A Comparative Study of Polar Code Constructions for the AWGN Channel

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Abstract—We present a comparative study of the performance of various polar code constructions in an additive white Gaussian noise (AWGN) channel. A polar code construction is any algorithm that selects $K$ best among $N$ possible polar bit-channels at the design signal-to-noise-ratio (design-SNR) in terms of bit error rate (BER). Optimal polar code construction is hard and therefore many suboptimal polar code constructions have been proposed at different computational complexities. Polar codes are also non-universal meaning the code changes significantly with the design-SNR. However, it is not known which construction algorithm at what design-SNR constructs the best polar codes. We first present a comprehensive survey of all the well-known polar code constructions along with their full implementations. We then propose a heuristic algorithm to find the best design-SNR for constructing best possible polar codes from a given construction algorithm. The proposed algorithm involves a search among several possible design-SNRs. We finally use our algorithm to perform a comparison of different construction algorithms using extensive simulations. We find that all polar code construction algorithms generate equally good polar codes in an AWGN channel, if the design-SNR is optimized.

Keywords—Bhattacharyya bounds, bit-channels, Gaussian approximation, polar codes

I. INTRODUCTION

Polar codes have been the subject of active research in recent times, mainly due to the fact that they are the first ever provably capacity achieving codes, with explicit construction and very low complexity of encoding and decoding. The polar codes were invented by Erdal Arikan [1], using a novel concept called channel polarization. Soon after, both the concept of channel polarization as well as polar codes have been extended to a number of applications and generalizations [13]–[27].

Let us consider a binary input discrete memoryless symmetric (BI-DMS) channel. Channel polarization is a technique by which one manufactures $N$ polarized channels (called bit-channels) out of $N$ identical independent copies of BI-DMS channels. The channels are polarized without any loss of capacity, in the sense that they are either extremely noisy or noiseless as $N \to \infty$. Then one can easily achieve a rate of transmission close to capacity, simply by choosing to transmit over only the good bit-channels. However, at any finite blocklength $N$ and rate $R \triangleq K/N$, a ranking algorithm for the bit-channels according to their bit error rate (BER) becomes necessary to select $K$ good channels out of $N$. Here, $K$ is the number of information bits in each code word of length $N$. This selection of bit-channels completely defines a polar code and therefore is called the polar code construction.

The polar code construction is critical to obtain the best performance at finite blocklengths. As we mentioned, the polar code construction has an explicit definition in theory. It is challenging in practice because precise estimation of the bit-channels is intractable. Therefore, a wide range of approximate construction methods are proposed in [1]–[12].

An important characteristic of polar codes is their non-universality. That is, different polar codes are generated depending on the specified value of signal-to-noise ratio (SNR), known as the design-SNR. A change in operating SNR is possible in practice but a change in code according to SNR is not desirable. Therefore we wish to construct a polar code at one design-SNR and use it for a range of possible SNRs. As we see later, the choice of design-SNR is critical for the performance at all SNRs of interest. Unfortunately, there has been no study to identify the best design-SNR for any polar code construction.

A major issue with polar codes has been their inferior BER performance at finite blocklengths, compared to the state-of-the-art LDPC and Turbo codes of similar blocklength [28]. Several ideas have been proposed to overcome this issue. One may use a list decoder [28] to overcome this problem but this comes at the expense of an increase in decoding complexity.

In this paper, we aim at finding the best polar code construction algorithm among a range of available algorithms over a binary input additive white Gaussian noise (BI-AWGN) channel. We first propose a simple search algorithm to find the best design-SNR for each construction algorithm. Then we can find the best code construction algorithm among all. We find in our extensive simulations that all construction algorithms produce equally good polar codes when design-SNR is optimized.

The rest of the paper is organized as follows. In Section II, we present brief introduction to polar codes and outline our notation and channel model. In Section II, we describe the encoding, successive cancellation decoding and the construction of polar codes. In Section III, we review four main polar code construction algorithms which form the basis for several other variations. In Section IV, we discuss the non-universality of polar codes. Our discussion includes a proposal to fairly compare all construction algorithms. In Section V, we give a detailed description along with pseudocode for efficient implementations of polar code constructions. We finally present our simulation results in Section VI and conclude in Section VII.

II. POLAR CODES

Given any subset of indices $I$ of elements of a vector $x$, we denote the corresponding sub-vector as $x_I$. Similarly, when $I$ denotes the indices of columns of a matrix $A$, the corresponding sub-matrix is denoted $A_I$.

A polar code may be specified completely by $(N, K, F)$,
It maps \( \tilde{\text{SCD}} \) — the Successive Cancellation Decoder (SCD) — example in [1]. Then the bit decisions are made at the left of likelihood transformation equations, as illustrated with an example in [1]. The likelihoods of the classic belief propagation algorithm. The likelihoods of polar code. A compact alternative form is of Arikan’s standard polarizing kernel \( F \) of information bits encoded per codeword, and \( F \) is the set of non-frozen bit indices and where \( \mu \) is a set of information bits. The encoding operation for a vector of information bits \( \mu \) is the \( N \) \( 1 \) \( F \) \( N,K,F \). At first it was proposed to find the bit channels using a Monte-Carlo simulation of the bit-channels. This can be applied to a wide range of channels including finite and infinite alphabet channels such as AWGN. Although the bounds are loose for many of the bit-channels and more accurate methods are devised later, the codes designed using such bounds exhibit good performance. This construction enjoys the least \( O(N) \) complexity among all (excluding the selection of \( K \) best among \( N \) metrics obtained). This includes \( 2N - 1 \) transformation operations corresponding to the transformation of upper bounds by polarizing kernel. This well-known method was also studied in a recent paper [7].

Another earlier construction is proposed in [1], based on a Monte-Carlo simulation of the bit-channels. This can be applied to a wide range of channels including finite and infinite alphabet channels such as AWGN. However, the algorithm has the greatest complexity \( O(MN \log N) \) among all, where \( M \) is the number of iterations of the Monte-Carlo simulation.

A more recent construction algorithm was proposed by Tal and Vardy [3] based on an earlier proposal from Mori and Tanaka [5], [6]. At first it was proposed to find the bit channels by evaluating of their full finite alphabet distributions. The algorithm becomes intractable due to the explosion of the alphabet size to a power of \( N \) by the end of \( n \) channel transformations. This problem is specifically addressed in [3] by employing a novel low complexity close-to-optimal quantizer. In addition, they provide theoretical guarantees for the loss of performance due to the quantization. Note that, some channels are better estimated by simple bounds on Bhattacharyya parameters [1], [2]. Hence, in [3] the authors conclude a final algorithm by improving the BER estimates with the Bhattacharyya parameters whenever they are better. The final algorithm is considered by far the most accurate construction algorithm available with theoretical guarantees.

The algorithm is extendable to infinite output channels by using a quantization algorithm. In [3], the authors propose a quantization algorithm for AWGN channels. When the channel output has \( \mu \) symbols, the final complexity of the algorithm is \( O(N \cdot \mu^2 \log \mu) \) (excluding the selection of \( K \) best among \( N \) metrics obtained). This contains \( 2N - 1 \) number of \( O(\mu^2) \) complexity bit channel convolutions plus \( 2N - 1 \) number of \( O(\mu^2 \log \mu) \) complexity quantizer operations. The

\[
(N, K, F) = (8, 5, \{0, 2, 4\})
\]

\[
\begin{array}{cccccccc}
  d_0 &=& 0 & & & & & \\
  d_1 &=& u_0 & & & & & \\
  d_2 &=& 0 & & & & & \\
  d_3 &=& u_1 & & & & & \\
  d_4 &=& 0 & & & & & \\
  d_5 &=& u_2 & & & & & \\
  d_6 &=& u_3 & & & & & \\
  d_7 &=& u_4 & & & & & \\
\end{array}
\]

Fig. 1. Illustration of Arikan’s \( O(N \log_2 N) \) complexity encoder implementation of (2) with \( (N, K, F) = (8, 5, \{0, 2, 4\}) \) where \( N \) is the length of a code word in bits, \( K \) is the number of information bits encoded per codeword, and \( F \) is a set of \( N - K \) integer indices called frozen bit locations from \( \{0, 1, \ldots, N - 1\} \).

Encoding — For an \( (N, K, F) \) polar code we describe below the encoding operation for a vector of information bits \( u \) of length \( K \). The rate of the code is \( R = K/N \). Let \( n \triangleq \log_2(N) \) and \( F \supseteq n = F \otimes \cdots \otimes F \) (\( n \) copies) be the \( n \)-fold Kronecker product of Arikan’s standard polarizing kernel \( F \triangleq \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \).

Then, a codeword is generated as

\[
x = G \cdot u = (F \otimes n)_{|F^c} \cdot u,
\]

where \( F^c \triangleq \{0, 1, \ldots, N - 1\} \setminus F \) corresponds to the set of non-frozen bit indices and \( G \triangleq (F \otimes n)_{|F^c} \) is the generator matrix of polar code. A compact alternative form is

\[
x = F \otimes n \cdot d
\]

where \( d \in \{0, 1\}^N \) is such that \( d_F = 0 \) and \( d_{F^c} = u \).

Note that \( d_F \) is the set of frozen bits as defined in Arikan’s original formulation [1] and is taken here as zeros. Arikan also proposed an efficient implementation of complexity \( O(N \log N) \) of the encoding (2) as shown in Fig. 1.

Modulation and Channel Model — Our channel is a BI-AWGN channel, with zero mean and variance \( N_0^{-1} \). Bits in \( x \) are modulated as \( \hat{x} \) using binary phase shift keying (BPSK). It maps \( 0 \to -\sqrt{RE_b} \) and \( 1 \to +\sqrt{RE_b} \), where \( E_b \) denotes the energy spent per each information bit. We thus obtain the following channel

\[
y = \hat{x} + n.
\]

Without loss of generality, we normalize the noise variance to be unity for the AWGN in all our future discussions.

Successive Cancellation Decoder (SCD) — The SCD algorithm [1] essentially follows the same encoder diagram in Fig. 1 using decoding operations that resemble one iteration of the classic belief propagation algorithm. The likelihoods evolve in the reverse direction from right-to-left, using a pair of likelihood transformation equations, as illustrated with an example in [1]. Then the bit decisions are made at the left end of the circuit and broadcasted to the rest of the circuit. A complete pseudocode for implementing an SCD is available in [29]. The overall complexity is only \( O(N \log_2 N) \).

The Polar Code Construction — The choice of the set \( F \) is a critical step in polar coding (i.e., the polar code construction). This corresponds to the selection of best \( K \) bit-channels among \( N \), in terms of the bit error rate (BER) at a given value of \( (RE_b/N_0) \) defined as the design-SNR.

Since the exact BER of bit-channels is intractable, several approximations are used. This leads to many different constructions reviewed in Section III. The detailed algorithms are given later in Section V.

III. Overview of Current Literature on Polar Code Constructions

The earliest construction of polar codes is based on the evolution of simple bounds on the Bhattacharyya parameters of bit channels [2]. Though these bounds are proved in [1] only for BI-DMS channels, they may also be extended for infinite alphabet channels such as BI-AWGN. Although the bounds are loose for many of the bit-channels and more accurate methods are devised later, the codes designed using such bounds exhibit good performance. This construction enjoys the least \( O(N) \) complexity among all (excluding the selection of \( K \) best among \( N \) metrics obtained). This includes \( 2N - 1 \) transformation operations corresponding to the transformation of upper bounds by polarizing kernel. This well-known method was also studied in a recent paper [7].

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quantizer uses an intelligent data-structure that combines a heap and a list. The initialization of the algorithm involves the AWGN channel quantization to $\mu$ symbols, which takes an additional $O(\mu)$ complexity. Overall, this algorithm has the second largest complexity, only next to the earlier Monte-Carlo based construction algorithm by Arikan.

For AWGN channels, the estimation of bit-channels based on Gaussian approximation is proposed in [4]. This enables to use the Gaussian distribution approximations on the intermediate likelihoods. This was found to well-approximate the actual bit-channels of polar codes [8], [9]. A similar algorithm after the original proposal in [4] was studied in [8], [9].

The Gaussian approximation algorithm takes a complexity of $O(N)$ function computations (excluding the selection of $K$ best among $N$ metrics obtained) similar to the Bhattacharyya bounds based algorithm, but involves relatively higher complexity function computations. Overall, this construction algorithm enjoys the second least complexity.

There also exist several other heuristic constructions, extended or inspired from the above constructions, e.g. [10]–[12]. However, these methods are less interesting due to poorer performance, higher complexity, and having no theoretical guarantees. Finally, constructions for different channels are also available in [21], [30]–[32], for different input alphabets [33], different kernels [34]–[36] and concatenated codes [37], [38], which fall out of the scope of this paper.

IV. NON-UNIVERSALITY OF POLAR CODES

In coding theory, most of the codes are universal in the sense that their definition is independent of the channel SNR, but polar codes are different. Arikan defines the set $F$ of polar codes such that the block error rate (BLER) of polar codes is minimum under SCD. Since BLER is a function of SNR, it is not very surprising that the polar code changes with the given design-SNR. Later in our simulations, we see that the change is very significant in terms of performance. There are a few recent attempts to design universal polar codes [27], [32], [39] but they come at a cost of much higher complexity at decoder and/or encoder. Further development of the theory of universal polar codes is required for their practical significance. In this paper, we restrict our attention towards Arikan’s original polar codes only. We aim to design a polar code at a particular design-SNR and use it for a range of SNRs due to the following reasons.

1) In a number of experiments it is evident that the performance of polar codes constructed at one design-SNR is good for a range of SNRs (see for e.g. [28]).

2) The construction algorithms are not optimizing the performance exactly at the design-SNR. That is, better performance at a given SNR may be obtained by constructing the code at a slightly different design-SNR (see Section VI). This means even if we update the code dynamically with SNR, the performance may not be optimal.

As a result, the problem reduces to finding at what design-SNR, we should design the polar code. Unfortunately, to the best of our knowledge, there is no such study of polar codes in this direction. Many research works often consider a heuristic choice of design-SNR.

In Section VI we see that the choice of design-SNR is indeed critical for its performance at all SNRs. Further, we find that such a performance depends on many parameters such as rate, blocklength and the algorithm used for the construction. This observation makes an exhaustive search for the design-SNR inevitable. The following simple search algorithm is proposed to find the best design-SNR.

1) Consider a set of SNRs $\{S_1, S_2, \ldots, S_m\}$ that covers the range of SNRs of interest.

2) Design $m$ polar codes at design-SNRs equal to $S_i$, $i = 1, \ldots, m$ (i.e. find an $F$ at each $S_i$), using any given construction method.

3) Plot the performance curves BER vs. SNR or BLER vs. SNR of all above polar codes.

4) Select the curve that best suits the needs of the target application and declare the corresponding SNR for its construction.

We may use the search for different construction algorithms and make a fair comparison of performance among all. Extensive simulations of this comparison strategy are presented in Section VI.

V. THE POLAR CODE CONSTRUCTION ALGORITHMS

In this section we review all four important polar code construction methods denoted PCC-0, PCC-1, PCC-2, PCC-3, and provide full pseudocode implementations. The corresponding bit-channel metrics generated from each algorithm are denoted $z^{(0)}$, $z^{(1)}$, $z^{(2)}$, $z^{(3)}$ respectively.

We should use logarithmic domain (or log-domain in short) calculations especially at high blocklengths such as $N \geq 256$ to avoid underflow. For simplicity we mention linear domain formulas only.

A. PCC-0: Arikan’s Bhattacharyya bounds of bit channels

This earliest construction is from Arikan [1], [2], using Bhattacharyya parameters. He proved in [1, Appendix-D] that a pair of upper bounds on the Bhattacharyya parameters of bit-channels evolve as simply as $\{z, z\} \rightarrow \{2z - z^2, z^2\}$ at each polarizing transform $F$. Due to its simplicity, this construction has been widely used, and produced good polar codes.

There is an important modification to be used over [2], which is due to the non-universality of polar codes as explained in [8]. The original recursive algorithm requires an initial value and this was proposed by Arikan as 0.5 [2], corresponding to the worst BER. This initial is actually the Bhattacharyya parameter of the underlying BI-AWGN channel, therefore it may be replaced with $\exp(-RE_b/N_0)$ [8]. Now, the original initial 0.5 will be obtained at $RE_b = -1.5917$dB. The final algorithm is given below as Algorithm PCC-0.

A sample run of PCC-0 reveals that some of the bounds quickly increase and may even increase beyond 0.5 (but always < 1). This suggests a very inaccurate channel estimation, which motivated several alternate constructions. However, the advantage of the alternate constructions was not characterized.
well in literature.

**Algorithm PCC-0**: The Bhattacharyya bounds

| INPUT | \(N, K,\) and design-SNR \(EdB = (RE_0/N_0)\) in dB |
| OUTPUT | \(F \subset \{0, 1, \ldots, N-1\}\) with \(|F| = N - K\) |
|---|---|
| 1. | \(S = 10^{EdB/10}\) and \(n = \log_2 N\) |
| 2. | \(z(0) \in \mathbb{R}^N,\) initialize \(z(0)[0] = \exp(-S)\) |
| 3. | for \(j = 1 : n\) do \(\triangleright\) for each stage in Fig. 1, right-to-left |
| 4. | \(u = 2^j\) |
| 5. | for \(t = 0 : u - 1\) do \(\triangleright\) For each connection |
| 6. | \(T = z(u \lceil T \rceil)\) |
| 7. | \(z(u \lceil t \rceil) = 2T - T^2\) \(\triangleright\) Upper channel |
| 8. | \(z(u \lceil u/2 + t \rceil) = T^2\) \(\triangleright\) Lower channel |
| 9. | end |
| 10. | end |
| 11. | \(F = \text{indices of greatest elements}(z(N), N - K)\) |
| 12. | Return \(F\) |

**Function indices_of_greatest_elements(v, l)**

| INPUT | Vector \(v\) of dimension \(|v| \times 1\) and integer \(l\) |
| OUTPUT | \(I,\) an \(l \times 1\) integer vector containing \(l\) indices in \(\{0, 1, \ldots, |v| - 1\}\) |
|---|---|
| 1. | \([v, idx] = \text{Sort}(v, \text{`descending'})\) |
| 2. | \(I = idx[0 : l - 1]\) \(\triangleright\) Store the first \(l\) indices |
| 3. | Return \(I\) |

**B. PCC-1: Arikan’s Monte-Carlo estimation of bit channels**

A Monte-Carlo estimation of the bit-channel metrics is proposed in [1]. As a simulation based algorithm, it can be applied to a variety of channels.

We consider two specific improvements to the original proposal in [1] as follows.

1) We simulate all-zero codeword transmission only, since the polar code is a linear code. Each iteration is now equivalent to an SCD with all bits treated as frozen.

2) We calculate BER of bit-channels rather than their Bhattacharya parameters [1, Eq. (54) and (80)].

The first modification simplifies the algorithm by avoiding the encoding operation for each simulated transmission, in addition to avoiding the step of updating the bit decisions in SCD. The second modification improves the precision of the estimate, simply because we use the exact BERs. This reflects in the overall BLER equation [9, Eq. (3)] which is relatively more accurate than [1, Eq. (54) and (65)]. The complexity reduces to half due to these modifications. Also, as mentioned earlier the likelihood operations (step-7, step-12 of UpdateL and step-7, step-11 of PCC-1) are given in linear domain, but it is preferable to perform these operations in log-domain.

One particular disadvantage of this construction is the accuracy of the construction, which is limited severely by the Monte-Carlo iterations \(M\). More specifically, the channels with BER less than or close to \(1/M\) receive highly unreliable estimates. There exist many bit-channels that have extremely low BER, due to the polarization effect. Most of such good channels will receive a zero estimate, which makes a comparison difficult.

When the rate is high enough, the construction works well, since the good channels are always chosen for the information transmission. This avoids the need of any comparison among the good channels. On the other hand, when the rate is very low, the choice will only be among channels that tend to be very good. In that case, any choice would result in approximately the same performance.

**Algorithm PCC-1**: The Monte-Carlo estimation

| INPUT | \(N, K,\) the design-SNR \(EdB = (RE_0/N_0)\) in dB, \(M =\) Monte-Carlo size, \(\text{randn}()\) — a standard Gaussian pseudo random number generator |
| OUTPUT | \(F \subset \{0, 1, \ldots, N-1\}\) with \(|F| = N - K\) |
|---|---|
| 1. | Allocate and make visible to the other functions \(N \times (n + 1)\) matrices \(B, L\) and set \(B = 0\) |
| 2. | Bit-channel metrics \(c \in \mathbb{R}^N,\) initialize \(c = 0\) |
| 3. | \(n = \log_2 N\) and \(S = 10^{EdB/10}\) |
| 4. | for \(t = 1 : M\) do |
| 5. | \(y = -\sqrt{S} + \text{randn}(N, 1)\) \(\triangleright\) normalized channel, all-zero tx |
| 6. | \(L = \text{NaN}\) \(\triangleright\) Reset \(L\) |
| 7. | Initialize the last column of \(L\): \(L[j][n] = \text{Pr}(y_j) / \text{Pr}(y_j) = \exp(-2y_j \sqrt{S})\) \(\forall j\) |
| 8. | for \(i = 0, 1, \ldots, N - 1\) do \(\triangleright\) The SCD with all frozen |
| 9. | \(l = \text{bitreversal}(i)\) |
| 10. | UpdateL\((l, 0)\) \(\triangleright\) Update \(L, \) esp. \(L[l][0]\) |
| 11. | \(\tilde{d}[l] = \begin{cases} 0, & \text{if } L[l][0] \geq 1 \\ 1, & \text{else} \end{cases}\) |
| 12. | end |
| 13. | \(c = c + \tilde{d}\) \(\triangleright\) Real addition |
| 14. | end |
| 15. | \(z(1) = c/M\) \(\triangleright\) Monte-Carlo averaging for BERs (not necessary) |
| 16. | \(F = \text{indices of highest elements}(z(1), N - K)\) |
| 17. | Return \(F\) |

**C. PCC-2: Tal & Vardy’s estimation of bit-channel TPMs**

Tal & Vardy’s construction algorithm [3], attempts to estimate the full transition probability matrices (TPMs) of bit-channels, instead of only estimating BERs. The desired BERs may be estimated from the TPMs. In spite of this effort, some estimates may be looser than the simple Bhattacharya bounds calculated in PCC-0. This motivates to combine these two methods and come up with a hybrid algorithm that gives better estimates of the bit-channels compared to the individual algorithms. The final algorithm is denoted as Tal & Vardy’s construction for polar codes [3]. Like before, we compute BERs of bit-channels instead of Bhattacharya parameters for improved accuracy.

When the channel is symmetric we can consider only the half of the channel. So when we say that the output alphabet is of size \(\mu\) and its TPM is of dimension \(2 \times \mu\), we are actually referring to a symmetric channel of \(2\mu\) output symbols and its
the conditional densities are:

The quantization is performed such that the channel’s BER can be used whenever the alphabet size exceeds a threshold \( \mu \). The quantization is proposed to control its size. It will then be used whenever the alphabet size exceeds a threshold \( \mu \). The quantization is performed such that the channel’s BER can be used whenever the alphabet size exceeds a threshold \( \mu \). The quantization is proposed to control its size.

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A full pseudocode implementation of the overall quantization of AWGN channel to \( \mu \) symbols is given below.

\[
\text{Function discretizeAWGN(\( \mu \))}
\]

\[
\begin{align*}
\text{INPUT :} & \quad \text{Quantization size } \mu, \text{ design-SNR } EdB = RE_b/N_0 \text{ in dB} \\
\text{OUTPUT:} & \quad \text{TPM } P \text{ of size } 2 \times \mu \\
1. & \quad S = 10^{EdB/10} \\
2. & \quad \lambda(y) \triangleq \exp \left( -2y\sqrt{2S} \right) \text{ and } \quad \triangleright \text{ Eq. (6)} \\
3. & \quad C(x) \triangleq 1 - \log_2 (1 + x) + x \log_2(x)/(1 + x) \quad \triangleright \text{ Eq. (4)} \\
4. & \quad a \in \mathbb{R}^N, \text{ initialize } a[0] = 0 \text{ and } a[\mu] = \infty \\
5. & \quad \text{for } j = 1 : \mu - 1 \text{ do} \\
6. & \quad \quad a[j] = \text{solve}\{C(\lambda(y)) = j/\mu\} \\
7. & \quad \quad \text{end} \\
8. & \quad \text{for } j = 0 : \mu - 1 \text{ do} \\
9. & \quad \quad P[0][j] = Q\left( \sqrt{2S} + a[j] \right) - Q\left( \sqrt{2S} + a[j + 1] \right) \\
10. & \quad \quad P[1][j] = Q\left( -\sqrt{2S} + a[j] \right) - Q\left( -\sqrt{2S} + a[j + 1] \right) \\
11. & \quad \quad \text{end} \\
12. & \quad \text{Return } P \\
\end{align*}
\]

The bit-channel convolutions — Each recursive use of the basic kernel \( F = \{0, 1\} \) generates two polarized channels from a pair of identical channels \( W : \mathcal{X} \to \mathcal{Y} \). All the channels are represented by TPMs of two rows.

The pair of polarized channels are obtained by two self convolution operators \( \oplus \) and \( \ominus \), referred to as upper-convolution and lower-convolution, respectively (\( \oplus \) denotes usual binary EX-OR). The corresponding definitions are below [1], [3].

\[
W \oplus W(y_1, y_2 | i) = \frac{1}{2} \left\{ W(y_1|0)W(y_2|0 \oplus i) + W(y_1|1)W(y_2|1 \oplus i) \right\}, \forall y_1, y_2 \in \mathcal{Y} \quad (9)
\]
\[ W \equiv W(y_1, y_2, b | i) = \frac{1}{2} W(y_1 | i) W(y_2 | b \oplus i) \]

\[ \forall y_1, y_2 \in Y \text{ and } b \in \{0, 1\} \quad (10) \]

Proper care must be taken in applying these convolutions, to have all columns with likelihoods > 1 (symmetric half of the channel). Whenever this gets violated, we simply swap the two probability values in the column, which brings its symmetric symbol in place. Another issue is whenever there are two or more columns with the same LR, we merge them to one by simply adding the columns.

Finally, we can see that the size of output alphabet of both the new channels is \(O(\mu^2)\). This increase in alphabet size is indeed high, which soon becomes intractable as we apply this transformation recursively. Hence the quantizer algorithms are essential.

### Function `upperConolve()`

**INPUT**: A 2 \(\times\) \(\mu\) TPM \(P\) of channel to get convolved  
**OUTPUT**: TPM \(Q\) of size 2 \(\times\) \(\mu'\), \(\mu' \leq \mu + 1\)  
1. Allocate \(Q \in \mathbb{R}^{2 \times \mu + 1}/2\) and initialize \(idx = -1\)  
2. for \(i = 0 : \mu - 1\) do  
   3. \(idx = idx + 1\)  
   4. \(Q[0][idx] = (P[0][i]^2 + P[1][i]^2)/2\)  
   5. \(Q[1][idx] = P[0][i]P[1][i]\)  
   6. for \(j = i + 1 : \mu - 1\) do  
      7. \(idx = idx + 1\)  
      8. \(Q[0][idx] = P[0][i]P[0][j] + P[1][i]P[1][j]\)  
      9. \(Q[1][idx] = P[0][i]P[1][0] + P[1][i]P[0][0]\)  
      10. if \(Q[0][idx] < Q[1][idx]\) then  
          11. swap\(Q[0][idx], Q[1][idx]\); end;  
   end  
12. end  
13. Merge the columns of \(Q\) with same LR (add columns)  
14. Return \(Q\)

The quantizer algorithm — A critical part of the construction algorithm is to find a good quantization algorithm, which was given in [3], as discussed below. This quantization algorithm requires a special data-structure which is a combination of a heap and a linked list. We call this data-structure a heaplist.

A heaplist is a data-structure that holds \(L\) symbol probabilities (columns) from the TPM as a list and \(L - 1\) loss-of-capacity values as a heap. These values in heap result from several symbol-merger operations, defined below.

One main objective of the data-structure is to enable a low complexity \(O(\log L)\) operation for extracting the minimum of the values in heap, while simultaneously maintaining the list of \(L\) symbols in a sorted order of likelihoods.

A symbol-merger operation is the critical component of this quantization algorithm, and is a part of the heaplist. This operation is allowed only on two consecutive symbols in the list. A merger of two columns in the list replaces the two columns by a single column equal to their sum corresponding to one new symbol, simultaneously adjusting the heap accordingly. Each merger reduces the size of the heaplist by one element and changes the heaplist.

Overall, a heaplist is simply a data-structure providing three main operations described as follows.

- **initialize_heaplist()**: Given a \(2 \times L\) TPM, initialize the list part of the data-structure with its columns in the increasing order of likelihoods, all \(\geq 1\). Involves a sorting operation with \(O(L \log L)\) complexity. The heap part of the data-structure is stored with the \(L - 1\) values of loss in capacity when two consecutive symbols in the list are merged. These \(L - 1\) values in heap are tightly attached to the first \(L - 1\) values in the list.
- **minloss_index()**: Find the value of minimum loss-of-capacity, and return the index of the column in the list attached to the minimum loss. The index indicates the optimal symbol-merger, which merges the column and its next.
- **merge_at_index()**: Perform the merger operation of the two elements in list, present at the given index and its next. This will result in the reduction of size of the heap and the list, by one element. One entry in the list gets updated and one entry gets removed. Accordingly, a few entries in the heap will be updated.
- **Miscellaneous**:
  - size() — to know the number of columns of the TPM currently maintained within the heaplist.
  - TPM() — to extract the TPM from heap, in a two-row matrix format.
  - has_duplicates() — true or false based on whether there are columns in list with same likelihood ratio.

The Overall Implementation and Pseudocode — The following few steps are involved in the main algorithm to generate \(N\) symmetric bit-channel TPMs.
Function Quantize_to_size()

INPUT : TPM P of size $2 \times L$, quantization size $\mu < L$
OUTPUT: Quantized TPM of size $2 \times \mu', \mu' \leq \mu$

1: Instantiate a heaplist object $H$
2: $H$.initialize_heaplist($P$)
3: while $H$.size() > $\mu$ or $H$.has_duplicates() do
   4:     $idx = H$.minloss_index()
   5:     $H$.merge_at_index($idx$)
4: end
6: Return $H$.TPM()

1) Initialize the list with one finite alphabet channel equal to the quantized version of the AWGN channel having alphabet size $\mu$.
2) For each finite alphabet channel in the list, repeat the following:
a) Perform upper and lower convolutions of the channel with itself to generate two new channels of higher size.
b) Quantize the two resultant channels to the size $\mu$ and replace the one original channel.
3) If the number of channels is less than $N$, then repeat step 2, otherwise go to step 4.
4) Calculate the BER of each channel and compare with its Bhattacharyya parameter calculated using PCC-0. Declare the least value as the bit-channel metric.

The pseudocode of the same is provided as algorithm PCC-2. This represents the full construction algorithm of Tal & Vardy [3], except that we compute exact-BER of the bit-channels instead of their Bhattacharyya bounds for the comparison. This improves the precision of bit-channel metrics before they are compared and the greatest $N-K$ values are selected.

D. PCC-3: Trifonov’s Gaussian approximation of bit channels

Gaussian approximation for polar code construction was first proposed by Trifonov [4] and rediscovered in [8], [9].

The basic idea is to estimate all the log-likelihood ratios (LLR’s) at intermediate stages as Gaussian variables. This simplifies the analysis by requiring only to calculate the mean and variance of the LLRs at each stage of decoding. The bit channel metrics in this case are different. We estimate a bit-channel metric proportional to the argument of a Q-function, which represents its BER under Gaussian approximation [9]. So while being sufficient for comparison, the metrics in $z^{(3)}$ are not BERs but are inversely proportional to the BERs of bit-channels unlike the metrics from other constructions.

The following interpolation function and its inverse are essentially used to perform this construction (as in [42]).

$$\phi(x) \triangleq \begin{cases} \exp(-0.4527 x^{0.86} + 0.0218) & \text{if } 0 < x \leq 10 \\ \sqrt{\pi/2} (1 - \psi(0)) \exp(-x/4) & \text{if } x > 10 \end{cases}$$ (11)

Full pseudocode to implement this algorithm is given below. We may employ bisection method to find the inverse of the function $\phi(x)$ (we observed that for this special function, bisection method works faster than Newton-Raphson method).

Algorithm PCC-2: The full TPM estimation

INPUT : $N$, $K$ and design-SNR $EdB = (RE_b/N_0)$ in dB;
$z^{(0)}$ is vector of bit-channel metrics from PCC-0

OUTPUT: $F \subset \{0, 1, \ldots, N-1\}$ with $|F| = N-K$

1: Allocate channels, an array of $N$ TPMs, each of dimension $2 \times \mu$.
2: $n = \log_2 N$
3: $channels[0] = \text{discretize AWGN}(\mu, EdB)$ //Initialize
4: for $j = 1 : n$ do $\triangleright$ for each stage in Fig. 1, right-to-left
   5:     $u = 2^j$
6:     for $t = 0 : \frac{n}{2} - 1$ do
5:         $ch1 = \text{upperConvolve}(channels[t])$
6:         $ch2 = \text{lowerConvolve}(channels[t])$
7:     $channels[t] = \text{Quantize_to_size}(ch1, \mu)$
8:     $channels[\mu/2 + t] = \text{Quantize_to_size}(ch2, \mu)$
9: end
10: Return $F$
11: $F = \text{indices_of_greatest_elements}(z^{(2)}, N-K)$ // Find indices of the greatest $N-K$ elements
22: Return $F$

Algorithm PCC-3: The Gaussian approximation

INPUT : $N$, $K$ and design-SNR $EdB = (RE_b/N_0)$ in dB

OUTPUT: $F \subset \{0, 1, \ldots, N-1\}$ with $|F| = N-K$

1: $S = 10^{EdB/10}$ and $n = \log_2 N$
2: $z^{(3)} \in \mathbb{R}^N$, initialize $z^{(3)}[0] = 4S$
3: for $j = 1 : n$ do $\triangleright$ for each stage in Fig. 1, right-to-left
4:     $u = 2^j$
5:     for $t = 0 : \frac{n}{2} - 1$ do
6:         $T = z^{(3)}[t]$
7:         $z^{(3)}[t] = \phi^{-1} (1 - (1 - \phi(T))^2)$ $\triangleright$ Upper channel
8:     $z^{(3)}[\mu/2 + t] = 2T$ $\triangleright$ Lower channel
9: end
10: $F = \text{indices_of_least_elements}(z^{(3)}, N-K)$ // Find indices of the least $N-K$ elements
12: Return $F$

Function indices_of_least_elements($v$, $l$)

INPUT : Vector $v$ of dimension $|v| \times 1$ and integer $l$
OUTPUT: An $l \times 1$ integer vector containing $l$ indices in $\{0, 1, \ldots, |v| - 1\}$

1: Return $\left(\text{indices_of_greatest_elements}(-v, l)\right)$

VI. SIMULATIONS AND DISCUSSION

In this section we consider comparing the performance of polar codes from various polar code constructions. Figs. 2
to 5 present the performance of polar codes produced by each of algorithms PCC-0 to PCC-3 at $N = 2048$ and $R = 0.5$. Different curves in each figure represent the polar codes constructed at different design-SNRs. Clearly, design-SNR is critical for all construction algorithms to generate polar codes with a good performance. Interestingly, PCC-3 shows high performance variations with design-SNR.

We may observe that at high design-SNRs, the performance degrades with an increase in design-SNR for all the construction algorithms. In fact in the current example, PCC-2 and PCC-3 followed this more precisely and performed the best at the least design-SNRs. The least design-SNR we considered is $-1.5917$dB, at which PCC-0 takes the worst-case initial 0.5.

As discussed in Section IV, the optimal design-SNR is a function of all possible parameters such as rate, block-length and construction algorithm. We therefore select the best design-SNR for each construction algorithm from the above graphs and then make a fair comparison. This comparison is shown in Fig. 6. We see that if we can find the optimal design-SNR, any polar code construction algorithm produces polar codes of equally good performance.

VII. CONCLUSIONS

We have presented a comprehensive survey and full pseudocode implementations of all the well-known construction algorithms. We then proposed a simple discrete search to find the best design-SNR for any given polar code construction algorithm. We then compared various polar code constructions and concluded that all are equally good in AWGN if the design-SNR is optimized for the best performance. Thus in future, we may use simple algorithms only.

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