ON THE $u$-INARIANT OF FUNCTION FIELDS OF CURVES OVER COMPLETE DISCRETELY VALUED FIELDS

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Abstract. Let $K$ be a complete discretely valued field with residue field $\kappa$. If $\text{char}(K) = 0$, $\text{char}(\kappa) = 2$ and $[\kappa : \kappa^2] = d$, we prove that there exists an integer $N$ depending on $d$ such that the $u$-invariant of any function field in one variable over $K$ is bounded by $N$. The method of proof is via introducing the notion of uniform boundedness for the $p$-torsion of the Brauer group of a field and relating the uniform boundedness of the $2$-torsion of the Brauer group to the finiteness of the $u$-invariant. We prove that the $2$-torsion of the Brauer group of function fields in one variable over $K$ is uniformly bounded.

Let $K$ be a complete discretely valued field with residue field $\kappa$ and $F$ a function field in one variable over $K$. Suppose $\text{char}(\kappa) \neq 2$. A bound for the $u$-invariant of $F$ in terms of the $u$-invariant of function fields in one variable over $\kappa$ was obtained by Harbater-Hartmann-Krashen [7] using patching techniques. This recovers the $u$-invariant of function fields of non-dyadic $p$-adic curves ([17]). Leep ([13]), using results of Heath-Brown ([9]), proved that the $u$-invariant of function fields of all $p$-adic curves (including dyadic curves) is 8. An alternate proof for function fields of dyadic curves is given in ([18]). In fact more generally we proved that if $\text{char}(K) = 0$, $\text{char}(\kappa) = 2$ and $\kappa$ is perfect, then $u(F) \leq 8$. If $[\kappa : \kappa^2]$ is infinite it is easy to construct anisotropic quadratic forms over $K$ and hence over $F$ of arbitrarily large dimension. The question remained open whether the $u$-invariant of $F$ is finite if $\text{char}(\kappa) = 2$ and $[\kappa : \kappa^2]$ is finite. The aim of this article is to give an affirmative answer to this question. More precisely we prove the following (4.3)

**Theorem 1.** Let $K$ be complete discretely valued field with residue field $\kappa$. Suppose that $\text{char}(K) = 0$, $\text{char}(\kappa) = 2$ and $[\kappa : \kappa^2]$ is finite. Then there exists an integer $M$ which depends only on $[\kappa : \kappa^2]$ such that for any finite extension $F$ of $K(t)$, $u(F) \leq M$.

It was conjectured in ([18]) that $u(F)$ is at most $8[\kappa : \kappa^2]$. The bound we give for $u(F)$ is not effective and far from the conjectural bound.

Let $L$ be a field of characteristic not equal to 2 with $H^M(L, \mu_2^{\otimes M}) = 0$ for some $M \geq 1$. Suppose that there exists an integer $N$ such that for all finite extensions $E$ of $L$ and for any $\alpha \in H^n(E, \mu_2^{\otimes n})$, $n \geq 2$, there exists an extension $E'$ of $E$ of degree at most $N$ such that $\alpha \otimes_E E' = 0$. Then a theorem of Krashen (1.5) asserts that the $u$-invariant of $L$ is finite. Our aim is to prove that if $K$ is a complete discretely valued field with residue field $\kappa$ of characteristic 2 and $[\kappa : \kappa^2]$ finite and $F$ a function field in one variable over $K$, then such an integer $N$ exists for $F$, thereby proving the finiteness of the $u$-invariant of $F$.

We introduce the notion of uniform boundedness for the $\ell$-torsion of the Brauer group $\text{Br}(L)$ of $L$, where $L$ is any field. We say that the Brauer group of $L$ is uniformly $\ell$-bounded if there exists an integer $N$ such that for any finite extension $E$ of $L$ and for any set of finitely many elements $\alpha_1, \ldots, \alpha_n \in \ell \text{Br}(E)$, there is a
finite extension $E'$ of $E$ of degree at most $N$ such that $\alpha_i \otimes E E' = 0$ for $1 \leq i \leq n$. Using the result of Krashen (1.5), we show that if $L$ is a field of characteristic not equal to 2 with $H^M(L, \mu_2^\otimes M) = 0$ for some $M \geq 1$ and with the Brauer group of $L$ is 2-uniformly bounded, then $u(L)$ is finite (1.8). It looks plausible that there are fields $L$ of finite $u$-invariant with the Brauer group of $L$ not uniformly 2-bounded.

The main result of the paper is to prove the uniform $p$-boundedness for the Brauer group of any function field $F$ in one variable over a complete discretely valued field $K$ with residue field $\kappa$, where $\text{char}(\kappa) = p$, $\text{char}(F) \neq p$ and $[\kappa : \kappa^p]$ is finite. We also prove the uniform $\ell$-boundedness for the Brauer group of any function field in one variable over a complete discretely valued field with residue field $\kappa$ and $\ell \neq \text{char}(\kappa)$ under the assumption that the Brauer groups of $\kappa$ and $\kappa(t)$ are uniformly $\ell$-bounded. This result for function fields of $p$-adic curves ($p \neq \ell$) is due to Saltman ([19]). To prove our theorems we use the patching techniques of Harbater-Hartmann-Krashen and results of ([18]).

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1. Galois cohomology, Symbol length, $u$-invariant and Uniform Bound

In this section we recall the recent results of Krashen and Saltman connecting the symbol length and effective index in Galois cohomology with the $u$-invariant of a field.

Let $K$ be a field and $\ell$ a prime not equal to the characteristic of $K$. Let $\mu_\ell$ denote the Galois module of $\ell$th roots of unity and $H^n(K, \mu_\ell^\otimes m)$ denote the $n$th Galois cohomology group with values in $\mu_\ell^\otimes m$. We have $H^1(K, \mu_\ell) \simeq K^*/K^{*\ell}$. For $a \in K^*$, let $(a) \in H^1(K, \mu_\ell)$ denote the image of $aK^{*\ell}$. Let $a_1, \ldots, a_n \in K^*$. The cup product $(a_1) \cdot (a_2) \cdots (a_n) \in H^n(K, \mu_\ell^\otimes n)$ is called a symbol in $H^n(K, \mu_\ell^\otimes n)$. A theorem of Voevodsky ([22]) asserts that every element in $H^n(K, \mu_\ell^\otimes n)$ is a sum of symbols.

Let $\alpha \in H^n(K, \mu_\ell^\otimes n)$. The symbol length of $\alpha$, denoted by $\lambda(\alpha)$, is defined as the smallest $m$ such that $\alpha$ is a sum of $m$ symbols in $H^n(K, \mu_\ell^\otimes m)$. For any $\alpha \in H^n(K, \mu_\ell^\otimes m)$, the effective index of $\alpha$, denoted by $\text{eind}(\alpha)$, is defined to be the minimum of the degrees of finite field extensions $E$ of $K$ with $\alpha_E = 0$, where $\alpha_E$ is the image of $\alpha$ in $H^n(E, \mu_\ell^\otimes m)$. Since $H^2(K, \mu_\ell) \simeq \ell \text{Br}(K)$, for $\alpha \in \ell \text{Br}(K)$, $\text{eind}(\alpha)$ is equal to the index of a central simple algebra $A$ over $K$ representing $\alpha$. The following lemma asserts that this definition of effective index coincides with the definition in ([12]).

**Lemma 1.1.** Let $K$ be a field and $\ell$ a prime not equal to $\text{char}(K)$. Let $\alpha \in H^n(K, \mu_\ell^\otimes m)$. Suppose that exists an extension $L$ of $K$ of degree at most $N$ with $\alpha \otimes_K L = 0$. Then there exists a separable field extension $E$ of $K$ of degree at most $N$ such that $\alpha \otimes_K E = 0$.

**Proof.** Let $L$ be an extension of $K$ of degree at most $N$ with $\alpha \otimes_K L = 0$. Let $E$ be the separable closure of $K$ in $L$. Let $p$ be the characteristic of $K$. Suppose $p > 0$. Then $L/E$ is of degree $p^r$ for some $r \geq 0$. Since $\ell \neq p$, the restriction map $H^n(E, \mu_\ell^\otimes m) \to H^n(L, \mu_\ell^\otimes m)$ is injective ([21, Cor. on p.12]). Hence $\alpha \otimes_K E = 0$. $\square$

We now recall a theorem of Krashen ([12, 4.2], cf. [20]).
Theorem 1.2. Let $K$ be a field and $\ell$ a prime not equal to the characteristic of $K$. Let $n \geq 1$. Suppose that there exists an integer $N$ such that for every finite extension $L$ of $K$ and for every element $\beta \in H^d(L, \mu_2^{\otimes d})$, $1 \leq d \leq n-1$, $\mathrm{eind}(\beta) \leq N$. Then for any $\alpha \in H^n(K, \mu_2^{\otimes n})$, $\lambda(\alpha)$ is bounded in term of $\mathrm{eind}(\alpha)$, $N$ and $n$.

The following is a consequence of the above theorem.

Corollary 1.3. Let $K$ be a field and $\ell$ a prime not equal to the characteristic of $K$. Let $n \geq 1$. Suppose that there exists an integer $N$ such that for all finite extensions $L$ of $K$ and for all $\alpha \in H^d(L, \mu_2^{\otimes d})$, $1 \leq d \leq n$, $\mathrm{eind}(\alpha) \leq N$. Then there exists an integer $N'$ which depends only on $N$ and $n$ such that $\lambda(\alpha) \leq N'$ for all finite extensions $L$ of $K$ and $\alpha \in H^n(K, \mu_2^{\otimes n})$.

Let $K$ be a field of characteristic not equal to 2. The $u$-invariant of $K$ is defined to be the supremum of dimensions of anisotropic quadratic forms over $K$. The following theorem is a consequence of a theorem of Orlov, Vishik, and Voevodsky ([15]) on the Milnor conjecture (cf. [10], [16]).

Theorem 1.4. Let $K$ be a field of characteristic not equal to 2. Suppose that there exist integers $M \geq 1$ and $N$ such that $H^M(K, \mu_2) = 0$ and $\lambda(\alpha) \leq N$ for all $\alpha \in H^d(K, \mu_2^{\otimes d})$, $1 \leq d < M$. Then the $u$-invariant is bounded by a function of $M$ and $N$.

The following follows from (1.4) and (1.3) (cf. [12, 5.5]).

Corollary 1.5. Let $K$ be a field of characteristic not equal to 2. Suppose that there exist integers $N \geq 1$ and $M \geq 1$ such that for all finite extensions $L$ of $K$, $H^M(L, \mu_2) = 0$ and for all $n \geq 1$ and $\alpha \in H^n(L, \mu_2)$, $\mathrm{eind}(\alpha) \leq N$. Then there exists an integer $N'$, which depends only on $N$ and $M$, such that $u(K) \leq N'$.

Let $K$ be a field and $\ell$ a prime not equal to the characteristic of $K$. We say that $K$ is $(n, \ell)$-uniformly bounded if there exists an integer $N$ such that for any finite extension $L$ of $K$ and $\alpha_1, \cdots, \alpha_m \in H^n(L, \mu_2^{\otimes n})$ there is an extension $E$ of $L$ with $[E : L] \leq N$ and $\alpha_i \otimes \ell, E = 0$ for $1 \leq i \leq m$. Such an $N$ is called an $(n, \ell)$-uniform bound for $K$. We note that if $N$ is an $(n, \ell)$-uniform bound for $K$, then $N$ is also an $(n, \ell)$-uniform bound for any finite extension $L$ of $K$.

In view of a theorem of Voevodsky ([22]) on the Bloch-Kato conjecture, every element in $H^n(K, \mu_2^{\otimes n})$ is a sum of symbols. In particular, $N$ is an $(n, \ell)$-uniform bound of $K$ if and only if for given symbols $\alpha_1, \cdots, \alpha_m \in H^n(K, \mu_2^{\otimes n})$ there is an extension $L$ of $K$ with $[L : K] \leq N$ and $\alpha_i \otimes \ell = 0$ for $1 \leq i \leq m$.

Lemma 1.6. Let $K$ be a field and $\ell$ a prime not equal to the characteristic of $K$. If $N$ is an $(n, \ell)$-uniform bound for $K$, then $N$ is also a $(d, \ell)$-uniform bound for $K$ for all $d \geq n$.

Proof. Suppose $N$ is an $(n, \ell)$-uniform bound for $K$. It is enough to prove the lemma for $d = n + 1$. Let $L$ be a finite extension of $K$ and $\alpha_1, \cdots, \alpha_m \in H^{n+1}(L, \mu_2^{\otimes (n+1)})$ be symbols. Then $\alpha_i = \beta_i \cdot (a_i)$ for some symbols $\beta_i \in H^n(L, \mu_2^{\otimes n})$ and $a_i \in L^*$, $1 \leq i \leq m$. Since $N$ is an $(n, \ell)$-uniform bound for $K$, there exists a field extension $E$ of $L$ with $[E : L] \leq N$ and $\beta\otimes \ell = 0$ for $1 \leq i \leq m$. Then clearly $\alpha_i \otimes \ell = 0$ for $1 \leq i \leq m$. Thus $N$ is also an $(n + 1, \ell)$-uniform bound for $K$. □

Corollary 1.7. Let $K$ be a field and $\ell$ a prime not equal to the characteristic of $K$. Suppose that $N$ is a $(2, \ell)$-uniform bound for $K$. Then for every $n \geq 2$, there
exists an integer $N_n$, which depends only on $N$ and $n$ such that $\lambda(\alpha) \leq N_n$ for all $\alpha \in H^n(K, \mu_\ell^{\otimes n})$.

Proof. Since $N$ is $(2, \ell)$-uniform bound for $K$, by (1.6), $N$ is an $(n, \ell)$-uniform bound for $K$ for all $n \geq 2$. Let $\alpha \in H^n(K, \mu_\ell^{\otimes n})$. Then, by (1.2), $\lambda(\alpha)$ is bounded in terms of $\text{eind}(\alpha)$, $N$ and $n$. Since $\text{eind}(\alpha) \leq N$, $\lambda(\alpha)$ is bounded in terms of $N$ and $n$. \qed

Corollary 1.8. Let $K$ be a field of characteristic not equal to 2. Suppose that there exists an integer $M$ such that for all finite extensions $L$ of $K$, $H^M(L, \mu_2) = 0$ and $N$ is a $(2,2)$-uniform bound for $K$. Then there exists $N'$ which depends only on $N$ and $M$ such that for any finite extension $L$ of $K$, $u(L) \leq N'$.

Proof. Since the conditions on $N$ are also satisfied by any finite extension of $K$, it is enough to find an $N'$ which depends only on $N$ and $M$ such that $u(K) \leq N'$.

Since $N$ is a $(2,2)$-uniform bound for $K$, by (1.7) there exist integers $N_n$ for $1 \leq n < M$, which depends only on $N$ and $n$ such that for all $\alpha \in H^n(K, \mu_\ell^{\otimes n})$, $\lambda(\alpha) \leq N$. Let $N'$ be the maximum of $N_n$ for $1 \leq n < M$. Thus the corollary follows from (1.4). \qed

Let $R$ be a discrete valuation ring with field of fractions $K$ and residue field $\kappa$. Let $\ell$ be a prime not equal to $\text{char}(\kappa)$. Then there is a residue homomorphism $\partial : H^n(K, \mu_\ell^{\otimes m}) \to H^{n-1}(\kappa, \mu_\ell^{\otimes (m-1)})$ with kernel $H^n_{\text{et}}(R, \mu_\ell^{\otimes m})$.

Let $A$ be an integral domain with field of fractions $K$. Let $\ell$ be a prime which is a unit in $A$. We have the natural map $H^n_{\text{et}}(A, \mu_\ell^{\otimes m}) \to H^n(K, \mu_\ell^{\otimes m})$. An element $\alpha$ of $H^n(K, \mu_\ell^{\otimes m})$ is said to be unramified on $A$ if $\alpha$ is in the image of $\iota$. Suppose that $A$ is a regular ring. For each height one prime ideal $P$ of $A$, we have the residue homomorphism $\partial_P : H^n(K, \mu_\ell^{\otimes m}) \to H^{n-1}(\kappa(P), \mu_\ell^{\otimes (m-1)})$, where $\kappa(P)$ is the residue field at $P$. We have the following

Theorem 1.9. ([5, 7.4]) Let $A$ be a regular two dimensional integral domain with field of fractions $K$ and $\ell$ a prime which is a unit in $A$. The sequence

$$0 \to H^2_{\text{et}}(A, \mu_\ell) \to H^2(K, \mu_\ell) \to \oplus_{P \in \text{Spec}(A)^{(1)}} H^1(\kappa(P), \mathbb{Z}/\ell\mathbb{Z})$$

is exact, where $\text{Spec}(A)^{(1)}$ is the set of height one prime ideals of $A$.

We now recall a few notations from ([7]). Let $R$ be a complete discrete valuation ring with field of fractions $K$ and residue field $\kappa$. Let $F$ be the function field of a curve over $K$. Let $\mathcal{X}$ be a regular proper model of $F$ over $R$ and $X$ its reduced special fiber. For any codimension one point $\eta$ of $\mathcal{X}$, let $F_\eta$ be the completion of $F$ at the discrete valuation of $F$ given by $\eta$ and $\kappa(\eta)$ the residue field at $\eta$. For a closed point $P$ of $\mathcal{X}$, let $F_P$ be the field of fractions of the completion of the local ring at $P$ and $\kappa(P)$ the residue field at $P$. Let $U$ be an open subset of $X$. Let $R_U$ be the ring of all those functions in $F$ which are regular on $U$. Then $R \subset R_U$. Let $t$ be a parameter in $R$. Let $\hat{R}_U$ be the completion of $R_U$ at the ideal $(t)$. Let $\hat{F}_U$ be the field of fractions of $\hat{R}_U$.

Let $A$ be a regular integral domain with field of fractions $F$. For a maximal ideal $m$ of $A$, let $\hat{A}_m$ denote the completion of the local ring $A_m$ and $F_m$ the field of fractions of $\hat{A}_m$. 
2. Uniform bound - bad characteristic case

Let $K$ be a complete discretely valued field with residue field $\kappa$. Let $p = \text{char}(\kappa)$. In this section we show that there is a $(2, p)$-uniform bound for $K(t)$ which depends only on $[\kappa : \kappa^p]$.

First we recall the following two results from ([18]).

**Theorem 2.1.** ([18, 2.4]) Let $R$ be a complete discrete valuation ring with field of fractions $K$ and residue field $\kappa$. Suppose that $\text{char}(K) = 0$, $\text{char}(\kappa) = p > 0$ and $[\kappa : \kappa^p] = p^d$. Let $\pi \in R$ be a parameter and $u_1, \ldots, u_d \in R^*$ units such that $\kappa = \kappa^p(\overline{u}_1, \ldots, \overline{u}_d)$, where for any $u \in R$, $\overline{u}$ denotes the image of $u$ in $\kappa$. Suppose that $K$ contains a primitive $p^d$th root of unity. Then any $\alpha \in H^2(K, \mu_p)$ splits over $K(\sqrt[1/p]{\pi}, \sqrt[1/p]{u_1}, \ldots, \sqrt[1/p]{u_{d-1}}, \sqrt[1/p]{u_d})$. In particular if $d > 0$, $p^{2d}$ is a $(2, p)$-uniform bound for $K$ and if $d = 0$, $p$ is a $(2, p)$-uniform bound for $K$.

**Proposition 2.2.** ([18, 3.5]) Let $A$ be a complete regular local ring of dimension 2 with field of fractions $F$ and residue field $\kappa$. Suppose that $\text{char}(F) = 0$, $\text{char}(\kappa) = p > 0$ and $[\kappa : \kappa^p] = p^d$. Let $\pi, \delta \in A$ and $u_1, \ldots, u_d \in A^*$ such that the maximal ideal $m$ of $A$ is generated by $\pi$ and $\delta$, and $\kappa = \kappa^p(\overline{u}_1, \ldots, \overline{u}_d)$, where for any $u \in A$, $\overline{u}$ denotes the image of $u$ in $\kappa$. Suppose that $F$ contains a primitive $p^d$th root of unity. Then any $\alpha \in H^2(F, \mu_p)$ which is unramified on $A$ except possibly at $(\pi)$ and $(\delta)$ splits over $F(\sqrt[1/p]{\pi}, \sqrt[1/p]{\delta}, \sqrt[1/p]{\overline{u}_1}, \ldots, \sqrt[1/p]{\overline{u}_d})$.

**Lemma 2.3.** Let $\mathcal{X}$ be a regular integral two dimensional scheme and $F$ its function field. Suppose $C$ and $E$ are regular curves on $\mathcal{X}$ with normal crossings. Let $\mathcal{P}$ be a set of closed points of $\mathcal{X}$ with field of fractions $F$. Then there exist $f, g \in F^*$ such that the maximal ideal at $P$ is generated by $f$ and $g$ for each $P \in \mathcal{P}$ and $f$ (resp. $g$) defines $C$ (resp. $E$) at each $P \in \mathcal{P} \cap C$ (resp. $P \in \mathcal{P} \cap E$).

**Proof.** Let $R$ be the semi-local ring at all $P \in \mathcal{P}$. Since $R$ is a unique factorisation domain, there exist $f_1, g_1 \in R$ such that $\text{div}_{\text{Spec}(R)}(f_1) = C \mid_{\text{Spec}(R)}$ and $\text{div}_{\text{Spec}(R)}(g_1) = E \mid_{\text{Spec}(R)}$. For each $P \in \mathcal{P}$, let $m_P$ be the maximal ideal of the local ring $R_P$ at $P$. Then for each $P \in C \cap E \cap \mathcal{P}$, $m_P = (f_1, g_1)$. Let $P \in \mathcal{P}$. Suppose $P \not\in C$. Then $P \in E$. Since $E$ is regular on $\mathcal{X}$, by the choice of $g_1$, there exists $\theta_P \in m_P$ such that $m_P = (\theta_P, g_1)$. By the Chinese remainder theorem, there exists $\pi_P \in m_P$ such that $\pi_P \not\in m_Q$ for all $Q \in \mathcal{P}, Q \neq P$ and $\pi_P = \theta_P$ module $m_Q^2$. Then $m_P = (\pi_P, g_1)$. Similarly for each $P \not\in E$, choose $\delta_P \in m_P$ such that $m_P = (f_1, \delta_P)$ and $\delta_P \not\in m_Q$ for all $Q \in \mathcal{P}, Q \neq P$. Let

$$f_2 = \prod_{P \in \mathcal{P} \cap C} \pi_P, \quad f = f_1 f_2$$

and

$$g_2 = \prod_{P \in \mathcal{P} \cap E} \delta_P, \quad g = g_1 g_2.$$ 

Then $f_2$ and $g_2$ are units at all $P \in C \cap E$. We claim that $f$ and $g$ have the required properties. Let $P \in \mathcal{P}$. Suppose $P \in C \cap E$. Then by the choices $f_2$ and $g_2$, they are units at $P$ and $m_P = (f_1, g_1)$. In particular $m_P = (f, g)$ and $f, g$ define $C$ and $E$ respectively at $P$. Suppose that $P \not\in C$. Then $f_1$ and $g_2$ are units at $P$ and $f_2 = \pi_P u_P$ for some unit $u_P$ at $P$. Since $m_P = (\pi_P, g_1)$, we have $m_P = (f, g)$. Since $g_1$ defines $E$ at $P$ and $g_2$ is a unit at $P$, $g$ defines $E$ at $P$. Similarly if $P \not\in E$, then $m_P = (f, g)$ and $f$ defines $C$ at $P$. \qed
Theorem 2.4. Let $K$ be a complete discretely valued field with residue field $\kappa$. Suppose that $\text{char}(K) = 0$ and $\text{char}(\kappa) = p > 0$ and $[\kappa : \kappa^p] = p^d$. Assume that $K$ contains a primitive $p^d$th root of unity. Then $p^{4d+4}$ is a $(2,p)$-uniform bound for $K(t)$.

Proof. Let $F$ be a finite extension of $K(t)$. Let $\alpha_1, \cdots, \alpha_m \in H^2(F, \mu_p)$. Let $\mathcal{X}$ be a regular proper model of $F$ over the ring of integers $R$ of $K$ such that the support of $\text{ram}(\alpha_i)$ for all $i$ and the special fiber is contained in $C \cup E$, where $C$ and $E$ are regular curves on $\mathcal{X}$ having only normal crossings. Let $\eta$ be the generic point of an irreducible component $X_\eta$ of the special fiber of $\mathcal{X}$. Then $\kappa(\eta)$ is a function field in one variable over $\kappa$. Since $[\kappa : \kappa^p] = p^d$, $\kappa(\eta) : \kappa(\eta)^p = p^{d+1}$ ([3, A.V.135, Corollary 3]). Let $\pi_\eta$ be a parameter at $\eta$ and $u_{\eta,1}, \cdots, u_{\eta,d+1} \in F^*$ be lifts of a $p$-basis of $\kappa(\eta)$. Then, by (2.1), $\alpha_i \otimes F_\eta(\sqrt[p]{u_{\eta,1}}, \cdots, \sqrt[p]{u_{\eta,d}}, \sqrt[p]{u_{\eta,d+1}}, \sqrt[p]{\pi_\eta}) = 0$ for all $i$.

Let $f \in F^*$ be chosen such that $\nu_\eta(f) = 1$ for all $\eta$. By the Chinese remainder theorem, choose $u_1, \cdots, u_{d+1} \in F^*$ units at each $\eta$ such that $u_i = u_{\eta,j} \in \kappa(\eta)$. Then $\alpha_i \otimes F_\eta(\sqrt[p]{u_1}, \sqrt[p]{u_2}, \cdots, \sqrt[p]{u_{d+1}}) = 0$ for all $i$. By ([8, 5.8], [11, 1.17]), there exists a non-empty open set $U_\eta$ of the component $X_\eta$ of the special fiber, such that $\alpha_i \otimes F_{U_\eta}(\sqrt[p]{u_1}, \sqrt[p]{u_2}, \cdots, \sqrt[p]{u_{d+1}}) = 0$ for all $i$.

Let $\mathcal{P}$ be the finite set of closed points of $\mathcal{X}$ which are not in $U_\eta$ for any $\eta$. Let $A$ be the semi-local ring at the points of $\mathcal{P}$. For $P \in \mathcal{P}$, let $A_P$ be the local ring at $P$. Since the ramifications of $\alpha_i$ for all $i$ are in normal crossings, for each $P \in \mathcal{P}$, the maximal ideal $m_P$ at $P$ is $(\pi_P, \delta_P)$ for some $\pi_P$ and $\delta_P$ such that $\alpha_i$ is unramified on $A_P$ except possibly at $(\pi_P)$ and $(\delta_P)$. Since the residue field $\kappa(P)$ at $P$ is a finite extension of $\kappa$ and $[\kappa : \kappa^p] = p^d$, $\kappa(P) : \kappa(P)^p = p^d$ ([3, A.V.135, Corollary 3]). Let $v_{P,1}, \cdots, v_{P,d} \in A^*_P$ be lifts of a $p$-basis of $\kappa(P)$. By the Chinese remainder theorem, choose $h_1, \cdots, h_d \in A^*$ such that $h_i = v_{P,i}$ modulo the maximal ideal of $P$ for all $P \in \mathcal{P}$ and $1 \leq i \leq d$. By (2.3), there exist $g_1, g_2 \in F^*$ such that for any point $P \in \mathcal{P}$, we have $m_P = (g_1, g_2)$ and $g_1$ defines $C$ at all $P \in \mathcal{P} \cap C$, and $g_2$ defines $E$ at all $P \in \mathcal{P} \cap E$. In particular, each $\alpha_i$ is unramified on $A_P$ except possibly at $(g_1)$ and $(g_2)$.

Let $L = F(\sqrt[p]{u_1}, \sqrt[p]{u_2}, \cdots, \sqrt[p]{u_{d+1}} \subseteq \mathcal{Y} \to \mathcal{X}$ be a proper birational morphism such that $\mathcal{Y}$ is regular. Then $\mathcal{Y}$ is a regular proper model of $L$. Let $Y$ be the special fiber of $\mathcal{Y}$ and $\phi : Y \to X$ be the induced morphism. Let $y$ be a point of $Y$. Suppose $\phi(y) = \eta \in X$ is the generic point of an irreducible component of $X$. Then by the choice of $L$, $F_\eta(\sqrt[p]{u_1}, \sqrt[p]{u_2}, \cdots, \sqrt[p]{u_{d+1}}) \subseteq L_Y$. Since $\alpha_i \otimes F_{\eta}(\sqrt[p]{u_1}, \sqrt[p]{u_2}, \cdots, \sqrt[p]{u_{d+1}}) = 0$ for all $i$, $\alpha_i \otimes L_y = 0$ for all $i$. Suppose $\phi(y) = P$ is a closed point of $X$. Let $P \in U_\eta$ for some $\eta$. Then $F_{U_\eta} \subset F_P$. By the choice of $L$, $F_{U_\eta}(\sqrt[p]{u_1}, \sqrt[p]{u_2}, \cdots, \sqrt[p]{u_{d+1}}) \subset F_P(\sqrt[p]{u_1}, \sqrt[p]{u_2}, \cdots, \sqrt[p]{u_{d+1}}) \subseteq L_y$. Since $\alpha_i \otimes F_{U_\eta}(\sqrt[p]{u_1}, \sqrt[p]{u_2}, \cdots, \sqrt[p]{u_{d+1}}) = 0$ for all $i$, $\alpha_i \otimes L_y = 0$ for all $i$. Suppose $P \notin U_\eta$ for all $\eta$. Then $P \in \mathcal{P}$. Since $F_P(\sqrt[p]{h_1}, \cdots, \sqrt[p]{h_d}, \sqrt[p]{g_1}, \sqrt[p]{g_2}) \subseteq L_y$ and $\alpha_i \otimes F_P(\sqrt[p]{h_1}, \cdots, \sqrt[p]{h_d}, \sqrt[p]{g_1}, \sqrt[p]{g_2}) = 0$ for all $i$, $\alpha_i \otimes L_y = 0$ for all $i$.

In particular, by ([8, 9.12]), $\alpha_i \otimes L = 0$ for all $i$. Since $[L : F] \leq p^{4d+4}$, the theorem follows. \qed

Corollary 2.5. Let $K$ be a complete discretely valued field with residue field $\kappa$. Suppose that $\text{char}(K) = 0$ and $\text{char}(\kappa) = p > 0$ and $[\kappa : \kappa^p] = p^d$. Let $\zeta$ be a primitive $p^d$th root of unity. Then $[K(\zeta) : K]p^{4d+4}$ is a $(2,p)$-uniform bound for $K(t)$.
Proof. Let $K' = K(\zeta)$. Then $K'$ is a complete discretely valued field with residue field $\kappa$. Let $F$ be a finite extension of $K(t)$. Let $\alpha_1, \ldots, \alpha_m \in H^2(F, \mu_{p^\infty})$. Since $F' = F(\zeta)$ is also a function field over $K'$, by (2.4), there exists an extension $L$ of $F'$ of degree at most $p^{4d+4}$ such that $\alpha_i \otimes L = 0$ for all $i$. Since $[L : F] = [L : F'][F' : F]$, the corollary follows.

The above corollary and (1.6) give the following

Corollary 2.6. Let $K$ be a complete discretely valued field with residue field $\kappa$. Suppose that $\text{char}(K) = 0$ and $\text{char}(\kappa) = p > 0$ and $[\kappa : \kappa^p] = p^d$. Let $\zeta$ be a primitive $p^d$th root of unity. Then $[K(\zeta) : K] p^{4d+4}$ is an $(n, p)$-uniform bound for $K(t)$ for all $n \geq 2$.

3. Uniform bound - good characteristic case

Let $F$ be the function field of a $p$-adic curve. In ([19]), Saltman proved that if $\ell$ is a prime not equal to $p$ and $\alpha_1, \ldots, \alpha_m \in H^2(F, \mu_{\ell})$, then there exists an extension $L$ of $F$ such that $[L : F] \leq \ell^2$ and $\alpha_i \otimes L = 0$ for all $i$, i.e., $F$ is $(2, \ell)$-uniformly bounded. Let $K$ be a complete discretely valued field with residue field $\kappa$. Let $\ell$ be a prime not equal to $\text{char}(\kappa)$. In this section we show that $K(t)$ is $(2, \ell)$-uniformly bounded under some conditions on $\kappa$.

Theorem 3.1. Let $K$ be a complete discretely valued field with residue field $\kappa$. Let $\ell$ be a prime not equal to $\text{char}(\kappa)$ and $n \geq 1$. If $N$ is an $(n, \ell)$-uniform bound for $\kappa$, then $N$ is an $(n, \ell)$-uniform bound for $K$.

Proof. Let $L$ be a finite extension of $K$. Then $L$ is a complete discretely valued field with residue $\kappa'$. Let $R$ be the valuation ring of $L$ and $\pi \in R$ be a parameter. Let $\alpha_1, \ldots, \alpha_m \in H^n(L, \mu_{\ell^m})$. Let $S$ be the integral closure of $R$ in $L(\sqrt[p]{\pi})$. Then $S$ is also a complete discrete valuation ring with residue field $\kappa'$. Since $S/R$ is ramified, $\alpha_i \otimes L(\sqrt[p]{\pi})$ is unramified at $S$ for each $i$. Hence there exists $\beta_i \in H^n(S, \mu_{\ell^m})$ such that $\beta_i \otimes S \pi^{\ell^m} = \alpha_i$. Since $N$ is an $(n, \ell)$-uniform bound for $\kappa$, there exists an extension $L_0$ of $\kappa'$ of degree at most $N$ such that $\beta_i \otimes L_0 = 0$ for all $i$. Let $E$ be the extension of $L$ of degree equal to $[L_0 : \kappa']$ with residue field $L_0$. Let $T$ be the integral closure of $R$ in $E(\sqrt[p]{\pi})$. Then $T$ is a complete discrete valued ring with residue field $L_0$ and $S \subset T$. Since $\beta_i \otimes L_0 = 0$, $\beta_i \otimes S T = 0$ for all $i$. In particular $\alpha_i \otimes E(\sqrt[p]{\pi}) = 0$ for all $i$. Since the degree of $E$ over $L$ is equal to the degree of $L_0$ over $\kappa'$ and $[L_0 : \kappa'] \leq N$, we have $[E(\sqrt[p]{\pi}) : L] \leq \ell N$. \hfill $\square$

Lemma 3.2. Let $A$ be a regular local ring with residue field $\kappa$ and maximal ideal $m = (\pi, \delta)$. Let $F$ be the field of fractions of $A$, $\ell$ a prime not equal to $\text{char}(\kappa)$. Let $B$ (resp. $B'$) be the integral closure of $A$ in $F(\sqrt[p]{\pi}, \sqrt{\delta})$ (resp. $F(\sqrt[p]{\pi})$). Let $\alpha \in H^2(F, \mu_{\ell^m})$. If $\alpha$ is unramified on $A$ except possibly at $(\pi)$ and $(\delta)$ (resp. except possibly at $(\pi)$), then $\alpha \otimes F(\sqrt[p]{\pi}, \sqrt{\delta})$ is unramified on $B$ (resp. $\alpha \otimes F(\sqrt[p]{\pi})$ is unramified on $B'$).

Proof. By ([18, 3.3]), $B$ is a regular local ring of dimension 2 with residue field $\kappa$. Let $P$ be a height one prime ideal of $B$ and $Q = P \cap A$. Then $Q$ is a height one prime ideal of $A$. If $Q \neq (\pi)$ and $(\delta)$, then $\alpha$ is unramified at $Q$ and hence $\alpha \otimes F(\sqrt[p]{\pi}, \sqrt{\delta})$ is unramified at $P$. Suppose $Q = (\pi)$ or $(\delta)$. Then $Q$ is ramified in $B$ and hence $\alpha \otimes F(\sqrt[p]{\pi}, \sqrt{\delta})$ is unramified at $P$. Since $B$ is a regular local ring of dimension 2 and $\alpha \otimes F(\sqrt[p]{\pi}, \sqrt{\delta})$ is unramified at every height one prime ideal of $B$, $\alpha \otimes F(\sqrt[p]{\pi}, \sqrt{\delta})$ is unramified on $B$ (cf. 1.9). The other case follows similarly. \hfill $\square$
Theorem 3.3. Let $R$ be a complete discrete valuation ring with field of fractions $K$ and residue field $\kappa$. Let $F$ be the function field of a curve over $K$. Let $\ell$ be a prime not equal to $\text{char}(\kappa)$ and $\alpha_1, \cdots, \alpha_r \in H^2(K, \mu_\ell^\otimes)$. Then there exist $f, g, h \in F^*$ such that each $\alpha_i \otimes F(\sqrt[\ell]{f}, \sqrt[\ell]{g}, \sqrt[\ell]{h})$ is unramified at all codimension one points of any regular proper model of $F(\sqrt[\ell]{f}, \sqrt[\ell]{g}, \sqrt[\ell]{h})$.

Proof. Let $\mathcal{X}$ be a regular proper model of $F$ over $R$ such that the union of the support of ramification locus of $\alpha_i$, for $1 \leq i \leq r$, is contained in the union of regular curves $C$ and $E$ with $C \cup E$ having only normal crossings. Let $f \in F^*$ be such that

$$\text{div}_{\mathcal{X}}(f) = C + E + J$$

for some divisor $J$ on $\mathcal{X}$ which does not contain any irreducible component of $C \cup E$ and does not pass through any point of $C \cap E$. Let $g \in F^*$ be such that

$$\text{div}_{\mathcal{X}}(g) = C + G$$

for some divisor $G$ on $\mathcal{X}$ which does not contain any irreducible component of $C \cup E \cup J$ and does not pass through any point of $C \cap E$, $C \cap J$ and $E \cap J$. Let $h \in F^*$ be such that

$$\text{div}_{\mathcal{X}}(h) = E + H$$

for some divisor $H$ on $\mathcal{X}$ which does not contain any irreducible component of $C \cup E \cup J \cup G$ and does not pass through any point of $C \cap E$, $C \cap J$, $C \cap G$, $E \cap J$, $E \cap G$ and $J \cap G$.

Let $L = F(\sqrt[\ell]{f}, \sqrt[\ell]{g}, \sqrt[\ell]{h})$ and $\mathcal{Y}$ be a regular proper model of $L$ over $R$. We claim that $\alpha_i \otimes_F L$ is unramified on $\mathcal{Y}$. Let $y \in \mathcal{Y}$ be a codimension one point of $\mathcal{Y}$. Then $y$ lies over a point $x$ of $\mathcal{X}$. If $x$ is not on $C$ or $E$, then each $\alpha_i$ is unramified at $x$ and hence $\alpha_i \otimes_F L$ is unramified at $y$. Assume that $x \in C \cup E$.

Suppose that $x$ is a codimension one point of $\mathcal{X}$. Since $x$ is on $C$ or $E$, by the choice of $f$, $f$ is a parameter at $x$ and hence $L/K$ is ramified at $x$. In particular $\alpha_i \otimes_F L$ is unramified at $y$.

Suppose that $x$ is a closed point of $\mathcal{X}$. Suppose $x \in C$ and $x \notin E$. Let $A_x$ be the local ring of $\mathcal{X}$ at $x$ and $S_y$ be the local ring of $\mathcal{Y}$ at $y$. Suppose $x \notin J$. Then the maximal ideal $m_x$ at $x$ is $(f, \delta_x)$ for some $\delta_x \in m_x$ and each $\alpha_i$ is unramified on the local ring at $x$ except possibly at $(f)$. Thus, by (3.2), each $\alpha_i \otimes_F F(\sqrt[\ell]{f})$ is unramified at the integral closure of $A_x$ in $F[\sqrt[\ell]{f}]$. Since the integral closure of $A_x$ in $F[\sqrt[\ell]{f}]$ is contained in $S_y$, each $\alpha_i \otimes_F L$ is unramified at $S_y$. Suppose $x \in J$. If $x \notin G$, then as above each $\alpha_i \otimes_F L$ is unramified at $S_y$. If $x \in J \cap G$, then by the choice $h$, $x \notin H$ and hence as above, each $\alpha_i \otimes_F L$ is unramified at $S_y$. Similarly if $x \in E$ and $x \notin C$, then each $\alpha_i \otimes_F L$ is unramified at $S_y$.

Suppose $x \in C \cap E$. Then $x \notin G$ and $x \notin H$. In particular, $m_x = (g, h)$ and each $\alpha_i$ is unramified on $A_x$ except possibly at $(g)$ and $(h)$. As above each $\alpha_i \otimes_F L$ is unramified at $S_y$. \hfill \Box

Lemma 3.4. Let $A$ be a semi-local regular domain with field of fractions $F$. For each maximal ideal $m$ of $A$, let $s(m)$ be a separable finite extension of the residue field $\kappa(m)$ at $m$ of degree $N_m$. Let $N$ be a common multiple of $N_m$, $m$ varying over all maximal ideals of $A$. Then there exists an extension $E$ of $F$ of degree at most $N$ such that for each maximal ideal $m$ of $A$ and for each maximal ideal $m'$ of the integral closure $B_m$ of $A_m$ in $E$, $B_m/m'$ contains a field isomorphic to $s(m)$.
Proof. Since $s(m)$ is a finite separable extension of $\kappa(m)$, there exists $\theta_m \in s(m)$ such that $s(m) = \kappa(m)(\theta_m)$. Let $f_m(X) \in \kappa(m)[X]$ be the minimal polynomial of $\theta_m$ over $\kappa(m)$. Then the degree of $f_m(X)$ is $N_m$. Let $f(X) \in A[X]$ be a monic polynomial of degree $N$ such that $f(X) = f_m(X)^{[\deg(f_m)]}$ modulo $m$ for each maximal ideal $m$ of $A$. Let $g(X)$ be any monic irreducible factor of $f(X)$ over $A$. Let $E = F[X]/(g(X))$. We claim that $E$ has the required property.

Let $m$ be a maximal ideal of $A$. By the choice of $f(X)$ and $g(X)$, we have $g(X) = f_m(X)^{r_m}$ modulo $m$ for some $r_m \geq 1$. Let $B_m$ be the integral closure of $A_m$ in $E$. Since $g(X)$ is monic, $A_m[X]/(g(X))$ is isomorphic to a subring of $B_m$. Let $g_m(X) \in A_m[X]$ be a monic polynomial with $g_m(X) = f_m(X)$ modulo $m$. Since $g(X) = f_m(X)^{r_m} = g_m(X)^{r_m}$ modulo $m$, the ideal $\bar{m}$ of $A_m[X]/(g(X))$ generated by $m$ and $g_m(X)$ is a maximal ideal with $(A_m[X]/(g(X)))/\bar{m} \simeq s(m)$. Since $B_m$ is integral over a subring isomorphic to $A_m[X]/(g(X))$, for every maximal ideal $m'$ of $B_m$, $B_m/m'$ contains a subfield isomorphic to $(A_m[X]/(g(X)))/\bar{m} \simeq s(m)$.

Corollary 3.5. Let $A$ be a semi-local regular domain with field of fractions $F$. Let $\ell$ be a prime. Suppose that $\ell$ is a unit in $A$. Let $\beta \in H^2_{\et}(A, \mu_\ell)$. Suppose that for every maximal ideal $m$ of $A$, there exists a finite separable extension $s(m)$ of $\kappa(m)$ of degree $N_m$ such that $\beta \otimes_A s(m) = 0$. Let $N$ be a common multiple of $N_m$, where $m$ varies over maximal ideals of $A$. Let $E$ be the field constructed in (3.4). Then for any maximal ideal $m$ of $A$, $\beta \otimes_A (E \otimes_F F_m) = 0$.

Proof. Let $B$ be the integral closure of $A$ in $E$. Let $m$ be a maximal ideal of $A$, and $A_m$ the completion of $A$ at $m$. Then, $B \otimes_A A_m$ is complete and by the choice of $E$, $B \otimes_A A_m$ modulo its radical is isomorphic to a product of fields with each of $A$ be a semi-local regular domain with field of fractions $F$. Let $\ell$ be a prime not equal to $\text{char}(\kappa)$. Suppose that $N_1$ is a $(2, \ell)$-uniform bound for $\kappa(t)$ and $N_2$ is a $(2, \ell)$-uniform bound for $\kappa$. Then $\ell^3(N_1)(N_2)$ is a $(2, \ell)$-uniform bound for $K(t)$.

Theorem 3.6. Let $K$ be complete discretely valued field with residue field $\kappa$. Let $\ell$ be a prime not equal to $\text{char}(\kappa)$. Suppose that $N_1$ is a $(2, \ell)$-uniform bound for $\kappa(t)$ and $N_2$ is a $(2, \ell)$-uniform bound for $\kappa$. Then $\ell^3(N_1)(N_2)$ is a $(2, \ell)$-uniform bound for $K(t)$.

Proof. Let $F$ be a finite extension of $K(t)$. Let $\alpha_1, \ldots, \alpha_m \in H^2(F, \mu_\ell^\otimes2)$. Then, by (3.3), there exist $f, g, h \in F^*$ such that $\alpha_i \otimes F(\sqrt{f}, \sqrt{g}, \sqrt{h})$ is unramified at every codimension one point of any regular proper model of $F(\sqrt{f}, \sqrt{g}, \sqrt{h})$ over the valuation ring $R$ of $K$. Let $L = F(\sqrt{f}, \sqrt{g}, \sqrt{h})$. Let $\mathcal{Y}$ be a regular proper model of $L$ over $R$ and $Y$ the reduced special fiber of $\mathcal{Y}$.

Let $A$ be the semi-local ring at the generic points of all irreducible components of the special fiber $Y$ of $\mathcal{Y}$. Then $A$ is a semi-local regular ring with field of fractions $L$. Since each $\alpha_i \otimes_F L$ is unramified on $\mathcal{Y}$, there exists $\beta_i \in H^2_{\et}(A, \mu_\ell^\otimes2)$ such that $\beta_i \otimes_A L = \alpha_i$ for $1 \leq i \leq m$.

Let $\eta$ be the generic point of an irreducible component $Y_\eta$ of $Y$ and $m_\eta$ be the maximal ideal of $A$ associated to $\eta$. Since $\kappa(\eta) = A/m_\eta$ is a finite extension of $\kappa(t)$ and $N_1$ is a $(2, \ell)$-uniform bound for $\kappa(t)$, there exists a finite extension $s(\eta)$ of $\kappa(\eta)$ of degree at most $N_1$ such that $\beta_i \otimes_A s(\eta) = 0$ for all $i$. By (1.1), we assume that $s(\eta)$ is separable over $\kappa(\eta)$. By (3.4), there exists a field extension $L_1$ of $L$ of degree at most $N_1$ such that for every maximal ideal $m'_\eta$ of the integral closure $B_{m_\eta}$ of $A_{m_\eta}$ in $L$, $B_{m_\eta}/m'_\eta$ contains a subfield isomorphic to $s(\eta)$. Hence, by (3.5), $\alpha_i \otimes (L_1 \otimes L_\eta) = \beta_i \otimes (L_1 \otimes L_\eta) = 0$ for all $i$. By ([8, 5.8], [11, 1.17]), there
exists a non-empty open set $U_\eta$ of the component $Y_\eta$ of the special fiber $Y$, such that $\alpha_i \otimes L_1 \otimes L_{U_\eta} = 0$ for all $i$.

Let $\mathcal{P}$ be the finite set of closed points of $\mathcal{Y}$ which are not in $U_\eta$ for any $\eta$. Let $A_{\mathcal{P}}$ be the regular semi-local ring at the closed points of $\mathcal{P}$. Since each $\alpha_i$ is unramified on $\mathcal{Y}$, there exists $\beta_i \in H^2_{\text{et}}(A_{\mathcal{P}}, \mu_\ell^2)$ such that $\beta_i \otimes L = \alpha_i \otimes L$ (cf. 1.9). Let $P \in \mathcal{P}$. Since the residue field $\kappa(P)$ at $P$ is a finite extension of $\kappa$, by the assumption on $\kappa$, there exists an extension $s(P)$ of $\kappa(P)$ of degree at most $N_2$ such that $\beta_i \otimes s(P) = 0$ for all $i$. Once again, by (1.1), we assume that each $s(P)$ is a separable extension of $\kappa(P)$. Then, as above, by applying (3.4, 3.5) to $A$, $s(P)$ and $\beta_i$, there exists a field extension $L_2$ of $L$ of degree at most $N_2$ such that $\alpha_i \otimes (L_2 \otimes L_P) = 0$ for all $i$ and for all $P \in \mathcal{P}$.

Let $E = LL_1L_2$. Then arguing as in (2.4), using ([8, 9.12]), we conclude that $\alpha_i \otimes E = 0$ for all $i$. Since $[E : F] \leq [L : F][L_1 : L][L_2 : L] \leq \ell^3(N_1!(N_2!)$, the theorem follows.

**Corollary 3.7.** Let $\kappa$, $\ell$, $N_1$ and $N_2$ be as in (3.6). Let $\zeta$ be a primitive $\ell$th root of unity. Then $[K(\zeta), K]^{\ell^3(N_1!(N_2!)}$ is a $(2, \ell)$-uniform bound of $K(t)$.

The above corollary and (1.6) gives the following

**Corollary 3.8.** Let $\kappa$, $\ell$, $N_1$ and $N_2$ be as in (3.6). Let $\zeta$ be a primitive $\ell$th root of unity. Then $[K(\zeta), K]^{\ell^3(N_1!(N_2!)}$ is an $(n, \ell)$-uniform bound of $K(t)$ for all $n \geq 2$.

### 4. Symbol length and $u$-invariant

**Theorem 4.1.** Let $K$ be a complete discretely valued field with residue field $\kappa$. Let $\ell$ be a prime not equal to $\operatorname{char}(\kappa)$. Suppose that there exist integers $N_1$ and $N_2$ such that $\kappa(t)$ is $(2, \ell)$-uniformly bounded by $N_1$ and $\kappa$ is $(2, \ell)$-uniformly bounded by $N_2$. Let $n \geq 2$. Then there exists an integer $M$, which depends only on $N_1$, $N_2$ and $n$ such that for every finite extension $F$ of $K(t)$ and for all $\alpha \in H^n(F, \mu_\ell^n)$, $\lambda(\alpha) \leq M_n$.

**Proof.** By (3.7), $K(t)$ is $(2, \ell)$-uniformly bounded by $N = (\ell - 1)^\ell(N_1!(N_2!$. Hence any finite extension $F$ of $K(t)$ is also $(2, \ell)$-uniformly bounded by $N$. The theorem follows from (1.7). \qed

**Theorem 4.2.** Let $K$ be a complete discretely valued field with residue field $\kappa$. Let $p = \operatorname{char}(\kappa)$. Suppose that $\operatorname{char}(K) = 0$, $p > 0$ and $[\kappa : \kappa^p] = p^d$. Then there exists an integer $M$, which depends only on $d$ such that for any finite extension $F$ of $K(t)$ and for all $\alpha \in H^n(F, \mu_p^{\otimes n})$, $n \geq 1$, $\lambda(\alpha) \leq M$.

**Proof.** By (2.5), $K(t)$ is $(2, p)$-uniformly bounded by $(p - 1)p^{4d + 4}$. Let $F$ be a finite extension of $K(t)$. Then $F$ is also $(2, p)$-uniformly bounded by $(p - 1)p^{4d + 4}$. Let $n \geq 1$. By (1.7), there exists an integer $N_n$, which depends only on $d$ and $n$ such that for all $\alpha \in H^n(F, \mu_p^{\otimes n})$, $\lambda(\alpha) \leq N_n$. Since the $p$-cohomological dimension of $K$ is at most $d + 2$ ([6]) and $F$ is a function field in one variable over $K$, the $p$-cohomological dimension of $F$ is $d + 3$. Hence $H^n(F, \mu_p^{\otimes n}) = 0$ for all $n \geq d + 4$. Let $N$ be the maximum of $N_n$ for $2 \leq n \leq d + 3$. Then $\lambda(\alpha) \leq N$ for all $\alpha \in H^n(F, \mu_p^{\otimes n})$ and $n \geq 2$. \qed

**Theorem 4.3.** Let $K$ be complete discretely valued field with residue field $\kappa$ and $F$ a function field of a curve over $K$. Suppose that $\operatorname{char}(K) = 0$, $\operatorname{char}(\kappa) = 2$ and $[\kappa : \kappa^2]$ is finite. Then there exists an integer $M$ which depends only on $[\kappa : \kappa^2]$ such that for any finite extension $F$ of $K(t)$, $u(F) \leq M$. 

Proof. The theorem follows from (4.2) and (1.8).

We end with the following

Question 4.4. Let \( L \) be a field of characteristic not equal to 2 with \( u(L) \) finite. Is the Brauer group of \( L \) uniformly 2-bounded?

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