THE LOG MINIMAL MODEL PROGRAM FOR KÄHLER 3-FOLDS

OMPROKASH DAS AND CHRISTOPHER HACON

Abstract. In this article we show that the log minimal model program for \( \mathbb{Q} \)-factorial dlt pairs \((X, B)\) on a compact Kähler 3-fold holds. More specifically, we show that after finitely many divisorial contractions and flips we obtain either a (log) minimal model or a Mori fiber space. We also prove a base point free theorem for Kähler 3-folds.

Contents

1. Introduction 2
2. Preliminaries 5
2.1. Projectivity criteria 10
2.2. Resolution of singularities and Kawamata-Viehweg vanishing theorem 10
2.3. MMP for Kähler 3-folds 14
2.4. Technical lemmas 17
3. Minimal model program for dlt pseudo-effective pairs 22
4. The minimal model program for uniruled pairs 28
4.1. Cone Theorem 31
4.2. Existence of divisorial contractions and flips 39
5. Towards the Existence of Mori fiber spaces 41
6. Main Theorems 47
References 53

2020 Mathematics Subject Classification. 14E30, 32J27, 32J17, 14J30.

O. Das was supported by the Start–Up Research Grant(SRG), Grant No. # SRG/2020/000348 of the Science and Engineering Board Research Board (SERB), Govt. Of India.

C. Hacon was partially supported by NSF research grants no: DMS-1952522, DMS-1801851, DMS-2301374, DMS-58503413 and by a grant from the Simons Foundation: Award Number: 256202.
1. Introduction

The minimal model program or MMP is one of the most important tools in the birational classification of complex projective varieties. It was fully established in dimension 3 in the 80’s and 90’s and recently extended to many cases in arbitrary dimension including the case of varieties of log general type [BCHM10].

There are many technical difficulties in adapting the minimal model program to compact Kähler manifolds. Some of the standard techniques used in the MMP for projective varieties fail for compact Kähler manifolds, for example, Mori’s Bend and Break technique for producing rational curves, the base point free theorem and the contraction of negative extremal rays fail on Kähler manifolds. Campana and Peternell investigated the existence of Mori contractions in [CP97, Pet98, Pet01], using the deformation theory of rational curves on smooth 3-folds developed in [Kol91, Kol96]. In [HP16], Peternell and Höring successfully established the minimal model program for compact Kähler 3-folds $X$ with terminal singularities and $K_X$ pseudo-effective. In a subsequent paper [HP15] they also proved the existence of Mori fiber spaces when $X$ has terminal singularities and $K_X$ is not pseudo-effective.

In [HP16] the authors introduced many new tools which enabled them to use several techniques from the projective MMP. Building on the work of [HP16, HP15, CHP16], in this article we show that the minimal model program on compact Kähler 3-folds works in much greater generality. More precisely, we show that this program holds for $\mathbb{Q}$-factorial dlt pairs $(X, B)$. The main results of this article are the following.

**Theorem 1.1.** Let $(X, B)$ be a dlt pair where $X$ is a $\mathbb{Q}$-factorial compact Kähler 3-fold. If $K_X + B$ is pseudo-effective, then there exists a finite sequence of flips and divisorial contractions

$$\phi : X \to X_1 \to \ldots \to X_n$$

such that $K_{X_n} + \phi_* B$ is nef.

This result is proved in [HP16] when $X$ has terminal singularities and $B = 0$.

**Theorem 1.2.** Let $(X, B)$ be a dlt pair where $X$ is a $\mathbb{Q}$-factorial compact Kähler 3-fold. If $K_X + B$ is not pseudo-effective, then there exists a finite sequence of flips and divisorial contractions

$$\phi : X \to X_1 \to \ldots \to X_n$$

and a Mori fiber space $\varphi : X_n \to S$, i.e. a morphism such that $-(K_{X_n} + \phi_* B)$ is $\varphi$-ample and $\rho(X_n/S) = 1$. 
This result is proved in [HP15] when $X$ has terminal singularities and $B = 0$.

We note that one of the main difficulties in proving the above theorems is proving the existence of divisorial contractions. In the pseudo-effective case, the existence of flips and divisorial contractions to a point has already been established by work of [CHP16], [HP16] and [DO24] (see Theorem 2.23), and so it remains to prove the existence of divisorial contractions to a curve. This is one of the key results of this article.

**Definition 1.3.** [HP16, Definition 4.3 and 7.1][CHP16, Notation 4.1] Let $X$ be a normal $\mathbb{Q}$-factorial compact Kähler 3-fold with rational singularities.

- We say that a curve $C \subset X$ is very rigid if $\dim_m \text{Chow}(X) = 0$ for all $m \geq 1$.
- Let $(X, B)$ be a log canonical pair. A $(K_X + B)$-negative extremal ray $R$ of $\overline{NA}(X)$ is called small if every curve $C \subset X$ with $[C] \in R$ is very rigid.
- An extremal ray $R$ as above is called divisorial type if it is not small.

**Notation 1.4.** Let $(X, B)$ be a dlt pair, where $X$ is a $\mathbb{Q}$-factorial compact Kähler 3-fold. Let $R$ be a $(K_X + B)$-negative extremal ray of divisorial type which is defined by a nef class $\alpha$ so that $R = \overline{NA}(X) \cap \alpha^\perp$. Let $S$ be the surface which is covered by and contains all the curves $C \subset X$ such that $[C] \in R$ (cf. [HP16, Lemma 7.5]). Let $\nu : \tilde{S} \to S$ be the normalization morphism. Consider the nef reduction $f : \tilde{S} \to T$ of the nef $(1,1)$-class $\nu^*(\alpha|_S)$ (see [HP15, Theorem 3.19]). Note that since $S$ is covered by a family of $\alpha$-trivial curves, the lifts of these curves on $\tilde{S}$ give a family of $\nu^*(\alpha|_S)$-trivial curves. Thus from the definition of nef dimension it follows that $\dim T \leq 1$, and in particular, $f : \tilde{S} \to T$ is a morphism (cf. [BCE+02, 2.4.4]). We define the notation $n(\alpha)$ as the nef dimension of $\nu^*(\alpha|_S)$, i.e.,

$$n(\alpha) := \nu_{\text{nef}}(\nu^*(\alpha|_S)) = \dim T \in \{0, 1\}$$

where $\nu_{\text{nef}}(\ldots)$ denotes the nef dimension.

**Theorem 1.5.** Let $(X, B)$ be a $\mathbb{Q}$-factorial dlt pair where $X$ is a compact Kähler 3-fold. Let $R$ be a $(K_X + B)$-negative extremal ray of divisorial type supported by a nef and big $(1,1)$-class $\alpha$ such that $\alpha - (K_X + B)$ is Kähler and the nef dimension is $n(\alpha) = 1$. Then the contraction $c_R : X \to Y$ of $R$ exists.

One of the main difficulties in proving a contraction theorem in the Kähler category is the lack of a base-point free theorem analogous to that of [KM98, Theorem 3.3] in the projective case. Note that, an exact analogue of [KM98,
Theorem 3.3] is impossible on a compact Kähler variety which is not projective, since the existence of a big divisor on a compact Kähler variety with rational singularities implies that it is projective by [Nam02, Theorem 1.6] (see Theorem 2.13). However, there is a base-point free conjecture in the Kähler category involving nef and big cohomology classes which can be thought of as an analogue of [KM98, Theorem 3.3]. This conjecture is stated in [Hör21, Conjecture 1.1] for manifolds.

Conjecture 1.6 (Base point freeness). Let \((X, B)\) be a klt pair, where \(X\) is a normal \(\mathbb{Q}\)-factorial compact Kähler variety, and \(\alpha \in H^{1,1}_{\text{BC}}(X)\) a nef class on \(X\). If \(\alpha - (K_X + B)\) is nef and big, then there exists a proper surjective morphism with connected fibers \(f : X \to Y\) to a normal compact Kähler variety \(Y\) with rational singularities and a Kähler class \(\alpha_Y \in H^{1,1}_{\text{BC}}(Y)\) such that \(\alpha = f^*\alpha_Y\).

As an application of Theorems 1.1 and 1.2 and other related results we prove this conjecture when \(X\) has dimension 3.

**Theorem 1.7.** Let \((X, B)\) be a log pair, where \(X\) is a normal \(\mathbb{Q}\)-factorial compact Kähler 3-fold, and \(\alpha \in H^{1,1}_{\text{BC}}(X)\) a nef class. Assume that one of the following conditions is satisfied:

(i) \((X, B)\) is klt and \(\alpha - (K_X + B)\) is nef and big, or

(ii) \((X, B)\) is dlt and \(\alpha - (K_X + B)\) is a Kähler class.

Then there exists a proper surjective morphism with connected fibers \(\psi : X \to Z\) to a normal Kähler variety \(Z\) with rational singularities and a Kähler class \(\alpha_Z \in H^{1,1}_{\text{BC}}(Z)\) on \(Z\) such that \(\alpha = \psi^*\alpha_Z\). In particular, in case (ii), \(\psi\) is a projective morphism.

When \(X\) has terminal singularities, \(B = 0\) and \(\alpha - K_X\) is a Kähler class, this theorem is proved in [Hör21, Theorem 1.3] (when \(\alpha\) is both nef and big) and in [TZ18, Theorem 2.7] (when \(\alpha\) is nef but not big). In fact our proof is a direct generalization of the techniques in [Hör21] and [TZ18] using more general MMP results such as Theorems 1.1 and 1.2.

This article is organized in the following manner. In Section 2 we collect some important technical results which are used throughout the article. In Section 3 we prove Theorems 1.5 and 1.1 under the additional hypothesis that \(X\) has strongly \(\mathbb{Q}\)-factorial singularities. Section 4 is dedicated to developing the cone and contraction theorems for non-pseudo-effective pairs; the contraction theorem is again proved under the additional hypothesis that \(X\) has strongly \(\mathbb{Q}\)-factorial singularities. These results are then used in Section 5 to prove some important technical results related to the existence of Mori fiber spaces. The main result of this section is Theorem 5.5, which is a special case of Theorem 1.7, namely the case when \(K_X + B\) is not pseudo-effective. In
Section 6 we prove all the main theorems in full generality, namely Theorems 1.1, 1.2, 1.5 and 1.7.

Acknowledgment. The authors would like to thank Sébastien Boucksom, Ved Datar, Andreas Höring, Sabyasachi Mukherjee, Mihai Păun, Valentino Tosatti, and Mingchen Xia for answering their questions and the anonymous referees for their many corrections and suggestions to improve this paper.

2. Preliminaries

An analytic variety or simply a variety is an irreducible and reduced complex space. A pair \((X, B)\) consists of a normal analytic variety \(X\) and an effective \(\mathbb{Q}\)-divisor \(B \geq 0\) such that \(K_X + B\) is \(\mathbb{Q}\)-Cartier. We define the singularities of the pair \((X, B)\) as in [KM98]. If \(B\) is not assumed to be effective, then we will call \((X, B)\) a sub-pair and the corresponding singularities of \((X, B)\) sub-klt, sub-dlt, etc.

Definition 2.1. [HP16, Definition 2.2] An analytic variety \(X\) is Kähler or a Kähler space if there exists a positive closed real \((1, 1)\)-form \(\omega \in \mathcal{A}^{1,1}(X)\) such that the following holds: for every point \(x \in X\) there exists an open neighborhood \(x \in U\) and a closed embedding \(\iota_U : U \to V\) into an open set \(V \subset \mathbb{C}^N\), and a strictly plurisubharmonic \(C^\infty\)-function \(f : V \to \mathbb{R}\) such that \(\omega|_{U \cap X_{\text{sm}}} = (i\partial\bar{\partial}f)|_{U \cap X_{\text{sm}}}\). Here \(X_{\text{sm}}\) is the smooth locus of \(X\).

In the following we collect some important definitions. For a more detailed discussion, we encourage the reader to consult [HP16, HP15, CHP16] and the references therein.

Definition 2.2. (i) A compact analytic variety \(X\) is Fujiki’s class \(C\) if one of the following equivalent conditions are satisfied:

(a) \(X\) is a meromorphic image of a compact Kähler variety \(Y\), i.e., there exists a dominant meromorphic map \(f : Y \to X\) from a compact Kähler variety \(Y\) (see [Fuj78, 4.3, page 34]).

(b) \(X\) is a holomorphic image of a compact Kähler manifold, i.e., there is a surjective morphism \(f : Y \to X\) from a compact Kähler manifold \(Y\) (see [Fuj78, Lemma 4.6]).

(c) \(X\) is bimeromorphic to a compact Kähler manifold (see [Var89, Theorem 3.2, page 51]).

(ii) On a normal compact analytic variety \(X\) we replace the use of Néron-Severi group \(\text{NS}(X)_{\mathbb{R}}\) by \(H^{1,1}_{\text{BC}}(X)\), the Bott-Chern cohomology of real closed \((1, 1)\)-forms with local potentials or equivalently, the closed bidegree \((1, 1)\)-currents with local potentials. See [HP16, Definition 3.1]
and 3.6] for more details. More specifically, we define
\[ N^1(X) := H^{1,1}_{BC}(X). \]

(iii) If \( X \) is in Fujiki’s class \( C \) and has rational singularities, then from [HP16, Eqn. (3)] we know that \( N^1(X) = H^{1,1}_{BC}(X) \subset H^2(X, \mathbb{R}) \). In particular, the intersection product can be defined in \( N^1(X) \) via the cup product of \( H^2(X, \mathbb{R}) \).

(iv) Let \( X \) be a normal compact analytic variety contained in Fujiki’s class \( C \). We define \( N_1(X) \) to be the vector space of real closed currents of bi-dimension \((1, 1)\) modulo the following equivalence relation:
\[ T_1 \equiv T_2 \text{ if and only if } T_1(\eta) = T_2(\eta) \]
for all real closed \((1, 1)\)-forms \( \eta \) with local potentials.

(v) We define \( \overline{NA}(X) \subset N_1(X) \) to be the closed cone generated by the classes of positive closed currents \( \Theta \geq 0 \) (see [Dem12, §1.C]). The Mori cone \( \overline{NE}(X) \subset \overline{NA}(X) \) is defined as the closure of the cone of currents of integration \( T_C \), where \( C \subset X \) is an irreducible curve.

(vi) Let \( X \) be a normal compact analytic variety and \( u \in H^{1,1}_{BC}(X) \). Then \( u \) is called pseudo-effective if it can be represented by a bi-degree \((1, 1)\)-current \( T \in D^{1,1}(X) \) which is locally of the form \( \partial \bar{\partial} f \) for some psh function \( f \). \( u \) is called nef if it can be represented by a form \( \alpha \) with local potentials such that for some positive \((1, 1)\)-form \( \omega \) on \( X \) and for every \( \epsilon > 0 \), there exists a \( C^\infty \)-function \( f_\epsilon \in A^0(X) \) such that \( \alpha + i\partial \bar{\partial} f_\epsilon \geq -\epsilon \omega \). See [Dem85] for more details.

(vii) The nef cone \( \operatorname{Nef}(X) \subset N^1(X) \) is the cone generated by nef cohomology classes. Let \( \mathcal{K} \) be the open cone in \( N^1(X) \) generated by the classes of Kähler forms. Note that the nef cone \( \operatorname{Nef}(X) \) is the closure of \( \mathcal{K} \), i.e. \( \operatorname{Nef}(X) = \overline{\mathcal{K}} \).

(viii) We say that a variety \( X \) is Q-factorial if for every Weil divisor \( D \subset X \), there is a positive integer \( k > 0 \) such that \( kD \) is a Cartier divisor, and for the canonical sheaf \( \omega_X \), there is a positive integer \( m > 0 \) such that \( (\omega_X^m)^{**} \) is a line bundle. It is well known that if \( X \) is a Q-factorial 3-fold and \( X \dashrightarrow X' \) is a flip or a divisorial contraction, then \( X' \) is also Q-factorial.

(ix) If \( X \) is a normal variety then we say that a coherent sheaf \( \mathcal{L} \) is divisorial if it is reflexive of rank 1. If \( U \subset X \) is Stein, then \( \mathcal{L}|_U \cong \mathcal{O}_U(D) \) for some Weil divisor \( D \) on \( U \). We say that a divisorial sheaf \( \mathcal{L} \) is Q-Cartier (or a Q-line bundle) if \( (\mathcal{L}^{\otimes m})^{**} \) is a line bundle for some \( m \in \mathbb{N} \). We will say that a variety \( X \) is strongly Q-factorial if every divisorial sheaf \( \mathcal{L} \) is a Q-line bundle. Note that a complex manifold is an example of a strongly Q-factorial variety (see Lemma 2.3).
Lemma 2.3. If $X$ is a complex manifold, then every reflexive rank 1 sheaf on $X$ is a line bundle. In particular, $X$ is strongly $\mathbb{Q}$-factorial.

Proof. Let $\mathcal{L}$ be a reflexive rank 1 sheaf on $X$. Since $X$ is a manifold, from [Kob87, Corollary V.5.20, page 160] it follows that there is an analytic subset $Z \subset X$ such that $\mathcal{L}|_{X \setminus Z}$ is a line bundle and $\text{codim}_X(Z) \geq 3$. Then from [Har74, Theorem 4] it follows that $\text{Pic}(X) \cong \text{Pic}(X \setminus Z)$. Thus $\mathcal{L}|_{X \setminus Z}$ extends to an unique line bundle on $X$, say $\mathcal{M}$. But since $\mathcal{L}$ is a reflexive sheaf, $\mathcal{L}|_{X \setminus Z} \cong \mathcal{M}|_{X \setminus Z}$ and $\text{codim}_X(Z) \geq 2$, it follows that $\mathcal{L} \cong \mathcal{M}$ is a line bundle.

Lemma 2.4. Let $f : Y \to X$ be a small bimeromorphic projective morphism of analytic varieties such that $X$ is strongly $\mathbb{Q}$-factorial. Then $f$ is an isomorphism.

Proof. Let $\mathcal{L}$ be a relatively ample line bundle. Since $X$ is strongly $\mathbb{Q}$-factorial, $(f_*\mathcal{L})^{**}$ is $\mathbb{Q}$-Cartier and so $\mathcal{M} := ((f_*\mathcal{L})^{**})$ is a line bundle for some integer $m > 0$. Since $f$ is small, there is an open subset $U \subset Y$ such that $\text{codim}_Y(Y \setminus U) \geq 2$ and $f|_U$ is an isomorphism. Then $(f_*\mathcal{M})|_U \cong \mathcal{L}^m|_U$, and so $f_*\mathcal{M} \cong \mathcal{L}^m$ is a relatively ample line bundle. Thus $f$ is an isomorphism.

Lemma 2.5. Let $X$ be a strongly $\mathbb{Q}$-factorial compact analytic variety, $(X, B)$ a dlt pair, and $f : X \dasharrow X'$ a $(K_X + B)$-flip or divisorial contraction. Then $X'$ is also strongly $\mathbb{Q}$-factorial.

Proof. This follows easily from the base point free theorem [Nak87, Theorem 4.8]. We may assume that $(X, B)$ is klt. If $f : X \to X'$ is a $(K_X + B)$-negative divisorial contraction with exceptional divisor $E$, then for any divisorial sheaf $\mathcal{L}$ on $X'$, let $\mathcal{M} = (f_*\mathcal{L})^{**}$. Then $\mathcal{M}$ is a divisorial sheaf on $X$. Recall that by assumption $X$ is strongly $\mathbb{Q}$-factorial, $\rho(X/X') = 1$ and $E$ generates $\mathcal{N}^1(X/X')$. Thus $\mathcal{M}$ is $\mathbb{Q}$-Cartier and there are integers $m > 0$ and $n \in \mathbb{Z}$ such that $(\mathcal{M}^m)^{**} \otimes \mathcal{O}_X(nE)$ is a $f$-numerically trivial line bundle. Working locally over $X'$, it follows from the base point free theorem [Nak87, Theorem 4.8] that $(\mathcal{M}^m)^{**} \otimes \mathcal{O}_X(nE) \cong f_*\mathcal{M}'$ for some line bundle $\mathcal{M}'$ on $X'$. Let $U := X' \setminus (f(E) \cup X'_\text{sing})$ and $j : U \to X'$ be the inclusion, then $U \subset X'$ is a big open subset, i.e. the complement of a finite union of analytic subvarieties of codimension at least 2. Then $(\mathcal{M}^m)^{**} \cong j_*((\mathcal{L}|_U)^m) \cong j_*((\mathcal{M}'|_U) \cong \mathcal{M}'$ and so $\mathcal{M}$ is $\mathbb{Q}$-Cartier as required.

A similar argument shows that $X'$ is strongly $\mathbb{Q}$-factorial when $f : X \dasharrow X'$ is a flip.

Remark 2.6. Note that if $D$ is a $\mathcal{Q}$-Cartier divisor on a variety $X$, then in algebraic geometry we say that $D$ is nef if $D \cdot C \geq 0$ for every irreducible curve $C \subset X$. However the cohomology class of the current of integration of $D$ is not necessarily nef in the sense defined in Definition 2.2 (vi). To avoid this
sort of confusion, temporarily we will call a \(\mathbb{Q}\)-Cartier divisor \(D\) \emph{algebraically nef} if \(D \cdot C \geq 0\) for every irreducible curve \(C \subset X\), and \(D\) is \emph{analytically nef} if the cohomology class of the current of integration associated to \(D\) is nef in the sense of Definition 2.2 (vi). We remark that if \(D = \sum d_iD_i, d_i \in \mathbb{Q}\), is a linear combination of effective Cartier divisors \(D_i\), then the corresponding current of integration is given by \(\int_D \ldots = \sum d_i \int_{D_i} \ldots\)

Note that if \(X\) is a compact Kähler variety, then analytically nef implies algebraically nef but the converse is not true in general, see [HP18, page 385] for a counterexample. However, the following two lemmas show that these two versions of nefness are equivalent in important cases: (i) \(X\) is a Moishezon space (see Lemma 2.7), and (ii) \(X\) is a Kähler 3-fold and \(D\) is an adjoint divisor (see Lemma 2.9).

**Lemma 2.7.** [Pau98, Corollary 1, page 418] Let \(X\) be a normal compact Moishezon variety, i.e. \(X\) is bimeromorphic to a projective variety. Let \(D\) be a \(\mathbb{Q}\)-Cartier divisor on \(X\). Then \(D\) is analytically nef if and only if it is algebraically nef.

**Remark 2.8.** Note that one can prove the above lemma more generally for \(\mathbb{R}\)-Cartier divisors by passing to a resolution of singularities \(\pi: \tilde{X} \to X\) such that \(\tilde{X}\) is a projective manifold and then use [DP04, Corollary 0.2], [Pau98, Theorem 1, page 412] and [DHP24, Lemma 2.38].

The proof of the next lemma relies on the MRC fibration introduced in [KMM92] and [Cam92]; see also [CH20, Remark 6.10] for remarks on the analytic version of the MRC fibration.

**Lemma 2.9.** Let \((X, B)\) be a dlt pair, where \(X\) is a \(\mathbb{Q}\)-factorial compact Kähler 3-fold. Assume that one of the following conditions is satisfied:

- (i) \(K_X + B\) is pseudo-effective, or
- (ii) \(K_X + B\) is not pseudo-effective and there exists a Kähler form \(\omega\) on \(X\) such that \(K_X + B + \omega\) is pseudo-effective.

Then \(K_X + B\) (resp. \(K_X + B + \omega\)) is analytically nef if and only if it is algebraically nef.

**Proof.** If \(K_X + B\) is pseudo-effective, then the result follows from a similar proof as in [HP16, Corollary 4.2] and [CHP16, Corollary 4.1]. For a complete proof in this case, see [DO24, Corollary 2.19]. In the second case, first replacing \(B\) by \((1 - \varepsilon)B\) and \(\omega\) by \(\omega + \varepsilon B\) we may assume that \((X, B)\) is klt. Now if the base of the MRC fibration of \(X\) has dimension \(\leq 1\), then from Lemma 2.42 and its proof it follows that \(X\) is projective and additionally \(\text{NS}(X)_{\mathbb{R}} = H^{1,1}_{\text{BC}}(X)\). Then the result follows from Lemma 2.7. If the base of the MRC fibration of \(X\) has dimension 2, then the result follows by a similar proof as in...
[HP15, Corollary 3.5]. Note that this proof uses [HP15, Lemma 3.4], which is replaced here by Claim 4.7.

Remark 2.10. If \( f : X \to Y \) is a proper surjective morphism between normal analytic varieties and \( \dim Y > 0 \), then we say that a \( \mathbb{Q} \)-Cartier \( \mathbb{Q} \)-divisor \( D \) on \( X \) is *nef over* \( Y \) or *\( f \)-nef* if \( D \cdot C \geq 0 \) for every irreducible (compact) curve \( C \subset X \) such that \( f(C) = \text{pt} \). Note that to be more precise we should call \( D \) *algebraically nef over* \( Y \) or *algebraically \( f \)-nef*, however, by Lemma 2.7 it is equivalent to *analytically nef* (over \( Y \)) if \( f \) is a Moishezon morphism (see [GPR94, Definition VIII.3.5 (2), page 334]). Since all the morphisms considered in this article are Moishezon, e.g. projective morphisms, proper bimeromorphic morphisms, etc., this will not create any confusion.

Let \( X \) be a normal compact variety and \( \omega \) a real closed \((1,1)\)-form on \( X \) with local potentials. Then we can define

\[
\lambda_\omega \in N_1(X)^*, \quad \text{via} \quad \lambda_\omega([T]) = T(\omega).
\]

This gives a well defined canonical map

\[
\Phi : N^1(X) \to N_1(X)^*, \quad [\omega] \mapsto \lambda_\omega.
\]

If in addition \( X \) belongs to Fujiki’s class \( C \) and has rational singularities, then \( \Phi \) is an isomorphism by [HP16, Proposition 3.9]. Moreover, if \( \dim X = 3 \), then \( \Phi(\text{Nef}(X)) = \text{NA}(X)^* \) by [HP16, Proposition 3.15].

We recall the following useful result from [HP16] for future reference.

Lemma 2.11. [HP16, Lemma 3.3] Let \( f : X \to Y \) be a proper bimeromorphic morphism between normal compact complex spaces in Fujiki’s class \( C \) with at most rational singularities. Then we have an injection

\[
f^* : H^{1,1}_{BC}(Y) = H^1(Y, H_Y) \hookrightarrow H^1(X, H_X) = H^{1,1}_{BC}(X)
\]

whose image is given by

\[
\text{Im}(f^*) = \{ \alpha \in H^1(X, H_X) \mid \alpha \cdot C = 0 \ \forall \ C \subset X \ \text{curve s.t.} \ f_* C = 0 \}.
\]

Furthermore, let \( \alpha \in H^1(X, H_X) \subset H^2(X, \mathbb{R}) \) be a class such that \( \alpha = f^* \beta \) with \( \beta \in H^2(Y, \mathbb{R}) \). Then there exists a smooth real closed \((1,1)\)-form with local potentials \( \omega_Y \) on \( Y \) such that \( \alpha = f^*[\omega_Y] \).

In general the push-forward of a cohomology class \([T]\) of a bi-degree \((1,1)\)-current \( T \) with local potentials may not be a cohomology class in \( H^{1,1}_{BC}(Y) \), since the current \( f_* T \) may not have local potentials on \( Y \). See [HP16, Lemma 3.4] for a sufficient condition when this does hold.
Remark 2.12. Note that if $X$ is a compact Kähler space and $\pi : X' \to X$ a projective morphism, then $X'$ is again Kähler (see [Var89, Prop. 1.3.1.(vi), page 24]). In particular, $X$ has a Kähler desingularisation. A subvariety of a Kähler space is also Kähler, see [Var89, Prop. 1.3.1.(i), page 24].

2.1. Projectivity criteria. Recall the following generalization of [Moi66] due to Namikawa.

**Theorem 2.13.** [Nam02, Theorem 1.6] Let $X$ be a compact Moishezon variety with 1-rational singularities. If $X$ is Kähler, then $X$ is projective.

**Remark 2.14.** Recall that $X$ has 1-rational singularities if it admits a resolution $\nu : X' \to X$ such that $R^1\nu_*\mathcal{O}_{X'} = 0$. In particular, by Lemma 2.31, any klt variety has rational and hence 1-rational singularities.

Next we recall the following of Kodaira’s projectivity criterion from [Kod54].

**Theorem 2.15.** Let $X$ be a compact Kähler variety with rational singularities such that $H^2(X, \mathcal{O}_X) = 0$, then $X$ is projective.

**Proof.** Let $\nu : X' \to X$ be a resolution of singularities, then $X'$ is a Kähler manifold and $R^i\nu_*\mathcal{O}_{X'} = 0$ for all $i > 0$. Thus $H^2(X', \mathcal{O}_{X'}) \cong H^2(X, \mathcal{O}_X) = 0$. By [Kod54], $X'$ is projective and hence $X$ is Moishezon. By Theorem 2.13, $X$ is projective. □

2.2. Resolution of singularities and Kawamata-Viehweg vanishing theorem. The existence of resolutions of singularities for analytic varieties and embedded resolutions are proved in [AHV77] and [BM97]. Unlike the case of algebraic varieties (of finite type over a field), for analytic varieties the resolution of singularities is not obtained via global blow ups of smooth centers, unless the variety is (relatively) compact. The following version of log resolution will be useful for us.

Let $X$ be a normal analytic variety and $D = \sum_{i=1}^n D_i$ a reduced Weil divisor on $X$. Then there exists a unique largest Zariski open subset $U$ of $X$ contained in its smooth locus such that $D|_U$ is a simple normal crossing divisor and $\text{codim}_X(X \setminus U) \geq 2$. The open subset $U$ is called the simple normal crossing locus of the pair $(X, D)$ and we denote it by $\text{SNC}(X, D)$. Also, recall that a locally compact topological space $X$ is called countable at infinity or $\sigma$-compact if it can be written as a countable union of compact subsets. Clearly, any compact space $X$ is $\sigma$-compact. Moreover, locally compact and second countable Hausdorff spaces are $\sigma$-compact.

**Theorem 2.16** (Log Resolution). [BM97, Theorems 13.2, 1.10 and 1.6] Let $X \subset W$ be a relatively compact open subset of an analytic variety $W$ and
THE LOG MINIMAL MODEL PROGRAM FOR Kähler 3-Folds

$D \subset X$ a pure codimension 1 reduced analytic subset of $X$. Then there exists a projective bimeromorphic morphism $f : Y \to X$ from a smooth variety $Y$ satisfying the following properties:

1. $f$ is a successive blow up of smooth centers contained in $X \setminus \text{SNC}(X, D)$,
2. $f^{-1}(\text{SNC}(X, D)) \cong \text{SNC}(X, D)$, and
3. $\text{Ex}(f)$ is a pure codimension 1 subset of $Y$ such that $\text{Ex}(f) \cup (f^{-1}_* D)$ has SNC support.

Proof. Since $W$ is a locally compact Hausdorff space and $X \subset W$ is a relatively compact open subset, there exists another relatively compact open subset $X_1 \subset W$ containing $X$ such that $X \subset X_1$. Since every point of an analytic variety has a second countable open neighborhood, it follows that $X_1$ is $\sigma$-compact. Thus by [BM97, Theorem 13.3], there is a projective bimeromorphic morphism $f_1 : Y_1 \to X_1$ from a smooth variety $Y_1$ obtained by finitely many blow ups of smooth centers contained in the singular locus of $X_1$. Note that $X_2 := f_1^{-1}(X) \subset Y_1$ is relatively compact. Now consider the non-SNC locus $Z := Y_1 \setminus \text{SNC}(Y_1, (f_1^{-1}_* D \cup \text{Ex}(f_1)))$; this is a closed analytic subset of $Y_1$. Note that $Z \cap X_2$ is a closed analytic subset of the manifold $X_2$ and also an open subset of $Z$. It then follows that $Z \cap X_2$ is a relatively compact open subset of $Z$. Therefore by [BM97, Theorem 13.2] applied to $Z \cap X_2 \subset X_2$ we obtain a projective bimeromorphic morphism $g : Y \to X_2$ (which is a composite of blow ups of smooth centers) from a smooth variety $Y$ such that $(Y, (f_1|_{X_2} \circ g)_*^{-1} D + \text{Ex}(f_1|_{X_2} \circ g))$ is a log smooth pair, $f_1|_{X_2} \circ g$ is an isomorphism over the SNC locus of $(X, D)$ and $\text{Ex}(f_1|_{X_2} \circ g)$ is a pure codimension 1 subset of $Y$. Then we conclude by setting $f := f_1|_{X_2} \circ g$.

Next we will state Chow’s lemma for analytic varieties due to Hironaka [Hir75]. Note that unlike Chow’s lemma for algebraic varieties (of finite type over a field), in the analytic category it does not hold for arbitrary proper morphism between analytic varieties; it only holds for proper bimeromorphic morphisms.

**Theorem 2.17 (Chow’s Lemma).** [Hir75, Corollary 2] Let $f : X \to Y$ be a proper bimeromorphic morphism between two complex spaces such that $Y$ is reduced and $\sigma$-compact. Then there exists a projective bimeromorphic morphism $\nu : X' \to X$ from a complex space $X'$ such that the composition $f' = f \circ \nu : X' \to Y$ is projective.

As an application of Chow’s lemma we prove the following useful result.

**Lemma 2.18 (Reducing Proper Morphism to Projective Morphisms).** Let $f : X \to S$ be a proper surjective morphism of analytic varieties, and let $L$ be
a $f$-big line bundle on $X$ and $D$ a $\mathbb{Q}$-divisor. Then over any relatively compact open subset $V \subset S$, there exists a proper bimeromorphic morphism $\alpha : W \to f^{-1}V$ from a smooth analytic variety $W$ such that $\beta = f|_{f^{-1}V} \circ \alpha : W \to V$ is a projective morphism and $(W, \alpha_*^{-1}(D|_{f^{-1}V}) + \text{Ex}(\alpha))$ is a log smooth pair.

Proof. Let $\phi : X \dasharrow Y$ be the relative Iitaka fibration of $L$ over $S$ and $g : Y \to S$ the induced projective morphism. Since $L$ is $f$-big, $\phi : X \dasharrow Y$ is bimeromorphic. Let $p : \Gamma \to X$ and $q : \Gamma \to Y$ be the resolution of indeterminacy of $\phi$ so that $p$ is proper (see [GPR94, Theorem VII.1.9]).

\begin{center}
\begin{tikzpicture}
\node (X) at (0,0) {$X$};
\node (Y) at (2,2) {$Y$};
\node (U) at (0,2) {$U$};
\node (S) at (2,0) {$S$};
\node (Gamma) at (1,1) {$\Gamma$};
\path[->]
(X) edge node[left]{$f$} (S)
(X) edge node[above]{$\phi$} (Y)
(Y) edge node[below]{$g$} (S)
(U) edge node[above]{$q$} (Y)
(U) edge node[left]{$p$} (X)
\end{tikzpicture}
\end{center}

Now fix a relatively compact open subset $V \subset S$. Choose another relatively compact open set $U \subset S$ containing $V$ such that $\overline{V} \subset U$. Note that $U$ is $\sigma$-compact, since it is relatively compact. Since $f$ and $g$ are both proper morphisms, it follows that $X_U := f^{-1}U$ and $Y_U := g^{-1}U$ are both $\sigma$-compact. Let $\Gamma_U := q^{-1}(g^{-1}U) = p^{-1}(f^{-1}U)$. Then from the commutative diagram above it follows that $q|_{\Gamma_U} : \Gamma_U \to g^{-1}U$ is a proper morphism. In particular, $\Gamma_U$ is $\sigma$-compact. Note that $q|_{\Gamma_U}$ is bimeromorphic. Therefore by Theorem 2.17 there is a projective bimeromorphic morphism $h : Z \to \Gamma_U$ from an analytic variety $Z$ such that $q|_{\Gamma_U} \circ h : Z \to Y_U$ is a projective bimeromorphic morphism. Since $g$ is projective, so is $Z \to U$.

Now we replace $U$ by our previously fixed open set $V$. Then $Z_V := (g \circ q \circ h)^{-1}V$ is a relatively compact open subset of $Z$. Let $r : W \to Z_V$ be the log resolution of $(Z_V, (p \circ h)^{-1}(D|_{f^{-1}V}))$ as in Theorem 2.16. Let $\alpha := p|_{\Gamma_V} \circ h|_{h^{-1}\Gamma_V} \circ r$ and $\beta := g|_{g^{-1}V} \circ q|_{\Gamma_V} \circ h|_{h^{-1}\Gamma_V} \circ r$, where $\Gamma_V := p^{-1}(f^{-1}V) = q^{-1}(g^{-1}V)$. Note that $\beta$ is a projective morphism, since it is a composition of projective morphisms over relatively compact bases. Then $\alpha : W \to f^{-1}V$ is a proper bimeromorphic morphism and $\beta : W \to V$ is a projective morphism such that $\beta = f|_{f^{-1}V} \circ \alpha$ and $(W, \alpha_*^{-1}(D|_{f^{-1}V}) + \text{Ex}(\alpha))$ is a log smooth pair. \hfill $\Box$

**Definition 2.19.** Let $f : X \to Y$ be a proper surjective morphism of analytic varieties and $L$ a line bundle on $X$. Then $L$ is called $f$-nef-big, if $c_1(L) \cdot C \geq 0$ for every irreducible curve $C \subset X$ such that $f(C) = pt$, and $\kappa(X/Y, L) = \dim X - \dim Y$ (see [Nak87, (B), page 554]). A $\mathbb{Q}$-Cartier divisor $D$ on $X$
is called $f$-nef-big if and only if so is $\mathcal{O}_X(mD)$ for some $m > 0$ sufficiently divisible.

The following is a version of the (relative) Kawamata-Viehweg vanishing theorem for proper morphisms between analytic varieties.

**Theorem 2.20.** [Nak87, Theorem 3.7][Fuj13, Corollary 1.4] Let $\pi : X \to S$ be a proper surjective morphism from a complex manifold $X$ onto an analytic variety $S$. Let $H$ be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $X$ such that it is $\pi$-nef-big and $\{H\}$ has SNC support. Then $R^i\pi_* (\omega_X \otimes \mathcal{O}_X([H])) = 0$ for all $i > 0$.

We prove the following variant which is more convenient for us.

**Theorem 2.21.** Let $\pi : X \to S$ be a proper surjective morphism of analytic varieties. Let $\Delta \geq 0$ be a $\mathbb{Q}$-divisor on $X$ such that $(X, \Delta)$ is klt, and $D$ is a $\mathbb{Q}$-Cartier integral Weil divisor on $X$ such that $D - (K_X + \Delta)$ is $\pi$-nef-big. Then

$$R^i\pi_* \mathcal{O}_X(D) = 0 \quad \text{for all} \quad i > 0.$$

**Proof.** First note that the question is local on the base. Then by Lemma 2.18, over a relatively compact Stein open subset $U \subset S$, there exists a proper bimeromorphic morphism $f : Y \to \pi^{-1}U$ from a smooth variety $Y$ such that $\pi|_{\pi^{-1}U} \circ f : Y \to U$ is a projective morphism and $(Y, f^{-1}(\text{Supp}(D) + \Delta)|_{\pi^{-1}U} + \text{Ex}(f))$ is a log smooth pair. Replacing $S, X$ and $\pi$ by $U, \pi^{-1}U$ and $\pi|_{\pi^{-1}U}$, respectively, we may assume that $S$ is a Stein space and $f : Y \to X$ is log resolution of $(X, \Delta + \text{Supp}(D))$ such that $\pi \circ f$ is projective.

Next observe that, since $S$ is (relatively compact and) Stein and $\pi \circ f$ is projective, every line bundle on $Y$ corresponds to a (non-unique) Cartier divisor. Indeed, if $\mathcal{M}$ is a line bundle on $Y$ and $H$ is a $(\pi \circ f)$-ample Cartier divisor on $Y$, then $\mathcal{M} \otimes \mathcal{O}_Y(mH)$ is relatively globally generated over $S$ for all $m \gg 0$. In particular, $f_* \mathcal{M} \otimes \mathcal{O}_Y(mH) \neq 0$ for all $m \gg 0$. Since $S$ is Stein, this implies that $H^0(\mathcal{M} \otimes \mathcal{O}_Y(mH)) \neq 0$ for all $m \gg 0$. Let $\Theta$ be an effective Cartier divisor defined by a non-zero element of $H^0(\mathcal{M} \otimes \mathcal{O}_Y(mH))$. Then $\mathcal{M} \cong \mathcal{O}_Y(\Theta - mH)$. In particular, the canonical line bundle $\omega_Y$ is given by a Cartier divisor, which we will denote by the usual notation $K_Y$.

Now write

$$K_Y + \Gamma = f^*(K_X + \Delta) + E$$

such that $\Gamma \geq 0$ and $E \geq 0$ do not share any common component and $f_*\Gamma = \Delta$ and $f_*E = 0$.

Let $A$ be a $\mathbb{Q}$-Cartier $(\pi \circ f)$-nef-big divisor on $Y$ such that $f^*D = f^*(K_X + \Delta) + A = K_Y + \Gamma - E + A$. Since $\{A\}$ has SNC support, by Theorem 2.20 we have

$$(2.1) \quad R^i f_* \mathcal{O}_Y(K_Y + [A]) = 0 \quad \text{and} \quad R^i(\pi \circ f)_* \mathcal{O}_X(K_Y + [A]) = 0 \quad \text{for all} \quad i > 0.$$
Now since \((X, \Delta)\) is a klt pair, the coefficients of \(\Gamma\) are contained in the interval \((0, 1)\), and thus
\[
K_Y + [A] = [f^*D + E - \Gamma] \geq [f^*D]
\]
so that \(f_*O_Y(K_Y + [A]) = O_X(D)\). Combining (2.1) with a standard Leray spectral sequence argument, it follows that \(R^i\pi_*O_X(D) = 0\) for all \(i > 0\).

\[\square\]

### 2.3. MMP for Kähler 3-folds

The following results are improvements of the important results from [CHP16] and [HP16].

**Theorem 2.22.** [DO24, Theorem 2.26] Let \((X, B)\) be a dlt pair, where \(X\) is a \(\mathbb{Q}\)-factorial compact Kähler 3-fold. If \(K_X + B\) is pseudo-effective, then there is a rational number \(d > 0\) and an at most countable set of curves \(\{\Gamma_i\}_{i \in I}\) such that
\[
0 < -(K_X + B) \cdot \Gamma_i \leq d
\]
and that
\[
\overline{\mathcal{N}}(X) = \overline{\mathcal{N}}(X)_{(K_X+B) \geq 0} + \sum_{i \in I} \mathbb{R}^+ [\Gamma_i].
\]

Note that if \(\omega\) is modified Kähler, then there are only finitely many \(i \in I\) such that \((K_X + B + \omega) \cdot \Gamma_i < 0\), see Remark 4.13 for a detailed discussion.

**Theorem 2.23.** Let \((X, B)\) be a dlt pair, where \(X\) is a \(\mathbb{Q}\)-factorial compact Kähler 3-fold, and \(K_X + B\) is pseudo-effective. Let \(R\) be a \((K_X + B)\)-negative extremal ray supported by a nef class \(\alpha\). Then

1. If \(R\) is small, then the contraction \(c_R : X \to Y\) of \(R\) exists and \(Y\) is a compact Kähler space.
2. If \(R\) is divisorial and \(n(\alpha) = 0\), then the contraction \(c_R : X \to Y\) of \(R\) exists and \(Y\) is a compact Kähler space with \(\mathbb{Q}\)-factorial singularities.
3. Assume that \(R\) is divisorial, \(n(\alpha) = 1\), and one of the following conditions is satisfied:
   a. \(X\) has terminal singularities and \(K_X \cdot C < 0\), where \(R = \mathbb{R}^+ \cdot [C]\) for some curve \(C \subset X\), or
   b. \(S\) has semi-log canonical singularities, where \(S\) is the unique surface covered by curves in the class \(R\).

Then the contraction \(c_R : X \to Y\) of \(R\) exists and \(Y\) is a compact Kähler space with \(\mathbb{Q}\)-factorial singularities.

**Proof.** (1) follows from [DO24, Theorem 2.28]. (2) follows from [DO24, Theorem 2.30]. (3) follows from [DO24, Proposition 2.31]. These results also follow from [DH23, Theorem 1.2].

\[\square\]
Theorem 2.24. [CHP16, Theorem 4.3] Let \((X, B)\) be a dlt pair, where \(X\) is a \(\mathbb{Q}\)-factorial compact \(\text{K"ahler}\) 3-fold. Let \(X \to Y\) be a \((K_X + B)\)-flipping contraction, then the flip \(X^+ \to Y\) exists so that \(X^+\) is a \(\mathbb{Q}\)-factorial compact \(\text{K"ahler}\) 3-fold and \((X^+, B^+)\) is dlt, where \(B^+ := \phi_* B\), and \(\phi : X \dashrightarrow X^+\) is the induced bimeromorphic map.

Theorem 2.25. [DO24, Theorem 1.12] Let \((X, B)\) be a dlt pair, where \(X\) is a \(\mathbb{Q}\)-factorial compact \(\text{K"ahler}\) 3-fold. Then any sequence of \((K_X + B)\)-flips is finite.

Proposition 2.26. Let \((X, B)\) be a \(\mathbb{Q}\)-factorial dlt pair and \(f : X \to Y\) a projective surjective morphism between two normal compact analytic varieties. If \(\dim X \leq 3\), then we can run a relative \((K_X + B)\)-MMP over \(Y\) which terminates either with a minimal model or a Mori fiber space, according to whether \(K_X + B\) is pseudo-effective over \(Y\) or not. Moreover, if \(Y\) is a \(\text{K"ahler}\) variety and \(X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_n \cdots\) are the steps of a \((K_X + B)\)-MMP over \(Y\), then every \(X_i\) is a \(\mathbb{Q}\)-factorial \(\text{K"ahler}\) variety for \(i \geq 0\); additionally, if \(\psi : X_n \to Y'\) is a Mori fiber space over \(Y\), then \(Y'\) is also \(\text{K"ahler}\).

Proof. Since \((X, B)\) is a \(\mathbb{Q}\)-factorial dlt pair, \((X, (1 - \varepsilon)B)\) is klt for any \(0 < \varepsilon \leq 1\). Recall that the (relative) Mori cone \(\overline{\text{NE}}(X/Y)\) is a strongly convex closed cone, and hence it is the convex hull of its extremal rays. In particular, if \(K_X + B\) is not nef over \(Y\), then there is a \((K_X + B)\)-negative extremal ray \(R\) of \(\overline{\text{NE}}(X/Y)\); note that at this stage we do not know whether \(R\) is generated by an irreducible curve or not. Then there is a \(0 < \delta \ll 1\) such that \((K_X + (1 - \delta)B) \cdot R < 0\). Since \((X, (1 - \delta)B)\) is klt, and the (relative) cone and contraction theorems (for projective morphisms) are known for klt pairs due to [Nak87, Theorem 4.12], it follows that \(R\) can be contracted by a projective morphism over \(Y\). Since \(\dim X \leq 3\), the existence of flips (over \(Y\)) follows from Theorem 2.24. The termination of flips (over \(Y\)) follows from Theorem 2.25. The proof of the fact that a \((K_X + B)\)-MMP over \(Y\) terminates with either a minimal model or a Mori fiber space according to whether \(K_X + B\) is pseudo-effective over \(Y\) or not, works exactly as in the algebraic case, since \(f : X \to Y\) is a projective morphism. The \(\mathbb{Q}\)-factoriality condition is preserved at each step as a formal consequence of the contraction theorem as in the algebraic case. Now if \(Y\) is \(\text{K"ahler}\), then by [Var89, Proposition 1.3.1, page 24], \(X = X_0\) is \(\text{K"ahler}\). If \(g_i : X_i \to Z_i\) is a contraction of a \((K_{X_i} + B_i)\)-negative extremal ray of \(\overline{\text{NE}}(X_i/Y)\), then by the relative base-point free theorem [Nak87, Theorem 4.10], it follows that the induced morphism \(h_i : Z_i \to Y\) is projective (note that arguing as above we may replace \((X, B)\) by \((X, (1 - \delta)B)\) and hence we may assume that \((X, B)\) is klt so that [Nak87, Theorem 4.10] applies). Then again from [Var89, Proposition 1.3.1, page 24] it follows that \(Z_i\) is \(\text{K"ahler}\). If
$g_i$ is a flipping contraction and $g_i^+ : X_{i+1} \to Z_i$ is the flip, then again $X_{i+1}$ is Kähler by the same argument. In the Mori fiber space case again by a similar argument it follows that $Y''$ is Kähler.

\[ \square \]

**Lemma 2.27.** Let $(X, B)$ be a klt pair, where $X$ is a compact Kähler 3-fold. Then the following morphisms exist:

1. a projective small bimeromorphic morphism $\nu : X' \to X$ such that $X'$ is strongly $\mathbb{Q}$-factorial, and
2. a projective bimeromorphic morphism $\nu : X' \to X$ such that $X'$ is strongly $\mathbb{Q}$-factorial and $(X', B_{X'})$ is a terminal pair such that $K_{X'} + B_{X'} = \nu^*(K_X + B)$.

**Proof.** (1) Let $\nu : X' \to X$ be a log resolution of the pair $(X, B)$. We may assume that $\nu$ is a projective morphism. Write $\nu^*(K_X + B) = K_{X'} + B' - E'$, where $B', E' \geq 0$, $\nu^*B' = B$ and $B'$ and $E'$ have no common components. Choose $\varepsilon \in \mathbb{Q}^+$ sufficiently small so that $(X', B' + \varepsilon \text{Ex}(\nu))$ is klt. Next we run a $(K_{X'} + B' + \varepsilon \text{Ex}(\nu))$-MMP over $X$ using Proposition 2.26. Replacing $X'$ by the output of this MMP, we may assume that $E' + \varepsilon \text{Ex}(\nu)$ is nef over $X$ and $X'$ is strongly $\mathbb{Q}$-factorial (see Lemmas 2.3 and 2.5). Then by the negativity lemma (see [Wan21, Lemma 1.3]), we have $E' + \varepsilon \text{Ex}(\nu) = 0$ and hence $\nu$ is small.

For a proof of (2), first replace $\nu$ by a higher log resolution if necessary so that $X'$ contains all exceptional divisors $E$ over $X$ such that the discrepancy $a(E, X, B) \leq 0$. Then we run a $(K_{X'} + B')$-MMP over $X$. Replacing $X'$ by the output of this MMP and applying the negativity lemma we obtain the required result.

\[ \square \]

**Lemma 2.28 (DLT Modification).** Let $X$ be compact Kähler 3-fold and $(X, B)$ a log canonical pair. Then there exists a projective bimeromorphic morphism $f : (X', B') \to (X, B)$ such that

1. $X'$ has strongly $\mathbb{Q}$-factorial terminal singularities,
2. $(X', B')$ is a dlt pair, and
3. $K_{X'} + B' = f^*(K_X + B)$.

**Proof.** This follows from [DO24, Corollary 1.30] and Lemma 2.5.

\[ \square \]

**Proposition 2.29.** Let $X$ be a compact Kähler 3-fold with klt singularities and $\mu : X' \to X$ a proper bimeromorphic morphism. Then every fiber of $\mu$ is rationally chain connected.

**Proof.** Let $\nu : \tilde{X} \to X$ be a resolution of singularities dominating $X'$. Then it suffices to show that every fiber of $\nu$ is rationally chain connected. Thus
replacing \( X' \) by \( \tilde{X} \) and \( \mu \) by \( \nu \) we may assume that \( X' \) is smooth and \( \mu \) is projective. The relative minimal model program holds in this context by Proposition 2.26, and thus the result now follows from [HM07, Theorem 1.2] (with \( S = X \)).

2.4. Technical lemmas. In this subsection we will prove some technical results which will be used in the rest of the article. Note that some of the results here are obvious for projective varieties but not necessarily so for analytic varieties. We will use these results throughout the article without further reference.

Remark 2.30. Recall that if \( f : X \to Y \) is a projective morphism between two analytic varieties and \( C \subset Y \) is a (compact) irreducible curve, then there is a compact irreducible curve \( \Gamma \subset X \) such that \( f(\Gamma) = C \). Moreover, if \( f : X \to Y \) is a proper bimeromorphic morphism between analytic varieties, then from Chow’s lemma (Theorem 2.17) it follows that for every \( y \in f(Ex(f)) \), \( f^{-1}(y) \) is covered by (compact) curves.

Lemma 2.31. If \((X, B)\) is dlt, then \( X \) has rational singularities.

Proof. See [Fuj22, Theorem 3.12].

Lemma 2.32. Let \( f : X' \to X \) be a proper bimeromorphic morphism between strongly \( \mathbb{Q} \)-factorial normal compact Kähler 3-folds with klt singularities. Then the \( f \)-exceptional divisors give a basis for \( N^1(X'/X) := N^1(X')/f^*N^1(X) \).

Proof. Let \( \nu : X'' \to X' \) be a resolution of singularities of \( X' \) such that \( f \circ \nu \) is a projective morphism (see Theorem 2.17). We will show that \( N^1(X''/X) := N^1(X'')/(f \circ \nu)^*N^1(X) \) is generated by the \((f \circ \nu)\)-exceptional divisors. To this end let \( f' = f \circ \nu \), then by Lemma 2.4, there is a reduced \( f' \)-exceptional divisor \( E \) so that \( \text{Supp} E = \text{Ex}(f') \). We run the \((K_{X''} + E)\)-MMP over \( X \) using Proposition 2.26. Note that since \( X \) has klt singularities, this MMP contracts all \( f' \)-exceptional divisors. Let \( X'' = X_0 \to X_1 \to \ldots \to X_n \) be the corresponding MMP over \( X \). Since \( X' \) is strongly \( \mathbb{Q} \)-factorial, so is \( X_n \) by Lemma 2.5, and then it follows from Lemma 2.4 that \( X_n = X \). By [CHP16, Proposition 3.1] it follows that if \( X_i \to X_{i+1} \) is a flip, then \( N^1(X_i) \cong N^1(X_{i+1}) \) and if \( X_i \to X_{i+1} \) is a divisorial contraction, then \( N^1(X_i)/N^1(X_{i+1}) \) is one dimensional and generated by the corresponding exceptional divisor. Thus \( N^1(X''/X) \) has a basis given by the \( f' \)-exceptional divisors. Similarly one sees that \( N^1(X''/X') \) has a basis given by the \( \nu \)-exceptional divisors. The lemma now follows easily.
Lemma 2.33. Let $S$ be a smooth projective surface such that $H^2(S, \mathcal{O}_S) = 0$. Let $\alpha$ be a nef $(1,1)$-class such that $\alpha^2 = 0$ and $K_S \cdot \alpha < 0$. Then $S$ is covered by $\alpha$-trivial curves.

Proof. Since $H^2(S, \mathcal{O}_S) = 0$, from the exponential sequence and [HP16, Eqn. (2) and (3), page 223], it follows that $\text{NS}(S)_{\mathbb{R}} = H^{1,1}_{\text{BC}}(S)$. In particular, $\alpha$ is a nef $\mathbb{R}$-divisor. Recall that on a projective surface the cone $\overline{N}\text{F}(S)$ of numerical classes of nef curves is equal to the cone of numerical classes of movable curves $\overline{N}\text{M}(S)$. Fix an ample $\mathbb{Q}$-divisor $A$. By the structure theorem for the cone of movable curves $\overline{N}\text{M}(S)$, see [Das20, Theorem 1.9] (also see [Ara10, Theorem 1.3] and [Leh12, Theorem 1.3]), for every $\epsilon > 0$ we have a decomposition $\alpha = C_\epsilon + \sum_{i=1}^k \lambda_i \epsilon M_{i, \epsilon}$, where $C_\epsilon$ is a pseudo-effective 1-cycle such that $(K_S + \epsilon A) \cdot C_\epsilon \geq 0$, $\lambda_i \epsilon \geq 0$, the $M_{i, \epsilon}$ are movable curves for all $1 \leq i \leq k$. Since $\alpha$ is nef, then $\alpha \cdot C_\epsilon \geq 0$ and $\alpha \cdot M_{i, \epsilon} \geq 0$. Since $\alpha^2 = 0$, then $\alpha \cdot M_{i, \epsilon} = \alpha \cdot C_\epsilon = 0$ for all $1 \leq i \leq k$. If $k > 0$, then as $M_{i, \epsilon}$ is movable the lemma is proved. Otherwise, $k = 0$ and $\alpha \equiv C_\epsilon$, so that $(K_S + \epsilon A) \cdot \alpha \geq 0$. Taking the limit as $\epsilon$ goes to 0, it follows that $K_S \cdot \alpha \geq 0$ contradicting our assumptions. □

Definition 2.34. [Bou04, Def. 2.2] Let $X$ be a normal compact Kähler variety.

(i) A closed positive bi-degree $(1,1)$-current $T$ with local potentials is called a Kähler current, if $T - \omega$ is a positive current for some Kähler form $\omega$ on $X$.

(ii) A class $\alpha \in H^{1,1}_{\text{BC}}(X)$ is called big if it contains a Kähler current $T$.

(iii) A big class $\alpha \in H^{1,1}_{\text{BC}}(X)$ is called a modified Kähler class if it contains a Kähler current $T$ such that the Lelong number $\nu(T, D) = 0$ for all prime Weil divisors $D$ on $X$.

The following lemma is a singular version of [Bou04, Proposition 2.3].

Lemma 2.35. Let $X$ be a normal compact Kähler variety and $\alpha \in H^{1,1}_{\text{BC}}(X)$. Then $\alpha$ is a modified Kähler class if and only if there is a projective birational morphism $\mu : X' \to X$ from a Kähler manifold $X'$ and a Kähler class $\alpha'$ such that $\mu^* \alpha = \alpha' + E$, where $E \geq 0$ is an effective and $\mu$-exceptional $\mathbb{R}$-divisor. In particular, here $-E$ is $\mu$-ample and $\text{Supp}(E) = \text{Ex}(f)$.

Proof. The if part is obvious, e.g., see the first part of the proof of [Bou04, Proposition 2.3]. We will now prove the converse statement, to that end, let $f : X' \to X$ be a resolution of singularities of $X$. Since $\alpha$ is a modified Kähler class, there is a Kähler current $T$ with (psh) local potentials in the class $\alpha$ such that $\nu(T, P) = 0$ for all prime Weil divisors $P$ on $X$. Since $T$ has psh local potentials, $f^* T$ can be defined by simply pulling back the local potentials. Since $T$ is a Kähler current and the exceptional locus of $f$ supports an effective
relative anti-ample divisor $F$, it follows that $f^*T - \varepsilon F$ is also a Kähler current in the class $f^*\alpha - \varepsilon F$ for $0 < \varepsilon \ll 1$. Choose a Kähler form $\omega'$ on $X'$ such that $f^*T - \varepsilon F \geq \omega'$ and let $T' := f^*T - \varepsilon F - \omega' \geq 0$. Let $T' = D + S'$ be the Siu decomposition of $T'$, where $D = \sum Q \nu(T', Q)Q$ and $S'$ is a closed positive bi-degree $(1, 1)$-current such that $\nu(S', Q) = 0$ for all prime Weil divisor $Q$ on $X'$. We claim that $D$ is an $\mathbb{R}$-divisor, i.e. $\nu(T', Q) = 0$ for all but finitely many prime Weil divisors $Q$. To see this, observe that

\[
\nu(f^*T, Q) = \nu(T' + \varepsilon F + \omega', Q) = \nu(T' + \varepsilon F, Q) + \nu(\omega', Q) = \nu(T' + \varepsilon F, Q) \geq \nu(T', Q),
\]

since $\nu(\omega', Q) = 0$ as $\omega'$ is a smooth form. Now from the definition of $T$ it follows that $\nu(f^*T, Q) = 0$ if $Q$ is not $f$-exceptional, and hence $D$ is an effective $f$-exceptional divisor. Thus we have $f^*T = D + \varepsilon F + U'$, where $U' = S' + \omega'$. Since $\nu(U', Q) = \nu(S' + \omega', Q) = 0$ for all prime Weil divisors $Q$ on $X'$, it follows that the $(1, 1)$-class $[U']$ is modified Kähler. By Demailly’s regularization theorem, there is a Kähler current $U_k$ with analytic singularities in the class $[U']$ such that $\nu(U_k, Q) \leq \nu(U', Q)$ for all prime Weil divisors $Q$ on $X'$. In particular, $\nu(U_k, Q) = 0$ for all prime Weil divisors $Q$ on $X'$. Let $g : X'' \to X'$ be a resolution of singularities of $U_k$ so that $g^*U_k = \Theta + G$, where $\Theta$ is a smooth form and $G$ is an effective $g$-exceptional $\mathbb{R}$-divisor. Note that since $U_k \geq \varepsilon \omega'$ for some $\varepsilon > 0$, we have $\Theta \geq \varepsilon g^*\omega'$. Then by [Bou02a, Lemma 2.9], there is an effective $g$-exceptional $\mathbb{R}$-divisor $E$ on $X''$ such that $\Theta - E$ is cohomologous to a Kähler form, i.e. $[\Theta - E] = [\omega'']$ for some Kähler form $\omega''$ on $X''$. In particular, $[g^*U_k] = [\omega''] + [E + G]$ and thus we have

\[
(f \circ g)^*\alpha = g^*([D + \varepsilon F] + [U_k]) = [\omega''] + [g^*(D + \varepsilon F) + E + G],
\]

where $g^*(D + \varepsilon F) + E + G$ is a $f \circ g$-exceptional effective $\mathbb{R}$-divisor. This completes our proof.

\[\square\]

**Corollary 2.36.** [Bou04, Proposition 2.4] Let $X$ be a $\mathbb{Q}$-factorial compact Kähler $3$-fold with klt singularities, and $\alpha \in H^{1, 1}_{BC}(X)$ a modified Kähler class. If $S \subset X$ is an irreducible surface and $\tilde{S} \to S$ its normalization, then $\alpha|_{\tilde{S}}$ is big.

**Proof.** By Lemma 2.35 there is a projective bimeromorphic morphism $\mu : X' \to X$ from a Kähler manifold $X'$, a Kähler class $\alpha'$ and an effective $\mu$-exceptional divisor $E \geq 0$ such that $\mu^*\alpha = \alpha' + E$. Let $S' = \mu^{-1}_*S$ and $\nu : S' \to S$. We may assume that $S'$ is smooth and hence $\nu$ factors through $\tilde{\nu} : S' \to \tilde{S}$. Since $S'$ is not contained in the support of $E$, then

\[
\tilde{\nu}^*(\alpha|_{\tilde{S}}) = (\mu^*\alpha)|_{S'} = (\alpha' + [E])|_{S'}
\]

is big and so $\alpha|_{\tilde{S}}$ is also big.

\[\square\]
Lemma 2.37. Let $\phi : X \to X'$ be either a divisorial contraction or a flip between two compact Kähler 3-folds with $\mathbb{Q}$-factorial klt singularities. If $\omega$ is a modified Kähler class on $X$, then $\omega' := \phi_\ast \omega$ is a modified Kähler class on $X'$.

Proof. This follows easily from Lemma 2.35.

Lemma 2.38. Let $(X, B)$ be a klt pair, where $X$ is a $\mathbb{Q}$-factorial compact Kähler 3-fold. Let $\alpha$ be a nef and big $(1,1)$-class on $X$. Then there exist a modified Kähler class $\Theta$ and an effective $\mathbb{Q}$-divisor $F \geq 0$ such that $\alpha = \Theta + F$ and $(X, B + F)$ is klt.

Proof. Let $f : X' \to X$ be a resolution, then $f^\ast \alpha$ is big. By [Bou02b, Theorem 1.4], passing to a higher resolution, we may assume that $f^\ast \alpha = \omega' + E$, where $\omega'$ is a Kähler form and $E \geq 0$ is an effective $\mathbb{R}$-divisor. Since being Kähler is an open condition, we may assume that $E \geq 0$ is an effective $\mathbb{Q}$-divisor.

We may also assume that $f$ is a projective. Then we can rewrite $\alpha$ as

$$\alpha = f_\ast (\varepsilon \omega' + (1 - \varepsilon) f^\ast \alpha) + \varepsilon f_\ast E \quad \text{for } 0 < \varepsilon < 1.$$ 

We may assume that $\varepsilon \in \mathbb{Q}$. Note that $\varepsilon \omega' + (1 - \varepsilon) f^\ast \alpha$ is a Kähler class on $X'$, since $\alpha$ is nef, and therefore $\Theta' := f_\ast (\varepsilon \omega' + (1 - \varepsilon) f^\ast \alpha)$ is a modified Kähler class on $X$. Observe that $(X, B + \varepsilon f_\ast E)$ is klt for sufficiently small $\varepsilon \in \mathbb{Q}^+$. □

Lemma 2.39 (Hodge Index Theorem). [BHPVdV04, Corollary IV.2.15 and Theorem IV.3.1] Let $S$ be a smooth compact Kähler surface and $\omega \in H^{1,1}(S)$ a $(1,1)$-class such that $\omega^2 > 0$. Let $\alpha \in H^{1,1}(S)$ be a $(1,1)$-class on $S$. If $\omega \cdot \alpha = 0$, then $\alpha^2 \leq 0$; moreover, if $\alpha \neq 0$, then $\alpha^2 < 0$.

Lemma 2.40. Let $f : S \to T$ be a proper morphism with connected fibers from a normal compact analytic surface to a smooth projective curve $T$. Let $\alpha \in H^{1,1}_{BC}(S)$ be a nef class and $C \subset S$ a curve such that $f(C) = T$ and $\alpha \cdot C = \alpha \cdot F = 0$, where $F$ is a general fiber of $f$. Then $\alpha \equiv 0$, i.e. $\alpha \cdot \Gamma = 0$ for all curves $\Gamma \subset S$; in particular, the nef dimension $\nu_{\text{nef}}(\alpha) = 0$.

Proof. This follows from the same arguments as in the proof of [BCE+02, Proposition 2.5] using Lemma 2.39 above.

Lemma 2.41. Let $X$ be a normal $\mathbb{Q}$-factorial Kähler 3-fold and $S$ a prime divisor with minimal resolution $\nu : S' \to S$. Then there is an effective $\mathbb{Q}$-divisor $E$ on $S'$ such that

$$\nu^\ast ((K_X + S)|_S) = K_{S'} + E.$$ 

Proof. See [CHP16, §2B]. □

Lemma 2.42. Let $X$ be a uniruled normal compact Kähler 3-fold with klt singularities such that the base of its MRC fibration has dimension less than 2, then $X$ is a projective variety and $H^2(X, \mathcal{O}_X) = 0$. □
Proof. Let $\pi : X \rightarrow Z$ be the MRC fibration (see [CH20, Remark 6.10]). By assumption $\dim Z = 0$ or $1$. First note that, since $X$ has rational singularities, by Theorem 2.13, $X$ is projective if and only if any resolution $\tilde{X}$ of the singularities of $X$ is projective. Note also that by Proposition 2.29, the fibers of $\nu : \tilde{X} \rightarrow X$ are rationally chain connected, thus it follows that $\tilde{X} \rightarrow Z$ is also a MRC fibration. Thus replacing $X$ by a resolution of singularities we may assume that $X$ is a compact Kähler manifold. Since the general fibers of $\pi$ are rationally connected, by [Deb01, Corollary 4.18] $H^0(F, \Omega^i_F) = 0$ for all $i \geq 1$ where $F$ is a general fiber. We claim that $H^0(X, \Omega^2_X) = 0$. If $\dim Z = 0$, then this is clear, so assume that $\dim Z = 1$. Then observe that the following exact sequence

$$
\pi^* \Omega_Z \rightarrow \Omega_X \rightarrow \Omega_{X/Z} \rightarrow 0
$$

is left exact over a Zariski open dense subset of $Z$. This follows from the generic smoothness of $\pi$ and the fact that the MRC fibration is an almost holomorphic map. Restricting this sequence to a general fiber $F$ of $\pi$ we get the following short exact sequence

$$
(2.2) \quad 0 \rightarrow \mathcal{O}_F \rightarrow \Omega_X|_F \rightarrow \Omega_F \rightarrow 0.
$$

Thus we have a short exact sequence

$$
(2.3) \quad 0 \rightarrow \Omega_F \rightarrow \Omega^2_X|_F \rightarrow \Omega^2_F \rightarrow 0.
$$

Since, as observed above, $H^0(F, \Omega^i_F) = 0$ for $i = 1, 2$, we have $H^0(F, \Omega^2_X|_F) = 0$. In particular, $s|_F = 0$ for any section $s \in H^0(X, \Omega^2_X)$. Since $\Omega^2_X$ is torsion free, it follows that $H^0(X, \Omega^2_X) = 0$. Then $H^2(X, \mathcal{O}_X) = H^0(X, \Omega^2_X) = 0$ and by Kodaira’s projectivity criterion (Theorem 2.15), we have that $X$ is projective. □

Lemma 2.43. Let $S$ be a smooth compact Kähler surface. If $K_S$ is not pseudoeffective, then $S$ is projective.

Proof. Since $K_S$ is not pseudoeffective and $S$ is Kähler, then $H^2(S, \mathcal{O}_S) = H^0(S, K_S)^* = 0$. The claim then follows from Kodaira’s criterion (Theorem 2.15). □

Lemma 2.44. Let $(X, B \geq 0)$ be a log pair, where $X$ is a normal compact analytic variety with $\mathbb{Q}$-Gorenstein singularities, i.e. $(\omega_X^{\otimes m})^{**}$ is a line bundle for some $m > 0$. Let $f : X \rightarrow Y$ be a proper surjective morphism to a normal compact analytic variety $Y$. Assume that one of the following conditions is satisfied:

(i) $(X, B)$ is dlt and $-(K_X + B)$ is $f$-ample, or
(ii) $(X, B)$ is klt and $-(K_X + B)$ is $f$-nef-big.
Then \( Y \) has rational singularities.

**Proof.** Since \( X \) is \( \mathbb{Q} \)-Gorenstein, \( B \) is a \( \mathbb{Q} \)-Cartier divisor and \( X \) has klt singularities. Thus by Lemma 2.31, \( X \) has rational singularities. In the dlt case, perturbing the coefficients of \( B \) we may assume that \((X, B)\) is klt and \(-(K_X + B)\) is still \( f \)-ample. Therefore by Theorem 2.21, in both of the cases above we have \( R^i f_* \mathcal{O}_X = 0 \) for all \( i > 0 \). Then by [Kov00, Theorem 1] \( Y \) has rational singularities. Note that the proof of [Kov00, Theorem 1] uses Grothendieck duality and the Grauert-Riemenschneider vanishing theorem, both of which are known in the analytic category due to [RR74, RRV71] and [Tak85, Corollary II], respectively.

\[ \square \]

3. **Minimal model program for dlt pseudo-effective pairs**

Throughout this section \((X, B)\) will be a dlt pair with \( X \) a strongly \( \mathbb{Q} \)-factorial compact Kähler 3-fold and \( \alpha = K_X + B + \omega \) a nef and big (1,1)-class, such that \( \omega \) is a Kähler class, the corresponding extremal ray \( R := \text{Null}(\alpha) \cap \overline{\text{NA}}(X) \) is of divisorial type and \( n(\alpha) = 1 \). Since \( \alpha - \epsilon \omega \) is also big for any \( 0 < \epsilon \ll 1 \), by [Bou04, Thm. 3.12 and Prop. 3.8] there exist positive real numbers \( \lambda_j > 0 \) and irreducible surfaces \( S_j \subset X \) such that

\[
K_X + B + (1 - \epsilon)\omega \equiv \sum \lambda_j S_j + N(K_X + B + (1 - \epsilon)\omega),
\]

where \( N = N(K_X + B + (1 - \epsilon)\omega) \) is a pseudo-effective class which is nef in codimension 1, i.e. \( N|_{S'} \) is pseudo-effective for any surface \( S' \subset X \). If \([C] \in R\) and \( C \) belongs to a positive dimensional family of curves, since \((K_X + B + (1 - \epsilon)\omega) \cdot C = -\epsilon \omega \cdot C < 0\), then \( S_j \cdot R < 0 \) for some \( j \). But then all curves \([C] \in R\) must be contained in \( S_j \) and \( S_j \) is covered by such curves. It follows that \( R \cdot S_k \geq 0 \) for all \( k \neq j \). Thus \( S_j \) is the unique such surface.

We will next prove Theorem 1.5 when \( X \) is strongly \( \mathbb{Q} \)-factorial and \( K_X + B \) is pseudo-effective. Since this is the most delicate proof of the paper, we give a brief explanation of the main ideas behind this proof. Let \((X, B)\) be a klt pair, \( R \) a \((K_X + B)\)-negative extremal ray of divisorial type cut out by a nef class \( \alpha \). The strategy is to construct the divisorial contraction \( X \to Z \) by running a minimal model program for a dlt pair \((X', \Delta')\) where every step is negative for some component of \([\Delta']\) and thus the corresponding flips and contractions exist by Theorems 2.23 and 2.24. The dlt pair \((X', \Delta')\) is constructed as follows. Let \( S \subset X \) be the surface covered by the curves \( C \subset X \) such that \([C] \in R\). Let \( b = \text{mult}_S B \). The pair \((X, \Delta := B + (1 - b)S)\) may not be dlt, so we take the corresponding dlt model \( \mu : (X', \Delta') \to (X, \Delta) \). Since \((X, B)\) is klt, \( K_{X'} + \Delta' - \mu^*(K_X + \Delta) \) is an effective divisor whose support coincides with the exceptional locus of \( \mu \). If the divisorial contraction \( f : X \to Z \) exists,
then by standard arguments, \( \alpha = f^* \alpha_Z \) for some Kähler class \( \alpha_Z \) on \( Z \) and \( Z \) is the output of the \((K_X + \Delta')\)-minimal model program over \( Z \) or equivalently of the \((K_X + \Delta' + t\mu^* \alpha)\)-minimal model program for any \( t \gg 0 \). Since we do not yet know that \( f : X \to Z \) exists, we mimic this strategy and we run the \( \mu^* \alpha \)-trivial \((K_X + \Delta')\)-minimal model program (each step of this minimal model program is a step of the \((K_X + \Delta' + t\mu^* \alpha)\)-minimal model program for \( t \gg 0 \)). We then show that the output of this minimal model program is indeed the required divisorial contraction.

**Theorem 3.1.** Theorem 1.5 holds when \( X \) is strongly \( \mathbb{Q} \)-factorial and \( K_X + B \) is pseudo-effective.

**Proof.** Recall that \( \alpha \) is a nef \((1,1)\)-class such that \( \alpha \cdot C = 0 \) if and only if \( [C] \in R \). Let \( S \) be the unique surface which is covered by and contains all the curves \( C \subset X \) such that \( [C] \in R \). Let \( \nu : S' \to S \) be the normalization morphism and \( \tilde{f} : S' \to T \) the nef reduction of \( \alpha|_{S'} := \nu^*(\alpha|_S) \) (see [HP15, Theorem 3.19]). Since \( n(\alpha) = 1 \), \( T \) is a smooth projective curve. Moreover, we have \( F \cdot \nu^*(\alpha|_S) = 0 \) for any fiber \( F \) of \( \tilde{f} \). We also have that \( \alpha \cdot C > 0 \) if \( C \subset X \) is not contained in \( S \) or if \( C \) is contained in \( S \) but dominates \( T \) (cf. [BCE+, Proposition 2.5]). By assumption \( \omega = \alpha - (K_X + B) \) is a Kähler class. Replacing \( B \) by \((1-\varepsilon)B\) and \( \omega \) by \( \omega + \varepsilon B \) for sufficiently small \( \varepsilon \in \mathbb{Q}^+ \), we may assume that \((X,B)\) is klt.

Let \( b = \text{mult}_S B \). For two divisors \( D \) and \( D' \) we say \( D \equiv_\alpha D' \) if and only if \( (D-D') \cdot C = 0 \) for any curve \( C \subset X \) such that \( \alpha \cdot C = 0 \). Since, as observed above, \( S \cdot R < 0 \) and \( \alpha \) supports the extremal ray \( R \), we have \( K_X + B \equiv_\alpha \alpha S \) for some \( \alpha > 0 \).

Let \( \mu : X' \to X \) be a log resolution of \((X,B)\) and set \( \Delta' = \mu_*^{-1}(B + (1-b)S) + \text{Ex}(\mu) \) and \( S' = \mu_*^{-1}S \). Since \( \mu \) is a projective morphism, after running a \((K_{X'} + \Delta')\)-MMP over \( X \) via Proposition 2.26, we may assume that \( K_{X'} + \Delta' \) is nef over \( X \). By Lemma 2.5, \( X' \) is strongly \( \mathbb{Q} \)-factorial, and thus by Lemma 2.32, if \( \eta \) is a Kähler class on \( X' \), then \( \eta \equiv_{X'} -F \), where \( F \) is an effective \( \mu \)-exceptional divisor. In particular the support of the exceptional locus equals \( \text{Supp}(F) \). If \( U = X \setminus S \) and \( U' = \mu^{-1}(U) \), then \( K_{U'} + \Delta'|_{U'} \equiv_U \sum a_j E_j|_{U'} \), where the left hand side is \( \mu|_{U'} \)-nef and the right hand side is an effective \( \mu|_{U'} \)-exceptional divisor whose support equals \( \text{Ex}(\mu|_{U'}) \). By the negativity lemma, \( \sum c_j E_j|_{U'} = 0 \) and hence \( \text{Ex}(\mu|_{U'}) = \emptyset \). Thus \( \mu \) is an isomorphism over the complement of \( S \). We then have

\[
(3.2) \quad K_{X'} + \Delta' = \mu^*(K_X + B) + \sum c_j E_j + (1-b)S' \equiv_{\alpha'} \sum d_j E_j + dS',
\]

where \( \alpha' := \mu^* \alpha, d_j \geq c_j > 0 \) and \( d > 1-b > 0 \). We will now run the \( \alpha' \)-trivial \((K_{X'} + \Delta')\)-MMP.
Claim 3.2. There exists a sequence of $\alpha'$-trivial $(K_{X'} + \Delta')$-flips and divisorial contractions

$$
\phi_n : X' = X'_0 \longrightarrow X'_1 \longrightarrow X'_2 \longrightarrow \cdots \longrightarrow X'_n
$$

such that there are no $\alpha'_n$-trivial, $(K_{X'_n} + \Delta'_n)$-negative curves on $X'_n$. Moreover, we have $K_{X'_n} + \Delta'_n \equiv_{\alpha'_n} \phi_{n,*}(\sum d_j E_j + dS')$, where $\alpha'_n = \phi_{n,*}\alpha'$ is a nef $(1, 1)$-class.

Proof of Claim 3.2. We will show that this MMP exists and each step preserves the above relation $K_{X'_i} + \Delta'_i \equiv_{\alpha'_i} \phi_{i,*}(\sum d_j E_j + dS')$, where $\alpha'_i = \phi_{i,*}\alpha'$ is an $(1, 1)$-class and $\phi_i : X'_i \longrightarrow X'_i$ is the induced bimeromorphic map. We proceed by induction. Let $R_i$ be an $\alpha'_i$-trivial, $(K_{X'_i} + \Delta'_i)$-negative extremal ray. From the above relation, it follows that $\phi_{i,*}(\sum d_j E_j + dS') \cdot R_i < 0$ and hence the contracted locus is contained in the support of $[\Delta'_i]$. Note that $R_i$ intersects some component $P$ of $[\Delta'_i]$ negatively and hence each contracted curve is contained in this component $P$. Since $(X', \Delta')$ is dlt, so is $(X'_i, \Delta'_i)$ and hence $(P, \Delta_P)$ is dlt where $K_P + \Delta_P = (K_{X'_i} + \Delta'_i)|_P$. In particular $P$ has semi-log canonical singularities. By Theorem 2.23 and Theorem 2.24, we may flip/contract $R_i$ via $X'_i \longrightarrow X'_{i+1}$. Let $g : X'_i \rightarrow Z$ be the contraction of $R_i$ ($Z = X'_{i+1}$ if $R_i$ is divisorial). By [CHP16, Proposition 3.1(5)], $\alpha'_i = g^*\alpha_Z$, where $\alpha_Z$ is a nef $(1, 1)$-class on $Z$ and $g_*(K_{X'_i} + \Delta'_i - \phi_{i,*}(\sum d_j E_j + dS'))$ is a $Q$-Cartier divisor such that

$$
g_*(K_{X'_i} + \Delta'_i - \phi_{i,*}(\sum d_j E_j + dS')) \equiv_{\alpha_Z} 0.
$$

Pulling back to $X'_{i+1}$ we have $K_{X'_{i+1}} + \Delta'_{i+1} \equiv_{\alpha_{i+1}} \phi_{i+1,*}(\sum d_j E_j + dS')$. By Lemma 2.5, $X'_{i+1}$ is strongly $Q$-factorial, and by Theorem 2.25 after finitely many steps we may assume that there are no $\alpha'_n$-trivial, $(K_{X'_n} + \Delta'_n)$-negative extremal rays. By the cone theorem (see Theorem 2.22) it follows that $(K_{X'_n} + \Delta'_n) \cdot C \geq 0$ for any $\alpha'_n$-trivial curve $C$. \(\square\)

Recall that $\mu$ is an isomorphism over the complement of $S$, and hence the support of $\sum E_j$ is equal to the support of $\mu^{-1}(S)$.

Claim 3.3. $\phi_n$ contracts $S'$ and every $E_j$ such that $\nu_{\text{nef}}(\mu^*\alpha|_{E_j}) = 1$.

Proof of Claim 3.3. Let $K_{X'} + B' = \mu^*(K_X + B)$ and $F$ an effective exceptional $Q$-divisor such that $-F$ is $\mu$-ample; note that $B'$ is not necessarily effective here. Replacing $B'$ by $B' + \epsilon F$ and letting $\omega' = \mu^*\omega - \epsilon F$ for some $0 < \epsilon \ll 1$, we may write $\alpha' = \mu^*\alpha = K_{X'} + B' + \omega'$, where $\omega'$ is a Kähler class on $X'$. Then $K_{X'} + \Delta' = K_{X'} + B' + \mathcal{E}'$, where $\mathcal{E}' \geq 0$ is a $Q$-divisor such that $\text{Supp}(\mathcal{E}') = S' + \sum E_j$. Thus on $X'_n$ we have

$$
\alpha'_n + \mathcal{E}'_n = K_{X'_n} + \Delta'_n + \omega'_n,
$$

where $\omega'_n \in H^{1,1}_{BC}(X'_n)$ is a modified Kähler class (see Lemma 2.37).
Claim 3.4. Let $\mathcal{F} := \mu^* S + \epsilon F$. For any $0 < \epsilon \ll 1$ and $t \gg 0$, $(-\mathcal{F} + t\alpha')|_{S'}$ and $(-\mathcal{F} + t\alpha')|_{E_j}$ are Kähler for all $j$.

Proof of Claim 3.4. First, note that $-aS|_{S'} \equiv \alpha -(K_X + B)|_{S'} \equiv \omega|_{S'}$ is ample over $T$. Since $-F$ is $\mu$-ample, then $(-\mu^* S - \epsilon F)|_{S'}$ is ample over $T$ for any $0 < \epsilon \ll 1$. Since $\alpha|_{S'} \equiv f^* \alpha_T$ where $\alpha_T$ is a Kähler class on $T$, it follows that $(-\mu^* S - \epsilon F + t\alpha')|_{S'} = (-\mathcal{F} + t\alpha')|_{S'}$ is Kähler for all $t \gg 0$. Next, we consider an exceptional divisor $E_j$. If $\dim \mu(E_j) = 0$, then $\alpha'|_{E_j} = 0 = \mu^* S|_{E_j}$ and as $-F$ is $\mu$-ample the claim follows. If $\dim \mu(E_j) = 1$, let $E_j \to V_j$ be the Stein factorization of $\mu|_{E_j}$, then $V_j$ is a smooth curve. If $\alpha \cdot \mu(E_j) > 0$, then $\alpha|_{V_j}$ is ample (since $V_j$ is a curve, $H^2(V_j, \mathcal{O}_{V_j}) = 0$ and thus every $(1,1)$-class on $V_j$ is represented by an $\mathbb{R}$-divisor) and it follows that $(-S + t\alpha)|_{V_j}$ is ample for any $t \gg 0$. Since $-F$ is $\mu$-ample, then $(\mu^* (-S + t\alpha) - \epsilon F)|_{E_j} = (-\mathcal{F} + t\alpha')|_{E_j}$ is ample, hence Kähler. Finally, if $\alpha \cdot \mu(E_j) = 0$, then $[\mu(E_j)] \in R$ and so $-S \cdot \mu(E_j) > 0$. Thus $-S|_{V_j}$ is ample, and since $-F$ is $\mu$-ample, then $-(\mu^* S + \epsilon F)|_{E_j}$ is ample for $0 < \epsilon \ll 1$. Since $\alpha'|_{E_j} \equiv 0$ in this case, $(-\mu^* S + \epsilon F + t\alpha')|_{E_j} = (-\mathcal{F} + t\alpha')|_{E_j}$ is ample (and hence Kähler) for any $t > 0$.

We will now show that $\phi_n$ is $\mathcal{F}$-non-positive. This means that if $p : W \to X'$ and $q : W \to X'_n$ is the normalization of the graph of $\phi_n : X' \dashrightarrow X'_n$, then $p^* \mathcal{F} \geq q^* \mathcal{F}_n$ where $\mathcal{F}_n := \phi_{n,*} \mathcal{F}$. By the negativity lemma, it suffices to show that $q^* \mathcal{F}_n - p^* \mathcal{F}$ is $q$-nef. Suppose that $C \subset W$ is a $q$-exceptional curve, then $C$ is not $p$-exceptional and in particular $\alpha' \cdot p_* C = p^* \alpha' \cdot C = \alpha_n \cdot q_* C = 0$. It follows that $p_* C$ is contained in $S'$ or in $\sum E_j$. Since $(-\mathcal{F} + t\alpha')|_{S'}$ and $(-\mathcal{F} + t\alpha')|_{E_j}$ are Kähler,

$$(q^* \mathcal{F}_n - p^* \mathcal{F}) \cdot C = -p^* \mathcal{F} \cdot C = -\mathcal{F} \cdot p_* C = (-\mathcal{F} + t\alpha') \cdot p_* C > 0$$

as required. Thus $\phi_n$ is $\mathcal{F}$-non-positive.

Abusing notation we let $E_0 = S'$. Assume by contradiction that $\phi_{n,*}(E_j) \neq 0$, where $E_j$ is a component of $\sum_{j \geq 0} E_j$ with $\nu_{\text{nef}}(\alpha'|_{E_j}) = 1$. Let $\lambda$ be the smallest positive rational number such that $\text{mult}_{E_{j,n}}(\phi_{n,*} (\mathcal{E}' - \lambda F)) \leq 0$, where $E_{j,n} := \phi_{n,*} E_j$, for every component $E_j$ of $\sum_{j \geq 0} E_j$ with $\nu_{\text{nef}}(\alpha'|_{E_j}) = 1$. Thus there is a component $E_{k,n}$ of $\phi_{n,*}(\sum E_j + S')$ with $\nu_{\text{nef}}(\alpha'|_{E_k}) = 1$ such that $\text{mult}_{E_{k,n}}(\phi_{n,*} (\mathcal{E}' - \lambda F)) = 0$. Since $\nu_{\text{nef}}(\alpha'|_{E_k}) = 1$, the nef reduction map $E_k \to W_k$ is a surjective morphism to a smooth curve $W_k$. Let $C_{k,w}$ be the fiber of the nef reduction map over a point $w \in W_k$. We claim that the induced dominant meromorphic map $E_{k,n} \dashrightarrow W_k$ is a morphism with connected fibers, $\alpha'_{n,}\Gamma = 0$ for all fibers $\Gamma$ of this morphism and $\alpha'_{n,}|_{E_{k,n}} \neq 0$, i.e. $\alpha'|_{E_{k,n}} \cdot C_0 \neq 0$ for some curve $C_0 \subset E_{k,n}$. Moreover, we also claim that if $w \in W_k$ is general, then $\phi_n$ is an isomorphism on a neighborhood of $C_{k,w}$. Proceeding by induction, suppose that the claim holds for $\phi_i$ and consider a
contracted curve $\Sigma \in X'_i$ (in the step $X'_i \rightarrow X'_{i+1}$) which is contained in $E_{k,i}$. Note that $E_{k,i} \subset X'_i$ is a normal surface, since it is a dlt center of the dlt pair $(X'_i, \Delta'_i)$. Since $\alpha'|_{E_{k,i}} \cdot \Sigma = \alpha'_i \cdot \Sigma = 0$ and $\alpha'_i|_{E_{k,i}} \neq 0$, from Lemma 2.40 it follows that $\Sigma$ is contained in a fiber of $E_{k,i} \rightarrow W_k$. Since $E_{k,i}$ is not contracted by $\psi_i : X'_i \rightarrow X'_{i+1}$, $\Sigma$ is not contained in the general fibers of $h_i : E_{k,i} \rightarrow W_k$, consequently, $\psi_i$, and hence $\phi_{i+1} : X'_i := X' \rightarrow X'_{i+1}$ is an isomorphism near the general fibers of $h_i$. In particular, the induced dominant meromorphic map $h_{i+1} : E_{k,i+1} \rightarrow W_k$ is almost holomorphic, i.e. over a dense Zariski open subset of $W_k$, the fibers of $h_{i+1}$ are all compact. This implies that $h_{i+1}$ is a morphism (as $W_k$ is a curve and $E_{k,i+1}$ is normal). The rest of the claim now follows immediately.

We will now identify $C_{k,w}$ with its image in $E_{k,n}$ for a general $w \in W_k$. By the definition of $\lambda$ and $k$ we have

$$
(\lambda'_n + \lambda'_n - \lambda \mathcal{F}_n) \cdot C_{k,w} = (\mathcal{E}'_n - \lambda \mathcal{F}_n) \cdot C_{k,w} \leq 0.
$$

Here we have used that fact if $\nu_{\text{red}}(\alpha'|_{E_j}) = 0$, then $E_j$ does not intersect $C_{k,w}$ as otherwise $E_j \cap E_k$ contains a curve $\Gamma$ dominating $W_k$ and hence such that $\alpha' \cdot \Gamma > 0$, contradicting the fact that $\alpha'|_{E_j} \equiv 0$.

Since $\omega'_n$ is a modified Kähler class, by Corollary 2.36, $\omega'_n|_{E_{k,n}}$ is big. In particular, $\omega'_n \cdot C_{k,w} > 0$. Since $\alpha'_n$ is nef and $\alpha'_n \cdot C_{k,w} = 0$, then by Claim 3.2

$$
(K_{X'_k} + \Delta'_n) \cdot C_{k,w} \geq 0.
$$

Then from (3.4) and (3.3) it follows that

$$
0 \geq (\alpha'_n + \lambda \mathcal{F}_n) \cdot C_{k,w} = (K_{X'_k} + \Delta'_n + \omega'_n - \lambda \mathcal{F}_n) \cdot C_{k,w} \geq -\lambda \mathcal{F}_n \cdot C_{k,w}.
$$

Since $\mathcal{F} - q^* \mathcal{F}_n$ is an effective divisor whose support does not contain $q^{-1}_* C_{k,w} = p^{-1}_* C_{k,w}$ (via above identification), then

$$
-\mathcal{F}_n \cdot C_{k,w} = -q^* \mathcal{F}_n \cdot q^{-1}_* C_{k,w} \geq -p^* \mathcal{F} \cdot p^{-1}_* C_{k,w} = -\mathcal{F} \cdot C_{k,w}.
$$

Since $(-\mathcal{F} + t \alpha')|_{E_k}$ is Kähler and $\alpha' \cdot C_{k,w} = 0$, it follows that

$$
-\mathcal{F} \cdot C_{k,w} = (-\mathcal{F} + t \alpha') \cdot C_{k,w} > 0.
$$

Putting together equations (3.5), (3.6), and (3.7), we obtain a contradiction.

\[ \square \]

\textbf{Claim 3.5.} $\phi_n^*(\sum E_j + S') = 0$.

\textbf{Proof.} Let $W$ be the normalization of the graph of the induced bimeromorphic map $\psi_n : X \rightarrow X'_n$, and $p : W \rightarrow X$ and $q : W \rightarrow X'_n$ the projections. Let $I$ and $I'$ be the sets of all indices of the $\mu$-exceptional divisors which are contracted and respectively, not contracted by $\phi_n$. Then $\phi_n$ contracts $S'$ and the divisor $\sum_{i \in I} E_i \subset \text{Ex}(\mu)$, and $\psi_n$ contracts $S$ and extracts $\sum_{i \in I'} E_i \subset \text{Ex}(\mu)$.

Set $G = q^* \phi_{n,*}(\sum d_j E_j + d S') \geq 0$. We claim that $G$ is $p$-exceptional, i.e. if $F$ is a component of $G$, then $p_* F = 0$. Let $W'$ be the normalization of the
Then there is a unique morphism \( \theta : W' \to X' \) and \( q' : W' \to X'_n \) the projections. Then there is a unique morphism \( \theta : W' \to X' \) such that \( p \circ \theta = \mu \circ p' \) and \( q' = q \circ \theta \).

\[
(3.8)
\]

Let \( F' \) be the normalization of \( \theta^{-1}_* F \), then \( F' \) is the normalization of a component of \( q'^* \phi_{n,*} (\sum d_j E_j + dS') \) and \( p(F') = \mu(p'(F')) \). So if \( p_* F \neq 0 \), then \( p_* F = \mu_* p'_* F' = S \), and hence \( p'_* F' = S' \); in particular, in this case we have

\[
(3.9) \quad \nu_{\text{nef}} ((\mu \circ p')^* \alpha|_{F'}) = \nu_{\text{nef}} (p'^* \alpha'|_{S'}) = 1.
\]

Now we will show that this equation leads to a contradiction by proving that in fact \( \nu_{\text{nef}} ((\mu \circ p')^* \alpha|_{F'}) = \nu_{\text{nef}} ((p \circ \theta)^* \alpha|_{F'}) = 0 \). To this end, first observe that if \( F' \) is the normalization of a component of the strict transform \( q'^{-1} (\phi_{n,*} (\sum d_j E_j + dS')) \), then from Claim 3.3 it follows that \( \nu_{\text{nef}} ((p \circ \theta)^* \alpha|_{F'}) = 0 \). Now assume that \( F' \) is the normalization of a \( q' \)-exceptional divisor. If \( q'(F') \) is a point, then clearly \( \nu_{\text{nef}} ((p \circ \theta)^* \alpha|_{F'}) = 0 \) as \( (\mu \circ p')^* \alpha = p'^* \alpha' = q'^* \alpha'_n \). If \( q'(F') = \Gamma' \) is a curve, then \( \Gamma' \) is contained in a component of \( \phi_{n,*} (\sum d_j E_j + dS') \) by our construction of \( F' \) and thus from the Claim 3.3 again it follows that \( \alpha'_n \cdot \Gamma' = 0 \), in particular, \( \nu_{\text{nef}} ((p \circ \theta)^* \alpha|_{F'}) = 0 \).

Thus \( p_* F = 0 \) must hold; in particular, the divisor \( G = q^* \phi_{n,*} (\sum d_j E_j + dS') \) is \( p \)-exceptional.

Now we will show that \( G \) is nef over \( X \). To this end assume by contradiction that \( G \cdot C < 0 \) for some curve \( C \subset W \) such that \( p(C) = \text{pt} \). Thus \( C \) is contained in \( F \), a component of \( G \). We have \( \alpha'_n \cdot q_* C = q^* \alpha'_n \cdot C = p^* \alpha \cdot C = 0 \). Since \( \phi_{n,*} (\sum d_j E_j + dS') \equiv \alpha'_n K_{X'_n} + \Delta'_n \), it follows that

\[
G \cdot C = \phi_{n,*} \left( \sum d_j E_j + dS' \right) \cdot q_* C = \left( K_{X'_n} + \Delta'_n \right) \cdot q_* C \geq 0,
\]

which is a contradiction. Thus \( G \) is nef over \( X \).

Then by the negativity lemma we have \( G = 0 \), and hence \( \phi_{n,*} (\sum E_j + S') = 0 \).

\[\square\]

Claim 3.6. \( \psi_n : X \to X'_n \) is a morphism.
Proof. By contradiction assume that \( \psi_n \) is not a morphism. Let \( W \) be a resolution of singularities of the graph of \( \psi_n \) and \( p : W \to X \) and \( q : W \to X'_n \) the induced morphisms. By Theorem 2.17, possibly replacing \( W \) by a higher resolution we may assume that \( p \) is projective. If \( \psi_n \) is not a morphism, then there is a curve \( C \) in \( W \) such that \( p_* C = 0 \) but \( q_* C = C'_n \neq 0 \). Let \( \omega_n \) be a Kähler form on \( X'_n \). Then \( \omega_n \cdot C'_n > 0 \). Note that since \( X \) is strongly \( \mathbb{Q} \)-factorial with klt singularities and \( p \) is projective, by Lemma 2.32, \( N^1(W/X) \) is generated by the \( p \)-exceptional divisors, say \( E_1, \ldots, E_r \). Then there exist real numbers \( e_1, \ldots, e_r \in \mathbb{R} \) such that \( [q^* \omega_n + \sum e_i E_i] = 0 \) in \( N^1(W/X) \). Thus there exists a \((1,1)\)-form \( \omega \) with local potentials on \( X \) such that \( [p^* \omega] = [q^* \omega_n + \sum e_i E_i] \). Now since \( S \cdot R < 0 \), there is a real number \( r \in \mathbb{R} \) such that \( (\omega + rS) \cdot R = 0 \).

Since \( \psi_{n+1} S = 0 \) by Claim 3.5, it follows that \( \sum e_i E_i + rp^* S \) is a \( q \)-exceptional divisor and \( [p^* (\omega + rS) - q^* \omega_n] = \sum e_i E_i + rp^* S \). Now we claim that \( [p^* (\omega + rS) - q^* \omega_n] \equiv_{X_n} 0 \). Indeed, if \( \gamma \subset W \) is a curve contracted by \( q \), then \( \alpha \cdot p_* \gamma = q^* \alpha_n \cdot \gamma = 0 \). In particular, if \( p_* \gamma \neq 0 \), then \( p_* \gamma \) is a curve contained in the extremal ray \( R \), and hence \( [\omega + rS] \cdot p_* \gamma = 0 \) and the claim follows. Then applying the negativity lemma we get \( \sum e_i E_i + rp^* S = 0 \), and hence we have \( [p^* (\omega + rS)] = [q^* \omega_n] \). Therefore

\[
0 < \omega_n \cdot C'_n = q^* \omega_n \cdot C = (\omega + rS) \cdot p_* C = 0,
\]

and this is a contradiction. \( \square \)

Thus \( X \to X'_n \) is a morphism of strongly \( \mathbb{Q} \)-factorial varieties with klt singularities which contracts a unique divisor \( S \). By Lemma 2.32, it follows that \( \rho(X/X'_n) = 1 \) and hence this is the desired divisorial contraction. \( \square \)

4. The Minimal Model Program for Uniruled Pairs

In this section we will consider the minimal model program for non-pseudo-effective dlt compact Kähler 3-fold pairs \((X, B)\). Since \( K_X + B \) is not pseudo-effective, neither is \( K_X \) and hence the MRC fibration \( X 
\to \to \to \) \( Z \) is non-trivial (see e.g. [HP15, Introduction]). Let \( \nu : X' \to X \) be any resolution, then since \((X, B)\) is dlt, the fibers of \( \nu \) are rationally chain connected (see Proposition 2.29) and hence \( X' 
\to \to \to \) \( Z \) is the MRC fibration of \( X' \). Recall that if \( \dim Z \leq 1 \), then by Lemma 2.42, \( X \) is projective and \( H^2(X, \mathcal{O}_X) = 0 \). Since this case is well understood, we will focus on the case where \( \dim Z = 2 \). Note that by [HP15, Remark 3.2], \( Z \) is not uniruled and hence \( K_Z \) is pseudo-effective. Moreover, from Definition 2.2(i) it follows that \( Z \) is in Fujiki’s class \( \mathcal{C} \). Then replacing \( Z \) by a resolution of singularities we may assume that \( Z \) is a smooth compact complex surface in Fujiki’s class \( \mathcal{C} \), and hence by [Fuj83, Remark 1.1, page 236], \( Z \) is Kähler.
**Definition 4.1.** Let \((X, B)\) be a log pair, where \(X\) is a \(\mathbb{Q}\)-factorial compact Kähler 3-fold. Suppose that the base of the MRC fibration \(f : X \to Z\) has dimension 2. Let \(X_z \cong \mathbb{P}^1\) be a general fiber of \(f\). Then a modified Kähler class \(\omega\) on \(X\) is called \((K + B)\)-normalized if \((K + B + \omega) \cdot X_z = 0\). Moreover, if \(\omega\) is a Kähler class, then we call it a \((K + B)\)-normalized Kähler class. Note that as \(\omega\) is a modified Kähler class, it is positive on general fibers \(X_z\), and hence \((K + B) \cdot X_z < 0\).

**Lemma 4.2.** With the same notations and hypothesis as in the definition above, assume that \((X, B)\) is dlt and \(\omega\) is a \((K + B)\)-normalized modified Kähler class. Then \(K + B + \omega\) is pseudo-effective.

**Proof.** Since \(\omega\) is a modified Kähler class, there is a bimeromorphic morphism from a compact Kähler manifold \(\mu : X' \to X\) and a Kähler class \(\omega' \in H^{1,1}_{BC}(X')\) represented by a Kähler form on \(X'\) such that \(\mu_* \omega' = \omega\). We may assume that there is an effective \(\mu\)-exceptional \(\mathbb{Q}\)-divisor \(F\) such that \(-F\) is \(\mu\)-ample.

We may write \(K_{X'} + B' = \mu^*(K + B) + E\) where \(E \geq 0, B' \geq 0, \mu_* B' = B\), and \(E\) and \(B'\) do not share any common component. Since \(\dim Z = 2\) and \(\dim \mu(\text{Ex}(\mu)) \leq 1\), it follows that \(X' \to X\) is an isomorphism on a neighborhood of a general fiber \(X_z\) of \(X \to Z\). Therefore \(\omega' = (K_{X'} + B')\)-normalized. Note that if \(K_{X'} + B' + \omega'\) is pseudo-effective, then so is \(K + B + \omega = \mu_*(K_{X'} + B' + \omega')\). Replacing \((X, B)\) by \((X', B')\) we may therefore assume that \(X\) is smooth and \(\omega\) is a Kähler class.

Since the pseudo-effective cone is closed, it is enough to show that \((K + (1 - \delta)B + (1 + \varepsilon)\omega)\) is pseudo-effective for all \(1 \gg \varepsilon \gg \delta > 0\). Since \(\omega\) is a \((K + B)\)-normalized Kähler class, it follows easily that \((K + (1 - \delta)B + (1 + \varepsilon)\omega)|_{X_z}\) is Kähler (or equivalently, it has positive degree). Let \(Z' \to Z\) be a resolution of singularities of \(Z\) and \(\mu : X' \to X\) a log resolution of \((X, B)\) such that the induced meromorphic map \(\varphi' : X' \to Z'\) is a morphism. We may assume that \(X_z = X'_z\) for general \(z \in Z\). Since \(\omega\) is a Kähler class, there exists an effective \(\mu\)-exceptional divisor \(F\) such that \(\omega' := \mu^* \omega - F\) is a Kähler class.

Set

\[K_{X'} + B'_{\delta, \varepsilon} := \mu^*(K_X + (1 - \delta)B) + \varepsilon F.\]

Then we have

\[\mu^*(K_X + (1 - \delta)B + (1 + \varepsilon)\omega) = K_{X'} + B'_{\delta, \varepsilon} + \omega'_\varepsilon,\]

where \(\omega'_\varepsilon = \mu^* \omega + \varepsilon \omega'\) is a Kähler class, since \(\omega\) is a Kähler class on \(X\).

Therefore it is enough to show that \(K_{X'} + (B'_{\delta, \varepsilon})^{\geq 0} + \omega'_\varepsilon\) is pseudo-effective. Let \(X'_{\varepsilon}\) be a general fiber of \(\varphi' : X' \to Z'\). Then \(X'_{\varepsilon} \cong \mathbb{P}^1\) and \(c_1((K_{X'} + (B'_{\delta, \varepsilon})^{\geq 0} + \omega'_\varepsilon)|_{X'_{\varepsilon}})\) is a Kähler class. Thus by [Gue20, Theorem], \(K_{X'/Z'} + \omega'_{\varepsilon} + \mu^* \omega - F\) is pseudo-effective.
\((B'_{d,\varepsilon})^{\geq 0} + \omega'_\varepsilon\) is pseudo-effective. Now since \(Z'\) is not uniruled, \(K_{Z'}\) is pseudo-effective. Therefore \(K_{X'} + (B'_{d,\varepsilon})^{\geq 0} + \omega'_\varepsilon\) is pseudo-effective as required.

\[\square\]

**Corollary 4.3.** Let \((X, B)\) be a klt pair, where \(X\) is a \(\mathbb{Q}\)-factorial compact Kähler 3-fold. Assume that \(X\) is uniruled and the dimension of the base of the MRC fibration \(X \rightarrow Z\) is 2. Let \(\omega\) be a nef and big \((1,1)\)-class on \(X\) such that \((K_X + B + \omega) \cdot F = 0\) for general fibers \(F\) of \(X \rightarrow Z\). Then \(K_X + B + \omega\) is pseudo-effective.

**Proof.** Since \((X, B)\) is klt and \(\omega\) is nef and big, by Lemma 2.38 we can write \(\omega = \Theta + F\), where \(\Theta\) is a modified Kähler class and \(F\) is an effective \(\mathbb{Q}\)-divisor such that \((X, B + F)\) klt. Thus replacing \(B\) by \(B + F\) and \(\omega\) by \(\Theta\) we may assume that \(\omega\) is a modified Kähler class. Then the result follows from Lemma 4.2.

\[\square\]

**Lemma 4.4.** Let \((X, B)\) be a dlt pair, where \(X\) is a \(\mathbb{Q}\)-factorial compact Kähler 3-fold. Suppose that \(X\) is uniruled and the base of the MRC fibration \(f : X \rightarrow Z\) has dimension 2. Let \(F(\cong \mathbb{P}^1)\) be a general fiber of \(f\). If \((K_X + B) \cdot F \geq 0\), then \(K_X + B\) is pseudo-effective.

**Proof.** The proof is similar to the proof of Lemma 4.2 and so we omit it.

\[\square\]

We will also need the following result.

**Lemma 4.5.** Let \((X, B)\) be a dlt pair, where \(X\) is a \(\mathbb{Q}\)-factorial compact Kähler 3-fold. Let \(\omega\) be a modified Kähler class on \(X\) such that \(\alpha = K_X + B + \omega\) is a nef and big \((1,1)\)-class. Let \(S \subset X\) be an irreducible surface such that \(S \subset \text{Null}(\alpha)\), i.e. \(\alpha^2 \cdot S = 0\). Then \(S\) is a Moishezon space and it is covered by \(\alpha\)-trivial curves.

**Proof.** Let \(\pi : S' \rightarrow S\) be the minimal resolution of \(S\) dominating the normalization \(\bar{S} \rightarrow S\). By Lemma 2.41 we have

\[K_{S'} + E = \pi^*((K_X + S)|_S),\]

where \(E \geq 0\) is an effective \(\mathbb{Q}\)-divisor.

Note that since \(\bar{S}\) is a Kähler surface, then so is \(S'\). We separate two cases based on the numerical dimension of \(\pi^*(\alpha|_S)\).

**Case I:** \(\pi^*(\alpha|_S) \equiv 0\). In this case \(-\pi^*((K_X + B)|_S) \equiv \pi^*(\omega|_S)\). Since \(\omega\) is a modified Kähler class, \(\omega|_S\) is a big \((1,1)\)-class by Corollary 2.36. Therefore
\(-\pi^*((K_X + B)|_S)\) is a big divisor on \(S'\); in particular, \(S'\) is a Moishezon space. Furthermore, since \(S'\) is smooth and Kähler, it is projective by [Moi66] (also see Theorem 2.13). Consequently, \(S'\) can be covered by a family of \(\pi^*(\alpha|_S)\)-trivial curves \(\{C_t\}_{t \in \mathcal{T}}\). Pushing forward these curves gives a covering family of \(\alpha|_S\)-trivial curves on \(S\).

**Case II:** \(\pi^*(\alpha|_S) \neq 0\). Since \(\omega\) is modified Kähler, \(\omega|_S\) is big. Since \(\alpha\) is nef, \(\omega \cdot \alpha \cdot S = (\omega|_S \cdot \alpha|_S) > 0\).

Note also that since \(\alpha\) is big, \(K_X + B + (1 - \epsilon)\omega\) is also big for \(0 < \epsilon \ll 1\) (recall in fact that the big cone is open cf. [Bou04, §2.3]). We may write

\[
K_X + B + (1 - \epsilon)\omega \equiv \sum_{j=1}^{r} \lambda_j S_j + P,
\]

where \(\lambda_j > 0\) for all \(j\) and \(P\) is nef in codimension 1.

Since \(\alpha^2 \cdot S = 0\), it follows that

\[
\left(\sum_{j=1}^{r} \lambda_j S_j + P\right) \cdot \alpha \cdot S = (K_X + B + (1 - \epsilon)\omega) \cdot \alpha \cdot S = -\epsilon \omega \cdot S \cdot \alpha < 0.
\]

Since \(\alpha\) is nef and \(P|_S\) is pseudo-effective, we must have \(S = S_j\) for some \(1 \leq j \leq r\) and \(\alpha \cdot S^2 < 0\).

Let \(B = aS + B'\) such that \(0 \leq a \leq 1\) and \(S\) is not contained in the support of \(B'\). Then we have

\[
\lambda_j S_j + P\bigg|_S = aS + B' \bigg|_S - \omega|_S + S - B' \bigg|_S \cdot \alpha \cdot S = a^2 \cdot S + (1 - a)\alpha \cdot S^2 - B' \bigg|_S \cdot \alpha|_S - \omega \cdot \alpha \cdot S < 0.
\]

Since \(\pi^*(\alpha|_S)\) is nef, this shows that \(K_{S'}\) is not pseudo-effective. Thus by Lemma 2.43, \(S'\) is projective, and hence \(S\) is a Moishezon space. Furthermore, by Lemma 2.33, \(S'\) is covered by \(\pi^*(\alpha|_S)\)-trivial curves. Pushing forward these curves on \(S\) we see that \(S\) is also covered \(\alpha|_S\)-trivial curves. \(\square\)

4.1. **Cone Theorem.** The purpose of this section is to prove the following cone theorem which is a direct generalization of the results of [HP15], [HP16], [CHP16] and [DO24]. The techniques that we use are all inspired by these papers.
\textbf{Theorem 4.6.} Let \((X, B)\) be a dlt pair, where \(X\) is a \(\mathbb{Q}\)-factorial compact Kähler 3-fold. Suppose that \(X\) is uniruled and the base of the MRC fibration \(X \twoheadrightarrow Z\) is a surface. Suppose that \((K_X + B) \cdot X_z < 0\) for general \(z \in Z\). Let \(\omega\) be a \((K_X + B)\)-normalized modified Kähler class. Then there exist a countable family of curves \(\Gamma_i\) on \(X\) and a positive number \(d\) such that \(0 < -(K_X + B + \omega) \cdot \Gamma_i \leq d\) and
\[
\text{NA}(X) = \text{NA}(X)_{(K_X + B + \omega) \geq 0} + \sum_{i \in I} \mathbb{R}^+[\Gamma_i].
\]

\textit{Proof.} By Lemma 4.2, \(K_X + B + \omega\) is pseudo-effective and so by [Bou04] (applied to a resolution of \(X\) and then pushing forward) we have a divisorial Zariski decomposition
\[
(4.4) \quad K_X + B + \omega \equiv \sum_{j=1}^r \lambda_j S_j + P,
\]
where the \(S_j\)'s are surfaces, \(\lambda_j \in \mathbb{R}^+\) for all \(j\) and \(P\) is a pseudo-effective class which is nef in codimension 1.

\textit{Claim 4.7.} Let \(S \subset X\) be a surface such that \((K_X + B + \omega)|_S\) is not pseudo-effective, then \(S = S_j\) for some \(1 \leq j \leq r\), \(S\) is Moishezon and any desingularization \(\hat{S} \to S\) is a uniruled projective surface.

\textit{Proof.} Since \(P\) is nef in codimension 1, then \(P|_S\) is pseudo-effective. If \(S \neq S_j\) for all \(1 \leq j \leq r\), then \((\sum_{j=1}^r \lambda_j S_j)|_S \geq 0\) and hence \((K_X + B + \omega)|_S\) is pseudo-effective, contradicting our assumptions. Thus, possibly reindexing, we may assume that \(S = S_1\). Let \(b = \text{mult}_S(B)\), then \(0 \leq b \leq 1\). We then have
\[
(4.5) \quad K_X + S + B - bS + \omega + \frac{1-b}{\lambda_1} \left( \sum_{j=2}^r \lambda_j S_j + P \right) \equiv \left( 1 + \frac{1-b}{\lambda_1} \right) (K_X + B + \omega).
\]
Since \((K_X + B + \omega)|_S\) is not pseudo-effective and \((B - bS + \omega + \frac{1-b}{\lambda_1} (\sum_{j=2}^r \lambda_j S_j + P))|_S\) is pseudo-effective, from the above equality it follows that \((K_X + S)|_S\) is not pseudo-effective. Let \(\pi : \hat{S} \to S\) be the minimal resolution of \(S\) dominating its normalization \(\bar{S}\), then by Lemma 2.41, there exists an effective divisor \(E \geq 0\) on \(\hat{S}\) such that \(K_{\hat{S}} = \pi^*((K_X + S)|_S) - E\). But then it follows that \(K_{\hat{S}}\) is not pseudo-effective, and hence \(\hat{S}\) is projective by Lemma 2.43, and \(\hat{S}\) is uniruled. In particular, \(S\) is Moishezon. Finally, observe that if \(S'\) is any resolution of singularities of \(S\), then it factors through the minimal resolution \(\hat{S}\), and hence \(S'\) is projective and uniruled. \(\square\)

Next we establish a form of bend and break.
Claim 4.8. There exists a number $d > 0$ such that if $C \subset X$ is a curve with $-(K_X + B + \omega) \cdot C > d$, then $[C] = [C_1] + [C_2]$, where $C_1$ and $C_2$ are two non-zero integral effective 1-cycles.

Proof. First using the arguments of [CHP16, Lemma 4.2] and passing to a dlt model as in Lemma 2.28, we may assume that $(X, B)$ is dlt and $X$ has terminal singularities. The proof of this claim involves two main steps, in the first step we will construct four sets $A, B, C$ and $D$ of finitely many curves on $X$. These sets will determine the number $d > 0$. Next we will show that if $C$ is a curve in $X$ such that $-(K_X + B + \omega) \cdot C > d$, then $\dim_{\mathbb{C}} \text{Chow}(X) > 0$. We proceed with the constructions of the sets.

Let $A$ be the set of all curves $C \subset X$ satisfying the following properties:

1. $(K_X + B) \cdot C < 0$,
2. $B \cdot C < 0$ and
3. $C$ is contained in a horizontal (over $Z$) component $T$ of $B$.

We claim that $A$ is a finite set. Indeed, if $T$ is a horizontal component of $B$, then $T$ is not uniruled, since the induced morphism $T \to Z$ is generically finite and $Z$ is not uniruled. Let $\tilde{T} : \hat{T} \to T$ be the minimal resolution of $T$. Then $\hat{T}$ is not uniruled, and hence $K_{\hat{T}}$ is pseudo-effective. Moreover, by the MMP and abundance for compact Kähler surfaces, there exists an effective $\mathbb{Q}$-divisor $D \geq 0$ on $\hat{T}$ such that $K_{\hat{T}} \sim_{\mathbb{Q}} D$. Now since the coefficients of $B$ are contained in the interval $(0, 1]$, there is a non-negative rational number $\lambda \geq 0$ such that $\text{mult}_T(1 + \lambda)B = 1$. Then from our hypothesis it follows that $(K_X + (1 + \lambda)B) \cdot C < 0$. Then by Lemma 2.41 there is an effective $\mathbb{Q}$-divisor $E \geq 0$ on $\hat{T}$ such that $\pi^*((K_X + (1 + \lambda)B)|_T) = K_{\hat{T}} + E \sim_{\mathbb{Q}} D + E$.

Therefore, from the projection formula, it follows that $\pi_*(D + E) \cdot C < 0$, i.e. $C$ is contained in the support of $\pi_*(D + E)$. In particular, the set $A$ is finite.

Now let $S$ be a component of $B$ which is vertical over $Z$ and write

$$K_S + E \sim_{\mathbb{Q}} \pi^*((K_X + S)|_S),$$

where $\pi : \hat{S} \to S$ is the minimal resolution of $S$.

If $\kappa(\hat{S}) \geq 0$, set $F := B(K_S)$, where $B(\cdot)$ denotes the stable base locus. If $\kappa(\hat{S}) = -\infty$, the we set $F := 0$. Let $B$ be the union of the curves $\pi_*(E + F)$ as $S$ varies over all components of $B$ that do not dominate $Z$.

Let $C$ be the finite set of curves $C \subset X$ which are contained in the singular locus of the support of $B + \sum S_j$.

Let $D$ be the set of all curves $C \subset X$ such that $\omega \cdot C \leq 0$. We claim that $D$ is a finite set. Indeed, since $\omega$ is a modified Kähler class, by Lemma 2.35, there exists a projective bimeromorphic morphism $f : Y \to X$ from a compact
Kähler manifold $Y$, a Kähler class $\omega_Y \in H^{1,1}_{BC}(Y)$, and an effective $\mathbb{R}$-divisor $E \geq 0$ such that $-E$ is $f$-ample, $\text{Supp}(E) = \text{Ex}(f)$ and $f^*\omega = \omega_Y + [E]$. If $C$ is a curve from the set $\mathcal{D}$ and $C' \subset Y$ is a curve in $Y$ such that $f(C') = C$, then $0 \geq \omega \cdot C = f^*\omega \cdot C' = (\omega_Y + E) \cdot C'$. This implies that $E \cdot C' < 0$, since $\omega_Y \cdot C' > 0$, as $\omega_Y$ is a Kähler class. In particular, $C'$ is contained in the support of $E$, since $E$ is effective. Therefore $C = f(C')$ is contained in $f(\text{Supp}E)$. Since $E$ is $f$-exceptional and dim $X = 3$, it follows that $\mathcal{D}$ is a finite set.

Now we define

$$d := \max\{4, -(K_X + B + \omega) \cdot C \mid C \subset X \text{ is a curve, and } C \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}\}.$$

**Claim 4.9.** For any curve $C \subset X$ such that $-(K_X + B + \omega) \cdot C > d$, we have $\dim_C \text{Chow}(X) > 0$.

**Proof.** By assumption $\omega \cdot C > 0$ and so $-(K_X + B) \cdot C > d$. We will separate two cases depending on whether $B \cdot C < 0$ or $B \cdot C \geq 0$.

**Case I:** Suppose that $B \cdot C < 0$. In this case there is a component $S$ of $B$ containing $C$. Moreover, we see that $C$ is not contained in any other component of $B$ or $\sum S_j$, since $C \notin \mathcal{C}$, and also that $S$ is not horizontal over $Z$, since $C \notin \mathcal{A}$. In particular, $S$ is vertical over $Z$. Now since $C \notin \mathcal{B}$, $C$ is not contained in $\pi(E)$, where $E$ and $\pi : \hat{S} \rightarrow S$ as in (4.6). Let $\hat{C} \subset \hat{S}$ be the strict transform of $C$ and $\text{mult}_S B = b$. Since $C \notin \mathcal{D}$, then $\omega \cdot C > 0$ and since $C$ is not contained in $\pi(E)$, then $E \cdot \hat{C} \geq 0$. Then as $S \cdot C < 0$ and $(B - bS) \cdot C \geq 0$, we have

$$K_{\hat{S}} \cdot \hat{C} \leq (K_{\hat{S}} + E) \cdot \hat{C}$$

$$= (K_X + S) \cdot C$$

$$= (K_X + B + (1 - b)S - (B - bS)) \cdot C$$

$$\leq (K_X + B) \cdot C$$

$$< (K_X + B + \omega) \cdot C$$

$$< -d.$$

Now if $\kappa(\hat{S}) \geq 0$, then as $\hat{C}$ is not contained in $F = B(K_{\hat{S}})$, it follows that $K_{\hat{S}} \cdot \hat{C} \geq 0$, which is a contradiction to the above inequality. Thus $\kappa(\hat{S}) < 0$ and then from MMP and abundance for smooth compact Kähler surfaces it follows that $K_{\hat{S}}$ is not pseudo-effective. Hence $\hat{S}$ is projective by Lemma 2.43.

Now since $\hat{S}$ is a smooth and projective surface and $-K_{\hat{S}} \cdot \hat{C} > d \geq 4$, by [Kol96, Theorem II.1.15] we have $\dim_C \text{Chow}(\hat{S}) > 0$, i.e. $\hat{C}$ deforms in $\hat{S}$, in particular its push-forward $C$ deforms in $S \subset X$, and hence $\dim_C \text{Chow}(X) >$
0.

**Case II:** Suppose now that \( B \cdot C \geq 0 \). Since \( C \notin \mathcal{D} \), \( \omega \cdot C > 0 \) and hence \( K_X \cdot C < (K_X + B + \omega) \cdot C < -d \leq -4 \). Since \( X \) has \( \mathbb{Q} \)-factorial terminal singularities, it follows from [HP16, Theorem 4.5] that \( C \) is not very rigid (see [HP16, Definition 4.3]). Let \( m \) be the smallest positive integer such that \( \dim_m \text{Chow}(X) > 0 \). Let \( \Gamma \to T \) be the corresponding family. Replacing \( \Gamma \) by an irreducible component which contains \( C \subset X \) we may assume that \( \Gamma \) is irreducible, and hence so is the locus covered by the family \((\Gamma_t)_{t \in T}\). Consequently, \( \Gamma_t \) is irreducible for \( t \in T \) very general. Then from the minimality of \( m \) it also follows that this family has no fixed component. Now by (4.4) there is a unique surface \( S_j \) covered by the \( \{\Gamma_t\}_{t \in T} \). Note that \((K_X + B + \omega)|_{S_j} \) is not pseudo-effective, since \((K_X + B + \omega)|_{S_j} \cdot \Gamma_t = m(K_X + B + \omega) \cdot C < 0 \). By Claim 4.7, the minimal resolution \( \hat{S} \) of \( S = S_j \) is projective and uniruled. Since \( C \notin \mathcal{C} \), we have that \( C \) is not contained in \( S_l \) for \( l \neq j \) and hence \( S_l \cdot C \geq 0 \) for \( l \neq j \). Since \( P|_{S_j} \) is pseudo-effective,

\[
P \cdot mC = P|_{S_j} \cdot mC = P|_{S_j} \cdot \Gamma_t \geq 0.
\]

Using the same notation as in Case I and its proof we see that \( E \cdot m\hat{C} \geq 0 \) and \( \omega \cdot mC > 0 \), where \( \hat{C} \) is the strict transform of \( C \) under \( \pi : \hat{S} \to S = S_j \). We also know that \( S_j \cdot C < 0 \). Therefore from (4.5) we have

\[
K_{\hat{S}} \cdot \hat{C} \leq (K_{\hat{S}} + E) \cdot \hat{C} = (K_X + S_j) \cdot C \leq \left( 1 + \frac{1-b}{\lambda_j} \right) (K_X + B + \omega) \cdot C < -d \leq -4.
\]

By [Kol96, Theorem II.1.15], we have \( \dim_{\mathbb{C}} \text{Chow}(\hat{S}) > 0 \), i.e. \( \hat{C} \) deforms in \( \hat{S} \). Thus by pushing forward \( \hat{C} \) we have that \( C \) deforms, i.e. \( \dim_{\mathbb{C}} \text{Chow}(X) > 0 \). □

We will now prove the bend and break property, i.e. Claim 4.8. If \( C \subset X \) is a curve satisfying \(- (K_X + B + \omega) \cdot C > d \), then by Claim 4.9 and its proof it follows that, the curve \( C \) deforms in a family \( \{\Gamma_t\}_{t \in T} \) covering a unique uniruled surface \( S \). Since \((K_X + B + \omega) \cdot C < 0 \), then \((K_X + B + \omega)|_{S} \) is not pseudo-effective. Since \( P|_{S} \) is pseudo-effective, by (4.4) it follows easily that \( S = S_j \). We also know that the curve \( C \) is contained in \( S \) but not in \( S_{\text{sing}} \). Moreover, if \( \pi : \hat{S} \to S \) is the minimal resolution of \( S \), then we know from Claim 4.7 that \( \hat{S} \) is a projective uniruled surface. Now from the proof of Claim 4.9 we have \( K_{\hat{S}} \cdot \hat{C} < -d \leq -4 \). Thus by [HP16, Lemma 5.5(b)] there is an effective 1-cycle \( \sum_{k=1}^{m} C_k \) with \( m \geq 2 \) such that \([\hat{C}] = \sum_{k=1}^{m} C_k \) and \( K_{\hat{S}} \cdot C_i < 0 \) for \( i = 1, 2 \). Since \( \pi : \hat{S} \to S \) is the minimal resolution, \( K_{\hat{S}} \) is \( \pi \)-nef and hence \( \pi_* C_j \neq 0 \) for \( j = 1, 2 \). In particular, we also have a decomposition \([C] = \sum_{k=1}^{m} \pi_* C_k \) with at least two non-zero terms. This concludes the proof.
The theorem now follows from Claim 4.8 and the arguments in the proof of [CHP16, Theorem 4.1].

**Corollary 4.10.** Let \((X, B)\) be a dlt pair, where \(X\) is a \(\mathbb{Q}\)-factorial compact Kähler 3-fold. Suppose that \(X\) is uniruled and the base of the MRC fibration \(X \to Z\) is a surface. Let \(\omega\) be a modified Kähler class such that \((K_X + B + \omega) \cdot X_z > 0\) for general \(z \in Z\), then there exist finitely many curves \(\{\Gamma_i\}_{i=1}^N\) such that

\[
\overline{\text{NA}}(X) = \overline{\text{NA}}(X_{(K_X + B + \omega) \geq 0}) + \sum_{i=1}^N \mathbb{R}^+[\Gamma_i].
\]

**Proof.** Pick \(t = -(K_X + B) \cdot X_z / \omega \cdot X_z\), then \(0 < t < 1\) and \((K_X + B + t\omega) \cdot X_z = 0\). By Theorem 4.6, there exists a countable family of curves \(\Gamma_i\) with \(0 < -(K_X + B + t\omega) \cdot \Gamma_i \leq d\) and

\[
\overline{\text{NA}}(X) = \overline{\text{NA}}(X_{(K_X + B + t\omega) \geq 0}) + \sum_{i=1}^N \mathbb{R}^+[\Gamma_i].
\]

**Claim 4.11.** \(\omega \cdot \Gamma_i > d/(1-t)\) for all but finitely many \(i\)’s.

Assuming the claim for the time being we will complete the proof first. Using the claim we have

\[(K_X + B + \omega) \cdot \Gamma_i = (K_X + B + t\omega) \cdot \Gamma_i + (1-t)\omega \cdot \Gamma_i > 0\]

for all but finitely many \(i\)’s. This concludes the proof.

**Proof of Claim 4.11.** Since \(\omega\) is a modified Kähler class, by Lemma 2.35 there exist a projective bimeromorphic morphism \(\nu : X' \to X\) from a Kähler manifold \(X'\), a Kähler class \(\omega' \in H^{1,1}(X')\) and an effective \(\nu\)-exceptional divisor \(F \geq 0\) such that \(\nu^*\omega = \omega' + [F]\). Note that from the negativity lemma it follows that \(\text{Supp}(F) = \text{Ex}(\nu)\). Now let \(\Gamma'_i\) be the strict transform of the curve \(\Gamma_i\) which is not contained in \(\nu(F)\). Note that since \(\dim \nu(F) \leq 1\), there are only finitely many \(\Gamma_i\)’s contained in \(\nu(F)\).

We will assume by contradiction that the claim is false, i.e. \(\omega \cdot \Gamma_i \leq d/(1-t)\) for infinitely many indices \(i\). Let \(\Lambda\) be the set of indices for all such curves \(\Gamma_i\). Without loss of generality we may assume that \(\Gamma_i\) is not contained in \(\text{Supp}(\nu(F))\) for any \(i \in \Lambda\). Observe that the \(\Gamma_i\)’s for \(i \in \Lambda\) belong to distinct equivalence classes in \(\overline{\text{NA}}(X)\); in particular, the strict transforms \(\Gamma'_i\) also belong to distinct equivalence classes in \(\overline{\text{NA}}(X')\) for all \(i \in \Lambda\). Now by the projection formula we have

\[
\omega' \cdot \Gamma'_i = (\nu^*\omega - F) \cdot \Gamma'_i \leq \omega \cdot \Gamma_i \leq d/(1-t)
\]

for all \(i \in \Lambda\).
Since $\omega'$ is a Kähler class, by [Tom16, Lemma 4.4] the curves $\Gamma'_i$ belong to a bounded family for all $i \in \Lambda$. In particular, the $\Gamma'_i$'s belong to finitely many distinct equivalence classes in $\text{NA}(X')$, this is a contradiction. □

Remark 4.12. Observe that in the settings of Corollary 4.10 if $(X, B)$ is klt and $\omega$ is only a nef and big class such that $(K_X + B + \omega) \cdot X > 0$ for general $z \in Z$ (this last condition is satisfied for example if $K_X + B + \omega$ itself is big), then the same finiteness conclusion holds. Indeed, it follows from Corollary 4.10 using Lemma 2.38.

Remark 4.13. We note that a similar finiteness result as in Corollary 4.10 also holds for pseudo-effective pairs. More specifically, if $(X, B)$ is a $\mathbb{Q}$-factorial compact Kähler 3-fold dlt (resp. klt) pair and $K_X + B$ is pseudo-effective, then for any modified Kähler (resp. nef and big) class $\omega$, there are only finitely many $(K_X + B + \omega)$-negative extremal rays of $\text{NA}(X)$. Indeed, by Theorem 2.22 there are countably many $(K_X + B)$-negative extremal rays generated by the curves $\{\Gamma_i\}_{i \in I}$ and a positive rational number $d > 0$ such that $0 < -(K_X + B) \cdot \Gamma_i \leq d$ for all $i \in I$. First we will deal with the modified Kähler case. Since $\omega$ is modified Kähler, as in the proof of Claim 4.11 (using the same notations) we have $\nu^* \omega = \omega' + [F]$. If $(K_X + B + \omega) \cdot \Gamma_i < 0$ and $\Gamma_i$ is not contained in $\nu(\text{Supp}(F))$, let $\Gamma'_i$ be the strict transform of $\Gamma_i$. Then we have $F \cdot \Gamma'_i \geq 0$ and $\omega' \cdot \Gamma'_i \leq (\omega' + F) \cdot \Gamma'_i = \omega \cdot \Gamma_i \leq d$. Since $\omega'$ is Kähler, this implies that there are only finitely many such curves $\Gamma_i$. Furthermore, since $\nu(\text{Supp}(F))$ contains only finitely many curves (as $\dim X = 3$), our claim follows.

Now if $(X, B)$ is klt and $\omega$ is a nef and big class, then using Lemma 2.38 we may assume that $\omega$ is a modified Kähler class, and thus we are done by the previous case.

Next we prove a technical result which will be used in the existence of small contractions (see Theorem 4.16) and also in Section 6.

Proposition 4.14. Let $(X, B)$ be a dlt pair, where $X$ is a $\mathbb{Q}$-factorial compact Kähler 3-fold. Assume that $X$ is uniruled, $K_X + B$ is not pseudo-effective, and the base of the MRC fibration $X \to Z$ has dimension 2. Let $R = \mathbb{R}^+[\Gamma_i]$ be an extremal ray of $\text{NA}(X)$ with a nef supporting class $\alpha$. Assume that there is a nef and big class $\eta$ such that $K_X + B + \eta$ is pseudo-effective and $(K_X + B + \eta) \cdot R < 0$. If $R$ is small and $S \subset X$ is an irreducible surface, then the following holds:

$$\alpha^2 \cdot S > 0.$$ 

Proof. First note that by a standard technique using Theorem 4.6 and Corollary 4.10 we can write $\alpha - (K_X + B + \eta) = \omega$, for some Kähler class $\omega$, i.e.
\( \alpha = K_X + B + \eta + \omega \). Assume by way of contradiction that \( \alpha^2 \cdot S = (\alpha|_S)^2 = 0 \).

First assume that \( \alpha|_S = 0 \). Then we have \(- (K_X + B)|_S = (\eta + \omega)|_S \). Thus \(- (K_X + B)|_S \) is an ample divisor on \( S \), in particular, \( S \) is projective. So we can cover \( S \) by a family of curves. But since \( \alpha|_S = 0 \), all these curves are contained in \( R \), this is a contradiction, since \( R \) is small.

Next assume that \( \alpha|_S \neq 0 \) but \( (\alpha|_S)^2 = 0 \). Then we have

\[
0 = \alpha^2 \cdot S = (K_X + B) \cdot \alpha \cdot S + (\eta + \omega) \cdot \alpha \cdot S
\]

and

\[
(\eta + \omega) \cdot \alpha \cdot S = (\eta + \omega)|_S \cdot \alpha|_S > 0, \quad \text{since } \alpha|_S \text{ is a non-zero nef class.}
\]

Therefore we have

\[
(4.7) \quad (K_X + B) \cdot \alpha \cdot S < 0.
\]

By a similar computation we also have

\[
(4.8) \quad (K_X + B + \eta) \cdot \alpha \cdot S < 0.
\]

In particular, \((K_X + B)|_S \) (resp. \((K_X + B + \eta)|_S \)) is not pseudo-effective, since \( \alpha|_S \) is a nef class.

Let \( \pi : \hat{S} \to S \) be the minimal resolution of \( S \) dominating the normalization of \( S \). We make the following claim.

**Claim 4.15.** There is an effective \( \mathbb{Q} \)-divisor \( E \geq 0 \) on \( \hat{S} \) such that

\[
(K_{\hat{S}} + E) \cdot \pi^*(\alpha|_S) < 0.
\]

Note that once we have this claim, the rest of the proof works exactly as in the proof of [CHP16, Proposition 4.4], since the only thing used there is this property of the nef class \( \pi^*(\alpha|_S) \), and it does not depend on whether \( K_X + B \) is pseudo-effective or not. In the following we will prove our claim.

**Proof of Claim 4.15.** We will split the proof into two cases.

**Case I:** Assume that \( B \cdot \alpha \cdot S = B|_S \cdot \alpha|_S < 0 \). Then \( S \) is contained in the support of \( B \), since \( \alpha|_S \) is nef. Then there exists a real number \( \lambda \geq 0 \) such that the coefficient of \( S \) in \((1 + \lambda)B\) is 1. Then using \((4.7)\) we have

\[
(K_X + (1 + \lambda)B) \cdot \alpha \cdot S \leq (K_X + B) \cdot \alpha \cdot S < 0.
\]

Thus by adjunction (see Lemma 2.41), there exists an effective \( \mathbb{Q} \)-divisor \( E \) on \( \hat{S} \) such that

\[
(K_{\hat{S}} + E) \cdot \pi^*(\alpha|_S) = \pi^*((K_X + (1 + \lambda)B)|_S) \cdot \pi^*(\alpha|_S) = (K_X + (1 + \lambda)B) \cdot \alpha \cdot S < 0.
\]

This proves our claim in this case.
Case II: Assume that \( B \cdot \alpha \cdot S \geq 0 \). Then we have

\[
K_X \cdot \alpha \cdot S \leq (K_X + B) \cdot \alpha \cdot S < 0.
\]

Now consider the Zariski decomposition of \( K_X + B + \eta \):

\[
K_X + B + \eta \equiv \sum_{j=1}^{r} \lambda_j S_j + P,
\]

where \( \lambda_j \geq 0 \) for all \( j \) and \( P \) is nef in codimension 1.

We claim that \( S = S_j \) for some \( j \). If not, then \( \sum \lambda_j S_j|_S + P|_S \) is pseudo-effective. But then from (4.8) we have

\[
0 > (K_X+B+\eta)\cdot\alpha \cdot S = \left( \sum \lambda_j S_j|_S \right) \cdot \alpha|_S + P|_S \cdot \alpha|_S \geq 0, \quad \text{a contradiction.}
\]

Next we claim that \( \alpha \cdot S^2 < 0 \). To see this first assume that \( S = S_1 \). Then we have

\[
0 > (K_X + B + \eta) \cdot \alpha \cdot S = \left( \sum_{j=2}^{r} \lambda_j S_j|_S \cdot \alpha|_S \right) + \lambda_1 \alpha \cdot S^2 + P|_S \cdot \alpha|_S.
\]

Since the first and the last term on the right hand side are non-negative, it follows that \( \alpha \cdot S^2 < 0 \). Combining this with (4.9) we have \( (K_X + S) \cdot \alpha \cdot S < 0 \). Then by adjunction (see Lemma 2.41) there exists an effective \( \mathbb{Q} \)-divisor \( E \) on \( \hat{S} \) such that

\[
(K_{\hat{S}} + E) \cdot \pi^*(\alpha|_S) = \pi^*((K_X + S)|_S) \cdot \pi^*(\alpha|_S) = (K_X + S) \cdot \alpha \cdot S < 0.
\]

This completes the proof of the claim.

4.2. Existence of divisorial contractions and flips.

**Theorem 4.16.** Let \( (X, B) \) be a dlt pair, where \( X \) is a strongly \( \mathbb{Q} \)-factorial compact Kähler 3-fold. Suppose that \( X \) is uniruled and the dimension of the base of the MRC fibration \( X \to Z \) is 2, and \( (K_X + B) \cdot F < 0 \) for a general fiber \( F \) of \( X \to Z \). Let \( \omega \) be a Kähler class (or more generally a nef and big class) such that \( K_X + B + \omega \) is pseudo-effective, and \( R \) a \( (K_X + B + \omega) \)-negative extremal ray. Then the contraction \( c_R : X \to Y \) of the ray \( R \) exists. Moreover, if \( c_R \) is a small contraction, then the flip of \( c_R \) exists.

Note that we will only use the strongly \( \mathbb{Q} \)-factorial hypothesis for the case of contractions of divisors to curves (i.e. \( n(\alpha) = 1 \)).
Proof. Since $R$ is $(K_X + B + \omega)$-negative, for any Kähler class $\omega'$, there is a number $0 < \varepsilon \ll 1$ such that $R$ is also $(K_X + B + \omega + \varepsilon \omega')$-negative. Since $\omega$ is nef, $\omega + \varepsilon \omega'$ is Kähler, and hence $\omega + \varepsilon \omega' + \delta B$ is Kähler for $0 < \delta \ll \varepsilon$. Thus replacing $B$ by $(1 - \delta)B$ and $\omega$ by $\omega + \varepsilon \omega' + \delta B$ we may assume that $(X, B)$ is klt and $\omega$ is a Kähler class. Note that $K_X + B + \omega$ is pseudo-effective and $(K_X + B + \omega) \cdot R < 0$. By a similar argument as in the proof of [CHP16, Proposition 4.3] (using Theorem 4.6 and Corollary 4.10) we may assume that the extremal ray $R$ is cut out by a nef $(1, 1)$-class $\alpha$. Re-scaling $\alpha$ if necessary, we see that $\eta := \alpha - (K_X + B + \omega)$ is positive on $\overline{\NA}(X) \setminus \{0\}$, and hence $\eta$ is a Kähler class by [HP16, Corollary 3.16]. Thus it follows that $\alpha$ is a nef and big class, since $K_X + B + \omega$ is pseudo-effective.

Suppose that $R$ is small, then the contraction $c_R$ exists by [CHP16, Theorem 4.2], see also [DH23, Theorem 4.12]. Note that even though [CHP16, Theorem 4.2] is stated for non-uniruled varieties, its proof works in our case if we replace [CHP16, Proposition 4.4] by Proposition 4.14. Since $\omega \cdot R > 0$ (as $\omega$ is Kähler), we have $(K_X + B) \cdot R < 0$ and so this is also a $(K_X + B)$-flipping contraction. The existence of the flip then follows from Theorem 2.24.

If $R$ is of divisorial type, then the corresponding irreducible divisor $S$ is covered by and contains all the curves $C \subset X$ such that $[C] \in R$. Recall from Notation 1.4 that $\nu : \tilde{S} \to S$ is the normalization and $f : \tilde{S} \to T$ is an $\alpha$-trivial fibration with $\dim T = n(\alpha) \in \{0, 1\}$.

If $n(\alpha) = 0$, then the existence of $c_R$ follows by the same arguments as in [HP16, Corollary 7.7], also see [DH23, Corollary 3.4]. Note in fact that in this case it suffices to show that if $mS$ is Cartier, then $(-mS)|_S$ is ample. This in turn is a consequence of the fact that $-mS \cdot R > 0$.

If $n(\alpha) = 1$, the existence of the corresponding divisorial contraction will follow from the proof of Theorem 3.1. Note that, the only place in the proof of Theorem 3.1 where we used that $K_X + B$ is pseudo-effective, is in the proof of Claim 3.2. We will now explain how to run the corresponding minimal model program in the non-pseudo-effective case. We will adopt the notation of Claim 3.2.

In the first part of the proof of this claim we must show that we can perform the required flips and divisorial contractions $X'_i \dashrightarrow X'_{i+1}$. Consider an $\alpha'_i$-trivial, $(K_{X'_i} + \Delta'_i)$-negative extremal ray $R_i$ such that $P \cdot R_i < 0$ for some component $P$ of $[\Delta'_i]$. By adjunction, $K_P + \Delta_P = (K_{X'_i} + \Delta'_i)|_P$ induces a dlt surface pair $(P, \Delta_P)$ such that $\alpha_P := \alpha'_i|_P$ is nef. Note however that, as $P \cdot R_i < 0$, all curves $[C] \in R_i$ are contained in $P$ and hence $\alpha_P$ is not Kähler. Consider the induced linear map $\iota : \overline{\NA}(P) \to \overline{\NA}(X'_i)$ and let $F = \iota^{-1}(R_i)$, then $F$ is a $(K_P + \Delta_P)$-negative extremal face of $\overline{\NA}(P)$. By the surface cone theorem and contraction theorem (see [DO24, Theorem 1.31] and [DHY23, Lem. 2.30 and Cor. 2.32(2)]), there is a corresponding contraction.
morphism $\gamma : P \to W$. By [HP16, Proposition 7.4] (see also [DH23, Theorem 4.7]), there exists a bimeromorphic morphism $g : X'_i \to Z$ such that $g|_P = \gamma$ and $g|_{X'_i \setminus P}$ is an isomorphism onto $Z \setminus W$. If $g$ is a flipping contraction, then the flip exists by Theorem 2.24 and termination follows from Theorem 2.25.

We may therefore assume that we have constructed $X'_n$ so that there are no $\alpha'_n$-trivial $(K_{X'_n} + \Delta'_n)$-negative extremal rays and we must conclude that there are no $\alpha'_n$-trivial, $(K_{X'_n} + \Delta'_n)$-negative curves. This follows from the cone theorem proved in Theorem 4.6 and Corollary 4.10 as we will now explain. Recall that $\omega'_n$ is modified Kähler and $\alpha'_n = K_{X'_n} + \Delta'_n + \omega'_n$ is nef and big. Choose $0 < \varepsilon \ll 1$ so that $K_{X'_n} + \Delta'_n + (1 - \varepsilon)\omega'_n$ is big, in particular, $(K_{X'_n} + \Delta'_n + (1 - \varepsilon)\omega'_n) \cdot X_z > 0$. Thus by Corollary 4.10 it follows that there exist finitely many curves $\{\Gamma_i\}_{i=1}^N$ such that

$$\mathcal{NA}(X'_n) = \mathcal{NA}(X'_n)_{(K_{X'_n} + \Delta'_n + (1 - \varepsilon)\omega'_n) \geq 0} + \sum_{i=1}^N \mathbb{R}^+[\Gamma_i].$$

Suppose that $\Sigma$ is an $\alpha'_n$-trivial $(K_{X'_n} + \Delta'_n)$-negative curve, then $\Sigma = \Sigma_1 + \Sigma_2$, where $\Sigma_i \in \mathcal{NA}(X)_{(K_{X'_n} + \Delta'_n + (1 - \varepsilon)\omega'_n) \geq 0}$ and $\Sigma_2 \in \sum_{i=1}^N \mathbb{R}^+[\Gamma_i]$. Since $\alpha'_n \cdot \Sigma = 0$ and $\alpha'_n$ is nef, we have $\alpha'_n \cdot \Sigma_1 = \alpha'_n \cdot \Sigma_2 = 0$. Observe that the rays $\mathbb{R}^+[\Gamma_i], i = 1, 2, \ldots, n$ are $(K_{X'_n} + \Delta'_n)$-negative. Then by our assumption above $\alpha'_n \cdot \Gamma_i > 0$ for all $i \in \{1, 2, \ldots, n\}$, and hence $\Sigma_2 = 0$, in particular, $\Sigma \in \mathcal{NA}(X)_{(K_{X'_n} + \Delta'_n + (1 - \varepsilon)\omega'_n) \geq 0}$. Since $\Sigma$ is $\alpha'_n$-trivial, it follows that $(K_{X'_n} + \Delta'_n) \cdot \Sigma \geq 0$, this is a contradiction to the fact that $(K_{X'_n} + \Delta'_n) \cdot \Sigma < 0$. This proves our claim.

\[\Box\]

5. Towards the Existence of Mori fiber spaces

In this section we will prove the key technical results needed for the proof of Theorem 1.2 in the next section. We start with the following log version of [HP15, Theorem 1.3].

**Theorem 5.1.** Let $(X, B)$ be a dlt pair, where $X$ is a strongly-$\mathbb{Q}$-factorial compact Kähler 3-fold. Suppose that $X$ is uniruled and the base of the MRC fibration $f : X \dasharrow Z$ has dimension 2 and $(K_X + B) \cdot F < 0$, where $F$ is a general fiber of $f$. Then there is a bimeromorphic map $\phi : X \dasharrow X'$ given by a sequence of $(K_X + B)$-flips and divisorial contractions such that for any $(K_{X'} + B')$-normalized Kähler class $\omega'$ on $X'$ (see Definition 4.1), the adjoint class $K_{X'} + B' + \omega'$ is nef, where $B' = \phi_* B$.

**Proof.** Suppose that there is a $(K_X + B)$-normalized Kähler class $\omega$ on $X$, such that the adjoint class $K_X + B + \omega$ is not nef. Replacing $B$ by $(1 - \varepsilon)B$ and $\omega$ by $\omega + \varepsilon \omega$ for $0 < \varepsilon \ll 1$, we may assume that $(X, B)$ is klt. By Theorem 4.6, there
is a \((K_X + B + \omega)\)-negative extremal ray \(R\). Note that \(K_X + B + \omega\) is pseudo-effective by Lemma 4.2, and thus by Theorem 4.16 we may flip or contract \(R\). Repeating this procedure, since every step is \((K_X + B)\)-negative, by Theorem 2.25 we obtain the required bimeromorphic map \(\phi : X \dasharrow X'\).

Following the same ideas as in [HP15] we prove a log version of [HP15, Theorem 1.4] below.

**Theorem 5.2.** Let \((X, B)\) be a terminal pair, where \(X\) is a \(\mathbb{Q}\)-factorial compact Kähler 3-fold. Suppose that \(X\) is uniruled and the base of the MRC-fibration \(f : X \dasharrow Z\) has dimension 2. Let \(\omega\) be a nef and big class on \(X\) such that \(K_X + B + \omega\) is nef and \((K_X + B + \omega) \cdot F = 0\), where \(F\) is a general fiber of \(f\). Then there exists a proper surjective morphism with connected fibers \(\varphi : X \rightarrow S\) onto a normal compact Kähler surface \(S\) such that \(K_X + B + \omega\) is \(\varphi\)-trivial, i.e. \((K_X + B + \omega)|_{X_s} \equiv 0\) for all \(s \in S\).

**Proof.** We will closely follow the proof of [HP15, Theorem 1.4] here. We will consider the nef dimension \(\nu_{\text{net}}(K_X + B + \omega)\). First note that, since a dense open subset of \(X\) is covered by \((K_X + B + \omega)\)-trivial curves, the nef dimension \(\nu_{\text{net}}(K_X + B + \omega) \leq 2\). We claim that \(\nu_{\text{net}}(K_X + B + \omega) = 2\). Indeed, if the nef dimension is 0, then \(-(K_X + B) \equiv \omega\), and thus \(-(K_X + B)\) is nef and big. In particular, \(X\) is a Moishezon space. Since \(X\) is also Kähler and has rational singularities, by Theorem 2.13 \(X\) is projective. Then by [Zha06, Theorem 1] \(X\) is rationally connected. This contradicts the fact that the base of the MRC fibration of \(X\) has dimension 2. If the nef dimension is 1, then there is a proper surjective morphism \(f : X \rightarrow C\) to a smooth projective curve \(C\) such that \((K_X + B + \omega)|_F \equiv 0\) for all general fibers \(F\) of \(f\) (see [BCE+02, 2.4.4]).

Now since \(X\) has terminal singularities, a general fiber \(F\) is smooth, and thus by adjunction we have \((K_X + B)|_F = K_F + B|_F\) such that \((F, B|_F \geq 0)\) has klt singularities. Moreover, from Lemma 5.3 we see that \(-(K_F + B|_F) \equiv \omega|_F\) is nef and big. Therefore by [Zha06, Theorem 1] the general fiber of \(f\) is rationally connected. This contradicts the fact that the base of the MRC fibration of \(X\) has dimension 2. Thus \(\nu_{\text{net}}(K_X + B + \omega) = 2\) as claimed.

There is an induced rational map \(Z \dasharrow \text{Chow}(X)\) sending the general points of \(Z\) to the points corresponding to the general fibers of the MRC fibration \(X \dasharrow Z\). Replace \(Z\) by an appropriate resolution so that \(Z \rightarrow \text{Chow}(X)\) is a morphism, and let \(\Gamma \rightarrow Z\) be the normalization of the pull-back of the universal family over \(\text{Chow}(X)\). Then \(\Gamma\) is a normal compact complex 3-fold with equi-dimensional fibers of dimension 1 over \(Z\). Let \(p : \Gamma \rightarrow X\) and \(q : \Gamma \rightarrow Z\) be the projection maps. Note that as observed in the proof of [HP15, Theorem 1.4], \(\Gamma\) is in Fujiki’s class \(C\) and hence by [Var86, Theorem 3], \(Z\) is in Fujiki’s class \(C\). Moreover, since \(Z\) is a smooth compact surface, by [Fuj83, Remark 1.1, page 236] \(Z\) is Kähler.
We claim that there is a nef and big $(1,1)$-class $\alpha$ on $Z$ such that
\begin{equation}
(5.1) \quad p^*(K_X + B + \omega) = q^*\alpha.
\end{equation}

We split our proof into 4 steps below as in the proof of [HP15, Theorem 1.4].

**Step 1:** This step shows the existence of a nef $(1,1)$-class $\alpha$ on $Z$ satisfying (5.1). The proof of this step is exactly same as the proof of Step 1 of [HP15, Theorem 1.4]. However, for the convenience of the reader, we include the details here. Since the general fiber of $q$ is a rational curve, by [Kol96, II, 2.8.6.2], it follows that $R^1q_*\mathcal{O}_\Gamma = 0$. Consider the exponential sequence
\[ 0 \to \mathcal{O}_Z \to \mathcal{O}_\Gamma \to \mathcal{O}_\Gamma^* \to 0. \]
Since $\mathcal{O}_Z = q_*\mathcal{O}_\Gamma \to q_*\mathcal{O}_\Gamma^* = \mathcal{O}_\Gamma^*$ is surjective, we have $R^1q_*\mathcal{O}_\Gamma = 0$. By the universal coefficient theorem $R^1q_*\mathbb{R} = 0$. We now consider the Leray spectral sequence $E^{i,j}_2 = H^i(Z, R^jq_*\mathbb{R})$ degenerating to $H^*(\Gamma, \mathbb{R})$. Since $E^{i,1}_2 = 0$ for all $i$, it follows that there is an exact sequence
\[ 0 \to H^2(Z, \mathbb{R}) \to H^2(\Gamma, \mathbb{R}) \to H^0(Z, R^2q_*\mathbb{R}). \]
To show that $[p^*(K_X + B + \omega)] = [q^*\alpha]$ it then suffices to show that if $s \in E^{0,2}_0 = H^0(Z, R^2q_*\mathbb{R})$ is the section defined by $s(z) = [p^*(K_X + B + \omega)]_{|z} \in H^2(\Gamma, \mathbb{R})$, then $s = 0$. Since $R^2q_*\mathbb{R}$ is constructible (see [EZS10, Proposition 3.5]), then the section $s$ vanishes if and only if it vanishes pointwisely. Note that since $\omega$ is a $(K_X + B)$-normalized Kähler class, the claim clearly holds for general $z \in Z$. Since $\Gamma \to Z$ has equi-dimensional fibers and $p^*(K_X + B + \omega)$ is nef, it follows that $p^*(K_X + B + \omega) \equiv 0$ on every irreducible component of every fiber. Note that since $q^*\alpha$ is nef, $\alpha$ is also nef by [Pau98, Theorem 1].

**Step 2:** With the notation as in the Step 2 of the proof of [HP15, Theorem 1.4] let $\mu : \hat{X} \to \Gamma$ be a resolution of singularities of $\Gamma$ such that the exceptional locus of $\hat{p} = p \circ \mu : \hat{X} \to X$ has pure codimension 1. Set $\hat{q} = q \circ \mu$. Then we have
\begin{equation}
(5.2) \quad \hat{q}^*\alpha \cdot \hat{D} = 0 \quad \text{in } N_1(\hat{X})
\end{equation}
for every irreducible component $\hat{D}$ of the exceptional locus of $\hat{p}$. Since $X$ has $\mathbb{Q}$-factorial terminal singularities, the proof of this step is also the same proof of Step 2 of [HP15, Theorem 1.4], so we skip the details here.

Next we claim that $\alpha$ is a big class on $Z$, i.e. $\alpha^2 > 0$. This is Step 3 below.

**Step 3:** The proof of this step is almost identical to the proof of Step 3 of [HP15, Theorem 1.4]. We include the details here for the convenience of the reader. Note that using Lemma 2.38 from now on we may assume that $\omega$ is a modified Kähler class; observe that we loose the nefness of $\omega$ here, but
it is not needed in the rest of the proof. Then by Lemma 2.35, replacing \( \hat{X} \) by a higher resolution of \( \Gamma \) if necessary, we may assume that there is an effective \( \hat{p} \)-exceptional \( \mathbb{R} \)-divisor \( F \geq 0 \) on \( \hat{X} \) such that \( \hat{p}^* \omega - F \) is a Kähler class. Since the Kähler cone is open, there is a Kähler class \( \eta_Z \) on \( Z \) such that \( \hat{p}^* \omega - F - \hat{q}^* \eta_Z \) is a Kähler class on \( \hat{X} \). Then by a similar argument as in the proof of Lemma 4.2 it follows that

\[
(5.3) \quad K_{\hat{X}/Z} + \hat{B} + \hat{p}^* \omega - F - \hat{q}^* \eta_Z \quad \text{is pseudo-effective,}
\]

where \( K_{\hat{X}} + \hat{B} = \hat{p}^*(K_X + B) + E, \hat{B} \geq 0, \hat{E} \geq 0, \hat{p}_* \hat{B} = B, \) and \( \hat{B} \) and \( E \) do not share any common component.

Now we have

\[
(5.4) \quad \hat{p}^*(K_X + B + \omega) = (K_{\hat{X}/Z} + \hat{B} + \hat{p}^* \omega - F - \hat{q}^* \eta_Z) - E + F + \hat{q}^* K_Z + \hat{q}^* \eta_Z.
\]

We will show that \( \hat{q}^* \alpha^2 = \hat{q}^* \alpha \cdot \hat{p}^*(K_X + B + \omega) \) is a non-zero class in \( \overline{NA}(\hat{X}) \). To that end first observe that, since \( \alpha \) is nef and \( K_{\hat{X}/Z} + \hat{B} + \hat{p}^* \omega - F - \hat{q}^* \eta_Z \) is pseudo-effective, the intersection product \( \hat{q}^* \alpha \cdot (K_{\hat{X}/Z} + \hat{B} + \hat{p}^* \omega - F - \hat{q}^* \eta_Z) \) is an element of \( \overline{NA}(\hat{X}) \). Since \( E \) and \( F \) are both \( \hat{p} \)-exceptional, from (5.2) it follows that \( \hat{q}^* \alpha \cdot (-E + F) = 0 \) in \( \overline{NA}(\hat{X}) \). Now since the surface \( Z \) is not uniruled, by classification \( K_Z \) is pseudo-effective; in particular, \( \hat{q}^* \alpha \cdot \hat{q}^* K_Z \) is an element of \( \overline{NA}(\hat{X}) \). Now recall that \( \alpha \neq 0 \), since \( K_X + B + \omega \neq 0 \). Since \( \eta_Z \) is a Kähler class and \( \alpha \) is a non-zero nef class, the Hodge index theorem yields \( \eta_Z \cdot \alpha > 0 \) (see Lemma 2.39). In particular, \( \hat{q}^* \alpha \cdot \hat{q}^* \eta_Z \) is a non-zero element of \( \overline{NA}(\hat{X}) \). Therefore \( \hat{q}^* \alpha^2 = \hat{q}^* \alpha \cdot \hat{p}^*(K_X + B + \omega) \) is a non-zero element of \( \overline{NA}(\hat{X}) \), and thus \( \alpha^2 \neq 0 \) in \( N^1(Z) \).

Step 4: Finally, Step 4 of the proof of [HP15, Theorem 1.4] shows the existence of a fibration \( \varphi : X \rightarrow S \) such that \( (K_X + B + \omega)|_{X_s} \equiv 0 \) for \( s \in S \). This step works here without any change, and completes our proof.

\[\square\]

**Lemma 5.3.** Let \( X \) be a normal compact Kähler variety and \( f : X \rightarrow C \) a proper surjective morphism to a smooth projective curve \( C \). Let \( \omega \in H^{1,1}_{BC}(X) \) be a nef and big class. Then the restriction \( \omega|_F \) is nef and big for general fibers \( F \) of \( f \).

**Proof.** The proof follows immediately from the definition of nef and big class.

\[\square\]

**Corollary 5.4.** Let \( (Y, B_Y) \) be a klt pair, where \( Y \) is a \( Q \)-factorial compact Kähler 3-fold. Suppose that \( Y \) is uniruled and the base of the MRC-fibration \( g : Y \rightarrow Z' \) has dimension 2. Let \( \omega_Y \) be a nef and big class on \( Y \) such that \( K_Y + B_Y + \omega_Y \) is nef and \( (K_Y + B_Y + \omega_Y) \cdot F = 0 \), where \( F \) is a general fiber
Then there exists a proper surjective morphism with connected fibers \( \psi : Y \to S \) onto a normal compact Kähler surface \( S \) such that \( K_Y + B_Y + \omega_Y \) is \( \psi \)-trivial, i.e. \( (K_Y + B_Y + \omega_Y)|_{Y_s} \equiv 0 \) for all \( s \in S \).

**Proof.** Let \( h : X \to Y \) be a terminalization of the pair \( (Y, B_Y) \) such that \( X \) is strongly \( \mathbb{Q} \)-factorial (cf. Lemma 2.27). Set \( K_X + B := h^*(K_Y + B_Y) \) and \( \omega = h^*\omega_Y \). Then we have

\[
K_X + B + \omega = h^*(K_Y + B_Y + \omega_Y).
\]

Note that since \( \dim Z' = 2 \), then general fibers of \( g : Y \to Z' \) and \( g \circ h : X \to Z' \) are isomorphic. In particular, if \( F \) is a general fiber of \( g \circ h : X \to Z' \), then \( (K_X + B + \omega) \cdot F = (K_Y + B_Y + \omega_Y) \cdot F = 0 \). Thus by Theorem 5.2 there is a proper surjective morphism with connected fibers \( \varphi : X \to S \) to a Kähler surface \( S \) such that \( K_X + B + \omega \) is \( \varphi \)-trivial. With the notations as in the proof of Theorem 5.2 we get the following commutative diagram.

Using the rigidity lemma now we will show that the fibration \( \varphi : X \to S \) factors through \( h : X \to Y \). Note that the fibers of \( h \) are covered by curves, so it is enough to work with the curves contained in the fibers of \( h \). Let \( C \) be a curve in \( X \) contracted by \( h \). Then \( (K_X + B + \omega) \cdot C = h^*(K_Y + B_Y + \omega_Y) \cdot C = 0 \).

Let \( \hat{C} \) be a curve in \( \hat{X} \) such that \( \hat{p}(\hat{C}) = C \). Recall from the proof of Theorem 5.2 that \( \hat{p}^*(K_X + B + \omega) = \hat{q}^*\alpha \) for some nef and big class \( \alpha \) on \( Z \). Thus we have

\[
\hat{q}^*\alpha \cdot \hat{C} = \hat{p}^*(K_X + B + \omega) \cdot \hat{C} = (K_X + B + \omega) \cdot C = 0.
\]

Therefore \( \alpha \cdot \hat{q}_*(\hat{C}) = 0 \). Now recall from the construction of \( \nu : Z \to S \) in Step 4 of the proof of [HP15, Theorem 1.4] that the morphism \( \nu : Z \to S \) contracts exactly the \( \alpha \)-trivial curves. Therefore, either \( \hat{C} \) is contracted by \( \hat{q} \), or its image is contracted by \( \nu \). In particular, from the diagram (5.5) it follows that the curve \( C \subset X \) is contracted by \( \varphi : X \to S \). Therefore, by
the rigidity lemma (see [BS95, Lemma 4.1.13]), there is a proper surjective morphism with connected fibers $\psi : Y \to S$ such that $\varphi = \psi \circ h$. It is then clear that $K_Y + B_Y + \omega_Y$ is $\psi$-trivial.

□

The following theorem is an application of Corollary 5.4 and also a generalization of [TZ18, Theorem 2.7]. This establishes Theorem 1.7 when $K_X + B$ is not pseudo-effective.

**Theorem 5.5.** Let $(X, B)$ be a dlt pair, where $X$ is a $\mathbb{Q}$-factorial compact Kähler 3-fold. Let $\omega$ be a nef and big class on $X$ such that $\alpha := K_X + B + \omega$ is nef but not big. If either $(X, B)$ is a klt pair or $\omega$ is Kähler, then there exists a proper surjective morphism with connected fibers $\psi : X \to Y$ onto a normal compact Kähler variety $Y$ with rational singularities and a Kähler class $\alpha_Y \in H^{1,1}_{BC}(Y)$ such that $\alpha = \psi^* \alpha_Y$.

**Proof.** Note that if $(X, B)$ is dlt and $\omega$ is Kähler, then replacing $B$ by $(1 - \epsilon)B$ and $\omega$ by $\omega + \epsilon B$, we may assume that $(X, B)$ is klt. We will closely follow the arguments in [TZ18, Theorem 2.7]. First observe that, since $\omega$ is a big class and $\alpha$ is not big, $K_X + B$ is not pseudo-effective; in particular $K_X$ is not pseudo-effective. Therefore $X$ is uniruled. Let $X \dashrightarrow T$ be the MRC fibration of $X$. If $\dim T \leq 1$, then from Lemma 2.42 and its proof it follows that $X$ is projective and $H^2(X, \mathcal{O}_X) = 0$. In particular, $\alpha$ is an $\mathbb{R}$-divisor in this case, and our result follows from the well known base-point free theorem. So from now on we assume that $\dim T = 2$. Now we claim that $\omega$ is a $(K_X + B)$-normalized nef and big class, i.e. $(K_X + B + \omega) \cdot F = 0$ for general fibers $F$ of $X \dashrightarrow T$. If not, then by Lemma 4.4 there exists a $0 < \mu < 1$ such that $(K_X + B + \mu \omega) \cdot F = 0$, where $F \equiv \mathbb{P}^1$ is a general fiber of the MRC fibration $X \dashrightarrow T$. Then by Corollary 4.3, $K_X + B + \mu \omega$ is pseudo-effective. Thus we have $\alpha = (K_X + B + \mu \omega) + (1 - \mu) \omega$ is a big class, a contradiction.

Now by Corollary 5.4 there exists a proper surjective morphism $f : X \to Y$ to a normal compact Kähler surface $Y$ such that $(K_X + B + \omega)|_y \equiv 0$ for all $y \in Y$. Also, as in the proof of Theorem 5.2, we get the following commutative diagram:

$$
\begin{array}{ccc}
\Gamma & \xrightarrow{q} & Z \\
\downarrow p & & \downarrow \nu \\
X & \xrightarrow{f} & Y 
\end{array}
$$

where $Z$ is a smooth compact Kähler surface, $\Gamma$ is a normal 3-fold in Fujiki’s class $\mathcal{C}$, and $p$ and $\nu$ are bimeromorphic.
Replacing $\Gamma$ by a resolution we may further assume that $\Gamma$ is a compact Kähler 3-fold (see Definition 2.2.(i)). As in the proof of Corollary 5.4 and Theorem 5.2 we have $p^*\alpha = q^*\beta$ for some nef and big $(1,1)$-class $\beta$ on $Z$. We claim that $\beta$ is pullback of a Kähler class from $Y$. To see this, first recall that from the Step 4 of the proof of [HP15, Theorem 1.4] it follows that $\text{Ex}(\nu) = \text{Null}(\beta)$ (see Definition 6.1 for $\text{Null}(\beta)$). Moreover, by Lemma 2.44, $Y$ has rational singularities. Thus by Lemma 2.11 there is a $(1,1)$-class $\gamma \in H^1_{\text{BC}}(Y)$ such that $\beta = \nu^*\gamma$. Then $\gamma$ is nef and big (see [Pau98, Theorem 1]), and by the projection formula we have $\gamma^2 = (\nu^*\gamma)^2 = \beta^2 > 0$. Moreover, if $C \subset Y$ is a curve and $C' \subset Z$ its strict transform, then $\beta \cdot C' > 0$, since $\text{Null}(\beta) = \text{Ex}(\nu)$ and $C'$ is not contained $\text{Ex}(\nu)$. Then again by the projection formula we have $\gamma \cdot C = \nu^*\gamma \cdot C' = \beta \cdot C' > 0$. Finally, since $Y$ is a normal surface, it has only finitely many (rational) singular points. Therefore by [Hör21, Lemma 2.1], $\gamma$ is a Kähler class on $Y$. This proves the claim.

Now from the commutativity of the diagram (5.6) we have $p^*(f^*\gamma - \alpha) = 0$. Since $p^* : H^{1,1}_{\text{BC}}(X) \to H^{1,1}_{\text{BC}}(\Gamma)$ is injective, we have $f^*\gamma - \alpha = 0$, i.e. $K_X + B + \omega = f^*\gamma$, where $\gamma$ is a Kähler class on $Y$. This completes the proof.

6. Main Theorems

In this section we prove all the main theorems in full generality. The key technical result of this section is the proof of Theorem 1.7 for $X$ strongly $\mathbb{Q}$-factorial and $K_X + B$ pseud-effective, see Theorem 6.4. We start with the following definition.

**Definition 6.1.** Let $X$ be a normal compact Kähler variety and $\alpha \in H^{1,1}_{\text{BC}}(X)$ a nef and big class. Then we define the null locus of $\alpha$ as follows:

\[
\text{Null}(\alpha) = \bigcup_{Z \subset X, \dim Z > 0, \alpha \cdot Z = 0} Z.
\]

A priori we do not know whether $\text{Null}(\alpha)$ is a closed analytic subset of $X$ or not.

**Proposition 6.2.** Let $X$ be a normal $\mathbb{Q}$-factorial compact Kähler 3-fold with klt singularities. Let $\alpha$ be a nef and big $(1,1)$-class on $X$. Assume that $\text{Null}(\alpha)$ consists of only finitely many curves of $X$. Then there exists a proper bimeromorphic morphism $\mu : X \to Z$ onto a normal analytic variety $Z$ such that every connected component of $\text{Null}(\alpha)$ is contracted to a point and $X \setminus \text{Null}(\alpha) \cong Z \setminus \mu(\text{Null}(\alpha))$.

**Proof.** The proof of [CHP16, Theorem 4.2] holds here without any change. We note that [CHP16, Theorem 4.2] is stated only for non-uniruled varieties,
however this hypothesis is only used to show that \( \text{Null}(\alpha) \) consists of finitely many curves (see [CHP16, Proposition 4.4]). This is however part of our hypothesis.

The following lemma provides a sufficient criteria in dimension 3 for a nef and big class to be Kähler. The same result in arbitrary dimension is established in [DHP24, Theorem 2.29].

**Lemma 6.3.** Let \( X \) be a normal compact analytic variety of dimension 3. Let \( \alpha \in H^{1,1}_{BC}(X) \) be a nef and big class such that \( \alpha^\dim V \cdot V > 0 \) for every positive dimensional subvariety \( V \subset X \). Then \( \alpha \) is a Kähler class.

**Proof.** Let \( f : Y \to X \) be a resolution of singularities of \( X \). Then \( f^*\alpha \) is a nef and big class on \( Y \), and by the projection formula and [CT15, Theorem 1.1] it follows that \( \text{EnK}(f^*\alpha) = \text{Null}(f^*\alpha) = \text{Ex}(f) \), where \( \text{EnK}(\cdot) \) denotes the non-Kähler locus, see [Bou04, Definition 3.16]. Next, by Demailly’s regularization theorem [Dem92] and [Bou04, Theorem 3.17(ii)], there exists a Kähler current \( T \) with analytic singularities contained in the class \( f^*\alpha \) such that \( T \) is singular (i.e. not a smooth form) precisely along the exceptional locus of \( f \). Therefore the current \( f_*T \) (contained in the class \( \alpha \)) is a Kähler current which is singular along the closed set \( f(\text{Ex}(f)) \). Since \( \dim X = 3 \), we have that \( \dim f(\text{Ex}(f)) \leq 1 \). Now, since by construction \( f_*T \) is a smooth form on the open set \( X \setminus f(\text{Ex}(f)) \), the Lelong numbers \( \nu(f_*T, x) = 0 \) for all \( x \in X \setminus f(\text{Ex}(f)) \). Thus for any positive real number \( c > 0 \), the Lelong sub-level sets \( E_c(f_*T) := \{ x \in X \ | \ \nu(f_*T, x) \geq c \} \) are contained in \( f(\text{Ex}(f)) \). By a theorem of Siu [Siu74], we know that \( E_c(f_*T) \) is a closed analytic subset of \( X \) for all \( c > 0 \). Therefore every irreducible component of \( E_c(f_*T) \) is either a projective curve or a point contained in \( f(\text{Ex}(f)) \) for all \( c > 0 \). Let \( C \subset f(\text{Ex}(f)) \) be an irreducible projective curve. Then first observe that if \( \nu : \hat{C} \to C \) is the normalization, then \( \text{NS}(\hat{C})_R = H^{1,1}_{BC}(\hat{C}) \), since \( H^2(\hat{C}, \mathcal{O}_{\hat{C}}) = 0 \). Therefore \( \alpha|_{\hat{C}} \) is a class of an \( \mathbb{R} \)-Cartier nef divisor on \( \hat{C} \) and in fact, it is an ample class, since \( \deg(\alpha|_{\hat{C}}) = \alpha \cdot C > 0 \) by hypothesis. Now pushing forward \( \alpha|_{\hat{C}} \) by \( \nu \), we see that \( \alpha|_C \) is a class of an \( \mathbb{R} \)-Cartier divisor on \( C \). Since \( \nu \) is finite, \( \alpha|_C \) is an ample class on \( C \), hence a Kähler class. Then by [DP04, Proposition 3.3(iii)] it follows that \( \alpha \in H^{1,1}_{BC}(X) \) is a Kähler class on \( X \).

\[ \square \]

The following result is a special case of Theorem 1.7.

**Theorem 6.4.** Let \((X, B)\) be a dlt pair, where \( X \) is a normal strongly \( Q \)-factorial compact Kähler 3-fold. Let \( \omega \) be a nef and big class on \( X \) such that \( \alpha := K_X + B + \omega \) is nef and big. If either \((X, B)\) is klt or \( \omega \) is Kähler, then there
exists a proper bimeromorphic morphism $\psi : X \to Z$ onto a normal compact Kähler 3-fold $Z$ with rational singularities and a Kähler class $\alpha_X \in H^{1,1}_B(Z)$ on $Z$ such that $\alpha = \psi^{\ast}\alpha_Z$.

**Proof.** Note that if $(X, B)$ is dlt and $\omega$ is Kähler, then replacing $B$ by $(1-\epsilon)B$ and $\omega$ by $\omega + \epsilon B$, we may assume that $(X, B)$ is klt. Therefore, in this proof, we will assume that $(X, B)$ is klt and $\omega$ is nef and big. We closely follow the arguments of [Hör21, Theorem 1.3]. For the convenience of the reader, we reproduce these arguments here indicating the necessary changes. The main idea is to construct $\psi$ by running an $\alpha$-trivial MMP and then to contract the remaining $\alpha$-trivial curves. First note that using Lemma 2.38 we may assume that $\omega$ is a modified Kähler class on $X$. Now if $H^2(X, \mathcal{O}_X) = 0$, then $H^{1,1}_B(X) \cong \text{NS}(X)_{\mathbb{R}}$ and $\alpha$ is represented by an $\mathbb{R}$-divisor. Therefore $X$ is a Moishezon space with rational singularities. Then by Theorem 2.13 $X$ is projective, and the result follows from a well known base-point free theorem for projective varieties. So assume that $H^2(X, \mathcal{O}_X) \neq 0$. Then either $X$ is not uniruled and hence $K_X$ is pseudo-effective, or $X$ is uniruled and the base of the MRC fibration $X \dasharrow Z$ has dimension 2 (see Lemma 2.42 and its proof).

**Claim 6.5.** We may run the $\alpha$-trivial $(K_X + B)$-MMP starting with $(X_0, B_0) := (X, B)$, $\omega_0 := \omega$ and $\alpha_0 := \alpha$ and ending with a bimeromorphic map $\phi : X \dasharrow X_n$ such that every $\alpha_n$-trivial curve is $(K_{X_n} + B_n)$-non-negative, where $\alpha_n = \phi_n^{\ast}\alpha$ and $B_n = \phi_n^{\ast}B$.

**Proof of Claim 6.5.** By induction assume that we have already constructed the first $i$ steps

$$\phi_i : X_0 \dasharrow X_1 \dasharrow X_2 \dasharrow \ldots \dasharrow X_i.$$

Note that $(X_i, B_i := \phi_i^{\ast}B)$ is a strongly $\mathbb{Q}$-factorial klt pair (by Lemma 2.5), $\omega_i = \phi_i^{\ast}\omega$ is modified Kähler and $\alpha_i = \phi_i^{\ast}\alpha$ is nef and big. Now suppose that there is an $\alpha_i$-trivial curve $C \subset X_i$ such that $(K_{X_i} + B_i) \cdot C < 0$ (and thus $\omega_i \cdot C > 0$). We will show that there is an $\alpha_i$-trivial flip or divisorial contraction $\mu_i : X_i \dasharrow X_{i+1}$.

If $K_{X_i} + B_i$ is not pseudo-effective (and hence the base of the MRC fibration $X_i \dasharrow Z_i$ has dimension 2), then by Lemma 4.4 it follows that $(K_{X_i} + B_i) \cdot F_i < 0$ for general fibers $F_i$ of $X_i \dasharrow Z_i$.

The existence of $\mu_i$ follows from Theorem 4.6, Corollary 4.10 and Theorem 4.16. To see this, pick $0 < \epsilon \ll 1$ so that $K_{X_i} + B_i + (1-\epsilon)\omega_i$ is big, in particular, $(K_{X_i} + B_i + (1-\epsilon)\omega_i) \cdot F_i > 0$. Then by Corollary 4.10

$$\mathrm{NA}(X_i) = \mathrm{NA}(X_i)(K_{X_i} + B_i + (1-\epsilon)\omega_i)_{\geq 0} + \sum_{j=1}^{N} \mathbb{R}^{\mathbb{Q}}[\Gamma_j],$$
where \((K_X + B_i + (1 - \epsilon)\omega_i) \cdot \Gamma_j < 0\). We decompose \(C = \eta + \sum_{j \in J} \lambda_j \Gamma_j\) accordingly, where \(J = \{1, \ldots, N\}\). Since \(\alpha_i \cdot C = 0\), we may assume that \(\alpha_i \cdot \Gamma_j = 0\) for all \(j \in J\). Since \((K_X + B_i + (1 - \epsilon)\omega_i) \cdot C < 0\), we may assume that \(J \neq \emptyset\) and hence for some \(j_0 \in J\), we have \(\lambda_{j_0} \neq 0\) and \((K_X + B_i + (1 - \epsilon)\omega_i) \cdot \Gamma_{j_0} < 0\). Note that Theorem 4.16 does not immediately apply here since \(\omega_i\) is no longer nef. We will instead argue as follows. Since \(\alpha_i \cdot \Gamma_{j_0} = 0\) and \((K_X + B_i + (1 - \epsilon)\omega_i) \cdot \Gamma_{j_0} < 0\), it follows that \(\omega_i \cdot \Gamma_{j_0} > 0\) and \((K_X + B_i) \cdot \Gamma_{j_0} < 0\). Since \(\Gamma_{j_0}\) generates an extremal ray, it is cut out by a nef class \(\gamma\). Since \(\alpha_i\) is nef and big, replacing \(\gamma\) by \(\gamma + \alpha_i\), we may assume that \(\gamma\) is nef and big.

By standard arguments, \(\eta := t\gamma - (K_X + B_i)\) is Kähler for \(t \gg 0\). Then \(K_X + B_i + \eta = t\gamma\) is nef and big, \(\text{Null}(t\gamma) \cap \overline{\text{NA}}(X) = R_{j_0}\) and \(\alpha_i \cdot R_{j_0} = 0\), where \(R_{j_0} = \mathbb{R}^+[\Gamma_{j_0}]\). Moreover, recall that if \(F\) is a general fiber of the MRC fibration \(X_i \to Z_i\), then \((K_X + B_i) \cdot F < 0\) (by Lemma 4.4). Now choose \(0 < \delta \ll 1\) so that \(K_X + B_i + (1 - \delta)\eta\) is still big (hence pseudo-effective); then \(R_{j_0}\) is a \((K_X + B_i + (1 - \delta)\eta)\)-negative extremal ray. Thus by Theorem 4.16 the contraction \(c_{R_{j_0}} : X_i \to Y\) of \(R_{j_0}\) exists. If \(c_{R_{j_0}}\) is divisorial, then we let \(X_{i+1} = Y\) and \(\mu_i = c_{R_{j_0}}\). If \(c_{R_{j_0}}\) is small, then the flip \(X_i \to X_i^+\) exists by Theorem 2.24. In this case we let \(X_{i+1} = X_i^+\) and \(\mu_i : X_i \to X_{i+1}\) the induced bimeromorphic map.

On the other hand if \(K_X + B_i\) is pseudo-effective, then the existence of \(\mu_i\) follows from Theorems 2.22, 2.23, 2.24 and 3.1. Note that \(\alpha_{i+1} := \mu_{i,*}\alpha_i\) is nef and big by [CHP16, Prp. 3.1, eqn. (5)], and \(\omega_{i+1} := \mu_{i,*}\omega_i\) is a modified Kähler class by Lemma 2.37. This MMP terminates after finitely many steps, say \(\phi : X \to X_n\) by Theorem 2.25. In particular, every \(\alpha_n\)-trivial curve on \(X_n\) is \((K_{X_n} + B_n)\)-non-negative, or equivalently, every \((K_{X_n} + B_n)\)-negative curve is \(\alpha_n\)-positive. \(\square\)

By Lemma 4.5, if \(\text{Null}(\alpha_n)\) contains a surface \(S\), then \(S\) is Moishezon and it is covered by a family of \(\alpha_n\)-trivial curves \(\{C_t\}_{t \in T}\). Since \(\omega_n\) is a modified Kähler class, \(\omega_n|_S\) is big and hence \(\omega_n \cdot C_t = \omega_n|_S \cdot C_t > 0\). But then from \(0 = \alpha_n \cdot C_t = (K_{X_n} + B_n + \omega_n) \cdot C_t\) it follows that \((K_{X_n} + B_n) \cdot C_t < 0\), this is a contradiction. Therefore \(\text{Null}(\alpha_n)\) is a union of curves.

Now let \(f : X' \to X_n\) be a resolution of singularities of \(X_n\). Then \(f^*\alpha_n\) is nef and big, and thus by [CT15, Theorem 1.1], the non-Kähler locus is \(\text{EnK}(f^*\alpha_n) = \text{Null}(f^*\alpha_n)\). If \(S' \subset \text{EnK}(f^*\alpha_n)\) is a divisor, then by the projection formula \(0 = (f^*\alpha_n)^2 \cdot S' = \alpha_n^2 \cdot f_*S'\). Thus \(S'\) is \(f\)-exceptional, since \(\text{Null}(\alpha_n)\) does not contain any surface, as proved above. Moreover, since \(\text{EnK}(f^*\alpha_n)\) is a proper closed analytic subset of \(X'\) (see [CT15, Theorem 2.2]), it follows that \(\text{Null}(f^*\alpha_n)\) is a finite union of curves and \(f\)-exceptional divisors. But since \(\text{Null}(\alpha_n) \subset \text{EnK}(\alpha_n) \subset f(\text{EnK}(f^*\alpha_n))\), it follows that \(\text{Null}(\alpha_n)\) is a
finite union of curves. Then by Proposition 6.2 there exists a proper bimeromorphic morphism \( \mu : X_n \to Z \) onto a normal compact analytic variety \( Z \) contracting each curve in \( \text{Null}(\alpha_n) \) to a point and inducing an isomorphism on the open sets \( X_n \setminus \text{Null}(\alpha_n) \cong Z \setminus \mu(\text{Null}(\alpha_n)) \).

Set \( \psi := \mu \circ \phi_n : X \dashrightarrow Z \). We claim that \( \psi \) is a morphism. To see this, we will use descending induction to show that the induced map \( \psi_i : X_i \dashrightarrow Z \) is a morphism. Note that \( \psi_n \) is a morphism as constructed above. Suppose that we have already shown that \( \psi_i : X_i \to Z \) is a morphism. If \( X_{i-1} \to X_i \) is a divisorial contraction, then clearly \( X_{i-1} \to Z \) is a morphism and we are done by induction on \( i \). If on the other hand \( X_{i-1} \to X_i \) is a flip, then let \( X_i \to Z_{i-1} \) be the flipped contraction. If \( Z_{i-1} \to Z \) is not a morphism, then by the rigidity lemma (see [BS95, Lemma 4.1.13]) there is a flipped curve \( C_i \subset X_i \) such that \( C := \psi_i^* C_i \neq 0 \). Let \( W \) be the normalization of the graph of the induced bimeromorphic map \( \pi_i : X_i \to X_i \), and \( p : W \to X_i \) and \( q : W \to X_n \) are the projections. Now by construction \( \alpha_i \) and \( \alpha_n \) are both nef classes. Moreover, since \( \pi_i : X_i \to X_n \) is a composition of a finite sequence of \( \alpha_i \)-trivial flips and divisorial contractions, by a repeated application of [CHP16, Proposition 3.1, eqn. (5)] it follows that \( p^* \alpha_i = q^* \alpha_n \). Let \( C_W \subset W \) be a curve such that \( p_* C_W = d C_i \) for some \( d > 0 \) and \( C_n := q_* C_W \subset X_n \). Since \( \psi_n (C_n) = C \), then \( \alpha_n \cdot C_n > 0 \). Then we have

\[
0 = \alpha_i \cdot d C_i = p^* \alpha_i \cdot C_W = q^* \alpha_n \cdot C_W = \alpha_n \cdot C_n > 0, \quad \text{a contradiction.}
\]

Therefore \( Z_{i-1} \to Z \) is a morphism, and since \( X_{i-1} \to Z_{i-1} \) is a flipping contraction, \( X_{i-1} \to Z \) is also a morphism. This concludes the proof that \( \psi : X \to Z \) is a morphism.

Next we claim that \( Z \) is a Kähler variety with rational singularities and there exists a Kähler class \( \alpha_Z \in H^{1,1}_{\text{BC}}(Z) \) such that \( \alpha_n = \mu^* \alpha_Z \). To this end, first observe that \( (X, B) \) has \( \mathbb{Q} \)-factorial klt singularities, and \( \alpha = K_X + B + \omega \) and \( \omega \) are both nef and big classes. Now recall that \( \mu : X_n \to Z \) contracts precisely the null locus \( \text{Null}(\alpha_n) \), and since this locus is 1 dimensional, \( \mu \) is \( \alpha_n \)-trivial. Since \( X \to X_n \) is a sequence of \( \alpha \)-trivial flips and divisorial contractions, it follows that \( \psi : X \to Z \) is also \( \alpha \)-trivial, i.e. the restriction of \( \alpha \) to every fiber of \( \psi \) is numerically trivial, so \( \alpha |_{X_z} \equiv 0 \) for all \( z \in Z \). Thus \( -(K_X + B) |_{X_z} \equiv \omega |_{X_z} \) for all \( z \in Z \). In particular, \( -(K_X + B) \) is \( \psi \)-nef, since \( \omega \) is nef by hypothesis. Moreover, since \( \psi \) is a bimeromorphic morphism, \( -(K_X + B) \) is \( \psi \)-big. By Lemma 2.44, \( Z \) has rational singularities.

Now from Definition 2.2.(i) it follows that \( Z \) is in Fujiki’s class \( \mathcal{C} \). Then by Lemma 2.11 there exists a \((1,1)\)-class \( \alpha_Z \in H^{1,1}_{\text{BC}}(Z) \) (represented by a real closed \((1,1)\)-form with local potentials) such that \( \alpha_n = \mu^* \alpha_Z \).

Next we claim that \( \alpha_Z \) is a Kähler class on \( Z \). Indeed, let \( V \subset Z \) be a subvariety of positive dimension and \( V' \) the strict transform of \( V \) under \( \mu \). By
the projection formula we have $(\alpha_Z)^{\dim V} \cdot V = (\alpha_n)^{\dim V'} \cdot V' > 0$, since $V'$ is not contained in Null($\alpha_n$). Then by Lemma 6.3, $\alpha_Z$ is a Kähler class on $Z$.

Finally notice that, since every step of the above MMP is $\alpha$-trivial, from the construction above it follows that $\alpha = \psi^*\alpha_Z$. This completes the proof.

We are now ready to prove our main theorems in full generality. We start with the base-point free Theorem 1.7.

**Proof of Theorem 1.7.** If $(X, B)$ is dlt and $\alpha - (K_X + B)$ is Kähler, then $(X, (1-\epsilon)B)$ is klt and $\alpha - (K_X + (1-\epsilon)B)$ is Kähler for any $0 < \epsilon \ll 1$. Therefore in either case we may assume that $(X, B)$ is klt and $\alpha - (K_X + B)$ is nef and big. Let $\nu : X' \to X$ be the small projective morphism given by Lemma 2.27, then $X'$ is strongly $\mathbb{Q}$-factorial, $K_{X'} + B' = \nu^*(K_X + B)$ is klt, $\alpha' = \nu^*\alpha$ is nef and $\alpha' - (K_{X'} + B')$ is nef and big. By Theorem 5.5 and Theorem 6.4 there exist a proper surjective morphism with connected fibers $\phi : X' \to Z$ to a normal Kähler variety $Z$ with rational singularities and a Kähler class $\alpha_Z \in H^{1,1}_{BC}(Z)$ such that $\alpha' = \phi^*\alpha_Z$. Let $C$ be a $\nu$-exceptional curve, then $\alpha' \cdot C = \alpha_Z \cdot \phi_*C = 0$. Since $\alpha_Z$ is Kähler, then $C$ is contracted by $\psi$. By the rigidity lemma (see [BS95, Lemma 4.1.13]), there is a morphism $\psi : X \to Z$ such that $\phi = \psi \circ \nu$. Thus $\alpha = \nu_*(\alpha') = \nu_*(\nu^*(\phi^*\alpha_Z)) = \psi^*\alpha_Z$; this completes our proof.

**Proof of Theorem 1.5.** This is an immediate corollary of Theorem 1.7.

**Proof of Theorem 1.1.** By Theorem 2.22 the cone theorem holds for $(X, B)$. By Theorem 2.23, Theorem 2.24 and Theorem 1.5 flips and divisorial contractions exist and hence we may run a MMP which terminates by Theorem 2.25.

**Proof of Theorem 1.2.** Since $K_X + B$ is not pseudo-effective, $K_X$ is not pseudo-effective. Thus by [Bru06, Corollary 1.2] applied to a resolution of $X$ it follows that $X$ is uniruled. By Lemma 2.42, we may assume that the dimension of the base of the MRC fibration $X \to Z$ is 2. Let $F$ be a general fiber of the MRC fibration $f : X \to Z$. By Lemma 4.4, if $(K_X + B) \cdot F \geq 0$, then $K_X + B$ is pseudo-effective, contradicting our assumption. Therefore $(K_X + B) \cdot F < 0$.

By Theorem 5.1, there is a $(K_X + B)$-MMP, $\phi : X \to X'$ such that for every $(K_{X'} + B')$-normalized Kähler class $\omega'$ (see Definition 4.1), the class $K_{X'} + B' + \omega'$ is nef, where $B' = \phi_*B$. Note that thanks to Theorem 1.5, Theorem 5.1 and its proof also applies to $\mathbb{Q}$-factorial pairs that are not necessarily strongly $\mathbb{Q}$-factorial.
Since $X'$ is Kähler and $(K_{X'} + B') \cdot F' < 0$, where $F' \cong F$ is a general fiber of the induced MRC fibration $X' \rightarrow Z$, we may pick a $(K_{X'} + B')$-normalized Kähler class, say, $\omega'$. Now choose $0 < \epsilon \ll 1$ so that $\omega' + \epsilon B'$ is a Kähler class. Then $(X', (1 - \epsilon) B')$ is klt, and thus by Corollary 5.4, there exists a holomorphic fibration $\psi : X' \rightarrow S'$ onto a normal compact Kähler surface $S'$ such that $(K_{X'} + B' + \omega')|_{X'_s} \equiv (K_{X'} + (1 - \epsilon) B' + (\omega' + \epsilon B'))|_{X'_s} \equiv 0$ for all $s' \in S'$, i.e. $K_{X'} + B' + \omega'$ is $\psi$-trivial. In particular, $\psi$ is a projective morphism and so the theorem now follows from the usual relative Minimal Model Program for projective morphisms as in Proposition 2.26. □

References

[AHV77] José M. Aroca, Heisuke Hironaka, and José L. Vicente, Desingularization theorems, Memorias de Matemática del Instituto “Jorge Juan” [Mathematical Memoirs of the Jorge Juan Institute], vol. 30, Consejo Superior de Investigaciones Científicas, Madrid, 1977. MR480502

[Ara10] Carolina Araujo, The cone of pseudo-effective divisors of log varieties after Batyrev, Math. Z. 264 (2010), no. 1, 179–193. MR2564937

[BCE+02] Thomas Bauer, Frédéric Campana, Thomas Eckl, Stefan Kebekus, Thomas Peternell, Sławomir Rams, Tomasz Szemberg, and Lorenz Wotzlaw, A reduction map for nef line bundles, Complex geometry (Göttingen, 2000), 2002, pp. 27–36. MR1922095

[BCHM10] Caucher Birkar, Paolo Cascini, Christopher D. Hacon, and James McKernan, Existence of minimal models for varieties of log general type, J. Amer. Math. Soc. 23 (2010), no. 2, 405–468. MR2601039 (2011f:14023)

[BHPVdV04] Wolf P. Barth, Klaus Hulek, Chris A. M. Peters, and Antonius Van de Ven, Compact complex surfaces, Second, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 4, Springer-Verlag, Berlin, 2004. MR2030225

[BM97] Edward Bierstone and Pierre D. Milman, Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant, Invent. Math. 128 (1997), no. 2, 207–302. MR1440306

[Bou02a] Sébastien Boucksom, On the volume of a line bundle, arXiv Mathematics e-prints (January 2002), math/0201031v1, available at math/0201031v1.

[Bou02b] Sébastien Boucksom, On the volume of a line bundle, Internat. J. Math. 13 (2002), no. 10, 1043–1063. MR1945706

[Bou04] Sébastien Boucksom, Divisorial Zariski decompositions on compact complex manifolds, Ann. Sci. École Norm. Sup. (4) 37 (2004), no. 1, 45–76. MR2050205

[Bru06] Marco Brunella, A positivity property for foliations on compact Kähler manifolds, Internat. J. Math. 17 (2006), no. 1, 35–43. MR2204838

[BS95] Mauro C. Beltrametti and Andrew J. Sommese, The adjunction theory of complex projective varieties, De Gruyter Expositions in Mathematics, vol. 16, Walter de Gruyter & Co., Berlin, 1995. MR1318687

[Cam92] F. Campana, Connexité rationnelle des variétés de Fano, Ann. Sci. École Norm. Sup. (4) 25 (1992), no. 5, 539–545. MR1191735
[CH20] Junyan Cao and Andreas Höring, *Rational curves on compact Kähler manifolds*, J. Differential Geom. **114** (2020), no. 1, 1–39. MR4047551

[CHP16] Frédéric Campana, Andreas Höring, and Thomas Peternell, *Abundance for Kähler threefolds*, Ann. Sci. Éc. Norm. Supér. (4) **49** (2016), no. 4, 971–1025. MR3552019

[CP97] Frédéric Campana and Thomas Peternell, *Towards a Mori theory on compact Kähler threefolds. I*, Math. Nachr. **187** (1997), 29–59. MR1471137

[CT15] Tristan C. Collins and Valentino Tosatti, *Kähler currents and null loci*, Invent. Math. **202** (2015), no. 3, 1167–1198. MR3425388

[Das20] Omprokash Das, *Finiteness of log minimal models and nef curves on 3-folds in characteristic* $p > 5$, Nagoya Math. J. **239** (2020), 76–109. MR4138896

[Deb01] Olivier Debarre, *Higher-dimensional algebraic geometry*, Universitext, Springer-Verlag, New York, 2001. MR1841091

[Dem12] Jean-Pierre Demailly, *Analytic methods in algebraic geometry*, Surveys of Modern Mathematics, vol. 1, International Press, Somerville, MA; Higher Education Press, Beijing, 2012. MR2978333

[Dem85] , *Mesures de Monge-Ampère et caractérisation géométrique des variétés algébriques affines*, Mém. Soc. Math. France (N.S.) **19** (1985), 124. MR813252

[Dem92] , *Regularization of closed positive currents and intersection theory*, J. Algebraic Geom. **1** (1992), no. 3, 361–409. MR1158622

[DH23] Omprokash Das and Christopher Hacon, *On the Minimal Model Program for Kähler 3-folds*, arXiv e-prints (June 2023), arXiv:2306.11708v1, available at 2306.11708v1.

[DHP24] Omprokash Das, Christopher Hacon, and Mihai Păun, *On the 4-dimensional minimal model program for Kähler varieties*, Adv. Math. **443** (2024), Paper No. 109615. MR4719824

[DHY23] Omprokash Das, Christopher Hacon, and José Ignacio Yáñez, *MMP for Generalized Pairs on Kähler 3-folds*, arXiv e-prints (April 2023), arXiv:2305.00524v1, available at 2305.00524v1.

[DO24] Omprokash Das and Wenhao Ou, *On the log abundance for compact Kähler threefolds*, Manuscripta Math. **173** (2024), no. 1-2, 341–404. MR4684351

[DP04] Jean-Pierre Demailly and Mihai Păun, *Numerical characterization of the Kähler cone of a compact Kähler manifold*, Ann. of Math. (2) **159** (2004), no. 3, 1247–1274. MR2113021

[EZS10] Fouad El Zein and Jawad Snoussi, *Local systems and constructible sheaves*, Arrangements, local systems and singularities, 2010, pp. 111–153. MR3025862

[Fuj13] Osamu Fujino, *A transcendental approach to Kollár’s injectivity theorem II*, J. Reine Angew. Math. **681** (2013), 149–174. MR3181493

[Fuj22] Osamu Fujino, *Minimal model program for projective morphisms between complex analytic spaces*, arXiv e-prints (January 2022), arXiv:2201.11315v1, available at 2201.11315v1.

[Fuj78] Akira Fujiki, *Closedness of the Douady spaces of compact Kähler spaces*, Publ. Res. Inst. Math. Sci. **14** (1978/79), no. 1, 1–52. MR0486648

[Fuj83] , *Kählerian normal complex surfaces*, Tohoku Math. J. (2) **35** (1983), no. 1, 101–117. MR695662

[GPR94] H. Grauert, Th. Peternell, and R. Remmert (eds.), *Several complex variables. VII*, Encyclopaedia of Mathematical Sciences, vol. 74, Springer-Verlag, Berlin,
1994. Sheaf-theoretical methods in complex analysis, A reprint of it Current problems in mathematics. Fundamental directions. Vol. 74 (Russian), Vseross. Inst. Nauchn. i Tekhn. Inform. (VINITI), Moscow. MR1326617

[Gue20] Henri Guenancia, Families of conic Kähler-Einstein metrics, Math. Ann. 376 (2020), no. 1-2, 1–37. MR4055154

[Hör21] A. Höring, Adjoint (1,1)-classes on threefolds, Izv. Ross. Akad. Nauk Ser. Mat. 85 (2021), no. 4, 215–224. MR4295001

[Har74] Reese Harvey, Removable singularities of cohomology classes in several complex variables, Amer. J. Math. 96 (1974), 498–504. MR364671

[Hir75] Heisuke Hironaka, Flattening theorem in complex-analytic geometry, Amer. J. Math. 97 (1975), 503–547. MR0393556

[HM07] Christopher D. Hacon and James McKernan, On Shokurov’s rational connectedness conjecture, Duke Math. J. 138 (2007), no. 1, 119–136. MR2309156

[HP15] Andreas Höring and Thomas Peternell, Mori fibre spaces for Kähler threefolds, J. Math. Sci. Univ. Tokyo 22 (2015), no. 1, 219–246. MR3329195

[HP16] , Minimal models for Kähler threefolds, Invent. Math. 203 (2016), no. 1, 217–264. MR3437871

[HP18] Andreas Höring and Thomas Peternell, Bimeromorphic geometry of Kähler threefolds, Algebraic geometry: Salt Lake City 2015, 2018, pp. 381–402. MR3821156

[KM98] János Kollár and Shigefumi Mori, Birational geometry of algebraic varieties, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original. MR1658959 (2000b:14018)

[KMM92] János Kollár, Yoichi Miyaoka, and Shigefumi Mori, Rational connectedness and boundedness of Fano manifolds, J. Differential Geom. 36 (1992), no. 3, 765–779. MR1189503

[Kob87] Shoshichi Kobayashi, Differential geometry of complex vector bundles, Publications of the Mathematical Society of Japan, vol. 15, Princeton University Press, Princeton, NJ; Princeton University Press, Princeton, NJ, 1987. Kanô Memorial Lectures, 5. MR909698

[Kod54] K. Kodaira, On Kähler varieties of restricted type (an intrinsic characterization of algebraic varieties), Ann. of Math. (2) 60 (1954), 28–48. MR068871

[Kol91] János Kollár, Extremal rays on smooth threefolds, Ann. Sci. École Norm. Sup. (4) 24 (1991), no. 3, 339–361. MR1100994

[Kol96] , Rational curves on algebraic varieties, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 32, Springer-Verlag, Berlin, 1996. MR1440180

[Kov00] Sándor J. Kovács, A characterization of rational singularities, Duke Math. J. 102 (2000), no. 2, 187–191. MR1749436

[Leh12] Brian Lehmann, A cone theorem for nef curves, J. Algebraic Geom. 21 (2012), no. 3, 473–493. MR2941401

[Moï66] B. G. Moïsezon, On n-dimensional compact complex manifolds having n algebraically independent meromorphic functions. I, Izv. Akad. Nauk SSSR Ser. Mat. 30 (1966), 133–174. MR216522

[Nak87] Noboru Nakayama, The lower semicontinuity of the plurigenera of complex varieties, Algebraic geometry, Sendai, 1985, 1987, pp. 551–590. MR946250
[Nam02] Yoshinori Namikawa, *Projectivity criterion of Moishezon spaces and density of projective symplectic varieties*, Internat. J. Math. 13 (2002), no. 2, 125–135. MR1891205

[Pau98] Mihai Paun, *Sur l’effectivité numérique des images inverses de fibrés en droites*, Math. Ann. 310 (1998), no. 3, 411–421. MR1612321

[Pet01] Thomas Peternell, *Towards a Mori theory on compact Kähler threefolds. III*, Bull. Soc. Math. France 129 (2001), no. 3, 339–356. MR1881199

[Pet98] _____, *Towards a Mori theory on compact Kähler threefolds. II*, Math. Ann. 311 (1998), no. 4, 729–764. MR1637984

[RR74] Jean-Pierre Ramis and Gabriel Ruget, *Résidus et dualité*, Invent. Math. 26 (1974), 89–131. MR352522

[RRV71] J. P. Ramis, G. Ruget, and J. L. Verdier, *Dualité relative en géométrie analytique complexe*, Invent. Math. 13 (1971), 261–283. MR308439

[Siu74] Yum Tong Siu, *Analyticity of sets associated to Lelong numbers and the extension of closed positive currents*, Invent. Math. 27 (1974), 53–156. MR352516

[Tak85] Kenshō Takegoshi, *Relative vanishing theorems in analytic spaces*, Duke Math. J. 52 (1985), no. 1, 273–279. MR791302

[Tom16] Matei Toma, *Bounded sets of sheaves on Kähler manifolds*, J. Reine Angew. Math. 710 (2016), 77–93. MR3437560

[TZ18] Valentino Tosatti and Yuguang Zhang, *Finite time collapsing of the Kähler-Ricci flow on threefolds*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 18 (2018), no. 1, 105–118. MR3783785

[Var86] J. Varouchas, *Sur l’image d’une variété kählérienne compacte*, Fonctions de plusieurs variables complexes, V (Paris, 1979–1985), 1986, pp. 245–259. MR926290

[Var89] Jean Varouchas, *Kähler spaces and proper open morphisms*, Math. Ann. 283 (1989), no. 1, 13–52. MR973802

[Wan21] Juanyong Wang, *On the Iitaka conjecture C_{n,m} for Kähler fibre spaces*, Ann. Fac. Sci. Toulouse Math. (6) 30 (2021), no. 4, 813–897. MR4350100

[Zha06] Qi Zhang, *Rational connectedness of log Q-Fano varieties*, J. Reine Angew. Math. 590 (2006), 131–142. MR2208131

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Navy Nagar, Colaba, Mumbai 400005

Email address: omdas@math.tifr.res.in
Email address: omprokash@gmail.com

Department of Mathematics, University of Utah, 155 S 1400 E, Salt Lake City, Utah 84112

Email address: hacon@math.utah.edu