Less eavesdropping losses induce more eavesdropping information gain

Zhanjun Zhang¹, Zhongxiao Man¹ and Yong Li²

¹ Wuhan Institute of Physics and Mathematics, The Chinese Academy of Sciences, Wuhan 430071, China
² Department of Physics, Center China Normal University, Wuhan 430079, China

E-mail: Zhangzj@wipm.ac.cn

(Dated: February 27, 2019)

The eavesdropping scheme proposed by Wójcik [Phys. Rev. Lett. 90, 157901(2003)] on the quantum communication protocol of Boström and Felbinger [Phys. Rev. Lett. 89, 187902(2002)] is improved by constituting a new set of attack operations. The improved scheme only induces half of the eavesdropping losses in Wójcik’s scheme, therefore, in a larger domain of the quantum channel transmission efficiency $\eta$, i.e., $[0, 75\%]$, the eavesdropper Eve can attack all the transmitted bits. Comparing to Wójcik’s scheme, in the improved scheme the eavesdropping (legitimate) information gain does not vary in the $\eta$ domain of $[0, 50\%]$, while in the $\eta$ domain of $(50\%, 75\%]$ the less eavesdropping losses induce more eavesdropping information gains, for Eve can attack all the transmitted bits and accordingly eavesdropping information gains do not decrease. Moreover, for the Boström-Felbinger protocol, the insecurity upper bound of $\eta$ presented by Wójcik is pushed up in this paper, that is, according to Wójcik’s eavesdropping scheme, the Boström-Felbinger protocol is not secure for transmission efficiencies lower than almost 60%, while according to the improved scheme, it is not secure for transmission efficiencies lower than almost 80%.

PACS number(s): 03.67.Hk, 03.65.Ud

Quantum key distribution (QKD) is an ingenious application of quantum mechanics, in which two remote legitimate users (Alice and Bob) establish a shared secret key through the transmission of quantum signals. Much attention has been focused on QKD after the pioneering work of Bennett and Brassard published in 1984 [1]. Till now there have been many theoretical QKDs [2-20]. Different from the QKDs, the deterministic secure direct communication protocol is to transmit directly the secret messages without first generating QKD to encrypt them. Hence it is very useful and usually desired, especially in some urgent time. However, the deterministic secure direct communication is more demanding on the security than QKDs. Therefore, only recently a few of deterministic secure direct protocols have been proposed [21-24]. One of them is the famous Boström-Felbinger protocol [22], which allows the generation of a deterministic key or even direct secret communication. In Ref [22] the protocol has been claimed to be secure and experimentally feasible. However, since the security of the Boström-Felbinger protocol can be impaired as far as considerable quantum losses are taken into account, very recently Wójcik has presented an undetectable eavesdropping scheme on the Boström-Felbinger protocol[25]. Wójcik’s eavesdropping scheme induces the eavesdropping losses at the level of 50% and the anticorrelation of the state of the home photon (kept by Bob, the legitimate receiver of secret messages)
with that of the travel photon (sent by Bob to Alice, the sender of secret messages). If the transmission efficiency $\eta$ of the quantum channel is not taken into account, the probability of the eavesdropper (i.e., Eve) being detected is zero due to the anticorrelation. However, in the case of the considerable quantum channel losses, it is possible for legitimate users to detect the eavesdropping by observing the quantum channel losses. That is, although Eve can attack all the transmitted bits and the eavesdropping losses can be hidden in the channel losses when $\eta \leq 50\%$, if she attacks all the transmitted bits when $\eta > 50\%$, then the eavesdropping losses is greater than the channel losses and accordingly the legitimate users can find Eve in the line by observing the channel losses. In fact, when $\eta > 50\%$, it is still possible for Eve to avoid the legitimate users’ detection, for she can eavesdrop only the fraction $\mu = 2(1 - \eta)$ of the transmitted bits to induce less eavesdropping losses, which can be completely hidden in the quantum channel losses. Hence, in [25] it is concluded that the Boström-Felbinger protocol is not secure for transmission efficiencies lower than almost 60\%, i.e., the insecurity upper bound of $\eta$ is 60\%. Nonetheless, in the case of $\eta > 50\%$, the eavesdropping (legitimate) information gain will surely decrease (increase) due to eavesdropping only a fraction of the whole transmitted bits. Then it is intriguing to ask, in a low-loss quantum channel (i.e., a high transmission efficiency channel), whether Eve can have more information gain in a way of reducing the eavesdropping losses such that she is able to eavesdrop all the transmitted bits in a larger domain of $\eta$. If so, then the insecurity upper bound of $\eta$ presented by Wójcik can be pushed up. To address the question, in this brief report, we improve the the Wójcik’s eavesdropping scheme [25] by constituting a new set of attack operations. Our improved eavesdropping scheme indeed induces less eavesdropping losses than that in [25], therefore, in a larger domain of $\eta$, Eve can attack all the transmitted bits. Since when $\eta = 1$ the eavesdropping (legitimate) information gain does not decrease (increase) with the decrease of the eavesdropping losses, the larger domain in which Eve can attack all the transmitted bits means that less eavesdropping losses may induce more eavesdropping information gain. Hence, for the Boström-Felbinger protocol, the insecurity upper bound of the transmission efficiency presented by Wójcik can be pushed up. One will see these later.

Let us start with the brief description of the Boström-Felbinger protocol [22]. Bob prepares two photons in the entangled state $|\Psi^+\rangle = (|0\rangle|1\rangle + |1\rangle|0\rangle)/\sqrt{2}$ of the polarization degrees of freedom. He stores one photon (home photon) in his lab and sends Alice the other one (travel photon) via a quantum channel. After receiving the travel photon Alice randomly switches between the control mode and the message mode. In the control mode Alice measures the polarization of the travel photon first and then announces publicly the measurement result and the measurement basis she used. After knowing Alice’s announcement Bob also switches to the control mode to measure the home photon in the same basis as that Alice used. Then he compares both measurement results. They should be perfectly anticorrelated in the absence of Eve. Therefore, the appearance of identical results is considered to be the evidence of eavesdropping, and if it occurs the transmission is aborted. In the other case, the transmission continues. In the message mode, Alice performs the $Z_j(\in \{0, 1\})$ operation on the travel photon to encode $j$ and sends it back to Bob, where $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$. After receiving the travel photon Bob measures the state of both photons in the Bell basis to decode the $j = 0(1)$ corresponding to the $|\Psi^+\rangle(|\Psi^-\rangle)$ result.
Now along the line of Wójcik’s eavesdropping scheme [25], we present detailedly the improved scheme, which never produces the identical results of the measurements performed by Bob and Alice in the control mode also but induces less eavesdropping losses than that in [25]. Let us first consider an ideal quantum channel (i.e., $\eta = 1$). Obviously, Eve has no access to the home photon but can manipulate the travel photon while it goes from Bob to Alice and back from Alice to Bob. Eve uses two auxiliary spatial modes $x, y$. She prepares a photon in the state $|0\rangle$ and lets the other one be an empty mode, e.g., in the state $|\text{vac}\rangle_x |0\rangle_y$. Accordingly, the state of the whole system including the entangled photon pair is

$$|\text{initial}\rangle = |\Psi^+_h\rangle |\text{vac}\rangle_x |0\rangle_y.$$  \hspace{1cm} (1)

When Bob sends the travel photon to Alice, Eve attacks the quantum channel by manipulating the travel photon through an unitary operation (referred as to be the $B-A$ attack hereafter) as follow,

$$T = N_{xy} C_{ytz} N_{ty} C_{txy} H_y,$$  \hspace{1cm} (2)

where $N$ represents the CNOT gate [25], $H$ is the Hadamard gate, and $C$ stands for the so-called three-mode CPBS gate [25], which is constructed by CNOT gates and a polarizing beam splitter transmitting (reflecting) photons in the state $|0\rangle(|1\rangle)$. When acting on the initial state, the $B-A$ attack transforms the whole system to the state $|B-A\rangle = T|\text{initial}\rangle$ of the form

$$|B-A\rangle = \frac{1}{2} |0\rangle_h (|\text{vac}\rangle_x |1\rangle_y + |1\rangle_x |0\rangle_y)$$

$$+ \frac{1}{2} |1\rangle_h (|0\rangle_x |\text{vac}\rangle_y + |0\rangle_x |\text{vac}\rangle_y).$$ \hspace{1cm} (3)

Suppose that Alice now switches to the control mode and measures the state of the mode $t$. According to equation 3, one can see that after the B-A attack Alice will detect no photon with the probability 1/4 or with the probability of 3/4 a photon whose state is perfectly anticorrelated with the state of the home photon. Therefore, the probability of eavesdropping detection based on the correlation observation equals zero. This point is completely same as that in Ref.[25]. Moreover, from equation 3 one can see that the $B-A$ attack in the present eavesdropping scheme also induces the eavesdropping losses. However, it is worthy to be mentioned that the eavesdropping losses level (25%) in the present scheme is only half of that (50%) in Ref.[25]. This implies that, comparing to the Wójcik’s eavesdropping scheme, in the present eavesdropping scheme, the domain in which Eve can attack all the transmitted bits is enlarged to be [0, 75%] from the [0, 50%] in Ref.[25]. Nonetheless, this does not certainly mean that, the insecurity upper bound of transmission efficiency presented by Wójcik can be pushed up, for now we still do not know the variation of the eavesdropping (legitimate) information gain. Let us now analyze the performance of the scheme in the case of Alice operating in the message mode. After Alice performs the $Z^j$ operation and sends the travel photon back to Bob, Eve performs her second attack (named as the $A-B$ attack hereafter) on the travel photon. The $A-B$ attack consists of the unitary operation $T^{-1}$. After the $A-B$ attack, the corresponding state of the whole system is

$$|A-B\rangle = T^{-1} Z^j |B-A\rangle$$
The final step of the eavesdropping scheme is a measurement of polarization performed on the $y$ photon, the result is denoted by $k$. The result of Bob’s Bell-state measurement on both photons is denoted by $m = 0(1)$ corresponding to the $|\Psi^+\rangle_{ht}$ state. Let $P(k, m|j)$ be the conditional probability of possible measurement outputs of $|A - B\rangle$ for a given value of $j$, then the only nonzero probabilities are,

$$P_1(0, 0|0) = 1, \quad P_1(0, 0|1) = P_1(1, 0|1) = P_1(1, 1|1) = 1/4. \quad (5)$$

Assume that in Alice’s secret messages, the occupation possibilities of the '0' and the '1' bits are $c_0$ and $1 - c_0$ respectively, then one can work out the mutual information between any two parties,

$$I_{AE} = I_{AB} = c_0 - \frac{1}{2}(1 - c_0) \log_2(1 - c_0) + (1 + c_0) \log_2(1 + c_0), \quad (6)$$

$$I_{BE} = -(1 + c_0) \log_2(1 + c_0) + \frac{1-c_0}{4} \log_2(1 - c_0) + \frac{1+3c_0}{4} \log_2(1 + 3c_0). \quad (7)$$

Specifically, when $c_0 = 1/2$, then

$$I_{AE} = I_{AB} = \frac{3}{4} \log_2 \frac{4}{3} \approx 0.311$$

$$I_{BE} = 1 - \frac{3}{2} \log_2 3 + \frac{5}{8} \log_2 5 \approx 0.074. \quad (8)$$

Same as Wójcik’s eavesdropping scheme, the present scheme is also not symmetric. If after the $A - B$ attack Eve performs an additional unitary operation $S = X_z Z_{t_y} X_t$ with the probability of $1/2$, where $X$ is an negation, then the asymmetry can be removed. If the $S$ is performed, the final state of the whole system is transformed into

$$|A - B\rangle^{(s)} = ST^{-1}Z_t^d |B - A\rangle$$

$$= \frac{1}{\sqrt{2}} ([0\rangle_k |1\rangle_z |j\rangle_y - |1\rangle_k |0\rangle_z |1\rangle_y]) \text{vac}_x$$

$$= \frac{1}{2} ([|\Psi^+\rangle_{ht} |j\rangle_y + |\Psi^-\rangle_{ht} |j\rangle_y - |\Psi^+\rangle_{ht} |1\rangle_y + |\Psi^-\rangle_{ht} |1\rangle_y]) \text{vac}_x. \quad (9)$$

Then the only nonzero $P(k, m|j)$’s are,

$$P_1^{(s)}(0, 0|0) = P_1^{(s)}(0, 1|0) = P_1^{(s)}(1, 0|0) = P_1^{(s)}(1, 1|0) = 1/4, \quad P_1^{(s)}(1, 1|1) = 1. \quad (10)$$

Considering the assumption that in Alice’s secret messages the occupation possibilities of the '0' and the '1' bits are $c_0$ and $1 - c_0$ respectively, one can work out the mutual information between any two parties,

$$I_{AE}^{(s)} = I_{AB}^{(s)} = 1 - c_0 - \frac{1}{2}[c_0 \log_2 c_0 + (2 - c_0) \log_2(2 - c_0)], \quad (11)$$

$$I_{BE}^{(s)} = -(2 - c_0) \log_2(2 - c_0) + \frac{c_0}{4} \log_2 c_0 + \frac{4 - 3c_0}{4} \log_2(4 - 3c_0). \quad (12)$$
Specifically, when $c_0 = 1/2$, then

\[
I^{(s)}_{AE} = I^{(s)}_{AB} = \frac{3}{4} \log_2 \frac{4}{3} \approx 0.311
\]

\[
I^{(s)}_{BE} = 1 - \frac{3}{2} \log_2 3 + \frac{5}{8} \log_2 5 \approx 0.074.
\]

Incidentally, it is easily found that, when $c_0 \neq 1/2$ (Note that $c_0 = 0$ or $c_0 = 1$ is meaningless), $I_{AE} \neq I^{(s)}_{AE}$, $I_{AB} \neq I^{(s)}_{AB}$ and $I_{BE} \neq I^{(s)}_{BE}$. Same as that in Ref.[25], later we only consider the case of $c_0 = 1/2$.

According to the viewpoint in Ref.[25], that is, since Eve knows exactly when each of the $S$ operations has been performed, the symmetrization procedure does not reduce the mutual information between Alice and Eve while it disturbs the communication between Alice and Eve in such a way the mutual information between Alice and Bob is reduced, in the present scheme after the $S$ operations the mutual information between Alice and Bob is also reduced to be $I_{AB} = \frac{4}{3} \log_2 3 - 1 \approx 0.189$.

By the way, one can easily find that in the present eavesdropping scheme the quantum bit error rate induced by the eavesdropping is also at the same level of $1/4$ as that in Ref.[25].

Thus far, we have improved Wójcik’s eavesdropping scheme. One can see that the improved scheme is almost same as Wójcik’s eavesdropping scheme except for the induced eavesdropping losses. Hence, all the discussions in Ref.[25] except for those related to the eavesdropping losses are also suitable for the present paper and we will not repeat them. Now let us discuss a very important property related with the eavesdropping losses. According to our above calculations of the mutual information, one can see that, when $\eta = 1$, comparing to Wójcik’s eavesdropping scheme, the eavesdropping (legitimate) information gain does not decrease (increase) with the decrease of the eavesdropping losses in the improved scheme. As mentioned before, in the improved scheme the $\eta$ domain in which Eve can attack all the transmitted bits is enlarged to be [0, 75%] from the [0, 50%] in Ref.[25]. In the $\eta$ domain of (50%, 75%), the eavesdropping information gain does not decrease in the improved scheme but in Wójcik’s eavesdropping scheme. Thus, in this sense, one can say that the less eavesdropping losses do induce more eavesdropping information gain. Moreover, in the improved scheme, when $\eta$ increases from 75%, though the mutual information between Alice and Eve (Bob) will decrease (increase) for Eve can only attack a fraction of the whole transmitted bits, the $I_{AE}$ is still able to exceed the $I_{AB}$ up to almost 80% transmission efficiency (cf. Fig.1).

In summary, we have improved Wójcik’s eavesdropping scheme on the Boström-Felbinger quantum communication protocol. The improved scheme only induces half of the eavesdropping losses in Wójcik’s scheme and accordingly in the $\eta$ domain of [0, 75%] Eve can attack all the transmitted bits. When $\eta = 1$ the eavesdropping (legitimate) information gain does not accordingly decrease (increase). Hence, in the $\eta$ domain of (50%,75%), the less eavesdropping losses do induce more eavesdropping information gain. Moreover, as for as the Boström-Felbinger protocol is concerned, the insecurity upper bound of the transmission efficiency has been pushed up from the 60% in Wójcik’s scheme to 80% in the present scheme.

Zhang thanks to Prof. Baiwen Li for his encouragement. This work is funded by the National Science Foundation of China under No.10304022.
FIG. 1: Mutual information between Alice and Eve (Bob) $I_{AE}(I_{AB})$ as a function of quantum channel transmission efficiency $\eta$. According to the figure 4 in Ref.[25], when $\eta$ increases from 75\%, $I_{AB}$ in this figure should increase not linearly but more slowly. We simplify it just because we only want to estimate an approximate upper bound.
[1] C. H. Bennett and G. Brassard, in *Proceedings of the IEEE International Conference on Computers, Systems and Signal Processings, Bangalore, India* (IEEE, New York, 1984), p175.
[2] N. Gisin, G. Ribordy, W. Tittel, and H. Zbinden, Rev. Mod. Phys. **74**, 145 (2002).
[3] A. K. Ekert, Phys. Rev. Lett. **67**, 661 (1991).
[4] C. H. Bennett, Phys. Rev. Lett. **68**, 3121 (1992).
[5] C. H. Bennett, G. Brassard, and N.D. Mermin, Phys. Rev. Lett. **68**, 557(1992).
[6] L. Goldenberg and L. Vaidman, Phys. Rev. Lett. **75**, 1239 (1995).
[7] B. Huttner, N. Imoto, N. Gisin, and T. Mor, Phys. Rev. A **51**, 1863 (1995).
[8] M. Koashi and N. Imoto, Phys. Rev. Lett. **79**, 2383 (1997).
[9] W. Y. Hwang, I. G. Koh, and Y. D. Han, Phys. Lett. A **244**, 489 (1998).
[10] P. Xue, C. F. Li, and G. C. Guo, Phys. Rev. A **65**, 022317 (2002).
[11] S. J. D. Phoenix, S. M. Barnett, P. D. Townsend, and K. J. Blow, J. Mod. Opt. **42**, 1155 (1995).
[12] H. Bechmann-Pasquinucci and N. Gisin, Phys. Rev. A **59**, 4238 (1999).
[13] A. Cabello, Phys. Rev. A **61**,052312 (2000); **64**, 024301 (2001).
[14] A. Cabello, Phys. Rev. Lett. **85**, 5635 (2000).
[15] G. P. Guo, C. F. Li, B. S. Shi, J. Li, and G. C. Guo, Phys. Rev. A **64**, 042301 (2001).
[16] G. L. Long and X. S. Liu, Phys. Rev. A **65**, 032302 (2002).
[17] F. G. Deng and G. L. Long, Phys. Rev. A **68**, 042315 (2003).
[18] J. W. Lee, E. K. Lee, Y. W. Chung, H. W. Lee, and J. Kim, Phys. Rev. A **68**, 012324 (2003).
[19] Daegene Song, Phys. Rev. A **69**, 034301(2004).
[20] X. B. Wang, Phys. Rev. Lett. **92**, 077902 (2004).
[21] A. Beige, B. G. Englert, C. Kurtsiefer, and H. Weinfurter, Acta Phys. Pol. A **101**, 357 (2002).
[22] Kim Bostrom and Timo Felbinger, Phys. Rev. Lett. **89**, 187902 (2002).
[23] F. G. Deng, G. L. Long, and X. S. Liu, Phys. Rev. A **68**, 042317 (2003).
[24] F. G. Deng and G. L. Long, Phys. Rev. A **x**, (2004).
[25] A. Wojcik, Phys. Rev. Lett. **90**, 157901 (2003).