Delta \ll 1$, the dressing will only increase the energy splitting between the dark and bright states to $\omega_q = 2\Omega^2 (1 / \Delta + 1 / \Delta_+)$.

This yields a spin qubit with $|\tilde{D}\rangle$ as the ground state and $|\tilde{B}\rangle$ as the excited state. The spin-CNT coupling Hamiltonian is accordingly transformed to $H_{int} \approx \sum_{k=1,2} \hbar g_k \sigma_q q_k$, where $\sigma_x = \sigma_+ + \sigma_-$, with $\sigma_x = |\tilde{D}\rangle \langle B| + |B\rangle \langle \tilde{D}|$ and $\sigma_+ = \sigma_1^\dagger$. When we further restrict our discussion to a dispersive regime $\omega_q > \omega_m \gg |g_k|$, the spin qubit becomes a quantum data bus, allowing for mechanical excitations to be exchanged between the CNTs. By using a time-averaging treatment [61–63], the unitary dynamics of the system is then described by an effective Hamiltonian [64],

$$H_{\text{eff}} = H_{\text{cnt}} \otimes \sigma_z,$$

and $\sigma_z = |B\rangle \langle B| - |\tilde{D}\rangle \langle \tilde{D}|$. The Hamiltonian $H_{\text{cnt}}$ includes a coherent spin-mediated CNT-CNT coupling in the beam-splitter form, which is conditioned on the spin state. Here, we neglect the direct CNT-CNT coupling much smaller than the spin-mediated coupling [64]. Furthermore, we find that the decoupling of one CNT from the spin gives rise to a spin-induced shift of the vibrational resonance of the other CNT. Hence, the dynamics described by $H_{\text{eff}}$ can be used to implement controlled Hadamard and phase gates.

Quantum delayed-choice experiment with mechanical resonators.—Let us first discuss the Hadamard gate. Having $I_k = I$ and $d_k = d$ gives a symmetric coupling $g_s = g$, and a mechanical beam-splitter coupling of strength $J = 2g^2 \omega_q / (\omega_m^2 - \omega_q^2)$. Unitary evolution for a time $\tau_0 = \pi / (4J)$ then leads to $b_1 (\tau_0) = (b_1 - ib_2) / \sqrt{2}$ and $b_2 (\tau_0) = (b_2 - ib_1) / \sqrt{2}$. For the phase gate, we can turn off the current, for example, of the second CNT, so that $g_1 = g$ and $g_2 = 0$. In this case, a dispersive
shift of \( \simeq J \) is imprinted into the vibrational resonance of the first CNT, which in turn introduces a relative phase \( \phi \simeq J \tau_1 \) after a time \( \tau_1 \) under unitary evolution. Note that, here, both Hadamard and phase gates are controlled operations conditional on the spin state, as mentioned before [64].

We now turn to the quantum delayed-choice experiment with the macroscopic CNTs. We assume that the hybrid system is initially prepared in the state \( |\Psi\rangle_i = (b^\dagger_1 \otimes I_2 |\text{vac}\rangle) \otimes |D\rangle \), where |\text{vac}\rangle \) refers to the phonon vacuum and \( I_2 \) is the identity operator for the second CNT. After the initialization, the currents are tuned to be \( I_k = I \), to drive the system for a time \( \tau_0 \), and the resulting Hadamard operation splits the single phonon into an equal superposition across both CNTs. Then, we turn off \( I_2 \) for a time \( \tau_1 \) to accumulate a relative phase between the CNTs. While achieving the desired phase \( \phi \), we turn on \( I_2 \) following a spin single-qubit rotation \( |D\rangle \to \cos (\phi) |0\rangle + \sin (\phi) |D\rangle \) [65–67] with \( \phi \) a rotation angle, and hold for another \( \tau_0 \) for a Hadamard operation. Therefore, this Hadamard gate is in a quantum superposition of being both present and absent. The three steps correspond, respectively, to the input beam splitter, the phase shifter and the quantum output beam splitter acting in sequence on a single photon in the Mach-Zehnder interferometer, as shown in Fig. 1(a). The final state of the system therefore becomes

\[
|\Psi\rangle_f = \cos (\phi) |\text{particle}\rangle |0\rangle + \sin (\phi) |\text{wave}\rangle |D\rangle,
\]

(2)

where \( |\text{particle}\rangle = \left\{ \exp (i\phi) b^\dagger_1 + ib^\dagger_2 \right\} |\text{vac}\rangle /\sqrt{2} \) and \( |\text{wave}\rangle = \left\{ \exp (i\phi) - 1 \right\} b^\dagger_1 + i \left\{ \exp (i\phi) + 1 \right\} b^\dagger_2 \right\} |\text{vac}\rangle /2 \) describe the particle and wave behaviors, respectively. We find from Eq. (2) that the mechanical phonon is in a quantum superposition of both a wave and a particle, and thus can exhibit both characteristics simultaneously. In the case of \( \phi = 0 \), the single phonon behaves completely as a particle, but as a wave for \( \phi = \pi/2 \). The morphing between them can also be observed by tuning the rotation angle \( \phi \). The probability, \( P_k \), of finding a phonon in the \( k \)th CNT is given by

\[
P_k = \frac{1}{2} + (-1)^k \frac{1}{2} \sin^2 (\phi) \cos (\phi),
\]

(3)

which includes two physical contributions, one from the particle nature and the other from the wave nature. Note that the spin in a mixed state \( \cos^2 (\phi) |0\rangle \langle 0| + \sin^2 (\phi) |D\rangle \langle D| \) is capable of reproducing the same measured statistics as in Eq. (3) [12, 68–70]. Thus, in order to exclude the classical interpretation and prove the existence of the coherent wave-particle superposition, the quantum coherence between the states |0\rangle and |D\rangle should be verified [20, 21, 25, 26]. Experimentally, such a verification can be implemented by performing quantum state tomography to show all elements of the density matrix of the spin [67].

![FIG. 2. Morphing between particle and wave characteristics of a CNT mechanical phonon. Phonon occupation (a) \( n_1 \) and (b) \( n_2 \) as a function of the relative phase \( \phi \) and the rotation angle \( \varphi \). The analytical results (colored surfaces) are in excellent agreement with the numerical simulations (black symbols). Here, in addition to \( \gamma_r / 2\pi = 200 \gamma_m / 2\pi = 80 \text{ Hz} \), we assume that \( g / 2\pi = 100 \text{ kHz} \), \( \omega_m / 2\pi = 2 \text{ MHz} \), \( \Omega = 10 \omega_m \), and \( \Delta_r = 142 \omega_m \), resulting in \( \omega_I \simeq 1.5 \omega_m \) and then \( J / 2\pi \simeq 12 \text{ kHz} \) and that \( n_{th} = 100 \), corresponding to an environmental temperature of \( \simeq 10 \text{ mK} \).](image-url)

Next, we consider how to initialize and measure the mechanical system. In order to excite the left CNT to a single-phonon state, we make the spin qubit transition frequency \( \omega_q \) close to the mechanical frequency \( \omega_m \), after the CNT is cooled into the quantum ground state [42, 71–75]. The spin-CNT coupling Hamiltonian is then approximately given by a Jaynes-Cummings-like Hamiltonian\n
\[
\hat{H}_{\text{int}} \simeq \hbar g \left( \sigma_+ b_1 + \sigma_- b_1^\dagger \right).
\]

When acting for a time of \( = \pi / (2g) \), such a Hamiltonian can, with the spin qubit in the excited state |B\rangle, transfer a mechanical excitation to the left CNT [76]. For the phonon number measurement, we still need \( \omega_q \simeq \omega_m \) as in the initialization, but the spin qubit is required to be in the ground state |D\rangle. In this situation, the Rabi frequency between the spin and the mechanical resonator depends on the number of phonons in the resonator [76–80]. Thus by directly measuring the occupation probability of |B\rangle, the phonon number in each CNT can be obtained.

Mechanical noise.—Before discussing the mechanical noise, we need to analyze the total operation time, \( \tau_T = 2\tau_0 + \tau_1 \), required for our quantum delayed-choice experiment. Note that during \( \tau_T \), we have neglected the spin single-qubit operation time due to the driving pulse length \( \sim \text{ ns} \) [81, 82]. Since \( 0 \leq \tau_1 \leq 2\pi / J \), we focus on the maximum \( \tau_T: \tau_T^\text{max} = 5\pi / (2J) \). A modest spin-CNT coupling \( g / 2\pi = 100 \text{ kHz} \), which can be obtained by tuning the current \( I \) and the distance \( d \) [64], is able to mediate an effective CNT-CNT coupling \( J / 2\pi \simeq 12 \text{ kHz} \), thus giving \( \tau_T^\text{max} \simeq 0.1 \text{ ms} \). The relaxation time \( T_1 \) of a single NV spin at low temperatures can reach up to a few minutes. Moreover, a single spin in an ultra-pure diamond example typically has a dephasing time \( T_2 \simeq 2 \text{ ms} \) even at room temperature [83, 84], corresponding to a dephasing rate \( \gamma_d / 2\pi \simeq 80 \text{ Hz} \). These justify neglecting the spin decoherence. In this case, the mechanical noise
FIG. 3. Signal visibility $\mathcal{R}$ as a function of the temperature $T$. The yellow shaded area represents the signal-resolved regime, where the morphing between wave and particle can be effectively observed in the fluctuation noise. The vertical line corresponds to the critical temperature $T_c$. The inset shows a linear increase in $T_c$ with increasing the ratio of the spin-mediated CNT-CNT coupling strength $J$ to the mechanical-mode decay rate $\gamma_m$. Here, all parameters are set to be the same as in Fig. 2.

dominates the dissipative processes. The dynamics of the system is therefore governed by the following master equation,

$$
\dot{\rho}(t) = \frac{i}{\hbar} [\rho(t), H(t)] - \frac{\gamma_m}{2} n_{\text{th}} \sum_{k=1,2} \mathcal{L}(b_k^\dagger) \rho(t) - \frac{\gamma_m}{2} (n_{\text{th}} + 1) \sum_{k=1,2} \mathcal{L}(b_k) \rho(t),
$$

where $\rho(t)$ is the density operator of the system, $\gamma_m$ is the mechanical decay rate, $n_{\text{th}} = [\exp(\hbar \omega_m/k_B T) - 1]^{-1}$ is the equilibrium phonon occupation at temperature $T$, and $\mathcal{L}(\rho) = \rho^{\dagger} \rho - \frac{1}{2} \rho^{\dagger} \rho^{\dagger} \rho$ is the Lindblad superoperator. Here, $H(t)$ is a binary Hamiltonian of the form,

$$
H(t) = \begin{cases} 
H_0, & 0 < t \leq \tau_0, \text{ and } \tau_0 + \tau_1 < t \leq \tau_T \\
H_1, & \tau_0 < t \leq \tau_0 + \tau_1,
\end{cases}
$$

with $H_0 = J \left( \sum_{k=1,2} b_k^\dagger b_k + b_1^\dagger b_1 + b_2^\dagger b_2 \right) \sigma_z$ and $H_1 = J b_1^\dagger b_1 \sigma_z$. In Eq. (5), we did not include the spin single-qubit operation before the third time interval because the length of the driving pulse is very short, as mentioned above. The master equation in Eq. (4) drives the phonon occupation of the $k$th CNT to be $n_k = \langle b_k^\dagger b_k \rangle (\tau_T) = P_k \exp(-\gamma_m \tau_T) + n_{\text{th}} [1 - \exp(-\gamma_m \tau_T)]$ at time $t = \tau_T$.

For a realistic CNT, we can set the mechanical linewidth to be $\gamma_m/2\pi = 0.4$ Hz [85], leading to a single-phonon lifetime of $\tau_m = 1/\gamma_m \simeq 400$ ms. In this situation, $\tau_m$ is much longer than the total operation time $\tau_T$, $\gamma_m \tau_T \ll 1$ and, thus, we obtain

$$
n_k = P_k + n_{\text{th}} \gamma_m \tau_T.
$$

This shows that, in addition to the coherent signal $P_k$, the final occupation has a thermal contribution $n_{\text{th}} \gamma_m \tau_T$. In Fig. 2, we demonstrate the morphing behavior between particle and wave at $T \simeq 10$ mK, according to Eq. (6). To confirm this, we also plot numerical simulations, which are in exact agreement with our analytical expression. The thermal occupation, $n_{\text{th}} \gamma_m \tau_T$, increases as the phase $\phi$, because such a phase arises from the dynamical accumulation as discussed above. However, an extremely long phonon lifetime causes it to become negligible even at finite temperatures, as shown in Fig. 2.

We now consider the fluctuation noise. In the limit $\gamma_m \tau_T \ll 1$, the fluctuation noise $\delta n_k^\text{noise}$ in the phonon occupation $n_k$ is expressed, according to the analysis in the Supplemental Material [64], as $(\delta n_k^\text{noise})^2 = P_k (2P_k - 1) \gamma_m \tau_m + (2P_k + 1) n_{\text{th}} \gamma_m \tau_T$, where the first term is the vacuum fluctuation, which can be neglected, and the second term is the thermal fluctuation, which increases with temperature. To quantitatively describe the ability to resolve the coherent signal from the fluctuation noise, we typically employ the signal-to-noise ratio defined as $\mathcal{R}_k = P_k / \delta n_k^\text{noise}$. The signal-resolved regime often requires $\mathcal{R}_k > 1$ for any $P_k$. However, the probability $P_k$ in the range zero to unity indicates that there always exist some $P_k$ such that $\mathcal{R}_k < 1$, in particular, at finite temperatures. Nevertheless, we find that the total fluctuation noise $S^2 = (\delta n_1^\text{noise})^2 + (\delta n_2^\text{noise})^2$ is kept below an upper bound $B^2 = \gamma_m \tau_T^\text{max} + 4n_{\text{th}} \gamma_m \tau_T^\text{max}$, and further that assuming $B^2 < 1/2$ can make either or both of $\mathcal{R}_1$ and $\mathcal{R}_2$ greater than 1. In this case, at least one CNT signal is resolved for each measurement. The conservation of the coherent phonon number equal to 1 ensures that the unresolved signal can be inferred from the resolved one, which allows the morphing between wave and particle to be effectively observed from the fluctuation noise. To quantify this, we define a signal visibility as,

$$
\mathcal{R} = \frac{\sqrt{2}}{2B},
$$

in analogy to the signal-to-noise ratio $\mathcal{R}_k$. The ratio $\mathcal{R}$ describes the visibility of the total signal rather than the single CNT signals. At zero temperature ($n_{\text{th}} = 0$), the noise originates only from the vacuum fluctuation, and this yields $\mathcal{R} \gg 1$. However, at finite temperatures, $n_{\text{th}}$ increases as $T$, causing a decrease in $\mathcal{R}$, as shown in Fig. 3. Therefore, the requirement of $\mathcal{R} > 1$ sets an upper bound on the temperature, and as a result, leads to a critical temperature,

$$
T_c = \frac{\hbar \omega_m}{k_B \ln[(1 + 15\pi \gamma_m/J) / (1 - 5\pi \gamma_m/J)]}.
$$
The critical temperature linearly increases with $J/\gamma_m$, as plotted in the inset of Fig. 3. For modest parameters of $J/2\pi = 12$ kHz and $\gamma_m/2\pi = 0.4$ Hz, a critical temperature $T_c$ of $\approx 47$ mK, which is routinely accessible in current experiments, can be achieved.

Conclusions.—We have presented a proposal for a quantum delayed-choice experiment with nanomechanical resonators, which enables a macroscopic test of an arbitrary quantum wave-particle superposition. The ability to tolerate the mechanical noise has also been given here, demonstrating that our proposal can be implemented with current experimental techniques. While we have chosen to focus on a spin-nanomechanical setup, the present method could be directly extended to other hybrid systems, for example, mechanical devices coupled to a superconducting atom [55, 76, 86]. We believe that this proposed quantum delayed-choice experiment of massive mechanical resonators not only leads to a better understanding of quantum theory at the macroscopic scale, but also indicates that, like the vertical and horizontal polarizations of photons, the mechanical wave-particle nature, as an additional degree of freedom of phonons, may be widely exploited for quantum information applications [14, 23].

W.Q. thanks Peng-Bo Li for valuable discussions. W.Q. and J.Q.Y. were supported in part by the National Key Research and Development Program of China (Grant No. 2016YFA0301200), the China Postdoctoral Science Foundation (Grant No. 2017M610752), and the NSFC (Grant No. 11774022). A.M. and F.N. acknowledge the support of a grant from the John Templeton Foundation. F. N. is supported in part by the MURI Center for Dynamic Magneto-Optics via the Air Force Office of Scientific Research (AFOSR) (FA9550-14-1-0040), Army Research Office (ARO) (Grant No. 73315PH), Asian Office of Aerospace Research and Development (AOARD) (Grant No. FA2386-18-1-4045), Japan Science and Technology Agency (JST) (the ImPACT program and CREST Grant No. JPMJCR1676), Japan Society for the Promotion of Science (JSPS) (JSPS-RFBR Grant No. 17-52-50023), RIKEN-AIST Challenge Research Fund, and the John Templeton Foundation.

Phys. Rev. A 88, 015005 (2016).

[1] N. Bohr, in Quantum Theory and Measurement, edited by J. A. Wheeler and W. H. Zurek (Princeton University Press, Princeton, NJ, 1984), pp. 9-49.

[2] J. A. Wheeler, in Mathematical Foundations of Quantum Theory, edited by A. R. Marlow (Academic Press, Cambridge, 1978), pp. 9-48.

[3] X.-s. Ma, J. Kohler, and A. Zeilinger, “Delayed-choice gedanken experiments and their realizations,” Rev. Mod. Phys. 88, 015005 (2016).

[4] T. Hellmuth, H. Walther, A. Zajonc, and W. Schleich, “Delayed-choice experiments in quantum interference,” Phys. Rev. A 35, 2532 (1987).

[5] Y.-H. Kim, R. Yu, S. P. Kulik, Y. Shih, and M. O. Scully, “Delayed “Choice” Quantum Eraser,” Phys. Rev. Lett. 84, 1 (2000).

[6] V. Jacques, E. Wu, F. Grosshans, F. Treussart, P. Grangier, A. Aspect, and J.-F. Roch, “Experimental Realization of Wheeler’s Delayed-Choice Gedanken Experiment,” Science 315, 966–968 (2007).

[7] V. Jacques, E. Wu, F. Grosshans, F. Treussart, P. Grangier, A. Aspect, and J.-F. Roch, “Delayed-Choice Test of Quantum Complementarity with Interfering Single Photons,” Phys. Rev. Lett. 100, 220402 (2008).

[8] A. G. Manning, R. I. Khakimov, R. G. Dall, and A. G. Truscott, “Wheeler’s delayed-choice gedanken experiment with a single atom,” Nat. Phys. 11, 539 (2015).

[9] R. Ionicioiu, R. B. Mann, and D. R. Terno, “Determinism, Independence, and Objectivity are Incompatible,” Phys. Rev. Lett. 114, 060405 (2015).

[10] Y. Liu, J. Lu, and L. Zhou, “Information gain versus interference in Bohr’s principle of complementarity,” Opt. Express 25, 202-211 (2017).

[11] F. Vedovato, C. Agnesi, M. Schiavon, D. Dequal, L. Calderaro, M. Tomasin, D. G. Marangon, A. Stanco, V. Luceri, G. Bianco, G. Vallone, and P. Villori, “Extending Wheeler’s delayed-choice experiment to space,” Sci. Adv. 3, e1701180 (2017).

[12] R. Chaves, G. B. Lemos, and J. Pienaar, “Causal Modeling the Delayed-Choice Experiment,” Phys. Rev. Lett. 120, 190401 (2018).

[13] R. Ionicioiu and D. R. Terno, “Proposal for a Quantum Delayed-Choice Experiment,” Phys. Rev. Lett. 107, 230406 (2011).

[14] G. Adesso and D. Girolami, “Quantum optics: Wave–particle superposition,” Nat. Photon. 6, 579 (2012).

[15] P. Shadbolt, J. C. F. Mathews, A. Laing, and J. L. O’Brien, “Testing foundations of quantum mechanics with photons,” Nat. Phys. 10, 278 (2014).

[16] S. S. Roy, A. Shukla, and T. S. Mahesh, “NMR implementation of a quantum delayed-choice experiment,” Phys. Rev. A 85, 022109 (2012).

[17] R. Auccaise, R. M. Serra, J. G. Filgueiras, R. S. Sarthour, I. S. Oliveira, and L. C. Céleri, “Experimental analysis of the quantum complementarity principle,” Phys. Rev. A 85, 032121 (2012).

[18] T. Xin, H. Li, B.-X. Wang, and G.-L. Long, “Realization of an entanglement-assisted quantum delayed-choice experiment,” Phys. Rev. A 92, 022126 (2015).

[19] J.-S. Tang, Y.-L. Li, X.-Y. Xu, G.-Y. Xiang, C.-F. Li, and G.-C. Guo, “Realization of quantum Wheeler’s delayed-choice experiment,” Nat. Photon. 6, 600–604 (2012).

[20] A. Peruzzo, P. Shadbolt, N. Brunner, S. Popescu, and J. L. O’Brien, “A Quantum Delayed-Choice Experiment,” Science 338, 643–637 (2012).

[21] F. Kaiser, T. Coudreau, P. Milman, D. B. Ostrowsky, and S. Tanzilli, “Entanglement-Enabled Delayed-Choice Experiment,” Science 338, 637–640 (2012).

[22] H. Yan, K. Liao, Z. Deng, J. He, Z.-Y. Xue, Z.-M. Zhang, and S.-L. Zhu, “Experimental observation of simultaneous wave and particle behavior in a narrowband single-photon wave packet,” Phys. Rev. A 91, 042132 (2015).
[23] A. S. Rab, E. Polino, Z.-X. Man, N. B. An, Y.-J. Xia, N. Spagnolo, R. L. Franco, and F. Sc}_ciarino, “Entanglement of photons in their dual wave-particle nature,” Nat. Commun. 8, 915 (2017).

[24] G.-L. Long, W. Qin, Z. Yang, and J.-L. Li, “Realistic interpretation of quantum mechanics and encounter-delayed-choice experiment,” Sci. China Phys. Mech. Astron. 61, 030311 (2018).

[25] S.-B. Zheng, Y.-P. Zhong, K. Xu, Q.-J. Wang, H. Wang, L.-T. Shen, C.-P. Yang, J. M. Martinis, A. N. Cleland, and S.-Y. Han, “Quantum Delayed-Choice Experiment with a Beam Splitter in a Quantum Superposition,” Phys. Rev. Lett. 115, 260403 (2015).

[26] K. Liu, Y. Xu, W. Wang, S.-B. Zheng, T. Roy, S. Kundu, M. Chand, A. Ranadive, R. Vijay, Y. Song, L. Duan, and L. Sun, “A twofold quantum delayed-choice experiment in a superconducting circuit,” Sci. Adv. 3, e1603159 (2017).

[27] M. Blencowe, “Quantum electromechanical systems,” Phys. Rep. 395, 159-222 (2004).

[28] Y.-X. Liu, A. Miranowicz, Y. B. Gao, J. Bajer, C. P. Sun, and F. Nori, “Qubit-induced phonon blockade as a signature of quantum behavior in nanomechanical resonators,” Phys. Rev. A 82, 032101 (2010).

[29] K. Stamnigel, P. Komar, S. J. M. Habraken, S. D. Bennett, M. D. Lukin, P. Zoller, and P. Rabl, “Optomechanical Quantum Information Processing with Photons and Phonons,” Phys. Rev. Lett. 109, 013603 (2012).

[30] S. D. Bennett, N. Y. Yao, J. Otterbach, P. Zoller, P. Rabl, and M. D. Lukin, “Phonon-Induced Spin-Spin Interactions in Diamond Nanostructures: Application to Spin Squeezing,” Phys. Rev. Lett. 110, 156402 (2013).

[31] J. Bochmann, A. Vainsencher, D. A. Wesschalom, and A. N. Cleland, “Nanomechanical coupling between microwave and optical photons,” Nat. Phys. 9, 712 (2013).

[32] M. Aspelmeyer, T. J. Kimpenberg, and F. Marquardt, “Cavity optomechanics,” Rev. Mod. Phys. 86, 1391 (2014).

[33] H. Jing, S. K. Özdemir, X.-Y. Lü, J. Zhang, L. Yang, and F. Nori, “PT-Symmetric Phonon Laser,” Phys. Rev. Lett. 113, 053604 (2014).

[34] X.-Y. Lü, Y. Wu, J. R. Johansson, H. Jing, J. Zhang, and F. Nori, “Squeezed Optomechanics with Phase-Matched Amplification and Dissipation,” Phys. Rev. Lett. 114, 093602 (2015).

[35] P.-B. Li and F.-L. Li, “Proposal for a quantum delayed-choice experiment with a spin-mechanical setup,” Phys. Rev. A 94, 042130 (2016).

[36] D. A. Golter, T. Oo, M. Amezcua, K. A. Stewart, and H. Wang, “Optomechanical Quantum Control of a Nitrogen-Vacancy Center in Diamond,” Phys. Rev. Lett. 116, 143602 (2016).

[37] A. P. Reed, K. H. Mayer, J. D. Teufel, L. D. Burkhart, W. Pfaff, M. Reagor, L. Sletten, X. Ma, R. J. Schoelkopf, E. Knill, and K. W. Lehnert, “Faithful conversion of propagating quantum information to mechanical motion,” Nat. Phys. 13, 1163 (2017).

[38] M. Cirio, K. Debnath, N. Lambert, and F. Nori, “Amplified Optomechanical Transduction of Virtual Radiation Pressure,” Phys. Rev. Lett. 119, 053601 (2017).

[39] M. Poot and H. S. J. van der Zant, “Mechanical systems in the quantum regime,” Phys. Rep. 511, 273-335 (2012).

[40] M. Arndt and K. Hornberger, “Testing the limits of quantum mechanical superpositions,” Nat. Phys. 10, 271 (2014).

[41] J.-Q. Liao and L. Tian, “Macroscopic Quantum Superposition in Cavity Optomechanics,” Phys. Rev. Lett. 116, 163602 (2016).

[42] X. Wang, A. Miranowicz, H.-R. Li, and F. Nori, “Hybrid quantum device with a carbon nanotube and a flux qubit for dissipative quantum engineering,” Phys. Rev. B 95, 205415 (2017).

[43] C. S. Muñoz, A. Lara, J. Puebla, and F. Nori, “Hybrid systems for the generation of non-classical mechanical states via quadratic interactions,” arXiv preprint arXiv:1802.01306 (2018).

[44] M. Bahrami, M. Paternostro, A. Bassi, and H. Ulbricht, “Proposal for a Noninterferometric Test of Collapse Models in Optomechanical Systems,” Phys. Rev. Lett. 112, 210404 (2014).

[45] L. Diósi, “Testing Spontaneous Wave-Function Collapse Models on Classical Mechanical Oscillators,” Phys. Rev. Lett. 114, 050403 (2015).

[46] P. Sekatski, M. Aspelmeyer, and N. Sangouard, “Macroscopic Optomechanics from Displaced Single-Photon Entanglement,” Phys. Rev. Lett. 112, 080502 (2014).

[47] R. Ghobadi, S. Kumar, B. Pepper, D. Bouwmeester, A. I. Lvovsky, and C. Simon, “Optomechanical Micro-Macro Entanglement,” Phys. Rev. Lett. 112, 080503 (2014).

[48] C. F. Ockeloen-Korpf, E. Damkïagg, J.-M. Pirkkalainen, M. Asjad, A. A. Clerk, F. Massel, M. J. Woolley, and M. A. Sillanpää, “Stabilized entanglement of massive mechanical oscillators,” Nature (London) 556, 478 (2018).

[49] J. R. Johansson, N. Lambert, I. Mahboob, H. Yamaguchi, and F. Nori, “Entangled-state generation and Bell inequality violations in nanomechanical resonators,” Phys. Rev. B 90, 174307 (2014).

[50] V. C. Vivoli, T. Barnea, C. Galland, and N. Sangouard, “Proposal for an Optomechanical Bell Test,” Phys. Rev. Lett. 116, 070405 (2016).

[51] S. G. Hofer, K. W. Lehnert, and K. Hammerer, “Proposal to Test Bell’s Inequality in Electromechanics,” Phys. Rev. Lett. 116, 070406 (2016).

[52] I. Marinkovic, A. Wallucks, R. Riedinger, S. Hong, M. Aspelmeyer, and S. Gröblacher, “An optomechanical Bell test,” arXiv preprint arXiv:1806.10615 (2018).

[53] S. Iijima, “Helical microtubules of graphitic carbon,” Nature (London) 354, 56 (1991).

[54] I. Buluta, S. Ashhab, and F. Nori, “Natural and artificial atoms for quantum computation,” Reports Prog. Phys. 74, 104401 (2011).

[55] Z. L. Xiang, S. Ashhab, J. Q. You, and F. Nori, “Hybrid quantum circuits: Superconducting circuits interacting with other quantum systems,” Rev. Mod. Phys. 85, 623 (2013).

[56] P. Rabl, P. Cappellaro, M. V. G. Dutt, L. Jiang, J. R. Maze, and M. D. Lukin, “Strong magnetic coupling between an electronic spin qubit and a mechanical resonator,” Phys. Rev. B 79, 041302 (2009).

[57] P. Rabl, S. J. Kolkowitz, F. H. L. Koppens, J. G. E. Harris, P. Zoller, and M. D. Lukin, “A quantum spin transducer based on nanoelectromechanical resonator arrays,” Nat. Phys. 6, 602 (2010).

[58] S. Kolkowitz, A. C. B. Jayich, Q. P. Unterreithmeier, S. D. Bennett, P. Rabl, J. G. E. Harris, and M. D. Lukin, “Coherent sensing of a mechanical resonator with...
a single-spin qubit,” Science **335**, 1603–1606 (2012).

[59] P.-B. Li, Z.-L. Xiang, P. Rabl, and F. Nori, “Hybrid Quantum Device with Nitrogen-Vacancy Centers in Diamond Coupled to Carbon Nanotubes,” Phys. Rev. Lett. **117**, 015502 (2016).

[60] P. Cao, R. Betzholz, S. Zhang, and J. Cai, “Entangling distant solid-state spins via thermal phonons,” Phys. Rev. B **96**, 245418 (2017).

[61] O. Gamel and D. F. V. James, “Time-averaged quantum dynamics and the validity of the effective Hamiltonian model,” Phys. Rev. A **82**, 052106 (2010).

[62] W. Qin, A. Miranovic, P.-B. Li, X.-Y. Liu, J. Q. You, and F. Nori, “Exponentially Enhanced Light-Matter Interaction, Cooperativities, and Steady-State Entanglement Using Parametric Amplification,” Phys. Rev. Lett. **120**, 093601 (2018).

[63] C. Leroux, L. C. G. Govia, and A. A. Clerk, “Enhancing Cavity Quantum Electrodynamics via Antisqueezing: Synthetic Ultracoupling,” Phys. Rev. Lett. **120**, 093602 (2018).

[64] See Supplementary Material at http://xxx for detailed derivations of our main results, which includes Refs. [87–94].

[65] P. Huang, X. Kong, N. Zhao, F. Shi, P. Wang, X. Rong, R.-B. Liu, and J. Du, “Observation of an anomalous decoherence effect in a quantum bath at room temperature,” Nat. Commun. **2**, 370 (2011).

[66] S. E. Lillie, D. A. Broadway, J. D. A. Wood, D. A. Simpson, A. Stacey, J.-P. Tetienne, and L. C. L. Hollenberg, “Environmentally Mediated Coherent Control of a Spin Qubit in Diamond,” Phys. Rev. Lett. **118**, 167204 (2017).

[67] J. Xing, Y.-R. Zhang, S. Liu, Y.-C. Chang, J.-D. Yue, H. Fan, and X.-Y. Pan, “Experimental investigation of quantum entropic uncertainty relations for multiple measurements in pure diamond,” Sci. Rep. **7**, 2563 (2017).

[68] H.-L. Huang, Y.-H. Luo, B. Bai, Y.-H. Deng, H. Wang, H.-S. Zhong, Y.-Q. Nie, W.-H. Jiang, X.-L. Wang, J. Zhang, L. Li, N.-L. Liu, T. Byrnes, J. P. Dowling, C.-Y. Lu, and J.-W. Pan, “A loophole-free Wheeler-delayed-choice experiment,” arXiv preprint arXiv:1806.00156 (2018).

[69] E. Polino, I. Agresti, D. Poderini, G. Carvacho, Gi. Milani, G. B. Lemos, R. Chaves, and F. Sciarrino, “Device independent certification of a quantum delayed choice experiment,” arXiv preprint arXiv:1806.00211 (2018).

[70] S. Yu, W. Liu, Y.-T. Wang, J.-S. Tang, C.-F. Li, and G.-C. Guo, “Experimental realization of causality-assisted Wheeler’s delayed-choice experiment using single photons,” arXiv preprint arXiv:1806.03689 (2018).

[71] F. Xue, Y. D. Wang, Y.-x. Liu, and F. Nori, “Cooling a micromechanical beam by coupling it to a transmission line,” Phys. Rev. B **76**, 205302 (2007).

[72] J. Q. You, Y.-x. Liu, and F. Nori, “Simultaneous Cooling of an Artificial Atom and its Neighboring Quantum System,” Phys. Rev. Lett. **100**, 047001 (2008).

[73] M. Grajcar, S. Ashhab, J. R. Johansson, and F. Nori, “Lower limit on the achievable temperature in resonator-based sideband cooling,” Phys. Rev. B **78**, 035406 (2008).

[74] Y. Ma, Z.-q. Yin, P. Huang, W. L. Yang, and J. F. Du, “Cooling a mechanical resonator to the quantum regime by heating it,” Phys. Rev. A **94**, 053836 (2016).

[75] J. B. Clark, F. Lecocq, R. W. Simmonds, J. Aumentado, and J. D. Teufel, “Sideband cooling beyond the quantum backaction limit with squeezed light,” Nature (London) **541**, 191–195 (2017).

[76] A. D. O’Connell, M. Hofheinz, M. Ansmann, R. C. Bialczak, M. Lenander, E. Lucero, M. Neeley, D. Sank, H. Wang, M. Weides, J. Wenner, J. M. Martinis, and A. N. Cleland, “Quantum ground state and single-phonon control of a mechanical resonator,” Nature (London) **464**, 697 (2010).

[77] M. O. Scully and M. S. Zubairy, Quantum Optics (Cambridge University Press, Cambridge, 1997).

[78] Y.-x. Liu, L. F. Wei, and F. Nori, “Generation of non-classical photon states using a superconducting qubit in a microcavity,” Europhys. Lett. **67**, 941 (2004).

[79] M. Hofheinz, E. M. Weig, M. Ansmann, R. C. Bialczak, E. Lucero, M. Neeley, A. D. O’Connell, H. Wang, J. M. Martinis, and A. N. Cleland, “Generation of Fock states in a superconducting quantum circuit,” Nature (London) **454**, 310 (2008).

[80] M. Hofheinz, H. Wang, M. Ansmann, R. C. Bialczak, E. Lucero, M. Neeley, A. D. O’Connell, D. Sank, J. Wenner, J. M. Martinis, and A. N. Cleland, “Synthesizing arbitrary quantum states in a superconducting resonator,” Nature (London) **459**, 546 (2009).

[81] G.-Q. Liu, Y.-R. Zhang, Y.-C. Chang, J.-D. Yue, H. Fan, and X.-Y. Pan, “Demonstration of entanglement-enhanced phase estimation in solid,” Nat. Commun. **6**, 6726 (2015).

[82] G.-Q. Liu, J. Xing, W.-L. Ma, P. Wang, C.-H. Li, H. C. Po, Y.-R. Zhang, H. Fan, R.-B. Liu, and X.-Y. Pan, “Single-Shot Readout of a Nuclear Spin Weakly Coupled to a Nitrogen-Vacancy Center at Room Temperature,” Phys. Rev. Lett. **118**, 150504 (2017).

[83] G. Basalabramanian, P. Neumann, D. Twitchen, M. Markham, R. Kolesov, N. Mizuochi, J. Isoya, J. Achard, J. Beck, J. Tissler, V. Jacques, P. R. Hemmer, F. Jelezko, and J. Wrachtrup, “Ultralong spin coherence time in isotopically engineered diamond,” Nat. Mater. **8**, 383–387 (2009).

[84] N. Bar-Gill, L. M. Pham, A. Jarmola, D. Budker, and R. L. Walsworth, “Solid-state electronic spin coherence time approaching one second,” Nat. Commun. **4**, 1743 (2013).

[85] J. Moser, A. Eichler, J. Güttinger, M. I. Dykman, and A. Bachtold, “ Nanotube mechanical resonators with quality factors of up to 5 million,” Nat. Nanotechnol. **9**, 1007–1011 (2014).

[86] X. Gu, A. F. Kockum, A. Miranovic, Y.-x. Liu, and F. Nori, “Microwave photonics with superconducting quantum circuits,” Phys. Rep. **718-719**, 1–102 (2017).

[87] V. Sazonova, Y. Yaish, H. Üstünel, D. Roundy, T. A. Arias, and P. L. McEuen, “A tunable carbon nanotube electromechanical oscillator,” Nature (London) **435**, 278 (2005).

[88] H. Üstünel, D. Roundy, and T. A. Arias, “Modeling a suspended nanotube oscillator,” Nano Lett. **5**, 523–526 (2005).

[89] D. García-Sánchez, A. San Paulo, M. J. Esplandiu, F. Perez-Murano, L. Forró, A. Aguasca, and A. Bachtold, “Mechanical Detection of Carbon Nanotube Resonator Vibrations,” Phys. Rev. Lett. **99**, 085501 (2007).

[90] Z. Y. Ning, T. W. Shi, M. Q. Fu, Y. Guo, X. L. Wei, S. Gao, and Q. Chen, “Transversally and axially tunable suspended nanotube oscillator,” Nano Lett. **8**, 383–387 (2008).
[91] S. Truax, S.-W. Lee, M. Muoth, and C. Hierold, “Axially tunable carbon nanotube resonators using co-integrated microactuators,” Nano Lett. 14, 6092–6096 (2014).

[92] I. Tsioutsios, A. Tavernarakis, J. Osmond, P. Verlot, and A. Bachtold, “Real-time measurement of nanotube resonator fluctuations in an electron microscope,” Nano Lett. 17, 1748–1755 (2017).

[93] J. R. Johansson, P. D. Nation, and F. Nori, “Qutip: An open-source Python framework for the dynamics of open quantum systems,” Comput. Phys. Commun. 183, 1760–1772 (2012).

[94] J. R. Johansson, P. D. Nation, and F. Nori, “Qutip 2: A Python framework for the dynamics of open quantum systems,” Comput. Phys. Commun. 184, 1234–1240 (2013).
Here, we, first, in Sec. S1 present more details of how to obtain the spin-controlled coherent coupling between separated mechanical resonators. Second, in Sec. S2, we show the detailed implementation of the controlled Hadamard gate, the phase gate, and the mechanical quantum delayed-choice experiment. Next, in Sec. S3, we derive in detail the phonon occupation of each CNT at finite temperatures. Then, Sec. S4 describes the detailed derivation of the fluctuation noise and the detailed analysis of the requirement of resolving the coherent signal from the environment-induced fluctuation. Finally, in Sec. S5 we show the method of the numerical simulation used in this work.

S1. Spin-controlled coherent coupling between separated mechanical resonators

![Supplemental Material](image)

**FIG. S1.** (Color online) (a) Schematic representation of a mechanical quantum delayed-choice experiment with an NV electronic spin and two carbon nanotubes (CNTs). The mechanical vibrations of the CNTs, labelled by \( k = 1, 2 \), are completely decoupled or coherently coupled, depending, respectively, on whether or not the intermediate spin is in the spin state \( |0\rangle \), with the dc current \( I_k \) through the \( k \)th CNT, and the distance \( d_k \) between the spin and the \( k \)th CNT. (b) Level structure of the driven NV spin in the electronic ground state. Here we have assumed that the Zeeman splitting between the spin states \( |\pm 1\rangle \) is eliminated by applying an external field.

The effective Hamiltonian \( H_{\text{eff}} \) in the article describes a spin-mediated CNT-CNT coupling conditioned on the NV spin state. This is the basic element underlying our proposal. To understand more explicitly the spin-controlled coupling between the CNTs, in this section we derive in detail the effective Hamiltonian. We consider a hybrid quantum system consisting of two parallel CNTs and an NV electronic spin (a qutrit), as depicted in Fig. S1(a). Here, for convenience, illustrations in Figs. 1(b) and 1(c) in the article are reproduced in Figs. S1(b) and S1(a), respectively. The CNTs, respectively, carry dc currents \( I_1 \) and \( I_2 \), both along the \( +\hat{x} \)–direction. A spin is placed between them, at a distance \( d_1 \) (\( d_2 \)) from the first (second) CNT. According to the Biot-Savart law, the CNTs can, at the position of the spin, generate a magnetic field \( \vec{B}(0)_{\text{cnt}} = B(0)_{\text{cnt}} \hat{z} \), where

\[
B(0)_{\text{cnt}} = \sum_{k=1,2} (-1)^{k-1} \frac{\mu_0 I_k}{2\pi d_k},
\]

\( \hat{\epsilon} (\epsilon = x, y, z) \) is a unit vector in the \( \hat{\epsilon} \)–direction, \( \mu_0 \) is the vacuum permeability, and the subscript “cnt” refers to the CNTs. When the CNTs vibrate along the \( \hat{y} \)–direction, the magnetic field is parametrically modulated by their mechanical displacements \( y_1 \) and \( y_2 \), and then is reexpressed, up to first order, as \( \vec{B}_{\text{cnt}} = \vec{B}_{\text{cnt}}^{(0)} + \vec{B}_{\text{cnt}}^{(1)}, \) where \( \vec{B}_{\text{cnt}}^{(1)} = B_{\text{cnt}}^{(1)}\hat{z} \) is a first-order modification, and where \( B_{\text{cnt}}^{(1)} = \sum_{k=1,2} G_k y_k \), with a magnetic-field gradient,

\[
G_k = \frac{\mu_0 I_k}{2\pi d_k^2}.
\]

Note that, here, \( y_1 > 0 \) (\( y_2 < 0 \)) indicates a decrease in \( d_1 \) (\( d_2 \)). Therefore, the sign, \((-1)^{k-1}\), in Eq. (S1) does not appear in Eq. (S2). Furthermore, an external magnetic field, \( \vec{B}_{\text{ext}} = B_x(t) \hat{x} + B_z \hat{z} \), is applied to the NV spin. We have assumed, as required below, that \( B_x(t) \) is a time-dependent component but \( B_z \) is a dc component. The
Hamiltonian governing the NV spin is therefore given by
\[ H_{NV} = \hbar D S_z^2 + \mu_B g_s \left( B^{(0)}_{\text{cnt}} + B_z \right) S_z + \mu_B g_s B_x (t) S_x + \mu_B g_s B^{(1)}_{\text{cnt}} S_z, \]  
(S3)

where \( g_s \approx 2 \) is the Landé factor, \( \mu_B \) the Bohr magneton, \( D \approx 2\pi \times 2.87 \text{ GHz} \) the zero-field splitting, and \( S_z \) the \( \epsilon \)-component of the spin operator \( \hat{S} \) (\( \epsilon = x,y,z \)). In terms of the eigenstates, \( \{ |m_x, m_z = 0, \pm 1 \rangle \} \), of \( S_z \), the operator \( S_x \) is expanded as
\[ S_x = \frac{1}{2} \left( \begin{array}{ccc} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{array} \right), \]  
(S4)

and accordingly, the Hamiltonian \( H_{NV} \) is transformed to
\[ H_{NV} = \{ \hbar D + \mu_B g_s \left( B^{(0)}_{\text{cnt}} + B_z \right) \} | + 1 \rangle \langle + 1 | + \{ \hbar D - \mu_B g_s \left( B^{(0)}_{\text{cnt}} + B_z \right) \} | - 1 \rangle \langle - 1 | \\
+ \frac{1}{\sqrt{2}} \mu_B g_s B_x (t) (| - 1 \rangle \langle 0 | + | + 1 \rangle \langle 0 | + \text{H.c.} \\
+ \mu_B g_s B^{(1)}_{\text{cnt}} (| + 1 \rangle \langle - 1 | - | - 1 \rangle \langle + 1 |). \]  
(S5)

We find that the magnetic field along the \( \hat{z} \)-direction causes different Zeeman shifts to be imposed, respectively, on the spin states \( | \pm 1 \rangle \), and also that the magnetic field along the \( \hat{x} \)-direction drives the transition between the spin states \( |0 \rangle \) and \( | \pm 1 \rangle \).

The quantum treatment of the mechanical motion demonstrates that the mechanical vibrations of the CNTs can be modelled by two single-mode harmonic oscillators with a Hamiltonian
\[ H_{\text{mev}} = \sum_{k=1,2} \hbar \omega_k b^\dagger_k b_k, \]  
(S6)

where \( \omega_k \) is the phonon frequency and \( b_k \) \((b^\dagger_k)\) is the phonon annihilation (creation) operator. Here, we have subtracted the constant zero-point energy \( \hbar \omega_k/2 \). The mechanical displacement \( y_k \) is accordingly expressed as
\[ y_k = y^{(k)}_{\text{zp}} \left( b_k + b^\dagger_k \right) \equiv y^{(k)}_{\text{zp}} q_k, \]  
(S7)

where \( q_k \) is the canonical phonon position operator, and \( y^{(k)}_{\text{zp}} = [\hbar / (2 m_k \omega_k)]^{1/2} \), with \( m_k \) being the effective mass, describes the zero-point (zp) motion. Combining Eqs. (S5), (S6), and (S7) gives the full Hamiltonian of the hybrid system,
\[ H_F = \sum_{k=1,2} \hbar \omega_k b^\dagger_k b_k + \{ \hbar D + \mu_B g_s \left( B^{(0)}_{\text{cnt}} + B_z \right) \} | + 1 \rangle \langle + 1 | \\
+ \{ \hbar D - \mu_B g_s \left( B^{(0)}_{\text{cnt}} + B_z \right) \} | - 1 \rangle \langle - 1 | \\
+ \frac{1}{\sqrt{2}} \mu_B g_s B_x (t) (| - 1 \rangle \langle 0 | + | + 1 \rangle \langle 0 | + \text{H.c.} \\
+ \sum_{k=1,2} \mu_B g_s G_k y^{(k)}_{\text{zp}} (| + 1 \rangle \langle + 1 | - | - 1 \rangle \langle - 1 |) q_k. \]  
(S8)

The last line in Eq. (S8) describes a magnetic coupling between the spin and the mechanical modes. In order to realize a tunable detuning between them, \( B_x (t) \) is chosen to be \( B_x (t) = B_0 \cos (\omega_0 t) \) with amplitude \( B_0 \) and frequency \( \omega_0 \). In a frame rotating at \( H_{\text{rot}} = \hbar \omega_0 (| - 1 \rangle \langle - 1 | + | + 1 \rangle \langle + 1 |), \) the full Hamiltonian can be divided into two parts, \( H_F = H_{\text{low}} + H_{\text{high}}, \) where
\[ H_{\text{low}} = \sum_{k=1,2} \hbar \omega_k b^\dagger_k b_k + \hbar \delta_+ | + 1 \rangle \langle + 1 | + \hbar \delta_- | - 1 \rangle \langle - 1 | \\
+ \hbar \Omega (| - 1 \rangle \langle 0 | + | + 1 \rangle \langle 0 | + \text{H.c.} \\
+ \sum_{k=1,2} \hbar g_k (| + 1 \rangle \langle + 1 | - | - 1 \rangle \langle - 1 |) q_k, \]  
(S9)

\[ H_{\text{high}} = \hbar \Omega [\exp (i 2 \omega_0 t) | - 1 \rangle \langle 0 | + \exp (i 2 \omega_0 t) | + 1 \rangle \langle 0 | + \text{H.c.}], \]  
(S10)
account for the low- and high-frequency components, respectively. Here, we have defined

\[ h\delta_\pm = hD \pm \mu_B g_s \left( B_{\text{nt}}^{(0)} + B_z \right) - \hbar \omega_0, \]

\[ h\Omega = \frac{1}{2\sqrt{2}} \mu_B g_s B_0, \]

\[ \hbar g_k = \mu_B g_s G_k y_{\text{sp}}^{(k)}. \] (S11)

Roughly, having \( \delta'_\pm = \delta_\pm + 2\omega_0 \gg \Omega \) allows one to make the rotating-wave approximation (RWA), and to straightforwardly remove \( H_{\text{high}} \). However, as demonstrated in Sec. S5, the accumulated error increases during the evolution, causing the dynamics driven by \( H_{\text{low}} \) to deviate largely from that driven by \( H_F \). Thus, we are not using the RWA here. In order to suppress the error accumulation, we need to analyze the effects of \( H_{\text{high}} \) in the limit \( \delta'_\pm \gg \Omega \). In such a limit, we can employ a time-averaging treatment for the high-frequency component \( H_{\text{high}} \) [S1–S3], and as a result, its effective behavior is described by the following time-averaged Hamiltonian,

\[
\begin{align*}
\mathcal{H}_{\text{high}} &= \hbar \left( \frac{2\Omega^2}{\delta_-} + \frac{\Omega^2}{\delta_+} \right) | -1 \rangle \langle -1 | + \hbar \left( \frac{\Omega^2}{\delta_-} + \frac{2\Omega^2}{\delta'_+} \right) | 1 \rangle \langle 1 | + \hbar \left( \frac{\Omega^2}{\delta_-} + \frac{1}{\delta'_+} \right) \left\{ \exp \left[ i \left( \delta_- + \delta'_- - \delta_+ + \delta'_+ \right) t \right] - 1 \right\} + \hbar \Omega + \hbar \Omega \langle 0 | [S1–S3], \text{ and as a result, its effective behavior is described by the following time-averaged Hamiltonian,}
\end{align*}
\]

\[
H_F \approx H_{\text{low}} + \mathcal{H}_{\text{high}}. \] (S13)

As seen in Sec. S5, the error accumulation is strongly suppressed when \( \mathcal{H}_{\text{high}} \) is included.

Tuning \( B_{\text{nt}}^{(0)} + B_z = 0 \) yields \( \delta_\pm = \delta_- = \Delta_- \) and \( \delta'_\pm = \delta'_- = \Delta_+ \), implying that the spin states \( | \pm 1 \rangle \) have the same Zeeman shift of \( \Delta = \Delta_- + 3\Omega^2/\Delta_+ \), as shown in Fig. S1(b). Therefore, we can define a bright state, \( | B \rangle = (| + 1 \rangle + | - 1 \rangle)/\sqrt{2} \), which is dressed by the spin state |0⟩, and a dark state, \( | D \rangle = (| + 1 \rangle - | - 1 \rangle)/\sqrt{2} \), which decouples from the spin state |0⟩. In terms of the states |B⟩ and |D⟩, the full Hamiltonian becomes

\[
H_F \approx \sum_{k=1,2} \hbar \omega_k b_k^\dagger b_k + \hbar \Delta (|B\rangle \langle B| + |D\rangle \langle D|) + \hbar \sqrt{2\Omega} (|0\rangle \langle B|) + |B\rangle \langle B|) \right) . \] (S14)

The dressing mechanism allows us to introduce two dressed states,

\[ |\Phi_-\rangle = \cos (\theta) |0\rangle - \sin (\theta) |B\rangle, \] (S15)

\[ |\Phi_+\rangle = \sin (\theta) |0\rangle + \cos (\theta) |B\rangle, \] (S16)

where \( \tan (\theta) = 2\sqrt{2}\Omega/\Delta \). Upon substituting them back into the full Hamiltonian in Eq. (S14) and then using the identity operator \( \mathcal{I} = |D\rangle \langle D| + |\Phi_-\rangle \langle \Phi_-| + |\Phi_+\rangle \langle \Phi_+| \), we can straightforwardly obtain

\[
H_F \approx \sum_{k=1,2} \hbar \omega_k b_k^\dagger b_k + \hbar \omega_+ |\Phi_+\rangle \langle \Phi_+| + \hbar \omega_D |D\rangle \langle D| \\
+ \sum_{k=1,2} \hbar \left[ g_k (-)|\Phi_-\rangle \langle D| + g_k (+)|D\rangle \langle \Phi_+| + \text{H.c.} \right] q_k \\
+ \hbar \Omega^2 \frac{\Delta_+}{\Delta_+} \left[ \cos (2\theta) |\Phi_+\rangle \langle \Phi_+| - \frac{1}{2} \sin (2\theta) (|\Phi_+\rangle \langle \Phi_+| + \text{H.c.}) - \cos^2 (\theta) |D\rangle \langle D| \right] . \] (S17)

Here,

\[ \omega_+ = \sqrt{\Delta^2 + 8\Omega^2}, \] (S18)

\[ \omega_D = \frac{1}{2} \left( \Delta + \sqrt{\Delta^2 + 8\Omega^2} \right), \] (S19)

\[ g_k (-) = - g_k \sin (\theta), \] (S20)

\[ g_k (+) = g_k \cos (\theta). \] (S21)
Under the assumption of $\Delta \gg \Omega$, we have $\theta \simeq 0$, such that $\sin (\theta) \simeq \sin (2\theta) \simeq 0$, $\cos (\theta) \simeq \cos^2 (\theta) \simeq \cos (2\theta) \simeq 1$, $\omega_+ \simeq \Delta + 4\Omega^2 / \Delta$, $\omega_D \simeq \Delta + 2\Omega^2 / \Delta$, and $|\Phi_+\rangle \simeq |B\rangle$. In this limit, the coupling between $|0\rangle$ and $|B\rangle$ only causes an energy splitting, of $\simeq 2\Omega^2 / \Delta$, between the states $|B\rangle$ and $|D\rangle$, so $|B\rangle$ and $|D\rangle$ can be used to define a spin qubit. Correspondingly, the full Hamiltonian is approximated as

$$H'_F = \sum_{k=1,2} \hbar \omega_k b_k^\dagger b_k + \frac{1}{2} \hbar \omega_q \sigma_z + \sum_{k=1,2} \hbar g_k \sigma_x g_k,$$  \hspace{1cm} (S22)

where $\omega_q = 2\Omega^2 / \Delta + 2\Omega^2 / \Delta_+$, $\sigma_z = |B\rangle \langle B| - |D\rangle \langle D|$, and $\sigma_x = \sigma_+ + \sigma_-$ with $\sigma_- = |D\rangle \langle B|$ and $\sigma_+ = \sigma_-^\dagger$. Modest parameters [S4–S9], $m_k = 1.0 \times 10^{-22}$ kg, $\omega_k/2\pi = 2$ MHz, $d_k \simeq 2$ nm, and $I_k \simeq 380$ nA, could result in a spin-CNT coupling of up to $g_k/2\pi \simeq 100$ kHz.

Furthermore, from Eq. (S22) it is found that the sequential actions of the terms $\sigma_+ b_1$ and $\sigma_- b_2^\dagger$, as well as of the counter-rotating terms $\sigma_- b_1$ and $\sigma_+ b_2^\dagger$, can transfer a mechanical phonon from the left to the right CNT, and the reverse process is caused by their Hermitian conjugates. When restricting our discussion to a dispersive regime, $\omega_q \pm \omega_k \gg |g_k|$, this phonon transfer becomes dominant. Hence, in the dispersive regime the dynamics described by $H'_F$ in Eq. (S22) enables a spin quantum bus for the mechanical phonons and can be used to realize a coherent CNT-CNT coupling.

In order to show more explicitly, we rewrite $H'_F$ in the interaction picture as

$$H'_F = \sum_{k=1,2} \hbar g_k \left\{ \sigma_+ b_k \exp \left[ i (\omega_q - \omega_k) t \right] + \sigma_- b_k^\dagger \exp \left[ i (\omega_q + \omega_k) t \right] + \text{H.c.} \right\}.$$  \hspace{1cm} (S24)

The condition in Eq. (S23) justifies to use a time-averaging treatment of the Hamiltonian $H'_F$ [S1, S2]. In the time-averaging treatment, all terms in Eq. (S24) are considered as high-frequency components and exhibit time-averaged behaviors. Based on this, the dynamics of the system can be determined by an effective Hamiltonian

$$H_{\text{eff}} = \frac{2\hbar \omega_q}{\omega_q^2 - \omega_m^2} \left[ \sum_{k=1,2} g_{k}^2 b_k^\dagger b_k + g_1 g_2 \left( b_1^\dagger b_2 + b_2^\dagger b_1 \right) \right] \otimes \sigma_z.$$  \hspace{1cm} (S25)

Here, we have assumed that $\omega_k = \omega_m$. As expected, Eq. (S25) shows a coherent spin-mediated CNT-CNT coupling, corresponding to the standard linear coupler transformation, which can give rise to a direct phonon exchange. Thus in this case, the spin qubit works as a quantum bus. At the same time, it also shows that the CNT-CNT coupling can be turned off if the intermediate spin is in the state $|0\rangle$. This is because the IV spin in the state $|0\rangle$ is decoupled from the CNTs, and the mechanical phonons can no longer be transferred from one CNT to another. Specifically, if the spin is in the state $|D\rangle$ or $|B\rangle$, the CNTs are coupled; however, if the spin is instead in the state $|0\rangle$, they are decoupled. Note that in Eq. (S25) ac Stark shifts caused to be imposed on the qubit have been excluded because we focus only on the quantum states of the CNTs.

In the last part of this section, we evaluate the direct coupling between the CNTs. For simplicity, we assume that $I_k = I$, $d_k = d$, and that the CNTs have the same length $L$. The attractive force acting on the $k$th CNT is

$$\vec{F}_k = (-1)^{k-1} F \hat{y},$$  \hspace{1cm} (S26)

where

$$F = \frac{\mu_0 LI^2}{2\pi (d - y_1 + y_2)}$$  \hspace{1cm} (S27)

is the force size. The work done by the force is given straightforwardly by

$$W = \frac{\mu_0 LI^2 (y_1 - y_2)}{2\pi (d - y_1 - y_2)}.$$  \hspace{1cm} (S28)

After applying a perturbation expansion and then a quantization, this direct CNT-CNT coupling is found to be

$$W = \hbar W^{(1)} (b_1 - b_2 + \text{H.c.}) + \hbar W^{(2)} \left[ (b_1 + b_1^\dagger)^2 + (b_2 + b_2^\dagger)^2 - 2 \left( b_1 + b_1^\dagger \right) \left( b_2 + b_2^\dagger \right) \right],$$  \hspace{1cm} (S29)
where
\[
W^{(1)} = \frac{\mu_0 LI^2 y_{xp}}{2\pi d\hbar}, \quad (S30)
\]
\[
W^{(2)} = \frac{\mu_0 LI^2 y_{xp}}{2\pi d^2\hbar}. \quad (S31)
\]
For a modest setup [S4–S9], \(m = 1.0 \times 10^{-22} \text{ kg}, \omega_m = 2\pi \times 2 \text{ MHz}, L = 10 \text{ nm}, d = 2 \text{ nm}, \text{ and } I = 380 \text{ nA}, \) we have
\[
W^{(1)} \simeq 2\pi \times 20 \text{ kHz}, \quad (S32)
\]
which is much smaller than the mechanical resonance frequency \(\omega_m, \) and also have
\[
W^{(2)} \simeq 2\pi \times 1 \text{ kHz}, \quad (S33)
\]
which is much smaller than the spin-mediated CNT-CNT coupling, for example, \(\simeq 2\pi \times 12 \text{ kHz, as shown in the section below. Therefore, the direct CNT-CNT coupling can be neglected in our setup.}

**S2. Controlled Hadamard gate, phase gate, and mechanical quantum delayed-choice experiment**

In order to implement a quantum delayed-choice experiment with macroscopic CNT mechanical resonators, we need a controlled Hadamard gate and a phase gate to act on the CNT mechanical modes. Below, we demonstrate how the effective Hamiltonian in Eq. (S25) can be used to make all required gates. Let us first consider the controlled Hadamard gate. Tuning the currents to be \(I_k = I \) and, at the same time, the distances to be \(d_k = d \) results in a symmetric coupling \(g_k = g. \) The effective Hamiltonian \(H_{\text{eff}} \) is accordingly reduced to
\[
H_{\text{eff}} = H_{\text{cnt}} \otimes \sigma_z, \quad (S34)
\]
is a beam-splitter-type interaction, and where
\[
J = \frac{2g^2\omega_q}{\omega_q^2 - \omega_m^2} \quad (S35)
\]
is an effective CNT-CNT coupling strength. In our discussion, the NV spin is restricted to a subspace spanned by \(\{|0\rangle, |D\rangle\} \), where the spin is a control qubit of a Hadamard gate. The spin in the state \(|D\rangle\) mediates the coherent coupling between the separated CNTs, and causes them to evolve under the Hamiltonian \(H_{\text{cnt}} \) in Eq. (S34). According to the Heisenberg equation of motion, \(b_k (t) = \exp (iH_{\text{cnt}} t/\hbar) b_k \exp (-iH_{\text{cnt}} t/\hbar), \) the unitary evolution for a time \(t = \tau_0 \equiv \pi/(4J) \) corresponds to a Hadamard-like gate,
\[
b_1 (\tau_0) = \frac{1}{\sqrt{2}} (b_1 - ib_2), \quad (S36)
\]
\[
b_2 (\tau_0) = \frac{1}{\sqrt{2}} (b_2 - ib_1). \quad (S37)
\]
However, when the spin state is \(|0\rangle\), the two CNTs decouple from each other. In this case, their quantum states remain unchanged under the unitary evolution, yielding
\[
b_1 (t) = b_1, \quad (S38)
\]
\[
b_2 (t) = b_2. \quad (S39)
\]
We have therefore achieved a spin-controlled Hadamard gate between the CNTs. That is, if the NV spin is in the state \(|D\rangle\), then the Hadamard operation is applied to the CNTs, and if the NV spin is in the state \(|0\rangle\), then the states of the CNTs are unchanged.

We next consider the phase gate. For the phase gate, we tune the currents to be \(I_1 \neq 0 \) and \(I_2 = 0, \) such that \(g_1 = g \) and \(g_2 = 0, \) causing the effective Hamiltonian in Eq. (S25) to become
\[
H_{\text{cnt}} = \hbar b_1^1 b_1 \sigma_z. \quad (S40)
\]
We find from Eq. (S40) that there exists a spin-induced shift, $J$, of the mechanical resonance. This dispersive shift can, in turn, introduce a dynamical phase, $\phi(t) = Jt$, onto the first CNT. With the spin being in the state $|D\rangle$, we solve the Heisenberg equations of motion for the CNTs, and then obtain a phase gate,

\begin{align}
    b_1(t) &= \exp[i\phi(t)] b_1, \\
    b_2(t) &= b_2.
\end{align}

(S41) (S42)

In fact, similar to the controlled Hadamard gate discussed above, the phase gate can also be controlled by the spin according to Eq. (S40).

Having achieved all required gates, we now turn to the detailed description of the macroscopic quantum delayed-choice experiment with CNT resonators. The hybrid system is initially prepared in the state $|\Psi\rangle_i \equiv |\Psi(0)\rangle = \left(b_1^\dagger \otimes I_2 \right) |\text{vac}\rangle \otimes |D\rangle$, where $|\text{vac}\rangle$ refers to the phonon vacuum of the CNTs and $I_2$ is the identity operator on the second CNT. First, we turn on the currents of the CNTs and ensure $I_k = I$. After a time $\tau_0$, a Hadamard operation is applied to the CNTs and accordingly, $|\Psi\rangle_i$ becomes

$$|\Psi(\tau_0)\rangle = \frac{1}{\sqrt{2}} \left(b_1^\dagger + i b_2^\dagger\right) |\text{vac}\rangle |D\rangle.$$  

(S43)

Then, we turn off the current of the second CNT for a phase accumulation for a time $\tau_1$. As a consequence, the system further evolves to

$$|\Psi(\tau_0 + \tau_1)\rangle = \frac{1}{\sqrt{2}} \left[\exp(i\phi) b_1^\dagger + i b_2^\dagger\right] |\text{vac}\rangle |D\rangle.$$ 

(S44)

While achieving the desired phase $\phi$, we make a spin single-qubit rotation $|D\rangle \rightarrow \cos(\varphi) |0\rangle + \sin(\varphi) |D\rangle$, and have

$$|\Psi(\tau_0 + \tau_1)\rangle = \frac{1}{\sqrt{2}} \left[\exp(i\phi) b_1^\dagger + i b_2^\dagger\right] |\text{vac}\rangle \left(\cos\varphi |0\rangle + \sin\varphi |D\rangle\right).$$ 

(S45)

Here, note that, we have ignored the length of the driving pulse of the spin rotation as being of the order of ns, and thus assumed that the state of the CNTs remains unchanged. At the end of the driving pulse, we turn on the current of the second CNT again and hold for another $\tau_0$ to perform a Hadamard gate. This gate is in a quantum superposition of being present and absent. The three operations on the mechanical phonon correspond to the actions, on a single photon, of the input beam splitter, the phase shifter, and the output beam splitter, respectively, in quantum delayed-choice experiments with a Mach-Zehnder interferometer. The final state is therefore given by

$$|\Psi\rangle_f \equiv |\Psi(2\tau_0 + \tau_1)\rangle = \cos(\varphi) |\text{particle}\rangle |0\rangle + \sin(\varphi) |\text{wave}\rangle |D\rangle,$$

(S46)

where

FIG. S2. (Color online) (a) Probability $P_1$ and (b) $P_2$ as a function of the rotation angle $\varphi$ and the relative phase $\phi$. This represents a continuous transition between a particle-type behavior ($\varphi = 0$) and a wave-type behavior ($\varphi = \pi/2$).
\begin{align}
|\text{particle}\rangle &= \frac{1}{\sqrt{2}} \left[ \exp(i\phi) b_1^\dagger + i b_2^\dagger \right] |\text{vac}\rangle, \quad (S47) \\
|\text{wave}\rangle &= \frac{1}{2} \left\{ [\exp(i\phi) - 1] b_1^\dagger + i [\exp(i\phi) + 1] b_2^\dagger \right\} |\text{vac}\rangle, \quad (S48)
\end{align}

describe particle and wave behaviors, respectively. This reveals that the CNT mechanical phonon is in a quantum superposition of both a particle and a wave. The probability of finding a single phonon in the $k$th CNT is expressed as

$$P_k = \frac{1}{2} + (-1)^k \frac{1}{2} \sin^2(\phi) \cos(\phi), \quad (S49)$$

according to Eq. (S46). In Fig. S2, we have plotted the probabilities $P_1$ and $P_2$ versus the rotation angle $\phi$ and the relative phase $\phi$. In this figure we find that the mechanical phonon shows a morphing behavior between particle ($\phi = 0$) and wave ($\phi = \pi/2$).

Note that the spin, in a classical mixed state of the form $\cos^2(\phi) |0\rangle \langle 0| + \sin^2(\phi) |D\rangle \langle D|$, would lead to the same measured statistics in Eq. (S49), that is, a local hidden variable model is capable of reproducing the quantum predictions. This is a loophole [S10–S13]. However, as discussed in Refs. [S14–S17], this loophole can be avoided as long as the second Hadamard operation is ensured to be in a truly quantum superposition of being present and absent. In our proposal, the second Hadamard operation is conditioned on the spin state. If the spin is in the $|0\rangle$ state, then the Hadamard operation is absent; if the spin is in the $|D\rangle$ state, then the Hadamard operation is present; if the spin is in a quantum superposition of the $|0\rangle$ and $|D\rangle$ states, then the Hadamard operation is in a quantum superposition of being present and absent. To confirm such a quantum superposition, in Fig. S3 we numerically calculate the fidelity, $F = \langle \Psi | \rho_{\text{actual}}(\tau_T) | \Psi \rangle$, between the desired state $|\Psi\rangle$ in Eq. (S46) and the actual state $\rho_{\text{actual}}(\tau_T)$ obtained from the exact master equation in Eq. (S120). From this figure, we find that the fidelity is very close to unity even for the finite temperature of $T \simeq 10$ mK. Furthermore, in experiments, in order to exclude the classical interpretation and prove the existence of the coherent wave-particle superposition, the quantum coherence between the states $|0\rangle$ and $|D\rangle$ should be verified. Experimentally, this coherence can be prepared by a spin single-qubit operation [S18–S20], and can be verified by performing quantum state tomography to show all the elements of the density matrix of the spin [S20].

**FIG. S3.** (Color online) (a) Fidelity $F$ as a function of the rotation angle $\phi$. All the results are numerically obtained by integrating the exact master equation in Eq. (S120). Here, in addition to $\gamma_s/2\pi = 200\gamma_m/2\pi = 80$ Hz, we have assumed that $g/2\pi = 100$ kHz, $\omega_m/2\pi = 2$ MHz, $\Omega = 10\omega_m$, and $\Delta = 142\omega_m$, resulting in $\omega_q \simeq 1.5\omega_m$ and then $J/2\pi \simeq 12$ kHz. We have also assumed that $n_{th} = 100$, which corresponds to an environmental temperature of $\simeq 10$ mK.
S3. Phonon occupation at finite temperatures

We begin by considering the total operation time, which is given by \( \tau_T = 2 \tau_0 + \tau_1 \), as discussed in Sec. S2. Here, \( \tau_0 = \pi / (4J) \) is the time for the Hadamard gate and \( \tau_1 \in [0, 2\pi / J] \) is the time for the phase gate. In a realistic setup, we can assume \( \omega_m / 2\pi \approx 2 \text{ MHz}, \omega_q / 2\pi \approx 3 \text{ MHz}, \) and \( g / 2\pi = 100 \text{ kHz} \), such that \( J / 2\pi \approx 12 \text{ kHz} \), yielding a maximum total time \( \tau_T^{\text{max}} = 2 \tau_0 + \tau_1^{\text{max}} \approx 0.1 \text{ ms} \), where \( \tau_1^{\text{max}} = 2\pi / J \) is the maximum phase gate time. Note that, the operation time \( \tau_T \) depends inversely on the CNT-CNT coupling strength \( J \), but the enhancement in \( J \) is limited by the validity of the effective Hamiltonian \( H_{\text{eff}} \).

The total decoherence in our setup can be divided into two parts, one from the spin and the other from the CNTs. The spin decoherence in general includes the relaxation and the dephasing. For an NV electronic spin, the relaxation can be governed by the following master equation

\[
\dot{\rho}(t) = i [\rho(t), H(t)] - \frac{\gamma_m}{2} n_{\text{th}} \sum_{k=1,2} \mathcal{L}(b_k^\dagger) \rho(t) - \frac{\gamma_m}{2} (n_{\text{th}} + 1) \sum_{k=1,2} \mathcal{L}(b_k) \rho(t),
\]

where \( \rho \) is the density operator of the system, \( \gamma_m \) is the mechanical decay rate, \( n_{\text{th}} = [\exp(h \omega_m / k_B T) - 1]^{-1} \) is the equilibrium phonon occupation at temperature \( T \), and \( \mathcal{L}(\sigma) \rho(t) = \sigma^\dagger \rho(t) \sigma - 2 \sigma \rho(t) \) is the Lindblad superoperator. Here, \( H(t) \) is a binary Hamiltonian of the form,

\[
H(t) = \begin{cases} 
H_0, & 0 < t \leq \tau_0, \text{ and } \tau_0 + \tau_1 < t \leq \tau_T \\
H_1, & \tau_0 < t \leq \tau_0 + \tau_1,
\end{cases}
\]

with

\[
H_0 = \hbar J \left( \sum_{k=1,2} b_k^\dagger b_k + b_1 b_2^\dagger + b_2 b_1^\dagger \right) \sigma_z,
\]

\[
H_1 = \hbar J b_1^\dagger b_1 \sigma_z.
\]

The three time intervals in Eq. (S51) correspond to the first Hadamard gate, the phase gate and the second Hadamard gate, respectively. Note that in Eq. (S51), we did not include the spin single-qubit rotation before the third interval because the length of the driving pulse is of the order of \( \hbar \). We can derive the system evolution step by step.

Let us now consider the first evolution interval \( 0 < t \leq \tau_0 \). During this interval, the coupling of the CNT mechanical modes introduces two delocalized phononic modes,

\[
c_{\pm} = \frac{1}{\sqrt{2}} (b_1 \pm b_2),
\]

such that \( H_0 \) is diagonalized to be

\[
H_0 = 2 \hbar J c_{\pm}^\dagger c_{\pm} \sigma_z,
\]

and the master equation in Eq. (S50) is reexpressed, in terms of the modes \( c_{\pm} \), as

\[
\dot{\rho} = i \left[ \rho, 2 \hbar J c_{\pm}^\dagger c_{\pm} \sigma_z \right] - \frac{\gamma_m}{2} n_{\text{th}} \sum_{\mu=1,2} \mathcal{L}(c_{\mu}^\dagger) \rho(t) - \frac{\gamma_m}{2} (n_{\text{th}} + 1) \sum_{\mu=1,2} \mathcal{L}(c_{\mu}) \rho(t).
\]

In order to calculate the phonon occupations at the end of the first interval, we need to obtain the equations of motion for \( \langle c_{\pm}^\dagger c_{\pm} \rangle, \langle c_{\pm}^\dagger c_{-} \rangle, \langle c_{\pm}^\dagger c_{-} \sigma_z \rangle, \) and \( \langle c_{\pm}^\dagger c_{-} \sigma_z^2 \rangle \). Here, \( \langle O \rangle \) represents the expectation value of the operator \( O \). Following
the master equation in Eq. (S56), we have

\[
\frac{d}{dt} \langle c_\pm \rangle = -\gamma_m \langle c_\pm \rangle + \gamma_m n_{\text{th}},
\]

\[
\frac{d}{dt} \langle c_\pm \rangle = i2J \langle c_\pm \sigma_z \rangle - \gamma_m \langle c_\pm \rangle,
\]

\[
\frac{d}{dt} \langle c_\pm \rangle = i2J \langle c_\pm \sigma_z^2 \rangle - \gamma_m \langle c_\pm \rangle,
\]

\[
\frac{d}{dt} \langle c_\pm \rangle = i2J \langle c_\pm \sigma_z^2 \rangle - \gamma_m \langle c_\pm \rangle,
\]

where we have used the relation \(\sigma_z^2 = \sigma_z\). We can straightforwardly solve the differential equation (S57) to find

\[
\langle c_\pm \rangle (t) = \left( \frac{1}{2} - n_{\text{th}} \right) \exp (-\gamma_m t) + n_{\text{th}}.
\]

Combining Eqs. (S59) and (S60) gives

\[
\langle c_\pm \rangle (t) = (-1)^j \frac{1}{2} \exp (-i2Jt) \exp (-\gamma_m t),
\]

for \(j = 1, 2\). Upon substituting Eq. (S62) back into Eq. (S58), we can then obtain

\[
\langle c_\pm \rangle (t) = \frac{1}{2} \exp (-i2Jt) \exp (-\gamma_m t).
\]

It is found, according to Eq. (S54), that in the localized-mode basis,

\[
\langle b_k \rangle (\tau_0) = \left( \frac{1}{2} - n_{\text{th}} \right) \exp (-\gamma_m \tau_0) + n_{\text{th}},
\]

\[
\langle b_k \rangle (\tau_0) = \frac{i}{2} \exp (-\gamma_m \tau_0).
\]

For the second evolution interval \(\tau_0 < t \leq \tau_0 + \tau_1\), we directly use the master equation in Eq. (S50) but with \(H(t)\) replaced by \(H_1\). When comparing with the master equation in Eq. (S56), we see that the equations of motion for \(\langle b_k \rangle\), \(\langle b_k \rangle\), \(\langle b_k \rangle\), and \(\langle b_k \rangle\) should have the same forms as in Eqs. (S57), (S58), (S59), and (S60), but with the substitutions \(c_+ \rightarrow b_1\), \(c_- \rightarrow b_2\) and \(2J \rightarrow J\). In combination with the initial conditions, given in Eqs. (S64) and (S65), we follow the same procedure as above to find

\[
\langle b_k \rangle (\tau_0 + \tau_1) = \left( \frac{1}{2} - n_{\text{th}} \right) \exp [-\gamma_m (\tau_0 + \tau_1)] + n_{\text{th}},
\]

\[
\langle b_k \rangle (\tau_0 + \tau_1) = \frac{i}{2} \exp (-iJ\tau_1) \exp [-\gamma_m (\tau_0 + \tau_1)].
\]

We now turn to the third evolution interval \(\tau_0 + \tau_1 < t \leq 2\tau_0 + \tau_1\). Before this interval or at the end of the second interval, we apply a single qubit rotation, \(|D \rangle \rightarrow \cos (\varphi) |0\rangle + \sin (\varphi) |D\rangle\), on the NV spin to engineer the subsequent Hadamard operation to be in a quantum superposition of being absent and present. In this situation, we still use the delocalized-mode basis and the corresponding master equation in Eq. (S56). According to Eqs. (S66) and (S67), the initial conditions of the last evolution can be rewritten, in terms of \(c_\pm\), as

\[
\langle c_\pm \rangle (\tau_0 + \tau_1) = \left[ \frac{1}{2} \pm \frac{1}{2} \sin (J\tau_1) - n_{\text{th}} \right] \exp [-\gamma_m (\tau_0 + \tau_1)] + n_{\text{th}},
\]

\[
\langle c_\pm \rangle (\tau_0 + \tau_1) = -\frac{i}{2} \cos (J\tau_1) \exp [-\gamma_m (\tau_0 + \tau_1)],
\]

\[
\langle c_\pm \rangle (\tau_0 + \tau_1) = (1)^j \sin^2 (\varphi) \langle c_\pm \rangle (\tau_0 + \tau_1),
\]

for \(j = 1, 2\). Then, as before, solving the differential equations in Eqs. (S57), (S58), (S59) and (S60) leads to

\[
\langle c_\pm \rangle (t) = \left[ \frac{1}{2} \pm \frac{1}{2} \sin (J\tau_1) - n_{\text{th}} \right] \exp (-\gamma_m t) + n_{\text{th}},
\]

\[
\langle c_\pm \rangle (t) = -\frac{i}{2} \cos (J\tau_1) \left\{ \cos^2 (\varphi) + \sin^2 (\varphi) \exp [-i2J (t - \tau_0 - \tau_1)] \right\} \exp (-\gamma_m t),
\]
which, in turn, gives
\[ n_k \equiv \langle b_k^\dagger b_k \rangle (\tau_T) = (P_k - n_{th}) \exp(-\gamma_m \tau_T) + n_{th}, \]
(S73)
which is the phonon occupation of the \( k \)th at the end of the third interval. For a realistic CNT, the mechanical linewidth can be set to \( \gamma_m / 2\pi = 0.4 \) Hz [S23], and then we obtain a phonon lifetime of \( \simeq 400 \) ms, which is much longer than the maximum total time \( \tau_T^{\max} \simeq 0.1 \) ms. This ensures \( \gamma_m \tau_T \ll 1 \), which results in
\[ n_k \simeq P_k + n_{th} \gamma_m \tau_T. \]
(S74)
This shows that the occupation for each CNT has two contributions: one from a coherent phonon signal and one from thermal excitations. Furthermore, we find from Eq. (S74) that the thermal excitations have equal contributions to \( n_1 \) and \( n_2 \). This is because the thermal excitations do not contribute to the interference. For an environmental temperature \( T = 10 \) mK, the equilibrium phonon occupation is \( n_{th} \simeq 100 \), yielding \( n_{th} \gamma_m \tau_T^{\max} \simeq 0.03 \), which can be neglected, as shown in Fig. 2 of the article.

**S4. Signal-to-noise ratio at finite temperatures**

In addition to the thermal occupation, \( n_{th} \gamma_m \tau_T \), in Eq. (S74), the desired signal \( P_k \) is also always accompanied by fluctuation noise. Such a noise includes vacuum fluctuations and thermal fluctuations. In particular, the latter increases with temperature, so that the signal can be completely drowned in the noise when the temperature is sufficiently high. In this case, it is very difficult to observe the morphing between wave and particle. Thus in this section, we analyze this fluctuation noise in detail, and demonstrate that, in order for the morphing behavior to be observed effectively, the total fluctuation noise of both CNTs should be limited by an upper bound, which leads to a critical temperature \( T_c \).

Specifically, we begin by deriving the fluctuation \( \delta n_k \) in the occupation \( n_k \), for \( k = 1, 2 \). This is defined by
\[ (\delta n_k)^2 = \langle (b_k^\dagger b_k)^2 \rangle \tau_T - \langle b_k^\dagger b_k \rangle^2 \tau_T = \langle b_k^\dagger b_k b_k^\dagger b_k \rangle \tau_T + n_k - n_k^2. \]
(S75)
In order to understand the fluctuation noise better, we need to derive an analytical expression of \( \delta n_k \). In Sec. S3, \( n_k \) has been given in Eq. (S74). Below, we derive the evolution of \( \langle b_k^\dagger b_k b_k^\dagger b_k \rangle \) in a step-by-step manner as in Sec. S3.

We now consider the first evolution interval \( 0 < \tau \leq \tau_0 \). During this interval, the delocalized modes \( c_{\pm} \) in Eq. (S54) are employed owing to the coupling of the CNT mechanical modes, and the dynamics is described by the master equation in Eq. (S56). To achieve \( \langle b_k^\dagger b_k b_k^\dagger b_k \rangle \) at time \( \tau_T \), the dynamical evolutions of \( \langle c_+^\dagger c_+ c_+ c_\pm \rangle \), \( \langle c_+^\dagger c_- c_- c_\pm \rangle \), \( \langle c_+^\dagger c_+ c_+ c_- \rangle \), \( \langle c_+^\dagger c_+ c_- c_- \rangle \), and \( \langle c_+^\dagger c_- c_- c_- \rangle \) are involved. The equations of motion for \( \langle c_+^\dagger c_\pm c_\pm c_\pm \rangle \) and \( \langle c_+^\dagger c_\pm c_\pm c_- \rangle \) are
\[ \frac{d}{dt} \langle c_+^\dagger c_\pm c_\pm c_\pm \rangle = 4\gamma_m n_{th} \langle c_+^\dagger c_\pm c_\pm c_\pm \rangle - 2\gamma_m \langle c_+^\dagger c_+ c_\pm c_\pm \rangle, \]
(S76)
\[ \frac{d}{dt} \langle c_+^\dagger c_+ c_\pm c_- \rangle = \gamma_m n_{th} \left( \langle c_+^\dagger c_+ c_\pm c_- \rangle - 2\gamma_m \langle c_+^\dagger c_- c_- \rangle \right). \]
(S77)
Substituting Eq. (S61) yields
\[ \langle c_+^\dagger c_\pm c_\pm c_\pm \rangle (\tau_0) = 2X (\tau_0), \]
(S78)
\[ \langle c_+^\dagger c_+ c_\pm c_- \rangle (\tau_0) = X (\tau_0), \]
(S79)
where
\[ X (t) = n_{th} (n_{th} - 1) \exp(-\gamma_m t) + n_{th} (1 - 2n_{th}) \exp(-\gamma_m t) + n_{th}^2. \]
(S80)
The equations of motion for \( \langle c_+^\dagger c_+ c_\pm c_- \rangle \) are found to be
\[ \frac{d}{dt} \langle c_+^\dagger c_+ c_\pm c_- \rangle = i2J \langle c_+^\dagger c_+ c_\pm c_- \rangle - 2\gamma_m n_{th} \langle c_+^\dagger c_+ c_\pm c_- \rangle - 2\gamma_m \langle c_+^\dagger c_\pm c_\pm c_- \rangle, \]
(S81)
\[ \frac{d}{dt} \langle c_-^\dagger c_+ c_\pm c_- \rangle = i2J \langle c_-^\dagger c_+ c_\pm c_- \rangle + 2\gamma_m n_{th} \langle c_-^\dagger c_+ c_\pm c_- \rangle - 2\gamma_m \langle c_-^\dagger c_\pm c_\pm c_- \rangle, \]
(S82)
\[ \frac{d}{dt} \langle c_+^\dagger c_+ c_- c_- \rangle = i2J \langle c_+^\dagger c_+ c_- c_- \rangle + 2\gamma_m n_{th} \langle c_+^\dagger c_- c_- c_- \rangle - 2\gamma_m \langle c_+^\dagger c_- c_- c_- \rangle, \]
(S83)
\[ \frac{d}{dt} \langle c_-^\dagger c_- c_- c_- \rangle = i2J \langle c_-^\dagger c_- c_- c_- \rangle + 2\gamma_m n_{th} \langle c_-^\dagger c_- c_- c_- \rangle - 2\gamma_m \langle c_-^\dagger c_- c_- c_- \rangle. \]
Together with Eq. (S62), solving straightforwardly the coupled differential equations (S82) and (S83) results in
\[ \langle c_+^\dagger c_+ c_+ \sigma_z \rangle (t) = -n_{th} \left[ 1 - \exp(-\gamma_m t) \right] \exp(-i2Jt) \exp(-\gamma_m t), \] (S84)
which, in turn, gives
\[ \langle c_+^\dagger c_+ c_+ \rangle (\tau_0) = -iY(\tau_0), \] (S85)
where
\[ Y(t) = n_{th} \left[ 1 - \exp(-\gamma_m t) \right] \exp(-\gamma_m t). \] (S86)
In a treatment similar to that used for \( \langle c_+^\dagger c_+^\dagger c_- \rangle \), we obtain
\[ \langle c_+^\dagger c_+^\dagger c_- \rangle (\tau_0) = -iY(\tau_0), \] (S87)
\[ \langle c_+^\dagger c_+^\dagger c_+ \rangle (\tau_0) = 0. \] (S88)
Upon combining Eqs. (S78), (S79), (S85), (S87), and (S88), this yields, after inversion back to the localized-mode basis,
\[ \langle b_+^\dagger b_+^\dagger b_+ b_k \rangle (\tau_0 + \tau_1) = 2X(\tau_0 + \tau_1), \] (S89)
\[ \langle b_+^\dagger b_+ b_2 \rangle (\tau_0 + \tau_1) = X(\tau_0 + \tau_1), \] (S90)
\[ \langle b_+^\dagger b_+ b_2 \rangle (\tau_0 + \tau_1) = \langle b_+^\dagger b_+^\dagger b_2 b_2 \rangle (\tau_0 + \tau_1) = i \exp(-iJ\tau_1) Y(\tau_0 + \tau_1), \] (S91)
\[ \langle b_+^\dagger b_+^\dagger b_2 b_2 \rangle (\tau_0 + \tau_1) = 0. \] (S92)
During the second evolution interval \( \tau_0 < t \leq \tau_0 + \tau_1 \), the dynamics of the system is driven by the master equation given in Eq. (S50), but with \( H(t) \) replaced by \( H_1 \). Thus, as mentioned in Sec. S3, the system has a dynamical evolution similar to what has already been discussed with the delocalized-mode basis in the first interval. We follow the same recipe as above and then find
\[ \langle b_+^\dagger b_+^\dagger b_+ b_k \rangle (\tau_0 + \tau_1) = 2X(\tau_0 + \tau_1), \] (S93)
\[ \langle b_+^\dagger b_+ b_2 \rangle (\tau_0 + \tau_1) = X(\tau_0 + \tau_1), \] (S94)
\[ \langle b_+^\dagger b_+ b_2 \rangle (\tau_0 + \tau_1) = \langle b_+^\dagger b_+^\dagger b_2 b_2 \rangle (\tau_0 + \tau_1) = i \exp(-iJ\tau_1) Y(\tau_0 + \tau_1), \] (S95)
\[ \langle b_+^\dagger b_+^\dagger b_2 b_2 \rangle (\tau_0 + \tau_1) = 0. \] (S96)
at the end of this interval.
For the third evolution interval \( \tau_0 + \tau_1 < t \leq \tau_T \), we return back to the master equation in Eq. (S56), and also back to the delocalized-mode basis. According to Eqs. (S93), (S94), (S95), and (S96), the evolution at this stage starts from
\[ \langle c_+^\dagger c_+^\dagger c_+^\dagger c_\pm \rangle (\tau_0 + \tau_1) = 2X(\tau_0 + \tau_1) \pm i2 \sin(J\tau_1) Y(\tau_0 + \tau_1), \] (S97)
\[ \langle c_+^\dagger c_+^\dagger c_+^\dagger c_- \rangle (\tau_0 + \tau_1) = X(\tau_0 + \tau_1), \] (S98)
\[ \langle c_+^\dagger c_+^\dagger c_+^\dagger c_- \rangle (\tau_0 + \tau_1) = \langle c_+^\dagger c_+^\dagger c_- \rangle (\tau_0 + \tau_1) = -i \cos(J\tau_1) Y(\tau_0 + \tau_1), \] (S99)
\[ \langle c_+^\dagger c_+^\dagger c_+^\dagger c_- \rangle (\tau_0 + \tau_1) = \langle c_+^\dagger c_+^\dagger c_- \rangle (\tau_0 + \tau_1) = i(-1)^{j+1} \sin^2 (\varphi) \cos(J\tau_1) Y(\tau_0 + \tau_1), \] (S100)
\[ \langle c_+^\dagger c_+^\dagger c_+^\dagger c_- \rangle (\tau_0 + \tau_1) = \langle c_+^\dagger c_+^\dagger c_- \rangle (\tau_0 + \tau_1) = 0. \] (S101)
where \( j = 1, 2 \). Note that, before this evolution, the spin state has already been transformed from \( |D\rangle \to \cos(\varphi) |0\rangle + \sin(\varphi) |D\rangle \) via a single-qubit rotation. Then, by following the same procedure as in the first interval, the last evolution ends with
\[ \langle c_+^\dagger c_+^\dagger c_+^\dagger c_\pm \rangle (\tau_T) = 2X(\tau_T) \pm 2 \sin(J\tau_1) Y(\tau_T), \] (S102)
\[ \langle c_+^\dagger c_+^\dagger c_+^\dagger c_- \rangle (\tau_T) = X(\tau_T), \] (S103)
\[ \langle c_+^\dagger c_+^\dagger c_- \rangle (\tau_T) = \langle c_+^\dagger c_+^\dagger c_- \rangle (\tau_T) = -i \cos(J\tau_1) \left[ \cos^2(\varphi) - i \sin^2(\varphi) \right] Y(\tau_T), \] (S104)
\[ \langle c_+^\dagger c_+^\dagger c_- \rangle (\tau_T) = 0. \] (S105)
and as a result, with

\[ \langle b_k^\dagger b_k^\dagger b_k b_k \rangle (\tau_T) = 2X (\tau_T) + 2 (-1)^j \sin^2 (\varphi) \cos (J\tau_1) Y (\tau_T). \]  

(S106)

It is seen that on the right-hand side of Eq. (S106), the first term arises from the particle behavior of a phonon and the second term arises from its wave behavior.

By substituting Eq. (S106) into Eq. (S75), the fluctuation \( \delta n_k \) in the occupation \( n_k \) is given by

\[
(\delta n_k)^2 = (n_{th}^2 - 2P_k n_{th} - P_k^2) \exp (-2\gamma_m \tau_T) \\
- (2n_{th} + 1)(n_{th} - P_k) \exp (-\gamma_m \tau_T) + n_{th} (n_{th} + 1).
\]

(S107)

Since \( \gamma_m \tau_T \ll 1 \), we have

\[
(\delta n_k)^2 \simeq (\delta n_k^{\text{signal}})^2 + (\delta n_k^{\text{noise}})^2,
\]

(S108)

where

\[
(\delta n_k^{\text{signal}})^2 = P_k (1 - P_k),
\]

(S109)

\[
(\delta n_k^{\text{noise}})^2 = P_k (2P_k - 1) \gamma_m \tau_T + n_{th} \gamma_m \tau_T (2P_k + 1).
\]

(S110)

Here, \( \delta n_k^{\text{signal}} \), the quantum fluctuation induced by the Heisenberg uncertainty principle, accounts for the coherent signal, and \( \delta n_k^{\text{noise}} \) represents the fluctuation noise, including the vacuum (the first term) and thermal (the second term) fluctuations. To confirm the predictions of Eq. (S108), we perform numerics, as shown in Fig. S4. Specifically, we plot the fluctuation noises \( \delta n_1^{\text{noise}} \) and \( \delta n_2^{\text{noise}} \) versus the relative phase \( \phi \). The analytical expression is in excellent agreement with our numerical simulations. Furthermore, the respective CNT signal-to-noise ratios can be defined as

\[
R_k = \frac{P_k}{\delta n_k^{\text{noise}}}.
\]

(S111)

Note that, here, we did not use \( \delta n_k \) to define \( R_k \) because \( \delta n_k^{\text{signal}} \) in \( \delta n_k \) results from quantum fluctuations of the desired signal, as mentioned previously; and therefore this is not the environmental noise. In order to resolve a signal from the fluctuation noise, the ratio \( R_k \) is required to be \( R_k > 1 \). However, Eq. (S111) demonstrates that this criterion is not always met for all values of \( P_k \), in particular, at finite temperatures. For example, \( P_k = 0 \) leads directly to \( R_k = 0 \). To address this problem, we now consider the total fluctuation noise,

\[
S^2 = (\delta n_1^{\text{noise}})^2 + (\delta n_2^{\text{noise}})^2.
\]

(S112)
We further assume that

\[ S^2 < P_1^2 + P_2^2. \]  

(S113)

Under this assumption, if \( R_k < 1 \), then \( R_{3-k} > 1 \) for \( k = 1, 2 \); otherwise \( R_1 > 1, R_2 > 1 \). This means that at least one of the signals, \( P_1 \) or \( P_2 \), is resolved for each measurement. Because the coherent phonon number equal to 1 is conserved, and therefore the signals in the two CNTs are complementary, the unresolved signal can be completely deduced from the resolved one. Thus, the criterion in Eq. (S113) ensures that the morphing behavior between wave and particle can be observed from the environment-induced fluctuation noise. In fact, for any value of \( P_k \), the total noise \( S \) is limited by an upper bound,

\[ S < B = \sqrt{\gamma_m \tau_T} + 4n_{th} \sqrt{\gamma_m \tau_T} \text{max}, \]  

(S114)

which is independent of \( P_k \). Meanwhile, \( \sqrt{P_1^2 + P_2^2} \) is also limited by a lower bound \( \sqrt{2}/2 \). Thereby, in order to meet the criterion given in Eq. (S113), it is required that

\[ B < \frac{\sqrt{2}}{2}, \]  

(S115)
Based on this condition, we can define a signal visibility

\[ \mathcal{R} = \frac{\sqrt{2}}{2B}, \]  

in analogy to \( R_k \). When \( \mathcal{R} > 1 \), the morphing between wave and particle can be observed, and cannot otherwise. This, in turn, leads to an upper bound on the equilibrium phonon occupation,

\[ n_{th} < \frac{1 - 2\gamma_m \tau_{T}^{\max}}{8\gamma_m \tau_{T}^{\max}}, \]  

and therefore an upper bound on the temperature,

\[ T < \frac{\hbar\omega_m}{k_B \ln \left( \frac{1 + 6\gamma_m \tau_{T}^{\max}}{(1 - 2\gamma_m \tau_{T}^{\max})} \right)}. \]  

Because \( \tau_{T}^{\max} \approx \frac{5\pi}{2J} \), the critical temperature is

\[ T_c = \frac{\hbar\omega_m}{k_B \ln \left( \frac{1 + 15\pi\gamma_m / J}{(1 - 5\pi\gamma_m / J)} \right)}. \]

In Fig. S5 we plot the signal-to-noise ratios \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) at the temperature \( T \approx 10 \) mK. We find that almost all signals can be resolved, and also, as expected, find that when the signal in one CNT is unresolved, the signal in the other CNT is resolved. In fact, the upper bound \( B \) is the fluctuation noise in the total phonon occupation \( \langle \hat{b}^\dagger_1 \hat{b}_1 + \hat{b}^\dagger_2 \hat{b}_2 \rangle \) at time \( \tau_T \). The criterion \( \mathcal{R} > 1 \) heralds that to resolve the morphing behavior, the fluctuation noise in \( \langle \hat{b}^\dagger_1 \hat{b}_1 + \hat{b}^\dagger_2 \hat{b}_2 \rangle (\tau_T) \) is required to be smaller than \( \sqrt{2}/2 \).

S5. Numerical simulations

In order to confirm our analytical results, we need to numerically simulate the dynamics with the full master
equation given by

$$\dot{\rho}(t) = \frac{i}{\hbar} [\rho(t), H_F] - \frac{\gamma_s}{2} \mathcal{L}(\sigma'_z) \rho(t)$$

$$- \frac{\gamma_m}{2} n_{\text{th}} \sum_{k=1,2} \mathcal{L}(b_k^\dagger) \rho(t) - \frac{\gamma_m}{2} (n_{\text{th}} + 1) \sum_{k=1,2} \mathcal{L}(b_k) \rho(t),$$

where \(\sigma'_z = |D\rangle\langle D| - |0\rangle\langle 0|\), and \(H_F\) is the full Hamiltonian of Eq. (S8). Here, we use the Python framework QuTiP [S24, S25] to set up this problem. However, the full Hamiltonian is time-dependent, and it takes a long time to integrate the corresponding Schrödinger equation or the master equation, in particular, for our case, where all quantum gates result from the deterministic time evolution of the system. Thus, in our numerical simulations, we replace \(H_F\) with \(H_{\text{low}} + \overline{H}_{\text{high}}\), as in Eq. (S13). This is a reasonable replacement because in our proposal \(\Omega\) (tens of MHz) is required to be much smaller than \(\Delta'\) (up to \(\sim\) GHz). In Fig. S6, we plot the unitary evolution of the phonon occupations, \(\langle b_1^\dagger b_1 \rangle\) and \(\langle b_2^\dagger b_2 \rangle\), of the CNTs. Symbols are the exact results from the full Hamiltonian \(H_F\) and solid curves are given by the approximate Hamiltonian \(H_{\text{low}} + \overline{H}_{\text{high}}\). We find an excellent agreement for a very long evolution time, and thus \(H_F\) can be very well approximated by \(H_{\text{low}} + \overline{H}_{\text{high}}\). For additional comparison, we also plot the phonon occupation evolution driven only by the low-frequency component \(H_{\text{low}}\), corresponding to dotted curves. As seen in Fig. S6, owing to the error accumulation, the dynamics of \(H_{\text{low}}\) deviates largely from the full dynamics of \(H_F\), even within one oscillation cycle. With the above replacement, we obtain the numerical simulations plotted in Fig. 2 of the article, and also in Fig. S4 of the Supplemental Material.

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[S1] O. Gamel and D. F. V. James, “Time-averaged quantum dynamics and the validity of the effective Hamiltonian model,” Phys. Rev. A 82, 052106 (2010).

[S2] W. Qin, A. Miranowicz, P.-B. Li, X.-Y. Liu, J. Q. You, and F. Nori, “Exponentially Enhanced Light-Matter Interaction, Cooperativities, and Steady-State Entanglement Using Parametric Amplification,” Phys. Rev. Lett. 120, 093601 (2018).

[S3] C. Leroux, L. C. G. Govia, and A. A. Clerk, “Enhancing Cavity Quantum Electrodynamics via Antisqueezing: Synthetic Ultrastrong Coupling,” Phys. Rev. Lett. 120, 093602 (2018).

[S4] V. Sazonova, Y. Yaish, H. Üstünel, D. Roundy, T. A. Arias, and P. L. McEuen, “A tunable carbon nanotube electromechanical oscillator,” Nature (London) 431, 284 (2004).

[S5] H. Üstünel, D. Roundy, and T. A. Arias, “Modeling a suspended nanotube oscillator,” Nano Lett. 5, 523–526 (2005).

[S6] D. Garcia-Sánchez, A. San Paulo, M. J. Esplandiu, F. Perez-Murano, L. Forró, A. Aguasca, and A. Bachtold, “Mechanical Detection of Carbon Nanotube Resonator Vibrations,” Phys. Rev. Lett. 99, 085501 (2007).

[S7] Z. Y. Ning, T. W. Shi, M. Q. Fu, Y. Guo, X. L. Wei, S. Gao, and Q. Chen, “Transversally and axially tunable carbon nanotube resonators in situ fabricated and studied inside a scanning electron microscope,” Nano Lett. 14, 1221–1227 (2014).

[S8] S. Truax, S.-W. Lee, M. Muroth, and C. Hierold, “Axially tunable carbon nanotube resonators using co-integrated microactuators,” Nano Lett. 14, 6092–6096 (2014).

[S9] I. Tsoutsios, A. Tavernarakis, J. Osmond, P. Verlot, and A. Bachtold, “Real-time measurement of nanotube resonator fluctuations in an electron microscope,” Nano Lett. 17, 1748–1755 (2017).

[S10] R. Chaves, G. B. Lemos, and J. Pienaar, “Causal Modeling the Delayed-Choice Experiment,” Phys. Rev. Lett. 120, 190401 (2018).

[S11] H.-L. Huang, Y.-H. Luo, B. Bai, Y.-H. Deng, H. Wang, H.-S. Zhong, Y.-Q. Nie, W.-H. Jiang, X.-L. Wang, J. Zhang, L. Li, N.-L. Liu, T. Byrnes, J. P. Dowling, C.-Y. Lu, and J.-W. Pan, “A loophole-free Wheeler-delayed-choice experiment,” arXiv preprint arXiv:1806.00156 (2018).

[S12] E. Polino, I. Agresti, D. Poderini, G. Carvacho, Gi. Milani, G. B. Lemos, R. Chaves, and F. Sciarrino, “Device independent certification of a quantum delayed choice experiment,” arXiv preprint arXiv:1806.00211 (2018).

[S13] S. Yu, W. Liu, Y.-T. Wang, J.-S. Tang, C.-F. Li, and G.-C. Guo, “Experimental realization of causality-assisted Wheeler’s delayed-choice experiment using single photons,” arXiv preprint arXiv:1806.03689 (2018).

[S14] A. Peruzzo, P. Shadbolt, N. Brunner, S. Popescu, and J. L. O’Brien, “A Quantum Delayed-Choice Experiment,” Science 338, 634–637 (2012).

[S15] F. Kaiser, T. Coudreau, P. Milman, D. B. Ostrowsky, and S. Tanzilli, “Entanglement-Enabled Delayed-Choice Experiment,” Science 338, 637–640 (2012).

[S16] S.-B. Zheng, Y.-P. Zhong, K. Xu, Q.-J. Wang, H. Wang, L.-T. Shen, C.-P. Yang, J. M. Martinis, A. N. Cleland, and S.-Y. Han, “Quantum Delayed-Choice Experiment with a Beam Splitter in a Quantum Superposition,” Phys. Rev. Lett. 115, 260403 (2015).
[S17] K. Liu, Y. Xu, W. Wang, S.-B. Zheng, T. Roy, S. Kundu, M. Chand, A. Ranade, R. Vijay, Y. Song, L. Duan, and L. Sun, “A twofold quantum delayed-choice experiment in a superconducting circuit,” Sci. Adv. 3, e1603159 (2017).

[S18] P. Huang, X. Kong, N. Zhao, F. Shi, P. Wang, X. Rong, R.-B. Liu, and J. Du, “Observation of an anomalous decoherence effect in a quantum bath at room temperature,” Nat. Commun. 2, 570 (2011).

[S19] S. E. Lillie, D. A. Broadway, J. D. A. Wood, D. A. Simpson, A. Stacey, J.-P. Tétienne, and L. C. L. Hollenberg, “Environmentally Mediated Coherent Control of a Spin Qubit in Diamond,” Phys. Rev. Lett. 118, 167204 (2017).

[S20] J. Xing, Y.-R. Zhang, S. Liu, Y.-C. Chang, J.-D. Yue, H. Fan, and X.-Y. Pan, “Experimental investigation of quantum entropic uncertainty relations for multiple measurements in pure diamond,” Sci. Rep. 7, 2563 (2017).

[S21] G. Balasubramanian, P. Neumann, D. Twitchen, M. Markham, R. Kolesov, N. Mizuochi, J. Isoya, J. Achar, J. Beck, J. Tissler, V. Jacques, P. R. Hemmer, F. Jelezko, and J. Wrachtrup, “Ultralong spin coherence time in isotopically engineered diamond,” Nat. Mater. 8, 383–387 (2009).

[S22] N. Bar-Gill, L. M. Pham, A. Jarmola, D. Budker, and R. L. Walsworth, “Solid-state electronic spin coherence time approaching one second,” Nat. Commun. 4, 1743 (2013).

[S23] J. Moser, A. Eichler, J. Güttinger, M. I. Dykman, and A. Bachtold, “Nanotube mechanical resonators with quality factors of up to 5 million,” Nat. Nanotechnol. 9, 1007–1011 (2014).

[S24] J. R. Johansson, P. D. Nation, and F. Nori, “Qutip: An open-source Python framework for the dynamics of open quantum systems,” Comput. Phys. Commun. 183, 1760–1772 (2012).

[S25] J. R. Johansson, P. D. Nation, and F. Nori, “Qutip 2: A Python framework for the dynamics of open quantum systems,” Comput. Phys. Commun. 184, 1234–1240 (2013).