ON THE EFFECT OF ROTATION ON THE LIFE-SPAN OF ANALYTIC SOLUTIONS TO THE 3D INVIScid PRIMITIVE EQUATIONS

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Abstract. We study the effect of the rotation on the life-span of solutions to the 3D hydrostatic Euler equations with rotation and the inviscid Primitive equations (PEs) on the torus. The space of analytic functions appears to be the natural space to study the initial value problem for the inviscid PEs with general initial data, as they have been recently shown to exhibit Kelvin-Helmholtz type instability. First, for a short interval of time that is independent of the rate of rotation $|\Omega|$, we establish the local well-posedness of the inviscid PEs in the space of analytic functions. In addition, thanks to a fine analysis of the barotropic and baroclinic modes decomposition, we establish two results about the long time existence of solutions. (i) Independently of $|\Omega|$, we show that the life-span of the solution tends to infinity as the analytic norm of the initial baroclinic mode goes to zero. Moreover, we show in this case that the solution of the 3D inviscid PEs converges to the solution of the limit system, which is governed by the 2D Euler equations. (ii) We show that the life-span of the solution can be prolonged unboundedly with $|\Omega| \to \infty$, which is the main result of this paper. This is established for “well-prepared” initial data, namely, when only the Sobolev norm (but not the analytic norm) of the baroclinic mode is small enough, depending on $|\Omega|$. Furthermore, for large $|\Omega|$ and “well-prepared” initial data, we show that the solution to the 3D inviscid PEs is approximated by the solution to a simple limit resonant system with the same initial data.

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1. Introduction

For large-scale oceanic and atmospheric dynamics, the vertical scale (a few kilometers for the ocean, 10-20 kilometers for the atmosphere) is much smaller than the horizontal scales (several thousands of kilometers). The following 3D viscous primitive equations (PEs) has been a standard framework for studying geostrophic adjustment of frontal anomalies in a rotating continuously stratified fluid of strictly rectilinear fronts and jets (see, e.g., [10, 30, 31, 35, 37, 47, 55, 57] and references therein):}

\[
\begin{align*}
\partial_t v + v \cdot \nabla v + w \partial_z v - \nu_h \Delta v - \nu_z \partial_{zz} v + \Omega v^\perp + \nabla p &= 0, \\
\partial_z p + T &= 0, \\
\partial_t T + v \cdot \nabla T + w \partial_z T - \kappa_h \Delta T - \kappa_z \partial_{zz} T &= 0, \\
\nabla \cdot v + \partial_z w &= 0,
\end{align*}
\] (1.1)
(1.2)
(1.3)
(1.4)

setting in the horizontal channel \{ \((x_1, x_2, z) : 0 \leq z \leq H, (x_1, x_2) \in T^2\) \}, subject to the following initial and boundary conditions:

\[
\begin{align*}
(v, T)|_{t=0} &= (v_0, T_0), \\
(v_z, w, T_z)|_{z=0, H} &= 0.
\end{align*}
\] (1.5)
(1.6)
Here the horizontal velocity field \( v = (v_1, v_2) \), the vertical velocity \( w \), the temperature \( T \), and the pressure \( p \) are the unknown quantities which are functions of the independent variables \( (x', z, t) = (x_1, x_2, z, t) \). The 2D horizontal gradient and Laplacian are denoted by \( \nabla = (\partial_{x_1}, \partial_{x_2}) \) and \( \Delta = \partial_{x_1x_1} + \partial_{x_2x_2} \), respectively. The non-negative constants \( \nu_h, \nu_v, \kappa_h \) and \( \kappa_v \) are the horizontal viscosity, the vertical viscosity, the horizontal diffusivity and the vertical diffusivity coefficients, respectively. The parameter \( \Omega \in \mathbb{R} \) stands for the speed of rotation in the Coriolis force, and \( v^\perp = (-v_2, v_1) \). The 3D viscous PEs is derived by performing a formal asymptotic limit of the small aspect ratio (the ratio of the depth or the height to the horizontal length scale) from the Rayleigh-Bénard (Boussinesq) system, and this limit is justified rigorously first by Azérad and Guilléen [2] in a weak sense then later by Li and Titi [51] in a strong sense with error estimates.

The global existence of strong solutions to the 3D PEs with full viscosity and full diffusion was first established by Cao and Titi in [19], and later by Kobelkov in [40], see also the subsequent articles of Kukavica and Ziane [45, 46] for different boundary conditions, as well as Hieber and Kashiwabara [36] for some progress towards relaxing the smoothness on the initial data by using the semigroup method. This result has been improved later by Cao, Li and Titi [15, 16, 17], where the authors proved global well-posedness for 3D PEs with only horizontal viscosity, i.e., with \( \nu_h > 0 \) and \( \nu_v = 0 \). On the other hand, with only vertical viscosity, i.e., \( \nu_h = 0 \) and \( \nu_v > 0 \), Cao, Lin and Titi established recently [18] the local well-posedness of the PEs in Sobolev spaces by considering an additional weak dissipation, which is the linear (Rayleigh-like friction) damping. This linear damping helps the system overcome the ill-posedness in Sobolev spaces established in [50]. See also [29] for a similar idea on the effect of this linear damping.

When \( \nu_h = \nu_v = 0 \), the inviscid PEs without coupling with the temperature is also called the hydrostatic Euler equations. In the absence of rotation (\( \Omega = 0 \)), the linear ill-posedness of the inviscid PEs, near certain shear-flows, has been established by Renardy in [56]. Later on, the nonlinear ill-posedness of the inviscid PEs without rotation was established by Han-Kwan and Nguyen in [34], where they built an abstract framework to show that the inviscid PEs are ill-posed in any Sobolev space. Moreover, it was proven that smooth solutions to the inviscid PEs, in the absence of rotation, can develop singularities in finite time (cf. Cao, Ibrahim, Nakanishi and Titi [14], and Wong [58]). It is shown in [58] that these results on the finite-time blowup and the ill-posedness can also be extended to the 3D inviscid PEs with rotation, i.e., \( \Omega \neq 0 \). By virtue of the finite-time blowup results, one can conclude that there is no hope to show the global well-posedness of the 3D inviscid PEs, even with fast rotation. The optimal result one can expect is that fast rotation prolongs the life-span of solutions to the 3D inviscid PEs.

The linear ill-posedness results mentioned above show that the linearized 2D inviscid PEs (as well as the 3D case [58]), around a special steady state background flow, has unstable solutions of the form \( u(t, x, z) = e^{2\pi i k_x x} e^{\sigma t} u_k(z) \), where \( \Re \sigma_k = \lambda k \) for some \( \lambda \in \mathbb{R} \) and \( \lambda \neq 0 \). Such Kelvin-Helmholtz type instability, which is similar to the one appearing in the context of vortex sheets (see, e.g., [13], the survey paper [8] and reference therein), precludes the construction of solutions in Sobolev spaces for general initial data. To overcome this strong instability, one should consider initial data \( u_0 \) that are strongly localized in Fourier, typically for which \( |\hat{u}_0(k, z)| \leq e^{-\delta|k|^{1/s}} \) with \( \delta > 0 \) and \( s \geq 1 \). Such localization condition corresponds to Gevrey class of order \( s \) in the \( x \) variable. Kelvin-Helmholtz type instability forces us to choose \( s = 1 \) for the well-posedness result, which is the space of analytic functions. This is consistent with positive results reported in [41] and in this paper. Notably, for the Prandtl equations, which have some similarities in its structure with the PEs, is shown in [28] that its linearization around a special background flow has unstable solutions of similar form, but with \( \Re \sigma_k \sim \lambda \sqrt{k} \) for \( k \gg 1 \) arbitrarily large and some positive \( \lambda \in \mathbb{R}_+ \). This implies that the optimal Gevrey class order \( s \) for Prandtl equation is \( s = 2 \), which is consistent with the positive results reported in [28, 50]. This shows that the linear instability of the inviscid PEs is “worse” than that of the Prandtl equations.
Due to the ill-posedness discussed above, in order to show the well-posedness of the inviscid PEs, one needs to assume either some special structures (local Rayleigh condition) on the initial data or real analyticity for general initial data \( \|H\|_{L^2(\Omega)} \). Indeed, the authors of \cite{[14]} establish the local well-posedness of the 3D inviscid PEs in the space of analytic functions for various boundary conditions including the periodic boundary condition. Their approach utilizes explicit estimates for the pressure, regardless of the underlying boundary conditions. These estimates depend explicitly on \( \Omega \), from which one concludes that the time of existence shrinks to zero as \( |\Omega| \) increases toward infinity. As we will describe below, this conclusion is in some sense counter intuitive at least in the absence of boundary for the 3D Euler or Navier-Stokes equations, i.e., in the case of periodic boundary condition. Indeed, Babin, Mahalov and Nicolaenko \cite{[9], [10], [11], [12]} have shown that in \( T^3 \), fast rotation displays a strong averaging mechanism that weakens the nonlinear effects. This mechanism gives the global regularity in the 3D Navier-Stokes case, and the prolongation of the life-span of the solutions in the case of 3D Euler equations (see also \cite{[21], [22], [23], [24], [25], [39], [41]} and references therein for the case of \( \mathbb{R}^3 \)). In addition, we refer to \cite{[3], [33], [42], [52]} for simple examples demonstrating the above averaging/dispersion mechanism. Our purpose here is to show that in \( T^3 \), the fast rotation delays the singularity formation, and thus prolongs the life-span of the solution of the 3D inviscid PEs.

For mathematical simplicity, we consider system \( (1.1) - (1.4) \) with \( T_0 = 0 \), which implies \( T \equiv 0 \) for smooth solutions. By considering the inviscid case, i.e., \( v_h = v_z = 0 \), in this paper, we are interested in the effect of rotation on the 3D inviscid PEs (hydrostatic Euler equations)

\[
\begin{align*}
\partial_t v + v \cdot \nabla v + w \partial_z v + \Omega v_\perp + \nabla p &= 0, \\
\partial_z p &= 0, \\
\nabla \cdot v + \partial_z w &= 0,
\end{align*}
\]

in three-dimensional unit torus \( T^3 \), subject to the following initial and boundary conditions:

\[
\begin{align*}
v|_{t=0} &= v_0, \\
v, w &\text{ are periodic in } (x', z) \text{ with period } 1, \\
v &\text{ is even in } z \text{ and } w \text{ is odd in } z.
\end{align*}
\]

Observe that the space of periodic functions with respect to \( z \) with the symmetry condition \( (1.3) \) is invariant under the dynamics of system \( (1.8)-(1.10) \). If \( H = \frac{1}{2} \), the solution to system \( (1.8)-(1.10) \) in \( T^3 \) subject to \( (1.1)-(1.3) \) restricted on the horizontal channel \( \{(x', z): 0 \leq z \leq \frac{1}{2}, x' \in T^2\} \) is the solution to system \( (1.8)-(1.10) \) subject to the physical boundary conditions, i.e., \( w|_{z=0} = 0 \) and \( v, w \) are periodic in \( x' \) with period 1, and initial condition \( v_0 \) being even extendable in \( z \) variable. Working in \( T^3 \) allows us to use Fourier analysis, and makes the mathematical presentation simpler and more elegant.

The paper is organized as follows. In section 2, we introduce the notation and collect some preliminary results. In section 3, we establish the local well-posedness of the 3D inviscid PEs \( (1.8)-(1.10) \) subject to \( (1.1)-(1.3) \) in the space of analytic functions in a short time interval uniform in \( \Omega \). In section 4, independently of \( \Omega \), we show that the life-span of the solution tends to infinity as the analytic norm of the initial baroclinic mode goes to zero. Moreover, we show in this case that the solution of the 3D inviscid PEs converges to the solution of the limit system, which is governed by the 2D Euler equations. The intuition stems from the observation that the 3D inviscid PEs is reduced to the 2D Euler equations when the baroclinic mode is zero initially. In section 5, we explore further the structure of the inviscid PEs with rotation and derive its formal limit resonant system when \( |\Omega| \to \infty \). Let us emphasize that this limit resonant system is not solely the 2D Euler equations when the initial baroclinic mode is not zero. Moreover, we investigate this limit resonant system and establish its global regularity in both Sobolev and the analytic functions spaces. In section 6, we establish the main result of this paper, namely, the life-span of the solution to the 3D inviscid PEs goes toward infinity, with \( |\Omega| \to \infty \). This is established...
for well-prepared initial data, namely, when only the Sobolev norm (but not the analytic norm) of the baroclinic mode is small enough, depending on $|\Omega|$. Furthermore, for large $|\Omega|$ and “well-prepared” initial data, we show that the solution to the 3D inviscid PEs is indeed approximated by the solution to the limit resonant system that is the main feature of section 5. We also discuss in this section the rational behind the need for the smallness condition in the well-prepared initial data. The last section is an appendix, which is devoted to stating and proving technical lemmas concerning key nonlinear estimates.

2. Preliminaries

In this section, we introduce the notation and collect some preliminary results that will be used in this paper. The universal constant $C$ appears in this paper may change from step to step. When we use subscript for $C$, e.g., $C_r$, it means that the constant depends only on $r$.

2.1. Functional Settings. We use the notation $x := (x', z) = (x_1, x_2, z) \in \mathbb{T}^3$, where $x'$ and $z$ represent the horizontal and vertical variables, respectively. $\mathbb{T}^3$ is the three-dimensional torus with unit length. Denote by $\|f\| := \|f\|_{L^2(\mathbb{T}^3)} = (\int_{\mathbb{T}^3} |f(x)|^2dx)^{\frac{1}{2}}$, associated with the inner product $\langle f, g \rangle = \int_{\mathbb{T}^3} f(x)g(x)dx$ for $f, g \in L^2(\mathbb{T}^3)$. For a function $f \in L^2(\mathbb{T}^3)$, $\hat{f}_k$ denotes its Fourier coefficient, so that $\hat{f}(x) = \sum_{k \in \mathbb{Z}^3} \hat{f}_k e^{2\pi ik \cdot x}$ and $\int_{\mathbb{T}^3} e^{-2\pi ik \cdot x} f(x)dx$. For $r \geq 0$, define the following Sobolev $H^r$ norm and $\dot{H}^r$ semi-norm

$$\|f\|_{H^r} := \left( \sum_{k \in \mathbb{Z}^3} (1 + |k|^{2r}) |\hat{f}_k|^2 \right)^{\frac{1}{2}}, \quad \|f\|_{\dot{H}^r} := \left( \sum_{k \in \mathbb{Z}^3} |k|^{2r} |\hat{f}_k|^2 \right)^{\frac{1}{2}}.$$

For more details about Sobolev spaces, see [1].

For $s > 0$, a function $f \in C^s(\mathbb{T}^3)$ is said to be in Gevrey class of order $s$, denoted by $f \in G^s(\mathbb{T}^3)$, if there exist constants $\rho > 0$ and $M > 0$ such that for every $x \in \mathbb{T}^3$ and $\alpha \in \mathbb{N}^3$, one has $|\partial^\alpha f(x)| \leq M \left( \frac{\rho^{|\alpha|}}{\rho^{\alpha}} \right)^s$. Denote by $A = \sqrt{-\Delta + \partial_z^2}$, subject to periodic boundary condition. For each $s > 0$ and $r \geq 0$, we define a family, parameterized by $\tau \geq 0$, of normed spaces

$$\mathcal{D}(e^{\tau A^{1/s}} : H^r(\mathbb{T}^3)) := \{ f \in H^r(\mathbb{T}^3) : \|e^{\tau A^{1/s}} f\|_{H^r} < \infty \},$$

where the norm is defined by

$$\|e^{\tau A^{1/s}} f\|_{H^r} := \left( \sum_{k \in \mathbb{Z}^3} (1 + |k|^{2r} e^{2\pi |k|^{1/s}}) |\hat{f}_k|^2 \right)^{\frac{1}{2}}.$$

Let us denote the semi-norm by

$$\|A^r e^{\tau A^{1/s}} f\| := \left( \sum_{k \in \mathbb{Z}^3} |k|^{2r} e^{2\pi |k|^{1/s}} |\hat{f}_k|^2 \right)^{\frac{1}{2}},$$

then it is easy to see that

$$\|e^{\tau A^{1/s}} f\|^2_{H^r} = \|A^r e^{\tau A^{1/s}} f\|^2 + \|f\|^2.$$

For more details about Gevrey class, we refer the readers to [26, 27, 49]. Observe that

$$G^s(\mathbb{T}^3) = \bigcup_{\tau > 0} \mathcal{D}(e^{\tau A^{1/s}} : H^r(\mathbb{T}^3)).$$

(2.1)

For the proof of (2.1), see [49]. The next lemma comes from [49] (see also [26], addressing an important property of the space $\mathcal{D}(e^{\tau A^{1/s}} : H^r(\mathbb{T}^3))$. 

Lemma 2.1. If \( s \geq 1, \tau \geq 0, \) and \( r > \frac{3}{2} \), then \( D(e^{\tau A^{1/s}} : H^r(\mathbb{T}^3)) \) is a Banach algebra, and for any \( f, g \in D(e^{\tau A^{1/s}} : H^r(\mathbb{T}^3)) \), we have

\[
\|e^{\tau A^{1/s}}(fg)\|_{H^r} \leq C_{r,s}\|e^{\tau A^{1/s}}f\|_{H^r}\|e^{\tau A^{1/s}}g\|_{H^r}.
\]

For the semi-norm, we also have a similar estimate

\[
\|A^r e^{\tau A^{1/s}}(fg)\| \leq C_{r,s} \left( |\hat{f}_0| + \|A^r e^{\tau A^{1/s}}f\| \right) \left( |\hat{g}_0| + \|A^r e^{\tau A^{1/s}}g\| \right).
\]

For the proof, we refer the readers to [26] for the case when \( s = 1 \), and to [51] for the case when \( s > 1 \).

Remark 1. Since the inviscid PEs is linearly ill-posed in Sobolev spaces and Gevrey class of order \( s > 1 \), we focus on Gevrey class of order \( s = 1 \), which is equivalent to the space of analytic function.

2.2. Projections and reformulation of the problem. In this paper, we assume that \( \int_{\mathbb{T}^3} v_0(x) dx = 0 \). This assumption is made to simplify the mathematical presentation. See Remark 3 for detailed explanation.

Integrating (1.15) in \( \mathbb{T}^3 \), by integration by parts, thanks to (1.10) and (1.12), we obtain

\[
\partial_t \int_{\mathbb{T}^3} v dx + \Omega \int_{\mathbb{T}^3} v^\perp dx = 0.
\]

Therefore, for any time \( t \geq 0 \), \( v \) has zero mean in \( \mathbb{T}^3 \):

\[
\int_{\mathbb{T}^3} v dx = \hat{v}_0 = 0. \tag{2.2}
\]

Denote by

\[
\hat{L}^2 := \left\{ \varphi \in L^2(\mathbb{T}^3, \mathbb{R}^2) : \int_{\mathbb{T}^3} \varphi(x) dx = 0 \right\}.
\]

The barotropic mode \( \mathbf{\tau} \) and baroclinic mode \( \mathbf{\bar{v}} \) are defined by

\[
\mathbf{\tau}(\mathbf{x}') := \int_{0}^{1} v(\mathbf{x}', z) dz = \sum_{k \in \mathbb{Z}^2, k_3 = 0} \hat{v}_k e^{2\pi i k \cdot x}, \quad \mathbf{\bar{v}}(\mathbf{x}) := v - \mathbf{\tau} = \sum_{k \in \mathbb{Z}^2, k_3 \neq 0} \hat{v}_k e^{2\pi i k \cdot x}.
\]

From boundary condition (1.13) and incompressible condition (1.10), one observes that

\[
\nabla \cdot \mathbf{\tau} = \int_{0}^{1} \nabla \cdot v(\mathbf{x}', z) dz = - \int_{0}^{1} \partial_z w(\mathbf{x}', z) dz = 0. \tag{2.3}
\]

Since \( \nabla \cdot \mathbf{\tau} = 0 \) and \( \mathbf{\tau} \) has zero mean over \( \mathbb{T}^2 \) due to (2.2), there exists a stream function \( \psi(\mathbf{x}') \in H^1(\mathbb{T}^2) \), defined uniquely up to a constant, such that \( \mathbf{\tau} = \nabla \perp \psi = (-\partial_{x_3} \psi, \partial_{x_1} \psi) \). Therefore, one has \( v \in S \) where

\[
S := \left\{ \varphi \in \hat{L}^2 : \nabla \cdot \mathbf{\bar{v}} = 0 \right\} = \left\{ \varphi \in \hat{L}^2 : \varphi = \nabla \perp \psi(\mathbf{x}') + \mathbf{\bar{v}}(\mathbf{x}) \text{ with } \psi \in H^1(\mathbb{T}^2) \right\}.
\]

For \( \varphi \in \hat{L}^2 \), the rotating matrix is

\[
\mathcal{J} \varphi := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (\varphi_1) = (-\varphi_2, \varphi_1) = \varphi^\perp.
\]

Denote the 2D Leray projection by \( P_2 : \hat{L}^2 \to S \) as \( P_2 \varphi := \mathbf{\bar{v}} + P_h \mathbf{\bar{v}} \). Moreover, define an operator \( P : S \to S \) as \( P \varphi := P_2(\mathcal{J} \varphi) \). A direct computation using \( \nabla \cdot \mathbf{\bar{v}} = 0 \) yields \( P \varphi = \mathbf{\bar{v}}^\perp \). It is easy to see that the kernel of \( P \) is

\[
\ker P = \left\{ \varphi \in S : \varphi^\perp = 0 \right\} = \left\{ \varphi \in S : \varphi = \mathbf{\tau} \right\}.
\]
Therefore, we define the projection $P_0 : \mathcal{S} \to \ker P$ as
\[
P_0 \varphi := \bar{\varphi} = \int_0^1 \varphi(x',z)dz,
\]
which actually projects any vector $\varphi \in \mathcal{S}$ to its barotropic mode. Now applying $P_\varphi$ to equation (1.8), thanks to (1.9), and since $v \in \mathcal{S}$, we get
\[
\partial_t v + P_\varphi(v \cdot \nabla v + w\partial_z v) + \Omega \tilde{w} = 0.
\]
Next, applying $P_0$ and $I - P_0$ to equation (2.4), by integration by parts, thanks to (1.13) and (2.3), we derive the evolution equations for the barotropic mode $\varphi$ and the baroclinic mode $\bar{\omega}$:
\[
\begin{align*}
\partial_t \varphi + P_k \left( \varphi \cdot \nabla \varphi \right) + P_k P_0 \left( (\nabla \cdot \bar{v}) \bar{v} + \bar{v} \cdot \nabla \bar{v} \right) &= 0, \\
\partial_t \bar{\omega} + \bar{v} \cdot \nabla \bar{\omega} + \bar{\omega} \cdot \nabla \bar{\omega} - P_0 \left( \bar{v} \cdot \nabla \bar{\omega} + (\nabla \cdot \bar{v}) \bar{\omega} \right) - \left( \int_0^s \nabla \cdot \bar{v}(x',s)ds \right) \partial_s \bar{\omega} + \Omega \bar{\omega} &= 0.
\end{align*}
\]
In summary, we have the following lemma.

**Lemma 2.2.** For $v \in \mathcal{S}$, system (1.8)–(1.10) is equivalent to system (2.4)–(2.6).

Notice that if we consider $v_0 \in \ker P$, i.e., consider $\bar{v}_0 = 0$, then from (2.6) we can see $\bar{v}$ remains zero. Therefore, system (2.4)–(2.6) reduces to the 2D Euler equations, which is globally well-posed. Based on this observation, we establish the first long time existence result in section 4 by assuming the analytic norm of $v_0$ is small. In order to investigate the effect of rotation, we further study the evolution of the baroclinic mode. This can be done by further decomposing the baroclinic mode in order to identify the resonant and non-resonant parts due to the rotation. Since the rotation matrix $J$ has eigenvalues $\pm i$, with corresponding eigenvectors $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \mp i \end{pmatrix}$, we can define
\[
P_{\pm} \varphi := \left\langle (I - P_0) \varphi, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm i \end{pmatrix} \right\rangle E \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm i \end{pmatrix} = \frac{1}{2} \left\langle \bar{\varphi} , \frac{1}{i}, \frac{1}{i} \right\rangle = \frac{1}{2}(\varphi \pm i \varphi^\perp).
\]
Here $\langle \cdot, \cdot \rangle_E$ denotes the usual Euclidean inner product. Similar ideas and projections for 3D rotating Euler equations can be found in [24, 41]. Observe that the operator $P$ has three eigenvalues, $0$ and $\pm i$. Therefore, the projections $P_0$ and $P_{\pm}$ project $v$ into the eigenspaces corresponding to $0$ and $\mp i$, respectively. Consequently, we have the following:

**Lemma 2.3.** For any $\varphi \in L^2(\mathbb{T}^3)$, we have
\[
\varphi = P_0 \varphi + P_+ \varphi + P_- \varphi \quad \text{and} \quad P_{\pm} P_{\mp} \varphi = P_{\pm} \varphi, \quad P_0 P_0 \varphi = P_0 \varphi, \quad P_\pm P_\mp \varphi = P_\pm P_\mp \varphi = P_\pm P_0 \varphi = 0.
\]

**Proof.** The proof is straightforward from the definition of $P_0$ and $P_\pm$, and the fact that $\bar{\varphi} = \bar{\varphi}$.

For projections $P_0, P_\pm$, we have the following properties. The proof is straightforward and we omit it.

**Lemma 2.4.** For $f, g \in L^2(\mathbb{T}^3)$, we have $\langle P_0 f, g \rangle = \langle f, P_0 g \rangle = \langle P_0 f, P_0 g \rangle$ and $\langle P_\pm f, g \rangle = \langle f, P_\mp g \rangle$. If $f \in H^r(\mathbb{T}^3)$ with $r \geq 0$, then for $|\alpha| \leq r$, we have $\partial^\alpha P_0 f = P_0 \partial^\alpha f$ and $\partial^\alpha P_{\pm} f = P_{\pm} \partial^\alpha f$. Furthermore, if $f \in D(e^{\tau A^{1/2}} : H^r(\mathbb{T}^3))$ with $s > 0$ and $r \geq 0$, one has $A^s e^{\tau A^{1/2}} P_0 f = P_0 A^s e^{\tau A^{1/2}} f$.

The Leray projection $P_h$ enjoys the following properties. For the proof, see, for example, [22].

**Lemma 2.5.** For $f, g \in L^2(\mathbb{T}^3)$, we have $\langle P_h f, g \rangle = \langle f, P_h g \rangle$ and $P_h P_0 f = P_0 P_h f$. If $f \in H^r(\mathbb{T}^3)$ with $r \geq 0$, then for $|\alpha| \leq r$, one has $\partial^\alpha P_h f = P_h \partial^\alpha f$. Moreover, if $f \in D(e^{\tau A^{1/2}} : H^r(\mathbb{T}^3))$ with $s > 0$ and $r \geq 0$, one gets $A^s e^{\tau A^{1/2}} P_h f = P_h A^s e^{\tau A^{1/2}} f$. 
For the relation between the norm of \( v \) and the norms of \( \bar{v}, \tilde{v} \) in \( L^2(\mathbb{T}^3) \) and \( D(e^{\tau A^{1/r}} : H^r(\mathbb{T}^3)) \), we have the following Lemma. The proof is straightforward and we omit it.

**Lemma 2.6.** Let \( v = P_0v + (I - P_0)v = \bar{v} + \tilde{v} \). Suppose that \( r \geq 0, s \geq 0, \) and \( \tau \geq 0, \) we have

\[
\|v\|^2 = \|\bar{v}\|^2 + \|\tilde{v}\|^2 \quad \text{and} \quad \|e^{\tau A^{1/r}}v\|^2_{H^r} = \|e^{\tau A^{1/r}}\bar{v}\|^2_{H^r} + \|e^{\tau A^{1/r}}\tilde{v}\|^2_{H^r}.
\]

Observe that \( \tilde{v}^\perp \) can be written as \( \tilde{v}^\perp = -i(P_+v - P_-v) \). Hence applying \( P_\pm \) to (2.6), one has

\[
\partial_t P_\pm v + P_\pm \left( \bar{v} \cdot \nabla \tilde{v} + \bar{v} \cdot \nabla \tau + \tau \cdot \nabla \bar{v} - P_0(\bar{v} \cdot \nabla \tilde{v} + (\nabla \cdot \bar{v})\tilde{v}) \right.
\]
\[
\left. - (\int_0^z \nabla \cdot \tilde{v}(x',s)ds) \partial_z \tilde{v} \right) \mp i\Omega P_\pm v = 0.
\]

(2.7)

By setting \( u_\pm = e^{i\Omega t}P_\pm v \), (2.7) can be rewritten as

\[
\partial_t u_\pm + e^{i\Omega t}P_\pm \left( \bar{v} \cdot \nabla \tilde{v} + \bar{v} \cdot \nabla \tau + \tau \cdot \nabla \bar{v} - P_0(\bar{v} \cdot \nabla \tilde{v} + (\nabla \cdot \bar{v})\tilde{v}) \right.
\]
\[
\left. - (\int_0^z \nabla \cdot \tilde{v}(x',s)ds) \partial_z \tilde{v} \right) = 0.
\]

(2.8)

For the \( u_+ \) part, thanks to Lemma 2.6, we have

\[
P_+(\bar{v} \cdot \nabla \tilde{v}) = \frac{1}{2}(\bar{v} \cdot \nabla \tilde{v} + i\bar{v} \cdot \nabla \tilde{v}^\perp) - \frac{1}{2}P_0(\bar{v} \cdot \nabla \tilde{v} + i\bar{v} \cdot \nabla \tilde{v}^\perp) = e^{i\Omega t} \left( \bar{v} \cdot \nabla u_+ - P_0(\bar{v} \cdot \nabla u_+) \right),
\]

\[
P_+(\bar{v} \cdot \nabla \tau) = \frac{1}{2}(\bar{v} \cdot \nabla \tau + i\bar{v} \cdot \nabla \tau^\perp) = \frac{1}{2} \bar{v} \cdot \nabla (\tau + i\tau^\perp),
\]

\[
P_+(\tau \cdot \nabla \bar{v}) = \frac{1}{2}(\bar{v} \cdot \nabla \bar{v} + \tau \cdot \nabla \bar{v}^\perp) = e^{i\Omega t}(\tau \cdot \nabla u_+),
\]

\[
P_+ P_0 \left( \bar{v} \cdot \nabla \tilde{v} + (\nabla \cdot \bar{v})\tilde{v} \right) = 0.
\]

Observe that by integration by parts one has

\[
P_+ \left( \int_0^z \nabla \cdot \tilde{v}(x',s)ds \partial_z \tilde{v} \right) = \frac{1}{2} \left( \int_0^z \nabla \cdot \tilde{v}(x',s)ds \partial_z \tilde{v} + i \int_0^z \nabla \cdot \tilde{v}(x',s)ds \partial_z \tilde{v} \right)
\]
\[
- \frac{1}{2} P_0 \left( \int_0^z \nabla \cdot \tilde{v}(x',s)ds \partial_z \tilde{v} + i \int_0^z \nabla \cdot \tilde{v}(x',s)ds \partial_z \tilde{v} \right)
\]
\[
= e^{i\Omega t} \int_0^z \nabla \cdot \tilde{v}(x',s)ds \partial_z u_+ + e^{i\Omega t} \left( (\nabla \cdot \bar{v})u_+ \right).
\]

Therefore, \( u_+ \) part in (2.8) becomes

\[
\partial_t u_+ = - \left( \bar{v} \cdot \nabla u_+ + \tau \cdot \nabla u_+ - P_0(\bar{v} \cdot \nabla u_+ + (\nabla \cdot \bar{v})u_+) \right) - \left( \int_0^z \nabla \cdot \tilde{v}(x',s)ds \partial_z u_+ \right)
\]
\[
- \frac{1}{2} e^{-i\Omega t} (\bar{v} \cdot \nabla)(\tau + i\tau^\perp).
\]

(2.9)

Using \( \tilde{v} = u_+ e^{i\Omega t} + u_- e^{-i\Omega t} \), we can furthermore rewrite (2.9) as

\[
\partial_t u_+ = - e^{i\Omega t} \left( u_+ \cdot \nabla u_+ - P_0(u_+ \cdot \nabla u_+ + (\nabla \cdot u_+)u_+) \right) - \left( \int_0^z \nabla \cdot u_+(x',s)ds \partial_z u_+ \right)
\]
\[
- \left( \nabla \cdot u_+ + \frac{1}{2} (u_+ \cdot \nabla)(\tau + i\tau^\perp) \right) - e^{-2i\Omega t} \frac{1}{2} (u_- \cdot \nabla)(\tau + i\tau^\perp)
\]
For Lemma 2.7.

Since the effect of rotation.

For the relation between the norm of \( \tilde{v} \) and the norms of \( u_\pm \) in \( L^2(\mathbb{T}^3) \) and \( \mathcal{D}(e^{rA^{1/s}}; H^r(\mathbb{T}^3)) \), we have the following Lemma. The proof is straightforward and we omit it.

**Lemma 2.8.** Let \( u_\pm = \frac{1}{r} e^{iT\Omega}(\tilde{v} \pm i\tilde{v}^\perp) \). Suppose that \( r \geq 0 \), \( s > 0 \), and \( \tau \geq 0 \), we have

\[
\|u_+\|^2 = \|u_-\|^2 = \frac{1}{2r^2} \|\tilde{v}\|^2 \quad \text{and} \quad \|e^{\tau A^{1/s}}u_+\|^2_{H^r} = \|e^{\tau A^{1/s}}u_-\|^2_{H^r} = \frac{1}{2r^2} \|e^{\tau A^{1/s}}\tilde{v}\|^2_{H^r}.
\]

In sections 3 and 4, we work with system (2.10)–(2.12) since the results are independent of the rate of rotation. On the other hand, in section 5 and 6, we work with system (2.10)–(2.12) since our focus is on the effect of rotation.

**3. Local in time Well-posedness**

In this section, we study the local in time well-posedness in the space of analytic functions system (2.5)–(2.6) in \( \mathbb{T}^3 \), subject to the following symmetry boundary conditions and initial conditions:

\[
\overline{\nabla}, \tilde{v} \text{ are periodic in } \mathbb{T}^3 \text{ and are even in } z;
\]
Assume that \( t = 0 \) and \( \nabla \cdot \Omega = 0 \). Observe that whenever \( v \in \mathcal{S} \) then \( \nabla \cdot \bar{v} \in \mathcal{S} \). We have the following result:

**Theorem 3.1.** Assume \( \mathcal{T}_0, \mathcal{T}_0^{\infty} \in \mathcal{S} \cap \mathcal{D}(e^{\tau_0 A} : H^r(\mathbb{T}^3)) \) with \( r > \frac{3}{2} \) and \( \tau_0 > 0 \). Let \( \Omega \in \mathbb{R} \) be arbitrary and fixed. Then there exist a time

\[
T = \frac{\tau_0}{1 + 2C_r(1 + \|e^{\tau_0 A} \mathcal{T}_0\|^2_{H^r} + \|e^{\tau_0 A} \mathcal{T}_0^{\infty}\|^2_{H^r})} > 0,
\]

and a function

\[
\tau(t) = \tau_0 - 2tC_r(1 + \|e^{\tau A} \mathcal{T}_0\|^2_{H^r} + \|e^{\tau A} \mathcal{T}_0^{\infty}\|^2_{H^r}),
\]

both independent of \( \Omega \), such that there exists a unique solution

\[
(v, \bar{v}) \in L^\infty(0, T; \mathcal{S} \cap \mathcal{D}(e^{\tau(t) A} : H^r(\mathbb{T}^3))) \cap L^2(0, T; \mathcal{D}(e^{\tau(t) A} : H^{r + \frac{1}{2}}(\mathbb{T}^3)))
\]

to system (2.5)–(2.6) on \([0, T]\). Moreover, the unique solution \((v, \bar{v})\) depends continuously on the initial data, in the sense of (3.23).

Observe that in the theorem above, the local time of existence is independent of \( \Omega \), unlike the situation in [14] under the periodic boundary condition. Thanks to Lemma 2.2 and Lemma 2.6, we have the following corollary for the original system (1.8)–(1.13).

**Corollary 3.2.** Assume \( v_0 \in \mathcal{S} \cap \mathcal{D}(e^{\tau_0 A} : H^r(\mathbb{T}^3)) \) with \( r > \frac{3}{2} \) and \( \tau_0 > 0 \). Let \( \Omega \in \mathbb{R} \) be arbitrary and fixed. Then there exist a time \( T \) defined in (3.3) and a function \( \tau(t) \) defined in (3.4), both independent of \( \Omega \), such that there exists a unique solution

\[
v \in L^\infty(0, T; \mathcal{S} \cap \mathcal{D}(e^{\tau(t) A} : H^r(\mathbb{T}^3))) \cap L^2(0, T; \mathcal{D}(e^{\tau(t) A} : H^{r + \frac{1}{2}}(\mathbb{T}^3)))
\]

to system (1.8)–(1.13) on \([0, T]\). Moreover, the unique solution \( v \) depends continuously on the initial data.

To show the existence of solutions, one can work on the Galerkin approximation of system (2.5)–(2.6) to establish an uniform energy estimate, then by using, in a nontraditional way (cf. [48]), the Aubin-Lions compactness theorem to pass to the limit and show the existence of solutions. For simplicity, we only do the formal energy estimates (for details, see [29]). Finally, we establish the uniqueness of solutions and its continuous dependence on the initial data.

**3.1. Energy Estimates.** In this section, we establish the formal energy estimates for system (2.5)–(2.6). By virtue of Lemma 2.4 and Lemma 2.5, and since \( \nabla \cdot \mathcal{T} = 0 \), we have the conservation of the \( L^2 \) energy

\[
\|\nabla(v (t))\|^2 + \|\bar{v}(t)\|^2 = \|\nabla(v_0)\|^2 + \|\bar{v}_0\|^2.
\]

Next, employing Lemma 2.2 and Lemma 2.5, we derive the following estimate for the analytic norm

\[
\frac{1}{2} \frac{d}{dt} \|A e^{\tau A} \mathcal{T}\|^2 = \dot{\tau} \|A e^{\tau A} \mathcal{T}\|^2 - \langle A e^{\tau A} (\nabla \cdot \bar{v}) e^{\tau A} \mathcal{T}\rangle - \langle A e^{\tau A} (v \cdot \nabla) e^{\tau A} \mathcal{T}\rangle,
\]

and

\[
\frac{1}{2} \frac{d}{dt} \|A e^{\tau A} \bar{v}\|^2 = \dot{\tau} \|A e^{\tau A} \bar{v}\|^2 - \langle A e^{\tau A} (\bar{v} \cdot \nabla) e^{\tau A} \bar{v}\rangle - \langle A e^{\tau A} (v \cdot \nabla) e^{\tau A} \bar{v}\rangle - \langle A e^{\tau A} (\nabla \cdot \bar{v}) e^{\tau A} \bar{v}\rangle + \langle A e^{\tau A} (\int_0^t (\nabla \cdot \bar{v}) (x', s) ds) \partial_x \bar{v}\rangle e^{\tau A} \bar{v}\rangle.
\]

By Lemma 2.2 and Lemma 2.3, since \( \nabla \) and \( \bar{v} \) having zero mean and thanks to Young’s inequality, we obtain

\[
\frac{1}{2} \frac{d}{dt} \left( \|A e^{\tau A} \mathcal{T}\|^2 + \|A e^{\tau A} \bar{v}\|^2 \right) + \left( \|A e^{\tau A} \mathcal{T}\|^2 + \|A e^{\tau A} e^{\tau A} \bar{v}\|^2 \right)
\]
\[ \leq \left( \hat{\tau} + C_r \left( \| A^\tau e^{\tau A T} \| + \| A^\tau e^{\tau A \bar{T}} \| + 1 \right) \left( \| A^{\tau + \frac{1}{2}} e^{\tau A T} \|^2 + \| A^{\tau + \frac{1}{2}} e^{\tau A \bar{T}} \|^2 \right) \right) \]

\[ \leq \left( \hat{\tau} + C_r \left( 1 + \| e^{\tau A T} \|_{H^r} + \| e^{\tau A \bar{T}} \|_{H^r} \right) \right) \left( \| A^{\tau + \frac{1}{2}} e^{\tau A T} \|^2 + \| A^{\tau + \frac{1}{2}} e^{\tau A \bar{T}} \|^2 \right). \]  

(3.7)

**Remark 2.** Here we add the term \( \| A^{\tau + \frac{1}{2}} e^{\tau A T} \|^2 + \| A^{\tau + \frac{1}{2}} e^{\tau A \bar{T}} \|^2 \) to both sides so that one can obtain the regularity in \( L^2(0, T; \mathcal{D}(e^{\tau(t) A} : H^{\tau + \frac{1}{2}}(T^3))) \).

Let \( \tau \) satisfy

\[ \hat{\tau} + 2C_r \left( 1 + \| e^{\tau_0 A T_0} \|_{H^r}^2 + \| e^{\tau_0 A T_0} \|_{H^r}^2 \right) = 0, \]

hence

\[ \tau(t) = \tau_0 - 2tC_r \left( 1 + \| e^{\tau_0 A T_0} \|_{H^r}^2 + \| e^{\tau_0 A T_0} \|_{H^r}^2 \right). \]

(3.9)

Denote by

\[ \mathcal{T} = \frac{\tau_0}{1 + 2C_r \left( 1 + \| e^{\tau_0 A T_0} \|_{H^r}^2 + \| e^{\tau_0 A T_0} \|_{H^r}^2 \right) > 0}, \]

therefore, \( \tau(t) \geq \tau(\mathcal{T}) = \frac{\tau_0}{1 + 2C_r \left( 1 + \| e^{\tau_0 A T_0} \|_{H^r}^2 + \| e^{\tau_0 A T_0} \|_{H^r}^2 \right) > 0} \) for \( t \in [0, \mathcal{T}] \). Here we require \( C_r \) to be large enough such that

\[ C_r \geq 2(\mathcal{C}_r + \mathcal{C}_r^{\frac{1}{2}}), \]

(3.11)

where \( \mathcal{C}_r \) appears in (3.24) and \( \mathcal{C}_r^{\frac{1}{2}} \) appears in (3.22). Thanks to (3.6), (3.7), and (3.8), one obtains that for \( t \in [0, \mathcal{T}] \),

\[ \| e^{\tau(t) A T} \|^2_{H^r} + \| e^{\tau(t) A \bar{T}} \|^2_{H^r} + 2 \int_0^t \| A^{\tau + \frac{1}{2}} e^{(s) A T} \|^2 + \| A^{\tau + \frac{1}{2}} e^{(s) A \bar{T}} \|^2 ds \]

\[ \leq \| e^{\tau_0 A T_0} \|^2_{H^r} + \| e^{\tau_0 A T_0} \|^2_{H^r}. \]

Moreover, it is easy to see that \( (\mathfrak{v}, \bar{\mathfrak{v}}) \in \mathfrak{S} \). Therefore, the solution \( (\mathfrak{v}, \bar{\mathfrak{v}}) \) satisfies (3.5).

For the estimates on \( \partial_t \mathfrak{v} \) and \( \partial_t \bar{\mathfrak{v}} \), by directly applying \( L^2 \) estimate on (2.5) and (2.6), thanks to Lemma 2.1 2.4 and 2.5 by the Hölder inequality and the Sobolev inequality, since \( r > \frac{\gamma}{2} \), one has

\[ \| \partial_t \mathfrak{v} \| \leq C_r (\| \mathfrak{v} \|^2_{H^r} + \| \bar{\mathfrak{v}} \|^2_{H^r}), \]

\[ \| \partial_t \bar{\mathfrak{v}} \| \leq C_r (\| \mathfrak{v} \|^2_{H^r} + \| \bar{\mathfrak{v}} \|^2_{H^r} + |\Omega| \| \bar{\mathfrak{v}} \|), \]

(3.13)

\[ \| A^{\tau + \frac{1}{2}} e^{\tau A T} \| \leq C_r \left( \| e^{\tau A T} \|^2_{H^r} + \| e^{\tau A \bar{T}} \|^2_{H^r} + |\Omega| \| e^{\tau A \bar{T}} \|^2_{H^{\tau + \frac{1}{2}}} \right); \]

(3.14)

By virtue of the bound (3.12), from (3.13)–(3.15), we have

\[ \partial_t \mathfrak{v} \in L^2(0, \mathcal{T}; D(e^{\tau(t) A} : H^{\tau + \frac{1}{2}})) \cap L^\infty(0, \mathcal{T}; L^2), \]

\[ \partial_t \bar{\mathfrak{v}} \in L^1(0, \mathcal{T}; D(e^{\tau(t) A} : H^{\tau + \frac{1}{2}})) \cap L^\infty(0, \mathcal{T}; L^2). \]

3.2. Uniqueness of Solutions and Continuous Dependence on the Initial Data. In this section, we show the uniqueness of solutions and the continuous dependence on the initial data. Let \( (\mathfrak{v}_1, \bar{\mathfrak{v}}_1) \) and \( (\mathfrak{v}_2, \bar{\mathfrak{v}}_2) \) be two strong solutions to system (2.6)–(2.10) with initial data \( ((\mathfrak{v}_0)_1, (\bar{\mathfrak{v}}_0)_1) \) and \( ((\mathfrak{v}_0)_2, (\bar{\mathfrak{v}}_0)_2) \), respectively. Assume the radius of analyticity for initial data \( ((\mathfrak{v}_0)_1, (\bar{\mathfrak{v}}_0)_1) \) is \( \tau_0 \), and for \( ((\mathfrak{v}_0)_2, (\bar{\mathfrak{v}}_0)_2) \) is \( \tau_20 \). Let \( \tau_0 = \min \{ \tau_{01}, \tau_{02} \} \), and

\[ M = \max \left\{ \| e^{\tau_0 A (\mathfrak{v}_0)_1} \|^2_{H^r} + \| e^{\tau_0 A (\bar{\mathfrak{v}}_0)_1} \|^2_{H^r}, \| e^{\tau_0 A (\mathfrak{v}_0)_2} \|^2_{H^r}, \| e^{\tau_0 A (\bar{\mathfrak{v}}_0)_2} \|^2_{H^r} \right\}. \]

(3.16)

Denote by \( \mathfrak{v} = \mathfrak{v}_1 - \mathfrak{v}_2 \) and \( \bar{\mathfrak{v}} = \bar{\mathfrak{v}}_1 - \bar{\mathfrak{v}}_2 \). By virtue of (3.9) and (3.10), we define

\[ \bar{\tau}(t) = \tau_0 - 2C_r (1 + M), \quad \bar{\mathcal{T}} = \frac{\tau_0}{1 + 2C_r (1 + M)}. \]  

(3.17)
Here $C_r$ satisfies (3.11). From previous sections, and by the definition of $\gamma_0$ and $M$, we know \[\|e^\gamma(t)A\bar{v}(t)\|_{2}^{2} + \|e^\gamma(t)A\bar{v}(t)\|_{H^r}^{2}\leq M \text{ for } i, 2 \text{ and } t \in [0, \bar{T}]\). From (2.5)–(2.6), it is clear that
\[
\begin{align*}
\partial_t \bar{\pi} + \mathcal{P}_k \left( \bar{\pi} \cdot \nabla \bar{\mathbf{v}}_1 + \bar{\pi}_2 \cdot \nabla \bar{\mathbf{v}} \right) + \mathcal{P}_k \mathcal{P}_0 \left( \left( \nabla \cdot \bar{\mathbf{v}} \right) \bar{\mathbf{v}}_1 + \left( \nabla \cdot \bar{\mathbf{v}}_2 \right) \bar{\mathbf{v}} + \bar{\mathbf{v}}_1 \cdot \nabla \bar{\mathbf{v}}_1 + \bar{\mathbf{v}}_2 \cdot \nabla \bar{\mathbf{v}}_2 + \tau A \bar{\pi} - \mathcal{P}_0 \left( \nabla \cdot \bar{\mathbf{v}} \right) \bar{\mathbf{v}}_1 + \left( \nabla \cdot \bar{\mathbf{v}}_2 \right) \bar{\mathbf{v}} \\
+v \cdot \nabla \bar{\mathbf{v}}_1 + \bar{\mathbf{v}}_2 \cdot \nabla \bar{\mathbf{v}} + \bar{\mathbf{v}}_1 \cdot \nabla \bar{\mathbf{v}} + \bar{\mathbf{v}}_2 \cdot \nabla \bar{\mathbf{v}} - P_0 \left( \nabla \cdot \bar{\mathbf{v}} \right) \bar{\mathbf{v}}_1 + \left( \nabla \cdot \bar{\mathbf{v}}_2 \right) \bar{\mathbf{v}} \\
+ \left( \int_0^z \nabla \cdot \bar{\mathbf{v}}(x',s)ds \right) \partial_s \bar{\mathbf{v}}_1 - \left( \int_0^z \nabla \cdot \bar{\mathbf{v}}_2(x',s)ds \right) \partial_s \bar{\mathbf{v}} + \Omega \bar{\mathbf{v}} = 0. \tag{3.19}
\end{align*}
\]

By the energy estimate, thanks to Lemma [2.3] and Lemma [2.5] we have
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \left( \|e^\gamma(t)A\bar{v}(t)\|_{H^r}^{2} + \|e^\gamma(t)A\bar{v}(t)\|_{H^r}^{2} \right)^{2} - \tilde{\tau} \left( \|A e^\gamma A\bar{v}\|^2 + \|A^r e^\gamma A\bar{v}\|^2 \right) \\
+ \left( \nabla \cdot \nabla \bar{v}_1 + \nabla \cdot \nabla \bar{v}_2 + \nabla \cdot \bar{v}_1 + \nabla \cdot \bar{v}_2 \right) \bar{v}_1 + \bar{v}_2 \cdot \nabla \bar{v}_1 + \bar{v}_2 \cdot \nabla \bar{v}_2 + \bar{v} \cdot \nabla \bar{v}_1 + \bar{v}_2 \cdot \nabla \bar{v}_2 \\
+ \left( A^{-\frac{1}{2}} e^\gamma A \left( \nabla \cdot \nabla \bar{v}_1 + \nabla \cdot \nabla \bar{v}_2 + \nabla \cdot \bar{v}_1 + \nabla \cdot \bar{v}_2 \right) \bar{v}_1 + \left( \nabla \cdot \bar{v}_2 \right) \bar{v} + \bar{v}_1 \cdot \nabla \bar{v}_1 + \bar{v}_2 \cdot \nabla \bar{v}_2 \right), A^{-\frac{1}{2}} e^\gamma A \bar{v} \\
+ \left( \int_0^z \nabla \cdot \bar{v}(x',s)ds \right) \partial_s \bar{\mathbf{v}}_1 - \left( \int_0^z \nabla \cdot \bar{\mathbf{v}}_2(x',s)ds \right) \partial_s \bar{\mathbf{v}} + \Omega \bar{\mathbf{v}} = 0. \tag{3.20}
\end{align*}
\]

Thanks to the Hölder inequality, Young’s inequality and the Sobolev inequality, since $r > \frac{5}{2}$, and noticing that \(\bar{\mathbf{v}}\) and \(\bar{v}\) have zero mean over $T^3$, one has
\[
\begin{align*}
\left| \left( \nabla \cdot \nabla \bar{\mathbf{v}}_1 + \nabla \cdot \nabla \bar{\mathbf{v}}_2 + (\nabla \cdot \bar{\mathbf{v}}) \bar{\mathbf{v}}_1 + (\nabla \cdot \bar{\mathbf{v}}_2) \bar{\mathbf{v}} + \bar{\mathbf{v}} \cdot \nabla \bar{\mathbf{v}}_1 + \bar{\mathbf{v}}_2 \cdot \nabla \bar{\mathbf{v}}, \bar{\mathbf{v}} \right) \\
+ \left( \nabla \cdot \bar{\mathbf{v}}_1 + \bar{\mathbf{v}}_2 \cdot \nabla \bar{\mathbf{v}} + \bar{\mathbf{v}} \cdot \nabla \bar{\mathbf{v}}_1 + \bar{\mathbf{v}}_2 \cdot \nabla \bar{\mathbf{v}} + \bar{\mathbf{v}} \cdot \nabla \bar{\mathbf{v}}_1 + \bar{\mathbf{v}}_2 \cdot \nabla \bar{\mathbf{v}} \\
- \left( \int_0^z \nabla \cdot \bar{\mathbf{v}}(x',s)ds \right) \partial_s \bar{\mathbf{v}}_1 - \left( \int_0^z \nabla \cdot \bar{\mathbf{v}}_2(x',s)ds \right) \partial_s \bar{\mathbf{v}} \right) \right| \\
\leq C_r \left( \|\bar{\mathbf{v}}_1\|_{H^r} + \|\bar{\mathbf{v}}_2\|_{H^r} + \|\bar{\mathbf{v}}\|_{H^r} + \|\bar{\mathbf{v}}_2\|_{H^r} \right) \left( \|\bar{\mathbf{v}}\|_{H^r}^{2} + \|\bar{\mathbf{v}}_2\|_{H^r}^{2} \right) \\
\leq C_r \left( \|\bar{\mathbf{v}}_1\|_{H^r} + \|\bar{\mathbf{v}}_2\|_{H^r} + \|\bar{\mathbf{v}}\|_{H^r} + \|\bar{\mathbf{v}}_2\|_{H^r} \right) \left( \|A e^\gamma A\bar{v}\|^2 + \|A^r e^\gamma A\bar{v}\|^2 \right). \tag{3.21}
\end{align*}
\]

where in the last step we apply the Poincaré inequality. Next, thanks to Lemma [A.4] and by Young’s inequality, we have
\[
\begin{align*}
\left| \left( A^{-\frac{1}{2}} e^\gamma A \left( \nabla \cdot \nabla \bar{\mathbf{v}}_1 + \nabla \cdot \nabla \bar{\mathbf{v}}_2 + (\nabla \cdot \bar{\mathbf{v}}) \bar{\mathbf{v}}_1 + (\nabla \cdot \bar{\mathbf{v}}_2) \bar{\mathbf{v}} + \bar{\mathbf{v}} \cdot \nabla \bar{\mathbf{v}}_1 + \bar{\mathbf{v}}_2 \cdot \nabla \bar{\mathbf{v}}, A^{-\frac{1}{2}} e^\gamma A \bar{\mathbf{v}} \right) \\
+ \left( A^{-\frac{1}{2}} e^\gamma A \left[ \nabla \cdot \nabla \bar{\mathbf{v}}_1 + \bar{\mathbf{v}}_2 \cdot \nabla \bar{\mathbf{v}} + \bar{\mathbf{v}} \cdot \nabla \bar{\mathbf{v}}_1 + \bar{\mathbf{v}}_2 \cdot \nabla \bar{\mathbf{v}}_2 + \bar{\mathbf{v}} \cdot \nabla \bar{\mathbf{v}}_1 + \bar{\mathbf{v}}_2 \cdot \nabla \bar{\mathbf{v}}_2 \right), A^{-\frac{1}{2}} e^\gamma A \bar{\mathbf{v}} \right) \right| \\
\leq C_r \frac{1}{4} \left( \|e^\gamma A\bar{v}_1\|_{H^r} + \|e^\gamma A\bar{v}_2\|_{H^r} + \|e^\gamma A\bar{v}\|_{H^r} + \|e^\gamma A\bar{v}_2\|_{H^r} \right) \left( \|A e^\gamma A\bar{v}\|^2 + \|A^r e^\gamma A\bar{v}\|^2 \right). \tag{3.22}
\end{align*}
\]

Combining (3.20)–(3.22) and thanks to (3.11), we have
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \left( \|e^\gamma A\bar{v}(t)\|^2_{H^r} + \|e^\gamma A\bar{v}(t)\|^2_{H^r} \right). \tag{3.23}
\end{align*}
\]
this additional term does not affect the higher order energy estimates. Thus, when

\[ 4.1 \]

Remark

completes the proof of Theorem 3.1.

In section 6, however, we will demonstrate that system (2.5)–(2.6) exhibits a different behavior for large

\[ \Omega \]

\[ \Omega \]

Consider the 2D Euler equations:

\[ \begin{align*}
\partial_t \nabla + \nabla \cdot \nabla \nabla + \nabla P &= 0, \\
\nabla \cdot \nabla &= 0, \\
\nabla (0) &= \nabla_0.
\end{align*} \]

Here \( \nabla \) depends only on the horizontal variables \( x' \). The global existence of solutions to system (4.1)–(4.3) in Sobolev spaces \( H^r \) with \( r \geq 3 \) is a classical result, see, e.g., [9]. Moreover, from equation (3.84) in [9], for \( r \geq 3 \), we have

\[ \frac{d}{dt} \| \nabla \|_{H^r} \leq C_r \| \nabla \|_{H^r} (1 + \ln^+ \| \nabla \|_{H^r}). \]
Let $\|V_0\|_{H^r} \leq M$ for some $M \geq 0$. Since $\ln x + 1 \leq 2 \ln(x + e)$, by setting $W(t) = \|V(t)\|_{H^r} + e$, from (4.4), we have $\frac{d}{dt}W \leq C_r W \ln W$. Therefore, we get the following bound:

$$\|V(t)\|_{H^r} \leq W(t) \leq W(0)e^{C_r t} = (\|V_0\|_{H^r} + e)e^{C_r t} \leq (M + e)e^{C_r t} =: \theta_{M, r}(t). \quad (4.5)$$

We need the following lemma from [49].

**Lemma 4.1.** For $f, g \in D(e^{rA}: H^{r+\frac{1}{2}})$ where $r > \frac{5}{2}$ and $\tau \geq 0$, one has

$$\left\langle A'e^{rA}(f \cdot \nabla g), A'e^{rA}g \right\rangle \leq C_r(\|A'f\|\|A'r\|g\|^2 + \|\nabla \cdot f\|_{L^\infty}\|A'e^{rA}g\|^2)$$

$$+ C_r \tau \|A'^{r+\frac{1}{2}}e^{rA}f\|\|A'^{r+\frac{1}{2}}e^{rA}g\|^2.$$

Moreover, if $r > 3$, then $\|A'^{r+\frac{1}{2}}e^{rA}f\|$ can be replaced by $\|A'e^{rA}f\|$.

Based on Lemma 4.1, the authors in [49] proved the global existence of solutions to system (4.1)–(4.3) for initial data in the space of analytic functions. For completion, we state it here, with slight difference from the original statement in [49].

**Proposition 4.2.** Assume $\nabla_0 \in S \cap D(e^{tA}: H^{r}(\mathbb{T}^3))$ with $r > 3$ and $\tau_0 > 0$, and suppose that $\|e^{tA}\nabla_0\|_{H^r} \leq M$ for some $M \geq 0$. Then there exists a non-increasing function

$$\tau(t) = \tau_0 \exp\left(-C_r \int_0^t h(s)ds\right), \quad (4.6)$$

where $h(t) := \|e^{tA}\nabla_0\|^2_{H^r} + C_r \int_0^t \|\nabla \cdot \theta_{M, r}(s)\|ds$, and $\theta_{M, r}(t)$ defined in (4.5), such that for any given time $T > 0$, there exists a unique solution $\nabla \in L^\infty(0, T; D(e^{rA}: H^{r}(\mathbb{T}^3)))$ to system (4.1)–(4.3). Moreover, there exist constants $C > 1$ and $C_r > 1$ such that

$$\|e^{rA(t)}\nabla(t)\|^2_{H^r} \leq h^2(t) \leq C^{\exp(C_r t)}. \quad (4.7)$$

4.2. **Long time existence of the 3D inviscid PEs.** The following is the main theorem of this section, which concerns the long time existence of solutions to system (2.5)–(2.6) in the case when the analytic norm of $\nabla_0$ is small.

**Theorem 4.3.** Assume $\nabla_0 \in S \cap D(e^{tA}: H^{r+1}(\mathbb{T}^3))$, $\bar{\nabla}_0 \in S \cap D(e^{tA}: H^{r}(\mathbb{T}^3))$ with $r > \frac{5}{2}$ and $\tau_0 > 0$. Let $\Omega \in \mathbb{R}$ be arbitrary and fixed. Let $M \geq 0$ and $\varepsilon > 0$, and suppose that $\|e^{tA}\nabla_0\|_{H^{r+1}} \leq M$ and $\|e^{tA}\bar{\nabla}_0\|_{H^r} \leq \varepsilon$. Then there are constants $C > 1$ and $C_r > 1$, and a function $K(t) = C^{\exp(C_r t)}$, such that if $T = T(\tau_0, \varepsilon, M, r)$ satisfies

$$\int_0^T e^{sK(s)}ds = \frac{\tau_0}{2\varepsilon}, \quad (4.8)$$

then the unique solution obtained in Theorem 3.2 satisfies $\bar{\nabla} \in L^\infty(0, T; S \cap D(e^{rA}: H^{r}(\mathbb{T}^3)))$, with

$$\tau(t) = e^{-\int_0^t K(s)ds}(\tau_0 - \varepsilon \int_0^t e^{-K(s)ds}). \quad (4.9)$$

In particular, from (4.8), $T \geq \ln(\ln(\ln(\frac{1}{\varepsilon}))) \to \infty$, as $\varepsilon \to 0^+$.

Thanks to Lemma 2.6, we immediately have the following corollary.

**Corollary 4.4.** Assume $\nabla_0 \in S \cap D(e^{tA}: H^{r+1}(\mathbb{T}^3))$, and the conditions of Theorem 4.3 hold. Then the unique solution obtained in Corollary 3.2 satisfies $v \in L^\infty(0, T; S \cap D(e^{rA}: H^{r}(\mathbb{T}^3)))$, with $T$ defined in (4.8) and $\tau$ defined in (4.9).
Remark 4. For the proof of Theorem 4.3, we only establish formal energy estimates. However, these formal estimates can be justified rigorously by establishing them first for the Galerkin approximation system and then passing to the limit using the Aubin-Lions compactness theorem.

Remark 5. The constants $C$ and $C_r$ in $K(t)$, in the proof below, may change from step to step, and are always taken to be larger than 1. When necessary, we use $K_1(t), K_2(t), \ldots$ to emphasize the changes. At the end, we choose some suitable and large enough $C$ and $C_r$ for the $K(t)$ in Theorem 4.3.

Proof of Theorem 4.3. Let $\overline{V}$ be the unique global solution to the 2D Euler equations (4.1)–(4.3) in the space $D(e^{\tau_1(t)A}_0 : H^{r+1}(\mathbb{T}^3))$, with initial condition $\overline{V}_0 = \overline{\tau}_0$ and $\tau_1(t)$ satisfying (4.6). Let $\overline{\phi} = \overline{\tau} - \overline{V}$. Then from (2.4)–(2.6) one has

$$\partial_t \overline{\phi} + \mathbb{P}_h \left( \nabla \overline{\phi} \cdot \nabla \phi + \overline{V} \cdot \nabla \phi \right) + \mathbb{P}_h P_0 \left( (\nabla \cdot \overline{\phi}) \overline{v} + \overline{v} \cdot \nabla \overline{\phi} \right) = 0,$$

$$\partial_t \overline{\phi} + \overline{v} \cdot \nabla \overline{\phi} + \overline{\phi} \cdot \nabla \overline{\phi} + \overline{V} \cdot \nabla \overline{\phi} + \overline{v} \cdot \nabla \overline{\phi} + \overline{\phi} \cdot \nabla \overline{\phi} - P_0 \left( (\nabla \cdot \overline{\phi}) \overline{v} + \overline{v} \cdot \nabla \overline{\phi} \right)$$

$$- \left( \int_0^Z \nabla \cdot \overline{\phi}(s) d\tau \right) \partial_\tau \overline{\phi} + \Omega \overline{\phi} = 0,$$

with initial condition

$$\overline{\phi}(0) = \overline{\tau}_0 - \overline{V}_0 = 0, \quad \overline{\phi}(0) = \overline{v}_0.$$

Thanks to Lemma 2.4 and Lemma 2.5, we have

$$\frac{1}{2} \frac{d}{dt} \left\| A^r e^{rA} \overline{\phi} \right\|^2 = \frac{\tau}{2} \left\| A^{r+\frac{3}{2}} e^{rA} \overline{\phi} \right\|^2 - \left\langle A^r e^{rA} \overline{\phi}, A^r e^{rA} \overline{\phi} \right\rangle - \left\langle A^r e^{rA} (\overline{\phi} \cdot \nabla \overline{\phi}), A^r e^{rA} \overline{\phi} \right\rangle$$

$$- \left\langle A^r e^{rA} (\overline{\phi} \cdot \nabla \overline{\phi}), A^r e^{rA} \overline{\phi} \right\rangle - \left\langle A^r e^{rA} (\overline{\phi} \cdot \nabla \overline{\phi}), A^r e^{rA} \overline{\phi} \right\rangle - \left\langle A^r e^{rA} ((\nabla \cdot \overline{\phi}) \overline{v} + \overline{v} \cdot \nabla \overline{\phi}) \overline{\phi}, A^r e^{rA} \overline{\phi} \right\rangle.$$

By using Lemma 4.1 and Lemma 4.3, we have

$$\left\langle A^r e^{rA} (\overline{\phi} \cdot \nabla \overline{\phi}), A^r e^{rA} \overline{\phi} \right\rangle + \left\langle A^r e^{rA} (\overline{\phi} \cdot \nabla \overline{\phi}), A^r e^{rA} \overline{\phi} \right\rangle + \left\langle A^r e^{rA} ((\nabla \cdot \overline{\phi}) \overline{v} + \overline{v} \cdot \nabla \overline{\phi}) \overline{\phi}, A^r e^{rA} \overline{\phi} \right\rangle$$

$$+ \left\langle A^r e^{rA} ((\nabla \cdot \overline{\phi}) \overline{v} + \overline{v} \cdot \nabla \overline{\phi}) \overline{\phi}, A^r e^{rA} \overline{\phi} \right\rangle$$

$$\leq C_r \left( \left\| A^r e^{rA} \overline{\phi} \right\| + \left\| A^r e^{rA} \overline{\phi} \right\| \right) \left( \left\| A^{r+\frac{3}{2}} e^{rA} \overline{\phi} \right\|^2 + \left\| A^{r+\frac{3}{2}} e^{rA} \overline{\phi} \right\|^2 \right).$$

Here we use the fact that $\overline{\phi}$ has zero mean value since $\overline{\tau}$ and $\overline{v}$ both have zero mean value, and $\overline{\phi}$ has zero mean value since $\overline{v} = 0$. By virtue of Lemma 4.1, since $\nabla \cdot \overline{V} = 0$, one obtains

$$\left\langle A^r e^{rA} (\overline{\phi} \cdot \nabla \overline{\phi}), A^r e^{rA} \overline{\phi} \right\rangle + \left\langle A^r e^{rA} (\overline{\phi} \cdot \nabla \overline{\phi}), A^r e^{rA} \overline{\phi} \right\rangle \leq C_r \left\| A^r e^{rA} \overline{\phi} \right\| \left( \left\| A^r e^{rA} \overline{\phi} \right\|^2 + \left\| A^r e^{rA} \overline{\phi} \right\|^2 \right).$$
From Lemma 2.1 thanks to the Cauchy–Schwarz inequality and since $\bar{v}$ and $\bar{\phi}$ have zero mean, we have
\[
\left| \int A^T e^{\tau A}(\bar{v} \cdot \nabla \bar{v}), A^T e^{\tau A}\bar{v}) \right| + \left| \int A^T e^{\tau A}(\bar{\phi} \cdot \nabla \bar{v}), A^T e^{\tau A}\bar{\phi}) \right| 
\leq C_r \|e^{\tau A}v\|_{H^{r+1}}(\|A' e^{\tau A}\bar{v}\|^2 + \|A' e^{\tau A}\bar{\phi}\|^2).
\]
Combining all the estimates above, we have
\[
\frac{1}{2} \frac{d}{dt}(\|A' e^{\tau A}\bar{v}\|^2 + \|A' e^{\tau A}\bar{\phi}\|^2)
\leq \left( \dot{\tau} + C_r(\|A' e^{\tau A}\bar{v}\| + \|A' e^{\tau A}\bar{\phi}\|) + C_r \tau \|e^{\tau A}v\|_{H^{r+1}} \right) \left( \|A' e^{\tau A}\bar{v}\|^2 + \|A' e^{\tau A}\bar{\phi}\|^2 \right) + C_r \|A' e^{\tau A}\bar{\phi}\|_{H^{r+1}}\left( \|A' e^{\tau A}\bar{v}\|^2 + \|A' e^{\tau A}\bar{\phi}\|^2 \right).
\]
(4.10)
As indicated in Remark 5 we will use $K_0, K_1, K_2, ...$ to indicate the change in $K(t)$ from step to step, and all of them are increasing double exponentially in $t$. Recall that $\tau_1$ is defined by (4.10). Indeed, there exists a function $K_0(t)$ such that $\tau_1(t) \geq \tau_0 e^{-\int_0^t K_0(s)ds}$. Let $\tau \leq \tau_1$. Recall from (4.11), we have
\[
\|e^{\tau(t)}A\bar{v}(t)\|_{H^{r+1}} \leq \|e^{\tau_1(t)}A\bar{v}(t)\|_{H^{r+1}} \leq C_{\exp(C_r,t)} =: K_1(t).
\]
Denote by
\[
F = \|A' e^{\tau A}\bar{v}\|^2 + \|A' e^{\tau A}\bar{\phi}\|^2, \quad G = \|A' e^{\tau A}\bar{v}\|^2 + \|A' e^{\tau A}\bar{\phi}\|^2.
\]
We can rewrite (4.11) as
\[
\frac{d}{dt} F \leq 2(\dot{\tau} + C_r F^{\frac{1}{2}} + \tau K_2)G + K_2 F.
\]
Notice that when $\tau$ satisfies
\[
\dot{\tau} + C_r F^{\frac{1}{2}} + \tau K_2 \leq 0,
\]
we have
\[
F(t) \leq F(0)e^{\int_0^t K_2(s)ds} \leq F(0)e^{K_3(t)},
\]
and therefore $C_r F(t)^{\frac{1}{2}} \leq F(0)^{\frac{1}{2}}e^{K_3(t)}$. Notice that $F(0) = \|A' e^{\tau_0 A}\bar{v_0}\|^2 \leq \|e^{\tau_0 A}\bar{v_0}\|^2 \leq \epsilon^2$. From (4.11), we require that
\[
\frac{d}{dt}(\tau_0 e^{\int_0^t K_2(s)ds}) + e^{\int_0^t K_2(s)ds}e^{K_4(t)} \leq 0.
\]
It is clear that $e^{\int_0^t K_2(s)ds}e^{K_4(t)} \leq e^{K_5(t)}$ for some new $K_5(t)$. Therefore, instead of (4.13), we require that
\[
\frac{d}{dt}(\tau_0 e^{\int_0^t K_2(s)ds}) + e^{K_5(t)} \leq 0.
\]
(4.14)
Integrating (4.14) from 0 to $t$ in time, we have
\[
\tau(t) e^{\int_0^t K_2(s)ds} \leq \tau_0 - \epsilon \int_0^t e^{K_5(s)}ds.
\]
(4.15)
Recall that we also need $\tau(t) \leq \tau_1(t)$ and we know that $\tau_1(t) \geq \tau_0 e^{-\int_0^t K_0(s)ds}$. Therefore, for a new and suitable function $K(t)$, we can set
\[
\tau(t) = e^{-\int_0^t K(s)ds}(\tau_0 - \epsilon \int_0^t e^{K(s)}ds)
\]
such that $\tau(t)$ satisfies the condition in (4.11) and also $\tau(t) \leq \tau_1(t)$. One can see $\tau(t) > 0$ on $t \in [0, T]$ when $T$ satisfies $\int_0^T e^{K(s)}ds = \frac{\tau_0}{2\epsilon}$. Since $K(t)$ is double exponential in time, and $\int_0^T e^{K(s)}ds \leq Te^{K(T)} \leq e^{2K(T)}$, we have $T \geq \ln(\ln(\frac{1}{2\epsilon})) \to \infty$ as $\epsilon \to 0^+$. 
From (4.12), since $\varphi$ and $\bar{v}$ have zero mean, we can apply the Poincaré inequality to obtain

$$\|e^{\tau(t)A\varphi(t)}\|^2_{H^r} + \|e^{\tau(t)A\bar{v}(t)}\|^2_{H^r} \leq e^{2e^{K(t)}}$$

(4.17) when $K(t)$ is chosen suitably, on $t \in [0, T]$, with $(\tau(t))$ defined by (4.10). From (4.7), and since $\tau \leq \tau_1$, we know $\|e^{\tau(t)A\varphi(t)}\|_{H^r}$ is also bounded on $t \in [0, T]$. By triangle inequality, we have

$$\|e^{\tau(t)A\varphi(t)}\|_{H^r} + \|e^{\tau(t)A\bar{v}(t)}\|_{H^r} \leq \|e^{\tau(t)A\varphi(t)}\|_{H^r} + \|e^{\tau(t)A\varphi(t)}\|_{H^r} < \infty$$

for $t \in [0, T]$. Therefore, the time of existence of the solution to system (2.5)–(2.6) satisfies (4.8).

4.3 Convergence to the 2D Euler equations. Based on Theorem 4.3, we have the following result concerning the convergence of solutions of the 3D inviscid PEs (2.5)–(2.6) to solutions of the 2D Euler equations (4.1)–(4.3) in the space of analytic functions.

**Theorem 4.5.** Assume a sequence of initial data $\{\bar{v}_n = v_0\}_{n \in \mathbb{N}} \subset \mathcal{S} \cap D(e^{\tau_n A} : H^{r+1}(T^3))$ and $\{\bar{v}_n\}_{n \in \mathbb{N}} \subset \mathcal{S} \cap D(e^{\tau_n A} : H^r(T^3))$ with $r > \frac{5}{2}$ and $\tau_0 > 0$. Let $\Omega \subset \mathbb{R}$ be arbitrary and fixed. Suppose $\|e^{\tau_n A\bar{v}_n}\|_{H^{r+1}} \leq M$ for some $M > 0$, and $\|e^{\tau_n A\bar{v}_n}\|_{H^r} \leq \epsilon_n$ with $\epsilon_n \to 0$, as $n \to \infty$. Then there are constants $C > 1, C_r > 1$, and a function $K(t) = C\exp(Cr\cdot t)$, such that for each $n \in \mathbb{N}$, if the function $\tau^n(t)$ and the time $T_n$ satisfy

$$\tau_n(t) = e^{-\int^t_0 K(s)ds}(\tau_0 - \epsilon_n \int^t_0 K(s)ds), \quad \int^t_0 K(s)ds = \frac{\tau_0}{2\epsilon_n},$$

the solution to system (2.2)–(2.7) with initial data $(\tau_0, \bar{v}_0)$ satisfies $(\tau^n, \bar{v}^n) \in L^\infty(0, T_n; \mathcal{S} \cap D(e^{\tau A} : H^r(T^3)))$. Let $\bar{v} \in L^\infty(0, \infty; \mathcal{S} \cap D(e^{\tau A} : H^r(T^3)))$ be the unique global solution to the 2D Euler equations (4.1)–(4.3) with initial data $\bar{v}(0) = v_0$. Then, $(\tau^n, \bar{v}^n)$ converges to $\bar{v}$ for $t \in [0, T_0]$, as $n \to \infty$, in the following sense:

$$\|e^{\tau(t)A} (\bar{v}^n + \bar{v} - \bar{v})(t)\|_{H^r} \leq \epsilon_n e^{K(t)} \to 0, \quad n \to \infty.$$  

(4.18)

**Proof.** Denote by $\bar{v} = v^n - \bar{v}$. By virtue of the proof of Theorem 4.3, we just need to prove the estimate (4.18). Since $\tau^n(t) \leq \tau^n(t)$ for any $n \in \mathbb{N}$, from (4.17), one has

$$\|e^{\tau(t)A\bar{v}^n(t)}\|_{H^r} + \|e^{\tau(t)A\bar{v}^n(t)}\|_{H^r} \leq \|e^{\tau(t)A\bar{v}^n(t)}\|_{H^r} + \|e^{\tau(t)A\bar{v}^n(t)}\|_{H^r} \leq \epsilon_n e^{K(t)}$$

when the function $K(t)$ is chosen suitably. Therefore, we have

$$\|e^{\tau(t)A} (\bar{v}^n + \bar{v} - \bar{v})(t)\|_{H^r} \leq \|e^{\tau(t)A\bar{v}^n(t)}\|_{H^r} + \|e^{\tau(t)A\bar{v}^n(t)}\|_{H^r} \leq \epsilon_n e^{K(t)} \to 0, \quad n \to \infty.$$  

5. Limit resonant system

In this section, we derive the formal resonant limit resonant system of the original system (2.5)–(2.6) as $|\Omega| \to \infty$, and establish some properties of the limit resonant system. Recall from (2.10), we have

$$\partial_t u_+ = -e^{it\Omega} \left( u_+ \cdot \nabla u_+ - P_0(u_+ \cdot \nabla u_+) + (\nabla \cdot u_+)(s)ds \right) \partial_z u_+$$

$$:= I_1$$

$$- \left( \nabla \cdot u_+ + \frac{1}{2}(u_+ \cdot \nabla)(u_+ - \bar{u}) \right)$$

$$:= I_2$$

$$- e^{-i\Omega} \left( u_- \cdot \nabla u_- - P_0(u_- \cdot \nabla u_-) + (\nabla \cdot u_-)(s)ds \right) \partial_z u_-$$

$$:= I_3$$
\[-e^{-2\Omega t} \frac{1}{2} (u_\cdot \nabla)(\nabla + \mathbf{i} \Omega) = -e^{\Omega t} I_1 - I_0 - e^{-\Omega t} I_{-1} - e^{-2\Omega t} I_{-2}. \tag{5.1}\]

Observe that \(I_0\) is a typical resonant term. Unlike the case of the 3D Euler equations where there are frequency selection resonances, all frequencies resonate in \(I_0\). We can rewrite (5.1) as
\[
\partial_t \left[ u_+ - \frac{i}{\Omega} \left( e^{\Omega t} I_1 - e^{-\Omega t} I_{-1} - \frac{1}{2} e^{-2\Omega t} I_{-2} \right) \right] = -\frac{i}{\Omega} \left( e^{\Omega t} \partial_t I_1 - e^{-\Omega t} \partial_t I_{-1} - \frac{1}{2} e^{-2\Omega t} \partial_t I_{-2} \right) - I_0.
\]

Denote by the formal limits of \(u_+, u_-, \nabla\) to be \(U_+, U_-, \nabla\). By taking limit \(\Omega \to \infty\) formally, since \(u_-\) is the complex conjugate of \(u_+\), we obtain the limit resonant equations of \(u_\pm\):
\[
\partial_t U_\pm = -(\nabla \cdot \nabla) U_\pm - \frac{1}{2} (U_\pm \cdot \nabla)(\nabla \pm \nabla^\perp). \tag{5.2}
\]

For the limit equation of \(\nabla\), recall from (2.12) that
\[
\partial_t \nabla + \mathbb{P}_h(\nabla \cdot \nabla \nabla) + e^{2\Omega t} \mathbb{P}_h P_0 \left( u_+ \cdot \nabla u_+ + (\nabla \cdot u_+) u_+ \right)
+ e^{-2\Omega t} \mathbb{P}_h P_0 \left( u_- \cdot \nabla u_- + (\nabla \cdot u_-) u_- \right) = 0.
\]

Observe that \(\mathbb{P}_h(\nabla \cdot \nabla \nabla)\) is a typical resonant term. Using the similar method in the derivation of \(U_+\), we can derive the limit resonant equation for \(\nabla\) as
\[
\partial_t \nabla + \mathbb{P}_h(\nabla \cdot \nabla \nabla) = 0. \tag{5.3}
\]

Observe that (5.3) is the 2D Euler system. Consider the initial conditions
\[
(\nabla_0, (U_+)_0, (U_-)_0) = (\nabla_0, \frac{1}{2}(\nabla + i \nabla^\perp), \frac{1}{2}(\nabla - i \nabla^\perp))
\]
for system (5.2)–(5.3). Since \(v_0 \in \mathcal{S}\), we have \(\nabla \cdot \nabla = 0\), \(P_0 \nabla = \nabla\), and \(P_0 U_\pm = 0\).

Besides the equations for \(U_\pm\) and \(\nabla\), we also want a baroclinic mode \(\tilde{V}\) similar as in the original system. Since initially \(U_\pm(0) = \frac{1}{2}(\nabla_0 \pm i \nabla^\perp_0)\), we define \(\tilde{V} := U_+ + U_-\) so that \(U_\pm = \frac{1}{2}(\nabla \pm i \nabla^\perp)\). From (5.2), we have
\[
\partial_t \tilde{V} + (\nabla \cdot \nabla) \tilde{V} + \frac{1}{2} (\tilde{V} \cdot \nabla \nabla - \tilde{V}^\perp \cdot \nabla^\perp) = 0. \tag{5.4}
\]

Since \(\nabla \cdot \nabla = 0\), (5.4) is equivalent to
\[
\partial_t \tilde{V} + \nabla \cdot \nabla \tilde{V} + \frac{1}{2} \tilde{V}^\perp (\nabla^\perp \cdot \nabla) = 0.
\]

Since \(P_0 U_\pm = 0\), we see \(P_0 \tilde{V} = 0\). Therefore, we consider the following limit resonant system
\[
\partial_t \tilde{V} + \mathbb{P}_h(\nabla \cdot \nabla \tilde{V}) = 0, \tag{5.5}
\]
\[
\partial_t \tilde{V} + \tilde{V} \cdot \nabla \tilde{V} + \frac{1}{2} \tilde{V}^\perp (\nabla^\perp \cdot \nabla) = 0, \tag{5.6}
\]
\[
\tilde{V}(0) = \nabla_0, \quad \tilde{V}(0) = \tilde{V}_0, \tag{5.7}
\]
with \(P_0 \nabla = \nabla\) and \(P_0 \tilde{V} = 0\). Observe that (5.5) is the 2D Euler system, and (5.6) is a linear transport equation with an additional stretching term.

Next, we establish the global well-posedness of limit resonant system (5.5)–(5.7) in both Sobolev spaces and the space of analytic functions. Recall that the global well-posedness of (5.5)–(5.7) has been established in Proposition 4.2.
Proposition 5.1. Assume $V_0 \in S \cap H^{r+1}(T^3)$ and $\tilde{V}_0 \in S \cap H^{r}(T^3)$ with $r > \frac{5}{2}$. Let $M \geq 0$, and suppose that $\|V_0\|_{H^{r+1}} \leq M$. Then there exist constants $C > 1$ and $C_r > 1$, and a function $K(t) := C e^{C(t)}$, such that for any given time $T > 0$, there exists a unique solution $V(t) \in L^\infty(0, T; S \cap H^{r+1}(T^3))$ and $\tilde{V} \in L^\infty(0, T; S \cap H^{r}(T^3))$ of system (5.3)–(5.7) on $[0, T]$, and satisfies
\[
\|V(t)\|_{H^{r+1}} \leq K(t), \quad \|\tilde{V}(t)\|_{H^{r}} \leq \|\tilde{V}_0\|_{H^{r}} e^{K(t)}.
\] (5.8)
Moreover, assume $\tilde{V}_0 \in D(e^{\tau A}) : H^{r+1}(T^3)$ and $\tilde{V} \in D(e^{\tau A}) : H^{r}(T^3)$ with $r > \frac{5}{2}$ and $\tau > 0$, and suppose that $\|e^{\tau A}V_0\|_{H^{r+1}} \leq M$. Then there exists a function
\[
\tau(t) = \tau_0 \exp(- \int_0^t K(s) ds),
\] (5.9)
such that for any given time $T > 0$, there exists a unique solution $\tilde{V} \in L^\infty(0, T; S \cap D(e^{\tau A}) : H^{r+1}(T^3))$ and $\tilde{V} \in L^\infty(0, T; S \cap D(e^{\tau A}) : H^{r}(T^3))$ of system (5.3)–(5.7) on $[0, T]$ such that
\[
\|e^{\tau A}V(t)\|_{H^{r+1}} \leq K(t), \quad \|e^{\tau A}\tilde{V}(t)\|_{H^{r}} \leq \|e^{\tau A}\tilde{V}_0\|_{H^{r}} e^{K(t)}.
\]

Proof. We will use the notation $K_1, K_2, \ldots$ as indicated in Remark 5. The global well-posedness of the 2D Euler equations in Sobolev spaces and corresponding growth estimate is classical, see [9]. From (4.13), we obtain that $\|\tilde{V}\|_{H^{r+1}} \leq K_1(t)$ for some function $K_1(t)$. For the growth of $\|\tilde{V}\|_{H^{r}}$, by standard energy estimate, since $\nabla \cdot V = 0$ and $r > \frac{5}{2}$, we have
\[
\frac{d}{dt} \|\tilde{V}\|_{H^{r}} \leq C_r \|\tilde{V}\|_{H^{r+1}} \|\tilde{V}\|_{H^{r}}^2.
\]
By the Grönwall inequality, and by virtue of the growth of $\|\tilde{V}\|_{H^{r+1}}$, we obtain that
\[
\|\tilde{V}(t)\|_{H^{r}} \leq \|\tilde{V}_0\|_{H^{r}} \exp\left(\frac{1}{2} C_r \int_0^t K_1(s) ds \right) \leq \|\tilde{V}_0\|_{H^{r}} e^{K(t)}
\]
for some suitable function $K(t)$, such that $\|\tilde{V}\|_{H^{r+1}} \leq K(t)$ also holds. By virtue of these formal energy estimates, the global well-posedness of system (3.3)–(3.7) in Sobolev spaces follows.

The global well-posedness of the 2D Euler equations in the space of analytic functions and the corresponding growth estimate are established in Proposition 4.2. From Proposition 4.2, we can first choose some suitable functions $K_1(t)$ and $K_2(t)$ such that $\tau(t) \leq \tau_0 \exp(- \int_0^t K_1(s) ds)$ and $\|e^{\tau A}V(t)\|_{H^{r+1}} \leq K_2(t)$. For the baroclinic mode $\tilde{V}$, first, it is easy to see the $L^2$ energy is conserved. Next, using Lemma 2.1 and Lemma 4.1, since $r > \frac{5}{2}$ and $\int_T^T \tilde{V}(x) dx = 0$, we have
\[
\frac{1}{2} \frac{d}{dt} \|A r e^{\tau A} \tilde{V}\|^2 = \tau \|A r \tilde{V}\|^2 - \left(\tilde{V}^t A r \tilde{V} \cdot \tilde{V}, A r \tilde{V}\right) = \frac{1}{2} \|A r e^{\tau A} \tilde{V}\|^2 + C_r \|\tilde{V}\|_{H^{r+1}} A r e^{\tau A} \tilde{V}\|^2.
\]
For suitable $K_1(t)$ and $K_2(t)$, we have $\tau + C_r \tau \|A r e^{\tau A} \tilde{V}\| \leq \tau (-K_1 + C_r K_2) \leq 0$. Therefore, by the Grönwall inequality, for some suitable function $K(t)$, we have
\[
\|A r e^{\tau A} \tilde{V}(t)\|^2 \leq \|A r e^{\tau A} \tilde{V}_0\|^2 \exp\left(\int_0^t C_r \|e^{\tau(s)} A \tilde{V}\|_{H^{r+1}} ds \right) \leq \|e^{\tau A} \tilde{V}_0\|_{H^{r+1}}^2 e^{K(t)}.
\]
Since $L^2$ energy is conserved, we have
\[
\|e^{\tau A} \tilde{V}(t)\|_{H^{r}} \leq \|e^{\tau A} \tilde{V}_0\|_{H^{r}} e^{K(t)}.
\]
We can choose $K(t)$ large enough such that $\tau(t) = \tau_0 \exp(- \int_0^t K(s) ds)$ and $\|e^{\tau A} \tilde{V}\|_{H^{r+1}} \leq K(t)$. Notice that $\tau(T) > 0$ for any finite time $T < \infty$. Therefore, the solution $(\tilde{V}, \tilde{V})$ exists in the space of analytic functions globally in time. \qed
Remark 6. The use of $K(t)$ above still follows Remark 5. The conclusion is that the growth of $\|\nabla (t)\|_{H^{r+1}}$ and $\|e^{\tau(t)}A(t)\|_{H^{r+1}}$ are double exponential in time, while the growth of $\|\nabla (t)\|_{H^r}$ and $\|e^{\tau(t)}A(t)\|_{H^r}$ are triple exponential in time.

Remark 7. Suppose $(\nabla, \nabla)$ is a solution of system (5.5)–(5.7). In the special case when $\nabla$ is uniformly bounded in time, i.e., $\|\nabla (t)\|_{H^{r+1}} \leq C_{M,r}$ and $\|e^{\tau(t)}A(t)\|_{H^{r+1}} \leq C_{M,r}$ for $t \in [0, \infty)$, it is easy to see that the growths of $\|\nabla (t)\|_{H^r}$ and $\|e^{\tau(t)}A(t)\|_{H^r}$ become only exponential in time. Moreover, $(0, \nabla_0)$ is always a steady state.

Remark 8. Since $U_{\pm} = \frac{1}{2}(\nabla + i\nabla)$, similar as Lemma 2.8 for $r \geq 0$ and $\tau \geq 0$, we have $\|U_{\pm}\|_r^2 = \|U_{\pm}\|_r^2 = \|e^{\tau(t)}U_{\pm}\|_r^2 = \frac{1}{2}\|e^{\tau(t)}V\|_r^2$. Therefore, the growing bounds of $\|\nabla (t)\|_{H^r}$ and $\|e^{\tau(t)}A(t)\|_{H^r}$ also apply to $\|U_{\pm}(t)\|_{H^r}$ and $\|e^{\tau(t)}A(t)\|_{H^r}$.

6. Effect of Rotation

In section 4, we see that by requiring $\|e^{\tau(t)}A\|_{H^r} \leq \epsilon$, the life-span of the solution to system (2.3)–(2.6) has a lower bound $T \geq \ln(\ln(\ln(\ln \frac{1}{\epsilon})))$, as $\epsilon \to 0^+$, and this result is uniform in $\Omega \in \mathbb{R}$. In this section, we establish the effect of the rate of rotation $|\Omega|$ on the life-span $T$. With the help of fast rotation, i.e., when $|\Omega|$ is large, we show that the time of existence of the solution in the space of analytic functions can be prolonged as long as the Sobolev norm $\|v_0\|_{H^r}$ is small depending on $\Omega$, while the analytic norm $\|e^{\tau(t)}v\|_{H^r}$ can be large (of order 1). We call such initial data as “well-prepared” initial data. The following theorem is the main result of this paper.

Theorem 6.1. Assume $\tau_0 \in \mathcal{S} \cap \mathcal{D}(e^{\tau(t)} : H^{r+3}(\mathbb{T}^3))$, $v_0 \in \mathcal{S} \cap \mathcal{D}(e^{\tau(t)} : H^{r+2}(\mathbb{T}^3))$ with $r > \frac{3}{2}$ and $\tau_0 > 0$. Let $M > 0$ and $\delta > 0$, then there exist constants $C_{\tau_0} > 1$, $C_M, \tau_0 > 1$, $C_r > 1$, $\bar{C}_M, \tau_0 > 1$, $\bar{C}_r > 1$, and functions $K(t) := e^{C_{M,\tau_0}}$, $K_0(t) := e^{C_{M,\tau_0}}$, with $K(t) > K_0(t)$. Suppose that $|\Omega| \geq C_{\tau_0}e^{K(t)}$, and that $\|e^{\tau(t)}v\|_{H^{r+3}} + \|e^{\tau(t)}v_0\|_{H^{r+2}} \leq M$ with $\|v_0\|_{H^{r+3}} \leq \frac{1}{M}$. Then there exists a time $T = T(\tau_0, |\Omega|, M, r) \geq 1$ satisfying

$$C_{\tau_0}e^{K(T)} = |\Omega|_0,$$

such that when $|\Omega| \geq |\Omega|_0$, the unique solution $(\tau, \nabla)$ to system (2.3)–(2.6) in Theorem 3.1 satisfies $(\tau, \nabla) \in L^\infty(0, T; \mathcal{S} \cap \mathcal{D}(e^{\tau(t)} : H^r(\mathbb{T}^3)))$, with

$$\tau(t) = \left(\tau_0 - \int_0^t \frac{K_0(s)}{|\Omega|_0} - e^{K_0(s)}ds - \int_0^t e^{K_0(s)}ds\right)e^{-\int_0^s K_0(s)ds} > 0.$$  

(6.2)

In particular, from (6.2), $T \geq \ln(\ln(\ln(\ln |\Omega|_0))) \to \infty$, as $|\Omega| \to \infty$.

Thanks to Lemma 2.2 and Lemma 2.6, we immediately have the following corollary.

Corollary 6.2. Suppose $v_0 \in \mathcal{S} \cap \mathcal{D}(e^{\tau(t)} : H^{r+3}(\mathbb{T}^3))$, and the conditions of Theorem 6.1 hold. Then the unique solution $v$ obtained in Corollary 6.2 satisfies $v \in L^\infty(0, T; \mathcal{S} \cap \mathcal{D}(e^{\tau(t)} : H^r(\mathbb{T}^3)))$, when $|\Omega| \geq |\Omega|_0$, with $T$ defined in (6.1) and $\tau$ defined in (6.2).
us the long time existence of the solution to system (2.10)–(2.12), and therefore the long time existence of the solution to system (2.5)–(2.6).

In section 6.1, we first give a rational behind the smallness of the initial baroclinic mode.

6.1. A rational behind the smallness of the initial baroclinic mode. The result of Theorem 6.1 is for “well-prepared” initial data, namely, for a given fixed \( \delta > 0 \), \( \| \tilde{v}_0 \|_{H^{3+\delta}} \leq \frac{1}{|\Omega_0|} \). Before we go into the proof of Theorem 6.1 we briefly rationalize, below, the reason behind this smallness condition on the baroclinic mode.

Consider the linear inviscid PEs:

\[
\begin{align*}
\partial_t V + \Omega V^\perp + \nabla p &= 0, \\
\partial_z p &= 0, \\
\nabla \cdot V + \partial_z w &= 0,
\end{align*}
\]

whose explicit solution is

\[
V(x, t) = \overline{V}_0(x') + \mathcal{R}(t)\tilde{V}_0(x),
\]

where

\[
\mathcal{R}(t) := \begin{pmatrix}
\cos(\Omega t) & \sin(\Omega t) \\
-\sin(\Omega t) & \cos(\Omega t)
\end{pmatrix}.
\]

We see there is no “decay” due to rotation in the linear level. This is different from the linearized 3D Euler equations with rotation, for which one can obtain certain decay due to dispersion/averaging mechanism, see, e.g., [24, 41].

Now let us look back to our nonlinear inviscid PEs (2.5)–(2.6). The first equation (2.5) is the evolution of the barotropic mode, which is the 2D Euler with source terms coming from the baroclinic mode. The second equation (2.6) is the evolution of the baroclinic mode, which is the Burger’s equations with rotation and other nonlinear coupling terms. For the Burger’s equations with rotation, it is shown in [3, 52] that when the rotation rate \( |\Omega| \) is large enough depending on the initial data, the solution exists globally in time because of the absence of resonance between the rotation and nonlinearity, which allows a very strong averaging mechanism that weakens the nonlinearity. In our case, however, the additional coupling nonlinear terms in (2.6) resonate with the rotation term, which does not allow for this simple scenario to take place. However, thanks to the smallness assumption on the initial baroclinic mode, the additional coupling nonlinear terms are initially small, which allows us to push this argument further.

Another reason behind this smallness assumption is indicated in [38], where a finite-time blowup of solutions to the inviscid PEs with rotation is established. Indeed, for the initial data

\[
v_0(x) = v_0(x, z) = \left( \lambda(-z^2 + \frac{1}{3})\sin x, -\Omega\sin x \right)
\]

with \( \lambda > 0 \), it is shown that \( \frac{\partial}{\partial x} \) is an upper bound for the blowup time. Notably here \( \overline{v}_0 = (0, -\Omega\sin x) \) and \( \tilde{v}_0 = (\lambda(-z^2 + \frac{1}{3})\sin x, 0) \). Therefore, when \( |\Omega| \gg 1 \), we have:

- when \( \lambda = |\Omega| \), the baroclinic mode satisfies \( \tilde{v}_0 \sim |\Omega| \), and the whole initial data satisfies \( v_0 \sim |\Omega| \).
- An upper bound of blowup time in this case satisfies \( T \sim \frac{1}{|\Omega|} \);
- when \( \lambda = 1 \), the baroclinic mode satisfies \( \tilde{v}_0 \sim 1 \), while the whole initial data satisfies \( v_0 \sim |\Omega| \).
- An upper bound of blowup time in this case satisfies \( T \sim 1 \);
- when \( \lambda = \frac{1}{|\Omega|} \), this implies a smallness condition on the baroclinic \( \tilde{v}_0 \sim \frac{1}{|\Omega|} \), while the whole initial data satisfies \( v_0 \sim |\Omega| \). An upper bound of blowup time in this case satisfies \( T \sim |\Omega| \).
The above, in particular, the last item suggest that the smallness condition on the baroclinic mode is required to guarantee the long time existence of solutions to the 3D inviscid PEs with fast rotation.

Further reasoning for the smallness condition on the initial baroclinic mode will be provided in Remark 11 and Remark 12 below.

6.2. The perturbed system around $|\Omega| = \infty$. Since the limit resonant system (5.2)–(5.3) is globally well-posed, the idea to show long time existence of the solution is to consider the difference between the original system (2.10)–(2.12) and the limit resonant system (5.2)–(5.3). Denote by $\overline{\phi} = \overline{\tau} - \overline{\nabla}$, and $\phi_\pm = u_\pm - U_\pm$. Taking the difference between system (2.10)–(2.12) and system (5.2)–(5.3), we obtain

$$
\partial_t \overline{\phi} + P_h \left[ \overline{\phi} \cdot \nabla \overline{\tau} + \overline{\phi} \cdot \nabla \overline{\nabla} + \overline{\nabla} \cdot \nabla \phi + e^{2i\mu t} P_0 (Q_{1,+,+} + Q_{2,++,+}) + e^{-2i\mu t} P_0 (Q_{1,--,+} + Q_{2,---,+}) \right] = 0,
$$

and

$$
\partial_t \phi_\pm + \overline{\phi} \cdot \nabla U_\pm + \overline{\phi} \cdot \nabla \phi_\pm + \overline{\nabla} \cdot \nabla \phi_\pm + \frac{1}{2} (\phi_\pm \cdot \nabla) (\overline{\tau} \pm i\overline{\nabla}) + \overline{\nabla} \cdot \nabla \phi_\pm + \frac{1}{2} (\phi_\pm \cdot \nabla)(\overline{\phi} \pm i\overline{\phi}^\perp)
$$

$$
+ \frac{1}{2} (U_\pm \cdot \nabla) (\overline{\phi} \pm i\overline{\phi}^\perp) + e^{\mp i\Omega t} \left( Q_{1,+,--} + Q_{2,++--} - P_0 Q_{2,++} - Q_{3,+++} \right)
$$

$$
+ e^{\mp 2i\Omega t} Q_{4,+,+} = 0,
$$

(6.3)

where

$$
Q_{1,+,--} = \phi_\pm \cdot \nabla U_\pm + \phi_\pm \cdot \nabla \phi_\pm + U_\pm \cdot \nabla \phi_\pm + U_\pm \cdot \nabla U_\pm,
$$

$$
Q_{2,++--} = (\nabla \cdot \phi_\pm) U_\pm + (\nabla \cdot \phi_\pm) \phi_\mp + (\nabla \cdot U_\pm) \phi_\mp + (\nabla \cdot U_\pm) U_\pm,
$$

$$
Q_{3,+++} = \left( \int_0^s \nabla \cdot \phi_\pm (x', s) ds \right) \partial_s U_\pm + \left( \int_0^s \nabla \cdot \phi_\pm (x', s) ds \right) \partial_s \phi_\mp
$$

$$
+ \left( \int_0^s \nabla \cdot U_\pm (x', s) ds \right) \partial_s \phi_\mp + \left( \int_0^s \nabla \cdot U_\pm (x', s) ds \right) \partial_s U_\pm,
$$

$$
Q_{4,+,+} = \frac{1}{2} \left[ (\phi_\pm \cdot \nabla) (\overline{\tau} \mp e^{i\nabla}) + (\phi_\pm \cdot \nabla)(\overline{\phi} \mp e^{i\phi^\perp}) \right]
$$

$$
+(U_\pm \cdot \nabla)(\overline{\phi} \mp e^{i\phi^\perp}) + (U_\pm \cdot \nabla)(\overline{\tau} \mp e^{i\nabla} \overline{\phi}^\perp).
$$

(6.4)

We supplement the initial conditions for the limit resonant system (5.2)–(5.3) as

$$
\overline{\tau}_0 = \overline{\tau}_0, \quad (U_\pm)_0 = (u_\pm)_0 = \frac{1}{2} (\overline{v}_0 \pm i\overline{v}_0^\perp).
$$

(6.5)

Therefore, the initial conditions for the perturbed system is

$$
\overline{\phi}_0 = 0, \quad (\phi_\pm)_0 = 0.
$$

(6.6)

6.3. Proof of Theorem 6.1. In this subsection, we prove Theorem 6.1. From Proposition 5.1 let $\overline{\nabla}$ and $U_\pm$ be the global solution in $S \cap D(e^{\tau(i)A}: H^{r+3}(T^3))$ and $S \cap D(e^{\tau(i)A}: H^{r+2}(T^3))$, respectively, to system (5.2)–(5.3), with initial data (6.3) and $\tau(i)$ defined by (5.9). Applying $A^* e^{\tau A} \phi$ to (6.3)–(6.4), and taking the $L^2$ inner product of (6.3) with $A^* e^{\tau A} \phi$, (6.4) with $2A^* e^{\tau A} \phi_\mp$, thanks to Lemma 2.4 and Lemma 2.8, we obtain

$$
\frac{1}{2} \frac{d}{dt} \|A^* e^{\tau A} \phi\|^2 = \frac{d}{dt} \|A^* e^{\tau A} \phi\|^2 - \left( A^* e^{\tau A} (\overline{\phi} \cdot \nabla \overline{\nabla}), A^* e^{\tau A} \phi \right) - \left( A^* e^{\tau A} (\overline{\phi} \cdot \nabla \overline{\phi}), A^* e^{\tau A} \phi \right)
$$

$$
- \left( A^* e^{\tau A} (\overline{\tau} \cdot \overline{\phi}), A^* e^{\tau A} \phi \right) - e^{2i\mu t} \left( A^* e^{\tau A} (Q_{1,+,+,+} + Q_{2,++,+}), A^* e^{\tau A} \phi \right).
$$

(6.7)
\[-e^{-2\Omega t}\left\langle \phi^+ e^{-\tau A}(Q_{1,+-} + Q_{2,-}), \phi^{-} e^{-\tau A}(\overline{\phi}) \right\rangle, \tag{6.7}\]

and

\[
\frac{d}{dt}(\|\phi^+ e^{-\tau A}(\overline{\phi})\|^2 + \|\phi^{-} e^{-\tau A}(\overline{\phi})\|^2) = 2\Omega(\|\phi^+ e^{-\tau A}(\overline{\phi})\|^2 + \|\phi^{-} e^{-\tau A}(\overline{\phi})\|^2) \\
- 2\left\langle \phi^+ e^{-\tau A}(\overline{\phi}) \cdot \nabla U^+, \phi^{-} e^{-\tau A}(\overline{\phi}) \right\rangle - 2\left\langle \phi^+ e^{-\tau A}(\overline{\phi}) \cdot \nabla \phi^-, \phi^{-} e^{-\tau A}(\overline{\phi}) \right\rangle \\
- 2\left\langle \phi^+ e^{-\tau A}(\overline{\phi}) \cdot \nabla \phi^-, \phi^{-} e^{-\tau A}(\overline{\phi}) \right\rangle - 2\left\langle \phi^+ e^{-\tau A}(\overline{\phi}) \cdot \nabla \phi^-, \phi^{-} e^{-\tau A}(\overline{\phi}) \right\rangle \\
- \left\langle \phi^+ e^{-\tau A}(\overline{\phi}) \cdot \nabla (\overline{\phi} + i\overline{\phi}), \phi^{-} e^{-\tau A}(\overline{\phi}) \right\rangle - \left\langle \phi^+ e^{-\tau A}(\overline{\phi}) \cdot \nabla (\overline{\phi} - i\overline{\phi}), \phi^{-} e^{-\tau A}(\overline{\phi}) \right\rangle \\
- \left\langle \phi^+ e^{-\tau A}(\overline{\phi}) \cdot \nabla (\overline{\phi} + i\overline{\phi}), \phi^{-} e^{-\tau A}(\overline{\phi}) \right\rangle - \left\langle \phi^+ e^{-\tau A}(\overline{\phi}) \cdot \nabla (\overline{\phi} - i\overline{\phi}), \phi^{-} e^{-\tau A}(\overline{\phi}) \right\rangle \\
- 2\left\langle \phi^+ e^{-\tau A}(Q_{1,+-} + Q_{3,++}), \phi^{-} e^{-\tau A}(\overline{\phi}) \right\rangle + \left\langle \phi^+ e^{-\tau A}(Q_{1,+-} + Q_{3,++}), \phi^{-} e^{-\tau A}(\overline{\phi}) \right\rangle \\
- 2\left\langle \phi^+ e^{-\tau A}(Q_{1,--} + Q_{3,--}), \phi^{-} e^{-\tau A}(\overline{\phi}) \right\rangle + \left\langle \phi^+ e^{-\tau A}(Q_{1,--} + Q_{3,--}), \phi^{-} e^{-\tau A}(\overline{\phi}) \right\rangle \\n- 2\left\langle \phi^+ e^{-\tau A}(Q_{1,--} + Q_{3,--}), \phi^{-} e^{-\tau A}(\overline{\phi}) \right\rangle - 2\left\langle \phi^+ e^{-\tau A}(Q_{1,--} + Q_{3,--}), \phi^{-} e^{-\tau A}(\overline{\phi}) \right\rangle. \tag{6.8}\]

There are totally 71 different nonlinear terms in (6.7) and (6.8). We separate them into the following four different types. We use $V$ to denote the velocity field of the limit resonant system, i.e., $\overline{V}$ and $U_{\pm}$, and use $\phi$ to denote the velocity filed of the perturbed system, i.e., $\overline{\phi}$ and $\phi_{\pm}$.

- Type 1: terms that are trilinear in $\phi$, e.g., $\left\langle \phi^+ e^{-\tau A}(\overline{\phi} \cdot \nabla \overline{\phi}), \phi^{-} e^{-\tau A}(\overline{\phi}) \right\rangle$.
- Type 2: terms that are bilinear in $\phi$ with no derivative of $\phi$, e.g., $\left\langle \phi^+ e^{-\tau A}(\overline{\phi} \cdot \overline{\nabla}), \phi^{-} e^{-\tau A}(\overline{\phi}) \right\rangle$.
- Type 3: terms that are linear in $\phi$, e.g., $e^{2\Omega t}\left\langle \phi^+ e^{-\tau A}(U_{+} \cdot \overline{\nabla} U_{+}), \phi^{-} e^{-\tau A}(\overline{\phi}) \right\rangle$.
- Type 4: terms that are bilinear in $\phi$ and a derivative of $\phi$, e.g., $\left\langle \phi^+ e^{-\tau A}(\overline{\nabla} \phi), \phi^{-} e^{-\tau A}(\overline{\phi}) \right\rangle$.

### 6.3.1. Estimates of Type 1 and Type 2 terms.

For type 1 nonlinear terms (19 terms), using Lemma $A.1$ and for type 2 nonlinear terms (15 terms), using Lemma $2.1$, since $\overline{\phi}$, $\phi_{\pm}$, $\overline{V}$ and $U_{\pm}$ all have zero mean value in $T_{-1}$, we have

\[
\begin{align*}
&\left| \left\langle \overline{\phi}^+ e^{-\tau A}(\nabla \overline{V}), \phi^{-} e^{-\tau A}(\overline{\phi}) \right\rangle \right| + \left| \left\langle \overline{\phi}^+ e^{-\tau A}(\nabla \overline{\phi}), \phi^{-} e^{-\tau A}(\overline{\phi}) \right\rangle \right| \\
&+ e^{2\Omega t}\left| \left\langle \phi^+ e^{-\tau A}(\overline{\phi} \cdot \nabla U_{+} + \phi_{+} \cdot \nabla \phi_{+} + (\nabla \cdot U_{+})\phi_{+} + (\nabla \cdot \phi_{+})\phi_{+} \right\rangle, \phi^{-} e^{-\tau A}(\overline{\phi}) \right\rangle \right| \\
&+ e^{-2\Omega t}\left| \left\langle \phi^+ e^{-\tau A}(\overline{\phi} \cdot \nabla U_{-} + \phi_{-} \cdot \nabla \phi_{-} + (\nabla \cdot U_{-})\phi_{-} + (\nabla \cdot \phi_{-})\phi_{-} \right\rangle, \phi^{-} e^{-\tau A}(\overline{\phi}) \right\rangle \right| \\
&+ 2\left| \left\langle \phi^+ e^{-\tau A}(\overline{\phi} \cdot \nabla U_{+}), \phi^{-} e^{-\tau A}(\overline{\phi}) \right\rangle \right| + 2\left| \left\langle \phi^+ e^{-\tau A}(\overline{\phi} \cdot \nabla U_{-}), \phi^{-} e^{-\tau A}(\overline{\phi}) \right\rangle \right| \\
&+ 2\left| \left\langle \phi^+ e^{-\tau A}(\overline{\phi} \cdot \nabla \phi_{+}), \phi^{-} e^{-\tau A}(\overline{\phi}) \right\rangle \right| + 2\left| \left\langle \phi^+ e^{-\tau A}(\overline{\phi} \cdot \nabla \phi_{-}), \phi^{-} e^{-\tau A}(\overline{\phi}) \right\rangle \right| \\
&+ \left| \left\langle \phi^+ e^{-\tau A}(\overline{\phi} \cdot \nabla (\overline{\phi} + i\overline{\phi})), \phi^{-} e^{-\tau A}(\overline{\phi}) \right\rangle \right| + \left| \left\langle \phi^+ e^{-\tau A}(\overline{\phi} \cdot \nabla (\overline{\phi} - i\overline{\phi})), \phi^{-} e^{-\tau A}(\overline{\phi}) \right\rangle \right| \\
&+ \left| \left\langle \phi^+ e^{-\tau A}(\overline{\phi} \cdot \nabla \phi_{+} + \overline{\phi} \cdot \nabla \phi_{+}), \phi^{-} e^{-\tau A}(\overline{\phi}) \right\rangle \right| + \left| \left\langle \phi^+ e^{-\tau A}(\overline{\phi} \cdot \nabla \phi_{-} - \overline{\phi} \cdot \nabla \phi_{+}), \phi^{-} e^{-\tau A}(\overline{\phi}) \right\rangle \right| \\
&+ \left| \left\langle \phi^+ e^{-\tau A}(\overline{\phi} \cdot \nabla \phi_{+} + i\overline{\phi} \cdot \nabla \phi_{+}), \phi^{-} e^{-\tau A}(\overline{\phi}) \right\rangle \right| + \left| \left\langle \phi^+ e^{-\tau A}(\overline{\phi} \cdot \nabla \phi_{-} - i\overline{\phi} \cdot \nabla \phi_{+}), \phi^{-} e^{-\tau A}(\overline{\phi}) \right\rangle \right|.
\end{align*}\]
For type 3 nonlinear terms (14 terms), when \( \Omega \neq 0 \), we first explain the idea on the sample term \( e^{2i\Omega t} \left\langle A^r e^{\tau A} (U_+ \cdot \nabla U_+), A^r e^{\tau A} \bar{\phi} \right\rangle \). Indeed, by differentiation by parts, we have

\[
e^{2i\Omega t} \left\langle A^r e^{\tau A} (U_+ \cdot \nabla U_+), A^r e^{\tau A} \bar{\phi} \right\rangle,
\]

\[
e^{2i\Omega t} \left\langle A^r e^{\tau A} (U_+ \cdot \nabla U_+), A^r e^{\tau A} \bar{\phi} \right\rangle = \frac{1}{2i\Omega} \partial_t \left( e^{2i\Omega t} \left\langle A^r e^{\tau A} (U_+ \cdot \nabla U_+), A^r e^{\tau A} \bar{\phi} \right\rangle \right) - \frac{1}{2i\Omega} e^{2i\Omega t} \partial_t \left( \left\langle A^r e^{\tau A} (U_+ \cdot \nabla U_+), A^r e^{\tau A} \bar{\phi} \right\rangle \right).
\]

We leave the first term until integrating in time. For the second term, we have

\[
- \frac{1}{2i\Omega} e^{2i\Omega t} \partial_t \left( \left\langle A^r e^{\tau A} (U_+ \cdot \nabla U_+), A^r e^{\tau A} \bar{\phi} \right\rangle \right)
\]

\[
\leq \frac{1}{|I|} \left| \left\langle A^{r+1} e^{\tau A} (U_+ \cdot \nabla U_+), A^r e^{\tau A} \bar{\phi} \right\rangle \right| + \frac{1}{2|I|} \left| \left\langle A^r e^{\tau A} \partial_t (U_+ \cdot \nabla U_+), A^r e^{\tau A} \bar{\phi} \right\rangle \right|
\]

\[
+ \frac{1}{2|I|} \left| \left\langle A^r e^{\tau A} (U_+ \cdot \nabla U_+), A^r e^{\tau A} \partial_t \bar{\phi} \right\rangle \right| := I_1 + I_2 + I_3.
\]

Thanks to the Cauchy–Schwarz inequality, Lemma 2.1 and Lemma 2.2, since \( \bar{\phi}, \phi, V, U_\pm \) all have zero mean value in \( \mathbb{T}^3 \), and since \( r > \frac{1}{2} \), from (5.2) and (6.8), we have

\[
I_1 \leq \frac{C_r}{|I|^2} \left| \partial_t \left| A^{r+1} e^{\tau A} U_+ \right| \right|^2 + C_r \left| A^{r+2} e^{\tau A} U_+ \right|^2 \left| A^r e^{\tau A} \bar{\phi} \right|^2,
\]

\[
I_2 \leq \frac{C}{|I|} \left| \left\langle A^r e^{\tau A} \left\{ \left( \nabla \cdot U_+ + \frac{1}{2} (U_+ \cdot \nabla) (V + i \nabla^2) \right) \cdot \nabla U_+ \right\}, A^r e^{\tau A} \bar{\phi} \right\rangle \right|
\]

\[
+ \left| \left\langle A^r e^{\tau A} \left\{ U_+ \cdot \nabla \left( \nabla \cdot U_+ + \frac{1}{2} (U_+ \cdot \nabla) (V + i \nabla^2) \right) \right\}, A^r e^{\tau A} \bar{\phi} \right\rangle \right|
\]

\[
\leq \frac{C_r}{|I|^2} \left| A^{r+2} e^{\tau A} U_+ \right|^2 \left| A^{r+2} e^{\tau A} V \right| \left| A^r e^{\tau A} \bar{\phi} \right|^2,
\]

\[
I_3 \leq \frac{C_r}{|I|^2} \left| A^{r+2} e^{\tau A} U_+ \right|^2 \left| A^{r+2} e^{\tau A} \nabla \right| \left| A^r e^{\tau A} \bar{\phi} \right|^2 + \frac{C_r}{|I|^2} \left| A^{r+2} e^{\tau A} U_+ \right|^2,
\]

and
Applying differentiation by parts to all the type 3 nonlinear terms (14 terms), one obtains

\[
I_3 \leq \frac{C}{\| \Omega \|}\left| \left\langle A^t e^{rA} P_h (U_+ \cdot \nabla U_+), A^t e^{rA} \left\{ \phi \cdot \nabla \nabla + \bar{\phi} \cdot \nabla \bar{\phi} + \nabla \cdot \nabla \bar{\phi} \right\} \right\rangle + e^{2i\Omega t} P_0 \left( Q_{1,+} + Q_{2,+} \right) + e^{-2i\Omega t} P_0 \left( Q_{1,-} + Q_{2,-} \right) \right| \\
\leq \frac{C}{\| \Omega \|}\left| \left\langle A^{t+1} e^{rA} P_h (U_+ \cdot \nabla U_+), A^{t-1} e^{rA} \left\{ \phi \cdot \nabla \nabla + \bar{\phi} \cdot \nabla \bar{\phi} + \nabla \cdot \nabla \bar{\phi} \right\} \right\rangle + e^{2i\Omega t} P_0 \left( Q_{1,+} + Q_{2,+} \right) + e^{-2i\Omega t} P_0 \left( Q_{1,-} + Q_{2,-} \right) \right| \\
\leq \frac{C_r}{\| \Omega \|} \left\| A^{t+2} e^{rA} U_+ \right\|^2 + \left\| A^r e^{rA} \nabla \right\|^2 + \left\| A^r e^{rA} U_+ \right\|^2 + \left\| A^r e^{rA} U_- \right\|^2 + \left\| A^r e^{rA} \phi_+ \right\|^2 + \left\| A^r e^{rA} \phi_- \right\|^2.
\]

Applying differentiation by parts to all the type 3 nonlinear terms (14 terms), one obtains

\[
-e^{2i\Omega t} \left[ \left\langle A^t e^{rA} (U_+ \cdot \nabla U_+), A^t e^{rA} \phi \right\rangle + \left\langle A^t e^{rA} \left( (\nabla \cdot U_+) U_+ \right), A^t e^{rA} \phi \right\rangle \right.
\]

\[
+ \left. \left\langle A^t e^{rA} \left( U_+ \cdot \nabla \right) (\nabla - i\nabla^\perp), A^t e^{rA} \phi_+ \right\rangle \right]
\]

\[
-e^{-2i\Omega t} \left[ \left\langle A^t e^{rA} (U_- \cdot \nabla U_-), A^t e^{rA} \phi \right\rangle + \left\langle A^t e^{rA} \left( (\nabla \cdot U_-) U_- \right), A^t e^{rA} \phi \right\rangle \right.
\]

\[
+ \left. \left\langle A^t e^{rA} \left( U_- \cdot \nabla \right) (\nabla + i\nabla^\perp), A^t e^{rA} \phi_- \right\rangle \right]
\]

\[
-2e^{i\Omega t} \left[ \left\langle A^t e^{rA} (U_+ \cdot \nabla U_+), A^t e^{rA} \phi \right\rangle - \left\langle A^{t-1} e^{rA} \left( \int_0^z \nabla \cdot U_+ (x', s) ds \right) \partial_z U_+, A^t e^{rA} \phi_- \right\rangle \right.
\]

\[
+ \left. \left\langle A^t e^{rA} (U_+ \cdot \nabla U_-), A^{t-1} e^{rA} \phi_+ \right\rangle - \left\langle A^{t+1} e^{rA} \left( \int_0^z \nabla \cdot U_+ (x', s) ds \right) \partial_z U_-, A^t e^{rA} \phi_+ \right\rangle \right. \]

\[
-2e^{-i\Omega t} \left[ \left\langle A^t e^{rA} (U_- \cdot \nabla U_+), A^t e^{rA} \phi \right\rangle - \left\langle A^{t-1} e^{rA} \left( \int_0^z \nabla \cdot U_- (x', s) ds \right) \partial_z U_+, A^t e^{rA} \phi_- \right\rangle \right.
\]

\[
+ \left. \left\langle A^t e^{rA} (U_- \cdot \nabla U_-), A^{t-1} e^{rA} \phi_+ \right\rangle - \left\langle A^{t+1} e^{rA} \left( \int_0^z \nabla \cdot U_- (x', s) ds \right) \partial_z U_-, A^t e^{rA} \phi_+ \right\rangle \right. \]

\[
= \frac{1}{2i\Omega} \partial_t \left[ e^{2i\Omega t} \left\langle A^t e^{rA} (U_+ \cdot \nabla U_+), A^t e^{rA} \phi \right\rangle + \left\langle A^t e^{rA} \left( \nabla \cdot U_+ U_+ \right), A^t e^{rA} \phi \right\rangle \right.
\]

\[
+ \left. \left\langle A^t e^{rA} \left( U_+ \cdot \nabla \right) (\nabla - i\nabla^\perp), A^t e^{rA} \phi_+ \right\rangle \right]
\]

\[
+ \frac{1}{2i\Omega} \partial_t \left[ e^{-2i\Omega t} \left\langle A^t e^{rA} (U_- \cdot \nabla U_-), A^t e^{rA} \phi \right\rangle + \left\langle A^t e^{rA} \left( \nabla \cdot U_- U_- \right), A^t e^{rA} \phi \right\rangle \right.
\]

\[
+ \left. \left\langle A^t e^{rA} \left( U_- \cdot \nabla \right) (\nabla + i\nabla^\perp), A^t e^{rA} \phi_- \right\rangle \right]
\]

\[
- \frac{2}{i\Omega} \partial_t \left[ e^{i\Omega t} \left\langle A^t e^{rA} (U_+ \cdot \nabla U_+), A^t e^{rA} \phi \right\rangle - \left\langle A^{t-1} e^{rA} \left( \int_0^z \nabla \cdot U_+ (x', s) ds \right) \partial_z U_+, A^t e^{rA} \phi_- \right\rangle \right.
\]

\[
+ \left. \left\langle A^t e^{rA} (U_+ \cdot \nabla U_-), A^{t-1} e^{rA} \phi_+ \right\rangle - \left\langle A^{t+1} e^{rA} \left( \int_0^z \nabla \cdot U_+ (x', s) ds \right) \partial_z U_-, A^t e^{rA} \phi_+ \right. \right. \]

\[
+ \frac{2}{i\Omega} \partial_t \left[ e^{-i\Omega t} \left\langle A^t e^{rA} (U_- \cdot \nabla U_+), A^t e^{rA} \phi \right\rangle - \left\langle A^{t-1} e^{rA} \left( \int_0^z \nabla \cdot U_- (x', s) ds \right) \partial_z U_+, A^t e^{rA} \phi_- \right\rangle \right.
\]

\[
+ \left. \left. \left\langle A^t e^{rA} (U_- \cdot \nabla U_-), A^{t-1} e^{rA} \phi_+ \right\rangle - \left\langle A^{t+1} e^{rA} \left( \int_0^z \nabla \cdot U_- (x', s) ds \right) \partial_z U_-, A^t e^{rA} \phi_+ \right. \right. \right. \right. \]
where $R$ corresponds the remaining terms. Using the similar estimates as \((6.10)\), thanks to Young’s inequality, when $|\Omega| > 1$, we have

$$
|R| \leq C_r \left( \|A^{r+2}e^{\tau A}\nabla\|^4 + \|A^{r+2}e^{\tau A}U_+\|^4 + \|A^{r+2}e^{\tau A}U_-\|^4 + 1 \right)
$$

$$
\times \left( \frac{1}{2} \|A^r e^{\tau A}\phi\|^2 + \|A^r e^{\tau A}\phi\|^2 \right)
$$

$$
+ \frac{C_r}{|\Omega|} \left( |\tau|^2 + \|A^{r+2}e^{\tau A}\nabla\|^4 + \|A^{r+2}e^{\tau A}U_+\|^4 + \|A^{r+2}e^{\tau A}U_-\|^4 + 1 \right).
$$

(6.11)

For $\partial_s N$, since $\bar{\phi}(0) = \phi_+(0) = \phi_-(0) = 0$, using Lemma 2.1 since $\nabla$ and $U_{\pm}$ have zero mean value in $T^3$, by Young’s inequality, we have

$$
\left| \int_0^t \partial_s N(s) ds \right| = |N(t)| \leq \frac{C_r}{|\Omega|} \left( \|A^{r+1}e^{\tau A}\nabla\|^2 + \|A^{r+1}e^{\tau A}U_+\|^2 + \|A^{r+1}e^{\tau A}U_-\|^2 \right)
$$

$$
\times \left( \|A^r e^{\tau A}\phi\| + \|A^r e^{\tau A}\phi_+\| + \|A^r e^{\tau A}\phi_-\| \right).
$$

(6.12)

6.3.3. Estimates of Type 4 terms. The difficulties are on the estimate of type 4 nonlinear terms (23 terms). Thanks to Lemma 4.4 since $\nabla \cdot \nabla = 0$, we have

$$
\left| \left\langle A^r e^{\tau A}(\nabla \cdot \nabla \phi), A^r e^{\tau A}\phi \right\rangle \right| \leq C_r \|A^r e^{\tau A}\nabla\| \|A^r e^{\tau A}\phi\|^2 + C_r \tau \|A^{r+rac{1}{2}} e^{\tau A}\nabla\| \|A^{r+rac{1}{2}} e^{\tau A}\phi\|^2.
$$

(6.13)

Thanks to Lemma A.4 by integration by parts, we have

$$
\left| \left\langle A^r e^{\tau A}(\nabla \cdot \nabla \phi_+), A^r e^{\tau A}\phi_+ \right\rangle \right| + \left| \left\langle A^r e^{\tau A}(\nabla \cdot \nabla \phi_-), A^r e^{\tau A}\phi_- \right\rangle \right|
$$

$$
\leq \left| \left\langle A^r e^{\tau A}(\nabla \cdot \nabla \phi_+), A^r e^{\tau A}\phi_+ \right\rangle \right| - \left| \left\langle \nabla \cdot \nabla A^r e^{\tau A}\phi_+, A^r e^{\tau A}\phi_- \right\rangle \right|
$$

$$
+ \left| \left\langle A^r e^{\tau A}(\nabla \cdot \nabla \phi_-), A^r e^{\tau A}\phi_- \right\rangle \right| - \left| \left\langle \nabla \cdot \nabla A^r e^{\tau A}\phi_-, A^r e^{\tau A}\phi_+ \right\rangle \right|
$$

$$
+ \left| \left\langle \nabla \cdot \nabla A^r e^{\tau A}\phi_+, A^r e^{\tau A}\phi_+ \right\rangle \right| + \left| \left\langle \nabla \cdot \nabla A^r e^{\tau A}\phi_-, A^r e^{\tau A}\phi_- \right\rangle \right|
$$

$$
\leq C_r \|A^r e^{\tau A}\nabla\| (\|A^r e^{\tau A}\phi_+\|^2 + \|A^r e^{\tau A}\phi_-\|^2)
$$

$$
+ C_r \tau \|A^{r+rac{1}{2}} e^{\tau A}\nabla\| (\|A^{r+rac{1}{2}} e^{\tau A}\phi_+\|^2 + \|A^{r+rac{1}{2}} e^{\tau A}\phi_-\|^2),
$$

(6.14)

where we have used $\left| \left\langle \nabla \cdot \nabla A^r e^{\tau A}\phi_+, A^r e^{\tau A}\phi_- \right\rangle \right| + \left| \left\langle \nabla \cdot \nabla A^r e^{\tau A}\phi_-, A^r e^{\tau A}\phi_+ \right\rangle \right| = 0$ by integration by parts and $\nabla \cdot \nabla = 0$. Thanks to Lemma A.4 and Lemma A.6 since $r > \frac{\delta}{2}$, by integration by parts and by the Sobolev inequality, we have

$$
e^{i\Omega t} \left| \left\langle A^r e^{\tau A}(U_+ \cdot \nabla \phi_+), A^r e^{\tau A}\phi_+ \right\rangle \right| - e^{i\Omega t} \left| \left\langle A^r e^{\tau A}(U_+ \cdot \nabla \phi_-), A^r e^{\tau A}\phi_- \right\rangle \right|
$$

$$
- e^{i\Omega t} \left| \left\langle A^r e^{\tau A}\left( \int_0^z \nabla \cdot U_+(x', s) ds \right) \partial_z \phi_+, A^r e^{\tau A}\phi_+ \right\rangle \right|
$$

$$
- e^{i\Omega t} \left| \left\langle A^r e^{\tau A}\left( \int_0^z \nabla \cdot U_+(x', s) ds \right) \partial_z \phi_- \right\rangle \right| \left( A^r e^{\tau A}\phi_+ \right)
$$

$$
\leq \left| \left\langle A^r e^{\tau A}(U_+ \cdot \nabla \phi_+), A^r e^{\tau A}\phi_- \right\rangle \right| - \left| \left\langle U_+ \cdot \nabla A^r e^{\tau A}\phi_+, A^r e^{\tau A}\phi_- \right\rangle \right|
$$

$$
+ \left| \left\langle A^r e^{\tau A}(U_+ \cdot \nabla \phi_), A^r e^{\tau A}\phi_+ \right\rangle \right| - \left| \left\langle U_+ \cdot \nabla A^r e^{\tau A}\phi_-, A^r e^{\tau A}\phi_+ \right\rangle \right|
$$

$$
+ \left| \left\langle A^r e^{\tau A}\left( \int_0^z \nabla \cdot U_+(x', s) ds \right) \partial_z \phi_+, A^r e^{\tau A}\phi_- \right\rangle \right|
$$

where $R$ corresponds the remaining terms. Using the similar estimates as \((6.10)\), thanks to Young’s inequality, when $|\Omega| > 1$, we have
− \left\langle \left( \int_0^z \nabla \cdot U_+(x', s) ds \right) A' e^{rA} t_\phi_\pm, A' e^{rA} t_\phi_\mp \right\rangle

+ \left\langle A' e^{rA} \left( \int_0^z \nabla \cdot U_+(x', s) ds \right) \partial_z \phi_\pm, A' e^{rA} t_\phi_\mp \right\rangle

− \left\langle \left( \int_0^z \nabla \cdot U_+(x', s) ds \right) A' e^{rA} \partial_z \phi_\pm, A' e^{rA} t_\phi_\mp \right\rangle

+ \left\langle U_+ \cdot \nabla A' e^{rA} t_\phi_\pm, A' e^{rA} t_\phi_\mp \right\rangle + \left\langle U_+ \cdot \nabla A' e^{rA} t_\phi_\pm, A' e^{rA} t_\phi_\mp \right\rangle

− \left\langle \left( \int_0^z \nabla \cdot U_+(x', s) ds \right) A' e^{rA} \partial_z \phi_\pm, A' e^{rA} t_\phi_\mp \right\rangle

− \left\langle \left( \int_0^z \nabla \cdot U_+(x', s) ds \right) A' e^{rA} \partial_z \phi_\pm, A' e^{rA} t_\phi_\mp \right\rangle

\leq C_r \left\| A'^{-1} e^{rA} U_\mp \right\| \left\| \left( \sum_{\alpha=0}^m A' e^{rA} \phi_\mp \right)^2 \right\|

+ C_r \left\| A'^{-1} e^{rA} U_\mp \right\| \left\| \left( \sum_{\alpha=0}^m A' e^{rA} \phi_\mp \right)^2 \right\|

(6.15)

where we have used

\left\langle U_+ \cdot \nabla A' e^{rA} t_\phi_\pm, A' e^{rA} t_\phi_\mp \right\rangle + \left\langle U_+ \cdot \nabla A' e^{rA} t_\phi_\pm, A' e^{rA} t_\phi_\mp \right\rangle

− \left\langle \left( \int_0^z \nabla \cdot U_+(x', s) ds \right) A' e^{rA} \partial_z \phi_\pm, A' e^{rA} t_\phi_\mp \right\rangle

− \left\langle \left( \int_0^z \nabla \cdot U_+(x', s) ds \right) A' e^{rA} \partial_z \phi_\pm, A' e^{rA} t_\phi_\mp \right\rangle = 0

by integration by parts. Similarly, we have

\left\| e^{-\alpha t} A' e^{rA} (U_\mp \cdot \nabla \phi_\pm), A' e^{rA} t_\phi_\pm \right\| + e^{-\alpha t} \left\langle A' e^{rA} (U_\mp \cdot \nabla \phi_\pm), A' e^{rA} t_\phi_\pm \right\rangle

− e^{-\alpha t} \left\langle A' e^{rA} \left( \int_0^z \nabla \cdot U_\mp (x', s) ds \right) \partial_z \phi_\pm, A' e^{rA} t_\phi_\pm \right\rangle

− e^{-\alpha t} \left\langle A' e^{rA} \left( \int_0^z \nabla \cdot U_\mp (x', s) ds \right) \partial_z \phi_\pm, A' e^{rA} t_\phi_\pm \right\rangle

\leq C_r \left\| A'^{-1} e^{rA} U_\mp \right\| \left\| \left( \sum_{\alpha=0}^m A' e^{rA} \phi_\mp \right)^2 \right\|

+ C_r \left\| A'^{-1} e^{rA} U_\mp \right\| \left\| \left( \sum_{\alpha=0}^m A' e^{rA} \phi_\mp \right)^2 \right\|

(6.16)

Next, since \(-iU_+ = U_+\), we have

\left\langle U_+ \cdot \nabla A' e^{rA} \phi_\pm, A' e^{rA} \phi_\pm \right\rangle + \left\langle (\nabla \cdot A' e^{rA} \phi_\pm) U_+, A' e^{rA} \phi_\pm \right\rangle

+ \left\langle U_+ \cdot A' e^{rA} (\nabla \phi_\pm - \bar{\phi}_\pm^{\perp}), A' e^{rA} \phi_\pm \right\rangle

\leq \left\langle U_+ \cdot \nabla A' e^{rA} \phi_\pm, A' e^{rA} \phi_\pm \right\rangle + \left\langle U_+ \cdot \nabla A' e^{rA} \phi_\pm, A' e^{rA} \phi_\pm \right\rangle

+ \left\langle (\nabla \cdot A' e^{rA} \phi_\pm) U_+, A' e^{rA} \phi_\pm \right\rangle + \left\langle U_+ \cdot \nabla A' e^{rA} \phi_\pm, A' e^{rA} \phi_\pm \right\rangle

\leq \left\langle (\nabla \cdot U_+) A' e^{rA} \phi_\pm, A' e^{rA} \phi_\pm \right\rangle + \left\langle A' e^{rA} \phi_\pm, \nabla U_+, A' e^{rA} \phi_\pm \right\rangle

+ \left\langle U_+ \cdot \nabla A' e^{rA} \phi_\pm, A' e^{rA} \phi_\pm \right\rangle - \left\langle A' e^{rA} \phi_\pm, \nabla A' e^{rA} \phi_\pm, U_+ \right\rangle.

Notice that

\left\langle U_+ \cdot \nabla A' e^{rA} \phi_\pm, A' e^{rA} \phi_\pm \right\rangle - \left\langle A' e^{rA} \phi_\pm, \nabla A' e^{rA} \phi_\pm, U_+ \right\rangle
Therefore, by the Sobolev inequality and the Hölder inequality, since \( r > \frac{3}{2} \), we have

\[
\left| \sum_{i=1}^{n} A_i^r e^{\tau A_i^r} \phi_+ + A^r e^{\tau A^r} \phi \right| \leq C_r \| A^r e^{\tau A^r} \phi \| \left( \sum_{i=1}^{n} A_i^r e^{\tau A_i^r} \right) + \| A_i^r e^{\tau A_i^r} \| \| A^r e^{\tau A^r} \phi \| \leq C_r \left( \sum_{i=1}^{n} A_i^r e^{\tau A_i^r} \right) + \| A^r e^{\tau A^r} \phi \|. 
\]

Based on this, thanks to Lemma \( A.4 \) and Lemma \( A.5 \), we have

\[
\left| \sum_{i=1}^{n} A_i^r e^{\tau A_i^r} \phi_+ + A^r e^{\tau A^r} \phi \right| \leq C_r \left( \sum_{i=1}^{n} A_i^r e^{\tau A_i^r} \right) + \| A^r e^{\tau A^r} \phi \|. 
\]

Similarly, we have

\[
\left| \sum_{i=1}^{n} A_i^r e^{\tau A_i^r} \phi_+ + A^r e^{\tau A^r} \phi \right| \leq C_r \left( \sum_{i=1}^{n} A_i^r e^{\tau A_i^r} \right) + \| A^r e^{\tau A^r} \phi \|. 
\]

For the rest parts in type 4, there is no cancellation as above. First, by the Hölder inequality, we have

\[
\left| \sum_{i=1}^{n} A_i^r e^{\tau A_i^r} \phi_+ + A^r e^{\tau A^r} \phi \right| \leq C_r \left( \sum_{i=1}^{n} A_i^r e^{\tau A_i^r} \right) + \| A^r e^{\tau A^r} \phi \|. 
\]

Based on this, using Lemma \( A.4 \) to \( A.7 \), we have

\[
\left| \sum_{i=1}^{n} A_i^r e^{\tau A_i^r} \phi_+ + A^r e^{\tau A^r} \phi \right| \leq C_r \left( \sum_{i=1}^{n} A_i^r e^{\tau A_i^r} \right) + \| A^r e^{\tau A^r} \phi \|. 
\]
Similarly, we have
\[
\left| \left( A^t e^{rA} \left( \int_0^t \nabla \cdot \phi + (x', s) ds \right) \right), A^r e^{rA} \phi \right| \\
\leq \left| \left( \partial_2 U_+ \right) A^r e^{rA} \left( \int_0^t \nabla \cdot \phi + (x', s) ds \right) \right| \\
+ \left| \left( A^r e^{rA} \partial_2 U_+ \right) A^r e^{rA} \phi \right| \\
\leq C_r \left( \| A^r e^{rA} \partial_2 U_+ \|_L^\infty + \| A^r e^{rA} \phi \|_L^2 \right) \\
+ C_r \left( \| A^r e^{rA} \partial_2 U_+ \| + \| A^r e^{rA} \phi \|_L^2 \right). 
\] (6.19)

Next, by the H"older inequality, we have
\[
\left| \left( A^t e^{rA} \left( \int_0^t \nabla \cdot \phi + (x', s) ds \right) \right), A^r e^{rA} \phi \right| \\
\leq \left| \left( \partial_2 U_+ \right) A^r e^{rA} \left( \int_0^t \nabla \cdot \phi + (x', s) ds \right) \right| \\
+ \left| \left( A^r e^{rA} U_+ \right) A^r e^{rA} \phi \right| \\
\leq C_r \left( \| \partial_2 U_+ \|_L^\infty + \| A^r e^{rA} \phi \|_L^2 \right) \\
+ C_r \left( \| A^r e^{rA} U_+ \| + \| A^r e^{rA} \phi \|_L^2 \right). 
\] (6.20)

Based on this, thanks to Lemma 4.4 to 4.7 we obtain
\[
\left| \left( A^t e^{rA} \left( \int_0^t \nabla \cdot \phi + (x', s) ds \right) \partial_2 U_+ \right), A^r e^{rA} \phi \right| \\
\leq \left| \left( \partial_2 U_+ \right) A^r e^{rA} \left( \int_0^t \nabla \cdot \phi + (x', s) ds \right) \right| \\
+ \left| \left( A^r e^{rA} U_+ \right) A^r e^{rA} \phi \right| \\
\leq C_r \left( \| A^r e^{rA} \partial_2 U_+ \|_L^\infty + \| A^r e^{rA} \phi \|_L^2 \right) \\
+ C_r \left( \| A^r e^{rA} U_+ \| + \| A^r e^{rA} \phi \|_L^2 \right) \\
\times \left( \| A^r e^{rA} \phi \|_L^2 + \| A^r e^{rA} \phi \|_L^2 \right). 
\] (6.21)
Therefore, by the Sobolev inequality, the Poincaré inequality, and Young’s inequality, since $U^+ \leq C + \|\tau A\|_{L^\infty}$, we obtain

\[
\frac{d}{dt} \left( \frac{1}{2} \|A^\tau e^{-\tau A} \phi\|^2 + \|A^\tau e^{-\tau A} \phi_+\|^2 + \|A^\tau e^{-\tau A} \phi_-\|^2 \right) - 5
\]

Therefore, the Sobolev inequality, the Poincaré inequality, and Young’s inequality, since $r > \frac{\tau}{2}$, $\tau \leq \tau_0$, and $U_\pm$ have zero mean value, we have

\[
\|A^\tau e^{-\tau A} \phi\|^2 + \|A^\tau e^{-\tau A} \phi_+\|^2 + \|A^\tau e^{-\tau A} \phi_-\|^2
\]

By Young’s inequality, the term $\|A^\tau e^{-\tau A} \phi\|^2$ can be combined with other terms, and we can rewrite (6.23) as

\[
\frac{d}{dt} \left( \frac{1}{2} \|A^\tau e^{-\tau A} \phi\|^2 + \|A^\tau e^{-\tau A} \phi_+\|^2 + \|A^\tau e^{-\tau A} \phi_-\|^2 \right) - 5
\]

Therefore, the Sobolev inequality, the Poincaré inequality, and Young’s inequality, since $r > \frac{\tau}{2}$, $\tau \leq \tau_0$, and $U_\pm$ have zero mean value, we have

\[
\|A^\tau e^{-\tau A} \phi\|^2 + \|A^\tau e^{-\tau A} \phi_+\|^2 + \|A^\tau e^{-\tau A} \phi_-\|^2
\]

By Young’s inequality, the term $\|A^\tau e^{-\tau A} \phi\|^2$ can be combined with other terms, and we can rewrite (6.23) as

\[
\frac{d}{dt} \left( \frac{1}{2} \|A^\tau e^{-\tau A} \phi\|^2 + \|A^\tau e^{-\tau A} \phi_+\|^2 + \|A^\tau e^{-\tau A} \phi_-\|^2 \right) - 5
\]

Therefore, the Sobolev inequality, the Poincaré inequality, and Young’s inequality, since $r > \frac{\tau}{2}$, $\tau \leq \tau_0$, and $U_\pm$ have zero mean value, we have

\[
\|A^\tau e^{-\tau A} \phi\|^2 + \|A^\tau e^{-\tau A} \phi_+\|^2 + \|A^\tau e^{-\tau A} \phi_-\|^2
\]

By Young’s inequality, the term $\|A^\tau e^{-\tau A} \phi\|^2$ can be combined with other terms, and we can rewrite (6.23) as

\[
\frac{d}{dt} \left( \frac{1}{2} \|A^\tau e^{-\tau A} \phi\|^2 + \|A^\tau e^{-\tau A} \phi_+\|^2 + \|A^\tau e^{-\tau A} \phi_-\|^2 \right) - 5
\]
Denote by
\[ F := \frac{1}{2} ||A^r e^{rA} \varphi||^2 + ||A^r e^{rA} \varphi_+||^2 + ||A^r e^{rA} \varphi_-||^2, \]
\[ G := ||A^{r/2} e^{rA} \varphi||^2 + 2 ||A^{r/2} e^{rA} \varphi_+||^2 + 2 ||A^{r/2} e^{rA} \varphi_-||^2, \]
\[ K(t) := e^{C_{M,\tau,0}}, \quad \bar{K}(t) := e^{\bar{K}(t)}, \]
which are double exponential and triple exponential in time. We will follow the rule on the use of notation as indicated in Remark 5. From Proposition 5.1 and thanks to Lemma 2.8, one has
\[ \|U\|_{H^{r+\delta}} \leq (\bar{K}_1(t) + e^{(\tau(t))A_U(t)}) \|U\|_{H^{\infty}} \leq \bar{K}_1(t), \] provided that \( t \) satisfies (6.9). Observe that in (6.24), \( \|U\|_{L^\infty}, \|A^{r/2} U\|_{L^\infty}, \|\partial_x U\|_{L^\infty}, \) and \( \|A^{r/2} \partial_x U\|_{L^\infty} \) are the terms force the smallness assumption on Sobolev norm of the baroclinic mode. For \( \delta > 0 \), by Proposition 5.1 and Lemma 2.8 thanks to the Sobolev inequality, when \( \|\tilde{V}_0\|_{H^{r+\delta}} = \|\tilde{V}_0\|_{H^{r+\delta}} \leq \frac{1}{|\Omega_0|}, \) we have
\[ \|\tilde{V}\|_{H^{r+\delta}} \leq \frac{C\bar{K}(t)}{|\Omega_0|}. \]

Since \( |\Omega| \geq |\Omega_0| \), we can rewrite (6.24) as
\[ \frac{dF}{dt} \leq (\bar{K}_2 + \frac{\bar{K}_2}{|\Omega_0|})G + \frac{\bar{K}_2}{|\Omega_0|} + \partial_t N. \]
By setting \( \bar{\tau} + C_r F = \bar{K}_2 + \frac{\bar{K}_2}{|\Omega_0|} = 0 \), it follows that \( \frac{dF}{dt} \leq \bar{K}_2 F + \frac{\bar{K}_2}{|\Omega_0|} + \partial_t N. \) By the Grönwall inequality,
\[ \frac{d}{dt}(F e^{-\int_0^t \bar{K}_2(s)ds}) \leq \frac{\bar{K}_2}{|\Omega_0|} + (\partial_t N)e^{-\int_0^t \bar{K}_2(s)ds}. \]
Integrating from 0 to \( t \), noticing that \( F(0) = 0 \), one obtains that
\[ F(t) e^{-\int_0^t \bar{K}_2(s)ds} \leq \frac{1}{|\Omega_0|} \int_0^t \bar{K}_2(s)ds + \int_0^t (\partial_x N(s)) e^{-\int_0^s \bar{K}_2(\xi) d\xi} ds. \]
From (6.12), we know \( |N(t)| \leq \frac{1}{|\Omega_0|} \bar{K}_3(t)F(t) \). Moreover, \( \frac{1}{|\Omega_0|} \bar{K}_3(t)F(t) \) is increasing in time. By integration by parts in time, thanks to the Cauchy–Schwarz inequality, since \( N(0) = 0 \), we have
\[ \int_0^t (\partial_x N(s)) e^{-\int_0^s \bar{K}_2(\xi)d\xi} ds \leq |N(t)| + \int_0^t |N(s)| \partial_x e^{-\int_0^s \bar{K}_2(\xi) d\xi} ds \]
\[ \leq \frac{1}{|\Omega_0|} \bar{K}_3 F + \frac{t}{|\Omega_0|} \bar{K}_3 F \bar{K}_2 \leq \frac{1}{|\Omega_0|} \bar{K}_4 + \frac{1}{|\Omega_0|} F. \]
Thus, one gets \( F(t) \leq \frac{1}{|\Omega_0|} e^{\bar{K}_5(t)} + \frac{1}{|\Omega_0|} e^{\bar{K}_5(t)} F(t) \), which is equivalent to
\[ F(t) \leq \frac{e^{\bar{K}_5(t)}}{|\Omega_0| - e^{\bar{K}_6(t)}}. \]
Plugging this back to \( \bar{\tau} + C_r F + \bar{K}_2 + \frac{\bar{K}_2}{|\Omega_0|} = 0 \), one can require that
\[ \bar{\tau} + \sqrt{|\Omega_0| - e^{\bar{K}_6(t)}} + \tau \bar{K}_6 + \frac{1}{|\Omega_0|} \bar{K}_6 \leq 0. \]
By the Grönwall inequality, one can require

\[ \frac{d}{dt} (Te^{\int_0^t \tilde{K}_0(s)ds}) \leq \frac{-e^{\tilde{K}_1(t)} - e^{\tilde{K}_2(t)}}{\sqrt{|\Omega_0| - e^{\tilde{K}_0(t)}}}. \]

Integrating from 0 to \( t \), for some suitable function \( \tilde{K}_0(t) \), one can require that

\[ \tau(t) = \left( \tau_0 - \int_0^t \frac{e^{\tilde{K}_0(s)}}{\sqrt{\Omega_0} - e^{\tilde{K}_0(s)}} ds - \int_0^t \frac{e^{\tilde{K}_0(s)}}{\Omega_0} ds \right) e^{-\int_0^t \tilde{K}_0(s)ds}. \] (6.27)

Notice that \( \tau \) in (6.27) also satisfies (5.9) when \( \tilde{K}_0(t) \) is chosen suitably. In order to have \( \tau(t) > 0 \), we just need to require that

\[ \tau_0 \geq \frac{3e^{\tilde{K}_0(t)}}{\sqrt{\Omega_0} - e^{\tilde{K}_0(t)}} \quad \text{and} \quad \tau_0 \geq \frac{3e^{\tilde{K}_0(t)}}{\Omega_0} \] (6.28)

for some suitable function \( \tilde{K}_0(t) > \tilde{K}_0(t) \). For some new \( \tilde{K}(t) > \tilde{K}_0(t) \) and the given \( \Omega_0 \), let \( T \) satisfy

\[ C_{\tau_0}e^{\tilde{K}(T)} = |\Omega_0|, \] (6.29)

then the two conditions in (6.28) are satisfied on \( t \in [0, T] \). Thus, \( \tau(t) > 0 \) on \( t \in [0, T] \). From (6.29), we know that \( e^{\tilde{K}(T)} \geq \frac{|\Omega_0|}{2C_{\tau_0}} \) and thus the time \( T \) satisfies

\[ T \geq \ln(\ln(\ln(\ln|\Omega_0|))) \rightarrow \infty, \] (6.30)

as \( |\Omega_0| \rightarrow \infty \).

When \( \tilde{K}(t) \) is chosen suitably, from (6.29), we know that

\[ \| A^r e^{\tau(t)A\phi(t)} \|_H^2 + \| A^r e^{\tau(t)A\phi\pm(t)} \|_H^2 + \| A^r e^{\tau(t)A\phi\pm(t)} \|_H^2 \leq \frac{2e^{\tilde{K}(t)}}{|\Omega_0| - e^{\tilde{K}(t)}} < \infty \] (6.31)

for \( t \in [0, T] \). Since \( \tilde{\phi} \) and \( \phi\pm \) have zero mean value in \( T^3 \), by the Poincaré inequality, the \( L^2 \) norm can be bounded by the higher order norm. Therefore, one has

\[ \| e^{\tau(t)A\tilde{\phi}(t)} \|_H^2 + \| e^{\tau(t)A\phi\pm(t)} \|_H^2 \leq \frac{2e^{\tilde{K}(t)}}{|\Omega_0| - e^{\tilde{K}(t)}} < \infty \] (6.31)

for \( t \in [0, T] \). Since \( \tau(t) \) satisfies (5.9), it follows that

\[ \| e^{\tau(t)A\tilde{\phi}(t)} \|_H^2 \leq \frac{2e^{\tilde{K}(t)}}{|\Omega_0| - e^{\tilde{K}(t)}} < \infty \] (6.31)

for \( t \in [0, T] \). Since \( \tilde{\tilde{\nu}} = \tilde{\phi} + \nabla \tilde{\phi} \) and \( \tilde{u}_\pm = \tilde{\phi}_\pm + \tilde{U}_\pm \), by triangle inequality, thanks to Lemma 2.8, we have \( \| e^{\tau(t)A\tilde{\phi}(t)} \|_H^2 + \| e^{\tau(t)A\tilde{\phi}(t)} \|_H^2 < \infty \) for \( t \in [0, T] \). Therefore, we obtain \( (\tilde{\tilde{\nu}}, \tilde{\phi}) \in L^\infty(0, T; D(e^{\tau(t)A}) : H^r(T^3)) \). This completes the proof of Theorem 6.1.

6.4. Approximation by the limit resonant system. As a consequence of the proof of Theorem 6.1, the following theorem describes the approximation of the solution to the original system (2.5)–(2.6) by the solution to the limit resonant system (5.13)–(5.20) in the space of analytic functions, for large rotation rate \( |\Omega| \) and small initial baroclinic mode in Sobolev norm.
Theorem 6.3. Suppose the conditions in Theorem 6.1 hold, and let \((\overline{\nabla}, \overline{\nabla})\) be the solution to system (5.5)–(5.7) with initial data \((\overline{\nabla}_0, \overline{\nabla}_0)\). Denote by \(\overline{\phi} = \nabla - \nabla\) and \(\overline{\phi} = \nabla - \nabla\), then, for \(|\nabla| \geq |\nabla_0|\), one has

\[
\|e^{\tau(t)}A\overline{\phi}(t)\|_{H^s} + \|e^{\tau(t)}A\overline{\phi}(t)\|_{H^r} \lesssim \frac{e^{\hat{K}(t)}}{|\nabla_0| - e^{-K(t)}},
\]

for \(t \in [0, T]\) with \(T\) given by (6.17) and \(\tau(t)\) given by (6.2).

Proof. The proof is an immediate consequence of (6.31). \(\square\)

6.5. Remarks and discussions.

Remark 9. To emphasize the difference between smallness in analytic norm and in Sobolev norm, for \(|\nabla| \gg 1\), consider \(\overline{\nabla}_0 = c_k e^{i \hat{k} \cdot x}\) with \(k_3 \neq 0\), \(|\hat{k}| = \left[\nabla_0^{-1} \ln \nabla\right]\) and \(|c_k| = (\ln \nabla)^{-\tau - 2}|\nabla|^{-1}\). When \(0 < \delta < 1\), since \(r > \frac{2}{3}\), we have \(\|\overline{\nabla}_0\|_{H^{s+\delta}} \lesssim \|\overline{\nabla}_0\|_{H^{s+\delta}} \sim |\nabla|^{-1}\). Therefore, one can construct a sequence of initial data

\[
\{\overline{\nabla}_0(\nabla)\} = c_{k,0}(\nabla) e^{i \hat{k}(\nabla) \cdot x},
\]

where \(|k(\nabla)| = \left[\nabla_0^{-1} \ln \nabla\right]\) and \(|c_{k,0}(\nabla)| = (\ln \nabla)^{-\tau - 2}|\nabla|^{-1}\). Then as \(|\nabla| \to \infty\), the existence time of solutions \(T \to \infty\), with initial condition \(\|e^{\hat{K}}(\nabla)\nabla\|_{H^{r+2}} \sim 1\). This result needs fast rotation, and is very different from Theorem 6.3.

Remark 10. In the view of Remark 4 when the solution to the 2D Euler equations with initial data \(\overline{\nabla}_0\) is uniformly bounded in time, the function \(\hat{K}(t)\) appears in the proof of Theorem 6.1 becomes only exponentially in time. This reduces two logarithms in (6.30) and one concludes that \(T \lesssim \ln(\ln |\nabla_0|)\). Moreover, when \(\overline{\nabla}_0 = 0\), the function \(\hat{K}(t)\) is uniformly bounded in time, hence \(T \gtrsim \ln |\nabla_0|\).

Remark 11. In estimate (6.19) we have the resonance term

\[
(U_+ \cdot \nabla)(\overline{\phi} + i \overline{\phi}^\perp) = (U_+ \cdot \nabla \overline{\phi} - U_+^\perp \cdot \nabla \overline{\phi}^\perp) = U_+^\perp (\nabla^\perp \cdot \overline{\phi}),
\]

which involves the vorticity \(\nabla^\perp \cdot \overline{\phi}\). Notice that in the limit resonant system (5.5)–(5.7), the evolution of the barotropic mode \(\overline{\nabla}\) is independent of the baroclinic mode \(\nabla\), and therefore we can control the vorticity \(\nabla^\perp \cdot \overline{\phi}\). However, for the original system (2.20)–(2.23) (or the perturbed system (6.3)–(6.4)), the evolution of the barotropic mode \(\nabla\) (or \(\overline{\phi}\)) depends also on the baroclinic mode \(\nabla\) (or \(\overline{\phi}\)). Therefore, we are unable to control (6.32) without the smallness condition on the initial baroclinic mode.

Remark 12. In estimate (6.21), we have the term \(e^{\hat{K}t}(\int_0^t \nabla \cdot \phi_+(x, s) ds) \partial_2 U_+\). Despite the oscillation, we are unable to apply similar methods as in type 3 due to the loss of derivative on the baroclinic mode. For this term, we do not have cancellation as other terms in type 4. Therefore, we are forced to require the smallness condition on the initial baroclinic mode.

APPENDIX A. ESTIMATES OF NONLINEAR TERMS

In this appendix, we list the estimates of nonlinear terms in the space of analytic functions. Lemma A.1–A.3 will be used to prove the local well-posedness.

First, we estimate nonlinear terms of the form \(f \cdot \nabla g\).

Lemma A.1. For \(f, g, h, \in D(e^{\tau A} : H^{r+\frac{1}{2}})\), where \(r > 2\) and \(\tau \geq 0\), one has

\[
\left| \left( A^r e^{\tau A}(f \cdot \nabla g), A^r e^{\tau A}h \right) \right| \leq C_r \left[ \|A^r e^{\tau A}f\| + |f_0| \right] \left[ \|A^r e^{\tau A}g\| + \|A^r e^{\tau A}h\| \right]
\]

\[
+ \left[ A^{r+\frac{1}{2}} e^{\tau A}f \right] \left[ \|A^r e^{\tau A}g\| + \|A^r e^{\tau A}h\| \right].
\]

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Proof. First, notice that \( \left\langle A^r e^{\tau A}(f \cdot \nabla g), A^s e^{\tau A}h \right\rangle = \left\langle f \cdot \nabla g, A^r e^{\tau A}H \right\rangle \), where \( H = A^s e^{\tau A}h \). We use Fourier representation of \( f, g \) and \( H \), in which we can write

\[
 f(x) = \sum_{j \in \mathbb{Z}^3} \hat{f}_j e^{2\pi i j \cdot x}, \quad g(x) = \sum_{k \in \mathbb{Z}^3} \hat{g}_k e^{2\pi i k \cdot x}, \quad A^r e^{\tau A}H(x) = \sum_{l \in \mathbb{Z}^3} |l|^r e^{\tau |l|} \hat{H}_l e^{2\pi i l \cdot x}.
\]

Therefore,

\[
 \left\langle f \cdot \nabla g, A^r e^{\tau A}H \right\rangle \leq \sum_{j+k+l=0} |\hat{f}_j| |\hat{g}_k| |l|^r |e^{\tau |l|} \hat{H}_l|.
\]

From \( |l| = |j + k| \leq |j| + |k| \) we have the following inequalities:

\[
 |l|^r \leq (|j| + |k|)^r \leq C_r (|j|^r + |k|^r), \quad e^{\tau |l|} \leq e^{\tau |j|} e^{\tau |k|}.
\]

Applying these inequalities, we have

\[
 \left\langle f \cdot \nabla g, A^r e^{\tau A}H \right\rangle \leq \sum_{j+k+l=0} C_r |\hat{f}_j| |\hat{g}_k| (|j|^r + |k|^r) e^{|j| \tau} e^{|k| \tau} |l|^r e^{\tau |l|} |\hat{H}_l|.
\]

Since \( |k|, |j|, |l| \) are all nonnegative, we have \( |k|^r \leq (|j| + |l|)^r \leq |j|^r + |l|^r \), therefore,

\[
 \left\langle f \cdot \nabla g, A^r e^{\tau A}H \right\rangle \leq \sum_{j+k+l=0} C_r |\hat{f}_j| |\hat{g}_k| (|j|^r + |k|^r) e^{|j| \tau} e^{|k| \tau} |l|^r e^{\tau |l|} |\hat{H}_l|.
\]

Thanks to the Cauchy–Schwarz inequality, since \( r > 2 \), we have

\[
 A_1 = \sum_{j+k+l=0} C_r |\hat{f}_j| |\hat{g}_k| e^{|j| \tau} e^{|k| \tau} |l|^r e^{\tau |l|} |\hat{H}_l| e^{|l| \tau} e^{\tau |l|} |\hat{H}_l|.
\]

Similarly, we have

\[
 A_2 = \sum_{j+k+l=0} C_r |\hat{f}_j| |\hat{g}_k| e^{|j| \tau} e^{|k| \tau} |l|^r e^{\tau |l|} |\hat{H}_l|.
\]

\[
 A_3 = \sum_{j+k+l=0} C_r |\hat{f}_j| |\hat{g}_k| e^{|j| \tau} e^{|k| \tau} |l|^r e^{\tau |l|} |\hat{H}_l|.
\]
For \( A_4 \), thanks to the Cauchy–Schwarz inequality, since \( r > 2 \), we have
\[
A_4 = \sum_{j+k+l=0} C_r |k|^{r+\frac{1}{2}}|\hat{f}_j||\hat{g}_k||\hat{h}_l|
\]
\[
= C_r \sum_{j \in \mathbb{Z}^3} e^{\pi j |\hat{f}_j|} \sum_{k \in \mathbb{Z}^3 \setminus \{0, -j\}} |k|^{r+\frac{1}{2}} |\hat{g}_k| e^{\pi j |k| |j+k|} |\hat{h}_{j-k}|
\]
\[
\leq C_r \left\{ |\hat{f}_0| + \left( \sum_{j \in \mathbb{Z}^3 \setminus \{0\}} \frac{1}{j} \right)^{\frac{1}{2}} \left( \sum_{j \in \mathbb{Z}^3 \setminus \{0\}} j^{2r} e^{\pi j |\hat{f}_j|^2} \right)^{\frac{1}{2}} \right\}
\]
\[
\times \sup_{j \in \mathbb{Z}^3} \left( \sum_{k \in \mathbb{Z}^3 \setminus \{0, -j\}} k^{2r+1} e^{\pi j |k|} |\hat{g}_k|^2 \right)^{\frac{1}{2}} \left( \sum_{k \in \mathbb{Z}^3 \setminus \{0, -j\}} (j+k)^{2r} e^{\pi j |k|} |\hat{h}_{j-k}|^2 \right)^{\frac{1}{2}}
\]
\[
\leq C_r \left( \| A^r e^{\tau A} f \| + |\hat{f}_0| \right) \| A^{r+\frac{1}{2}} e^{\tau A} g \| \| A^{r+\frac{1}{2}} e^{\tau A} h \|
\]

Combine the estimates for \( A_1 \) to \( A_4 \), and since \( \| A^r e^{\tau A} g \| \leq \| A^{r+\frac{1}{2}} e^{\tau A} g \| \), \( \| A^r e^{\tau A} h \| \leq \| A^{r+\frac{1}{2}} e^{\tau A} h \| \), we achieve the desired inequality.

Similarly, we estimate \((\nabla \cdot f)g\) in the following:

**Lemma A.2.** For \( f, g, h \in \mathcal{D}(e^{\tau A} : H^{r+\frac{1}{2}}) \), where \( r > 2 \) and \( \tau \geq 0 \), one has
\[
\left\langle A^r e^{\tau A} \left( (\nabla \cdot f)g, A^r e^{\tau A} h \right) \right\rangle \leq C_r \left( \| A^r e^{\tau A} g \| + |\hat{f}_0| \right) \| A^{r+\frac{1}{2}} e^{\tau A} f \| \| A^{r+\frac{1}{2}} e^{\tau A} h \|
\]
\[
+ \| A^{r+\frac{1}{2}} e^{\tau A} g \| \| A^{r+\frac{1}{2}} e^{\tau A} f \| \| A^{r+\frac{1}{2}} e^{\tau A} h \|.
\]

The proof of Lemma A.2 is almost the same as Lemma A.1 so we omit it.

Finally, we provide an estimate for \((\int_0^z \nabla \cdot f(x', s) ds) \partial_z g\) in the following:

**Lemma A.3.** For \( f, g, h \in \mathcal{D}(e^{\tau A} : H^{r+\frac{1}{2}}) \), where \( r > 2 \), \( \tau \geq 0 \), and \( \mathcal{T} = 0 \), one has
\[
\left\langle A^r e^{\tau A} \left( \left( \int_0^z \nabla \cdot f(x', s) ds \right) \partial_z g, A^r e^{\tau A} h \right) \right\rangle \leq C_r \left( \| A^r e^{\tau A} f \| \| A^{r+\frac{1}{2}} e^{\tau A} g \| \| A^{r+\frac{1}{2}} e^{\tau A} h \|
\]
\[
+ \| A^{r+\frac{1}{2}} e^{\tau A} g \| \| A^{r+\frac{1}{2}} e^{\tau A} f \| \| A^{r+\frac{1}{2}} e^{\tau A} h \| + \| A^{r+\frac{1}{2}} e^{\tau A} g \| \| A^{r+\frac{1}{2}} e^{\tau A} f \| \| A^{r+\frac{1}{2}} e^{\tau A} h \|\right).
\]

**Proof.** First, \( \left\langle A^r e^{\tau A} \left( \left( \int_0^z \nabla \cdot f(x', s) ds \right) \partial_z g, A^r e^{\tau A} h \right) \right\rangle = \left\langle \left( \int_0^z \nabla \cdot f(x', s) ds \right) \partial_z g, A^r e^{\tau A} H \right\rangle \). Since \( \mathcal{T} = 0 \), one has \( f(x) = \sum_{j \in \mathbb{Z}^3 \setminus \{0\}} \hat{f}_j e^{2\pi (ij' \cdot x' + ij_3 z)} \) where \( j' = (j_1, j_2) \). Then we have
\[
\int_0^z \nabla \cdot f(x', s) ds = \sum_{j \in \mathbb{Z}^3 \setminus \{0\}} \frac{1}{j_3} j' \cdot \hat{f}_j e^{2\pi (ij' \cdot x' + ij_3 z)} - \sum_{j \in \mathbb{Z}^3 \setminus \{0\}} \frac{1}{j_3} j' \cdot \hat{f}_j e^{2\pi ij_3 z}.
\]

Therefore,
\[
\left\langle \left( \int_0^z \nabla \cdot f(s) ds \right) \partial_z g, A^r e^{\tau A} H \right\rangle \leq \left\langle \left( \sum_{j \in \mathbb{Z}^3 \setminus \{0\}} \frac{1}{j_3} j' \cdot \hat{f}_j e^{2\pi (ij' \cdot x' + ij_3 z)} \right) \partial_z g, A^r e^{\tau A} H \right\rangle
\]
Let us estimate $I_2$ first. For $l = (l', l_3) = (-j' - k', -k_3)$, by using the inequalities

$$|j'|^{\frac{1}{2}} \leq C(|k|^{\frac{1}{2}} + |l|^{\frac{1}{2}}), \quad |k|^{\frac{1}{2}} \leq C(|j'|^{\frac{1}{2}} + |l|^{\frac{1}{2}}), \quad |l| \leq C_r(|j'| + |k|),$$

one has

$$I_2 \leq \sum_{j' + k' + l' = 0 \atop k_3, l_3 = 0 \atop j, k, j' \neq 0} C_r \left| \frac{1}{|j_3|} \left| j' \right| \left| k \right| \left| \hat{g}_k \right| \left( |j'| |r + \frac{1}{2}| |l| \right)^{\tau} e^{\tau |j'| |e^{\tau |l|}} \left| \hat{f}_j \right| \left| \hat{h}_l \right| \right|$$

$$\leq \sum_{j' + k' + l' = 0 \atop k_3, l_3 = 0 \atop j, k, j' \neq 0} C_r \left| \frac{1}{|j_3|} \left| j' \right| \left| k \right| \left| \hat{g}_k \right| \left( |j'| |r + 1| |k| \right) + \left| j' \right| |k| |r + 1| e^{\tau |j'| |e^{\tau |k|}} \left| \hat{f}_j \right| \left| \hat{h}_l \right| \right|$$

$$\leq \sum_{j' + k' + l' = 0 \atop k_3, l_3 = 0 \atop j, k, j' \neq 0} C_r \left| \frac{1}{|j_3|} \left| j' \right| \left| k \right| \left| \hat{g}_k \right| \left( |k|^{\frac{1}{2}} |j' |^{r + \frac{1}{2}} |l| + |k| |j' |^{r + \frac{1}{2}} |l|^{r + \frac{1}{2}} + |j' |^{r + \frac{1}{2}} |k|^{r + \frac{1}{2}} |l| \right)^{\tau} e^{\tau |j'| |e^{\tau |k|}} \left| \hat{f}_j \right| \left| \hat{h}_l \right| \right|$$

$$= B_1 + B_2 + B_3 + B_4.$$
The estimate for $B_2$ is similar as $B_1$, and we can get $B_2 \leq C_r \| A^r \frac{1}{2} e^{rA} f \| A^r e^{rA} g \| A^r \frac{1}{2} e^{rA} h \|$. For $B_3$, thanks to the Cauchy–Schwarz inequality, since $r > 2$, we have

$$B_3 = \sum_{j^* + k^* + l^* = 0 \atop j^*_3, k^*_3, l^*_3 \neq 0} C_r \sum_{j^*_3 \in \mathbb{Z}^3} \frac{1}{|j^*_3|} |j^*_3| |k^*_3| \| e^{-r|j^*|} e^{-r|k^*|} |\hat{g}_{k^*}| |\hat{f}_{j^*}| |\hat{h}_{l^*}|$$

$$= C_r \sum_{j^*_3 \in \mathbb{Z}^3} \frac{1}{|j^*_3|} |j^*_3| \| e^{-r|j^*|} e^{-r|k^*|} |\hat{g}_{k^*}| |\hat{f}_{j^*}| (|j^*_3| + |k^*_3|) r e^{-r|j^* + k^* + l^*|} |\hat{h}_{l^* - (j^* + k^* + l^*)}|$$

$$\leq C_r \left( \sum_{j^*_3 \in \mathbb{Z}^3 \atop j^*_3 \neq 0} \frac{1}{|j^*_3|} |j^*_3| \| e^{-r|j^*|} e^{-r|k^*|} |\hat{g}_{k^*}| |\hat{f}_{j^*}| \right)^{\frac{1}{2}} \left( \sum_{j^*_3 \in \mathbb{Z}^3 \atop j^*_3 \neq 0} |j^*_3| |k^*_3| \| e^{-r|j^* + k^* + l^*|} |\hat{h}_{l^* - (j^* + k^* + l^*)}| \right)^{\frac{1}{2}}$$

$$\leq C_r \| A^r \frac{1}{2} e^{rA} f \| A^r \frac{1}{2} e^{rA} g \| A^r \frac{1}{2} e^{rA} h \|.$$

The estimate for $B_3$ is similar as $B_3$, and we can get $B_3 \leq C_r \| A^r \frac{1}{2} e^{rA} f \| A^r \frac{1}{2} e^{rA} g \| A^r \frac{1}{2} e^{rA} h \|$. The estimates of $B_3$ to $B_4$ indicate that $I_2$ satisfies the desired inequality.

Now let us estimate on $I_1$. For $j^* + k^* + l^* = 0$, by using the inequalities

$$|j^*_3| \leq C(\|k^*_3\| + |l^*_3|), \quad |k^*_3| \leq C(|j^*_3| + |l^*_3|), \quad |l^*_3| \leq C_r (|j^*_3| + |k^*_3|),$$

we have

$$I_1 \leq \sum_{j^*_3 + k^*_3 + l^*_3 = 0 \atop j^*_3, k^*_3, l^*_3 \neq 0} C_r \sum_{j^*_3 \in \mathbb{Z}^3} \frac{1}{|j^*_3|} |j^*_3| |k^*_3| \| \hat{g}_{k^*} \| (|j^*_3| + |k^*_3|) e^{-r|j^*|} e^{-r|k^*|} |\hat{f}_{j^*}| \| e^{-r|l^*|} |\hat{h}_{l^*}|$$

$$\leq \sum_{j^*_3 + k^*_3 + l^*_3 = 0 \atop j^*_3, k^*_3, l^*_3 \neq 0} C_r \sum_{j^*_3 \in \mathbb{Z}^3} \frac{1}{|j^*_3|} |j^*_3| \| \hat{g}_{k^*} \| (|j^*_3| + |k^*_3|) e^{-r|j^*|} e^{-r|k^*|} |\hat{f}_{j^*}| \| e^{-r|l^*|} |\hat{h}_{l^*}|$$

$$\leq \sum_{j^*_3 + k^*_3 + l^*_3 = 0 \atop j^*_3, k^*_3, l^*_3 \neq 0} C_r \sum_{j^*_3 \in \mathbb{Z}^3} \frac{1}{|j^*_3|} (|j^*_3| |k^*_3| + |j^*_3| |l^*_3| + |k^*_3| |l^*_3| + |j^*_3| |k^*_3| |l^*_3|)$$

$$\leq \frac{1}{|j^*_3|} \sum_{j^*_3 \in \mathbb{Z}^3} \left( |j^*_3| \| \hat{g}_{k^*} \| e^{-r|j^*|} e^{-r|l^*|} \right) \left( |j^*_3| \| \hat{f}_{j^*} \| \right) \left( |k^*_3| \| \hat{h}_{l^*} \| \right) =: B_1 + B_2 + B_3 + B_4.$$

Thanks to the Cauchy–Schwarz inequality, since $r > 2$, we have

$$\tilde{B}_1 = \sum_{j^*_3 + k^*_3 + l^*_3 = 0 \atop j^*_3, k^*_3, l^*_3 \neq 0} C_r \sum_{j^*_3 \in \mathbb{Z}^3} \frac{1}{|j^*_3|} |j^*_3| \| e^{-r|j^*|} e^{-r|k^*|} |\hat{g}_{k^*}| \| e^{-r|l^*|} |\hat{f}_{j^*}| \| e^{-r|l^*|} |\hat{h}_{l^*}|$$

$$= C_r \sum_{j^*_3 \in \mathbb{Z}^3 \atop j^*_3 \neq 0} \frac{1}{|j^*_3|} |j^*_3| \| e^{-r|j^*|} e^{-r|k^*|} |\hat{g}_{k^*}| \| e^{-r|l^*|} |\hat{f}_{j^*}| \| e^{-r|l^*|} |\hat{h}_{l^* - (j^*_3 + k^*_3)}|$$

$$\leq C_r \sum_{j^*_3 \in \mathbb{Z}^3 \atop j^*_3 \neq 0} \frac{1}{|j^*_3|} |j^*_3| \| e^{-r|j^*|} e^{-r|k^*|} |\hat{g}_{k^*}| \| e^{-r|l^*|} \left( \sum_{j^*_3 \in \mathbb{Z}^3 \atop j^*_3 \neq 0} \frac{1}{|j^*_3|} |j^*_3| \| e^{-r|j^*|} \right)^{\frac{1}{2}}$$
\[ \times \left( \sum_{j_3 \neq 0} \frac{1}{|j_3|^2} \sum_{j' \in \mathbb{Z}^2} \left| (j' + k', j_3 + k_3) \right|^{2r} |e^{2\tau |j' + k', j_3 + k_3|} \right) \frac{1}{2} \leq C_r \left\| A^{r + \frac{1}{2}} e^{\tau A} f \right\| \left\| A^{r + \frac{1}{2}} e^{\tau A} g \right\| \left\| A^{r + \frac{1}{2}} e^{\tau A} h \right\|, \]

where in the second inequality, we use Fubini theorem to exchange the order of \( \sum_{j_3 \neq 0} \) and \( \sum_{k_3 \neq 0} \). The estimate for \( \overline{B}_2 \) is similar to \( \overline{B}_1 \), and we can get \( \overline{B}_2 \leq C_r \left\| A^{r + \frac{1}{2}} e^{\tau A} f \right\| \left\| A^{r + \frac{1}{2}} e^{\tau A} g \right\| \left\| A^{r + \frac{1}{2}} e^{\tau A} h \right\| \). For \( \overline{B}_3 \), thanks to the Cauchy–Schwarz inequality, since \( r > 2 \), we have

\[ \overline{B}_3 = \sum_{j + k + 1 = 0} C_r \left\| A^{r + \frac{1}{2}} e^{\tau A} f \right\| \left\| A^{r + \frac{1}{2}} e^{\tau A} g \right\| \left\| A^{r + \frac{1}{2}} e^{\tau A} h \right\|, \]

where in the first inequality we use \( |j' + k' - j \pm k| \) due to \( r > 2 \). The estimate for \( \overline{B}_4 \) is similar as \( \overline{B}_3 \), and we can get \( \overline{B}_4 \leq C_r \left\| A^{r + \frac{1}{2}} e^{\tau A} f \right\| \left\| A^{r + \frac{1}{2}} e^{\tau A} g \right\| \left\| A^{r + \frac{1}{2}} e^{\tau A} h \right\| \). The estimates of \( \overline{B}_1 \) to \( \overline{B}_4 \) indicate that \( I_1 \) satisfies the desired inequality. \( \square \)

**Lemma A.4**. For \( f, g, h \in \mathcal{D}(e^{\tau A} : H^{r + \frac{1}{2}}) \), where \( r > \frac{5}{2} \) and \( \tau \geq 0 \), one has

\[ \left( \int \langle A^{r + \frac{1}{2}} e^{\tau A} (f \cdot \nabla) g, A^{r} e^{\tau A} h \rangle - \langle f \cdot \nabla A^{r + \frac{1}{2}} e^{\tau A} g, A^{r} e^{\tau A} h \rangle \right) \leq C_r \left\| A^{r} f \right\| \left\| A^{r} g \right\| \left\| A^{r} h \right\| + C_r \left\| A^{r + \frac{1}{2}} e^{\tau A} f \right\| \left\| A^{r + \frac{1}{2}} e^{\tau A} g \right\| \left\| A^{r + \frac{1}{2}} e^{\tau A} h \right\|. \]

**Lemma A.5**. For \( f, g, h \in \mathcal{D}(e^{\tau A} : H^{r + \frac{1}{2}}) \), where \( r > \frac{5}{2} \) and \( \tau \geq 0 \), one has

\[ \left( \int \langle A^{r + \frac{1}{2}} e^{\tau A} (\nabla \cdot f) g, A^{r} e^{\tau A} h \rangle - \langle (\nabla \cdot A^{r} e^{\tau A} f) g, A^{r + \frac{1}{2}} e^{\tau A} h \rangle \right) \leq C_r \left\| A^{r} f \right\| \left\| A^{r} g \right\| \left\| A^{r} h \right\| + C_r \left\| A^{r + \frac{1}{2}} e^{\tau A} f \right\| \left\| A^{r + \frac{1}{2}} e^{\tau A} g \right\| \left\| A^{r + \frac{1}{2}} e^{\tau A} h \right\|. \]

The proof of Lemma A.4 is similar to that of Lemma 8 in [49] since it involves nonlinear term similar to that appearing in the Euler equations. The proof of Lemma A.5 is similarly to that of Lemma A.4. Therefore, they are omitted.
The next two lemmas provide the estimates for nonlinear terms which are specific to the structure of the PEs.

**Lemma A.6.** For \( f \in D(e^{rA} : H^{r+\frac{1}{2}}) \) and \( g, h \in D(e^{sA} : H^{s+\frac{1}{2}}) \), where \( r > \frac{s}{2}, s \geq 0, \) and \( \overline{f} = 0, \) one has

\[
\left| \left\langle A' e^{rA} \left( \int_0^z \nabla \cdot f(x', s) ds \right), A' e^{sA} h \right\rangle \right| - \left\langle \left( \int_0^z \nabla \cdot f(x', s) ds \right) A' e^{sA} \partial_z g, A' e^{sA} h \right\rangle 
\leq C_r \|A'^{1} f\| \|A' g\| \|A' h\| + C_r \tau \|A'^{\frac{3}{2}} e^{sA} f\| \|A'^{\frac{3}{2}} e^{sA} g\| \|A'^{\frac{3}{2}} e^{sA} h\|.
\]

**Lemma A.7.** For \( g \in D(e^{rA} : H^{r+\frac{1}{2}}) \) and \( f, h \in D(e^{sA} : H^{s+\frac{1}{2}}) \), where \( r > \frac{s}{2}, s \geq 0, \) and \( \overline{f} = 0, \) one has

\[
\left| \left\langle A' e^{rA} \left( \int_0^z \nabla \cdot f(x', s) ds \right), A' e^{sA} h \right\rangle \right| - \left\langle \partial_z g A' e^{rA} \left( \int_0^z \nabla \cdot f(x', s) ds \right), A' e^{sA} h \right\rangle 
\leq C_r \|A'^{1} g\| \|A' f\| \|A' h\| + C_r \tau \|A'^{\frac{3}{2}} e^{rA} g\| \|A'^{\frac{3}{2}} e^{sA} f\| \|A'^{\frac{3}{2}} e^{sA} h\|.
\]

Since the proof of Lemma A.6 is similar to that of Lemma A.7, we first focus below on the proof of Lemma A.6, and later we sketch the proof of Lemma A.7 with emphasis on the main differences.

**Proof.** (proof of Lemma A.7) First, denote by \( H = A' e^{rA} h, \) and let

\[
I := \left| \left\langle A' e^{rA} \left( \int_0^z \nabla \cdot f(x', s) ds \right), A' e^{sA} h \right\rangle \right| - \left\langle \partial_z g A' e^{rA} \left( \int_0^z \nabla \cdot f(x', s) ds \right), A' e^{sA} h \right\rangle 
= \left| \left\langle \left( \int_0^z \nabla \cdot f(x', s) ds \right) \partial_z g, A' e^{rA} H \right\rangle - \left\langle \partial_z g A' e^{rA} \left( \int_0^z \nabla \cdot f(x', s) ds \right), H \right\rangle \right|.
\]

Similar to the proof of Lemma A.3, using Fourier representation of \( f, \) since \( \overline{f} = 0, \) we have

\[
\int_0^z \nabla \cdot f(x', s) ds = \sum_{j \in \mathbb{Z}^3} \frac{1}{|j|} \hat{f}_j e^{2\pi i j \cdot x'},
\]

where \( j' = (j_1, j_2). \) Using Fourier representation of \( g \) and \( H, \) we have

\[
I \leq C \sum_{j+k+l=0 \atop j_1, k_1, l_1 \neq 0} \frac{1}{|j|} \left| \hat{g}_k \right| \left| \hat{H}_l \right| \left| j' \right| \left| l \right| \left| e^{r|l|} \right| - \left| j' \right| \left| e^{r|j|} \right| ,
\]

\[
+ C \sum_{j+k+l=0 \atop k_1, l_1 \neq 0} \frac{1}{|j|} \left| \hat{f}_j \right| \left| \hat{g}_k \right| \left| \hat{H}_l \right| \left| j' \right| \left| l \right| \left| e^{r|l|} \right| - \left| (j', 0) \right| \left| e^{r|(j', 0)|} \right| := I_1 + I_2.
\]

We estimate \( I_2 \) first. By virtue of the following observation [39]: For \( r \geq 1 \) and \( \tau \geq 0, \) and for all positive \( \xi, \eta \in \mathbb{R}, \) we have

\[
|\xi^r e^{\xi} - \eta^r e^{\eta}| \leq C_r |\xi - \eta| \left( |\xi - \eta|^r - \xi^{r-1} \eta^{r-1} + \tau |\xi - \eta|^r + |\eta|^r \right)e^{\tau |\xi - \eta|} \geq |\xi - \eta| \left( |\xi - \eta|^r - \xi^{r-1} \eta^{r-1} + \tau |\xi - \eta|^r + |\eta|^r \right)e^{\tau |\xi - \eta|},
\]

with \( |\xi| = |l|, \eta = |(j', 0)| = |j'|, \) and \( |\xi - \eta| = |l| - |(j', 0)| \leq \left| \left| l \right| - |j'| \right| \leq |l| = |k|, \) inequality (A.1) implies

\[
I_2 \leq C_r \sum_{j+k+l=0 \atop k_1, l_1 \neq 0} \frac{1}{|j|} \left| \hat{f}_j \right| \left| \hat{g}_k \right| \left| \hat{H}_l \right| \left| j' \right| \left| l \right| \left| e^{r|l|} \right| - \left| (j', 0) \right| \left| e^{r|(j', 0)|} \right|.
\]
By the definition of $H$, and since $e^x \leq 1 + xe^x$ for any $x \geq 0$, we have
\[ |\tilde{H}_t| = |u|^r e^{\tau|\tilde{H}|} |\tilde{h}_t| \leq |u|^r (1 + \tau |u| e^{\tau|h|}) |\tilde{h}_t| \leq |u|^r |\tilde{h}_t| + \tau (|j'| + |k|)|\tilde{H}_t|.\]

Therefore, one obtains that
\[ |\tilde{H}_t| \left( |k|^{r-1} + |j'|^{r-1} + \tau(|k| + |j'|) e^{|k| e^{\tau|j|}} \right) \]
\[ \leq \left( |u|^r |\tilde{h}_t| + \tau (|j'| + |k|)|\tilde{H}_t| \right) \left( |k|^{r-1} + |j'|^{r-1} \right) + |\tilde{H}_t| \left( \tau(|k| + |j'|) e^{|k| e^{\tau|j|}} \right) \]
\[ \leq |\tilde{h}_t||u|^r (|k|^{r-1} + |j'|^{r-1}) + \tau C_r |\tilde{H}_t|(|k|^r + |j'|^r) e^{|k| e^{\tau|j|}}.\]

Based on this, one has
\[ I_2 \leq C_r \sum_{j', k' + l' = 0} \frac{1}{|j_1^3|} |\tilde{f}_j||\tilde{h}_t||j'|||k|^2|u|^r (|k|^{r-1} + |j'|^{r-1}) \]
\[ + \tau C_r \sum_{j' + k' + l' = 0} \frac{1}{|j_1^3|} |\tilde{f}_j||\tilde{h}_t||j'|||k|^2 (|k|^r + |j'|^r) e^{|k| e^{\tau|j|}} := I_{21} + I_{22}.\]

Here
\[ I_{21} = C_r \left( \sum_{j', k', l' = 0} \frac{1}{|j_1^3|} |\tilde{f}_j||\tilde{h}_t||j'|||k|^{|r+1}|u|^r + \frac{1}{|j_1^3|} |\tilde{f}_j||\tilde{h}_t||j'|||k|^2|u|^r \right) := I_{211} + I_{212}.\]

Thanks to the Cauchy–Schwarz inequality, since $r > \frac{5}{2}$, we have
\[ I_{211} = C_r \sum_{j', k' + l' = 0} \frac{1}{|j_1^3|} |\tilde{f}_j||\tilde{h}_t||j'|||k|^{|r+1}|(j' + k', k_3)|^r|\tilde{h}_{-(j' + k', k_3)}| \]
\[ \leq C_r \left( \sum_{j', k' + l' = 0} \frac{1}{|j_1^3|} |\tilde{f}_j||\tilde{h}_t||j'|||k|^{|r+1}|(j' + k', k_3)|^r|\tilde{h}_{-(j' + k', k_3)}| \right)^\frac{1}{2} \]
\[ \times \sup_{j' \in \mathbb{Z}^3} \left( \sum_{k' + l' = 0} |(j' + k', k_3)|^{2r} |\tilde{h}_{-(j' + k', k_3)}|^2 \right)^\frac{1}{2} \leq C_r ||A^r f|| ||A^{r+1} g|| ||A^r h||, \]
and
\[ I_{212} = C_r \sum_{j', k' + l' = 0} \frac{1}{|j_1^3|} |\tilde{f}_j||\tilde{h}_t||j'|||k|^{|r+1}|(j' + k', k_3)|^r|\tilde{h}_{-(j' + k', k_3)}| \]
\[ \leq C_r \sum_{k' \in \mathbb{Z}^3} |(k', \pm 1)|^{1-r} \sum_{k_3 \neq 0} |k|^{r+1}|\tilde{h}_t||j'|||k|^{|r+1}|(j' + k', k_3)|^r|\tilde{h}_{-(j' + k', k_3)}| \]
\[ \leq C_r ||A^r f|| \sum_{k' \in \mathbb{Z}^2} |(k', \pm 1)|^{1-r} \left( \sum_{k_3 \neq 0} |k|^{2r+2} \right)^\frac{1}{2}.\]
\[
\times \left( \sum_{k_3 \neq 0} \sum_{j' \in \mathbb{Z}^2} |(j' + k', k_3)|^{2r} |\hat{h}_{-j' + k', k_3}|^2 \right)^{\frac{1}{2}}
\]
\[
\leq C_r \|A^r f\| \|A^r h\| \left( \sum_{k' \in \mathbb{Z}^2} |(k', \pm 1)|^{2-2r} \right)^{\frac{1}{2}} \left( \sum_{k' \in \mathbb{Z}^2} \sum_{k_3 \neq 0} |k|^{2r+2} |\hat{g}_k|^2 \right)^{\frac{1}{2}}
\]
\[
\leq C_r \|A^r f\| \|A^{r+1} g\| \|A^r h\|.
\]

Next, for \(I_{22}\), we have
\[
I_{22} = \tau C_r \sum_{j' + k' + t' = 0} \frac{1}{|j'|} |\hat{f}_{j'}| |\hat{g}_k| |\tilde{H}_l| |j'| |k|^{r+1} e^{\tau |k|} e^{\tau |j|}
\]
\[
+ \tau C_r \sum_{j' + k' + t' = 0} \frac{1}{|j'|} |\hat{f}_{j'}| |\hat{g}_k| |\tilde{H}_l| |j'| |k|^{r+1} e^{\tau |k|} e^{\tau |j|} := I_{221} + I_{222}.
\]

Noticing that \(|k|^\frac{1}{r} \leq C(|j'|^{\frac{1}{2r}} + |l|^{\frac{1}{2r}})\) and \(|j'|^{\frac{1}{2}} \leq C(|k|^\frac{1}{2r} + |l|^{\frac{1}{2r}})\), thanks to the Cauchy–Schwarz inequality, since \(r > \frac{5}{2}\), we have
\[
I_{221} = \tau C_r \sum_{j' + k' + t' = 0} \frac{1}{|j'|} |\hat{f}_{j'}| |\hat{g}_k| |\tilde{H}_l| |j'| |k|^{r+2} e^{\tau |k|} e^{\tau |j|}
\]
\[
\leq \tau C_r \sum_{j' + k' + t' = 0} \frac{1}{|j'|} |\hat{f}_{j'}| |\hat{g}_k| |\tilde{H}_l| |j'| |k|^{r+\frac{3}{2}} (|j'|^{\frac{1}{2r}} + |l|^{\frac{1}{2r}}) e^{\tau |k|} e^{\tau |j|} e^{\tau |l|}
\]
\[
\leq \tau C_r \sum_{j' + k' + t' = 0} \frac{1}{|j'|} |\hat{f}_{j'}| |\hat{g}_k| |\tilde{H}_l| |j'| |k|^{r+\frac{3}{2}} e^{\tau |k|} e^{\tau |j|} e^{\tau |l|}
\]
\[
\leq \tau C_r \left( \sum_{j' \in \mathbb{Z}^2} \frac{1}{|j'|^{2-2r}} \right)^{\frac{1}{2}} \left( \sum_{j' \in \mathbb{Z}^2} \sum_{k_3 \neq 0} |j'|^{2r+1} e^{2\tau |j'|} |\hat{f}_{j'}|^2 \right)^{\frac{1}{2}} \left( \sum_{k_3 \neq 0} |k|^{2r+3} e^{2\tau |k|} |\hat{g}_k|^2 \right)^{\frac{1}{2}}
\]
\[
\times \sup_{j' \in \mathbb{Z}^2} \left( \sum_{k_3 \neq 0} |j' + k', k_3|^{2r+1} e^{2\tau |j' + k', k_3|} |\hat{h}_{-j' + k', k_3}|^2 \right)^{\frac{1}{2}}
\]
\[
\leq \tau C_r \|A^{r+\frac{3}{2}} e^{-A} f\| \|A^{r+\frac{3}{2}} e^{-A} g\| \|A^{r+\frac{3}{2}} e^{-A} h\|, \text{ and}
\]
\[
I_{222} = \tau C_r \sum_{j' + k' + t' = 0} \frac{1}{|j'|} |\hat{f}_{j'}| |\hat{g}_k| |\tilde{H}_l| |j'|^{r+1} |k|^{r+1} e^{\tau |k|} e^{\tau |j|}
\]
\[
\leq \tau C_r \sum_{j' + k' + t' = 0} \frac{1}{|j'|} |\hat{f}_{j'}| |\hat{g}_k| |\tilde{H}_l| |j'|^{r+\frac{3}{2}} |k|^{r+\frac{3}{2}} (|k|^{\frac{1}{2}} + |l|^{\frac{1}{2}}) e^{\tau |k|} e^{\tau |j|} e^{\tau |l|}
\]
Therefore, $I_2$ satisfies the desired estimates.

To estimate $I_1$, we use (A.11) with $\xi = |l|$, $\eta = |j|$, and with $|\xi - \eta| = |l - j| = |k|$, to obtain

$$I_1 \leq C_r \sum_{j+k+l=0 \atop j,s,k,j' \neq 0} \frac{1}{|j|} |\hat{f}_j| |\hat{g}_k| |\hat{h}_l||j'||k|^{-r} \left( |k|^{-r+1} + |j|^{-r+1} + \tau (|k|^{r-1} + |j|^{r-1}) e^{-r|k|} e^{-r|j|} \right).$$

(A.3)

Thanks to (A.3), one obtains that

$$I_1 \leq C_r \sum_{j+k+l=0 \atop j,s,k,j' \neq 0} \frac{1}{|j|} |\hat{f}_j| |\hat{g}_k| |\hat{h}_l||j'||k|^{-r} (|k|^{-r+1} + |j|^{-r+1})$$

$$+ \tau C_r \sum_{j+k+l=0 \atop j,s,k,j' \neq 0} \frac{1}{|j|} |\hat{f}_j| |\hat{g}_k| |\hat{h}_l||j'||k|^{r-1} (|k|^{r-1} + |j|^{r-1}) e^{-r|k|} e^{-r|j|} := I_{11} + I_{12}.$$

Here

$$I_{11} \leq C_r \left( \sum_{j+k+l=0 \atop j,s,k,j' \neq 0} \frac{1}{|j|} |\hat{f}_j| |\hat{g}_k| |\hat{h}_l||j'||k|^{r+1} |l|^r + \frac{1}{|j|} |\hat{f}_j| |\hat{g}_k| |\hat{h}_l||j'||k|^{r-1} |l|^r \right) := I_{111} + I_{112}.$$

Thanks to the Cauchy–Schwarz inequality, since $r > \frac{5}{2}$, we have

$$I_{111} = C_r \sum_{j+k+l=0 \atop j,s,k,j' \neq 0} \frac{1}{|j|} |\hat{f}_j| \sum_{k \in \mathbb{Z}^3 \atop k,j' \neq 0} |k|^{r+1} |j + k||\hat{g}_k||\hat{h}_{-(j+k)}||l|^r$$

$$\leq C_r \left( \sum_{j+k+l=0 \atop j,s,k,j' \neq 0} |j|^{2-2r} \right)^{\frac{1}{2}} \left( \sum_{j+k+l=0 \atop j,s,k,j' \neq 0} |j|^{2r} |\hat{f}_j|^2 \right)^{\frac{1}{2}} \left( \sum_{k \in \mathbb{Z}^3 \atop k,j' \neq 0} |k|^{2r+2} |\hat{g}_k|^2 \right)^{\frac{1}{2}}$$
\[ \times \sup_{j \in \mathbb{Z}^3} \left( \sum_{k \in \mathbb{Z}^3 \atop k_3 \neq 0} |j + k|^2 |\hat{h}_{-j+k}|^2 \right)^{\frac{1}{2}} \leq C_r \| A^r f \| \| A^{r+1} g \| \| A^r h \|, \text{ and} \]

\[ I_{112} = C_r \sum_{k \in \mathbb{Z}^3 \atop k_3 \neq 0} |k|^2 |\hat{g}_k| \sum_{j \in \mathbb{Z}^3 \atop j_3 \neq j'} \frac{1}{|j_3|} |j|^r |\hat{f}_j| |j + k|^r |\hat{h}_{-j+k}| \]
\[ \leq C_r \left( \sum_{k \in \mathbb{Z}^3 \atop k_3 \neq 0} |k|^{2-2r} \right)^{\frac{1}{2}} \left( \sum_{j \in \mathbb{Z}^3 \atop j_3 \neq j'} |k|^{2r+2} |\hat{g}_k|^2 \right)^{\frac{1}{2}} \left( \sum_{j \in \mathbb{Z}^3 \atop j_3 \neq j'} |j|^{2r} |\hat{f}_j|^2 \right)^{\frac{1}{2}} \]
\[ \times \sup_{k \in \mathbb{Z}^3} \left( \sum_{j \in \mathbb{Z}^3} |j + k|^2 |\hat{h}_{-j+k}|^2 \right)^{\frac{1}{2}} \leq C_r \| A^r f \| \| A^{r+1} g \| \| A^r h \|. \]

Next, for \( I_{12} \), we have
\[ I_{12} \leq \tau C_r \sum_{j + k + l = 0 \atop j_3, k_3, j' \neq 0} \frac{1}{|j_3|} |\hat{f}_j| |\hat{g}_k| |\hat{h}_l||j|^r |k|^r e^{\tau |k|} |e^{\tau |j|} |l|^2 \]
\[ + \tau C_r \sum_{j + k + l = 0 \atop j_3, k_3, j' \neq 0} \frac{1}{|j_3|} |\hat{f}_j| |\hat{g}_k| |\hat{h}_l||j|^r |k|^r e^{\tau |k|} |e^{\tau |j|} |l|^2 \]
\[ \leq \tau C_r \sum_{j + k + l = 0 \atop j_3, k_3, j' \neq 0} \frac{1}{|j_3|} |\hat{f}_j| |\hat{g}_k| |\hat{h}_l||j|^r |k|^r e^{\tau |k|} |e^{\tau |j|} |l|^2 \]
\[ \leq \tau C_r \left( \sum_{j \in \mathbb{Z}^3 \atop j_3, j' \neq 0} |j|^{2-2r} \right)^{\frac{1}{2}} \left( \sum_{j \in \mathbb{Z}^3 \atop j_3, j' \neq 0} |j|^{2r+1} e^{2r |j|} |\hat{f}_j|^2 \right)^{\frac{1}{2}} \left( \sum_{k \in \mathbb{Z}^3 \atop k_3 \neq 0} |k|^{2r+3} e^{2r |k|} |\hat{g}_k|^2 \right)^{\frac{1}{2}} \]
\[ \times \sup_{k \in \mathbb{Z}^3} \left( \sum_{j \in \mathbb{Z}^3 \atop j_3, j' \neq 0} |j + k|^2 |\hat{h}_{-j+k}|^2 \right)^{\frac{1}{2}} \leq \tau C_r \| A^{r+\frac{1}{2}} e^{\tau A} f \| \| A^{r+\frac{1}{2}} e^{\tau A} g \| \| A^{r+\frac{1}{2}} e^{\tau A} h \|, \text{ and} \]
\[ I_{122} = \tau C_r \sum_{j + k + l = 0 \atop j_3, k_3, j' \neq 0} \frac{1}{|j_3|} |\hat{f}_j| |\hat{g}_k| |\hat{h}_l||j|^r |k|^2 e^{\tau |k|} |e^{\tau |j|} |l| \]
\begin{align*}
&\leq \tau C_r \sum_{j+k+l=0} \frac{1}{|j| |\hat{f}_j||\hat{g}_k||\hat{H}_l||j||l|^{\tau+\frac{1}{2}}|k|^2|l|^{\tau}} (|k|^\tau + |l|^{\tau} e^{\tau|j|} e^{\tau|l|}) \\
&\leq \tau C_r \sum_{j+k+l=0} \frac{1}{|j| |\hat{f}_j||\hat{g}_k||\hat{H}_l||j||l|^{\tau+\frac{1}{2}}|k|^2|l|^{\tau}} e^{\tau|j|} e^{\tau|l|} \\
&\leq \tau C_r \left( \sum_{k \in \mathbb{Z}^3} |k|^{2r-2} \right)^{\frac{1}{2}} \left( \sum_{k \in \mathbb{Z}^3} |k|^{2r+3} e^{2\tau|k|} |\hat{g}_k|^2 \right)^{\frac{1}{2}} \left( \sum_{j \in \mathbb{Z}^3} |j|^{2r+1} e^{2\tau|j|} |\hat{f}_j|^2 \right)^{\frac{1}{2}} \\
&\times \sup_{j \in \mathbb{Z}^3} \left( \sum_{j+k \neq 0} |j + k|^{2r+1} e^{2\tau|j+k|} |\hat{H}_{j+k}|^2 \right)^{\frac{1}{2}} \\
&\leq \tau C_r \|A^{r+\frac{1}{2}} e^{\tau A} f\| \|A^{r+\frac{1}{2}} e^{\tau A} g\| \|A^{r+\frac{1}{2}} e^{\tau A} h\|.
\end{align*}

Therefore, \( I_1 \) satisfies the desired estimates. The proof is completed. \qedhere

Finally, we sketch the proof of Lemma \( A.6 \).

Proof. (proof of Lemma \( A.6 \)) Similar to the proof of Lemma \( A.7 \) we have

\begin{align*}
I := \left| \left( \int_0^2 \nabla \cdot f(x', s) ds \right) A^{r} e^{\tau A} h \right| - \left| \left( \int_0^2 \nabla \cdot f(x', s) ds \right) A^{r+\frac{1}{2}} e^{\tau A} h \right|
\end{align*}

\begin{align*}
&= \left| \left( \int_0^2 \nabla \cdot f(x', s) ds \right) A^{r} e^{\tau A} \partial_x g, A^{r+\frac{1}{2}} e^{\tau A} h \right|
\end{align*}

\begin{align*}
&\leq C \sum_{j+k+l=0} \frac{1}{|j| |\hat{f}_j||\hat{g}_k||\hat{H}_l||j||l|^{\tau}} \| |k|^\tau \| \| |l|^\tau \| - |\hat{g}_k|^{2r} |\hat{g}_k|^{2r} |\hat{H}_l|^{2r} |\hat{H}_l|^{2r} \| |j||l|^{\tau} \| - |\hat{g}_k|^{2r} |\hat{g}_k|^{2r} |\hat{H}_l|^{2r} |\hat{H}_l|^{2r} \| \right)
\end{align*}

\begin{align*}
+ C \sum_{j+k+l=0} \frac{1}{|j| |\hat{f}_j||\hat{g}_k||\hat{H}_l||j||l|^{\tau}} \| |k|^\tau \| \| |l|^\tau \| - |\hat{g}_k|^{2r} |\hat{g}_k|^{2r} |\hat{H}_l|^{2r} |\hat{H}_l|^{2r} \| \right)
\end{align*}

For \( I_1 \), since \( j + k + l = 0 \), we use \( A.1 \) with \( \xi = |l|, \eta = |k| \) and \( |\xi - \eta| = |l| - |k| \leq |l| - |l| = |j| \), to conclude

\begin{align*}
I_1 \leq C_r \sum_{j+k+l=0} \frac{1}{|j| |\hat{f}_j||\hat{g}_k||\hat{H}_l||j||l|^{\tau}} \| |k|^\tau \| \| |l|^\tau \| - |\hat{g}_k|^{2r} |\hat{g}_k|^{2r} |\hat{H}_l|^{2r} |\hat{H}_l|^{2r} \| \right)
\end{align*}

For \( I_2 \), since \( j = 0 \), \( k + l = 0 \), we use \( A.1 \) with \( \xi = |l|, \eta = |k| \) and \( |\xi - \eta| = |l| - |l| \leq |l| - |l| = |j| \), to obtain

\begin{align*}
I_2 \leq C_r \sum_{j+k+l=0} \frac{1}{|j| |\hat{f}_j||\hat{g}_k||\hat{H}_l||j||l|^{\tau}} \| |k|^\tau \| \| |l|^\tau \| - |\hat{g}_k|^{2r} |\hat{g}_k|^{2r} |\hat{H}_l|^{2r} |\hat{H}_l|^{2r} \| \right)
\end{align*}

Observe that the difference between the sums in the right-hand sides of \( A.4 \) and \( A.3 \) is manifested in the factors \( |j||l|^{\tau} \| \) and \( |j||l|^{\tau} \| \), and between \( A.5 \) and \( A.2 \) is manifested in the factors \( |j||l|^{\tau} \| \) and
Therefore, one can follow the estimates of $I_1$ in (A.3) and $I_2$ in (A.2), and obtain that $I_1$ in (A.4) and $I_2$ in (A.5) satisfy the desired bound in Lemma A.6.

\[\square\]

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