CONDITIONS FOR THE EXISTENCE OF POSITIVE RADIAL SOLUTIONS FOR A CLASS OF QUASILINEAR SYSTEMS

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ABSTRACT. By using a monotone iterative scheme and Arzela-Ascoli theorem, we show the existence of positive radial solutions to the quasilinear systems

\[
\begin{align*}
\Delta_{\phi_1} u &= a_1(|x|)f_1(v), \quad x \in \mathbb{R}^N \ (N \geq 3), \\
\Delta_{\phi_2} v &= a_2(|x|)f_2(u), \quad x \in \mathbb{R}^N \ (N \geq 3),
\end{align*}
\]

under appropriate conditions on the functions \(\phi_1, \phi_2\), the weights \(a_1, a_2\) and to the nonlinearities \(f_1, f_2\). Moreover, we obtain a number of qualitative results concerning the behavior of solutions. We also point that the functions \(\phi_1\) and \(\phi_2\) includes special cases appearing in mathematical models in nonlinear elasticity, plasticity, generalized Newtonian fluids, and in quantum physics.

1. Introduction and Statement of the Main Results

In this paper, we are concerned with the existence of nonnegative solutions for a quasilinear system of the type

\[
\begin{align*}
\Delta_{\phi_1} u &= a_1(|x|)f_1(v), \quad x \in \mathbb{R}^N \ (N \geq 3), \\
\Delta_{\phi_2} v &= a_2(|x|)f_2(u), \quad x \in \mathbb{R}^N \ (N \geq 3),
\end{align*}
\] (1.1)

where \(\Delta_{\phi_i} u (i = 1, 2)\) stands for the \(\phi_i\)-Laplacian operator defined as \(\Delta_{\phi_i} u := \text{div}(\phi_i(|\nabla u|)\nabla u)\) and the \(C^1\)-functions \(\phi_1\) and \(\phi_2\) satisfy throughout the paper the following conditions:

\begin{itemize}
  \item \(\phi_i \in C^1((0, \infty), (0, \infty))\) and \(\lim\limits_{t \to 0} t\phi_i(t) = 0\);
  \item \(t\phi_i(t) > 0\) is strictly increasing for \(t > 0\);
  \item there exist positive constants \(k_i, \overline{k}_i\), the continuous and increasing functions \(\underline{\theta}_i, \overline{\theta}_i : [0, \infty) \to [0, \infty)\) and the continuous functions \(\underline{\psi}_i, \overline{\psi}_i : [0, \infty) \to [0, \infty)\) such that
\end{itemize}

\[
k_i \underline{\theta}_i(s_1)\underline{\psi}_i(s_2) \leq h_i^{-1}(s_1s_2) \leq \overline{k}_i \overline{\theta}_i(s_1)\overline{\psi}_i(s_2) \text{ for all } s_1, s_2 > 0,
\] (1.2)

where \(h_i^{-1}\) is the inverse function of \(h_i(t) = t\phi_i(t)\) for \(t > 0\).

The motivation for the present work stems from recent investigations of the authors [24], [12, 13]. We give a quick review here. Lair, [12] has considered...
entire large radial solutions for the elliptic system
\[
\begin{aligned}
\Delta u &= a_1 (|x|) v^\alpha, \\
\Delta v &= a_2 (|x|) u^\beta, \quad x \in \mathbb{R}^N \ (N \geq 3),
\end{aligned}
\] (1.3)
where \(0 < \alpha \leq 1, 0 < \beta \leq 1\), \(a_1\) and \(a_2\) are nonnegative continuous functions on \(\mathbb{R}^N\), and he proved that a necessary and sufficient condition for this system to have a nonnegative entire large radial solution (i.e., a nonnegative spherically symmetric solution \((u,v)\) on \(\mathbb{R}^N\) that satisfies \(\lim_{|x| \to \infty} u(x) = \lim_{|x| \to \infty} v(x) = \infty\)), is
\[
\int_0^\infty t a_1 (t) \left( t^{2-N} \int_0^t s^{N-3} Q (s) \, ds \right)^\alpha \, dt = \infty, \quad (1.4)
\]
\[
\int_0^\infty t a_2 (t) \left( t^{2-N} \int_0^t s^{N-3} P (s) \, ds \right)^\beta \, dt = \infty,
\] (1.5)
where \(P (r) = \int_0^r \tau a_1 (\tau) \, d\tau\) and \(Q (r) = \int_0^r \tau a_2 (\tau) \, d\tau\).

It is well known, see Yang [24], that if \(a : [0, \infty) \to [0, \infty)\) is a spherically symmetric continuous function and the nonlinearity \(f : [0, \infty) \to [0, \infty)\) is a continuous, increasing function with \(f (0) \geq 0\) and \(f (s) > 0\) for all \(s > 0\) which satisfies
\[
\int_1^\infty \frac{1}{f (t)} \, dt = \infty,
\] (1.6)
then the single equation
\[
\begin{aligned}
\Delta u &= a (|x|) f (u) \text{ for } x \in \mathbb{R}^N \ (N \geq 3), \\
\lim_{|x| \to \infty} u (|x|) &= \infty
\end{aligned}
\] (1.7)
has a nonnegative radial solution if and only if \(a\) satisfies
\[
\lim_{t \to \infty} A_a (t) = \infty, \quad A_a (t) := \int_0^t s^{N-1} \int_0^s z^{N-2} a (z) \, dz \, ds.
\]
After a simple computation, we can see that
\[
\lim_{t \to \infty} A_a (t) = \frac{1}{N-2} \int_0^\infty ra (r) \, dr.
\]
However, there is no equivalent results for systems (1.1), where \(f_1, f_2\) satisfy a condition of the form (1.6). One of the purpose of this paper is to fill this gap.

Subsequently, Lair [13] extended the result of [12] to a more general case by merely requiring \(\alpha \beta \leq 1\), and showed that if \(\alpha \beta > 1\), then (1.3) has an entire large solution if either (1.4) and (1.5) fails to hold, i.e., \(a_1\) and \(a_2\) satisfy (at least) one of the conditions
\[
\int_0^\infty t a_1 (t) \left( t^{2-N} \int_0^t s^{N-3} Q (s) \, ds \right)^\alpha \, dt < \infty, \quad (1.8)
\]
\[
\int_0^\infty t a_2 (t) \left( t^{2-N} \int_0^t s^{N-3} P (s) \, ds \right)^\beta \, dt < \infty. \quad (1.9)
\]
To summarises, if \(\alpha \beta > 1\), a sufficient condition to ensure the existence of a positive entire large solution for the system (1.3) is that \(a_1\) and \(a_2\) satisfy (1.8)
or (1.9). Therefore, it remains unknown whether this is a necessary condition. However, we know from the reference [24] that this is not true for the single equation (1.7). The second purpose of this paper is to prove that this does not happen the systems either.

Finally, we note that if \( a_1 \) and \( a_2 \) satisfy
\[
1) \quad \int_0^\infty r a_1 (r) \, dr = \infty, \quad 2) \quad \int_0^\infty r a_2 (r) \, dr = \infty,
\]
then they also satisfy both (1.4) and (1.5), and likewise, if they satisfy
\[
3) \quad \int_0^\infty r a_1 (r) \, dr < \infty, \quad 4) \quad \int_0^\infty r a_2 (r) \, dr < \infty,
\]
then they also satisfy (1.8) and (1.9). In both cases, however, the converse is not true. For further results, see for instance, [1, 16, 18, 19, 20] and the references therein.

In the present paper, we are interested in providing a proof to our goals for a more general class of quasilinear systems of the form (1.1). This, actually, is the third motivation of our paper since the \( \phi_1 \)-Laplacian operator appears in mathematical models in nonlinear elasticity, plasticity, generalized Newtonian fluids, and in quantum physics.

Several results concerning our goals were obtained by Gregorio [3], Franchi-Lanconelli and Serrin [5], Hamdy-Massar-Tsouli [7], Keller [8], Kon'kov [10], Jaroš-Takaši [11], Losev-Mazepa [16], Lieberman [14], Li-Zhang-Zhang [15], Luthey [17], Mazepa [18], Naito-Usami [19, 20], Osserman [21], Smooke [22], Zhang-Zhou [25] and Zhang [26].

We expect that our work, while currently focussed on a very specific problem, will lead to general insights and new methods with potential applications to a much wider class of problems.

Throughout the paper we let \( \alpha, \beta \in (0, \infty) \) be arbitrary parameters. We work under the following assumptions:

(A) \( a_1, a_2 : [0, \infty) \rightarrow [0, \infty) \) are spherically symmetric continuous functions (i.e., \( a_i (x) = a_i (|x|) \) for \( i = 1, 2 \));

(C1) \( f_1, f_2 : [0, \infty) \rightarrow [0, \infty) \) are continuous, increasing, \( f_1 (0) \cdot f_2 (0) \geq 0 \) and \( f_1 (s_1) \cdot f_2 (s_2) > 0 \) for all \( s_1, s_2 > 0 \);

(C2) there exist positive constants \( \overline{c}_1, \overline{c}_2 \), the continuous and increasing functions \( g_1, g_2 : [0, \infty) \rightarrow [0, \infty) \) and the continuous functions \( \overline{\xi}_1, \overline{\xi}_2 : [0, \infty) \rightarrow [0, \infty) \) such that
\[
\begin{align*}
&f_1 (t_1 \cdot w_1) \leq \overline{c}_1 g_1 (t_1) \cdot \overline{\xi}_1 (w_1) \quad \forall \ w_1 \geq 1 \quad \text{and} \quad \forall \ t_1 \geq M_1 \cdot \overline{g}_2 (f_2 (\alpha)) , \tag{1.12} \\
&f_2 (t_2 \cdot w_2) \leq \overline{c}_2 g_2 (t_2) \cdot \overline{\xi}_2 (w_2) \quad \forall \ w_2 \geq 1 \quad \text{and} \quad \forall \ t_2 \geq M_2 \cdot \overline{g}_1 (f_1 (\beta)) , \tag{1.13}
\end{align*}
\]
where \( M_1 \geq \max \left\{ 1, \overline{\xi}_2 \right\} \) and \( M_2 \geq \max \left\{ 1, \overline{\xi}_1 \right\} \); and

(C3) there are some constants \( \underline{c}_1, \underline{c}_2 \in (0, \infty) \) and the continuous functions \( \underline{\xi}_1, \underline{\xi}_2 : [0, \infty) \rightarrow [0, \infty) \) such that
\[
\begin{align*}
&f_1 (m_1 w_1) \geq \underline{c}_1 \underline{\xi}_1 (w_1) \quad \forall \ w_1 \geq 1, \quad \tag{1.14} \\
&f_2 (m_2 w_2) \geq \underline{c}_2 \underline{\xi}_2 (w_2) \quad \forall \ w_2 \geq 1, \quad \tag{1.15}
\end{align*}
\]
where \( m_1 \in (0, \min \{ \beta, \underline{g}_2 (f_2 (\alpha)) \} ) \) and \( m_2 \in (0, \min \{ \alpha, \underline{g}_1 (f_1 (\beta)) \} ) \).
2. Main Results

As announced we start with the formulation of our results. It is convenient to give some notations needed in the sequel. The reader may just as well glance through this paper and return to it when necessary.

\[
\mathcal{A}_i(t) = \int_0^t k_i \psi_i(s^{-N}) \int_0^s z^{N-1}a_i(z)dz ds, i = 1, 2
\]

\[
\mathcal{P}_{1,2}(r) = \int_0^r \psi_2 \left( \mathcal{C}_1 y^{1-N} \int_0^y t^{N-1}a_1(t)\xi_1 (1 + \mathcal{A}_2 (t)) dt \right) dy,
\]

\[
\mathcal{P}_{2,1}(r) = \int_0^r \psi_1 \left( \mathcal{C}_2 y^{1-N} \int_0^y t^{N-1}a_2(t)\xi_2 (1 + \mathcal{A}_1 (t)) dt \right) dy,
\]

\[
\mathcal{P}_{1,2}(\infty) = \lim_{r \to \infty} \mathcal{P}_{1,2}(r), \mathcal{P}_{2,1}(\infty) = \lim_{r \to \infty} \mathcal{P}_{2,1}(r)
\]

\[
\mathcal{A}_i(t) = \int_0^t k_i \psi_i(s^{-N}) \int_0^s z^{N-1}a_i(z)dz ds, i = 1, 2
\]

\[
\mathcal{P}_{1,2}(r) = \int_0^r h_1^{-1} \left( \mathcal{C}_1 y^{1-N} \int_0^y t^{N-1}a_1(t)\xi_1 (1 + \mathcal{A}_2 (t)) dt \right) dy,
\]

\[
\mathcal{P}_{2,1}(r) = \int_0^r h_2^{-1} \left( \mathcal{C}_2 y^{1-N} \int_0^y t^{N-1}a_2(t)\xi_2 (1 + \mathcal{A}_1 (t)) dt \right) dy,
\]

\[
\mathcal{P}_{1,2}(\infty) = \lim_{r \to \infty} \mathcal{P}_{1,2}(r), \mathcal{P}_{2,1}(\infty) = \lim_{r \to \infty} \mathcal{P}_{2,1}(r)
\]

\[
H_{1,2}(r) = \int_a^r \frac{1}{\theta_1(g_1 \left( M_1 \theta_2(f_2 (t)) \right))} dt, H_{1,2}(\infty) = \lim_{s \to \infty} H_{1,2}(s)
\]

\[
H_{2,1}(r) = \int_b^r \frac{1}{\theta_2(g_2 \left( M_2 \theta_1(f_1 (t)) \right))} dt, H_{2,1}(\infty) = \lim_{s \to \infty} H_{2,1}(s).
\]

Let us point that

\[
H'_{1,2}(r) = \frac{1}{\theta_1(g_1 \left( M_1 \theta_2(f_2 (r)) \right))} > 0 \text{ for } r > a,
\]

and

\[
H'_{2,1}(r) = \frac{1}{\theta_2(g_2 \left( M_2 \theta_1(f_1 (r)) \right))} > 0 \text{ for } r > b,
\]

and then \(H_{1,2}\) has the inverse function \(H_{1,2}^{-1}\) on \([0, H_{1,2}(\infty)]\) respectively \(H_{2,1}\) has the inverse function \(H_{2,1}^{-1}\) on \([0, H_{2,1}(\infty)]\).

Having all these notations clearly for the readers, we state the following first result:

**Theorem 2.1.** Assume that \(H_{1,2}(\infty) = H_{2,1}(\infty) = \infty\) and \((A)\), hold. Furthermore, if \(f_1\) and \(f_2\) satisfy the hypotheses \((C1)\) and \((C2)\) then the system \((1.1)\) has one positive radial solution

\[
(u, v) \in C^1([0, \infty)) \times C^1([0, \infty)) \text{ with } (u(0), v(0)) = (\alpha, \beta).
\]

If in addition, \(f_1\) and \(f_2\) satisfy the hypothesis \((C3)\), \(\mathcal{P}_{1,2}(\infty) = \infty\) and \(\mathcal{P}_{2,1}(\infty) = \infty\) then \(\lim_{r \to \infty} u(r) = \infty\) and \(\lim_{r \to \infty} v(r) = \infty\). Conversely, if \(\xi_i = \xi_i^*\)
includes all known results about the large solutions for (1.1) such that \((u(0), v(0)) = (\alpha, \beta)\), then

\[ a_1 \text{ and } a_2 \text{ satisfy } P_{1,2}(\infty) = P_{1,2}(\infty) = \infty \text{ and } P_{2,1}(\infty) = P_{2,1}(\infty) = \infty. \]

Our Theorem 2.1 includes all known results about the large solutions for (1.1) as well as all of the ‘mixed’ cases and therefore gives an answer for our first goal. Next, we are interested in the existence of entire bounded radial solutions for the system (1.1).

**Theorem 2.2.** Suppose that \(H_{1,2}(\infty) = H_{2,1}(\infty) = \infty\) and \(A\), hold. Furthermore, if \(f_1\) and \(f_2\) satisfy the hypotheses \((C1)\) and \((C2)\) then the system (1.1) has one positive radial solution

\[ (u, v) \in C^1([0, \infty)) \times C^1([0, \infty)) \text{ with } (u(0), v(0)) = (\alpha, \beta). \]

Moreover, if \(P_{1,2}(\infty) < \infty\) and \(P_{2,1}(\infty) < \infty\) then \(\lim_{r \to \infty} u(r) < \infty\) and \(\lim_{r \to \infty} v(r) < \infty\).

The next Theorem present the situation when one of the components is bounded while the other is large.

**Theorem 2.3.** Assume that \(H_{1,2}(\infty) = H_{2,1}(\infty) = \infty\) and \(A\), hold. Furthermore, if \(f_1\) and \(f_2\) satisfy the hypotheses \((C1)\) and \((C2)\) then the system (1.1) has one positive radial solution

\[ (u, v) \in C^1([0, \infty)) \times C^1([0, \infty)) \text{ with } (u(0), v(0)) = (\alpha, \beta). \]

Moreover, the following hold:

1.) If in addition, \(f_2\) satisfy the condition (1.15), \(P_{1,2}(\infty) < \infty\) and \(P_{2,1}(\infty) = \infty\) then \(\lim_{r \to \infty} u(r) < \infty\) and \(\lim_{r \to \infty} v(r) = \infty\).

2.) If in addition, \(f_1\) satisfy the condition (1.14), \(P_{1,2}(\infty) = \infty\) and \(P_{2,1}(\infty) < \infty\) then \(\lim_{r \to \infty} u(r) = \infty\) and \(\lim_{r \to \infty} v(r) < \infty\).

We now propose a more refined question concerning the solutions of system (1.1). In analogy with Theorems 2.1-2.3, we can also prove the following three theorems.

**Theorem 2.4.** Assume that the hypothesis \((A)\) holds. If \((C1)\), \((C2)\), \(P_{1,2}(\infty) < H_{1,2}(\infty) < \infty\) and \(P_{2,1}(\infty) < H_{2,1}(\infty) < \infty\) are satisfied, then the system (1.1) has one positive bounded radial solution

\[ (u, v) \in C^1([0, \infty)) \times C^1([0, \infty)) \text{ with } (u(0), v(0)) = (\alpha, \beta). \]

such that

\[
\begin{cases}
\alpha + P_{1,2}(r) \leq u(r) \leq H_{1,2}(1) P_{1,2}(r), \\
\beta + P_{2,1}(r) \leq v(r) \leq H_{2,1}(1) P_{2,1}(r).
\end{cases}
\]

**Theorem 2.5.** Assume that the hypothesis \((A)\) holds. If \((C1)\), \((C2)\), (1.14), \(H_{1,2}(\infty) = \infty\), \(P_{1,2}(\infty) = \infty\) and \(P_{2,1}(\infty) < H_{2,1}(\infty) < \infty\) are satisfied, then the system (1.1) has one positive radial solution

\[ (u, v) \in C^1([0, \infty)) \times C^1([0, \infty)) \text{ with } (u(0), v(0)) = (\alpha, \beta), \]

such that \(\lim_{r \to \infty} u(r) = \infty\) and \(\lim_{r \to \infty} v(r) < \infty\).
**Theorem 2.6.** Assume that the hypothesis (A) holds. If (C1), (C2), (1.15), $P_{2,1}(\infty) = \infty$, $H_{2,1}(\infty) = \infty$ and $\mathcal{H}_{1,2}(\infty) < H_{1,2}(\infty) < \infty$ are satisfied, then the system (1.1) has one positive radial solution

$$(u, v) \in C^1([0, \infty)) \times C^1([0, \infty))$$

with $(u(0), v(0)) = (\alpha, \beta)$, such that $\lim_{r \to \infty} u(r) < \infty$ and $\lim_{r \to \infty} v(r) = \infty$.

**Remark 2.7.** Our assumptions (O3), (C2) and (C3) are further discussed in the famous book of Krasnosel’skii and Rutickii [9] (see also Soria [23]). Moreover, the class of nonlinearities considered by Lair [12], [13] are also included.

**Remark 2.8.** (see [6, Lemma 2.1]) Suppose $\phi_i (i=1,2)$ satisfy (O1), (O2) and (O4) there exist $l_i, m_i > 1$ such that

$$l_i \leq \frac{\Phi_i' (t) \cdot t}{\Phi_i (t)} \leq m_i \ 	ext{for any} \ t > 0,$$

where $\Phi_i (t) = \int_0^t \phi_i (s) ds, t > 0$;

(O5) there exist $a_i, a_i^* > 0$ such that

$$a_i^* \leq \frac{\Phi_i'' (t) \cdot t}{\Phi_i (t)} \leq a_i \ 	ext{for any} \ t > 0.$$

Then, the assumption (1.2) holds with

$$\bar{\psi}_i = \psi_i = h_i^{-1}, \ k_i = \bar{k}_i = 1, \ \theta_i (t) = \min \{t^{1/m_i}, t^{1/l_i}\}, \ \bar{\theta}_i (t) = \max \{t^{1/m_i}, t^{1/l_i}\}.$$

We would like to point that:

**Remark 2.9** (see [6] for more information). The function $\Phi_i$ appears in a lot of physical applications, such as:

- **Nonlinear Elasticity:** $\Phi_i (t) = (1 + t^2)^p - 1$, $\phi_i (t) = 2p (1 + t^2)^{p-1}, t > 0$ and $p > \frac{1}{2}$;

- **Plasticity:** $\Phi_i (t) = t^p (\ln (1 + t))^q$, $\phi_i (t) = \frac{\ln (t + 1)}{t+1} [(pt^{p-1} + qt^{q-1}) \ln (t + 1) + qt^{p-1}], t > 0, p > 1$ and $q > 0$;

- **Generalized Newtonian fluids:** $\Phi_i (t) = \int_0^t s^{1-p} (\sinh s)^q ds$, $\phi_i (t) = t^{-p} \arcsinh^q t$;

- **Plasma Physics:** $\Phi_i (t) = \frac{v^p}{p} + \frac{v^q}{q}$, $\phi_i (t) = tp^{-2} + t^{q-2}$ where $t > 0$ and $1 < p < q$.

**Remark 2.10.** Let

$$M_1^+ = \sup_{t \in [0, \infty)} \int_0^t k_1 \psi_2 (s^{1-N}) \int_0^s z^{N-1} a_2 (z) dz ds$$

and

$$M_2^+ = \sup_{t \in [0, \infty)} \int_0^t k_1 \psi_1 (s^{1-N}) \int_0^s z^{N-1} a_1 (z) dz ds.$$

The following situations improve our theorems:

a) If $M_1^+ \in (0, \infty)$ then the condition (1.12) is not necessary but $H_{1,2} (r)$ must be replaced by

$$H_{1,2} (r) = \int_a^r \frac{1}{\bar{\theta}_1 (f_1 (M_1 \bar{\theta}_2 (f_2 (t))))} dt, M_1 \geq \max \left\{1, \frac{\beta}{\bar{\theta}_2 (f_2 (\alpha))} \right\} \cdot (1 + M_1^+),$$

(2.1)
and therefore $\overline{P}_{1,2}(r) = \int_0^r \varphi_2 \left( \varphi_1 y^{1-N} \int_0^y t^{N-1} a_1(t) dt \right) dy$.

b) If $M_2^+ \in (0, \infty)$ then the condition (1.13) is not necessary but $H_{2,1}(r)$ must be replaced by

$$H_{2,1}(r) = \int_b^r \frac{1}{\vartheta_2(f_2(M_2^+(f_1(t))))} dt, \quad M_2 \geq \max \left\{ 1, \frac{\alpha}{\vartheta_1(f_1(\beta))} \right\} \cdot (1 + M_2^+) .$$  \hfill (2.2)

and therefore $\overline{P}_{2,1}(r) = \int_0^r \varphi_2 \left( \varphi_1 y^{1-N} \int_0^y t^{N-1} a_2(t) dt \right) dy$.

c) If $M_1^+ \in (0, \infty)$ and $M_2^+ \in (0, \infty)$ then the conditions (1.12) and (1.13) are not necessary but $H_{1,2}(r)$ and $H_{2,1}(r)$ must be replaced by (2.1) and (2.2).

Here $\overline{P}_{1,2}(r)$ and $\overline{P}_{2,1}(r)$ are defined as in a), b).

d) If $m_1 \geq 1$ then $\xi_1 = 1$ and $\xi_1 = f_1$.

e) If $m_2 \geq 1$ then $\xi_2 = 1$ and $\xi_2 = f_2$.

f) If $m_1 \geq 1$ and $m_2 \geq 1$ then $\xi_1 = \xi_2 = 1$, $\xi_1 = f_1$ and $\xi_2 = f_2$.

3. PROOF OF THE MAIN RESULTS

The first important tool in our proof is a variant of the Arzelà–Ascoli Theorem.

3.1. The Arzelà–Ascoli Theorem. Let $r_1, r_2 \in \mathbb{R}$ with $r_1 \leq r_2$ and

$$(K = [r_1, r_2], d_K(x,y))$$

be a compact metric space, with the metric $d_K(x,y) = |x-y|$, and let

$$(C([r_1, r_2]) = \{ g : [r_1, r_2] \rightarrow \mathbb{R} \mid g \text{ is continuous on } [r_1, r_2] \}$$

denote the space of real valued continuous functions on $[r_1, r_2]$ and for any $g \in C([r_1, r_2])$, let

$$\|g\|_\infty = \max_{x \in [r_1, r_2]} |g(x)|$$

be the maximum norm on $C([r_1, r_2])$.

Remark 3.1. Let $g^1, g^2 \in C([r_1, r_2])$. If $d(g^1, g^2) = \|g^1 - g^2\|_\infty$ then $(C([r_1, r_2]), d)$ is a complete metric space.

Definition 3.2. We say that the sequence $\{g_n\}_{n \in \mathbb{N}}$ from $C([r_1, r_2])$ is bounded if there exists a positive constant $C < \infty$ such that $\|g_n(x)\|_\infty \leq C$ for each $x \in [r_1, r_2]$ (Equivalently: $|g_n(x)| \leq C$ for each $x \in [r_1, r_2]$ and $n \in \mathbb{N}$).

Definition 3.3. We say that the sequence $\{g_n\}_{n \in \mathbb{N}}$ from $C([r_1, r_2])$ is equicontinuous if for any given $\varepsilon > 0$, there exists a number $\delta > 0$ (which depends only on $\varepsilon$) such that

$$|g_n(x) - g_n(y)| < \varepsilon \text{ for all } n \in \mathbb{N} \quad \text{whenever } d_K(x,y) < \delta \text{ for every } x, y \in [r_1, r_2].$$

Definition 3.4. Let $\{g_n\}_{n \in \mathbb{N}}$ be a family of functions defined on $[r_1, r_2]$. The sequence $\{g_n\}_{n \in \mathbb{N}}$ converges uniformly to $g(x)$ if for every $\varepsilon > 0$ there is an $N$ (which depends only on $\varepsilon$) such that

$$|g_n(x) - g(x)| < \varepsilon \text{ for all } n > N \text{ and } x \in [r_1, r_2].$$
**Theorem 3.5** (Arzelà–Ascoli theorem). If a sequence \( \{g_n\}_{n \in \mathbb{N}} \) in \( C([r_1, r_2]) \) is bounded and equicontinuous then it has a subsequence \( \{g_{n_k}\}_{k \in \mathbb{N}} \) which converges uniformly to \( g(x) \) on \( C([r_1, r_2]) \).

3.2. **Proof of Theorems 2.1-2.3.** Radially symmetric solutions of the problem (1.1) correspond to solutions of the ordinary differential equations system

\[
\begin{align*}
\{ \begin{array}{l}
(r^{N-1} \phi_1(|u'(r)|))' &= r^{N-1}a_1(r) f_1(v(r)) \text{ on } [0, \infty), \\
(r^{N-1} \phi_2(|v'(r)|))' &= r^{N-1}a_2(r) f_2(u(r)) \text{ on } [0, \infty),
\end{array} \tag{3.1} \end{align*}
\]

subject to the initial conditions \( (u(0), v(0)) = (\alpha, \beta) \) and \( (u'(0), v'(0)) = (0, 0) \), since \( (u(r), v(r)) \) is a radially symmetric positive entire solution of the system (1.1). Integrating (3.1) from 0 to \( r \), we obtain

\[
\begin{align*}
\{ \begin{array}{l}
\phi_1(|u'(r)|)u'(r) &= \frac{1}{r^{N-1}} \int_0^r r^{N-1}a_1(r) f_1(v(s)) \, ds \text{ on } [0, \infty), \\
\phi_2(|v'(r)|)v'(r) &= \frac{1}{r^{N-1}} \int_0^r r^{N-1}a_2(r) f_2(u(s)) \, ds \text{ on } [0, \infty).
\end{array} \tag{3.2} \end{align*}
\]

Taking into account the equations (3.2), it is easy to see that \( u(r) \) is an increasing function on \([0, \infty)\) of the radial variable \( r \), and the same conclusion holds for \( v(r) \). Thus, for radial solutions of the system (3.1) we seek for solutions of the system of integral equations

\[
\begin{align*}
\{ \begin{array}{l}
u(r) &= \alpha + \int_0^r h_1^{-1}(t^{1-N} \int_0^t s^{N-1}a_1(s)f_1(v(s))ds)dt, \quad r \geq 0, \\
v(r) &= \beta + \int_0^r h_2^{-1}(t^{1-N} \int_0^t s^{N-1}a_2(s)f_2(u(s))ds)dt, \quad r \geq 0. \tag{3.3} \end{array} \end{align*}
\]

The system (3.3) can be solved by using successive approximation. We define inductively \( \{u_m\}_{m \geq 0} \) and \( \{v_m\}_{m \geq 0} \) on \([0, \infty)\) as follows

\[
\begin{align*}
\{ \begin{array}{l}
u_0(r) &= \alpha, \quad v_0(r) = \beta, \\
u_m(r) &= \alpha + \int_0^r h_1^{-1}(t^{1-N} \int_0^t s^{N-1}a_1(s)f_1(v_{m-1}(s))ds)dt, \quad r \geq 0, \\
v_m(r) &= \beta + \int_0^r h_2^{-1}(t^{1-N} \int_0^t s^{N-1}a_2(s)f_2(u_{m-1}(s))ds)dt, \quad r \geq 0. \tag{3.4} \end{array} \end{align*}
\]

Obviously, for all \( r \geq 0 \) and \( m \in \mathbb{N} \) it holds that \( u_m(r) \geq \alpha, \ v_m(r) \geq \beta \) and \( v_0 \leq v_1 \). Our assumptions yield \( u_1(r) \leq u_2(r) \), for all \( r \geq 0 \), so \( v_1(r) \leq v_2(r) \), for all \( r \geq 0 \). Continuing on this line of reasoning, we obtain that the sequences \( \{u_m\}_m \) and \( \{v_m\}_m \) are increasing on \([0, \infty)\).
We next establish bounds for the non-decreasing sequences \( \{u_m\}_m \) and \( \{v_m\}_m \). From (3.4) we obtain the following inequalities

\[
v_m(r) = \beta + \int_0^r h_2^{-1}(t^{1-N}) \int_0^{t} s^{N-1}a_2(s)f_1(u_m(s))ds \, dt
\]

\[
\leq \beta + \int_0^r h_2^{-1}(f_2(u_m(t)))t^{1-N} \int_0^{t} z^{N-1}a_2(z)dz \, dt
\]

\[
\leq \beta + \int_0^r \kappa_2 \overline{\theta}_2(f_2(u_m(t))) \overline{\psi}_2(t^{1-N}) \int_0^{t} z^{N-1}a_2(z)dz \, dt
\]

\[
\leq \beta + \overline{\theta}_2(f_2(u_m(r))) \int_0^r \kappa_2 \overline{\theta}_2(f_2(u_m(t))) \overline{\psi}_2(t^{1-N}) \int_0^{t} z^{N-1}a_2(z)dz \, dt \quad (3.5)
\]

\[
\leq \overline{\theta}_2(f_2(u_m(r))) \left( \frac{\beta}{\overline{\theta}_2(f_2(u_m(r))} + \mathcal{A}_a_2(r) \right)
\]

\[
\leq \overline{\theta}_2(f_2(u_m(r))) \left( \frac{\beta}{\overline{\theta}_2(f_2(\alpha))} + \mathcal{A}_a_2(r) \right)
\]

\[
\leq M_1 \overline{\theta}_2(f_2(u_m(r))) (1 + \mathcal{A}_a_2(r))
\]

and, in the same vein

\[
u_m(r) = \alpha + \int_0^r h_1^{-1}(t^{1-N}) \int_0^{t} s^{N-1}a_1(s)f_1(v_m-1(s))ds \, dt
\]

\[
\leq \alpha + \int_0^r h_1^{-1}(t^{1-N}) \int_0^{t} s^{N-1}a_1(s)f_1(v_m(s))ds \, dt \quad (3.6)
\]

\[
\leq M_2 \overline{\theta}_1(f_1(v_m(r))) (1 + \mathcal{A}_a_1(r)).
\]

Moreover, using (3.5), by an elementary computation it follows that

\[
u'_m(r) \leq h_1^{-1} \left( r^{1-N} \int_0^r s^{N-1}a_1(s)f_1(v_m(s))ds \right)
\]

\[
\leq h_1^{-1} \left( r^{1-N} \int_0^r s^{N-1}a_1(s)f_1(M_1 \overline{\theta}_2(f_2(u_m(s))) (1 + \mathcal{A}_a_2(s)))ds \right)
\]

\[
\leq h_1^{-1} \left( r^{1-N} \int_0^r s^{N-1}a_1(s)\tau_1 g_1(M_1 \overline{\theta}_2(f_2(u_m(s))) \overline{\xi}_1 (1 + \mathcal{A}_a_2(s)))ds \right)
\]

\[
\leq h_1^{-1} \left( g_1(M_1 \overline{\theta}_2(f_2(u_m(r)))) \overline{\tau}_1 r^{1-N} \int_0^r s^{N-1}a_1(s)\overline{\xi}_1 (1 + \mathcal{A}_a_2(s))ds \right)
\]

\[
\leq \kappa_1 \overline{\theta}_1(g_1(M_1 \overline{\theta}_2(f_2(u_m(r)))) \overline{\psi}_1 \left( \overline{\tau}_1 r^{1-N} \int_0^r s^{N-1}a_1(s)\overline{\xi}_1 (1 + \mathcal{A}_a_2(s))ds \right)
\]

Arguing as above, but now with the second inequality (3.6), one can show that

\[
v'_m(r) = h_2^{-1} \left( r^{1-N} \int_0^r s^{N-1}a_2(s)f_1(u_{m-1}(s))ds \right) \quad (3.8)
\]

\[
\leq \kappa_2 \overline{\theta}_2(g_2(M_2 \overline{\theta}_1(f_1(v_m(r)))) \overline{\psi}_1 \left( \kappa_2 r^{1-N} \int_0^r s^{N-1}a_2(s)\overline{\xi}_2 (1 + \mathcal{A}_a_1(s))ds \right)
\]
Combining the previous relations (3.7) and (3.8), we further obtain
\[
\frac{(u_1^m (r))'}{\theta_1 (g_1 (M_1 \theta_2 (f_2 (u_m (r)))))} \leq k_1 \psi_2 \left( c_1 r^{1-N} \int_0^r s^{N-1} a_1 (s) \xi_1 (1 + A_{a_2} (s)) \, ds \right)
\]
\[
\frac{(u_2^m (r))'}{\theta_2 (g_2 (M_2 \theta_1 (f_1 (v_m (r)))))} \leq k_2 \psi_2 \left( c_2 r^{1-N} \int_0^r s^{N-1} a_2 (s) \xi_2 (1 + A_{a_1} (s)) \, ds \right)
\]

Integrating the inequalities (3.9) and (3.10) from 0 to \( r \), yields that
\[
\int_a^{u_m (r)} \frac{1}{\theta_1 (g_1 (M_1 \theta_2 (f_2 (t))))} \, dt \leq \mathcal{P}_{1,2} (r) , \quad \int_b^{v_m (r)} \frac{1}{\theta_2 (g_2 (M_2 \theta_1 (f_1 (t))))} \, dt \leq \mathcal{P}_{2,1} (r).
\]
(3.11)

Also, going back to the setting of \( H_{1,2} \) and \( H_{2,1} \), we rewrite (3.11) as
\[
H_{1,2} (u_m (r)) \leq k_1 \mathcal{P}_{1,2} (r) \quad \text{and} \quad H_{2,1} (v_m (r)) \leq k_2 \mathcal{P}_{2,1} (r),
\]
(3.12)
which plays a basic role in the proof of our main results. Since \( H_{1,2} \) (resp. \( H_{2,1} \)) is a bijection with the inverse function \( H_{1,2}^{-1} \) (resp. \( H_{2,1}^{-1} \)) strictly increasing on \([0, \infty)\), the inequalities (3.12) can be reformulated as
\[
u_m (r) \leq H_{1,2}^{-1} (k_1 \mathcal{P}_{1,2} (r)) \quad \text{and} \quad v_m (r) \leq H_{2,1}^{-1} (k_2 \mathcal{P}_{2,1} (r)).
\]
(3.13)
So, we have found upper bounds for \( \{u_m (r)\}_{m \geq 1} \) and \( \{v_m (r)\}_{m \geq 1} \) which are dependent of \( r \). We point to the reader that the corresponding estimates (3.13) are sometimes essential.

Next we prove that the sequences \( \{u_m (r)\}_{m \geq 1} \) and \( \{v_m (r)\}_{m \geq 1} \) are bounded and equicontinuous on \([0, c_0]\) for arbitrary \( c_0 > 0 \). To do this, we take
\[
C_1 = H_{1,2}^{-1} (k_1 \mathcal{P}_{1,2} (c_0)) \quad \text{and} \quad C_2 = H_{2,1}^{-1} (k_2 \mathcal{P}_{2,1} (c_0))
\]
and since \( (u_m (r))' \geq 0 \) and \( (v_m (r))' \geq 0 \) it follows that
\[
u_m (r) \leq u_m (c_0) \leq C_1 \quad \text{and} \quad v_m (r) \leq v_m (c_0) \leq C_2.
\]
We have proved that \( \{u_m (r)\}_{m \geq 1} \) and \( \{v_m (r)\}_{m \geq 1} \) are bounded on \([0, c_0]\) for arbitrary \( c_0 > 0 \). Using this fact in (3.7) and (3.8) we show that the same is true for \( (u_m (r))' \) and \( (v_m (r))' \). By construction we verify that
\[
u'_m (r) = h^{-1}_1 \left( r^{1-N} \int_0^r s^{N-1} a_1 (s) f_1 (v_{m-1} (s)) \, ds \right)
\]
\[
\leq h^{-1}_1 \left( r^{1-N} \int_0^r s^{N-1} a_1 (s) f_1 (v_m (s)) \, ds \right)
\]
\[
\leq h^{-1}_1 \left( \int_0^r a_1 (s) f_1 (v_{m-1} (s)) \, ds \right)
\]
\[
\leq h^{-1}_1 \left( \|a_1\|_{\infty} \int_0^r f_1 (v_{m-1} (s)) \, ds \right)
\]
\[
\leq h^{-1}_1 \left( \|a_1\|_{\infty} f_1 (C_2) \int_0^r ds \right)
\]
\[
\leq h^{-1}_1 (\|a_1\|_{\infty} f_1 (C_2) c_0) \quad \text{on} \quad [0, c_0].
\]
We follow the argument used in (3.14) to obtain
\[(v_m(r))' \leq h_2^{-1}(\|a_2\|_\infty f_2(C_1)c_0) \text{ on } [0, c_0].\]

Summarizing, we have found that
\[
\begin{aligned}
(u_1^m(r))' &\leq h_1^{-1}(\|a_1\|_\infty f_1(C_2)c_0) \text{ on } [0, c_0], \\
(v_m(r))' &\leq h_2^{-1}(\|a_2\|_\infty f_2(C_1)c_0) \text{ on } [0, c_0].
\end{aligned}
\]

Finally, it remains to prove that \(\{u_m(r)\}_{m \geq 1}\) and \(\{v_m(r)\}_{m \geq 1}\) are equicontinuous on \([0, c_0]\) for arbitrary \(c_0 > 0\). Let \(\varepsilon_1, \varepsilon_2 > 0\) be arbitrary. To verify equicontinuity on \([0, c_0]\) observe that the mean value theorem yields
\[
\begin{aligned}
|u_m(x) - u_m(y)| &= |(u_m(\zeta_1))'| |x - y| \leq h_1^{-1}(\|a_1\|_\infty f_1(C_2)c_0) |x - y| , \\
|v_m(x) - v_m(y)| &= |(v_m(\zeta_2))'| |x - y| \leq h_2^{-1}(\|a_2\|_\infty f_2(C_1)c_0) |x - y| ,
\end{aligned}
\]
for all \(n \in \mathbb{N}\) and all \(x, y \in [0, c_0]\) and for some \(\zeta_1, \zeta_2\). Then it suffices to take
\[
\delta_1 = \frac{\varepsilon_1}{h_1^{-1}(\|a_1\|_\infty f_1(C_2)c_0)} \quad \text{and} \quad \delta_2 = \frac{\varepsilon_2}{h_2^{-1}(\|a_2\|_\infty f_2(C_1)c_0)}
\]
to see that \(\{u_m(r)\}_{m \geq 1}\) and \(\{v_m(r)\}_{m \geq 1}\) are equicontinuous on \([0, c_0]\).

Since \(\{u_m(r)\}_{m \geq 1}\) and \(\{v_m(r)\}_{m \geq 1}\) are bounded and equicontinuous on \([0, c_0]\) we can apply the Arzelà–Ascoli theorem with \([r_1, r_2] = [0, c_0]\). Thus, there exists a subsequence, denoted \(\{(u_{m_1}(r), v_{m_1}(r))\}\) that converges uniformly on \([0, 1]\) \(\times\) \([0, 1]\). Let
\[
(u_{m_1}(r), v_{m_1}(r)) \xrightarrow{(m_1, m_1')} (u_1(r), v_1(r)) \text{ uniformly on } [0, 1].
\]
Likewise, the subsequence \(\{(u_{m_1}(r), v_{m_1}(r))\}\) is bounded and equicontinuous on the interval \([0, 2]\). Hence, it must contain a convergent subsequence
\[
\{(u_{m_2}(r), v_{m_2}(r))\},
\]
that converges uniformly on \([0, 2] \times [0, 2]\). Let
\[
(u_{m_2}(r), v_{m_2}(r)) \xrightarrow{(m_2, m_2')} (u_2(r), v_2(r)) \text{ uniformly on } [0, 2] \times [0, 2].
\]
Note that
\[
\{u_{m_2}(r)\} \subseteq \{u_{m_1}(r)\} \subseteq \{u_m(r)\}_{m \geq 2} \quad \text{and} \quad \{v_{m_2}(r)\} \subseteq \{v_{m_1}(r)\} \subseteq \{v_m(r)\}_{m \geq 2}.
\]
These imply
\[
u_2(r) = u_1(r) \quad \text{and} \quad v_2(r) = v_1(r) \text{ on } [0, 1].
\]
Proceeding in this fashion we obtain a countable collection of subsequences such that
\[
\{u_{m_1}\} \subseteq \ldots \subseteq \{u_{m_1}(r)\} \subseteq \{u_m(r)\}_{m \geq n}
\]
and
\[
\{v_{m_1}\} \subseteq \ldots \subseteq \{v_{m_1}(r)\} \subseteq \{v_m(r)\}_{m \geq n}.
\]
and a sequence \( \{(u_n(r), v_n(r))\} \) such that
\[
(u_n(r), v_n(r)) \in C[0, n] \times C[0, n] \quad \text{for} \quad n = 1, 2, 3, ...
\]
\[
(u_n(r), v_n(r)) = (u_1(r), v_1(r)) \quad \text{for} \quad r \in [0, 1]
\]
\[
(u_n(r), v_n(r)) = (u_2(r), v_2(r)) \quad \text{for} \quad r \in [0, 2]
\]
\[
\ldots \quad \ldots
\]
\[
(u_n(r), v_n(r)) = (u_{n-1}(r), v_{n-1}(r)) \quad \text{for} \quad r \in [0, n - 1].
\]
Together, these observations show that there exists a sequence \( \{(u_n(r), v_n(r))\} \) that converges to \((u(r), v(r))\) on \([0, \infty)\) satisfying
\[
(u_n(r), v_n(r)) = (u(r), v(r)) \quad \text{if} \quad 0 \leq r \leq n.
\]
This convergence is uniformly on bounded intervals, implying \((u(r), v(r)) \in C[0, \infty) \times C[0, \infty), and moreover, the family \( \{(u_n(r), v_n(r))\} \) is also equicontinuous. The solution \((u(r), v(r))\) constructed in this way is radially symmetric.

Going back to the system (3.1), the radial solutions of (1.1) are the solutions of the ordinary differential equations system (3.1). We conclude that radial solutions of (1.1) with \(u(0) = \alpha, v(0) = \beta\) satisfy:
\[
u(r) = \alpha + \int_0^r h_1^{-1}(t^{1-N}) \int_0^t s^{N-1} a_1(s) f_1(v(s)) ds dt, \quad r \geq 0, \quad (3.15)
\]
\[
v(r) = \beta + \int_0^r h_2^{-1}(t^{1-N}) \int_0^t s^{N-1} a_2(s) f_2(u(s)) ds dt, \quad r \geq 0. \quad (3.16)
\]
We are now ready to give a complete proof of the Theorems 2.1-2.3.

3.2.1. **Proof of Theorem 2.1 completed:** From (3.16) we obtain the following inequalities
\[
v(r) = \beta + \int_0^r h_2^{-1}(t^{1-N}) \int_0^t s^{N-1} a_2(s) f_2(u(s)) ds dt
\]
\[
\geq \beta + \int_0^r h_2^{-1}(f_2(\alpha) z^{1-N}) \int_0^z s^{N-1} a_2(s) ds dz
\]
\[
\geq \beta + \mathcal{A}_2(f_2(\alpha)) \mathcal{A}_2(r)
\]
\[
\geq m_1(1 + \mathcal{A}_2(r)),
\]
and, in the same vein
\[
u(r) = \alpha + \int_0^r h_1^{-1}(t^{1-N}) \int_0^t s^{N-1} a_1(s) f_1(v(s)) ds dt
\]
\[
\geq m_2(1 + \mathcal{A}_1(r)).
If \( P_{1,2}(\infty) = P_{2,1}(\infty) = \infty \), we observe that
\[
\begin{align*}
  u(r) &= \alpha + \int_0^r h_1^{-1}(t^{1-N}) \int_0^t s^{N-1} a_1(s) f_1(v(s)) ds dt \\
  & \geq \alpha + \int_0^r h_1^{-1} \left( y^{1-N} \int_0^y t^{N-1} a_1(t) f_1 \left( m_1(1 + A_{a_2}(t)) \right) dt \right) dy \\
  & \geq \alpha + \int_0^r h_1^{-1} \left( \xi_1 y^{1-N} \int_0^y t^{N-1} a_1(t) \xi_1 \left( 1 + A_{a_2}(t) \right) dt \right) dy \\
  & = \alpha + P_{1,2}(r).
\end{align*}
\]
Analogously, we refine the strategy above to prove:
\[
\begin{align*}
  v(r) & \geq \beta + \int_0^r h_2^{-1} \left( \xi_2 y^{1-N} \int_0^y t^{N-1} a_2(t) \xi_2 \left( 1 + A_{a_1}(t) \right) dt \right) dy \\
  & = \beta + P_{2,1}(r),
\end{align*}
\]
and passing to the limit as \( r \to \infty \) in (3.17) and in the above inequality we conclude that
\[
\lim_{r \to \infty} u(r) = \lim_{r \to \infty} v(r) = \infty,
\]
which yields the result. In order to prove the converse let \((u, v)\) be an entire large radial solution of (1.1) such that \((u, v) = (\alpha, \beta)\). Then, \((u, v)\) satisfy
\[
\begin{align*}
  u(r) &= \alpha + \int_0^r h_1^{-1} \left( t^{1-N} \int_0^t s^{N-1} a_1(s) f_1(v(s)) ds \right) dt, \ r \geq 0, \\
  v(r) &= \beta + \int_0^r h_2^{-1} \left( t^{1-N} \int_0^t s^{N-1} a_2(s) f_2(u(s)) ds \right) dt, \ r \geq 0,
\end{align*}
\]
and, so
\[
H_{1,2}(u(r)) \leq k_1 P_{1,2}(r) \quad \text{and} \quad H_{2,1}(v(r)) \leq k_2 P_{2,1}(r). \tag{3.18}
\]
By passing to the limit as \( r \to \infty \) in (3.18) we find that \( a_1 \) and \( a_2 \) satisfy
\[
\bar{P}_{1,2}(\infty) = \bar{P}_{2,1}(\infty) = \infty,
\]
since \((u, v)\) is large and \( H_{1,2}(\infty) = H_{2,1}(\infty) = \infty \). This completes the proof. We next consider:

3.2.2. **Proof of Theorem 2.2 completed:** If \( \bar{P}_{1,2}(\infty) < \infty \) and \( \bar{P}_{2,1}(\infty) < \infty \), then using the same arguments as in (3.15) and (3.16) we can see that
\[
\begin{align*}
  u(r) \leq H_{1,2}^{-1} \left( k_1 \bar{P}_{1,2}(\infty) \right) < \infty \quad \text{and} \quad v(r) \leq H_{2,1}^{-1} \left( k_2 \bar{P}_{2,1}(\infty) \right) < \infty
\end{align*}
\]
for all \( r \geq 0 \). Hence \((u, v)\) is bounded and this completes the proof.

3.2.3. **Proof of Theorem 2.3 completed: Case 1:** By an analysis similar to the Theorems 2.1 and 2.3 above, we have that
\[
\begin{align*}
  u(r) & \leq H_{1,2}^{-1} \left( k_1 \bar{P}_{1,2}(\infty) \right) < \infty \quad \text{and} \quad v(r) \geq b + k_2 \bar{P}_{2,1}(r).
\end{align*}
\]
So, if
\[
\bar{P}_{1,2}(\infty) < \infty \quad \text{and} \quad \bar{P}_{2,1}(\infty) = \infty
\]
then
\[ \lim_{r \to \infty} u(r) < \infty \text{ and } \lim_{r \to \infty} v(r) = \infty. \]

In order, to complete the proofs it remains to proceed to the

**Case 2):** In this case, we invoke the proof of Theorem 2.2. An easy computation yields that
\[ u(r) \geq \alpha + \mathcal{K}_1 P_{1,2}(r) \text{ and } v(r) \leq H_{2,1}^{-1}(\mathcal{K}_2 P_{2,1}(r)). \]

Our conclusion follows by letting \( r \to \infty \) in (3.19).

3.3. **Proof of Theorems 2.4-2.6.**

3.3.1. **Proof of Theorem 2.4 completed:** We deduce from (3.12) and the conditions of the theorem that
\[
H_{1,2}(u_m(r)) \leq \mathcal{K}_1 P_{1,2}(\infty) < \mathcal{K}_1 H_{1,2}(\infty) < \infty,
\]
\[
H_{2,1}(v_m(r)) \leq \mathcal{K}_2 P_{2,1}(\infty) < \mathcal{K}_2 H_{2,1}(\infty) < \infty.
\]

On the other hand, since \( H_{1,2}^{-1} \) and \( H_{2,1}^{-1} \) are strictly increasing on \([0, \infty)\), we find out that
\[ u_m(r) \leq H_{1,2}^{-1}(\mathcal{K}_1 P_{1,2}(\infty)) < \infty \text{ and } v_m(r) \leq H_{2,1}^{-1}(\mathcal{K}_2 P_{2,1}(\infty)) < \infty, \]
and then the non-decreasing sequences \( \{u_m(r)\}_{m \geq 1} \) and \( \{v_m(r)\}_{m \geq 1} \) are bounded above for all \( r \geq 0 \) and all \( m \). Putting these two facts together yields
\[ (u_m(r), v_m(r)) \to (u(r), v(r)) \text{ as } m \to \infty \]
and the limit functions \( u \) and \( v \) are positive entire bounded radial solutions of system (1.1).

3.3.2. **Proof of Theorem 2.5 and 2.6 completed:** It is a straightforward adaptation of the above proofs.

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