On $[A, A]/[A, [A, A]]$ and on a $W_n$-action on the consecutive commutators of free associative algebra

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Abstract

We consider the lower central filtration of the free associative algebra $A_n$ with $n$ generators as a Lie algebra. We consider the associated graded Lie algebra. It is shown that this Lie algebra has a huge center which belongs to the cyclic words, and on the quotient Lie algebra by the center there acts the Lie algebra $W_n$ of polynomial vector fields on $\mathbb{C}^n$. We compute the space $[A_n, A_n]/[A_n, [A_n, A_n]]$ and show that it is isomorphic to the space $\Omega^2_{\text{closed}}(\mathbb{C}^n) \oplus \Omega^4_{\text{closed}}(\mathbb{C}^n) \oplus \Omega^6_{\text{closed}}(\mathbb{C}^n) \oplus \ldots$.

Introduction

Let $A$ be an associative algebra. A free resolution $R^*$ of $A$ is a free graded differential algebra $R^* = \bigoplus R^i, i \in \mathbb{Z}_{\leq 0}$, with differential $Q$ has degree +1 such that the cohomology of $Q$ is only in degree zero, and is canonically isomorphic to $A$ as algebra. Such a resolution can be used for calculation of ”higher derived functors” for $A$. For example, higher cyclic homology of $A$ is the higher derived functor for the functor

$$A \to A/[A, A].$$

It means that for the calculation of cyclic homology of $A$ we have to take an arbitrary free resolution $R^*$ of $A$ and consider the quotient $R^*/[R^*, R^*]$. The differential $Q$ acts in $R^*/[R^*, R^*]$, and cohomology of $Q$ is the higher derived functor of $A \to A/[A, A]$.

It is natural to try to calculate ”higher derived” for other functors. Surely there is a lot of interesting and important functors, but unfortunately in the most cases it is rather hard to calculate the higher derived functors. The cyclic homology is an exception, because it can be expressed in the terms of usual Hochschild homology. Another relatively simple case is the functor of abelianization

$$A \to A^{ab} \simeq A/J,$$
where $J$ is the two-sided ideal generated by the brackets $[a, b]$, $a, b \in A$. Let $A$ be the polynomial algebra $\mathbb{C}[x_1, \ldots, x_n]$. A resolution $\mathcal{R}^\bullet$ of $A$ can be constructed in terms of the dual Grassmannian algebra $\Lambda^\bullet(\xi_1, \ldots, \xi_n)$. Let $A_+^\vee$ be the kernel of the augmentation map

$$\Lambda^\bullet(\xi_1, \ldots, \xi_n) \to \mathbb{C}.$$  

Then $A_+^\vee$ is a graded algebra and the resolution $\mathcal{R}^\bullet$ of $A$ is a free graded algebra generated by the dual shifted space $(A_+^\vee)^*[-1]$. The differential in $\mathcal{R}^\bullet$ is given by the coproduct

$$(A_+^\vee)^* \to (A_+^\vee)^* \otimes (A_+^\vee)^*.$$  

On the quotient

$$\mathcal{R}^\bullet / \mathcal{R}^\bullet[\mathcal{R}^\bullet, \mathcal{R}^\bullet] \mathcal{R}^\bullet$$

the differential acts by zero. Therefore the "higher abelianization" of $\mathbb{C}[x_1, \ldots, x_n]$ is an algebra of functions on super vector space $A_+^\vee[-1]$ (for example, the "higher abelianization" of $\mathbb{C}[x_1, x_2]$ is the algebra $\mathbb{C}[u_1, u_2, u_{1,2}]$, deg $u_i = 0$, deg $u_{1,2} = -1$).

In this paper we study the functor

$$A \to A/[A, [A, A]].$$

In order to say something about the higher functors we need to know what is the quotient $A/[A, [A, A]]$ for a free algebra $A$. The related question is to determine the higher functors for

$$A \to A/A[A, [A, A]]A$$

Our first result is an explicit calculation of the last quotient for the free algebra $T(V)$ of an $n$-dimensional vector space $V$; we also denote this algebra by $A_n$. Let

$$\Omega^\bullet = S^\bullet(V) \otimes \Lambda^\bullet(V)$$

be the de Rham complex of the polynomial differential forms on the dual space $V^\ast$. Differential $d$ on $\Omega^\bullet$ determines the bivector field $d \wedge d = \nu$. Note that

$$d^2 = \frac{1}{2} [d, d] = 0,$$

therefore $[\nu, \nu] = 0$ and $\Omega^\bullet$ with the bracket

$$[\omega_1, \omega_2] = (-1)^{\deg \omega_1} \cdot 2d \omega_1 \wedge d \omega_2$$

is a $\mathbb{Z}_2$-graded Poisson algebra. A quantization of $(\Omega, \nu)$ may be given by the very simple formula:

$$\omega_1 \ast \omega_2 = \omega_1 \wedge \omega_2 + (-1)^{\deg \omega_1} d \omega_1 \wedge d \omega_2.$$  

Denote the result of quantization by $\Omega(V)_\ast$. Then $\Omega(V)_\ast$ is a $\mathbb{Z}_2$-graded associative algebra and the algebra of even forms with the quantized product $\Omega^{\text{even}}(V)_\ast$ is a subalgebra of $\Omega(V)_\ast$ generated by the space $V \hookrightarrow \Omega_0(V)$.  

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**Proposition.** The quotient algebra

\[ B = A_n/A_n[A_n, [A_n, A_n]]A_n \]

is isomorphic to \( \Omega^{\text{even}}(V)_* \). The map \( B \to \Omega^{\text{even}}(V)_* \) restricted to the space of generators \( V \) is the identity map.

The second commutator

\[ [\Omega^{\text{even}}(V)_*, [\Omega^{\text{even}}(V)_*, \Omega^{\text{even}}(V)_*]] \]

vanishes, so the natural homomorphism \( A_n \to \Omega^{\text{even}}(V)_* \) induces a map

\[ \theta : [A_n, A_n]/[A_n, [A_n, A_n]] \to [\Omega^{\text{even}}(V)_*, \Omega^{\text{even}}(V)_*]. \]

It is easy to see that \( [\Omega^{\text{even}}(V)_*, \Omega^{\text{even}}(V)_*] \) coincides with the space of exact \((=\text{closed of degree } > 0)\) even forms. We prove the map \( \theta \) is an isomorphism

\[ \theta : [A_n, A_n]/[A_n, [A_n, A_n]] \to \Omega^{\text{even}>0}_*(V). \]

This result is equivalent to the fact that

\[ [A_n, [A_n, A_n]] = [A_n, A_n] \cap (A_n \cdot [A_n, [A_n, A_n]] \cdot A_n) \]

Let us summarize. We consider the following functors on the category of associative algebras:

\[ F_1 : B \to B/[B, B], \]
\[ F_2 : B \to B/[B, [B, B]], \]
\[ G_1 : B \to B/B[B, B][B, \]
\[ G_2 : B \to B/B[B, [B, B]]B, \]
\[ F_{1,2} : B \to [B, B]/[[B, B], B], \]
\[ G_{1,2} : B \to B/(B[B, [B, B]]B + [B, B]) \]

Higher derived functors for \( F_1, F_2, G_1, G_2, F_{1,2}, G_{1,2} \) can be found without a lot of problems when we know the higher abelianization and cyclic homology.

For example, let \( B = \mathbb{C}[x_1, x_2] \). We know that a free resolution of \( \mathbb{C}[x_1, x_2] \) is the algebra generated by \( u_1, u_2 \) and \( u_{1,2} \). Differential \( Q \) is defined by

\[ Q(u_i) = 0, \quad Q(u_{1,2}) = u_1u_2 - u_2u_1. \]

Functor \( G_2 \) applied to the resolution gives us the even part of the algebra of forms as a superspace:

\[ \mathbb{C}[u_1, u_2, u_{1,2}; du_1, du_2, du_{1,2}]. \]

The differential \( Q \) acts nontrivially only on \( u_{1,2} \): \( Qu_{1,2} = du_1 \wedge du_2 \). So the higher derived functor for the functor \( G_2 \) is just the cohomology of this differential.
Now consider the exact sequence of functors

\[ 0 \to F_{1,2} \to F_2 \to F_1 \to 0. \]

Using it we can find "higher derived" for \( F_2 \).

In the beginning of this work we tried to analyze the numerical results on the dimensions \( A_{n,k}^{\ell} \) of graded components for algebra \( A_n \) for \( n = 2, 3 \) and small \( k, \ell \), obtained by Eric Rains on MAGMA (see (17) below). Here \( A_{n,k}^{\ell} \) is the quotient space of \( k \)-commutators of \( A_n \) modulo \( k + 1 \)-commutators, and \( A_{n,k}^{\ell,1} \) is the component of \( A_{n,k}^{\ell} \) of monomials of the length \( \ell \). For example, \( A_{n,1} = A_n/[A_n, A_n] \) is the space of cyclic words on \( n \) variables, and the dimensions \( A_{n,k}^{\ell,1} \) grow exponentially as \( \ell \) tends to \( \infty \). Our first observation was an unexpected phenomena that the spaces \( A_{n,k}^{\ell,1} \) for \( k \geq 2 \) grow polynomially on \( \ell \). We saw it from (17) for small \( k, \ell \) and \( n = 2, 3 \). In general, it is a conjecture till now.

Then, if they grow polynomially, we tried to think about them as about some tensor fields, more precisely, (maybe not irreducible) \( W_n \)-modules. The present paper is the result of our attempt to understand these two phenomena—the polynomial growth of \( A_{n,k}^{\ell,1} \), and a structure of \( W_n \)-module on it.

Another strange thing which appeared from our results is that the space \([A_n, A_n]/[A_n, [A_n, A_n]]\) is an associative algebra. The algebra structure is very unclear from this definition, but it follows from the description of the last space as \( \Omega_{\text{closed}}(V) \). It is a commutative algebra with the usual wedge product of differential forms.

At the moment we can not find such a theory for higher \( A_{n,k}, k > 2 \). We have some conjectures which hopefully will be published somewhere.

Now let us outline the contents of the paper:

In Section 1 we prove "by hands" that 
\[ [A_n, A_n]/[A_n, [A_n, A_n]] \simeq \Omega^2_{\text{closed}}(\mathbb{C}^n) \]
for \( n = 2, 3 \).

In Section 2 we develop our main techniques, aimed to find \( A_n/(A_n[A_n, [A_n, A_n]]A_n) \) and \([A_n, A_n]/[A_n, [A_n, A_n]]\) for general \( n \), we prove our results here modulo Lemma 2.2.2.2 which is proven in Section 3. To prove this Lemma in Section 3 we use a theorem describing all irreducible \( W_n \)-modules of a reasonable class.

In Section 4 we define a \( W_n \)-action on the quotient (by the center) \( \tilde{g}r(A_n) \) of the associated graded Lie algebra of \( A_n \) with respect to the lower central filtration. We also consider many examples here.

1 The isomorphism \([A_n, A_n]/[A_n, [A_n, A_n]] \simeq \Omega^2_{\text{closed}}(\mathbb{C}^n)\) for 
\( n = 2, 3 \)

In this Section we compute "by hands" the quotient \([A_n, A_n]/[A_n, [A_n, A_n]]\) for \( n = 2 \) and \( n = 3 \). To make the exposition more clear, we first define the concept of a non-commutative 1-form.
1.1 Non-commutative 1-forms

Let $A$ be an associative algebra. A 1-form on $A$ is a finite sum of the expressions $a \cdot db \cdot c$, where $a, b, c \in A$ modulo the following two relations:

(i) the cyclicity: $t \cdot a \cdot db \cdot c = a \cdot db \cdot c \cdot t$ for any $a, b, c, t \in A$,

(2) the Leibniz rule: $d(a \cdot b) = (da) \cdot b + a \cdot db$.

We say that a 1-form on $A$ is exact if it has the form $\omega = da$, $a \in A$.

Consider the space $\Omega^1_A/d\Omega^0_A$ of 1-forms on $A$ modulo the exact 1-forms. We can reduce any 1-form to an expression $a \cdot db$ using the cyclicity. Next, modulo the exact forms $a \cdot db + b \cdot da = 0$. Thus, there is a map of the space $\Lambda^2(A) \rightarrow \Omega^1_A/d\Omega^0_A$ which is clearly surjective. What is its kernel?

We have the relation:

\[ a_1 d(a_2 a_3) = a_1 d(a_2) a_3 + a_1 a_2 d(a_3) = a_3 a_1 d(a_2) + a_1 a_2 d(a_3) \]

which is

\[ a_1 \land (a_2 a_3) = (a_3 a_1) \land a_2 + (a_1 a_2) \land a_3 \quad (1) \]

or, in more symmetric form,

\[ (a_1 a_2) \land a_3 + (a_2 a_3) \land a_1 + (a_3 a_1) \land a_2 = 0 \quad (2) \]

It is clear that there are no other relations. We proved the following result:

Lemma. For any associative algebra $A$, the space $\Omega^1_A/d\Omega^0_A$ is isomorphic to $\Lambda^2(A)/(relations(2))$.

\[ \square \]

1.2 A Lemma

In the case when $A = A_n$, the free associative algebra with $n$ generators over $\mathbb{C}$, the space $\Lambda^2(A_n)/(relations(2))$ is isomorphic to the commutator $[A_n, A_n]$. We have the following Lemma:

Lemma. (i) $\Lambda^2(A_n)/(relations(2)) = [A_n, A_n]$,

(ii) $\Lambda^2(A_n/[A_n, A_n])/(relations \ (2)) = [A_n, A_n]/[A_n, [A_n, A_n]]$

Proof. For any associative algebra $A$, we have the short exact sequence:

\[ 0 \rightarrow HC_1(A) \rightarrow (\Lambda^2 A) / (relations \ [2]) \rightarrow [A, A] \rightarrow 0 \quad (3) \]

where the last map is the commutator map: $a \land b \mapsto [a, b]$. The correctness of this map follows from the following relation for any associative algebra $A$:

\[ [a, bc] + [b, ca] + [c, ab] = 0 \]
The kernel of this map is the first cyclic homology $HC^1(A)$. Now the statement (i) of lemma follows from the fact that $HC^1(A_n) = 0$.

Consider $A$ as a Lie algebra with the bracket $[a, b] = a \cdot b - b \cdot a$. Then the short exact sequence \([\mathfrak{g}]\) is a sequence of $A$-modules, and the action of $A$ on $HC^1(A)$ is trivial. Consider the corresponding long exact sequence in Lie algebra homology:

$$\cdots \to HC_1(A) \to \left( \Lambda^2(A/[A, A]) \right) / (a \wedge (b \cdot c) + b \wedge (c \cdot a) + c \wedge (a \cdot b) = 0) \to [A, A]/[A, [A, A]] \to 0$$

One should explain the middle term. It is not true that $\Lambda^2(A)/[A, \Lambda^2(A)]$ is $\Lambda^2(A/[A, A])$, and we should use the identity $a \wedge (b \cdot c) + b \wedge (c \cdot a) + c \wedge (a \cdot b) = 0$. We have by this identity:

$$[a, b \wedge c] = [a, b] \wedge c + b \wedge [a, c] = a \wedge [b, c]$$

which explains the middle term. Again, the statement (ii) of lemma follows from the fact that $HC^1(A_n) = 0$. \qed

**Remark.** The definition of a $k$-form on an associative algebra $A$, generalizing the definition of 1-form here, is given in [K2]. It is proven there that the de Rham complex of the algebra $A_n$ obeys the Poincare lemma. It is also proven that the space of closed two-forms on $A_n$ is $[A_n, A_n]$. From this point of view, the statement (i) of the Lemma above is the Poincare lemma at the first term.

### 1.3 The case $n = 2$

We denote $A_{n,2} = [A_n, A_n]/[A_n, [A_n, A_n]]$, and we denote by $A^\ell_{n,2}$ the graded component in $A_{n,2}$ consisting from the monomials of the length $\ell$. We prove here the following theorem:

**Theorem.** $\dim A^\ell_{2,2} = \ell - 1$

**Proof.** By Lemma 1.2 we know that $[A_n, A_n]/[A_n, [A_n, A_n]] \simeq \left( \Lambda^2(A_n/[A_n, A_n]) \right) / \text{(relations \[\mathfrak{g}\])}$. The idea is to show that any element in $\left( \Lambda^2(A_2/[A_2, A_2]) \right) / \text{(relations \[\mathfrak{g}\])}$ is equivalent to an element of the form $x_1^k \wedge x_2^m$, where $k, m \geq 1$.

On the other hand, the space $\Omega^2_{\text{closed}}(\mathbb{C}^2)$ of closed 2-forms of "length" $\ell$ (the length is the eigenvalue of the operator $\text{Lie}_e$ where $e = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}$ is the Euler vector field) on the 2-dimensional vector space has also dimension $\ell - 1$. Denote by $\tilde{m}$ the image of a monomial $m \in A_2$ to the commutative algebra $\mathbb{C}[x_1, x_2] = A_2/A_2[A_2, A_2]A_2$. Then we have the map $[m_1, m_2] \mapsto d(\tilde{m}_1) \wedge d(\tilde{m}_2)$, which clearly defines a map $[A_2, A_2]/[A_2, [A_2, A_2]] \to \Omega^2_{\text{closed}}(\mathbb{C}^2)$. We want to prove that this map is an isomorphism.

We start with the isomorphism $\Omega^1_{A_2}/d\Omega^0_{A_2} \simeq [A_2, A_2]$. Now we consider the space of non-commutative 1-forms $\text{adb}$ modulo exact forms and modulo forms of the types
using the equation (4), we see that it is the same that
\[ \ell \]
property (4) we see that
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of the corresponding closed 2-forms on
\[ n \]
Indeed, \( (dx_1) a(x_1, x_2) \) and
\[ d(x_2) b(x_1, x_2) \]
where \( a \) and \( b \) are non-commutative monomials. Consider the form
\( (dx_1) a(x_1, x_2) \). Suppose \( a = a_1 \cdot a_2 \), then in our quotient space
\[ (dx_1) \cdot a_1 \cdot a_2 = (dx_1) \cdot a_2 \cdot a_1 \] \hspace{1cm} (4)
Using this operation, we can suppose that \( a(x_1, x_2) \) has form \( a = x_1^k \cdot a_1 \) where \( a_1 \) starts
with \( x_2 \) (or is equal to 1). We have in our quotient space:
\[ (dx_1) \cdot x_1^k \cdot a_1 = \frac{1}{k+1} (dx_1^{k+1}) \cdot a_1 \] \hspace{1cm} (5)
Indeed, \( (dx_1^{k+1}) \cdot a_1 = \sum_{i=0}^{k} x_i \cdot dx_1 \cdot x_1^{k-i} \cdot a_1 \). Consider a summand \( x_i \cdot dx_1 \cdot x_1^{k-i} \cdot a_1 \).
Using the cyclic symmetry of 1-forms, the latter is the same that \( dx_1 \cdot x_1^{k-i} \cdot a_1 \cdot x_i \). Now
using the equation (4), we see that it is the same that \( dx_1 \cdot x_1^k \cdot a_1 \). Equation (5) is proven.

Now we proceed in the same way: we represent \( a_1 \) as \( a_1 = x_2^m \cdot x_1^n \cdot a_2 \). Using the
property (4) we see that \( dx_1 \cdot x_2^m \cdot x_1^n \cdot a_2 = dx_1 \cdot x_1^n \cdot a_2 \cdot x_2^m = \frac{1}{n} d(x_1^{k+1+n}) \cdot (a_2 \cdot x_2^m) \),
and so on. Finally, we obtain that \( (dx_1) \cdot a(x_1, x_2) \) is equivalent to a form of the type
\( d(x_1^n) \cdot x_2^m \), which proves the theorem.

1.4 The case \( n = 3 \)

Theorem. The dimension of the space \( A_{3,2}^\ell \) is \( \ell^2 - 1 \).

Proof. Again, it is the dimension of the polynomial closed two-forms on \( \mathbb{C}^3 \) of the length \( \ell \). Our proof is analogous to the proof in the case \( n = 2 \).

First we reduce a non-commutative 1-form to the form \( d(x_1^k) \cdot a(x_2, x_3) \). Then, modulo
exact forms, it is \( -da(x_2, x_3) \cdot x_1^k \). We can proceed as above to reduce any form to the
type \( d(x_1^{k_1}) \cdot x_2^{k_2} \cdot x_3^{k_3} \) where \( x_2^{k_2} \) and \( x_3^{k_3} \) commute (\( \{i, j, m\} = \{1, 2, 3\} \)).

We can not apply this proof for \( n > 3 \). On the other hand, computations showed that, starting from \( n = 4 \), the dimension of the space \( A_{n,2}^\ell \) is greater than the dimension of the corresponding closed 2-forms on \( \mathbb{C}^n \) (see (17) in Section 4.2). We give the answer
in the next Section.

2 The quotient \([A_n, A_n]/[A_n, [A_n, A_n]]\) for general \( n \)

Consider the following algebra structure on the (commutative) even forms on an \( n \)-dimensional vector space \( \Omega_{\text{even}}(\mathbb{C}^n) \):
\[ \omega_1 \circ \omega_2 = \omega_1 \wedge \omega_2 + d\omega_1 \wedge d\omega_2 \] \hspace{1cm} (6)
where \( d \) is the de Rham differential. Notice that this product is not commutative neither skew-commutative. Later on, we consider only this algebra structure on \( \Omega^{\text{even}} \).

**Remark.** For any (not necessarily commutative) differential graded associative algebra \( A^\bullet \) we can define a new algebra \( A^\bullet \ast \) with the product

\[
a \ast b = a \cdot b + (-1)^{\deg a}(da) \cdot (db)
\]

(\( a, b \) are homogeneous) which is also associative.

We have a map \( \varphi_n : A_n \to \Omega^{\text{even}}(\mathbb{C}^n)_* \) which maps \( x_i \in A_n \) to \( x_i \in \Omega^0(\mathbb{C}^n) \), and we extend it to \( A_n \) in the unique way to get a map of algebras.

2.1

2.1.1

**Lemma.** The map \( \varphi_n : A_n \to \Omega^{\text{even}}(\mathbb{C}^n)_* \) is surjective.

**Proof.** Prove first that any monomial on the coordinates \( \{x_i\}'s \) belongs to the image. Let \( M_1 \) and \( M_2 \) be two such monomials which belong to the image of \( \varphi_n \), we prove that \( M_1 \cdot M_2 \) also does. If \( M_1 = \varphi_n(R_1) \), and \( M_2 = \varphi_n(R_2) \), then \( M_1 \cdot M_2 = \frac{1}{2} \varphi_n(R_1 \cdot R_2 + R_2 \cdot R_1) \). It proves that any monomial on \( \{x_i\}'s \) belongs to the image because linear monomials \( x_1, \ldots, x_n \) belong to the image by definition. Analogously we prove that any even monomial on \( \{dx_i\}'s \) belongs to the image, and the general statement. \( \square \)

2.1.2

**Lemma.**

(i) The map \( \varphi_n \) maps the commutator \([A_n, A_n]\) to the closed (=exact) forms of degree \( > 0 \) \( \Omega^{\text{even+}}_{\text{closed}}(\mathbb{C}^n) \), and the map \( \varphi_n : [A_n, A_n] \to \Omega^{\text{even+}}_{\text{closed}}(\mathbb{C}^n) \) is surjective,

(ii) the triple commutator \([A_n, [A_n, A_n]]\) is mapped by \( \varphi_n \) to 0,

(iii) the kernel of the map \( \varphi_n \) is \( K_n = A_n \cdot [A_n, [A_n, A_n]] \cdot A_n \).

**Proof.**

(i): it is clear that \( [\omega_1, \omega_2] = 2d\omega_1 \wedge d\omega_2 \). Therefore, the image \([A_n, A_n]\) belongs to closed forms. Surjectivity can be proved analogously with the lemma above,

(ii) it is clear from (i),

(iii) it follows from (ii) that \( K_n \) belongs to the kernel of \( \varphi_n \), because \( \varphi_n \) is a map of associative algebras, and its kernel is a two-sided ideal. Now it is sufficiently to prove that the algebra \( A_n/K_n \) is isomorphic under \( \varphi_n \) to \( \Omega^{\text{even}}_{\text{closed}}(\mathbb{C}^n)_* \). It follows from the following presentation of the commutative algebra \( \Omega^{\text{even}}(\mathbb{C}^n) \) by generators and relations: it is generated by \( \{x_i\} \) and \( \{dx_i \wedge dx_j\} \) with the usual commutativity relations and the relation

\[
(dx_i \wedge dx_j) \cdot (dx_k \wedge dx_l) = -(dx_i \wedge dx_k) \cdot (dx_j \wedge dx_l)
\]
Therefore, $\Omega^{\text{even}}(\mathbb{C}^n)$ is a commutative algebra generated by $\{x_i\}$ and $\{\eta_{i,j}\}$, where $\eta_{i,j} = -\eta_{j,i}$ and with the relations

$$\eta_{i,j} \cdot \eta_{k,l} + \eta_{i,k} \cdot \eta_{j,l} = 0$$

(8)

Now let us consider the algebra $A_n/K_n$. It is generated by $\{x_i\}$ and $\{[x_i, x_j]\}$. We should check the relations (8), that is

$$[x_i, x_j] \cdot [x_k, x_l] + [x_i, x_k] \cdot [x_j, x_l] \in K_n$$

(9)

It follows from the following identity in the free algebra:

$$[x_i, x_j] \cdot [x_k, x_l] + [x_i, x_k] \cdot [x_j, x_l] = [x_i x_j, x_k x_l] + [x_i x_k, x_j x_l] + [x_i x_l, x_j x_k]$$

(10)

Lemma is proven. \(\square\)

2.2 The main theorem

We prove here the following theorem:

**Theorem.** The map $\varphi_n : [A_n, A_n]/[A_n, [A_n, A_n]] \to \Omega_{\text{even}+}^{\text{closed}}$.

By Lemma 2.1.2 above, the Theorem follows from the following Lemma:

**Key-Lemma.** The intersection $[A_n, A_n] \cap (A_n \cdot [A_n, [A_n, A_n]] \cdot A_n) = [A_n, [A_n, A_n]]$.

We prove this Lemma and the Theorem in the rest of this Section and in Section 3.

2.2.1

Consider the space $[A_n, A_n \cdot [A_n, [A_n, A_n]]]$. It is clear that this space belongs to the intersection $[A_n, A_n] \cap (A_n \cdot [A_n, [A_n, A_n]] \cdot A_n)$. We first prove the Key-Lemma in this particular case.

**Lemma.** The space $[A_n, A_n \cdot [A_n, [A_n, A_n]]]$ belongs to $[A_n, [A_n, A_n]]$.

**Proof.** Let $t_1, t_2, t_3, t_4, t_5$ be arbitrary monomials in $A_n$. We need to prove that

$$[t_1, t_2 \cdot [t_3, [t_4, t_5]]] \in [A_n, [A_n, A_n]]$$

(11)

We have:

$$[t_1, t_2 [t_3, [t_4, t_5]]] = [t_1, t_2 t_5 [t_4, t_5]] - [t_1, t_2 [t_4, t_5] t_3]$$

$$= [t_1, t_2 t_5 [t_4, t_5]] - [t_1, t_3 t_2 [t_4, t_5]] + [t_1, [t_3, t_2 [t_4, t_5]]]$$

(12)
Notice that the third summand in the last line belongs to \([A_n, [A_n, A_n]]\). Now we apply the identity:

\[
[a, bc] + [b, ca] + [c, ab] = 0
\]  

(13)

We set \(a = t_1, b = t_2t_3\) or \(t_3t_2\), and \(c = [t_4, t_5]\). By (13) we have:

\[
[t_1, t_2[t_3, [t_4, t_5]]] = [t_1, t_2t_3[t_4, t_5]] - [t_1, t_3t_2[t_4, t_5]] + [t_1, [t_3, t_2[t_4, t_5]]]
\]

\[
= -[t_2t_3, [t_4, t_5]t_1] - [[t_4, t_5], t_1t_2t_3]
\]

\[
+ [t_3t_2, [t_4, t_5]t_1] + [[t_4, t_5], t_1t_3t_2]
\]

\[
+ [t_1, [t_3, t_2[t_4, t_5]]]
\]

\[
= [[t_3, t_2], [t_4, t_5]t_1] - [[t_4, t_5], t_1t_2t_3] + [[t_4, t_5], t_1t_3t_2] + [t_1, [t_3, t_2[t_4, t_5]]]
\]  

(14)

### 2.2.2 The proof of the Theorem

By Lemma 1.2, we have the isomorphism \(\theta: (\Lambda^2(A_n/[A_n, A_n]))/(relations\ (2)) \xrightarrow{\sim} [A_n, A_n]/[A_n, [A_n, A_n]]\). The map \(\theta\) is induced by the map \(\theta: a \wedge b \mapsto [a, b]\).

Recall that we denote by \(K_n\) the kernel of the map of algebras \(\varphi_n: A_n \to \Omega_{\text{even}}^n(\mathbb{C}^n)_\ast\), that is, the space \(K_n = A_n \cdot [A_n, [A_n, A_n]] \cdot A_n\) by Lemma 2.1.2(iii). By Lemma 2.2.1, the bracket \([K_n, A_n] \in [A_n, [A_n, A_n]]\), and, therefore, the map \(\theta\) defines a map

\[
\theta: (\Lambda^2(A_n/(K_n + [A_n, A_n])))/(relations\ (2)) \xrightarrow{\sim} [A_n, A_n]/[A_n, [A_n, A_n]]
\]  

(15)

Here in the last formula we reduce the relation \(a \wedge bc + b \wedge ca + c \wedge ab = 0\) modulo \(K_n + [A_n, A_n]\).

2.2.2.1.

**Lemma.** The space \(A_n/(K_n + [A_n, A_n])\) is isomorphic to \(\Omega_{\text{even}}^n(\mathbb{C}^n)/\text{Imd}\), and the isomorphism is given by the map \(\varphi_n\).

**Proof.** It follows from Lemma 2.1.1 and 2.1.2 //

2.2.2.2. Now we deduce our Theorem to the following result:

**Lemma.** The space \(\Lambda^2(\Omega_{\text{even}}^n/\text{Imd})/(relations\ \omega_1 \wedge (\omega_2 \wedge \omega_3) + \omega_2 \wedge (\omega_3 \wedge \omega_1) + \omega_3 \wedge (\omega_1 \wedge \omega_2) = 0\) is isomorphic to \(\Omega_{\text{closed}}^n\), and the isomorphism is given by the formula \(\alpha \wedge \beta \mapsto da \wedge db\).

We prove Lemma 2.2.2.2 in the next Section.
3 A proof of Lemma 2.2.2.2

3.1 The irreducible $W_n$-modules of the class $C$

We need to prove that the space $\Lambda^2(\Omega^{\text{even}}/\text{Imd})/(\text{relations } \omega_1 \wedge (\omega_2 \wedge \omega_3) + \omega_2 \wedge (\omega_3 \wedge \omega_1) + \omega_3 \wedge (\omega_1 \wedge \omega_2) = 0)$ is isomorphic to $\Omega^{\text{even+}}$. The both sides are modules over the Lie algebra $W_n$ of polynomial vector fields on an $n$-dimensional vector space $\mathbb{C}^n$. We are going to define some (a very general) class $C$ of $W_n$-modules; all $W_n$-modules we meet here belong to this class.

Consider the vector field $e = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i}$. We characterize the class $C$ of $W_n$-modules $L$ by the two conditions:

(i) the operator $e$ is semisimple on $L$ with finite-dimensional eigenspaces,

(ii) the eigenvalues of $e$ are bounded from below on $L$.

We are going to describe all irreducible modules over $W_n$ of the class $C$. Let $W_n^0$ be the Lie subalgebra of $W_n$ of vector fields vanishing at the origin. Then $W_n^0$ has the subalgebra $W_n^{00}$ of vector fields vanishing at the origin with zero of at least second order. Actually $W_n^{00}$ is an ideal in $W_n^0$, and $W_n^0/W_n^{00} \cong gl_n$.

Let $D$ be a Young diagram, and let $F_D$ be the corresponding $gl_n$-module (see [Ful]). Denote by $F_D$ the coinduced module $F_D = \text{Hom}_{U(W_n^0)}(U(W_n), F_D)$.

**Theorem.** (i) The all representations $F_D$ are irreducible except the case when $D$ is just a one column, that is $F_D = \Lambda^i(\mathbb{C}^n)^*$, $F_D = \Omega^i(\mathbb{C}^n)$; in the last case $F_D$ contains the image of the de Rham differential $d\Omega^{i-1}(\mathbb{C}^n)$, which is irreducible,

(ii) the modules $F_D$ for $D$ not a 1 column, $\Omega^i(\mathbb{C}^n)/d\Omega^{i-1}(\mathbb{C}^n)$, and the trivial representation exhaust all irreducible $W_n$-modules of the class $C$.

3.2

Let $V$ be an $n$-dimensional vector space, we prefer to work in not-coordinate way. Then $\Omega^i(V)$ is coinduced from $\Lambda^i(V^*)$.

**Lemma.** The map $i: V^* \otimes (\Omega^{\text{even}}(V)/\text{Imd}) \rightarrow \Lambda^2(\Omega^{\text{even}}/\text{Imd})/(\text{relations } \omega_1 \wedge (\omega_2 \wedge \omega_3) + \omega_2 \wedge (\omega_3 \wedge \omega_1) + \omega_3 \wedge (\omega_1 \wedge \omega_2) = 0)$ is surjective. Here we consider $V^*$ as linear 0-forms, and the map $i$ is the composition of the inclusion with the subsequent factorization.

**Proof.** Denote by $\omega_1 \wedge \omega_2$ the class of $\omega_1 \wedge \omega_2$ in the quotient space $\Lambda^2(\Omega^{\text{even}}/\text{Imd})/(\text{relations } \omega_1 \wedge (\omega_2 \wedge \omega_3) + \omega_2 \wedge (\omega_3 \wedge \omega_1) + \omega_3 \wedge (\omega_1 \wedge \omega_2) = 0)$. Write $\omega_1 = x_i \wedge x_1$. We have: $x_i x_1 \wedge x_2, (x_1 \wedge x_2) \wedge x_i, x_1 \wedge x_2, (x_1 \wedge x_2) \wedge x_i, x_1 \wedge x_2, (x_1 \wedge x_2) \wedge x_i, x_1 \wedge x_2$. The second summand $(x_1 \wedge x_2) \wedge x_i$ belongs to $V^* \otimes (\Omega^{\text{even}}(V)/\text{Imd})$. Thus, modulo this image, we can freely move all $x_i$’s from $\omega_1$ to $\omega_2$. We can do it many times. Finally, instead of $\omega_1$ we will have a form without $x_i$’s, that is, a form
\( dx_i \wedge \cdots \wedge dx_i \). This form is exact, and therefore is zero in \( \Lambda^2 (\Omega^{\text{even}}/\text{Im}d) \). We are done. \( \square \)

### 3.3

Thus, we have a surjective map \( i \) of \( V \otimes \Omega^{2k}(V)/\text{Im}d \) to \( \Lambda^2 (\Omega^{\text{even}}/\text{Im}d) \) \((\text{relations } \omega_1 \wedge (\omega_2 \wedge \omega_3) + \omega_2 \wedge (\omega_3 \wedge \omega_1) + \omega_3 \wedge (\omega_1 \wedge \omega_2) = 0)\), and the latter is mapped to \( \Omega^{2k+2}_{\text{closed}}(V) \) by the formula \( \omega_1 \wedge \omega_2 \mapsto d\omega_1 \wedge d\omega_2 \). This map \( j \) is clearly surjective. We need to prove that the induced map \( (V \otimes \Omega^{2k}(V)/\text{relations}) \mapsto \Omega^{2k+2}_{\text{closed}}(V) \) is an isomorphism. Our tool is Theorem 4.1.

The main point is that \( (V \otimes \Omega^{2k}(V)/\text{Im}d) \) is not a \( W_n \)-module, only \( (V \otimes \Omega^{2k}(V)/\text{relations}) \) is. But it is still a \( \mathfrak{gl}_n \)-module. Therefore, we should use the representation theory of \( \mathfrak{gl}_n \)-modules.

First of all, we describe the \( \Omega^k(V) \) as a \( \mathfrak{gl}_n \)-module. The answer is given in Figure 1. It follows from the Littlewood-Richardson rule applied to \( \Lambda^k(V^*) \otimes S^N(V^*) \). As a

![Figure 1: The space \( \Omega^k(V) \) as a \( \mathfrak{gl}_n \)-module](image)

\( W_n \)-module, \( \Omega^k(V) \) has a submodule \( \text{Im}k = \Omega^k(V)_{\text{closed}} = d\Omega^{k-1}(V) \), and the quotient module \( \Omega^k(V)/\text{Im}k \simeq \text{Im}k+1 \) by the Poincare lemma. It is clear that as a \( \mathfrak{gl}_n \)-module,
the submodule $Im_k$ is the first line in Figure 1, while the quotient-module $\Omega^k(V)/Im_k \simeq Im_{k+1}$ is the second line.

Now we should pass from $\Omega^k(V)/d\Omega^{k-1}(V)$ to $V \otimes (\Omega^k(V)/d\Omega^{k-1}(V))$. The space $\Omega^k(V)/d\Omega^{k-1}(V)$ is the second line in Figure 1, and now we add a 1 new box to each Young diagram by the Littlewood-Richardson rule. We obtain a number of diagrams, but the only one among them will be a column (of the height $k + 2$).

Now consider the situation of Lemma 2.2.2.2 ($k$ is even, $k = 2m$). The only column of the height $2m + 2$ maps isomorphically to the corresponding component in $\Omega^{2m+2}(V)_{\text{closed}}$ under the map of Lemma 2.2.2.2. Therefore, there are no columns in the kernel of the map $i: V^* \otimes (\Omega^{\text{even}}(V)/\text{Imd}) \to \Lambda^2 (\Omega^{\text{even}}/\text{Imd})/(\text{relations } \omega_1 \wedge (\omega_2 \wedge \omega_3) + \omega_2 \wedge (\omega_3 \wedge \omega_1) + \omega_3 \wedge (\omega_1 \wedge \omega_2) = 0)$. But when we consider the map $\tilde{i}: (V^* \otimes (\Omega^{\text{even}}(V)/\text{Imd}))/\text{relations} \to \Lambda^2 (\Omega^{\text{even}}/\text{Imd})/(\text{relations } \omega_1 \wedge (\omega_2 \wedge \omega_3) + \omega_2 \wedge (\omega_3 \wedge \omega_1) + \omega_3 \wedge (\omega_1 \wedge \omega_2) = 0)$, then its kernel is smaller than the kernel of the map $i$, and is a $W_n$-module. It does not contain any column in the $gl_n$-decomposition. We know from Theorem 3.1 the description of all $W_n$-modules of the class $C$. It could not be $d\Omega^2(V)$ because it does not contain any column. Therefore, it is the coinduced module with some Young diagram $D$ which is not a column. It is clear by the Littlewood-Richardson rule that $D \otimes S^*(V^*)$ is bigger than we have from the second line of Figure 1 multiplied by $V^*$.

Lemma 2.2.2.2 is proven \qed

Theorem 2.2 and the Key-Lemma 2.2 are proven. \qed

4 A $W_n$-action on $\text{gr}A_n$

4.1 The Theorem on $W_n$-action

Consider the Lie algebra $\text{gr}A_n$. Consider its first component $A_n/[A_n, A_n]$. Consider the image under the canonical projection $p: A_n \to A_n/[A_n, A_n]$ of the kernel $K_n = A_n[A_n, [A_n, A_n]]A_n$. Denote this image by $Z$.

Lemma. $Z$ belongs to the center of the Lie algebra $\text{gr}A_n$.

Proof. It follows from Lemma 2.2.1. \qed

Denote the quotient Lie algebra $\text{gr}A_n/Z$ by $\text{gr}A_n$. It is $\mathbb{Z}_{\geq 1}$-graded Lie algebra.

Theorem. On each graded component of $\text{gr}A_n$ there acts the Lie algebra $W_n$ of polynomial vector fields on an $n$-dimensional vector space. The bracket is $W_n$-equivariant.

Proof. Consider the Lie algebra $\text{Der}(A_n)$ of the derivations of $A_n$ as of associative algebra. This Lie algebra can be easily described: a derivation of a free associative algebra is uniquely defined by its values on the generators $\{x_i\}_{i=1,\ldots,n}$, and these values may be arbitrary. This Lie algebra acts on $A_n$, and it induces the action on any quotient like $A_n/[A_n, A_n]$, $A_n/A_n[A_n, A_n]A_n$, etc. In particular, it acts on $A_n/A_n[A_n, [A_n, A_n]]A_n$. 13
The last space is isomorphic to $\Omega_{\text{even}}(\mathbb{C}^n)_*$ as algebra (considered as an algebra in degree 0) by Lemmas 2.1.1 and 2.1.2(iii). Then we have a map $\varphi: \text{Der}(A_n) \to \text{Der}(\Omega_{\text{even}}(\mathbb{C}^n)_*)$. Denote by $\Im$ the image of this map.

**Lemma.** The Lie algebra $\Im$ acts on $\tilde{gr}A_n$, and the bracket in $\tilde{gr}A_n$ is $\Im$-equivariant.

**Proof.** We need to prove that $\Im$ acts on each consecutive quotient. For this we need to prove that if we apply a derivation such that all generators $x_i$ are mapped to $A_n[A_n, [A_n, A_n]]A_n$ to a $k$-commutator in $A_n$, the image will belong to $(k+1)$-commutator. It follows (for any $k$) immediately from Lemma 2.2.1.

Now we just should construct a Lie subalgebra isomorphic to $W_n$ in $\Im$. This is the Lie algebra $W_n$ which acts in the natural way on all even forms in $\Omega_{\text{even}}(\mathbb{C}^n)_*$. It is clear that this subalgebra belongs to the image $\Im$. Indeed, each derivation of $A_n/A_n[A_n, [A_n, A_n]]A_n$ is defined by its values on the generators $\{x_i\}$. These values are defined up to $A_n/A_n[A_n, [A_n, A_n]]A_n$. Take an arbitrary lift of each value in $A_n$, we get a derivation of $A_n$. This speculation proves also that the map $\text{Der}(A_n) \to \text{Der}(A_n/A_n[A_n, [A_n, A_n]]A_n)$ is surjective, and $\Im = \text{Der}(\Omega_{\text{even}}(\mathbb{C}^n)_*)$. Consider the Lie algebra $W_n$ acting on commutative forms $\Omega_{\text{even}}(\mathbb{C}^n)$ in the natural way. Then it acts on the quantized algebra $\Omega_{\text{even}}(\mathbb{C}^n)_*$ as well. In particular, the canonical $W_n$ acts on $\tilde{gr}A_n$. □

### 4.2 Examples

**Example.** The first grading component of the Lie algebra $\tilde{gr}^1(A_n) = \Omega_{\text{even}}/\text{Im}d$. The bracket $\Lambda^2(\tilde{gr}^2(A_n)) \to \tilde{gr}^2(A_n) = [A_n, A_n]/[A_n, [A_n, A_n]] = \Omega_{\text{even+}}$ of the map $\omega_1 \wedge \omega_2 \to (d\omega_1) \wedge (d\omega_2)$ which is clearly $W_n$-equivariant.

**Example.** The following computation was made using MAGMA by Eric Rains.

Let $F_1 = A_n$, and $F_k = [A_n, F_{k-1}]$ for $k > 1$. Denote $A_{n,k} = F_k/F_{k+1}$, and denote by $A_{n,k}^\ell$ the graded component of $A_{n,k}$ consisting from the monomials of degree $\ell$. Consider the bigraded Hilbert series for $A_n$:

$$H_n = \sum_{\ell \geq 0, k \geq 1} \dim A_{n,k}^\ell u^k t^\ell$$

(16)
For \( n = 2 \) and for \( n = 3 \) the bigraded Hilbert series are:

\[
H_2(u, t) = \\
(u) \\
+ (2u)t \\
+ (3u + u^2)t^2 \\
+ (4u + 2u^2 + 2u^3)t^3 \\
+ (6u + 3u^2 + 4u^3 + 3u^4)t^4 \\
+ (8u + 4u^2 + 6u^3 + 8u^4 + 6u^5)t^5 \\
+ (14u + 5u^2 + 8u^3 + 13u^4 + 15u^5 + 9u^6)t^6 \\
+ (20u + 6u^2 + 10u^3 + 18u^4 + 26u^5 + 30u^6 + 18u^7)t^7 \\
+ (36u + 7u^2 + 12u^3 + 23u^4 + 37u^5 + 54u^6 + 57u^7 + 30u^8)t^8 \\
+ (60u + 8u^2 + 14u^3 + 28u^4 + 48u^5 + 80u^6 + 108u^7 + 110u^8 + 56u^9)t^9 \\
+ \mathcal{O}(t^{10})
\] (17)

\[
H_3(u, t) = \\
(u) \\
+ (3u)t \\
+ (6u + 3u^2)t^2 \\
+ (11u + 8u^2 + 8u^3)t^3 \\
+ (24u + 15u^2 + 24u^3 + 18u^4)t^4 \\
+ (51u + 24u^2 + 48u^3 + 72u^4 + 48u^5)t^5 \\
+ (130u + 35u^2 + 80u^3 + 162u^4 + 206u^5 + 116u^6)t^6 \\
+ \mathcal{O}(t^7)
\]

We will find the consecutive quotients as \( W_n \)-modules of the class \( C \), for small \( k \), and \( n = 2, 3 \). We have already proved in the paper that \( A_{n,2} \) is a \( W_n \)-module. Consider the space \( A_{n,3} \) for \( n = 2 \). We are going to show that the dimensions \( \dim A_{\ell,2} \) are exactly like the character of a \( W_2 \)-module. Indeed, we know from (17) that \( \dim A_{\ell,2} = 2(\ell - 2) \) for \( 3 \leq \ell \leq 9 \). Consider a \( W_2 \)-module coinduced from a Young diagram \( D_1 \) showed in Figure 2 with a 2-dimensional \( \mathfrak{gl}_2 \)-module on the level \( \ell = 3 \). Then on a level \( \ell \) this \( W_2 \)-module should have dimension \( 2S_{k-3,2} \) where \( S_{k,2} \) is dimension of symmetric polynomials in 2 variables of degree \( k \). We have: \( S_{k,2} = k + 1 \). This shows that the spaces \( A_{\ell,2} \) have the character of a \( W_2 \)-module for \( \ell \leq 9 \).

Consider now \( A_{2,4} \). Here the polynomial for \( A_{2,4} \) is \( 3t^4 + 8t^5 + 13t^6 + 18t^7 + 23t^8 + \)
First take the difference with $3t^4 + 6t^5 + \cdots + 3(k + 1)t^{k+1} + \cdots$ which is the character of a $W_2$-module. The difference is $\sum_{k \geq 0} 2(k + 1)t^{k+5}$ which is a character of $W_2$-module.

The case $A_{2,5}$ is analogous.

It is more interesting to consider the case $n = 3$. Consider $A_{3,3}$. The polynomial for

\[ A_{3,3}^\ell = 8t^3 + 24t^4 + 48t^5 + 80t^6 + \mathcal{O}(t^7). \]

The first coefficient 8 is the space generated by Lie words $[x_i, [x_j, x_k]]$ on 3 letters $x_1, x_2, x_3$ (some of $i, j, k$ may coincide). This space has dimension 8, and as $\mathfrak{gl}_3$-module it is corresponded to the Young diagram $D_1$ (see Figure 2). The irreducible $\mathfrak{gl}_3$-module corresponded to this Young diagram has dimension 8, and it can be easily computed by the character formula. Consider the $W_3$-module coinduced from this $\mathfrak{gl}_3$-module. This coinduced character is $8 \sum_{k \geq 0} (k+1)(k+2)t^{k+3}$. It is exactly our $p_{3,3}$ up to $t^6$.

Finally, consider $A_{3,4}$. Here we have: $p_{3,4} = 18t^4 + 72t^5 + 162t^6 + \mathcal{O}(t^7)$. The first space with dimension 18 is the space of Lie brackets $[x_i, [x_j, [x_k, x_l]]]$ on 3 letters

\[ p_{3,4} = 18t^4 + 72t^5 + 162t^6 + \mathcal{O}(t^7). \]
As \( \mathfrak{gl}_3 \)-module, it is the direct sum of two representations with the Young diagrams \( D_2 \) and \( D_3 \) (see Figure 3). The \( \mathfrak{gl}_3 \)-module with the left diagram, \( D_2 \), has dimension 3, and the representation with the right diagram, \( D_3 \), has dimension 15. The coinduced \( W_3 \)-module from the direct sum of these two representations on the level 4 has the character \( 18 \sum_{k \geq 0} \frac{(k+1)(k+2)}{2} t^{k+4} \). Subtract this character from \( p_{3,4} \). The difference is \( 18t^5 + 54t^6 + O(t^7) \) which again has the character of the coinduced module from \( D_4 \oplus D_5 \) on the level 5 (see Figure 4). The irreducible \( \mathfrak{gl}_3 \)-module corresponding to the Young diagram \( D_4 \) has dimension 3, and the irreducible \( \mathfrak{gl}_3 \)-module corresponding to \( D_5 \) has dimension 15. The number of boxes in the Young diagram should be equal to the length of the words in the corresponding representation of \( \mathfrak{gl}_3 \), that is, to the level \( \ell \).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{young_diagrams}
\caption{The Young diagrams \( D_4 \) (left) and \( D_5 \) (right)}
\end{figure}

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