On the relation between Auslander-Reiten \((d + 2)\)-angles and Serre duality

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Abstract

Let \( \mathcal{C} \) be an \((d + 2)\)-angulated category with an \(d\)-suspension functor \(\Sigma^d\). Our main results show that every Serre functor on \( \mathcal{C} \) is an \((d + 2)\)-angulated functor. We also show that \( \mathcal{C} \) has a Serre functor \(S\) if and only if \( \mathcal{C} \) has Auslander-Reiten \((d + 2)\)-angles. Moreover, \( \tau_d = S\Sigma^{-d} \) where \( \tau_d \) is \(d\)-Auslander-Reiten translation. These results generalize the works by Bondal-Kapranov and Reiten-Van den Bergh. In addition, we prove that for a strongly functorially finite subcategory \( \mathcal{X} \) of \( \mathcal{C} \), the quotient category \( \mathcal{C}/\mathcal{X} \) is an \((d + 2)\)-angulated category if and only if \( (\mathcal{C}, \mathcal{C}) \) is an \(\mathcal{X}\)-mutation pair, and if and only if \( \tau_d \mathcal{X} = \mathcal{X} \). This generalizes a result by Jørgensen who proved the equivalence between the first and the third conditions for triangulated categories.

Key words: \((d + 2)\)-angulated categories; Auslander-Reiten \((d + 2)\)-angles; Serre duality.

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1 Introduction

Unless otherwise specified \( k \) will be an algebraically closed field and all categories in this article will be \(k\)-linear Hom-finite. Let \( \mathcal{C} \) be an additive category. We say that \( \mathcal{C} \) has a Serre functor \(S\), that is, an auto-equivalence for which there are a natural equivalence
\[
\text{Hom}_\mathcal{C}(X, Y) \simeq \text{Hom}_\mathcal{C}(Y, SX)^*
\]
for any \( X, Y \in \mathcal{C} \), where \((-)^* = \text{Hom}_k(-, k) \) is the \( k \)-linear duality functor.

Bondal and Kapranov showed the following result. Van den Bergh has also given other proof of this result in the appendix of this article [B, Theorem A.4.4].

**Theorem 1.1.** [BK, Proposition 3.3] Let \( \mathcal{C} \) be a triangulated category. Then every Serre functor is a triangulated functor.

Reiten and Van den Bergh gave a connection between Auslander-Reiten triangles and Serre duality.

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**Theorem 1.2.** [RV, Theorem I.2.4] Let $\mathcal{C}$ be a triangulated category with shift functor $\Sigma$. Then $\mathcal{C}$ has a Serre functor $S$ if and only if $\mathcal{C}$ has Auslander-Reiten triangles. Moreover, $\tau = S\Sigma^{-1}$ where $\tau$ is Auslander-Reiten translation.

Quotient categories come in a number of different flavours. The one to be considered here is probably the most basic. Let $\mathcal{C}$ be an additive category with a subcategory $\mathcal{X}$. For objects $A, B \in \mathcal{C}$, denote by $\mathcal{X}(A, B)$ all the morphisms from $A$ to $B$ which factor through an object of $\mathcal{X}$. Then the quotient category $\mathcal{C}/\mathcal{X}$ has the same objects as $\mathcal{C}$, and its morphism spaces are defined by

$$\text{Hom}_{\mathcal{C}/\mathcal{X}}(A, B) = \text{Hom}_{\mathcal{C}}(A, B)/\mathcal{X}(A, B).$$

Jørgensen gave the necessary and sufficient condition for quotient categories to be triangulated categories.

**Theorem 1.3.** [J, Theorem 3.3] Let $\mathcal{C}$ be a triangulated category with a Serre functor and $\mathcal{X}$ a functorially finite subcategory of $\mathcal{C}$. Then the quotient category $\mathcal{C}/\mathcal{X}$ is a triangulated category if and only if $\tau_{\mathcal{X}} = \mathcal{X}$.

In [GKO], Geiß, Keller and Oppermann introduced $(d + 2)$-angulated categories. These are generalizations of triangulated categories, in the sense that triangles are replaced by $(d + 2)$-angles, that is, morphism sequences of length $d + 2$. Thus a 1-angled category is precisely a triangulated category. Iyama and Yoshino [IY] defined Auslander-Reiten $(d + 2)$-angle in $(d + 2)$-angulated categories. Later, Fedele [F] defined Auslander-Reiten $(d + 2)$-angles in additive subcategories of $(d + 2)$-angulated categories closed under $d$-extensions, an example of which are wide subcategories.

In this article we will generalise these results into the higher homological case. Moreover, our proof is not far from the usual triangulated case. We hope that our work would motivate further study on $(d + 2)$-angulated categories.

Our first main result is following.

**Theorem 1.4.** (see Theorem 3.3 for details) Let $\mathcal{C}$ be an $(d + 2)$-angulated category with an $d$-suspension functor $\Sigma^d$ and a right Serre functor $F$. Then there exists a natural isomorphism $\zeta: F\Sigma^d \rightarrow \Sigma^d F$ such that $F: \mathcal{C} \rightarrow \mathcal{C}$ is an $(d + 2)$-angulated functor.

Our second main result is following.

**Theorem 1.5.** (see Theorem 4.5 for details) Let $\mathcal{C}$ be a Krull-Schmidt $(d + 2)$-angulated category with an $d$-suspension functor $\Sigma^d$. Then $\mathcal{C}$ has Auslander-Reiten $(d + 2)$-angles if and only if $\mathcal{C}$ has a Serre functor $S$. Moreover $\tau_d = S\Sigma^{-d}$ where $\tau_d$ is $d$-Auslander-Reiten translation.

Our third main result is following.

**Theorem 1.6.** (see Theorem 5.6 for details) Let $\mathcal{C}$ be a Krull-Schmidt $(d + 2)$-angulated category with an $d$-suspension functor $\Sigma^d$ and a Serre functor $S$, and $\mathcal{X}$ be a strongly functorially finite subcategory of $\mathcal{C}$. Then the following statements are equivalent:
(1) \((\mathcal{C}, \mathcal{C}')\) is an \(\mathcal{X}'\)-mutation pair.

(2) The quotient category \(\mathcal{C}/\mathcal{X}\) is an \((d+2)\)-angulated category.

(3) \(\tau_d \mathcal{X}' = \mathcal{X}'\).

The article is organised as follows: In Section 2, we review some elementary definitions that we need to use, including \((d+2)\)-angulated categories and Auslander-Reiten \((d+2)\) angles. In Section 3, we show the first main result. In Section 4, we prove the second main result and give an example illustrating this result. In Section 5, we show the third main result.

2 Preliminaries

2.1 (Right) \((d+2)\)-angulated categories

Let \(\mathcal{C}\) be an additive category with an endofunctor \(\Sigma^d : \mathcal{C} \to \mathcal{C}\). An \((d+2)\)-\(\Sigma^d\)-sequence in \(\mathcal{C}\) is a sequence of morphisms

\[
A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{d-1}} A_d \xrightarrow{f_d} A_{d+1} \xrightarrow{f_{d+1}} \Sigma^d A_0.
\]

Its left rotation is the \((d+2)\)-\(\Sigma^d\)-sequence

\[
A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{d}} A_{d+1} \xrightarrow{f_{d+1}} \Sigma^d A_0 \xrightarrow{(-1)^d \Sigma^d f_0} \Sigma^d A_1.
\]

A morphism of \((d+2)\)-\(\Sigma^d\)-sequences is a sequence of morphisms \(\varphi = (\varphi_0, \varphi_1, \cdots, \varphi_{d+1})\) such that the following diagram commutes

\[
\begin{array}{cccccccc}
A_0 & \xrightarrow{f_0} & A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_d} & A_{d+1} & \xrightarrow{f_{d+1}} & \Sigma^d A_0 \\
B_0 & \xrightarrow{g_0} & B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & \cdots & \xrightarrow{g_d} & B_{d+1} & \xrightarrow{g_{d+1}} & \Sigma^d B_0 \\
\varphi_0 & & \varphi_1 & & \varphi_2 & & \cdots & & \varphi_{d+1} & & \Sigma^d \varphi_0
\end{array}
\]

where each row is an \((d+2)\)-\(\Sigma^d\)-sequence. It is an isomorphism if \(\varphi_0, \varphi_1, \varphi_2, \cdots, \varphi_{d+1}\) are all isomorphisms in \(\mathcal{C}\), and a weak isomorphism if \(\varphi_i\) and \(\varphi_{i+1}\) are isomorphisms for some \(0 \leq i \leq d+1\) (with \(\varphi_{d+2} := \Sigma \varphi_0\)). Note that the composition of two weak isomorphisms need not be a weak isomorphism.

We recall the notion of a right \((d+2)\)-angulated category from [L2, Definition 2.1].

**Definition 2.1.** A right \((d+2)\)-angulated category is a triple \((\mathcal{C}, \Sigma^d, \Theta)\), where \(\mathcal{C}\) is an additive category, \(\Sigma^d\) is an endofunctor of \(\mathcal{C}\) (\(\Sigma^d\) is called the \(d\)-suspension functor), and \(\Theta\) is a class of \((d+2)\)-\(\Sigma^d\)-sequences (whose elements are called right \((d+2)\)-angles), which satisfies the following axioms:

(N1) (a) The class \(\Theta\) is closed under isomorphisms, direct sums and direct summands.

(b) For each object \(A \in \mathcal{C}\) the trivial sequence

\[
A \xrightarrow{\text{Id}_A} A \to 0 \to 0 \to \cdots \to 0 \to \Sigma^d A
\]
belongs to Θ.

(c) Each morphism \( f_0 : A_0 \to A_1 \) in \( \mathcal{C} \) can be extended to a right \( (d + 2) \)-angle:

\[
A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{d-1}} A_d \xrightarrow{f_d} A_{d+1} \xrightarrow{f_{d+1}} \Sigma^d A_0.
\]

(N2) If an \( (d + 2) \)-\( \Sigma^d \)-sequence belongs to \( \Theta \), then its left rotation belongs to \( \Theta \).

(N3) Each solid commutative diagram

\[
\begin{array}{ccccccccc}
A_0 & \xrightarrow{f_0} & A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_{d-1}} & A_d & \xrightarrow{f_d} & A_{d+1} & \xrightarrow{f_{d+1}} & \Sigma^d A_0 \\
\phi_0 & \downarrow & \phi_1 & \downarrow & \phi_2 & \downarrow & \cdots & \downarrow & \phi_{d+1} & \downarrow & \Sigma^d \phi_0 \\
B_0 & \xrightarrow{g_0} & B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & \cdots & \xrightarrow{g_{d-1}} & B_d & \xrightarrow{g_d} & B_{d+1} & \xrightarrow{g_{d+1}} & \Sigma^d B_0
\end{array}
\]

with rows in \( \Theta \) can be completed to a morphism of \( (d + 2) \)-\( \Sigma^d \)-sequences.

(N4) Given a commutative diagram

\[
\begin{array}{ccccccccc}
A_0 & \xrightarrow{f_0} & A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_{d-1}} & A_d & \xrightarrow{f_d} & A_{d+1} & \xrightarrow{f_{d+1}} & \Sigma^d A_0 \\
\phi_1 & \downarrow & \phi_2 & \downarrow & \phi_3 & \downarrow & \cdots & \downarrow & \phi_{d+1} & \downarrow & \Sigma^d \phi_0 \\
B_0 & \xrightarrow{g_0} & B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & \cdots & \xrightarrow{g_{d-1}} & B_d & \xrightarrow{g_d} & B_{d+1} & \xrightarrow{g_{d+1}} & \Sigma^d B_0
\end{array}
\]

whose top rows and second column belong to \( \Theta \). Then there exist morphisms \( \varphi_i : A_i \to B_i \) \( (i = 2, 3, \cdots, d + 1) \), \( \psi_j : B_j \to C_j \) \( (j = 2, 3, \cdots, d + 1) \) and \( \phi_k : A_k \to C_{k-1} \) \( (k = 3, 4, \cdots, d + 1) \) with the following two properties:

(I) The sequence \((1_{A_1}, \varphi_1, \varphi_2, \cdots, \varphi_{d+1})\) is a morphism of \( (d + 2) \)-\( \Sigma^d \)-sequences.

(II) The \( (d + 2) \)-\( \Sigma^d \)-sequence

\[
A_2 \xrightarrow{(f_2, \psi_2)} A_3 \oplus B_2 \xrightarrow{(-f_3 \ g_3 \ \psi_3)} A_4 \oplus B_3 \oplus C_2 \xrightarrow{(-f_4 \ g_4 \ \psi_4 \ \phi_4)} A_5 \oplus B_4 \oplus C_3
\]
On the relation between Auslander-Reiten \((d + 2)\)-angles and Serre duality

\[
\begin{pmatrix}
-f_2 & 0 & 0 \\
\varphi_5 & -g_4 & 0 \\
\varphi_d & -g_{d-1} & 0
\end{pmatrix}
\rightarrow \cdots
\rightarrow
\begin{pmatrix}
-f_d & 0 & 0 \\
\varphi_{d+1} & -g_{d-1} & 0 \\
\varphi_d & -g_{d-1} & h_{d-2}
\end{pmatrix}
\rightarrow A_{d+1} \oplus B_d \oplus C_{d-1}
\]

\[
\begin{pmatrix}
(1)^{d+1} \psi_{d+1} & -g_d & 0 \\
\varphi_{d+1} & -g_{d-1} & 0
\end{pmatrix}
\rightarrow
B_{d+1} \oplus C_d \rightarrow C_{d+1} \rightarrow \Sigma^d f_0 \circ g_{d+1} = \Sigma^d A_2
\]

belongs to \(\Theta\), and \(h_{d+1} \circ \psi_{d+1} = \Sigma^d f_0 \circ g_{d+1}\).

The notion of a left \((d + 2)\)-angulated category is defined dually.

If \(\Sigma^d\) is an automorphism, it is easy to see that the converse of an axiom (N2) also holds, thus the right \((d + 2)\)-angulated category \((\mathcal{C}, \Sigma^d, \Theta)\) is an \((d + 2)\)-angulated category in the sense of Geiss-Keller-Oppermann [GKO] Definition 1.1] and in the sense of Bergh-Thaule [BT1, Theorem 4.4]. If \((\mathcal{C}, \Sigma^d, \Theta)\) is a right \((d + 2)\)-angulated category, \((\mathcal{C}, \Omega, \Phi)\) is a left \((d + 2)\)-angulated category, \(\Omega\) is a quasi-inverse of \(\Sigma^d\) and \(\Theta = \Phi\), then \((\mathcal{C}, \Sigma^d, \Theta)\) is an \((d+2)\)-angulated category.

Assume that \(\Sigma^d\) is an automorphism. Now modify axiom (N1) into a new axiom (N1*).

(N1*) (a) The class \(\Theta\) is closed under weak isomorphisms.

(b) For each object \(A \in \mathcal{C}\) the trivial sequence

\[
A \xrightarrow{\text{Id}_A} A \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow \cdots \rightarrow 0 \rightarrow \Sigma^d A
\]

belongs to \(\Theta\).

(c) Each morphism \(f_0: A_0 \rightarrow A_1\) in \(\mathcal{C}\) can be extended to an \((d + 2)\)-angle:

\[
A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{d-1}} A_d \xrightarrow{f_d} A_{d+1} \xrightarrow{f_{d+1}} \Sigma^d A_0.
\]

Remark 2.2. [BT1, Theorem 3.2] If \(\Theta\) is a collection of \((d + 2)\)-\(\Sigma^d\)-sequences satisfying the axioms (N2) and (N3), and \(\Sigma^d\) is an automorphism. then the following are equivalent:

(1) \(\Theta\) satisfies (N1);

(2) \(\Theta\) satisfies (N1)*.

2.2 Auslander-Reiten \((d + 2)\)-angles

We denote by \(\text{rad}_\mathcal{C}\) the Jacobson radical of \(\mathcal{C}\). Namely, \(\text{rad}_\mathcal{C}(A, A)\) coincides with the Jacobson radical of the endomorphism ring \(\text{End}_\mathcal{C}(A)\) for any \(A \in \mathcal{C}\).

Definition 2.3. [IY, Definition 3.8] and [E, Definition 5.1] Let \(\mathcal{C}\) be an \((d + 2)\)-angulated category. An \((d + 2)\)-angle

\[
A_\bullet : A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} A_{d+1} \xrightarrow{\alpha_{d+1}} \Sigma^d A_0
\]

in \(\mathcal{C}\) is called an \textit{Auslander-Reiten \((d + 2)\)-angle} if \(\alpha_0\) is left almost split, \(\alpha_d\) is right almost split and when \(d \geq 2\), also \(\alpha_1, \alpha_2, \cdots, \alpha_{d-1}\) are in \(\text{rad}_\mathcal{C}\).
Remark 2.4. [F, Remark 5.2] Assume $A_\bullet$ as in the above definition is an Auslander-Reiten $(d + 2)$-angle. Since $\alpha_0$ is left almost split implies that $\text{End}(A_0)$ is local and hence $A_0$ is indecomposable. Similarly, since $\alpha_d$ is right almost split, then $\text{End}(A_{d+1})$ is local and hence $A_{d+1}$ is indecomposable. Moreover, when $d = 1$, we have $\alpha_0$ and $\alpha_1$ in $\text{rad} C$, so that $\alpha_d$ is right minimal and $\alpha_0$ is left minimal. When $d \geq 2$, since $\alpha_{d-1} \in \text{rad} C$, we have that $\alpha_d$ is right minimal and similarly $\alpha_0$ is left minimal.

Remark 2.5. [F, Lemma 5.3] Let $C$ be an $(d + 2)$-angulated category and

$$A_\bullet : A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} A_{d+1} \xrightarrow{\alpha_{d+1}} \Sigma^d A_0$$

be an $(d + 2)$-angle in $C$. Then the following are equivalent:

1. $A_\bullet$ is an Auslander-Reiten $(d + 2)$-angle;
2. $\alpha_0, \alpha_1, \cdots, \alpha_{d-1}$ are in $\text{rad} C$ and $\alpha_d$ is right almost split;
3. $\alpha_1, \alpha_2, \cdots, \alpha_d$ are in $\text{rad} C$ and $\alpha_0$ is left almost split.

Lemma 2.6. [F, Lemma 5.4] Let $C$ be an $(d + 2)$-angulated category and

$$A_\bullet : A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} A_{d+1} \xrightarrow{\alpha_{d+1}} \Sigma^d A_0$$

be an $(d + 2)$-angle in $C$. Assume that $\alpha_d$ is right almost split and if $d \geq 2$, also that $\alpha_1, \alpha_2, \cdots, \alpha_{d-1}$ are in $\text{rad} C$. Then the following are equivalent:

1. $A_\bullet$ is an Auslander-Reiten $(d + 2)$-angle;
2. $\text{End}(A_0)$ is local;
3. $\alpha_{d+1}$ is left minimal;
4. $\alpha_0$ is in $\text{rad} C$.

3 Every Serre functor is an $(d + 2)$-angulated functor

Let $C$ be a $k$-linear Hom-finite additive category. When there is no danger of confusion, we will sometimes instead of $\text{Hom}_C(X, A) \xrightarrow{\text{Hom}_C(X, f)} \text{Hom}_C(X, B)$ write one of the following simplified forms:

$$\text{Hom}(X, A) \xrightarrow{\text{Hom}(X, f)} \text{Hom}(X, B)$$

$$(X, A) \xrightarrow{(X, f)} (X, B).$$

A right Serre functor is an additive functor $F : C \to C$ together with isomorphisms $\eta_{A,B} : \text{Hom}(A, B) \to \text{Hom}(B, FA)^*$ for any $A, B \in C$ which are natural in $A$ and $B$, where $(\cdot)^* := \text{Hom}_k(\cdot, k)$. 

A left Serre functor is a functor $G : \mathcal{C} \to \mathcal{C}$ together with isomorphisms

$$\zeta_{A,B} : \text{Hom}(A, B) \to \text{Hom}(GB, A)^*$$

for any $A, B \in \mathcal{C}$ which are natural in $A$ and $B$.

Let $\eta_A : \text{Hom}(A, FA) \to k$ be given by $\eta_{A,A}(\text{id}_A)$ and let $f \in \text{Hom}(A, B)$. Looking at the following commutative diagram (which follows from the naturality of $\eta_{A,B}$ in $B$)

$$\begin{array}{ccc}
\text{Hom}(A, A) & \xrightarrow{\eta_{A,A}} & \text{Hom}(A, FA)^* \\
\downarrow & & \downarrow \\
\text{Hom}(A, B) & \xrightarrow{\eta_{A,B}} & \text{Hom}(B, FA)^*
\end{array}$$

we find for $g \in \text{Hom}(B, FA)$

$$\eta_{A,B}(f)(g) = \eta_A(gf).$$

Similarly by the naturality of $\eta_{A,B}$ in $A$ we obtain a commutative diagram

$$\begin{array}{ccc}
\text{Hom}(B, B) & \xrightarrow{\eta_{B,B}} & \text{Hom}(B, FB)^* \\
\downarrow & & \downarrow \\
\text{Hom}(A, B) & \xrightarrow{\eta_{A,B}} & \text{Hom}(B, FA)^*
\end{array}$$

This yields for $g \in \text{Hom}(B, FA)$ the formula

$$\eta_{A,B}(f)(g) = \eta_B(F(f)g). \tag{3.1}$$

**Remark 3.1.** [RV, Lemma I.1.5] $\mathcal{C}$ has a Serre functor if and only it has both a right and a left Serre functor if and only it has a right Serre functor which is an auto-equivalence.

**Definition 3.2.** [I, BT2] Let $(\mathcal{C}, \Sigma^d)$ and $(\mathcal{C}', \Omega^d)$ be two $(d + 2)$-angulated categories. An (covariant) additive functor $F : \mathcal{C} \to \mathcal{C}'$ is called $(d + 2)$-angulated if it has the following properties:

1. There exists a natural isomorphism $\phi : F\Sigma^d \to \Omega^d F$;
2. $F$ preserves $(d + 2)$-angles, that is to say, if

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{d+1}} A_{d+1} \xrightarrow{f_{d+1}} \Sigma^d A_0$$

is an $(d + 2)$-angle in $\mathcal{C}$, then

$$FA_0 \xrightarrow{F(f_0)} FA_1 \xrightarrow{F(f_1)} FA_2 \xrightarrow{F(f_2)} \cdots \xrightarrow{F(f_{d+1})} FA_{d+1} \xrightarrow{\phi_{A_0} \circ F(f_{d+1})} \Omega^d FA_0$$

is an $(d + 2)$-angle in $\mathcal{C}'$.

Bondal and Kapranov [BK, Proposition 3.3] prove that the Serre functor is a triangulated functor in a triangulated category. Later, Van den Bergh [B, Theorem A.4.4] gives other proof
methods. The following result show that the Serre functor is an \((d+2)\)-angulated functor in an \((d+2)\)-angulated category. This generalizes the work by Bondal and Kapranov. Our proof is an adaptation of the proof of Van den Bergh. For more details, see also \[1\] Theorem 10.5.1.

**Theorem 3.3.** Let \(\mathcal{C}\) be a \(k\)-linear Hom-finite Krull-Schmidt \((d+2)\)-angulated category with a right Serre functor \(F\). Then there exists a natural isomorphism \(\zeta: F\Sigma^d \to \Sigma^d F\) such that \(F: \mathcal{C} \to \mathcal{C}\) is an \((d+2)\)-angulated functor.

**Proof.** Assume that \((F, \eta_{X,Y})\) is a right Serre functor. Put

\[
\eta_X := \eta_{X,X}(\text{Id}_X) \in \text{Hom}(X, F^* X), \quad \forall X \in \mathcal{C}.
\]

**Step 1:** We claim that there exists a natural isomorphism \(\zeta: F\Sigma^d \to \Sigma^d F\) satisfying

\[
\eta_X(\Sigma^{-d}\zeta_X \circ \Sigma^{-d}f) = (-1)^d \eta_{\Sigma^d X}(f), \quad \forall f \in \text{Hom}(\Sigma^d X, F(\Sigma^d X)) \text{ and } X \in \mathcal{C}.
\]  

By the following two isomorphisms

\[
\text{Hom}(\Sigma^d X, F(\Sigma^d X))^* \xrightarrow{\sim} \text{Hom}(X, F\Sigma^{-d}(\Sigma^d X))^*
\]

\[
\eta^*_{X, F\Sigma^{-d}(\Sigma^d X)}: \text{Hom}(X, F\Sigma^{-d}(\Sigma^d X))^* \xleftarrow{\sim} \text{Hom}(F\Sigma^{-d}(\Sigma^d X), F^* X)
\]

and \((-1)^d \eta_{\Sigma^d X} \in \text{Hom}(\Sigma^d X, F(\Sigma^d X))^*\), there exists \(\zeta_X \in \text{Hom}(F(\Sigma^d X), \Sigma^d (F^* X))\) such that

\[
\eta^*_{X, F\Sigma^{-d}(\Sigma^d X)}(\Sigma^{-d}\zeta_X)(g) = (-1)^d \eta_{\Sigma^d X}(\Sigma^d g), \quad \forall g: X \to F\Sigma^{-d}(\Sigma^d X).
\]  

Using Serre duality, the above equality \((3.3)\) is equivalent to

\[
\eta_{X, F\Sigma^{-d}(\Sigma^d X)}(\Sigma^{-d}\zeta_X)(g) = (-1)^d \eta_{\Sigma^d X}(\Sigma^d g).
\]  

By the equality \((3.1)\), the above equality \((3.4)\) is identified with

\[
\eta_X(\Sigma^{-d}\zeta_X \circ \Sigma^{-d}f) = (-1)^d \eta_{\Sigma^d X}(f), \quad \forall f \in \text{Hom}(\Sigma^d X, F(\Sigma^d X)),
\]

as desired. That is to say, the equality \((3.2)\) holds.

It remains to show that \(\zeta\) is a natural isomorphism.

By the following two isomorphisms

\[
\eta^*_{\Sigma^d X, \Sigma^d (F^* X)}: \text{Hom}(\Sigma^d (F^* X), F(\Sigma^d X))^* \xrightarrow{\sim} \text{Hom}(\Sigma^d X, \Sigma^d (F^* X))^*
\]

\[
\text{Hom}(\Sigma^d X, \Sigma^d (F^* X))^* \xleftarrow{\sim} \text{Hom}(X, F^* X)^*
\]

and \((-1)^d \eta_X \in \text{Hom}(X, F^* X)^*\), there exists \(\theta_X \in \text{Hom}(\Sigma^d (F^* X), F(\Sigma^d X))\) such that

\[
\eta^*_{\Sigma^d X, \Sigma^d (F^* X)}(\theta_X)(g) = (-1)^d \eta_X(\Sigma^{-d}g), \quad \forall g: \Sigma^d X \to \Sigma^d (F^* X).
\]  

Using Serre duality, the above equality \((3.5)\) is equivalent to

\[
\eta^*_{\Sigma^d X, \Sigma^d (F^* X)}(\theta_X) = (-1)^d \eta_X(\Sigma^{-d}g).
\]  

\[
\text{Hom}(\Sigma^d X, \Sigma^d (F^* X))^* \xrightarrow{\sim} \text{Hom}(X, F^* X)^*,
\]

and \((-1)^d \eta_X \in \text{Hom}(X, F^* X)^*\), there exists \(\theta_X \in \text{Hom}(\Sigma^d (F^* X), F(\Sigma^d X))\) such that

\[
\eta^*_{\Sigma^d X, \Sigma^d (F^* X)}(\theta_X)(g) = (-1)^d \eta_X(\Sigma^{-d}g), \quad \forall g: \Sigma^d X \to \Sigma^d (F^* X).
\]  

Using Serre duality, the above equality \((3.5)\) is equivalent to

\[
\eta^*_{\Sigma^d X, \Sigma^d (F^* X)}(\theta_X) = (-1)^d \eta_X(\Sigma^{-d}g).
\]
By the equality (3.1), the above equality (3.6) is identified with
\[ \eta_{\Sigma^d X}(\theta_X \Sigma^d g) = (-1)^d \eta_X(g), \quad \forall g \in \text{Hom}(X, FX). \tag{3.7} \]

Now we show that \( \zeta \) is an isomorphism. For any morphism \( g: X \to FX \), we have
\[ \eta_X(\Sigma^{-d}(\zeta_X \theta_X)g) \overset{(3.2)}{=} (-1)^d \eta_{\Sigma^d X}(\theta_X \Sigma^d g) \overset{(3.7)}{=} \eta_X(g). \]

By the bilinear
\[ (-, -): \text{Hom}(X, FX) \times \text{Hom}(FX, FX) \to k \]

is non-degenerate, we obtain \( \Sigma^{-d}(\zeta_X \theta_X) = \text{Id}_{FX} \) and then \( \zeta_X \theta_X = \text{Id}_{\Sigma^dFX} \).

Similarly, for any morphism \( g: \Sigma^d X \to F(\Sigma^d X) \), we have
\[ \eta_{\Sigma^d X}(\theta_X \zeta_X g) \overset{(3.7)}{=} (-1)^d \eta_X(\Sigma^{-d}(\zeta_X g)) \overset{(3.2)}{=} \eta_{\Sigma^d X}(g). \]

So \( \theta_X \zeta_X = \text{Id}_{F(\Sigma^d X)} \). This shows that \( \zeta_X \) is an isomorphism.

Now we show that \( \zeta \) is a natural transformation. Namely, for any morphism \( f: X \to Y \), we have the following commutative diagram:

\[
\begin{array}{ccc}
F(\Sigma^d X) & \xrightarrow{\zeta_X} & \Sigma^d (FX) \\
\downarrow & & \downarrow \quad \Sigma^d (Ff) \\
F(\Sigma^d Y) & \xrightarrow{\zeta_Y} & \Sigma^d (FY).
\end{array}
\]

Note that \( \Sigma^{-d} \zeta_X \in \text{Hom}(F \Sigma^{-d}(\Sigma^d X), FX) \) and \( \Sigma^{-d} \zeta_Y \in \text{Hom}(F \Sigma^{-d}(\Sigma^d Y), FY) \), for any morphism \( h \in \text{Hom}(Y, F \Sigma^{-d}(\Sigma^d X)) \), we have
\[
\text{Hom}(f, -)^*(\eta^*_X, F \Sigma^{-d}(\Sigma^d X))(\Sigma^{-d} \zeta_X)(h) \overset{\text{duality}}{=} \eta^*_X, F \Sigma^{-d}(\Sigma^d X)(\Sigma^{-d} \zeta_X)(h) \space{31} \eta^*_X, F \Sigma^{-d}(\Sigma^d X)(h)(\Sigma^{-d} \zeta_Y) \\
\overset{\text{duality}}{=} \eta^*_X, F \Sigma^{-d}(\Sigma^d X)(h)(\Sigma^{-d} \zeta_Y) \overset{(3.1)}{=} \eta_X(\Sigma^{-d} \zeta_X) \overset{(3.2)}{=} (-1)^d \eta_{\Sigma^d X}(\Sigma^d h \circ \Sigma^d f) \\
\overset{(3.1)}{=} (-1)^d \eta_{\Sigma^d X}(\Sigma^d h \circ \Sigma^d f) \overset{(3.1)}{=} (-1)^d \eta_{\Sigma^d Y}(F(\Sigma^d f))(\Sigma^d h) \\
\overset{(3.2)}{=} (-1)^d \eta_{\Sigma^d Y}(\Sigma^{-d} \zeta_Y F(\Sigma^{-d} \Sigma^d f) h) \\
\overset{(3.1)}{=} (-1)^d \eta^*_Y, F \Sigma^{-d}(\Sigma^d Y)(\Sigma^{-d} \zeta_Y)(F(\Sigma^{-d} \Sigma^d f) h) \\
\overset{\text{duality}}{=} (-1)^d \eta^*_Y, F \Sigma^{-d}(\Sigma^d Y)(\Sigma^{-d} \zeta_Y)(F(\Sigma^{-d} \Sigma^d f) h) \overset{\text{duality}}{=} \text{Hom}(\Sigma^{-d}(\Sigma^d f)^*(\eta^*_Y, F \Sigma^{-d}(\Sigma^d Y)(\Sigma^{-d} \zeta_Y))(h).
Consider the following commutative diagram:

\[
(F\Sigma^{-d}(\Sigma^dX), FX) \xrightarrow{\eta_X, \varphi_{\Sigma^{-d}(\Sigma^dX)}} (X, F\Sigma^{-d}(\Sigma^dX))^* \\
(\cdot, Ff) \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 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\[ \omega_{d+1}\beta_d = \mathbb{F}\alpha_d \circ \omega_d \] (3.9)

**Claim I:** The equality (3.8) holds if and only if

\[ \eta_{A_{d+1}}(\omega_{d+1} \circ \Sigma^d \varphi \circ \alpha_{d+1}) = (-1)^d \eta_{A_0}(\Sigma^{-d} \beta_{d+1} \circ \varphi), \quad \forall \varphi \in \text{Hom}(A_0, \Sigma^{-d} B_{d+1}). \] (3.10)

The equality (3.9) holds if and only if

\[ \eta_{A_{d+1}}(\omega_{d+1} \beta_d \phi) = \eta_{A_d}(\omega_d \phi \alpha_d), \quad \forall \phi \in \text{Hom}(A_{d+1}, B_d). \] (3.11)

We first show that the equality (3.9) holds if and only if the equality (3.10) holds.

Suppose that the equality (3.9) holds. For any morphism \( \varphi \in \text{Hom}(A_0, \Sigma^{-d} B_{d+1}) \), by the equality (3.8), we have

\[ \Sigma^{-d}(\zeta_{A_0} \mathbb{F} \alpha_{d+1} \circ \omega_{d+1}) \circ \varphi = \Sigma^{-d} \beta_{d+1} \circ \varphi : A_0 \to \mathbb{F} A_0. \]

It follows that

\[ (-1)^d \eta_{A_0}(\Sigma^{-d} \beta_{d+1} \circ \varphi) = (-1)^d \eta_{A_0}(\Sigma^{-d}(\zeta_{A_0} \mathbb{F} \alpha_{d+1} \circ \omega_{d+1}) \circ \varphi) \]

\[ = \eta_{\Sigma^d A_0}(\mathbb{F} \alpha_{d+1} \circ \omega_{d+1} \circ \Sigma^d \varphi) \]

\[ = \eta_{A_{d+1}}(\omega_{d+1} \Sigma^d \varphi \circ \alpha_{d+1}). \]

This shows that the equality (3.10) holds.

Conversely, assume that the equality (3.10) holds. For any morphism \( \varphi \in \text{Hom}(A_0, \Sigma^{-d} B_{d+1}) \), by the equalities (3.10), (3.1) and (3.2), we have

\[ \eta_{A_0}(\Sigma^{-d} \beta_{d+1} \circ \varphi) = \eta_{A_0}(\Sigma^{-d}(\zeta_{A_0} \mathbb{F} \alpha_{d+1} \circ \omega_{d+1}) \circ \varphi). \]

By the bilinear

\[ \text{Hom}(A_0, \Sigma^{-d} B_{d+1}) \times \text{Hom}(\Sigma^{-d} B_{d+1}, \mathbb{F} A_0) \longrightarrow k, \quad (\varphi, g) \longmapsto \eta_{A_0}(g \varphi) \]

is non-degenerate, we obtain \( \zeta_{A_0} \mathbb{F} \alpha_{d+1} \circ \omega_{d+1} = \beta_{d+1} \). Namely, the equality (3.8) holds.

Now we show that the equality (3.9) holds if and only if the equality (3.11) holds.

Assume that the equality (3.9) holds. For any morphism \( \phi \in \text{Hom}(A_{d+1}, B_d) \), by the equality (3.9), we have

\[ \omega_{d+1} \beta_d \phi = \mathbb{F} \alpha_d \circ \omega_d \phi : A_{d+1} \to \mathbb{F} A_{d+1}. \]

It follows that

\[ \eta_{A_{d+1}}(\omega_{d+1} \beta_d \phi) = \eta_{A_{d+1}}(\mathbb{F} \alpha_d \circ \omega_d \phi) \]

\[ = \eta_{A_d}(\omega_d \phi \alpha_d). \]

This shows that the equality (3.11) holds.

Conversely, assume that the equality (3.11) holds. For any morphism \( \phi \in \text{Hom}(A_{d+1}, B_d) \), we have

\[ \eta_{A_{d+1}}(\omega_{d+1} \beta_d \phi) \]

\[ \eta_{A_d}(\omega_d \phi \alpha_d) \]

\[ \eta_{A_{d+1}}(\mathbb{F} \alpha_d \circ \omega_d \phi). \]
By the bilinear
\[ \text{Hom}(A_{d+1}, B_d) \times \text{Hom}(B_d, F A_{d+1}) \rightarrow k, \quad (\phi, h) \rightarrow \eta_{A_{d+1}}(h \phi) \]
is non-degenerate, we obtain \( \omega_{d+1} \beta_n = F \alpha_d \circ \omega_d \). That is to say, the equality (3.3) holds.

**Claim II:** The equalities (3.10) and (3.11) hold \( \iff \) If there are morphisms \( \varphi: A_0 \rightarrow \Sigma^{-d} B_{d+1} \) and \( \phi: A_{d+1} \rightarrow B_d \) such that \( \Sigma^d \varphi \circ \alpha_{d+1} = \beta_d \phi \), then
\[ \eta_{A_{d+1}}(\omega_d \phi \alpha_d) = (-1)^d \eta_{A_0}(\Sigma^{-d} \beta_{d+1} \circ \varphi). \] (3.12)

If the equalities (3.10) and (3.11) hold, obviously, the equality (3.12) also holds.

Conversely, assume that the equality (3.12) holds. Put
\[ V_1 := \{ \Sigma^d \varphi \circ \alpha_{d+1} \mid \varphi \in \text{Hom}(A_0, \Sigma^{-d} B_{d+1}) \} \subseteq \text{Hom}(A_{d+1}, B_{d+1}), \]
\[ V_2 := \{ \beta_d \phi \mid \phi \in \text{Hom}(A_{d+1}, B_d) \} \subseteq \text{Hom}(A_{d+1}, B_{d+1}). \]
Take a set of bases \( a_1, a_2, \cdots, a_m \) of \( V_1 \cap V_2 \), and expand them to a set of bases
\[ a_1, \cdots, a_m, a_{m+1}, \cdots, a_{m+t} \]
of \( V_1 \). At the same time, it is also extended to a set of bases
\[ b_1 = a_1, \cdots, b_m = a_m, b_{m+1}, \cdots, b_{m+s} \]
of \( V_2 \). Take a set of bases \( c_1, \cdots, c_n \) of \( \text{Hom}(B_{n+1}, F A_{n+1}) \). Write
\[ \eta_{A_{n+1}}(c_j a_i) = a_{ij}, \quad i = 1, \cdots, m + t; \quad j = 1, \cdots, n, \]
\[ \eta_{A_{n+1}}(c_j b_i) = b_{ij}, \quad i = 1, \cdots, m + s; \quad j = 1, \cdots, n, \]
\[ M = (a_{ij})_{(m+t) \times n}, \quad N = (b_{ij})_{(m+s) \times n}. \]
Thus the matrices \( M \) and \( N \) are the same as the first \( m \) rows.

Since \( a_1, \cdots, a_m, a_{m+1}, \cdots, a_{m+t} \) is a set of bases of \( V_1 \), there are morphisms
\[ \varphi_1, \cdots, \varphi_m, \varphi_{m+1}, \cdots, \varphi_{m+t} \in \text{Hom}(A_0, \Sigma^{-d} B_{d+1}) \]
such that
\[ a_1 = \Sigma^d \varphi_1 \circ \alpha_{d+1}, \cdots, a_m = \Sigma^d \varphi_m \circ \alpha_{d+1}, a_{m+1} = \Sigma^d \varphi_{m+1} \circ \alpha_{d+1}, \cdots, a_{m+t} = \Sigma^d \varphi_{m+t} \circ \alpha_{d+1}. \]
Put
\[ d_i = (-1)^d \eta_{A_0}(\Sigma^{-d} \beta_{d+1} \circ \varphi), \quad i = 1, \cdots, m + t. \]
Similarly, there are morphisms
\[ \phi_1, \cdots, \phi_m, \phi_{m+1}, \cdots, \phi_{m+t} \]
such that
\[ b_1 = \beta_d \phi_1, \ldots, b_m = \beta_d \phi_m, \ldots, b_{m+1} = \beta_d \phi_{m+1}, \ldots, b_{m+s} = \beta_d \phi_{m+s}. \]

Put
\[ e_i = \eta_{A_d}(\omega_d \phi_i \alpha_d), \quad i = 1, \ldots, m + s. \]

Since \( \beta_d \phi_i = b_i = a_i = \Sigma^d \phi_i \circ \alpha_{d+1}, \quad i = 1, \ldots, m. \) By the equality \((3.12)\), we have
\[ e_i = \eta_{A_d}(\omega_d \phi_i \alpha_d) = (-1)^d(\Sigma^{-d} \beta_{d+1} \circ \phi_i) = d_i, \quad i = 1, \ldots, m. \]

It follows that there exists a morphism \( \omega_{d+1}: B_{d+1} \to FA_d \) such that the equalities \((3.10)\) and \((3.11)\) hold if and only if the following two linear equations have solutions
\[
M \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} d_1 \\ \vdots \\ d_{m+t} \end{pmatrix},
\]
\[
N \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} e_1 \\ \vdots \\ e_{m+s} \end{pmatrix}.
\]

Note that the matrices \( M \) and \( N \) are the same as the first \( m \) rows, we take \( L = \begin{pmatrix} M \\ N' \end{pmatrix} \) which is the matrix of \((m + t + s) \times n\), where \( N' \) is the last \( s \) line of \( N \). Therefore, the necessary and sufficient condition for the above two linear equations to have a solution is that the following linear equations have solutions
\[
L \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} d_1 \\ \vdots \\ d_{m+t} \\ e_{m+s} \end{pmatrix}.
\] (3.13)

By the bilinear \( \text{Hom}(A_{d+1}, B_d) \times \text{Hom}(B_d, FA_{d+1}) \to k, \quad (u, v) \mapsto \eta_{A_{d+1}}(vu) \) is non-degenerate, we obtain that the matrix \( L \) is row full rank, so that the above linear equations \((3.13)\) have solutions. Our assertion is true.

**Step 3:** We show that the equality \((3.12)\) holds.

Assume that there are morphisms \( \varphi: A_0 \to \Sigma^{-d}B_{d+1} \) and \( \phi: A_{d+1} \to B_d \) such that
\[ \Sigma^d \varphi \circ \alpha_{d+1} = \beta_d \phi. \]

Consider the following morphisms of \((d + 2)\)-angles in \( \mathcal{C} \):

\[
\begin{array}{cccccccccccc}
A_1 & \overset{\alpha_1}{\longrightarrow} & A_2 & \overset{\alpha_2}{\longrightarrow} & A_3 & \cdots & \overset{\alpha_{d-1}}{\longrightarrow} & A_{d} & \overset{\alpha_d}{\longrightarrow} & A_{d+1} & \overset{\alpha_{d+1}}{\longrightarrow} & \Sigma^d A_0 \overset{(-1)^d \Sigma^d \alpha_d}{\longrightarrow} & \Sigma^d A_1 \\
\downarrow & \Delta_1 & \downarrow & \Delta_2 & \cdots & \downarrow & \Delta_d & \downarrow & \Delta_{d+1} & \downarrow & \phi & \cdots & \\
FA_0 & \overset{\beta_1}{\longrightarrow} & FA_1 & \overset{\beta_2}{\longrightarrow} & B_2 & \cdots & \overset{\beta_{d-2}}{\longrightarrow} & B_{d-1} & \overset{\beta_{d-1}}{\longrightarrow} & B_d & \overset{\beta_{d+1}}{\longrightarrow} & \Sigma^d FA_0
\end{array}
\]
where \( \delta_1, \delta_2, \ldots, \delta_d \) exist by the axiom (N3). It follows that

\[
\eta_{A_d}(\omega_d \phi_{d-1}) = \eta_{A_{d-1}}(\omega_d \beta_{d-1} \delta_{d-1}) = \eta_{A_{d-1}}(F \alpha_{d-1} \circ \omega_d \delta_d)
\]

\[
\eta_{A_{d-1}}(\omega_{d-1} \delta_d \alpha_{d-1}) = \eta_{A_{d-1}}(\omega_{d-1} \beta_{d-2} \delta_{d-1}) = \eta_{A_{d-1}}(F \alpha_{d-2} \circ \omega_{d-2} \delta_{d-1})
\]

\[
\eta_{A_d}(\omega_{d-2} \delta_{d-1} \alpha_{d-2})
\]

\[
\vdots
\]

\[
\eta_{A_2}(\omega_2 \delta_{1} \alpha_2) = \eta_{A_2}(\omega_2 \beta_1 \delta_2) = \eta_{A_1}(F \alpha_1 \circ \delta_2)
\]

\[
\eta_{A_1}(\delta_2 \alpha_1) = \eta_{A_1}(F \alpha_0 \circ \delta_1)
\]

\[
\eta_{A_0}(\delta_1 \alpha_0) = (-1)^d \eta_{A_0}(\Sigma^{-d} \beta_{d+1} \circ \varphi).
\]

as required.

This completes the proof. \( \square \)

4 Connection between Auslander-Reiten \((d+2)\)-angles and Serre duality

In this section, we build a link between Auslander-Reiten \((d+2)\)-angles and Serre duality.

**Lemma 4.1.** Let \( \mathcal{C} \) be a \( k \)-linear Hom-finite Krull-Schmidt \((d+2)\)-angulated category,

\[
A_\bullet : A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} A_{d+1} \xrightarrow{\alpha_{d+1}} \Sigma^d A_0
\]

and

\[
B_\bullet : B_0 \xrightarrow{\beta_0} B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_{d-1}} B_d \xrightarrow{\beta_d} B_{d+1} \xrightarrow{\beta_{d+1}} \Sigma^d B_0
\]

be two Auslander-Reiten \((d+2)\)-angles in \( \mathcal{C} \). Then \( A_0 \simeq B_0 \) if and only if \( A_{d+1} \simeq B_{d+1} \).

**Proof.** Assume that \( \varphi_{d+1} : A_{d+1} \to B_{d+1} \) is an isomorphism. Then \( \varphi_{d+1} \alpha_d \) is not split epimorphism, otherwise, \( \alpha_d \) is a split epimorphism and this is impossible. Since \( \alpha_d \) is right almost split, there exists a morphism \( \varphi_d : A_d \to B_d \) such that \( \varphi_{d+1} \alpha_d = \beta_d \varphi_d \). Thus we have the following commutative diagram

\[
\begin{array}{ccccccccc}
A_0 & \xrightarrow{\alpha_0} & A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & \cdots & \xrightarrow{\alpha_{d-1}} & A_d & \xrightarrow{\alpha_d} & A_{d+1} & \xrightarrow{\alpha_{d+1}} & \Sigma^d A_0 \\
\downarrow \varphi_0 & & \downarrow \varphi_1 & & \downarrow \varphi_2 & & \cdots & & \downarrow \varphi_d & & \downarrow \varphi_{d+1} & & \downarrow \Sigma^d \varphi_0 \\
B_0 & \xrightarrow{\beta_0} & B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & \cdots & \xrightarrow{\beta_{d-1}} & B_d & \xrightarrow{\beta_d} & B_{d+1} & \xrightarrow{\beta_{d+1}} & \Sigma^d B_0 \\
\end{array}
\]

of \((d+2)\)-angles in \( \mathcal{C} \).

If \( \varphi_0 \) is not an isomorphism, since \( A_0 \) and \( B_0 \) are indecomposable, we have that \( \varphi_0 \) is not split monomorphism. Since \( \alpha_0 \) is left almost split, there exists a morphism \( h : A_1 \to B_0 \) such that \( \varphi_0 = h \alpha_0 \). Hence \( \beta_{d+1} \varphi_{d+1} = (\Sigma^d \varphi_0) \alpha_{d+1} = \Sigma^d h (\Sigma^d \alpha_0) \alpha_{d+1} = 0 \). This implies that \( \beta_{d+1} = 0 \). This is a contradiction since \( B_\bullet \) is an Auslander-Reiten \((d+2)\)-angles in \( \mathcal{C} \). Therefore \( \varphi_0 : A_0 \to B_0 \) is an isomorphism. \( \square \)
Remark 4.2. Let $\mathcal{C}$ be a $k$-linear Hom-finite Krull-Schmidt $(d+2)$-angulated category, and

$$
A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} A_{d+1} \xrightarrow{\alpha_{d+1}} \Sigma^d A_0
$$

be an Auslander-Reiten $(d+2)$-angles in $\mathcal{C}$. By Lemma 4.1, we know that $A_0$ is uniquely determined up to isomorphisms. In this case, we write $A_0 = \tau_d A_{d+1}$.

In order to prove the main result of this section, we need the following lemma.

Lemma 4.3. Let $\mathcal{C}$ be a $k$-linear Hom-finite Krull-Schmidt $(d+2)$-angulated category. Assume that

$$
\tau_d C \xrightarrow{\alpha_0} C_1 \xrightarrow{\alpha_1} C_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-1}} C_d \xrightarrow{\alpha_d} C \xrightarrow{\alpha_{d+1}} \Sigma^d \tau_d C
$$

is an Auslander-Reiten $(d+2)$-angles in $\mathcal{C}$ and $B$ is a indecomposable object in $\mathcal{C}$. Then the following hold:

1. For any non-zero $g \in \text{Hom}_\mathcal{C}(B, \tau_d \Sigma^d C)$, there is $f \in \text{Hom}_\mathcal{C}(C, B)$ such that $\alpha_{d+1} = gf$.
2. For any non-zero $f \in \text{Hom}_\mathcal{C}(C, B)$, there is $g \in \text{Hom}_\mathcal{C}(B, \tau_d \Sigma^d C)$ such that $\alpha_{d+1} = gf$.

Proof. (1) For any non-zero morphism $g: B \rightarrow \tau_d \Sigma^d C$, there exists an $(d+2)$-angle

$$
\tau_d C \xrightarrow{\beta_0} B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_{d-1}} B_d \xrightarrow{\beta_d} B \xrightarrow{g} \Sigma^d \tau_d C
$$

in $\mathcal{C}$. Since $g$ is non-zero, then $\beta_0$ is not split monomorphism. It follows that there exists a morphism $h: C_1 \rightarrow B_1$ such that $\beta_0 = h \alpha_0$. Thus we have the following commutative diagram

$$
\tau_d C \xrightarrow{\alpha_0} C_1 \xrightarrow{\alpha_1} C_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-1}} C_d \xrightarrow{\alpha_d} C \xrightarrow{\alpha_{d+1}} \Sigma^d \tau_d C
$$

of $(d+2)$-angles in $\mathcal{C}$. We obtain $\alpha_{d+1} = gf$.

(2) For any non-zero morphism $f: C \rightarrow B$, there exists an $(d+2)$-angle

$$
A_0 \xrightarrow{\gamma_0} A_1 \xrightarrow{\gamma_1} \cdots \xrightarrow{\gamma_{d-2}} A_{d-1} \xrightarrow{\gamma_{d-1}} C \xrightarrow{f} B \xrightarrow{\gamma} \Sigma^d A_0
$$

in $\mathcal{C}$. Since $f \gamma_{d-1} = 0$ and $f \neq 0$, we have that $\gamma_{d-1}$ is not split epimorphism. Since $\alpha_d$ is right almost split, there exists a morphism $u: A_{d-1} \rightarrow C_d$ such that $\alpha_d u = \gamma_{d-1}$ and then

$$
\alpha_{d+1} \gamma_{d-1} = \alpha_{d+1} \alpha_d u = 0.
$$

So there exists a morphism $g: B \rightarrow \Sigma^d \tau_d C$ such that $\alpha_{d+1} = gf$. \qed

The knowledge of using linear algebra has the following result which will be used later, and we omit the proof.

Lemma 4.4. Let $\mathcal{C}$ be a $k$-linear Hom-finite Krull-Schmidt additive category. If $\mathcal{F}: \mathcal{A} \rightarrow \mathcal{A}$ is a map, where $\mathcal{A}$ is a class consisting of isomorphic classes of indecomposable objects in $\mathcal{C}$,
and for any indecomposable object \( X \in \mathcal{A} \), there exists a \( k \)-linear map \( \eta_X : \text{Hom}(X, F_X) \to k \) such that the bilinear
\[
(-, -) : \text{Hom}(X, Y) \times \text{Hom}(Y, F_X) \to k, \quad (f, g) = \eta_X(gf)
\]
is non-degenerate, where \( Y \in \mathcal{A} \) is any indecomposable object. Then \( F \) can be viewed as a right Serre functor, where
\[
\eta_{X,Y} : \text{Hom}(X, Y) \to \text{Hom}(Y, F_Y)^*
\]
given by
\[
\eta_{X,Y}(f)(g) = \eta_X(gf), \quad \forall f \in \text{Hom}(X, Y), g \in \text{Hom}(Y, F_X).
\]

Now we state and prove our second main result. This generalizes the work by Reiten and Van den Bergh [RV, Theorem I.2.4] for triangulated categories.

**Theorem 4.5.** Let \( \mathcal{C} \) be a \( k \)-linear Hom-finite Krull-Schmidt \((d + 2)\)-angulated category with an \( d \)-suspension functor \( \Sigma^d \). Then \( \mathcal{C} \) has Auslander-Reiten \((d + 2)\)-angles if and only if \( \mathcal{C} \) has a Serre functor \( S \).

If either of these properties holds, then the action of the Serre functor on objects coincides with \( \tau_{d} \Sigma^d \), namely \( \tau_{d} S \Sigma^{-d} \). In this case, \( \tau_{d} \) is called \( d \)-Auslander-Reiten translation.

**Proof.** We first show the ‘only if’ part. Assume that \( \mathcal{C} \) has Auslander-Reiten \((d + 2)\)-angles. For any indecomposable object \( C \in \mathcal{C} \), put \( FC = \tau_{d} \Sigma^{d} C \). Then there exists an Auslander-Reiten \((d + 2)\)-angle
\[
\tau_{d} C \xrightarrow{\alpha_0} C_1 \xrightarrow{\alpha_1} C_2 \xrightarrow{-\alpha_2} \cdots \xrightarrow{-\alpha_{d-1}} C_d \xrightarrow{\alpha_d} C \xrightarrow{\alpha_{d+1}} \Sigma^{d} \tau_{d} C
\]
Thus \( \alpha_{d+1} \neq 0 \). For the indecomposable object \( C \), take a \( k \)-linear map
\[
\eta_C : \text{Hom}(C, FC) \to k
\]
such that \( \eta_C(\alpha_{d+1}) \neq 0 \). By Lemma 4.3, for any indecomposable object \( B \in \mathcal{C} \), the pairing
\[
\text{Hom}(C, B) \times \text{Hom}(B, FC) \to k
\]
given by \( (f, g) = \eta_C(gf) \) is a non-degenerate \( k \)-bilinear map. By Lemma 4.4, \( F \) can be viewed as a right Serre functor. Dually, we can prove that \( \mathcal{C} \) has a left Serre functor.

To prove the ‘if’ part. Assume that \( \mathcal{C} \) has a Serre functor. Let \( F \) be a right Serre functor. Then for any indecomposable object \( C \), there exists a \( k \)-linear isomorphism
\[
(\eta^*_C)^{-1} : \text{End}(C) \to \text{Hom}(C, FC).
\]
Since \( \text{End}(C) \) is local and \( k \) is an algebraically closed field, we have \( \text{End}(C)/\text{radEnd}(C) \cong k \). Put \( \theta_C : \text{End}(C) \to k \) is a natural epimorphism, \( \text{Ker} \theta_C = \text{radEnd}(C) \). Take
\[
\delta = (\eta^*_C)^{-1}(\theta_C) \in \text{Hom}(C, FC),
\]
we have $\delta \neq 0$ and it can be embedded into an $(d + 2)$-angle

$$\Sigma^{-d} C \overset{\alpha_0}{\longrightarrow} C_1 \overset{\alpha_1}{\longrightarrow} C_2 \overset{\alpha_2}{\longrightarrow} \cdots \overset{\alpha_{d-1}}{\longrightarrow} C_d \overset{\alpha_d}{\longrightarrow} C \overset{\delta}{\longrightarrow} FC$$

in $\mathcal{C}$. When $d \geq 2$, we can choose $\alpha_1, \alpha_2, \cdots, \alpha_{d-1}$ in $\text{rad} \mathcal{C}$.

We claim that

$$\Sigma^{-d} C \overset{\alpha_0}{\longrightarrow} C_1 \overset{\alpha_1}{\longrightarrow} C_2 \overset{\alpha_2}{\longrightarrow} \cdots \overset{\alpha_{d-1}}{\longrightarrow} C_d \overset{\alpha_d}{\longrightarrow} C \overset{\delta}{\longrightarrow} FC$$

is an Auslander-Reiten $(d + 2)$-angle in $\mathcal{C}$. In fact, let $\beta : B \rightarrow C$ is not a split epimorphism, where $B$ is an indecomposable object in $\mathcal{C}$. Then the composition

$$\text{Hom}(C, B) \xrightarrow{\text{Hom}(C, \beta)} \text{End}(C) \xrightarrow{\theta_Z} k$$

is zero. Consider the following commutative diagram

$$\begin{array}{ccc}
\text{Hom}(C, C) & \xrightarrow{(\eta_{C,C}^*)^{-1}} & \text{Hom}(C, FC) \\
\downarrow & & \downarrow \\
\text{Hom}(C, \beta)^* & \xrightarrow{(\eta_{C,B}^*)^{-1}} & \text{Hom}(\beta, FC)
\end{array}$$

we can obtain

$$\delta \beta = (\eta_{C,C}^*)^{-1}(\theta_Z)\beta = (\eta_{C,B}^*)^{-1}\text{Hom}(C, \beta)^*(\theta_C) = (\eta_{C,B}^*)^{-1}(\theta_C)\text{Hom}(C, \beta)) = 0.$$

So there exists a morphism $h : C \rightarrow C_d$ such that $\beta = \alpha_d h$. This shows that $\alpha_d$ is a right almost split.

Since $C$ is an indecomposable object, we have that $\Sigma^{-d} FC$ is also indecomposable implies that $\text{End}(\Sigma^{-d} FC)$ is local. By Lemma 5.2, we know that

$$\Sigma^{-d} FC \overset{\alpha_0}{\longrightarrow} C_1 \overset{\alpha_1}{\longrightarrow} C_2 \overset{\alpha_2}{\longrightarrow} \cdots \overset{\alpha_{d-1}}{\longrightarrow} C_d \overset{\alpha_d}{\longrightarrow} C \overset{\delta}{\longrightarrow} FC$$

is an Auslander-Reiten $(d + 2)$-angle in $\mathcal{C}$.

Dually, we can show that for any indecomposable object $C \in \mathcal{C}$, there exists an Auslander-Reiten $(d + 2)$-angle:

$$C \overset{\beta_0}{\longrightarrow} C_1 \overset{\beta_1}{\longrightarrow} C_2 \overset{\beta_2}{\longrightarrow} \cdots \overset{\beta_{d-1}}{\longrightarrow} C_d \overset{\beta_d}{\longrightarrow} G\Sigma^d C \overset{\beta_{d+1}}{\longrightarrow} \Sigma^d C$$

where $G$ is a left Serre functor.

Now we give an example illustrating our main result in this section.

**Example 4.6.** We first recall the standard construction of $(d + 2)$-angulated categories given by Geiß-Keller-Oppermann [GKO, Theorem 1]. Let $\mathcal{C}$ be a triangulated category and $\mathcal{T}$ an $d$-cluster tilting subcategory which is closed under $\Sigma^d$, where $\Sigma$ is the shift functor of $\mathcal{C}$. Then
\((\mathcal{T}, \Sigma^d, \Theta)\) is an \((d + 2)\)-angulated category, where \(\Theta\) is the class of all sequences

\[
A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{d-1}} A_d \xrightarrow{f_d} A_{d+1} \xrightarrow{f_{d+1}} \Sigma^d A_0
\]

such that there exists a diagram

\[
\begin{array}{c}
A_0 \\
\Rightarrow
\end{array} \quad
\begin{array}{c}
A_1 \quad f_1 \\
\Rightarrow
\end{array} \quad
\begin{array}{c}
A_2 \\
\Rightarrow
\end{array} \quad \cdots \quad \begin{array}{c}
A_d \quad f_d \\
\Rightarrow
\end{array}
\begin{array}{c}
A_0 \xleftarrow{A_{1,5}} A_1 \xleftarrow{A_{2,5}} \cdots A_{d-0,5} \xleftarrow{A_{d+1}}
\end{array}
\]

with \(A_i \in \mathcal{T}\) for all \(i \notin \mathbb{Z}\), such that all oriented triangles are triangles in \(\mathcal{C}\), all non-oriented triangles commute, and \(f_{d+1}\) is the composition along the lower edge of the diagram.

Assume that \(\mathcal{C}\) has a Serre functor \(S\). It is clear that the \((d + 2)\)-angulated category \((\mathcal{T}, \Sigma^d, \Theta)\) has also a Serre functor \(S\). By Theorem 4.5, we know that \(\mathcal{C}\) has Auslander-Reiten \((d + 2)\)-angles and \(\tau_d = \Sigma^{-d}\).

5 \((d + 2)\)-angulated quotient categories

Let \(\mathcal{C}\) be an additive category, and \(\mathcal{X}\) be a subcategory of \(\mathcal{C}\). Recall that we say a morphism \(f: A \to B\) in \(\mathcal{C}\) is an \(\mathcal{X}\)-monic if

\[
\text{Hom}_\mathcal{C}(f, X): \text{Hom}_\mathcal{C}(B, X) \to \text{Hom}_\mathcal{C}(A, X)
\]

is an epimorphism for all \(X \in \mathcal{X}\). We say that \(f\) is an \(\mathcal{X}\)-epic if

\[
\text{Hom}_\mathcal{C}(X, f): \text{Hom}_\mathcal{C}(X, A) \to \text{Hom}_\mathcal{C}(X, B)
\]

is an epimorphism for all \(X \in \mathcal{X}\). Similarly, we say that \(f\) is a left \(\mathcal{X}\)-approximation of \(B\) if \(f\) is an \(\mathcal{X}\)-monoic and \(A \in \mathcal{X}\). We say that \(f\) is a right \(\mathcal{X}\)-approximation of \(A\) if \(f\) is an \(\mathcal{X}\)-epic and \(B \in \mathcal{X}\).

A subcategory \(\mathcal{X}\) is called contravariantly finite if any object in \(\mathcal{C}\) admits a right \(\mathcal{X}\)-approximation. Dually we can define covariantly finite subcategory.

**Definition 5.1.** Let \(\mathcal{C}\) be an \((d + 2)\)-angulated category. A subcategory \(\mathcal{X}\) of \(\mathcal{C}\) is called strongly contravariantly finite, if for any object \(C \in \mathcal{C}\), there exists an \((d + 2)\)-angle

\[
B \to X_1 \to X_2 \to \cdots \to X_{d-1} \to X_d \xrightarrow{g} C \to \Sigma^d B
\]

where \(g\) is a right \(\mathcal{X}\)-approximation of \(C\) and \(X_1, \ldots, x_d \in \mathcal{X}\). Dually we can define (strongly covariantly finite subcategory.

A strongly contravariantly finite and strongly covariantly finite subcategory is called strongly functorially finite.

The dual version of the following result is also true, and we have omitted it here.
Lemma 5.2. Let $\mathcal{C}$ be an $(d+2)$-angulated category and $\mathcal{X}$ a covariantly finite subcategory of $\mathcal{C}$. Then the quotient category $\mathcal{C}/\mathcal{X}$ is a right $(d+2)$-angulated category with the following endofunctor and right $(d+2)$-angles:

1. For any object $A \in \mathcal{C}$, we take an $(d+2)$-angle
   
   $$A \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \ldots \xrightarrow{f_{d-1}} X_d \xrightarrow{f_d} H A \xrightarrow{f_{d+1}} \Sigma^d A$$

   with $f_0$ is a left $\mathcal{X}$-approximation of $A$ and $X_1, X_2, \ldots, X_d \in \mathcal{X}$. Then $H$ gives a well-defined endofunctor of $\mathcal{C}/\mathcal{X}$.

2. For any $(d+2)$-angle
   
   $$A_0 \xrightarrow{g_0} A_1 \xrightarrow{g_1} A_2 \xrightarrow{g_2} A_3 \xrightarrow{g_3} \ldots \xrightarrow{g_{d-1}} A_d \xrightarrow{g_d} A_{d+1} \xrightarrow{g_{d+1}} \Sigma^d A_0$$

   with $g_0$ is an $\mathcal{X}$-monic, take the following commutative diagram of $(d+2)$-angles.

   $$
   \begin{array}{c}
   A_0 \xrightarrow{g_0} A_1 \xrightarrow{g_1} A_2 \xrightarrow{g_2} A_3 \xrightarrow{g_3} \ldots \xrightarrow{g_{d-1}} A_d \xrightarrow{g_d} A_{d+1} \xrightarrow{g_{d+1}} \Sigma^d A_0 \\
   \downarrow \phi_1 \quad \downarrow \phi_2 \quad \downarrow \phi_3 \quad \ldots \quad \downarrow \phi_{d-1} \quad \downarrow \phi_d \quad \downarrow \phi_{d+1} \\
   A_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \ldots \xrightarrow{f_{d-1}} X_d \xrightarrow{f_d} H A_0 \xrightarrow{f_{d+1}} \Sigma^d A_0 
   \end{array}
   $$

   Then we have a complex

   $$A_0 \xrightarrow{\tau} A_1 \xrightarrow{\tau_1} A_2 \xrightarrow{\tau_2} \ldots \xrightarrow{\tau_{d-1}} A_d \xrightarrow{\tau_d} A_{d+1} \xrightarrow{\tau_{d+1}} H A_0.$$

   We define right $(d+2)$-angles in $\mathcal{C}/\mathcal{X}$ as the complexes which are isomorphic to complexes obtained in this way.

Proof. See [L1] Theorem 3.7] and [L2] Remark 3.8].

The notion of mutation pairs of subcategories in an $(d+2)$-angulated category was defined by Lin [L1], Definition 3.1]. We recall the definition here.

Definition 5.3. Let $\mathcal{C}$ be an $(d+2)$-angulated category, and $\mathcal{X} \subseteq A$ be two subcategories of $\mathcal{C}$. The pair $(A, \mathcal{X})$ is called a $\mathcal{X}$-mutation pair if it satisfies the following conditions:

1. For any object $A \in \mathcal{A}$, there exists an $(d+2)$-angle
   
   $$A \xrightarrow{x_0} X_1 \xrightarrow{x_1} X_2 \xrightarrow{x_2} \ldots \xrightarrow{x_{d-1}} X_d \xrightarrow{x_d} B \xrightarrow{x_{d+1}} \Sigma^d A$$

   where $X_i \in \mathcal{X}, B \in \mathcal{A}, x_0$ is a left $\mathcal{X}$-approximation of $A$ and $x_d$ is a right $\mathcal{X}$-approximation of $B$.

2. For any object $C \in \mathcal{A}$, there exists an $(d+2)$-angle
   
   $$D \xrightarrow{x'_0} X'_1 \xrightarrow{x'_1} X'_2 \xrightarrow{x'_2} \ldots \xrightarrow{x'_{d-2}} X'_{d-1} \xrightarrow{x'_{d-1}} X'_d \xrightarrow{x'_d} C \xrightarrow{x'_{d+1}} \Sigma^d D$$

   where $X'_i \in \mathcal{X}, D \in \mathcal{A}, x'_0$ is a left $\mathcal{X}$-approximation of $D$ and $x'_d$ is a right $\mathcal{X}$-approximation of $C$. 
Let $\mathcal{C}$ be an $(d+2)$-angulated category. Recall that a subcategory $\mathcal{A}$ of $\mathcal{C}$ is called extension closed if for any morphism $\alpha_{d+1}: A_{d+1} \to \Sigma A_0$ with $A_0, A_{n+1} \in \mathcal{A}$, there exists an $(n+2)$-angle

$$A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-2}} A_{d-1} \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} A_{d+1} \xrightarrow{\alpha_{d+1}} \Sigma^d A_0,$$

where each $A_i \in \mathcal{A}$. It is clear that $\mathcal{C}$ is extension closed in $\mathcal{C}$.

**Lemma 5.4.** ([1], Theorem 3.7] Let $\mathcal{C}$ be an $(d+2)$-angulated category and $\mathcal{X} \subseteq \mathcal{A}$ be two subcategories of $\mathcal{C}$. If $(\mathcal{A}, \mathcal{A})$ is an $\mathcal{X}$-mutation pair and $\mathcal{A}$ is extension closed, then the quotient category $\mathcal{A}/\mathcal{X}$ is an $(d+2)$-angulated category.

**Lemma 5.5.** Let $\mathcal{C}$ be a $k$-linear Hom-finite Krull-Schmidt $(d+2)$-angulated category with a Serre functor $\mathcal{S}$, and $\mathcal{X}$ is a subcategory of $\mathcal{C}$. Given an $(d+2)$-angle

$$A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-1}} A_{d-1} \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} A_{d+1} \xrightarrow{\alpha_{d+1}} \Sigma^d A_0 \quad (5.1)$$

If $\tau_d \mathcal{X} = \mathcal{X}$, then $\alpha_0$ is an $\mathcal{X}$-monic if and only if $\alpha_d$ is an $\mathcal{X}$-epic.

**Proof.** Apply the functor $\text{Hom}(X, -)$ with $X \in \mathcal{X}$ to the $(d+2)$-angle (5.1), we have the following long exact sequence:

$$\text{Hom}(X, A_d) \to \text{Hom}(X, A_{d+1}) \to \text{Hom}(X, \Sigma^d A_0) \to \text{Hom}(X, \Sigma^d A_1)$$

For $\alpha_d: A_d \to A_{d+1}$ to be an $\mathcal{X}$-epic is the same as for the first arrow in the long exact sequence always to be epimorphism. This is the same as for the second arrow always to be zero, which is again the same as for the third arrow always to be injective.

Using Serre duality, the third arrow can be identified with

$$\text{Hom}(A_0, \Sigma^{-d} \mathcal{S} X)^\ast \to \text{Hom}(A_1, \Sigma^{-d} \mathcal{S} X)^\ast$$

which is monomorphism if and only if

$$\text{Hom}(A_1, \Sigma^{-d} \mathcal{S} X) \longrightarrow \text{Hom}(A_0, \Sigma^{-d} \mathcal{S} X)$$

$$\text{Hom}(A_1, \tau_d X) \longrightarrow \text{Hom}(A_0, \tau_d X)$$

is epimorphism. For this always to be epimorphism is the same as for $\alpha_0: A_0 \to A_1$ to be a $(\tau \mathcal{X})$-monic, that is, an $\mathcal{X}$-monic.

**Theorem 5.6.** Let $\mathcal{C}$ be a $k$-linear Hom-finite Krull-Schmidt $(d+2)$-angulated category with a Serre functor $\mathcal{S}$, and $\mathcal{X}$ a strongly functorially finite subcategory of $\mathcal{C}$. Then the following statements are equivalent:

1. $(\mathcal{C}, \mathcal{C})$ is an $\mathcal{X}$-mutation pair.
2. The quotient category $\mathcal{C}/\mathcal{X}$ is an $(d+2)$-angulated category.
3. $\tau_d \mathcal{X} = \mathcal{X}$. 

\[\blacksquare\]
Proof. (1) $\implies$ (2). By Lemma [5.4] we have that $\mathcal{C}/\mathcal{X}$ is an $(d+2)$-angulated category.

(2) $\implies$ (1). Since $\mathcal{X}$ is strongly covariantly finite, for any object $A \in \mathcal{C}$, there exists an $(d+2)$-angle in $\mathcal{C}$:

$$A \xrightarrow{\alpha_0} X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-1}} X_d \xrightarrow{\alpha_d} \mathbb{H}A \xrightarrow{\alpha_{d+1}} \Sigma^d A$$

where $X_1, \ldots, X_d \in \mathcal{X}$ and $\alpha_0$ is a left $\mathcal{X}$-approximation of $A$. Since $\mathcal{X}$ is strongly contravariantly finite, for the object $\mathbb{G}A \in \mathcal{C}$, there exists an $(d+2)$-angle in $\mathcal{C}$:

$$\mathbb{LH}A \xrightarrow{\beta_0} Y_1 \xrightarrow{\beta_1} Y_2 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_{d-1}} Y_d \xrightarrow{\beta_d} \mathbb{H}A \xrightarrow{\beta_{d+1}} \Sigma^d A$$

where $Y_1, \ldots, Y_d \in \mathcal{X}$ and $\beta_d$ is a right $\mathcal{X}$-approximation of $\mathbb{H}A$.

Since $\beta_d$ is a right $\mathcal{X}$-approximation of $\mathbb{H}A$ and $X_d \in \mathcal{X}$, then there exists a morphism $\varphi_d: X_d \to Y_d$ such that $\beta_d \varphi_d = \alpha_d$. Thus we have the following morphisms of $(d+2)$-angles

$$A \xrightarrow{\alpha_0} X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-1}} X_d \xrightarrow{\alpha_d} \mathbb{H}A \xrightarrow{\alpha_{d+1}} \Sigma^d A$$

$$\mathbb{LH}A \xrightarrow{\beta_0} Y_1 \xrightarrow{\beta_1} Y_2 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_{d-1}} Y_d \xrightarrow{\beta_d} \mathbb{H}A \xrightarrow{\beta_{d+1}} \Sigma^d \mathbb{H}A.$$}

Since $\mathcal{C}/\mathcal{X}$ is an $(d+2)$-angulated category, then the morphism $\varphi: A \to \mathbb{LH}A$ is invertible in $\mathcal{C}/\mathcal{X}$. It follows that there exists a morphism $\varphi': \mathbb{LH}A \to A$ such that $\varphi' \circ \varphi = \text{Id}_A$ and then $\text{Id}_A - \varphi' \varphi$ factorizes through an object in $\mathcal{X}$. So it factorizes through the left $\mathcal{X}$-approximation $\alpha_0: A \to X_1$ of $A$. Namely there exists a morphism $u: X_1 \to A$ such that $\text{Id}_A - \varphi' \varphi = u \alpha_0$. Note that $(\text{Id}_A - \varphi' \varphi) \circ \Sigma^{-d} \alpha_{d+1} = u(\alpha_0 \circ \Sigma^{-d} \alpha_{d+1}) = 0$ and then

$$\Sigma^{-d} \alpha_{d+1} = (\varphi' \varphi) \circ \Sigma^{-d} \alpha_{d+1} = \varphi' \circ \Sigma^{-d}(\beta_{d+1}).$$

Hence $\alpha_{d+1} = \Sigma^d \varphi' \circ \beta_{d+1}$ and we also have the following morphisms of $(d+2)$-angles

$$\mathbb{LH}A \xrightarrow{\beta_0} Y_1 \xrightarrow{\beta_1} Y_2 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_{d-1}} Y_d \xrightarrow{\beta_d} \mathbb{H}A \xrightarrow{\beta_{d+1}} \Sigma^d \mathbb{LH}A$$

$$A \xrightarrow{\alpha_0} X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-1}} X_d \xrightarrow{\alpha_d} \mathbb{H}A \xrightarrow{\alpha_{d+1}} \Sigma^d A.$$}

Now we show that $\alpha''_d$ is a right $\mathcal{X}$-approximation of $\mathbb{H}A$. Indeed, for any morphism $a: X \to \mathbb{H}A$ with $X \in \mathcal{X}$, since $\beta_d$ is a right $\mathcal{X}$-approximation of $\mathbb{H}A$, there exists a morphism $b: X \to Y_d$ such that $\beta_d b = a$. It follows that $a = \beta_d b = \alpha_d(\phi_d b)$. This shows that $\alpha_d$ is a right $\mathcal{X}$-approximation of $\mathbb{H}A$.

Similarly, one can show that for any object $C \in \mathcal{C}$, there exists an $(d+2)$-angle in $\mathcal{C}$:

$$D \xrightarrow{\gamma_0} Z_1 \xrightarrow{\gamma_1} Z_2 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_{d-1}} Z_d \xrightarrow{\gamma_d} C \xrightarrow{\gamma_{d+1}} \Sigma^d A$$

where $Z_1, \ldots, Z_d \in \mathcal{X}$, $\gamma_0$ is a left $\mathcal{X}$-approximation and $\alpha_d$ is a right $\mathcal{X}$-approximation.

This shows that $(\mathcal{C}, \mathcal{C})$ is an $\mathcal{X}$-mutation pair.
(1) \implies (3). It suffices to show that \( \mathcal{X} \subseteq \mathbb{S}^{-d} \mathcal{X} \) and \( \Sigma^{-d} X \subseteq \mathcal{X} \). We only show that \( \mathcal{X} \subseteq \mathbb{S}^{-d} \mathcal{X} \), dually, we can show that \( \Sigma^{-d} \mathcal{X} \subseteq \mathcal{X} \).

For any indecomposable object \( X \in \mathcal{X} \), then \( \Sigma^{-d} X \) is also indecomposable object. Since \( \mathcal{C} \) has a Serre functor \( \mathbb{S} \), then there exists an Auslander-Reiten \((d+2)\)-angle

\[
X \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{d-2}} A_{d-1} \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} \Sigma^{-d} X \xrightarrow{\alpha_{d+1}} \Sigma X
\]

Since \((\mathcal{C}, \mathcal{X})\) is an \( \mathcal{X} \)-mutation pair, then the object \( \Sigma^{-d} X \) is also indecomposable. Since \( C \) has a Serre functor \( \mathbb{S} \), then there exists an Auslander-Reiten \((d+2)\)-angle

\[
B \xrightarrow{\beta_0} X_1 \xrightarrow{\beta_1} X_2 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_{d-1}} X_d \xrightarrow{\beta_d} \Sigma^{-d} X \xrightarrow{\beta_{d+1}} \Sigma B
\]

where \( X_1, \cdots, X_d \in \mathcal{X}, \beta_0 \) is a left \( \mathcal{X} \)-approximation and \( \beta_d \) is a right \( \mathcal{X} \)-approximation.

If \( \Sigma^{-d} X \not\in \mathcal{X} \), then \( \alpha_d: X_d \to \Sigma^{-d} X \) is not a split epimorphism. Thus there exists a morphism \( u: X_n \to A_n \) such that \( \beta_d = \alpha_d u \). It follows that \( \alpha_{d+1} \beta_d = \alpha_{d+1} \alpha_d u = 0 \). So there exists a morphism \( v: \Sigma B \to \Sigma X \) such that \( v \beta_{d+1} = \alpha_{d+1} \). Since \( \beta_0 \) is a left \( \mathcal{X} \)-approximation and \( X \in \mathcal{X} \), there exists a morphism \( w: X_1 \to X \) such that \( w \beta_0 = \Sigma^{-d} v \) and then \( v = \Sigma^{-d} w \circ \Sigma^{-d} \beta_0 \). Hence \( \alpha_{d+1} = v \beta_{d+1} = \Sigma^{-d} w \circ (\Sigma^{-d} \beta_0 \circ \beta_{d+1}) = 0 \). This is a contradiction.

Thus we obtain \( \Sigma^{-d} X \in \mathcal{X} \) and \( X \in \mathbb{S}^{-d} \mathcal{X} \). Since \( \mathcal{C} \) is a Krull-Schmidt category we infer that \( \mathcal{X} \subseteq \mathbb{S}^{-d} \mathcal{X} \).

(3) \implies (1). This follows from Lemma 5.5. \qed

In Theorem 5.6 if \( n = 1 \), we have the following.

**Corollary 5.7.** [J Theorem 3.3] Let \( \mathcal{C} \) be a triangulated category with an Auslander-Reiten translation \( \tau \), and \( \mathcal{X} \) a functorially finite subcategory of \( \mathcal{C} \). Then the quotient category \( \mathcal{C}/\mathcal{X} \) is a triangulated category if and only if \( \tau \mathcal{X} = \mathcal{X} \).

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