ON THE JUMPING LINES OF BUNDLES OF LOGARITHMIC VECTOR FIELDS ALONG PLANE CURVES

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Abstract. For a reduced curve $C : f = 0$ in the complex projective plane $\mathbb{P}^2$, we study the set of jumping lines for the rank two vector bundle $T(C)$ on $\mathbb{P}^2$, whose sections are the logarithmic vector fields along $C$. We point out the relations of these jumping lines with the Lefschetz type properties of the Jacobian module of $f$ and with the Bourbaki ideal of the module of Jacobian syzygies of $f$. Classical general results by W. Barth resurface in the study of this special class of rank two vector bundles, where in addition explicit computations using computer algebra softwares are available.

1. Introduction

Let $C : f = 0$ be a reduced curve of degree $d$ in $X = \mathbb{P}^2$, $S = \mathbb{C}[x, y, z]$ with the usual grading, and $AR(f)$ be the graded $S$-module of Jacobian syzygies of $f$, see equation (2.1) below. Let $E_C$ be the locally free sheaf on $X$ corresponding to the graded module $AR(f)$, and recall that

\[(1.1) \quad E_C = T(C)(-1),\]

where $T(C)$ is the sheaf of logarithmic vector fields along $C$ as considered for instance in [1, 11, 18]. For a line $L$ in $X$, the pair of integers $(d^L_1, d^L_2)$ such that $d^L_1 \leq d^L_2$ (resp. without the condition $d^L_1 \leq d^L_2$), and such that $E_C|_L = \mathcal{O}_L(-d^L_1) \oplus \mathcal{O}_L(-d^L_2)$ is called the (ordered) splitting type (resp. the unordered splitting type) of $E_C$ along $L$, see for instance [14, 19]. Unless we say the opposite, in this paper we use the (ordered) splitting type. For a generic line $L_0$, the corresponding splitting type $(d^{L_0}_1, d^{L_0}_2)$ is known to be constant, see Theorem 2.2 (2) below.

When the graded $S$-module $AR(f)$ is free (equivalently, when $E_C$ splits as a direct sum of two line bundles on $X$), which can be considered as the simplest case, then the corresponding curve is called free, a notion going back to K. Saito [20]. When the minimal resolution of the graded $S$-module $AR(f)$ is slightly more complicated, we get the nearly free curves considered in [1, 2, 12, 18].

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A line \( L \) in \( X \) is called a jumping line for \( E_C \) or, equivalently, for \( T\langle C \rangle \), if \( d^k_L < d^l_1 \). When \( C \) is a free curve, there are no jumping lines for \( E_C \), while for a nearly free curve \( C \) the jumping lines for \( E_C \), if they exist, form a line \( \mathcal{L} \) in the dual projective space \( \mathbb{P}(S_1) \), determined by the jumping point \( P(C) \) associated to \( C \) by S. Marchesi and J. Vallès in [18]. In this note we study the set of jumping lines for \( E_C \) for any reduced plane curve \( C \). Even when \( C \) is a line arrangement \( \mathcal{A} \) in \( X \), the relation between the combinatorics of \( \mathcal{A} \) and the possible splitting types of the corresponding vector bundle \( T\langle \mathcal{A} \rangle \) is quite mysterious, see for instance [5, Question 7.12].

In section 2 we start by recalling some basic notions and results, in particular Theorem 2.2 which determines completely the generic splitting type \((d^l_1, d^l_2)\) in terms of the minimal degree \( r = mdr(f) \) of a Jacobian syzygy and the degree \( d \) of the curve \( C \). The invariant \( r \) also decides whether the vector bundle \( E_C \) is stable: the stability holds if and only if \( 2r \geq d \), see [21] or the discussion in the proof of Corollary 3.5. The new results in the first section are Theorems 2.5 and 2.7 giving new insight on the Hilbert function \( \{ k \mapsto h^1(\mathbb{P}^2, E_C(k)) \} \).

In section 3 we relate the integer \( d^k_L \) to some Lefschetz type properties of the multiplication by an equation \( \alpha_L \) of the line \( L \), acting on the Jacobian module \( N(f) \), see Theorem 3.1. Then we define and establish the first properties of the \( k \)-th jumping locus \( V_k(C) \) of the curve \( C \), which consists of all lines \( L \) in \( X \) such that \( d^k_L \leq k \), see Theorem 3.2. The main results in the second sections are Corollaries 3.5 and 3.6. We note that the claim in Corollary 3.6 (1) fits perfectly well with a general result of W. Barth about the pure dimensionality of some sets of jumping lines, see Remark 3.7.

In section 4 we recall the definition of the Bourbaki ideal \( B(C, \rho_1) \subset S \) associated to the curve \( C \) and to a minimal degree Jacobian syzygy \( \rho_1 \) for \( f \), see Theorem 4.1. This allows us to present the vector bundle \( E_C \) as an extension of the ideal of a codimension 2 locally complete intersection by a line bundle. The general construction of this type goes back to Serre [22], and it was widely used to construct rank 2 vector bundles on \( \mathbb{P}^n \) for \( n \geq 3 \), see [19, Chapter 1, section 5] and the many references given there. When \( mdr(f) \leq d/2 \) (resp. \( mdr(f) < d/2 \)), a line \( L \) is not a jumping line if (resp. if and only if) it avoids the support of the subscheme \( Z(C, \rho_1) \) of \( \mathbb{P}^2 \) defined by the ideal \( B(C, \rho_1) \), see Theorem 4.3, Theorem 4.9 and Corollary 4.1. This result generalizes the result of S. Marchesi and J. Vallès in [18] concerning nearly free curves. When \( mdr(f) > d/2 \), a line \( L \) avoiding the support of the subscheme \( Z(C, \rho_1) \) may be a jumping line, but it satisfies \( d^k_L \geq d - mdr(f) - 1 \), see Theorem 4.6 and this lower bound seems to be strict in many cases. The strong dependence of the ideal \( B(C, \rho_1) \) and of the scheme \( Z(C, \rho_1) \) of the choice of the syzygy \( \rho_1 \) is illustrated in Example 5.6.

We conclude with four examples. In the first one, we discuss the case of smooth curves, and point out in particular, that for \( d = 2d' + 1 \) odd, the geometry of the
jumping locus curve \( V_{d-1}(C) \) is quite interesting. This is a special case of Barth’s result in [3, Theorem 2 and Section 7], saying that a rank 2 stable vector bundle on \( \mathbb{P}^2 \) with even second Chern class is determined by the associated net of quadrics, having the curve \( V_{d-1}(C) \) as its discriminant.

The other three examples discuss singular curves and satisfy all \( d_{L_0} = 2 \) and hence \( V_2(C) = \mathbb{P}(S_1) \), the set of all lines in \( \mathbb{P}^2 \). In Example 5.3 \( C \) is a singular quintic, the first jumping locus \( V_1(C) \) is a smooth conic, hence the 1-dimensional irreducible components of the jumping loci are not necessarily lines, and this is related to another general result by W. Barth, on the smoothness of some sets of jumping lines, see Remark [5.4]. In Example 5.5 \( C \) is Zariski sextic with 6 cusps on a conic, the first jumping locus \( V_1(C) \) is the union of a line \( L \) and two points, and hence it is not pure dimensional. In Example 5.6 \( C \) is another singular sextic, the first jumping locus \( V_1(C) \) consists of 11 points, and the 0-th jumping locus \( V_0(C) \) is one of the points in \( V_1(C) \). In all these four examples, the corresponding vector bundles \( E_C \) are stable, and hence the structure of the jumping loci can be rather subtle even in the class of stable rank two vector bundles on \( \mathbb{P}^2 \).

As shown by these examples, the jumping loci \( V_k(C) \) for any plane curve \( C \) can be determined explicitly using a Computer Algebra software, in our case we have used the package SINGULAR, see [6].

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2. Preliminaries

For the coordinate ring \( S = \mathbb{C}[x, y, z] \) and a graded \( S \)-module \( M \), let \( M_k \) denote the homogeneous degree \( k \)-part of \( M \) and, for an integer \( m \), define the shifted graded \( S \)-module \( M(m) \) by the condition \( M(m)_k = M_{m+k} \) for any \( k \). For \( g \in S \), let \( g_x, g_y, g_z \) denote the partial derivative of \( g \) with respect to \( x, y, z \). Then the graded \( S \)-module \( AR(f) = AR(C) \subset S^3 \) of all relations is defined by

\[
(2.1) \quad AR(f)_k := \{ (a, b, c) \in S^3_k \mid af_x + bf_y + cf_z = 0 \}.
\]

Its sheafification \( E_C := \widehat{AR(f)} \) is a rank two vector bundle on \( \mathbb{P}^2 \), see [1] [20] [21] for details. More precisely, one has \( E_C = T(C)(-1) \), where \( T(C) \) is the sheaf of logarithmic vector fields along \( C \) as considered for instance in [1], [11], [18]. We set

\[
ar(f)_m = \dim AR(f)_m = \dim H^0(\mathbb{P}^2, E_C(m - 1))
\]

for any integer \( m \). We have the following, see [1] [12].
Definition 2.1. (1) A curve $C$ is free if the graded $S$-module $AR(f)$ is free, say with a basis $\rho_1, \rho_2$. If $\deg \rho_i = d_i \ (i = 1, 2)$, the multiset of integers $(d_1, d_2)$ is called the exponents of a free curve $C$.

(2) A curve $C$ is nearly free if the graded $S$-module $AR(f)$ has a minimal generator system of syzygies $\rho_1, \rho_2, \rho_3$, such that the degrees $\deg \rho_i$ satisfy $d_1 \leq d_2 = d_3$ and there is a relation

$$h\rho_1 + \ell_2\rho_2 + \ell_3\rho_3 = 0,$$

for $h \in S$ and independent linear forms $\ell_2, \ell_3 \in S$. The multiset $(d_1, d_2)$ is called the exponents of a nearly free curve $C$.

Let $mdr(f) := \min\{k \mid AR(f)_k \neq (0)\}$ be the minimal degree of a Jacobian syzygy for $f$. In this paper we assume that $mdr(f) \geq 1$, unless otherwise specified. Let $N(f) = \hat{J}_{f}/J_f$, with $J_f$ the Jacobian ideal of $f$ in $S$, spanned by the partial derivatives $f_x, f_y, f_z$ of $f$, and $\hat{J}_f$ the saturation of the ideal $J_f$ with respect to the maximal ideal $\mathfrak{m} = (x, y, z)$ in $S$. The quotient module $N(f)$ coincides with $H^0_{\mathfrak{m}}(S/J_f)$ and is called the Jacobian module of $f$, or of the plane curve $C$, see [21]. The quotient $M(f) = S/J_f$ is called the Jacobian algebra of $f$, and we denote

$$m(f)_k = \dim M(f)_k$$

for any integer $k$. Let $\nu(C) = \dim N(f)_{[T/2]}$, where $T = 3(d - 2)$. It is known that the curve $C : f = 0$ is free (resp. nearly free) if and only if $\nu(C) = 0$ (resp. $\nu(C) = 1$), see [8, 10, 12]. Recall the definition of the global Tjurina number

$$\tau(C) = \sum_{p \in C} \tau(C, p)$$

of the curve $C$, where $\tau(C, p)$ is the Tjurina number of the singularity $(C, p)$, and the fact that $\tau(C)$ is the degree of the Jacobian ideal $J_f$. This numerical invariant of $C$ occurs in the following formulas for the Chern numbers of the vector bundle $T(C)(k) = E_C(k - 1)$, namely one has

$$c_1(T(C)(k)) = 3 - d + 2k, \quad c_2(T(C)(k)) = d^2 - (k + 3)d + k^2 + 3k + 3 - \tau(C),$$

see for instance in [11] Equation (3.2)]. The following result was established in [1], see Theorem 1.1, Proposition 3.1 and Proposition 3.2.

Theorem 2.2. With the above notation, set $r = mdr(f)$. Then the following hold, where the line $L_0$ is generic and the line $L$ is arbitrary.

1. $d_1^2 + d_2^2 = d - 1$;
2. $d_1^L \geq d_L$;
3. One has $d_1^L = r$ if $r < (d - 2)/2$, and $d_1^L = \lfloor (d - 1)/2 \rfloor$ otherwise.
4. $\max(r - \nu(C), 0) \leq d_1^L \leq d_1^{L_0} \leq \min(r, \lfloor (d - 1)/2 \rfloor)$.
5. $(d - 1)^2 - d_1^{L_0}d_2^{L_0} = \tau(C) + \nu(C)$. 

Let $\alpha_L$ be the defining equation of the line $L$ in $X$. Then one has an exact sequence

$$0 \to \mathcal{O}_X(-1) \xrightarrow{\alpha_L} \mathcal{O}_X \to \mathcal{O}_L \to 0,$$

where the first non-trivial morphism is induced by multiplication by the linear form $\alpha_L$. Let $k$ be an integer and tensor the above exact sequence by the vector bundle $E_C(k)$. We get

$$0 \to E_C(k-1) \xrightarrow{\alpha_L} E_C(k) \to E_C(k)|_L \to 0,$$

with $E_C(k)|_L \simeq \mathcal{O}_L(k-d_1^L) \oplus \mathcal{O}_L(k-d_2^L)$, since we assume as in the Introduction that $E_C|_L \simeq \mathcal{O}_L(-d_1^L) \oplus \mathcal{O}_L(-d_2^L)$. Then we have the following.

**Proposition 2.3.** The long exact sequence of cohomology groups of the short exact sequence above starts as follows:

$$0 \to AR(f)_{k-1} \xrightarrow{\alpha_L} AR(f)_k \xrightarrow{\pi_L} H^0(L, \mathcal{O}_L(k-d_1^L) \oplus \mathcal{O}_L(k-d_2^L)) \to N(f)_{k+d-2} \xrightarrow{\alpha_L} N(f)_{k+d-1} \to \cdots.$$  

Moreover, for $k = -1$, the corresponding morphism $N(f)_{d-3} \xrightarrow{\alpha_L} N(f)_{d-2}$ is injective and $d_i^L \geq 0$ for any line $L$.

**Proof.** This is exactly as in the proof of [11, Theorem 5.7]. The key point is the identification $H^1(\mathbb{P}^2, E_C(k)) = N(f)_{k+d-1}$, valid for any integer $k$, for which we refer to [21, Proposition 2.1]. For the last claim, note that $N(f)_{d-3} \subset S_{d-3}$ and $N(f)_{d-2} \subset S_{d-2}$, as the Jacobian ideal is generated in degree $d-1$. \qed

Recall also the following result, saying that the Jacobian module $N(f)$ enjoys a Lefschetz type property, see [10].

**Theorem 2.4.** If $L_0 : \alpha_{L_0} = 0$ is a generic line in $X$, then the morphism

$$N(f)_{s-1} \xrightarrow{\alpha_{L_0}} N(f)_s,$$

induced by the multiplication by $\alpha_{L_0}$, is injective for $s < \lceil T/2 \rceil$, and surjective for $s \geq \lceil T/2 \rceil$.

Using the formula (2.2) above and the formulas in [8], (2.3), we get

$$ar(f)_{k+1} + ar(f)_{d-k} + \left( \frac{d+k+2}{2} \right) - 3 \left( \frac{k+3}{2} \right) = n(f)_{d+k} + \tau(C),$$

for any integer $k$. We set $k+1 = d-r-e$, for an integer $e \geq 0$, and we note that

$$ar(f)_{d-r-e} = \dim S_{d-2r-e} + \alpha(C, e) = \left( \frac{d-2r-e+2}{2} \right) + \alpha(C, e),$$

for any integer $e \geq 0$. We set $k+1 = d-r-e$, for an integer $e \geq 0$, and we note that

$$ar(f)_{d-r-e} = \dim S_{d-2r-e} + \alpha(C, e) = \left( \frac{d-2r-e+2}{2} \right) + \alpha(C, e),$$

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for any integer $e \geq 0$. We set $k+1 = d-r-e$, for an integer $e \geq 0
for some integer $\alpha(C, e) \geq 0$, if $r = \text{mdr}(f)$ and we assume that $2r \leq d$, $e \leq 2$. Since $d - 5 - k = r + e - 4 \leq r - 2$, we see that $ar(f)_{d-5-k} = 0$ and a direct computation transforms the equation \((2.4)\) into

\[(2.6) \quad n(f)_{2d-r-e-1} + \tau(C) - \alpha(C, e) = (d - 1)^2 - r(d - r - 1) - \frac{(e - 2)(e - 3)}{2}.
\]

We know that for $r = \text{mdr}(f) < d/2$, one has

\[(2.7) \quad \tau(C) + \nu(C) = (d - 1)^2 - r(d - r - 1),
\]

see \cite[Theorem 1.2]{21}. Hence we have establish the following result.

**Theorem 2.5.** If $r = \text{mdr}(f) < d/2$ and $0 \leq e \leq 2$, then the following holds

\[n(f)_{d+r+e-5} = n(f)_{2d-r-e-1} = \nu(C) - \frac{(e - 2)(e - 3)}{2} + \alpha(C, e),\]

where $\alpha(C, e) \geq 0$. In particular, we have the following.

1. For $e = 2$, we get $\alpha(C, e) = 0$ and

\[n(f)_{j} = \nu(C) \quad \text{for any} \quad j \in [d + r - 3, 2d - r - 3].\]

2. For $e = 1$, either $\alpha(C, e) = 1$, and then $C$ is free and $n(f)_{d+r-4} = n(f)_{2d-r-2} = \nu(C) = 0$, or $\alpha(C, e) = 0$ and then $C$ is not free and $n(f)_{d+r-4} = n(f)_{2d-r-2} = \nu(C) - 1$.

**Proof.** Use the above formula \((2.7)\) and the well known duality result for $N(f)$ implying that $n(f)_{j} = n(f)_{T-j}$ for any integer $j$, see \cite{21}.

\[\square\]

**Example 2.6.** (i) For $C : f = (x^8 + y^8)(x + 2y + z)(x - 3y + 7z) = 0$, we have $d = 11$, $r = \text{mdr}(f) = 3$, $\nu(C) = 3$ and the nonzero dimensions $n(f)_m$ are the following: $n(f)_{10} = n(f)_{17} = 2$ and $n(f)_{j} = 3$ for $j \in [11, 16]$.

(ii) For $C : f = (x^8 + y^8)(x + 2y + z)(x - 3y + 7z)(x + 5y - 11z) = 0$, we have $d = 12$, $r = \text{mdr}(f) = 4$, $\nu(C) = 6$ and the nonzero dimensions $n(f)_m$ are the following: $n(f)_{11} = n(f)_{19} = 3$, $n(f)_{12} = n(f)_{18} = 5$ and $n(f)_{j} = 6$ for $j \in [13, 17]$.

In both cases the stabilization $n(f)_{j} = \nu(C)$ starts at $j = d + r - 3$ as predicted by Theorem \[2.5\].

**Theorem 2.7.** If $r = \text{mdr}(f) \geq d/2$ and $f_s = x^d + y^d + z^d$, then the following hold.

1. For $d = 2d' + 1$ odd and $j \in \{3d' - 2, 3d' - 1\}$, one has $T = 6d' - 3$,

\[n(f)_{j} = n(f_s)_{j} - \tau(C) = m(f)_{s} - \tau(C) \quad \text{and} \quad n(f)_{3d' - 2} = n(f)_{3d' - 1}.
\]

2. For $d = 2d'$ even and $j \in \{3d' - 4, 3d' - 3\}$, one has $T = 6d' - 6$,

\[n(f)_{j} = n(f_s)_{j} - \tau(C) = m(f)_{s} - \tau(C) \quad \text{and} \quad n(f)_{3d' - 4} = n(f)_{3d' - 3} - 1.
\]
Proof. First note that $M(f_s) = N(f_s)$, and that one has $m(f)_k = m(f)_{T-k}$ for any $k$. It follows from [8, Equation (1.2)] that one has $m(f)_k = m(f_s)_k$, for any $k \leq d - 2 + mdr(f)$. Using now [12, Formula (2.8)], which says that

$$n(f)_k = m(f)_k + m(f)_{T-k} - m(f_s)_k - \tau(C),$$

we get all the claims, except the last one. It is easy to show that

$$m(f)_{3d-4} = m(f)_{3d-3} - 1,$$

e.g. by using the formulas for these dimensions given in [25, Proposition 2.1], and the explicit form for $n = 2$ given just after the proof.

□

Example 2.8. In Examples 5.5 and 5.6 below, we consider two singular sextics with $\tau = 12$. Note that $T = 12$ in this case and $m(f)_5 = 18$, $m(f)_6 = 19$. Hence Theorem 2.7 (2) implies that for both curves on has $n(f)_5 = 6$, $n(f)_6 = 7$, which is confirmed by using SINGULAR and producing explicit bases in Examples 5.5 and 5.6.

3. Jumping lines and Lefschetz type properties for the Jacobian module

The following result generalizes the claim in Theorem 2.2 (3) above from a generic line $L_0$ to any line $L$.

Theorem 3.1. For any line $L : \alpha_L = 0$ in $X$, we have $d_1^L = \min\{mdr(f), k(f, L)\}$, where

$$k(f, L) = \min\{k \in \mathbb{N} : N(f)_{k+d-2} \overset{\alpha_L}{\to} N(f)_{k+d-1} \text{ is not injective} \}.$$

Proof. If $k < \min\{mdr(f), k(f, L)\}$, the exact sequence (2.3) implies that $k < d_1^L$. Hence $\min\{mdr(f), k(f, L)\} \leq d_1^L$. If $k = mdr(f)$ or if $k = k(f, L)$, the same exact sequence implies $k \geq d_1^L$. Hence $d_1^L \leq \min\{mdr(f), k(f, L)\}$, which proves our claim.

□

The above proof also implies the following.

Corollary 3.2. Let $C : f = 0$ be a reduced plane curve of degree $d$ and set $r = mdr(f)$. Then the following hold.

1. If $d_1^L = r$, then $L$ is not a jumping line, $ar(f)_r \leq 2$, and the equality is possible only when $C$ is free with exponents $(d_1, d_1)$, and $d = 2d_1 + 1$ is odd.
2. If $d_1^L < r < d_2^L$, then $ar(f)_r \leq r - d_1^L + 1$.
3. If $d_2^L \leq r$, then $ar(f)_r \leq 2r - d + 3$.
The equality $ar(f)_r = 2r - d + 3$ occurs when $C$ is a nearly free curve with $d = 2d_1$ even and exponents $(d_1, d_1)$, and in many other cases, see Examples 5.3 and 5.6 below.

**Proof.** For the first claim note that $d_1^{T_0} \leq r$ by Theorem 2.2 (4) or by Theorem 3.1 and hence $L$ is not a jumping line. The inequality $ar(f)_r \leq 2$ follows from the exact sequence (2.3) since $AR(f)_{r-1} = 0$. If equality $ar(f)_r = 2$ holds, it follows that $f$ has two linearly independent Jacobian syzygies, both of degree $r$. Hence the sum of their degrees is $2r = 2d_1^L \leq d_1^L + d_2^L = d - 1$. This is possible only when there are equalities everywhere and the curve $C$ is free with exponents $(r, r)$ by [23] Lemma (1.1). The remaining claims follow along the same lines. \[ \square \]

**Remark 3.3.** If the morphism $N(f)_{s-1} \xrightarrow{\alpha_k} N(f)_s$ is not injective and $s \leq \lceil T/2 \rceil$, then the morphism $N(f)_s \xrightarrow{\alpha_k} N(f)_{s+1}$ is also not injective. Indeed, let $u \in N(f)_{s-1}$ be a non-zero element, such that $u \cdot \alpha_L = 0$.

Then, for a generic line $L_0 = \alpha_{L_0}$, the element $u_0 = u \cdot \alpha_{L_0} \in N(f)_s$ is non-zero by Theorem 2.3. On the other hand, it is clear that

$$u_0 \cdot \alpha_L = u \cdot \alpha_{L_0} \cdot \alpha_L = u \cdot \alpha_L \cdot \alpha_{L_0} = 0.$$ 

In other words, the injective morphism $N(f)_{s-1} \xrightarrow{\alpha_k} N(f)_s$ sends $K(\alpha_L)_{s-1}$ into $K(\alpha_L)_s$, where

$$K(\alpha_L)_m = \ker \{ N(f)_m \xrightarrow{\alpha_k} N(f)_{m+1} \}.$$ 

Now we investigate the jumping lines of $E_C$, namely the lines $L$ in $X$ such that $d_1^L < d_1^{T_0}$. Any line $L$ in $X$ corresponds clearly to a point in $\mathbb{P}(S_1)$, corresponding to a defining linear form $\alpha_L$. For any integer $k < mdr(f)$, consider the linear map

$$\lambda_k : S_1 \to Hom(N(f)_{d-2+k}, N(f)_{d-1+k}),$$

sending a linear form $\alpha_L \in S_1$ to the morphism of multiplication by $\alpha_L$. We assume that $d - 2 + k < T/2$, i.e. $k < (d - 2)/2$, and hence

$$n(f)_{d-2+k} \leq n(f)_{d-1+k},$$

by Theorem 2.4 where $n(f)_m = \dim N(f)_m$ for any integer $m$. Let

$$\Sigma_k \subset Hom(N(f)_{d-2+k}, N(f)_{d-1+k}),$$

denote the affine variety of linear maps which are not of maximal rank. Recall that

$$\text{codim} \Sigma_k = n(f)_{d-1+k} - n(f)_{d-2+k} + 1,$$

when $n(f)_{d-2+k} > 0$, and $\Sigma_k = \emptyset$ when $n(f)_{d-2+k} = 0$.

We define the $k$-th jumping locus of the curve $C : f = 0$ to be the set

$$V_k(C) = \{ L \in \mathbb{P}(S_1) : d_1^L \leq k \}.$$ 

**Theorem 3.4.** If $k \geq mdr(f)$, then $V_k(C) = \mathbb{P}(S_1)$. On the other hand, for $k < mdr(f)$, the following hold.
(1) If \( n(f)_{d-2+k} = 0 \), then \( V_k(C) = \emptyset \).

(2) If \( n(f)_{d-2+k} > 0 \), then \( V_k(C) = (\lambda^{-1}_k(\Sigma_k) \setminus \{0\})/\mathbb{C}^* \) is a determinantal subvariety in \( \mathbb{P}(S_1) = (S_1 \setminus \{0\})/\mathbb{C}^* \).

(3) \( \emptyset = V_{-1}(C) \subset V_0(C) \subset \ldots \subset V_{d^0-1}(C) \subset V_{d^0}(C) = \mathbb{P}(S_1) = \mathbb{P}^2 \).

(4) If \( \delta_k = n(f)_{d-1+k} = n(f)_{d-2+k} > 0 \), then \( V_k(C) \) is a curve of degree at most \( \delta_k \).

(5) If \( \delta_k = n(f)_{d-1+k} = n(f)_{d-2+k} + 1 > 1 \), then \( V_k(C) \) is either 1-dimensional, or 0-dimensional and \( |V_k(C)| \leq \delta_k(\delta_k - 1)/2 \) in this latter case.

**Proof.** The first two claims follow from the exact sequence (2.3). The third claim follows from the inequality \( d_1^2 \leq d_1^0 \), see Theorem 2.2 (2). To prove (4), note that in this case \( \Sigma_k \) is a hypersurface of degree \( \delta_k \) and \( 0 \notin \Sigma_k \). Note that \( \Lambda_k = \text{im} \lambda_k \) is a linear space not contained in \( \Sigma_k \) by Theorem 2.4. It follows that \( \lambda_k^{-1}(\Sigma_k) \) is a (possibly non-reduced) surface in \( S_1 = \mathbb{C}^3 \), defined by a homogeneous polynomial of degree \( \delta_k \). The proof of the last claim is similar, in this case \( \Sigma_k \) has codimension 2, and hence \( \lambda_k^{-1}(\Sigma_k) \) has codimension either 1 or 2, i.e. it cannot consist only of the origin 0. When \( \lambda_k^{-1}(\Sigma_k) \) has codimension 1, it consists of a number of lines, bounded by the degree of the determinantal variety \( \Sigma_k \). This degree is known to be \( \delta_k(\delta_k - 1)/2 \), see [15, Example 19.10]. □

**Corollary 3.5.** Let \( C : f = 0 \) be a reduced plane curve of degree \( d \) which is neither free nor nearly free, and assume that \( r = \text{mdr}(f) \) satisfies \( r < d/2 \). Then the vector bundle \( E_C \) is not stable, and it is semistable exactly when \( d = 2d^r + 1 \) is odd and \( r = d^r \). Moreover, the following hold.

1. \( d_1^0 = r \) and hence \( V_r(C) = \mathbb{P}(S_1) = \mathbb{P}^2 \).
2. \( V_{r-1}(C) \) is a curve of degree at most \( \nu(C) \) in \( \mathbb{P}(S_1) \).
3. \( V_{r-2}(C) \) is either 1-dimensional, or 0-dimensional and in this latter case \( |V_{r-2}(C)| \leq \nu(C)(\nu(C) - 1)/2 \).

In Example 3.2 we have \( d = 5 > 4 = 2r \), \( \nu(C) = 3 \) and \( V_0(C) \) consists of 3 points, hence the bound in Corollary 3.5 (3) is sharp in this case. The same example shows that the curve \( V_{r-1}(C) \) is not necessarily smooth, since \( V_1(C) \) consists of 3 lines in this case.

**Proof.** Recall that the inequality \( 2r \geq d \) holds if and only if the bundle \( E_C \) is stable, see [21, Proposition 2.4]. Note that for \( d \) even, \( E_C \) is semistable if and only if it is stable, while for \( d = 2d^r + 1 \) odd, \( E_C \) is semistable if and only if \( 2r \geq d - 1 \). To see this, use the characterization of (semi)stable rank 2 vector bundles on \( \mathbb{P}^n \) given by [19, Lemma 1.2.5, page 84].

The first claim in Corollary 3.5 follows from Theorem 2.2 (3), the second claim from Theorem 2.5 (1) and Theorem 3.4 (4) for \( k = r - 1 \), and the final claim from Theorem 2.5 (2) and Theorem 3.4 (5) for \( k = r - 2 \). □
Corollary 3.6. Let $C : f = 0$ be a reduced plane curve of degree $d$ which is neither free nor nearly free, and assume that $r = \text{mdr}(f)$ satisfies $r \geq d/2$. Then the vector bundle $E_C$ is stable and the following hold.

1. For $d = 2d' + 1$, one has $d_1^{10} = d'$ and $V_{d'-1}(C)$ is a curve of degree at most $\nu(C)$ in $\mathbb{P}(S_1)$.
2. For $d = 2d'$, one has $d_1^{10} = d' - 1$ and $V_{d'-2}(C)$ is either 1-dimensional, or 0-dimensional and in this latter case $|V_{d'-2}(C)| \leq \nu(C)(\nu(C) - 1)/2$.

In Example 5.1 we have $d = 4 < 6 = 2r$ and $V_0(C)$ is the union of 3 lines, hence a pure 1-dimensional variety. In Example 5.5 we have $d = 6 = 2r$ and $V_1(C)$ is the union of a line and a point, hence it is 1-dimensional, but not pure 1-dimensional. On the other hand, in Example 5.6 we have $d = 6 < 8 = 2r$, $\nu(C) = 7$ and $V_1(C)$ consists of 11 points.

Proof. The first claim follows from Theorem 2.7 (1) and Theorem 3.4 (4) for $k = d' - 1$, and the final claim from Theorem 2.7 (2) and Theorem 3.4 (5) for $k = d' - 2$. □

Remark 3.7. The claims above saying that some jumping sets $V_k(C)$ are pure 1-dimensional are related to Barth’s Theorem (applied to our setting), see [19, Theorem 2.2.4] as well as [19, pp. 118-119], saying that if $\mathcal{E}$ is a rank 2 vector bundle on $\mathbb{P}^2$, which is semistable and has an even Chern class $c_1(\mathcal{E})$, then the set of jumping lines of $\mathcal{E}$ is pure 1-dimensional. In our case, the condition $c_1(E_C) = -(d-1)$ even is the same as the condition $d = 2d' + 1$ odd. Moreover the condition $E_C$ semistable is equivalent to the condition $\text{mdr}(f) \geq d' = (d-1)/2$, see the proof of Corollary 3.5 above. In this situation, the equation of the curve $V_{d'-1}$ is by the determinant of the mapping $N(f)^{3d'-2} \xrightarrow{\alpha_L} N(f)^{3d'-1}$.

For semistable rank 2 vector bundles $\mathcal{E}$ with odd Chern class, i.e. $d = 2d'$, the corresponding result to Barth’s Theorem fails. An example of this situation for our bundles $E_C$ is given below in Example 5.5. In this case, K. Hulek has introduced in [17] the notion of a jumping line $L$ of the second kind, which means that the mapping $N(f)^{3d'-4} \xrightarrow{\alpha_L^2} N(f)^{3d'-2}$ has not maximal rank. Theorem 2.7 (2) implies that the set of jumping lines of the second kind is defined by the vanishing of the determinant of this latter mapping, and hence is a (possibly non-reduced) curve $C(E_C)$ of degree $2(\nu(C) - 1)$, since this determinant is not identically zero by [17, Theorem 3.2.2]. See Example 5.1 below for a situation where this curve (considered with reduced structure) has degree $< 2(\nu(C) - 1)$.

Example 3.8. Let $C : f = 0$ be a nearly free curve of degree $d$, with exponents $d_1 \leq d_2$. When $d_1 = d_2$, it is known that there are no jumping lines and the generic splitting type is $(d_1^{10}, d_2^{10}) = (d_1 - 1, d_1)$. The corresponding vector bundles $E_C$ is isomorphic to $T_X(-d_1 - 1)$, the shifted tangent bundle of $X$, see for details [11, 18]. Consider now the case $d_1 < d_2$, when it is known that the generic splitting type
is \((d_1^{(b)}, d_2^{(a)}) = (d_1, d_2 - 1)\), and a jumping line \(L\) has a splitting type \((d_1^1, d_2^2) = (d_1 - 1, d_2)\), see [11, 18]. Apply now Theorem 3.4 to this situation. By [12, Corollary 2.17], we know that \(n(f)_m = 1\) if \(d + d_1 - 3 \leq m \leq d + d_2 - 3\) and \(n(f)_m = 0\) otherwise. If we apply Theorem 3.4 (1) for \(V\) [18], and a generalization of this result is discussed in our next section, see Theorem 4.3.

Apply now Theorem 3.4 (4) for \(S\) (4.1) for any syzygy \(\rho\) of the 3 matrix \(a, b, c\) as third row and a homogeneous ideal \(B\). Proposition 2.1.

The following result, except the claim (2), was stated for line arrangements in [13, 18], and we define thus a new morphisms of graded \(S\)-modules

\[(4.1)\quad S(-r) \xrightarrow{u} AR(f), \quad u(h) = h \cdot \rho_1.\]

For any syzygy \(\rho = (a, b, c) \in AR(f)_m\), consider the determinant \(\Delta(\rho) = \det M(\rho)\) of the \(3 \times 3\) matrix \(M(\rho)\) which has as first row \(x, y, z\), as second row \(a_1, b_1, c_1\) and as third row \(a, b, c\). Then it turns out that \(\Delta(\rho)\) is divisible by \(f\), see [8], and we define thus a new morphisms of graded \(S\)-modules

\[(4.2)\quad AR(f) \xrightarrow{v} S(r - d + 1), \quad v(\rho) = \Delta(\rho)/f,\]

and a homogeneous ideal \(B(C, \rho_1) \subset S\) such that \(1m v = B(C, \rho_1)(r - d + 1)\). The following result, except the claim (2), was stated for line arrangements in [13, Proposition 2.1].

**Theorem 4.1.** Let \(C : f = 0\) be a reduced plane curve of degree \(d\). For any choice of a nonzero syzygy \(\rho_1 = (a_1, b_1, c_1) \in AR(f)_r\), where \(r = mdr(f)\), we get a morphisms of graded \(S\)-modules

\[(0 \to S(-r) \xrightarrow{u} AR(f) \xrightarrow{v} B(C, \rho_1)(r - d + 1) \to 0),\]

and the following hold.

1. The ideal \(B(C, \rho_1)\) is saturated, defines a subscheme \(Z(C, \rho_1) = V(B(C, \rho_1))\) of \(\mathbb{P}^2\) of dimension at most 0, and its degree is given by

\[\deg B(C, \rho_1) = (d - 1)^2 - r(d - r - 1) - \tau(C)\]

2. The ideal \(B(C, \rho_1)\) and the codimension 2 subscheme \(Z(C, \rho_1)\) are locally complete intersections.
3. The ideal \(B(C, \rho_1)\) and the subscheme \(Z(C, \rho_1)\) do not depend on the choice of \(\rho_1\) when \(\dim AR(f)_r = 1\).
4. The curve \(C\) is free if and only if \(B(C, \rho_1) = S\).
Remark 4.2. Let \( I \subset \mathbb{Z}_2 \). Since the scheme \( X \) belongs to this support, the surjectivity of \( \tilde{v} \) of the syzygy \( \rho \) and hence the support of \( I \) is easy to see that the support \( I \) is unique (up to a nonzero factor).

It follows that \( B(C, \rho_1) \) is a Bourbaki ideal for the syzygy module \( AR(f) \), see [4], Chapitre 7, §4, Thm. 6, as well as section 3 in [24]. A similar construction for surfaces in \( \mathbb{P}^3 \) was given in [1]. The dependence of the ideal \( B(C, \rho_1) \) and of the scheme \( Z(C, \rho_1) \) of the choice of the syzygy \( \rho_1 \) is illustrated in Example 5.6.

**Proof.** We let the reader check that the proof given for [13, Proposition 2.1] works as well in this more general setting. As for the new claim (2), we proceed as follows.

If we sheafify the exact sequence of graded \( S \)-modules from Theorem 4.1, we get an exact sequence

\[
0 \to \mathcal{O}_X(-r) \xrightarrow{\tilde{u}} E_C \xrightarrow{\tilde{v}} I(r - d + 1) \to 0.
\]

Here \( I \) is the sheaf ideal in \( \mathcal{O}_X \) associated to the Bourbaki ideal \( B(C, \rho_1) \), and hence the support of \( \mathcal{O}_X/I \) coincides with the support of the scheme \( Z(C, \rho_1) \). If \( p \) belongs to this support, the surjectivity of \( \tilde{v}_p \) implies that the corresponding ideal \( I_p \) is generated by at most two elements. Indeed, \( E_{C,p} \) is a free \( \mathcal{O}_{X,p} \)-module of rank 2. Since the scheme \( Z(C, \rho_1) \) is 0-dimensional, this yields the claim (2).

\[\square\]

**Remark 4.2.** Let \( I(\rho_1) \) be the ideal in \( S \) generated by the components \( a_1, b_1, c_1 \) of the syzygy \( \rho_1 \) and let \( Z(I(\rho_1)) \) be the corresponding subscheme in \( \mathbb{P}^2 \). Then it is easy to see that the support \( |Z(I(\rho_1))| \) of \( Z(I(\rho_1)) \) coincides with the support \( |Z(C, \rho_1)| \) of \( Z(C, \rho_1) \) outside \( C \). The example \( C : f = x^5 y^2 z^2 + x^9 + y^9 = 0 \), where \( \rho_1 = (-2xy^2z, 0, 9x^4 + 5y^2z^2), \) \( |Z(I(\rho_1))| = \{(0 : 1 : 0), (0 : 0 : 1)\} \) and \( |Z(C, \rho_1)| = \{(0 : 1 : 0)\} \), shows that these two supports do not coincide in general.

Note that in this example \( r = mdr(f) = 4 \) and \( \dim AR(f)_4 = 1 \), so the choice of \( \rho_1 \) is unique (up to a nonzero factor).

**Theorem 4.3.** Let \( C : f = 0 \) be a reduced plane curve of degree \( d \), set \( r = mdr(f) \) and consider the subscheme \( Z(C, \rho_1) \) introduced above. Any line \( L \) in \( \mathbb{P}^2 \) which avoids the support of \( Z(C, \rho_1) \) is not a jumping line if \( 2r \leq d \). More precisely, the (unordered) splitting type of \( E_C \) along \( L \) is \( (r, d-1-r) \).

**Proof.** If we tensor the exact sequence \( (1.3) \) by \( \mathcal{O}_L \), for \( L \) a line disjoint from the support of \( Z(C, \rho_1) \), we get the following exact sequence

\[
0 \to \mathcal{O}_L(-r) \xrightarrow{\alpha} E_C|_L \xrightarrow{\beta} \mathcal{O}_L(r - d + 1) \to 0.
\]
The isomorphism classes of such extensions of $\mathcal{O}_L(r-d+1)$ by $\mathcal{O}_L(-r)$ are classified by

$$\text{Ext}^1(\mathcal{O}_L(r-d+1), \mathcal{O}_L(-r)) = \text{Ext}^1(\mathcal{O}_L, \mathcal{O}_L(d-1-2r)) = H^1(L, \mathcal{O}_L(d-1-2r)) = 0,$$

see [16 Section III.6], which proves our claim.

**Corollary 4.4.** Let $C : f = 0$ be a reduced plane curve of degree $d$, such that $r = mdr(f) \leq d/2$. Then the set of jumping lines for the vector bundle $E_C$ is contained in a union of at most $(d-1)^2 - r(d-r-1) - \tau(C)$ lines in $\mathbb{P}(S_1)$.

**Remark 4.5.** The condition $2r \leq d$ in Theorem 4.3 is necessary, as Example 5.3 below shows.

**Theorem 4.6.** Let $C : f = 0$ be a reduced plane curve of degree $d$ and consider the subscheme $Z(C, \rho_1)$ introduced above. Then, if $r = mdr(f) > d/2$, the splitting type $(d^L_1, d^L_2)$ along any line $L$ in $\mathbb{P}^2$ which avoids the support of $Z(C, \rho_1)$ satisfies $d^L_1 \geq d - 1 - r$. In particular, if $2r - d \in \{1, 2\}$, then $d^L_1 \in \{d^{L_0} - 1, d^{L_0}_1\}$.

Examples in the next section shows that this lower bound is sharp in many cases, e.g. in the situation of the last claim, both values for $d^L_1$ are obtained, see the final parts of Examples 5.3 and 5.6.

**Proof.** We use the same notation as in the proof of Theorem 4.3. In the exact sequence (4.4) we have $E_C|_L = \mathcal{O}_L(-d^L_1) \oplus \mathcal{O}_L(-d^L_2)$. The surjective morphism $\beta$ is induced by a pair of homogeneous polynomials $(A_1, A_2) \in S_{a_1} \times S_{a_2}$, where $a_i = r - d + 1 + d^L_i$ for $i = 1, 2$, satisfying the condition $G.C.D.(A_1, A_2) = 1$. Indeed, at the level of sections, the morphism $\beta$ is given by

$$(s_1, s_2) \mapsto A_1s_1 + A_2s_2.$$ 

Note that $a_1 \leq a_2$. If $A_1 \neq 0$, then $a_1 \geq 0$, and this yields the claim of our Theorem. If $A_1 = 0$, it follows that $A_2$ is a non-zero constant, and hence $a_2 = 0$. This implies

$$d^L_2 = d - 1 - r < \frac{d - 1}{2},$$

which is a contradiction. Indeed, $d^L_2 \geq d^L_1$ implies

$$d^L_2 \geq \frac{d - 1}{2}.$$ 

The last claim follows by checking that, in these two situations, one has

$$d - r - 1 = d^{L_0} - 1.$$ 

□
4.7. On lines meeting the support of the jumping subscheme $Z(C, \rho_1)$. Let $L$ be a line in $\mathbb{P}^2$ such that $L \cap |Z(C, \rho_1)| = \{p_1, ..., p_s\}$. For each such point $p_k$ we define its multiplicity as follows. Consider a system of local coordinates $(u, v)$ centered at $p_k$ such that the equation of the line $L$ is given by $u = 0$. The localized ideal $\mathcal{I}_{p_k} \subset \mathcal{O}_{X, p_k} = \mathbb{C}\{u, v\}$, being a complete intersection, is generated by two analytic germs, say $g(u, v)$ and $h(u, v)$. Then we set

$$m_k = \dim_{\mathbb{C}} \frac{\mathbb{C}\{u, v\}}{(u, g(u, v), h(u, v))} = \dim_{\mathbb{C}} \frac{\mathbb{C}\{v\}}{(g(0, v), h(0, v))}.$$

Then clearly $1 \leq m_k < +\infty$ and one has

$$\frac{\mathbb{C}\{u, v\}}{(u)} \otimes_{\mathbb{C}\{u, v\}} \frac{\mathbb{C}\{u, v\}}{(g(u, v), h(u, v))} = \frac{\mathbb{C}\{u, v\}}{(u, g(u, v), h(u, v))},$$

and hence the latter ring can be regarded as the local ring of the point $p_k$ in the scheme theoretic intersection $Z(C, \rho_1) \cap L$. The ideal $\mathcal{I}_{p_k} = (g(u, v), h(u, v)) \subset \mathcal{O}_{X, p_k}$, being a complete intersection, we have a free resolution

$$0 \to \mathcal{O}_{X, p_k} \to \mathcal{O}^2_{X, p_k} \to \mathcal{I}_{p_k} \to 0,$$

where the non-trivial morphisms are given by the pair $(g(u, v), h(u, v))$. When we tensor by $\mathcal{O}_{L, p_k}$, we get the following exact sequence

$$0 \to \mathcal{O}_{L, p_k} \to \mathcal{O}^2_{L, p_k} \to \mathcal{I}_{p_k} \otimes \mathcal{O}_{L, p_k} \to 0,$$

and the corresponding morphisms are given by the pair $(g(0, v), h(0, v)) \neq (0, 0)$. It follows that the first morphism is injective, and up-to a change of basis in $\mathcal{O}^2_{L, p_k} = \mathbb{C}\{v\}^2$ is given by the pair $(v^{m_k}, 0)$. It follows that

$$\mathcal{I}_{p_k} \otimes \mathcal{O}_{L, p_k} = \mathbb{C}\{v\} \oplus \frac{\mathbb{C}\{v\}}{(v^{m_k})}.$$

If we tensor now the exact sequence (4.3) by $\mathcal{O}_L$, we get, keeping track of the twists and using the above local computations, the following result. When the points $p_k \in Z(C, \rho_1) \cap L$ are all simple points, then this result is already in [14], see equation (7).

**Proposition 4.8.** With the above notation, there is an exact sequence

$$0 \to \mathcal{O}_L(-r) \to E_C|_L \to \mathcal{O}_L(r - d + 1 - m_L) \oplus \left( \bigoplus_{k=1,s} \frac{\mathcal{O}_{L, p_k} M_{p_k}^{m_k}}{M_{p_k}} \right) \to 0,$$

where $m_L = \sum_{k=1,s} m_k$ and $M_{p_k} \subset \mathcal{O}_{L, p_k}$ denotes the corresponding maximal ideal.

Using this Proposition, we can prove the following result.
Theorem 4.9. Let $C : f = 0$ be a reduced plane curve of degree $d$, set $r = mdr(f)$ and consider the subscheme $Z(C, \rho_1)$ introduced above. Any line $L$ in $\mathbb{P}^2$ which meets the support of $Z(C, \rho_1)$ is a jumping line if $2r \leq d - 1$. More precisely, the splitting type of $E_C$ along $L$ is $(r - m_L, d - 1 - r + m_L)$, and hence, in particular, $m_L \leq r$ and $V_{r-1}(C)$ is a line arrangement consisting of at most $\nu(C)$ lines, dual to the support of the subscheme $Z(C, \rho_1)$.

Proof. It is clear that the splitting type of $E_C$ along $L$ is $(r - h, d - 1 - r + h)$ for some $0 \leq h \leq r$. If $0 \leq h < m_L$, then we have $-r + h \geq r - d + 1 - h > r - d + 1 - m_L$, and hence there is no surjective morphism from $E_C|_L$ to $\mathcal{O}_L(r - d + 1 - m_L)$, which is a contradiction in view of Proposition 4.8. It follows that $h \geq m_L$. Assume now that $h > m_L$. Then $-r > r - d + 1 - h$, and hence the first nontrivial morphism in the exact sequence from Proposition 4.8 is given by a pair $(H, 0)$, where $H$ is a homogeneous polynomial of degree $h > m_L$. This implies that the torsion part of the cokernel of this morphism has dimension equal to $h > m_L$, a contradiction.

Since the degree of the subscheme $Z(C, \rho_1)$ is $\nu(C)$ for $2r \leq d - 1$ by Theorem 4.1 (1) and Theorem 2.2 (3) and (5), the last claim follows as well.

A computation of the splitting type using this approach can be seen in Example 5.2.

Remark 4.10. The example of a nearly free curve $C$ with exponents $(d_1, d_1)$ discussed in Example 3.8, when there are no jumping lines but the scheme $Z(C, \rho_1)$ consists of a simple point, shows that a line $L$ meeting the support of $Z(C, \rho_1)$ may not be a jumping line if $r = mdr(f) \geq d/2$. A similar situation is described in Examples 5.3 and 5.5 below. Note that Example 5.5 shows that the set of jumping lines described in Corollary 4.4 is not necessarily pure 1-dimensional, i.e. it may consists of lines and isolated points, even when $r \leq d/2$.

5. Some examples

First we consider the smooth curves.

Example 5.1. Let $C : f = 0$ be a smooth curve of degree $d \geq 3$. Then $r = mdr(f) = d - 1$ and the graded $S$-module $AR(f)$ is generated by the Koszul type syzygies

$$\rho_1 = (f_y, -f_x, 0), \quad \rho_2 = (f_z, 0, -f_x) \quad \text{and} \quad \rho_3 = (0, f_z, -f_y).$$

With this choice, the Bourbaki ideal $B(C, \rho_1)$ is spanned by $v(\rho_2) = d \cdot f_x$ and $v(\rho_3) = d \cdot f_y$, hence it is a global complete intersection. For the Fermat type curve $C : f_s = x^d + y^d + z^d = 0$,

the support of the scheme $Z(C, \rho_1)$ is the multiple point $p = (0 : 0 : 1)$. The line $L : z = 0$ does not pass through this point and Theorem 3.1 implies that
\(d^r_1 = 0 = d - r - 1\), i.e. for this line we get equality in the inequality given by Theorem 4.6.

**Case** \(d = 2d' + 1\) **odd.** Then Corollary 3.6 implies that \(V_{d-1}(C)\) is a curve in \(\mathbb{P}(S_1)\). The geometry of these curves \(V_{d-1}(C)\) depends on the equation \(f\). For instance, in the case of a plane cubic

\[
C : f = x^3 + y^3 + z^3 + 3txyz = 0, \text{ where } t \in \mathbb{C}, \ t^3 \neq -1,
\]

an easy direct computation shows that

\[
(5.1) \quad V_{d-1}(C) : t(a^3 + b^3 + c^3) + (2 - t^3)abc = 0,
\]

where \((a : b : c)\) are the coordinates on \(\mathbb{P}(S_1)\). In other words, the jumping variety \(V_{d-1}(C)\) determines the complex structure of \(C\) up to finite indeterminacy in this case. This is related to Barth’s result in [3, Theorem 2 and Section 7], saying that a rank 2 stable vector bundle on \(\mathbb{P}^2\) with even second Chern class is determined by the associated net of quadrics, having the curve \(V_{d-1}(C)\) as its discriminant.

**Case** \(d = 2d'\) **even.** Then Corollary 3.6 implies \(V_{d-2}(C)\) is nonempty. For \(f = x^4 + y^4 + z^4\) and using the usual monomial bases for \(N(f) = M(f)\), we get \(V_0(C) : abc = 0\), hence the union of 3 lines. In particular, \(V_0(C)\) is pure 1-dimensional in this case. Note that the determinant of the mapping \(N(f)_2 \xrightarrow{\alpha_L^4} N(f)_4\), where \(\alpha_L = ax + by + cz\), is given by \(a^4b^4c^4\). Hence the curve of jumping lines of second order \(C(E_C)\) is given by the equation \(a^4b^4c^4 = 0\), and hence its support coincides with \(V_0(C)\) in this case. In other words, we have equality in [17, Proposition 9.1].

The computations in the following examples were all done using the computer algebra software SINGULAR, see [6]. The Chern classes of \(E_C\) can be computed in each case using (2.2) above, since we give in each example the corresponding global Tjurina number \(\tau(C)\).

**Example 5.2.** Let \(C : f = 0\), where \(f = x^5 + y^5 + (x^4 + y^4)z\). Then \(d = 5\), \(\tau(C) = 9\), and \(r = mdr(f) = 2\). Hence Theorem 2.2(3) implies that the corresponding generic splitting type of \(E_C\) is \((d_1^0, d_2^a) = (2, 2)\). The Jacobian ideal \(J_f\) is spanned by \(f_x, f_y, f_z\), and its saturation \(\hat{J}_f\) is spanned by \(x^3, y^3\). The only non-zero dimensions \(n(f)_m\) are in this case \(n(f)_4 = n(f)_5 = 3\) and \(n(f)_3 = n(f)_6 = 2\). Moreover, a vector space basis of \(N(f)_3\) (resp. of \(N(f)_4\)) is given by \(x^3, y^3\) (resp. \(x^4, x^3y, xy^3\)). With respect to these bases, the multiplication \(\{N(f)_3 \xrightarrow{\alpha_L^4} N(f)_4\}\), where \(\alpha_L = ax + by + cz\), is given by

\[
(ax + by + cz) \cdot x^3 = (a - \frac{5c}{4})x^4 + bx^3y
\]

and

\[
(ax + by + cz) \cdot y^3 = (\frac{5c}{4} - b)x^4 + axy^3.
\]
It follows that \( V_0(C) \) consists of 3 points, namely \((0 : 0 : 1), (0 : 5 : 4), (5 : 0 : 4)\). 
Since \( \nu(C) = 3 \), it follows that we have equality in Corollary 3.5(3), hence the bound there is sharp in this situation. Similarly, a basis for \( N(f)_5 \) is given by \( x^5, x^3y^2, x^2y^3 \) and the multiplication \( \{ N(f)_4 \to N(f)_5 \} \) is given by \((ax+by+cz) \cdot x^4 = (a+b-\frac{5c}{4})x^5, (ax+by+cz) \cdot x^3y = (a-\frac{5b}{4}-b)x^5+bx^3y^2 \) and \((ax+by+cz) \cdot xy^3 = -(b-\frac{5c}{4})x^5+ax^2y^3 \).

It follows that \( V_1(C) \) consists of 3 lines, namely \( \mathcal{L}_1 : a = 0, \mathcal{L}_2 : b = 0 \) and \( \mathcal{L}_3 : 4(a+b)-5c = 0 \). The \( S \)-module \( AR(f) \) has 4 generating syzygies, of degrees 2, 4, 4, and a direct computation shows that the scheme \( Z(C, \rho_1) \), which does not depend on the choice of the syzygy \( \rho_1 \), consists of the simple points \( P_1 = (1 : 0 : 0), P_2 = (0 : 1 : 0) \) and \( P_3 = (4 : 4 : -5) \). It follows that the line \( \mathcal{L}_j \subset \mathbb{P}(S_1) \) above consists of all the lines in \( \mathbb{P}^2 \) passing through the point \( P_j \), for \( j = 1, 2, 3 \). Note that the corresponding lines \( L = L_{i,j} \) in \( V_0(C) \) pass through the points \( P_i, P_j \) in the support of \( Z(C, \rho_1) = \{ P_1, P_2, P_3 \} \), and one has \( m_L = r = 2 \) in this case, as predicted by Theorem 4.9. More precisely, one has \( L_{1,2} : z = 0, L_{1,3} : 5y+4z = 0 \) and \( L_{2,3} : 5x+4z = 0 \).

**Example 5.3.** Let \( C : f = 0 \), where \( f = 2x^5+2y^5+5x^2y^2z \). Then \( d = 5, \tau(C) = 10 \), and we see that the \( S \)-module \( AR(f) \) is generated by 4 syzygies \( \rho_i, i = 1, \ldots, 4 \), all of degree \( r = mdr(f) = 3 \). Hence Theorem 2.2(3) implies that the corresponding generic splitting type of \( E_C \) is \((d^2, d^2) = (2, 2) \). The Jacobian ideal \( J_f \) is spanned by \( f_x, f_y, f_z \), and its saturation \( \bar{J}_f \) is spanned by \( f_x, f_y, f_z, x^3y, xy^3 \). The only non-zero dimensions \( n(f)_m \) are in this case \( n(f)_4 = n(f)_5 = 2 \). Moreover, a vector space basis of \( N(f)_4 \) (resp. of \( N(f)_5 \)) is given by \( x^3y, xy^3 \) (resp. \( x^4y, xy^4 \)). With respect to these bases, the multiplication \( \{ N(f)_4 \to N(f)_5 \} \), where \( \alpha_L = ax+by+cz \), is given by

\[
(ax+by+cz) \cdot x^3y = ax^4y - cxy^4
\]

and

\[
(ax+by+cz) \cdot xy^3 = -cx^4y + bxy^4.
\]

Using Theorem 3.4(4) for \( k = 1 \), we get that \( V_1(C) \), the set of jumping lines for \( E_C \), is the smooth conic \( Q : ab-c^2 = 0 \) in \( \mathbb{P}(S_1) \).

Hence in this case we have

\[
\emptyset = V_{-1}(C) = V_0(C) \subset V_1(C) = Q \subset V_2(C) = \mathbb{P}(S_1).
\]

Indeed, Theorem 3.4(1) implies that \( V_0(C) = \emptyset \). If we choose

\[
\rho_1 = (0, x^2y, -2(y^3+x^2z)) \in AR(f)_3,
\]

then the corresponding Bourbaki \( B(C, \rho_1) \) is the ideal \((xz, y^2, xy)\), and hence the scheme \( Z(C, \rho_1) \) consists of two points, a simple one at \((1 : 0 : 0)\), given in local coordinates by an ideal \((u,v)\), and a double point at \((0 : 0 : 1)\), given in local coordinates by an ideal \((u,v^2)\).
Among the lines on $Q$, only the lines $x = 0$ and $y = 0$ meet the support of $Z(C, \rho_1)$. For the other lines in $Q$, the bound given by Theorem 4.6 is $d_i^2 \geq d - r - 1 = 1$. In fact, we have equality, hence this bound is sharp in this situation.

**Remark 5.4.** The smooth conic $Q$ above is one of the smooth degree $n$ curves occurring as jumping loci, predicted by Barth for stable rank 2 vector bundles $\mathcal{E}$ on $\mathbb{P}^2$, with $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E}) = n$, see [3, Application 1, section 5.4]. Indeed, note that the normalization of our vector bundle $E_C$ is $\mathcal{E}_C = E_C(2)$ and it satisfies $c_1(\mathcal{E}_C) = 0$ and $c_2(\mathcal{E}_C) = 2$. For the computation of these Chern numbers, one can use (2.2) above. Similar remarks apply for the cubic curve in (5.1), which is smooth for $t^3 \notin \{-1, 0, 8\}$.

**Example 5.5.** Let $C : f = 0$, where $f = (x^2 + y^3)^3 + (y^3 + z^3)^2$, i.e. $C$ is a Zariski sextic with 6 cusps on a conic. Then $d = 6$, $\tau(C) = 12$, and we see that the $S$-module $AR(f)$ is generated by 4 syzygies $\rho_1$, $i = 1, ..., 4$, of degrees $r = mdr(f) = 3 = d_1 < d_2 = d_3 = d_4 = 5$. Hence Theorem 2.2 (4) implies that the corresponding generic splitting type of $E_C$ is $(d_1^{\alpha_1}, d_2^{\alpha_2}) = (2, 3)$. The Jacobian ideal $J_f$ is spanned by $f_x, f_y, f_z$, and its saturation $\tilde{J}_f$ is spanned by $g = y^3 + z3$ and $h = (x^2 + y^2)^2$. The only non-zero dimensions $n(f)_{\alpha}$ are in this case $n(f)_3 = n(f)_9 = 1$, $n(f)_4 = n(f)_8 = 4$, $n(f)_5 = n(f)_7 = 6$ and $n(f)_6 = 7$. Moreover, a vector spaces basis of $N(f)_5$ (resp. of $N(f)_6$) is given by

$$x^2g, y^2g, xyg, xzg, yzg, zh,$$

and respectively by

$$x^3g, x^2yg, y^3g, x^2zg, y^2zg, xyzg, z^2h.$$

With respect to these bases, the multiplication $\{N(f)_5 \xrightarrow{\alpha_L} N(f)_6\}$, where $\alpha_L = ax + by + cz$, is given by the matrix

$$M(L) = \begin{pmatrix}
  a & 0 & 0 & 0 & 0 \\
  b & 0 & a & 0 & 0 \\
  0 & b & 0 & 0 & 0 \\
  c & 0 & 0 & a & 0 \\
  0 & c & 0 & 0 & b \\
  0 & 0 & c & b & a \\
  0 & 0 & 0 & 0 & c
\end{pmatrix}$$

Using Theorem 3.4 (5) for $k = 1$, we get that $V_1(C)$, the set of jumping lines for $E_C$, is the set of lines $L$ such that rank $M(L) < 6$. A direct computation, shows that $V_1(C)$ consists of the line $\mathcal{L} : a = 0$ and one points, namely $P_1 = (1 : 0 : 0)$. A vector basis for $N(f)_4$ is given by $xg, yg, zg, h$, and using the given bases, the multiplication $\{N(f)_4 \xrightarrow{\alpha_L} N(f)_5\}$, where $\alpha_L = ax + by + cz$, is given by the matrix

$$\begin{pmatrix}
  a & 0 & 0 & 0 & 0 \\
  b & 0 & a & 0 & 0 \\
  0 & b & 0 & 0 & 0 \\
  c & 0 & 0 & a & 0 \\
  0 & c & 0 & 0 & b \\
  0 & 0 & c & b & a \\
  0 & 0 & 0 & 0 & c
\end{pmatrix}$$
Let $M'(L) = \begin{pmatrix}
a & 0 & 0 & 0 \\
0 & b & 0 & -b \\
b & a & 0 & 0 \\
c & 0 & a & 0 \\
0 & c & b & 0 \\
0 & 0 & 0 & c
\end{pmatrix}$

Using Theorem [3.1] it follows that $V_0(C)$ is the set of lines $L$ such that rank $M'(L) < 4$, which implies that $V_0(C) = \{P_1, P_2, P_3\}$, where $P_1$ is as above, $P_2 = (0 : 1 : 0)$ and $P_3 = (0 : 0 : 1)$. Hence in this case we have

$$\emptyset = V_{-1}(C) \subset V_0(C) = \{P_1, P_2, P_3\} \subset V_1(C) = \{P_1\} \cup \mathcal{L} \subset V_2(C) = \mathbb{P}(S_1).$$

Since $\dim AR(f)_3 = 1$, there is essentially a unique choice

$$\rho_1 = (y^2, -xz^2, xy^2) \in AR(f)_3.$$ 

Then the corresponding Bourbaki $B(C, \rho_1)$ is the ideal $(xy^2, xz^2, y^2)$, and hence the scheme $Z(C, \rho_1)$ consists of three points, one nonreduced at $p_1 = (1 : 0 : 0)$, given in local coordinates by an ideal $(u^2, v^2)$, the other nonreduced at $p_2 = (0 : 1 : 0)$, given in local coordinates by an ideal $(u, v^2)$, and a reduced point at $p_3 = (0 : 0 : 1)$ given by $(u, v)$. Note that the line $\mathcal{L}$ consists of all the lines passing through the point $p_1$, the line $L_1 : x = 0$, corresponding to the point $P_1$, passes through the points $p_2$ and $p_3$, the line $L_2 : y = 0$, corresponding to the point $P_2$, passes through the points $p_1$ and $p_3$, and the line $L_3 : z = 0$, corresponding to the point $P_3$, passes through the points $p_1$ and $p_2$. None of the points $p_i$ is situated on the sextic $C$.

**Example 5.6.** Let $C : f = 0$, where $f = x^6 + y^6 + 3x^2y^2z^2$. Then $d = 6$, $\tau(C) = 12$, and we see that the $S$-module $AR(f)$ is generated by 5 syzygies $\rho'_i$, $i = 1, ..., 5$, of degrees $r = mdr(f) = 4 = d_1 = d_2 < d_3 = d_4 = d_5 = 5$, see their expressions given below. Hence Theorem [2.2] (4) implies that the corresponding generic splitting type of $E_C$ is $(d_1^{\alpha_1}, d_2^{\alpha_2}) = (2, 3)$. The Jacobian ideal $J_f$ is spanned by $f_x, f_y, f_z$, and its saturation $\hat{J}_f$ is spanned by $f_x, f_y, f_z, x^3y, x^2y^2, xy^3$. The only non-zero dimensions $n(f)_m$ are in this case $n(f)_4 = n(f)_8 = 3$, $n(f)_5 = n(f)_7 = 6$ and $n(f)_6 = 7$. Moreover, a vector spaces basis of $N(f)_5$ (resp. of $N(f)_6$) is given by

$$xy^4, x^2y^3, x^3y^2, x^4y, xy^3z, x^3yz,$$

and respectively by

$$xy^5, x^2y^4, x^3y^3, x^4y^2, x^5y, xy^4z, x^4yz.$$

With respect to these bases, the multiplication $\{N(f)_5 \to N(f)_6\}$, where $\alpha_L = ax + by + cz$, is given by the matrix

$$\begin{pmatrix}
a & 0 & 0 & 0 \\
0 & b & 0 & -b \\
b & a & 0 & 0 \\
c & 0 & a & 0 \\
0 & c & b & 0 \\
0 & 0 & 0 & c
\end{pmatrix}$$
Using Theorem 3.4 (5) for $\alpha$ where $V_3$, which implies that $V$ by Theorem 3.1, it follows that $V$.

The software SINGULAR gives the following minimal system of generators for the $\rho$-module $\rho = (0 : 1 : 1)$.

$$\rho_1 = (0, -x^2yz, y^4 + x^2z^2), \quad \rho_2 = (-xy^2z, 0, x^4 + y^2z^2), \quad \rho_3 = (xyz^3, -x^4z, x^2y^3 - yz^4),$$

$$\rho_4 = (-y^4z, xyz^3, x^3y^2 - xz^4) \text{ and } \rho_5 = (-y^5 - x^2yz^2, x^5 + xy^2z^2, 0).$$

Since now $\dim AR(f) = 2$, there are several choices for the syzygy $\rho_1$ in Theorem 4.1. We discuss three choices.

**Choice 1.** If we choose $\rho_1 = \rho_1'$, then the corresponding Bourbaki ideal $B(C, \rho_1)$ is spanned by $g_2 = v(\rho_2') = -xyz$, $g_3 = v(\rho_3') = xz^3$, $g_4 = v(\rho_4') = -y^3z$ and $g_5 = v(\rho_5') = -y^4 - x^2z^2$, where $v$ is the morphism defined in (4.2). Hence the
scheme $Z(C, ρ_1)$ consists of two points, both nonreduced, one at $p_1 = (1 : 0 : 0)$, given in local coordinates $u, v$ by an ideal $(uv, v^2 + u^4)$, and another at $p_2 = (0 : 0 : 1)$, given in local coordinates by an ideal $(u, v^3)$.

Choice 2. If we choose $ρ_1 = ρ_2^2$, then the corresponding Bourbaki ideal $B(C, ρ_1)$ is spanned by $h_1 = v(ρ_1) = xyz$, $h_3 = v(ρ_3) = x^3z$, $h_4 = v(ρ_4) = yz^3$ and $h_5 = v(ρ_5) = −x^4 − y^2z^2$. Hence the support of the scheme $Z(C, ρ_1)$ consists of two points, one at $q_1 = (0 : 1 : 0)$, and another at $p_2 = (0 : 0 : 1)$, the same point as in Choice 1.

Choice 3. If we choose $ρ_1 = ρ_1' + tρ_2^2$, where $t ∈ C^*$, then the corresponding Bourbaki ideal $B(C, ρ_1)$ is spanned by $k_1 = v(ρ_1') = txyz$, $k_2 = v(ρ_2') = −xyz$, $k_3 = v(ρ_3') = xz(z^2 + tx^2)$, $k_4 = v(ρ_4') = −yz(y^2 + tz^2)$ and $k_5 = v(ρ_5') = −y^4 − x^2z^2 − t(x^4 + y^2z^2) = −y^2(y^2 + tz^2) − x^2(tx^2 + z^2)$. If we take $t = −s^4$ for $s ∈ C^*$, then the support of the scheme $Z(C, ρ_1)$ consists of the following 9 points:

i) $z_j(s) = (ε_j : s : 0)$ for $j = 1, 2, 3, 4$, where $ε_j$ are the 4 roots of the equation $ε^4 = 1$;

ii) $z_j(s) = (0 : s^2 : (−1)^j)$, where $j = 5, 6$;

iii) $z_j(s) = ((−1)^j : 0 : s^2)$, where $j = 7, 8$ and

iv) $z_9(s) = p_2 = (0 : 0 : 1)$.

Theorem 4.1 (1) tells us that $\deg B(C, ρ_1) = 9$, and hence all these points $z_j(s)$ are simple points. When $s → 0$, we see that the points 6 points $z_j(s)$ for $j ∈ \{1, 2, 3, 4, 7, 8\}$ converge to the point $p_1$, and the 2 points $z_j(s)$ for $j ∈ \{5, 6\}$ converge to the point $p_2 = z_9(s)$. Similarly, when $|s| → +∞$, the 6 points $z_j(s)$ for $j ∈ \{1, 2, 3, 4, 5, 6\}$ converge to the point $q_1$, and the 2 points $z_j(s)$ for $j ∈ \{7, 8\}$ converge to the point $p_2 = z_9(s)$. Moreover, the line $L_7 : z = 0$, corresponding to the point $P_7$, contains the 4 points $z_j(s)$ for $j ∈ \{1, 2, 3, 4\}$ for any $s$, the maximal number of collinear points among the points $z_j(s)$.

Note that the line $L_1 : x + y + z = 0$ is disjoint from the support of the scheme $Z(C, ρ_1)$ for most choices of $ρ_1$, and the bound given by Theorem 4.1 is $d^r_1 ≥ d − r − 1 = 1$. In fact, we have equality, hence this bound is sharp in this situation as well.

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