Classes of hypercomplex polynomials of discrete variable based on the quasi-monomiality principle

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Abstract

The umbral calculus formalism will be embody within a Clifford-algebraic structure of radial type to derive the interplay between Appell sequences and ladder operator representations in the context of discrete hypercomplex variables.

The importance of this quasi-monomiality formalism provided by the Lie-algebraic representation of raising and lowering operators lies in the fact that the Appell sequences may be derived from the classical exponential generating function (EGF) carrying the continuous Dirac operator \( D = \sum_{j=1}^n e_j \partial_{x_j} \).

This in turn yields an operational scheme to derive, in an ease and naturalness way, new families of hypercomplex polynomials of discrete variable from methods already known in continuum.

Keywords: Clifford algebras, finite difference operators, monomiality principle, umbral calculus

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1. Introduction

1.1. State of Art

The modern language of Appell/Sheffer sets based on the quasi-monomiality formalism (see e.g. [6], [2] and the references given there) plays a central role in the construction of polynomial solutions for finite difference equations possessing Lie-algebraic symmetries (cf. [20], [9] & [21]).

Although this formalism has been fully developed and popularized by Ben Cheikh, Dattoli, Srivastava et all (see e.g. [6], [25], [6], [4] and the references given there), the fundamentals of such theory has its roots in the former paper of Di Bucchianico-Loeb-Rota [8].

The construction of polynomial solutions based on operational methods has a long history that started with the study of classical equations of motion from the quantum

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mechanical side by Wigner (cf. [30]). More recently and probably the core topic in the study of Lie-algebraic representations from the classical mechanics side is what Turbiner-Ushveridze [28], Gagnon-Winternitz [15] and numerous followers across called quasi-exact solvability. That consists into the construction of polynomial spaces as finite-dimensional irreducible Fock spaces which are invariant under the action of a degree-preserving operator, the so-called number operator in the language of quantum mechanics (cf. [4], [29]).

The main goal of this paper is the hypercomplex extension of the quasi-monomiality formalism to discrete variables in interrelationship with Lie-algebraic representations of finite difference operators which are Clifford-vector-valued. In continuum this is currently an ongoing research topic (cf. [5], [10], [19]).

1.2. The Lie-algebraic background through hypercomplex variables

Motivated from [12] the main idea towards Lie-algebraic discretization through hypercomplex variables consists into start from a given algebra of radial type acting on the space of Clifford-vector-valued polynomials \( P = \mathbb{R}[x] \otimes Cl_{0,n} \), a Hilbert module generated from the tensor product between the ring of multi-variable polynomials \( \mathbb{R}[x] (x \in \mathbb{R}^n) \) and the Clifford algebra \( Cl_{0,n} \) of signature \((0,n)\). The rigorous formulation of this approach is based on the study of left representations within the algebra of endomorphisms \( \text{End}(P) \) through the canonical correspondence

\[
\text{End}(P) \cong \text{Alg} \{ L_j, M_j, e_j : j = 1, \ldots, n \},
\]

Hereby \( L_1, L_2, \ldots, L_n, M_1, M_2, \ldots, M_n \) together with \( I : f(x) \mapsto f(x) \) are assumed to be the canonical generators of the Weyl-Heisenberg algebra of dimension \( 2n + 1 \) generated from the graded commuting relations

\[
[L_j, L_k] = 0, \quad [M_j, M_k] = 0, \quad [L_j, M_k] = \delta_{jk} I
\]

while \( e_1, e_2, \ldots, e_n \) is fixed as an orthogonal basis of \( \mathbb{R}^n \) satisfying the graded anti-commuting relations

\[
e_j e_k + e_k e_j = -2\delta_{jk}.
\]

In this context, the canonical description of \( \text{End}(P) \) provided by (1) corresponds to the Lie-algebraic counterpart of the radial algebra introduced by Sommen in [27]. Moreover, the left endomorphisms \( L = \sum_{j=1}^{n} e_j L_j \) and \( M = \sum_{j=1}^{n} e_j M_j \) encode the symmetries of the orthosymplectic Lie algebra \( \text{osp}(1|2) \) (cf. [12], Subsection 2.3).

As examples of Appell systems that can be derived from this scheme one can mention the hypercomplex extension of factorial powers considered in [11], Subsection 3.2 as the discrete counterpart of the Clifford-vector-valued homogeneous polynomials that yields the Fischer decomposition carrying the finite difference Dirac operators \( D_n^\pm_h \) (see also [12], Subsection 3.1). The resulting degree-preserving spaces of Clifford-vector-valued polynomials that results into Appell sequences carrying both difference discretizations \( D_n^\pm_h \) of the Dirac operator \( D = \sum_{j=1}^{n} e_j \partial_{x_j} \) was recently embody on Howe’s invariant theory framework [17] by the author in [13], Section 3 as irreducible representations that encode the Howe dual pair \((SO(n), su(1,1))\).
The description of the action of the operators $L_j$ and $M_j$ in terms of quasi-
monomiality principle provides an amalgamation of the substantial work already done in
studying multi-variable Appell sequences through continuous and discrete variables (cf. [3], [6],
[2], [29]). Indeed, if the set $\{m_\alpha(x) : \alpha \in \mathbb{N}_0^n\}$ determined by the condition
$m_0(x) = 1 (0 \in \mathbb{R}^n)$ and by the set of quasi-monomiality constraints

$$L_j m_\alpha(x) = \alpha_j m_{\alpha-e_j}(x) \quad \text{and} \quad M_j m_\alpha(x) = m_{\alpha+e_j}(x)$$

with $\alpha = \sum_{j=1}^n \alpha_j e_j$ has

$$\sum_{|\alpha|=0}^\infty m_\alpha(x) \frac{y^\alpha}{\alpha!} = \prod_{j=1}^n \frac{1}{\kappa(\ell^{-1}(y_j))} \exp \left( x_j \ell^{-1}(y_j) \right)$$

as exponential generating function (EGF) then the associated ladder operators $L_j$ and
$M_j$ given by (cf. [4, Section 3])

$$L_j = \ell (\partial_{x_j}) \quad \text{and} \quad M_j = \left( x_j - \kappa (\partial_{x_j}) \kappa (\partial_{x_j})^{-1} \right) \ell (\partial_{x_j})^{-1}$$

satisfy the Weyl-Heisenberg graded commuting relations (2).

The implicit mathematical conditions corresponding to the above characterization
(see also [23, Theorem 2.5.3] and [23, Theorem 3.7.1]) are the isomorphism between the
algebra of formal power series and the algebra of linear functionals carrying the algebra
of multi-variable polynomials (cf. [23, Theorem 2.1.1]) and the shift-invariant property
(cf. [23, Corollary 2.2.8]) carrying the operators $\kappa (\partial_{x_j})$ and $\ell (\partial_{x_j})$ obtained from the substitutions $t \to \partial_{x_j}$ on

$$\kappa(t) = \sum_{k=0}^\infty a_k \frac{t^k}{k!} \quad \text{and} \quad \ell(t) = \sum_{k=1}^\infty b_k \frac{t^k}{k!}.$$

The isomorphism between the algebra of linear functionals carrying the algebra of
multi-variable polynomials also shows that $\ell' (\partial_{x_j})$ is also shift-invariant and coincides
with $L'_j := [L_j, x_j]$, the so-called Pincherle derivative (cf. [9, Section 2]). The existence
of $\ell' (\partial_{x_j})^{-1}$ is then assured by the condition $\ell' (\partial_{x_j})^{-1} 1 \neq 0$ (cf. [20, Subsection 2.1]).

1.3. The Scope of this paper

The core purpose of this framework consists into describe the set of Clifford-vector-
valued polynomials on the lattice based on representations of the algebra of endomorphisms
$\text{End} (\mathcal{P})$ through the lattice $h\mathbb{Z}^n$.

2From the border view of quantum field theory, the quasi-
monomiality principle yields a meaningful
interpretation of the quantum field lemma associated to the second quantization method.
3In the operational form, the EGF may be represented as the action of the exponentiation operator
$\exp \left( \sum_{j=1}^n y_j M_j \right)$ on $m_0(x) = 1$.
4In the book [23] the Pincherle derivative associated to $\ell' (\partial_{x_j})$ can be found on Section 6 of
Chapter 2 under the name of umbral shift. To keep the consistence with former framework developed
in [12], one will adopt the terminology Pincherle derivative along this paper.
With the aim of making this paper self-contained and easy to digest, the main results enclosed between Section 2 and Section 3 are intertwined by several examples and remarks. To underpin this approach with other approaches available on the literature one summarize in Section 4 the main contributions and the forthcoming directions on which this operational framework can be useful.

The idea besides this paper is not to claim a new theory to study hypercomplex polynomials on the lattice but instead to propose an approach that makes intuitive and fully rigorous the study special functions and integral transforms on the lattice from methods already known in continuum.

2. Quasi-Monomiality through discrete hypercomplex variables

In this section some basic notations and properties carrying the discrete Clifford setting will be collected. Special emphasize will given to the explicit description of the ladder operators on the lattice based on the knowledge of the multi-variable EGF.

Further details enclosed concerning the definition and properties of Clifford algebras may be found in [27], [16, Chapter 1] and [22, Chapter 2]. For an overview of finite difference discretizations of Dirac operators one refer to [16, Chapter 5] and [11, Section 2]. For the construction of finite difference discretizations on the lattice based on the interplay between finite difference calculus and Lie-algebraic symmetries one refer to [13, Subsection 2.1] and [12, Section 2].

2.1. Discrete Clifford calculus

Let \( \mathbb{R}[x] \) be the ring of multi-variable polynomials in the variable \( x \in \mathbb{R}^n \) and \( e_1, e_2, \ldots, e_n \) an orthogonal basis of \( \mathbb{R}^n \). The Clifford algebra of signature \((0,n)\), denoted by \( C_{0,n} \), corresponds to an algebra of dimension \( 2^n \) generated from the graded commuting relations (3). Under the linear space isomorphic between \( C_{0,n} \) and the exterior algebra \( \Lambda^* (\mathbb{R}^n) \) determined from the mapping

\[
e_{j_1} e_{j_2} \ldots e_{j_r} \mapsto dx_{j_1} dx_{j_2} \ldots dx_{j_r},
\]

with \( 1 \leq j_1 < j_2 < \ldots < j_r \leq n \), every \( a \in C_{0,n} \) may be represented by means of linear combinations involving \( r \)-multivector basis of the form \( e_{j_1} e_{j_2} \ldots e_{j_r} \), labeled by subsets \( J = \{j_1, j_2, \ldots, j_r\} \) of \( \{1, 2, \ldots, n\} \) i.e.

\[
a = \sum_{r=0}^{n} \sum_{|J|=r} a_J e_J, \quad \text{with } e_J = e_{j_1} e_{j_2} \ldots e_{j_r}.
\]

This in particular allows us to represent any vector \( x \in \mathbb{R}^n \) as \( x = \sum_{j=1}^{n} x_j e_j \in C_{0,n} \), and moreover the translations \( (x_1, x_2, \ldots, x_j \pm h, \ldots, x_n) \) on the lattice \( h\mathbb{Z}^n \subset \mathbb{R}^n \) with mesh width \( h > 0 \) as displacements of the form \( x \pm he_j \). In the same order of ideas every multi-index \( \alpha \in \mathbb{N}_0^n \) may be represented as \( \alpha = \sum_{j=1}^{n} \alpha_j e_j \).

The Clifford algebra \( C_{0,n} \) is in fact an universal algebra over \( \mathbb{R} \) that can generated as an algebra of radial type whose center is generated from the anti-commutator \( xy + yx \) between the vectors \( x = \sum_{j=1}^{n} x_j e_j \) and \( y = \sum_{j=1}^{n} x_j e_j \). This in turn results into the inner product relation

\[
\sum_{j=1}^{n} x_j y_j = -\frac{1}{2} (xy + yx)
\]
that will be denoted along this paper by $x \cdot y$.

Next, for each Clifford-vector-valued function $f(x) = \sum_{r=0}^{n} \sum_{|j|=r} f_{j}(x)e_{j}$, with $f_{j}(x)$ real-valued, one define the forward/backward discretizations of the partial derivatives $\partial_{x_{j}}$ on the lattice $h\mathbb{Z}^{n}$ as

$$\left(\partial_{h}^{+j}f\right)(x) = \frac{f(x + he_{j}) - f(x)}{h} \quad \text{and} \quad \left(\partial_{h}^{-j}f\right)(x) = \frac{f(x) - f(x - he_{j})}{h}. \quad (7)$$

These finite difference operators, interrelated by translation operators of the form $(T^{\pm_{j}}_{h}f)(x) = f(x \pm he_{j})$:

$$T^{3}_{h}(\partial_{h}^{-j}f)(x) = (\partial_{h}^{-j}f)(x) \quad \text{and} \quad T^{j+3}_{h}(\partial_{h}^{-j}f)(x) = (\partial_{h}^{-j}f)(x). \quad (8)$$

satisfy the product rules

$$\partial_{h}^{+j}(g(x)f(x)) = (\partial_{h}^{+j}g(x)f(x + he_{j}) + g(x)(\partial_{h}^{+j}f)(x))$$

$$\partial_{h}^{-j}(g(x)f(x)) = (\partial_{h}^{-j}g(x)f(x - he_{j}) + g(x)(\partial_{h}^{-j}f)(x)). \quad (9)$$

Based on (7) one can introduce the forward/backward discretizations $D^{\pm}_{h}$ of the continuum Dirac operator $D = \sum_{j=1}^{n} e_{j}\partial_{x_{j}}$ as

$$D^{+}_{h} = \sum_{j=1}^{n} e_{j}\partial_{h}^{+j} \quad \text{and} \quad D^{-}_{h} = \sum_{j=1}^{n} e_{j}\partial_{h}^{-j}. \quad (10)$$

Let us now restrict ourselves for Clifford-vector-valued functions belonging to the space of Clifford-vector-valued polynomials $\mathcal{P} = \mathbb{R}[x] \otimes \mathbb{C}l_{0,n}$. From the product rules (9) one can see that simplest representation carrying the algebra of endomorphisms $\text{End}(\mathcal{P})$ described by (10) are the forward/backward discretizations $\partial_{h}^{+j}$ resp. $\partial_{h}^{-j}$ and the multiplication operators $x_{j}T^{3}_{h} : f(x) \mapsto x_{j}f(x - he_{j})$ resp. $x_{j}T^{j+3}_{h} : f(x) \mapsto x_{j}f(x + he_{j})$. Indeed, a simple computation based in (8) results into the graded commuting rules

$$[\partial^{+j}_{h}, x_{k}T_{h}^{-k}] = \delta_{jk}I \quad \text{and} \quad [\partial^{-j}_{h}, x_{k}T_{h}^{+k}] = \delta_{jk}I$$

that in turn yield the set of Weyl-Heisenberg relations (2).

Thus, the operators $X_{h}$ and $X_{-h}$ defined via the coordinate formula

$$X_{\varepsilon} : f(x) \mapsto \sum_{j=1}^{n} e_{j}x_{j}f(x - \varepsilon e_{j}) \quad (11)$$

are the Fourier duals of $D^{+}_{h}$ and $D^{-}_{h}$, respectively.

Although the Weyl-Heisenberg relations (2) offer us a reliable way to construct the Fourier duals of $D^{\pm}_{h}$ from a wide range of raising operators on the lattice $h\mathbb{Z}^{n}$ in such a way that the quasi-monomialy constraints (4) fulfills (see e.g. the multiplication operators $W^{\pm}_{h}$ considered in (13), the existence of a EGF of the form (5) assures that the raising operators $M_{j}$ provided from (6) are uniquely determined (cf. (23) Theorem 3.7.1)).

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5In the language of Clifford analysis it is also common to use the terminology Fischer duality (see e.g. (19)) instead of the Fourier duality terminology arising in the setting of invariant theory (cf. (17)).
2.2. Quasi-monomiality formulation on the lattice

Without loss of generality one will consider the forward differences $\partial^+_h$ as the corresponding lowering operators embody in (4) and the forward difference Dirac operator $D^+_h$ as the corresponding discretization of $D = \sum_{j=1}^{n} e_j \partial_{x_j}$.

From the definition of $\partial^+_h$ and $D^+_h$ labeled by (7) and (10), respectively, one can see that the multi-variable function of the form

$$G_h(x, y) = \prod_{j=1}^{n} (1 + hy_j)^{\frac{x_j}{h}}$$

converges asymptotically to $G(x, y) = \exp(x \cdot y)$, as $h$ approaches to zero, and satisfies the eigenvalue equation $D^+_h G_h(x, y) = y G_h(x, y)$.

Now let us denote by $|\alpha| = \sum_{j=1}^{n} \alpha_j$ the sum of all the components of the multi-index $\alpha = \sum_{j=1}^{n} \alpha_j e_j$ and by $\alpha! = \alpha_1! \alpha_2! \ldots \alpha_n!$ the multi-index factorial. In terms of the multi-index generalization of the falling factorials, defined as

$$(x; h)_{\alpha} = \prod_{j=1}^{n} \left(x_j - jh\right)$$

the EGF (12) may be represented as a multi-variable hypergeometric series expansion of the form

$$G_h(x, y) = \prod_{j=1}^{n} \exp\left(\frac{x_j}{h} \log(1 + hy_j)\right) = \sum_{k=0}^{\infty} \sum_{|\alpha| = k} (x; h)_\alpha \frac{y^\alpha}{\alpha!}.$$ 

Based on the operational identity

$$(x; h)_{\alpha} = \prod_{j=1}^{n} \left(x_j T_{-h}^{-1}\right)^{\alpha_j} 1$$

one can conclude that the falling factorials (13) are the simplest quasi-monomial counterpart of the classical multi-variable monomials $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$ on the lattice $h\mathbb{Z}^n$ that can be constructed from the constraint (4).

On the other hand, the Taylor series expansion

$$f(x + he_j) = \sum_{k=0}^{\infty} \frac{1}{k!} (\partial_{x_j})^k f(x)$$

gives rise to the operational representation $T^{+j}_h = \exp(h \partial_{x_j})$ at the level of $\mathcal{P}$, and moreover, to the formal inversion formula

$$\partial_{x_j} = \frac{1}{h} \log \left(1 + h \partial_{x_j}^+\right)$$

written in terms of the logarithmic function $\log(t) = \int_{1}^{t} \frac{ds}{s}$ that follows straightforwardly from [7] and [23, Theorem 2.1.1]. This in turn shows that the raising/lowering operators considered above are a particular example of (6). The functions $\kappa(t)$ and $\ell(t)$ are then given by $\kappa(t) = 1$ and

$$\ell(t) = \exp(ht) - \frac{1}{h}.$$
Remark 2.1. When the forward finite differences $\partial^{+j}_h$ are replaced by the backward finite differences $\partial^{-j}_h$, one can see in terms of [12] resp. [13] that $G_{-h}(x,y)$ resp. $\{(x; -h) : \alpha \in \mathbb{N}_0^n\}$ are the corresponding EGF resp. Sheffer set of polynomials that encode the raising operators $x_j T^{+j}_h$.

The quasi-monomials on the lattice $h\mathbb{Z}^n$ constructed from the quasi-monomial operational representation

$$m_{\alpha}(x) = \prod_{j=1}^{n} (M_j)^{\alpha_j} 1$$

allows us to compute most of the families of multi-variable polynomial sequences of binomial type, already considered in [26], [8], [18], [3], [20], [4] and [29].

In particular, taking the formal series representation $\partial x_j = \frac{1}{h} \log T^{+j}_h$ that follows from [14] and a general function $\kappa(t)$, described in terms of the power series expansion

$$\kappa(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \text{ with } a_0 \neq 0,$$

one can see that the EGF of the form

$$G_h(x,y;\kappa) = \prod_{j=1}^{n} \frac{1}{\kappa \left( \frac{1}{h} \log (1 + hy_j) \right)} (1 + hy_j)^{\frac{\kappa}{a_j}}$$

yield the set of operators $L_j = \partial^{+j}_h$ and $M_j = \left( x_j - \kappa'(\partial x_j) \kappa (\partial x_j)^{-1} \right) T^{-j}_h$ as generators of the Weyl-Heisenberg algebra of dimension $2n + 1$ satisfying (2).

One will finish this subsection with three examples that illustrate the applicability of quasi-monomiality approach on the lattice $h\mathbb{Z}^n$. The first two examples involve the Poisson-Charlier polynomials and Bernoulli polynomials of second kind. On the last example one will sketch how can we derive for $\kappa(t) = 1$ the multi-variable quasi-monomials carrying the central finite difference operators $L_j = \frac{1}{2} \left( \partial^{+j}_h + \partial^{-j}_h \right)$.

Example 2.1 (Poisson-Charlier polynomials). For a parameter $a \in \mathbb{R}$ one can construct the following generalization of multi-variable Poisson-Charlier polynomials $c_{\alpha}(x; h, a)$ for a lattice with mesh width $h > 0$ from $\kappa(t) = \exp \left( \frac{2}{h} (\exp(ht) - 1) \right)$.

This results into the following expressions for the EGF, logarithmic derivative and raising operators:

- **EGF:**
  $$G_h(x,y;\kappa) = \prod_{j=1}^{n} \exp(-ay_j) (1 + hy_j)^{\frac{\kappa}{a_j}}.$$

- **Logarithmic derivative:** $\frac{\kappa'(t)}{\kappa(t)} = a \exp(ht)$.

- **Raising operators:** $M_j = x_j T^{-j}_h - aI$. 

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Example 2.2 (Bernoulli polynomials of second kind). Based on the series expansion

\[
\frac{1}{t} = \sum_{m=0}^{\infty} (1 - t)^m = \sum_{m=0}^{\infty} \sum_{k=0}^{m} \binom{m}{k} (-1)^k t^k
\]

that fulfils for values of \( t \) in the range \( 0 < t < 2 \), one can derive from \( \kappa(t) = \frac{h t}{\exp(h t) - 1} \) the corresponding multi-variable Bernoulli polynomials \( b_\alpha(x; h) \) of second kind on \( h \mathbb{Z}^n \).

The EGF, logarithmic derivative and raising operators are then described in the following way:

- **EGF:**
  \[
  G_h(x, y; \kappa) = \prod_{j=1}^{n} \log \left( 1 + \frac{h y_j}{h y_j} \right) \left( 1 + h y_j \right)^{\frac{x_j}{h}}.
  \]

- **Logarithmic derivative:**
  \[
  \frac{\kappa'(t)}{\kappa(t)} = \frac{1}{t} - \frac{h}{1 - \exp(-ht)}.
  \]

- **Raising operators:**
  \[
  M_j = x_j T_h^{-j} - \sum_{m=0}^{\infty} \sum_{k=0}^{m} \binom{m}{k} (-1)^k \left( \frac{1}{h} \log T_h^{+j} \right)^k - h \left( T_h^{-j} \right)^m.
  \]

Herewith \( \left( \frac{1}{h} \log T_h^{+j} \right)^k \) is nothing else than the formal representation of the iterated partial derivative \( (\partial_{x_j})^k \) while \( \left( T_h^{-j} \right)^m = \exp(-mh\partial_{x_j}) \).

Example 2.3 (Quasi-monomials carrying central differences). When the forward differences \( \partial_h^{+j} \) are replaced by the central differences \( L_j = \frac{1}{2} \left( \partial_h^{+j} + \partial_h^{-j} \right) \) one gets from the combination of the formal power series representation of \( L_j \):

\[
L_j = \frac{1}{h} \sinh (h\partial_{x_j})
\]

with [23, Theorem 2.1.1] that the EGF of the form

\[
\prod_{j=1}^{n} \exp \left( \frac{x_j}{h} \sinh^{-1}(h y_j) \right) = \prod_{j=1}^{n} \left( h y_j + \sqrt{1 + h^2 y_j^2} \right)^{\frac{x_j}{h}}.
\]

encode the set of ladder operators \( L_j \) and \( M_j = x_j [L_j, x_j]^{-1} \), with \([L_j, x_j] = \cosh(h\partial_{x_j})\).

Here one recall that the right hand side of the above formula follows from the fact that \( \log \left( t + \sqrt{1 + t^2} \right) \) is the inverse function of \( \sinh(t) \).

In terms of the vector-field \( u : y \mapsto u(y) \) defined componentwise via the transformations

\[
u_j(y) = y_j - \frac{1}{h} + \sqrt{\frac{1}{h^2} + y_j^2}
\]

one can see that the above multi-variable EGF equals to \( G_h(x, u(y)) \), where \( G_h(x, y) \) stands the multi-variable EGF [72].
Thus, the quasi-monomials $m_\alpha(x)$ generated from the operational rule \[15\] can be determined from the action of the multi-index derivative
\[
(\partial_y)^\alpha = (\partial_{y_1})^{\alpha_1} (\partial_{y_2})^{\alpha_2} \ldots (\partial_{y_n})^{\alpha_n}
\]
on $G_h(x, u(y))$. In concrete, from the hypergeometric series expansion \[14\] one gets
\[
m_\alpha(x) = [(\partial_y)^\alpha G_h(x, u(y))]_{y=0} = \frac{\gamma_\alpha}{\alpha!} (x; h)_\alpha
\]
with $\gamma_\alpha = [(\partial_y)^\alpha u(y)^\alpha]_{y=0}$ and $(x; h)_\alpha$ the falling factorials defined viz \[14\].

**Remark 2.2.** In contrast with \[9, Example 4 of Section 2\], where the quasi-monomials carrying central finite difference operators were computed by a binomial convolution formula, the quasi-monomials derived in Example 2.3 correspond to the 'Taylor coefficients' of the EGF $G_h(x, u(y))$.

**Remark 2.3.** The Bernoulli polynomials of the second kind considered in Example 2.2 may also be formulated based on the formal series representation of multi-variable integral operators.

One suggest to the interested readers to take a look for the sequence of examples explored in \[8, Subsection 4.2\] on which the corresponding formal power series representations as hypergeometric series expansions of $\hypergeom{0}{1}$—type allows to describe, in particular, families of Bernoulli polynomials of the second kind carrying the central difference operators as series expansions generated from Bessel functions.

3. The hypercomplex approach

3.1. Classes of Clifford-vector-valued raising operators

One now want to construct the hypercomplex extension of the quasi-monomial basis \[15\]. To that end one will start with the following proposition, corresponding to the Lie-algebraic description of the Clifford-vector-valued raising operators on the lattice $h\mathbb{Z}^n$.

**Proposition 3.1.** Let $\kappa(t)$ defined as above and $X_h$ the multiplication operator defined viz \[11\]. If there is a multi-variable function $\lambda(y)$ ($y \in \mathbb{R}^n$) such that
\[
\lambda \left( \frac{D_h^+ \exp(x \cdot y)}{\exp(x \cdot y)} \right) = \prod_{j=1}^n \kappa(y_j)
\]
then the Fourier dual $\Lambda_h$ of $D_h^+$ is given by
\[
\Lambda_h = X_h - [\log \lambda \left( D_h^+ \right), x].
\]

**Proof:** First, recall that from \[23, Theorem 3.6.5\] and from the isomorphism between the algebra of formal power series and the algebra of linear functionals (cf. \[23, Theorem 2.1.1\]) there is a one-to-one correspondence between the logarithmic derivative
\[
\kappa'(y_j) \quad \kappa(y_j) = \left[ \frac{d \log(\kappa(t))}{dt} \right]_{t=y_j}
\]
and the Pincherle derivative $[\log (\kappa (\partial_x)) , x_j] = \kappa'(\partial_x) \kappa(\partial_x)^{-1}$.

In the same order of ideas, for $L_j = \partial^{h_j}$ there is a one-to-one correspondence between the backward shift $T^{-j}_h = \left( T^{h_j}_h \right)^{-1}$ that yields from the product rules and the exponentiation relation $\exp(-h y_j) = \exp(h y_j)^{-1}$. Then the Fourier dual $\Lambda_h = \sum_{j=1}^n e_j M_j$ constructed from have the following Lie-algebraic representation in the algebra $\text{End}(\mathcal{P})$:

$$\Lambda_h = X_h - \sum_{j=1}^n e_j \kappa'(\partial_x) \kappa(\partial_x)^{-1} T^{-j}_h. \quad (17)$$

Now let us take a close look from the commutator $[\log(\lambda(D^+_h)), x]$. A short computation based on the identity $\exp((x + h e_j) \cdot y) = \exp(x \cdot y) \exp(h y_j)$ shows that

$$\frac{D^+_h \exp(x \cdot y)}{\exp(x \cdot y)} = \sum_{j=1}^n e_j \exp(h y_j) - \frac{1}{h}$$

corresponds to the representation of $D^+_h$ in the algebra of formal power series (cf. [23, Theorem 2.1.1]). Combination of the chain rule

$$\partial_{y_j} \left( \log(\lambda(D^+_h)) \right) \left( \frac{D^+_h \exp(x \cdot y)}{\exp(x \cdot y)} \right) \exp(h y_j) = \frac{\kappa'(y_j)}{\kappa(y_j)}$$

with [23, Theorem 3.6.5] result into the sequence of identities

$$[\log(\lambda(D^+_h)), x] = \sum_{j=1}^n e_j \left[ \log(\lambda(D^+_h)), x_j \right]$$

$$= \sum_{j=1}^n e_j \left[ \log \left( \kappa(\partial_x) \right), x_j \right] \exp(-h \partial_x)$$

$$= \sum_{j=1}^n e_j \kappa'(\partial_x) \kappa(\partial_x)^{-1} T^{-j}_h.$$

From the above identity one conclude that (17) is therefore equivalent to $\Lambda_h = X_h - [\log \lambda(D^+_h), x]$. ■

Recall that from Proposition 3.1 the set of formal inversion formulae (14), interrelating $\partial_x$ and $\partial^{h_j}$, together with [23, Theorem 2.1.1] shows that the multi-variable function $\lambda(y)$ ($y \in \mathbb{R}^n$) always exists and it is given by

$$\lambda(y) = \prod_{j=1}^n \kappa \left( \frac{1}{h} \log(1 + h y_j) \right). \quad (18)$$

So, one looks for the ladder operators $\Lambda_h$ as covariant versions of the $X_h$ defined viz equation (11) on which the operator $\lambda(D^+_h)$ is obtained from the substitutions $y_j \rightarrow \partial^{h_j}$ on the right hand side of (18).

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Regardless this construction one can easily see that multi-variable functions of the form \( \lambda(y) = \prod_{j=1}^{n} (1 + hy_j)^{d_j} \) \((d_j \in \mathbb{R})\) gives rise to \( \Lambda_h = \sum_{j=1}^{n} e_j (x_j - d_j h) T^{-1}_{h,j} \), a non-trivial class of finite difference discretizations for the multiplication operator \( X = \sum_{j=1}^{n} e_j x_j I \) whereas in case where \( \lambda(y) \) is determined from the constraint

\[
\log \lambda \left( \frac{D_h^+ \exp(x \cdot y)}{\exp(x \cdot y)} \right) = \sum_{j=1}^{n} \frac{\exp(hy_j) - hy_j}{h^2}
\]

results into \( [\log \lambda(D_h^+), x] = D_h^- \), where \( D_h^- \) stands the backward discretization of the Dirac operator defined in [10].

This yields \( \Lambda_h = X_h - D_h^- \) as the corresponding finite difference discretization for the Clifford-Hermite operator

\[
X - D = -\exp \left( \frac{|x|^2}{2} \right) D \exp \left( -\frac{|x|^2}{2} \right)
\]

(19)
on the lattice \( h\mathbb{Z}^n \).

Here one would also like to emphasize that under the choice of \( \lambda(y) \) one can also obtain alternative finite difference discretizations for the Clifford-Hermite operator [19].

This will be the purpose of the next example:

Example 3.1 (A non-trivial discretization for the Clifford-Hermite operator).

For the multi-variable function

\[
\lambda(y) = \prod_{j=1}^{n} \exp \left( \frac{1 + hy_j}{2h^2} + \frac{1}{(2 + 2hy_j)h^2} \right)
\]
on the lattice \( h\mathbb{Z}^n \) one has

\[
\lambda \left( \frac{D_h^+ \exp(x \cdot y)}{\exp(x \cdot y)} \right) = \prod_{j=1}^{n} \exp \left( \frac{\cosh(hy_j)}{h^2} \right).
\]

From the identities \( \partial_{y_j} \cosh(hy_j) = h \sinh(hy_j) \) and

\[
\frac{\sinh(hy_j)}{h} \exp(-hy_j) = \frac{1 - \exp(-2hy_j)}{2h}
\]
on one gets \( [\log \lambda(D_h^+), x] = \partial_{x_j}^{-1} \), the backward discretization of the partial derivative \( \partial_{x_j} \) on the lattice with mesh width \( 2h > 0 \). Therefore, the summand

\[
[\log \lambda(D_h^+), x] = \sum_{j=1}^{n} e_j [\log \lambda(D_h^+), x_j]
\]
equals to \( D_{2h}^- \), the finite difference Dirac operator of backward type defined on the coarse lattice \((2h)\mathbb{Z}^n\) of \( h\mathbb{Z}^n \).
3.2. Appell set formulation for finite difference Dirac operators

Once developed over the previous subsection the key tools to formulate the hypercomplex extension of the monomial principle (4), one have now the minimal amount of tools required to study the Appell set property carrying the finite difference Dirac operator $D^+_h$.

One say that the set \( \{ w_k(x; h; \lambda) : k \in \mathbb{N}_0 \} \) of Clifford-vector-valued polynomials is an Appell set carrying $D^+_h$ if $w_0(x; h; \lambda) = a$ is a Clifford number and $D^+_h w_k(x; h; \lambda)$ is a Clifford-vector-valued polynomial of degree $k - 1$ satisfying the Appell set property

\[
D^+_h w_k(x; h; \lambda) = k w_{k-1}(x; h; \lambda).
\] (20)

Equivalently, the construction of Appell sets may be formulated as a time-evolution problem in the space-time domain $h \mathbb{Z}^n \times \mathbb{R}$. In concrete, this may be formulated in the following way: Find for each $(x, t) \in h \mathbb{Z}^n \times \mathbb{R}$ a function $G_h(x, t; \lambda)$ satisfying the set of equations

\[
\begin{cases}
D^+_h G_h(x, t; \lambda) = t G_h(x, t; \lambda) & \text{for } (x, t) \in h \mathbb{Z}^n \times \mathbb{R} \setminus \{0\} \\
G_h(x, 0; \lambda) = a & \text{for } x \in h \mathbb{Z}^n.
\end{cases}
\] (21)

Based on the embedding of $(x, t) \in \mathbb{R}^{n+1}$ in $C\ell_{0,n}$ given by the paravector representation $t + x = t + \sum_{j=1}^n e_j x_j$, one can determine in the same order of ideas of [5, Section 5] the solution of (21) as an hypercomplex version of the Taylor series expansion. Indeed, if $G_h(x, t; \lambda)$ is a $C^\infty$-function with respect to $t \in \mathbb{R}$ such that

\[
[ (\partial_t)^k G_h(x, t; \lambda) ]_{t=0} = w_k(x; h; \lambda)
\]

is a Clifford-vector-valued polynomial of degree $k$, then the Taylor series expansion of $G_h(x, t; \lambda)$ around $x \in h \mathbb{Z}^n$ given by

\[
G_h(x, t; \lambda) = \sum_{k=0}^{\infty} \frac{t^k}{k!} w_k(x; h; \lambda)
\] (22)

is a solution of the evolution problem (21) that uniquely determines the hypercomplex EGF carrying $D^+_h$.

Now let us take a close look for the monomiality principle from the ladder operators $\Lambda_h$. Based on Fock space formalism one can construct each $w_k(x; h; \lambda)$ by means of the operational rule

\[
w_k(x; h; \lambda) = \mu_k (\Lambda_h)^k a.
\] (23)

The constants $\mu_k \in \mathbb{R}$ ($k \in \mathbb{N}_0$) provided from (23) are thus determined from the condition $w_0(x; h; \lambda) = a$ (normalization condition) and the Appell set constraint (20).

Thus the EGF (22) equals to $G_h(x, t; \lambda) = G(\Lambda_h, t)a$, where $G(x, t)$ given by

\[
G(x, t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mu_k x^k
\] (24)
corresponds to the EGF carrying the classical Dirac operator \( D = \sum_{j=1}^{n} e_j \partial_{x_j} \).

One will turn to the series representation of the hypercomplex EGF \( G \) and \( G_h \) in Subsection 3.4 with the purpose of determine explicitly the coefficients \( \mu_k \) and the hypercomplex counterpart of the hypergeometric series expansion (14) as well.

The bold notations \( G \) and \( G_h \) will be adopted to make distinction between the hypercomplex EGF of the above form and the multi-variable EGF \( G \) and \( G_h \) considered in Subsection 2.2 whereas the shorthand notations \( w_k(x;h) \) and \( G_h(x,t) \) will be used to denote the hypercomplex Appell polynomial of degree \( k \) and the hypercomplex EGF, respectively, carrying the constant function \( \lambda(y) = 1 \), that is, the Clifford-vector-valued polynomials resp. EGF generated from the ladder operator \( X_h \) defined in (11).

The next example, on which one will obtain multinomial representations for \( w_k(x;h) \), will be of special interest in the forthcoming subsections.

**Example 3.2 (The hypercomplex version of the falling factorials).** The hypercomplex extension of the falling factorials \( (x;h)_{\alpha} \) of order \( |\alpha| = k \) provided by (13) are represented, for a given \( a \in \mathbb{C}_{n} \), through the operational formula

\[
 w_k(x;h) = \mu_k (X_h)^k a.
\]

From the Weyl-Heisenberg graded commuting relations (2) carrying the operators \( M_j = x_j T_h^{-j} \), is it clear that \((X_h)^2\) is a scalar-valued operator given by the summation formula

\[
(X_h)^2 = - \sum_{j=1}^{n} \left( x_j T_h^{-j} \right)^2.
\]

A straightforward computation based on the multinomial formula give rise to

\[
 w_{2m}(x;h) = \mu_{2m} \left( - \sum_{j=1}^{n} \left( x_j T_h^{-j} \right)^2 \right)^m a
\]

\[
 = (-1)^m \mu_{2m} \sum_{s=0}^{m} \sum_{|\alpha|=s} \frac{m!}{\alpha!} (x;h)_{2\alpha} a
\]

\[
 w_{2m+1}(x;h) = \frac{\mu_{2m+1}}{\mu_{2m}} \sum_{j=1}^{n} e_j x_j T_h^{-j} w_{2m}(x;h)
\]

\[
 = (-1)^m \mu_{2m+1} \sum_{j=1}^{n} \sum_{s=0}^{m} \sum_{|\alpha|=s} \frac{m!}{\alpha!} (x;h)_{2\alpha+e_j} e_j a.
\]

Incidentally, one can represent the summand \( \sum_{s=0}^{m} \sum_{|\alpha|=s} \frac{m!}{\alpha!} (x;h)_{2\alpha} \) through the inverse of the \( n \)-dimensional Weierstraß transform on \( \mathcal{P} \):

\[
 (Wf)(x) = \frac{1}{(4\pi)^{n/2}} \int_{\mathbb{R}^n} f(x-y) \exp \left( -\frac{|y|^2}{4} \right) dy = \exp \left( -\frac{1}{2} D^2 \right) f(x),
\]

where \( \exp \left( -\frac{1}{2} D^2 \right) = \prod_{j=1}^{n} \exp \left( \frac{1}{2} (\partial_{x_j})^2 \right) \) denotes the \( n \)-dimensional Weierstraß operator (cf. [8, Section 5.3]).
Recall that the action of \( \exp \left( \frac{1}{2} D^2 \right) \) on the multi-index polynomial \( x^\beta = x_1^{\beta_1} x_2^{\beta_2} \ldots x_n^{\beta_n} \) yields an multi-variable Hermite polynomial \( H_\beta(x) \) (cf. [23, Example 4, pp.30]).

In terms of the multi-index falling factorials [13], each \( H_\beta(x) \) may be represented as

\[
H_\beta(x) = \exp \left( \frac{1}{2} D^2 \right) x^\beta = \sum_{k=0}^{\left\lceil \frac{|\beta|}{2} \right\rceil} \sum_{|\alpha| = k} \frac{1}{\alpha!} \left( -\frac{1}{2h^2} \right)^{|\alpha|} (\beta h; h)_{2\alpha} x^{\beta - \alpha}.
\]

Based on this and under the 1–norm constraint

\[
\|x\|_1 := \sum_{j=1}^{n} |x_j| = 2mh,
\]

one gets, in terms of the vector-fields \( e := \sum_{j=1}^{n} e_j \) and \( |x|_1 := \sum_{j=1}^{n} e_j |x_j| \), the meaningful relation for \( x \neq 0 \):

\[
\sum_{s=0}^{m} \sum_{|\alpha| = s} \frac{m!}{\alpha!} (x; h)_{2\alpha} = m! \left( 2h^2 \right)^{2m} H_{\|x\|_1} \left( -\frac{1}{2h^2} e \right).
\]

Based on the identity \( H_\beta(x) = (W^{-1} y^\beta)(x) \), one can therefore recast \( w_{2m}(x; h) \) as

\[
w_{2m}(x; h) = (-1)^m \mu_{2m} m! \left( 2h^2 \right)^{2m} \left( W^{-1} y^{|x|_1} \right) \left( -\frac{1}{2h^2} e \right) a
\]

and \( w_{2m+1}(x; h) \) from the combination of the above expression with the recursive relation

\[
w_{2m+1}(x; h) = \frac{\mu_{2m+1}}{\mu_{2m}} w_{2m}(x; h).
\]

3.3. Classes of discrete hypercomplex polynomials

With the aim of describing the classes of hypercomplex polynomials of discrete variable generated from the operational rule (23) in terms of the the multiplication operator \( X_h \) defined viz (11) one will use a minimal amount of well known facts from Lie algebra theory.

The subsequent lemma will play an important role in the determination of a Rodrigues-type representation formula involving the construction of an operator \( \sigma(D_h^n) \) which intertwines the ladder operators \( \Lambda_h = X_h - \left[ \log \lambda(D_h^n), x \right] \) and \( X_h \).

**Lemma 3.1.** If the three generators \( A, B, C \) of a certain Lie algebra satisfy the graded commutation rules \( [A, B] = C \) and \( [C, B] = 0 \), then for the exponentiation operator

\[
\exp(B) = \sum_{k=0}^{\infty} \frac{1}{k!} B^k
\]

one gets \( [A, \exp(B)] = C \exp(B) \).
Proof: First, recall the following summation formula that holds for every $A, B$ and $k \in \mathbb{N}$:

$$[A, B^k] = \sum_{j=0}^{k-1} B^j [A, B] B^{k-1-j}.$$ 

Under the conditions $[A, B] = C$ and $[C, B] = 0$ one get that the above summation formula equals to $[A, B^k] = CB^{k-1}$. This leads to the set of identities

$$[A, \exp(B)] = \sum_{k=0}^{\infty} \frac{1}{k!} [A, B^k] = \sum_{k=1}^{\infty} \frac{1}{(k-1)!} CB^{k-1} = C \exp(B).$$

Proposition 3.2 (Rodrigues-type formula). Let $\kappa(t)$ defined as above, $\lambda(y)$ the multi-variable function given by Proposition 3.1 and $x \cdot D := \sum_{j=1}^{n} x_j \partial_{x_j}$ the radial derivative.

Then the operator $\sigma(D^+_h) \in \text{End}(\mathbb{P})$ defined via the relation

$$\sigma(D^+_h) = \lambda \left[ \exp(-x \cdot D) D^+_h \exp(x \cdot D) \right]$$

satisfies

$$X_h - \log \lambda(D^+_h), x] = \sigma(D^+_h)^{-1} X_h \sigma(D^+_h).$$

Moreover, the set of Clifford-vector-valued polynomials $w_k(x; h; \lambda)$ determined from

$$\sigma(D^+_h) = \kappa(0)^n \sigma(D^+_h)^{-1} w_k(x; h).$$

Proof: Under the conditions of Proposition 3.1 one can see that the multi-variable $\lambda(y)$ satisfies

$$\lambda \left( \frac{D^+_h \exp(x \cdot y)}{\exp(x \cdot y)} \right) = \prod_{j=1}^{n} \kappa(y_j).$$

Thus, under the substitutions $y_j \to \partial_{x_j}$ on both sides of the above equation, one immediately gets that $\sigma(D^+_h) = \lambda \left[ \exp(-x \cdot D) D^+_h \exp(x \cdot D) \right]$ is a shift-invariant operator satisfying the log-relation

$$\log \sigma(D^+_h) = \sum_{j=1}^{n} \log \kappa \left( \partial_{x_j} \right).$$

Based on the same order of ideas considered in the proof of Proposition 3.1 the following set of graded commuting relations

$$[x_j, - \log \sigma(D^+_h)] = [\log \kappa \left( \partial_{x_j} \right), x_j] = \kappa^\prime \left( \partial_{x_j} \right) \kappa \left( \partial_{x_j} \right)^{-1}$$
follows straightforwardly for each $j = 1, 2, \ldots, n$. Hence, from direct application of Lemma 3.1 for the substitutions $A = x_j I$, $B = \exp (-\log \sigma(D_h^+))$ and $C = \kappa' (\partial_{x_j}) \kappa (\partial_{x_j})^{-1}$ results into

$$[x_j, \sigma(D_h^+)^{-1}] = [x_j, \exp (-\log \sigma(D_h^+))] = \kappa' (\partial_{x_j}) \kappa (\partial_{x_j})^{-1} \sigma(D_h^+)^{-1}. \tag{1}$$

Multiplying both sides of each above identity on the right by $T_h^{-j} = \exp (-h \partial_{x_j})$ one obtains, after a straightforward computation, the following set of intertwining properties

$$\left( x_j - \kappa' (\partial_{x_j}) \kappa (\partial_{x_j})^{-1} \right) T_h^{-j} \sigma(D_h^+)^{-1} = \sigma(D_h^+)^{-1} x_j T_h^{-j}. \tag{2}$$

Multiplying both sides of the above identity on the right by the operator $\sigma(D_h^+)$, it follows by linearity arguments that the operator $\Lambda_h = X_h - [\log \lambda(D_h^+), x]$ admits the representation formula

$$\Lambda_h = \sigma(D_h^+)^{-1} X_h \sigma(D_h^+). \tag{3}$$

The statements for $w_k(x; h; \lambda) = \kappa(0)^n \sigma(D_h^+)^{-1} w_k(x; h)$ are then immediate from the previous formula and from the relation

$$\prod_{j=1}^n \kappa (\partial_{x_j}) a = \prod_{j=1}^n \kappa(0) a = \kappa(0)^n a. \tag{4}$$

Based on the integral representation

$$\sigma(D_h^+)^{-1} = \int_0^\infty \exp (-\sigma(D_h^+)^s) \, ds \tag{5}$$

resulting from the substitution $r \to \sigma(D_h^+)$ on the identity $r^{-1} = \int_0^\infty \exp (-rs) \, ds$, the next corollary is rather obvious:

**Corollary 3.1 (Integral representation).** Let $\kappa(t)$ defined as above, $\lambda(y)$ be the multivariable function given by Proposition 3.1 and $\sigma(D_h^+)$ the operator defined in Proposition 3.2.

Then each $w_k(x; h; \lambda)$ admits the following integral representation formula

$$w_k(x; h; \lambda) = \kappa(0)^n \int_0^\infty \exp (-s \sigma(D_h^+)) [w_k(x; h)] \, ds. \tag{6}$$

**Example 3.3 (Hypercomplex extension of Poisson-Charlier polynomials).** Let us consider again the function $\kappa(t) = \exp (\frac{t}{h} (\exp(ht) - 1))$ used in Example 2.2 to characterize the multi-variable Poisson-Charlier polynomials $c_n(x; h; a)$ of order $|a|$.

Following the same order of ideas of Example 3.2, the Clifford-vector-valued operator of the form $\Lambda_h = \sum_{j=1}^n e_j (x_j T_h^{-j} - a I)$ gives rise to the formulae

$$w_{2m}(x; h; \lambda) = \mu_{2m} \left( - \sum_{j=1}^n (x_j T_h^{-j} - a)^2 \right)^m \tag{7}$$

$$= (-1)^m \mu_{2m} \sum_{s=0}^m \sum_{|\alpha|=s} \frac{m!}{\alpha!} c_{2s}(x; h; a) \tag{8}$$
\[ w_{2m+1}(x; h; \lambda) = \frac{\mu_{2m+1}}{\mu_{2m}} \sum_{j=1}^{n} e_j (x_j T_h^{-j} - a) w_{2m}(x; h; \lambda) \]

\[ = (-1)^m \mu_{2m+1} \sum_{j=1}^{n} \sum_{s=0}^{m} \sum_{|\alpha| = s} \frac{m!}{\alpha!} e_{2\alpha + e_j}(x; h; a) e_j a. \]

Alternatively, based on the properties \( \exp\left( a\partial_x^{+j}\right) a = a, \exp\left( -a\partial_x^{+j}\right) = \exp\left( a\partial_x^{+j}\right)^{-1} \)
and
\[ \prod_{j=1}^{n} \exp\left( -a\partial_x^{+j}\right) = \exp\left( -a \sum_{j=1}^{n} \partial_x^{+j}\right) \]
it follows, from direct application of Proposition 3.1 and Proposition 3.2, that each \( w_k(x; h; \lambda) \) admits operational representation

\[ w_k(x; h; \lambda) = \exp\left( -a \sum_{j=1}^{n} \partial_x^{+j}\right) w_k(x; h). \]

**Remark 3.1 (The Poisson-Charlier connection).** It is straightforward to see that one can recast \( \Lambda_h = X_h - D_h^- \) as

\[ \Lambda_h = \sum_{j=1}^{n} e_j \left( \left( x_j + \frac{1}{h}\right) T_h^{-j} - \frac{1}{h} I \right), \]
and so, the polynomials \( w_k(x; h; \lambda) \) determined from the operatorial rule (23) may be described in terms hypercomplex Poisson-Charlier polynomials obtained in Example 3.3 carrying the parameter \( a = \frac{1}{h} \) and the displaced vector variable \( x + \frac{1}{h} e \), with \( e = \sum_{j=1}^{n} e_j \).

### 3.4. Hypergeometric series representations

In this subsection one shall construct the hypercomplex extension of the multi-variable EGF \( G_h(x, y; \kappa) \) considered by (16) based on hypercomplex quasi-monomials representations for \( w_k(x; h; \lambda) \) obtained on the previous subsections.

Based on the operational formula \( G_h(x, t; \lambda) = G(\Lambda_h, t) a \), one shall compute, first of all, the constants \( \mu_k \) assigned by the operational formula (24) based on the constraint

\[ D G(x, t) = t G(x, t). \]

Hence, in order to solve this equation in order to \( \mu_k \) \((k \in \mathbb{N}_0)\) one must take the ansatz (24) and compute the action of the Dirac operator \( D = \sum_{j=1}^{n} e_j \partial_x \) on the Clifford-vector-valued monomials \( x^k \).

Recall that \( x^k \) equals to \( x^{2m} = (-1)^m |x|^{2m} \) \((k = 2m)\) and \( x^{2m+1} = (-1)^m |x|^{2m} \) \((k = 2m+1)\). The following lemma, a particular case of cf. [12, Lemma 3.1], is the key ingredient used in the proof in the subsequent proposition:
Lemma 3.2. The Clifford-vector-valued monomials satisfy the recursive relations

\[ D^{x^k} = \begin{cases} 
-2m x^{2m-1}, & \text{if } k = 2m \in \mathbb{N}_0 \\
-(2m + n)x^{2m}, & \text{if } k = 2m + 1 \in \mathbb{N}
\end{cases} \]

(25)

Proposition 3.3. The EGF \((24)\) satisfies \(D G(x, t) = t G(x, t)\) if and only if the constants \(\mu_k\) equals

\[ \mu_{2m} = (-1)^m \frac{\left(\frac{1}{2}\right)_m}{(2m)_m} \quad \text{and} \quad \mu_{2m+1} = (-1)^m \frac{\left(\frac{3}{2}\right)_m}{(2m+1)_m}. \]

Moreover

\[ G(x, t) = {}_0F_1 \left( \frac{n}{2}; -\frac{t^2}{4} x^2 \right) + tx {}_0F_1 \left( \frac{n}{2} + 1; -\frac{t^2}{4} x^2 \right) \]

Proof: Applying Lemma 3.2 at each summand of \((24)\) one gets the splitting formula

\[ DG(x, t) = \sum_{m=0}^{\infty} -2m \mu_{2m} x^{2m-1} \frac{t^{2m}}{(2m)!} + \sum_{m=0}^{\infty} -(2m + n) \mu_{2m+1} x^{2m} \frac{t^{2m+1}}{(2m+1)!} \]

Therefore \(G(x, t)\) is a solution of the equation \(D G(x, t) = t G(x, t)\) if and only if

\[ \frac{-2m \mu_{2m}}{(2m)!} = \frac{\mu_{2m-1}}{(2m-1)!} \quad \text{and} \quad \frac{-(2m + n) \mu_{2m+1}}{(2m+1)!} = \frac{\mu_{2m}}{(2m)!}, \]

that is

\[ \mu_{2m} = -\mu_{2m-1} \quad \text{and} \quad \mu_{2m+1} = -\frac{2m + 1}{2m + n} \mu_{2m-1}. \]

Induction over \(m \in \mathbb{N}_0\) shows that, in terms of the Pochhammer symbol \((a)_m = \frac{\Gamma(a + m)}{\Gamma(a)}\),

\[ \mu_{2m} = (-1)^m \frac{\left(\frac{1}{2}\right)_m}{(2m)_m} \quad \text{and} \quad \mu_{2m+1} = (-1)^m \frac{\left(\frac{3}{2}\right)_m}{(2m+1)_m}. \]

Hence, a short computation based on the properties

\[ \frac{\left(\frac{1}{2}\right)_m}{(2m)_m} = \frac{\left(\frac{1}{2}\right)_m}{m!} = \frac{\left(\frac{3}{2}\right)_m}{(2m+1)!} \]

gives in turn above hypergeometric series splitting of type \( {}_0F_1 \):

\[ G(x, t) = \sum_{m=0}^{\infty} (-1)^m \frac{\left(\frac{1}{2}\right)_m}{(2m)_m} x^{2m} \frac{t^{2m}}{(2m)!} + \sum_{m=0}^{\infty} (-1)^m \frac{\left(\frac{3}{2}\right)_m}{(2m+1)_m} x^{2m+1} \frac{t^{2m+1}}{(2m+1)!} \]

\[ = {}_0F_1 \left( \frac{n}{2}; -\frac{t^2}{4} x^2 \right) + tx {}_0F_1 \left( \frac{n}{2} + 1; -\frac{t^2}{4} x^2 \right). \]

From direct combination of the above proposition with Proposition 3.2, the next corollary follows naturally:
Corollary 3.2. The EGF $G_h(x, t)$ defined via the Taylor series expansion (22) has the formal hypergeometric series representation

$$G_h(x, t) = {}_0F_1\left(\frac{n}{2}; -\frac{t^2}{4} (X_h)^2\right) a + tX_h \ {}_0F_1\left(\frac{n}{2} + 1; -\frac{t^2}{4} (X_h)^2\right) a.$$

Moreover for the function $\kappa(t)$ defined as above and for the multi-variable function $\lambda(y)$ defined in Proposition 3.1 we then have

$$G_h(x, t, \lambda) = \kappa(0) \sigma(D_h)_{-1} [G_h(x, t)].$$

Remark 3.2. Based on Example 3.2 the even powers $(X_h)^{2m} a$ are described, for each $x \in hZ$ satisfying $\|x\|_1 = 2mh$, by the set of identities

$$(X_h)^{2m} a = (-1)^m \left(\frac{h}{2}\right)_m w_{2m}(x; h)$$

$$= (-1)^m m! \left[ -\frac{1}{2h} \text{sgn}(x) \right]^{-\frac{|\beta|}{2}} H_{\|x\|_1} \left( -\frac{1}{2h} \text{sgn}(x) \right) a.$$

On the above formula $H_\beta(y) = \exp\left(\frac{1}{2} D^2\right) y^\beta$ denotes the multi-variable Hermite polynomials of order $|\beta| = 2m$.

Therefore, the hypergeometric functions of the form $\ {}_0F_1\left( s + 1; -\frac{t^2}{4} (X_h)^2\right) a$ provided by Corollary 3.2 are explicitly given by

$$\ {}_0F_1\left( s + 1; -\frac{t^2}{4} (X_h)^2\right) a =$$

$$= \sum_{m=0}^{\infty} \left(\frac{h^2 t}{2}\right)^{2m} (s + 1) \sum_{\|x\|_1 = 2mh} H_{\|x\|_1} \left( -\frac{1}{2h^2} e \right) a.$$

4. Concluding Remarks

4.1. Outlook of the Results

The set of results enclosed in Section 3 interrelating the construction of Appell sets through quasi-monomial representation (23) with the hypercomplex EGF, gives us a notable extension of the multi-variable approach ‘revisited’ in a self-contained way along Subsection 2.2. Note that Corollary 3.2 corresponds to the hypercomplex extension of the multi-variable EGF $G_h(x, y; \kappa)$ considered in (16).

The $\ {}_0F_1$-hypergeometric series representations obtained on Subsection 3.4 shows that the hypercomplex EGF $G_h(x, t; \lambda)$ obtained from the substitution $x \rightarrow \Lambda_h$ on $G(x, t) a$ are of Bessel type and close to the monogenic exponential function obtained in (5, Theorem 2). Indeed, based on the hypergeometric representation for the Bessel function $J_s(u)$ of order $s$:

$$J_s(u) = \frac{1}{\Gamma(s + 1)} \left(\frac{u}{2}\right)^s \ {}_0F_1\left( s + 1; -\frac{u^2}{4}\right),$$

19
the EGF obtained in Proposition 3.3 is equivalent to

\[ G(x, t) = \Gamma \left( \frac{n}{2} \right) \left( \frac{tx}{2} \right)^{-\frac{n}{2}+1} \left( J_{\frac{n}{2}-1}(tx) + n J_{\frac{n}{2}}(tx) \right). \]

Notice that the hypercomplex extension of the ladder operator construction provided by Proposition 3.1 is based on the knowledge of the multi-variable EGF obtained in Subsection 2.2. This was indeed the main novelty of this approach against other recent approaches towards this direction (see e.g. [5, Sections 4 & 5] and [10, Section 4]). In addition, the construction of the hypercomplex extension of the EGF based on the Weyl-Heisenberg symmetries (2) encoded on multi-variable EGF \( G_h(x, y; \kappa) \) labeled by equation (16) avoids a-priori the construction of quasi-monomials in terms permutational products involving ‘regular variables’ (see e.g. [16, Subsection 2.2] on which such construction was revisited and discussed and [21, Section 2] where such framework was applied to compute an hypercomplex extension of the ‘generalized’ Bernoulli polynomials).

The operational characterization given by Proposition 3.2 allows us to compute, from the knowledge of the hypercomplex extension of the falling factorial considered in Example 3.2, several classes of hypercomplex polynomials of discrete variable a symbolic way like e.g. the hypercomplex extension of the Poisson-Charlier polynomials illustrated in Example 3.3. Part of this characterization implies that the hypercomplex Poisson-Charlier polynomials carrying the parameter \( a > 0 \) are solutions of the set of differential-difference equations

\[
\begin{align*}
\partial_a f(x, a) &= -a \sum_{j=1}^{n} \partial^j \partial^{-j} f(x, a) \quad \text{for} \quad (x, a) \in h\mathbb{Z}^n \times [0, \infty) \\
f(x, 0) &= w_k(x; h) \quad \text{for} \quad x \in h\mathbb{Z}^n.
\end{align*}
\]  

(26)

whereas for \( a < 0 \), the hypercomplex Charlier polynomials may be recovered from the mapping transformation \( f(x, a) \mapsto f(x, -a) \) on (26).

From a general perspective, the integral representation given by Corollary 3.1 allows us to determine each \( w_k(x; h; \lambda) \) as \( w_k(x; h; \lambda) = \int_0^\infty f(x, s) ds \), where \( f(x, s) \) is a solution of the differential-difference time-evolution problem on the space-time domain \( h\mathbb{Z}^n \times [0, \infty) \)

\[
\begin{align*}
\partial_s f(x, s) &= -s \sigma(D^+_h) f(x, s) \quad \text{for} \quad (x, s) \in h\mathbb{Z}^n \times (0, \infty) \\
f(x, 0) &= w_k(x; h) \quad \text{for} \quad x \in h\mathbb{Z}^n.
\end{align*}
\]  

(27)

4.2. Further directions

Another perspective for the integral representation of \( w_k(x; h; \lambda) \) based on Corollary 3.1 may be obtained from the representation of the exponentiation operator \( \exp \left(-s \sigma(D^+_h) \right) \) on the momentum space on \( Q_h = \left( -\frac{\pi}{h}, \frac{\pi}{h} \right)^n \), the corresponding Brioullin zone\(^6\) of the

\(^6\)Roughly speaking, the Brioullin zone corresponds to the cellular decomposition of \( \mathbb{R}^n \) determined from a lattice such that there is a one-to-one correspondence between each point of the lattice with the boundary points the each cell in the momentum space.
lattice \( h\mathbb{Z}^n \), through the ‘discrete Fourier’ transform (cf. [16] Subsection 5.2)

\[
(F_h g)(y) = \begin{cases} 
\frac{h^n}{(2\pi)^n} \sum_{x \in h\mathbb{Z}^n} g(x) \exp(ix \cdot y) & \text{for } y \in Q_h \\
0 & \text{for } y \in \mathbb{R}^n \setminus Q_h.
\end{cases}
\]

Here we recall that the ‘discrete Fourier’ transform is a unitary operator from \( \ell_2(h\mathbb{Z}^n) \) onto \( L_2(Q_h) \) whose inverse is given by the restriction of the Fourier transform \( F : L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n) \) to the lattice \( h\mathbb{Z}^n \), that is \( F_h^{-1} = \mathcal{R}_h F \) where \( \mathcal{R}_h g(x) \) stands the restriction of the function \( g(x) \) to \( h\mathbb{Z}^n \) and

\[
(F g)(y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} g(y) \exp(-ix \cdot y) dy.
\]

On the other hand, from the unitary one-to-one correspondence \( \kappa(\partial_{x_j}) \mapsto \kappa(-i\partial_{x_j}) \) provided by Fourier transform (28) one infers from the construction of \( \lambda(D_h^x) \) and \( \sigma(D_h^x) \) in Proposition 3.1 and Proposition 3.2 respectively, the following one-to-one correspondence

\[
F : \exp(-s\sigma(D_h^x)) g(x) \mapsto \exp\left(-s\lambda\left(D_h^x \frac{\exp(-ix \cdot y)}{\exp(-ix \cdot y)}\right)\right) (F g)(y).
\]

Based on this one can infer that the integral description of \( w_k(x; h; \lambda) \), provided by Corollary 3.1 corresponds to the following multiple integral representation over \( Q_h \times [0, \infty) \):

\[
\frac{\kappa(0)^n}{(2\pi)^n} \int_0^\infty \int_{Q_h} \exp\left(-s\lambda\left(D_h^x \frac{\exp(-ix \cdot y)}{\exp(-ix \cdot y)}\right) - ix \cdot y\right) (F_h w_k)(y; h) dy ds.
\]

Integral representations of the above type hints an additional possible application in the study of spectral problems within discrete quantum mechanics (cf. [9] Section 6) involving finite difference discretizations of \( D = \sum_{j=1}^n e_j \partial_{x_j} \). Although the eigenfunctions any finite difference Dirac operator can be computed as formal power series representations (for example, the hypercomplex formulation of EGF provided by equation (21)) that recovers the full spectrum of \( D \) as \( h \to 0 \). The existence of ‘doublers’ within the Brillouin zone is notoriously the major difficult in case that self-adjoint discretizations such as \( \frac{1}{2} \left(D_{h/2}^x + D_{h/2}^y\right) \) are considered (cf. [24] Chapter 4).

Besides the hypercomplex extension of multi-variable falling factorials \( (x; h)_\alpha \) defined in [13], the construction obtained in Example 3.2 gives us some interesting insights concerning applications in sampling theory and integral transforms.

Part of the construction provided on this example shows that the hypercomplex polynomials \( w_k(x; h) \) may be reconstructed from the sampling points determined from the

---

7 The ‘discrete Fourier’ transform is a particular case of a Fourier quadrature rule used to represent a lattice function on the momentum space. Further details may be found in [13].

8 From the Fourier analysis side, the presence of ‘doublers’ is equivalent to say that the symbol of the finite difference Dirac operator has many zeroes inside the Brillouin zone.
intersection of the lattice $h\mathbb{Z}^n$ with the level curves determined from the 1-norm constraint

$$\sum_{j=1}^{n} |x_j| = 2 \left\lfloor \frac{k}{2} \right\rfloor h,$$

as it is depicted in Figure 1.

Based on this observation, one can compute the Taylor series approximation of a continuous function $f(x)$ defined on a bounded domain $\Omega$ of $\mathbb{R}^n$ by simply take the $m$-term truncation of $\sum_{j=1}^{n} g(x + h e_j, t) + g(x - h e_j, t) - 2g(x, t) h^2$, $(x, t) \in h\mathbb{Z}^n \times [0, \infty)$

$g(x, 0) = m_\alpha(x)$

as solutions of the differential-difference equation

$$\partial_t g(x, t) = \sum_{j=1}^{n} \left( \frac{g(x + h e_j, t) + g(x - h e_j, t) - 2g(x, t)}{h^2} \right), (x, t) \in h\mathbb{Z}^n \times [0, \infty)$$

This point of view, roughly considered in [7], was applied recently in [1] to compute a discrete counterpart of the heat kernel as solutions of the differential-difference equation

---

9i.e. the kernel of $t^{-\frac{n}{2}} (Wf) \left( \frac{x}{\sqrt{t}} \right)$, where $W$ stands the Weierstraß transform considered in Example 3.2.
On this direction one would like to observe, from one hand, that the construction considered in [7] shall be understood as special case of a more general framework on which the interpolating functions are determined upon the action of an integral transform on the quasi-monomials \( m_\alpha(x) \) determined from the operational rule (15).

On the other hand, the action of the propagator \( \exp \left( -\frac{t}{4} \left( D_{h/2}^+ + D_{h/2}^- \right)^2 \right) \) on each \( m_\alpha(x) \) results into solutions of the above differential-difference equation. Hereby the multi-variable quasi-polynomial \( m_\alpha(x) \) may be taken from the range of polynomials explored in Example 2.3 (or even in [9, Example 4 of Section 2]) through the substitution \( h \to \frac{h}{2} \).

Obviously, this approach could be applied and extended to other shapes. For example, the inspiring work of Kisil [18] shows in turn the potentiality that this quasi-monomiality approach can bring as a new insight against other well-known path integral formalisms based on the construction of Euclidean Green’s functions (cf. [21 Chapter 2]). Its feasibility will be discussed on further research.

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