Interior capacities of condensers with infinitely many plates in a locally compact space

Natalia Zorii

Abstract. The study deals with the theory of interior capacities of condensers in a locally compact space, a condenser being treated here as a countable, locally finite collection of arbitrary sets with the sign +1 or −1 prescribed such that the closures of opposite-signed sets are mutually disjoint. We are motivated by the known fact that, in the noncompact case, the main minimum-problem of the theory is in general unsolvable, and this occurs even under very natural assumptions (e.g., for the Newtonian, Green, or Riesz kernels in \( \mathbb{R}^n \), \( n \geq 2 \), and closed condensers of finitely many plates). Therefore it was particularly interesting to find statements of variational problems dual to the main minimum-problem (and hence providing some new equivalent definitions of the capacity), but now always solvable (e.g., even for nonclosed, unbounded condensers of infinitely many plates). For all positive definite kernels satisfying B. Fuglede’s condition of consistency between the strong and the vague (= weak*) topologies, problems with the desired properties are posed and solved. Their solutions provide a natural generalization of the well-known notion of interior capacitary distributions associated with a set. We give a description of those solutions, establish statements on their uniqueness and continuity, and point out their characteristic properties.

Mathematics Subject Classification (2000): 31C15.

Key words: Minimal energy problems, interior capacities of condensers, interior capacitary distributions associated with a condenser, consistent kernels, completeness theorem for signed Radon measures.

1. Introduction

The present work is devoted to further development of the theory of interior capacities of condensers in a locally compact space. A condenser will be treated here as a countable, locally finite collection of arbitrary (noncompact or even nonclosed) sets with the sign +1 or −1 prescribed such that the closures of opposite-signed sets are mutually disjoint. For a background of the theory for condensers of finitely many plates we refer the reader to [Z1]–[Z6]; see also [O], where the condensers were additionally assumed to be compact.

The reader is expected to be familiar with the principal notions and results of the theory of measures and integration on a locally compact space; its exposition can be found in [B2, E2] (see also [F1, Z2] for a brief survey).

The theory of interior capacities of condensers provides a natural extension of the well-known theory of interior capacities of sets, developed by H. Cartan [C] and Vallée-Poussin [VP] for classical kernels in \( \mathbb{R}^n \) and later on generalized by B. Fuglede [F1] for general kernels in a locally compact space \( X \). However, those
two theories — for sets and, on the other hand, condensers — are drastically
different. To illustrate this, it is enough to note that, in the noncompact case,
the main minimum-problem of the theory of interior capacities of condensers is
in general unsolvable, and this phenomenon occurs even under very natural as-
sumptions (e.g., for the Newtonian, Green, or Riesz kernels in \( \mathbb{R}^n \), \( n \geq 2 \), and
closed condensers of finitely many plates); compare with [C, F1]. Necessary and
sufficient conditions for the problem to be solvable have been given in [Z3, Z5];
see Sec. 5.1 below for a brief survey.

Therefore it was particularly interesting to find statements of variational prob-
lems dual to the main minimum-problem of the theory of interior capacities of
condensers, but in contrast to the last one, now always solvable — e.g., even for
nonclosed, unbounded condensers of infinitely many plates. (When speaking on
duality of variational problems, we mean their extremal values to be equal.)

In all that follows, \( X \) denotes a locally compact Hausdorff space, and \( \mathcal{M} = \mathcal{M}(X) \)
the linear space of all real-valued Radon measures \( \nu \) on \( X \) equipped with the
vague (= weak\( ^* \)) topology, i.e., the topology of pointwise convergence on the
class \( C_0(X) \) of all real-valued continuous functions on \( X \) with compact support.

A kernel \( \kappa \) on \( X \) is meant to be a lower semicontinuous function \( \kappa : X \times X \to (-\infty, \infty] \). In order to avoid certain difficulties, we follow [F1] in assuming that
\( \kappa \geq 0 \) unless the space \( X \) is compact.

The energy and the potential of a measure \( \nu \in \mathcal{M} \) with respect to a kernel \( \kappa \) are
defined by

\[
\kappa(\nu, \nu) := \int \kappa(x, y) \, d(\nu \otimes \nu)(x, y)
\]

and

\[
\kappa(x, \nu) := \int \kappa(x, y) \, d\nu(y), \quad x \in X,
\]

respectively, provided the corresponding integral above is well defined (as a finite
number or \( \pm \infty \)). Let \( \mathcal{E} \) denote the set of all \( \nu \in \mathcal{M} \) with \( -\infty < \kappa(\nu, \nu) < \infty \).

In the present study we shall be concerned with minimal energy problems over
certain subclasses of \( \mathcal{E} \), properly chosen. For all positive definite kernels satisfying
B. Fuglede’s condition of consistency between the strong and the vague topologies
on \( \mathcal{E} \) (see Sec. 2 below), those variational problems are shown to be dual to
the main minimum-problem of the theory of interior capacities of condensers
(and hence providing some new equivalent definitions of the capacity), but now
always solvable. See Theorems 2–4 and Corollaries 10, 12. Their solutions
provide a natural generalization of the well-known notion of interior capacitary
distributions associated with a set (see [F1]). We give a description of those
solutions, establish statements on their uniqueness and continuity, and point out
their characteristic properties; see Sec. 7–10. The results obtained hold true,
e.g., for the Newtonian, Green or Riesz kernels in \( \mathbb{R}^n \), \( n \geq 2 \), as well as for the
restriction of the logarithmic kernel in \( \mathbb{R}^2 \) to an open unit ball.
2. Preliminaries: topologies, consistent and perfect kernels

Recall that a measure $\nu \geq 0$ is said to be concentrated on $E$, where $E$ is a subset of $X$, if the complement $\complement E := X \setminus E$ is locally $\nu$-negligible; or, equivalently, if $E$ is $\nu$-measurable and $\nu = \nu_E$, where $\nu_E$ denotes the trace of $\nu$ upon $E$.

Let $\mathcal{M}^+(E)$ be the convex cone of all nonnegative measures concentrated on $E$, and $\mathcal{E}^+(E) := \mathcal{M}^+(E) \cap \mathcal{E}$. We also write $\mathcal{M}^+ := \mathcal{M}^+(X)$ and $\mathcal{E}^+ := \mathcal{E}^+(X)$.

From now on, the kernel under consideration is always assumed to be positive definite, which means that it is symmetric (i.e., $\kappa(x, y) = \kappa(y, x)$ for all $x, y \in X$) and the energy $\kappa(\nu, \nu)$, $\nu \in \mathcal{M}$, is nonnegative whenever defined. Then $\mathcal{E}$ is known to be a pre-Hilbert space with the scalar product

$$\kappa(\nu_1, \nu_2) := \int \kappa(x, y) d(\nu_1 \otimes \nu_2)(x, y)$$

and the seminorm $\|\nu\| := \sqrt{\kappa(\nu, \nu)}$; see [F1]. A (positive definite) kernel is called strictly positive definite if the seminorm $\|\cdot\|$ is a norm.

A measure $\nu \in \mathcal{E}$ is said to be equivalent in $\mathcal{E}$ to a given $\nu_0 \in \mathcal{E}$ if $\|\nu - \nu_0\| = 0$; the equivalence class, consisting of all those $\nu$, will be denoted by $[\nu_0]_\mathcal{E}$.

In addition to the strong topology on $\mathcal{E}$, determined by the above seminorm $\|\cdot\|$, it is often useful to consider the weak topology on $\mathcal{E}$, defined by means of the seminorms $\nu \mapsto |\kappa(\nu, \mu)|$, $\mu \in \mathcal{E}$ (see [F1]). The Cauchy-Schwarz inequality

$$|\kappa(\nu, \mu)| \leq \|\nu\| \|\mu\|, \quad \nu, \mu \in \mathcal{E},$$

implies immediately that the strong topology on $\mathcal{E}$ is finer than the weak one.

In [F1], B. Fuglede introduced the following two properties of consistency between the induced strong, weak, and vague topologies on $\mathcal{E}^+$:

(C) Every strong Cauchy net in $\mathcal{E}^+$ converges strongly to every its vague cluster point;

(CW) Every strongly bounded and vaguely convergent net in $\mathcal{E}^+$ converges weakly to the vague limit;

in [F2], the properties (C) and (CW) were shown to be equivalent.

Definition 1. Following B. Fuglede, we call a kernel $\kappa$ consistent if it satisfies either of the properties (C) and (CW), and perfect if, in addition, it is strictly positive definite.

Remark 1. One has to consider nets or filters in $\mathcal{M}^+$ instead of sequences, for the vague topology in general does not satisfy the first axiom of countability. We follow Moore’s and Smith’s theory of convergence, based on the concept of nets (see [MS]; cf. also [E2, Chap. 0] and [K, Chap. 2]). However, if a locally compact
space $X$ is metrizable and countable at infinity, then $\mathcal{M}^+$ satisfies the first axiom of countability (see [F1, Lemma 1.2.1]) and the use of nets may be avoided.

**Theorem 1** [F1]. A kernel $\kappa$ is perfect if and only if $E^+$ is strongly complete and the strong topology on $E^+$ is finer than the vague one.

**Examples.** In $\mathbb{R}^n$, $n \geq 3$, the Newtonian kernel $|x - y|^{2-n}$ is perfect [C]. So are the Riesz kernel $|x - y|^{\alpha - n}$, $0 < \alpha < n$, in $\mathbb{R}^n$, $n \geq 2$ (see [D1, D2]), and the logarithmic kernel $-\log |x - y|$ in $\mathbb{R}^2$, restricted to an open unit ball [L]. Furthermore, if $D$ is an open set in $\mathbb{R}^n$, $n \geq 2$, and its generalized Green function $g_D$ exists (see, e.g., [HK, Th. 5.24]), then the Green kernel $g_D$ is perfect as well [E1].

**Remark 2.** As is seen from Theorem 1, the concept of consistent or perfect kernels is an efficient tool in minimal energy problems over classes of nonnegative measures with finite energy. Indeed, the theory of capacities of sets has been developed in [F1] exactly for those kernels. We shall show below that this concept is still efficient in minimal energy problems over classes of signed measures associated with a condenser. This is guaranteed by a theorem on the strong completeness of proper subspaces of $E$, to be stated in Sec. 11 below.

3. Condensers of countably many plates. Measures associated with a condenser; their energies and potentials

3.1. Let $I^+$ and $I^-$ be countable (finite or infinite) disjoint sets of indices $i \in \mathbb{N}$, the latter being allowed to be empty, and let $I$ denote their union. Assume that to every $i \in I$ there corresponds a nonempty set $A_i \subset X$.

**Definition 2.** A collection $A = (A_i)_{i \in I}$ is called an $(I^+, I^-)$-condenser (or simply a condenser) in $X$ if every compact subset of $X$ might have points in common with only a finite number of $A_i$ and, moreover,

\[ A_i \cap A_j = \emptyset \quad \text{for all} \quad i \in I^+, \ j \in I^- . \tag{1} \]

The sets $A_i$, $i \in I^+$, and $A_j$, $j \in I^-$, are said to be the positive and, respectively, the negative plates of an $(I^+, I^-)$-condenser $A = (A_i)_{i \in I}$. Note that any two equal-signed plates of a condenser might intersect each other (or even coincide). Given $I^+$ and $I^-$, let $\mathcal{C} = \mathcal{C}(I^+, I^-)$ be the class of all $(I^+, I^-)$-condensers in $X$. A condenser $A \in \mathcal{C}$ is called closed or compact if all $A_i$, $i \in I$, are closed or, respectively, compact. Similarly, we call it universally measurable if all the plates are universally measurable — that is, measurable with respect to every $\nu \in \mathcal{M}^+$. Next, $A = (A_i)_{i \in I}$ is said to be finite if so is $I$.

Given $A = (A_i)_{i \in I}$, write $\overline{A} := (\overline{A_i})_{i \in I}$. Then, due to (1), $\overline{A}$ is a (closed) $(I^+, I^-)$-condenser. In the sequel, also the following notation will be required:

\[ A := \bigcup_{i \in I} A_i, \quad A^+ := \bigcup_{i \in I^+} A_i, \quad A^- := \bigcup_{i \in I^-} A_i. \]
Note that both $A^+$ and $A^-$ might be noncompact even for a compact $A$.

### 3.2. With the preceding notation, write

$$\alpha_i := \begin{cases} +1 & \text{if } i \in I^+, \\ -1 & \text{if } i \in I^- . \end{cases}$$

Given $A \in \mathcal{C}$, let $\mathcal{M}(A)$ consist of all (finite or infinite) linear combinations

$$\mu := \sum_{i \in I} \alpha_i \mu^i, \quad \text{where } \mu^i \in \mathcal{M}^+(A_i).$$

Any two $\mu_1$ and $\mu_2$ in $\mathcal{M}(A)$,

$$\mu_1 = \sum_{i \in I} \alpha_i \mu^i_1 \quad \text{and} \quad \mu_2 = \sum_{i \in I} \alpha_i \mu^i_2,$$

are regarded to be identical ($\mu_1 \equiv \mu_2$) if and only if $\mu^i_1 = \mu^i_2$ for all $i \in I$. Observe that, under the relation of identity thus defined, the following correspondence between $\mathcal{M}(A)$ and the Cartesian product $\prod_{i \in I} \mathcal{M}^+(A_i)$ is one-to-one:

$$\mathcal{M}(A) \ni \mu \mapsto (\mu^i)_{i \in I} \in \prod_{i \in I} \mathcal{M}^+(A_i).$$

We call $\mu \in \mathcal{M}(A)$ a measure associated with $A$, and $\mu^i$, $i \in I$, its $i$-coordinate.

For measures associated with a condenser, it is therefore natural to introduce the following concept of convergence, actually corresponding to the vague convergence by coordinates. Let $S$ denote a directed set of indices, and let $\mu_s$, $s \in S$, and $\mu_0$ be given elements of the class $\mathcal{M}(\mathcal{A})$.

**Definition 3.** A net $(\mu_s)_{s \in S}$ is said to converge to $\mu_0$ $\mathcal{A}$-vaguely if

$$\mu^i_s \to \mu^i_0 \quad \text{vaguely for all } i \in I.$$

Then $\mathcal{M}(\mathcal{A})$, equipped with the topology of $\mathcal{A}$-vague convergence, and the product space $\prod_{i \in I} \mathcal{M}^+(\mathcal{A}_i)$ become homeomorphic. Since the space $\mathcal{M}(X)$ is Hausdorff, so are both $\mathcal{M}(\mathcal{A})$ and $\prod_{i \in I} \mathcal{M}^+(\mathcal{A}_i)$ (see, e.g., [K, Chap. 3, Th. 5]).

Similarly, a set $\mathfrak{F} \subset \mathcal{M}(\mathcal{A})$ is called $\mathcal{A}$-vaguely bounded if all its $i$-projections are vaguely bounded — that is, if for all $\varphi \in C_0(X)$ and $i \in I$,

$$\sup_{\mu \in \mathfrak{F}} |\mu^i(\varphi)| < \infty.$$

**Lemma 1.** If $\mathfrak{F} \subset \mathcal{M}(\mathcal{A})$ is bounded and closed in the $\mathcal{A}$-vague topology, then it is $\mathcal{A}$-vaguely compact.
Proof. Since by [B2, Chap. III, §2, Prop. 9] any vaguely bounded and closed part of $\mathfrak{M}$ is vaguely compact, the lemma follows immediately from Tychonoff’s theorem on the product of compact spaces (see, e.g., [K, Chap. 5, Th. 13]). □

3.3. Fix a linear combination $\mu \in \mathfrak{M}(\mathcal{A})$. Since each compact subset of $X$ might intersect with only finite number of $A_i$, $i \in I$, for every $\varphi \in C_0(X)$ only finite number of $\mu^i(\varphi)$, $i \in I$, are nonzero. This yields that to every $\mu \in \mathfrak{M}(\mathcal{A})$ there corresponds a unique Radon measure $R\mu$ such that

$$R\mu(\varphi) = \sum_{i \in I} \alpha_i \mu^i(\varphi) \quad \text{for all } \varphi \in C_0(X);$$

its positive and negative parts in Jordan’s decomposition, $R\mu^+$ and $R\mu^-$, can be written in the form

$$R\mu^+ = \sum_{i \in I^+} \mu^i, \quad R\mu^- = \sum_{i \in I^-} \mu^i.$$

Of course, the mapping $R : \mathfrak{M}(\mathcal{A}) \to \mathfrak{M}$ thus defined is in general non-injective, i.e., one may choose $\mu' \in \mathfrak{M}(\mathcal{A})$ so that $\mu' \not\equiv \mu$, while $R\mu' = R\mu$. (It would be injective if all $A_i$, $i \in I$, were mutually disjoint.) We shall call $\mu, \mu' \in \mathfrak{M}(\mathcal{A})$ equivalent in $\mathfrak{M}(\mathcal{A})$, and write $\mu \cong \mu'$, whenever their $R$-images coincide.

Lemma 2. The $\mathcal{A}$-vague convergence of $(\mu_s)_{s \in S}$ to $\mu_0$ implies the vague convergence of $(R\mu_s)_{s \in S}$ to $R\mu_0$.

Proof. This is obvious in view of the fact that the support of any $\varphi \in C_0(X)$ may have points in common with only a finite number of $\overline{A_i}$, $i \in I$.

Remark 3. The statement of Lemma 2 in general can not be inverted. However, if all $\overline{A_i}$, $i \in I$, are mutually disjoint, then the vague convergence of $(R\mu_s)_{s \in S}$ to $R\mu_0$ implies the $\mathcal{A}$-vague convergence of $(\mu_s)_{s \in S}$ to $\mu_0$. This can be seen by using the Tietze-Urysohn extension theorem (see, e.g., [E2 Th. 0.2.13]).

3.4. We next proceed to define energies and potentials of $\mu \in \mathfrak{M}(\mathcal{A})$. A proper definition is based on the mapping $R : \mathfrak{M}(\mathcal{A}) \to \mathfrak{M}$ and the following assertion.

Lemma 3. Fix $\mu \in \mathfrak{M}(\mathcal{A})$ and a lower semicontinuous function $\psi$ on $X$ such that $\psi \geq 0$ unless $X$ is compact. If the integral $\int \psi dR\mu$ is well defined, then

$$\int \psi dR\mu = \sum_{i \in I} \alpha_i \int \psi d\mu^i,$$

and it is finite if and only if the series on the right is absolutely convergent.

Proof. We can certainly assume $\psi$ to be nonnegative, for if not, we replace $\psi$ by a function $\psi'$ obtained by adding to $\psi$ a suitable constant $c > 0$:

$$\psi'(x) := \psi(x) + c \geq 0,$$
which is always possible since a lower semicontinuous function is bounded from below on a compact space. Hence, for every \( N \in \mathbb{N} \),
\[
\int \psi \, dR\mu^+ \geq \sum_{i \in I^+, \ i \leq N} \int \psi \, d\mu^i.
\]

On the other hand, the sum of \( \mu^i \) over all \( i \in I^+ \) that do not exceed \( N \) approaches \( R\mu^+ \) vaguely as \( N \to \infty \); consequently (see, e.g., [F1])
\[
\int \psi \, dR\mu^+ \leq \lim_{N \to \infty} \sum_{i \in I^+, \ i \leq N} \int \psi \, d\mu^i.
\]

Combining the last two inequalities and then letting \( N \) tend to \( \infty \) yields
\[
\int \psi \, dR\mu^+ = \sum_{i \in I^+} \int \psi \, d\mu^i.
\]

Since the same holds true for \( R\mu^- \) and \( I^- \) instead of \( R\mu^+ \) and \( I^+ \), respectively, the lemma follows. \( \square \)

**Corollary 1.** Given \( \mu, \mu_1 \in \mathcal{M}(\mathcal{A}) \) and \( x \in X \), then
\[
\kappa(x, R\mu) = \sum_{i \in I} \alpha_i \kappa(x, \mu^i),
\]
(3)
\[
\kappa(R\mu, R\mu_1) = \sum_{i,j \in I} \alpha_i \alpha_j \kappa(\mu^i, \mu_1^j),
\]
(4)
each of the identities being understood in the sense that its right-hand side is well defined whenever so is the left-hand one and then they coincide. Furthermore, the left-hand side in (3) or in (4) is finite if and only if the corresponding series on the right absolutely converges.

**Proof.** Relation (3) is a direct consequence of (2), while (4) follows from Fubini’s theorem and Lemma 3 on account of the fact that \( \kappa(x, \nu) \), where \( \nu \in \mathcal{M}^+ \) is given, is lower semicontinuous on \( X \) (see, e.g., [F1]). \( \square \)

**Definition 4.** Given \( \mu, \mu_1 \in \mathcal{M}(\mathcal{A}) \), then
\[
\kappa(x, \mu) := \kappa(x, R\mu)
\]
is called the value of the **potential** of \( \mu \) at a point \( x \in X \), and
\[
\kappa(\mu, \mu_1) := \kappa(R\mu, R\mu_1)
\]
the **mutual energy** of \( \mu \) and \( \mu_1 \) — of course, provided the right-hand side of the corresponding relation is well defined. For \( \mu \equiv \mu_1 \) we get the **energy** \( \kappa(\mu, \mu) \) of \( \mu \); i.e., if \( \kappa(R\mu, R\mu) \) is well defined, then
\[
\kappa(\mu, \mu) := \kappa(R\mu, R\mu) = \sum_{i,j \in I} \alpha_i \alpha_j \kappa(\mu^i, \mu^j).
\]
(5)
Corollary 2. For \( \mu \in \mathcal{M}(A) \) to be of finite energy, it is necessary and sufficient that so be all \( \mu_i \), \( i \in I \), and
\[
\sum_{i \in I} \| \mu_i \|^2 < \infty.
\]

Proof. This follows immediately from (5) and Corollary 1 due to the inequality
\[
2 \kappa(\nu_1, \nu_2) \leq \| \nu_1 \|^2 + \| \nu_2 \|^2, \quad \nu_1, \nu_2 \in \mathcal{E}.
\]

Remark 4. Given \( \mu \in \mathcal{M}(A) \), then the series in (5) actually defines the energy of the vector measure \( (\mu_i)_{i \in I} \) relative to the infinite interaction matrix of the form \( (\alpha_i \alpha_j)_{i,j \in I} \); compare with \[GR\], \[NS\] Chap. 5, § 4. Our approach, however, is based on the fact that, due to the specific interaction matrix, this value can also be obtained as the energy of the corresponding Radon measure \( R\mu \).

Remark 5. Since we make no difference between \( \mu \in \mathcal{M}(A) \) and \( R\mu \) when dealing with their energies or potentials, we shall sometimes call a measure associated with a condenser simply a measure — certainly, if this causes no confusion.

3.5. Let \( \mathcal{E}(A) \) consist of all \( \mu \in \mathcal{M}(A) \) of finite energy \( \kappa(\mu, \mu) =: \| \mu \|^2 \). Since \( \mathcal{M}(A) \) forms a convex cone, it is seen from Corollary 2 that so does \( \mathcal{E}(A) \). Let us treat \( \mathcal{E}(A) \) as a semimetric space with the semimetric
\[
\| \mu_1 - \mu_2 \| := \| R\mu_1 - R\mu_2 \|, \quad \mu_1, \mu_2 \in \mathcal{E}(A);
\]
then \( \mathcal{E}(A) \) and its \( R \)-image become isometric. The topology on \( \mathcal{E}(A) \) defined by means of the semimetric (6) will be called strong. Two elements of \( \mathcal{E}(A) \), \( \mu_1 \) and \( \mu_2 \), are called equivalent in \( \mathcal{E}(A) \) if \( \| \mu_1 - \mu_2 \| = 0 \). If, in addition, the kernel \( \kappa \) is assumed to be strictly positive definite, then the equivalence in \( \mathcal{E}(A) \) implies that in \( \mathcal{M}(A) \), namely then \( \mu_1 \cong \mu_2 \).

4. Interior capacities of condensers; elementary properties

4.1. Let \( \mathcal{H} \) be a set in the pre-Hilbert space \( \mathcal{E} \) or in the semimetric space \( \mathcal{E}(A) \), an \( (I^+, I^-) \)-condenser \( A \) being given. In either case, let us introduce the quantity
\[
\| \mathcal{H} \|^2 := \inf_{\nu \in \mathcal{H}} \| \nu \|^2,
\]
interpreted as \( +\infty \) if \( \mathcal{H} \) is empty. If \( \| \mathcal{H} \|^2 < \infty \), one can consider the variational problem on the existence of \( \lambda = \lambda(\mathcal{H}) \in \mathcal{H} \) with minimal energy
\[
\| \lambda \|^2 = \| \mathcal{H} \|^2;
\]
such a problem will be referred to as the \( \mathcal{H} \)-problem. The \( \mathcal{H} \)-problem is called solvable if a minimizing measure \( \lambda(\mathcal{H}) \) exists.

The following elementary lemma is a slight generalization of \[FI\] Lemma 4.1.1.
Lemma 4. Suppose $\mathcal{H}$ is convex and $\lambda = \lambda(\mathcal{H})$ exists. Then for any $\nu \in \mathcal{H},$
\[ \|\nu - \lambda\|^2 \leq \|\nu\|^2 - \|\lambda\|^2. \tag{7} \]

Proof. Assume $\mathcal{H} \subset \mathcal{E}$. For every $t \in [0, 1]$, the measure $\mu := (1 - t)\lambda + t\nu$ belongs to $\mathcal{H}$, and therefore $\|\mu\|^2 \geq \|\lambda\|^2$. Evaluating $\|\mu\|^2$ and then letting $t$ tend to zero, we get $\kappa(\nu, \lambda) \geq \|\lambda\|^2$, and (7) follows (see [F1]).

Suppose now $\mathcal{H} \subset \mathcal{E}(\mathcal{A})$. Then $R\mathcal{H} := \{R\nu : \nu \in \mathcal{H}\}$ is a convex subset of $\mathcal{E}$, while $R\lambda$ is a minimizer in the $R\mathcal{H}$-problem. What has just been shown therefore yields
\[ \|R\nu - R\lambda\|^2 \leq \|R\nu\|^2 - \|R\lambda\|^2, \]
which gives (7) when combined with (6). □

We shall be concerned with the $\mathcal{H}$-problem for various specific $\mathcal{H}$ related to the notion of interior capacity of an $(I^+, I^-)$-condenser (in particular, of a set); see Sec. 4.2 and Sec. 7 below for the definitions.

4.2. Fix a continuous function $g : X \to (0, \infty)$ and a numerical vector $a = (a_i)_{i \in I}$ with $a_i > 0$, $i \in I$. Given an $(I^+, I^-)$-condenser $\mathcal{A}$ in $X$, write
\[ \mathcal{M}^+(A_i, a_i, g) := \left\{ \nu \in \mathcal{M}^+(A_i) : \int g \, d\nu = a_i \right\}, \]
and let $\mathcal{M}(\mathcal{A}, a, g)$ consist of all $\mu \in \mathcal{M}(\mathcal{A})$ with $\mu^i \in \mathcal{M}^+(A_i, a_i, g)$ for all $i \in I$.

Given a kernel $\kappa$, also write
\[ \mathcal{E}^+(A_i, a_i, g) := \mathcal{M}^+(A_i, a_i, g) \cap \mathcal{E}, \quad \mathcal{E}(\mathcal{A}, a, g) := \mathcal{M}(\mathcal{A}, a, g) \cap \mathcal{E}(\mathcal{A}). \]

Definition 5. We shall call the value
\[ \text{cap} \mathcal{A} := \text{cap} (\mathcal{A}, a, g) := \frac{1}{\|\mathcal{E}(\mathcal{A}, a, g)\|^2} \tag{8} \]
the (interior) capacity of an $(I^+, I^-)$-condenser $\mathcal{A}$ (with respect to $\kappa$, $a$, and $g$).

Here and in the sequel, we adopt the convention that $1/0 = +\infty$. It follows immediately from the positive definiteness of the kernel that
\[ 0 \leq \text{cap} (\mathcal{A}, a, g) \leq \infty. \]

Remark 6. If $I$ is a singleton, then any $(I^+, I^-)$-condenser consists of just one set, say $A_1$. If moreover $g = 1$ and $a_1 = 1$, then the notion of interior capacity of a condenser, defined above, certainly reduces to the notion of interior capacity of a set (see [F1]). We denote it by $C(\cdot)$ as well, i.e., $C(A_1) := \|\mathcal{E}^+(A_1, 1, 1)\|^{-2}$.

Remark 7. In the case of the Newtonian kernel $|x - y|^{-1}$ in $\mathbb{R}^3$, the notion of capacity of a condenser $\mathcal{A}$ has an evident electrostatic interpretation. In the
framework of the corresponding electrostatics problem, the function \( g \) serves as a characteristic of nonhomogeneity of the conductors \( A_i, i \in I \).

4.3. On \( \mathcal{C} = \mathcal{C}(I^+, I^-) \), it is natural to introduce an ordering relation \( \prec \) by declaring \( A' \prec A \) to mean that \( A'_i \subset A_i \) for all \( i \in I \). Here, \( A' = (A'_i)_{i \in I} \). Then \( \text{cap}(\cdot, a, g) \) is a nondecreasing function of a condenser, namely

\[
\text{cap}(A', a, g) \leq \text{cap}(A, a, g) \quad \text{whenever} \quad A' \prec A.
\]

Given \( A \in \mathcal{C} \), denote by \( \{ \mathcal{K} \}_A \) the increasing ordered family of all compact condensers \( \mathcal{K} = (K_i)_{i \in I} \in \mathcal{C} \) such that \( \mathcal{K} \prec A \).

**Lemma 5.** If \( \mathcal{K} \) ranges over \( \{ \mathcal{K} \}_A \), then

\[
\text{cap}(A, a, g) = \lim_{\mathcal{K} \uparrow A} \text{cap}(\mathcal{K}, a, g). \tag{10}
\]

**Proof.** We can certainly assume \( \text{cap}(A, a, g) \) to be nonzero, since otherwise (10) follows at once from (9). Then the set \( \mathcal{E}(A, a, g) \) must be nonempty; fix \( \mu \), one of its elements. Given \( \mathcal{K} \in \{ \mathcal{K} \}_A \) and \( i \in I \), let \( \mu^i_K \) denote the trace of \( \mu^i \) upon \( K_i \), i.e., \( \mu^i_K := \mu^i_{K_i} \). Applying Lemma 1.2.2 from [F1], we conclude that

\[
\int g \, d\mu^i = \lim_{\mathcal{K} \uparrow A} \int g \, d\mu^i_K, \quad i \in I, \tag{11}
\]

\[
\kappa(\mu^i, \mu^j) = \lim_{\mathcal{K} \uparrow A} \kappa(\mu^i_K, \mu^j_K), \quad i, j \in I. \tag{12}
\]

Fix \( \varepsilon > 0 \). It follows from (11) and (12) that for every \( i \in I \) one can choose a compact set \( K^0_i \subset A_i \) so that

\[
\frac{a_i}{\int g \, d\mu^i_{K^0_i}} < 1 + \varepsilon i^{-2}, \tag{13}
\]

\[
\left| \left| \mu^i \right|^2 - \left| \mu^i_{K^0_i} \right|^2 \right| < \varepsilon^2 i^{-4}. \tag{14}
\]

Having denoted \( \mathcal{K}^0 := (K^0_i)_{i \in I} \), for every \( \mathcal{K} \in \{ \mathcal{K} \}_A \) that follows \( \mathcal{K}^0 \) we therefore have \( \int g \, d\mu^i_K \neq 0 \) and

\[
\hat{\mu}_K := \sum_{i \in I} \frac{\alpha_i a_i}{\int g \, d\mu^i_K} \mu^i_K \in \mathcal{E}(\mathcal{K}, a, g),
\]

the finiteness of the energy being obtained from (14) and Corollary 2. This yields

\[
\|\hat{\mu}_K\|^2 \geq \|\mathcal{E}(\mathcal{K}, a, g)\|^2. \tag{15}
\]

We next proceed to show that

\[
\|\mu\|^2 = \lim_{\mathcal{K} \uparrow A} \|\hat{\mu}_K\|^2. \tag{16}
\]
To this end, it can be assumed that $\kappa \geq 0$; for if not, then $\mathcal{A}$ must be finite since $X$ is compact, and (16) follows from (11) and (12) when substituted into (5). Therefore, for every $K$ that follows $K_0$ and every $i \in I$ we get

$$\|\mu^i_K\| \leq \|\mu^i\| \leq \|R\mu^+ + R\mu^-\|,$$

(17)

$$\|\mu^i - \mu^i_K\| < \varepsilon i^{-2},$$

(18)

the latter being clear from (14) because of $\kappa(\mu^i_K, \mu^i - \mu^i_K) \geq 0$. Also observe that, by (5),

$$\left| \|\mu\|^2 - \|\mu_K\|^2 \right| \leq \sum_{i,j \in I} \left| \kappa(\mu^i, \mu^j) - \frac{a_i}{\int g \, d\mu_K} \frac{a_j}{\int g \, d\mu^j_K} \kappa(\mu^i_K, \mu^j_K) \right|$$

$$\leq \sum_{i,j \in I} \left[ \kappa(\mu^i - \mu^i_K, \mu^j) + \kappa(\mu^j_K, \mu^j - \mu^j_K) + \left( \frac{a_i}{\int g \, d\mu_K} \frac{a_j}{\int g \, d\mu^j_K} - 1 \right) \kappa(\mu^j_K, \mu^j_K) \right].$$

When combined with (13), (17), and (18), this yields

$$\left| \|\mu\|^2 - \|\mu_K\|^2 \right| \leq M \varepsilon \quad \text{for all } K \succ K_0,$$

where $M$ is finite and independent of $K$, and the required relation (16) follows. Substituting (15) into (16), in view of the arbitrary choice of $\mu \in \mathcal{E}(\mathcal{A}, a, g)$ we get

$$\|\mathcal{E}(\mathcal{A}, a, g)\|^2 \geq \lim_{K \uparrow A} \|\mathcal{E}(K, a, g)\|^2.$$

Since the converse inequality is obvious from (9), the proof is complete.

Let $\mathcal{E}^0(\mathcal{A}, a, g)$ denote the class of all $\mu \in \mathcal{E}(\mathcal{A}, a, g)$ such that, for every $i \in I$, the support $S(\mu^i)$ of $\mu^i$ is compact and contained in $A_i$.

**Corollary 3.** The capacity $\text{cap} (\mathcal{A}, a, g)$ remains unchanged if the class $\mathcal{E}(\mathcal{A}, a, g)$ in its definition is replaced by $\mathcal{E}^0(\mathcal{A}, a, g)$. In other words,

$$\|\mathcal{E}(\mathcal{A}, a, g)\|^2 = \|\mathcal{E}^0(\mathcal{A}, a, g)\|^2.$$

**Proof.** We can certainly assume $\|\mathcal{E}(\mathcal{A}, a, g)\|^2$ to be finite, since otherwise the corollary follows immediately from $\mathcal{E}^0(\mathcal{A}, a, g) \subset \mathcal{E}(\mathcal{A}, a, g)$. Then, by (9) and (10), for every $\varepsilon > 0$ there exists a compact condenser $K \prec \mathcal{A}$ such that

$$\|\mathcal{E}(K, a, g)\|^2 \leq \|\mathcal{E}(\mathcal{A}, a, g)\|^2 + \varepsilon.$$

This leads to the claimed assertion when combined with the relation

$$\|\mathcal{E}(K, a, g)\|^2 \geq \|\mathcal{E}^0(\mathcal{A}, a, g)\|^2 \geq \|\mathcal{E}(\mathcal{A}, a, g)\|^2.$$

□
4.4. Unless explicitly stated otherwise, in all that follows it is assumed that
\[ \text{cap}(\mathcal{A}, a, g) > 0; \]  
see below for necessary and sufficient conditions for this to occur.

**Lemma 6.** For (19) to hold, it is necessary and sufficient that either of the following three equivalent conditions be satisfied:

(i) \( \mathcal{E}(\mathcal{A}, a, g) \) is nonempty;
(ii) there exist \( \nu_i \in \mathcal{E}^+(A_i, a_i, g), i \in I, \) such that \( \sum_{i \in I} \| \nu_i \| < \infty; \)
(iii) \( \sum_{i \in I} \| \mathcal{E}^+(A_i, a_i, g) \|^2 < \infty. \)

**Proof.** Indeed, the equivalency of (19) and (i) is obvious, while that of (i) and (ii) can be obtained directly from Corollary 2. If (iii) holds, then one can choose \( \nu_i \in \mathcal{E}^+(A_i, a_i, g), i \in I, \) so that \( \| \nu_i \|^2 < \| \mathcal{E}^+(A_i, a_i, g) \|^2 + \delta^{-2}, \) and (ii) follows. Since (ii) obviously results (iii), the proof is complete. \( \square \)

**Corollary 4.** For (19) to be satisfied, it is necessary that
\[ C(A_i) > 0 \quad \text{for all} \ i \in I. \]  
If \( \mathcal{A} \) is finite, then (19) and (20) are actually equivalent.

**Proof.** For Lemma 6, (ii) to hold, it is necessary that, for every \( i \in I, \) there exists a nonzero nonnegative measure of finite energy concentrated on \( A_i, \) which in turn is equivalent to (20) by [F1, Lemma 2.3.1]. Since the former implication can obviously be inverted whenever \( \mathcal{A} \) is finite, the proof is complete. \( \square \)

Let \( g_{\inf} \) and \( g_{\sup} \) denote respectively the infimum and the supremum of \( g \) over \( A. \)

**Corollary 5.** Assume \( 0 < g_{\inf} \leq g_{\sup} < \infty. \) Then (19) holds if and only if
\[ \sum_{i \in I} \frac{a_i^2}{C(A_i)} < \infty. \]

**Proof.** Lemma 6, (iii) implies the corollary when combined with the inequalities
\[ \frac{a_i^2}{g_{\sup}^2 C(A_i)} \leq \| \mathcal{E}^+(A_i, a_i, g) \|^2 \leq \frac{a_i^2}{g_{\inf}^2 C(A_i)}, \ i \in I, \]  
(21)
to be proved below by reasons of homogeneity.

To verify (21), fix \( i \in I. \) One can certainly assume \( C(A_i) \) to be nonzero, for otherwise Corollary 4 with \( I = \{i\} \) shows that each of the three parts in (21) equals \( +\infty. \) Therefore, there exists \( \theta \in \mathcal{E}^+(A_i, 1, 1). \) Since
\[ \theta' := \frac{a_i \theta}{\int g \ d\theta} \in \mathcal{E}^+(A_i, a_i, g), \]
12
we get
\[ a_i^2 \|\theta\|^2 \geq g_{\inf}^2 \|\theta'\|^2 \geq g_{\inf}^2 \|\mathcal{E}^+(A_i, a_i, g)\|^2, \]
and the right-hand side of (21) is obtained by letting \( \theta \) range over \( \mathcal{E}^+(A_i, 1, 1) \).

To verify the left-hand side, fix \( \omega \in \mathcal{E}^+(A_i, a_i, g) \). Then
\[ 0 < a_i g_{\inf}^{-1} \leq \omega(X) \leq a_i g_{\sup}^{-1} < \infty. \]
Hence, \( \omega(X)^{-1} \omega \in \mathcal{E}^+(A_i, 1, 1) \) and
\[ \|\omega\|^2 \geq \frac{a_i^2}{g_{\sup}} \|\mathcal{E}^+(A_i, 1, 1)\|^2. \]

In view of the arbitrary choice of \( \omega \in \mathcal{E}^+(A_i, a_i, g) \), this completes the proof. \( \square \)

4.5. In the following choice of \( \omega \in \mathcal{E}^+(A_i, a_i, g) \), this completes the proof.

**Lemma 7.** For \( \text{cap} (A, a, g) \) to be finite, it is necessary that
\[ C(A_j) < \infty \quad \text{for some} \quad j \in I. \quad (22) \]
This condition is also sufficient if it is additionally assumed that \( \sum_{i \in I} a_i < \infty \), \( g_{\sup} < \infty \), \( A \) is closed, while \( \kappa \) is bounded from above on \( A^+ \times A^- \) and perfect.

**Proof.** Let \( \text{cap} A < \infty \) and assume, on the contrary, that
\[ C(A_i) = \infty \quad \text{for all} \quad i \in I. \quad (23) \]
Given \( \varepsilon > 0 \), then for every \( i \) one can choose \( \nu_i \in \mathcal{E}^+(A_i, 1, 1) \) with compact support so that
\[ \|\nu_i\| \leq \varepsilon a_i^{-1-i^{-2}} g_{\inf}. \]
Since then
\[ \nu := \sum_{i \in I} \frac{\alpha_i a_i \nu_i}{\int g d\nu_i} \in \mathcal{E}(A, a, g) \]
and
\[ \|\nu\| \leq \varepsilon \sum_{i \in I} i^{-2}, \]
we arrive at a contradiction by letting \( \varepsilon \) tend to 0.
Assume now all the conditions of the remaining part of the lemma to be satisfied, and let (22) be true — say, for \( j \in I^+ \). Consider the (finite) condenser \( B \) with the positive plates \( B_1 \) and \( B_2 \) and the negative one \( B_3 \), where
\[ B_1 := A_j, \quad B_2 := \bigcup_{i \in I^+ \setminus \{j\}} A_i, \quad B_3 := A^- . \]

13
Also write \( b := (b_1, b_2, b_3) \), where
\[
\begin{align*}
b_1 &:= a_j, \\
b_2 &:= \sum_{i \in I^+ \setminus \{j\}} a_i, \\
b_3 &:= \sum_{i \in I^-} a_i.
\end{align*}
\]
(If either of the sets \( B_2 \) and \( B_3 \) is empty, it should just be dropped, as well as the corresponding coordinate of \( b \).) Then for every \( \mu \in \mathcal{E}(\mathcal{A}, a, g) \) there exists \( \nu \in \mathcal{E}(\mathcal{B}, b, g) \) such that \( R\mu = R\nu \), and therefore
\[
\|\mathcal{E}(\mathcal{B}, b, g)\|^2 \leq \|\mathcal{E}(\mathcal{A}, a, g)\|^2.
\]
Furthermore, Lemma 13 from [Z4] shows that, under the stated assumptions, there exists \( \zeta \in \mathcal{E}(\mathcal{B}) \) such that \( \int gd\zeta^1 = b_1 \) (hence, \( \zeta \neq 0 \)) and
\[
\|\zeta\|^2 = \|\mathcal{E}(\mathcal{B}, b, g)\|^2.
\]
Since \( \kappa \) is strictly positive definite, this implies that \( \|\mathcal{E}(\mathcal{B}, b, g)\|^2 \) is nonzero. Hence, so is \( \|\mathcal{E}(\mathcal{A}, a, g)\|^2 \), as was to be proved. \( \square \)

5. On the solvability of the main minimum-problem

5.1. Because of (19), we are naturally led to the \( \mathcal{E}(\mathcal{A}, a, g) \)-problem (cf. Sec. 4.1), i.e., the problem on the existence of \( \lambda \in \mathcal{E}(\mathcal{A}, a, g) \) with minimal energy
\[
\|\lambda\|^2 = \|\mathcal{E}(\mathcal{A}, a, g)\|^2;
\]
the \( \mathcal{E}(\mathcal{A}, a, g) \)-problem might certainly be regarded as the main minimum-problem of the theory of interior capacities of condensers. The collection (possibly empty) of all minimizing measures \( \lambda \) in this problem will be denoted by \( \mathcal{S}(\mathcal{A}, a, g) \).

If moreover \( \text{cap} (\mathcal{A}, a, g) < \infty \), let us look, as well, at the \( \mathcal{E}(\mathcal{A}, a \text{ cap } \mathcal{A}, a, g) \)-problem. By reasons of homogeneity, both the \( \mathcal{E}(\mathcal{A}, a, g) \)- and the \( \mathcal{E}(\mathcal{A}, a \text{ cap } \mathcal{A}, g) \)-problems are simultaneously either solvable or unsolvable, and their extremal values are related to each other by the following law:
\[
\frac{1}{\|\mathcal{E}(\mathcal{A}, a, g)\|^2} = \|\mathcal{E}(\mathcal{A}, a \text{ cap } \mathcal{A}, g)\|^2. \tag{24}
\]

Assume for a moment that \( \mathcal{A} \) is compact. Since the mapping
\[
\nu \mapsto \int g \, d\nu, \quad \nu \in \mathcal{M}^+(K),
\]
where \( K \subset X \) is a compact set, is vaguely continuous, \( \mathcal{M}(\mathcal{A}, a, g) \) is compact in the \( \mathcal{A} \)-vague topology. Therefore, if \( \mathcal{A} \) is additionally assumed to be finite, while \( \kappa \) is continuous on \( A^+ \times A^- \) (which, due to [1], is always the case for either of the classical kernels), then \( \|\mu\|^2 \) is \( \mathcal{A} \)-vaguely lower semicontinuous on \( \mathcal{E}(\mathcal{A}) \), and the solvability of both the problems immediately follows (cf. [O] Th. 2.30)).
But if $\mathcal{A}$ is noncompact, then the class $\mathcal{M}(\mathcal{A}, a, g)$ is no longer $\mathcal{A}$-vaguely compact and the problems become quite nontrivial. Moreover, it has recently been shown by the author that, in the noncompact case, the problems are in general unsolvable and this occurs even under very natural assumptions (e.g., for the Newtonian, Green, or Riesz kernels in $\mathbb{R}^n$, $n \geq 2$, and finite, closed condensers).

In particular, it was proved in [Z3] that, if $\mathcal{A}$ is finite and closed, $\kappa$ is perfect, and bounded and continuous on $A^+ \times A^-$, and satisfies the generalized maximum principle (see, e.g., [L, Chap. VI]), while $0 < g_{\text{inf}} \leq g_{\text{sup}} < \infty$, then for either of the $\mathcal{E}(\mathcal{A}, a, g)$- and the $\mathcal{E}(\mathcal{A}, a \cap \mathcal{A}, g)$-problems to be solvable for any vector $a$, it is necessary and sufficient that

$$C(A_i) < \infty \quad \text{for all } i \in I.$$ 

If moreover there exists $i_0 \in I$ such that

$$C(A_{i_0}) = \infty,$$

then both the problems are unsolvable for all $a = (a_i)_{i \in I}$ with $a_{i_0}$ large enough. In [Z3, Th. 1], the last statement was sharpened. It was shown that if, in addition to all the preceding assumptions, for all $i \neq i_0$,

$$C(A_i) < \infty \quad \text{and} \quad A_i \cap A_{i_0} = \emptyset,$$

while $\kappa(\cdot, y) \to 0$ (as $y \to \infty$) uniformly on compact sets, then there exists a number $\Lambda_{i_0} \in [0, \infty)$ such that the problems are unsolvable if and only if

$$a_{i_0} > \Lambda_{i_0}.$$ 

**Remark 8.** It was actually shown in [Z5] that

$$\Lambda_{i_0} = \int g \, d\tilde{\lambda}^{i_0},$$

where $\tilde{\lambda}$ is a minimizer (it exists) in the auxiliary $\mathcal{H}$-problem for

$$\mathcal{H} := \{ \mu \in \mathcal{E}(\mathcal{A}) : \mu_i \in \mathcal{E}^+(A_i, a_i, g) \quad \text{for all } i \neq i_0 \}.$$ 

**Remark 9.** The mentioned results were actually obtained in [Z3, Z5] for the energy evaluated in the presence of an external field.

5.2. In view of the results reviewed in Sec. 5.1, it was particularly interesting to find statements of variational problems dual to the $\mathcal{E}(\mathcal{A}, a \cap \mathcal{A}, g)$-problem (and hence providing new equivalent definitions of $\text{cap} \mathcal{A}$), but now solvable for any $(I^+, I^-)$-condenser $\mathcal{A}$ (e.g., even nonclosed or infinite) and any vector $a$. We have succeeded in this under the following conditions, which will always be tacitly assumed.
From now on, in addition to (19), the following standing assumptions will be always required. The kernel $\kappa$ is assumed to be consistent, and either
\[ I^- = \emptyset, \]
or the following three conditions are satisfied:
\begin{align*}
g_{\inf} & > 0, \quad (25) \\
\sup_{x \in A^+, \ y \in A^-} \kappa(x, y) & < \infty, \quad (26) \\
|a| & := \sum_{i \in I} a_i < \infty. \quad (27)
\end{align*}

**Remark 10.** These assumptions on a kernel are not too restrictive. In particular, they all are satisfied by the Newtonian, Riesz, or Green kernels in $\mathbb{R}^n$, $n \geq 2$, provided the Euclidean distance between $A^+$ and $A^-$ is nonzero, as well as by the restriction of the logarithmic kernel in $\mathbb{R}^2$ to an open unit ball.

### 6. $\mathcal{A}$-vague and strong cluster sets of minimizing nets

To formulate the results obtained, we shall need the following notation.

6.1. Denote by $\mathcal{M}(\mathcal{A}, a, g)$ the class of all $(\mu_t)_{t \in T} \subset \mathcal{E}^0(\mathcal{A}, a, g)$ such that
\[
\lim_{t \in T} \|\mu_t\|^2 = \|\mathcal{E}(\mathcal{A}, a, g)\|^2. \quad (28)
\]

This class is not empty, which is clear from (19) in view of Corollary 3.

Let $\mathcal{M}(\mathcal{A}, a, g)$ (respectively, $\mathcal{M}'(\mathcal{A}, a, g)$) consist of all limit points of the nets $(\mu_t)_{t \in T} \in \mathcal{M}(\mathcal{A}, a, g)$ in the $\mathcal{A}$-vague topology of the space $\mathcal{M}(\mathcal{A})$ (respectively, in the strong topology of the semimetric space $\mathcal{E}(\mathcal{A})$). Also write
\[
\mathcal{E}(\mathcal{A}, \leq a, g) := \left\{ \mu \in \mathcal{E}(\mathcal{A}) : \int gd\mu^i \leq a_i \text{ for all } i \in I \right\}.
\]

With the preceding notation and under our standing assumptions (see Sec. 5.2), there holds the following lemma, to be proved in Sec. 12 below.

**Lemma 8.** Given $(\mu_t)_{t \in T} \in \mathcal{M}(\mathcal{A}, a, g)$, there exist its $\mathcal{A}$-vague cluster points; hence, $\mathcal{M}(\mathcal{A}, a, g)$ is nonempty. Moreover,
\[
\mathcal{M}(\mathcal{A}, a, g) \subset \mathcal{M}'(\mathcal{A}, a, g) \cap \mathcal{E}(\mathcal{A}, \leq a, g). \quad (29)
\]

Furthermore, for every $\chi \in \mathcal{M}'(\mathcal{A}, a, g)$,
\[
\lim_{t \in T} \|\mu_t - \chi\|^2 = 0, \quad (30)
\]
and hence \( M'(A, a, g) \) forms an equivalence class in \( \mathcal{E}(\overline{A}) \).

It follows from (28) – (30) that
\[
\|\zeta\|^2 = \|\mathcal{E}(A, a, g)\|^2 \quad \text{for all} \quad \zeta \in M(A, a, g).
\]

Also observe that, if \( A = K \) is compact, then moreover \( M(K, a, g) \subset \mathcal{M}(K, a, g) \), which together with the preceding relation proves the following assertion.

**Corollary 6.** If \( A = K \) is compact, then the \( \mathcal{E}(K, a, g) \)-problem is solvable.

Actually,
\[
S(K, a, g) = M(K, a, g).
\]  

(31)

6.2. When approaching \( A \) by the increasing family \( \{K\}_A \) of the compact condensers \( K \prec A \), we shall always suppose all those \( K \) to be of capacity nonzero. This involves no loss of generality, which is clear from (19) and Lemma 5.

Then Corollary 6 enables us to introduce the (nonempty) class \( \mathcal{M}_0(A, a, g) \) of all nets \( (\lambda_K)_{K \in \{K\}_A} \), where \( \lambda_K \in S(K, a, g) \) is arbitrarily chosen. Let \( \mathcal{M}_0(A, a, g) \) consist of all \( A \)-vague cluster points of those nets. Since, by Lemma 5,
\[
\mathcal{M}_0(A, a, g) \subset \mathcal{M}(A, a, g),
\]
application of Lemma 8 yields the following assertion.

**Corollary 7.** The class \( \mathcal{M}_0(A, a, g) \) is nonempty, and
\[
\mathcal{M}_0(A, a, g) \subset \mathcal{M}(A, a, g) \subset \mathcal{M}'(A, a, g).
\]

**Remark 11.** Each of the cluster sets, \( \mathcal{M}_0(A, a, g) \), \( \mathcal{M}(A, a, g) \) and \( \mathcal{M}'(A, a, g) \), plays an important role in our study. However, if \( \kappa \) is additionally assumed to be strictly positive definite (hence, perfect), while \( \overline{A}_i, i \in I \), are mutually disjoint, then all these three classes coincide and consist of just one element.

6.3. Also the following notation will be required. Given \( \chi \in \mathcal{M}'(A, a, g) \), write
\[
\mathcal{M}'_\mathcal{E}(A, a, g) := [R\chi]_\mathcal{E}.
\]

This equivalence class does not depend on the choice of \( \chi \), which is clear from Lemma 8. Lemma 8 also yields that, for any \( (\mu_t)_{t \in T} \in \mathcal{M}(A, a, g) \) and any \( \nu \in \mathcal{M}'_\mathcal{E}(A, a, g) \), \( R\mu_t \rightarrow \nu \) in the strong topology of the pre-Hilbert space \( \mathcal{E} \).

7. Extremal problems dual to the main minimum-problem

Throughout Sec. 7, as usual, we are keeping all our standing assumptions, stated in Sec. 5.2.
7.1. A proposition $R(x)$ involving a variable point $x \in X$ is said to subsist \textit{nearly everywhere} (n. e.) in $E$, where $E$ is a given subset of $X$, if the set of all $x \in E$ for which $R(x)$ fails to hold is of interior capacity zero. See, e.g., [F1].

If $C(E) > 0$ and $f$ is a universally measurable function bounded from below nearly everywhere in $E$, write

$$\inf_{x \in E} f(x) := \sup \left\{ q : f(x) \geq q \text{ n. e. in } E \right\}.$$ 

Then

$$f(x) \geq \inf_{x \in E} f(x) \text{ n. e. in } E.$$ 

This follows immediately from the fact, to be often used in what follows, that the union of a sequence of sets $U_n \cap E$ with $C(U_n \cap E) = 0$ is of interior capacity zero as well, provided $U_n$, $n \in \mathbb{N}$, are universally measurable whereas $E$ is arbitrary (see the corollary to Lemma 2.3.5 in [F1] and the remark attached to it).

7.2. Let $\hat{\Gamma} = \hat{\Gamma}(A, a, g)$ denote the class of all Radon measures $\nu \in \mathcal{E}$ such that there exist real numbers $c_i(\nu)$, $i \in I$, satisfying the relations

$$\alpha_i a_i \kappa(x, \nu) \geq c_i(\nu) g(x) \text{ n. e. in } A_i, \quad i \in I, \quad \sum_{i \in I} c_i(\nu) \geq 1.$$  

(32) \hspace{1cm} (33)

\textbf{Remark 12.} Given $\nu \in \hat{\Gamma}$, then the series in (33) must be \textit{absolutely convergent}. Indeed, due to (19) and Corollary 3, there exists $\mu \in \mathcal{E}_0(A, a, g)$; then, by [F1, Lemma 2.3.1], the inequality in (32) holds $\mu^i$-almost everywhere. In view of $\int g \, d\mu^i = a_i$, this gives

$$\kappa(\alpha_i \mu^i, \nu) \geq c_i(\nu), \quad i \in I.$$ 

Since, by Fubini’s theorem and Lemma 3, $\sum_{i \in I} \kappa(\alpha_i \mu^i, \nu)$ absolutely converges, the required conclusion follows.

We also observe that the class $\hat{\Gamma}(A, a, g)$ is \textit{convex}, which can easily be seen from the property of sets of interior capacity zero mentioned just above.

The following assertion, to be proved in Sec. 15 below, holds true.

\textbf{Theorem 2.} Under the standing assumptions,

$$\|\hat{\Gamma}(A, a, g)\|^2 = \text{cap}(A, a, g).$$  

(34)

If $\|\hat{\Gamma}(A, a, g)\|^2 < \infty$, we are interested in the $\hat{\Gamma}(A, a, g)$-\textit{problem} (cf. Sec. 4.1), i.e., the problem on the existence of $\hat{\omega} \in \hat{\Gamma}(A, a, g)$ with minimal energy

$$\|\hat{\omega}\|^2 = \|\hat{\Gamma}(A, a, g)\|^2;$$
the collection of all those \( \hat{\omega} \) will be denoted by \( \hat{\mathcal{G}} = \hat{\mathcal{G}}(A,a,g) \).

A minimizing measure \( \hat{\omega} \) can be shown to be \textit{unique} up to a summand of semi-norm zero (and, hence, it is unique whenever the kernel under consideration is strictly positive definite). Actually, the following stronger result holds true.

**Lemma 9.** If \( \hat{\omega} \) exists, \( \hat{\mathcal{G}}(A,a,g) \) forms an equivalence class in \( \mathcal{E} \).

**Proof.** Since \( \hat{\Gamma} \) is convex, Lemma 4 yields that \( \hat{\mathcal{G}} \) is contained in an equivalence class in \( \mathcal{E} \). To prove that \( \hat{\mathcal{G}} \) actually coincides with that equivalence class, it suffices to show that, if \( \nu \) belongs to \( \hat{\Gamma} \), then so do all measures equivalent to \( \nu \) in \( \mathcal{E} \). But this follows at once from the property of sets of interior capacity zero mentioned in Sec. 7.1 and the fact that the potentials of any two equivalent in \( \mathcal{E} \) measures coincide nearly everywhere in \( X \) [F1, Lemma 3.2.1]. \( \square \)

7.3. Assume for a moment that \( \text{cap} \langle A, a, g \rangle \) is finite. When combined with (8) and (24), Theorem 2 shows that the \( \hat{\Gamma}(A,a,g) \)-problem and, on the other hand, the \( \mathcal{E}(A,a \text{cap} A,g) \)-problem have the same infimum, equal to the capacity \( \text{cap} A \), and so these two variational problems are \textit{dual}.

But what is surprising is that their infimum, \( \text{cap} A \), turns out to be always an actual minimum in the former extremal problem, while this is not the case for the latter one (see Sec. 5.1). In fact, the following statement on the solvability of the \( \hat{\Gamma}(A,a,g) \)-problem, to be proved in Sec. 15 below, holds true.

**Theorem 3.** Under the standing assumptions, if moreover \( \text{cap} A < \infty \), then the class \( \hat{\mathcal{G}}(A,a,g) \) is nonempty and can be given by the formula

\[
\hat{\mathcal{G}}(A,a,g) = \mathcal{M}_c(A,a \text{cap} A,g).
\]

The numbers \( c_i(\hat{\omega}), i \in I \), satisfying both (32) and (33) for \( \hat{\omega} \in \hat{\mathcal{G}}(A,a,g) \), are determined uniquely, do not depend on the choice of \( \hat{\omega} \), and can be written in either of the forms

\[
c_i(\hat{\omega}) = \alpha_i \text{cap} A^{-1} \kappa(\zeta^i, \zeta),
\]

\[
c_i(\hat{\omega}) = \alpha_i \text{cap} A^{-1} \lim_{s \in S} \kappa(\mu^i_s, \mu_s),
\]

where \( \zeta \in \mathcal{M}(A,a \text{cap} A,g) \) and \( \mu_s \in \mathcal{M}(A,a \text{cap} A,g) \) are arbitrarily given.

The following two assertions, providing additional information about \( c_i(\hat{\omega}), i \in I \), can be obtained directly from the preceding theorem.

**Corollary 8.** Given \( \hat{\omega} \in \hat{\mathcal{G}}(A,a,g) \), it follows that

\[
c_i(\hat{\omega}) = \inf_{x \in A_i} \frac{\alpha_i a_i \mathcal{K}(x,\hat{\omega})}{g(x)}, \quad i \in I.
\]
Corollary 9. The inequality (33) for \( \hat{\omega} \in \mathcal{G}(A, a, g) \) is actually an equality; i. e.,

\[
\sum_{i \in I} c_i(\hat{\omega}) = 1. \tag{39}
\]

Remark 13. Assume for a moment that \( C(A_j) = 0 \) for some \( j \in I \). Then \( \text{cap} \ A = 0 \) according to Corollary 4. On the other hand, the measure \( \nu_0 = 0 \) belongs to \( \hat{\Gamma}(A, a, g) \) since it satisfies both (32) and (33) with \( c_i(\nu_0), i \in I \), where

\[
c_j(\nu_0) \geq 1 \quad \text{and} \quad c_i(\nu_0) = 0, \quad i \neq j.
\]

This implies that the identity (34) actually holds true in the degenerate case \( C(A_j) = 0 \) as well, and then \( \mathcal{G}(A, a, g) \) consists of all \( \nu \in \mathcal{E} \) of seminorm zero. What then, however, fails to hold is the statement on the uniqueness of \( c_i(\hat{\omega}) \).

7.4. Let \( \hat{\Gamma}_*(A, a, g) \) consist of all \( \nu \in \hat{\Gamma}(A, a, g) \) for which the inequality (33) is actually an equality. By arguments similar to those that have been applied above, one can see that \( \hat{\Gamma}_*(A, a, g) \) is convex, and hence all the solutions to the minimal energy problem over this class form an equivalence class in \( \mathcal{E} \). Combining this with Theorems 2, 3 and Corollary 9 leads to the following assertion.

Corollary 10. Under the standing assumptions,

\[
\|\hat{\Gamma}_*(A, a, g)\|^2 = \text{cap} \ (A, a, g).
\]

If moreover \( \text{cap} \ (A, a, g) < \infty \), then the \( \hat{\Gamma}_*(A, a, g) \)-problem is solvable and the class \( \hat{\mathcal{G}}_*(A, a, g) \) of all its solutions is given by the formula

\[
\hat{\mathcal{G}}_*(A, a, g) = M'_c(A, a \text{cap} A, g).
\]

Remark 14. Theorem 2 and Corollary 10 (cf. also Theorem 4 and Corollary 12 below) provide new equivalent definitions of the capacity \( \text{cap} \ (A, a, g) \). Note that, in contrast to the initial definition (cf. Sec. 4.2), no restrictions on the supports and total masses of measures from the classes \( \hat{\Gamma}(A, a, g) \) or \( \hat{\Gamma}_*(A, a, g) \) have been imposed; the only restriction involves their potentials. These definitions of the capacity are actually new even in the simplest case of a finite, compact condenser; compare with [O]. They are not only of obvious academic interest, but seem also to be important for numerical computations.

7.5. Our next purpose is to formulate an \( \mathcal{H} \)-problem such that it is still dual to the \( \mathcal{E}(A, a \text{ cap} A, g) \)-problem and solvable, but now with \( \mathcal{H} \) consisting of measures associated with a condenser.

Let \( \Gamma(A, a, g) \) consist of all \( \mu \in \mathcal{E}(\overline{A}) \) for which both the relations (32) and (33) hold (with \( \mu \) in place of \( \nu \)). In other words,

\[
\Gamma(A, a, g) := \left\{ \mu \in \mathcal{E}(\overline{A}) : \ R\mu \in \hat{\Gamma}(A, a, g) \right\}.
\]
Observe that the class \( \Gamma(A, a, g) \) is convex and
\[
\|\Gamma(A, a, g)\|^2 \geq \|\hat{\Gamma}(A, a, g)\|^2. \tag{40}
\]
We proceed to show that the inequality (40) is actually an equality, and that the minimal energy problem, if considered over the class \( \Gamma(A, a, g) \), is still solvable.

**Theorem 4.** Under the standing assumptions,
\[
\|\Gamma(A, a, g)\|^2 = \text{cap} (A, a, g). \tag{41}
\]
If moreover \( \text{cap} (A, a, g) < \infty \), then the \( \Gamma(A, a, g) \)-problem is solvable and the class \( \mathcal{G}(A, a, g) \) of all its solutions \( \omega \) is given by the formula
\[
\mathcal{G}(A, a, g) = \mathcal{M}'(A, a \text{ cap } A, g). \tag{42}
\]

**Proof.** We can certainly assume \( \text{cap} A \) to be finite, for if not, (41) is obtained directly from (34) and (40). Then, according to Lemma 8 with \( a \text{ cap } A \) instead of \( a \), the class \( \mathcal{M}'(A, a \text{ cap } A, g) \) is nonempty; fix \( \chi \), one of its elements. It is clear from its definition and the identity (35) that \( \chi \in \mathcal{E}(\overline{A}) \) and \( R\chi \in \hat{\mathcal{G}}(A, a, g) \). Hence, \( \chi \in \Gamma(A, a, g) \), and therefore
\[
\|\hat{\Gamma}(A, a, g)\|^2 = \|\chi\|^2 \geq \|\Gamma(A, a, g)\|^2.
\]
in view of (34) and (40), this proves (41) and, as well, the inclusion
\[
\mathcal{M}'(A, a \text{ cap } A, g) \subset \mathcal{G}(A, a, g).
\]
But the right-hand side of this inclusion is an equivalence class in \( \mathcal{E}(\overline{A}) \), which follows from the convexity of \( \Gamma(A, a, g) \) and Lemma 4 in the same manner as in the proof of Lemma 9. Since, by Lemma 8, also the left-hand side is an equivalence class in \( \mathcal{E}(\overline{A}) \), the two sets must actually be equal. \( \square \)

**Corollary 11.** If \( A = K \) is compact and \( \text{cap} (K, a, g) < \infty \), then any solution to the \( \mathcal{E}(K, a \text{ cap } K, g) \)-problem gives, as well, a solution to the \( \Gamma(K, a, g) \)-problem.

**Proof.** This follows from (42), when combined with (29) and (31) for \( a \text{ cap } K \) in place of \( a \). \( \square \)

**Remark 15.** Assume \( \text{cap} A < \infty \), and fix \( \omega \in \hat{\mathcal{G}}(A, a, g) \) and \( \hat{\omega} \in \hat{\mathcal{G}}(A, a, g) \). Since, by (35) and (42), \( \kappa(x, \omega) = \kappa(x, \hat{\omega}) \) nearly everywhere in \( X \), the numbers \( c_i(\omega), i \in I \), satisfying (32) and (33) for \( \nu = \omega \), are actually equal to \( c_i(\hat{\omega}) \). This implies that relations (36) – (39) do hold, as well, for \( \omega \) in place of \( \hat{\omega} \).

**Remark 16.** Observe that, in all the preceding assertions, we have not imposed any restrictions on the topology of \( A_i, i \in I \). So, all the \( \hat{\Gamma}(A, a, g)_- \), \( \hat{\Gamma}_+(A, a, g)_- \), and \( \Gamma(A, a, g)_- \)-problems are solvable even for a nonclosed, infinite condenser \( A \).
Remark 17. If $I = \{1\}$ and $g = 1$, Theorems 2–4 and Corollary 10 can be derived from [F11]. Moreover, then one can choose $\gamma \in \mathcal{G}(\mathcal{A}, a, g)$ so that

$$\gamma(X) = a_1 C(A_1),$$

and exactly this kind of measures was called by B. Fuglede interior capacitary distributions associated with the set $A_1$. However, this fact in general can not be extended to a condenser $\mathcal{A}$ consisting more than one plate; that is, then

$$\mathcal{G}(\mathcal{A}, a, g) \cap \mathcal{E}(\mathcal{A}, \text{cap} \mathcal{A}, g) = \emptyset,$$

which is seen from the unsolvability of the $\mathcal{E}(\mathcal{A}, \text{cap} \mathcal{A}, g)$-problem.

8. Interior capacitary constants associated with a condenser

8.1. Throughout Sec. 8, it is always required that $\text{cap} (\mathcal{A}, a, g) < \infty$. Due to the uniqueness statement in Theorem 3, the following notion naturally arises.

Definition 6. The numbers

$$C_i := C_i(\mathcal{A}, a, g) := c_i(\hat{\omega}), \quad i \in I,$$

satisfying both the relations (32) and (33) for $\hat{\omega} \in \hat{\mathcal{G}}(\mathcal{A}, a, g)$, are said to be the (interior) capacitary constants associated with an $(I^+, I^-)$-condenser $\mathcal{A}$.

Corollary 12. The interior capacity $\text{cap} (\mathcal{A}, a, g)$ equals the infimum of $\kappa(\nu, \nu)$, where $\nu$ ranges over the class of all $\nu \in \mathcal{E}$ (similarly, $\nu \in \mathcal{E}(\bar{\mathcal{A}})$) such that $\alpha_i a_i \kappa(x, \nu) \geq C_i(\mathcal{A}, a, g) g(x)$ n. e. in $A_i$, $i \in I$.

The infimum is attained at any $\hat{\omega} \in \hat{\mathcal{G}}(\mathcal{A}, a, g)$ (respectively, $\omega \in \mathcal{G}(\mathcal{A}, a, g)$), and hence it is an actual minimum.

Proof. This follows immediately from Theorems 2–4 and Remark 15.

8.2. Some properties of the interior capacitary constants $C_i(\mathcal{A}, a, g)$, $i \in I$, have already been provided by Theorem 3 and Corollaries 8, 9. Also observe that, if $I = \{1\}$, then certainly $C_1(\mathcal{A}, a, g) = 1$ (cf. [F11 Th. 4.1]).

Corollary 13. $C_i(\cdot, a, g)$, $i \in I$, are continuous under exhaustion of $\mathcal{A}$ by the increasing family of all compact condensers $\mathcal{K} \subset \mathcal{A}$. Namely,

$$C_i(\mathcal{A}, a, g) = \lim_{\mathcal{K} \uparrow \mathcal{A}} C_i(\mathcal{K}, a, g).$$

Proof. Under our assumptions, $0 < \text{cap} \mathcal{K} < \infty$ for every $\mathcal{K} \in \{\mathcal{K}\}_A$, and hence there exists $\lambda_\mathcal{K} \in S(\mathcal{K}, \text{cap} \mathcal{K}, g)$. Substituting $\lambda_\mathcal{K}$ into (36) yields

$$C_i(\mathcal{K}, a, g) = \alpha_i \text{cap} \mathcal{K}^{-1} \kappa(\lambda_\mathcal{K}, \lambda_\mathcal{K}), \quad i \in I.$$

(43)
On the other hand, it follows from Lemma 5 that the net
\[ \text{cap } A \cap K^{-1} \lambda_K, \quad \text{where } K \in \{K\}_{A}, \]
belongs to the class \( \mathcal{M}(A, a \text{ cap } A, g) \). Substituting it into (37) and then combining the relation obtained with (43), we get the corollary. □

**Corollary 14.** Assume \( C(A_j) = \infty \) for some \( j \in I \). If moreover \( g_{\inf} > 0 \), then
\[ C_j(A, a, g) \leq 0. \]

**Proof.** Assume, on the contrary, that \( C_j > 0 \). Given \( \hat{\omega} \in \hat{G}(A, a, g) \), then
\[ \alpha_j a_j \kappa(x, \hat{\omega}) \geq C_j g_{\inf} > 0 \text{ n.e. in } A_j, \]
and therefore, by [F1, Lemma 3.2.2],
\[ C(A_j) \leq a_j^2 \|\hat{\omega}\|^2 C_j^{-2} g_{\inf}^{-2} < \infty, \]
which is a contradiction. □

**Remark 18.** Observe that the necessity part of Lemma 7, which has been proved above with elementary arguments, can also be obtained as a consequence of Corollary 14. Indeed, if (23) were true, then by Corollary 14 the sum of \( C_i \), where \( i \) ranges over \( I \), would be not greater than 0, which is impossible.

**9. Interior capacitary distributions associated with a condenser**

As always, we are keeping all our standing assumptions, stated in Sec. 5.2. Throughout Sec. 9, it is also required that \( \text{cap } A < \infty \).

Our next purpose is to introduce a notion of interior capacitary distributions \( \gamma_A \) associated with a condenser \( A \) such that the distributions obtained possess properties similar to those of interior capacitary distributions associated with a set. Fuglede’s theory of interior capacities of sets [F1] serves here as a model case.

**9.1.** If \( A = K \) is compact, then, as follows from Theorem 4, Corollary 11 and Remark 15, any minimizer \( \lambda_K \) in the \( E(K, a \text{ cap } K, g) \)-problem has the desired properties, and so \( \gamma_K \) might be defined as
\[ \gamma_K := \lambda_K, \quad \text{where } \lambda_K \in S(K, a \text{ cap } K, g). \]

However, as is seen from Remark 17, in the noncompact case the desired notion can not be obtained as just a direct generalization of the corresponding one from the theory of interior capacities of sets. Having in mind that, similar to our model case, the required distributions should give a solution to the \( \Gamma(A, a, g) \)-problem
and be strongly and \( \mathcal{A} \)-vaguely continuous under exhaustion of \( \mathcal{A} \) by compact condensers, we arrive at the following definition.

**Definition 7.** We shall call \( \gamma_{\mathcal{A}} \in \mathcal{E}(\overline{\mathcal{A}}) \) an (interior) capacitary distribution associated with \( \mathcal{A} \) if there exists a subnet \( (\mathcal{K}_s)_{s \in S} \) of \( (\mathcal{K})_{\mathcal{K} \in \mathcal{K}(\mathcal{A})} \) and

\[
\lambda_{\mathcal{K}_s} \in \mathcal{S}(\mathcal{K}_s, \text{cap}\mathcal{K}_s, g), \quad s \in S,
\]

such that \( (\lambda_{\mathcal{K}_s})_{s \in S} \) converges to \( \gamma_{\mathcal{A}} \) in both the \( \mathcal{A} \)-vague and the strong topologies. Let \( \mathcal{D}(\mathcal{A}, a, g) \) denote the collection of all those \( \gamma_{\mathcal{A}} \).

Application of Lemmas 5 and 8 enables us to rewrite the above definition in the following, apparently weaker, form:

\[
\mathcal{D}(\mathcal{A}, a, g) = \mathcal{M}_0(\mathcal{A}, \text{a cap} \mathcal{A}, g). \tag{44}
\]

**Theorem 5.** \( \mathcal{D}(\mathcal{A}, a, g) \) is nonempty, contained in an equivalence class in \( \mathcal{E}(\overline{\mathcal{A}}) \), and compact in the induced \( \mathcal{A} \)-vague topology. Furthermore,

\[
\mathcal{D}(\mathcal{A}, a, g) \subset \mathcal{G}(\mathcal{A}, a, g) \cap \mathcal{E}(\overline{\mathcal{A}}, \leq \text{a cap} \mathcal{A}, g). \tag{45}
\]

Given \( \gamma := \gamma_{\mathcal{A}} \in \mathcal{D}(\mathcal{A}, a, g) \), then

\[
\|\gamma\|^2 = \text{cap} \mathcal{A}, \tag{46}
\]

\[
\alpha_i a_i \kappa(x, \gamma) \geq C_i g(x) \quad \text{n. e. in} \ A_i, \quad i \in I, \tag{47}
\]

where \( C_i = C_i(\mathcal{A}, a, g) \), \( i \in I \), are the interior capacitory constants. Actually,

\[
C_i = \frac{\alpha_i \kappa(\gamma^i, \gamma)}{\text{cap} \mathcal{A}} = \inf_{x \in A_i^+} \frac{\alpha_i a_i \kappa(x, \gamma)}{g(x)}, \quad i \in I. \tag{48}
\]

If \( I^- \neq \emptyset \), assume moreover that the kernel \( \kappa(x, y) \) is continuous on \( \overline{\mathcal{A}}^+ \times \overline{\mathcal{A}}^- \), while \( \kappa(\cdot, y) \to 0 (\text{as} \ y \to \infty) \) uniformly on compact sets. Then, for every \( i \in I \),

\[
\alpha_i a_i \kappa(x, \gamma) \leq C_i g(x) \quad \text{for all} \ x \in S(\gamma^i), \tag{49}
\]

and hence

\[
\alpha_i a_i \kappa(x, \gamma) = C_i g(x) \quad \text{n. e. in} \ A_i \cap S(\gamma^i).
\]

Also note that \( \mathcal{D}(\mathcal{A}, a, g) \) is contained in an equivalence class in \( \mathcal{M}(\overline{\mathcal{A}}) \) provided the kernel \( \kappa \) is strictly positive definite, and it consists of a unique element \( \gamma_{\mathcal{A}} \) if, moreover, all \( \overline{\mathcal{A}_i}, i \in I \), are mutually disjoint.

**Remark 19.** As is seen from the preceding theorem, the properties of interior capacitary distributions associated with a condenser are quite similar to those of
interior capacitary distributions associated with a set (cf. [F1, Th. 4.1]). The only important difference is that the sign $\leq$ in the inclusion (45) in general can not be omitted — even for a finite, closed, noncompact condenser. Cf. Remark 17.

Remark 20. Like as in the theory of interior capacities of sets, in general none of the i-coordinates of $\gamma_A$ is concentrated on $A_i$ (unless $A_i$ is closed). Indeed, let $X = \mathbb{R}^n$, $n \geq 3$, $\kappa(x,y) = |x-y|^{2-n}$, $g = 1$, $I^+ = \{1\}$, $I^- = \{2\}$, $a_1 = a_2 = 1$, and let $A_1 = \{x : |x| < r\}$ and $A_2 = \{x : |x| > R\}$, where $0 < r < R < \infty$. Then it can be shown that

$$\gamma_A = \gamma_{\mathcal{A}} = \left[\theta^+ - \theta^-\right] \text{cap} \mathcal{A},$$

where $\theta^+$ and $\theta^-$ are obtained by the uniform distribution of unit mass over the spheres $S(0, r)$ and $S(0, R)$, respectively. Hence, $|\gamma_A|(A) = 0$.

9.2. The purpose of this section is to point out characteristic properties of the interior capacitary distributions and the interior capacitary constants.

**Proposition 1.** Assume $\mu \in \mathcal{E}(\mathcal{A})$ has the properties

$$\|\mu\|^2 = \text{cap} (\mathcal{A}, a, g),$$

$$\alpha_ia_i\kappa(x,\mu) \geq \frac{\alpha_i\kappa(\mu_i,\mu)}{\text{cap} \mathcal{A}} g(x) \quad n. e. \text{ in } A_i, \quad i \in I.$$

Then $\mu$ is equivalent in $\mathcal{E}(\mathcal{A})$ to every $\gamma_A \in \mathcal{D}(\mathcal{A}, a, g)$ and, for all $i \in I$,

$$C_i(\mathcal{A}, a, g) = \frac{\alpha_ia_i\kappa(x,\mu)}{\text{cap} \mathcal{A}} = \left(\inf_{x \in A_i}\frac{\alpha_ia_i\kappa(x,\mu)}{g(x)}\right).$$

Actually, there holds the following stronger result, to be proved in Sec. 17 below.

**Proposition 2.** Let $\nu \in \mathcal{E}(\mathcal{A})$ and $\tau_i \in \mathbb{R}$, $i \in I$, satisfy the relations

$$\alpha_ia_\kappa(x,\nu) \geq \tau_i g(x) \quad n. e. \text{ in } A_i, \quad i \in I,$$

$$\sum_{i \in I} \tau_i = \frac{\text{cap} \mathcal{A} + \|\nu\|^2}{2 \text{cap} \mathcal{A}}.\quad (51)$$

Then $\nu$ is equivalent in $\mathcal{E}(\mathcal{A})$ to every $\gamma_A \in \mathcal{D}(\mathcal{A}, a, g)$ and, for all $i \in I$,

$$\tau_i = C_i(\mathcal{A}, a, g) = \left(\inf_{x \in A_i}\frac{\alpha_ia_i\kappa(x,\nu)}{g(x)}\right).$$

Thus, under the conditions of Proposition 1 or 2, if moreover $\kappa$ is strictly positive definite and all $A_i$, $i \in I$, are mutually disjoint, then the measure under consideration is actually the (unique) interior capacitary distribution $\gamma_A$. 25
10. On continuity of the interior capacities, capacitary distributions, and capacitary constants

10.1. Given $A_n = (A^n_i)_{i \in I}$, $n \in \mathbb{N}$, and $A$ in $\mathcal{C} = \mathcal{C}(I^+, I^-)$, we write $A_n \uparrow A$ if $A_n \prec A_{n+1}$ for all $n$ and

$$A_i = \bigcup_{n \in \mathbb{N}} A^n_i, \quad i \in I.$$ 

Following [B1, Chap. 1, §9], we call a locally compact space countable at infinity if it can be written as a countable union of compact sets.

**Theorem 6.** Suppose that either $g_{\text{inf}} > 0$ or the space $X$ is countable at infinity. If $A_n, n \in \mathbb{N}$, are universally measurable and $A_n \uparrow A$, then

$$\text{cap} (A, a, g) = \lim_{n \in \mathbb{N}} \text{cap} (A_n, a, g). \tag{53}$$

Assume moreover $\text{cap} (A, a, g)$ to be finite, and let $\gamma_n := \gamma_{A_n}, n \in \mathbb{N}$, denote an interior capacitary distribution associated with $A_n$. If $\gamma$ is an $A$-vague limit point of $(\gamma_n)_{n \in \mathbb{N}}$ (such a $\gamma$ exists), then $\gamma$ is actually an interior capacitary distribution associated with the condenser $A$, and

$$\lim_{n \in \mathbb{N}} \|\gamma_n - \gamma\|^2 = 0.$$

Furthermore,

$$C_i(A, a, g) = \lim_{n \in \mathbb{N}} C_i(A_n, a, g), \quad i \in I. \tag{54}$$

Thus, if $\kappa$ is additionally assumed to be strictly positive definite (hence, perfect) and all $A_i, i \in I$, are mutually disjoint, then the (unique) interior capacitary distribution associated with $A_n$ converges both $A$-vaguely and strongly to the (unique) interior capacitary distribution associated with $A$.

**Remark 21.** Theorem 6 remains true if $(A_n)_{n \in \mathbb{N}}$ is replaced by the increasing ordered family of all compact condensers $K$ such that $K \prec A$. Moreover, then the assumption that either $g_{\text{inf}} > 0$ or $X$ is countable at infinity can be omitted. Cf., e.g., Lemma 5 and Corollary 13.

**Remark 22.** If $I = \{1\}$ and $g = 1$, Theorem 6 has been proved in [F1, Th. 4.2].

10.2. The remainder of the article is devoted to proving the results formulated in Sec. 6–10 and is organized as follows. Theorems 2, 3, 5, and 6 are proved in Sec. 15, 16, and 18. Their proofs utilize a description of the potentials of measures from the classes $\mathcal{M}'(A, a, g)$ and $\mathcal{M}_0(A, a, g)$, to be given in Sec. 13 and 14 by Lemmas 12 and 13. In turn, Lemmas 12 and 13 use a theorem on the strong completeness of proper subspaces of $\mathcal{E}(A)$, which is a subject of Sec. 11.
11. On the strong completeness

11.1. Keeping all our standing assumptions on \( \kappa, g, a, \) and \( \mathcal{A} \), stated in Sec. 5.2, we consider \( \mathcal{E}(\mathcal{A}, \leq a, g) \) to be a topological subspace of the semimetric space \( \mathcal{E}(\mathcal{A}) \); the induced topology is likewise called the strong topology.

**Theorem 7.** Suppose \( \mathcal{A} \) is closed. Then the semimetric space \( \mathcal{E}(\mathcal{A}, \leq a, g) \) is complete. In more detail, if \( (\mu_s)_{s \in S} \subset \mathcal{E}(\mathcal{A}, \leq a, g) \) is a strong Cauchy net and \( \mu \) is its \( \mathcal{A} \)-vague cluster point (such a \( \mu \) exists), then \( \mu \in \mathcal{E}(\mathcal{A}, \leq a, g) \) and

\[
\lim_{s \in S} \|\mu_s - \mu\|^2 = 0. \tag{55}
\]

Assume, in addition, that the kernel is strictly positive definite and all \( A_i \), \( i \in I \), are mutually disjoint. If moreover \( (\mu_s)_{s \in S} \subset \mathcal{E}(\mathcal{A}, \leq a, g) \) converges strongly to \( \mu_0 \in \mathcal{E}(\mathcal{A}) \), then actually \( \mu_0 \in \mathcal{E}(\mathcal{A}, \leq a, g) \) and \( \mu_s \rightarrow \mu_0 \) \( \mathcal{A} \)-vaguely.

**Remark 23.** This theorem is certainly of independent interest since, according to the well-known counterexample by H. Cartan \([C]\), the pre-Hilbert space \( \mathcal{E} \) is strongly incomplete even for the Newtonian kernel \( |x - y|^2 - n \) in \( \mathbb{R}^n \), \( n \geq 3 \).

**Remark 24.** Assume the kernel is strictly positive definite (hence, perfect). If moreover \( I^- = \emptyset \), then Theorem 7 remains valid for \( \mathcal{E}(\mathcal{A}) \) in place of \( \mathcal{E}(\mathcal{A}, \leq a, g) \) (cf. Theorem 1). A question still unanswered is whether this is the case if \( I^+ \) and \( I^- \) are both nonempty. We can however show that this is really so for the Riesz kernels \( |x - y|^\alpha - n, 0 < \alpha < n, \) in \( \mathbb{R}^n, n \geq 2 \) (cf. \([Z1, Th. 1]\)). The proof utilizes Deny’s theorem \([D1]\) stating that, for the Riesz kernels, \( \mathcal{E} \) can be completed with making use of distributions of finite energy.

11.2. We start by auxiliary assertions to be used in the proof of Theorem 7.

**Lemma 10.** \( \mathcal{E}(\mathcal{A}, \leq a, g) \) is \( \mathcal{A} \)-vaguely bounded.

**Proof.** Fix \( i \in I \), and let a compact set \( K \subset A_i \) be given. Since \( g \) is positive and continuous, the inequalities

\[
a_i \geq \int g d\mu^i \geq \mu^i(K) \min_{x \in K} g(x), \quad \text{where} \quad \mu \in \mathcal{E}(\mathcal{A}, \leq a, g),
\]

yield

\[
\sup_{\mu \in \mathcal{E}(\mathcal{A}, \leq a, g)} \mu^i(K) < \infty,
\]

and the lemma follows. \( \square \)

**Lemma 11.** Suppose \( \mathcal{A} \) is closed. If a net \( (\mu_s)_{s \in S} \subset \mathcal{E}(\mathcal{A}, \leq a, g) \) is strongly bounded, then its \( \mathcal{A} \)-vague cluster set is nonempty and contained in \( \mathcal{E}(\mathcal{A}, \leq a, g) \).

**Proof.** We begin by showing that the nets \( (R\mu^+_s)_{s \in S} \) and \( (R\mu^-_s)_{s \in S} \) are strongly bounded as well, i.e.,

\[
\sup_{s \in S} \|R\mu^\pm_s\|^2 < \infty, \quad i \in I. \tag{56}
\]
Of course, this needs to be proved only when $I^- \neq \emptyset$; then, in accordance with the standing assumptions, all the relations $(25)$, $(26)$, and $(27)$ hold. Since

$$\int g \, d\mu^i_s \leq a_i, \quad i \in I,$$

$(25)$ implies

$$\sup_{s \in S} \mu^i_s(X) \leq a_i g^{-1}_{\inf} < \infty, \quad i \in I.$$  

(58)

Hence, by $(27)$,

$$\sup_{s \in S} R\mu^\pm_s(X) \leq |a| g^{-1}_{\inf} < \infty.$$  

When combined with $(26)$, this shows that $\kappa(R\mu^+_s, R\mu^-_s)$ is bounded from above on $S$, and hence so do $\|R\mu^+_s\|^2$ and $\|R\mu^-_s\|^2$.

Moreover, according to Lemmas 1 and 10, there exists an $A$-vague cluster point $\mu$ of the net $(\mu_s)_{s \in S}$. Denoting by $(\mu_d)_{d \in D}$ a subnet of $(\mu_s)_{s \in S}$ such that

$$\mu^i_d \to \mu^i \text{ vaguely for all } i \in I,$$

we get from Lemma 2

$$R\mu^\pm_d \to R\mu^\pm \text{ vaguely.}$$

(60)

It remains to show that $R\mu^+$ and $R\mu^-$ are both of finite energy and that $(57)$ holds true for $\mu^i$ in place of $\mu^i_s$. To this end, recall that, if $Y$ is a locally compact Hausdorff space and $\psi$ is a lower semicontinuous function on $Y$ such that $\psi \geq 0$ unless its support is compact, then the map

$$\nu \mapsto \int \psi \, d\nu, \quad \nu \in \mathcal{M}^+(Y),$$

is lower semicontinuous in the induced vague topology (see, e.g., [F1]). Applying this to $Y = X \times X$, $\psi = \kappa$ and, subsequently, $Y = A_i$, $\psi = g|_{A_i}$ and using $(56)$, $(60)$ and, respectively, $(57)$ and $(59)$, we arrive at the required assertions. \hfill $\Box$

**Corollary 15.** Assume a net $(\mu_s)_{s \in S} \subset \mathcal{E}(A, \leq a, g)$ is strongly bounded. Then for every $i \in I$, $\|\mu^i_s\|^2$ and $\kappa(\mu^i_s, \mu^i_s)$ are bounded on $S$.

**Proof.** In view of $(56)$, the required relation

$$\sup_{s \in S} \|\mu^i_s\|^2 < \infty \quad \text{for all } i \in I,$$

(61)

will be proved once we show that

$$\sum_{i,j \in I^\pm} \kappa(\mu^i_s, \mu^j_s) \geq C > -\infty,$$

(62)

where $C$ is independent of $s$. Since $(62)$ is obvious when $\kappa \geq 0$, one can assume $X$ to be compact. Then $\kappa$, being lower semicontinuous, is bounded from below
on $\mathbf{X}$ (say by $-c$, where $c > 0$), while $A$ and, hence, $|a|$ are finite. Furthermore, then $g_{\text{inf}} > 0$; therefore, (58) holds true. This implies that
\[
\kappa(\mu_s^i, \mu_s^j) \geq -a_i a_j g_{\text{inf}}^2 c \quad \text{for all } i, j \in I,
\]
and (62) follows.

The above arguments also show that $\kappa(\mu_s^i, R\mu_s^+) \text{ and } \kappa(\mu_s^i, R\mu_s^-)$, where $i \in I$ is given, are both bounded from below on $S$. Since these functions of $s$ are bounded from above as well, which is clear from (56) and (61) by the Cauchy-Schwarz inequality, the required boundedness of $\kappa(\mu_s^i, \mu_s^j)$ on $S$ follows. □

11.3. Proof of Theorem 7. Suppose $A$ is closed, and let $(\mu_s)_{s \in S}$ be a strong Cauchy net in $\mathcal{E}(A, \leq a, g)$. Since such a net converges strongly to every its strong cluster point, $(\mu_s)_{s \in S}$ can certainly be assumed to be strongly bounded. Then, by Lemma 11, there exists an $A$-vague cluster point $\mu$ of $(\mu_s)_{s \in S}$, and
\[
\mu \in \mathcal{E}(A, \leq a, g).
\] (63)

We next proceed to verify (55). Of course, there is no loss of generality in assuming $(\mu_s)_{s \in S}$ to converge $A$-vaguely to $\mu$. Then, by Lemma 2, $(R\mu_s^+)_{s \in S}$ and $(R\mu_s^-)_{s \in S}$ converge vaguely to $R\mu^+$ and $R\mu^-$, respectively. Since, by (56), these nets are strongly bounded in $\mathcal{E}^+$, the property $(CW)$ (see Sec. 2) shows that they approach $R\mu^+$ and $R\mu^-$, respectively, in the weak topology as well, and so
\[
R\mu_s \rightarrow R\mu \quad \text{weakly}.
\]

This gives
\[
\|\mu_s - \mu\|^2 = \|R\mu_s - R\mu\|^2 = \lim_{l \in S} \kappa(R\mu_s - R\mu, R\mu_s - R\mu_l),
\]
and hence, by the Cauchy-Schwarz inequality,
\[
\|\mu_s - \mu\|^2 \leq \|\mu_s - \mu\| \lim_{l \in S} \inf \|\mu_s - \mu_l\|,
\]
which proves (55) as required, because $\|\mu_s - \mu_l\|$ becomes arbitrarily small when $s, l \in S$ are both large enough.

Suppose now that $\kappa$ is strictly positive definite, while all $A_i, i \in I$, are mutually disjoint, and let the net $(\mu_s)_{s \in S}$ converge strongly to some $\mu_0 \in \mathcal{E}(A)$. Given an $A$-vague limit point $\mu$ of $(\mu_s)_{s \in S}$, then we conclude from (55) that $\|\mu_0 - \mu\| = 0$, hence $\mu_0 \equiv \mu$ since $\kappa$ is strictly positive definite, and finally $\mu_0 \equiv \mu$ because $A_i, i \in I$, are mutually disjoint. In view of (63), this means that $\mu_0 \in \mathcal{E}(A, \leq a, g)$, which is a part of the desired conclusion. Moreover, $\mu_0$ has thus been shown to be identical to any $A$-vague cluster point of $(\mu_s)_{s \in S}$. Since the $A$-vague topology is Hausdorff, this implies that $\mu_0$ is actually the $A$-vague limit of $(\mu_s)_{s \in S}$ (cf. [31] Chap. I, § 9, n° 1, cor.), which completes the proof. □
12. Proof of Lemma 8

Fix any \((\mu_s)_{s \in S}\) and \((\nu_t)_{t \in T}\) in \(\mathbb{M}(\mathcal{A}, a, g)\). It follows by standard arguments that
\[
\lim_{(s,t) \in S \times T} \|\mu_s - \nu_t\|^2 = 0,
\]
(64)
where \(S \times T\) is the directed product of the directed sets \(S\) and \(T\) (see, e.g., [K, Chap. 2, § 3]). Indeed, by the convexity of the class \(\mathcal{E}(\mathcal{A}, a, g)\),
\[
2 \|\mathcal{E}(\mathcal{A}, a, g)\| \leq \|\mu_s + \nu_t\| \leq \|\mu_s\| + \|\nu_t\|,
\]
and hence, by (28),
\[
\lim_{(s,t) \in S \times T} \|\mu_s + \nu_t\|^2 = 4 \|\mathcal{E}(\mathcal{A}, a, g)\|^2.
\]

Then the parallelogram identity gives (64) as claimed.

Relation (64) implies that \((\mu_s)_{s \in S}\) is strongly fundamental. Therefore Theorem 7 shows that there exists an \(A\)-vague cluster point \(\mu_0\) of \((\mu_s)_{s \in S}\), and moreover \(\mu_0 \in \mathcal{E}(\overline{\mathcal{A}}, \leq a, g)\) and \(\mu_s \to \mu_0\) strongly. This means that \(\mathcal{M}(\mathcal{A}, a, g)\) and \(\mathcal{M}'(\mathcal{A}, a, g)\) are both nonempty and satisfy the inclusion (29).

What is left is to prove that \(\mu_s \to \chi\) strongly, where \(\chi \in \mathcal{M}'(\mathcal{A}, a, g)\) is given. But then one can choose a net in \(\mathbb{M}(\mathcal{A}, a, g)\), say \((\nu_t)_{t \in T}\), convergent to \(\chi\) strongly, and repeated application of (64) gives immediately the desired conclusion. □

13. Potentials of strong cluster points of minimizing nets

13.1. The aim of this section is to provide a description of the potentials of measures from the class \(\mathcal{M}'(\mathcal{A}, a, g)\). As usual, we are keeping all our standing assumptions, stated in Sec. 5.2.

Lemma 12. There exist \(\eta_i \in \mathbb{R}, i \in I\), such that, for every \(\chi \in \mathcal{M}'(\mathcal{A}, a, g)\),
\[
\alpha_i a_i \kappa(x, \chi) \geq \alpha_i \eta_i g(x) \quad \text{n. e. in } A_i, \quad i \in I,
\]
(65)
\[
\sum_{i \in I} \alpha_i \eta_i = \|\mathcal{E}(\mathcal{A}, a, g)\|^2.
\]
(66)

These \(\eta_i, i \in I\), are determined uniquely and given by either of the formulas
\[
\eta_i = \kappa(\zeta_i^t, \zeta),
\]
(67)
\[
\eta_i = \lim_{s \in S} \kappa(\mu_s^t, \mu_s),
\]
(68)
where \(\zeta \in \mathcal{M}(\mathcal{A}, a, g)\) and \((\mu_s)_{s \in S} \in \mathbb{M}(\mathcal{A}, a, g)\) are arbitrarily chosen.
Proof. Throughout the proof, we shall assume every net of the class $M(A, a, g)$ to be strongly bounded, which certainly involves no loss of generality.

Fix $\zeta \in M(A, a, g)$ and choose $(\mu_t)_{t \in T} \in M(A, a, g)$ that converges $A$-vaguely to $\zeta$. We begin by showing that

$$\kappa(\zeta^i, \zeta) = \lim_{t \in T} \kappa(\mu^i_t, \mu_t), \quad i \in I. \quad (69)$$

Since, by Corollary 15, $\|\mu^i_t\|$ is bounded from above on $T$ (say by $M_1$), while $\mu^i_t \to \zeta^i$ vaguely, the property $(CW)$ yields that $\mu^i_t$ approaches $\zeta^i$ also weakly. Hence, for every $\varepsilon > 0$,

$$|\kappa(\zeta^i - \mu^i_t, \zeta)| < \varepsilon$$

whenever $t \in T$ is large enough. Furthermore, by the Cauchy-Schwarz inequality,

$$|\kappa(\mu^i_t, \zeta) - \kappa(\mu^i_t, \mu_t)| = |\kappa(\mu^i_t, R\zeta - R\mu_t)| \leq M_1 \|\zeta - \mu_t\|, \quad t \in T.$$ 

Since, by Lemma 8, $\mu_t \to \zeta$ strongly, the last two relations combined give (69).

We next proceed to show that $\eta_i, i \in I$, defined by (67), satisfy both (65) and (66), where $\chi \in M'(A, a, g)$ is given. Since (66) is obtained directly from

$$\sum_{i \in I} \alpha_i \kappa(\zeta^i, \zeta) = \|\zeta\|^2 = \|E(A, a, g)\|^2,$$

suppose, contrary to (65), that there exist $j \in I$ and a set $E_j \subset A_j$ of interior capacity nonzero such that

$$\alpha_j a_j \kappa(x, \chi) < \alpha_j \eta_j g(x) \quad \text{for all } x \in E_j. \quad (70)$$

Then one can choose $\nu \in E^+$ with compact support so that $S(\nu) \subset E_j$ and

$$\int g \, d\nu = a_j.$$

Integrating the inequality in (70) with respect to $\nu$ gives

$$\alpha_j [\kappa(\chi, \nu) - \eta_j] < 0. \quad (71)$$

To get a contradiction, for every $\tau \in (0, 1]$ write

$$\tilde{\mu}_t^i := \begin{cases} \mu^i_t - \tau(\mu^i_t - \nu) & \text{if } i = j, \\ \mu^i_t & \text{otherwise.} \end{cases}$$

Clearly,

$$\tilde{\mu}_t := \sum_{i \in I} \alpha_i \tilde{\mu}_t^i \in E(A, a, g), \quad t \in T,$$
and consequently
\[ \|E(A, a, g)\|^2 \leq \|\tilde{\mu}_t\|^2 = \|\mu_t\|^2 - 2\alpha_j \tau \kappa(\mu_t, \mu^j_t - \nu) + \tau^2 \|\mu^j_t - \nu\|^2. \quad (72) \]
The coefficient of \(\tau^2\) is bounded from above on \(T\) (say by \(M_0\)), while by Lemma 8
\[ \lim_{t \to T} \|\mu_t - \chi\|^2 = 0. \]
Combining (67), (69) and substituting the result obtained into (72) therefore gives
\[ 0 \leq M_0 \tau^2 + 2\alpha_j \tau \left[ \kappa(\chi, \nu) - \eta_j \right]. \]
By letting here \(\tau\) tend to 0, we arrive at a contradiction to (71), which proves (65).
To prove the statement on uniqueness, consider some other \(\eta'_i, i \in I\), satisfying both (65) and (66). Then they are necessarily finite, and for every \(i\),
\[ \alpha_i a_i \kappa(x, \chi) \geq \max\{\alpha_i \eta_i, \alpha_i \eta'_i\} g(x) \text{ n.e. in } A_i, \quad (73) \]
which follows from the property of sets of interior capacity zero mentioned in Sec. 7.1. Since \(\mu^j_i\) is concentrated on \(A_i\) and has finite energy and compact support, application of [F1, Lemma 2.3.1] shows that the inequality in (73) holds \(\mu^j_i\)-almost everywhere in \(X\). Integrating it with respect to \(\mu^j_i\) and then summing up over all \(i \in I\), in view of \(\int g \, d\mu^j_i = a_i\) we have
\[ \kappa(\mu_t, \chi) \geq \sum_{i \in I} \max\{\alpha_i \eta_i, \alpha_i \eta'_i\}, \quad t \in T. \]
Passing here to the limit as \(t\) ranges over \(T\), we get
\[ \|\chi\|^2 = \lim_{t \to T} \kappa(\mu_t, \chi) \geq \sum_{i \in I} \max\{\alpha_i \eta_i, \alpha_i \eta'_i\} \geq \sum_{i \in I} \alpha_i \eta_i = \|E(A, a, g)\|^2, \]
and hence
\[ \max\{\alpha_i \eta_i, \alpha_i \eta'_i\} = \alpha_i \eta_i, \quad i \in I, \]
for the extreme left and right parts of the above chain of inequalities are equal.
Applying the same arguments again, but with the roles of \(\eta_i\) and \(\eta'_i\) reversed, we conclude that \(\eta_i = \eta'_i\) for all \(i \in I\), as claimed.
It remains to show that \(\eta_i\) can be written in the form (68), where \((\mu_s)_{s \in S} \in \mathcal{M}(A, a, g)\) is given. By Corollary 15, for every \(i \in I\), \(\kappa(\mu^j_i, \mu_s)\) is bounded on \(S\).
Fix \(j \in I\) and choose a cluster point \(\eta^0_j\) of \(\{\kappa(\mu^j_s, \mu_s) : s \in S\}\); then, in view of Lemmas 1 and 10, one can select an \(A\)-vaguely convergent subnet \((\mu_d)_{d \in D}\) of \((\mu_s)_{s \in S}\) such that
\[ \eta^0_j = \lim_{d \to D} \kappa(\mu^j_d, \mu_d). \]
But what has been proved just above implies immediately that \(\eta^0_j = \eta_j\). Since this means that any cluster point of the net \(\kappa(\mu^j_s, \mu_s), s \in S\), coincides with \(\eta_j\), (68) follows. \(\square\)
13.2. In what follows, $\eta_i =: \eta_i(A, a, g)$, $i \in I$, will always denote the numbers appeared in Lemma 12. They are uniquely determined by relation (65), where $\chi \in \mathcal{M}'(A, a, g)$ is arbitrarily chosen, taken together with (66). This statement on uniqueness can actually be strengthened as follows.

Lemma 12'. Given $\chi \in \mathcal{M}'(A, a, g)$, choose $\eta'_i \in \mathbb{R}$, $i \in I$, so that
$$
\sum_{i \in I} \alpha_i \eta'_i \geq \| \mathcal{E}(A, a, g) \|^2.
$$
If there holds (65) for $\eta'_i$ in place of $\eta_i$, then $\eta'_i = \eta_i$ for all $i \in I$.

Proof. This follows in the same manner as the uniqueness statement in Lemma 12. □

13.3. The following assertion is specifying Lemma 12 for a compact condenser $K$.

Corollary 16. Let $A = K$ be compact. Given $\lambda_K \in S(K, a, g)$, then for every $i$,
$$
\alpha_i a_i \kappa(x, \lambda_K) \geq \alpha_i \kappa(\lambda^i_K, \lambda_K) g(x) \quad \text{n. e. in } K_i,
$$
and hence
$$
\alpha_i a_i \kappa(x, \lambda_K) = \kappa(\lambda^i_K, \lambda_K) g(x) \quad \lambda^i_K-\text{almost everywhere.}
$$

Proof. In view of (61) and (67), $\eta_i(K, a, g)$, $i \in I$, can be written in the form
$$
\eta_i(K, a, g) = \kappa(\lambda^i_K, \lambda_K),
$$
which leads to (74) when substituted into (65). Since $\lambda^i_K$ has finite energy and is supported by $K_i$, the inequality in (74) holds $\lambda^i_K$-almost everywhere in $X$. Hence, (75) must be true, for if not, we would arrive at a contradiction by integrating the inequality in (74) with respect to $\lambda^i_K$. □

14. Potentials of $A$-vague cluster points of minimizing nets

In this section we shall restrict ourselves to measures $\xi$ of the class $\mathcal{M}_0(A, a, g)$. It is clear from Corollary 7 that their potentials have all the properties described in Lemmas 12 and 12'. Our purpose is to show that, under proper additional restrictions on the kernel, that description can be sharpened as follows.

Lemma 13. In the case where $I^- \neq \emptyset$, assume moreover that $\kappa(x, y)$ is continuous on $A^+ \times A^-$, while $\kappa(\cdot, y) \to 0$ (as $y \to \infty$) uniformly on compact sets. Given $\xi \in \mathcal{M}_0(A, a, g)$, then for all $i \in I$,
$$
\alpha_i a_i \kappa(x, \xi) \geq \alpha_i \kappa(\xi^i, \xi) g(x) \quad \text{n. e. in } A_i, \quad (76)
$$
$$
\alpha_i a_i \kappa(x, \xi) \leq \alpha_i \kappa(\xi^i, \xi) g(x) \quad \text{for all } x \in S(\xi^i), \quad (77)
$$
and hence

\[ a_i \kappa(x, \xi) = \kappa(\xi^i, \xi) g(x) \quad \text{n. e. in } A_i \cap S(\xi^i). \]

**Proof.** Choose \( \lambda_K \in \mathcal{S}(\mathcal{K}, a, g) \) such that \( \xi \) is an \( \mathcal{A} \)-vague cluster point of the net \((\lambda_K)_{K \in \{K\}_A}\). Since this net belongs to \( \mathcal{M}(\mathcal{A}, a, g) \), from (67) and (68) we get

\[ \eta_i = \kappa(\xi^i, \xi) = \lim_{K \in \{K\}_A} \kappa(\lambda^i_K, \lambda_K), \quad i \in I. \]

Substituting this into (65) with \( \xi \) in place of \( \chi \) gives (76) as required.

We next proceed to prove (77). To this end, fix \( i \) (say \( i \in I^+ \)) and \( x_0 \in S(\xi^i) \). Without loss of generality it can certainly be assumed that

\[ \lambda_K \to \xi \quad \text{\( \mathcal{A} \)-vaguely,} \quad (78) \]

since otherwise we shall pass to a subnet and change the notation. Then, due to (75) and (78), one can choose \( x_K \in S(\lambda^i_K) \) so that

\[ x_K \to x_0 \quad \text{as } K \uparrow A, \quad (79) \]

\[ a_i \kappa(x_K, \lambda_K) = \kappa(\lambda^i_K, \lambda_K) g(x_K). \]

Taking into account that, by [FII, Lemma 2.2.1], the map \((x, \nu) \mapsto \kappa(x, \nu)\) is lower semicontinuous on the product space \( \mathcal{X} \times \mathcal{M}^+ \) (where \( \mathcal{M}^+ \) is equipped with the vague topology), we conclude from what has already been shown that the desired relation (77) will follow once we prove

\[ \kappa(x_0, R\xi^-) = \lim_{K \in \{K\}_A} \kappa(x_K, R\lambda^-_K). \quad (80) \]

The case we are thus left with is \( I^- \neq \emptyset \). Then, according to our standing assumptions, \( g_{\text{int}} > 0 \) and \( |a| < \infty \), and therefore there is \( q \in (0, \infty) \) such that

\[ R\lambda^-_K(\mathcal{X}) \leq q \quad \text{for all } K \in \{K\}_A. \quad (81) \]

Since, by (78) and Lemma 2, \( R\lambda^-_K \to R\xi^- \) vaguely, we thus get

\[ R\xi^- (\mathcal{X}) \leq q. \quad (82) \]

Fix \( \varepsilon > 0 \). Under the assumptions of the lemma, one can choose a compact neighborhood \( W_{x_0} \) of the point \( x_0 \) in \( \overline{A^-} \) and a compact neighborhood \( F \) of \( W_{x_0} \) in \( \mathcal{X} \) so that

\[ F_* := F \cap \overline{A^-} \neq \emptyset \]

and

\[ |\kappa(x, y)| < q^{-1} \varepsilon \quad \text{for all } (x, y) \in W_{x_0} \times \mathcal{C}F. \quad (83) \]
In the remainder, $\tilde{C}$ and $\partial\tilde{A}$ denote respectively the complement and the boundary of a set relative to $A^-$ (where $A^-$ is regarded to be a topological subspace of $X$).

Having observed that $\kappa|_{W_{x_0} \times \overline{A^-}}$ is continuous, we proceed to construct a function

$$\varphi \in C_0(W_{x_0} \times \overline{A^-})$$

with the following properties:

$$\varphi|_{W_{x_0} \times F_*} = \kappa|_{W_{x_0} \times F_*},$$

$$|\varphi(x, y)| \leq q^{-1}\varepsilon \quad \text{for all } (x, y) \in W_{x_0} \times \tilde{C}F_*.$$  \hspace{1cm} (84)

To this end, consider a compact neighborhood $V_*$ of $F_*$ in $\overline{A^-}$ and write

$$f := \begin{cases} \kappa & \text{on } W_{x_0} \times \tilde{\partial}F_*, \\ 0 & \text{on } W_{x_0} \times \tilde{\partial}V_* \end{cases}$$

Note that $E := (W_{x_0} \times \tilde{\partial}F_*) \cup (W_{x_0} \times \tilde{\partial}V_*)$ is a compact subset of the Hausdorff and compact, hence normal, space $W_{x_0} \times V_*$ and $f$ is continuous on $E$. By using the Tietze-Urysohn extension theorem (see, e.g., [E2, Th. 0.2.13]), we deduce from (83) that there exists a continuous function $\hat{f} : W_{x_0} \times V_* \to [-\varepsilon q^{-1}, \varepsilon q^{-1}]$ such that $\hat{f}|_E = f|_E$. Thus, the function in question can be defined as follows:

$$\varphi := \begin{cases} \kappa & \text{on } W_{x_0} \times F_*, \\ \hat{f} & \text{on } W_{x_0} \times (V_* \setminus F_*), \\ 0 & \text{on } W_{x_0} \times \tilde{C}V_* \end{cases}$$

Furthermore, since the function $\varphi$ is continuous on $W_{x_0} \times \overline{A^-}$ and has compact support, one can choose a compact neighborhood $U_{x_0}$ of $x_0$ in $W_{x_0}$ so that

$$|\varphi(x, y) - \varphi(x_0, y)| < q^{-1}\varepsilon \quad \text{for all } (x, y) \in U_{x_0} \times \overline{A^-}. \hspace{1cm} (86)$$

Given an arbitrary measure $\nu \in M^+(\overline{A^-})$ with the property that $\nu(X) \leq q$, we conclude from (83)–(86) that, for all $x \in U_{x_0}$,

$$|\kappa(x, \nu|_{\tilde{C}F})| \leq \varepsilon; \quad \hspace{1cm} (87)$$

$$\kappa(x, \nu|_F) = \int \varphi(x, y) d(\nu - \nu|_{\tilde{C}F})(y), \quad \hspace{1cm} (88)$$

$$\left| \int \varphi(x, y) d\nu|_{\tilde{C}F}(y) \right| \leq \varepsilon; \quad \hspace{1cm} (89)$$

$$\left| \int [\varphi(x, y) - \varphi(x_0, y)] d\nu(y) \right| \leq \varepsilon. \quad \hspace{1cm} (90)$$

35
Finally, choose $K_0 \in \{K\}_A$ so that for all $K \succ K_0$ there hold $x_K \in U_x$ and
\[ \left| \int \varphi(x_0, y) d(R\lambda_K - R\xi^-)(y) \right| < \varepsilon; \]
such a $K_0$ exists because of (78) and (79). Applying now relations (87) – (90) to each of the measures $R\lambda_K - R\xi^-$, which is possible due to (81) and (82), for all $K \succ K_0$ we therefore get
\[ |\kappa(x_K, R\lambda_K) - \kappa(x_0, R\xi^-)| \leq |\kappa(x_K, R\lambda_K) - \kappa(x_0, R\xi^-)| + 2\varepsilon \]
\[ \leq \left| \int \varphi(x_K, y) dR\lambda_K(y) - \int \varphi(x_0, y) dR\xi^-(y) \right| + 4\varepsilon \]
\[ \leq \left| \int \left[ \varphi(x_K, y) - \varphi(x_0, y) \right] dR\lambda_K(y) \right| + \left| \int \varphi(x_0, y) d(R\lambda_K - R\xi^-)(y) \right| + 4\varepsilon \]
\[ \leq \varepsilon + \varepsilon + 4\varepsilon = 6\varepsilon, \]
and (80) follows by letting $\varepsilon$ tend to 0. The proof is complete. \qed

15. Proof of Theorems 2 and 3

We begin by showing that
\[ \text{cap } (A, a, g) \leq \|\hat{\Gamma}(A, a, g)\|^2. \] (91)
To this end, $\|\hat{\Gamma}(A, a, g)\|^2$ can certainly be assumed to be finite. Then there are $\nu \in \hat{\Gamma}(A, a, g)$ and $\mu \in \mathcal{E}(A, a, g)$, the existence of $\mu$ being clear from (19) and Corollary 3. By [F1, Lemma 2.3.1], the inequality in (32) holds $\mu^i$-almost everywhere. Integrating it with respect to $\mu^i$ and then summing up over all $i \in I$, in view of $\int g \, d\mu^i = a_i$ we get
\[ \kappa(\nu, \mu) \geq \sum_{i \in I} c_i(\nu), \]
hence $\kappa(\nu, \mu) \geq 1$ by (33), and finally
\[ \|\nu\|^2 \|\mu\|^2 \geq 1 \]
by the Cauchy-Schwarz inequality. The last relation, being valid for arbitrary $\nu \in \hat{\Gamma}(A, a, g)$ and $\mu \in \mathcal{E}(A, a, g)$, implies (91), which in turn yields Theorem 2 provided $\text{cap } A = \infty$. We are thus left with proving both Theorems 2 and 3 in the case where $\text{cap } A$ is finite. Then the $\mathcal{E}(A, a \text{ cap } A, g)$-problem can be considered as well.

Taking (5) and (24) into account, we deduce from Lemmas 8 and 12 with $a$ replaced by $a \text{ cap } A$ that, for every $\chi \in \mathcal{M}'(A, a \text{ cap } A, g)$,
\[ \|\chi\|^2 = \text{cap } A \] (92)
and there exist unique \( \tilde{\eta}_i \in \mathbb{R}, \ i \in I \), such that
\[
\alpha_i a_i \kappa(x, \chi) \geq \tilde{\eta}_i g(x) \quad \text{n.e. in } A_i, \quad i \in I, \quad (93)
\]
\[
\sum_{i \in I} \tilde{\eta}_i = 1. \quad (94)
\]
Actually,
\[
\tilde{\eta}_i = \alpha_i \operatorname{cap} A^{-1} \eta_i(A, a \cap A, g), \quad i \in I, \quad (95)
\]
where \( \eta_i(A, a \cap A, g), \ i \in I, \) are the numbers uniquely determined in Sec. 13.
Using the property of sets of interior capacity zero mentioned in Sec. 7.1 and the fact that the potentials of equivalent in \( \mathcal{E} \) measures coincide nearly everywhere in \( \mathbf{X} \), we conclude from (93) and (94) that
\[
\mathcal{M}'_\mathcal{E}(A, a \cap A, g) \subset \hat{\Gamma}(A, a, g). \quad (96)
\]
Together with (91) and (92), this implies that, for every \( \sigma \in \mathcal{M}'_\mathcal{E}(A, a \cap A, g) \),
\[
\operatorname{cap} A = \| \sigma \|^2 \geq \| \hat{\Gamma}(A, a, g) \|^2 \geq \operatorname{cap} A,
\]
which completes the proof of Theorem 2. The last two relations also yield
\[
\mathcal{M}'_\mathcal{E}(A, a \cap A, g) \subset \hat{\mathcal{G}}(A, a, g).
\]
As both the sides of this inclusion are equivalence classes in \( \mathcal{E} \) (see Lemma 9), they must actually be equal, and (35) follows.
Applying Lemma 12′ for \( a \cap A \) in place of \( a \), we deduce from (35) that \( c_i(\hat{\omega}), \ i \in I, \) satisfying (32) and (33) for \( \nu = \hat{\omega} \in \hat{\mathcal{G}}(A, a, g) \), are determined uniquely, do not depend on the choice of \( \hat{\omega} \), and are actually equal to \( \tilde{\eta}_i \). Therefore, substituting (67) and, subsequently, (68) for \( a \cap A \) in place of \( a \) into (95), we get (36) and (37). This proves Theorem 3.

### 16. Proof of Theorem 5

We start by observing that \( \mathcal{D}(A, a, g) \) is nonempty, contained in an equivalence class in \( \mathcal{E}(A) \), and satisfies the inclusions
\[
\mathcal{D}(A, a, g) \subset \mathcal{M}(A, a \cap A, g) \subset \mathcal{M}'(A, a \cap A, g) \cap \mathcal{E}(A, a \cap A, g). \quad (96)
\]
Indeed, this follows from (44), Corollary 7, and Lemma 8, the last two being taken for \( a \cap A \) in place of \( a \). Substituting (42) into (96) gives (45) as required.
Since, by (45), every \( \gamma \in \mathcal{D}(A, a, g) \) is a minimizer in the \( \Gamma(A, a, g) \)-problem, the claimed relations (46) and (47) are obtained directly from Theorem 3 and 4 in view of Definition 6. To show that \( C_i(A, a, g), \ i \in I, \) can actually be given by means of (48), one only needs to substitute \( \gamma \) instead of \( \zeta \) into (36) — which is possible due to (96) — and use Corollary 8.
Assume for a moment that, if $I^{-} \neq \emptyset$, then the kernel $\kappa(x, y)$ is continuous on $A^{\tau} \times A^{\tau}$, while $\kappa(\cdot, y) \rightarrow 0$ (as $y \rightarrow \infty$) uniformly on compact sets. In order to establish (19), it suffices to apply Lemma 13 (with $a \cap A$ in place of $a$) to $\gamma$, which can be done because of (14), and then substitute (18) into the result obtained.

To prove that $D(A, a, g)$ is $A$-vaguely compact, fix $(\gamma_{s})_{s \in S} \subset D(A, a, g)$. Then the theorem on iterated limits from $K$, Chap. 2, §4] yields that this net is $A$-vaguely bounded and hence, by Lemma 1, $A$-vaguely relatively compact. Let $\gamma_{0}$ denote one of its $A$-vague cluster points, and let $(\gamma_{t})_{t \in T}$ be a subnet of $(\gamma_{s})_{s \in S}$ that converges $A$-vaguely to $\gamma_{0}$. In view of (14), the proof will be completed once we show that

$$\gamma_{0} \in \mathcal{M}_{0}(A, a \cap A, g).$$

(97)

By (14), for every $t \in T$ there exist a subnet $(K_{st})_{s_{t} \in S_{t}}$ of the net $(K)_{K \in \{K\}_{A}}$ and

$$\lambda_{s_{t}} \in S(K_{st}, a \cap A, g), \quad s_{t} \in S_{t},$$

such that $\lambda_{s_{t}}$ approaches $\gamma_{t}$, $A$-vaguely as $s_{t}$ ranges over $S_{t}$. Consider the Cartesian product $\prod\{S_{t} : t \in T\}$ — that is, the collection of all functions $\psi$ on $T$ with $\psi(t) \in S_{t}$, and let $D$ denote the directed product $T \times \prod\{S_{t} : t \in T\}$ (see, e.g., $K$, Chap. 2, §3]). Given $(t, \psi) \in D$, write

$$K_{(t, \psi)} := K_{\psi(t)} \quad \text{and} \quad \lambda_{(t, \psi)} := \lambda_{\psi(t)}.$$

Then the theorem on iterated limits from $K$, Chap. 2, §4] yields that $(\lambda_{(t, \psi)})_{(t, \psi) \in D}$ converges $A$-vaguely to $\gamma_{0}$. Since, as can be seen from the above construction, $(K_{(t, \psi)})_{(t, \psi) \in D}$ forms a subnet of $(K)_{K \in \{K\}_{A}}$, this proves (97) as required. \[\square\]

17. Proof of Proposition 2

Consider $\nu \in E(A)$ and $\tau_{i} \in \mathbb{R}$, $i \in I$, satisfying both the assumptions (50) and (51), and fix arbitrarily $\gamma_{A} \in D(A, a, g)$ and $(\mu_{t})_{t \in T} \in M(A, a \cap A, g)$. Since $\mu_{t}^{i}$ is concentrated on $A_{t}$ and has finite energy and compact support, the inequality in (50) holds $\mu_{t}^{i}$-almost everywhere. Integrating it with respect to $\mu_{t}^{i}$ and then summing up over all $i \in I$, in view of (46) and (51) we obtain

$$2 \kappa(\mu_{t}, \nu) \geq \|\gamma_{A}\|^{2} + \|\nu\|^{2}, \quad t \in T.$$

But $(\mu_{t})_{t} \in T$ converges to $\gamma_{A}$ in the strong topology of the semimetric space $E(A)$, which is clear from (96) and Lemma 8 with $a \cap A$ instead of $a$. Therefore, passing in the preceding relation to the limit as $t$ ranges over $T$, we get

$$\|\nu - \gamma_{A}\|^{2} = 0,$$

which is a part of the conclusion of the proposition. In turn, the preceding relation implies that, actually, the right-hand side in (51) is equal to 1, and that $\nu \in M'(A, a \cap A, g)$. Since, in view of Theorem 3, the latter means that

$$R
\nu \in \hat{G}(A, a, g),$$

38
the claimed relation (52) follows.

\[ 18. \text{Proof of Theorem 6} \]

To establish (53), fix \( \mu \in E(A, a, g) \). Under the assumptions of the theorem, either \( g_{\inf} > 0 \), and consequently \( \mu^i(X) < \infty \) for all \( i \in I \), or \( X \) is countable at infinity; in any case, every \( A_i, i \in I \), is contained in a countable union of \( \mu^i \)-integrable sets. Therefore, by [B2, E2] (cf. also the appendix below),

\[
\int g \, d\mu^i = \lim_{n \in \mathbb{N}} \int g \, d\mu_{A_n}^i, \quad i \in I,
\]

\[
\kappa(\mu^i, \mu^j) = \lim_{n \in \mathbb{N}} \kappa(\mu_{A_n}^i, \mu_{A_n}^j), \quad i, j \in I,
\]

where \( \mu_{A_n}^i \) denotes the trace of \( \mu^i \) upon \( A_n^i \). Applying the same arguments as in the proof of Lemma 5, but now based on the preceding two relations instead of (11) and (12), we arrive at (53) as required.

By (19) and (53), for every \( n \in \mathbb{N} \), \( \text{cap} (A_n, a, g) \) can certainly be assumed to be nonzero. Suppose moreover that \( \text{cap} (A_n, a, g) \) is finite; then, by (9), so is \( \text{cap} (A_n, a, g) \). Hence, according to Theorem 5, there exists

\[
\gamma_n := \gamma_{A_n} \in D(A_n, a, g). \quad (98)
\]

Observe that \( R\gamma_n \) is a minimizer in the \( \hat{\Gamma}(A_n, a, g) \)-problem, which is clear from (35), (42), and (45). Since, furthermore,

\[
\hat{\Gamma}(A_{n+1}, a, g) \subset \hat{\Gamma}(A_n, a, g),
\]

application of Lemma 4 to \( H = \hat{\Gamma}(A_n, a, g) \), \( \nu = R\gamma_{n+1} \), and \( \lambda = R\gamma_n \) gives

\[
\| \gamma_{n+1} - \gamma_n \|^2 \leq \| \gamma_{n+1} \|^2 - \| \gamma_n \|^2.
\]

Also note that \( \| \gamma_n \|^2, n \in \mathbb{N}, \) is a Cauchy sequence in \( \mathbb{R} \), because, by (53), its limit exists and, being equal to \( \text{cap} A \), is finite. The preceding inequality therefore yields that \( (\gamma_n)_{n \in \mathbb{N}} \) is a strong Cauchy sequence in the semimetric space \( E(\overline{A}) \).

Besides, since \( \text{cap} A_n \leq \text{cap} A \), we derive from (45) that

\[
(\gamma_n)_{n \in \mathbb{N}} \subset E(\overline{A}, \leq \text{a cap} A, g).
\]

Hence, by Theorem 7, there exists an \( A \)-vague cluster point \( \gamma \) of \( (\gamma_n)_{n \in \mathbb{N}} \), and

\[
\lim_{n \in \mathbb{N}} \| \gamma_n - \gamma \|^2 = 0.
\]

Let \( (\gamma_t)_{t \in T} \) denote a subnet of the sequence \( (\gamma_n)_{n \in \mathbb{N}} \) that converges \( A \)-vaguely and strongly to \( \gamma \). We next proceed to show that

\[
\gamma \in D(A, a, g). \quad (99)
\]

39
For every $t \in T$, consider the ordered family $\{K_t\}_{A_t}$ of all compact condensers $K_t \prec A_t$. By (98), there exist a subnet $(K_{st})_{st \in S_t}$ of $(K_t)_{K_t \in \{K_t\}_{A_t}}$ and

$$\lambda_{st} \in S(K_{st}, a \cap K_{st}, g)$$

such that $(\lambda_{st})_{st \in S_t}$ converges both strongly and $A$-vaguely to $\gamma_t$. Consider the Cartesian product $\prod \{S_t : t \in T\}$, that is, the collection of all functions $\psi$ on $T$ with $\psi(t) \in S_t$, and let $D$ denote the directed product $T \times \prod \{S_t : t \in T\}$. Given $(t, \psi) \in D$, write

$$K_{(t, \psi)} := K_{\psi(t)} \quad \text{and} \quad \lambda_{(t, \psi)} := \lambda_{\psi(t)}.$$ 

Then the theorem on iterated limits from [K, Chap. 2, §4] yields that $(\lambda_{(t, \psi)})_{(t, \psi) \in D}$ converges both strongly and $A$-vaguely to $\gamma$. Since $(K_{(t, \psi)})_{(t, \psi) \in D}$ forms a subnet of $(K)_{K_t \in \{K_t\}_{A_t}}$, this proves (99) as required.

What is finally left is to prove (54). By Corollary 13, for every $n \in \mathbb{N}$ one can choose a compact condenser $K^0_n \prec A_n$ so that

$$\left| C_i(A_n, a, g) - C_i(K^0_n, a, g) \right| < n^{-1}, \quad i \in I.$$ 

This $K^0_n$ can certainly be chosen so large that the sequence obtained, $(K^0_n)_{n \in \mathbb{N}}$, forms a subnet of $(K)_{K_t \in \{K_t\}_{A_t}}$; therefore, repeated application of Corollary 13 yields

$$\lim_{n \in \mathbb{N}} C_i(K^0_n, a, g) = C_i(A, a, g).$$

This leads to (54) when combined with the preceding relation. □

19. Acknowledgments

The author is greatly indebted to P. Dragnev, D. Hardin, and E. B. Saff for several valuable comments concerning this study.

20. Appendix

Let $\nu \in \mathcal{M}^+(X)$ be given. As in [E2, Chap. 4, §4.7], a set $E \subset X$ is called $\nu$-$\sigma$-finite if it can be written as a countable union of $\nu$-integrable sets.

The following assertion, related to the theory of measures and integration, has been used in Sec. 18. Although it is not difficult to deduce it from [B2, E2], we could not find there a proper reference.

**Lemma 14.** Consider a lower semicontinuous function $\psi$ on $X$ such that $\psi \geq 0$ unless the space $X$ is compact, and let $E$ be the union of an increasing sequence of $\nu$-measurable sets $E_n$, $n \in \mathbb{N}$. If moreover $E$ is $\nu$-$\sigma$-finite, then

$$\int \psi \, d\nu_E = \lim_{n \in \mathbb{N}} \int \psi \, d\nu_{E_n}.$$
**Proof.** Without loss of generality, we can certainly assume \( \psi \) to be nonnegative. Then for every \( \nu-\sigma \)-finite set \( Q \),

\[
\int \psi \, d\nu_Q = \int \psi \varphi_Q \, d\nu,
\]

(100)

where \( \varphi_Q(x) \) equals 1 if \( x \in Q \), and 0 otherwise. Indeed, this can be concluded from [E2, Chap. 4, § 4.14] (see Propositions 4.14.1 and 4.14.6).

On the other hand, since \( \psi \varphi_{E_n}, n \in \mathbb{N} \), are nonnegative and form an increasing sequence with the upper envelope \( \psi \varphi_E \), [E2] Prop. 4.5.1 gives

\[
\int \psi \varphi_E \, d\nu = \lim_{n \in \mathbb{N}} \int \psi \varphi_{E_n} \, d\nu.
\]

Applying (100) to both the sides of this equality, we obtain the lemma. \( \square \)

**References**

[B1] N. Bourbaki, *Topologie générale, Chap. I–II*, Actualités Sci. Ind., 1142, Paris (1951).

[B2] N. Bourbaki, *Intégration, Chap. I–IV*, Actualités Sci. Ind., 1175, Paris (1952).

[C] H. Cartan, *Théorie du potentiel newtonien: énergie, capacité, suites de potentiels*, Bull. Soc. Math. France 73 (1945), 74–106.

[D1] J. Deny, *Les potentiels d’énergie finite*, Acta Math. 82 (1950), 107–183.

[D2] J. Deny, *Sur la définition de l’énergie en théorie du potential*, Ann. Inst. Fourier Grenoble 2 (1950), 83–99.

[E1] R. Edwards, *Cartan’s balayage theory for hyperbolic Riemann surfaces*, Ann. Inst. Fourier 8 (1958), 263–272.

[E2] R. Edwards, *Functional analysis. Theory and applications*, Holt. Rinehart and Winston, New York (1965).

[F1] B. Fuglede, *On the theory of potentials in locally compact spaces*, Acta Math. 103 (1960), 139–215.

[F2] B. Fuglede, *Caractérisation des noyaux consistants en théorie du potentiel*, Comptes Rendus 255 (1962), 241–243.

[GR] A. A. Gonchar, E. A. Rakhmanov, *On the equilibrium problem for vector potentials*, Uspekhi Mat. Nauk 40:4 (1985), 155–156; English transl. in: Russian Math. Surveys 40:4 (1985), 183–184.
[HK] W. K. Hayman, P. B. Kennedy, *Subharmonic functions*, Academic Press, London (1976).

[K] J. L. Kelley, *General topology*, Princeton, New York (1957).

[L] N. S. Landkof, *Foundations of modern potential theory*, Nauka, Fizmatlit, Moscow (1966); English trans., Springer-Verlag, Berlin (1972).

[MS] E. H. Moore, H. L. Smith, *A general theory of limits*, Amer. J. Math. 44 (1922), 102–121.

[NS] E. M. Nikishin, V. N. Sorokin, *Rational approximations and orthogonality*, Nauka, Fizmatlit, Moscow (1988); English trans., Translations of Mathematical Monographs 44, Amer. Math. Soc., Providence, RI 1991.

[O] M. Ohtsuka, *On potentials in locally compact spaces*, J. Sci. Hiroshima Univ. Ser. A-1 25 (1961), 135–352.

[VP] Ch. de la Valée-Poussin, *Le potentiel logarithmique, balayage et représentation conforme*, Louvain–Paris (1949).

[Z1] N. Zorii, *A noncompact variational problem in the Riesz potential theory. I; II*, Ukrain. Math. Zh. 47 (1995), 1350–1360; 48 (1996), 603–613 (in Russian); English transl. in: Ukrain. Math. J. 47 (1995); 48 (1996).

[Z2] N. Zorii, *Extremal problems in the theory of potentials in locally compact spaces. I; II; III*, Bull. Soc. Sci. Lettr. Łódź 50 Sér. Rech. Déform. 31 (2000), 23–54; 55–80; 81–106.

[Z3] N. Zorii, *On the solvability of the Gauss variational problem*, Comput. Meth. Funct. Theory 2 (2002), 427–448.

[Z4] N. Zorii, *Equilibrium problems for potentials with external fields*, Ukrain. Math. Zh. 55 (2003), 1315–1339 (in Russian); English transl. in: Ukrain. Math. J. 55 (2003).

[Z5] N. Zorii, *Necessary and sufficient conditions for the solvability of the Gauss variational problem*, Ukrain. Math. Zh. 57 (2005), 60–83 (in Russian); English transl. in: Ukrain. Math. J. 57 (2005).

[Z6] N. Zorii, *On capacities of condensers in locally compact spaces*, Bull. Soc. Sci. Lettr. Łódź 56 Sér. Rech. Déform. 50 (2006), 125–142.