Asymptotic Behavior of Positive Solutions of the Equation

\[ \Delta u + Ku^{\frac{n+2}{n-2}} = 0 \text{ in } \mathbb{R}^n \]

and Positive Scalar Curvature

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Abstract

We study asymptotic behavior of positive smooth solutions of the conformal scalar curvature equation in \( \mathbb{R}^n \). We consider the case when the scalar curvature of the conformal metric is bounded between two positive numbers outside a compact set. It is shown that the solution has slow decay if the radial change is controlled. For a positive solution with slow decay, the corresponding conformal metric is found to be complete if and only if the total volume is infinite. We also determine the sign of the Pohozaev number in some situations and show that if the Pohozaev is equal to zero, then either the solution has fast decay, or the conformal metric corresponding to the solution is complete and the corresponding solution in \( \mathbb{R} \times S^{n-1} \) has a sequence of local maxima that approach the standard spherical solution.

KEY WORDS: positive solutions, critical Sobolev exponent, Pohozaev’s identity, positive scalar curvature.

1991 AMS MS Classifications: Primary 35J60; Secondary 58G03.

1. Introduction

For \( n \geq 3 \), we consider positive smooth solutions \( u \) of the conformal scalar curvature equation

\[ \Delta u + Ku^{\frac{n+2}{n-2}} = 0 \text{ in } \mathbb{R}^n, \]

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where $\Delta$ is the standard Laplacian on $\mathbb{R}^n$ and $K$ is a smooth function. Equation (1.1) is studied extensively by many authors and remarkable results are obtained. Some of the recent publications on this equation provide excellent descriptions of the development. Here we are concerned with the case when $K$ is bounded between two positive numbers outside a compact subset of $\mathbb{R}^n$. That means that the conformal metric $g = u^{4/(n-2)}g_o$ has bounded positive scalar curvature outside a compact subset, where $g_o$ is Euclidean metric. Accordingly, positive smooth solutions of equation (1.1) can be broadly classified as whether the conformal metric $g$ is complete or not, and if it is not complete, whether it can be realized as a smooth metric on $S^n$ after a stereographic projection.

The total volume of the Riemannian manifold $(\mathbb{R}^n, g)$ is given by

$$\int_{\mathbb{R}^n} u^{2n/(n-2)}(x) \, dx.$$  

One can also classify positive smooth solutions of equation (1.1) according to whether the total volume is finite or not. These geometric classifications are closely related to the decay of $u$ near infinity. A positive smooth solution $u$ of equation (1.1) in $\mathbb{R}^n$ is said to have slow decay if there exist positive constants $C_s$ and $r_s$ such that

$$u(x) \leq C_s |x|^{-\frac{n+2}{n-2}} \text{ for } |x| \geq r_s. \tag{1.3}$$

It is said to have fast decay if there exist positive constants $C_f$ and $r_f$ such that

$$u(x) \leq C_f |x|^{-(n-2)} \text{ for } |x| \geq r_f. \tag{1.4}$$

(Note that the definition of slow decay is different from the one in [1] and [3].) If $u$ has fast decay, then $g$ can be realized as a metric on $S^n$ (cf. lemma 2 in [13], see also [3]). Using a result of Brezis and Kato [3], it is shown in [17] that if the total volume of $(\mathbb{R}^n, g)$ is finite, then $u$ has fast decay.

We may infer that the cases of slow decay, completeness and infinite total volume are related to each other. They also pertain to the asymptotic dimension of $(\mathbb{R}^n, g)$ (cf. the conjecture propounded by Gromov in [14], see also [17]). For slow decay means that the conformal metric $g$ is asymptotically a bounded perturbation of the standard product metric on $\mathbb{R}^+ \times S^{n-1}$ [17]. When $K$ is equal to a positive constant outside a compact set, Caffarelli, Gidas and Spruck in their celebrated work [4] show that either $g$ can be realized as a smooth metric on $S^n$, or $u$ is asymptotic to one of a one-parameter family of radial solutions and has slow decay. In [5], using the method of moving planes, Chen and Lin show that if there exist positive constants $c_1, c_2, c_3, l$ and $R_o$, with $c_3 > 1$, such that

$$0 < c_1 \leq |\nabla K(x)||x|^{l+1} \leq c_2 \text{ for } |x| \geq R_o, \tag{1.5}$$
\[
\lim_{|x| \to \infty} K(x) = K_\infty > 0 \quad \text{and} \quad K(y) \leq K(x) \quad \text{for} \quad |y| \geq c_3 |x| \quad \text{and} \quad |x| \geq R_o,
\]
then \(u\) has slow decay. In \([17]\), the second author shows that if \(K\) is bounded between two positive numbers in \(\mathbb{R}^n\), \(x \cdot \nabla K(x) \leq 0\) for large \(|x|\), \(u\) is bounded from above and \(u\) is \textit{radially dominating} (see \([17]\)), then \(u\) has slow decay. In this paper we obtain the slow decay by assuming a control on the growth in the radial direction.

**Theorem A.** Assume that \(K\) is bounded between two positive constants outside a compact subset of \(\mathbb{R}^n\). Assume also that there exist positive constants \(r_o, c\) and \(C\) such that
\[
\frac{\partial K}{\partial r}(r, \theta) \geq -Ce^{-cr} \quad \text{for} \quad \theta \in S^{n-1} \quad \text{and} \quad r \geq r_o,
\]
where \((r, \theta)\) is the polar coordinates on \(\mathbb{R}^n\). Let \(u\) be a positive smooth solution of equation (1.1) in \(\mathbb{R}^n\). If there exist positive constants \(C'\) and \(r'\) such that
\[
C' \left| \frac{\partial u}{\partial r}(r, \theta) \right| \leq C' u(r, \theta) \quad \text{for} \quad \theta \in S^{n-1} \quad \text{and} \quad r \geq r',
\]
then \(u\) has slow decay.

It is observed that if there exist positive constants \(C_l\) and \(r_l\) such that
\[
u(x) \geq C_l |x|^{-\frac{n^2}{2}} \quad \text{for} \quad |x| \geq r_l,
\]
then \(g\) is a complete metric on \(\mathbb{R}^n\). Inequality (1.10) is related to the \textit{Pohozaev number} of \(u\), denoted by \(P(u)\) (see section 2). For a positive smooth solution \(u\) with slow decay, \(P(u) \neq 0\) implies that (1.10) holds. Using the results of Korevaar, Mazzeo, Pacard and Schoen in \([15]\), if \(K\) is equal to a positive constant outside a compact set and \(g\) is complete, then (1.10) holds and \(P(u) < 0\). Chen and Lin \([4]\) show that, under the assumptions (1.5), (1.6), (1.7) and in addition, if it is also assumed that \(x \cdot \nabla K(x) \leq 0\) for large \(|x|\), then the conformal metric \(g\) can be realized as a smooth metric on \(S^n\) if and only if \(P(u) = 0\), and it can be deduced that \(g\) is complete (or (1.10) holds) if and only if \(P(u) < 0\). The sign of the Pohozaev number can also be determined in the following situation.

**Theorem B.** Assume that \(\lim_{|x| \to \infty} K(x) = K_\infty > 0\) and \(|\nabla K| \leq C_o\) in \(\mathbb{R}^n\) for a positive number \(C_o\). Let \(u\) be a positive smooth solution of (1.1) in \(\mathbb{R}^n\) with slow decay. Assume also that \(K\) satisfies one of the conditions in lemma 2.4 so that \(P(u)\) exists. Then we have \(P(u) \leq 0\). In addition, if it is also assumed that
\[ x \cdot \nabla K(x) \leq 0 \text{ for large } |x|, \text{ then } P(u) = 0 \text{ implies that } u \text{ has fast decay.} \]

Similar results are obtained by Chen and Lin in [5] and [8], and by Korevaar, Mazzeo, Pacard and Schoen in [15]. Indeed, our contribution to theorem B consists mainly of showing that the arguments used in their works can be applied in a general setting. It is rather natural to relate the condition that \( P(u) = 0 \) with fast decay, due to the results mentioned above, and also to the Kazdan-Warner identity which says that if the conformal metric \( u^{4/(n-2)} g_o \) can be realized as a smooth metric on \( S^n \), then \( P(u) = 0 \). An interesting example of Chen and Lin (theorem 1.6 in [8]) shows that the relation is actually quite subtle. Let

\begin{equation}
(1.11) \quad v(s, \theta) = r^{n-2} u(r, \theta), \quad r = e^s, \quad r > 0, \quad s \in \mathbb{R} \quad \text{and} \quad \theta \in S^{n-1}.
\end{equation}

We show that, under the conditions in the first part of theorem B, if \( P(u) = 0 \), then either \( u \) has fast decay, or the conformal metric \( g \) is complete and

\[ \liminf_{|x| \to \infty} |x|^{n-2} u(x) = 0 \]

and there exists a sequence \( \{s_j\} \subset \mathbb{R} \) of local maxima of

\begin{equation}
(1.12) \quad \bar{v} = \int_{S^{n-1}} v \, d\theta,
\end{equation}

\( s_j \to +\infty \) as \( j \to \infty \), such that the sequence \( \{v_j\} \) defined by

\[ v_j(s, \theta) = v(s_j + s, \theta) \quad \text{for} \quad s \in \mathbb{R}, \quad \theta \in S^{n-1} \quad \text{and} \quad j = 1, 2, \ldots, \]

converges uniformly in \( C^2 \)-norm on compact subsets of \( \mathbb{R} \times S^{n-1} \) to \( C (\cosh s)^{(2-n)/2} \) for a positive constant \( C \) depending on \( K_\infty \). The example in [8] suggests that the second situation in the above statement may occur. We note that the conformal factor \( (\cosh s)^{-2} \) transforms the cylinder \( \mathbb{R} \times S^{n-1} \) into \( S^n \setminus \{p, -p\} \) [15]. It is also noteworthy that even if \( P(u) = 0 \), the conformal metric \( g \) may still be complete.

**Theorem C.** Assume that \( K \) is bounded between two positive numbers outside a compact subset of \( \mathbb{R}^n \). Let \( u \) be a positive smooth solution of equation (1.1) in \( \mathbb{R}^n \) with slow decay. Then either \( u \) has fast decay or the conformal metric \( g = u^{4/(n-2)} g_o \) is complete. Furthermore, the conformal metric \( g \) is complete if and only if the total volume of \( (\mathbb{R}^n, g) \) is infinite.

When \( K \) may not have a limit at infinity, for a particular class of positive solutions, we can determine the sign of the Pohozaev number.
Theorem D. Assume that $K$ is bounded between two positive numbers outside a compact subset of $\mathbb{R}^n$. Let $u$ be a positive smooth solution of equation (1.1) in $\mathbb{R}^n$. Assume that

$$\lim_{r \to \infty} \int_{S_r} r \left[ \frac{\partial u}{\partial r} + \frac{n - 2}{2} \frac{u}{r} \right]^2 dS = 0.$$  

If $P(u)$ exists, then $P(u) \leq 0$. Furthermore, $P(u) = 0$ implies that

$$\liminf_{|x| \to \infty} |x|^{(n-2)/2} u(x) = 0.$$  

In theorem D, we do not assume that $u$ has slow decay. The meaning of condition (1.13) is explained in more detail in (2.26) and section 4. We outline the content of each section. Section 2 reviews the Pohozaev identity, gradient estimate and spherical Harnack inequality. Theorem A is proved in section 3. Section 4 is devoted to a study of the Pohozaev number in general and the proofs of theorem B, C and D in particular. We use $C$, $C_1$, ..., $c$, $c_1$, ... to denote various constants which may be different from section to section according to the context. Moreover, throughout this article, $n \geq 3$ is an integer.

2. Preliminaries

Let $u$ be a positive smooth solution of equation (1.1) and

$$P(u, r) = \frac{n - 2}{2n} \int_{B_o(r)} x \cdot \nabla K(x) u^{2n/(n-2)}(x) \, dx \quad \text{for} \quad r > 0,$$

where $B_o(r)$ is the open ball with center at the origin and radius $r > 0$. The Pohozaev identity [26] shows that

$$P(u, r) = \int_{S_r} \left[ r \left( \frac{\partial u}{\partial r} \right)^2 - \frac{r}{2} |\nabla u|^2 + \frac{n - 2}{2n} rK u^{2n/(n-2)} + \frac{n - 2}{2} \frac{u}{r} \frac{\partial u}{\partial r} \right] dS$$

for $r > 0$, where $S_r = \partial B_o(r)$ is the sphere of radius $r$. The Pohozaev number of $u$ is given by

$$P(u) = \lim_{r \to \infty} P(u, r)$$

provided the limit exists.
Lemma 2.4. Let $u$ be a positive smooth solution of equation (1.1) with slow decay. Then the limit in (2.3) exists if we assume anyone of the following conditions.

(I) There exists a number $m > 1$ such that

\begin{equation}
 x \cdot \nabla K = r \frac{\partial K}{\partial r} \in L^m(\mathbb{R}^n \setminus B_0(1)) .
\end{equation}

(II) There exist positive numbers $C, \varepsilon$ and $r_o$ such that

\begin{equation}
 \left| \frac{\partial K}{\partial r} \right| \leq \frac{C}{r (\ln r)^{1+\varepsilon}} \text{ for } r \geq r_o .
\end{equation}

(III) $x \cdot \nabla K(x)$ does not change sign for large $|x|$ and there exist positive constants $\alpha$ and $\beta$ such that $-\alpha^2 \leq K \leq \beta^2$ in $\mathbb{R}^n$.

Proof. We note that

\begin{equation*}
 \left| \int_{\mathbb{R}^n \setminus B_o(1)} \frac{\partial K}{\partial r} u^{\frac{m}{n}} \, dx \right| \leq C_o \left[ \int_{\mathbb{R}^n \setminus B_o(1)} \left| \frac{r}{\ln r} \frac{\partial K}{\partial r} \right|^m \, dx \right]^{\frac{1}{m}} \left[ \int_{\mathbb{R}^n \setminus B_o(1)} |x|^{-nl} \, dx \right]^{\frac{1}{l}} \to 0
\end{equation*}

as $R \to \infty$, where $l > 1$ is the number such that $1/m + 1/l = 1$. Here $C_o$ is a positive constant. Likewise,

\begin{equation*}
 \left| \int_{\mathbb{R}^n \setminus B_o(1)} |x| \frac{\partial K}{\partial r} u^{\frac{2a}{n+2}} \, dx \right| \leq C'_o \int_{R}^{\infty} \frac{dr}{r (\ln r)^{1+\varepsilon}} \to 0
\end{equation*}

as $R \to \infty$, where $C'_o$ is a positive constant. Hence the limit in (2.3) exists if we assume either (I) or (II). Assume that $x \cdot \nabla K(x) \leq 0$ for large $|x|$. The integral in (2.1) is a decreasing function for large $r$. For $R > R_o$ large enough, we have

\begin{equation*}
 \int_{R_o}^{R} \int_{S_r} r \frac{\partial K}{\partial r} u^{\frac{2a}{n+2}} \, dS \, dr \geq C_1 \int_{R_o}^{R} \int_{S^{n-1}} \frac{\partial K}{\partial r} \, d\theta \, dr = C_1 \int_{R_o}^{R} \frac{d}{dr} \left( \int_{S^{n-1}} K(r, \theta) \, d\theta \right) \, dr \\
\geq -C_1 \omega_n (\alpha^2 + \beta^2) ,
\end{equation*}

where $C_1$ is a positive constant and $\omega_n$ is the volume of $S^{n-1}$. Hence the integral in (2.1) is bounded from below and the limit in (2.3) exists. The other case is similar.

We assume that there exist positive constants $a, b$ and $r_o$ such that

\begin{equation}
 0 < a^2 \leq K(x) \leq b^2 \quad \text{for } |x| \geq r_o .
\end{equation}

$u$ is said to satisfy a spherical Harnack inequality if there exists a positive constant $C_h$ such that the inequality

\begin{equation}
 \sup_{S_r} u \leq C_h \inf_{S_r} u
\end{equation}

for some $r > r_o$. 

\[\square\]
holds for all $r > 0$. In [23], the following relations are observed (see also [28]). A version of the result is proved in [8]. It is certainly well-known among the experts, but for the sake of completeness, we describe the argument below.

**Lemma 2.9.** Assume that $K$ satisfies (2.7). Let $u$ be a positive smooth solution of (1.1) in $\mathbb{R}^n$. Then the following statements are equivalent.

(a) $u$ has slow decay.

(b) $u$ satisfies a spherical Harnack inequality.

(c) There exists a positive constant $C_g$ such that
\[ r |\nabla u(r, \theta)| \leq C_g u(r, \theta) \quad \text{for} \quad r > 0 \quad \text{and} \quad \theta \in S^{n-1}. \]

**Proof.** Assuming slow decay, write equation (1.1) as $\Delta u(x) + f(x) u(x) = 0$ in $\mathbb{R}^n$, where $f(x) = K(x) u^{4/(n-2)}(x)$. We have the following scaling property for $f$:
\[ 0 \leq R^2 f^2(Rx) \leq \frac{b^2 C_s^{n-2}}{|x|^2} \quad \text{for} \quad R |x| \geq r_1, \]

where $r_1 = \max \{r_s, r_o\}$. Here $C_s, r_s$ are the constants in (1.3) and $r_o$ is the constant in (2.7). Using a scaling as above and the Harnack inequality (theorem 8.20 in [13]), there exist positive constants $C_1$ and $R_1$ such that
\[ \sup_{B_{2R} \setminus B_R} u \leq C_1 \inf_{B_{2R} \setminus B_R} u \quad \text{for} \quad R \geq R_1. \]

It follows that we have a spherical Harnack inequality for spheres of large radius. As $u$ is positive, therefore inside a big ball $B_o(R_2)$ we have
\[ \sup_{B_o(R_2)} u \leq C_2 \inf_{B_o(R_2)} u \]

for a positive constant $C_2$ that depends on $R_2$ and $u$. Hence we have (b). We obtain (c) by using the gradient estimate in [13] (pp. 37), equation (1.1) and (2.8). On the other hand, (c) implies that, for a fixed $r > 0$,
\[ |\nabla u(r, \theta)| \leq C' u(r, \theta) \quad \text{for} \quad \theta \in S^{n-1}, \]

where $C'$ is a positive constant independent on $r$. After an integration we have
\[ \sup_{S_r} u \leq e^{C' \pi} \inf_{S_r} u \quad \text{for} \quad r > 0, \]

which is (b). From (b) and the fact that there exist positive constants $C_2$ and $r_2$ such that
\[ \int_{S^{n-1}} u(r, \theta) d\theta \leq C_2 r^{-(n-2)/2} \quad \text{for} \quad r \geq r_2 \]
(see, for example, [7]), we conclude that \( u \) has slow decay.

The following result is a direct consequence of Pohozaev identity (2.2) and lemma 2.9.

**Lemma 2.13.** Assume that \( K \) satisfies (2.7). Let \( u \) be a positive smooth solution of equation (1.1) in \( \mathbb{R}^n \) with slow decay. Assume also that \( K \) satisfies one of the conditions in lemma 2.4 so that \( P(u) \) exists. If \( P(u) \neq 0 \), then

\[
(2.14) \quad u(x) \geq C_1|x|^{-(n-2)/2}
\]

for large \( |x| \) and for a positive constant \( C_1 \).

**Proof.** Suppose that (2.14) does not hold. Then there exists a sequence \( \{x_i\} \subset \mathbb{R}^n \) such that \( \lim_{i \to \infty} |x_i| = \infty \) and \( u(x_i)|x_i|^{-(n-2)/2} \to 0 \) as \( i \to \infty \). Let \( r_i = |x_i| \) for \( i = 1, 2, ... \). Using spherical Harnack inequality (2.8) and the gradient estimate in lemma 2.9 we have

\[
\frac{r_i^{(n-2)/2}}{S_{r_i}} \max_{S_{r_i}} u \to 0 \quad \text{and} \quad \frac{r_i^{n/2}}{S_{r_i}} \max_{S_{r_i}} |\bigtriangledown u| \to 0 \quad \text{as} \quad i \to \infty.
\]

From (2.2) we have \( P(u) = 0 \), which is a contradiction. Hence (2.14) holds.  \( \Box \)

Without the assumption of slow decay, we have the following estimates.

**Theorem 2.15.** Assume that \( K \) satisfies (2.7). Let \( u \) be a positive smooth solution of equation (1.1) in \( \mathbb{R}^n \). If there exists a positive constant \( \delta \) such that

\[
(2.16) \quad |P(u, r)| \geq \delta^2
\]

for large \( r \), then there exist positive constants \( C' \) and \( C'' \) such that

\[
(2.17) \quad \int_{B_{r}(r)} u^{\frac{2n}{n-2}}(x) \, dx \geq C' \ln r \quad \text{and} \quad \int_{B_{r}(r)} |\bigtriangledown u|^2 \, dx \geq C'' \ln r
\]

for large \( r \). In particular, if \( P(u) \) exists and is non-zero, then the total volume of \( (\mathbb{R}^n, g) \) is infinite.

**Proof.** Applying Young’s inequality we have

\[
(2.18) \quad \int_{S_{r}} u \frac{\partial u}{\partial r} \, dS \leq \int_{S_{r}} r \left| \frac{\partial u}{\partial r} \right|^2 \, dS + \int_{S_{r}} \frac{u^2}{r} \, dS \leq \int_{S_{r}} r |\bigtriangledown u|^2 \, dS + C_1 \int_{S_{r}} r^{\frac{n}{n-2}} \, dS + \frac{\delta^2}{n-2}
\]

for large \( r \), where \( C_1 \) is a positive constant depending on \( \delta \) and \( n \). From Pohozaev identity (2.2) and (2.18), there exist positive constant \( C'_2 \) and \( C'_3 \) such that

\[
(2.19) \quad C'_2 \int_{S_{r}} |\bigtriangledown u|^2 \, dS + C'_3 \int_{S_{r}} u^{\frac{2n}{n-2}} \, dS \geq \frac{\delta^2}{2r}
\]
for large $r$. It follows that
\begin{equation}
\int_{B_o(r)} |\nabla u|^2 \, dx + \int_{B_o(r)} u^{\frac{2n}{n-2}} \, dx \geq C_4 \ln r
\end{equation}
for large $r$ and for a positive constant $C_4$. We can modify the argument in the proof of theorem 3.1 in [17] for our case so as to obtain positive constants $C_5$ and $D_1$ such that
\begin{equation}
C_5 \int_{B_o(R)} u^{\frac{2n}{n-2}} \, dx + D_1 \geq \int_{B_o(R)} |\nabla u|^2 \, dx
\end{equation}
for all large $R$. (2.20) and (2.21) imply that
\begin{equation}
\int_{B_o(r)} u^{\frac{2n}{n-2}} \, dx \geq C' \ln r
\end{equation}
for large $r$ and for a positive constant $C'$. Similarly we obtain the other estimate in (2.17).

\begin{remark}
Partial results regarding the estimates in (2.17) are obtained in [9] and [17]. The arguments in [9] and [17] yield the stronger estimates that
\[
\int_{S_r} u^{\frac{2n}{n-2}} \, dS \geq \frac{C_6}{r} \quad \text{and} \quad \int_{S_r} |\nabla u|^2 \, dS \geq \frac{C_7}{r}
\]
for large $r$ and for some positive constants $C_6$ and $C_7$.
\end{remark}

Let $v$ be defined as in (1.11). Then $v$ satisfies the equation
\begin{equation}
\frac{\partial^2 v}{\partial s^2} + \Delta_\theta v - \left(\frac{n-2}{2}\right)^2 v + K v^{\frac{n+2}{n-2}} = 0 \quad \text{in} \quad \mathbb{R} \times S^{n-1},
\end{equation}
where $\Delta_\theta$ is the standard Laplacian on $S^{n-1}$ (see, for example, [17]). The Pohozaev identity (2.2) becomes
\begin{equation}
P(v, s) = \frac{n-2}{2n} \int_{-\infty}^{s} \int_{S^{n-1}} \frac{\partial K}{\partial t} v^{\frac{2n}{n-2}} \, d\theta \, dt
\end{equation}
for $s \in \mathbb{R}$. With the notations, condition (1.13) in theorem D can also be written as
\begin{equation}
\lim_{s \to \infty} \int_{S^{n-1}} \left[\frac{\partial v}{\partial s}(s, \theta)\right]^2 \, d\theta = 0.
\end{equation}
3. Slow decay

Proof of Theorem A. By differentiating the function

\[ \int_{S_r} u^{\frac{2n}{n-2}} \, dS = \int_{S_{n-1}} u^{\frac{2n}{n-2}}(r, \theta) \, r^{n-1} \, d\theta \quad \text{for} \quad r > r_o \]

and using (1.9), there exist positive constants \( C_o \) and \( m \) such that

(3.1) \[ \int_{S_r} u^{\frac{2n}{n-2}} \, dS \leq C_o \, r^m \quad \text{for} \quad r \geq r_o. \]

By (1.8) and (3.1) we have

\[ \frac{n - 2}{2n} \int_{B_o(R)} r \frac{\partial K}{\partial r} u^{\frac{2n}{n-2}} \, dx \geq -C_1 - C_2 \int_{r_1}^R e^{-cr \, r^m+1} \, dr \geq -C_3 \quad \text{for} \quad R \geq r_1, \]

where \( C_1, C_2 \) and \( C_3 \) are positive constants and \( r_1 \geq \max \{ r_o, r' \} \). Together with (1.9) and Pohozaev identity (2.2) we obtain

(3.2) \[ \int_{S_R} R | \nabla u |^2 \, dS \leq 2C_3 + C_4 \int_{B_o(R)} \frac{u^2}{r} \, dx + \frac{n - 2}{n} \int_{B_o(R)} K u^{\frac{2n}{n-2}} \, dS \]

for large \( R \), where \( C_4 \) is a positive constant. By choosing a larger \( C_3 \) if necessary, we may assume that (3.2) holds for all \( R > 0 \). Similarly, as \( u > 0 \) in \( \mathbb{R}^n \), we may assume that (1.9) holds for \( r > 0 \). Integrating both sides of (3.2) we obtain

(3.3) \[ \int_{B_o(R)} \frac{r}{n}| \nabla u |^2 \, dx \leq 2C_3 R + C_4 \int_{B_o(R)} \frac{u^2}{r} \, dx + \frac{n - 2}{n} \int_{B_o(R)} K u^{\frac{2n}{n-2}} \, dx \]

for \( R > 0 \). Using equation (1.1) and (1.9) we have

(3.4) \[ \int_{B_o(R)} r K u^{\frac{2n}{n-2}} \, dx = \int_{B_o(R)} ru \left( K u^{\frac{n+2}{n-2}} \right) \, dx = \int_{B_o(R)} (ru)(-\Delta u) \, dx \]

\[ = \int_{B_o(R)} \frac{r}{n}| \nabla u |^2 \, dx + \int_{B_o(R)} \frac{\partial u}{\partial r} \, dx - \int_{S_R} R u \frac{\partial u}{\partial r} \, dS \]

\[ \leq \int_{B_o(R)} \frac{r}{n}| \nabla u |^2 \, dx + C_5 \int_{B_o(R)} \frac{u^2}{r} \, dx + C_6 \int_{S_R} u^2 \, dS \]

for large \( R \), where \( C_5 \) and \( C_6 \) are positive constants. It follows from (3.3) and (3.4) that

(3.5) \[ \frac{2}{n} \int_{B_o(R)} r K u^{\frac{2n}{n-2}} \, dx \]

\[ \leq 2C_3 R + C_7 \int_{B_o(R)} \frac{u^2}{r} \, dx + C_8 \int_{S_R} u^2 \, dS \]

\[ \leq 2C_3 R + C_7 \epsilon \int_{S_R} r^2 u^{\frac{2n}{n-2}} \, dS + C_8 \epsilon \int_{B_o(R)} ru^{\frac{2n}{n-2}} \, dx + C_9 \epsilon^{-(n-2)/2} R \]
for large \( R \), where we use Young’s inequality and \( C_7, C_8 \) and \( C_9 \) are positive constants. From (1.9) we have

\[
\frac{d}{dr} \left( \int_{S_r} r^2 u^{\frac{2n}{n-2}} \, dS \right) \leq C_{10} \int_{S_r} ru^{\frac{2n}{n-2}} \, dS \quad \text{for } r \geq r_o,
\]

where \( C_{10} \) is a positive constant. Hence

\[
\int_{S_r} r^2 u^{\frac{2n}{n-2}} \, dS = \int_0^r \left( \int_{S_s} s^2 u^{\frac{2n}{n-2}} \, dS \right) \, ds \\
\quad \leq \ C_{11} + \int_{r_o}^r \left( \int_{S_s} s^2 u^{\frac{2n}{n-2}} \, dS \right) \, ds \\
\quad \leq \ C_{11} + C_{10} \int_{r_o}^r su^{\frac{2n}{n-2}} \, dSds \\
\quad \leq \ C_{12} \int_{B_{o}(r)} ru^{\frac{2n}{n-2}} \, dx \quad \text{for } r \geq 2r_o,
\]

where \( C_{11} \) and \( C_{12} \) are positive constants that depends on \( r_o \) and \( u \). If

\[
\lim_{R \to \infty} \int_{B_{o}(R)} u^{\frac{n}{n-2}} \, dx = \infty,
\]

then by using theorem 3.1 in [17], which can be modified for \( K \) satisfying (2.7) by making the constants there larger if necessary, together with a result of Brezis and Kato [3], we have

\[
u(x) \leq C_{13}|x|^{-(n-2)}
\]

for large \( |x| \) and for a positive constant \( C_{13} \) (cf. the proof of theorem 3.16 in [17]). In particular, \( u \) has slow decay. So we may assume that

\[
\lim_{R \to \infty} \int_{B_{o}(R)} u^{\frac{n}{n-2}} \, dx = \infty.
\]

As \( K(x) \geq a^2 \) for large \( |x| \), it follows from (3.8) that

\[
\int_{B_{o}(R)} rKu^{\frac{2n}{n-2}} \, dx \geq \frac{a^2}{2} \int_{B_{o}(R)} ru^{\frac{2n}{n-2}} \, dx
\]

for large \( R \). If we choose \( \epsilon \) to be small, then (3.5), (3.6) and (3.9) imply that

\[
\int_{B_{o}(R)} ru^{\frac{2n}{n-2}} \, dx \leq C_{14}R \quad \text{for } R \geq r_o
\]

for large \( R \) and

\[
\int_{S_R} R^2 u^{\frac{2n}{n-2}} \, dS \leq C_{15}R
\]

for large \( R \), where \( C_{14} \) and \( C_{15} \) are positive constants. The last inequality can also be written as

\[
\int_{S^{n-1}} u^{\frac{2n}{n-2}}(r, \theta) \, d\theta \leq C_{15}r^{-n}
\]
for large $r$. Using lemma 4.39 and 4.40 in [17], we obtain spherical Harnack inequality (2.8) for spheres of large radius. Together with (2.12) or (3.10) we conclude that $u$ has slow decay.

\[ \square \]

**Remark 3.11.** Condition (1.8) in theorem A is used only in obtaining a lower bound on $P(u, r)$. Therefore theorem A remains valid if we replace (1.8) by the assumption that $P(u, r) \geq -\delta^2$ for large $r$ and for a constant $\delta$.

### 4. Asymptotic behavior

**Theorem 4.1.** Assume that $\lim_{|x| \to \infty} K(x) = K_\infty > 0$ and $|\nabla K| \leq C_o$ in $\mathbb{R}^n$ for a positive number $C_o$. Let $u$ be a positive smooth solution of equation (1.1) in $\mathbb{R}^n$ with slow decay. Assume also that $K$ satisfies one of the conditions in lemma 2.4 so that $P(u)$ exists. Then we have $P(u) \leq 0$. If $P(u) = 0$, then either $u$ has fast decay, or the conformal metric $u^{4/(n-2)}g_o$ is complete and

\[ \lim_{\|x\| \to \infty} |x|^{-\frac{n-2}{2}} u(x) = 0 \]

and there exists a sequence $\{s_l\} \subset \mathbb{R}$ of local maxima of $\bar{v}$, $s_l \to +\infty$ as $l \to +\infty$, such that the sequence $\{v_l\}$ defined by

\[ v_l(s, \theta) = v(s_l + s, \theta) \quad \text{for} \quad s \in \mathbb{R}, \quad \theta \in S^{n-1} \quad \text{and} \quad l = 1, 2, ..., \]

converges uniformly in $C^2$-norm on compact subsets of $\mathbb{R} \times S^{n-1}$ to $C(\cosh s)^{(2-n)/2}$ for a positive constant $C$ that depends on $K_\infty$. Here $v$ and $\bar{v}$ are defined in (1.11) and (1.12) respectively.

**Proof.** Suppose that $P(u) > 0$. Lemma 2.13 implies that there exist positive constants $c_1$, $c_2$ and $s_o$ such that

\[ c_1 \leq v(s, \theta) \leq c_2 \quad \text{for} \quad s \geq s_o \quad \text{and} \quad \theta \in S^{n-1}. \]

We have $P(u) = P(v) = \lim_{s \to +\infty} P(v, s)$. For $t > 0$, let

\[ v_t(s, \theta) = v(s + t, \theta) \quad \text{for} \quad \theta \in S^{n-1} \quad \text{and} \quad s \geq s_o - t. \]

Using the equation

\[ \frac{\partial^2 v_t}{\partial s^2} + \Delta_\theta v_t = F_t \quad \text{for} \quad s \geq s_o - t, \]

where

\[ F_t(s, \theta) = \left( \frac{n-2}{2} \right)^2 v_t(s, \theta) - K(s + t, \theta) v_t^{\frac{n+4}{n-2}}(s, \theta) \]
for $s \in \mathbb{R}$ and $\theta \in S^{n-1}$, (4.2), the boundedness of $|\nabla K|$, and elliptic estimates (cf. [13]), there exists a sequence of positive numbers $t_i \to \infty$ as $i \to \infty$ and a $C^2$-function $v_\infty$ defined on $\mathbb{R} \times S^{n-1}$ such that $v_{t_i}$ converges in $C^2$-norm on compact subsets of $\mathbb{R} \times S^{n-1}$ to $v_\infty$ as $i \to \infty$. Furthermore, $v_\infty$ satisfies the equation

$$
(4.5) \quad \frac{\partial^2 v_\infty}{\partial s^2} + \Delta_\theta v_\infty - \left(\frac{n-2}{2}\right)^2 v_\infty + K v_\infty \frac{n+2}{n-2} = 0 \quad \text{in} \quad \mathbb{R} \times S^{n-1},
$$

and $c_1 \leq v_\infty \leq c_2$ in $\mathbb{R} \times S^{n-1}$. By a result of Caffarelli, Gidas, Spruck in [4], $v_\infty$ is independent on $\theta$. It follows that the Pohozaev number $P(v_\infty)$ is negative (cf. section 2.1 in [13]). Hence $P(v_\infty, s) < 0$ for large $s$. Fixed a large number $s_0$ so that $P(v_\infty, s_0) < 0$. As $v_{t_i}$ converges in $C^2$-norm to $v_\infty$ in a compact neighborhood of $s_0 \times S^{n-1}$, we have $P(v_{t_i}, s_0) < 0$ for $i$ large. But $P(v_{t_i}, s_0) = P(v, t_i + s_0) \to P(v) > 0$ as $i \to \infty$. Therefore we have a contradiction. Hence $P(u) \leq 0$. Assume that $P(u) = 0$. It follows from the above argument that $v$ cannot be bounded away from zero in $\mathbb{R}^+ \times S^{n-1}$, that is,

$$
\lim \inf_{|x| \to \infty} |x|^{-\frac{n+2}{n-2}} u(x) = 0.
$$

Define

$$
(4.6) \quad \bar{v}(s) = \int_{S^{n-1}} v(s, \theta) d\theta \quad \text{for} \quad s \in \mathbb{R}.
$$

Then $\bar{v}$ is bounded from above in $\mathbb{R}$. Either $\bar{v}'(s) \geq 0$ or $\bar{v}'(s) \leq 0$ for all large $s$, or $\bar{v}'$ keeps on changing sign near positive infinity. In the first two cases we have $\lim_{s \to +\infty} \bar{v}(s) = \nu_o$ exists, as $\bar{v}(s)$ is bounded positive for large $s$. If $\nu_o > 0$, then using spherical Harnack inequality (2.8), there exists a positive constant $c_3$ such that $v(s, \theta) \geq c_3^2$ for large $s$ and for $\theta \in S^{n-1}$. Hence we may apply the above argument to show that $P(u) < 0$, which is a contradiction. Assume that $\nu_o = 0$. Spherical Harnack inequality (2.8) implies that

$$
\lim_{s \to +\infty} \max_{\theta \in S^{n-1}} v(s, \theta) = 0.
$$

It seems standard to conclude that $u$ has fast decay. We show this in the appendix. Therefore we need only to consider the case when $\bar{v}'$ keeps on changing sign near positive infinity. If this is the case, then there exists a sequence of numbers $\{s_i\}$ such that $\lim_{i \to \infty} s_i = +\infty$ and $\bar{v}$ achieves local maximum at $s_i$ for $i = 1, 2, \ldots$

From (A.8) below, there exists a positive constant $C_2$ such that

$$
(4.7) \quad \bar{v}''(s) - \left(\frac{n-2}{2}\right)^2 \bar{v}(s) + C_2 \bar{v}^{\frac{n+2}{n-2}}(s) \geq 0
$$

for large positive $s$. Using (4.7) and the fact that $\bar{v}''(s_i) \leq 0$ we have

$$
(4.8) \quad \bar{v}(s_i) \geq \left[ \frac{1}{C_2} \left(\frac{n-2}{2}\right)^2 \right]^{\frac{1}{n+2}}
$$
for large $i$. It follows from spherical Harnack inequality (2.8) that there exists a positive constant $c_4$ such that

$$v(s_i, \theta) \geq c_4^2 \quad \text{for} \quad \theta \in S^{n-1}$$

and for large $i$. Define

$$v_i(s, \theta) = v(s + s_i, \theta) \quad \text{for} \quad s \in \mathbb{R}, \quad \theta \in S^{n-1}$$

and $i = 1, 2, ...$. As above, a subsequence of $\{v_i\}$ converges in $C^2$-norm on compact subsets of $\mathbb{R} \times S^{n-1}$ to a non-negative $C^2$ function $v_\infty$ on $\mathbb{R} \times S^{n-1}$ which satisfies the equation

$$\frac{\partial^2 v_\infty}{\partial s^2} + \Delta_\theta v_\infty = v_\infty \left[ \left( \frac{n-2}{2} \right)^2 - K_\infty \frac{4}{v_\infty^{n-4}} \right] \quad \text{in} \quad \mathbb{R} \times S^{n-1}.$$  

Furthermore, from (4.9) we have

$$v_\infty(0, \theta) \geq c_4^2 > 0 \quad \text{for} \quad \theta \in S^{n-1}.$$ 

Let

$$u_\infty(x) = u_\infty(r, \theta) = r^{-(n-2)/2} v_\infty(s, \theta) \quad \text{for} \quad |x| > 0,$$

where $x = (r, \theta)$ and $r = e^s$. Then $u_\infty$ satisfies the equation

$$\Delta u_\infty = -K_\infty u_\infty^{n+2} \quad \text{in} \quad \mathbb{R}^n \setminus \{0\}.$$ 

Furthermore $u_\infty(x) \geq 0$ for $|x| > 0$ and $u_\infty(|x|) \geq c_4^2 > 0$ for $|x| = 1$. It follows from equation (4.12) and the maximum principle that $u_\infty(x) > 0$ for $|x| > 0$. Hence $v_\infty$ is positive in $\mathbb{R} \times S^{n-1}$. Because $v_\infty$ may not be bounded away from zero, we can only concluded that $P(v_\infty) \leq 0$. The above argument shows that if $P(v_\infty) < 0$, then $P(u) < 0$. Hence $P(v_\infty) = 0$. By a result of Caffarelli, Gidas and Spruck ([4], see also [15]), $v_\infty$ is radial, and together with $v'_\infty(0) = 0$ we have

$$v_\infty(s) = C(cosh(s)^{(2-n)/2}) \quad \text{for} \quad s \in \mathbb{R} \quad \text{and for a positive number C depending on} \quad K_\infty.$$ 

It remains to show that the conformal metric corresponding to $u$ is complete in this case. From (4.7) and the fact that $v$ is bounded from above, there exists a positive constant $c_5$ such that

$$v''(s) \geq -c_5^2 \quad \text{for} \quad s \in \mathbb{R}.$$ 

At $s_i$ we have $v'(s_i) = 0$, therefore $v'(s)$ is not too negative for $s$ close to $s_i$ for $i = 1, 2, ...$ By using (4.8) and (4.13), there exists a positive number $\epsilon$ independent on $i$ such that

$$v(s) \geq \frac{1}{2} \left[ \frac{1}{C_2} \left( \frac{n-2}{2} \right)^2 \right]^{1/2} \quad \text{for} \quad s \in [s_i, s_i + \epsilon], \quad i = 1, 2, ...$$
Using spherical Harnack inequality (2.8) there exists a positive constant $c_6$ such that

\begin{equation}
(4.15) \quad v(s, \theta) \geq c_6^2 \quad \text{for } s \in [s_i, s_{i+1}], \ \theta \in S^{n-1}
\end{equation}

and $i = 1, 2, \ldots$ Without loss of generality, we may assume that $s_1 > 0$ and $s_{i+1} > s_i$ for $i = 1, 2, \ldots$ For any fixed $\theta \in S^{n-1}$, the length of the curve $r \mapsto (r, \theta), \ r \in (1, \infty)$, in the conformal metric $u^{4/(n-2)}g_o$ is given by

\[
\int_1^\infty u^{\frac{2}{n-2}}(r, \theta) \, dr = \int_0^\infty v^{\frac{2}{n-2}}(s, \theta) \, ds \geq \sum_{i=1}^{\infty} \int_{s_i}^{s_{i+\epsilon}} v^{\frac{2}{n-2}}(s, \theta) \, ds = \infty.
\]

Hence the conformal metric is complete. \hfill \Box

**Corollary 4.16.** Assume that $K$ satisfies (2.7). Let $u$ be a positive smooth solution of equation (1.1) in $\mathbb{R}^n$ with slow decay. Then either $u$ has fast decay or the conformal metric $g = u^{4/(n-2)}g_o$ is complete. Furthermore, the conformal metric $g$ is complete if and only if the total volume of $(\mathbb{R}^n, g)$ is infinite.

**Proof.** Consider $\bar{v}$ as defined in (4.6) and $\lim_{s \to \infty} \bar{v}(s)$. If the limit exists and is equal to zero, then $u$ has fast decay (see appendix A). If the limit is non-zero, then $u(x) \geq c|x|^{(2-n)/2}$ for large $|x|$ and for a positive constant $c$, and hence the conformal metric $g$ is complete. In case the limit does not exist, then $\bar{v}'$ keeps changing signs near positive infinity. The argument in the last part of the proof of theorem 4.1 shows that the conformal metric is also complete in this case. Finally, if the total volume of $(\mathbb{R}^n, g)$ is infinite, then $u$ does not have fast decay and hence by above $g$ is complete. If $g$ is complete, then either $\lim_{s \to +\infty} \bar{v}(s)$ exists and is positive, or (4.15) holds. In either cases we have

\[
\int_{\mathbb{R}^n \setminus B_o(1)} u^{\frac{2n}{n-2}}(x) \, dx = \int_0^\infty \int_{S^{n-1}} v^{\frac{2n}{n-2}} \, d\theta \, ds = \infty.
\]

That is, the total volume of $(\mathbb{R}^n, g)$ is infinite. \hfill \Box

**Theorem 4.17.** Assume that $K$ satisfies the conditions in theorem 4.1 and $x \cdot \nabla K(x) \leq 0$ for large $|x|$. Let $u$ be a positive smooth solution of equation (1.1) in $\mathbb{R}^n$ with slow decay. If $P(u) = 0$, then $u$ has fast decay.

**Proof.** Let $\bar{v}$ be defined as in (4.6). Suppose that $P(u) = 0$ and $u$ does not have fast decay. It follows from the argument in the proof of theorem 4.1 that there exists a sequence $\{s_j\} \subset \mathbb{R}$ such that $s_j \to +\infty$ as $j \to \infty$ and each $s_j$ is a local minimum for $\bar{v}$ and $\lim_{j \to \infty} \bar{v}(s_j) = 0$. Let

\begin{equation}
(4.18) \quad w_j(s, \theta) = \frac{v(s + s_j, \theta)}{\bar{v}(s_j)} \quad \text{for } s \in \mathbb{R}, \ \theta \in S^{n-1} \text{ and } j = 1, 2, \ldots
\end{equation}
Given $S > 0$, we claim that there exist positive numbers $C_s$ and $k_s$ such that

$$w_j(s, \theta) \leq C_s \quad \text{for} \quad s \in [-S, S], \quad \theta \in S^{n-1} \quad \text{and} \quad j \geq k_s.$$  

From (A.7) and (A.8) in the appendix we have

$$\left(\frac{n-2}{2}\right)^2 \tilde{v}(s) - C_1 \tilde{v}^{n+2}(s) \geq \tilde{v}''(s) \geq \left(\frac{n-2}{2}\right)^2 \tilde{v}(s) - C_2 \tilde{v}^{n+2}(s)$$

for large $s$ and for some positive constants $C_1$ and $C_2$. It follows that there exists a positive constant $\varepsilon_o$ such that

$$0 \leq \tilde{v}''(s) \leq \frac{1}{2} \left(\frac{n-2}{2}\right)^2 \tilde{v}(s) \quad \text{for} \quad \tilde{v}(s) \leq \varepsilon_o \quad \text{and} \quad \text{large} \quad s.$$  

Choose the number $k_s$ such that

$$\tilde{v}(s) e^{\frac{1}{2} \left(\frac{n-2}{2}\right)^2 s^2} < \varepsilon_o \quad \text{for} \quad j \geq k_s.$$  

We may also assume that for $j \geq k_s$, $s_j$ is large. For $j \geq k_s$ and $s$ close to $s_j$, $\tilde{v}''(s)$ satisfies (4.20). Hence we obtain

$$\tilde{v}'(s) \leq \frac{1}{2} \left(\frac{n-2}{2}\right)^2 \int_{s_j}^{s} \tilde{v}(t) \, dt \leq \frac{1}{2} \left(\frac{n-2}{2}\right)^2 \tilde{v}(s)(s - s_j)$$

for $s > s_j$ close to $s_j$, as $\tilde{v}'(s) \geq 0$ for $s \geq s_j$ close to $s_j$. Therefore we have

$$\frac{\tilde{v}(s)}{\tilde{v}(s_j)} \leq e^{\frac{1}{2} \left(\frac{n-2}{2}\right)^2 (s - s_j)^2} \quad \text{for} \quad s \in [s_j, s_j + S].$$

Likewise, we obtain a similar inequality on $[s_j - S, s_j]$. By using spherical Harnack inequality (2.8), we obtain the desired constant $C_s$ and the bound in (4.19). It follows from (4.19) that there exists a subsequence of $\{w_j\}$ which converges in $C^2$-norm on compact subsets of $\mathbb{R} \times S^{n-1}$ to a solution $w$ of the equation

$$\frac{\partial^2 w}{\partial s^2} + \Delta_\theta w - \left(\frac{n-2}{2}\right)^2 w = 0 \quad \text{in} \quad \mathbb{R} \times S^{n-1}.$$  

The associated function $h$ related to $w$ as in (1.11) is, by the maximum principle and the above equation, a positive harmonic function on $\mathbb{R}^n \setminus \{0\}$. Therefore $h(x) = a|x|^{2-n} + b$ and

$$w(s, \theta) = a e^{-\frac{n-2}{2} s} + b e^{\frac{n-2}{2} s} \quad \text{for} \quad s \in \mathbb{R} \quad \text{and} \quad \theta \in S^{n-1}$$

for some positive constants $a$ and $b$. As $\tilde{w}$ has a critical point at $s = 0$, we have $a = b > 0$. As in [13] we obtain

$$\lim_{j \to \infty} \int_{S^{n-1}} \left[ \frac{1}{2} \left( \frac{\partial w_j}{\partial s} \right)^2 (0, \theta) - \frac{1}{2} |\nabla_\theta w_j|^2 (0, \theta) - \frac{1}{2} \left(\frac{n-2}{2}\right)^2 w_j^2 (0, \theta) \right]$$

as $\varepsilon_o \to 0$. 

\[\text{(4.23)}\]
\[
+ \bar{v}_{\frac{n-2}{2}}(s_j) \frac{n-2}{2n} K w_j^{\frac{2n}{n-2}}(0, \theta) \]
\[
= \int_{S^{n-1}} \left[ \frac{1}{2} \left( \frac{\partial w}{\partial s} \right)^2 (0, \theta) - \frac{1}{2} \left( \frac{n-2}{2} \right)^2 w^2(0, \theta) \right] d\theta
\]
\[
= -\omega_n \left( \frac{n-2}{2} \right)^2 ab < 0.
\]

Using the assumption that \((\partial K/\partial s)(s, \theta) \leq 0\) for large \(s\) and \(\theta \in S^{n-1}\), Pohozaev identity (2.25) and \(\lim_{s \to +\infty} P(v, s) = 0\), we see that \(P(v, s) \geq 0\) for large \(s\). On the other hand, by (4.23) we have

\[
\frac{P(v, s_j)}{\bar{v}^2(s_j)} = \int_{S^{n-1}} \left[ \frac{1}{2} \left( \frac{\partial w_j}{\partial s} \right)^2 (0, \theta) - \frac{1}{2} |\nabla w_j|^2(0, \theta) - \frac{1}{2} \left( \frac{n-2}{2} \right)^2 w_j^2(0, \theta)
+ \bar{v}_{\frac{n-2}{2}}(s_j) \frac{n-2}{2n} K w_j^{\frac{2n}{n-2}}(0, \theta) \right] d\theta < 0
\]

for \(j\) large. Hence we have a contradiction. Therefore if \(P(u) = 0\), then \(u\) has fast decay. \(\square\)

**Remark 4.24.** One can also prove theorem 4.17 by modifying the argument of Chen and Lin in section 3 of [5].

**Theorem 4.25.** Assume that \(K\) satisfies (2.7). Let \(u\) be a positive smooth solution of equation (1.1) in \(\mathbb{R}^n\). Assume also that

\[
(4.26) \quad \lim_{r \to \infty} \int_{S_r} r \left[ \frac{\partial u}{\partial r} + \frac{n-2}{2} \frac{u}{r} \right]^2 dS = 0.
\]

If \(P(u)\) exists, then \(P(u) \leq 0\). Furthermore, \(P(u) = 0\) implies that

\[
(4.27) \quad \lim \inf_{|x| \to \infty} |x|^{(n-2)/2} u(x) = 0.
\]

**Proof.** Let

\[
(4.28) \quad \omega(r) = r^{n-2} \int_{S^{n-1}} u^2(r, \theta) d\theta \quad \text{for} \quad r > 0.
\]

We have

\[
(4.29) \quad \omega'(r) = 2 r^{n-2} \int_{S^{n-1}} u \left[ \frac{\partial u}{\partial r} + \frac{n-2}{2} \frac{u}{r} \right] d\theta \quad \text{for} \quad r > 0.
\]

Using equation (1.1) we obtain

\[
(4.30) \quad \frac{\omega''(r)}{4} = \frac{n-3}{2} r^{n-3} \int_{S^{n-1}} u \left[ \frac{\partial u}{\partial r} + \frac{n-2}{2} \frac{u}{r} \right] d\theta + \frac{r^{n-2}}{2} \left[ \int_{S^{n-1}} |\nabla u|^2 d\theta - \int_{S^{n-1}} Ku^{\frac{2n}{n-2}} d\theta \right]
\]
for \( r > 0 \). From Pohozaev identity (2.2) we also have

\[
(4.31) \quad P(u, r) = \int_{S_r} r \left[ \frac{\partial u}{\partial r} + \frac{n-2}{2} \frac{u}{r} \right]^2 dS - \frac{n-2}{2} \int_{S_r} u \left[ \frac{\partial u}{\partial r} + \frac{n-2}{2} \frac{u}{r} \right] dS \\
- \frac{1}{2} \int_{S_r} r |\nabla u|^2 dS + \frac{n-2}{2n} \int_{S_r} r K u^\frac{2n}{n-2} dS
\]

for \( r > 0 \). Under condition (4.26) and \( P(u) \geq 0 \), we claim that \( \omega'(r) \) cannot be non-negative for all large \( r \). In order to obtain a contradiction, suppose that \( \omega'(r) \geq 0 \) for large \( r \). From (4.29) we have

\[
(4.32) \quad \omega'(r) \leq 2 \left( \int_{S_r} \frac{u^2}{r} dS \right)^\frac{1}{2} \left( \int_{S_r} \frac{1}{r} \left[ \frac{\partial u}{\partial r} + \frac{n-2}{2} \frac{u}{r} \right]^2 dS \right)^\frac{1}{2} \leq \frac{C_1 \omega(r)}{r}
\]

for large \( r \), where \( C_1 \) is a positive constant. If there exists a positive constant \( c' \) such that \( \omega'(r) \geq c' \) for large \( r \), then we have

\[
(4.33) \quad \omega(r) \geq \frac{c' r}{2}
\]

for large \( r \). It follows from (4.32) that

\[
(4.34) \quad \omega'(r) \leq \frac{C_2 \omega(r)}{r^\frac{n}{2}}
\]

for large \( r \) and a positive constant \( C_2 \). Integrating both sides of (4.34) we conclude that \( \omega \) is bounded from above in \( \mathbb{R}^+ \), which contradicts (4.33). Therefore we may assume that

\[
(4.35) \quad \liminf_{r \to \infty} \omega'(r) = 0.
\]

Let \( c \) to be a positive constant to be fixed later. Suppose that the inequality

\[
(4.36) \quad \omega''(r) \leq -c \frac{\omega(r)}{r^2}
\]

does not hold for all large \( r \). Then there exists a sequence of positive numbers \( r_j \to \infty \) as \( j \to \infty \) such that

\[
(4.37) \quad \omega''(r_j) > -c \frac{\omega(r_j)}{r_j^2}
\]

for \( j = 1, 2, ... \). From (4.30) we have

\[
(4.38) \quad \frac{1}{2} \int_{S_{r_j}} r_j |\nabla u|^2 dS + \frac{n-3}{2} \int_{S_{r_j}} u \left[ \frac{\partial u}{\partial r} + \frac{n-2}{2} \frac{u}{r_j} \right] dS + \frac{c \omega(r_j)}{4} \geq \frac{1}{2} \int_{S_{r_j}} r_j K u^\frac{2n}{n-2} dS
\]
for \( j = 1, 2, \ldots \) On the other hand, from (4.31) we have

\[
\frac{1}{2} \int_{S_{r_j}} r_j K u^{\frac{2n}{n-2}} dS = P(u, r_j) + \frac{1}{2} \int_{S_{r_j}} |\nabla u|^2 dS \\
+ \frac{n-3}{2} \int_{S_{r_j}} u \left[ \frac{\partial u}{\partial r} + \frac{n-2}{2} \frac{u}{r_j} \right] dS + \frac{1}{2} \int_{S_{r_j}} u \left[ \frac{\partial u}{\partial r} + \frac{n-2}{2} \frac{u}{r_j} \right] dS \\
- \int_{S_{r_j}} r_j \left[ \frac{\partial u}{\partial r} + \frac{n-2}{2} \frac{u}{r_j} \right]^2 dS + \frac{1}{n} \int_{S_{r_j}} r_j K u^{\frac{2n}{n-2}} dS
\]

for \( j = 1, 2, \ldots \) Using the Hölder inequality and the Young inequality we obtain

\[
\omega(r_j) = \int_{S_{r_j}} \frac{u^2}{r_j} dS \leq \omega_n^{2/n} \left( \int_{S_{r_j}} r_j u^{\frac{2n}{n-2}} dS \right)^{\frac{n-2}{n}}
\]

and

\[
\omega(r_j) \leq \frac{n-2}{n} \int_{S_{r_j}} r_j u^{\frac{2n}{n-2}} dS + \frac{2}{n} \omega_n
\]

for \( j = 1, 2, \ldots \) As \( \omega'(r) \geq 0 \) for large \( r \) and \( K(x) \geq a^2 \) for large \( |x| \), it follows from (4.40) that there exists a positive number \( c_o \) such that

\[
\int_{S_{r_j}} r_j K u^{\frac{2n}{n-2}} dS \geq c_o^2
\]

for \( j = 1, 2, \ldots \) Using (4.26), (4.29), (4.41) and (4.42) we have

\[
\frac{1}{n} \int_{S_{r_j}} r_j K u^{\frac{2n}{n-2}} + P(r, r_j) + \frac{1}{2} \int_{S_{r_j}} u \left[ \frac{\partial u}{\partial r} + \frac{n-2}{2} \frac{u}{r_j} \right] dS \\
- \int_{S_{r_j}} r_j \left[ \frac{\partial u}{\partial r} + \frac{n-2}{2} \frac{u}{r_j} \right]^2 dS > \frac{c \omega(r_j)}{4}
\]

for large \( J \), if we choose \( c \) to be small enough. (4.38), (4.39) together with (4.43) give a contradiction. Hence (4.36) holds for a small \( c \). Integrating both sides of (4.36) and using the assumption that \( \omega(r) \) is non-decreasing for large \( r \), we obtain

\[
\omega'(R) - \omega'(r) \leq -c \int_r^R \frac{\omega(s)}{s^2} ds \leq -c \omega(r) \int_r^R \frac{ds}{s^2} = c \omega(r) \left[ \frac{1}{R} - \frac{1}{r} \right]
\]

for large \( R \) and \( r \) with \( R > r \). Letting \( R \) approaches infinity by a sequence which satisfies (4.35) we have

\[
\omega'(r) \geq \frac{c \omega(r)}{r} \quad \text{for} \quad r \geq r_o.
\]

Integrating both sides of (4.45), there exists a positive constant \( C_3 \) such that

\[
\omega(r) \geq C_3 r^c \quad \text{for} \quad r \geq r_o.
\]
Together with (4.32) we have
\begin{equation}
\omega'(r) \leq \frac{C_4 \omega(r)}{r^{1+\frac{2}{n}}} 
\end{equation}
for large \( r \), where \( C_4 \) is a positive constant. Integrating both sides of (4.47) we conclude again that \( \omega \) is bounded from above in \( \mathbb{R}^+ \), which contradicts (4.46). Hence we may dismiss the case that \( \omega'(r) \) is non-negative for all large \( r \).

Next, assume that \( \omega \) achieves a local minimum at \( r_o \). Therefore we have \( \omega'(r_o) = 0 \) and \( \omega''(r_o) \geq 0 \). It follows from (4.29), (4.30) and (4.31) that
\begin{equation}
\frac{1}{2} \int_{S_{r_o}} r_o \nabla u^2 \, dS \geq \frac{1}{2} \int_{S_{r_o}} r_o Ku^{\frac{2n}{n-2}} \, dS \quad \text{and} \\
\frac{1}{2} \int_{S_{r_o}} r_o Ku^{\frac{2n}{n-2}} \, dS = P(u, r_o) + \frac{1}{2} \int_{S_{r_o}} r_o \nabla u^2 \, dS - \int_{S_{r_o}} r_o \left[ \frac{\partial u}{\partial r} + \frac{n-2}{2} u \right]^2 \, dS \\
+ \frac{1}{n} \int_{S_{r_o}} r_o Ku^{\frac{2n}{n-2}} \, dS .
\end{equation}
Therefore if \( P(u) > 0 \), or \( P(u) = 0 \) and \( \omega(r) \geq c_1^2 > 0 \) for large \( r \) and for a positive constant \( c_1 \), then \( \omega \) has no local minimum on \((r_1, \infty)\) for large \( r_1 \). Hence \( \omega'(r) \leq 0 \) for large \( r \) and \( \omega(r) \to C_0 \geq 0 \) as \( r \to \infty \). Using a result in \( [4] \), there exist a sequence \( \{r_i\} \) of positive numbers such that \( \lim_{i \to \infty} r_i = \infty \),
\begin{equation}
(4.48) \quad r_i \omega'(r_i) \to 0 \quad \text{and} \quad r_i [r_i \omega''(r_i) + \omega'(r_i)] \to 0 \quad \text{as} \quad i \to \infty .
\end{equation}
The first limit in (4.48) shows that
\begin{equation}
(4.49) \quad \int_{S_{r_i}} u \left[ \frac{\partial u}{\partial r} + \frac{n-2}{2} \frac{u}{r_i} \right] \, dS \to 0 \quad \text{as} \quad i \to \infty .
\end{equation}
The second limit in (4.48) together with (4.49) imply that
\begin{equation}
(4.50) \quad \int_{S_{r_i}} r_i \nabla u^2 \, dS - \int_{S_{r_i}} r_i Ku^{\frac{2n}{n-2}} \, dS \to 0 \quad \text{as} \quad i \to \infty .
\end{equation}
It follows from (4.26), (4.31), (4.49) and (4.50) that
\begin{equation}
(4.51) \quad P(u, r_i) \to -\frac{1}{n} \int_{S_{r_i}} r_i Ku^{\frac{2n}{n-2}} \, dS \leq 0 \quad \text{as} \quad i \to \infty .
\end{equation}
Hence \( P(u) \leq 0 \), which is a contradiction. From (4.40) and (4.51), if \( \omega(r) \geq c_1^2 \) for large \( r \), then \( P(u) < 0 \). Therefore if \( P(u) = 0 \), then \( \liminf_{r \to \infty} \omega(r) = 0 \), which in turns implies that \( \liminf_{|x| \to \infty} |x|^{(n-2)/2} u(x) = 0 \).

Let \( u_o = C r^{-(n-2)/2} \) for \( r \geq 1 \), where \( C \) is a positive constant. \( u_o \) satisfies equation (1.1) in \( \mathbb{R}^n \setminus B_o(1) \) with \( K = [(n - 2)/2] C^{-(n-4)/(n-2)} \). Furthermore,
\[ \frac{\partial u_o}{\partial r} + \frac{n-2}{2} \frac{u_o}{r} = 0 \quad \text{in} \quad \mathbb{R}^n \setminus B_o(1) . \]
Therefore condition (4.26) can also be viewed as \( u \) is asymptotically close to \( u_\omega \).
The results obtained here suggest that equation (1.1) favors \( x \cdot \nabla K(x) \) to be negative rather than positive. Indeed, Bianchi shows that if \( K \) is radially symmetric and \( K' \geq 0, \neq 0 \) in \( \mathbb{R}^+ \), then equation (1.1) has no positive solutions in \( \mathbb{R}^n \).

**Appendix**

**Theorem A.1.** Assume that \( a^2 \leq K(x) \leq b^2 \) for large \( |x| \) and for positive constants \( a \) and \( b \). Let \( u \) be a positive smooth solution of equation (1.1) in \( \mathbb{R}^n \). If \( \lim_{|x| \to \infty} |x|^{(n-2)/2}u(x) = 0 \), then there exist positive constants \( c_1 \) and \( c_2 \) such that \( c_1 |x|^{2-n} \leq u(x) \leq c_2 |x|^{2-n} \) for large \( |x| \). In addition, if \( u \) is radial, then \( \lim_{r \to \infty} r^{n-2}u(r) \) exists and is positive.

**Proof.** Let

\[
\bar{u}(r) = \int_{S^{n-1}} u(r, \theta) \, d\theta \quad \text{for} \quad r > 0.
\]

We have

\[
\int_{S^{n-1}} K(r, \theta) u^\frac{n+2}{n-2}(r, \theta) \, d\theta \geq a^2 \omega_n^{\frac{4}{n+2}} \bar{u}^\frac{n+2}{n-2}(r)
\]

for large \( r \). It follows from equation (1.1) that

\[
\bar{u}''(r) + \frac{n-1}{r} \bar{u}'(r) + C_1 \bar{u}^\frac{n+2}{n-2} \leq 0
\]

for large \( r \), where \( C_1 = a^2 \omega_n^{-4/(n+2)} \) is a positive constant. Using spherical Harnack inequality (2.8) we have

\[
\int_{S^{n-1}} K(r, \theta) u^\frac{n+2}{n-2}(r, \theta) \, d\theta \leq b^2 \omega_n \left[ \sup_{S_r} u \right]^\frac{n+2}{n-2} = b^2 \omega_n^{-4} \left[ \omega_n \sup_{S_r} u \right]^\frac{n+2}{n-2}
\]

\[
\leq b^2 \omega_n^{-4} \left[ C_h \int_{S^{n-1}} u(r, \theta) \, d\theta \right]^\frac{n+2}{n-2}
\]

for large \( r \). Therefore we obtain

\[
\bar{u}''(r) + \frac{n-1}{r} \bar{u}'(r) + C_2 \bar{u}^\frac{n+2}{n-2} \geq 0
\]

for large \( r \), where \( C_2 = b^2 C_h^\frac{n+2}{n-2} \omega_n^{-4} \). Let

\[
\bar{v}(s) = r^\frac{n-2}{2} \bar{u}(r), \quad \text{where} \quad r = e^s, \quad r \gg 1.
\]
From (A.4) and (A.5) we have

\[\begin{align*}
(A.7) & \quad \dddot{v}(s) - \left(\frac{n-2}{2}\right)^2 \ddot{v}(s) + C_1 \dot{v}^{\frac{n+2}{n-2}}(s) \leq 0 \quad \text{and} \\
(A.8) & \quad \dddot{v}(s) - \left(\frac{n-2}{2}\right)^2 \ddot{v}(s) + C_2 \dot{v}^{\frac{n+2}{n-2}}(s) \geq 0
\end{align*}\]

for large positive \(s\). As \(\dddot{v}(s) \to 0\) when \(s \to +\infty\), (A.8) implies that \(\dddot{v}(s) > 0\) for large \(s\). Hence \(\dddot{v}(s) < 0\) for large \(s\). That is,

\[\begin{align*}
(A.9) & \quad -r \dddot{u}(r) > \frac{n-2}{2} \ddot{u}(r)
\end{align*}\]

for large \(r\). Multiplying both sides of (A.9) by \(C_2^{-1}[2/(n-2)] [r^{(n-2)/2} \dddot{u}(r)]^{4/(n-2)}\) and using the fact that \(r^{(n-2)/2} \dddot{u}(r) \to 0\) as \(r \to \infty\), for any \(\varepsilon > 0\), we have

\[\begin{align*}
(A.10) & \quad -\varepsilon \dddot{u}(r) > C_2 \dddot{r} \dddot{u}^{\frac{n+2}{n-2}}(r)
\end{align*}\]

for large \(r\). As \(\dddot{u}(r) \leq 0\) for large \(r\), from (A.10) we obtain

\[\begin{align*}
(A.11) & \quad \varepsilon \dddot{u}(r) > C_2 \int_r^\infty t \dddot{u}^{\frac{n+2}{n-2}}(t) \, dt
\end{align*}\]

for large \(r\). From (A.5) we have

\[\begin{align*}
(A.12) & \quad [r\dddot{u}(r)]' + (n-2) \dddot{u}(r) + C_2 r \dddot{u}^{\frac{n+2}{n-2}}(r) \geq 0
\end{align*}\]

for large \(r\). Integrating both sides of (A.12) from \(r\) to \(R\), letting \(R \to \infty\) and using the gradient estimate in lemma 2.9, we obtain

\[\begin{align*}
(A.13) & \quad r\dddot{u}(r) + (n-2) \dddot{u}(r) \leq C_2 \int_r^\infty t \dddot{u}^{\frac{n+2}{n-2}}(t) \, dt < \varepsilon \dddot{u}(r)
\end{align*}\]

for large \(r\). It follows from (A.13) that for any positive number \(m < (n-2)\), there exists a positive constant \(C_3\) such that

\[\dddot{u}(r) \leq C_3 r^{-m}\]

for large \(r\). Similarly we have

\[\begin{align*}
(A.15) & \quad r\dddot{u}(r) + (n-2) \dddot{u}(r) \geq C_1 \int_r^\infty t \dddot{u}^{\frac{n+2}{n-2}}(t) \, dt
\end{align*}\]

for large \(r\). It follows from (A.4) that

\[\begin{align*}
(A.16) & \quad r^{n-1} \dddot{u}(r) \leq r_o^{n-1} \dddot{u}(r_o) - C_1 \int_{r_o}^r t^{n-1} \dddot{u}^{\frac{n+2}{n-2}}(t) \, dt
\end{align*}\]

for \(r_o\) and \(r\) large, with \(r > r_o\). We note that the integral in (A.16) makes sense because of (A.14). From (A.15) we have

\[\begin{align*}
(A.17) & \quad r^{n-1} \dddot{u}(r) \geq C_1 r^{n-2} \int_r^\infty t \dddot{u}^{\frac{n+2}{n-2}}(t) \, dt - (n-2)r^{n-2} \dddot{u}(r)
\end{align*}\]
for large $r$. Fix $r_o$ to be large enough so that by (A.9) $u'(r_o) \leq 0$. Substituting (A.17) into (A.16) we obtain

$$r^{n-2} \bar{u}(r) \geq \frac{1}{n-2} \left[ C_1 \int_{r_o}^r t^{n-1} \bar{u}_{n/2}^n (t) dt + C_1 r^{n-2} \int_r^\infty t \bar{u}_{n/2}^n (t) dt - r_o^{n-1} \bar{u}'(r_o) \right] \geq C_4^2$$

for large $r$ and for some positive constant $C_4$. Likewise, using (A.13) we have

$$r^{n-2} \bar{u}(r) \leq \frac{1}{n-2} \left[ C_2 \int_{r_o}^r t^{n-1} \bar{u}_{n/2}^n (t) dt + C_2 r^{n-2} \int_r^\infty t \bar{u}_{n/2}^n (t) dt - r_o^{n-1} \bar{u}'(r_o) \right] \leq C_5^2$$

for large $r$ and for some positive constant $C_5$. Using spherical Harnack inequality (2.8), we obtain the desired bounds. We observe that in case $K$ and $u$ are radially symmetric, the above argument shows that

$$r^{n-2} u(r) = \frac{1}{n-2} \left[ \int_{r_o}^r K(t) t^{n-1} u_{n/2}^n (t) dt + r^{n-2} \int_r^\infty K(t) t u_{n/2}^n (t) dt - r_o^{n-1} u'(r_o) \right]$$

for large $r$. Hence $\lim_{r \to \infty} r^{n-2} u(r) = c_o > 0$, where

$$c_o = \frac{1}{n-2} \left[ \int_{r_o}^\infty K(t) t^{n-1} u_{n/2}^n (t) dt - r_o^{n-1} u'(r_o) \right]$$

by using (A.18).

It follows from lemma A.1 and (2.12) that the only possible indices $\kappa$ such that

$$c_1 \leq |x|^\kappa u(x) \leq c_2 \quad \text{for large} \quad |x|$$

and for some positive numbers $c_1$ and $c_2$ are $\kappa = (n-2)/2$ or $\kappa = n-2$.

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