Dynamical and thermodynamical stability of isothermal distributions in the HMF model

P.H. Chavanis

1 Laboratoire de Physique Théorique (IRSAMC), CNRS and UPS, Université de Toulouse, F-31062 Toulouse, France

E-mail: chavanis@irsamc.ups-tlse.fr

To be included later

Abstract. We provide a new derivation of the conditions of dynamical and thermodynamical stability of homogeneous and inhomogeneous isothermal distributions in the Hamiltonian Mean Field (HMF) model. This proof completes the original thermodynamical approach of Inagaki [Prog. Theor. Phys. 90, 557 (1993)]. Our formalism, based on variational principles, is simple and the method can be applied to more general situations. For example, it can be used to settle the dynamical stability of polytropic distributions with respect to the Vlasov equation [Chavanis & Campa, arXiv:1001.2109]. For isothermal distributions, the calculations can be performed fully analytically, providing therefore a clear illustration of the method.

PACS. 0 5.20.-y Classical statistical mechanics - 05.45.-a Nonlinear dynamics and chaos - 05.20.Dd Kinetic theory - 64.60.De Statistical mechanics of model systems

1 Introduction

The statistical mechanics of systems with long-range interactions has recently been the object of an intense activity [1234]. A system with long-range interactions is characterized by a binary potential $u(r)$ which decreases at large distances $r$ slower than $r^{-d}$ where $d$ is the dimension of space. Systems with long-range interactions are numerous in nature and include for example self-gravitating systems, two-dimensional vortices, bacterial populations experiencing chemotaxis, neutral and non neutral plasmas, wave-particle systems, free-electron lasers,... These systems exhibit a very interesting dynamics and thermodynamics.

On a dynamical point of view, they display robust and long-lived quasi-stationary states (QSSs) that are non-Boltzmannian. These distributions are stable steady states of the Vlasov equation on the coarse-grained scale resulting from a process of violent relaxation [5]. On a thermodynamical point of view, their Boltzmannian statistical equilibrium states can display numerous types of phase transitions due to ensembles inequivalence [67]. Ensembles inequivalence is generic for systems with long-range interactions, as first evidenced in astrophysics (see reviews in [8910]), but not compulsory.

A simple toy model of systems with long-range interactions, called the Hamiltonian Mean Field (HMF) model, has received a particular attention. This model consists of $N$ particles of unit mass moving on a ring and interacting via a cosine potential $u = N^{-1} \cos(\theta_i - \theta_j)$ where $\theta_i$ denotes the angle that makes particle $i$ with an axis of reference. This model was first introduced by Messer & Spohn [11] and called the cosine model. It was reintroduced independently ten years later by different groups of researchers [1213141516] and was very much studied since then (see a short history in [17]). In particular, the seminal paper of Antoni & Ruffo [16] inspired many works on the subject.

The HMF model exhibits two successive types of relaxation. For short timescales, the distribution function is governed by the Vlasov equation that ignores correlations (or “collisions”) between particles. In this regime, the system can experience a collisionless violent relaxation towards a steady state of the Vlasov equation on the coarse-grained scale. In principle, this quasistationary state (QSS) can be predicted by using Lynden-Bell’s statistical theory of violent relaxation [1819202122]. However, this prediction may fail because collisionless relaxation can be incomplete [222324]. Since the Vlasov equation admits an infinite number of steady solutions, the prediction of the QSS actually reached by the system in case of incomplete relaxation is very difficult, or even impossible. Nevertheless, it can be useful to derive general stability criteria in order to determine which distributions are stable or unstable with respect to the Vlasov equation. This is the problem of Vlasov dynamical stability [13151617232526272829]. Of course, only stable states are relevant to characterize QSSs in the context of violent relaxation. On a longer timescale, the system is expected to achieve a Boltzmannian statistical equilibrium state due to the development of correlations between particles (finite $N$ effects or graininess). This statistical equilibrium state corresponds to the distribution that maximizes the Boltzmann entropy at fixed mass and energy. Only global or local entropy maxima are relevant...
(minima or saddle points must be discarded). This is the problem of dynamical and thermodynamical stability. Messer & Spohn \cite{11} considered a potential energy of the form $E = N^{-1} \sum_{i<j} V(x_i, x_j)$ and proved rigorously that the mean field approximation is exact for $N \to +\infty$ and that the statistical equilibrium state in the canonical ensemble corresponds to the global minimum of free energy $F[f]$ at fixed mass $M$. Considering specifically the cosine model, they showed that it displays a second order phase transition from a homogeneous phase to a clustered phase below a critical temperature $T_c$. Inagaki \cite{14} studied the thermodynamical stability of the modified Konishi-Kaneko \cite{12} system in the microcanonical ensemble by considering the maximization of entropy $S[f]$ at fixed mass $M$ and energy $E$. By using the theory of Poincaré on linear series of equilibria (see, e.g., \cite{10,13,31} for details), or by studying the sign of the second order variations of entropy (for the homogeneous phase), he showed that the system displays a microcanonical second order phase transition at a critical energy $E_c$, corresponding to the critical temperature $T_c$. Similar results were obtained by Pichon \cite{15} who showed in addition that this type of phase transitions could explain the formation of bars in disk galaxies. Antoni & Ruffo \cite{16} studied the statistical equilibrium state of the HMF model in the canonical ensemble directly from the partition function. They simplified it by using the Hubbard-Stratonovich transformation and the saddle point approximation valid for $N \to +\infty$. They evidenced a second order phase transition at $T = T_c$ in the canonical ensemble and performed numerical simulations in the microcanonical ensemble. These simulations show a discrepancy with the theoretical results close to the transition energy $E_c$, but this discrepancy is not due to ensembles inequivalence but to nonequilibrium effects \cite{16,24,32,33}. More recently, Barré et al. \cite{30} studied the statistical mechanics of the HMF model by applying large deviation technics. They confirmed the existence of microcanonical and canonical second order phase transitions and the equivalence of statistical ensembles. Finally, Chavanis et al. \cite{17} pursued the thermodynamical approach of Inagaki \cite{14} based on variational principles. In particular, they reduced the stability problem to the study of an eigenvalue equation and solved this eigenvalue equation numerically for any energy and analytically close to the critical point $(E_c, T_c)$. They proved by this method that the statistical ensembles are equivalent and that the homogeneous states are stable for $E > E_c$ (or $T > T_c$) and unstable for $E < E_c$ (or $T < T_c$). Below that critical energy or critical temperature, they are replaced by inhomogeneous states that are always stable. This method has the advantage of showing which type of perturbation

is able to trigger the instability of the homogeneous phase below $E_c$ or $T_c$.

Let us now review the results concerning the dynamical stability of steady states of the Vlasov equation in the context of the HMF model. We first consider the linear dynamical stability problem. Inagaki & Konishi \cite{13} and Pichon \cite{15} studied the linear dynamical stability of the isothermal (Maxwell) distribution with respect to the Vlasov equation. They considered the homogeneous phase and derived the dispersion relation using the methods of plasma physics and stellar dynamics. They showed that the system becomes dynamically unstable below the critical temperature $T_c$ or the critical energy $E_c$ (the same as the ones arising in the thermodynamical approach) leading to an instability similar to the Jeans instability in self-gravitating systems. They showed that only the modes $n = \pm 1$ grow, leading to the formation of a single cluster. Inagaki & Konishi \cite{14} also compared their theoretical results with direct numerical simulations, finding a good agreement. Antoni & Ruffo \cite{16} studied the linear dynamical stability of the spatially homogeneous waterbag distribution and determined the critical temperature $T_c$ and the critical energy $E_c$ marking the separation between stable and unstable states. More recently, Choi & Choi \cite{26}, Chavanis et al. \cite{17} and Jain et al. \cite{27} completed these studies and obtained explicit expressions for the growth rate and pulsation period of isothermal, polytropic and waterbag distributions. On the other hand, Chavanis & Delfini \cite{28} performed an exhaustive study of the linear dynamical stability of the HMF model by using the Nyquist method. They considered various types of distributions (single and double humped) and derived general stability criteria and stability diagrams for both attractive (ferromagnetic) and repulsive (antiferromagnetic) interactions.

Let us finally review the results concerning the formal nonlinear dynamical stability of a steady state of the Vlasov equation. A distribution function is said to be formally stable \cite{33} if it is a local minimum or a local maximum of an energy-Casimir functional (i.e. the second variations of the energy-Casimir functional are positive definite or negative definite). Formal stability implies linear stability, but the converse is wrong in general. Yamaguchi et al. \cite{24} derived a necessary and sufficient condition of formal nonlinear dynamical stability for spatially homogeneous distribution functions of the HMF model. They observed that linear stability and formal stability criteria coincide in that case. Chavanis et al. \cite{17} reconsidered the formal stability problem on a new angle (see also \cite{28,34} for more details) which can be extended to more general situations \cite{34}. They showed that the variational problem for the distribution function $f(\theta, v)$ is equivalent to a simpler variational problem for the density $\rho(\theta)$. This equivalence is valid for both homogeneous and inhomogeneous distributions. For spatially homogeneous distributions, they showed that the condition of formal stability can be written as a condition on the velocity of sound in the corresponding barotropic gas. This is equivalent to the criterion

\[ \frac{\partial^2 E}{\partial v^2} = 0 \]

where $E$ is the energy-Casimir functional and $v$ is the velocity. This method is related to the Antonov first law in astrophysics \cite{35,36}.
derived by Yamaguchi et al. but expressed in a different manner. For spatially inhomogeneous distributions, they reduced the formal stability problem to the study of an eigenvalue equation. Chavanis & Delfini showed that this eigenvalue equation can be solved analytically at the point of neutral stability, providing therefore an explicit condition to locate this point in the series of equilibria. They also generalized the preceding results to arbitrary potentials of interaction $u(r,r')$ and discussed more refined criteria of dynamical stability that are obtained by conserving a larger class of Vlasov invariants.

Recently, Campa & Chavanis derived a very general criterion of linear dynamical stability valid for homogeneous and inhomogeneous distributions. They also derived sufficient conditions of linear and formal dynamical stability that are weaker than the general criterion but more explicit. For spatially homogeneous distributions, they proved that the criteria of formal and linear stability coincide. As a by-product, their results also return the thermodynamical stability criteria obtained previously by different methods.

Let us finally mention that the Vlasov dynamical stability of polytropic distributions has been studied by Chavanis & Campa by plotting the series of equilibria and using the Poincaré criterion. On the other hand, the Vlasov dynamical stability and the thermodynamical stability of the Lynden-Bell distributions have been studied by Antoniazzi et al. and Stanisca et al. by solving the variational problem numerically. These studies exhibit a rich phase diagram with several types of phase transitions showing the complexity of the stability problem in the general case.

In this paper, we shall present a new method to determine the dynamical and thermodynamical stability of isothermal distributions of the HMF model. The idea is to start from the fundamental variational problems and transform them into equivalent but simpler variational problems until a point at which they can be explicitly solved. This completes the thermodynamical approach of Inagaki by proving analytically the stability of the inhomogeneous phase which was not done in Inagaki's paper. An interest of our approach is its simplicity and generality. Indeed, it can be extended to solve Vlasov and Lynden-Bell stability problems. For example, it has been used recently to settle the Vlasov dynamical stability of polytropic distributions (see Sec. 8 of [24]). However, in that case, the calculations are less explicit than for isothermal distributions. It is therefore interesting to develop the calculations in detail in the case of isothermal distributions (where they are fully analytical) in order to clearly illustrate the method. Although we rederive known results in a different manner, we think that the present approach is interesting and potentially useful to tackle more general problems. In addition, our approach not only determines the strict equilibrium state (global maximum of entropy) but it can also say whether a critical point of entropy is metastable (local entropy maximum) or unstable (saddle point of entropy). Although isothermal distributions of the HMF model do not display metastable states, this information can be valuable in more general situations where our method can be applied.

2 Series of equilibria

The HMF model is a system of $N$ particles of unit mass $m = 1$ moving on a circle and interacting via a cosine potential. The dynamics of these particles is governed by the Hamilton equations:

$$
\dot{\theta}_i = \frac{\partial H}{\partial v_i}, \quad \dot{v}_i = -\frac{\partial H}{\partial \theta_i},
$$

where $\theta_i \in [-\pi, \pi]$ and $-\infty < v_i < +\infty$ denote the position (angle) and the velocity of particle $i$ and $k$ is the coupling constant (we assume here that $k > 0$). The thermodynamic limit corresponds to $N \to +\infty$ in such a way that the rescaled energy $\varepsilon = 8\pi E/kM^2$ remains of order unity. We can take $k \sim 1/N$ which is the Kac prescription. In that case, the energy is extensive, $E/N \sim 1$, but non-additive. For $N \to +\infty$, the mean field approximation is exact and the $N$-body distribution function is a product of $N$ one body distributions: $P_N(\theta_1, v_1, ..., \theta_N, v_N) = P_1(\theta_1, v_1, t)...P_1(\theta_N, v_N, t)$. Let us introduce the distribution function $f(\theta, v, t) = N P_1(\theta, v, t)$. For a fixed interval of time and $N \to +\infty$, the evolution of the distribution function $f(\theta, v, t)$ is governed by the Vlasov equation:

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial \theta} - \frac{\partial \Phi}{\partial \theta} \frac{\partial f}{\partial v} = 0,$$

where

$$\Phi(\theta, t) = -\frac{k}{2\pi} \int_0^{2\pi} \cos(\theta - \theta') \rho(\theta', t) d\theta',$$

is the self-consistent potential generated by the density of particles $\rho(\theta, t) = \int f(\theta, v, t) dv$. The mean force acting on a particle located in $\theta$ is $(\mathcal{F}) = -\partial \Phi/\partial \theta(\theta, t)$. Expanding the cosine function in equation (3), we obtain

$$\Phi(\theta, t) = B_x \cos \theta + B_y \sin \theta,$$

where

$$B_x = \frac{k}{2\pi} \int \rho(\theta, t) \cos \theta d\theta,$$

$$B_y = \frac{k}{2\pi} \int \rho(\theta, t) \sin \theta d\theta,$$
are proportional to the magnetization (with the opposite sign). The magnetization can be viewed as the order parameter of the HMF model.

Let us introduce the mass
\[ M[\rho] = \int \rho \, d\theta, \]
and the mean field energy
\[ E[f] = \frac{1}{2} \int f v^2 \, d\theta \, dv + \frac{1}{2} \int \rho \delta \, d\theta = K + W, \]
where \( K \) is the kinetic energy and \( W \) the potential energy. Using equations (11)-(13), the potential energy can be expressed in terms of the magnetization as
\[ W = -\frac{\pi B^2}{k}. \]

We also introduce the Boltzmann entropy
\[ S[f] = -\int f \ln \left( \frac{f}{N} \right) \, d\theta \, dv, \]
and the Boltzmann free energy
\[ F[f] = E[f] - TS[f], \]
where \( T = 1/\beta \) is the temperature. In the microcanonical ensemble, the statistical equilibrium state is determined by the maximization problem
\[ \max \{ S[f] \mid E[f] = E, M[f] = M \}. \]

In the canonical ensemble, the statistical equilibrium state is determined by the minimization problem
\[ \min \{ F[f] \mid M[f] = M \}. \]

The Boltzmann entropy functional (10) and the maximum entropy principle (12) can be justified from a standard combinatorial analysis (see, e.g., [10,11,17]). The distribution function \( f(\theta, v) \) that is solution of (12) is the most probable macroscopic state, i.e. the macrostate that is the most represented at the microscopic level, assuming that the accessible microstates (with the proper values of mass and energy) are equiprobable (see Appendix C).

We shall first determine the critical points of these variational problems. This will allow us to set the notations that will be needed in the following. The critical points of the maximization problem (12) are determined by the variational principle
\[ \delta S - \beta \delta E - \alpha \delta M = 0, \]
where \( \beta = 1/T \) and \( \alpha \) are Lagrange multipliers associated with the conservation of energy and mass. The critical points of the minimization problem (13) are determined by the variational principle
\[ \delta F + \alpha T \delta M = 0, \]
where \( \alpha \) is a Lagrange multiplier associated with the conservation of mass. Since \( T \) is fixed in the canonical ensemble, it is clear that equation (13) is equivalent to equation (14). Therefore, the optimization problems (12) and (13) have the same critical points. Performing the variations in equations (14) and (15), we find that the critical points are given by the mean field Maxwell-Boltzmann distribution
\[ f(\theta, v) = A' e^{-\beta \left( \frac{\theta^2}{2} + \Phi(\theta) \right)}, \]
where \( A' = e^{-1-\alpha} \) is a constant. Integrating over the velocity, we get the mean field Boltzmann distribution
\[ \rho(\theta) = A e^{-\beta \Phi(\theta)}, \]
where \( A = (2\pi/\beta)^{1/2} A' \). Using the expression (11) of the potential, the distribution function (10) can be rewritten
\[ f(\theta, v) = A' e^{-\beta \left( \frac{\theta^2}{2} + B_x \cos \theta + B_y \sin \theta \right)}. \]

It is convenient to write \( B_x = B \cos \phi \) and \( B_y = B \sin \phi \) with \( B = (B_x^2 + B_y^2)^{1/2} \). In that case, the foregoing expression takes the form
\[ f(\theta, v) = A' e^{-\beta \left( \frac{\theta^2}{2} + B \cos(\theta - \phi) \right)}. \]
The corresponding density profile is
\[ \rho(\theta) = A e^{-\beta B \cos(\theta - \phi)}. \]
The amplitude \( A \) and the magnetization \( B \) are determined by substituting equation (20) in equations (15), (9) and (7). This yields
\[ A = \frac{M}{2\pi I_0(\beta B)}, \]
and
\[ \frac{2\pi B}{kM} = \frac{I_1(\beta B)}{I_0(\beta B)}, \]
where \( I_n(x) \) is the modified Bessel function of order \( n \). Equation (22) determines the magnetization \( B \) as a function of the temperature \( T \). Then, \( A \) is given by equation (21). Note that the critical points are degenerate. There exists an infinity of critical points which differ only by their phase \( \phi \), i.e. by the position of the maximum of the density profile. They have the same value of entropy or free energy (see below). In the following, we shall take \( \phi = 0 \) without loss of generality. In that case, \( B_x = B \) and \( B_y = 0 \). Then, the distribution function and the density can be written
\[ f(\theta, v) = \left( \frac{\beta}{2\pi} \right)^{1/2} \rho(\theta) e^{-\beta \frac{\theta^2}{2}}, \]
\[ \rho(\theta) = \frac{M}{2\pi I_0(\beta B)} e^{-\beta B \cos \theta}, \]
where \( B \) is determined in terms of \( T \) by equation (22). The study of the self-consistency relation (22) is classical:
B = 0 is always solution while solutions with B ≠ 0 only exist for T < Tc = kM/4π [14,17].

Let us now determine the expressions of the energy, entropy and free energy. For the Maxwell-Boltzmann distribution [23], the kinetic energy is

$$K = \frac{1}{2}MT.$$  

(25)

Combining this relation with equation 23, we find that the total energy \(E = K + W\) is given by

$$E = \frac{1}{2}MT - \frac{\pi B^2}{k},$$  

(26)

The series of equilibria giving T as a function of E is determined by equations 22 and 24 by eliminating B. The relation that gives the magnetization B as a function of the energy E is determined by equations 22 and 27 by eliminating T. Finally, using equations 11 and 26, the entropy is given by

$$S = \frac{1}{2}M \ln T - \int \rho \ln \rho \, d\theta, \quad (27)$$

up to a term \(\frac{1}{2}M + \frac{3}{2}M \ln(2\pi) + M \ln M\). Using equation 24, it can be rewritten

$$S = \frac{1}{2}M \ln T + M \ln I_0(\beta B) - \frac{2\pi B^2}{kT},$$  

(28)

up to a term \(\frac{1}{2}M + \frac{3}{2}M \ln(2\pi)\). The relation between the entropy S and the energy E is determined by equations 28, 29 and 22. Using equations 26 and 28, the free energy is given by

$$F = \frac{1}{2}MT - \frac{1}{2}MT \ln T - MT \ln I_0(\beta B) + \frac{\pi B^2}{k},$$  

(29)

up to a term \(-\frac{1}{2}MT + \frac{3}{2}MT \ln(2\pi)\). The relation between the free energy F and the temperature T is determined by equations 29 and 22.

It is convenient to write these equations in parametric form by introducing the parameter \(x = \beta B\). Then, we have

$$b \equiv \frac{2\pi B}{kM} = \frac{I_1(x)}{I_0(x)},$$  

(30)

$$\eta \equiv \frac{\beta kM}{4\pi} = \frac{x}{2b(x)},$$  

(31)

$$\epsilon \equiv \frac{8\pi E}{kM^2} = \frac{1}{\eta(x)} - 2b(x)^2,$$  

(32)

$$s \equiv \frac{S}{M} = \frac{1}{2} \ln \eta(x) + \ln I_0(x) - 2\eta(x)b(x)^2,$$  

(33)

$$f \equiv \frac{8\pi F}{kM^2} = \epsilon(x) - \frac{2}{\eta(x)}s(x).$$  

(34)

These relations apply to the inhomogeneous states \((b \neq 0)\). For the homogeneous states \((b = 0)\), we have

$$\epsilon = \frac{1}{\eta}, \quad s = \frac{1}{2} \ln \epsilon, \quad f = \frac{1}{\eta} + \frac{1}{\eta} \ln \eta.$$  

(35)

The magnetization \(b\) takes values between 0 and 1, the inverse temperature \(\eta\) between 0 and \(+\infty\) and the energy \(\epsilon\) between \(\epsilon_{\text{min}} = -2\) and \(+\infty\). The homogeneous states exist for any \(\eta \geq 0\) and for any \(\epsilon \geq 0\). The inhomogeneous states exist for any \(\eta \geq \eta_c = 1\) and any \(\epsilon_{\text{min}} \leq \epsilon \leq \epsilon_c = 1\).

The bifurcation point is located at

$$\epsilon_c \equiv \frac{8\pi E_c}{kM^2} = 1, \quad \eta_c \equiv \frac{\beta kM}{4\pi} = 1.$$  

(36)

Close to the bifurcation point \((\eta \to \eta_c = 1^+, \epsilon \to \epsilon_c = 1^-)\), we get (see Appendix A):

$$b \simeq \sqrt{2(\eta - 1)}, \quad \eta \simeq \frac{1}{2} + \frac{1}{2}(1 - \epsilon),$$  

(37)

$$s \simeq \frac{1}{2}(1 - \epsilon), \quad f \simeq 1 + \frac{2}{3}(\eta - 1)^2.$$  

(38)

Close to the ground state \((\eta \to +\infty, \epsilon \to \epsilon_{\text{min}} = -2)\), we get (see Appendix A):

$$b \simeq 1 - \frac{1}{4\eta}, \quad b \simeq 1 - \frac{\epsilon + 2}{8}, \quad \eta \simeq \frac{2}{\epsilon + 2}.$$  

(39)

$$s \simeq \ln(\epsilon + 2), \quad f \simeq -2 + \frac{2}{\eta} \ln \eta.$$  

(40)

From these relations, we can obtain the curves \(T(E), B(T), B(E), S(E)\) and \(F(T)\) in parametric form. These curves are plotted in Figures 15 for completeness. These results are well-known and they have been derived in many papers [14,16,17,30] using different methods. The present approach, that complements the original approach
of Inagaki [14], is the most direct and the most complete. Indeed, these relations characterize all the critical points of (12) and (13). Now, we must select among these critical points those that are (local) maxima of $S$ at fixed $E$ and $M$ (microcanonical ensemble) and those that are (local) minima of $F$ at fixed $M$ (canonical ensemble). This is the object of the next sections.

**Remark:** using the Poincaré theorem (see, e.g. [10,31]), we can directly conclude from the series of equilibria that the homogeneous states are microcanonically (resp. canonically) stable for $E \geq E_c$ (resp. $T \geq T_c$) while they are microcanonically (resp. canonically) unstable for $E \leq E_c$ (resp. $T \leq T_c$). On the other hand, the inhomogeneous states are always stable. The caloric curve therefore exhibits a microcanonical second order phase transition marked by the discontinuity of $\beta'(E) = S''(E)$ at $E = E_c$ and a canonical second order phase transition marked by the discontinuity of $E'(\beta) = (\beta F)'(\beta)$ at $\beta = \beta_c$. The ensembles are equivalent. In order to illustrate our method, which remains valid in more general situations, we shall however treat both canonical and microcanonical ensembles.

### 3 Canonical ensemble

#### 3.1 The functionals $F[f]$ and $F[\rho]$

The minimization problem [13] has several interpretations:

(i) It determines the statistical equilibrium state of the HMF model in the canonical ensemble. In that thermodynamical interpretation $S$ is the Boltzmann entropy, $F$ is the Boltzmann free energy and $T$ is the thermodynamical temperature. The minimization problem (13) can therefore be interpreted as a criterion of canonical thermodynamical stability. For isolated systems that evolve at fixed energy, like the HMF model, the canonical ensemble is not physically justified. For such systems, the proper ensemble to consider is the microcanonical ensemble and the statistical equilibrium state is given by (12). However, the canonical ensemble always provides a sufficient condition of microcanonical stability [3] and it can be useful in that respect.

Indeed, if a system is canonically stable

---

3 For systems with short-range interactions, the ensembles are equivalent and we can choose the one that is the most convenient to make the calculations. For systems with long-range interactions, the ensembles may not be equivalent: grand canonical stability implies canonical stability which itself im-
at the temperature $T$, then it is automatically granted to be microcanonically stable at the corresponding energy $E = E(T)$. Therefore, we can start by this ensemble and consider the microcanonical ensemble only if the canonical ensemble does not cover the whole range of energies. On the other hand, for systems in contact with a thermal bath fixing the temperature, like the Brownian Mean Field (BMF) model [17], the canonical ensemble is the proper ensemble to consider and the statistical equilibrium state is given by [13].

(ii) It determines a particular steady state of the Vlasov equation that is formally nonlinearly dynamically stable. The minimization problem [13] can therefore be interpreted as a sufficient condition of dynamical stability. In that dynamical interpretation, $S$ is a particular Casimir (pseudo entropy), $F$ is an energy-Casimir functional (pseudo free energy) and $T$ is a positive constant. It can be convenient to develop a thermodynamical analogy [34] to study this dynamical stability problem and use a common vocabulary. In this way, the methods developed in thermodynamics can be applied to the dynamical stability context.

It is shown in Appendix A.2. of [37] that the solution of [13] is given by

$$ f(\theta, v) = \left( \frac{\beta}{2\pi} \right)^{1/2} \rho(\theta) e^{-\beta \frac{v^2}{2}}, \quad (41) $$

where $\rho(\theta)$ is the solution of

$$ \min_{\rho} \{ F[\rho] \mid M[\rho] = M \}, \quad (42) $$

where

$$ F[\rho] = \frac{1}{2} \int \rho^2 d\theta + T \int \rho \ln \rho d\theta. \quad (43) $$

Therefore, the minimization problems [13] and [12] are equivalent:

$$ [13] \iff [12]. \quad (44) $$

This equivalence holds for global and local minimization [37]: (i) $f(\theta, v)$ is the global minimum of [13] iff $\rho(\theta)$ is the global minimum of [12] and (ii) $f(\theta, v)$ is a local minimum of [13] iff $\rho(\theta)$ is a local minimum of [12]. We are therefore led to considering the minimization problem [12] which is simpler to study since it involves the density $\rho(\theta)$ instead of the distribution function $f(v, \theta)$.

Before that, let us compare the conditions of stability issued from [13] and [12]. The critical points of [13] are given by equations [28], [29] and [22] where $\beta$ is prescribed. A critical point of $F[\rho]$ at fixed mass is a (local) minimum iff

$$ \delta^2 F = \frac{1}{2} \int \delta \rho \delta \Phi d\theta + T \int \frac{(\delta f)^2}{2f} d\theta dv > 0, \quad (45) $$

for all perturbations $\delta f$ that do not change the mass: $\delta M = 0$. On the other hand, the critical points of [12] are given by equations [24] and [22] where $\beta$ is prescribed. A critical point of $F[\rho]$ at fixed mass is a (local) minimum iff

$$ \delta^2 F = \frac{1}{2} \int \delta \rho \delta \Phi d\theta + T \int \frac{(\delta \rho)^2}{2\rho} d\theta > 0, \quad (46) $$

for all perturbations $\delta \rho$ that do not change the mass: $\delta M = 0$. This stability criterion is equivalent to the stability criterion [45] but it is simpler because it is expressed in terms of the density instead of the distribution function.

**Remark 1:** the thermodynamical approach of Messer & Spohn [11] in the canonical ensemble directly leads to the minimization problem [12] for the density, and justifies it rigorously. Our approach recovers this result by another method. It also shows that this minimization problem provides a sufficient condition of dynamical stability with respect to the Vlasov equation.

**Remark 2:** considering the BMF model [17], the minimization problem [13] determines stable steady states of the mean field Kramers equation and the minimization problem [12] determines stable steady states of the mean field Smoluchowski equation. According to the equivalence [44], we conclude that a distribution function $f(\theta, v)$ is a stable steady state of the mean field Kramers equation if the corresponding density field $\rho(\theta)$ is a stable steady state of the mean field Smoluchowski equation (see [35] for a more general statement).

**Remark 3:** the equivalence between [13] and [12] can be extended to a larger class of functionals of the form $S[f] = -\int C(f) \, d\theta dv$ where $C$ is convex. Such functionals (Casimirs) arise in the Vlasov dynamical stability problem [28,29,33]. We refer to [11,34,39] for a detailed discussion of this equivalence.

### 3.2 The function $F(B)$

#### 3.2.1 Global minimization

The equivalence [44] is valid for an arbitrary potential of interaction $u(r, r')$. Now, for the HMF model, the problem can be further simplified. Indeed, the potential energy is given by equation [9] so that the free energy [13] can be rewritten

$$ F[\rho] = -\pi B^2 \frac{B^2}{k} + T \int \rho \ln \rho d\theta. \quad (47) $$

We shall first determine the global minimum of free energy at fixed mass. To that purpose, we shall reduce the minimization problem [17] to an equivalent but simpler minimization problem.

To solve the minimization problem [17], we proceed in two steps: we first minimize $F[\rho]$ at fixed $M$ and $B_\pi$ and

---

5 This is a refined condition of formal stability with respect to the usual criterion [33] since the mass is treated here as a constraint (see [28,29] for a more detailed discussion).

6 We have used this method in different situations (see, e.g. [34,36,37]).
Writing the variational principle as

$$
\delta \left( \int \rho \ln \rho \, d\theta \right) + \alpha \delta M + \mu_x \delta B_x + \mu_y \delta B_y = 0,
$$
(48)

we obtain

$$
\rho_1(\theta) = Ae^{-\lambda_x \cos \theta - \lambda_y \sin \theta},
$$
(49)

where $A = e^{-1-\alpha}$, $\lambda_x = -\frac{1}{\pi} \mu_x$ and $\lambda_y = -\frac{1}{\pi} \mu_y$. The Lagrange multipliers are determined by the constraints $M$, $B_x$ and $B_y$. If we write $B_x = B \cos \phi$ and $B_y = B \sin \phi$ then we find that $\lambda_x = \lambda \cos \phi$ and $\lambda_y = \lambda \sin \phi$ with $\lambda = (\lambda_x^2 + \lambda_y^2)^{1/2}$. Equation (49) can be rewritten as

$$
\rho_1(\theta) = Ae^{-\lambda \cos(\theta - \phi)}. 
$$
(50)

Finally, $A$ and $\lambda$ are determined in terms of $M$ and $B$ through the equations

$$
A = \frac{M}{2\pi I_0(\lambda)},
$$
(51)

and

$$
b \equiv \frac{2\pi B}{kM} = \frac{I_1(\lambda)}{I_0(\lambda)},
$$
(52)

Equation (50) is the (unique) global minimum of $F[\rho]$ with the previous constraints since $\delta^2 F = \frac{1}{\pi} T \int \frac{d\theta}{\rho^2} > 0$ (the constraints are linear in $\rho$ so that their second variations vanish). Then, we can express the free energy $F[\rho]$ as a function of $B$ by writing $F(B) \equiv F[\rho_1]$. After straightforward calculations, we obtain

$$
F(B) = -\frac{\pi B^2}{k} + T \frac{2\pi B}{k} - MT \ln I_0(\lambda),
$$
(53)

where $\lambda(B)$ is given by equation (52). Finally, the minimization problem (42) is equivalent to the minimization problem

$$
\min_B \left\{ F(B) \right\},
$$
(54)

in the sense that the solution of (42) is given by equations (50), (51) and (52) where $B$ is the solution of (53). Note that the mass constraint is taken into account implicitly in the variational problem (53). Therefore, (42) and (54) are equivalent for global minimization. However, (53) is much simpler because, for given $T$ and $M$, we just have to determine the minimum of a function $F(B)$ instead of the minimum of a functional $F[\rho]$ at fixed mass.

Let us therefore study the function $F(B)$ defined by equations (50) and (52). Its first derivative is

$$
F'(B) = -\frac{2\pi B}{k} + T \frac{2\pi B}{k} + MT \left( \frac{2\pi B}{kM} - \frac{I_0(\lambda)}{I_0(\lambda)} \right) \frac{d\lambda}{dB}.
$$
(55)

Using the identity $I_0'(\lambda) = I_1(\lambda)$ and the relation (52), we see that the term in parenthesis vanishes. Then, we get

$$
F'(B) = \frac{2\pi}{k} (T \lambda - B),
$$
(56)

The critical points of $F(B)$, satisfying $F'(B) = 0$, correspond therefore to

$$
\lambda = x \equiv \beta B. 
$$
(57)

Substituting this result in equation (52), we obtain the self-consistency relation

$$
\frac{2\pi B}{kM} = \frac{I_1(\beta) B}{I_0(\beta)},
$$
(58)

which determines the magnetization $B$ as a function of the temperature $T$. This returns the results of Sec. 2.

Now, a critical point of $F(B)$ is a minimum if $F''(B) > 0$ and a maximum if $F''(B) < 0$. Differentiating equation (56) with respect to $B$, we find that

$$
F''(B) = \frac{2\pi}{k} \left( T \frac{d\lambda}{dB} - 1 \right).
$$
(59)

Therefore, a critical point is a minimum if

$$
\frac{d\lambda}{dB} < T,
$$
(60)

and a maximum when the inequality is reversed.

We can determine the minimum of the function $F(B)$ by a simple graphical construction. To that purpose, we plot $b \equiv 2\pi B/kM$ as a function of $\lambda$ according to equation (52). This is represented in Figure 6. We find that $2\pi B/kM \to 1$ for $\lambda \to +\infty$. On the other hand, $2\pi B/kM \sim \lambda/2$ for $\lambda \to 0$. Therefore, the magnetization $B$ takes values between 0 and $B_{\text{max}} = \frac{kM}{2\pi}$. According to equation (57), the critical points of $F(B)$ are determined by the intersection of this curve with the straight line $b = \lambda/(2\eta)$. The critical point is a minimum (resp. maximum) of $F(B)$ if the slope of the curve $b(\lambda)$ at that point is smaller (resp. larger) than the straight line $b = \lambda/(2\eta)$.

![Graphical construction determining the critical points of $F(B)$ and their stability.](image)

Fig. 6. Graphical construction determining the critical points of $F(B)$ and their stability. The critical points are determined by the intersection(s) between the curve $b = b(\lambda)$ defined by equation (52) and the straight line $b = \lambda/(2\eta)$. The critical point is a minimum (resp. maximum) of $F(B)$ if the slope of the curve $b(\lambda)$ at that point is smaller (resp. larger) than the straight line $b = \lambda/(2\eta)$.
solution $B = 0$ and an inhomogeneous solution $B(T) \neq 0$. According to inequality (60), a solution is a minimum of $F(B)$ if $d(2\pi B/kM)/d\lambda < 2\pi T/kM$ and a maximum if $d(2\pi B/kM)/d\lambda > 2\pi T/kM$. Therefore, a critical point of free energy $F(B)$ is a minimum (resp. maximum) if the slope of the main curve is lower (resp. higher) than the slope of the straight line at the point of intersection. From this criterion, we easily conclude that: for $T > T_c$, the homogeneous solution $B = 0$ is the global minimum of $F(B)$; for $T < T_c$, the inhomogeneous solution $B(T) \neq 0$ is the global minimum of $F(B)$ while the homogeneous solution $B = 0$ is a local maximum.

Now, using the equivalence between (42) and (54), we conclude that the global minimum of $\beta(E)$ is monotonic (see Figure 1), we conclude that the maximum of entropy at fixed mass and energy is the homogeneous solution for $E > E_c$ and the inhomogeneous solution for $E < E_c$. Since we cover all the accessible range of energies, we conclude that the ensembles are equivalent. We shall, however, treat the microcanonical ensemble specifically in Sec. 4 since the method can be useful in other contexts where the ensembles are not equivalent (see, e.g. [24]).

3.2.2 Local minimization

We shall now show that the minimization problems (42) and (54) are also equivalent for local minimization. To that purpose, we shall relate the second order variations of $F[\rho]$ to the second variations of $F(B)$ by using a suitable decomposition.

A critical point of (42) is determined by the variational principle

$$\delta F + \alpha T \delta M = 0,$$

where $\alpha$ is a Lagrange multiplier accounting for the conservation of mass. This leads to the distribution

$$\rho = \frac{M}{2\pi I_0(\beta B)} e^{-\beta B \cos \theta},$$

where we have taken $\phi = 0$ without loss of generality (see Sec. 2). The magnetization $B$ is obtained by substituting equation (54) in equation (3) leading to the self-consistency relation (42). Using equations (3) and (43), this critical point is a (local) minimum of $F$ at fixed mass iff

$$\delta^2 F = \frac{\pi}{k} ((\delta B_x)^2 + (\delta B_y)^2) + \frac{1}{2} T \int \frac{\delta^2 \rho}{\rho} d\theta > 0,$$

for all perturbations $\delta \rho$ that conserve mass: $\int \delta \rho d\theta = 0$. The corresponding variations of magnetization are

$$\delta B_z = -\frac{k}{2\pi} \int \delta \rho \cos \theta \ d\theta,$$

This method was previously used in the context of the statistical mechanics of the 2D Euler [40, 41] and Vlasov [34] equations.
The conditions (69), (70) and (71) lead to the relations

\[ \delta B_y = -\frac{k}{2\pi} \int \delta \rho \sin \theta \, d\theta. \]  

(67)

We can always write the perturbations in the form

\[ \delta \rho = \delta \rho_\parallel + \delta \rho_\perp, \]  

(68)

where \( \delta \rho_\parallel = (\mu + \nu_x \cos \theta + \nu_y \sin \theta) \rho \) and \( \delta \rho_\perp \equiv \delta \rho - \delta \rho_\parallel \). The second condition ensures that all the perturbations are considered. We shall now choose the constants \( \mu, \nu_x \) and \( \nu_y \) such that

\[ \int \delta \rho_\perp \, d\theta = 0, \]  

(69)

\[ \delta B_x = -\frac{k}{2\pi} \int \delta \rho_\parallel \cos \theta \, d\theta, \]  

(70)

\[ \delta B_y = -\frac{k}{2\pi} \int \delta \rho_\parallel \sin \theta \, d\theta. \]  

(71)

This implies that

\[ \int \delta \rho_\perp \, d\theta = 0, \]  

(72)

\[ \int \delta \rho_\perp \cos \theta \, d\theta = 0, \]  

(73)

\[ \int \delta \rho_\perp \sin \theta \, d\theta = 0. \]  

(74)

The conditions (69), (70) and (71) lead to the relations

\[ \mu M - \frac{k}{2\pi} I \nu_x B = 0, \]  

(74)

\[ \delta B_x = \mu B = \frac{k}{2\pi} I \nu_x, \]  

(75)

\[ \delta B_y = \frac{k}{2\pi} \nu_y (M - I), \]  

(76)

where we have defined

\[ I \equiv \int \rho \cos^2 \theta \, d\theta. \]  

(77)

This forms a system of three algebraic equations that determines the three constants \( \mu, \nu_x \) and \( \nu_y \). Using the equilibrium distribution (64), we find after simple algebra that

\[ I = M \left[ 1 - \frac{1}{\beta B} \frac{I_1(\beta B)}{I_0(\beta B)} \right], \]  

(78)

where we have used the identity \( I_0(x) - I_2(x) = \frac{2}{\pi} I_1(x) \). Let us define the function \( B(\lambda) \) by the relation

\[ \frac{2\pi B(\lambda)}{kM} = \frac{I_1(\lambda)}{I_0(\lambda)}. \]  

(79)

For \( \lambda = \beta B \), according to the self-consistency relation (22), we have \( B(\lambda) = B \). Differentiating equation (79) with respect to \( \lambda \), we get

\[ \frac{2\pi}{kM} B'(\lambda) = \frac{I_1(\lambda) I_0(\lambda) - I_0(\lambda) I_1(\lambda)}{I_0(\lambda)^2}. \]  

(80)

Using the identities \( I_0(\lambda) = I_1(\lambda) \), \( I_1(\lambda) = \frac{1}{2} (I_0(\lambda) + I_2(\lambda)) \) and \( I_0(x) - I_2(x) = \frac{2}{\pi} I_1(x) \), and recalling equation (79), the foregoing relation can be rewritten

\[ \frac{2\pi}{kM} B'(\lambda) = 1 - \frac{1}{\lambda} \frac{I_1(\lambda)}{I_0(\lambda)} = \frac{4\pi^2 B(\lambda)^2}{k^2 M^2}. \]  

(81)

Taking \( \lambda = \beta B \) and introducing the function \( \lambda(B) \), which is the inverse of \( B(\lambda) \), we obtain the identity

\[ \frac{2\pi}{kM} \frac{1}{\lambda'B(B)} = 1 - \frac{1}{\beta B} \frac{I_1(\beta B)}{I_0(\beta B)} = \frac{4\pi^2 B^2}{k^2 M^2}. \]  

(82)

Comparing this relation with equation (78), we obtain

\[ I = \frac{2\pi}{k \lambda'(B)} + \frac{4\pi^2 B^2}{k^2 M}. \]  

(83)

Solving equations (74), (75) and (76) for \( \mu, \nu_x \) and \( \nu_y \), and using the result (83), we find that

\[ \nu_x = -\lambda'(B) \delta B_x, \quad \mu = -\frac{2\pi B \lambda'(B)}{kM} \delta B_z, \]  

(84)

\[ \nu_y = \frac{2\pi}{k} \frac{\delta B_y}{M - I}. \]  

(85)

Therefore, the perturbation \( \delta \rho_\parallel \) takes the form

\[ \delta \rho_\parallel = -\lambda'(B) \left( \frac{2\pi B}{kM} + \cos \theta \right) \rho(\theta) \delta B_x - \frac{2\pi}{k} \sin \theta \rho(\theta) \frac{\delta B_y}{M - I}. \]  

(86)

By construction, \( \delta \rho_\parallel \) and \( \delta \rho_\perp \) are orthogonal for the scalar product weighted by \( 1/\rho \) in the sense that

\[ \int \frac{\delta \rho_\parallel \delta \rho_\perp}{\rho} \, d\theta = 0. \]  

(87)

Indeed, we have

\[ \int \frac{\delta \rho_\parallel \delta \rho_\perp}{\rho} \, d\theta = \int (\mu + \nu_x \cos \theta + \nu_y \sin \theta) \delta \rho_\perp \, d\theta = 0, \]  

(88)

where we have used equations (72) and (73) to get the last equality. As a result, we obtain

\[ \int \frac{(\delta \rho)^2}{\rho} \, d\theta = \int \left( \frac{\delta \rho_\parallel}{\rho} \right)^2 \, d\theta + \int \left( \frac{\delta \rho_\perp}{\rho} \right)^2 \, d\theta. \]  

(89)
Using equations (86) and (83), we obtain after simplification
\[
\int \frac{(\delta \rho_\perp)^2}{\rho} \, d\theta = \frac{2\pi}{k} \mathcal{N}(B)(\delta B_\perp)^2 + \frac{4\pi^2}{k^2} \frac{1}{M-1} (\delta B_\parallel)^2.
\]
(90)

Therefore, the second order variations of free energy given by equation (95) can be written
\[
\delta^2 F = \frac{\pi}{k} \left[ (T\mathcal{N}(B) - 1)(\delta B_\parallel)^2 + \frac{2}{k} \frac{1}{M-1} \int \left( \frac{\delta \rho_\perp}{\rho} \right)^2 \, d\theta \right].
\]
(91)

Using identity (59), we obtain
\[
\delta^2 F = \frac{1}{2} F''(B)(\delta B_\parallel)^2 + \frac{1}{2} T \int \left( \frac{\delta \rho_\perp}{\rho} \right)^2 \, d\theta.
\]
(92)

For the inhomogeneous phase \(B \neq 0\), using equation (78) and the self-consistency relation (22), we get
\[
I = M - \frac{2\pi}{k} T.
\]
(93)

In that case, equation (92) reduces to
\[
\delta^2 F = \frac{1}{2} F''(B)(\delta B_\parallel)^2 + \frac{1}{2} T \int \left( \frac{\delta \rho_\perp}{\rho} \right)^2 \, d\theta.
\]
(94)

On the other hand, for the homogeneous phase \(B = 0\), equation (78) leads to
\[
I = \frac{M}{2}.
\]
(95)

Using equation (200), equation (92) reduces to
\[
\delta^2 F = \frac{1}{2} F''(0) \left[ (\delta B_\parallel)^2 + (\delta B_\perp)^2 \right] + \frac{1}{2} T \int \left( \frac{\delta \rho_\perp}{\rho} \right)^2 \, d\theta.
\]
(96)

These relations show that \(\rho\) is a local minimum of \(F[\rho]\) at fixed mass iff \(B\) is a local minimum of \(F(B)\). Indeed, if \(F''(B) > 0\) then \(\delta^2 F > 0\) since the last term is positive. On the other hand, if \(F''(B) < 0\) it suffices to consider a perturbation of the form (86) with \(\delta \rho_\perp = 0\) and \(\delta \rho_\parallel\) given by equation (58) to conclude that \(\delta^2 F < 0\) for this perturbation. This implies that \(\rho\) is not a local minimum of \(F[\rho]\) since there exists a particular perturbation that decreases the free energy. This is the case for the homogeneous solutions when \(T < T_c\) since they are local maxima of \(F(B)\). Therefore, (12) and (51) are equivalent for local minimization. Combining all our results, we conclude that the variational problems (13), (42) and (54) are equivalent for local and global minimization:
\[
\text{(13)} \iff \text{(12)} \iff \text{(54)}.
\]
(97)

4 Microcanonical ensemble

4.1 The functionals \(S[f]\) and \(S[\rho]\)

The maximization problem (12) has several interpretations:

(i) It determines the statistical equilibrium state of the HMF model in the microcanonical ensemble. In that thermodynamical interpretation \(S\) is the Boltzmann entropy. The maximization problem (12) can therefore be interpreted as a criterion of microcanonical thermodynamical stability.

(ii) It determines a particular steady state of the Vlasov equation that is formally nonlinearly dynamically stable. The maximization problem (12) can therefore be interpreted as a sufficient condition of dynamical stability. In that dynamical interpretation, \(S\) is a particular Casimir (pseudo entropy). As explained previously, it is convenient to develop a thermodynamical analogy (31) to study this dynamical stability problem and use a common vocabulary.

It is shown in Appendix A.1. of (37) that the solution of (12) is given by
\[
f(\theta, v) = \left( \frac{\beta}{2\pi} \right)^{1/2} \rho(\theta) e^{-\beta \frac{v^2}{2}},
\]
where \(\beta = 1/T\) is determined by the energy constraint
\[
E = \frac{1}{2} M T + W,
\]
and \(\rho(\theta)\) is the solution of
\[
\max_\rho \{ S[\rho] \mid M[\rho] = M \},
\]
where
\[
S[\rho] = \frac{1}{2} M \ln T - \int \rho \ln \rho \, d\theta.
\]
(101)

Eliminating the temperature thanks to the constraint (99), we can write the entropy in terms of \(\rho\) alone as
\[
S[\rho] = - \int \rho \ln \rho \, d\theta + \frac{1}{2} M \ln (E - W[\rho]).
\]
(102)

Therefore, the maximization problems (12) and (100) are equivalent:
\[
\text{(12)} \iff \text{(100)}.
\]
(103)

This equivalence holds for global and local maximization (37): (i) \(f(\theta, v)\) is the global maximum of (12) iff \(\rho(\theta)\) is the global maximum of (100) and (ii) \(f(\theta, v)\) is a local maximum of (12) iff \(\rho(\theta)\) is a local maximum of (100). We are led therefore to considering the maximization problem

---

8 This is a refined condition of formal stability with respect to the usual criterion (54) since both the mass and the energy are treated here as constraints (see [28, 29] for a more detailed discussion).
which is simpler to study since it involves the density $\rho(\theta)$ instead of the distribution function $f(v, \theta)$.

Before that, let us compare the conditions of stability issued from \[12\] and \[104\]. The critical points of \[12\] are given by equations (23), (24), (22) and (20) where $E$ is prescribed. Furthermore, a critical point of $S$ at fixed mass and energy is a (local) maximum iff (see Appendix A):

$$\delta^2 S = -\int \frac{(\delta f)^2}{2f} \, d\theta dv - \frac{1}{2} \int \delta \rho \delta \Phi \, d\theta < 0, \quad (104)$$

for all perturbations $\delta f$ that do not change the mass and the energy at first order: $\delta M = \delta E = 0$. On the other hand, the critical points of \[104\] are given by equations (101), (25) and (26) where $E$ is prescribed (see Appendix A.1 of [37]). Furthermore, a critical point of $S$ at fixed mass is a (local) maximum (if see Appendix A.1 of [37]):

$$\delta^2 S = -\int \frac{(\delta \rho)^2}{2\rho} \, d\theta - \frac{1}{2MT^2} \left( \int \delta \rho \delta \Phi \, d\theta \right)^2 < 0, \quad (105)$$

for all perturbations $\delta \rho$ that conserve mass: $\delta M = 0$. This stability criterion is equivalent to the stability criterion \[104\] but it is simpler because it is expressed in terms of the density instead of the distribution function.

**Remark 1:** the thermodynamical approach of Kiessling [12] in the microcanonical ensemble rigorously justifies the maximization problem \[12\].

**Remark 2:** comparing the stability criteria \[15\] and \[104\], we see that canonical stability implies microcanonical stability in the sense that a (local) minimum of $F$ at fixed mass is necessarily a (local) maximum of $S$ at fixed mass and energy. Indeed, if inequality \[15\] is satisfied for all perturbations that conserve mass, then inequality \[104\] is satisfied a fortiori for all perturbations that conserve mass and energy. However, the reciprocal is wrong in case of ensembles inequivalence. Therefore, we just have the implication \[15\] $\Rightarrow$ \[12\]. This result can also be obtained by comparing the stability criteria \[15\] and \[104\]. Indeed, since the last term in equation \[104\] is negative, it is clear that if inequality \[15\] is satisfied, then inequality \[104\] is automatically satisfied. In general this is not reciprocal and we may have ensembles inequivalence. However, if we consider a spatially homogeneous system for which $\Phi$ is uniform, the last term in equation \[104\] vanishes (since the mass is conserved) and the stability criteria \[15\] and \[104\] coincide. Therefore, for spatially homogeneous systems, we have ensembles equivalence.

**Remark 3:** according to the two interpretations of \[12\] recalled at the beginning of this section, we note that thermodynamical stability implies dynamical stability (for isothermal distributions). However, the converse is wrong since \[12\] provides just a sufficient condition of dynamical stability. More refined stability criteria are discussed in [23, 29].

**Remark 4:** the equivalence between \[12\] and \[104\] can be extended to a larger class of functionals of the form $S[f] = -\frac{1}{2} \int (f^q - f) \, d\theta dv$ associated to polytropic distributions. Such functionals (Casimirs) arise in the Vlasov dynamical stability problem. We refer to [23, 43] for a detailed discussion of this equivalence.

### 4.2 The function $S(B)$

#### 4.2.1 Global maximization

The equivalence \[103\] is valid for an arbitrary potential of interaction $u(r, r')$. Now, for the HMF model, the problem can be simplified further. Indeed, the potential energy is given by equation \[9\] so that the energy \[19\] and the entropy \[102\] can be rewritten

$$E = \frac{1}{2} MT - \frac{\pi B^2}{k}, \quad (106)$$

$$S[\rho] = -\int \rho \ln \rho \, d\theta + \frac{1}{2} M \ln \left( E + \frac{\pi B^2}{k} \right), \quad (107)$$

We shall first determine the global maximum of entropy at fixed mass. To that purpose, we shall reduce the maximization problem \[104\] to an equivalent but simpler maximization problem.

To solve the maximization problem \[104\], we proceed in two steps: we first maximize $S[\rho]$ at fixed $M$ and $B_x$ and $B_y$. Writing the variational problem as

$$- \int \delta \left( \int \rho \ln \rho \, d\theta \right) - \alpha \delta M - \mu_x \delta B_x - \mu_y \delta B_y = 0, \quad (108)$$

and proceeding as in Sec. 3.2.1, we obtain

$$\rho_1(\theta) = A e^{-\lambda \cos(\theta - \phi)}, \quad (109)$$

where $A$ and $\lambda$ are determined by the constraints $M$ and $B$ through

$$A = \frac{M}{2 \pi I_0(\lambda)}, \quad (110)$$

and

$$b = \frac{2 \pi B}{kM} = \frac{I_1(\lambda)}{I_0(\lambda)}, \quad (111)$$

Equation \[109\] is the (unique) global maximum of $S[\rho]$ with the previous constraints since $\delta^2 S = -\frac{1}{2} \int \frac{(\delta \rho)^2}{\rho} \, d\theta < 0$ (the constraints are linear in $\rho$ so that their second order variations vanish). Then, we can express the entropy $S$ as a function of $B$ by writing $S(B) \equiv S[\rho_1]$. After straightforward calculations, we obtain

$$S(B) = M \ln I_0(\lambda) - \frac{2 \pi B}{k} \lambda + \frac{M}{2} \ln \left( E + \frac{\pi B^2}{k} \right), \quad (112)$$
where $\lambda(B)$ is given by equation (111). Finally, the maximization problem (100) is equivalent to the maximization problem

$$\max_B \{ S(B) \},$$  \hspace{1cm} (113)

in the sense that the solution of (100) is given by equations (109) and (111) where $B$ is the solution of (113). Note that the energy and mass constraints are taken into account implicitly in the variational problem (113). Therefore, (109) and (113) are equivalent for global maximization. However, (113) is much simpler because, for given $E$ and $M$, we just have to determine the maximum of a function $S(B)$ instead of the maximum of a functional $S[\rho]$ at fixed mass and energy.

Let us therefore study the function $S(B)$ defined by equations (112) and (111). Its first derivative is

$$S'(B) = M \left( \frac{I_1(\lambda)}{I_0(\lambda)} - \frac{2\pi B}{kM} \right) \frac{d\lambda}{dB} - \frac{2\pi B}{k} \lambda + \frac{\pi B}{k} \frac{M}{E + \frac{\pi B^2}{k}}. \hspace{1cm} (114)$$

Using the identity $I_0'(x) = I_1(x)$ and the relation (111), we see that the term in parenthesis vanishes. Then, we get

$$S'(B) = \frac{2\pi B}{k} \left( \frac{B}{T} - \lambda \right), \hspace{1cm} (115)$$

where $T$ is determined by equation (109). The critical points of $S(B)$, satisfying $S'(B) = 0$, correspond to

$$\lambda = x \equiv \beta B. \hspace{1cm} (116)$$

Recalling equations (106) and (111), this leads to the self-consistency relations

$$\frac{2\pi B}{kM} = \frac{I_1(\beta B)}{I_0(\beta B)}, \hspace{1cm} (117)$$

$$E = \frac{1}{2} \frac{MT}{k} - \frac{\pi B^2}{k}. \hspace{1cm} (118)$$

which determine the magnetization as a function of the energy. This returns the relationships of Sec. 2.

Now, a critical point of $S(B)$ is a maximum if $S''(B) < 0$ and a minimum if $S''(B) > 0$. Differentiating equation (115) with respect to $B$, and recalling that the temperature $T$ is a function of $B$ given by equation (106), we find that

$$S''(B) = \frac{2\pi B}{k} \left( \frac{1}{T} - \frac{d\lambda}{dB} \right) \frac{4\pi B^2}{kMT^2}. \hspace{1cm} (119)$$

Using equation (59), we note that

$$S''(B) = -\frac{F''(B)}{T} \frac{8\pi^2 B^2}{k^2 M T^2}. \hspace{1cm} (120)$$

Therefore, a critical point of $S(B)$ is a maximum if

$$\frac{d\lambda}{dB} > \frac{1}{T} - \frac{4\pi B^2}{kMT^2}. \hspace{1cm} (121)$$

and a minimum if the inequality is reversed. Since the last term in equation (120) is negative, we recover the fact that canonical stability implies microcanonical stability. Indeed, if the critical point is a minimum of free energy ($F''(B) > 0$), then it is a fortiori a maximum of entropy ($S''(B) < 0$). Furthermore, we know that the series of equilibria $E(T)$ is monotonic (see Sec. 2). Therefore, the homogeneous solution is a maximum of $S(B)$ for $E > E_c$ since it is a minimum of $F(B)$ for $T > T_c$. On the other hand, the inhomogeneous solution is a maximum of $S(B)$ for $E < E_c$ since it is a minimum of $F(B)$ for $T < T_c$.

Finally, for the homogeneous solution $B = 0$, we have $S''(B) = -F''(B)/T$. Therefore, the homogeneous solution is a local minimum of $S(B)$ for $E < E_c$ since it is a local maximum of $F(B)$ for $T < T_c$. Using the equivalence between (109) and (113) for global maximization, we conclude that the maximum entropy state at fixed mass and energy is the homogeneous state for $E > E_c$ and the inhomogeneous state for $E < E_c$. On the other hand, since the ensembles are equivalent for homogeneous solutions (see Remark 2 in Sec. 4.1) and since we have established that the homogeneous solution is a saddle point of free energy for $T < T_c$ (see Sec. 5.2.2), we conclude that it is a saddle point of entropy for $E < E_c$. This result is shown by another method in the next section.

To complete our analysis, it can be useful to plot the function $S(B)$ for prescribed mass and energy. Using equations (112) and (111), the normalized entropy $s \equiv S/M$ can be expressed in terms of $\lambda$ according to

$$s(\lambda) = \ln I_0(\lambda) - \lambda \frac{I_1(\lambda)}{I_0(\lambda)} + \frac{1}{2} \ln \left( \epsilon + \frac{2I_1(\lambda)^2}{I_0(\lambda)^2} \right). \hspace{1cm} (122)$$

Eliminating $\lambda$ between expressions (122) and (111), we obtain the entropy $s(b)$ as a function of the magnetization $b$ for a fixed value of the energy $E$ (more precisely, for given $\epsilon$, these equations determine $s(b)$ in a parametric form). For $E > E_c$ and $E < E_c$, this function displays the two behaviors described above, as illustrated in Figure 8. For $E \rightarrow E_c$ so that $\lambda, b \rightarrow 0$, we find that the entropy
takes the approximate form
\[ s(b) \approx \frac{1}{2} \ln \epsilon + \left( \frac{1}{\epsilon - 1} \right) b^2 - \frac{\epsilon^2 + 4}{4\epsilon^2} b^4, \quad (\epsilon \to 1). \]
(123)

For \( \epsilon < 1 \), we explicitly check that the maximum satisfying \( s'(b) = 0 \) and \( s''(b) < 0 \) is given by the second relation in equation (106).

Remark: In Appendix E we plot the second variations of entropy \( S''(B(E)) \) (related to the variance of the magnetization) as a function of energy and recover the previous conditions of stability.

4.2.2 Local maximization

We shall now show that the maximization problems (100) and (113) are also equivalent for local maximization. To that purpose, we shall relate the second order variations of \( S[\rho] \) and \( S(B) \) by using a suitable decomposition.

A critical point of (100) is determined by the variational principle
\[ \delta S - \alpha \delta M = 0, \]
(124)
where \( \alpha \) is a Lagrange multiplier accounting for the conservation of mass. This leads to the distribution (see Appendix A.1. of [37]):
\[ \rho = \frac{M}{2\pi I_0(\beta B)} e^{-\beta B \cos \theta}, \]
(125)
where the temperature is determined by the energy according to equation (105). The magnetization \( B \) is obtained by substituting equation (125) in equations (10)–(9) leading to the self-consistency relation (22). Using equations (105) and (6), a critical point of \( S[\rho] \) at fixed mass is a (local) maximum iff
\[ \delta^2 S = -\int \frac{(\delta \rho)^2}{2\rho} \, d\theta + \pi \int \frac{((\delta B_x)^2 + (\delta B_y)^2)}{kT} \, d\theta - \frac{4\pi^2 B^2}{k^2 M T^2} (\delta B_x)^2 < 0, \]
(126)
for all perturbations \( \delta \rho \) that conserve mass: \( \int \delta \rho \, d\theta = 0 \). We note that the second order variations of entropy (124) are related to the second order variations of free energy (6) by
\[ \delta^2 S = -\frac{1}{T} \delta^2 F = -\frac{4\pi^2 B^2}{k^2 M T^2} (\delta B_x)^2. \]
(127)

Writing the perturbation \( \delta \rho \) in the form (18) with equation (80) and using expressions (41) and (65), we obtain for \( B \neq 0 \):
\[ \delta^2 S = -\frac{1}{2} \int \frac{E''(B)}{T} \left( \frac{8\pi^2 B^2}{k^2 M T^2} \right) (\delta B_x)^2 - \frac{1}{2} \int \frac{(\delta \rho_\perp)^2}{\rho} \, d\theta. \]
(128)

and for \( B = 0 \):
\[ \delta^2 S = -\frac{1}{2} \int \frac{F''(0)}{T} \left[ (\delta B_x)^2 + (\delta B_y)^2 \right] - \frac{1}{2} \int \frac{(\delta \rho_\perp)^2}{\rho} \, d\theta. \]
(129)

Finally, using identity (120), we arrive at
\[ \delta^2 S = \frac{1}{2} \int S''(B)(\delta B_x)^2 - \frac{1}{2} \int \frac{(\delta \rho_\perp)^2}{\rho} \, d\theta, \]
(130)
for \( B \neq 0 \) and
\[ \delta^2 S = \frac{1}{2} \int S''(B) [(\delta B_x)^2 + (\delta B_y)^2] - \frac{1}{2} \int \frac{(\delta \rho_\perp)^2}{\rho} \, d\theta, \]
(131)
for \( B = 0 \). These relations imply that \( \rho \) is a local maximum of \( S[\rho] \) at fixed mass iff \( B \) is a local maximum of \( S(B) \). Indeed, if \( S''(B) < 0 \), then \( \delta^2 S < 0 \) since the last term is negative. On the other hand, if \( S''(B) > 0 \), it suffices to consider a perturbation of the form \( \delta B_\perp = 0 \) and \( \delta B_\parallel \) given by equation (84) to conclude that \( \delta^2 S > 0 \) for this perturbation. This implies that \( \rho \) is not a local maximum of \( S[\rho] \) since there exists a particular perturbation that increases the entropy. This is the case for homogeneous solutions when \( E < E_c \). Therefore, (100) and (113) are equivalent for local maximization. Combining all our results, we conclude that the variational problems (12), (100), and (113) are equivalent for local and global maximization

\[ 12 \Leftrightarrow 100 \Leftrightarrow 113. \]
(132)

5 Conclusion

In this paper, we have presented a new method to settle the stability of homogeneous and inhomogeneous isothermal distributions in the HMF model. This method starts from general variational principles and transforms them into equivalent but simpler variational principles until a point at which the problem can be easily solved. For isothermal distributions, this method returns, as expected, the same results as those obtained in the past by different procedures [14, 29, 30], but we would like to emphasize why our approach is interesting and complementary to other methods.

First of all, it is based on general optimization problems: the maximization of \( S[f] \) at fixed mass \( M \) and energy \( E \) or the minimization of \( F[f] \) at fixed mass \( M \). These optimization problems provide either conditions of thermodynamical stability (in microcanonical and canonical ensembles respectively) or sufficient conditions of dynamical stability (more or less refined) with respect to the Vlasov equation. Therefore, our approach allows us to treat thermodynamical and dynamical stability problems with the same formalism. This is not possible if we follow an approach starting directly from the density of states.
g(E) or the partition function Z(β) which only applies to the thermodynamical problem \([11,16,30]\). As a consequence, our procedure still works if the Boltzmann functional is replaced by more general functionals of the form \(S[f] = - \int \mathcal{C}(f) \text{d} r \text{d} v\), where \(\mathcal{C}(f)\) is convex. In that case, the maximization of \(S\) at fixed \(M\) and \(E\) or the minimization of \(F\) at fixed \(M\) provide conditions of formal dynamical stability for arbitrary steady states of the Vlasov equation of the form \(f = f(\epsilon)\) with \(f'(\epsilon) < 0\) [29]. This dynamical stability problem has been studied recently for polytropic distributions in [21]. Our procedure could also be employed for the Lynden–Bell entropy [18], although the calculations would be more complicated. We have chosen to treat here the isothermal case in detail since the calculations are fully analytical. It offers therefore a simple illustration of the method.

Secondly, the optimization problems [12] and [13] on which our approach is based are both intuitive and rigorous. The thermodynamical approach is intuitive because the Boltzmann entropy \(S[f]\) can be obtained from a simple combinatorial analysis. In that case, the entropy is proportional to the logarithm of the disorder where the disorder measures the number of microstates associated with a given macrostate. Therefore, maximizing entropy \(S[f]\) at fixed mass and energy amounts to selecting the most probable macroscopic state consistent with the dynamical constraints [4]. In the context of the HMF model, this thermodynamical approach has been initiated in [11,17] but was not performed to completion since the stability of the inhomogeneous phase was not proven (at least analytically) by this method. This has been done in the present paper. The dynamical approach is also intuitive because we qualitatively understand that the stability of a dynamical system is linked to the fact that the system is in the minimum of a certain “potential”. For infinite dimensional systems, the nonlinear dynamical stability of a steady state of the Vlasov equation relies on the energy-Casimir method and its generalizations [29]. In the context of the HMF model, this approach has been followed in [17,25,29]. The optimization problems [12] and [13] are also rigorous because they have been given a precise justification by mathematicians. In statistical mechanics, the canonical criterion [13] has been justified rigorously in [11] and the microcanonical criterion [12] in [22]. Ensembles inequivalence has been formalized in [6]. On the other hand, the formal and nonlinear dynamical stability of a steady state of the Vlasov equation have been discussed extensively in the mathematical literature. We refer to [23] for a survey on the standard energy-Casimir method and to [35] for refined stability criteria (in the context of 2D turbulence).

Thirdly, our method determines not only the strict caloric curve \(\beta(E)\) (corresponding to global maxima of \(S\) at fixed \(E\) and \(M\), or global minima of \(F\) at fixed \(M\)) but it also provides the whole series of equilibria containing all the critical points of \(S\) at fixed \(E\) and \(M\), or the critical points of \(F\) at fixed \(M\). In particular, it allows us to determine metastable states that correspond to local maxima of \(S\) at fixed \(E\) and \(M\), or local minima of \(F\) at fixed \(M\). There are no such metastable states in the HMF model for isothermal distributions, but they can exist in other situations [22,24]. Our approach can be used to determine the sign of the second order variations of the thermodynamical potential in order to settle whether the critical point is a global maximum, a local maximum or a saddle point. In particular, we have been able to relate the second variations of the different functionals in order to show that the equivalence between the optimization problems is not only global but also local.

For these reasons, the approach developed in the present paper is an interesting complement to other methods [11,16,29,30] and it could find application and usefulness in more general situations.

### A Asymptotic expansions

In this section, we give the asymptotic expansions of \(b(x)\), \(\eta(x)\), \(\epsilon(x)\), \(s(x)\) and \(f(x)\) for \(x \to 0\) (corresponding to the bifurcation point) and \(x \to +\infty\) (corresponding to the ground state).

For \(x \to 0\), we have

\[
b(x) = \frac{x}{2} - \frac{x^3}{16} + \frac{x^5}{96} + o(x^6),
\]

\[
\eta(x) = 1 + \frac{x^2}{8} - \frac{x^4}{192} + o(x^6),
\]

\[
\epsilon(x) = 1 - \frac{5x^2}{8} + \frac{7x^4}{48} + o(x^6),
\]

\[
s(x) = -\frac{5x^2}{16} + \frac{41x^4}{768} + o(x^6),
\]

\[
f(x) = 1 - \frac{5x^4}{128} + o(x^5).
\]

For \(x \to +\infty\), we have

\[
b(x) = \frac{x}{2} - \frac{1}{2x} - \frac{1}{8x^2} + o(x^{-3}),
\]

\[
\eta(x) = \frac{x}{2} + \frac{1}{4} + \frac{3}{16x} + \frac{3}{16x^2} + o(x^{-3}),
\]
\[ \epsilon(x) = -2 + \frac{4}{x} - \frac{1}{x^2} + o(x^{-3}), \quad (140) \]

\[ s(x) = -\ln x + \frac{1}{2}(1 - \ln \pi) + \frac{1}{16x^2} + o(x^{-3}), \quad (141) \]

\[ f(x) = -2 + \frac{2 + 2 \ln x + 4 \ln x}{x^2} \ln \pi + \frac{2 \ln x}{x^2} + o(x^{-3}). \quad (142) \]

**B Derivation of inequality (104)**

We consider a perturbation \( \delta f \) around a distribution \( f \). The exact variations of mass \( (7) \) and energy \( (8) \) for any perturbation are

\[ \Delta M = \int \delta f \, d\theta \, dv, \quad (143) \]

\[ \Delta E = \int \delta f \, \frac{v^2}{2} \, d\theta \, dv + \int \Phi \delta \rho \, d\theta + \frac{1}{2} \int \delta \rho \delta \Phi \, d\theta \, dv. \quad (144) \]

Considering small perturbations \( \delta f \), the variations of entropy \( (10) \) up to second order is

\[ \Delta S = - \int (\ln f + 1) \delta f \, d\theta \, dv - \int \frac{(\delta f)^2}{2f} \, d\theta \, dv. \quad (145) \]

Let us now assume that \( f \) is a critical points of entropy at fixed mass and energy. It is determined by the variational principle \( (14) \), leading to

\[ \ln f + 1 = -\beta \left( \frac{v^2}{2} + \Phi \right) - \alpha. \quad (146) \]

Substituting this relation in equation \( (145) \), we obtain

\[ \Delta S = \int \left[ \beta \left( \frac{v^2}{2} + \Phi \right) + \alpha \right] \delta f \, d\theta \, dv - \int \frac{(\delta f)^2}{2f} \, d\theta \, dv. \quad (147) \]

Now, using the conservation of mass and energy \( \Delta M = \Delta E = 0 \), we get

\[ \Delta S = - \int \frac{(\delta f)^2}{2f} \, d\theta \, dv - \frac{1}{2} \int \delta \rho \delta \Phi \, d\theta \, dv. \quad (148) \]

Requiring that the critical point be a maximum of entropy at fixed mass and energy leads to inequality \( (104) \). We refer to \( 29,10 \) for generalizations of this result to a larger class of functionals in the Vlasov dynamical stability context.

**C Connection between statistical mechanics and variational principles**

In this section, we discuss the connection between the statistical mechanics of systems with long-range interactions and variational principles.

In the microcanonical ensemble, the accessible configurations (those having the proper value of energy) are equiprobable. Therefore, the density probability of the configuration \( (\theta_1,v_1,...,\theta_N,v_N) \) is \( P_N(\theta_1,v_1,...,\theta_N,v_N) = \frac{1}{g(2N)} \delta(E - H) \) where \( g(E) \) is the density of states

\[ g(E) = \int \delta(E - H) \, d\theta_1 \, dv_1 \cdots d\theta_N \, dv_N. \quad (149) \]

In other words, \( g(E)dE \) gives the number of microstates with energy between \( E \) and \( E + dE \). The entropy is defined by \( S(E) = \ln g(E) \). Let us introduce the (coarse-grained) one-body distribution function \( f(\theta,v) \). A macrostate is determined by the specification of the exact positions and velocities \( \{\theta_i,v_i\} \) of the \( N \) particles. A macrostate is determined by the specification of the density \( \{f(\theta,v)\} \) of particles in each cell \( [\theta, \theta + d\theta] \times [v,v + dv] \) irrespectively of their precise position in the cell. Let us call \( \Omega[f] \) the unconditional number of microstates corresponding to the macrostate \( f \). For \( N \to +\infty \), equation \( (149) \) can be formally rewritten

\[ g(E) \simeq \int \Omega[f] \delta(E[f] - E) \delta(M[f] - M) \, d\!f, \quad (150) \]

where \( E[f] \) is the mean field energy \( (8) \) and \( M[f] \) is the mass \( (7) \). The entropy of the macrostate \( f \) is defined by the Boltzmann formula \( S[f] = \ln \Omega[f] \). The Boltzmann entropy can be obtained by a standard combinatorial analysis leading, for \( N \to +\infty \), to equation \( (10) \). Therefore, using \( \Omega[f] = e^{S[f]} \), equation \( (150) \) can be rewritten

\[ g(E) \simeq \int e^{S[f]} \delta(E[f] - E) \delta(M[f] - M) \, d\!f. \quad (151) \]

The unconditional density probability of the distribution \( f \) is \( p_0[f] = 1 \frac{e^{S[f]}}{\mathcal{A}} \) (where \( \mathcal{A} \) is the hypervolume of the system in phase space). The microcanonical density probability of the distribution \( f \) is \( P[f] = \frac{1}{g(2N)} e^{S[f]} \delta(E[f] - E) \delta(M[f] - M) \).

Integrating over the velocities in equation \( (149) \), a classical calculation leads to

\[ g(E) = \frac{2\pi^{N/2}}{\Gamma(N/2)} \int [2(E - U)]^{N - 1} d\theta_1 \cdots d\theta_N, \quad (152) \]

where \( U(\theta_1,...,\theta_N) \) is the potential energy (second term in the r.h.s. of equation \( (11) \)). Let us introduce the (coarse-grained) one-body density \( \rho(\theta) \) and denote by \( \Omega[\rho] \) the unconditional number of microstates \( \{\theta_i\} \) corresponding to the macrostate \( \rho \). For \( N \to +\infty \), equation \( (152) \) can be
formally rewritten \(^1\)

\[
g(E) \simeq \int e^{-\frac{k}{2} \ln(E - W[\rho])} \Omega[\rho] \delta(M[\rho] - M) \, d\rho, \tag{153}
\]

where \(W[\rho]\) is the mean field potential energy (second term in the r.h.s. of equation \(^3\)). The unconditional number of microstates \(\Omega[\rho]\) can be obtained by a classical combinatorial analysis leading, for \(N \to +\infty\), to

\[
\Omega[\rho] = e^{-\int \rho \ln \rho \, d\rho}. \tag{155}
\]

Therefore, equation \((153)\) can be rewritten

\[
g(E) \simeq \int e^{S[\rho]} \delta(M[\rho] - M) \, d\rho, \tag{154}
\]

where \(S[\rho]\) is given by equation \((102)\). The unconditional density probability of the distribution \(\rho\) is \(P_0[\rho] = \frac{1}{\Omega} e^{-\int \rho \ln \rho \, d\rho}\) (where \(A\) is the volume of the system in physical space). The microcanonical density probability of the distribution \(\rho\) is \(P_\rho[\rho] = \frac{1}{\Omega} e^{S[\rho]} \delta(M[\rho] - M)\).

For the HMF model, let us introduce the magnetization vector

\[
B_x = -\frac{k}{2\pi} \sum_{i=1}^{N} \cos \theta_i, \quad B_y = -\frac{k}{2\pi} \sum_{i=1}^{N} \sin \theta_i. \tag{155}
\]

The potential energy can be expressed in terms of the magnetization as

\[
U = -\frac{\pi B^2}{k} + \frac{kN}{4\pi}. \tag{156}
\]

Therefore, the density of states \((152)\) can be rewritten

\[
g(E) = \frac{2\pi^{N/2}}{\Gamma(\frac{N}{2})} \int \left[ 2 \left( E + \frac{\pi B^2}{k} - \frac{kN}{4\pi} \right) \right]^{\frac{N-1}{2}}
\]

\[
\times \delta \left( \frac{2\pi B_x}{k} + \sum_i \cos \theta_i \right) \delta \left( \frac{2\pi B_y}{k} + \sum_i \sin \theta_i \right)
\]

\[
\times d \left( \frac{2\pi B_x}{k} \right) d \left( \frac{2\pi B_y}{k} \right) d\theta_1...d\theta_N, \tag{157}
\]

or, equivalently,

\[
g(E) = \frac{2\pi^{N/2}}{\Gamma(\frac{N}{2})} \int \left[ 2 \left( E + \frac{\pi B^2}{k} - \frac{kN}{4\pi} \right) \right]^{\frac{N-1}{2}}
\]

\[
\times \Omega \left( \frac{2\pi B_x}{k}, \frac{2\pi B_y}{k} \right) d \left( \frac{2\pi B_x}{k} \right) d \left( \frac{2\pi B_y}{k} \right), \tag{158}
\]

where \(\Omega(B)\) denotes the unconditional number of microstates \(\{\theta\}\) corresponding to the macrostate \(B\). For \(N \to +\infty\), we have

\[
g(E) \simeq \int e^{\frac{\pi}{2} \ln(E + \frac{\pi B^2}{k})} \Omega(B) \, d\Omega. \tag{159}
\]

The computation of \(\Omega(B)\) is classical \((4)\) and is briefly reproduced in Appendix \((3)\) with some complements, for the sake of self-consistency. For \(N \to +\infty\), we have

\[
\Omega(B) = e^{-\frac{\pi B^2}{k} + M \ln I_0(\lambda)}, \tag{160}
\]

Therefore, the density of states \((159)\) can finally be written

\[
g(E) \simeq \int e^{S(B)} \, d\Omega, \tag{161}
\]

where \(S(B)\) is given by equation \((112)\). The unconditional density probability of the magnetization \(\Omega\) is \(P_\Omega[\Omega] = \frac{1}{\Omega} e^{-\frac{\pi B^2}{k} + M \ln I_0(\lambda)}\). The microcanonical density probability of the magnetization \(\Omega\) is \(P_{\Omega}[\Omega] = \frac{1}{\Omega} e^{-\beta S[\Omega]} \delta(M[\Omega] - M)\).

In the canonical ensemble, the density probability of the configuration \((\theta_1, v_1, ..., \theta_N, v_N)\) is \(P_N(\theta_1, v_1, ..., \theta_N, v_N) = \frac{1}{Z(\beta)} e^{-\beta H}\) where \(Z(\beta)\) is the partition function

\[
Z(\beta) = \int e^{-\beta H} \, d\theta_1 d\theta_1 ... d\theta_N d\nu_N. \tag{162}
\]

The free energy is defined by \(F(\beta) = -\frac{1}{\beta} \ln Z(\beta)\). Introducing the unconditional number of microstates \(\Omega[\rho]\) corresponding to the macrostate \(f\), we obtain for \(N \to +\infty:\)

\[
Z(\beta) \simeq \int e^{-\beta E[f]} \Omega[f] \delta(M[f] - M) \, d\rho
\]

\[
\simeq \int e^{S[f] - \beta E[f]} \delta(M[f] - M) \, d\rho
\]

\[
\simeq \int e^{-\beta F[f]} \delta(M[f] - M) \, d\rho, \tag{163}
\]

where \(F[f] = E[f] - TS[f]\) is the Boltzmann free energy \((11)\). The canonical density probability of the distribution \(f\) is \(P[f] = \frac{1}{Z(\beta)} e^{-\beta F[f]} \delta(M[f] - M)\).

Integrating over the velocities in equation \((162)\), we get

\[
Z(\beta) = \left( \frac{2\pi}{\beta} \right)^{N/2} \int e^{-\beta U} \, d\theta_1 ... d\theta_N. \tag{164}
\]

Introducing the unconditional number of microstates \(\Omega[\rho]\) corresponding to the macrostate \(\rho\), we obtain for \(N \to +\infty:\)

\[
Z(\beta) \simeq \int e^{-\beta W[\rho]} \Omega[\rho] \delta(M[\rho] - M) \, d\rho
\]

\[
\simeq \int e^{-\beta F[\rho]} \delta(M[\rho] - M) \, d\rho
\]

\[
\simeq \int e^{-\beta E[\rho]} \delta(M[\rho] - M) \, d\rho, \tag{165}
\]

where \(F[\rho]\) is given by equation \((13)\). The canonical density probability of the distribution \(\rho\) is \(P[\rho] = \frac{1}{Z(\beta)} e^{-\beta F[\rho]} \delta(M[f] - M)\).

\(^{10}\) Like in the main part of the paper, for the sake of conciseness, we do not explicitly write the constant terms that are independent on \(\rho\) in the expression of the entropy (the term in the exponential). They can be restored easily.
For the HMF model for which the potential energy can be expressed in terms of the magnetization, the partition function \[ \Omega \] can be rewritten

\[
Z(\beta) = \left( \frac{2\pi}{\beta} \right)^{N/2} \int e^{-\beta U} \times \delta \left( \frac{2\pi B_x}{k} + \frac{1}{2} \sum \cos \theta_i \right) \delta \left( \frac{2\pi B_y}{k} + \frac{1}{2} \sum \sin \theta_i \right) \times d \left( \frac{2\pi B_x}{k} \right) d \left( \frac{2\pi B_y}{k} \right) d\theta_1...d\theta_N, \]

(166)
or, equivalently,

\[
Z(\beta) = \left( \frac{2\pi}{\beta} \right)^{N/2} \int e^{\beta \left( \frac{2\pi^2}{\beta} - \frac{1}{2} \sum \theta_i \right)} \times \Omega \left( \frac{2\pi B_x}{k} + \frac{1}{2} \sum \theta_i \right) d \left( \frac{2\pi B_x}{k} \right) d \left( \frac{2\pi B_y}{k} \right). \]

(167)

For \( N \rightarrow +\infty \), we get

\[
Z(\beta) \simeq \int e^{\beta \frac{2\pi^2}{\beta^2}} \Omega(B) dB \\
\simeq \int e^{\beta \frac{2\pi^2}{\beta^2}} e^{-\frac{\beta}{\beta^2} \lambda + M \ln I_0(\lambda)} dB \\
\simeq \int e^{-\beta F(B)} dB, \]

(168)

where \( F(B) \) is given by equation (53). The canonical density probability of the magnetization \( B \) is \( P(B) = \frac{1}{Z(\beta)} e^{-\beta F(B)} \).

Let us now denote by \( \phi \) a generic global variable such as \( f(\theta, v) \), \( \rho(\theta) \) or \( B \). We also recall that for systems with long-range interactions, for which the mean field approximation is exact in the proper thermodynamic limit \( N \rightarrow +\infty \), we have the extensive scalings \( S(\phi) = N s(\phi), E[\phi] = N e[\phi], F[\phi] = N f[\phi] \). Accordingly, the preceding results can be formally written

\[
g(E) \simeq \int e^{N s[\phi]} \delta(e_{MCE}[\phi] - e_{MCE}) \mathcal{D}\phi, \]

(169)
and

\[
Z(\beta) \simeq \int e^{-\beta N f[\phi]} \delta(c_{MCE}[\phi] - c_{MCE}) \mathcal{D}\phi, \]

(170)

where the \( \delta \)-functions take into account the constraints as described above. The microcanonical distribution of microstates corresponding to the macrostate \( \phi \) is \( \Omega[\phi] = e^{N s[\phi]} \delta(c_{MCE}[\phi] - c_{MCE}) \) and the microcanonical probability of the macrostate \( \phi \) is \( \Omega[\phi] = \frac{1}{Z(\beta)} e^{N s[\phi]} \delta(c_{MCE}[\phi] - c_{MCE}) \). Similarly, the canonical distribution of microstates corresponding to the macrostate \( \phi \) is \( \Omega[\phi] = e^{-\beta N f[\phi]} \delta(c_{C}[\phi] - c_{C}) \) and the canonical probability of the macrostate \( \phi \) is \( \Omega[\phi] = \frac{1}{Z(\beta)} e^{-\beta N f[\phi]} \delta(c_{C}[\phi] - c_{C}) \). For \( N \rightarrow +\infty \), we can make the saddle point approximation. In the microcanonical ensemble, we obtain

\[
g(E) = e^{S(E)} \simeq e^{N s[\phi_*]}, \]

(171)
i.e.

\[
\lim_{N \rightarrow +\infty} \frac{1}{N} S(E) = s[\phi_*], \]

(172)
where \( \phi_* \) is the solution of the maximization problem

\[
\max_{\phi} \{ s[\phi] \mid E, M \}. \]

(173)

This leads to the variational problems \[ 12 \], \[ 100 \] and \[ 113 \]. In the canonical ensemble, we obtain

\[
Z(\beta) = e^{-\beta F(\beta)} \simeq e^{-\beta N f[\phi_*]}, \]

(174)
i.e.

\[
\lim_{N \rightarrow +\infty} \frac{1}{N} F(\beta) = f[\phi_*], \]

(175)
where \( \phi_* \) is the solution of the minimization problem

\[
\min_{\phi} \{ f[\phi] \mid M \}. \]

(176)

This leads to the variational problems \[ 13 \], \[ 12 \] and \[ 53 \].

The preceding discussion shows the connection between the statistical mechanics of systems with long-range interactions (based on the calculation of the density of states and of the partition function) and variational principles (based on the maximization of entropy or minimization of free energy). It also shows how the variational problems \[ 12 \], \[ 100 \], \[ 113 \] and \[ 13 \], \[ 12 \], \[ 53 \] are related to each other. These results can be made rigorous by using the theory of large deviations. We refer to Ellis \[ 17 \] for a mathematical presentation of this theory and to Barré et al. \[ 22 \] and Touchette \[ 15 \] for its application to physical problems. In the present paper, we have considered a different approach. We started from the fundamental variational problems \[ 12 \] and \[ 53 \] that can be motivated by a simple combinatorial analysis. This is the historical approach of the problem finding its roots in Boltzmann’s work. This is also the traditional approach used by physicists to determine the statistical equilibrium state of self-gravitating systems (see, e.g., \[ 43,44,53 \]), two-dimensional point vortices (see, e.g., \[ 1,51 \]) and the HMF model (see, e.g., \[ 14,17 \]). Therefore, describing the statistical mechanics of systems with long-range interactions from the fundamental variational problems \[ 12 \] and \[ 13 \] and reducing them to simpler but equivalent forms \[ 100 \], \[ 113 \], \[ 12 \] and \[ 53 \] as we have done here is an interesting presentation that complements the one followed in \[ 11 \]. A bonus of this approach is that it remains valid when the variational problems \[ 12 \] and \[ 13 \] have a dynamical interpretation in relation to the (formal) nonlinear stability of the system with respect to the Vlasov equation \[ 20 \]. In that case, \( S[f] \) is a Casimir functional of the form \( S[f] = -\int C(f) d\theta dv \) (sometimes called a pseudo
entropy) that is more general than the Boltzmann functional arising in the thermodynamical approach. In particular, in the dynamical stability problem, the variational problems \[12\] and \[13\] cannot be obtained from a theory of large deviations since their physical interpretations are completely different.

**Remark 1:** The density of states \[14\] and the partition function \[15\] can also be calculated from field theoretical methods (see e.g. Horwitz & Katz \[52\] and de Vega & Sanchez \[53\] for self-gravitating systems and Antoni & Ruffo \[16\] for the HMF model). These approaches are valuable but they are also considerably more abstract than the one based on variational principles.

**Remark 2:** The mean field Boltzmann distribution \[16\] can also be obtained from the first equation of the Yvon-Bogoliubov-Green (YBG) hierarchy \[54\], but the condition of stability (related to the correlation functions appearing in the next equations of the YBG hierarchy) is more difficult to obtain than with the approach based on variational principles.

**Remark 3:** For \( N \to +\infty \), the density of states (resp. partition function) is dominated by the global maximum of entropy at fixed mass and energy (resp. minimum of free energy at fixed mass) according to \[17\] (resp. \[17\]). Nevertheless, local entropy maxima (resp. free energy minima), i.e. metastable states, are also fully relevant because they have very long lifetimes scaling like \( \sim \delta N \) \[55,56\]. This is particularly true in the case of classical self-gravitating systems for which there is no global entropy maximum (resp. free energy minimum) \[57\].

**D Distribution of the magnetization**

In this Appendix, we determine the distribution of the magnetization by a direct calculation and show its connection with the entropy \( S(B) \) and free energy \( F(B) \). The density probability of the magnetization vector defined by equation \[15\] is

\[
P \left( \frac{2\pi B_x}{k}, \frac{2\pi B_y}{k} \right) = \int \delta \left( \frac{2\pi B_x}{k} + \sum_i \cos \theta_i \right)
\times \delta \left( \frac{2\pi B_y}{k} + \sum_i \sin \theta_i \right) P_N(\theta_1, ..., \theta_N) d\theta_1...d\theta_N.
\]

In the microcanonical ensemble, the \( N \)-body distribution function is \( P_N(\theta_1, v_1, ..., \theta_N, v_N) = \frac{1}{g(E)} \delta(E - H) \). Integrating over the velocities, a classical calculation gives

\[
P_N(\theta_1, ..., \theta_N) = \frac{1}{g(E)} \frac{2\pi^{N/2}}{\Gamma(\frac{N}{2})} \left[ 2(E - U) \right]^{\frac{N-2}{2}}.
\]

where \( U \) is the potential energy (second term in the r.h.s. of equation \[14\]). In the canonical ensemble, the \( N \)-body distribution function is \( P_N(\theta_1, v_1, ..., \theta_N, v_N) = \frac{1}{Z(\beta)} e^{-\beta H} \). Integrating over the velocities, we obtain

\[
P_N(\theta_1, ..., \theta_N) = \frac{1}{Z(\beta)} \left( \frac{2\pi}{\beta} \right)^{N/2} e^{-\beta U}.
\]

Recalling that the potential energy can be expressed in terms of the magnetization according to equation \[15\], the density probability of the magnetization in microcanonical and canonical ensembles is given by

\[
P \left( \frac{2\pi B_x}{k}, \frac{2\pi B_y}{k} \right) = \frac{1}{g(E)} \frac{2\pi^{N/2}}{\Gamma(\frac{N}{2})} \times \left[ 2 \left( E + \frac{\pi B^2}{k} - \frac{kN}{4\pi} \right) \right]^{\frac{N-2}{2}}
\times \Omega \left( \frac{2\pi B_x}{k}, \frac{2\pi B_y}{k} \right),
\]

\[
P \left( \frac{2\pi B_x}{k}, \frac{2\pi B_y}{k} \right) = \frac{1}{Z(\beta)} \left( \frac{2\pi}{\beta} \right)^{N/2}
\times e^{\beta \left( \frac{\pi B^2}{k} - \frac{kN}{4\pi} \right)} \Omega \left( \frac{2\pi B_x}{k}, \frac{2\pi B_y}{k} \right),
\]

where

\[
\Omega \left( \frac{2\pi B_x}{k}, \frac{2\pi B_y}{k} \right) = \int \delta \left( \frac{2\pi B_x}{k} + \sum_i \cos \theta_i \right)
\times \delta \left( \frac{2\pi B_y}{k} + \sum_i \sin \theta_i \right) d\theta_1...d\theta_N,
\]

is the unconditional number of microstates with magnetization \( B \). The calculation of this integral is classical \[4\]. Using the Fourier representation of the \( \delta \)-function

\[
\delta(x) = \int_{-\infty}^{+\infty} e^{iqx} dq, \quad (183)
\]

we obtain

\[
\Omega = (2\pi)^{N-2} \int dq_x dq_y e^{N\hbar(q_x - q_y)}, \quad (184)
\]

where

\[
h(q_x, q_y) = i \frac{2\pi}{kM} (q_x B_x + q_y B_y) + \ln J_0(q), \quad (185)
\]

and \( q = \sqrt{q_x^2 + q_y^2} \). Recalling that \( k \sim 1/N \), the function \( h \) does not depend on \( N \). For \( N \to +\infty \), we can make the saddle point approximation

\[
\Omega \sim e^{N\hbar(q^{*}_x - q^{*}_y)}, \quad (186)
\]

where \( (q^{*}_x, q^{*}_y) \) corresponds to the maximum of \( h(q_x, q_y) \). The vanishing of \( \partial h/\partial q_x \) and \( \partial h/\partial q_y \) leads to

\[
\frac{i}{kM} B_x - \frac{J_1(q)}{J_0(q)} \frac{q_x}{q} = 0, \quad (187)
\]

\[
\frac{i}{kM} B_y - \frac{J_1(q)}{J_0(q)} \frac{q_y}{q} = 0, \quad (188)
\]
where we have used \( J_0'(x) = -J_1(x) \). Setting \( q_x = -i\lambda x \) and \( q_y = -i\lambda y \), we find that \( q = i\lambda \) where \( \lambda = \sqrt{\lambda_x^2 + \lambda_y^2} \). Substituting these expressions in equations (187) and (188) and using \( J_1(i\lambda) = iI_1(\lambda) \) and \( J_0(i\lambda) = I_0(\lambda) \), we find that \( \lambda \) is determined by

\[
\frac{2\pi B}{kM} = \frac{I_1(\lambda)}{I_0(\lambda)}, \tag{189}
\]

Then, we get

\[
q_x^* = i\lambda \frac{B_x}{B}, \quad q_y^* = i\lambda \frac{B_y}{B}, \tag{190}
\]

Substituting these values in equations (185) and (186), we finally obtain

\[
\Omega(B) \sim e^{N \left[ -\frac{\pi B}{2kM} \lambda + \ln I_0(\lambda) \right]}, \tag{191}
\]

Therefore, for \( N \to +\infty \), the unconditional density probability of the magnetization, i.e. the one corresponding to a Poissonian (uncorrelated) distribution of angles, is

\[
P_0(B) = \frac{1}{A} e^{N \left[ -\frac{\pi B}{2kM} \lambda + \ln I_0(\lambda) \right]}, \tag{192}
\]

This result can also be obtained from the theory of large deviations by a direct application of the Cramer theorem (see equations (6) and (7) in [180]).

We now take into account the correlations by using the \( N \)-body distribution functions (178) and (179). According to equations (189) and (191), the distribution of the magnetization in the microcanonical ensemble is given, for \( N \to +\infty \), by

\[
P_{MCE}(B) = \frac{1}{g(E)} e^{S(B)} \left( \frac{2\pi B}{kM} \lambda + \ln I_0(\lambda) \right). \tag{193}
\]

It can be written

\[
P_{MCE}(B) = \frac{1}{g(E)} e^{S(B)} \tag{194}
\]

where \( S(B) \) is the entropy defined in equation (112). According to equations (188) and (192), the distribution of the magnetization in the canonical ensemble is given, for \( N \to +\infty \), by

\[
P_{CE}(B) = \frac{1}{Z(\beta)} e^{\frac{\pi B^2}{M} k - \frac{\pi B}{k} \lambda + \ln I_0(\lambda)}. \tag{195}
\]

It can be written

\[
P_{CE}(B) = \frac{1}{Z(\beta)} e^{-\beta F(B)} \tag{196}
\]

where \( F(B) \) is the free energy defined in equation (58).

### 20. Variance of the magnetization

In this Appendix, we show the connection between the variance of the magnetization in canonical and microcanonical ensembles and the second order derivatives of entropy and free energy.

The unconditional density probability of the magnetization (192) can be written

\[
P_0(B) = \frac{1}{A} e^{Nh(B)}, \tag{197}
\]

where we have introduced the function

\[
h(B) = -\frac{2\pi B}{kM} \lambda + \ln I_0(\lambda), \tag{198}
\]

where \( h(B) \) is defined by equation (189). Note that \( h(B) \) can be interpreted as the entropy of the magnetization for a Poissonian distribution of angles. For \( N \to +\infty \), the distribution is strongly peaked around its maximum. The most probable value of the magnetization \( B \) corresponds to the maximum of \( h(B) \). Using equation (189), the first derivative of \( h(B) \) is

\[
h'(B) = -\frac{2\pi \lambda}{kB}, \tag{199}
\]

so that the most probable value of the magnetization is \( \lambda = B = 0 \) (we will see that it corresponds indeed to a maximum). Let us expand \( h(B) \) around its maximum \( h(0) = 0 \). The second derivative of \( h(B) \) is

\[
h''(B) = -\frac{4\pi}{kB^2} \tag{200}
\]

Using identity (82), we obtain

\[
h''(0) = -\frac{8\pi^2}{k^2M^2} < 0, \tag{201}
\]

justifying that \( B = 0 \) really is the maximum of \( h(B) \). Therefore, for \( \sqrt{N}B \sim 1 \), the distribution of the magnetization is the Gaussian:

\[
P_0(B) = \frac{4\pi}{k^2M} e^{-\frac{\pi B^2}{4\pi M^2}}. \tag{202}
\]

This result can be directly obtained from the central limit theorem. The variance of the unconditional distribution of magnetization is

\[
\langle B^2 \rangle_0 = \frac{k^2M}{4\pi^2}. \tag{203}
\]

Let us now consider the distribution of the magnetization in the canonical ensemble given by equation (195). The most probable value of \( B \) corresponds to the minimum of free energy \( F(B) \) as studied in section 3.2.1. The vanishing of \( F'(B) \) as studied in section 3.2.1. The second derivative of \( F(B) \) at the extremum point is given by equation (63) where \( \lambda'(B) \) is given by equation (84).
In the homogeneous phase $B = 0$, the variance of the magnetization is

$$\langle B^2 \rangle_{CE} = \frac{2}{3F''(0)}.$$  (204)

Using equation (204), we obtain

$$\lambda'(0) = \frac{4\pi}{kM}.$$  (205)

According to equation (59), we have

$$\beta F''(0) = \frac{8\pi^2}{k^2M} \left(1 - \frac{T_c}{T}\right).$$  (206)

Therefore, using equation (206), we obtain

$$\langle B^2 \rangle_{CE} = \frac{k^2M}{4\pi^2} \frac{1}{1 - \frac{T_c}{T}}.$$  (207)

We first note that, according to equation (206), the homogeneous phase is a minimum of free energy ($F''(0) > 0$) for $T > T_c$ and a maximum of free energy ($F''(0) < 0$) for $T < T_c$ in agreement with the graphical construction of section 4.2.4. On the other hand, for $T \to +\infty$, i.e. $\beta = 0$, the distribution of angles becomes uniformly distributed and uncorrelated (see equation (179)) so that we recover the result (208) valid for a Poissonian distribution. Finally, we note that the variance diverges for $T \to T_c^+$. This result was previously obtained in [17,54] (expressed in terms of the variance of the force $\langle F^2 \rangle = \langle B^2 \rangle/2$) from the second equation of the YBG hierarchy.

In the inhomogeneous phase $B \neq 0$, the variance of the magnetization is

$$\langle (\Delta B)^2 \rangle_{CE} = \frac{1}{3\beta F''(B)}.$$  (208)

Using equations (204) and (205), we obtain

$$\frac{1}{\lambda'(B)} = \frac{kM}{2\pi} - \frac{2\pi B^2}{kM}.$$  (209)

According to equation (59), we have

$$\beta F''(B) = \frac{8\pi^2}{k^2M} \left(2 - \frac{T_c}{T} - \frac{8\pi^2 B^2}{k^2M^2} - \frac{T_c}{T}\right).$$  (210)

Therefore, the variance of the magnetization is given by equations (208) and (210) where $B(T)$ is given by equation (59). Figure 7 shows that $F''(B(T))$ is always positive so that the inhomogeneous phase is always a maximum of free energy in agreement with the graphical construction of section 4.2.4.

Let us finally consider the distribution of the magnetization in the microcanonical ensemble given by equation (113). The second derivative of $S(B)$ is related to the second derivative of free energy by equation (120).

In the homogeneous phase $B = 0$, the variance of the magnetization is

$$\langle B^2 \rangle_{MCE} = -\frac{2}{S''(0)}. $$  (211)

Using equations (120) and (206), we obtain

$$S''(0) = -\beta F''(0) = -\frac{8\pi^2}{k^2M} \left(1 - \frac{T_c}{T}\right).$$  (212)

Using equation (113) with $B = 0$, this can be expressed in terms of the energy as

$$S''(0) = \frac{8\pi^2}{k^2M} \left(1 - \frac{E_c}{E}\right).$$  (213)

Therefore, using equation (211), we obtain

$$\langle B^2 \rangle_{MCE} = \frac{k^2M}{4\pi^2} \frac{1}{1 - \frac{E_c}{E}}.$$  (214)

We note that, according to equation (214), the homogeneous phase is a maximum of entropy ($S''(0) < 0$) for $E > E_c$ and a minimum of entropy ($S''(0) > 0$) for $E < E_c$ in agreement with the discussion of section 4.2.4. On the other hand, the variances of the magnetization in canonical and microcanonical ensembles coincide: $\langle B^2 \rangle_{CE} = \langle B^2 \rangle_{MCE}$.

In the inhomogeneous phase $B \neq 0$, the variance of the magnetization is

$$\langle (\Delta B)^2 \rangle_{MCE} = -\frac{1}{S''(E)}.$$  (215)

Using equations (120) and (210), we obtain

$$S''(B) = -\frac{8\pi^2}{k^2M} \left(2 - \frac{1}{T_c} - \frac{8\pi^2 B^2}{k^2M^2} - \frac{T_c}{T} + \frac{B^2}{T^2}\right).$$  (216)

Therefore, the variance of the magnetization is given by equations (215), (216) where $B(T)$ is given by equation (117). It can be expressed in terms of the energy by using equation (113). Figure 10 shows that $S''(B)$ is always negative so that the inhomogeneous phase is always a maximum of entropy in agreement with the results of section 4.2.4. The variances of the magnetization in canonical and microcanonical ensembles do not coincide in the inhomogeneous phase: $\langle (\Delta B)^2 \rangle_{CE} \neq \langle (\Delta B)^2 \rangle_{MCE}$.

In order to represent these results graphically (see Figures 6 and 10), it is convenient to introduce the dimensionless variables defined in section 2. The variance of the magnetization in the homogeneous phase is

$$N(b^2) = \frac{1}{1 - \eta} = \frac{1}{1 - 1/e}.$$  (217)
both in canonical and microcanonical ensembles. The variance of the magnetization in the inhomogeneous phase is

\[ N((\Delta b)^2)_{CE} = \frac{1}{1 - \frac{1}{2\eta} - 2\eta}, \quad (218) \]

in the canonical ensemble and

\[ N((\Delta b)^2)_{MCE} = \frac{1}{1 - \frac{1}{4\eta^2} - 2\eta + 8b^2\eta^2}, \quad (219) \]

in the microcanonical ensemble. In the microcanonical ensemble, it can be expressed in terms of the energy, using \( \epsilon = 1/\eta - 2b^2 \). Using the asymptotic expansions of Appendix A, we obtain close to the bifurcation point \((\epsilon, \eta) \to (1^-, 1^+)\):

\[ N((\Delta b)^2)_{CE} \sim \frac{1}{4(\eta-1)} \sim \frac{5}{4(1-\epsilon)}, \quad (220) \]

\[ N((\Delta b)^2)_{MCE} \sim \frac{1}{20(\eta-1)} \sim \frac{1}{4(1-\epsilon)}, \quad (221) \]

On the other hand, close to the ground state \((\epsilon, \eta) \to (-2^+, +\infty)\):

\[ N((\Delta b)^2)_{CE} \sim \frac{1}{8\eta^2} \sim \frac{(\epsilon + 2)^2}{32}, \quad (222) \]

\[ N((\Delta b)^2)_{MCE} \sim \frac{1}{16\eta^2} \sim \frac{(\epsilon + 2)^2}{64}. \quad (223) \]

We emphasize that although the ensembles are equivalent regarding the caloric curve \( \beta(E) \), the variance of the magnetization in the inhomogeneous phase differs in the two ensembles. Therefore, numerical simulations of the isolated HMF model (microcanonical ensemble) and of the dissipative BMF model (canonical ensemble) should lead to different values of \((\langle \Delta B \rangle^2)\).

References

1. Dynamics and thermodynamics of systems with long range interactions, edited by T. Dauxois et al., Lecture Notes in Physics 602, (Springer, 2002)
2. Dynamics and thermodynamics of systems with long range interactions: Theory and experiments, edited by A. Campa et al., AIP Conf. Proc. 970 (AIP, 2008).
3. Long-Range Interacting Systems, edited by T. Dauxois, S. Ruffo and L. Cugliandolo, Les Houches Summer School 2008, (Oxford: Oxford University Press, 2009)
4. A. Campa, T. Dauxois, S. Ruffo, Physics Reports 480, 57 (2009)
5. D. Lynden-Bell, Mon. Not. R. astr. Soc. 136, 101 (1967)
6. R. Ellis, K. Haven, B. Turkington, J. Stat. Phys. 101, 999 (2000)
7. F. Bouchet, J. Barré, J. Stat. Phys. 118, 1073 (2005)
8. T. Padmanabhan, Phys. Rep. 188, 285 (1990)
9. J. Katz, Found. Phys. 33, 223 (2003)
10. P.H. Chavanis, Int. J. Mod. Phys. B 20, 3113 (2006)
11. J. Messer, H. Spohn, J. Stat. Phys. 29, 561 (1982)
12. T. Konishi, K. Kaneko, J. Phys. A 25, 6283 (1992)
13. S. Inagaki, T. Konishi, Publ. Astron. Soc. Japan 45, 733 (1993)
14. S. Inagaki, Prog. Theor. Phys. 90, 557 (1993)
15. C. Pichon, PhD thesis, Cambridge (1994)
16. M. Antoni, S. Ruffo, Phys. Rev. E 52, 2361 (1995)
17. P.H. Chavanis, J. Vatteville, F. Bouchet, Eur. Phys. J. B 46, 61 (2005)
18. P.H. Chavanis, Eur. Phys. J. B 53, 487 (2006)
19. A. Antoniazzi, D. Fanelli, J. Barré, P.H. Chavanis, T. Dauxois, S. Ruffo, Phys. Rev. E 75, 011112 (2007)
20. A. Antoniazzi, D. Fanelli, S. Ruffo, Y. Yamaguchi, Phys. Rev. Lett. 99, 040601 (2007)
21. P.H. Chavanis, G. De Ninno, D. Fanelli, S. Ruffo, Out of equilibrium phase transitions in mean field Hamiltonian dynamics in Chaos, Complexity and Transport: Theory and Applications, edited by C. Chandre, X. Leoncini, G. Zaslavsky (World Scientific 2008)
22. F. Staniscia, P.H. Chavanis, G. de Ninno, D. Fanelli, Phys. Rev. E 80, 021138 (2009)
23. P.H. Chavanis, Physica A 365, 102 (2006)
24. P.H. Chavanis, A. Campa, to appear in EPJB [arXiv:1001.2109]
25. Y.Y. Yamaguchi, J. Barré, F. Bouchet, T. Dauxois, S. Ruffo, Physica A 337, 36 (2004)
26. M.Y. Choi, J. Choi, Phys. Rev. Lett. 91, 124101 (2003)
27. K. Jain, F. Bouchet, D. Mukamel, J. Stat. Mech. P11008 (2007)
28. P.H. Chavanis, L. Delfini, Eur. Phys. J. B 69, 389 (2009)
29. A. Campa, P.H. Chavanis, J. Stat. Mech. P06001 (2010)
30. J. Barré, F. Bouchet, T. Dauxois, S. Ruffo, J. Stat. Phys. 119, 677 (2005)
31. J. Katz, Mon. Not. R. astr. Soc. 183, 765 (1978)
32. V. Latora, A. Rapisarda, C. Tsallis, Physica A 305, 129 (2002)
33. D.D. Holm, J.E. Marsden, T. Ratiu, A. Weinstein, Phys. Rep. 123, 1 (1985)
34. P.H. Chavanis, AIP Conf. Proc. 970, 39 (2008)
35. J. Binney, S. Tremaine, Galactic Dynamics (Princeton Series in Astrophysics, 1987)
36. P.H. Chavanis, A&A 451, 109 (2006)
37. P.H. Chavanis, L. Delfini, Phys. Rev. E 81, 051103 (2010)
38. P.H. Chavanis, Eur. Phys. J. B 62, 179 (2008)
39. P.H. Chavanis, [arXiv:1002.0291]
40. F. Bouchet, Physica D 237, 1978 (2008)
41. P.H. Chavanis, Eur, Phys. J. B 70, 73 (2009)
42. M. Kiessling, Rev. Math. Phys. 21, 1145 (2009)
43. P.H. Chavanis, C. Sire, Phys. Rev. E 69, 016116 (2004)
44. P.H. Chavanis, Physica A 361, 81 (2006)
45. R.S. Ellis, K. Haven, B. Turkington, Nonlinearity 15, 239 (2002).
46. J.R. Ipser, G. Horwitz, Astrophys. J. 232, 863 (1979)
47. R.S. Ellis, Entropy, Large Deviations, and Statistical Mechanics (Springer-Verlag, New York 1985)
48. H. Touchette, Phys. Rep. 478, 1 (2009)
49. V.A. Antonov, Vest. Leningr. Gos. Univ. 7, 135 (1962).
50. D. Lynden-Bell, R. Wood, Mon. not. R. astron. Soc. 138, 495 (1968)
51. D. Montgomery, G. Joyce, Phys. Fluids 17, 1139 (1974)
52. G. Horwitz, J. Katz, Astrophys. J. 211, 226 (1977)
53. de Vega, H.J., Sanchez, N.: Nucl. Phys. B 625, 409 (2002)
54. P.H. Chavanis, Physica A 361, 55 (2006)
55. M. Antoni, S. Ruffo, A. Torcini, Europhys. Lett., 66, 645 (2004)
56. P.H. Chavanis, Astron. Astrophys. 432, 117 (2005)