PROOF OF A CONJECTURE OF DAVILA AND KENTER REGARDING A LOWER BOUND FOR THE ZERO FORCING NUMBER IN TERMS OF GIRTH AND MINIMUM DEGREE

RANDY DAVILA$^1$ AND THOMAS KALINOWSKI$^2$ AND SUDEEP STEPHEN$^2$

ABSTRACT. In this note, we study a dynamic coloring of vertices in a simple graph $G$. In particular, one may color an initial set of vertices black, with all other vertices white. Then, at each discrete time step, a black vertex with exactly one white neighbor will force its white neighbor to become black. The initial set of black vertices is called a zero forcing set if by iterating this aforementioned process, all of the vertices in $G$ become black. The zero forcing number of $G$ is the cardinality of a minimum zero forcing set in $G$, and is denoted by $Z(G)$. Davila and Kenter [Bounds for the zero forcing number of a graph with large girth. Theory and Applications of Graphs, 2(2) (2015)] conjectured that the zero forcing number satisfies $Z(G) \geq (g - 3)(\delta - 2) + \delta$ where $g$ and $\delta$ denote the girth and the minimum degree of the graph, respectively. This conjecture has been proven for graphs with girth $g \leq 10$. In this note, we prove it for all graphs with girth $g \geq 11$ and for all values of $\delta \geq 2$, thereby settling the conjecture.

1. Introduction

For a two-coloring of the vertex set of a simple graph $G = (V, E)$ consider the following color-change rule: a white vertex $u$ is converted to black if it is the only white neighbor of some black vertex $v$. We call such a black vertex $v$ a forcing vertex and say $v$ forces $u$. Given a two-coloring of $G$, the derived set is the set of black vertices obtained by applying the color-change rule until no more changes are possible. A zero forcing set for $G$ is a subset of vertices $S \subseteq V$ such that if initially the vertices in $S$ are colored black and the remaining vertices are colored white, then the derived set is the complete vertex set $V$. The minimum cardinality of a zero forcing set for the graph $G$ is called the zero forcing number of $G$, denoted by $Z(G)$. This concept was introduced by the AIM Minimum Rank – Special Graphs Work Group in [3] as a tool to bound the minimum rank of matrices associated with the graph $G$. Since its introduction the zero-forcing number has been studied as an interesting graph invariant with various applications [4, 5, 6, 9, 11, 12, 14]. Moreover, it has been established that the zero forcing problem is $NP$-complete [11], which motivates the search for easily computable bounds for $Z(G)$.
**Graph Terminology.** For the entirety of this note we will restrict ourselves to undirected finite simple graphs. Let $G = (V, E)$ be a graph. We will denote the order and size of $G$ by $n = |V|$ and $m = |E|$, respectively. Two vertices $v, w \in V$ will be called neighbors, or adjacent vertices, whenever $vw \in E$. The *open neighborhood* of $v \in V$ is the set of neighbors of $v$, denoted by $N(v) = N_G(v)$, whereas the *closed neighborhood* of $v$ is $N[v] = N_G[v] = N_G(v) \cup \{v\}$. The *degree* of $v \in V$ is the cardinality of its open neighborhood, and is denoted by $\deg_G(v) = |N(v)|$. The maximum and minimum vertex degrees in $G$ are denoted by $\Delta(G)$ and $\delta(G)$, respectively. The *distance* $\text{dist}_G(u, v)$ between two vertices $u$ and $v$ in a graph $G$ is the length of the shortest path between $u$ and $v$. A cycle of length $\ell$ is denoted as $C_\ell$.

Given a set of vertices $S \subseteq V$, the open neighborhood of $S$ is defined as $N(S) = N_G(S) = \bigcup_{v \in S} N(v)$. The closed neighborhood of $S$ is defined as $N[S] = N_G[S] = N(S) \cup S$. The girth of $G$, denoted $g = g(G)$, is the size of a smallest cycle which is contained in $G$ as a subgraph.

**Necessary Tools.** The maximum number of edges in a simple graph of order $n$ and girth at least $\ell + 1$ is denoted by $\text{ex}(n; \{C_3, C_4, \ldots, C_\ell\})$, often referred to as *extremal function*. The following theorem will be essential for the proof of our main result.

**Theorem 1.** [2 Theorem 1] Let $\ell \geq 4$ and $\ell + 1 \leq n \leq 2\ell$ be integers. Then

$$
\text{ex}(n; \{C_3, C_4, \ldots, C_\ell\}) = \begin{cases} 
n & \text{if } \ell + 1 \leq n \leq \lfloor 3\ell/2 \rfloor, \\
n + 1 & \text{if } \lfloor 3\ell/2 \rfloor + 1 \leq n \leq 2\ell - 1, \\
n + 2 & \text{if } n = 2\ell.
\end{cases}
$$

This statement will be used in the form of the following corollaries.

**Corollary 1.** For $g \geq 11$, $\text{ex}(g - 2, \{C_3, C_4, \ldots, C_{g-6}\}) = \begin{cases} 
g - 1 & \text{for } g \in \{11, 12, 13\}, \\
g - 2 & \text{for } g \geq 14.
\end{cases}$

**Corollary 2.** For $g \geq 11$, $\text{ex}(g - 2, \{C_3, C_4, \ldots, C_{g-4}\}) = g - 2$.

2. **Main Result**

In this section, we prove the following conjecture posed by Davila and Kenter [8].

**Conjecture 1.** [8] If $G$ is a graph with girth $g \geq 3$ and minimum degree $\delta \geq 2$, then

$$
Z(G) \geq \delta + (\delta - 2)(g - 3).
$$

Gentner et al. [11] and Gentner and Rautenbach [12] and Davila and Henning [7] have shown that inequality (1) is true for girth $g \leq 10$. Our aim in this note is to prove that (1) is true for all graphs with girth $g \geq 11$ and minimum degree $\delta \geq 2$. By doing so, we settle the conjecture. We state our main result as follows.

**Theorem 2.** Let $G$ be a graph with girth $g \geq 11$ and minimum degree $\delta \geq 2$. Then (1) is true.

**Remark 1.** Note that an alternative proof of Conjecture [1] which does not depend on the previous results has recently appeared in [10].
Proof. Let \( G \) be a graph with minimum degree \( \delta \geq 2 \) and girth \( g \geq 11 \). Suppose \( S \subseteq V \) is a zero forcing set with cardinality \( |S| \leq \delta + (\delta - 2)(g - 3) - 1 \). Let \( x_1, \ldots, x_t \) be a chronological list of forcing vertices resulting in all of \( \bar{V} \) becoming black starting with \( S \) as an initial set of black vertices. Let \( \bar{S} = V \setminus S \) be the set of initially white vertices. Since \( G \) is a graph with minimum degree \( \delta \geq 2 \), and girth \( g \geq 5 \), we have \( n \geq g(\delta - 1) \). Hence, we obtain the chain of inequalities

\[
|\bar{S}| = n - |S| \geq g(\delta - 1) - (\delta - 2)(g - 3) - \delta + 1 = g + 2\delta - 5 \geq g - 1.
\]

We next set \( X = \{x_1, \ldots, x_{g-2}\} \), and slightly modifying the notation of \([7]\), we set \( S_1 = S \cap N(x_1) \) and

\[
S_i = S \cap \left( N(x_i) \setminus \bigcup_{j=1}^{i-1} N[x_j] \right) \quad \text{for } i = 2, \ldots, g - 2,
\]

\[
S_X^* = \bigcup_{i=1}^{g-2} S_i,
\]

\[
S_X = X \cap (S \setminus S_X^*).
\]

Then \( S_X^* \) and \( S_X \) are disjoint subsets of \( S \), and consequently

\[
(2) \quad |S| \geq |S_X| + |S_X^*|.
\]

Since \( x_1 \in S \cap X \), and \( x_1 \notin S_i \) for all \( i \in [g - 2] \), we have \( x_1 \in S_X \) and thus \( |S_X| \geq 1 \). Let \( H = (X, E') \), be the graph with vertex set \( X \) and edge set

\[
E' = \{\{x_i, x_j\} : \{x_i, x_j\} \in E(G) \text{ or } v \in N(x_i) \cap N(x_j) \text{ for some } v \in V \setminus X\}.
\]

Let \( m' = |E'| \) be the size of \( H \).

Lemma 1. \( |S_X^*| \geq (\delta - 1)(g - 2) - m' \).

Proof. Since \( x_i \) is a forcing vertex in step \( i \), we have

\[
|N(x_i) \cap \left( S \cup \bigcup_{j=1}^{i-1} N[x_j] \right)| = \deg_G(x_i) - 1.
\]

Note that for every edge \( \{x_j, x_i\} \in E' \) with \( j < i \), we have either

- \( \{x_j, x_i\} \in E \) and \( N(x_i) \cap N(x_j) = \emptyset \), or
- \( \{x_j, x_i\} \notin E \) and \( |N(x_i) \cap N(x_j)| = 1 \).

This implies

\[
|N(x_i) \cap \bigcup_{j=1}^{i-1} N[x_j]| \leq \{|x_j : j < i \text{ and } \{x_i, x_j\} \in E'\}|,
\]

and consequently

\[
|S| = \left| S \cap \left( N(x_i) \setminus \bigcup_{j=1}^{i-1} N[x_j] \right) \right| \geq \deg_G(x_i) - 1 - \{|x_j : j < i \text{ and } \{x_i, x_j\} \in E'\}|.
\]
Using the fact that the sets $S_i$ are pairwise disjoint, we obtain

$$|S^*_X| = \sum_{i=1}^{g-2} |S_i| \geq \sum_{i=1}^{g-2} (\deg_G(x_i) - 1 - |\{x_j : j < i \text{ and } \{x_i, x_j\} \in E\}|) \geq (\delta - 1)(g - 2) - m',$$

where we used $\sum_{i=1}^{g-2} |\{x_j : j < i \text{ and } \{x_i, x_j\} \in E\}| = m'$. \hfill $\square$

**Lemma 2.** If $x \in X \setminus S_X$, then $\text{dist}_G(x, x') = 1$ for some $x' \in X$.

**Proof.** If $x \in X \setminus S_X$, then $x \in \tilde{S}$ or $x \in S^*_X$. If $x \in \tilde{S}$, then at some point in the forcing process a vertex $x' \in X$ forces $x$, which implies $\text{dist}_G(x, x') = 1$. If $x \in S^*_X$, then $x$ belongs to some $S_i$, $i \in [g - 2]$, which implies $x \in N(x_i)$, i.e., $\text{dist}_G(x, x_i) = 1$. \hfill $\square$

Next observe that Lemma[1] inequality[2], and our assumption on the cardinality of $S$, together provide the inequality

$$(\delta - 2)(g - 3) + \delta - 1 \geq |S| \geq |S_X| + |S^*_X| \geq |S_X| + (\delta - 1)(g - 2) - m'$$

which implies

$$m' \geq g - 3 + |S_X|.$$

Note that the girth of $H$ is at least $g/2$, since every edge in $H$ corresponds to an edge or a path of length 2 in $G$, and thus

$$m' \leq \text{ex}(g - 2, \{C_3, \ldots, C_{[(g-1)/2]}\}).$$

Since $g - 2 \leq 2[(g-1)/2]$, it follows by Theorem[1] that $m' \leq g$. Thus, $m' \in \{g - 2, g - 1, g\}$. That is, we have three separate cases to consider. We handle these cases next.

**Case 1:** If $m' = g - 2$ then $|S_X| = 1$, hence $|X \setminus S_X| = g - 3$ and by Lemma[2] at least $g - 3$ edges of $H$ correspond to edges of $G$. Consequently, a cycle of length $k$ in $H$ leads to a cycle of length at most $k + 1$ in $G$ (as there is at most one edge in the cycle that corresponds to a path of length two in $G$). Therefore, $H$ does not contain any cycle, hence $m' \leq g - 3$, which is the required contradiction.

**Case 2:** If $m' = g - 1$ then $|S_X| \leq 2$, hence $|X \setminus S_X| \geq g - 4$ and by Lemma[2] at least $g - 4$ edges of $H$ correspond to edges of $G$. Consequently, a cycle of length $k$ in $H$ leads to a cycle of length at most $k + 3$ in $G$, and therefore the girth of $H$ is at least $g - 3$. By Corollary[2] this implies $m' \leq g - 2$, which is the required contradiction.

**Case 3:** If $m' = g$ then $|S_X| \leq 3$, hence $|X \setminus S_X| \geq g - 5$ and by Lemma[2] at least $g - 5$ edges of $H$ correspond to edges of $G$. Consequently, a cycle of length $k$ in $H$ leads to a cycle of length at most $k + 5$ in $G$, and therefore the girth of $H$ is at least $g - 5$. By Corollary[1] this implies $m' \leq g - 1$ which is the required contradiction.

This completes the proof of the theorem, and the conjecture presented in [8] is resolved in the affirmative. \hfill $\square$

### 3. Concluding remarks

Let $f(g, \delta)$ denote the minimum zero forcing number over all graphs of girth $g$ and minimum degree $\delta$. Theorem[2] provides a lower bound for $f$, and from [8] we know that this bound is tight in the following cases:

- $f(g, 2) = 2$ for all $g \geq 3$ (the $g$-cycle),
• $f(3, \delta) = \delta$ for all $\delta \geq 1$ (the complete graph $K_{\delta+1}$),
• $f(4, \delta) = 2\delta - 2$ for all $\delta \geq 2$ (the complete bipartite graph $K_{\delta,\delta}$),
• $f(4, 3) = 4$ (the 3-cube),
• $f(5, 3) = 5$ (the Petersen graph),
• $f(6, 3) = 6$ (the Heawood graph).

Consequently, the smallest open cases are the following.

**Question 1.** We know $7 \leq f(7, 3) \leq 8$ and $8 \leq f(8, 3) \leq 10$. Can we close these gaps?

**Question 2.** We know $f(5, 4) \geq 8$. What is the best upper bound we can come up with?

In general the bound $f(g, \delta) \geq \delta + (g-3)(\delta-2)$ is not sharp. For instance, using essentially the same argument as in the proof of Theorem 2, one can prove $f(g, \delta) \geq \delta + (g-3)(\delta-2) + 1$ for $g \geq 14$, $\delta \geq 3$, and more generally, for large values of $\delta$ and $g$ the exponential lower bound established in [13] is stronger than the bound from the present note. This motivates the following questions.

**Question 3.** What are upper bounds for $f(g, \delta)$?

**Question 4.** What can be said about the asymptotic behaviour of $f(g, \delta)$?

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