On the solvable multi–species reaction–diffusion processes

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Abstract
A family of one–dimensional multi–species reaction–diffusion processes on a lattice is introduced. It is shown that these processes are exactly solvable, provided a nonspectral matrix equation is satisfied. Some general remarks on the solutions to this equation, and some special solutions are given. The large–time behavior of the conditional probabilities of such systems are also investigated.

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1 Introduction

In recent years, the asymmetric exclusion process and the problems related to it, including for example the bipolymerization [1], dynamical models of interface growth [2], traffic models [3], the noisy Burgers equation [4], and the study of shocks [5, 6], have been extensively investigated. The dynamical properties of this model have been studied in [6–8]. As the results obtained by approaches like mean field are not reliable in one dimension, it is useful to introduce solvable models and analytic methods to extract exact physical results. Among these methods is the coordinate Bethe–ansatz, which was used in [9] to solve the asymmetric simple exclusion process on a one–dimensional lattice. In [10], a similar technique was used to solve the drop–push model [11], and a generalized one–parameter model interpolating between the asymmetric simple exclusion model and the drop–push model. In [12], this family was further generalized to a family of processes with arbitrary left- and right- diffusion rates. All of these models were lattice models. Finally, the behaviour of latter model on continuum was investigated in [13]. The continuum models of this kind are also investigated in [14, 15]. In [16] a generalization of such processes was studied which contained annihilation of particles as well.

In all of these, people have been mainly concerned with the so–called single–species processes, in which only one kind of particles exist and move on the lattice (or the continuum). Another interesting problem is the study of multi–species systems in them several kinds of particles move and interact on a lattice. In [17], single–species systems have been characterized for them the equations governing the evolution of the N–point functions contain N- or less- point functions. This has been done for multi–species systems in [18]. In [19], two–species reaction–diffusion systems have been introduced that are solvable in the sense that the S matrix corresponding to them is factorizable into the two–particle S matrices. It is found there that the the criterion for this is that the interactions (which are of the nearest–neighbor type) are to be so that the S matrix satisfy a kind of spectral Yang–Baxter equation.

We follow the same line. That is, we investigate interactions which can be written as boundary conditions for the probability functions. These interactions preserve the total number of particles, so that if one begins with N particles, knowing the N–point probabilities is enough to know everything about the system. Using the coordinate Bethe–ansatz, it is found that for this ansatz to be consistent, the S matrix should satisfy a kind of spectral Yang–Baxter equation [19]. However, the S matrix is of a special form containing the boundary conditions (or interactions), and not every solutions of the spectral Yang–Baxter equation can be used to construct such a solvable model. We investigate the spectral equation the S matrix should satisfy and show that this is equivalent to a non–spectral equation for the boundary conditions. This is independent of the number of species.

The scheme of the paper is the following. In section 2, the it is shown that the prescription of investigating multi–species reaction–diffusion systems in terms of diffusion systems equipped with suitable boundary conditions is studied. In
section 3, the Bethe–ansatz solution for such (solvable) systems is obtained and its large–time behavior is investigated. In section 4, the solvability criterion is obtained and it is shown that this criterion is a nonspectral matrix equation. In section 5, some general properties of the solutions of the solvability criterion is studied. Finally, in section 6 some special solutions of the solvability equation are studied.

2 Multi–species reaction–diffusion systems and the boundary conditions

Consider a system consisting of \( N \) particles on a lattice, drifting to the right with unit rate if the right neighboring site is empty, and interacting with each other only if two of them are adjacent. Suppose that there are \( n \) kinds (or species) of particles and the interaction between the particles is just in the form that if two particles \( A^\alpha \) and \( A^\beta \) are adjacent to each other, they may change to \( A^\gamma \) and \( A^\delta \) with the rate \( b^\gamma_\delta \). That is, the allowed processes are

\[
\begin{align*}
A^\alpha \emptyset & \rightarrow \emptyset A^\alpha, & \text{with rate 1} \\
A^\alpha A^\beta & \rightarrow A^\gamma A^\delta, & \text{with rate } b^\gamma_\delta
\end{align*}
\]

These processes result in the master equation

\[
\dot{P}^{\alpha_1,\ldots,\alpha_N}(x_1,\ldots,x_N; t) = P^{\alpha_1,\ldots,\alpha_N}(x_1-1,\ldots,x_N; t) + \cdots
\]

\[
+ P^{\alpha_1,\ldots,\alpha_N}(x_1,\ldots,x_N-1; t)
\]

\[
- NP^{\alpha_1,\ldots,\alpha_N}(x_1,\ldots,x_N; t),
\]

if \( x_i < x_{i+1} - 1 \). The symbol \( P^{\alpha_1,\ldots,\alpha_N}(x_1,\ldots,x_N; t) \) denotes the probability of finding a particle of type \( \alpha_1 \) in \( x_1 \), a particle of type \( \alpha_2 \) in \( x_2 \), \ldots at the time \( t \). The so–called physical region consists of the points satisfying \( x_i < x_{i+1} \). If \( x_i = x_{i+1} - 1 \), the interactions change the equation. For clarity, let’s write the evolution equation for the two–particle sector:

\[
\dot{P}^{\alpha\beta}(x, x+1) = P^{\alpha\beta}(x-1, x+1)
\]

\[
+ \sum_{(\gamma\delta) \neq (\alpha\beta)} b^\gamma_\delta P^{\gamma\delta}(x, x+1) - B^{\alpha\beta} P^{\alpha\beta}(x, x+1)
\]

\[
- P^{\alpha\beta}(x, x+1),
\]

where

\[
B^{\alpha\beta} := \sum_{(\gamma\delta) \neq (\alpha\beta)} b^\gamma_\delta
\]

Defining the diagonal elements of \( b \) as

\[
b^{\alpha\beta}_{\alpha\beta} := 1 - \sum_{(\gamma\delta) \neq (\alpha\beta)} b^\gamma_\delta
\]
it is seen that (3) can be written as
\[ \dot{P}_{\alpha\beta}(x, x+1) = P_{\alpha\beta}(x, x-1) + b_{\gamma\delta}^{\alpha\beta} P_{\gamma\delta}(x, x+1) - 2P_{\alpha\beta}(x, x+1), \] (6)
where summation is implied on repeated indices. Comparing this with (2), it is seen that it can be written as (2) provided one introduces the boundary condition
\[ P_{\alpha\beta}(x, x) = b_{\gamma\delta}^{\alpha\beta} P_{\gamma\delta}(x, x+1), \] (7)
or, in a more compact form,
\[ |P(x, x)| = b |P(x, x+1)|. \] (8)

The matrix \( b \) should satisfy two criteria. First, its non–diagonal elements should be nonnegative (since they are rates). Second, the sum of the elements of each of its columns should be one. This can be written in a compact form as
\[ \langle s | \otimes \langle s | b = \langle s | \otimes \langle s |, \] (9)
where
\[ s_{\alpha} := 1. \] (10)

Note that if the number of species is one, the asymmetric simple exclusion process \( \ref{eq:2} \) is obtained.

### 3 The Bethe–ansatz solution

As in \( \ref{eq:19} \), one can write a Bethe–ansatz solution for (2) with the boundary condition
\[ |P(\cdots, x_k = x, x_{k+1} = x, \cdots)\rangle = b_{k,k+1} |P(\cdots, x_k = x, x_{k+1} = x+1, \cdots)\rangle, \] (11)
where
\[ b_{k,k+1} := 1 \otimes \cdots \otimes 1 \otimes \underbrace{b}_{k,k+1} \otimes 1 \otimes \cdots \otimes 1. \] (12)

We take the ansatz
\[ |P(\chi; t)\rangle = e^{Et} |\Psi(\chi)\rangle, \] (13)
and it is seen that \( |\Psi(\chi)\rangle \) should satisfy
\[ E |\Psi(x_1, \cdots, x_N)\rangle = |\Psi(x_1 - 1, \cdots, x_N)\rangle + \cdots + |\Psi(x_1, \cdots, x_N - 1)\rangle - N |\Psi(x_1, \cdots, x_N)\rangle, \] (14)
The Bethe–ansatz is that, one takes the following form for $|\Psi(x)\rangle$.

$$|\Psi(x)\rangle = \sum_{\sigma} A_{\sigma} e^{i\sigma(p)\cdot x} |\psi\rangle,$$  

(16)

where $|\psi\rangle$ is an arbitrary vector and the summation runs over the elements of the permutation group. Plugging this in (14) results in

$$E = \sum_{k=1}^{N} \left( e^{-ip_k} - 1 \right).$$  

(17)

The boundary condition (15) yields

$$\left[ 1 - e^{i\sigma(p_{k+1})} b_{k,k+1} \right] A_{\sigma} + \left[ 1 - e^{i\sigma(p_k)} b_{k,k+1} \right] A_{\sigma\sigma_k} = 0.$$  

(18)

Here $\sigma$ is that element of the permutation group which only interchanges $p_k$ and $p_{k+1}$. From this, one obtains

$$A_{\sigma\sigma_k} = S_{k,k+1}[\sigma(p_k),\sigma(p_{k+1})] A_{\sigma},$$  

(19)

where the matrix $S$ is defined through

$$S(p_1, p_2) := -(1 - z_1 b)^{-1}(1 - z_2 b),$$  

(20)

and the definition of $S_{k,k+1}$ is similar to that of $b_{k,k+1}$ in (12). we have also used the definition

$$z_j := e^{ip_j}.$$  

(21)

This shows that one can construct $A_{\sigma}$’s from $A_1$ by writing $\sigma$ as a product of $\sigma_k$’s. But these elements of the permutation group satisfy

$$\sigma_k \sigma_{k+1} \sigma_k = \sigma_{k+1} \sigma_k \sigma_{k+1}.$$  

(22)

This means that

$$A_{\sigma_k \sigma_{k+1} \sigma_k} = A_{\sigma_{k+1} \sigma_k \sigma_{k+1}},$$  

(23)

or

$$S_{k,k+1}(p_{k+1}, p_{k+2}) S_{k+1,k+2}(p_k, p_{k+2}) S_{k,k+1}(p_k, p_{k+1}) = S_{k+1,k+2}(p_k, p_{k+1}) S_{k,k+1}(p_k, p_{k+2}) S_{k+1,k+2}(p_{k+1}, p_{k+2}).$$  

(24)
This can be written as

\[ [S(p_2, p_3) \otimes 1][1 \otimes S(p_1, p_3)][S(p_1, p_2) \otimes 1] = [1 \otimes S(p_1, p_2)][S(p_1, p_3) \otimes 1] \times [1 \otimes S(p_2, p_3)]. \] (25)

Writing the \( S \) matrix as the product of the permutation matrix \( \Pi \) and an \( R \) matrix:

\[ S_{k,k+1} =: \Pi_{k,k+1}R_{k,k+1}, \] (26)

(25) is transformed to

\[ R_{23}(p_2, p_3)R_{13}(p_1, p_3)R_{12}(p_1, p_2) = R_{12}(p_1, p_2)R_{13}(p_1, p_3)R_{23}(p_2, p_3). \] (27)

This is the spectral Yang–Baxter equation.

Provided this condition is satisfied, it is easy to see that the conditional probability (the propagator) is

\[ U(x; t|y; 0) = \int \frac{d^N p}{(2\pi)^N} e^{-i p \cdot y} \sum_{\sigma} A_\sigma e^{i \sigma(p) \cdot x} e^{t E(p)}, \] (28)

where the integration region for each \( p_i \) is from \([0, 2\pi]\), and we have taken \( A_e = 1 \). (\( e \) is the identity of the permutation group.) Note that (4), and the condition of nonnegativity of the nondiagonal elements of \( b \) ensure that the absolute values of the eigenvalues of \( b \) don’t exceed 1. So there is no singularity in \( S(p_1, p_2) \) except at \( p_1 = 0 \), and this is removed by setting \( p_j \to p_j + i \epsilon \), where one should consider the limit \( \epsilon \to 0^+ \). This is the same as what has been done in [9] and [10], for example. Using this propagator, one can of course write the probability at the time \( t \) in terms of the initial value of the probability:

\[ |P(x; t)\rangle = \sum_y U(x; t|y; 0)|P(y; 0)\rangle \] (29)

For the two–particle sector, it is not difficult to obtain \( U \). In fact, as there is only one matrix \( (b) \) in the expression for \( U \), one can treat it as a \( c \)-number and the problem is reduced to that of [16], with \( \lambda \) replaced by \( b \). So,

\[ U(x; t|y; 0) = e^{-2t} \frac{t^{x_1-y_1}}{(x_1 - y_1)! (x_2 - y_2)!} + e^{-2t} \sum_{l=0}^{\infty} \frac{t^{l+x_2-y_1}}{(l + x_2 - y_1)!} \frac{t^{x_1-y_2}}{(x_1 - y_2)!} b^l \left( -1 + \frac{t b}{x_1 - y_2 + 1} \right). \] (30)

One can decompose the vector space on which \( b \) acts into a subspace on which \( b = 1 \) (eigenspace of \( b \) corresponding to eigenvalue 1) and another invariant subspace. This is done by decomposing the unit matrix into two projectors:

\[ 1 = Q + R, \] (31)
where $Q$ and $R$ are projections satisfying

$$QR = RQ = 0. \tag{32}$$

$Q$ is projection on the eigenspace of $b$ corresponding to the eigenvalue 1, and $R$ is projection on the other invariant subspace of $b$. Using this, one can write $U$ as

$$U(x; t | y; 0) = \left[ e^{-2t} \frac{t^{x_1-y_1}}{(x_1-y_1)!} \frac{t^{x_2-y_2}}{(x_2-y_2)!} + e^{-2t} \sum_{l=0}^{\infty} \frac{t^{l+x_2-y_1}}{(l+x_2-y_1)!} \frac{t^{x_1-y_2}}{(x_1-y_2)!} \left( -1 + \frac{t}{x_1-y_2+1} \right) \right] Q$$

$$+ \left[ e^{-2t} \frac{t^{x_1-y_1}}{(x_1-y_1)!} \frac{t^{x_2-y_2}}{(x_2-y_2)!} + e^{-2t} \sum_{l=0}^{\infty} \frac{t^{l+x_2-y_1}}{(l+x_2-y_1)!} \frac{t^{x_1-y_2}}{(x_1-y_2)!} b^l \left( -1 + \frac{t b}{x_1-y_2+1} \right) \right] R. \tag{33}$$

Here we have used

$$b = b(Q + R) = Q + bR. \tag{34}$$

As the eigenvalues of $b$, other than 1, are assumed to have moduli less than one, the second term in (33) is the same as (33) in [16], that is, a term obtained from the boundary condition corresponding to annihilation ($\lambda < 1$ in [16]). The first term corresponds to an asymmetric simple exclusion process [9]. The large–time behavior of these two terms are also simply obtained. The large–time behavior of the first was obtained in [13], and that of the second in [16]. At large times, the second term is found to be independent of $b$ (or $\lambda$) and vanishing faster than $1/t$. Also, the summation of this term vanishes as $t$ tends to infinity. In fact, using [16] it is seen that

$$\text{the second term} = \frac{1}{2\pi t} \left\{ e^{-[(x_1-y_1-t)^2+(x_2-y_2-t)^2]/(2t)} - e^{-[(x_1-y_1-t)^2+(x_2-y_2-t)^2]/(2t)} \right\}, \quad t \to \infty. \tag{35}$$

So at large times only the first term of (33) survives. This means that at large times, the propagator is proportional to the projection on the eigenspace of $b$ corresponding to the eigenvalue 1 (the projection on the equilibrium subspace of $b$) and the proportionality constant is simply the propagator of the asymmetric simple exclusion process.

To conclude, for large times the two–particle conditional probability is that of an asymmetric simple exclusion process projected on the eigenspace of $b$ corresponding to its unit eigenvalue.
4 Solvability criteria for the boundary conditions

From (20), it is seen that $S(p_1, p_2)$ is a binomial of degree one with respect to $z_2 := e^{yp_2}$. Putting this in (25), one arrives at a quadratic expression with respect to $z_3$. The coefficients of this expression are, of course, matrices depending on $z_1$ and $z_2$. It is easy to find the roots of this expression for $z_3$. In fact, putting $z_3 = z_1$ in (25), one arrives at the identity

$$[S(p_2, p_1) \otimes 1] [S(p_1, p_2) \otimes 1] \equiv [1 \otimes S(p_1, p_2)] [1 \otimes S(p_2, p_1)].$$

(We note that $S(p_1, p_2) S(p_2, p_1) \equiv 1$.) Also, putting $z_3 = z_2$, another identity is obtained:

$$[1 \otimes S(p_1, p_2)] [S(p_1, p_2) \otimes 1] \equiv [1 \otimes S(p_1, p_2)] [S(p_1, p_2) \otimes 1].$$

These two identities show that the roots of the quadratic expression for $z_3$ are $z_1$ and $z_2$. That is, one can write that expression as

$$(z_3 - z_1)(z_3 - z_2)Q(z_1, z_2) = 0.$$  

(38)

So, (25) is equivalent to $Q = 0$, which itself is obtained by putting $z_3 = 0$ in (25):

$$[(1 - z_2 b)^{-1} \otimes 1][1 \otimes (1 - z_1 b)^{-1}][(1 - z_1 b)^{-1} (1 - z_2 b) \otimes 1]$$

$$= [1 \otimes (1 - z_1 b)^{-1}] [(1 - z_2 b) [(1 - z_1 b)^{-1} \otimes 1][1 \otimes (1 - z_2 b)^{-1}].$$

(39)

Inverting both sides, one arrives at

$$[(1 - z_2 b)^{-1} (1 - z_1 b) \otimes 1][1 \otimes (1 - z_1 b)] [(1 - z_2 b) \otimes 1]$$

$$= [1 \otimes (1 - z_2 b)] [(1 - z_1 b) \otimes 1][1 \otimes (1 - z_2 b)^{-1} (1 - z_1 b)].$$

(40)

This is a quadratic expression in terms of $z_1$. For $z_1 = 0$, (40) gives the identity

$$[(1 - z_2 b)^{-1} \otimes 1][(1 - z_2 b) \otimes 1] \equiv [1 \otimes (1 - z_2 b)][1 \otimes (1 - z_2 b)^{-1}],$$

(41)

while for $z_1 = z_2$, the identity

$$[1 \otimes (1 - z_2 b)][(1 - z_2 b) \otimes 1] \equiv [1 \otimes (1 - z_2 b)][1 \otimes (1 - z_2 b) \otimes 1]$$

(42)

is obtained. So, the quadratic expression corresponding to (40) is equivalent to

$$z_1(z_1 - z_2)\tilde{Q}(z_2) = 0,$$

(43)

and to find $\tilde{Q}$, one simply uses the coefficient of $z_2^2$ in (40). This is

$$(1 - z_2 b_{12})^{-1} b_{12} b_{23} (1 - z_2 b_{12}) = (1 - z_2 b_{23}) b_{12} b_{23} (1 - z_2 b_{23})^{-1},$$

(44)
or
\[ b_{12}b_{23}(1 - z_2b_{12})(1 - z_2b_{23}) = (1 - z_2b_{12})(1 - z_2b_{23})b_{12}b_{23}. \] (45)

This is a quadratic expression in \( z_2 \). But the coefficients of \( z_2^0 \) and \( z_2^2 \) are identities. So the only remaining equation is
\[ b_{12}b_{23}(b_{12} + b_{23}) = (b_{12} + b_{23})b_{12}b_{23}, \] (46)
or
\[ b_{12}[b_{12}, b_{23}] = [b_{12}, b_{23}]b_{23}. \] (47)

(47) is equivalent to (23). But it is seen that (47) is non–spectral, whereas (23) is spectral. So it is far simpler to seek the solutions to (47) than to seek those of (23).

To summarize, a matrix \( b \), or the reactions (1), correspond to an exactly solvable reaction diffusion system on a one–dimensional lattice, provided \( b \) satisfies (47) and (5) (or (9), equivalently), and the non–diagonal elements of \( b \) are nonnegative.

5 General properties of the solutions to the solvability criteria

Solutions to (47) enjoy two general properties. First, if \( b \) is a solution, then
\[ b' := \alpha b + \beta, \] (48)
is another solution for constant \( \alpha \) and \( \beta \). If \( b \) satisfies (4), then
\[ \langle s \rangle \otimes \langle s \rangle b' = (\alpha + \beta)\langle s \rangle \otimes \langle s \rangle. \] (49)
So, putting \( \beta := 1 - \alpha \), ensures that \( b' \) satisfies (4). If \( \alpha > 0 \), then the nondiagonal elements of \( b' \) are nonnegative provided the nondiagonal elements of \( b \) are nonnegative. So
\[ b' := \alpha b + (1 - \alpha) \] (50)
corresponds to a solvable system (for \( \alpha > 0 \)) if \( b \) does. It is easy to see that the meaning of this transformation is simply to multiply the reaction rates by \( \alpha \).

Second, if \( b \) is a solution to (47), then
\[ b' := u \otimes u b u^{-1} \otimes u^{-1} \] (51)
satisfies (47) as well. Here \( u \) is an arbitrary (nonsingular) matrix. This transformation, however, does not necessarily respects the conditions (4) and nonnegativity of the rates. So, another problem arises. Suppose \( b \) is a solution to
and we want to obtain a solvable system using the transformation (51). We must have

\[ \langle s \rangle \otimes \langle s \rangle u \otimes u b = \langle s \rangle \otimes \langle s \rangle u \otimes u. \] (52)

This means that \( u \) must change \( \langle s \rangle \) to some \( \langle s' \rangle \) so that \( \langle s' \rangle \otimes \langle s' \rangle \) is a left eigenvector of \( b \) corresponding to a unit eigenvalue. One may search in the eigenvectors of \( b \) to find whether there is an eigenvector of the form \( \langle s' \rangle \otimes \langle s' \rangle \). If there is such an eigenvector, then any matrix \( u \) which changes \( \langle s \rangle \) to \( \langle s' \rangle \) can be used to obtain \( b' \) according to (51). This \( b' \) satisfies (47) and (49). But its diagonal elements may be nonnegative or not; this should be checked separately. If none of the eigenvalues of \( b \) are of the form \( \langle s' \rangle \otimes \langle s' \rangle \), then this method cannot be used to obtain a solvable system. This method resembles very much to that used in [20].

6 some special cases

**case I:** \( b^2 = \alpha + \beta b \) (\( \alpha \) and \( \beta \) are numbers). In this case, one can define

\[ b' := b + \gamma, \] (53)

with \( \gamma \) satisfying

\[ \gamma^2 + \beta \gamma - \alpha = 0, \] (54)

to obtain

\[ b'^2 = (\beta + 2\gamma)b'. \] (55)

Putting this \( b' \) in (17), one obtains the braid equation for \( b' \):

\[ b'_{12}b'_{23}b'_{12} = b'_{23}b'_{12}b'_{23}. \] (56)

From (55), it is seen that \( b' \) either can be scaled to a projection \( (b'^2 = b') \), or is nilpotent. One concludes then that any nilpotent or projection solution to the (nonspectral) braid equation is a solution to (17). One can then use any linear combination of this solution with the unit matrix as another solution to that equation. Note, however, that these solutions of (17) do not necessarily satisfy other criteria of the solvable system, that is nonnegativity of the nondiagonal elements and (49). An inspection of the solutions obtained in [18] shows that solutions 1—15, and 17 are of this type. As mentioned in the previous section, one can of course take a linear combination of each solution with the unit matrix to obtain another solution.

**case II:** \( b = u \otimes v \). Here (17) takes the form

\[ u^2 \otimes v[v, u] \otimes v = u \otimes [v, u]u \otimes v^2. \] (57)
A simple way to satisfy this is to set
\[ [u, v] = 0. \] (58)

So, using any two commuting matrices \(u\) and \(v\) one can construct a solution to (47). If the elements of one of these matrices are nonnegative, and the nondiagonal elements of the other are also nonnegative, then the nondiagonal elements of \(b\) are nonnegative. If
\[ \langle s | u = \langle s | v = \langle s |, \] (59)
then \(b\) satisfies (57) as well. Of course, having found a solution of this type one can use a linear combination of it with the unit matrix as another solution. Solutions 1, 4, 7, 14, 17, 20, 21, 22, 25, 26, and 28 of [19] are of this kind.

It is possible to have other solutions to (57). In this case, let’s also use (9). This shows that one may rescale \(u\) and \(v\) so that (59) is satisfied. One then arrives at
\[
\begin{align*}
  u^2 &= u \\
  v^2 &= v \\
  uvu &= uvu,
\end{align*}
\] (60)
if \([u, v] \neq 0\). From these, it is seen that \(u\), \(v\), \(1 - u\), and \(uvu\) are projections. Moreover,
\[
(1 - u)uvu = uvu(1 - u) = 0. \] (61)

This shows that \(1 - u\) and \(uvu\) can be simultaneously diagonalized. The diagonal form of them will be
\[
1 - u = \begin{pmatrix}
  0 & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 1
\end{pmatrix}
\]
\[
uvu = \begin{pmatrix}
  1 & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 0
\end{pmatrix}
\] (62)

Here the elements of the above matrices are matrices themselves, and 1 is the unit matrix of the appropriate dimension. Writing an ansatz for \(v\):
\[
v = \begin{pmatrix}
  v_{11} & v_{12} & v_{13} \\
  v_{21} & v_{22} & v_{23} \\
  v_{31} & v_{32} & v_{33}
\end{pmatrix}
\] (63)
and putting it in (60), one finally arrives at the following forms for \( u \) and \( v \).

\[
\begin{align*}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\end{align*}
\]

\[
\begin{align*}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & w & 0 \\
0 & w' & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\end{align*}
\]

(64)

where all of the entries in the above matrices are matrices, and \( w \) and \( w' \) should satisfy

\[
ww' = w'w = 0.
\]

(65)

Each of the diagonal blocks of these matrices may be zero dimensional, except the first. It should, at least, be one dimensional. The reason is that \( u \) and \( v \) have at least one common left eigenvector, \( \langle s \rangle \), corresponding to the unit eigenvalue. Also the dimension of each block of \( u \) is equal to that of the corresponding block in \( v \). Also note that if the dimension of \( u \) and \( v \) is 2 (there are two kinds of particles) then there will no space left for \( w \) and \( w' \), and \( u \) and \( v \) must be commuting.

The final result is that in two dimension no new solution exists (\( u \) and \( v \) must be commuting), and in more than two dimensions, \( u \) and \( v \) must be of the form (64). Of course any similarity transformation on (14) gives another solution to (47). In fact, one has to use a similarity transformation to make \( \langle s \rangle \) a left eigenvector of \( u \) and \( v \) with unit eigenvalue.

Two very simple subcases are \( b = 1 \otimes v \) and \( b = u \otimes 1 \). These describe reactions

\[
A^\alpha A^\beta \rightarrow A^\alpha A^\delta, \quad \text{with rate } v_{\beta}^\delta,
\]

(66)

and

\[
A^\alpha A^\beta \rightarrow A^\gamma A^\beta, \quad \text{with rate } u_{\alpha}^\gamma,
\]

(67)

respectively. That is, in each case only one of the particles change, and the rate of change is independent of the type of the other particle.
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