The maximum number of paths of length three in a planar graph

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Abstract
Let \( f(n, H) \) denote the maximum number of copies of \( H \) possible in an \( n \)-vertex planar graph. The function \( f(n, H) \) has been determined when \( H \) is a cycle of length 3 or 4 by Hakimi and Schmeichel and when \( H \) is a complete bipartite graph with smaller part of size 1 or 2 by Alon and Caro. We determine \( f(n, H) \) exactly in the case when \( H \) is a path of length 3.

KEYWORDS
extremal graphs, planar graphs, turan number

1 | INTRODUCTION AND MAIN RESULT

In recent times, generalized versions of the extremal function \( \text{ex}(n, H) \) have received considerable attention. For graphs \( G \) and \( H \), let \( \mathcal{N}(H, G) \) denote the number of subgraphs of \( G \) isomorphic to \( H \) (referred to as copies of \( H \)). Let \( \mathcal{F} \) be a family of graphs, then a graph \( G \) is said to be \( \mathcal{F} \)-free if it contains no graph from \( \mathcal{F} \) as a subgraph. Alon and Shikhelman [3] introduced the following generalized extremal function (stated in higher generality in [4]),

\[
\text{ex}(n, H, \mathcal{F}) = \max\{\mathcal{N}(H, G) : G \text{ is an } \mathcal{F}\text{-free graph on } n \text{ vertices}\}.
\]
If $F = \{F\}$, we simply write $\text{ex}(n, H, F)$. The earliest result of this type is due to Zykov [28] (and also independently by Erdős [8]), who determined $\text{ex}(n, K_s, K_t)$ exactly for all $s$ and $t$. Erdős conjectured that asymptotically $\text{ex}(n, C_3, C_3) = \left(\frac{n}{5}\right)^5$ (where the lower bound comes from considering a blown up $C_3$). This conjecture was finally verified a quarter of a century later by Hatami et al. [20] and independently by Grzesik [17]. Recently, the asymptotic value of $\text{ex}(n, C_k, C_{k-2})$ was determined for every odd $k$ by Grzesik and Kielak [18]. In the opposite direction, the extremal function $\text{ex}(n, C_3, C_5)$ was considered by Bollobás and Győri [5]. Their results were subsequently improved in the papers [3], [9], and [10], but the problem of determining the correct asymptotic remains open. The problem of maximizing $P_e$ copies in a $P_k$-free graph was investigated in [15].

It is interesting that although maximizing copies of a graph $H$ in the class of $F$-free graphs has been investigated heavily, maximizing $H$-copies in other natural graph classes has received less attention. In the setting of planar graphs such a study was initiated by Hakimi and Schmeichel [19]. Let $f(n, H)$ denote the maximum number of copies of $H$ possible in an $n$-vertex planar graph. Observe that $f(n, H)$ is equal to $\text{ex}(n, H, \mathcal{F})$, where $\mathcal{F}$ is the family of $K_{3,3}$ or $K_5$ subdivisions [24]. The case when $H$ is a clique and $\mathcal{F}$ is a family of clique minors has also been investigated (see, e.g., [11], [23], and [26]).

Hakimi and Schmeichel determined the function $f(n, H)$ when $H$ is a triangle or cycle of length four. Moreover, they classified the extremal graphs attaining this bound (a small correction to their result was given in [1]).

**Theorem 1** (Hakimi and Schmeichel [19]). Let $G$ be a maximal planar graph with $n \geq 6$ vertices, then $\mathcal{N}(C_3, G) \leq 3n - 8$. This bound is attained if and only if $G$ is a graph obtained from $K_3$ by recursively placing a vertex inside a face and joining the vertex to the three vertices of that face (graphs constructed in this way are referred to as Apollonian networks).

**Theorem 2** (Hakimi and Schmeichel [19] and Alameddine [1]). Let $G$ be a maximal planar graph with $n \geq 5$ vertices, then $\mathcal{N}(C_4, G) \leq \frac{1}{2}(n^2 + 3n - 22)$. For $n \neq 7, 8$, the bound is attained if and only if $G$ is the graph shown in Figure 1A. For $n = 7$, the bound is attained if and only if $G$ is the graph in Figure 1A,B. For $n = 8$, the bound is attained if and only if $G$ is the graph in Figure 1A,C.

Thus, we have $f(n, C_3) = 3n - 8$ when $n \geq 6$ and $f(n, C_4) = \frac{1}{2}(n^2 + 3n - 22)$ for $n \geq 5$. In [16], the last five authors extended the results of Hakimi and Schmeichel by determining

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**Figure 1** Planar graphs maximizing the number of cycles of length 4

(A) $F_{n_3}$

(B)

(C)
$f(n, C_3)$ for all $n$. Asymptotic results for $f(n, C_{2k})$ with $k = 3, 4, 5, 6$, have recently been obtained by Cox and Martin [6,7].

In the case when $H$ is a complete bipartite graph, Alon and Caro [2] determined the value of $f(n, H)$ exactly. They obtained the following results.

**Theorem 3** (Alon and Caro [2]). For all $k \geq 2$ and $n \geq 4$,

$$f(n, K_{1,k}) = 2 \binom{n-1}{k} + 2 \binom{3}{k} + (n-4) \binom{4}{k}.$$  

**Theorem 4** (Alon and Caro [2]). For all $k \geq 2$ and $n \geq 4$,

$$f(n, K_{2,k}) = \begin{cases} 
\binom{n-2}{k}, & \text{if } k \geq 5 \text{ or } k = 4 \text{ and } n \neq 6; \\
3, & \text{if } (k, n) = (4, 6); \\
\binom{n-2}{3}, & \text{if } k = 3, n \neq 6; \\
12, & \text{if } (k, n) = (3, 6); \\
\binom{n-2}{2} + 4n - 14, & \text{if } k = 2.
\end{cases}$$

Other results in this direction include a linear bound on the maximum number of copies of a 3-connected planar graph by Wormald [27] and independently Eppstein [12]. The exact bound on the maximum number of copies of $K_4$ was obtained independently by Alon and Caro [2] and by Wood [25]. Let $P_k$ denote the path on $k$ vertices. It is well-known that $f(n, P_3) = 3n - 6$, and it follows from Theorem 3 that $f(n, P_3) = n^2 + 3n - 16$ for $n \geq 4$. The order of magnitude of $f(n, H)$ when $H$ is a fixed tree was determined in [14] and for general $H$ (and in arbitrary surfaces) by Huynh et al. [22] (see also [21] for results in general sparse settings). In particular, for a path on $k$ vertices, we have $f(n, P_k) = \Theta(n^{\frac{k-1}{2}})$.

In this paper, we determine $f(n, P_4)$, the maximum number of copies of a path on four vertices possible in $n$-vertex planar graph. Our main result is the following.

**Theorem 5.** We have,

$$f(n, P_4) = \begin{cases} 
12, & \text{if } n = 4; \\
147, & \text{if } n = 7; \\
222, & \text{if } n = 8; \\
7n^2 - 32n + 27, & \text{if } n = 5, 6 \text{ and } n \geq 9.
\end{cases}$$

For integers $n \in \{4, 5, 6\}$ and $n \geq 9$, the only $n$-vertex planar graph attaining the value $f(n, P_3)$ is the graph $F_n$. For $n = 7$ and $n = 8$ the graphs pictured in Figure 1B,C, respectively are the only graphs attaining the value $f(n, P_4)$. 

We also note that the asymptotic value of $f(n, P_5)$ was determined to be $n^3$ in [13], and Cox and Martin [6] obtained the asymptotic result $f(n, P_7) = \frac{4}{27}n^4 + O(n^{4-1/3})$.

2 NOTATION AND PRELIMINARIES

Let $G$ be a planar graph. We denote the vertex and the edge sets of $G$ by $V(G)$ and $E(G)$, respectively. For a vertex $v \in V(G)$, $d_G(v)$ denotes the degree of $v$. We omit the subscript whenever there is no ambiguity about which graph we are referring to. We denote by $N(v)$ the set of neighbors of $v$. For two vertices $x, y \in V(G)$, we denote the set of vertices which are adjacent to both $x$ and $y$ by $N(x, y)$. We also denote the size of $N(x, y)$ by $d(x, y)$. The minimum degree of $G$ is denoted by $\delta(G)$. For simplicity, we refer to a path of length three as a 3-path. We denote the number of $P_4$’s in $G$ by $P_4(G)$. For $x \in V(G)$, the number of $P_4$’s in $G$ containing $x$ is denoted by $P_4(G, x)$. We denote the $n$-vertex graph obtained by joining every vertex from a path with $n - 2$ vertices to two additional adjacent vertices by $F_n$ (pictured in Figure 1A).

For any maximal planar graph $G$ on $n$ vertices ($n \geq 3$) it can be shown that $3 \leq \delta(G) \leq 5$. Moreover for a vertex $v$ in $V(G)$, if $d(v) = k$ and $N(v) = \{x_1, x_2, x_3, \ldots, x_k\}$, then $N(v)$ induces a unique cycle of length $k$. We may choose a drawing of $G$ so that $v$ is contained in the interior of the cycle. Without loss of generality, we may assume that we have a cycle $C$ with vertex sequence $x_1, x_2, x_3, \ldots, x_k, x_1$. Let us denote the edge $\{x_i, v\}$ by $e_i$ for $i = 1, 2, 3, \ldots, k$ (see Figure 2).

We partition the set of 3-paths containing $v$ into three different classes, depending on the location of their middle edge.

A Type-I, 3-path with respect to a vertex $v$ is a 3-path which contains an edge $e_i$ as its middle edge (see Figure 3).
A Type-II, 3-path with respect to a vertex \( v \) is a 3-path which starts with vertices \( v, x_i, x_j \). Furthermore, if the middle edge is an edge of the cycle \( C \), then we call such a 3-path a Type-II (A), 3-path. Otherwise, we call it a Type-II(B), 3-path (see Figure 4).

A Type-III, 3-path with respect to a vertex \( v \) is a 3-path which starts at the vertex \( v \) such that its middle edge connects a vertex from \( N(v) \) to a vertex from \( V(G) \setminus (N(v) \cup \{v\}) \). Furthermore, if the last vertex is not from \( N(v) \), then we call such a 3-path a Type-III(A), 3-path. Otherwise, we call it a Type-III(B), 3-path (see Figure 5).

It is easy to see that each of the 3-paths containing the vertex \( v \) is in exactly one of the three classes which we have defined. For simplicity, we will sometimes write Type-(I), (II), (III), 3-path with respect to a vertex \( v \), when the vertex under consideration is clear.

We will use the following two lemmas in our proof of the main theorem. The first lemma gives the number of 3-paths in a given graph \( G \).

**Lemma 1.** For a graph \( G \), the number of 3-paths in \( G \) is

\[
P_3(G) = \sum_{\{x, y\} \in E(G)} (d(x) - 1)(d(y) - 1) - 3N(C_3, G).
\]

**Proof.** Consider an edge \( \{x, y\} \in E(G) \) and count the number of 3-paths containing \( x \) as the second and \( y \) the third vertex of the 3-path. There are \( d(x) - 1 \) possibilities to choose the first vertex and \( d(y) - 1 \) possibilities to choose the last vertex of the path. Since the first and the last vertex of the 3-path need to be different, from the total number of \( (d(x) - 1)(d(y) - 1) \) possibilities we need to subtract the number of triangles containing the edge \( \{x, y\} \), which is \( d(x, y) \).

Therefore,
\[ P_4(G) = \sum_{(x,y) \in E(G)} ((d(x) - 1)(d(y) - 1) - d(x,y)) = \sum_{(x,y) \in E(G)} (d(x) - 1)(d(y) - 1) - 3N(C_3, G), \]

as each triangle is counted three times in the sum. This completes the proof of Lemma 1. \[ \square \]

With this lemma we can prove the following lemma.

**Lemma 2.** For every \( n \)-vertex planar graph \( G \) with \( \delta(G) \geq 4 \) we have

\[ P_4(G) < 7n^2 - 36n + 50. \]

**Proof.** Without loss of generality we may assume that \( G \) is a maximal planar graph with \( 3n - 6 \) edges and \( 2n - 4 \) triangular faces. In particular it contains at least \( 2n - 4 \) triangles.

From Lemma 1 the total number of 3-paths in \( G \) is equal to

\[ P_4(G) = \sum_{(x,y) \in E(G)} (d(x) - 1)(d(y) - 1) - 3N(C_3, G) \leq \sum_{(x,y) \in E(G)} (d(x) - 1)(d(y) - 1) - 3(2n - 4) = \frac{1}{2} \sum_{x \in V(G)} (d(x) - 1) \left( \sum_{y \in N(x)} d(y) - d(x) \right) - 6n + 12. \]

Since \( \delta(G) \geq 4 \) and the sum of the degrees of all the vertices is equal to \( 2e(G) = 6n - 12 \), for each vertex \( x \) we have

\[ \sum_{y \in N(x)} d(y) = 6n - 12 - d(x) - \sum_{y \not\in N(x)} d(y) \leq 6n - 12 - d(x) - 4(n - 1 - d(x)) = 3d(x) + 2n - 8. \]

This gives us the following bound:

\[ P_4(G) \leq \frac{1}{2} \sum_{x \in V(G)} (d(x) - 1)(2d(x) + 2n - 8) - 6n + 12 = \sum_{x \in V(G)} d^2(x) + (n - 5) \sum_{x \in V(G)} d(x) - n(n - 4) - 6n + 12 \leq ((n - 1)^2 + (n - 3)^2 + 4^2(n - 2)) + (n - 5)(6n - 12) - n^2 - 2n + 12 = 7n^2 - 36n + 50, \]

where the last inequality comes from convexity.

It remains to notice that, since \( \delta(G) \geq 4 \) and \( G \) is a planar graph, we have \( n \geq 6 \), hence \( 7n^2 - 36n + 50 < 7n^2 - 32n + 27. \) \[ \square \]
3 | PROOF OF THEOREM 5

We are going to prove the theorem by induction on the number of vertices. The base cases, when \( n \leq 9 \), will be discussed later.

Let \( G \) be a planar graph on \( n \) vertices. Then we have \( \Delta(G) \leq 5 \). From Lemma 2 we may assume \( \Delta(G) = 3 \). We are going to prove the rest by induction, after removing a vertex of degree 3.

Let \( v \) be a vertex of degree 3 and \( N(v) = \{x_1, x_2, x_3\} \). Our goal is to show that \( P_4(G, v) \leq 14n - 39 \). Indeed, by deleting the vertex \( v \) we obtain a maximal planar graph \( G' \) on \( (n - 1) \) vertices, and by the induction hypothesis we have \( P_4(G') \leq 7(n - 1)^2 - 32(n - 1) + 27 \). Therefore,

\[
P_4(G) \leq 7(n - 1)^2 - 32(n - 1) + 27 + 14n - 39 = 7n^2 - 32n + 27.
\]

Notice that the vertices \( x_1, x_2, x_3 \) induce a triangle. Denote the edges \( \{x_i, v\} \) by \( e_i \), \( i \in \{1, 2, 3\} \). The number of Type-I, 3-paths with \( e_i \) in the middle is \( 2d(x_i) - 4 \) for all \( i \in \{1, 2, 3\} \). Thus we have \( 2\sum_{i=1}^{3} d(x_i) - 12 \) Type-I, 3-paths. The number of Type-II, 3-paths starting at \( v \) and continuing to a vertex \( x_i \), \( i \in \{1, 2, 3\} \), is \( d(x_1) + d(x_2) + d(x_3) - d(x_i) - 4 \). Thus, we have \( 2\sum_{i=1}^{3} d(x_i) - 12 \) Type-II, 3-paths. It remains to count the number of Type-III, 3-paths with respect to the vertex \( v \). For this we need to consider two subcases.

**Case 1.1:** \( N(x_1) \cap N(x_2) \cap N(x_3) = \{v\} \)

For each edge \( e \) which is not incident to the triangle, we can have at most four Type-III(A), 3-paths with respect to the vertex \( v \) (see Figure 6). Since there are at most \( (3n - 6) - (\sum_{i=1}^{3} d(x_i) - 3) \) such edges which are not incident to the triangle, it follows that the number of Type-III(A), 3-paths is at most \( 4(3n - 6 - (\sum_{i=1}^{3} d(x_i) - 3)) \).

The remaining 3-paths are Type-III(B), 3-paths. Recall that in this case each vertex \( v' \neq v \) can be adjacent to at most two vertices of the triangle induced by \( N(v) \). Thus for each such vertex \( v' \), \( v' \notin \{x_1, x_2, x_3, v\} \), we have at most two Type-III(B), 3-paths (see Figure 7). Hence we have at most \( 2(2n - 4) \) Type-III(B), 3-paths. Thus we get,

\[
P_4(G, v) \leq 4\sum_{i=1}^{3} d(x_i) - 24 + 4 \left(3n - 3 - \sum_{i=1}^{3} d(x_i)\right) + 2(n - 4) = 14n - 44.
\]

**FIGURE 6** Four Type-III(A), 3-paths for a fixed edge \( e \)

**FIGURE 7** Type-III(B), 3-paths for a fixed vertex \( v', v' \notin \{v, x_1, x_2, x_3\} \)
Therefore, $R_i(G, v) \leq 14n - 44 < 14n - 39$ and we have no extremal graph in this case.

**Case 1.2:** There exists a vertex $u, u \neq v$, such that $N(x_1) \cap N(x_2) \cap N(x_3) = \{v, u\}$

We consider the three regions formed by vertices $u, x_1, x_2, $ and $x_3$. Let the region defined by the vertices $u, x_1, $ and $x_2$ which does not contain $x_3$ be $R_1$, the region defined by the vertices $u, x_2, $ and $x_3$ which does not contain $x_1$ be $R_2$, and lastly the region defined by the vertices $u, x_1, $ and $x_3$ and not containing $x_2$ be $R_3$, as shown in Figure 8.

From the planarity of $G$, notice that there is at most one edge $e_1$ with end vertices $u$ and $v$ such that $v$ lies inside the region $R_1$ and $v$ is adjacent to both $x_1$ and $x_2$. Similarly there is at most one edge $e_2$ and $e_3$ with respect to the regions $R_2$ and $R_3$, respectively meeting the conditions stated for $e_1$. We refer to the edges $e_1, e_2, $ and $e_3$ as star edges of $G$ with respect to the vertex $v$.

Take an edge $e$ such that $V(e) \cap \{x_1, x_2, x_3\} = \emptyset$. Then there are at most five Type-III(A), 3-paths with respect to the vertex $v$, containing the edge $e$, since $G$ is planar. Furthermore, for each star edge (if one exists) in the three regions there are five Type-III(A), 3-paths. Figure 9 shows an edge $e$ in region $R_1$ with all five possible 3-paths of this kind.

Notice that for each vertex $w$ inside the regions, one can have at most two Type-III(B), 3-paths containing $w$. For the vertex $u$, we have six Type-III(B), 3-paths containing the vertex $u$ (see Figure 10).
1. If there is no star edge in each of the three regions, then we have at most
\[
4 \left( 3n - 6 - \left( \sum_{i=1}^{3} d(x_i) - 3 \right) \right) + 2(n - 5) + 6 = -4 \sum_{i=1}^{3} d(x_i) + 14n - 16
\]
Type-III, 3-paths containing the vertex v. Thus, we have
\[
P_4(G, v) \leq 4 \sum_{i=1}^{3} d(x_i) - 24 - 4 \sum_{i=1}^{3} d(x_i) + 14n - 16 \leq 14n - 40.
\]
Therefore, \( P_4(G, v) < 14n - 39 \).

2. If there is only one star edge, then we have at most
\[
4 \left( 3n - 6 - \left( \sum_{i=1}^{3} d(x_i) - 3 \right) - 1 \right) + 5 + 2(n - 5) + 6 = -4 \sum_{i=1}^{3} d(x_i) + 14n - 15
\]
Type-III, 3-paths with respect to the vertex v. Therefore \( P_4(G, v) \leq 14n - 39 \).

**Remark 1.** Equality holds if we have a vertex v of degree three and a vertex u which is adjacent to all of the vertices incident to v. All the other vertices share exactly two neighbors with v, and we have exactly one star edge.

3. If there are exactly two star edges, then we have two regions containing them. Without loss of generality, let the regions be \( R_1 \) and \( R_2 \). The third region, \( R_3 \), may or may not contain a vertex.

3.1 If there is a vertex in \( R_3 \), then at least one vertex in \( R_3 \) is a neighbor of the vertex u, hence this vertex is not a neighbor of at least one of the vertices \( x_1 \) or \( x_3 \). Otherwise, we would have another star edge. It follows that there is no Type-III(B), 3-path containing this vertex. Thus, the number of Type-III, 3-paths with respect to the vertex v is at most
\[
4 \left( 3n - 6 - \left( \sum_{i=1}^{3} d(x_i) - 3 \right) - 2 \right) + 10 + 2(n - 6) + 6 = -4 \sum_{i=1}^{3} d(x_i) + 14n - 16.
\]
So we have \( P_4(G, v) \leq 14n - 40 \). Therefore, \( P_4(G, v) < 14n - 39 \).

3.2 If there is no vertex in the region \( R_3 \), then at least one of the regions \( R_1 \) or \( R_2 \) contains at least two vertices, since \( n \geq 10 \). Without loss of generality, suppose \( R_1 \) contains at least two vertices. Let \( f_1 \) be the star edge in the region. This edge is incident to u, and we denote the other vertex it is incident to by \( u_1 \). We have \( u_1 \in N(x_1) \cap N(x_2) \). If there is a vertex in the region \( R \), defined by the vertices \( x_1, u_1 \), and u not containing \( x_3 \), then there is an edge \( \{u_1, u'_1\} \) in the region \( R \), where \( u'_1 \notin \{x_1, x_2, x_3\} \). This edge is in at most three Type-III(A),
3-paths. Moreover $u'_1$ is not incident to the vertex $x_2$. Hence $u'_1$ is not in any of the Type-III (B) paths. Therefore we have at most

$$4 \left( 3n - 6 - \sum_{i=1}^{3} d(x_i) - 3 \right) - 2 - 1 + 13 + 2(n - 6) + 6$$

$$= -4 \sum_{i=1}^{3} d(x_i) + 14n - 17$$

Type-III, 3-paths with respect to the vertex $v$. Consequently, we have $P_4(G, v) \leq 14n - 41$. Therefore, $P_4(G, v) < 14n - 39$.

Similarly the region defined by the vertices $x_2, u_1, u$ not containing $x_1$ is also empty, otherwise we are done by induction.

Thus the vertices must be in the region $R'$, defined by the vertices $x_1, x_2, u_1$ not containing $u$. Consider an edge $f_2 = \{u_1, u_2\}$ in the region $R'$. If $u_2$ is the only vertex in the region $R'$, then $N(u_2) = \{x_1, x_2, u_1\}$, and we are done by induction, since we have a vertex $u_2$ of degree three with at most one star edge, which was settled in Cases 1.2(1) and 1.2(2) (see Figure 11).

If the vertex $u_2$ is not a neighbor of one of the vertices $x_1$ or $x_2$, then the edge $f_2$ is not incident to the triangle and is contained in at most three Type-III(A), 3-paths. Moreover $u_2$ is in none of the Type-III(B) paths. Therefore, we have at most

$$4 \left( 3n - 6 - \sum_{i=1}^{3} d(x_i) - 3 \right) - 2 - 1 + 13 + 2(n - 6) + 6$$

$$= -4 \sum_{i=1}^{3} d(x_i) + 14n - 17$$

Type-III, 3-paths. Consequently, we have $P_4(G, v) \leq 14n - 41$. Therefore, $P_4(G, v) < 14n - 39$.

We have that there are at least two vertices in the region $R'$, and $u_2$ is incident with both of the vertices $x_1$ and $x_2$.

**FIGURE 11** A vertex $u_2$ with the property that two of the corresponding regions have no vertex inside
A similar argument to the one given in Case 1.2(3.1) gives us that there is no vertex in the region defined by the vertices \( x_1, u_2, u_1 \) not containing \( x_2 \) and, likewise, in the region defined by the vertices \( x_2, u_2, u_1 \) not containing \( x_1 \). Thus, all the vertices must be in the region defined by the vertices \( x_1, u_2, x_2 \) not containing \( x_1 \); let us denote this region by \( R'' \). Consider an edge \( f = \{ u_2, u_3 \} \) in the region \( R'' \). Thus we proceed with a similar argument as before, this time applied to the region \( R'' \) and the corresponding vertex \( u_3 \). Notice that \( N(u_3) = \{ x_1, x_2, u_{k-1} \} \) and \( u_k \) is incident with at most one star edge, which was settled in Case 1.2(1) and Case 1.2(2).

4. Suppose there are three star edges. Let \( u, y \) be the star edge in the region \( R_i \), for each \( i \in \{1, 2, 3\} \). Since \( n \geq 10 \), one of the regions \( R_1, R_2, \) or \( R_3 \) contains at least one additional vertex other than \( y \). Without loss of generality, let \( R_i \) be such a region. If there is a vertex in the region \( x_1, y_1, u \) not containing \( x_2 \), then we have at least one edge, say \( \{ y_1, y_1' \} \), for some \( y_1' \) inside the region bounded by \( x_1, y_1, u \) and \( x_2 \). The edge \( \{ y_1, y_1' \} \) is in at most three Type-III(A), 3-paths. Moreover, the vertex \( y_1' \) is not incident to \( x_2 \) and \( x_3 \). Hence it is not in any Type-III(B) paths. Thus, we have at most

\[
4 \left( 3n - 6 - \left( \sum_{i=1}^{3} d(x_i) - 3 \right) - 4 \right) + 18 + 2(n - 6) + 6 = -4 \sum_{i=1}^{3} d(x_i) + 14n - 16
\]

Type-III, 3-paths. Hence we have \( P_4(G, v) \leq 14n - 40 \). Therefore, \( P_4(G, v) < 14n - 39 \).

Similarly, the region defined by \( x_2, y_1, u \) not containing the vertex \( x_1 \) must be empty. Otherwise, we are done by induction.

If the region obtained from the vertices \( x_1, y_1, x_2 \) not containing \( u \) contains only one vertex \( u' \), then we have a degree three vertex \( u' \), and there is at most one star edge corresponding to the vertex \( u' \). Hence, we are done by induction as in Case 1.2(1) or Case 1.2(2) for the vertex \( u' \). Otherwise, if the region obtained by the vertices \( x_1, y_1, x_2 \) not containing \( u \) contains more than one vertex, then we are done by similar arguments given in Case 1.2(3.2).

3.1 | Basis for the induction

Here we are going to find the maximum number of paths of length three in a planar graph with at most nine vertices. This will form the basis for the induction. We are going to recall some facts from the previous calculations. Let \( G \) be a maximal planar graph on \( n \) vertices, and \( v \in V(G) \) be a vertex of minimum degree. By Lemma 2 we have \( d(v) = 3 \).

- Suppose there is no vertex other than \( v \) adjacent to all the neighbors of \( v \), then from Case 1.1 we have
• Suppose there is a vertex other than \( v \) which is adjacent to all the neighbors of \( v \), then we consider the following cases:
  - If there is no star edge with respect to the vertex \( v \), then from Case 1.2.1 we have
    \[
P_3(G, v) \leq 14n - 40.
    \]
  - If there is only one star edge with respect to the vertex \( v \), then from Case 1.2.2 we have
    \[
P_3(G, v) \leq 14n - 39.
    \]
  - If there are two star edges with respect to the vertex \( v \), then in this case we cannot use Case 1.2.3, since \( n \) is not at least 10. However, by similar calculations we have a weaker result for all \( n \).
    \[
P_3(G, v) \leq 4 \sum_{i=1}^{3} d(x_i) - 24 + 4 \left( 3n - 6 - \left( \sum_{i=1}^{3} d(x_i) - 3 \right) - 2 \right) + 10 + 2(n - 5) + 6
    = 14n - 38.
    \]
  - If there are three star edges with respect to the vertex \( v \), then in this case we cannot use Case 1.2.4, since \( n \leq 9 \). However, by similar calculations we have a weaker result for all \( n \).
    \[
P_3(G, v) \leq 4 \sum_{i=1}^{3} d(x_i) - 24 + 4 \left( 3n - 6 - \left( \sum_{i=1}^{3} d(x_i) - 3 \right) - 3 \right) + 15 + 2(n - 5) + 6
    = 14n - 37.
    \]

**Claim 1.** \( f(4, P_3) = 12 \) and \( f(5, P_3) = 42 \).

**Proof.** The maximal planar graphs with four and five vertices are unique. The graphs are \( K_4 \) and \( K_5 - \), respectively. It is easy to verify that \( f(4, P_3) = 12 \) and \( f(5, P_3) = 42 \).

**Claim 2.** \( f(6, P_3) = 87 \).

**Proof.** Let \( G \) be a maximal planar graph on six vertices. First we prove the following claim.

**Claim 3.** There is a vertex different from \( v \) which is adjacent to every neighbor of \( v \). Moreover, there is one star edge with respect to \( v \).
Proof. Let $N(v) = \{x_1, x_2, x_3\}$ and the remaining two vertices other than $v, x_1, x_2,$ and $x_3$ be $y_1$ and $y_2$. By maximality, every edge of $G$ must be incident to exactly two triangular faces. Thus each of the edges in $\{x_1x_2, x_2x_3, x_3x_1\}$ must be incident to a triangular face which is not incident to $v$. The number of vertices contained in the triangular region bounded by $x_1, x_2,$ and $x_3$ not containing $v$ is 2, namely $y_1$ and $y_2$. Thus, two of the edges, say $x_1x_2$ and $x_2x_3$, must use one of the vertices in $\{y_1, y_2\}$, say $y_1$, such that the $x_1x_2y_1$ and $x_2x_3y_1$ are triangular faces incident to the two edges. From the property that every face of a maximal planar graph is of size 3, necessarily $y_1$ and $y_2$ must be adjacent. Hence we obtain that the vertex $y_1$ is adjacent to every neighbor of $v$. Moreover, the edge $y_1y_2$ is the only star edge of $G$ with respect to $v$. □

Now we proceed in proving Claim 2. Deleting the vertex $v$, we get a maximal planar graph on five vertices which contains 42 3-paths.

Therefore, using Claim 3 we have that the number of 3-paths which contain the vertex $v$ is at most 45, from (2) and (3). Thus $P_3(G) \leq 42 + 45 = 87$ and we have a unique extremal graph $F_6$ with 87 3-paths. □

Claim 4. \(f(7, P_3) = 147\).

Proof. Let $G$ be a maximal planar graph on seven vertices. Deleting this vertex we get a maximal planar graph with six vertices and containing at most 87 3-paths. Since the number of vertices is seven, there are at most two star edges. Therefore using (1), (2), (3), and (4), the maximum number of 3-paths containing the vertex is 60. Hence $P_3(G) \leq 147$ and equality holds if we deleted a vertex with two star edges and the graph we obtained was $F_6$. There are only two faces in $F_6$ where we can place the deleted vertex to have two star edges, in both cases we get the same graph which pictured in Figure 1B. □

Claim 5. \(f(8, P_3) = 222\).

Proof. Let $G$ be a maximal planar graph on eight vertices. After deleting the vertex $v$ from $G$, we get a seven vertex maximal planar graph containing at most 147 paths of length three. However, from (1), (2), (3), (4), and (5), the maximum number of 3-paths that contain the vertex $v$ is at most 75. Thus $P_3(G) \leq 222$ and equality holds if we have deleted a vertex with incident three star edges and the graph we got was also extremal (Figure 1B). There is a unique face of the graph in Figure 1B where we can place a deleted vertex to have three star edges. This leads us to the unique extremal graph pictured in Figure 1C. □

Claim 6. \(f(9, P_3) = 306\).

Proof. Let $G$ be a maximal planar graph on nine vertices. If there is no other vertex incident to all the vertices incident to the vertex $v$, then using (1) we have at most 82 3-paths that contain $v$. Since deleting the vertex $v$ results in an eight vertex maximal planar graph, it contains at most 222 3-paths, from Claim 5. Thus we have $P_3(G) \leq 304$.

Now assume that the neighbors of $v$ have a common adjacent vertex other than $v$. Consider the three regions obtained as in Figure 8.
(i) If each of the three regions is nonempty, then there is a unique maximal planar graph of this kind (see Figure 12). Using Lemma 1 one can compute that this planar graph contains 303 3-paths.

(ii) If two of the regions contain two vertices each, then the remaining region contains no vertex. The two nonempty regions contain a star edge. If in each of the two regions, we have a vertex which is incident to exactly one vertex of the triangle $N(v)$, then we have at most

$$4(3n - 6 - \sum_{i=1}^{3} d(x_i) - 3) - 2 - 2 + 16 + 2(n - 7) + 6 = 14n - 44 = 82$$

3-paths that contain the vertex $v$. Since deleting the vertex $v$ results in an eight vertex maximal planar graph, which contains at most 222, 3-paths, from Claim 5, we get $P_4(G) \leq 304$.

If only one of the two regions contain a vertex which is incident to exactly one vertex of the triangle, then there are only two such maximal planar graphs (see Figure 13). The number of 3-paths they contain are, respectively, 290 and 297.

If in each of the two regions there is no vertex incident to exactly one vertex of the triangle, then the planar graph is unique (see Figure 14). The number of 3-paths in this graph is 296.

(iii) Assume one of the regions contains a vertex and another contains three vertices (the third one is empty).

Suppose there is only one star edge, then the number of 3-paths that contain the vertex $v$ is at most
Since one of the vertices of the triangle will be of degree 4, after removing the vertex \( v \), we will not have the unique extremal graph in Figure 1C, since it does not contain a vertex of degree four. Thus, in this case, we have 

\[ P_4(G) < 222 + 84 = 306. \]

After removing the vertex \( v \) we will not get the unique extremal graph in Figure 1C. Thus, in this case, we have 

\[ P_4(G) < 222 + 84 = 306. \]

If there are two star edges, and there are two vertices which are incident to exactly one of the vertices of the triangle, then we have at most 

\[
4 \sum_{i=1}^{3} d(x_i) - 24 + 4 \left( 3n - 6 - \left( \sum_{i=1}^{3} d(x_i) - 3 \right) - 1 - 1 \right) + 8 \\
+ 2(n - 6) + 6 = 14n - 42 = 84.
\]

3-paths containing the vertex \( v \). Therefore 

\[ P_4(G) \leq 304. \]

If there are two star edges, and there are at least three vertices incident to two of the vertices of the triangle, then Figure 15 shows all possible nine vertex planar graphs. There are 300, 289, 292, 299, and 302, 3-paths in those graphs, respectively.

(iv) Assume all four vertices are in the same region.

Suppose there is no star edge, then we have at most
\[
4 \sum_{i=1}^{3} d(x_i) - 24 + 4 \left(3n - 6 - \left(\sum_{i=1}^{3} d(x_i) - 3\right)\right) + 2(n - 6) + 6
= 14n - 42 = 84
\]

3-paths containing the vertex \(v\). After removing the vertex \(v\) we will not get the unique extremal graph in Figure 1C, since for each face of the graph in Figure 1C has a star edge. Thus, in this case, we have \(P_4(G) < 222 + 84 = 306\).

Suppose there is a star edge and there is exactly one vertex which is not incident to two of the vertices of the triangle. Then that vertex must be incident to a vertex of the triangle. That vertex of the triangle has degree 8, therefore after deleting the vertex \(v\), we will get a vertex of degree 7. Since the graph in Figure 1C does not contain a vertex of degree 7, the number of 3-paths not containing \(v\) is at most 221. The number of 3-paths containing the vertex \(v\) is at most

\[
4 \sum_{i=1}^{3} d(x_i) - 24 + 4 \left(3n - 6 - \left(\sum_{i=1}^{3} d(x_i) - 3\right) - 1 - 1\right) + 5 + 3
+ 2(n - 6) + 6 = 14n - 42 = 84.
\]

Thus \(P_4(G) < 306\).

Suppose there is a star edge and there is more than one vertex which is not incident to two of the vertices of the triangle. Then the number of 3-paths containing the vertex \(v\) is at most

\[
4 \sum_{i=1}^{3} d(x_i) - 24 + 4 \left(3n - 6 - \left(\sum_{i=1}^{3} d(x_i) - 3\right) - 1 - 2\right) + 5 + 6
+ 2(n - 7) + 6 = 14n - 42 = 81.
\]

Thus \(P_4(G) < 306\), since after deleting the vertex \(v\) we get a maximal planar graph on eight vertices and \(f(8, P_4) = 222\).

Finally, if there is a star edge and all four vertices are incident to two of the vertices of the triangle, then the maximal planar graph is uniquely defined, see Figure 16 which is \(F_9\). It contains 306 paths of length three. Therefore \(f(9, P_4) = 306\), and the unique extremal planar graph on nine vertices is \(F_9\).

**Figure 16** A maximal planar graph with nine vertices, containing maximum number of 3-paths.
So far we have determined \( f(n, P_4) \) for all integers \( n \). We also have proven that for all \( n, n < 10 \), the planar graph maximizing number of \( P_4 \)'s is unique. Even more we have shown that the unique extremal graph is \( F_9 \), for \( n = 9 \).

In the remaining part of this section, we are going to show that for all \( n, n \geq 9 \), the only planar graph maximizing the number of 3-paths is \( F_n \). For this we are going to use a proof by induction on the number of vertices. The base case for \( n = 9 \) is complete. Let us assume that \( G \) is an \( n, n \geq 10 \), vertex graph with \( f(n, P_4) \) 3-paths, then we are going to show that \( G = F_n \) under the assumption that the only extremal planar graph with \( (n - 1) \) vertices is \( F_{n-1} \). From the proof of the upper bound, we know that to have \( f(n, P_4) \) paths of length three, we have one of two possibilities as outlined in Remark 1.

From Lemma 2 we have the minimum degree of \( G \) is 3, we also have that for any vertex of degree three, say \( v \), all other vertices share at least two neighbors with \( v \). After removing the vertex \( v \), we obtain the unique extremal graph \( F_{n-1} \) in this case. Therefore there are only two such faces in \( F_{n-1} \), namely the faces with two high degree vertices and a vertex of degree three (the outer face and bottom face from Figure 1A). In both settings, after placing \( v \) in the proper face and adding all three edges, we obtain the graph \( F_n \). Therefore we have the desired result \( G = F_n \). □

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