Traversability of Multi-Boundary Wormholes

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ABSTRACT: We generalize the Gao-Jafferis-Wall construction of traversable two-sided wormholes to multi-boundary wormholes. In our construction, we take the background spacetime as multi-boundary black holes in AdS\textsubscript{3} in the hot limit where the dual CFT state in certain regions locally resembles the thermofield double state. Furthermore, in these regions, the causal shadow is exponentially small in this limit. Based on these two features of the hot limit, and with the three-boundary wormhole as our main example, we show that traversability between any two asymptotic regions in a multi-boundary wormhole can be triggered using a double-trace deformation. In particular, the two boundary region need not have the same temperature and angular momentum. We discuss the non-trivial angular dependence of traversability in our construction, as well as the effect of the causal shadow region.
1 Introduction

Wormholes have long been of interest since the time of Einstein and Rosen [1]. Although Einstein-Rosen bridges, or more generally, multi-boundary wormholes connect different asymptotic regions of spacetime, topological censorship [2, 3] forbids their traversability when only classical matter fields are present. However in some cases, quantum matter fields can cause violations of the averaged null energy condition (ANEC), and wormholes could thus be made traversable. ANEC says that the integral of stress tensor along any complete null geodesic is non-negative,

$$\int_{\gamma} T_{ab}k^{a}k^{b} \geq 0. \quad (1.1)$$
In recent years, there have been many approaches to constructing traversable wormholes from ANEC violations, see [4–11]. In particular, in the seminal paper by Gao, Jafferis and Wall [4], the authors construct a traversable wormhole using a two-sided BTZ black hole as the background, where the dual CFT state is the thermofield double (TFD) state. With an appropriate sign of coupling, a double-trace deformation that directly couples the two boundary CFTs can cause the violation of ANEC. Under this coupling, the horizons shift and allow certain timelike geodesics that connect the two asymptotic boundaries. In [5], this construction is generalized to rotating BTZ black holes. This protocol was interpreted in [9] as a quantum teleportation protocol between entangled quantum systems. This connection with quantum information aspects of the traversability protocol have been of great interest recently (e.g. see [12–17]) as a concrete realization of the ER=EPR idea [18].

In this paper, we generalize this construction to any pair of asymptotic regions in a multi-boundary black hole (non-rotating or rotating) in AdS$_3$. For a general multi-boundary black hole, there exists a causal shadow region between horizons of different asymptotic regions, which makes it hard to traverse. In our construction, we focus on the the hot limit considered in [19], where the temperatures related to all horizons are large. In that limit, for any two horizons, there exists a region where the causal shadow between them is exponentially small, so a double-trace deformation can easily render the wormhole traversable. As we will see, the hot limit will also give us convenience in doing the calculations, which otherwise would be difficult to perform.

There are some interesting features in our construction compared with [4] and [5]. The first is that the pair of boundaries in our traversable wormhole construction is quite general in that they can have different temperatures and angular momenta. Besides, there is a non-trivial angular dependence of traversability. From one asymptotic region, signals can reach another particular asymptotic region only when it is shot in some range of the angular coordinate. Otherwise it may traverse to other asymptotic regions, or end up in the black hole interior.

In section 2, we review the construction of multi-boundary wormholes in AdS$_3$ and their important properties that will be useful in later sections. The geometry of these wormholes in the hot limit is also discussed, as well as the entanglement structure of the dual CFT state. Then, a general review of the Gao-Jafferis-Wall construction is given in section 3, where emphasis is given on a general type of coupling between the boundaries that can induce traversability. Based on these two ingredients, we construct the multi-boundary traversable wormhole in section 4. We summarize our findings and discuss their implications and connections with recent work in the literature in section 5. A number of technical details and supporting calculations are left to the appendices.

Note that, while the terms can be used interchangeably, our choice is to use “multi-boundary black holes” when the context refers to the background spacetime, and use “multi-boundary wormholes” when the context refers to traversability in particular.
2 Multi-boundary black holes in AdS

In this section, we will first review how to construct multi-boundary black holes by quotienting empty AdS with isometries, following an algebraic approach [20–25]. Then we discuss fixed points of those isometries, (renormalized) geodesic distances in different conformal frames, and how they behave in the hot limit. Those results will be useful in our construction of multi-boundary traversable wormholes. Finally, we briefly describe the CFT states that are dual to these geometries.

2.1 Quotients of AdS

In three-dimensional Einstein gravity, the Ricci tensor completely specifies the Riemann tensor. The consequence of this is that all gravity solutions are locally isometric to AdS, which is the Lorentzian, maximally-symmetric spacetime with constant negative curvature and isometry group $SO(2,2) \simeq SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$. Besides pure AdS, other solutions to the equations of motion are locally AdS but differ globally from it and can be obtained by quotienting AdS by a discrete subgroup $\Gamma$ of $SO(2,2)$. Throughout the paper, we take the AdS radius $L_{\text{AdS}} = 1$. The spacetime AdS can be defined as the submanifold of $\mathbb{R}^{2,2} = \{ p = \begin{pmatrix} U + X & -V + Y \\ V + Y & U - X \end{pmatrix} \}$, where we defined the 4-vector $\bar{x}^a = (U,V,X,Y)$ and metric $\bar{\eta}_{ab} = \text{diag}(-1,-1,1,1)$. In global coordinates, this hyperboloid is parametrized by the intrinsic coordinates $(t,r,\phi)$ defined by

$$X = r \cos \phi, \quad Y = r \sin \phi, \quad U = \sqrt{1 + r^2} \cos t, \quad V = \sqrt{1 + r^2} \sin t$$

which gives the induced metric

$$ds^2 = -(1 + r^2)dt^2 + \frac{dr^2}{1 + r^2} + r^2 d\phi^2$$

where $t \sim t + 2\pi$ and $\phi \sim \phi + 2\pi$. The connected part of the group $SO_c(2,2)$ is $SL(2,\mathbb{R}) \otimes SL(2,\mathbb{R})/\mathbb{Z}_2$. The group elements $(g_L,g_R) \in SO_c(2,2)$ act on a point $p$ according to

$$p \rightarrow g_L p g_R^t.$$

From this, we see that the $\mathbb{Z}_2$ symmetry correspond to the equivalence relation $(g_L,g_R) \sim (-g_L, -g_R)$. A convenient basis of generators $\{J_1,J_2,J_3\} \times \{\bar{J}_1, \bar{J}_2, \bar{J}_3\}$ of the isometry group

\footnote{For construction of these geometries using explicit forms of the Killing vectors, see [26].}

\footnote{$dp$ is the matrix defined by taking the differential of every element of the matrix $p$.}

\footnote{Usually the universal cover of $t$ is taken by unwrapping it, but as we will see, it is not necessary here since the wormhole constructions will automatically remove closed timelike curves.}
\( SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \) is

\[
J_1 \equiv -\frac{1}{2} (J_{XU} - J_{YV}), \quad \tilde{J}_1 \equiv -\frac{1}{2} (J_{XU} + J_{YV}) \\
J_2 \equiv -\frac{1}{2} (J_{YU} + J_{XV}), \quad \tilde{J}_2 \equiv -\frac{1}{2} (J_{YU} - J_{XV}) \\
J_3 \equiv -\frac{1}{2} (J_{UV} - J_{XY}), \quad \tilde{J}_3 \equiv \frac{1}{2} (J_{UV} + J_{XY})
\]

(2.5)

where the Killing vectors \( J_{ab} = \bar{x}_a \partial_b - \bar{x}_b \partial_a \) obey the \( SO(2, 2) \) algebra

\[
[J_{ab}, J_{cd}] = \bar{\eta}_{ac} J_{bd} - \bar{\eta}_{ad} J_{bc} - \bar{\eta}_{bd} J_{ac}
\]

(2.6)

In matrix representation, the generators are expressed as

\[
J_1 = -\frac{1}{2} \gamma_1, \quad J_2 = -\frac{1}{2} \gamma_2, \quad J_3 = -\frac{1}{2} \gamma_3
\]

(2.7)

where

\[
\gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

(2.8)

and similarly for \( \tilde{J}_i \).

To understand the action of the group elements \((g_L, g_R)\), we will describe \(AdS_3\) as the group manifold of \(SL(2, \mathbb{R})\), with the Penrose diagram shown in figure 1. The action of group elements \(g \in SL(2, \mathbb{R})\) on the identity element \(e\) is shown there, according to which they are classified into conjugacy classes depending on where the point \(e \to ge g_t = gg_t\) lies,

- Hyperbolic \( \text{Tr} \, g > 2 \) \( gg_t \in I \)
- Hyperbolic \( \text{Tr} \, g < -2 \) \( gg_t \in II \)
- Elliptic \( |\text{Tr} \, g| < 2 \) \( gg_t \in III, IV \)
- Parabolic \( |\text{Tr} \, g| = 2 \) \( gg_t \in \text{light cones} \)

We will focus on the action of subgroups \( \Gamma \subseteq SO_c(2, 2) \) with \( \text{Tr} \, g > 2 \) hyperbolic elements, whose fixed points are on the boundary of \(AdS_3\). This is because it ensures that \(AdS_3/\Gamma\) is free of conical singularities and closed timelike curves \([20, 22]\). Removing from the spacetime the past and future of those fixed points yields the restricted spacetime \(\tilde{AdS}_3\) where the action of the quotient on the spacetime is free of pathologies and leads to a spacetime \(\tilde{AdS}_3/\Gamma\). We will illustrate this process by reviewing the construction of \(\tilde{AdS}_3/\Gamma\) in the case of BTZ black holes \([27, 28]\) and three-boundary black holes \([20–22]\). We also discuss generalizations to \(n\)-boundary black holes with and without non-trivial topologies \([20, 22, 24]\). A Cauchy slice of these geometries is a Riemannian manifold of genus \(g\) and boundary number \(n\). So, we can

5Our matrix representation of \(p\) is different from that defined in \([24, 25]\), which causes the generators to be slightly different.

6In matrix representation, \(\tilde{J}_i\) takes the same matrix form as \(J_i = -\frac{1}{2} \gamma_i\) but the infinitesimal transformations on \(p\) are different from those of \(J_i\)'s, since \(J_i : p \to -\frac{1}{2} \gamma_i p\) while \(\tilde{J}_i : p \to -\frac{1}{2} p \gamma_i^t\).
classify the black hole geometries by a 2-tuple \((n, g)\). In the non-rotating case, the number of parameters (or in other words, dimension of the moduli space) needed to specify the \((n, g)\) geometry is equal to 1 for \((2, 0)\) and is \(6g - 6 + 3n\) otherwise. In the rotating case, this number is doubled.

Before reviewing the construction of these geometries, we will give general formulas for calculating the geodesic distance. The group manifold representation allow us to easily calculate the geodesic distances \(d(p, q)\) between two arbitrary points, \(p\) and \(q\) [23]. In particular, if \(p\) and \(q\) are connected by a spacelike geodesic, then

\[
d(p, q) = \cosh^{-1} \left( \frac{\text{Tr} \left( p^{-1} q \right)}{2} \right). \tag{2.9}
\]

With a timelike geodesic connecting \(p\) and \(q\), the geodesic distance is

\[
d(p, q) = \cos^{-1} \left( \frac{\text{Tr} \left( p^{-1} q \right)}{2} \right). \tag{2.10}
\]

When \(\text{Tr} \left( p^{-1} q \right) < -2\), there is no geodesic connecting \(p\) and \(q\).

**BTZ black hole**

In this case, the subgroup \(\Gamma\) is generated by a single element

\[
\gamma_{BTZ} = (g_{L, BTZ}, g_{R, BTZ}) = \left( e^{\xi_{L, BTZ}}, e^{\xi_{R, BTZ}} \right) \tag{2.11}
\]
Figure 2: A Cauchy slice of a BTZ black hole shown as a quotient of AdS$_3$. The action of $\gamma$ identifies the two blue geodesics, and the region between them is the fundamental domain of the quotient. The minimal geodesic $H$ separating the two coincides with the event horizon of the black hole. In the non-rotating case, this slice is at $t = 0$. But in the case of rotation, there is a relative boost between the two identified geodesics.

and a convenient choice for $\xi_{L,\text{BTZ}}$ and $\xi_{R,\text{BTZ}}$ is

$$\xi_{L,\text{BTZ}} = -J_2, \quad \xi_{R,\text{BTZ}} = -\tilde{J}_2$$

with $\ell = 2\pi(r_+ + r_-)$ and $\tilde{\ell} = 2\pi(r_+ - r_-)$ being two positive real parameters. In matrix representation, this gives

$$g_{L,\text{BTZ}} = \begin{pmatrix} \cosh\left(\frac{\ell}{2}\right) \sinh\left(\frac{\ell}{2}\right) \\ \sinh\left(\frac{\ell}{2}\right) \cosh\left(\frac{\ell}{2}\right) \end{pmatrix}, \quad g_{R,\text{BTZ}} = \begin{pmatrix} \cosh\left(\frac{\tilde{\ell}}{2}\right) \sinh\left(\frac{\tilde{\ell}}{2}\right) \\ \sinh\left(\frac{\tilde{\ell}}{2}\right) \cosh\left(\frac{\tilde{\ell}}{2}\right) \end{pmatrix}.$$  

The isometry $\gamma$ has two fixed points at the boundary given by $t = 0, \phi = \pi/2$ and $t = 0, \phi = 3\pi/2$. Removing the past and future regions of these fixed points gives the restricted space $\text{AdS}_3$. Any two geodesics that are related by the isometry $\gamma_{\text{BTZ}}$ are identified, and we can choose a region that is bounded by such a pair of geodesics as the fundamental domain of $\text{AdS}_3/\Gamma$, see figure 2. The minimal length between these two geodesics is uniquely determined by $r_+$ and $r_-$, and is the intersection of the geodesic connecting the fixed points with the fundamental domain. This defines the two-sided BTZ black hole, where each side is covered by the usual BTZ coordinates

$$ds^2 = -\frac{(r_B^2 - r_+^2)(r_B^2 - r_-^2)}{r_B^2} dt_B^2 + \frac{r_B^2}{(r_B^2 - r_+^2)(r_B^2 - r_-^2)} dr_B^2 + (d\phi_B - \frac{r_+ - r_-}{r_B^2} dt_B)^2$$

(2.14)
where the subscript B means that we are using BTZ coordinates. The thermodynamic quantities related to the black hole are

\[
M = \frac{r_+^2 + r_-^2}{8G_N} = \ell^2 + \tilde{\ell}^2, \quad J = \frac{r_+ r_-}{4G_N} = \frac{\ell^2 - \tilde{\ell}^2}{4G_N},
\]

\[
T_H = \frac{1}{\beta} = \frac{r_+^2 - r_-^2}{2\pi r_+} = \frac{\ell^2}{2\pi^2 (\ell + \tilde{\ell})}, \quad \Omega_H = \frac{r_-^2}{r_+} = \frac{\ell - \tilde{\ell}}{\ell + \tilde{\ell}}.
\]

(2.15)

By writing the point \(p\) in (2.1) in terms of the BTZ coordinates using the transformation

\[
U = \sqrt{\frac{r_B^2 - r_-^2}{r_+^2 - r_-^2}} \cosh (r_+ \phi_B + r_- t_B), \quad X = \sqrt{\frac{r_B^2 - r_+^2}{r_+^2 - r_-^2}} \cosh (r_+ t_B + r_- \phi_B),
\]

\[
V = \sqrt{\frac{r_B^2 - r_+^2}{r_+^2 - r_-^2}} \sinh (r_+ t_B + r_- \phi_B), \quad Y = \sqrt{\frac{r_B^2 - r_-^2}{r_+^2 - r_-^2}} \sinh (r_+ \phi_B + r_- t_B)
\]

(2.16)

one can show that the action of \(\gamma_{BTZ}\) on \(p\) is simply to map \(\phi_B \rightarrow \phi_B + 2\pi\). The length of the bifurcation surface (horizon length) generated by \(\gamma\) can be found from (2.9) to be [23]

\[
h = \cosh^{-1} \left( \frac{\Tr g_{L,BTZ}}{2} \right) + \cosh^{-1} \left( \frac{\Tr g_{R,BTZ}}{2} \right)
\]

(2.17)

From (2.11), we see that this gives the expected horizon length of \(\ell + \tilde{\ell} = 2\pi r_+\).

**Three-boundary black hole**

The subgroup \(\Gamma\) in this case is generated by two elements \(\gamma_i = (g_{iL}, g_{iR}), i = 1, 2\). We choose the first one to be the same as the isometry used to construct the BTZ black hole\(^7\)

\[
\gamma_1 = (g_{1L}, g_{1R}) = (e^{\ell_1 \xi_{1L}}, e^{\tilde{\ell}_1 \xi_{1R}})
\]

(2.18)

where \(\xi_{1L} = -J_2\) and \(\xi_{1R} = -\tilde{J}_2\). The second element is given by

\[
\gamma_2 = (g_{2L}, g_{2R}) = (e^{\ell_2 \xi_{2L}}, e^{\tilde{\ell}_2 \xi_{2R}})
\]

(2.19)

where \(\xi_{2L} = -(J_2 \cosh \alpha + J_3 \sinh \alpha)\) and \(\xi_{2R} = -(\tilde{J}_2 \cosh \tilde{\alpha} + \tilde{J}_3 \sinh \tilde{\alpha})\). In matrix representation, this is

\[
g_{2L} = \begin{pmatrix}
\cosh \left( \frac{\ell_2}{2} \right) & e^\alpha \sinh \left( \frac{\ell_2}{2} \right) \\
e^{-\alpha} \sinh \left( \frac{\ell_2}{2} \right) & \cosh \left( \frac{\ell_2}{2} \right)
\end{pmatrix}, \quad g_{2R} = \begin{pmatrix}
\cosh \left( \frac{\tilde{\ell}_2}{2} \right) & e^{\tilde{\alpha}} \sinh \left( \frac{\tilde{\ell}_2}{2} \right) \\
e^{-\tilde{\alpha}} \sinh \left( \frac{\tilde{\ell}_2}{2} \right) & \cosh \left( \frac{\tilde{\ell}_2}{2} \right)
\end{pmatrix}
\]

(2.20)

These two isometries define the first and second asymptotic regions, with the event horizons of these regions lying along the geodesics connecting the fixed points of \(\gamma_1\) and \(\gamma_2\), respectively.

\(^7\)Note that, here, the choice of generators \(\gamma_i\) is not unique. Other choices could be used, as long as they fall in certain conjugacy classes. Our choice here is convenient for calculation, but as we will see, it suffers from the issue that the third boundary region apparently vanishes in the hot limit. In appendix A, we give an example of another construction of the same geometry and discuss how it differs from the one used here.
Figure 3: A Cauchy slice of the three-boundary black hole shown as a quotient of AdS$_3$. The action of $\gamma_1$ identifies the two blue geodesics while $\gamma_2$ identifies the two red geodesics. The event horizons of the three boundaries $H_1$, $H_2$, and $H_3 = H'_3 \cup H''_3$ are also shown, where each of them coincide with the geodesic connecting the fixed points of the isometries $\gamma_1$, $\gamma_2$, and $\gamma_3$, respectively. Note that $\gamma_3$ has four fixed points instead of two, because it defines the third asymptotic region as the union of two separate regions in the Cauchy slice. In the case of no rotation, this slice is that of $t=0$.

The isometries that define the third asymptotic region are not independent of the above two. They are $\gamma'_3 = -\gamma_1 \gamma_2^{-1} \Rightarrow (g'_{3L}, g'_{3R}) = (-g_{1L} g_{2L}^{-1}, -g_{1R} g_{2R}^{-1})$ and $\gamma''_3 = -\gamma_1^{-1} \gamma_2 \Rightarrow (g''_{3L}, g''_{3R}) = (-g_{1L} g_{2L}, -g_{1R} g_{2R})^8$, corresponding to the two parts of the third boundary region as seen from the covering space. The resulting spacetime is a black hole with three asymptotic boundaries, as shown in figure 3. The spacetime in each asymptotic region is isometric to the exterior region of a BTZ black hole. Hence, each asymptotic region can be covered by the same metric (2.14) for $r_B > r_+$. The lengths of the horizons generated by these isometries can be found from (2.17) to be

$$h_1 = \frac{\ell_1 + \tilde{\ell}_1}{2}, \quad h_2 = \frac{\ell_2 + \tilde{\ell}_2}{2}, \quad \text{and} \quad h_3 = \frac{\ell_3 + \tilde{\ell}_3}{2}. \quad (2.21)$$

where we have defined

$$\ell_3 \equiv 2 \cosh^{-1} \left( \frac{\text{Tr} g_{3L}}{2} \right), \quad \text{and} \quad \tilde{\ell}_3 \equiv 2 \cosh^{-1} \left( \frac{\text{Tr} g_{3R}}{2} \right). \quad (2.22)$$

The parameter $\alpha$ can in turn be expressed using $\ell_i, i = 1, 2, 3$:

$$\cosh \alpha = \frac{\cosh \frac{\ell_1}{2} + \cosh \frac{\ell_2}{2} \cosh \frac{\ell_3}{2}}{\sinh \frac{\ell_1}{2} \sinh \frac{\ell_2}{2}} \quad (2.23)$$

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8Although $\gamma'_3$ and $\gamma''_3$ are both isometries defining the third region, for simplicity of notation, later we will refer to them collectively as $\gamma_3$. 

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and similarly for $\tilde{\alpha}$. Each asymptotic region can be associated with independent thermodynamic parameters (2.15). The angular velocity associated to a horizon generated by an isometry $\gamma_i$ can be given in terms of the isometry elements as [25]

$$\Omega_i = \frac{\cosh^{-1} \left( \frac{\text{Tr} g_{iL}}{2} \right) - \cosh^{-1} \left( \frac{\text{Tr} g_{iR}}{2} \right)}{\cosh^{-1} \left( \frac{\text{Tr} g_{iL}}{2} \right) + \cosh^{-1} \left( \frac{\text{Tr} g_{iR}}{2} \right)}$$

(2.24)

which gives

$$\Omega_1 = \frac{\ell_1 - \tilde{\ell}_1}{\ell_1 + \tilde{\ell}_1}, \quad \Omega_2 = \frac{\ell_2 - \tilde{\ell}_2}{\ell_2 + \tilde{\ell}_2}, \quad \text{and} \quad \Omega_3 = \frac{\ell_3 - \tilde{\ell}_3}{\ell_3 + \tilde{\ell}_3}.$$  

(2.25)

for the three boundaries. From this and the fact that the horizon lengths $h_i$ are given by $2\pi r_{+,-,i}$, we can relate the geometric parameters $\ell_i$ and $\tilde{\ell}_i$ for each boundary to the inner and outer horizon lengths of the corresponding black hole using

$$r_{+,-,i} = \frac{\ell_i \pm \tilde{\ell}_i}{4\pi}$$

(2.26)

for $i = 1, 2, 3$. We see that setting $\tilde{\ell}_i = 0$ corresponds to the extremal case\footnote{Here we have implicitly chosen a direction of spinning. For the other choice, $\ell_i = 0$ would correspond to an extremal black hole.}, while setting $\ell_i = \tilde{\ell}_i$ corresponds to the non-rotating case. The unique feature of (3, 0) geometry (and any geometry $(n, g)$ other than BTZ) is the existence of a region between the horizons $H_1$, $H_2$, and $H_3$ that does not intersect the causal past and future of any asymptotic region. This region is called the causal shadow of the spacetime [29], and it will be important in our discussion of traversability below. The causal shadow region is bounded by closed geodesics, which allow us to calculate its area using the Gauss-Bonnet theorem, giving $A_{CS} = 2(n - 2 + 2g)\pi$ for general $(n, g)$ spacetimes [19]. This shows that the causal shadow region exists for all geometries except (2, 0).

General $(n, g)$ black holes

More general black hole geometries can be constructed following the same method as discussed above. For the case without rotations, general $(n, g)$ geometries could be constructed using a cut-and-paste procedure [20, 22], and this could be easily generalized to cases with rotations, as we review below.

The simplest way to see this is to note that any $(n, g)$ black hole can be constructed from $2g + n - 2$ copies of the (3, 0) geometry (so-called “pair-of-pants” geometry) through a process of cutting, twisting, and gluing. Since the (3, 0) geometry is everywhere locally AdS3, the geometry that results from a process of cutting, twisting, and gluing different copies of it is also locally AdS3 and, therefore, is a solution of Einstein gravity. We will illustrate this process in the case of $n$ asymptotic regions and in case of genus $g$.

For instance, to construct the rotating (4, 0) geometry, we need two pairs of pants, each having 6 parameters (i.e. the mass and angular momentum of each asymptotic region). We
Figure 4: Construction of the $(4,0)$ and $(1,1)$ geometries using two and one pairs of pants, respectively. The dashed lines represent horizons of asymptotic regions. Note that each pair of pants is constructed from the process shown in figure 3, but here the shape of the Riemann surface is shown explicitly.

consider the Cauchy slices where both pairs are of the form shown in figure 3. As shown in figure 4a, if we cut only one asymptotic region in each of the pair of pants and glue the horizons together, this forces the lengths and orientations of the glued horizons to be equal (the $ℓ$’s and $\tilde{ℓ}$’s of the two glued regions) and introduces two new twist parameters. So, the total number of parameters is 12, which is the correct dimension of the moduli space of the rotating $(4,0)$ geometry. From the resulting Cauchy slice, we can time evolve and obtain the whole required geometry. Similarly, to construct general rotating $(n,0)$ geometries, we need $n - 2$ pairs of pants. By cutting $2n - 6$ asymptotic regions and gluing them together, we can construct a Cauchy slice of the rotating $(n,0)$ spacetime from which the whole geometry can be obtained by time evolution. One can easily check that the number of parameters in the resulting geometry is the correct dimension of the moduli space, which is $2 (3n - 6)$.

In the case of non-zero genus, we consider the simple case of rotating $(1,1)$ spacetime, which was first constructed in [24]. Using a Cauchy slice of a single rotating $(3,0)$ geometry, we can cut two asymptotic regions and then glue their horizons together. The remaining asymptotic region is now the exterior of a rotating BTZ black hole with the topology of a
torus behind the horizon, as shown in figure 4b. One can easily check that this process gives 
the correct number of dimensions of the moduli space, which is 6 in the case of rotating (1, 1) 
spacetime.

2.2 Fixed points and the conformal boundary

We now discuss the action of isometries $\gamma \in \Gamma$ on the conformal boundary of AdS$_3$, following 
the method discussed in [24]. Here we will be using the conformal frame

$$ds^2_{\text{global}} = -dt^2 + d\phi^2$$

which is naturally related to the global coordinates.

Taking $r \to \infty$ for a bulk point $p$ (2.1) gives a boundary point $p_\partial$. Up to a diverging 
factor, it is

$$p_\partial \propto (\cos \phi + \cos t, \sin \phi - \sin t) = 2(\cos \frac{u}{2} \cos \frac{v}{2} - \cos \frac{u}{2} \sin \frac{v}{2}) = 2\vec{v} \vec{u}$$

where

$$\vec{v} = \begin{pmatrix} \cos \frac{v}{2} \\ \sin \frac{v}{2} \end{pmatrix}, \quad \vec{u} = \begin{pmatrix} \cos \frac{u}{2} \\ -\sin \frac{u}{2} \end{pmatrix}$$

and $v = t + \phi$ and $u = t - \phi$ are the null coordinates at the boundary. The isometries of 
interest $\gamma = (g_L, g_R) \in \Gamma$ are hyperbolic elements with their fixed points at the boundary of 
AdS$_3$. Being a fixed point amounts to

$$p_\partial = g_L p_\partial g_R^t \Rightarrow \vec{v} \vec{u} = g_L \vec{v}(g_R \vec{u})^t,$$

where the equality holds up to an overall factor, since we are on the conformal boundary.

This means that we could find fixed points by finding eigenvectors of $g_L$ and $g_R$. In 
general, $g_L$ and $g_R$ each have two eigenvectors, and combinations of them give “corners” of 
the “boundary diamond” of $\gamma$ where the action of $\gamma$ takes place. Next, we will illustrate these 
notions for the BTZ black hole and the three-boundary black hole. Analysis of fixed points 
for general $(n, g)$ geometries could be performed in a similar manner.

For the BTZ black hole, all elements of $\Gamma$ are integer powers of $\gamma_{\text{BTZ}}$. Both $g_{L, \text{BTZ}}$ and 
$g_{R, \text{BTZ}}$ have two eigenvectors

$$g_{L, \text{BTZ}} \vec{v}_\pm = e^{\pm \ell/2} \vec{v}_\pm, \quad g_{R, \text{BTZ}} \vec{u}_\pm = e^{\pm \ell/2} \vec{u}_\pm$$

where

$$\vec{v}_\pm = \frac{1}{\sqrt{2}} \begin{pmatrix} \pm 1 \\ 1 \end{pmatrix}, \quad \vec{u}_\pm = \frac{1}{\sqrt{2}} \begin{pmatrix} \pm 1 \\ 1 \end{pmatrix}$$

As shown in figure 5, there only two boundary diamonds for the BTZ black hole, with their 
left and right corners at $(t = 0, \phi = \pi/2)$ and $(t = 0, \phi = 3\pi/2)$. Inside each diamond, there
Figure 5: Boundary diamonds for the BTZ black hole, where $\phi \sim \phi + 2\pi$. As we can see, there are two diamonds, each containing one asymptotic boundary of the fundamental domain.

are infinitely many copies of the fundamental domain, or in other words, the fundamental domain and its images.

For the three-boundary black hole, we could find the fixed points and boundary diamonds in a similar manner. But in this case, we have infinitely many fixed points (and diamonds) since the group $\Gamma$ not only contains elements like $\gamma_i^m$, $i = 1, 2$ but also more general “words” like $\gamma_1^m \gamma_2^n \gamma_1^k$... etc. For $\gamma_i$, $i = 1, 2$ we have

$$g_{iL}\vec{v}_{+,i} = e^{\pm \ell_i/2} \vec{v}_{+,i}, \quad g_{iR}\vec{u}_{+,i} = e^{\pm \ell_i/2} \vec{u}_{+,i}$$

with $\vec{v}_{+,1}$ and $\vec{u}_{+,1}$ the same as those of the BTZ black hole, and

$$\vec{v}_{\pm,2} = \frac{1}{\sqrt{1 + e^{2\alpha}}} \begin{pmatrix} e^{\alpha} \\ 1 \end{pmatrix}, \quad \vec{u}_{\pm,2} = \frac{1}{\sqrt{1 + e^{2\alpha}}} \begin{pmatrix} e^{\tilde{\alpha}} \\ 1 \end{pmatrix}.$$ (2.34)

For the three-boundary black hole, the three asymptotic boundaries of the fundamental domain are contained in the diamonds which we call “fundamental diamonds” generated by $\gamma_i$, $i = 1, 2, 3$. Other diamonds will be dubbed “image diamonds”. In figure 6, we show the fundamental diamonds of the three-boundary black hole. The corners of the fundamental diamonds can be found from

$$p_{+,i} = \vec{v}_{+,i} \vec{u}_{+,i}^d, \quad p_{+,i} = \vec{v}_{+,i} \vec{u}_{+,i}^d, \quad p_{-,i} = \vec{v}_{-,i} \vec{u}_{+,i}^d, \quad p_{-,i} = \vec{v}_{-,i} \vec{u}_{-,i}^d.$$ (2.35)

where again $i = 1, 2, 3$.

For any point $p_\theta$ on the $i^{th}$ asymptotic region of the fundamental domain, there are two types of image points under the group action:

1. Points that are in the same fundamental diamond as $p_\theta$: these points are generated by acting on $p_\theta$ with isometries that only involve integer powers of $\gamma_i$;
2. Points that are in the image diamonds: these points are generated by acting with other kinds of isometries on \( p_\partial \).

Although it is hard to find all the image diamonds, we are sure that they must lie between diamond 1 and 2, and must be spacelike separated from both of them. The boundary distance from the left corner of diamond 1 (\( p_{++1} \)) to the right corner of diamond 2 (\( p_{++2} \)) is

\[
d_{\text{bdy}}(p_{++1}, p_{++2}) = \sqrt{\frac{\pi}{2} - 2 \tan^{-1} e^{-\alpha}} \left( \frac{\pi}{2} - 2 \tan^{-1} e^{-\tilde{\alpha}} \right)
\]

(2.36)

When \( \alpha \) and \( \tilde{\alpha} \) are small (i.e. \( \ell_i \) and \( \tilde{\ell}_i \) are large), to leading order, the distance is

\[
d_{\text{bdy}}(p_{++1}, p_{++2}) = (\alpha \tilde{\alpha})^{1\over 2} + O((\alpha \tilde{\alpha})^{3\over 2}).
\]

(2.37)

Given a choice of the boundary conformal frame, we can also define the regularized geodesic distance through the bulk between boundary points. First, note that for any \( 2 \times 2 \) matrix \( p \) with \( \det p = 1 \),

\[
p^{-1} = R_\perp p^t R_\perp^t, \quad \text{where} \quad R_\perp = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]

(2.38)

Also, the elements of a matrix \( p \) of any bulk point scales linearly with \( r \). So, in the limit \( r \to \infty \)

\[
d_{\text{bulk}}(p_1, p_2) = \cosh \left( \frac{\text{Tr} \left( p_1^{-1} p_2 \right)}{2} \right)
\]
\[ \cosh^{-1} \left( \frac{\text{Tr} \left( R_1 p_1^t R_1^t p_2 \right)}{2} \right) \]

\[ = \log \left( r^2 \right) + \log \left( \text{Tr} \left( R_1 p_1^t R_1^t p_2 \right) \right) + \mathcal{O} \left( r^{-2} \right) \]

\[ = \log \left( r^2 \right) + \log \left( 4 \text{Tr} \left( R_1 \left( \vec{v}_1 \vec{u}_1^t \right) R_1^t \left( \vec{v}_2 \vec{u}_2^t \right) \right) \right) + \mathcal{O} \left( r^{-2} \right) \quad (2.39) \]

To find the renormalized boundary geodesic distance, we subtract \( \log \left( r^2 \right) \) then take the \( r \to \infty \) limit, giving

\[ d_{\text{ren}}^{\text{global}} \left( p_{1\partial}, p_{2\partial} \right) = \log \left( 4 \left( \vec{u}^\perp_1 \cdot \vec{v}_2^\perp \right) \left( \vec{v}_1^\perp \cdot \vec{v}_2^\perp \right) \right) \quad (2.40) \]

where

\[ \vec{u}^\perp = R_1 \vec{u} \quad \text{and} \quad \vec{v}^\perp = R_1 \vec{v} \quad (2.41) \]

Similarly, the renormalized geodesic distance between a bulk point \( p \) and a boundary point \( q_{\partial} = 2 \vec{v}^\perp \vec{u}^\perp \) is given by

\[ d_{\text{ren}}^{\text{global}} \left( p, q_{\partial} \right) = \log \left( \text{Tr} \left( p^{-1} q_{\partial} \right) \right) = \log \left( 2 \text{Tr} \left( p^{-1} \vec{v}^\perp \vec{u}^\perp \right) \right) \quad (2.42) \]

An important question is finding the corresponding expressions to the renormalized geodesic distances (2.40)-(2.42) for the boundary of an asymptotic region that is in the BTZ conformal frame \( ds^2_{\text{BTZ}} = -dt^2_B + d\phi_B^2 \). This question is resolved in subsection 2.3.

### 2.3 Geodesic distances in the BTZ conformal frame

In this subsection, we calculate the renormalized geodesic distance from a bulk point \( p \) to a boundary point \( q_{\partial} \) that is in the BTZ conformal frame. We assume that \( q_{\partial} \) is on the boundary of the fundamental domain, so it is in one of those fundamental diamonds defined in section 2.2. In that diamond, we choose the BTZ conformal frame, and the renormalized distance we calculate here is compatible with that frame. We also assume that \( p \) and \( q_{\partial} \) are spacelike separated so that we use (2.9) rather than (2.10) to calculate the distance.

First let us work out the conformal transformation between the AdS global conformal frame and the BTZ frame. For simplicity, we first study a boundary diamond of the BTZ black hole, as shown in figure 5. Then we convert our results to smaller diamonds using isometries.

Recall that global AdS\(_3\) and the BTZ coordinates are related to the embedding coordinates via (2.2) and (2.16). On the boundary where both radial coordinates go to infinity,

\[ Y/X = \tan \phi = \frac{\sinh \frac{\tilde{u}_B + \ell v_B}{2\pi}}{\cosh \frac{\tilde{u}_B + \ell v_B}{2\pi}}, \quad V/U = \tan t = \frac{\sinh \frac{\tilde{u}_B + \ell v_B}{2\pi}}{\cosh \frac{\tilde{u}_B + \ell v_B}{2\pi}} \quad (2.43) \]

where \( u_B = t_B - \phi_B, v_B = t_B + \phi_B \). Then, using null coordinates on the global AdS\(_3\) boundary, \( u = t - \phi \) and \( v = t + \phi \), the above equations simplify to

\[ u = \tan^{-1} \sinh \frac{\tilde{u}_B}{2\pi}, \quad v = \tan^{-1} \sinh \frac{\tilde{v}_B}{2\pi}. \quad (2.44) \]
Based on that, we can get the conformal transformation between the two conformal frames,

\[ ds_{\text{global}}^2 = -dudv = \Omega^2(0) = \Omega^2_u \Omega^2_v (-du_B dv_B) = \Omega^2_u \Omega^2_v ds_{\text{BTZ}}^2 \]  

(2.45)

where the conformal factor \( \Omega^2 \) factorizes into the “left-moving” and “right-moving” conformal factors

\[ \Omega^2_u = \frac{\ell}{2\pi \cosh \frac{t_{UB}}{2\pi}} = \frac{\ell}{2\pi} \cos u, \quad \Omega^2_v = \frac{\ell}{2\pi \cosh \frac{t_{VB}}{2\pi}} = \frac{\ell}{2\pi} \cos v. \]  

(2.46)

As we can see, when \( u = \pm \frac{\pi}{2} \) or \( v = \pm \frac{\pi}{2} \), \( u_B \) or \( v_B \) diverge, and the conformal factors vanish. This marks the boundary of the “boundary diamond” we are in. Also, the conformal factors reach their maximal value at the “center” of the diamond, i.e. where \( u = 0 \) and \( v = 0 \).

For the BTZ black hole, its two asymptotic regions actually have the largest boundary diamonds that we can have in any wormhole construction. If we want to study an arbitrary boundary diamond with size \( \Delta u \) and \( \Delta v \) in the two null directions, we can perform \( \text{AdS}_3 \) isometries on the boundary to move and change the size of the above “big diamond.”

Recall the following generators of \( \text{AdS}_3 \) isometries and their actions on the boundary

\[ 2J_1 = -(J_{XU} - J_{YV}) = \sin v \partial_v \equiv \partial_x, \quad 2\tilde{J}_1 = -(J_{XU} + J_{YV}) = \sin u \partial_u \equiv \partial_y \]  

(2.47)

where we have defined

\[ x = \log \tan \frac{v}{2}, \quad y = \log \tan \frac{u}{2}. \]  

(2.48)

Their actions, which are written here as translations in \( x \) and \( y \), could be used to change the size of the boundary diamond. Since we have a factorized structure here, we only study the \( v \) direction, and calculations in the other direction follow.

In the \( v \) direction, translating \( x \) by \( x_0 = \log \tan \frac{\pi}{2} \) amounts to changing the diamond boundaries from \( v = \pm \frac{\pi}{2} \) to \( v = \pm v_0 \). Denoting the left-moving coordinate in the new diamond by \( v' \) we have

\[ \tan \frac{v'}{2} = \tan \frac{v}{2} \tan \frac{v_0}{2}. \]  

(2.49)

Here \( v' = \pm v_0 = \pm \frac{\Delta v}{2} \) are the boundaries of the new diamond, and they are where \( v = \pm \frac{\pi}{2} \) are mapped to, and we assume \( v_0 < \frac{\pi}{2} \). Based on this relation, we have

\[ dv' = \frac{1 - \cos v' \cos v_0}{\sin v_0} dv. \]  

(2.50)

The left-moving conformal factor then becomes

\[ \Omega^2_v = \left( \frac{\ell}{2\pi \cos v} \right) \left( \frac{1 - \cos v' \cos v_0}{\sin v_0} \right) = \frac{\ell}{2\pi} \frac{\cos \frac{v}{2} - \cos v_0}{\sin v_0} \]  

(2.51)

Inside a diamond, it is bounded by

\[ \Omega^2_v \leq \frac{\ell}{2\pi} \tan \frac{v_0}{2} = \frac{\ell}{2\pi} \tan \frac{\Delta v}{4}, \]  

(2.52)
where the equality holds at \( v' = 0 \). When a diamond has a small size, this bound is approximately
\[
\Omega_v^2 \leq \frac{\ell v_0}{4\pi} = \frac{\ell \Delta v}{8\pi}.
\]
(2.53)

Also inside a diamond, when the point is close to one of the diamond boundary (i.e. \( v' \) is close to \( v_{\text{bdy}} = v_0 \) or \(-v_0\)), \( \Omega_v^2 \) has the following expansion:
\[
\Omega_v^2 = \frac{\ell}{2\pi} (|v' - v_{\text{bdy}}|) + \mathcal{O}((v' - v_{\text{bdy}})^2).
\]
(2.54)

Similar relations hold for the \( u \) direction. For diamonds that are not centred at \( v = 0, u = 0 \), we can use the boundary isometries \( \partial_v \) and \( \partial_u \) to do the translations. The bounds above should also apply.

As discussed in section 2.2, the geodesic distance between a bulk point \( p \) and a boundary point \( q_\partial \) is
\[
d_{\text{bulk}}(p, q) = \cosh^{-1} \left( \frac{\text{Tr} (p^{-1} q)}{2} \right)
= \log(r) + \log \left( \text{Tr} (p^{-1} q_\partial) \right) + \mathcal{O} \left( r^{-2} \right).
\]
(2.55)

To renormalize the distance in the BTZ conformal frame, we should subtract it by \( \log r_B \) where \( r_B \) is a properly chosen radial coordinate associated to the boundary diamond that \( q_\partial \) is in.

In Fefferman-Graham coordinates, when we transform between the global and BTZ conformal frames, to leading order in \( z \), we have \( z_B = z/|\Omega| \). Also, to leading order, \( z \sim 1/r \) and \( z_B \sim 1/r_B \), so we have \( r_B \sim r/|\Omega| = r |\Omega_u \Omega_v| \). Therefore a properly defined renormalized geodesic distance is
\[
d_{\text{ren}}^{\text{BTZ}} (p, q_\partial) = \log \left( \text{Tr} (p^{-1} q_\partial) \right) - \log |\Omega_u \Omega_v| = d_{\text{ren}}^{\text{global}} (p, q_\partial) - \log |\Omega_u \Omega_v|.
\]
(2.56)

2.4 The hot limit of multi-boundary wormholes

In order to construct multi-boundary traversable wormholes in section 4, a certain limit is taken such that there are two desired features: 1) for two nearby asymptotic regions, the horizons are exponentially close over a sufficiently large region, and 2) for a point \( q_\partial \) on the boundary of the fundamental domain, the conformal factors \( \Omega^2 = \Omega_u^2 \Omega_v^2 \) associated with its image points are exponentially small. For reasons that will be clear below, we call the limit that satisfy these properties for any \( (n, g) \) geometry the “hot limit”.

For multi-boundary wormholes with trivial topologies, we choose to take a limit where all \( \ell_i \) and \( \tilde{\ell}_i \) are large, with \( \ell_i/\tilde{\ell}_i \) fixed (i.e. \( M_i/J_i \) fixed)\(^\text{10}\). In the case without rotation, this is exactly the “hot limit” considered in [19]. In the case with rotation, this is also a limit where the temperatures in all asymptotic regions are large. It also implies that all horizon lengths

\(^\text{10}\)For wormholes with internal parameters (i.e. non-trivial topologies or with \( n > 3 \)), the proper limit will also involve taking certain internal parameters to be large, in addition to having \( \ell_i \) and \( \tilde{\ell}_i \) large, with \( \ell_i/\tilde{\ell}_i \) fixed. We will discuss this briefly in section 5.
are large compared to the AdS scale (although the converse is not necessarily true). So, we will keep using the same terminology. Next, we explain the two features we advertised, with the three-boundary wormhole being our main example.

First, we study the minimal distance between two neighbouring horizons. For non-rotating \((3,0)\) geometries, this has been computed in [19] by focusing on the half-plane of the \(t=0\) slice. The minimal distance \(d_{ij}\) between horizons \(H_i\) and \(H_j\) depends on the horizon lengths, and is given by

\[
cosh d_{ij} = \frac{\cosh (h_i/2) \cosh (h_j/2) + \cosh (h_k/2)}{\sinh (h_i/2) \sinh (h_j/2)}
\]

(2.57)

Applying (2.57) to horizons \(H_1\) and \(H_2\) in our construction, we have from (2.23) that

\[
d_{12} = \alpha = \tilde{\alpha},
\]

(2.58)

In appendix B, we generalize (2.57) to the case with rotations, where the minimal distance between horizons \(H_1\) and \(H_2\) was shown to be given simply by

\[
d_{12} = \alpha + \tilde{\alpha}.
\]

(2.59)

Other minimal horizon distances can be found from this expression by simple permutations. It can be easily shown that, in the hot limit, \(\alpha\) and \(\tilde{\alpha}\) are exponentially small, and so is \(d_{ij}\). As a special case, when all \(\ell_i = \ell\) and \(\tilde{\ell}_i = \tilde{\ell}\) are large, we have \(\alpha \sim 2e^{-\ell/4}\), \(\tilde{\alpha} \sim 2e^{-\tilde{\ell}/4}\) and \(d_{ij} \sim e^{-\ell/4} + e^{-\tilde{\ell}/4}\). Furthermore, in this limit, it was found [19] that the distance between the horizons is exponentially small over a large subset \(D_\phi\) of the angular domain that is of order the AdS length. In appendix B, we show that this feature also applies in the rotating case. Additionally, we show there that this is no longer the case when only either \(\ell_i\) or \(\tilde{\ell}_i\) are taken to be large while the other is held fixed, which is a large horizons limit but not a hot limit\(^1\).

These results also hold in the case of a general \(n\)-boundary black hole. As discussed in section 2.1, a general \((n,0)\) spacetime with \(n \geq 3\) can be constructed from \(n-2\) copies of \((3,0)\) geometry. Thus, to calculate the minimal distance \(d_{ij}\) between any two horizons \(H_i\) and \(H_j\), we assume without loss of generality that both horizons were part of a single copy of \((3,0)\) geometry. The third horizon \(H_k\) in this copy will become part of the causal shadow of the new \((n,0)\) geometry and its length \(h_k\) will be one of the parameters of the moduli space associated with the causal shadow region. Therefore, the same minimal distance \(d_{ij}\) between horizons \(H_i\) and \(H_j\) as in the \((3,0)\) geometry will hold. Choosing \(h_k \ll h_i + h_j\) in the hot limit, \(d_{ij}\) will also be exponentially small in this case\(^2\).

The other important feature of the geometry in the hot limit is that, for a point \(q_\partial\) on the boundary of the fundamental domain, its image points \(q_\partial^\text{image}\) all have conformal factors

\(^{11}\)This has some interesting consequences for the extremal limit that we briefly discuss in section 5.

\(^{12}\)This result can also be shown to hold when \(g \neq 0\), though we will not discuss it in detail here. See section 5 for some comments in the case with genus.
that are exponentially small. Recall, from equation (2.56), we found that the renormalized distance between \( p \) and \( q_\partial \text{image} \) in BTZ frame is

\[
d_{\text{ren}}^{\text{BTZ}}(p, q_\partial \text{image}) = \log \left( \frac{\text{Tr} (p^{-1} q_\partial \text{image})}{\Omega_u \Omega_v} \right) = d_{\text{ren}}^{\text{global}}(p, q_\partial \text{image}) - \log |\Omega_u \Omega_v|,
\]

where \( \Omega_u \) and \( \Omega_v \) are the conformal factors associated with \( q_\partial \text{image} \). Therefore, when we have a bulk point \( p \) that is in the same asymptotic region as \( q_\partial \), in the BTZ frame, \( d^{\text{BTZ}}_{\text{ren}}(p, q_\partial \text{image}) > d^{\text{BTZ}}_{\text{ren}}(p, q_\partial) \) in the hot limit, and their difference is linear in \( \ell_i \) and \( \tilde{\ell}_i \), as will be shown below.

To show that the conformal factor is exponentially small for image points in the hot limit, recall that, in section 2.2, the image points are classified into two types. For those in the image diamonds, the image diamonds are in general exponentially small in the hot limit. In 2.3, the conformal factors were found to satisfy the bounds

\[
\Omega_u^2 \leq \frac{\tilde{\ell} \Delta u^{\text{image}}}{8\pi} \quad \text{and} \quad \Omega_v^2 \leq \frac{\ell \Delta v^{\text{image}}}{8\pi}
\]

where \( \Delta u^{\text{image}} \) and \( \Delta v^{\text{image}} \) determine the size of the diamond to which \( q_\partial^{(m)} \) belongs. Hence, what (2.61) means is that the conformal factors of the image diamonds are upper bounded by the size of the diamond. It also turns out that the whole product is bounded by the boundary distance between the left and right fixed point of the image diamond, since

\[
d_{\text{bdy}}(p_+^{\text{image}}, p_-^{\text{image}}) = \sqrt{\Delta u^{\text{image}} \Delta v^{\text{image}}}.
\]

Take the (3, 0) geometry as our example, there all the image diamonds lie between diamonds 1 and 2 and are spacelike separated from them. Then, using (2.37), we have in the hot limit

\[
d_{\text{bdy}}(p_+^{\text{image}}, p_-^{\text{image}}) < d_{\text{bdy}}(p_{++,1}, p_{++,2}) \sim \sqrt{\alpha \tilde{\alpha}}.
\]
Therefore

\[ \Omega^2 = \Omega_u^2 \Omega_v^2 \lesssim \frac{\ell \tilde{\ell}}{64\pi^2 \alpha \tilde{\alpha}}. \]  

(2.63)

In the hot limit, \( \Omega^2 \) is exponentially small. As a special case when \( \ell_i = \ell \) and \( \tilde{\ell}_i = \tilde{\ell} \), we have \( \Omega^2 \lesssim e^{-(\ell+\tilde{\ell})/4} \), and since \( d_{\text{ren}}^{\text{global}} = \mathcal{O}(1) \), we have \( d_{\text{ren}}^{\text{BTZ}} \gtrsim \ell + \tilde{\ell} \).

When \( q_{\text{image}}^{\text{image}} \) belongs to the same fundamental diamond as \( q_{\partial} \), then, in the hot limit, \( q_{\text{image}}^{\text{image}} \) is exponentially close to at least one of the fixed points of the fundamental diamond. Recall from (2.54) that, when this is the case, the conformal factors can be approximated as

\[ \Omega_u^2 \simeq \frac{\tilde{\ell}}{2\pi} \left( |u - u_{\text{bdy}}| \right) \quad \text{and/or} \quad \Omega_v^2 \simeq \frac{\ell}{2\pi} \left( |v - v_{\text{bdy}}| \right). \]

(2.64)

Next, we will explicitly show in this case that \( \Omega^2 = \Omega_u^2 \Omega_v^2 \) is exponentially small in the \((3,0)\) geometry. A similar argument also holds for other geometries. In [30], the authors calculated the size of the fundamental domain of the non-rotating BTZ black hole on the boundary. On the \( t = 0 \) slice, the boundaries of the fundamental domain are at

\[ \phi = \pi \pm \sin^{-1} \left( \tanh \left( \pi r_+ \right) \right). \]

(2.65)

This means that the maximal boundary distance between the fundamental domain and the left/right corner of the diamond, denoted below by \( d_{\partial} \), is

\[ d_{\partial} = \cos^{-1} \left( \tanh \left( \pi r_+ \right) \right). \]

(2.66)

In the case of rotation, one can show that this expression generalizes to\(^{13}\)

\[ d_{\partial} = \sqrt{ \left( \cos^{-1} \tanh \frac{\ell}{2} \right) \left( \cos^{-1} \tanh \frac{\tilde{\ell}}{2} \right) } \]

(2.67)

In the hot limit, \( d_{\partial} \sim 2e^{-(\ell+\tilde{\ell})/4} \). Furthermore, it is easy to see that this equation reduces to (2.66) when \( \ell = \tilde{\ell} \), using (2.17).

In our three-boundary wormhole construction, the fundamental diamond 1 is the same as a diamond in the BTZ black hole since both are generated by the same isometry, as discussed in section 2.1, so we should expect the same formula to hold. Without loss of generality, we assume that \( \ell_1 \geq \ell_2, \ell_3 \) and \( \tilde{\ell}_1 \geq \tilde{\ell}_2, \tilde{\ell}_3 \). So, from figure 3, the distance between the fundamental domain and the diamond is largest for diamond 1, which is given by (2.67) with \( \ell \) and \( \tilde{\ell} \) replaced by \( \ell_1 \) and \( \tilde{\ell}_1 \), respectively. Therefore, if \( \epsilon \) is the distance between \( q_{\partial}^{(m)} \) and the fixed point of the fundamental diamond, then \( \epsilon < d_{\partial} \). Furthermore, from (2.64), we have \( \Omega^2 \sim \epsilon^2 \). This provides a lower bound on \( d_{\text{ren}}^{\text{BTZ}}(p, q_{\text{image}}^{\text{image}}) \) that in the hot limit is

\[ d_{\text{ren}}^{\text{BTZ}}(p, q_{\text{image}}^{\text{image}}) \geq - \log d_{\partial} \gtrsim \ell_1 + \tilde{\ell}_1 \]

(2.68)

\(^{13}\)The idea is to realize that, since \( \gamma_{\text{BTZ}} \) defined in (2.11) maps the two boundaries of the fundamental domain to each other, then \( \gamma_{\text{BTZ}}^{1/2} \) will map the boundary centre of the fundamental domain to its boundary. This centre point, in global coordinates, is \((t = 0, \phi = \pi)\). Acting on this point with \( \gamma_{\text{BTZ}}^{1/2} \) gives the coordinates of the boundary of the fundamental domain, from which we calculate \( d_{\partial} \).
To summarize, we showed that the conformal factors associated with $q_\partial$ in the hot limit are exponentially small, whether $q_\partial$ is in an image diamond or in the fundamental diamond. As a consequence, $d_{\text{ren}}^{\text{BTZ}}(p, q_\partial) \gtrsim \ell + \tilde{\ell}$.

2.5 The CFT dual of $(n, g)$ geometries

The bulk $(n, g)$ spacetime is dual to a CFT state $|\Sigma_{n, g}\rangle \in \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$, where $\mathcal{H}_i$ is the Hilbert space of a CFT state on a circle. In the energy eigenbasis, this state can be expressed as

$$|\Sigma_{n, g}\rangle = \sum_{i_1, \ldots, i_n} A_{i_1, \ldots, i_n} |i_1\rangle_1 \cdots |i_n\rangle_n$$

where the coefficient $A_{i_1, \ldots, i_n}$ is a function of the $2(6g - 6 + 3n)$ moduli of rotating $(n, g)$ geometry. A Cauchy slice of $(n, g)$ spacetime is a Riemann surface $\Sigma_{n, g}$ with $n$ boundaries and genus $g$. Suppose that the state of the CFTs at the $n$ boundaries is $|\phi_1 \ldots \phi_n\rangle \in \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$. In the large temperature limit, the gravitational path integral over the Euclidean Riemann surface with boundary conditions fixed by $|\phi_1 \ldots \phi_n\rangle$ is dominated by the fully-connected bulk geometry, which by Wick rotation gives a Cauchy slice $\Sigma_{n, g}$ that can give the full $(n, g)$ spacetime by Lorentzian time-evolution - see [19, 30, 31] for details. Varying the moduli changes the dominant bulk geometry in the gravitational path integral, which induces first-order phase transitions that generalize the Hawking-Page transition [32] in the $(2, 0)$ spacetime. For example, for sufficiently large temperatures, the CFT state dual to the BTZ black hole is a thermofield-double state and (2.69) becomes

$$|\Sigma_{2, 0}\rangle = \sum_i e^{-\beta_i E_i / 2} |i\rangle_1 |i\rangle_2$$

In general, determining the coefficients $A_{i_1, \ldots, i_n}$ from the path integral over an arbitrary $\Sigma_{n, g}$ is difficult. However, the CFT dual of $\Sigma_{n, g}$ in the puncture limit where $h_i \ll 1$ was investigated in [30]. It was found that in this case (2.69) becomes

$$|\Sigma_{n, 0}\rangle = \sum_{i_1, \ldots, i_n} C_{i_1 \ldots i_n} e^{-\tilde{\beta}_1 E_1 / 2} \cdots e^{-\tilde{\beta}_n E_n / 2} |i_1\rangle_1 \cdots |i_n\rangle_n$$

where $C_{i_1 \ldots i_n}$ depend on the $n$-point function of the CFTs and the moduli parameters,

$$\tilde{\beta}_i = \beta_i - \log r_d - 2 \log 3$$

$\beta_i$ is the inverse temperature of the BTZ geometry in the exterior of the $i$th asymptotic region, and $r_d$ is an undetermined constant that is independent from the moduli parameters for $(3, 0)$ geometry but in general depends on the internal moduli for $n > 3$ (see [30]).

In the hot limit, the entanglement structure of $|\Sigma_{n, 0}\rangle$ was investigated in [19]. In particular, it was found that the bipartite entanglement between any two CFTs at different

\[14\] Note that, for simplicity of notation, we are ignoring rotation for a moment. However, these equations can easily be generalized to the case of rotation.
boundaries, up to exponentially small corrections, is that of the thermofield-double state over a large region of AdS scale size\textsuperscript{15}. Thus, the CFT state dual to the local geometry in this particular region that extends between the $i^{th}$ and $j^{th}$ asymptotic regions through the causal shadow is well approximated by $|\Sigma_{2,0,i,j}⟩ = |\text{TFD}_{i,j}⟩$. This result will be important below in making hot multi-boundary wormholes traversable.

3 Traversability in BTZ black holes

In this section, we give a general review of the construction of traversable wormholes in BTZ black holes via double trace deformations\textsuperscript{4}, including the case with rotation\textsuperscript{5} and nontrivial dependence on the transverse coordinate (following\textsuperscript{6}).

In general, traversable wormholes result from violations of the averaged null energy condition (ANEC). We will first review the relation between such a violation and its backreaction on the metric using perturbation theory. Then we will review how a double trace deformation can cause such a violation.

3.1 Metric perturbation

The metric of a rotating BTZ black hole in the co-rotating coordinates is obtained by substituting for the co-rotating transverse coordinate $x = \phi - \frac{r}{r_+}t$ in (2.14), giving\textsuperscript{16}

$$ds^2 = -\frac{(r^2 - r_+^2)}{r^2}(r^2 - r_-^2)dt^2 + \frac{r^2}{(r^2 - r_+^2)(r^2 - r_-^2)}dr^2 + r^2(N(r)dt + dx)^2$$

(3.1)

where

$$N(r) = \frac{r_- r^2 - r_+^2}{r^2}.$$  

(3.2)

We can pass to Kruskal coordinates by defining the right- and left-moving null coordinates. In the right exterior region, they are defined as

$$U = e^{\kappa u}, \quad V = -e^{-\kappa v}$$

(3.3)

where $\kappa = (r_+^2 - r_-^2)/r_+$ is the surface gravity, $u,v = t \pm r_*$ are the outgoing/ingoing coordinates, and the tortoise coordinate $r_*$ is

$$r_* = \frac{1}{2\kappa} \log \frac{\sqrt{r^2 - r_-^2} - \sqrt{r_+^2 - r_-^2}}{\sqrt{r^2 - r_+^2} + \sqrt{r_+^2 - r_-^2}}.$$  

(3.4)

\textsuperscript{15}This is the same region denoted by $D_\phi$ in section 2.4 where the distance $d_{ij}$ between the two horizons $H_i$ and $H_j$ is exponentially small.

\textsuperscript{16}In sections 3 and 4, for simplicity, we use coordinates without subscript for BTZ coordinates. This should not be confused with the global AdS$_3$ coordinates in section 2.
This gives the metric

\[
\mathrm{d}s^2 = \frac{1}{(1 + UV)^2} \left\{ -4U \mathrm{d}V + 4r_-(U \mathrm{d}V - V \mathrm{d}U) \mathrm{d}x + \left[ r_+^2(1 - UV)^2 + 4UVr_+^2 \right] \mathrm{d}x^2 \right\}.
\]  

(3.5)

As we can see, the asymptotic boundary in Kruskal coordinates is located at \( UV = -1 \).

To linear order, the geodesic equation implies that a null ray starting from the left boundary in the far past (where \( V = 0 \) and \( U = -\infty \)) to have

\[
V(U) = -(2g_{UV}(V = 0))^{-1} \int_{-\infty}^{U} \mathrm{d}U h_{kk} = \frac{1}{4} \int_{-\infty}^{U} \mathrm{d}U h_{kk} \tag{3.6}
\]

where \( h_{kk} \) is the norm of \( k^a = (\partial/\partial U)^a \) after first-order backreaction from the quantum stress tensor. To get \( h_{kk} \) from the stress tensor, we use the linearized Einstein equations:

\[
8\pi G_N \left\langle T_{kk} \right\rangle = -\frac{1}{2r_+^2} \left[ (r_+^2 - r_-^2) h_{kk} + 2r_- \partial_x h_{kk} + \partial_x^2 h_{kk} \right]
+ (r_+^2 - r_-^2) \partial_U (Uh_{kk}) - 2\partial_U \partial_x h_{kk} + \partial_x^2 h_{xx} \right)
\]

(3.7)

where \( T_{kk} = T_{ab} k^a k^b \). To find the shift \( \Delta V \) at \( U = +\infty \), one merely need to integrate this equation over all \( U \),

\[
8\pi G_N \int_{-\infty}^{+\infty} \left\langle T_{kk} \right\rangle \mathrm{d}U = -\frac{1}{2r_+^2} \left[ (r_+^2 - r_-^2) + 2r_- \partial_x + \partial_x^2 \right] \int_{-\infty}^{+\infty} h_{kk} \mathrm{d}U \tag{3.8}
\]

where asymptotic AdS boundary conditions have been used.

In [4, 5], the authors consider boundary couplings that are independent of the transverse coordinate for simplicity. In that case, \( h_{kk} \) is independent of \( x \), and equation (3.8) could be simplified a lot:

\[
8\pi G_N \int \left\langle T_{kk} \right\rangle \mathrm{d}U = \frac{r_+^2 - r_-^2}{2r_+^2} \int h_{kk} \mathrm{d}U. \tag{3.9}
\]

and the shift of \( V \) coordinate at \( U = +\infty \) is

\[
\Delta V(+\infty) = \frac{1}{4} \int_{-\infty}^{+\infty} \mathrm{d}U h_{kk} = \frac{4\pi G_N r_+^2}{r_+^2 - r_-^2} \int \left\langle T_{kk} \right\rangle \mathrm{d}U. \tag{3.10}
\]

But in general we could add a boundary coupling that has nontrivial dependence on the transverse coordinate. Then we could solve (3.8) using a Green’s function \( H \) [6]

\[
\left( \int \mathrm{d}U h_{kk} \right) (x) = 8\pi G_N \int \mathrm{d}x' H(x - x') \int \mathrm{d}U \left\langle T_{kk} \right\rangle (x') \tag{3.11}
\]

with

\[
H(x - x') = \begin{cases} 
\frac{r_+ e^{-(r_+ - r_-)(x' - x)}}{1 - e^{-2\pi(r_+ - r_-)}} + \frac{r_+ e^{(r_+ - r_-)(x' - x)}}{e^{2\pi(r_+ - r_-)} - 1} & x' \geq x \\
\frac{r_+ e^{-(r_+ - r_-)(2\pi - x + x')}}{1 - e^{-2\pi(r_+ - r_-)}} + \frac{r_+ e^{(r_+ - r_-)(2\pi - x + x')}}{e^{2\pi(r_+ - r_-)} - 1} & x' \leq x.
\end{cases}
\]  

(3.12)
in position space where \(x, x' \in [0, 2\pi)\). In Fourier space, \(H\) takes the form

\[
H(x - x') = \sum_q e^{iq(x-x')} H_q, \quad H_q = \frac{1}{2\pi} \frac{2r_+^2}{r_+^2 - r_-^2 - 2iqr_+ + q^2}
\]  

(3.13)

If we are working with planar BTZ black holes, \(H\) takes the following form,

\[
H(x - x') = \begin{cases} 
  r_+ e^{-(r_- + r_+)(x'-x)} & x' \geq x \\
  r_+ e^{-(r_+ - r_-)(x-x')} & x' \leq x,
\end{cases}
\]  

(3.14)

where \(x\) and \(x'\) can take value on the whole real axis, and in Fourier space one should just adapt the sum in the compact case to an integral.

Note that, in particular, the zero-mode Green’s function diverges in the extremal limit. This means that our perturbation theory breaks down in that limit, although this still suggests that the wormhole will be open for quite a long time, as will be shown below.

In contrast, the non-zero modes of \(H_q\) remains finite at extremality. So in the extremal limit, it suffices to study only the zero mode. Recalling that the BTZ temperature is given by

\[
T_H = \frac{r_+^2 - r_-^2}{4\pi G_N},
\]

we have

\[
\frac{\pi T_H}{r_+} \int h_{kk} dU dx = 8\pi G_N \int \langle T_{kk} \rangle dU dx,
\]

so that (3.6) gives the average shift \(\Delta V(U) \equiv V(U) - V(-\infty)\) as

\[
T_H \Delta V_{\text{average}} (U) = 2G_N r_+ \int_{-\infty}^{U} \int_0^{2\pi} \langle T_{kk} \rangle dU dx.
\]

(3.15)

But in any case, we could use (3.6) and (3.11) to calculate the shift \(\Delta V(U)\). In particular, the shift at \(U = +\infty\) is given by

\[
\Delta V(+\infty) = \frac{1}{4} \int_{-\infty}^{\infty} dU h_{kk} = 2\pi G_N \int dx' H(x - x') \int_{-\infty}^{\infty} dU \langle T_{kk} \rangle (x')
\]

(3.17)

By choosing the boundary conformal frame \(ds_{\text{BTZ}}^2 = -dt^2 + d\phi^2 = -dt^2 + (dx + \frac{r_-}{r_+} dt)^2\), one could relate the boundary time with the \(V\) coordinate

\[
t = -\frac{r_+}{r_+^2 - r_-^2} \log (\pm V).
\]

(3.18)

where the sign is + for the left boundary and is − for the right boundary. The shortest transit time \(t_*\) from left to right boundary is realized by the geodesic that leaves the left boundary at \(V = -|\Delta V|/2\) and arrives at the right boundary at \(|\Delta V|/2\),

\[
t_* = -\frac{2r_+}{r_+^2 - r_-^2} \log \left( \frac{|\Delta V|}{2} \right).
\]

(3.19)

Besides, we can also calculate the shift of the angular coordinate on the boundary. Since on the horizon we are following a particular generator where \(x\) is constant, on the boundary the change in \(\phi\) is

\[
\phi_* = -\frac{2r_-}{r_+^2 - r_-^2} \log \left( \frac{|\Delta V|}{2} \right).
\]

(3.20)
3.2 Violation of ANEC from a double trace deformation

In AdS/CFT, the eternal BTZ black hole is dual to the thermofield double (TFD) state

$$|\Psi\rangle = \frac{1}{\sqrt{Z(\beta, \Omega_H)}} \sum_n e^{-\beta (E_n - \Omega_H J_n)/2} |E_n, J_n\rangle_L |E_n, J_n\rangle_R.$$  \hfill (3.21)

Traversability is achieved by coupling the two boundaries with a double-trace deformation

$$\delta S = \int dt dx \, h(t, x) \mathcal{O}_R(t, x) \mathcal{O}_L(-t, x) = -\int dt \, \delta H$$  \hfill (3.22)

where $\mathcal{O}_{L/R}$ is a scalar operator living in the left/right CFT, and we choose its scaling dimension to be $\Delta = \frac{d}{2} - \sqrt{\left(\frac{d}{2}\right)^2 + m^2}$ in order to have a relevant deformation [4]. The boundary operator $\mathcal{O}_{L/R}$ is dual to a bulk scalar field $\Phi_{L/R}$ with mass $m$. To make the wormhole traversable, $h(t, x)$ needs to be of some definite sign for a period of time, which we denote as $[t_0, t_f]$. Next we show how ANEC is violated by such a boundary coupling.

The starting point is to evaluate the bulk two-point function along $V = 0$

$$G(U, U') \equiv \langle \Phi_R(U, x) \Phi_R(U', x) \rangle$$  \hfill (3.23)

In perturbative expansion in the boundary coupling, the one-loop contribution to the two-point function is [4]

$$G_h = 2 \sin(\pi \Delta) \int_{t_0}^{t_f} dt_1 dx_1 \, h(t_1, x_1) \mathcal{K}(r', t', x'; -t_1 + i\beta/2, x_1) \mathcal{K}_{ret}(r, t, x; t_1, x_1) + (t \leftrightarrow t')$$  \hfill (3.24)

where $\mathcal{K}$ is the bulk-to-boundary propagator, and $\mathcal{K}_{ret}$ is the retarded bulk-to-boundary propagator. Since the BTZ black hole is just quotiented AdS3, the propagators take the same form as those in AdS3, but with an image sum. The bulk-to-boundary propagator in the right exterior region in rotating BTZ metric is [4, 5]

$$\mathcal{K}(z, t, x; t_1, x_1) = \left(\frac{r'_+ - r_+}{2\Delta + 1}\right)^{\frac{d}{2}} \sum_{n = -\infty}^{\infty} \left[ -\sqrt{z - 1} \cosh(\kappa\delta t - r_- \delta x_n) + \sqrt{z} \cosh(r_+ \delta x_n) \right]^{-\Delta}$$  \hfill (3.25)

where

$$z = \frac{r'^2 - r^2}{r'^2 - r^2}, \quad \delta t = t - t_1, \quad \delta x_n = x - x_1 + 2\pi n.$$  \hfill (3.26)

Then we convert this to the Kruskal coordinates in the right exterior region based on the following relations:

$$t = \frac{1}{2\kappa} \log \left( -\frac{U}{V} \right), \quad z = \left( \frac{1 - UV}{1 + UV} \right)^2.$$  \hfill (3.27)

Evaluated along $V = 0$, $\mathcal{K}$ becomes

$$\mathcal{K}(U, 0, x; U_1, x_1) = \left(\frac{r'_+ - r_+}{2\Delta + 1}\right)^{\frac{d}{2}} \sum_{n = -\infty}^{\infty} \left[ -\frac{U}{U_1} e^{-r_- \delta x_n} + \cosh(r_+ \delta x_n) \right]^{-\Delta}.$$  \hfill (3.28)
The other ingredient in $G_h$ is the retarded bulk-to-boundary propagator

$$
K_{\text{ret}}(z,t,x; t_1, x_1) = |K(z,t,x; t_1, x_1)| \theta(\delta t) \theta \left( \sqrt{z - 1} \cosh (\kappa \delta t - r \delta x) - \sqrt{z} \cosh (r \delta x) \right).$$

(3.29)

Now we are ready to write down $G_h(U,U')$:

$$
G_h(U,U') = C_0 \sum_{n=-\infty}^{\infty} \int_0^{2\pi} dx_n \int_U^0 \frac{dU_1}{\kappa U_1} h \left( \frac{\log(U_1)}{\kappa}, x_n \right) \left[ \left( e^{-r_\delta x_n} U_1 U' + \cosh (r_+ \delta x_n) \right) \left( e^{-r_\delta x_n} \frac{U}{U_1} - \cosh (r_+ \delta x_n) \right) \right]^{-\Delta} + (U \leftrightarrow U')
$$

(3.30)

where $C_0 = \frac{r_\kappa \Delta \sin(\pi \Delta)}{2(2\Delta \pi)^2}$, and we have used the fact that on the right boundary $t = \frac{\log(U)}{\kappa}$.

For BTZ black holes, we could discard the image sum and simply extend the range of the $x_1$ integral to the whole real axis [4]. But one should not forget the constraint imposed by the $\theta$-function in the retarded propagator, which requires

$$
e^{-r_\delta x U} - U_1 \cosh (r_+ \delta x) \geq 0.
$$

(3.31)

With the Green’s function at hand, the bulk stress tensor associated with the scalar field is

$$
\langle T_{\mu\nu} \rangle = \lim_{x \to x'} \left( \partial_\mu \partial_\nu G(x,x') - \frac{1}{2} g_{\mu\nu} \partial_\rho \partial_\sigma G(x,x') - \frac{1}{2} g_{\mu\nu} m^2 G(x,x') \right)
$$

(3.32)

When evaluated along the horizon at $V = 0$, the $g_{UU}$ component of the unperturbed metric vanishes, so to leading order we have

$$
\langle T_{kk} \rangle = \lim_{U' \to U} \partial_U \partial_{U'} G_h(U,U')
$$

(3.33)

Finally one can compute the opening of the traversable wormhole by inserting (3.30) and (3.33) into (3.17).

4 Traversability of multi-boundary wormholes in AdS$_3$

As shown in [19], for non-rotating multi-boundary wormholes in the hot limit, the boundary state locally resembles the thermofield double state in region $D_\phi$ discussed in section 2.4. This could be easily generalized to rotating wormholes by adding an angular potential. In regions that we call $D_x$ (since $x$ is a more well-defined coordinate on the horizon in the rotating case), the horizons are exponentially close to each other, and the state locally looks like TFD

$$
|\Psi\rangle = \frac{1}{\sqrt{Z(\beta_{\text{TFD}}, \Omega_{\text{TFD}})}} \sum_n e^{-\beta_{\text{TFD}}(E_n - \Omega_{\text{TFD}} J_n)/2} |E_n, J_n\rangle_L |E_n, J_n\rangle_R.
$$

(4.1)

Since this state is only locally TFD, $\beta_{\text{TFD}}$ and $\Omega_{\text{TFD}}$ can take any value depending on the conformal frame we choose and should not be confused with the actual black hole inverse
temperature and angular velocity. In the hot limit, one expects that such wormholes can be made traversable by the approach described in section 3. Next, we will show this, taking the three-boundary wormhole as our example, and focusing on the traversing process from boundary 1 to boundary 2 without loss of generality.

We will first set up the stage, describing and justifying the planar BTZ coordinates that we are going to use. In these coordinates, our calculations will be very similar to that of [4]. Then, we show that in the hot limit, the image sum in the Green’s function could be well approximated by the leading term, thus simplifying our calculation. Finally, we calculate the wormhole opening with a double-trace deformation, which must be larger than the local causal shadow region.

4.1 Planar BTZ coordinates and the boundary coupling

In region $D_x$, one is able to use planar BTZ coordinates to describe the spacetime to good approximation [19]. Here, we use the following planar coordinates to describe both sides of the wormhole:

$$ds^2 = - (\tilde{r}^2 - \tilde{r}_-^2) d\tilde{t}^2 + \frac{d\tilde{r}^2}{\tilde{r}^2 - \tilde{r}_+^2} + \tilde{r}^2 d\tilde{x}^2$$

(4.2)

where $\tilde{x}$ is non-compact and can take value on the whole real axis, and the choice of $\tilde{r}_+$ is arbitrary. The corresponding Kruskal metric is

$$ds^2 = \frac{1}{(1 + U V)^2} \left( -4d\tilde{U}d\tilde{V} + \tilde{r}_+^2 (1 - \tilde{U} V)^2 d\tilde{x}^2 \right).$$

(4.3)

Although there is a causal shadow between the two horizons in the hot limit, it is exponentially small in $\ell$ and $\tilde{\ell}$. So we could put the origin of the Kruskal coordinates at the bifurcation surface of horizon 1 or 2, or any place between them, with an exponentially small error. This justifies using the metric (4.3) for $D_x$. We will come back to this in section 4.3.

Note that, in the planar BTZ metric, the horizon size parameters can be scaled arbitrarily, as long as we change the definition of coordinates accordingly. To be more concrete, there are two kinds of coordinate transformations that we can make (they are expressed in the ordinary angular coordinate $\phi$ for now and we will come back to the co-rotating $x$ later):

1. “Adjusting the temperature” (rescaling $r_+$ and $r_-$ by the same amount):

$$\tilde{r} = \lambda r, \quad \tilde{t} = \frac{t}{\lambda}, \quad \tilde{\phi} = \frac{\phi}{\lambda}$$

(4.4)

with the new horizon parameters $\tilde{r}_\pm = \lambda r_\pm$;

2. “Changing the angular momentum” (changing the relative size of $r_+$ and $r_-$):

$$\left( \tilde{t}, \tilde{\phi} \right) = \left( t \cosh \gamma + \phi \sinh \gamma, t \sinh \gamma + \phi \cosh \gamma \right)$$

$$\tilde{r}^2 = r^2 + r_+^2 - r_-^2.$$  

(4.5)
with the new horizon parameters \( \tilde{r}_+ = r_+ \cosh \gamma + r_- \sinh \gamma \) and \( \tilde{r}_- = r_+ \sinh \gamma + r_- \cosh \gamma \). As a special case, we could set \( \tilde{r}_- = 0 \) by choosing \( \gamma = -\tanh^{-1}\frac{r_-}{r_+} \). In this case we have

\[
(\tilde{t}, \tilde{\phi}) = \left( r_+ t - r_- \phi, r_+ \phi - r_- t \right)/\sqrt{\tilde{r}_+^2 - \tilde{r}_-^2}
\]

with \( \tilde{r}_+^2 = r_+^2 - r_-^2 \).

Note that we are not changing the actual temperature and angular momentum (that is why they are in quotes) because for planar black holes the temperature and angular momentum only depend on the boundary conformal frame, not the bulk metric. Besides, we only expect this kind of coordinate transformation to be valid outside (or on) the horizon, since that is the region we care about in our traversable wormhole construction.

For simplicity, we would like to choose \( \tilde{r}_- = 0 \) and \( \tilde{r}_+ \) be some fixed \( O(1) \) number while \( r_+,i \)'s are large. To clarify notations, from here on, we use the tilded quantities as those associated to the bulk planar BTZ coordinates with \( \tilde{r}_- = 0 \), and the untilded ones as those related to the chosen conformal frame in some asymptotic region, sometimes with labels to denote which boundary we are referring to.

Combining (4.4) and (4.6), the coordinate transformations we will use on boundaries 1 and 2 are

\[
(\tilde{t}, \tilde{\phi}) = \left( r_{+,i} t_i - r_{-,i} \phi_i, r_{+,i} \phi_i - r_{-,i} t_i \right)/\tilde{r}_+
\]

\[
\tilde{r}_+^2 = \frac{r_{+,i}^2 - r_{-,i}^2}{r_{+,i}^2 - r_{-,i}^2}
\]  

where \( i = 1, 2 \) indicate different asymptotic regions, and this should be understood as two different transformations with the same resulting metric. We should also expect from this that if the two boundaries are different, on the two boundaries we could in general measure different times for the traversing process.

Sometimes the inverse transformation for \( t \) and \( \phi \) is also needed:

\[
(t_i, \phi_i) = \frac{\tilde{r}_+}{\tilde{r}_+^2 - r_{-,i}^2} (r_{+,i} \tilde{t} + r_{-,i} \tilde{\phi}, r_{+,i} \tilde{\phi} - r_{-,i} \tilde{t})
\]

In terms of the co-rotating coordinates, the transformations and inverse transformations for \((t, x)\) and \((\tilde{t}, \tilde{x})\) are

\[
\tilde{t} = \frac{\kappa_i t_i - r_- x_i}{\tilde{r}_+}, \quad \tilde{x} = \frac{r_+ \tilde{t} - r_- \tilde{x}}{\tilde{r}_+}
\] 

\[
t_i = \frac{\tilde{r}_+}{\kappa_i} \left( \tilde{t} + \frac{r_-}{r_+} \tilde{x} \right), \quad x_i = \frac{r_+}{r_-} x_i.
\]

The new and old Kruskal null coordinates are related by

\[
\tilde{U} = e^{-r_- x_i} U_i, \quad \tilde{V} = e^{r_+ x_i} V_i.
\]
From the planar coordinates we use, it is tempting to conclude that our setup can be directly reduced to that of [4], reviewed in section 3. But, here, the subtlety is that the boundary coupling is not naturally defined in the conformal frame related to our bulk metric. To perform calculations, we need to first look at the conformal transformations and how they act on boundary operators.

The boundary metric in the \( i \)th asymptotic region is

\[
 ds_i^2 = -dt_i^2 + d\phi_i^2 = \frac{\tilde{r}_i^2}{r_{+,i} - r_{-,i}} \left( -d\tilde{t}^2 + d\tilde{\phi}^2 \right).
\]

(4.12)

The double-trace deformation that we would like to add is\(^{17}\)

\[
 \delta S = \int dt_1 dt_2 dx_1 dx_2 \ f(t_1, t_2, x_1, x_2) \mathcal{O}_1(t_1, x_1) \mathcal{O}_2(t_2, x_2).
\]

(4.13)

and for a general bi-local coupling, \( f \) is

\[
 f(t_1, t_2, x_1, x_2) = h(t_1, x_1) \ \delta^{(2)}(x_2 - \eta(x_1))
\]

(4.14)

where \( x_i = (t_i, x_i) \), \( i = 1, 2 \), and the function \( \eta \) specifies how the points on the two boundaries are coupled to each other. Integrating out the delta function gives the coupling expressed in terms of only one set of boundary coordinates. Here for computational convenience, we choose the functions \( h \) and \( \eta \) such that the double-trace deformation takes the following form when expressed in the conformal frame associated to the bulk tilded coordinates

\[
 \delta S = \int d\tilde{t} d\tilde{x} \ \tilde{h}(\tilde{t}, \tilde{x}) \left( \frac{\tilde{r}_{+,1}^2 - \tilde{r}_{-,1}^2}{\tilde{r}_{+,1}^2} \right)^{\Delta_1 - \frac{1}{2}} \left( \frac{\tilde{r}_{+,2}^2 - \tilde{r}_{-,2}^2}{\tilde{r}_{+,2}^2} \right)^{\Delta_2 - \frac{1}{2}} \tilde{\mathcal{O}}_1(\tilde{t}, \tilde{x}) \tilde{\mathcal{O}}_2(\tilde{t}, \tilde{x}).
\]

(4.15)

where \( \tilde{\mathcal{O}}_{1/2} \) are \( \mathcal{O}_{1/2} \) in this conformal frame and we have included the conformal factors as in (4.12) to the correct powers in order to account for the transformations of the boundary operators with conformal dimension \( \Delta \) and also the Jacobian from the change of the integration variables.

Next, we choose the explicit form of \( \tilde{h}(\tilde{t}, \tilde{x}) \). For simplicity, we turn it on at some time \( \tilde{t}_0 \) and turn off at some later time \( \tilde{t}_f \). For every \( \tilde{t} \), for example, we could choose a constant coupling,

\[
 \tilde{h}(\tilde{t}, \tilde{x}) = h\lambda^{2-2\Delta}
\]

(4.16)

or a Gaussian to make it localize near some angular position \( \tilde{x}_0 \)

\[
 \tilde{h}(\tilde{t}, \tilde{x}) = h\lambda^{2-2\Delta} \exp \left( -\frac{\tilde{r}_1^2 (\tilde{x}_1 - \tilde{x}_0)^2}{\sigma^2} \right)
\]

(4.17)

where \( \lambda \) is some fixed quantity with dimension of temperature and \( h \) is a small and dimensionless parameter. Note that in \([4, 5]\), the authors consider the boundary coupling that has temperature dependence. But, in principle, we can have a fixed boundary coupling as long as the dimensions are correct.

\(^{17}\)Different from section 3 (e.g. in (3.22)), here we take the boundary times to run in the same direction on both sides.
4.2 Image sum in the hot limit

Next, we show that the image sum in $G_h$ can be well approximated by keeping only the leading term. Note that $G_h$ is built from two bulk-to-boundary propagators, so let us study them first.

According to the extrapolating dictionary, the bulk-to-boundary propagator can be obtained from the two-point function

$$\mathcal{K}(p, q_{\partial}) = \lim_{r' \to \infty} r'^{\Delta} G(p, q) = \lim_{r' \to \infty} r'^{\Delta} G(r, t, x; r', t', x')$$

(4.18)

where $p$ and $q$ are two points in the AdS$_3$ bulk, and we use unprimed and primed coordinates to denote those related to points $p$ and $q$ respectively.

In AdS$_3$, the two-point function for a free scalar field is given by

$$G(p, q) = G_{\text{AdS}_3}(Z) = \frac{1}{4\pi} \left( Z^2 - 1 \right)^{-1/2} \left( Z + (Z^2 - 1)^{1/2} \right)^{1-\Delta}$$

(4.19)

where $Z = 1 + \frac{\sigma(p,q)}{2}$ and $\sigma(p,q)$ is the (squared) distance between $p$ and $q$ in the four dimensional embedding space (sometimes called “chordal distance” [34]), and with all fractional powers of positive real numbers defined by using the positive real branch. The chordal distance is related to the geodesic distance $d(p, q)$ in AdS space by

$$\sigma(p,q) = 4 \sinh^2 \left( \frac{d(p,q)}{2} \right).$$

(4.20)

When $Z$ is large, the two-point function has the following expansion:

$$G_{\text{AdS}_3}(p, q) = \frac{Z^{-\Delta}}{4\pi} \left( 2^{1-\Delta} + \frac{1+\Delta}{2^{1+\Delta}} Z^{-2} + O(Z^{-3}) \right).$$

(4.21)

In AdS$_3$, the (unrenormalized) distance between a bulk point $p$ and a boundary point $q_{\partial}$ has the divergent part $\log r'$, so $G_{\text{AdS}_3}(x, x')$ decays as $(r')^{-\Delta}$. The $(r')^{\Delta}$ in the extrapolating dictionary (4.18) precisely cancels this. Therefore, equivalently, we could get the bulk-to-boundary propagator by plugging in a “renormalized distance” between $p$ and $q_{\partial}$ into (4.19) to get the bulk-to-boundary propagator. According to our analysis in section 2.2, in the conformal frame associated with the global coordinates, this “renormalized distance” is defined by subtracting the unrenormalized distance by $\log r'$.

However, when we are using conformal frames that are not naturally related to the AdS$_3$ global coordinates in the bulk, this $(r')^{\Delta}$ factor in the extrapolating dictionary should be otherwise chosen according to the conformal frame. The new dictionary looks like

$$K = \lim_{r' \to \infty} r'^{\Delta} G(r, t, x; r', t', x')$$

(4.22)

where $\tilde{r}' = r'|\Omega|$ and $\Omega^2$ is the conformal factor such that the boundary metric $ds_{\Omega}^2$ satisfies $ds^2 = -dt^2 + d\phi^2 = \Omega^2 ds_{\Omega}^2$. Equivalently, we could plug in a renormalized length that is compatible with our conformal frame in (4.19) to obtain the bulk-to-boundary propagator.
For the three-boundary wormholes we constructed in section 2, just like the two-sided BTZ case, the bulk-to-boundary propagators have contributions from image points. To be more precise, for points \( p \) and \( q_\theta \) in the fundamental domain, we need to take into account the contributions from the point pairs \((p, g_L q_\theta g_R)\), where \( g_L \) and \( g_R \) are any “words” that are combinations of the group elements chosen to make the wormhole.

We would like to locate these image points and find how they contribute to the bulk-to-boundary propagator in the hot limit. Recall from section 2.2 that there are two types of fixed points of the diamond. For those inside the image diamonds, those image diamonds lie between two large boundary diamonds and are exponentially small in the hot limit.

The definition of the fundamental domain requires that there must exist a bulk Cauchy slice that passes through \( p \) and all the fixed points, so the bulk point \( p \) must be spacelike separated from those image points. This means that we use (2.9) rather than (2.10) to calculate the geodesic distance between spacelike separated bulk and boundary points in the BTZ frame. Applying that to the image points, we found in section 2.4 that the geodesic distance between two large boundary diamonds is at least linearly large in \((\ell_i + \ell_i')\) in the hot limit. Therefore, from (4.19) and (4.20), the contributions to the bulk-to-boundary propagator from the image points are exponentially suppressed, and thus can be ignored.

### 4.3 Traversing the causal shadow

In this subsection we show that in the hot limit, \(|\Delta V|\) induced by the boundary coupling is larger than the gap between horizons \(|\Delta V_{CS}|\) due to the existence of the causal shadow region (see figure 8), so that \(|\Delta V| - |\Delta V_{CS}|\) is positive, and therefore the wormhole is traversable.

Based on the above two subsections, the one-loop contribution to the Green’s function is

\[
\tilde{G}_h(\tilde{U}, \tilde{U'}) = \tilde{C}_0 \int d\tilde{x}_1 \int_{\tilde{U}_0}^{\tilde{U}} d\tilde{U}_1 \tilde{h} \left( \frac{\log \tilde{U}_1}{\tilde{r}_+} \right) \left( \tilde{U}_1 \tilde{U} + \cosh (\tilde{r}_+ \delta \tilde{x}) \right) \left( \frac{\tilde{U}}{\tilde{U}_1} - \cosh (\tilde{r}_+ \delta \tilde{x}) \right)^{-\Delta}
\]

\[
+ (\tilde{U} \leftrightarrow \tilde{U'}),
\]

(4.23)

where \( \tilde{U}_0 = e^{\tilde{x}_i \tilde{I}_0} \), \( \delta \tilde{x} = \tilde{x} - \tilde{x}_1 \) and

\[
\tilde{C}_0 = \frac{\tilde{r}_+^{2\Delta} \sin(\pi \Delta)}{2 (2\Delta \pi)^2} \left( \frac{r_{+,1}^2 - r_{-,1}^2}{\tilde{r}_+^2} \right) \left( \frac{r_{+,2}^2 - r_{-,2}^2}{\tilde{r}_+^2} \right) \Delta_1 \tilde{r}_+^2 \Delta_2 \left( \frac{r_{+,1}^2 - r_{-,1}^2}{\tilde{r}_+^2} \right) \left( \frac{r_{+,2}^2 - r_{-,2}^2}{\tilde{r}_+^2} \right) \Delta_1 \tilde{r}_+^2 \Delta_2 \frac{\sin(\pi \Delta)}{2 (2\Delta \pi)^2}.
\]

(4.24)

The limits of the \( \tilde{x} \) integral above is set by the theta function \( \theta \left( \frac{\tilde{U}}{\tilde{U}_1} - \cosh (\tilde{r}_+ \delta \tilde{x}) \right) \).

From this we can calculate the stress tensor,

\[
\langle \tilde{T}_{kk} \rangle = \lim_{\tilde{U} \to \tilde{U'}} \partial_{\tilde{U}} \partial_{\tilde{U}'} \tilde{G}_h(\tilde{U}, \tilde{U'}).
\]

(4.25)
Figure 8: The Penrose diagram of a black hole spacetime with causal shadow. In particular, this could represent the causal structure of a section that contains two asymptotic regions in the three-boundary wormhole geometry. In the figure, we mark the two bifurcation surfaces $H_1$ and $H_2$, and $\Delta V_{CS}$ caused by the causal shadow. In the hot limit that we consider in the text, $\Delta V_{CS}$ is exponentially small in $\ell$ and $\tilde{\ell}$ in region $D_x$.

Then the shift of $\tilde{V}$ coordinate at $\tilde{U} = +\infty$ is

$$
\Delta \tilde{V}(\tilde{x}) = 2\pi G_N \int_{-\infty}^{+\infty} d\tilde{x}' \tilde{H}(\tilde{x} - \tilde{x}') \left( \int_{-\infty}^{+\infty} d\tilde{U} \langle \tilde{T}_{kk} \rangle (\tilde{x}') \right)
$$

(4.26)

where $\tilde{H}(\tilde{x} - \tilde{x}')$ is the Green’s function (3.14) for non-compact $\tilde{x}$ and $\tilde{x}'$ when $\tilde{r}_- = 0$,

$$
H(\tilde{x} - \tilde{x}') = \tilde{r}_+ e^{-\tilde{r}_+ |\tilde{x}' - \tilde{x}|}.
$$

(4.27)

Then in the coordinates related to the $i$th boundary, $\Delta V_i$ is

$$
\Delta V_i(x_i) = e^{-r_{-i} x_i} \Delta \tilde{V}(\tilde{x}).
$$

(4.28)

From the above calculation, we could see that there is no exponential suppression in $\Delta V_i$ when we take $\ell_i$ and $\tilde{\ell}_i$ to be large. What we get at most is a polynomial suppression. Because the metric is nearly flat near the horizon, $\Delta V_{CS}$ is of the same order as the size of the causal shadow, which is exponentially small. So $|\Delta V_i|$ is indeed bigger than $|\Delta V_{CS}|$, and signals can traverse through the wormhole. Besides, to good approximation, $|\Delta V_{CS}|$ could be ignored and $\Delta V_{total} \simeq |\Delta V|$.

As a consistency check, here we show that the physical quantity $\Delta V_i$ does not depend on the fictitious parameter $\tilde{r}_+$ that we have been using to simplify the calculations.

Our starting point is (4.23). We write $\tilde{G}_h \equiv F + F'$ where $F$ is the term explicitly shown in (4.23)

$$
F(\tilde{U}, \tilde{U}') = \tilde{C}_0 \int d\tilde{x}_1 \int_{\tilde{U}_0}^{\tilde{U}} \frac{d\tilde{U}_1}{\tilde{r}_+ \tilde{U}_1} \tilde{h} \left( \log \frac{\tilde{U}_1}{\tilde{r}_+}, \tilde{x}_1 \right) \left[ \left( \tilde{U}_1 \tilde{U}' + \cosh (\tilde{r}_+ \delta \tilde{x}) \right) \left( \frac{\tilde{U}}{\tilde{U}_1} - \cosh (\tilde{r}_+ \delta \tilde{x}) \right) \right]^{-\Delta}
$$

(4.29)
and $F'$ is the term with $\tilde{U}$ and $\tilde{U}'$ exchanged. Based on the symmetry, $(\tilde{T}_{kk})$ is calculated by

$$
\langle \tilde{T}_{kk} \rangle = 2 \lim_{\tilde{U}' \rightarrow \tilde{U}} \partial_{\tilde{U}'} \partial_{\tilde{U}} F(\tilde{U}, \tilde{U}')
$$

(4.30)

Next, we change the integration variables to make the dependence on $\tilde{r}_+$ clear. First we define a new integration variable $y \equiv \cosh(\tilde{r}_+ \delta \tilde{x}) = \cosh[\tilde{r}_+(\tilde{x} - \tilde{x}_1)]$, and now $F$ is

$$
F(\tilde{U}, \tilde{U}') = \frac{2 \tilde{C}_0}{\tilde{r}_+^2} \int_{\tilde{U}_0}^{\tilde{U}} \frac{d\tilde{U}_1}{\tilde{U}_1} \int_{\tilde{U}_1}^{\tilde{U}} \frac{dy}{\sqrt{y^2 - 1}} \tilde{h} \left( \log \frac{\tilde{U}_1}{\tilde{r}_+}, \tilde{x}_1 \right) \left( \tilde{U}_1 \tilde{U}' + y \right) \left( \frac{\tilde{U}}{\tilde{U}_1} - 1 \right)^{-\Delta}
$$

(4.31)

where the limits of the $y$ integral are determined by the theta function $	heta \left( \frac{\tilde{U}_1}{\tilde{r}_+} - \cosh (\tilde{r}_+ \delta \tilde{x}) \right)$, and the argument $\tilde{x}_1$ in the function $\tilde{h}$ should be implicitly treated as a function of $y$.

As we can see, all the $\tilde{r}_+$ dependence in the prefactor $\frac{2 \tilde{C}_0}{\tilde{r}_+^2}$ cancels out. Then, recall the relations (4.9)

$$
\tilde{r}_+ \tilde{x} = r_+ x_i, \quad \tilde{r}_+ \tilde{t} = \kappa_t t_i - r_{-i} x_i
$$

(4.32)

and on the horizon $V = 0$,

$$
\tilde{U} = e^{\tilde{r}_+ \tilde{t}} = e^{\kappa_t t_i - r_{-i} x_i}.
$$

(4.33)

Similar relations hold for $\tilde{U}$, $\tilde{U}'$ and $\tilde{U}_0$ in the integration limits, and they can be expressed in terms of purely boundary quantities. Also, we should not introduce any $\tilde{r}_+$ dependence in $\tilde{h}$ by hand. This means that, when choosing the form of $\tilde{h}$, the argument $\tilde{x}_1$ and $\tilde{t}_1$ in $\tilde{h}$ should both come with a factor of $\tilde{r}_+$, since the combination $\tilde{r}_+ \tilde{t}_1$ and $\tilde{r}_+ \tilde{x}_1$ can be converted by (4.32) to something that only involves parameters and coordinates related to some boundary. In terms of the new variable $y$, this means that we must have the combination $(\tilde{r}_+ \tilde{x} - \cosh^{-1} y)$ independent of $\tilde{r}_+$. Therefore, $F$ is also independent of $\tilde{r}_+$.

The physical observable $\Delta V_i$ on one boundary is

$$
\Delta V_i(x_i) = e^{-r_{-i} x_i} \frac{2 \pi G_N}{2} \int_{-\infty}^{+\infty} d\tilde{x}' \tilde{r}_+ e^{-\tilde{r}_+ |\tilde{x}'|} \left( \int_{-\infty}^{+\infty} d\tilde{U} \langle \tilde{T}_{kk} \rangle \right) (\tilde{x}')
$$

(4.34)

From $F$ to $\int d\tilde{U} \langle \tilde{T}_{kk} \rangle$, no extra dependence on $\tilde{r}_+$ is introduced and, from our previous argument, $\int d\tilde{U} \langle \tilde{T}_{kk} \rangle$ as a function of $\tilde{x}'$ should only depend on the combination $\tilde{r}_+ \tilde{x}'$. As we can see, all other parts involving tilded coordinates in (4.34) all come with a factor of $\tilde{r}_+$, so the physical quantity $\Delta V_i$ will not have any $\tilde{r}_+$ dependence.

### 4.4 Numerical results

In this subsection, we present some numerical results in order to illustrate our construction. Below, we will consider two types of boundary coupling: for every $\tilde{t}$, 1) the coupling is constant, as in (4.16) and 2) a Gaussian distribution centered at some point, as in (4.17).

In our numerical calculations below, the boundary coupling will be turned on at $\tilde{t}_0 = 0$ and never shut off. We also take $h = 1$ and $\lambda = 1$ in the boundary coupling, and $G_N = 1$ for simplicity. Furthermore, without loss of generality, we only consider some subspace of the
Figure 9: For the case of constant coupling, the averaged null energy $\int \tilde{T}_{kk}d\tilde{U}$ (left) and the horizon shift $\Delta V_1$ at $x_1 = 0$ (right). In both panels, we choose $h = 1$, $\lambda = 1$, $G_N = 1$, $r_{+,2} = 100$, $r_{-,2} = 20$ and $r_{+,1} = 100$.

Figure 10: For the case of Gaussian coupling, the averaged null energy $\int \tilde{T}_{kk}d\tilde{U}$ at $x_1 = 0$ (left) and its profile for general $x_1$ (right). In both panels, we choose $h = 1$, $\lambda = 1$, $G_N = 1$, $r_{+,2} = 100$, $r_{-,2} = 20$ and $r_{+,1} = 100$, $\sigma = 0.2$ and $x_0 = 0$. In the right panel we also choose $\Delta = 0.6$.

Figure 11: For the case of Gaussian coupling, the shift of horizon $\Delta V_1$ at $x_1 = 0$ (left) and its profile for general $x_1$ (right). In both panels, we choose $h = 1$, $\lambda = 1$, $G_N = 1$, $r_{+,2} = 100$, $r_{-,2} = 20$ and $r_{+,1} = 100$, $\sigma = 0.2$ and $\tilde{x}_0 = 0$. In the right panel we also choose $\Delta = 0.6$.

whole parameter space: $r_{+,2} = 100$, $r_{-,2} = 20$ and $r_{+,1} = 100$. We let $r_{-,1}$ (or equivalently the ratio between angular momentum and mass $J_1/M_1$ on boundary 1) and the scaling dimension $\Delta$ be free and see how various quantities change with them.

The quantities we are going to show here are the averaged null energy $\int \tilde{T}_{kk}d\tilde{U}$ and the
shift of the horizon $\Delta V_1$ as measured on boundary 1. Note that here $\int \tilde{T}_{kk}d\tilde{U}$ is not a physical quantity since we could choose any kind of “tilted coordinates”, but we show it here because its negativity is important for traversability. For convenience we choose $\tilde{r}_+ = 1$.

Results for the case of constant coupling are shown in figure 9. There we show $\int \tilde{T}_{kk}d\tilde{U}$ and $\Delta V_1$ at $x_1 = 0$ (or equivalently $\tilde{x} = 0$) for different $\Delta$ and $J_1/M_1$. As we can see, both quantities are negative, and are diverging near extremality.

For Gaussian coupling, we choose $\sigma = 0.2$ and $\tilde{x}_0 = 0$. In figure 10 we show $\int \tilde{T}_{kk}d\tilde{U}$ at $x_1 = 0$ (or equivalently $\tilde{x} = 0$) and its angular dependence for some choices of parameters, while results about $\Delta V_1$ are shown in figure 11.

5 Discussion

In this work, we have extended the Gao-Jafferis-Wall traversability protocol [4] to multi-boundary wormholes. The main difficulty with achieving traversability in this case is the existence of the causal shadow region between the horizons, as well as calculating the image sum of the Green’s function. Our main result is that, in the hot limit, both of these difficulties can be circumvented and the wormholes can be made traversable. As shown in section 2, this is because there exists a region of order the AdS length in size where the horizons of any two boundaries are exponentially close to each other. As a consequence of this, the bipartite CFT state in this region is approximately given by the TFD state, which allow us to couple them in the same manner as in [4]. Furthermore, the distance between a bulk point in this limit and its image under the isometries of the geometry is large, which makes the contribution of the images to the Green’s function exponentially suppressed compared to the point in the fundamental domain. This greatly simplifies the calculation of the Green’s function required to calculate the average null energy along the horizon.

Although we presented explicit calculations only for the three-boundary wormhole geometry, generalization to the case of $n$-boundary wormholes (i.e. $(n,0)$ geometries) is straightforward. In the case of non-trivial topology behind the horizons, one has to be more careful in realizing the hot limit; in addition to choosing external parameters that give large temperatures, one also has to choose the internal moduli parameters such that there exists a large enough region where the horizons are exponentially close to each other, which is always possible [19][18] (e.g. see figure 12). Furthermore, the CFT dual to the bulk region where the horizons are exponentially close to each other is well-approximated by the TFD state; so, the traversability analysis reduces to exactly the same one we used for the case without genus.

In the extremal limit, we showed in appendix B that the minimal distance $d_{ij}$ between the horizons diverges logarithmically. However, from (4.23) and (4.26), we see that the time advance $\Delta V$ induced by the double trace deformation diverges polynomially, which is also illustrated in figures 9 and 11. For this reason, we expect that the wormhole is still traversable.

18The arguments presented in [19] were for the static case. However, as discussed in section 2, these arguments can easily be generalized to the rotating case using an appropriate choice of Cauchy slices.
in the extremal limit even though, as discussed in section 3, the perturbative analysis that allowed us to calculate $\Delta V$ is not strictly valid in the extremal limit\(^{19}\).

Recall that, in the ER=EPR proposal\(^{18}\), entanglement between two (non-interacting) quantum systems is geometrically realized by a non-traversable wormhole (i.e. Einstein-Rosen bridge) connecting them. When the two systems are allowed to dynamically interact with each other via a quantum interaction like the double trace deformation, a quantum teleportation protocol becomes possible and quantum information can be teleported between them through the wormhole that now becomes traversable. As pointed out in\(^{4}\), this is distinct from the standard quantum teleportation protocol where only classical interactions are allowed between the two entangled systems (though see\(^{12}\) for connections with standard quantum teleportation). On the one hand, this provided a concrete mechanism for recovery of quantum information via the Hayden-Preskill protocol\(^{36}\) from the Hawking radiation of old black holes\(^{9}\). One the other hand, it inspired a number of experimental proposals (e.g.\(^{14, 17}\)) for quantum teleportation via quantum interactions between two entangled systems\(^{20}\). Looked at from this perspective, and although in our construction there is a very small amount of multipartite entanglement and only bipartite entanglement is utilized for teleportation, our work is a first step toward a generalization of the quantum teleportation protocol to quantum systems with multipartite entanglement. Since the CFT state dual to a general $(n, g)$ geometry is not known for general values of the moduli parameters, one can focus on the hot limit where locally the entanglement is mainly bipartite and is approximately a TFD state. It would be interesting to realize such a quantum state in the lab and perform the quantum teleportation protocol on it. As discussed in this work, the main new features in this case are the causal shadow region as well as the non-trivial angular dependence. It would be interesting to understand how these features are realized in an experimental set-up of quantum teleportation in the case of quantum circuits with multipartite entanglement. We expect that, in this case, the traversability protocol will occur on a mixed TFD state and that the “size” of the causal shadow region will provide an upper bound on the fidelity of the\(^{18}\)For further discussion on traversable wormholes in the extremal limit, see\(^{35}\).

\(^{20}\)The proposal\(^{14}\) was experimentally realized in\(^{37}\) using an ion trap quantum computer.
teleported state.

As discussed in [9], the experience of an observer traversing through the wormhole connecting a two-sided black hole is that of a smooth free fall through flat space. For an observer traversing through a multi-boundary wormhole, the experience is as pleasant as the two-sided black hole only for particular angular domains. Therefore, one has to be sure of the accuracy of their trajectory into the black hole to avoid the possibility of entrapment in the causal shadow region.

There are several directions for future investigations. First, it would be interesting to extend this work to higher dimensions, where gravity is more interesting than in three dimensions. Second, as shown in [6, 8], it is possible to construct traversable wormholes in flat space. It would be interesting to extend our construction to traversable multi-boundary wormholes in flat space, possibly using the hot limit to achieve that. Furthermore, as discussed above, this work can be interpreted as a quantum teleportation circuit with multipartite entanglement as a resource. Therefore, one can extend the analysis of [14, 17] to this case and characterize how multipartite entanglement affects the properties and conditions of teleportation.

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A An alternative construction of the three-boundary black hole

We have constructed a three-boundary black hole in section 2.1 by choosing some isometries and taking the quotient. Although the representation there is convenient for calculation, the asymptotic region 3 (whose horizon is generated by $\gamma_1^{-1} \gamma_2$ and $\gamma_1 \gamma_2^{-1}$) does not seem to be on the same footing as the other two. It will vanish as we take the hot limit. To show that it is only an artifact of our choice of isometries, here we give an alternative representation where the third boundary does not vanish completely in the hot limit. We will only focus on the non-rotating case, which is generated by a diagonal subgroup of isometries, where $\gamma_L = \gamma_R \equiv \gamma$, and this could be easily generalized to the rotating case.

First let us consider the most general form of a $SL(2, \mathbb{R})$ generator:

$$\xi = x_1 J_1 + x_2 J_2 + x_3 J_3$$  \hspace{1cm} (A.1)

This generator is hyperbolic when $x_1^2 + x_2^2 - x_3^2 > 0$ (derived from $\text{Tr} e^\xi > 2$). The length of horizon generated by $\gamma = e^\xi$ is

$$\ell = 2 \cosh^{-1} \frac{\text{Tr} \gamma}{2} = \sqrt{x_1^2 + x_2^2 - x_3^2}. \hspace{1cm} (A.2)$$
Figure 13: The three-boundary wormhole in the hot limit under our alternative construction with $\alpha_1 = -\alpha_2 = \alpha, \beta_1 = -\beta_2 = \beta = \frac{\pi}{4}$, where $H_1$, $H_2$ and $H_3$ are the three horizons. As we can see, on the boundary, the fixed points become close to each other, but the asymptotic region 3 does not completely vanish.

From this, it is natural to parametrize our generator as

$$\xi = \ell (\cosh \alpha \sin \beta J_1 + \cosh \alpha \cos \beta J_2 - \sinh \alpha J_3) \equiv \ell (\vec{a} \cdot \vec{J}) \quad (A.3)$$

where the generator is written as an inner product taken with signature $(++-)$, where $\vec{a} = (\cosh \alpha \sin \beta, \cosh \alpha \cos \beta, \sinh \alpha)$, and $\vec{J} = (J_1, J_2, J_3)$.

To make a three-boundary wormhole, we choose two such generators

$$\xi_1 = \ell_1 (\cosh \alpha_1 \sin \beta_1 J_1 + \cosh \alpha_1 \cos \beta_1 J_2 - \sinh \alpha_1 J_3) = \ell_1 (\vec{a}_1 \cdot \vec{J}) \quad (A.4)$$

$$\xi_2 = \ell_2 (\cosh \alpha_2 \sin \beta_2 J_1 + \cosh \alpha_2 \cos \beta_2 J_2 - \sinh \alpha_2 J_3) = \ell_2 (\vec{a}_2 \cdot \vec{J}). \quad (A.5)$$

and the corresponding group elements are $\gamma_1 = e^{\xi_1}$ and $\gamma_2 = e^{\xi_2}$. Then the group element related to the third asymptotic region is $\gamma_3 = -\gamma_1^{-1} \gamma_2$. The horizon length of the third region are related to our parameters by

$$\cosh \frac{\ell_3}{2} = -\cosh \frac{\ell_1}{2} \cosh \frac{\ell_2}{2} + \sinh \frac{\ell_1}{2} \sinh \frac{\ell_2}{2} (\vec{a}_1 \cdot \vec{a}_2) \quad (A.6)$$

This is consistent with the dimension of our moduli space. Although we have six parameters in $\xi_1$ and $\xi_2$, only the inner product of $\vec{a}_1$ and $\vec{a}_2$ matters and there are only three independent parameters. We can choose either $\{\ell_1, \ell_2, \vec{a}_1 \cdot \vec{a}_2\}$ or $\{\ell_1, \ell_2, \ell_3\}$ as our parameters.

Our previous representation corresponds to the choice $\vec{a}_1 = (0, -1, 0)$ and $\vec{a}_2 = (0, -\cosh \alpha, -\sinh \alpha)$. We can reproduce our previous results by plugging in the two vectors. Note that $J_1$ is not involved in this representation.
But in general we could have a quite wide range of parameter choices to make a three-boundary wormhole, as long as it does not become a one-boundary torus wormhole as in [24].

For example, we restrict ourselves to a family of isometries that satisfy

\[ \alpha_1 = -\alpha_2 = \alpha, \quad \beta_1 = -\beta_2 = \beta = \frac{\pi}{4}, \]  

and we could still tune the horizon lengths by tuning \( \ell_1, \ell_2, \alpha \). Then as we did in section 2.2, we could calculate the eigenvectors of the \( \gamma_i \)'s and analyze the fixed points on the boundary in the hot limit, and those fixed points are also endpoints of the horizons. For the non-rotating case, all the fixed points are on the \( t = 0 \) slice, and here we take \( \phi \in [0, 2\pi) \). As shown in figure 13, in the hot limit, endpoints of \( H_1 \) approach \( \phi = \frac{3\pi}{2} \) and \( \phi = \pi \), endpoints of \( H_2 \) approach \( \phi = \frac{3\pi}{2} \) and \( \phi = 0 \), and endpoints of \( H_3 \) approach \( \phi = 0 \) and \( \phi = \pi \). As seen in AdS_3 covering space, although one of the two connected part of \( H_3 \) vanishes, there is still part of \( H_3 \) that remains.

This construction should be easily generalized to other reasonable choices of parameters, and also to the rotating case.

### B  Minimal distance between horizons in the hot limit

We will generalize (2.57) to the case of rotating (3, 0) geometry. We focus on the distance \( d_{12} \) between \( H_1 \) and \( H_2 \) since it is the simplest in our representation of the geometry. Due to the symmetry of the construction, the point on \( H_1 \) that is closest to \( H_2 \) sits at the origin of global coordinates. Furthermore, if the point on \( H_2 \) that is closest to \( H_1 \) has coordinates \((t_m, r_m, \phi_m)\), then \( t_m = 0 \) by left-right symmetry (see figure 6b) and we can set the angular coordinate such that \( \phi_m = 0 \). Recall that any geodesic in AdS_3 can be viewed as the intersection of a plane in the embedding space (2.1) that passes through the origin with the hyperboloid of AdS_3.

The idea here is to find the two vectors that span the plane defining \( H_2 \), then use them to find \( r_m \). Using the geodesic distance equation (2.9), we can then find \( d_{12} \).

Suppose that the left and right corners of the diamond of \( H_2 \) have coordinates \((-t_0, -\phi_0)\) and \((t_0, \phi_0)\), respectively, at the boundary. Using (2.35) and (2.34), it is straightforward to show that

\[ t_0 = \tan^{-1} e^{-\alpha} - \tan^{-1} e^{-\tilde{\alpha}} \]  

\[ \phi_0 = \tan^{-1} e^{-\alpha} + \tan^{-1} e^{-\tilde{\alpha}}. \]  

Then, in embedding space, the vectors \( \vec{v}_i = (X_i, Y_i, U_i, V_i) \) that point from the origin to the points \((-t_0, -\phi_0)\) and \((t_0, \phi_0)\) at the boundary can be found using (2.2) to be

\[ \vec{v}_L = (\cos \phi_0, -\sin \phi_0, \cos t_0, -\sin t_0) \quad \text{and} \quad \vec{v}_R = (\cos \phi_0, \sin \phi_0, \cos t_0, \sin t_0) \]  

The difference is that, to make a (3, 0) wormhole, in the bulk, the geodesic connecting the fixed points of \( \gamma_1 \) must not cross that connecting the fixed points of \( \gamma_2 \), while they cross each other in the (1, 1) wormhole construction.
The vector connecting the origin with \((0, r_m, 0)\) is parallel to \(\vec{v}_L + \vec{v}_R\). From this, it is easy to show that

\[
\frac{r_m}{\sqrt{1 + r_m^2}} = \frac{\cos \phi_0}{\cos t_0}
\]  
(B.4)

The matrix representation of \((0, r_m, 0)\) is

\[
p_m = \begin{pmatrix}
\sqrt{1 + r_m^2} + r_m & 0 \\
0 & \sqrt{1 + r_m^2} - r_m
\end{pmatrix}
\]  
(B.5)

So, using (2.9), the minimal distance between \(H_1\) and \(H_2\) is the geodesic distance between \(p_m\) and the origin and is given by

\[
d_{12} = \cosh^{-1} \left( \frac{\text{Tr} p_m}{2} \right) = \cosh^{-1} \left( \sqrt{1 + r_m^2} \right)
\]  
(B.6)

Combining this with (B.4) gives

\[
d_{12} = \tanh^{-1} \left( \frac{\cos \phi_0}{\cos t_0} \right)
\]  
(B.7)

After some algebra, this can be simplified into

\[
d_{12} = \frac{\alpha + \tilde{\alpha}}{2}.
\]  
(B.8)

As a consistency check, note that in the non-rotating case where \(\ell_i = \tilde{\ell}_i\), we have

\[
\alpha = \tilde{\alpha} \Rightarrow d_{12} = \alpha
\]  
(B.9)

which is precisely (2.58) that was quoted in section 2.4. Other minimal geodesic distances (i.e. \(d_{23}\) and \(d_{13}\)) can be obtained from (B.8) by simple permutations. This completes our generalization of the minimal geodesic distance equation to the rotating case. That the angular domain \(D_\phi\) over which \(d_{12}\) is exponentially small is also of AdS length scale in the rotating case follows from the same analysis in the non-rotating case in [19] through an appropriate choice of the Cauchy slice on which the distance is calculated.

**B.1 The large horizon limit near extremality**

This is the limit where

\[
\ell_i \to \infty \quad \text{and} \quad \tilde{\ell}_i \to 0 \quad \Leftrightarrow \quad h_i \to \infty \quad \text{and} \quad T_{H,i} \to 0
\]  
(B.10)

From (B.8), it is easy to see that this limit implies

\[
\alpha \to 0 \quad \text{and} \quad \tilde{\alpha} \to \infty \quad \Rightarrow \quad d_{ij} \to \infty
\]  
(B.11)

This shows that the minimal geodesic distance between the horizons in the extremal limit will diverge. In particular, one can show that the divergence is logarithmic \(d_{ij} \sim \log \left(2/\pi T_H\right) + \)
$O(T_H^2)$. Note however that in the hot limit that we are interested in in this paper and claim that it can induce traversability, we have

$$\ell_i \to \infty \quad \text{and} \quad \tilde{\ell}_i \to \infty \iff h_i \to \infty \quad \text{and} \quad T_{H,i} \to \infty$$

(B.12)

implying that

$$\alpha \to 0 \quad \text{and} \quad \tilde{\alpha} \to 0 \quad \Rightarrow \quad d_{ij} \to 0$$

(B.13)

What this shows is that the hot limit implies the large horizon limit, but the large horizon limit does not imply the hot limit near extremality. It also shows that the exponentially small local causal shadow region exists only for the hot limit.
References

[1] A. Einstein and N. Rosen, The Particle Problem in the General Theory of Relativity, Phys. Rev. 48 (1935) 73–77.

[2] J. L. Friedman, K. Schleich and D. M. Witt, Topological censorship, Phys. Rev. Lett. 71 (Sep, 1993) 1486–1489.

[3] G. Galloway, K. Schleich, D. Witt and E. Woolgar, Topological censorship and higher genus black holes, Phys. Rev. D 60 (1999) 104039, [gr-qc/9902061].

[4] P. Gao, D. L. Jafferis and A. C. Wall, Traversable Wormholes via a Double Trace Deformation, JHEP 12 (2017) 151, [1608.05687].

[5] E. Caceres, A. S. Misobuchi and M.-L. Xiao, Rotating traversable wormholes in AdS, JHEP 12 (2018) 005, [1807.07239].

[6] Z. Fu, B. Grado-White and D. Marolf, A perturbative perspective on self-supporting wormholes, Class. Quant. Grav. 36 (2019) 045006, [1807.07917].

[7] Z. Fu, B. Grado-White and D. Marolf, Traversable Asymptotically Flat Wormholes with Short Transit Times, Class. Quant. Grav. 36 (2019) 245018, [1908.03273].

[8] J. Maldacena, A. Milekhin and F. Popov, Traversable wormholes in four dimensions, 1807.04726.

[9] J. Maldacena, D. Stanford and Z. Yang, Diving into traversable wormholes, Fortsch. Phys. 65 (2017) 1700034, [1704.05333].

[10] J. Maldacena and X.-L. Qi, Eternal traversable wormhole, 1804.00491.

[11] G. T. Horowitz, D. Marolf, J. E. Santos and D. Wang, Creating a Traversable Wormhole, Class. Quant. Grav. 36 (2019) 205011, [1904.02187].

[12] L. Susskind and Y. Zhao, Teleportation through the wormhole, Phys. Rev. D 98 (2018) 046016, [1707.04354].

[13] R. van Breukelen and K. Papadodimas, Quantum teleportation through time-shifted AdS wormholes, JHEP 08 (2018) 142, [1708.09370].

[14] B. Yoshida and N. Y. Yao, Disentangling Scrambling and Decoherence via Quantum Teleportation, Phys. Rev. X 9 (2019) 011006, [1803.10772].

[15] D. Bak, C. Kim and S.-H. Yi, Bulk view of teleportation and traversable wormholes, JHEP 08 (2018) 140, [1805.12349].

[16] B. Freivogel, D. A. Galante, D. Nikolakopoulou and A. Rotundo, Traversable wormholes in AdS and bounds on information transfer, JHEP 01 (2020) 050, [1907.13140].

[17] A. R. Brown, H. Gharibyan, S. Leichenauer, H. W. Lin, S. Nezami, G. Salton et al., Quantum Gravity in the Lab: Teleportation by Size and Traversable Wormholes, 1911.06314.

[18] J. Maldacena and L. Susskind, Cool horizons for entangled black holes, Fortsch. Phys. 61 (2013) 781–811, [1306.0533].

[19] D. Marolf, H. Maxfield, A. Peach and S. F. Ross, Hot multiboundary wormholes from bipartite entanglement, Class. Quant. Grav. 32 (2015) 215006, [1506.04128].
[20] S. Aminneborg, I. Bengtsson, D. Brill, S. Holst and P. Peldan, *Black holes and wormholes in (2+1)-dimensions*, *Class. Quant. Grav.* 15 (1998) 627–644, [gr-qc/9707036].

[21] D. R. Brill, *Multi-black hole geometries in (2+1)-dimensional gravity*, *Phys. Rev. D* 53 (1996) 4133–4176, [gr-qc/9511022].

[22] D. Brill, *Black holes and wormholes in (2+1)-dimensions*, *Lect. Notes Phys.* 537 (2000) 143, [gr-qc/9904083].

[23] H. Maxfield, *Entanglement entropy in three dimensional gravity*, *JHEP* 04 (2015) 031, [1412.0687].

[24] S. Aminneborg, I. Bengtsson and S. Holst, *A Spinning anti-de Sitter wormhole*, *Class. Quant. Grav.* 16 (1999) 363–382, [gr-qc/9805028].

[25] K. Krasnov, *Analytic continuation for asymptotically AdS 3-D gravity*, *Class. Quant. Grav.* 19 (2002) 2399–2424, [gr-qc/0111049].

[26] E. Caceres, A. Kundu, A. K. Patra and S. Shashi, *A Killing Vector Treatment of Multiboundary Wormholes*, *JHEP* 02 (2020) 149, [1912.08793].

[27] M. Banados, C. Teitelboim and J. Zanelli, *The Black hole in three-dimensional space-time*, *Phys. Rev. Lett.* 69 (1992) 1849–1851, [hep-th/9204099].

[28] M. Banados, M. Henneaux, C. Teitelboim and J. Zanelli, *Geometry of the (2+1) black hole*, *Phys. Rev. D* 48 (1993) 1506–1525, [gr-qc/9302012].

[29] M. Headrick, V. E. Hubeny, A. Lawrence and M. Rangamani, *Causality & holographic entanglement entropy*, *JHEP* 12 (2014) 162, [1408.6300].

[30] V. Balasubramanian, P. Hayden, A. Maloney, D. Marolf and S. F. Ross, *Multiboundary Wormholes and Holographic Entanglement*, *Class. Quant. Grav.* 31 (2014) 185015, [1406.2663].

[31] H. Maxfield, S. Ross and B. Way, *Holographic partition functions and phases for higher genus Riemann surfaces*, *Class. Quant. Grav.* 33 (2016) 125018, [1601.00980].

[32] S. Hawking and D. N. Page, *Thermodynamics of Black Holes in anti-De Sitter Space*, *Commun. Math. Phys.* 87 (1983) 577.

[33] J. M. Maldacena, *Eternal black holes in anti-de Sitter*, *JHEP* 04 (2003) 021, [hep-th/0106112].

[34] J. Louko, D. Marolf and S. F. Ross, *On geodesic propagators and black hole holography*, *Phys. Rev. D* 62 (2000) 044041, [hep-th/0002111].

[35] S. Fallows and S. F. Ross, *Making near-extremal wormholes traversable*, 2008.07946.

[36] P. Hayden and J. Preskill, *Black holes as mirrors: Quantum information in random subsystems*, *JHEP* 09 (2007) 120, [0708.4025].

[37] K. Landsman, C. Figgatt, T. Schuster, N. Linke, B. Yoshida, N. Yao et al., *Verified Quantum Information Scrambling*, *Nature* 567 (2019) 61–65, [1806.02807].