Abstract. In this paper, we study different variations of minimum width color-spanning annulus problem among a set of points $P = \{p_1, p_2, \ldots, p_n\}$ in $\mathbb{R}^2$, where each point is assigned with a color in $\{1, 2, \ldots, k\}$. We present algorithms for finding a minimum width color-spanning axis-parallel square annulus (CSSA), minimum width color spanning axis-parallel rectangular annulus (CSRA), and minimum width color-spanning equilateral triangular annulus of fixed orientation (CSETA). The time complexities of computing (i) a CSSA is $O(n^3 + n^2 k \log k)$ which is an improvement by a factor $n$ over the existing result on this problem, (ii) that for a CSRA is $O(n^4 \log n)$, and for (iii) a CSETA is $O(n^3 k)$. The space complexity of all the algorithms is $O(k)$.

1 Introduction

The motivation for studying color-spanning objects stems from the facility location problems, where we may have different types of facilities, and the objective is to identify a location of desired shape with at least one copy of each facility and the measure of the region is optimized. In this paper, we study the minimum width color-spanning annulus problem for different objects.

A point set $P = \{p_1, \ldots, p_n\}$ is given in $\mathbb{R}^2$. Each point $p \in P$ is assigned a color from the set $\chi = \{1, 2, \ldots, k\}$ of $k$ distinct colors. There exists at least one point of each color. A region is color-spanning if it contains at least one point of each color. An annulus $A$ is a region bounded by two co-centric homothetic closed curves $C_{\text{in}}$ (inner curve) and $C_{\text{out}}$ (outer curve). The (common) center $c$ of $C_{\text{in}}$ and $C_{\text{out}}$ is referred to as the annulus-center, and the width of the annulus is the Euclidean distance between two closest points on the boundary of $C_{\text{in}}$ and $C_{\text{out}}$ respectively. In this paper, we are interested to find square, rectangular and triangular annulus The objective is to minimize the width of the annulus.

Related Work: The minimum width annulus problem is well studied in the literature. The most common variation of this problem is the color-spanning circle. Abellanas et al. [1] showed that the smallest color spanning circle can be computed in $O(kn \log n)$ time. Abellanas et al. [2] also showed that the narrowest color-spanning strip and smallest axis-parallel color-spanning rectangle can be found in $O(n^2 \alpha(k) \log k)$ and $O(n(n - k) \log^2 n)$ time respectively. Das
et al. [8] improved the time complexity of narrowest color spanning strip problem to $O(n^2 \log n)$, and smallest color-spanning axis-parallel rectangle problem to $O(n(n - k) \log k)$. They also provided a solution for the arbitrary oriented color-spanning rectangle problem in $O(n^3 \log k)$ time using $O(n)$ space. Recently, Khanteimouri et al. [12] presented an algorithm for color spanning square in $O(n \log^2 n)$ time. Khanteimouri et al. [11] also presented a solution for the color spanning axis-parallel equilateral triangle in $O(n \log n)$ time.

On the other hand, the problem of computing the minimum width annulus is also studied in the literature. Given a set of $n$ points, computing the minimum width circular annulus containing all the points was independently addressed in [9,14,16]. All of their methods result in a time complexity of $O(n^2)$. There are also sub-quadratic time algorithms for the circular annulus problem. Using parametric searching technique, Agarwal et al. [5] presented an $O(n^{7/2})$ time algorithm. Agarwal et al. [3] also presented a randomized algorithm for the same problem which runs in $O(n^{2+\epsilon})$ time. To talk about the variations other than circular annulus for a general point set in $\mathbb{R}^2$, the well known results are an $O(n)$ time optimum algorithm for the axis parallel rectangular annulus by Ablenas et al. [3], and an $O(n \log n)$ time optimum algorithm for the axis parallel square annulus by Gluchshenko et al. [10]. Mukherjee et al. [13] proposed an algorithm for computing the minimum width axis parallel rectangular annulus for a point set in $\mathbb{R}^d$ in $O(nd)$ time. They also proposed an algorithm for arbitrary oriented minimum width rectangular annulus in $\mathbb{R}^2$ that runs in $O(n^3 \log n)$ time using $O(n)$ space. Recently, Bae [7] proposed the minimum width square annulus of arbitrary orientation in $O(n^3 \log n)$ time. The color spanning annulus problem is comparatively new in the literature. Acharyya et al. [4] presented two algorithms for finding the minimum width color spanning circular and axis-parallel square annulus. Both the algorithms run in $O(n^4 \log n)$ time using $O(n)$ space.

**Main Contribution:** In this paper, we propose algorithms to compute the minimum width color-spanning annulus where $C_{\text{in}}$ and $C_{\text{out}}$ are (i) axis-parallel squares ($CSSA$), (ii) axis-parallel rectangles ($CSRA$), and (iii) equilateral triangles of fixed orientation ($CSETA$). The time complexities of the proposed algorithms are: (i) for a $CSSA$ is $O(n^3 + n^2 k \log k)$, (ii) for a $CSRA$ is $O(n^4 \log n)$, and (iii) for a $CSETA$ is $O(n^3 k)$. The space complexity of all the algorithms is $O(k)$. The algorithm for $CSSA$ is an improvement of the existing result of [4] on this problem by a factor of $n$. Moreover, if $k$ is constant, then the improvement factor is $n \log n$.

## 2 Preliminaries

Interior of an annulus $A$, defined by $\text{INT}(A)$, is the region inside $A$ excluding $C_{\text{in}}$ and $C_{\text{out}}$. 

Observation 1. The points of distinct color lying on $C_{in}$ and $C_{out}$ are said to define an annulus $A$, and $\text{INT}(A)$ does not contain any point of color same as those defining the annulus $A$.

Throughout the paper, we use $h_p$ and $v_p$ to denote the horizontal and vertical lines passing through point $p$. $d(p_i, p_j)$, $d_\infty(p_i, p_j)$ denote the distance between the pair of points $p_i, p_j \in P$ in $L_2$ and $L_\infty$ norm respectively. The closest distance of a line segment $\ell$ from a point $p_i \in P$ will be denoted by $d(\ell, p_i)$. We also denote $\text{color}(p)$ as the color of a point $p \in P$.

3 Axis Parallel Color-spanning Square Annulus

An axis-parallel color-spanning square annulus (CSSA) is a color-spanning annulus $A$ bounded by two co-centric axis-parallel squares $C_{out}$ and $C_{in}$. In [4], it is shown that either $C_{in}$ or $C_{out}$ of a minimum width square annulus (CSSA) has two points of different colors in its two boundaries. We prove a stronger claim.

Lemma 1. Either $C_{in}$ or $C_{out}$ of a CSSA has two points of distinct color on its two mutually parallel boundaries.

Proof. For a contradiction, let only the two mutually perpendicular boundaries (say top and left boundaries) of the $C_{out}$ of a CSSA contain two points of different colors. We can reduce the size of $C_{out}$ (as well as $C_{in}$) by moving the bottom boundary upward and the right boundary to the left of both $C_{out}$ and $C_{in}$ with the same speed until the bottom boundary or the right boundary of $C_{out}$ touches a point. Observe that, during this movement if the bottom and/or right boundaries of $C_{in}$ encounter some point, those points will enter in the annular region, but no point goes out from the inside to the outside of the annular region. Thus, the annulus remains color-spanning and its width does not increase.

If two mutually perpendicular boundaries of $C_{in}$ contain two points of different colors, then also both $C_{in}$ and $C_{out}$ can be expanded keeping the annular region color-spanning such that the width of the annulus does not increase until the lemma is satisfied by $C_{in}$. Thus the lemma follows. \qed

We consider each pair of bi-colored points $p, q \in P$ to define the mutually parallel boundaries of $C_{out}$ and compute the minimum width annulus. Similar method works for defining the mutually parallel boundaries of $C_{in}$ with bi-colored pair of points in $P$.

Lemma 2. If $C_{out}$ of a minimum width axis-parallel square annulus $\mathcal{A}$, is defined by two points $p, q \in P$ on its two parallel boundaries, and its annulus-center is fixed at a point $c$, then its $C_{in}$ must pass through a point $r$ with minimum $L_\infty$ distance among the farthest points of every color $i \in \{1, 2, \ldots, k\} \setminus \{\text{color}(p), \text{color}(q)\}$ from the annulus-center $c$. 

Proof. As the annulus-center is fixed at the point $c$ and its radius is also fixed $d_\infty(c, p)$, $C_{out}$ is fixed. Now, for each color $i \in \{1, 2, \ldots, k\} \setminus \{\text{color}(p), \text{color}(q)\}$, the point of color $i$ having farthest distance must lie inside the annular region. Since all these $k - 2$ points need to be included in the annular region, $C_{in}$ will be defined by one such point which is closest to $c$.

Let $y(p) > y(q)$, and consider the horizontal strip of width $\delta = y(p) - y(q)$, defined by the horizontal lines $h_p$ and $h_q$. If $|x(p) - x(q)| > \delta$, then $p,q$ can not define $C_{out}$. Otherwise the points $p, q$ define $C_{out}$ whose center lies on a horizontal interval $C = [a, b]$, where $a = (\min(x(p), x(q)) + \frac{\delta}{2}, \frac{y(p)+y(q)}{2})$ and $b = (\max(x(p), x(q)) - \frac{\delta}{2}, \frac{y(p)+y(q)}{2})$. This configuration always holds as shown in [7][10].

Consider the left boundary of the square centered at $a$ and the right boundary of the square centered at $b$, both of radius $\frac{\delta}{2}$. These boundaries along with the horizontal lines through $p, q$ defines a rectangle $R$ (see Fig. 1(a)). All possible feasible $C_{out}$ are contained in $R$. We consider the subset $P' \subseteq P$ within $R$ and verify whether $P'$ is color spanning by using a linear scan. If $P'$ is not color spanning, then we can not have any CSSA contained in $R$. Hence we discard the horizontal strip defined by the pair $p, q$. Otherwise, for each point $r \in P'$ ($\text{color}(r) \notin \{\text{color}(p), \text{color}(q)\}$), we plot its distance in $L_\infty$ metric from different points of line segment $C$. Each of these distance functions $f(r)$ is a combination of line segments with slopes in $\{1, 0, -1\}$ as described in [4][7] (see Fig. 1(b)).

Lemma 3. [4][7] The distance curves $f(r)$ and $f(s)$ for the points $r, s \in P \cap R$ respectively intersect at exactly one point.

We consider the functions $F_i$ for all points each color $i \in \{1, 2, \ldots, k - 2\}$ separately. Let $\Gamma(i)$ be the upper envelope of the functions of $F_i$, $i = \{1, 2, \ldots, k\}$. For each point $\alpha \in C$, if the vertical line drawn at $\alpha$ intersects $\Gamma(i)$ at a point $\beta$, and $\beta$ lies on $f(r) \in F_i$, then the point $r$ of $\text{color}(i)$ is closest to $C_{out}$ centered at $\alpha$ among all points of $P$ having $\text{color}(i)$ inside $R$. Observe that, each $\Gamma(i)$ is one
of the forms listed in Figure 2 and their corresponding symmetric forms. This follows from the fact that the upper envelope $\Gamma(i)$ of the curves in $F(i)$ consists of at most one line segment of slopes \{-1, 0, 1\}.

![Figure 2: Nature of $\Gamma(i)$](image)

Now, to compute the furthest points of each color $i$ from $C$, we need to consider the lower envelope $\Gamma$ of $\Gamma(i)$, $i \in \{1, 2, \ldots, k\} \setminus \{\text{color}(p), \text{color}(q)\}$. Again to minimize the width of the annulus, we choose the point on $\Gamma$ having maximum vertical distance from $C$. Its projection $c$ on $C$ is the center of the CSSA with $p, q$ on the two parallel sides of its $C_{\text{out}}$ (see Fig. 1(b)).

Lemma 4. Given the rectangle $R$, the set of points within $R$, we can find the annulus-center $c$ in $O(n \log n)$ time.

Proof. Let $\text{color}(p) = k$, $\text{color}(q) = k - 1$, and there are $n_i$ points of color $i = \{1, 2, \ldots, k - 2\}$. By Lemma 3, the computation of upper envelope $\Gamma(i)$ takes $O(n_i \log n_i)$ time [15]. Thus, the total time for computing $\Gamma(i)$ for all $i = 1, 2, \ldots, k - 2$ is $O(n \log n)$. Now, we have $k$ totally defined functions $\Gamma(i)$ in the interval domain $C = [a, b]$, where each pair of functions $\Gamma(i)$ and $\Gamma(j)$ intersect in at most two points. The size of the lower envelope $\Gamma$ of the functions $\Gamma(i)$, $i = 1, 2, \ldots, k - 2$ is $\lambda_2(k - 2)$ (the Davenport Schinzel sequence of order 2), and it can be computed in $O(\lambda_2(k - 2) \log k)$ time [15]. Finally, we compute the point having maximum y-coordinate on $\Gamma$ by inspecting all its vertices. Since $\lambda_2(k - 2) = 2k - 5$, the total time required for processing $\Gamma(i)$, $i = 1, 2, \ldots, k - 2$ is $O(k \log k)$. Thus, the total time complexity is dominated by computing $\Gamma(i)$, $i = 1, 2, \ldots, k - 2$, which is $O(n \log n)$.

Lemma 5. Given a set of $n$ points, each assigned with one of the $k$ given colors, the minimum width CSSA can be computed in $O(n^3 \log n)$ time.

Proof. We consider $O(n^2)$ pairs of bi-colored points. For each pair of such points $(p, q)$, we can construct the rectangle $R$ in $O(1)$ time. With the help of a linear search we can verify whether $R$ is color-spanning. If $R$ is color-spanning, then using lemma 4 we can determine the optimum square annulus contained in $R$ in $O(n \log n)$ time. Thus the time complexity result follows.
We can further improve the time complexity in the following way using a total $O(k)$ amount of extra space. We choose a pair of points $p, q \in P$ and let $|x(p) - x(q)| > \delta$, where $\delta = y(p) - y(q)$. Now, we can construct the rectangle $R$ and verify its color-spanning property as was done earlier. If it is color spanning, then for each color we maintain the upper envelope $\Gamma_i$. As mentioned earlier, $\Gamma_i$’s are of constant complexity. Thus, each $\Gamma_i$ can be maintained using $O(1)$ space. For each point $r \in P$, if $r$ is inside $R$, then we consider its distance curve $f(r)$ from $C$, and update $\Gamma_i$ for $i = \text{color}(r)$ considering the intersection of $f(r)$ and the existing $\Gamma_i$. This can be done in $O(1)$ time. Thus considering all points in $P$, we can construct the $\Gamma_i, i = \{1, 2, \ldots, k - 2\}$ in $O(n)$ time using $O(k)$ space. At the end, we consider the lower envelope $\Gamma$ of these $\Gamma_i$’s and return the maximum point. As in Lemma 4, this can be done in $O(k \log k)$ time using $O(k)$ space. Considering $O(n^2)$ pairs of bi-colored points, we have the following result:

**Theorem 1.** Given a set of $n$ points, each assigned with one of the $k$ given colors, the minimum width CSSA can be computed in $O(n^3 + n^2k \log k)$ time using $O(k)$ extra space.

4  **Axis Parallel Color-spanning Rectangular Annulus**

An axis-parallel color-spanning rectangular annulus (CSRA) is a color-spanning annulus $A$ bounded by two co-centric axis-parallel rectangles $C_{out}$ and $C_{in}$. The top (resp. bottom, left, right) boundaries of $C_{in}$ and $C_{out}$ are said to be similar sides of these two rectangles. The width of CSRA is half of the difference of lengths (widths) of $C_{out}$ and $C_{in}$ (see Fig. 3).

![Fig. 3: Rectangular Annulus](image_url)

**Lemma 6.** The necessary and sufficient condition for $A$ to be a minimum width CSRA is that (i) all the four edges of $C_{out}$ must contain at least one point, and at least one edge of $C_{in}$ must contain at least one point, or (ii) all the four edges of $C_{in}$ must contain at least one point, and at least one edge of $C_{out}$ must contain at least one point. In both the cases, these five points of $P$ are of different colors.

**Proof.** Part (i): For a contradiction let us assume that three edges of $C_{out}$ contains three points, one edge of $C_{in}$ contains a point, and the colors of these four
points are different satisfying Observation 1. Let \( e \) be the edge of \( C_{out} \) containing no point. We start moving the edge \( e \) of \( C_{out} \) containing no point towards the annulus-center \( c \) in the self-parallel manner. To maintain the same width of \( A \) we need to move the similar side of \( C_{in} \) of \( e \) towards \( c \) simultaneously with \( e \) until \( e \) hits a point of \( P \) having color different from the colors of all three points on \( C_{out} \) (see Fig. 3). Note that, all the points lying in the annular region of \( A \) remains in the annular region of \( A' \) formed with the new positions of \( e \) (surely, a few more points may enter in the annular region).

If none of the edges of \( C_{in} \) contains a point, then we can reduce the width of the annulus by moving the four edges of \( C_{in} \) away from the annulus-center \( c \) in self-parallel manner until at least one edge of \( C_{in} \) hits a point inside the annular region \( A \) satisfying Observation 1.

**Part (ii):** Similar proof holds to show that the width of an annulus defined by four points on the four edges of \( C_{in} \) and one edge of \( C_{out} \) containing a point.

Thus, the lemma follows. \( \Box \)

Lemma 6 leads to the following result:

**Lemma 7.** In an optimum CSRA a pair of similar sides (of \( C_{in} \) and \( C_{out} \)) will contain two points of different colors.

We now discuss the algorithm for rectangular annulus based on the Lemma 7. Assume that the points in \( P \) are available in sorted order with respect to \( x \)- and \( y \)-coordinates in two arrays \( P_x \) and \( P_y \) respectively. Consider a pair of points \( p, q \in P \) of different colors. We test whether a CSRA is possible with \( p \) and \( q \) on the top boundaries of \( C_{out} \) and \( C_{in} \) respectively. The width of such a CSRA, if exists, will be \( \delta = y(p) - y(q) \). Similar method is adopted to find the existence of a CSRA with \( p, q \) in the bottom, left or right boundaries of \( C_{in} \) and \( C_{out} \).

Fig. 4: Illustration of rectangular annulus construction
Let us consider \(h_p\) and \(h_q\). In a linear scan we can find a point \(s \in P_y\) such that the horizontal strip defined by \(h_p\) and \(h_s\) is color spanning. If \(y(p) - y(s) \leq 2\delta\), then an annulus of width \(\delta\) is trivially obtained with \(C_{in} = \emptyset\). Thus, we assume that \(y(p) - y(s) > 2\delta\).

Observe that, for all points \(r \in P\) with \(y(r) \leq y(s)\), the horizontal strip \(H\) defined by \(h_p\) and \(h_r\) will be color spanning. We now compute a minimum width \(CSRA\) inside the strip \(H\) with points \(p\) and \(r\) lying respectively on the top and bottom boundary of \(C_{out}\), and \(q\) lying on the top boundary of \(C_{in}\).

Fix a point \(a \in P_y\) with \(x(a) < \min(x(p), x(q), x(r))\) inside the strip \(H\) having color different from that of \(p, q\) and \(r\). In a linear scan in the array \(P_x\), we can get a point \(b_x\) such that the rectangle defined by the lines \(h_p, h_r, v_a, v_b\) is color-spanning. Thus for all points \(c \in P_s\) satisfying \(y(c) \in [y(p), y(r)]\) and \(x(c) \geq x(b)\), the rectangle defined by the lines \(h_p, h_r, v_a, v_c\) will be color-spanning. In a line sweep inside the strip \(H\), we choose those points \(c\) with \(x(c) \geq x(b)\) and having color different from that of \(p, q, r, a\). Now, the rectangle \(R\) formed by \(h_p, h_r, v_a, v_c\) defines \(C_{out}\). The corresponding \(C_{in}\) will be co-centric with \(C_{out}\), its length and width will be \((x(c) - x(a)) - 2\delta\) (see Fig. 4). We can test whether the created annulus is color-spanning or not by inspecting the points in \(P\) in another linear scan. Thus, we have the following result:

**Lemma 8.** Given a set of \(n\) points, each assigned with one of the \(k\) given colors, the minimum width \(CSRA\) can be computed in \(O(n^6)\) time using \(O(n)\) space.

Proof. For each pair of points \(p, q \in P\), we execute four loops: (i) choosing the points \(r\) using a horizontal line sweep, (ii) choosing the points \(a\) using a vertical line sweep inside the strip \(H\), (iii) choosing the points \(c\) using a vertical line sweep from \(a\) towards right, and then (iv) testing whether the created annulus is color-spanning by testing the points in the rectangle \(R = C_{out}\).

We can improve the time complexity by merging the two loops (iii) and (iv) mentioned in the proof of Lemma 3 as follows:

For each \(a \in H\), we start sweeping two vertical lines \(L_1\) and \(L_2\) simultaneously by using an array \(D\) of size \(k\), and a scalar variable \(Z\). \(D[i]\) indicates the number of points of color \(i\) in the annulus, and \(Z\) indicates the number of colors absent in the annulus. We initialize \(D[i] = 0\) for all \(i = 1, 2, \ldots, k\), \(D[\text{color}(p)] = D[\text{color}(q)] = D[\text{color}(r)] = D[\text{color}(a)] = 1\), and \(Z = k - 4\). We also initialize the starting position of \(L_2\) as the index of the rightmost point \(d \in P_x\) with \(x(d) < x(a) + \delta\) and \(y(d) \in y(p, y(r))\). The sweep of \(L_1\) is implemented by considering the points in \(P_x\) in order from the point \(a\). For each encountered point \(c\) of color \(i\) (say), if \(y(c) \notin [y(p), y(r)]\), then \(c\) is not feasible to be within the rectangular annulus currently under construction. Otherwise, we do the following:

\[This\ also\ can\ also\ be\ tested\ in\ poly-logarithmic\ time\ by maintaining\ \(k\)\ range\ trees\ with\ points\ corresponding\ to\ \(k\)\ colors\ separately\ and\ performing\ emptiness\ queries\ for\ four\ axis-parallel\ rectangles\ in\ all\ those\ range\ trees;\ this\ will\ increase\ in\ the\ space\ complexity\ to\ \(O(n \log n)\).]
\[ c \text{ enters in the annulus: } \] we set \( D[i] = D[i] + 1 \). If \( D[i] = 1 \) (a point of color \( i \) is newly obtained in the annulus), then \( Z \) is decremented by 1. Now, if \( x(c) - x(a) \leq 2\omega \) [indicating \( C_{in} = \emptyset \) (see Fig. 5(a))], we need not have to do anything. If \( x(c) - x(a) > 2\omega \) [indicating \( C_{in} \neq \emptyset \) (see Fig. 5(b))], we start sweeping \( L_2 \) from its present position up to \( x(c) - \delta \). For all points \( d \) encountered by \( L_2 \), if \( y(q) > y(d) > y(r) + \delta \) then \( D[j] = D[j] - 1 \), where \( j = \text{color}(d) \). If \( D[j] = 0 \) (indicating no point of color \( j \) in the annulus) then \( Z \) is incremented by 1.

**Check whether the annulus \( A \) is color-spanning:** If \( Z = 0 \), then report “success”, and stop sweeping of \( L_1 \).

Since sweeping of \( L_1 \) and \( L_2 \) needs \( O(n) \) time, the time complexity of the algorithm reduces to \( O(n^3) \). Here, it needs to be mentioned that, for each point \( p \), we are testing whether there exists an annulus of width \( \delta = y(p) - y(q) \) by choosing all possible points \( q \) satisfying \( y(q) < y(p) \). Thus, for each point \( p \), we choose \( O(n) \) points as \( q \) in the worst case. But, this can be improved using a simple binary search technique. We can find minimum value of \( \delta \) (the minimum width of an annulus) with \( p \) on the top boundary of \( C_{out} \) by choosing \( q \) using binary search among the points in \( P_y \) having \( y \)-coordinate less than \( y(p) \) and of different color. Thus for each point \( p \), we need to choose \( O(\log n) \) points as \( q \), which leads to a total time complexity of \( O(n^4 \log n) \). Now, this algorithm can also be implemented in inplace manner using \( O(k) \) extra space. For each choice of \( p \in P \) (in decreasing order), we first get \( P_y \) in \( O(n \log n) \) time (in the array \( P \) itself) to choose an appropriate member \( q \in P \). All the members in \( P \) below \( q \) can serve the role of the point \( r \in P \). Now, for \( p, q, r \in P \), we sort \( P \) again to get \( P_x \) (in the array \( P \)), and the sweep is performed to choose \( a \in P \) to compute minimum width \( \text{CSRA} \), if exists. Thus, we need to store \( p, q, r \) (using \( O(1) \) variables) to identify the next triple \( p, q', r' \in P \) for processing. This leads to the following result:
Theorem 2. Given a set of \( n \) points, each assigned with one of the \( k \) given colors, the minimum width CSRA can be computed in \( O(n^4 \log n) \) time using \( O(k) \) extra space.

5 Color Spanning Equilateral Triangular Annulus

A color-spanning equilateral triangular annulus (CSETA) is a color-spanning annulus \( A \) bounded by two co-centric equilateral triangles \( C_{\text{out}} \) and \( C_{\text{in}} \) where the common center for both the triangles is termed as annulus-center \( c \). We assume that the base of \( C_{\text{in}} \) and \( C_{\text{out}} \) are parallel to the \( x \)-axis. Two such types of annulus is possible depending on whether the apex of \( C_{\text{in}} \) and \( C_{\text{out}} \) is above or below the base of the corresponding triangle. We will explain the method assuming that the apex is above the base. The other case can be similarly processed.

\[ \text{Wedge:} \quad \text{Consider a pair of half-lines emanating from a point } p \text{ having angles } 60^\circ \text{ and } 120^\circ \text{ with the horizontal line at } p. \text{ The point } p \text{ is said to be the vertex of the wedge. The (open) areas created by these two half-lines above and below } p \text{ are termed as the upward wedge and downward wedge respectively for the point } p \text{ (see Fig. 6).} \]

\[ \text{Similar vertex and edge:} \quad \text{Each pair of vertices, one of } C_{\text{in}} \text{ and one of } C_{\text{out}}, \text{ that are collinear with the annulus-center } c, \text{ are said to be similar vertices. Similarly the edges of } C_{\text{in}} \text{ and } C_{\text{out}} \text{ that are parallel to each other, are said to be similar edges. In Fig. 7(a), } x \text{ and } u \text{ are similar vertices, and } xy \text{ and } uv \text{ are similar edges.} \]

\[ \text{Fig. 6: Basic constructions} \]

\[ \text{Fig. 7: Two types of boundary points configuration} \]

\[ ^2 \text{the point of intersection of three medians of the equilateral triangle} \]
**Width of Annulus:** The width of a triangular annulus is the difference in the length of the line segments perpendicular on two similar edges from its annulus-center $c$ (see Fig. 7(a)).

**Lemma 9.** The distance among a pair of similar vertices in $C_{in}$ and $C_{out}$ of a CSETA $A$ is twice the width of that CSETA.

**Proof.** Consider Fig 7(a), where $C_{out} = \triangle xyz$ and $C_{in} = \triangle uvw$. The line containing $\overline{uv}$ bisects both $\angle yxz$ and $\angle uvw$. The width of $A$ is the perpendicular distance from $u$ to the line $\overline{yx}$, which is equal to $|um|$. Thus, $\triangle xum$ is right angled with $\angle mxu = 30^\circ$, and $|xu| = 2 \times |um|$. \hfill $\Box$

**Lemma 10.** The necessary and sufficient condition for $A$ to be a minimum width CSETA is that (i) all the three edges of $C_{out}$ must contain at least one point, and at least one edge of $C_{in}$ must contain at least one point or (ii) all the three edges of $C_{in}$ must contain at least one point, and at least one edge of $C_{out}$ must contain at least one point. In both the cases, these four points of $P$ are of different colors.

**Corollary 1** If a point lies on a vertex of $C_{in}$ or $C_{out}$, then a CSETA can be defined by three points (instead of four points) on boundary$^3$ (see Fig. 7(b)).

### 5.1 Algorithm

Here, we describe the general framework of the algorithm to find CSETA of minimum width where $C_{out}$ is defined by three points of distinct colors (see Lemma 10). The same method works to find a CSETA of minimum width where $C_{in}$ is defined by three points. We sort the points in $P$ with respect to their $y$-coordinates. Consider each bi-colored pair of points $p, q \in P$ with $x(p) < x(q)$. Consider a downward wedge $W_{p,q}$ at a vertex $v_{p,q}$ by drawing a line of angle $60^\circ$ through the point $p$ and a line of angle $120^\circ$ through the point $q$ (see Fig. 8). In linear time we can find the points $P_{p,q} \subseteq P$ lying in $W_{p,q}$ sorted with respect to their $y$-values. If $W_{p,q}$ is not color-spanning, $C_{out}$ cannot be defined by $W_{p,q}$. So, let us assume that $W_{p,q}$ is color-spanning and the pair $(p, q)$ defines two boundaries of $C_{out}$. Let $\ell_{p,q}$ be a vertical line through $v_{p,q}$. Now, the following results are important.

**Lemma 11.** The base of $C_{out}$ corresponds to a horizontal line through a point $r \in P_{p,q}$, where $y(r) < \min(y(p), y(q))$ color$(r) \notin \{\text{color}(p), \text{color}(q)\}$.

For each point $r \in P_{p,q}$, we define two lines, namely

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$^3$ it can be shown in a similar way as in lemma 10

$^4$ similar idea works when it define $C_{in}$ by considering the points outside the wedge to define $C_{out}$ instead of considering inner wedge points.
**Fig. 8: Construction of wedges and lines**

$h_r$: the horizontal line through the point $r$, and  

$w_r$: the wedge line, which is a line of angle $60^\circ$ (resp. $120^\circ$) through the point $r$ depending on whether $r$ is to the left (resp. right) of $\ell_{p,q}$.

By Lemma 11 we set $r = p$ or $q$ depending on whether $y(p) < y(q)$ or $y(p) > y(q)$ (consider the minimum one), and start sweeping a horizontal line. The points in $W_{p,q}$ encountered by the sweep line are the event-points. At each event-point $r$, if the triangle $\Delta$ defined by the vertex $v_{p,q}$ and the line $h_r$ in the wedge $W_{p,q}$ is color-spanning, $r$ defines the base of $C_{out}$ and we compute $C_{in}$ as follows:

- Create an array $D$ of size $k$ to store the distance of the closest point of each color $i$ from the boundary of $C_{out}$. Initialize $D[i] = \infty$ for all $i = 1, 2, \ldots, k$.
- Let $\theta$ be the point of intersection of $h_r$ and $\ell_{p,q}$.
- For every point $s \in \Delta$, do the followings:
  - Let $\alpha_s$ and $\beta_s$ be the points of intersection of $w_s$ and $h_s$ with $\ell_{p,q}$. We will use the term $\alpha$-point and $\beta$-point of $s$ to denote the points $\alpha_s$ and $\beta_s$ respectively.
  - Compute the distances $d(\theta, \beta_s)$ and $d(v_{p,q}, \alpha_s)$.
  - The distance of $s$ from the boundary of $C_{out}$ is $\mu = \min(d(\theta, \beta_s), \frac{d(v_{p,q}, \alpha_s)}{2})$ (see Observation 9). If $s$ is of color $i$ then store $\min(\mu, D[i])$ in $D[i]$.  
- The width of the annulus $\delta = \max_{i=1}^{k} D[i]$, which can be computed by a linear scan in the array $D$. Thus, $C_{in}$ is determined.

**Lemma 12.** The overall time complexity of this simple scheme is $O(n^4)$. 
Proof. For each of the $O(n^2)$ pair of points $(p, q)$, the sweep considers $O(n)$ event points in $P_{p,q}$. For each event point $r \in P_{p,q}$, we need to spend $O(n)$ time in the worst case to inspect all the points in $\Delta$. Thus the result follows.

We can improve the time complexity by maintaining two AVL trees $G$ and $H$ (instead of the array $D$) while processing each pair of points $(p, q) \in P$. Here, $G[i]$ stores the closest point $\alpha$-point of each color $i$ from $v_{p,q}$ and $H$ stores the minimum $y$-coordinate of the $\beta$-points of each color $i$. Each element of $G$ and $H$ is attached with the corresponding color index. Auxiliary arrays $G'$ and $H'$ are maintained, where $G[i]$ (resp. $H[i]$) stores the position (address of location) containing color $i$ in $G$ (resp. $H$). Thus, the size of both $G$, $H$, $G'$ and $H'$ are $O(k)$.

We have the points $P$ in decreasing order of their $y$-coordinates. When a new point $s \in W_{p,q}$ of color $i \in \{1, 2, \ldots, k\} \setminus \{\text{color}(p), \text{color}(q)\}$ is faced by the sweep line, we process $s$. The position of color $i$ in the array $G$ and $H$ are $j_1 = G'[i]$ and $j_2 = H'[i]$ respectively. $G[j_1]$ is updated with $\min\left(\frac{d(v_{p,q}, \alpha_s)}{2}, G[j_1]\right)$, and $H[j_2]$ is updated with $y(s)$. Next, both the AVL trees $G$ and $H$ are adjusted in $O(\log k)$ time. Now, define $r = s$ (i.e., the base line at point $s$), and scan both $G$ and $H$ in increasing order of their values using a merge like pass to find the required $C_{in}$ so that the annular region $A$ is of minimum width and contains every color from either $G$ or $H$. Thus, we have the following result:

**Theorem 3.** Given a set $P$ of $n$ points, each point is assigned with one of the $k$ possible colors, the minimum width CSETA can be computed in $O(n^3k)$ time using $O(k)$ additional space.

Proof. The correctness of the algorithm follows from Lemma 10 its corollary, and the algorithm where we always maintain the nearest points of each color corresponding to the wedge boundaries and the base line in two $O(k)$ sized AVL trees. The time complexity is analyzed considering the fact that we need to consider $\binom{n}{2}$ pairs of bi-colored points $(p, q) \in P$ to define the wedge $W_{p,q}$. During the line sweep for the processing $W_{p,q}$, at each event point $r \in P_{p,q}$ we spend $O(k)$ time\(^5\). Hence, the time complexity result follows. The extra space requirement follows from the size of $G$, $H$, $G'$ and $H'$.

Corollary 11 suggested that we need to consider $C_{out}$ with each point on one of its three vertices. When a point $p \in P$ is considered as apex, the wedge is defined by the two lines of angle $60^\circ$ and $120^\circ$ with the axis through the point $p$ and similar method is executed to get the optimum CSETA. The case when the point $p \in P$ is the left (resp. right) endpoint of the base, has already been considered in our algorithm due to Lemma 11 Thus the result follows.

\(^5\) The adjustment of the AVL-trees $G$ and $H$ needs $O(\log k)$ time. But, to find the elements from $G$ and $H$ to determine the minimum width annulus $A$ needs a linear scan in $G$ and $H$. 

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