A Finite State Model for Time Travel

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A time machine that sends information back to the past may, in principle, be built using closed time-like curves. However, the realization of a time machine must be congruent with apparent paradoxes that arise from traveling back in time. Using a simple model to analyze the consequences of time travel, we show that several paradoxes, including the grandfather paradox and Deutsch’s unproven theorem paradox, are precluded by basic axioms of probability. However, our model does not prohibit traveling back in time to affect past events in a self-consistent manner.

I. BACKGROUND

The possibility of building a time machine has been proposed by many authors [1–14]. Two common approaches are through closed time-like curves (CTC) [1–9] and quantum phenomena [10–13]. Although the general theory of relativity allows for CTCs, it is not clear if the laws of physics permit their existence [15,16]. Hence the possibility of traveling back to the distant past remains an open question. Paradoxical thought experiments have been devised to suggest that traveling back in time may lead to violations of causality, and hence is not possible. The most famous paradox is the grandfather paradox, in which an agent travels back in time to kill his grandfather before his father was conceived. In this case, the agent will not exist at the current time and hence cannot travel back in time to kill his grandfather. An alternative version of the grandfather paradox is autoinfanticide, where an agent travels back in time to kill himself as an infant. This paradox plays a central role in the argument against traveling back in time. Another paradox is the Deutsch’s unproven theorem paradox [1], in which an agent travels back in time to reveal the proof of a mathematical theorem. The proof is then recorded in a document that the agent reads in future time. Another version of Deutsch’s unproven paradox is what we call the chicken-and-egg paradox. A hen travels back in time to lay an egg. The egg hatches into the hen herself. Without the egg, the hen would not exist but without the hen traveling back in time, the egg would not be laid.

In this paper, a simple model is used in an attempt to solve time travel paradoxes and help set the logical foundations of traveling back in time. Our approach is quite different from approaches that focus on how a time machine can be built (in principle) [11]. We suppose that a time machine can be built and then analyze what could be possible (or impossible) in time travel. We use a simple directed cyclic graph to explain causal relationships in different scenarios of time travel. Our conclusion is that, assuming traveling back in time is feasible, an agent who travels back in time is unable to kill himself although he may be able to alter the past in other ways; in a self-consistent manner.

The self-consistency principle was proposed by Wheeler and Feynman [12], Novikov et al [13] and Lloyd et al [11]. It states that traveling back in time may be possible, but it cannot happen in a way that violates causality. Causality in this case includes events that happen in the future affecting the past. This principle precludes time travel paradoxes but does not forbid traveling back in time. Due to space limitations, the reader is referred to [11,14,19] for detailed discussion of the self-consistency principle.

II. MODEL

Our model can be considered as a simple case of graphical models. Graphical models have been extensively studied and are applicable in many fields such as in econometric models, social sciences, artificial intelligence and even in medical studies. Publications on graphical models are so numerous that we can only provide a non-exhaustive list [20–32]. Although directed acyclic graphs have been at the center stage of graphical models, directed cyclic graphical models have also received significant attention [28–32]. Two important components in graphical models are intervention and the do calculus. The theory of graphical models has few constraints built in on what is physically possible. This leaves the theory very general.

\[ \sigma_1 \rightarrow \sigma_2 \rightarrow \sigma_3 \rightarrow \cdots \rightarrow \sigma_i \rightarrow \cdots \rightarrow \sigma_k \rightarrow \cdots \rightarrow \sigma_n \]

FIG. 1. A simple graphical model for a Markov Chain

We use a simple directed cyclic graph to study traveling back in time. First, we build constraints into our model as follows. Consider physical states evolving on a timeline as shown in Fig. 1. The graph is a one dimensional chain, and branching is excluded. Traveling back in time introduces a loop as in Fig. 2. We do not include intervention and do calculus because this enables us to simplify our analysis, while capturing the important physics for a closed system.

\[ \sigma_1 \rightarrow \sigma_2 \rightarrow \sigma_3 \rightarrow \cdots \rightarrow \sigma_i \rightarrow \cdots \rightarrow \sigma_k \rightarrow \cdots \rightarrow \sigma_n \]

FIG. 2. A simple cyclic graph to model traveling back in time from \( t = k \) to \( t = i \).

At each time \( t \), the state of the system \( \sigma_t \) is a random variable. Time is also discretized and the arrows connect
events at neighboring times \( \sigma_t \rightarrow \sigma_{t+1} \). The probability of transition from \( \sigma_t \) to \( \sigma_{t+1} \) is given by \( T_{t+1}(\sigma_{t+1}|\sigma_t) \). In this case, the conditional probabilities can be interpreted as a transition matrix, and the graph as a Markov Chain. The following assumptions are used based on physical considerations:

1. The statistical time flows in the same direction as the physical time.

2. Local normalization constraint is enforced, i.e. \( \sum_{\sigma_{t+1}} T_{t+1}(\sigma_{t+1}|\sigma_t) = 1 \). Given that the system is in a state \( \sigma_t \) at time \( t \), the system has to take on a state at \( t+1 \). In general, we can condition on more than one variable, e.g. \( T_{t+1}(\sigma_{t+1}|\sigma_t, \sigma_j, \cdots) \), then the local normalization condition is \( \sum_{\sigma_{t+1}} T_{t+1}(\sigma_{t+1}|\sigma_t, \sigma_j, \cdots) = 1 \).

3. Basic probability axioms are satisfied. Let \( A_i \) be a set of states and \( P(A_i) \) be its probability measure, then,

\[
P(\Omega) = 1, \quad 0 \leq P(A_i) \leq 1 \tag{2}
\]

\[
P(A_i \cup A_j) = P(A_i) + P(A_j), \tag{3}
\]

\( \Omega \) is the set of all possible states and \( A_i \) and \( A_j \) are mutually exclusive. Clearly, for discrete events if \( \sigma_i \in \Omega \) and \( \sigma_j \in \Omega \), \( \sigma_i \neq \sigma_j \), then \( P(\sigma_i \cup \sigma_j) = P(\sigma_i) + P(\sigma_j) \). Here, we use a shorthand notation \( \sigma_i \equiv \{\sigma_i\} \).

A sequence of states \( \pi_n \) is shown in Fig. 1. If the set of all possible states is given by \( \Omega \), then the set of all possible sequences is given by \( \Pi = \Omega^n \). The probability of obtaining \( \pi_n \) is,

\[
P_{mc}(\pi_n) = p(\sigma_1)T_2(\sigma_2|\sigma_1)T_3(\sigma_3|\sigma_2) \cdots T_n(\sigma_n|\sigma_{n-1}) \tag{4}
\]

\( p(\sigma_1) \) is the probability of sampling the initial state \( \sigma_1 \).

The conditional probabilities encode the physics of how the system evolve from state to state. It can be shown that for \( P_{mc}(\pi_n) \), basic axioms of probabilities hold.

In the case of traveling back in time, the causal relationship has an arrow that loops back into the past (Fig. 2). To model traveling back in time, we condition on two states instead of one, \( \tilde{T}_i(\sigma_i|\sigma_{i-1}, \sigma_k) \) where \( \sigma_i \) is an event in the future with respect to time \( i \). In this case,

\[
P(\pi_n) = p(\sigma_1)T_2(\sigma_2|\sigma_1) \cdots \tilde{T}_i(\sigma_i|\sigma_{i-1}, \sigma_k) \cdots T_n(\sigma_n|\sigma_{n-1}) \tag{5}
\]

All the conditional probabilities \( T_j(\sigma_j|\sigma_{j-1}) \) are the same as in Eq. 4 except for \( \tilde{T}_i(\sigma_i|\sigma_{i-1}, \sigma_k) \). Making such a generalization is non-trivial because we need to check that the basic axioms of probabilities continue to hold. At this point, we would like to emphasize some key points that are important in this paper,

1. Time travel consists of sending a signal back to the past. The signal causes an effect only at one time point \( t = i \) as in Fig. 2. The signal could contain a set of instructions to carry out some task or be an agent that travels back in time.

2. The conditional probabilities \( T_{j}, j = 1, 2, \ldots, j \neq i \) in Eq. 4 are determined by the physics of how the system evolves forward in time.

3. The term \( \tilde{T}_i(\sigma_i|\sigma_{i-1}, \sigma_k) \) is special as it is the only term in Eq. 21 that encodes the effects of traveling back in time.

4. Our framework is probabilistic, in which many sequences of states can happen with non-zero probability, in contrast to a deterministic view where only one sequence is possible. Given any sequence \( \pi_n \), its probability of occurrence can be calculated using Eq. 21.

5. A paradox be represented by many different sequences of states. Our objective is to show that either all these sequences happen with zero probability, or they result in violation of the basic axioms of probability.

Consider \( \tilde{T}_i \) to be a function of three discrete variables, \( \sigma_{i-1}, \sigma_i \) and \( \sigma_k \). This function has to satisfy,

\[
0 \leq \tilde{T}_i(\sigma_i|\sigma_{i-1}, \sigma_k) \leq 1 \tag{6}
\]

\[
\sum_{\pi_n} P(\pi_n) = 1 \tag{7}
\]

\[
\sum_{\sigma_i} \tilde{T}_i(\sigma_i|\sigma_{i-1}, \sigma_k) = 1 \tag{8}
\]

The first two conditions are analogous to Eq. 1 and 2. The last condition is the local normalization condition. Eq. 7 can be reduced to,

\[
\sum_{\sigma_i, \sigma_k} \tilde{T}_i(\sigma_i|\sigma_{i-1}, \sigma_k)V(\sigma_k|\sigma_i) = 1 \tag{9}
\]

\( V(\sigma_k|\sigma_i) \) is the conditional probability of \( \sigma_k \) given \( \sigma_i \) summed over all possible intermediate states \( \sigma_{i+1} \cdots \sigma_k \). Detailed derivation of Eq. 9 is given in Appendix A. This is an important equation. We will use this equation together with Eq. 6 and 8 to show that the grandfather paradox, Deutsch’s unproven theorem paradox and chicken-and-egg paradox have to be precluded in time travel.

### A. Two-state system

For a two-state system, \( \sigma \) takes the values \( \{0, 1\} \). Using Eq. 9 and 8 and summing over four combinations \( \sigma_{i+1}, \sigma_k \in \{0, 1\} \), we obtain,

\[
[\tilde{T}_i(0|\sigma_{i-1}, 1) - \tilde{T}_i(0|\sigma_{i-1}, 0)][V(1|0) - V(1|1)] = 0 \tag{10}
\]
We must have \( V(1) = V(1|1) \) or \( \hat{T}_i(0|\tilde{\sigma}_i, 1) = \hat{T}_i(0|\tilde{\sigma}_i, 0) \). For the case when \( V(1) \neq V(1|1) \), the transition matrix \( \hat{T}_i \) does not depend on \( \sigma_i \). In this case, the backward loop in Fig. 2 has no effect. We can’t change the probability distribution of the past. For the case \( V(1) = V(1|1) \), we could have \( \hat{T}_i(0|\tilde{\sigma}_{i-1}, 1) \neq \hat{T}_i(0|\tilde{\sigma}_{i-1}, 0) \) and the transition probabilities at \( t = i \) could be affected by a signal from future time \( (t = k) \).

### B. Grandfather paradox in a two-state system

The grandfather paradox can be used to illustrate the physical implications of Eq. (10). The basic assumptions we will use are (i) resurrection is impossible, and (ii) basic axioms of probabilities must be satisfied.

Consider an agent sending a signal back in time to kill himself. Let us denote the dead state as \( \sigma = 0 \) and alive state as \( \sigma = 1 \). No resurrection implies that \( V \) is of the form, \( \alpha \rightarrow \beta \rightarrow \gamma \). Let \( \hat{T}_i(\sigma_i|\tilde{\sigma}_{i-1}, 1) = S(\sigma_i|\tilde{\sigma}_{i-1}) \) be the transition probabilities for the scenario in which the agent sends a signal from the future to kill himself. Let \( \hat{T}_i(\sigma_i|\tilde{\sigma}_{i-1}, 0) = N(\sigma_i|\tilde{\sigma}_{i-1}) \) be the transition probabilities for the sequences of events the agent is dead at \( t = k \) and hence no signal is sent from the future to kill himself. Hence \( S \) (the “killing” matrix) and \( N \) are of the form,

\[
S = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad N = \begin{pmatrix} 1 & b^* \\ 0 & b \end{pmatrix}
\]

(11)

\( b^* = 1 - b \) is the probability of dying at \( t = i \). Substituting values of \( N, S \) and \( V \) into Eq. (10), we obtain \( [1 - b^*]b = 0 \). Either \( b^* = 1 \) or \( b = 0 \). When \( b^* = 1 \), the agent dies at \( t = k \) with probability 1. When \( b = 0 \), the agent dies sometime between \( t = i \) and \( t = k \) with probability 1. In either case, the sequence in which the agent is alive at \( t = k \) and thus able to send the signal occurs with zero probability. Note that we are analyzing probabilities rather than specific events.

The conclusion comes about because resurrection is impossible \( V(1|0) = 0 \). Suppose resurrection is possible, \( V(1|0) = \alpha^* < 1 \), the paradox goes away when \( \alpha^* = \beta \). Intuitively, if we allow resurrection, the agent could send a signal back in time from \( t = k \) to kill himself at \( t = i \). Between the time \( t = i \) and \( t = k \), the agent is resurrected and hence could again send the signal at \( t = k \). There is no contradiction in this case.

Another way to resolve the paradox is to relax the assumption that the agent always succeeds to kill himself.

In this case, the matrix \( S \) is \( \begin{pmatrix} 1 & \lambda^* \\ 0 & \lambda \end{pmatrix} \), \( \lambda > 0 \).

\[
\hat{T}_i(\sigma_i|\tilde{\sigma}_{i-1}, 0) = N(\sigma_i|\tilde{\sigma}_{i-1}) \] gives, \( \beta(\lambda - b) = 0 \). If \( \beta = 0 \), then the agent dies sometime between \( t = i \) and \( t = k \). If \( \lambda = b \) then \( S = N \), the signal from the future could not change the transition probability at \( t = i \). The agent cannot change his own fate by sending a signal to the past.

### C. Deutsch’s unproven theorem paradox

An agent sends a signal containing the proof of a mathematical theorem back in time. The signal is encoded in a document that the agent reads in future time. Denote the existence of the proof as \( \sigma = 0 \) and absence of the proof as \( \sigma = 1 \). A general form of \( V \) is, \( V = (\alpha \beta^* \beta) \), \( \alpha^* = 1 - \alpha \), \( \beta^* = 1 - \beta \). The basic assumptions we use are (i) the transition from \( \sigma = 1 \) to \( \sigma = 0 \) (transition of absence of proof to existence of proof) happens solely through the signal traveling back in time, and (ii) the transition from \( \sigma = 0 \) to \( \sigma = 1 \) happens with zero probability (once the proof is obtained, it never gets lost). Hence \( \beta = 1 \) and \( \alpha^* = 0 \). The transition probabilities are \( \hat{T}_i(\sigma_i|0 |\tilde{\sigma}_{i-1} = 1, \sigma_k = 0) = 0 \) representing no signal sent if proof does not exist at \( t = k \). \( \hat{T}_i(\sigma_i|0 |\tilde{\sigma}_{i-1} = 1, \sigma_k = 0) = 1 \) represents a signal being sent when the proof exists at \( t = k \). These basic assumptions contradict with Eq. (10), \( \hat{T}_i(0|1,1) - \hat{T}_i(0|1,0)(\alpha^* - \beta) = (0 - 1)(0 - 1) \neq 0 \). Hence the assumptions are false and Deutsch’s unproven theorem paradox is precluded.

The paradox can be resolved if we relax the assumptions. Suppose we allow the possibility that the proof can get lost \( (\alpha^* \geq 0) \) and that the proof can be derived by some brilliant mathematician \( \beta \leq 1 \). Then Eq. (10) can be satisfied if \( \alpha^* = \beta \). There is no paradox here because the proof can be sent back in time and subsequently be lost. It can be re-derived again and be sent back to the past.

### D. Three-state system

For a three-state system, \( \sigma \) takes the values \{0, 1, 2\}. For simplicity, let \( \hat{T}_i(\sigma_i|\tilde{\sigma}_{i-1}, 0) = N(\sigma_i|\tilde{\sigma}_{i-1}) \) and \( \hat{T}_i(\sigma_i|\tilde{\sigma}_{i-1}, 1) = \hat{T}_i(\sigma_i|\tilde{\sigma}_{i-1}, 2) = S(\sigma_i|\tilde{\sigma}_{i-1}) \). Using Eq. (5) and (9),

\[
[N(1|\tilde{\sigma}_{i-1}) - S(1|\tilde{\sigma}_{i-1})][V(0|1) - V(0|0)] + (12)
\]

![FIG. 3. Shaded region shows the possible values of](image-url)
We have from Eq. (12),
\[ [N(2\tilde{\sigma}_{-1}) - S(2\tilde{\sigma}_{-1})][V(0)|2] - V(0)|0] = 0 \]
this is an equation of the form \( ax + yb = 0 \) given \( a = [V(0)|1] - V(0)|0] \) and \( b = [V(0)|2] - V(0)|0] \), \( x = [N(1\tilde{\sigma}_{-1}) - S(1\tilde{\sigma}_{-1})] \) and \( y = [N(2\tilde{\sigma}_{-1}) - S(2\tilde{\sigma}_{-1})] \) can be solved. There are in general infinitely many solutions. From Eq. (9), the range of \([N(1\tilde{\sigma}_{-1}) - S(1\tilde{\sigma}_{-1})] \) and \([N(2\tilde{\sigma}_{-1}) - S(2\tilde{\sigma}_{-1})] \) is bounded by the shaded region in Fig. 3. Given \( a \) and \( b \), the set of solutions for \( x \) and \( y \) contains all the points on the line shown in Fig. 3. The slope of the line is given by \(-a/b \). \( N \neq S \) implies that transition to the state \( \sigma_t \) depends on future state \( \sigma_k \), that is, signals from the future can affect the probability distribution of the past.

E. The grandfather paradox in a three-state system

Consider the three states represent healthy (\( \sigma = 2 \)), sick (\( \sigma = 1 \)) and dead (\( \sigma = 0 \)). First, we lay down our assumptions,

1. Assume resurrection is impossible so that transition from \( \sigma = 0 \) to \( \sigma \neq 0 \) happens with zero probability. Then the matrix \( V \) is of the form,
\[
V = \begin{pmatrix}
1 & \alpha_0 & \beta_0 \\
0 & \alpha_1 & \beta_1 \\
0 & \alpha_2 & \beta_2 \\
\end{pmatrix}
\]
with \( \alpha_0 + \alpha_1 + \alpha_2 = 1 \) and \( \beta_0 + \beta_1 + \beta_2 = 1 \).

2. The agent is able to send a signal back in time to kill himself only if he is not dead at \( t = k \).
\[
\hat{T}_i(\sigma_i|\sigma_{i-1},1) \text{ and } \hat{T}_i(\sigma_i|\sigma_{i-1},2) \]
are the conditional probabilities that the agent is alive and sends a signal back in time to kill himself. Let, \( \hat{T}_i(\sigma_i|\sigma_{i-1},1) = \hat{T}_i(\sigma_i|\sigma_{i-1},2) = S(\sigma_i|\sigma_{i-1}). \)
\[
\hat{T}_i(\sigma_i|\sigma_{i-1},0) \]
is the conditional probability that the agent is dead at \( t = k \) and can not send a signal back in time to kill himself. Let \( \hat{T}_i(\sigma_i|\sigma_{i-1},0) = N(\sigma_i|\sigma_{i-1}). \)
Hence \( S \) (the “killing” matrix) \( N \) are,
\[
S = \begin{pmatrix}
1 & 1 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix}
\text{ and } N = \begin{pmatrix}
1 & a_0 & b_0 \\
0 & a_1 & b_1 \\
0 & a_2 & b_2 \\
\end{pmatrix}
\]

We have from Eq. (12),
\[
\begin{align*}
a_1(1 - a_0) + a_2(1 - \beta_0) & = 0 \\
b_1(1 - \alpha_0) + b_2(1 - \beta_0) & = 0
\end{align*}
\]
There are four cases in which Eq. (15) is satisfied.

1. \( \alpha_0 = 1 \) and \( \beta_0 = 1 \). Then \( V = S \) which means the agent is dead at \( t = k \) with probability 1 (recall that \( S \) is the killing matrix).

2. \( \alpha_0 = 1 \) and \( \beta_0 < 1 \). To satisfy Eq. (15), \( a_2 = b_2 = 0 \). In this case the agent is dead at \( t = k \) with probability 1 (see Appendix B for the proof).

3. \( \alpha_0 < 1 \) and \( \beta_0 = 1 \). To satisfy Eq. (15), \( a_1 = b_1 = 0 \). In this case the agent is dead at \( t = k \) with probability 1 (see Appendix B for the proof).

4. \( \alpha_0 < 1 \) and \( \beta_0 < 1 \). Then \( a_1 = a_2 = b_1 = b_2 = 0 \) and \( N = S \) which means the agent is dead at \( t = i \) with probability 1.

In all cases, the agent is dead with probability 1 at \( t = k \) and hence never has a chance to send a signal back in time to kill himself. Suppose \( S \) is not the killing matrix (Eq. (14)) or resurrection is possible, then this argument does not hold, and the agent is able to alter his fate by changing the probability of being healthy, sick or dead.

F. The chicken-and-egg paradox

Consider the chicken-and-egg paradox in which at time \( t = k \), a hen travels back in time to \( t = i \) to lay an egg. The egg hatches into the hen herself. At this time point, there are two copies of the hen, the older self and the younger self (the chick). As both copies travel to time \( t = k \), the chick grow older and travels back in time to lay the egg. This paradox seems “self-consistent” in the sense that there is no contradiction in existence of the hen and chick from one time point to another. However the problem is the hen seems to pop out from nowhere.

There are three possible states, hen and chick (\( \sigma = 0 \)), hen only (\( \sigma = 1 \)) and no hen and no chick (\( \sigma = 2 \)). We exclude the state of chick only, otherwise we would need four states.

There are no hen and no chick initially, hence \( \tilde{\sigma}_{-1} = 2 \). Let \( \hat{T}_i(\sigma_i|\tilde{\sigma}_{i-1},1) = \hat{T}_i(\sigma_i|\tilde{\sigma}_{i-1},2) = N(\sigma_i|\tilde{\sigma}_{i-1}). \) This is the case when no chick travels back in time and hence there remains no hen and no chick at \( t = i \). Let \( \hat{T}_i(\sigma_i|\tilde{\sigma}_{i-1},0) = \hat{T}_i(\sigma_i|\tilde{\sigma}_{i-1}), \) the chick travels back in time from \( t = k \) to \( t = i \). The matrices \( S \) and \( N \) are,
\[
S = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 0 \\
\end{pmatrix}
\text{ and } N = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
\]

The matrix \( V \) is of the form,
\[
V = \begin{pmatrix}
\alpha_0 & \beta_0 & 0 \\
\alpha_1 & \beta_1 & 0 \\
\alpha_2 & \beta_2 & 1 \\
\end{pmatrix}
\]
The first two columns are general expressions with \( \sum_{j=0}^2 \alpha_j = 1 \) and \( \sum_{j=0}^2 \beta_j = 1 \). The last column is \((0, 0, 1)^T \) because when there is no hen and no chick at time \( t = i \), then there will be no hen and no chick at \( t = k \). Now consider the probability,
\[
P(\tilde{\sigma}_{i-1}, \sigma_i, \sigma_k) = p(\tilde{\sigma}_{i-1})\hat{T}_i(\sigma_i|\tilde{\sigma}_{i-1}, \sigma_k)V(\sigma_k|\sigma_i)
\]
\( p(\tilde{\sigma}_{i-1}) \) is the probability of sampling the state \( \tilde{\sigma}_{i-1} \). Since \( \tilde{\sigma}_{i-1} = 2 \), \( p(\tilde{\sigma}_{i-1}) = \delta_{\tilde{\sigma}_{i-1},2} \). We remind the reader
that the probability distribution \( V \) is the sum of probabilities over all possible intermediate sequences. The chicken-and-egg paradox requires both hen and chick to be present at \( t = k \) (\( \sigma_k = 0 \)) and the chick to appear at \( t = 1 \) (\( \sigma_1 = 1 \)), all intermediate states can take arbitrary values. Reading off entries from matrices \( S \) and \( V \),

\[
P(\sigma_{i-1} = 2, \sigma_i = 1, \sigma_k = 0) = \beta_0
\]  

Using Eq. (12) we can calculate what \( \beta_0 \) should be,

\[
[1 - 0](\beta_0 - \alpha_0) - [0 - 1]\alpha_0 = 0
\]  

\( \Rightarrow \beta_0 = 0 \)

The sum of probabilities of all possible sequences of states that represent the chicken-and-egg paradox equals zero. Therefore the chicken-and-egg event happens with zero probability.

III. DISCUSSION

We have shown, using a graphical model with a loop back into the past, that the grandfather paradox, Deutsch's unproven theorem paradox and the chicken-and-egg paradox are precluded in time travel. We have also demonstrated that changing the probability distributions of the past is possible when no contradicting events are present. For the paradoxes we discussed in this paper, we gave scenarios in which the paradoxes are resolved. Our analysis is based on isolated two- and three-state systems.

For future work, it would be useful to generalize our formalism to arbitrary systems. Lastly, in cases when the causal relationship between events at different times are very complex, the existence of time travel paradoxes in these cases may be very subtle. We hope that our mathematical framework can be used to uncover new time travel paradoxes, especially those that are embedded in complex interactions of events and are not obvious.

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IV. APPENDIX A: DERIVATION OF EQ. (10)

Probability of a sequence \( \pi_n \) is given by,

\[
P(\pi_n) = p(\sigma_1)T_2(\sigma_2|\sigma_1) \cdots \hat{T}_i(\sigma_i|\sigma_{i-1}, \sigma_k) \cdots T_n(\sigma_n|\sigma_{n-1})
\]

Summing over all sequences,

\[
\sum_{(\pi_n)} P(\pi_n) = \sum_{\sigma_1, \sigma_2, \cdots, \sigma_n} p(\sigma_1)T_2(\sigma_2|\sigma_1) \cdots \hat{T}_i(\sigma_i|\sigma_{i-1}, \sigma_k) \cdots T_n(\sigma_n|\sigma_{n-1})
\]

Since \( \sum_{\sigma_j} T_j(\sigma_j|\sigma_{j-1}) = 1 \forall j \), summation can be evaluated recursively between \( \sigma_{k+1} \) and \( \sigma_n \). That is,

\[
\sum_{\sigma_{k+1}, \cdots, \sigma_n} T_{k+1}(\sigma_{k+1}|\sigma_k) \cdots T_n(\sigma_n|\sigma_{n-1}) = 1
\]

Next define,

\[
U(\sigma_{i-1}) = \sum_{\sigma_1, \cdots, \sigma_{i-2}} p(\sigma_1)T_2(\sigma_2|\sigma_1) \cdots T_{i-1}(\sigma_{i-1}|\sigma_{i-2})
\]

\[
V(\sigma_k|\sigma_i) = \sum_{\sigma_{i+1}, \cdots, \sigma_{k-1}} T_{i+1}(\sigma_{i+1}|\sigma_i) \cdots T_k(\sigma_k|\sigma_{k-1})
\]

Then Eq. (22) becomes,

\[
\sum_{(\pi_n)} P(\pi_n) = \sum_{\sigma_{i-1}, \sigma_i, \sigma_k} U(\sigma_{i-1})\hat{T}_i(\sigma_i|\sigma_{i-1}, \sigma_k)V(\sigma_k|\sigma_i)
\]

The objective is to find the conditions in which \( \sum_{(\pi_n)} P(\pi_n) = 1 \). \( U(\sigma_{i-1}) \) is the probability of sampling the state \( \sigma_{i-1} \), it depends on the conditional probabilities \( T_j, j \leq i-1 \) and the initial condition \( p(\sigma_1) \). We therefore have the freedom to choose \( U \) for example, by choosing different initial conditions. Holding \( T \) and \( V \) fixed, we require \( \sum_{(\pi_n)} P(\pi_n) = 1 \) for different choices of \( U \), in which we arrive at,

\[
\sum_{\sigma_{i-1}, \sigma_k} \hat{T}_i(\sigma_i|\sigma_{i-1}, \sigma_k)V(\sigma_k|\sigma_i) = 1
\]

V. APPENDIX B: THE GRANDFATHER PARADOX IN A THREE-STATE SYSTEM

We present the proof that for the grandfather paradox in a three-state system, the probability that the agent is dead at \( t = k \) is one. We consider cases II and III in which Eq. (15) is satisfied,

A. Case II: \( \alpha_0 = 1 \) and \( \beta_0 < 1 \)

In this case, \( a_2 = b_2 = 0 \) and,

\[
V = \begin{pmatrix} 1 & 1 & \beta_0 \\ 0 & 0 & \beta_1 \\ 0 & 0 & \beta_2 \end{pmatrix}
\]

\[
N = \begin{pmatrix} 1 & a_0 & b_0 \\ 0 & a_1 & b_1 \\ 0 & 0 & 0 \end{pmatrix}
\]

We calculate the probability that the agent is dead,

\[
P(\sigma_k = 0) = \sum_{\sigma_1, \cdots, \sigma_{k-1}} p(\sigma_1)T_2(\sigma_2|\sigma_1) \cdots
\]

\[
= \sum_{\sigma_{i-1}, \sigma_i} U(\sigma_{i-1})\hat{T}_i(\sigma_i|\sigma_{i-1}, 0)V(0|\sigma_i)
\]

\[
= \sum_{\sigma_{i-1}, \sigma_i} U(\sigma_{i-1})N(\sigma_i|\sigma_{i-1})V(0|\sigma_i)
\]

Reading off the entries of matrices \( V \) and \( N \) in Eq. (28) and (29), we get \( \sum_{\sigma_i} N(\sigma_i|\sigma_{i-1})V(0|\sigma_i) = 1 \) for all \( \sigma_{i-1} \). Hence \( P(\sigma_k = 0) = 1 \).
B. Case III: $\alpha_0 < 1$ and $\beta_0 = 1$

In this case, $a_1 = b_1 = 0$ and,

$$V = \begin{pmatrix} 1 & \alpha_0 & 1 \\ 0 & \alpha_1 & 0 \\ 0 & \alpha_2 & 0 \end{pmatrix} \quad (31)$$

$$N = \begin{pmatrix} 1 & a_0 & b_0 \\ 0 & 0 & 0 \\ 0 & a_2 & b_2 \end{pmatrix} \quad (32)$$

We calculate the probability that the agent is dead, using Eq. (30) and reading off the entries of matrices $V$ and $N$ in Eq. (31) and (32), we get \[ \sum_{\sigma_i} N(\sigma_i|\sigma_{i-1})V(0|\sigma_i) = 1 \] for all $\sigma_{i-1}$. Hence $P(\sigma_k = 0) = 1$.

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