ON THE WEIGHTED FRACTIONAL POINCARÉ-TYPE INEQUALITIES

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ABSTRACT. Weighted fractional Poincaré-type inequalities are proved on John domains whenever the weights defined on the domain depend on the distance to the boundary and to an arbitrary compact set in the boundary of the domain.

1. Introduction

In this article we study a version of the classical fractional Poincaré-type inequality where the domain in the double integral in the Gagliardo seminorm is replaced by a smaller one:

\[
\left( \int_\Omega |u(x) - u_\Omega|^p \, dx \right)^{1/p} \leq C \left( \int_\Omega \int_{B(x, \tau d(x))} \frac{|u(x) - u(y)|^p}{|x - y|^{n + sp}} \, dy \, dx \right)^{1/p},
\]

(1.1)

The parameter \( \tau \) in the double integral belongs to \((0, 1)\) and \(d(x)\) denotes the distance from \(x\) to \(\partial \Omega\). The inequality (1.1) was introduced in [4]. It is well-known that the fractional classical Poincaré inequality is valid for any bounded domain, while this new version (1.1) depends on the geometry of the domain. In [4] it was proved that the inequality (1.1) is valid on John domains and, hence, in particular on Lipschitz domains. An example of a domain where the inequality (1.1) is not valid was also given. We refer the reader to [3] and [2] where the fractional Sobolev-Poincaré versions of (1.1) are considered. For a weighted version of (1.1) where weights are power functions to the boundary we refer to [3].

The main result of our paper is the following theorem where the distance to an arbitrary set of the boundary has been added as a weight.

**Theorem 1.1.** Let \( \Omega \) in \( \mathbb{R}^n \) be a bounded John domain and \( 1 < p < \infty \). Given a compact set \( F \) in \( \partial \Omega \), and the parameters \( \beta \geq 0 \) and \( s, \tau \in (0, 1) \), there exists a constant \( C \) such that

\[
\left( \int_\Omega |u(x) - u_{\Omega, \omega}|^p d_F^{p \beta}(x) \, dx \right)^{1/p} \leq C \left( \int_\Omega \int_{B(x, \tau d(x))} \frac{|u(x) - u(y)|^p}{|x - y|^{n + sp}} d_F^{p \beta}(x) dy \, dx \right)^{1/p},
\]

(1.2)

for all functions \( u \in L^p(\Omega, d(x)^{p \beta}) \), where \( d(x) \) and \( d_F(x) \) denote the distance from \( x \) to \( \partial \Omega \) and \( F \) respectively, and \( u_{\Omega, \omega} \) is the weighted average \( \frac{1}{d_F^{p \beta}(\Omega)} \int_\Omega u(z) d_F^{p \beta}(z) \, dz \).

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In addition, the constant $C$ in (1.2) can be written as
\[ C = C_{n,p,\beta} \tau^{s-n} K^{n+\beta}, \]
where $K$ is the geometric constant introduced in (5.1).

We would like to emphasize two points in this result: The first one is that no extra conditions are required for the compact set $F$ in $\partial \Omega$. The second point is that the estimate shows how the constant depends on the given $\tau$ and a certain geometric condition of the domain.

Some of the essential auxiliary parts for the proofs for weighted inequalities are from [7] and [8] where a useful decomposition technique was introduced by the second author. Our work was stimulated by the papers of Augusto C. Ponce, [10], [11], [12], where more general fractional Poincaré inequalities for functions defined on Lipschitz domains were investigated.

The paper is organized as follows: In Section 2, we introduce some definitions and preliminary results. In Section 3, we show how to use decompositions of functions to extend the validity of certain inequalities on “simple domains”, such as cubes, to more complex ones. We are interested in extending the results from cubes to John domains. In Section 4, we apply the results obtained in the previous section to estimate the constant in the unweighted version of (1.2) on cubes. Especially we are interested in how the constant depends on $\tau$. This result is auxiliary of our main theorem but it might be of independent interest. In Section 5, we show the validity of the weighted fractional Poincaré inequality studied in this paper with the estimate of the constant and a generalization to the type of inequalities considered by Ponce.

2. Notation and preliminary results

Throughout the paper $\Omega$ in $\mathbb{R}^n$ is a bounded domain with $n \geq 2$, $1 < p < \infty$, and $1 < q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, unless otherwise stated. Moreover, given $\eta : \Omega \to \mathbb{R}$ a weight (i.e., a positive measurable function) and $1 \leq r \leq \infty$, we denote by $L^r(\Omega, \eta)$ the space of Lebesgue measurable functions $u : \Omega \to \mathbb{R}$ equipped with the norm
\[ \|u\|_{L^r(\Omega, \eta)} := \left( \int_{\Omega} |u(x)|^r \eta(x) \, dx \right)^{1/r} \]
if $1 \leq r < \infty$, and
\[ \|u\|_{L^\infty(\Omega, \eta)} := \text{ess sup}_{x \in \Omega} |u(x)\eta(x)|. \]

Finally, given a set $A$ we denote by $\chi_A(x)$ its characteristic function.

**Definition 2.1.** Let $\mathcal{C}$ be the space of constant functions from $\mathbb{R}^n$ to $\mathbb{R}$ and $\{U_t\}_{t \in \Gamma}$ a collection of open subsets of $\Omega$ that covers $\Omega$ except for a set of Lebesgue measure zero; $\Gamma$ is an index set. It also satisfies the additional requirement that for each $t \in \Gamma$ the set $U_t$ intersects a finite number of $U_s$ with $s \in \Gamma$. This collection $\{U_t\}_{t \in \Gamma}$ is called an open covering of $\Omega$. Given $g \in L^1(\Omega)$ orthogonal to $\mathcal{C}$ (i.e., $\int g \varphi = 0$ for all $\varphi \in \mathcal{C}$), we say that a collection of functions $\{g_t\}_{t \in \Gamma}$ in $L^1(\Omega)$ is a $\mathcal{C}$-orthogonal decomposition of $g$ subordinate to $\{U_t\}_{t \in \Gamma}$ if the following three properties are satisfied:

1. $g = \sum_{t \in \Gamma} g_t$.
2. $\text{supp}(g_t) \subset U_t$.
3. $\int_{U_t} g_t \varphi = 0$, for all $\varphi \in \mathcal{C}$ and $t \in \Gamma$. 

We also refer to this collection of functions by a \( C \)-decomposition. We say that \( \{g_t\}_{t \in \Gamma} \) is a finite \( C \)-decomposition if \( g_t \neq 0 \) only for a finite number of \( t \in \Gamma \).

Inequality \([1,2]\), and similar Poincaré type inequalities, can be written in terms of a distance to the space of constant functions \( C \) by replacing its left hand side by
\[
\inf_{\alpha \in C} \left( \int_\Omega |u(x) - \alpha|^p d\mu(x) \right)^{1/p}.
\]
The technique used in this paper may also be considered when the distance to other vector spaces \( V \) are involved, in which case, a \( V \)-orthogonal decomposition of functions is required. We direct the reader to [9] where a generalized version of the Korn inequality is studied by using decomposition of functions.

Let us denote by \( G = (V, E) \) a graph with vertices \( V \) and edges \( E \). Graphs in this paper have neither multiple edges nor loops and the number of vertices in \( V \) is at most countable.

A rooted tree (or simply a tree) is a connected graph \( G \) in which any two vertices are connected by exactly one simple path, and a root is simply a distinguished vertex \( a \in V \). Moreover, if \( G = (V, E) \) is a rooted tree with a root \( a \), it is possible to define a partial order “\( \preceq \)" in \( V \) as follows: \( s \preceq t \) if and only if the unique path connecting \( t \) with the root \( a \) passes through \( s \). The height or level of any \( t \in V \) is the number of vertices in \( \{s \in V : s \preceq t \text{ with } s \neq t\} \). The parent of a vertex \( t \in V \) is the vertex \( s \) satisfying that \( s \preceq t \) and its height is one unit smaller than the height of \( t \). We denote the parent of \( t \) by \( t_p \). It can be seen that each \( t \in V \) different from the root has a unique parent, but several elements in \( V \) could have the same parent. Note that two vertices are connected by an edge (adjacent vertices) if one is the parent of the other.

**Definition 2.2.** Let \( \Omega \) be in \( \mathbb{R}^n \) be a bounded domain. We say that an open covering \( \{U_t\}_{t \in \Gamma} \) is a tree covering of \( \Omega \) if it also satisfies the properties:

1. \( \chi_\Omega(x) \leq \sum_{t \in \Gamma} \chi_{U_t}(x) \leq N \chi_\Omega(x) \), for almost every \( x \in \Omega \), where \( N \geq 1 \).
2. \( \Gamma \) is the set of vertices of a rooted tree \((\Gamma, E)\) with a root \( a \).
3. There is a collection \( \{B_t\}_{t \neq a} \) of pairwise disjoint open cubes with \( B_t \subseteq U_t \cap U_{t_p} \).

**Definition 2.3.** Given a tree covering \( \{U_t\}_{t \in \Gamma} \) of \( \Omega \) we define the following Hardy-type operator \( T \) on \( L^1 \)-functions:
\[
Tg(x) := \sum_{a \neq t \in \Gamma} \frac{\chi_t(x)}{|W_t|} \int_{W_t} |g|,
\]
where
\[
W_t := \bigcup_{s \preceq t} U_s,
\]
and \( \chi_t \) is the characteristic function of \( B_t \) for all \( t \neq a \).

We may refer to \( W_t \) by the shadow of \( U_t \).

Note that the definition of \( T \) is based on the a-priori choice of a tree covering \( \{U_t\}_{t \in \Gamma} \) of \( \Omega \). Thus, whenever \( T \) is mentioned in this paper there is a tree covering \( \{U_t\}_{t \in \Gamma} \) of \( \Omega \) explicitly or implicitly associated to it.

The following fundamental result was proved in [8, Theorem 4.4], which shows the existence of a \( C \)-decomposition of functions subordinate to a tree covering of the domain.
Theorem 2.4. Let $\Omega$ in $\mathbb{R}^n$ be a bounded domain with a tree covering $\{U_t\}_{t \in \Gamma}$. Given $g \in L^1(\Omega)$ such that $\int_{\Omega} g \varphi = 0$, for all $\varphi \in C$, and $\text{supp}(g) \cap U_s \neq \emptyset$ for a finite number of $s \in \Gamma$, there exists a $C$-decompositions $\{g_t\}_{t \in \Gamma}$ of $g$ subordinate to $\{U_t\}_{t \in \Gamma}$ (refer to Definition 2.1).

Moreover, let $t \in \Gamma$. If $x \in B_s$ where $s = t$ or $s_p = t$ then

$$|g_t(x)| \leq |g(x)| + \frac{|W_t|}{|U_t|} T g(x), \quad (2.3)$$

where $W_t$ denotes the shadow of $U_t$ defined in (2.2). Otherwise

$$|g_t(x)| \leq |g(x)|. \quad (2.4)$$

Remark 2.5. The $C$-decomposition stated in Theorem 2.4 is finite. This fact is not in the statement of [8, Theorem 4.4] but it is easily deduced from its proof.

In the next lemma, the continuity of the operator $T$ is shown. We refer the reader to [7, Lemma 3.1] for its proof.

Lemma 2.6. The operator $T : L^q(\Omega) \to L^q(\Omega)$ defined in (2.1) is continuous for any $1 < q \leq \infty$. Moreover, its norm is bounded by

$$\|T\|_{L^q \to L^q} \leq 2 \left( \frac{qN}{q-1} \right)^{1/q}. \quad (2.5)$$

Here $N$ is the overlapping constant from Definition 2.2.

If $q = \infty$, the previous inequality means $\|T\|_{L^\infty \to L^\infty} \leq 2$. Actually, for being $T$ an averaging operator, it can be easily observed that $\|T\|_{L^\infty \to L^\infty} = 1$, but it does not affect our work. Notice that $L^q(\Omega, \omega^{-q}) \subset L^1(\Omega)$ if the weight $\omega : \Omega \to \mathbb{R}_{>0}$ satisfies that $\omega^p \in L^1(\Omega)$. Then, the operator $T$ introduced in Definition 2.3 for functions in $L^1(\Omega)$ is well-defined in $L^q(\Omega, \omega^{-q})$.

Lemma 2.7. Let $\Omega$ in $\mathbb{R}^n$ be a bounded domain, $\{U_t\}_{t \in \Gamma}$ a tree covering of $\Omega$ and $\omega : \Omega \to \mathbb{R}$ a weight which satisfies $\omega^p \in L^1(\Omega)$. If $\omega$ satisfies that

$$\text{ess sup}_{y \in W_t} \omega(y) \leq C_2 \text{ess inf}_{x \in B_t} \omega(x), \quad (2.6)$$

for all $a \neq t \in \Gamma$, then the Hardy-type operator $T$ defined in (2.1) and subordinate to $\{U_t\}_{t \in \Gamma}$ is continuous from $L^q(\Omega, \omega^{-q})$ to itself. Moreover, its norm for $1 < q < \infty$ is bounded by

$$\|T\|_{L^q \to L^q} \leq 2 \left( \frac{qN}{q-1} \right)^{1/q} C_2, \quad (2.7)$$

where $L$ denotes $L^q(\Omega, \omega^{-q})$, and $N$ is the overlapping constant from Definition 2.2.

Proof. Given $g \in L^q(\Omega, \omega^{-q})$ we have

$$\int_{\Omega} |T g(x)|^q \omega^{-q}(x) \, dx \leq \int_{\Omega} \omega^{-q}(x) \left| \sum_{a \neq t \in \Gamma} \frac{\chi_t(x)}{|W_t|} \int_{W_t} |g(y)| \, dy \right|^q \, dx$$

$$= \int_{\Omega} \left| \sum_{a \neq t \in \Gamma} \frac{\chi_t(x)}{|W_t|} \omega^{-1}(x) \int_{W_t} |g(y)| \omega^{-1}(y) \omega(y) \, dy \right|^q \, dx.$$
Now, condition (2.5) implies that \( \omega(y) \leq C_2 \omega(x) \) for almost every \( x \in B_t \) and \( y \in W_t \). Thus,

\[
\int_{\Omega} |Tg(x)|^q \omega^{-q}(x) \, dx
\leq \int_{\Omega} \left| \sum_{a \notin \Gamma} \frac{\chi_t(x)}{|W_t|} \omega^{-1}(x) \int_{W_t} g(y) |\omega^{-1}(y)| \, dy \right|^q \, dx
\]

\[
= C_2^q \int_{\Omega} \left| \sum_{a \notin \Gamma} \frac{\chi_t(x)}{|W_t|} \int_{W_t} g(y) |\omega^{-1}(y)| \, dy \right|^q \, dx
\]

\[
= C_2^q \int_{\Omega} |T(g\omega^{-1})|^q \, dx.
\]

Finally, \( g\omega^{-1} \) belongs to \( L^q(\Omega) \) and \( T \) is continuous from \( L^q(\Omega) \) to itself; we refer to Lemma 2.6 hence

\[
\int_{\Omega} |Tg(x)|^q \omega^{-q}(x) \, dx \leq \left( 2^q \frac{qN}{q-1} \right) C_2^q \|g\|_{L^q(\Omega, \omega^{-q})}^q.
\]

\[\square\]

3. A DECOMPOSITION AND FRACTIONAL POINCARÉ INEQUALITIES

Let \( \Omega \) in \( \mathbb{R}^n \) be an arbitrary bounded domain and \( \{U_t\}_{t \in \Gamma} \) an open covering of \( \Omega \). The weight \( \omega : \Omega \to \mathbb{R}_{>0} \) satisfies that \( \omega^p \in L^1(\Omega) \). In addition, \( u_\Omega \) denotes the average

\[
\frac{1}{|\Omega|} \int_{\Omega} u(z) \, dz.
\]

For weighted spaces of functions, \( u_{\Omega, \omega} \) represents the weighted average

\[
\frac{1}{\omega(\Omega)} \int_{\Omega} u(z) \omega(z) \, dz,
\]

where \( \omega(\Omega) := \int_{\Omega} \omega(z) \, dz \).

Now, given a bounded domain \( U \) in \( \mathbb{R}^n \) and a nonnegative measurable function \( \mu : U \times U \to \mathbb{R} \) we introduce the following Poincaré type inequality

\[
\inf_{c \in \mathbb{R}} \|u - c\|_{L^p(U, \omega^p)} \leq C \left( \int_{U} \int_{U} |u(x) - u(y)|^p \mu(x, y) \, dy \, dx \right)^{1/p},
\]

where \( u \in L^p(U, \omega^p) \). Notice that the right hand side in this inequality might be infinite. The validity of \( (3.6) \) depends on \( U, p, \mu \) and \( \omega \). The function \( \mu(x, y) \) might be zero, however, \( \omega(x) \) is strictly positive almost everywhere in \( \Omega \).

Let us mention three examples.

**Examples 3.1.**

1. The weighted fractional Poincaré inequality with \( \mu(x, y) = \frac{1}{|x-y|^{n+sp}} \), where \( s \in (0, 1) \), is the classical fractional Poincaré inequality which is clearly valid for any arbitrary bounded domain.

2. If \( \mu(x, y) = \frac{\chi_{B_x}(y)}{|x-y|^{n+sp}} \), where \( B_x \) is the ball centered at \( x \) with radius \( r_\tau d(x) \) for \( s, \tau \in (0, 1) \), then the inequality represents a more recently studied fractional Poincaré inequality whose validity depends on the geometry of the domain (refer to [4] for details).

3. Finally, \( \mu(x, y) = \frac{\rho(|x-y|)}{|x-y|^p} \), where \( \rho \) is a certain nonnegative radial function, is another inequality which has also been studied recently (refer to [10] for details).
Inequality (3.6) deals with an estimation of the distance to \( C \) of an arbitrary function \( u \) in \( L^p(\Omega, \omega^p) \). The local-to-global argument used in this paper to study this Poincaré type inequalities is based on the fact that \( L^p(\Omega, \omega^p) \) is the dual space of \( L^q(\Omega, \omega^{-q}) \) and the existence of decompositions of functions in \( L^q(\Omega, \omega^{-q}) \) orthogonal to \( C \). Let us properly define this set and a subspace:

\[
W := \{ g \in L^q(\Omega, \omega^{-q}) : \int g \varphi = 0 \text{ for all } \varphi \in C \}
\]

(3.7)

\[
W_0 := \{ g \in W : \text{supp}(g) \text{ intersects a finite number of } U_t \}. \quad (3.8)
\]

The integrability of \( \omega^p \) implies that \( L^q(\Omega, \omega^{-q}) \subset L^1(\Omega) \), then \( W \) and \( W_0 \) are well-defined. Following Remark 2.5, the \( C \)-decomposition of functions in \( W_0 \) stated in Theorem 2.4 is finite, which is not valid in general for functions in \( W \). This property verified by the functions in \( W_0 \) simplifies the proof of Lemma 3.3, which motivates the definition of this space.

Now, we introduce the spaces

\[
W \oplus \omega^p C = \{ g + \alpha \omega^p / g \in W \text{ and } \alpha \in C \}
\]

\[
S := W_0 \oplus \omega^p C = \{ g + \alpha \omega^p / g \in W_0 \text{ and } \alpha \in C \}. \quad (3.9)
\]

It is not difficult to observe that \( L^q(\Omega, \omega^{-q}) = W \oplus \omega^p C \) and \( S \) is a subspace of \( L^q(\Omega, \omega^{-q}) \). The following lemma, which was proved in [8, Lemma 3.1], states that \( S \) is also dense in \( L^q(\Omega, \omega^{-q}) \) and uses in its proof the requirement that says that for each \( t \in \Gamma \) the set \( U_t \) intersects a finite number of \( U_s \) with \( s \in \Gamma \).

**Lemma 3.2.** The space \( S \) is dense in \( L^q(\Omega, \omega^{-q}) \). Moreover, if \( g + \alpha \omega^p \) is an element in \( S \) then

\[
\|g\|_{L^q(\Omega, \omega^{-q})} \leq 2\|g + \alpha \omega^p\|_{L^q(\Omega, \omega^{-q})}.
\]

**Lemma 3.3.** If there exists an open covering \( \{U_t\}_{t \in \Gamma} \) of \( \Omega \) such that (3.6) is valid on \( U_t \) for all \( t \in \Gamma \), with a uniform constant \( C_1 \), and there exists a finite \( C \)-orthogonal decomposition of any function \( g \) in \( W_0 \) subordinate to \( \{U_t\}_{t \in \Gamma} \), with the estimate

\[
\sum_{t \in \Gamma} \|g_t\|_{L^q(U_t, \omega^{-q})}^q \leq C_0 \|g\|_{L^q(\Omega, \omega^{-q})}^q,
\]

then, there exists a constant \( C \) such that

\[
\|u - u_{\Omega, \omega}\|_{L^p(\Omega, \omega^p)} \leq C \left( \sum_{t \in \Gamma} \int_{U_t} \int_{U_t} |u(x) - u(y)|^p \mu(x, y) \, dy \, dx \right)^{1/p}
\]

is valid for any \( u \in L^p(\Omega, \omega^p) \). Moreover, the constant \( C = 2C_0 C_1 \) holds in (3.10).

**Proof.** Without loss of generality we can assume that \( u_{\Omega, \omega} = 0 \). We estimate the norm on the left hand side of the inequality by duality. Thus, let \( g + \omega^p \psi \) be an arbitrary function in \( S \), we refer to Lemma 3.2. Then, by using the finite \( C \)-orthogonal decomposition of \( g \) we conclude that

\[
\int_{\Omega} u(g + \alpha \omega^p) = \int_{\Omega} ug = \sum_{t \in \Gamma} \int_{U_t} ug_t = \sum_{t \in \Gamma} \int_{U_t} u - c_t g_t.
\]

(3.11)

Notice that the identity in the second line is valid for any \( t \in \Gamma \) and \( c_t \in \mathbb{R} \).
Next, by using the Hölder inequality in (3.11), the fact that (3.6) is valid on $U_t$ with a uniform constant $C_1$ and, finally, the Hölder inequality over the sum, we obtain
\[
\int_{\Omega} u(g + \alpha \omega^p) \leq \sum_{t \in \Gamma} \inf_{c \in \mathbb{R}} \| u - c \|_{L^p(U_t, \omega^p)} \| g_t \|_{L^q(U_t, \omega^{-q})}
\]
\[
\leq C_1 \sum_{t \in \Gamma} \left( \int_{U_t} \int_{U_t} |u(x) - u(y)|^p \mu(x, y) \, dy \, dx \right)^{1/p} \| g_t \|_{L^q(U_t, \omega^{-q})}
\]
\[
\leq C_1 \left( \sum_{t \in \Gamma} \int_{U_t} \int_{U_t} |u(x) - u(y)|^p \mu(x, y) \, dy \, dx \right)^{1/p} \left( \sum_{t \in \Gamma} \| g_t \|_{L^q(U_t, \omega^{-q})}^q \right)^{1/q}
\]
\[
\leq C_0 C_1 \left( \sum_{t \in \Gamma} \int_{U_t} \int_{U_t} |u(x) - u(y)|^p \mu(x, y) \, dy \, dx \right)^{1/p} \| g \|_{L^q(U, \omega^{-q})}
\]
\[
\leq 2C_0 C_1 \left( \sum_{t \in \Gamma} \int_{U_t} \int_{U_t} |u(x) - u(y)|^p \mu(x, y) \, dy \, dx \right)^{1/p} \| g + \alpha \omega^p \|_{L^q(U, \omega^{-q})}.
\]

Finally, as $S$ is dense in $L^q(\Omega, \omega^{-q})$, by taking the supremum over all the functions $g + \alpha \omega^p$ in $S$ with $\| g + \alpha \omega^p \|_{L^q(\Omega, \omega^{-q})} \leq 1$ we prove the result. \qed

4. On fractional Poincaré inequalities on cubes

In this section, we use the results stated in the previous two sections to show a certain fractional Poincaré inequality on an arbitrary cube $Q$. Thus, in order to show the existence of the $C$-decomposition, which is used later to apply Lemma 3.3, we define a tree covering $\{U_t\}_{t \in \Gamma}$ of $Q$. This covering is only used in this section and for cubes. In the following section, we work with a different bounded domain, an arbitrary bounded John domain, which requires a different covering. However, let us warn the reader that we will keep the notation $\{U_t\}_{t \in \Gamma}$ used in Section 3.

The validity of the local inequality stated in the following proposition is well-known. We refer the reader to [3] for its proof.

**Proposition 4.1.** The fractional Poincaré inequality
\[
\inf_{c \in \mathbb{R}} \| u(x) - c \|_{L^p(U)} \leq \left( \frac{\text{diam} (U)^{n+sp}}{|U|} \int_U \int_U \frac{|u(y) - u(x)|^p}{|y - x|^{n+sp}} \, dy \, dx \right)^{1/p}
\]
holds for any bounded domain $U$ in $\mathbb{R}^n$ and $1 \leq p < \infty$.

The following proposition is a special case of [3] Lemma 2.2. In the present paper, we give a different proof which let us estimate the dependance of the constant with respect to $\tau$.

**Proposition 4.2.** Let $Q$ in $\mathbb{R}^n$ be a cube with side length $l(Q) = L$, $1 < p < \infty$ and $\tau \in (0, 1)$. Then, the following inequality holds
\[
\inf_{c \in \mathbb{R}} \| u(x) - c \|_{L^p(Q)} \leq C_{n,p} \tau^{s-n} L^s \left( \int_Q \int_{Q \cap B(x, \tau L)} \frac{|u(y) - u(x)|^p}{|y - x|^{n+sp}} \, dy \, dx \right)^{1/p},
\]
where $C_{n,p}$ depends only on $n$ and $p$. 
Proof. This result follows from Lemma 3.3 on the cube \( Q \), where \( \mu(x, y) = \frac{1}{|x-y|^{n+p}} \) and \( \omega \equiv 1 \). So, let us start by defining an appropriate tree covering of \( Q \) to obtain, via Theorem 2.4 and Remark 2.5, a finite \( C \)-decomposition of any functions in \( W_0 \). Let \( m \in \mathbb{N} \) be such that \( \frac{n+3}{\tau} < m \leq 1 + \frac{n+3}{\tau} \) and \( \{ A_t \}_{t \in \Gamma} \) the regular partition of \( Q \) with \( m^n \) open cubes. The side length of each cube is \( l(A_t) = \frac{L}{m} \). In the example shown in Figure 1 \( m = 4 \) and the index set \( \Gamma \) has 16 elements.

![Figure 1. A tree covering of Q](image)

The tree covering of \( Q \) that we are looking for will be defined by enlarging the sets in the covering \( \{ A_t \}_{t \in \Gamma} \) in an appropriate way but keeping the tree structure of \( \Gamma \), which is introduced in the following lines. Indeed, we pick a cube \( A_a \), whose index will be the root, and inductively define a tree structure in \( \Gamma \) such that the unique chain connecting \( t \) with \( a \) is associated to a chain of cubes connecting \( Q_t \) with \( Q_a \), with minimal number of cubes, such that two consecutive cubes share a \( n-1 \) dimensional face. In Figure 1, the cube \( A_a \) is in the lower left corner and the tree structure is represented using black arrows that “descend” to the root. Now that \( \Gamma \) has a tree structure, we define the tree covering \( \{ U_t \}_{t \in \Gamma} \) of \( Q \) with the rectangles \( U_t := (A_t \cup A_{t_p})^\circ \) if \( t \neq a \) and \( U_a := A_a \). In order to have a better understanding of the construction, notice that \( U_t \cap U_{t_p} = A_{t_p} \) for all \( t \neq a \). Moreover, the index set \( \Gamma \) in the example with its tree structure has 7 levels, from level 0 to level 6 (refer to page 3 for definitions), with only one index of level 6, whose rectangle \( U_t \) appears in Figure 1 in a different color.

Now, let us define the collection \( \{ B_t \}_{t \neq a} \) of pairwise disjoint open cubes \( B_t \subseteq U_t \cap U_{t_p} \) or equivalently \( B_t \subseteq A_{t_p} \). Given \( t \neq a \), we split \( A_{t_p} \) into \( 3^n \) cubes with the same size. The open set \( B_t \) is the cube in the regular partition of \( A_{t_p} \) whose closure intersects the \( n-1 \) dimensional face \( A_{t_p} \) in the intersection \((A_t \cap A_{t_p})^\circ \). There are \( 3^{n-1} \) cubes with that property but we pick \( B_t \) to be the one which does not share any part of any other \( n-1 \) dimensional face of \( A_{t_p} \).

The cubes in \( \{ B_t \}_{t \neq a} \) have side length equal to \( \frac{L}{3m} \) and are represented in Figure 1 by the 15 grey gradient small cubes. By its construction, it is easy to check that \( \{ B_t \}_{t \neq a} \) is a collection of pairwise disjoint open cubes \( B_t \subseteq U_t \cap U_{t_p} \), hence, \( \{ U_t \}_{t \in \Gamma} \) is a tree covering of \( Q \) with \( N = 2n \) (it could also be less).

By Theorem 2.4 there is a finite \( C \)-decomposition of functions \( \{ g_t \}_{t \in \Gamma} \) subordinate to \( \{ U_t \}_{t \in \Gamma} \) which satisfies (2.3) and (2.4). Moreover, it can be seen that

\[
\frac{|W_s|}{|B_s|} \leq \frac{|Q|}{|B_s|} = (3m)^n,
\]
for all \( s \in \Gamma \), thus,
\[
|g_t(x)| \leq |g(x)| + (3m)^n T g(x),
\]
for all \( t \in \Gamma \) and \( x \in U_t \). Next, using the continuity of \( T \) stated in Lemma 2.6 and some straightforward calculations we conclude
\[
\sum_{t \in \Gamma} \|g\|_{L^q(U_t)}^q \leq 2^{q-1} N \left( 1 + (3m)^n 2^q \frac{qN}{q-1} \right) \|g\|_{L^q(Q)}^q
\]
\[
\leq \frac{2^{2q+2^q} q^2}{q-1} (3m)^n \|g\|_{L^q(Q)}^q
\]
\[
\leq \frac{2^{2q+2^q} n^2 q}{q-1} \left( 1 + \sqrt{n+3} \right)^n \tau^{-n} \|g\|_{L^q(Q)}^q.
\]
Hence, we have a finite \( C \)-decomposition of any function in \( \mathcal{W}_0 \) subordinate to \( \{U_t\}_{t \in \Gamma} \) with the constant in the estimate equal to
\[
C_0 = \left( \frac{2^{2q+2^q} n^2 q}{q-1} \right)^{1/q} \left( 1 + \sqrt{n+3} \right)^n \tau^{-n}.
\]
Now, from Proposition 4.1 and using that \( m > \frac{\sqrt{n+3}}{\tau} \), we can conclude that inequality (3.6) is valid on each \( U_t \) with an uniform constant
\[
C_1 = (n+3)^{n/2p} (\tau L)^s.
\]
Thus, using Lemma 3.3 we can claim that
\[
\|u - u_Q\|_{L^p(Q)} \leq 2 C_0 C_1 \left( \sum_{t \in \Gamma} \int_{U_t} \int_{U_t} \frac{|u(x) - u(y)|^p}{|x-y|^{pn+sp}} \, dy \, dx \right)^{1/p}.
\]
Finally, \( \text{diam}(U_t) \leq \sqrt{n+3} \frac{L}{m} \leq \tau L \), thus \( U_t \subset B(x, \tau L) \) for any \( x \in U_t \), thus, using the control on the overlapping of the tree covering given by \( N = 2n \), it follows that
\[
\|u - u_Q\|_{L^p(Q)} \leq C_{n,p} \tau^{-n} (\tau L)^s \left( \int_Q \int_{Q \cap B(x, \tau L)} \frac{|u(x) - u(y)|^p}{|x-y|^{pn+sp}} \, dy \, dx \right)^{1/p},
\]
where
\[
C_{n,p} = 2 \left( \frac{2^{2q+2^q} n^2 q}{q-1} \right)^{1/q} \left( 1 + \sqrt{n+3} \right)^n (n+3)^{n/2p} (2n)^{1/p}.
\]

5. On fractional Poincaré inequalities on John domains

In this section, we apply the results obtained in the previous sections on an arbitrary bounded John domain \( \Omega \). Its definition is recalled below. The weight \( \omega(x) \) is defined as \( d_F(x)^\beta \), where \( d_F(x) \) denotes the distance from \( x \) to an arbitrary compact set \( F \) in \( \partial \Omega \) and \( \beta \geq 0 \). In the particular case where \( F = \partial \Omega \), \( d_{\partial \Omega}(x) \) is simply denoted as \( d(x) \). Notice that \( \omega^p \) belongs to \( L^1(\Omega) \) for being \( \Omega \) bounded and \( \beta \) nonnegative.

A Whitney decomposition of \( \Omega \) is a collection \( \{Q_t\}_{t \in \Gamma} \) of closed pairwise disjoint dyadic cubes, which verifies
\[
(1) \quad \Omega = \bigcup_{t \in \Gamma} Q_t,
(2) \quad \text{diam}(Q_t) \leq \text{dist}(Q_t, \partial \Omega) \leq 4 \text{diam}(Q_t),
(3) \quad \frac{1}{4} \text{diam}(Q_s) \leq \text{diam}(Q_t) \leq 4 \text{diam}(Q_s), \text{ if } Q_s \cap Q_t \neq \emptyset.
\]
Here, \( \text{dist}(Q_t, \partial \Omega) \) is the Euclidean distance between \( Q_t \) and the boundary of \( \Omega \), denoted by \( \partial \Omega \). The diameter of the cube \( Q_t \) is denoted by \( \text{diam}(Q_t) \) and the side length is written as \( \ell(Q_t) \).

Two different cubes \( Q_s \) and \( Q_t \) with \( Q_s \cap Q_t \neq \emptyset \) are called \textit{neighbors}. This kind of covering exists for any proper open set in \( \mathbb{R}^n \) (refer to [13, VI 1] for details). Moreover, each cube \( Q_t \) has less than or equal to \( 12^n \) neighbors. And, if we fix \( 0 < \epsilon < \frac{1}{4} \) and define \((1 + \epsilon)Q_t\) as the cube with the same center as \( Q_t \) and side length \((1 + \epsilon)\) times the side length of \( Q_t \), then \((1 + \epsilon)Q_t\) touches \((1 + \epsilon)Q_s\) if and only if \( Q_t \) and \( Q_s \) are neighbors.

Given a Whitney decomposition \( \{Q_t\}_{t \in \Gamma} \) of \( \Omega \) we refer by an expanded Whitney decomposition of \( \Omega \) to the collection of open cubes \( \{Q^*_t\}_{t \in \Gamma} \) defined by

\[
Q^*_t := \frac{9}{8} Q_t^c.
\]

Observe that this collection of cubes satisfies that

\[
\chi_\Omega(x) \leq 12^n \sum_{t \in \Gamma} \chi_{Q^*_t}(x) \leq (12^n)^2 \chi_\Omega(x)
\]

for all \( x \in \mathbb{R}^n \).

We recall the definition of a \textit{bounded John domain}. A bounded domain \( \Omega \) in \( \mathbb{R}^n \) is a John domain with constants \( a \) and \( b \), \( 0 < a \leq b < \infty \), if there is a point \( x_0 \) in \( \Omega \) such that for each point \( x \) in \( \Omega \) there exists a rectifiable curve \( \gamma_x \) in \( \Omega \), parametrized by its arc length written as \( \text{length}(\gamma_x) \), such that

\[
\text{dist}(\gamma_x(t), \partial \Omega) \geq \frac{a}{\text{length}(\gamma_x)} t \quad \text{for all } t \in [0, \text{length}(\gamma_x)]
\]

and

\[
\text{length}(\gamma_x) \leq b.
\]

Examples of John domains are convex domains, uniform domains, and also domains with slits, for example \( B^2(0,1) \setminus [0,1) \). The John property fails in domains with zero angle outward spikes. John domains were introduced by Fritz John in [6] and they were renamed by O. Martio and J. Sarvas as John domains later.

There are other equivalent definitions of John domains. In these notes, we are interested in a definition of the style of Boman chain condition (see [1]) in terms of Whitney decompositions and trees. This equivalent definition is introduced in [8].

**Definition 5.1.** A bounded domain \( \Omega \) in \( \mathbb{R}^n \) is a John domain if for any Whitney decomposition \( \{Q_t\}_{t \in \Gamma} \) of \( \Omega \), there exists a constant \( K > 1 \) and a tree structure of \( \Gamma \), with a root \( a \), that satisfies

\[
Q_s \subseteq K Q_t,
\]

for any \( s, t \in \Gamma \) with \( s \geq t \). In other words, the shadow of \( Q_t \) written as \( W_t \) is contained in \( K Q_t \); refer to (2.2). Moreover, the intersection of the cubes associated to adjacent indices, \( Q_t \) and \( Q_{tp} \), is an \( n - 1 \) dimensional face of one of these cubes.

Now, given a Whitney decomposition \( \{Q_t\}_{t \in \Gamma} \) of a bounded John domain \( \Omega \) in \( \mathbb{R}^n \), with constant \( K \) in the sense of (5.1), we define the tree covering \( \{U_t\}_{t \in \Gamma} \) of expanded Whitney cubes such that

\[
U_t := Q^*_t.
\]
The overlapping is bounded by \( N = 12^n \). Now, each open cube \( B_t \) in the collection \( \{ B_t \}_{t \neq a} \) shares the center with the \( n-1 \) dimensional face \( Q_t \cap Q_{tp} \) and has side length \( l_t \), where \( l_t \) is the side length of \( Q_t \). It follows from the third condition in the Whitney decomposition, and some calculations, that this collection is pairwise disjoint and

\[
B_t \subset Q^*_t \cap Q^*_{tp} = U_t \cap U_{tp},
\]

Moreover, it can be seen that

\[
\frac{|W_t|}{|B_t|} \leq \left( \frac{K_9^9}{l_t^9} \right)^n = 72^n K^n, \tag{5.3}
\]

for all \( t \in \Gamma \), with \( t \neq a \).

**Lemma 5.2.** Let \( \Omega \) in \( \mathbb{R}^n \) be a John domain with the constant \( K \) in the sense of \( (5.1) \), \( F \) in \( \partial \Omega \) a compact set and \( d_F(x) \) the distance from \( x \) to \( F \). Then,

\[
\sup_{y \in W_t} d_F(y) \leq 3K \sqrt{n} \inf_{x \in B_t} d_F(x),
\]

for all \( t \in \Gamma \).

A similar inequality is also valid if we consider the weight \( d^\beta_F(x) \) with a nonnegative power of the distance to \( F \). Thus, this lemma implies, via Lemma 2.7, the continuity of the operator \( T \) from \( L^q(\Omega, d^{\beta}_F) \) to itself with an estimation of its constant. Then, there exists a \( C \)-decomposition with a weighted estimate for a certain weight.

**Proof.** Given \( t \in \Gamma \), with \( t \neq a \), \( x \in B_t \) and \( y \in W_t := \cup_{s \succeq t} U_s \), we have to prove that \( d_F(y) \leq 3K d_F(x) \). Notice that \( d(x) \leq d_F(x) \) for all \( x \in \Omega \). Moreover, \( Q_s \subseteq KQ_t \) for all \( s \succeq t \), then \( W_t \subseteq KU_t \). In addition,

\[
d_F(y) = |y - x| + d_F(x) \leq \text{diam}(W_t) + d_F(x) \\
\leq K \text{diam}(U_t) + d_F(x) \\
= K^9 \text{diam}(Q_t) + d_F(x).
\]

Finally, using the second property stated in the Whitney decomposition it follows that \( 3Q_t \subset \Omega \). Then, as

\[
\text{dist}(Q^*_t, \partial \Omega) \geq \text{dist}(Q^*_t, (3Q_t)^c) \geq \frac{15}{16} l_t,
\]

doing some calculations we can assert that

\[
\text{diam}(Q_t) \leq \frac{16}{15} \sqrt{n} \text{dist}(Q^*_t, \partial \Omega) \leq \frac{16}{9} \sqrt{n} \text{dist}(Q^*_t, \partial \Omega).
\]

Thus,

\[
d_F(y) \leq 2K \sqrt{n} \text{dist}(Q^*_t, \partial \Omega) + d_F(x) \\
\leq 2K \sqrt{n} d(x) + d_F(x) \leq 2K \sqrt{n} d_F(x) + d_F(x).
\]

\( \square \)

Now we are able to prove Theorem 1.1 and also to give the dependence of the constant \( C \) on the given value of \( \tau \) and the constant \( K \) from \( (5.1) \).
Proof of Theorem 1.1. This result follows from Lemma 3.3 with the tree covering \( \{ U_t \}_{t \in \Omega} \) of \( \Omega \) defined in (5.2), \( \omega(x) := d_F^\beta(x) \) and

\[
\mu(x, y) := \frac{d_F^{\beta p}(x) d_F^{\beta p}(y) \chi_{B(x, \tau d(x))}(y)}{|x - y|^{n+sp}}.
\] (5.4)

Notice that \( \omega^p \) belongs to \( L^1(\Omega) \), the condition assumed at the beginning of Section 3. The validity of (3.6) on a cube \( U_t \), with a uniform constant \( C_1 \), follows from Proposition 4.2. Indeed, by using the fact that \( U_t \) is an expanded Whitney cube by a factor 9/8 and \( F \subseteq \partial \Omega \), it follows that

\[
\sup_{x \in U_t} d_F^\beta(x) \leq 2^\beta \inf_{x \in U_t} d_F^\beta(x).
\]

Thus, we have

\[
\inf_{c \in \mathbb{R}} \| u(x) - c \|_{L^p(U_t, d_F^\beta)} \leq C_{n,p} \tau^{s-n} L_t^\beta 2^\beta \left( \int_{U_t} \int_{U_t} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} d_F^{\beta p}(x) \chi_{B(x, \tau L_t)}(y) \, dy \, dx \right)^{1/p},
\]

where \( L_t \) is the side length of \( U_t \) and \( C_{n,p} \) is the constant in (4.1). Now, observe that \( L_t \leq d(x) \) for all \( x \in U_t \). Indeed, if \( x \in Q_t \) then

\[
L_t = \frac{9}{8} l_t < \sqrt{n} l_t = \text{diam}(Q_t) \leq \text{dist}(Q_t, \partial \Omega) \leq d(x),
\]

where \( l_t \) is the side length of \( Q_t \). Now, if \( x \in U_t \setminus Q_t \) then

\[
\sqrt{n} l_t \leq \text{dist}(Q_t, \partial \Omega) \leq \text{dist}(U_t, \partial \Omega) + \frac{1}{16} \sqrt{n} l_t,
\]

hence, \( \frac{15}{16} \sqrt{n} l_t \leq \text{dist}(U_t, \partial \Omega) \) and

\[
L_t = \frac{9}{8} l_t < \frac{15}{16} \sqrt{n} l_t \leq \text{dist}(U_t, \partial \Omega) \leq d(x).
\]

Then, the validity of \( L_t \leq d(x) \) for all \( x \in U_t \) implies (3.6) for all \( U_t \), where \( \mu(x, y) \) is the function defined in (5.4), and the uniform constant

\[
C_1 = C_{n,p} \tau^{s-n} 2^\beta.
\] (5.5)

Next, by Theorem 2.4, there is a finite \( C \)-decomposition of functions \( \{ g_t \}_{t \in \Gamma} \) subordinate to \( \{ U_t \}_{t \in \Gamma} \) of any function \( g \) in \( W_0 \) which satisfies (2.3) and (2.4). Moreover, using (5.3), it can be seen that

\[
|g_t(x)| \leq |g(x)| + (72K)^n T g(x),
\]

for all \( t \in \Gamma \) and \( x \in U_t \).

Now, \( \omega(x) := d_F^\beta(x) \) fulfills the hypothesis of Lemma 2.7 where the constant in (2.5) is \( C_2 = (3K \sqrt{n})^\beta \). The last assertion uses Lemma 5.2. Thus, the operator \( T \) is continuous from \( L := L^q(\Omega, d_F^{-q\beta}) \) to itself with the norm

\[
\| T \|_{L \to L} \leq 2 \left( \frac{qN}{q - 1} \right)^{1/q} (3K \sqrt{n})^\beta.
\]
Hence,
\[
\sum_{t \in \Gamma} \|g_t\|_{L^q(U_t, d_x^{-q\beta})}^q \\
\leq 2^{q-1} \left\{ \left( \sum_{t \in \Gamma} \int_{U_t} |g(x)|^q d_x^{-q\beta}(x) \right) + (72K)^{qn} \left( \sum_{t \in \Gamma} \int_{U_t} |Tg(x)|^q d_x^{-q\beta}(x) \right) \right\} \\
\leq 2^{q-1} N \left\{ \int_{\Omega} |g(x)|^q d_x^{-q\beta}(x) dx + (72K)^{qn} \int_{\Omega} |Tg(x)|^q d_x^{-q\beta}(x) dx \right\} \\
\leq 2^{q-1} N \left\{ 1 + (72K)^{qn} 2^q \left( \frac{qN}{q-1} \right) (3K^n)\beta \right\} \|g\|_{L^q(\Omega, d_x^{-q\beta})}^q \\
\leq 4^q N^2 (72K)^{qn} \left( \frac{q}{q-1} \right) (3K^n)^{\beta} \|g\|_{L^q(\Omega, d_x^{-q\beta})}^q \\
= 4^q 12^{2n} 72^{qn} (3\sqrt{n})^{\beta} \left( \frac{q}{q-1} \right) K^{q(n+\beta)} \|g\|_{L^q(\Omega, d_x^{-q\beta})}^q.
\]

Therefore, we have a \( C \)-decomposition subordinate to \( \{U_t\}_{t \in \Gamma} \) with constant
\[
C_0 = 4(12)^{2n/q} (72)^{n} (3\sqrt{n})^{\beta} \left( \frac{q}{q-1} \right)^{1/q} K^{n+\beta}.
\]
(5.6)

Finally, inequality (3.10) and the control on the overlapping of the tree covering by \( N = 12^n \) implies (1.2). \( \Box \)

Remark 5.3. Notice that the proof of Theorem 1.1 provides an explicit constant \( C = 2C_0 C_1 \) for inequality (1.2), where \( C_0 \) and \( C_1 \) are described respectively in (5.6) and (5.5).

The next result, similar to Proposition 4.1, follows from the H"older inequality (equivalently, from Minkowski’s integral inequality).

Proposition 5.4. Let \( \rho : \mathbb{R}^n \setminus \{0\} \to \mathbb{R} \) be a positive radial Lebesgue measurable function which is increasing with respect to the radius. Then, the fractional Poincaré type inequality
\[
\|u(x) - u_U\|_{L^p(U)} \leq \frac{\text{diam}(U)^{n/p} \rho(\text{diam}(U))}{|U|^{1/p}} \left( \int_U \int_U |y - x|^n (\rho |y - x|)^p \, dy \, dx \right)^{1/p}
\]
holds for any bounded domain \( U \) in \( \mathbb{R}^n \) and \( 1 < p < \infty \), where \( u_U := \frac{1}{|U|} \int_U u(y) \, dy \).

Proof.
\[
\int_U |u(x) - u_U|^p \, dx = \int_U \left| \frac{1}{|U|} \int_U u(x) - u(y) \, dy \right|^p \, dx \leq \frac{\text{diam}(U)^n \rho(\text{diam}(U))}{|U|} \int_U \int_U |y - x|^n (\rho |y - x|)^p \, dy \, dx.
\]

Remark 5.5. If \( \rho(x) = |x|^s \), with \( s \in (0, 1) \), we recover the classical fractional Poincaré inequality.
We generalize the fractional Poincaré inequality stated in Theorem 1.1 by replacing
the fractional derivatives given by the power functions $|x|^s$, with $0 < s < 1$, by general
increasing and positive radial functions $\rho(x)$.

**Theorem 5.6.** Let $\Omega$ in $\mathbb{R}^n$ be a bounded John domain and $1 < p < \infty$. Given an
arbitrary compact set $F$ in $\partial \Omega$, a parameter $\beta \geq 0$ and a positive radial Lebesgue
measurable function $\rho : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ increasing with respect to the radius, there exists
a constant $C$ such that

$$\left(\int_{\Omega} |u(x) - u_{\Omega,\omega}|^p d_F^\beta(x) dx\right)^{1/p}$$

$$\leq C \left(\int_{\Omega} \int_{\Omega \cap B(x, d_F(x))} \frac{|u(x) - u(y)|^p}{|x - y|^n (\rho(x - y))^p} [\rho(2d_F(x))]^p d_F^\beta(x) dy dx\right)^{1/p}$$

(5.8)

for all function $u \in L^p(\Omega, d_F^\beta)$. We denote by $d_F(x)$ and $d_F(x)$ the distance from $x$
to $\partial \Omega$ and $F$ respectively, and by $u_{\Omega,\omega}$ the weighted average

$$\frac{1}{d_F^\beta(\Omega)} \int_{\Omega} u(z) d_F^\beta(z) dz.$$

In addition, the constant $C$ in (5.8) can be written as

$$C = C_{n,p,\beta} K^{n+\beta},$$

where $K$ is the geometric constant introduced in (5.1).

**Proof.** This proof mimics the one of Theorem 1.1 with Proposition 5.4 instead of Propo-
sition 4.2. Indeed, we will use again the tree covering $\{U_t\}_{t \in \Gamma}$ of $\Omega$ defined in (5.2)
and the weight $\omega(x) = d_F^\beta(x)$, however, in this case $\mu(x,y)$ is defined as

$$\mu(x,y) := \frac{[\rho(2d_F(x))]^p d_F^\beta(x)}{|x - y|^n (\rho(x - y))^p}.$$

We only have to show that (3.6) is verified on $U_t$, for all $t$, with uniform constant. This
fact follows from (5.7) by using the inequality $\text{diam}(U_t) \leq 2d_F(x)$ for all $x \in U_t$. \qed

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