FAMILIES OF CANONICAL LOCAL PERIODS ON SPHERICAL VARIETIES

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Abstract. We consider the variation of canonical local periods on spherical varieties proposed by Sakellaridis-Venkatesh in families. We formulate conjectures for the rationality and meromorphic property of canonical local periods and establish these conjectures for strongly tempered spherical $G$-varieties without type $N$-roots when $G$ is split.

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1. Introduction

1.1. Zeta integrals in families. The variation of local zeta integrals for smooth representations in families is extensively considered in literature and has various applications.

In the influential paper [10], Cogdell-Piatetski-Shapiro consider the family $\{\pi_x = I_G^P \sigma \otimes \chi_x \mid x \in X\}$. Here $G$ is the product of two general linear groups over a $p$-adic field $F$, $P \subset G$ is a parabolic subgroup with Levi factor $M$, $\sigma$ is a square-integrable irreducible (complex) $M(F)$-representation with $I_G^P$ the normalized parabolic induction, $X$ is the (complex) torus parameterizing unramified characters of $M$ and $\chi_x$ is the unramified character corresponding to $x$. Then they construct a Whittaker model $W(\pi)$ consisting of functions on $X \times G(F)$ which interpolates the Whittaker model of $\pi_x$ for each $x \in X$ and show the Rankin-Selberg local zeta integral $Z_{RS}(W, s)$ is meromorphic on $X \times \mathbb{C}$ for each $W \in W(\pi)$. As an application, this meromorphy property allows one to express $L(s, \pi)$ in term of $\sigma$.

Note that the family $\{\pi_x\}$ is actually the specializations of some $O_X[G(F)]$-module $\pi$. Taking this algebraic viewpoint, one can consider smooth admissible $G(F)$-representations over general reduced Noetherian coefficient ring $R$, and establish the theory of zeta integrals, in particular the meromorphy property, in this algebraic setting. For general linear groups, Moss [27, 28] algebraizes the Rankin-Selberg zeta integral machinery and obtains the meromorphy property for the co-Whittaker $R[G(F)]$-modules introduced by Emerton-Helm [15] and Helm [20]. For classical groups, Girsch [17] algebraizes the doubling zeta integral machinery and establishes the meromorphy property. One key ingredient is the admissibility of Jacquet modules of smooth admissible $R[G(F)]$-modules.

The variation of local zeta integrals in families plays an important role in the study of $p$-adic $L$-functions. For example, Disegni [12, 14] applied the meromorphy property of local zeta integrals at places $v \nmid p$ to construct the desired $p$-adic $L$-function appearing in the representation theoretic $p$-adic Gross-Zagier formula of ordinary families.
1.2. Canonical local periods in families. In a broad context, Sakellaridis-Venkatesh \cite{32} introduces the canonical period on spherical varieties to study the special (local) $L$-value, which in some sense can be viewed as a vast generalization of local zeta integrals.

In this paper, we shall consider the variation of canonical local periods in families, which we expect to behave similarly as zeta integrals. In this introduction, we shall only consider the geometric quotient $Y := H \backslash G$ for reductive groups $H \subset G$ over $F$ which is spherical and strongly tempered in the sense that the matrix coefficients of any tempered $G(F)$-representation are absolutely integrable over $H(F)$. Then the canonical local period $\alpha_x$ for any tempered $G(F)$-representation $\pi$ associated to $Y$ is the bilinear form in $\text{Hom}_{H(F)^{\alpha}}(\pi \boxtimes \pi^\vee, \mathbb{C})$ defined by integrating matrix coefficients of $\pi$ over $H$

$$\alpha_x(\varphi_1, \varphi_2) = \int_{H(F)} (\pi(h)\varphi_1, \varphi_2) dh, \quad \varphi_1, \varphi_2 \in \pi, \varphi_2 \in \pi^\vee.$$ 

Here $\pi^\vee$ is the contragredient representation of $\pi$ and $\langle \cdot, \cdot \rangle$ is a non-degenerate $G(F)$-invariant pairing on $\pi \times \pi^\vee$. For example (\cite[Proposition 4.10]{37}), in the case $Y = \Delta \text{GL}_n \backslash \text{GL}_n \times \text{GL}_{n+1}$,

$$\alpha_x (W_1, W_2) = Z_{RS} \left( W_1 \frac{1}{2}, W_2 \frac{1}{2} \right), \quad W_1 \in \mathcal{W}(\pi, \psi), W_2 \in \mathcal{W}(\pi^\vee, \psi^{-1})$$

for any tempered $\text{GL}_n(F) \times \text{GL}_{n+1}(F)$-representation $\pi$ under proper normalizations on the Haar measure and the $G$-invariant pairing $\langle -, - \rangle$. Here, $\mathcal{W}(\pi, \psi)$ is the Whittaker model of $\pi$ with respect to a nontrivial additive character $\psi$.

To fix ideas, we consider the following set-up. Let $E$ be a field embeddable into $\mathbb{C}$, $R$ be a reduced $E$-algebra of finite type and $\Sigma \subset \text{Spec}(R)$ be a fixed Zariski dense subset of closed points. Let $\pi$ be a finitely generated smooth admissible torsion-free left $R[G(F)]$-module (See Section 2.3 for details). Assume that

(a) for any $x \in \Sigma$, the specialization $\pi|_x := \pi \otimes_R k(x)$ is absolutely irreducible and tempered in the sense that the following set of embeddings is nonempty

$$\mathcal{E}(\pi|_x) : = \{ \tau : k(x) \hookrightarrow \mathbb{C} | (\pi|_x)_\tau := \pi|_x \otimes_{k(x)} \tau, \mathbb{C} \text{ is tempered} \};$$

(b) there exists a finitely generated smooth admissible torsion-free $R[G(F)]$-module $\tilde{\pi}$ and a $G(F)$-invariant $R$-bilinear pairing

$$\langle \cdot, \cdot \rangle : \pi \times \tilde{\pi} \rightarrow R$$

such that for each $x \in \Sigma$, $\tilde{\pi}|_x \cong (\pi|_x)^\vee$ and $\langle \cdot, \cdot \rangle$ induces a non-degenerate $G(F)$-invariant pairing

$$\langle \cdot |_x, \cdot |_x \rangle : \pi|_x \times \tilde{\pi}|_x \rightarrow k(x).$$

This set-up is mainly motivated by our intended global application to study $p$-adic $L$-functions on eigenvarieties: $\text{Spec}(R)$ (resp. $\Sigma$) plays the role of eigenvarieties (resp. classical points), $\pi$ is the local component of the universal automorphic representation on the eigenvariety and the above assumptions hold in many interesting cases (See \cite[Chapter 4]{13} for a detailed discussion for the case $G = \text{GL}_2$).

**Conjecture 1.1** (Meromorphy and rationality for canonical period integrals). Assume $Y$ is a strongly tempered spherical $G$-variety. Let $\pi$ be a $R[G(F)]$-module as above. Then

(1) (Rationality) For any $x \in \Sigma$, there exists a unique bi-$H(F)$-invariant pairing

$$\alpha_{\pi|_x} : \pi|_x \times \tilde{\pi}|_x \rightarrow k(x)$$

such that for any $\tau \in \mathcal{E}(\pi|_x)$, the following diagram commutes

$$\begin{array}{ccc}
\pi|_x \times \tilde{\pi}|_x & \overset{\alpha_{\pi|_x}}{\longrightarrow} & k(x) \\
\downarrow{\tau} & & \downarrow{\tau} \\
(\pi|_x)_\tau \times (\tilde{\pi}|_x)_\tau & \overset{\alpha(\pi|_x)_\tau}{\longrightarrow} & \mathbb{C}.
\end{array}$$

Here $\alpha_{(\pi|_x)_\tau}$ is defined with respect to the linear extension of $\langle \cdot, \cdot \rangle|_x$.

(2) (Meromorphy) Up to shrinking $\text{Spec}(R)$ to an open subset containing $\Sigma$, there exists a bi-$H(F)$-invariant $R$-linear pairing

$$\alpha_x : \pi \times \tilde{\pi} \rightarrow R$$

2
such that for any \( x \in \Sigma \), the following diagram is commutative

\[
\begin{array}{c}
\pi \times \tilde{\pi} \\
\downarrow \alpha_x \\
\pi|_x \times \tilde{\pi}|_x
\end{array} \xrightarrow{\alpha_{\pi|x}}
\begin{array}{c}
R \\
\downarrow \kappa(x)
\end{array}
\]

Here both vertical arrows are the specialization maps.

For general spherical varieties \( Y \), one can extract the canonical local period \( \alpha_x \) from the conjectural Plancherel decomposition for the unitary \( G(F) \)-representation \( L^2(Y) \). Moreover, to make the canonical local period independent of the invariant pairing \( \langle \cdot, \cdot \rangle \), we consider the quotient of \( \alpha_x \) by \( \langle \cdot, \cdot \rangle \). In this way, the existence of the \( G(F) \)-invariant \( R \)-bilinear pairing in condition (b) of Conjecture 1.1 can be dropped. For the precise formulation, we refer to Conjectures 3.7 and 3.8 for details.

The following is the main result of this paper.

**Theorem 1.2** (Theorem 4.18 and Theorem 4.22). Let \( Y = H \setminus G \) be a homogeneous spherical \( G \)-variety with \( G \) split. Assume \( Y \) is strongly tempered without type \( N \)-roots. Let \( \pi \) be a \( R[G(F)] \)-module as in Conjecture 1.1. Then

- the rationality conjecture holds for \( \pi \),
- the meromorphy conjecture holds for \( \pi|_U \) where \( U \subset \text{Spec}(R) \) is a Zariski dense open subset; moreover the meromorphy conjecture holds for \( \pi \) if the discrete support of \( \pi \) is rigid around each \( x \in \Sigma \) (see Definition 4.17).

Roughly speaking, the discrete support of \( \pi \) is rigid around \( x \in \Sigma \) means that if \( \pi \rightarrow I^G_\sigma \) for some discrete series \( \sigma \), then for \( y \in \Sigma \) around \( x \), \( \pi|_y \rightarrow I^G_\sigma \otimes \chi|_y \) for some character \( \chi|_y \). We expect this property to hold in a reasonable generality. In fact, if \( G \) is a product of general linear groups, the discrete support of \( \pi \) is rigid around each \( x \in \Sigma \) for all the \( R[G(F)] \)-modules \( \pi \) in consideration (see Proposition 4.21) and for general \( G \), the discrete support of \( \pi \) is rigid around \( x \in \Sigma \) if \( \pi|_x = I^G_\sigma \) with \( \sigma \) regular supercuspidal (see Proposition 4.20).

We record an immediate corollary, which studies distinguished representations in an unramified twisting family. Let \( P = MN \subset G \) be a parabolic subgroup with Levi factor \( M \) and denote by \( X \) the torus of unramified complex characters on \( M(F) \).

**Corollary 1.3.** Let \( Y = H \setminus G \) be a homogeneous spherical \( G \)-variety with \( G \) split. Assume \( Y \) is strongly tempered without type \( N \)-roots and wavefront. Let \( \sigma \) be a discrete series (complex) representation on \( M(F) \) and assume the parabolic induction \( I^G_\sigma \) is irreducible and distinguished, i.e. \( m(I^G_\sigma) := \dim \text{Hom}_{\text{par}}(I^G_\sigma, \mathbb{C}) \neq 0 \). Then all the representations \( \pi|_x = I^G_\sigma \otimes \chi|_x \) (not necessarily irreducible), \( x \in X \) in the unramified twisting family are distinguished. Here for \( x \in X \), \( \chi|_x \) is the unramified character corresponding to \( x \).

**Proof.** Fix a nonzero invariant pairing \( \langle \cdot, \cdot \rangle_{\sigma} \) on \( \sigma \boxtimes \sigma^\vee \) and choose a good maximal open compact subgroup \( K \subset G(F) \) such that \( G(F) = P(F)K \). Then by the identifications

\[
\pi|_x \xrightarrow{\sim} V := I^K_{K \cap P \sigma}\quad (\text{resp. } \pi|_x^\vee \xrightarrow{\sim} V^\vee := I^K_{K \cap P \sigma^\vee}), \quad f \mapsto f|_K
\]

and [8, Proposition 3.1.3], the bi-linear pairing

\[
V \times V^\vee \rightarrow \mathbb{C}; \quad \langle v_1, v_2 \rangle = \int_K \langle v_1(k), v_2(k) \rangle_{\sigma} dk
\]

induces a non-degenerate invariant pairing on \( \pi|_x \times (\pi|_x)^\vee \) for each \( x \in X \). Moreover, for any \( v_1 \in V \), \( v_2 \in V^\vee \) and any \( g \in G \), the function

\[
x \mapsto \langle \pi|_x(g)v_1, v_2 \rangle
\]

is regular. In particular, the family \( \pi|_x \) satisfies the assumptions in Conjecture 1.1.

By [32, Theorem 6.4.1], \( \alpha_{I^G_\sigma} \neq 0 \) since \( I^G_\sigma \) is distinguished. By Theorem 1.2, the canonical local period is meromorphic and thus \( \alpha_{\pi|_x} \neq 0 \) for \( x \) in an open subset of \( X \). In this open subset, \( m(\pi|_x) \geq 1 \) and consequently by the upper-semicontinuity of \( m(\pi|_x) \) established in [16, Appendix D], one deduces \( m(\pi|_x) \geq 1 \) for all \( x \in X \).

\( \square \)

**Remark 1.4.** In [2], for symmetric pairs \((G, H)\), meromorphic families of \( H \)-distinguished linear functionals over complex torus are constructed. By generalizing this result to reduced Noetherian coefficient rings and comparing the square of these linear functionals with the canonical local periods, one may
obtain the meromorphy of canonical period integrals (Conjectures 3.7 and 3.8) for symmetric pairs. We plan to consider this problem in another paper.

Remark 1.5. Attached to the canonical local period $\alpha_{\pi}$, there is the spherical character

$$J_{\pi}(f) := \sum_{i} \alpha_{\pi}(\pi(f)\varphi_{i} \otimes \varphi^{i}), \quad f \in \mathcal{S}(G(F), \mathbb{C})$$

on the Hecke algebra $\mathcal{S}(G(F), \mathbb{C})$ of complex-valued compactly supported functions on $G(F)$. Here $\{\varphi_{i}\}$ and $\{\varphi^{i}\}$ are arbitrary dual basis of $\pi$ and $\pi^{\vee}$ with respect to $\langle \cdot, \cdot \rangle$. Sometimes (e.g. in the relative trace formula framework), it is more convenient to consider the spherical characters.

In an old version of this paper, we also formulate parallel conjectures for spherical characters (See [6, Conjecture 3.12 and Conjecture 3.13]). However, we don’t know whether the two versions (local periods/spherical characters) are equivalent. Under the condition that the fibre ranks of representations are locally constant, the conjecture for canonical local periods implies that for spherical characters (See [6, Lemma 3.14]).

In that version, we obtain the meromorphy property for spherical characters in the Gan-Gross-Prasad unitary group case (See [6, Theorem 5.8]) using a completely different method. We apply the spherical character identity of Beuzart-Plessis to relate the spherical character on unitary groups to the Rankin-Selberg and Asai zeta integrals. The meromorphy property follows from that of Rankin-Selberg and Asai zeta integrals.

Before explaining the strategy to Theorem 1.2, we explain our global motivation.

Let $F$ be a number field and $H \subset G$ be a spherical pair of reductive groups over $F$ satisfying certain conditions (e.g. the multiplicity one property). For any cuspidal unitary automorphic representation $\pi$ of $G(\mathbb{A})$, Sakellaridis-Venkatesh [32] conjectured that the global period

$$P_{H}(\varphi) = \int_{H(F) \backslash H(\mathbb{A})} \varphi(h)dh, \quad \varphi \in \pi$$

has a conjectural Eulerian decomposition into products of local periods

$$P_{H}(\varphi_{1})P_{H}(\varphi_{2}) = L_{Y}^{(S)}(\pi) \prod_{v \in S} \alpha_{\pi_{v}}(\varphi_{1,v} \otimes \varphi_{2,v}), \quad \varphi_{1} \in \pi, \varphi_{2} \in \pi^{\vee}$$

where $L_{Y}^{(S)}(\pi)$ is certain $L$-value associated to $Y := H \backslash G$ and $\pi$. Examples of such decomposition include the Ichino formula for the triple product case (see [23]) and the Ichino-Ikeda conjecture for the unitary Gan-Gross-Prasad case (proved in [4, 1] recently).

The conjectural decomposition $(\ast)$ can be used to construct $p$-adic $L$-functions. Using geometric methods, one can study the variation of (proper modification of) the global periods $P_{H}(\varphi)$ when $\pi$ varies in a $p$-adic family. Instead of choosing the test vector $\varphi$ carefully (so that the local period can be computed explicitly) and then viewing the period integral $P_{H}(\varphi)$ as the $p$-adic $L$-function (see [3] for the Waldspurger toric case, [22] for the triple case, [19] for the Gan-Gross-Prasad case), we prefer to apply the meromorphy property of canonical local periods at places $v \nmid p$ to view the ratio

$$\frac{P_{H}(\varphi_{1})P_{H}(\varphi_{2})}{\prod_{v \in S,v \nmid p} \alpha_{\pi_{v}}(\varphi_{1,v} \otimes \varphi_{2,v})}, \quad \varphi_{1} \in \pi, \varphi_{2} \in \pi^{\vee},$$

which is independent of the choice of test vectors, as the candidate for the conjectural $p$-adic $L$-function $L_{Y}^{(S)}(\pi)$ of the $p$-adic family $\pi$. In the Waldspurger toric period case, Liu-Zhang-Zhang [26] implemented this strategy to construct a $p$-adic $L$-function and established a special value formula of this $p$-adic $L$-function.

In some cases (including the triple product and the Gan-Gross-Prasad case), the conjectural decomposition $(\ast)$ has an arithmetic counterpart, which relates the Beilinson-Bloch height pairing of certain cycles to the product of certain derivative $L$-values and the same local periods. In a similar vein, this arithmetic formula can be used to study the derivative special values of $p$-adic $L$-functions of different nature. For results in this direction, see [13] for the $p$-adic Gross-Zagier formula.

1.3. Sketch of the proof. Now we return to the proof of Theorem 1.2. Let $\pi$ be a finitely generated smooth admissible torsion-free $R[G(F)]$-module as in Conjecture 1.1. In the same spirit as [27, 28, 17], the basic strategy to prove Theorem 1.2 is reducing the meromorphy of $\alpha_{\pi}$ to showing certain formal power series are actually rational functions. For this, we study the asymptotic behavior of the matrix coefficients of $\pi$ along the spherical subgroup $H$. Two key ingredients in the proof are
Admissibility of Jacquet modules ([11, Corollary 1.5], see also Theorem 4.1). For any parabolic subgroup \( P = MN \subset G \) with Levi factor \( M \), the Jacquet module

\[
J_N(\pi) := \pi/\pi(N), \quad \pi(N) := \{ \langle \pi(n)v - v | v \in \pi, n \in N(F) \} \subset \pi
\]
is a smooth admissible \( R[M(F)] \)-module.

Asymptotic behavior of matrix coefficients (Corollary 4.3). This is a consequence of Casselman’s canonical pairing theorem for \( G(F) \)-representations (recorded as Theorem 4.2). Fix a minimal parabolic subgroup \( P_{\theta_G} = M_{\theta_G}N_{\theta_G} \subset G \) with \( A_{\theta_G} \subset M_{\theta_G} \) a maximal split torus. Let \( \Delta_G \) be the set of simple \( G \)-roots with respect to \( (\theta_G, A_{\theta_G}) \). Then for any \( \Theta_G \subset \Delta_G \), any \( v \in \pi(N_{\theta_G}) \) and any \( \tilde{v} \in \tilde{\pi} \), there exists \( \epsilon > 0 \) such that for any \( a \in A_{\theta_G} \) with \( |\alpha(a)| < \epsilon \) for all \( \alpha \in \Delta_G - \Theta_G \), \( \langle \pi(a)v, \tilde{v} \rangle = 0 \)

\[
A_{\theta_G}^- := \{ a \in A_{\theta_G}(F) | |\alpha(a)| \leq 1 \ \forall \ \alpha \in \Delta_G \}.
\]

We now take the triple product case and the \((GL_3 \times GL_2, GL_2)\)-case to explain the ideas.

We first consider the (split) triple product case: \( H := G_m \setminus GL_2 \subset G := G_m \setminus GL_2^3 \). Here, \( H \) embeds into \( G \) diagonally. Note that for any \( x \in \Sigma \), by the Cartan decomposition (Theorem 4.4)

\[
GL_2(F) = \bigsqcup_{a \geq b} K t(a, b)K, \quad K = GL_2(O_F), \quad t(a, b) = \begin{pmatrix} \omega^a & 0 \\ 0 & \omega^b \end{pmatrix}
\]

the canonical local period

\[
\alpha_{\pi|x}(v, \tilde{v}) = \sum_{a \geq 0} \text{Vol}(Kt(a, 0)K) \int_{K \times K} \langle \pi(x(k t(a, 0)k_2)v, \tilde{v}) \rangle_{x} dk_1 dk_2.
\]

with

\[
\text{Vol}(Kt(a, 0)K) = \begin{cases} (1 + q^{-1})q^a & a > 0 \\ 1 & a = 0. \end{cases}
\]

Here, we assume \( \text{Vol}(K) = 1 \) and \( q \) is the cardinality of the residue field of \( O_F \).

It suffices to show that for any \( v \in \pi \) and \( \tilde{v} \in \tilde{\pi} \), the formal sum

\[
\tilde{F}_{\theta_H,\tilde{v}}(T) := \sum_{a \geq 1} \langle \pi(t(a, 0))v, \tilde{v} \rangle T^a
\]

is actually rational over \( R \) and then evaluate at \( T = q \).

Identify the set of simple roots \( \Delta_G \) with \( \Delta_{GL_2} \sqcup \Delta_{GL_2} \sqcup \Delta_{GL_2} \), where \( \Delta_{GL_2} \) consists of

\[
\alpha : A_2(F) = (F^\times)^2 \to F^\times, \quad (t_1, t_2) \mapsto t_1/t_2.
\]

Here \( A_n \subset GL_n \) is the diagonal torus. Consider \( \Theta_G = \emptyset_G \) in the above theorem for asymptotic behavior of matrix coefficients. Since \( A_{\emptyset_G} = \emptyset_G \setminus A_2 \times A_2 \times A_2 \), one finds that \( \tilde{F}_{\theta_H,\tilde{v}}(T) \) is a polynomial for \( v \in \pi(N_{\emptyset_G}) \). Moreover,

\[
\pi(t(1, 0)) \in \text{End}_{R[A_{\emptyset_G}]}J_{N_{\emptyset_G}}(\pi) \hookrightarrow \text{End}_{R}J_{N_{\emptyset_G}}(\pi)^{K_{A_{\emptyset_G}}}
\]

where \( K_{A_{\emptyset_G}} \) is any open compact subgroup of \( A_{\emptyset_G}(F) \) such that \( J_{N_{\emptyset_G}}(\pi)^{K_{A_{\emptyset_G}}} \) containing a (finite) system of generators of \( J_{N_{\emptyset_G}}(\pi) \). By the admissibility of Jacquet modules, \( J_{N_{\emptyset_G}}(\pi)^{K_{A_{\emptyset_G}}} \) is finitely generated over \( R \). Thus there exists a non-zero polynomial \( P(X) = \sum_{n \geq 0} c_n X^n \in R[X] \) such that for any \( v \in \pi \), \( P(\pi(t(1, 0)))v \in \pi(N_{\emptyset_G}) \). Since

\[
\tilde{F}_{\theta_H, \pi(t(1, 0)))v, \tilde{v}}(T) = P(T^{-1})\tilde{F}_{\theta_H, v, \tilde{v}}(T) - \sum_{n \geq 1} \sum_{a = 1}^{n} c_n \langle \pi(t(a, 0))v, \tilde{v} \rangle T^{a-n} \in R[T],
\]

one immediately deduces \( \tilde{F}_{\theta_H, v, \tilde{v}}(T) \) is a rational function of the form \( \frac{Q(T)}{T^{(T)}} \) with \( Q(T) \in R[T] \).

At each point \( x \in \Sigma \), the evaluation \( \tilde{F}_{\theta_H, v, \tilde{v}}|_x(q) \) of \( \tilde{F}_{\theta_H, v, \tilde{v}}(q) \) at \( T = q \) gives the desired value, namely the absolutely convergent sum

\[
\sum_{a \geq 0} \langle \pi(t(a, 0))v, \tilde{v} \rangle q^a.
\]

by the absolutely convergence of \( \alpha_{\pi|x}(v, \tilde{v}) \). When the discrete support of \( \pi \) is rigid at \( x \in \Sigma \), we explicitly construct the polynomial \( P(T) \) and then deduce \( \pi|_x(q^{-1}) \neq 0 \) from the absolutely convergence of \( \alpha_{\pi|x}(v, \tilde{v}) \). Consequently, \( \tilde{F}_{\theta_H, v, \tilde{v}}(q)|_x = \tilde{F}_{\theta_H, v, \tilde{v} |_x(q)} \) and the meromorphy conjecture for \( \pi \) holds.
Now we turn to the rank two case

\[ H = \text{GL}_2 \hookrightarrow G := \text{GL}_3 \times \text{GL}_2, \quad g \mapsto \left( \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}, g \right). \]

Again by the Cartan decomposition and the volume formula, it suffices to show that for any \( v \in \pi \) and \( \tilde{v} \in \tilde{\pi} \), the formal power series

\[
\tilde{F}^+_{\Theta_H,v,\tilde{v}}(T_1, T_2) := \sum_{a > b, \pm b > 0} \langle \pi(t(a,b))v, \tilde{v} \rangle T_1^{a-b} T_2^{b}, \quad \tilde{F}^+_{\Delta_H,v,\tilde{v}}(T_2) := \sum_{\pm b > 0} \langle \pi(t(a,a))v, \tilde{v} \rangle T_2^{[a]},
\]

\[
\tilde{F}^0_{\Theta_H,v,\tilde{v}}(T_1) := \sum_{a > 0} \langle \pi(t(a,0))v, \tilde{v} \rangle T_1^{a}, \quad \tilde{F}^0_{\Delta_H,v,\tilde{v}} := \langle v, \tilde{v} \rangle
\]

are all rational over \( R \) and then sum up the evaluations at \( T_1 = q \) and \( T_2 = 1 \). Here \( \tilde{F}^+_{\Theta_H,v,\tilde{v}} \) deals with summations over the “2-dimensional cones” \( \{ a > b, \pm b > 0 \} \) while \( \tilde{F}^0_{\Theta_H,v,\tilde{v}} \) and \( \tilde{F}^+_{\Delta_H,v,\tilde{v}} \) deal with the “1-dimensional cones” \( \{ a > b, b = 0 \} \) and \( \{ a = b, b > 0 \} \).

The evaluation part is similar to the triple case. For the rationality part, we proceed by induction. We reduce the rationality of the series over “2-dimensional cones” to that of the series over “1-dimensional cones” which can be handled similarly as the triple product case.

Identify \( \Delta_G \) with \( \text{GL}_3 \uplus \text{GL}_3 \) where \( \text{GL}_3 = \{ \beta_1, \beta_2 \} \) with

\[ \beta_1 : A_3(F) = (F^\times)^3 \to F^\times; \quad (t_1, t_2, t_3) \mapsto t_1/t_{2+i}. \]

Consider the subset \( \Theta_G = \{ \beta_1, \alpha \} \) of \( \Delta_G \) and \( \pi(t(1,1)) \in \text{End}_{\text{R}(M_n_G(F))/J_{n_G}}(\pi) \). Then by the aforementioned asymptotic behavior of matrix coefficients, the rationality of \( \tilde{F}^0_{\Theta_H,v,\tilde{v}} \) reduces to that of \( \tilde{F}^0_{\Theta_H,v,\tilde{v}} \).

And the rationality of \( \tilde{F}^0_{\Theta_H,v,\tilde{v}} \) can be obtained by considering the subset \( \{ \beta_2, \alpha \} \) and the endomorphism induced by \( \pi(t(1,1)) \).

To deal with \( \tilde{F}^0_{\Theta_H,v,\tilde{v}} \), we further decompose it into formal series over the “2-dimensional cones” \( \{ a > b, b > 0 \} \) and \( \{ a > b, b > 0 \} \). Then by considering the action of the Weyl group of \( G \), one can reduce the rationality of the series over \( \{ a > b, b > 0 \} \) and \( \{ 0 > a > b \} \) to that of \( \tilde{F}^0_{\Theta_H,v,\tilde{v}} \).

The strategy is available for very general situations. We introduce the notion of reduction structures in Definition 4.14 which abstracts the geometric properties validating the reduction process above. Then we show the meromorphic property of \( \alpha_\pi \) holds for all strongly tempered spherical \( G \)-varieties admitting reduction structures in Theorem 4.18. Finally, in Theorem 4.22, we prove all split strongly tempered spherical varieties without type \( N \)-roots admit reduction structures case by case using the classification result (listed in [36, Section 1.1]).

Remark 1.6. Actually, inner forms and low rank variations of split strongly tempered spherical \( G \)-varieties without type \( N \)-roots also admit reduction structures. See Proposition 4.23 for the Gan-Gross-Prasad case for unitary groups.

Acknowledgement We express our sincere gratitude to Prof. Y. Tian for his consistent encouragement, and to Prof. A. Burungale for carefully reading earlier versions of the article. We are grateful to the anonymous referee whose comments helped us improve the article. In particular, we thank the referee for pointing out a gap in an earlier version and Prof. G. Moss for generously suggesting Proposition 4.20 to fix it. We thank Prof. J.-F. Dat for kindly answering our question on Jacquet modules and thank Prof. J. Yang for discussions on global applications.

L. Cai is partially supported by NSFC grant No.11971254.

2. Notation and conventions

In the rest of this paper,

- \( F \) is a finite extension of \( \mathbb{Q}_p \) with ring of integers \( \mathcal{O}_F \), uniformizer \( \varpi \), residue field \( k \) and normalized norm

\[ | \cdot | : F \to (\mathbb{Z}/p)^\times \cup \{ 0 \}; \quad \varpi \mapsto q^{-1}, \quad q = \varpi. \]

- \( E \) is a field embeddable into \( \mathbb{C} \) and \( E \) is an algebraic closure of \( E \),

- \( X/E \) is a locally of finite type reduced scheme and for any \( x \in X \), \( k(x) \) is the residue field \( \mathcal{O}_{X,x}/m_x \mathcal{O}_{X,x} \),

- \( \Sigma \subset X \) is a Zariski dense subset of closed points.
2.1. Fibres of quasi-coherent sheaves. Let $\mathcal{F}$ be a quasi-coherent sheaf on $X$. A non-zero section $s \in \mathcal{F}(X)$ is torsion if there exists an open affine subset $U = \text{Spec} \, (R) \subset X$ such that $\text{Ann}_R(s|_U)$ contains non-zero divisors. The sheaf $\mathcal{F}$ is torsion-free if $\mathcal{F}(U)$ contains no torsion elements for any open affine subset $U \subset X$.

For any $x \in X$, the fiber $\mathcal{F}|_x$ of $\mathcal{F}$ at $x$ is the $k(x)$-vector space $\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} k(x)$. For any $s \in \mathcal{F}(X)$, let $s(x) \in \mathcal{F}_x$ be the image of $s$ under the natural map $\mathcal{F}(X) \to \mathcal{F}_x \to \mathcal{F}|_x$. When $\mathcal{F}$ is coherent, the function
\[
\phi_x : X \to \mathbb{N}, \quad x \mapsto \dim_{k(x)} \mathcal{F}|_x
\]
is upper semi-continuous, i.e. $\{ x \in X \mid \phi_x(x) \leq n \}$ is open for any $n \in \mathbb{N}$ (The upper semi-continuous theorem, see [18, Example 12.7.2]).

- If $\phi_x$ is locally constant, $\mathcal{F}$ is finite projective (note that $X$ is reduced, see [34, Lemma 0FWG]),
- the zero locus $\{ x \in X | s(x) = 0 \} \subset X$ of a non-zero section $s \in \mathcal{F}(X)$ is constructible by Chevalley theorem and hence $s$ is torsion if the zero locus contains a Zariski dense subset.

Let $\mathcal{F}^* \subset \mathcal{F}$ be an subsheaf. For any section $s \in \mathcal{F}^*(X)$, let $s|_x \in \mathcal{F}^*|_x$ be the image of $s$ under the natural map $\mathcal{F}^*(X) \to \mathcal{F}(X) \to \mathcal{F}|_x$ and let $\mathcal{F}^*|_x \subset \mathcal{F}|_x$ be the subset consisting of all $s|_x$. Note that when $\mathcal{F}^* \subset \mathcal{F}$ is a $\mathcal{O}_X$-submodule, then $\mathcal{F}^*|_{\mathcal{F},x}$ (resp. $s|_x$) is the image of $\mathcal{F}^*|_x$ (resp. $s(x)$) via the natural map $\mathcal{F}^*|_x \to \mathcal{F}|_x$. When no confusion arises, we will omit $\mathcal{F}$ in the notations.

We introduce the following notations. Let $K_X$ be the sheaf of total quotient rings of $\mathcal{O}_X$. Note that (see [34, Lemma 02OW]) for any $x \in X$, $K_{X,x}$ is the total quotient ring of $\mathcal{O}_{X,x}$.

- A family $\{ f_x \in \mathcal{F}^*|_x \}_{x \in \Sigma}$ is called meromorphic if there exists an open subset $X' \subset X$ containing $\Sigma$ and a section $F \in \mathcal{F}^*(X')$ such that for any $x \in \Sigma$, $F(x) = f_x$. When $\mathcal{F}$ is torsion-free, the section $F$ is unique in the sense that for any open subset $X''$ containing $\Sigma$ and any section $G \in \mathcal{F}^*(X'')$ such that $G(x) = f_x$, $\mathcal{F}|_{X''} \cong G|_{X''}^\times$.
- A family $\{ f_x \in \text{Map}(\mathcal{F}^*, k) \}_{x \in \Sigma}$ is called meromorphic if there exists an open subset $X' \subset X$ containing $\Sigma$ and a sheaf morphism $F : \mathcal{F}^*|_{X'} \to K_{X'}$ such that for any $x \in \Sigma$, $F_x(\mathcal{F}^*_x) \subset O_{X,x}$ and the following diagram commutes
\[
\begin{array}{ccc}
\mathcal{F}^*|_{X'} & \xrightarrow{F_x} & \mathcal{O}_{X,x} \\
\downarrow & & \downarrow \\
\mathcal{F}^*|_x & \xrightarrow{f_x} & k(x)
\end{array}
\]
Such a morphism $F$ is unique and extends to $X$ by $\mathcal{F}^*(X) \to \mathcal{F}^*(X') \to K_{X'} \cong K_X(X)$.

In both cases, we say $F$ interpolates $f_x$.

For any $\mathcal{O}_{X,x}$-subsheaf $\mathcal{F}^* \subset \mathcal{F}$, let $\mathcal{F}^*^{-1}$ denote the sheaf $\mathcal{F}^*$ with inverted $\mathcal{O}_{X,x}$-action
\[
c \cdot f := c^{-1} f, \quad c \in \mathcal{O}_{X,x}, \quad f \in \mathcal{F}^*
\]
Let $Q_{\mathcal{F}^*}(\mathcal{F})$ denote the sheaf $\mathcal{F} \times \mathcal{F}^*^{-1}$ and let $\mathcal{H} \in Q_{\mathcal{F}^*}(\mathcal{F})(X)$ denote the section $(f, g) \in Q_{\mathcal{F}^*}(\mathcal{F})(X)$. We will omit $\mathcal{F}$ in notations when no confusion arises.

2.2. Measures on $G$-varieties. Let $G$ be a reductive group over $F$. A $G$-variety $Z$ over $F$ is a geometrically integral and separated $F$-scheme of finite type together with an $F$-algebraic $G$-action $G \times Z \to Z$.

Assume $Z$ is a smooth $n$-dimensional $G$-variety over $F$ equipped with a $G$-equivariant $F$-rational differential $n$-form $\omega$. Cover $Z(F)$ by local charts $(U, \quad U \xrightarrow{\omega} \mathcal{O}_F^n)$. Note that the standard differential $n$-form $dx_1 \wedge \cdots \wedge dx_n$ on $F^n$ defines a Haar measure $dx_1 \cdots dx_n$ on $F^n$ such that $\text{Vol}(\mathcal{O}_F^n) = 1$. Assume
\[
w(x_1, \ldots, x_n) := f(x_1, \ldots, x_n)dx_1 \wedge \cdots \wedge dx_n.
\]
Then the measure $i_U^!(\{ f(x_1, \ldots, x_n) \}) \cdot dx_1 \cdots dx_n$ on $U$ glues to a $G(F)$-equivariant measure $|\omega|$ on $Z(F)$.

Note that for any $\mathbb{Q}$-algebra $A$, $f_{Z(F)}^!(|\omega|) = A$ for any $f \in S(Z(F), A)$, the space of $A$-valued compact supported locally constant functions on $Z(F)$.

Let $H \subset G$ be a closed $F$-subgroup. By [31, Section 3.8], there exists a $G$-equivariant top degree differential form on the geometric quotient $H \backslash G$ if and only if the modulus characters of $G$ and of $H$ coincide on $H$. In particular, when $H$ is reductive, the above construction is applicable to $H \backslash G(F)$ and we will denote the resulting measure by $d\mu_{H \backslash G}$ when the top differential is insignificant.
2.3. Smooth admissible modules. Let $G$ be a reductive group over $F$. For any Noetherian ring $R$, a $R$-module $M$ equipped with a $R$-linear action of $G(F)$ is called

- smooth if any $v \in M$ is fixed by some open compact subgroup of $G(F)$;
- admissible if for any compact open subgroup $K \subset G(F)$, the submodule $M^K \subset M$ of $K$-fixed elements is finitely generated over $R$;
- finitely generated if $M$ is finitely generated as a $R[G(F)]$-module;
- torsion-free if $M$ is torsion-free as a $R$-module.

A quasi-coherent $O_X$-module $\pi$ equipped with a group morphism $G(F) \to \text{Aut}_{O_X}(\pi)$ is called an $O_X[G(F)]$-module. It is moreover called smooth/admissible/finitely generated/torsion-free if for any $x \in X$, there exists an open affine neighborhood $x \in U = \text{Spec } (R) \subset X$ such that $\pi(U)$ is smooth/admissible/finitely generated/torsion-free as $R[G(F)]$-module. Note that the fiber $\pi|_x$ is a smooth/admissible/finitely generated $k(x)[G(F)]$-module for any $x \in X$ if so is $\pi$.

When $X = \text{Spec } (C)$ for some field extension $C/E$, a smooth $C[G(F)]$-module $\pi$ is called irreducible (resp. absolutely irreducible) if it (resp. its base change to any (hence all) algebraically closed field) contains no proper non-zero submodule. Unless otherwise specified, all irreducible representation in this paper would be non-zero.

**Lemma 2.1.** Assume $X = \text{Spec } (R)$ and let $p \subset R$ be a minimal prime with corresponding generic point $\eta$. Let $\pi$ be a finitely generated smooth admissible torsion-free $R[G(F)]$-module. Then if $\pi|_x$ is irreducible (resp. absolutely irreducible) for all $x \in \Sigma$, $\pi|_\eta$ is irreducible (resp. absolutely irreducible).

**Proof.** Let $\{p_1 = p, p_2, \ldots, p_n\}$ be the set of minimal primes of $R$ and let $\sigma_i$ be the maximal torsion-free quotient of the $R/p_i[G(F)]$-module $\pi/p_i\pi$. By the argument of [15, Lemma 6.3.6], the diagonal map $\pi \to \prod_i \sigma_i$ is an injection. Let $X_i = \text{Spec } (R/p_i) \subset X = \text{Spec } (R)$. Then for any $x \in U := X - \cup_i \neq 1 X_i$, the surjection

$$\pi|_x \to (\pi/p\pi)|_x \to \sigma|_x$$

is actually an isomorphism. Note that $\Sigma \cap U$ is non-empty. As $\sigma$ is torsion-free, we have $\sigma|_x \neq 0$ for any $x \in \Sigma|\eta$ by [15, Lemma 2.1.7] and consequently, $\pi|_x \cong \sigma|_x$ for any $x \in \Sigma \cap \eta$.

Up to replacing $(\pi, R)$ by $(\sigma, R/p)$, we may assume $R$ is an integral domain. Take an exact sequence of $k(\eta)[G(F)]$-representations

$$0 \to V|_\eta \to \pi|_\eta \to W|_\eta \to 0$$

with $W|_\eta$ irreducible. Let $V := \pi \cap V|_\eta$ and $W := \pi/V$. Then by localization at $\eta$, the natural exact sequence

$$0 \to V \to \sigma \to W \to 0$$

recovers

$$0 \to V|_\eta \to \pi|_\eta \to W|_\eta \to 0.$$

By the upper semicontinuous theorem, there exists an open subset $X'_K \subset X$ for any open compact subgroup $K \subset G(F)$ such that for any $x \in X'_K$,\n
$$\dim_{k(\eta)} V^K|_x = \dim_{k(\eta)} V^K|_\eta; \quad \dim_{k(\eta)} \pi^K|_x = \dim_{k(\eta)} \pi^K|_\eta; \quad \dim_{k(\eta)} W^K|_x = \dim_{k(\eta)} W^K|_\eta.$$

Note that $\pi|_x \cong W|_x$ for any $x \in \Sigma \cap X'_K$ when $K$ is small enough, so

$$\dim_{k(\eta)} \pi^K|_\eta = \dim_{k(\eta)} W^K|_x = \dim_{k(\eta)} W^K|_\eta.$$

Consequently, $V|_{\eta} = 0$ and $\pi|_\eta$ is irreducible.

Assume $\pi|_x$ is absolutely irreducible for all $x \in \Sigma$. To show $\pi|_{\eta}$ is absolutely irreducible, it suffices to show $\pi|_{\eta \otimes k(\eta)} C$ is irreducible for any finite field extension $C/k(\eta)$. Take any $R$-subalgebra $\hat{R} \subset C$ which is a finitely generated $R$-module such that $\text{Fract}(\hat{R}) = C$ and consider $\hat{\pi} = \pi \otimes R \hat{R}$. Let $\hat{\Sigma}$ (resp. $\hat{\eta}$) be the pre-image of $\Sigma$ (resp. $\eta$) along $\text{Spec } (\hat{R}) \to X$. Then $\hat{\Sigma}$ is Zariski dense and for any $\hat{x} \in \hat{\Sigma}$, $\hat{\pi}|_{\hat{x}}$ is irreducible. Hence $\hat{\pi}|_{\eta} = \pi|_{\eta \otimes k(\eta)} C$ is irreducible and we are done. \hfill $\square$

For any smooth $O_X[G(F)]$-module $\pi, \pi^* := \text{Hom}_{O_X}(\pi, O_X)$ has an $O_X[G(F)]$-module structure

$$(g \cdot \ell)(v) := \ell(g^{-1} \cdot v); \quad \forall \ell \in \pi^*, \ v \in \pi, \ g \in G(F).$$

The $O_X[G(F)]$-submodule $\pi^\vee := \{\ell \in \pi^* \ | \ \ell \text{ smooth}\} \subset \pi^*$ is called the smooth dual of $\pi$. Let $\pi, \tilde{\pi}$ be finitely generated smooth admissible torsion-free $O_X[G(F)]$-modules together with a $G(F)$-equivariant pairing $\langle -,- \rangle : \pi \times \tilde{\pi} \to O_X$ such that $\pi|_x$ and $\tilde{\pi}|_x$ is irreducible for each $x \in \Sigma$ and the specialization $\langle -,- \rangle|_x$ of $\langle -,- \rangle$ at each $x \in \Sigma$ is non-degenerate. For latter applications, we shall compare $\pi^\vee$ and $\tilde{\pi}$. 


Lemma 2.2. Notations as above. Then \((-,-)\) induces an injection \(\tilde{\pi} \mapsto \pi'\). In particular, \(\tilde{\pi}|_\eta \cong (\pi|_\eta)'\) for each generic point \(\eta \in X\).

Proof. By the Hom-Tensor adjunction, the pairing \((-,-)\) gives an element \(f \in \text{Hom}_{\mathcal{O}_X(G(F))}(\tilde{\pi}, \pi')\) such that for each \(x \in \Sigma\), the induced morphism
\[
f|_x : \tilde{\pi}|_x \to \pi'|_x \to (\pi|_x)'
\]
is an isomorphism. Then for any generic point \(\eta \in X\),
\[
f_\eta \in \text{Hom}_{k[\eta]}(\tilde{\pi}|_\eta, \pi'|_\eta) = \text{Hom}_{\mathcal{O}_X(G(F))}(\tilde{\pi}, \pi') \otimes_{\mathcal{O}_X} k(\eta)
\]
is non-zero. Thus by Lemma 2.1, \(f_\eta : \tilde{\pi}|_\eta \to \pi'|_\eta\) is an isomorphism and \(\text{Ker}(f)_\eta = 0\). As \(\text{Ker}(f)\) is torsion-free, one has \(\text{Ker}(f) = 0\) and we are done. \(\square\)

We remark that if \(\pi|_x\) is irreducible for all \(x \in \Sigma\), one can construct a finitely generated smooth admissible torsion-free \(\mathcal{O}_X(G(F))\)-module \(\tilde{\pi}\) from \(\pi\) such that for each \(x \in \Sigma\), \(\tilde{\pi}|_x \cong (\pi|_x)'\) by the MVW involution at least for \(G\) classical (see [30] for details). When \(G = \text{GL}_n\), one can even define a \((G(F))\)-invariant pairing \((-,-) : \pi \times \tilde{\pi} \to \mathcal{O}_X\) which induces non-degenerate pairing \((-,-)|_x : \pi|_x \times \tilde{\pi}|_x \to k(x)\) at each \(x \in \Sigma\) upon shrinking \(X\) to an open subset containing \(\Sigma\) (see [12, Proposition 5.2.6] for details).

3. Local periods in families

In this section, we formulate several conjectures about canonical local periods for spherical varieties.
Throughout this section, let \(G\) be a reductive group over \(F\) and \(Y\) be a \(G\)-variety over \(F\) which is
- homogeneous affine i.e. \(G(F)\) acts transitively on \(Y(F)\) and the stabilizers of points on \(Y(F)\) are reductive.
- spherical i.e. \(Y\) is normal and there is a Borel subgroup of \(G_F\) whose orbit on \(Y_F\) is Zariski open.
Moreover, we assume \(Y(F) \neq \emptyset\) and let \(H = G_{y_0}\) for a fixed \(y_0 \in Y(F)\). Note that \(H\) is a reductive \(F\)-subgroup of \(G\) and \(Y\) is isomorphic to the geometric quotient \(H\backslash Y\).

3.1. Plancherel decompositions on spherical varieties. Fix a Haar measure \(d\mu_Y\) on \(Y(F)\) such that \(\int_{Y(F)} f d\mu_Y \in \mathbb{Q}\) for any \(f \in S(Y(F), \mathbb{Q})\). Then the Hilbert space \(L^2(Y(F))\) of square-integrable functions with respect to \(\mu\) is naturally an unitary \((G(F))\)-representation. As \((G(F))\) is a postliminal locally compact group, there is an isomorphism of unitary \((G(F))\)-representations, which is unique in a suitable sense (for details, see [25, Section 3.2])
\[
L^2(Y(F)) \cong \int_{\tilde{\mathcal{G}}} \mathcal{H}_\pi d\mu(\pi).
\]
Here,
- \(\tilde{\mathcal{G}}\) is the space of unitary irreducible \((G(F))\)-representations with the Fell topology and \(\mu\) is a Radon measure on \(\tilde{\mathcal{G}}\).
- \(\{\mathcal{H}_\pi\}_\pi\) is a measurable field of Hilbert spaces and for each \(\pi\), \(\mathcal{H}_\pi\) is a unitary representation isomorphic to a direct sum of at most countably many copies of \(\pi\).

Determining the Plancherel decomposition is a fundamental problem in harmonic analysis.

When \(G = H \times H\) and \(Y = \Delta(H) \backslash H \times H(\cong H)\), then
\[
L^2(H(F)) \cong \int_{\tilde{H}} \pi \otimes \tilde{\pi} d\mu_H^{PL}(\pi)
\]
where \(\pi \otimes \tilde{\pi} \cong \text{End}(\pi)\) is endowed with the Hilbert-Schmidt norm and \(\mu_H^{PL}\) is the Plancherel measure on \(\tilde{H}\). Here, \(\tilde{\pi}\) is the complex conjugation of \(\pi\). Note that Harish-Chandra proved that the support of \(\mu_H^{PL}\) is the tempered representations.

When \(Y\) is strongly tempered in the sense that the matrix coefficients of any tempered \((G(F))\)-representation \(\pi\) are absolutely integrable over \(H(F)\), the measure \(\mu\) is the Plancherel measure \(\mu_G^{PL}\) (See [32, Theorem 6.2.1]).

When \(G\) is split, Sakellaridis-Venkatesh gives a conjectural description of the Plancherel decomposition. Let \(G'\) be the dual group of \(G\). By Sakellaridis-Venkatesh [32, Chapter 2] and Knop-Schalke [24], there is a dual group \(G'_Y\) over \(\mathbb{C}\) equipped with a morphism \(f : G'_Y \times \text{SL}_2(\mathbb{C}) \to G'\). Denote by \(G'_Y\) the split group over \(F\) dual to \(G'_Y\). It is expected (see [32, Section 17.3]) that there is a measure \(\mu_{G'_Y}^{PL}\) on the
$G_Y^\vee$-conjugacy classes of tempered Langlands parameters on $G_Y$ such that the Plancherel decomposition has the form

$$L^2(G_Y(F)) \xrightarrow{\sim} \int_{[\phi]} \mathcal{H}_{[\phi]}d\mu_{G_Y}^\text{PL}([\phi]), \quad \mathcal{H}_{[\phi]} = \bigoplus \pi \boxotimes \bar{\pi}$$

where for each $[\phi]$, $\pi$ runs over the $L$-packet associated to $[\phi]$ and the Hilbert structure on each $\pi \boxotimes \bar{\pi}$ is given by an integer multiple of the Hilbert-Schmidt norm.

Let $W_F$ be the Weil group of $F$ and $L_F = W_F \times \text{SL}_2(\mathbb{C})$ be the Langlands group. A (local) Arthur parameter is a homomorphism

$$\psi : L_F \times \text{SL}_2(\mathbb{C}) \rightarrow G^\vee$$

such that $\psi \mid_{L_F}$ is a tempered (i.e. bounded on $L_F$) Langlands parameter and $\psi \mid_{\text{SL}_2(\mathbb{C})}$ is algebraic. Given any Arthur parameter, the associated Langlands parameter is given by

$$L_F \rightarrow L_F \times \text{SL}_2(\mathbb{C}), \quad w \mapsto \left( w, \begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix} \right).$$

It is conjectured that to each $G_Y^\vee$-conjugacy class of Arthur parameter $[\psi]$, we may “naturally” associate a finite set of unitary $G(F)$-representations, the so-called Arthur packet of $[\psi]$. It contains the $L$-packet of the associated Langlands parameter.

A $Y$-distinguished Arthur parameter (see [32, Section 16.2]) is a commutative diagram

$$\begin{CD}
G_Y^\vee \times \text{SL}_2(\mathbb{C}) @>>> G^\vee \\
\langle \phi, \text{id} \rangle @>>> f @>>> \psi \\
L_F \times \text{SL}_2(\mathbb{C}) @>>> G^\vee
\end{CD}$$

where $\psi : L_F \times \text{SL}_2(\mathbb{C}) \rightarrow G^\vee$ is an Arthur parameter and $\phi : L_F \rightarrow G_Y^\vee$ is a tempered Langlands parameter. The local conjecture predicts the Plancherel decomposition of $L^2(Y(F))$ is given by the $Y$-distinguished Arthur parameters of $G$.

Conjecture 3.1 (Weak form of the local conjecture [32, Conjecture 16.2.2]). There is a direct integral decomposition

$$L^2(Y(F)) \xrightarrow{\sim} \int_{[\psi]} \mathcal{H}_{[\psi]}d\mu_{G_Y}^\text{PL}([\psi])$$

where

- $[\psi]$ runs over $G_Y^\vee$-conjugacy classes of $Y$-distinguished Arthur parameter.
- $\mathcal{H}_{[\psi]}$ is isomorphic to a (possibly empty) direct sum of irreducible representations belonging to the Arthur packet of $[\psi]$.

Remark 3.2. In general, Conjecture 3.1 states only necessary conditions for a representation to appear in the Plancherel decomposition of $L^2(Y(F))$. When $Y$ is strongly tempered and $G$ is split, it is claimed that $G_Y^\vee = G^\vee$ in [32, Page 7] and the Plancherel decomposition in [32, Theorem 6.2.1] is compatible with the form in Conjecture 3.1.

3.2. Canonical local periods. We now construct a “canonical” bi-$H(F)$-invariant pairing on $\pi \otimes \pi^\vee$ for $\mu_{G_Y}^\text{PL}$-almost all $\pi$ in the Plancherel decomposition of $L^2(Y(F))$.

Let $C^\infty(Y(F))$ (resp. $\mathcal{S}(Y(F))$) be the space of complex valued smooth (resp. Schwartz) functions on $Y(F)$ equipped with the $G(F)$-action given by right translation. The $G(F)$-invariant bilinear pairing

$$\mathcal{S}(Y(F)) \times C^\infty(Y(F)) \rightarrow \mathbb{C}, \quad (f, g) \mapsto \int_{Y(F)} f(y)g(y)d\mu_Y(y)$$

induces an injection of $G(F)$-modules

$$\mathcal{S}(Y(F))^\vee \hookrightarrow C^\infty(Y(F))$$

identifying $\mathcal{S}(Y(F))^\vee$ with vectors in $C^\infty(Y(F))$ which are smooth under the action of $G(F)$.

For any irreducible smooth admissible $G(F)$-representation $\pi$, denote by

$$\mathcal{S}(Y(F))_\pi := \mathcal{S}(Y(F))/\mathcal{N}_\pi, \quad \mathcal{N}_\pi := \bigcap_{f \in \text{Hom}_{G(F)}(\mathcal{S}(Y(F)), \pi)} \ker f$$

the maximal $\pi$-isotypic quotient of $\mathcal{S}(Y(F))$. 

10
Assume now $G$ is split and Conjecture 3.1 holds. Then by the Gelfand-Kostyuchenko method (see [25]), the decomposition ($\ast$) determines a $G(F)$-invariant positive semi-definite Hermitian form $(\langle \cdot, \cdot \rangle_\pi : \mathcal{S}(Y(F)) \times \mathcal{S}(Y(F)) \rightarrow \mathbb{C})$ which factors through $\mathcal{S}(Y(F))_\pi$ for $\mu_{G,F}^\text{PL}$-almost all $\pi$ such that

$$
\int_{Y(F)} f_1(x)\overline{f_2(x)} \, dx = \int_{[\psi]} \sum_{\pi \in [\psi]} \langle f_1, f_2 \rangle_\pi \, d\mu_{G,F}^\text{PL}([\psi]), \quad \forall f_1, f_2 \in \mathcal{S}(Y(F)).
$$

It is expected (see [32, Section 17.3]) and we will assume that $(\langle \cdot, \cdot \rangle_\pi$ is well-defined for all the unitary representation $\pi$ appearing in the decomposition ($\ast$).

Note that the irreducibility of $\pi$ factors through $\langle \cdot, \cdot \rangle_\pi$ implies

$$\dim_{\mathbb{C}} \text{Hom}_{G(F)}(S(Y(F)), \pi) = 1.$$

The following lemma is straightforward.

**Lemma 3.3.** Any non-degenerate $G(F)$-invariant linear functional $\ell$ on $\pi \otimes \bar{\pi}$ induces an isomorphism

$$\text{Hom}_{G(F)}(S(Y(F))_\pi \otimes S(Y(F))_{\bar{\pi}}, \pi \otimes \bar{\pi}) \cong \text{Hom}_{G(F)}(S(Y(F))_\pi \otimes S(Y(F))_{\bar{\pi}}, \mathbb{C}).$$

**Proof.** By [29, Theorem 3.1], one has

$$\text{Hom}_{G(F)}(S(Y(F))_\pi \otimes S(Y(F))_{\bar{\pi}}, \pi \otimes \bar{\pi}) \cong \text{Hom}(V(\pi) \otimes V(\bar{\pi}), \mathbb{C}) \otimes \text{Hom}_{G(F)}(\pi \otimes \bar{\pi}, \pi \otimes \bar{\pi})$$

$$\cong \text{Hom}(V(\pi) \otimes V(\bar{\pi}), \mathbb{C}) \otimes \text{Hom}_{G(F)}(\pi \otimes \bar{\pi}, \mathbb{C})$$

$$\cong \text{Hom}_{G(F)}(S(Y(F))_\pi \otimes S(Y(F))_{\bar{\pi}}, \mathbb{C})$$

where the second $\cong$ is given by composing with $\ell$.

Fix a non-degenerate $G(F)$-invariant pairing $\ell$ on $\pi \otimes \bar{\pi}$. Let

$$p_\ell : S(Y(F))_\pi \otimes S(Y(F))_{\bar{\pi}} \rightarrow S(Y(F))_\pi \otimes S(Y(F))_{\bar{\pi}} \rightarrow \pi \otimes \bar{\pi}$$

be the unique $G^2(F)$-equivariant map such that $\ell \circ p_\ell = \ell_\pi$ by the above lemma. Taking dual, we obtain a $G^2(F)$-equivariant map

$$p_\ell^* : \pi \otimes \bar{\pi} \rightarrow (\pi \otimes \bar{\pi})^\vee \rightarrow (S(Y(F))^\vee)^\vee \rightarrow C^\infty(Y(F))^2.$$

Composing with the $H^2(F)$-equivariant evaluation map

$$e_{y_0,y_0} : C^\infty(Y(F))^2 \rightarrow \mathbb{C}, \quad f \mapsto f(y_0, y_0),$$

one obtains a bi-$H(F)$-invariant linear functional

$$P_{\pi,\ell} : \pi \otimes \bar{\pi} \rightarrow C^\infty(Y(F))^2 \otimes_{y_0,y_0} \mathbb{C}.$$

View $\ell$ as a $G(F)$-invariant pairing on $\pi \otimes \bar{\pi}$ via the natural identification

$$\pi \otimes \bar{\pi} = \pi \otimes \bar{\pi}, \quad a \circ b \mapsto b \otimes a.$$

By construction, it is straightforward to check (see 2.1 for notations)

- The $C^\infty$-subset
  
  $$\{(\pi \otimes \bar{\pi})^\ast \, \ell, \quad \forall \ell(v) \neq 0\}$$
  
  is independent of the choice of $\ell$.

- Denote the space $Q_{(\pi \otimes \bar{\pi})}$ by $Q_{(\pi \otimes \bar{\pi})}$. Then the map

  $$Q_{\pi \otimes \bar{\pi}} : Q(\pi \otimes \bar{\pi}) \rightarrow \mathbb{C}, \quad v \mapsto \frac{P_{\pi,\ell}(v)}{\ell(w)}$$

  is independent on the choice of $\ell$.  

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Note that $\pi \cong \pi^\vee$ as $G(F)$-representations.

**Definition 3.4.** Let $Q_\pi : Q(\pi \otimes \pi^\vee) \to \mathbb{C}$ be the map $Q_{\pi \otimes \pi}$ precomposed with any isomorphism $\kappa : \pi^\vee \cong \tilde{\pi}$.

By the above discussions, we have the following lemma.

**Lemma 3.5.** The map $Q_\pi$ is independent of the choice of the isomorphism in Plancherel decomposition $(\ast)$, the $G(F)$-invariant linear form $\ell$ and the isomorphism $\kappa$.

When the spherical variety $Y$ is strongly tempered, we can proceed in a same manner to define $Q_\pi$ by taking the measure $\mu^F_G$ in the Plancherel decomposition. Moreover, in this case, there is a down-to-earth description of $Q_\pi$.

**Proposition 3.6** (Theorem 6.2.1 [32]). Assume $Y$ is a strongly tempered spherical $G$-variety. Then for any tempered $G(F)$-representation $\pi$,

$$Q_\pi \left( \frac{v}{w} \right) = \frac{\int_{H(F)} \ell(\pi(h)v)dh}{\ell(w)}, \quad v, w \in \pi \otimes \pi^\vee, \quad \ell(w) \neq 0.$$  

Here, $\ell : \pi \otimes \pi^\vee \to \mathbb{C}$ is any non-trivial $G(F)$-invariant linear form and the Haar measure on $H(F)$ is determined by those measures on $Y(F)$ and $G(F)$.

### 3.3. Families of local periods

We start with the rationality property of $Q_\pi$. For a smooth admissible representation $\pi$ over $E$, let $\mathcal{E}(\pi)$ be the set of field embeddings $\tau : E \to \mathbb{C}$ such that $\pi_\tau$ is irreducible and appears in the Plancherel decomposition of $L^2(Y(F))$.

**Conjecture 3.7** (Rationality $Q$). Let $\pi$ be a smooth admissible $G(F)$-representation over $E$ with non-empty $\mathcal{E}(\pi)$. Let $Q(\pi \otimes_E \pi^\vee) := Q_{(\pi \otimes_E \pi^\vee)^*}(\pi \otimes_E \pi^\vee)$ where

$$(\pi \otimes_E \pi^\vee)^* = \left\{ \sum_i v_i \otimes v^i : v \in \pi \otimes \pi^\vee \right\}$$

Then there exists a unique $E$-linear map $Q_\pi : Q(\pi \otimes_E \pi^\vee) \to E$ such that for any $\tau \in \mathcal{E}(\pi)$, the following diagram commutes

$$\begin{array}{ccc}
Q(\pi \otimes_E \pi^\vee) & \xrightarrow{Q_\pi} & E \\
\downarrow{\tau} & & \downarrow{\tau} \\
Q(\pi_\tau \otimes \pi^\vee) & \xrightarrow{Q_{\pi_\tau}} & \mathbb{C}.
\end{array}$$

Assuming the rationality conjecture, we now discuss the behavior of $Q_\pi$ when $\pi$ varies in families.

**Conjecture 3.8** (Meromorphy $Q$). Let $\pi, \tilde{\pi}$ be finitely generated smooth admissible torsion-free $O_K \left[ G(F) \right]$-modules such that for any $x \in \Sigma$, $\mathcal{E}(\pi|_x) \neq \emptyset$ and $\tilde{\pi}|_x \cong (\pi|_x)^\vee$. Let $(\pi \otimes \tilde{\pi})^* \subset \pi \otimes \tilde{\pi}$ be the subsheaf such that for any open subset $U \subset X$,

$$(\pi \otimes \tilde{\pi})^*(U) = \left\{ s \in (\pi \otimes \tilde{\pi})(U) \mid s(x) \in (\pi \otimes \tilde{\pi})|_x^* \right\}, \quad \forall x \in \Sigma \cap U$$

and set $Q(\pi \otimes \tilde{\pi}) := Q_{(\pi \otimes \tilde{\pi})^*}(\pi \otimes \tilde{\pi})$. Then the family $\{ Q_{\pi_x} : Q(\pi \otimes \tilde{\pi})|_x \to k(x) \}_{x \in \Sigma}$ is meromorphic.

### 4. Strongly tempered spherical varieties without type $N$-roots

In this section, we prove Conjectures 1.1 for strongly tempered spherical varieties without type $N$-roots using the admissibility of Jacquet modules and the asymptotic behaviour of matrix coefficients. We shall assume $X = \text{Spec}(R)$ where $R$ is a reduced $E$-algebra of finite type.

We need some preliminary results.

Firstly, we record the admissibility of Jacquet modules established in [11, Corollary 1.5].

**Theorem 4.1.** Let $P = MN \subset G$ be any parabolic subgroup with Levi factor $M$. Then for any smooth admissible $R[G(F)]$-module $\pi$, the Jacquet module

$$J_N(\pi) := \pi/\pi(N), \quad \pi(N) := \langle \pi(n)v - v|v \in \pi, n \in N(F) \rangle \subset \pi$$

is a smooth admissible $R[M(F)]$-module.
Secondly, we need the following consequence of Casselman’s canonical pairing for $G(F)$-representations. Fix a minimal parabolic subgroup $P_{\emptyset_G} = M_{\emptyset_G} N_{\emptyset_G} \subset G$. Let $A_{\emptyset_G} \subset M_{\emptyset_G}$ be a maximal split torus and $\Delta_G$ be the set of simple roots with respect to $(P_{\emptyset_G}, A_{\emptyset_G})$. Let
\[
A_{\emptyset_G}^\times := \{ a \in A_{\emptyset_G}(F) \mid |\alpha(a)| \leq 1 \ \forall \ \alpha \in \Delta_G \}.
\]
Now we record Casselman’s canonical pairing theorem [8, Theorem 4.3.3]:

**Theorem 4.2.** Let $\pi$ be a smooth admissible $G(F)$-representation over a characteristic zero field $K$. Then for any $\Theta_G \subset \Delta_G$, there exists a $M_{\emptyset_G}$-invariant non-degenerate pairing
\[
(\cdot, \cdot)_{N_{\emptyset_G}} : J_{N_{\emptyset_G}}(\pi) \times J_{N_{\emptyset_G}}(\pi^\vee) \to K
\]
and a constant $\epsilon > 0$ such that for any $a \in A_{\emptyset_G}^\times$ with $|\alpha(a)| < \epsilon$ for all $\alpha \in \Delta_G - \Theta_G$,
\[
(\pi(a)v, v^\vee)_{N_{\emptyset_G}} = (\pi(a)v, v^\vee)_{N_{\emptyset_G}}, \quad \forall \ v \in \pi, v^\vee \in \pi^\vee.
\]
Here $(\cdot, \cdot)$ is the contraction on $\pi \times \pi^\vee$, $N_{\emptyset_G}$ is the unipotent subgroup opposite to $N_{\emptyset_G}$ and in the right hand side, $\pi(a)v$ and $v^\vee$ are viewed as elements in $J_{N_{\emptyset_G}}(\pi)$ and $J_{N_{\emptyset_G}}(\pi)$ respectively.

The following immediate consequence is crucial for our strategy:

**Corollary 4.3.** Let $\pi$ be a finitely generated smooth admissible $R[G(F)]$-module satisfying the assumptions in Conjecture 1.1. Then for any $\Theta_G \subset \Delta_G$, any $v \in \pi(N_{\emptyset_G})$ and any $\tilde{v} \in \tilde{\pi}$, there exists $\epsilon > 0$ such that for any $a \in A_{\emptyset_G}^\times$ with $|\alpha(a)| < \epsilon$ for all $\alpha \in \Delta_G - \Theta_G$, $(\pi(a)v, \tilde{v}) = 0$.

**Proof.** By Lemma 2.1 and Lemma 2.2, it suffices to prove the statement for the contraction
\[
\pi|_G \times (\pi|_G)^\vee \to k(\eta)
\]
for the finitely many generic points $\eta \in X = \text{Spec} \ (R)$. This immediately follows from Theorem 4.2. \qed

Thirdly, we need the Cartan decomposition for $H$. Let $P_{\emptyset_H} = M_{\emptyset_H} N_{\emptyset_H} \subset H$ be a minimal parabolic subgroup. Let $A_{\emptyset_H} \subset M_{\emptyset_H}$ be a maximal split torus and $\Delta_H$ be the set of simple roots with respect to $(P_{\emptyset_H}, A_{\emptyset_H})$. For any $\Theta_H \subset \Delta_H$, denote by
\[
A_{\emptyset_H} := \{ x \in A_{\emptyset_H} \mid \alpha(x) = 1 \ \forall \ \alpha \in \Theta_H \}.
\]
Set $T_{\emptyset_H} := A_{\emptyset_H}^\times / A_{\emptyset_H}^0$ for $? = -,-,-$, where
\[
A_{\emptyset_H}^\times := \{ x \in A_{\emptyset_H}(F) \mid |\alpha(x)| \leq 1 \ \forall \ \alpha \in \Delta_H - \Theta_H \};
\]
\[
A_{\emptyset_H}^- := \{ x \in A_{\emptyset_H}(F) \mid |\alpha(x)| < 1 \ \forall \ \alpha \in \Delta_H - \Theta_H \};
\]
\[
A_{\emptyset_H}^0 := \bigcap_{\alpha \in \text{Rat}(A_{\emptyset_H})} \ker |\alpha| = A_{\emptyset_H}(O_F).
\]
Here, $\text{Rat}(A_{\emptyset_H})$ denotes the set of rational characters on $A_{\emptyset_H}$.

**Theorem 4.4** (Cartan Decomposition). There exists a maximal compact open subgroup $K_H \subset H(F)$, a finite subset $\omega \subset M_{\emptyset_H}$ normalizing $K_H$ such that
\[
H(F) = \bigsqcup_{\omega \in \omega \in T_{\emptyset_H}^-} K_H \mathfrak{w} t K_H.
\]
Moreover, if the center $Z_H$ of $H$ is anisotropic or $H$ is unramified, then one can choose $\omega = \{ e \}$.

**Proof.** See [5, Section 4.4]. For the case $Z_H$ is isotropic, see also [8, Lemma 1.4.5]. The case $H$ unramified is also explained in [9, Section 1]. \qed

Finally, we record the following volume formula.

**Proposition 4.5.** For any $\Theta_H \subset \Delta_H$, there exists a constant $C_{\Theta_H} \in \mathbb{Q}^\times$ such that
\[
\text{Vol}(K_H \mathfrak{w} t K_H) = C_{\Theta_H} \delta_{P_{\emptyset_H}}^{-1}(t), \quad \forall \ t \in T_{\emptyset_H}^-.
\]
Here $\delta_{P_{\emptyset_H}}$ is the modulus character of $P_{\emptyset_H}$.

**Proof.** When $H$ is unramified, see [9, Proposition 1.6]. In general, one can deduce the result from [7, Theorem 4.2]. \qed
Now we introduce the notion of admitting a reduction structure for spherical varieties, which is crucial for Theorem 1.2. The reduction structure deals with the regions $\mathcal{T}_{\Theta_H}^-$, $\Theta_H \subset \Delta_H$. Recall the following facts for $\mathcal{T} = \mathbb{R}, \mathbb{C}$ and $\Theta_H \subset \Delta_H$:

- $\mathcal{T}_{\Theta_H}$ is naturally an abelian semigroup. Moreover any $t \in \mathcal{T}_{\Theta_H}$ induces an injection
  $$T_t : \mathcal{T}_{\Theta_H} \hookrightarrow \mathcal{T}_{\Theta_H}$$
- the natural map $\mathcal{T}_{\Theta_H}^0 \to \mathcal{T}_{\Theta_H}^0$ is an injection for any $\Theta_H \subset \Theta_H' \subset \Delta_H$. Moreover,
  $$\mathcal{T}_{\Theta_H} = \bigsqcup_{\Theta_H \subset \Theta_H'} \mathcal{T}_{\Theta_H}^-.$$

For each region $\mathcal{T}_{\Theta_H}^-$, the set $\Delta_H - \Theta_H$ of simple roots induces the map:
  $$\mathcal{T}_{\Theta_H}^- \to \mathbb{Z}_{\geq 1}^{\Delta_H - \Theta_H}, \quad t \mapsto (\nu_\alpha(t) := \text{val}(\alpha(t)))_{\alpha \in \Delta_H - \Theta_H}.$$  

To control the kernel, we fix a finite subset $\mathcal{C}_H \subset \text{Rat}(A_{\Theta_H})$ such that $\Delta_H \cap \mathcal{C}_H = \emptyset$ while $\Delta_H \cup \mathcal{C}_H$ is a basis of $\text{Rat}(A_{\Theta_H}) \otimes \mathbb{Z} \mathbb{Q}$. Note that the cardinality of $\mathcal{C}_H$ is just the split rank of the center $Z_H$ of $H$ and one has the embedding
  $$\mathcal{T}_{\Theta_H}^- \to \mathbb{Z}_{\geq 1}^{\Delta_H - \Theta_H} \times \mathbb{Z}^{\mathcal{C}_H}, \quad t \mapsto ((\nu_\alpha(t))_{\alpha \in \Delta_H - \Theta_H}, (\nu_\beta(t))_{\beta \in \mathcal{C}_H}). \quad (4.1)$$

In fact, for the pairs $(G, H)$ we shall consider, the split rank of $Z_H$ is at most 1 and $\mathcal{C}_H$ is either the empty set or a singleton. For the convenience of readers, we now introduce the reduction structure for these two cases separately.

For now on, assume $A_{\Theta_H} \subset A_G$. Firstly, we consider the case $\mathcal{C}_H = \emptyset$.

**Definition 4.6** (Case $\mathcal{C}_H = \emptyset$) A reduction structure for $\Theta_H \subset \Delta_H$ with respect to $G$ is a finite set
  $$S = S(\Theta_H) = \{(\Theta, w, s(\Theta, w))\}$$

where

- $\Theta \subset \Delta_G$,
- $w \in W_G := Z_G(A_{\Theta_G}) \backslash N_G(A_{\Theta_G})$, the Weyl group of $A_{\Theta_G}$,
- $s = s(\Theta, w) \in w^{-1}A_{\Theta_G}w \cap A_{\Theta_H}^0$ whose image in $\mathcal{T}_{\Theta_H}^-$ belongs to $\mathcal{T}_{\Theta_H}^- - \{e\}$

satisfying the following two finite conditions

(F1) for any $(\Theta, w, s) \in S$ and any $n \in \mathbb{N}$, $\mathcal{T}_{\Theta_H}^- - T_n(\mathcal{T}_{\Theta_H}^-)$ is a finite disjoint union of sets of the form $T_t^0(\mathcal{T}_{\Theta_H}^-)$ with $t \in \mathcal{T}_{\Theta_H}^0$ and $\Theta_H \subset \Theta_H'$. (When $\Theta_H = \Delta_H$, $\mathcal{T}_{\Theta_H}^- = T_0(\mathcal{T}_{\Theta_H}^-)$ automatically);

(F2) for any $0 < \epsilon < 1$, the image of
  $$\bigcap_{(\Theta, w, s) \in S} \left\{a \in A_{\Theta_H}^- \mid \text{if } waw^{-1} \in A_{\Theta_G}, \exists \alpha \in \Delta_G - \Theta, |\alpha(waw^{-1})| \geq \epsilon \right\} \in \mathcal{T}_{\Theta_H}^-$$

is finite, while the image is not finite when $S$ is replaced by any proper subset.

Here if $S = \emptyset$, (F1) is empty and (F2) reads that $\mathcal{T}_{\Theta_H}^-$ is finite (in fact, a singleton).

The reduction structure is introduced to analyze the rationality of the following formal power series.

Let $\pi$ a finitely generated smooth admissible $R[\mathcal{G}(F)]$-module satisfying the assumptions in Conjecture 1.1. In particular, there is a $R[\mathcal{G}(F)]$-module $\hat{\pi}$ with an invariant $R$-bilinear pairing $\langle \cdot, \cdot \rangle$ on $\pi \times \hat{\pi}$. For any subset $\Theta_H \subset \Delta_H$, $v \in \pi A_{\Theta_H}^0$ and $\tilde{v} \in \hat{\pi} A_{\Theta_H}^0$, consider
  $$\tilde{F}_{\Theta_H, v, \tilde{v}}(T) := \sum_{t \in \mathcal{T}_{\Theta_H}^-} \langle \pi(t)v, \tilde{v} \rangle T^{v(t)}, \quad T^{v(t)} := \prod_{\alpha \in \Delta_H - \Theta} T_{\alpha}^{v_{\alpha}(t)}.$$

**Example 4.7** ($H$-anisotropic case). When $H$ is anisotropic, then $\Delta_H = C_H = \emptyset$. In this case, $\Delta_H$ admits the reduction structure $\emptyset$ and $\tilde{F}_{\Delta_H, v, \tilde{v}} = \langle v, \tilde{v} \rangle$ is constant.

**Example 4.8** (The triple product case). Let $H = \text{PGL}_2$ embed into $G = \mathbb{G}_m \backslash \text{GL}_2^{\text{diag}}$ diagonally. One has $C_H = \emptyset$ and $\Delta_H = \{\alpha\}$ with $\alpha \left[ \begin{array}{cc} t_1 & 0 \\ 0 & t_2 \end{array} \right] = t_1/t_2$. Then $\tilde{F}_{\Delta_H, v, \tilde{v}} = \langle v, \tilde{v} \rangle$ and
  $$\tilde{F}_{\Delta_H, v, \tilde{v}}(T) = \sum_{a \geq 1} \langle \pi(t(a,0))v, \tilde{v} \rangle T^a, \quad t(a,0) = \left[ \begin{array}{cc} a^a & 0 \\ 0 & 1 \end{array} \right].$$

}
We now discuss the rationality of $\tilde{F}_{\emptyset_H,v,\bar{v}}(T)$. With respect to $G$, $\emptyset_H$ admits the reduction structure

$$S = \{(\emptyset_G, e, s = t(1,0))\}$$

since

(F1) for each $n$, we have the finite decomposition

$$T_{\emptyset_H}^- = T_n^s(T_{\emptyset_H}^-) = \bigcup_{i=1}^n T_i^s(T_{\emptyset_H}^-)$$

(F2) for any $0 < \epsilon \leq 1$,

$$\left\{ a \in A_{\emptyset_H}^- \mid \alpha(a) > \epsilon \right\} / A_{\emptyset_H}^0$$

is finite.

By the admissibility of Jacquet modules, there is a non-zero polynomial $P(X) = \sum_{n \geq 0} c_n X^n$ such that $P(\pi(s))v \in \pi(N_\emptyset)$ for all $v \in \pi$. By (F1), one has the equation of formal power series

$$\tilde{F}_{\emptyset_H} P(\pi(s))v,\bar{v} (T) = P(T^{-1})\tilde{F}_{\emptyset_H,v,\bar{v}} - T_n^s\sum_{n \geq 1} c_n(\pi(t(a,0))v,\bar{v})T_n^a - n.$$  

Corollary 4.3 and (F2) imply the rationality of $\tilde{F}_{\emptyset_H} P(\pi(s))v,\bar{v}$: The rationality of $\tilde{F}_{\emptyset_H,v,\bar{v}}$ then follows.

Now we turn to the case $C_H$ is a singleton, say $C_H = \{\beta\}$. For each $\Theta_H \subset \Delta_H$, consider the domain $Z_{\geq 1}\Delta_H^{\emptyset H} \times Z$ embedding 4.1. All the coordinates are contained in the “cone” $Z_{\geq 1}$ except the last one which corresponds to $\beta$. To define the right object $\tilde{F}_{\emptyset_H,v,\bar{v}}$ (so it is a formal power series), we further decompose $T_{\emptyset_H}$ into disjoint unions of “cones”:

$$T_{\emptyset_H}^s = T_{\emptyset_H}^s T_{\emptyset_H}^s \bigcup_{\emptyset_H \subset \emptyset_H} T_{\emptyset_H}^-$$

where for $* = +, 0, -$,

For $* = +, 0, -$ we have

$$T_{\emptyset_H}^- = \bigcup_{\emptyset_H \subset \emptyset_H} T_{\emptyset_H}^-$$

and for any $t \in T_{\emptyset_H}^-,$

$$T_t(T_{\emptyset_H}^s) \subset T_{\emptyset_H}^s.$$  

Let $A_{\emptyset_H}^0$ be the pre-image of $T_{\emptyset_H}^s$ in $A_{\emptyset_H}^0$.

**Definition 4.9** (Case $ZC_H = 1$). For any $\Theta_H \subset \Delta_H$ and $* = +, 0, -$, a reduction structure with respect to $G$ for the pair $(\Theta_H, *)$ is a finite set

$$S = S(\Theta_H,*) = \{(\Theta, w, s(\Theta, w))\}$$

where

- $\Theta \subset \Delta_G$ and $w \in W_G$;
- $s = s(\Theta, w) \in w^{-1} A_{\emptyset_H} w \cap A_{\emptyset_H}^-$, whose image in $T_{\emptyset_H}^-$ belongs to $T_{\emptyset_H}^{*,0} \cup T_{\emptyset_H}^{*,*} - \{e\}$

satisfying the following two finite conditions

(F1) for any $(\Theta, w, s) \in S$ and $n \in N$, $T_{\emptyset_H}^- = T_n^s(T_{\emptyset_H}^-)$ is a finite disjoint union of sets of the form

- $T_t T_{\emptyset_H}^s$ with $t \in T_{\emptyset_H}^s$ and $\Theta_H \subset \emptyset_H$ if $* = 0$;
- $T_t T_{\emptyset_H}^s$ with $t \in T_{\emptyset_H}^s$ and $\Theta_H \subset \emptyset_H'$ or $T_t T_{\emptyset_H}^{*,0}$ with $t_1 \in T_{\emptyset_H}^-$. if $* = +, -$ (both types of sets may occur in the union)

(F2) for any $0 < \epsilon \leq 1$, the image of

$$\bigcap_{(\Theta, w, s) \in S} \left\{ a \in A_{\emptyset_H}^- \mid \text{if } waw^{-1} \in A_{\emptyset_H}^- \text{ then } a(\Delta_G - \Theta, \alpha(waw^{-1})) \geq \epsilon \right\}$$

in $T_{\emptyset_H}^- *$ is finite, while the image is not finite when $S$ is replaced by any proper subset.
Here if $S = \emptyset$, (F1) is empty and (F2) means $T_{\Theta_H}^{-,-1}$ is finite.

For $\Theta_H$, if $S(\Theta_H,*)$ is a reduction structure for $(\Theta_H,*)$, then we call
\[
S = S(\Theta_H) = \bigsqcup_{* = +, 0, -} S(\Theta_H,*)
\]
a reduction structure for $\Theta_H$.

These reduction structures are introduced to analyze the following formal power series:
\[
\tilde{F}_{\Theta_H,v,\tilde{v}}(T) := \sum_{v \in \pi A_{x_H}^{G}, \tilde{v} \in \pi A_{x_H}^{G}} \tilde{F}_{\Theta_H,v,\tilde{v}}(T), \quad v \in \pi A_{x_H}^{G} , \tilde{v} \in \pi A_{x_H}^{G}
\]
\[
\tilde{F}_{\Theta_H,v,\tilde{v}}^*(T) := \sum_{t \in T_{\Theta_H}^{*,-,-1}} \langle \pi(t) v, \tilde{v} \rangle T^{[v(t)]}, \quad T^{[v(t)]} := T_{\beta}^{\langle v(t) \rangle} \prod_{\alpha \in \Delta_H - \Theta_H} T_{\alpha}(v(t)).
\]

Here we put $| \cdot |$ on the power of $T$ (make difference only when $* = -$) to obtain formal power series.

**Example 4.10** (The case $G_m \subset \text{PGL}_2$). Consider the pair
\[
H = G_m \hookrightarrow G = \text{PGL}_2, \quad t \mapsto \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}
\]
and choose $C_H = \{ \text{id} \}$. Then $\tilde{F}_{\emptyset_H,v,\tilde{v}} = \langle v, \tilde{v} \rangle$ and
\[
\tilde{F}_{\emptyset_H,v,\tilde{v}}(T) = \sum_{a \geq 1} \langle \pi(w^a) v, \tilde{v} \rangle T^a
\]
The pairs $(\emptyset_H, \pm)$ have the reduction structures
\[
S(\emptyset_H, +) = \{(G, e, w)\}, \quad S(\emptyset_H, -) = \{(G, w, w^{-1})\}, \quad w := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]
since
\begin{itemize}
  \item for any $0 < \epsilon \leq 1$, \{ $a \in F^x | \epsilon \leq |a| < 1$ \}/$O_{F}^x$ and \{ $a \in F^x | 1 \leq |a| \leq \epsilon^{-1}$ \}/$O_{F}^x$ are finite;
  \item for $1 \leq n \in \mathbb{N}$,
\end{itemize}
\[
T_{\emptyset_H}^{-,-,-1} = T_{\emptyset_H}^{-,1} T_{\emptyset_H}^{-,-1} = \bigsqcup_{i=1}^{n} T_{\emptyset_H}^{-,i} T_{\emptyset_H}^{-,-1}.
\]
Take any non-zero polynomial $P(X)$ (resp. $P_{w}(X)$) such that for all $v \in \pi$, $P(\pi(w)) v \in \pi(N_{0,G})$ (resp. $P_{w}(\pi(w^{-1})) v \in \pi(w N_{0,G} w^{-1})$). Then
\[
\tilde{F}_{\emptyset_H,v,\tilde{v}}^*(T) = P(T^{-1}) \tilde{F}_{\emptyset_H,v,\tilde{v}}(T) + G_{+}(T^{-1}),
\]
\[
\tilde{F}_{\emptyset_H,v,\tilde{v}}^*(T) = P_{w}(T^{-1}) \tilde{F}_{\emptyset_H,v,\tilde{v}}(T) + G_{-}(T^{-1})
\]
for some polynomials $G_{+}$ and $G_{-}$. By Corollary 4.3 and (F2), the summations in the left hand are actually finite. The rationality of $\tilde{F}_{\emptyset_H,v,\tilde{v}}(T)$ then follows.

**Remark 4.11.** actually the above decomposition can be generalized to the case $\sharp C_H \geq 2$ so that the following proposition still holds. Precisely, one can decompose
\[
T_{\emptyset_H}^{\sigma} = \bigsqcup_{I, \epsilon_I} T_{\emptyset_H, I}^{\epsilon_I}, \quad I \subset C_H, \quad \epsilon_I = (\epsilon_\beta) \in \{ + \}^{C_H - I}
\]
and for $? = -$, ..., $T_{\emptyset_H, I}^{\epsilon_I} := A_{\emptyset_H, I}^{\epsilon_I}/A_{\emptyset_H}^{0}$ where
\[
A_{\emptyset_H, I}^{\epsilon_I} := \{ x \in A_{\emptyset_H}^{0} | v_{\beta}(x) = 0, \forall \beta \in I, \epsilon_{\beta} v_{\beta}(x) > 0, \forall \beta \in C_H - I \}.
\]
Then we can define a reduction structure for the triple $(\Theta_H, I, \epsilon_I)$ in a similar manner and consider the rationality of the formal powers
\[
\tilde{F}_{\emptyset_H, I, v, \tilde{v}}^{\epsilon_I}(T) := \sum_{t \in T_{\emptyset_H, I}^{-,-1}} \langle \pi(t) v, \tilde{v} \rangle T^{[v(t)]}.
\]
From now on, we assume $\sharp C_H \leq 1$. 16
Proposition 4.12. Take $* = 0, +, 0, -$ and $\Theta_H \subset \Delta_H$ such that any $\Theta_H \subset \Theta'_H \subset \Delta_H$ admits a reduction structure $S(\Theta'_H, *)$. Then for any $v \in \pi A^s_{\Theta}$ and $\bar{v} \in \pi A^s_{\Theta}$, the formal power series $\tilde{F}^s_{\Theta_H, v, \bar{v}}$ is rational.

Moreover, for each $(\Theta, w, s) \in \bigsqcup_{\Theta_H \subset \Theta} S(\Theta_H, *),$ there exists a one variable polynomial $P_{\Theta, w, s}$ which is independent of $v$ and $\bar{v}$, and whose first and last coefficients are units such that

$$\tilde{F}^s_{\Theta_H, v, \bar{v}}(T) = \frac{N(T)}{M(T) \prod_{(\Theta, w, s)} P_{\Theta, w, s}(T^{*-v(s)}(t))}$$

for some polynomial $N(T) \in \mathbb{Z}[T]$ and monomial $M(T) \in \mathbb{Z}[T]$. Here $(\Theta, w, s)$ runs over $\bigsqcup_{\Theta_H \subset \Theta} S(\Theta_H, *)$.

Proof. For any $\Theta \subset \Delta_G$, let $P_\Theta$ be the standard parabolic with standard Levi $M_\Theta = Z P_\Theta(A_\Theta)$ and unipotent radical $N_\Theta$. We will treat the case $\Theta_H = 1$ here. The case $\Theta_H = \emptyset$ is similar and easier. By construction, it suffices to show $\tilde{F}^s_{\Theta_H, v, \bar{v}}$ is rational for each $* = +, 0, -$. The idea is to make induction on the size of $\Delta_H - \Theta_H$.

We start with $* = 0$ (the treatment when $C_H = \emptyset$ is similar to this case). For $\Theta_H = \Delta_H$, $T^{*-0}_{\Theta_H}$ is a singleton and $\tilde{F}^0_{\Theta_H, v, \bar{v}}$ is a constant. For general $\Theta_H \subset \Delta_H$, we make the induction hypothesis that $\tilde{F}^0_{\Theta_H, v, \bar{v}}$ is rational for any $\Theta_H \subset \Theta_H \subset \Delta_H$, $v$ and $\bar{v}$. Now we prove that $\tilde{F}^0_{\Theta_H, v, \bar{v}}$ is rational.

Note that for $(\Theta, w, s) \in S(\Theta_H, 0)$, $\pi(s)$ induces an invertible element in $\text{End}_{R[w^{-1} M_{\Theta}(F)]}(J_{w^{-1} N_{\Theta}(w)}(\pi))$. Since $\pi$ is a finitely generated smooth admissible $R[G(F)]$-module, $J_{w^{-1} N_{\Theta}(w)}(\pi)$ is a finitely generated smooth admissible $R[w^{-1} M_{\Theta}(F)]$-module by Theorem 4.1. Thus by [12, Lemma 4.1], the $R$-module $\text{End}_{R[w^{-1} M_{\Theta}(F)]}(J_{w^{-1} N_{\Theta}(w)}(\pi))$ is finite and as explained in [27, Page 1980], there exists

$$P_{\Theta, w, s}(X) = \sum_{i=0}^N a_i X^i \in R[X], \quad a_0, a_N \in R^*$$

such that $P_{\Theta, w, s}(\pi(s)v) \in \pi(w^{-1} N_{\Theta}(w))$ for all $v \in \pi$. For $0 \leq i \leq N$, assume that

$$T^{*-0}_{\Theta_H} - T_{s^i}^* \left( T^{*-0}_{\Theta_H} \right) = \bigsqcup_{\Theta_H \subseteq \Theta_H \subseteq \Delta_H} \bigcup_{i \in I_{\Theta_H, i}} T_{s^i} \left( T^{*-0}_{\Theta_H} \right)$$

(4.2)

where each $I_{\Theta_H, i} \subset T^{*-0}_{\Theta_H}$ is a finite subset. Then one has

$$T^{uv(s)} \tilde{F}^0_{\Theta_H, \pi(s)v, \bar{v}} = \tilde{F}^0_{\Theta_H, v, \bar{v}} - \sum_{v' \in v} T^{uv(t)} \tilde{F}^0_{\Theta_H, \pi(t)v, \bar{v}}, \quad T^{uv(t)} := \prod_{\alpha \in \Delta_H - \Theta_H} T_{\alpha}^{v(t)}$$

Consequently,

$$\tilde{F}^0_{\Theta_H, P_{\Theta, w, s}(\pi(s)v), \bar{v}}(v) = \tilde{F}^0_{\Theta_H, \pi(s)v, \bar{v}} - \sum_{i=1}^N \sum_{v' \in v} T^{uv(t)-v(s)} \tilde{F}^0_{\Theta_H, \pi(t)v, \bar{v}}$$

(4.3)

Then by the induction hypothesis, $\tilde{F}^0_{\Theta_H, v, \bar{v}}$ is rational if $\tilde{F}^0_{\Theta_H, P_{\Theta, w, s}(\pi(s)v), \bar{v}}$ is rational.

For any $v \in \pi$, the element

$$v := \prod_{(\Theta, w, s) \in S} P_{\Theta, w, s}(\pi(s(\Theta, w)))v$$

lies in $\bigcap_{(\Theta, w, s) \in S} \pi(w^{-1} N_{\Theta}(w))$. Then by Corollary 4.3, $\tilde{F}^0_{\Theta_H, v, \bar{v}}$ is a polynomial in the variables $T_{\alpha}, \alpha \in \Delta_H - \Theta_H$.

If $S$ has only one element so that $v' = P_{\Theta, w, s}(\pi(s(\Theta, w)))v$, we obtain the rationality of $\tilde{F}^0_{\Theta_H, v, \bar{v}}$. If $S$ has more than one element, say, $v' = P_{\Theta, w, s}(\pi(s(\Theta, w)))v_1$ for some $v_1 \in \pi$, then replacing $v$ by $v_1$ in Equation 4.3, we obtain the rationality of $\tilde{F}^0_{\Theta_H, v_1, \bar{v}}$ (note that the induction hypothesis includes any $v \in \pi A^s_{\Theta}$ and any $\bar{v} \in \pi A^s_{\Theta}$). As $S$ is finite, one finally deduces $\tilde{F}^0_{\Theta_H, v, \bar{v}}$ is rational.

By a similar induction argument, the case $* = \pm$ is reduced to the case $* = 0$. Precisely, similar to Decomposition 4.2 for the case $* = 0$, for each $i$, we have

$$T^{*-0}_{\Theta_H} - T_{s^i}^* \left( T^{*-0}_{\Theta_H} \right) = \bigsqcup_{\Theta_H \subseteq \Theta_H \subseteq \Delta_H} \bigcup_{i \in I_{\Theta_H, i}} T_{s^i} \left( T^{*-0}_{\Theta_H} \right) \bigsqcup_{i \in I_{\Theta_H, i}} T_{s^i} \left( T^{*-0}_{\Theta_H} \right).$$

(4.4)
where $I_{\Theta_H,i}$ (resp. $I_{\Theta_H,i}$) is a finite subset of $T_{\Theta_H}^{-,\ast}$ (resp. $T_{\Theta_H}^{-,\ast}$). Then, similar to Equation 4.3, we have

$$\tilde{F}_{\Theta_H,v} = P_{\Theta_H,v}(T - |v(s)|) F_{\Theta_H,v} + \sum_{i=1}^{N} \sum_{t \in I_{\Theta_H, i}} a_i T^{(|v(t)| - |v(s)|)} \tilde{F}_{\Theta_H,\pi(t)v}$$

(4.5)

Note that the rationality of $\tilde{F}_{\Theta_H,\pi(t)v}$ is already established. The same argument in the $\ast = 0$ case gives the rationality for $\tilde{F}_{\Theta_H,\pi(t)v}$.

For the "Moreover" part, we just apply Equations 4.3 and 4.5 by making induction on the size of $\Delta_H - \Theta_H$. We are done.

Let $n_{\Theta_H}$ be the Lie algebra of $N_{\Theta_H}$ and assume

$$\det(Ad_A_{\Theta_H})|_{n_{\Theta_H}} = \sum_{\alpha \in \Delta_H} N_{\alpha} \alpha; \ N_{\alpha} \in \mathbb{N}.$$ 

We will need to evaluate $\tilde{F}_{\Theta_H,\tilde{v}}$ at $T_{\alpha} = q^{N_{\alpha}}$, $\alpha \in \Delta_H - \Theta_H$ and $T_{\beta} = 1$, $\beta \in C_H$. To this end, it would be convenient to change the formal power series in multiple variables to formal power series in one variable. We record the following corollary of Proposition 4.12. Note that for any $t \in A_{\Theta_H}$,

$$\delta_{F_{\Theta_H,v}}^{-1}(t) = \prod_{\alpha \in \Delta_H - \Theta_H} q^{N_{\alpha} v_{\alpha}(t)}.$$ 

For each $N \in \mathbb{N}$ and $\ast = 0, +, 0, -$ and $v \in \pi^{A_{\Theta_H}}, \tilde{v} \in \pi^{A_{\Theta_H}}$, the formal power series

$$F_{\Theta_H,v}^\ast(S) := \sum_{N \geq 0} \left( \sum_{t \in T_{\Theta_H,v}^{-,\ast}} (\pi(t)v, \tilde{v}) \delta_{F_{\Theta_H,v}}^{-1}(t) \right) S^N$$

is rational. Moreover, for each $(\Theta_H, \ast)$, there exists polynomials $Q(S), P(S) \in \mathbb{Z}[S]$ with the first and last coefficients of $P(S)$ being units such that

$$F_{\Theta_H,v}^\ast(S) = \frac{Q(S)}{P(S)}$$

**Proof.** The inner summation in the definition of $F_{\Theta_H,v}^\ast$ is finite and thus as formal power series, $F_{\Theta_H,v}^\ast(S)$ is just the evaluation of $\tilde{F}_{\Theta_H,v}^\ast(T)$ at $T_{\alpha} = q^{N_{\alpha}} S$, $\alpha \in \Delta_H - \Theta_H$ and $T_{\beta} = S$, $\beta \in C_H$.

Take the expression

$$\tilde{F}_{\Theta_H,v}(T) = \frac{N(T)}{M(T)}$$

as in Proposition 4.12. Since $\sum_{\alpha \in \Delta_H - \Theta_H \cup C_H} v_{\alpha}(s) > 0$, the evaluation of $P_{\Theta_H,v}(T - |v(s)|)$ at $T_{\alpha} = q^{N_{\alpha}} S$, $\alpha \in \Delta_H - \Theta_H$ and $T_{\beta} = S$, $\beta \in C_H$ is a polynomial in $S$ whose first and last coefficients are units. Consequently, $F_{\Theta_H,v}^\ast(S)$ is also rational. \qed

**Definition 4.14.** The $G$-spherical variety $Y$, or spherical pair $(G, H)$, admits a reduction structure if any $\Theta_H \subset \Delta_H$ admits a reduction structures with respect to $G$.

**Remark 4.15.** We remark that if $(G, H)$ and $(G', H')$ are spherical pairs such that
• there is a subgroup $Z$ of the center of $G$ such that $G' = G/Z$, $H' = H/(Z \cap H)$ and $Z \cap H$ is anisotropic;
• or there is an isogeny $\phi : G \to G'$ mapping $H$ to $H'$,

then the quotient map (resp. $\phi$) identifies $\Delta_H$, $\text{Rat}(A_{\theta_H})$, $\Delta_G$ with $\Delta_{H'}$, $\text{Rat}(A_{\theta_{H'}})$, $\Delta_{G'}$. Under these identifications, a reduction structure for $\Theta_H$ with respect to $G$ is sent to a reduction structure for $\Theta_{H'} = \Theta_H$ with respect to $G'$. In particular, admitting a reduction structure is preserved when modifying spherical pairs by the operations above.

Now we evaluate $F_{\Theta_H,v,v}$ at $S = 1$ when the pair $(G, H)$ is strongly tempered. Normalize the Haar measure $dh$ on $H(F)$ so that $\text{Vol}(K_H, dh) = 1$ and take $\omega$, $C_{\Theta_H}$ as in Theorem 4.4 and Proposition 4.5. We start with the following result for complex tempered representations (which will apply to $\pi_x$, $x \in \Sigma$).

**Lemma 4.16.** Assume the spherical pair $(G, H)$ is strongly tempered and admits a reduction structure. Let $\pi, \tilde{\pi}$ be irreducible tempered complex $G(F)$-representations equipped with a non-degenerate bi-$H(F)$-invariant pairing $\langle \cdot , \cdot \rangle$. Take any $v \in \pi_K^H$ and $\tilde{v} \in \tilde{\pi}_K^H$. Then for $\Theta_H \subset \Delta_H$, $F_{\Theta_H,v,v}(S)$ is regular at $S = 1$ and

$$F_{\Theta_H,v,v}(1) = \sum_{t \in T_{\Theta_H}} \langle \pi(t)v, \tilde{v} \rangle \delta_{FB_H}^{-1}(t).$$

Moreover, one has

$$I(v, \tilde{v}) := \sum_{w \in \omega} \sum_{\Theta_H} \sum_{t \in T_{\Theta_H}} C_{\Theta_H} F_{\Theta_H,v,v}(w^{-1}) \delta_{FB_H}^{-1}(1) = \int_{H(F)} \langle \pi(h)v, \tilde{v} \rangle dh.$$

**Proof.** We deal with the case $\pi C_H = 1$ here. Since the irreducible complex representation $\pi$ (hence $\tilde{\pi}$) is tempered, the period integral

$$\int_{H(F)} \langle \pi(h)v, \tilde{v} \rangle dh$$

is absolutely convergent. Then by Theorem 4.4 and Proposition 4.5, the summation

$$\sum_{w \in \omega} \sum_{\Theta_H \subset \Delta_H} \sum_{t \in T_{\Theta_H}} C_{\Theta_H} \delta_{FB_H}^{-1}(1) \int_{K_H \times K_H} \langle \pi(t)\pi(k_2)v, \tilde{\pi}(w^{-1})\tilde{\pi}(k_1^{-1})\tilde{v} \rangle dk_1 dk_2$$

$$= \sum_{w \in \omega} \sum_{\Theta_H \subset \Delta_H} \sum_{t \in T_{\Theta_H}} C_{\Theta_H} \langle \pi(t)v, \tilde{\pi}(w^{-1})\tilde{v} \rangle \delta_{FB_H}^{-1}(t)$$

is absolutely convergent and equals to the period integral (see [33, Page 149]). Thus the summation

$$\sum_{t \in T_{\Theta_H}} \langle \pi(t)v, \tilde{v} \rangle \delta_{FB_H}^{-1}(t) = \sum_{N \geq 0} \sum_{t \in T_{\Theta_H}, N} \langle \pi(t)v, \tilde{v} \rangle \delta_{FB_H}^{-1}(t)$$

(4.6)

is also absolutely convergent for each $\Theta_H \subset \Delta_H$ and $* = +, 0, -$. By the absolute convergence of (4.6), one immediately deduces that the formal power series $F^*_{\Theta_H,v,v}$ converges absolutely when $|S| \leq 1$. As it is rational, one must have $F^*_{\Theta_H,v,v}$ is regular at $S = 1$ and

$$F^*_{\Theta_H,v,v}(1) = \sum_{t \in T_{\Theta_H}} \langle \pi(t)v, \tilde{v} \rangle \delta_{FB_H}^{-1}(t).$$

$\square$

Now we turn to Conjecture 1.1. We need the following notation:

**Definition 4.17.** Let $\hat{\Sigma}$ be the pre-image of $\Sigma$ via the natural map $\text{Spec } (R \otimes_E \hat{E}) \to \text{Spec } (R)$. Let $\pi$ be a finitely generated smooth admissible $R[G(F)]$-module such that for any $x \in \hat{\Sigma}$,

$$\mathcal{E}(\hat{\pi}|_x) := \{ \tau : \hat{E} \hookrightarrow \mathbb{C} | \hat{\pi}|_x \tau := \hat{\pi}|_x \otimes_{E, \tau} \mathbb{C} \text{ is tempered} \neq \emptyset, \hat{\pi} := \pi \otimes_E \hat{E}. \}

We say the discrete support of $\hat{\pi}$ is rigid around $x \in \hat{\Sigma}$ if there exist

• an open neighborhood $x \in V \subset \text{Spec } (R \otimes_E \hat{E})$,
• a smooth character $\chi : M(F) \to \mathcal{O}_V^\times$ such that $\chi|_x$ is the trivial character,
• a parabolic subgroup $P = MN \subset G$ with Levi factor $M$,
• an irreducible smooth admissible $M(F)$-representation $\sigma$ over $\hat{E}$ such that $\sigma|_x$ is a discrete series for some $\tau \in \mathcal{E}(\hat{\pi}|_x)$.
such that $\tilde{\pi}|_U \subset I_{F}^{G} \sigma \otimes \chi|_U$ for each $y \in \tilde{\Sigma} \cap V$. By abuse of notation, we say the discrete support of $\pi$ is rigid around each $x \in \Sigma$ if the discrete support of $\tilde{\pi}$ is rigid around each $x \in \tilde{\Sigma}$.

**Theorem 4.18.** Assume the spherical $G$-variety $Y$ is strongly tempered and admits a reduction structure. Let $\pi$ be a finitely generated smooth admissible $R[G(F)]$-module as in Conjecture 1.1. Then

- the rationality conjecture holds for $\pi$,
- the meromorphy conjecture holds for $\pi|_{V}$ where $U \subset \text{Spec}(R)$ is a Zariski dense open subset; moreover the meromorphy conjecture holds for $\pi$ if the discrete support of $\pi$ is rigid around each point $x \in \Sigma$.

**Proof.** For any $v \in \pi$ (resp. $\tilde{v} \in \pi$), let

$$v' = \int_{K_H} \pi(k)vdk \quad \text{resp.} \quad \tilde{v}' = \int_{K_H} \tilde{\pi}(k)\tilde{v}dk$$

For any $\Theta_H \subset \Delta_H$, write $F_{\Theta_H,w,v}(S) = \frac{Q(S)}{P(S)}$ with polynomials

$$P(S) = \sum_{n=0}^{N} a_n S^n, \quad Q(S) = \sum_{n=0}^{N} b_n (S-1)^n \in R[S]$$

as in Corollary 4.13. In particular, $P(S)$ is independent of $v$ and $\tilde{v}$ and $a_n, b_n \in R^{\times}$. Moreover by Lemma 4.16, for any $x \in \Sigma$, if $r := \text{ord}_{x=1} P_{x} \leq N$ is the order of $P_{x}$ at $S = 1$, then $r \leq \text{ord}_{x=1} Q_{x}$. Thus, the evaluation $F_{\Theta_H,w,v}(S)|_{x}(1)$ of $F_{\Theta_H,w,v}(S)|_{x}$ at $S = 1$ is just $\frac{b_{r}(S)}{a_{r}(S)}$.

Firstly we prove the rationality conjecture. For any $x \in \Sigma$ and $\tau \in \mathcal{E}(\pi|_{x})$, $\frac{Q(S)}{P(S)}$ represents the rational function $F_{\Theta_H,\tau(w'|x),\tau(v'|x)}$ defined for $\tau|_{x}$. Set

$$I|_{x}(v|_{x}, \tilde{v}|_{x}) := \sum_{\Theta_H \subset \Delta_H} C_{\Theta_H} F_{\Theta_H,w',\tilde{v}(w'-1)\tilde{v}}(1)$$

Then by Proposition 3.6 and Lemma 4.16, the rationality conjecture holds for $\pi|_{x}$ and $I|_{x}$ is the desired bi-$H(F)$-invariant bilinear form on $\pi|_{x} \times \pi|_{x}$.

Now we consider the meromorphy conjecture. Let $p_{1}, \ldots, p_{k}$ be the minimal primes of $R$. For each $1 \leq i \leq k$, there exists $0 \leq r(i) \leq N$ such that

$$a_{0}, \ldots, a_{r(i)-1} \in p_{i}, \quad a_{r(i)} \notin p_{i}.$$ 

Set $U = \text{Spec}(R) - Z$ where $Z = \bigcup \Theta_H, i \in \Theta_H, i$ with $Z(\Theta_H, i) = V(p_{i}) \cap V(a_{r(i)}) \subset \text{Spec}(R), 1 \leq i \leq k$. Let $	ext{Frac}(R) = \prod_{p_{i}} R_{p_{i}}$ be the total quotient ring of $R$. Set

$$I(v, \tilde{v}) := \sum_{\Theta_H} \sum_{w, w'} C_{\Theta_H} F_{\Theta_H,w',\tilde{v}(w-1)\tilde{v}}(1) \epsilon \text{Frac}(R), \quad F_{\Theta_H,w',\tilde{v}(w-1)\tilde{v}}(1) := \left( \frac{b_{r}(S)}{a_{r}(S)} \right)_{i} \epsilon \text{Frac}(R).$$

Then for any $x \in \Sigma \cap U$, one has

$$I(v|_{x}, \tilde{v}|_{x}) = I|_{x}(v|_{x}, \tilde{v}|_{x}), \quad \forall v \in \pi, \tilde{v} \in \pi.$$ 

Since $\Sigma \cap U$ is Zariski dense in $\text{Spec}(R)$, $I(-,-)$ is forced to be bi-$H(F)$-invariant and bilinear by the interpolation property. Thus, the meromorphy conjecture holds for $\pi|_{U}$.

For the meromorphy conjecture of $\pi$, it suffices to show $I(-,-)$ actually interpolates $I|_{x}(-,-)$ for each $x \in \Sigma$. For this, one can base change from $R$ to $R \otimes_{E} \tilde{E}$ and consider the corresponding result for $\tilde{\pi}$. Clearly for $x \in \tilde{\Sigma}$, $I(-,-)$ (defined for $\tilde{\pi}$) interpolates $I|_{x}(-,-)$ if for all $\Theta_H \subset \Delta_H$ and $* = +, 0, -$ (resp. 0) if $2 \mathcal{C}_{H} = 1$ (resp. $\mathcal{C}_{H} = 0$), $F_{\Theta_H,w,v}(1)$ (defined for $\tilde{\pi}$) is regular at $x$. By Proposition 4.12 and (the proof) of Corollary 4.13, $F_{\Theta_H,w,v}(1)$ is regular at $x$ if one can choose $\Theta_{w, s}$ such that

$$P_{\Theta,w,s}|_{x}(\delta_{\Theta_{w,s}}(s)) \neq 0, \quad \forall (\Theta, w, s) \in \bigcup_{\Theta \subset \Theta_{H}} S(\Theta_{H}, *).$$

Assume the discrete support of $\tilde{\pi}$ is rigid around each $x \in \tilde{\Sigma}$ and take $V, \sigma$ and $\chi$ as in Definition 4.17. Then by the theory of Jacquet modules (see [8, Section 6.3]), for any $(\Theta, w, s)$

- the central characters of irreducible sub-quotients of $J_{w-1-N \Theta w}(I_{F}^{G} \sigma \otimes \chi|_{w})$ are interpolated by $\mathcal{O}_{Y}$-valued characters, say $\{\omega_{i}\}_{i}^{I}$;
- the multiplicity of irreducible sub-quotients of $J_{w-1-N \Theta w}(I_{F}^{G} \sigma \otimes \chi|_{w})$ with the same central character are uniformly bounded, say $\{N_{i}\}_{i}$. 

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when \( y \) varies in \( V \). Then locally around \( x \), \( \prod_{i \in I} (\hat{\pi}(s) - \omega_i(s))^{N_i} \) kills \( J_{w^{-1}N_0w}(\hat{\pi}) \) and we can take \( P_{\Theta,w,s}(T) = \prod_{i \in I} (T - \omega_i(s))^{N_i} \). Since \( I_P^0 \sigma \) is a direct sum of tempered representations for some \( \tau \in \mathcal{E}(\hat{\pi}[s]) \), one has \( P_{\Theta,w,s}[x](\delta_{\Theta_H}(s)) \neq 0 \) by Lemma 4.19 below.

**Proof.** Let \((G, H)\) be a strongly tempered spherical pair and \( \pi \) be an irreducible tempered complex \( G(F)\)-representation. Let \( \ast \) be one of +, 0, or \(-\) if \( \not\exists C_H = 1 \) or \( \not\exists C_H = 0 \). Let \( \Theta_H \subset \Delta_H \) and consider a triple \((\Theta, w, s)\) where \( \Theta \subset \Delta_G \), \( w \in W \) and \( s \in w^{-1}A_{\Theta_H}w \cap A_{\Theta_H}^- \) with image in \( T_{\Theta_H} \) belongs to \( T_{\Theta_H} - \{ \epsilon \} \). Assume that for some \( \epsilon > 0 \) small enough,

\[
\left\{ a \in A_{\Theta_H}^- : \text{if} \ waw^{-1} \in A_{\Theta_H}^- \text{ then } |a(a \omega w^{-1})| > \epsilon \right\} \neq A_{\Theta_H}^-.
\]

Then any eigenvalue for the action of \( \pi(s) \) on the Jacquet module \( J_{w^{-1}N_0w}(\pi) \) has absolute value strictly smaller than \( \delta_{\Theta_H}(s) \).

**Proof.** Let \( \{ \chi_i \}_{i \in I} \) be the set of central characters of irreducible sub-quotients of the finite length smooth admissible \( w^{-1}M_{\Theta}(F)\)-representation \( J_{w^{-1}N_0w}(\pi) \). For each \( \chi_i \), let \( N_i \) be the multiplicity of the irreducible sub-quotients with central character \( \chi_i \). Consider the generalized eigenspace

\[
V_i := \left\{ v \in J_{w^{-1}N_0w}(\pi) \left| (\pi(a) - \chi_i(a))^{N_i} v = 0, \forall a \in w^{-1}A_{\Theta_H}w \right. \right\}.
\]

Then as \( w^{-1}A_{\Theta_H}w \)-representations,

\[
J_{w^{-1}N_0w}(\pi) = \bigoplus_i V_i.
\]

By assumption, there exists \( a \in A_{\Theta_H}^- \) such that \( waw^{-1} \in A_{\Theta_H}^- \) and for any \( a \in \Delta_G - \Theta \), \( |a(a \omega w^{-1})| < \epsilon \).

Let \((-,-)\) be the contraction on \( \pi \times \pi^\vee \). By Theorem 4.2,

\[
(\pi(as^n)v, v^\vee) = (\pi(as^n)v, v^\vee)_{w^{-1}N_0w}, \forall n \geq 0, v \in \pi, v^\vee \in \pi^\vee.
\]

Assume that \( v \) (as an element in \( J_{w^{-1}N_0w}(\pi) \)) decomposes as

\[
v = \sum_i v_i, \quad v_i \in V_i.
\]

On one hand, similar to Lemma 4.16, the summation

\[
\sum_{n \geq 0} (\pi(as^n)v, v^\vee) \delta_{\Theta_H}^{-1}(as^n)
\]

converges absolutely as \( \pi \) is tempered and \((G, H)\) is strongly tempered. On the other hand,

\[
(\pi(as^n)v_i, v^\vee)_{w^{-1}N_0w} = \chi_i(s)^{N_i} \sum_{j = 0}^{N_i - 1} \binom{n}{j} \chi_i^{N_i - j} \left( \pi(s) - \chi_i(s) \right)^j \pi(a)v_i, v^\vee), \quad n \geq N_i.
\]

Thus for each \( i \), one must have

\[
|\chi_i(s)\delta_{\Theta_H}^{-1}(s)| < 1.
\]

We are done.

We expect the discrete support of \( \pi \) to be rigid around each \( x \in \Sigma \) in practice. The following proposition is due to Prof. G. Moss.

**Proposition 4.20.** Assume \( E = \mathcal{E} \) and let \( \pi \) be a finitely generated smooth admissible torsion-free \( \mathcal{E}[G(F)]\)-module such that

- \( \pi|_x \) is irreducible tempered for each \( x \in \Sigma \),
- the natural morphism \( R \to \text{End}_{\mathcal{E}[G(F)]}(\pi) \) is an isomorphism.

Then the discrete support of \( \pi \) is rigid around \( x \in \Sigma \) if \( \pi|_x = I_P^0 \sigma \) where \( P = MN < G \) is a parabolic subgroup with Levi factor \( M \) and \( \sigma \) is an irreducible regular supercuspidal \( M(F)\)-representation over \( E \).

**Proof.** Base change along any \( \tau \in \mathcal{E}(\pi|_x) \), we may assume \( E = \mathbb{C} \). Since \( \pi|_x \) is tempered, \( \pi|_x \to I_P^0 \tau \) for some parabolic subgroup \( P' = M'N' \subset G \) and discrete series \( M'(F)\)-representation \( \tau \). The \( M'(F)\)-representation \( \tau \) also has cuspidal support \( \sigma \). Thus by [8, Theorem 6.5.1] for \( \tau \), one has \( \sigma \) is unitary (hence a discrete series). Let \( Z(G) \) and \( Z(M) \) be the Bernstein center for smooth \( G(F) \) and \( M(F) \)-representations. Since \( R \cong \text{End}_{\mathcal{E}[G(F)]}(\pi) \), there exists a morphism \( Z(G) \to R \) such that for any \( y \in \text{Spec}(R) \), the composition

\[
Z(G) \to R \to k(y)
\]
factors through the action of $Z(G)$ on $\pi|_y$. Let $s$ (resp. $t$) be the $G$-inertial (resp. $M$-inertial) class of $[M,\sigma]$ and $Z(G)_s$ (resp. $Z(M)_t$) be the corresponding component. Then up to replacing $\text{Spec}(R)$ by its connected component containing $x$, we may assume $Z(G) \to R$ factors through $Z(G)_s \to R$.

The Harish-Chandra morphism $Z(G) \to Z(M)$ (see [11, Theorem 4.1]) induces a finite morphism $Z(G)_s \to Z(M)_t$, which is locally étale at the point $\sigma$ by the regularity assumption. Locally around $\sigma$, we can take a section and by abuse of notation, we denote this section by $Z(M)_t \to Z(G)_s$. Then the composition

$$Z(M)_t \to Z(G)_s \to R \to k(x)$$

is given by the action of $Z(M)_t$ on $\sigma$. Let $M_0(F) \subset M(F)$ be the intersection of the kernels of all unramified characters of $M(F)$. Then $M(F)/M_0(F)$ is a free abelian group of finite rank. Let $\chi : M(F) \to \mathbb{C}[M(F)/M_0(F)]; \ m \mapsto [m]$ be the universal unramified character of $M(F)$. Since $\sigma$ is supercuspidal, $Z(M)_t$ is just the ring of functions on the orbit of $\sigma$ under unramified twisting. Hence there exists an open neighborhood $x \in V \subset \text{Spec}(R)$ such that for any $y \in V$, the composition

$$Z(M)_t \to Z(G)_s \to R \to k(y)$$

is simply given by the action of $Z(M)_t$ on $\sigma \otimes \chi|_y$. Then by the definition of the Harish-Chandra morphism, the cuspidal support of $\pi|_y$ is $\sigma \otimes \chi|_y$ for $y \in V$. Then for $y \in \Sigma \cap V$, $\pi|_y \to \mathcal{I}_{\Sigma}^\oplus \sigma \otimes \chi|_y$ and the desired result follows.

When $G$ is a product of general linear groups, the above result can be improved thanks to the theory of extended Bernstein varieties.

**Proposition 4.21.** Assume $G$ is a product of general linear groups and $E = \tilde{E}$. Let $\pi$ be a finitely generated smooth admissible torsion-free $R[G(F)]$-module such that $\pi|_x$ is irreducible and tempered at any $x \in \Sigma$. Then the discrete support of $\pi$ is rigid around each $x \in \Sigma$.

**Proof.** We will briefly indicate how to deduce this result from the theory of co-Whittaker modules and extended Bernstein varieties summarized in [12]. We will use the notations in loc.cit. Note that $\pi|_x$ is generic at each $x \in \Sigma$. Assume $G = \prod_i G_i$ with $G_i = \text{GL}_{n_i}$ for some $n_i \in \mathbb{N}$. Let $\pi_i$ be the top derivative of $\pi$ with respect to $\prod_{j \neq i} G_j$. Then by [15, Lemma 3.1.5], $\pi_i$ is a finitely generated smooth admissible torsion-free $O_X[G_i]$-module for each $i$. By [12, Lemma 4.2.1.4.2.3.4.2.5], up to shrinking $\text{Spec}(R)$ to an open subset containing $\Sigma$, $\pi$ and $\pi_i$ are all co-Whittaker. Moreover by [12, Lemma 4.2.6], one has $\pi = \bigotimes_{i=1}^n \pi_i$, up to further shrinking the open subset. Thus we are reduced to the case $G = \text{GL}_{n_i}$, which we assume below.

By [12, Lemma 4.2.5], $\text{End}_{R[G(F)]}(\pi) = R$. For each generic point $\eta \in \text{Spec}(R)$, $\pi|_{\eta}$ is absolutely irreducible by Lemma 2.1 and thus determines an inertial type $[s]$ and a multi-partition $[t]$ by [12, Section 2.2]. Then by [12, Section 3.3], up to replacing $\text{Spec}(R)$ by its connected component containing $x$, there is a map $\alpha$ from $\text{Spec}(R)$ to the component $X'[s,[t]]$ of the extended Bernstein variety. Let $\mathfrak{N}$ be the torsion-free co-Whittaker module, whose construction is explained in the proof of [12, Lemma 5.2.2], over $X'[s,[t]]$. Now by [12, Lemma 4.2.6], $\alpha^* \mathfrak{N} \cong \pi$ on an open neighborhood of $x \in \Sigma$.

By the construction of $\mathfrak{N}$ and the classification results of $\text{GL}_{n_i}(F)$-representations, we are done. 

The most important source of spherical varieties admitting reduction structures is strongly tempered spherical varieties without type $N$-roots.

**Theorem 4.22.** For $G$ split, all strongly tempered spherical $G$-varieties without type $N$-roots admit reduction structures.

**Proof.** According to [36, Section 1.1], a strongly tempered spherical pair without type $N$-roots for $G$-split is essentially one of the following

$$\text{GL}_n \subset \text{GL}_{n+1} \times \text{GL}_n; \quad \text{SO}_n \subset \text{SO}_{n+1} \times \text{SO}_n; \quad \Delta_{G_n} \setminus \text{GL}_2 \subset \Delta_{G_n} \setminus \text{GL}_4 \times \text{GL}_2; \quad \text{Sp}_4 \times \text{Sp}_2 \subset \text{Sp}_6 \times \text{Sp}_4;$$

or the pair $G_n \subset \text{PGL}_2$, which is already dealt with in Example 4.10. Before analyzing the remaining four pairs case by case, we explain the rough idea to produce a reduction structure for $\Theta_H \subset \Delta_H$ and $\ast = +, 0, -$ (resp. $\ast = 0$) if $2C_H = 1$ (resp. $C_H = \emptyset$):
• firstly, choose elements $w \in W$ such that $A^\leftarrow_{\Theta_H}$ can be decomposed into a disjoint union $\bigsqcup_w A^-_{\Theta_H,w}$ where $A^-_{\Theta_H,w} \neq \emptyset$ and $w^{-1}A^-_{\Theta_H,w} \subset A^-_{\Theta_G}$. For example, one can take the single element $w = e$ if and only if $A^-_{\Theta_H,w} \subset A^-_{\Theta_G}$;

• secondly, choose several $\Theta \subset \Delta_G$ for each $w$ such that for $\epsilon > 0$ small enough, the set
  $$\bigcap_{\Theta} \left\{ a \in A^-_{\Theta_H,w} \mid \exists \alpha \in \Delta_G - \Theta, |\alpha(waw^{-1})| \geq \epsilon \right\}$$

is finite modulo $A^\circ_{\Theta_H}$. In this step, we always choose $\Theta$ as large as possible.

• lastly, choose the element $s$ for each pair $(\Theta, w)$.

(1) The $(\text{GL}_{n+1} \times \text{GL}_n, \text{GL}_n)$-case. Consider the pair

$$H := \text{GL}_n < G := \text{GL}_{n+1} \times \text{GL}_n, \quad g \mapsto (\text{diag}(g, 1), g).$$

Let $B_n$ be the Borel subgroup of $\text{GL}_n$ of upper-triangular matrices with $A_n \subset B_n$ the subgroup of diagonal matrices. Let $P_{0_G} = B_{n+1} \times B_n$, $A_{0_G} = A_{n+1} \times A_n$ and $P_{0_H} = B_n$, $A_{0_H} = A_n \subset A_{0_G}$. Note that $\Delta_{\text{GL}_n} = \{ \alpha_i | 1 \leq i \leq n-1 \}$ where for $1 \leq i \leq n-1$

$$\alpha_i : A_n \to \mathbb{G}_m; \quad (t_1, \cdots, t_n) \mapsto t_i/t_{i+1}.$$ Identify $\Delta_G$ with $\Delta_{\text{GL}_{n+1}} \cup \Delta_{\text{GL}_n}$. Let $\beta_i$ (resp. $\alpha_i$) be the $i$-th simple root of $\text{GL}_{n+1}$ (resp. $\text{GL}_n$). Let $w_i \in W_G$, $1 \leq i \leq n+1$ be the element

$$w_i(\text{diag}(t_1, \cdots, t_{i-1}, t_i, \cdots, t_{n+1}), \text{diag}(a_1, \cdots, a_n))w_i^{-1} = (\text{diag}(t_1, \cdots, t_{i-1}, t_{i+1}, \cdots, t_n), \text{diag}(a_1, \cdots, a_n))$$

Note that $w_{n+1} = e$. Let $T(i, \varpi) := \text{diag}(\varpi, \cdots, \varpi, 1, \cdots, 1) \in A_n(F)$ where the first $i$-terms are $\varpi$.

Let $C_H$ consists of the character

$$A_n \to \mathbb{G}_m; \quad (t_1, \cdots, t_n) \mapsto t_n.$$

With respect to this choice, $A^\leftarrow_{\Theta_H} \cup A^\rightarrow_{\Theta_H} \subset A^\leftarrow_{\Theta_G}$. Now we give a reduction structure $S(\Theta_H, \ast)$ for each $\Theta_H \subset \Delta_H$ and $\ast = +, 0, -$. For $\ast = 0$ and $\Theta_H \subset \Delta_H$, the set

$$S(\Theta_H, 0) = \{ (\Delta_G - \{ \beta_i, \alpha_i \}, w_{n+1}, T(i, \varpi)) | \alpha_i \in \Delta_H - \Theta_H \}$$

is a reduction structure since

• for any $0 < \epsilon \leq 1$,

$$\bigcap_{(\Theta, w, s) \in S(\Theta_H, 0)} \left\{ a \in A^\leftarrow_{\Theta_H} \mid waw^{-1} \in A^\rightarrow_{\Theta_G}, \exists \alpha \in \Delta_G - \Theta, |\alpha(waw^{-1})| \geq \epsilon \right\} = \left\{ a \in A^\leftarrow_{\Theta_H} \mid |a_i/a_{i+1}| \geq \epsilon, \alpha_i \in \Delta_H - \Theta_H \right\}$$

has finite image in $T^{-\rightarrow}_{\Theta_H}$;

• for each $(\Theta, w, s) \in S(\Theta_H, 0)$, $s \in w^{-1}A^-_{\Theta_H} \cap A^\rightarrow_{\Theta_H}$ and for any $n \in \mathbb{N}$, $T^{-\rightarrow}_{\Theta_H}(T^{-\rightarrow}_{\Theta_H})$ is a finite disjoint union of translations of $T^{-\rightarrow}_{\Theta_H \cup \{\alpha_i\}}$.

For $\ast = +$ and $\Theta_H \subset \Delta_H$, the set

$$S(\Theta_H, +) = \{ (\Delta_G - \{ \beta_i, \alpha_i \}, w_{n+1}, T(i, \varpi)) | \alpha_i \in \Delta_H - \Theta_H \}$$

is a reduction structure since

• for any $0 < \epsilon \leq 1$,

$$\bigcap_{(\Theta, w, s) \in S(\Theta_H, +)} \left\{ a \in A^\leftarrow_{\Theta_H} \mid waw^{-1} \in A^\rightarrow_{\Theta_G}, \exists \alpha \in \Delta_G - \Theta, |\alpha(waw^{-1})| \geq \epsilon \right\} = \left\{ a \in A^\leftarrow_{\Theta_H} \mid |a_i/a_{i+1}| \geq \epsilon, \alpha_i \in \Delta_H - \Theta_H, |a_n| \geq \epsilon \right\}$$

has finite image in $T^{-\rightarrow}_{\Theta_H}$;

• for each $(\Theta, w, s) \in S(\Theta_H, +)$, $s \in w^{-1}A^-_{\Theta_H} \cap A^\leftarrow_{\Theta_H}$ and for any $n \in \mathbb{N}$, $T^{-\rightarrow}_{\Theta_H}(T^{-\rightarrow}_{\Theta_H})$ is a finite disjoint union of translations of $T^{-\rightarrow}_{\Theta_H \cup \{\alpha_i\}}$ (resp. $T^{-\rightarrow}_{\Theta_H}$) if $\Theta \neq \Delta_G - \{ \beta_n \}$ (resp. $\Theta = \Delta_G - \{ \beta_n \}$).
For the case \( \ast = - \), let \( S = S(\Theta_H, -) \) be the disjoint union of the set

\[
\{(\Delta_G = \{\beta_j, \alpha_j\}, w_j, T(i, \varpi))\}_{\alpha_j \in \Delta_H - \Theta_H} \bigcup \{(\Delta_G = \{\beta_{j+1}, \alpha_j\}, w_j, \varpi^{-1}T(i, \varpi))\}_{\alpha_j \in \Delta_H - \Theta_H}
\]

where \( 2 \leq j \leq n \) and \( \alpha_{j-1} \notin \Theta_H \), and the set

\[
\{(\Delta_G = \{\beta_{j+1}, \alpha_j\}, w_1, \varpi^{-1}T(i, \varpi))\}_{\alpha_j \in \Delta_H - \Theta_H} \bigcup \{(\Delta_G = \{\beta_1\}, w_1, \varpi^{-1}T(0, \varpi))\}
\]

Here the terms with \( w = w_j \) are designed to deal with the region \( \{a \in A_{\Theta_H}^- | |a_{j-1}| \leq 1 < |a_j|\} \). The set \( S(\Theta_H, -) \) is a reduction structure since

- for any \( 0 < \epsilon \leq 1 \),

\[
\bigcup_{(\Theta, w, s) \in S} \{a \in A_{\Theta_H}^- | \text{if } waw^{-1} \in A_{\Theta_H}^-; \exists \alpha \in \Delta_G - \Theta, |\alpha(waw^{-1})| \geq \epsilon\}
\]

- for each \( (\Theta, w, s) \in S, s \in w^{-1}A_{\Theta_H}^w \cap A_{\Theta_H}^+ \) and for any \( n \in \mathbb{N} \), \( T_{\Theta_H}^n - |T_{\Theta_H}^{n-1}|(\Theta) \) is a finite disjoint union of translations of \( T_{\Theta_H}^n - \beta_1 \) (resp. \( T_{\Theta_H}^n - 0 \)) if \( \Theta \neq \Delta_G - \{\beta_1\} \) (resp. \( \Theta = \Delta_G - \{\beta_1\} \))

The \( (SO_{n+1} \times SO_n, SO_n)\)-case. Let \( V_{n+1} \) be a non-degenerate quadratic \( F \)-space of dimension \( n+1 \) and \( v \in V_{n+1} \) be an anisotropic vector. Let \( V_n = (v)^\perp \subset V_{n+1} \). Let \( SO_{n+1} = SO(V_{n+1}) \) and identify \( SO_n = SO(V_n) \) with the stabilizer of \( v \). Let \( G = SO_{n+1} \times SO_n \) and embed \( H = SO_n \) into \( G \) diagonally.

When \( n = 1 \), \( H \) is anisotropic. When \( n = 2 \), we can proceed as in Example 4.10 as \( SO(3) \cong \text{PGL}(2) \) and \( SO(2) \) is a torus. For \( n \geq 3 \), fix an orthogonal decomposition

\[
V_n = X_n \oplus W_n \oplus Y_n; \quad V_{n+1} = X_{n+1} \oplus W_{n+1} \oplus Y_{n+1}
\]

such that \( X_n \subset X_{n+1} \) and \( Y_n \subset Y_{n+1} \) are totally isotropic subspaces of the same dimension and \( W_n, W_{n+1} \) are anisotropic. By fixing bases of \( X_n \) and \( X_{n+1} \) compatibly, we obtain minimal parabolic subgroups \( P_n \subset SO_n \) and \( P_{n+1} \subset SO_{n+1} \) with maximal split tori \( A_n \) and \( A_{n+1} \) respectively such that \( P_n \subset P_{n+1} \) and \( A_n \subset A_{n+1} \). Take \( A_{\Theta} = A_{n+1} \times A_n \subset P_{\Theta} = P_{n+1} \times P_n, A_{\Theta_H} = A_n \subset P_{\Theta_H} = P_n \).

Let \( r_i := \dim_F X_i \). Then for \( i = n, n+1 \),

- the Levi factor of \( P_i \) is isomorphic to \( (F^\times)^r_i \times SO(W_i) \) and \( A_i \) is isomorphic to \( (F^\times)^{r_i} \);
- the set of simple roots \( \Delta_{SO} \) consists of characters \( \alpha^j_i : A_i \to \mathbb{G}_m; \quad (a_1, \ldots, a_r) \mapsto a_j/a_{j+1} \quad \forall 1 \leq j \leq r_i - 1 \)

with

- the character \( \alpha^j_i(a) = a_{r_j} \) if \( \dim_F W_i = 1 \);
- the character \( \alpha^j_i(a) = a_{r_j} \) if \( \dim_F W_i = 0 \).

Identify \( \Delta_G \) with \( \Delta_{SO_{n+1}, \Theta} \cap \Delta_{SO_n} \) and write \( \alpha^{j+1} \) (resp. \( \alpha^j \)) as \( \beta_j \) (resp. \( \alpha_j \)). For any \( 1 \leq j \leq r_n \), let \( T(j, \varpi) = (\varpi, \ldots, \varpi, 1, \ldots, 1) \in A_{\Theta_H}^- \) with the first \( j \)-terms being \( \varpi \).

When \( r_{n+1} = r_n + 1 \), or equivalently \( \dim_F W_n = 1 \), one has \( A_{\Theta}^- \subset A_{\Theta}^0 \). In this case, \( \Theta_H \subset \Delta_H \) admits the reduction structure

\[
\{(\Delta_G - \{\beta_j, \alpha_j\}, e, T(j, \varpi))_{\alpha_j \in \Delta_H - \Theta_H, j < r_n}, (\Delta_G - \{\beta_{r_n}, \alpha_{r_n+1}, \alpha_{r_n}\}, e, T(r_n, \varpi))_{\alpha_{r_n} \in \Delta_H - \Theta_H}\}
\]

When \( r_{n+1} = r_n \), or equivalently \( \dim_F W_n = 0 \), let \( w \in W_G \) be the element such that

\[
w^{-1}(a_1, \ldots, a_{r_n-1}, a_{r_n}, b_1, \ldots, b_{r_n}) w = ((a_1, \ldots, a_{r_n-1}, a_{r_n}^{-1}), (b_1, \ldots, b_{r_n}))
\]

Then for \( \Theta_H \subset \Delta_H \),

- if \( \alpha_{r_n-1} \in \Theta_H \), then \( A_{\Theta_H}^- \subset A_{\Theta_H}^0 \) and \( \Theta_H \) has the reduction structure

\[
\{(\Delta_G - \{\beta_j, \alpha_j\}, e, T(j, \varpi))_{\alpha_j \in \Delta_H - \Theta_H}\}
\]
• If $\alpha_{r_n-1} \notin \Theta_H$ and $\alpha_r \in \Theta_H$, then $w^{-1}A_{\Theta_H}^+ w \subset A_{\Theta_H}^-$ and $\Theta_H$ has the reduction structure

$$\{(\Delta_G - \{\beta, j\}, w, T(j, \omega))_{\alpha_j \in \Delta_H - \Theta_H, j \leq r_n-2}, (\Delta_G - \{\beta, r_{r_n-1}\}, w, T(r_n - 1, 1))\}$$

where $T(r_n - 1, 1) = (\omega, \cdots, \omega, \omega^{-1}) \in A_{\Theta_H}^+$ with the first $(r_n - 1)$-terms being $\omega$,

• If $\alpha_{r_n-1} \notin \Theta_H$ and $\alpha_r \notin \Theta_H$, then $S(\Theta_H)$ admits the reduction structure

$$\{(\Delta_G - \{\beta, j\}, e, T(j, \omega)), (\Delta_G - \{\beta\, j\}, w, T(j, \omega))_{\alpha_j \in \Delta_H - \Theta_H, j \leq r_n-2}$$

$$\{(\Delta_G - \{\beta, r_{r_n-1} - \{\alpha_r, \alpha_{r_n-1}\}, e, T(r_n - 1, \omega)), (\Delta_G - \{\beta, r_{r_n-1} - \{\alpha_r, \alpha_{r_n-1}\}, w, T(r_n - 1, \omega))\}$$

$$(\Delta_G - \{\beta, r_{r_n-1} - \{\alpha_r, \alpha_{r_n-1}\}, e, T(r_n, \omega)), (\Delta_G - \{\beta, r_{r_n-1} - \{\alpha_r, \alpha_{r_n-1}\}, w, T(r_n - 1, \omega))\}$$

(3) The $\Delta G_m \setminus GL_2 \times GL_2 \subset \Delta G_m \setminus GL_4 \times GL_2$-case. For the pair

$$(H := \Delta G_m \setminus GL_2 \times GL_2, G := \Delta G_m \setminus GL_4 \times GL_2, \quad (g_1, g_2) \mapsto (\text{diag}\{g_1, g_2\}, g_2)),$$

take $A_{\Theta_H}^0 = \Delta G_m \setminus A_2 \times A_2 \subset P_{\Theta_H} = \Delta G_m \setminus B_4 \times B_2$ and $A_{\Theta_H}^+ = \Delta G_m \setminus A_2 \times A_2 \subset P_{\Theta_H} = \Delta G_m \setminus B_2 \times B_2$. Identify $\Delta_G$ with $\Delta GL_4 \sqcup \Delta GL_4$ and $\Delta_H$ with $\Delta GL_2 \sqcup \Delta GL_2$. For $i = 1, 2, 3$, let $\beta_i \in \Delta GL_4 \subset \Delta_G$ be the $i$-th simple root. Let $\alpha$ (resp. $\alpha_i$, $i = 1, 2$) be the unique simple root in $\Delta GL_2 \subset \Delta_G$ (the $i$-th copy of $\Delta GL_2$ in $\Delta_H$). For any permutation $abcd$ of 1234, let $w_{abcd}$ be the element such that

$$w_{abcd}((t_1, t_2, t_3, t_4), (a_1, a_2)) w_{abcd}^{-1} = ((t_1, t_6, t_c, t_d), (a_1, a_2)).$$

Denote by $a_j \in A_2(F)$, $i = 0, 1, 2$ with

$$a_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad a_1 = \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}, \quad a_2 = \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}$$

and $s_{ij} = (a_i, a_j) \in A_{\Theta_H}^+(F), 0 \leq i, j \leq 2$. Let $C_H$ consist of the character

$$\mathbb{G}_m \setminus A_2 \times A_2 \to \mathbb{G}_m, \quad ((t_1, t_2), (t_3, t_4)) \mapsto t_2/t_3.$$

With respect to this choice, $A_{\Theta_H}^{-0} \sqcup A_{\Theta_H}^{-+} \subset A_{\Theta_H}^\pm$. Then for any $\Theta_H \subset \Delta_H$ and $* = +, 0, -$ one can choose the following reduction structure: the terms with $w = w_{abcd}$ are designed to deal with the region $\{(t_1, t_2), (t_3, t_4)\} \in A_{\Theta_H}^{-*} | |t_1| \leq |t_6| \leq |t_c| \leq |t_d| \}$

| $\Theta_H$ | Members in $S(\Theta_H, 0)$ | Members in $S(\Theta_H, +)$ |
| --- | --- | --- |
| $\Delta_H$ | $(\Delta_G - \{\beta\}, w_{1234}, s_{20})$ | $(\Delta_G - \{\beta\}, w_{1234}, s_{20})$ |
| $\{\alpha_1\}$ | $(\Delta_G - \{\beta_1, \alpha\}, w_{1234}, s_{21})$ | $(\Delta_G - \{\beta_1, \alpha\}, w_{1234}, s_{21})$ |
| $\{\alpha_2\}$ | $(\Delta_G - \{\beta_1\}, w_{1234}, s_{10})$ | $(\Delta_G - \{\beta_1\}, w_{1234}, s_{10})$ |
| $\emptyset_H$ | $(\Delta_G - \{\beta_1\}, w_{1234}, s_{10})$ | $(\Delta_G - \{\beta_1\}, w_{1234}, s_{10})$ |

(4) The $Sp_4 \times Sp_2 \subset Sp_6 \times Sp_4$-case. Let $w_n \in GL_n(F)$ be the anti-diagonal matrix with entries 1. For $n = 2m$, define the split symplectic group

$$Sp_n = \{g \in GL_{2m} | gJ_{2m}g = J_{2m} \} \quad J_{2m} = \begin{pmatrix} 0 & -w_m \\ w_m & 0 \end{pmatrix}.$$
Table 2. Reduction Structure for $\Delta G_m \backslash GL_2 \times GL_2 \subset \Delta G_m \backslash GL_4 \times GL_2$, $\ast = -$ 

| $\Theta_H$ | Members in $S(\Theta_H)$ (each line with the same Weyl element) |
|-----------|------------------------------------------------------------------|
| $\Delta_H$ | $(\Delta G - \{\beta_2\}, w_{3412}, s_{02})$                     |
| $\{\alpha_1\}$ | $(\Delta G - \{\beta_1, \alpha\}, w_{3124}, s_{01}), (\Delta G - \{\beta_3, \alpha\}, w_{3124}, s_{21})$ |
| $\{\alpha_2\}$ | $(\Delta G - \{\beta_1, \alpha\}, w_{3142}, s_{10}), (\Delta G - \{\beta_3, \alpha\}, w_{3142}, s_{12})$ |
| $\Theta_H$ | $(\Delta G - \{\beta_1, \alpha\}, w_{3124}, s_{01}), (\Delta G - \{\beta_3, \alpha\}, w_{3124}, s_{21})$ |

For the pair

$$H := Sp_4 \times Sp_2 \leftrightarrow G := Sp_6 \times Sp_4,$$

$$\left( h_1, h_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \mapsto \left( \begin{pmatrix} a & 0 & b \\ 0 & h_1 & 0 \\ c & 0 & d \end{pmatrix}, h_1 \right),$$

take $A_{\Theta_H} = A_4 \times A_2 \subset P_{\Theta_H} = B_4 \times B_2$ and $A_{\Theta_H} = A_6 \times A_4 \subset P_{\Theta_H} = B_6 \times B_4$. Identify $\Delta_H$ with $\Delta_{Sp_4} \cup \Delta_{Sp_2}$ and $\Delta_G$ with $\Delta_{Sp_6} \cup \Delta_{Sp_4}$. Denote by $\gamma_i$ (resp. $\beta_1$, resp. $\alpha_i$) the $i$-th simple root of $A_6$ (resp. $A_4$, resp. $A_2$). Let $w_1 = e$ and $w_2, w_3 \in W_G$ such that

$$w_2((t_1, t_2, t_3), (a_1, a_2))w_2^{-1} = ((t_2, t_1, t_3), (a_1, a_2)),$$

$$w_3((t_1, t_2, t_3), (a_1, a_2))w_3^{-1} = ((t_2, t_3, t_1), (a_1, a_2)).$$

Denote by $a_i \in A_4(F)$, $i = 0, 1, 2$ with $a_0 = (1, 1), a_1 = (\varpi, 1), a_2 = (\varpi, \varpi)$ and $s_{ij} = (a_i, \varpi^j) \in A_{\Theta_H}(F), i = 0, 1$ and $j = 0, 1$. Then for any $\Theta_H \subset \Delta_H$, one can choose the following reduction structure: the terms with $w = w_i, i = 1, 2, 3$ are designed to deal with the region

$$\{(t_1, t_2, t_3) \in A_{\Theta_H} \mid |t_3| \leq |t_1| \leq |t_2| \leq 1, \text{ resp. } |t_1| \leq |t_3| \leq |t_2| \leq 1, \text{ resp. } |t_1| \leq |t_2| \leq |t_3| \leq 1\}$$

Table 3. Reduction Structure for $Sp_4 \times Sp_2 \subset Sp_6 \times Sp_4$

| $\Theta_H$ | Members in $S(\Theta_H)$ (each line with the same Weyl element) |
|-----------|------------------------------------------------------------------|
| $\Theta_H \supset \{\alpha_1\}$ | $(\Delta G - \{\gamma_i, \beta_i\}, w_3, s_{0})$ with $i$ satisfying $\beta_i \in \Delta_H - \Theta_H$ |
| $\{\beta_1, \beta_2\}$ | $(\Delta G - \{\gamma_1\}, e, s_{01})$ |
| $\{\beta_1\}$ | $(\Delta G - \{\gamma_1\}, e, s_{01}), (\Delta G - \gamma_3, \beta_2), e, s_{21})$ |
| $\{\beta_2\}$ | $(\Delta G - \{\gamma_1\}, e, s_{01}), (\Delta G - \gamma_2, \beta_1), e, s_{11})$ |
| $\Theta_H$ | $(\Delta G - \{\gamma_1\}, e, s_{01}), (\Delta G - \{\gamma_2, \beta_1\}, e, s_{21}), (\Delta G - \{\gamma_3, \beta_1\}, e, s_{21})$ |

Proposition 4.22 deals with split spherical varieties. For non-split ones, we record the following:

**Proposition 4.23.** Let $K/F$ be a quadratic field extension. Let $V_{n+1}$ be a non-degenerate Hermitian space with respect to $K/F$ of dimension $n + 1$. Let $U_{n+1} = U(V_{n+1})$ be the unitary group of $V_{n+1}$. Let $v \in V_{n+1}$ be an anisotropic vector and $V_n = \langle v \rangle^\perp \subset V_{n+1}$. Identify $U_n = U(V_n)$ with the stabilizer of
v. Let $G = U_{n+1} \times U_n$ and embed $H = U_n$ into $G$ diagonally. Then all the statements in Theorem 1.2 holds for the strongly tempered spherical pair $(G, H)$.

Proof. One can show that $(G, H)$ admits a reduction structure similarly as the case $(SO_{n+1} \times SO_n, SO_n)$.

 Declarations On behalf of all authors, the corresponding author states that there is no conflict of interest.

 Statements Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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