On continuous symmetries of second-order homogeneous linear ordinary differential equations

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Abstract. In this work, a method to extract continuous symmetries of general second-order linear ordinary differential equation is presented. The formalism is illustrated by two examples.

1. Introduction

Ordinary differential equations (ODEs) appear in many fields of Physics [1] and the analysis of their symmetries plays an important role to extract information about the solutions of those equations [2-4]. In this work, we have applied the Anderson-Kumei-Wulfman method [5-8] to extract continuous symmetries of general second-order linear ordinary differential equation.

In general, we consider general homogeneous linear ODEs represented by the action of the differential operator $\hat{A}(x)$ on a function $f(x)$

$$\hat{A}(x) f(x) = 0, \quad (1)$$

where the differential operator $\hat{A}(x)$ is give by

$$\hat{A}(x) = \alpha_0(x) + \alpha_1(x) \partial_x + \alpha_2(x) \partial_{xx} + ... \quad (2)$$

In this work we are interested in second order ODEs, then $\alpha_i = 0$, for $i \geq 3$.

A symmetry operator $\hat{Q}$ of equation (1) is defined as a differential operator that maps solution of equation (1) into solution of the same equation, i.e.,

$$\hat{A}(x) g(x) = 0, \quad (3)$$

where

$$g(x) = \hat{Q}(x) f(x). \quad (4)$$

After assuming a particular form for the operators $\hat{Q}$, condition (3) under constrain (1) defines the symmetries of the ODE (1).
2. Symmetry extraction

Let us start by considering a general second-order homogeneous linear ODE:

$$\frac{d^2 y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y(x) = 0. \tag{5}$$

After substituting

$$y(x) = f(x) \exp \left[ -\frac{1}{2} \int x p(\xi) d\xi \right] \tag{6}$$

in equation (5), it is easy to see that the function $f(x)$ satisfies the following differential equation:

$$\frac{d^2 f}{dx^2} + v(x)f(x) = 0, \tag{7}$$

where

$$v(x) = q(x) - \frac{1}{4}p^2(x) - \frac{1}{2}\frac{dp}{dx}. \tag{8}$$

We are interested in continuous symmetry generators of the form

$$\hat{Q}(x) = \alpha(x) + \beta(x) \frac{d}{dx} \tag{9}$$

for the equation (7). In other words, if $f(x)$ is a solution of equation (7) then we need to find the functions $\alpha(x)$ and $\beta(x)$ so that:

$$g(x) = \hat{Q}(x)f(x) \tag{10}$$
is also a solution of equation (7), i.e.

$$\frac{d^2 g}{dx^2} + v(x)g(x) = 0. \tag{11}$$

Then differentiating equation (10), replacing in (11) and using that $f(x)$ is lineally independent, we obtain the following equation system:

$$\alpha'' - v'\beta - 2v\beta' = 0, \tag{12}$$
$$2\alpha' + \beta'' = 0. \tag{13}$$

The above system determines the conditions for $\alpha(x)$ and $\beta(x)$ function.

Working on equations (12) and (13), we obtain:

$$\beta''' + 4v\beta' + 2v'\beta = 0. \tag{14}$$

Equation (14) may be solved if $v(x)$ function is known.

Let us assume that we know two particular solutions $u_1(x)$ and $u_2(x)$ for equation (7), i.e.

$$u_1'' + vu_1 = 0 \quad \text{and} \quad u_2'' + vu_2 = 0. \tag{15}$$

Now, if we consider the following definition

$$\phi(x) = C_1u_1^2 + C_2u_2^2 + C_3u_1u_2, \tag{16}$$

It is easy to show that $\phi(x)$ function satisfies the equation:

$$\phi''' + 4v\phi' + 2v'\phi = 0. \tag{17}$$
Therefore, equation (14) defines the $\beta(x)$ function as

$$\beta(x) = C_1u_1^2 + C_2u_2^2 + C_3u_1u_2,$$

where the functions $u_1(x)$ and $u_2(x)$ are two lineally independent particular solutions of equation (7):

$$u'' + vu_1 = 0 \quad \text{and} \quad u'' + vu_2 = 0.$$

However, it is only necessary to know one solution since the second linear independent solution of equation (7) is obtained by the relation

$$u_2(x) = u_1(x) \int_x^\infty \frac{d\xi}{[u_1(\xi)]^2}.$$  

Finally, having $\beta(x)$ it is possible to obtain $\alpha(x)$ expansion coefficient by:

$$\alpha'' = -\frac{1}{2}\beta''.$$  

3. Examples
The symmetry extraction method will be exemplified by two simple equations:

3.1. Example 1
Let us consider the simplest second order ODE:

$$f_{xx} = 0.$$  

Step 1: Find a particular solution of (21)

$$u_1 = C_1x.$$  

Step 2: Use equation (19), to build a second independent solution

$$u_2 = C_1x \int_x^\infty \frac{d\xi}{[C_1\xi]^2} = C_2.$$  

Step 3: Use equation (18) to obtain $\beta(x)$ function

$$\beta(x) = B_1x^2 + B_2 + B_3x.$$  

Step 4: Use equation (13) to obtain $\alpha(x)$ function

$$\alpha(x) = -\frac{1}{2} \int_x^\infty \beta''(\xi)d\xi = -B_1x.$$  

Step 5: Build the symmetry generators

$$\hat{Q}_1 = \frac{d}{dx}, \quad \hat{Q}_2 = x\frac{d}{dx}, \quad \hat{Q}_3 = x - x^2 \frac{d}{dx}.$$  

Step 6: Check the symmetry property

$$\hat{A}\hat{Q}_1f = 0, \quad \hat{A}\hat{Q}_2f = 0, \quad \hat{A}\hat{Q}_3f = 0.$$  

Step 7: Find the algebra

$$[\hat{Q}_1, \hat{Q}_2] = \hat{Q}_1, \quad [\hat{Q}_1, \hat{Q}_3] = I - 2\hat{Q}_2, \quad [\hat{Q}_2, \hat{Q}_3] = \hat{Q}_3.$$  

And, introducing the new definition:

$$\hat{A}_0 = \hat{Q}_2 - 1/2, \quad \hat{A}_- = \hat{Q}_1, \quad \hat{A}_+ = \hat{Q}_3,$$

the following commutation relations are obtained:

$$[\hat{A}_0, \hat{A}_\pm] = \pm \hat{A}_\pm, \quad [\hat{A}_+, \hat{A}_-] = -2\hat{A}_0.$$
3.1.1. **Symmetry Visualization**: With the symmetry generators we can obtain the action of this generators on the solution of the original ODE.

The general solution of ODE (21) is:

\[ y = a + bx, \quad (31) \]

where \( a \) and \( b \) are arbitrary constants. Then the actions of \( \hat{A}_+ \), \( \hat{A}_- \) and \( \hat{A}_0 \) over (31) are

\[
\begin{align*}
\exp(\theta \hat{A}_0)(a + bx) &= \bar{a} + \bar{b}x, & \bar{a} &= e^{-\theta/2}a, & \bar{b} &= e^{\theta/2}b \\
\exp(\theta \hat{A}_-)(a + bx) &= \bar{a} + bx, & \bar{a} &= a + \theta b \\
\exp(\theta \hat{A}_+)(a + bx) &= a + bx, & \bar{b} &= b + \theta a
\end{align*}
\]

Figure 1 shows the plot of (31) for particular values of \( a \) and \( b \) constants. Figures 2, 3 and 4 show the action of \( \hat{A}_+ \), \( \hat{A}_- \) and \( \hat{A}_0 \) symmetry operators on solution (31) respectively.

![Figure 1](image1.png)

![Figure 2](image2.png)

![Figure 3](image3.png)

![Figure 4](image4.png)

3.2. **Example 2**

Let us consider the second order ODE:

\[ f_{xx} + k^2 f = 0. \quad (32) \]

**Step 1**: Find a particular solution of (32)

\[ u_1 = \cos(kx). \quad (33) \]

**Step 2**: Use equation (19) to build a second independent solution

\[ u_2 = \cos(kx) \int x \frac{d\xi}{[\cos(kx)]^2} = \frac{\sin(kx)}{k}. \quad (34) \]
Step 3: Use equation (18) to obtain $\beta(x)$ function

$$\beta(x) = C_1 \cos^2(kx) + C_2 \frac{\sin^2(kx)}{k^2} + C_3 \frac{\sin(kx)}{k} \cos(kx).$$  \hspace{1cm} (35)$$

Step 4: Use equation (13) to obtain $\alpha(x)$ function

$$\alpha(x) = C_1 \frac{k \sin(2kx)}{2} - C_2 \frac{\sin(2kx)}{2k} + C_3 \frac{1}{2} (1 - \cos(2kx)) + C_4.$$  \hspace{1cm} (36)$$

Step 5: Build the symmetry generators

$$\hat{Q}_1 = \frac{1}{2} \sin(2kx) + \frac{1}{2} (1 + \cos(2kx)) \frac{d}{dx},$$

$$\hat{Q}_2 = \frac{1}{2} - \frac{1}{2} \cos(2kx) + \frac{1}{2k} \sin(2kx) \frac{d}{dx},$$

$$\hat{Q}_3 = \frac{1}{2k} \sin(2kx) + \frac{1}{2k^2} (\cos(2kx) - 1) \frac{d}{dx}.$$  \hspace{1cm} (37-39)$$

Step 6: Check the symmetry property

$$\hat{A} \hat{Q}_1 f = 0 \quad \hat{A} \hat{Q}_2 f = 0 \quad \hat{A} \hat{Q}_3 f = 0.$$  \hspace{1cm} (40)$$

Step 7: Find the algebra

$$[\hat{Q}_1, \hat{Q}_2] = \hat{Q}_1, \quad [\hat{Q}_1, \hat{Q}_3] = I - 2 \hat{Q}_2, \quad [\hat{Q}_2, \hat{Q}_3] = \hat{Q}_3.$$  \hspace{1cm} (41)$$

And, introducing the new definitions:

$$\hat{A}_0 = \hat{Q}_2 - 1/2, \quad \hat{A}_- = \hat{Q}_1, \quad \hat{A}_+ = \hat{Q}_3,$$  \hspace{1cm} (42)$$

the following commutation relations are obtained:

$$[\hat{A}_0, \hat{A}_\pm] = \pm \hat{A}_\pm, \quad [\hat{A}_+, \hat{A}_-] = -2 \hat{A}_0.$$  \hspace{1cm} (43)$$

3.2.1. Matching examples 1 and 2: From the equation (33) and (34) we have the general solution of EDO (32)

$$y = C_1 \cos(kx) + \frac{C_2}{k} \sin(kx).$$  \hspace{1cm} (44)$$

Taking the limit $k \to 0$ on equation (32), we obtain equation (21). In same way, this limit reduces solution (44) to the solution of equation (21).

$$y = C_1 + C_2 x.$$  \hspace{1cm} (45)$$

On the other hand, taking the limit $k \to 0$ on symmetry generators(37), (38) and (39), we obtain

$$\lim_{k \to 0} (\hat{Q}_1) = \lim_{k \to 0} \left( \frac{k}{2} \sin(2kx) + \frac{1}{2} (1 + \cos(2kx)) \frac{d}{dx} \right) = \frac{d}{dx},$$

$$\lim_{k \to 0} (\hat{Q}_2) = \lim_{k \to 0} \left( \frac{1}{2} - \frac{1}{2} \cos(2kx) + \frac{1}{2k} \sin(2kx) \frac{d}{dx} \right) = x \frac{d}{dx},$$

$$\lim_{k \to 0} (\hat{Q}_3) = \lim_{k \to 0} \left( \frac{1}{2k} \sin(2kx) + \frac{1}{2k^2} (\cos(2kx) - 1) \frac{d}{dx} \right) = x - x^2 \frac{d}{dx}.$$  \hspace{1cm} (46-48)$$

Showing the compatibility between the symmetry operators obtained in example 1 and 2 respectively. It is noteworthy that commutations relations (28) remain invariant under the limit $k \to 0$. 

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4. Conclusion
A general method to find continuous symmetries of second-order linear ODEs has been presented. To obtain the symmetry generators a particular solution of the ODE under study is required. The method has been illustrated by two examples.

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