Another Form for LAMBDA Method

ZHOU Yangmei LIU Jingnan

1 Introduction

Whether the full cycle ambiguities are correctly resolved is key in GPS precise positioning. Generally speaking, the existing methods for resolving GPS ambiguity are roughly divided into three classes: the classical least-square (LS) search method, the ambiguity function method and the integer least-square method. When search method is used, the ambiguity search space is defined on the basis of the variance of the float solution and the integer property of ambiguity[1]. This approach has a very slow search speed and very low calculation efficiency, and it is also difficult to study the statistical properties of the integer solution for ambiguities. The ambiguity function method, by means of the function constructed by the integer property of ambiguity, can determine their integer estimates[2]. This method makes it possible to find the position parameters rapidly in the case of real-time kinematic (RTK) positioning. But its disadvantages are that the discrimination between the proper solution and false solutions is not as robust as the LS search method and that it does not deal with the statistical properties of the integer solution. However, the integer least-squares (LS) method is stricter than the former two[3]. The LAMBDA approach, on the basis of this method, was proposed in 1993 by Teunissen[4-6]. This LAMBDA approach has a very small search space and a very fast resolving speed. Furthermore, its statistical properties can be studied through the rigorous theory. Nowadays a full theory on the probabilities of integer estimation has been established. Hence, in this paper, in accordance with the integer LS method and analysis for the structure of the
LAMBDA approach, we present another form for LAMBDA approach.

We make frequent use of the following symbols. The weighted norm of a vector $\xi$ will be denoted as $\|x\|_{M}$, thus $\|x\|_{M} = x^{T} M^{-1} x$. The $m$-dimensional real space will be denoted as $\mathbb{R}^{m}$ and the $m$-dimensional integer space as $\mathbb{Z}^{m}$.

2 Traditional LAMBDA method

In the light of the GNSS observation system, we have the following mathematical model:

\[ y = Az + Bx + \epsilon \]

\[ E(y) = Az + Bx \]  \hspace{1cm} (1)

where $y$ is the phase observable vector; $z \in \mathbb{Z}^{n}$ is the unknown integer ambiguity vector; $x \in \mathbb{R}^{n}$ is the unknown position parameter vector; $A$ and $B$ are the given coefficient matrices of full column rank; $\epsilon$ is the random vector which contains all residual errors plus the observational noise. By solving the linearized observation Eq. (1) with general least-square adjustment, we can directly obtain real-valued estimates and their corresponding variance-covariance matrices for both the ambiguity vector $z$ and the position parameter vector $x$.

\[ D_{z} = \begin{bmatrix} D_{x} & D_{x} \\ D_{x} & D_{x} \end{bmatrix} \]

(2)

Here, $z \in \mathbb{Z}^{n}$ is the integer estimate satisfying the objective function (3). Since $D_{z}$ is not a diagonal matrix, the search speed with the objective Eq. (3) is very slow. In order to raise the resolving efficiency, in 1993, Teunissen[8-6] proposed a LAMBDA method, which is based on both the sequential conditional LS ambiguity technique and the lower-triangular Cholesky decomposition.

2.1 Sequential conditional LS ambiguity technique

Let the expectation and dispersion of $\xi_{i} = \left( \xi_{1}, \cdots, \xi_{i-1} \right)^{T} \in \mathbb{R}^{i-1}$ and $\xi_{i} \in \mathbb{R}$ be given as:

\[ E(\xi_{i}) = \mu_{i} \]

(5)

Then the LS estimate of $\xi_{i}$, when $\mu_{i}$ is constrained to the fixed vector $z_{i}$, is given as

\[ \xi_{i} = z_{i} - D_{i} D_{i}^{-1} \left( z_{i} - z_{i} \right) \]  \hspace{1cm} (i = 2, \cdots, m)

(3)

Here, $\xi_{i}$ is a shorthand notation for $\xi_{(i-1)\cdots,1}$. The estimate $\xi_{i}$ refers to the conditional LS ambiguity estimate. It is conditioned on fixing the previous ambiguities to the values $z_{j}, j = 1, \cdots, i - 1$. Note that $\xi_{i}$ and $\xi_{i-1}$ are uncorrelated. This is an important property that will be used repeatedly below.

For $i = 2$ in Eq. (3), we have

\[ \xi_{21} = \xi_{2} - \sigma_{21} \sigma_{2}^{2} (\xi_{2} - z_{2}) \]  \hspace{1cm} (7)

where $\xi_{21}$ is uncorrelated with $\xi_{2}$. For $i = 3$, the conditional LS estimate $\xi_{32}$ follows from fixing the two ambiguities $\mu_{3}$ and $\mu_{2}$ to the values $z_{3}$ and $z_{2}$. Note, however, that since $\xi_{32}$ is invariant to any regular transformation of $\xi_{3}$ and $\xi_{2}$, we may as well fix $\xi_{3}$ and $\xi_{2}$ to the values $z_{3}$ and $z_{2}$. At this time, $D_{i} = \text{diag} \{ \sigma_{i}^{2} \}$ is a diagonal matrix. Hence, we may obtain

\[ \xi_{32} = \xi_{3} - \sigma_{3} \sigma_{3}^{2} (\xi_{3} - z_{3}) \]  \hspace{1cm} (8)

Here, $\xi_{32}$ is uncorrelated with both $\xi_{3}$ and $\xi_{2}$. We may continue this process in this way till the last ambiguity, thus the formula of the sequential conditional LS ambiguity is summarized as follows:
\[ \varepsilon_{i|j} = \tilde{z}_i - \sum_{j=1}^{i-1} \sigma_{i,j} \sigma_{j,j}^{-1} (\varepsilon_{j,j} - z_j) \quad (i = 2, \ldots, m) \tag{9} \]

where \( \sigma_{i,j} \) denotes the covariance between \( z_i \) and \( \varepsilon_{j,j} \), and \( \sigma_{j,j} \) is the variance of \( \varepsilon_{j,j} \). For \( i = 1 \), \( \varepsilon_{i|1} \) is just equal to \( \tilde{z}_1 \).

### 2.2 Lower-triangular Cholesky decomposition and traditional LAMBDA method

Let two sides of Eq. (9) minus \( z_i \) simultaneously, we have

\[ (\varepsilon_i - z_i) = (\varepsilon_{i|i} - z_i) + \sum_{j=1}^{i-1} \sigma_{i,j} \sigma_{j,j}^{-1} (\varepsilon_{j,j} - z_j) \quad (i = 2, \ldots, m) \tag{10} \]

In vector-matrix form, it will be

\[
\begin{bmatrix}
\varepsilon_1 - z_1 \\
\varepsilon_2 - z_2 \\
\vdots \\
\varepsilon_m - z_m \\
\end{bmatrix} =
\begin{bmatrix}
1 & \vdots & \vdots \\
1 & \ddots & \vdots \\
l_{m1} & \cdots & 1 \\
\end{bmatrix}
\begin{bmatrix}
\varepsilon_1 - z_1 \\
\varepsilon_2 - z_2 \\
\vdots \\
\varepsilon_m - z_m \\
\end{bmatrix}
\]

\[
(\mathbf{L})_i = \begin{cases} 
1 & \text{for } i = j \\
\sigma_{i,j} \sigma_{j,j}^{-1} & \text{for } 1 \leq j < i \leq m 
\end{cases}
\]

(11)

with \( l_{ij} = \sigma_{i,j} \sigma_{j,j}^{-1} \) (\( 1 \leq j < i \leq m \)). Let \( \mathbf{z} = (\varepsilon_1, \ldots, \varepsilon_m)^T \), \( \mathbf{z}_i = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_i)^T \), and

\[ (\mathbf{L})_i = \begin{cases} 
1 & \text{for } i = j \\
\sigma_{i,j} \sigma_{j,j}^{-1} & \text{for } 1 \leq j < i \leq m 
\end{cases}
\]

(12)

And let \( \mathbf{D} = \text{diag}(\sigma_{1,1}, \sigma_{2,2}, \ldots) \). Since the sequential conditional LS ambiguities are mutually uncorrelated, their variance-covariance matrix is diagonal. When applying the error propagation law to Eq. (11), the variance-covariance matrix of \( \mathbf{z} \) gives a lower-triangular decomposition. Therefore, \( \mathbf{z} \) and \( \mathbf{z}_i \), and their variance-covariance matrices are related as

\[ \mathbf{z} - \mathbf{z} = \mathbf{L}(\mathbf{z} - \mathbf{z}) \quad \text{and} \quad \mathbf{D}_i = \mathbf{L} \mathbf{D} \mathbf{L}^T \tag{12} \]

According to Eq. (12), the objective Eq. (3) is transformed into the following minimizing problem:

\[ \min_{\mathbf{z}} \| \mathbf{z} - \mathbf{z} \|_2 = \min_{\mathbf{z}} \sum_{i=1}^{m} (\varepsilon_{i|1} - z_i) / \sigma_{i|i} \tag{13} \]

Through continuously searching for satisfying the objective Eq. (13), we may obtain the integer LS estimate \( \mathbf{z} \in \mathbb{I}^{m} \) of ambiguity vector \( \mathbf{z} \).

### 3 Another form for LAMBDA method

In fact, the LAMBDA method proposed by Teunissen has another form. This new form is based on both the back-sequential conditional LS ambiguity technique and the upper-triangular Cholesky decomposition.

#### 3.1 Back-sequential conditional LS ambiguity technique

Let the expectation and dispersion of \( \varepsilon_i = (\varepsilon_{i|2}, \ldots, \varepsilon_{i|m})^T \in \mathbb{R}^{m-i} \) and \( \varepsilon_i \in \mathbb{R} \) be given as:

\[
E\left[ \begin{bmatrix} \varepsilon_i \\
\mu_i \\
\end{bmatrix} \right] = \begin{bmatrix} \mu_i \\
\mu_i \\
\end{bmatrix},
D\left[ \begin{bmatrix} \varepsilon_i \\
\mu_i \\
\end{bmatrix} \right] = \begin{bmatrix} \sigma_i & D_{i2} \\
D_{i1} & D_{i1} \\
\end{bmatrix}
\]

(14)

Then the LS estimate of \( \varepsilon_i \), when \( \mu_i \) is constrained to the fixed vector \( \varepsilon_i \), is given as

\[ \tilde{z}_{i|i} = \tilde{z}_i - D_{i1} D_{i1}^{-1} (\tilde{z}_i - \varepsilon_i) \quad (i = m-1, \ldots, 1) \tag{15} \]

Here, \( \varepsilon_{i|j} \) denotes a shorthand notation for \( \tilde{z}_{i|j} \). The estimate \( \tilde{z}_{i|i} \) is conditioned on fixing the following ambiguities to the values \( z_j, j = i+1, \ldots, m \). Note that likewise \( \tilde{z}_{i|i} \) and \( \varepsilon_i \) are uncorrelated and that although Eq. (15) has the same form as Eq. (6), some of their symbols have different meanings.

For \( i = m-1 \) in Eq. (15), we have

\[ \tilde{z}_{(m-1)|m} = \tilde{z}_{m-1} - \sigma_{m-1|m} \sigma_{m-1|m}^{-1} (\varepsilon_{m} - z_m) \quad (m-1) \tag{16} \]

in which \( \tilde{z}_{(m-1)|m} \) is uncorrelated with \( z_m \). For \( i = m-2 \), the conditional LS estimate \( \tilde{z}_{(m-2)|m} \) always follows from fixing the two ambiguities \( \mu_{m} \) and \( \mu_{m-1} \) to the values \( z_m \) and \( z_{m-1} \). Note, however, that since \( \tilde{z}_{(m-2)|m} \) is invariant to any regular transformation of \( z_m \) and \( z_{m-1} \), we may fix \( z_m \) and \( z_{m-1} \) to the values \( z_m \) and \( z_{m-1} \). At this time, \( D_i = \text{diag}(\sigma_i, \sigma_{i-1|m}) \) is diagonal. Hence, we may obtain

\[ \tilde{z}_{(m-2)|m} = \tilde{z}_{m-2} - \sigma_{m-2|m} \sigma_{m-2|m}^{-1} (\varepsilon_{m} - z_m) \quad (m-2) \tag{17} \]

Here, \( \tilde{z}_{(m-2)|m} \) is uncorrelated with both \( z_m \) and \( z_{m-1} \). We may continue this process in this way till the first ambiguity, thus the formula of the back-sequential conditional LS ambiguity is summarized as follows:

\[ \tilde{z}_{i|i} = \tilde{z}_i - \sum_{j=i+1}^{m} \sigma_{i,j} \sigma_{j,j}^{-1} (\varepsilon_{j,j} - z_j) \quad (i = m-1, \ldots, 1) \tag{18} \]

where \( \sigma_{i,j} \) denotes the covariance between \( z_i \) and \( \varepsilon_{j,j} \), and \( \sigma_{j,j} \) is the variance of \( \varepsilon_{j,j} \). For
3.2 Upper-triangular Cholesky decomposition and another form for LAMBDA method

Likewise let two sides of Eq. (18) minus \( z_i \) simultaneously, it will be

\[
(z_i - z_i) = (z_{ij} - z_i) + \sum_{j=m}^{i} \sigma_{ij} \sigma_{ij}^{-1} (z_{ji} - z_i)
\]

\((i = m - 1, \ldots, 1)\) (19)

In vector-matrix form, we have

\[
\begin{bmatrix}
\hat{z}_1 - z_1 \\
\hat{z}_2 - z_2 \\
\vdots \\
\hat{z}_m - z_m
\end{bmatrix} =
\begin{bmatrix}
1 & u_{i2} & \cdots & u_{im} \\
& 1 & \cdots & u_{jm} \\
& & \ddots & \vdots \\
& & & 1
\end{bmatrix}
\begin{bmatrix}
\hat{z}_{11} - z_1 \\
\hat{z}_{12} - z_2 \\
\vdots \\
\hat{z}_{1m} - z_m
\end{bmatrix}
\]

\[= (1 \leq i < j \leq m)\] (20)

with \( u_{ij} = \sigma_{ij} \sigma_{ij}^{-1} \) (1 \leq i < j \leq m). Let \( \hat{z} = (\hat{z}_1, \ldots, \hat{z}_m)^T \), \( \hat{z}_i = (\hat{z}_{i1}, \hat{z}_{i2}, \ldots, \hat{z}_{im})^T \),

and

\[
(U)_\sigma = \begin{cases}
0 & \text{for } 1 \leq j < i \leq m \\
1 & \text{for } i = j \\
\sigma_{ij} \sigma_{ij}^{-1} & \text{for } 1 \leq i < j \leq m
\end{cases}
\]

And let \( D_x = \text{diag}(\sigma_{11}, \ldots, \sigma_{mm}) \). Since the back-sequential conditional LS ambiguities are mutually uncorrelated, their variance-covariance matrix is diagonal. When applying the error propagation law to Eq. (20), the variance-covariance matrix of \( \hat{z} \) gives a upper-triangular decomposition. Therefore, \( \hat{z} \) and \( \hat{z} \), and their variance-covariance matrices are related as

\[
\hat{z} - z = U (\hat{z}_i - z) \quad \text{and} \quad D_i = UD_{x}U^T (21)
\]

According to Eq. (21), the objective Eq. (3) is transformed into the following minimizing problem:

\[
\text{min}_{\hat{z}} \| \hat{z} - z \|_{\sigma} = \text{min}_{\hat{z}} \sum_{i=1}^{m} (\hat{z}_{ii} - z_i) / \sigma_{ii}
\]

(22)

Note that although Eq. (22) has the same form as Eq. (13), some of their symbols have different meanings. Through continuously searching for satisfying the objective Eq. (22), we may obtain the integer LS solution \( \hat{z} \in \mathbb{Z}^m \) of ambiguity vector \( z \).

4 Some remarks

The above two LAMBDA methods are both such methods that by it, through transforming the variance-covariance of ambiguity float solution into a diagonal matrix, the speed on resolving the integer LS estimate \( \hat{z} \in \mathbb{I}^m \) of ambiguity vector \( z \) is raised. In practical calculation, not according to the above process of sequential conditional LS technique or of back-sequential conditional LS technique, we perform the traditional LAMBDA method or ALAMBDA method. But through the lower-triangular or upper-triangular Cholesky decomposition for the variance-covariance matrix of \( \hat{z} \), we obtain the unit lower-triangular matrix \( L \) and the corresponding diagonal matrix \( D \) or the unit upper-triangular matrix \( U \) and the corresponding diagonal matrix \( D_x \). This is expressed as:

\[
D = LDL^T \quad \text{or} \quad D_x = UD_xU^T
\]

Thus according to Eq. (13) or Eq. (22), we can obtain the integer LS solution for the ambiguity vector.

5 Numerical results

For the comparison between the above two methods, we give an example in the case of short baseline. With the LAMBDA method and the new method proposed in this paper, we process the observations that include \( L_1 \) and \( L_2 \) phases at the same time. The epoch number is 31. The calculation results by the LAMBDA method are shown in Fig. 1 and those of the new method in Fig. 2. According to these two figures, it is found that five epochs are needed when the correct ambiguity is resolved by the LAMBDA method, while four epochs are needed by the new method. This indicates that the new LAMBDA method has equivalent calculation efficiency to the LAMBDA method. In addition, we also find that these two methods have the same fixed baseline solution after the correct ambiguity solution is resolved. This indicates that this new LAMBDA method is completely correct.
is based on both the lower-triangular Cholesky decomposition and the sequential conditional LS technique, while another form for LAMBDA method proposed in this paper is based on both the upper-triangular Cholesky decomposition and the back-sequential conditional LS technique. Their sequence resolving ambiguities are just opposite. Theoretically speaking, both these two methods belong to integer LS method. They have identical principle and equivalent calculation efficiency. Therefore the integer LS solution of these two methods should be identical.

References
1 Frei E, Beutler G (1990) Rapid static positioning based on the fast ambiguity resolution approach “FARA”, theory and first results. Manuscript Geodesy, 15:325-356
2 Mader G L (1992) Rapid static and kinematic global positioning system solutions using the ambiguity function technique. Journal of Geophysics Research, 97(B3):3 271-3 283
3 Teunissen P J G (1993) Least-squares estimation of the integer GPS ambiguities. IAG General Meeting, Beijing.
4 Teunissen P J G (1995) The least-squares ambiguity decorrelation adjustment, a method for fast GPS integer ambiguity estimation. Journal of Geodesy, 70:65-82
5 Jonge P, Tiberius C (1996) The LAMBDA method for integer ambiguity estimation: implementation aspects. Tech. Rep., Delft Geodetic Computing Center, Delft University of Technology, Delft.
6 Teunissen P J G (1994) A new method for fast carrier phase Ambiguity Estimation. PLANS’94, Las Vegas.

6 Conclusions

The LAMBDA method proposed by Teunissen

Fig. 1 Comparison of the float solution with the fixed solution for LAMBDA method

Fig. 2 Comparison of the float solution with the fixed solution for the new method