A CONTROLLED LOCAL-GLOBAL THEOREM FOR SIMPLICIAL COMPLEXES

SPIROS ADAMS-FLOROU

ABSTRACT. In this paper we prove that a simplicial map of finite-dimensional locally finite simplicial complexes has contractible point inverses if and only if it is an $\epsilon$-controlled homotopy equivalence for all $\epsilon > 0$ if and only if $f \times \text{id}_X$ is a bounded homotopy equivalence measured in the open cone over the target. This confirms for such a space $X$ the slogan that arbitrarily fine control over $X$ corresponds to bounded control over the open cone $O(X_+)$. For the proof a one parameter family of cellulations $\{X'_\epsilon\}_{\epsilon < \epsilon(X)}$ is constructed which provides a retracting map for $X$ which can be used to compensate for sufficiently small control.

1. INTRODUCTION

A homeomorphism has point inverses which are all points. If a map $f$ is homotopic to a homeomorphism it is reasonable to suppose that $f$ might have point inverses that are ‘close’ to being points in some suitable sense. Controlled topology takes ‘close’ to mean small with respect to a metric. One then studies maps with small point inverses and attempts to prove that such a map is homotopic to a homeomorphism.

This approach has many successes in the literature: as a consequence of Chapman and Ferry’s $\alpha$-approximation theorem ([CF79]) a map between closed metric topological manifolds with sufficiently small point inverses is homotopic to a homeomorphism through maps with small point inverses. One can also consider maps where the point inverses all have the homotopy groups of a point, i.e. are contractible. In the non-manifold case Cohen proves in [Coh67] that a p.l. map of finite polyhedra with contractible point inverses is a simple homotopy equivalence.

When doing controlled topology it is desirable that the space we consider, $X$, comes equipped with a metric. In the absence of a metric it is sufficient that $X$ has at least a map $p : X \to M$ to a metric space $(M, d)$, called a control map, which then allows us to measure distances in $M$. In general to be able to detect information about $X$ from the control map and the metric on $M$ we would ideally like $p$ to be highly connected.

Let $f : X \to Y$ be a map of spaces equipped with control maps $p : X \to M$, $q : Y \to M$ to a metric space $(M, d)$. We say that $f : (X, p) \to (Y, q)$ is $\epsilon$-controlled if $f$ commutes with the control maps $p$ and $q$ up to a discrepancy of $\epsilon$, i.e. for all $x \in X$,

$$d(p(x), q(f(x))) < \epsilon.$$ 

We say that $f : (X, p) \to (Y, q)$ is an $\epsilon$-controlled homotopy equivalence if there exists a homotopy inverse $g$ and homotopies $h_1 : g \circ f \sim \text{id}_X$ and $h_2 : f \circ g \sim \text{id}_Y$ such that all of $f : (X, p) \to (Y, q)$, $g : (Y, q) \to (X, p)$, $h_1 : (X \times \mathbb{R}, p \times \text{id}_{\mathbb{R}}) \to (X, p)$ and $h_2 : (Y \times \mathbb{R}, q \times \text{id}_{\mathbb{R}}) \to (Y, q)$ are $\epsilon$-controlled maps.

Note that an $\epsilon$-controlled homotopy equivalence $f$ and its inverse $g$ do not move points more than a distance $\epsilon$ when measured in $M$ and that the homotopy tracks are no longer than $\epsilon$ when measured in $M$. If $X$ and $Y$ are also metric spaces it is perfectly possible that the homotopy tracks are large in $X$ or $Y$ and only become small after mapping to $M$. 

1
Controlled topology is not functorial because the composition of two maps with control less than $\epsilon$ is a map with control less than $2\epsilon$. This motivates Pedersen’s development in [Ped84b] and [Ped84a] of bounded topology, where the emphasis is no longer on how small the control is but rather just that it is finite. A map $f : (X, p) \to (Y, q)$ is called bounded if $f$ commutes with the control maps up to a finite discrepancy $B$. Similarly a bounded homotopy equivalence is one where all the maps and homotopies are bounded. Bounded topology is functorial as the sum of two finite bounds remains finite.

In [FP95] Ferry and Pedersen suggest a relationship between controlled topology on a space $X$ and bounded topology on the open cone $O(X_+)$ when they write in a footnote:

“It is easy to see that if $Z$ is a Poincaré duality space with a map $Z \to K$ such that $Z$ has $\epsilon$-Poincaré duality for all $\epsilon > 0$ when measured in $K$ (after subdivision), e.g. a homology manifold, then $Z \times \mathbb{R}$ is an $O(K_+)$-bounded Poincaré complex. The converse (while true) will not concern us here.”

For $X$ a proper subset of $S^n$ the open cone $O(X_+) \subset \mathbb{R}^{n+1}$ is the union of all rays from the origin $0 \in \mathbb{R}^{n+1}$ through points in $X_+ = X \cup \{x_0\}$ together with the subspace metric. There is a natural map $j_X : X \times \mathbb{R} \to O(X_+)$, called the coning map, given by

$$j_X(x, t) := \begin{cases} tx, & t \geq 0, \\ -tx_0, & t \leq 0. \end{cases}$$

See section 2 for a more general definition of the open cone and the coning map for more general metric spaces.

The footnote above leads one to conjecture that $f : (X, qf) \to (Y, q)$ is an $\epsilon$-controlled homotopy equivalence for all $\epsilon > 0$ if and only if

$$f \times \text{id}_\mathbb{R} : (X \times \mathbb{R}, j_Y(f \times \text{id}_\mathbb{R})) \to (Y \times \mathbb{R}, j_Y)$$

is a bounded homotopy equivalence. In this paper we prove this conjecture for the case of a simplicial map of finite-dimensional locally finite (henceforth f.d. l.f.) simplicial complexes measured in the target. We may measure in the target since such complexes come naturally equipped with a path metric. We prove

**Theorem 1.** Let $f : X \to Y$ be a simplicial map of f.d. l.f. simplicial complexes with $Y$ equipped with the path metric. Then the following are equivalent:

(i) $f$ has contractible point inverses,

(ii) $f : (X, f) \to (Y, \text{id}_Y)$ is an $\epsilon$-controlled homotopy equivalence for all $\epsilon > 0$,

(iii) $f \times \text{id}_\mathbb{R} : (X \times \mathbb{R}, j_Y(f \times \text{id}_\mathbb{R})) \to (Y \times \mathbb{R}, j_Y)$ is a bounded homotopy equivalence.

Working with simplicial maps makes life much easier - one needs only check that the point inverses of the barycentres are contractible:

**Proposition 2.** Let $f : X \to Y$ be a simplicial map of l.f. f.d. simplicial complexes. Then

(i) for all simplices $\sigma \in Y$, there is a p.l. isomorphism $f^{-1}(\hat{\sigma}) \cong f^{-1}(\hat{\sigma}) \times \hat{\sigma}$,

(ii) $f$ has contractible point inverses if and only if $f^{-1}(\hat{\sigma})$ is contractible for all $\sigma \in Y$.

Moreover, simplicial maps allow us to 'lift' certain properties of the target space to the preimage, in particular the fact that open stars deformation retract onto open simplices:
**Proposition 3.** Let \( f : X \to Y \) be a simplicial map of f.d. l.f. simplicial complexes. Then for all \( \sigma \in Y \), \( f^{-1}(\text{st}(\sigma)) \) p.l. deformation retracts onto \( f^{-1}(\text{st}(\sigma)) \).

If, as in the theorem, we additionally suppose that a simplicial map \( f : X \to Y \) has contractible point inverses, then \( f \) turns out to have the approximate homotopy lifting property: for all \( \epsilon > 0 \), the lifting problem

\[
Z \times \{0\} \xrightarrow{h} X \quad \text{with} \quad f \circ h = X \quad \text{and} \quad f \circ h \rightarrow Y
\]

has a solution \( \tilde{H}_\epsilon : Z \times I \to X \) such that the diagram commutes up to \( \epsilon \), i.e.

\[
d_Y(H(z,t), f(\tilde{H}_\epsilon(z,t))) < \epsilon
\]

for all \( (z,t) \in Z \times I \), where \( d_Y \) is the metric on \( Y \). This is precisely the definition of an approximate fibration given by Coram and Duvall in [CD77].

The key ingredient in proving \((i) \Rightarrow (iii)\) and obtaining the approximate homotopy lifting property is the construction and use of the fundamental \( \epsilon \)-subdivision cellulation \( X'_\epsilon \) of an f.d. l.f. simplicial complex \( X \). The \( X'_\epsilon \) are a family of cellulations with \( \lim_{\epsilon \to 0} X'_\epsilon = X \) similar to the family of cellulations obtained by taking slices \( ||X|| \times \{t\} \) of the prism \( ||X|| \times [0,1] \) triangulated so that \( ||X|| \times \{0\} \) is given the triangulation \( X \) and \( ||X|| \times \{1\} \) the barycentric subdivision \( Sd X \). The key difference is that the cellulations \( X'_\epsilon \) are defined in such a way as to guarantee that the homotopy from \( X'_\delta \) to \( X \) through \( X'_{\delta'} \) for \( \delta' \in (0,\epsilon) \) has control \( \epsilon \). These cellulations provide retracting maps that compensate for \( \epsilon \)-control when proving squeezing results. This is precisely what is missing when trying to prove such results for a more general class of spaces.

Section 2 recap some necessary preliminaries. In section 3 the fundamental \( \epsilon \)-subdivision cellulation of an f.d. l.f. simplicial complex is defined and a few useful properties explained. In section 4 Propositions 2 and 3 are proved and consequently a direct proof of Theorem 1 is given.

**Acknowledgement.** This work is partially supported by Prof. Michael Weiss’ Humboldt Professorship.

### 2. Preliminaries

In this paper only locally finite finite-dimensional simplicial complexes will be considered. Such a space \( X \) shall be given a metric \( d_X \), called the standard metric, as follows. First define the standard \( n \)-simplex \( \Delta^n \) in \( \mathbb{R}^{n+1} \) as the join of the points \( e_0 = (1,0,\ldots,0), \ldots, e_n = (0,\ldots,0,1) \in \mathbb{R}^{n+1} \). \( \Delta^n \) is given the subspace metric \( d_{\Delta^n} \) of the standard \( \ell_2 \)-metric on \( \mathbb{R}^{n+1} \). The locally finite finite-dimensional simplicial complex \( X \) is then given the path metric whose restriction to each \( n \)-simplex is \( d_{\Delta^n} \). Distances between points in different connected components are thus \( \infty \). See §4 of Bar03 or Definition 3.1 of [HR95] for more details.

Let \( p : Y \to X \) be a simplicial map of locally-finite simplicial complexes equipped with standard metrics. For \( \sigma \) a simplex in \( Y \), the diameter of \( \sigma \) measured in \( X \) is

\[
diam(\sigma) := \sup_{x, y \in \sigma} d_X(p(x), p(y)).
\]

\(^1\)By a deformation retract we mean a strong deformation retract.
The radius of $\sigma$ measured in $X$ is
$$\text{rad}(\sigma) := \inf_{x \in d\sigma} d_X(p(\tilde{\sigma}), p(x)).$$

The mesh of $X$ measured in $Y$ is
$$\text{mesh}(X) := \sup_{\sigma \in X} \{\text{diam}(\sigma)\}.$$

The comesh of $X$ measured in $Y$ is
$$\text{comesh}(X) := \inf_{\sigma \in X} \{\text{rad}(\sigma)\}.$$

Using the standard metric on $X$ and $\text{id}_X : X \to X$ as the control map $\text{diam}(\sigma) = \sqrt{2}$ and $\text{rad}(\sigma) = \frac{1}{\sqrt{|\sigma|(|\sigma|+1)}}$ for all $\sigma \in X$, so consequently $\text{mesh}(X) = \sqrt{2}$ and if $X$ is $n$-dimensional $\text{comesh}(X) = \frac{1}{\sqrt{n(n+1)}}$.

The open star $\text{st}(\sigma)$ of a simplex $\sigma \in X$ is defined by
$$\text{st}(\sigma) := \bigcup_{\tau \supset \sigma} \hat{\tau}.$$  

The open cone was first considered by Pedersen and Weibel in [PWS9] where it was defined for subsets of $S^n$. This definition was extended to more general spaces by Anderson and Munkholm in [AM90]. We make the following definition: For a complete metric space $(M, d)$ the open cone $O(M_+)$ is defined to be the identification space $M \times \mathbb{R}/\sim$ with $(m, t) \sim (m', t')$ for all $m, m' \in M$ if $t \leq 0$. We define a metric $d_{O(M_+)}$ on $O(M_+)$ by setting
$$d_{O(M_+)}((m, t), (m', t')) = \begin{cases} \text{td}(m, m'), & t \geq 0, \\ 0, & t \leq 0, \end{cases}$$
$$d_{O(M_+)}((m, t), (m, s)) = |t - s|$$

and defining $d_{O(M_+)}((m, t), (m', s))$ to be the infimum over all paths from $(m, t)$ to $(m', s)$, which are piecewise geodesics in either $M \times \{r\}$ or $\{n\} \times \mathbb{R}$, of the length of the path. i.e.
$$d_{O(M_+)}((m, t), (m', s)) = \max\{\min\{t, s\}, 0\} d_X(m, m') + |t - s|.$$  

This metric is carefully chosen so that
$$d_{O(M_+)}|_{M \times \{t\}} = \begin{cases} \text{td}_{O(M_+)}|_{M \times \{1\}}, & t \geq 0, \\ 0, & t \leq 0. \end{cases}$$

This is precisely the metric used by Anderson and Munkholm in [AM90] and also by Siebenmann and Sullivan in [SS79], but there is a notable distinction: we do not necessarily require that our metric space $(M, d)$ has a finite bound.

There is a natural map $j_X : X \times \mathbb{R} \to O(X_+)$ given by the quotient map
$$X \times \mathbb{R} \to X \times \mathbb{R}/\sim$$
$$(x, t) \mapsto [(x, t)].$$

We call this the coning map.

For $M$ a proper subset of $S^n$ with the subspace metric, the open cone $O(M_+)$ can be thought of as the subset of $\mathbb{R}^{n+1}$ consisting of all the points in the rays out of the origin through points in $M_+ := M \cup \{pt\}$ with the subspace metric. This is not the same as the metric we just defined above but it is Lipschitz equivalent.
3. Subdivision cellulations

In this section we construct a controlled 1-parameter family of subdivision cellulations of $X$ which shall be used later in constructing controlled homotopies. This 1-parameter family is defined in analogy to the 1-parameter family of subdivision cellulations obtained by restricting a triangulation of the prism $X \times I$ to the slices $\{X \times \{t\}\}_{0 < t < 1}$.

Given an f.d. l.f. simplicial complex $X$ and its barycentric subdivision $SdX$, we may triangulate the prism $\|X\| \times I$ so that $\|X\| \times \{0\}$ has triangulation $X$ and $\|X\| \times \{1\}$ has triangulation $SdX$.

**Definition 3.1.** The canonical triangulation of $\|X\| \times I$ from $X$ to $SdX$ is defined to have one $(|\sigma| + n + 1)$-simplex

$$(\sigma \times \{0\}) \ast (\hat{\sigma}_0 \ldots \hat{\sigma}_n \times \{1\})$$

in $\|X\| \times I$ for every chain of inclusions in $X$ of the form

$$\sigma \leq \sigma_0 < \ldots < \sigma_n.$$

With a slight abuse of terminology we shall call such a chain of inclusions a flag in $X$ of length $n$. It may easily be verified that this indeed gives a triangulation.

**Example 3.2.** Let $X$ be a 2-simplex. Figure 1 illustrates the canonical triangulation of $\|X\| \times I$ and what the induced cellulations of the slice $\|X\| \times \{0.5\}$ is.

![Figure 1. Obtaining cellulations from the prism.](image)

The slices $\{\|X\| \times \{t\}\}_{0 < t < 1}$ form a continuous family of cellulations of $\|X\|$ from $X$ to $SdX$. Mapping cells identically to corresponding cells and taking the limit as $t \to 0$ there is a straight line homotopy on $\|X\|$ sending the cellulation of $\|X\| \times \{t\}$ to $X$ by mapping through the cellulations $(\|X\| \times \{s\})_{0 < s < t}$. We adapt this procedure to give a family of cellulations, $X'_\epsilon$, where the straight line homotopy from $X'_\epsilon$ to $X$ has control at most $\epsilon$ measured in $X$.

**Definition 3.3.** Define the flag cellulation of $X$ by

$$\chi(X) := \bigcup_{m=0}^{\dim(X)} \bigcup_{\sigma \leq \sigma_0 < \ldots < \sigma_m} \sigma \times \hat{\sigma}_0 \ldots \hat{\sigma}_m \subset X \times SdX.$$
Observe that $\chi(X)$ has the same cellulation as that inherited by $||X|| \times \{t\}$ for any $t \in (0, 1)$ from the canonical triangulation of the prism from $X$ to $SdX$. We now construct a 1-parameter family of p.l. isomorphisms $\Gamma_\epsilon : \chi(X) \to X$ which shall be used to give $X$ a 1-parameter family of cellulations.

**Definition 3.4.** For $0 \leq \epsilon < \text{comesh}(X)$ define a map $\Gamma_\epsilon : \chi(X) \to X$ by

$$
\begin{align*}
\Gamma_\epsilon(v \times \hat{v}) &:= v, \quad \text{for all vertices } v \in X, \\
\Gamma_\epsilon(v \times \hat{\tau}) &:= \partial B_\epsilon(v) \cap \hat{\tau}, \quad \text{for all inclusions of a vertex } v < \tau,
\end{align*}
$$

where $\partial B_\epsilon(v)$ is the sphere of radius $\epsilon$ centred at the vertex $v$ and $\partial B_0(v) := v$.

Extend $\Gamma_\epsilon$ piecewise linearly over each cell of $\chi$ by

$$
\Gamma_\epsilon : \sigma \times \hat{\sigma}_0 \ldots \hat{\sigma}_m \to X, \\
(s_0, \ldots, s_n, t_0, \ldots, t_m) \mapsto \sum_{i=0}^n \sum_{j=0}^m s_i t_j \Gamma_\epsilon(v_i \times \hat{\sigma}_j),
$$

where $\sigma = v_0 \ldots v_n$ with barycentric coordinates $(s_0, \ldots, s_n)$ and $\hat{\sigma}_0 \ldots \hat{\sigma}_m$ has barycentric coordinates $(t_0, \ldots, t_m)$.

We call the image under $\Gamma_\epsilon$ of the flag cellulation the fundamental $\epsilon$-subdivision cellulation of $X$ and denote it by $X'_\epsilon$. We use the following notation for the cells of $X'_\epsilon$:

$$
\begin{align*}
\Gamma_{\sigma_0, \ldots, \sigma_m}(\sigma) &:= \Gamma_\epsilon(\sigma \times \hat{\sigma}_0 \ldots \hat{\sigma}_m), \\
\Gamma_{\sigma_0, \ldots, \sigma_m}(\hat{\sigma}) &:= \Gamma_\epsilon(\hat{\sigma} \times \hat{\sigma}_0 \ldots \hat{\sigma}_m),
\end{align*}
$$

for all flags $\sigma \leq \sigma_0 < \ldots < \sigma_n$. □

**Example 3.5.** Let $X$ be the simplex $\sigma = v_0v_1v_2$ with faces labelled $\tau_0 = v_0v_1$, $\tau_1 = v_1v_2$ and $\tau_2 = v_0v_2$, then the fundamental $\epsilon$-subdivision cellulation of $X$ is as in Figure 2. Each $\Gamma_{\sigma_0, \ldots, \sigma_1}(\tau)$ is the closed cell pointed to by the arrow. □

![Figure 2. The cellulation $X'_\epsilon$ for a 2-simplex.](image-url)
Remark 3.6. Note that for all $0 < \epsilon < \text{comesh}(X)$, $\Gamma_\epsilon$ is a p.l. isomorphism and that $\Gamma_0 = \text{pr}_1 : X \times Sd X \to X$. Hence

$$
\Gamma_\delta \circ \Gamma_\epsilon^{-1} : X'_\epsilon \to X'_\delta
$$

is a p.l. isomorphism for all $0 < \epsilon, \delta < \text{comesh}(X)$. Further, for $0 < \epsilon < \text{comesh}(X)$ the cellulation $X'_\epsilon$ is homotopic to $X$ via the straight line homotopy

$$
h_{2,\epsilon} : Y \times I \to Y (y, t) \mapsto \Gamma_\epsilon(1-t)\Gamma_\epsilon^{-1}(y).
$$

This homotopy sends each vertex $\Gamma_\tau(v)$ to the point $v$ along a straight line of length precisely $\epsilon$. Convexity of the cells of $Y'_\epsilon$ guarantees that all homotopy tracks are of length at most $\epsilon$. Hence $h_{2,\epsilon}$ has control $\epsilon$.

4. Proof of main theorem

In this section we prove the main theorem which we restate for convenience.

Theorem 1. Let $f : X \to Y$ be a simplicial map of f.d. l.f. simplicial complexes equipped with their path metrics and let $j_Y : Y \times \mathbb{R} \to O(Y_+)$ be the coning map. Then the following are equivalent:

(i) $f$ has contractible point inverses,
(ii) $f : (X, f) \to (Y, \text{id}_Y)$ is an $\epsilon$-controlled homotopy equivalence for all $\epsilon > 0$,
(iii) $f \times \text{id}_\mathbb{R} : (X \times \mathbb{R}, j_Y(f \times \text{id}_\mathbb{R})) \to (Y \times \mathbb{R}, j_Y)$ is a bounded homotopy equivalence.

To facilitate the proof of the main theorem we first require two propositions.

Proposition 2. Let $f : X \to Y$ be a simplicial map of l.f. f.d. simplicial complexes. Then

(i) for all simplices $\sigma \in Y$, there is a p.l. isomorphism $f^{-1}(\bar{\sigma}) \cong f^{-1}(\bar{\sigma}) \times \bar{\sigma}$,
(ii) $f$ has contractible point inverses if and only if $f^{-1}(\bar{\sigma})$ is contractible for all $\sigma \in Y$.

Proof. (i): If $\bar{\sigma}$ is not in the image of $f$ then the result holds as $f^{-1}(\bar{\sigma}) = f^{-1}(\bar{\sigma}) = \emptyset$.

Let $\sigma = w_0 \ldots w_m$ be some simplex in $Y$. Suppose there is a $\bar{\sigma} \in X$ such that $f(\bar{\sigma}) = \sigma$. Let $f_{\bar{\sigma}} := f|_{\bar{\sigma}} : \bar{\sigma} \to \sigma$. Since $\sigma$ is the join of its vertices we have that

$$
\tau = \bigstar_{i=0}^m f^{-1}_{\bar{\sigma}}(w_i)
$$

with $f^{-1}_{\bar{\sigma}}(x) \cong \prod_{i=0}^m f^{-1}_{\bar{\sigma}}(w_i) \cong f^{-1}_{\bar{\sigma}}(\bar{\sigma})$ for all $x \in \bar{\sigma}$. Whence $f^{-1}_{\bar{\sigma}}(\bar{\sigma}) \cong f^{-1}_{\bar{\sigma}}(\bar{\sigma}) \times \bar{\sigma}$.

Suppose $\tau_0 < \tau_1$ are such that $f(\tau_i) = \sigma$ for $i = 0, 1$. Then

$$
f^{-1}_{\tau_0}(\bar{\sigma}) \subset f^{-1}_{\tau_1}(\bar{\sigma}) \subset \bigstar_{i=0}^m f^{-1}_{\bar{\sigma}}(w_i)
$$

and consequently

$$
f^{-1}(\bar{\sigma}) \cong \bigcup_{\tau : f(\tau) = \sigma} f^{-1}(\bar{\sigma}) = \bigcup_{\tau : f(\tau) = \sigma} f^{-1}_{\tau_0}(\bar{\sigma}) \times \bar{\sigma} = f^{-1}_{\tau_1}(\bar{\sigma}) \times \bar{\sigma}.
$$

(ii): Clear from the fact that $f^{-1}(x) \cong f^{-1}(\bar{\sigma})$ for $x \in \bar{\sigma}$. □
This proposition tells us that for a simplicial map \(f\) with contractible point inverses, the restriction over each simplex, \(f_i: f^{-1}(\hat{\sigma}) \to \hat{\sigma}\), is a trivial fibre bundle with fibre \(f^{-1}(\hat{\sigma}) \cong \ast\). We will see that we can define a section over each simplex interior and the contractibility of each \(f^{-1}(\hat{\sigma})\) allows us to piece these local sections together by homotopies that are large in \(X\) but can be made arbitrarily small in \(Y\). This yields a global homotopy inverse \(g_\epsilon\), for all \(\epsilon > 0\), that is an approximate section in the sense that \(f \circ g_\epsilon \simeq \text{id}_Y\) via homotopy tracks of diameter < \(\epsilon\). This approximate section can be used to approximately lift homotopies, hence we see that \(f\) is an approximate fibration.

**Proposition 3.** Let \(f: X \to Y\) be a simplicial map of f.d. f.l. simplicial complexes. Then for all \(\sigma \in Y, f^{-1}(\text{st}(\sigma))\) p.l. deformation retracts onto \(f^{-1}(\hat{\sigma})\).

**Proof.** If \(f^{-1}(\hat{\sigma})\) is empty then so is \(f^{-1}(\hat{\rho})\) for all \(\rho \geq \sigma\) and hence \(f^{-1}(\text{st}(\sigma))\) is empty so the result holds vacuously.

Suppose instead that \(f^{-1}(\hat{\sigma}) \cong f^{-1}(\hat{\sigma}) \times \hat{\sigma}\) is non-empty. For every \(\rho > \sigma\) let \(\sigma_\rho^C \in Y\) be the unique simplex such that \(\rho = \sigma \ast \sigma_\rho^C\). For all \(\tau \in X\) with \(f(\tau) = \rho\) let \(f_{\tau} := f|_\tau: \tau \to \rho\) so that \(\tau = f_{\tau}^{-1}(\sigma) \ast f_{\tau}^{-1}(\sigma_\rho^C)\). Every \(x \in \tau \cup f_{\tau}^{-1}(\hat{\sigma})\) can be written uniquely as \(x = (1 - t)x_\sigma + tx_{\sigma_\rho^C}\), for \(x_\sigma \in f_{\tau}^{-1}(\hat{\sigma}), x_{\sigma_\rho^C} \in f_{\tau}^{-1}(\sigma_\rho^C), t \in [0, 1)\). Thus letting the \(t\) parameter go to 0 at unit speed and staying there thereafter defines a linear (strong) deformation retraction of \(\tau \cup f_{\tau}^{-1}(\hat{\sigma})\) onto \(f_{\tau}^{-1}(\hat{\sigma})\). The deformation retractions defined like this for different simplices surjecting onto \(\rho\) agree on intersections and so glue to give a p.l. deformation retraction of \(f^{-1}(\hat{\rho} \cup \hat{\sigma})\) onto \(f^{-1}(\hat{\sigma})\). These glue together to give the desired deformation retraction of \(f^{-1}(\text{st}(\sigma))\) onto \(f^{-1}(\hat{\sigma})\).

**Proof of Theorem** \((i) \Rightarrow (iii)\): Let \(f: X \to Y\) be a simplicial map of f.d. f.l. simplicial complexes with contractible point inverses. Then \(f\) is necessarily surjective as contractible point inverses are non-empty. We seek to define a one parameter family of homotopy inverses

\[
\{g_\epsilon: Y \to X\}_{0 < \epsilon < \text{comesh}(Y)}
\]

and homotopies

\[
\{h_{1,\epsilon}: \text{id}_X \simeq g_\epsilon \circ f\}_{0 < \epsilon < \text{comesh}(Y)}, \quad \{h_{2,\epsilon}: \text{id}_Y \simeq f \circ g_\epsilon\}_{0 < \epsilon < \text{comesh}(Y)}
\]

parametrised by control. Given such families we obtain a bounded homotopy inverse \(g\) to

\[
f \times \text{id}_\mathbb{R}: (X \times \mathbb{R}, j_\mathbb{R} \circ (f \times \text{id}_\mathbb{R})) \to (Y \times \mathbb{R}, j_\mathbb{R})
\]

defined by

\[
g: Y \times \mathbb{R} \to X \times \mathbb{R};
\]

\[
(y, t) \mapsto g_\alpha(t)(y)
\]

and bounded homotopies

\[
h_1: \text{id}_{X \times \mathbb{R}} \simeq g \circ (f \times \text{id}_\mathbb{R}): X \times \mathbb{R} \times I \to X \times \mathbb{R};
\]

\[
(x, t, s) \mapsto h_{1,\alpha(t)}(x, s),
\]

\[
h_2: \text{id}_{Y \times \mathbb{R}} \simeq (f \times \text{id}_\mathbb{R}) \circ g: Y \times \mathbb{R} \times I \to Y \times \mathbb{R};
\]

\[
(y, t, s) \mapsto h_{2,\alpha(t)}(y, s),
\]

where \(\alpha: \mathbb{R} \to (0, \text{comesh}(Y))\) is the function

\[
\alpha: t \mapsto \begin{cases} \text{comesh}(Y), & t \leq 1/\text{comesh}(Y), \\ 1/t, & t \geq 1/\text{comesh}(Y). \end{cases}
\]
Give $Y$ the fundamental $\epsilon$-subdivision cellulation $Y'_\epsilon$ as defined in Definition 3.4. We define $g_\epsilon$, $h_{1,\epsilon}$, and $h_{2,\epsilon}$ by induction. First, define a map $\gamma : \chi(Y) \to X$ by induction on the flag length of cells in $\chi(Y)$. Let

$$\gamma_{\hat{\sigma} \times \sigma} : \hat{\sigma} \to f^{-1}(\hat{\sigma})$$

be any map, then define $\gamma$ on $\hat{\sigma} \times \sigma$ as the closure of the map

$$\gamma_{\hat{\sigma} \times \sigma} \times \text{id}_\sigma : \hat{\sigma} \times \sigma \to f^{-1}(\hat{\sigma}) \times \sigma \cong f^{-1}(\hat{\sigma}).$$

Let $\Phi_{\tau,\sigma} : f^{-1}(\hat{\sigma}) \to f^{-1}(\hat{\tau})$ denote the maps obtained in the closure of $\gamma_{\hat{\sigma} \times \hat{\tau}}$ for $\tau < \sigma$ such that

$$\gamma_{\hat{\sigma} \times \hat{\tau}} = (\Phi_{\tau,\sigma} \circ \gamma_{\hat{\sigma} \times \hat{\sigma}}) \times \text{id}_\sigma : \hat{\sigma} \times \hat{\tau} \to f^{-1}(\hat{\sigma}) \times \hat{\tau} \cong f^{-1}(\hat{\tau}).$$

Now suppose that we have continuously defined $\gamma$ on all cells of $\chi(Y)$ of flag length at most $n$ and that the map takes the form

$$\gamma_{\hat{\sigma}_0 \ldots \hat{\sigma}_n} : \hat{\sigma}_0 \ldots \hat{\sigma}_n \to f^{-1}(\hat{\sigma}_0) \times \cdots \times f^{-1}(\hat{\sigma}_n)$$

on each cell for $i \leq n$ for some maps

$$\gamma_{\hat{\sigma}_0 \ldots \hat{\sigma}_i : \hat{\sigma}_0 \ldots \hat{\sigma}_i \to f^{-1}(\hat{\sigma}_0)}.$$

These maps define a map

$$\gamma_{\hat{\sigma}_0 \ldots \hat{\sigma}_{n+1} : \hat{\sigma}_0 \ldots \hat{\sigma}_{n+1} \to f^{-1}(\hat{\sigma}_0)}$$

which extends to a map

$$\gamma_{\hat{\sigma}_0 \ldots \hat{\sigma}_{n+1} \times \hat{\sigma}_0} : \hat{\sigma}_0 \ldots \hat{\sigma}_{n+1} \times \hat{\sigma}_0 \to f^{-1}(\hat{\sigma}_0)$$

by the contractibility of $f^{-1}(\hat{\sigma}_0)$. Define $\gamma$ on the cell $\hat{\sigma}_0 \ldots \hat{\sigma}_{n+1} \times \hat{\sigma}_0$ as the closure of the map

$$\gamma_{\hat{\sigma}_0 \ldots \hat{\sigma}_{n+1} \times \hat{\sigma}_0} : \hat{\sigma}_0 \ldots \hat{\sigma}_{n+1} \times \hat{\sigma}_0 \to f^{-1}(\hat{\sigma}_0) \times \hat{\sigma}_0 \cong f^{-1}(\hat{\sigma}_0).$$

By induction this defines the map $\gamma$.

For all $0 < \epsilon < \text{comesh}(Y)$, set

$$g_\epsilon := \gamma \circ \Gamma_\epsilon^{-1} : Y \to \chi(Y) \to X.$$

We claim that $\{g_\epsilon\}_{0 < \epsilon < \text{comesh}(Y)}$ is a one parameter family of homotopy inverses to $f$ parametrised by control.

Consider first the composition $f \circ g_\epsilon$.

$$f \circ \gamma = pr_2 \circ (\gamma_{\hat{\sigma}_0 \ldots \hat{\sigma}_n} \times \text{id}_{\hat{\sigma}_0}) = pr_2 : \hat{\sigma}_0 \ldots \hat{\sigma}_n \times \hat{\sigma}_0 \to f^{-1}(\hat{\sigma}_0) \times \hat{\sigma}_0 \to \hat{\sigma}_0.$$

Hence $f \circ \gamma : \chi(Y) \subset SdY \times X \to Y$ is just projection onto $Y$, i.e. the map $\Gamma_0 = \lim_{\tau \to 0} \Gamma_\tau$. Thus $f \circ g_\epsilon = (f \circ \gamma) \circ \Gamma_\epsilon^{-1} = \Gamma_0 \circ \Gamma_\epsilon^{-1}$. Choosing $h_{2,\epsilon}$ precisely as in Remark 3.6, we have $h_{2,\epsilon} : \text{id}_X = \Gamma_\epsilon \circ \Gamma_\epsilon^{-1} \cong \Gamma_0 \circ \Gamma_\epsilon^{-1} = f \circ g_\epsilon$ is an $\epsilon$-controlled homotopy and in fact

$$\{h_{2,\epsilon}\}_{0 < \epsilon < \text{comesh}(Y)}$$

is a one parameter of homotopies parametrised by control.

Now consider the other composition: $g_\epsilon \circ f = \gamma \circ \Gamma_\epsilon^{-1} \circ f$. Define a homotopy

$$h_{1,\epsilon} : X \times I \to X$$

by

$$h_{1,\epsilon} = \text{id}_{f^{-1}(\hat{\sigma})} \times h_{2,\epsilon} : f^{-1}(\hat{\sigma}) \times \hat{\sigma} \times [0,1) \to f^{-1}(\hat{\sigma})$$

with $h_{1,\epsilon}(-,1) := \lim_{t \to 1} h_{1,\epsilon}(-,t)$. This homotopy is sent by $f$ to $h_{2,\epsilon}$:

$$f(h_{1,\epsilon}(x,t)) = h_{2,\epsilon}(f(x),t), \quad \forall (x,t) \in X \times I.$$
Hence $h'_{1,\epsilon}$ has control $\epsilon$.

We now seek a homotopy $h''_{1,\epsilon} : h'_{1,\epsilon}(-,1) \simeq g_\epsilon \circ f$ with zero control. Looking at $f^{-1}(\Gamma_{\epsilon}(\rho \times \hat{\sigma}_0))$ for $\rho = \hat{\sigma}_0 \ldots \hat{\sigma}_n$ observe that $h'_{1,\epsilon}(-,1)$ is the closure of the map

$$\Phi_{\sigma_0,\sigma_n} \times h_{2,\epsilon}(-,1) = \Phi_{\sigma_0,\sigma_n} \times (\Gamma_0 \circ \Gamma_{\epsilon}^{-1}) : f^{-1}(\hat{\sigma}_n) \times \Gamma_{\epsilon}(\rho \times \hat{\sigma}_0) \to f^{-1}(\hat{\sigma}_0) \times \hat{\sigma}_0,$$

whereas $g_\epsilon \circ f$ is the closure of the map

$$(\gamma_{\sigma_0} \ldots \sigma_n \times \hat{\sigma}_0 \times id_{\hat{\sigma}_0}) \circ \Gamma_{\epsilon}^{-1} \circ pr_2 : f^{-1}(\hat{\sigma}_n) \times \Gamma_{\epsilon}(\rho \times \hat{\sigma}_0) \to f^{-1}(\hat{\sigma}_0) \times \hat{\sigma}_0.$$

The component of this map from $\Gamma_{\epsilon}(\hat{\rho})$ to $\hat{\sigma}_0$ is $\Gamma_0 \Gamma_{\epsilon}^{-1}$ and so agrees with the component of $h'_{1,\epsilon}(-,1)$ to $\hat{\sigma}_0$. We now find inductively a homotopy $h''_{1,\epsilon} : h'_{1,\epsilon}(-,1) \simeq g_\epsilon \circ f$ which only moves things in the fibre direction and hence has $0$ control. This is achieved precisely as before using the contractibility of the fibres. The concatenation $h_{1,\epsilon} := h''_{1,\epsilon} * h'_{1,\epsilon}$ is an $\epsilon$-controlled homotopy $id_Y \simeq g_\epsilon \circ f$. As we use the same homotopies in the fibre direction for all $0 < \epsilon < \text{comesh}(Y)$ this gives a one parameter family \( \{ h_{1,\epsilon} : id_Y \simeq g_\epsilon \circ f \}_{0 < \epsilon < \text{comesh}(Y)} \) parametrised by control as required.

Note also that $f : f^{-1}(\tau) \to \tau$ is a homotopy equivalence for all $\tau \in Y$ by restricting $g_\epsilon, h_{1,\epsilon}$ and $h_{2,\epsilon}$. We call such a homotopy equivalence a $Y$-triangular homotopy equivalence. It is an open conjecture that $f : X \to Y$ is homotopic to a $Y$-triangular homotopy equivalence if and only if $f$ is homotopic to an $\epsilon$-controlled homotopy equivalence for all $\epsilon > 0$. $Y$-triangular homotopy equivalences are discussed in [Ada13].

(iii) $\Rightarrow$ (ii): Let $f \times id$ have homotopy inverse $g$ and homotopies $h_1 : id_{X \times \mathbb{R}} \simeq g \circ (f \times id_{\mathbb{R}})$ and $h_2 : id_{Y \times \mathbb{R}} \simeq (f \times id_{\mathbb{R}}) \circ g$ all with bound at most $B < \infty$. Let $p_t : \mathbb{R} \to \{ t \}$ be projection onto $t \in \mathbb{R}$.

Let $g_t := (id_X \times p_t) \circ g|_{Y \times \{ t \}} : Y \times \{ t \} \to X \times \mathbb{R} \to X \times \{ t \}$. This is a homotopy inverse to $f \times id_{\{ t \}} : X \times \{ t \} \to Y \times \{ t \}$ with homotopies

$$(id_X \times p_t) \circ h_1|_{X \times \{ t \}} : (id_X \times p_t) \circ id_{X \times \{ t \}} \simeq (id_X \times p_t) \circ (g \circ (f \times id_{\mathbb{R}}))|_{X \times \{ t \}} = (id_X \times p_t) \circ g|_{Y \times \{ t \}} \circ (f \times id_{\{ t \}}) = g_t \circ (f \times id_{\{ t \}})$$

and

$$(id_X \times p_t) \circ h_2|_{Y \times \{ t \}} : (id_Y \times p_t) \circ id_{Y \times \{ t \}} \simeq (id_Y \times p_t) \circ ((f \times id_{\mathbb{R}}) \circ g)|_{Y \times \{ t \}} = (id_Y \times p_t) \circ (f \times id_{\{ t \}}) \circ g|_{Y \times \{ t \}} = (f \times id_{\{ t \}}) \circ g_t.$$
for $\epsilon < \epsilon'$ as the homotopy tracks for the point $y$ must travel a distance of at least $\epsilon'$.

This is a contradiction and so $f$ is surjective.

Each point $y \in Y$ is contained in a unique simplex interior and hence in that simplex’s open star: $\sigma \subset st(\sigma)$. Since the star is open there is an $\epsilon'$ such that $B_{\epsilon'}(y) \subset st(\sigma)$. By hypothesis we can find an $\epsilon'$-controlled homotopy inverse, $g_{\epsilon'}$, to $f$. Thus $f^{-1}(y)$ is homotopic to $g_{\epsilon'}(y)$ within $f^{-1}(st(\sigma))$. By Proposition\[ f^{-1}(st(\sigma)) \text{ deformation retracts onto } f^{-1}(\sigma) \text{. By Proposition\[ this is p.l. isomorphic to } f^{-1}(\sigma) \times \sigma \text{ which in turn deformation retracts onto } f^{-1}(\sigma) \times \{y\} = f^{-1}(y). \text{ Applying these two deformation retractions to the homotopy } f^{-1}(y) \cong g_{\epsilon'}(y) \text{ gives a contraction of } f^{-1}(y). \text{ Hence } f \text{ has contractible point inverses.} \qed

We conclude with an example illustrating the construction in the proof of $(i) \Rightarrow (iii)$.

**Example 4.1.** Let $\underline{0} = (0, 0, 0)$, $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$ be points in $\mathbb{R}^3$. Define $Y$ to be the simplicial complex with the following 2-simplices: $\sigma_1 := \underline{0} \ast e_1 \ast (e_1 + e_2)$ and $\sigma_2 := \underline{0} \ast e_2 \ast (e_1 + e_2)$. Define $X$ to be the simplicial complex with the following 2-simplices: $\tau_1 := \underline{0} \ast e_1 \ast (e_1 + e_2)$, $\tau_2 := e_3 \ast (e_2 + e_3) \ast (e_1 + e_2 + e_3)$, $\tau_3 := \underline{0} \ast e_3 \ast (e_1 + e_2 + e_3)$ and $\tau_4 := \underline{0} \ast (e_1 + e_2) \ast (e_1 + e_2 + e_3)$. The projection map $f : X \rightarrow Y; (x, y, z) \mapsto (x, y, 0)$ is simplicial and has contractible point inverses. Give $Y$ the cellulation $Y_{\epsilon'}$ for some small $\epsilon > 0$ as pictured in Figure 3.

![Figure 3. $\epsilon$-subdivision cellulations.](image)

We define $g_{\epsilon}$ as in the proof by first defining maps $\gamma_{\hat{\rho} \times \hat{\rho}} : \hat{\rho} \rightarrow f^{-1}(\hat{\rho})$ for all $\rho \in Y$. We define

$$
\gamma_{\hat{\rho} \times \hat{\rho}} = \begin{cases} 
0, & \hat{\rho} \subset \sigma_1 \setminus \sigma_2, \\
1/2, & \hat{\rho} \subset \sigma_1 \cap \sigma_2, \\
1, & \hat{\rho} \subset \sigma_2 \setminus \sigma_1.
\end{cases}
$$

Then, for all $\rho \in \sigma_1 \cap \sigma_2$ we choose the maps

$$
\gamma_{\hat{\rho} \hat{\sigma}_1} : \hat{\rho} \hat{\sigma}_1 \rightarrow f^{-1}(\hat{\rho})
$$

for $i = 1, 2$ as follows:

$$
\gamma_{\hat{\rho} \hat{\sigma}_1 \times \hat{\rho}}(t_0, t_1) = \frac{1}{2} t_0,
$$

$$
\gamma_{\hat{\rho} \hat{\sigma}_2 \times \hat{\rho}}(t_0, t_1) = \frac{1}{2} t_0 + t_1
$$

where $(t_0, t_1)$ are barycentric coordinates.
Finally for $v$ either vertex of $\sigma_1 \cap \sigma_2$ and $\rho$ the 1-simplex of $\sigma_1 \cap \sigma_2$ we define the maps

$$\gamma_{\hat{v}\hat{\rho}\hat{v}_i} : \hat{v}\hat{\rho}\hat{v}_i \to f^{-1}(v)$$

for $i = 1, 2$ as follows:

$$\gamma_{\hat{v}\hat{\rho}\hat{v}_1}(t_0, t_1, t_2) = \frac{1}{2}t_0 + \frac{1}{2}t_1,$$

$$\gamma_{\hat{v}\hat{\rho}\hat{v}_2}(t_0, t_1, t_2) = \frac{1}{2}t_0 + \frac{1}{2}t_1 + t_2.$$

The resulting map $g_\epsilon$ is illustrated in Figure 4 where it is exaggerated to show where each cell of $Y_\epsilon^\prime$ is sent.

![Figure 4. Constructing the map $g_\epsilon$.](image)

REFERENCES

[Ada13] S. Adams-Florou, Triangular homotopy equivalences, ArXiv e-prints (2013), 1310.2768.

[AM90] D. R. Anderson and H. J. Munkholm, Geometric modules and algebraic $K$-homology theory, $K$-Theory 3(6), 561–602 (1990).

[Bar03] A. Bartels, Squeezing and higher algebraic $K$-theory, K-Theory 28(1), 19–37 (2003).

[CD77] D. S. Coram and P. F. Duvall, Approximate fibrations, Rocky Mountain J. Math. 7(2), 275–288 (1977).

[CF79] T. A. Chapman and S. Ferry, Approximating homotopy equivalences by homeomorphisms, Amer. J. Math. 101(3), 583–607 (1979).

[Cob67] M. M. Cohen, Simplicial structures and transverse cellularity, Ann. of Math. (2) 85, 218–245 (1967).

[FP95] S. Ferry and E. K. Pedersen, Epsilon surgery theory, in Novikov conjectures, index theorems and rigidity, Vol. 2 (Oberwolfach, 1993), volume 227 of London Math. Soc. Lecture Note Ser., pages 167–226, Cambridge Univ. Press, Cambridge, 1995.

[HR95] N. Higson and J. Roe, On the coarse Baum-Connes conjecture, in Novikov conjectures, index theorems and rigidity, Vol. 2 (Oberwolfach, 1993), volume 227 of London Math. Soc. Lecture Note Ser., pages 227–254, Cambridge Univ. Press, Cambridge, 1995.

[Ped84a] E. K. Pedersen, $K_\epsilon$-invariants of chain complexes, in Topology (Leningrad, 1982), volume 1060 of Lecture Notes in Math., pages 174–186, Springer, Berlin, 1984.

[Ped84b] E. K. Pedersen, On the $K_\epsilon$-functors, J. Algebra 90(2), 461–475 (1984).

[PW89] E. K. Pedersen and C. A. Weibel, $K$-theory homology of spaces, in Algebraic topology (Arcata, CA, 1986), volume 1370 of Lecture Notes in Math., pages 346–361, Springer, Berlin, 1989.
[SS79] L. Siebenmann and D. Sullivan, On complexes that are Lipschitz manifolds, in Geometric topology (Proc. Georgia Topology Conf., Athens, Ga., 1977), pages 503–525, Academic Press, New York, 1979.