Minimal Affinizations of Representations
of Quantum Groups:
the Simply-laced Case

Vyjayanthi Chari\textsuperscript{1},
Department of Mathematics,
University of California, Riverside, CA 92521, USA.

Andrew Pressley,
Department of Mathematics,
King’s College, Strand, London WC2R 2LS, UK.

Introduction

In \cite{2}, we defined the notion of an affinization of a finite-dimensional irreducible representation \( V \) of the quantum group \( U_q(\mathfrak{g}) \), where \( \mathfrak{g} \) is a finite-dimensional complex simple Lie algebra and \( q \in \mathbb{C}^\times \) is transcendental. An affinization of \( V \) is an irreducible representation \( \hat{V} \) of the quantum affine algebra \( U_q(\hat{\mathfrak{g}}) \) which, regarded as a representation of \( U_q(\mathfrak{g}) \), contains \( V \) with multiplicity one, and is such that all other irreducible components of \( \hat{V} \) are strictly smaller than \( V \), with respect to a certain natural partial order on the set of isomorphism classes of finite-dimensional representations of \( U_q(\mathfrak{g}) \). In general, a given representation \( V \) has finitely many affinizations up to \( U_q(\mathfrak{g}) \)-isomorphism (always at least one), and it is natural to look for the minimal one(s). We refer the reader to the introduction to \cite{2} for a discussion of the significance of the notion of an affinization.

In \cite{2}, we show that, if \( \mathfrak{g} \) has rank 2, every \( V \) has a unique minimal affinization. In this paper, we consider the case when \( \mathfrak{g} \) is a simply-laced algebra of arbitrary rank. If \( \mathfrak{g} \) is of type A, there is again a unique minimal affinization (this result is, in fact, contained in \cite{4}). But, if \( \mathfrak{g} \) is of type D or E, and if the highest weight of \( V \) is not too singular, we show that \( V \) has precisely three minimal affinizations. In all cases, the minimal affinization(s) are described precisely in terms of the parametrization of the finite-dimensional irreducible representations of \( U_q(\hat{\mathfrak{g}}) \) given in \cite{3} (in the \( sl_2 \) case), in \cite{5} (in the \( sl_n \) case), and in \cite{6} (in the general case).

\textsuperscript{1}Both authors were partially supported by the NSF, DMS-9207701.
1 Quantum affine algebras and their representations

In this section, we collect the results about quantum affine algebras which we shall need later.

Let $\mathfrak{g}$ be a finite–dimensional complex simple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and Cartan matrix $A = (a_{ij})_{i,j \in I}$. Fix coprime positive integers $(d_i)_{i \in I}$ such that $(d_i a_{ij})$ is symmetric. Let $P = \mathbb{Z}^I$ and let $P^+ = \{ \lambda \in P \mid \lambda(i) \geq 0 \text{ for all } i \in I \}$. Let $R$ (resp. $R^+$) be the set of roots (resp. positive roots) of $\mathfrak{g}$. Let $\alpha_i$ ($i \in I$) be the simple roots and let $\theta$ be the highest root. Define a non-degenerate symmetric bilinear form $(\ , \ )$ on $\mathfrak{h}^*$ by $(\alpha_i, \alpha_j) = d_i a_{ij}$, and set $d_0 = \frac{1}{2}(\theta, \theta)$. Let $Q = \oplus_{i \in I} \mathbb{Z} \alpha_i \subset \mathfrak{h}^*$ be the root lattice, and set $Q^+ = \sum_{i \in I} \mathbb{N} \alpha_i$. Define a partial order $\geq$ on $P$ by $\lambda \geq \mu$ iff $\lambda - \mu \in Q^+$.

Let $q \in \mathbb{C}^\times$ be transcendental, and, for $r, n \in \mathbb{N}$, $n \geq r$, define

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}},$$

$$[n]_q! = [n]_q [n-1]_q \ldots [2]_q [1]_q,$$

$$\left[ \begin{array}{c} n \\ r \end{array} \right]_q = \frac{[n]_q!}{[r]_q! [n-r]_q!}.$$

**Proposition 1.1.** There is a Hopf algebra $U_q(\mathfrak{g})$ over $\mathbb{C}$ which is generated as an algebra by elements $x_i^\pm, k_i^\pm$ ($i \in I$), with the following defining relations:

$$k_i k_i^{-1} = k_i^{-1} k_i = 1, \quad k_i k_j = k_j k_i,$$

$$k_ix_j^\pm k_i^{-1} = q_i^{\pm a_{ij}} x_j^\pm,$$

$$[x_i^+, x_j^-] = \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}},$$

$$\sum_{r=0}^{1-a_{ij}} \left[ \begin{array}{c} 1-a_{ij} \\ r \end{array} \right]_q (x_i^\pm)^r x_j^\pm (x_i^\pm)^{1-a_{ij}-r} = 0, \quad i \neq j.$$

The comultiplication $\Delta$, counit $\epsilon$, and antipode $S$ of $U_q(\mathfrak{g})$ are given by

$$\Delta(x_i^+) = x_i^+ \otimes k_i + 1 \otimes x_i^+,$$

$$\Delta(x_i^-) = x_i^- \otimes 1 + k_i^{-1} \otimes x_i^-,$$

$$\Delta(k_i^\pm) = k_i^{\pm 1} \otimes k_i^{\pm 1},$$

$$\epsilon(x_i^\pm) = 0, \quad \epsilon(k_i^\pm) = 1,$$

$$S(x_i^+) = -x_i^{+1}, \quad S(x_i^-) = -k_i x_i^-; \quad S(k_i^\pm) = k_i^{\mp 1},$$

for all $i \in I$. $\square$

The Cartan involution $\omega$ of $U_q(\mathfrak{g})$ is the unique algebra automorphism of $U_q(\mathfrak{g})$ which takes $x_i^\pm \mapsto -x_i^\mp$, $k_i^{\pm 1} \mapsto k_i^{\mp 1}$, for all $i \in I$.

Let $\hat{I} = I \sqcup \{ 0 \}$ and let $\hat{A} = (a_{ij})_{i,j \in \hat{I}}$ be the extended Cartan matrix of $\mathfrak{g}$, i.e. the generalized Cartan matrix of the (untwisted) affine Lie algebra $\hat{\mathfrak{g}}$ associated to $\mathfrak{g}$. Let $a_0 = e^{d_0}$.
Theorem 1.2. Let $U_q(\mathfrak{g})$ be the algebra with generators $x_i^\pm$, $k_i^{\pm 1}$ ($i \in \hat{I}$) and defining relations those in 1.1, but with the indices $i$, $j$ allowed to be arbitrary elements of $\hat{I}$. Then, $U_q(\mathfrak{g})$ is a Hopf algebra with comultiplication, counit and antipode given by the same formulas as in 1.1 (but with $i \in \hat{I}$).

Moreover, $U_q(\mathfrak{g})$ is isomorphic to the algebra $A_q$ with generators $x_{i,r}^\pm$ ($i \in I$, $r \in \mathbb{Z}$), $k_i^{\pm 1}$ ($i \in I$), $h_{i,r}$ ($i \in I$, $r \in \mathbb{Z}\backslash\{0\}$) and $c^{\pm 1/2}$, and the following defining relations:

$$c^{\pm 1/2} \text{ are central,}$$
$$k_i k_i^{-1} = k_i^{-1} k_i = 1,$$
$$c^{1/2} c^{-1/2} = c^{-1/2} c^{1/2} = 1,$$
$$k_i k_j = k_j k_i, \quad k_i h_{j,r} = h_{j,r} k_i,$$
$$k_i x_{j,r} k_i^{-1} = q_i^{\pm a_{ij}} x_{j,r}^{\pm},$$
$$[h_{i,r}, x_{j,s}^\pm] = \pm \frac{1}{r} [r a_{ij}] q_i c^{\pm |r|/2} x_{j,r+s}^\pm,$$
$$x_{i,r+1}^\pm x_{j,s}^\mp - q_i^{\pm a_{ij}} x_{j,s}^\pm x_{i,r+1}^\mp = q_i^{\pm a_{ij}} x_{j,r+s+1}^\pm - x_{j,s+1}^\pm x_{i,r}^\mp,$$
$$[x_{i,r}^+, x_{j,s}^-] = \delta_{ij} \frac{c^{(r-s)/2} x_{i,r+s}^+ - c^{-(r-s)/2} x_{i,r+s}^-}{q_i - q_i^{-1}}$$

$$\sum_{\pi \in \Sigma_m} \sum_{k=0}^{m} (-1)^k \left[ \begin{array}{c} m \\ k \end{array} \right] q_i x_{i,r(\pi(1))}^\pm \cdots x_{i,r(\pi(k))}^\pm x_{i,r(\pi(k+1))}^\pm \cdots x_{i,r(\pi(m))}^\pm = 0, \quad i \neq j,$$

for all sequences of integers $r_1, \ldots, r_m$, where $m = 1 - a_{ij}$, $\Sigma_m$ is the symmetric group on $m$ letters, and the $\phi_{i,r}^\pm$ are determined by equating powers of $u$ in the formal power series

$$\sum_{r=0}^{\infty} \phi_{i,r}^\pm u^r = k_i^{\pm 1} \text{exp} \left( \pm (q_i - q_i^{-1}) \sum_{s=1}^{\infty} h_{i,s} u^s \right).$$

If $\theta = \sum_{i \in I} m_i \alpha_i$, set $k_\theta = \prod_{i \in I} k_i^{m_i}$. Suppose that the root vector $x_\theta^+$ of $\mathfrak{g}$ corresponding to $\theta$ is expressed in terms of the simple root vectors $x_i^+$ ($i \in I$) of $\mathfrak{g}$ as

$$x_\theta^+ = \lambda [x_{i_1}^+, x_{i_2}^+, \ldots, x_{i_k}^+, x_j^+] \cdots$$

for some $\lambda \in \mathbb{C}^\times$. Define maps $w_i^\pm : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ by

$$w_i^\pm(a) = x_{i,0}^\pm a - k_i^{\pm 1} a k_i^{\pm 1} x_{i,0}^\pm.$$

Then, the isomorphism $f : U_q(\mathfrak{g}) \to A_q$ is defined on generators by

$$f(k_0) = k_\theta^{-1}, \quad f(k_i) = k_i, \quad f(x_i^+) = x_{i,0}^+, \quad (i \in I),$$
$$f(x_0^+) = \mu w_{i_1}^+ \cdots w_{i_k}^+ (x_{j,-1}^-) k_\theta^{-1},$$
$$f(x_0^-) = \lambda k_\theta w_{i_1}^+ \cdots w_{i_k}^+ (x_{j,-1}^+),$$

where $\mu \in \mathbb{C}^\times$ is determined by the condition

$$[x_0^+, x_0^-] = \frac{k_0 - k_0^{-1}}{\mu}.$$
See [1], [5] and [7] for further details.

Note that there is a canonical homomorphism \( U_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \) such that \( x_i^\pm \mapsto x_i^\pm \), \( k_i^{\pm 1} \mapsto k_i^{\pm 1} \) for all \( i \in I \). Thus, any representation of \( U_q(\mathfrak{g}) \) may be regarded as a representation of \( U_q(\mathfrak{g}) \).

Let \( \hat{U}^\pm \) (resp. \( \hat{U}^0 \)) be the subalgebra of \( U_q(\mathfrak{g}) \) generated by the \( x_i^\pm \) (resp. by the \( \phi_i^\pm \)) for all \( i \in I, r \in \mathbb{Z} \). Similarly, let \( U^\pm \) (resp. \( U^0 \)) be the subalgebra of \( U_q(\mathfrak{g}) \) generated by the \( x_i^\pm \) (resp. by the \( k_i^{\pm 1} \)) for all \( i \in I \).

**Proposition 1.3.** (a) \( U_q(\mathfrak{g}) = U^-U^0U^+ \).
(b) \( U_q(\mathfrak{g}) = \hat{U}^-\hat{U}^0\hat{U}^+ \). □

See [5] or [8] for details.

We shall make use of the following automorphisms of \( U_q(\mathfrak{g}) \):

**Proposition 1.4.** (a) For all \( t \in \mathbb{C}^\times \), there exists a Hopf algebra automorphism \( \tau_t \) of \( U_q(\mathfrak{g}) \) such that

\[
\begin{align*}
\tau_t(x_{i,r}^\pm) &= t^r x_{i,r}^\pm, \quad \tau_t(h_{i,r}) = t^r h_{i,r}, \\
\tau_t(k_i^{\pm 1}) &= k_i^{\pm 1}, \quad \tau_t(c^{\pm 1/2}) = c^{\mp 1/2}.
\end{align*}
\]

(b) There is a unique algebra involution \( \hat{} \) of \( U_q(\mathfrak{g}) \) given on generators by

\[
\begin{align*}
\hat{x}_{i,r}^\pm &= -x_{i,-r}^\mp, \quad \hat{h}_{i,r} = -h_{i,r}, \\
\hat{\phi}_{i,r}^\pm &= \phi_{i,-r}^\mp, \quad \hat{k}_i^{\pm 1} = k_i^{\mp 1}, \\
\hat{c}^{\pm 1/2} &= c^{\mp 1/2}.
\end{align*}
\]

Moreover, we have

\[
(\hat{} \otimes \hat{}) \circ \Delta = \Delta^{op} \circ \hat{},
\]

where \( \Delta^{op} \) is the opposite comultiplication of \( U_q(\mathfrak{g}) \). □

See [2] for the proof. Note that \( \hat{} \) is compatible, via the canonical homomorphism \( U_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \), with the Cartan involution \( \omega \) of \( U_q(\mathfrak{g}) \).

A representation \( W \) of \( U_q(\mathfrak{g}) \) is said to be of type 1 if it is the direct sum of its weight spaces

\[
W_\lambda = \{ w \in W \mid k_i w = q_i^{\lambda(i)} w, \quad (\lambda \in P) \}.
\]

If \( W_\lambda \neq 0 \), then \( \lambda \) is a weight of \( W \). A vector \( w \in W_\lambda \) is a highest weight vector if \( x_i^+ w = 0 \) for all \( i \in I \), and \( W \) is a highest weight representation with highest weight \( \lambda \) if \( W = U_q(\mathfrak{g}).w \) for some highest weight vector \( w \in W_\lambda \). Lowest weight vectors and representations are defined similarly, by replacing \( x_i^+ \) by \( x_i^- \).

For a proof of the following proposition, see [5] or [8].

**Proposition 1.5.** (a) Every finite–dimensional representation of \( U_q(\mathfrak{g}) \) is completely reducible.

(b) Every finite–dimensional irreducible representation of \( U_q(\mathfrak{g}) \) can be obtained from a type 1 representation by twisting with an automorphism of \( U_q(\mathfrak{g}) \).

(c) Every finite–dimensional irreducible representation of \( U_q(\mathfrak{g}) \) of type 1 is both highest and lowest weight. Assigning to such a representation its highest weight defines a bijection between the set of isomorphism classes of finite–dimensional irreducible type 1 representations of \( U_q(\mathfrak{g}) \) and \( B^+ \).
(d) The finite-dimensional irreducible representation $V(\lambda)$ of $U_q(\mathfrak{g})$ of highest weight $\lambda \in P^+$ has the same character as the irreducible representation of $\mathfrak{g}$ of the same highest weight.

(e) The multiplicity $m_\nu(V(\lambda) \otimes V(\mu))$ of $V(\nu)$ in the tensor product $V(\lambda) \otimes V(\mu)$, where $\lambda, \mu, \nu \in P^+$, is the same as in the tensor product of the irreducible representations of $\mathfrak{g}$ of the same highest weight (this statement makes sense in view of parts (a) and (c)). □

A representation $V$ of $U_q(\mathfrak{g})$ is of type 1 if $c^{1/2}$ acts as the identity on $V$, and if $V$ is of type 1 as a representation of $U_q(\mathfrak{g})$. A vector $v \in V$ is a highest weight vector if

\[ x_{i,r} v = 0, \quad \phi_{i,r}^\pm v = \Phi_{i,r}^\pm v, \quad c^{1/2} v = v, \]

for some complex numbers $\Phi_{i,r}^\pm$. A type 1 representation $V$ is a highest weight representation if $V = U_q(\hat{\mathfrak{g}}) v$, for some highest weight vector $v$, and the pair of $(I \times \mathbb{Z})$-tuples $(\Phi_{i,r}^\pm)_{i \in I, r \in \mathbb{Z}}$ is its highest weight. Note that $\Phi_{i,r}^+ = 0$ (resp. $\Phi_{i,r}^- = 0$) if $r < 0$ (resp. $r > 0$), and that $\Phi_{i,0}^+ \Phi_{i,0}^- = 1$. (In [5], highest weight representations of $U_q(\hat{\mathfrak{g}})$ are called ‘pseudo-highest weight’.) Lowest weight vectors and representations of $U_q(\hat{\mathfrak{g}})$ are defined similarly.

If $\lambda \in P^+$, let $\mathcal{P}^\lambda$ be the set of all $I$-tuples $(P_i)_{i \in I}$ of polynomials $P_i \in \mathbb{C}[u]$, with constant term 1, such that $\deg(P_i) = \lambda(i)$ for all $i \in I$. Set $\mathcal{P} = \cup_{\lambda \in P^+} \mathcal{P}^\lambda$.

**Theorem 1.6.** (a) Every finite-dimensional irreducible representation of $U_q(\hat{\mathfrak{g}})$ can be obtained from a type 1 representation by twisting with an automorphism of $U_q(\hat{\mathfrak{g}})$.

(b) Every finite-dimensional irreducible representation of $U_q(\hat{\mathfrak{g}})$ of type 1 is both highest and lowest weight.

(c) Let $V$ be a finite-dimensional irreducible representation of $U_q(\hat{\mathfrak{g}})$ of type 1 and highest weight $(\Phi_{i,r}^\pm)_{i \in I, r \in \mathbb{Z}}$. Then, there exists $\mathbf{P} = (P_i)_{i \in I} \in \mathcal{P}$ such that

\[ \sum_{r=0}^{\infty} \Phi_{i,r}^+ u^r = q_i^{\deg(P_i)} \frac{P_i(q_i^{-2} u)}{P_i(u)} = \sum_{r=0}^{\infty} \Phi_{i,r}^- u^{-r}, \]

in the sense that the left- and right-hand terms are the Laurent expansions of the middle term about 0 and $\infty$, respectively. Assigning to $V$ the $I$-tuple $\mathbf{P}$ defines a bijection between the set of isomorphism classes of finite-dimensional irreducible representations of $U_q(\hat{\mathfrak{g}})$ of type 1 and $\mathcal{P}$.

(d) Let $\mathbf{P}, \mathbf{Q} \in \mathcal{P}$ be as above, and let $v_{\mathbf{P}}$ and $v_{\mathbf{Q}}$ be highest weight vectors of $V(\mathbf{P})$ and $V(\mathbf{Q})$, respectively. Then, in $V(\mathbf{P}) \otimes V(\mathbf{Q})$,

\[ x_{i,r}^+(v_{\mathbf{P}} \otimes v_{\mathbf{Q}}) = 0, \quad \phi_{i,r}^\pm(v_{\mathbf{P}} \otimes v_{\mathbf{Q}}) = \Psi_{i,r}^\pm(v_{\mathbf{P}} \otimes v_{\mathbf{Q}}), \]

where the complex numbers $\Psi_{i,r}^\pm$ are related to the polynomials $P_iQ_i$, as the $\Phi_{i,r}^\pm$ are related to $P_i$ in (5). In particular, if $\mathbf{P} \otimes \mathbf{Q}$ denotes the $I$-tuple $(P_iQ_i)_{i \in I}$, then $V(\mathbf{P} \otimes \mathbf{Q})$ is isomorphic to a quotient of the subrepresentation of $V(\mathbf{P}) \otimes V(\mathbf{Q})$ generated by the highest weight vectors. □

See [5] for further details. If the highest weight $(\Phi_{i,r}^\pm)_{i \in I, r \in \mathbb{Z}}$ of $V$ is given by an $I$-tuple $\mathbf{P}$ as in part (c), we shall often abuse notation by saying that $V$ has highest weight $\mathbf{P}$.

We shall need the following result from [2].
Lemma 1.7. Let \( \rho : U_q(\hat{\mathfrak{g}}) \to \text{End}(V) \) be a finite-dimensional irreducible representation of type 1 with highest weight \( P = (P_i)_{i \in I} \). For any \( t \in \mathbb{C}^\times \), denote by \( \tau_t^*(V) \) the representation \( \rho \circ \tau_t \). Then, \( \tau_t^*(V) \) has highest weight \( P^t = (P_i^t)_{i \in I} \), where

\[
P_i^t(u) = P_i(tu).
\]

Following [2], we say that a finite-dimensional irreducible representation \( V \) of \( U_q(\hat{\mathfrak{g}}) \) is an affinization of \( \lambda \in P^+ \) if \( V \cong V(\mathbf{P}) \) as a representation of \( U_q(\hat{\mathfrak{g}}) \), for some \( \mathbf{P} \in \mathcal{P}^\lambda \). Two affinizations of \( \lambda \) are equivalent if they are isomorphic as representations of \( U_q(\hat{\mathfrak{g}}) \); we denote by \([V]\) the equivalence class of \( V \). Let \( \mathcal{Q}^\lambda \) be the set of equivalence classes of affinizations of \( \lambda \).

The following result is proved in [2].

Proposition 1.8. If \( \lambda \in P^+ \) and \([V], [W] \in \mathcal{Q}^\lambda \), we write \([V] \preceq [W]\) iff, for all \( \mu \in P^+ \), either,

(i) \( m_\mu(V) \leq m_\mu(W) \), or 
(ii) there exists \( \nu > \mu \) with \( m_\nu(V) < m_\nu(W) \).

Then, \( \preceq \) is a partial order on \( \mathcal{Q}^\lambda \). □

An affinization \( V \) of \( \lambda \) is minimal if \([V]\) is a minimal element of \( \mathcal{Q}^\lambda \) for the partial order \( \preceq \), i.e. if \([W] \in \mathcal{Q}^\lambda \) and \([V] \preceq [W]\) implies that \([V] = [W]\). It is proved in [2] that \( \mathcal{Q}^\lambda \) is a finite set, so minimal affinizations certainly exist.

2 Diagram subalgebras

In this section, \( \mathfrak{g} \) is any finite-dimensional complex simple Lie algebra.

Let \( J \) be any non-empty connected subset of \( I \), and let \( U_q(\mathfrak{g}_J) \) be the Hopf subalgebra of \( U_q(\mathfrak{g}) \) defined by the generators and relations in 1.1 for which all the indices \( i, j \in J \). Similarly, let \( U_q(\hat{\mathfrak{g}}_J) \) be the subalgebra of \( U_q(\hat{\mathfrak{g}}) \) defined by the generators and relations in 1.2 for which all the indices \( i, j \in J \). Let \( P_J \) be the set of weights of \( U_q(\mathfrak{g}_J) \), \( R^+_J \) the set of positive roots, etc. If \( \lambda \in P \), let \( \lambda_J \) be the restriction of \( \lambda : I \to \mathbb{Z} \) to \( J \). Similarly, if \( \mathbf{P} = (P_i)_{i \in I} \in \mathcal{P} \) is an \( I \)-tuple of polynomials in \( \mathbb{C}[u] \) with constant term 1, let \( \mathbf{P}_J \in \mathcal{P}_J \) be the \( J \)-tuple \((P_i)_{i \in J}\).

Let \( \Delta_J \) be the comultiplication of \( U_q(\hat{\mathfrak{g}}_J) \). Note that \( U_q(\hat{\mathfrak{g}}_J) \) is not a Hopf subalgebra of \( U_q(\hat{\mathfrak{g}}) \) in general. However, we do have

Lemma 2.1. Let \( \emptyset \neq J \subseteq I \) be connected, and let \( \rho_J : U_q(\hat{\mathfrak{g}}_J) \to U_q(\hat{\mathfrak{g}}) \) be the canonical homomorphism of algebras. Then, for all \( i \in J \),

\[
\Delta(x^\pm_{i, \pm 1}) - (\rho_J \otimes \rho_J)(\Delta_J(x^\pm_{i, \pm 1})) \in \bigoplus_{\eta', \eta''} U_q(\hat{\mathfrak{g}})_{\eta'} \otimes U_q(\hat{\mathfrak{g}})_{\eta''},
\]

where the sum is over those \( \eta', \eta'' \in Q \setminus Q_J \) such that \( \eta' + \eta'' = \pm \alpha_i \), and

\[
U_q(\hat{\mathfrak{g}})_{\eta} = \{ u \in U_q(\hat{\mathfrak{g}}) | k_j u k_j^{-1} = q^{\eta(j)} u \text{ for all } j \in I \}.
\]

The proof of this lemma can be deduced in a straightforward manner from [1].
Fix a non-empty connected subset $J \subseteq I$. Let $\lambda \in P^+$, $P \in P^\lambda$, and let $M$ be a highest weight representation of $U_q(\hat{g})$ with highest weight $P$ and highest weight vector $m$. Let $M_J = U_q(\hat{g})_J.m$. Then, it follows from 1.3 that

$$(1) \quad M_J = \bigoplus_{\eta \in Q_J^+} M_{\lambda, \eta}.$$

Similarly, let $\mu \in P^+$, $Q \in P^\mu$, let $N$ be a highest weight representation of $U_q(\hat{g})$ of highest weight $Q$ and highest weight vector $n$, and let $N_J = U_q(\hat{g})_J.n$. Then, we have

$$(2) \quad M_J \otimes N_J = \bigoplus_{\eta \in Q_J^+} (M \otimes N)_{\lambda + \mu, \eta}.$$

Indeed, it is obvious that the left-hand side of (2) is contained in the right-hand side. On the other hand,

$$(M \otimes N)_{\lambda + \mu, \eta} = \bigoplus_{\eta', \eta''} M_{\lambda, \eta'} \otimes N_{\mu, \eta''},$$

where the sum is over those $\eta', \eta'' \in Q^+$ such that $\eta' + \eta'' = \eta$. But, since $\eta \in Q_J^+$, this clearly forces $\eta', \eta'' \in Q_J^+$, so by (1), $(M \otimes N)_{\lambda + \mu, \eta} \subseteq M_J \otimes N_J$. This proves (2).

Now, $M_J \otimes N_J$ admits an obvious action of $U_q(\hat{g})_J$ by using $\Delta_J$; we denote this representation by $M_J \otimes J N_J$. On the other hand, for weight reasons, the action of the $\Delta(x_{i,r}^\pm)$, $\Delta(x_{i,r}^\pm)$, for all $i \in J$, $r \in \mathbb{Z}$, obviously preserves $\otimes_{\eta \in Q_J^+} (M \otimes N)_{\lambda + \mu - \eta}$. This gives another representation of $U_q(\hat{g})_J$ on $M_J \otimes N_J$, using $\Delta$, which we denote by $M_J \otimes N_J$.

**Proposition 2.2.** The identity map $M_J \otimes N_J \rightarrow M_J \otimes N_J$ is an isomorphism of representations of $U_q(\hat{g})_J$.

**Proof.** The map obviously commutes with the action of $U_q(\hat{g})_J$. From 1.2, it follows that $U_q(\hat{g})_J$ is generated as an algebra by the elements of $U_q(\hat{g})_J$, the $x_{i,r}^\pm$ for $i \in J$, $r = \pm 1$, and the $c^{\pm 1/2}$. Since $c^{1/2}$ acts as the identity on $M$ and $N$, it suffices to prove that, for all $m' \in M_J$, $n' \in N_J$, $i \in J$, $r = \pm 1$,

$$(3) \quad \Delta(x_{i,r}^\pm). (m' \otimes n') - (\rho_J \otimes \rho_J)(\Delta_J(x_{i,r}^\pm)). (m' \otimes n') = 0.$$

The left-hand side of (3) obviously belongs to $M_J \otimes N_J$, since both terms involved do. On the other hand, by 2.1, the left-hand side also belongs to

$$\bigoplus_{\eta', \eta''} U_q(\hat{g})_{\eta', m' \otimes U_q(\hat{g})_{\eta''}, n'},$$

where the sum is over those $\eta', \eta'' \in Q \setminus Q_J$ such that $\eta' + \eta'' = \pm \alpha_i$. We may assume that $m' \in M_{\lambda - \xi'}$, $n' \in N_{\mu - \xi''}$, where $\xi', \xi'' \in Q_J^+$. Then, the weight of the first factor in a typical non-zero term in the above sum is $\lambda - \xi' + \eta'$. On the other hand, by (1), its weight must be of the form $\lambda - \eta$ for some $\eta \in Q_J^+$. Thus,

$$\eta' = \xi' - \eta.$$

But this is impossible, since $\xi' - \eta \in Q_J^+$ but $\eta' \notin Q_J^+$. Hence, the left-hand side of (3) is zero. \qed
Lemma 2.3. Let \( \emptyset \neq J \subseteq I \) define a connected subdiagram of the Dynkin diagram of \( \mathfrak{g} \). Let \( P \in \mathcal{P} \), and let \( v_P \) be a \( U_q(\hat{g}) \)-highest weight vector in \( V(P) \). Then, \( U_q(\hat{g}).v_P \) is an irreducible representation of \( U_q(\hat{g}) \) with highest weight \( P_J \).

Proof. Let \( W \) be a non-zero irreducible \( U_q(\hat{g}) \)-subrepresentation of \( U_q(\hat{g}).v_P \). Since \( U_q(\hat{g}).v_P \) is obviously preserved by the action of \( k_i \) for all \( i \in I \), it follows by 1.3 and 1.6(b) that we can choose \( 0 \neq w \in W \cap V(P)_\mu \), for some \( \mu \in \lambda - Q_J^+ \), such that

\[
x_{i,r}.w = 0, \\
\phi_{i,r}^\pm w = \Phi_{i,r}^\pm w,
\]

for some \( \Phi_{i,r}^\pm \in \mathbb{C} \) and all \( i \in J, r \in \mathbb{Z} \). Since \( \mu \in \lambda - Q_J^+ \), we see that (1) actually holds for all \( i \in I, r \in \mathbb{Z} \). Let \( W^+ \) be the linear subspace spanned by all elements \( w \in U_q(\hat{g}).v_P \cap V(P)_\mu \) satisfying (4) and (5) for fixed \( \Phi_{i,r}^\pm \). The relations in 1.2 show that the \( \phi_{i,r}^\pm \) preserve \( W^+ \) for all \( i \in I, r \in \mathbb{Z} \). Since the \( \phi_{i,r}^\pm \) act as commuting operators on \( V(P) \), and so on \( W^+ \), there exists \( w' \in W^+ \) satisfying both (4) and (5) for all \( i \in I, r \in \mathbb{Z} \). This means that \( w' \) must be a scalar multiple of \( v_P \), and so \( \mu = \lambda \). Thus, \( W^+ = \mathbb{C}.v_P \) and the lemma is established. \( \square \)

Lemma 2.4. Let \( \emptyset \neq J \subseteq I \) define a connected subdiagram of the Dynkin diagram of \( \mathfrak{g} \). Let \( \lambda \in P^+, \ P \in \mathcal{P}^\lambda, \) and \( \mu \in \lambda - Q_J^+ \). Then, if \( M \) is any highest weight representation of \( U_q(\hat{g}) \) with highest weight \( P \) and highest weight vector \( m \), we have

\[
m_\mu(M) = m_{\mu,j}(M_J),
\]

where, \( M_J = U_q(\hat{g}).m \).

Proof. If \( V \) is any type 1 representation of \( U_q(\hat{g}) \), and \( \mu \in P \), set

\[
V_\mu^+ = \{ v \in V_\mu \mid x_{i,0}.v = 0 \text{ for all } i \in I \}.
\]

Similarly, if \( W \) is any type 1 representation of \( U_q(\hat{g}) \), and \( \nu \in P_J \), define \( W^+_\nu \) in the obvious way. It is clear that

\[
m_\mu(M) = dim(M^+_\mu), \quad m_{\mu,j}(M_J) = dim((M_J)^+_\mu,j).
\]

Thus, it suffices to prove that

\[
M^+_\mu = (M_J)^+_\mu,j.
\]

If \( v \in M^+_\mu \), then, by 1.3(b), \( v \in \hat{U}^-\lambda - \mu.m \), where, for any \( \eta \in Q^+ \),

\[
\hat{U}^-\eta = \{ u \in \hat{U}^- \mid k_i u k_i^{-1} = e_i^{\eta(i)} u \text{ for all } i \in I \}.
\]

Since \( \lambda - \mu \in Q^+_J \), it follows that \( v \in \hat{U}^-\lambda - \mu.m \), and hence that \( v \in (M_J)^-_\mu,j \). Conversely, since conjugation by \( k_i \) clearly preserves \( (M_J)^+_\mu,j \subseteq M \) for all \( i \in I \), it suffices to prove that every \( U_q(\mathfrak{g}) \)-weight vector \( v \in (M_J)^-_\mu,j \) belongs to \( M^+_\mu \). If \( v \in M_\nu \), then \( \nu_J = \mu_J \), and \( \nu \in \lambda - Q^+_J \) by 1.3(b). This implies that \( \nu = \mu \), since restriction to \( J \) is injective on \( Q^+_J \). That \( x_{i,0}.v = 0 \) for all \( i \in I \setminus J \) is now clear, and the converse is proved. \( \square \)

The assumption that \( J \) is connected in 2.3 and 2.4 guaranteed that \( \mathfrak{g}_J \) was simple, and hence standard results about \( U_q(\mathfrak{g}) \) and \( U_q(\hat{g}) \) could be applied to \( U_q(\mathfrak{g}) \) and \( U_q(\hat{g}) \). The next two lemmas describe some consequences of restricting to disconnected subdiagrams.
Lemma 2.5. Let $J_1, J_2 \subseteq I$ be non-empty subsets for which $a_{ij} = 0$ if $i \in J_1$, $j \in J_2$ (in particular, $J_1 \cap J_2 = \emptyset$). Let $\lambda \in P^+$ and assume that $\lambda_{J_2} = 0$. If $P \in \mathcal{P}^\lambda$ and $\mu$ is a weight of $V(P)$ in $\lambda - Q_{J_1 \cup J_2}^+$, then $\mu \in \lambda - Q_{J_i}^+$.

Proof. By 1.3, every vector in $V(P)_\mu$ is a linear combination of vectors of the form

$$v = \sum \alpha_i v_i,$$

where $i, j \in J_1 \cup J_2$, $r_1, r_2, \ldots, r_k \in \mathbb{Z}$, $k \geq 1$. Since $a_{ij} = 0$ if $i \in J_1$, $j \in J_2$, the relations in 1.2 tell us that

$$[x_{i,r}^- x_{j,s}^-] = 0$$

if $i \in J_1$, $j \in J_2$, $r, s \in \mathbb{Z}$. Hence, we may assume that, in any expression (7), all of the $x_{i,r}^-$’s with $i \in J_2$ occur to the right of all $x_{i,r}^-$’s with $i \in J_1$. Since $\lambda_{J_2} = 0$, it follows that $x_{i,r}^- \cdot v = 0$ if $i \in J_2$, $r \in \mathbb{Z}$, so an expression of type (7) vanishes unless $i_1, \ldots, i_k$ all belong to $J_1$. □

If $\emptyset \neq J \subseteq I$, $\lambda \in P$, let $\lambda^J \in P$ be defined by

$$\lambda^J(i) = \begin{cases} \lambda(i) & \text{if } i \in J, \\ 0 & \text{if } i \notin J. \end{cases}$$

Similarly, if $P = (P_i)_{i \in I} \in \mathcal{P}$, let $P^J \in \mathcal{P}$ have $i^{th}$ component equal to $P_i$ if $i \in J$, and equal to 1 otherwise.

Lemma 2.6. Let

$$I = J_1 \amalg \{i_0\} \amalg J_2$$

(disjoint union), where $J_1$ and $J_2$ are such that $a_{ij} = 0$ if $i \in J_1$, $j \in J_2$. Let $\lambda \in P^+$, $P \in \mathcal{P}^\lambda$, and let $\mu \in P^+$ be of the form

$$\mu = \lambda - \sum_{j \in I, j \neq i_0} r_j \alpha_j, \quad (r_j \in \mathbb{N}).$$

Then, any $U_q(g)$–highest weight vector $v$ in $(V(P_{J_1 \amalg \{i_0\}}) \otimes V(P_{J_2}))_{\mu}$ (resp. in $(V(P_{J_1}) \otimes V(P_{J_2 \amalg \{i_0\}}))_{\mu}$) can be written

$$v = \sum w_t \otimes w'_t,$$

where $w_t \in V(P_{J_1 \amalg \{i_0\}})$, $w'_t \in V(P_{J_2})$ (resp. $w_t \in V(P_{J_1})$, $w'_t \in V(P_{J_2 \amalg \{i_0\}})$), and $w_t$, $w'_t$ are $U_q(g)$–highest weight vectors of weights $\lambda_{J_1 \amalg \{i_0\}} - \sum_{j \in J_1} r_j \alpha_j$ and $\lambda_{J_2} - \sum_{j \in J_2} r_j \alpha_j$ (resp. $\lambda_{J_1} - \sum_{j \in J_1} r_j \alpha_j$ and $\lambda_{J_2 \amalg \{i_0\}} - \sum_{j \in J_2} r_j \alpha_j$).

Proof. We consider the tensor product $V(P_{J_1 \amalg \{i_0\}}) \otimes V(P_{J_2})$ (the proof in the other case is similar). We can obviously write $v$ in the form (8) for some non–zero $U_q(g)$ weight vectors $w_t$ and $w'_t$, of weights $\mu_t$ and $\mu'_t$, say. We may assume, without loss of generality, that the $w'_t$ are linearly independent. Since $\mu_t + \mu'_t = \mu$ for all $t$, it now follows from 2.5 that $\mu_t \in \lambda_{J_1 \amalg \{i_0\}} - Q_{J_1}^+$, $\mu'_t \in \lambda_{J_2} - Q_{J_2}^+$. For weight reasons, it is clear that

$$\alpha^+_i w_t = 0 \text{ if } i \in J_1 \amalg \{i_0\}, \quad \alpha^+_i w'_t = 0 \text{ if } i \in J_1 \amalg \{i_0\}.$$
Hence, if \( j \in J_1 \), we have
\[
x_j^+.v = \sum_t (x_j^+.w_t \otimes k_j.w_t' + w_t \otimes x_j^+.w_t') = 0,
\]
so
\[
\sum_t q_j^{q(i,j)} x_j^+.w_t \otimes w_t' = 0.
\]
Since the \( w_t' \) are linearly independent, it follows that \( x_j^+.w_t = 0 \) for all \( j \in J_1 \).

Hence, each \( w_t \) is a \( U_q(\mathfrak{g}) \)-highest weight vector. Interchanging the roles of \( w_t \) and \( w_t' \) one shows that the \( w_t' \) are also \( U_q(\mathfrak{g}) \)-highest weight vectors, thus proving the lemma. □

3 The \( sl_{n+1}(\mathbb{C}) \) case

If \( \mathfrak{g} \) is of type \( A_n \), we take \( I = \{1, \ldots, n\} \), where \( a_{ii} = 2, a_{ij} = -1 \) if \( |i - j| = 1 \), and \( a_{ij} = 0 \) otherwise. The following result describes the minimal affinizations of \( \lambda \), for all \( \lambda \in P^+ \), in this case.

By the \( q \)-segment of length \( r \in \mathbb{N} \) and centre \( a \in \mathbb{C}^\times \), we mean the set of complex numbers \( \{aq^{-r+1}, aq^{-r+3}, \ldots, aq^{r-1}\} \).

**Theorem 3.1.** Let \( \mathfrak{g} = sl_{n+1}(\mathbb{C}) \), and let \( \lambda \in P^+ \). Then, \( Q^{\lambda} \) has a unique minimal element. Moreover, this element is represented by \( V(\mathbf{P}) \), for \( \mathbf{P} \in \mathcal{P}^{\lambda} \), if and only if, for all \( i \in I \) such that \( \lambda(i) > 0 \), the roots of \( P_i \) form the \( q \)-segment with centre \( a_i \), for some \( a_i \in \mathbb{C}^\times \), and length \( \lambda(i) \), where either
(a) for all \( i < j \), such that \( \lambda(i) > 0 \) and \( \lambda(j) > 0 \),
\[
\frac{a_i}{a_j} = q^{\lambda(i)+2(\lambda(i+1)+\cdots+\lambda(j-1))+j-i},
\]
or
(b) for all \( i < j \), such that \( \lambda(i) > 0 \) and \( \lambda(j) > 0 \),
\[
\frac{a_j}{a_i} = q^{\lambda(i)+2(\lambda(i+1)+\cdots+\lambda(j-1))+j-i}.
\]

In both cases, \( V(\mathbf{P}) \cong V(\lambda) \) as representations of \( U_q(\mathfrak{g}) \).

**Proof.** By Theorem 2.9 in [4], if \( \mathbf{P} \in \mathcal{P}^{\lambda} \), then \( V(\mathbf{P}) \) is irreducible as a representation of \( U_q(sl_{n+1}) \) if and only if the conditions in 3.1 hold. It is obvious that \([V(\mathbf{P})] \) is then the unique minimal element of \( Q^{\lambda} \). □

As an immediate consequence, we have

**Corollary 3.2.** Let \( \mathfrak{g} = sl_{n+1}(\mathbb{C}) \), and let \( \emptyset \neq J \subseteq I \) define a connected subdiagram of the Dynkin diagram of \( \mathfrak{g} \) (which is therefore of type \( A_{|J|} \)). Let \( \lambda \in P^+ \) and \( \mathbf{P} \in \mathcal{P}^{\lambda} \) be such that \( V(\mathbf{P}) \) is a minimal affinization of \( \lambda \). Then:
(a) \( V(\mathbf{P}_J) \) is a minimal affinization of \( \lambda_J \), and
(b) \( V(\mathbf{P}^J) \) is a minimal affinization of \( \lambda^J \). □

The following result is of crucial importance in the next section.
Proposition 3.3. Let $\mathfrak{g} = sl_{n+1}(C)$, let $\lambda \in P^+$, and let $P \in P^\lambda$ be such that
(a) $V(P)$ is not a minimal affinization of $\lambda$, and
(b) $V(P_{\Delta \{i\}})$ is a minimal affinization of $\lambda_{\Delta \{i\}}$, for $i = 1, n$.
Then, $m_{\lambda-\eta}(V(P)) > 0$.

Proof. As a representation of $U_q(\mathfrak{g})$, we have, by 1.5(a),

$$V(P) = V_0 \oplus \bigoplus_i V_i,$$

where $V_0 \cong V(\lambda)$, $V_i \cong V(\lambda - \eta_i)$, and $\eta_i \in Q^+$, $\eta_i \neq 0$ (the $\eta_i$ are not necessarily distinct). Let $v^+_P$ be a $U_q(\mathfrak{g})$–highest weight vector in $V(P)$, and $v^-_P$ a $U_q(\mathfrak{g})$–lowest weight vector. We claim that either $x^+_0.v^+_P \notin V_0$ or $x^-_0.v^-_P \notin V_0$. Indeed, suppose the contrary and let $v \in V_0$. Then,

$$v = x^- . v^+_P = x^+. v^-_P,$$

for some $x^\pm \in U^\pm$. Since $[x^+_0, x^\pm] = 0$ by the relations in 1.1, it follows that

$$x^+_0.v^+_P = x^\pm . x^+_0.v^+_P \in x^\mp . V_0 \subseteq V_0.$$

But, since $k_0$ acts on $V(P)$ as $(k_1 k_2 \ldots k_n)^{-1}$, the algebra of operators on $V(P)$ defined by the action of $U_q(\mathfrak{g})$ is generated by the action of $U_q(\mathfrak{g})$ and $x^\pm_0$. It follows that $V_0$ is a $U_q(\mathfrak{g})$–subrepresentation of $V(P)$, and hence that $V(P) = V_0$, contradicting 3.3(i).

Write $v_P$ for $v^+_P$ from now on, and assume, without loss of generality, that $x^+_0.v_P \notin V_0$. Then, $x^+_0.v_P$ must have non–zero component, with respect to the decomposition (9), in some $V_i$ with $\eta_i \neq 0$. Then, $\eta_i \leq \theta$, and it suffices to prove that $\eta_i = \theta$.

Suppose for a contradiction that $\eta_i < \theta$. Then,

$$\eta_i = \sum_{i=1}^n r_i \alpha_i,$$

where each $r_i = 0$ or 1, and at least one $r_i = 0$. If $r_1 = 0$ (resp. $r_n = 0$), applying 2.3 and 2.4 with $J = I \Delta \{1\}$ (resp. $J = I \Delta \{n\}$) gives

$$m_{\lambda-\eta}(V(P)) = m_{(\lambda-\eta)_J}(V(P_J)),$$

which vanishes by 3.1 because $V(P_J)$ is a minimal affinization of $\lambda_J$ by 3.2(b). But this is impossible, since $m_{\lambda-\eta}(V(P)) > 0$.

Thus, $r_i = 0$ for some $1 < i < n$. Let

$$J_1 = \{1, 2, \ldots, i-1\}, \quad J_2 = \{i+1, i+2, \ldots, n\}.$$

By 2.6, any $U_q(\mathfrak{g})$–highest weight vector $v$ in $(V(P_{J_1 \Delta \{i\}}) \otimes V(P_{J_2}))(\lambda-\eta)$ is of the form

$$v = \sum_r w_r \otimes w'_r,$$

where the $w_r$ and the $w'_r$ are $U_q(\mathfrak{g})$–highest weight vectors of weights $\lambda^{J_1 \Delta \{i\}} - \sum_{j<i} r_j \alpha_j$ and $\lambda^{J_2} - \sum_{j>i} r_j \alpha_j$, respectively. But, by 3.2(b) and 3.3(b), both $V(P_{J_1 \Delta \{i\}})$ and $V(P_{J_2})$ are minimal affinizations, so, by 3.1, we have $r_j = 0$ for all $j < i$ and for all $j > i$. But then $\eta_i = 0$, a contradiction. $\square$

We isolate the result in the $\alpha_i$ case; this was proved in [4].
Proposition 3.4. Let \( g = sl_2(\mathbb{C}) \). For any \( r \in \mathbb{N} \), \( Q^{r\lambda_1} \) has a unique minimal element. This element is represented by \( V(P) \), where \( P \) is any polynomial of degree \( r \) whose roots form a \( q \)-segment. If \( [W] \in Q^{r\lambda_1} \) is not minimal, then \( m_{(r-2)\lambda_1}(W) > 0 \). \( \square \)

4 The main reduction

In this section, we continue to assume that \( g \) is an arbitrary finite-dimensional complex simple Lie algebra. We show (see Proposition 4.2) that minimal affinizations remain minimal on restriction to certain ‘admissible’ subdiagrams of the Dynkin diagram of \( g \). To explain the meaning of ‘admissible’, suppose temporarily that \( g \) is of type \( D \) or \( E \). Let \( i_0 \in I \) be the unique node of the Dynkin diagram of \( g \) which is linked to three nodes other than itself. The set \( I \) can then be written as a disjoint union
\[
I = I_1 \amalg I_2 \amalg I_3 \amalg \{i_0\}
\]
such that
(i) \( I_r \cup \{i_0\} \) is of type \( A \), for \( r = 1, 2, 3 \),
(ii) for each \( r = 1, 2, 3 \), there exists exactly one \( i \in I_r \) such that \( a_{i_0i} \neq 0 \), and
(iii) \( a_{ij} = 0 \) if \( i \in I_r, j \in I_s, r \neq s \).
Clearly, \( I_1, I_2, I_3 \) are uniquely determined, up to a permutation.

Definition 4.1. Let \( J \) be a non-empty subset of \( I \). If \( g \) is not of type \( D \) or \( E \), \( J \) is admissible iff \( J \) is of type \( A \). If \( g \) is of type \( D \) or \( E \), then \( J \) is admissible iff the following two conditions are satisfied:
(i) \( J \subseteq I_r \cup \{i_0\} \) for some \( r = 1, 2, 3 \), and
(ii) \( J \) is connected (or, equivalently, \( J \) is of type \( A \)).

Proposition 4.2. Let \( J \subseteq I \) be admissible, let \( \lambda \in P^+ \), and let \( P = (P_i)_{i \in I} \in \mathcal{P}^{\lambda} \). If \( V(P) \) is a minimal affinization of \( \lambda \), then \( V(P_J) \) is a minimal affinization of \( \lambda_J \).

Remark. This result is definitely false if \( J \) is not admissible, as will become clear in Theorem 6.1.

Proof of 4.2. The proof proceeds by induction on \( |J| \). If \( |J| = 1 \), we must prove, in view of 3.4, that the roots of of each \( P_i \) form a \( q_i \)-segment.

Assume first that \( i \) is linked to exactly one other node in \( I \), and suppose for a contradiction that the roots of \( P_i \) do not form a \( q_i \)-segment. Let \( Q_i \) be any polynomial with constant term 1 such that \( deg(Q_i) = deg(P_i) \), and whose roots do form a \( q_i \)-segment. Let \( Q \) be the \( I \)-tuple which is equal to \( P \) except in the \( i^{th} \) place, where it equals \( Q_i \). We prove that \( [V(Q)] \prec [V(P)] \), giving the desired contradiction to the minimality of \( V(P) \).

Note that, by taking \( \mu = \lambda - \alpha_i \), \( J = \{i\} \) in 2.4, and using 2.3 and the second part of 3.4, it follows that
\[
m_{\lambda-\alpha_i}(V(P)) > 0, \quad m_{\lambda-\alpha_i}(V(Q)) = 0.
\]
Thus, \( [V(P)] \neq [V(Q)] \). To prove that \( [V(Q)] \prec [V(P)] \), we must prove that, for all \( \mu \in P \), either 1.8(i) or 1.8(ii) holds. We may assume that \( \mu = \lambda - \sum_{i} a_i \alpha_i \),
\[ s_j \geq 0, \text{ since otherwise } m_\mu(V(P)) = m_\mu(V(Q)) = 0. \] Suppose first that \( s_i > 0. \) We have just shown that, if \( \mu = \lambda - \alpha_i, \) then \( 1.8(\text{i}) \) holds, while if \( \mu < \lambda - \alpha_i, \) then \( 1.8(\text{ii}) \) holds with \( \nu = \lambda - \alpha_i. \) On the other hand, if \( s_i = 0, \) then applying \( 2.4 \) with \( J = I \setminus \{i\}, \) we have

\[ m_\mu(V(P)) = m_{\mu_j}(V(P)) = m_{\mu_j}(V(Q)) = m_\mu(V(Q)), \]

and so \( 1.8(\text{i}) \) holds (note that \( I \setminus \{i\} \) is connected because of our assumption on \( i. \))

Suppose now that node \( i \) is linked to two other nodes, and assume for a contradiction that the roots of \( P_i \) do not form a \( q_i \)-segment. It is easy to see that there exist subsets \( J_1, J_2 \subseteq I \) such that

(a) \( I = J_1 \amalg \{i\} \amalg J_2 \) (disjoint union),
(b) \( J_1 \cup \{i\} \) defines a diagram of type \( A, \)
(c) \( J_2 \) is connected, and
(d) \( a_{jk} = 0 \) if \( j \in J_1, k \in J_2. \)

Let \( P' \in \mathcal{P}^{\lambda_{J_1} \cup \{i\}} \) be such that \( V(P') \) is a minimal affinization of \( \lambda_{J_1} \cup \{i\}, \) and let \( Q = (Q_j)_{j \in I} \) be defined by

\[
Q_j = \begin{cases} P_j & \text{if } j \in J_2, \\ P'_j & \text{if } j \in J_1 \cup \{i\}. \end{cases}
\]

We claim that \( [V(Q)] < [V(P)] \), giving a contradiction as before.

As in the first part of the proof, we see that \( [V(Q)] \neq [V(P)] \) and that, in proving that \( [V(Q)] \preceq [V(P)] \), we need only consider weights \( \mu \in P \) of the form \( \mu = \lambda - \sum_{j \in I} s_j \alpha_j, \) where \( s_j \geq 0 \) for all \( j \in I, \) \( s_i = 0, \) and \( m_\mu(V(Q)) > 0. \) We show that, for such \( \mu, \)

\[ m_\mu(V(Q)) = m_\mu(V(P)), \]

establishing \( 1.8(\text{i}) \) and proving our claim.

We make use of the following lemma, which will also be needed later.

**Lemma 4.3.** Let \( i \in I \) be such that

\[ I = J_1 \amalg \{i\} \amalg J_2 \]

(disjoint union), where \( J_1 \) is of type \( A, \) \( J_2 \) is connected, and \( a_{jk} = 0 \) if \( j \in J_1, k \in J_2 \). Let \( \lambda \in P^+, \) \( Q \in \mathcal{P}^\lambda, \) and assume that \( V(Q_{J_1}) \) is a minimal affinization of \( \lambda_{J_1}. \) Let \( \mu \in P \) be of the form \( \mu = \lambda - \sum_{j \in I} s_j \alpha_j, \) where \( s_j \geq 0 \) for all \( j \), and \( s_i = 0. \) If \( m_\mu(V(Q)) > 0, \) then \( s_j = 0 \) for all \( j \in J_1 \) (and so \( \mu \in \lambda - Q_{J_2}^+ \)).

Assuming this lemma for the moment, we see that, if \( m_\mu(V(Q)) > 0, \) then \( \mu \in \lambda - Q_{J_2}^+. \) Since \( P_{J_2} = Q_{J_2}, \) \( 2.4 \) implies, as desired, that \( m_\mu(V(P)) = m_\mu(V(Q)). \)

We have now proved \( 4.2 \) when \( |J| = 1. \) For the inductive step, assume that \( |J| = r > 1 \) and suppose that the result is known when \( |J| < r. \) Proceeding by contradiction, we suppose that \( V(P_{J'}) \) is a non-minimal affinization of \( \lambda_J. \) Define a subset \( J' \subseteq I \) and a node \( j_0 \in J \) as follows:

(i) if \( J \) contains an element \( j \) that is linked to exactly one other element in \( I, \) choose \( j_0 = j \) and \( J' = \emptyset; \)

(ii) otherwise, choose \( J' \) to be disjoint from \( J \) such that \( J \cup J' \) is admissible and \( I \setminus (J \cup J') \) is connected, and let \( j_0 \) be the unique element of \( J \) that is connected to an element of \( J'. \)
See the diagrams on the next page.

By the induction hypothesis, $V(P_{J \setminus \{j_0\}})$ is a minimal affinization. Hence, by 3.1, we may choose $P'_{j}$, for $j \in J' \cup \{j_0\}$, such that $\deg(P_{j}) = \deg(P'_{j})$, and such that, if we define the $(J \cup J')$–tuple $R = (R_{j})_{j \in J \cup J'}$ by

$$R_{j} = \begin{cases} P_{j} & \text{if } j \in J \setminus \{j_0\}, \\ P'_{j} & \text{if } j \in J' \cup \{j_0\}, \end{cases}$$

then $V(R)$ is a minimal affinization of $\lambda_{J \cup J'}$. Now define $Q = (Q_{j})_{j \in I \in P_{\lambda}}$ by

$$Q_{j} = \begin{cases} P'_{j} & \text{if } j \in J' \cup \{j_0\}, \\ P_{j} & \text{otherwise}. \end{cases}$$

We prove that $[V(Q)] < [V(P)]$, giving the usual contradiction.

Note first that, by 3.2, $V(Q_{J})$ is a minimal affinization of $\lambda_{J}$, but by assumption, $V(P_{J})$ is not minimal. By 3.3,

$$m_{\lambda_{J}} - \sum_{i \in J} \alpha_{i}(V(P_{J})) > 0, \quad m_{\lambda_{J}} - \sum_{i \in J} \alpha_{i}(V(Q_{J})) = 0.$$

By 2.3 and 2.4,

$$m_{\lambda} - \sum_{i \in J} \alpha_{i}(V(P)) > 0, \quad m_{\lambda} - \sum_{i \in J} \alpha_{i}(V(Q)) = 0.$$

Hence, $[V(P)] \neq [V(Q)]$.

To prove that $[V(Q)] < [V(P)]$, we need only consider, as usual, weights $\mu$ such that $m_{\mu}(V(Q)) > 0$ and $\mu = \lambda - \eta$, where $\eta = \sum_{j \in I} s_{j} \alpha_{j}$ and each $s_{j} \geq 0$. By the second equation in (10), $\eta \neq \sum_{j \in J} \alpha_{j}$. If $\eta > \sum_{j \in J} \alpha_{j}$, then 1.8(ii) holds with $\nu = \lambda - \sum_{j \in J} \alpha_{j}$. Hence, we may assume that $s_{j_1} = 0$ for some $j_1 \in J$. Define a subset $J''$ of $J$ as follows:

(i) $J'' = J$ if $j = j_1$.
(ii) if \( j_0 \neq j_1 \), then \( J'' \) is the maximal connected subset of \( J \) containing \( j_0 \) but not \( j_1 \).

See the diagrams on the next page.

Set \( J_1 = J' \cup J'' \), \( J_2 = I \backslash (J_1 \cup \{j_1\}) \). Note that \( J_1 \) is of type \( A \) and \( J_2 \) is connected. Applying 4.3, we see that \( \mu \in \lambda - Q_{J_2}^+ \). Since \( P_{J_2} = Q_{J_2} \) it follows as usual from 2.3 and 2.4 that

\[
m_\mu(V(P)) = m_\mu(V(Q)),
\]

thus completing the proof of the inductive step. □

All that remains is to give the

**Proof of 4.3.** By 1.6(d), \( V(Q) \) is isomorphic to a subquotient of the tensor product \( V(Q^{J_1}) \otimes V(Q_{J_2 \cup \{i_0\}}) \); a fortiori, \( m_\mu(V(Q^{J_1}) \otimes V(Q_{J_2 \cup \{i_0\}})) > 0 \). By 2.6, if \( v \in (V(Q^{J_1}) \otimes V(Q_{J_2 \cup \{i_0\}}))_\mu \) is any \( U_q(\mathfrak{g}) \)-highest weight vector, then

\[
v = \sum_t w_t \otimes w'_t,
\]

where \( w_t \in V(Q^{J_1}) \) is a \( U_q(\mathfrak{g}) \)-highest weight vector of weight \( \lambda^{J_1} - \sum_{j \in J_1} s_j \alpha_j \), and \( w'_t \in V(Q_{J_2 \cup \{j_0\}}) \) is a \( U_q(\mathfrak{g}) \)-highest weight vector of weight \( \lambda^{J_2 \cup \{j_0\}} - \sum_{j \in J_2} s_j \alpha_j \). Since \( V(Q^{J_1}) \) is a minimal affinization of \( \lambda^{J_1} \), 3.1 implies that \( s_j = 0 \) for all \( j \in J_1 \) and hence \( \mu = \lambda - Q_{J_2}^+ \). □

5 Twisting with the Cartan involution

In this section, \( \mathfrak{g} \) is an arbitrary finite-dimensional complex simple Lie algebra. If \( V \) is any representation of \( U_q(\hat{\mathfrak{g}}) \), given by a homomorphism \( \rho : U_q(\hat{\mathfrak{g}}) \to \text{End}(V) \),
say, we denote by $\hat{\omega}^*(V)$ the representation $\rho \circ \hat{\omega}$, where $\hat{\omega}$ is the involution of $U_q(\hat{g})$ defined in 1.4(b). Let $V^*$ be the $U_q(\hat{g})$-representation dual to $V$: recall that the action of $U_q(\hat{g})$ on $V^*$ is defined by

$$(x.f)(v) = f(S(x).v),$$

where $f \in V^*$, $x \in U_q(\hat{g})$, and $S : U_q(\hat{g}) \to U_q(\hat{g})$ is the antipode. It is clear that, if $V$ is an irreducible representation of $U_q(\hat{g})$, then $V^*$ and $\hat{\omega}^*(V)$ are both irreducible representations as well. The purpose of this section is to give the defining polynomials of $\hat{\omega}^*(V)$ and $V^*$ in terms of the defining polynomials of $V$. We need this result in the next section to prove the uniqueness of certain minimal affinizations.

Let $w_0$ be the longest element of the Weyl group of $g$, and let $i \rightarrow i$ be the bijection $I \rightarrow I$ such that $w_0(\alpha_i) = -\alpha_i$. It is well known that

$$\omega^*(V(\lambda)) \cong V(-w_0(\lambda)), \quad V(\lambda)^* \cong V(-w_0(\lambda)),$$

for all $\lambda \in P^+$, where $\omega$ is the Cartan involution of $U_q(g)$.

**Proposition 5.1.** Let $\lambda \in P^+$, $P = (P_i)_{i \in I} \in \mathcal{P}^\lambda$, and let

$$P_i(u) = \prod_{r=1}^{\lambda(i)} (1 - a_{r,i}^{-1}u), \quad (a_{r,i} \in \mathbb{C}^\times).$$

(a) Define $P^\hat{\omega} = (P_i^\hat{\omega})_{i \in I} \in \mathcal{P}^{-w_0(\lambda)}$ by

$$P_i^\hat{\omega}(u) = \prod_{r=1}^{\lambda(i)} (1 - q_r^2 a_{r,i}u).$$

Then, there exists $t \in \mathbb{C}^\times$, independent of $i \in I$, such that

$$\hat{\omega}^*(V(P)) \cong \tau_t^*(V(P^\hat{\omega}))$$

as representations of $U_q(\hat{g})$.

(b) Define $P^* = (P_i^*)_{i \in I} \in \mathcal{P}^{-w_0(\lambda)}$ by

$$P_i^*(u) = \prod_{r=1}^{\lambda(i)} (1 - a_{r,i}^{-1}u).$$

Then, there exists $t^* \in \mathbb{C}^\times$ such that, as representations of $U_q(\hat{g})$,

$$V(P)^* \cong \tau_{t^*}^*(V(P^*))$$

**Proof.** We first prove that it suffices to establish the proposition in the case when $\lambda$ is fundamental. We do this for part (b); the proof for part (a) is similar (see also [2], where the corresponding result was proved for rank two algebras). By 1.6(d), we see that $V(P)$ is the unique irreducible subquotient of

$$\bigotimes \bigotimes V(\lambda_i, a_{i,r}).$$
which contains the tensor product of the highest weight vectors (the tensor product of the representations can be taken in any order). It is not hard to see that

$$V(\lambda_i, a_{i,r})^* \cong V(\lambda_i^*, a_{i,r}^*),$$

for some $a_{i,r}^* \in \mathbb{C}^\times$ (this follows from Proposition 3.3 in [2]). Hence, $V(P)^*$ is the unique irreducible subquotient of

$$\bigotimes_{i \in I} \bigotimes_{r=1}^{\lambda(i)} V(\lambda_i^*, a_{i,r}^*)$$

containing the tensor product of the highest weight vectors. Thus, by 1.6(d), it suffices to calculate the $a_{i,r}^*$.

The proof of 5.1 in the fundamental case is a consequence of the following lemma.

**Lemma 5.2.** Suppose that $a_{i,j} \neq 0$, $i \neq j$, and that $a_i, a_j \in \mathbb{C}^\times$. Then,

(a) $m_{\lambda_i + \lambda_j - \alpha_i - \alpha_j}(V(\lambda_i, a_i) \otimes V(\lambda_j, a_j)) = 1$;

(b) if $v_i \in V(\lambda_i, a_i)$, $v_j \in V(\lambda_j, a_j)$ are $U_q(g)$-highest weight vectors, and $M = U_q(g). (v_i \otimes v_j) \subset V(\lambda_i, a_i) \otimes V(\lambda_j, a_j)$, then $m_{\lambda_i + \lambda_j - \alpha_i - \alpha_j}(M) = 0$ iff

$$\frac{a_i}{a_j} = q^{-(3d_i + d_j - 1)};$$

(c) Let $v \in (V(\lambda_i, a_i) \otimes V(\lambda_j, a_j))_{\lambda_i + \lambda_j - \alpha_i - \alpha_j}$ be a $U_q(g)$-highest weight vector. Then, $v$ is also $U_q(g)$-highest weight iff

$$\frac{a_i}{a_j} = q^{3d_j + d_i - 1}.$$

Assuming this lemma, 5.1(a) is proved as follows. Using the notation introduced in 5.2, we have

$$\hat{\omega}^*(M) \subseteq \hat{\omega}^*(V(\lambda_j, a_j)) \otimes \hat{\omega}^*(V(\lambda_i, a_i)).$$

As in the proof of Proposition 5.5 in [2],

$$\hat{\omega}^*(V(\lambda_i, a_i)) \cong V(\lambda_i^*, \overline{\alpha_i})$$

for some $\overline{\alpha_i} \in \mathbb{C}^\times$. Identifying the two representations above, we thus have

$$\hat{\omega}^*(M) \subseteq V(\lambda_i^*, \overline{\alpha_i}) \otimes V(\lambda_j^*, \overline{\alpha_j}).$$

Now, since $m_{\lambda_i + \lambda_j}(M) = 1$, we have $m_{\lambda_i + \lambda_j}(\hat{\omega}^*(M)) = 1$ by the discussion preceding 5.1. Hence, $\hat{\omega}^*(M)$ contains $U_q(g). (\nu_j \otimes \nu_i) \subseteq V(\lambda_j^*, \overline{\alpha_j}) \otimes V(\lambda_i^*, \overline{\alpha_i})$. Assume now that $a_i/a_j = q^{-(3d_i + d_j - 1)}$. Then, by 5.2(b), $m_{\lambda_i + \lambda_j - \alpha_i - \alpha_j}(M) = 0$, hence $m_{\lambda_i + \lambda_j - \alpha_i - \alpha_j}(\hat{\omega}^*(M)) = 0$. A fortiori, $m_{\lambda_i + \lambda_j - \alpha_i - \alpha_j}(U_q(g). (\nu_j \otimes \nu_i)) = 0$. By 5.2(b) again, $\overline{\alpha_j}/\overline{\alpha_i} = q^{-(3d_j + d_i - 1)}$. Since $d_j = d_i$ for all $i \in I$, we get

$$q_2^2 \overline{\alpha_j} a_j = q_i^2 \overline{\alpha_i} a_i,$$

from which 5.1(a) follows for fundamental representations.
We now prove 5.1(b). We continue to assume that

\[ \frac{a_i}{a_j} = q^{-(3d_i + d_j - 1)}. \]

Let \( a_i^* \in \mathbb{C}^\times \) be such that

\[ V(\lambda_i, a_i)^* = V(\lambda_i^*, a_i^*). \]

By standard properties of duals, \( M^* \) is a quotient of \( V(\lambda_i^*, a_i^*) \). Since \( m_{\lambda_i,\lambda_j - \alpha_i - \alpha_j}(M) = 0 \), we have \( m_{\lambda_i + \lambda_j - \alpha_i - \alpha_j}(M^*) = 0 \). Applying 5.2(c), we see that

\[ \frac{a_j^*}{a_i^*} = q^{3d_i + d_j - 1} = q^{3d_j + d_j - 1}. \]

This gives

\[ \frac{a_i}{a_j} = \frac{a_j}{a_i}, \]

from which 5.1(b) follows. \( \square \)

**Proof of 5.2(a).** It suffices to prove that, if \( a_{ij} \neq 0, i \neq j, \) and \( a_i \in \mathbb{C}^\times \), then

\[ m_{\lambda_i - \alpha_i}(V(\lambda_i, a_i)) = m_{\lambda_i - \alpha_i - \alpha_j}(V(\lambda_i, a_i)) = 0. \]

For, this result clearly implies that

\[ m_{\lambda_i,\lambda_j - \alpha_i - \alpha_j}(V(\lambda_i, a_i) \otimes V(\lambda_j, a_j)) = m_{\lambda_i + \lambda_j - \alpha_i - \alpha_j}(V(\lambda_i \otimes V(\lambda_j)), \]

and it easy to see that the last multiplicity is one.

It suffices to prove \( (11) \) when \( g \) is of rank 2. For, if \( J = \{i, j\} \subseteq I, \) then, by the rank 2 case, \( m_{(\lambda_i - \alpha_i - \alpha_j)}(V((\lambda_i, a_i)) = 0, \) so, by 2.4, \( m_{\lambda_i - \alpha_i - \alpha_j}(V(\lambda_i, a_i)) = 0. \)

If \( g \) is of type \( A_2, \) \( (11) \) is obvious, since, by 3.1, \( V(\lambda_i, a_i) \) is an irreducible representation of \( U_q(\mathfrak{g}). \)

If \( g \) is of type \( C_2 \) or \( G_2, \) this was proved in [2], Proposition 5.4(i). \( \square \)

**Proof of 5.2(b), (c).** Taking \( J = \{i, j\} \) we see that, by Proposition 2.2, it suffices to prove this result in the rank two case. If \( \mathfrak{g}_J \) is of type \( A_2, \) both parts (b) and (c) are established in the proof of Lemma 4.1 in [4].

If \( \mathfrak{g} \) is of type \( C_2 \) or \( G_2, \) then \( i = 7 \) for \( i = 1, 2. \) Part (b) was established in Proposition 5.4(c) in [2]. To prove (c), notice that, by (a), \( v \) is a \( U_q(\hat{\mathfrak{g}}) \)-highest weight vector in \( V(\lambda_i, a_i) \otimes V(\lambda_j, a_j) \) iff \( m_{\lambda_i + \lambda_j - \alpha_i - \alpha_j}(M^*) = 0, \) where \( M^* = U_q(\hat{\mathfrak{g}})(v_j \otimes v_i) \subseteq V(\lambda_j, a_j)^* \otimes V(\lambda_i, a_i)^*. \) Writing \( V(\lambda_i, a_i)^* \cong V(\lambda_i, a_i^*), \) we see from part (b) that

\[ \frac{a_j}{a_i} = q^{-(3d_j + d_i - 1)}. \]

A direct calculation in the rank two case now gives that

\[ a_i^* = ta_i, \]

and combining with \( (12) \) gives the desired result.
6 The simply--laced case

In this section, we assume that \( g \) is of type \( D \) or \( E \). Let \( I_1, I_2, I_3 \subset I \), and \( i_0 \in I \), be as defined at the beginning of Section 4. If \( \lambda \in P \), define subsets \( I_r(\lambda) \subseteq I_r \), \( r = 1, 2, 3 \), by the following conditions:

(i) \( \lambda_{I_r(\lambda)} = 0 \),
(ii) \( I_r(\lambda) \) is connected,
(iii) \( I_r(\lambda) \cup \{i_0\} \) is of type \( A \), and
(iv) \( I_r(\lambda) \) is maximal with respect to properties (i)--(iii).

Note that \( I_r(\lambda) \) may be empty.

The following theorem is the main result of this paper.

**Theorem 6.1.** Let \( g \) be of type \( D \) or \( E \). Let \( \lambda \in P^+ \) and assume that \( \lambda(i_0) \neq 0 \).

(a) If \( I_r(\lambda) = I_r \) for some \( r \in \{1, 2, 3\} \), then \( Q^\lambda \) has a unique minimal element. This element is represented by \( V(P) \), where \( P \in P^\lambda \), if and only if \( V(P_{I_1 \setminus I}) \) is a minimal affinization of \( \lambda_{I_1 \setminus I} \).

(b) Suppose that, for all \( r \in \{1, 2, 3\} \), \( I_r(\lambda) \neq I_r \). Then, \( Q^\lambda \) has exactly three minimal elements. In fact, if \( P \in P^\lambda \), then \( V(P) \) is minimal if and only if there exists \( r, s \in \{1, 2, 3\} \), \( r \neq s \), such that \( V(P_{I_1 \setminus I_r}) \) and \( V(P_{I_1 \setminus I_s}) \) are minimal affinizations of \( \lambda_{I_1 \setminus I_r} \) and \( \lambda_{I_1 \setminus I_s} \), respectively.

**Remarks.** 1. Note that, for any \( r \in \{1, 2, 3\} \), \( I_1 \setminus I_r \) is of type \( A \), so we know from the results of Section 3 precisely when \( V(P_{I_1 \setminus I_r}) \) is minimal.

2. It might be helpful to illustrate this theorem diagrammatically. First, if \( g \) is of type \( A \), \( \lambda \in P^+ \), \( P = (P_i)_{i \in I} \in P^\lambda \), and if the roots of \( P_i \) form a \( q \)-segment with centre \( a_i \) for all \( i \in I \), then we draw an arrow above the Dynkin diagram of \( g \),

\begin{center}
or
\end{center}

according as the \( a_i \) satisfy condition (a) or condition (b) in 3.1, respectively. If \( g \) is of type \( D \) or \( E \), the theorem says that, under the hypotheses of 6.1(a), the minimal element of \( Q^\lambda \) is given by the diagram

\begin{center}
and under the hypotheses of 6.1(b), the three minimal elements of \( Q^\lambda \) are given by the diagrams
\end{center}
Proof of 6.1. Suppose first that $I_r(\lambda) = I_r$ for all $r$. Then, if $V(\mathbf{P})$ is minimal, by 4.2 the roots of $P_i$ form a $q_i$-segment, and obviously $P_i = 1$ if $i \neq i_0$. By 1.7, $V(\mathbf{P})$ is unique up to twisting with an automorphism $\tau_t$, for some $t \in \mathbb{C}^\times$. In particular, the element $[V(\mathbf{P})] \in Q^\lambda$ is unique and part (a) is proved in this case.

Suppose next that $I_r(\lambda) = I_r$ for exactly two values of $r$, say $r = 1, 2$, without loss of generality. If $V(\mathbf{P})$ is a minimal affinization of $\lambda$, then, by 4.2, $V(\mathbf{P}_{I_3 \cup \{i_0\}})$ is a minimal affinization of $\lambda_{I_3 \cup \{i_0\}}$. By 3.1, for all $i \in I_3 \cup \{i_0\}$ such that $\lambda(i) > 0$, the roots of $P_i$ form a $q_i$-segment with centre $a_i$, say, where $a_i/a_{i_0}$ satisfies either condition (a) or condition (b) in 3.1. By 5.1, $V(\mathbf{P})$ satisfies condition (a) iff $(\hat{\omega}(V(\mathbf{P})))^*$ satisfies condition (b). Since $[V(\mathbf{P})] = [\hat{\omega}(V(\mathbf{P})))^*$ it follows that the equivalence class of $V(\mathbf{P})$ is uniquely determined.

For the remainder of the proof of 6.1(a), and also for the proof of 6.1(b), we introduce the following notation. If $r \in \{1, 2, 3\}$, let $i_r \in I_r$ be the unique index such that

(i) $\lambda(i_r) \neq 0$, and
(ii) $\{i_r\} \cup \{i_0\} \cup I_r(\lambda)$ is of type $A$.

Note that, if $I_r(\lambda) \neq I_r$, then $i_r$ and $i_0$ are the nodes of $\{i_r\} \cup \{i_0\} \cup I_r(\lambda)$ which are connected to only one other node (and $i_r = i_0$ if $I_r(\lambda) = I_r$).

Define $\theta_r(\lambda) = \sum_{i \in I_r(\lambda)} \alpha_i \in Q^+$. 

Proposition 6.2. Let $\lambda \in P^+$ satisfy $\lambda(i_0) > 0$, and let $\mathbf{P} \in \mathcal{P}^\lambda$. Assume that $V(\mathbf{P}_{I_r \cup \{i_0\}})$ is minimal for $r = 1, 2, 3$.

(i) Let $\{r, s, t\} = \{1, 2, 3\}$. The following statements are equivalent:
(a) $V(\mathbf{P}_{I_r \cup \{i_0\}})$ is a minimal affinization of $\lambda_{I_r \cup \{i_0\}}$;
(b) $V(\mathbf{P}_{I_r(\lambda) \cup \{i_0, i_r, i_s\} \cup I_r(\lambda)})$ is a minimal affinization of $\lambda_{I_r(\lambda) \cup \{i_0, i_r, i_s\} \cup I_r(\lambda)}$;
(c) $m_{\lambda - \alpha_{i_0} - \alpha_{i_s}} - \theta(\lambda) - \theta_r(\lambda) \in V(\mathbf{P})$.

(ii) Let $0 \neq \eta = \sum_j s_j \alpha_j \in Q^+$ be such that $m_{\lambda - \eta} V(\mathbf{P}) > 0$. Then,
(a) $s_{i_0} \neq 0$;
(b) if $j \in I_r$ is such that $s_j > 0$, and if $J \subseteq I_r \cup \{i_0\}$ is the connected subset of type $A$ which has $j$ and $i_0$ as its ‘end’ nodes, then $s_i > 0$ for all $i \in J$;
(c) if $I_r \neq I_r(\lambda)$ then either $s_j > 0$ for all $j \in I_r(\lambda)$ or $s_j = 0$ for all $j \in I_r \setminus I_r(\lambda)$.

Proof of 6.2. (i) The equivalence (a) \(\Leftrightarrow\) (b) is obvious from 3.1. The equivalence (b) \(\Leftrightarrow\) (c) follows from 2.4 and 2.3.
(ii) Suppose that \( m_{\mu}(V(P)) > 0 \). Write \( \mu = \lambda - \eta \), where \( \eta = \sum_j s_j \alpha_j \). Suppose that \( s_{i_0} = 0 \). Let \( \{r, s, t\} = \{1, 2, 3\} \). Since \( V(P_{I_r \cup \{i_0\}}) \) is minimal of type A, it follows from 2.4 and 3.1 that \( m_{\nu}(V(P_{I_r \cup \{i_0\}})) = 0 \) where \( \nu = \lambda_{I_r \cup \{i_0\}} - \eta' \), and \( \eta' \in Q^+_{I_r \cup \{i_0\}} \). Applying 2.6 to the decomposition \( I = I_r \cup \{i_0\} \cup (I_s \cup I_t) \) now shows that \( s_i = 0 \) for all \( i \in I_r, r = 1, 2, 3 \). Hence, \( \eta = 0 \), contradicting our assumption. This proves (a).

Let \( j \in I_r \) be such that \( s_j > 0 \) and let \( J \subseteq I_r \cup \{i_0\} \) be the type A subset which has \( j \) and \( i_0 \) as its ‘end’ nodes. Suppose that \( s_i = 0 \) for some \( i \in J \), say \( i = j' \). We have a unique decomposition

\[
I = J' \amalg \{j'\} \amalg J''
\]

(disjoint union), where \( j \in J' \subset I_r, i_0 \in J'' \cup \{j'\} \), \( J' \) is of type A and \( a_{rs} = 0 \) if \( r \in J', s \in J'' \). Applying 2.6, 2.4 and 3.1 again gives that \( s_i = 0 \) for all \( i \in J' \), contradicting \( s_j \neq 0 \). This proves (b).

Part (c) now follows by considering separately the cases \( s_i > 0 \) and \( s_i = 0 \).

We now return to the proof of 6.1(a) in the case \( I_1(\lambda) = I_1, I_r(\lambda) \neq I_r, r = 2, 3 \). Suppose for a contradiction that \( V(P_{I_1 \setminus I_1}) \) is not minimal. By 6.2(i) this means that

\[
\tag{13}
m_{\lambda - \theta_2(\lambda) - \theta_3(\lambda) - \alpha_{i_2} - \alpha_{i_3} - \alpha_{i_0}}(V(P)) > 0.
\]

By 3.1, there exists a unique \( Q = (Q_i)_{i \in I} \in \mathcal{P}^\lambda \) such that

(i) \( Q_i = 1 \) if \( i \in I_1 \);
(ii) \( Q_i = P_i \) if \( i \in I_2 \cup \{i_0\} \);
(iii) \( V(Q_{I_1 \setminus I_1}) \) is a minimal affinization of \( \lambda_{I_1 \setminus I_1} \).

We prove that \( [V(Q)] < [V(P)] \), contradicting the minimality of \( [V(P)] \).

Clearly, \( [V(Q)] \neq [V(P)] \), since, by 6.2(i),

\[
\tag{14}
m_{\lambda - \theta_2(\lambda) - \theta_3(\lambda) - \alpha_{i_2} - \alpha_{i_3} - \alpha_{i_0}}(V(Q)) = 0.
\]

Suppose that \( \mu \in P^+ \) is such that \( m_{\mu}(V(Q)) > 0 \), and let \( \mu = \lambda - \eta, \eta \in Q^+ \).

Write \( \eta = \sum_j s_j \alpha_j \). If \( s_{i_2} > 0 \) and \( s_{i_3} > 0 \), it follows from 6.2(ii)(a) that \( \eta > \theta_2(\lambda) + \theta_3(\lambda) + \alpha_{i_0} + \alpha_{i_2} + \alpha_{i_3} \). Equations (13) and (14) now show that condition 1.8(ii) is satisfied with \( \nu = \lambda - \theta_2(\lambda) - \theta_3(\lambda) - \alpha_{i_2} - \alpha_{i_3} - \alpha_{i_0} \).

If \( s_{i_2} \geq 0 \) and \( s_{i_3} = 0 \), let \( J = I_1 \cup I_2 \cup I_3(\lambda) \cup \{i_0\} \). By 2.4 and the fact that \( P_J = Q_J \), we get

\[
m_{\mu}(V(Q)) = m_{\mu,J}(V(Q_J)) = m_{\mu,J}(V(P_J)) = m_{\mu}(V(P)),
\]

so 1.8(i) is satisfied. If \( s_{i_2} = 0 \) and \( s_{i_3} > 0 \), let \( J' = I_1 \cup I_2(\lambda) \cup I_3 \cup \{i_0\} \). By 2.4, it suffices to show that \( m_{\mu,J'}(V(P_{J'})) = m_{\mu,J'}(V(Q_{J'})) \). Note that \( P_i = Q_i = 1 \) if \( i \in J' \setminus (I_3 \cup \{i_0\}) \), and that, if \( i \in I_3 \cup \{i_0\} \), then, by 4.2 and 3.1, there exists \( a_i, \gamma \in \mathbb{C}^\times \) such that the roots of \( P_i \) (resp. \( Q_i \)) form a \( q_i \)-segment with centre \( a_i \) (resp. \( \gamma a_i^{-1} \)). It follows from 5.1 that

\[
(\tau^*(V(P_{J'})))^* \cong \tau^*(V(Q_{J'}))
\]
for some $t \in \mathbb{C}^\times$ (here, $\hat{\omega}$ and $\tau_t$ are the appropriate automorphisms of $U_q(\hat{\mathfrak{g}}_{ir})$). This proves our assertion.

We have now shown that, if $V(P)$ is a minimal affinization of $\lambda$, then $V(P_{I \setminus I_1})$ is a minimal affinization of $\lambda_{I \setminus I_1}$. Conversely, suppose that $V(P_{I \setminus I_1})$ is minimal. Choose $Q \in \mathcal{P}^\lambda$ such that $V(Q)$ is minimal and $[V(Q)] \leq [V(P)]$. By the first part of the proof, $V(Q_{I \setminus I_1})$ is minimal. By 3.1, there exists $\gamma \in \mathbb{C}^\times$ such that either

(i) for all $i \in I \setminus I_1$, there exists $a_i \in \mathbb{C}^\times$ such that the roots of $P_i$ (resp. $Q_i$) form a $q_i$-segment with centre $a_i$ (resp. $\gamma a_i$),

or

(ii) for all $i \in I \setminus I_1$, there exists $a_i \in \mathbb{C}^\times$ such that the roots of $P_i$ (resp. $Q_i$) form a $q_i$-segment with centre $a_i$ (resp. $\gamma a_i^{-1}$).

Since $P_i = Q_i = 1$ for $i \in I_1$, it follows from 1.7 and 5.1 that either

(i) $V(P) \cong \tau^* \nu(V(Q))$,

or

(ii) $V(P) \cong (\hat{\omega}^* \tau^* \nu(V(Q)))^*$,

for some $t \in \mathbb{C}^\times$. But, in both cases, $[V(P)] = [V(Q)]$, so $[V(P)]$ is minimal.

This completes the proof of 6.1(a).

Suppose now, for 6.1(b), that $I_r(\lambda) \neq I_r$ for all $r$, that $V(P)$ is a minimal affinization of $\lambda$, but that neither $V(P_{\setminus I_2})$ nor $V(P_{\setminus I_3})$ is minimal. By 3.1 and 4.2, it follows that $V(P_{I \setminus I_1})$ is not minimal either (this is clear from the diagrams in the second remark following the statement of 6.1). By 6.2(i),

\[
(15) \quad m_{\lambda - \alpha_{i_0} - \alpha_{i_r} - \alpha_{i_3} - \theta_r(\lambda) - \theta_3(\lambda)}(V(P)) > 0
\]

for all $r \neq s \in \{1, 2, 3\}$. By 3.1 again, there exists $Q \in \mathcal{P}^\lambda$ such that $V(Q_{I \setminus I_1})$ is minimal and $Q_i = P_i$ if $i \in I_1 \cup I_2$. Notice that then $V(Q_{I \setminus I_2})$ is also a minimal affinization of $\lambda_{I \setminus I_2}(\lambda)$ and hence by 6.2(i)

\[
(16) \quad m_{\lambda - \alpha_{i_0} - \alpha_{i_r} - \alpha_{i_3} - \theta_r(\lambda) - \theta_3(\lambda)}(V(Q)) = 0, \quad r = 1, 2
\]

By (15), $[V(Q)] \neq [V(P)]$ and we prove next that $[V(P)] \prec [V(Q)]$.

Suppose that $\mu = \lambda - \eta$, where $\eta \in Q^+$ is such that $m_\mu(V(Q)) > 0$.

If either $s_{i_1}, s_{i_2} > 0$ or $s_{i_2}, s_{i_3} > 0$, then 6.2(ii)(a), together with equations (15) and (16), shows that 1.8(ii) holds with

\[
\nu = \lambda - \alpha_{i_0} - \alpha_{i_1} - \alpha_{i_3} - \theta_r(\lambda) - \theta_3(\lambda)
\]

for $r = 1$ or 2.

If $s_{i_3} = 0$, set $J = I_1 \cup I_2 \cup \{i_0\} \cup I_3(\lambda)$. By 6.2(ii)(b), $\eta \in Q^+_J$, and noting that $P_J = Q_J$, we get from 2.4 that

\[
m_\mu(V(P)) = m_\mu(V(P_{I})) = m_\mu(V(Q_{I})) = m_\mu(V(Q)).
\]

Finally if $s_{i_3} > 0$ and $s_{i_1} = s_{i_2} = 0$, take $J = I_1(\lambda) \cup I_2(\lambda) \cup \{i_0\} \cup I_3$. By 6.2(ii)(b), $\eta \in Q^+_J$. The same argument used in this case in 6.1(a) shows that, for some $\gamma \in \mathbb{C}^\times$,

\[
V(P_J) \cong \tau^*(V(Q_J)) \quad \text{or} \quad V(P_J) \cong (\hat{\omega}^* \tau^*(V(Q_J)))^*.
\]
In both cases, $m_\mu(V(P)) = m_\mu(V(Q))$, so 1.8(i) is satisfied.

We have now shown that, if $V(P)$ is minimal, then, for some $r \neq s$ in $\{1, 2, 3\}$, $P \in \mathcal{P}_{r,s}^\lambda$, where

$$\mathcal{P}_{r,s}^\lambda = \{P \in \mathcal{P}^\lambda \mid V(P_{I\setminus I_r}) \text{ and } V(P_{I\setminus I_s}) \text{ are both minimal}\}.$$ 

Note that, by 1.7, 3.1 and 5.1, if $P, Q \in \mathcal{P}_{r,s}^\lambda$, then $[V(P)] = [V(Q)]$. Moreover, if $P \in \mathcal{P}_{r,s}^\lambda$ and $t \in \{1, 2, 3\}\{r, s\}$, then, by 3.1, $V(P_{I\setminus I_t})$ is not minimal, and hence by 6.2(i),

$$m_{\lambda - \alpha_{i_0} - \alpha_{i_1} - \alpha_{i_2} - \theta_r(\lambda) - \theta_s(\lambda)}(V(P)) > 0,$$

$$m_{\lambda - \alpha_{i_0} - \alpha_{i_1} - \alpha_{i_2} - \theta_r(\lambda) - \theta_t(\lambda)}(V(P)) = 0,$$

$$m_{\lambda - \alpha_{i_0} - \alpha_{i_1} - \alpha_{i_2} - \theta_t(\lambda) - \theta_s(\lambda)}(V(P)) = 0.$$

It follows that, if $P^{r,s} \in \mathcal{P}_{r,s}^\lambda$, then the $[V(P^{r,s})]$ for $r < s$ in $\{1, 2, 3\}$, are distinct elements of $Q_\lambda$. We prove that all three elements are minimal. For this, it suffices to show that none of them is strictly less than the other two.

Suppose, for example, that $[V(P^{1,2})] \prec [V(P^{1,3})]$. Since

$$m_{\lambda - \alpha_{i_0} - \alpha_{i_1} - \alpha_{i_2} - \theta_1(\lambda) - \theta_2(\lambda)}(V(P^{1,2})) > 0,$$

$$m_{\lambda - \alpha_{i_0} - \alpha_{i_1} - \alpha_{i_2} - \theta_1(\lambda) - \theta_2(\lambda)}(V(P^{1,3})) = 0,$$

there exists $\eta \in Q^+$ such that $\eta < \alpha_{i_0} + \alpha_{i_1} + \alpha_{i_2} + \theta_1(\lambda) + \theta_2(\lambda)$ and $m_{\lambda - \eta}(V(P^{1,3})) > m_{\lambda - \eta}(V(P^{1,2}))$. But this is impossible, since $V(P^{1,3}_{I\setminus I_3})$ is minimal, so by 2.4 and 3.3, $m_{\lambda - \eta}(V(P^{1,3})) = 0$.

The proof of Theorem 6.1 is complete. □

References

1. Beck, J., Braid group action and quantum affine algebras, preprint, MIT, 1993.
2. Chari, V., Minimal affinizations of representations of quantum groups: the rank 2 case, preprint, 1994.
3. Chari, V. and Pressley, A. N., Quantum affine algebras, Commun. Math. Phys. 142 (1991), 261-83.
4. Chari, V. and Pressley, A. N., Small representations of quantum affine algebras, Lett. Math. Phys. 30 (1994), 131-45.
5. Chari, V. and Pressley, A. N., A Guide to Quantum Groups, Cambridge University Press, Cambridge, 1994.
6. Chari, V. and Pressley, A. N., Quantum affine algebras and their representations, preprint, 1994.
7. Drinfel’d, V. G., A new realization of Yangians and quantized affine algebras, Soviet Math. Dokl. 36 (1988), 212-6.
8. Lusztig, G., Introduction to Quantum Groups, Progress in Mathematics 110, Birkhäuser, Boston, 1993.