There is no isolated interface edge in very supercritical percolation

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Abstract

We consider the Bernoulli bond percolation model in a box Λ (not necessarily parallel to the directions of the lattice) in the regime where the percolation parameter is close to 1. We condition the configuration on the event that two opposite faces of the box are disconnected. We couple this configuration with an unconstrained percolation configuration. The interface edges are the edges which differ in the two configurations. We prove that, typically, each interface edge is within a distance of order \(\ln |\Lambda|\) of another interface edge or of a pivotal edge. We derive an estimate for the law of an edge which is far from the cut and the interface edges.

1 Introduction

We pursue here the study of the structure of large interfaces in the percolation model. We consider the Bernoulli bond percolation model in a box Λ (not necessarily parallel to the directions of the lattice) in the regime where the percolation parameter is close to 1. We condition the configuration on the event \(\{T \leftrightarrow B\}\) that two opposite faces \(T\) and \(B\) of the box are disconnected and we wish to gain some insight into the resulting configuration. Since \(p\) is close to 1, there will be a lot of pivotal edges, that is closed edges whose opening would create a connection between the faces \(T\) and \(B\). However, the effect of the conditioning is complex and is not limited

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to the presence of the pivotal edges. In [1], we constructed a coupling between two percolation configurations which allows to keep track of the effect of the conditioning. Let us sum up briefly the strategy of this construction. We consider the classical dynamical percolation process in the box $\Lambda$. We start with an initial configuration $X_0$. At each step, we choose one edge uniformly at random, and we update its state with a coin of parameter $p$. This process is denoted by $(X_t)_{t\in\mathbb{N}}$. Of course all the random choices are independent. Next, we duplicate the initial configuration $X_0$, thereby getting a second configuration $Y_0$. We use the same random variables as before to update this second configuration, with one essential difference. In the second configuration, we prohibit the opening of an edge if this opening creates a connection between the top $T$ and the bottom $B$. We denote by $\mu_p$ the invariant probability of the process $(X_t,Y_t)_{t\in\mathbb{N}}$. Let now $(X,Y)$ be a pair of percolation configurations distributed according to $\mu_p$. We define the set $\mathcal{P}$ of the pivotal edges

$$\mathcal{P} = \{ e \subset \Lambda : e \text{ is pivotal in } Y \text{ for } T \leftrightarrow B \}$$

and the set $\mathcal{I}$ of the interface edges

$$\mathcal{I} = \{ e \subset \Lambda : X(e) \neq Y(e) \}.$$

The effect of the conditioning is precisely encoded in the set of the interface edges $\mathcal{I}$. A standard Peierls estimate and the BK inequality yield that, typically, each pivotal edge is within distance of order $\ln |\Lambda|$ of another pivotal edge (see Proposition 1.4 of [1]). Our first main result is that the same is still true for the union of the pivotal edges and the interface edges. The box $\Lambda$ in the next theorem is centred at the origin but its sides are not necessarily parallel to the axis of $\mathbb{Z}^d$.

**Theorem 1.1.** There exists $\tilde{p} < 1$ and $\kappa \geq 2d$, such that for $p \geq \tilde{p}$, $c \geq 1$ and any box $\Lambda$ satisfying $|\Lambda| \geq \max\{2^{2d} c, 3^{6d}\}$, we have

$$\mu_p \left( \exists e \subset \mathcal{P} \cup \mathcal{I} \quad d(e, \Lambda^c \cup \mathcal{P} \cup \mathcal{I} \setminus \{e\}) \geq \kappa c \ln |\Lambda| \right) \leq \frac{1}{|\Lambda|^c}.$$

At first sight, theorem 1.1 looks like a minor improvement of theorem 1.1 of [1]. However, neither result implies the other. To control the distance up to order $\ln |\Lambda|$ instead of $\ln^2 |\Lambda|$ requires two major new ingredients compared to the previous argument. The factor $\ln^2 |\Lambda|$ in theorem 1.1 of [1] was due to the fact that the speed estimates could only be obtained on a time interval of order $|\Lambda|$, however an edge of the interface has a typical lifetime of order $|\Lambda| \ln |\Lambda|$. Here we obtain a speed estimate on a time interval of order $|\Lambda| \ln |\Lambda|$.
by studying a new type of space-time path which connects a pivotal edge at
time $t$ to an edge of $\mathcal{P}_s \cup \mathcal{I}_s$ at a time $s < t$. The length of this type of
space-time paths has an exponential decay property during a time interval
of order $|\Lambda| \ln |\Lambda|$. As a drawback, we have to replace $\mathcal{P}$ by $\mathcal{P} \cup \mathcal{I}$ due to the
construction of this new space-time path. Another complication is that, in
order to construct such a space-time path, we have to reverse the time in the
process $(X_t, Y_t)_{t \in \mathbb{N}}$. Unfortunately, the process $(X_t, Y_t)_{t \in \mathbb{N}}$ is not reversible
because an edge of the interface can only be created at a time $t$ from an edge
which is closed in both configurations $X_t$ and $Y_t$ but when it disappears, it
can be transformed into an open edge in both configurations. To deal with
this point, we introduce another process $(X_t, Y^{\varepsilon}_t)_{t \in \mathbb{N}}$, parametrised by $\varepsilon > 0$,
which is reversible and which converges towards the process $(X_t, Y_t)_{t \in \mathbb{N}}$ when $\varepsilon$
converges to 0.

Apart from the two crucial ingredients just described, the strategy of our
proof follows the general ideas in [1]. We control first the probability of
having an isolated pivotal edge. We then construct a space-time path which
represents the movement of the set $\mathcal{P} \cup \mathcal{I}$. The study of the space-time path
gives us a control on the speed of the movement. Combined with the fact
that an edge of the interface has a limited lifetime, we obtain a control on
the distance between an edge of the interface and the set $\mathcal{P} \cup \mathcal{I}$.

Ideally, we would like to estimate the law of an edge conditioned by the
existence of a cut set at a distance of order $\ln |\Lambda|$. We give next a result in
this direction, but in order to use the previous theorem, we have to add a
condition on the edges of the interface in the conditioning event.

**Theorem 1.2.** There exists $\tilde{p} < 1$ and $\kappa \geq 2d$, such that for $p \geq \tilde{p}$, $c \geq 1$
and any box $\Lambda$ such that $|\Lambda| \geq \max\{3^{6d} e^{4dc}, e^{36d}\}$ and any edge $e$ at distance
more than $2c \ln |\Lambda|$ from the boundary of $\Lambda$, we have

$$
\mu_p \left( e \in \mathcal{I} \mid \exists C \in \mathcal{C}, d(e, C \cup \mathcal{I} \setminus \{e\}) \geq 4dke \ln |\Lambda| \right) \leq \frac{1}{|\Lambda|^c}.
$$

The proof of theorem 1.2 is more delicate than for theorem 1.1 because
we have to take care of the conditioning factor. The initial steps are similar.
We rely on the same new type of space-time path as in theorem 1.1, we use
the reversible process $(X_t, Y^{\varepsilon}_t)_{t \in \mathbb{N}}$ in order to reverse the time. In the central
step involving the estimate on the space-time path, we condition on the
configuration at time $t$ in order to factorise the closing events which occur
strictly after time $t$. Ideally, we would like to obtain an estimate with a
conditioning event involving only the cut and not the interface, but we have
not succeeded so far.

The paper is organised as follows. In section 2, we define the model and
the notations which will be used in the rest of our study. In section 3, we
introduce the coupling which approaches the process \((X_t, Y_t)_{t \in \mathbb{N}}\). In section 4, we study the case where a pivotal edge is isolated. In section 5, we construct the new space-time path which will be used in the proofs. We control the speed of the set \(\mathcal{P} \cup \mathcal{I}\) with the help of this space-time path. Finally, the proofs of theorems 1.1 and 1.2 are respectively presented in sections 7 and 8.

2 The model and notations

We will reuse most of the notations in [1], which we recall briefly.

2.1 Geometric definitions

We give some standard geometric definitions.

The edges \(\mathbb{E}^d\). The set \(\mathbb{E}^d\) is the set of pairs \(\{x, y\}\) of points in \(\mathbb{Z}^d\) which are at Euclidean distance 1.

The usual paths. We say that two edges \(e\) and \(f\) are neighbours if they have one endpoint in common. A usual path is a sequence of edges \((e_i)_{1 \leq i \leq n}\) such that for \(1 \leq i < n\), \(e_i\) and \(e_{i+1}\) are neighbours.

The box \(\Lambda\). We will mostly work in a closed box \(\Lambda\) centred at the origin. The top side of the box is denoted by \(T\) and the bottom side is denoted by \(B\). The box might be tilted, i.e., its sides are not necessarily parallel to the axis of \(\mathbb{Z}^d\).

The cuts. We say that \(S\) is a cut if there is no usual path included in \(\Lambda \cap \mathbb{E}^d \setminus S\) which connects \(T\) and \(B\). Notice that the box might be tilted with respect to the lattice \(\mathbb{Z}^d\).

The \(*\)-paths. In order to study the cuts in any dimension \(d \geq 2\), we use \(*\)-connectedness on the edges as in [2]. We consider the supremum norm on \(\mathbb{R}^d\):

\[
\forall x = (x_1, \ldots, x_d) \in \mathbb{R}^d \quad \| x \|_{\infty} = \max_{i=1,\ldots,d} |x_i|.
\]

For \(e\) an edge in \(\mathbb{E}^d\), we denote by \(m_e\) the center of the unit segment associated to \(e\). We say that two edges \(e\) and \(f\) of \(\mathbb{E}^d\) are \(*\)-neighbours if \(\| m_e - m_f \|_{\infty} \leq 1\). A \(*\)-path is a sequence of edges \((e_1, \ldots, e_n)\) such that, for \(1 \leq i < n\), the edge \(e_i\) and \(e_{i+1}\) are \(*\)-neighbours.
2.2 The dynamical percolation

We define the dynamical percolation and the space-time paths.

**Percolation configurations.** A configuration in $\Lambda$ is a map from the set of edges in $\Lambda$ to $\{0, 1\}$. In a configuration $\omega$, an edge $e$ is said to be open if $\omega(e) = 1$ and closed if $\omega(e) = 0$.

**Probability measures.** We denote by $P_p$ the law of the Bernoulli bond percolation in $\Lambda$ with parameter $p$. We also define $P_D$ as the probability measure $P_p$ conditioned by the event $\{T \leftrightarrow B\}$, i.e.,

$$P_D(\cdot) = P_p(\cdot \mid T \leftrightarrow B).$$

**Probability space.** Throughout the paper, we assume that all the random variables used in the proofs are defined on the same probability space $\Omega$. For instance, this space contains the random variables used in the graphical construction presented below, as well as the random variables generating the initial configurations of the Markov chains. We denote simply by $P$ the probability measure on $\Omega$.

**Graphical construction.** We construct the dynamical percolation in $\Lambda$ as a discrete time Markov chain $(X_t)_{t \in \mathbb{N}}$. We will need an i.i.d. sequence of random edges in $\Lambda$, denoted by $(E_t)_{t \in \mathbb{N}}$, with uniform distribution over the edges of $\Lambda$. We also need an i.i.d. sequence of uniform variables in the interval $[0, 1]$, denoted by $(U_t)_{t \in \mathbb{N}}$. The sequences $(E_t)_{t \in \mathbb{N}}$, $(U_t)_{t \in \mathbb{N}}$ are independent. We build the process $(X_t)_{t \in \mathbb{N}}$ iteratively. At time 0, we start from a configuration $X_0$ and at time $t$, we set

$$X_t(e) = \begin{cases} X_{t-1}(e) & \text{if } E_t \neq e \\ 1_{\{U_t \leq p\}} & \text{if } E_t = e \end{cases}.$$ 

**The space-time paths.** We introduce the space-time paths which generalise both the usual paths and the $*$-paths to the dynamical percolation. A space-time path is a sequence of pairs, called time-edges, $(e_i, t_i)_{1 \leq i \leq n}$, such that, for $1 \leq i \leq n - 1$, we have either $e_i = e_{i+1}$, or $(e_i, e_{i+1}$ are neighbours and $t_i = t_{i+1}$). We define also space-time $*$-paths, by using edges which are $*$-neighbours in the above definition. For $s, t$ two integers, we define

$$s \land t = \min(s, t), \quad s \lor t = \max(s, t).$$
A space-time path \((e_i, t_i)_{1 \leq i \leq n}\) is open in the dynamical percolation process \((X_t)_{t \in \mathbb{N}}\) if
\[
\forall i \in \{1, \ldots, n\} \quad X_{t_i}(e_i) = 1
\]
and
\[
\forall i \in \{1, \ldots, n-1\} \quad e_i = e_{i+1} \implies \forall t \in [t_i \wedge t_{i+1}, t_i \vee t_{i+1}] \quad X_t(e_i) = 1.
\]
In the same way, we can define a closed space-time path by changing 1 to 0 in the previous definition. In the remaining of the article, we use the abbreviation STP to design a space-time path. Moreover, unless otherwise specified, the closed paths (and the closed STPs) are defined with the relation \(\ast\) and the open paths (and the open STPs) are defined with the usual relation. This is because the closed paths come from the cuts, while the open paths come from existing connexions.

We shall define the space projection of a STP. Given \(k \in \mathbb{N}\) and a sequence \(\Gamma = (e_i)_{1 \leq i \leq k}\) of edges, we say that it has length \(k\), which we denote by \(\text{length}(\Gamma) = k\), and we define its support
\[
\text{support}(\Gamma) = \{ e \subset \Lambda : \exists i \in \{1, \ldots, k\} \quad e_i = e \}.
\]
Let \(\gamma = (e_i, t_i)_{1 \leq i \leq n}\) be a STP, the space projection of \(\gamma\) is obtained by removing one edge in every time change in the sequence \((e_i)_{1 \leq i \leq n}\). More precisely, let \(m\) be the number of time changes in \(\gamma\). We define the function \(\phi : \{1, \ldots, n-m\} \to \mathbb{N}\) by setting \(\phi(1) = 1\) and
\[
\forall i \in \{1, \ldots, n-m\} \quad \phi(i + 1) = \begin{cases} 
\phi(i) + 1 & \text{if } e_{\phi(i)} \neq e_{\phi(i)+1} \\
\phi(i) + 2 & \text{if } e_{\phi(i)} = e_{\phi(i)+1}
\end{cases}.
\]
The sequence \((e_{\phi(i)})_{1 \leq i \leq n-m}\) is the space projection of \(\gamma\), denoted by \(\text{Space}(\gamma)\).

2.3 The interfaces by coupling.

As in [1], we define the interface with the help of a coupling between two processes of dynamical percolation. We start with the graphical construction \((X_t, E_t, U_t)_{t \in \mathbb{N}}\) of the dynamical percolation. We define a further process \((Y_t)_{t \in \mathbb{N}}\) as follows: at time 0, we start from an initial condition \((X_0, Y_0)\) belonging to the set
\[
\mathcal{E}_0 = \{ (\omega_1, \omega_2) \in \{0, 1\}^{\mathbb{Z}^d \cap \Lambda} \times \{ T \leftrightarrow B \} : \forall e \subset \Lambda \quad \omega_1(e) \geq \omega_2(e) \}.
\]
for all \( t \geq 1 \), we set
\[
\forall e \subset \Lambda \quad Y_t(e) = \begin{cases} 
Y_{t-1}(e) & \text{if } e \neq E_t \\
0 & \text{if } e = E_t \text{ and } U_t > p \\
1 & \text{if } e = E_t, U_t \leq p \text{ and } T \xleftarrow{Y_{t-1}^{E_t}} B \\
0 & \text{if } e = E_t, U_t \leq p \text{ and } T \xrightarrow{Y_{t-1}^{E_t}} B
\end{cases}
\]
where, for a configuration \( \omega \) and an edge \( e \), the notation \( \omega^e \) means the configuration obtained by opening \( e \) in \( \omega \). The set of the configurations satisfying \( \{ T \xleftrightarrow{X} B \} \) is irreducible and the process \((X_t)_{t \in \mathbb{N}}\) is reversible. Therefore, the process \((Y_t)_{t \in \mathbb{N}}\) is the dynamical percolation conditioned to satisfy the event \( \{ T \xleftrightarrow{X} B \} \). According to corollary 1.10 of [4], the invariant probability measure of \((Y_t)_{t \in \mathbb{N}}\) is \( P_D \), the probability \( P_p \) conditioned by the event \( \{ T \xleftrightarrow{X} B \} \). The set \( E_0 \) is irreducible and aperiodic. In fact, each configuration of \( E_0 \) communicates with the configuration where all edges are closed. The state space \( E_0 \) is finite, therefore the Markov chain \((X_t, Y_t)_{t \in \mathbb{N}}\) admits a unique equilibrium distribution \( \mu_p \). We now present a definition of the interface between \( T \) and \( B \) for a coupled process \((X_t, Y_t)_{t \in \mathbb{N}}\).

**Definition 2.1.** The interface at time \( t \) between \( T \) and \( B \), denoted by \( I_t \), is the set of the edges in \( \Lambda \) that differ in the configurations \( X_t \) and \( Y_t \), i.e.,
\[
I_t = \{ e \subset \Lambda : X_t(e) \neq Y_t(e) \}.
\]

We define next the set \( P_t \) of the pivotal edges for the event \( \{ T \xleftrightarrow{X} B \} \) in the configuration \( Y_t \).

**Definition 2.2.** The set \( P_t \) of the pivotal edges in \( Y_t \) is the collection of the edges in \( \Lambda \) whose opening would create a connection between \( T \) and \( B \), i.e.,
\[
P_t = \{ e \subset \Lambda : T \xrightarrow{Y_t} B \}.
\]

### 3 The coupling \((X, Y^\varepsilon)\)

#### 3.1 The graphical construction and the reversibility

Notice that the process \((X_t, Y_t)_{t \in \mathbb{N}}\) is not reversible. To see this point, we shall observe the transitions that can be realised in one step by the process \((X_t, Y_t)_{t \in \mathbb{N}}\). We introduce the following notation: let \( a, b, c, d \in \{0, 1\} \), since only one edge is modified at a time, we can represent the transitions at a time \( t \) as
\[
(a, b) \rightarrow (c, d),
\]
where $a, b, c, d$ satisfy

$$X_{t-1}(E_t) = a, \quad Y_{t-1}(E_t) = b, \quad X_t(E_t) = c, \quad Y_t(E_t) = d.$$ 

The transitions of type

$$
(0, 0) \rightarrow (1, 1) \quad (0, 0) \rightarrow (1, 0) \\
(1, 0) \rightarrow (1, 1) \quad (1, 0) \rightarrow (0, 0) \\
(1, 1) \rightarrow (0, 0)
$$

are realisable by the process $(X_t, Y_t)_{t \in \mathbb{N}}$. The transition $(1, 1) \rightarrow (1, 0)$ is impossible because closing the edge $E_t$ in $Y_t$ implies the closing of this edge in $X_t$. However, in the reversed process, the transition $(1, 1) \rightarrow (1, 0)$ is possible. To obtain a reversible process, we change the graphical construction by introducing another coupling involving an additional parameter $\varepsilon$. We use the same sequences $(U_t)_{t \in \mathbb{N}}, (E_t)_{t \in \mathbb{N}}$ as in the construction of $(X_t, Y_t)_{t \in \mathbb{N}}$. We begin by defining an additional process $(X^\varepsilon_t)_{t \in \mathbb{N}}$, which is the dynamical percolation with parameter $p - \varepsilon$, i.e.,

$$
\forall t \geq 1 \quad X^\varepsilon_t(e) = \begin{cases} 
X^\varepsilon_{t-1}(e) & \text{if } E_t \neq e \\
1_{\{U_t \leq p - \varepsilon\}} & \text{if } E_t = e 
\end{cases}.
$$

The process $(X_t, X^\varepsilon_t)_{t \in \mathbb{N}}$ is aperiodic, irreducible and reversible. Moreover, any configuration $(\omega, \omega')$ such that $\omega$ dominates $\omega'$ can be realised. We denote by $E$ the set of these configurations, i.e.,

$$E = \{ (\omega, \omega') : \forall e \subset \Lambda \quad \omega(e) \geq \omega'(e) \}.$$

We reuse then the same construction as for $(X_t, Y_t)$ and we define the process $(Y^\varepsilon_t)_{t \in \mathbb{N}}$ as the process $(X^\varepsilon_t)$ conditioned by the disconnection event $\{ T \leftrightarrow B \}$. The whole process $(X_t, Y_t, X^\varepsilon_t, Y^\varepsilon_t)_{t \in \mathbb{N}}$ is determined by the sequence $(E_t, U_t)_{t \in \mathbb{N}}$ and the initial condition. We consider finally the coupling $(X_t, Y^\varepsilon_t)_{t \in \mathbb{N}}$, which is an irreducible and aperiodic process. Moreover, the transition $(1, 1) \rightarrow (1, 0)$, which was impossible in $(X_t, Y_t)_{t \in \mathbb{N}}$, can now be realised in this process. This yields the desired reversibility property, which we prove next.

**Lemma 3.1.** The process $(X_t, Y^\varepsilon_t)_{t \in \mathbb{N}}$ is reversible with respect to its invariant probability measure $\mu^\varepsilon$.

**Proof.** The process $(X_t, Y^\varepsilon_t)_{t \in \mathbb{N}}$ is in fact obtained by conditioning the coupled process $(X_t, X^\varepsilon_t)_{t \in \mathbb{N}}$ to stay in the subset

$$E \cap \left( \{0, 1\}^\mathbb{Z} \times \{ T \leftrightarrow B \} \right).$$
This subset is irreducible and the process \((X_t, Y_t^\varepsilon)_{t \in \mathbb{N}}\) is aperiodic, by corollary 1.10 of [4], the process \((X_t, Y_t^\varepsilon)_{t \in \mathbb{N}}\) has a unique invariant probability measure denoted by \(\mu^\varepsilon\), and it is reversible with respect to \(\mu^\varepsilon\).

We also denote by \(P^\varepsilon_n\) the law of the process \((X_t, Y_t^\varepsilon)_{t \in \mathbb{N}}\) started from a random initial configuration \((X_0, Y_0^\varepsilon)\) of law \(\mu^\varepsilon\).

### 3.2 The approximation to \((X, Y)\)

The second important result on the modified coupling \((X, Y^\varepsilon)\) is that, when the parameter \(\varepsilon\) approaches 0, the invariant measure \(\mu^\varepsilon\) converges to \(\mu\).

**Lemma 3.2.** The following convergence holds:

\[
\lim_{\varepsilon \to 0} \mu^\varepsilon = \mu.
\]

**Proof.** These probability measures are defined on a finite state space. We denote by \((M(\varepsilon))_{i,j} = 1 \leq i,j \leq N\) (respectively \((A_{i,j})_{i,j \leq N}\)) the transition matrix of the Markov chain \((X_t, Y_t^\varepsilon)_{t \in \mathbb{N}}, (X_t, Y_t)_{t \in \mathbb{N}}\).

The only differences between \(A\) and \(M(\varepsilon)\) occur on the indices \((i, j)\) corresponding to the transitions of type \((1, 1) \to (1, 0)\), for which we have \(A_{i,j} = 0\) whereas the coefficient \(M(\varepsilon)_{i,j}\) is equal \(\varepsilon/|\mathbb{E}^d \cap \Lambda|\). Therefore the matrix \(M(\varepsilon)\) converges to \(A\) as \(\varepsilon\) approaches 0. Let us consider a sequence \((\varepsilon_n)_{n \in \mathbb{N}}\) converging to 0. The sequence \((\mu^\varepsilon_n)_{n \in \mathbb{N}}\) belongs to the compact set \([0, 1]^{\{0,1\}^\mathbb{E}^d \cap \Lambda}\).

Let \(\phi(n)\) be an increasing subsequence such that \(\mu^{\varepsilon_{\phi(n)}}\) converges to a certain point denoted by \(\mu^*\). For \(n \in \mathbb{N}\), we have

\[
\mu^{\varepsilon_n} M(\varepsilon_n) = \mu^{\varepsilon_n}.
\]

By continuity of the product and by the convergence of \((M(\varepsilon_n))_{n \in \mathbb{N}}\), we have

\[
\mu^* A = \mu^*.
\]

Since the process \((X_t, Y_t)_{t \in \mathbb{N}}\) has a unique invariant probability \(\mu\), we obtain that \(\mu^* = \mu\). Therefore the sequence \((\mu^{\varepsilon_n})_{n \in \mathbb{N}}\) converges to \(\mu\). This is true for any subsequence \(\phi(n)\). Hence the probability \(\mu^\varepsilon\) converges towards \(\mu^*\). \(\square\)
4 Distance between pivotal edges

We show first that, under \( P^\varepsilon \), the probability of an isolated pivotal edge in \( Y_t^\varepsilon \) is small. More precisely, we have the following proposition.

**Proposition 4.1.** There exist \( \tilde{p} < 1 \), \( \tilde{\varepsilon} > 0 \) and \( \kappa > 1 \) such that for \( p \geq \tilde{p} \) and \( \varepsilon \leq \tilde{\varepsilon} \), for any \( c > 0 \) and any \( \Lambda \) satisfying \( |\Lambda| \geq 3^{6d} \), we have

\[
P_{p-\varepsilon}\left( \exists e \in \mathcal{P}, d(e, \Lambda^\varepsilon \cup \mathcal{P} \setminus \{e\}) \geq \kappa c \ln |\Lambda| \iff |T \leftrightarrow X_B| \right) \leq \frac{1}{|\Lambda|^c}.
\]

The proof is the same as the proposition 1.4 of [1], by changing \( p \) into \( p - \varepsilon \). The idea is to find a closed \(*\)-path in \( Y_t^\varepsilon \) which connects \( e \) to \( \mathcal{P}_t \setminus \{e\} \) and which is also disjoint from a cut. Then we apply the BK inequality (see [3]) and we use the exponential decay for the probability of a closed \(*\)-path.

5 The construction of the STP

We will construct a STP which connects an edge \( e \in \mathcal{P}_t \) at time \( t \) and the set \( \mathcal{P}_s \cup \mathcal{I}_s \) at time \( s < t \). Before starting the construction, we define first some relevant properties of a STP, which will be enjoyed by our construction.

**Definition 5.1.** A STP \((e_1, t_1), \ldots, (e_n, t_n)\) is increasing (respectively decreasing) if

\[
t_1 \leq \cdots \leq t_n \text{ (resp. } t_1 \geq \cdots \geq t_n)\) .
\]

If a STP is increasing or decreasing, we say that it is monotone.

**Definition 5.2.** A STP \((e_1, t_1), \ldots, (e_n, t_n)\) in \( X \) (respectively \( Y \)) is called simple if each edge is visited only once or its status changes at least once between any two consecutive visits, i.e., for any \( i, j \) in \( \{1, \ldots, n\} \) such that \( |i - j| \neq 1 \),

\[
(e_i = e_j \quad t_i < t_j) \implies \exists s \in [t_i, t_j] \quad X_s(e_i) \neq X_t(e_i) \text{ (resp. } Y_s(e_i) \neq Y_t(e_i))\) .
\]

**Definition 5.3.** In a STP \((e_1, t_1), \ldots, (e_n, t_n)\), for \( 1 \leq i < n \), we say that the edge \( e_i \) is a time-change edge if \( e_i = e_{i+1} \).

We define next two properties of a STP related to the time-change edges.

**Definition 5.4.** A STP \((e_1, t_1), \ldots, (e_n, t_n)\) is impatient if every time-change is followed by an edge which is updated, i.e.,

\[
\forall i \in \{1, \ldots, n - 2\} \quad e_i = e_{i+1} \Rightarrow E_{t_i+1} = e_{i+2}.
\]
Definition 5.5. A STP \((e_1, t_1), \ldots, (e_n, t_n)\) is called \(X\)-closed-moving (resp. \(Y\)-closed-moving) if all the edges which are not time-change edges are closed in \(X\) (resp. in \(Y\)), i.e.,

\[ \forall i \in \{1, \ldots, n - 1\} \quad e_i \neq e_{i+1} \Rightarrow X_{t_i}(e_i) = 0 \quad (\text{resp. } Y_{t_i}(e_i) = 0). \]

We now construct a specific STP satisfying some of these properties.

Proposition 5.6. Let \(s < t\) be two times and \(e \in \mathcal{P}_t\). There exists a decreasing simple impatient STP which connects the time-edge \((e, t)\) to an edge of the set \(\mathcal{P}_s \cup \mathcal{I}_s \setminus \{e\}\) at time \(s\) or an edge \(f\) intersecting the boundary of \(\Lambda\) after time \(s\). Moreover this STP is \(X\)-closed-moving except on the edge \(e\).

Proof. The proof of this proposition is done in two steps. The first step is to construct a STP which connects certain edges. In the second step, we modify the STP obtained in the first step to get a simple and impatient STP.

Step 1. At time \(t\), the edge \(e\) belongs to a cut. Therefore, there exists a path \(\gamma_1\) which connects \(e\) to the boundary of \(\Lambda\). We start at the edge \(e\) and we follow the path \(\gamma_1\). If the path \(\gamma_1\) does not encounter an edge \(f \in \mathcal{I}_t \cup \mathcal{P}_{t-1} \setminus \{e\}\) then the STP

\[(e, t), (\gamma_1, t)\]

connects \(e\) to the boundary of \(\Lambda\), where the notation \((\rho, t)\), for a path \(\rho = (e_i)_{1 \leq i \leq n}\) and a time \(t\), means the sequence of time-edges \((e_i, t)_{1 \leq i \leq n}\). Suppose next that there exists an edge of \(\mathcal{I}_t \cup \mathcal{P}_{t-1} \setminus \{e\}\) in \(\gamma_1\). We enumerate the edges of \(\gamma_1\) in the order they are visited when starting from \(e\) and we consider the first edge \(e_1\) in \(\gamma_1\) which belongs to the set \(\mathcal{I}_t \cup \mathcal{P}_{t-1}\). We denote by \(\rho_1\) the sub-path of \(\gamma_1\) visited between \(e\) and \(e_1\). We then consider the time \(\eta(t)\) defined as follows:

\[ \eta(t) = \max \left\{ r < t : e_1 \in \mathcal{P}_r, e_1 \notin \mathcal{P}_{r-1} \right\}. \]

Since \(e_1 \in \mathcal{I}_t \cup \mathcal{P}_{t-1}\), the time \(\eta(t)\) when it becomes pivotal is strictly less than \(t\). If the time \(\eta(t)\) is before the time \(s\) then, at time \(s\), the edge \(e_1\) belongs to the set \(\mathcal{I}_s \cup \mathcal{P}_s \setminus \{e\}\). Therefore the STP

\[(e, t), (\rho, t), (e_1, t), (e_1, s)\]

satisfies the conditions in the proposition. If we have \(\eta(t) > s\), then we repeat the above argument starting from the edge \(e_1\) at time \(\eta(t)\). We obtain either a path \(\gamma_2\) which connects \(e_1\) to the boundary of \(\Lambda\) at time \(\eta(t)\) or a path \(\rho_2\)
which connects $e_1$ to an edge $e_2 \in \mathcal{I}_{\eta(t)} \cup \mathcal{P}_{\eta(t)-1} \setminus \{e\}$ and a time $\eta^2(t) < \eta(t)$. We proceed in this way until we reach a time edge $(e_k, \eta^k(t))$ with $\eta^k(t) \leq s$. Since $\eta(t) < t$, the sequence of times

$$\eta(t), \eta^2(t), \ldots, \eta^k(t)$$

decreases strictly through this procedure and this procedure terminates after a finite number of iterations. The concatenation of the paths obtained

$$(e, t), (\rho_1, t), (e_1, \eta(t)), \ldots, (\rho^k, \eta^{k-1}(t)), (f_k, s)$$

connects $e$ to an edge of $\mathcal{P}_s \cup \mathcal{I}_s \setminus \{e\}$. Since the sequence $(\eta^i(t))_{1 \leq i \leq k}$ is decreasing, this is a decreasing STP. Each time when the STP meets an edge of $\mathcal{I}$ which is different from $e$, there is a time change to the time before it opened in $X$, therefore each movement in space except on the edge $e$ is done through a closed edge in $X$ and the STP is $X$-closed-moving.

**Step 2.** We use two iterative procedures to transform the STP in the step 1 into a simple and impatient STP. To get a simple STP, we use the same procedure as in the proof of proposition 4.4 in [1]. Let us denote by $(e_i, t_i)_{0 \leq i \leq N}$ the STP obtained previously. Starting with the edge $e_0$, we examine the rest of the edges one by one. Let $i \in \{0, \ldots, N\}$. Suppose that the edges $e_0, \ldots, e_{i-1}$ have been examined and let us focus on $e_i$. We encounter three cases:

- For every index $j \in \{i+1, \ldots, N\}$, we have $e_j \neq e_i$. Then, we don’t modify anything and we start examining the edge $e_{i+1}$.
- There is an index $j \in \{i+1, \ldots, N\}$ such that $e_i = e_j$, but for the first index $k > i + 1$ such that $e_i = e_k$, there is a time $\alpha \in ]t_k, t_i]$ when $X_{\alpha}(e_i) = 1$. Then we don’t modify anything and we start examining the next edge $e_{i+1}$.
- There is an index $j \in \{i + 1, \ldots, N\}$ such that $e_i = e_j$ and for the first index $k > i + 1$ such that $e_i = e_k$, we have $X_{\alpha}(e_i) = 0$ for all $\alpha \in ]t_k, t_i]$. In this case, we remove all the time-edges whose indices are strictly between $i$ and $k$. We then have a simple time change between $t_i$ and $t_k$ on the edge $e_i$. We continue the procedure from the index $k$.

The STP becomes strictly shorter after every modification, and the procedure will end after a finite number of modifications. We obtain in the end a simple path in $X$. Since the procedure doesn’t change the order of the times $t_i$, we still have a decreasing path. In order to obtain an impatient STP, we modify the simple decreasing STP obtained above and we use another
iterative procedure as follows. We denote again by \((e_i, t_i)_{0 \leq i \leq n}\) the simple STP obtained above. We start by examining the time-edge \((e_0, t_0)\) and then the rest of the time edges of the STP one by one as illustrated in the figure 1. Suppose that we have examined the indices \(i \leq k\) and that we are checking the index \(k\). If the edge \(e_{k+1}\) is different from \(e_k\), we don’t modify at this stage and we continue the procedure from \((e_{k+1}, t_{k+1})\). If the edge \(e_{k+1}\) is equal to \(e_k\), then the time-edge \((e_k, t_k)\) belongs to a time change. Since the STP is \(X\)-closed-moving, then the edge \(e_{k+2}\) is closed at time \(t_{k+1}\). Let \([\alpha, \beta]\) be the biggest interval containing \(t_{k+1}\) during which the edge \(e_{k+2}\) is closed in \(X\). If \(\beta > t_{k+2}\), we replace the sub-sequence \((e_{k+1}, t_{k+1}), (e_{k+2}, t_{k+2})\) by \((e_{k+1}, t_k \land \beta), (e_{k+2}, t_k \land \beta), (e_{k+2}, t_{k+2})\) and we continue the STP at the time-edge \((e_{k+2}, t_k \land \beta)\). If \(\beta = t_{k+2}\), then we don’t modify the STP and we continue the procedure from the time-edge \((e_{k+2}, t_{k+2})\). The STP obtained after the modification procedure is decreasing, \(X\)-closed-moving and impatient. Moreover, between two consecutive visits of an edge \(f\) of the STP, there exists a time when the edge \(f\) is open. Therefore, this STP is also simple.
6 Speed estimates

We show here that the set $P \cup I$ cannot move too fast. Typically, during an interval of size $|\Lambda| \ln |\Lambda|$, the set $P \cup I$ can at most move a distance of order $\ln |\Lambda|$. This result relies on an estimate for the STP constructed in proposition 5.6 which we state in the following proposition.

**Lemma 6.1.** Let $e$ be an edge in $\Lambda$ and $\ell \in \mathbb{N}^*$. Let $(\varepsilon_1, \ldots, \varepsilon_n)$ a sequence of edges such that $|\text{support}(\varepsilon_1, \ldots, \varepsilon_n)| = \ell$. We have the following inequality:

$$\exists \tilde{p} < 1 \quad \forall p \geq \tilde{p} \quad \forall s, t \quad 0 < t - s \leq \ell|\Lambda|$$

$$P_\mu \left( \exists \gamma \text{ decreasing simple impatient} \right.$$

$$\left. X\text{-closed-moving STP except on } e, \gamma \text{ starts from } (e, t) \text{ and ends after } s, \text{ space}(\gamma) = (\varepsilon_1, \ldots, \varepsilon_n) \right) \leq \left( 1 + \frac{1}{|\Lambda|} \right)^{\ell|\Lambda|} (4 - 4p)^n.$$  

**Proof.** Let us fix a STP $\gamma$ satisfying the conditions stated in the probability. We denote by $(e_i, t_i)_{i \in I}$ the sequence of the time-edges of $\gamma$. We denote by $k$ the number of the time changes in $\gamma$ and by $T$ the set of the indices of the time changes, i.e.,

$$T = \{ i \in I : e_i = e_{i+1}, \ t_i \neq t_{i+1} \}.$$ 

We shall obtain an upper bound of the probability

$$P \left( (e_i, t_i)_{i \in I} \text{ decreasing simple impatient} \right.$$

$$\left. X\text{-closed-moving STP except on } e \right),$$  

which depends only upon the integer $n$ and the number of time changes $k$. In order to bound the probability appearing in the lemma, we shall sum over the choices of the set of the $k$ times, denoted by $K$, in the interval $\{s, \ldots, t\}$, over the choices of set of the $k$ edges, denoted by $A$, where the time changes occur and the number $k$ from 1 to $n$. The probability appearing in the lemma is less or equal than

$$\sum_{1 \leq k \leq n} \sum_{A \subseteq \{1, \ldots, n\}, |A| = k} \sum_{K \subseteq \{s, \ldots, t\}, |K| = k} P \left( (e_i, t_i)_{i \in I} \text{ decreasing simple impatient} \right.$$

$$\left. X\text{-closed-moving STP except on } e \right).$$  

Let us obtain an upper bound for this probability. The STP is impatient and $X$-closed-moving, therefore for any $i \in T$, the edge $e_{i+2}$ becomes open at time $t_{i+1} + 1$. Moreover, the STP is simple, thus for any pair of indices $(p, q) \in I \setminus T$, if $e_p = e_q$ and $t_p > t_q$, there exists a time $r \in ]t_q, t_p[$, such that
the edge \( e_p \) is open at time \( r \). We can rewrite the probability inside the sum as
\[
P_\mu \left( \begin{array}{l}
\forall i \in T \quad E_{t_{i+1} + 1} = e_{i+2} \\
\forall i \in I \setminus T \quad X_{t_i}(e_i) = 0 \\
\forall p, q \in I \setminus T \text{ s.t. } e_p = e_q, t_p > t_q \\
\exists r \in ]t_q, t_p[ \quad X_r(e_p) = 1
\end{array} \right). \tag{6.3}
\]
Since the times \( t_i \) are fixed, this probability can be factorised as a product over the edges. In fact, the event in the probability depends only on the process \((X_t)_{t \in \mathbb{N}}\). We introduce, for an edge \( f \subset \Lambda \), the subset \( J(f) \) of \( I \):
\[
J(f) = \{ i \in I : e_i = f \}.
\]
Let us denote by \( S \) the set support(\( \gamma \)). The previous probability is less or equal than
\[
\prod_{f \in S \setminus \{e\}} P_\mu \left( \begin{array}{l}
\forall i \in J(f) \cap (T + 2) \quad E_{t_{i+1} + 1} = f \\
\forall i \in J(f) \setminus T \quad X_{t_i}(f) = 0 \\
\forall p, q \in J(f) \setminus T \text{ s.t. } p < q \\
\exists r \in ]t_q, t_p[ \quad X_r(f) = 1
\end{array} \right). \tag{6.4}
\]
Let us consider one term of the product. For a fixed edge \( f \), we can order the set \( \{ t_i : i \in J(f) \setminus T \} \) in an increasing sequence \((\tau_i)_{1 \leq i \leq m_f} \), where \( m_f = |J(f) \setminus T| \). Let us denote by \( T(f) \) the set of the indices among \( \{1, \ldots, m_f\} \) which correspond to the end of a time change, i.e., the set corresponding to \( J(f) \cap (T + 2) \) before the reordering. Since the STP is simple, between two consecutive visits at time \( \tau_i \) and \( \tau_{i+1} \) of \( f \), there is a time \( \theta_i \) when \( f \) is open. Moreover the STP is impatient, so for each index \( i \in T(f) \), the edge \( f \) becomes open at time \( \tau_i + 1 \). Therefore, each term of the product \((6.4)\) is less or equal than
\[
P_\mu \left( \begin{array}{l}
\forall i \in \{1, \ldots, m_f\} \quad X_{\tau_i}(f) = 0 \\
\forall i \in T(f) \quad X_{\tau_{i+1}}(f) = 1 \\
\forall i \in \{1, \ldots, m_f - 1\} \quad \exists \theta_i \in ]\tau_i, \tau_{i+1}[ \quad X_{\theta_i}(f) = 1
\end{array} \right). \tag{6.5}
\]
In order to simplify the notations, we define, for a time \( r \), the event
\[
\mathcal{E}(r) = \left\{ \begin{array}{l}
\forall i \in \{1, \ldots, m_f\} \text{ such that } \tau_i \leq r \quad X_{\tau_i}(f) = 0 \\
\forall i \in T(f) \text{ such that } \tau_i + 1 \leq r \quad X_{\tau_{i+1}}(f) = 1 \\
\forall i \in \{1, \ldots, m_f - 1\} \text{ such that } \tau_i \leq r \\
\exists \theta_i \in ]\tau_i, \tau_{i+1}[ \quad X_{\theta_i}(f) = 1
\end{array} \right\}.
\]
The status of the edge \( f \) evolves according to a Markov chain on \( \{0, 1\} \). The sequence \((\tau_i)_{1 \leq i \leq m_f} \) being fixed, if \( m_f \in T(f) \), we condition \((6.5)\) by the events
before time $\tau_{m_f}$, we have

$$P_\mu\left(\begin{array}{l}
\forall i \in \{1, \ldots, m_f\} \hspace{1cm} X_{\tau_i}(f) = 0 \\
\forall i \in T(f) \hspace{1cm} X_{\tau_{i+1}}(f) = 1 \hspace{1cm} m_f \in T(f) \\
\forall i \in \{1, \ldots, m_f - 1\} \hspace{1cm} \exists \theta_i \in ]\tau_i, \tau_{i+1}[ \hspace{1cm} X_{\theta_i}(f) = 1
\end{array}\right) = \frac{P_\mu(\mathcal{E}(\tau_{m_f}))}{|\Lambda|}.$$

If $m_f \notin T(f)$, the probability

$$P_\mu\left(\begin{array}{l}
\forall i \in \{1, \ldots, m_f\} \hspace{1cm} X_{\tau_i}(f) = 0 \\
\forall i \in T(f) \hspace{1cm} X_{\tau_{i+1}}(f) = 1 \hspace{1cm} m_f \notin T(f) \\
\forall i \in \{1, \ldots, m_f - 1\} \hspace{1cm} \exists \theta_i \in ]\tau_i, \tau_{i+1}[ \hspace{1cm} X_{\theta_i}(f) = 1
\end{array}\right)$$

is equal to $P_\mu(\mathcal{E}(\tau_{m_f}))$. We then condition $P_\mu(\mathcal{E}(\tau_{m_f}))$ by the events before time $\tau_{m_f-1}$. We shall distinguish two cases according to whether $m_f - 1$ belongs to $T(f)$ or not. If $m_f - 1 \notin T(f)$, we have

$$P_\mu(\mathcal{E}(\tau_{m_f})) = P_\mu\left(\begin{array}{l}
X_{\tau_{m_f}}(f) = 0 \\
X_{\tau_{m_f-1}+1}(f) = 1 \hspace{1cm} \mathcal{E}(\tau_{m_f-1})
\end{array}\right) P_\mu(\mathcal{E}(\tau_{m_f-1})), $$

and if $m_f - 1 \notin T(f)$, we have

$$P_\mu(\mathcal{E}(\tau_{m_f})) = P_\mu\left(\begin{array}{l}
\exists \theta_{m_f} \in ]\tau_{m_f-1}, \tau_{m_f}[ \\
X_{\theta_{m_f}}(f) = 1
\end{array}\right) P_\mu(\mathcal{E}(\tau_{m_f-1})). $$

We condition successively the event $P_\mu(\mathcal{E}(\tau_i))$ by $\mathcal{E}(\tau_{i-1})$, we obtain

$$P_\mu(\mathcal{E}(\tau_{m_f})) = P_\mu(\mathcal{E}(\tau_1)) \prod_{1 \leq i < m_f, i \in T(f)} P_\mu\left(\begin{array}{l}
X_{\tau_i+1}(f) = 0 \\
X_{\tau_i+1}(f) = 1 \hspace{1cm} \mathcal{E}(\tau_i)
\end{array}\right) \times \prod_{1 \leq i < m_f, i \notin T(f)} P_\mu\left(\begin{array}{l}
X_{\tau_{i+1}}(f) = 0 \\
\exists \theta_i \in ]\tau_i, \tau_{i+1}[ \hspace{1cm} \mathcal{E}(\tau_i)
\end{array}\right) \times P_\mu\left(\begin{array}{l}
X_{\tau_{m_f}}(f) = 0 \\
\exists \theta_{m_f} \in ]\tau_{m_f-1}, \tau_{m_f}[ \hspace{1cm} X_{\theta_m}(f) = 1 \hspace{1cm} \mathcal{E}(\tau_{m_f-1})
\end{array}\right). \ (6.6)$$

By the Markov property, each term in the second product is equal to

$$P_\mu\left(\begin{array}{l}
X_{\tau_{i+1}}(f) = 0 \\
\exists \theta_i \in ]\tau_i, \tau_{i+1}[ \hspace{1cm} X_{\theta_i}(f) = 1 \hspace{1cm} \mathcal{E}(\tau_i)
\end{array}\right).$$
Since this probability is invariant by translation in time, it is equal to

\[ P_0 \begin{pmatrix} X_{\tau'}(f) = 0 \\ \exists \theta \in [0, \tau'] \\ X_\theta(f) = 1 \end{pmatrix}, \]

where we have set \( \tau' = \tau_{i+1} - \tau_i \) and \( P_0 \) is the law of the Markov chain \( (X_t(f))_{t \in \mathbb{N}} \) starting from a closed edge. By considering the stopping time \( \theta' \) defined as the first time after 0 when \( f \) is open, we have by strong Markov property

\[ P_0 \begin{pmatrix} X_{\tau'}(f) = 0 \\ \exists \theta \in [0, \tau'] \\ X_\theta(f) = 1 \end{pmatrix} \leq P_\mu(X_{\tau'}(f) = 0 \mid X_\theta(f) = 1) = P_1(X_{\tau' - \theta'}(f) = 0). \]

Notice that for \( r \geq 1 \), we have

\[ P_1(X_r(f) = 0) \leq P_\mu(X_r(f) = 0) = 1 - p. \]

Therefore we have

\[ P_0 \begin{pmatrix} X_{\tau'}(f) = 0 \\ \exists \theta \in [0, \tau'] \\ X_\theta(f) = 1 \end{pmatrix} \leq 1 - p. \]

As for the probabilities in the first product of (6.6), we can also replace \( \mathcal{E}(\tau_i) \) by \( \{X_{\tau_i}(f) = 0\} \) in the conditioning. The difference between the previous case is that we don’t have to consider the stopping time \( \theta' \), because we have \( \{X_1(f) = 1\} \). We have

\[ P_\mu \left( \begin{array}{c} X_{\tau_{i+1}}(f) = 0 \\ X_{\tau_{i+1}}(f) = 1 \end{array} \bigg| \mathcal{E}(\tau_i) \right) \leq P_1(X_{\tau_{i+1}}(f) = 0) P_0(X_1(f) = 1) \leq \frac{1-p}{|\Lambda|}. \]

Combining the upper bounds for each term of the product above, we have the following upper bound for \( P_\mu(\mathcal{E}(\tau_{m_f})) \):

\[ P_\mu(\mathcal{E}(\tau_{m_f})) \leq \frac{(1-p)^{m_f}}{|\Lambda||T(f) \cap \{1, \ldots, m_f - 1\}|}, \]

where

\[ |T(f) \cap \{1, \ldots, m_f - 1\}| = \begin{cases} |J(f) \cap (T + 2)| & \text{if } m_f \notin T(f) \\ |J(f) \cap (T + 2)| - 1 & \text{if } m_f \in T(f) \end{cases}. \]

In both cases, we have the following upper bound for (6.5):

\[ P_\mu \left( \begin{array}{c} \forall i \in \{1, \ldots, m_f\} X_{\tau_i}(f) = 0 \\ \forall i \in T(f) X_{\tau_{i+1}}(f) = 1 \\ \forall i \in \{1, \ldots, m_f - 1\} \exists \theta_i \in [\tau_i, \tau_{i+1}] X_{\theta_i}(f) = 1 \end{array} \right) \leq \frac{2(1-p)^{m_f}}{|\Lambda||J(f) \cap (T + 2)|}. \]
We obtain an upper bound for (6.3) by multiplying this inequality over the edges \( f \) in support(\( \gamma \)):

\[
P_{\mu} \left( \begin{array}{c}
\forall i \in T \quad E_{b+1}^i = e_{i+2} \\
\forall i \in I \setminus T \quad X_{t_i}(e_i) = 0 \\
\forall p, q \in I \setminus T \text{ s.t. } e_p = e_q, t_p > t_q \\
\exists r \| t_q, t_p \| X_r(e_p) = 1
\end{array} \right) \leq \frac{2^{|S|}(1-p)^{|S|}}{|\Lambda|^{|J(f) \cap (T+2)|}} \leq \frac{2^{|S|}(1-p)^{|I|-k}}{|\Lambda|^k}. \tag{6.7}
\]

Since \(|I| - k \geq n\), and \(|S| \leq n\), for \( k \) fixed and \((t_i)_{i \in I} \) fixed, we have the following upper bound for \((6.1)\),

\[
P \left( (e_i, t_i)_{i \in I} \text{ decreasing simple impatient X-closed-moving STP except on } e \right) \leq \frac{(2 - 2p)^n}{|\Lambda|^k}.
\]

Finally, we use this upper bound in \((6.2)\) and we have

\[
P_{\mu} \left( \begin{array}{c}
\exists \gamma \text{ decreasing simple impatient X-closed-moving STP except on } e, \\
\gamma \text{ starts from } (e, t) \text{ and ends after } s, \\
\text{space}(\gamma) = (\varepsilon_1, \ldots, \varepsilon_n)
\end{array} \right) \leq \sum_{1 \leq k \leq n} \sum_{A \subset \{1, \ldots, n\}, |A| = k} \sum_{K \subset \{s, \ldots, t\}, |K| = k} \left( \frac{2 - 2p}{|\Lambda|^k} \right)^n \leq \sum_{1 \leq k \leq n} \binom{n}{k} \binom{\ell|\Lambda|}{k} \left( \frac{2 - 2p}{|\Lambda|^k} \right)^n \leq \sum_{1 \leq k \leq n} \left( \frac{\ell|\Lambda|}{k} \right)^n \left( \frac{4 - 4p}{|\Lambda|^k} \right)^n \leq \left( 1 + \frac{1}{|\Lambda|^k} \right)^{\ell|\Lambda|} (4 - 4p)^n.
\]

This yields the desired result. \(\square\)

We use next proposition 5.6 and lemma 6.1 to show that the pivotal edges cannot move too fast.

**Proposition 6.2.** There exists \( \tilde{p} < 1 \), such that for \( p \geq \tilde{p} \), for \( \ell \geq 1, t \in \mathbb{N}, s \in \mathbb{N}, s \leq \ell|\Lambda| \) and any edge \( e \) at distance at least \( \ell \) from the boundary of \( \Lambda \),

\[
P_{\mu} \left( e \in \mathcal{P}_{t+s}, d(e, \mathcal{P}_t \cup \mathcal{I}_t \setminus \{e\}) \geq \ell \right) \leq \exp(-\ell).
\]

**Proof.** By proposition 5.6 there exists a STP which is decreasing simple impatient and X-closed-moving except on \( e \) which starts from the edge \( e \) at time \( t + s \) and ends at an edge of \( \mathcal{P}_t \cup \mathcal{I}_t \setminus \{e\} \) or an edge intersecting the
boundary of $\Lambda$ after the time $t$. In both cases, this STP has a length at least $\ell$. Therefore, we have the inequality

$$P_\mu\left(e \in \mathcal{P}_{t+s}, d(e, \mathcal{P}_t \cup \mathcal{I}_t \setminus \{e\}) \geq \ell\right) \leq P_\mu\left(\exists \gamma \text{ decreasing simple impatient } \mathcal{X}\text{-closed-moving STP except on } e, \gamma \text{ starts from } (e, t+s) \text{ and ends after } t, |\text{length}(\gamma)| \geq \ell\right).$$

Let us fix a path $(e_1, \ldots, e_\ell)$ with $n = \ell$ starting from $e$. By lemma [6.1] for $\ell \geq 1$, we have

$$P_\mu\left(\exists \gamma \text{ decreasing simple impatient } \mathcal{X}\text{-closed-moving STP except on } e, \gamma \text{ starts from } (e, t+s) \text{ and ends after } t, \text{space}(\gamma) = (e_1, \ldots, e_\ell)\right) \leq \left(1 + \frac{1}{|\Lambda|}\right)^{\ell|\Lambda|} (4 - 4p)^\ell.$$

We sum over the number of the choices for the path $(e_1, \ldots, e_\ell)$ and we obtain

$$P_\mu\left(\exists \gamma \text{ decreasing simple impatient } \mathcal{X}\text{-closed-moving STP except on } e, \gamma \text{ starts from } (e, t+s) \text{ and ends after } t, |\text{length}(\gamma)| \geq \ell\right) \leq \left(1 + \frac{1}{|\Lambda|}\right)^{\ell|\Lambda|} \beta(d)^\ell (4 - 4p)^\ell,$$

where $\beta(d)$ is the number of the $\ast$-neighbours of an edge in dimension $d$. There exists a $\tilde{p} < 1$ such that for $p \geq \tilde{p}$, we have

$$\left(1 + \frac{1}{|\Lambda|}\right)^{\ell|\Lambda|} \beta(d)^\ell (4 - 4p)^\ell \leq e^{-\ell}.$$

This gives the desired upper bound. \hfill \Box

7 Distance between the edges of $\mathcal{P} \cup \mathcal{I}$

We now prove theorem [1.1] with the help of proposition [6.2] and the observation that an edge of the interface cannot survive a too long time.

Proof of theorem 1.1. Let $c$ be a constant bigger than 1 and $\kappa$ be a constant which will be chosen later. Since $\mu_p$ is the invariant probability of the couple $(X_t, Y_t)_{t \in \mathbb{N}}$, we can choose an arbitrary time $t$ and rewrite the probability in the statement of the theorem as

$$P_\mu\left(\exists e \in \mathcal{P}_t \cup \mathcal{I}_t, d(e, \Lambda^c \cup \mathcal{P}_t \cup \mathcal{I}_t \setminus \{e\}) \geq \kappa c \ln |\Lambda|\right).$$

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We shall distinguish the two cases $e \in \mathcal{P}_t$ and $e \in \mathcal{I}_t \setminus \mathcal{P}_t$. We consider the first case where $e$ is a pivotal edge at time $t$ and we estimate the probability

$$P_\mu\left( \exists e \in \mathcal{P}_t, d(e, \Lambda^c \cup \mathcal{P}_t \cup \mathcal{I}_t \setminus \{e\}) \geq \kappa c \ln |\Lambda| \right).$$

This probability is less than

$$P_\mu\left( \exists e \in \mathcal{P}_t, d(e, \Lambda^c \cup \mathcal{P}_t \setminus \{e\}) \geq \kappa c \ln |\Lambda| \right).$$

Since this last probability is determined by the configuration $Y_t$, we can replace $P_\mu$ by the second marginal law of $\mu$ which is $P_\mu(\cdot | T \leftrightarrow B)$. By proposition 4.1 in the case where $\varepsilon = 0$, there exists $p_1 < 1$, $\kappa_1 \geq 0$ such that for $p \geq p_1$, we have for any $\Lambda$ such that $|\Lambda| \geq 3^d$ and $c \geq 1$,

$$P_\mu\left( \exists e \in \mathcal{P}_t, d(e, \Lambda^c \cup \mathcal{P}_t \setminus \{e\}) \geq \kappa_1 c \ln |\Lambda| \right) \leq \frac{1}{|\Lambda|^c}. \quad (7.1)$$

Let us fix an edge $e$ in $\Lambda$ and let us focus on the second case where the edge $e$ belongs to $\mathcal{I}_t \setminus \mathcal{P}_t$. Such an edge was pivotal when it became an edge of the interface and it became non pivotal at a later time. We consider the last time when it was pivotal before $t$ and we define the random integer $s$ such that

$$s = \inf \{ r \geq 0 : e \in \mathcal{P}_{t-r} \}.$$

The edge $e$ is not pivotal during the time interval $[t - s + 1, t]$ and it belongs to the interface. Moreover, it cannot be chosen to be modified during this interval since it must remain different in the two processes. Therefore, for any $r \in [t - s + 1, t]$, we have $E_r \neq e$. We conclude that the number $s$ cannot be too big because the sequence $(E_t)_{t \in \mathbb{N}}$ is a sequence of i.i.d. random edges chosen uniformly in $\Lambda$. More precisely, we have the following inequality:

$$P_\mu\left( s \geq 2dc|\Lambda|\ln|\Lambda| \right) \leq P\left( \forall r \in [t - 2dc|\Lambda|\ln|\Lambda| + 1, t], E_r \neq e \right) \leq \left( 1 - \frac{1}{2d|\Lambda|\ln|\Lambda|} \right)^{2dc|\Lambda|\ln|\Lambda|} \leq \frac{1}{|\Lambda|^c}. \quad (7.2)$$

We study next the case where $s < 2dc|\Lambda|\ln|\Lambda|$. Let us fix a $s < 2dc|\Lambda|\ln|\Lambda|$ and let us estimate the probability

$$P_\mu\left( e \in \mathcal{P}_{t-s}, e \in \mathcal{I}_t \setminus \mathcal{P}_t, d(e, \Lambda^c \cup \mathcal{P}_t \cup \mathcal{I}_t \setminus \{e\}) \geq \kappa c \ln |\Lambda| \right).$$

In order to use proposition 6.2, we would like to replace $t - s$ by $t + s$. However, the process $(X_t, Y_t)_{t \in \mathbb{N}}$ is not reversible under his stationary measure $\mu$. On
the other hand, by lemma 3.1, the process \((X_t, Y_t^\varepsilon)_{t \in \mathbb{N}}\) is reversible under \(\mu^\varepsilon\) and by lemma 3.2, the measure \(\mu^\varepsilon\) converges to \(\mu\) when \(\varepsilon\) converges to 0. Notice that the previous probability involves only events occurring during the time interval \([t - s, t]\) and we supposed that \(s < 2dc|\Lambda| \ln |\Lambda|\). Therefore, the probability

\[
P^\varepsilon_\mu \left( e \in \mathcal{P}_{t-s}, e \in \mathcal{I}_t \setminus \mathcal{P}_t, d(e, \Lambda^c \cup \mathcal{P}_t \cup \mathcal{I}_t \setminus \{e\}) \geq \kappa c \ln |\Lambda| \right)
\]

converges to

\[
P_\mu \left( e \in \mathcal{P}_{t-s}, e \in \mathcal{I}_t \setminus \mathcal{P}_t, d(e, \Lambda^c \cup \mathcal{P}_t \cup \mathcal{I}_t \setminus \{e\}) \geq \kappa c \ln |\Lambda| \right)
\]

when \(\varepsilon\) converges to 0. Moreover, for any \(\varepsilon > 0\), the process \((X_t, Y_t^\varepsilon)_{t \in \mathbb{N}}\) is reversible under \(\mu^\varepsilon\). Thus, we can reverse the time and we have

\[
P^\varepsilon_\mu \left( e \in \mathcal{P}_{t+s}, e \in \mathcal{I}_t \setminus \mathcal{P}_t, d(e, \Lambda^c \cup \mathcal{P}_t \cup \mathcal{I}_t \setminus \{e\}) \geq \kappa c \ln |\Lambda| \right) = P^\varepsilon_\mu \left( e \in \mathcal{P}_{t+s}, e \in \mathcal{I}_t \setminus \mathcal{P}_t, d(e, \Lambda^c \cup \mathcal{P}_t \cup \mathcal{I}_t \setminus \{e\}) \geq \kappa c \ln |\Lambda| \right).
\]

This last probability converges to

\[
P_\mu \left( e \in \mathcal{P}_{t+s}, e \in \mathcal{I}_t \setminus \mathcal{P}_t, d(e, \Lambda^c \cup \mathcal{P}_t \cup \mathcal{I}_t \setminus \{e\}) \geq \kappa c \ln |\Lambda| \right)
\]

when \(\varepsilon\) converges to 0. Therefore, we have

\[
P_\mu \left( e \in \mathcal{P}_{t-s}, e \in \mathcal{I}_t \setminus \mathcal{P}_t, d(e, \Lambda^c \cup \mathcal{P}_t \cup \mathcal{I}_t \setminus \{e\}) \geq \kappa c \ln |\Lambda| \right) = P_\mu \left( e \in \mathcal{P}_{t+s}, e \in \mathcal{I}_t \setminus \mathcal{P}_t, d(e, \Lambda^c \cup \mathcal{P}_t \cup \mathcal{I}_t \setminus \{e\}) \geq \kappa c \ln |\Lambda| \right).
\]

By proposition 6.2, there exists a \(p_2 < 1\) such that, by letting \(\kappa = 2d\), for \(p \geq p_2\) and \(c \geq 1\), the last probability is less than

\[
2 \exp \left( -2dc \ln |\Lambda| \right) = \frac{2}{|\Lambda|^{2dc}}.
\]

We then sum over the number of the choices for the edge \(e\) and of the number \(s\) from 1 to \(2dc|\Lambda| \ln |\Lambda|\). We obtain

\[
P_\mu \left( \exists e \in \mathcal{I}_t \setminus \mathcal{P}_t \quad d(e, \Lambda^c \cup \mathcal{P}_t \cup \mathcal{I}_t \setminus \{e\}) \geq \kappa c \ln |\Lambda| \quad \exists s \leq 2dc|\Lambda| \ln |\Lambda| \quad e \in \mathcal{P}_{t-s} \right) \leq \frac{8d^2c \ln |\Lambda|}{|\Lambda|^{2dc-2}}.
\]
Finally, we sum together the three cases (7.1), (7.2) and (7.3). Let
\[ \kappa = \max(\kappa_1, 2d), \quad \tilde{p} = \max(p_1, p_2), \]
we obtain for \( p \geq \tilde{p} \) and \( c \geq 1 \),
\[ P_\mu(\exists e \in \mathcal{P}_t \cup \mathcal{I}_t, d(e, \Lambda^c \cup \mathcal{P}_t \cup \mathcal{I}_t \setminus \{e\}) \geq 4dc\kappa \ln |\Lambda|) \leq \frac{2}{|\Lambda|^c} + \frac{8d^2c \ln |\Lambda|}{|\Lambda|^{2dc-2}}. \]
For \( \Lambda \) such that \( |\Lambda| \geq \max\{e^{2d^2c}, 3^{6d}\} \), we have
\[ \frac{2}{|\Lambda|^c} + \frac{8d^2c \ln |\Lambda|}{|\Lambda|^{2dc-2}} \leq \frac{1}{|\Lambda|^{c-1}}. \]
We can replace \( c \) with \( c+1 \) by replacing \( \kappa \) with \( \kappa(c+1)/c \) and we obtain the desired result. 

8 Law of an edge far from a cut and the interface

Now we prove the second main result theorem 1.2. This result shows that, for a fixed edge \( e \), conditionally on the existence of a cut far from \( e \) and on \( e \) being also far from the interface, its state in the configuration \( Y_t \) differs little from that in the configuration \( X_t \) which, at equilibrium, follows a Bernoulli variable with parameter \( p \).

Proof of theorem 1.2. We start with the same discussion as in the proof of theorem 1.1 to replace the probability in the theorem by
\[ P_\mu(e \in \mathcal{I}_t \mid \exists C \in \mathcal{C}_t, d(e, C \cup \mathcal{I}_t \setminus \{e\}) \geq 4dc\kappa \ln |\Lambda|). \]
We introduce again the time
\[ s = \inf \{ r \geq 0 : e \in \mathcal{P}_{t-r} \}. \]
By the same argument as in the proof of theorem 1.1 we have
\[ P_\mu(s \geq 2dc|\Lambda| \ln |\Lambda|) \leq \frac{1}{|\Lambda|^c}. \]
We then concentrate on the case \( s < 2dc|\Lambda| \ln |\Lambda| \). We define the event
\[ D = \{ \exists C \in \mathcal{C}_t, d(e, C \cup \mathcal{I}_t \setminus \{e\}) \geq 4dc\kappa \ln |\Lambda| \} \]
and we rewrite the conditioned probability as
\[
P_\mu \left( e \in \mathcal{I}_t, D, s < 2dc|\Lambda| \ln |\Lambda| \right) \frac{P_\mu(D)}{P_\mu(D)}.
\] (8.1)

Let us fix a \( s < 2dc|\Lambda| \ln |\Lambda| \), the numerator of (8.1) is less than
\[
\sum_{1 \leq s < 2dc|\Lambda| \ln |\Lambda|} P_\mu(e \in \mathcal{P}_{t-s}, D).
\]

We approach each term in the sum by \( P_\mu^\varepsilon(e \in \mathcal{P}_{t-s}, D) \), which, by reversibility is equal to \( P_\mu^\varepsilon(e \in \mathcal{P}_{t+s}, D) \). Therefore, by sending \( \varepsilon \) to 0, we have
\[
P_\mu(e \in \mathcal{I}_t, D) \leq \sum_{1 \leq s < 2dc|\Lambda| \ln |\Lambda|} P_\mu(e \in \mathcal{P}_{t+s}, D).
\]

By proposition [5.6] there exists a decreasing simple impatient \( X \)-closed-moving except on \( e \) STP which starts from \((e, t+s)\) and ends at an edge of \( \mathcal{P}_t \cup \mathcal{I}_t \setminus \{e\} \) or an edge intersecting the boundary of \( \Lambda \) after the time \( t \). In both cases, there exists a STP which starts from \((e, t+s)\) and which travels a distance at least \( 2dc\kappa \ln |\Lambda| \). By stopping after the first \( \lfloor \kappa c \ln |\Lambda| \rfloor \) edges, we can suppose that the length of this STP is \( \lfloor \kappa c \ln |\Lambda| \rfloor \). Therefore, the last probability is less than
\[
P_\mu \left( \exists \gamma \text{ decreasing simple impatient} \right.
X\text{-closed-moving STP except on } e
\gamma \text{ starts from } (e, t+s) \text{ and ends after } t
|\text{space}(\gamma)| = \lfloor \kappa c \ln |\Lambda| \rfloor
\exists C \in \mathcal{C}_t, d(e, C \cup \mathcal{I} \setminus \{e\}) \geq 4dc\kappa \ln |\Lambda| \right).
\]

Like in the proof of lemma [6.1] we fix a path \((\varepsilon_1, \ldots, \varepsilon_m)\) with \( m = \lfloor \kappa c \ln |\Lambda| \rfloor \) and we will sum over the paths at the end. To simplify the notation, we define the event
\[
\Gamma(e) = \left\{ \exists \gamma \text{ decreasing simple impatient} \right.
X\text{-closed-moving STP except on } e
\gamma \text{ starts from } (e, t+s) \text{ and ends after } t
|\text{space}(\gamma)| = (\varepsilon_1, \ldots, \varepsilon_m) \left. \right\}.
\]

The previous probability is less than
\[
\sum_{(\varepsilon_1, \ldots, \varepsilon_m) \text{ path from } e} P_\mu \left( \Gamma(e) \cap D \right).
\] (8.2)
We reuse the approximation of \( P_\mu \) by \( P_\varepsilon \) to calculate the probability. We concentrate on the following probability:

\[
P_\mu^e \left( \Gamma(e) \cap D \right). \tag{8.3}
\]

We condition the probability \( 8.3 \) by a configuration \((X_t, Y_t^e)\) in \( D \):

\[
P_\mu^e \left( \Gamma(e) \cap D \right) = \sum_{\omega \in D} P_\mu^e \left( \Gamma(e) \mid (X_t, Y_t^e) = \omega \right) P_\mu^e \left( (X_t, Y_t^e) = \omega \right).
\]

Let us fix a sequence of edges \((e_1, \ldots, e_n)\) and consider a STP \((e_1, t_1), \ldots, (e_n, t_n)\)

starting from \((e, t+s)\), ending after \( t \), such that

\[
\text{space}((e_i, t_i)_{1 \leq i \leq n}) = (\varepsilon_1, \ldots, \varepsilon_m).
\]

For an index \( i \in \{1, \ldots, n\} \) such that \( e_i \neq e \), we define a time interval \( T_i = [\theta(i), t_i] \), where

\[
\theta(i) = \sup \{ r \leq t_i : E_r = e_i, U_r \geq p \}.
\]

Since \( \theta(i) \) is the last time before \( t_i \) when \( e_i \) closes, the fact that \( \gamma \) is simple implies that

\[
\forall i, j \in \{1, \ldots, n\} \quad (i \neq j, e_i = e_j) \Rightarrow T_i \cap T_j = \emptyset.
\]

Moreover, if \( \theta(i) > t \), then the interval \( T_i \) depends on the variables \((E_r, U_r)_{r \geq t}\) and it is therefore independent from the configuration at time \( t \). The STP \( \gamma \) is \( X \)-closed-moving except on \( e \), therefore

\[
\forall i \in \{1, \ldots, n\} \quad \forall r \in T_i \quad X_r(e_i) = 0.
\]

For a subset \( N \) of \( \{1, \ldots, n\} \), we define

\[
A(e, N) = \left\{ \forall i \in \{1, \ldots, n\} \text{ such that } e_i \neq e \quad \exists T_i = [a_i, b_i] \quad (e_i, b_i)_{1 \leq i \leq n} \text{ decreasing impatient STP} \forall i \in N, \quad a_i = \sup \{ r \leq b_i : E_r = e_i, U_r \geq p \} > t \quad \forall r \in T_i \quad X_r(e_i) = 0 \quad \forall i, j \in \{1, \ldots, n\} \text{ such that } i \neq j, e_i = e_j \quad T_i \cap T_j = \emptyset \right\}
\]

and

\[
B(e, N) = \{ \forall i \in \{1, \ldots, n\} \setminus N \quad X_t(e_i) = 0 \}.
\]
We distinguish different cases according to the intervals $T_i$ which contain the time $t$. We define the deterministic set of indices 

$$L = \{ i \in \{1, \ldots, n\} : \forall j \in \{i+1, \ldots, n\} \ e_j \neq e_i \}$$

which correspond to the indices of the last visits. We also define the random set 

$$I = \{ i : t \in T_i \}.$$ 

We have therefore 

$$P_{\mu^e}(\Gamma(e) | (X_t, Y_t^\varepsilon) = \omega) = \sum_{N \subseteq L} P_{\mu^e}(\Gamma(e), I = N | (X_t, Y_t^\varepsilon) = \omega).$$

For $N \subseteq L$, the event $A(e, N)$ depends on the variables $(X_{r(e_i)})_{i \in N, r > t}$ and the event $B(e, N)$ depends on the variables $(X_t(e_i))_{i \in \Lambda}$. Thus, under $P_{\mu^e}$, the event $A(e, N)$ is independent from $(X_t, Y_t^\varepsilon)$ and from $B(e, N)$. We obtain 

$$P_{\mu^e}(\Gamma(e), I = N | (X_t, Y_t^\varepsilon) = \omega) \leq P_{\mu^e}(A(e, N) \cap B(e, N) | (X_t, Y_t^\varepsilon) = \omega) \leq P_{\mu^e}(A(e, N)) \times P_{\mu^e}(B(e, N) | (X_t, Y_t^\varepsilon) = \omega).$$

We resum next over $\omega \in D$ and we get 

$$P_{\mu^e}(\Gamma(e), D) \leq \sum_{\omega \in D} \sum_{N \subseteq L} P_{\mu^e}(\Gamma(e), I = N | (X_t, Y_t^\varepsilon) = \omega) P_{\mu^e}((X_t, Y_t^\varepsilon) = \omega) \leq \sum_{N \subseteq L} P_{\mu^e}(A(e, N)) P_{\mu^e}(B(e, N) \cap D).$$

The last probability depends only on the configuration $(X_t, Y_t^\varepsilon)$ at time $t$ and we can replace $P_{\mu^e}$ by $\mu^e$. By definition of $\mu^e$, we can rewrite this probability as 

$$P^e\left(B(e, N) \cap D \mid T \leftrightarrow B \text{ in } X^\varepsilon,\right),$$

where $P^e$ is the stationary measure of the couple $(X_t, X_t^\varepsilon)_{t \in \Lambda}$ introduced in the section [3.1]. Notice that this probability is a product measure over the edges of $\Lambda$. Moreover, the event $B(e, N)$ depends on the edges which are of distance less than $2\kappa c \ln |\Lambda|$ from the edge $e$, whereas the event $D$ depends on the edges that are at distance more than $4d c \kappa \ln |\Lambda|$ from $e$ or the edges which are open in $X_t$. We rewrite $D$ as 

$$D = D_1 \cap D_2$$
where
\[ D_1 = \{ \exists C \in C_t, d(e, C) \geq 4dc \kappa \ln |\Lambda| \} \]
and
\[ D_2 = \{ \forall f \subset \Lambda, d(f, e) \leq 2\kappa c \ln |\Lambda|, f \notin T \}. \]
The event \( D_1 \) is independent from \( D_2 \) and \( B(e, N) \) under \( P^\epsilon \). We have
\[ P^\epsilon(B(e, N) \cap D_1 \cap D_2) = P^\epsilon(B(e, N) \cap D_2) P^\epsilon(D_1). \]
Let us consider the probability \( P^\epsilon(B(e, N) \cap D_2) \), we write
\[ P^\epsilon(B(e, N) \cap D_2) = P^\epsilon(B(e, N) | D_2) P^\epsilon(D_2). \]
The condition \( D_2 \) implies that the uniform random variables
\[ \{ U_f : d(f, e) \leq 2\kappa c \ln |\Lambda| \} \]
in \([0, 1]\) which we use to construct \( P^\epsilon \) are not in the interval \([p - \epsilon, p]\). We have
\[ P^\epsilon(B(e, N) | D_2) \leq \frac{P_p(B(e, N))}{(1 - \epsilon)^M}, \]
where \( M \) is the number of edges at distance less than \( 2\kappa c \ln |\Lambda| \) from \( e \). We have therefore
\[ \mu^\epsilon(B(e, N) \cap D) \leq \frac{P_p(B(e, N)) P^\epsilon(D_1) P^\epsilon(D_2)}{P^\epsilon(T \leftrightarrow B \in X^\epsilon) (1 - \epsilon)^M} \mu^\epsilon(D). \]
We obtain
\[ P^\mu(\Gamma(e), D) \leq \frac{1}{(1 - \epsilon)^M} \sum_{N \subset L} P^\mu(A(e, N)) P_p(B(e, N)) \mu^\epsilon(D). \]
When \( \epsilon \) converges to 0, we have
\[ P^\mu(\Gamma(e), D) \leq \sum_{N \subset L} P^\mu(A(e, N)) P_p(B(e, N)) \mu(D). \] (8.4)
We then distinguish two cases according to the size of \( N \). If \( |N| \leq n/2 \), then there is a subset of \( (e_1, \ldots, e_n) \) of cardinal at least \( n/2 \) which is closed in \( X_t \). Therefore, we have
\[ \sum_{N \subset L, |N| \leq n/2} P^\mu(A(e, N)) P_p(B(e, N)) \mu(D) \leq \mu(D) \sum_{N \subset L, |N| \leq n/2} P_p(B(e, N)) \leq \mu(D)(4 - 4p)^{n/2}. \]
If $|N| > n/2$, then there are at least $n/2$ edges which are closed when being visited by the STP. Therefore, we can write

$$
\sum_{N \subset L, |N| \geq n/2} P \left( A(e, N) \right) P \left( B(e, N) \right) \mu(D)
\leq \mu(D) \sum_{N \subset L, |N| \geq n/2} P \left( A(e, N) \right)
\leq \mu(D) \sum_{N \subset L, |N| \geq n/2} P \left( (e_i, t_i)_{1 \leq i \leq n} \text{ decreasing simple impatient STP} \right).
$$

For each $N$ fixed, we reapply the techniques used in the proof of lemma 6.1. We fix at first the times $t_1, \ldots, t_n$ and we factorise the probability as a product over the edges of support$(e_1, \ldots, e_n)$. For each edge $f \in$ support$(e_1, \ldots, e_n)$, we define

$$
J(f) = \{ i \in N : e_i = f \}.
$$

We define in the same way $m_f$, $E(r)$ with the new definition of $J(f)$. The only difference is that the last visit of $f$ is not necessarily closed with the new definition of $J(f)$ so the equations (6.6) and (6.7) still hold. Since

$$
\sum_f m_f = |N| \geq n/2
$$

and $s < 2dc|\Lambda| \ln |\Lambda|$ we have

$$
P \left( \exists t_1, \ldots, t_n \in \{ t + 1, \ldots, t + s \} \right.
\left. (e_i, t_i)_{1 \leq i \leq n} \text{ decreasing simple impatient STP} \right.
\forall i \in N \ X_{t_i}(e_i) = 0
\leq \left( 1 + \frac{1}{|\Lambda|} \right)^{2dc|\Lambda| \ln |\Lambda|} (4 - 4p)^{n/2}.
$$

We sum over the choices of $N$ and we have

$$
\sum_{N \subset L, |N| \geq n/2} P \left( A(e, N) \right) \leq \left( 1 + \frac{1}{|\Lambda|} \right)^{2dc|\Lambda| \ln |\Lambda|} 2^n (4 - 4p)^{n/2}.
$$

Combining the two previous cases, we obtain for (8.4),

$$
P \left( \Gamma(e), D \right) \leq \mu(D) \left( (4 - 4p)^{n/2} + \left( 1 + \frac{1}{|\Lambda|} \right)^{2dc|\Lambda| \ln |\Lambda|} 2^n (4 - 4p)^{n/2} \right).
$$
We then sum over the choices of the path \((\varepsilon_1, \ldots, \varepsilon_m)\) in (8.2) and we obtain

\[
P_{\mu}\left( \begin{array}{l}
\exists \gamma \text{ decreasing simple impatient} \\
X\text{-closed-moving STP except on } e \\
\text{\gamma starts from } (e, t + s) \text{ and ends after } t \\
|\text{length}(\gamma)| = c \ln |\Lambda| \\
\exists C \in C_t, d(e, C \cup I_t \setminus \{e\}) \geq 4dc \ln |\Lambda|
\end{array} \right)
\leq \sum_{\varepsilon_1, \ldots, \varepsilon_m} \mu(D) \left( (4 - 4p)^{n/2} + \left(1 + \frac{1}{|\Lambda|}\right)^{2dc|\Lambda| \ln |\Lambda|} 2^n (4 - 4p)^{n/2} \right)
\leq \mu(D) \beta(d)^m \left( (4 - 4p)^{n/2} + \left(1 + \frac{1}{|\Lambda|}\right)^{2dc|\Lambda| \ln |\Lambda|} 2^n (4 - 4p)^{n/2} \right).\]

Since \(m = \lfloor \kappa c \ln |\Lambda| \rfloor\) and \(m \leq n\), there exists a \(\tilde{p} < 1\) and \(\kappa \geq 1\) such that for \(p \geq \tilde{p}\), we have

\[
\beta(d)^m \left( (4 - 4p)^{n/2} + \left(1 + \frac{1}{|\Lambda|}\right)^{2dc|\Lambda| \ln |\Lambda|} 2^n (4 - 4p)^{n/2} \right) \leq \frac{1}{|\Lambda|^c}.
\]

We sum over \(s\) from 1 to \(2dc|\Lambda| \ln |\Lambda|\). We have therefore the following upper bound for (8.1):

\[
\frac{2dc \ln |\Lambda|}{|\Lambda|^{c-1}}.
\]

Combined with the case \(s > 2dc|\Lambda| \ln |\Lambda|\), we have

\[
P_{\mu}(e \in I_t \mid D) \leq \frac{2dc \ln |\Lambda|}{|\Lambda|^{c-1}} + \frac{1}{|\Lambda|^c}.
\]

For a box \(\Lambda\) such that \(|\Lambda| \geq e^{4dc}\), we have

\[
\frac{2dc \ln |\Lambda|}{|\Lambda|^{c-1}} + \frac{1}{|\Lambda|^c} \leq \frac{1}{|\Lambda|^{c-2}}.
\]

We can replace \(c - 2\) with \(c\) by replacing \(\kappa\) with \(\kappa(c + 2)/c\) and we obtain the desired result.

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