EXISTENCE OF NONOSCILLATORY RELATIVELY BOUNDED SOLUTIONS OF SECOND ORDER NEUTRAL DIFFERENTIAL EQUATIONS

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ABSTRACT
In this paper some sufficient conditions are obtained to insure the existence of positive solutions which is relatively bounded from one side for nonlinear neutral differential equations of second order. We used the Krasnoselskii’s fixed point theorem and Lebesgue’s dominated convergence theorem to obtain new sufficient conditions for the existence of a Nonoscillatory one side relatively bounded solutions. These conditions are more applicable than some known results in the references. Three examples included to illustrate the results obtained.

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1. INTRODUCTION

This paper is concerned with the existence of a positive relatively bounded solution of the neutral differential equations of the form

\[ (a(t)(x(t) - q(t)x(\tau(t))))' - p(t)f(t, x(t), x(\sigma(t)), x'(t), x'(\sigma(t))) = 0. \]  

(1.1)

with respect to equation (1.1), throughout we shall assume the following:

(i) \( p, q \in C([t_0, \infty), \mathbb{R}^+), a \in C([t_0, \infty), (0, \infty)), t \geq t_0 \), \( t \geq t_0, t_0 \in \mathbb{R} \).

(ii) \( \tau, \sigma \in C([t_0, \infty), \mathbb{R}), \tau(t) \leq t, \sigma(t) \leq t, \lim_{t \to \infty} \tau(t) = \infty, \lim_{t \to \infty} \sigma(t) = \infty \)

(iii) \( f \in C([t_0, \infty) \times \mathbb{R}^4, \mathbb{R}) \), \( f \) is nondecreasing function, and \( xf(t, x(t), x(\sigma(t)), x'(t), x'(\sigma(t))) > 0, x \neq 0 \).

By a solution of Eq. (1.1) we mean a function \( x \in C([t_1, \infty), \mathbb{R}) \), \( \rho(t_1) = \min\{\tau(t), \sigma(t)\} \), for some \( t_1 \geq t_0 \), such that \( a(t)(x(t) - q(t)x(\tau(t))) \) is continuously differentiable on \([t_1, \infty)\) and such that \( x(t) \) satisfy Eq. (1.1) for \( t \geq t_1 \). A solution \( x(t) \) is said to be nonoscillatory if it is either eventually positive or eventually negative that is there exists \( t_*, t_0 \), such that either \( x(t) > 0 \) or \( x(t) < 0 \) for all \( t \geq t_* \), otherwise is said oscillatory [10].

Recently there have been a lot of activities concerning the existence of nonoscillatory solutions for neutral differential equations. In 1999, S. Tanaka [12] study the first order differential equations:

\[ \frac{d}{dt}[x(t) + h(t)x(\tau(t))] + \sigma f(t, x(g(t))) = 0 \]

and established some sufficient conditions to insure the existence of positive solution of previous equation. In 2002, Y. Zhou, B. Zhang [14], found some sufficient conditions for the existence of nonoscillatory solutions the following equation:

\[ \frac{d^n}{dt^n}[x(t) + cx(t - \tau)] + (-1)^{n+1}[P(t)x(t - \sigma)] - Q(t)x(t - \delta) = 0. \]

In 2005, Y. Yu, H. Wang [13], studied the nonoscillatory solutions of a class of second-order nonlinear neutral delay differential equations with positive and negative coefficients of the form:

\[ r(t)(x(t) + P(t)x(t - \tau))' + Q_1(t)f(x(t - \sigma_1)) - Q_2(t)g(x(t - \sigma_2)) = 0 \]

In 2009, B. Dorociakova and R. Olach [5] studied the first order delay differential equations:

\[ x'(t) + p(x(t))x(t) + q(t)x(\tau(t)) = 0. \]

In the same year I. Culkov, L. Hanutiakov, R. Olach [3] studied the second order nonlinear neutral differential equations

\[ \frac{d^2}{dt^2}[x(t) + a(t)x(t - \tau)] = p(t)f(x(t - \sigma)). \]

In 2011, R. Olach etc. al [4], studied the first order neutral differential equations:

\[ \frac{d}{dt}[x(t) - a(t)x(t - \tau)] = p(t)f(x(t - \sigma)) \]

In 2012, L. Lietc. al [9], studied the existence of a bounded nonoscillatory positive solution for the equation.

\[ \frac{d}{dt}[x(t) + a(t)x(t - \tau)] + p(t)f(x(t - \sigma)) + q(t)g(x(t - \beta)) = 0 \]

In 2013, T. Canadan [1], obtained sufficient conditions for first-order nonlinear neutral differential equations to have nonoscillatory solutions for different ranges of \( p_1(t) \) and \( p_2(t, \xi) \) of the forms:

\[ [(x(t) - p_1(t)x(\tau(t)))]' + Q_1(t)g(x(t - \sigma)) = 0 \]

\[ [(x(t) - p_1(t)x(\tau(t)))]' + \int_{\xi}^{\tau} Q_2(t)g(x(t - \xi))d\xi = 0 \]

and

\[ \left[x(t) + \int_{\alpha}^{\beta} p_2(t, \xi)x(t - \xi)d\xi \right]' + \int_{\alpha}^{\tau} Q_2(t)g(x(t - \xi))d\xi = 0 \]

In 2017, F. Kong [8], studied the Existence of nonoscillatory solutions of a kind of first-order neutral differential equation:

\[ \frac{d}{dx}[x(t) + P_1(t)x(t - \tau_1) + P_2(t)x(t + \tau_2)] + Q_1(t)g_1(x(t - \sigma_1)) - Q_2(t)g_2(x(t + \sigma_2)) = 0 \]

In 2018, B. Çına and M. Tamer Şenel [2], obtained some sufficient conditions for the existence of positive solutions of variable coefficient nonlinear second order neutral differential equation with distributed deviating arguments of the form:

\[ \left(x(t) - \int_{\alpha_1}^{\beta_1} P(t, \xi)x(t - \xi)d\xi \right)' + \int_{\alpha_2}^{\beta_2} f(t, x(\sigma(t, \xi)))d\xi = 0 \]

In this paper we prove that the existence of solution of Eq. (1.1) is relatively bounded, and we show that the solution is bounded from one side by function from above and below by function and ratio function respectively. some sufficient conditions for this purpose are obtained.

**Definition 1.1** A function \( x(t) \) is said to be relatively bounded from below (above) if there is a function \( y(t) \) and constant \( k \) such that \( y(t) \leq x(t) \leq k(y(t)) \).

The following fixed point theorem and Lebesgue’s dominated convergence theorem will be used to prove the main results in the next section.

**Lemma 1.2** [7] (Krasnoselskii’s Fixed Point Theorem).

Let \( X \) be a Banach space, let \( \Omega \) be a bounded closed convex subset of \( X \), and let \( S_1, S_2 \) be maps of \( \Omega \) into \( X \) such that \( S_1x + S_2y \in \Omega \) for every pair \( x, y \in \Omega \). If \( S_1 \) is...
contractive and $S_2$ is completely continuous, then the equation $S_2 x + S_1 x = x$ has a solution in $\Omega$.

**Theorem 1.3** [11] (The Lebesgue Dominated Convergence Theorem)
Let $\{f_n\}$ be a sequence of measurable functions on $E$. Suppose there is a function $g$ that is integrable over $E$ and dominates $\{f_n\}$ on $E$ in the sense that $|f_n(x)| \leq g(x)$ on $E$ for all $n$. If $\{f_n\} \to \{f\}$ pointwise a.e. on $E$, then $f$ is integrable over $E$ and

$$
\lim_{n \to \infty} \int_E f_n = \int_E f \quad \text{where } E \text{ is a finite measurable set.}
$$

2. **Existence of One Side Relatively Bounded Solutions:**

In this section we will establish several sufficient conditions to insure the existence of a nonoscillatory solutions which are one side relatively bounded by functions and a ratio of positive functions on $[t_1, \infty)$ of Eq.(1.1), $t_1 \geq t_0$. Without loss of generality we will discuss the existence of eventually positive solution and the existence of eventually negative solution can be discussed in similar way.

The following conditions will be used in the next results:

**H1.** $0 < q(t) \leq c < 1$

**H2.** $M_1 \leq f(t, \cdot) \leq M_2, \quad M_1, M_2 \neq 0,$ are constants.

**H3.** $m_1 x(t) \leq f(t, \cdot) \leq m_2 x(t), \quad m_1, m_2 \neq 0,$ are constants

**Theorem 2.1.** Suppose that H1, H3 hold, and there exist bounded function $u \in C^1([t_0, \infty), [0, \infty))$, a constant $N^* > 0$, and $\rho(t) \geq t_0$ such that

$$
u(t) \leq \frac{q(t)u(t)}{q(t)} \quad (2.1)
$$

$$
u(t) - q(t)u(t) \leq \int_t^{\infty} \int_s^{\infty} \frac{p(\xi)}{a(\xi)} d\xi ds \leq \frac{1}{m_2} (1 - q(t)) \quad (2.2)
$$

Then Eq.(1.1) has a nonoscillatory relatively bounded from below.

**Proof.** Let $C([t_0, \infty), \mathbb{R})$ be the set of all continuous bounded functions with the norm $\|x\| = \sup_{t \geq t_0} |x(t)|$. Then $C([t_0, \infty), \mathbb{R})$ is a Banach space. We define a closed, bounded, and convex subset $\Omega$ of $C([t_0, \infty), \mathbb{R})$ as follows:

$$
\Omega = \{x = x(t) \in C([t_0, \infty), \mathbb{R}): u(t) \leq x(t) \leq N^*, \quad N^* > 0, \quad t \geq t_0\}. \quad (2.3)
$$

For simplicity let

$$
f(t, x(t)) = f(t, x(t), x(t), x'(t), x'(t - \sigma(t))).
$$

Now we define two maps $S_1$ and $S_2: \Omega \to C([t_0, \infty), \mathbb{R})$ as follows:

$$
(S_1 x)(t) = \begin{cases} 
q(t)x(t), & t \geq t_1, \\
(S_1 x)(t_1), & t_0 \leq t \leq t_1,
\end{cases}
$$

$$
(S_2 x)(t) = \begin{cases} 
\left\{ \begin{array}{ll}
\int_t^{\infty} \int_s^{\infty} \frac{p(\xi)}{a(\xi)} f(\xi, x(\xi)) d\xi ds, & t \geq t_1 \\
(u(t) - q(t)u(t_1)), & t_0 \leq t \leq t_1
\end{array} \right. 
\end{cases}
$$

(2.4)

We will show that for any $x, y \in \Omega$ we have $S_1 x + S_2 y \in \Omega$.

From condition (2.2) it follows that for $t \geq t_1$

$$
\int_t^{\infty} \int_s^{\infty} \frac{p(\xi)}{a(\xi)} d\xi ds < \infty. \quad (2.5)
$$

For every $x, y \in \Omega$ and $t \geq t_1$, with regard (2.2) we obtain

$$
(S_1 x)(t) + (S_2 y)(t) = q(t)x(t) + \int_t^{\infty} \int_s^{\infty} \frac{p(\xi)}{a(\xi)} f(\xi, y(\xi)) d\xi ds \\
\leq q(t)N^* + m_2 N^* \int_t^{\infty} \int_s^{\infty} \frac{p(\xi)}{a(\xi)} d\xi ds \\
\leq N^*(q(t) + m_2 N^*) \int_t^{\infty} \int_s^{\infty} \frac{p(\xi)}{a(\xi)} d\xi ds \leq N^*.
$$

For $t \in [t_0, t_1]$, we have

$$
(S_1 x)(t) + (S_2 y)(t) = (S_1 x)(t_1) + u(t) - q(t_1)u(t_1) \\
\leq q(t_1)x(t_1) + N^*(1 - q(t_1)) \\
\leq q(t_1)N^* + N^*(1 - q(t_1)) = N^*.
$$

Furthermore, for $t \geq t_1$, with regard (2.2) we obtain

$$
(S_1 x)(t) + (S_2 y)(t) = q(t)x(t) + \int_t^{\infty} \int_s^{\infty} \frac{p(\xi)}{a(\xi)} f(\xi, y(\xi)) d\xi ds, \\
\geq q(t)u(t) + m_1 \int_t^{\infty} \int_s^{\infty} \frac{p(\xi)}{a(\xi)} f(\xi, y(\xi)) d\xi ds, \\
\geq q(t)u(t) + m_1 \int_t^{\infty} \int_s^{\infty} \frac{p(\xi)}{a(\xi)} y(\xi) d\xi ds, \\
\geq q(t)u(t) + m_1 \int_t^{\infty} \int_s^{\infty} \frac{p(\xi)}{a(\xi)} u(\xi) d\xi ds, \\
\geq q(t)u(t) + m_1 \min\{u(t)\} \int_t^{\infty} \int_s^{\infty} \frac{p(\xi)}{a(\xi)} d\xi ds \\
\geq u(t).
$$

65
Let \( t \in [t_0, t_1] \), from Eq.(2.4), with regard to (2.1) we get

\[
(S_1x)(t) + (S_2y)(t) = (S_1x)(t_1) + u(t) - q(t_1)u(\tau(t_1)) = q(t_1)x(\tau(t_1)) + u(t) - q(t_1)u(\tau(t_1)),
\]

\[
\leq q(t_1)u(\tau(t_1)) + u(t) - q(t_1)u(\tau(t_1)) = u(t).
\]

Thus, we have proved that \( S_1x + S_2y \in \Omega \), for any \( x, y \in \Omega \).

We will show that \( S_1 \) is a contraction mapping on \( \Omega \). For \( x, y \in \Omega \) and \( t \geq t_1 \) we have

\[
\|S_1x - S_1y\| = \sup_{t \geq t_1} \|(S_1x)(t) - (S_1y)(t)\| = \sup_{t \geq t_1} \|q(t_1)x(\tau(t_1)) - q(t_1)y(\tau(t_1))\| = \sup_{t \geq t_1} \|x(\tau(t_1)) - y(\tau(t_1))\| \
\leq c \sup_{t \geq t_1} \|x(\tau(t)) - y(\tau(t))\| = c\|x - y\|.
\]

Also for \( t \in [t_0, t_1] \).

\[
\|S_1x - S_1y\| = \sup_{t \in [t_0, t_1]} \|(S_1x)(t) - (S_1y)(t)\| = \|S_1x - S_1y\| \leq c\|x - y\|.
\]

Hence

\[
\|S_1x - S_1y\| \leq c\|x - y\|.
\]

Thus \( S_1 \) is a contraction mapping on \( \Omega \).

To show that \( S_2 \) is completely continuous. First we will show that \( S_2 \) is uniformly bounded. By (2.5) and H2 it follows:

\[
\int_{t_1}^{\infty} \int_{t}^{\infty} \frac{p(\xi)}{a(\xi)} f(s, x(\xi)) d\xi ds 
\leq \int_{t_1}^{\infty} \int_{t}^{\infty} \frac{p(\xi)}{a(\xi)} \|f(s, x(\xi)) - f(s, x(\xi))\| d\xi ds.
\]

Since

\[
\|f(s, x(\xi)) - f(s, x(\xi))\| \to 0 \text{ as } k \to \infty,
\]

by applying the Lebesgue dominated convergence theorem, we obtain

\[
\lim_{k \to \infty} (S_2x_k)(t) - (S_2x)(t) = 0
\]

This means that \( S_2 \) is continuous.

Now to prove \( S_2 \) is relatively compact, we have to show that \( \{S_2x : x \in \Omega\} \) is uniformly bounded and equicontinuous on \([t_0, \infty)\) according to Arzelâ€s-Ascoli theorem [6]. It is clear that from (2.3) we get \( \{S_2x : x \in \Omega\} \) is uniformly bounded. To show \( \{S_2x : x \in \Omega\} \) is equicontinuous on \([t_0, x)\). Let \( x \in \Omega \) and any \( \varepsilon > 0 \), with regard to (2.8), there exists \( t_1 \geq t_1 \) large enough so that

\[
\int_{t}^{\infty} \int_{t_1}^{\infty} \frac{p(\xi)}{a(\xi)} f(s, x(\xi)) d\xi ds < \frac{\varepsilon}{2} \quad t \geq t_1.
\]

Then, for \( x \in \Omega \), \( T_2 > T_1 \geq t_1 \), we have

\[
\|(S_2x_k)(T_2) - (S_2x)(T_1)\| = \sup_{T_1 \leq T_2} \|(S_2x_k)(T_2) - (S_2x)(T_1)\| \
\leq \|(S_2x_k)(T_2)\| + \|(S_2x)(T_1)\| \
\leq \int_{T_1}^{\infty} \int_{T_1}^{\infty} \frac{p(\xi)}{a(\xi)} f(s, x(\xi)) d\xi ds < \frac{\varepsilon}{2} \quad T_2 \geq t_1.
\]

For \( x \in \Omega \) and \( t_1 \leq T_1 < T_2 \leq t_3 \), we get

\[
\|(S_2x(T_2) - (S_2x)(T_1)\| = \sup_{T_1 \leq T_2 \leq t_3} \|(S_2x(T_2) - (S_2x)(T_1)\| \
\leq \int_{T_1}^{\infty} \int_{T_1}^{\infty} \frac{p(\xi)}{a(\xi)} f(s, x(\xi)) d\xi ds < \varepsilon.
\]

Thus there exists \( \delta_1 = \frac{\varepsilon}{m_2\max_{\xi \in [t_0, x]} \frac{p(\xi)}{a(\xi)}} \), such that

\[
\|(S_2x(T_2) - (S_2x)(T_1)\| < \varepsilon \text{ if } 0 < T_2 - T_1 < \delta_1.
\]
Finally, for any $x \in \Omega$, $t_0 \leq T_1 < T_2 \leq t_1$, and for any
\[\varepsilon > 0,\]
\[|(S_2 x)(T_2) - (S_2 x)(T_1)| = 0 < \varepsilon, \text{ if } 0 < T_2 - T_1 < \delta_2.\]
Hence, $S_2 \Omega$ is relatively compact. By Lemma 1.2 then Eq.(1.1) has a nonoscillatory relatively bounded from below. The proof is complete.

The next theorem we will give another new sufficient conditions to prove that the Eq.(1.1) has a nonoscillatory relatively bounded from above by $v(t)$.

**Theorem 2.2.** Suppose that H1, H3 hold, and there exist bounded function $v \in C^1(\{t_0, +\infty\}, [0, \infty))$, $\rho(t) \geq t_0$ such that
\[
\frac{1}{m_1} [1 - q(t)] \leq \int_{t_0}^{\infty} \int_{t}^{\infty} \frac{p(\xi)}{a(\xi)} d\xi ds
\leq \frac{1}{m_2 \max\{v(t)\}} [v(t) - q(t)v(r(t))], t \geq t_1. \tag{2.10}
\]
Then Eq.(1.1) has a nonoscillatory relatively bounded from above.

**Proof.** Let $C(\{t_0, +\infty\}, \mathbb{R})$ be the set of all continuous bounded functions with the norm $||x|| = \sup_{t \geq t_0} |x(t)|$. Then $C(\{t_0, +\infty\}, \mathbb{R})$ is a Banach space. Let $\Omega$ be a closed bounded, and convex subset of $C(\{t_0, +\infty\}, \mathbb{R})$ defined as
\[\Omega = \{x = x(t) \in C(\{t_0, +\infty\}, \mathbb{R}) : N_x \leq x(t) \leq v(t), t \geq t_0, \}
and the two maps $S_1$ and $S_2 : \Omega \rightarrow C(\{t_0, +\infty\}, \mathbb{R})$ defined as
\[
(S_1 x)(t) = \begin{cases} 
q(t)x(r(t)), & t \geq t_1, \\
\big\{q(t)x(r(t)), (S_2 x)(t_1), & t_0 \leq t \leq t_1,
\end{cases}
\]
\[
(S_2 x)(t) = \begin{cases} 
\int_{t_0}^{\infty} \int_{s}^{\infty} \frac{p(\xi)}{a(\xi)} f(\xi, x(\xi)) d\xi ds, & t \geq t_1, \\
v(t) - q(t_1)v(r(t_1)), & t_0 \leq t \leq t_1.
\end{cases} \tag{2.12}
\]
We will show that for any $x, y \in \Omega$ we have $S_1 x + S_2 y \in \Omega$.

For every $x, y \in \Omega$ and $t \geq t_1$, we obtain
\[
(S_1 x)(t) + (S_2 y)(t) = \begin{cases} 
q(t)x(r(t)) + \int_{t_0}^{\infty} \int_{s}^{\infty} \frac{p(\xi)}{a(\xi)} f(\xi, y(\xi)) d\xi ds, & t \geq t_1, \\
q(t_1)x(r(t_1)) + v(t) - q(t_1)v(r(t_1)) \leq q(t_1)x(r(t_1)) + v(t) - q(t_1)v(r(t_1)) &= v(t).
\end{cases}
\]
Furthermore, for $t \geq t_1$, we get
\[
(S_1 x)(t) + (S_2 y)(t) = q(t)x(r(t)) + \int_{t_0}^{\infty} \int_{s}^{\infty} \frac{p(\xi)}{a(\xi)} f(\xi, y(\xi)) d\xi ds
\geq q(t)N_x + m_1 \int_{t_0}^{\infty} \int_{s}^{\infty} \frac{p(\xi)}{a(\xi)} y(\xi) d\xi ds
\geq q(t)N_x + m_1 N_x \int_{t_0}^{\infty} \int_{s}^{\infty} \frac{p(\xi)}{a(\xi)} d\xi ds
\geq q(t)N_x + N_x (1 - q(t)) \geq N_x.
\]
Then for $t \in [t_0, t_1]$, from equation (2.11) and (2.12), we have
\[
(S_1 x)(t) + (S_2 y)(t) = (S_1 x)(t_1) + v(t) - q(t_1)v(r(t_1)) 
\geq q(t_1)x(r(t_1)) + v(t) - q(t_1)v(r(t_1))
\geq q(t_1)N_x + N_x - q(t_1)N_x = N_x.
\]
Thus we have proved that $S_1 x + S_2 y \in \Omega$ for any $x, y \in \Omega$. We can treat the rest of the proof in similar way as in the proof of theorem (2.1). By Lemma 1.2 there is an $x_0 \in \Omega$ such that $S_1 x_0 + S_2 x_0 = x_0$. We conclude that $x_0(t)$ is a positive solution of (2.2). The proof is complete.

In the next theorem we will give another new sufficient conditions to prove that the Eq.(1.1) has a nonoscillatory one side relatively bounded from below by ratio function $u(t) / a(t)$.

**Theorem 2.3.** Suppose that H1, H2 hold, and there exist bounded function $u \in C^1(\{t_0, +\infty\}, [0, \infty))$, $\rho(t) \geq t_0$ such that
\[
\frac{u(t)}{a(t)} \geq \frac{u(t)}{a(t)} \geq \frac{v(t)}{a(t)}, \quad t_0 \leq t \leq t_1. \tag{2.13}
\]
We define a closed bounded, and convex subset $\Omega$ of $C(\{t_0, +\infty\}, \mathbb{R})$ as follows:
\[
\Omega = \{x = x(t) \in C(\{t_0, +\infty\}, \mathbb{R}) : u(t) \leq x(t) \leq N^*, t \geq t_0, \}
\]
We will show that for any $x, y \in \Omega$ we have $S_1 x + S_2 y \in \Omega$.

For every $x, y \in \Omega$ and $t \geq t_1$, we obtain
\[
(S_1 x)(t) + (S_2 y)(t) = q(t)x(r(t)) + \int_{t_0}^{\infty} \int_{s}^{\infty} \frac{p(\xi)}{a(\xi)} f(\xi, y(\xi)) d\xi ds
\geq q(t)v(r(t)) + m_2 \int_{t_0}^{\infty} \int_{s}^{\infty} \frac{p(\xi)}{a(\xi)} v(\xi) d\xi ds
\geq q(t)v(r(t)) + m_2 \max\{v(t)\} \int_{t_0}^{\infty} \int_{s}^{\infty} \frac{p(\xi)}{a(\xi)} d\xi ds
\geq q(t)v(r(t)) + v(t) - q(t_1)v(r(t_1)) = v(t).
\]
Furthermore, for $t \geq t_1$, we get
\[
(S_1 x)(t) + (S_2 y)(t) = q(t)x(r(t)) + \int_{t_0}^{\infty} \int_{s}^{\infty} \frac{p(\xi)}{a(\xi)} f(\xi, y(\xi)) d\xi ds
\geq q(t)N_x + m_1 \int_{t_0}^{\infty} \int_{s}^{\infty} \frac{p(\xi)}{a(\xi)} y(\xi) d\xi ds
\geq q(t)N_x + m_1 N_x \int_{t_0}^{\infty} \int_{s}^{\infty} \frac{p(\xi)}{a(\xi)} d\xi ds
\geq q(t)N_x + N_x (1 - q(t)) \geq N_x.
\]
Then Eq.(1.1) has a nonoscillatory relatively bounded from below.

**Proof.** Let $C(\{t_0, +\infty\}, \mathbb{R})$ be the set of all continuous bounded functions with the norm $||x|| = \sup_{t \geq t_0} |x(t)|$. Then $C(\{t_0, +\infty\}, \mathbb{R})$ is a Banach space. We define a closed, bounded, and convex subset $\Omega$ of $C(\{t_0, +\infty\}, \mathbb{R})$ as follows:
\[
\Omega = \{x = x(t) \in C(\{t_0, +\infty\}, \mathbb{R}) : u(t) \leq x(t) \leq N^*, t \geq t_0, \}
\]
The two maps $S_1$ and $S_2 : \Omega \rightarrow C(\{t_0, +\infty\}, \mathbb{R})$ defined as
\[
(S_1 x)(t) = \{(q(t)x(r(t)), (S_2 x)(t_1), t_0 \leq t \leq t_1, \\
(S_2 x)(t) = \int_{t_0}^{\infty} \int_{s}^{\infty} \frac{p(\xi)}{a(\xi)} f(\xi, x(\xi)) d\xi ds, t \geq t_1, \tag{2.16}
\]
We will show that for any $x, y \in \Omega$ we have $S_1 x + S_2 y \in \Omega$. The proof is complete.
For every \( x, y \in \Omega \) and \( t \geq t_1 \), we obtain
\[
(S_1x)(t) + (S_2y)(t) = q(t)x(\tau(t)) + \int_t^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} f(\xi, y(\xi)) d\xi ds
\]
\[
\leq q(t)N^* + \frac{M_2}{a(t)} \int_t^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} y(\xi) d\xi ds
\]
\[
\leq q(t)N^* + \frac{M_2N^*}{a(t)} \int_t^\infty \frac{p(\xi)}{a(\xi)} d\xi ds
\]
\[
\leq q(t)N^* + (N^* - q(t)N^*) = N^*.
\]
For \( t \in [t_0, t_1] \), we have
\[(S_1x)(t) + (S_2y)(t) = (S_1x)(t_1) + (S_2y)(t_1) \leq N^*.
\]
Furthermore, for \( t \geq t_1 \), we get
\[
(S_1x)(t) + (S_2y)(t) = q(t)x(\tau(t)) + \int_t^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} f(\xi, y(\xi)) d\xi ds
\]
\[
\geq \frac{q(t)}{a(\tau(t))} u(\tau(t)) + q(t)u(\tau(t)) + \frac{q(t)}{a(\tau(t))} u(\tau(t)) - q(t)a(t)u(\tau(t))
\]
\[
= u(\tau(t)) + \frac{a(\tau(t))}{a(t)} u(\tau(t))
\]
\[
= \frac{u(\tau(t))}{a(t)}
\]
\[
\]
Let \( t \in [t_0, t_1] \). Using (2.13), we get
\[(S_1x)(t) + (S_2y)(t) = (S_1x)(t_1) + (S_2y)(t_1)
\]
\[
\geq \frac{u(t_1)}{a(t_1)} u(t_1)
\]
\[
\]
Thus we have proved that \( S_1x + S_2y \in \Omega \) for any \( x, y \in \Omega \). We can treat the rest of the proof in similar way as in the proof of theorem (2.1). By Lemma 1.2 there is an \( x_0 \in \Omega \) such that \( S_1 x_0 + S_2 x_0 = x_0 \). We conclude that \( x_0(t) \) is a positive solution of Eq.(1.1). The proof is complete.
In the next theorem we will give another new sufficient conditions to prove that the Eq.(1.1) has a nonoscillatory relatively bounded from below by ratio function \( \frac{u(t)}{a(t)} \).

**Theorem 2.4.** Suppose that \( H1, H3 \) hold, and there exist bounded function \( v \in C^1([t_0, \infty), [0, \infty)), \mu \) \((t_1) \geq t_0 \) such that
\[
\frac{1}{m_1}(1 - q(t)) \leq \int_t^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} d\xi ds
\]
\[
\leq \frac{a(\tau(t))v(\tau(t)) - q(t)a(t)v(\tau(t))}{m_2^2 \max_{t \in [t_0, t_1]} [v(t)] a(t)} , t \geq t_1.
\]
Then Eq.(1.1) has a nonoscillatory relatively bounded from above.

**Proof.** Let \( C([t_0, \infty), \mathbb{R}) \) be the set of all continuous bounded functions with the norm \( ||x|| = \sup_{t \geq t_0} |x(t)| \). Then \( C([t_0, \infty), \mathbb{R}) \) is a Banach space. We define a closed, bounded, and convex subset \( \Omega \) of \( C([t_0, \infty), \mathbb{R}) \)as follows:
\[
\Omega = \left\{ x = x(t) \in C([t_0, \infty), \mathbb{R}) : N \leq x(t) \leq \frac{v(t)}{a(t)} , t \geq t_0 , N > 0 \right\}.
\]
We now define two maps \( S_1 \) and \( S_2 : \Omega \rightarrow C([t_0, \infty], \mathbb{R}) \) as follows:
\[(S_1x)(t) = \begin{cases} q(t)x(\tau(t)), & t \geq t_1 \\
(S_1x)(t_1), & t_0 \leq t \leq t_1
\end{cases}
\]
\[(S_2y)(t) = \begin{cases} \int_t^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} f(\xi, x(\xi)) d\xi ds, & t \geq t_1 \\
\int_{t_0}^t \int_s^\infty \frac{p(\xi)}{a(\xi)} f(\xi, x(\xi)) d\xi ds, & t_0 \leq t \leq t_1
\end{cases}
\]
We will show that for any \( x, y \in \Omega \) we have \( S_1x + S_2y \in \Omega \).
For every \( x, y \in \Omega \) and \( t \geq t_1 \), we obtain
\[
(S_1x)(t) + (S_2y)(t) = q(t)x(\tau(t)) + \int_t^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} f(\xi, y(\xi)) d\xi ds
\]
\[
\leq \frac{q(t)}{a(\tau(t))} v(\tau(t)) + m_2^2 \max_{t \in [t_0, t_1]} [v(t)] a(t)
\]
\[
\leq \frac{q(t)}{a(\tau(t))} v(\tau(t)) + \frac{m_2^2 \max_{t \in [t_0, t_1]} [v(t)]}{a(t)} v(\tau(t))
\]
\[
= \frac{v(\tau(t))}{a(t)}
\]
\[
\]
For \( t \in [t_0, t_1] \), we have
\[
(S_1x)(t) + (S_2y)(t) = q(t)x(\tau(t)) + \frac{v(t)}{a(t)} - q(t)_1 v(\tau(t)_1)
\]
\[
\leq q(t)_1 x(\tau(t)_1) + \frac{v(t)}{a(t)} - q(t)_1 v(\tau(t)_1)
\]
\[
\leq q(t)_1 v(\tau(t)_1) + \frac{v(t)}{a(t)} - q(t)_1 v(\tau(t)_1) = v(t)
\]

Furthermore, for \( t \geq t_1 \), we get
\[
(S_1x)(t) + (S_2y)(t) = q(t)x(\tau(t)) + \int_t^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} f(\xi, y(\xi)) d\xi ds
\]
\[
\geq q(t)N + \frac{M_1}{a(t)} \int_t^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} d\xi ds
\]
\[
\geq q(t)N + N(1 - q(t)) = N.
\]
Let \( t \in [t_0, t_1] \). From Eqs. (3.21) and (3.22), we get
\[
(S_1x)(t) + (S_2y)(t) = q(t)_1 x(\tau(t)_1) + \frac{v(t)}{a(t)} - q(t)_1 v(\tau(t)_1)
\]
\[
= q(t)_1 x(\tau(t)_1) + \frac{v(t)}{a(t)} - q(t)_1 v(\tau(t)_1) = v(t)
\]

86
Thus we have proved that for any $x, y \in \Omega$ we can treat the rest of the proof in similar way as in the proof of theorem 2.1. By Lemma 1.2 there is an $x_0 \in \Omega$ such that $S_1x_0 + S_2x_0 = x_0$. We conclude that $x_0(t)$ is a positive solution of Eq. (1.1). The proof is complete.

**Example 2.5.** Consider the following nonlinear Neutral differential equation

$$
\frac{d}{dt} \left( d \frac{d}{dx} \left( a(t)(x(t) - q(t)x(t - 2)) \right) \right) - p(t) \left( \frac{2}{t} + 1 \right) = 0.
$$

(2.20)

Where $a(t) = e^{-1st}$, $p(t) = \frac{1}{s} e^{-2t^2}$, and $q(t) = 0.3$, Let

$u(t) = e^{-0.5t}, 1 \leq f(t, x(t)) = \frac{2}{t} + 1 \leq 3, t \geq t_1 = 1$.

Solution: It is clear that condition (2.1) holds, since $e^{-0.5t} \leq e^{-0.5(t - 2)}, t_0 \leq t \leq t_1$.

To show condition (2.2) of theorem (2.1) verified:

$$
\frac{1}{M_1} (u(t) - q(t)u(t)) \leq \int_t^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} d\xi ds
\leq \frac{1}{M_1} (1 - q(t)), \quad t \geq 1,
$$

Let $R_1(t) = \frac{1}{M_1} (u(t) - q(t)u(t)), R_2(t) = 0.5e^{-0.5t}$, and

$R_3(t) = \frac{1}{M_2} (1 - q(t)).$

Then $R_1(t) \leq R_2(t) \leq R_3(t)$, for $t \geq 1$ so all conditions of Theorem 2.1 hold, by Theorem there exists a positive solution of Eq. (2.20).

**Example 2.6.** Consider the following nonlinear Neutral differential equation

$$
\frac{d}{dt} \left( \frac{d}{dx} \left( a(t)(x(t) - q(t)x(t - 2)) \right) \right) - p(t) \left( \frac{1}{s} \sin t + 1.5 \right) = 0, \quad t \geq 0, \quad (2.21)
$$

where $a(t) = \frac{-(10 + 5t)}{s}$, $p(t) = \frac{2}{(2t + 1)^2}$, $q(t) = 0.5$, and $\nu(t) = 2 - e^{-t} \leq f(t, x(t)) = \frac{1}{s} \sin t + 1.5 \leq 2$, $t_1 = 2$.

**Solution.** To show condition (2.10) of theorem (2.2) verified:

$$
\frac{1}{m_1} \left[ 1 - q(t) \right] \leq \int_t^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} d\xi ds
\leq \frac{[\nu(t) - q(t)\nu(t)]}{m_2 \max[v(t)]}, \quad t \geq t_1.
$$

Let $R_1(t) = \frac{1}{m_1} \left[ 1 - q(t) \right], R_2(t) = \frac{1 + 0.8t}{2 + t}$ and

$R_3(t) = \frac{[\nu(t) - q(t)\nu(t)]}{m_2 \max[v(t)]}.$

Then $R_1(t) \leq R_2(t) \leq R_3(t)$, for $t \geq 1$ so all conditions of Theorem 2.2 hold, by Theorem there exists a positive solution of Eq. (2.21).

**Example 2.7.** Consider the following nonlinear Neutral differential equation

$$
\frac{d}{dt} \left( \frac{d}{dx} \left( a(t)(x(t) - q(t)x(t - 1)) \right) \right) - p(t)(\cos t + 1) = 0, \quad t \geq 0, \quad (2.22)
$$

Where $a(t) = 2^{-1.5t}, p(t) = 2^{-3t - 3}$, and $q(t) = 0.5$, and $u(t) = 2^{-0.5t}, 1 \leq f(t, x(t)) = \cos t + 1 \leq 2, t_1 = 2$.

**Solution:** It is clear that condition (2.3) holds, since

$$
\frac{u(t)}{a(t)} - q(t) = 2^{-0.5t} \frac{1}{2^{-1.5t} - 2^{-3t - 3}} \geq 0, 0 \leq t \leq 2.
$$

To show condition (2.14) of theorem (2.3) verified:

$$
\frac{a(\tau)u(t) - q(t)a(t)u(\tau(t))}{M_1a(t)a(\tau(t))} \leq \int_t^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} d\xi ds
\leq \frac{1}{M_2} (1 - q(t)), t \geq t_1.
$$

Let $R_1(t) = \frac{a(\tau)u(t) - q(t)a(t)u(\tau(t))}{M_1a(t)a(\tau(t))}.$
\[ R_2(t) = \frac{2^{-0.5t-3}}{\ln(2)} \]

and

\[ R_3(t) = \frac{1}{M_2} \left( 1 - q(t) \right). \]

Then \( R_1(t) \leq R_2(t) \leq R_3(t) \), for \( t \geq 1 \) so all conditions of Theorem 2.3 hold, by Theorem there exists a positive solution of Eq. (2.22).

![Graph of \( R_1(t), R_2(t), \) and \( R_3(t) \) of theorem (2.3).](image)

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وجد الحل المقيد النسبي الغير متذبذب للمعادلات التفاضلية المحايدة من الرتبة الثانية

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المستخلص:
في هذا البحث حصلنا على الشروط الكافية لوجود الحل الموجب المقيق النسبي من جهة واحدة للمعادلات التفاضلية المحايدة من الرتبة الثانية (Krasnoselskii) ومبرهنة التقارب المهيمنة لليبيك للحصول على شروط كافية لوجود الحلول النسبية المعقدة من جهة واحدة. هذه الشروط أكثر تطبيق من النتائج المعروفة في المصادر. ثلاثة أمثلة لبرهنة النتائج التي حصلنا عليها.