Separable bimodules and approximation*

S. Caenepeel  
Faculty of Applied Sciences  
Free University of Brussels, VUB  
B-1050 Brussels, Belgium  
scaenepe@vub.ac.be

Bin Zhu†  
Department of Mathematical Sciences  
Tsinghua University  
100084 Beijing, China  
bzhu@math.tsinghua.edu.cn

Abstract

Using approximations, we give several characterizations of separability of bimodules. We also discuss how separability properties can be used to transfer some representation theoretic properties from one ring to another one: contravariant finiteness of the subcategory of (finitely generated) left modules with finite projective dimension, finitistic dimension, finite representation type, Auslander algebra, tame or wild representation type.

0 Introduction

The notions of approximation and contravariantly finite subcategory were introduced and studied by Auslander and Smalø [3] in connection with the study of the existence of almost split sequences in a subcategory. It turns out that these notions are important in the study of representation theory of Artin algebras. For example, Auslander and Reiten (cf. [1], [2]) proved that certain contravariantly finite subcategories of a module category are in one-to-one correspondence to cotilting modules. Auslander and Reiten ([1], [2]) showed the image of a functor having a right adjoint is contravariantly finite, we refer to [20] for a more general result. Now let $R$ and $T$ be rings, and $M$ a $(T,R)$-bimodule. Then we have a pair of adjoint functors between the categories of $R$-modules and $T$-modules, and it follows from the Auslander-Reiten result that the evaluation map $u_M : M \otimes_R *M \to T$ is a right Im$(F)$—approximation of $T$. This observation enables us to study separable bimodules and separable extensions from the point of view of homological finiteness theory. Separable bimodules have been introduced by Sugano [18]; there has been a revived interest recently, see for example [4], [5], [10] and [11].

In this note, we will apply approximation theory to study ring extensions. Another aim is to study representation theoretic properties that are shared by rings connected by a bimodule.

Let $A$ be an Artin algebra. If $P^e_{\infty}(A)$, the category of finitely generated left $A$-modules with finite projective dimension, is contravariantly finite in the category $A$-mod of finitely generated left $A$-modules, then the finitistic dimension of $A$ is finite (see [8], [9], [10]). Bass conjectured that the the finitistic dimension of $A$ is finite, if $A$ is a finite dimensional algebra over a field $k$ (cf. [8]). The problem is that $P^e_{\infty}(A)$ is

---

*Research supported by the bilateral project BIL99/43 “New computational, geometric and algebraic methods applied to quantum groups and differential operators” of the Flemish and Chinese governments.

†The second author was partially supported by NSF 10001017 and by the Scientific Foundation for returned overseas Chinese scholars, Ministry of education. The second author also wishes to thank the Free University of Brussels for its warm hospitality during his visit to Brussels.
not always contravariantly finite in $A$-mod (see [2] and [6]), so it is important to find algebras for which $P^\infty_S(A)$ is contravariantly finite. If $T$ is a biseparable extension of $R$, then the following properties are shared by $T$ and $R$: contravariant finiteness of the category of finitely generated modules with finite projective dimension; finitistic and Finitistic dimension. In the situation where $R$ and $T$ are Artin algebras, we have that $R$ is an Auslander algebra if and only if $T$ is an Auslander algebra. If $R$ and $T$ are Artin algebras connected by a biseparable $(T,R)$-bimodule, then $T$ is of finite representation type if and only if $R$ is of finite representation type; this generalizes a result of Jans [8] and of Higman [7]. If two finite dimensional algebras $R$ and $T$ over an algebraically closed field are connected by a biseparable bimodule, the $T$ is of tame (resp. wild) representation type if and only if $R$ is of tame (resp. wild) representation type. Some of these results have been proved in [14], in the case of skew group ring extensions; a skew group ring extension is a biseparable extension if the order of the group is invertible (compare to [16, Theorem 1.4]).

Our paper is organized as follows: in Section 1, we recall some preliminary results. In Section 2, we present some characterizations of separability of bimodules, using approximations, and we discuss how separable bimodules can be used to construct new separable bimodules. Section 3 is devoted to the study of representation theoretic properties of rings connected by a (separable) bimodule. In Section 4, we show that approximations are reflected by separable functors and we show that the conditional expectation of a Frobenius extension is an approximation.

1 Preliminary results

Let $R$ and $T$ be rings (associative with unit), and let $M \in \mathcal{T}_M T$ be a $(T,R)$-bimodule. Then the right and left duals

$$M^* = \text{Hom}_R(M,R) \quad \text{and} \quad ^*M = \text{THom}(M,R)$$

are both $(R,T)$-bimodules; the left and right action are respectively given by

$$rf(t)(m) = rf(tm) \quad \text{and} \quad (m)rgt = ((mr)g)t$$

for all $r \in R$, $t \in R$, $m \in M$, $f \in M^*$ and $g \in ^*M$.

A ring extension $R/S$ is a ring homomorphism $i : S \to R$. $R$ is then naturally an $S$-bimodule. $R/S$ is called separable if the multiplication map $R \otimes_S R \to R$ splits as a map of $R$-bimodules.

For a ring $T$, we consider the following full subcategories of the category of left $T$-modules $\mathcal{T}_M T$:

- $T$-mod, consisting of finitely generated left $T$-modules;
- $\mathcal{P}_T (T)$ consisting of modules with finite projective dimension;
- $\mathcal{R}_T (T)$ consisting of finitely generated left $T$-modules with finite projective dimension.

Let $k$ be a commutative Artin ring. Recall that a $k$-algebra $A$ is called an Artin algebra if $A$ is finitely generated as a $k$-module. An Artin algebra is of finite representation type if there are only finitely many isomorphism classes of finitely generated indecomposable left modules. Now we recall the definition of finitistic dimension of an Artin algebra [8]:

$$\text{fin.dim}(A) = \sup\{\text{proj.dim}(M) \mid M \in \mathcal{P}_T^\infty (T)\}$$

$$\text{Fin.dim}(A) = \sup\{\text{proj.dim}(M) \mid M \in \mathcal{P}_T^\infty (T)\}$$

A conjecture of Bass states that the finitistic dimension of a finite dimensional algebra over a field $k$ is finite, see [2] and [6] for an introduction and some partial results.
Now we recall some definitions from [1] and [2] that we will need in the sequel. Recall first that a covariant functor $F : C \to \text{Sets}$ is called finitely generated if and only if there exists an object $X \in C$ and a surjective natural transformation $C(X, \bullet) \to F$. A contravariant functor is finitely generated if the corresponding covariant functor $C^{\text{op}} \to \text{Sets}$ is finitely generated.

**Definition 1.1** Let $C$ be a full subcategory of the category $\mathcal{D}$.

(i) $C$ is called contravariantly finite in $\mathcal{D}$ if for all $X \in \mathcal{D}$, the representable functor $\text{Hom}_{\mathcal{D}}(\bullet, X)$ restricted to $C$ is finitely generated as a functor on $C$;

(ii) $C$ is called covariantly finite in $\mathcal{D}$ if for all $Y \in \mathcal{D}$, the representable functor $\text{Hom}_{\mathcal{D}}(Y, \bullet)$ restricted to $C$ is finitely generated as a functor on $C$;

(iii) $C$ is called functorially finite in $\mathcal{D}$ if $C$ is co- and contravariantly finite in $\mathcal{D}$.

$C$ is a contravariantly finite subcategory of $\mathcal{D}$ if and only if the following holds: for each $X \in \mathcal{D}$, there exists $X_1 \in C$ and a morphism $f : X_1 \to X$ such that $\text{Hom}_C(\bullet, f) : \text{Hom}_C(\bullet, X_1) \to \text{Hom}_D(\bullet, X)$ is surjective. This means that every map $\psi : C \to X$, with $C \in C$, factors through $f$:

$$\psi = f \circ \varphi : C \xrightarrow{\varphi} X_1 \xrightarrow{f} X$$

for some $\varphi : C \to X_1$. The map $f$ is then called a right $C$-approximation of $X$. Observe that a right $C$-approximation is not unique. Left $C$-approximations are defined dually. Pairs of adjoint functors induce approximations:

**Lemma 1.2** ([1], [2]) Suppose that $F : C \to \mathcal{D}$ has a right adjoint $G$. Then $\text{Im}(F)$, the full subcategory of $\mathcal{D}$, consisting of objects of the form $F(C)$ with $C \in C$, is contravariantly finite in $\mathcal{D}$. For any $X \in \mathcal{D}$, the counit map $\varepsilon_X : FG(X) \to X$ is a right $\text{Im}(F)$-approximation of $X$. $\text{Im}(G)$ is covariantly finite in $C$.

**Proof.** Let $\eta : 1_C \to GF$ be the unit of the adjunction. Then for all $C \in C$, we have

$$\varepsilon_{F(C)} \circ F(\eta_C) = 1_{F(C)}$$

Consider $f : F(C) \to X$ in $\mathcal{D}$. $\varepsilon$ is a natural transformation, so we have a commutative diagram

$$\begin{array}{ccc}
FGF(C) & \xrightarrow{FG(f)} & FG(X) \\
| & & | \\
\varepsilon_{F(C)} & & \varepsilon_X \\
\downarrow & & \downarrow \\
F(C) & \xrightarrow{f} & X
\end{array}$$

We then compute

$$f = f \circ \varepsilon_{F(C)} \circ F(\eta_C) = \varepsilon_X \circ FG(f) \circ F(\eta_C)$$

and this is exactly the factorization that we need.

Lemma [1.3] has been generalized in [20]: let $T$ be a full subcategory contravariantly finite of $C$. Then $F(T)$, the full category of $\mathcal{D}$ consisting of objects isomorphic to some $F(T)$, with $T \in T$, is a contravariantly finite subcategory of $\mathcal{D}$.
2 Separable bimodules

The aim of this section is to produce some new separable bimodules from given separable bimodules. Separable bimodules were introduced by Sugano [13] and studied recently in [3], [5], [10] and [11], among others. We recall the definition from [5]. Let \( R \) and \( T \) be rings. Given a bimodule \( \tau M_R \), there is a natural \( T \)-bimodule homomorphism,

\[
u_M : M \otimes_R \ast M \to T; \quad \nu_M(m \otimes f) = (m)f
\]

**Definition 2.1** \( M \) is called a separable bimodule, or \( T \) is called \( M \)-separable over \( R \), if \( u_M \) is a split \( T \)-\( T \)-epimorphism.

**Remark 2.2** It is easy to see that \( M \) is separable if and only if there exists \( e = \sum m_i \otimes f_i \in M \otimes_R \ast M \) such that \( u_M(e) = 1_T \) and \( te = et \), for all \( t \in T \). \( e \) is then called a separable element of \( M \) \([10]\).

**Definition 2.3** A bimodule \( \tau M_R \) is called biseparable if \( M \) and \( M^* \) are separable and \( \tau M, M_R \) are finitely generated projective modules.

Assume that \( M_R \) is finitely generated projective. Then the evaluation map \( M \to \ast(M^*) = \rho \text{Hom}(M^*, R) \) is an isomorphism. Identifying \( M \) and \( \ast(M^*) = \rho \text{Hom}(M^*, R) \), we find that \( u_{M^*} : M^* \otimes_T M \to R \) is the evaluation map given by

\[
u_{M^*}(f \otimes m) = f(m)
\]

**Definition 2.4** A ring extension \( R/S \) is called biseparable if \( _R\rho \text{Tor} \) and \( _S\rho \text{R} \) are biseparable bimodules.

To a bimodule \( \tau M_R \), we can associate an adjoint pair of functors \( (F = M \otimes_R \ast, G = \tau \text{Hom}(M, \ast)) \) between the categories \( \lambda \mathcal{M} \) and \( \tau \mathcal{M} \) of respectively left \( R \)-modules and left \( T \)-modules. The same formula defines an adjoint pair of functors between the categories of bimodules \( \lambda \mathcal{M}_T \) and \( \tau \mathcal{M}_T \). Using approximations, we now easily find the following characterization of the separability of a bimodule.

**Theorem 2.5** Let \( \tau M_R \) be a bimodule. Then the following statements are equivalent.

1) \( \tau M_R \) is separable, that is, \( u_M : M \otimes R \ast M \to T \) is a split \( T \)-\( T \)-epimorphism;
2) there exists a split epimorphism of \( T \)-bimodules \( \phi : M \otimes R \ast M \to T; \)
3) there exists a split epimorphism of \( T \)-bimodules \( \phi : M \otimes_R X \to T \) for some bimodule \( _RX_T \);
4) \( T \) is a direct summand \( M \otimes_R \ast M \) as a \( T \)-bimodule.

**Proof.** The implications 1) \( \Rightarrow \) 2) \( \Rightarrow \) 3) and 1) \( \Rightarrow \) 4) \( \Rightarrow \) 2) are obvious, and we are done if we can show that 3) implies 1). We have seen in Section \([\text{I}]) that \( \text{Im}(F) \) is a contravariantly generated subcategory of \( \tau \mathcal{M}_T \), and

\[
u_M : M \otimes R \ast M \to T; \quad \nu_M(m \otimes f) = (m)f
\]

is a right \( \text{Im}(F) \)-approximation of \( \tau T_T \). This means that for any \( T \)-bimodule morphism \( \phi : M \otimes_R X \to T \), there exists a \( T \)-bimodule map \( \phi_1 : M \otimes_R X \to M \otimes R \ast M \) such that \( \phi = \nu_M \circ \phi_1 \). If \( \phi \) is split, then \( u_M \) is also split, and \( \tau M_R \) is separable. \( \square \)

Let \( A/S \) be a ring extension (in other words, we have a ring homomorphism \( i : S \to A \)). Then we have two bimodules \( _AA_S \) and \( _SA_A \), and \( A/S \) is a separable extension if and only if \( _AA_S \) is separable, while \( A/S \) is a split extension if and only if \( _SA_A \) is separable (see \([10]\)). From Theorem 2.5 we immediately obtain the following result.
Corollary 2.6 A ring extension $A/S$ is separable if and only if $A$ is a direct summand of $A \otimes_S A$ as an $A$-bimodule; $A/S$ is split if and only if $S$ is a direct summand of $A$ as an $S$-bimodule.

Let $X$ be a $T$-bimodule. An element $x \in X$ is called faithful if $x \cdot t = 0$ implies $t = 0$. Denote

$$TZ^f(X)_T = \{x \in X \mid x \text{ is faithful and } x \cdot t = t \cdot x, \forall t \in T\}$$

Lemma 2.7 Let $M$ be a $T$-$R$-bimodule. Then $M \otimes_R ^*M$ contains a submodule $N$ which is isomorphic to $T$ as a $T$-bimodule if and only if $TZ^f(M \otimes_R ^*M)_T \neq \Phi$

Proof. Suppose that $N$ is a submodule of $T M \otimes_R ^*M_T$ and $\phi : T \rightarrow N$ is a $T$-bimodule isomorphism. It is easy to see that $e = \phi(1)$ is a casimir element of $M \otimes_R ^*M$ [10]. If $et = 0$, then $\phi(t) = 0$, and then $t = 0$. Therefore $e \in TZ^f(M \otimes_R ^*M)_T$.

Conversely, let $e \in TZ^f(M \otimes_R ^*M)_T$. Then $Te$ is a $T$-subbimodule of $M \otimes_R ^*M$. It is easy to see that $Te$ is isomorphic to $T$ as a $T$-bimodules. \hfill $\square$

We will now discuss how to produce separable bimodules from given separable bimodules.

Theorem 2.8 Let $TM_R$ be separable and $RN_S$ a bimodule such that the evaluation map

$$u^*_N : N \otimes_R ^*M_R \rightarrow ^*M, u^*_N (m \otimes f) = (m)f$$

is a split $R$-$T$-epimorphism. Then $TM_R \otimes_R RN_S$ is separable.

Proof. We have an $S$-$T$-bimodule isomorphism

$$\tau \text{Hom}(M \otimes_R RN_S, T) \cong \tau \text{Hom}(N, \tau \text{Hom}(M, T))$$

It follows that we also have a $T$-$S$-bimodule isomorphism

$$(M \otimes_R RN_S) \otimes_S ^* (M \otimes_R RN_S) \cong M \otimes_R \tau \text{Hom}(N, ^*M)$$

From the fact that $u^*_N$ is a split $R$-$T$-epimorphism, it follows that $^*M$ is an $R$-$T$-direct summand of $N \otimes_R ^*M_R \tau \text{Hom}(N, ^*M)$. Then $T$ is a $T$-$T$-direct summand of of $(M \otimes_R RN_S) \otimes_S ^*(M \otimes_R RN_S)$, and Theorem 2.8 tells us that $M \otimes_R RN_S$ is separable. \hfill $\square$

As a special case, we have the following consequence.

Corollary 2.9 Let $TM_R$ and $RN_S$ be separable bimodules. Then $TM_R \otimes_R RN_S$ is also a separable bimodule.

Proof. Let $RN_S$ be separable. It is easy to see that $u^*_N : N \otimes_R ^*M_R \rightarrow ^*M, u^*_N (m \otimes f) = (m)f$ is a split $R$-$T$-epimorphism (see also the proof of Lemma [3,4]), and it follows from Theorem 2.8 that $TM_R \otimes_R RN_S$ is separable. \hfill $\square$

Theorem 2.10 Let $TM_R$ be separable and $SX_R$ biseparable. Then Hom$_R(M, X)$ is a separable $T$-$S$-bimodule.
Proof. Set $N = \text{Hom}_R(X, M)$, $^*N = T\text{Hom}(N, T)$. We will prove that $N \otimes S^*N$ contains $T$ as a $T$-bimodule direct summand, and then, by Theorem 2.5, $N$ is separable. It is easy to see that $X \otimes_T \text{Hom}(M, T)$ is an $S$-$T$-bimodule. If we can prove that $N \otimes_S (X \otimes_T \text{Hom}(M, T))$ contains $T$ as a $T$-bimodule direct summand, then it follows from Theorem 2.3 that $T$ is a direct summand of $TN \otimes_S N$. From the biseparability of $S_X^R$, it follows that
\[
\phi : \text{Hom}_R(X, M) \otimes_S X \to M, \quad \phi(f \otimes x) = (x)f
\]
is a split epimorphism, hence $TM_R$ is a direct summand of $\text{Hom}_R(X, M) \otimes_S X$. Then $(\text{Hom}_R(X, M) \otimes_S X) \otimes_T \text{Hom}(M, T)$ contains $M \otimes_R \text{Hom}(M, T)$ as a $T$-bimodule direct summand. Since $TM_R$ is separable, $TM_T$ is a direct summand of $M \otimes_T \text{Hom}(M, T)$, and then it is a direct summand of $(\text{Hom}_R(X, M) \otimes_S X) \otimes_T \text{Hom}(M, T)$. Since
\[
N \otimes_S (X \otimes_T \text{Hom}(M, T)) \cong (\text{Hom}_R(X, M_R) \otimes_S X) \otimes_T \text{Hom}(M, T),
\]
$TM_T$ is a direct summand of $N \otimes_S (X \otimes_T \text{Hom}(M, T))$. The proof is finished. \qed

Theorem 2.11 Let $TM_R$ be a separable bimodule and $N$ an $T$-$R$-bimodule. Then $T(M \oplus N)_R$ is separable.

Proof. We have an $R$-$T$-bimodule isomorphism
\[
(M \oplus N) \cong M \oplus N
\]
Therefore we have $T$-bimodule isomorphisms
\[
(M \oplus N) \otimes_R (M \oplus N) \cong (M \oplus N) \otimes_R (M \oplus N)
\]
\[
\cong (M \otimes_R M \oplus M \otimes_R N) \oplus (N \otimes_R M) \oplus (N \otimes_R N)
\]
If $M$ is separable, then $T$ is a $T$-bimodule direct summand of $M \otimes_R M$, by Theorem 2.5. Then $T$ is also a $T$-bimodule direct summand of $(M \oplus N) \otimes_R (M \oplus N)$, and, again by Theorem 2.5, $T(M \oplus N)_R$ is separable. \qed

Let $n$ be a positive integer and $M$ a module. The direct sum of $n$ copies of $M$ is denoted by $M^n$.

Theorem 2.12 $TM_R$ is separable if and only if $TM_R^n$ is separable.

Proof. One implication is a direct consequence of Theorem 2.11. Conversely, assume that $TM_R^n$ is separable. $^*(M^n) \cong (^*M)^n$, so we have an isomorphism
\[
M^n \otimes_R ^*(M^n) \cong (M \otimes_R ^*M)^n
\]
It follows from the separability of $M^n$ that the map
\[
u_{M^n} : (M \otimes_R ^*M)^n \to T, \quad u_{M^n}(m^i \otimes f^i) = \sum (m^i)f^i
\]
is a split epimorphism. Here $m^i \otimes f^i \in M \otimes_R ^*M$ denotes the $i$-th component of an element $(M \otimes_R ^*M)^n$. Let $e = (e^i)$ be a separability element of $(M \otimes_R ^*M)^n$. It is easy to see that the sum $\sum e^i$ of all entries of $e$ is a separability element of $M$. Therefore $M$ is a separable bimodule. \qed
3 Representations of rings related by a bimodule

There have been various studies of properties shared by rings $R$ and $T$ related by a bimodule $\tau M_R$. A precursor of these studies is the Higman’s Theorem \cite{7}, stating that a finite group has finite representation type in characteristic $p$ if and only if its Sylow $p$-subgroup is cyclic. This result appeared later as a Corollary of Jans’ Theorem \cite{9}: for an Artin algebra $R \supseteq T$ in a split separable extension, $R$ has finite representation type if and only if $T$ has finite representation type.

More results of this type can be found in \cite{5}. In this Section, we are mainly interested in representation theoretic properties shared by rings related by a bimodule, such as: contravariantly finiteness of the subcategory of modules with finite projective dimension, Finitistic (or finitistic) dimension and representation theoretic properties shared by rings related by a bimodule, such as: contravariantly finiteness of the subcategory of modules with finite projective dimension, Finitistic (or finitistic) dimension and representation types, Auslander algebras. Some of these were discussed in \cite{16}, in the special case of a skew group ring extension. We will generalize Jans’ result to biseparable bimodules. We will prove the following result, for finite dimensional algebras $R$ and $T$ over an algebraically closed field: if there exists a bimodule $\tau M_R$, then $T$ is of tame (resp. wild) representation type if and only if its Sylow $p$-subgroup is cyclic. This result appeared later as a precursor of these studies is the Higman’s Theorem \cite{7}, stating that a finite group has finite representation type in characteristic $p$ if and only if its Sylow $p$-subgroup is cyclic. This result appeared later as a Ramification of Jans’ Theorem \cite{9}: for an Artin algebra $R \supseteq T$ in a split separable extension, $R$ has finite representation type if and only if $T$ has finite representation type.

The following elementary Lemma will be a key tool in our subsequent results.

Lemma 3.1 Let $\tau M_R$ be a separable bimodule. Then $\tau M = \text{DSIm}(F)$.

Proof. For a left $T$-module $N$, consider the left $R$-module

$$^*M^N = \tau \text{Hom}(M, N)$$

We have a left $T$-module homomorphism

$$u^N : M \otimes_R ^*M^N \rightarrow N, \quad u^N(m \otimes f) = (m)f$$

Let $e = \sum m_i \otimes f_i$ be a separability element of $M$, and consider the map

$$v_N : N \rightarrow M \otimes_R ^*M^N, \quad v_N(x) = \sum m_j \otimes f_j \cdot x$$

Here $f_j \cdot x \in ^*M^N$ is defined by the formula $(m)(f_j \cdot x) = ((m)f_j)x$, for all $m \in M$. Now we claim:

1) $v_N$ is left $T$-linear. Indeed, for all $t \in T$ and $n \in N$, we have

$$v_N(t \cdot x) = \sum m_j \otimes f_j \cdot (tx) = \left(\left(\sum m_j \otimes f_j\right) \cdot t\right) \cdot x$$

$$= t\left(\sum m_j \otimes f_j\right) \cdot x = t\left(\sum m_j \otimes f_j \cdot x\right) = tv_N(x)$$

2) $u^N \circ v_N = id_N$: for all $x \in N$, we have

$$(u^N \circ v_N)(x) = \sum (m_j)f_j \cdot x = x$$

This means that $u^N$ is a split epimorphism of left $T$-modules and $N$ is a direct summand of $M \otimes_R ^*M^N$. □
Let \( T/R \) be a ring extension, and consider the adjoint pair \( (F = T \otimes_R \bullet, G) \), where \( F \) is the induction functor, and \( G \) is the restriction of scalars functor, between the categories of left \( R \)-modules and left \( T \)-modules. We have a second adjoint pair \( (F' = rT \otimes_T \bullet, G' = r\text{Hom}(T, \bullet)) \) between the categories of left \( T \)-modules and left \( R \)-modules.

**Proposition 3.2** Let \( T/R \) be a separable extension with \( T \) projective as a right \( R \)-module and assume that \( \text{proj.dim}(rT) < \infty \). Then \( DSF(P^\omega(R)) = P^\omega(T) \). Moreover, if \( P^\omega(R) \) is contravariantly finite in \( R\mathcal{M} \), then \( P^\omega(T) \) is contravariantly finite in \( T\mathcal{M} \).

**Proof.** Let \( X \in R\mathcal{M} \) with projective dimension \( m \). Since \( T_R \) is projective, \( \text{proj.dim}(T \otimes_R X) \leq m \). Then we have \( F(P^\omega(R)) \subseteq P^\omega(T) \), and then \( DSF(P^\omega(R)) \subseteq P^\omega(T) \) because \( P^\omega(T) \) is closed under taking direct summands. Now let \( Y \in P^\omega(T) \). By the change of rings Theorem ([19, section 4.3]), \( \text{proj.dim}(rY) \leq \text{proj.dim}(rT) + \text{proj.dim}(rT) < \infty \). \( T_R \) is separable, so we can apply Lemma 3.1, and we find that \( Y \) is a direct summand of \( F(Y) \), proving the first assertion.

Assume that \( P^\omega(R) \) is contravariantly finite in \( R\mathcal{M} \). By [20, Theorem 2.1], \( F(P^\omega(R)) \) is contravariantly finite in \( T\mathcal{M} \). We will next show that \( DSF(P^\omega(R)) \) is contravariantly finite in \( T\mathcal{M} \). Let \( Y \) be a left \( T \)-module, and \( Y_0 \to Y \) be a right \( F(P^\omega(R)) \)-approximation of \( Y \). We verify that \( Y_0 \to Y \) is also a right \( DSF(P^\omega(R)) \)-approximation of \( Y \). Take \( Z \in DSF(P^\omega(R)) \), and let a morphism \( g : Z \to Y \). \( Z \in DSF(P^\omega(R)) \), so there exists \( Z_1 \in F(P^\omega(R)) \) and a split \( T \)-monomorphism \( i : Z_1 \to Z_1 \). Let \( \pi \) be a left inverse of \( i \). Then there is a \( T \)-homomorphism \( h : Z_1 \to Y_0 \), with \( g \circ \pi = f \circ h \). It follows that \( g = g \circ \pi \circ i = f \circ h \circ i \), and \( g \) factors through \( f \). Then \( DSF(P^\omega(R)) \) is contravariantly finite, and therefore \( P^\omega(T) \) is contravariantly finite in \( T\mathcal{M} \).

**Theorem 3.3** Let \( T/R \) be a biseparable extension. Then

1. \( P^\omega(R) \) is contravariantly finite in \( R\mathcal{M} \) if and only if \( P^\omega(T) \) is contravariantly finite in \( T\mathcal{M} \);
2. \( P^\omega(R) \) is contravariantly finite in \( R\mathcal{M} \) (resp. in \( R\text{-mod} \)) if and only \( P^\omega(T) \) is contravariantly finite in \( T\mathcal{M} \) (resp. in \( T\text{-mod} \));
3. \( \text{Fin.dim} T = \text{Fin.dim} R \);
4. \( \text{fin.dim} T = \text{fin.dim} R \).

**Proof.** One implication of 1) follows immediately from Proposition 3.2. We prove the converse direction. First we show that

\[
DSF'(P^\omega(T)) = P^\omega(R)
\]

It is easy to show that \( DSF'(P^\omega(T)) \subseteq P^\omega(R) \). Conversely, take \( Y \in P^\omega(R) \). \( T \) is projective as a right \( R \)-module, and \( \text{proj.dim}(T R \otimes_R Y) < \infty \), so \( \text{proj.dim}(rT \otimes_T (T R \otimes_R Y)) < \infty \). Using the fact that \( rT_R \) is separable, we obtain that \( Y \) is an \( R \)-module direct summand of \( F'(T R \otimes_R Y) \), that is, \( Y \in DSF'(P^\omega(T)) \).

Now suppose that \( P^\omega(T) \) is contravariantly finite. By [20, Theorem 2.1], \( F'(P^\omega(T)) \) is contravariantly finite in \( R\mathcal{M} \), and then \( DSF'(P^\omega(T)) \) is contravariantly finite (compare to the proof of Proposition 3.3). Consequently \( P^\omega(R) \) is contravariantly finite in \( R\mathcal{M} \).

The proof of 2) is similar to the proof of 1), because \( T \) is finitely generated and projective as a left and right \( R \)-module. We omit the details.

Now we prove 3). Take \( X \in P^\omega(T) \) with projective dimension \( m \). \( \text{proj.dim}(rT) = 0 \), hence \( \text{proj.dim}(rX) \leq m \) and we claim that we have equality: the projective dimension of \( F(X) \) is smaller than the projective dimension of \( X \) as an \( R \)-module, and, by Lemma 3.1, \( X \) is a \( T \)-direct summand of \( F(X) \).
It follows that \( m \leq \text{Fin.dim } R \), and \( \text{Fin.dim } T \leq \text{Fin.dim } R \). We are done if we can show that \( \text{Fin.dim } T \geq \text{Fin.dim } R \). Take \( Y \in \mathcal{M}_R \) with projective dimension \( n \). It is easy to see that \( \text{proj.dim}(T \otimes_R Y) \leq n \) and \( \text{proj.dim}(R \otimes_T (T \otimes_R Y)) \leq n \). If \( \text{proj.dim}(T \otimes_R Y) < n \), then \( \text{proj.dim}(R \otimes_T (T \otimes_R Y)) < n \), and \( Y \) being a direct summand, has projective dimension stricty smaller than \( n \), contradicting the assumption on \( Y \). We conclude that \( \text{proj.dim}(T \otimes_R Y) = n \) and \( n \leq \text{Fin.dim } T \). This shows that \( \text{Fin.dim } T \geq \text{Fin.dim } R \).

The proof of 4) is similar to the proof of 3), using the fact that \( T \) is finitely generated as a left and right \( R \)-module. □

**Theorem 3.4** Let \( T/R \) be a biseparable extension of Artin algebras. Then

1. \( \text{dom.dim } T = \text{dom.dim } R \);
2. \( T \) is an Auslander algebra if and only if \( R \) is an Auslander algebra.

**Proof.** 1) Let

\[
0 \rightarrow T \rightarrow I_0 \rightarrow I_1 \rightarrow \cdots \rightarrow I_i \rightarrow \cdots
\]

be a minimal injective resolution of \( T \). The restriction of scalars functor \( G \) preserves injectives, so this resolution is also an injective resolution of \( T \) as an \( R \)-module. Now \( R \) is a direct summand of \( RT \), so we have an injective resolution of \( R \)

\[
0 \rightarrow R \rightarrow I'_0 \rightarrow I'_1 \rightarrow \cdots \rightarrow I'_i \rightarrow \cdots
\]

with \( I'_j \) an \( R \)-module direct summand of \( I_j \), for all \( j \).

If \( \text{dom.dim } T = \infty \), that is, every \( I_j \) is projective, then \( I'_j \) is projective since \( I_j \) is projective in \( \mathcal{M}_R \), and \( \text{dom.dim } R = \infty \).

If \( \text{dom.dim } T = n \), then \( I_n \) is not projective as a \( T \)-module, and the same argument as in the case where \( \text{dom.dim } T = \infty \) shows that \( \text{dom.dim } R \geq n \). If \( \text{dom.dim } R > n \), then the \( R \)-injective resolution of \( T \) has the property that \( I_n \) is projective as an \( R \)-module. This implies that \( T \otimes_R I_n \) is a projective \( T \)-module, and \( I_n \) is a projective \( T \)-module, since it is a \( T \)-direct summand of \( T \otimes_R I_n \). This is a contradiction, so it follows that \( \text{dom.dim } R = n \), finishing the proof of part 1).

2) Recall that an Artin algebra \( T \) is an Auslander algebra if and only if \( \text{glob.dim } T \leq 2 \) and \( \text{dom.dim } T \geq 2 \). From [5, Theorem 2.6], we know that the global dimensions of \( R \) and \( T \) are equal. Combining this with part 1), we find 2). □

**Remark 3.5** Theorem 3.4 has been proved in [16], in the case of skew group ring extensions.

**Proposition 3.6** Let \( R \) and \( T \) be Artin algebras, and assume that there exists a biseparable \( (T,R) \)-bimodule \( M \). Then \( R \) is of finite representation type if and only if \( T \) is of finite representation type.

**Proof.** Assume that \( R \) is of finite representation type. It follows from the separability of \( M \) and Lemma 3.1 that the full subcategory \( \text{ind}(TM) \) of \( TM \) consisting of finitely generated indecomposable modules coincides with the full subcategory \( \text{ind}(DSIm(F)) \) of \( DSIm(F) \) consisting of finitely generated indecomposable modules. \( \text{ind}(DSIm(F)) \) is of finite representation type since \( R \) is of finite representation type, and it follows that \( \text{ind}(T) \) is of finite representation type. The converse implication follows from the fact that the \( (R,T) \)-bimodule \( M^* \) is biseparable. □
Corollary 3.7 Let $A$ be a finite dimensional algebra over a field $k$. If there is a separable bimodule $\Lambda M_k$, then $A$ is of finite representation type.

Proof. We note that $\text{ind}(\Lambda M)$ contains just one object. It then follows from the proof of Proposition 3.6 that $A$ is of finite representation type.

From now on we assume that $R$ and $T$ are finite dimensional Artin algebras over an algebraically closed field $k$. Any finite dimensional $k$-algebra is Morita equivalent to a basic algebra of the form $kQ/I$. $Q$ is called the Gabriel quiver of $A$ (cf. [17]). Recall that a quiver $Q$ is a pair $(Q_0, Q_1)$, where $Q_0$ is the set of vertices and $Q_1$ is the set of arrows. A $k$-algebra $A = kQ/I$ is of tame representation type if for each dimension vector $\mathbf{z} \in \mathbb{N}^{Q_0}$, there exist finitely many parametrizing $A$-$k[t]$-bimodules $M_1, \ldots, M_s$ satisfying the two following conditions:

1. every $M_i$ is finitely generated and free as a right $k[t]$-module;
2. every indecomposable $A$-module $X$ for which $\dim X = \mathbf{z}$ is isomorphic to a module of the form $M_i \otimes (k[t]/(t - \lambda))$, with $i \in \{1, \ldots, s\}$ and $\lambda \in k$.

It was proved in [12],[13] that $A$ is of tame representation type if and only if $A$ is weakly tame. This means that for every $\mathbf{z} \in \mathbb{N}^{Q_0}$, there is a family of finitely generated $A$-$k[t]$-bimodules $M_1, \ldots, M_s$ such that each indecomposable $A$-module $X$ with $\dim X = \mathbf{z}$ is a direct summand of $M_i \otimes_{k[t]} S$ for some $i \in \{1, \ldots, s\}$ and a simple $k[t]$-module $S$.

Proposition 3.8 Let $R$ and $T$ be finite dimensional algebras over an algebraically closed field. Then $R$ is of tame representation type if and only if $T$ is of tame representation type.

Proof. We divide the proof into two parts. First, we show that we can restrict attention to the situation where $T$ and $R$ are basic. Then we prove the Theorem for basic algebras $R$ and $T$.

Let $T'$ be the basic algebra of $T$. Then there is a $(T',T)$-bimodule $X$ that induces an equivalence $F_1 = X \otimes_T \bullet : T' M \to T' \cdot M$. We claim that $T' X \otimes_T M_R$ is biseparable. Firstly, since $T' X_T$ and $T M_R$ are separable, by Corollary 2.9 we have that $F_1(M)$ is separable. Secondly, we have an isomorphism of $(R,S)$-bimodules:

$$\text{Hom}_R(X \otimes_T M, R) \cong \text{Hom}_T(X, \text{Hom}(M, R))$$

$T' X_T$ is biseparable, because it induces a Morita equivalence. It follows from Theorem 2.10 that $\text{Hom}_T(X, \text{Hom}_R(M, R))$ is separable. Third, we can easily verify that $F_1(M)_{\cdot R}$ and $T F_1(M)$ are finitely generated projective. Then we have that $T' X \otimes_T M_R$ is biseparable. Dually, let $R'$ be the basic algebra of $R$ and $Y_{R'}$ the bimodule inducing an equivalence $G_1 = \bullet \otimes_R Y_{R'} : M_R \to M_{R'}$. A similar argument shows that $F_1(M) \otimes_R Y$ is a biseparable $(T', R')$-bimodule. So without loss of generality, we can assume that $T$ and $R$ are basic algebras and $T M_R$ is biseparable.

Assume that $R = kQ/I$ is of tame representation type. We will prove that $T = k\Gamma/J$ is weakly tame. Let $\mathbf{w} = (w(i))_{i \in I_0}$ be a dimension vector. We prove that there are only finitely many dimension vectors $\mathbf{v} = (\nu(i))_{i \in I_0}$ with $\mathbf{v} = \dim G(N)$ for some $T$-module $N$ with $\dim N = \mathbf{w}$. Let $P(j)$ be the indecomposable projective $R$-module corresponding to the vertex $j$. Then we have an isomorphism

$$\text{Hom}_R(P(j), G(N)) \cong \text{Hom}_T(M \otimes_R P(j), N)$$

$\text{Hom}_T(M \otimes_R P(j), N)$ is a direct summand of $\text{Hom}_T(M, N)$, and it follows that

$$\dim \text{Hom}_R(P(j), G(N)) \leq \dim \text{Hom}_T(M, N)$$
for all $j$. The left hand side has an upper bound $\dim k\Hom_1(M, k^{\Sigma w(i)})$. Therefore there are only finitely many dimension vectors $v = (v(j))_{j \in Q_0}$ with $v = \dim G(N)$ for some $T$-module $N$ with $\dim N = w$. Let $\ind_R(v)$ denote the subcategory of $R\mathcal{M}$ consisting of indecomposable modules with dimension vector $v$. Since $R$ is tame, there are finitely many $(R, k[t])$-bimodules $M_i$ which are finitely generated free right $k[t]$-modules and parametrize all $\ind_R(v)$, where $\mathcal{Z}$ fullfills the above estimation. Let $M_1^t = M \otimes_R M_i$. Then we have that any indecomposable left $T$-module $N$ with $\dim N = z$ is isomorphic to a direct summand of $M_1^t \otimes_{k[t]} S$ for some simple $k[t]$-module $S$. This proves that $T$ is weakly tame. For the converse direction, we use the separability of the $(R, T)$-bimodule $M^*$ and the tameness of $T$. The arguments are then the duals of the ones presented above. 

Combining Propositions [3.6] and [3.8] with Drozd's tame-wild dichotomy theorem [8], we obtain the following result.

**Theorem 3.9** Let $R$ and $T$ be finite dimensional algebras over an algebraically closed field, and $T\mathcal{M}_R$ a biseparable bimodule. Then

1. $R$ is of finite representation type if and only if $T$ is of finite representation type;
2. $R$ is of tame and infinite representation type if and only if $T$ is of tame and infinite representation type;
3. $R$ is of wild representation type if and only if $T$ is of wild representation type.

## 4 Right approximations

It is a difficult problem to decide which subcategories are contravariantly finite, and to find a right approximation (see [1], [2], [20]). In this Section, we will see that separable functors reflect approximations. Then we give some descriptions of split extensions and Frobenius extensions.

Let $F : \mathcal{C} \to \mathcal{D}$ be a covariant functor. $F$ induces a natural transformation

$$\mathcal{F} : \Hom_{\mathcal{C}}(\bullet, \bullet) \to \Hom_{\mathcal{D}}(F(\bullet), F(\bullet)), \quad \mathcal{F}_{C,C'}(f) = F(f)$$

Recall from [14] that $F$ is called separable if $\mathcal{F}$ splits as a natural transformation, that is, there exists a natural transformation

$$\mathcal{P} : \Hom_{\mathcal{D}}(F(\bullet), F(\bullet)) \to \Hom_{\mathcal{C}}(\bullet, \bullet)$$

such that $\mathcal{P} \circ \mathcal{F}$ is the identity natural transformation on $\Hom_{\mathcal{C}}(\bullet, \bullet)$. For a detailed study of separable functors, we refer the reader to [8].

**Proposition 4.1** Let $F : \mathcal{C} \to \mathcal{D}$ be a separable functor, and $\mathcal{T}$ a full subcategory of $\mathcal{C}$. Let $C_1 \in \mathcal{T}$ and $C \in \mathcal{C}$, and a morphism $f : C_1 \to C$. If $F(f) : F(C_1) \to F(C)$ is a right (resp. a left) $F(\mathcal{T})$-approximation of $F(C)$, then $f : C_1 \to C$ is a right (resp. a left) $\mathcal{T}$-approximation of $C$.

**Proof.** Assume that $F(f) : F(C_1) \to F(C)$ is a right $F(\mathcal{T})$-approximation of $F(C)$. Let $g : B \to C$ be a morphism in $\mathcal{C}$. Then there exists a morphism $h : F(B) \to F(C)$ in $\mathcal{D}$ such that the following diagram
commutes:

\[
\begin{array}{ccc}
F(B) & \xrightarrow{h} & F(C_1) \\
\downarrow F(1_B) & & \downarrow F(f) \\
F(B) & \xrightarrow{F(g)} & F(C)
\end{array}
\]

that is, \(F(g) = F(f) \circ h\). It follows from the separability of \(F\) that we have the following commutative diagram in \(C\):

\[
\begin{array}{ccc}
B & \xrightarrow{\mathcal{P}(h)} & C_1 \\
\downarrow 1_B & & \downarrow f \\
B & \xrightarrow{g} & C
\end{array}
\]

or \(g = f \circ \mathcal{P}(h)\). This means that \(f : C_1 \to C\) is a right \(\mathcal{T}\)-approximation of \(C\). The proof in the case of a left approximation is similar.

We will now study ring extensions from the point of view of approximations. Consider a ring extension \(R/S\). We use the following notation for the restriction of scalars functors:

\[
G' : \mathcal{M}_R \to \mathcal{M}_S \quad \text{and} \quad G : \mathcal{S}_R \mathcal{M}_R \to \mathcal{S}_S \mathcal{M}_S
\]

It is well-known that \(G\) and \(G'\) have a right adjoint and a left adjoint, and it follows that \(\text{Im}(G)\) (resp. \(\text{Im}(G')\)) is covariantly and contravariantly finite im \(\mathcal{S}_S \mathcal{M}_S\) (resp. \(\mathcal{M}_S\)) (see Section 1), and we can construct left and right approximations. In particular, a right \(\text{Im}(G)\)-approximation of \(\mathcal{S}_S\) is the map \(\phi : R^* \to S\), mapping \(f \in R^*\) to \(f(1)\). \(\phi\) is also a right \(\text{Im}(G')\)-approximation of \(S\).

**Proposition 4.2** If \(\phi : sR_R \to \text{Hom}_S(R, S)\) is an isomorphism of \((S,R)\)-bimodules, then \(E = \phi(1) : sR_S \to S\) is a right \(\text{Im}(G)\)-approximation of \(sS_S\), and also a right \(\text{Im}(G')\)-approximation of \(S_S\).

**Proof.** We know that \(u_S : sR^* \otimes_R R_S \to S\) is a right \(\text{Im}(G)\)-approximation of \(sS_S\). We have the following commutative diagram of \(S\)-bimodules:

\[
\begin{array}{ccc}
R & \xrightarrow{E} & S \\
\cong \downarrow & = & \downarrow \\
R \otimes_R R & \xrightarrow{\phi \otimes 1_R} & S \\
\phi \otimes 1_R & = & u_S \\
R^* \otimes_R R & \xrightarrow{u_S} & S
\end{array}
\]

All the vertical maps are isomorphisms, so \(E\) is a right \(\text{Im}(G)\)-approximation of \(S\). \(u_S\) is also a right \(\text{Im}(G')\)-approximation of \(S\), and the same is true for \(E\). □
Corollary 4.3  If \( R/S \) is a Frobenius extension, with Frobenius system \( \{ E, x, y \} \), then \( E : R \twoheadrightarrow S \) is a right \( \text{Im}(G') \)-approximation of \( sS \).

A right \( \text{Im}(G') \)-approximation \( E : R \twoheadrightarrow S \) of \( S \) is called non-degenerate if \( \ker(E) \) contains no non-zero right ideal of \( R \).

Theorem 4.4  Let the \( S \)-bimodule map \( E : R \twoheadrightarrow S \) be a right \( \text{Im}(G') \)-approximation of \( S \) and \( A = \text{End}_S(R) \). Then

\[
R^* = \text{Hom}_S(R,S) = E \circ A
\]

as \((S,R)\)-bimodules. If \( E \) is non-degenerate, then \( R \cong E \circ A \) as \((S,R)\)-bimodules. Conversely, if \( R \cong E \circ A \) as \((S,R)\)-bimodules, then there exists a non-degenerate \( \text{Im}(G') \)-approximation \( E_1 : sR \twoheadrightarrow S \) of \( S \).

Proof.  Let \( E : R \twoheadrightarrow S \) be a right \( \text{Im}(G') \)-approximation of \( S \). For any \( f \in \text{Hom}_S(R,S) \), there is an \( h : R \twoheadrightarrow S \) such that \( f = E \circ h \), and it follows that \( R^* = E \circ A \), as right \( A \)-modules. Observe that \( E \circ A \) is a left \( S \)-module, since \( s \cdot (E \circ f) = E \circ (s \cdot f) \), for all \( s \in S \) and \( f \in A \). Indeed, for all \( x \in R \), we have

\[
s \cdot (E \circ f)(x) = s \cdot E(f(x)) = E(sf(x)) = (E \circ (s \cdot f))(x)
\]

Finally, we have a monomorphism \( R \twoheadrightarrow A \), mapping \( r \) to \( m_r : R \twoheadrightarrow R, m_r(x) = rx \), and we have an \((S,R)\)-bimodule isomorphism \( R^* = E \circ A \).

Now suppose that \( E \) is non-degenerate. It is easy to see that \( E \circ R \) is an \((S,R)\)-subbimodule of \( R^* \). The map \( \alpha : R \twoheadrightarrow E \circ R \), \( \alpha(x) = E \circ m_r \), is surjective. It is injective, since \( E \) is non-degenerate: if \( x \in \ker(\alpha) \), then \( E(xr) = 0 \), for all \( r \in R \), and \( xr \) is a non-zero right ideal of \( R \) contained in \( \ker E \). Thus \( \alpha \) is an isomorphism of right \((S,R)\)-bimodules.

Conversely, let \( \alpha : R \twoheadrightarrow E \circ R \) be an isomorphism of \((S,R)\)-bimodules, with inverse \( \alpha^{-1} \). Let \( \alpha(1) = E \circ a \) and \( \alpha^{-1}(E) = b \). Then

\[
E = \alpha(\alpha^{-1}(E)) = \alpha(b) = \alpha(1)b = E \circ m_a \circ m_b = E \circ m_{ab}
\]

This implies that \( E(abx) = E(x) \), and \( E((ab-1)x) = 0 \), for all \( x \in R \). Then \( ab = 1 \), otherwise \( R(ab-1) \) is a non-zero right ideal contained in \( \ker E \). Also

\[
1 = \alpha^{-1}(\alpha(1)) = \alpha^{-1}(E \circ m_a) = \alpha^{-1}(E)a = ba
\]

\( E \circ m_a \) is an \( S \)-bimodule map since

\[
(E \circ m_a)(sx) = \alpha(1)(sx) = (\alpha(1) \cdot s)(x) = \alpha(s)(x) = s(\alpha(1))(x) = s(E \circ m_a)(x)
\]

for all \( x \in R \) and \( s \in S \). \( E = E \circ m_{ab} = E \circ m_a \circ m_b \) is a right \( \text{Im}(G') \)-approximation of \( S \), so \( E \circ m_a \) is also a right \( \text{Im}(G') \)-approximation of \( S \). Finally, \( E \circ m_a \) is non-degenerate: assume that there exists \( x \in R \) such that \( (E \circ m_a)(xr) = 0 \) for all \( r \in R \). Then

\[
\alpha(x)(r) = (\alpha(1)x)(r) = (E \circ m_a)(xr) = 0
\]

for all \( r \in R \), hence \( \alpha(x) = 0 \), and \( x = 0 \), since \( \alpha \) is bijective. \( \square \)
References

[1] M. Auslander and I. Reiten, Homological finite subcategories, in “Representation Theory of Algebras and Related Topics”, London Math. Soc. Lecture Notes Ser. 168, Cambridge University Press, Cambridge, 1992, 1–42.

[2] M. Auslander and I. Reiten, Applications of contravariantly finite subcategories, Adv. Math. 86 (1991), 111–152.

[3] M. Auslander and S. Smalø, Almost split sequences in subcategories, J. Algebra 69 (1981), 426–454.

[4] S. Caenepeel, G. Militaru and Shenglin Zhu, “Frobenius and separable functors for generalized module categories and nonlinear equations”, Lect. Notes in Math. 1787, Springer Verlag, Berlin, 2002.

[5] S. Caenepeel and L. Kadison, Are biseparable extensions Frobenius?, K-Theory 24 (2001), 361–383.

[6] W. Crawley-Boevey, On tame algebras and bocses, Proc. London Math. Soc. 56 (1988), 451–483.

[7] D. Higman, Representations of finite groups at characteristic $p$, Duke Math. J. 21 (1954), 377–381.

[8] B. Zimmermann-Huisgen, The finitistic dimension conjectures—a tale of 3.5 decades, in “Abelian groups and Modules”, A. Facchini and C. Menini (Eds.), Kluwer Academic Publishers, Dordrecht, 1995, p. 501–517.

[9] J. Jans, Representation type of subalgebras and algebras, Canad. J. Math. 10 (1957), 39–44.

[10] L. Kadison, “New examples of Frobenius extensions”, University Lect. Series 14, Amer. Math. Soc., Providence, 1999.

[11] L. Kadison, Separability and the twisted Frobenius bimodules, Algebras and Representation Theory 2 (1999), 397–414.

[12] J. A. de la Peña, Functors preserving tameness, Fundamenta Math. 137 (1991), 177–185.

[13] J. A. de la Peña, Constructible functors and the notion of tameness, Comm. Algebra 24 (1996), 1939–1955.

[14] C. Năstăsescu, M. Van den Bergh and F. Van Oystaeyen, Separable functors applied to graded rings, J. Algebra 123 (1989), 397–413.

[15] R. Pierce, “Associative algebras”, Grad. Text in Math. 88, Springer Verlag, Berlin, 1982.

[16] I. Reiten and C. Riedtmann, Skew group algebras in the representation theory of Artin algebras, J. Algebra 92 (1985), 224–282.

[17] C. Ringel, “Tame algebras and integral quadratic form”, Lect. Notes in Math. 1099, Springer Verlag, Berlin, 1984.

[18] K. Sugano, Note on separability of endomorphism rings, Hokkaido Math. J. 11 (1982), 111–115.

[19] C. Weibel, “An introduction to homological algebra”, Cambridge studies in adv. math. 38, Cambridge University Press, Cambridge, 1994.

[20] B. Zhu, Contravariantly finite subcategories and adjunctions, Algebra Coll. 8:3 (2001), 307–314.