IN Variant BASIS NUMBER FOR $C^\ast$-ALGEBRAS

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Abstract. We develop the ring-theoretic notion of Invariant Basis Number in the context of unital $C^\ast$-algebras and their Hilbert $C^\ast$-modules. Characterization of $C^\ast$-algebras with Invariant Basis Number is given in $K$-theoretic terms, closure properties of the class of $C^\ast$-algebras with Invariant Basis Number are given, and examples of $C^\ast$-algebras both with and without the property are explored. For $C^\ast$-algebras without Invariant Basis Number, we determine structure in terms of a “Basis Type” and describe a class of $C^\ast$-algebras which are universal in an appropriate sense. We conclude by investigating properties which are strictly stronger than Invariant Basis Number.

1. Introduction

Leavitt [8], [9] investigated unital rings $R$ with the property that any free module $X$ over $R$ has a fixed basis size. Rings with this property are said to have Invariant Basis Number and examples of such include commutative and Noetherian rings. Leavitt characterizes [9, Corollary 1] rings with Invariant Basis Number in the following manner: a ring $R$ has Invariant Basis Number if and only if there exists another ring $R'$ with Invariant Basis Number and a unital homomorphism $\phi : R \rightarrow R'$. For rings without Invariant Basis Number, Leavitt assigns [9, Theorem 1] a pair of positive integers he terms the “module type” of the ring. Constructions [7], [8], [9] of rings, termed Leavitt algebras $L_K(m,n)$, with arbitrary module type are given.

The fundamental structure of the Leavitt algebras has appeared in some surprising contexts. The algebra $L_K(1,n)$ given by Leavitt [9, Section 3] is the purely algebraic analogue of the Cuntz $C^\ast$-algebra $O_n$ and pre-dates Cuntz’s investigations. Indeed, the close connection between Leavitt algebras and
Cuntz algebras inspired the formulation of Leavitt Path Algebras associated to graphs, which act as analogues to graph $C^*$-algebras. General Leavitt algebras $L_K(m,n)$ have been investigated by Ara and Goodearl [3] in the context of “separated” Leavitt Path Algebras. Several $C^*$-algebraic versions of the Leavitt algebras $L_K(m,n)$ have been recently used in the work of Ara and Exel [1], [2] related to dynamical systems.

In this paper, we will formulate the property of Invariant Basis Number in the context of $C^*$-algebras and their Hilbert $C^*$-modules. Using $K$-theoretic tools, we are able to formulate an improved characterization of $C^*$-algebras with Invariant Basis Number in Theorem 3.2. We reproduce in Theorem 4.1 Leavitt’s type-classification for $C^*$-algebras without Invariant Basis Number and prove in Theorem 5.1 that each Basis Type is possible for some $C^*$-algebra. In Section 5, we determine that the $C^*$-algebras $U_{m,n}$ studied by McClanahan [10] are universal objects for $C^*$-algebras without Invariant Basis Number and, as such, are the correct analogue of the Leavitt algebras $L_K(m,n)$. Finally, we will investigate several stronger variations of Invariant Basis Number as proposed in the purely algebraic case by Cohn [4].

2. $C^*$-module preliminaries

We will always assume our $C^*$-algebras to be unital and denote the unit by 1 or $1_A$. A $C^*$-module $X$ over a $C^*$-algebra $A$ (more briefly, an $A$-module) is a complex vector space which is a right $A$-module and is equipped with an $A$-valued inner-product $\langle \cdot, \cdot \rangle : X \times X \to A$ which is $A$-linear in the second coordinate and $A$-adjoint-linear in the first coordinate. If $X$ is complete with respect to the norm $\|x\| = \|\langle x, x \rangle\|_A^{\frac{1}{2}}$ then it is termed a Hilbert $A$-module. We will use Wegge-Olsen [16, Chapter 15] as a reference for basic Hilbert $C^*$-module results.

The space of adjointable $A$-module homomorphisms between two $A$-modules $X$ and $Y$ will be denoted $L(X,Y)$. An adjointable homomorphism $\phi$ is unitary if it is bijective and isometric, that is, $\langle x, x' \rangle_X = \langle \phi(x), \phi(x') \rangle_Y$ for all $x, x' \in X$. We will say that $X$ and $Y$ are unitarily equivalent, and write $X \simeq Y$, if there exists a unitary in $L(X,Y)$.

An $A$-module $X$ is algebraically finitely generated if there exist $x_1, \ldots, x_n \in X$ such that $X = \text{span}_A(x_1, \ldots, x_n)$. We will never consider the weaker notion of topological finite generation, and so will omit the term “algebraically” in the remainder. An $A$-module $X$ is projective if it is a direct summand of a free $A$-module. It is a known result ([16, Theorem 15.4.2] for example) that a finitely generated projective $A$-module is isomorphic (as an $A$-module) to a Hilbert $A$-module. Further, the finitely generated projective Hilbert $A$-modules are all of the form $pA^n$ for some $n \geq 1$ and some matrix projection $p \in M_n(A)$. 
We will denote the set of projections in $M_n(A)$ by $P_n(A)$. For $p \in P_n(A)$ and $q \in P_m(A)$ we will set $p \oplus q = \text{diag}(p,q) \in P_{n+m}(A)$. We will say $p$ and $q$ are stably equivalent if there is a matrix projection $r$ for which $p \oplus r \sim q \oplus r$, where "\sim" denotes (Murray–von Neumann) equivalence in $P_\infty(A) = \bigcup_{n=1}^{\infty} P_n(A)$. The stable equivalence class of $p$ will be denoted $[p]_0$ and considered as an element of the group $K_0(A)$. The (additive) order of an element $[p]_0 \in K_0(A)$ will be denoted $| [p]_0 |_{K_0(A)}$ or $| [p]_0 |$ if the $C^*$-algebra $A$ is clear from context.

3. Invariant Basis Number

Let $A$ be a unital $C^*$-algebra. The finitely generated free $A$-module of rank $n$ is $A^n := A \oplus \cdots \oplus A$ where there are $n$ summands. The action of $A$ on $A^n$ is coordinate-wise multiplication on the right and the inner-product is given by $\langle (a_1,\ldots,a_n),(b_1,\ldots,b_n) \rangle = a_1^*b_1 + \cdots + a_n^*b_n$. Although we write them as tuples, that is, row vectors, it is often beneficial to view elements of $A^n$ instead as column vectors. The coordinate projections $\pi_i : A^n \to A$ defined by $\pi_i(a_1,\ldots,a_n) = a_i$ are bounded, contractive, adjointable $A$-module homomorphisms. Therefore, a Cauchy sequence in $A^n$ is Cauchy in each coordinate and hence, as $A$ itself is complete, converges in each coordinate. Thus $A^n$ is a complete (i.e. Hilbert) $A$-module. In keeping with the literature, free Hilbert $A$-modules will henceforth be referred to as standard $A$-modules, where the completeness is understood.

The fundamental question we will consider is this: under what conditions are the standard modules distinct from one another? We will make this notion of distinctness precise with the next definition.

**Definition 3.1.** A $C^*$-algebra $A$ has Invariant Basis Number (hereafter, has IBN) if

$$A^n \simeq A^m \iff n = m.$$  

Unitary equivalence is, in general, a stronger condition than $A$-module isomorphism. In fact, unitaries are precisely the isometric $A$-module isomorphisms. However, in the case of standard modules every $A$-module homomorphism $\phi : A^n \to A^m$ may be represented as a $m \times n$ matrix with elements in $A$ and so is automatically adjointable. Therefore, if $\phi : A^n \to A^m$ is an $A$-module isomorphism then the Polar Decomposition [16, Theorem 15.3.7] yields a unitary in $L(A^n,A^m)$. We have formulated the definition in terms of unitary equivalence, rather than module isomorphism, to emphasize the Hilbert structure of the standard modules.

A matrix $U \in M_{m,n}(A)$ will be termed a unitary if $UU^* = I_n$ and $U^*U = I_m$. As noted above, we may identify $L(A^n,A^m)$ with $M_{m,n}(A)$ and a unitary homomorphism in $L(A^n,A^m)$ corresponds to a unitary matrix in $M_{m,n}(A)$. The definition of Invariant Basis Number may thus be rephrased as follows: $A$ has IBN if and only if every unitary matrix over $A$ is square.
Example. It is not hard to verify that a matrix with entries in an commutative algebra is invertible if and only if it is square. Hence, commutative $C^*$-algebras have Invariant Basis Number.

The connection between matrices and Invariant Basis Number gives our first main result.

Theorem 3.2. A $C^*$-algebra $A$ has IBN if and only if the group element $[1_A]_0 \in K_0(A)$ has infinite order.

Proof. If $A$ does not have IBN, then $A^n \simeq A^m$ for some $n > m > 0$ and hence there is a unitary matrix in $M_{m,n}(A)$. This unitary implements the (Murray–von Neumann) matrix equivalence of the projections $I_m$ and $I_n$ and consequently we have

$$I_{n-m} \oplus I_m \sim I_n \sim I_m \sim 0 \oplus I_m.$$  

Thus, $I_{n-m}$ is stably equivalent to 0, that is, $(n-m)[1_A]_0 = [I_{n-m}]_0 = 0$, and so $[1_A]_0$ has finite order.

Conversely, if $[1_A]_0$ has finite order $k$ then $I_k$ is stably equivalent to 0, that is, there exists $N > 0$ and $p \in P_N(A)$ such that $p \oplus I_k \sim p \oplus 0 \sim p$. As $I_N \sim p \oplus (I_N - p)$ we have

$$I_N \oplus I_k \sim (I_N - p) \oplus p \oplus I_k \sim (I_N - p) \oplus p \sim I_N$$  

and so $I_{N+k} \sim I_N$. The matrix implementing this equivalence is unitary and thus corresponds to a unitary homomorphism from $A^N$ to $A^{N+k}$. Since $k > 0$ we must conclude that $A$ does not have IBN. □

It is hinted in the above proof that when a $C^*$-algebra does not have IBN the order of $[1_A]_0$ yields information about equivalence of standard modules. We shall make this connection clear in Section 4 when we turn our attention fully to $C^*$-algebras without IBN.

The $K$-theoretic description of IBN immediately expands the class of $C^*$-algebras with that property beyond the commutative. In particular, it is well-known (see [14], for example) that stably-finite $C^*$-algebras, that is, those without any proper matrix isometries, have a totally ordered $K_0$ group. Further, in this case the element $[1_A]_0$ is an order unit for $K_0$ in the sense that for any $g \in K_0$ there is a positive integer $k$ for which $-k[1_A]_0 < g < k[1_A]$. It follows that $[1_A]_0$ cannot have a finite order and, applying Theorem 3.2, we conclude that a stably-finite $C^*$-algebra must have IBN. We would like to remark that this could also be inferred from the matricial description of IBN, as any rectangular unitary could be “cut down” to a square proper isometry.

The functorial properties of $K_0$ also yield the following result which will be used extensively to demonstrate closure properties for the class of $C^*$-algebras with IBN.
Proposition 3.3. A $C^*$-algebra $A$ has IBN if and only if there exists a $C^*$-algebra $B$ which has IBN and a unital $*$-homomorphism $\phi : A \to B$.

Proof. Necessity is easily satisfied by letting $B = A$ and $\phi = id_A$.

To show sufficiency, we note that the functorial properties of $K_0$ induce a group homomorphism $K_0(\phi) : K_0(A) \to K_0(B)$. Since $\phi$ is unital we have $K_0(\phi)[1_A]_0 = [1_B]_0$. If $B$ has IBN then $[1_B]_0$ has infinite order in $K_0(B)$ and so its preimage $[1_A]_0$ must have infinite order in $K_0(A)$. Thus $A$ has IBN. □

The above statement mirrors the purely algebraic characterization of rings with IBN given by Leavitt [9, Corollary 1].

The proposition has immediate consequences for the closure properties of the class of $C^*$-algebras with Invariant Basis Number.

Corollary 3.4. IBN is preserved under direct sums and unital extensions.

Proof. Suppose that $A$ is a $C^*$-algebra with IBN. If $B$ is a unital $C^*$-algebra then the coordinate map $a \oplus b \mapsto a$ is a unital $*$-homomorphism and thus $A \oplus B$ has IBN.

If $B$ is any unital extension of $A$, then there exists a $C^*$-algebra $C$ and a short exact sequence

$$0 \to C \to B \xrightarrow{\phi} A \to 0.$$ 

Of course $\phi$ is a surjective $*$-homomorphism, hence is unital, and thus $B$ has IBN. □

Note that a direct sum inherits IBN even if only one of the summands has that property. We conclude our discussion of $C^*$-algebras with IBN by leveraging the results to find non-commutative, non-stably-finite $C^*$-algebras which have IBN.

Example. Consider the Cuntz algebra $O_{\infty}$, the universal $C^*$-algebra generated by a countable family of isometries with pairwise disjoint ranges. Since $O_{\infty}$ contains proper isometries it is certainly neither commutative nor (stably) finite. However, it is a classical result of Cuntz [5, Corollary 3.11] that $K_0(O_{\infty}) = \mathbb{Z}$ and is generated by $[1]_0$. Thus by, Theorem 3.2, $O_{\infty}$ has IBN.

Example. On the opposite end of the spectrum, consider the Toeplitz algebra $T$, the universal $C^*$-algebra generated by a single non-unitary isometry. Of course $T$ is neither commutative nor (stably) finite but is well known to be an extension of the commutative $C^*$-algebra $C(\mathbb{T})$ by the compact operators $K$. Thus by Corollary 3.4, $T$ has IBN.

3.1. A remark on the non-unital case. It is a perfectly legitimate criticism that we are dealing solely with unital $C^*$-algebras. Let us briefly describe why we wish to avoid the nonunital case.

Suppose that $A$ is a nonunital $C^*$-algebra. Unlike in the unital case, the adjointable $A$-module homomorphisms in $L(A^n, A^m)$ are not identified with
$M_{m,n}(A)$, but rather with $m \times n$ matrices over the multiplier algebra of $A$, which we'll denote by $\mathcal{M}(A)$. Of course $\mathcal{M}(A)$ is, practically by definition, unital. The unitary equivalence $A^n \simeq A^m$ thus implies the existence of a unitary matrix in $M_{m,n}(\mathcal{M}(A))$ and so $\mathcal{M}(A)^n \simeq \mathcal{M}(A)^m$. It is not hard to see that the logic is reversible and so $A^n \simeq A^m$ if and only if $\mathcal{M}(A)^n \simeq \mathcal{M}(A)^m$.

As a consequence of the above reasoning, we see that the statement "$A^n \simeq A^m$ if and only if $n = m$" is equivalent to "$\mathcal{M}(A)$ has IBN." This is what we believe should be the working definition of IBN for nonunital $C^*$-algebras. In fact, since $\mathcal{M}(A) = A$ when $A$ is unital, it agrees with our unital definition.

Unfortunately, we do not feel this definition to be particularly useful. First, many nice properties of a $C^*$-algebra are not preserved in its multiplier algebra. Separability being a prime example. Second, we do not know of a method, outside a very few special cases, to detect information about $K_0(\mathcal{M}(A))$ based on information about $A$. Since our main tools are $K$-theoretic this is a major stumbling block.

4. $C^*$-algebras without Invariant Basis Number

We now turn our attention to those unital $C^*$-algebras which lack the Invariant Basis Number property. By Theorem 3.2, we may conclude that $C^*$-algebras $A$ without IBN are characterized by having a finite order for the element $[1_A]_0 \in K_0(A)$. A particularly tractable case is when $[1_A]_0$ has order 1, i.e. is the zero element of $K_0(A)$.

Example. When $H$ is an infinite dimensional Hilbert space $B(H)$ does not have IBN because $K_0(B(H)) = \{0\}$.

Example. The Cuntz algebra $\mathcal{O}_2$ is the universal $C^*$-algebra generated by two isometries $v_1$ and $v_2$ satisfying $v_1 v_1^* + v_2 v_2^* = 1$ and $v_1^* v_2 = v_2^* v_1 = 0$. A result of Cuntz [5, Theorem 3.7] is that $K_0(\mathcal{O}_2) = \{0\}$ and so $\mathcal{O}_2$ does not have IBN. In fact, we can concretely see the equivalence $\mathcal{O}_2 \simeq \mathcal{O}_2^2$ via the map $(a,b) \mapsto v_1 a + v_2 b$ which extends to a unitary homomorphism and corresponds to the $1 \times 2$ unitary matrix $[v_1 v_2]$.

Example. For a slightly less trivial example, consider the Cuntz algebra $\mathcal{O}_3$. We have that $K_0(\mathcal{O}_3) = \mathbb{Z}/2\mathbb{Z}$ and is in fact generated by $[1]_0$. Thus $\mathcal{O}_3$ does not have IBN. Much like for $\mathcal{O}_2$ we can in fact write down a $1 \times 3$ unitary matrix $[v_1 v_2 v_3]$ which gives the unitary equivalence $\mathcal{O}_3 \simeq \mathcal{O}_3^3$. Of course in general we have $K_0(\mathcal{O}_n) = \mathbb{Z}/(n-1)\mathbb{Z}$ and so no Cuntz algebra has IBN.

Recalling the definition of Invariant Basis Number, a $C^*$-algebra lacks IBN precisely when two or more standard modules with differing ranks are equivalent. The restrictions on when such equivalence may occur give some structural information for $C^*$-algebras without IBN. The precise nature of that information is contained in our next main result.
Theorem 4.1. If $A$ is a $C^*$-algebra without IBN, then there exists a unique largest positive integer $N$ and a unique smallest positive integer $K$ satisfying:

1. if $n, m \geq 1$, $n < N$, and $A^n \simeq A^m$ then $n = m$, and
2. if $n, m \geq 1$ and $A^n \simeq A^m$ then $(n - m) \equiv 0 \mod K$.

This result is comparable to [9, Theorem 1]. The first condition characterizes $N$ as the least rank for which distinctness of the standard $A$-modules fails: all standard $A$-modules of rank less than $N$ are distinct. The second condition characterizes $K$ as the minimum "jump" in rank possible between equivalent standard $A$-modules.

Definition 4.2. If $A$ is a $C^*$-algebra without IBN, then the pair $(N, K)$ given by Theorem 4.1 is the Basis Type of $A$. For notational purposes we may write $\text{type}(A) = (N, K)$ or $(N_A, K_A)$.

Proof of Theorem 4.1. Since $A$ does not have IBN there are at least two distinct ranks $n, m$ for which $A^n \simeq A^m$. In particular, the set $E := \{j \geq 0 : \exists k \neq j \text{ s.t. } A^j \simeq A^k\}$ is nonempty and so $N := \min\{n : n \in E\}$ is well defined. If $n \lt N$ then $n \notin E$ and so $A^n \simeq A^m$ only if $m = n$. So our choice of $N$ satisfies the first condition. That our $N$ is the largest possible is immediate, since if $N' > N$ then there is at least one rank $(N$ itself) less than $N'$ for which the first condition does not hold.

Let $N$ be as above and define $K = \min\{k > 0 : A^N \simeq A^{N+k}\}$, which exists by our choice of $N$. Note that for any $n \geq N + K$ we have

$A^n = A^{n-N-K+N+K} \simeq A^{n-N-K} \oplus A^{N+K} \simeq A^{n-N-K} \oplus A^N \simeq A^{n-K}$.

Through iteration of this process, we obtain an integer $n'$ satisfying $N \leq n' < N + K$, $n' \equiv n \mod K$, and $A^{n'} \simeq A^n$. Because of this, it is enough to check a simpler version of the second condition: if $A^n \simeq A^m$ for $N \leq n, m < N + K$ then $n = m$. (Note this will guarantee the minimality of $K$.) Suppose that $n, m$ are two ranks satisfying the simplified hypothesis but with $m > n$. Then

$A^n \simeq A^{N+K} \simeq A^{N+K-m} \oplus A^m \simeq A^{N+K-m} \oplus A^n \simeq A^{N+K-(m-n)}$

and, as $K - (m-n) < K$, we have contradicted the minimality of $K$. \hfill \Box

The Basis Type of a $C^*$-algebra determines the equivalences of standard modules. In particular, if $\text{type}(A) = (N, K)$ then there are precisely $N + K$ unitary equivalence classes of standard modules: the distinct ones of rank less than $N$ and the $K$ classes for ranks $N, N+1, \ldots, N + K - 1$.

Example. Revisiting the examples from the beginning of the section, we find that $B(H)$ and $O_2$ both have Basis Type $(1,1)$. The Cuntz algebra $O_3$ is of Basis Type $(1,2)$ since (as may be checked) $O_3 \not\simeq O_3^2$ but $O_3 \simeq O_3^3$.

Recalling that $K_0(O_2) = K_0(B(H)) = 0$ while $K_0(O_3) = \mathbb{Z}/2\mathbb{Z}$ the following proposition is perhaps unsurprising.
Proposition 4.3. If $A$ is a $C^*$-algebra with Basis Type $(N, K)$, then the order of $[1_A]_0$ in $K_0(A)$ is equal to $K$.

Proof. Since $A$ does not have IBN the element $[1_A]_0$ must have some finite order $J$. Since $A^N \simeq A^{N+K}$ by definition of the Basis Type we conclude that $I_N$ and $I_{N+K}$ are (Murray–von Neumann) equivalent matrix projections; consequently we have $K[1_A]_0 = [I_K]_0 = 0$ in $K_0(A)$ and thus $K \equiv 0 \mod J$. Re-examination of the proof for Theorem 3.2 yields that as $J[1_A]_0 = 0$ there exists some $M$ such that $I_{M+J} \sim I_M$, i.e. $A^M \simeq A^{M+J}$. Thus, by definition of $K$, we have $J \equiv 0 \mod K$. We must then conclude that $J = K$, as desired. □

Following Leavitt [9, Section 2], we will give the Basis Types a lattice structure as follows:

$$\begin{align*}
(N_1, K_1) \leq (N_2, K_2) & \iff N_1 \leq N_2 \text{ and } K_2 \equiv 0 \mod K_1, \\
(N_1, K_1) \lor (N_2, K_2) & = (\max(N_1, N_2), \text{lcm}(K_1, K_2)), \\
(N_1, K_1) \land (N_2, K_2) & = (\min(N_1, N_2), \text{gcd}(K_1, K_2)).
\end{align*}$$

We are able to relate this lattice structure to various algebraic operations primarily through the following proposition.

Proposition 4.4. Let $A$ and $B$ be $C^*$-algebras, $A$ without IBN, and $\phi : A \to B$ a unital *-homomorphism. Then $B$ is without IBN and $\text{type}(B) \leq \text{type}(A)$.

Proof. Note that by Proposition 3.3 $B$ cannot have IBN. Let $\text{type}(A) = (N_A, K_A)$ and $\text{type}(B) = (N_B, K_B)$. The functoriality of $K_0$ induces a group homomorphism $K_0(\phi) : K_0(A) \to K_0(B)$ which takes $[1_A]_0$ to $[1_B]_0$. Being a group homomorphism, it follows that the order of $K_0(\phi)[1_A]_0 \in K_0(B)$ must divide the order of $[1_A]_0 \in K_0(A)$. We thus have

$$[1_A]_0 \mid_{K_0(A)} \equiv 0 \mod [1_B]_0 \mid_{K_0(B)},$$

which combines with Proposition 4.3 to give us $K_A \equiv 0 \mod K_B$.

We may amplify $\phi$ to $\phi^{(m,n)} : M_{m,n}(A) \to M_{m,n}(B)$ by applying $\phi$ entry-wise. Since $\phi$ is unital any unitary matrix in $M_{m,n}(A)$ is sent, via $\phi^{(m,n)}$, to a unitary matrix in $M_{m,n}(B)$. Thus if $A^n \simeq A^m$ then so too $B^n \simeq B^m$; in particular we have $B^{NA} \simeq B^{NA+KA}$. By construction (see Theorem 4.1) $N_B = \min\{n : \exists j \neq n \text{ s.t. } B^n \simeq B^j\}$ and so we conclude that $N_B \leq N_A$. □

The primary utility of the previous proposition is to prove various closure properties of the class of $C^*$-algebras without IBN.

Corollary 4.5. If $A$ does not have IBN and $B$ is a quotient of $A$, then $B$ does not have IBN.

This is Proposition 4.4 applied to the quotient map.
Corollary 4.6. If $A$ and $B$ are $C^*$-algebras without IBN then $\text{type}(A \oplus B) = \text{type}(A) \vee \text{type}(B)$.

Proof. Proposition 4.4 applied to the coordinate projections $(a, b) \mapsto a$ and $(a, b) \mapsto b$ has us conclude that $\text{type}(A) \leq \text{type}(A \oplus B)$ and $\text{type}(B) \leq \text{type}(A \oplus B)$, and so $\text{type}(A) \vee \text{type}(B) \leq \text{type}(A \oplus B)$.

As $K_0(A \oplus B) = K_0(A) \oplus K_0(B)$ we use Proposition 4.3 to conclude that $K_{A \oplus B} = \text{lcm}(K_A, K_B)$.

Suppose, without loss of generality, that $\max(N_A, N_B) = N_A$. With $K_{A \oplus B} = \text{lcm}(K_A, K_B)$, we have

$$A^{N_A} \simeq A^{N_A + K_A} \simeq A^{N_A + 2K_A} \simeq \ldots \simeq A^{N_A + K_{A \oplus B}}$$

and, as $B^{N_A} \simeq B^{N_A - N_B} \oplus B^{N_B} \simeq B^{N_A - N_B} \oplus B^{N_B + K_B} \simeq B^{N_A + K_B}$, we have also

$$B^{N_A} \simeq B^{N_A + K_B} \simeq B^{N_A + 2K_B} \simeq \ldots \simeq B^{N_A + K_{A \oplus B}}.$$  

Consequently

$$(A \oplus B)^{N_A} = A^{N_A} \oplus B^{N_A} \simeq A^{N_A + K_{A \oplus B}} \oplus B^{N_A + K_{A \oplus B}} \simeq (A \oplus B)^{N_A + K_{A \oplus B}}.$$  

We conclude that $N_{A \oplus B} \leq N_A = \max(N_A, N_B)$. As $\text{type}(A) \wedge \text{type}(B) \leq \text{type}(A \oplus B)$, that is, $\max(N_A, N_B) \leq N_{A \oplus B}$, we have equality.

In conclusion, $N_{A \oplus B} = \max(N_A, N_B)$ and $K_{A \oplus B} = \text{lcm}(K_A, K_B)$ and so $\text{type}(A \oplus B) = \text{type}(A) \wedge \text{type}(B)$. \qed

In contrast to Corollary 3.4, it is quite necessary that neither summand of $A \oplus B$ has IBN. It is natural to suspect that the remaining lattice operation will correspond to tensor products.

Corollary 4.7. If $A$ and $B$ are $C^*$-algebras without IBN, then $\text{type}(A \otimes B) \leq \text{type}(A) \wedge \text{type}(B)$.

The proof of this corollary is nothing but Proposition 4.4 applied to the embeddings $a \mapsto a \otimes 1_B$ and $b \mapsto 1_A \otimes b$. Two remarks are in order: first, that the result holds for any norm structure we may place on $A \otimes B$; second, that it is unknown (even, to our knowledge, in the purely algebraic case) whether inequality ever occurs.

Corollary 4.8. If $\{A_i, \phi_i\}$ is an inductive system of $C^*$-algebras, each without IBN, and each $\phi_i$ is unital, then the direct limit $C^*$-algebra $A$ of the system does not have IBN.

The proof of this corollary is Proposition 4.4 applied to the universal maps $\psi_i : A_i \to A$, which are unital.

Finally, we will demonstrate that the class of $C^*$-algebras without Invariant Basis Number is unfortunately not closed under Morita equivalence. A good reference for the theory of Morita equivalence is [11]. Our motivating example is the algebra $\mathcal{O}_\infty$ and the fact that the identity of a corner $C^*$-algebra $p\mathcal{O}_\infty p$ is the projection $p$. 
Proposition 4.9. Let $A$ be a infinite simple unital $C^*$-algebra, then there is a $C^*$-algebra $B$ Morita equivalent to $A$ which does not have IBN.

Proof. If $A$ is infinite, then there exists a proper isometry $v \in A$. As $vv^* \sim v^*v = 1_A$ we have

$$[1_A]_0 = [A - vv^*]_0 + [vv^*]_0 = [1_A - vv^*]_0 + [1_A]_0$$

and so $[1_A - vv^*]_0 = 0$ in $K_0(A)$. Now consider the full corner $B = (1_A - vv^*)A(1_A - vv^*)$, which is Morita-equivalent to $A$ [11, Example 3.6], and note that $1_B = 1_A - vv^*$. Thus, $[1_B]_0 = 0$ in $K_0(B)$ and so $B$ does not have IBN.

Returning to the concrete example, $O_{\infty}$ is a unital simple infinite $C^*$-algebra. We have seen before that $O_{\infty}$ has IBN but now, by the above proposition, it contains many full corners which does not have IBN.

5. Universal algebras for Basis Types

A natural question stemming from the discussion of Basis Type is this: are all pairs $(N, K)$ of positive integers realized as the Basis Types of $C^*$-algebras? We shall answer this in the affirmative and further we will exhibit $C^*$-algebras which are “universal” for their Basis Type.

Our investigation will be motivated by the situation for the Basis Types $(1, K)$. If $\text{type}(A) = (1, K)$ then necessarily $A \simeq A^{K+1}$ and so there is a unitary $1 \times (K + 1)$ matrix, that is, a row unitary. The elements of such a matrix are isometries satisfying the Cuntz relations and so there is an induced unital $*$-homomorphism (in fact, an embedding) of $O_{K+1}$ into $A$. Now as $O_{K+1} \simeq O_{K+1}^1$ and $K_0(O_{K+1}) = \mathbb{Z}/K\mathbb{Z}$ we conclude via Proposition 4.3 that $\text{type}(O_{K+1}) = (1, K)$. We consider the Cuntz algebra $O_{K+1}$ “universal” for Basis Type $(1, K)$ in this sense: whenever $\text{type}(A) = (1, K)$ there is an induced unital $*$-homomorphism $\phi : O_{K+1} \to A$. We use the term universal loosely because this homomorphism is not necessarily unique. For example, when $A$ is itself a Cuntz algebra then $\phi$ can be given by any permutation of the generating isometries.

In [10], McClanahan investigated $C^*$-algebras $U^{nc}_{m,n}$ defined as follows:

$$U^{nc}_{m,n} := C^*(u_{ij} : U = [u_{ij}] \in M_{m,n} \text{ satisfies } UU^* = I_m, U^*U = I_n).$$

The $C^*$-algebra $U^{nc}_{m,n}$ has the universal property that whenever $A$ is a $C^*$-algebra with elements $\{a_{ij}\}$ such that $[a_{ij}] \in M_{m,n}(A)$ is unitary then there is a unital $*$-homomorphism $\phi : U^{nc}_{m,n} \to A$ with $\phi(u_{ij}) = a_{ij}$. Since there is a natural identification of $U^{nc}_{m,n}$ with $U^{nc}_{n,m}$ (taking $u_{ij}$ to $u_{ji}^*$) we shall only consider the cases where $n > m$.

Suppose that $A$ is a $C^*$-algebra with $\text{type}(A) = (N, K)$. Then by definition $A^N \simeq A^{N+K}$ and so there is an $N \times (N + K)$ unitary matrix over $A$. By the universal property we have a unital $*$-homomorphism $\phi : U^{nc}_{N,N+K} \to A$. 
Thus we may recast the universal property enjoyed by the $U^{nc}_{m,n}$ as follows: if $A$ is a $C^*$-algebra of Basis Type $(m, n - m)$ then there is a unital $\ast$-homomorphism $\phi : U^{nc}_{m,n} \to A$. McClanahan proved that $U^{nc}_{1,n} = \mathcal{O}_n$ and so there is no conflict with our previous discussion. He further demonstrated that $U^{nc}_{m,n}$ is not simple whenever $m > 0$ (there is always a unital $\ast$-homomorphism $\phi : U^{nc}_{m,n} \to \mathcal{O}_{n-m+1}$) and so, unlike for the Cuntz algebras, the universal property does not guarantee an embedding of $U^{nc}_{m,n}$ into a $C^*$-algebra when $m > 1$.

Since $U^{nc}_{m,n}$, by definition, has a unitary $m \times n$ matrix we conclude that its standard modules of ranks $n$ and $m$ are equivalent, and so $U^{nc}_{m,n}$ does not have IBN. Ara and Goodearl have recently shown in [3] that $K_0(U^{nc}_{m,n}) = \mathbb{Z}/(n-m)\mathbb{Z}$ (and is generated by $1$) and so by Proposition 4.3 we have that $\text{type}(U^{nc}_{m,n}) = (N, n - m)$ for some $N \leq m$. To prove that we have $N = m$, we shall exploit the universal property of $U^{nc}_{m,n}$ together with our next main result.

**Theorem 5.1.** For each pair $(N, K)$ of positive integers there is a $C^*$-algebra $A$ with $\text{type}(A) = (N, K)$.

**Proof.** We have already seen that for $K > 0$, $\text{type}(\mathcal{O}_{K+1}) = (1, K)$. As $(1, K) \lor (N, 1) = (N, K)$ we conclude by Corollary 4.6 that it is enough, given $N > 0$, to exhibit a $C^*$-algebra of Basis Type $(N, 1)$.

By combining [13, Theorem 3.5] and [12, Theorem 5.3] we may, for fixed $N > 0$, obtain a unital $C^*$-algebra $A$ with the following properties:

1. for $n < N$ the $C^*$-algebras $M_n(A)$ are finite,
2. for $m \geq N$ the $C^*$-algebras $M_m(A)$ are properly infinite, and
3. $K_0(A) = 0$.

Since $K_0(A) = 0$ it follows that from Theorem 3.2 and Proposition 4.3 that $A$ does not have IBN and has basis type $(N', 1)$ for some $N' > 0$. Since $K_0(M_N(A)) = K_0(A) = 0$ and $M_N(A)$ is properly infinite there is an embedding (see [15, Proposition 4.2.3]) of $\mathcal{O}_2$ into $M_N(A)$. Thus there is a $1 \times 2$ unitary matrix (with entries consisting of the images of the Cuntz isometries) over $M_N(A)$ which, viewed in a different light, is an $N \times 2N$ unitary matrix over $A$ itself. Thus $A^N \simeq A^{2N}$ and we conclude that $N' \leq N$. Suppose that $N' < N$. As $\text{type}(A) = (N', 1)$ we have $A^{N'} \simeq A^{N'+1}$ and so there is a unitary $N' \times (N' + 1)$ matrix. Deleting any one column from this matrix yields a $N' \times N'$ proper isometry, contradicting the fact that $M_{N'}(A)$ is finite. Hence, $N' = N$ and $\text{type}(A) = (N, 1)$. $\square$

We emphasize that the $C^*$-algebras in Theorem 5.1 (obtained from [12] and [13]) are not simple. Since the $C^*$-algebras $U^{nc}_{m,n}$ are also not simple in general, it is a question of some interest to us if Basis Types beyond $(1, K)$ are possible for simple $C^*$-algebras.

**Corollary 5.2.** $\text{type}(U^{nc}_{m,n}) = (m, n - m)$. 
This is obtained from Theorem 5.1, Proposition 4.4, and the universal property of \( U_{m,n}^{nc} \).

**Corollary 5.3.** \( U_{m,n}^{nc} = U_{m',n'}^{nc} \) if and only if \( n = n' \) and \( m = m' \).

Note that the Basis Types are able to distinguish the \( C^* \)-algebras \( U_{m,n}^{nc} \) and \( U_{m+1,n+1}^{nc} \) while the \( K \)-theory cannot: they share the same \( K_0 \) group, \( \mathbb{Z}/(n-m)\mathbb{Z} \), and both have trivial \( K_1 \) (see [3, Section 5]).

Finally, we are able to use the \( C^* \)-algebras \( U_{m,n}^{nc} \) to prove that IBN is preserved under inductive limits. In [10, Remark, p. 1066] McClanahan notes that \( U_{m,n}^{nc} \) is semiprojective in the sense of [6, Section 3]: that whenever \( \{ B_i \} \) is an inductive system of \( C^* \)-algebras with limit \( B \) and \( \phi : U_{m,n}^{nc} \to B \) is a unital \( * \)-homomorphism then there exists a unital \( * \)-homomorphism \( \phi_k : U_{m,n}^{nc} \to B_k \) for some \( k \).

**Proposition 5.4.** If \( \{ A_i, \phi_i \} \) is an inductive family of \( C^* \)-algebras, each with IBN and each \( \phi_i \) unital, then the \( C^* \)-algebraic direct limit \( A \) of the system has IBN.

**Proof.** If the limit \( A \) did not have IBN, then it must have some Basis Type \( (N,K) \). By the universal property there is a unital \( * \)-homomorphism \( \psi : U_{N,N+K}^{nc} \to A \) and hence also, because of the semiprojectivity, a unital \( * \)-homomorphism \( \psi_n : U_{N,N+K}^{nc} \to A_n \) for some \( n \). But, as \( A_n \) has IBN, we would then conclude by Proposition 3.3 that \( U_{N,N+K}^{nc} \) has IBN, a clear contradiction. □

### 6. Stronger notions

In [4], Cohn considered two ring-theoretic properties strictly stronger than Invariant Basis Number. The \( C^* \)-algebraic analogues are formulated below.

**Definition 6.1.** A \( C^* \)-algebra has IBN\(_1\) if, whenever \( n, m \) are integers and \( X \) an \( A \)-module, \( A^n \cong A^m \oplus X \) implies \( n \geq m \).

**Definition 6.2.** A \( C^* \)-algebra \( A \) has IBN\(_2\) if for all \( n > 0 \), \( A^n \cong A^n \oplus X \) for some \( A \)-module \( X \) implies \( X = 0 \).

The next proposition is nearly immediate.

**Proposition 6.3.** IBN\(_2\) \( \Rightarrow \) IBN\(_1\) \( \Rightarrow \) IBN.

**Proof.** Suppose \( A \) has IBN\(_2\). If \( n < m \) and \( A^n \cong A^m \oplus X \) for some \( A \)-module \( X \) then \( A^n \cong A^n \oplus A^{m-n} \oplus X \) and we conclude by IBN\(_2\) that \( A^{m-n} \oplus X = 0 \), i.e. \( m - n = 0 \) a contradiction. Suppose that \( A \) has IBN\(_1\). If \( A^n \cong A^m \) for \( n > m \) then \( A^m \cong A^n \oplus 0 \) and so \( n \leq m \), a contradiction. □

Our main goal for this section is twofold: first, to demonstrate that these properties are distinct; and second, to better characterize \( C^* \)-algebras satisfying the properties IBN\(_1\) and IBN\(_2\). This goal is easily accomplished for the property IBN\(_2\).
Theorem 6.4. A $C^*$-algebra $A$ has $\text{IBN}_2$ if and only if $A$ is stably finite.

Proof. Suppose that $A$ is not stably finite, that is, there is a proper isometry $V \in M_n(A)$ for some $n \geq 1$. Note that $I_n \sim VV^*$ and $I_n \sim I_n - VV^* \oplus VV^* \sim I_n - VV^* \oplus I_n$. Thus, $A^n \simeq A^n \oplus (I - VV^*)A^n$ where $(I_n - VV^*)A^n \neq 0$ as $V$ is proper. Thus, $A$ does not have $\text{IBN}_2$.

Suppose that $A$ does not have $\text{IBN}_2$. Then $A^n \simeq A^n \oplus X$ for some $n \geq 1$ and nontrivial $A$-module $X$. Note that the embedding $\iota : A^n \to A^n \oplus X$ is an adjointable $A$-module homomorphism which is isometric in the sense that $\iota^* \iota = I_n$. Let $U \in L(A^n \oplus X, A^n)$ be a unitary, then $V = U \circ \iota : A^n \to A^n$ is an adjointable $A$-module homomorphism with $V^*V = I_n$ and $VV^* = U(I_n \oplus 0)U^* \neq I_n$. Thus, $V$ corresponds to a $n \times n$ proper matrix isometry and $M_n(A)$ is not finite. \hfill \square

Since there are $C^*$-algebras with $\text{IBN}$ which are not stably finite (for example, the Toeplitz algebra) we conclude that $\text{IBN}_2$ is strictly stronger than $\text{IBN}$.

Although we do not yet know of a better characterization for $C^*$-algebras with $\text{IBN}_1$, we are nevertheless able to conclude that it is a distinct property from $\text{IBN}$.

Example. Consider the $C^*$-algebra $\mathcal{T}_2$ which is the universal algebra for two isometries $v_1$ and $v_2$ satisfying $v_1^*v_2 = v_2^*v_1 = 0$ and $v_1v_1^* + v_2v_2^* < 1$. Note that $V = [v_1, v_2] \in M_{1,2}(\mathcal{T}_2)$ is a proper matrix isometry in the sense that $V^*V = I_2$ and $VV^* < 1$. Since $V$ is adjointable the submodule $VT_2^2 \subset \mathcal{T}_2$ is complementable (with complement ker $V^*$) and so

$$\mathcal{T}_2 = VT_2^2 \oplus \ker V^* \simeq T_2^2 \oplus \ker V^*.$$  

Thus, $\mathcal{T}_2$ does not have $\text{IBN}_1$ but Cuntz [5, Proposition 3.9] has shown $K_0(\mathcal{T}_2) = \mathbb{Z}$ and is generated by $[1]_0$, hence $\mathcal{T}_2$ does have $\text{IBN}$.

Indeed, the relationship $A \simeq A^2 \oplus X$ guarantees a unital $*$-homomorphism $\phi : \mathcal{T}_2^2 \to A$ in much the same way the relationship $A \simeq A^2$ guarantees an embedding $\psi : O_2 \to A$.

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