Relation between the autocorrelation and Wigner functions

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Abstract

We show a simple mechanism to measure the Wigner function of a harmonic oscillator. For this system we also show that autocorrelation and Wigner functions are equivalent.

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Non classical states of ions [1] and cavity fields [2] have been produced recently in experiments around the world [3–7]. Once a given nonclassical state has been produced, it is important to count with mechanisms that allow us to measure them, making the gathering of such information a key problem in quantum mechanics. Among this mechanisms one can count with the fact that the passage of atoms through a cavity may indicate the photon statistics of the cavity field [2]. For instance, information about the position or momentum allows us to look for non classicality of the system. However, it is possible to obtain full information from a system by measuring, not some of its observables, but directly the density matrix [8, 9], i.e. obtaining information about all possible observables. One of the possible ways of obtaining such information is via a quasiprobability distribution function, that may be related to the density matrix by using the equation [10, 11]

\[ F(\alpha, s) = \frac{2}{\pi(1-s)} \sum_{k=0}^{\infty} \left( \frac{s+1}{s-1} \right)^k \langle \alpha, k | \rho | \alpha, k \rangle \]  

with \( s \) the quasiprobability function’s parameter that indicates which is the relevant distribution (\( s = -1 \) Husimi [12], \( s = 0 \) Wigner [13] and \( s = 1 \) Glauber-Sudarshan [14, 15] distribution functions), \( \rho \) the density matrix and the states \( |\alpha, k\rangle \) are the so-called displaced number states [16].

It is well known that the Glauber-Sudarshan \( P \)-function is highly singular (note the term \( s - 1 \) in the denominator). It may be used to measure non-classicality of states [17]. It is not the purpose of the present contribution to discuss about when is this function well-behaved, however, it may be used to write the density in a (diagonal) coherent state basis

\[ \rho = \frac{1}{\pi} \int d^2 \alpha P(\alpha) |\alpha\rangle \langle \alpha| = \frac{1}{\pi} \int d^2 \alpha F(\alpha, 1) |\alpha\rangle \langle \alpha|, \]  

that may be used to derive Fokker-Planck equations (partial differential equations) from master equations (equations that involve superoperators) [18].

The (Husimi) \( Q \)-function may be obtained from (1) by taking \( s = -1 \). In such a case, the only term that survives in the sum is \( k = 0 \), that allow us to write

\[ Q(\alpha) = F(\alpha, -1) = \frac{1}{\pi} \langle \alpha | \rho | \alpha \rangle. \]  

Moreover, besides applications in classical optics, it has been shown that these phase space distributions can be expressed, in thermofield dynamics, as overlaps between the state
of the system and *thermal* coherent states [19], that is probably the reason by which, systems subject to decay may still be "measured" [20].

Wineland’s [8] and Haroche’s [9] groups used the above expression to measure the Wigner function \((s = 0)\) case of the quantized motion of an ion and the quantized cavity field, respectively. It is somehow simple to obtain a quasiprobability distribution function from experimental data from the above equation as there is already there a direct recipe. Let us write equation (1) as

\[
F(\alpha, s) = \frac{2}{\pi(1 - s)} \sum_{k=0}^{\infty} \left( \frac{s + 1}{s - 1} \right)^k \langle k | D^\dagger(\alpha) \rho D(\alpha) | k \rangle \tag{4}
\]

where \(D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)\), with \(a\) and \(a^\dagger\) the annihilation and creation operators respectively, is the Glauber displacement operator. Note that, in order to obtain a quasiprobability distribution function we need to do the following: displace the system by an amplitude \(\alpha\) and then measure the diagonal elements of the displaced density matrix.

Equation (4) may be rewritten as

\[
F(\alpha, s) = \frac{2}{\pi(1 - s)} Tr \left\{ \left( \frac{s + 1}{s - 1} \right)^{a^\dagger a} D^\dagger(\alpha) \rho D(\alpha) \right\}. \tag{5}
\]

By using the commutation properties under the symbol of trace, and the system in a *pure* state \(|\psi\rangle\), the above equation may be casted into

\[
F(\alpha, s) = \frac{2}{\pi(1 - s)} Tr \left\{ D(\alpha) \left( \frac{s + 1}{s - 1} \right)^{a^\dagger a} D^\dagger(\alpha) \rho D(\alpha) \right\} = \frac{2}{\pi(1 - s)} \langle\psi| D(\alpha) \left( \frac{s + 1}{s - 1} \right)^{a^\dagger a} D^\dagger(\alpha)|\psi\rangle \tag{6}
\]

Consider now a displaced harmonic oscillator with frequency \(\omega\)

\[
H = \omega a^\dagger a + \beta a^\dagger + \beta^* a \tag{7}
\]

with \(\beta\) the amplitude of the displacement. One can directly write the evolved wave function as (we set \(\hbar = 1\))

\[
|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle = D^\dagger(\beta/\omega)e^{-i\omega t a^\dagger a}D(\beta/\omega)|\psi(0)\rangle \tag{8}
\]

From equation (6) we may obtain the *autocorrelation function* [21]

\[
A(t) = \langle\psi(0)|\psi(t)\rangle = \langle\psi(0)|D^\dagger(\beta/\omega)e^{-i\omega t a^\dagger a}D(\beta/\omega)|\psi(0)\rangle \tag{9}
\]
that is very similar to equation (6). In fact, if we choose $t = \pi/\omega$ in the above equation, it produces a term

$$e^{-i\pi a^\dagger a} = (-1)^a^\dagger a,$$

(10)

that is essential in the production of the Wigner function (the alternating term), so that by setting $s = 0$ in equation (6), the Wigner and autocorrelation functions become proportional:

$$F(\beta/\omega, 0) = W(\beta/\omega) = \frac{2}{\pi} A(\pi/\omega)$$

(11)

which is not surprising as the Wigner function is the generating function for all spatial autocorrelation functions of the wave function [22].

Thus, an eigenstate of the harmonic oscillator, namely, a number state $|n\rangle$, may be easily measured, simply by choosing as initial state $|\psi(0)\rangle = |n\rangle$, and projecting it with the same number state. This can be done for instance in cavity QED, by writing the evolved wavefunction as a density matrix and then measuring its diagonal elements by passing atoms through the cavity [20]. Note however, that, for every displacement of the harmonic oscillator, a single value of the Wigner function is obtained. Therefore for the reconstruction of the Wigner function it is necessary a big number of experiments in order to fill the phase space up.

Note that such systems may be emulated in classical light propagation through waveguide arrays [23, 24] due to the analogy between linear lattices and the atom-field interaction [25]. Therefore, experiments leading to measurements of quasiprobability distribution functions may be easier to implement in classical optical systems.

In conclusion, we have shown a simple method to reconstruct the Wigner function for the harmonic oscillator and have shown that for this system, the autocorrelation function is proportional to the Wigner function.
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