Novel discrete symmetries in the general $\mathcal{N} = 2$ supersymmetric quantum mechanical model

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Abstract In addition to the usual supersymmetric (SUSY) continuous symmetry transformations for the general $\mathcal{N} = 2$ SUSY quantum mechanical model, we show the existence of a set of novel discrete symmetry transformations for the Lagrangian of the above SUSY quantum mechanical model. Out of all these discrete symmetry transformations, a unique discrete transformation corresponds to the Hodge duality operation of differential geometry and the above SUSY continuous symmetry transformations (and their anticommutator) provide the physical realizations of the de Rham cohomological operators of differential geometry. Thus, we provide a concrete proof of our earlier conjecture that any arbitrary $\mathcal{N} = 2$ SUSY quantum mechanical model is an example of a Hodge theory where the cohomological operators find their physical realizations in the language of symmetry transformations of this theory. Possible physical implications of our present study are pointed out, too.

Keywords $\mathcal{N} = 2$ supersymmetric quantum mechanics · superspace approach · continuous and discrete symmetries · de Rham cohomological operators · Hodge duality operation · a physical model for the Hodge theory

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1 Introduction

The supersymmetric (SUSY) models of quantum mechanics (QM) provide one of the most fertile grounds for the growth of ideas, germinating out from the branches of mathematics and physics, in a physically meaningful manner (see, e.g. [1-5]). A decisive feature of these models is the observation that, at the classical level, the commuting variables of the theory transform to their anticommuting counterparts and vice-versa due to the presence of a fermionic SUSY symmetry in the theory. At the quantum level, SUSY QM admits two Hamiltonians (and corresponding states) which are connected to each-other, in a specific manner, because of the presence of the above quoted fermionic SUSY symmetry. The existence of the latter is one of the hallmarks of any arbitrary SUSY quantum mechanical theory in any arbitrary dimension of spacetime.

A very special class of the above SUSY QM is the one which is characterized by the existence of two fermionic ($Q^2 = \bar{Q}^2 = 0$) charges ($Q$ and $\bar{Q}$) and the (bosonic) Hamiltonian ($H$) of the theory which obey a specific $\mathcal{N} = 2$ SUSY algebra. All these charges generate continuous symmetry transformations that also satisfy the above algebra in their operator form. These sets of SUSY quantum mechanical systems are known as the general $\mathcal{N} = 2$ SUSY models. Some of the physical examples of such a class of theories have been recently shown to be endowed with a novel set of discrete symmetry transformations. The interplay of the discrete and usual continuous symmetry transformations has led to establish that the $\mathcal{N} = 2$ SUSY models provide the physical examples of Hodge theory [4, 5].

In a very recent paper [5], we also conjectured that any arbitrary $\mathcal{N} = 2$ SUSY quantum mechanical model would provide a physical example for the Hodge theory where the de Rham cohomological operators, Hodge duality operation, degree of a form, etc., of differential geometry would find their physical realizations in the language of symmetry properties of the above SUSY systems. The purpose of our present investigation is to give the proof of the above conjecture for any arbitrary $\mathcal{N} = 2$ SUSY model with any arbitrary superpotential. We show that this general model is endowed with discrete symmetry transformations which, together with the usual continuous symmetry transformations, provide the physical realizations of all the cohomological operators of differ-
ential geometry. In fact, there exists a novel duality symmetry in the theory and all aspects of the cohomological operators are realized in the language of symmetry properties and conserved charges (and their eigenvalues).

In our earlier works [6–10], we have shown that the 2D usual (non-)Abelian 1-form gauge theories, 4D Abelian 2-form gauge theory and 6D Abelian 3-form gauge theory provide perfect models for the Hodge theory within the framework of Becchi-Rouet-Stora-Tyutin (BRST) formalism. Furthermore, exploiting the Hodge decomposition theorem and choosing the physical state to be the harmonic state, we have shown that the 2D (non-)Abelian 1-form gauge theories provide a new model for the topological field theory (TFT) [6,11] which captures a part of the key aspects of Witten-type TFTs [12] and some salient features of Schwarz-type TFTs [13]. None of the above theories [6–11] are, however, supersymmetric in nature. The central goal of our present endeavour is to first study the general \( \mathcal{N} = 2 \) SUSY quantum mechanical system and gain deep insights into its mathematical and physical structures and comment on the generalization of these ideas, if possible, to the study of \( \mathcal{N} = 2 \) SUSY gauge theories of phenomenological interest.

The following factors have propelled us to pursue our present investigation. First and foremost, it is of utmost importance for us to provide a concrete proof of our earlier conjecture that any arbitrary \( \mathcal{N} = 2 \) SUSY quantum mechanical model would provide a tractable physical example of a Hodge theory. Second, it is always an important endeavour to state a general rule for the solution of a given problem. In our present work, we have provided a general answer to a general question of physical importance. Finally, we are very hopeful that we shall be able to apply our current ideas to \( \mathcal{N} = 2 \) SUSY gauge theories which might turn out to be the field theoretic model for a (quasi-)topological field theory as well as an example of a Hodge theory.

In the broader perspective, our present study is essential besides the above cited factors of motivations. It is quite possible that our present study would enable us to count the correct degrees of freedom associated with the fields of a \( \mathcal{N} = 2 \) SUSY quantum gauge theory in a given dimension of spacetime. To achieve the above goal, we shall have to exploit the Hodge decomposition theorem in the quantum Hilbert space of this specific SUSY gauge theory under consideration. Since the most symmetric state would turn out to be the harmonic state, we shall be forced to choose it as the physical state of the theory. The annihilation of this state by the supercharges would lead to the correct counting of the degrees of freedom. It is gratifying to state that we have already performed such kind of analysis in our earlier works on the usual 2D free (non-)Abelian gauge theories [6–11]. We are very hopeful that our ideas of the earlier works [6–11] would persist even in the case of \( \mathcal{N} = 2 \) SUSY quantum gauge theories in a specific dimension of spacetime where we shall be able to obtain the perfect SUSY quantum gauge models for the Hodge theory as well as, possibly, a set of SUSY examples of TFTs.

The contents of our present paper are organized as follows. In Sect. 2, we concisely recapitulate the bare essentials of superspace approach to \( \mathcal{N} = 2 \) SUSY QM and derive the Lagrangian and its associated continuous SUSY transformations. Our Sect. 3 deals with the derivation of conserved charges from the Noether theorem. We discuss various discrete symmetry transformations of the Lagrangian of our theory in Sect. 4. We devote time on the algebraic structures of various symmetry transformations in Sect. 5 and show their relevance to the cohomological aspects of differential geometry. We establish mapping between the conserved charges and cohomological operators in our Sect. 6. Finally, we make a few concluding remarks in Sect. 7.

In our Appendix, we provide the proof for the specific \( \mathcal{N} = 2 \) algebra amongst the supercharges and Hamiltonian of our present SUSY theory in a simpler form.

### 2 Preliminaries: superspace approach to the description of \( \mathcal{N} = 2 \) SUSY quantum mechanical model

We begin with a \( \mathcal{N} = 2 \) supervariable \( X(t, \theta, \bar{\theta}) \) on a (1,2)-dimensional supermanifold which is characterized by the superspace coordinates \( Z^M = (t, \theta, \bar{\theta}) \) where \( t \) is the bosonic \( (t^2 \neq 0) \) variable and \( (\theta, \bar{\theta}) \) are a pair of Grassmann variables (with \( \theta^2 = \bar{\theta}^2 = 0, \theta \bar{\theta} + \bar{\theta} \theta = 0 \)). We note that, later on, the ordinary variable \( t \) would turn out to be the evolution parameter for the SUSY quantum mechanical system. The above supervariable can be expanded, along the Grassmannian directions, as \([14,15]\)

\[
X(t, \theta, \bar{\theta}) = x(t) + i \theta \psi(t) + i \bar{\theta} \bar{\psi}(t) + \theta \bar{\theta} A(t),
\]

where \(x(t), \psi(t), \bar{\psi}(t)\) are the basic dynamical variables and \(A(t)\) is an auxiliary variable for our SUSY quantum mechanical system. We note that the variables \(x(t), A(t)\) form a bosonic pair and \(\psi(t), \bar{\psi}(t)\) (with \(\psi^2 = \bar{\psi}^2 = 0, \psi \bar{\psi} + \bar{\psi} \psi = 0\)) are their SUSY counterparts (i.e. fermionic pair of variables). All these variables are function of the evolution parameter \(t\).

The two supercharges \(Q\) and \(\bar{Q}\) (with \(Q^2 = 0, \bar{Q}^2 = 0\)) for the above \( \mathcal{N} = 2 \) SUSY general quantum mechanical theory, are defined as \([14,15]\)

\[
Q = \frac{\partial}{\partial \theta} + i \theta \frac{\partial}{\partial t}, \quad \bar{Q} = \frac{\partial}{\partial \bar{\theta}} + i \bar{\theta} \frac{\partial}{\partial t},
\]

where \(\partial_M = \partial/\partial Z^M = (\partial/\partial t, \partial/\partial \theta, \partial/\partial \bar{\theta})\) are the partial derivatives defined on the (1,2)-dimensional supermanifold. The latter turn out to be the generators for the translations along \((t, \theta, \bar{\theta})\) directions as illustrated below:

\[
t \rightarrow t' = t + \epsilon \bar{\theta} + \bar{\epsilon} \theta, \quad \theta \rightarrow \theta' = \theta + \epsilon, \quad \bar{\theta} \rightarrow \bar{\theta}' = \bar{\theta} + \bar{\epsilon},
\]

(3)
where ε and ℓ are the infinitesimal shift transformation parameters along the Grassmannian directions of the (1,2)-dimensional supermanifold. Thus, they are fermionic (ε² = 0, ℓ² = 0, ε ℓ + ℓ ε = 0) in nature.

The SUSY transformation (δ) on the supervariable can be expressed in terms of the supercharges Q and Q as illustrated above. The transformation (δ) can be divided into two infinitesimal transformations δ₁ and δ₂ because of the presence of N = 2 SUSYQM. These are juxtaposed as:

\[ \dot{δ}_1 x = i d \psi, \quad \ddot{δ}_1 \psi = -ε (x + i A), \quad \dot{δ}_1 A = -ε \psi, \quad \ddot{δ}_1 = 0, \]
\[ \dot{δ}_2 x = i ε \bar{ψ}, \quad \ddot{δ}_2 \psi = -ε (x - i A), \quad \dot{δ}_2 A = ε \bar{ψ}, \quad \ddot{δ}_2 = 0, \]

where we have defined \( x = dx/dt \), \( \psi = d\psi/dt \), \( \bar{ψ} = d\bar{ψ}/dt \). It can be readily checked that δ₁ and δ₂ are off-shell nilpotent of order two (i.e. \( \ddot{δ}_1^2 = \ddot{δ}_2^2 = 0 \)).

The general Lagrangian for the N = 2 SUSY quantum mechanical model can be written, in terms of D, \( \bar{D} \), and W, as (see, e.g. [13, 14] for details)

\[ \mathcal{L} = \int d\theta d\bar{\theta} \left[ \frac{1}{2} \bar{D} X(t, \theta, \bar{\theta}) D X(t, \theta, \bar{\theta}) - W(X(t, \theta, \bar{\theta})) \right], \]

where D and \( \bar{D} \) are the following supercovariant derivatives

\[ D = \frac{\partial}{\partial \theta} - i \theta \frac{\partial}{\partial t}, \quad \bar{D} = \frac{\partial}{\partial \bar{\theta}} - i \bar{\theta} \frac{\partial}{\partial t}, \]

and W(X) is the superpotential which is an arbitrary function of the supervariable X(t, θ, \( \bar{\theta} \)). One can expand the above superpotential W(X(t, θ, \( \bar{\theta} \)), by Taylor expansion, around the ordinary space variable x, as follows:

\[ W(X(t, \theta, \bar{\theta})) = W(x + i \theta \bar{ψ} + i \bar{ψ} \psi + \theta \bar{δ} A) = W(x) + (i \theta \bar{ψ} + i \bar{ψ} \psi + \theta \bar{δ} A) W'(x) + \frac{1}{2!} (i \theta \bar{ψ} + i \bar{ψ} \psi + \theta \bar{δ} A)^2 W''(x). \]

We note that there will be no further higher order terms in the above expansion. Finally, performing the proper Grassmannian integration in (6), we obtain the following physical Lagrangian for the N = 2 SUSY model:

\[ L = \frac{1}{2} \bar{ψ} \bar{ψ}(t) + \frac{1}{2} \bar{ψ} \bar{ψ}(t) - \bar{ψ} \bar{ψ}(t) + A(t) W'(x) + \frac{1}{2} A^2(t) + \frac{1}{2} \bar{ψ} \bar{ψ}(t) - \bar{ψ} \bar{ψ}(t) W''(x), \]

where W(x) is an arbitrary superpotential and its first- and second-order derivatives are: W'(x) = \( \frac{d}{dx} W(x) \), W''(x) = \( \frac{d^2}{dx^2} W(x) \). The above Lagrangian can be simplified, modulo a total derivative, as:

\[ L = \frac{1}{2} \bar{ψ} \bar{ψ}(t) + W'(x) A(t) + \frac{1}{2} A^2(t) \]
\[ + W''(x) \bar{ψ}(t), \]

because of the fact that \( \bar{ψ} \bar{ψ} + \bar{ψ} \bar{ψ} = 0 \). We shall focus on the above Lagrangian (10) for our further discussions (in the rest of our discussions).

3 Continuous symmetries: conserved charges

The continuous SUSY transformations δ₁ and δ₂ (cf. (5)) can be re-expressed in terms of the fermionic (s₁² = 0, s₂² = 0) symmetry transformations if we identify \( \ddot{δ}_1 = \bar{ε} s_1 \) and \( \ddot{δ}_2 = ε s_2 \). In explicit form, these transformations are

\[ s_1 x = i \bar{ψ}, \quad s_2 x = i ψ, \quad s_1 \psi = -(\dot{x} + i A), \quad s_2 ψ = -(\dot{x} - i A), \quad s_1 A = -\bar{ψ}, \quad s_2 A = ψ, \quad s_1 \bar{ψ} = 0, \quad s_2 ψ = 0. \]

We note that s₁ and s₂ are off-shell nilpotent of order two. In other words, we observe the validity of s₁² = 0, s₂² = 0 in their operator form where any equation of motion, emerging from (10), is not used. The Lagrangian (10) transforms, under infinitesimal continuous transformations s₁ and s₂, as

\[ s_1 L = \frac{d}{dt} [-W'(x)], \quad s_2 L = \frac{d}{dt} [i \bar{ψ} (\dot{x} - i A) + \bar{ψ} W']. \]

This observation establishes that the action integral \( S = \int L dt \) remains invariant under the continuous SUSY transformations s₁ and s₂.

One of the key ingredients of a N = 2 SUSY theory is the fact that two successive SUSY transformations must generate the spacetime translation in a given dimension of spacetime. In our one-dimensional case, we have the following relationship for the generic variable Φ, namely:

\[ s_1 Φ \equiv \{ s_1, s_2 \} Φ = -2 i Φ, \]
\[ Φ = x, \bar{ψ}, ψ, A, \bar{ψ}, ψ, W, W', W''. \]

Thus, modulo \(-2i\) factor, we have the time translation for a variable if we apply the two successive SUSY continuous symmetry transformations. In other words, we have the following symmetry transformation:

\[ s_1 s_2 s_1 L = (s_1 s_2 + s_2 s_1) L \equiv \frac{d}{dt} L. \]

It is obvious, at this juncture, that we have three standard continuous symmetry transformations in the theory. Two of them are fermionic (s₁² = 0, s₂² = 0) and one of them (sₜ) is bosonic. The latter is obtained from the anticommutator between the above fermionic symmetry transformations (i.e. sₜ = \{ s₁, s₂ \}).
By exploiting the standard techniques of Noether theorem, one can compute the conserved charges, corresponding to the above continuous symmetry transformations. The expressions, for the complete set of these charges, are

\[ Q = (i\dot{x} - A)\psi, \quad \bar{Q} = \bar{\psi}(i\dot{x} + A), \]

\[ Q_w = \frac{1}{2}p^2 - \frac{1}{2}A^2 - AW' - W''\bar{\psi}\psi \]

\[ \equiv \frac{1}{2}p^2 - \frac{1}{2}A^2 - AW' - W''\psi = H, \quad (15) \]

where \( H \) is the Hamiltonian of the theory and \( p = \dot{x} \) is the momentum w.r.t. \( x \). The conservation of the above charges (i.e. \( \dot{Q} = 0, \ddot{Q} = 0, \dot{Q}_w = 0 \)) can be proven by exploiting the following Euler-Lagrange equations of motion:

\[ \ddot{x} - AW'' - W'''\bar{\psi}\psi = 0, \quad A = -W', \]

\[ \psi - iW''\bar{\psi}\psi = 0 \Rightarrow \ddot{\psi} + iW''\dot{\psi}\bar{\psi} + (W'')^2\bar{\psi}\psi = 0, \]

\[ \ddot{\psi} + iW''\dot{\psi} = 0 \Rightarrow \ddot{\psi} - iW''\dot{\psi} + (W'')^2\bar{\psi}\psi = 0, \quad (16) \]

which emerge from the Lagrangian (10) of the theory. The other way of proving the conservation law is to exploit the canonical (anti)commutators from the Lagrangian (10) and check that the commutators \([H, Q] = 0, [H, \bar{Q}] = 0, [H, \dot{Q}] = 0\) are trivially satisfied. From the Heisenberg’s equation of motion, this will, ultimately, imply the conservation law \((\dot{Q} = \ddot{Q} = H = 0)\).

4 Discrete symmetries: duality transformations for the \( \mathcal{N} = 2 \) SUSY quantum mechanical model

We focus here on a set of discrete symmetries of the Lagrangian (10) for our present general \( \mathcal{N} = 2 \) SUSY model of QM. We observe that under the following discrete transformations:

\[ x \to -x, \quad t \to +t, \quad \psi(t) \to \pm i\bar{\psi}(t), \]

\[ \frac{\partial}{\partial t} \psi(t) \to \mp i\bar{\psi}(t), \quad W''(t) \to +W'(t), \quad W''(x) \to -W''(x), \quad A(t) \to +A(t), \quad (17) \]

the Lagrangian (10) remains invariant. It is to be noted that, in the above, we have primarily \( \pi \) discrete symmetry transformations for the Lagrangian (10) where \( \psi \to \pm i\bar{\psi} \) means \( \psi(t) \to \bar{\psi}(t) = \pm i\bar{\psi}(t) \) and \( W'(x) \to +W'(x) \) explicitly implies \( W'(x) \to W'(-x) = +W'(x) \). As a consequence, the first derivative on the potential function \( W'(x) \) is even under the parity transformation (i.e. \( \dot{P}W'(-x) = W'(x) \)). We note further that there is a parity symmetry in the theory but there is no non-trivial time-reversal symmetry (as \( t \to -t \)). It is to be emphasized that here the prime on \( \psi(t) \) does not mean the space derivative. Rather, the prime here corresponds to an internal discrete transformation like: \( \psi(t) \to \bar{\psi}(t) = e^{\pm i\pi/2} \bar{\psi}(t) \). We repeat that there are two hidden symmetries \((\psi \to \pm i\bar{\psi})\) in the above discrete symmetry transformations.

One can invoke the above cited time-reversal symmetry in the theory by checking that the following discrete transformations:

\[ x \to -x, \quad t \to -t, \quad \psi(t) \to \pm i\bar{\psi}(t), \]

\[ \bar{\psi}(t) \to \mp i\bar{\psi}(t), \quad W'(x) \to -W'(x), \]

\[ W''(x) \to +W''(x), \quad A(t) \to -A(t), \quad (18) \]

also leave the Lagrangian invariant (i.e. \( L \to \bar{L} \)). We explicitly mean, by the above transformations, the following

\[ \tilde{P} : x \to -x, \quad \tilde{P}W'(x) \equiv W'(-x) = -W'(x), \]

\[ \tilde{i} : t \to -t, \quad \tilde{i}\psi(t) \equiv \psi(-t) = \pm i\bar{\psi}(t). \quad (19) \]

Similarly, the other transformations can be interpreted. Thus, we figure out that there are parity and time-reversal symmetries together in (18).

There is yet another discrete symmetry in the theory which, as we shall see later on, plays a crucial role in our further discussions. The following discrete symmetry transformations:

\[ x \to -x, \quad t \to -t, \quad \psi(t) \to + \pm \bar{\psi}(t), \]

\[ \bar{\psi}(t) \to \mp \psi(t), \quad W'(x) \to -W'(x), \]

\[ W''(x) \to +W''(x), \quad A(t) \to -A(t), \quad (20) \]

leave the Lagrangian of our theory, yet again, invariant (i.e. \( L \to \bar{L} \)). Thus, we lay emphasis on the fact that we have parity as well as time-reversal symmetry in the theory. The key difference between (18) and (20) is associated with the transformations of the fermionic variables \( \psi \) and \( \bar{\psi} \) under the time-reversal symmetry transformations \( t \to -t \). Whereas in (18), there is an \( i \) factor in the transformations of the above variables, there is no \( i \) factor in the latter transformations. At the level of signatures, too, there is a difference between the two transformations if we take a close look at them.

We wrap up this section with the remark that we have listed only three types of discrete symmetries in the above. However, in principle, there might be existence of more such symmetries. We shall see later, in our forthcoming sections, that the discrete symmetry transformation (20) would play a very crucial role in our present endeavour of establishing a connection between the symmetries and the cohomological operators and it would also allow only SUSY potentials that are even under parity (i.e. \( W(-x) = W(x) \)). The latter correspond to the square integrable eigenfunctions for SUSY QM (2, 3). We further note that, in all our three discrete symmetries (cf. (17), (18), (20)), the fermionic variable \( \psi(t) \) transforms to \( \bar{\psi}(t) \) and vice-versa. Thus, at present stage, these variables are ‘dual’ to each-other. We shall see the consequences of this observation, later on, when we shall focus more on the presence of the duality transformations.
5 Algebraic structures: cohomological aspects

The continuous symmetry transformations $s_1$, $s_2$, and $s_w$ (cf. (11), (13)), in their operator form, obey the following algebraic structure:

\[ s_1^2 = 0, \quad s_2^2 = 0, \quad s_w = \{s_1, s_2\} = s_1 s_2 + s_2 s_1, \]
\[ [s_1, s_2] = 0, \quad [s_w, s_1] = 0, \quad [s_w, s_2] = 0, \quad s_w = (s_1 + s_2)^2. \]

(21)

This algebraic structure is exactly same as the algebra of de Rham cohomological operators $(d, \delta, \Delta)$ of differential geometry where $(\delta)d$ are the (co-)exterior derivatives and $\Delta$ is the Laplacian operator. In explicit form, the algebra, satisfied by these cohomological operators, are

\[ d^2 = 0, \quad \delta^2 = 0, \quad \Delta = \{d, \delta\} = d \delta + \delta d, \]
\[ [\Delta, d] = 0, \quad [\Delta, \delta] = 0, \quad \Delta = (d + \delta)^2. \]

(22)

A close look at (21) and (22) tempts us to identify $(d, \delta, \Delta)$ with the set of continuous symmetries $(s_1, s_2, s_w)$. However, there are other decisive properties that are also associated with $(d, \delta, \Delta)$. These properties have to be captured in the language of symmetry transformations of the Lagrangian if we wish to establish a perfect analogy between the cohomological operators and symmetries. For instance, first of all, we know that the nilpotent $(\delta^2 = d^2 = 0)$ (co-)exterior derivatives $(\delta)d$ are connected with each-other by the Hodge duality $(\ast)$ operation defined on a given manifold. This important relationship is mathematically expressed as follows:

\[ \delta = \pm \ast d \ast, \quad \delta^2 = 0, \quad d^2 = 0, \]

(23)

where $(\pm)$ signs are determined by the inner product of $p$-forms in a given dimension of the manifold without a boundary. For an even-dimensional manifold $\delta = - \ast d \ast$ and, for the odd-dimensional manifold, the signatures are decided by the degree of the forms, involved in the inner product, on that particular odd-dimensional manifold (see, e.g. [16–18] for more details).

Within the realm of theoretical physics, the relationships (23) are expressed in the language of symmetry properties [19]. It can be seen that the interplay of the continuous and discrete symmetry transformations (cf. Sects. 3 and 4) leads to a relationship that is exactly similar in form as (23). For instance, we have the validity of the following relationship for a generic variable $\Phi$ of the Lagrangian (10), namely;

\[ s_1 \Phi = \pm s_2 \ast \Phi, \quad s_1^2 = s_2^2 = 0, \]
\[ \Phi = x, \psi, \bar{\psi}, A, W', W''. \]

(24)

where $\ast$ is the discrete symmetry transformations of Sect. 4. The $(\pm)$ signs in (23) are dictated by two successive operations of the discrete symmetry transformations on the generic variable $\Phi$ (see, e.g. [19] for details):

\[ \ast (\ast \Phi) = \pm \Phi, \quad \Phi = x, \psi, \bar{\psi}, A, W', W''. \]

(25)

Furthermore, for our present model, there exists an inverse relationship (i.e. $s_2 \Phi = \mp \ast s_1 \ast \Phi$) corresponding to the relationship given in (24).

In view of the sacrosanct statements, made above about the duality-invariant theory, let us study the sanctity of the discrete symmetries (17), (18) and (20). In the case of (17), it can be checked that $\ast (\ast \Phi) = \pm \Phi$ for $\Phi = x, \psi, \bar{\psi}, A, W', W''$. As it turns out, it can be explicitly verified that the relationships $s_1 \Phi = + \ast s_2 \ast \Phi$ (and/or $s_2 \Phi = - \ast s_1 \ast \Phi$) are not satisfied. Physically, too, it is not allowed (see, e.g. [2–3]) because it corresponds to the superpotentials that are odd under parity (i.e. $W(-x) = -W(x)$). Similarly, in the case of (18), we check that $\ast (\ast \Phi_1) = + \Phi_1$ where $\Phi_1 = x, A, W', W''$ and $\ast (\ast \Phi_2) = - \Phi_2$ for $\Phi_2 = \psi, \bar{\psi}$. Here we have taken the generic variable $\Phi = (\Phi_1, \Phi_2)$. It is interesting to point out that the relations $s_1 \Phi_1 = + \ast s_2 \ast \Phi_1$ (or its inverse $s_2 \Phi_1 = - \ast s_1 \ast \Phi_1$) are not satisfied. In addition, the relations $s_1 \Phi_2 = + \ast \ast \Phi_2$ (or its inverse $s_2 \Phi_2 = + \ast \ast \Phi_2$) are also not satisfied for the transformations (18). Thus, we shall discard both these sets of discrete symmetry transformations as they do not obey the sanctity of the strictures laid down by the rules of a duality invariant physical theory (see, e.g. [19] for details).

Now let us concentrate on the discrete symmetry transformations (20) with the upper signature (i.e. $x \rightarrow -x$, $t \rightarrow -t$, $\psi \rightarrow +\bar{\psi}$, $\bar{\psi} \rightarrow -\psi$, $A \rightarrow -A$, $W' \rightarrow -W'$, $W'' \rightarrow +W''$). It can be checked explicitly that

\[ \ast (\ast x) = +x, \quad \ast (\ast \psi) = -\psi, \quad \ast (\ast \bar{\psi}) = +\bar{\psi}, \quad \ast (\ast A) = +A, \quad \ast (\ast W') = -W', \quad \ast (\ast W'') = +W''. \]

(26)

With these inputs, we readily verify (with $\Phi = (\Phi_1, \Phi_2)$)

\[ s_1 \Phi_1 = + \ast s_2 \ast \Phi_1, \quad s_2 \Phi_1 = - \ast s_1 \ast \Phi_1, \]
\[ s_1 \Phi_2 = + \ast s_2 \ast \Phi_2, \quad s_2 \Phi_2 = + \ast s_1 \ast \Phi_2, \]
\[ \Phi_1 = x, A, W', W'', \quad \Phi_2 = \psi, \bar{\psi}. \]

(27)

Thus, we conclude that the discrete transformations (20), with the upper signature, are physically well-defined transformations that correspond to the Hodge duality $(\ast)$ operation of differential geometry as the relations (27) provide a physical realization of the relationship $\delta = \pm \ast d \ast$ between the (co-)exterior derivatives (of differential geometry) in terms of symmetries.

We close this section with the remark that the lower signature of the discrete symmetry transformations (20) does not lead to the correct relationships $s_1 \Phi_1 = + \ast s_2 \ast \Phi_1$ as well as $s_1 \Phi_2 = + \ast s_2 \ast \Phi_2$ (and their corresponding reverse relations). Thus, a unique set of discrete symmetry transformations for our duality invariant physical theory, namely;

\[ x \rightarrow -x, \quad t \rightarrow -t, \quad \psi \rightarrow +\bar{\psi}, \quad \bar{\psi} \rightarrow -\psi, \quad A \rightarrow A, \quad W' \rightarrow -W', \quad W'' \rightarrow +W'', \]

(28)

is the one, we shall be concentrating on, for the rest of our discussions. First of all, we observe that under the above discrete symmetry transformations:

\[ \ast \hat{Q} = + \hat{Q}, \quad \ast \hat{Q} = - \hat{Q}, \quad \ast H = +H, \quad \ast L = +L, \]

(29)
where $*$ is nothing but the transformations (28). We point out that the transformations: $Q \rightarrow + \bar{Q}$, $\bar{Q} \rightarrow - \bar{Q}$ are analogous to the electromagnetic duality transformations for the source-free Maxwell equations where $E \rightarrow + B$ and $B \rightarrow - E$. This is the reason that, at the end of Sect. 4, we have claimed that the variables $\psi$ and $\bar{\psi}$ are “dual” to each-other because these variables distinguish the two SUSY nilpotent and conserved charges ($Q, \bar{Q}$). The second point to be noted is the fact that the transformations (28) allow superpotentials that are even under parity (i.e. $W(-x) = W(x)$) and these are the ones that are physically interesting (see, e.g. [2, 3]).

These relations would play very important roles in our further discussions as would become clear when we shall discuss about the ladder operators in the language of the conserved charges.

Let us now define an eigenstate $|\xi\rangle_p$ w.r.t. the operator $(\bar{Q}Q/H)$ (i.e. $(\bar{Q}Q/H)|\xi\rangle_p = p|\xi\rangle_p$) where $p$ is the eigenvalue. By exploiting the algebra (31), it is straightforward to check that

$$
\begin{align*}
&\left(\frac{\bar{Q}Q}{H}\right)Q|\xi\rangle_p = (p + 1)Q|\xi\rangle_p, \\
&\left(\frac{\bar{Q}Q}{H}\right)\bar{Q}|\xi\rangle_p = (p - 1)\bar{Q}|\xi\rangle_p, \\
&\left(\frac{\bar{Q}Q}{H}\right)H|\xi\rangle_p = pQ|\xi\rangle_p.
\end{align*}
$$

(32)

The above equation establishes the fact that the states $Q|\xi\rangle_p$, $\bar{Q}|\xi\rangle_p$ and $H|\xi\rangle_p$ have the eigenvalues $(p + 1)$, $(p - 1)$, $p$ respectively. Thus, if we identify the eigenvalue $p$ with the degree of a form, then, it is clear that the following mapping between the conserved charges and cohomological operators of differential geometry emerge, namely:

$$(Q, \bar{Q}, H) \longleftrightarrow (d, \delta, \Delta).$$

(33)

We point out that the property of (lowering)raising of a given differential form by the operations of the (co-)exterior derivatives is also captured in the language of the conserved charges when we identify the eigenvalue of the operator $(\bar{Q}Q/H)$ with the degree of a given differential form.

There is yet another representation of $(d, \delta, \Delta)$ in the language of the eigenvalues of a set of conserved charges $(Q, \bar{Q}, H)$. To this end in mind, we define an eigenstate $|\chi\rangle_q$ with the eigenvalue $q$, as:

$$
\left(\frac{\bar{Q}Q}{H}\right)|\chi\rangle_q = q|\chi\rangle_q.
$$

(34)

Exploiting the structure of (31), it is straightforward to verify the following consequences that ensue due to (34), namely;

$$
\begin{align*}
&\left(\frac{\bar{Q}Q}{H}\right)Q|\chi\rangle_q = (q - 1)Q|\chi\rangle_q, \\
&\left(\frac{\bar{Q}Q}{H}\right)\bar{Q}|\chi\rangle_q = (q + 1)\bar{Q}|\chi\rangle_q, \\
&\left(\frac{\bar{Q}Q}{H}\right)H|\chi\rangle_q = qQ|\chi\rangle_q.
\end{align*}
$$

(35)

which establishes that the states $\bar{Q}|\chi\rangle_q$, $Q|\chi\rangle_q$ and $H|\chi\rangle_q$ have the eigenvalues $(q + 1)$, $(q - 1)$, $q$ respectively. Thus, as far as the lowering and raising property of $(d, \delta, \Delta)$ is concerned, we have the following mapping between the conserved charges $(Q, \bar{Q}, H)$ and the cohomological operators:

$$(\bar{Q}, Q, H) \longleftrightarrow (d, \delta, \Delta).$$

(36)

Thus, ultimately, we conclude that if the degree of a given form is identified with the eigenvalue of a given state (in

6 Conserved charges and cohomological operators: mappings

There is still one issue which has not yet been settled as far as the perfect analogy between the de Rham cohomological operators $(d, \delta, \Delta)$ and the conserved charges $(Q, \bar{Q}, H)$ is concerned. We know that the operation of $d$ on a differential form $(f_n)$ of degree $n$ raises it to a form $(f_{n+1})$ of degree $(n+1)$ (i.e. $df_n \sim f_{n+1}$) where $n = 0, 1, 2, \ldots$. On the contrary, the action of the co-exterior derivative on a form $(\omega_n)$ of degree $n$ lowers the degree of the form by one (i.e. $\delta f_n \sim f_{n-1}$) where $n = 1, 2, 3, \ldots$. Due to $\Delta = (d + \delta)^2 \equiv d\delta + \delta d$, it is clear that the Laplacian operator $\Delta$ does not affect the degree of a form on which it operates (i.e. $\Delta f_n \sim f_n$) (see, e.g. [16–18]).

To capture the above properties, in the language of symmetry generators, we note the following straightforward algebra:

$$
\begin{align*}
[Q \bar{Q}, Q] &= + QH, & [\bar{Q} Q, \bar{Q}] &= - H \bar{Q}, \\
[\bar{Q} Q, Q] &= - H Q, & [\bar{Q} Q, \bar{Q}] &= + \bar{Q} H.
\end{align*}
$$

(30)

It is to be emphasized that we have normalized the expressions for the supercharges $Q, \bar{Q}$ with some constant factors so that the basic SUSY algebra: $Q^2 = \bar{Q}^2 = 0, [Q, \bar{Q}] = H, [H, \bar{Q}] = [H, Q] = 0$ is satisfied. We point out the fact that $\bar{Q} = - i[Q, H] = 0$ and $\bar{Q} = - i[\bar{Q}, H] = 0$ imply that $HQ = QH, \bar{Q}H = \bar{Q}H$. The latter relations lead to: $H^{-1}Q = QH^{-1}, H^{-1}\bar{Q} = \bar{Q}H^{-1}$ if the inverse of the Hamiltonian exists. Since we are focusing on the non-singular Hamiltonian (in the matrix form), we presume that the Casimir operator $H$ has its well-defined inverse. The latter, too, would be the Casimir operator for the whole algebra (i.e. $(H^{-1}, \bar{Q}, Q)$).

As a consequence, we have an algebraically suitable form of (30), as follows

$$
\begin{align*}
&\left[\frac{\bar{Q} Q}{H}, Q\right] = + Q, & \left[\frac{\bar{Q} Q}{H}, \bar{Q}\right] = - \bar{Q}, \\
&\left[\frac{\bar{Q} Q}{H}, Q\right] = - Q, & \left[\frac{\bar{Q} Q}{H}, \bar{Q}\right] = + \bar{Q}.
\end{align*}
$$

(31)

The details of these aspects of the algebraic structures could be found in our Appendix A where the charges have been redefined suitably.
the quantum Hilbert space) corresponding to the operator \(\hat{Q}Q/H\), then, the consequences ensuing from the operation of the set of charges \(\{\hat{Q},Q,H\}\) is identical to the operation of the set \(\{d,\delta,\Delta\}\) on the degree of a given form. This is why the mapping (36) is correct. We close this section with the final remark that we have provided two physical realizations of the cohomological operators in terms of the conserved charges of the general \(N=2\) SUSY quantum mechanical model at the algebraic level.

7 Conclusions

In our present investigation, we have given a concrete proof of our earlier conjecture that any arbitrary \(N=2\) SUSY quantum mechanical model would provide a cute physical example of the Hodge theory. We have derived the general Lagrangian for the \(N=2\) SUSY quantum mechanical theory by exploiting the basic tenets of (i) the \(N=2\) SUSY supervariable and its expansion along the Grassmannian directions of a \((1,2)\)-dimensional supermanifold (ii) the idea of supercovariant derivatives, and (iii) the Taylor expansion of an arbitrary superpotential (cf. (8)). We have also demonstrated that the discrete as well as continuous symmetry properties of this Lagrangian (and their corresponding generators) provide the physical realizations of the cohomological operators. As a consequence, the general \(N=2\) SUSY model is a physical example of the Hodge theory because all the cohomological operators find their physical realizations in the language of the interplay between the underlying discrete as well as continuous symmetry transformations of our present theory.

In addition to the well-known continuous symmetry transformations, generated by the conserved charges \(\{Q,Q,\hat{Q},H\}\), we have discussed various kinds of discrete symmetries in our present endeavour (cf. (17), (18), (20)). Out of these three discrete symmetry transformations, only one (i.e. (20)) is the perfect symmetry for a duality-invariant theory. We have shown that, under the perfect discrete symmetry transformations (20), the supercharges \(Q\) and \(\hat{Q}\) transform in exactly the same manner as the duality transformations for the electric and magnetic fields of a source-free Maxwell equations. Further, the above unique discrete symmetry transformation turns out to provide a physical realization of the Hodge duality operation of differential geometry. Thus, for our present general \(N=2\) SUSY model, we have also been able to provide the physical realizations of the relationships (i.e. \(\delta = \pm * d *\)) between the nilpotent \((d^2 = \delta^2 = 0)\) of order two (co-)exterior derivatives \((\delta(d))\) of the differential geometry in the language of symmetries.

\[\delta = \pm * d *\]

As pointed out, in the concluding remark at the fag end of Sect. 4, there might exist many discrete symmetries in the theory, under which, the Lagrangian (10) would remain invariant. The decisive feature of a physically relevant discrete symmetry would always be, however, the validity of the relations \(s_1 \Phi = \pm * s_2 \Phi, s_2 \Phi = \mp * s_1 \Phi\) for a generic variable \(\Phi\) where \((\pm)\) signs would be dictated by the signatures of \(* (\Phi)\). In this context, we note that, under the following two sets of discrete symmetry transformations

\[x \to -x, \quad t \to +t, \quad \psi \to +\psi, \quad \bar{\psi} \to +\bar{\psi},\]
\[A(t) \to A(t), \quad W'(x) \to -W'(x), \quad \bar{W}''(x) \to W''(x),\]

the Lagrangian (10) remains invariant (i.e. \(L \to L\)). However, these do not satisfy the sacrosanct relations \(s_1 \Phi = \pm * s_2 \Phi, s_2 \Phi = \mp * s_1 \Phi\). In addition, the above discrete symmetry transformations imply that the superpotentials must be odd under parity (i.e. \(W(-x) = -W(x)\)). However, such kind of potentials are not allowed by the SUSY QM as they do not always lead to the square integrable eigenfunctions. Hence, the above sets of discrete symmetry transformations are not physically interesting at all to us. Furthermore, the correct transformations of \(Q\) and \(\hat{Q}\), under the discrete symmetry transformations, also shed light on the correctness of a chosen set of discrete symmetry transformations for our present SUSY quantum mechanical theory.

It would be very nice endeavour to exploit our present observations in the context of the study of \(N=2\) SUSY (non-)Abelian gauge theories (in any arbitrary dimension of spacetime) where there is a possibility of the appearance of cohomological structure. Such theories might be shown to be the perfect models for the Hodge theory as well as new models for TFTs. The latter are expected to turn out to be different from the Witten-type TFTs [12] as well as Schwarz-type TFTs [13]. We have performed such kind of study in the case of usual 2D (non-)Abelian gauge theories (see, e.g. [6,11]) where the full strength of the symmetries of the Hodge theory has been exploited in its full blaze of glory. Our work can be possibly extended to contain exotic discrete symmetries that have been mentioned in [21,22]. This exercise will definitely enrich the mathematical structure of our present analysis where the existence of discrete symmetries plays an important role. Furthermore, we plan to explore the possibility of existence of the cohomological structure in the \(N=4\) SUSY quantum mechanical theories [23]. In fact, we expect many possibilities of the physical realizations of the cohomological operators in this case. Our present work can also be extended to higher dimensional (e.g. 2D and 3D) SUSY quantum mechanical models.

\[\delta = \pm * d *\]
following the work done in [24]. We are devoting time on the above cited problems and our results would be reported, later on, in our future publications [25].

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Appendix A: On the derivation of $\mathcal{N} = 2$ SUSY algebra

The whole of algebraic structures in Sect. 6 are based on the basic $\mathcal{N} = 2$ SUSY algebra $\mathcal{Q}^2 = \mathcal{Q}^2 = 0, \{\mathcal{Q}, \mathcal{Q}\} = H, [H, \mathcal{Q}] = [H, \bar{Q}] = 0$ which is satisfied on the on-shell. To corroborate this statement, first of all, auxiliary variable $A$ in the Lagrangian (10) is replaced by $(-W')$ due to the equation of motion $A = -W'$ (which emerges from (10) itself). Furthermore, the symmetry transformations $s_1$ and $s_2$ (cf. (11)) are modified a bit by the overall constant factors. Thus, we have the following different looking Lagrangian

$$L_0 = \frac{1}{2} \dot{x}^2 + i \bar{\psi}(\bar{\psi} - \frac{1}{2}(W')^2 + W'' \bar{\psi}) \psi,$$

which remains invariant under the following transformations

$$s_1x = -\frac{1}{\sqrt{2}} i \psi, \quad s_1\psi = \frac{1}{\sqrt{2}} (\dot{x} - i W'), \quad s_1\psi = 0,$$

$$s_2x = \frac{1}{\sqrt{2}} i \bar{\psi}, \quad s_2\psi = \frac{1}{\sqrt{2}} (\dot{x} + i W'), \quad s_2\bar{\psi} = 0.$$

The above transformations are nilpotent of order two (i.e. $s_1^2 = s_2^2 = 0$) only when the equations of motion $\dot{\psi} - i W'' \psi = 0$, $\dot{\bar{\psi}} + i W'' \bar{\psi} = 0$ are used. It can be checked that $s_1L_0 = d/dt \left( W' \psi / \sqrt{2} \right)$ and $s_2L_0 = d/dt \left( i \bar{\psi} \psi / \sqrt{2} \right)$. Hence, the action integral $\mathcal{S} = \int dt L_0$ remains invariant under $s_1$ and $s_2$.

The conserved Noether charges, that emerge corresponding to (A.2), are

$$\mathcal{Q} = -\frac{1}{\sqrt{2}} (i \bar{\psi} + W') \psi, \quad \bar{\mathcal{Q}} = \frac{1}{\sqrt{2}} \bar{\psi} (i \dot{x} - W').$$

These charges are same as quoted in (15) except the fact that $A$ has been replaced by $(-W')$ (due to the equation of motion from the Lagrangian (10)) and the constant factors $(\pm 1/\sqrt{2})$ have been included for the algebraic convenience. It can be readily checked that the above charges are the generator for the transformations (A.2) because we have the following relationships:

$$s_1 \Phi = \pm i \left[ \Phi, \mathcal{Q} \right]_{\pm}, \quad s_2 \Phi = \pm i \left[ \Phi, \bar{\mathcal{Q}} \right]_{\pm},$$

where the generic variable $\Phi$ corresponds to the variables $x, \psi, \bar{\psi}$ and the subscripts $(\pm)$ on square brackets stand for the (anti)commutator depending on the generic variable $\Phi$ being fermionic/bosonic in nature. The $(\pm)$ signs, in front of the brackets, are also chosen judiciously (see, e.g. [20] for details).

The structure of the specific $\mathcal{N} = 2$ SUSY algebra now follows when we exploit the basic relationship (A.4). In other words, we observe the following

$$s_1 \mathcal{Q} = i \left[ \mathcal{Q}, \mathcal{Q} \right] = 0 \Rightarrow \mathcal{Q}^2 = 0,$$

$$s_1 \bar{\mathcal{Q}} = i \left[ \bar{\mathcal{Q}}, \bar{\mathcal{Q}} \right] = 0 \Rightarrow \bar{\mathcal{Q}}^2 = 0,$$

$$s_1 \mathcal{Q} = i \left[ \mathcal{Q}, \bar{\mathcal{Q}} \right] = iH \Rightarrow \{ \mathcal{Q}, \bar{\mathcal{Q}} \} = H,$$

$$s_2 \mathcal{Q} = i \left[ \mathcal{Q}, \mathcal{Q} \right] = iH \Rightarrow \{ \mathcal{Q}, \mathcal{Q} \} = H,$$

where $H$ is the Hamiltonian (corresponding to the Lagrangian (A.1)). The explicit form of $H$ can be mathematically expressed as:

$$H = \frac{1}{2} \dot{x}^2 + \frac{1}{2} (W')^2 - W'' \bar{\psi} \psi \equiv \frac{1}{2} p^2 + \frac{1}{2} (W')^2 - W'' \bar{\psi} \psi,$$

where $p = \dot{x}$ is the momentum corresponding to the variable $x$. We also lay emphasis on the fact that we have exploited the inputs from equations of motion $\dot{\psi} - i W'' \psi = 0$, $\dot{\bar{\psi}} + i W'' \bar{\psi} = 0$ in the derivation of $H$ from the Legendre transformation $H = x \dot{p} + \bar{\psi} \Pi_\psi + \psi \Pi_\bar{\psi} - L$ where $\Pi_\psi = -i \dot{\psi}$ and $\Pi_\bar{\psi} = 0$. The derivation of specific $\mathcal{N} = 2$ SUSY algebra (cf. (A.5)) is very straightforward because we have used only (A.2) and (A.3) in the calculation of l.h.s. of (A.5) from which, the results of the r.h.s. (i.e. specific $\mathcal{N} = 2$ SUSY algebra) trivially ensue.

We wrap up this Appendix with the remarks that the specific $\mathcal{N} = 2$ SUSY algebra $\mathcal{Q}^2 = \bar{\mathcal{Q}}^2 = 0, \{ \mathcal{Q}, \bar{\mathcal{Q}} \} = H$, listed in (A.5), is valid only on the on-shell where the validity of the Euler-Lagrange equations of motion is taken into account. Furthermore, it may be trivially noted that, for the choices $W' = \alpha x$ and $W' = \alpha f(x)$ in the Lagrangian (A.1), we obtain the Lagrangians for the SUSY harmonic oscillator and its generalization in [3]. For the description of the motion of a charged particle in the $X - Y$ plane under the influence of a magnetic field along $Z$-direction, the choice for $W'$ could be found in the standard books on SUSY quantum mechanics and relevant literature (see, e.g. [2, 3]).

References

1. See, e.g., E. Witten, Nucl. Phys. B 188, 513 (1981)
2. See, e.g., F. Cooper, A. Khare, U. Sukhatme, Phys. Rep. 251, 264 (1995)
3. See, e.g., A. Das, Field Theory: A Path Integral Approach (World Scientific, Singapore, 1993)
4. R. Kumar, R. P. Malik, Euro. Phys. Lett. 98, 11002 (2012)
5. R. P. Malik, A. Khare, Ann. Phys. 334, 142 (2013)
6. R. P. Malik, Int. J. Mod. Phys. A 22, 3521 (2007)
7. R. P. Malik, Mod. Phys. Lett. A 15, 2079 (2000), ibid. A 16 477 (2001)
8. S. Gupta, R. P. Malik, Eur. Phys. J. C 58, 517 (2008)
9. R. Kumar, S. Krishna, A. Shukla, R. P. Malik, Eur. Phys. J. C 72, 2188 (2012)
10. R. Kumar, S. Krishna, A. Shukla, R. P. Malik, arXiv:1203.5519 [hep-th]
11. R. P. Malik, J. Phys. A: Math. Gen. 41, 4167 (2001)
12. E. Witten, Comm. Math. Phys. 17, 353 (1988)
13. A. S. Schwarz, Lett. Math. Phys. 2, 247 (1978)
14. See, e.g., F. Cooper, B. Freedman, Ann. Phys. 146, 262 (1983)
15. See, e.g., A. Lahiri, P. K. Roy, B. Bagchi, Int. J. Mod. Phys. A 5, 1383 (1990)
16. T. Eguchi, P. B. Gilkey, A. Hanson, Phys. Rep. 66, 213 (1980)
17. See, e.g., S. Mukhi, N. Mukunda, Introduction to Topology, Differential Geometry and Group Theory for Physicists (Wiley Eastern Private Limited, New Delhi, 1990).
18. K. Nishijima, Prog. Theor. Phys. 80, 897 (1988)
19. S. Deser, A. Gomberoff, M. Henneaux, C. Teitelboim, Phys. Lett. B 400, 80 (1997)
20. S. Gupta, R. Kumar, R. P. Malik, arXiv:0908.2561 [hep-th].
21. F. Correa, V. Jakubsky, L. Nieto, M. S. Plyushchay, Phys. Rev. Lett. 101, 030403 (2008)
22. F. Correa, V. Jakubsky, M. S. Plyushchay, J. Phys. A 41, 485303 (2008)
23. M. de Crombrugghe, V. Rittenberg, Ann. Phys. 151, 99 (1983)
24. A. Khare, J. Maharana, Nucl. Phys. B 244, 409 (1984)
25. R. P. Malik, et al., in preparation