Analysis wavefront propagating in free space based on the Zernike polynomials and Gauss-Laguerre modes expansion

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Abstract. The Zernike orthogonal polynomials are often used to describe the wavefront. The coefficients of the wavefront expansion in terms of Zernike polynomials allow us to determine the root-mean-square error of the deviation from the ideal wavefront. For visual assessment of image quality a spread point function is used. In this paper, we use the Fresnel transform to propagate the initial wavefront in free space and show how the coefficients of the Zernike polynomials and Gauss-Laguerre modes change. In modern ophthalmology, aberrometers are used to measure the wavefront distortions on the surface of elements of the optical system of a human eye. In this paper, we investigate the possibility of reconstructing the original field on the basis of the wavefront obtained at a certain distance.

1. Introduction

The problem of measuring the distortions of the wavefront is often encountered in optics, for example, in the construction of terrestrial astronomical telescopes, in optical communication systems, in industrial laser technology, in medicine, etc. [1-9]. The measurement of distortions of the wavefront is performed with the aim of compensating them, in particular by means of adaptive or active optics [10-15]. The wave front sensor is one of the main elements of the adaptive adjustment system. Its task is the measurement of the wavefront aberration and transfer these measurements to the processing device. The main causes of wavefront aberrations are: atmospheric turbulence, nonideality of the optical elements of the system, errors in the alignment of the system, etc. Today, there is a wide variety of wave front sensors, but the most common are the interference method [16] and the Shack-Hartman method [17, 18].

Phase recovering can be realized also using diffractive optical elements (DOE) which performing optical decomposition of the light field by orthogonal basis [19-22]. A generally accepted representation of the wave front is the basis of Zernike polynomials [23-26]. Earlier, for the direct measurement of the wavefront aberration coefficients, DOEs were matched with a set of Zernike polynomials [21, 22, 27-29], which can be successfully applied to wavefront analysis with small aberrations [29].

Let us take into account that Zernike polynomials are not invariant to distribution in free space and passing through the lens systems [30-32], and there are attempts to analyze the wavefront aberration on other bases [4, 33-35]. Laser beams that are invariant to propagation in free space and lens transformations are the Gauss-Laguerre modes [36]. Multiple order DOEs matched with the Gauss-Laguerre mode set were also successfully used for optical analysis of laser fields [37, 38].

In this paper, a wavefront propagation in free space is simulated using the Fresnel transform and it is shown how Zernike and Gauss-Laguerre expansion coefficients vary.
2. Theoretical foundations
This paper discusses the Zernike function as follows (Fig. 1):

\[ Z_{nm}(r, \varphi) = \sum_{m=-\infty}^{\infty} R_m(r) \left[ \cos(m\varphi) \right], \]
\[ R_m(r) = \frac{(-1)^n (n+m)!}{n! (n-m)!} \left( \frac{r}{R} \right)^{n-m}. \]

where \( R_m(r) \) – radial polynomials Zernike:

Figure 1. Pictures of Zernike functions.

Wavefront aberrations occurring in optical systems are usually described in terms of Zernike functions as follows:

\[ W(r, \varphi) = \exp \left[ 2\pi i \psi(r, \varphi) \right], \]
\[ \psi(r, \varphi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} c_{mn} Z_{nm}(r, \varphi). \]

We used the simplest Fourier-correlator optical system to construct the pictures of the point-scattering function (PSF) in the presence of typical aberrations.

However, there is an alternative possibility of analyzing the aberrations of the original field. In this paper, we use the Fresnel transform to propagate the wave front in free space and show how the weight coefficients of the Zernike polynomials and the Gauss-Laguerre modes change.

For example, the Gauss-Laguerre basis is used to analyze the mode composition of laser radiation, the Zernike polynomials for the analysis of the wave front, and the basis-harmonics and functions of their containing (Gauss-Laguerre, Zernike) for the rotation-invariant decomposition.

Gauss-Laguerre modes are invariant to free space propagation and transformation by a lens unlike Zernike polynomials. The Gauss-Laguerre mode energy in both the object and spectral planes is concentrated in a limited region, although, strictly speaking, these functions are invariant to the Fourier transform in infinite limits, and all radially symmetric light fields possess the property of invariance to rotation.
In this paper, the Gauss-Laguerre modes are used in the following form (Fig. 2):

$$GL_n^m(r, \varphi) = K_{nm}(\sqrt{2}r/s)^m L_n^m\left((r/s)^2\right) \exp\left[i\pi \varphi\right],$$

(4)

where the Laguerre polynomials are given by the recurrence formula:

$$L_n^m(x) = 1/n(2n+m-1-x)L_{n-1}^m(x)-(n+m-1)L_{n-2}^m(x),$$

where $L_0^m(x) = \exp\left(-x^2/2\right)$, $L_1^0(x) = (1 + m - x) \exp\left(-x^2/2\right)$.

![Figure 2. Gauss-Laguerre modes, s=0.15, K_{nm}=1.](image)

Fresnel transformation is an expansion of a parabolic waves:

$$U(u,v,z) = \frac{-ik}{2\pi z} \exp[ikz] \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) \exp\left[\frac{ik}{2z}(x-u)^2 + (y-v)^2\right] dx dy,$$

(5)

provided that $(x-u)^2 + (y-v)^2 \ll z$, and $k = 2\pi/\lambda$, $\lambda$ - wavelength, $z$ - distance for which distributed wavefront.

In the framework of this paper, the Fresnel transform is realized through the Fourier transform (\( T \)) as follows:

$$U(u,v,z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) \exp\left[\frac{ik}{2z}(u^2 + v^2)\right] \exp[2\pi i(ux + vy)] dx dy = \mathcal{F}\{f(x,y)\exp\left[\frac{ik}{2z}(u^2 + v^2)\right]\}.$$

(6)

3. Numerical simulation

We shall numerically simulate the propagation in the free space of a certain wave front $W(r, \varphi)$ over a distance equal to $z$, with phase $\psi(r, \varphi)$ equal to one Zernike polynomial $Z_{nm}$ with a weighting coefficient $C_{nm}$, which corresponds to a certain aberration. (the simulation results for each of the previously considered aberrations are presented in Table 1). Then we expand the wave front at various distances $z_i$, and also expand the PSF by Zernike polynomials and the Gauss-Laguerre modes, and then try to reconstruct the initial phase of the wave front.
We carry out the expansion of the wave front at a distance equal to $z_i$ by Zernike polynomials and Laguerre Gaussian modes, using Zernike analyzer and Laguerre-Gaussian. For an objective evaluation of the coefficients obtained, we shall calculate the mean square deviation (RMS) of the expansion coefficients after the propagation of the wave front $C_i^z$ and ideal $C_k$ according to the formula

$$s = \sqrt{\frac{\Delta C_i^z}{c}}, \Delta C = \sum_{k=1}^{n} (C_i^z - C_k)^2, c = \sum_{k=1}^{n} C_k^2.$$ 

The RMS was obtained for all aberrations, which are considered in Table 1. The values of the RMS are presented in Table 2.

### Table 1. Wavefront propagation in free space.

| $z_i$ | $C_{00}$ | $C_{11}$ | $C_{20}$ | $C_{31}$ | $C_{33}$ | $C_{40}$ | $C_{42}$ | $C_{44}$ |
|-------|---------|---------|---------|---------|---------|---------|---------|---------|
| 200   | ![Image](image1.png) | ![Image](image2.png) | ![Image](image3.png) | ![Image](image4.png) | ![Image](image5.png) | ![Image](image6.png) | ![Image](image7.png) | ![Image](image8.png) |
| 1100  | ![Image](image9.png) | ![Image](image10.png) | ![Image](image11.png) | ![Image](image12.png) | ![Image](image13.png) | ![Image](image14.png) | ![Image](image15.png) | ![Image](image16.png) |
| infinity | ![Image](image17.png) | ![Image](image18.png) | ![Image](image19.png) | ![Image](image20.png) | ![Image](image21.png) | ![Image](image22.png) | ![Image](image23.png) | ![Image](image24.png) |

The average value of the RMS of the expansion coefficients for the Zernike basis is 0.093 and 0.111 for the Gauss-Laguerre modes.

Let us consider an example in which the phase of the wave front is given by a superposition of three basis functions of Zernike polynomials with the following weight coefficients $C_{20}=0.8$; $C_{31}=1.6$; $C_{42}=0.4$ (Fig.3).

The value of the RMS of the expansion coefficients for the Zernike basis is 0.110 and 0.086 for the Gauss-Laguerre modes. Let's try to restore some field by the formula 3.

### Table 2. RMS coefficients of expansion by Zernike basis (Z) and Gauss-Laguerre modes (GL).

| Type of aberration | $Z_{00}$ | $Z_{11}$ | $Z_{20}$ | $Z_{22}$ | $Z_{31}$ | $Z_{33}$ | $Z_{40}$ | $Z_{42}$ | $Z_{44}$ |
|-------------------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| $S_Z$             | 0.078   | 0.165   | 0.065   | 0.114   | 0.125   | 0.077   | 0.029   | 0.105   | 0.077   |
| $S_{GL}$          | 0.026   | 0.149   | 0.040   | 0.171   | 0.111   | 0.175   | 0.037   | 0.117   | 0.174   |
Fig. 3. Amplitude and phase: (a) – phase of the wavefront, (b) – wavefront (WF), wave front in the plane \( z_1 = 200 \) (c), \( z_2 = 1100 \) (d), (e) – Fourier plane (PSF).

Figure 4. The expansion coefficients (absolute values) of the field by Zernike functions (a) Gauss-Laguerre modes (b) \( \xi(r, \phi) \).

Figure 5. Amplitude and phase: reconstructed wave front by complex Zernike coefficients at a distance \( z_1 = 200 \) (a), \( z_2 = 1100 \) (b), (c) – infinity (Fourier plane (PSF)).

It can be seen from figure 5 that the Zernike analyzer carried out the reconstruction of the wave front in some plane incorrectly. This is explained by the fact that the basis functions of Zernike are not invariant to the Fresnel and Fourier distribution operators. Figure 5 (a, b, c) shows a poorly informative and highly scaled reconstructed amplitude and phase of the wave front, which is shown in Fig. 3 (c, d, e).

We will try to recover the field according to the Gauss-Laguerre modes, as
\[
\xi(r, \phi) = \sum_{n=0}^{N} \sum_{m=0}^{N} K_{nm} GL^n_m (r, \phi).
\]
The reconstructed wave front is not very informative in view of the fact that not every superposition of the Gauss-Laguerre modes is invariant to the rotation. It was proposed to use the absolute values of the Zernike coefficients to reconstruct the phase of a given wave front $\psi(r, \phi)$ and the absolute values of the coefficients of the Gauss-Laguerre modes to reconstruct the wave front $\xi(r, \phi)$ in some plane at a distance equal to $z_i$.

The visual similarity between the restored phase (Figure 7b) and the initial phase (Figure 3a.) is quite obvious. This result is expected, and is confirmed by the low RMS value of the coefficients of the wave front expansion in the plane $z_2 = 1100$.

We calculate the RMS for the coefficients of wavefront expansion in planes at a distance $z_1 = 200$, $z_2 = 1100$, $z_3 = \infty$ and field $\xi(r, \phi)$. For this necessary to expand the field $\xi(r, \phi)$ by the basis of Zernike polynomials.

As a result, the RMS for all three cases is approximately 0.1. This indicates the possibility of correct reconstruction of the wavefront by the Gauss-Laguerre modes at any distance, both in the near and in the far zone.
4. Conclusion
The propagation of the wave front in free space was carried out using the Fresnel transform and it was shown how the weight coefficients of the expansion by the basis of the Zernike polynomials and by the Gauss-Laguerre modes change.

They were recognized by the coefficients of the expansion in the Zernike polynomials and the Gauss-Laguerre modes using the Zernike and Gauss-Laguerre analyzer. It was found that the MSE for each of the considered aberrations is less than 0.174, and the maximum values appear in the case where aberration is described by Zernike functions with equal radial and meridional index. So the aberration of the "tilt" type $Z_{11}$ has an RMS of decomposition coefficients of 0.165 and 0.149, respectively, the aberration of the «defocus» type $Z_{22} - 0.114 \pm 0.171$, «terfoil» $-0.077 \pm 0.175$, «quadrofoil» $-0.077 \pm 0.174$.

In addition, within the framework of this paper, a study was made of the possibility of reconstructing the phase features of the wave front using DOE, which make it possible to perform an optical expansion of the light field by the basis of the Zernike functions and the Gauss-Laguerre modes. It is proposed to use the absolute values of the coefficients for reconstructing the phase of the wave front. It is shown both the visual similarity of the restored and the initial phase, as well as the objective evaluation of the RMS.

As a result of the experiments, it was revealed that the field $\xi(r, \phi)$ reconstructed by Gauss-Laguerre modes has a visual similarity with the PSF of the wavefront under investigation, and its decomposition coefficients have a relatively small RMS. Moreover, this field is sufficiently invariant both to the rotation and to the Fresnel and Fourier transforms.

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