BIRATIONAL MAPS OF MODULI OF BRILL-NOETHER PAIRS

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ABSTRACT. Let $C$ be a smooth projective irreducible curve of genus $g$. And let $G_\alpha(n, d, l)$ be the moduli space of $\alpha$ stable pairs of a vector bundle of rank $n$, deg $d$ and a subspace of $H^0(C, E)$ of dim = $l$. We find an explicit birational map from $G_\alpha(n, d, n+1)$ to $G_\alpha(1, d, n+1)$ for $C$ general, $\frac{1}{\alpha} \gg 0$ and $g \geq n^2 - 1$. Because of this and other examples, we conjecture $G_\alpha(a, d, a) \rightarrow G_\alpha(z, d, a+z)$ for $\frac{1}{\alpha} \gg 0$ and $C$ general with $g > 2$.

INTRODUCTION

Let $C$ be a smooth projective irreducible curve of genus $g$ defined over $\mathbb{C}$. And for a vector bundle $E$ over $C$ let $\mu(E) = \frac{\text{deg}(E)}{\text{rank}(E)}$. We say $E$ is stable (or semistable) if for every subbundle $S \subset E$ we have $\mu(S) < \mu(E)$ (or $\leq$). The chief advantage of stable bundles is that they form a coarse moduli space $M = M_s(n, d)$ of rank $n$, deg $d$ stable bundles. Contained in $M$ is the Brill-Noether locus $W = W_{n,d}^r$ of $E \in M$ such that $h^0(C, E) \geq r+1$. It is also known that $W$ is locally a determinantal variety with ‘theoretical’ dimension given by $\rho(r, n, d) = n^2(g-1) + 1 - (r+1)(r+1 - d + n(g-1))$. And hence, if $W$ is not empty it has the above minimal dimension. It is easily seen that $\rho(a+z-1, a, d) = \rho(a+z-1, z, d)$. This suggest some sort of relationship between $W_{a,d}^{a+z-1}$ and $W_{z,d}^{a+z-1}$ — perhaps a birational isomorphism. A candidate for that birational isomorphism is the following. Let $E \in W_{a,d}^{a+z-1}$ be generated by global sections, and consider the sequence of vector bundles with $M_E$ the kernel of the evaluation map.

$0 \rightarrow M_E \rightarrow H^0(C, E) \otimes \mathcal{O}_C \rightarrow E \rightarrow 0.$
Now if $E$ has no trivial summands $H^0(C, E)^* \subseteq H^0(C, M^*_E)$. So we hope $M^*_E \in W_{a+z-1}^{a+z}$. Furthermore, if $H^0(C, E)^* = H^0(C, M^*_E)$, then there is an inverse map using the dual of the exact sequence above. This is a special case of the dual span map which we will define shortly.

There are problems with the map $E$ goes to $M^*_E$.

1. $E$ may not be spanned.
2. $M^*_E$ may not be stable.
3. $h^0(C, E)$ may be $> a + z$.
4. $h^0(C, M^*_E)$ may be $> a + z$.

Problems 1 and 2 are serious. They are solved under some conditions in this paper and are conjectured here to be solvable under ‘most’ conditions. Sometimes they fail and nothing can be done. But 3 and 4 have been solved by Raghavendra and Vishwanath. Rather than looking at bundles, we look at Brill-Noether pairs $(E, V)$ which are pairs of a vector bundle $E$ and a space of sections $V \subseteq H^0(C, E)$. A pair $(E, V)$ is of type $(n, d, l)$ if $\text{rank}(E) = n$, $\text{deg}(E) = d$ and $\text{dim}(V) = l$.

A morphism of Brill-Noether pairs $(F, W) \to (E, V)$ is a morphism $F \to E$ such that the natural morphism $W \to H^0(C, E)$ factors through $V$. A morphism is a subbundle if $F$ is a subbundle of $E$. One should note that if $F = E$, an automorphism of the bundle may induce an isomorphism of pairs $(E, V)$ and $(E, W)$ with $V \neq W$ as subspaces of $H^0(C, E)$. This never happens to simple bundles (such as stable bundles) because the only automorphisms are scalars.

To form a moduli space of pairs requires a notion of stability. Following King and Newstead [3], we choose a rational number $\alpha > 0$. Then we define slope.

$$\mu_\alpha = \frac{\text{deg}(E) + \alpha \text{dim}(V)}{\text{rank}(E)}.$$ 

Now we have the usual definition of stability. $(E, V)$ is $\alpha$ stable (or $\alpha$ semistable) if for
every subbundle \((F, W), \mu_\alpha(F, W) < \mu_\alpha(E, V)\) (or \(\leq\)).

**Definition.** The set of Brill-Noether pairs of type \((n, d, l)\) with \(\alpha\) stability is \(G_\alpha(n, d, l) = \) the set of isomorphism classes of \(\alpha\) stable pairs \((E, V)\) with \(\text{rank}(E) = n, \deg(E) = d\) and \(\dim(V) = l\). And if no \(\alpha\) appears we are considering the set (of not necessarily stable) Brill-Noether pairs.

We will use \(\alpha\) with \(\frac{1}{\alpha} \gg 0\). To see how this works assume \(\text{rank } E = n\) is fixed and \(h^0(C, E) = h\) is fixed. For \(F\) a subbundle of \(E\), \(\mu(E) - \mu(F) \geq \frac{1}{n^2}\). So set \(\alpha < \frac{1}{n^2}/h\). Now if \(E\) is stable, \((E, V)\) is stable. And if \(E\) is unstable the pair \((E, V)\) is unstable. But if \(E\) is semistable, three things can happen. Suppose \(\mu(F) = \mu(E)\) and \(F \subsetneq E\). and \((F, W)\) is a subpair of \((E, V)\). If for all such \(F\), \(\frac{\dim W}{\text{rank } F} < \frac{\dim V}{\text{rank } E}\) (or \(\leq\)), then the pair \((F, W)\) is \(\alpha\) stable (or \(\alpha\) semistable). Otherwise the pair is \(\alpha\) unstable.

Raghavendra and Vishwanath [6] construct a moduli space for Brill Noether pairs with \(\alpha\) small. And even though we do not use it, we note King and Newstead [3] have constructed a coarse moduli space of \(\alpha\) stable Brill Noether pairs for \(\alpha > 0\) and rational on a polarised curve (which need not be reduced).

Now back to the dual span.

**Definition.** The set of spanning pairs is the set of pairs \((E, V)\), where \(V\) spans \(E\) and has no trivial summands. The set of \(\alpha\) stable spanning pairs is:

\[ S_\alpha(n, d, l) = \{(E, V) \in G_\alpha(n, d, l) \text{ such that } E \text{ is spanned by } V\} \]

Now we revisit the dual span map. Let \((E, V)\) be a spanning pair which contains no trivial summand (but is not necessarily stable). We have an exact sequence:

\[ 0 \to M_{V, E} \to V \otimes \mathcal{O}_C \to E \to 0. \]

where \(M_{V, E}\) is just the kernel of the evaluation map. And the dual gives us:

\[ 0 \to E^* \to V^* \otimes \mathcal{O}_C \to M_{V, E}^* \to 0. \]
Notice the pair \((M^*_V,E^{\ast}), V^{\ast}\) is a (not necessarily stable) spanning pair. So this map needs a name.

**Definition.** The dual span map is the bijection of the set of spanning pairs which takes \((E, V)\) to \((M^*_E, V^{\ast}), V^{\ast}\), the dual of the kernel of the evaluation map. If \(V = H^0(C, E)\) we may write \(M_E\).

If the type of \((E, V)\) is \((n, d, l)\) then the type of the dual span is \((l - n, d, l)\). Furthermore, the dual span of \((M^*_V,E^{\ast}), V^{\ast}\) is just \((E, V)\). The problem then is stability. If \((V, E)\) is \(\alpha\) stable, is its dual span \(\alpha\) stable? Not always. But perhaps often. And is a general \(\alpha\) stable Brill-Noether pair a spanning pair for a given type \((n, d, l)\) with \(l > n\)? Where by general we mean an Zariski open set which is dense in each irreducible component.

Now the main result.

**Main Theorem.** If \(C\) is a general curve, \(\frac{1}{\alpha} \gg 0\), and \(g \geq n^2 - 1\) or \(g = n = 2\), then

\[
S_\alpha(1, d, n + 1)_{\text{red}} \cong S_\alpha(n, d, n + 1)_{\text{red}}.
\]

**Remark 1.** We could state a stronger result for \(n = 2\) because Teixidor i Bigas [8] has proven that \(W^3_{2,d}\) is reduced and irreducible. And Tan [7] has proven the locus is non-empty. The problem with extending this to the case \(n > 2\) is showing a general pair in the space \(G_\alpha(n, d, n + 1)\) is spanned. Although we show some component is generally spanned. Nothing is known about the scheme structure in general. And the condition \(g \geq n^2 - 1\) seems to be a flaw in our proof and not a part of nature.

**Conjecture 1.** Fix \((n, d, l)\) with \(l > n\), let \(C\) be a general curve with \(g > 2\), and choose \(\alpha\) so \(\frac{1}{\alpha} \gg 0\). Then \(S_\alpha(n, d, l)\) is dense in \(G_\alpha(n, d, l)\).

**Conjecture 2.** Fix \((n, d, l)\) with \(l > n\) and let \(C\) be a general curve with \(g > 2\). For a general \((E, V) \in S_\alpha(n, d, l)\) with \(\frac{1}{\alpha} \gg 0\), \(M_{V,E}\) is \(\alpha\) stable. and the morphism \(S_\alpha(n, d, l) \to S_\alpha(l - n, d, l)\) is birational (at least after reducing the schemes).
§1 Stability Results

Now we prove that in some cases, the bundle $M_E$ is stable. A trick for rank 1 allows us to show some $M_{L,V}$ with $V \neq H^0(C,L)$ are stable on a general curve. We shall also indicate why we need general bundles on general curves.

The first result that needs mentioning is that if $E$ is stable and $\mu(E) > 2g$ then $M_E$ is stable [1, theorem 1.2]. If $l > gn$ then we get a morphism:

$$G_\alpha(n, l + gn, l + n) \rightarrow G_\alpha(l, l + gn, l + n).$$

In [5], Mercat proves the above is an isomorphism. This beautiful result provides evidence for the conjecture. But this should not mislead the reader to believe we will have an isomorphism in general. Or that we should get a birational map on special curves.

It can be shown [1, proposition 1.5 and example 2.6] that for a small number $\epsilon > 0$ there is a bundle (on any curve) with $\mu(E) \geq 2g - \epsilon$ and $M_E$ unstable. This necessitates the conjecture’s assumption that we deal with a general pair $(E, V)$.

It is also assumed that the curve is general. Suppose to the contrary that $C$ is special, in fact hyperelliptic, with hyperelliptic bundle $A$. If $\mu(E) > g + 1$ then $h^0(C, A^* \otimes E) > 0$. So $A \subset E$ and $M_A \subset M_E$. Now $\mu(M_A) = -2$, and if $h^1(C, E) = 0$ and $\mu(E) < 2g$, we have $\mu(M_E) = \frac{-\mu(E)}{\mu(E) - g} < -2$. So $M_E$ is unstable. A similar construction applies to any fixed gonality $\beta$ when $g \gg 0$.

Now we get a more systematic theorem for $E = L$ a line bundle on a general curve.

**Theorem 2.** Let $C$ be a smooth projective irreducible curve of genus $g$. And assume that if $\rho(r,d,1) < 0$, then $W_{r,d,1}^r = \emptyset$. If $L$ is generated by sections $M_L$ is semistable. In fact, it fails to be stable iff all the following hold.

1. $h^1(C, L) = 0$.
2. $\deg L = g + r$ and $r | g$.
3. There is an effective divisor $Z$ with $h^0(C, L(-Z)) = h^0(C, L) - 1$ and $\deg Z = 1 + \frac{2}{r}$. 
Remark 2. Assuming that $C$ has the property that $\dim(W_d^r) = \rho(g, r, d)$. If $g \geq 3$ then by a dimension count, a general $L$ satisfying 1 and 2 does not satisfy 3. And for any genus a general $L \in W_d^r$ but not in $W_d^{r+1}$ is spanned by a dimension count.

Proof of Theorem 2. Given $L$ generated by sections and $S$ a subbundle of $M_L$, there is a commutative diagram:

\[
\begin{array}{cccccc}
0 & \rightarrow & O & \rightarrow & V \otimes O_C & \rightarrow & E & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & S & \rightarrow & V \otimes O_C & \rightarrow & E & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & M_L & \rightarrow & H^0(C, L) \otimes O_C & \rightarrow & L & \rightarrow & 0 \\
\end{array}
\]

$S = M_{E,V}$. It would be simpler if $E$ was a line bundle. It is possible to take hyperplane sections and get $S = M_{W,A}$ where $A$ is the determinant of $E$. But there is an obvious question. Is $W = H^0(C, A)$? The answer is generally no. If $E = A$ (and hence $A$ is a subbundle of $L$), then the answer is yes. But suppose $E \neq A$. In taking hyperplane sections, we get

\[
0 \rightarrow \oplus O_C \rightarrow E \rightarrow A \rightarrow 0.
\]

And this sequence is exact on global sections. Now if we tensor by $\omega_C$ we get a sequence which is not exact on global sequences. This is because $E$ has no trivial summands and hence no degree 0 quotient bundles, and hence $E \otimes \omega_C$ has not quotient bundles of degree $\leq 2g - 2$. That means $E \otimes \omega_C$ is non special. But the subbundles $\omega_C$ are special.

The failure of the sequence tensored by $\omega_C$ means $H^0(C, A) \otimes H^0(C, \omega_C) \rightarrow H^0(C, A \otimes \omega_C)$ does not surject. But by a theorem of Green [2, Theorem 4.b.2] this only happens if the image of the morphism induced by $A$ is a rational normal curve. If it is a pencil the base point pencil trick applies. So it must have image of rational normal curve of deg $\geq 2$.

Given our assumption on generality of the curve, the gonality of the curve is $\geq \frac{2g+2}{2}$ so the
only possibility is if \( \deg(A) = g + 2 \) and \( h^0(C, A) = 3 \) and hence \( A \) is nonspecial. So a subbundle of the form \( M_A \) for \( A \) a special bundle must be induced by a subbundle \( A \subseteq L \). And all other special bundles are of the form \( M_{A,W} \) for \( W \) not complete.

Now suppose \( h^0(C, L) = r + 1 \). Then a simple calculation shows \( \mu(M_L) \geq \frac{-g-r}{r} \) with equality iff \( L \) is nonspecial. Now if \( M_L \) is destabilized by a subbundle of the form \( M_{A,W} \), \( \dim(W) = s \leq r - 1 \) and \( \deg(A) < \deg(L) \). Since \( \mu(M_{A,W}) = -\deg(A)/(\dim(W) - 1) \), we want \( \deg(A) = \delta \) as small as possible for a given \( s \). So we use the Brill-Noether numbers. We have:

\[
g - (s + 1)(s + 1 - \delta + g - 1) \geq 0
\]

\[
\delta \geq s + g - \frac{g}{s + 1}.
\]

If the last inequality is exact, then \((s + 1)|g\). Now we claim

\[
\mu(M_L) \geq \frac{-g-r}{r} \geq \frac{-s-g}{s} + \frac{g}{(s+1)s} \geq \mu(M_{A,V})
\]

\[
\frac{-g}{r} \geq \frac{g}{s} + \frac{g}{s(s+1)}
\]

\[
-g(s(s+1)) \geq -gr(s + 1) + gr
\]

\[
-gs^2 - gs \geq -gs^2 - g(r-s)s.
\]

So \( \mu(M_L) \geq \mu(M_{A,W}) \) with equality iff \( L \) is not special, \( s = r - 1 \), \( r|g \), \( A \) is special and hence \( A = L(-Z) \) for some effective divisor \( Z \). \( \square \)

\[ \text{§2 Main Results} \]

*Proof of Main Theorem.* We need to construct a family parameterizing sequences:

\[
0 \rightarrow M_{L,V} \rightarrow V \otimes O_C \rightarrow L \rightarrow 0.
\]

The line bundles \( L \) are parameterized by the Poincaré bundle \( \mathcal{P} \) on the degree \( d \) Jacobian \( J^d \). Consider \( X = \text{Spec}(\text{Sym}(V^* \otimes \mathcal{P})) \). There is a canonical section of \( V^* \otimes \mathcal{P} \) and this
gives rise to a sequence
\[ 0 \to M_{P,V} \to V \otimes \mathcal{O}_C \to P \to 0 \]
over \(X \times C\) which parameterizes the above sequences once we throw away degenerate sequences (those where \(V\) drops rank or does not span \(L\)).

We also need to parameterize sequences by using a ‘Poincare’ bundle on the space of rank = \(n\) stable bundles. This is a problem because there is no such bundle if \(n\) and \(d\) are not coprime. We can however, find finitely many Zariski open subsets which cover the moduli space, and construct a Poincare bundle over an etale cover of each cover so that the universal map coincides with the projection map. Using this we can construct our family as above.

Now let \(\delta\) be the least integer such that \(\rho(n,\delta,1) \geq 0\). For \(\delta \leq d \leq g + n\) we have a family of linebundles \(L\) which are spanned by global sections with \(\dim h^0(C, L) = n + 1\). And by Theorem 2 the dual span is generically stable. Now as above we get a family of sequences giving a spanning pair and it’s dual.

But what if \(d > g + n\)? Then we must use an incomplete space of sections and Theorem 2 does not apply. The way out of this mess is to start with the rank \(n\) vector bundles, show they are stable and spanned, and their kernel will be a line bundle which must be stable.

Since \(g \geq n^2 - 1\), \(\rho(n,g+1,n) \geq 0\) and hence \(S_\alpha(n,g+1,n+1)\) is not empty. Let \(E\) be a stable bundle with \((E,V)\) an element of that space. Consider a point \(p \in C\). \(E(p) \in S_\alpha(n,g+1+n,n+1)\). So the latter space is non-empty. It has a component on which a general element is generically spanned and whose spanned subbundle has no trivial summands. We now do a dimension count to show a general bundle is spanned. If a generic bundle has spanned subbundle \(F\) with \(\deg(F) = (g+1+n-a)\), then the space of these bundles has \(\dim = \rho(n,g+1+n,n) - (n+1)\). The dimension of bundles each subbundle can fit into is \(a(n)\) as found by counting possible elementary transformations. So the dimension of bundles we started with is \(\rho(n,g+1+n,n) - a\). Hence \(a\) is zero because the minimum dimension is given by \(\rho\) (assuming the space is not empty). In conclusion,
the generic bundle is spanned. Furthermore, this technique works for all \(d > g + n\).

Now that the family is constructed we use it to obtain a birational map from \(S_n = S_\alpha(n, d, n + 1)\) to \(S_1 = S_\alpha(1, d, n + 1)\). There is a natural map from \(X\) to \(S_1 \times S_n\); we call the image \(S\). Since the map is given by a dual span, if \(x \in S\) and \(y \in S\) and the image of \(x\) and \(y\) correspond in \(S_1\) then they correspond in \(S_n\), and vice versa. So \(S\) maps injectively into \(S_1\) and \(S_n\). The maps are then birational on the reduced schemes.

All that remains is the case \(g = n = 2\). We do that as an example. □

**Example 1.** Let \(g = n = 2\). If \(h^0(C, L) = 3\), then \(\deg(L) = 4 = 2g\). By Theorem 2, \(M_L\) is semistable but not stable. In particular \(M_\omega_C = \omega_C^*\) is a subbundle of \(M_L\) and hence \(\omega_C\) is a quotient bundle of \(M_L^*\). Assume for the moment that \(L \neq \omega_C^\otimes 2\), and let \(A = L \otimes \omega_C^*\). There is a sequence

\[
0 \to A \to M_L^* \to \omega_C \to 0.
\]

\(A\) is the only subbundle with \(\mu(A) = \mu(M_L)\). But \(h^0(C, A) = 1\) and so \((H^0(C, L)^*, M_L^*)\) is \(\alpha\) stable. So now let \(A = \omega_C\). We have a sequence

\[
0 \to \omega_C \to M_L^* \to \omega_C \to 0.
\]

If \(M_L^*\) is spanned, so is the cokernel \(\omega_C\). That means the sequence is exact on global sections and \(h^0(C, M_L^*) \geq 4\). Furthermore, the endomorphisms (given by scalar multiplication and surjection onto the subbundle \(\omega_C\) of \(M_L^*)\) has dimension 2 if the bundle is indecomposable, and 4 otherwise. In the first case we have a 3 dimensional family of subspaces of the space of sections. This is acted upon by the group of endomorphism. Modding out by scalars we get a 2 dimensional family of stable pairs. This is impossible because there is only one dual span. So the bundle decomposes. Now there is a three dimensional family of subspaces acted on by a 3 dimension group of automorphisms and we get a unique pair (up to isomorphism). Furthermore, the pair is spanned and hence any subbundle has only a 1 dimensional family of sections (since the cokernel, which is spanned, has a two dimensional space of sections).
Now the unspanned bundles all have subbundles $\omega_C$. So it is easily seen that they are $\alpha$ unstable.

Now all of this gives us our isomorphism for $d = 4$. What about $d > 4$. We have no stable bundles to tensor by a linebundle. (And semistable does no good because the determinantal loci is defined only for stable bundles.) So we consider elementary transformations. We make an ad hoc definition and call a primitive transformation of $E$ at a reduced point $p$ to be the kernel of a sequence:

$$0 \to E_p \to E \to O_p \to 0.$$

We can dualize $E$ take a primitive transformation and dualize again to get:

$$0 \to E \to E_p \to O_p \to 0.$$

By abuse of terminology and notation we call this a primitive transformation. If we can show a primitive transformation is stable, then we can do the dimension count done in the proof of the Main Theorem.

The proof that primitive transformations are stable is given by Lange and Narasimhan [4, Lemma 4.3]. Each rank 2 vector bundle can be thought of as a ruled surface. The surface has a minimal section (meaning a section with minimal self intersection $s$). The self intersection is $> 0$ (or $\geq 0$) iff $E$ is stable (or semistable). The minimal section is not unique but with $g \geq 2$ there are only finitely many unless $s \geq 2$ or $E$ is trivial.

The point is that a primitive transformation raises $s$ unless the transformation corresponds to a point on a minimal section. If $s = 1$ or 0, and $E$ is not trivial, there are only finitely many minimal sections, and therefore, $s$ rises for a general point and $E_p$ is stable. If $s \geq 2$ it may drop, but only by one, which would leave it stable.

The upshot is that primitive transformations show $G_\alpha(2, d, 3)$ is non-empty for $d \geq 5$. And furthermore, the dimension count in the proof of the Main Theorem applies to prove $S_\alpha(2, d, 3)$ is also non-empty.
Theorem 3. Assume $C$ is general in the sense that if $\rho(g, r, d) < 0$, then $W^r_d = \emptyset$. Then it follows that for $\rho(g, n, n, d) < 0$, then $W^n_{n,d} = \emptyset$.

To prove this we need an elementary lemma.

Lemma 1. Let $E$ be a vector bundle of rank $n$. $\rho(g, n, n, d) \geq 0$ is equivalent to the inequality $\mu(E) = \frac{d}{n} \geq 1 + \frac{g}{n+1} \overset{\text{def}}{=} f(n)$.

Proof of Lemma 1. Use $\rho(g, n, n, d) = n^2(g - 1) + 1 - (n + 1)(n + 1 - d + n(g - 1))$ and solve for $\frac{d}{n}$. □

Remark 3. The function $f(n)$ is strictly decreasing.

Proof of Theorem 3. Assume $\rho(g, n, n, d) < 0$ and hence $\rho(g, n, 1, d) < 0$ which implies $W^n_{1,d} = \emptyset$. Now assume $E \in W^n_{n,d}$. We have three cases:

(1) $E$ spanned,

(2) the global sections span a proper subbundle of the same rank, and

(3) the global sections span a subbundle of smaller rank.

Case 1. If $E$ is spanned, then the dual span is a line bundle $L$ with $L \in W^n_{1,d} = \emptyset$. So this is impossible.

Case 2. Let $F$ be the proper subbundle spanned by the sections of $E$. We have two subcases. A) $F$ has no trivial summands, and B) $F = \oplus O_C \oplus G$ where $G$ has no trivial summands. Case A follows from the proof of Case 1. As for B we note that $h^0(C, G) \geq \text{rank}(G) + 1$. But stability of $E$ and the fact that $f(n)$ is strictly decreasing gives

$$\mu(G) \leq \mu(E) < f(n) < f(\text{rank}(G)).$$

So by induction, case 2B follows.

Case 3. Argue as in 2B. □
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