Effect of disorder on topological charge pumping in the Rice-Mele model

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Recent experiments with ultracold quantum gases have successfully realized integer-quantized topological charge pumping in optical lattices. Motivated by this progress, we study the effects of static disorder on topological Thouless charge pumping. We focus on the half-filled Rice-Mele model of free spinless fermions and consider random diagonal disorder. In the instantaneous basis, we compute the polarization, the entanglement spectrum, and the local Chern marker. As a first main result, we conclude that the space-integrated local Chern marker is best suited for a quantitative determination of topological transitions in a disordered system. In the time-dependent simulations, we use the time-integrated current to obtain the pumped charge in slowly periodically driven systems. As a second main result, we observe and characterize a disorder-driven breakdown of the quantized charge pump. There is an excellent agreement between the static and the time-dependent ways of computing the pumped charge. The topological transition occurs well in the regime where all states are localized on the given system sizes and is therefore not tied to a delocalization-localization transition of Hamiltonian eigenstates. For individual disorder realizations, the breakdown of the quantized pumping occurs for parameters where the spectral bulk gap inherited from the band gap of the clean system closes, leading to a globally gapless spectrum. As a third main result and with respect to the analysis of finite-size systems, we show that the disorder average of the bulk gap severely overestimates the stability of quantized pumping. A much better estimate is the typical value of the distribution of energy gaps, also called mode of the distribution.

I. INTRODUCTION

The physics of topological charge pumps [1,2] has attracted intensified interest due to recent quantum-gas experiments realizing charge pumps with fermions [3] or strongly interacting bosons [4], and a spin pump [5]. These experiments observe quantized pumping over a number of cycles despite the fact that finite particle numbers and inhomogeneous systems were considered. Current theoretical efforts investigate the stability of charge pumps in optical lattices in genuine many-body systems [6,17], in noninteracting systems with disorder [18,23], with dissipation [24], or as a proximity effect [25].

A common notion is that topological properties remain protected against small amounts of disorder [24,26,29]. Considering a topological insulator, one expects the integer quantization of a topological invariant to remain robust as long as the bulk gap stays intact. Therefore, disorder drawn from a bounded disorder distribution (e.g., a box distribution), whose width is significantly smaller than the many-body gap, does not lead to a breakdown of topological quantization (see, e.g., Ref. [26]). Remarkably, disorder can also induce topological properties into a system, leading to the so-called topological Anderson insulator [30], which has been investigated experimentally [31,32]. Different from the scenario studied here, Titum et al. [33] introduced the case of an anomalous Floquet-Anderson insulator that is characterized by a winding number rather than a Chern number and possesses fully localized Floquet eigenstates.

Another interesting direction concerns topology in quasicrystals [34]. A very recent experiment [32] has investigated the stability of a topological charge pump with noninteracting fermions in a quasiperiodic disorder potential that is akin to the Aubry-André model [35]. Both the breakdown of topology in a sufficiently strong disorder potential as well as disorder-induced quantized pumping have been observed [32].

Aubry-André-like systems have also been realized in further quantum-gas experiments [36,39], while more general forms of quasicrystals are at the heart of recent experimental efforts [40,42]. A number of theoretical studies has addressed the stability of topological properties against this type of quasi-disorder, including, e.g., the Su-Schrieffer-Heeger (SSH) model [43]. The effect of disorder on an SSH model has also been investigated in a recent quantum-gas experiment [31].

Motivated by these experimental developments, we theoretically investigate the stability of topological properties of charge pumps against disorder in systems of noninteracting fermions. We concentrate on the half-filled Rice-Mele model with periodic boundary conditions (see the sketch in Fig. 1) and introduce random disorder in the onsite potentials. The problem of charge pumping in the presence of disorder has previously been addressed in a number of related studies [18,32,44]. In Ref. [18], a different model has been investigated using open boundary conditions, with an emphasis on the pumping of charge between edge states. As a result, they report the ex-
istence of a disorder-driven topological transition into a state without quantized pumping. A very appealing perspective on a disordered charge pump, modeled by the Rice-Mele model, has been put forward in Ref. [44]. While the Hamiltonian eigenstates are fully localized in one dimension with random onsite disorder for any disorder strength [45], there is a delocalization-localization transition in the Floquet eigenstates at the point where quantized pumping breaks down. Our work extends the analysis and theoretical understanding in several ways. First and complementary to the approach of [18], we use periodic boundary conditions that circumvent the complication of identifying edge states. Their unambiguous identification is complicated in the presence of randomness due to possible Anderson-localization of all single-particle states. Instead, we characterize the charge pump by four bulk quantities: The polarization, the entanglement spectrum, the local Chern marker and its integral as a measure for the bulk Chern number, and, finally, the integrated time-dependent current. The first three quantities are studied in the instantaneous eigenbasis, and the integrated current is extracted from time-dependent simulations with finite-period pump cycles.

While the polarization is a well-established object for the description of adiabatic transport [40], we argue that the representation of the polarization as an angular variable simplifies its analysis. The entanglement spectrum contains information about quantized charge pumping via the spectral flow as has been discussed in [11]. Here, we show that the quantized winding is preserved in the presence of weak disorder, while symmetries protecting the topology of the SSH model [47] that is visited twice in a clean system are broken.

Generally, in an inhomogeneous system, one cannot rely on the classification and expressions for topological invariants of lattice-translational invariant systems and hence a need for real-space version arises, as was emphasized in [18]. The usual path is to work with many-body wave functions and twisted boundary conditions, similarly to the description of the integer Quantum Hall effect. The local Chern marker provides an alternative tool for the calculation of an invariant in inhomogeneous systems. The local Chern marker has originally been introduced to capture aspects of topology in spatially inhomogeneous and/or finite systems [49] such as they arise in the presence of a trapping potential or interfaces (see, e.g., [50]). Here, we use the local Chern marker to obtain the global Chern number of our disordered systems by integration over the spatial coordinate. For this purpose, we generalize the expression for the local Chern marker to a time-periodic situation in the adiabatic limit.

For the critical disorder strength, we observe an excellent agreement between the integrated local Chern marker and the pumped charge obtained from sufficiently slow time-dependent simulations. We stress that our work focuses on finite systems as they are realized in ultracold quantum gases.

The analysis of the fluctuations of the local Chern marker around the bulk Chern number provides a useful quantitative measure for the topological transition. Comparing these fluctuations, the integrated local Chern marker, and the time-integrated current provides a complementary and consistent picture for the transition. In particular, the pumped charge, which is equivalent to the time-integrated current, can readily be accessed in cold-atom experiments [3, 4, 44].

Since most quantum-gas experiments measure an average over many one-dimensional realizations with varying disorder potentials, it is important to carefully analyze the full distribution of the relevant measures. We show that the disorder average of the bulk gap overestimates the critical disorder strength, while the mode, the most likely value, agrees very well with the point at which the disorder-averaged measures for topological charge pumping start to deviate from integer values. This insight may be relevant for the analysis of experimental data [44] as well as of numerical studies of finite systems [44].

The plan of this exposition is the following. In Sec. [11] we introduce the Rice-Mele model and the types of disorder studied in this work. Section [11] discusses the set of measures used to characterize the quantized charge pumping. Our results are presented in Sec. [11] where we discuss static measures and the results of time-dependent simulations. We conclude with a summary and a discussion in Sec. [11].
II. MODEL

The Rice-Mele model \cite{Rice-Mele1982} describes a one-dimensional lattice system of spinless fermions with alternating hopping-matrix elements and a staggered onsite potential, illustrated schematically in Fig. 1(a). It can be written as:

\[ \hat{H} = \sum_{j=0}^{L-1} \left[ -J_j c_j^\dagger c_{j+1} + \text{h.c.} + V_j \hat{n}_j \right]. \]

Here, \( j \) is the site index, \( J_j = J(1 + (-1)^j \delta) \) is an alternating hopping rate \( J \) is the nearest neighbor hopping, and \( V_j = \left( \frac{(-1)^j \Delta}{2} \right) \) is the onsite staggered potential. For simplicity, we parameterize the alternating tunneling with \( J \) and the dimensionless dimerization parameter \( \delta \). To characterize the topological properties of a system with translational invariance, the polarization \( P \) is usually formulated in terms of the cell-periodic polarization 4-potential \( \phi \). The operator \( c_j^\dagger \) creates a fermion on site \( j \) and \( \hat{n}_j = c_j^\dagger c_j \). We use a smooth pumping scheme of the form

\[ \left( \Delta / J, \delta \right) = (R_\delta \sin \theta, R_\delta \cos \theta). \]

Note that at \( \theta = 0(\pi) \), the SSH model is realized in its topological(trivial) phase.

We consider uniform, bounded diagonal disorder that yields a modification of the onsite potential:

\[ V_j \to V_j + \epsilon_j. \]

Here, \( \epsilon_j \) is an energy drawn from the uniform distribution \( \epsilon_j \sim \mathcal{U}[-\epsilon, \epsilon] \), where \( \epsilon \) is the disorder strength.

III. METHODS AND OBSERVABLES

A. Polarization

To characterize the topological properties of a charge pump, one can consider the evolution of the many-body polarization \( P(t) \) over the course of the adiabatic driving of a time-dependent Hamiltonian \[H(t)\]. The integral over the time derivative of \( P \) yields the pumped charge:

\[ \Delta Q = \int_0^\Delta t \partial_t P(t). \]

For a system with translational invariance, the polarization is usually formulated in terms of the cell-periodic part of the momentum eigenstates in the unit cell, here denoted by \( |u(k)\rangle \), leading to

\[ P = \frac{i}{2\pi} \int dk \langle u(k) | \partial_k u(k) \rangle. \]

In the case of a disordered system, the quasimomentum \( k \) is no longer a good quantum number and the topological invariants must be constructed in the position basis.

For a finite system with periodic boundary conditions, we use the exponentiated operator, which is, following Resta \cite{Resta1989}, given by \( \hat{X}^\epsilon = \exp \left( i \frac{2\pi}{Q_a} \hat{X} \right) \). Then, we obtain:

\[ P(t) = \frac{Q_a}{2\pi} \text{Im} \ln \langle \Psi(t) | \hat{X}^\epsilon | \Psi(t) \rangle \quad (\text{mod} \ Q_a), \]

where \( |\Psi(t)\rangle \) is a many-body wavefunction. Importantly, \( P \) is only defined modulo \( Q_a \), which has units of a dipole moment, while \( a \) is the lattice spacing, set to unity in the following, and \( Q \) symbolizes the charge. This expression for the many-body polarization reduces to the usual form for noninteracting fermions for a filled band \cite{Wang2009, Yang2012, Mross2012}.

For a many-body state that is a Slater determinant composed of single-particle states \( |\phi_i\rangle \), the polarization can be written as

\[ P(t) = \frac{Q_a}{2\pi} \text{Im} \ln \det \Psi^{(3)} \Psi. \]

Here, \( \Psi_{ij} = \phi_i(j) \) is different from the many-body wavefunction in Eq. \( \Psi \), and \( |\phi_i\rangle \) are the occupied single-particle states. The total transported charge \( \Delta Q \) is related to the polarization via Eq. \( \Delta Q \).

For a charge pump, we see that the quantization of the pumped charge corresponds to the winding of the polarization. This winding is necessarily quantized, which seems like a contradiction as it would preclude non-quantized pumping. This can be reconciled by realizing that, for gapless states, the polarization becomes non-analytic \cite{Essler2010}.

B. Local Chern Marker

The local Chern marker \cite{Essler2010} (LCM) is a local observable that can capture some aspects of topology from 'local' (real-space) properties of a system. In the present work, it is mainly used as a tool for computing the Chern number of inhomogeneous systems with periodic boundary conditions. Moreover, we will analyze the information contained in the spatial fluctuations of the local Chern marker \( C(j) \). An alternative to the local Chern marker for the description of spatially inhomogeneous systems has recently been used in \cite{Resta1989} using twisted boundary conditions.

We start from the expression for the Chern number of a spatially two-dimensional translational invariant system with quasimomentum \( (k_x, k_y) \)

\[ C = -\frac{1}{\pi} \text{Im} \int dk_x \int dk_y \sum_{p \in \mathcal{B}} \sum_{q,q' \in \mathcal{B}} \langle p, k_x, k_y | \partial_{k_x} | q, k_x, k_y \rangle \langle q, k_x, k_y | \partial_{k_y} | p, k_x, k_y \rangle, \]

where \( \mathcal{B} \) is a filled band, or set of occupied states. To arrive at a real-space representation, we need to identify the projection operators onto the occupied and unoccupied single-particle subspaces. Inserting a resolution of the identity by summing over an additional set of occupied single-particle states \( |n, k_x, k_y \rangle \) yields (omitting \( k_x \) and
The result is the bulk Chern number

\[ C = -\frac{1}{\pi} \text{Im} \int dk_x \int dk_y \sum_{n \in \mathcal{B}} \sum_{p \in \mathcal{B}} \sum_{j} \langle p|n\rangle \langle n|\partial_{k_y} q|q\rangle \partial_{k_y} \langle q|p\rangle. \tag{9} \]

By replacing the \( k_y \) integral with a sum over states, one can rewrite this as:

\[ C = \frac{2}{L} \text{Im} \sum_{k_x} \int dk_y \sum_n \sum_{p \in \mathcal{B}} \sum_{p \notin \mathcal{B}} \sum_{j} \langle p|n\rangle \langle n|\partial_{k_y} q|q\rangle \partial_{k_y} \langle q|p\rangle \]
\[ = -\frac{2}{L} \int dk_y \text{Im} \text{Tr} \left[ \hat{P} \partial_{k_x} \hat{Q} \partial_{k_y} \hat{P} \right], \tag{11} \]

where we use the cyclic property of the trace, and the trace is here expressed via

\[ \text{Tr} \cdot = \sum_n \sum_{k_x} \langle n, k_x, k_y | \cdot | n, k_x, k_y \rangle \tag{12} \]

and

\[ \partial_{k_y} \hat{P} = \sum_{p \in \mathcal{B}} \left( \partial_{k_y} |p, k_x, k_y\rangle \right) \langle p, k_x, k_y \rangle \]
\[ + |p, k_x, k_y\rangle \left( \partial_{k_y} |p, k_x, k_y\rangle \right) . \tag{13} \]

Here, \( \hat{P} \) and \( \hat{Q} = 1 - \hat{P} \) are projection operators onto the occupied and unoccupied states, respectively. The local Chern marker \( C(r) \) is found by inserting a complete set of basis states \( |r\rangle \) in the position basis:

\[ C_{\text{LCM}} = -\frac{2}{L} \text{Im} \sum_r C(r) = \frac{2}{L} \text{Im} \sum_r \langle r| \hat{P} \partial_{k_x} \hat{Q} \partial_{k_y} \hat{P} |r\rangle. \tag{14} \]

The result is the bulk Chern number \( C_{\text{LCM}} \), where the subindex LCM indicates that it is computed from the local Chern marker. By omitting the sum over \( r \), Eq. \((14)\) reduces to the usual expression for the local Chern marker for a spatially inhomogeneous system in two spatial dimensions.

Using the correspondence between an adiabatic periodic system with pumping frequency \( \Omega \) and a synthetic dimension, under which \( \Omega' = \theta \leftrightarrow k_y \), we can find an analogous expression for a one-dimensional charge pump, starting from Eq. \((11)\). However, as there is no analogous real-space operator for the time dimension, the derivative and integral with respect to time have to be computed explicitly while the trace over the site index \( j \) is omitted in Eq. \((11)\):

\[ C(j) = -\frac{2}{L} \text{Im} \int_{0}^{2\pi} d\theta \langle j| \hat{P} \partial_{\theta} \hat{Q} \partial_{\theta} \hat{P} |j\rangle. \tag{15} \]

For the \( k \) derivative, we have

\[ \partial_k (\hat{P} \hat{Q}) = (\partial_k \hat{P}) \hat{Q} + \hat{P} (\partial_k \hat{Q}) = 0 \tag{16} \]

since \( \hat{P} \hat{Q} = 0 \). For a finite number of momentum states, it is replaced by the differential quotient, acting to the left:

\[ \hat{P} \partial_{\theta} \hat{Q} = \frac{\hat{P}(k + dk) - \hat{P}(k)}{dk} \hat{Q}(k) = \frac{\hat{P}(k + dk)}{dk} \hat{Q}(k) \]
\[ = \frac{L}{2\pi} \hat{X}^{\epsilon \delta} \hat{P}(k) \hat{X}^{\epsilon \delta} \hat{Q}(k). \tag{17} \]

The LCM for a charge pump with periodic boundary conditions is thus:

\[ C(j) = \text{Im} \frac{2\pi}{\pi} \int_{0}^{2\pi} d\theta \langle j| \hat{X}^{\epsilon \delta} \hat{P}(\theta) \hat{X}^{\epsilon \delta} \hat{Q}(\theta) (\partial_{\theta} \hat{P}(\theta)) |j\rangle. \tag{18} \]

Translation-invariant systems (with periodic boundary conditions) will have a constant LCM. The introduction of disorder breaks the translational symmetry, leading to a fluctuating, position-dependent LCM. However, as long as the system has a band gap, the sum \( C_{\text{LCM}} = \sum_j C(j) \) of the LCM is expected to be quantized.

In the case of a gapless system, the kernel of Eq. \((18)\) is discontinuous. This leads to non-quantized values for \( C_{\text{LCM}} \) when the system is gapless, indicating a breakdown of quantized pumping.

To compute the local Chern marker, we use a discretized form with periodic boundary conditions. Using that \( \hat{Q} \hat{P} \hat{P} = \hat{Q} (\hat{P} \hat{P}) \hat{P} = \lim_{d\theta \to 0} Q(\theta) P(\theta + d\theta) P(\theta) / d\theta \), we arrive at

\[ C(j) = \frac{1}{\pi} \sum_{n=0}^{N_{\theta} - 1} \text{Im} \langle j| \hat{X}^{\epsilon \delta} \hat{P}(\theta_n) \hat{X}^{\epsilon \delta} \hat{Q}(\theta_n) \hat{P}(\theta_{n+1}) | \hat{P}(\theta_n) | j \rangle. \tag{19} \]

Here, \( \theta_n = 2\pi n / N_{\theta} \), and \( N_{\theta} \) is the number of points in the discretization of \( \theta \) with a step size \( d\theta = \theta_{n+1} - \theta_n \).

Our numerical analysis shows that in the topological phase, this converges to a fixed value for \( C(j) \) as \( d\theta \) is decreased. The projector \( \hat{P}(\theta) \) is not necessarily continuous. However, the integral over this function is well-defined and converges with decreasing \( d\theta \). The sum \( C_{\text{LCM}} := \sum_j C(j) \) yields the Chern number for a translationally invariant system as our discussion shows.

In the simulations, we use \( d\theta / 2\pi = 10^{-3} \) which gives a Chern number which is quantized with an accuracy of 1–\( C_{\text{LCM}} = \pm 10^{-3} \) at \( w = 0 \) and for individual realizations for \( w / J \lesssim 2 \), which is shown in Fig. \( \text{2} \) where we scan a range of \( 10^{-4} < d\theta / 2\pi < 0.1 \). In the regime \( w / J > 3 \), a sizable dependence on \( d\theta \) of \( C_{\text{LCM}} \) can persist even up to \( d\theta / 2\pi = 10^{-3} \) for individual realizations (data not shown). This can lead to a smaller accuracy of \( 10^{-1} \) for the \( C_{\text{LCM}} \). Therefore, in this regime, the quantitative results for the LCM are less reliable for single disorder realizations. However, due to the disorder average, the mean of the integrated LCM \( C_{\text{LCM}} \) is two orders of magnitude more reliable when compared to single realizations.

We will apply this approach to inhomogeneous systems in this work and the results and comparison with other measures will lead to a consistent picture for integer quantization of \( C_{\text{LCM}} \) in the topological phase.
C. Entanglement spectrum

The entanglement spectrum is given by the eigenvalues of the entanglement Hamiltonian \( \hat{H}_E \) and can be defined with the reduced density matrix \( \hat{\rho}_A \) of a spatial bipartition of the system into two halves (A and B) [56].

\[
\frac{1}{Z} \hat{\rho}_A = e^{-\hat{H}_E} . \tag{20}
\]

\( \hat{H}_E \) is a dimensionless free-fermion operator that is strictly positive due to the normalization requirement of the reduced density matrix \( \hat{\rho}_A \). The entanglement eigenvalues (EEVs) are the eigenvalues of \( \hat{H}_E \). The EEVs \(-\log \Lambda^2_{\mu}\) are related to the Schmidt values \( \Lambda_{\mu} \) that are found in a Schmidt decomposition of the two halves of the system. We can compute these values of the free-fermion Hamiltonian directly from the reduced density matrix, which can be obtained from the single-particle correlation matrix \( C_{ij} = \langle \hat{c}_i^\dagger \hat{c}_j \rangle \) computed in the many-body ground state, as described by Peschel [57]. Writing \( \hat{H}_E \) in its single-particle eigenbasis, \( \hat{H}_E = \sum_{\alpha=1}^{L} \epsilon_{\alpha} \hat{d}_{\alpha}^\dagger \hat{d}_{\alpha} \), the partition function is \( Z = \prod_{\alpha=1}^{L} \frac{1}{1 - e^{-\epsilon_{\alpha}}} \). The eigenvalues \( \Lambda_{\mu}^2 \) in the full many-body space of the reduced density matrix \( \hat{\rho}_A \) result from fixing a set of occupations \( n = \{ n_{\alpha} \} \). Every choice for these occupations corresponds to a many-body state \( |\mu\rangle \) indexed by \( \mu \). They are given by

\[
-\ln(\Lambda_{\mu}^2) = \sum_{\alpha=1}^{L} \left[ \ln(v_{\alpha}) n_{\alpha} - \ln(1 - v_{\alpha})(1 - n_{\alpha}) \right] , \tag{21}
\]

where \( v_{\alpha} \) are the eigenvalues of the single-particle correlation matrix \( \langle \hat{c}_i^\dagger \hat{c}_j \rangle \) when \( i, j \) are restricted to one half of the system. As the states are all localized for finite disorder, the entanglement across one cut in the system is independent of the boundary conditions for a sufficiently large system. The same is true for zero disorder due to translational invariance.

The eigenvalues \( v_{\alpha} \) are symmetrically spread around 0.5 and the low-lying eigenstates of the entanglement Hamiltonian are strongly localized at the chosen cut, see [57]. We assign a particle-number imbalance \( \Delta N_\mu \) to a state defined by \( n \) as follows: Focusing on the states of the correlation matrix (when restricted to a subsystem) localized on the left side of the subsystem, eigenvalues \( v_{\alpha} < 0.5(>0.5) \) correspond to states on the left(right) side of the cut and are assigned a label \( N_\alpha = -1(+1) \). In order to calculate particle imbalances of the full many-body spectrum of \( \hat{\rho}_A \), states on the left(right) side of the cut are counted as unoccupied(occupied) in Eq. (21), which is then evaluated for all possible occupations \( n \). The total particle imbalance of the many-body state \( |\mu\rangle \) is

\[
\Delta N_\mu = \sum_{\alpha=1}^{L} N_\alpha n_\alpha . \tag{22}
\]

The imbalance \( \Delta N_0 = 0 \) is assigned to the unique lowest EEV \( \mu = 0 \) for \( \theta \in (-2\pi, 0) \).

Figure 3 shows the entanglement spectrum of the clean system \((w/J = 0)\) over the course of one pump cycle \((R_\delta = 0.5, R_\Delta = 2.3)\) [11]. Under an adiabatic modulation of the pumping parameters, the entanglement spectrum shows a continuous flow as long as the system is gapped. As was discussed in [11], the spectral flow and its nontrivial winding structure is a hallmark of the topological nature of the charge pump. After one pump cycle, the spectrum returns to itself but the particle imbalances...
with a time step of $\Delta t_J$.

We drive periodically with a period $T$ then reads $\hat{U}$ via the Crank-Nicholson method (see, e.g., [58] and references therein).

For the Rice-Mele model, the total current operator is given by:

$$\hat{J} = i \sum_{j=1}^{L} (J_j \hat{c}_j^\dagger \hat{c}_{j+1} - \text{h.c.}),$$

identifying the $0^{\text{th}}$ with the $L^{\text{th}}$ site. In praxis, we compute $\langle \hat{J} \rangle / L$, which, in the topological regime, should yield $\Delta Q = 1$ when integrated over one pump cycle. This is a physical quantity that allows us to connect to experiments. The expectation value of the groundstate current is identically zero. The quantized pumping arises from the virtual admixtures into the higher bands, which are proportional to $O(1/T)$. In this way, the integrated current over one pump cycle converges to a constant in the $T \gg J$ limit.

The effect of non-adiabatic pumping has been studied in several works, see, e.g., [58, 59]. For sufficiently slow pumps, it has been shown that the corrections to the exact quantization scale according to $O(1/T^2)$.

E. Numerical methods

For the static calculations, we directly diagonalize the single-particle Hamiltonian at each $\theta$. For the discretization of the LCM, we take $d\theta/2\pi = 10^{-3}$.

For the real-time simulations, we propagate all single-particle states of the lower half of the spectrum via the Crank-Nicholson method (see Sec. III E).

As long as the bands are clearly separated, $C_{\text{LCM}}$ of each band calculated via the integrated local Chern marker at half filling (see Secs. III B and IV E). As long as the bands are clearly separated, $C_{\text{LCM}}$ remains quantized.

The gap closing can occur at any point in the pump cycle. Relevant information about a potential topological breakdown is therefore encoded in the minimum gap over the entire pump cycle:

$$\Delta \epsilon_{\text{min}} := \min_{\theta} \Delta \epsilon(\theta)/J.$$  \hspace{1cm} (25)

Figure 5 shows the distribution of the minimum gap over $n = 500$ disorder realizations for $w/J = 2$, 2.75, 2.95. For $w/J = 2$, 2.75, the distributions are symmetric and their mean is a useful quantity. For $w/J = 2.95$, the distribution of minimum gaps has its maximum, the mode, at zero, and the form of the distribution starts to evolve towards an exponentially decaying distribution. In Fig. 5(b), we show the minimum gap as a function of $\epsilon$. The bands are colored based on $C_{\text{LCM}}$ at half filling.
disorder strength $w/J$. Black dots indicate single realizations. The red line and blue shaded area denote the mean and standard deviation over all realizations, respectively. The orange line shows the mode of the minimum gap distributions. For every $w$, we use a different disorder seed to generate the onsite potentials to preempt a dependence on single realizations. While the mean gap closes gradually at around $w/J \approx 3.25$, the mode of the gap shows a very sharp transition to zero at $w/J \approx 2.95$. We demonstrate below (Sec. IV G) that the mode and not the mean gap is the relevant quantity to predict the topological transition of the local Chern maker and the pumped charge.

In the absence of disorder, the minimum band gap is $\Delta = 2J$ at $\theta = 0$ for the chosen cycle and parameters used in this paper. We will later see that quantized charge pumping breaks down at $w/J \approx 3$. Therefore, this illustrates that the disorder strength can in fact exceed the gap-width of the single-particle spectrum without significantly affecting the quantized pumping behavior.

Since the single-particle gap of the clean system at $\theta = \pi/2$ is $\Delta$, there is, in principle, a disorder realization for which the gap closes at that point for $w \geq \Delta$, where the disorder realization is exactly the (oppositely) staggered potential with strength $-\Delta$. A single, individual realization, however, does not affect the mean of the $C_{\text{LCM}}$. The fraction of small-gap realizations increases beyond $w = 2.3J$ but leads to a significant deviation of $C_{\text{LCM}}$ at $w \approx 3J$ (see the discussion in Sec. IV E).

![Diagram](image_url)

**FIG. 5.** *Energy gap for diagonal disorder.* (a): Distribution of the minimum gap found along the pump cycle, $\Delta_{\text{min}}$, for $w/J = 2, 2.75, 2.95$ over $n = 500$ disorder samples. The most likely value, the mode, of the minimum gap becomes zero at around $w/J \approx 2.95$. (b): We show the minimum gap as a function of $w/J$. The red line and shaded region correspond to the mean and the standard deviation, respectively, computed for $L = 400$. The orange line indicates the mode, which vanishes at $w/J \approx 2.95$. We use a linear $\theta$-grid of $d\theta/2\pi = 10^{-4}$ that limits the accuracy of finding the minimum gap in a given realization. The dashed line corresponds to a vanishing gap.

![Diagram](image_url)

**FIG. 6.** *Minimum inverse participation ratio $I_{\text{min}}$ for diagonal disorder.* We plot the minimum IPR over $\theta$ and over 40 disorder realizations as a function of $w$, for various system sizes. At $w/J \approx 1.5$, $I_{\text{min}}$ becomes independent of system size for systems $L \geq 200$, indicating that all single-particle states are localized with a localization length much smaller than system size.

### B. Localization of single-particle states

To demonstrate that our numerically simulated system sizes are sufficient to capture the localized nature of the single-particle states, we compute the inverse participation ratio. Full localization at any $w > 0$ is the expected behavior in one dimension for random diagonal disorder.

The inverse participation ratio $I$ is a common measure to quantify the degree of localization for a state:

$$I(|\psi\rangle) = \frac{\sum_i |\langle i|\psi\rangle|^4}{\langle \phi|\phi\rangle^2}.$$  \hspace{1cm} (26)

Here, $I$ is calculated in the real-space basis, where $|\phi\rangle$ is a single-particle eigenstate and $\langle \phi|\phi\rangle$ is the amplitude on a site $i$. For a completely localized state that has weight on only one site, $I = 1$, whereas for a totally delocalized one, $I = 1/L$, where $L$ is the system size.

Figure 6 shows the minimum value $I_{\text{min}} := \min_{\theta,s} I$ of the entire spectrum over one pump cycle and over all given disorder realizations indexed by $s$, corresponding to the most delocalized state. The observed behavior is compatible with the expectation of all states being localized for any finite disorder strength. At low disorder, there are states which appear delocalized across the entire system because of the localization length exceeding the system size, indicated by the $L$ dependence of $I_{\text{min}}$.

For all finite $w$, $I_{\text{min}}$ is much larger than $1/L$, which is the typical value for a fully delocalized state. For
\[ w = J, \quad (b) \quad w = 3J, \quad (c) \quad w = 3.75J, \] computed for \( L = 400 \) sites. We show data for one fixed disorder configuration and vary \( w \).

\[ w/J > 1.5, \quad \langle I_{\text{min}} \rangle \text{ is no longer dependent on system size for } L \geq 200, \text{ indicating a localization length much shorter than system size. Crucially, even in this region, quantized pumping persists, as will be shown below. This agrees with the analysis of [44], where the breakdown of quantized pumping has been linked to a delocalization-localization transition of Floquet eigenstates, which occurs deep in the regime of localized single-particle Hamiltonian eigenstates.}

### C. Entanglement spectrum

Disorder in the onsite potentials breaks the chiral symmetry at \( \theta = 0, \pi \) which correspond to the topological and the trivial phase of the SSH model, respectively. We see in Figs. 7(a) and (b) that, for small disorder, the entanglement spectrum is perturbed as \( w \) becomes finite, but the topological winding structure of the states is preserved. The degeneracies in the entanglement spectrum persist at weak disorder yet, since chiral symmetry is broken, they are shifted to arbitrary positions along the pump cycle, depending on the specific disorder realization. At large disorder strength, there are discontinuities in the entanglement spectrum as exemplified by the data in Fig. 7(c). These signal the breakdown of quantized charge pumping. The position and number of these discontinuities is highly dependent on the particular disorder realization chosen. Therefore, it is more cumbersome to extract the exact transition at the breakdown of the quantized pumping from the entanglement spectrum compared to other measures discussed here.

### D. Polarization

Figure 8 shows data for the polarization \( P \) for \( n = 40 \) disorder realizations as a function of the pump parameter \( \theta \in [0, 2\pi) \) for various disorder strengths. Due to \( P \) being defined modulo \( 2\pi \), it is convenient to use a polar plot with \( P \) being the angular variable and \( \theta \) being the radial.
one. The colored dots show the circular mean
\[ \overline{P}(\theta) = \text{atan}2 \left( \frac{1}{n} \sum_{j=1}^{n} \sin P(\theta)_j, \frac{1}{n} \sum_{j=1}^{n} \cos P(\theta)_j \right) \] (27)
for each value of the pumping parameter \( \theta \), where \( \text{atan}2 \) is the two-argument arctangent. Single realizations are shown as blue lines. For low disorder strengths, all realizations lie on top of each other and the circular mean exhibits a smooth dependence on \( \theta \) with winding 1. For \( w/J = 2.8 \) and beyond, some realizations display discontinuities as indicated by the jumps in the blue lines. At \( w/J = 2.8 \), only a few realizations have such jumps, whereas for \( w/J = 3.4 \), almost all polarizations are discontinuous, with individual realizations showing multiple discontinuities of arbitrary size. The circular mean becomes visually discontinuous at \( w/J = 3.2 \), indicating that quantization breaks down in a significant number of realizations. We associate this with the breakdown of quantized charge pumping. In this case, the winding number of the polarization of an individual realization becomes undefined: It is impossible to know whether the jump in the polarization is clockwise or counterclockwise for single realizations. Despite this fact, the circular mean retains a sense of direction in its winding and winds around exactly once, meaning that on average, charge is transported across the system in a well-defined direction. For a topologically trivial pump cycle, the circular mean, as expected, shows no winding (data not shown). This suggests that, even beyond the critical disorder strength, the circular mean of the polarization still preserves information on the quantized pumping that exists in the neighboring topological phase.

E. Chern number from integrated local Chern marker

Local Chern marker and its fluctuations. The local Chern marker \( C(j) \) is shown for \( w/J = 2, 3, 3.6 \) in Figs. 9(a)-(c) for a single disorder realization. For zero disorder (data not shown here), the local Chern marker is \( C(j) = 1/L \) (within our numerical accuracy) as expected. As \( w \) increases, fluctuations around this value emerge and increase as \( w \) grows. For \( w/J = 3.6 \), quantization clearly no longer holds.

We next demonstrate that the spatial fluctuations of the local Chern marker defined via
\[ \delta C_j = 1 - LC(j) \] (28)
contain relevant information and can be used to pinpoint the transition. A local Chern marker \( LC(j) > 1 \) \( (\delta C_j < 0) \) implies that an excess amount of charge crosses site \( j \) per pump cycle, while for \( LC(j) < 1 \) \( (\delta C_j > 0) \) a lesser amount moves through. In order for the system to possess an integer-quantized Chern number, these fluctuations have to precisely cancel out when summing over the whole sample. Our analysis shows that this happens for individual realizations in the topological regime.

The respective distributions of \( \delta C_j \) obtained from \( n = 500 \) disorder configurations are plotted in Figs. 9(d)-(f). In the topological regime [see Fig. 9(d)], \( \Pr(\delta C_j) \) is symmetric and sharply peaked around zero, while the variance gradually increases as the transition point is approached. For \( w = 3.6J \), the distribution is skewed towards \( \delta C_j > 0 \), resulting in \( LC_{\text{CM}} < 1 \). In order to determine the breakdown of the quantized pumping, we compute the skewness \( \gamma \) of the distributions from
\[ \gamma = \frac{n}{\pi L \sum_{i=1}^{nL} (x_i - \overline{x})^3}{\left( \frac{1}{\pi L \sum_{i=1}^{nL} (x_i - \overline{x})^2} \right)^{3/2}} \] (29)
with \( x_i \) being \( C(j) \) computed for a given disorder realization (recall that \( n \) is the number of disorder realizations). The dependence of \( \gamma \) on the disorder strength is shown in

FIG. 9. Local Chern marker. (a)-(c): \( C(j) L \) versus position \( j \) for a single realization and for disorder strengths (a) \( w = 2J \) with \( CLCM = 1.00 \), (b) \( w = 3J \) with \( CLCM = 1.00 \), and (c) \( w = 3.6J \) with \( CLCM = 0.97 \), computed for \( L = 400 \). (d)-(f): Normalized distribution of the fluctuations of the local Chern marker \( \delta C_j = 1 - LC(j) \) for (d) \( w = 2J \), (e) \( w = 3J \), and (f) \( w = 3.6J \) from 500 disorder realizations for \( L = 400 \). Note that the distribution in (f) is slightly skewed towards positive values, see Fig. 10.
Fig. 10 for $L = 100, 200, 400$. Remarkably, the skewness exhibits a strong signature at the transition. It first decreases but consistently, and for all $L$, increases sharply as the Chern number begins to significantly deviate from one. This suggests that the transition point is at $w/J \approx 3.0$. There is no detectable $L$-dependence for the transition point for the system sizes considered here within the used grid of $\Delta w = 0.1 J$ around the transition region. We conclude that the analysis of the full distribution of local Chern marker fluctuations and of its moments is very useful to obtain a quantitative picture of the breakdown, as will be substantiated by the following comparison with the pumped charge from the time-integrated current.

**Integrated local Chern marker and breakdown of quantized pumping.** Figure 11 shows the evolution of the integrated local Chern marker as a function of $w$ for different system sizes $L$. Black dots indicate single realizations. The red line and the blue shaded region are the disorder averages and standard deviation, respectively.

The integrated local Chern marker $C_{\text{LCM}}$ and thus the total Chern number remains quantized up to $w \approx 3J$ within our numerical accuracy and for the $w$-grid used to compute the $C_{\text{LCM}}$. To provide a quantitative figure, for $L = 400$, $1 - C_{\text{LCM}} > 10^{-2}$ for $w > 3J$ while for $w \leq 3J$, $1 - C_{\text{LCM}} < 10^{-3}$.

At strong disorder $w/J \gtrsim 3$, the system becomes typically gapless and hence the Chern number is ill-defined. We therefore expect a significant breakdown of the quantized charge pumping. This is consistent with the numerical data, where we observe a smooth onset of deviations from $C_{\text{LCM}}$. Note that, even for $w/J > 3$, the results for $C_{\text{LCM}}$ nonetheless spread around a central mean value over all realizations. Moreover, the distributions do not show a bimodal behavior with realizations clustering around zero and one, as might have been expected. Its fluctuations around the mean value furthermore decrease as $L$ increases. Note that the standard deviation around the disorder-averaged Chern number $\overline{C_{\text{LCM}}}$ increases just at the point where $\overline{C_{\text{LCM}}}$ ceases to be quantized.

An interesting question concerns the system-size scaling of the mean Chern number in the transition region (see the discussion and further references in [48]). Since our work is concerned with the properties of ensembles of finite-size systems, as realized in quantum-gas experiments, we leave this for future research. We further note that finite-size corrections for the pumped charge were studied in [62].

For single realizations, the notion that a topological quantity cannot change without closing a gap is confirmed by Fig. 12. There, we plot the integrated local Chern marker $C_{\text{LCM}}$ versus the minimum gap $\Delta \epsilon_{\text{min}}/J$ for individual realizations and different disorder strengths. Points on the left with $\Delta \epsilon_{\text{min}}/J \lesssim 10^{-5}$ indicate realizations where the energy gap closes. The $C_{\text{LCM}}$ for these realizations exhibit non-integer values with a spread that increases for larger $w/J$. The points on the right side of the figure correspond to realizations that do not close the gap and have an integer-quantized $C_{\text{LCM}}$. This confirms the expected behavior that, on the level of individual realizations, a topological quantity cannot change without a gap closing.

For large $w/J$, the ratio of gap-closing realizations increases sharply. We thus conclude that the breakdown of topological charge pumping on *ensembles of finite systems* occurs due to sufficiently many individual realizations acquiring close-to-zero gaps which happens well before the mean (disorder-averaged) minimum gap closes. This scenario applies to the finite-size quantum-gas experiments where typically, $L \sim 100$, and averages over many one-dimensional systems are measured. We have also studied the parameters from [44] using the integrated local Chern marker and consistently observe a breakdown at $w/J \approx 3$, confirming by Fig. 12. There, we plot the integrated local Chern marker $C_{\text{LCM}}$ as a function of $w$. (a) $L = 100$, (b) $L = 200$, (c) $L = 400$. Main panels: 40 samples at each disorder strength. Insets: Zoom into the transition region with 500 samples. The red line indicates the mean value for each disorder strength and the standard deviation is represented by the shaded region.
in agreement with [44].

F. Time-dependent case and integrated current

In this section, we study the dependence of the time-integrated current on the finite period of the pump cycle. In the adiabatic limit, any finite system will ultimately have an exactly quantized integrated current, as the system is gapped. However, the gaps of some systems may become exponentially small in system size. On the other hand, a finite period $T$ can lead to non-quantized behavior due to Landau-Zener tunneling [58], even in the presence of a band gap.

In Fig. 13, we show the distribution of pumped charge, calculated from the time-integrated current over 500 disorder realizations and for various pumping periods $T$. For $T = 10J$, the pumped charge is not quantized, presumably due to Landau-Zener tunneling into the second band. For $T = 100J$ and $T = 1000J$, the average pumped charge closely follows the Chern number computed via the LCM. Quantization breaks down at the critical disorder strength $w/J \approx 3$.

G. Comparison between instantaneous and time-dependent measures

Figure 14 shows a comparison between the disorder average of the minimum energy gap, the disorder-averaged integrated local Chern marker $C_{\text{LCM}}$, and the skewness of the local Chern marker distributions, and the time-integrated local current as a function of $w/J$.

Interestingly, the pumped charge $\overline{\Delta Q}$ obtained from the time-dependent simulations and $C_{\text{LCM}}$ are very similar to each other even in the gapless regime with non-quantized $C_{\text{LCM}}$. The observed difference could be the result of an intricate combination of slow convergence of the LCM (see Sec. III B), finite-size effects, and statistics.

We note that the breakdown of the integer-quantization of the mean of the Chern number occurs at $w/J \approx 3$, whereas the mean minimum gap closes at $w/J \approx 3.3$. As Fig. 14 clearly shows, the onset of significant deviations of $C_{\text{LCM}}$ and of the pumped charge $\overline{\Delta Q}$ from one does, however, coincide with the closing of the typical value of the gap, the mode (vertical shaded region), and the onset of a significant skewness $\gamma$ in the distribution of local Chern markers. These observations further support our assertion that the most likely gap, rather than the disorder-averaged gap, should be considered when quantifying the transition point.

In conclusion, all three quantities computed in the instantaneous basis and the integrated current suggest the stability of quantized charge pumping at weak disorder. A breakdown of integer quantization is observed for $w/J \gtrsim 3$ and is signalled by the behavior of the skewness of the distributions of local Chern-marker fluctuations consistent with the closing of the most likely energy gap.

V. SUMMARY AND DISCUSSION

In this work, we studied the Rice-Mele model of non-interacting, spinless fermions in the presence of random diagonal disorder. We used exact diagonalization to compute a set of static measures to characterize the properties
of a charge pump, including the polarization, the entanglement spectrum, and the integrated local Chern marker. These quantities were computed in the instantaneous eigenbasis. As disorder is introduced into the system, we consistently observe robustness of integer-quantized pumping. We demonstrated that all these measures indicate a breakdown of the quantized charge pumping at sufficiently strong disorder. The breakdown of quantized pumping manifests itself as a breakdown of winding in the spectral flow of the entanglement spectrum. Plotting the polarization as an angular variable makes the breakdown particularly transparent in this quantity. The integrated local Chern marker and the skewness extracted from the LCM’s full distribution appear to be the best suited for determining the transition point quantitatively.

In particular, the fluctuations of the local Chern marker around the bulk Chern number provide relevant information about the breakdown. It would be very desirable to develop a qualitative interpretation of these fluctuations. For instance, it remains open whether nonlocal adiabatic processes play a role in topological charge pumping. These effects were described by Khemani et al. as a consequence of adiabatic variations of a single onsite potential in an Anderson insulator. The situation in a charge pump is not entirely different even though the variation of onsite potentials here happens in a correlated fashion.

In each individual realization, the critical disorder strength obtained from the integrated local Chern marker agrees well with the point at which the bulk gap closes. For ensembles of finite systems, however, we emphasized the importance of sample-to-sample fluctuations. In particular, we demonstrated that the typical value of the minimum bulk gap along the pump cycle is much better suited than the disorder-averaged gap to describe the distribution obtained from finite systems. In particular, the vanishing of the mode is linked to the breakdown of topological properties on finite systems. This may not be surprising, given the known existence of Lifshitz tails at the edges of spectra of disordered systems (see, e.g., [64]).

The topological transition occurs deep in an Anderson insulator, showing that topological charge pumping is robust against localization, consistent with the results of [32].

We complemented the analysis with time-dependent simulations of the time-periodic pump process and observe deviations from integer-quantized pumping due to a breakdown of adiabaticity for fast pumping. For sufficiently slow pumping, we find agreement with the critical disorder strength obtained from the bulk Chern number computed in the instantaneous basis.

In this work, we presented a comprehensive comparison of several measures using the instantaneous basis and direct time-dependent simulations. Developing a physical picture for local transport processes complementary to the Floquet-localization scenario of Ref. [44] would be an interesting next direction. In this regard, the limit of frequency going to zero in the Floquet picture might be subtle on finite systems, as one should recover the behavior of the instantaneous-basis behavior (see [44]).

Several studies have already theoretically addressed the question of charge pumping in an interacting system [6,17], which should be realizable in state-of-the-art quantum-gas experiments [3,5,32]. Conceptually and from a methodological point of view, the question arises which approaches are best suited to compute the Chern number for a charge pump in a many-body system. The direct calculation of the current in time-dependent simulations will always work, yet requires making the pump period large. The extension of the Floquet picture of Ref. [44] to the many-body case would be interesting, yet the development of controlled Floquet expansions for the many-body case is still the topic of ongoing research [65]. An extension of the local Chern marker to interacting systems would be desirable (see [66] for recent work in this direction), while polarization and entanglement spectrum can be computed in the many-body case as well (see, e.g., [11]).

Another, related future direction would be to combine the effects of disorder and interactions and to investigate the possibility of topological charge pumping in both a disordered ergodic and in the many-body localized phase [67,68] (see also the discussion in [32,44]). Finally, the stability of topological pumping in quasiperiodic potentials will be interesting as well (see, e.g., [69,70]).

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