CLASSIFICATION OF SYLOW CLASSES OF PARABOLIC AND REFLECTION SUBGROUPS IN UNITARY REFLECTION GROUPS

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Abstract. Let $\ell$ be a prime divisor of the order of a finite unitary reflection group. We classify up to conjugacy the parabolic and reflection subgroups that are minimal with respect to inclusion, subject to containing an $\ell$-Sylow subgroup. The classification assists in describing the $\ell$-Sylow subgroups of unitary reflection groups up to group isomorphism. This classification also relates to the modular representation theory of finite groups of Lie type. We observe that unless a parabolic subgroup minimally containing an $\ell$-Sylow subgroup is $G$ itself, the reflection subgroup within the parabolic minimally containing an $\ell$-Sylow subgroup is the whole parabolic subgroup.

1. Introduction

Throughout let $G$ be a finite unitary reflection group acting on the complex vector space $V$ of dimension $n$. For each prime $\ell$ dividing the order of $G$, we classify the parabolic and reflection subgroups up to $G$-conjugacy that are minimal with respect to inclusion, subject to containing a $\ell$-Sylow subgroup. This classification has been done previously in [12] for finite real reflection groups via an algorithm based on the Borel-de Siebenthal algorithm [2]. It is clear that a parabolic/reflection subgroup minimally contains an $\ell$-Sylow subgroup of $G$ if and only if its order has the same $\ell$-adic valuation as $|G|$ and none of its parabolic/reflection subgroups have this property.

It is sufficient to consider irreducible $G$ since the parabolic/reflection subgroups of $G = G_1 \times \ldots \times G_k$ are of the form $H = H_1 \times \ldots \times H_k$ where each $H_i$ is a parabolic/reflection subgroup of $G_i$. The classification of irreducible unitary reflection groups was given by Shephard and Todd in [9], which can be found in Lehrer and Taylor [8, Chap. 8]. In [11] by Taylor, a classification of parabolic and reflection subgroups of unitary reflection groups is given up to conjugacy. Our classification will follow by a direct proof for the infinite family of reflection groups $G(m, p, n)$ and case by case computations for the 34 exceptional cases.

Recall a reflection subgroup of $G$ is a subgroup generated by reflections and a parabolic subgroup is the pointwise stabiliser $G_U$ of some subspace $U$ of $V$. By a theorem of Steinberg [10, Thm. 1.5], a parabolic subgroup is a reflection subgroup. The conjugacy class of parabolic subgroups minimally containing $\ell$-Sylow subgroups is unique since the class of parabolic subgroups is closed under conjugation and intersection. However, since the class of reflection subgroups is not closed under intersection, we do not necessarily have uniqueness of the conjugacy class of reflection subgroups minimally containing $\ell$-Sylow subgroups.

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Definition 1.1. Call the conjugacy class of parabolic subgroups minimally containing the $\ell$-Sylow subgroups the $\ell$-Sylow conjugacy class of parabolic subgroups. Refer to such a parabolic subgroup in the conjugacy class as $P_\ell$. In the case that $G = P_\ell$ we say that $\ell$ is cuspidal for $G$.

Definition 1.2. Call a conjugacy class of reflection subgroups minimally containing the $\ell$-Sylow subgroups a $\ell$-Sylow conjugacy classes of reflection subgroups. Refer to such a reflection subgroup as $R_\ell$ in one of these conjugacy classes. If $G = R_\ell$ we say that $\ell$ is supercuspidal for $G$.

We will first classify the $P_\ell$ up to $G$-conjugacy. It is then sufficient to consider only the cuspidal cases while classifying the $R_\ell$ up to $G$-conjugacy, since the parabolic closure of a $R_\ell$ is a $P_\ell$ by [12, Cor. 2.5].

The motivations for studying these minimal classes containing $\ell$-Sylow subgroups are various. A major application is in its assistance in describing the $\ell$-Sylow subgroups of unitary reflection groups. It is assists since the $\ell$-Sylow subgroup of some irreducible $G$ is the $\ell$-Sylow subgroup of some supercuspidal reflection group up to group isomorphism. We describe the $\ell$-Sylow subgroups of the supercuspidal cases in Section §6, with the supercuspidal cases summarised in Table 5. This gives a complete description of $\ell$-Sylow subgroups of unitary reflection groups up to group isomorphism.

Another application is in the modular representation theory of finite reductive groups. In particular, [5, Thm. 4.2] states the following:

Theorem 1.3 (Geck-Hisse-Malle). Let $G$ be a connected reductive group with connected centre and some $\mathbb{F}_q$ structure where $q$ is a power of a prime different from $\ell$. Let $F$ be the Frobenius morphism of $G$.

(i) If there does not exist an $F$-stable Levi subgroup $L$ of $G$ such that the $\ell$-adic valuations of $|G^F|$ and $|L^F|$ are equal, then the Steinberg character of $G^F$ is cuspidal.

(ii) Let $L$ be an $F$-stable Levi subgroup of $G$ such that the $\ell$-adic valuations of $|G^F|$ and $|L^F|$ are equal, and let $D$ be an $\ell$-Sylow subgroup of $L^F$. If $C_G(D)$ is a subgroup of $L$, then the semisimple vertex of the Steinberg character of $G^F$ is contained in $L^F$.

This theorem raises the question of finding the minimal Levi subgroups containing $\ell$-Sylow subgroups in finite reductive groups. In the case of Weyl groups, the classification of Sylow classes of parabolic subgroups answers a special case of this question, as will be seen in a forthcoming paper. Furthermore, in Enguehard and Michel [4, Thm. 3.2] it is seen that the $\ell$-Sylow subgroups of a finite reductive group depend partly on the $\ell$-Sylow subgroups of a unitary reflection group.

In §2 we introduce the basic definitions of unitary reflection groups and notation regarding the classification of reflection subgroups as seen in §1. In §3 we investigate some properties of these $\ell$-Sylow classes of parabolic and reflection subgroups. In §4 we classify the $\ell$-Sylow conjugacy class of parabolic subgroups. In §5 we classify the $\ell$-Sylow conjugacy classes of reflection subgroups for the cuspidal cases, allowing us to deduce $R_\ell$ in all cases. In §6 we use our classification of the supercuspidal cases seen in Table 5 to describe the $\ell$-Sylow subgroups of unitary reflection groups. In the tables of §8 we present the classification of $\ell$-Sylow class of parabolic subgroups, the cuspidal cases, the $\ell$-Sylow classes of reflection subgroups, the supercuspidal cases and the cases where the
\( \ell \)-Sylow classes of reflection subgroups are not unique. By inspection of these tables we are able to note the following observation.

**Observation 1.4.** If \( \ell \) is not cuspidal for an irreducible \( G \), it is supercuspidal for \( P_\ell \).

We can also observe from the tables which cases that the \( \ell \)-Sylow class of reflection subgroups is not unique. They are collected in Table 3 for clarity. It is interesting to note that in the real reflection group case the non-unique classes were made of isomorphic reflection groups generated by dual root systems as seen in [12]. However, extending to the unitary reflection groups, there are non-unique classes that are not made of isomorphic groups as seen in the cases \( G_9 \) with \( \ell = 3 \), \( G_{17} \) with \( \ell = 3 \), \( G_{18} \) with \( \ell = 2 \), \( G_{21} \) with \( \ell = 5 \) and \( G_{26} = M_3 \) with \( \ell = 3 \).

## 2. Notation and Preliminaries

A complex reflection is a linear transformation of \( V \) with finite order and fixed space is a hyperplane. A unitary reflection group or finite complex reflection group \( G \) on \( V \) is a finite group generated by complex reflections. The name unitary reflection group comes from the fact that every finite subgroup of \( GL(V) \) preserves a positive definite hermitian form \( (\cdot, \cdot) \) on \( V \) and so the group is unitary with respect to this hermitian form.

The imprimitive unitary reflection group \( G(m, p, n) \) introduced by Shephard and Todd is defined in [11] Section 3 in the following way. Let \([n] := \{1, 2, ..., n\}\) and \(\{e_i \mid i \in [n]\}\) be an orthonormal basis for the \( n \) dimensional complex vector space \( V \). Let \( \mu_m \) be the group of \( m \)th roots of unity. We write \( \theta \) as the linear transformation that maps \( e_i \) to \( \theta(i) e_i \) for each \( 1 \leq i \leq n \), where \( \theta : [n] \to \mu_m \). Let \( p \mid m \), define \( A(m, p, n) \) to be the group of all linear transformations \( \hat{\theta} \) such that \( \prod_{i=1}^{n} \theta(i)^{\frac{mp}{p}} = 1 \). We also define the action of \( \pi \in \text{Sym}(n) \) on \( V \) to be \( \pi(e_i) = e_{\pi(i)} \). The group \( G(m, p, n) \) is the semidirect product of \( A(m, p, n) \) by \( \text{Sym}(n) \). Shephard and Todd [3] proved every irreducible unitary reflection group belongs to a list \( G_k \) for \( 1 \leq k \leq 37 \). We call the \( k \) the Shephard-Todd number of the irreducible unitary reflection group. The Shephard-Todd numbering can be found in [3]. For example, \( G_1 = G(1, 1, n) \cong \text{Sym}(n) \) with \( n \geq 2 \), \( G_2 = G(m, p, n) \) with \( m > 1 \) and \( n > 1 \), \( G_3 = G(m, 1, 1) \cong C_m \) with \( m > 1 \), \( G_4 = G_{22} \) are the primitive rank two groups and \( G_{23} = G_{37} \) are the primitive groups of rank greater than two.

We now introduce some notation seen in [11]. This is necessary to understand the classifications of parabolic subgroups and reflection subgroups of \( G(m, p, n) \) up to conjugacy.

**Definition 2.1.** Call \((m', p', n')\) a feasible triple for \( G(m, p, n) \) if \( m, p \) and \( n \) are positive integers and \( n' \leq n \), \( p' \mid m', m' \mid m \) and \( \frac{mp}{p'} \mid \frac{m'}{p'} \).

By [11] Lemma 3.3] we have \((m', p', n')\) is a feasible triple if and only if \( G(m', p', n') \) arises as a reflection subgroup of \( G(m, p, n) \). We define a total order on the feasible triples by writing \((m_1, p_1, n_1) \geq (m_2, p_2, n_2)\) if \( n_1 > n_2 \); or \( n_1 = n_2 \) and \( m_1 > m_2 \); or \( n_1 = n_2, m_1 = m_2 \) and \( p_1 \geq p_2 \).

**Definition 2.2.** A partition \( \lambda \) of \( n \), written \( \lambda \vdash n \), is a sequence \( \lambda = (n_1, n_2, ..., n_d) \) where \( n_1, n_2, ..., n_d \) are positive integers such that \( n = \sum_{i=1}^{d} n_i \) and \( n_1 \geq n_2 \geq ... \geq n_d \).
Theorem 2.4. Any reflection subgroup of $G(m, p, n)$ is conjugate to $G_{\Delta}^\alpha$ for some augmented partition $\Delta$ and $\alpha \in \mu_m$. The parabolic subgroups of $G(m, p, n)$ are of the form

(i) $G_{\Delta} = G(m, p, n_j) \times \prod_{i \neq j} G(1, 1, n_i)$, where $(n_i) \vdash n$,

(ii) $G_{\Delta}^\alpha = \hat{\theta}_\alpha G_{\Delta} \hat{\theta}_\alpha^{-1}$, where all feasible triples have $m_i = p_i = 1$ and $\alpha \in \mu_m$.

3. Properties of $P_\ell$ and $R_\ell$

In this section we note some properties of the $\ell$-Sylow class of parabolic subgroups and the $\ell$-Sylow classes of reflection subgroups.

The next proposition involves the characterisation of unitary reflection group $G$ in terms of its ring of invariants being a polynomial algebra, where the product of the degrees of the algebraically independent homogeneous generating polynomials is the order of $G$ (see [8] Chap. 3-4]).

Proposition 3.1. Let $G$ be an irreducible unitary reflection group on $V$ and $d_1, ..., d_n$ be the degrees of the homogeneous algebraically independent polynomials generating the $G$-invariant polynomial algebra. If $\ell \mid d_i$ for all $1 \leq i \leq n$ then $\ell$ is cuspidal for $G$.

Proof. By [8] Corollary 3.24 the centre $Z(G)$ has order $\gcd(d_1, ..., d_n)$. Then by Schur’s Lemma $Z(G)$ acts on $V$ as scalar multiplication, so is a cyclic group acting on $V$ as the $\gcd(d_1, ..., d_n)$th roots of unity. Let $P$ be a proper parabolic subgroup of $G$, hence fixing some nonzero subspace $U$ of $V$. Then $P \cap Z(G) = \{1\}$, as this is the only element of $Z(G)$ that fixes $U$ pointwise. Therefore, $P$ cannot contain a $\ell$-Sylow subgroup if $|Z(G)| = \gcd(d_1, ..., d_n)$ is divisible by $\ell$. \qed
The following propositions give some structural properties of the normaliser of elements from these \( \ell \)-Sylow classes.

**Lemma 3.2.** Let \( S \) and \( T \) be subgroups of \( G \) both containing a \( \ell \)-Sylow subgroup. If \( T \) is a subgroup of \( N_G(S) \), then \( S \cap T \) contains an \( \ell \)-Sylow subgroup of \( G \).

**Proof.** Since \( S \) is a normal subgroup of \( N_G(S) \), we have \( ST \) is a subgroup of \( N_G(S) \) with \( |ST| = \frac{|S||T|}{|S \cap T|} \). Since \( |ST| \) divides \( |G| \), the \( \ell \)-adic valuations of \( |S \cap T| \) and \( |G| \) must be equal and so the lemma follows. \( \square \)

**Proposition 3.3.** \( N_G(P_\ell) \) contains only \( P_\ell \) from the \( \ell \)-Sylow class of parabolic subgroups.

**Proof.** Let \( P_\ell' \) be an element of the \( \ell \)-Sylow class of parabolic subgroups and also a subgroup of \( N_G(P_\ell) \). By Lemma 3.2, \( P_\ell \cap P_\ell' \) contains an \( \ell \)-Sylow subgroup. Since the class of parabolic subgroups is closed under intersection \( P_\ell \cap P_\ell' \) is a parabolic subgroup containing an \( \ell \)-Sylow subgroup, and so by minimality of \( P_\ell \) and \( P_\ell' \) we have \( P_\ell = P_\ell' \). \( \square \)

This statement is not true for the \( \ell \)-Sylow classes of reflection subgroups as the intersection of reflection subgroups is not necessarily a reflection subgroup. A counterexample for the above proposition in the case of reflection subgroups is the dihedral group of order 12 with \( \ell = 3 \). There are two non-conjugate reflection subgroups of type \( A_2 \) minimally containing 3-Sylow subgroups. Meanwhile, the normaliser of one of these reflection subgroups is the full group.

**Proposition 3.4.** The only elements of the \( \ell \)-Sylow classes of reflection subgroups that \( N_G(P_\ell) \) contains are those contained in \( P_\ell \).

**Proof.** Let \( R \) be a reflection subgroup in the \( \ell \)-Sylow classes of reflection subgroups and in \( N_G(P_\ell) \). By Lemma 3.2, \( P_\ell \cap R \) contains an \( \ell \)-Sylow subgroup. Since the intersection of a parabolic subgroup and a reflection subgroup is a reflection subgroup [12, Lemma 2.2], the minimality of \( R \) implies that \( R \) is a subgroup of \( P_\ell \). \( \square \)

**Proposition 3.5.** The only \( \ell \)-Sylow subgroups that \( N_G(P_\ell) \) contains are those contained in \( P_\ell \).

**Proof.** This is clear from Lemma 3.2. \( \square \)

**Proposition 3.6.** The only \( \ell \)-Sylow subgroups that \( N_G(R_\ell) \) contains are those contained in \( R_\ell \).

**Proof.** Same as above, replacing \( P_\ell \) by \( R_\ell \). \( \square \)

4. **Classifying the \( \ell \)-Sylow Class of Parabolic Subgroups**

We will now classify the \( \ell \)-Sylow class of parabolic subgroups for each irreducible \( G \). We define the \( \ell \)-adic valuation of a positive integer as \( \nu_\ell(n) := \max\{v \in \mathbb{N} \mid \ell^v | n\} \). Since the \( \ell \)-Sylow class of parabolic subgroups is always unique, it is made up of the smallest order parabolic subgroups whose order have the same \( \ell \)-adic valuation as \( |G| \).

For the exceptional cases \( G_4 - G_{27} \) we have the classification of parabolic subgroups of rank greater than two primitive unitary reflection groups from [11] and can deduce...
for the rank two primitive unitary reflection groups from [8] Chapter 6 and Table D.1. This gives the classification for $G_k$ for $4 \leq k \leq 37$ as seen in Table 1.

For $G_1 = G(1, 1, n)$, we have already done the classification in [12] Table 1 as it is the same as the classification for the real reflection group of type $A_{n-1}$. For $G_3 = G(m, 1, 1)$, where $m > 1$ the classification is simply the whole group as it has no proper parabolic subgroup other than the trivial group by Theorem 2.4. These classifications of $\ell$-Sylow classes of parabolic subgroups are found in [1] for the $G_2$ case we will need a result on the $\ell$ divisibility of factorials since $\nu_\ell(\ell G(m, p, n)) = n\nu_\ell(m) - \nu_\ell(p) + \nu_\ell(n!)$. Now we state a form of Kummer’s Theorem for multinomial coefficients.

**Lemma 4.1.** If $\lambda \vdash n$, then $\nu_\ell(n!) - \sum_{i=0}^{k} \nu_\ell(\lambda_i !)$ is equal to the number of carries when summing $\lambda_i$ in base $\ell$.

Furthermore, we have the following Corollary of Lemma 4.1 proved in [12] Lemma 2.6.

**Corollary 4.2.** Let the base-$\ell$ expression of $n$ be $(b_r, b_{r-1} \ldots b_1 b_0)_\ell$. Then the partition $\lambda \vdash n$ that provides the minimum of the set

$$\left\{\prod_{i=1}^{k} \lambda_i ! \mid \nu_\ell(n!) = \sum_{i=1}^{k} \nu_\ell(\lambda_i !)\right\}$$

is given by the join $\lambda = \lambda^r \cup \lambda^{r-1} \cup \ldots \cup \lambda^1 \cup \lambda^0$ where $\lambda^i = (\ell^i, \ldots, \ell^i)$ with length $b_j$.

**Theorem 4.3.** Let $\ell$ be a prime divisor of the order of $G_2 = G(m, p, n)$, where $m > 1$ and $n > 1$. Let the base-$\ell$ expression of $n$ be $(b_r, b_{r-1} \ldots b_1 b_0)_\ell$.

- (i) If $\ell \mid m$, then the $\ell$-Sylow class of parabolic subgroups of $G_2$ has a unique element $G(m, p, n)$.
- (ii) If $\ell \nmid m$, then the $\ell$-Sylow class of parabolic subgroups of $G_2$ is the $G_2$-conjugates of $\prod_{i=1}^{k} G(1, 1, \ell^i)^b_i$.

**Proof.** Suppose $\ell \mid m$. Suppose $P_\ell = G_\Delta$. Now $G_\Delta$ must be contained in the reflection subgroup $K := G(m, p, n_j) \times \prod_{i \neq j} G(\ell^{n_i}(m), \ell^{n_i}(p), n_i)$. By Kummer’s Theorem we know $\sum_{i=1}^{d} \nu_\ell(n_i !) \leq \nu_\ell(n !)$ with equality if and only if there are no carries when summing the $n_i$ in base $\ell$. By inspection of the order of $K$ and $G_2$ it is clear we require the no carries case to preserve the $\ell$-adic valuation from $G_2$ to $K$. Furthermore we require $n_i \nu_\ell(m) - \nu_\ell(p) = 0$ for each $i \neq j$ so that we preserve the $\ell$-adic valuation from $K$ to $G_\Delta$. Since $\nu_\ell(m) \geq \nu_\ell(p)$, the only solution is when $\nu_\ell(m) = \nu_\ell(p)$ and $n_i = 1$ for all $i \neq j$. To preserve the $\ell$-adic valuation from $G_2$ to $G_\Delta$ we must have $n_j \nu_\ell(m) - \nu_\ell(p) = n \nu_\ell(m) - \nu_\ell(p)$, and so $n_j = n$ and so the set of $i \neq j$ must be empty. Hence, the $\ell$-Sylow class of parabolic subgroups of $G_2$ is itself $G(m, p, n)$ when $n \mid m$.

Now suppose $\ell \nmid m$. It is clear that $G(1, 1, n)$ contains an $\ell$-Sylow subgroup of $G$, reducing to the case of $G_1$. \qed
SYLOW CLASSES OF REFLECTION SUBGROUPS

5. CLASSIFYING THE ℓ-SYLOW CLASS OF REFLECTION SUBGROUPS

We classify the ℓ-Sylow conjugacy classes of reflection subgroups of unitary reflection groups for the cuspidal cases seen in Table 2 in [11] a classification of reflection subgroups is given up to conjugacy. Using the tables in [11] we can deduce the classification of ℓ-Sylow conjugacy classes of reflection subgroups for G4 − G37. These can be found in Tables 3 and 4. The ℓ-Sylow conjugacy classes of reflection subgroups of G1 seen in Table 3 follows from [12]. We now classify for the cases G2 and G3.

Theorem 5.1. Let ℓ be a prime divisor of the order of G2 = G(m, p, n), where m > 1 and n > 1. Let the base-ℓ expression of n be (bkbk−1...b1b0)ℓ.

(i) If ℓ | p, then the ℓ-Sylow classes of reflection subgroups are the conjugacy classes of \( \hat{G} \hat{\theta}_0 G(ℓμ(m), ℓμ(p), n)\hat{θ}_0^{-1} \), where \( α \in \{ e^{\frac{ik}{m}} | 0 ≤ k ≤ \gcd(\frac{p}{pμ(p)}, n) − 1 \} \).

(ii) If ℓ ∤ p, then the ℓ-Sylow class of reflection subgroups is the unique conjugacy class of \( \prod_{i=0}^{k} G(ℓμ(m), 1, ℓ^i b_i) \).

Proof. By the cuspidal cases seen in Table 2 we only need to consider when ℓ | m. Let \( R_ℓ \) be such that it is GΔ = \( \prod_{i=1}^{d} G(m_i, p_i, n_i) \) or \( G^a_Δ \) for some augmented partition \( Δ \) and \( α \in \mu_m \). The \( m_i \) and \( p_i \) must be powers of ℓ, as otherwise \( G_Δ \) contains a proper reflection subgroup with the same ℓ-adic valuation giving a contradiction of minimality. Therefore, \( G_Δ \) is a reflection subgroup of \( H := \prod_{i=1}^{d} G(ℓμ(m), ℓμ(p), n_i) \). By assumption \( ν_ℓ(|H|) = ν_ℓ(|G|) \), so we have

\[ nν_ℓ(m) − dν_ℓ(p) + \sum_{i=1}^{d} ν_ℓ(n_i!) = nν_ℓ(m) − ν_ℓ(p) + ν_ℓ(n!) \]

Then by Kummer’s theorem we have \( dν_ℓ(p) ≤ ν_ℓ(p) \). Since \( d ≥ 1 \) and \( ν_ℓ(p) ≥ 0 \), we have either \( ν_ℓ(p) = 0 \) or \( d = 1 \).

Hence, we now also assume that \( ℓ | p \), so \( d = 1 \). This gives \( G_Δ = G(m_1, p_1, n) \) a reflection subgroup of \( H = G(ℓμ(m), ℓμ(p), n) \). Now \( ν_ℓ(|G_Δ|) = ν_ℓ(|H|) \) gives

\[ nν_ℓ(m_1) − ν_ℓ(p_1) = nν_ℓ(m) − ν_ℓ(p) \]

Also, since \( \frac{m_1}{p} | \frac{m}{p} \) we have

\[ ν_ℓ(m_1) − ν_ℓ(p_1) ≤ ν_ℓ(m) − ν_ℓ(p) \]

Subtracting this from the above equation gives \( ν_ℓ(m_1) ≥ ν_ℓ(m) \), since \( n > 1 \). Combining this with \( m_1 | m \) we have \( ν_ℓ(m_1) = ν_ℓ(m) \), and then \( ν_ℓ(p_1) = ν_ℓ(p) \) follows. So by Theorem 3.8 we know that there are \( \gcd(\frac{p}{pμ(p)}, n) \) ℓ-Sylow classes of reflection subgroups, each class made up of \( G_2 \)-conjugates of \( G_Δ = G(ℓμ(m), ℓμ(p), n) \), where \( α \in \{ e^{\frac{ik}{m}} | 0 ≤ k ≤ \gcd(\frac{p}{pμ(p)}, n) − 1 \} \).

We now assume \( ℓ ∤ p \). Hence, \( G_Δ = \prod_{i=1}^{d} G(m_i, p_i, n_i) \) is a subgroup of \( H = G(ℓμ(m), 1, n_i) \). If \( ν_ℓ(m_i) = ν_ℓ(m) \) for some \( i \) then \( ν_ℓ(p_i) = 0 \) so the group orders have equal ℓ-adic valuations. Now if \( ν_ℓ(m_i) = ν_ℓ(m) − x_i \), for some \( x_i ∈ \mathbb{Z}^+ \), then \( ν_ℓ(p_i) = −n_ix_i \), giving a contradiction. Hence, \( G_Δ = \prod_{i=1}^{d} G(ℓμ(m), 1, n_i) \) for some partition \( (n_i) ⊢ n \). Corollary 4.2 gives us the reflection subgroup \( \prod_{i=0}^{k} G(ℓμ(m), 1, ℓ^i b_i) \), which is clearly also minimal with respect to containment. The conjugacy class of groups of
this type is unique by Theorem 3.8 and any reflection subgroup preserving the ℓ-adic valuation of $G_3$ will contain an element in this conjugacy class. Hence, the result follows.

Theorem 5.2. Let $\ell$ be a prime divisor of the order of $G_3 = G(m, 1, 1)$, where $m > 1$. The $\ell$-Sylow reflection subgroup of $G_3$ is $G(\ell^{\nu(m)}, 1, 1)$.

Proof. Trivial.

6. Sylow subgroups of Unitary reflection groups

Our classification of $\ell$-Sylow classes of reflection subgroups significantly reduces the work required to describe the group isomorphism types of the $\ell$-Sylow subgroups of unitary reflection groups. To describe all $\ell$-Sylow subgroups of unitary reflection subgroups up to group isomorphism, it is sufficient to describe them for the supercuspidal cases as seen in Table 5. This is true because we can firstly reduce to describing for irreducible unitary reflection group $G$, as the $\ell$-Sylow subgroup of a general unitary reflection group is the direct product of the $\ell$-Sylow subgroups of its components. Then for each irreducible $G$, its $R_\ell$ will contain an $\ell$-Sylow subgroup of $G$. We will now list the $\ell$-Sylow subgroups up to group isomorphism for all the supercuspidal cases. The description of the isomorphism classes of the unitary reflection groups can be found in Chap. 6 §2-§4 and Chap. 8 §10. We will require the following well known result regarding the $\ell$-Sylow subgroups of the Symmetric group that can be found on Pg. 82.

Proposition 6.1. The $\ell$-Sylow subgroups of $\text{Sym}(n)$ are isomorphic to $\prod_{i=1}^{k} [C_{\ell^i}]^{a_i}$, where $n = (a_k a_{k-1} \ldots a_1 a_0)_{\ell}$ and $C_{\ell^i}$ is the wreath product of $i$ copies of the cyclic group of order $\ell^i$.

We can now describe the $\ell$-Sylow subgroups up to isomorphism for the supercuspidal cases:

- For $G(1, 1, \ell^i)$: the group is isomorphic to $\text{Sym}(\ell^i)$, so by Proposition 6.1 the $\ell$-Sylow subgroups are isomorphic to $C_{\ell^i}$.
- For $G(\ell^i, \ell^j, n)$: the group is isomorphic to the semidirect product $A(\ell^i, \ell^j, n) \rtimes \text{Sym}(n)$, it is clear by Proposition 6.1 that the $\ell$-Sylow subgroups are isomorphic to $A(\ell^i, \ell^j, n) \rtimes \prod_{\zeta=1}^{k} [C_{\ell^j}]^{a_{\zeta}}$, where $n = (a_k a_{k-1} \ldots a_1 a_0)_{\ell}$.
- For $G(\ell^i, 1, \ell^j)$: the group is isomorphic to the semidirect product $A(\ell^i, 1, \ell^j) \rtimes \text{Sym}(\ell^j)$, it is clear by Proposition 6.1 that the $\ell$-Sylow subgroups are isomorphic to $A(\ell^i, 1, \ell^j) \rtimes C_{\ell^j}$.
- For $G(\ell^i, 1, 1)$ the $\ell$-Sylow subgroup is trivially the full group.
- For $G_4 = L_2$ and $\ell = 2$: the $2$-Sylow subgroups are isomorphic to quaternion groups.
- For $G_8$ and $\ell = 3$: the $3$-Sylow subgroup is trivially the cyclic group of order 3
- For $G_{12}$ and $\ell = 2$: which is isomorphic to the binary octahedral group, the $2$-Sylow subgroup is given by the generalised quaternion group.
For $G_{16}$ and $\ell = 2$: which is isomorphic to the direct product of a cyclic group of order 5 and the binary icosahedral group. Hence, the 2-Sylow subgroup of $G_{16}$ is the 2-Sylow subgroup of binary icosahedral group, which is the quaternion group.

- For $G_{16}$ and $\ell = 3$: the 3-Sylow subgroup is trivially the cyclic group of order 3.
- For $G_{20}$ and $\ell = 5$: the 5-Sylow subgroup is trivially the cyclic group of order 5.
- For $G_{24} = J_3^4$ and $\ell = 7$: the 7-Sylow subgroup is trivially the cyclic group of order 7.
- For $G_{25} = L_3$ and $\ell = 3$: this is a subgroup of $Sp_4(3)$, which has the same 3-divisibility. Let $N \cong C_3^3$ be the normal subgroup of $Sp_4(3)$ consisting of all unitriangular matrices in $Sp_4(3)$ such that the only nonzero entries above the diagonal are in the first row. This has a complementary subgroup of $H \cong C_3$ that consists of the unitriangular matrices in $Sp_4(3)$ such that the only nonzero entry above the diagonal is in the $(2,3)$ position. The conjugation action of $H$ on $N$ is given by $d(a, b, c) = (a, b - ad, c)$. Hence, the 3-Sylow subgroup of $E_6$ is $C_3^3 \rtimes C_3$ with this defined conjugation action of $C_3$ on $C_3^3$.
- For $G_{32} = L_4$ and $\ell = 2$: the group is isomorphic to the direct product of the cyclic group of order 3 and $Sp_4(3)$. From [3, Table 8.12] and [1, Proposition 4.10], $Sp_4(3)$ has a maximal subgroup isomorphic to $(SL_2(3) \times SL_2(3)) \rtimes C_2$, where the $C_2$ acts by swapping the factors. Hence the 2-Sylow subgroup of $L_4$ can be described as $(Q_8 \times Q_8) \rtimes C_2$ where $C_2$ acts by swapping the factors and $Q_8$ is the quaternion group.
- For $G_{32} = L_4$ and $\ell = 5$: the 5-Sylow subgroup is trivially the cyclic group of order 5.
- For $G_{35} = E_6$ and $\ell = 3$: the group has a subgroup isomorphic to $L_3$, which contains a 3-Sylow subgroup of $E_6$. Hence, the 3-Sylow subgroup is described in the same as $L_3$.

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8. Tables

We use the same notation as in [11] Section 5 and [8] Chapter 6. We note [11] Section 5 omits defining $B_n^{(3)} := G(3, 1, n)$.

| $G$ | $|G|$ | $\ell$ | $P_\ell$ | $|P_\ell|$ | $\ell$ cuspidal |
|-----|------|-------|----------|---------|----------------|
| $G_1 = G(1, 1, n)$, $n \geq 2$ | $n!$ | $\ell | n!$ | $\prod_{i=1}^{k} G(1, 1, \ell)^{b_i}$ | $\prod_{i=1}^{k} (\ell)^{b_i}$ | $n = \ell^q$ for $q \geq 2$ |
| $G_2 = G(m, p, n)$, $m > 1, n > 1$ | $\frac{m \cdot n!}{p}$ | $\ell | \frac{m \cdot n!}{p}$ | $\prod_{i=1}^{k} G(m, p, n)$ for $\ell | m$ | $\prod_{i=1}^{k} (\ell)^{b_i}$ for $\ell | m$ | $\ell | m$ |
| $G_3 = G(m, 1, 1)$, $m > 1$ | $m$ | $\ell | m$ | $G(m, 1, 1)$ | $m$ | $\ell | m$ |
| $G_4 = L_2$ | $2^k - 3$ | 2 | $L_2$ | $2^k - 3$ | 2 |
| $G_5 = C_1 \circ L_2$ | $2^k - 3$ | 2 | $G_4$ | $2^k - 3$ | 2 |
| $G_6 = L_3$ | $2^k - 3$ | 2 | $L_3$ | $2^k - 3$ | 2 |
| $G_7 = M_3$ | $2^k - 3$ | 2 | $M_3$ | $2^k - 3$ | 2 |
| $G_8 = J_1^{(3)}$ | $2^k - 3$ | 2 | $J_1^{(3)}$ | $2^k - 3$ | 2 |
| $G_9 = J_2^{(3)}$ | $2^k - 3$ | 2 | $J_2^{(3)}$ | $2^k - 3$ | 2 |
| $G_{10} = L_4$ | $2^k - 3$ | 2 | $L_4$ | $2^k - 3$ | 2 |
| $G_{11} = M_4$ | $2^k - 3$ | 2 | $M_4$ | $2^k - 3$ | 2 |
| $G_{12} = J_3^{(3)}$ | $2^k - 3$ | 2 | $J_3^{(3)}$ | $2^k - 3$ | 2 |
| $G_{13} = K_4$ | $2^k - 3$ | 2 | $K_4$ | $2^k - 3$ | 2 |
| $G_{14} = K_5$ | $2^k - 3$ | 2 | $K_5$ | $2^k - 3$ | 2 |
| $G_{15} = K_6$ | $2^k - 3$ | 2 | $K_6$ | $2^k - 3$ | 2 |
| $G_{16} = K_7$ | $2^k - 3$ | 2 | $K_7$ | $2^k - 3$ | 2 |
| $G_{17} = K_8$ | $2^k - 3$ | 2 | $K_8$ | $2^k - 3$ | 2 |

Table 1. Type of $P_\ell$ in $G$
| $G$ | Cuspidal $\ell$ | $G |$ |
|-----|----------------|---------|
| $G(1,1,\ell^i)$ for $i \in \mathbb{Z}^+$ | $\ell$ | $(\ell^i)!$ |
| $G(m,p,n)$ for $\ell | m$ | $\ell$ | $\frac{m!n!}{\ell^p}$ |
| $G(m,1,1)$ | $\ell$ | $m$ |
| $G_4 = L_2$ | 2 | $2^4 \cdot 3$ |
| $G_6$ | 2 | $2^4 \cdot 3$ |
| $G_k$, $k = 5$ and $7 \leq k \leq 22$ | $\ell | |G_k|$ | $|G_k|$ |
| $G_{23} = H_3$ | 2 | $2^4 \cdot 3 \cdot 5$ |
| $G_{24} = J_3^{(4)}$ | 7 | $2^4 \cdot 3 \cdot 7$ |
| $G_{25} = L_3$ | 3 | $2^3 \cdot 3^4$ |
| $G_{26} = M_3$ | 2, 3 | $2^4 \cdot 3^4$ |
| $G_{27} = J_3^{(6)}$ | 2, 3 | $2^4 \cdot 3^3 \cdot 5$ |
| $G_{28} = F_4$ | 2, 3 | $2^4 \cdot 3^2$ |
| $G_{29} = N_4$ | 2, 5 | $2^6 \cdot 3 \cdot 5$ |
| $G_{30} = H_4$ | 2, 3, 5 | $2^{10} \cdot 3^2 \cdot 5$ |
| $G_{31} = O_4$ | 2, 3, 5 | $2^4 \cdot 3^3 \cdot 5$ |
| $G_{32} = L_4$ | 2, 5 | $2^7 \cdot 3^2 \cdot 5$ |
| $G_{33} = K_5$ | 2 | $2^8 \cdot 3^4 \cdot 5$ |
| $G_{34} = K_6$ | 2, 3, 7 | $2^9 \cdot 3^2 \cdot 5 \cdot 7$ |
| $G_{35} = E_6$ | 3 | $2^8 \cdot 3^4 \cdot 5$ |
| $G_{36} = E_7$ | 2 | $2^{10} \cdot 3^4 \cdot 5 \cdot 7$ |
| $G_{37} = E_8$ | 2, 3, 5 | $2^{14} \cdot 3^3 \cdot 5^2 \cdot 7$ |
Table 3. Type of $R_2$ in $G$ excluding primitive $G$ of rank 2

| $G_r$ | $|G_r|$ | $\ell$ | $R_2$ | $|R_2|$ |
|-------|--------|-------|-------|-------|
| $G_1 = G(1,1,n)$, $n \geq 2$ | $n!$ | $\ell | n!$ | $\prod_{i=1}^{k} G(1,1,\ell^i)^{b_i}$ | $\prod_{i=1}^{k} (\ell^i)^{b_i}$ |
| $G_2 = G(m,p,n)$, $m > 1, n > 1$ | $\frac{n! m!}{p}$ | $\ell | n m$ | \[ \theta, G(\rho^{m/n}, \rho^{m/n}, n) \neq 1 \] | $\prod_{i=1}^{k} G(\rho^{m/n}, \rho^{m/n}, \ell^i)^{b_i}$ for $\ell | p$ | $\prod_{i=1}^{k} (\rho^{m/n})(\ell^i)^{b_i}$ for $\ell | m$ and $\ell \nmid p$. |
| $G_3 = G(m,1,1)$, $m > 1$ | $m!$ | $\ell | m$ | $\prod_{i=1}^{k} G(1,1,\ell^i)^{b_i}$ | $\prod_{i=1}^{k} (\ell^i)^{b_i}$ for $\ell | m$. |

Notes:
- $\theta$ denotes a primitive element of $\mathbb{Z}_m$.
- $\rho^{m/n}$ denotes a primitive element of $\mathbb{Z}_{m/n}$.
- $\theta, G(\rho^{m/n}, \rho^{m/n}, n) \neq 1$ indicates a specific condition on the elements $\theta$ and $\rho^{m/n}$.
- The notation $\prod_{i=1}^{k} G(\rho^{m/n}, \rho^{m/n}, \ell^i)^{b_i}$ for $\ell | p$ and $\prod_{i=1}^{k} (\rho^{m/n})(\ell^i)^{b_i}$ for $\ell | m$ and $\ell \nmid p$ specifies different conditions based on divisibility of $\ell$.
| $G$       | $|G|$   | $\ell$ | $R_\ell$ | $|R_\ell|$ |
|----------|--------|--------|----------|----------|
| $G_4 = L_2$ | $2^3 \cdot 3$ | 2 | $L_2$ | $2^3 \cdot 3$ |
| $G_5 = C_4 \times T$ | $2^3 \cdot 3^2$ | 2 | $L_2$ and $L_2$ | $2^3 \cdot 3$ |
| $G_6 = C_4 \circ L_2$ | $2^4 \cdot 3$ | 2 | $D_2^{(4)}$ | $2^4$ |
| $G_7 = C_3 \times (C_4 \circ T)$ | $2^4 \cdot 3^2$ | 2 | $D_2^{(4)}$ | $2^4$ |
| $G_8 = TC_4$ | $2^5 \cdot 3$ | 2 | $G(4,1,2)$ | $2^5$ |
| $G_9 = C_9 \circ O$ | $2^6 \cdot 3$ | 2 | $G(8,2,2)$ | $2^6$ |
| $G_{10} = C_3 \times TC_4$ | $2^5 \cdot 3^2$ | 2 | $G(4,1,2)$ | $2^5$ |
| $G_{11} = C_4 \times (C_8 \circ O)$ | $2^5 \cdot 3^2$ | 2 | $G(8,2,2)$ | $2^6$ |
| $G_{12} \cong GL_2(F_3)$ | $2^4 \cdot 3$ | 2 | $G_{12}$ | $2^4 \cdot 3$ |
| $G_{13} = C_4 \circ O$ | $2^5 \cdot 3$ | 2 | $G(8,4,2)$ | $2^5$ |
| $G_{14} = C_3 \times G_{12}$ | $2^4 \cdot 3^2$ | 2 | $G_{12}$ | $2^4 \cdot 3$ |
| $G_{15} = C_3 \times (C_4 \circ O)$ | $2^4 \cdot 3^2$ | 2 | $G(8,4,2)$ | $2^5$ |
| $G_{16} = C_5 \times T$ | $2^4 \cdot 3 \cdot 5^2$ | 2, 3 | $G_{16}$ | $2^4 \cdot 3 \cdot 5^2$ |
| $G_{17} = C_5 \times (C_4 \circ T)$ | $2^4 \cdot 3 \cdot 5^2$ | 2 | $D_2^{(4)}$ | $2^4$ |
| $G_{18} = C_{15} \times T$ | $2^4 \cdot 3^2 \cdot 5^2$ | 2 | $D_2^{(4)}$ | $2^4$ |
| $G_{19} = C_{15} \times (C_4 \circ T)$ | $2^4 \cdot 3^2 \cdot 5^2$ | 2 | $G(5,1,1)^2$ | $5^2$ |
| $G_{20} = C_3 \times T$ | $2^4 \cdot 3^2 \cdot 5$ | 2 | $G_{20}$ | $2^4 \cdot 3 \cdot 5$ |
| $G_{21} = C_3 \times (C_4 \circ T)$ | $2^4 \cdot 3^2 \cdot 5$ | 2 | $D_2^{(4)}$ | $2^4$ |
| $G_{22} = C_4 \circ T$ | $2^4 \cdot 3 \cdot 5$ | 2 | $D_2^{(5)}$ and $G_{20}$ | $2^5$, $2 \cdot 5$ and $2^4 \cdot 3^2 \cdot 5$ |
Table 5. Supercuspidal cases of $G$

| $G$          | $\ell$-Sylow prime | $|G|$          |
|--------------|---------------------|----------------|
| $G(1, 1, \ell^i)$ for $i \in \mathbb{Z}^+$ | $\ell$ | $(\ell^i)!$ |
| $G(\ell^j, \ell^k, n)$ for $j \leq i \in \mathbb{Z}^+, n > 1$ | $\ell$ | $\ell^{m-j}n!$ |
| $G(\ell^i, 1, \ell^j)$ for $i, j \in \mathbb{Z}^+$ | $\ell$ | $\ell^{i+j}(\ell^i)!$ |
| $G(\ell^i, 1, 1)$ for $i \in \mathbb{Z}^+$ | $\ell$ | $\ell^i$ |
| $G_4 = L_2$ | 2 | $2^3 \cdot 3$ |
| $G_8$ | 3 | $2^5 \cdot 3$ |
| $G_{12}$ | 2 | $2^4 \cdot 3$ |
| $G_{16}$ | 2, 3 | $2^3 \cdot 3 \cdot 5^2$ |
| $G_{20}$ | 5 | $2^3 \cdot 3^2 \cdot 5$ |
| $G_{24} = J_3^{(4)}$ | 7 | $2^4 \cdot 3 \cdot 7$ |
| $G_{25} = L_3$ | 3 | $2^4 \cdot 3^4$ |
| $G_{32} = L_4$ | 3 | $2^8 \cdot 3^3 \cdot 5$ |
| $G_{35} = E_6$ | 3 | $2^4 \cdot 3^4 \cdot 5$ |

Table 6. Cases where $R_\ell$ is not unique up to conjugacy

| $G$          | $\ell$ | $R_\ell$ | $|R_\ell|$ | Conjugacy Classes |
|--------------|--------|----------|-----------|-------------------|
| $G_2 = G(m, p, n)$ | $\ell|p$ | $G(\ell^\nu(m), \ell^\nu(p), n)$ | $\ell^{\nu(m)\nu(p)}n!$ | $\gcd\left(\frac{p}{\ell^\nu(\ell p)}, n\right)$ |
| $G_5$ | 2 | $L_2$ | $2^3 \cdot 3$ | 2 |
| $G_9$ | 3 | $A_2$ | 2 | 2 |
| $G_{12}$ | 3 | $A_2$ | 2 | 2 |
| $G_{13}$ | 3 | $A_2$ | 2 | 2 |
| $G_{17}$ | 3 | $A_2$ | 2 | 2 |
| $G_{18}$ | 2 | $L_2$ | $2^3 \cdot 3$ | 2 |
| $G_{20}$ | 2 | $L_2$ | $2^4 \cdot 3$ | 2 |
| $G_{21}$ | 5 | $D_2^{(3)}$ | 2 | 2 |
| $G_{26} = M_3$ | 3 | $L_3$ | $2^4 \cdot 3^4$ | 1 |
| $G_{28} = F_4$ | 2 | $B_4$ | 2 | 2 |
| $G_{29} = N_4$ | 5 | $A_4$ | $2^4 \cdot 3^4$ | 1 |
| $G_{31} = O_4$ | 5 | $A_4$ | $2^4 \cdot 3^4$ | 1 |
| $G_{34} = K_6$ | 7 | $A_6$ | $2^4 \cdot 3^4 \cdot 7$ | 2 |
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