Generation of helical magnetic fields from inflation

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Abstract. The generation of helical magnetic fields during single field inflation due to an axial coupling of the electromagnetic field to the inflaton has been discussed. We have found that such a coupling always leads to a blue spectrum of magnetic fields during slow roll inflation. Though the helical magnetic fields further evolve during the inverse cascade in the radiation era after inflation, we have concluded that the magnetic fields generated by such an axial coupling can not lead to observed field strength on cosmologically relevant scales.

1. Introduction

There exists persuasive observational evidence for large scale magnetic fields with typical strength of $\mu$ Gauss in various cosmic structures from stars to galaxies and galaxy clusters [1, 2, 3]. Recently, using Fermi and HESS datasets, an interesting lower bound of $B \geq 10^{-16}$ Gauss has been obtained on the strength of magnetic fields in the intergalactic medium [4].

All such observations naturally lead to the question of the origin of these magnetic fields. Various mechanisms of magnetic field generation can be broadly classified into two categories: Early universe or primordial, and late universe or astrophysical. Among the former ones, magnetic fields produced during inflation are most interesting as an arbitrary spectrum for the magnetic fields can be produced, and therefore, can lead to sufficient amplitude on large scales to provide seeds for the observed fields in galaxies and clusters [5, 6].

In what follows, we have briefly discussed the magnetic field generation during inflation due to an axial coupling and shown that contrary to a non-helical coupling, a helical coupling always leads to a magnetic spectral index $n = 1$ during slow roll [7]. We have further shown that the backreaction turns out to be negligible as the magnetic energy density at the end of inflation is too small. Even though, the inverse cascade in the radiation era after inflation transfers power from small to large scales, the resulting field strength on cosmologically interesting scales is still insufficient to provide seeds for the observed magnetic fields in cosmic structures [7].

2. Axial coupling and the Fourier mode equation

Since the standard electromagnetic (EM) action is conformally invariant, the EM field fluctuations are not amplified in the conformally flat expanding background of inflation. In order to generate magnetic fields, one needs to break the conformal invariance of the EM field, e.g., by coupling the EM field to a scalar or a pseudo-scalar field or to a curvature invariant. Here, we have investigated the first possibility and studied a helical coupling given by:

$$\mathcal{L}_I(\phi, A_\mu) = \frac{1}{4} f(\phi) F_{\alpha\beta} \tilde{F}^{\alpha\beta}.$$  \hspace{1cm} (1)
It describes a coupling of the scalar field $\phi$ or the inflaton to the parity-violating term $F\tilde{F}$, where $F$ is the dual of the EM field tensor, and is defined as $F_{\mu\nu} \equiv \frac{1}{2} \eta_{\mu\nu\alpha\beta} F^{\alpha\beta}$, where $F_{\mu\nu} \equiv \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$, and $\eta^{\mu\nu\alpha\beta}$ is the totally anti-symmetric tensor with $\eta^{0123} = (-g)^{-1/2}$.

Adopting the Coulomb gauge, where $A^{\mu} = (0, A^t)$ with $\partial_t A^i = 0$, we have found that the inhomogeneous Maxwell’s equation with the axial current becomes:

$$\tilde{A}_i - \nabla^2 A_i = -f'(\phi) \phi \epsilon_{ijk} \partial_j A_k,$$

(2)

where $f'(\phi) \equiv df/d\phi$, and the overdots denote derivative with respect to the conformal time $t$. Note that, for a constant axial coupling, $f'(\phi) = 0$, the sourced Maxwell’s equation reduces to the standard free wave equation and no EM fluctuations are amplified during inflation. After quantization of the vector potential, we have found that the Fourier modes $A_h(t, k)$ corresponding to the helicity states $h = \pm$ satisfy the wave equation [7]:

$$\tilde{A}_h + \left[ k^2 + h k f'(\phi) \right] A_h = 0.$$

(3)

The power spectrum of a primordial stochastic magnetic field, statistically homogeneous and isotropic, is given by two scalar functions and can be written as:

$$\langle \tilde{B}_i(t, k) \tilde{B}_i^*(t, q) \rangle \equiv \frac{(2\pi)^3}{2} \delta(k - q) \left\{ (\delta_{ij} - \hat{k}_i \hat{k}_j) P_S(t, k) - i \epsilon_{ijk} \hat{k}_j P_A(t, k) \right\},$$

(4)

where $\tilde{B} \equiv \tilde{a}^2 B$ is the re-scaled magnetic field by the expansion factor $a$, and $P_{S/A}$ are the symmetric/anti-symmetric parts of the power spectrum respectively. With respect to the helicity basis, the spectra can directly be written as:

$$P_{S/A}(t, k) = k^2 \left( |A_+(t, k)|^2 \pm |A_-(t, k)|^2 \right),$$

(5)

and the re-scaled magnetic energy density per logarithmic wave number is given by:

$$\frac{d\rho_B}{d\ln k}(t, k) = \frac{k^3}{(2\pi)^2} P_S(t, k).$$

(6)

3. The magnetic power spectra in slow roll inflation

In order to treat the theory perturbatively, the strength of the interaction between the scalar field and the EM field should be small at all times, which then leads to $|f_N| \leq 1$, where $f_N$ is the dimensionless part of the coupling term, defined as: $f_N \equiv df/d\ln a = f'(\phi) \phi / \mathcal{H}$. In the slow roll regime, where $f_N$ is approximately constant, the mode Eq. (3) can be solved analytically in terms of Coulomb wave functions. By requiring the solution to approach the free wave solution as initial condition, the full normalized solution can be written as:

$$A_h(k, t) = \frac{1}{\sqrt{2k}} \left[ G_0(-hf_N/2, -kt) + iF_0(-hf_N/2, -kt) \right].$$

(7)

Using the asymptotic behaviour of this solution at late times, we have found that the power spectra are given by [7]:

$$P_S(k) = k \frac{\sinh(\pi f_N)}{\pi f_N} \quad \text{and} \quad P_A(k) = k \frac{\cosh(\pi f_N) - 1}{\pi f_N}.$$

(8)

Note that both the spectra are proportional to $k$, and only their amplitude changes with the coupling strength $f_N$, which tend to $(2\pi f_N)^{-1} \exp(\pi f_N)$ for large values of $f_N$. Using the
symmetric power spectrum, the magnetic energy density per logarithmic wave number at the end of inflation can directly be computed to be:

\[
\frac{d\rho_B}{d\ln k}(t_{\text{end}}, k) = \frac{k^4}{a_{\text{end}}^4} \frac{\sinh(\pi f_N)}{4\pi^3} = \frac{k^4}{a_{\text{end}}^4} S^2(f_N),
\]

which then leads to:

\[
\rho_B(t_{\text{end}}) \simeq \frac{1}{4} H_{\text{end}}^4 S^2(f_N),
\]

where \(H_{\text{end}}\) is the Hubble parameter at the end of inflation. Now, using the Friedmann equation, we can write:

\[
\Omega_B(t_{\text{end}}) \equiv \frac{\rho_B}{\rho_{\text{tot}}} \simeq \frac{S^2(f_N)}{12} \left(\frac{H_{\text{end}}}{m_P}\right)^2.
\]

The generic condition for back-reaction to be negligible is \(\Omega_B < 1\) which then leads to:

\[
S(f_N) \lesssim \frac{m_P}{H_{\text{end}}} \simeq \left(\frac{m_P}{T_*}\right)^2 \simeq 1.5 \times 10^{10} \left(\frac{10^{14} \text{ GeV}}{T_*}\right)^2,
\]

where \(T_*\) is the reheating temperature, given by \(T_*^4 \simeq m_P^2 H_{\text{end}}^2\). In terms of \(f_N\), the back-reaction bound becomes:

\[
f_N \lesssim 17 \left(\frac{10^{14} \text{ GeV}}{T_*}\right).
\]

Therefore, no back-reaction on the background evolution is expected if inflation ends well below the Planck scale [7]. Typically, the effective coupling term can be large in a situation wherein the EM field is coupled to a large number of pseudo-scalar fields [8].

**4. Final strength of the magnetic fields**

After inflation, due to helicity conservation, the magnetic field evolves by inverse cascade, which can move power from small scales to large scales without affecting the spectral shape of the spectrum on large scales [9]. On the other hand, on small scales, the magnetic fields are exponentially damped due to viscosity of the plasma. A sketch of the process of inverse cascade is shown in Figure 1. Taking into account the maximal amplification of the magnetic field due to inverse cascade for our case, we have found that the final strength of the magnetic field is given by:

\[
\tilde{B}(k) \simeq 3 \times 10^{-19} \text{ Gauss } S(f_N) \left(\frac{k}{10^{12} \text{/Mpc}}\right)^2 \left(\frac{T_*}{10^{14} \text{ GeV}}\right)^{9/11}.
\]

At a typical scale of \(\lambda = 0.1\) Mpc or \(k = 10/\text{Mpc}\), the field strength becomes:

\[
\tilde{B}(k = 10/\text{Mpc}) \simeq 3 \times 10^{-41} \text{ Gauss } S(f_N) \left(\frac{T_*}{10^{14} \text{ GeV}}\right)^{9/11}.
\]

which is by far insufficient for subsequent dynamo amplification, which requires seed fields of the order of at least \(10^{-20}\) Gauss [10]. Furthermore, to estimate how large this field can become for our best case, we require \(S(f_N) < (m_P/T_*)^2\) and therefore,

\[
\tilde{B}(k = 10/\text{Mpc}) \leq 10^{-32} \text{ Gauss } \left(\frac{10^{14} \text{ GeV}}{T_*}\right)^{13/11}.
\]

In order to obtain the minimal necessary field for dynamo amplification of about \(10^{-20}\) Gauss, we would require \(T_* \leq 10^3 \text{ GeV}\), which corresponds to a rather low inflation scale, but not completely excluded [7].
5. Conclusions
We have discussed the generation of helical magnetic fields during inflation induced by an axial coupling term. During slow roll inflation, we have shown that the magnetic field power spectrum is always blue with spectral index $n = 1$. Since the helical coupling is active only around horizon crossing, and is nearly constant during slow roll, all modes are coherently amplified, which leaves the shape of the spectrum invariant with respect to the initial one. Although, after inflation and reheating, an inverse cascade can transfer power from small to large scales, it turns out that for typical reheating temperatures, the magnetic fields are of $\mathcal{O}(10^{-40})$ Gauss, which are largely insufficient for dynamo amplification [10]. Recently, we have shown that even the resonant amplification of magnetic fields due to an oscillating coupling fails to provide sufficient amplitude for the observed field strength due to extremely large back-reaction [11].

Acknowledgement
We acknowledge the financial support from the Swiss National Science Foundation.

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