Nonperturbative late time asymptotics for heat kernel in gravity theory

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Abstract

Recently proposed nonlocal and nonperturbative late time behavior of the heat kernel is generalized to curved spacetimes. Heat kernel trace asymptotics is dominated by two terms one of which represents a trivial covariantization of the flat-space result and another one is given by the Gibbons-Hawking integral over asymptotically-flat infinity. Nonlocal terms of the effective action generated by this asymptotics might underly long-distance modifications of the Einstein theory motivated by the cosmological constant problem. New mechanisms of the cosmological constant induced by infrared effects of matter and graviton loops are briefly discussed.

1. Introduction

In this paper we continue the studies of nonperturbative infrared behavior in field-theoretical models initiated in [1]. In contrast to ultraviolet properties incorporating renormalized coupling constants – coefficients of local invariants in the action, infrared behavior manifests itself in nonlocal structures responsible for long-distance phenomena. The growing interest in such phenomena, especially in context of the gravitational theory, arises due to recent attempts of resolving the cosmological constant problem.
by means of long-distance modifications of Einstein theory. Moreover, these modifications often call for nonperturbative treatment in view of the nonlinear aspects of van Dam-Veltman-Zakharov discontinuity problem [2] and the presence of a hidden nonperturbative scale in gravitational models with extra dimensions [3]. On the other hand, nonlocalities also (and, moreover, primarily) arise in virtue of fundamental quantum effects of matter and graviton loops which, for instance, can play important role in gravitational radiation theory [4, 5] and cosmology [6]. Therefore, they can successfully compete with popular phenomenological mechanisms of infrared modifications, induced, say, by braneworld scenarios with extra dimensions [7, 8] or other models [9]. This makes nonperturbative analysis of nonlocal quantum effects very interesting and promising.

Nonlocal quantum effects can be described by the Schwinger-DeWitt proper time method based on the heat kernel

\[ K(s \mid x, y) = \exp \left[ s F(\nabla) \right] \delta(x, y), \]  

(1.1)

which is a main building block of Feynman diagrams with the inverse propagator \( F(\nabla) \) – the differential operator of field disturbances on some matter and gravitational field background [10, 11]. The infrared physics is then determined by the late time behavior of \( K(s) = \exp [s F(\nabla)] \) and by its functional trace that generates the one-loop effective action [10, 11, 1]

\[ \Gamma_{\text{one-loop}} \equiv \frac{1}{2} \text{Tr} \ln F(\nabla) = -\frac{1}{2} \int_0^\infty \frac{ds}{s} \text{Tr} K(s), \]  

(1.2)

\[ \text{Tr} K(s) \equiv \int dx K(s \mid x, x). \]  

(1.3)

In particular, nonlocal terms of \( \Gamma_{\text{one-loop}} \) arise as a contribution of the upper limit in the proper-time integral (1.2), which makes the late time asymptotics of \( \text{Tr} K(s) \) most important\(^1\).

The heat kernel trace, including its late time asymptotics, was first studied within the covariant nonlocal curvature expansion in [12, 14, 15, 16]. Then its nonperturbative asymptotics was obtained in [1] for a particular case of flat spacetime and used to derive new nonlocal and essentially nonlinear terms in the effective action (1.2). These terms represent a generalization of the logarithmic Coleman-Weinberg potential to the case

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\(^1\)Here the effective action is defined in Euclidean space with positive-signature metric. Its application in physical spacetime with Lorentzian signature is based on analytic continuation methods which range from a conventional Wick rotation in scattering theory (for in-out matrix elements) to a special retardation prescription in a wide class of problems for a mean field (in-in expectation value) [12, 13]. These methods nontrivially apply to nonlocal terms and extend from the usual perturbation theory to its partial resummation corresponding to the nonperturbative technique of the present work.
of nonconstant fields vanishing at infinity and, thus, generating special type of nonlocal behavior. The goal of this paper is to generalize it to curved spacetime with generic asymptotically-flat geometry. In the next section we begin by formulating our main results after briefly recapitulating the setting of the problem and conclusions of [1].

2. The setting of the problem and main results

In [1] it was shown that the heat kernel of the differential operator $F(\nabla)$ with generic potential $V(x)$ in flat Euclidean (positive signature) spacetime with $d$ dimensions,

$$F(\nabla) = \Box - V(x), \quad \Box = \nabla^\mu \nabla_\mu,$$

has a nonperturbative in potential late time expansion

$$K(s|x, y) = \frac{1}{(4\pi s)^{d/2}} \exp \left(-\frac{|x - y|^2}{4s}\right) \left[ \Phi(x) \Phi(y) + O\left(\frac{1}{s}\right) \right], \quad s \to \infty. \quad (2.2)$$

Its leading order behaviour is determined in terms of a special function $\Phi(x)$ – the solution of the homogeneous equation with unit boundary condition at infinity

$$F(\nabla) \Phi(x) = 0,$$

$$\Phi(x) \to 1, \quad |x| \to \infty. \quad (2.3)$$

This solution can be represented in a closed form in terms of the Green’s function of (2.1) with zero boundary conditions at $|x| \to \infty$, $G(x, y)$,

$$\Phi(x) = 1 + \frac{1}{\Box - V} V(x) \equiv 1 + \int dy G(x, y) V(y). \quad (2.4)$$

As a byproduct of this result it was also shown that the functional trace of this heat kernel has an asymptotic $1/s$-expansion beginning with

$$\text{Tr} K(s) = \frac{1}{(4\pi s)^{d/2}} \int dx \left[ -s V \Phi - 2 \nabla_\mu \Phi \frac{1}{\Box - V} \nabla^\mu \Phi + 1 + O\left(\frac{1}{s}\right) \right]. \quad (2.5)$$

Important property of this functional trace that was understood in [1] is that this expansion cannot be obtained directly by integrating the coincidence limit of the expansion (2.2). This happens because the latter is not uniform in $|x| \to \infty$ and, therefore, yields erroneous contribution when integrating over infinite spacetime. This explains, in particular, why the leading behaviour of (2.5) is $O(s/s^{d/2})$ in contrast to that of (2.2),
Nevertheless, (2.5) can be recovered from (2.2) by functionally integrating the variational equation

\[
\frac{\delta \text{Tr} K(s)}{\delta V(x)} = -s K(s|x, x). \tag{2.6}
\]

The leading term of the asymptotic expansion (2.5) was obtained in [1] exactly by this procedure. This result was also verified in [1] by a direct summation of the covariant perturbation theory developed for the heat kernel trace in [14]. The subleading term of (2.5) was derived entirely by the second method, because the corresponding term of the heat kernel (2.2) was not yet known.

In this paper we generalize the results of [1] to the case of the scalar operator (2.1) in curved spacetime with the covariant d’Alembertian (Laplacian in Euclidean space) defined with respect to generic asymptotically flat metric \(g_{\mu\nu}(x)\)

\[
\Box = g^{\mu\nu}(x) \nabla_{\mu} \nabla_{\nu} = \frac{1}{g^{1/2}(x)} \frac{\partial}{\partial x^\mu} g^{1/2}(x) g^{\mu\nu}(x) \frac{\partial}{\partial x^\nu}. \tag{2.7}
\]

We obtain the heat kernel in the first two orders of the \(1/s\)-expansion. It has the form

\[
K(s|x, y) = \frac{1}{(4\pi s)^{d/2}} \exp \left[ -\frac{\sigma(x, y)}{2s} \right] \times \left\{ \Phi(x) \Phi(y) + \frac{1}{s} \Omega_1(x, y) + O\left(\frac{1}{s^2}\right) \right\} g^{1/2}(y), \tag{2.8}
\]

where \(\sigma(x, y)\) is the the world function – one half of the geodesic distance between the points \(x\) and \(y\). The leading order is again defined by the function (2.4) which is determined in terms of the Green’s function,

\[
G(x, y) = \frac{1}{F(\nabla)} \delta(x, y) \tag{2.9}
\]

of the curved space operator \(F(\nabla)\) with the covariant d’Alembertian (2.7) and with Dirichlet boundary conditions at infinity\(^2\). The subleading term is more complicated, and the expression for \(\Omega_1(x, y)\) is presented in Sect.4 below.

We also derive the asymptotics of the functional trace of the heat kernel corresponding to (2.8)

\[
\text{Tr} K(s) = \frac{1}{(4\pi s)^{d/2}} \left\{ s W_0 + W_1 + O\left(\frac{1}{s}\right) \right\}. \tag{2.10}
\]

\(^2\)We define the \(\delta(x, y)\)-function as a scalar with respect to the first argument \(x\) and as a density of unit weight with respect to the second one – \(y\). Correspondingly the heat kernel defined by Eq.(1.1) and the kernel of the Green’s function \(G(x, y)\) have the same weights of their arguments. This asymmetry in \(x\) and \(y\) explains the presence of the factor \(g^{1/2}(y)\) in (2.8) and a biscalar nature of \(\Omega_1(x, y)\).
The leading term here turns out to be a covariantized (curved space) version of the same term in the flat-space trace (2.5) plus the surface integral of the local function of metric and its first-order derivatives at spacetime infinity,

\[
W_0 = -\int dx \, g^{1/2} V \Phi(x) + \frac{1}{6} \Sigma[g_\infty], \tag{2.11}
\]

\[
\Sigma[g_\infty] = \int |x| \to \infty d\sigma^\mu \delta^{\alpha\beta} \left( \partial_\alpha g_{\beta\mu} - \partial_\mu g_{\alpha\beta} \right). \tag{2.12}
\]

This surface integral over the sphere of radius $|x| \to \infty$ is written here in cartesian coordinates and involves only the flat-space asymptotics of the metric

\[
g_\infty^{\mu\nu}(x) \equiv g_{\mu\nu}^{(x)} \bigg|_{|x| \to \infty} = \delta_{\mu\nu} + O \left( \frac{1}{|x|^{d-2}} \right). \tag{2.13}
\]

Its covariant version in the form of the Gibbons-Hawking surface integral of the extrinsic curvature of the boundary is discussed in Sect. 6. We also demonstrate that the subleading term $W_1$ confirms the result of the direct summation of perturbation series in potential [1] in the flat-space case (2.5).

The organization of the paper is as follows. In Sect. 3 we derive the technique of recurrent equations for the coefficients of the $1/s$-expansion of the heat kernel and discuss the peculiarities of setting their boundary value problem. We apply this technique in the leading order of the asymptotic expansion and derive the $V$-dependent part of the algorithm (2.11). This is achieved by functionally integrating the variational equation (2.6) in the leading order of $1/s$-expansion. In Sect. 4 this technique is extended to the subleading order, and it is shown how it reproduces in flat spacetime the third term of (2.5) – the result obtained in [1] by tedious summation of nonlocal perturbation series. In Sect. 5 we perform a major check on the correctness of the metric dependence of the late-time asymptotics of $\text{Tr} K(s)$. The functional integration of Eq. (2.6) with respect to a potential determines $\text{Tr} K(s)$ only up to an arbitrary functional of the metric independent of $V$. So we derive the metric variational equation analogous to (2.6) and show that the bulk part of $W_0$ in (2.11) exactly satisfies this equation. However, local functional derivative with respect to $g_{\mu\nu}(x)$ in the bulk (that is for finite $|x|$) does not feel the asymptotic surface term of the form (2.12), so in order to establish the latter we compare in Sect. 6 the nonperturbative asymptotics of $\text{Tr} K(s)$ with its covariant curvature expansion of [15, 16]. This comparison confirms the bulk structure of the algorithms (2.11) and also fixes the additional surface integral – the Gibbons-Hawking term (2.12). As a byproduct of this procedure we establish a new representation for this surface term in the form of the bulk integral of the nonlocal Lagrangian which is expanded in covariant curvature series and explicitly independent of such auxiliary quantities as extrinsic curvature of the boundary. In the concluding section we list the
omissions of the proposed formalism, the prospects of its extension beyond the leading order and its generalizations to spacetimes with other than asymptotically-flat boundary conditions. We also briefly discuss the status of the cosmological constant induced by nonperturbative effective action which originates from the late time asymptotics of the above type. In the appendix we present the variational formalism used in the subleading order of the late time expansion.

3. Heat kernel and heat kernel trace at late times

To find late time asymptotics of the heat kernel in curved space we use the ansatz

\[ K(s|x,y) = \frac{1}{(4\pi s)^{d/2}} \exp \left[ -\frac{\sigma(x,y)}{2s} \right] \Omega(s|x,y) g^{1/2}(y), \tag{3.1} \]

\[ \Omega(s|x,y) = \Omega_0(x,y) + \frac{1}{s} \Omega_1(x,y) + O \left( \frac{1}{s^2} \right). \tag{3.2} \]

Here \( \sigma(x,y) \) is a world function – one half of the geodesic distance between the points \( x \) and \( y \) – satisfying the equation

\[ \frac{1}{2} g^{\mu\nu}(x) \nabla_\mu \sigma(x,y) \nabla_\nu \sigma(x,y) = \sigma(x,y). \tag{3.3} \]

This ansatz is motivated by the small time limit of the heat kernel in which \( \Omega(s|x,y) \) has a regular Schwinger-DeWitt expansion in powers of \( s \), \( \Omega(s|x,y) = \Delta^{1/2}(x,y) \left[ 1 + O(s) \right] \), where the overall factor \( \Delta^{1/2}(x,y) = g^{-1/4}(x) \left[ \det \partial_\mu \partial_\nu \sigma(x,y) \right] g^{-1/4}(y) \) is the (dedensitized) Pauli-Van Vleck-Morette determinant.

As we will see in what follows, disentangling of \( \Delta^{1/2}(x,y) \) as a separate factor in (3.1) is not useful for the purposes of late time expansion. However, the quantity is rather important and related to a serious simplifying assumption which underlies our results. The assumption we make is the absence of focal points in the congruence of geodesics determining the world function \( \sigma(x,y) \). We assume that for all pairs of points \( x \) and \( y \), \( \Delta(x,y) \neq 0 \), which guarantees that \( \sigma(x,y) \) is globally and uniquely defined on the asymptotically-flat spacetime in question. This assumption justifies the ansatz (3.1)-(3.2) which should be globally valid because the coefficients of the expansion (3.2) will satisfy elliptic boundary-value problems with boundary conditions at infinity. This requirement is independent of the asymptotic flatness because the presence of caustics in the geodesic flow, \( \Delta(x,y) = 0 \), might depend on local properties of the gravitational field, unrelated to its long-distance behavior. Roughly, the gravitational field should not be too strong to guarantee the geodesic convexity of the whole spacetime. This assumption might be too strong to incorporate physically interesting situations, but we believe that the main result will survive the presence of caustics (though, maybe
by the price of additional contributions which are essentially nonperturbative and go beyond the scope of this paper\(^3\).

Substituting the ansatz (3.1) in the heat equation
\[
\frac{\partial}{\partial s} K(s \mid x,y) = F(\nabla_x) \, K(s \mid x,y) \tag{3.4}
\]
one obtains the equation for the unknown function \(\Omega(s \mid x,y)\)
\[
\frac{\partial \Omega}{\partial s} + \frac{1}{s} \left( \sigma^\mu \nabla_\mu + \frac{1}{2} \Box \sigma - \frac{d}{2} \right) \Omega = F(\nabla) \Omega, \tag{3.5}
\]
where \(\sigma^\mu \equiv \nabla^\mu \sigma(x,y)\), and \(\Box \sigma \equiv \Box_x \sigma(x,y)\).

Assuming the validity of the \(1/s\)-expansion (3.2) for \(\Omega(s \mid x,y)\) at \(s \to \infty\) (which follows, in particular, from the perturbation theory for \(K(s \mid x,y)\) \([14, 1]\) – there is no nonanalytic terms in \(1/s\) like \(\ln(1/s)\)), one easily obtains the series of recurrent equations for the coefficients of this expansion. They start with
\[
F(\nabla) \Omega_0(x,y) = 0, \tag{3.6}
\]
\[
F(\nabla) \Omega_1(x,y) = \left( \sigma^\mu \nabla_\mu + \frac{1}{2} \Box \sigma - \frac{d}{2} \right) \Omega_0(x,y). \tag{3.7}
\]

An obvious difficulty with the choice of their concrete solution is that they do not form a well posed boundary value problem. Indeed, natural zero boundary conditions at infinity for the original kernel \(K(s \mid x,y)\) do not impose any boundary conditions on the function \(\Omega(s \mid x,y)\) except the restriction on the growth of \(\Omega(s \mid x,y)\) to be slower than \(\exp \left[ + \sigma(x,y) / 2s \right] \) (in view of the exponential factor in (3.1)). On the other hand, this freedom in choosing non-decreasing at \(|x| \to \infty\) solutions facilitates their existence. In particular, the elliptic equation (3.6) with positive definite operator \(F(\nabla)\) (which we assume) would not have nontrivial solutions decaying at spacetime infinity. Thus, the only remaining criterion for the selection of solutions in (3.6)-(3.7) is the requirement of their symmetry in the arguments \(x\) and \(y\). As we will see now, this criterion taken together with certain assumptions of naturalness result in concrete solutions which will be further checked on consistency by different methods including perturbation theory, the variational equation for the heat kernel trace (2.6) and its metric analogue, etc.

The way this strategy works in the leading order of the \(1/s\)-expansion was demonstrated in [1] and is as follows. Make a natural assumption that \(\Omega_0(x,y)\) at \(|x| \to \infty\) is

\(^3\)This hope is based on a simple fact that the leading order of the \(1/s\)-expansion – the primary object of this paper – is not sensitive to the properties of the world function at all (see Eq. (3.6) below, which does not involve \(\sigma(x,y)\)). Beyond this order the main object of interest, \(\text{Tr} K(s)\), involves the coincidence limit of the world function \(\sigma(x,x) = 0\), while its asymptotic coefficients in (3.2) nonlocally depend on global geometry and can acquire from caustics additional contributions analogous to those of multiple geodesics connecting the points \(x\) and \(y\) beyond the geodesically convex neighborhood [17].
not growing and independent of the angular direction \( n^\mu = x^\mu / |x| \) quantity \( C(y) \) – the function of only \( y \). Then the solution of the corresponding boundary value problem

\[
F(\nabla) \Omega_0(x, y) = 0, \\
\Omega_0(x, y) \bigg|_{|x| \to \infty} = C(y),
\]

is unique and reads \( \Omega_0(x, y) = \Phi(x) C(y) \), where \( \Phi(x) \) is a special function (2.4) solving the homogeneous equation subject to unit boundary conditions at infinity. Then, the requirement of symmetry in \( x \) and \( y \) implies that \( \Omega_0(x, y) = C \Phi(x) \Phi(y) \), where the value of the numerical normalization coefficient \( C = 1 \) follows from the comparison with the exactly known heat kernel in flat spacetime with vanishing potential \( V(x) = 0 \). Thus

\[
\Omega_0(x, y) = \Phi(x) \Phi(y). \tag{3.8}
\]

This answer was checked in [1] in few lowest orders of perturbation theory in powers of the potential.

Substituting the expansion (2.8) for the coincidence limit \( K(s | x, x) \) in the variational equation for \( \text{Tr} K(s) \) (2.6) one has the corresponding variational equation for \( W_0 \) in Eq.(2.10),

\[
\frac{\delta W_0}{\delta V(x)} = -g^{1/2}(x) \Omega_0(x, x) = -g^{1/2}(x)\Phi^2(x). \tag{3.9}
\]

Its integrability – the symmetry of the variation of its right-hand side with respect to \( V(y) \) in \( x \) and \( y \) – can be checked with the use of the following variational derivative

\[
\frac{\delta \Phi(x)}{\delta V(y)} = G(x, y) \Phi(y) \tag{3.10}
\]

which, in its turn, follows from the variation of the inverse operator

\[
\frac{\delta}{\delta V(y)} \frac{1}{F(\nabla)} = -\frac{1}{F(\nabla)} \frac{\delta F(\nabla)}{\delta V(y)} \frac{1}{F(\nabla)}. \tag{3.11}
\]

Applying (3.10) in the right-hand of (3.9) one finds

\[
\frac{\delta}{\delta V(y)} g^{1/2}(x) \Omega_0(x, x) = 2g^{1/2}(x) \Phi(x) G(x, y) \Phi(y), \tag{3.12}
\]

which is symmetric in \( x \) and \( y \) in view of the symmetry of the Green’s function \(^4\),

\[
g^{1/2}(x) G(x, y) = g^{1/2}(y) G(y, x). \tag{3.13}
\]

\(^4\)Which follows from the hermiticity of the operator \( F(\nabla) \) in the measure \( g^{1/2}(x) \) and the assumption that \( G(x, y) \) is a density with respect to \( y \).
Thus, the equation is integrable and its explicit solution (2.11) can be checked by direct variation again with the use of (3.10),

\[-\frac{\delta}{\delta V(y)} \int dx \, g^{1/2}(x) \Phi(x) = -g^{1/2}(x) \Phi(y) \left( 1 + \int dx \, G(y, x) \Phi(x) \right) \]

\[= -g^{1/2}(y) \Omega_0(y, y). \] (3.14)

4. Subleading order: particular case of flat spacetime

In the subleading order of $1/s$-expansion the situation is more complicated. The next coefficient $\Omega_1(x, y)$ satisfies the inhomogeneous equation (3.7) the right hand side of which can be rewritten in the form

\[F(\nabla) \Omega_1(x, y) = \frac{1}{2} \left[ \vec{F}(\nabla_x) \Phi(x) \sigma(x, y) - d \Phi(x) \right] \Phi(y) \] (4.1)

in view of the equation for $\Phi$, $F(\nabla) \Phi = 0$. A natural solution $\Omega_1(x, y) = \psi(x, y)/2$ with

\[\psi(x, y) = \frac{1}{F(\nabla_x)} \left[ \vec{F}(\nabla_x) \Phi(x) \sigma(x, y) - d \Phi(x) \right] \Phi(y) \] (4.2)

is not, however, correct because it violates the symmetry in $x$ and $y$. Symmetric solution differs from this one by some solution of the homogeneous equation. The latter can be obtained by projecting a rather generic two-point function $v(x, y)$ onto the space of solutions by the nonlocal projector $\Pi(\nabla_x)$

\[\Pi(\nabla) = 1 - \frac{1}{F(\nabla)} \vec{F}(\nabla). \] (4.3)

Here the arrow indicates the action of the differential operator in the direction opposite to its Green’s function, $1/F(\nabla)$, written in the operator form. That is, the action of this projector on $v(x, y)$ in

\[\Omega_1(x, y) = \frac{1}{2} \psi(x, y) + \Pi(\nabla_x) v(x, y) \] (4.4)

implies that

\[\Pi(\nabla_x) v(x, y) = v(x, y) - \int dz \, G(x, z) \vec{F}(\nabla_z) v(z, y), \] (4.5)

and the integration by parts that would reverse the action of $F(\nabla_x) \Phi(x)$ of $G(x, z)$ (and, thus, would lead to a complete cancellation of the first term) is impossible without generating nontrivial surface terms.
The needed symmetry of $\Omega_1(x, y)$ can be attained by choosing $v(x, y) = \psi(y, x)/2$ in (4.4) such that the special solution of the homogeneous equation takes the form

$$\Pi(\nabla_x) v(x, y) = \frac{1}{2} \psi(y, x) - \frac{1}{2} \psi(y, x) \frac{\vec{F}(\nabla_x) \frac{\nabla}{F(\nabla_x)}}{2}$$ (4.6)

(here we again use the operator notations for the Green’s function and the operator $\vec{F}(\nabla_x)$ acting, this time, on $\psi(y, x)$ from the right). Remarkably, in view of the structure of the function (4.2) and the equation $F(\nabla) \Phi(x) = 0$ the second term here turns out to be symmetric in $x$ and $y$. Therefore by adding (4.6) to the solution of the inhomogeneous equation $\psi(x, y)$ we finally obtain the needed symmetry of $\Omega_1(x, y)$

$$\Omega_1(x, y) = \frac{1}{2} \psi(x, y) + \frac{1}{2} \psi(y, x) - \frac{1}{2} \frac{1}{F(\nabla_x)} \vec{F}(\nabla_x) [\Phi(x) \sigma(x, y) \Phi(y)] \vec{F}(\nabla_y) \frac{\nabla}{F(\nabla_y)}$$ (4.7)

Interestingly, the analogue of the variational equation (3.9) for the subleading term of the $1/s$-expansion of $\text{Tr} K(s)$

$$\frac{\delta W_1}{\delta V(x)} = -g^{1/2}(x) \Omega_1(x, x)$$ (4.8)

also satisfies the integrability condition and has a formal solution in terms of the Green’s function of $F(\nabla)$. As shown in Appendix A it reads as

$$W_1 = \frac{1}{2} \int dx \, dy \, g^{1/2}(y) [\vec{F}(\nabla_x) \Phi(x) \sigma(x, y) \Phi(y) \vec{F}(\nabla_y)] G(y, x),$$ (4.9)

where the operators in square brackets are acting in the directions indicated by arrows on the arguments of $\Phi(x) \sigma(x, y) \Phi(y)$.

Unfortunately, however, the validity of the algorithms (4.7) and (4.9) can at the moment be rigorously established only in flat spacetime. Problem is that the nonlocal function $\psi(x, y)$ is well (and uniquely) defined only when the expression in square brackets of (4.2) sufficiently rapidly goes to zero at spacetime infinity. This expression has two terms

$$\vec{F}(\nabla_x) \Phi(x) \sigma(x, y) - d \Phi(x) = 2\sigma^{\mu}(x, y) \nabla_{\mu} \Phi(x) + \Phi(x) [\Box \sigma(x, y) - d].$$ (4.10)

The first term has a power law falloff $1/|x|^{d-2}$ at $|x| \to \infty$ in view of the behavior of $\sigma^{\mu}(x, y) \sim |x|$ and $\nabla_{\mu} \Phi(x) \sim 1/|x|^{d-1}$. This makes the contribution of this term (convolution with the kernel of Green’s function in (4.2)) well defined at least in dimensions
4. On the contrary, the second term is proportional to the deviation of geodesics
\[ \square \sigma(x, y) - d \] which has the following rather moderate falloff
\[ \square \sigma(x, y) - d \sim \frac{1}{|x|}, \quad |x| \to \infty. \] (4.11)

Therefore a purely metric contribution to (4.2) turns out to be quadratically divergent in the infrared. Tracing the origin of this difficulty back to the equation (3.7) we see that the source term in its right hand side is \( O(1/|x|) \), so that the solution \( \Omega_1(x, y) \sim |x| \) is not vanishing at infinity and, therefore, is not uniquely fixed by Dirichlet boundary conditions. Some principles of fixing this ambiguity would certainly regularize the integral in the definition of \( \psi(x, y) \) and uniquely specify all quantities in the subleading order. Unfortunately, we do not have these principles at the moment. That is why in what follows we will restrict the consideration of this order to the flat-space case where this problem does not arise at all.

In flat spacetime the geodesic deviation scalar (4.11) is identically vanishing, because
\[ g_{\mu \nu} = \delta_{\mu \nu}, \quad \sigma(x, y) = \frac{1}{2} |x - y|^2, \quad \sigma^\mu(x, y) = (x - y)^\mu, \]
\[ \square \sigma(x, y) = d. \] (4.12)

Therefore, the expression for \( \psi(x, y) \) becomes well defined. Correspondingly, in the square brackets of (4.9) only one term containing \( \nabla_\mu \nabla_\nu \sigma(x, y) = -\delta^{\mu \nu} \) survives and yields
\[ \bar{F}(\nabla_x) \Phi(x) \sigma(x, y) \Phi(y) \bar{F}(\nabla_y) = -4 \nabla_\mu \Phi(x) \nabla^\mu \Phi(y), \] so that \( \Omega_1(x, y) \) and the subleading term of the functional trace \( W_1 \) considerably simplify
\[ \Omega_1(x, y) = \frac{1}{\square - V} (x - y)^\mu \nabla_\mu \Phi(x) \Phi(y) + (x \leftrightarrow y) \]
\[ + 2 \frac{1}{\square - V} \nabla_\mu \Phi(x) \frac{1}{\square - V} \nabla^\mu \Phi(y), \] (4.13)
\[ W_1 = -2 \int dx \, dy \, \nabla_\mu \Phi(x) \nabla^\mu \Phi(y) G(y, x) = -2 \int dx \, \nabla_\mu \Phi \frac{1}{\square - V} \nabla^\mu \Phi(x). \] (4.14)

The last expression coincides with the second term of (2.5) obtained in [1] by direct summation of perturbation series and, thus, confirms the present nonperturbative (in the potential \( V \)) method.

5. Metric dependence

In this section we perform a major check on the validity of the asymptotics (2.11) in curved spacetime. It is determined by its functional derivative with respect to \( V(x) \).
only up to arbitrary metric functional. This functional can be determined from the metric variational derivative of \( \text{Tr} K(s) \) – the analogue of Eq.(2.6). So we derive the corresponding equation below and show that the asymptotics (2.11) indeed satisfies it, which confirms the spacetime integral (bulk) part of (2.11). Local variational derivative \( \delta \/ \delta g_{\mu \nu}(x) \) at finite \(|x|\) cannot probe possible surface integrals (of local combinations of metric and its derivatives) at spacetime infinity, so that the additional surface term (2.12) will be recovered in the next section by another method.

From the operator definition of the heat kernel (2.1) it follows that its metric variation reads

\[
\delta g \text{Tr} K(s) = -s \text{Tr} (\delta g F K(s)) = -s \int dx \delta g F(\nabla_x) K(s \mid x, x') \bigg|_{x'=x}, \tag{5.1}
\]

where the variation of the operator coincides with that of the covariant d’Alembertian acting on scalars (2.7) and equals

\[
\delta g F(\nabla) = \delta g \Box = -\delta g_{\mu \nu} \nabla^\mu \nabla^\nu - \frac{1}{2}(\nabla^\lambda \delta g_{\mu \nu})(\delta^\mu_{\lambda} \nabla^\nu + \delta^\nu_{\lambda} \nabla^\mu - g^{\mu \nu} \nabla^\lambda). \tag{5.2}
\]

The corresponding variational derivative can be rewritten in the form of the following integral bilinear in two test functions \( \varphi(x) \) and \( \psi(x) \)

\[
\int dx \ g^{1/2}(x) \ \frac{\delta F(\nabla)}{\delta g_{\mu \nu}(y)} \varphi(x) = -g^{1/2} f^{\mu \nu}(\nabla_x, \nabla_y) \varphi(x) \psi(y) \bigg|_{x=y} = -g^{1/2} \psi(y) f^{\mu \nu}(\nabla_y, \nabla_y) \varphi(y). \tag{5.3}
\]

The kernel of this local form is given by the differential operator \( f^{\mu \nu}(\nabla_x, \nabla_y) \) with covariant derivatives acting on two different arguments \( x \) and \( y \) (or correspondingly to the right and to the left as indicated above by arrows)

\[
f^{\mu \nu}(\nabla_x, \nabla_y) = -\nabla_x^\mu \nabla_y^\nu + \frac{1}{2} g^{\mu \nu} \Box_x + \frac{1}{2} g^{\mu \nu} \nabla_x^\lambda \nabla_x^\lambda. \tag{5.4}
\]

Using the expression (3.1) for \( K(s \mid x, y) \) and a simple relation \( f^{\mu \nu}(\nabla_x, \nabla_y) \sigma(x, y) \mid_{y=x} = g^{\mu \nu} \) one has

\[
\frac{\delta \text{Tr} K(s)}{\delta g_{\mu \nu}(x)} = -s g^{1/2}(x) \ f^{\mu \nu}(\nabla_x, \nabla_y) K(s \mid x, y) \bigg|_{x=y} = g^{1/2}(x) \bigg[ \frac{1}{2} g^{\mu \nu} \Omega(s \mid x, x) - s f^{\mu \nu}(\nabla_x, \nabla_y) \Omega(s \mid x, y) \bigg|_{y=x} \bigg] \tag{5.5}
\]

where the first term arises from the action of \( f^{\mu \nu}(\nabla_x, \nabla_y) \) on the exponential in \( K(s \mid x, y) \). Therefore the metric variational derivatives of \( \text{Tr} K(s) \) in the first two
orders of the $1/s$-expansion become

\[
\frac{\delta W_0}{\delta g_{\mu\nu}} = -g^{1/2} f^{\mu\nu}(\nabla_x, \nabla_y) \Omega_0(x,y) \Big|_{y=x}, \quad (5.6)
\]

\[
\frac{\delta W_1}{\delta g_{\mu\nu}} = \frac{1}{2} g^{1/2} g^{\mu\nu} \Phi^2(x) - g^{1/2} f^{\mu\nu}(\nabla_x, \nabla_y) \Omega_1(x,y) \Big|_{y=x}, \quad (5.7)
\]

where we took into account that $\Omega_0(x,x) = \Phi^2(x)$.

In the rest of this section we will focus at checking the relation (5.6). For this purpose let us first calculate its right hand side. After substituting the expression $\Omega_0(x,y) = \Phi(x) \Phi(y)$ and taking into account the relation $\Box \Phi = V \Phi$ one finds

\[-f^{\mu\nu}(\nabla_x, \nabla_y) \Omega_0(x,y) \Big|_{y=x} = -\frac{1}{2} \left( g^{\mu\nu} V \Phi^2 - 2 \nabla^\mu \Phi \nabla^\nu \Phi + g^{\mu\nu} \nabla^\lambda \Phi \nabla_\lambda \Phi \right). \quad (5.8)\]

To find the metric variational derivative in the left hand side of (5.6), we first note that the surface integral (2.12) does not contribute to it for any finite $|x|$. Then write down the variational derivative of $\Phi(x) = \Phi(x)[g_{\mu\nu}]$. The variation of the nonlocal Green’s function in (2.4) gives

\[
\frac{\delta \Phi(x)}{\delta g_{\mu\nu}(y)} = -\frac{1}{F(\nabla)} \frac{\delta F(\nabla)}{\delta g_{\mu\nu}(y)} V(x) = -\frac{1}{F(\nabla)} \frac{\delta F(\nabla)}{\delta g_{\mu\nu}(y)} (\Phi(x) - 1), \quad (5.9)
\]

where we used the relation $\left(1/F(\nabla)\right)V(x) = \Phi(x) - 1$. Then in view of the expression (5.3) for $\delta F(\nabla)/\delta g_{\mu\nu}$

\[
\frac{\delta \Phi(x)}{\delta g_{\mu\nu}(y)} = G(x, y) f^{\mu\nu}(\nabla_y, \nabla_y) \Phi(y). \quad (5.10)
\]

Let us integrate this equation over $x$ with the (densitized) potential $g^{1/2} V(x)$. Then using the symmetry of the Green’s function of $F(\Box)$ one has

\[
\int dx \ g^{1/2}(x) V(x) \frac{\delta \Phi(x)}{\delta g_{\mu\nu}(y)} = g^{1/2}(y) \ (\Phi(y) - 1) \ f^{\mu\nu}(\nabla_y, \nabla_y) \Phi(y), \quad (5.11)
\]

or in view of the expression for $f^{\mu\nu}(\nabla_y, \nabla_y)$

\[
\int dx \ g^{1/2}(x) V(x) \frac{\delta \Phi(x)}{\delta g_{\mu\nu}(y)} = \frac{1}{2} g^{1/2} \left( - g^{\mu\nu} V \Phi + g^{\mu\nu} V \Phi^2 - 2 \nabla^\mu \Phi \nabla^\nu \Phi + g^{\mu\nu} \nabla^\lambda \Phi \nabla_\lambda \Phi \right)(y). \quad (5.12)
\]

Thus finally

\[
\frac{\delta}{\delta g_{\mu\nu}(y)} \int dx \ g^{1/2} \left( - V \Phi \right) = -\frac{1}{2} g^{1/2} \left( g^{\mu\nu} V \Phi^2 - 2 \nabla^\mu \Phi \nabla^\nu \Phi + g^{\mu\nu} \nabla^\lambda \Phi \nabla_\lambda \Phi \right), \quad (5.13)
\]

the first term in the right hand side of (5.12) being cancelled by the variation of $g^{1/2}$ in the integration measure. Comparison with (5.8) finally confirms the relation (5.6).
6. Comparison with perturbation theory

Comparison with perturbation theory in flat space has actually been done in Sect. 4. There the leading and subleading orders of \( \text{Tr} K(s) \) were shown to coincide with those of (2.5), which in turn were obtained in [1] by direct summation of the covariant perturbation series in potential. Here we will make a similar check for the metric part of \( W_0 \) and, in particular, reveal the metric surface term (2.12).

The leading order of the \( 1/s \)-expansion for \( \text{Tr} K(s) \) was obtained up to cubic order in curvature and potential \( \Re = (V, R_{\mu\nu}) \) in [15, 16]. For a scalar operator (2.1), (2.7), it looks like

\[
\text{Tr} K(s) = \frac{s}{(4\pi s)^{d/2}} \int dx g^{1/2} \left\{ P - P \frac{1}{\Box} P + \frac{1}{3} P \frac{1}{\Box} R - \frac{1}{6} R_{\mu\nu} \frac{1}{\Box} R^{\mu\nu} + \frac{1}{18} R \frac{1}{\Box} R \\
+ P \left( \frac{1}{\Box} P \right) \frac{1}{\Box} P - \frac{1}{6} R \left( \frac{1}{\Box} P \right) \frac{1}{\Box} P - \frac{1}{3} P \left( \frac{1}{\Box} P \right) \frac{1}{\Box} R \\
+ \frac{1}{36} P \left( \frac{1}{\Box} R \right) \frac{1}{\Box} R + \frac{1}{18} R \left( \frac{1}{\Box} R \right) \frac{1}{\Box} P - \frac{1}{216} R \left( \frac{1}{\Box} R \right) \frac{1}{\Box} R \\
+ \frac{1}{12} R \left( \frac{1}{\Box} R^{\mu\nu} \right) \frac{1}{\Box} R_{\mu\nu} - \frac{1}{6} R^{\mu\nu} \left( \frac{1}{\Box} R_{\mu\nu} \right) \frac{1}{\Box} R \\
+ \frac{1}{6} \left( \frac{1}{\Box} R^{\alpha\beta} \right) \left( \nabla_\alpha \frac{1}{\Box} R \right) \nabla_\beta \frac{1}{\Box} R \\
- \frac{1}{3} \left( \nabla_\mu \frac{1}{\Box} R^{\mu\alpha} \right) \left( \nabla_\nu \frac{1}{\Box} R_{\mu\alpha} \right) \frac{1}{\Box} R \\
- \frac{1}{3} \left( \frac{1}{\Box} R^{\mu\nu} \right) \left( \nabla_\mu \frac{1}{\Box} R^{\alpha\beta} \right) \nabla_\nu \frac{1}{\Box} R_{\alpha\beta} + O[\Re^4] \right\} + O \left( \frac{1}{s^{d/2}} \right), \quad s \to \infty.
\]

(6.1)

Here \( P(x) \) is the redefined potential term of the operator,

\[
P(x) = \frac{1}{6} R(x) - V(x),
\]

(6.2)

and every Green’s functions of the covariant curved-space d’Alembertian \( \Box = g^\mu\nu \nabla_\mu \nabla_\nu \), \( 1/\Box \), is acting on the nearest curvature or a potential standing to the right of it. The tensor nature of the Green’s function is not explicitly specified here by assuming that it is always determined by the nature of the quantity acted upon by \( 1/\Box \). To clarify how efficiently these condensed notations allow one to simplify the presentation, we explicitly write as an example one of the nonlocal factors above, \( (1/\Box)R_{\mu\nu}(x) \). Manifestly it reads as

\[
\frac{1}{\Box} R_{\mu\nu}(x) \equiv \int dy G_{\mu\nu}^{\alpha\beta}(x, y) R_{\alpha\beta}(y),
\]

(6.3)
where $G_{\mu\nu}^{\alpha\beta}(x, y), \Box G_{\mu\nu}^{\alpha\beta}(x, y) = \delta_{\mu\nu}^{\alpha\beta}(x, y), G_{\mu\nu}^{\alpha\beta}(x, y)|_{|x|\to\infty} = 0$, is the Green’s function of $\Box$ acting on a second-rank symmetric tensor field with zero boundary conditions at infinity.

Using (6.2) in (6.1) one finds that the leading order term of $\text{Tr} K(s)$ consists of two parts – one explicitly featuring only the original potential $V$ acted upon by Green’s functions of the curved-space $\Box$’s and another purely metric one

$$W_0 = -\int dx \frac{1}{\Box} V(x) \left( \frac{1}{\Box} V(x) \right)^2 = \int dx \frac{1}{\Box} V(x) \frac{1}{\Box} V(x),$$

(6.5)

where now all Green’s function $1/\Box$ are acting to the right. The expression (6.4) should now be compared with the expansion of (2.11) in powers of $\mathcal{R} = (V, R_{\mu\nu})$.

The nonlocal expansion of $\Phi(x)$ in (2.11)

$$\Phi(x) = 1 + \frac{1}{\Box} V(x) + \frac{1}{\Box} V(x) + O\left(V^3\right)$$

(6.6)

obviously recovers the first integral of (6.4) explicitly containing only powers of potential with metric-dependent nonlocalities. The second integral in (6.4) of purely metric nature seems to be completely missing in the expression (2.11) for $W_0$. We have to clarify why this term does not violate the metric variational equation (5.6) that was directly checked above.

A crucial observation is that this term is a topological invariant independent of local metric variations in the interior of spacetime – exactly in this class of $\delta g_{\mu\nu}(x)$ the functional derivative of (5.6) was calculated in Sect.5. Direct expansion in powers of the metric perturbation $h_{\mu\nu}, g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}$, on flat-space background in cartesian coordinates shows that this term reduces to the surface integral at spacetime infinity. For the class of asymptotically flat metrics with $h_{\mu\nu}(x) \sim 1/|x|^{d-2}, |x| \to \infty$, this surface integral is linear in perturbations (contributions of higher powers of $h_{\mu\nu}$ to this
integral vanish) and involves only a local asymptotic behavior of the metric \( g_\infty \equiv \delta_{\mu\nu} + h_{\mu\nu}(x) |_{|x|\to\infty} \),

\[
\int dx \, g^{1/2} \left\{ R - R_{\mu\nu} \frac{1}{R} R_{\mu\nu} + \frac{1}{2} \frac{1}{R} \right\}
\]
\[
+ \frac{1}{2} R \left( \frac{1}{R} R_{\mu\nu} \right) \frac{1}{R_{\mu\nu}} - R_{\mu\nu} \left( \frac{1}{R} R_{\mu\nu} \right) \frac{1}{R}
\]
\[
+ \left( \frac{1}{R} R_{\alpha\beta} \right) \left( \nabla_\alpha \frac{1}{R} R_{\beta} \right)
\]
\[
- 2 \left( \nabla_\mu \frac{1}{R} R_{\alpha\Gamma} \right) \left( \nabla_\nu \frac{1}{R} R_{\mu\alpha} \right) \frac{1}{R}
\]
\[
- 2 \left( \frac{1}{R} R_{\mu\nu} \right) \left( \nabla_\mu \frac{1}{R} R_{\alpha\beta} \right) \nabla_\nu \frac{1}{R} R_{\alpha\beta} + O[R_{\mu\nu}^4]
\]
\[
= \int |x|\to\infty d\sigma^\mu \left( \partial^\nu h_{\mu\nu} - \partial_\mu h \right) \equiv \Sigma[g_\infty]. \tag{6.7}
\]

Here \( d\sigma^\mu \) is the surface element on the sphere of radius \(|x| \to \infty\), \( \partial^\mu = \delta^\mu_{\nu} \partial_\nu \) and \( h = \delta^\mu_{\nu} h_{\mu\nu} \). Covariant way to check this relation is to calculate the metric variation of this integral and show that its integrand is the total divergence which yields the surface term of the above type linear in \( \delta g_{\mu\nu}(x) = h_{\mu\nu}(x) \). Thus, the correct expression for \( W_0 \) modified by the metric functional integration “constant” \( \Sigma[g_\infty] \) is indeed given by Eqs.(2.11)-(2.12), and this constant does not contribute to the metric variational derivative \( \delta W_0 / \delta g_{\mu\nu}(x) \) at any finite \(|x| \).

For asymptotically-flat metrics with a power-law falloff at infinity \( h_{\mu\nu}(x) \sim M / |x|^{d-2} \); \(|x| \to \infty\), the contribution of \( \Sigma[g_\infty] \) is finite and nonvanishing. For example, for \((d+1)\)-dimensional Einstein action foliated by asymptotically-flat \(d\)-dimensional spatial surfaces this surface integral yields exactly the ADM energy \( M \) of the gravitational system. In a covariant form it can also be rewritten as a Gibbons-Hawking term \( S_{\text{GH}}[g] = \Sigma[g_\infty] \) – the double of the extrinsic curvature trace \( K \) on the boundary (with a properly subtracted infinite contribution of the flat-space background) \cite{18}

\[
\Sigma[g_\infty] = -2 \int_{\infty} d^{d-1}\sigma \left( g^{(d-1)} \right)^{1/2} \left( K - K_0 \right). \tag{6.8}
\]

Thus, this is the surface integral of the local function of the boundary metric and its normal derivative. The virtue of the relation (6.7) is that it expresses this surface integral in the form of the spacetime (bulk) integral of the nonlocal functional of the bulk metric. The latter does not explicitly contain auxiliary structures like the vector field normal to the boundary, though these structures are implicitly encoded in boundary conditions for nonlocal operations in the bulk integrand of (6.7). It should be mentioned here that the nontrivial equation (6.7) enlarges the list of relations between
nonlocal invariants derived in [19]. The difference of this relation from those of [19] is that it is an infinite series in curvatures and forms a nonvanishing topological invariant, while the relations of [19] are homogeneously cubic in curvatures and hold only for low spacetime dimensionalities $d < 6$.

Note also, in passing, that the definition of the topological invariant (6.7) can be rewritten as the nonlocal curvature expansion of the (Euclidean) Einstein-Hilbert action [13]. It is important that this expansion begins with the quadratic order in the curvature

$$- \int dx \, g^{1/2} R(g) - 2 \int_{\infty}^\infty d^{d-1} \sigma \left( g^{(d-1)} \right)^{1/2} (K - K_0)$$

$$= \int dx \, g^{1/2} \left\{ -\left( R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) \frac{1}{\Box} R_{\mu\nu} + O \left[ R^3_{\mu\nu} \right] \right\}, \quad (6.9)$$

and the corresponding quadratic form is linear in the Einstein tensor – the fact that was earlier observed, up to surface terms, in [20] (see Eq.(112) in this reference). This observation can serve as a basis for covariantly consistent nonlocal modifications of Einstein theory [13] motivated by the cosmological constant and cosmological acceleration problems [21].

To summarize this section, we conclude that perturbation theory confirms, up to the local surface term, the nonperturbative algorithm for the leading order of the $1/s$-expansion. Apparently, this surface term can also be grasped by the variational technique of Sect.5 which will be done elsewhere\textsuperscript{5}.

7. Discussion: induced cosmological constant and nonlocal effective action

Thus we have generalized the heat-kernel asymptotics of [1] to curved asymptotically-flat spacetimes. Together with the trivial covariantization of the flat-space bulk integral (2.5) this generalization includes the Gibbons-Hawking surface integral (2.12) of the extrinsic curvature of the boundary.

Apart from this integral the leading asymptotics vanishes in the absence of the potential $V$ which encodes non-gravitational (or matter) fields of the system. This has a simple qualitative explanation. Pure gravity has two derivatives in the interaction vertex, which improves its infrared behavior – graviton scattering amplitudes have no

\textsuperscript{5}To attain the surface term in (2.11) one should remember that the variational equation (5.1) is based on the cyclic property of the operator product under the sign of the functional trace. This, in turn, is equivalent to integration by parts without extra surface terms. This property is violated in the lowest (first) order in the curvature [14] which gives rise to the surface term of (2.11).
infrared divergences even despite the massless nature of the field. This fully agrees with the vanishing of the leading asymptotics of $\text{Tr} K(s)$ for the effects probing local geometry, providing better convergence properties of the integral (1.2) at $s \to \infty$.

The contribution of the Gibbons-Hawking term probes only global quantities like the ADM energy defined by the integral over infinitely remote boundary. Therefore, it seems to be robust against ultraviolet structure of the theory and is likely to be universal for a wide class of models independently of their microscopic nature. Apparently, this serves as a justification for the phenomenological long-distance modifications of gravity theory motivated by the cosmological constant problem [21]. The nonlocal representation of the Einstein-Hilbert action (6.9) plays important role in such modifications because it underlies the construction of their covariant actions [13].

These modifications might arise not only within braneworld theories like GRS [7] or DGP [8] models. Rather, they can be mediated by new nonperturbative nonlocal contributions to the quantum effective action [1]. In their turn, these contributions originate from the infrared asymptotics of the above type. As soon as the results of [1] are generalized to curved spacetime, these effects can be directly analyzed in gravitational models of interest and are currently under study [22]. In connection with this it is worth sketching possible directions of the further research. Clearly, they incorporate possible generalizations of the obtained results and should provide closing the loopholes in our formalism above.

One important generalization consists in overstepping the limits of the asymptotically-flat spacetime. The simplest thing to do is to consider the asymptotically deSitter boundary conditions. On the one hand, they are strongly motivated by the cosmological acceleration phenomenon and, on the other hand, by the dS/CFT-correspondence conjecture inspired from string theory [23]. This generalization implies essential modification of both perturbative and nonperturbative techniques for the heat kernel, the generalization of the Gibbons-Hawking term to asymptotically dS-spacetimes, etc. Another generalization concerns the inclusion of fields of higher spins with the covariant derivatives in the d'Alembertian involving not only the metric connection but the gauge field connection as well.

Open issues include the modification due to possible violation of geodesic convexity in curved spacetime and the extension to higher orders of late time expansion. Interestingly, both (seemingly different) issues might be related because they both involve the geodesic deviation of Eq.(4.11). Possible contribution of caustics, briefly discussed in Sect. 3 above, might be important because it is likely to give qualitatively new terms originating from summation over multiple geodesics [17]. These terms cannot be reached by perturbations in contrast to the partial summation of perturbation series underlying our present results.
On the other hand, higher orders of the 1/s-expansion can be important within the cosmological constant problem. In particular, the subleading order $O(1/s^{d/2})$ incorporates the cosmological term of the quantum effective action (1.2). Indeed, this term is expected to appear as a covariantization of the third term ($\sim 1$) in the flat-space asymptotics (2.5) of $\text{Tr} \, K(s)$,

$$\frac{1}{s^{d/2}} \int dx \times 1 \to \frac{1}{s^{d/2}} \int dx \, g^{1/2}(x) \times 1.$$  

(7.1)

Interestingly, it has the same form also in the limit of $s \to 0$, determined by the first coefficient $a_0(x, x) = 1$ of the Schwinger-DeWitt expansion [10, 1]. Via the integral (1.2) it generates the ultraviolet-divergent cosmological term

$$\Gamma_\Lambda = \Lambda_\infty \int dx \, g^{1/2}, \quad \Lambda_\infty = -\frac{1}{2(4\pi)^{d/2}} \int_0^\infty \frac{ds}{s^{1+d/2}}.$$  

(7.2)

In fact, this expression is also infrared divergent in the coordinate sense – the volume integral $\int dx \, g^{1/2}$ for asymptotically-flat spacetime diverges at $|x| \to \infty$.

Of course, the abundance of divergences indicates that the cosmological constant cannot consistently arise in asymptotically-flat spacetime. The contribution (7.2) in massless theories does not carry any sensible physical information and is cancelled due to a number of interrelated mechanisms. First, its cancellation is guaranteed by the contribution of the local path-integral measure to the effective action, which annihilates strongest (volume) divergences under appropriate regularization of the path integral [24]. Another mechanism is based on the use of the dimensional regularization which puts to zero all power-like divergences. Interestingly, in the latter case this happens due to exact cancellation of the ultraviolet divergence of (7.2) at $s = 0$ against its infrared counterpart at $s \to \infty$. This may be regarded as a well-known statement that the cosmological constant problem is of both infrared and ultraviolet nature\(^6\). All these mechanisms, however, stop working for massive theories or for theories with spontaneously broken symmetry, where the induced vacuum energy presents a real hierarchy problem [25].

Preliminary results of Sect. 4 for $W_1$ allow one to look at the above mechanisms from a somewhat different viewpoint. To begin with, the cosmological term structure $\int dx \, g^{1/2}$ in $\text{Tr} \, K(s)$ behaves differently at late times and at $s \to 0$. In contrast to the $s \to 0$ limit, this term is completely absent at $s \to \infty$ – the functional $W_1$ given by (4.9) does not contain the part of zeroth order in the curvature and potential (indeed, in the absence of the potential the expression (4.9) is linear in $\Box_x \Box_y \sigma(x, y)$ and vanishes

---

\(^6\)In dimensional regularization the integral (7.2) is analytically continued to the domain of $d$ where it is convergent either at the lower (ultraviolet) or upper (infrared) limits. The pole parts of these two complimentary divergences are opposite in sign and cancel one another.
for flat spacetime). The functional $W_1$ was recovered from the variational derivative with respect to $V(x)$, Eq. (2.6), so one could have expected that the cosmological term should have been added to (4.9) as a functional integration "constant". But this is not the case, because $W_1$ exactly satisfies the metric variational equation (5.7) (this will be shown elsewhere [22]).

On the other hand, it was mentioned that in view of the slow falloff properties of the geodesic deviation (4.11), the expression (4.9) cannot be trusted beyond flat spacetime. However, the fact that it formally passes a subtle check of Eq.(5.7) suggests that under certain regularization of the divergent integrals the algorithm (4.9) will survive the transition to curved spacetime. In contrast to (2.5), it does not contain the cosmological term integral $\sim \int dx g^{1/2}$. This does not, however, indicate major contradiction with the covariant curvature expansion of [14, 15, 16], because this integral is formal and infrared divergent, which reflects the continuity of the spectrum of the operator $F(\nabla)$ in asymptotically-flat spacetime\textsuperscript{7}.

Altogether, this might qualitatively alter the mechanisms of induced cosmological constant and, in particular, exclude exact cancellation of its ultraviolet and infrared contributions occurring in the dimensional regularization case above. This alteration is likely to result in a nonlocal effective action of the type suggested in [1]. In fact, the origin of nonlocality is similar to the nonlocal representation of the Einstein action (6.9) generated by the subtraction of the linear in metric perturbation part of the bulk integral. Effective subtraction due to $W_1$ is currently under study. We expect that this might bring to light interesting interplay between the cosmological constant problem and infrared asymptotics of the heat kernel and nonlocal effective action.

**Appendix A. Tr $K(s)$ in subleading order**

The check of the integrability condition for (4.8) in the subleading order is based on (3.10). It gives

$$\frac{\delta}{\delta V(y)} g^{1/2}(x) \Omega_1(x,x) = g^{1/2}(x) G(x,y) \psi(y,x) + (x \leftrightarrow y)$$

$$- g^{1/2}(x) \Phi(x) \left[ G(x,y) \sigma(x,y) + d G^2(x,y) \right] \Phi(y)$$

$$- g^{1/2}(x) G(x,y) \frac{1}{F_x} \Rightarrow F_x \left[ \Phi(x) \sigma(x,y) \Phi(y) \right] \Rightarrow F_y \frac{1}{F_y}, \quad (A.1)$$

\textsuperscript{7}The lowest order of this expansion is responsible for this infrared divergent integral, and the subtlety in its treatment was clearly emphasized in [14]. Reconsidering its contribution with a special emphasis on surface integrals at spacetime infinity is currently under study both within the perturbation theory and the nonperturbative technique of the present paper [22].
where for brevity we denoted by $F_x = F(\nabla_x)$ and
\[
G^2(x,y) \equiv \frac{1}{F(\nabla_x)} G(x,y) = \frac{1}{F^2(\nabla_x)} \delta(x,y). \tag{A.2}
\]
The symmetry of this expression in $x$ and $y$ guarantees the existence of the solution $W_1$. Direct verification of this solution given by Eq.(4.9) looks as follows.

To begin with, the expression (4.9) can be rewritten in the form
\[
W_1 = \frac{1}{2} \int dx \, g^{1/2}(x) \frac{1}{F(\nabla_x)} \widehat{F}(\nabla_x) \Phi(x) \sigma(x,y) \Phi(y) \left. \frac{\nabla}{\nabla} F_y \right|_{y=x}^{y=\infty}, \tag{A.3}
\]
where the Green’s function is represented in the operator form as $1/F(\nabla_x)$ acting on the $x$-argument of $\widehat{F}(\nabla_x) \Phi(x) \sigma(x,y) \Phi(y) \frac{\nabla}{\nabla} F_y$ with a subsequent identification of $y$ and $x$. Then its variational derivative equals
\[
\frac{\delta W_1}{\delta V(x)} = -\Phi(x) \sigma(x,y) \Phi(y) \left. \frac{\nabla}{\nabla} F_y \right|_{y=x} \left[ \frac{1}{2} \int dy dz \, \frac{\nabla}{\nabla} \left[ \Phi(x) \sigma(z,y) \Phi(y) \right] \frac{\nabla}{\nabla} F_y \right] g^{1/2}(y) G(y,z)
\]
\[
+ \frac{1}{2} \int dy dz \, \frac{\nabla}{\nabla} \left[ \Phi(x) \sigma(x,y) \Phi(y) \right] \frac{\nabla}{\nabla} F_y \left. \frac{\nabla}{\nabla} F_y \right|_{y=x} \left[ \Phi(x) \sigma(z,y) \Phi(y) \right] \frac{\nabla}{\nabla}, \tag{A.4}
\]
where the first two terms arise from the variation of operators $(F(\nabla_x), F(\nabla_y))$ and functions $(\Phi(x), \Phi(y))$ in Eq.(A.3), while the third term corresponds to the variation of the Green’s function. In the second term one can integrate by parts without extra surface terms so that $F_z = F(\nabla_z)$ would act on $G(y,z)$, because $\left[ G(z,x) \Phi(x) \sigma(z,y) \Phi(y) \right]$ vanishes at $|z| \to \infty$. This removes integration over $z$ and yields the coincidence limit $F_y \sigma(y,z)\big|_{z=y} = d$,
\[
\int dy dz \, G(y,z) \frac{\nabla}{\nabla} \left[ \Phi(x) \sigma(z,y) \Phi(y) \right] \frac{\nabla}{\nabla} F_y \left. g^{1/2}(y) \right|_{y=z} = \int dy \, g^{1/2}(y) G(y,x) \Phi(x) \left[ \sigma(z,y) \Phi(y) \right] \frac{\nabla}{\nabla} F_y \left|_{z=y} \right.
\]
\[
= d \, g^{1/2} \Phi(x) \left. \frac{\nabla}{\nabla} F_x \right|_{z=y}. \tag{A.5}
\]
The action of the Green’s function on $\Phi(x)$ here is not a well defined operation because $\Phi(x) \to 1$ at infinity and the integral is infrared divergent. However, the first term of (A.4) is also divergent, and together with (A.5) it forms the expression $-g^{1/2} \psi(x,x)$ (which is well-defined at least for a flat spacetime). Therefore,
\[
\frac{\delta W_1}{\delta V(x)} = -g^{1/2} \psi(x,x) + \frac{1}{2} \int dy \, G^{1/2} \frac{1}{F_x} \left. \frac{\nabla}{\nabla} \left[ \Phi(x) \sigma(x,y) \Phi(y) \right] \frac{\nabla}{\nabla} F_y \right|_{y=x}
\]
\[
= -g^{1/2} \Omega_1(x,x), \tag{A.6}
\]
which finally proves the needed equation (4.8).

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