MOLLIFICATION AND NON-VANISHING OF AUTOMORPHIC L-FUNCTIONS ON GL(3)

BINGRONG HUANG, SHENHUI LIU, AND ZHAO XU

Abstract. We prove a non-vanishing result for central values of L-functions on GL(3), by using the mollification method and the Kuznetsov trace formula.

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1. INTRODUCTION

There has been vast research on the non-vanishing of central L-values for families of automorphic forms, since the pioneering work of Duke [5] and Iwaniec–Sarnak [12, 13]. To get positive-proportional non-vanishing results in families, one typically turns to the method of moments and the mollification method à la Selberg (see, for example, [17, 26, 18, 19, 20, 24, 21, 1, 15, 23], and others). In the current work we follow this approach and go beyond families of GL(2) forms (and symmetric-square lifts of GL(2) forms), and study the central L-values of Maass forms on GL(3) and prove a non-vanishing result of such values (Theorem 1.1), which is a positive-proportional result in the sense of Remark 1.2.

To state our result, we introduce a few notations and refer the reader to § 2.1 for certain details. Pick an orthogonal basis \{φ_j\} of Hecke–Maass forms for Γ = SL(3, Z). Each φ_j has spectral parameter ν_j = (ν_j,1, ν_j,2, ν_j,3), the Langlands parameter µ_j = (µ_j,1, µ_j,2, µ_j,3), and the Hecke eigenvalues A_j(n, 1). The main objects under investigation are the L-functions

\[ L(s, φ_j) = \sum_{n=1}^{∞} \frac{A_j(1, n)}{n^s} \quad \text{for } \Re(s) > 1. \]

A simple observation is that there is no trivial reason for \( L(\frac{1}{2}, φ_j) \) to vanish, since every φ_j is necessarily even and the sign of the functional equation of \( L(s, φ_j) \) is positive. In fact, one expects many of \( L(\frac{1}{2}, φ_j) \) to be nonzero. As in Blomer–Buttcane [2], we consider the generic case in short interval. Let \( µ_0 = (µ_0,1, µ_0,2, µ_0,3) \) and \( ν_0 = (ν_0,1, ν_0,2, ν_0,3) \), and satisfy the corresponding relations (2.2) and (2.1). We also assume

\[ |µ_{0,i}| \asymp |ν_{0,i}| \asymp T := \|µ_0\|, \quad 1 \leq i \leq 3. \]

Let \( M = T^θ \) for any fixed 0 < θ < 1. Define a test function \( h_{T,M}(µ) \) (depending on \( µ_0 \)) for \( µ = (µ_1, µ_2, µ_3) \in \mathbb{C}^3 \) by

\[ h_{T,M}(µ) := P(µ)^2 \left( \sum_{w \in W} \psi \left( \frac{w(µ) - µ_0}{M} \right) \right)^2, \]

where...
where
\[ \psi(\mu) = \exp\left(\mu_1^2 + \mu_2^2 + \mu_3^2\right) \]
and
\[ P(\mu) = \prod_{0 \leq n \leq A} \prod_{k=1}^{3} \frac{\left(\nu_k - \frac{1}{4}(1 + 2n)\right)\left(\nu_k + \frac{1}{4}(1 + 2n)\right)}{|\nu_{0,k}|^2} \]
for some fixed large \( A > 0 \). Here
\[ W := \left\{ I, \ w_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \ w_3 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \ w_4 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \ w_5 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \ w_6 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\} \]
is the Weyl group of \( SL(3, \mathbb{R}) \). The function \( h(\mu) \) has the localizing effect at a ball of radius \( M \) about \( w(\mu_0) \) for each \( w \in W \), and other nice properties stated in \( \S 2.1 \). Then with the normalizing factor
\[ N_j := ||\phi_j||^2 \prod_{k=1}^{3} \cos\left(\frac{3}{2} \pi \nu_{j,k}\right), \]
our main result is as follows.

**Theorem 1.1.** We have
\[ \sum_{j \geq 1} \frac{h_{T,M}(\mu_j)}{N_j} \gg T^3 M^2. \]

**Remark 1.2.** By a stronger form of the \( GL(3) \) spectral large sieve inequality obtained by Young ([27, Theorem 1.1]), one can get the following weighted Weyl law:
\[ \sum_{j \geq 1} \frac{h_{T,M}(\mu_j)}{N_j} \ll T^3 M^2. \]
In fact, we can replace the above “\( \ll \)” by “\( \asymp \)”.

Next we outline the structure of the paper and give the proof of Theorem 1.1. In \( \S 2.1 \) we briefly review facts of Maass forms and their \( L \)-functions, as well as the main analytic tool, the \( GL(3) \) Kuznetsov trace formula (Lemma 2.9). Define the mollifier \( M_j \) for \( L(\frac{1}{2}, \phi_j) \) by
\[ M_j := \sum_{\ell \leq L} \frac{A_j(1, \ell)}{\ell^{1/2}} x_{\ell}, \]
where
\[ x_{\ell} := \mu(\ell) \frac{1}{2\pi i} \int_{(3)} \frac{(L/\ell)^s}{\Gamma(3)} \frac{dx}{s^2 \log L} = \begin{cases} \mu(\ell) \frac{\log(L/\ell)}{\log L}, & \ell \leq L, \\ 0, & \ell > L, \end{cases} \]
and \( L = T^\delta \) for some small \( \delta > 0 \). Then we study the mollified moments of the central \( L \)-values and prove the following two propositions in \( \S 3 \) and \( \S 4 \), respectively.

**Proposition 1.4.** We have
\[ \sum_j \frac{h_{T,M}(\mu_j)}{N_j} |L(\frac{1}{2}, \phi_j) M_j|^2 \ll T^3 M^2, \]
provided \( 0 < \delta < 11/78 \).

**Proposition 1.5.** We have
\[ \sum_j \frac{h_{T,M}(\mu_j)}{N_j} L(\frac{1}{2}, \phi_j) M_j \asymp T^3 M^2, \]
provided \( 0 < \delta < 11/78 \).
Remark 1.6. The above results can be improved. The restriction of $\delta$ comes from the contribution of Eisenstein series, which can be refined if we use the subconvexity bounds for $GL(1)$ and $GL(2)$ $L$-functions or the average Lindelöf bound of the related families of $L$-functions.

Throughout the paper, $\varepsilon$ is an arbitrarily small positive number and $B$ is a sufficiently large positive number which may not be the same at each occurrence.

2. Preliminaries

In this section we review essential facts and tools, required for later development.

2.1. Hecke–Maass cusp forms and their $L$-functions. Let $G = GL(3, \mathbb{R})$ with maximal compact subgroup $K = O(3, \mathbb{R})$ and center $Z \cong \mathbb{R}^\times$. Let $\mathbb{H}_3 = G/(K \cdot Z)$ be the generalized upper half-plane. For $0 \leq c \leq \infty$, let

$$\Lambda_c = \left\{ \mu = (\mu_1, \mu_2, \mu_3) \in \mathbb{C}^3 \Big| \mu_1 + \mu_2 + \mu_3 = 0, \ |\Re \mu_k| \leq c, \ k = 1, 2, 3 \right\},$$

and

$$\Lambda'_c = \left\{ \mu \in \Lambda_c \mid \{-\mu_1, -\mu_2, -\mu_3\} = \{\overline{\mu_1}, \overline{\mu_2}, \overline{\mu_3}\} \right\}.$$

Consider a Hecke–Maass form $\phi$ in $L^2(SL(3, \mathbb{Z}) \backslash \mathbb{H}_3)$. Here $\mu$ will be the Langlands parameters. Define the spectral parameters

$$\nu_1 = \frac{1}{3}(\mu_1 - \mu_2), \quad \nu_2 = \frac{1}{3}(\mu_2 - \mu_3), \quad \nu_3 = \frac{1}{3}(\mu_3 - \mu_1).$$

We have

$$\mu_1 = 2\nu_1 + \nu_2, \quad \mu_2 = \nu_2 - \nu_1, \quad \mu_3 = -\nu_1 - 2\nu_2.$$

We will simultaneously use $\mu$ and $\nu$. By unitarity and the standard Jacquet–Shalika bounds, the Langlands parameter of an arbitrary irreducible representation $\mu$ is contained in $\Lambda'_{1/2} \subseteq \Lambda_{1/2}$, and the non-exceptional parameters are in $\Lambda'_0 = \Lambda_0$.

Let $\phi$ be a Hecke–Maass cusp form for $\Gamma = SL(3, \mathbb{Z})$ with Fourier coefficients $A_\phi(m, n) \in \mathbb{C}$ for $(m, n) \in \mathbb{N}^2$. The standard $L$-function of $\phi$ is given by

$$L(s, \phi) := \sum_{n=1}^{\infty} \frac{A_{\phi}(1, n)}{n^s} \text{ for } \Re(s) > 1.$$ 

For the dual form $\bar{\phi}$ the coefficients of $L(s, \bar{\phi})$ are $A_{\phi}(1, n) = \overline{A_{\phi}(n, 1)}$. The functional equation of $L(s, \phi)$ is

$$L(s, \phi) \prod_{j=1}^{3} \Gamma_R(s - \mu_j) = L(1 - s, \bar{\phi}) \prod_{j=1}^{3} \Gamma_R(1 - s + \mu_j),$$

where $\Gamma_R(s) = \pi^{-s/2} \Gamma(s/2)$.

2.2. The minimal Eisenstein series and its Fourier coefficients. Let

$$U_3 = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\} \cap \Gamma.$$ 

For $z \in \mathbb{H}_3$ and $\Re(\nu_1), \Re(\nu_2)$ sufficiently large, we define the minimal Eisenstein series

$$E(z; \mu_1, \mu_2) := \sum_{\gamma \in U_3 \setminus \Gamma} I_{\nu_1, \nu_2}(\gamma z),$$

where

$$I_{\nu_1, \nu_2}(z) := y_1^{1+\nu_1+2\nu_2} y_2^{1+2\nu_1+\nu_2},$$

and

$$z = \begin{pmatrix} y_1 y_2 & y_1 x_2 & x_3 \\ 0 & y_1 & x_1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x_2 & x_3 \\ 0 & 1 & x_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 y_2 & 0 & 0 \\ 0 & y_1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

with $x_1, x_2, x_3 \in \mathbb{C}$. Note that $x_1, x_2, x_3$ are defined up to a factor of $\pm 1$. 
with \( y_1, y_2 > 0 \) and \( x_1, x_2, x_3 \in \mathbb{R} \). It has meromorphic continuation in \( \nu_1 \) and \( \nu_2 \). The Fourier coefficients \( A_\mu(m_1, m_2) \) is defined by (see Goldfeld [7, Theorems 10.8.6])

\[
A_\mu(m, 1) = \sum_{d_1, d_2 \mid m} d_1^{\mu_1} d_2^{\mu_2} a_3^{\mu_3},
\]

and the symmetry and Hecke relation (see Goldfeld [7, Theorems 6.4.11])

\[
A_{\mu_1, \mu_2}(m, 1) = A_{\mu_1, \mu_2}(1, m), \\
A_{\mu_1, \mu_2}(m, n) = \sum_{d \mid (m, n)} \mu(d) A_{\mu_1, \mu_2}(m/d, 1) A_{\mu_1, \mu_2}(1, n/d).
\]

Hence we have

\[
|A_\mu(m_1, m_2)| \ll_\varepsilon (m_1 m_2)^{\varepsilon}, \quad \text{if } \mu \in (i\mathbb{R})^2.
\]

In order to state the Kuznetsov trace formula in the §2.5, we introduce

\[
N_\mu = \frac{1}{16} \prod_{k=1}^{3} |\zeta(1 + 3\mu_k)|^2
\]

corresponding to the minimal Eisenstein series \( E(z; \mu_1, \mu_2) \), where \( \mu = (\mu_1, \mu_2, \mu_3) \). Recall that (see [25]) we have

\[
\frac{1}{\zeta(1 + it)} \ll \log(1 + |t|),
\]

which implies that

\[
\frac{1}{N_\mu} \ll \sum_{k=1}^{3} \log^2(1 + |\nu_k|).
\]

2.3. The maximal Eisenstein series and its Fourier coefficients. Let

\[
P_{2,1} = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \right\} \cap \Gamma.
\]

Let \( \mu \in \mathbb{C} \) have sufficiently large real part, and let \( g : SL(2, \mathbb{Z}) \backslash \mathbb{H} \rightarrow \mathbb{C} \) be a Hecke–Maass cusp form with \( \|g\| = 1 \), Langlands parameter \( \mu_g \in i\mathbb{R} \) and Hecke eigenvalue \( \lambda_g(m) \). The maximal Eisenstein series twisted by a Maass form \( g \) is defined by

\[
E(z; \mu; g) := \sum_{\gamma \in P_{2,1} \backslash \Gamma} \det(\gamma z)^{1/2 + \mu} g(m_{P_{2,1}}(\gamma z)),
\]

where \( z \) is defined as in (2.4), and

\[
m_{P_{2,1}} : \mathbb{H}_3 \rightarrow \mathbb{H}_2, \\
\begin{pmatrix} y_1 y_2 & y_1 x_2 & x_3 \\ y_1 & x_1 & 1 \end{pmatrix} \mapsto \begin{pmatrix} y_2 & x_2 \\ 1 & 1 \end{pmatrix}
\]

is the restriction to the upper left corner. It has a meromorphic continuation in \( \mu \). The Fourier coefficients are determined by

\[
B_{\mu,g}(m, 1) = \sum_{d_1, d_2 \mid m} \lambda_g(d_1) d_1^{\mu_1} d_2^{-2\mu_2},
\]

and the symmetry and Hecke relation as above (see Goldfeld [7, Proposition 10.9.3 and Theorem 6.4.11]). Recall that we have the following Kim–Sarnak bound for \( GL(2) \) Fourier coefficients (see Kim [16, Appendix 2])

\[
|\lambda_g(n)| \ll_\varepsilon n^{7/64 + \varepsilon}.
\]

Hence we have

\[
|B_{\mu,g}(m_1, m_2)| \ll_\varepsilon (m_1 m_2)^{7/64 + \varepsilon}, \quad \text{if } \mu \in i\mathbb{R}.
\]

We also introduce

\[
N_{\mu,g} = 8L(1, Ad^2 g)|L(1 + 3\mu, g)|^2,
\]
where $L(s, \text{Ad}^2 g)$ is the adjoint square $L$-function of $g$, and $L(s, g)$ is the $L$-function of $g$. We have the lower bounds

$$L(1, \text{Ad}^2 g) \gg (1 + |\mu_g|)^{-\varepsilon}, \quad L(1 + it, g) \gg (1 + |t| + |\mu_g|)^{-\varepsilon}.$$  

These lower bounds follow from [9, 10, 14], and [6]. Therefore, for $\mu \in i\mathbb{R}$, it follows that

(2.8) \quad \frac{1}{N_{\mu, g}} \ll (1 + |\mu| + |\mu_g|)^{\varepsilon}.

2.4. The Kloosterman sums. For $n_1, n_2, m_1, m_2, D_1, D_2 \in \mathbb{N}$, we need the relevant Kloosterman sums

$$\hat{S}(n_1, n_2, m_1; D_1, D_2) := \sum_{C_1 \pmod{D_1}, C_2 \pmod{D_2}} e \left( n_2 \frac{C_1 C_2}{D_1} + m_1 \frac{C_2}{D_2/D_1} + n_1 \frac{C_1}{D_1} \right)$$

for $D_1 \mid D_2$, and

$$S(n_1, m_2, m_1, n_2; D_1, D_2) := \sum_{B_1, C_1 \pmod{D_1}} e \left( n_1 B_1 + m_1 (Y_1 D_2 - Z_1 B_2) \frac{D_1}{D_2} + m_2 B_2 + n_2 (Y_2 D_1 - Z_2 B_1) \frac{D_2}{D_1} \right),$$

where $Y_j B_j + Z_j C_j \equiv 1 \pmod{D_j}$ for $j = 1, 2$.

2.5. The Kuznetsov trace formula. We first introduce some notation. Define the spectral measure on the hyperplane $\mu_1 + \mu_2 + \mu_3 = 0$ by

$$d_{\text{spec}} \mu = \text{spec}(\mu) d\mu,$$

where

$$\text{spec}(\mu) := \prod_{k=1}^3 \left( 3 \nu_k \tan \left( \frac{3\pi}{2} \nu_k \right) \right) \quad \text{and} \quad d\mu = d\mu_1 d\mu_2 d\mu_3 = d\mu_1 d\mu_2 d\mu_3.$$

Following [3, Theorems 2 & 3], we define the following integral kernels in terms of Mellin–Barnes representations. For $s \in \mathbb{C}$, $\mu \in \Lambda_\infty$ define the meromorphic function

$$\hat{G}^{\pm}(s, \mu) := \frac{\pi^{-3s}}{12288\pi^{7/2}} \prod_{k=1}^3 \Gamma \left( \frac{4}{3}(s - \mu_k) \right) \pm i \prod_{k=1}^3 \Gamma \left( \frac{4}{3}(1 + s - \mu_k) \right),$$

and for $s = (s_1, s_2) \in \mathbb{C}^2$, $\mu \in \Lambda_\infty$ define the meromorphic function

$$G(s, \mu) := \frac{1}{\Gamma(s_1 + s_2)} \prod_{k=1}^3 \Gamma(s_1 - \mu_k) \Gamma(s_2 + \mu_k).$$

The latter is essentially the double Mellin transform of the GL(3) Whittaker function. We also define the following trigonometric functions

$$S^{++}(s, \mu) := \frac{1}{24\pi^2} \prod_{k=1}^3 \cos \left( \frac{3}{2} \pi \nu_k \right),$$

$$S^{+-}(s, \mu) := \frac{1}{32\pi^2} \cos \left( \frac{3}{2} \pi \nu_1 \right) \sin \left( \frac{3}{2} \pi \nu_2 \right) \sin \left( \frac{3}{2} \pi \nu_3 \right) \sin \left( \pi (s_1 - \mu_1) \right) \sin \left( \pi (s_2 + \mu_2) \right) \sin \left( \pi (s_2 + \mu_3) \right),$$

$$S^{-+}(s, \mu) := \frac{1}{32\pi^2} \cos \left( \frac{3}{2} \pi \nu_2 \right) \sin \left( \frac{3}{2} \pi \nu_1 \right) \sin \left( \pi (s_1 - \mu_1) \right) \sin \left( \pi (s_1 - \mu_2) \right) \sin \left( \pi (s_2 + \mu_3) \right),$$

$$S^{--}(s, \mu) := \frac{1}{32\pi^2} \cos \left( \frac{3}{2} \pi \nu_3 \right) \sin \left( \frac{3}{2} \pi \nu_1 \right) \sin \left( \pi (s_1 - \mu_1) \right) \sin \left( \pi (s_1 - \mu_2) \right) \sin \left( \pi (s_2 + \mu_2) \right).$$
For \( y \in \mathbb{R} \setminus \{0\} \) with \( \text{sgn}(y) = \epsilon \), let
\[
K_{w_4}(y; \mu) := \int_{-i\infty}^{i\infty} \frac{|y|^{-s} \mathcal{G}(s, \mu)}{2\pi i} \frac{ds}{s}.
\]
For \( y = (y_1, y_2) \in (\mathbb{R} \setminus \{0\})^2 \) with \( \text{sgn}(y_1) = \epsilon_1 \), \( \text{sgn}(y_2) = \epsilon_2 \), let
\[
K_{w_4}^{\epsilon_1, \epsilon_2}(y; \mu) := \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} |4\pi^2 y_1|^{-s_1} |4\pi^2 y_2|^{-s_2} \mathcal{G}(s, \mu) S^{\epsilon_1, \epsilon_2}(s, \mu) \frac{ds_1 ds_2}{(2\pi i)^2}.
\]

We can now state the Kuznetsov trace formula in the version of Butt cane\cite{Butt}, Theorems 2, 3, 4].

**Lemma 2.9.** Let \( n_1, n_2, m_1, m_2 \in \mathbb{N} \) and let \( h \) be a function that is holomorphic on \( \Lambda_{1/2+k} \) for some \( \delta > 0 \), symmetric under the Weyl group \( \mathcal{W} \), of rapid decay when \( |\text{Im} \mu_j| \to \infty \), and satisfies
\[
h(3\nu_j \pm 1) = 0, \quad j = 1, 2, 3.
\]
Then we have
\[
\mathcal{C} + \mathcal{E}_{\text{min}} + \mathcal{E}_{\text{max}} = \Delta + \Sigma_4 + \Sigma_5 + \Sigma_6,
\]
where
\[
\mathcal{C} := \sum_j \frac{h(\mu_j)}{\mathcal{N}_j} A_j(m_1, m_2) A_j(n_1, n_2),
\]
\[
\mathcal{E}_{\text{min}} := \frac{1}{24(2\pi)^2} \sum_{\mu, \mu, \mu, \mu, \mu, \mu} \frac{h(\mu)}{\mathcal{N}_\mu} A_\mu(m_1, m_2) A_\mu(n_1, n_2) d\mu_1 d\mu_2,
\]
\[
\mathcal{E}_{\text{max}} := \sum_g \frac{1}{2\pi i} \int_{\Re(\mu)=0} h(\mu + \mu_1, \mu - \mu_2, -2\mu_3) B_{\mu,g}(m_1, m_2) B_{\mu,g}(n_1, n_2) \frac{d\mu_1 d\mu_2}{\mathcal{N}_\mu g},
\]
and
\[
\Delta := \delta_{n_1, m_1} \delta_{n_2, m_2} \frac{1}{192\pi^4} \int_{\Re \mu = 0} h(\mu) d_{\text{spec}} \mu,
\]
\[
\Sigma_4 := \sum_{\epsilon = \pm 1} \sum_{D_2|D_1} \frac{\hat{S}(-e n_2, m_2, m_1; D_2, D_1)}{D_1 D_2} \Phi_{w_4} \left( \frac{e_1 m_2 n_2}{D_1 D_2} \right),
\]
\[
\Sigma_5 := \sum_{\epsilon = \pm 1} \sum_{D_2|D_1} \frac{\hat{S}(e n_1, m_1, m_2; D_1, D_2)}{D_1 D_2} \Phi_{w_5} \left( \frac{e_1 m_1 m_2}{D_1 D_2} \right),
\]
\[
\Sigma_6 := \sum_{\epsilon, \epsilon = \pm 1} \sum_{D_2|D_1} \frac{S(e_2 n_2, e_1 n_1, m_1, m_2; D_1, D_2)}{D_1 D_2} \Phi_{w_6} \left( -\frac{e_2 m_2 n_2 D_2}{D_1^2}, -\frac{e_2 m_2 n_1 D_1}{D_2^2} \right),
\]
with
\[
\Phi_{w_4}(y) := \int_{\Re \mu = 0} h(\mu) K_{w_4}(y; \mu) d_{\text{spec}} \mu,
\]
\[
\Phi_{w_5}(y) := \int_{\Re \mu = 0} h(\mu) K_{w_4}(-y; -\mu) d_{\text{spec}} \mu,
\]
\[
\Phi_{w_6}(y_1, y_2) := \int_{\Re \mu = 0} h(\mu) K_{w_6}^{\text{sgn}(y_1), \text{sgn}(y_2)} ((y_1, y_2); \mu) d_{\text{spec}} \mu.
\]

In the first moment, we will use \( h(\mu) = h_{T,M}(\mu) \) to be the test function which is same as Blomer–Butt cane\cite{Butt}, and in the second moment, we will use \( h_2(\mu) = h_{T,M}(\mu) \mathcal{W}_{\mu,N}(m_1 m_2) \) (see (3.2)) to be the test function. The function \( h(\mu) \) localizes at a ball of radius \( M \) about \( w(\mu_0) \) for each \( w \in \mathcal{W} \). We have
\[
\mathcal{D}_k h_{T,M}(\mu) \ll_k M^{-k},
\]
for any differential operator $\mathcal{D}_k$ of order $k$, which we use frequently when we integrate by parts, and sufficiently many differentiations can save arbitrarily many powers of $T$. Moreover, by trivial estimate, we have
\begin{equation}
\int_{\text{Re}\mu=0} h_{T,M}(\mu) d_{\text{spec}}\mu \asymp T^3 M^2.
\end{equation}

2.6. **The weight functions.** For the weight functions, we will need the following results in Blomer–Buttcane [2, Lemma 1, Lemma 8, and Lemma 9].

**Lemma 2.14.** For some large enough constant $B > 0$, we have
\[ \Phi_{w_4}(y) \ll |y|^{1/10} T^B, \quad \Phi_{w_5}(y) \ll |y|^{1/10} T^B, \quad \Phi_{w_6}(y_1, y_2) \ll |y_1 y_2|^{3/5} T^B. \]

**Lemma 2.15.**

(i) If $0 < |y| \leq T^{3-\varepsilon}$, then for any constant $B > 0$, we have
\[ \Phi_{w_4}(y) \ll_{\varepsilon, B} T^{-B}. \]

(ii) If $T^{3-\varepsilon} < y$, then
\[ y^k \Phi_{w_4}^{(k)}(y) \ll_{k, \varepsilon} T^{3-2\varepsilon} M^2 (T + |y|^{1/3})^k. \]

**Lemma 2.16.** Let $T := \min\{|y_1|^{1/3}|y_2|^{1/6}, |y_1|^{1/6}|y_2|^{1/3}\}$. If $T \ll T^{1-\varepsilon}$, then we have
\[ \Phi_{w_6}(y_1, y_2) \ll_{\varepsilon, B} T^{-B}. \]

There is a slight difference that we use (2.12) and (2.13) instead of [2, (3.7) and (3.8)]. This have no influence in the proof. We define $\Phi_{2,w_4}(y)$, $\Phi_{2,w_5}(y)$, and $\Phi_{2,w_6}(y)$ as in (2.11) by using the test function $h_2(\mu)$. In the proof of the above three Lemmas, the only two properties of $h_{T,M}(\mu)$ which are used is (2.12) and (2.13). Here, we remark that $h_2(\mu)$ also satisfies these two inequalities by using properties of $W_{\mu,N}$ (see §3). So, $\Phi_{2,w_4}(y)$, $\Phi_{2,w_5}(y)$, and $\Phi_{2,w_6}(y)$ also satisfies the corresponding bounds in the above three Lemmas.

3. **The mollified second moment**

Let $G(s) = (\cos \frac{\pi s}{2})^{-100A}$, where $A$ is a positive integer. For $|L(\frac{1}{2}, \phi_j)|^2$, we will use the following approximate functional equation.

**Lemma 3.1.** Let $\phi_j$ be a $GL(3)$ Maass form with Langlands parameters $\mu_j \in \Lambda_{1/2}'$. We have
\[ |L(\frac{1}{2}, \phi_j)|^2 = 2 \sum_{m_1, m_2} \frac{A_j(m_1, m_2)}{(m_1 m_2)^{1/2}} W_j (m_1 m_2), \]
where
\[ W_j(y) := \frac{1}{2\pi i} \int_{(3)} \zeta(1 + 2s)(\pi^3 y)^{-s} \prod_{k=1}^{3} \prod_{\pm} \frac{\Gamma\left(\frac{s + 1/2 \pm \mu_{j,k}}{2}\right)}{\Gamma\left(\frac{1/2 \pm \mu_{j,k}}{2}\right)} G(s) \frac{ds}{s}. \]
Moreover, we have
\[ y^i W_j^{(i)}(y) \ll_{i,B} \left(1 + \frac{y}{\prod_{k=1}^{3} (1 + |\mu_{j,k}|)}\right)^B, \]
for any non-negative integer $i$, and any large positive integer $B$.

**Proof.** See e.g. Iwaniec–Kowalski [11, §5.2].

Note that the sum in Proposition 1.4 is essentially supported on the generic forms which satisfy
\[ |\mu_{j,k}| \asymp |v_{j,k}| \asymp T, \quad 1 \leq k \leq 3. \]
So we assume $\phi_j$ also satisfy the above relation. By Stirling’s formula, if $|s| \ll |z|^{1/2}$, then
\[ \frac{\Gamma(z + s)}{\Gamma(z)} = z^s \left(1 + \sum_{n=1}^{N} \frac{P_n(s)}{z^n} + O\left(\frac{(1 + |s|)^{2N+2}}{|z|^{N+1}}\right)\right), \]
for certain polynomials $P_n(s)$ of degree $2n$. Since $G(s)$ has exponential decay, we may truncate at $|\text{Im } s| \ll T^\varepsilon$ with only a small error in $W_j(y)$. Based on the above arguments, together with $\mu_j \in \Lambda_{1/2}$, we have

$$W_j(y) = \frac{1}{2\pi i} \int_{(3)} \zeta(1+2s) (\pi^3 y)^{-s} \prod_{k=1}^{3} \prod_{\pm} \left( \frac{1/2 + \mu_j k}{2} \right)^{s/2} \cdot \left( 1 + \sum_{n=1}^{N} \frac{2^n P_n(s)}{(1/2 + \mu_j) n} + O\left( \frac{\left( 1 + |s| \right)^{2N+2}}{|\mu|^{N+1}} \right) \right) G(s) \frac{ds}{s}.$$ 

On the other hand, by Hecke multiplicity relations we have

$$|M_j|^2 = \sum_{\ell_1} \sum_{\ell_2} \frac{1}{\ell_1^{1/2} \ell_2^{1/2}} x_{\ell_1} x_{\ell_2} A_j(\ell_1,1) A_j(\ell_2,1)$$

$$= \sum_{d} \sum_{\ell_1} \sum_{\ell_2} \frac{1}{d \ell_1^{1/2} \ell_2^{1/2}} x_{d \ell_1} x_{d \ell_2} A_j(\ell_1,\ell_2).$$

By Lemma 3.1 and Hecke relations again, we have

$$\sum_{j} \frac{h_{T,M}(\mu_j)}{N_j} |L(\frac{1}{2}, \phi_j M_j)|^2$$

$$= 2 \sum_{j} \frac{h_{T,M}(\mu_j)}{N_j} \sum_{d} \sum_{\ell_1} \sum_{\ell_2} \frac{x_{d \ell_1} x_{d \ell_2}}{d \ell_1^{1/2} \ell_2^{1/2}} A_j(\ell_1, \ell_2) \sum_{m_1, m_2} \frac{1}{m_1^{1/2} m_2^{1/2}} W_j(m_1 m_2) A_j(m_1, m_2)$$

$$= 2 \sum_{d} \frac{1}{d} \sum_{\ell_1, \ell_2} \frac{x_{d \ell_1} x_{d \ell_2}}{d \ell_1^{1/2} \ell_2^{1/2}} \sum_{m_1, m_2} \frac{1}{m_1^{1/2} m_2^{1/2}} \sum_{j} \frac{h_{T,M}(\mu_j) W_j(m_1 m_2)}{N_j} A_j(\ell_1, \ell_2) A_j(m_1, m_2).$$

Let

$$W_{j,N}(y) = \frac{1}{2\pi i} \int_{(3)} \zeta(1+2s) (\pi^3 y)^{-s} \prod_{k=1}^{3} \prod_{\pm} \left( \frac{1/2 + \mu_j k}{2} \right)^{s/2} \cdot \left( 1 + \sum_{n=1}^{N} \frac{2^n P_n(s)}{(1/2 + \mu_j) n} \right) G(s) \frac{ds}{s}.$$ 

Then we can replace $W_j$ by $W_{j,N}$ with a negligible error if we choose $N$ to be large enough. Now we use the Kuznetsov trace formula (see Lemma 2.9) with the test function

$$h_2(\mu) = h_{T,M}(\mu) W_{\mu,N}(m_1 m_2),$$

where $W_{\mu,N}$ is defined the same as $W_{j,N}$ with $\mu$ replacing $\mu_j$. It turns out we are led to estimate

$$\sum_{d} \frac{1}{d} \sum_{\ell_1, \ell_2} \frac{x_{d \ell_1} x_{d \ell_2}}{d \ell_1^{1/2} \ell_2^{1/2}} \sum_{m_1, m_2} \frac{1}{m_1^{1/2} m_2^{1/2}} \left( \Delta^{(2)} + \Sigma_4^{(2)} + \Sigma_5^{(2)} + \Sigma_6^{(2)} - \varepsilon_{\max}^{(2)} - \varepsilon_{\min}^{(2)} \right),$$

where

$$\Delta^{(2)} := \delta_{\ell_1, m_1} \delta_{\ell_2, m_2} \frac{1}{192 \pi^5} \int_{\text{Re } \mu = 0} h_2(\mu) d_{\text{spec }} \mu,$$

$$\Sigma_4^{(2)} := \sum_{\ell} \sum_{m_2} \frac{\tilde{S}(-e \ell_2, m_2; D_2, D_1)}{D_1 D_2} \Phi_{2, w_4} \left( \frac{e m_1 \ell_2}{D_1 D_2} \right),$$

$$\Sigma_5^{(2)} := \sum_{\ell} \sum_{m_2} \frac{\tilde{S}(e \ell_2, m_2; D_2, D_1)}{D_1 D_2} \Phi_{2, w_5} \left( \frac{e m_1 \ell_2}{D_1 D_2} \right),$$

$$\Sigma_6^{(2)} := \sum_{\ell_1, \ell_2} \sum_{m_1} \sum_{m_2} \frac{\tilde{S}(e_1 \ell_1, e_2 \ell_2, m_1, m_2; D_1, D_2)}{D_1 D_2} \Phi_{2, w_6} \left( -\frac{e_2 m_1 \ell_2 D_2}{D_1^2}, -\frac{e_1 m_2 \ell_1 D_1}{D_2^2} \right),$$

$$\Sigma_7^{(2)} := \sum_{\ell_1, \ell_2} \sum_{m_1} \sum_{m_2} \frac{\tilde{S}(e_1 \ell_1, e_2 \ell_2, m_1, m_2; D_1, D_2)}{D_1 D_2} \Phi_{2, w_7} \left( -\frac{e_2 m_1 \ell_2 D_2}{D_1^2}, -\frac{e_1 m_2 \ell_1 D_1}{D_2^2} \right),$$

$$\Sigma_8^{(2)} := \sum_{\ell_1, \ell_2} \sum_{m_1} \sum_{m_2} \frac{\tilde{S}(e_1 \ell_1, e_2 \ell_2, m_1, m_2; D_1, D_2)}{D_1 D_2} \Phi_{2, w_8} \left( -\frac{e_2 m_1 \ell_2 D_2}{D_1^2}, -\frac{e_1 m_2 \ell_1 D_1}{D_2^2} \right).$$
with \( \Phi_{2,w_1}(y) \), \( \Phi_{2,w_2}(y) \), and \( \Phi_{2,w_3}(y) \) defined as in (2.11) by using the new test function \( h_2(\mu) \) given by (3.2), respectively; and

\[
\zeta^{(2)}_{\text{max}} := \sum_{\gamma} \frac{1}{2\pi i} \int_{\text{Re}(\mu) = 0} \frac{h_2(\mu + \mu_\gamma, \mu - \mu_\gamma, -2\mu)}{N_{\mu,\gamma}} B_{\mu_\gamma}(m_1, m_2) B_{\mu_\gamma}(\ell_1, \ell_2) \, d\mu,
\]

\[
\zeta^{(2)}_{\text{min}} := \frac{1}{24(2\pi i)^2} \int_{\text{Re}(\mu) = 0} \frac{h_2(\mu)}{N_{\mu}} A_{\mu}(m_1, m_2) A_{\mu}(\ell_1, \ell_2) \, d\mu_1 \, d\mu_2.
\]

3.1. **The diagonal term.** Note that we have \( \ell_1 = m_1 \), and \( \ell_2 = m_2 \). Thus we infer that the diagonal term in (3.3) is

\[
\frac{1}{96\pi^3} \sum_d \sum_{\ell_1} \sum_{\ell_2} \frac{x_{d_\ell_1} x_{d_\ell_2}}{\ell_1 \ell_2} \int_{\text{Re} \mu = 0} h_{T,M}(\mu) W_{\mu,N}(\ell_1 \ell_2) \, d_{\text{spec}} \mu.
\]

By the definition of \( W_{\mu,N} \), we only need to deal with the leading term, which contributes

\[
\frac{1}{96\pi^3} \int_{\text{Re} \mu = 0} h_{T,M}(\mu) S(\mu) \, d_{\text{spec}} \mu,
\]

where

\[
S(\mu) := \sum_d \frac{\mu^2(d)}{d} \sum_{(\ell_1,d) = 1} \sum_{(\ell_2,d) = 1} \frac{\mu(\ell_1) a_{d\ell_1} \mu(\ell_2) a_{d\ell_2}}{\ell_1 \ell_2} W_{\mu}(\ell_1 \ell_2),
\]

with

\[
a_{\ell} := \frac{1}{2\pi i} \int_{(3)} \left( \frac{L/\ell}{s^2} \right)^{s} ds \log L \frac{\log(L/\ell)}{\log L}, \quad \ell \leq L,
\]

\[
W_{\mu}(y) := \frac{1}{2\pi i} \int_{(3)} \zeta(1 + 2s) \left( \frac{\pi^3 y}{|\mu_1 \mu_2 \mu_3|} \right)^{s} G(s) \frac{ds}{s}.
\]

By moving the line of integration in \( W_{\mu} \) to left, we pass the double pole at \( s = 0 \). Hence, by the residue theorem, we infer that

\[
W_{\mu}(\ell_1 \ell_2) = c_1 \log |\mu_1 \mu_2 \mu_3| + c_2 \log \ell_1 \ell_2 + c_3 + O(T^{-B}),
\]

where \( c_1 \), \( c_2 \), and \( c_3 \) are constants. The main contribution from \( W_{\mu}(\ell_1 \ell_2) \) comes from the first two terms, which can be treated similarly. For convenience, we only give the details of the first term. Note that the goal is to show

\[
\sum_d \frac{\mu^2(d)}{d} \sum_{(\ell_1,d) = 1} \sum_{(\ell_2,d) = 1} \frac{\mu(\ell_1) a_{d\ell_1} \mu(\ell_2) a_{d\ell_2}}{\ell_1 \ell_2} \int_{\text{Re} \mu = 0} h_{T,M}(\mu) \log |\mu_1 \mu_2 \mu_3| \, d_{\text{spec}} \mu \ll T^3 M^2.
\]

For the \( \ell_1 \)-sum, we have

\[
\sum_{(\ell_1,d) = 1} \frac{\mu(\ell_1) a_{d\ell_1}}{\ell_1} = \frac{1}{2\pi i} \int_{(2)} \prod_{p|d} \left( 1 - \frac{1}{p^{1+s}} \right)^{-1} \frac{(L/(d_\ell_1))^s}{s^2 \zeta(1+s) \log L} \, ds.
\]

Recall that in the region \( \text{Re} s \geq 1 - c / \log(|\text{Im} s| + 3) \) (here \( c \) is some positive constant), \( \zeta(s) \) is analytic except for a single pole at \( s = 1 \), and has no zeros and satisfies \( \zeta(s)^{-1} \ll \log(|\text{Im} s| + 3), \zeta'(s)/\zeta(s) \ll \log(|\text{Im} s| + 3) \) (see e.g. [25, (3.11.7) and (3.11.8)]). We move the line of integration in (3.6) to

\[
\mathcal{C}_\varepsilon := \{ ix : |x| \geq \varepsilon \} \cup \{ i\varepsilon i^\theta : \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2} \},
\]

and \( \varepsilon \) is sufficiently small. It follows that

\[
(3.6) = \frac{1}{\log L} \prod_{p|d} \frac{p}{p-1} + \int_{\mathcal{C}_\varepsilon} \prod_{p|d} \left( 1 - \frac{1}{p^{1+s}} \right)^{-1} \frac{(L/(d_\ell_1))^s}{s^2 \zeta(1+s) \log L} \, ds.
\]
We have the similar expression for the $\ell_2$-sum. Inserting these into (3.5), we consider the resulting $d$-sum. A typical term is

\begin{equation}
\sum_{d \leq L} \frac{\mu(d)}{d} \prod_{p | d} \left( \frac{p}{p-1} \right)^2 \ll \log T
\end{equation}

which implies (3.5) by trivial computation. For the term involving $\log(\ell_1 \ell_2) = \log \ell_1 + \log \ell_2$, we have

\begin{equation}
\sum_{\ell_1 \geq 1 \atop (\ell_1, d) = 1} \frac{\mu(\ell_1)}{\ell_1^{1+s}} \log \ell_1 = -\left\{ \frac{1}{\zeta(1+s)} \prod_{p | d} \left( 1 - \frac{1}{p^{s+1}} \right)^{-1} \right\}^T.
\end{equation}

A similar argument shows that its contribution to (3.4) is $\ll T^3M^2$.

### 3.2. The $w_4$ and $w_5$ terms

We only deal with the $w_4$-term, since the $w_5$-term is very similar. Inserting a smooth unity to $m_1, m_2$ sums, we are led to estimate

\begin{equation}
\sum_{\epsilon = \pm 1} \sum_d \sum_{\ell_1, \ell_2} \frac{1}{\sqrt{\ell_1 \ell_2}} \Sigma_4^{(2)}(\epsilon, d, \ell_1, \ell_2, M_1, M_2),
\end{equation}

where $M_1, M_2 \gg 1, M_1M_2 \ll T^{3+\epsilon}$,

\begin{equation}
\Sigma_4^{(2)}(\epsilon, d, \ell_1, \ell_2, M_1, M_2) := \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \frac{W_1(\frac{m_1}{M_1})W_2(\frac{m_2}{M_2})}{\sqrt{m_1m_2}} \sum_{\substack{D, \delta \geq 1 \atop m_2 = \ell_1D}} \tilde{S}(\epsilon m_2, m_1; D, D\delta) \Phi_{2,w_4}(\frac{em_1m_2\ell_2}{D^2\delta}),
\end{equation}

and $W_i(x)$ ($i = 1, 2$) are compactly supported in $[1, 2]$ and satisfy $x^jW_i^{(j)}(x) \ll 1$. By Lemma 2.14, we can truncate the $D, \delta$ sums at some $T^cB$ for some large $B$ at the cost of a negligible error. Then by Lemma 2.15, we can truncate the sums, again with a negligible error, at

\begin{equation}
\frac{m_1m_2\ell_2}{D^2\delta} \geq T^{3-\epsilon},
\end{equation}

or in other words,

\begin{equation}
D^2\delta \leq \frac{M_1M_2L}{T^{3-\epsilon}} \ll LT^{\epsilon}.
\end{equation}

Note that we have $m_2 = \ell_1D$ now, which implies $M_2$ is small. That is,

\begin{equation}
M_2 \leq LD/\delta \leq (M_1M_2/T^{3-\epsilon})^{1/2}L^{3/2}/\delta^{3/2} \ll L^{3/2}T^{\epsilon}/\delta^{3/2}.
\end{equation}

And by (3.11) we have

\begin{equation}
M_1 \gg T^{3-\epsilon}D^2\delta/(m_2\ell_2) = T^{3-\epsilon}D^2/(\ell_1\ell_2) \gg T^{3-\epsilon}/L^2.
\end{equation}

We apply the Poisson summation formula in the $m_1$-variable, getting

\begin{align*}
\sum_{m_1=1}^{\infty} \frac{W_1(\frac{m_1}{M_1})}{\sqrt{m_1}} \tilde{S}(\epsilon m_2, m_1; D, D\delta) \Phi_{2,w_4}(\frac{em_1m_2\ell_2}{D^2\delta}) &= \sum_{a \equiv \ell_1 \pmod{\delta}} \tilde{S}(\epsilon m_2, m_1; D, D\delta) \sum_{m_1 \equiv a \pmod{\delta}} \frac{W_1(\frac{m_1}{M_1})}{\sqrt{m_1}} \Phi_{2,w_4}(\frac{em_1m_2\ell_2}{D^2\delta}) \\
&= \frac{M_1^{1/2}}{\delta} \sum_{a \equiv \ell_1 \pmod{\delta}} \tilde{S}(\epsilon m_2, m_1; D, D\delta) \sum_{m_1 \in \mathbb{Z}} e\left(\frac{am}{\delta}\right) \\
&\quad \cdot \int_{-\infty}^{\infty} W_1(x) \Phi_{2,w_4}(\frac{em_1m_2\ell_2}{D^2\delta}) e\left(-\frac{mM_1x}{\delta}\right) dx.
\end{align*}
Integration by parts in connection with Lemma 2.15 (ii) and the above bounds for $M_1$ and $M_2$ shows that the integral is negligible unless $m = 0$. When $m = 0$, by opening $\tilde{S}(-\ell \ell_2, m_2, a; D, D\delta)$ and compute the $a$ sum, we obtain

$$
\sum_{a \pmod{\delta}} \tilde{S}(-\ell \ell_2, m_2, a; D, D\delta) = \sum_{a \pmod{\delta}} \sum_{C_1 \pmod{D}, C_2 \pmod{D}} e \left( m_2 \frac{\tilde{C}_1 C_2}{D} + a \frac{\tilde{C}_2}{\delta} - \ell \ell_2 \frac{C_1}{D} \right) = 0
$$

unless $\delta = 1$. With the help of this and $m_2 = \ell_1 D$, we see that the contribution from $m = 0$ to $\Sigma_4^{(2)}(\epsilon, \ell_1, \ell_2, M_1, M_2)$ is

$$
M_1^{1/2} \ell_1^{-1/2} \sum_{D = 1}^{\infty} W_2 \left( \frac{\ell_1 D}{M_2} \right) \frac{1}{D^{3/2}} \sum_{C_1 \pmod{D} \atop (C_1, D) = (C_2, D) = 1} e \left( -\ell \ell_2 C_1 / D \right) \int_{-\infty}^{\infty} W_1(x) \Phi_{w_4} \left( \frac{\ell_1 \ell_2 M_1 x}{D} \right) dx.
$$

By inserting the definitions of $\Phi_{w_4}(x)$ and $K_w(y; \mu)$, the above $x$-integral becomes

$$
\int_{-\infty}^{\infty} W_1(x) \frac{dx}{x^{1/2}} \Re_{\mu = 0} h_2(\mu) K_w \left( \frac{\ell_1 \ell_2 M_1 x}{D} ; \mu \right) d_{\text{spec}} \mu dx
$$

$$
= \int_{\Re \mu = 0} h_2(\mu) \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} x^{-s - 1/2} W_1(x) dx \right) \left| \frac{\ell_1 \ell_2 M_1}{D} \right|^{-s} \tilde{G}^s(s, \mu) \frac{ds}{2\pi i} d_{\text{spec}} \mu
$$

where the Mellin transform $\tilde{W}_1$ is entire and rapidly decaying, which means that we can restrict the $s$-integral to $|\text{Im } s| \leq T^\varepsilon$. Inserting the definition of $\tilde{G}^s(s, \mu)$ into (3.12), we obtain the corresponding $\mu$-integral:

$$
\int_{\Re \mu = 0} h(\mu) \prod_{j=1}^{3} \Gamma \left( \frac{1}{2} (s - \mu_j) \right) \prod_{j=1}^{3} \Gamma \left( \frac{1}{2} (1 + s + \mu_j) \right) \pm i \prod_{j=1}^{3} \Gamma \left( \frac{1}{2} (1 + s + \mu_j) \right) d_{\text{spec}} \mu.
$$

We only need to consider the first part, since the second part is very similar. For fixed $\sigma \in \mathbb{R}$, $t \in \mathbb{R}$ and $|t| \geq 1$, we have the Stirling formula

$$
\Gamma(\sigma + it) = \sqrt{2\pi e^{-\pi t/2}} |t|^{\sigma-1/2} e^{\pi t} \left( 1 + O(|t|^{-1}) \right),
$$

where $\lambda = 1$, if $t \geq 1$, and $\lambda = -1$, if $t \leq -1$. For convenience, we denote $\mu_1 = it_1$, $\mu_2 = it_2$, $\mu_3 = -i(t_1 + t_2) = it_3$, and $\text{Im } s = t$. Here, in the essential integrated range, we have $|t_1| \approx |t_k - t_k'| \approx T$ for $1 \leq k, k' \leq 3$ and $k \neq k'$, since we are considering the generic case. Without loss of generality, we assume $t_1 > t_2 > 0$, and $t_3 < 0$, and get

$$
\prod_{j=1}^{3} \frac{\Gamma \left( \frac{1}{2} (s - \mu_j) \right) }{\Gamma \left( \frac{1}{2} (1 + s + \mu_j) \right) } = 2^{-3/2} e^{(-3t(2 + 1) + \pi/2)} \frac{e^{i(t-t_1) \log(t_1-t) + (t-t_2) \log(t_2-t) + i(t_1+t_2+t) \log(t_1+t_2+t)}}{(t_1-t)^{1/2} (t_2-t)^{1/2} (t_1+t_2+t)^{1/2}} + O(T^{-5/2}).
$$

By trivial estimate, the error term from the above to (3.10) is $O(T^{2+\varepsilon} M^2 L)$.

For the main term in (3.13), we use partial integration to prove its contribution is small. Actually, for the $\mu_1$-integral, we denote the phase function

$$
\phi(t_1) := -(t - t_1) \log(t_1 - t) + (t_1 + t_2 + t) \log(t_1 + t_2 + t).
$$

We see that

$$
\phi'(t_1) = \frac{t_1 + t_2 + t}{t_1 - t} \gg 1.
$$

And hence, by partial integration many times, the integral is negligible. We finally prove that the contribution from the $w_4$-term is $O(T^{2+\varepsilon} M^2 L)$. 

3.3. The \( w_6 \) term. By Lemma 2.14, we can truncate the \( D_1, D_2 \) sums at some \( T^B \) for some large \( B \) at the cost of a negligible error. Then by Lemma 2.16, we can truncate the sum further at

\[
\frac{(m_1 \ell_2)^{1/3} (m_2 \ell_1)^{1/6}}{D_1^{1/2}} \geq T^{1-\varepsilon}, \quad \text{and} \quad \frac{(m_1 \ell_2)^{1/6} (m_2 \ell_1)^{1/3}}{D_2^{1/2}} \geq T^{1-\varepsilon},
\]

which means

\[
\frac{(m_1 m_2 \ell_1 \ell_2)^{1/2}}{(D_1 D_2)^{1/2}} \geq T^{2-\varepsilon}.
\]

This gives us

\[
D_1 D_2 \ll \frac{m_1 m_2 \ell_1 \ell_2}{T^{4-\varepsilon}},
\]

which is impossible provided that \( L \leq T^{1/2-\varepsilon} \), since the essential sums of \( m_1 \) and \( m_2 \) are truncated at \( m_1 m_2 \leq T^{3+\varepsilon} \). Thus, the contribution of the \( w_6 \) term is \( O(T^{-B}) \).

3.4. The contribution from the Eisenstein series. We only treat the contribution of the maximal Eisenstein series, since the minimal Eisenstein series can be handled similarly and the contribution will be smaller. We have the Weyl law on \( GL_2 \) (see [8]),

\[
\# \{ g : t_g \leq T \} = \frac{T^2}{12} \frac{T \log T}{2\pi} + C_0 T + O \left( \frac{T}{\log T} \right),
\]

where \( C_0 \) is a constant. Combining this together with definitions of \( h(\mu) \) and \( h_2(\mu) \), and note that we are considering the generic case, we see that the essential region of the integration on \( \mu \) is of length \( \asymp M \), and the essential number of the sum of \( \mu_g \) is of size \( \asymp TM \). Hence, by the bound (2.7), (2.8) and the above argument, we have

\[
\mathcal{E}_{\max}^{(2)} \ll \varepsilon T M^2 (T^3 L^2)^{7/64} T^\varepsilon.
\]

Hence the contribution of the right hand side of (3.3) of the maximal Eisenstein series is

\[
\ll \varepsilon L T^{3/2} T M^2 (T^3 L^2)^{7/64} T^\varepsilon \ll \varepsilon T^3 M^2 T^{-11/64+\varepsilon} L^{39/32} \ll \varepsilon T^3 M^2,
\]

provided \( L \ll T^{11/78-\varepsilon} \).

4. The mollified first moment

Let \( G(s) \) be defined as in \( \S 3 \), and \( T_0 = T^{1+\varepsilon} \). Consider the integral

\[
\frac{1}{2\pi i} \int_{(3)} G(s) L \left( s + \frac{1}{2}, \phi_j \right) \frac{T_0^3 s}{s} ds.
\]

By moving the line of integration to \( \text{Re}(s) = -3 \), and using the functional equation (2.3), we get

\[
L \left( s + \frac{1}{2}, \phi_j \right) = \sum_{m \geq 1} \frac{A_j(1, m)}{m^{1/2}} V(m, T_0) + \sum_{m \geq 1} \frac{A_j(m, 1)}{m^{1/2}} V_j(m, \mu_j),
\]

where

\[
V(y, T_0) := \frac{1}{2\pi i} \int_{(3)} G(s) \left( \frac{y}{T_0^3} \right)^{-s} \frac{ds}{s},
\]

and

\[
V_j(y, \mu_j) := \int_{(3)} G(s) (y T_0^3)^{-s} \prod_{k=1}^3 \Gamma \left( \frac{s+1/2+\mu_{j,k}}{2} \right) \frac{ds}{s}.
\]

To see the properties of \( V \) and \( V_j \), we use the strategy in [11]. Obviously, we have \( V(y, T_0) = 1 + O \left( (y/T_0^3)^B \right) \), and \( V(y, T_0) \ll (y/T_0^3)^{-B} \), by moving the integration line to \( \text{Re } s = -B \) and
\[ \text{Re } s = B \text{ respectively. For } V_j, \text{ we have } V_j(y, \mu_j) \ll (yT)^{-B} \text{ for any } y \geq 1, \text{ by using the Stirling formula and moving the integration line to } \text{Re } s = B/(3\varepsilon). \text{ With the help of these, we infer that} \]

\[
\sum_j \frac{h_{T,M}(\mu_j)}{N_j} L\left(\frac{1}{2}, \phi_j \right) M_j = \sum_j \frac{h_{T,M}(\mu_j)}{N_j} \sum_{\ell} \frac{x_{\ell}}{\ell^{1/2}} A_j(1, \ell) \sum_m \frac{1}{m^{1/2}} V(m, T_0) A_j(m, 1) + O(T^{-B})
\]

\[
= \sum_{\ell} \frac{x_{\ell}}{\ell^{1/2}} \sum_m \frac{V(m, T_0)}{m^{1/2}} \sum_j \frac{h_{T,M}(\mu_j)}{N_j} A_j(1, \ell) A_j(m, 1) + O(T^{-B}).
\]

Then, by the Kuznetsov trace formula, we have

\[
\sum_{\ell} \frac{x_{\ell}}{\ell^{1/2}} \sum_m \frac{V(m, T_0)}{m^{1/2}} \sum_j \frac{h_{T,M}(\mu_j)}{N_j} A_j(1, \ell) A_j(m, 1)
\]

\[
= \sum_{\ell} \frac{x_{\ell}}{\ell^{1/2}} \sum_m \frac{V(m, T_0)}{m^{1/2}} \left( \Delta^{(1)} + \Sigma_4^{(1)} + \Sigma_5^{(1)} + \Sigma_6^{(1)} - \varepsilon_{\max}^{(1)} - \varepsilon_{\min}^{(1)} \right),
\]

where

\[
\Delta^{(1)} := \delta_{\ell_1, \ell_2} \frac{1}{192 \pi^5} \int_{\text{Re } \mu = 0} h_{T,M}(\mu) \, d_{\text{spec } \mu},
\]

\[
\Sigma_4^{(1)} := \sum_{\epsilon = \pm 1} \sum_{D_1, D_2} \frac{S(\epsilon \ell, 1, m; D_1, D_2)}{D_1 D_2} \Phi_{w_4} \left( \frac{\epsilon m}{D_1 D_2} \right),
\]

\[
\Sigma_5^{(1)} := \sum_{\epsilon = \pm 1} \sum_{D_1, D_2} \frac{S(\epsilon \ell, 1, m; D_1, D_2)}{D_1 D_2} \Phi_{w_5} \left( \frac{\epsilon m}{D_1 D_2} \right),
\]

\[
\Sigma_6^{(1)} := \sum_{\epsilon_1, \epsilon_2 = \pm 1} \sum_{D_1, D_2} \frac{S(\epsilon_1 \ell, 1, m; D_1, D_2)}{D_1 D_2} \Phi_{w_6} \left( -\frac{\epsilon_1 m D_1}{D_1^2}, -\frac{\epsilon_1 m D_2}{D_2^2} \right),
\]

and

\[
\varepsilon_{\max}^{(1)} := \sum_g \frac{1}{2 \pi i} \int_{\text{Re } \mu = 0} \frac{h_{T,M}(\mu + \mu_2, \mu - \mu_1, -2\mu)}{N_{\mu, \nu}} \Phi_{w_4}(1, m) B_{\mu, \nu}(\ell, 1) \, d\mu,
\]

\[
\varepsilon_{\min}^{(1)} := \frac{1}{24 (2\pi i)^2} \int_{\text{Re } \mu = 0} \frac{h_{T,M}(\mu)}{N_{\mu}} A_{\mu}(1, m) A_{\mu}(\ell, 1) \, d\mu_1 \, d\mu_2.
\]

4.1. The diagonal term. By trivial estimation, the contribution of the diagonal term is

\[
V(1, T_0) \frac{1}{192 \pi^5} \int_{\text{Re } \mu = 0} h_{T,M}(\mu) \, d_{\text{spec } \mu} = \frac{1}{192 \pi^5} \int_{\text{Re } \mu = 0} h_{T,M}(\mu) \, d_{\text{spec } \mu} + O(T^{-B}) \approx T^3 M^2.
\]

4.2. The other terms. The treatments of the other terms are actually similar and in fact easier than those in the mollified second moment. So we omit the arguments here.

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School of Mathematics, Shandong University, Jinan, Shandong 250100, China
E-mail address: binguong@mail.sdu.edu.cn

231 W 18th Ave, MW 549, Columbus, OH 43210, USA
E-mail address: liu.2076@osu.edu

School of Mathematics, Shandong University, Jinan, Shandong 250100, China
E-mail address: zhxu@sdu.edu.cn