Sync in Complex Dynamical Networks: Stability, Evolution, Control, and Application

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Abstract

In the past few years, the discoveries of small-world and scale-free properties of many natural and artificial complex networks have stimulated significant advances in better understanding the relationship between the topology and the collective dynamics of complex networks. This paper reports recent progresses in the literature of synchronization of complex dynamical networks including stability criteria, network synchronizability and uniform synchronous criticality in different topologies, and the connection between control and synchronization of complex networks as well. The economic-cycle synchronous phenomenon in the World Trade Web, a scale-free type of social economic networks, is used to illustrate an application of the network synchronization mechanism.

keywords: Synchronization, small-world, scale-free, pinning control, economic-cycle

1 Introduction

Synchronization is a long-lasting fundamental concept and is, in fact, a universal phenomenon in all areas of science and technology. There are many interesting synchronization phenomena seen in our daily life, including for example fireflies flashing in unison, crickets chirping in synchrony, and heart cells beating in rhythm.

One of the most significant and interesting properties of a dynamical network is the synchronous output motion of its network elements (nodes). Synchronization in coupled dynamical networks and systems has been studied for many years within a common framework based on nonlinear dynamical system theories, due to its ubiquity in many technological fields such as coupled laser systems, biochemical systems, and communication networks. In recent years, synchronization in a network of coupled chaotic systems has become a topic of great interest. It has been observed, however, that most existing work have been concentrated on networks with completely regular topological structures such as chains, grids, lattices, and fully-connected graphs. Two typical cases are the discrete-time coupled map lattices.
and the continuous-time cellular nonlinear networks [16]. The main benefit of these simple architectures is that it allows one to focus on the complexity caused by the nonlinear dynamics of the nodes, without considering additional complexity caused by the network structure itself. Another appealing feature is the possibility of realizing such regularly coupled networks by integrated circuits, with the obvious advantage of compactness, for potential engineering and technological applications.

The topology of a network often affects its functional behaviors. For instance, the topology of a social network affects the spreading of information and also disease, while an unsuitable topology of a power grid can damage its robustness and stability. In fact, we are confronting all kinds of networks with complex structures everyday, where handy instances are the Internet and the World Wide Web (WWW). Therefore, people are particularly interested in the question of how to model such complex networks. Traditionally, the topology of a complex network is described by a completely random network generated by the so-called Erdős-Rényi (ER) model [24], which is at the opposite end of the spectrum from a completely regular network, and is one of the oldest and perhaps also the best tools for study. However, with the increasing interest in trying to understand the essence of various real-life complex networks such as the Internet, the World Wide Web, the metabolic networks, and various social and economic networks like the scientific-collaboration network and the World Trade Web, etc. [2, 19, 41, 42, 55, 62, 64], people have discovered the so-called small-world phenomena and scale-free feature, which cannot be well described by the ER random graph theory, and therefore stimulate a strong desire of building new network models [2, 19, 39, 55, 56].

Watts, Strogatz and Newman (WSN), for example, introduced their small-world network models, which generate networks having short average path lengths along with large clusters [55, 71, 72]. These new models show a transition from a regular network to a random network. Both the ER random graph and WSN small-world network models have a common Poisson connectivity distribution and are homogenous in nature: each node in such networks has about the same number of connections. Another significant discovery is the scale-free feature in a number of real-life complex networks, whose connectivity distributions are in the power-law form as first pointed out by Barabási and Albert (BA) [7, 8]. Owing to the non-homogenous topology, i.e., most nodes have very few connections and only very few nodes have many connections, scale-free networks show a unique characteristic: robustness and yet fragility (see [3, 17, 43, 67] for more precise interpretations). The aforementioned small-world and scale-free features have also been empirically verified to fit many real-life complex networks: they are largely clustered with a short path length, following a power-law degree distribution.

The discovery of the small-world effect and scale-free feature of most complex networks has led to a fascinating set of common problems concerning how a network structure facilitates and constrains the network collective behaviors. In particular, a lot of work have been concentrated on synchronization in small-world and scale-free dynamical networks, including small-world networks of oscillators [9, 28, 48, 68, 72], small-world neural networks [31, 45], small-world circle map lattices [10], coupled map lattices with small-world interactions [26, 33], and scale-free networks of oscillators [11, 38, 58, 67] and maps [6, 32, 44]. Not limited to the above, synchronous phenomena in complex networks with time delays [5, 18, 37, 49], time-varying coupling [11, 12, 45, 46, 47], and weighted coupling [14, 52, 53] have been extensively investigated, along with which many network evolving factors have been understood to play an important role of synchrony improvements [21, 22, 28, 29, 51, 58, 77, 78]. Of particular interest in real applications is the synchronization phenomenon in the scale-free World Trade Web studied [41], which will be further introduced in this article below.

Although this survey can only cover a small part of recent progresses in this fast developed literature, we wish it could show the audience that current wide studies of complex networks will continuously motivate more efforts devoted to the understanding of synchronization and many other aspects of various complex dynamical networks and systems.
2 Preliminaries: several models of complex network topologies

2.1 ER model: random networks

The ER model of random networks [20] is defined as a random graph (network) having \( N \) labelled nodes connected by \( n \) edges, which are chosen randomly from all the \( \frac{N(N-1)}{2} \) possible edges. The network evolves as follows: Start with \( N \) nodes, and every pair of nodes are connected with the same probability \( p \), as shown in Figure 1.

![Figure 1: Evolution of an ER random network. One starts with isolated nodes in (a), and connects every pair of nodes with probability (b) \( p = 0 \), (c) \( p = 0.15 \), and (d) \( p = 0.25 \), respectively (After [66]).](image)

The main goal of the random graph theory is to determine, under what connectivity probability \( p \), a particular property of a graph (network) will likely arise. For a large \( N \), the ER model generates a homogenous random network, whose connectivity approximately follows a Poisson distribution described...
by (Fig. 2)

\[ P(k) \approx e^{-\langle k \rangle} \frac{(\langle k \rangle)^k}{k!} \]  \hspace{1cm} (1)

where \( \langle k \rangle \) is the average of \( k_i \), the degree of node \( i \), over all nodes in the network. With this connectivity distribution, nodes in the network are quite uniformly spread out, which is known as a homogenous feature of the distribution.

### 2.2 WS and NW models: small-world networks

Recall the evolving algorithm of the WS model [71] described as follows: (I) Start with a \( K \)-nearest-neighbor coupled network consisting of \( N \) nodes arranged in a ring, and each node \( i \) is adjacent to its neighbor nodes \( i \pm 1, i \pm 2, \ldots, i \pm K/2 \), where \( K \) is even. In order to have a sparse but connected network, assume \( 1 < k \ll N \) in general. (II) Randomly rewire each edge of the network with probability \( p \). Rewiring in this context means shifting one end of the edge to a new node chosen at random from the whole network, with the constraint that no two different nodes can have more than one edge in between, and no node can have an edge with itself. There are three prominent network evolving phases in the WS small-world model: when \( p = 0 \), the WS evolving network stays at the original state of a regular ring; when \( p = 1 \), the WS network evolves to a completely random network; only when \( 0 < p \ll 1 \), a small-world network emerges from the WS model.

To avoid leading to the formulation of isolated clusters, which may happen in the WS model, Newman and Watts proposed their NW small-world model as a variant of the WS model [57], which evolves as follows: (I) Start with a \( N \)-node regular ring in which every node is connected to its first \( K \) neighbors (\( K/2 \) on each side). Here also \( 1 < K \ll N \). (II) Instead of rewiring in the WS model, add a new long-range edge (short-cut) into the network with probability \( 0 < p \ll 1 \) between randomly chosen pair of nodes.

![Figure 3: Network evolution of small-world models: (a) The WS model with the rewiring of edges. (b) The NW model with the addition of edges. (After [66]).](image)

It has been well known that only for sufficient large \( N \) and very small \( p \), the NW model is then equivalent to the WS model, and networks so-generated are hereby in the category of small-world networks having

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both short average distances and large clusters (Fig. 3). Both the WS model and the NW model generate small-world networks having the same degree distribution as that of random networks shown in Fig. 4.

2.3 BA model: scale-free networks

The algorithm of the Barabási-Albert (BA) scale-free model is in the following [8]: (I) Growth: Starting with a small number \(m_0\) of nodes, at every time step, add a new node with \(m \leq m_0\) edges that link the new node to \(m\) different nodes already presented in the network. (II) Preferential attachment: When choosing the nodes to which the new node connects, assume that the probability \(\Pi_i\) that a new node will be connected to node \(i\) depends on the degree \(k_i\) of node \(i\), in such a way that

\[
\Pi_i = \frac{k_i}{\sum_j k_j}
\]

After \(t\) time steps, we get a network having \(m_0 + t\) nodes and \(mt\) edges. This network arrives at a scale-invariant state with the probability that a node has \(k\) edges following a power-law distribution \(P(k) \sim 2m^2k^{-\gamma_{BA}}\) with an exponent \(\gamma_{BA} = 3\) (Fig. 4), where the scaling exponent \(\gamma_{BA}\) is independent of parameter \(m\) and the network scale \(N\), and in this scale-invariant sense the network is said to be scale-free. It has been argued that the BA scale-free network model has captured two basic mechanisms, growth and preferential attachment, responsible for the scale-free feature and "rich gets richer" phenomenon in many real-life and large-scale complicated networks [8].

![Figure 4: Degree distribution \(P(k)\) of scale-free networks generated by the BA model, with \(N = m_0 + t = 10000\) and \(m = m_0 = 1, 3, 5, 7\), respectively.](image)
3 Stability criteria of dynamical network synchronization

3.1 The criterion of $\lambda_N/\lambda_2$

Consider a network having $N$ identical, diffusively coupled nonlinear oscillators as follows, in which each node is an $n$-dimensional continuous-time dynamical system:

$$\dot{x}_i = f(x_i) + c \sum_{j=1}^{N} a_{ij} H(x_j), \quad i = 1, 2, \cdots, N,$$

(3)

where $x_i = [x_{i1}, x_{i2}, \ldots, x_{in}]^T \in \mathbb{R}^n$ are the state variables of node $i$, the constant $c > 0$ represents the coupling strength of the network, and $H \in \mathbb{R}^{n \times n}$ is the output matrix of each node. The coupling matrix $A = (a_{ij}) \in \mathbb{R}^{N \times N}$ represents the coupling configuration of the network; if there is a connection between node $i$ and node $j$, then $a_{ij} = 1$; otherwise, $a_{ij} = 0$ $(i \neq j)$. Moreover, $a_{ii} = -k_i$.

The coupled network (3) is said to achieve (asymptotical) synchronization if

$$x_1(t) = x_2(t) = \cdots = x_N(t) \rightarrow s(t), \quad \text{as } t \rightarrow \infty,$$

(4)

where $s(t) \in \mathbb{R}^n$ can be an equilibrium point, a periodic orbit, and even a chaotic orbit, depending on the interest of study.

Suppose that the network is connected without isolate clusters, whose coupling matrix $A$ is therefore a symmetric irreducible matrix. In this case, zero is an eigenvalue of $A$ with multiplicity 1 and all the other real eigenvalues of $A$ are strictly negative, denoted by $0 = \lambda_1 > \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_N$.

Diagonalize the variation equation of network (3), which yields

$$\dot{\xi}_i = [Df(s) + c\lambda_i DH(s)]\xi_i, \quad i = 1, 2, \cdots, N,$$

(5)

where $\xi_i \in \mathbb{R}^n$ is the transversal error of node $i$ to the synchronized states (4).

Denote $c\lambda_i = \alpha + i\beta$, where $\alpha > 0$, $\beta = 0$. The variation equation (5) comes to the so-called Master Stability Equation of network (3):

$$\dot{\xi}_i = [Df(s) + \alpha DH(s)]\xi_i, \quad i = 1, 2, \cdots, N,$$

(6)

whose largest Lyapunov exponent $L_{max}$ is a function of $\alpha$ (and $\beta = 0$), the Master Stability Function [9, 59].

If for every nonzero eigenvalue $\lambda_i$, $i = 2, \cdots, N$, the corresponding $L_{max} < 0$, and the synchronized states (4) are then stable. Having scaling transformations with the ordered eigenvalues $\lambda_i$, $i = 2, \cdots, N$, we finally arrive at the following group of inequalities:

$$\begin{cases} 
    c\lambda_N > \alpha_2^2 = \alpha_2 \\
    c\lambda_2 < \alpha_1^N = \alpha_1 
\end{cases}$$

(7)

Therefore, to achieve the stable synchronized states (4), Barahona and Pecora stated that the following inequality should be satisfied [3]:

$$\frac{\lambda_N}{\lambda_2} < \frac{\alpha_2}{\alpha_1}$$

(8)
3.2 The criterion of $\lambda_2$

Wang, Li, and Chen considered the network (3) of $N$ identical, linearly and diffusively coupled nodes [40, 67, 68]:

$$\dot{x}_i = f(x_i) + c \sum_{j=1}^{N} a_{ij} \Gamma x_j, \quad i = 1, 2, \cdots, N,$$

(9)

where the inner linking matrix $\Gamma = \text{diag}(r_1, \cdots, r_n)$ is a constant 0-1 diagonal matrix linking the coupled variables, i.e., the matrix $H$ in network (3) is a linear diagonal matrix $\Gamma$.

Wang and Chen [67] showed the audience the following theorem in the sense of Lyapunov stability.

**Theorem 1** Suppose there exists an $n \times n$ diagonal matrix $P > 0$ and two constants $\bar{d} < 0$ and $\tau > 0$, such that

$$[Df(s(t)) + d\Gamma]^T P + P[Df(s(t)) + d\Gamma] \leq -\tau I_n$$

for all $d \leq \bar{d}$, where $I_n \in \mathbb{R}^n$ is a unity matrix. If

$$c > \frac{\bar{d}}{|\lambda_2|},$$

(10)

then the synchronized states of dynamical network (3) are exponentially stable.

Further specify $\Gamma = I_n$, and assume each node (an $n$ dimensional continuous-time dynamical system) is chaotic, whose largest Lyapunov exponent is $h_{\text{max}} > 0$. The following theorem pointed out the analytic value of constant $\bar{d} = -h_{\text{max}}$ [40].

**Theorem 2** Consider the network (3) having $N$ identical chaotic systems, where $\Gamma = I_n$. If

$$c > \frac{h_{\text{max}}}{|\lambda_2|},$$

(11)

then the synchronized states of dynamical network (3) are exponentially stable.

Two criteria of $\lambda_N/\lambda_2$ and $\lambda_2$ are not contradictory. In fact, when we set $H = I_n$ in the variational equation (5), there is a bridge between the largest Lyapunov exponent $L_{\text{max}}$ of network (3) and the largest Lyapunov exponent $h_{\text{max}}$ of each chaotic node in the network:

$$L_{\text{max}}(\lambda_i) = h_{\text{max}} + c\lambda_i, \quad i = 1, 2, \cdots, N,$$

(12)

which finally reaches the same $\lambda_2$ criterion of Eq. (11). Extended from these two synchronization criteria, networks are proposed to be categorized to two classes with their corresponding synchronizability [35].

3.3 The case of discrete-time coupled networks

Consider the following discrete-time form of a complex network with coupled maps:

$$x_i(k+1) = f(x_i(k)) + c \sum_{j=1}^{N} a_{ij} f(x_j), \quad i = 1, 2, \cdots, N,$$

(13)

where corresponding variables are defined in previous sections. Similarly, the discrete-time coupled network (13) is said to achieve (asymptotical) synchronization if

$$x_1(k) = x_2(k) = \cdots = x_N(k) \rightarrow s(k), \quad \text{as } k \rightarrow \infty.$$  

(14)

The counterpart of Theorem 2 for discrete-time coupled network (13) is as follows [40].
Theorem 3  Consider the chaotic network (13) having \( N \) identical coupled maps. If

\[
\frac{1 - e^{-h_{\text{max}}}}{\lambda_2} < c < \frac{1 + e^{-h_{\text{max}}}}{\lambda_N}.
\]

(15)

then the synchronized states (14) of dynamical network (13) are exponentially stable.

Therefore, condition (15) can be re-written in the form of master stability function (8), with \( \alpha_1 = 1 - e^{-h_{\text{max}}} \), \( \alpha_2 = 1 + e^{-h_{\text{max}}} \) in this specified case.

Some latest extensions of network (9) with time-delays \([37, 45]\) and time-varying embedments \([46, 47]\) have been investigated, whose stability conditions still belong to the \( \lambda_2 \) criterion as that of the origin (9). More closely, some researchers have noticed the synchronization stability in the framework of graph theory \([4, 11, 12]\), which, although not covered in this paper, deserves more attention and future explorations.

### 4 Synchronizability of complex evolving networks

#### 4.1 Synchronizability of small-world networks

Given the dynamics of an isolate node and the inner linking structural matrix, the synchronizability of the dynamical network (9), with respect to a specific coupling configuration, is said to be strong if the network can synchronize with a small value of the coupling strength \( c \) in condition (10).

The second-largest eigenvalue of the coupling matrix in a globally coupled network is \(-N\). This implies that for any given and fixed nonzero coupling strength \( c \), a globally coupled network will synchronize as long as its size \( N \) is large enough. On the other hand, the second-largest eigenvalue of the coupling matrix of a nearest-neighbor coupled network tends to zero as \( N \to \infty \), which implies that for any given and fixed nonzero coupling strength \( c \), a nearest-neighbor coupled network cannot synchronize if its size \( N \) is sufficiently large \([67, 68]\).

Now, consider the dynamical network (9) with the NW small-world connections. Let \( \lambda_{2_{\text{sw}}} \) be the second-largest eigenvalue of the coupling matrix \( A \). It was found that \([68]\), for any given coupling strength \( c \): (i) for any \( N > |d|/c \), there exists a critical value \( \bar{p} \) such that if \( \bar{p} \leq p \leq 1 \) then the small-world network will synchronize; (ii) for any given \( p \in (0, 1] \), there exists a critical value \( \bar{N} \) such that if \( N > \bar{N} \) then the small-world network will synchronize. These results imply that the ability of achieving synchronization in a large-size nearest-neighbor coupled network can be greatly enhanced by just adding a tiny fraction of distant links, thereby making the network a small-world. This also reveals an advantage of small-world networks for achieving synchronization, if desired.

#### 4.2 Synchronizability of scale-free networks: Robust and yet fragile

Consider the dynamical network (9) again, but with BA scale-free connections instead. It was found \([67]\) that the second-largest eigenvalue of the corresponding coupling matrix is very close to \(-1\), which actually is the second-largest eigenvalue of star-shaped coupled networks. This implies that the synchronizability of a scale-free network is about the same as that of a star-shaped coupled network. It may be due to the extremely inhomogeneous connectivity distribution of such networks: a few “hubs” in a scale-free network play a similar important role as a single center in a star-shaped coupled network.

The robustness of synchronization in a scale-free dynamical network has also been investigated \([67]\), against either random or specific removal of a small fraction \( \delta \) of nodes from the network. Obviously, the removal of some nodes from network (9) will change the coupling matrix. If the second-largest eigenvalue of the coupling matrix remains unchanged, then the synchronization stability of the network will remain
unchanged after the removal of such a node. Let $A \in \mathbb{R}^{N \times N}$ and $\tilde{A} \in \mathbb{R}^{N \times N}$ be the coupling matrices of the original network with $N$ vertices and the new network after removal of $[\delta N]$ vertices, respectively. Denote $\lambda_2$ and $\tilde{\lambda}_2$ as the second-largest eigenvalues of $A$ and $\tilde{A}$, respectively.

It was found [67] that even when as many as 5% of randomly chosen nodes are removed, the second-largest eigenvalue of the coupling matrix, $\tilde{\lambda}_2$, remains almost unchanged; therefore, the ongoing synchronization is not altered. On the other hand, although the scale-free structure is particularly well-suited to tolerate random errors, it is also particularly vulnerable to deliberate attacks. In particular, under an intentional attack, the magnitude of $\tilde{\lambda}_2$ decreases rapidly; for example, it almost decreases to one half of its original value of $\lambda_2$ in magnitude, when only 1% fraction of the highly connected nodes were removed. Therefore, it is very reasonable to believe that the error tolerance and attack vulnerability of synchronizability in scale-free networks are rooted in their extremely inhomogeneous connectivity patterns.

### 4.3 Synchronizability preference and network robustness

However, the preference attachment in the BA model of scale-free networks may not directly result in its fragility to specific removal of a small fraction of (highly connected) nodes. We make further exploring with the introduction of synchronous preferential attachment mechanism into the BA model, and the probability $\Pi_i$ with which the new node is connected to node $i$ depends on the synchronizability (characterized by $\lambda_{2i}$) of the new network if the new node had connected to node $i$, such that [21]

$$\Pi_i = \frac{\lambda_{2i}}{\sum_j \lambda_{2j}}.$$ (16)

After $t \ll m_0$ time steps, we obtain a synchronization-preferential network with $m_0 + t$ nodes and $mt$ edges.

An extreme case is that instead of the preferential attachment (16), when a new node is added to the network, the criterion for choosing the $m$ nodes to which the new node connects is to maximize the
synchronizability of the obtained network, or equivalently, to minimize the second-largest eigenvalue of the corresponding coupling matrix, which comes to a synchronization-optimal network \[22\].

Now we consider the robustness of synchronization in dynamical network \[9\] against either random or specific removal of a small fraction \(\delta\) of vertices in these newly proposed networks. In extensive numerical simulations, it has been shown that for the synchronizability of a synchronization-preferential network, its error tolerance to random removal is a little better than that of a BA scale-free network. The value of \(\lambda_2\) decreased from -1.2493 to -0.800 when 5% of the vertices were randomly removed, which has none significant reduction when more nodes were randomly removed. This implies that the proposed synchronization-preferential model is robust against random failures.

Move to the synchronization fragility of the synchronization-preferential network with respect to deliberate attacks. At every time step, we remove the nodes with the highest degree and found that the second-largest eigenvalue of coupling matrix decreased from -1.2493 to -0.3802 when as many as 5% of the most connected nodes were removed on purpose. Furthermore, the decreased magnitude of \(\lambda_2\) of a synchronization-preferential network is smaller than that of a BA network. Therefore, a synchronization-preferential network is less vulnerable to specific removal of those most connected nodes than a BA scale-free network.

![Figure 6: Network synchronization fragility against specific attacks: changes of \(\lambda_2\) of the largest cluster in the BA scale-free networks (solid line with circles), the synchronization-optimal networks (dotted line with diamonds), the ER random networks (solid line with stars), and the synchronization-preferential networks (dash-dotted line with squares), when a fraction \(\delta\) of the most connected nodes are purposely removed. All curves are averaged from 5 groups of networks in simulations.](image)

These conclusions could be visualized through the comparisons of the robustness and fragility of synchronizability among the ER model, the BA model, the synchronization-optimal model, and the synchronization-preferential model (Figs. 5-6). The illustrations imply that the preferential attachment mechanism does not necessarily lead to the robust-and-yet-fragile “Achilles heel” of (BA) scale-free networks.

4.4 Synchronizability of growing networks

The existence of lower bound and upper bound of the coupling strength \(c\) in inequality \[15\] for synchronizing discrete-time networks is similar to the master stability synchronization in continuous-time
network. Here, through the instance of discrete-time dynamical networks, we investigate the effect of growing mechanism on the network synchronization.

Define $R := \frac{\lambda_1 - \lambda_2}{\lambda_2 - \lambda_N}$, which measures the distance from the first eigenvalue to the main part of the spectral density $\rho(\lambda)$ normalized by the extension of the main part [23]. Therefore, the invalidity of condition (15) equals to the following inequality is satisfied

$$\frac{1}{R} < \frac{2e^{-h_{\text{max}}}}{1 - e^{-h_{\text{max}}}}$$

i.e., to synchronize such a coupled dynamical network of maps, the connection density of the network should satisfy the bound determined by its individual nodes.

As an example of the basic BA model for scale-free networks, with the growing mechanism of adding new nodes one by one, there exists a maximum synchronous network scale $N_{\text{max}}$ (Fig. 7). A network generated by the BA model will not synchronize if the network further expands to have more than $N_{\text{max}}$ nodes, because the condition (15) is then no longer satisfied [40]. For example, if each node is a Hénon map,

$$y(k) = -1.4y(k - 1)^2 + 0.3y(k - 2) + 1.0,$$

the maximum network scale $N_{\text{max}} = 4$ for the positive Lyapunov exponent of the map: $h_{\text{max}} = 0.418$ (Fig. 8).

![Figure 7: Distribution of the maximum scale-free network size $N_{\text{max}}$ versus $h_{\text{max}}$ in [0.1, 1.0] (averaged output of 20 random groups) by the BA model with $m = m_0 = 1$.](image)

Therefore, there are two conditions to be satisfied for synchronizing a discrete-time dynamical network: the network scale $N \leq N_{\text{max}}$ and the inequality (15) [40]. For example, as stated before, the value $N_{\text{max}}$ of a network of coupled Hénon maps for achieving synchronization is 4. Given a 4-node network of coupled Hénon maps in a star-shaped structure, condition (15) is further changed to

$$\frac{1 - e^{-h_{\text{max}}}}{\lambda_2} = 0.3416 < c < \frac{1 + e^{-h_{\text{max}}}}{\lambda_N} = 0.4146.$$

So, if the coupling strength $c$ is less than 0.3416 or larger than 0.4146, this 4-node star of coupled Hénon maps cannot synchronize although its network scale $N \leq N_{\text{max}} = 4$. However, if the coupling
Figure 8: Synchronization area of the network size for a scale-free network of coupled Hénon maps (averaged output of 20 random groups). The dotted line is $\frac{1 + e^{-h_{\text{max}}}}{|\lambda_N|}$, and the solid line is $\frac{1 - e^{-h_{\text{max}}}}{|\lambda_2|}$.

strength is suitable for synchronization, i.e., $0.3416 < c < 0.4146$, but a new Hénon node is added to the synchronized 4-node star, thereby obtaining a 5-node network (thus, $N > N_{\text{max}} = 4$), which cannot achieves synchronization finally.

## 5 From synchronization to control

Now we want to stabilize network (18) onto a homogeneous stationary state defined by

$$x_1 = x_2 = \cdots = x_N = \bar{x}, \quad f(\bar{x}) = 0. \quad (19)$$

To achieve such a goal, we apply feedback controllers on a small fraction $\delta (0 < \delta \ll 1)$ of the nodes in network (18). Suppose that nodes $i_1, i_2, \ldots, i_l$ are selected, where $l = \lfloor \delta N \rfloor$ stands for the smaller but nearest integer to the real number $\delta N$. This controlled network can be described in the re-ordered form:

$$\begin{cases}
\dot{x}_i = f(x_i) + c \sum_{j=1}^{N} a_{ij} \Gamma x_j - cd \Gamma (x_i - \bar{x}), & i = 1, 2, \cdots, l \\
\dot{x}_i = f(x_i) + c \sum_{j=1}^{N} a_{ij} \Gamma x_j, & i = l+1, l+2, \cdots, N
\end{cases} \quad (20)$$

where feedback control gain $d > 0$. Generally, the number of controllers is preferred to be very small compared with the whole network scale $N$, i.e., $l \ll N$, and the feedback control method in controlled network (20) is the so-called pinning control [43, 69]. Through this section, we mainly show a bridge over the pinning control of complex networks and the network synchronization [43].

Owing to the local error-feedback nature of each pinned node, it is guaranteed that, as the feedback control gain $d \to \infty$, the states of the $l$ controlled nodes can be pinned to the homogenous target state $\bar{x}$. Hence, the pinning control stability of network (20) is converted to that of the following virtually
controlled network, and those pinned nodes function as virtual controllers to those un-pinned nodes:

\[
\begin{align*}
\mathbf{x}_i &= \bar{\mathbf{x}}, & i &= 1, 2, \ldots, l \\
\dot{\mathbf{x}}_i &= \mathbf{f}(\mathbf{x}_i) + c \sum_{j=l+1}^{N} \tilde{b}_{ij} \mathbf{x}_j + \tilde{u}_i, & i &= l + 1, l + 2, \ldots, N
\end{align*}
\]  \tag{21}

where the virtual control laws are taken as

\[
\tilde{u}_i = -\tilde{d}_i (\mathbf{x}_i - \bar{\mathbf{x}}), & i &= l + 1, l + 2, \ldots, N
\]  \tag{22}

with the virtual control feedback gains \( \tilde{d}_i = \sum_{j=1}^{l} a_{ij} \), and \( \tilde{B} = (\tilde{b}_{ij}) \in \mathbb{R}^{(N-l) \times (N-l)} \) defined as

\[
\begin{align*}
\tilde{b}_{ij} &= a_{ij}, & j &\neq i, j &= l + 1, \ldots, N, \quad i &= l + 1, l + 2, \ldots, N \\
\tilde{b}_{ii} &= -\sum_{j=l+1, j \neq i}^{N} a_{ij}
\end{align*}
\]  \tag{23}

Figure 9: An illustration of virtual control (dotted line) in specifically pinning control of a network generated by the BA model with \( N = 10, m = m_0 = 3 \).

Figure 9 is an illustration of the virtual control process in pinning controlling a network generated by the BA model with \( N = 10, m = m_0 = 3 \). It can be observed that while the pinned nodes are stabilized onto the homogenous state \( \bar{\mathbf{x}} \), their dynamics spread in the network as shown by the (red) dotted lines, which functions as the virtual control to those nodes which were not placed controllers. At the same time, it can be found in Fig. 9(b) that the whole 10-node original network was broken into two parts: An isolated 1\textsuperscript{st} node, which is only connected with the 4\textsuperscript{th} node, and the rest 8-node subnetwork. While increasing the pinning fraction of nodes, this subnetwork continues to shrink and finally reaches the case as shown in Fig. 9(d): every unpinned node is 'isolated' and virtually controlled by other pinned and also stabilized nodes.
Denote
\[ \tilde{A} = \tilde{B} + \text{diag} \left( \tilde{d}_{l+1}, \tilde{d}_{l+2}, \cdots, \tilde{d}_N \right), \]
and set \( \Gamma = I_n \). We have the following theorem on the pinning controlled network.\[ \text{(21)} \]

**Theorem 4** Assume \( f(x) \) is Lipschitz continuous in \( x \) with a Lipschitz constant \( L_f > 0 \), and the chaotic node \( \dot{x}_i = f(x_i) \), for all \( i = 1, 2, \cdots, N \), has the largest positive Lyapunov exponent \( h_{\text{max}} > 0 \). If \( \tilde{A} \) is irreducible, then the dynamical network (21) with virtual controllers (22) is globally (or locally asymptotically) stable about homogenous state \( \bar{x} \), provided that \( \Gamma = I_n \) and
\[ c > \frac{Q}{\lambda_{\text{min}}(-A)} \quad \text{(24)} \]
with \( Q \) being the Lipschitz constant \( L_f \) (or the largest Lyapunov exponent \( h_{\text{max}} \)).

It is very easy to verify from Theorem 2 that condition (24) is exactly the same stable condition for synchronizing network (25):
\[ \dot{x}_i = f(x_i) + c \sum_{j=l+1}^{N} a_{ij} \Gamma x_j, \quad i = l + 1, l + 2, \ldots, N. \quad \text{(25)} \]

Figure 10: Specifically pinning the biggest node of 19 degrees in a 50-node Chen network generated by the BA model: (a)-(d) are stabilizing phases with different coupling strengths. The homogenous stationary state is \( \bar{x} = [7.9373 \ 7.9373 \ 21]^T \), an unstable fixed point of Chen system [15].

Hence, the whole pinned network, controlled by a very small amount of local feedback controllers and those virtual controllers, is stabilized onto the homogenous state \( \bar{x} \) via synchronization [15]. This phenomenon is illustrated in Fig. 10 when the biggest node in a 50-node Chen network generated from the
BA model was specially pinned with suitable coupling strength $c$ and feedback gain $d$, over which other 49 coupled chaotic Chen systems \cite{15} (i.e., nodes) in the network were synchronized to the designer-preferred unstable fixed point $\mathbf{x} = [7.9373 \hspace{0.5cm} 7.9373 \hspace{0.5cm} 21]^T$ of Chen system.

6 Sync criticality stories on Kuramoto models

Collective phase synchronization in a large population of oscillators having natural different frequencies is another main branch in the literature of network synchronization, and the Kuramoto model is one of the most representative models of coupled phase oscillators. After Wiener first described the mathematical connection between the nonlinear dynamics and statical physics \cite{73}, which was later fruitfully studied by Winfree \cite{74}. Kuramoto refined the model, and formalized the solution to a network of globally coupled limit-cycle oscillators \cite{36}, uncovering the situation why the oscillators are completely de-synchronized until the coupling strength overcomes a criticality $C_{syn}$. Many variations and extensions of the original Kuramoto model appeared later have been recently surveyed in \cite{1} (and references therein).

We consider a network of $N$ coupled limit-cycle oscillators whose phases $\theta_i, i = 1, 2, \cdots, N$, evolve as

$$\frac{d\theta_i}{dt} = \omega_i + \sum_{j=1}^{N} c_{ij} a_{ij} \sin(\theta_j - \theta_i),$$
(26)

where $c_{ij}$ is the coupling strength between node (oscillator) $i$ and node (oscillator) $j$. Frequencies $\omega_i, i = 1, 2, \cdots, N$, are randomly distributed following the given frequency distribution $g(\omega)$, which is assumed that $g(\omega) = g(-\omega)$. The network coupling matrix $A = (a_{ij}) \in \mathbb{R}^{N \times N}$ is defined as in previous sections.

If we select the identical coupling scheme for a globally coupled network as

$$c_{ij} = \frac{c}{N}$$
(27)

Eq. (26) is the classical Kuramoto model \cite{36}, whose synchronization criticality is $C_{syn} = \frac{2}{\pi g(0)}$. With similar constant identical coupling schemes, Kuramoto model has been investigated its criticality $C_{syn}$ over small-world and (or) scale-free networks. Hong et al. reported their synchronization observations of a larger criticality of coupling strength on small-world networks than that of globally coupled networks \cite{28}. And, the latest investigation stated the absence of critical coupling strength in frequency synchronization of a swarm of oscillators connected as a scale-free network having a power-law exponent $2 < \gamma \leq 3$ \cite{30}. These results strongly suggest that the collective synchronous behaviors of complex networks depend on the network topologies.

However, the situation is completely different when the coupling scheme is non-identical and asymmetric. Define the non-identical asymmetric coupling scheme \cite{38}:

$$c_{ij} = c_j = \frac{c}{k_i}, \hspace{0.5cm} i, j = 1, \cdots, N,$$
(28)

where $k_i$ fits the given degree distribution $P(k)$ of a network. Therefore, we have

$$\frac{d\theta_i}{dt} = \omega_i + \frac{c}{k_i} \sum_{j=1}^{N} a_{ij} \sin(\theta_j - \theta_i).$$
(29)

If for every node $i$, its degree $k_i = N, i = 1, 2, \cdots, N$, model \cite{29} is degraded to the classical Kuramoto model for globally coupled networks. An interesting finding \cite{38} is that, with the non-identical and asymmetric coupling scheme \cite{29}, there does exist a uniform criticality of coupling strength, $C_{syn} = \frac{2}{\pi g(0)}$, to synchronize diversely random networks of limit-cycle oscillators \cite{29}, which is surprisingly the same as the critical coupling strength of globally coupled networks.
Figure 11: The average order parameter $r_{av}$ vs the coupling strength $C$ for networks generated by the ER model (dashed line with circle markers), the Local-World model (dotted line with square markers), the BA model (solid line with diamond markers), and the GKK model having power-law exponent $\gamma = 6$ (dash-dot line with right triangle markers). All networks have the same scale $N = 2048$ and the same average degree $\langle k \rangle = 6$.

Two main categories of degree distributions of random complex networks are in the forms of $P_{\text{power}}(k) \propto k^{-\gamma}$ and $P_{\text{exp}}(k) \propto e^{-k}$. Therefore, to verify the independence of $C_{\text{syn}}$ on different degree distributions $P(k)$ of network topologies, in numerical studies we select the famous ER model [20] to generate networks having an exponential degree distribution $P_{\text{exp}}(k) \propto e^{-k}$, and the BA model [8] and the GKK model [27] to generate networks having the scale-invariant power-law degree distribution $P_{\text{power}}(k) \propto k^{-\gamma}$ with $\gamma = 3, 6$, respectively. The proposed local-world evolving model [39], which owns a transition between the exponential degree distribution and the power-law degree distribution, is adopted as the final prototype with the parameters $M = 10$, $m = m_0 = 3$.

We fix the network scale $N = 2048$, and the average degree $\langle k \rangle = 6$. Therefore, all networks have the same number of nodes and edges with different connectivity patterns in the following simulation studies. Define the average order parameter as

$$ r_{av} = \left\langle \left[ \frac{\sum_{i=1}^{N} k_i e^{i\theta_i}}{\sum_{i=1}^{N} k_i} \right] \right\rangle, $$

where $\langle \cdots \rangle$ and $[\cdots]$ denote the averages over different realizations of intrinsic networks and over different realizations of intrinsic frequencies, respectively. In all the simulations, $r_{av}$ is averaged over 10 groups of networks satisfying the same degree distribution $P(k)$, and each network has 5 sets of frequencies with the distribution $g(\omega)$. We further specify

$$ g(\omega) = \begin{cases} 0.5, & \text{if } -1 < \omega < 1 \\ 0, & \text{otherwise} \end{cases} $$

Therefore, $C_{\text{syn}} \approx 1.273$ in this case.

As shown in Fig. 11, there is a common critical coupling strength of the value $C_{\text{syn}} = \frac{2}{g(0)\pi}$ in these four categories of networked limit-cycle oscillators, and the average order parameter $r_{av}$ increased
sharply when $C_{syn} \leq c$. In other words, the significant difference among complex network topologies does not show effect on the collective synchrony of the non-identically asymmetrically coupled limit-cycle oscillators \cite{24}. However, as concluded from \cite{28, 51, 30}, there is no such a same critical coupling strength of collective synchrony in different random complex networks of identically symmetrically coupled limit-cycle oscillators.

7 An application example: Synchronization in the World Trade Web

Recent studies of the World Trade Web (WTW) \cite{41, 62} have shown that there is also a significant scale-free feature in this economic network (Fig. 12). In such a World Trade Web, every country is a dynamical node, and the connections (edges) between any pair of countries (nodes) are their imports and exports. It was pointed out that the United States is the biggest node in the WTW \cite{41}. As mentioned above, the complexity of the network topology usually dominates the dynamical behaviors of a network, and the stability of a few ‘big’ nodes determine the synchronizability and stability of a scale-free dynamical network \cite{43, 67}. Therefore, it is interesting to ask to what extent the economy of the United States affects the economic development in other relatively ‘smaller’ countries. In more subtle details, we are interested in finding whether there is synchronization of economic cycles existing between the United States and the other countries.

Figure 12: Cumulative import and export weighted degree distributions of the WTW with 188 nodes and 12413 export links and 12669 import links. The dashed line is the power-law form $P(k) \sim k^{-\gamma}$ with $\gamma = 0.6 \sim 1.6$.

The economic-cycle synchronization is characterized by the correlation between the cyclical components of outputs of two arbitrarily chosen countries, $i$ and $j$ \cite{24, 25}:

$$corr(y^c_i, y^c_j) = \frac{\text{cov}(y^c_i, y^c_j)}{\sqrt{\text{var}(y^c_i)\text{var}(y^c_j)}}, \quad i, j = 1, 2, \ldots, n,$$

where $y^c_i$ is the cyclical component of output of country $i$, which can be specified here as the real Gross
Domestic Product (GDP). A positive correlation means that synchronization exists in the economic cycles between countries \( i \) and \( j \), and a higher correlation implies a higher degree of synchronization of economic cycles between two countries.

The economy development in a country is a very complex issue, and may be affected by many unpredictable factors such as significant changes of economic or political regimes occurred in a country (like USSR and some eastern European countries), the shocks from those speculated money firms (such as the Quantum Fund of George Soros), the continuation of a country’s economic policy, the change of exchange rate policy, and local wars, etc. Hence, in studying the synchronization phenomenon of economic cycles, to decrease the perturbations on economy development from unreasonable ‘noise’ to the minimum, we should probe those countries where stable and peaceful political situations and mature market economic systems have been well maintained. With the statistical data of real GDP in 1975-2000 that we can collected at that time, an appropriate choice contains 22 developed countries, i.e., the United States, the United Kingdom, Germany, Japan, France, Canada, Australia, Austria, Belgium, Norway, Italy, Finland, Denmark, Greece, Ireland, Iceland, Netherlands, Portugal, Spain, Sweden, New Zealand and Luxemburg (Fig. 13). It can be observed from the figure that in 1975-2000, totally 18 developed countries did have significant economic-cycle synchronization with the United States, where 5 countries (the United Kingdom, Australia, Canada, Finland, Sweden) show much stronger synchronization of economic cycles, except only 3 countries: Japan, Germany and Austria.

![Figure 13: 22 developed countries’ economic cycles synchronization phenomena. The positive real GDP correlation means synchronous economic cycles of those indicated countries with the United States, and the negative correlation means asynchronous economic cycles of the indicated countries with the United States.](image)

8 To probe further

In the past few years, advances in complex dynamical networks have uncovered some amazing similarities among many diverse large-scale natural and artificial systems such as the Internet, the world wide web, the world trade web, cellular nonlinear networks, metabolic systems, and even the collaborations of the Hollywood movie stars. Researchers from inter-disciplines have been inspired by the newly discovered small-world and scale-free features, and significant progress has been gained in the studies of the effect
of complex network topology on network dynamical behaviors, particularly the network synchronization phenomenon.

In this article, we have reported our recent research work on the synchronization of complex dynamical networks, both from the theoretical studies and economic applications. Even with the inspiring advances of today, there are still many important theoretical and technical explorations on modelling, evolution, analysis, control, and synchronization of complex dynamical networks left for future persistent research.

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