CONTROLLABILITY FOR A POPULATION EQUATION WITH INTERIOR DEGENERACY

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Abstract. We deal with a degenerate model in divergence form describing the dynamics of a population depending on time, on age and on space. We assume that the degeneracy occurs in the interior of the spatial domain and we focus on null controllability. To this aim, first we prove Carleman estimates for the associated adjoint problem, then, via cut off functions, we prove the existence of a null control function localized in the interior of the space domain. We consider two cases: either the control region contains the degeneracy point $x_0$, or the control region is the union of two intervals each of them lying on one side of $x_0$. This paper complement some previous results, concluding the study of the subject.

Dedicated to Irena, on the occasion of her 70th birthday, with sincere esteem

1. Introduction

We consider the following degenerate population model describing the dynamics of a single species:

\begin{align}
\frac{\partial y}{\partial t} + \frac{\partial y}{\partial a} - (k_y x) + \mu(t, a, x)y = f(t, a, x)\chi_{\omega} & \quad \text{in } Q, \\
y(t, a, 1) = y(t, a, 0) = 0 & \quad \text{on } Q_{T,A}, \\
y(0, a, x) = y_0(a, x) & \quad \text{in } Q_{A,1}, \\
y(t, 0, x) = \int_0^A \beta(a, x) y(t, a, x) da & \quad \text{in } Q_{T,1}
\end{align}

where $Q := (0, T) \times (0, A) \times (0, 1)$, $Q_{T,A} := (0, T) \times (0, A)$, $Q_{A,1} := (0, A) \times (0, 1)$ and $Q_{T,1} := (0, T) \times (0, 1)$. Here $y(t, a, x)$ is the distribution of certain individuals at location $x \in (0, 1)$, at time $t \in (0, T)$, where $T$ is fixed, and of age $a \in (0, A)$. $A$ is the maximal age of life, while $\beta$ and $\mu$ are the natural fertility and the natural death rate, respectively. Thus, the formula $\int_0^A \beta y da$ denotes the distribution of newborn individuals at time $t$ and location $x$. The function $k$, which is the dispersion coefficient, depends on the space variable $x$ and we assume that it degenerates in an interior point $x_0$ of the state space. In particular, we say that

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Definition 1.1. The function $k$ is weakly degenerate (WD) if there exists $x_0 \in (0, 1)$ such that $k(x_0) = 0$, $k > 0$ on $[0, 1] \setminus \{x_0\}$, $k \in W^{1,1}(0, 1)$ and there exists $M \in (0, 1)$ so that $(x - x_0)k' \leq Mk$ a.e. in $[0, 1]$.

Definition 1.2. The function $k$ is strongly degenerate (SD) if there exists $x_0 \in (0, 1)$ such that $k(x_0) = 0$, $k > 0$ on $[0, 1] \setminus \{x_0\}$, $k \in W^{1,\infty}(0, 1)$ and there exists $M \in [1, 2)$ so that $(x - x_0)k' \leq Mk$ a.e. in $[0, 1]$.

For example, as $k$ one can consider $k(x) = |x - x_0|^\alpha$, $\alpha > 0$.

Finally, in the model, $\chi_\omega$ is the characteristic function of the control region $\omega \subset (0, 1)$ which can contain $x_0$ or can be the union of two intervals each of them lying on different sides of the degeneracy point, more precisely:

$$\omega = \omega_1 \cup \omega_2$$

where

$$\omega_i = (\lambda_i, \beta_i) \subset (0, 1), i = 1, 2, \text{ and } \beta_1 < x_0 < \lambda_2.$$
We underline that here, contrary to \cite{5}, the function $k$ is less regular, the control region $\omega$ not only can contain $x_0$, but can be also the union of two intervals each of them lying on one side of $x_0$ and $T$ can be greater than $A$. Moreover, contrary to \cite{1}, where $T > A$ and $k$ degenerates at the boundary, here we assume that $T$ can be smaller than $A$ and $k$ degenerates at $x_0 \in (0, 1)$. Hence, this paper is the completion of all the previous ones. Moreover, the technique used in Theorem 4.10 can be also applied either when $k$ degenerates at the boundary of the domain, completing \cite{8}, or when $k$ is in non divergence form and $k$ degenerates at the boundary or in the interior of the domain, completing \cite{9}. Finally, observe that in this paper, as in \cite{3} or in \cite{9}, we do not consider the positivity of the solution, even if it is clearly an interesting question to face: this problem is related to the minimum time, i.e. $T$ cannot be too small (see \cite{16} for related results in non degenerate cases. This topic will be the subject of further investigations.

A final comment on the notation: by $c$ or $C$ we shall denote universal strictly positive constants, which are allowed to vary from line to line.

2. Well posedness results

For the well posedness of the problem, we assume the following hypotheses on the rates $\mu$ and $\beta$:

**Hypothesis 2.1.** The functions $\mu$ and $\beta$ are such that

\begin{equation}
\begin{aligned}
\bullet \beta &\in C(\bar{Q}_{A,1}) \text{ and } \beta \geq 0 \text{ in } Q_{A,1}, \\
\bullet \mu &\in C(\bar{Q}) \text{ and } \mu \geq 0 \text{ in } Q.
\end{aligned}
\end{equation}

To prove well posedness of \eqref{1.1}, we introduce, as in \cite{11}, the following Hilbert spaces

\[
H^1_k(0, 1) := \left\{ u \in W^{1,1}_0(0, 1) : \sqrt{k}u' \in L^2(0, 1) \right\}
\]

and

\[
H^2_k := \left\{ u \in H^1_k(0, 1) : ku_x \in H^1(0, 1) \right\}.
\]

We have, as in \cite{11}, that the operator

\[
A_0 u := (ku_x)_x, \quad D(A_0) := H^2_k(0, 1)
\]

is self–adjoint, nonpositive and generates an analytic contraction semigroup of angle $\pi/2$ on the space $L^2(0, 1)$.

As in \cite{3}, setting $A_\alpha u := \frac{\partial u}{\partial \alpha}$, we have that

\[
A u := A_\alpha u - A_0 u,
\]

for

\[
u \in D(A) = \left\{ u \in L^2(0, A; D(A_0)) : \frac{\partial u}{\partial \alpha} \in L^2(0, A; H^1_k(0, 1)), \right\}
\]

\[
u(0, x) = \int_0^A \beta(a, x)u(a, x)da.
\]
generates a strongly continuous semigroup on \( L^2(Q_{A,1}) := L^2(0, A; L^2(0, 1)) \) (see also [4]). Moreover, the operator \( B(t) \) defined as

\[
B(t)u := \mu(t, a, x)u,
\]

for \( u \in D(A) \), can be seen as a bounded perturbation of \( A \) (see, for example, [2]); thus also \((A + B(t), D(A))\) generates a strongly continuous semigroup.

Setting \( L^2(Q) := L^2(0, T; L^2(Q_{A,1})) \), the following well posedness result holds (see [8] for the proof):

**Theorem 2.1.** Assume that \( k \) is weakly or strongly degenerate at 0 and/or at 1. For all \( f \in L^2(Q) \) and \( y_0 \in L^2(Q_{A,1}) \), the system (1.1) admits a unique solution \( y \in U := C([0, T]; L^2(Q_{A,1})) \cap L^2(0, T; H^1(0, A; H^1_k(0, 1))) \) and

\[
\sup_{t \in [0, T]} \|y(t)\|_{L^2(Q_{A,1})} + \int_0^T \int_0^A \|\sqrt{k}y_x\|_{L^2(0, 1)}^2 \, da \, dt 
\leq C \|y_0\|_{L^2(Q_{A,1})}^2 + C \|f\|_{L^2(Q)}^2,
\]

where \( C \) is a positive constant independent of \( k, y_0 \) and \( f \).

In addition, if \( f \equiv 0 \), then \( y \in C^1([0, T]; L^2(Q_{A,1})) \).

3. CARLEMAN ESTIMATES

In this section we show degenerate Carleman estimates for the following adjoint system associated to (1.1):

\[
\begin{aligned}
\frac{\partial z}{\partial t} + \frac{\partial z}{\partial a} + (k(x)z_x)_x - \mu(t, a, x)z &= f, \quad (t, a, x) \in Q, \\
z(t, a, 0) &= z(t, a, 1) = 0, \quad (t, a) \in Q_{T, A}, \\
z(t, A, x) &= 0, \quad (t, x) \in Q_{T, 1}.
\end{aligned}
\]

On \( k \) we make additional assumptions:

**Hypothesis 3.1.** The function \( k \) is (WD) or (SD). Moreover, if \( M > \frac{4}{3} \), then there exists a constant \( \theta \in (0, M] \) such that

\[
x \mapsto \frac{k(x)}{|x - x_0|^{\theta}} \quad \begin{cases}
\text{is non increasing on the left of } x = x_0, \\
\text{is non decreasing on the right of } x = x_0.
\end{cases}
\]

In addition, when \( M > \frac{3}{2} \) the function in (3.2) is bounded below away from 0 and there exists a constant \( \Gamma > 0 \) such that

\[
|k'(x)| \leq \Gamma |x - x_0|^{2\theta - 3} \text{ for a.e. } x \in [0, 1].
\]

Now, let us introduce the weight function

\[
\varphi(t, a, x) := \Theta(t, a)\psi(x),
\]
where
\[
(3.5) \quad \Theta(t,a) := \frac{1}{[t(T-t)]^4 a^4} \quad \text{and} \quad \psi(x) := c_1 \left[ \int_{x_0}^{x} \frac{y-x_0}{k(y)} dy - c_2 \right].
\]

The following estimate holds:

**Theorem 3.1.** Assume that Hypothesis \([3.1]\) is satisfied. Then, there exist two strictly positive constants \(C\) and \(s_0\) such that every solution \(v\) of (3.1) in
\[
V := L^2(Q_{T,A}; H^2_k(0,1)) \cap H^1(0,T; H^1(0,A; H^1_k(0,1)))
\]
satisfies, for all \(s \geq s_0\),
\[
\int_{Q} \left( s \Theta(v_x)^2 + s^3 \Theta \frac{(x-x_0)^2}{k} v^2 \right) e^{2s\varphi} dx \, dt \\
\leq C \left( \int_{Q} f^2 e^{2s\varphi} dx \, dt + sc_1 \int_{0}^{T} \int_{A} [k \Theta e^{2s\varphi}(x-x_0)(v_x)^2 dx]_{x=1}^{x=0} \, dt \right)
\]

Clearly the previous Carleman estimate holds for every function \(v\) that satisfies (3.1) in \((0,T) \times (0,A) \times (B,C)\) as long as \((0,1)\) is substituted by \((B,C)\) and \(k\) satisfies Hypothesis \([3.1]\) in \((B,C)\).

**Proof of Theorem 3.1.** The proof of Theorem 3.1 follows the ideas of the one of [8, Theorem 3.1] or [9, Theorem 3.6] (for the non divergence case). As in the previous papers, we consider, first of all, the case when \(\mu \equiv 0\): for every \(s > 0\) consider the function
\[
\begin{align*}
\begin{cases}
\text{(3.6)} & \\
w(t,a,x) := e^{s\varphi(t,a,x)} v(t,a,x),
\end{cases}
\end{align*}
\]
where \(v\) is any solution of (3.1) in \(V\), so that also \(w \in V\), since \(\varphi < 0\). Moreover, \(w\) satisfies
\[
\begin{align*}
\begin{cases}
(e^{-s\varphi} w)_t + (e^{-s\varphi} w)_a + (k(e^{-s\varphi} w)_x)_x = f(t,a,x), & (t,x) \in Q, \\
\quad w(0,a,x) = w(T,a,x) = 0, & (a,x) \in Q_{A,1}, \\
\quad w(t,A,x) = w(t,0,x) = 0, & (t,x) \in Q_{T,1}, \\
\quad w(t,a,0) = w(t,a,1) = 0, & (t,a) \in Q_{T,A},
\end{cases}
\end{align*}
\]
and [8 Lemma 3.1] still holds. In particular, setting
\[
\begin{align*}
\begin{cases}
L^+_s w := (kw_x)_x - s(\varphi_t + \varphi_a)w + s^2 k\varphi_x^2 w, \\
L^-_s w := w_t + w_a - 2sk\varphi_x w_x - s(k\varphi_x)_x w,
\end{cases}
\end{align*}
\]
we have
Lemma 3.2. [see [8] Lemma 3.1] Assume Hypothesis 3.1 The following identity holds (3.7)
\[
\langle L_s^+ w, L_s^- w \rangle_{L^2(Q)} = \frac{s}{2} \int_Q (\varphi_t + \varphi_{aa}) w^2 dxdadt \\
+ s \int_Q k(x)(k(x)\varphi_x)_{xx} w w_x dxdadt \\
- 2s^2 \int_Q k\varphi_x \varphi_{tx} w^2 dxdadt - 2s^2 \int_Q k\varphi_x \varphi_{xa} w^2 dxdadt \\
+ s \int_Q (2k^2 \varphi_{xx} + kk' \varphi_x) w_x^2 dxdadt \\
+ s^3 \int_Q (2k\varphi_{xx} + k' \varphi_x) k\varphi_x^2 w^2 dxdadt \\
+ s \int_Q \varphi_{at} w^2 dxdadt.
\]
\{D.T.\}

\[
\begin{align*}
\int_{Q_{T,A}} [kw_x w_t]_0^1 dadt &+ \int_{Q_{T,A}} [kw_x w_a]_0^1 dadt \\
- \frac{s}{2} \int_{Q_{A,1}} [\varphi_a w^2]_0^T dxa.
\end{align*}
\{B.T.\}

\[
\begin{align*}
\int_{Q_{T,A}} [k\varphi_x \varphi_{tx} w w_x]_0^1 dadt &+ s^2 \int_{Q_{T,A}} [k\varphi_x \varphi_{xa} w]_0^1 dadt \\
- \frac{1}{2} \int_{Q_{T,1}} [kw_x^2]_0^A dxdt + \frac{1}{2} \int_{Q_{T,1}} [(s^2 k\varphi_x^2]_0^A dxdt \\
- s(\varphi_t + \varphi_a)] w^2]_0^A dxdt.
\end{align*}
\]

We underline the fact that in this case all integrals and integrations by parts are justified by the definition of \(D(A)\) and the choice of \(\varphi\), while, if the degeneracy is at the boundary of the domain as in [8], they were guaranteed by the choice of Dirichlet conditions at \(x = 0\) or \(x = 1\), i.e. where the operator is degenerate.

As a consequence of the definition of \(\varphi\), one has the next estimate:

Lemma 3.3. Assume Hypothesis 3.1 There exist two strictly positive constants \(C\) and \(s_0\) such that, for all \(s \geq s_0\), all solutions \(w\) of (3.6) satisfy the
following estimate

\[
sC \int_Q \Theta k w_x^2 dx\,dt + s^3 C \int_Q \Theta^3 \frac{(x - x_0)^2}{k} w^2 dx\,dt \leq \{\text{D.T.}\}.
\]

Proof. Using the definition of \(\phi\), the distributed terms given in Lemma 3.2 take the form

\[
\{\text{D.T.}\} = \left\{ \begin{array}{l}
\frac{s}{2} \int_Q (\Theta_{tt} + \Theta_{aa}) \psi w^2 dx\,dt - 2s^2 c_1 \int_Q \Theta \Theta_t \frac{(x - x_0)^2}{k} w^2 dx\,dt \\
- 2s^2 c_1 \int_Q \Theta \Theta_a \frac{(x - x_0)^2}{k} w^2 dx\,dt \\
+ sc_1 \int_Q \Theta \left( 2 - \frac{k'}{k} (x - x_0) \right) k(w_x)^2 dx\,dt \\
+ s^3 c_3 \int_Q \Theta^3 \left( 2 - \frac{k'}{k} (x - x_0) \right) \frac{(x - x_0)^2}{k} w^2 dx\,dt \\
+ s \int_Q \Theta_{ta} \psi w^2 dx\,dt.
\end{array} \right.
\]

Because of the choice of \(\phi(x)\), one has, as in [11],

\[
2 - \frac{(x - x_0)k'}{k} \geq 2 - M \quad \text{a.e.} \quad x \in [0, 1].
\]

Thus, there exists \(C > 0\) such that, the distributed terms satisfy the estimate (3.8)

\[
\{\text{D.T.}\} \geq \frac{s}{2} \int_Q (\Theta_{tt} + \Theta_{aa}) \psi w^2 dx\,dt - s^2 C \int_Q |\Theta \Theta_t| \frac{(x - x_0)^2}{k} w^2 dx\,dt \\
- s^2 C \int_Q |\Theta \Theta_a| \frac{(x - x_0)^2}{k} w^2 dx\,dt \\
+ s C \int_Q \Theta(w_x)^2 dx\,dt + s^3 C \int_Q \Theta^3 \frac{(x - x_0)^2}{k} w^2 dx\,dt \\
+ s \int_Q \Theta_{ta} \psi w^2 dx\,dt.
\]

By [9, Lemma 3.5], we conclude that, for \(s\) large enough,

\[
s^2 C \int_Q (|\Theta_t| + |\Theta_a|) \frac{(x - x_0)^2}{k} w^2 dx\,dt \leq C s^2 \int_Q \Theta^3 \frac{(x - x_0)^2}{k} w^2 dx\,dt \\
\leq \frac{C^3}{4} s^3 \int_Q \Theta^3 \frac{(x - x_0)^2}{k} w^2 dx\,dt.
\]
Again as in [11, Lemma 4.1], we get
\[
\left| \frac{s}{2} \int_Q (\Theta_{tt} + \Theta_{aa}) \psi w^2 \, dx \, dt \right| \leq sC \int_Q \Theta^{3/2} w^2 \, dx \, dt 
\]
(3.9)
\[
\leq \frac{C}{4} s \int_Q \Theta k(w_x)^2 \, dx \, dt 
\]
\[+ \frac{C^3}{4} s^3 \int_Q \Theta^3 \left( \frac{x - x_0}{k} \right)^2 w^2 \, dx \, dt. \]

Analogously, one has that the last term in (3.8), i.e. \( s \int_Q \Theta_t \psi w^2 \, dx \, dt \) satisfies
\[
\left| s \int_Q \Theta_t \psi w^2 \, dx \, dt \right| \leq \frac{C}{4} s \int_Q \Theta k(w_x)^2 \, dx \, dt 
\]
\[+ \frac{C^3}{4} s^3 \int_Q \Theta^3 \left( \frac{x - x_0}{k} \right)^2 w^2 \, dx \, dt. \]

Summing up, we obtain
\[
\{D.T.\} \geq - \frac{C}{4} s \int_Q \Theta(w_x)^2 \, dx \, dt - \frac{C^3}{4} s^3 \int_Q \Theta^3 \left( \frac{x - x_0}{k} \right)^2 w^2 \, dx \, dt 
\]
\[+ \frac{C^3}{4} s^3 \int_Q \Theta^3 \left( \frac{x - x_0}{k} \right)^2 w^2 \, dx \, dt 
\]
\[+ sC \int_Q \Theta(w_x)^2 \, dx \, dt + sC \int_Q \Theta^3 \left( \frac{x - x_0}{k} \right)^2 w^2 \, dx \, dt 
\]
\[+ \frac{C}{4} s \int_Q \Theta(w_x)^2 \, dx \, dt - \frac{C^3}{4} s^3 \int_Q \Theta^3(w_x)^2 \, dx \, dt 
\]
\[+ \frac{C^3}{4} s^3 \int_Q \Theta^3 \left( \frac{x - x_0}{k} \right)^2 w^2 \, dx \, dt. \]

\[\square\]

Proceeding as in [8] and in [11], one has for the boundary terms the following lemma:

**Lemma 3.4.** Assume Hypothesis 3.1. The boundary terms in (3.7) reduce to
\[
-sC \int_0^T \int_0^A \Theta(t) k \left[ (x - x_0)(w_x)^2 \right]_{x=0}^{x=1} \, dx \, dt.
\]

By Lemmas 3.2-3.4 there exist \( C > 0 \) and \( s_0 > 0 \) such that all solutions \( w \) of (3.6) satisfy, for all \( s \geq s_0 \),
\[
s \int_Q \Theta k w_x^2 \, dx \, dt + s^3 \int_Q \Theta^3 \left( \frac{x - x_0}{k} \right)^2 w^2 \, dx \, dt 
\]
\[\leq C \left( \int_Q f^2 e^{2s} \, dx \, dt + sC \int_0^T \int_0^A [\Theta k(x)(x - x_0)(w_x)^2]_{x=0}^{x=1} \, dx \, dt \right).\]
Hence, if $\mu \equiv 0$, Theorem 3.1 follows recalling the definition of $w$ and the fact that

$$L^+_s w + L^-_s w = e^{s\varphi} f,$$

If $\mu \neq 0$, we consider the function $\mathcal{F} = f + \mu v$. Hence, there are two strictly positive constants $C$ and $s_0$ such that, for all $s \geq s_0$, the following inequality holds

$$(3.10)$$

$$\int_Q \left( s\Theta k(v_x)^2 + s^3 \Theta^3 \frac{(x-x_0)^2}{k} v^2 \right) e^{2s\varphi} dxdadt \leq C \left( \int_Q \mathcal{F}^2 e^{2s\varphi} dxdadt + s \int_0^T \int_0^A \left[ k\Theta e^{2s\varphi} (x-x_0)(v_x)^2 dadt \right]_{x=0}^{x=1} dadt \right).$$

On the other hand, we have

$$(3.11)$$

$$\int_Q \mathcal{F}^2 e^{2s\varphi} dxdadt \leq 2 \left( \int_Q |f|^2 e^{2s\varphi} dxdadt + \int_Q |\mu|^2 |v|^2 e^{2s\varphi} dxdadt \right).$$

Now, setting $\nu := e^{s\varphi} v$, we obtain

$$(3.12)$$

$$\int_Q |\mu|^2 |v|^2 e^{2s\varphi} dxdadt \leq \|\mu\|_{\infty}^2 \int_0^1 \nu^2 dx$$

$$= \|\mu\|_{\infty}^2 \int_0^1 \left( \frac{k^{1/3}}{|x-x_0|^{2/3}} \nu^2 \right)^{3/4} \left( \frac{|x-x_0|^2}{k} \nu^2 \right)^{1/4}$$

$$\leq C \int_0^1 \frac{k^{1/3}}{|x-x_0|^{2/3}} \nu^2 dx + C \int_0^1 \frac{|x-x_0|^2}{k} \nu^2 dx.$$

As in [34], proceeding as in [11] and applying the Hardy-Poincaré inequality proved in [10] to the function $\nu$ with weight $p(x) = |x-x_0|^{4/3}$, if $K \leq \frac{4}{3}$, or $p(x) = (k(x)|x-x_0|^{4})^{1/3}$, if $K > 4/3$, we can prove that

$$\int_0^1 \frac{k^{1/3}}{|x-x_0|^{2/3}} \nu^2 dx \leq C \int_0^1 k(v_x)^2 dx$$

$$\leq C \int_Q k(x)e^{2s\varphi} v_x^2 dxdadt$$

$$+ Cs^2 \int_Q \Theta^2 e^{2s\varphi} \frac{(x-x_0)^2}{k} v^2 dxdadt.$$

$$(3.13)$$
In any case, by (3.11), (3.12) and (3.13), we have
\( (3.14) \)
\[
\int_Q |\bar{f}|^2 e^{2s\varphi} \, dx \, da \leq 2 \int_Q |f|^2 e^{2s\varphi} \, dx \, da + C \int_Q k(x) e^{2s\varphi} v_x^2 \, dx \, da \\
+ C s^2 \int_Q \Theta^2 e^{2s\varphi} (x-x_0)^2 \frac{v^2}{k} \, dx \, da \\
\leq C \int_Q |f|^2 e^{2s\varphi} \, dx \, da + C \int_Q \Theta(x) e^{2s\varphi} v_x^2 \, dx \, da \\
+ C s^2 \int_Q \Theta^3 e^{2s\varphi} (x-x_0)^2 \frac{v^2}{k} \, dx \, da.
\]

Substituting in (3.10), one can conclude
\[
\int_Q \left( s \Theta k v_x^2 + s^3 \Theta^3 \frac{(x-x_0)^2}{k} v^2 \right) e^{2s\varphi} \, dx \, da \leq C \left( \int_Q |f|^2 e^{2s\varphi} \, dx \, da \right) \\
+ s \int_0^T \int_A \left[ k \Theta e^{2s\varphi} (x-x_0)(v_x)^2 \, da \, dt \right]_{x=x_0}^{x=1} \, dt,
\]
for all \( s \) large enough.

### 4. Observability and Controllability

In this section we will prove, as a consequence of the Carleman estimates established in Section 3, observability inequalities for the associated adjoint problem of (1.1):
\( (4.1) \)
\[
\begin{align*}
\frac{\partial v}{\partial t} + \frac{\partial v}{\partial a} + (k(x)v_x)_x - \mu(t, a, x)v + \beta(a, x)v(t, 0, x) &= 0, \quad (t, x, a) \in Q, \\
v(t, a, 0) &= v(t, a, 1) = 0, \quad (t, a) \in Q_{T,A}, \\
v(T, a, x) &= v_T(a, x) \in L^2(Q_{A,1}), \quad (a, x) \in Q_{A,1}, \\
v(t, A, x) &= 0, \quad (t, x) \in Q_{T,1}.
\end{align*}
\]

From now on, we assume that the control set \( \omega \) is such that
\( (4.2) \)
\[ x_0 \in \omega = (\alpha, \rho) \subset \subset (0, 1), \]
or
\( (4.3) \)
\[ \omega = \omega_1 \cup \omega_2, \]
where
\( (4.4) \)
\[ \omega_i = (\lambda_i, \rho_i) \subset (0, 1), \ i = 1, 2, \ \text{and} \ \rho_1 < x_0 < \lambda_2. \]

**Remark 4.1.** Observe that, if (1.2) holds, we can find two subintervals \( \omega_1 = (\lambda_1, \rho_1) \subset \subset (\alpha, x_0), \omega_2 = (\lambda_2, \rho_2) \subset \subset (x_0, \rho) \).

Moreover, on \( \beta \) we assume the following assumption:
Hypothesis 4.1. Suppose that there exists $\bar{a} < A$ such that
\begin{equation}
\beta(a, x) = 0 \text{ for all } (a, x) \in [0, \bar{a}] \times [0, 1].
\end{equation}

Observe that Hypothesis 4.1 has a biological meaning. Indeed, $\bar{a}$ is the minimal age in which the female of the population become fertile, thus it is natural that before $\bar{a}$ there are no newborns. For other comments on Hypothesis 4.1 we refer to [9].

In order to prove the desired observability inequality for the solution $v$ of (4.1) we proceed, as usual, using a density argument. To this purpose, we consider, first of all the space
\begin{equation}
W := \left\{ v \text{ solution of (4.1)} \mid v_T \in D(A^2) \right\},
\end{equation}
where $D(A^2) = \left\{ u \in D(A) \mid Au \in D(A) \right\}$. Clearly $D(A^2)$ is densely defined in $D(A)$ (see, for example, [6, Lemma 7.2]) and hence in $L^2(Q_{A,1})$ and
\[W = C^1([0, T] ; D(A))\]
\[\subset V := L^2(Q_{T,A}; H_0^2(0,1)) \cap H^1(0,T; H^1(0,A; H_0^1(0,1))) \subset U.\]

Proposition 4.2 (Caccioppoli’s inequality). Let $\omega'$ and $\omega$ two open subintervals of $(0,1)$ such that $\omega' \subset\subset \omega \subset (0,1)$ and $x_0 \notin \bar{\omega}'$. Let $\psi(t,x) := \Theta(t,a,\Psi(x))$, where
\begin{equation}
\Theta(t,a) = \frac{1}{t^4(T-t)^4a^4}
\end{equation}
and $\Psi \in C([0, 1], (-\infty,0)) \cap C^1([0, 1] \setminus \{x_0\}, (-\infty,0))$ is such that
\begin{equation}
|\Psi_x| \leq \frac{c}{\sqrt{k}} \text{ in } [0, 1] \setminus \{x_0\}.
\end{equation}

Then, there exist two strictly positive constants $C$ and $s_0$ such that, for all $s \geq s_0$,
\begin{equation}
\int_0^T \int_0^A \int_{\omega'} v_x^2e^{2s\psi} dx dt \leq C \left( \int_0^T \int_0^A \int_{\omega} v^2 dx dt + \int_Q f^2 e^{2s\psi} dx dt \right),
\end{equation}
for every solution $v$ of (4.1).

The proof of the previous proposition is similar to the one given in [8, Proposition 4.2] and [10, Proposition 4.2], so we omit it.

Moreover, the following non degenerate inequality proved in [9] is crucial:

Theorem 4.3. [see [9], Theorem 3.2] Let $z \in Z$ be the solution of (3.1), where $f \in L^2(Q)$, $k \in C^1([0,1])$ is a strictly positive function and
\[Z := L^2(Q_{T,A}; H^2(0,1) \cap H_0^1(0,1)) \cap H^1(0,T; H^1(0,A; H_0^1(0,1))).\]
Then, there exist two strictly positive constants $C$ and $s_0$, such that, for any $s \geq s_0$, $z$ satisfies the estimate
\begin{equation}
\int_Q \left( s^3 \phi^2 z^2 + s \phi z_x^2 \right) e^{2s \Phi} \, dx \, dt \leq C \int_Q f^2 e^{2s \Phi} \, dx \, dt
\end{equation}
(4.9)
\[ - C s \kappa \int_0^T \int_0^A \left[ ke^{2s \Phi} \phi(z_x)^2 \right] \, d t \, d a t. \]

Here the functions $\phi$ and $\Phi$ are defined as follows
\begin{equation}
\phi(t, a, x) = \Theta(t, a) e^{\kappa \sigma(x)}, \\
\Phi(a, t, x) = \Theta(t, a) \Psi(x), \\
\Psi(x) = e^{\kappa \sigma(x)} - e^{2 \kappa ||\sigma||_\infty},
\end{equation}
(4.10)
where $(t, a, x) \in Q$, $\kappa > 0$, $\sigma(x) := \mathcal{O} \int_x^1 \frac{1}{k(t)} \, dt$, $\mathcal{O} = ||k'||_{L^\infty(0, 1)}$ and $\Theta$ is given in (4.6).

Remark 4.4. The previous Theorem still holds under the weaker assumption $k \in W^{1, \infty}(0, 1)$ without any additional assumption.

On the other hand, if we require $k \in W^{1,1}(0, 1)$ then we have to add the following hypothesis: there exist two functions $g \in L^1(0, 1)$, $h \in W^{1,\infty}(0, 1)$ and two strictly positive constants $g_0$, $h_0$ such that $g(x) \geq g_0$ and
\begin{equation}
- \frac{k'(x)}{2 \sqrt{k(x)}} \left( \int_x^1 g(t) \, dt + h_0 \right) + \sqrt{k(x)} g(x) = h(x) \quad \text{for a.e. } x \in [0, 1].
\end{equation}
(4.11)

In this case, i.e. if $k \in W^{1,1}(0, 1)$, the function $\Psi$ in (4.10) becomes
\begin{equation}
\Psi(x) := - r \left[ \int_0^x \frac{1}{k(t)} \int_t^1 g(s) \, ds \, dt + \int_0^x \frac{h_0}{\sqrt{k(t)}} \, dt \right] - \mathcal{C},
\end{equation}
(4.12)
where $r$ and $\mathcal{C}$ are suitable strictly positive functions. For other comments on Theorem 4.3 we refer to [9].

In the following, we will apply Theorem 4.3 in the intervals $[\lambda, 1]$ and $[-\rho_1, \rho_1]$ under these weaker assumptions. In particular, on $k$ we assume:

Hypothesis 4.2. The function $k$ satisfies Hypothesis 3.1. Moreover, if $k \in W^{1,1}(0, 1)$, then there exist two functions $g \in L^\infty_{\text{loc}}([-\rho_1, 1] \setminus \{x_0\})$, $h \in W^1_{\text{loc}}([-\rho_1, 1] \setminus \{x_0\}, L^\infty(0, 1))$ and two strictly positive constants $g_0$, $h_0$ such that $g(x) \geq g_0$ and
\begin{equation}
- \frac{\tilde{k}'(x)}{2 \sqrt{\tilde{k}(x)}} \left( \int_x^B g(t) \, dt + h_0 \right) + \sqrt{\tilde{k}(x)} g(x) = h(x, B)
\end{equation}
(4.13)
for a.e. $x \in [-\rho_1, 1], B \in [0, 1]$ with $x < B < x_0$ or $x_0 < x < B$, where
\begin{equation}
\tilde{k}(x) := \begin{cases} 
  k(x), & x \in [0, 1], \\
  k(-x), & x \in [-1, 0].
\end{cases}
\end{equation}
(4.14)
With the aid of Theorems 3.1, 4.3 and Proposition 4.2, we can now show \( \omega \)-local Carleman estimates for (3.1).

**Theorem 4.5.** Assume Hypothesis 4.2. Then, there exist two strictly positive constants \( C \) and \( s_0 \) such that every solution \( v \) of (3.1) in \( V \) satisfies, for all \( s \geq s_0 \),

\[
\int_Q \left( s\Theta k v_x^2 + s^3 \Theta^3 \frac{(x-x_0)^2 v^2}{k} \right) e^{2s\phi} \, dx \, dt \leq C \int_Q f^2 e^{2s\phi} \, dx \, dt
\]

\[
+ C \int_0^T \int_0^A \int_\omega v^2 \, dx \, dt
\]

**Proof.** First assume that \( \omega \) satisfies (4.2) and take \( w_i, i = 1, 2 \), as in Remark 4.1. Now, fix \( \bar{\lambda}_i, \bar{\rho}_i \in \omega_i = (\lambda_i, \rho_i), i = 1, 2 \), such that \( \bar{\lambda}_i < \bar{\rho}_i \) and consider a smooth function \( \xi : [0,1] \to [0,1] \) such that

\[
\xi(x) = \begin{cases} 
0 & x \in [0, \bar{\lambda}_1], \\
1 & x \in [\bar{\lambda}_1, \bar{\lambda}_2], \\
0 & x \in [\bar{\rho}_2, 1], 
\end{cases}
\]

where \( \bar{\lambda}_i = (\bar{\lambda}_i + \bar{\rho}_i)/2, i = 1, 2 \). Define \( w := \xi v \), where \( v \) is any fixed solution of (3.1). Then \( w \) satisfies

\[
\begin{align*}
&w_t + w_a + (kw_x)_x - \mu w = \xi f + (k\xi_x v)_x + \xi x k v_x =: h, \quad (t,a,x) \in Q, \\
&w(t,a,0) = w(t,a,1) = 0, \quad (t,a) \in Q_{T,A}.
\end{align*}
\]

Thus, applying Theorem 3.1 Proposition 4.2 and proceeding as in [8], we have

\[
\int_0^T \int_0^A \int_{\bar{\lambda}_2}^{\bar{\lambda}_1} \left( s\Theta k v_x^2 + s^3 \Theta^3 \frac{(x-x_0)^2}{k} v^2 \right) e^{2s\phi} \, dx \, dt
\]

\[
= \int_0^T \int_0^A \int_{\bar{\lambda}_2}^{\bar{\lambda}_1} \left( s\Theta k w_x^2 + s^3 \Theta^3 \frac{x^2}{k} w^2 \right) e^{2s\phi} \, dx \, dt
\]

\[
\leq C \left( \int_Q f^2 e^{2s\phi} \, dx \, dt + \int_0^T \int_0^A \int_\omega v^2 \, dx \, dt \right).
\]

Now, consider a smooth function \( \eta : [0,1] \to [0,1] \) such that

\[
\eta(x) = \begin{cases} 
0 & x \in [0, \bar{\lambda}_2], \\
1 & x \in [\bar{\lambda}_2, 1], 
\end{cases}
\]

and define \( z := \eta v \). Then \( z \) satisfies

\[
\begin{align*}
&z_t + z_a + (kz_x)_x - \mu z = \eta f + (k\eta_x v)_x + \eta x k v_x =: h, \quad (t,a,x) \in Q_{T,A} \times (\lambda_2, 1), \\
z(t,a,\lambda_2) = z(t,a,1) = 0, \quad t \in Q_{T,A},
\end{align*}
\]
Clearly the equation satisfied by $z$ is not degenerate, thus applying Theorem 4.3 and [14, Lemma 4.1] on $(\lambda_2, 1)$, one has

$$
\int_0^T \int_0^A \int_{\lambda_2}^1 (s^3 \phi^3 z^2 + s\phi z_x^2)e^{2s\phi} \, dx \, da \, dt \leq C \int_0^T \int_0^A \int_{\lambda_2}^1 h^2 e^{2s\phi} \, dx \, da \, dt
$$

$$
\leq C \left( \int_Q f^2 e^{2s\phi} \, dx \, da \, dt + \int_0^T \int_0^A \int_\omega v^2 \, dx \, da \, dt \right).
$$

Hence

$$
\int_0^T \int_0^A \int_{\lambda_2}^1 (s^3 \phi^3 v^2 + s\phi v_x^2)e^{2s\phi} \, dx \, da \, dt = \int_0^T \int_0^A \int_{\lambda_2}^1 (s^3 \phi^3 z^2 + s\phi z_x^2)e^{2s\phi} \, dx \, da \, dt
$$

$$
\leq C \left( \int_Q f^2 e^{2s\phi} \, dx \, da \, dt + \int_0^T \int_0^A \int_\omega v^2 \, dx \, da \, dt \right),
$$

for a strictly positive constant $C$. Proceeding, for example, as in [11] one can prove the existence of $\varsigma > 0$, such that, for all $(t, a, x) \in [0, T] \times [0, A] \times [\lambda_2, 1]$, we have

$$
e^{2s\phi} \leq \varsigma e^{2s\phi}, \quad \left(\frac{x - x_0}{k(x)}\right)^2 e^{2s\phi} \leq \varsigma e^{2s\phi}.
$$

Thus, for a strictly positive constant $C$,

$$
\int_0^T \int_0^A \int_{\lambda_2}^1 \left(s\Theta k v_x^2 + s^3 \Theta^3 \left(\frac{x - x_0}{k}\right)^2 v^2\right)e^{2s\phi} \, dx \, da \, dt
$$

$$
\leq C \left( \int_Q f^2 e^{2s\phi} \, dx \, da \, dt + \int_0^T \int_0^A \int_\omega v^2 \, dx \, da \, dt \right).
$$

Hence,

$$
\int_0^T \int_0^A \int_{\lambda_2}^1 \left(s\Theta k v_x^2 + s^3 \Theta^3 \left(\frac{x - x_0}{k}\right)^2 v^2\right)e^{2s\phi} \, dx \, da \, dt
$$

$$
\leq C \left( \int_Q f^2 e^{2s\phi} \, dx \, da \, dt + \int_0^T \int_0^A \int_\omega v^2 \, dx \, da \, dt \right).
$$

To complete the proof it is sufficient to prove a similar inequality for $x \in [0, \lambda_1]$. To this aim, we use the reflection procedure as in [9]; thus we consider the functions

$$
W(t, a, x) := \begin{cases} 
v(t, a, x), & x \in [0, 1], \\
-v(t, a, -x), & x \in [-1, 0], 
\end{cases}
$$

$$
\tilde{f}(t, a, x) := \begin{cases} 
f(t, a, x), & x \in [0, 1], \\
-f(t, a, -x), & x \in [-1, 0], 
\end{cases}
$$
\[
\tilde{\mu}(t, a, x) := \begin{cases} 
\mu(t, a, x), & x \in [0, 1], \\
\mu(t, a, -x), & x \in [-1, 0],
\end{cases}
\]
so that \( W \) satisfies the problem
\[
\begin{cases}
W_t + W_a + (\tilde{k}W_x)_x - \tilde{\mu}W = \tilde{f}, & (t, x) \in Q_{T,A} \times (-1, 1), \\
W(t, a, -1) = W(t, a, 1) = 0, & t \in Q_{T,A},
\end{cases}
\]
(by the way, observe that in [9] there is a misprint in the definition of \( \mu \); it clearly must be defined in this way, otherwise \( W \) is not the solution of the associated problem). Now, consider a cut off function \( \zeta : [-1, 1] \to [0, 1] \) such that
\[
\zeta(x) = \begin{cases} 
0 & x \in [-1, -\tilde{\rho}_1], \\
1 & x \in [-\tilde{\lambda}_1, \tilde{\lambda}_1], \\
0 & x \in [\tilde{\rho}_1, 1],
\end{cases}
\]
and define \( Z := \zeta W \). Then \( Z \) satisfies
\[
(4.20) \quad \begin{cases}
Z_t + Z_a + (\tilde{k}Z_x)_x - \tilde{\mu}Z = \tilde{h}, & (t, x) \in Q_{T,A} \times (-\rho_1, \rho_1), \\
Z(t, a, -\rho_1) = Z(t, a, \rho_1) = 0, & t \in Q_{T,A},
\end{cases}
\]
where \( \tilde{h} = \zeta \tilde{f} + (\tilde{k}\zeta x W)_x + \zeta \tilde{k}W_x \). Now, applying the analogue of Theorem 4.3 on \((-\rho_1, \rho_1)\) in place of \((0, 1)\), using the definition of \( W \), the fact that \( Z_x(t, a, -\rho_1) = Z_x(t, a, \rho_1) = 0 \) and since \( \zeta \) is supported in \([-\tilde{\rho}_1, -\tilde{\lambda}_1] \cup [\tilde{\lambda}_1, \tilde{\rho}_1]\), we get
\[
\begin{align*}
\int_0^T \int_0^A \int_0^{\tilde{\lambda}_1} \left( s\Theta k(W_x)^2 + s^3\Theta^3 \frac{(x-x_0)^2}{k} W^2 \right) e^{2s\varphi} dx da dt \\
= \int_0^T \int_0^A \int_0^{\tilde{\lambda}_1} \left( s\Theta k(Z_x)^2 + s^3\Theta^3 \frac{(x-x_0)^2}{k} Z^2 \right) e^{2s\varphi} dx da dt \\
\leq C \int_0^T \int_0^A \int_{-\rho_1}^{\rho_1} \left( s\Theta (Z_x)^2 + s^3\Theta^3 Z^2 \right) e^{2s\varphi} dx da dt \\
\leq C \int_0^T \int_0^A \int_{-\rho_1}^{\rho_1} \left( s\Theta (Z_x)^2 + s^3\Theta^3 Z^2 \right) e^{2s\varphi} dx da dt
\end{align*}
\]
\[
\begin{align*}
&\leq C \int_0^T \int_0^A \int_{-\rho_1}^{\rho_1} \tilde{h}^2 e^{2s\Phi} dx \, dt \\
&+ C \int_0^T \int_0^A \int_{-\rho_1}^{\rho_1} (W^2 + (W_x)^2) e^{2s\Phi} dx \, dt \\
&+ C \int_0^T \int_0^A \int_{-\rho_1}^{\rho_1} \tilde{f}^2 dx \, dt + C \int_0^T \int_0^A \int_{-\rho_1}^{\rho_1} W^2 dx \, dt \\
&\leq C \int_0^T \int_0^A \int_{-\rho_1}^{\rho_1} \tilde{f}^2 dx \, dt + C \int_0^T \int_0^A \int_{-\rho_1}^{\rho_1} W^2 dx \, dt \\
&\leq C \int_0^T \int_0^A \int_{-\rho_1}^{\rho_1} \tilde{f}^2 dx \, dt + C \int_0^T \int_0^A \int_{-\rho_1}^{\rho_1} W^2 dx \, dt \\
&\leq C \int_0^T \int_0^A \int_{-\rho_1}^{\rho_1} \tilde{f}^2 dx \, dt + C \int_0^T \int_0^A \int_{-\rho_1}^{\rho_1} W^2 dx \, dt
\end{align*}
\]  
(by \cite{14} Lemma 4.1) and since \( \tilde{f}(t,a,x) = -f(t,a,-x) \), for \( x < 0 \)

for some strictly positive constants \( C \) and \( s \) large enough. Here \( \Phi \) is related to \((-\rho_1, \rho_1)\).

Hence, by definitions of \( Z, W \) and \( \zeta \), and using the previous inequality one has

\[
\begin{align*}
&\int_0^T \int_0^A \int_0^{\tilde{\lambda}_1} \left( s\Theta k(v_x)^2 + s^3 \Theta^3 \frac{(x-x_0)^2}{k} v^2 \right) e^{2s\varphi} dx \, dt \\
&= \int_0^T \int_0^A \int_0^{\tilde{\lambda}_1} \left( s\Theta k(W_x)^2 + s^3 \Theta^3 \frac{(x-x_0)^2}{k} W^2 \right) e^{2s\varphi} dx \, dt \\
&\leq C \left( \int_Q f^2 dx \, dt + \int_0^T \int_0^A \int_{\omega} v^2 dx \, dt \right).
\end{align*}
\]

Moreover, by (4.19) and (4.21), the conclusion follows.

Nothing changes in the proof if \( \omega = \omega_1 \cup \omega_2 \) and each of these intervals lie on different sides of \( x_0 \), as the assumption implies. \( \square \)

**Remark 4.6.** Observe that the results of Theorem 4.5 still holds true if we substitute the domain \((0,T) \times (0,A)\) with a general domain \((T_1, T_2) \times (\delta, A)\), provided that \( \mu \) and \( \beta \) satisfy the required assumptions. In this case, in place of the function \( \Theta \) defined in (4.16), we have to consider the weight function

\[
\hat{\Theta}(t,a) := \frac{1}{(t-T_1)^4(T_2-t)^4(a-\delta)^4}.
\]

Using the previous local Carleman estimates one can prove the next observability inequalities.

**Theorem 4.7.** Assume Hypotheses \(4.1\) with \( \bar{a} < T \leq A \), and \(4.2\). Then, for every \( \delta \in (0, A) \), there exists a strictly positive constant \( C = C(\delta) \) such
that every solution \( v \) of (4.1) in \( \mathcal{V} \) satisfies
\[
\int_0^A \int_0^1 v^2(T - \bar{a}, a, x) \, dx \, da \leq C \int_0^T \int_0^\delta \int_0^1 v^2(t, a, x) \, dx \, da \, dt + C \left( \int_0^T \int_0^1 v_T^2(a, x) \, dx \, da + \int_0^T \int_0^A \int_\omega v^2 \, dx \, da \, dt \right).
\]

Moreover, if \( v_T(a, x) = 0 \) for all \( (a, x) \in (0, T) \times (0, 1) \), one has
\[
\int_0^A \int_0^1 v^2(T - \bar{a}, a, x) \, dx \, da \leq C \int_0^T \int_0^\delta \int_0^1 v^2(t, a, x) \, dx \, da \, dt + C \int_0^T \int_0^A \int_\omega v^2 \, dx \, da \, dt.
\]

Observe that in [9] Theorem 4.4], which is the analogue of Theorem 4.7 in the non divergence case, there is a mistake in the statement. Indeed, we assumed \( \frac{k'}{\sqrt{k}} \in L^\infty_{\text{loc}}([0, 1] \setminus \{x_0\}) \), which was a consequence of (4.24) below (see the remark after (46) in [9]); the precise assumption is:

there exist two functions \( g \in L^\infty_{\text{loc}}([-\rho_1, 1] \setminus \{x_0\}) \), \( h \in W^{1,\infty}_{\text{loc}}([-\rho_1, 1] \setminus \{x_0\}, L^\infty(0, 1)) \) and two strictly positive constants \( g_0, h_0 \) such that \( g(x) \geq g_0 \) and
\[
\frac{k'(x)}{2\sqrt{k(x)}} \left( \int_x^B g(t) \, dt + h_0 \right) + \sqrt{k(x)} g(x) = h(x, B)
\]
for a.e. \( x \in [-\rho_1, 1], B \in [0, 1] \) with \( x < B < x_0 \) or \( x_0 < x < B \), where \( \bar{k} \) is defined in (4.11). Indeed, in order to prove [9] Theorem 4.4], we use [9] Theorem 4.3] which holds under (4.24). On the other hand, the statement of [9] Corollary 4.1], which is also a consequence of [9] Theorem 4.4], is correct.

**Proof of Theorem 4.7.** The proof follows the one of [8] Theorem 4.4], but we repeat here in a briefly way for the reader’s convenience underlining the differences since in [8] Theorem 4.4] \( k \) degenerates at the boundary of the domain, while here it degenerates in the interior.

As in [9], using the method of characteristic lines, one can prove the following implicit formula for \( v \) solution of (4.1):
\[
S(T - t)v_T(T + a - t, \cdot),
\]
if \( t \geq \bar{T} + a \) and
\[
v(t, a, \cdot) = \begin{cases} S(T - t)v_T(T + a - t, \cdot) + \int_{a}^{T+a-t} S(s-a)\beta(s,\cdot) v(s + t - a, 0, \cdot) \, ds, & \Gamma = \bar{a} \\ \int_{a}^{T} S(s-a)\beta(s,\cdot) v(s + t - a, 0, \cdot) \, ds, & \Gamma = \Gamma_{A,T} \end{cases},
\]
otherwise. Here \( (S(t))_{t \geq 0} \) is the semigroup generated by the operator \( A_0 - \mu Id \) for all \( u \in D(A_0) \) (\( Id \) is the identity operator), \( \Gamma_{A,T} := A - a + t - \bar{T} \).
and
\[(4.27) \quad \Gamma := \min\{\bar{a}, \Gamma_{A,T}\}.
\]
In particular, it results
\[(4.28) \quad v(t, 0, \cdot) := S(T - t)v_T(T - t, \cdot).
\]
Proceeding as in [8, Theorem 4.4], with suitable changes, one has that there exists a positive constant $C$ such that:
\[(4.29) \quad \int_{QA,1} v^2(\tilde{T}, a, x) dx \, da \leq C \int_{T}^{3T} \int_{QA,1} v^2(t, a, x) dx \, da \, dt.
\]
Take $\delta \in (0, A)$. By the previous inequality, we have
\[(4.30) \quad \int_{QA,1} v^2(\tilde{T}, a, x) dx \, da \leq C \int_{T}^{3T} \int_{\delta}^{A} \int_{0}^{1} v^2(t, a, x) dx \, da \, dt.
\]
Now, we will estimate the term $\int_{T}^{3T} \int_{\delta}^{A} \int_{0}^{1} v^2(t, a, x) dx \, da \, dt$. It results that
\[(4.31) \quad \int_{0}^{1} v^2 dx \leq C \left( \int_{0}^{1} \frac{k v^2 x}{k} dx + \int_{0}^{1} \frac{(x - x_0)^2}{k} v^2 \, dx \right),
\]
for a strictly positive constant $C$. Indeed, using the Young’s inequality to the function $v$, we obtain
\[(4.32) \quad \int_{0}^{1} |v|^2 \, dx \leq C \int_{0}^{1} \left( \frac{k^{1/3}}{(x - x_0)^{2/3}} v^2 \right)^{3/4} \left( \frac{(x - x_0)^2}{k} v^2 \right)^{1/4} \, dx
\]
\[\leq C \int_{0}^{1} \frac{k^{1/3}}{(x - x_0)^{2/3}} v^2 \, dx + C \int_{0}^{1} \frac{(x - x_0)^2}{k} v^2 \, dx.
\]
Now, consider the term
\[\int_{0}^{1} \frac{k^{1/3}}{(x - x_0)^{2/3}} v^2 \, dx.
\]
If $M > \frac{4}{3}$, take the function $\gamma(x) = (k(x)|x - x_0|^4)^{1/3}$. Clearly, $\gamma(x) = k(x) \left( \frac{(x - x_0)^2}{k(x)} \right)^{2/3} \leq Ck(x)$ and $\frac{k^{1/3}}{(x - x_0)^{2/3}} = \frac{\gamma(x)}{(x - x_0)^2}$. Moreover, using Hypothesis 3.1, one has that the function $\frac{\gamma(x)}{|x - x_0|^q} = \left( \frac{k(x)}{|x - x_0|^q} \right)^{\frac{1}{3}}$, where $q := \frac{4 + \vartheta}{3} \in (1, 2)$, is non increasing on the left of $x = x_0$ and non decreasing on the right of $x = x_0$. Hence, by the Hardy-Poincaré inequality given in [10] Proposition 2.6]
\[\int_{0}^{1} \frac{k^{1/3}}{(x - x_0)^{2/3}} v^2 \, dx = \int_{0}^{1} \frac{\gamma(x)}{(x - x_0)^2} v^2 \, dx \leq C \int_{0}^{1} kv^2 \, dx.
\]
Thus, if $M > \frac{4}{3}$, by (4.32), (4.31) holds. Now, assume $M \leq \frac{4}{3}$ and introduce the function $p(x) = |x - x_0|^{4/3}$. Obviously, there exists $q \in \left(1, \frac{4}{3}\right)$ such that the function $x \mapsto \frac{p(x)}{|x - x_0|^q}$ is nonincreasing on the left of $x = x_0$ and nondecreasing on the right of $x = x_0$. Thus, applying again [10, Proposition 2.6], one has

\begin{align}
\int_0^1 \frac{k^{1/3}}{|x - x_0|^{2/3}} v^2 \, dx &\leq \max_{[0,1]} k^{1/3} \int_0^1 \frac{1}{|x - x_0|^{2/3}} v^2 \, dx \\
&= \max_{[0,1]} k^{1/3} \int_0^1 \frac{p}{(x-x_0)^2} v^2 \, dx \\
&\leq \max_{[0,1]} k^{1/3} C \int_0^1 p(v_x)^2 \, dx \\
&= \max_{[0,1]} k^{1/3} C \int_0^1 k \frac{|x - x_0|^{4/3}}{k} (v_x)^2 \, dx \\
&\leq \max_{[0,1]} k^{1/3} C \int_0^1 (v_x)^2 \, dx.
\end{align}

Hence, (4.31) still holds and

\begin{align}
\int_\frac{T}{4}^T \int_\delta^A \int_0^1 v^2(t,a,x) \, dx \, da \, dt &\leq C \int_\frac{T}{4}^T \int_\delta^A \int_0^1 \tilde{\Theta} v_x^2 e^{2s\varphi} \, dx \, da \, dt \\
&+ C \int_\frac{T}{4}^T \int_\delta^A \int_0^1 \tilde{\Theta}^3 \frac{(x-x_0)^2}{k} v^2 e^{2s\varphi} \, dx \, da \, dt.
\end{align}

The rest of the proof follows as in [8, Theorem 4.4], so we omit it.

\[\square\]

**Corollary 4.8.** Assume Hypotheses 4.1 with $\tilde{a} = T < A$, and 4.2. Then, for every $\delta \in (0, A)$, there exists a strictly positive constant $C = C(\delta)$ such that every solution $v$ of (4.1) in $V$ satisfies

\[
\int_0^A \int_0^1 v^2(0,a,x) \, dx \, da \leq C \int_0^T \int_0^\delta \int_0^1 v^2(t,a,x) \, dx \, da \, dt \\
+ C \left( \int_0^T \int_0^\delta \int_0^1 v_x^2(a,x) \, dx \, da + \int_0^T \int_0^A \int_\omega v^2 \, dx \, da \right).
\]

Moreover, if $v_T(a,x) = 0$ for all $(a,x) \in (0,T) \times (0,1)$, one has

\[
\int_0^A \int_0^1 v^2(0,a,x) \, dx \, da \leq C \left( \int_0^T \int_0^\delta \int_0^1 v^2(t,a,x) \, dx \, da \, dt + \int_0^T \int_0^A \int_\omega v^2 \, dx \, da \right).
\]
Proceeding as in Theorem 4.7, one can prove the analogous result in the case $T > A$. Indeed, with suitable changes, one can prove again (4.25), if $t \geq \tilde{T} + \bar{a}$, and (4.26), otherwise. In particular, we have again (4.28). Thus:

**Theorem 4.9.** Assume Hypotheses 4.1 with $\bar{a} < A < T$, and 4.2. Then, for every $\delta \in (0, A)$, there exists a strictly positive constant $C = C(\delta)$ such that every solution $v$ of (4.1) in $V$ satisfies (4.22). Moreover, if $v_T(a, x) = 0$ for all $(a, x) \in (0, T) \times (0, 1)$, one has (4.23).

Actually, proceeding as in [9] with suitable changes, we can improve the previous results in the following way:

**Theorem 4.10.** Assume Hypotheses 4.1 and 4.2. If $T < A$, then, for every $\delta \in (T, A)$, there exists a strictly positive constant $C = C(\delta)$ such that every solution $v$ of (4.1) in $V$ satisfies

$$
\int_0^A \int_0^1 v^2(T - \bar{a}, a, x) dxd\!a \leq C \left( \int_0^\delta \int_0^1 v^2_t(a, x) dxd\!a + \int_T^T \int_0^A \int_\omega v^2 dxd\!adt \right).$

If $A < T$, then, for every $\delta \in (\bar{a}, A)$, there exists a strictly positive constant $C = C(\delta)$ such that every solution $v$ of (4.1) in $V$ satisfies

$$
\int_0^T v^2_T(a, x) dxd\!a \leq \int_0^{\delta - \bar{a}} \int_0^1 v^2(t, a, x) dxd\!a + \int_0^\delta \int_0^A \int_\omega v^2 dxd\!adt.$$

**Proof.** If $T < A$ the proof of the previous theorem is analogous to the one of [9] Theorem 4.6, with suitable changes, so we omit it.

Now, consider the case $A < T$ and fix $\delta \in (\bar{a}, A)$. We distinguish between the two cases $\delta < 2\bar{a}$ and $\delta \geq 2\bar{a}$.

First of all, consider $\delta < 2\bar{a}$: as in [8] Theorem 4.4, we can prove

$$
\int_{Q_{\bar{a}, 1}} v^2(T - \bar{a}, a, x) dxd\!a \leq \int_{Q_{\bar{a}, 1}} v^2(t, a, x) dxd\!a. \tag{4.36}
$$

Then, integrating over $[T - \bar{a}, T + \delta - 2\bar{a}]$, we have the following inequality:

$$
\int_{Q_{\bar{a}, 1}} v^2(T - \bar{a}, a, x) dxd\!a \leq C \int_{T - \bar{a}}^{T + \delta - 2\bar{a}} \left( \int_{\delta - \bar{a}}^\delta + \int_{\delta - \bar{a}}^A \right) \int_0^1 v^2(t, a, x) dxd\!adt. \tag{4.37}
$$

Using Theorem 4.5 we can prove

$$
\int_{T - \bar{a}}^{T + \delta - 2\bar{a}} \int_{\delta - \bar{a}}^A \int_0^1 v^2(t, a, x) dxd\!adt \leq C \int_0^\delta \int_0^1 v^2_t(a, x) dxd\!a + C \int_0^T \int_0^A \int_\omega v^2 dxd\!adt. \tag{4.38}
$$
Indeed, by (4.34) applied to $[T - \bar{a}, +\delta - 2\bar{a}]$ and Theorem 4.5, we have

\[
\int_{T - \bar{a}}^{T + \delta - 2\bar{a}} \int_{0}^{A} \int_{0}^{1} v^2(t, a, x) dx dt \leq C \int_{T - \bar{a}}^{T + \delta - 2\bar{a}} \int_{0}^{A} \int_{0}^{1} \tilde{\Theta}v^2 e^{2s\varphi} dx dt + C \int_{T - \bar{a}}^{T + \delta - 2\bar{a}} \int_{0}^{A} \int_{0}^{1} f^2 dx dt
\]

where, in this case, $f(t, a, x) := -\beta(a, x)v(t, 0, x)$. Hence,

\[
\int_{T - \bar{a}}^{T + \delta - 2\bar{a}} \int_{\delta}^{A} \int_{0}^{1} v^2(t, a, x) dx dt \leq C \int_{T - \bar{a}}^{T + \delta - 2\bar{a}} \int_{0}^{A} \int_{0}^{1} v^2(t, 0, x) dx dt + C \int_{T - \bar{a}}^{T + \delta - 2\bar{a}} \int_{0}^{A} \int_{0}^{1} \tilde{\Theta}v^2 x e^{2s\varphi} dx dt
\]

\[
\leq C \left( \int_{T - \bar{a}}^{T + \delta - 2\bar{a}} \int_{0}^{A} \int_{0}^{1} f^2 dx dt + \int_{T - \bar{a}}^{T + \delta - 2\bar{a}} \int_{0}^{A} \int_{0}^{1} \tilde{\Theta}v^2 dx dt \right)
\]

\[
= C \left( \int_{T - \bar{a}}^{T + \delta - 2\bar{a}} \int_{0}^{A} \int_{0}^{1} f^2(T - t, x) dx dt + \int_{T - \bar{a}}^{T + \delta - 2\bar{a}} \int_{0}^{A} \int_{0}^{1} \tilde{\Theta}v^2 dx dt \right)
\]

\[
\leq C \left( \int_{0}^{\delta - 2\bar{a}} \int_{0}^{A} \int_{0}^{1} f^2(a, x) dx da + \int_{0}^{\delta - 2\bar{a}} \int_{0}^{A} \int_{0}^{1} \tilde{\Theta}v^2 dx da \right)
\]

Hence (4.38) follows.

It remains to estimate the following integral:

\[
\int_{T - \bar{a}}^{T + \delta - 2\bar{a}} \int_{0}^{\delta - \bar{a}} \int_{0}^{1} v^2(t, a, x) dx dt.
\]

Observe that, since $t \leq T + \delta - 2\bar{a}$, $t - T + \bar{a} < \delta - \bar{a}$, hence

(4.39)

\[
\int_{T - \bar{a}}^{T + \delta - 2\bar{a}} \int_{0}^{\delta - \bar{a}} \int_{0}^{1} v^2(t, a, x) dx dt = \int_{T - \bar{a}}^{T + \delta - 2\bar{a}} \int_{0}^{\delta - \bar{a}} \int_{0}^{1} v^2(t, a, x) dx dt
\]

\[
+ \int_{T - \bar{a}}^{T + \delta - 2\bar{a}} \int_{t - T + \bar{a}}^{\delta - \bar{a}} \int_{0}^{1} v^2(t, a, x) dx dt.
\]
Now, by (4.25) and by the boundedness of \((S(t))_{t \geq 0}\),

\[
\int_{T - \bar{a}}^{T + \delta - 2\bar{a}} \int_{0}^{t - T + \bar{a}} \int_{0}^{1} v^2(t, a, x) dxdadt \\
= \int_{T - \bar{a}}^{T + \delta - 2\bar{a}} \int_{0}^{t - T + \bar{a}} \int_{0}^{1} (S(T - t)v_T(T + a - t, x))^2 dxdadt \\
\leq C \int_{T - \bar{a}}^{T + \delta - 2\bar{a}} \int_{0}^{t - T + \bar{a}} \int_{0}^{1} v_T^2(T + a - t, x) dxdadt \\
(4.40) \\
= C \int_{-\delta + 2\bar{a}}^{-\bar{a}} \int_{0}^{\bar{a} - z} \int_{0}^{1} v_T^2(z + a, x) dxdadz \\
= C \int_{-\delta + 2\bar{a}}^{-\bar{a}} \int_{z}^{\bar{a}} \int_{0}^{1} v_T^2(\sigma, x) dxd\sigma dz \\
\leq C \int_{-\delta + 2\bar{a}}^{-\bar{a}} \int_{0}^{\bar{a} - z} \int_{0}^{1} v_T^2(\sigma, x) dxd\sigma dz \\
\leq C \int_{0}^{\bar{a}} \int_{0}^{1} v_T^2(\sigma, x) dxd\sigma 
\]

On the other hand, if \(t \geq T - \bar{a}\) and \(a \in (t - T + \bar{a}, \delta - \bar{a})\), it results that \(T - t < A - a\), thus \(\Gamma = \bar{a}\) (to this purpose recall that \(\delta \in (\bar{a}, A)\) and \(\Gamma\) is defined in (4.27)). Hence in (4.26) we have to consider the first formula, i.e.

\[
v(t, a, \cdot) = S(T - t)v_T(T + a - t, \cdot) + \int_{a}^{T + a - t} S(s - a)\beta(s, \cdot)v(s + t - a, 0, \cdot)ds. 
\]

It follows that, proceeding as in (4.40),

\[
\int_{T - \bar{a}}^{T + \delta - 2\bar{a}} \int_{0}^{t - T + \bar{a}} \int_{0}^{1} v^2(t, a, x) dxdadt \\
\leq C \int_{T - \bar{a}}^{T + \delta - 2\bar{a}} \int_{0}^{t - T + \bar{a}} \int_{0}^{1} v_T^2(T + a - t, x) dxdadt \\
+ C \int_{T - \bar{a}}^{T + \delta - 2\bar{a}} \int_{t - T + \bar{a}}^{0} \int_{0}^{1} \left( \int_{a}^{T + a - t} v^2(s + t - a, 0, x) ds \right) dxdadt \\
= C \int_{-\delta + 2\bar{a}}^{-\bar{a}} \int_{0}^{\bar{a} - z} \int_{0}^{1} v_T^2(z + a, x) dxdadz \\
+ C \int_{T - \bar{a}}^{T + \delta - 2\bar{a}} \int_{0}^{t - T + \bar{a}} \int_{0}^{1} \left( \int_{a}^{T + a - t} v_T^2(T - s - t + a, x) ds \right) dxdadt \\
= C \int_{-\delta + 2\bar{a}}^{-\bar{a}} \int_{z}^{\bar{a}} \int_{0}^{1} v_T^2(\sigma, x) dxd\sigma dz \\
+ C \int_{T - \bar{a}}^{T + \delta - 2\bar{a}} \int_{0}^{t - T + \bar{a}} \int_{0}^{1} \left( \int_{a}^{T - a - t} v_T^2(a + z, x) dz \right) dxdadt 
\]
Hence, by (4.37) - (4.41), (4.35) follows.

Using the fact that in the first integral \( z \in (-\delta + 2\bar{a}, \bar{a}) \) and in the second one \( t \geq T - \bar{a} \), one has \( z + \delta - \bar{a} \leq \delta \) and \( T - t \leq \bar{a} \), respectively, this implies (4.41)

\[
\int_{T-\bar{a}}^{T+\delta-2\bar{a}} \int_{t+\bar{a}}^{t+\delta} \left( \int_{0}^{1} v_2^2(t, a, x) dx \right) dt \leq C \int_{T-\bar{a}}^{T+\delta} \int_{t}^{t+\delta} \int_{0}^{1} v_2^2(t, a, x) dx \sigma \left[ T - \bar{a}, T + \delta - 2\bar{a} \right]
\]

\[
+ C \int_{T-\bar{a}}^{T+\delta-2\bar{a}} \int_{t+\bar{a}}^{t+\delta} \left( \int_{0}^{1} v_2^2(t, a, x) dx \right) dt \leq C \int_{0}^{1} v_2^2(t, a, x) dx \sigma.
\]

Hence, by (4.37) - (4.41), (4.35) follows.

Now, consider the case \( \delta \geq 2\bar{a} \) and, in place of \([T - \bar{a}, T + \delta - 2\bar{a}]\), take the interval \([T - \bar{a}, T - \frac{\bar{a}}{4}]\). Hence we have

\[
\int_{Q_{A,1}} v^2(T - \bar{a}, a, x) dx \leq C \int_{T-\bar{a}}^{T+\frac{\bar{a}}{4}} \left( \int_{0}^{1} v_2^2(t, a, x) dx \right) dt.
\]

Proceeding as before, we can prove the analogous of (4.38), i.e.

\[
\int_{T-\bar{a}}^{T+\frac{\bar{a}}{4}} \int_{t}^{t+\frac{\bar{a}}{4}} \left( \int_{0}^{1} v_2^2(t, a, x) dx \right) dt \leq C \left( \int_{0}^{1} v_2^2(a, x) dx + \int_{0}^{T} \int_{0}^{\omega} v^2 dx \right).
\]

It remains to estimate

\[
\int_{T-\bar{a}}^{T+\frac{\bar{a}}{4}} \int_{0}^{1} v^2(t, a, x) dx dt.
\]

Also in this case, since \( t \in [T - \bar{a}, T - \frac{\bar{a}}{4}] \), it follows that \( t - T + \bar{a} \leq \delta - \bar{a} \) (recall that we are in the case \( \delta \geq 2\bar{a} \)). Proceeding as before, one has

\[
\int_{T-\bar{a}}^{T+\frac{\bar{a}}{4}} \int_{0}^{1} v^2(t, a, x) dx dt = \int_{T-\bar{a}}^{T-\frac{\bar{a}}{4}} \int_{0}^{T+\bar{a}} \int_{0}^{1} v^2(t, a, x) dx dt + \int_{T-\bar{a}}^{T+\frac{\bar{a}}{4}} \int_{t+\bar{a}}^{T+\bar{a}} \int_{0}^{1} v^2(t, a, x) dx dt.
\]
As for (4.40) and (4.41), we have:

\[(4.44) \int_{T-\bar{a}}^{T-\bar{\alpha}} \int_{t-T+\bar{a}}^{t-T+\bar{\alpha}} \int_{0}^{1} v^2(t, a, x) dx dt \leq C \int_{T-\bar{a}}^{T-\bar{\alpha}} \int_{t-T+\bar{a}}^{t-T+\bar{\alpha}} \int_{0}^{1} v^2_T(T + a - t, x) dx dt \]

\[= C \int_{T-\bar{a}}^{\bar{a}} \int_{0}^{z-\bar{a}} \int_{0}^{1} v^2_T(z + a, x) dx dz \]

\[\leq C \int_{T-\bar{a}}^{\bar{a}} \int_{0}^{\bar{a}} \int_{0}^{1} v^2_T(\sigma, x) dx d\sigma dz \]

and

\[(4.45) \int_{T-\bar{a}}^{T-\bar{\alpha}} \int_{1-T+\bar{a}}^{1-T+\bar{\alpha}} \int_{0}^{1} v^2(t, a, x) dx dt \leq C \int_{T-\bar{a}}^{T-\bar{\alpha}} \int_{1-T+\bar{a}}^{1-T+\bar{\alpha}} \int_{0}^{1} v^2_T(T + a - t, x) dx dt \]

\[+ C \int_{T-\bar{a}}^{T-\bar{\alpha}} \int_{1-T+\bar{a}}^{1-T+\bar{\alpha}} \int_{0}^{1} \left( \int_{0}^{T+a-t} v^2(s + t - a, 0, x) ds \right) dx dt \]

\[= C \int_{\bar{a}}^{\bar{a}} \int_{\bar{a}}^{\bar{a}+\bar{\alpha}} \int_{0}^{1} v^2_T(\sigma, x) dx d\sigma dz \]

\[+ C \int_{T-\bar{a}}^{T-\bar{\alpha}} \int_{1-T+\bar{a}}^{1-T+\bar{\alpha}} \int_{0}^{1} \left( \int_{-a}^{T-a} v^2_T(a + z, x) dz \right) dx dt \]

\[\leq C \int_{\bar{a}}^{\bar{a}} \int_{0}^{1} v^2_T(\sigma, x) dx d\sigma dz \]

\[+ C \int_{T-\bar{a}}^{T-\bar{\alpha}} \int_{1-T+\bar{a}}^{1-T+\bar{\alpha}} \int_{0}^{1} \left( \int_{0}^{T-t} v^2_T(\sigma, x) d\sigma \right) dx dt \]

\[\leq C \int_{0}^{\delta} \int_{0}^{1} v^2_T(\sigma, x) dx d\sigma + C \int_{T-\bar{a}}^{\bar{a}} \int_{1-T+\bar{a}}^{1-T+\bar{\alpha}} \int_{0}^{1} \left( \int_{0}^{\bar{a}} v^2_T(\sigma, x) d\sigma \right) dx dt \]

\[\leq C \int_{0}^{\delta} \int_{0}^{1} v^2_T(\sigma, x) dx d\sigma \]

By (4.42)-(4.45), (4.35) follows.

By Theorem 4.10 and using a density argument, one can deduce the following observability result:

**Proposition 4.11.** Assume Hypotheses 4.11 and 4.12. If $T < A$, then, for every $\delta \in (T, A)$, there exists a strictly positive constant $C = C(\delta)$ such that
every solution $v \in U$ of (4.1) satisfies
\[
\int_0^A \int_0^1 v^2(T-\bar{a},a,x)dxda \leq C \left( \int_0^\delta \int_0^1 v_T^2(a,x)dxda + \int_0^T \int_0^A \int_\omega v^2 dx dadt \right).
\]
If $A < T$, then, for every $\delta \in (\bar{a},A)$, there exists a strictly positive constant $C = C(\delta)$ such that every solution $v$ of (4.1) satisfies (4.46).

Here $v_T(a,x)$ is such that $v_T(A,x) = 0$ in $(0,1)$.

Observe that in the statements of the analogous results given in [9] for the non divergence case there is a misprint. Indeed the constant $C$ depends on $\delta$, as one can deduce by the proofs. The right statement is

...for every $\delta \in (T,A)$, there exists $C = C(\delta)$ such that...

We underline that the results are correct and in the correct way they are used to prove [9] Theorems 4.7 and 4.8.

As a consequence of Proposition 4.11 one can prove, as in [8] Theorem 4.7, the following null controllability result:

**Theorem 4.12.** Assume Hypotheses 4.1 and 4.2 and take $y_0 \in L^2(Q_{A,1})$. Then for every $\delta \in (T,A)$, if $T < A$, or for every $\delta \in (\bar{a},A)$, if $A < T$, there exists a control $f_\delta \in L^2(Q)$ such that the solution $y_\delta$ of (1.1) satisfies
\[
y_\delta(T,a,x) = 0 \quad \text{a.e.} \quad (a,x) \in (\delta,A) \times (0,1).
\]
Moreover, there exists $C = C(\delta) > 0$

(4.48) \[ \|f_\delta\|_{L^2(Q)} \leq C\|y_0\|_{L^2(Q_{A,1})}. \]

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**References**

[1] B. Ainseba, Y. Echarroudi, L. Maniar Null controllability of population dynamics with degenerate diffusion, Differential Integral Equations (2013), 1397–1410.

[2] F. Alabau-Boussouira, P. Cannarsa, G Fragnelli, Carleman estimates for degenerate parabolic operators with applications to null controllability, J. Evol. Equ. 6 (2006), 161–204.

[3] S. Anituta, Analysis and control of age-dependent population dynamics, Mathematical Modelling: Theory and Applications 11 (2000), Kluwer Academic Publishers, Dordrecht.

[4] V. Barbu, M. Iannelli, M. Martcheva, On the controllability of the Lotka-McKendrick model of population dynamics, J. Math. Anal. Appl. 253 (2001), 142-165.

[5] I. Boutaayamou, Y. Echarroudi, Null controllability of a population dynamics with interior degeneracy, accepted in Journal of Mathematics and Statistical Science.

[6] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, Springer Science+Business Media, LLC 2011.
[7] Y. Echarroudi, L. Maniar, *Null controllability of a model in population dynamics*, Electron. J. Differential Equations 2014 (2014), 1–20.

[8] G. Fragnelli, *Null controllability for a degenerate population model in divergence form via Carleman estimates*, submitted.

[9] G. Fragnelli, *Carleman estimates and null controllability for a degenerate population model*, Journal de Mathématiques Pures et Appliquées, 115 (2018), 74-126.

[10] G. Fragnelli, D. Mugnai, *Carleman estimates and observability inequalities for parabolic equations with interior degeneracy*, Advances in Nonlinear Analysis 2 (2013), 339–378.

[11] G. Fragnelli, D. Mugnai, *Carleman estimates, observability inequalities and null controllability for interior degenerate non smooth parabolic equations*, Mem. Amer. Math. Soc., 242 (2016), v+84 pp. Corrigendum, to appear.

[12] G. Fragnelli, D. Mugnai, *Carleman estimates for singular parabolic equations with interior degeneracy and non smooth coefficients*, Adv. Nonlinear Anal., 6 (2017), 61–84.

[13] G. Fragnelli, D. Mugnai, *Controllability of strongly degenerate parabolic problems with strongly singular potentials*, Electron. J. Qual. Theory Differ. Equ., 50 (2018), 1–11.

[14] G. Fragnelli, D. Mugnai, *Controllability of degenerate and singular parabolic problems: the double strong case with Neumann boundary conditions*, Opuscula Math. 39 (2019), 207-225.

[15] Y. He, B. Ainseba, *Exact null controllability of the Lobesia botrana model with diffusion*, J. Math. Anal. Appl. 409 (2014), 530–543.

[16] D. Maity, M. Tucsnak and E. Zuazua, *Controllability and positivity constraints in population dynamics with age structuring and diffusion*, Journal de Mathématiques Pures et Appliquées. In press, 10.1016/j.matpur.2018.12.006.

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