STRINGS AND NON-TOPOLOGICAL SOLITONS*

R. Fiore†

Dipartimento di Fisica, Università della Calabria
Istituto Nazionale di Fisica Nucleare, Gruppo collegato di Cosenza
Arcavacata di Rende, I-87036 Cosenza, Italy

D. Galeazzi, L. Masperi‡, A. Megevand

Centro Atómico Bariloche and Instituto Balseiro
(Comisión Nacional de Energía Atómica and Universidad de Cuyo)
8400 S. C. de Bariloche, Argentina

Abstract

We have numerically calculated topological and non-topological solitons in
two spatial dimensions with Chern-Simons term. Their quantum stability, as
well as that of the Maxwell vortex, is analyzed by means of bounce instantons
which involve three-dimensional strings and non-topological solitons.

*Work supported in part by the Ministero dell’Università e della Ricerca Scientifica
e Tecnologica of Italy and in part by the Consejo Nacional de Investigaciones Ci-
entíficas y Técnicas of Argentina

†email address: 40330::FIORE
FIORE@COSENZA.INFN.IT

‡email address: MASPERI@ARIB51.BITNET
1. Introduction

Gauge models of a scalar field with Chern-Simons term in two spatial dimensions are an interesting theoretical laboratory for the study of both topological and non-topological solitons. Their existence has been suggested in the thin-wall approximation beyond the favourable region of parameters for the potential giving metastable configurations which provide an inhomogeneous nucleation mechanism for first-order phase transition. The importance of these solitons depends on their quantum stability that is affected by bounce instantons which involve the existence of three-dimensional defects.

We have proved by numerical computation the existence of the two-dimensional solitons with the above anticipated features. At the same time we have examined the possibility of their quantum decay as well as that of the Maxwell vortex. This analysis in thin-wall approximation requires the existence of strings which correspond to a model enlarged with a second scalar field for which well-known three-dimensional $Q$-balls are possible, producing quantum decays of the two-dimensional solitons.

The $3 + 1$ generalization of the Chern-Simons theory may be interpreted as an axion toy model whose strings might give an inhomogeneous nucleation mechanism for the phase transition of a simplified electroweak symmetry breaking.

In section 2 we give a description using thin-wall approximation of the two and three dimensional defects which will be relevant for our discussion. Section 3 contains the numerical solutions of two dimensional Chern-Simons topological and non-topological solitons. Section 4 treats the quantum decays of Maxwell and Chern-Simons two-dimensional defects using instantons in thin-wall approximation. Some outlook is contained in section 5.

2. Defects of abelian gauge model for scalar fields

We begin describing in the thin-wall approximation the solitons which appear for a theory with two scalar fields coupled to independent $U(1)$ gauge fields, which will be relevant for the subsequent discussion. Let us consider the Lagrangian

$$\mathcal{L} = (D_\mu \phi)^* D^\mu \phi + (D_\mu \sigma)^* D^\mu \sigma - V(|\phi|, |\sigma|) + T_{\text{kin}},$$  \hspace{1cm} (1)$$

where

$$D_\mu \phi = (\partial_\mu + ig'Z_\mu) \phi, \quad D_\mu \sigma = (\partial_\mu + igA_\mu) \sigma$$
and it is necessary that the potential has two minima for different values of $|\phi|$ and $|\sigma|$, being e.g. of the form

$$V(|\phi|, |\sigma|) = h\left(|\phi|^2 - v^2\right)^2 + g|\sigma|^4 + f|\phi|^2|\sigma|^2 - M^2|\sigma|^2.$$  \hspace{1cm} (2)

The kinetic term $T_{kin}$ for gauge fields will be specified below.

If the two minima correspond to

$$|\phi| = v, \quad \sigma = 0$$  \hspace{1cm} (3)

and to

$$\phi = 0, \quad |\sigma| = \sigma_0$$  \hspace{1cm} (4)

for the case that the stable vacuum is represented by eq.(3), in two spatial dimensions one may build a vortex with $\phi$ such that

$$\phi \to v e^{i\theta}, \quad \sigma \to 0 \quad \text{for} \quad r \to \infty$$  \hspace{1cm} (5)

together with an angular component of gauge field $Z_\theta$ to make the energy finite. To minimize the energy it is convenient that in the vortex core $\phi \sim 0, \sigma \sim \sigma_0$.

Choosing the Maxwell kinetic term for the relevant gauge field:

$$T_{kin}^M = -\frac{1}{4\pi} (\partial_\mu Z_\nu - \partial_\nu Z_\mu)^2,$$  \hspace{1cm} (6)

the vortex energy, or energy per unit string length if the third spatial dimension $z$ is added, for static configurations will be

$$E_v = aR^2 + bR + \frac{c}{R^2}$$  \hspace{1cm} (7)

in terms of core radius $R$. Here the coefficients $a$, $b$ and $c$ are positive, the first term comes from the false vacuum region, the second one from the interface contribution whereas the third one arises from the magnetic energy with the constraint of constant flux $\pi R^2 B$.

The minimization of eq.(7) with respect to $R$ gives a classically stable configuration. If we additionally consider excitations of the core,

$$\sigma = \exp[i\nu(z,t)]\sigma_0,$$
where also the second gauge field $A$ will be present, which we may take as the electromagnetic one, the superconducting cosmic strings \[5\] emerge stabilized by constancy of the phase change along a closed string:

$$\oint dl \frac{d\nu}{dl} = 2\pi N.$$  

For the Lagrangian \(\mathcal{L}\) without gauge fields and with stable vacuum \(\mathcal{P}\) it is also possible to have classically stable configurations of real $\phi$ in three spatial dimensions where, inside a localized region $\phi \sim 0$, $\sigma \sim \sigma_0 e^{-i\omega t}$. Now, again in the thin-wall approximation, the energy will have the radius dependence

$$E_\sigma = aR^3 + bR^2 + \frac{c}{R^3},$$  

where the last term comes from the kinetic term of $\sigma$ with the condition of fixed charge. The minimization of the energy with respect to $R$ produces a non topological soliton called $Q$-ball which is proved to be stable for a charge larger than a threshold \([4]\).

If the parameters of the potential \(\mathcal{P}\) are changed so that the minimum \(\mathcal{P}\) is the absolute one, one has $a < 0$ and the string and the $Q$-ball, in case they survive, become bubbles of the corresponding stable phase in the metastable sea characterized by eq. \(\mathcal{P}\). Therefore, they may give way to an inhomogeneous nucleation mechanism for the transition from the $|\phi| = v$ phase to the $\phi = 0$ one as noticed for Maxwell vortices \([6]\).

Taking the Lagrangian \(\mathcal{L}\) with a complex $\phi$ coupled to $Z_\mu$ and a real $\sigma$ new types of defects may appear if we replace the Maxwell kinetic term in $3 + 1$ dimensions by

$$T_{kin}^{C-S} = \frac{1}{2} \gamma_{\mu\nu\rho\sigma} \partial_\mu Z_\nu \partial_\rho Z_\sigma,$$  

where notations will be justified below. The energy for static configurations will be

$$E = \int d^3x \left[ |\tilde{B}\phi|^2 + (\tilde{\nabla} \sigma)^2 + V + \frac{\gamma^2}{4} \left( \frac{\tilde{\nabla} \sigma \cdot \tilde{B}}{\phi} \right)^2 \right],$$  

where minimization on $Z_0$ has been performed.

We must note that in $2 + 1$ dimensions $\tilde{B} = B_z$ and $\partial_z \sigma$ may be replaced by a constant. In this way the problem described by eq.s \(\mathcal{P}\) and \(\mathcal{L}\) reduces to
that with Chern-Simons term which has been studied \[1\] for a sextic potential in $\phi$ without $\sigma$ and with two degenerate minima showing the existence of topological vortices for $|\phi| \to v$ and non topological solitons for $\phi \to 0$ when $r \to \infty$. The latter are possible, at variance from the Maxwell case, because the last term of eq.\[1\] requires that $B$ is confined in the region where $\phi$ is nonvanishing. To pass from the potential \(2\) to an effective one $V_{\text{eff}}(\phi)$ depending only on $\phi$ formally requires a minimization with respect to $\sigma$. This shows that for $|\phi| \sim v$, if the reasonable condition $f v^2 > M^2$ is satisfied, $v$ is a minimum of $V_{\text{eff}} = 0$, whereas for $\phi \sim 0$ $V_{\text{eff}}(0) < V_{\text{eff}}(v)$ if $h v^4 < \frac{M^4}{4g}$.

The analysis of these two-dimensional solitons in the thin-wall approximation shows \[2\] that topological vortices are favoured when $|\phi| = v$ corresponds to the absolute minimum but that they survive when $\phi = 0$ becomes the true vacuum up to a critical difference between both minima. Conversely the non-topological solitons prefer the situation where $\phi = 0$ is the absolute minimum but survive as metastable states when the broken symmetry phase becomes stable up to a critical difference with the symmetric false vacuum. Therefore, these non-topological solitons provide an inhomogeneous nucleation mechanism for the first-order transition from the "over-frozen" symmetric phase to the stable broken-symmetry one when the potential energy difference is larger than the critical value and the solutions become classically unstable increasing their radius indefinitely. The existence of these solutions has been confirmed numerically as will be discussed in the next section.

Returning to the 3+1 dimensional case, the generalization of two dimensional solitons is not immediate as in the case of the Maxwell string, because $\partial_z \sigma$ must be nonvanishing but cannot be constant for all values of $r$. Therefore interesting solutions are $z$-dependent configurations which, due to the equations for gauge field $Z_\mu$ coupled to the current $J_\mu$:

$$\frac{\gamma}{2} \vec{\nabla} \sigma \cdot \vec{B} = J_\theta , \quad \frac{\gamma}{2} \vec{\nabla} \sigma \times \vec{E} = \vec{J} ,$$

(12)
together with the additional radial dependence of $\sigma$, $Z_\theta$ and $Z_0$, give charge and angular $J_\theta$ current. If we wish $\sigma \sim 0$ in the core and $\sigma \sim \sigma_0$ outside, charge and current will be concentrated on the string surface.

To avoid values of $z$ for which the string size vanishes it is necessary that $\sigma$ is an increasing function of $z$. This is possible if $\sigma$ is originally a phase as occurs in the
Peccei-Quinn axion model \([7]\). In this case the renormalizable potential \((2)\) must be replaced by e.g.

\[
V(|\phi|, |\sigma|) = h\left(|\phi|^2 - v^2\right)^2 + \left[m_a^2 f_{PQ}^2 + k\left(|\phi|^2 - v^2\right)\right] \left(1 - \cos \frac{\sigma}{f_{PQ}}\right)
\]

which has a similar form for small \(\sigma\) if \(kv^2 > m_a^2 f_{PQ}^2\). Eq. \((13)\) corresponds to the axion effective potential \([8]\) when \(|\phi| = v\) whereas for \(\phi = 0\) has its minimum for \(\sigma/f_{PQ} = \pi\). This may be qualitatively interpreted by the fact that before the electroweak symmetry breaking transition there is no contribution of \(\arg \det \mathcal{M}\), being \(\mathcal{M}\) the fermion mass-matrix, to the strong \(CP\) violating parameter.

To understand the non-topological string in the thin-wall approximation one must consider the width \(\Delta\) around an average radius \(R\) in which the transition between the core at \(\sigma = 0\) and the outside region at \(\sigma/f_{PQ} = \pi\) may occur. Moreover one has to keep in mind that the true variable is \(U = \exp(i\frac{\sigma}{f_{PQ}})\) which passes from 1 to \(-1\). As indicated in Fig.1, starting from a value of \(z\) for which the transition occurs just below \(R + \frac{\Delta}{2}\), \((a)\), increasing \(z\) we lower the transition point almost down to \(R - \frac{\Delta}{2}\), \((b)\). At this moment for the remaining very small region still at \(\sigma/f_{PQ} = 0\) there is a jump to \(2\pi\) \((c)\). Then the transition point moves to the right and when it almost reaches \(R + \frac{\Delta}{2}\), \((d)\), the remaining small region still at \(\pi\) jumps to \(3\pi\) completing the cycle and returning to \((a)\).

For a closed string, along its length \(L\) an integer number of cycles must be performed which gives a sort of topological stability to the configuration. In first approximation the average radius of the core can be evaluated as in the two-dimensional case \([4]\) replacing the parameter \(\mu\) of the Chern-Simons term by \(\gamma \partial_z \sigma \approx \gamma f_{PQ} \cdot 2\pi \frac{N}{4}\). It is clear that the two discontinuities per cycle described above produce divergences of \(\partial_z \sigma\) which affect the second and last terms of the integrand of eq. \((11)\). This, however, gives only finite contributions to the energy due to the vanishingly small regions of \(r\) and \(z\) in which they appear. The jumps in \(\sigma\) are only possible as quantum effects which must be penalized by probability factors as occurs in cosmic strings for abrupt change of phase \([5]\). These jumps are the most economic ones for what concerns the energy of the configuration and allow the building of the non-topological string with only a gentle oscillation of transverse size around \(R\) going along its length.
3. Numerical solutions of Chern-Simons solitons

As indicated in the previous section, the two-dimensional Chern-Simons topological and non-topological solitons may provide an inhomogeneous nucleation mechanism for the transition to the symmetric or to the symmetry-broken phase respectively.

For that it is important to prove numerically the existence of the solution outside the favourable region of the potential parameters. Taking the sextic potential

\[ V_{\text{eff}} = \frac{1}{\mu^2} \left( |\phi|^2 - v^2 \right)^2 \left( |\phi|^2 + \lambda v^2 \right) \] (14)

which has two minima for \(-1 < \lambda < \frac{1}{2}\), and representing the relevant fields as

\[ \phi(r, \theta) = v \rho(r) e^{i n \theta}, \quad Z_\theta(r) = \frac{1}{r} [n - \zeta(r)] , \] (15)

the static classical solutions must satisfy \[3\]

\[
\frac{d^2 \zeta}{dx^2} - \frac{1}{x} \frac{d \zeta}{dx} - \frac{2}{\rho} \frac{d \rho}{dx} \frac{d \zeta}{dx} - \frac{4 \rho^2 \zeta}{\mu^2} = 0, \\
\frac{d^2 \rho}{dx^2} + \frac{1}{x} \frac{d \rho}{dx} + \frac{\mu^2}{4 x^2 \rho^3} \left( \frac{d \zeta}{dx} \right)^2 - \frac{1}{x^2} \zeta^2 \rho - \rho \left( \rho^2 - 1 \right) \left( 3 \rho^2 + 2 \lambda - 1 \right) = 0, \] (16)

where \(x = v^2 r\) and we have included the coupling constant \(g’\) into \(Z_\theta\).

This highly non-linear system of equations has been numerically solved for the topological

\[ \rho = 1, \quad \zeta = 0 \quad \text{for} \quad r \to \infty, \]

\[ \rho = 0, \quad \zeta = n \quad \text{for} \quad r \to 0 \] (17)

asymptotic behaviours taking \(n = 1\), and for the non-topological ones

\[ \rho = 0, \quad \zeta = -\alpha \quad \text{for} \quad r \to \infty, \]

\[ \rho = 1, \quad \zeta = 0 \quad \text{for} \quad r \to 0 \] (18)

taking \(n = 0\) and \(\alpha = -0.8\), choosing in all cases \(\mu^2 = 1\).

The natural region for the topological solutions corresponds to \(\lambda > 0\) where the absolute minimum of \(V_{\text{eff}}\) is at \(|\phi| = v\). But solutions appear also for \(\lambda < 0\), the
criterium being that $B$ decreases for large values of $x$ where $\rho$ has already attained its asymptotic limit in order to ensure finiteness of energy. Acceptable and non acceptable solutions for $\lambda < 0$ are shown in Fig. 2 (a) and (b) respectively. From the numerical results we may establish that $-0.1 > \lambda_{\text{crit}}^T > -0.2$.

On the other hand the natural region for non-topological solitons is $\lambda < 0$ where the absolute minimum of $V_{\text{eff}}$ is at $\phi = 0$. Acceptable solutions appear also for $\lambda > 0$ (Fig.3(a)) up to a critical value beyond which incorrect behaviours for $\zeta$ and $B$ at small $x$ force us to reject them (Fig.3(b)) which allows us to estimate that $\lambda_{\text{crit}}^{NT} \simeq 0.14$.

4. Quantum stability of Maxwell vortices and Chern-Simons non-topological solitons

The quantum stability of two-dimensional solitons is related to the three-dimensional ones which appear as bounce type instantons in Euclidean metric \cite{3} giving a decay probability $e^{-S}$ with $S$ bounce action taken as difference between instantons and unperturbed string actions.

In order to compare with our Chern-Simons solitons we start from what occurs to Maxwell vortices. Apart from the possibility of a monopole-antimonopole pair produced at higher scale if a hierarchy symmetry exists which would allow an intermediate state without flux, another channel corresponds to a bounce which includes $Q$-balls existing in the same generalization to 3+1 dimensions of the model. In fact for Lagrangian \cite{1} with potential \cite{2}, if the true vacuum corresponds to eq.(3) for which the vortex is favoured, the three-dimensional string may end on a $Q_\sigma$-ball of mass $E_\sigma$ made by the charge of field $\sigma$ which is stable in this case. Since beyond it the intermediate configuration has $\phi = v$, the magnetic flux is dispersed in an unconfined region of vanishingly small field (Fig.4(a)) that corresponds to the broken-symmetry vacuum to which the vortex may decay. The bounce action will be estimated as the mass of the $Q_\sigma$-ball pair and its Coulomb interaction

$$S = 2E_\sigma - \frac{Q_\sigma^2}{l} - E_v l ,$$

where the last term is the missing string energy with $E_v$ energy per unit length, and the unstable maximum with respect to $l$ must be taken.

If the potential is such that eq.(4) corresponds to stable vacuum, the possible
$Q$-ball is that inside which there is the charge corresponding to field $\phi$. This defect, where fields $\phi$ and $\sigma$ are exchanged respect to the discussion of section 2, will be denoted as $Q_\sigma$-ball. Now the vortex is clearly metastable so that the needed instanton requires a small $Q_\phi$-ball acting as an obstacle that deviates slightly the magnetic flux (Fig.4(b)) and giving an intermediate configuration of larger size which corresponds to the state reached by tunnel effect through the wall of eq.(7). The bounce action is similar to that of eq.(13) but the mass $E_\phi$ of the $Q_\phi$-ball is smaller and smaller for deeper minimum of eq.(2) at $\phi = 0$, making the decay more probable.

For the Chern-Simons non-topological soliton in two dimensions the potential (13) must be taken to allow the extension to three dimensions. Now the situation is somehow reversed compared to the Maxwell vortex. When the stable vacuum corresponds to eq.(4) the stability of non-topological soliton is favoured, the only instanton which allows its decay being that where the string ends in a $Q_\phi$-ball which disperses the flux since magnetic field must vanish for $\phi = 0$ (Fig.5(a)). On the contrary, when the stable vacuum becomes that of eq.(3), the obstacle of a small $Q_\sigma$-ball (2) inside which $\phi = 0$ enlarges the size of the defect(Fig.5(b)) in correspondence to tunnel effect from the metastable initial state.

5. Conclusions

We have numerically shown the existence of two-dimensional Chern-Simons solitons also outside their favoured range of potential parameters, confirming what anticipated by the thin-wall approximation. The faster decay of the metastable case has been understood by the extension of the model to 3+1 dimensions. The three-dimensional non-topological string appears in a model which can be understood as that of a scalar Higgs field in interaction with an electrically neutral gauge field additionally coupled to an axion field. Compared to the standard model, the $SU(2)$ gauge coupling constant has to be put equal to zero, apart from taking no care about the phenomenological values of the axion sector.

Making the model more realistic and including temperature effects, studying numerically the three-dimensional non-topological strings and analyzing their quantum stability it would be possible to compute their relevance as an inhomogeneous nucleation mechanism for the phase transition of the electroweak symmetry break-
ing.

**Acknowledgement:** We are indebted to A. Quartarolo for his collaboration in technical aspects of the manuscript. L.M. wishes to thank the hospitality at the Dipartimento di Fisica dell’Università della Calabria where part of this work has been performed.
References

[1] R.Jackiw and S.-Y.Pi, Phys. Rev. Lett. 64 (1990) 1969; R.Jackiw and E.Weinberg, Phys. Rev. Lett. 64 (1990) 2234; R.Jackiw, K.Lee and E.Weinberg, Phys. Rev. D42 (1990) 3488.

[2] L.Masperi, Mod.Phys. Lett. A7 (1992) 3245.

[3] J.Preskill and A.Vilenkin, ”Decay of Metastable Topological Defects”. Harvard Univ. preprint HUTP-92/A018; A.Vilenkin, Nucl. Phys. B196 (1982) 240.

[4] S.Coleman, Nucl. Phys. B262 (1985) 263; R.Friedberg, T.D.Lee and A.Sirlin, Phys. Rev. D13 (1976) 2739.

[5] E.Witten, Nucl. Phys. B249 (1985) 557.

[6] P.J.Steinhardt, Nucl. Phys. B190 (1981) 583; L.G.Jensen and P.J.Steinhardt, Phys. Rev. B27 (1983) 5549.

[7] R.Peccei and H.Quinn, Phys. Rev. Lett. 38 (1977) 1440; J.Kim, Phys. Rep. 150 (1987) 1.

[8] C.Vafa and E.Witten, Nucl. Phys. B234 (1984) 173; Phys. Rev. Lett. 53 (1984) 535.

[9] D.Bazeia, Phys. Rev. D43 (1991) 4074.
Figure Captions

Fig.1: Evolution of U along the non-topological string. Arrows indicate mapping of $R - \Delta r < r < R + \Delta r$. Triangles show values of U taken by most of the radial interval.

Fig.2: Numerical solution, (a) acceptable ($\lambda = -0.05$) and (b) non-acceptable ($\lambda = -0.25$), for the two-dimensional Chern-Simons topological vortex. We define $\beta = \frac{B}{v^2}$ and $x = rv^2$.

Fig.3: Numerical solution, (a) acceptable ($\lambda = 0.1$) and (b) non-acceptable ($\lambda = 0.2$), for the two-dimensional Chern-Simons non-topological soliton. We define $\beta = \frac{B}{v^2}$ and $x = rv^2$.

Fig.4: Instanton describing the decay of the two-dimensional Maxwell vortex. In case (a) the stable vacuum corresponds to $|\phi| = v$, $\sigma = 0$. The equivalent string ends on $Q_\sigma$-ball. The intermediate configuration indicates the decay to the vacuum. In case (b) the stable vacuum corresponds to $\phi = 0$, $|\sigma| = \sigma_0$. The equivalent string finds obstacle of $Q_\phi$. The intermediate configuration indicates the decay to a broader vortex.

Fig.5: Instanton describing the decay of the two-dimensional Chern-Simons non-topological soliton. The stable vacuum corresponds (a) to $\phi = 0$, $|\sigma| = \sigma_0$, (b) to $|\phi| = v$, $\sigma = 0$. 

This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-th/9310039v1
This figure "fig1-2.png" is available in "png" format from:

http://arxiv.org/ps/hep-th/9310039v1
This figure "fig1-3.png" is available in "png" format from:

http://arxiv.org/ps/hep-th/9310039v1
This figure "fig1-4.png" is available in "png" format from:

http://arxiv.org/ps/hep-th/9310039v1
Fig. 1
Fig. 2

Fig. 3
Fig. 5
