Edgeworth Expansions for Centered Random Walks on Covering Graphs of Polynomial Volume Growth

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Abstract

Edgeworth expansions for random walks on covering graphs with groups of polynomial volume growths are obtained under a few natural assumptions. The coefficients appearing in this expansion depend on not only geometric features of the underlying graphs but also the modified harmonic embedding of the graph into a certain nilpotent Lie group. Moreover, we apply the rate of convergence in Trotter’s approximation theorem to establish the Berry–Esseen-type bound for the random walks.

Keywords Berry–Esseen-type bound · Edgeworth expansion · Covering graph · Centered random walk

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1 Introduction and Main Results

It is well known that central limit theorems (CLTs, in short) are fundamental topics among a lot of branches of mathematics including, e.g., probability theory, graph theory, geometric analysis and harmonic analysis. They have been studied under various settings intensively and extensively. In particular, long-time asymptotics of random walks on infinite graphs are affected by not only properties of random walks themselves but also geometric features of underlying spaces such as periodicities and volume growths.

We mainly deal with covering graphs as our discrete models of interest in the present paper. Given a finitely generated group $\Gamma$, an infinite connected graph $X = (V, E)$ is called a $\Gamma$-covering graph if $X$ admits free $\Gamma$-actions on itself and the quotient base graph $X_0 := \Gamma \backslash X = (V_0, E_0)$ is a finite graph. In particular, $X$ is called a $\Gamma$-crystal...
lattice if $\Gamma$ is abelian, which includes classical periodic lattices such as square lattices, triangular lattices and hexagonal lattices. We here emphasize that the study of covering graphs is closely related to, e.g., crystallography, tiling theory and material sciences. See [29] for more details with extensive references therein.

There exist several papers in which some long-time asymptotics of random walks on crystal lattices are discussed. In general, it is not possible to apply the usual techniques to obtain such asymptotics directly to the case where the underlying space is a graph, since the notion of scale changes on graphs is not defined properly. Kotani and Sunada studied long-time asymptotics of random walks on a crystal lattice $X$ by placing a special emphasis on the geometric feature of $X$, e.g., [17,18]. In particular, they introduced the notion of harmonic realization of $X$, which is a discrete harmonic map $\Phi_0 : X \to \Gamma \otimes \mathbb{R} = \mathbb{R}^d$ to characterize an equilibrium configuration of $X$. Among their studies, they developed a new mathematical branch called the \textit{discrete geometric analysis} and it has been effectively applied to capture some asymptotics of random walks on $X$. See, e.g., [14,19].

On the other hand, there exist several studies on long-time asymptotics of symmetric or centered random walks on finitely generated groups. We know that the notion of volume growth of groups plays a crucial role in obtaining such results. On the other hand, it is not possible to characterize a finitely generated group itself in terms of its volume growth. However, there is a celebrated theorem of Gromov (cf. [7]) on a characterization of groups of polynomial volume growth, which asserts that such a group is \textit{virtually nilpotent}. Namely, there exists a nilpotent subgroup of $\Gamma$ with finite index. Hence, a number of results on long-time asymptotics of random walks on nilpotent groups have been known. We refer to Raugi [27], Pap [24] and Alexopoulos [1] for CLTs on nilpotent Lie groups and Breuillard [4] and Hough [8] for local CLTs on nilpotent Lie groups.

Afterward, a hybrid model between crystal lattices and groups of polynomial volume growths is introduced. Namely, we consider a $\Gamma$-covering graph $X = (V, E)$ of a finite graph $X_0 = (V_0, E_0)$ whose covering transformation group $\Gamma$ is finitely generated and of polynomial volume growth. By virtue of the Gromov theorem, we may assume that $\Gamma$ is nilpotent without loss of generality. So we simply call $X$ a $\Gamma$-\textit{nilpotent covering graph} in what follows. It is known that there exists a connected and simply connected nilpotent Lie group $(G, \bullet)$ in which $\Gamma$ is a cocompact lattice (see [22]). Therefore, we come to define the notion of harmonic realizations $\Phi_0 : X \to G$ in the same way as that of crystal lattices $\Phi_0 : X \to \mathbb{R}^d$. We note that the harmonicity of $\Phi_0 : X \to G$ determines the most natural configurations of $X$ corresponding to components of $G/[G, G]$, the abelianization of $G$. See (3.1) for its precise definition. Ishiwata studied long-time asymptotics of symmetric random walks on $X$ and obtained a CLT for them. Moreover, a rate of convergence of the $n$-step transition probability $p(n, x, y)$ to the heat kernel on $G$ was obtained in a geometric point of view. See [12,13]. Ishiwata et al. obtained in their successive papers [15,16] that functional CLTs hold for even non-symmetric random walks on $X$ under some natural conditions, which give the process-level convergences of such random walks to certain diffusion processes on $G$.

For an edge $e \in E$, we denote by $o(e)$, $t(e)$ and $\overline{e}$ the origin, the terminus and the inverse edge of $e$, respectively. We put $E_x = \{ e \in E \mid o(e) = x \}$ for $x \in V$. 
Consider a transition probability \( p : E \rightarrow [0, 1] \) satisfying \( \sum_{e \in E_x} p(e) = 1 \) for \( x \in V \) and \( p(e) + p(\bar{e}) > 0 \) for \( e \in E \). The transition probability \( p \) is supposed to be invariant under \( \Gamma \)-actions, that is, \( p(\gamma e) = p(e) \) for \( \gamma \in \Gamma \) and \( e \in E \). Then, \( p \) yields an \( X \)-valued random walk \( (\Omega_\chi(X), \mathbb{P}_\chi, \{w_n\}_{n=0}^\infty) \) starting at \( x \in V \), where \( \mathbb{P}_\chi \) is the probability measure on \( \Omega_\chi(X) \) induced from \( p \) and \( w_n(c) := o(e_{n+1}) \) for \( n \in \mathbb{N} \cup \{0\} \) and \( c = (e_1, e_2, \ldots, e_n, \ldots) \in \Omega_\chi(X) \). Let \( \pi : X \rightarrow X_0 \) be the covering map. Then, we can consider the transition probability \( p \) on \( X_0 \) defined by \( p(\pi(e)) = p(e) \) for \( e \in E \). Namely, \( p \) on \( X \) is the lift of \( p \) on \( X_0 \). Then, \( p \) on \( X_0 \) also induces an \( X_0 \)-valued random walk \( (\Omega_{\pi(x)}(X_0), \mathbb{P}_{\pi(x)}, \{w_n = \pi(w_n)\}_{n=0}^\infty) \) starting at \( \pi(x) \in V_0 \). In what follows, we assume that the random walk \( \{w_n\}_{n=0}^\infty \) on \( X_0 \) associated with \( p \) is irreducible. Then, there exists the unique normalized invariant measure \( m : V_0 \rightarrow (0, 1) \) which satisfies

\[
\sum_{x \in V_0} m(x) = 1, \quad \sum_{e \in (E_0)_x} p(\bar{e})m(t(e)), \quad x \in V_0,
\]

by applying the Perron–Frobenius theorem.

Suppose that the nilpotent Lie algebra \( \mathfrak{g} \) of \( G \) is decomposed as \( \mathfrak{g} = \mathfrak{g}_1^{(1)} \oplus \mathfrak{g}_2^{(2)} \oplus \cdots \oplus \mathfrak{g}_r^{(r)} \) for some \( r \in \mathbb{N} \) and satisfy some nice conditions (see (A1)). Denote by \( C_\infty(G) \) (resp. \( C_\infty(X) \)) the set of all continuous functions on \( G \) (resp. on \( X \)) vanishing at infinity. We also denote by \( C_\infty^c(G) \) the set of all compactly supported smooth functions on \( G \). We define the transition operator \( \mathcal{L} : C_\infty(X) \rightarrow C_\infty(X) \) and the approximation operator \( \mathcal{P}_\epsilon^H : C_\infty(G) \rightarrow C_\infty(X), \ \epsilon > 0 \), by

\[
\mathcal{L}f(x) := \sum_{e \in E_x} p(e) f(t(e)), \quad \mathcal{P}_\epsilon^H f(x) := f(\tau_\epsilon(\Phi_0(x))), \quad x \in X,
\]

respectively, where \( p \) is a \( \Gamma \)-invariant non-symmetric transition probability on \( X \), the map \( \Phi_0 : X \rightarrow G \) is the modified harmonic realization of \( X \) defined by (3.1), and \( \tau_{n^{-1/2}} : G \rightarrow G \) is the dilation of order \( n^{-1/2} \). See Sect. 2 for its definition.

Let \( \{a_1, a_2, \ldots, a_{d_1}\} \) and \( \{a_{d_1+1}, \ldots, a_{d_1+d_2}\} \) be fixed bases in \( \mathfrak{g}^{(1)} \) and \( \mathfrak{g}^{(2)} \), respectively, where \( d_k, \ k = 1, 2, \) is the dimension of \( \mathfrak{g}^{(k)} \). We put

\[
\sigma_i(\Phi_0) := \sum_{e \in E_0} p(e)m(o(e)) \log \left( \Phi_0(o(\bar{e}))^{-1} \bullet \Phi_0(t(\bar{e})) \right)_{a_i}, \quad i = 1, 2, \ldots, d_1,
\]

and

\[
\beta_i(\Phi_0) := \sum_{e \in E_0} p(e)m(o(e)) \log \left( \Phi_0(o(\bar{e}))^{-1} \bullet \Phi_0(t(\bar{e})) \right)_{a_i}, \quad i = d_1 + 1, \ldots, d_1 + d_2,
\]

where \( \bar{e} \in E \) is a lift of \( e \in E_0 \) to \( X \) and \( \log : G \rightarrow \mathfrak{g} \) is the inverse map of \( \exp \).

The following is a simple version of the CLT for a non-symmetric random walk \( \{w_n\}_{n=0}^\infty \) on the nilpotent covering graph \( X \) under the so-called centered condition, whose precise meaning will be given later soon.
Proposition 1.1 (cf. [15, Theorem 2.1]) Assume the centered condition. Then, for any function \( f \in C^\infty(G) \) and any \( t > 0 \), we have

\[
\| L^{[nt]} P^{H}_{n^{-1/2} f} - P^{H}_{n^{-1/2} e^{t/\mathcal{A}(\Phi_0)} f} \|_\infty \to 0 \quad (n \to \infty),
\]

(1.1)

where \( (e^{t/\mathcal{A}(\Phi_0)})_{t \geq 0} \) is a \( C_0 \)-semigroup whose infinitesimal generator \( \mathcal{A}(\Phi_0) \) is a second order sub-elliptic operator on \( C^\infty_c(G) \) given by

\[
\mathcal{A}(\Phi_0) = \frac{1}{2} \sum_{i,j=1}^{d_1} \sigma_i(\Phi_0)\sigma_j(\Phi_0)a_i a_j + \sum_{i=d_1+1}^{d_1+d_2} \beta_i(\Phi_0)a_i.
\]

(1.2)

This proposition means that the sequence of discrete semigroups generated by the \( G \)-valued random walk \( \{\Phi_0(w_n)\}_{n=0}^\infty \) converges to a heat semigroup \( (e^{t/\mathcal{A}(\Phi_0)})_{t \geq 0} \) on \( G \) under the usual CLT-scaling. The point of interest is that the symmetry of the random walk \( \{w_n\}_{n=0}^\infty \) implies \( \beta_i(\Phi_0) = 0 \) for \( i = d_1 + 1, \ldots, d_1 + d_2 \), whereas the converse does not hold in general. Therefore, we observe that each drift coefficient \( \beta_i(\Phi_0) \) is affected by the non-symmetry of the given random walk.

As a further interesting problem, it is natural to ask whether the precise rate of convergence of (1.1) is obtained or not. Namely, we would like to find out a function \( \Psi = \Psi(n) \) satisfying

\[
\| L^{[nt]} P^{H}_{n^{-1/2} f} - P^{H}_{n^{-1/2} e^{t/\mathcal{A}(\Phi_0)} f} \|_\infty \leq C \Psi(n), \quad n \in \mathbb{N},
\]

(1.3)

for some positive constant \( C > 0 \) independent of \( n \) and any function \( f : G \to \mathbb{R} \) with some regularity. Such a bound is usually called the Berry–Esseen-type bound for the discrete semigroup \( L^{[nt]} P^{H}_{n^{-1/2} f} \) associated with the transition probability \( p \). A more difficult problem than (1.3) is known as the Edgeworth expansions, which roughly means the asymptotic expansions on the left-hand side of (1.3). We refer to Götzte and Hipp [10] for Edgeworth expansions in the CLT for random walks on \( \mathbb{R}^d \) under some moment conditions, and Götzte [9] for an extension of the Edgeworth expansions to the infinite-dimensional cases. Bentkus and Pap [2] extended Raugi’s early result of CLTs on nilpotent Lie groups (cf. [27, Theorem 4.5]). Namely, they obtained not only the Berry–Esseen-type bounds but also the Edgeworth expansions on nilpotent Lie groups. However, their expansions are valid only up to the second order. Afterward, Pap showed that the Edgeworth expansions above are also valid up to arbitrary order in [25], which basically motivates our study.

The purpose of the present paper is to obtain the Edgeworth expansions of a certain class of non-symmetric random walks \( \{w_n\}_{n=0}^\infty \) on \( X \) up to arbitrary order as a refinement of (1.1), by noting geometric features of \( X \) including modified harmonicity of realization maps. We note that the underlying random walks are supposed to be generated by independently and identically distributed (i.i.d., in short) random variables in both papers [2,25]. On the other hand, the random walks on a nilpotent covering graph \( X \) are not always generated by i.i.d. random variables since \( X \) has inhomogeneous local structures. Therefore, we need careful examinations of techniques used in the present paper.
Let \( \Phi_0 : X \to G \) denote the modified harmonic realization of \( X \). In the following, we impose two technical but quite natural assumptions.

**(A1):** The nilpotent Lie group \( G \) is stratified of step \( r \in \mathbb{N} \). Namely, the corresponding Lie algebra \( \mathfrak{g} \) admits a direct sum decomposition \( \mathfrak{g} = \mathfrak{g}^{(1)} \oplus \mathfrak{g}^{(2)} \oplus \cdots \oplus \mathfrak{g}^{(r)} \) satisfying that \( [\mathfrak{g}^{(k)}, \mathfrak{g}^{(\ell)}] \subset \mathfrak{g}^{(k+\ell)} \) if \( k + \ell \leq r \) and the subspace \( \mathfrak{g}^{(1)} \) generates the whole \( \mathfrak{g} \) in the sense that \( \mathfrak{g}^{(k)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(k-1)}] \) for \( k = 2, 3, \ldots, r \).

**(A2):** Random walks on \( X \) is centered. In other words, it holds that the law of large numbers

\[
\lim_{n \to \infty} \frac{1}{n} \log(\Phi_0(w_n))|_{\mathfrak{g}^{(1)}} = 0, \quad \text{a.s.,}
\]

where \( \log(\Phi_0(w_n))|_{\mathfrak{g}^{(1)}} \) is the projection of \( \log(\Phi_0(w_n)) \) onto \( \mathfrak{g}^{(1)} \)-component.

We consider the class of stratified nilpotent Lie groups, which is more treatable than that of general ones. Moreover, this class should be sufficiently rich to reflect the geometric structures of general nilpotent Lie groups. This is why we may impose (A1) in the present paper. On the other hand, Proposition 1.1 also holds without imposing (A2) (cf. [15, Theorem 2.1]). However, we need to consider a slight extension of the function space to handle a certain average of the random walks when (A2) is not assumed. Therefore, (A2) is just imposed to avoid the cumbersomeness of arguments. We emphasize that the non-symmetry of the random walks readily implies (A2), whereas there exists a non-symmetric random walk on \( X \) which satisfies (A2). See Sect. 3 for more details. We also emphasize that there is a case where \( \beta_i(\Phi_0) \neq 0 \) even under (A2).

Then, the main result in the present paper is stated as follows:

**Theorem 1.2** (Edgeworth expansion) Assume (A1) and (A2). Let \( N \geq 2 \) and \( f \in C^3_b(\mathbb{N}^{N+1}) \), which is defined by (2.3). Then, we have

\[
\mathcal{L}^{[n]}\mathcal{P}^H_{n^{-1/2}} f(x) = \mathcal{P}^H_{n^{-1/2}} e^{\mathcal{A}(\Phi_0)} f(x) + \sum_{j=1}^{N-1} \frac{\xi_j}{n^{j/2}} + O(n^{-N/2}), \quad x \in V, \quad (1.4)
\]

as \( n \to \infty \), where the coefficient \( \xi_j, j = 1, 2, \ldots, N-1 \), depends on \( x, G, f, p, \Phi_0 \) and \( t \).

We also obtain the precise representations of the coefficients \( \{\xi_j\}_{j=1}^{N-1} \) in (1.4) by applying the multidimensional version of the so-called Euler–Maclaurin formula (see Proposition 6.2). For example, the first coefficient \( \xi_1 \) is written as

\[
\xi_1 = \sum_{d(1)=3} m^l_{\Phi_0} \int_{t + s = 1} \int_{t, s \geq 0} \int_{G^2} \widehat{S}^l_{(2)} f(x \cdot x_1 \cdot 1_G \cdot x_2) v^0_t \Phi_0(dx_1) v^0_s \Phi_0(dx_2) dt ds,
\]

where \( \sum_{d(1)=3} \) means the sum which runs over all multi-index of weighted length 3, the operator \( \widehat{S}^l_{(2)} \) is the symmetrization of the differential operator of the form \( X^l \) with respect to the second variable, and \( m^l_{\Phi_0} \) is a “moment” of the \( G \)-valued random walk \( X \).
\{\Phi_0(w_n)\}_{n=1}^\infty \) of order \( I \). Moreover, \((\nu_i^{\Phi_0})_{t \geq 0}\) is the centered Gaussian semigroup on \( G \) associated with the infinitesimal generator \( A(\Phi_0) \) defined by (1.2), that is, \((\nu_i^{\Phi_0})_{t \geq 0}\) is the unique convolution semigroup on \( G \) satisfying

\[
A(\Phi_0)f(x) = \lim_{t \downarrow 0} \frac{1}{t} \int_G (f(x \cdot y) - f(x)) \nu_i^{\Phi_0}(dy), \quad f \in \text{Dom}(A), \ x \in G.
\]

See Sect. 2.2 for general properties of Gaussian semigroups on \( G \).

We emphasize that the error term \( O(n^{-N/2}) \) heavily depends on the sup-norms of the derivatives of the function \( f \) of order up to \( 3(N + 1)r \), as is seen in the proof of Theorem 1.2. What is remarkable is that the Edgeworth expansion holds even when the underlying process is not always symmetric and the underlying space has non-commutative and inhomogeneous structures. As far as we know, there have not been such results. Hence, we believe that this result gives a new insight into the study of long-time asymptotics of random walks.

The rest of the present paper is organized as follows: In Sect. 2, we review basics of stratified nilpotent Lie groups including several useful formulas such as the Campbell–Baker–Hausdorff formula and the stratified Taylor formula. Moments of Gaussian measures on nilpotent Lie groups are also discussed. The basics of discrete geometric analysis on graphs due to Kotani and Sunada are presented in Sect. 3. We also introduce the notion of modified harmonic realizations, which play a crucial role in the proof of Theorem 1.2. In Sect. 4, an ergodic theorem for iterated transition operators is established. Before giving a proof of Theorem 1.2, we give a rough observation for the validity of the Edgeworth expansion by using explicit calculations of moments of random walks in Sect. 5. We show Theorem 1.2 and give the representations of the coefficients appearing in the expansions (Proposition 6.2) in Sect. 6. By a simple application of the Edgeworth expansion, we know that the Berry–Esseen-type bounds are also obtained. In Sect. 7, we see that the Berry–Esseen-type bound can be obtained by employing an extension of celebrated Trotter’s approximation theorem (Theorem 7.3). Moreover, we see that the Berry–Esseen-type bound is also established when the realization map of \( X \) is not always supposed to be modified harmonic (Theorem 7.5). For this sake, an estimation of the so-called corrector, which measures the difference between a realization and the modified harmonic one, plays an important role in the proof.

Notations. Several notations frequently used in the present paper are given below.

- For \( x \in \mathbb{R} \), the symbol \([x]\) stands for the greatest integer less than or equal to \( x \).
- The cardinality of a set \( A \) is denoted by \(|A|\).
- We denote by \( C \) a positive constant that may change from line to line.
- One writes \( f(n) = O(g(n)) (n \to \infty) \) if and only if there exists a positive real number \( M \) and \( n_0 \in \mathbb{N} \) such that \(|f(n)| \leq Mg(n)\) for \( n \geq n_0 \).
2 Basics of Nilpotent Lie Groups

2.1 Nilpotent Lie Groups

We give basic notions on nilpotent Lie groups, our continuous models of interest, and present fundamental calculus on them. In general, nilpotent Lie groups are classified by the structures of corresponding nilpotent Lie algebras. It is well known that there is a class of nilpotent Lie algebras which admits finite direct sum decompositions with some suitable Lie brackets. See (A1) in the previous section. The class is said to be stratified and contains a lot of important examples of nilpotent Lie algebras such as the Heisenberg ones.

Example 2.1 Let \( m \in \mathbb{N} \). The Heisenberg group \( \mathbb{H}^{2m+1}(\mathbb{R}) := (\mathbb{R}^{2m+1}, \cdot) \) defined by

\[
(x_1, x_2, \ldots, x_{2m}, x_{2m+1}) \cdot (y_1, y_2, \ldots, y_{2m}, y_{2m+1}) = (x_1 + y_1, x_2 + y_2, \ldots, x_{2m} + y_{2m}, x_{2m+1} + y_{2m+1} + \sum_{k=1}^{m} x_k y_{k+m})
\]

is the simplest example of stratified nilpotent Lie group of step 2. Indeed, the corresponding Lie algebra \( g = (\mathbb{R}^{2m+1}, [\cdot, \cdot]) \) spanned by \( \{a_1, a_2, \ldots, a_{2m}, a_{2m+1}\} \) with

\[
[a_i, a_j] = \begin{cases} a_{2m+1} & \text{if } j = i + m, \ i = 1, 2, \ldots, m \\ -a_{2m+1} & \text{if } j = i - m, \ i = m + 1, m + 2, \ldots, 2m \\ 0 & \text{otherwise} \end{cases}
\]

is decomposed as \( g = g^{(1)} \oplus g^{(2)} \equiv \text{Span}_{\mathbb{R}}\{a_1, a_2, \ldots, a_{2m}\} \oplus \text{Span}_{\mathbb{R}}\{a_{2m+1}\} \).

Consider a connected and simple connected nilpotent Lie group \( (G, \cdot) \) whose Lie algebra \( (g, [\cdot, \cdot]) \) is stratified. Let \( d_k \) be the dimension of \( g^{(k)} \) for \( k = 1, 2, \ldots, r \) and \( D := d_1 + d_2 + \cdots + d_r \). We fix a basis \( \{a_1, a_2, \ldots, a_D\} \) of \( g \) such that \( \{a_{d_1+\cdots+d_{k-1}+1}, \ldots, a_{d_1+\cdots+d_{k-1}+d_k}\} \) is a basis of \( g^{(k)} \) for \( k = 1, 2, \ldots, r \), where \( d_0 = 0 \) in convention. For \( x \in G \), we introduce the global coordinate of the first kind given by \( x = (x_1, x_2, \ldots, x_D) \) when \( x = \exp \left( \sum_{i=1}^{D} x_i a_i \right) \). It is known that there is a family of diffeomorphisms \( (\tau_\varepsilon)_{\varepsilon > 0} \) behaving as if it were scalar multiplications on \( G \). Let \( x \in G \) be written as \( x = \exp(a^{(1)} + a^{(2)} + \cdots + a^{(r)}) \), \( a^{(k)} \in g^{(k)}, \ k = 1, 2, \ldots, r \). Then, the dilation map \( \tau_\varepsilon : G \rightarrow G \) is defined by

\[
\tau_\varepsilon(x) := \exp(\varepsilon a^{(1)} + \varepsilon^2 a^{(2)} + \cdots + \varepsilon^r a^{(r)}), \quad \varepsilon \geq 0.
\]

We also define a continuous function \(| \cdot | : G \rightarrow [0, \infty)\) by

\[
|x| := \sum_{i=1}^{D} |x_i|^{1/\sigma_i}, \quad x = (x_1, x_2, \ldots, x_D) \in G,
\]
where $\sigma_i = k$ if $i = d_1 + \cdots + d_{k-1} + 1, \ldots, d_1 + \cdots + d_{k-1} + d_k$ for $k = 1, 2, \ldots, r$. We easily verify that $| \cdot |$ behaves like a norm on $G$, however, it has the homogeneity in the sense that $|\tau_\varepsilon(x)| = \varepsilon |x|$ for $\varepsilon > 0$ and $x \in G$. Therefore, $| \cdot |$ is called a homogeneous norm on $G$. An element $a \in \mathfrak{g}$ may be extended to left and right invariant $C^\infty$-vector fields on $G$, that is,

$$
af(x) = \lim_{t \to 0} \frac{1}{t} \left( f(x \bullet \exp(ta)) - f(x) \right),
$$

$$
\hat{a}f(x) = \lim_{t \to 0} \frac{1}{t} \left( f(\exp(ta) \bullet x) - f(x) \right), \quad f \in C^1(G).
$$

In order to discuss basic calculus such as Taylor’s formula on the nilpotent Lie group $G$, we now introduce the notion of multi-indices. We put $I := (i_1, i_2, \ldots, i_D) \in (\mathbb{N} \cup \{0\})^D$. For $I = (i_1, i_2, \ldots, i_D) \in I$ and $x = (x_1, x_2, \ldots, x_D) \in G$, we put $x^I := x_1^{i_1} x_2^{i_2} \cdots x_D^{i_D}$. Similarly, we put $a^I := a_1^{i_1} a_2^{i_2} \cdots a_D^{i_D}$ for $a \in \mathfrak{g}$.

We define $|I| := i_1 + i_2 + \cdots + i_D$ and $d(I) := \sigma_1 i_1 + \sigma_2 i_2 + \cdots + \sigma_D i_D$ for $I = (i_1, i_2, \ldots, i_D) \in I$. For $I = (i_1, i_2, \ldots, i_D)$ and $J = (j_1, j_2, \ldots, j_D)$ in $I$, we put $I + J = (i_1 + j_1, i_2 + j_2, \ldots, i_D + j_D)$. We also denote by $[j]$ the multi-index with 1 in the $j$-th component and 0 in the others.

The Campbell–Baker–Hausdorff formula, which is known to be the most important tool in calculus on nilpotent Lie groups, is given as

$$
(x \bullet y)^I = x^I + y^I + \sum_{d(J)+d(K)=d(I)} C_{JK} x^J y^K, \quad x, y \in G. \tag{2.1}
$$

For a multi-index $I \in I$, we define the left and right invariant differential operator $S^I$ by

$$
S^I := \frac{1}{|I|!} \sum_{[j_1]+[j_2]+\cdots+[j_{|I|]}=I} a_{j_1} a_{j_2} \cdots a_{j_{|I|}},
$$

$$
\hat{S}^I := \frac{1}{|I|!} \sum_{[j_1]+[j_2]+\cdots+[j_{|I|]}=I} \hat{a}_{j_1} \hat{a}_{j_2} \cdots \hat{a}_{j_{|I|}},
$$

which are regarded as the symmetrizations of $a^I$ and $\hat{a}^I$, respectively.

We are now ready for stating left stratified Taylor’s formula on $G$, which plays a key role in proving main results (see [6, Section 1] or [3, Section 20] for details). We note that right stratified Taylor’s formula on $G$ is also stated in the same way as the left one.

**Lemma 2.2** Let $N \in \mathbb{N} \cup \{0\}$, $f \in C^N(G)$ and $x \in G$. Then, we have

$$
f(x \bullet y) = \sum_{d(I) \leq N} S^I f(x)y^I + R^f_{N+1}(x, y), \quad y \in G,
$$

where
where the remainder term $R^f_{N+1}(x, y)$ satisfies

$$|R^f_{N+1}(x, y)| \leq C|y|^{N+1} \sup \left\{ |a^I f(x \bullet z)| : d(I) = N + 1, |z| \leq b^{N+1}|y| \right\}$$

(2.2)

for some positive constants $C, b > 0$ depending only on $G$.

For $N = 1, 2, 3, \ldots$, we define

$$C^N_b(G) := \left\{ f \in C_b(G) \mid a^I f \in C_b(G), I \in I, d(I) \leq N \right\}.$$  

(2.3)

Given $f \in C^N_b(G)$, we put

$$\|D^N f\|_{\infty} := \|f\|_{\infty} + \sum_{d(I) \leq N} \|a^I f\|_{\infty},$$

where $\| \cdot \|_{\infty}$ denotes the usual sup-norm. The following is useful in the proof of Theorem 1.2. See [25, Lemma 1].

**Lemma 2.3** Let $\ell \in \mathbb{N}$ and $I_1, I_2, \ldots, I_\ell \in I$. We put $d := d(I_1) + d(I_2) + \cdots + d(I_\ell)$ and take $f \in C^{rd}_b(G)$. Then, we have

$$|\hat{S}_{(2)}^I \cdots \hat{S}_{(4)}^{I_2} \hat{S}_{(2)}^{I_1} f(x_1 \bullet x_2 \bullet \cdots \bullet x_{2\ell+1})| \leq C(G) \left( 1 + |x_1 \bullet x_2 \bullet \cdots \bullet x_{2\ell}|^{(r-1)d} \right) \|D^{(rd)} f\|_{\infty},$$

where the function $f(x_1 \bullet x_2 \bullet \cdots \bullet x_{2\ell+1})$ is understood as $f : G^{2\ell+1} \to \mathbb{R}$ given by

$$f(x_1, x_2, \ldots, x_{2\ell+1}) := f(x_1 \bullet x_2 \bullet \cdots \bullet x_{2\ell+1})$$

and $\hat{S}_{(k)}^I$, $k = 1, 2, \ldots, 2\ell + 1$, is regarded as a differential operator with respect to the $k$-th variable $x_k$.

### 2.2 Gaussian Measures on Nilpotent Lie Groups

In this section, we discuss several properties of Gaussian measures on a nilpotent Lie group $G$. At the beginning, we give the definition of Gaussian semigroups and measures on $G$. We refer to, e.g., [2, 26] for more details.

A family $(\nu_t)_{t \geq 0}$ of probability measures on $G$ is called a **convolution semigroup** if $\nu_s \ast \nu_t = \nu_{s+t}$ for $s, t \geq 0$ and $\lim_{t \searrow 0} \nu_t = \nu_0 = \delta_{1_G}$ hold, where $\delta_{1_G}$ denotes the delta measure at the unit $1_G$. The infinitesimal generator $(\mathcal{A}, \text{Dom}(\mathcal{A}))$ of the convolution
semigroup $(\nu_t)_{t \geq 0}$ is defined by

$$\text{Dom}(\mathcal{A}) := \left\{ f \in (C_b(G), \| \cdot \|_{\infty}) \mid \lim_{t \searrow 0} \frac{1}{t} \int_G (f(x \cdot y) - f(x)) \nu_t(dy) \text{ exists} \right\},$$

$$\mathcal{A}f := \lim_{t \searrow 0} \frac{1}{t} \int_G (f(x \cdot y) - f(x)) \nu_t(dy), \quad f \in \text{Dom}(\mathcal{A}).$$

**Definition 2.4** A convolution semigroup $(\nu_t)_{t \geq 0}$ on $G$ is called a Gaussian semigroup if $\nu_t, \ t > 0,$ is non-degenerate and $t^{-1} \nu_t(G \setminus U) \to 0$ as $t \searrow 0$ holds for any neighborhood $U$ of the unit $1_G.$ A non-degenerate probability measure $\nu$ on $G$ is said to be a Gaussian measure if there is a Gaussian semigroup $(\nu_t)_{t \geq 0}$ such that $\nu_0 = \nu.$

We emphasize that, if $(\mu_t)_{t \geq 0}$ and $(\nu_t)_{t \geq 0}$ are two Gaussian semigroups with $\mu_1 = \nu_1,$ then we have $\mu_t = \nu_t$ for every $t \geq 0$ (see [26, Section 4]). If $(\nu_t)_{t \geq 0}$ is a Gaussian semigroup on $G,$ then the corresponding infinitesimal generator $\mathcal{A}$ is of the form

$$\mathcal{A} = \sum_{i,j=1}^{m} A_{ij} a_i a_j + \sum_{i=1}^{m} b_i a_i$$

(2.4)

for some $m \in \mathbb{N},$ where $\{b_i\}_{i=1}^{m} \subset \mathbb{R}$ and $(A_{ij})_{i,j=1}^{m} \in \mathbb{R}^{m} \otimes \mathbb{R}^{m}$ is a symmetric positive semidefinite matrix. Conversely, if $\{b_i\}_{i=1}^{m} \subset \mathbb{R}$ and $(A_{ij})_{i,j=1}^{m} \in \mathbb{R}^{m} \otimes \mathbb{R}^{m}$ is a symmetric positive semidefinite matrix, then there exists a unique convolution semigroup $(\nu_t)_{t \geq 0}$ on $G$ whose infinitesimal generator $\mathcal{A}$ coincides with (2.4) (see, e.g., [11]).

For a probability measure $\mu$ on $G,$ $I \in \mathcal{I}$ and $\alpha > 0,$ we put

$$m_\mu^I := \int_G x^I \mu(dx), \quad m_\mu^\alpha := \int_G |x|^\alpha \mu(dx).$$

A probability measure $\mu$ on $G$ is called centered if $m_\mu^I = 0$ for $I = [i], \ i = 1, 2, \ldots, d_1.$ A Gaussian semigroup $(\nu_t)_{t \geq 0}$ on $G$ is centered with $\nu_t = \tau_{t/2} \nu_1$ for $t > 0$ if and only if the corresponding infinitesimal generator is of the form

$$\mathcal{A} = \frac{1}{2} \sum_{i,j=1}^{d_1} A_{ij} a_i a_j + \sum_{i=d_1+1}^{d_1+d_2} b_i a_i,$$

(2.5)

where $(A_{ij})_{i,j=1}^{d_1} \in \mathbb{R}^{d_1} \otimes \mathbb{R}^{d_1}$ is a symmetric positive semidefinite matrix and $\{b_i\}_{i=d_1+1}^{d_1+d_2} \subset \mathbb{R}.$ We note that a Gaussian measure $\nu$ has finite moments of arbitrary order. As for moments of $\nu$ on $G,$ the following is well known. The proof is given for the sake of completeness.

**Lemma 2.5** (cf. [2, Lemma 2]) Let $(\nu_t)_{t \geq 0}$ be the Gaussian semigroup on $G$ whose infinitesimal generator is given by (2.5). We put $\nu = \nu_1.$ Then, the following hold.

1. For a multi-index $I$ satisfying that $d(I)$ is odd, we have $m_\nu^I = 0.$
(2) For a multi-index $I$ with $d(I) = 2$, we have

$$m_v^I = \begin{cases} b_i & \text{if } I = [i], \ i = d_1 + 1, d_1 + 2, \ldots, d_1 + d_2, \\ A_{ij} & \text{if } I = [i] + [j], \ i, j = 1, 2, \ldots, d_1 \end{cases}.$$ 

(3) For a multi-index $I$ satisfying that $d(I)$ is even and $d(I) \geq 4$, we have

$$m_v^I = \frac{1}{(d(I)/2)!} \sum_{d(J_1) = \ldots = d(J_{d(I)/2}) = 2} C_{J_1 \ldots J_{d(I)/2}} m_v^{J_1} \cdots m_v^{J_{d(I)/2}}.$$ 

(2.6)

**Proof** An identity $\nu = (\tau_1/\sqrt{2}(\nu))^2$ and (2.1) yield

$$m_v^I = \int \int_{G \times G} (xy)^I \tau_1/\sqrt{2}v(dx) \tau_1/\sqrt{2}v(dy)$$

$$= \frac{1}{2^{d(I)/2}} \int \int_{G \times G} (xy)^I v(dx) v(dy)$$

$$= \frac{1}{2^{d(I)/2}} \left( 2m_v^I + \sum_{d(J) + d(K) = d(I) \atop d(J), d(K) \geq 1} C_{JK} m_v^J m_v^K \right)$$

for all $I \in \mathcal{I}$. The identity above immediately leads to

$$m_v^I = \frac{1}{2^{d(I)/2} - 2} \sum_{d(J) + d(K) = d(I) \atop d(J), d(K) \geq 1} C_{JK} m_v^J m_v^K, \quad I \in \mathcal{I}.$$ 

(2.7)

(1) Suppose that $d(I) = 1$. Then, (2.7) gives rise to $m_v^I = m_v^I/\sqrt{2}$, which means $m_v^I = 0$.

(2) If $d(I)$ is odd and $d(J) + d(K) = d(I)$ with $d(J), d(K) \geq 1$, then either $d(J)$ or $d(K)$ is odd and less than $d(I)$. Therefore, by induction, we easily obtain $m_v^I = 0$.

(3) If $d(I) = 4$, Eq. (2.7) gives

$$m_v^I = \frac{1}{2} \sum_{d(J) = d(K) = 2} C_{JK} m_v^J m_v^K.$$ 

Suppose that (2.6) is true for all multi-indices with $d(I) \leq 2k$. Then, for a multi-index $I$ with $d(I) = 2k + 2$, one has

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Let $\Omega_1$ be a torsion-free, finitely generated nilpotent group and $X = (V, E)$ a $\Gamma$-nilpotent covering graph of a finite graph $X_0 = (V_0, E_0)$. We define the set of paths in $X$ starting at $x \in V$ by

$\Omega_{x,n}(X) := \{ c = (e_1, e_2, \ldots, e_n) \mid o(e_{i+1}) = t(e_i), \ i = 1, 2, \ldots, n - 1 \}, \quad n \in \mathbb{N} \cup \{ \infty \}.$

We give a transition probability $p : E_0 \to [0, 1]$ satisfying $\sum_{e \in (E_0)_x} p(e) = 1$ for $x \in V_0$ and $p(e) + p(\overline{e}) > 0$ for $e \in E_0$. Then, $p$ yields an $X_0$-valued random walk $(\Omega_1(X_0), \mathbb{P}_x, \{ w_n \}_{n=0}^\infty)$ starting at $x \in V_0$, where $\mathbb{P}_x$ is the probability distribution on $\Omega_1(X_0)$.

For any $\alpha \geq 0$ and any probability measure $\mu$ on $G$, we put

$\Lambda_\alpha(\mu) := n^{(2-\alpha)/2} \int_{|x| \geq n^{1/2}} |x|^\alpha \mu(dx), \quad L_\alpha(\mu) := n^{(2-\alpha)/2} \int_{|x| < n^{1/2}} |x|^\alpha \mu(dx).$

We note that, if $m_\mu^{k+2} < \infty$ and $\alpha \leq k + 2$ for some $k \in \mathbb{N}$, then it holds that

$\Lambda_\alpha(\mu) \leq n^{-k/2} m_\mu^{k+2}, \quad L_\alpha(\mu) \leq n^{-(\alpha-2)/2} m_\mu^\alpha.$ \hfill (2.8)

The estimates of these truncated moments of measures on $G$ play a crucial role in the proof of Theorem 1.2.
measure on $\Omega_\times(X_0)$ induced from $p$ and $w_n(c) := o(e_{n+1})$ for $n \in \mathbb{N} \cup \{0\}$ and $c = (e_1, e_2, \ldots, e_n, \ldots) \in \Omega_\times(X_0)$. Similarly, we denote by $\mathbb{P}_{x,n}$ for the projection of $\mathbb{P}_x$ onto $\Omega_{x,n}(X_0)$. In what follows, we assume that the random walk $\{w_n\}_{n=0}^\infty$ associated with $p$ is irreducible. Then, there exists the unique normalized invariant measure $m : V_0 \to (0, 1]$ which satisfies

$$\sum_{x \in V_0} m(x) = 1, \quad \sum_{e \in (E_0)_x} p(\overline{e}) m\left(t(e)\right), \quad x \in V_0,$$

by applying the Perron–Frobenius theorem. We put $\tilde{m}(e) := p(e)m(o(e))$ for $e \in E_0$. Then, the symmetry and the non-symmetry of a random walk is defined as follows:

**Definition 3.1** A random walk is said to be $(m)$-symmetric if it satisfies $\tilde{m}(e) = \tilde{m}(\overline{e})$ for $e \in E_0$. Otherwise, it is said to be $(m)$-non-symmetric.

A random walk on $X$ is given by a $\Gamma$-invariant lift of the random walk on $X_0$. Namely, the transition probability, say also $p : E \to [0, 1]$, satisfies $p(\gamma e) = p(e)$ for $\gamma \in \Gamma$ and $e \in E$. We also write $\mathbb{P}_x$ and $\mathbb{P}_{x,n}$ the probability measures on $\Omega_\times(X)$ and $\Omega_{x,n}(X)$ induced by $p : E \to [0, 1]$, respectively. If $c = (e_1, e_2, \ldots, e_n) \in \Omega_{x,n}(X)$, then we put $p(c) = p(e_1)p(e_2)\ldots p(e_n)$.

It is known that the boundary map

$$\partial : C_1(X_0, \mathbb{R}) = \left\{ \sum_{e \in E_0} a_e e \mid a_e \in \mathbb{R}, \overline{\varepsilon} = -\varepsilon \right\} \to C_0(X_0, \mathbb{R}) = \left\{ \sum_{x \in V_0} a_x x \mid a_x \in \mathbb{R} \right\}$$

between two chain groups of $X_0$ is defined by the linear map $\partial(e) := t(e) - o(e)$ for $e \in E_0$. Then, the first homology group of $X_0$ is defined by $H_1(X_0, \mathbb{R}) := \text{Ker}(\partial)$. We now introduce the homological direction of $X_0$ defined by

$$\gamma_p := \sum_{e \in E_0} \tilde{m}(e)e \in H_1(X_0, \mathbb{R}),$$

which indicates the homological drift of the random walk on $X_0$. The following is easily verified.

**Lemma 3.2** (cf. [19, Section 2]) A random walk on $X_0$ is $(m)$-symmetric if and only if $\gamma_p = 0$.

Let $\rho_{\mathbb{R}} : H_1(X_0, \mathbb{R}) \to \Gamma/[\Gamma, \Gamma] \otimes \mathbb{R} \cong g^{(1)}$ be the canonical surjective linear map induced by the canonical surjective homomorphism $\rho : \pi_1(X_0) \to \Gamma$, where $\pi_1(X_0)$ is the fundamental group of $X_0$. A piecewise smooth map $\Phi : V \to G$ is called a $\Gamma$-equivariant realization if $\Phi(\gamma x) = \gamma \cdot \Phi(x)$ for $\gamma \in \Gamma$ and $x \in V$. In particular, the notion of the modified harmonic realization is introduced in [15], which describes the most natural realization of the nilpotent covering graph $X$ in a geometric point of view.

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Definition 3.3 (modified harmonic realization, cf. [15]) A $\Gamma$-equivariant realization $\Phi_0 : V \to G$ is said to be modified harmonic if

$$\sum_{e \in E_x} p(e) \log \left( d\Phi_0(e) \right) \big|_{g^{(1)}} = \rho_{\mathbb{R}}(\gamma_p), \quad x \in V,$$

where $d\Phi_0(e) := \Phi_0(o(e))^{-1} \cdot \Phi_0(t(e))$ for $e \in E$ and $(\cdot)|_{g^{(1)}}$ is the projection onto $g^{(1)}$.

Note that such $\Phi_0$ is uniquely determined up to $g^{(1)}$-translation. The quantity appearing on the right-hand side of (3.1) is called the asymptotic direction of the random walk $\{w_n\}_{n=0}^\infty$. It is in fact regarded as a mean of the projection of the $G$-valued random walk $\xi_n := \Phi_0(w_n), \ n = 0, 1, 2, \ldots$, to components corresponding to $g^{(1)}$. See [15, Section 3] for details.

Lemma 3.4 (properties on $\rho_{\mathbb{R}}(\gamma_p)$)

1. A law of large numbers holds for the projected random walk $\{\log(\xi_n)|_{g^{(1)}}\}_{n=0}^\infty$. Namely, we have

$$\lim_{n \to \infty} \frac{1}{n} \log(\xi_n)|_{g^{(1)}} = \rho_{\mathbb{R}}(\gamma_p), \quad \mathbb{P}_x\text{-a.s.}$$

2. If the random walk $\{w_n\}_{n=0}^\infty$ is (m-)symmetric, that is, $\gamma_p = 0$, then one has $\rho_{\mathbb{R}}(\gamma_p) = 0$. However, the converse does not hold in general.

The random walk $\{w_n\}_{n=0}^\infty$ is said to be centered if $\rho_{\mathbb{R}}(\gamma_p) = 0$. The lemma above asserts that a symmetric random walk always satisfies (A2). However, there exists a non-symmetric and centered random walk on $X$. Thus, it is meaningful to impose (A2) in the present paper.

4 Ergodic Theorems for Transition Operators

Let us put $\ell^2(X_0) := \{ f : V_0 \to \mathbb{C} \}$, which is equipped with

$$\langle f, g \rangle_{\ell^2(X_0)} := \sum_{x \in V_0} f(x) \overline{g(x)}, \quad \| f \|_{\ell^2(X_0)} := \left( \sum_{x \in V_0} |f(x)|^2 \right)^{1/2}, \quad f, g \in \ell^2(X_0).$$

We denote by $K_0$, $K_0 = 1, 2, \ldots, \leq |V_0|$, for the period of the given random walk on $X_0$. Put $\alpha_k := e^{2\pi ki/K_0}$ for $k = 0, 1, \ldots, K_0 - 1$. Then, the Perron–Frobenius theorem implies that the transition operator $L : \ell^2(X_0) \to \ell^2(X_0)$ has the maximal simple eigenvalues $\alpha_0, \alpha_1, \ldots, \alpha_{K_0 - 1}$ with the corresponding normalized right eigenvectors $\phi_0, \phi_1, \ldots, \phi_{K_0 - 1}$ and left eigenfunctions $\psi_0, \psi_1, \ldots, \psi_{K_0 - 1}$. In particular, we see that $\phi_0(x) \equiv |V_0|^{-1/2}$ and $\psi(x) = |V_0|^{1/2} m(x)$ for $x \in V_0$. We put

$$\ell^2_{K_0}(X_0) := \{ f \in \ell^2(X_0) | \langle f, \psi_j \rangle_{\ell^2(X_0)} = 0 \text{ for } j = 0, 1, \ldots, K_0 - 1 \}. \quad \square$$
We note that $L$ preserves the set $\ell^2_{K_0}(X_0)$, that is, $f \in \ell^2_{K_0}(X_0)$ implies that $Lf \in \ell^2_{K_0}(X_0)$. Therefore, every function $f \in \ell^2(X_0)$ can be decomposed as

$$f = \langle f, m \rangle_{\ell^2(X_0)} + \sum_{j=1}^{K_0-1} \langle f, \psi_j \rangle_{\ell^2(X_0)} \phi_j + f_{\ell^2_{K_0}(X_0)} \in \ell^2_{K_0}(X_0).$$

(4.1)

The following is the fundamental ergodic theorems for the transition operator $L$. We refer to Ishiwata et al. [14, Theorem 3.2] for the proof.

**Proposition 4.1** (ergodic theorem I) Let $L$ be the transition operator acting on $\ell^2(X_0)$. Then, we have

$$\frac{1}{n} \sum_{k=0}^{n-1} L^k f(x) = \sum_{x \in V_0} m(x) f(x) + \frac{1}{n} A[f]_n(x)$$

for $n \in \mathbb{N}$, $f \in \ell^2(X_0)$ and $x \in V_0$, where

$$A[f]_n(x) := \sum_{j=1}^{K_0-1} \langle f, \psi_j \rangle_{\ell^2(X_0)} \sum_{k=0}^{n-1} \alpha^k_j \phi_j(x) + \sum_{k=0}^{n-1} L^k f_{\ell^2_{K_0}(X_0)}(x), \quad n \in \mathbb{N}, \quad x \in V_0$$

(4.2)

and it satisfies that $\|A[f]_n\|_{\ell^2(X_0)} = O(1)$ as $n \to \infty$.

As an extension of Proposition 4.1, we easily obtain the following.

**Proposition 4.2** (ergodic theorem II) Let $N \in \mathbb{N}$ and $f_1, f_2, \ldots, f_N \in \ell^2(X_0)$. Then, we have

$$\frac{1}{n^N} \sum_{\ell_N=0}^{n-N} \cdots \sum_{\ell_2=0}^{\ell_3} \sum_{\ell_1=0}^{\ell_2} L^{\ell_1} f_1(x) L^{\ell_2+1} f_2(x) \cdots L^{\ell_N+(N-1)} f_N(x)$$

$$= \frac{1}{N!} \prod_{\ell=1}^{N} \left( \sum_{x \in V_0} m(x) f_\ell(x) \right) + \sum_{\ell=1}^{N} \frac{1}{n^\ell} A[f_1, f_2, \ldots, f_N]^{(\ell)}_n(x)$$

(4.3)

for $x \in V_0$ and sufficiently large $n \in \mathbb{N}$, where the function $A[f_1, f_2, \ldots, f_N]^{(\ell)}_n$, $\ell = 1, 2, \ldots, N$, satisfies that $\|A[f_1, f_2, \ldots, f_N]^{(\ell)}_n\|_{\ell^2(X_0)} = O(1)$ as $n \to \infty$.

We only give a proof of Proposition 4.2 in the case $N = 2$ for the readers’ convenience in Appendix, because the proof in the case $N \geq 3$ is cumbersome but same as the case $N = 2$. 

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5 Validity of the Edgeworth Expansion; a Rough Observation

In this section, we give explicit calculations of all centralized moments of the scaled random walks \( \{\tau_{n-1/2} (\xi_n)\}_{n=1}^\infty \), where a moment of the random walk means the expectation

\[
\mathbb{E}^{x,n}_{\Phi_0} \left[ \left( \tau_{n-1/2} (\Phi_0(x)^{-1} \cdot \xi_n) \right)^I \right], \quad x \in V, \ n \in \mathbb{N}, \ I \in \mathcal{I}.
\]

Since the explicit representations of such moments have not been given yet in any references, it is worthwhile obtaining them here. We emphasize that an extension of the ergodic theorem for the transition operator \( \mathcal{L} \) (Proposition 4.2) plays a crucial role in the calculations. Moreover, we roughly observe the validity of the Edgeworth expansion of the scaled random walk by applying the representations of moments, which is not rigorous but helpful argument for the readers.

For a multi-index \( I \in \mathcal{I} \), we introduce a function \( F^I_{\Phi_0} : V \to \mathbb{R} \) by

\[
F^I_{\Phi_0}(x) := \sum_{e \in E_x} \frac{1}{d \Phi_0(e)^{I}} \Phi_0(e), \quad x \in V.
\]

where we recall \( d \Phi_0(e) := \frac{1}{(o(e))^{-1} \Phi_0(t(e))} \) for \( e \in E \). Since the function \( F^I_{\Phi_0} \) is \( \Gamma \)-invariant in the sense that \( F^I_{\Phi_0}(\gamma x) = F^I_{\Phi_0}(x) \) for \( \gamma \in \Gamma \) and \( x \in V \), it can be regarded as a function defined on the base graph \( V_0 \). We also write it for the same symbol \( F^I_{\Phi_0} : V_0 \to \mathbb{R} \). We put

\[
m^I_{\Phi_0} := \sum_{x \in V_0} m(x) F^I_{\Phi_0}(x) = \sum_{e \in E_0} \tilde{m}(e) \left( d \Phi_0(e)^{I} \right), \quad I \in \mathcal{I}.
\]

By definition, it is easily seen that

\[
m^I_{\Phi_0} = \begin{cases} 
0 & \text{if } d(I) = 1 \\
b_i(\Phi_0) & \text{if } I = [i], \ i = d_1 + 1, d_1 + 2, \ldots, d_1 + d_2 \\
\sigma_i(\Phi_0)\sigma_j(\Phi_0) & \text{if } I = [i] + [j], \ i, j = 1, 2, \ldots, d_i
\end{cases}
\]

5.1 Moments of Random Walks

We begin with the cases of low steps, that is, \( d(I) \leq 3 \).

Proposition 5.1 (Moments with \( d(I) \leq 3 \))

1. For \( I \in \mathcal{I} \) with \( d(I) = 1 \), we have \( \mathbb{E}^{x,n,I}_{\Phi_0} = 0 \).
2. For \( I \in \mathcal{I} \) with \( d(I) = 2 \), we have

\[
\mathbb{E}^{x,n,I}_{\Phi_0} = m^I_{\Phi_0} + \frac{1}{n} A[F^I_{\Phi_0}]_n(x),
\]

where \( A \) is the adjacency matrix.
where $A[F^I_{\Phi_0}]_n(x)$ is a function defined by (4.2).

(3) For $I \in I$ with $d(I) = 3$, we have
\[
E_{\Phi_0}^{x,n,I} = \frac{1}{n^{1/2}} m^n_{\Phi_0} + \frac{1}{n^{3/2}} A[F^I_{\Phi_0}]_n(x).
\]

**Proof** (1) By the modified harmonicity of $\Phi_0$, we easily have
\[
E_{\Phi_0}^{x,n,I} = \frac{1}{n^{1/2}} \sum_{c \in \Omega_{x,n-1}(X)} p(c) \left( \Phi_0(x)^{-1} \cdot \xi_{n-1}(c) \right)^I + \frac{1}{n^{1/2}} \sum_{e \in E_{i(c)}} p(e) \log \left( d \Phi_0(e) \right) |_{a^I} = \cdots = 0.
\]

(2) Consider the case where $I = [i]$, $i = d_1 + 1, d_1 + 2, \ldots, d_1 + d_2$. By using the Campbell–Baker–Hausdorff formula (2.1) and the modified harmonicity of $\Phi_0$, we have
\[
E_{\Phi_0}^{x,n,I} = \frac{1}{n} \sum_{c \in \Omega_{x,n-1}(X)} p(c) \sum_{e \in E_{i(c)}} p(e) \left\{ \left( \Phi_0(x)^{-1} \cdot \xi_{n-1}(c) \right)^{[i]} + \left( d \Phi_0(e) \right)^{[i]} \right\}
\]
\[
+ \sum_{d(J) = d(K) = 1} c_{JK} \left( \Phi_0(x)^{-1} \cdot \xi_{n-1}(c) \right)^J \left( d \Phi_0(e) \right)^K
\]
\[
= \frac{1}{n} \sum_{c \in \Omega_{x,n-1}(X)} p(c) \left( \Phi_0(x)^{-1} \cdot \xi_{n-1}(c) \right)^{[i]} + \frac{1}{n} L^{n-1} F^i_{\Phi_0}(x) = \cdots
\]
\[
= \frac{1}{n} \sum_{k=0}^{n-1} L^k F^i_{\Phi_0}(x).
\]

Since the function $F^i_{\Phi_0}$ is regarded as a function on $X_0$, we can apply Proposition 4.1 to obtain
\[
\frac{1}{n} \sum_{k=0}^{n-1} L^k F^i_{\Phi_0}(x) = \sum_{x \in V_0} m(x) F^i_{\Phi_0}(x) + \frac{1}{n} A[F^i_{\Phi_0}](x) = m^i_{\Phi_0} + \frac{1}{n} A[F^i_{\Phi_0}](x).
\]

As for the case $I = [i] + [j]$, $i, j = 1, 2, \ldots, d_1$, the proof can be done in the same way as above.

(3) Equation (2.1) yields that
\[
E_{\Phi_0}^{x,n,I} = \frac{1}{n^{3/2}} \sum_{c \in \Omega_{x,n-1}(X)} p(c) \sum_{e \in E_{i(c)}} p(e) \left\{ \left( \Phi_0(x)^{-1} \cdot \xi_{n-1}(c) \right)^I + \left( d \Phi_0(e) \right)^I \right\}
\]
\[
+ \sum_{d(J) = d(K) \geq 3} c_{JK} \left( \Phi_0(x)^{-1} \cdot \xi_{n-1}(c) \right)^J \left( d \Phi_0(e) \right)^K \]
By noting that the random variables \((\Phi_0(x)^{-1} \cdot \xi_{n-1}(c))^J\) and \((d\Phi_0(e))^K\) are independent, we have

\[
\sum_{c \in \Omega_{x,n-1}(X)} p(c) \sum_{e \in E(c)} p(e) \sum_{d(J)+d(K)=3, d(J),d(K) \geq 1} c_{J,K} \left( \Phi_0(x)^{-1} \cdot \xi_{n-1}(c) \right)^J \left( d\Phi_0(e) \right)^K
\]

\[
= \sum_{d(J)+d(K)=3, d(J),d(K) \geq 1} c_{J,K} \left\{ \sum_{c \in \Omega_{x,n-1}(X)} p(c) \left( \Phi_0(x)^{-1} \cdot \xi_{n-1}(c) \right)^J \right\} \mathcal{L}^{n-1} F^K_{\Phi_0}(x) = 0,
\]

where we used Item (1). Therefore, we inductively have

\[
\mathcal{E}_{\Phi_0}^{x,n,I} = \frac{1}{n^{3/2}} \left\{ \sum_{c \in \Omega_{x,n-1}(X)} p(c) \left( \Phi_0(x)^{-1} \cdot \xi_{n-1}(c) \right)^I + \mathcal{L}^{n-1} F^{I}_{\Phi_0}(x) \right\}
\]

\[
= \frac{1}{n^{3/2}} \sum_{k=0}^{n-1} \mathcal{L}^{k} F^{I}_{\Phi_0}(x).
\]

Then, it follows from Proposition 4.1 that

\[
\frac{1}{n} \sum_{k=0}^{n-1} \mathcal{L}^{k} F^{I}_{\Phi_0}(x) = m^{I}_{\Phi_0} + \frac{1}{n} A[F^{I}_{\Phi_0}]_{n}(x).
\]

By combining (5.1) with (5.2) and (5.3), we obtain

\[
\mathcal{E}_{\Phi_0}^{x,n,I} = \frac{1}{n^{1/2}} m^{I}_{\Phi_0} + \frac{1}{n^{3/2}} A[F^{I}_{\Phi_0}]_{n}(x),
\]

which completes the proof of Proposition 5.1. \(\square\)

We next discuss the cases of higher steps, that is, \(d(I) \geq 4\). For \(N \in \mathbb{N}\), \(f_1, f_2, \ldots, f_N \in \ell^2(X_0)\) and sufficiently large \(n \geq N\), we put

\[
Q^{(N)}_n[f_1, f_2, \ldots, f_N](x)
\]

\[
:= \sum_{\ell_N=0}^{n-N} \sum_{\ell_3=0}^{\ell_N} \sum_{\ell_2=0}^{\ell_3} \mathcal{L}^{\ell_1} f_1(x) \mathcal{L}^{\ell_2+1} f_2(x) \cdots \mathcal{L}^{\ell_N+(N-1)} f_N(x), \quad x \in V_0.
\]

By using the Campbell–Baker–Hausdorff formula (2.1), we easily obtain the following. The proof is so cumbersome but is straightforward as in the previous lemma.
Lemma 5.2 If a multi-index $I$ satisfies $d(I) \geq 4$, we have

$$
\mathbb{E}^{x,n} \left[ \left( \phi_0(x)^{-1} \cdot \xi_n \right)^T \right] = Q_n^{(I)} [F_{\phi_0}^I] (x) + \sum_{q_1=1}^{[d(I)/2]-1} \left( \sum_{d(J_1) = d(I) - 2q_1}^{d(J_1) = 2q_1} + \sum_{d(K_1) = d(I) - 2q_1}^{d(J_1) = 2q_1+1} \right) c_{J_1K_1} \left( Q_n^{(2)} [F_{\phi_0}^{J_1} F_{\phi_0}^{K_1}] (x) \right)
$$

$$
+ \sum_{q_2=1}^{[d(J_2)/2]-1} \left( \sum_{d(J_2) = d(I) - 2q_2}^{d(J_2) = 2q_2} + \sum_{d(K_2) = d(I) - 2q_2}^{d(J_2) = 2q_2+1} \right) c_{J_2K_2} \left( Q_n^{(3)} [F_{\phi_0}^{J_2} F_{\phi_0}^{J_2} F_{\phi_0}^{K_1}] (x) \right) + \cdots
$$

$$
+ \sum_{q_{d(I)/2} = 1}^{[d(I)/2]-1} \left( \sum_{d(J) = d(I) - 2q_{d(I)/2}}^{d(J) = 2q_{d(I)/2}} + \sum_{d(K) = d(I) - 2q_{d(I)/2}+1}^{d(J) = 2q_{d(I)/2}-1+1} \right) c_{Jd(I)/2,Kd(I)/2} Q_n^{d(I)/2} [F_{\phi_0}^{d(I)/2-1} F_{\phi_0}^{d(I)/2-1} F_{\phi_0}^{K_{d(I)/2}}} (x) (x)] + \cdots \right) \right) (5.4)
$$

for $x \in V_0$ and sufficiently large $n \in \mathbb{N}$, where the summation $\sum_{d(J) = a, d(K) = 1} d(J)$ is regarded as zero for $a = 3, 4, \ldots$ due to the modified harmonicity of $\phi_0$.

By using Proposition 4.2 and Lemma 5.2, we immediately obtain the following.

Theorem 5.3 (Moments with $d(I) \leq 4$) Let $I$ be a multi-index with $d(I) \geq 4$ and $n \in \mathbb{N}$ sufficiently large. Then, we have

$$
\mathcal{E}_{\phi_0}^{x,n,I} = \frac{a_{\phi_0}^{x,n,I} ([d(I)/2])}{n^{d(I)/2-[d(I)/2]}} + \cdots + \frac{a_{\phi_0}^{x,n,I} (1)}{n^{d(I)/2-1}} + \frac{a_{\phi_0}^{x,n,I} (0)}{n^{d(I)/2}}, \quad x \in V_0, \quad (5.5)
$$

where each coefficient $a_{\phi_0}^{x,n,I} (i)$, $i = 0, 1, 2, \ldots, [d(I)/2]$ is given by (5.7)–(5.9). Moreover, it holds that

$$
\|a_{\phi_0}^{x,n,I} (k)\| = O(1) \quad (n \to \infty), \quad k = 0, 1, 2, \ldots, [d(I)/2]. \quad (5.6)
$$
Proof It follows from Proposition 4.2 and Lemma 5.2 that

\[
\mathcal{E}_{\Phi_0}^{x,n,I} = \frac{1}{n^{d(I)/2-1}} \left( m_{\Phi_0}^I + \frac{1}{n} A[F_{\Phi_0}^I]_n(x) \right) \\
+ \sum_{q_1=1}^{[d(I)/2]-1} \left( \sum_{d(J_1)=2q_1}^{d(J_1)=d(I)-2q_1} \sum_{d(K_1)=d(I)-2q_1}^{} c_{J_1K_1} \right) + \frac{1}{n^{q_1+1}} A[F_{\Phi_0}^{J_1K_1}](x) + \frac{1}{n^2} A[F_{\Phi_0}^{J_1K_1}](x) \\
+ \cdots + \sum_{q_{d(I)/2]-1=1}^{d(I)/2-2} \left( \sum_{d(J_{d(I)/2]-1)=2q_{d(I)/2]-1}}^{d(I)/2-2} \sum_{d(K_{d(I)/2]-1)=d(I)/2-1}^{d(I)/2-2} \right) \\
\times \frac{1}{n^{d(I)/2-2-[d(I)/2]}} \left( \sum_{\ell=1}^{[d(I)/2]} \frac{1}{n^\ell} A[F_{\Phi_0}^{J_{d(I)/2]-1}, \ldots, F_{\Phi_0}^{K_{d(I)/2]-1}, \ldots, F_{\Phi_0}^{K_{d(I)/2]-2}, \ldots, F_{\Phi_0}^{K_{d(I)/2]}}}(x) \right) \\
= \frac{a_{x,n,I}(0)}{n^{d(I)/2-[d(I)/2]}+ \cdots + \frac{a_{x,n,I}(1)}{n^{d(I)/2-1} + \frac{a_{x,n,I}(0)}{n^{d(I)/2}-1} 
,
\]
\[ \times A[F_{\Phi_0}^{J_{(d(t)/2)\ell}^{-1}}, F_{\Phi_0}^{K_{(d(t)/2)\ell}^{-1}}, F_{\Phi_0}^{K_{(d(t)/2)\ell}^{-2}}, \ldots, F_{\Phi_0}^{K_{(d(t)/2)\ell}^{-1}}((d(t)/2)\ell)(x))] \ldots. \] (5.8)

and

\[
a_{\Phi_0}^{x,n,1}(\ell) = \sum_{q_1} \sum_{J_1, K_1} c_{J_1} \left( \ldots \sum_{q_{\ell-1}} \sum_{J_{\ell-1}, K_{\ell-1}} c_{J_{\ell-1} K_{\ell-1}} \left( \frac{1}{\ell!} m_{\Phi_0}^{J_{\ell-1} K_{\ell-1}} \ldots m_{\Phi_0}^{K_{\ell-2}} \ldots m_{\Phi_0}^{K_1} \right) \right) \ldots
\]

for \( \ell = 2, 3, \ldots, [d(t)/2] \). Equation (5.6) is then clearly obtained by definition. This completes the proof of Lemma 5.3. \( \square \)

### 5.2 Validity of the Edgeworth Expansion via a Rough Observation

In this subsection, we intuitively observe that the Edgeworth expansion of the scaled random walk \( \{\tau_{n^{-1/2}}(\xi_n)\}_{n=1}^{\infty} \) is valid by using Proposition 5.1 and Theorem 5.3. They are roughly obtained by the direct application of the stratified Taylor expansion formula to each expectations \( \mathcal{L}^n \mathcal{P}_{n^{-1/2}}^H f \) and \( \mathcal{P}_{n^{-1/2}}^H e^{\mathcal{A}(\Phi_0)} f \).

In what follows, we suppose that the function \( f : G \to \mathbb{R} \) is bounded and analytic. For the Gaussian semigroup \( (v_{\tau}^{\Phi_0})_{t \geq 0} \) associated with \( \mathcal{A}(\Phi_0) \), we write the corresponding Gaussian measure as \( \nu = v_{\tau}^{\Phi_0} \). We apply the right stratified Taylor formula to the function \( f \) at \( \tau_{n^{-1/2}}(\Phi_0(x)) \). Then, we have the Taylor series expansion

\[
\mathcal{L}^n \mathcal{P}_{n^{-1/2}}^H f(x) = \mathbb{E}^{x,n} \left[ f \left( \tau_{n^{-1/2}}(\xi_n) \right) \right] = \mathbb{E}^{x,n} \left[ f \left( \tau_{n^{-1/2}}(\Phi_0(x) \bullet (\Phi_0(x)^{-1} \bullet \xi_n)) \right) \right] = \sum_{d(t)=1} \mathcal{S}^1 f \left( \tau_{n^{-1/2}}(\Phi_0(x)) \right) \mathcal{E}^{x,n,1}_\Phi(x), \quad x \in V, \ n \in \mathbb{N}.
\]

On the other hand, it also follows from the right stratified Taylor formula that

\[
\mathcal{P}_{n^{-1/2}}^H e^{\mathcal{A}(\Phi_0)} f(x) = \mathbb{E}^\nu \left[ f \left( \tau_{n^{-1/2}}(\Phi_0(x)) \bullet X_1 \right) \right] = \sum_{d(t)=1} \mathcal{S}^1 f \left( \tau_{n^{-1/2}}(\Phi_0(x)) \right) \mathbb{E}^\nu[(X_1)^I]
\]
\[
= \sum_{d(I)=1}^{\infty} \mathcal{S}^I f\left(\tau_{n^{-1/2}}(\Phi_0(x))\right)m^I_v, \quad x \in V, \ n \in \mathbb{N},
\]

where \((X_t)_{t \geq 0}\) denotes the \(G\)-valued diffusion process whose infinitesimal generator is given by \(\mathcal{A}\). Then, for \(x \in V\) and \(n \in \mathbb{N}\), we have

\[
\mathcal{L}^n \Phi_{n^{-1/2}} f(x) - \Phi_{n^{-1/2}} e^{\mathcal{A}(\Phi_0)} f(x) = \sum_{d(I)=1}^{\infty} \mathcal{S}^I f\left(\tau_{n^{-1/2}}(\Phi_0(x))\right) \left(\mathcal{E}_{\Phi_0}^{x,n,I} - m^I_v\right).
\]

(5.10)

Let us consider the difference \(Z^{x,n,I} := \mathcal{E}_{\Phi_0}^{x,n,I} - m^I_v\) for \(x \in V, \ n \in \mathbb{N}\) and \(I \in I\), appearing on the right-hand side of (5.10). At first, suppose that \(d(I) = 1\). Then, Lemma 5.1-(1) and \(m^I_v = 0\) immediately imply that \(Z^{x,n,I} = 0\). We next consider the case where \(d(I) = 2\). By applying Lemma 5.1-(2) and an identity \(m^I_{\Phi_0} = m^I_v\) for \(d(I) = 2\), we have

\[
Z^{x,n,I} = \frac{1}{n} A[F_{\Phi_0}^I]_n(x).
\]

If the multi-index \(I\) satisfies \(d(I) = 3\), it follows from Lemma 5.1-(3) and \(m^I_v = 0\) that

\[
Z^{x,n,I} = \frac{1}{n^{1/2}} m^I_{\Phi_0} + \frac{1}{n^{3/2}} A[F_{\Phi_0}^I]_n(x).
\]

Next suppose that \(d(I)\) is odd and \(d(I) \geq 5\). By using Lemma 5.3 and \(m^I_v = 0\), we have

\[
Z^{x,n,I} = a_{\Phi_0}^{x,n,I}\left((d(I) - 1)/2\right) + \cdots + a_{\Phi_0}^{x,n,I}(1) + \frac{a_{\Phi_0}^{x,n,I}(0)}{n^{d(I)/2 - 1}} + \frac{a_{\Phi_0}^{x,n,I}(0)}{n^{d(I)/2}}.
\]

The rest is the case where \(d(I)\) is even and \(d(I) \geq 4\). The again use of Lemma 5.3 implies that

\[
Z^{x,n,I} = a_{\Phi_0}^{x,n,I}\left((d(I)/2\right) + \frac{a_{\Phi_0}^{x,n,I}(d(I)/2 - 1)}{n} + \cdots + \frac{a_{\Phi_0}^{x,n,I}(1)}{n^{d(I)/2 - 1}} + \frac{a_{\Phi_0}^{x,n,I}(0)}{n^{d(I)/2}} - m^I_v.
\]

By noting \(m^I_{\Phi_0} = m^I_v\) for a multi-index \(I\) with \(d(I) = 2\), we have
\[ d_{\Phi_0}^{x,n,I} (d(I)/2) = \sum_{d(J_i) = d(I)/2} C_{J_1} K_1 + \sum_{d(J_2) = d(I)/2 - 4} C_{J_2} K_2 + \cdots + \sum_{d(J_{d(I)/2-1}) = 2} C_{J_{d(I)/2-1}} K_{d(I)/2-1} \]
\[ \times \frac{1}{(d(I)/2)!} m_{\Phi_0} K_{d(I)/2-1} K_{d(I)/2-2} \cdots K_1 \]
\[ = \frac{1}{(d(I)/2)!} \sum_{d(J_1) = \ldots = d(J_{d(I)/2}) = 2} C_{J_1} \ldots J_{d(I)/2} m_v \]

where we used (2.6) for the final line. Therefore, we have

\[ \mathcal{Z}^{x,n,I} = \frac{d_{\Phi_0}^{x,n,I} (d(I)/2 - 1)}{n} + \cdots + \frac{d_{\Phi_0}^{x,n,I} (1)}{n^{d(I)/2-1}} + \frac{d_{\Phi_0}^{x,n,I} (0)}{n^{d(I)/2}}. \]

By combining (5.10) with the calculations of \( \mathcal{Z}^{x,n,I} \) above, we obtain

\[ \mathcal{L}^n \varphi_{n-1/2}^H f(x) - \mathcal{L}^n \varphi_{n-1/2}^H e_{\mathcal{A}} f(x) = \sum_{d(I) = 1}^{\infty} \mathcal{S}^I f\left( \tau_{n^{1/2}} (\Phi_0(x)) \right) \mathcal{Z}^{x,n,I} = \sum_{j=1}^{\infty} \frac{c_j}{n j^{1/2}}, \]

where each coefficient \( c_j = c_j(x, f, p, \Phi_0) \). \( j = 1, 2, \ldots \) is given in the following way. We put \( S_k := \{ k, k+2, k+4, \ldots \} \) for \( k = 1, 2, \ldots \). Then, one has

\[ c_1^n = \sum_{d(I) = 3} \mathcal{S}^I f\left( \tau_{n^{1/2}} (\Phi_0(x)) \right) m_{\Phi_0} \]
\[ + \sum_{s \in S_3} \sum_{d(I) = s} \mathcal{S}^I f\left( \tau_{n^{1/2}} (\Phi_0(x)) \right) a_{\Phi_0}^{x,n,I} \left( \frac{s-1}{2} \right), \]

\[ c_2^n = \sum_{d(I) = 2} \mathcal{S}^I f\left( \tau_{n^{1/2}} (\Phi_0(x)) \right) A[F_{\Phi_0}] n(x) \]
\[ + \sum_{s \in S_4} \sum_{d(I) = s} \mathcal{S}^I f\left( \tau_{n^{1/2}} (\Phi_0(x)) \right) a_{\Phi_0}^{x,n,I} \left( \frac{s-2}{2} \right), \]

\[ c_3^n = \sum_{d(I) = 3} \mathcal{S}^I f\left( \tau_{n^{1/2}} (\Phi_0(x)) \right) A[F_{\Phi_0}] n(x) \]
\[ + \sum_{s \in S_5} \sum_{d(I) = s} \mathcal{S}^I f\left( \tau_{n^{1/2}} (\Phi_0(x)) \right) a_{\Phi_0}^{x,n,I} \left( \frac{s-3}{2} \right), \]

and

\[ c_j^n = \sum_{s \in S_j} \sum_{d(I) = s} \mathcal{S}^I f\left( \tau_{n^{1/2}} (\Phi_0(x)) \right) a_{\Phi_0}^{x,n,I} \left( \frac{s-k}{2} \right), \quad j = 4, 5, \ldots \]
Though each coefficient $c^n_j = c^n_j(x, f, p, \Phi_0)$, $j = 1, 2, 3, \ldots$, depends on the choice of $n \in \mathbb{N}$, we can still expect that each $c^n_j$ satisfies that $\|c^n_j\| = O(1)$ as $n \to \infty$ for $j = 1, 2, 3, \ldots$, by applying (5.6). Thus, Eq. (5.11) should imply the possibility of the validity of the Edgeworth expansion of the scaled random walk $(\tau_n - 1/2(\xi_n))_{n=1}^\infty$. However, the argument in this subsection does not imply the rigorous proof of Theorem 1.2. Indeed, several technical difficulties appear when we deal with the remainder terms of the Taylor expansions of both $L_n^p \Phi_0^{-1/2} f$ and $\Phi_0^{-1/2} e^{A(\Phi_0)} f$. Therefore, we need another technical approach for the proof, while the observation above must be important.

6 Proof of Theorem 1.2

The aim of this section is to give a proof of Theorem 1.2, the main result of the present paper.

6.1 Proof of Theorem 1.2

Let $x \in V$ and $\{W_i\}_i^{\infty}$ be a sequence of random variables on $\Omega_k(X)$ with values in $G$ defined by $W_i(c) := d\Phi_0(e_i)$ for $c = (e_1, e_2, \ldots, e_i, \ldots) \in \Omega_k(X)$. Note that $\{W_i\}_i^{\infty}$ is independent but not identically distributed in general by definition and satisfies that

$$w_n = x \bullet W_1 \bullet W_2 \cdots \bullet W_n, \quad n = 0, 1, 2, \ldots.$$ 

We put $P_m := \sum_{x \in V_m} \mathbb{P}_x m(x)$ and $\mu^{(i)} := P_m \circ (W_i)^{-1}$, $i = 1, 2, \ldots$, which is the image probability measure of $P_m$. Then, we know that the law of each $w_n, n = 1, 2, \ldots$, is written by the convolution power $\mu^{(1)} \ast \mu^{(2)} \ast \cdots \ast \mu^{(n)}$. Moreover, we observe

$$m_{\mu^{(k)}}^I = \int_G x^I \mu^{(k)}(d x) = \sum_{e \in E_0} \tilde{m}(e)(d\Phi_0(e))^I = m_\Phi^I, \quad I \in I, \quad k = 1, 2, \ldots,$$

and $m_{\mu^{(k)}}^N < \infty$ for $k, N = 1, 2, \ldots$. We now give a proof of Theorem 1.2 only in the case of $t = 1$, since general cases are shown similarly to the case of $t = 1$.

Proof of Theorem 1.2 We basically follows the argument given by Bentkus and Pap [2] and Pap [25], although we need a careful examination to follow it. Indeed, random walks discussed in both Bentkus and Pap [2] and Pap [25] are independently and identically distributed, while our random walks are not always so. Moreover, there is a geometric constraint in terms of the modified harmonicity of the realization $\Phi_0$ in
our setting. The basic idea for the proof is to make use of the identity

$$\tau_{n-1/2}(\mu^{(1)} * \mu^{(2)} * \cdots * \mu^{(n)})$$

$$= \nu + \sum_{k=1}^{n} \tau_{n-1/2} v^{k-1} * (\tau_{n-1/2} \mu^{(k)} - \tau_{n-1/2} \nu) * \tau_{n-1/2} (\mu^{(k+1)} * \cdots * \mu^{(n)}) \quad (6.1)$$

for \(n = 1, 2, \ldots\), where we again recall that \(\nu = \nu^{\Phi_0}\) is the centered Gaussian measure corresponding to \((\nu^\Phi)^{t \geq 0}\), satisfying \(\tau_{n-1/2} v^* \nu = \nu\). In what follows, we always omit the convolution symbol \(*\) for the simplicity of notations. By using (6.1), we have

$$L^n P^n_{n-1/2} f(x) - P^n_{n-1/2} e^{^\Lambda(\Phi_0)} f(x)$$

$$= \sum_{k=1}^{n} \iiint_{B^d} f(x \cdot x_1 \cdot y_1 \cdot x_2) \tau_{n-1/2} v^{k-1} (dx_1)(\tau_{n-1/2} \mu^{(k)} - \tau_{n-1/2} \nu)(dy_1)$$

$$\times \tau_{n-1/2} (\mu^{(k+1)} \cdots \mu^{(n)})(dx_2), \quad f \in C^{(N+1)r}_b(G), \quad x \in V. \quad (6.2)$$

We put

$$I_1 := \sum_{k=1}^{n} \iiint_{|y_1| \geq 1} f(x \cdot x_1 \cdot y_1 \cdot x_2)$$

$$\times \tau_{n-1/2} v^{k-1} (dx_1)(\tau_{n-1/2} \mu^{(k)} - \tau_{n-1/2} \nu)(dy_1) \tau_{n-1/2} (\mu^{(k+1)} \cdots \mu^{(n)})(dx_2),$$

$$I_2 := \sum_{k=1}^{n} \iiint_{|y_1| < 1} f(x \cdot x_1 \cdot y_1 \cdot x_2)$$

$$\times \tau_{n-1/2} v^{k-1} (dx_1)(\tau_{n-1/2} \mu^{(k)} - \tau_{n-1/2} \nu)(dy_1) \tau_{n-1/2} (\mu^{(k+1)} \cdots \mu^{(n)})(dx_2).$$

We split the proof into seven steps.

**Step 1.** It follows from (2.8) that

$$|I_1| \leq \frac{1}{n} \|f\|_{\infty} \sum_{k=1}^{n} \left( \int_{|y_1| \geq 1} \tau_{n-1/2} \mu^{(k)}(dy_1) + \int_{|y_1| \geq 1} \tau_{n-1/2} \nu(dy_1) \right)$$

$$= \frac{1}{n^2} \|f\|_{\infty} \sum_{k=1}^{n} \left( \Lambda_0(\mu^{(k)}) + \Lambda_0(\nu) \right)$$

$$\leq \frac{1}{n^2} \|f\|_{\infty} n^{-N/2} \sum_{k=1}^{n} (m_{\mu^{(k)}}^{N+2} + m_{\nu}^{N+2}) = O(n^{-N/2}),$$

which means that the contribution of \(I_1\) is nothing but up to \(O(n^{-N/2})\).

**Step 2.** At the rest steps, we concentrate on looking into the integral \(I_2\). On the set \(||y_1| < 1\|\), we start with applying the right stratified Taylor formula (Lemma 2.2) up to the order \(N + 1\) to the function \(f(x \cdot x_1 \cdot y_1 \cdot x_2)\) at \(1_G\) with respect to \(y_1\). Then,
for \( x_1, x_2 \in G \), we have

\[
f(x \cdot x_1 \cdot y_1 \cdot x_2) = \sum_{d(I_1) \leq N+1} \widehat{S}^I_{(2)} f(x \cdot x_1 \cdot 1_G \cdot x_2) y_1^{I_1} + R^f_{N+2}(x_1, y_1, x_2)
\]

(6.3)

with the estimate

\[
|R^f_{N+2}(x_1, y_1, x_2)| \\
\leq C(G)|y_1|^{N+2} \text{ sup } \{ |\widehat{a}^{I_1} f(x \cdot x_1 \cdot y'_1 \cdot x_2)| : d(I_1) = N + 2, |y'_1| \leq b(G)^{N+2} |y_1| \}.
\]

By applying Lemma 2.3, one can give the estimation

\[
|R^f_{N+2}(x_1, y_1, x_2)| \leq C(G)|y_1|^{N+2} (1 + |x \cdot x_1|^{(N+2)(r-1)}) \|D^{(N+2)r} f\|_\infty. \tag{6.4}
\]

Therefore, (2.8) and (6.4) yield

\[
\begin{align*}
&\left| \sum_{k=1}^{n} \iint_{|y_1|<1} R^f_{N+2}(x_1, y_1, x_2) \tau_{n-1/2} v^{k-1} (dx_1)(\tau_{n-1/2} \mu^{(k)}) \\
&\quad \quad - \tau_{n-1/2} v)(dy_1) \tau_{n-1/2} (\mu^{(k+1)} \cdots \mu^{(n)}) (dx_2) \right| \\
&\leq \frac{1}{n} C(G) \|D^{(N+2)r} f\|_\infty \sum_{k=1}^{n} \left( \frac{1}{n} (L_{N+2}(\mu^{(k)}) \\
&\quad + L_{N+2}(v)) \right) (1 + |x|^{(N+2)(r-1)} + m_{\nu,k-1}) \\
&\leq \frac{1}{n^2} C(G) \|D^{(N+2)r} f\|_\infty \sum_{k=1}^{n} n^{-N/2} \left( m_{\nu}^{N+2}(\mu^{(k)}) \\
&\quad + m_{\nu}^{N+2}(1 + |x|^{(N+2)(r-1)} + m_{\nu,k-1}^{(N+2)(r-1)}) = O(n^{-N/2}),
\end{align*}
\]

where we have shown that the contribution of the remainder term in (6.3) is up to \( O(n^{-N/2}) \).

**Step 3.** We split the summation on the right-hand side of (6.3) into

\[
\sum_{d(I_1) \leq N+1} \widehat{S}^I_{(2)} f(x \cdot x_1 \cdot 1_G \cdot x_2) y_1^{I_1} = \left( \sum_{d(I_1) \leq 2} + \sum_{3 \leq d(I_1) \leq N+1} \right) \widehat{S}^I_{(2)} f(x \cdot x_1 \cdot 1_G \cdot x_2) y_1^{I_1}.
\]

Let us consider the case \( d(I_1) \leq 2 \). Since it holds that \( m_{\nu}^{I_1}(\mu^{(k)}) = m_{\nu}^{I_1} \) for \( k = 1, 2, \ldots, n \) when \( d(I_1) = 1 \), we have

\[
\left| \int_{|y_1|<1} y_1^{I_1} (\tau_{n-1/2} \mu^{(k)} - \tau_{n-1/2} v)(dy_1) \right| = \left| \int_{|y_1| \geq 1} y_1^{I_1} (\tau_{n-1/2} \mu^{(k)} - \tau_{n-1/2} v)(dy_1) \right| \\
\leq \frac{1}{n} (\Lambda_2(\mu^{(k)}) + \Lambda_2(v))
\]
for $k = 1, 2, \ldots, n$. Therefore, Lemma 2.3 and (2.8) imply that

$$
\left| \sum_{k=1}^{n} \sum_{d(I_1) \leq 2} \int \int \int_{|y_1| < 1} \int \int \int_{|y_2| \geq 1} \mathcal{S}_{\mathbf{I}} \left( x \cdot x_1 \cdot 1_G \cdot x_2 \cdot y_2 \cdot x_3 \right) \lambda^H(x) \right|
$$

$$
\leq \frac{1}{n} C(G) \| D^{2r} f \|_\infty \sum_{k=1}^{n} \sum_{d(I_1) \leq 2} \left( \frac{1}{n} \left( \Lambda_2(\mu^{(k)}) + \Lambda_2(v) \right) \right) (1 + |x|^{2(r-1)} + m_v^{2(r-1)})
$$

$$
\leq \frac{1}{n^2} C(G) \| D^{2r} f \|_\infty \sum_{k=1}^{n} \sum_{d(I_1) \leq 2} n^{-N/2} (m^{N+2} + m_v^{N+2}) (1 + |x|^{2(r-1)} + m_v^{2(r-1)})
$$

$$
= O(n^{-N/2}).
$$

**Step 4.** We consider the case where $3 \leq d(I_1) \leq N+1$. In the same way as (6.1), we have

$$
\sum_{k=1}^{n} \tau_{n-1/2} v^{k-1} (\tau_{n-1/2} \mu^{(k)}) - \tau_{n-1/2} v^{(k+1)} \mu^{(k+1)} \cdots \mu^{(n)}
$$

$$
= \sum_{k=1}^{n} \tau_{n-1/2} v^{k-1} (\tau_{n-1/2} \mu^{(k)}) - \tau_{n-1/2} v^{n-k}
$$

$$
+ \sum_{k=1}^{n} \sum_{\ell=1}^{n-k} \tau_{n-1/2} v^{k-1} (\tau_{n-1/2} \mu^{(k)}) - \tau_{n-1/2} v^{n-k}
$$

$$
\times \tau_{n-1/2} v^{\ell-1} (\tau_{n-1/2} \mu^{(k+\ell)}) - \tau_{n-1/2} v^{n-k}
$$

$$
\times \tau_{n-1/2} v^{\ell-1} (\tau_{n-1/2} \mu^{(k+\ell+1)} \cdots \mu^{(n)})
$$

$$
=: \mathcal{M}^{(n)}_1 + \mathcal{M}^{(n)}_2.
$$

Then, we obtain

$$
\left| \sum_{3 \leq d(I_1) \leq N+1} \int \int \int \int_{|y_1| < 1, |y_2| \geq 1, |y_3| \geq 1} \mathcal{S}_{\mathbf{I}} \left( x \cdot x_1 \cdot 1_G \cdot x_2 \cdot y_2 \cdot x_3 \right) \lambda^H(x) \right|
$$

$$
\leq \frac{1}{n^{3/2}} C(G) \sum_{k=1}^{n} \sum_{\ell=1}^{n-k} \sum_{3 \leq d(I_1) \leq N+1} \| D^{d(I_1) N} f \|_\infty
$$

$$
\left( \frac{1}{n} \left( \Lambda_2(\mu^{(k)}) + \Lambda_2(v) \right) \right) \left( \frac{1}{n} \left( \Lambda_0(\mu^{(k+\ell)}) + \Lambda_0(v) \right) \right)
$$

$$
\leq \frac{1}{n^{3/2}} C(G) \sum_{k=1}^{n} \sum_{\ell=1}^{n-k} \sum_{3 \leq d(I_1) \leq N+1} \| D^{d(I_1) N} f \|_\infty n^{-(d(I_1)-2)/2} n^{-N/2}
$$

$$
\leq \frac{1}{n^{3/2}} C(G) \sum_{k=1}^{n} \sum_{\ell=1}^{n-k} \sum_{3 \leq d(I_1) \leq N+1} \| D^{d(I_1) N} f \|_\infty n^{-(d(I_1)-2)/2} n^{-N/2}
$$

$$
\leq \frac{1}{n^{3/2}} C(G) \sum_{k=1}^{n} \sum_{\ell=1}^{n-k} \sum_{3 \leq d(I_1) \leq N+1} \| D^{d(I_1) N} f \|_\infty n^{-(d(I_1)-2)/2} n^{-N/2}
$$
by applying Lemma 2.3 and (2.8).

**Step 5.** On the set \( \{|y_1| < 1, |y_2| < 1\} \), we apply the right stratified Taylor formula up to the order \( N + 3 - d(I_1) \) to the function \( \widetilde{S}_{(2)}^{l_1}f(x \bullet x_1 \bullet 1_G \bullet x_2 \bullet y_2 \bullet x_3) \) at \( 1_G \) with respect to the variable \( y_2 \). Then, for fixed \( x_1, x_2, x_3 \in G \), we have

\[
\begin{align*}
\widetilde{S}_{(2)}^{l_1}f(x \bullet x_1 \bullet 1_G \bullet x_2 \bullet y_2 \bullet x_3) &= \sum_{d(I_2) \leq N+3-d(I_1)} \widetilde{S}_{(4)}^{l_1}f(x \bullet x_1 \bullet 1_G \bullet x_2 \bullet 1_G \bullet x_3) y_2^{l_2} + R_{N+3-d(I_1)},
\end{align*}
\]

where the remainder term \( R_{N+3-d(I_1)} \) satisfies

\[
| R_{N+3-d(I_1)}(x_1, x_2, y_2, x_3) | \leq C(G) |y_2|^{N+4-d(I_1)} \sup \left\{ |\tilde{R}_{(4)}^{l_1}f(x \bullet x_1 \bullet 1_G \bullet x_2 \bullet y_2' \bullet x_3)| \right\},
\]

Since it follows from Lemma 2.3 that

\[
| R_{N+3-d(I_1)}(x_1, x_2, y_2, x_3) | \leq C(G) |y_2|^{(N+4-d(I_1))(r-1)} \| D^{(N+4-d(I_1))r} f \|_\infty.
\]

we have

\[
\begin{align*}
\left| \sum_{3 \leq d(I_1) \leq N+1} \int \int \int \int_{|y_1| < 1, |y_2| < 1} R_{N+3-d(I_1)}(x_1, x_2, y_2, x_3) y_2^{l_1} \right| \\
\leq \frac{1}{n^{3/2}} C(G) \sum_{3 \leq d(I_1) \leq N+1} \sum_{k=1}^{n} \sum_{\ell=1}^{n-k} \| D^{(N+4-d(I_1))r} f \|_\infty \\
\times \left( \frac{1}{n} \left( L_{d(I_1)}(\mu^{(k)}) + L_{d(I_1)}(\nu) \right) \right) \left( \frac{1}{n} \left( L_{N+4-d(I_1)}(\mu^{(k+\ell)}) + L_{N+4-d(I_1)}(\nu) \right) \right) \\
\times \left\{ 1 + |x_1|^{(N+4-d(I_1))(r-1)} + m_{\mu_{k-1}}^{(N+4-d(I_1))(r-1)} + m_{\nu_{\ell-1}}^{(N+4-d(I_1))(r-1)} \right\} \\
\leq \frac{1}{n^{3/2}} C(G) \sum_{3 \leq d(I_1) \leq N+1} \sum_{k=1}^{n} \sum_{\ell=1}^{n-k} \| D^{(N+4-d(I_1))r} f \|_\infty n^{-(N+2-d(I_1))/2} n^{-(d(I_1)-2)/2} \\
\times (m_{\mu_{k+\ell}}^{N+4-d(I_1)} + m_{\nu_{\ell}}^{N+4-d(I_1)}) \left\{ 1 + |x_1|^{(N+4-d(I_1))(r-1)} + m_{\mu_{k-1}}^{(N+4-d(I_1))(r-1)} \right\}.
\end{align*}
\]
by using Lemma 2.3 and (2.8).

**Step 6.** We split the summation $\sum d(I_2) \leq N + 3 - d(I_1)$ as in (6.6) into $\sum d(I_2) \leq 2$ and $\sum 3 \leq d(I_2) \leq N + 3 - d(I_1)$. Then, in the same way as **Step 3**, we obtain

$$
\left| \sum_{3 \leq d(I_1) \leq N + 1} \sum_{d(I_2) \leq 2} \int \int \int \int_{|y_1| < 1, |y_2| < 1} \hat{S}_1 \hat{S}_2 f(x \bullet x_1 \bullet 1G \bullet x_2 \bullet 1G \bullet x_3) \times y_1^{I_1} y_2^{I_2} \mathcal{M}_2^{(n)}(dx_1 dy_2 dx_1 dy_2 dx_3) \right| = O(n^{-1/2}).
$$

As for the terms corresponding to the summations over $3 \leq d(I) \leq N + 1$ and $3 \leq d(I_2) \leq N + 3 - d(I_1)$, that is, over $d(I_1) + d(I_2) \leq N + 3$ with $d(I_1), d(I_2) \geq 3$, their estimates can be done by applying the similar idea to the one at **Step 4**. Namely, we replace the measure $\tau_{n-1/2}(\mu^{(k+\ell+1)} \cdots \mu^{(n)})$ in (6.5) by

$$
\tau_{n-1/2}(\mu^{(k+\ell+1)} \cdots \mu^{(n)}) = \tau_{n-1/2} v^{n-k-\ell} + \sum_{j=1}^{n-k-\ell} \tau_{n-1/2} v^{j-1} \tau_{n-1/2}(\mu^{(k+\ell+j)} - \nu) \tau_{n-1/2}(\mu^{(k+\ell+j+1)} \cdots \mu^{(n)}).
$$

We consider the integration with respect to the measure given by the second term in the decomposition above. Then, the same argument as the one at **Step 5** can be done here. It is clearly observed that this procedure finishes at $N$ times and then the integrand is given by

$$
\hat{S}_{I_N-1} \hat{S}_{I_N-2} \cdots \hat{S}_{I_2} f(x \bullet x_1 \bullet 1G \bullet x_2 \bullet 1G \bullet \cdots \bullet 1G \bullet x_N) y_1^{I_1} y_2^{I_2} \cdots y_{I_N-1}^{I_{N-1}}
$$

by repeating the use of the right stratified Taylor formula, where $d(I) = d(I_2) = \cdots = d(I_{N-1}) = 3$ and the domain of the integral is $\{|y_1| < 1, |y_2| < 1, \ldots, |y_{N-1}| < 1\}$.

In this procedure, the estimates of integrals with respect to the measures

$$
\sum_{i_1 + i_2 + \cdots + i_{\ell} = n-\ell+1} \tau_{n-1/2} v^{i_1} \tau_{n-1/2}(\mu^{(i_1+1)} - \nu) \tau_{n-1/2} v^{i_2} \tau_{n-1/2}(\mu^{(i_1+i_2+2)} - \nu) \times \cdots \times \tau_{n-1/2} v^{i_{\ell-1}} \tau_{n-1/2}(\mu^{(n-i_\ell)} - \nu) \tau_{n-1/2} v^{i_\ell}, \quad \ell = 2, 3, \ldots, N
$$

including, e.g., $\mathcal{M}_1^{(n)}$ have not been done, though the rest term is easily shown to be up to $O(n^{-1/2})$ by applying the right stratified Taylor formula up to order 2 and the same way as that of **Step 2**.
Step 7. At this final step, we consider the terms written by

\[
\sum_{i_1 + i_2 + \cdots + i_{\ell} = n - \ell + 1}^{d(I_1) + \cdots + d(I_{\ell-1}) \leq N + 2\ell - 3} \int \cdots \int_G^{\ell-1} \tilde{S}_{(2(\ell-1))}^{I_{\ell-1}} \cdots \tilde{S}_{(2(\ell-2))}^{I_{\ell-2}} \cdots \tilde{S}_{(2(\ell-1))}^{I_{\ell-1}} f(x \bullet x_1)
\]

for \( \ell = 2, 3, \ldots, N \). Since it holds that

\[
\int_{|y_k| < 1} y_k^l \tau_{n-1/2} (\mu^{(i_1+\cdots+i_k+k)} - \nu)(dy_k)
\]

\[
= \int_G y_k^l \tau_{n-1/2} (\mu^{(i_1+\cdots+i_k+k)} - \nu)(dy_k) - \int_{|y_k| \geq 1} y_k^l \tau_{n-1/2} (\mu^{(i_1+\cdots+i_k+k)} - \nu)(dy_k)
\]

\[
= n^{-d(I_k)/2} (m_{\Phi_0}^l - m_v^l) - \int_{|y_k| \geq 1} y_k^l \tau_{n-1/2} (\mu^{(i_1+\cdots+i_k+k)} - \nu)(dy_k)
\]

and

\[
\prod_{k=1}^{\ell-1} \left| \int_{|y_k| \geq 1} y_k^l \tau_{n-1/2} (\mu^{(i_1+\cdots+i_k+k)} - \nu)(dy_k) \right|
\]

\[
\leq \prod_{k=1}^{\ell-1} n^{-d(I_k)/2} \cdot \frac{1}{n} \left( \Lambda_{d(I_k)} (\mu^{(i_1+\cdots+i_k+k)}) + \Lambda_{d(I_k)} (\nu) \right)
\]

\[
\leq n^{-3(\ell-1)/2} n^{-(\ell-1)} \prod_{k=1}^{\ell-1} n^{-N/2} (m_{\mu_{i_1+\cdots+i_k+k}}^{N+2} + m_v^{N+2})
\]

\[
= n^{-(N+5)(\ell-1)/2} \prod_{k=1}^{\ell-1} (m_{\mu_{i_1+\cdots+i_k+k}}^{N+2} + m_v^{N+2}) = O(n^{-N/2})
\]

for \( k = 1, 2, \ldots, \ell \) and \( \ell = 2, 3, \ldots, N \), we may replace each integral

\[
\int_{|y_k| < 1} y_k^l \tau_{n-1/2} (\mu^{(i_1+\cdots+i_k+k)} - \nu)(dy_k), \quad k = 1, 2, \ldots, \ell, \quad \ell = 2, 3, \ldots, N
\]
in (6.7) by $n^{-d(I_{k})/2}(m^{I_{k}}_{\Phi_{0}} - m^{I_{k}}_{v})$. Therefore, we obtain

$$
\sum_{\ell=2}^{N} \sum_{i_{1}+i_{2}+\cdots+i_{\ell}=n-\ell+1} \sum_{d(I_{1}) \cdots d(I_{\ell-1}) \leq N+2\ell-3} \int_{G^{\ell}} \cdots \int_{G^{\ell}} S^{\ell-1}_{(2(\ell-1))} S^{\ell-2}_{(2(\ell-2))} \cdots S^{1}_{(2)} f(x \bullet x_{1})
\cdot 1_{G} \cdots 1_{G} (d(x_{1}) v^{\Phi_{0}}_{i_{1}/n} (d(x_{2}) v^{\Phi_{0}}_{i_{2}/n}) \cdots v^{\Phi_{0}}_{i_{\ell}/n} (d(x_{\ell})) \prod_{k=1}^{\ell-1} n^{-d(I_{k})/2}(m^{I_{k}}_{\Phi_{0}} - m^{I_{k}}_{v})
= \frac{\xi_{1}}{n^{1/2}} + \frac{\xi_{2}}{n^{3/2}} + \cdots + \frac{\xi_{N-1}}{n^{(N-1)/2}}.
(6.8)
$$

where we used $\tau_{n-1/2} v^{i} = v^{\Phi_{0}}_{i/n}$ for $i = 1, 2, \ldots$ and the each coefficient $\xi_{j}$, $j = 1, 2, \ldots, N - 1$, depends on $x, G, f, p$ and $\Phi_{0}$. The explicit representation of each $\xi_{j}$, $j = 1, 2, \ldots, N - 1$, is discussed in the next subsection. By putting it all together, we obtain the desired expansion (1.4) and this completes the proof of Theorem 1.2. □

### 6.2 Explicit Representations of Coefficients in the Edgeworth Expansion

We give explicit representations of the coefficients $\xi_{j} = \xi_{j}(x, G, f, p, \Phi_{0})$, $j = 1, 2, \ldots, N - 1$, appearing in the Edgeworth expansion in Theorem 1.2. To obtain them, we need to pick up the terms of order $n^{-j/2}$ on the left-hand side of (6.8). For this sake, we need to use the following multidimensional version of the so-called Euler–Maclaurin summation formula. We refer to Bentkus and Pap [2] for more details.

**Lemma 6.1 (cf. [25, Theorem 1])** Let $\ell \geq 2$ and $\Delta(\ell)$ be a simplex defined by

$$
\Delta(\ell) := \{ t = (t_{1}, t_{2}, \ldots, t_{\ell}) \in \mathbb{R}^{\ell} | t_{1} + t_{2} + \cdots + t_{\ell} = 1, t_{1}, t_{2}, \ldots, t_{\ell} \geq 0 \}.
$$

Suppose that a function $F : \Delta(\ell) \rightarrow \mathbb{R}$ has continuous partial derivatives of all orders $J \in \mathcal{I}$ with $|J| \leq (\ell - 1)s$ for some $s \in \mathbb{N}$. Then, we have

$$
\frac{1}{n^{\ell-1}} \sum_{i_{1}+i_{2}+\cdots+i_{\ell}=n-\ell+1} F(i_{1}/n, i_{2}/n, \ldots, i_{\ell}/n)
= \sum_{i=2}^{\ell} \sum_{k=0}^{s-1} \frac{1}{k! n^{k}} \int_{\Delta(i)} B^{(\ell,i)}_{k} (\partial_{1}, \partial_{2}, \ldots, \partial_{\ell}) F(t, 0, 0, \ldots, 0) dt + R^{(\ell,s)}_{n},
$$

where each $B^{(\ell,i)}_{j}$ is a polynomial defined by the following generating function

$$
\sum_{j=0}^{\infty} \frac{B^{(\ell,i)}_{j}(t_{1}, t_{2}, \ldots, t_{\ell})}{j!} x^{j} = (-1)^{\ell-1} x^{\ell-1} \sum_{q=1}^{\ell} \prod_{\alpha=1,2,\ldots,k, \alpha \neq q} (t_{1} - t_{q}) (t_{2} - t_{q}) \cdots (t_{i} - t_{q}) (e^{xt_{1}} - e^{xt_{q}}).
$$

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and the remainder $R_n^{(\ell, s)}$ satisfies $|R_n^{(\ell, s)}| \leq C(\ell, s, F)n^{-s}$ for some positive constant $C(\ell, s, F) > 0$.

We now apply Lemma 6.1 to (6.8) in the case where

$$
F(\frac{i_1}{n}, \frac{i_2}{n}, \ldots, \frac{i_\ell}{n})
= \sum_{d(l_1)+\cdots+d(l_\ell)\leq N+2\ell-3}
\int_{G^\ell} \hat{S}^{l_{\ell-1}}(2(l-1)) \hat{S}^{l_{\ell-2}}(2(l-2)) \cdots \hat{S}^{l_1}(2) f(x \cdot x_1 \cdot 1_G \cdot \cdots \cdot 1_G \cdot x_\ell)
\times v_{i_1/n}(dx_1)v_{i_2/n}(dx_2) \cdots v_{i_\ell/n}(dx_\ell).
$$

Then, we have the following explicit representations of coefficients $\xi_1, \xi_2, \ldots, \xi_{N-1}$ in terms of the centered Gaussian semigroup $(\nu/\Phi_1)_{t \geq 0}$ associated with the infinitesimal generator $\mathcal{A}(\Phi_0)$.

**Proposition 6.2** Let $\{\xi_j = \xi_j(x, G, f, p, \Phi_0)\}_{j=1}^{N-1}$ be as in Theorem 1.2. Then, for every $j = 1, 2, \ldots, N - 1$, we obtain

$$
\xi_j = \sum_{i=1}^{j} \sum_{\ell=2}^{j} \sum_{q=i-\ell+1}^{j} \sum_{d(l_1)+\cdots+d(l_\ell)=j-2q+2\ell-2}
\int_{\Delta(i)} \int_{G^\ell} \mathcal{D}^{l_1,l_2,\ldots,l_\ell}_{i,\ell,q} f(x \cdot x_1 \cdot 1_G \cdot \cdots \cdot 1_G \cdot x_\ell \cdot 1_G \cdot 1_G \cdot \cdots \cdot 1_G \cdot 1_G)
\times v_{i_1}(dx_1)v_{i_2}(dx_2) \cdots v_{i_\ell}(dx_\ell) dt_1 dt_2 \cdots dt_\ell \times \prod_{k=1}^{i} (m^l_{\Phi_0} - m^l_v)
$$

where each $\mathcal{D}^{l_1,l_2,\ldots,l_\ell}_{i,\ell,q}$ is a differential operator defined by

$$
\mathcal{D}^{l_1,l_2,\ldots,l_\ell}_{i,\ell,q} = \frac{1}{q!} B_q^{(i+1, \ell)} (\mathcal{A}(1), \mathcal{A}(3), \ldots, \mathcal{A}(2i+1)) \hat{S}^{l_1}_{(2i)} \hat{S}^{l_{\ell-1}}_{(2(i-1))} \cdots \hat{S}^{l_\ell}_{(2)}.
$$

### 7 Berry–Esseen-Type Bound via Trotter’s Approximation Theorem

By applying Theorem 1.2 in the case where $N = 2$, we immediately establish the so-called Berry–Esseen-type bound for the scaled random walk $\{\tau_{n^{-1/2}}(\xi_n)\}_{n=1}^{\infty}$, which gives the rate of convergence of the discrete semigroup generated by $\{\tau_{n^{-1/2}}(\xi_n)\}_{n=1}^{\infty}$ to the heat semigroup $(e^{t\mathcal{A}(\Phi_0)})_{t \geq 0}$ whose infinitesimal generator is $\mathcal{A}(\Phi_0)$.

**Corollary 7.1** (Berry–Esseen-type bound) For every $f \in C^0(G)$, we have

$$
\|L^{[n]} \mathcal{P}_{n^{-1/2}} f - \mathcal{P}_{n^{-1/2}} e^{t\mathcal{A}(\Phi_0)} f \|_\infty \leq \frac{C}{n^{1/2}}, \quad n \in \mathbb{N},
$$

for some positive constant $C = C(G, f, p, \Phi_0, t) > 0$. 


In this section, we give an alternative proof of Corollary 7.1 from functional analytic point of view. Recall that the proof of Proposition 1.1 given in [15] heavily depends on the celebrated Trotter’s semigroup approximation theorem. It provides a sufficient condition for the convergence of a sequence of operator semigroups in terms of the corresponding sequence of infinitesimal generators. See [21,30] for more details.

Let \((B_n, \| \cdot \|_{B_n}), n \in \mathbb{N}\), and \((E, \| \cdot \|_E)\) be Banach spaces. Let \(P_n : E \rightarrow B_n, n \in \mathbb{N}\), be a bounded linear operator satisfying \(\| f_n - P_n f \|_{B_n} \rightarrow 0\) as \(n \rightarrow \infty\). We say that the sequence of pairs \(\{(B_n, P_n)\}_{n=1}^\infty\) approximates \(E\) if \(\| P_n f \|_{B_n} \rightarrow \| f \|_E\) as \(n \rightarrow \infty\) for every \(f \in E\). Then, Trotter’s approximation theorem can be stated as follows:

**Proposition 7.2** (cf. [21,30]) Let \(T_n, n \in \mathbb{N}\), be a bounded linear operator on \(B_n\) with \(\| T_n \| \leq 1\). Let \(\{\ell(n)\}_{n=1}^\infty\) be a sequence of positive numbers and \(\mathfrak{A}_n := (T_n - I)/\ell(n)\) for \(n \in \mathbb{N}\). Suppose that \(\ell(n) \rightarrow 0\) as \(n \rightarrow \infty\) and \(\mathfrak{A}\) is defined by the closure of the limit \(\lim_{n \rightarrow \infty} \mathfrak{A}_n\). If the domain \(\text{Dom}(\mathfrak{A})\) is dense in \(E\) and the range \(\text{Ran}(\lambda_0 - \mathfrak{A})\) is dense in \(E\) for some \(\lambda_0 > 0\), then there exists a \(C^0\)-semigroup \((T_t)_{t \geq 0}\) on \(E\) such that

\[
\lim_{n \rightarrow \infty} \| T_n^{[t/\ell(n)]} P_n f - P_n T_t f \|_{B_n} = 0, \quad t \geq 0.
\]

Afterward, Campiti and Tacelli [5, Theorem 1.1] determined a rate of convergence of the Trotter’s theorem in the case where \(B_n \equiv E\) for \(n \in \mathbb{N}\). The following theorem is a certain refinement of Proposition 7.2 and is also regarded as an extension of Campiti and Tacelli [5, Theorem 1.1].

**Theorem 7.3** ([23, Theorem 1.3]) Let \(T_n, n \in \mathbb{N}\), be a bounded linear operator on \(B_n\) satisfying

\[
\| T_n^k \| \leq Me^{\omega k/n}, \quad n, k \in \mathbb{N},
\]

for some \(M \geq 1\) and \(\omega \geq 0\). Suppose that \(\mathfrak{D}\) is a dense subspace of \(E\) and \(\mathfrak{A} : (\mathfrak{D} \subset \text{Dom}(\mathfrak{A})) \rightarrow E\) is a linear operator. If \(\text{Ran}(\lambda - \mathfrak{A})\) is dense in \(E\) for some \(\lambda > \omega\), then the closure of \((\mathfrak{A}, \mathfrak{D})\) generates a \(C^0\)-semigroup \((T_t)_{t \geq 0}\) on \(E\) satisfying \(\| T_t \| \leq M e^{\omega t}\) for \(t \geq 0\). Moreover, suppose that

\[
\| n(T_n - I) P_n f \|_{B_n} \leq \varphi_n(f), \quad f \in \mathfrak{D},
\]

and the following estimate of the Voronovskaja-type formula holds:

\[
\| n(T_n - I) P_n f - P_n \mathfrak{A} f \|_{B_n} \leq \psi_n(f), \quad f \in \mathfrak{D},
\]

where \(\varphi_n, \psi_n : \mathfrak{D} \rightarrow [0, \infty)\) are seminorms on the subspace \(\mathfrak{D}\) with \(\lim_{n \rightarrow \infty} \psi_n(f) = 0\) for \(f \in \mathfrak{D}\). Then, for every \(t \geq 0\) and for every increasing \(\{k(n)\}_{n=1}^\infty\) of positive
integers, we have

\[ \| T_n^{k(n)} P_n f - P_n T_n f \|_{B_n} \]
\[ \leq M \exp(2\alpha k(n)/n) \left( \frac{\omega}{n} + \frac{k(n)}{n} \right) \varphi_n(f) + M \exp(\omega t e^{\alpha n}/n) \left| \frac{k(n)}{n} - t \right| \varphi_n(f) \]
\[ + M \exp(\omega t e^{\alpha n}/n) \int_0^t \exp(-\omega t e^{\alpha n}/n) \psi_n(T_s f) \, ds \]  
(7.4)

for all \( f \in \mathcal{D}_0 := \{ g \in \mathcal{D} \mid T_t g \in \mathcal{D}, \ t \geq 0 \} \), where we put \( t_n := \max \{ t, k(n)/n \} \).

See Namba [23] for some typical applications as well as its complete proof. We here give a proof of the following Proposition 7.5 by a simple application of Theorem 7.3, though the function space should be supposed to be

\[ \mathcal{D} := C_\infty(G) = \bigcap_{k=1}^\infty \left\{ f \in C_\infty(G) : \lim_{|x| \to \infty} a^I f(x) = 0, \ I \in I, \ d(I) = k \right\} \].

In this sense, Proposition 7.5 is weaker than Corollary 7.1. Nevertheless, it is worth mentioning here since it is proved by using not any probabilistic techniques but functional analytic ones.

Proposition 7.4  For every \( f \in \mathcal{D} = C_\infty(G) \), we have

\[ \| \mathcal{L}^{n^t} \mathcal{P} \mathcal{H}^{n-1/2} f - \mathcal{P} \mathcal{H}^{n-1/2} e^{t/\mathcal{A}(\Phi_0)} f \|_\infty \leq C_{n^{1/2}}, \ n \in \mathbb{N}, \]  
(7.5)

for some positive constant \( C = C(G, f, p, \Phi_0, t) > 0 \).

Proof  Let us take \( (B_n, \| \cdot \|_{B_n}) = (C_\infty(X), \| \cdot \|_\infty) \) for \( n \in \mathbb{N} \) and \( (E, \| \cdot \|_E) = (C_\infty(G), \| \cdot \|_\infty) \). Then, \( \{ (C_\infty(X), \mathcal{P} \mathcal{H}^{n-1/2}) \}_{n=1}^\infty \) approximates the Banach space \( C_\infty(G) \). We also take \( \mathcal{D} = C_\infty(G) \), which is a dense subspace of \( C_\infty(G) \). We define a sequence of bounded linear operators \( \{ T_n \}_{n=1}^\infty \) on \( C_\infty(X) \) by \( T_n = \mathcal{L} \) for \( n \in \mathbb{N} \). Note that \( \| \mathcal{L}^p \| \leq 1 \) for \( n \in \mathbb{N} \), that is, \( M = 1 \) and \( \omega = 0 \). Moreover, we take \( \mathfrak{A} = \mathcal{A}(\Phi_0) \), which satisfies that \( \text{Ran}(\lambda - \mathfrak{A}(\Phi_0)) \) is dense in \( C_\infty(G) \) for some \( \lambda > 0 \) (cf. [28, page 304]).

We now show that

\[ \| n(\mathcal{L} - I) \mathcal{P} \mathcal{H}^{n-1/2} f \|_\infty \leq \varphi_n(f) = \| \mathcal{A}(\Phi_0) f \|_\infty + \frac{C}{n^{1/2}} \left( \max_{e \in E_0} |d \Phi_0(\tilde{e})| \right) \| D^3 f \|_\infty \]  
(7.6)

and

\[ \| n(\mathcal{L} - I) \mathcal{P} \mathcal{H}^{n-1/2} f - \mathcal{P} \mathcal{H}^{n-1/2} \mathcal{A}(\Phi_0) f \|_\infty \leq \psi_n(f) = \frac{C}{n^{1/2}} \left( \max_{e \in E_0} |d \Phi_0(\tilde{e})| \right) \| D^3 f \|_\infty \]  
(7.7)
for every \( f \in \mathcal{D} \). Indeed, we apply the right stratified Taylor formula up to order 2 to the function \( f \) at \( \tau_{n-1/2}(\Phi_0(x)) \). Then, it follows from \( m_I^f = 0 \) for \( I \in \mathcal{I} \) with \( d(I) = 1 \) (Proposition 5.1) that

\[
 n(\mathcal{L} - I)\mathcal{P}^{H}_{n-1/2} f(x) = n^{1/2} \sum_{d(I) = 1} \widehat{S}^I f\left(\tau_{n-1/2}(\Phi_0(x))\right) m_I^f \\
+ \sum_{d(I) = 2} \widehat{S}^I f\left(\tau_{n-1/2}(\Phi_0(x))\right) m_I^f + n\mathbb{E}^{x,1}[R^f_3]
\]

\[
= \sum_{d(I) = 2} \widehat{S}^I f\left(\tau_{n-1/2}(\Phi_0(x))\right) m_I^f + n\mathbb{E}^{x,1}[R^f_3], \quad x \in V,
\]

where the remainder term \( R^f_3 \) satisfies

\[
\mathbb{E}^{x,1}[|R^f_3|] \leq \frac{C}{n^{3/2}} \mathbb{E}^{x,1}\left[|d\Phi_0(e)|^3 \sup \left\{|a^I f\left(\tau_{n-1/2}(\Phi_0(x))\right) \cdot z| : d(I) = 3, |z| \leq \frac{b^3}{n^{1/2}} |d\Phi_0(e)|^3\right\}\right]
\]

\[
\leq \frac{C}{n^{3/2}} \left(\max_{e \in E_0} |d\Phi_0(e)|^3\right) \|D^3 f\|_\infty.
\]

for some positive constants \( C > 0 \). We observe that

\[
\sum_{d(I) = 2} \widehat{S}^I f\left(\tau_{n-1/2}(\Phi_0(x))\right) m_I^f = \left(\frac{1}{2}\sum_{i,j=1}^{d_1} \sigma_i(\Phi_0)\sigma_j(\Phi_0) a_i a_j + \sum_{i=d_1+1}^{d_1+d_2} \beta(\Phi_0)|a_i|^2\right) f\left(\tau_{n-1/2}(\Phi_0(x))\right)
\]

\[
= \mathcal{P}^{H}_{n-1/2} \mathcal{A}(\Phi_0) f(x), \quad x \in V.
\]

Hence, we obtain

\[
\|n(\mathcal{L} - I)\mathcal{P}^{H}_{n-1/2} f\|_\infty \leq \|\mathcal{A}(\Phi_0) f\|_\infty + \frac{C}{n^{1/2}} \left(\max_{e \in E_0} |d\Phi_0(e)|^3\right) \|D^3 f\|_\infty,
\]

\[
\|n(\mathcal{L} - I)\mathcal{P}^{H}_{n-1/2} f - \mathcal{P}^{H}_{n-1/2} \mathcal{A}(\Phi_0) f\|_\infty \leq \frac{C}{n^{1/2}} \left(\max_{e \in E_0} |d\Phi_0(e)|^3\right) \|D^3 f\|_\infty.
\]
Therefore, we can apply Theorem 7.3 in the case where \( k(n) = [nt] \) for \( n \in \mathbb{N} \). By using (7.6) and (7.7) and by noting (e\(^tA(\Phi_0)\))(\( D \)) \( \subset D \) for \( t \geq 0 \), we obtain

\[
\| L^{[nt]} P_{n-1/2}^H f - P_{n-1/2}^H e^{tA(\Phi_0)} f \|_{\infty} \\
\leq \frac{\sqrt{[nt]}}{n} \varphi_n(f) + \left| \frac{[nt]}{n} - t \right| \varphi_n(f) + \int_0^t \psi_n(e^{sA(\Phi_0)} f) \, ds \\
\leq \frac{t^{1/2}}{n^{1/2}} \varphi_n(f) + \frac{1}{n} \varphi_n(f) + \int_0^t \psi_n(e^{sA(\Phi_0)} f) \, ds \leq \frac{C}{n^{1/2}}, \quad t \geq 0,
\]

for some positive constant \( C = C(G, f, p, \Phi_0, t) > 0 \). This means that we have established (7.5) for every \( f \in D \). This completes the proof. \( \square \)

Before closing this section, we mention the case where the \( \Gamma \)-equivariant realization \( \Phi : X \to G \) is not always modified harmonic. Since the proof of Theorem 1.2 heavily depends on the modified harmonicity of the realization, we do not expect to establish the precise Edgeworth expansions for the random walks on \( X \) without imposing the modified harmonicity. However, we now see that the Berry–Esseen-type bound for arbitrary \( \Gamma \)-equivariant realization \( \Phi : X \to G \) is immediately established. Namely, we obtain the following.

**Theorem 7.5** (Berry–Esseen-type bound for \( \Gamma \)-equivariant realizations) Let \( \Phi : X \to G \) be a \( \Gamma \)-equivariant realization of \( X \) and \( P_{n-1/2} : C_\infty(G) \to C_\infty(G), n \in \mathbb{N}, \) be the approximation operator defined by

\[
P_{n-1/2} f(x) := f \left( \tau_{n-1/2} (\Phi(x)) \right), \quad x \in V.
\]

For every \( f \in C_\infty(G) \), we have

\[
\| L^{[nt]} P_{n-1/2} f - P_{n-1/2} e^{tA(\Phi_0)} f \|_{\infty} \leq \frac{C}{n^{1/2}}, \quad n \in \mathbb{N},
\]

for some positive constant \( C = C(G, f, p, \Phi, \Phi_0, t) > 0 \).

**Proof** We fix a reference point \( x_0 \in V \). Then, we may put \( \Phi(x_0) = \Phi_0(x_0) = 1_G \) without loss of generality. By using the triangular inequality and Corollary 7.1, we have

\[
\| L^{[nt]} P_{n-1/2} f - P_{n-1/2} e^{tA(\Phi_0)} f \|_{\infty} \\
\leq \| L^{[nt]} P_{n-1/2} f - L^{[nt]} P_{n-1/2}^H f \|_{\infty} + \| L^{[nt]} P_{n-1/2}^H f - P_{n-1/2}^H e^{tA(\Phi_0)} f \|_{\infty} \\
+ \| P_{n-1/2} e^{tA(\Phi_0)} f - P_{n-1/2} e^{tA(\Phi_0)} f \|_{\infty} \\
\leq \| P_{n-1/2} f - P_{n-1/2}^H f \|_{\infty} + \| P_{n-1/2}^H e^{tA(\Phi_0)} f - P_{n-1/2} e^{tA(\Phi_0)} f \|_{\infty} + \frac{C}{n^{1/2}}.
\]
Recall that there is an intrinsic left invariant metric on $G$ called the Carnot–Carathéodory metric given by

$$d_{CC}(g, h) := \inf \left\{ \int_0^1 \| \dot{c}(t) \|_{g(t)} \, dt \mid c(0) = g, \ c(1) = h, \ \dot{c}(t) \in g^{(1)}_c(t) \right\}, \quad g, h \in G,$$

where $\| \cdot \|_{g(t)}$ denotes a fixed norm on $g^{(1)}_c(t)$ and $g^{(1)}_c(t)$ is the evaluation of $g^{(1)}$ at $c(t)$. Due to $f \in C^\infty_c(G)$, we find a positive constant $C > 0$ such that

$$|P_n(\tau_n - 1/2 f(x)) - P^H_n(\tau_n - 1/2 f(x))| \leq C d_{CC}(\tau_n - 1/2 (\Phi(x)), \tau_n - 1/2 (\Phi_0(x))) = C n^{1/2} d_{CC}(\Phi(x), \Phi_0(x)), \quad x \in V.$$

Since $d_{CC}(\Phi(x), \Phi_0(x)) = d_{CC}(\Phi(\gamma x), \Phi_0(\gamma x))$ for $x \in V$ and $\gamma \in \Gamma$, the function

$$x \mapsto d_{CC}(\Phi(x), \Phi_0(x))$$

can be regarded as a function defined on the base graph $X_0$. Therefore, we have

$$|P_n(\tau_n - 1/2 f(x)) - P^H_n(\tau_n - 1/2 f(x))| \leq C n^{1/2} \max_{x \in V_0} d_{CC}(\Phi(x), \Phi_0(x)), \quad x \in V.$$

Similarly, since $e^{t A_0} f$ is also Lipschitz, we have

$$|P^H_n(\tau_n - 1/2 e^{t A_0} f(x)) - P_n(\tau_n - 1/2 e^{t A_0} f(x))|_\infty \leq C n^{1/2} \max_{x \in V_0} d_{CC}(\Phi(x), \Phi_0(x)), \quad x \in V.$$

for some positive constant $C > 0$. By putting it all together, we obtain the desired bound. This completes the proof. 

The difference between $\Phi(x)$ and $\Phi_0(x)$ with respect to $d_{CC}$ is called the corrector, which is also applied effectively to the proof of a functional CLT for non-symmetric random walks on $X$ in, e.g., [15, Theorem 2.3]. We note that the terminology “corrector” is frequently used in the context of homogenization theory (cf. [20]).

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### Appendix A. Proof of Proposition 4.2

We here give a proof of Proposition 4.2 in the case $N = 2$. 

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Proof of Proposition 4.2 Throughout the proof, \((\cdot, \cdot)_{\ell^2(X_0)}\) and \(\|\cdot\|_{\ell^2(X_0)}\) are abbreviated as \((\cdot, \cdot)\) and \(\|\cdot\|\), respectively. By virtue of the decomposition (4.1), we have

\[
\frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{\ell=0}^{k} \mathcal{L}^\ell f(x) \mathcal{L}^{k+1} g(x)
= \frac{1}{2} \left(1 + \frac{1}{n}\right) \langle f, m \rangle \langle g, m \rangle + \frac{1}{n^2} \langle f, m \rangle \sum_{k=1}^{n} \sum_{j=1}^{K_0-1} (g, \psi_j) \alpha_j^k \phi_j(x)
+ \frac{1}{n^2} \langle f, m \rangle \sum_{k=0}^{n-1} \sum_{\ell=0}^{k} \mathcal{L}^\ell g_{\ell^2_{k_0}(X_0)}(x) + \frac{1}{n^2} \langle g, m \rangle \sum_{k=0}^{n-1} \sum_{\ell=0}^{k} \sum_{j=1}^{K_0-1} (f, \psi_j) \alpha_j^\ell \phi_j(x)
+ \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{\ell=0}^{k} \left( \sum_{j=1}^{K_0-1} (f, \psi_j) \alpha_j^k \phi_j(x) \right) \left( \sum_{j=1}^{K_0-1} (g, \psi_j) \alpha_j^\ell \phi_j(x) \right)
+ \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{\ell=0}^{k} \mathcal{L}^{k+1} g_{\ell^2_{k_0}(X_0)}(x) \sum_{\ell=0}^{k} \left( \sum_{j=1}^{K_0-1} (f, \psi_j) \alpha_j^\ell \phi_j(x) \right)
+ \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{\ell=0}^{k} \mathcal{L}^\ell f_{\ell^2_{k_0}(X_0)}(x) \mathcal{L}^{k+1} g_{\ell^2_{k_0}(X_0)}(x)
=: \frac{1}{2} \left(1 + \frac{1}{n}\right) \langle f, m \rangle \langle g, m \rangle + I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8
\]

for \(n \in \mathbb{N}\) and \(x \in V_0\). In particular, the terms \(I_1, I_3, I_4\) and \(I_5\) are calculated as follows:

\[
I_1 = -\frac{1}{n} \langle f, m \rangle \sum_{j=1}^{K_0-1} \frac{\alpha_j^{n+1}}{1 - \alpha_j} \phi_j(x) + \frac{1}{n^2} \langle f, m \rangle \sum_{j=1}^{K_0-1} \alpha_j (1 - \alpha_j^n) \langle g, \psi_j \rangle \frac{\phi_j(x)}{(1 - \alpha_j)^2},
\]

\[
I_3 = \frac{1}{n} \langle g, m \rangle \sum_{j=1}^{K_0-1} \frac{\langle f, \psi_j \rangle}{1 - \alpha_j} \phi_j(x) - \frac{1}{n^2} \langle g, m \rangle \sum_{j=1}^{K_0-1} \alpha_j (1 - \alpha_j^n) \langle f, \psi_j \rangle \frac{\phi_j(x)}{(1 - \alpha_j)^2},
\]

\[
I_4 = \frac{1}{n^2} \sum_{i,j=1}^{K_0-1} \frac{\langle f, \psi_i \rangle \langle g, \psi_j \rangle}{1 - \alpha_i} \left( \frac{\alpha_j (1 - \alpha_j^n)}{1 - \alpha_j} - \frac{\alpha_i \alpha_j (1 - \alpha_i^n \alpha_j^n)}{1 - \alpha_i \alpha_j} \right) \phi_i(x) \phi_j(x),
\]

\[
I_5 = \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{\ell=0}^{k} \mathcal{L}^{k+1} g_{\ell^2_{k_0}(X_0)}(x) \sum_{\ell=0}^{k} \left( \sum_{j=1}^{K_0-1} (f, \psi_j) \alpha_j^\ell \phi_j(x) \right).
\]
We note that the Perron–Frobenius theorem implies that there exists some \( \lambda \in (0, 1) \) such that \( \| L |_{\ell^2_{K_0}(X_0)} \| \leq \lambda \). Hence, we see that

\[
\left\| \sum_{k=0}^{n-1} L^k f \right\| \leq \sum_{k=0}^{n-1} \lambda^k \| f \| \leq \| f \| \frac{1}{1-\lambda} = O(1),
\]

\[
\left\| \sum_{k=1}^{n} k L^k f \right\| \leq n \sum_{k=1}^{n} \lambda^k \| f \| = O(n), \quad \text{and}
\]

\[
\left\| \sum_{k=0}^{n-1} \sum_{\ell=0}^{k} L^\ell f \right\| \leq \sum_{k=0}^{n-1} \sum_{\ell=0}^{k} \lambda^k \| f \| = O(n)
\]

(A.1)

for \( f \in \ell^2_{K_0}(X_0) \). We now set

\[
A[f, g]_n^{(1)}(x) = \frac{1}{2} \langle f, m \rangle \langle g, m \rangle - \langle f, m \rangle \sum_{j=1}^{K_0-1} \frac{\alpha_j^{n+1} \langle g, \psi_j \rangle}{1-\alpha_j} \phi_j(x)
\]

\[
+ \frac{1}{n} \langle f, m \rangle \sum_{k=0}^{n-1} \sum_{\ell=0}^{k} L^\ell g_{\ell, K_0}(X_0)(x)
\]

\[
+ \langle g, m \rangle \sum_{j=1}^{K_0-1} \frac{\langle f, \psi_j \rangle}{1-\alpha_j} \phi_j(x) + \frac{1}{n} \langle g, m \rangle \sum_{k=1}^{n} k L^k f_{\ell, K_0}(X_0)(x),
\]

(A.2)

and

\[
A[f, g]_n^{(2)}(x) = \langle f, m \rangle \sum_{j=1}^{K_0-1} \frac{\alpha_j (1-\alpha_j^n) \langle g, \psi_j \rangle}{(1-\alpha_j)^2} \phi_j(x)
\]

\[
- \langle g, m \rangle \sum_{j=1}^{K_0-1} \frac{\alpha_j^2 (1-\alpha_j^n) \langle f, \psi_j \rangle}{(1-\alpha_j)^2} \phi_j(x)
\]

\[
+ \sum_{i, j=1}^{K_0-1} \frac{\langle f, \psi_i \rangle \langle g, \psi_j \rangle}{1-\alpha_i} \left( \frac{\alpha_j (1-\alpha_j^n)}{1-\alpha_j} - \frac{\alpha_i\alpha_j (1-\alpha_i^n)}{1-\alpha_i\alpha_j} \right) \phi_i(x) \phi_j(x)
\]

\[
+ \sum_{k=0}^{n-1} L^{k+1} g_{\ell, K_0}(X_0)(x) \sum_{j=1}^{K_0-1} \frac{\langle f, \psi_j \rangle (1-\alpha_j^{k+1})}{1-\alpha_j} \phi_j(x)
\]

\[
+ \sum_{k=0}^{n-1} \sum_{j=1}^{K_0-1} \frac{\langle g, \psi_j \rangle \alpha_j^{k+1} \phi_j(x)}{1-\alpha_j} \sum_{\ell=0}^{k} L^\ell f_{\ell, K_0}(X_0)(x)
\]
\[ + \sum_{k=0}^{n-1} \sum_{\ell=0}^{k} \mathcal{L}_k f_{\mathcal{K}_0}(x) \mathcal{L}_k^2 g_{\mathcal{K}_0}(x). \]  

(A.3)

By noting (A.1), \(|\alpha_j| \leq 1, j = 1, 2, \ldots, K_0 - 1,\) and an inequality \(\|fg\|_2^2 \leq |V_0|\|f\|_2^2\|g\|_2^2\) for \(f, g \in \ell^2(X_0),\) we conclude that \(\|A[f, g](1)\| = O(1)\) and \(\|A[f, g](2)\| = O(1)\) as \(n \to \infty.\) This completes the proof of Proposition 4.2. 

\[ \square \]

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