Diffeomorphisms with a Generalized Lipschitz Shadowing Property

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Abstract. Shadowing property and structural stability are important dynamics with close relationship. Pilyugin and Tikhomirov proved that Lipschitz shadowing property implies the structural stability[5]. Todorov gave a similar result that Lipschitz two-sided limit shadowing property also implies structural stability for diffeomorphisms[10]. In this paper, we define a generalized Lipschitz shadowing property which unifies these two kinds of Lipschitz shadowing properties, and prove that if a diffeomorphism $f$ of a compact smooth manifold $M$ has this generalized Lipschitz shadowing property then it is structurally stable. The only if part is also considered.

1. Introduction. In differentiable dynamical systems, one of the main goals is to study the structural stability of the orbit structures for given systems. A system is said to be structurally stable if the orbit structure is unchanged by a small perturbation, that is, the perturbed system is conjugate to the original system.

The shadowing property is closely related to the structural stability. Roughly speaking, a diffeomorphism $f$ has the shadowing property if, every pseudo orbit in the manifold can be shadowed by a real orbit. In the last few years, dynamicists have studied various shadowing properties, and its relationship between structural stability and hyperbolicity ([8, 6, 9, 12]).

Let $M$ be a compact smooth Riemannian manifold without boundary and $f : M \to M$ be a diffeomorphism. Recall that $f$ is said to be $C^1$-structurally stable if there exists a neighborhood $U$ of the diffeomorphism $f$ in the $C^1$-topology such

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that any diffeomorphism \( g \in \mathcal{U} \) is topologically conjugate to \( f \), i.e., there is a homeomorphism \( h : M \to M \) such that \( h \circ f = g \circ h \).

Now let’s recall the classical Lipschitz shadowing property. Given \( d > 0 \), a sequence \( \{x_i\}_{i \in \mathbb{Z}} \) of points in \( M \) is called a \( d \)-pseudo orbit of \( f \) if the following inequality is satisfied
\[
d(x_{k+1}, f(x_k)) < d,
\]
for all \( k \in \mathbb{Z} \). We say that \( f \) has the Lipschitz shadowing property on a compact invariant set \( \Lambda \), if there exist constants \( L, d_0 > 0 \) with the following property: for any \( 0 < d \leq d_0 \) and any \( d \)-pseudo orbit \( \{x_i\}_{i \in \mathbb{Z}} \) in \( \Lambda \), there exists a point \( p \in M \) such that
\[
d(x_k, f^k(p)) \leq Ld, \quad \forall \ k \in \mathbb{Z}.
\]
It is well known that every compact hyperbolic invariant set has the Lipschitz shadowing property. Pilyugin and Tikhomirov ([5]) showed that the Lipschitz shadowing property on the whole manifold is equivalent to the structural stability.

As a consequence of Pilyugin and Tikhomirov’s work, Todorov ([10]) introduced a notion of Lipschitz two-sided limit shadowing property. Given a nonnegative real number \( \gamma \), we say that a sequence \( \{x_i\}_{i \in \mathbb{Z}} \) of points in \( M \) is a \( \gamma \)-weighted \( d \)-pseudo orbit of \( f \) if
\[
d(x_{k+1}, f(x_k)) < \frac{d}{(|k| + 1)^\gamma}, \quad \forall \ k \in \mathbb{Z}.
\]
A diffeomorphism \( f \) has Lipschitz two-sided limit shadowing property with exponent \( \gamma \) if there exist positive constants \( d_0, L \) such that for any \( 0 < d \leq d_0 \) and any \( \gamma \)-weighted \( d \)-pseudo orbit \( \{x_k\} \), there exists a point \( p \in M \) such that
\[
d(x_k, f^k(p)) \leq \frac{Ld}{(|k| + 1)^\gamma}, \quad \forall \ k \in \mathbb{Z}.
\]
Todorov showed that a diffeomorphism \( f \) is structurally stable if and only if \( f \) has Lipschitz two-sided limit shadowing property.

In this paper, we introduce a new notion of Lipschitz shadowing property by using a function \( \zeta \) which is called by slowly decreasing bounded sequence. It is a generalized notion than the definitions mentioned in [5] and [10]. And we will show that \( f \) has this kind of Lipschitz shadowing property with respect to \( \zeta \) if and only if \( f \) is structurally stable.

**Definition 1.1.** A function \( \zeta : \mathbb{Z} \to \mathbb{R}^+ \) is called a slowly decreasing bounded sequence if the followings are satisfied:

1. \( \{\zeta(i) : i \in \mathbb{Z}\} \) is bounded, that is, there is \( \kappa > 0 \) such that \( |\zeta(i)| < \kappa \) for all \( i \in \mathbb{Z} \);
2. \( \{\frac{i\zeta(i)}{i+1} : i \in \mathbb{Z}\} \) has positive upper and lower bounds, that is, there are \( 0 < c_1 < c_2 \) such that \( c_1 < \frac{i\zeta(i)}{i+1} < c_2 \) for all \( i \in \mathbb{Z} \);
3. \( \lim_{i \to \infty} \frac{1}{|i|} \log \zeta(i) = 0 \).

Fix \( \zeta : \mathbb{Z} \to \mathbb{R}^+ \) to be a slowly decreasing bounded sequence. Let \( \epsilon > 0 \) be given. We say that a sequence \( \{x_i : i \in \mathbb{Z}\} \) in \( M \) is a \( \zeta \)-weighted \( \epsilon \)-pseudo orbit if
\[
d(f(x_i), x_{i+1}) < \epsilon \cdot \zeta(i+1)
\]
for any \( i \in \mathbb{Z} \).

Now, we define a generalized Lipschitz shadowing property.
Definition 1.2. Fix $\zeta : \mathbb{Z} \to \mathbb{R}^+$ to be a slowly decreasing bounded sequence. We say that $f$ has the Lipschitz shadowing property with respect to $\zeta$ if there are $L > 1$, $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$ and any $\zeta$-weighted $\epsilon$-pseudo orbit $\{x_i\}_{i \in \mathbb{Z}}$, there is a point $y \in M$ such that
\[ d(f^i(y), x_i) < L \epsilon \cdot \zeta(i) \]
for any $i \in \mathbb{Z}$.

In the article, we will show the following theorem.

Theorem 1.3. Let $\zeta : \mathbb{Z} \to \mathbb{R}^+$ be a slowly decreasing bounded sequence. If a diffeomorphism $f$ has the Lipschitz shadowing property with respect to $\zeta$, then $f$ is structurally stable.

It is easy to check that once a bounded function $\zeta : \mathbb{Z} \to \mathbb{R}^+$ has the property
\[ \lim_{i \to \infty} \frac{\zeta(i + 1)}{\zeta(i)} = 1, \]
then it is a slowly decreasing bounded sequence. At the same time, we can easily see that the sequence $\zeta$ defined by $\zeta(i) \equiv 1, (i \in \mathbb{Z})$ or by $\zeta(i) = (|i| + 1)^{-\gamma}, (i \in \mathbb{Z})$ for some $\gamma \geq 0$ is bounded and
\[ \lim_{i \to \infty} \frac{\zeta(i + 1)}{\zeta(i)} = 1. \]

If we choose $\zeta \equiv 1$ then the Lipschitz shadowing property with respect to $\zeta$ is the classical Lipschitz shadowing property studied by Pilyugin and Tikhomirov (5). On the other hand, if we choose $\zeta(i) = (|i| + 1)^{-\gamma}, \forall i \in \mathbb{Z}$ for some fixed $\gamma \geq 0$ then the Lipschitz shadowing property with respect to $\zeta$ is the Lipschitz two-side limit shadowing property studied by Todorov (10).

In the article, we also prove the following theorem which can be seen as a generalization of the results in [5] and [10].

Theorem 1.4. Let $\zeta : \mathbb{Z} \to \mathbb{R}^+$ be any given bounded function with
\[ \lim_{i \to \infty} \frac{\zeta(i + 1)}{\zeta(i)} = 1. \]
A diffeomorphism $f$ has the Lipschitz shadowing property with respect to $\zeta$ if and only if $f$ is structurally stable.

2. Preliminary. Let $n$ be the dimension of manifold $M$. Let
\[ \mathcal{A} = \{A_k : k \in \mathbb{Z}\} \]
be a sequence of linear isomorphisms $A_k : E_k \to E_{k+1} (k \in \mathbb{Z})$ where $E_k$ are $n$-dimensional Euclidean spaces. In the following, we only consider the sequence $\mathcal{A} = \{A_k : k \in \mathbb{Z}\}$ with
\[ \sup_{k \in \mathbb{Z}} \max\{\|A_k\|, \|A_k^{-1}\|\} < \infty, \]
that is, there exists a constant $C_0 > 1$ such that $\max\{\|A_k\|, \|A_k^{-1}\|\} \leq C_0$ for any $k \in \mathbb{Z}$.

We say that the sequence $\mathcal{A} = \{A_k : k \in \mathbb{Z}\}$ has the Perron property if for any bounded sequence $\{f_k \in E_k : k \in \mathbb{Z}\}$, the equation
\[ x_{k+1} = A_k x_k + f_{k+1}, \quad k \in \mathbb{Z} \]
has a bounded solution.

The following proposition is a combination of Theorem 2.1.1 and Theorem 2.1.2 in [6] which was firstly proved by Pliss.

**Proposition 2.1** (Pliss [7]). If the sequence \( A = \{A_k : k \in \mathbb{Z}\} \) has the Perron property, then there are two subspaces \( U_0^-, S_0^+ \subset E_0 \) with the following properties:

(i) \( U_0^- + S_0^+ = \mathbb{R}^n \);

(ii) there are constants \( C \geq 1 \) and \( 0 < \lambda < 1 \) such that

\[
\|A_{k-1} \circ A_{k-2} \circ \cdots \circ A_0|_{S_0^+}\| \leq C\lambda^k,
\]

\[
\|A_{-1}^{-1} \circ A_{-k+1}^{-1} \circ \cdots \circ A_{-1}^{-1}|_{U_0^-}\| \leq C\lambda^k
\]

for all \( k \geq 1 \).

Denote by

\[
D^s(x) = \{v \in T_xM : \lim_{k \to +\infty} \|Df^k(x)v\| = 0\},
\]

\[
D^u(x) = \{v \in T_xM : \lim_{k \to -\infty} \|Df^k(x)v\| = 0\}
\]

for all \( x \in M \).

The following theorem was proved by Mañé [2] and Liao [1] independently. One can also find a detailed proof in the chapter 6 of [11].

**Proposition 2.2** (Mañé [2]). If \( D^s(x) + D^u(x) = T_xM \) for all \( x \in M \), then \( f \) is \( C^1 \)-structurally stable.

The converse of the above theorem is also true and it is a positive answer to the stability conjecture in the \( C^1 \) case. Here we have the following equivalent statement.

**Proposition 2.3** (Mañé [3]). If \( f \) is \( C^1 \)-structurally stable, then there are \( D^s(x), D^u(x) \subset T_xM \) for any \( x \in M \) with the following properties:

1. \( D^s(x) + D^u(x) = T_xM \) for all \( x \in M \);

2. there are \( C \geq 1 \) and \( \lambda \in (0, 1) \) such that for any \( x \in M \) and \( n \geq 1 \), one has

\[
\|Df^n|_{D^s(x)}\| \leq C\lambda^n,
\]

\[
\|Df^{-n}|_{D^u(x)}\| \leq C\lambda^n.
\]

3. **Proof of Theorem 1.3.** Let \( \zeta : \mathbb{Z} \to \mathbb{R}^+ \) be a slowly decreasing bounded sequence. We will show that if \( f \) has the Lipschitz shadowing property with respect to \( \zeta \) then for any \( x \in M \), the sequence

\[
\{A_i : T^iM \to T^{i+1}M, i \in \mathbb{Z}\}
\]

defined by

\[
A_i = \frac{\zeta(i)}{\zeta(i+1)}Df^i(x), \text{ for any } i \in \mathbb{Z}
\]

has the Perron property. By the Proposition 2.1, we can get two subspaces \( U_0^- \) and \( S_0^+ \) in \( T_xM \), then by proving that \( U_0^- \subset D^u(x), S_0^+ \subset D^s(x) \) we can end the proof of Theorem 1.3.

**Lemma 3.1.** Let \( A = \{A_k : E_k \to E_{k+1} : k \in \mathbb{Z}\} \) be a sequence of isomorphisms on \( n \)-dimensional Euclidean spaces with \( \sup_{k \in \mathbb{Z}}(\max\{\|A_k\|, \|A_k^{-1}\|\}) < \infty \). If there is a constant \( K > 0 \) such that for any positive integer \( j \) and any sequence \( \{w_k \in E_k : -j < k \leq j + 1\} \) with \( \|w_k\| < 1 \), there is a sequence of \( \{z_k : -j \leq k \leq j + 1\} \) such that
(i) $z_{k+1} = A_kz_k + w_{k+1}, -j \leq k \leq j$;
(ii) $\|z_k\| \leq K, -j \leq k \leq j + 1$,
then $A$ has the Perron property.

Proof. Noting that once $(\cdots, x_{-1}, x_0, x_1, \cdots)$ is a solution of equations $x_{k+1} = A_kx_k + w_{k+1}$, then $(\cdots, ax_{-1}, ax_0, ax_1, \cdots)$ is also a solution of equations $x_{k+1} = A_kx_k + aw_{k+1}$, we just need to show that for any sequence $\{w_k \in E_k : k \in \mathbb{Z}\}$ with all $\|w_k\| < 1$, equations $x_{k+1} = A_kx_k + w_{k+1}$ ($k \in \mathbb{Z}$) has a bounded solution. Let $\{w_k : k \in \mathbb{Z}\}$ with $\|w_k\| < 1$ be given. For any $j \in \mathbb{Z}^+$, by applying hypothesis of the lemma, we have a sequence $\{z_k^j : -j \leq k \leq j + 1\}$ with the properties

1. $z_{k+1}^j = A_kz_k^j + w_{k+1}$
2. $\|z_k^j\| \leq K$, for any $-j \leq k \leq j + 1$.

For any $k \in \mathbb{Z}$, we have a sequence $\{z_k^j : j = |k|, |k| + 1, \cdots \}$. We can find a sequence of natural numbers $n_1 < n_2 < \cdots$ such that

$$\lim_{l \to \infty} z_k^{n_l} = x_k$$

exists for every $k \in \mathbb{Z}$.

It is easy to see that

1. $x_{k+1} = A_kx_k + w_{k+1}$; and
2. $\|x_k\| \leq K$, for all $k \in \mathbb{Z}$,

and then $(\cdots, x_{-1}, x_0, x_1, \cdots)$ is a bounded solution of equations $x_{k+1} = A_kx_k + w_{k+1}$. This shows that $A$ has the Perron property. \hfill \Box

Now we fix $x \in M$ and let $\{A_i : T^{f_{i+1}}_xM \to T^{f_i}_xM, i \in \mathbb{Z}\}$ be the sequence defined by

$$A_i = \frac{\zeta(i)}{\zeta(i+1)} D_{f_{i+1}} f, \quad \text{for any } i \in \mathbb{Z}. $$

Denote by

$$B(y, r) = \{z \in M : d(y, z) < r\}$$

and

$$T_x y M(r) = \{v \in T_x y M : \|v\| < r\}$$

for $r > 0$ and $y \in M$. Let $\exp_y$ be the exponential map at $y \in M$. We can find $r_0 > 0$ such that for any $y \in M$, one has

(i) $\exp_y : T_y y M(r_0) \to B(y, r_0)$ is a diffeomorphism,
(ii) $d(\exp_y(v), \exp_y(w)) \leq 2\|v - w\|$ for any $v, w \in T_y y M(r_0)$; and
(iii) $\|\exp_y^{-1}(z_1) - \exp_y^{-1}(z_2)\| \leq 2d(z_1, z_2)$ for any $z_1, z_2 \in B(y, r_0)$.

Note that we assume $\{\zeta(i) : i \in \mathbb{Z}\}$ is bounded by a constant $\kappa$. Let $r = r_0/\kappa$. Then for any $x \in M$, the map

$$F_k : T^{f_{k+1}}_xM(r) \to T^{f_{k+1}}_xM$$

can be defined by

$$F_k(v) = \exp^{-1}_{f_{k+1} x}(f(\exp_{f_{k+1} x} (\zeta(k) v)))$$

for any $k \in \mathbb{Z}$. Clearly, we have

$$DF_k(0) = \frac{\zeta(k)}{\zeta(k+1)} D_{f_{k+1} x} f = A_k.$$
Since $f$ is $C^1$ and $M$ is compact we know that there is a bound for $\{|D_x f|, |D_x f^{-1}| : x \in M\}$. By the assumption that $\{\frac{\zeta(i)}{\zeta(k)} : i \in \mathbb{Z}\}$ has positive lower and upper bounds, we know that $\{\max\{|A_i|, |A_i^{-1}|\} : i \in \mathbb{Z}\}$ is bounded, that is, there is $C_0 > 1$ such that for any $i \in \mathbb{Z}$, one has $|A_i| \leq C_0$ and $|A_i^{-1}| \leq C_0$.

By the chain rule, it is easy to check that

$$D_v F_k = \frac{\zeta(k)}{\zeta(k+1)} D_f(exp_{f^{-1}}(\zeta(k)v)) \exp_{f}^{-1} D_{exp_{f^{-1}}(\zeta(k)v)} f \circ D_{\zeta(k)v} \exp f.$$ 

Since $\frac{\zeta(k)}{\zeta(k+1)}$ and $\zeta(k)$ are bounded and $D_v f$ varies uniformly continuously on $M$, for any $\mu > 0$, there is $\delta > 0$ depending on the bounds of $\frac{\zeta(k)}{\zeta(k+1)}$ and $\zeta(k)$ and $f$, such that for any $k \in \mathbb{Z}$ and any $v \in T_{f(k)} M$ with $|v| < \delta$, one has $|D_v F_k - A_k| < \mu$.

By the generalized mean value theorem, we know that for any $k \in \mathbb{Z}$ and any $v \in T_{f(k)} M$ with $|v| < \delta$, one has $|D_v F_k(v) - A_k(v)| < \mu |v|$.

In the followings, we assume that $f$ has the Lipschitz shadowing property with respect to $\zeta$ associated with constants $L, \epsilon_0$.

**Lemma 3.2.** There is a constant $K = 8L+1 > 0$ such that for any positive integer $j$ and any sequence $\{w_k \in T_{f(k)} M : -j < k \leq j+1\}$ with $\|w_k\| < 1$, there is a sequence $\{z_k : -j \leq k \leq j+1\}$ such that

1. $z_{k+1} = A_k z_k + w_{k+1} - j \leq k \leq j$,
2. $\|z_k\| \leq K$ for all $-j \leq k \leq j+1$.

**Proof.** From the previous discussion we know that there is $C_0$ only depending on $f$ and $\zeta$ such that $\max\{|A_k|, |A_k^{-1}|\} < C_0$ for any $k \in \mathbb{Z}$. Now let $Q = C_0^{2j+1} + C_0^{2j-1} + \cdots + C_0 + 1$ and then take $\mu_1 > 0$ such that $Q \mu_1 < 1$ and then $\mu_1 > \mu > 0$ such that $(8L + Q) \mu \leq \mu_1$. There is $\delta > 0$ such that for any $x \in M$ and any $k \in \mathbb{Z}$ and any $v \in T_{f(k)} M$ with $|v| < \delta$, one has $|F_k(v) - A_k(v)| < \mu |v|$.

Let $\{w_k : -j < k \leq j+1\}$ with $\|w_k\| < 1 (-j < k \leq j+1)$ be a given sequence. We define a sequence $\{\Delta_k \in T_{f(k)} M : -j < k \leq j+1\}$ inductively as following

$$\Delta_{-j} = 0, \Delta_{k+1} = A_k \Delta_k + w_{k+1}$$

for $-j \leq k \leq j$.

By the fact $|A_k| < C_0$ for all $-j \leq k \leq j$ we can see that

$$|\Delta_{-j+1}| = |A_{-j} \Delta_{-j} + w_{-j+1}| \leq 1,$n

$$|\Delta_{-j+2}| \leq |A_{-j+1} \Delta_{-j+1}| + |w_{-j+2}| \leq C_0 + 1,$n

and then

$$|\Delta_{-j+i}| \leq C_0^{i-1} + C_0^{i-2} + \cdots + 1,$n

for all $1 \leq i \leq 2j+1$ inductively. Hence we have

$$|\Delta_k| \leq C_0^{2j+1} + C_0^{2j-1} + \cdots + 1 = Q$$

for all $-j \leq k \leq j+1$. Let $d > 0$ be chosen with $d(8L + Q) < \delta$ and $4d \epsilon_0$. We can define a sequence $\{x_k \in M : k \in \mathbb{Z}\}$ by letting

$$x_k = \begin{cases} f^k, & \text{if } k < -j, \\ \exp_{f^k}(d\zeta(k) \Delta_k), & \text{if } -j \leq k \leq j+1, \\ f^{k-j-1}(\exp_{f^{k+1}}(d\zeta(j+1) \Delta_{j+1})), & \text{if } k > j+1. \end{cases}$$

**Claim.** $d(f(x_k), x_{k+1}) \leq 4d \zeta(k+1)$ for all $k \in \mathbb{Z}$.
Proof. Note that $d(f(x_k), x_{k+1}) = 0$ if $k < -j$ and $k > j + 1$. Now we consider the case of $-j \leq k \leq j + 1$. By the choice of $d$ we have

\[ d(f(x_k), x_{k+1}) < 2\|\exp_{f_{k+1}}^{-1}(f(x_k)) - \exp_{f_k}^{-1}(x_{k-1})\| \]

\[ = 2\|\zeta(k + 1)F_k(\zeta^{-1}(k)\exp_{f_k}(x_k)) - d\zeta(k + 1)\| \]

\[ = 2\|\zeta(k + 1)F_k(\zeta^{-1}(k)\exp_{f_k}(x_k)) - d\zeta(k + 1)\Delta_k\| \]

\[ = 2\zeta(k + 1)\cdot \|F_k(d\Delta_k) - d\Delta_k\| \]

\[ = 2\zeta(k + 1)\cdot \|F_k(d\Delta_k) - A_k(d\Delta_k) + A_k(d\Delta_k) - d\Delta_k\| \]

\[ \leq 2\zeta(k + 1)\cdot (\|F_k(d\Delta_k) - A_k(d\Delta_k)\| + d\|A_k(d\Delta_k) - d\Delta_k\|) \]

\[ \leq 2\zeta(k + 1)\cdot (\mu dQ + d) = 2d(1 + \mu Q)\zeta(k + 1) \]

\[ < 4d\zeta(k + 1). \]

This ends the proof of Claim. \(\square\)

From the above claim we know that $\{x_k\}_{k \in \mathbb{Z}}$ is a $\zeta$-weighted $4d$-pseudo orbit. Since we have chosen $4d < \epsilon_0$, by the assumption that $f$ has Lipschitz shadowing property with respect to $\zeta$ we know that there is a point $y \in M$ such that

\[ d(x_k, f^k(y)) < 4Ld\zeta(k) \]

for all $k \in \mathbb{Z}$. Now take $t_k = (d\zeta(k))^{-1}\exp_{f^k_y}^{-1}(f^k(y))$ for all $-j \leq k \leq j + 1$.

Claim. $\|t_k - \Delta_k\| \leq 8L$ for all $-j \leq k \leq j + 1$.

Proof. We can check that

\[ \|t_k - \Delta_k\| = \|(d\zeta(k))^{-1}\exp_{f^k_y}^{-1}(f^k(y)) - \Delta_k\| \]

\[ = \|(d\zeta(k))^{-1}\exp_{f^k_y}^{-1}(f^k(y)) - (d\zeta(k))^{-1}\exp_{f^k_y}^{-1}(x_k)\| \]

\[ = (d\zeta(k))^{-1}\|\exp_{f^k_y}^{-1}(f^k(y)) - \exp_{f^k_y}^{-1}(x_k)\| \]

\[ \leq 2(d\zeta(k))^{-1}d(f^k(y), x_k) \]

\[ \leq 8L. \]

This ends the proof of the claim. \(\square\)

Let $\{b_k : -j \leq k \leq j + 1\}$ be defined by $b_{-j} = t_{-j}$, $b_{k+1} = A_kb_k$ for all $-j \leq k \leq j$. Denote by $c_k = t_k - b_k$ for all $-j \leq k \leq j + 1$.

Claim. $\|c_k\| < 1$ for all $-j \leq k \leq j + 1$.

Proof. At first, let us calculate $\|t_{k+1} - A_k t_k\|$. For any $-j \leq k \leq j$, we have

\[ \|t_{k+1} - A_k t_k\| = \|(d\zeta(k + 1))^{-1}\exp_{f^{k+1}_y}^{-1}(f^{k+1}(y)) - A_k(t_k)\| \]

\[ = \|(d\zeta(k + 1))^{-1}\exp_{f^{k+1}_y}^{-1}(f^{k+1}(y)) - A_k(t_k)\| \]

\[ = \|d^{-1}F_k((\zeta(k))^{-1}\exp_{f^k_y}(f^k(y)) - d^{-1}A_k(dt_k))\| \]

\[ = \|d^{-1}F_k(dt_k) - d^{-1}A_k(dt_k)\| \]

\[ \leq d^{-1}\mu\|dt\| \]

\[ = \mu\|t\| \leq \mu(\|t - \Delta_k\| + \|\Delta_k\|) \leq \mu(8L + Q) \leq \mu_1. \]

Note $c_k = t_k - b_k$, $-j \leq k \leq j + 1$. Denoted by $\theta_k = t_{k+1} - A_k t_k$, $-j \leq k \leq j$, then we have
(1) \( \|\theta_k\| \leq \mu_1 \) for all \(-j \leq k \leq j\);
(2) \( c_{k+1} = t_{k+1} - b_{k+1} = A_k c_k + \theta_k, -j \leq k \leq j \).

By the fact that \( \|A_k\| \leq C_0 \) for all \( k \in \mathbb{Z} \), we can check that \( c_{-j} = 0, \|c_{-j+1}\| = \|A_{-j}c_{-j} + \theta_{-j}\| = \|\theta_{-j}\| \leq \mu_1, \|c_{-j+2}\| = \|A_{-j+1}c_{-j+1} + \theta_{-j+1}\| \leq C_0\mu_1 + \mu_1, \ldots, \|c_{-j+k}\| \leq (C_0^{j-1} + \cdots + 1)\mu_1, \ldots, \|c_{j+1}\| \leq (C_0^{j} + \cdots + 1)\mu_1 \). By the choice of \( \mu_1 \) we know that \( \|c_k\| < 1 \) for all \(-j \leq k \leq j+1 \). The claim is proved. \( \square \)

Now take \( z_k = \Delta_k - b_k \) for \(-j \leq k \leq j+1 \). By the fact that \( \Delta_{k+1} = A_k \Delta_k + w_{k+1} \) and \( b_{k+1} = A_k b_k \) we can see that \( z_{k+1} = A_k z_k + w_{k+1}, -j \leq k \leq j \). By the fact that \( \|\Delta_k - t_k\| < 8L \) and \( \|t_k - b_k\| < 1 \) we know that \( \|z_k\| < 8L+1, -j \leq k \leq j+1 \). This ends the proof of Lemma 3.2. \( \square \)

**End of the proof of Theorem 1.3.**

Remind that \( A_k = \frac{\zeta(k)}{\zeta(k+1)} D_{f,x} f \) for all \( k \in \mathbb{Z} \). From Lemma 3.1 and 3.2 we know that \( \{A_k : k \in \mathbb{Z}\} \) satisfies Perron property. By applying Pliss’ theorem, there are two subspaces \( U_0^{-}, S_0^{+} \subset \mathcal{T}_Z M \) such that

1. \( U_0^{-} + S_0^{+} = \mathcal{T}_Z M \),
2. there are \( C > 0 \) and \( \lambda \in (0,1) \) such that

\[
\|A_{k-1} \circ A_{k-2} \circ \cdots \circ A_0|_{S_0^{+}} \| \leq C\lambda^k,
\]

\[
\|A_{k-1} \circ A_{k+1}^{-1} \circ \cdots \circ A_{-1}^{-1}|_{U_0^{-}} \| \leq C\lambda^k
\]

for all \( k \geq 0 \).

**Claim.** \( S_0^{+} \subset D^s(x), U_0^{-} \subset D^u(x) \) for any \( x \in M \).

**Proof.** Let \( v \in S_0^{+} \). We have \( \|A_{k-1} \circ \cdots \circ A_0(v)\| \leq C\lambda^k\|v\| \). By the relationship between \( A_k \) and \( D_{f,x} f \) we can see

\[
Df^k(x)(v) = D_{f_{k-1}x} \circ D_{f_{k-2}x} \circ \cdots \circ D_x f(v) = \frac{\zeta(k)}{\zeta(k-1)} A_{k-1} \circ \frac{\zeta(k-1)}{\zeta(k-2)} A_{k-2} \circ \cdots \circ \frac{\zeta(1)}{\zeta(0)} A_0(v)
\]

Hence we have

\[
\|Df^k(x)(v)\| \leq \frac{\zeta(k)}{\zeta(0)} C\lambda^k\|v\|
\]

for all \( k \geq 0 \). By the assumption that \( \{\zeta(k)\} \) is bounded, we can see that

\[
\|Df^k(x)(v)\| \to 0
\]
as \( k \to +\infty \), and then \( v \in D^s(x) \).

Now let \( v \in U_0^{-} \), similarly we have

\[
Df^{-k}(x)(v) = \frac{\zeta(0)}{\zeta(-k)} A_{-k}^{-1} \circ A_{-k+1}^{-1} \circ \cdots \circ A_1^{-1}(v).
\]

Then we have

\[
\|Df^{-k}(x)(v)\| \leq \frac{\zeta(0)}{\zeta(-k)} C\lambda^k\|v\|
\]

for any \( k \geq 0 \). By the assumption that \( \lim_{k \to -\infty} \frac{1}{k} \log \zeta(k) = 0 \) we can also get

\[
\|Df^{-k}(x)(v)\| \to 0
\]
as $k \to +\infty$. This proves $v \in D^u(x)$.  

By the above claim and the fact that $U_0^- + S_0^+ = T_xM$, we know that $D^s(x) + D^u(x) = T_xM$ for all $x \in M$. Then our main theorem is a direct conclusion of Proposition 2.2.

4. Proof of Theorem 1.4. Now let us fix $\zeta : \mathbb{Z} \to \mathbb{R}^+$ to be a bounded sequence with the property

$$\lim_{i \to \infty} \frac{\zeta(i + 1)}{\zeta(i)} = 1.$$ 

Since $\zeta$ is a slowly decreasing bounded sequence, we know that if $f$ has the Lipschitz shadowing property with respect to $\zeta$, then $f$ is structurally stable. To prove Theorem 1.4, we just need to show that if $f$ is structurally stable, then $f$ has the Lipschitz shadowing property with respect to $\zeta$. Here we follow the proof of Theorem 2.2.7 of [4].

Proposition 4.1. $f$ has Lipschitz shadowing property with respect to $\zeta$ if and only if $f^k$ has Lipschitz shadowing property with respect to $\zeta(k \cdot)$ for some positive integer $k$.

Proof. Firstly we prove the necessary part. Assume that $f$ has Lipschitz shadowing property with respect to $\zeta$. Let $L, \epsilon_0$ be the constants corresponding to the Lipschitz shadowing property of $f$. Let $k$ be a given positive integer.

Let $\epsilon < \epsilon_0$. Take a $\zeta(k \cdot)$-weighted $\epsilon$-pseudo orbit $\{y_j\}$. Then we know that the sequence $\{x_i\}$ given by $x_{ki+j} = f^j(y_i)$ for any $i \in \mathbb{Z}$ and $j = 0, 1, \ldots, k-1$ is a $\zeta$-weighted $\epsilon$-pseudo orbit. Since $f$ has Lipschitz shadowing property with respect to $\zeta$, there is $\delta \in M$ such that

$$d(f^i(z), x_i) < L \epsilon \cdot \zeta(i)$$

for any $i \in \mathbb{Z}$. Especially we have

$$d(f^{ki}(z), y_i) < L \epsilon \cdot \zeta(ki)$$

for all $i \in \mathbb{Z}$. Hence the $\zeta(k \cdot)$-weighted $\epsilon$-pseudo orbit $\{y_j\}$ can be $L \epsilon$ shadowed by $\text{Orb}(z)$ with weight $\zeta(k \cdot)$. This shows that $f^k$ has Lipschitz shadowing property with respect to $\zeta(k \cdot)$.

Secondly we prove the sufficient part. Assume $f^k$ has Lipschitz shadowing property with respect to $\zeta(k \cdot)$ for some positive integer $k$. Let $L, \epsilon_0$ be the constants corresponding to Lipschitz shadowing property of $f^k$. Since $M$ is compact, we can find $K_1 > 0$ such that $\|D_p f\| < K_1$ for all $p \in M$. Since we have

$$\lim_{i \to \infty} \frac{\zeta(i + 1)}{\zeta(i)} = 1, \lim_{i \to \infty} \frac{\zeta(i + 2)}{\zeta(i)} = 1, \ldots, \lim_{i \to \infty} \frac{\zeta(i + k - 1)}{\zeta(i)} = 1,$$

there exist $K_2 > 1$ such that

$$\frac{1}{K_2} < \frac{\zeta(j + i)}{\zeta(j)} < K_2$$

for all $j \in \mathbb{R}$ and $i = 0, 1, \ldots, k-1$. Let $K = (1 + K_1 + \cdots + K_1^{k-1})K_2$ and $\epsilon_0 = \frac{\epsilon}{K}$. Let $0 < \epsilon < \epsilon_0$ and $\{x_i\}$ be a $\zeta$-weighted $\epsilon$-pseudo orbit. Then we have
Lemma 4.2. Let $f : M \to M$ be a $C^1$ structurally stable diffeomorphism on $M$. Then there exist two subspaces $S_p, U_p \subset T_p M$ for every $p \in M$ with following properties:

1. $S_p \oplus U_p = T_p M$ for every $p \in M$.
2. $Df(S_p) \subset S_{f(p)}$, $Df^{-1}(U_p) \subset U_{f^{-1}(p)}$ for every $p \in M$.
3. There exist constants $C, \lambda \in (0, 1)$ such that for any $p \in M$ and $n \geq 0$,
   \[ \|Df^n|_{S_p}\| < C\lambda^n \]
   \[ \|Df^{-n}|_{U_p}\| < C\lambda^n. \]
4. Denote by $P_p^s$ the projection from $T_p M$ to $S_p$ along the direction $U_p$. There is $N > 0$ such that
   \[ \|P_p^s\| < N, \quad \|Id - P_p^s\| < N \]
   for all $p \in M$.
5. Given any $\beta > 0$ and any positive integer $k$, there exists $\alpha > 0$ such that for any $p, q \in M$, if $d(f^k(p), q) < \alpha$ and $d(f^{-k}(q), p) < \alpha$, then there exist $\Pi_{p,q} : T_p M \to T_q M$ and $\Pi_{q,p} : T_p M \to T_p M$ such that
   \[ \|\Pi_{p,q} - Id\| \leq \beta, \quad \|\Pi_{q,p} - Id\| \leq \beta, \]
   \[ \Pi_{p,q}(D_{f^k(p)}\exp_q^{-1} Df^k(S_p)) \subset S_q, \quad \Theta_{q,p}(D_{f^{-k}(q)}\exp_p^{-1} Df^{-k}(U_q)) \subset U_p. \]
Proposition 4.3. Let $C > 1, \lambda \in (0, 1)$ be given. Assume that $\zeta : \mathbb{Z} \to \mathbb{R}^+$ is a bounded sequence with
\[
\lim_{i \to \infty} \frac{\zeta(i+1)}{\zeta(i)} = 1.
\]
There exist a positive integer $k$ and $\mu \in (0, 1)$ such that
\[
C\lambda^k \frac{\zeta(i+k)}{\zeta(i)} < \mu, \quad C\lambda^k \frac{\zeta(i)}{\zeta(i+k)} < \mu
\]
for all $i \in \mathbb{R}$.

Proof. Take $\lambda < \mu' < 1$. By the assumption of $\zeta$ we know that there is a positive integer $N$ such that for any $i$ with $|i| > N$,
\[
\frac{\lambda}{\mu'} < \frac{\zeta(i + 1)}{\zeta(i)} < \frac{\mu'}{\lambda}.
\]
Let
\[
C' = \max\{\zeta(i)/\zeta(j) : i, j \in [-N, N]\}.
\]
Then we have
\[
\frac{1}{C'} \left(\frac{\lambda}{\mu'} \right)^{j+2N} \leq \frac{\zeta(i + j)}{\zeta(i)} \leq C' \left(\frac{\mu'}{\lambda} \right)^{j+2N}
\]
for any $i \in \mathbb{Z}$ and positive integer $j$. Thus
\[
C\lambda^n \frac{\zeta(i + j)}{\zeta(i)} < C'C(\frac{\mu'}{\lambda})^{j+2N} \mu'^j.
\]
Since $\mu' \in (0, 1)$, we can take $k$ big enough such that $C'C(\frac{\mu'}{\lambda})^{-2N} \mu'^k = \mu < 1$. Thus we have
\[
C\lambda^k \frac{\zeta(i)}{\zeta(i+k)} < \mu.
\]
Similarly we have
\[
C\lambda^k \frac{\zeta(i + k)}{\zeta(i)} < \mu.
\]
\qed

The following is Theorem 1.3.1 of [4].

Proposition 4.4. Let $H_j, j \in \mathbb{R}$ be a sequence of Euclidean spaces of same dimension and $\phi_j : H_j \to H_{j+1}$ be a sequence of mappings with formulas $\phi_j(v) = A_j(v) + w_j(v)$ where $A_j$ are linear mappings. Assume that

1. There are constants $N > 0, \eta \in (0, 1)$ and splittings $H_j = S_j \oplus U_j$ at every $j$ (denote by $P_j$ the projection from $H_j$ to $S_j$ along $U_j$), such that
   (a1) $\|B_j\| < N$ and $\|I - P_j\| < N$ for all $j \in \mathbb{Z}$.
   (a2) $\|A_jS_j\| < \eta$, $A_j(S_j) \subset S_{j+1}$ for all $j \in \mathbb{Z}$.
2. If $U_{j+1} \neq \{0\}$, then there is a linear mapping $B_j : U_{j+1} \to H_j$ such that $B_j(U_{j+1}) \subset U_j$, $\|B_j\| < \eta$, $A_j B_j |U_j = I_d$.
3. There are constants $\rho > 0, r_0 > 0$ with $\rho N_1 < 1$ such that for any $v, v' \in H_j$ with $\|v\|, \|v'\| < r_0$,
\[
\|w_j(v) - w_j(v')\| \leq \rho \|v - v'\|
\]
holds, where
\[
N_1 = \frac{N^{1+\eta}}{1-\eta}.
\]
Then if we have \( \|w_j(0)\| \leq d \leq r_0/L \) for all \( j \in \mathbb{Z} \), then we can find points \( v_j \in H_j \) with \( \|v_j\| \leq Ld \) such that \( \phi_j(v_j) = v_{j+1} \) for all \( j \in \mathbb{Z} \), where
\[
L = \frac{N_1}{1 - \rho N_1}.
\]

**Proof of Theorem 1.4.** Assume \( f \) is structurally stable. We will show that \( f \) has Lipschitz shadowing property with respect to the weight \( \zeta \).

Let \( S_p, U_p, C, \lambda, N \) be given as in Lemma 4.2. By Lemma 4.3, we can find a positive integer \( k > 0 \) and \( \mu \in (0, 1) \) such that
\[
CA^k \frac{\zeta(k + 1)}{\zeta(k)} < \mu, \quad CA^k \frac{\zeta(k)}{\zeta(k + 1)} < \mu,
\]
for all \( i \in \mathbb{R} \). We just need to prove that \( f^k \) has Lipschitz shadowing property with respect to \( \zeta(k) \cdot \zeta(k) \) for positive integer \( k \) by Proposition 4.1. Note that we still have
\[
\lim_{i \to \infty} \frac{\zeta(k + 1)}{\zeta(k)} = 1.
\]
To simplify notations, we can still use \( f \) to denote \( f^k \) and \( \zeta \) to denote \( \zeta(k) \cdot \zeta(k) \). Note that if \( f \) has Lipschitz shadowing property with respect to \( \zeta \), then for any real number \( c > 0 \), \( f \) has Lipschitz shadowing property with respect to \( c \zeta \). Without loss of generality, we can assume that \( \zeta(i) \leq 1 \) for all \( i \in \mathbb{Z} \).

Now we take \( \nu_0 \in (0, 1) \) such that \( \eta = (1 + \nu_0)^2 \mu < 1 \). Let \( N_1 \) be fixed as in Proposition 4.4 associated to constants \( \eta, N \). Take \( \rho > 0 \) such that \( \rho N_1 < 1 \) and then take \( L \) as in Proposition 4.4.

Since \( M \) is compact, there is \( K_1 > 0 \) such that
\[
\|D_xf\| < K_1, \quad \|D_xf^{-1}\| < K_1
\]
for all \( x \in M \). Since \( \lim_{i \to \infty} \frac{\zeta(i+1)}{\zeta(i)} = 1 \), we can find \( K_2 > 0 \) such that
\[
\frac{1}{K_2} < \frac{\zeta(i+1)}{\zeta(i)} < K_2
\]
for all \( i \in \mathbb{Z} \). Now take \( 0 < \nu < \nu_0 \) such that
\[
N(2K_1K_2(1+\nu)+K_2)\nu < \frac{\rho}{2}.
\]
By the compactness of \( M \) and Proposition 4.2 we can find \( \epsilon_0 > 0 \) such that,
(a) for any \( x, y \in M \) with \( d(x, y) < \epsilon_0 \) we have
\[
\|D_{\exp_x^{-1}(y)} \exp_x\| < 1 + \nu, \quad \|D_y \exp_x^{-1}\| < 1 + \nu;
\]
(b) for any \( x, z \in M \) with \( d(f(x), z) < \epsilon_0 \) and any \( v \in T_xM \) with \( \|v\| < \epsilon_0 \), we have
\[
\|D_v(\exp_x^{-1} o f \circ \exp_z) - D_0(\exp_x^{-1} o f \circ \exp_z)\| < \nu;
\]
(c) for any \( x, z \in M \) with \( d(z, f(x)) < \epsilon_0 \), we have
\[
\|\Pi_{x,z} - Id\| < \nu, \quad \|\Theta_{z,x} - Id\| < \nu, \quad \|\Theta_{x,z}^{-1} - Id\| < \nu,
\]
where \( \Pi_{x,z} \) and \( \Theta_{z,x} \) are given as in Proposition 4.2.

Now let \( r_0 = \epsilon_0/K_1 \). Let us consider a \( \zeta \)-weighted \( \epsilon \)-pseudo orbit \( \{x_i\}_{i \in \mathbb{Z}} \) with
\[
0 < \epsilon < r_0/L.
\]
We set \( H_j = T_{x_j}M \) and define maps \( \phi_j : T_{x_j}M(r_0) \to T_{x_{j+1}}M \) as
\[
\phi_j(v) = (\zeta(j + 1))^{-1} \exp_{x_{j+1}}^{-1} \circ f \circ \exp_{x_j}(\zeta(j)v)
\]
for any \(v \in T_x M\) with \(\|v\| < r_0\). Noting that we have assumed \(\zeta(j) \leq 1\) for all \(j\), by the fact that \(d(x_{j+1}, f(x_j)) < \epsilon \cdot \zeta(j + 1) < \epsilon < \epsilon_0\) we know
\[
\|\phi_j(0)\| \leq (\zeta(j + 1)^{-1}d(x_{j+1}, f(x_j)) < \epsilon
\]
and
\[
d(f^{-1}(x_{j+1}), x_j) < K_1\epsilon \zeta(j + 1) \leq K_1\epsilon < \epsilon_0,
\]
for all \(j \in \mathbb{Z}\).

Denote by
\[
T_j = D_0\phi_j = \frac{\zeta(j)}{\zeta(j + 1)}D_{f(x_j)} \exp_{x_{j+1}}^{-1} \circ D_{x_j} f,
\]
\[
T_j' = D_{(\zeta(j))^{-1} \exp_{x_j}^{-1}(f^{-1}(x_{j+1}))}\phi_j = \frac{\zeta(j)}{\zeta(j + 1)}D_{f^{-1}(x_{j+1})} f \circ D_{\exp_{x_j}^{-1}(f^{-1}(x_{j+1}))} \exp_{x_j}.
\]
By the choice of \(\epsilon_0\) we know that \(\max\{\|T_j\|, \|T_j^{-1}\|, \|T_j'\|, \|T_j'^{-1}\|\} < K_1K_2(1 + \nu)\).

For any \(v \in H_j\) with \(\|v\| < r_0\), we have
\[
\|D_0\phi_j - T_j\| = \|D_0\phi_j - D_0\phi_j\| = \frac{\zeta(j)}{\zeta(j + 1)}\|D_{(\zeta(j))^{-1} \exp_{x_j}^{-1}(f^{-1}(x_{j+1}))} f - D_0(\exp_{x_j}^{-1} \circ f)\| \leq K_2\nu < \frac{\rho}{2}.
\]
Let \(S_j = S_{x_j}, U_j = U_{x_j}\) and \(P_j\) be the projections from \(T_x M\) to \(S_{x_j}\) along \(U_{x_j}\) for all \(j \in \mathbb{Z}\). By Proposition 4.2 we know
\[
\|P_j\| < N, \quad \|I - P_j\| < N
\]
for all \(j\). Now we take
\[
A_j^s = \Pi_{x_j,x_{j+1}} \circ T_j \circ P_j,
\]
\[
A_j^s = T_j' \circ \Theta_{x_j,x_{j+1}}^{-1} \circ (I - P_j)
\]
and \(A_j = A_j^s + A_j^s\). By the constructions of \(\Pi_{x_j,x_{j+1}}\) we know that \(A_j(S_j) \subset S_{j+1}\).

We also have
\[
\|A_j\|_{S_j} = \|A_j^s\|_{S_j} \leq \|\Pi_{x_j,x_{j+1}}\| \cdot \|T_j\|_{S_j} \|D_{f(x_j)} \exp_{x_{j+1}}^{-1}\| \cdot \|D_{x_j} f\|_{S_j} = (1 + \nu_0)^2\mu = \eta.
\]

By the constructions of \(\Theta_{x_j,x_{j+1}}\) we have
\[
A_j(U_j) = A_j^s(U_j) = T_j'(\Theta_{x_j,x_{j+1}}^{-1}(U_j)) \supset T_j'(D_{f^{-1}(x_{j+1})} \exp_{x_j}^{-1} \cdot D_{x_{j+1}} f^{-1}(U_{j+1})) = U_{j+1}.
\]
If we take \(B_j = \Theta_{x_j,x_{j+1}} \circ T_j'^{-1}\) then we have \(A_jB_j = Id, B_j(U_{j+1}) \subset U_j\) and
\[
\|B_j\| = \|\Theta_{x_j,x_{j+1}}\| \cdot \|T_j'^{-1}\|_{U_{j+1}} \leq \frac{\zeta(j + 1)}{\zeta(j)}\|\Theta_{x_j,x_{j+1}}\| \cdot \|D_{f^{-1}(x_{j+1})} \exp_{x_j}^{-1}\| \cdot \|D_{x_{j+1}} f^{-1}(U_{j+1})\| < (1 + \nu_0)^2\mu = \eta.
\]
Set \( w_j = \phi_j - A_j : H_j \to H_{j+1} \) for all \( j \in \mathbb{R} \). By noting that \( \|T_j - T'_j\| < \nu \), we have
\[
\|T_j - A_j\| = \|T_j \circ P_j - A_j^* + T_j \circ (Id - P_j) - A_j^*\|
\leq \|T_j \circ P_j - A_j^*\| + \|T_j \circ (Id - P_j) - A_j^*\|
\leq \|\Pi_{x_j, x_{j+1}} - Id\| \cdot \|T_j\| \cdot \|P_j\| + \|T_j - T'_j \circ \Theta_{x_j, x_{j+1}}^{-1} \| \cdot \|Id - P_j\|
\leq NK_1K_2(1 + \nu)\nu + N(\|T_j - T'_j\| + \|T'_j \circ \| \cdot \|\Theta_{x_j, x_{j+1}}^{-1} - Id\|)
\leq N(2K_1K_2(1 + \nu) + K_2)\nu < \frac{\rho}{2}.
\]
For any \( v \in H_j \) with \( \|v\| < r_0 \), by the fact that \( \|D_v\phi_j - T_j\| < \frac{\rho}{2} \), we have
\[
\|D_v\phi_j - A_j\| \leq \|D_v\phi_j - T_j\| + \|T_j - A_j\| < \rho.
\]
Thus by the generalized mean value theorem, we have
\[
\|w_j(v) - w_j(v')\| < \rho\|v - v'\|
\]
for all \( v, v' \in H_j \) with \( \|v\|, \|v'\| < r_0 \). By applying Proposition 4.4, we can find a sequence \( v_j \in H_j = T_{x_j}M \) with \( \|v_j\| < Le \) such that for any \( j \in \mathbb{Z} \), one has \( \phi_j(v_j) = v_{j+1} \). Let \( y_j = \exp_{x_j}(\zeta(j)v_j) \), then we have \( f(y_j) = y_{j+1} \) and
\[
d(y_j, x_j) = \|\zeta(j)v_j\| < Le \cdot \zeta(j).
\]
This proves that \( f \) has Lipschitz shadowing with respect to \( \zeta \) and \( L, r_0/L \) are the constants corresponding to the shadowing property.

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