DYNAMICAL BEHAVIORS OF
A GENERALIZED LORENZ FAMILY

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Abstract. In this paper, the ultimate bound set and globally exponentially attractive set of a generalized Lorenz system are studied according to Lyapunov stability theory and optimization theory. The method of constructing Lyapunov-like functions applied to the former Lorenz-type systems (see, e.g., Lorenz system, Rossler system, Chua system) is not applicable to this generalized Lorenz system. We overcome this difficulty by adding a cross term to the Lyapunov-like functions that used for the Lorenz system to study this generalized Lorenz system. The authors in [D. Li, J. Lu, X. Wu, G. Chen, Estimating the ultimate bound and positively invariant set for the Lorenz system and a unified chaotic system, Journal of Mathematical Analysis and Applications 323 (2006) 844-853] obtained the ultimate bound set of this generalized Lorenz system but only for some cases with $0 \leq \alpha < \frac{1}{29}$. The ultimate bound set and globally exponential attractive set of this generalized Lorenz system are still unknown for $\alpha \not\in \left[0, \frac{1}{29}\right)$. Comparing with the best results in the current literature [D. Li, J. Lu, X. Wu, G. Chen, Estimating the ultimate bound and positively invariant set for the Lorenz system and a unified chaotic system, Journal of Mathematical Analysis and Applications 323 (2006) 844-853], our new results fill up the gap of the estimate for the case of $\frac{1}{29} \leq \alpha < \frac{14}{173}$. Furthermore, the estimation derived here contains the results given in [D. Li, J. Lu, X. Wu, G. Chen, Estimating the ultimate bound and positively invariant set for the Lorenz system and a unified chaotic system, J. Math. Anal. Appl. 323 (2006) 844-853] as special case for the case of $0 \leq \alpha < \frac{1}{29}$.

1. Introduction. Chaos, as a very interesting nonlinear phenomenon with complex and unpredictable behavior, has been intensively studied in the last decades. The well-known Lorenz system, Rössler system, Chua’s circuit system, Shimizu-Morioka system, Glukhovsky-Dolzhansky system, Chen system, Li system, the unified chaotic system and other chaotic dynamical systems have motivated a large number of investigations on three-dimensional chaotic dynamical systems [1-2, 5-12]. Mathematicians, physicists, and engineers from various fields have studied the essence of chaos, characteristics of chaotic systems, bifurcations, routes to chaos, many other related topics and a lot of successful results have been reported [4, 8-11, 18-22, 24, 26, 27-30].

Ultimate boundedness of chaotic dynamical systems is one of the fundamental concepts in dynamical systems, which plays an important role in investigating the stability of the equilibrium, estimating the Lyapunov dimension of attractors and the Hausdorff dimension of attractors, the existence of periodic solutions, chaos control, chaos synchronization [2,5,11]. Ultimate boundedness of the Lorenz system has been investigated by Leonov et al. in a series of articles [7,12]. Since then other studies have developed ultimate bounds of similar chaotic dynamical systems [22, 24, 26-30]. However, the approach taken in each is only suitable for that particular system. It is very difficult to propose a universal approach to estimate ultimate bounds for an arbitrary chaotic dynamical system. To this end, we will study ultimate bounds of the new generalized Lorenz system in this paper.

The former Lorenz-type equations [22, 24, 26, 28-30] that we are searching for a global bounded region have a common characteristic: the elements of the main diagonal of the matrix $A$ are all negative [22, 24, 26, 28-30], where the matrix $A$ is the Jacobian matrix $\frac{df}{dx}$ of a continuous-time dynamical system defined by $\frac{dx}{dt} = f(x), x \in R^3$, evaluated at the origin $(0, 0, 0)$. However, there are positive numbers in the elements of main diagonal of matrix $C$, where matrix $C$ is the Jacobian matrix of the generalized Lorenz system evaluated at the origin $(0, 0, 0)$. The origin $(0, 0, 0)$ is an equilibrium point of this generalized Lorenz system. The method
of constructing Lyapunov-like functions applied to the former three-dimensional Lorenz-type dynamical systems [22, 24, 26, 28-30] isn’t applicable to this generalized Lorenz system. We overcome this difficulty by adding a cross term $xy$ to the Lyapunov-like function
\[
V(x, y, z) = a_1(x - a_2)^2 + b_1(y - b_2)^2 + c_1(z - c_2)^2, a_1 > 0, b_1 > 0, c_1 > 0, a_2 \in R, b_2 \in R, c_2 \in R
\]
to study this generalized Lorenz system. Using these Lyapunov-like functions, the present paper obtains the globally attractive sets for this generalized Lorenz system for several parameter ranges. The results obtained in this paper contain the existing results in [22] as special cases.

The rest of the paper is organized as follows. Section 2 gives the mathematical model of the generalized Lorenz system and the main results of this paper. Conclusions are drawn in Sect. 3.

2. Dynamical systems model and main results. The unified system is described by [14, 22]:
\[
\begin{align*}
\frac{dx}{dt} &= (10 + 25\alpha)(y - x), \\
\frac{dy}{dt} &= (28 - 35\alpha)x - xz + (29\alpha - 1)y, \\
\frac{dz}{dt} &= xy - \frac{a + 8}{3}z.
\end{align*}
\]
where $\alpha \in R$ is the system parameter. The system (1) is considered as a transition system between the Lorenz system [6] and the Chen system [3]. The system (1) reduces to the original Lorenz system with $\alpha = 0$, and system (1) is the original Chen system with $\alpha = 1$ [22].

For simplicity, let us denote $10 + 25\alpha = a_\alpha$, $28 - 35\alpha = b_\alpha$, $1 - 29\alpha = c_\alpha$, $\frac{a + 8}{3} = d_\alpha$, then system (1) takes the form
\[
\begin{align*}
\frac{dx}{dt} &= a_\alpha(y - x), \\
\frac{dy}{dt} &= b_\alpha x - c_\alpha y - xz, \\
\frac{dz}{dt} &= xy - d_\alpha z,
\end{align*}
\]
(2)

Some dynamics of the generalized Lorenz systems (2) were studied in [14, 22]. In the following, we will discuss the ultimate bound set and global attractive set of the generalized Lorenz systems (2).

Consider the system
\[
\frac{dX}{dt} = f(X),
\]
where $X = (x_1, x_2, \ldots, x_n) \in R^n, f : R^n \to R^n, t_0 \geq 0$ is the initial time, $X_0 \in R^n$ is an initial value and $X(t, t_0, X_0)$ is a solution to the system (3) satisfying $X(t_0, t_0, X_0) = X_0$ which for simplicity is denoted as $X(t)$. Assume $\Omega \subset R^n$ is a compact set. Define the distance between a point $X(t, t_0, X_0)$ and the set $\Omega$ as
\[
\rho(X(t, t_0, X_0), \Omega) = \inf_{Y \in \Omega} \| X(t, t_0, X_0) - Y \|.
\]

We will give the following definitions and introduce Lemma 1 which will be used in Theorem 1.

**Definition 1**. ([15, 16, 22, 23, 25, 26]). Suppose that there exists a compact set $Q \subseteq R^n$ such that
\[
\forall X_0 \in R^n/Q, \lim_{t \to +\infty} \rho(X(t, t_0, X_0), Q) = 0,
\]
then the set $Q$ is called an ultimate bound set of system (3).
Define the following generalized positively definite and radially unbounded Lyapunov-like function \( V(X) \), where \( L > 0 \) and \( \beta > 0 \) such that \( \forall X_0 \in R^n \), when \( V(X(t_0)) > L \), \( V(X(t)) > L \), the solution of system (3), \( X(t) \) along the \( V(X(t)) \) satisfies \( |V(X(t)) - L| \leq |V(X(t_0)) - L| e^{-\beta(t-t_0)} \), then the system (3) is said to have a global exponential attractive set \( \Omega = \{ X \mid V(X) \leq L \} \).

**Theorem 1.** Suppose that \( \forall \lambda > 0, \forall m > 0 \) and \( \alpha \in (\frac{-2}{3}, \frac{1}{29}) \). Then the following set

\[
\Omega_{\lambda,m} = \left\{ (x, y, z) \mid \lambda x^2 + my^2 + m \left[ z - \left( b_\alpha + \frac{\lambda}{m} a_\alpha \right) \right]^2 \leq R^2 \right\}, \forall \lambda > 0, \forall m > 0
\]

is an ultimate bound set and positively invariant set of generalized Lorenz system (2), where

\[
R^2 = \left\{ \begin{array}{ll}
\frac{(d_\alpha)^2}{4m a_{\alpha}(d_\alpha - a_\alpha)} & , \quad - \frac{2}{5} \leq \alpha \leq - \frac{52}{139} , \\
\frac{(d_\alpha)^2}{4m a_{\alpha}(d_\alpha + m b_\alpha)} & , \quad - \frac{52}{139} < \alpha < - \frac{1}{29} , \\
\frac{(d_\alpha)^2}{4m a_{\alpha}m b_\alpha} & , \quad - \frac{1}{29} < \alpha < - \frac{2}{175} .
\end{array} \right.
\]

**Proof.** Define the following generalized positively definite and radially unbounded Lyapunov-like function

\[
V_{\lambda,m}(X) = V_{\lambda,m}(x, y, z) = \lambda x^2 + my^2 + m \left[ z - \left( b_\alpha + \frac{\lambda}{m} a_\alpha \right) \right]^2, \forall \lambda > 0, \forall m > 0 ,
\]

Differentiating the above Lyapunov-like function \( V_{\lambda,m}(x, y, z) \) in (6) with respect to time \( t \) along the trajectory of system (2) yields

\[
\frac{dV_{\lambda,m}(X)}{dt} = 2\lambda x \frac{dx}{dt} + 2my \frac{dy}{dt} + 2m \left( z - \frac{\lambda}{m} a_\alpha - b_\alpha \right) \frac{dz}{dt},
\]

\[
= 2\lambda x (a_\alpha y - a_\alpha x) + 2my (b_\alpha x - xz - c_\alpha y) + 2m \left( z - \frac{\lambda}{m} a_\alpha - b_\alpha \right) (xy - d_\alpha z),
\]

\[
= -2\lambda a_\alpha x^2 - 2mc_\alpha y^2 - 2md_\alpha (z - \frac{\lambda}{m} a_\alpha + b_\alpha)^2 + \frac{md_\alpha}{2} \left( \frac{\lambda}{m} a_\alpha + b_\alpha \right)^2 ,
\]

\[
= \left( \lambda \frac{dx}{dt} + my \frac{dy}{dt} + m \left( z - \frac{\lambda}{m} a_\alpha - b_\alpha \right) \frac{dz}{dt} \right)^2,
\]

\[
\leq R^2.
\]
When $-\frac{2}{5} < \alpha < \frac{1}{29}$, we can get

$$a_\alpha > 0, b_\alpha > 0, c_\alpha > 0, d_\alpha > 0.$$ 

Obviously, we can see that $\Gamma_1$ that defined by

$$\Gamma_1 = \left\{ (x, y, z) \left| \frac{\lambda x^2}{4a_\alpha m} + \frac{my^2}{4c_\alpha m} + \frac{m(z - \lambda a_\alpha + mb_\alpha)}{(\lambda a_\alpha + mb_\alpha)^2} = 1 \right\}, \quad (7)$$

is an ellipsoid in $\mathbb{R}^3$ for $\forall \lambda > 0, \forall m > 0, \alpha \in (\frac{-2}{5}, \frac{1}{29})$. Outside $\Gamma_1$, $\frac{dV_{\lambda,m}(X)}{dt} < 0$, while inside $\Gamma_1$, $\frac{dV_{\lambda,m}(X)}{dt} > 0$. Thus, the maximum of $V_{\lambda,m}(X)$ can only be reached on $\Gamma_1$. Since the $V_{\lambda,m}(X)$ is a continuous function and $\Gamma_1$ is a bounded closed set, then the function (6) $V_{\lambda,m}(X)$ can reach its maximum value $\max V_{\lambda,m}(X) = R^2, (X \in \Gamma_1)$ on the surface $\Gamma_1$ defined in (7). Obviously, $\{(x, y, z) | V_{\lambda,m}(X) \leq \max V_{\lambda,m}(X), X \in \Gamma_1\}$ contains the solutions of the generalized Lorenz system (2).

By solving the following conditional extremum problem, one can get the maximum value of the function (6):

$$\left\{ \begin{array}{l}
\max V_{\lambda,m}(X) = \max \left\{ \frac{\lambda x^2 + my^2 + m(z - (b_\alpha + \frac{\lambda}{m} a_\alpha))^2}{(\lambda a_\alpha + mb_\alpha)^2} \right\}, \\
s.t. \frac{\lambda x^2}{4a_\alpha m} + \frac{my^2}{4c_\alpha m} + \frac{m(z - \lambda a_\alpha + mb_\alpha)}{(\lambda a_\alpha + mb_\alpha)^2} = 1,
\end{array} \right. \quad (8)$$

Let us take

$$\sqrt{\lambda} x = x_1, \sqrt{m} y = y_1, \sqrt{m} z = z_1, \lambda a_\alpha + mb_\alpha = c,$$

$$\frac{d_\alpha(\lambda a_\alpha + mb_\alpha)^2}{4ma_\alpha} = a_1, \frac{d_\alpha(\lambda a_\alpha + mb_\alpha)^2}{4mc_\alpha} = b_1.$$ 

By solving the following conditional extremum problem of $V_{\lambda,m}(X)$ in (8), one can easily get the conditional extremum problem

$$\left\{ \begin{array}{l}
\max V(x_1, y_1, z_1) = \max \left\{ x_1^2 + y_1^2 + (z_1 - 2c)^2 \right\}, \\
s.t. \frac{x_1^2}{a_1^2} + \frac{y_1^2}{b_1^2} + \frac{(z_1 - c)^2}{c^2} = 1.
\end{array} \right. \quad (9)$$

According to Lemma 1, we can easily get the above conditional extremum problem (9) as:

$$R^2 = \begin{cases}
\frac{(d_\alpha)^2(\lambda a_\alpha + mb_\alpha)^2}{4ma_\alpha(d_\alpha - a_\alpha)}, & -\frac{2}{5} \leq \alpha < -\frac{52}{149}, \\
\frac{(d_\alpha)^2(\lambda a_\alpha + mb_\alpha)^2}{4mc_\alpha(d_\alpha - c_\alpha)}, & -\frac{2}{175} \leq \alpha < \frac{1}{29}, \\
\frac{52}{149} < \alpha < -\frac{2}{175}. 
\end{cases}$$

This completes the proof.

\[ \square \]

**Remark 1.**

1) Let us take $\forall \lambda > 0, \forall m > 0$ in Theorem 1, then we can get a series of ultimate bound sets and positively invariant sets of the generalized Lorenz system (2) according to Theorem 1.

2) Let us take $m = 1$ in Theorem 1, then we can get that

$$\Omega_{\lambda,1} = \left\{ (x, y, z) | \lambda x^2 + y^2 + (z - b_\alpha - \lambda a_\alpha)^2 \leq r^2, \forall \lambda > 0 \right\}$$
is an ultimate bound set and positively invariant set of generalized Lorenz system (2), where

\[
p^2 = \begin{cases} 
\frac{(d_α)^2(αa_α+b_α)^2}{4αa_α}, & \frac{2}{5} \leq α \leq \frac{52}{175} \\
\frac{(d_α)^2(αa_α+b_α)^2}{4αa_α}, & \frac{2}{175} \leq α < \frac{52}{175} \\
(αa_α+b_α)^2, & \frac{52}{175} < α < -\frac{2}{175}.
\end{cases}
\]

Although in [22], the authors construct the generalized Lyapunov-like function \( V(x, y, z) = λx^2 + y^2 + (z - b_α - αa_α)^2 \), \( ∀λ > 0 \) and prove that there exists the ultimate bound set and positively invariant set for the generalized Lorenz system (2) for \( 0 ≤ α < \frac{1}{29} \). In particular, let us take \( m = 1 \) in Theorem 1, then we can get the conclusion that obtained in [22]. The results presented in Theorem 1 contain the existing results in [22] as special cases.

iii) Let us take \( m = 1, λ = 1 \) in Theorem 1, then we can get that

\[
Ω_{1,1} = \left\{ (x, y, z)| x^2 + y^2 + (z - b_α - αa_α)^2 \leq l^2 \right\}
\]

is an ultimate bound set and positively invariant set of generalized Lorenz system (2), where

\[
l^2 = \begin{cases} 
\frac{(d_α)^2(αa_α+b_α)^2}{4αa_α}, & \frac{2}{5} \leq α \leq \frac{52}{175} \\
\frac{(d_α)^2(αa_α+b_α)^2}{4αa_α}, & \frac{2}{175} \leq α < \frac{52}{175} \\
(αa_α+b_α)^2, & \frac{52}{175} < α < -\frac{2}{175}.
\end{cases}
\]

Though Theorem 1 gives the ultimate bound set and positively invariant set of the generalized Lorenz system (2), it does not gives the global exponential attractive set of the generalized Lorenz system (2). The global exponential attractive set of the generalized Lorenz system (2) is described by the following Theorem 2.

**Theorem 2.** Suppose \( ∀λ > 0, m > 0, α ∈ \left( -\frac{2}{5}, \frac{1}{29} \right) \), and let

\[
V_{λ,m}(X) = V_{λ,m}(x, y, z) = λx^2 + my^2 + m \left[ z - \left( \frac{b_α + \frac{λ}{m}a_α}{m} \right) \right]^2,
\]

\[
L_{λ,m} = \frac{d_α(λa_α + mb_α)}{mθ},
\]

\[θ = \min \{a_α, c_α, d_α\} > 0, X(t) = (x(t), y(t), z(t)), X(t₀) = (x(t₀), y(t₀), z(t₀)).\]

Then the estimation

\[
V_{λ,m}(X(t)) - L_{λ,m} ≤ [V_{λ,m}(X(t₀)) - L_{λ,m}] e^{-θ(t-t₀)}
\]

holds for system (2), and thus the set

\[
Ψ_{λ,m} = \left\{ X|V_{λ,m}(X) ≤ L_{λ,m} \right\} = \left\{ (x, y, z)| λx^2 + my^2 + m \left[ z - \left( \frac{b_α + \frac{λ}{m}a_α}{m} \right) \right]^2 ≤ L_{λ,m}, ∀λ > 0, ∀m > 0 \right\},
\]

is a global exponential attractive set of the generalized Lorenz system (2).

**Proof.** When \( -\frac{2}{5} < α < \frac{1}{29} \), we can get

\[a_α > 0, b_α > 0, c_α > 0, d_α > 0.\]

Define the following generalized positively definite and radially unbounded Lyapunov-like function

\[
V_{λ,m}(X) = V_{λ,m}(x, y, z) = λx^2 + my^2 + m \left[ z - \left( \frac{b_α + \frac{λ}{m}a_α}{m} \right) \right]^2, ∀λ > 0, ∀m > 0,
\]
Differentiating the above Lyapunov-like function $V_{\lambda,m}(X)$ with respect to time $t$ along the trajectory of system (2) yields

\[
\frac{dV_{\lambda,m}(X)}{dt}
= 2\lambda x \frac{dx}{dt} + 2my \frac{dy}{dt} + 2m(z - \frac{\lambda}{m}a - b) \frac{dz}{dt},
\]

\[
= 2\lambda x (a, y - a, x) + 2my (b, x - xz - c, y) + 2m (z - \frac{\lambda}{m}a - b) (xy - d, z),
\]

\[
= -2\lambda a, x^2 - 2mc, y^2 - 2md, z^2 + 2d, (\lambda a, + mb, z),
\]

\[
\leq -\lambda a, x^2 - mc, y^2 - md, z^2 + 2d, (\lambda a, + mb, z),
\]

\[
\leq -2 \lambda a, x^2 - mc, y^2 - md, z^2 + \frac{d, (\lambda a, + mb, z)}{m},
\]

\[
\leq -\theta V_{\lambda,m}(X) + \frac{d, (\lambda a, + mb, z)}{m},
\]

\[
= -\theta (V_{\lambda,m}(X) - L_{\lambda,m}).
\]

That is equivalent to say that

\[
\frac{dV_{\lambda,m}(X)}{dt}
\leq -\theta (V_{\lambda,m}(X) - L_{\lambda,m}).
\]  

Thus, we have

\[
[
V_{\lambda,m}(X(t)) - L_{\lambda,m}]
\leq [V_{\lambda,m}(X(t_0)) - L_{\lambda,m}] e^{-\theta(t-t_0)}.
\]

and

\[
\lim_{t \to +\infty} V_{\lambda,m}(X(t)) \leq L_{\lambda,m},
\]

which clearly shows that

\[
\Psi_{\lambda,m} = \left\{ X | \lim_{t \to +\infty} V_{\lambda,m}(X(t)) \leq L_{\lambda,m} \right\},
\]

\[
= \left\{ (x, y, z) | \lambda x^2 + my^2 + m(z - \frac{\lambda}{m}a + b) \leq L_{\lambda,m}, \forall \lambda > 0, \forall m > 0 \right\},
\]

is a global exponential attractive set of system (2).

This completes the proof.

Let us introduce Lemma 2 which will be used in the following parts of this paper.

**Lemma 2.** Suppose that $\alpha \in [\frac{1}{29}, \frac{14}{173}]$. Then we have the following inequality for system (2)

\[
0 \leq -2c, < d, < 2a,.
\]

**Proof.** When $\alpha \geq \frac{1}{29}$, we can get

\[
c, = 1 - 29, \leq 0.
\]

When $\alpha < \frac{14}{173}$, we can get

\[
d, + 2c, = \frac{\alpha + 8}{3} + 2 (1 - 29, ) = \frac{1}{3} (14 - 173, ) > 0.
\]

When $\alpha > -\frac{52}{173}$, we can get

\[
2a, - d, = 2 (10 + 25, ) - \frac{\alpha + 8}{3} = \frac{1}{3} (149, + 52) > 0.
\]

Therefore, to summarize what has been mentioned above, we can get Lemma 2. This completes the proof.

**Lemma 3.** Suppose that $\alpha \in [\frac{1}{29}, \frac{14}{173}]$. Then we can get the following inequality for system (2)

\[
\lim_{t \to +\infty} [z^2 - 2a, z] \leq 0.
\]  

Proof. Let us define

\[ V(x, z) = x^2 - 2a_\alpha z. \]

Then, its derivative along the orbits of system (2) is

\[ \frac{dV(x, z)}{dt} = 2x \frac{dx}{dt} - 2a_\alpha \frac{dz}{dt} = 2a_\alpha x (y - x) - 2a_\alpha (xy - d_\alpha z) = -2a_\alpha x^2 + 2a_\alpha d_\alpha z, \]

and,

\[ \frac{dV}{dt} + d_\alpha V = (d_\alpha - 2a_\alpha) x^2. \]

When \( \alpha \in \left[ \frac{1}{29}, \frac{14}{173} \right) \), we have the following inequality according to Lemma 2

\[ 0 \leq -2c_\alpha < d_\alpha < 2a_\alpha. \]

Thus, we have

\[ \frac{dV}{dt} + d_\alpha V = (d_\alpha - 2a_\alpha) x^2 \leq 0. \]

For any initial value \( V(t_0) = V_0 \), we have

\[ V(t) \leq V_0 e^{-d_\alpha (t-t_0)} \rightarrow 0 (t \rightarrow +\infty). \]

Thus,

\[ \lim_{t \rightarrow +\infty} V(t) = \lim_{t \rightarrow +\infty} [x^2 - 2a_\alpha z] \leq 0. \]

This completes the proof. \( \square \)

Remark 2: The method to prove the inequality in Lemma 3 is using the method in [12]. As early as in 1987, G. A. Leonov et al. have given the method to prove this kind of inequalities for the Lorenz system in the excellent paper [12].

Lemma 4. Suppose \( \alpha \in \left[ \frac{1}{29}, \frac{14}{173} \right) \), and let

\[ \eta = \frac{d_\alpha - c_\alpha}{3a_\alpha}, \varepsilon = \frac{d_\alpha + 2c_\alpha}{6} > 0, \]

\[ c_1 = c_\alpha + a_\alpha \eta, a_1 = a_\alpha (1 - \eta), c_2 = b_\alpha - c_\alpha \eta + a_\alpha \eta (1 - \eta). \]

Then we can obtain

\[ 0 < \eta < 1, c_2 > 0, 0 < \varepsilon < \min \{d_\alpha - 2a_\alpha \eta, a_1\}, \varepsilon \leq c_1. \]

Proof. When \( \alpha \in \left[ \frac{1}{29}, \frac{14}{173} \right) \), we have the following inequality according to Lemma 2

\[ 0 \leq -2c_\alpha < d_\alpha < 2a_\alpha. \]

So, we have

\[ 0 < \eta = \frac{d_\alpha - c_\alpha}{3a_\alpha} < \frac{2a_\alpha + a_\alpha}{3a_\alpha} = 1. \]

\[ d_\alpha - 2a_\alpha \eta - \varepsilon = d_\alpha - 2a_\alpha \frac{d_\alpha - c_\alpha}{3a_\alpha} - \frac{d_\alpha + 2c_\alpha}{6} = \frac{d_\alpha + 2c_\alpha}{6} > 0. \]

\[ a_1 - \varepsilon = a_\alpha (1 - \eta) - \frac{d_\alpha + 2c_\alpha}{6} = \frac{3a_\alpha - 2d_\alpha - c_\alpha}{3} = \frac{1}{9} (310\alpha + 71) > 0. \]

\[ c_1 - \varepsilon = c_\alpha + a_\alpha \eta - \frac{d_\alpha + 2c_\alpha}{6} = c_\alpha + a_\alpha \frac{d_\alpha - c_\alpha}{3a_\alpha} - \frac{d_\alpha + 2c_\alpha}{6} \geq 0. \]

And we can also get

\[ -c_\alpha \geq 0, 0 < \eta < 1, b_\alpha = 28 - 35\alpha > 0. \]
Therefore, we can get
\[ c_2 = b_\alpha - c_\alpha \eta + a_\alpha \eta (1 - \eta) > 0. \]
This completes the proof. \qed

For simplicity, let us simplify model (2) with the following reversible linear transform: \( x = x, y_1 = y - \eta x, z = z. \) Then model (2) takes the form
\[
\begin{align*}
\frac{dx}{dt} &= -a_1 x + a_\alpha y_1, \\
\frac{dy_1}{dt} &= -c_1 y_1 + c_2 x - xz, \\
\frac{dz}{dt} &= -d_\alpha z + xy_1 + \eta x^2.
\end{align*}
\] (13)

where
\[ \eta = \frac{d_\alpha - c_\alpha}{3a_\alpha}, a_1 = a_\alpha (1 - \eta), c_1 = c_\alpha + a_\alpha \eta, c_2 = b_\alpha - c_\alpha \eta + a_\alpha \eta (1 - \eta) > 0. \]

The global attractive sets of system (2) for parameter \( \alpha \in \left[ \frac{1}{29}, \frac{14}{173} \right] \) is described by the following Theorem 3 according to the above lemmas.

**Theorem 3.** Suppose \( \forall \lambda > 0, \forall m > 0, \alpha \in \left[ \frac{1}{29}, \frac{14}{173} \right], \) and let
\[ c_2 = b_\alpha - c_\alpha \eta + a_\alpha \eta (1 - \eta) > 0, \eta = \frac{d_\alpha - c_\alpha}{3a_\alpha}, \tau_{\lambda,m} = \frac{\lambda}{m}a_\alpha + c_2 > 0, \]
\[ \varepsilon = \frac{d_\alpha + 2c_\alpha}{6} > 0, M_{\lambda,m} = \left( 1 + \frac{4(d_\alpha - c_\alpha)^2}{(d_\alpha + 2c_\alpha)^2} \right) m\tau^2_{\lambda,m} > 0, \]
\[ V_{\lambda,m} (X) = V_{\lambda,m} (x, y_1, z) = \frac{1}{2} \left[ \lambda x^2 + m y_1^2 + m(z - \tau_{\lambda,m})^2 \right], \forall \lambda > 0, \forall m > 0. \]

Then the estimation
\[ \left[ V_{\lambda,m} (X (t)) - \frac{M_{\lambda,m}}{2} \right] \leq \left[ V_{\lambda,m} (X_0) - \frac{M_{\lambda,m}}{2} \right] e^{-2\varepsilon(t-t_0)}, \]
holds for system (13), and thus
\[ \Phi_{\lambda,m} = \left\{ (x, y_1, z) \mid \lambda x^2 + y_1^2 + (z - \tau_{\lambda,m})^2 \leq M_{\lambda,m} \right\}, \]

is a global exponential attractive set of system (13).

Thus,
\[ \Delta_{\lambda,m} \]
\[ = \left\{ (x, y, z) \mid 2\lambda x^2 + m \left( y - \frac{d_\alpha - c_\alpha}{3a_\alpha} x \right)^2 + m(z - \tau_{\lambda,m})^2 \leq M_{\lambda,m}, \forall \lambda > 0, m > 0 \right\} \]
is a global exponential attractive set of system (2).

**Proof.** Let us define
\[ f(z) = -m (d_\alpha - 2a_\alpha \eta - \varepsilon) z^2 + m\tau_{\lambda,m} (d_\alpha - 2\varepsilon) z + m\varepsilon \tau^2_{\lambda,m}, \] (14)
where
\[ \varepsilon = \frac{d_\alpha + 2c_\alpha}{6} > 0, \eta = \frac{d_\alpha - c_\alpha}{3a_\alpha} > 0, \tau_{\lambda,m} = \frac{\lambda}{m}a_\alpha + c_2; \]
\[ \forall \lambda > 0, \forall m > 0, c_2 = b_\alpha - c_\alpha \eta + a_\alpha \eta (1 - \eta). \]

According to Lemma 4, we can get
\[ d_\alpha - 2a_\alpha \eta - \varepsilon = d_\alpha - 2a_\alpha \frac{d_\alpha - c_\alpha}{3a_\alpha} - \frac{d_\alpha + 2c_\alpha}{6} = \frac{d_\alpha + 2c_\alpha}{6} > 0. \]
Therefore, we can get
\[
\max_{z \in \mathbb{R}} f(z) = \frac{(d_\alpha + 2c_\alpha)^2 + 4(d_\alpha - c_\alpha)^2}{6(d_\alpha + 2c_\alpha)} m \tau_{\lambda,m}^2 > 0.
\]
Define the following generalized positively definite and radially unbounded Lyapunov-like function
\[
V_{\lambda,m}(X) = V_{\lambda,m}(x, y_1, z) = \frac{1}{2} \left[ \lambda x^2 + my_1^2 + m(z - \tau_{\lambda,m})^2 \right], \forall \lambda > 0, \forall m > 0.
\]
According to Lemma 3 and Lemma 4, we have
\[
\lim_{t \to +\infty} [x^2 - 2a_\alpha z] \leq 0, 0 < \eta < 1, c_2 > 0, 0 < \varepsilon < \min \{a_\alpha, 2a_\alpha \eta, a_1\}, \varepsilon \leq c_1.
\]
Since \( \lim_{t \to +\infty} [x^2 - 2a_\alpha z] \leq 0 \), so there exists a positive constant \( T_0 > 0 \), when \( t > T_0 \), combining with Lemma 3 and Lemma 4 we have
\[
\frac{dV_{\lambda,m}(X)}{dt} \bigg|_{(13)} = \lambda x \frac{dx}{dt} + my_1 \frac{dy_1}{dt} + m(z - \tau_{\lambda,m}) \frac{dz}{dt},
\]
\[
= \lambda x(-a_1 x + a_1 y_1) + my_1(-c_1 y_1 + c_2 x - x z)
\]
\[
+ m(z - \tau_{\lambda,m})(-d_\alpha x + xy_1 + \eta x^2).
\]
Thus, we have
\[
\left[ V_{\lambda,m}(X(t)) - \frac{M_{\lambda,m}}{2} \right] \leq \left[ V_{\lambda,m}(X_0) - \frac{M_{\lambda,m}}{2} \right] e^{-2\varepsilon(t-t_0)}.
\]
By the definition, taking upper limit on both sides of the above inequality (15) as \( t \to +\infty \) results in
\[
\lim_{t \to +\infty} V_{\lambda,m}(X(t)) \leq \frac{M_{\lambda,m}}{2}.
\]
Namely, the set
\[
\Phi_{\lambda,m} = \left\{ (x, y_1, z) | \lambda x^2 + my_1^2 + m(z - \tau_{\lambda,m})^2 \leq M_{\lambda,m}, \forall \lambda > 0, m > 0 \right\},
\]
is a global exponential attractive set of system (13).
Thus, \[
\Delta_{\lambda,m} = \left\{ (x, y, z) \mid \lambda x^2 + m \left( y - \frac{d_\alpha - c_\alpha}{3a_\alpha} x \right)^2 + m(z - \tau_{\lambda,m})^2 \leq M_{\lambda,m}, \forall \lambda > 0, m > 0 \right\}
\]
is a global exponential attractive set of the generalized Lorenz system (2).

This completes the proof. \(\square\)

**Remark 3.** 1) Let us take \(\forall \lambda > 0, \forall m > 0\), then we can get a series of global exponential attractive sets of system (2) according to Theorem 3.

2) Let us take \(m = 1\), then we can get

\[
\Delta_{\lambda,1} = \left\{ (x, y, z) \mid \lambda x^2 + m \left( y - \frac{d_\alpha - c_\alpha}{3a_\alpha} x \right)^2 + m(z - \tau_{\lambda,1})^2 \leq M_{\lambda,1}, \forall \lambda > 0 \right\}
\]
is a global exponential attractive set of the generalized Lorenz system (2), where

\[
\tau_{\lambda,1} = \lambda a_\alpha + c_2, c_2 = b_\alpha - c_\alpha\eta + a_\alpha\eta (1 - \eta), \eta = \frac{d_\alpha - c_\alpha}{3a_\alpha},
\]

\[
M_{\lambda,1} = \left( 1 + \frac{4(d_\alpha - c_\alpha)^2}{(d_\alpha + 2c_\alpha)^2} \right) \tau_{\lambda,1}^2 > 0.
\]

3) Let us take \(\lambda = 1\), then we can get

\[
\Delta_{1,m} = \left\{ (x, y, z) \mid x^2 + m \left( y - \frac{d_\alpha - c_\alpha}{3a_\alpha} x \right)^2 + m(z - \tau_{1,m})^2 \leq M_{1,m}, \forall m > 0 \right\}
\]
is a global exponential attractive set of the generalized Lorenz system (2), where

\[
\tau_{1,m} = \frac{a_\alpha}{m} + c_2, c_2 = b_\alpha - c_\alpha\eta + a_\alpha\eta (1 - \eta), \eta = \frac{d_\alpha - c_\alpha}{3a_\alpha},
\]

\[
M_{1,m} = \left( 1 + \frac{4(d_\alpha - c_\alpha)^2}{(d_\alpha + 2c_\alpha)^2} \right) m\tau_{1,m}^2 > 0.
\]

4) Let us take \(\lambda = 1, m = 1\), then we can get

\[
\Delta_{1,1} = \left\{ (x, y, z) \mid x^2 + m \left( y - \frac{d_\alpha - c_\alpha}{3a_\alpha} x \right)^2 + m(z - \tau_{1,1})^2 \leq M_{1,1} \right\}
\]
is a global exponential attractive set of the generalized Lorenz system (2), where

\[
\tau_{1,1} = a_\alpha + c_2, c_2 = b_\alpha - c_\alpha\eta + a_\alpha\eta (1 - \eta), \eta = \frac{d_\alpha - c_\alpha}{3a_\alpha},
\]

\[
M_{1,1} = \left( 1 + \frac{4(d_\alpha - c_\alpha)^2}{(d_\alpha + 2c_\alpha)^2} \right) \tau_{1,1}^2 > 0.
\]

**Theorem 4.** Suppose that \(\alpha \in \left[ \frac{1}{29}, \frac{14}{173} \right] \), \(L_0 = \left( 1 + \frac{4(d_\alpha - c_\alpha)^2}{(d_\alpha + 2c_\alpha)^2} \right) (c_2)^2 > 0\), and \(c_2 = b_\alpha - c_\alpha\eta + a_\alpha\eta (1 - \eta), \eta = \frac{d_\alpha - c_\alpha}{3a_\alpha}\). Let \((x(t), y(t), z(t))\) be an arbitrary solution of system (2). Then we have the estimate

\[
\lim_{t \to +\infty} \left( y - \frac{d_\alpha - c_\alpha}{3a_\alpha} x \right)^2 + (z - c_2)^2 \leq L_0, \lim_{t \to +\infty} x^2(t) \leq \frac{3a_\alpha L_0}{(3a_\alpha - 2d_\alpha + 2c_\alpha)}.\]
Proof. Let us define
\[ V_0 (X) = V_0 (y_1, z) = y_1^2 + (z - c_2)^2. \]
Similarly to Theorem 3, we can get
\[ y_1^2 + (z - c_2)^2 \leq L_0, \tag{19} \]
where
\[ L_0 = \left( 1 + \frac{4(b_\alpha - d_\alpha)^2}{(b_\alpha + 2d_\alpha)^2} \right) \left( \frac{c_2}{c_2} \right)^2, c_2 = b_\alpha - c_\alpha \eta + a_\alpha \eta (1 - \eta), \eta = \frac{d_\alpha - c_\alpha}{3a_\alpha}. \]
From (19), we can get
\[ y_1^2 \leq L_0. \]
Let us define
\[ V (x) = \frac{1}{2} x^2, \]
Differentiating the above Lyapunov function \( V (x) \) with respect time \( t \) along the trajectory of system (13) yields
\[
\left. \frac{dV(x)}{dt} \right|_{(13)} = x \frac{dx}{dt},
\]
\[
\leq -a_1 x^2 + a_\alpha |x| |y_1|,
\]
\[
\leq -a_1 x^2 + \frac{a_\alpha}{2} x^2 + \frac{a_\alpha}{2} y_1^2,
\]
\[
\leq -\left( a_\alpha - \frac{d_\alpha - c_\alpha}{3a_\alpha} \right) x^2 + \frac{a_\alpha}{2} x^2 + \frac{a_\alpha L_0}{2},
\]
\[
\leq -\frac{3a_\alpha - 2d_\alpha + 2c_\alpha}{3} \left( V (x) - \frac{3a_\alpha L_0}{2(3a_\alpha - 2d_\alpha + 2c_\alpha)} \right). \]
Thus, we have
\[
\left[ V (X (t)) \right] \leq \left[ V (X_0) \right] - \frac{3a_\alpha L_0}{2(3a_\alpha - 2d_\alpha + 2c_\alpha)} e^{-\frac{(3a_\alpha - 2d_\alpha + 2c_\alpha)(t - t_0)}{3}}. \tag{20}
\]
For any \( \alpha \in \left[ \frac{1}{29}, \frac{14}{173} \right] \), we can get
\[ 3a_\alpha - 2d_\alpha + 2c_\alpha = \frac{49\alpha + 80}{3} > 0. \]
By the definition, taking upper limit on both sides of the above inequality (20) as \( t \to +\infty \) results in
\[
\lim_{t \to +\infty} V (X (t)) \leq \frac{3a_\alpha L_0}{2(3a_\alpha - 2d_\alpha + 2c_\alpha)}. \]
That is equivalent to say,
\[
\lim_{t \to +\infty} x^2 (t) \leq \frac{3a_\alpha L_0}{(3a_\alpha - 2d_\alpha + 2c_\alpha)}. \]
This completes the proof. \qed
3. Conclusions. In this paper, we have extended the method developed in [22], [24], [26], [28]-[30] to study the globally exponentially attractive set and positive invariant set for a more general Lorenz family. It has been shown that such a system indeed has globally exponentially attractive set and positive invariant set, and contains all the existing relative results as special cases. Exponential estimation is explicitly derived. The approach presented in this paper may be applied to study other dynamical systems in [13], [17]. The results that obtained in this paper offer theoretical support to study the Hausdorff dimension of attractors for this generalized Lorenz system. These theoretical results are also important and useful in chaos control, chaos synchronization.

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