On some random walks driven by spread-out measures

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Abstract
Let $G$ be a finitely generated group equipped with a symmetric generating $k$-tuple $S$. Let $| \cdot |$ and $V$ be the associated word length and volume growth function. Let $
u$ be a probability measure such that $
u(g) \simeq [(1 + |g|)^2V(|g|)]^{-1}$. We prove that if $G$ has polynomial volume growth then $
u^{(n)}(e) \simeq V(\sqrt{n \log n})^{-1}$. We also obtain assorted estimates for other spread-out probability measures.

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1 Introduction

This work is concerned with questions related to a number of recent studies where “stable-like” processes and random walks are considered. We focus on random walks on groups, mostly nilpotent groups and groups of polynomial volume growth, associated with various type of spread-out probability measures. Here, spread-out is used in a non-technical sense to convey the idea that these measures do not have finite support.

Given a probability measure $
u$ on a (finitely generated) group $G$, we consider the discrete time random walk $(X_n)_{n \geq 0}$ driven by $
u$ and started at $X_0 = e$. This means that $X_n = \xi_1 \cdots \xi_n$, $n \geq 1$, where $(\xi)^\infty$ is a i.i.d. sequence of $G$-random variables with common law $\nu$. The distribution of $X_n$ is the convolution power $\nu^{(n)}$. We also consider the associated continuous time random walk $X_t$ whose distribution is given by

$$p_t(g) = e^{-t} \sum_{n=0}^{\infty} \frac{t^n}{n!} \nu^{(n)}(g).$$

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This continuous time process will serve as a tool in the study of the discrete random walk driven by $\nu$, a technique that has been used by many authors before.

The question addressed in the present work is the following. Assuming good upper bounds on $\mu^{(n)}(e)$, under which circumstances can one prove matching lower bounds? Further, can one describe (in a certain sense) the region where, for a given $n$, $\nu^{(n)}(g) \simeq \nu^{(n)}(e)$? We provide answers for measures $\nu$ that are quite natural and for which well understood existing techniques are insufficient and/or need to be modified.

### 1.1 Main definitions

**Definition 1.1.** We say that $\| \cdot \| : G \to [0, \infty)$ is a norm on $G$ if $\|g\| = 0$ if and only if $g = e$ and, for all $g, h \in G$, $\|gh\| \leq \|g\| + \|h\|$. Given a norm $\| \cdot \|$, we say that $V(r) = \# \{ g \in G : \|g\| \leq r \}$ is the associated volume function.

The simplest and most common example is of a norm is provided by the word-length associated to a given finite symmetric set of generators. We will encounter other norms as well.

The properties studied in this work are the following.

**Definition 1.2.** Let $\mu$ be a symmetric probability measure on a group $G$. Let $\| \cdot \|$ be a norm with volume function $V$. Let $r : (0, \infty) \to (0, \infty), t \mapsto r(t)$, be a non-decreasing function. Let $(X_n)_0^\infty$ be the random walk on $G$ driven by $\mu$. We say that $\mu$ is $(\| \cdot \|, r)$-controlled if the following properties are satisfied:

1. For all $n$, $\mu^{(2n)}(e) \simeq V(r(n))^{-1}$.
2. For all $\epsilon > 0$ there exists $\gamma \in (0, \infty)$ such that

\[
P_e \left( \sup_{0 \leq k \leq n} \{ \|X_k\| \} \geq \gamma r(n) \right) \leq \epsilon.
\]

The first of these two properties is rather straightforward and self-explanatory. It provides a two-sided estimate for the probability of return of the random walk. In more general contexts, this property is also known as a two-sided “on-diagonal” bound. The second property is related to the first in so far as it actually easily implies the lower bound $\mu^{(2n)}(e) \geq V(cr(n))^{-1}$. It also provides a weak control of the behavior of $\mu^{(n)}(g)$ away from the neutral element $e$.

**Definition 1.3.** Let $\mu$ be a symmetric probability measure on a group $G$. Let $\| \cdot \|$ be a norm with volume function $V$. Let $r : (0, \infty) \to (0, \infty), t \mapsto r(t)$, be an increasing continuous function with inverse $\rho$. Let $(X_n)_0^\infty$ denote the random walk on $G$ driven by $\mu$. We say that $\mu$ is strongly $(\| \cdot \|, r)$-controlled if the following properties are satisfied:

1. There exists $C \in (0, \infty)$ and, for any $\kappa > 0$, there exists $c(\kappa) > 0$ such that, for all $n \geq 1$ and $g$ with $\|g\| \leq kr(n)$,

\[
c(\kappa)V(r(n)))^{-1} \leq \mu^{(2n)}(g) \leq CV(r(n))^{-1}.
\]
2. There exists $\epsilon, \gamma_1, \gamma_2 \in (0, \infty)$, $\gamma_2 \geq 1$, such that, for all $n, \tau$ such that $\frac{1}{2} \rho(\tau/\gamma_1) \leq n \leq \rho(\tau/\gamma_1)$

$$\inf_{x: \|x\| \leq \tau} \left\{ P_x \left( \sup_{0 \leq k \leq n} \{|X_k|, \|X_n\| \leq \tau \} \right) \right\} \geq \epsilon. \tag{1.1}$$

Strong control implies the following useful estimate. The last section of this paper gives an application of this estimate to random walks on wreath products.

**Proposition 1.4.** Assume that $r$ is continuous increasing with inverse $\rho$ and that the symmetric probability measure $\mu$ is strongly $(\| \cdot \|, r)$-controlled. Then, for any $n$ and $\tau$ such that $\gamma_1 r(2n) \geq \tau$, we have

$$\inf_{x: \|x\| \leq \tau} \left\{ P_x \left( \sup_{0 \leq k \leq n} \{|X_k|, \|X_n\| \leq \tau \} \right) \right\} \geq \epsilon^{1+2n/\rho(\tau/\gamma_1)}. \tag{1.2}$$

**Proof.** By induction on $\ell \geq 1$ such that $1 \leq 2n/\rho(\tau/\gamma_1) < (\ell + 1)$, we are going to prove that

$$\inf_{x: \|x\| \leq \tau} \left\{ P_x \left( \sup_{0 \leq k \leq n} \{|X_k|, \|X_n\| \leq \tau \} \right) \right\} \geq \epsilon^{1+\ell}.$$ 

This easily yields the desired result. For $\ell = 1$, the inequality follows from the strong control assumption. Assume the property holds for some $\ell \geq 1$. Let $n, \tau$ be such that $(\ell + 1) \leq 2n/\rho(\tau/\gamma_1) < (\ell + 2)$. Choose $n'$ such that $n - n' = \lfloor \rho(\tau/\gamma_1)/2 \rfloor$ and note that $2n' \in [1, (\ell + 1)\rho(\tau/\gamma_1))$. Write $Z_n = \sup_{k \leq n}\{||X_k||\}$ and, for any $x$ such that $\|x\| \leq \tau$,

$$P_x (Z_n \leq \gamma_2 \tau; \|X_n\| \leq \tau) \geq P_x (Z_n \leq \gamma_2 \tau; \|X_{n'}\| \leq \tau; \|X_n\| \leq \tau) \geq P_x \left( Z_{n'} \leq \gamma_2 \tau; \|X_{n'}\| \leq \tau; \sup_{n' \leq k \leq n} \{|X_k|\} \leq \gamma_2 \tau; \|X_n\| \leq \tau \right) = E_x (1\{Z_{n'} \leq \gamma_2 \tau; \|X_{n'}\| \leq \tau\} P_{X_{n'}} (Z_{n-n'} \leq \gamma_2 \tau; \|X_{n-n'}\| \leq \tau)) \geq \epsilon P_x (Z_{n'} \leq \gamma_2 \tau; \|X_{n'}\| \leq \tau) \geq \epsilon^{2+\ell}.$$ 

This gives the desired property for $\ell + 1$. \hfill \Box

### 1.2 Word-length radial measures

Let the group $G$ be equipped with a generating $k$-tuple

$$S = (s_1, \ldots, s_k)$$

and the associated finite symmetric set of generators $S = \{s_1^{\pm 1}, \ldots, s_k^{\pm 1}\}$. Let $|g|$ be the associated word length, that is, the minimal $k$ such that $g = u_1 \ldots u_k$ with $u_i \in S$, $1 \leq i \leq k$. By definition, the identity element $e$ has length 0.
Hence, $| \cdot |$ is a norm and $(x,y) \mapsto |x^{-1}y|$ is a left-invariant distance function on $G$. Let

$$V_S(r) = \# \{ g : |g| \leq r \}$$

be the volume of the ball of radius $r$. We say that $G$ has polynomial volume growth of degree $D$ if $V_S(r) \simeq r^D$ in the sense that the ratio $V_S(r)/r^D$ is bounded away from 0 and $\infty$ for $r \geq 1$. Finitely generated nilpotent groups have polynomial volume growth and, by Gromov’s theorem, any finitely generated group with polynomial volume growth contains a nilpotent subgroup of finite index. More precisely, any finitely generated group $G$ such that there exist constants $C,A$ and a sequence $n_k$ with $V(n_k) \leq C n_k^A$ contains a nilpotent subgroup of finite index and thus has polynomial volume growth of degree $D$ for some integer $D$. See, e.g., [8].

**Example 1.1.** Let $G$ be equipped with a word-length function $| \cdot |$ associated with a symmetric finite generating subset. Assume that $G$ has polynomial volume growth. The main results of [10] imply that, for any symmetric probability measure $\mu$ with finite generating support, $\mu$ is strongly $(| \cdot |, t \mapsto t)$-controlled. The main results of [1, 2] show that, if $\nu_\beta$ is symmetric and satisfies

$$\nu_\beta(g) \simeq [(1 + |g|)^2 V(|g|)]^{-1}$$

with $\beta \in (0,2)$, then $\nu_\beta$ is strongly $(| \cdot |, t \mapsto t^{1/\beta})$-controlled. See also [3, 14].

One example that motivates the present work is the case of the measure

$$\nu_2(g) = \frac{c}{(1 + |g|)^2 V(|g|)}.
$$

Can one provide good estimates for $\nu_2^{(n)}(e)$ on groups of polynomial volume growth? The following theorem gives a very satisfactory answer to this question and covers not only this particular example but the full range of cases passing through the classical threshold corresponding to the second moment condition.

**Theorem 1.5.** Let $G$ be equipped with a word-length function $| \cdot |$ associated with a symmetric finite generating subset. Let $V$ be the associated volume function and assume that $G$ has polynomial volume growth. Let $\phi : [0, \infty) \rightarrow [1, \infty)$ be a continuous regularly varying function of positive index. Let $r$ be the inverse function of

$$t \mapsto t^2 / \int_0^t s ds / \phi(s).
$$

Let $\nu_\phi$ be a symmetric probability measure such that

$$\nu_\phi(g) \simeq \frac{1}{\phi(|g|)V(|g|)}. \tag{1.3}$$

Then $\nu_\phi$ is strongly $(| \cdot |, r)$-controlled.

**Example 1.2.** Assume that $\phi(t) = (1 + t)^\beta \ell(t)$ with $\ell$ positive continuous and slowly varying (we refer the reader to [4, Chap. I] for the definition and basic properties of slowly and regularly varying functions. The scaling function $r$ of Theorem 1.5 can be described more explicitly as follows.

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• If $\beta > 2$, $r(t) \simeq t^{1/2}$.

• If $\beta < 2$, we have $t^2/\int_0^t \frac{\sigma(s)}{\phi(s)} \simeq c_\phi \phi(t)$ and $r$ is essentially the inverse of $\phi$, namely,
  
  $$r(t) \simeq \ell^{1/\beta} \ell_{\#}^{1/\beta}(t^3)$$

where $\ell_{\#}$ is the de Bruijn conjugate of $\ell$. See [4, Prop. 1.5.15]. For example, if $\ell$ has the property that $\ell(t^a) \simeq \ell(t)$ for all $a > 0$ then $\ell_{\#} \simeq 1/\ell$.

• The case $\beta = 2$ is more subtle and the proof is more difficult. The function $\psi : t \mapsto \int_0^t s \sigma(s) \phi(s) \simeq c \phi \phi(t)$ and $r$ is essentially the inverse of $\varphi$, namely,
  
  $$r(t) \simeq \frac{t^{1/\beta}}{\ell_{\#}^{1/\beta}}$$

where $\ell_{\#}$ is the de Bruijn conjugate of $\ell$. See [4, Prop. 1.5.15]. For example, if $\ell \equiv 1$, we have $\psi(t) \simeq \log t$ and $r(t) \simeq (t \log t)^{1/2}$. When $\ell(t) = (\log t)^{\gamma}$ with $\gamma \in \mathbb{R}$ then
  
  - If $\gamma > 1$, $\psi(t) \simeq 1$ and $r(t) \simeq t^{1/2}$;
  - If $\gamma = 1$, $\psi(t) \simeq \log \log t$ and $r(t) \simeq (t \log t)^{1/2}$;
  - If $\gamma < 1$, $\psi(t) \simeq (\log t)^{1-\gamma}$ and $r(t) \simeq (t \log t)^{1-\gamma^{1/2}}$.

1.3 Measures supported by the powers of the generators

In the critical case when $\phi$ is regularly varying of index 2 and $\nu_{\phi}$ has infinite second moment (i.e., $\sum_{g} |g|^2 \nu_{\phi}(g) = \infty$), the proof of Theorem 1.5 makes essential use of some of the results from [18] which are related to variations on the following class of examples. Recall that $G$ is equipped with the generating $k$-tuple $S = (s_1, \ldots, s_k)$. For any $k$-tuple $a = (\alpha_1, \ldots, \alpha_k) \in (0, \infty)^k$, and consider the probability measure $\mu_{S,a}$ supported on the powers of the generators $s_1, \ldots, s_k$ and defined by

\[
\mu_{S,a}(g) = \frac{1}{k} \sum_{m \in \mathbb{Z}} \sum_{i=1}^k \frac{\kappa_i}{(1 + |m|)^{1+\alpha_i}} 1_{s_i^m}(g). \tag{1.4}
\]

Set $\bar{\alpha}_i = \min\{\alpha_i, 2\}$ and $\alpha_* = \max\{\bar{\alpha}_i, 1 \leq i \leq k\}$.

Define

\[
\|g\|_{S,a} = \min \left\{ r : g = \prod_{j=1}^m s_i^j : \epsilon_j = \pm 1, \#\{j : i_j = i\} \leq r^{\alpha_*/\bar{\alpha}_i} \right\}. \tag{1.5}
\]

Note that $g \mapsto \|g\|_{S,a} : G \to [0, \infty)$ is a norm. Consider also the measure

\[
\nu_{S,a,\beta}(g) = \frac{c(G, a, \beta)}{(1 + \|g\|_{S,a})^{2\beta} V_{S,a}(\|g\|_{S,a})} \tag{1.6}
\]

with $\beta \in (0, 2)$. 

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Under the assumption that $G$ is nilpotent and $\{s_i : \alpha_i \in (0, 2)\}$ generates a subgroup of finite index in $G$, it is proved in [18] that there exists a positive real $D_{S,a}$ such that

$$Q_{S,a}(r) = \#\{\|g\|_{S,a} \leq r^{1/\alpha_*}\} \sim r^{D_{S,a}}$$

and

$$\mu_{a}^{(n)}(e) \leq C_{S,a} n^{-D_{S,a}}, \quad \nu_{S,a,\beta}^{(n)}(e) \leq C_{S,a,\beta} n^{-\alpha_* D(S,a)/\beta}.$$

Here we prove the following complementary result.

**Theorem 1.6.** Let $G$ be a finitely generated nilpotent group equipped with a generating $k$-tuple $S = (s_1, \ldots, s_k)$. Referring to the notation introduced above, fix $a \in (0, \infty)^k$ and assume that $\{s_i : \alpha_i \in (0, 2)\}$ generates a subgroup of finite index in $G$.

- The probability measure $\mu_{S,a}$ is strongly $(\| \cdot \|_{S,a}, t \mapsto t^{1/\alpha_*})$-controlled.
- For any $\beta \in (0, 2)$, $\nu_{S,a,\beta}$ is strongly $(\| \cdot \|_{S,a}, t \mapsto t^{1/\beta})$-controlled.

**Remark 1.7.** In [18], a detailed analysis of the sub-additive function $\| \cdot \|_{S,a}$ and the associated geometry is given. This analysis is key to the above result and to its proper understanding. For instance, it is important to understand that the parameter $\alpha_*$ is not necessarily a significant parameter. It is the quantity $\| \cdot \|_{S,a}^{\alpha_*}$ that is the important expression. Indeed, for any given nilpotent group $G$, [18] describes conditions on two pairs of tuples $(S, a), (S', a')$,

$$S = (s_i)_{i=1}^k \in G^k, a = (\alpha_i)_{i=1}^k \in (0, \infty)^k, S' = (s'_i)_{i=1}^{k'} \in G^{k'}, a' = (\alpha'_i)_{i=1}^{k'} \in (0, \infty)^{k'},$$

such that $\| \cdot \|_{S,a}^{\alpha_*} \sim \| \cdot \|_{S', a'}^{\alpha_*}$. Since the geometry $\| g \|_{S,a}$ is studied and described rather explicitly in [18], the above results give rather concrete controls of the random walks driven with $\mu_{S,a}$ or $\nu_{S,a,\beta}$.

On the one hand, in the case of the measures $\nu_{S,a,\beta}$ and with much more work, it is possible to improve upon the statement of Theorem 1.6 and obtain two side pointwise bound on $\nu_{S,a,\beta}$. Indeed, based on the results of [18], it is proved in [14] that, for all $g \in G$ and $n \geq 1$,

$$\nu_{S,a,\beta}^{(n)}(g) \sim \frac{n}{(n^{1/\beta} + \|g\|_{S,a})^{\alpha_* D_{S,a}/\beta}} \sim \min \left\{ \frac{1}{n^{\alpha_* D_{S,a}/\beta}}, \frac{n}{\|g\|_{S,a}^{\alpha_* D_{S,a}/\beta}} \right\}.$$

On the other hand, in the case of the measures $\mu_{S,a}$, Theorem 1.6 provides the most detailed result available at this time. Indeed, available techniques do not seem to be adequate to provide a sharp two-sided bound for $\mu_{S,a}^{(n)}(g)$. 

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1.4 A short guide

Section 2 is based on well-known variations of the celebrated Davies off-diagonal upper bound technique. Our key observation is that, even in cases where we do not expect to obtain full off-diagonal upper bounds, Davies technique provides enough information to prove control in the sense of Definition 1.2.

Section 3 describes the notion of pointwise pseudo-Poincaré inequality (a variation on the idea introduced in [6]) and shows how, with the help of the underlying group structure, a pseudo-Poincaré inequality allow us to upgrade control to strong control.

Section 4 applies the earlier results to a family of probability measures and random walks introduced in [18]. These measures are supported on the powers of the given generators. They provide examples for which no good off-diagonal upper bounds are known at this time. Nevertheless, the results developed here apply and capture useful properties of the associated random walks.

Section 5 is concerned with radial type measures where radial refers to a given norm on the group $G$. The simplest and most interesting case is when this norm is taken to be the usual word-length associated with a finite symmetric set of generators and, in this case, we prove Theorem 1.5.

Section 6 describes the applications to a class of random walks on wreath products. The notion of strong control (on the base group of the wreath product) leads to lower bounds on the probability of return on the wreath product.

2 Davies method, tightness and control

2.1 Davies method for the truncated process

In this section, we review how Davies’ method applies to the continuous time process associated with truncated jumping kernels. We follow [13, Section 5] rather closely even so our setup is somewhat different. The first paper treating jump kernels by Davies method is [5].

Throughout this section $G$ is a discrete group equipped with its counting measure. Fix a norm $g \mapsto \|g\|$ with volume function $V$ and set $d(x,y) = \|x^{-1}y\|$. Note that $d$ is a distance function on $G$. Consider the left-invariant symmetric jumping kernel

$$J(x,y) = \nu(x^{-1}y)$$

associated to a given symmetric probability measure $\nu$ on $G$. For $R > 0$, define

$$\delta_R := \sum_{\|x\| > R} \nu(x) \quad \text{and} \quad G(R) = \sum_{\|x\| \leq R} \|x\|^2 \nu(x),$$

and

$$J_R(x,y) := J(x,y)1_{\{d(x,y) \leq R\}}, \quad J''_R(x,y) := J(x,y)1_{\{d(x,y) > R\}}.$$
Denote by \( p(t, x, y) \) and \( p_R(t, x, y) \) the transition densities of the continuous time processes associated to \( J \) and \( J_R \), respectively. In particular,

\[
p(t, x, y) = p_t(x^{-1}y) = e^{-t} \sum_{n=0}^{\infty} \frac{t^n}{n!} \nu^{(n)}(x^{-1}y).
\]

Let

\[
E(f, f) = E_\nu(f, f) = \frac{1}{2} \sum_{x, y} (f(x) - f(y))^2 J(x, y)
\]

(2.2)

be the corresponding Dirichlet form and set also

\[
E_R(f, f) = \frac{1}{2} \sum_{x, y} (f(x) - f(y))^2 J_R(x, y).
\]

Note that

\[
E(f, f) - E_R(f, f) = \frac{1}{2} \sum_{x, y: d(x, y) > R} |f(x) - f(y)|^2 J(x, y)
\]

\[
\leq \sum_{x, y: d(x, y) > R} (f(x)^2 + f(y)^2) J(x, y) \leq 2 \|f\|^2 \delta_R
\]

Consider the on-diagonal upper bound given by

\[
\forall x \in G, \ t > 0, \ p_t(e) \leq m(t),
\]

(2.3)

where \( m : [0, \infty) \to [0, \infty) \) is continuous regularly varying function of negative index at infinity and \( m(0) < \infty \). Since the function \( t \mapsto m(t) \) may present a slowly varying factor, we follow [13]. The starting point is the log-Sobolev inequality

\[
\sum f^2 \log f \leq \epsilon E_R(f, f) + (2\epsilon \delta_R + \log m(\epsilon)) \|f\|^2 + \|f\|^2 \log \|f\|_2
\]

(2.4)

with \( \epsilon > 0 \) which follows from [2,3] by [7, Theorem 2.2.3]. The following technical proposition is the key to most of the results obtained in later sections.

**Proposition 2.1.** Assume that the on-diagonal upper bound (2.3) holds with \( m \) regularly varying of negative index. Then there is a constant \( C \) such that, for all \( R, t > 0 \) and \( x \in G \) we have

\[
p_R(t, e, x) \leq C e^{4\delta_R t} m(t) \left( \frac{t}{R^2/\mathcal{G}(R)} \right)^{\|x\|/3R}.
\]

Remark 2.2. The bound in this proposition is better than the uniform bound \( p_R(t, e, x) \leq C e^{4\delta_R t} m(t) \) only when \( t < R^2/\mathcal{G}(R) \).
Proof. It suffices to consider the case $t < R^2/\mathcal{G}(R)$. Starting with (2.4), we apply Davies method, as described in [13, Section 5.1] to estimate $p_R(t,e,x)$. Let

$$\Lambda_R(\psi) = \max \{ \| e^{-2\psi} \Gamma_R(e^\psi,e^\psi) \|_\infty, \| e^\psi \Gamma_R(e^{-\psi},e^{-\psi}) \|_\infty \}$$

with

$$\Gamma_R(\psi)(x) = \sum_y (|\psi(x) - \psi(y)|^2 J_R(x,y)).$$

then by [13, Corollary 5.3],

$$p_R(t,e,x) \leq Cm(t) \exp \left(4 \delta_R t + 72 \Lambda_R(\psi)^2 t - \psi(y) + \psi(x)\right).$$

Consider the case $x = x_0$ and $y = e$. For $\lambda > 0$, set $\psi(z) = \lambda (\|x_0\| - \|z\|)^+$ and write

$$e^{-2\psi(z)} \Gamma_R(e^\psi,e^\psi)(z) = \sum_y (e^{\psi(z)} - \psi(y))^2 J_R(z,y) \leq e^{2\lambda R} \sum_y (\psi(z) - \psi(y))^2 J_R(z,y) \leq \lambda^2 e^{2\lambda R} \sum_{\|y\| \leq R} \|y\|^2 d\nu \leq R^{-2} e^{3\lambda R} \mathcal{G}(R).$$

Since $\psi(e) = \lambda \|x_0\|$, we obtain

$$p_R(t,e,x_0) \leq Cm(t) \exp \left(4 \delta_R t + 72 t R^{-2} e^{3\lambda R} \mathcal{G}(R) - \lambda \|x_0\| \right).$$

Since $t < R^2/\mathcal{G}(R)$, we can set

$$\lambda = \frac{1}{3R} \log \frac{R^2}{t \mathcal{G}(R)}$$

so that the second term $72 t R^{-2} e^{3\lambda R} \mathcal{G}(R)$ is a constant. This yields the stated upper bound. \qed

2.2 Control

Meyer’s construction is a useful technique to construct the process $X^R_s$ by adding big jumps to $X^R_s$. See, e.g., [12] and [2, Lemma 3.1]. In this section, we combine the off-diagonal upper bound in Proposition 2.1 with Meyer’s construction to derive control type results for the process with jumping kernel $J$. Our goal is to show that there is a certain choice of continuous increasing function $r(t)$, for any $\varepsilon > 0$, there exists constant $\gamma > 1$ such that

$$P_e \left( \sup_{s \leq t} \|X_s\| \geq \gamma r(t) \right) \leq \varepsilon.$$

Let $X^R_s$ denote the process with truncated kernel $J^R$. It follows from Meyer’s construction that (see [12] and [2, Lemma 3.1])

$$P_e( X_s \neq X^R_s \text{ for some } s \leq t) \leq t \delta_R.$$

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For any $r > 0$, $\gamma > 1$, both to be specified later, we have

$$
P_e \left( \sup_{s \leq t} \| X_s \| \geq \gamma r \right)
\leq P_e \left( \sup_{s \leq t} \| X_s \| \geq \gamma r \right) + P_e \left( X_s \neq X_s^R \text{ for some } s \leq t \right)
\leq P_e \left( \sup_{s \leq t} \| X_s \| \geq \gamma r \right) + t\delta_R
\leq 2 \sup_{s \leq t} \left\{ P_e \left( \| X_s \| \geq \frac{\gamma}{2} r \right) \right\} + t\delta_R \tag{2.5}
$$

This will be helpful in deriving the following result.

**Proposition 2.3.** Assume that for all $\rho > 0$, $V(2\rho) \leq C_V V(\rho)$. Assume also that $\nu$ is such that (2.3) holds where $m$ is regularly varying of negative index.

For $\epsilon > 0$, fix a function $R(t)$ such that

$$2t\delta_R(t) < \epsilon \text{ and } \frac{t}{R(t)^2/G(R(t))} < e^{-1}.$$

Let $r(t) \geq R(t)$ be a positive continuous increasing function such that

$$\sup_{t>0} \left\{ m(t)V(r(t))e^{-r(t)/6R(t)} \right\} < \infty.$$

Then, for any $\epsilon > 0$ there exists a constant $\gamma \geq 1$ such that

$$P_e \left( \sup_{s \leq t} \| X_s \| \geq \gamma r(t) \right) < \epsilon$$

In particular, we have

$$p(t, e, e) \geq \frac{1 - \epsilon}{V(\gamma r(t))}.$$

If, in addition $V(r(t)) \simeq m(t)$, then the measure $\nu$ is $(\| \cdot \|, r)$-controlled in continuous time.

**Proof.** Proposition 2.1 implies that for $s \leq t$,

$$p_R(s, e, x) \leq Cm(s) \left( \frac{R^2}{G(R)} \right)^{\frac{s}{7} \| x \| / 3R} \leq Cm(s) \left( \frac{t}{R^2/G(R)} \right)^{\frac{t}{7} \| x \| / 3R}.$$

Fix $R = R(t)$, $r = r(t) \geq R$, decompose $\{ x : \| x \| \geq \frac{\gamma}{2} r \}$ into dyadic annuli.
\{x : \|x\| \simeq 2^i \gamma r\} and write
\[
P_e \left( \|X^R_s\| \geq \frac{\gamma r}{2} \right)
\leq C \sum_{i=0}^{\infty} m(s) \left( \frac{s}{t} \right)^{2^{i-1} - \gamma/3} e^{-2^{i-1} \gamma r/3R} V(2^i \gamma r(t))
\]
\[
= C m(t) V(\gamma r) \sum_{i=0}^{\infty} \frac{m(s)}{m(t)} \left( \frac{s}{t} \right)^{2^{i-1} - \gamma/3} e^{-2^{i-1} \gamma r/3R} \left( \frac{V(2^i \gamma r)}{V(\gamma r)} \right).
\]

Let \(C_{VD}\) denotes the volume doubling constant of \((G,d)\), then
\[
V(\gamma r) \leq C_{1+\log \gamma} V(r), \quad \frac{V(2^i \gamma r)}{V(\gamma r)} \leq C_{i_{VD}}.
\]
Recall that \(m(t)\) is a regularly varying function with negative index. Hence, for \(\gamma\) large enough, we have
\[
M = \sup_{0<s \leq t, i \in \mathbb{N}} \left\{ \frac{m(s)}{m(t)} \left( \frac{s}{t} \right)^{2^{i-1} - \gamma/3} \right\} < \infty.
\]
Therefore
\[
P_e \left( \|X^R_s\| \geq \frac{\gamma r}{2} \right) \leq C_1 m(t) V(r) e^{-r/6R} \sum_{i=0}^{\infty} e^{-2^i \gamma/12} C_{i_{VD}}
\]
By assumption, \(r = r(t)\) and \(R = R(t)\) satisfy
\[
\sup_{t>0} \left\{ m(t) V(r(t)) e^{-r(t)/3R(t)} \right\} < \infty.
\]
It follows that for \(\gamma\) sufficiently large, we have
\[
P_e \left( \|X^R_s\| \geq \frac{\gamma r}{2} r(t) \right) < \frac{\varepsilon}{4}.
\]
Plugging this estimate into (2.5), we obtain \(P_e \left( \sup_{s \leq t} \|X_s\| \geq \gamma r(t) \right) < \varepsilon. \)

**Corollary 2.4.** Under the hypotheses of Proposition 2.3, for any \(\varepsilon > 0\) there exists \(\gamma > 0\) such that
\[
P_e \left( \sup_{s \leq t} \|X_s\| \leq 2\gamma r(t), \|X_t\| \leq \gamma r(t) \right) \geq 1 - \varepsilon.
\]
**Proof.** Write
\[
P_e \left( \sup_{s \leq t} \|X_s\| \leq 2\gamma r(t), \|X_t\| \leq \gamma r(t) \right)
\]
\[
= P_e (\|X_t\| \leq \gamma r(t)) - P_e \left( \sup_{s \leq t} \|X_s\| \geq 2\gamma r(t), \|X_t\| \leq \gamma r(t) \right)
\]
\[
\geq 1 - 2P_e \left( \sup_{s \leq t} \|X_s\| \geq \gamma r(t) \right).
\]
Note that we have used the fact that, because of space homogeneity (i.e., group invariance), $X_t$ cannot escape to infinity in finite time.

Remark 2.5. The conclusions of Proposition 2.3 and Corollary 2.4 apply to the associated discrete time random walk. To see this, fix a regularly varying function $m$ and note that (up to changing $m$ to $cm$ for some constant $c$), (2.3) is equivalent to $\nu(2^n)(e) \leq m(n)$. Further, it is easy to control the difference between $P_e(\|X_t\| \geq r)$ and $P_e(\|X_n\| \geq r)$ with $n = \lfloor t \rfloor$ as long as $n$ is large enough. It follows that the proof above applies the discrete random walk result as well.

3 Pseudo-Poincaré inequality and strong control

3.1 Pseudo-Poincaré inequality

With some work, the results of the previous section can be extended to the more general context of graphs and discrete spaces. The results presented below make a more significant use of the underlying group structure.

Definition 3.1. Let $G$ be discrete group equipped with a symmetric probability measure $\nu$, a sub-additive function $\| \cdot \|$ and a positive continuous increasing function $r$ with inverse $\rho$. We say that $\nu$ satisfies a pointwise ($\| \cdot \|, r$)-pseudo-Poincaré inequality if, for any $f$ with finite support on $G$,

$$\forall g \in G, \sum_{x \in G} |f(xg) - f(x)|^2 \leq C\rho(\|g\|)E_\nu(f,f) \quad (3.1)$$

Here $E_\nu$ is the Dirichlet form of $\nu$ defined at (2.2).

Theorem 3.2. Assume that $(G, \| \cdot \|)$ is such that $V$ is doubling. Let $\nu$ be a symmetric probability measure such that $\nu(e) > 0$. Assume that $r$ is a positive doubling continuous increasing function such that

$$\nu(2^n)(e) \simeq V(r(n))^{-1}.$$

Assume further that $\nu$ satisfies the ($\| \cdot \|, r$)-pseudo-Poincaré inequality. Then there exists $\eta > 0$ such that for all $n$ and $g$ with $\|g\| \leq \eta r(n)$ we have

$$\nu(n)(g) \simeq V(r(n))^{-1}.$$

Proof. The hypothesis (3.1) and the argument of [10, Theorem 4.2] gives

$$|\nu(2^n+N)(x) - \nu(2^n+N)(e)| \leq C \left( \frac{\rho(\|x\|)}{N} \right)^{1/2} \nu(2^n)(e).$$

Fix $x$ and $n$ such that $\rho(\|x\|) \leq \eta n$ and use the above inequality with $N = 2n$ to obtain

$$\nu(4n)(x) \geq \left( 1 - (2C^n\eta)^{1/2} \right) \nu(4n)(e).$$
Hence, we can choose $\eta > 0$ such that

$$\nu^{(4n)}(x) \geq c\nu^{(4n)}(e).$$

Since $\nu(e) > 0$, this also holds for $4n + i, i = 1, 2, 3$, at the cost of changing the value of the positive constant $c$.

\[ \square \]

### 3.2 Strong control

**Definition 3.3.** We say that $\| \cdot \|$ is well-connected if there exists $b \in (0, \infty)$ such that, for any $r > 0$ and $x \in G$ with $\| x \| \leq r$ there exists a finite sequence of points $(x_0)^N \in G$ with $\| x_i \| \leq 2r, \| x_i^{-1}x_{i+1} \| \leq b$, $x_0 = e$ and $x_N = x$.

Note that in this definition, the number $N$ of points in the sequence is finite but that it depend on $x$ and no upper bound in terms of $r$ is required.

**Lemma 3.4.** Assume that $\| \cdot \|$ is well-connected and $V$ is doubling. Then for any fixed $\epsilon > 0$ there exists $M_\epsilon$ such that for any $r \geq 8b/\epsilon$ and any $\| x \| \leq r$ we can find $(z_i)^M, z_0 = e, z_M = x, M \leq M_\epsilon$, such that $\| z_i^{-1}z_{i+1} \| \leq \epsilon r$.

**Proof.** Let $\{ y_i : 1 \leq i \leq M' \}$ be a maximal $\epsilon r/4$-separated set of points in $B(e, 2r) = \{ \| g \| \leq 2r \}$. The ball $B_i = \{ y_i, \epsilon r/9 \}$ are disjoint and have volume $V(\epsilon r/9)$ comparable to $V(2r)$. Hence $M' \leq M'_{\epsilon}$ for some finite $M'_{\epsilon}$ independent of $r$. The union of the balls $B'_i = \{ y_i, \epsilon r/4 \}$ covers $B(2r)$ (otherwise, $\{ y_i : 1 \leq i \leq M' \}$ would not be maximal). In particular, these balls cover the sequence $(x_i)^N$ and we can extract a sequence $B'_i = B'_j, 1 \leq i \leq M \leq M'_{\epsilon}$ such that $B'_i \ni e, B'_M \ni x$ and

$$\inf \{ \| h^{-1}g \| : h \in B'_i, g \in B'_{i+1} \} \leq b.$$ 

Set $z_0 = e, z_i = y_j, 1 \leq i \leq N, z_{N+1} = z$. Then $\| z_{i-1}^{-1}z_i \| \leq 2\epsilon r/4 + b \leq \epsilon r$ as desired.

\[ \square \]

**Proposition 3.5.** Assume that the norm $\| \cdot \|$ is such that $V$ is doubling and $\| \cdot \|$ is well-connected. Let $r$ be a positive continuous increasing doubling function. Let $\nu$ be a symmetric probability measure that is $(\| \cdot \|, r)$-controlled and satisfies $\nu(\epsilon) > 0$ and a pointwise $(\| \cdot \|, r)$-pseudo-Poincaré inequality. Then $\nu$ is also strongly $(\| \cdot \|, r)$-controlled.

**Proof.** First, we show that for any $\kappa > 0$ there exists $c_\kappa > 0$ such that $\| x \| \leq \kappa r(\nu)$ implies

$$\nu^{(\kappa)}(x) \geq c V(r(\nu))^{-1}.$$ 

By Theorem 3.2 there exists $\eta$ such that $\nu^{(\kappa)}(x) \geq c_\kappa V(r(\nu))^{-1}$ for all $\| x \| \leq \eta r(\nu)$. By Lemma 3.4 for any fixed $\kappa$ there exists $M_\kappa$ such that for any $\| x \| \leq \kappa r(\nu)$ we can find $(z_i)^M, z_0 = e, z_M = x, M \leq M_\kappa$, such that $\| z_i^{-1}z_{i+1} \| \leq \eta r(\nu)/4$. Write $B_i = \{ \| z_i^{-1}g \| \leq \eta r(\nu)/4 \}$ and

$$\nu^{(\kappa M)}(x) \geq \sum_{(y_1, \ldots, y_M) \in B_i} \nu^{(\kappa)}(y_1) \cdot \nu^{(\kappa)}(y_{i+1}) \cdots \nu^{(\kappa)}(y_M^{-1}x) \geq c_1 M^{1+1} V(\eta r(\nu)/4)^M V(r(\nu))^{-M-1} \approx c_1' V(r(\nu))^{-1}.$$ 

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Since $\nu(e) > 0$, this shows that $\|x\| \leq \kappa r(n)$ implies $\nu^{(n)}(x) \geq cV(r(n))^{-1}$ as stated. In particular, for any fixed $\kappa$, there exists $\epsilon > 0$ such that for any $x, n$ with $\kappa r(n) \leq \tau$ and $\|x\| \leq \tau$,

$$P_x(\|X_n\| \leq \tau) \geq \epsilon.$$ 

Now, fix $\gamma_1 \in (1, \infty)$. Let $\epsilon_0 > 0$ be such that, for any $x, n, \tau$ with $\|x\| \leq \tau \leq \gamma_1 r(2n)$, we have $P_x(\|X_n\| \leq \tau) \geq \epsilon_0$. Let $\gamma \geq 1$ be given by Definition 1.2 so that $P_e(\sup_{k \leq n} \{\|X_k\| \geq \gamma t(r(n))\}) \geq \epsilon_0 / 2$.

Set $\gamma_2 = \gamma / \gamma_1 + 1$ and, for any $x, n, \tau$ with $\|x\| \leq \tau$ and $\frac{1}{2} \rho(\tau / \gamma_1) \leq n \leq \rho(\tau / \gamma_1)$, write

$$P_x \left( \sup_{k \leq n} \{\|X_k\| \leq \gamma_2 \tau, \|X_n\| \leq \tau \} \right) = P_x (\|X_t\| \leq \tau) - P_x \left( \sup_{k \leq n} \{\|X_k\| \geq \gamma_2 \tau, \|X_n\| \leq \tau \} \right) \geq \epsilon_0 - P_e \left( \sup_{k \leq n} \{\|X_k\| \geq \gamma \tau / \gamma_1 \} \right) \geq \epsilon_0 / 2.$$

This proves that $\nu$ is strongly $(\| \cdot \|, r)$-controlled.

As a simple illustration of these techniques, consider the case of an arbitrary symmetric measure $\nu$ with generating support and finite second moment (with respect to the word-length $|\cdot|$) on a group with polynomial volume growth of degree $D(G)$. It follows from [16] that $\nu^{(n)}(e) \simeq n^{-D(G)/2}$ and satisfies a pointwise classical pseudo-Poincaré inequality (with $\rho(t) = t^2$). Proposition 3.5 yields the following result.

**Theorem 3.6.** Let $G$ be a finitely generated group with polynomial volume growth with word-length $|\cdot|$. Assume that $\nu$ is symmetric, satisfies $\mu(e) > 0$, has generating support and satisfies $\sum |g|^2 \nu(g) < \infty$. Then $\nu$ is strongly $(|\cdot|, t \mapsto t^{1/2})$-controlled.

### 4 Measures supported on powers of generators

#### 4.1 The measure $\mu_{S,a}$

In this subsection we consider the special case when $G$ is a nilpotent group equipped with a generating $k$-tuple $S = (s_1, \ldots, s_k)$ and

$$J(x, y) = \mu_{S,a}(x^{-1} y), \ a = (\alpha_1, \ldots, \alpha_k) \in (0, \infty)^k$$ \hspace{1cm} (4.1)
with $\mu_{S,a}$ given by (1.4). Our aim is to prove the first statement in Theorem 1.6. The study of the random walks driven by this class of measure was initiated by the authors in [18] and we will refer to and use some of the main results of [18].

Following [18, Definition 1.3], let $w$ be the power weight system on the formal commutators on the alphabet $S$ associated with setting $w_i = 1/\tilde{\alpha}_i$, $\tilde{\alpha}_i = \min\{\alpha_i, 2\}$. Namely, The weight of any commutator $c$ using the sequence of letters $(s_{i_1}, \ldots, s_{i_m})$ from $S$ (or their formal inverse) is $w(c) = \sum_{1}^{m}w_i$. In [18], the authors proved the following result.

**Theorem 4.1** ([18]). Referring to the above setting and notation, assume that the subgroup of $G$ generated by $\{s_i : \alpha_i < 2\}$ is of finite index. Then there exists a real $D_{S,a} = D(S, w)$ such that

$$Q_{S,a}(r) \simeq r^{D_{S,a}}, \quad \mu_{S,a}(n) \simeq n^{-D_{S,a}}.$$  

The real $D_{S,a} = D(S, w)$ is given by [18, Definition 1.7]. Further, there exists a $k$-tuple $b = (\beta_1, \ldots, \beta_k) \in (0, 2)^k$ such that $\beta_i = \alpha_i < 2$, $D(S,a) = D(S,b)$, and

$$\forall g \in G, \|g\|_{S,a}^{\sigma_\beta} \simeq \|g\|_{S,\beta}^{\beta_*}.$$  

In addition, $\mu_{S,a}$ satisfies a pointwise ($\| \cdot \|_{S,a}, t \mapsto t^{1/\alpha_*}$)-pseudo-Poincaré inequality.

The volume estimate $Q_{S,a}(r) \simeq Q_{S,b}(r) \simeq r^{D_{S,a}}$ shows, in particular, that $(G, \| \cdot \|_{S,b})$ has the volume doubling property. The upper bound $\mu_{S,a}(n) \leq Cn^{-D_{S,a}}$ implies that the continuous time process with jump kernel $J$ defined above satisfies

$$\forall t > 0, x \in G, \quad p(t, x, x) \leq m(t) = Ct^{-D_{S,a}}.$$  

Note that $\| \cdot \|$ is clearly well-connected (Definition 3.3). In order to apply Propositions 2.3 and 3.5 to the present case and prove Theorem 1.6, it clearly suffices to prove the following lemma which provides estimates for $\delta_R$ and $G(R)$.

**Lemma 4.2.** Referring to the setting and hypotheses of Theorem 4.1 for $J$ given by (4.1), let $\| \cdot \| = \| \cdot \|_{S,b}$, $D = D_{S,b} = D_{S,a}$, we have

$$\begin{align*}
V(r) &= \# \{ g \in G : \|g\| \leq r \} \simeq r^{D_{\beta_*}}, \\
\delta_R &\simeq R^{-\beta_*}, \\
G(R) &\simeq R^{2-\beta_*}.
\end{align*}$$

**Proof.** The volume estimate follows immediately from Theorem 4.1. Let $w$ be the power weight system associated with $b$ (in particular, $v_i = 1/\beta_i > 1/2$). By [18, Proposition 2.17], for each $i$ there exists $0 < \beta_i' \leq \beta_i \leq \beta_* < 2$ such that

$$\|s_i^n\| \simeq |n|^{\beta_i'}/\beta_*.$$
Indeed, \( \alpha_i = \frac{\beta_i}{\beta_*} \). We have

\[
\delta_R = \sum_{\|x\| > R} \mu_{S,a}(x) = \sum_{i=1}^{k} \sum_{\|s^n\| > R} \frac{\kappa_i}{(1 + |n|)^{1 + \alpha_i}} \approx \sum_{i=1}^{k} \sum_{n > R^{\beta_i/\beta_*}} \frac{\kappa_i}{(1 + |n|)^{1 + \alpha_i}} \approx \sum_{i=1}^{k} R^{-\beta_i/\beta_*} \approx R^{-\beta_*}.
\]

The last estimate use that fact that there must be some \( i \in \{1, \ldots, k\} \) such that \( \alpha_i = \beta'_i \) and that, always, \( \alpha_i \geq \beta'_i \).

Similarly, since \( \alpha_i \geq \beta'_i \) and \( \beta_* < 2 \), we have \( 2\beta_i/\beta_* - \alpha_i > 0 \). This yields

\[
G(R) = \sum_{\|x\| \leq R} \|x\|^2 \mu_{S,a}(x) = \sum_{i=1}^{k} \sum_{\|s^n\| \leq R} \frac{\kappa_i\|s^n\|^2}{(1 + |n|)^{1 + \alpha_i}} \approx \sum_{i=1}^{k} R^{2 - \beta_i/\beta_*} \approx R^{2 - \beta_*}.
\]

This proves Lemma 4.2.

4.2 Some regular variation variants of \( \mu_{S,a} \)

Consider the class of measure \( \mu \) of the form

\[
\mu(g) = \frac{1}{K} \sum_{i=1}^{k} \sum_{m \in \mathbb{Z}} \frac{\kappa_i \ell_i(|m|)}{(1 + |m|)^{1 + \alpha_i}} \quad (4.2)
\]

where each \( \ell_i \) is a positive slowly varying function satisfying \( \ell_i(t^b) \approx \ell_i(t) \) for all \( b > 0 \) and \( \alpha_i \in (0, 2) \). For each \( i \), let \( F_i \) be the inverse function of \( r \mapsto r^{\alpha_i}/\ell_i(r) \).

Note that \( F_i \) is regularly varying of order \( 1/\alpha_i \) and that \( F_i(r) \approx [r\ell_i(r)]^{1/\alpha_i} \), \( r \geq 1 \), \( i = 1, \ldots, k \). We make the fundamental assumption that the functions \( F_i \) have the property that for any \( 1 \leq i, j \leq k \), either \( F_i(r) \leq CF_j(r) \) or \( F_j(r) \leq CF_i(r) \). For instance, this is clearly the case if all \( \alpha_i \) are distinct.

Set \( a = (\alpha_1, \ldots, \alpha_k) \in (0, 2)^k \) and consider also the power weight system \( v \) generated by \( v_i = 1/\alpha_i, \ 1 \leq i \leq k \), as in [18] Definition 1.3. Fix \( \alpha_0 \in (0, 2) \) such that

\[
\alpha_0 > \max \{ \alpha_i : 1 \leq i \leq k \}
\]

and \( \alpha_0/\alpha_i \notin \mathbb{N}, \ i = 1, \ldots, k \). Observe that there are convex functions \( K_i \geq 0, \ i = 0, \ldots, k \), such that \( K_i(0) = 0 \) and

\[
\forall r \geq 1, \ F_i(r^{\alpha_0}) \approx K_i(r). \quad (4.3)
\]

Indeed, \( r \mapsto F_i(r^{\alpha_0}) \) is regularly varying of index \( \alpha_0/\alpha_i \) with \( 1 < \alpha_0/\alpha_i \notin \mathbb{N} \).

By [4] Theorems 1.8.2-1.8.3 there are smooth positive convex functions \( K_i \) such
that \( \tilde{K}_i(r) \sim F_i(r^{\alpha_0}) \). If \( \tilde{K}_i(0) > 0 \), it is easy to construct a convex function \( K_i : [0, \infty) \to [0, \infty) \) such that \( K_i \simeq \tilde{K}_i \) on \([1, \infty)\) and \( K_i(0) = 0 \). Let use \( \mathcal{K} \) to denote the collection \((K_i)_{i=1}^{r_{1}}\).

Now, set
\[
\|g\| = \|g\|_{\mathcal{K}} = \min \left\{ r : g = \prod_{j=1}^{m} s_{i_j}^{\epsilon_j} : \epsilon_j = \pm 1, \ \# \{ j : i_j = i \} \leq K_i(r) \right\}.
\]

Because of the convexity property of the \( K_i, \| \cdot \| \) is a norm. Note also that it is well-connected. The following theorem is proved in [18].

**Theorem 4.3.** Referring to the above notation and hypothesis, there exist a real \( D = D_{S,a} = D(S,v) \) and a positive slowly varying function \( L \) (explicitly given in [18, Theorem 5.15] and which satisfies \( L(t^a) \simeq L(t) \) for all \( a > 0 \)) such that:

- For all \( r \geq 1, V(r) = \# \{ g : \|g\| \leq r \} \simeq r^{\alpha_0}DL(r) \)
- For each \( 1 \leq i \leq k \), there exists a regularly varying function \( \bar{F}_i \) such that \( \|s_i\|^{\alpha_0} \leq CF_i^{-1}(n) \) where \( \bar{F}_i \geq F_i \) and with equality for some \( 1 \leq i \leq r \).
- For all \( n \geq 1, \mu^{(2n)}(e) \leq C(nDL(n))^{-1} \).
- The measure \( \mu \) satisfies a pointwise \((\| \cdot \|, t \mapsto t^{1/\alpha_0})\)-pseudo-Poincaré inequality.

Here, we prove the following result.

**Theorem 4.4.** Let \( G \) be a finitely generated nilpotent group equipped with a generating \( k \)-tuple \((s_1, \ldots, s_k)\). Assume that \( \mu \) is a probability measure on \( G \) of the form \((4.2)\). Let \( \ell_i, F_i, L, D = D_{S,a}, \alpha_0 \in (0,2) \) and \( \| \cdot \| \) be as described above. Then \( \mu \) is strongly \((\| \cdot \|, t \mapsto t^{1/\alpha_0})\)-controlled.

**Proof.** It suffices to estimate the quantities \( \delta_R \) and \( \mathcal{G}(R) \) in the present context. For \( \delta_R \), we have
\[
\delta_R \simeq \sum_{i=1}^{k} \sum_{n \geq \tilde{F}_i(R^{\alpha_0})} \frac{1}{nF_i^{-1}(n)} \simeq \sum_{i=1}^{k} \frac{1}{F_i^{-1} \circ \tilde{F}_i(R^{\alpha_0})} \simeq R^{-\alpha_0}.
\]

A similar computation gives
\[
\mathcal{G}(R) \simeq \sum_{i=1}^{k} \frac{R^2}{F_i^{-1} \circ \tilde{F}_i(R^{\alpha_0})} \simeq R^{2-\alpha_0}.
\]
4.3 The critical case when $\alpha_i = 2$, $1 \leq i \leq k$

When $a = 2 = (2, \ldots, 2)$, that is, $\alpha_i = 2$ for all $1 \leq i \leq k$, we work with the usual word-length function $|g|$ associated with the generating set $S = \{s_1^{\pm 1}, \ldots, s_k^{\pm 1}\}$. In this case, $V(r) = \# \{g : |g| \leq r\} \approx r^{D(G)}$ where $D(G)$ is the classical degree of polynomial growth for the nilpotent group $G$. It is proved in [18] that $\mu_{S,2}^{(n)}(e) \leq C(n \log n)^{-D/2}$ and that $\mu_{S,2}$ satisfies a pointwise ($|\cdot|, t \mapsto (t \log t)^{1/2}$)-pseudo-Poincaré inequality. Further, $|s_i^n| \approx |n|^{1/\beta_i}$ with $\beta_i \geq 1$ and $\beta_i = 1$ for some $i$. From this it easily follows that

$$\delta_R \approx R^{-2}, \quad G(R) \approx \log R.$$

Applying Proposition 3.5 with $r(t) = (t \log t)^{1/2}$ yields the following theorem.

**Theorem 4.5.** Let $G$ be a finitely generated nilpotent group equipped with a generating $k$-tuple $(s_1, \ldots, s_k)$. Let $D(G)$ be the volume growth degree of $G$. Then $\mu_{S,2}$ is strongly ($|\cdot|, t \mapsto (t \log t)^{1/2}$)-controlled.

5 Norm-radial measures

In this section we assume that $G$ is a finitely generated group with polynomial volume growth of degree $D(G)$ and we consider norm-radial symmetric probability measures.

5.1 Radial measures with stable-like exponent $\alpha \in (0, 2)$

This subsection treats probability measures of the form

$$\nu_\alpha(x) \approx \frac{1}{(1 + \|x\|^\alpha V(\|x\|))}, \quad J(x,y) \approx \nu_\alpha(x^{-1}y),$$

where $\alpha \in (0, 2]$, $\|\cdot\|$ is a norm on $G$ and $V(r) = \# \{g : \|g\| \leq r\}$. The case when $\alpha \in (0, 2)$ and $V(r) \approx r^d$ for some $d$ is treated in [1][2][14] where global matching upper and lower bounds are obtained. We note that [1][2][14] are set in more general contexts where the group structure play no role. We start with the following easy observation.

**Lemma 5.1.** Referring the situation described above, assume that $V$ satisfies $V(2r) \leq C_D V(r)$ for all $r > 0$. Then $\delta_R \approx R^{-\alpha}$ and

$$G(R) \approx \begin{cases} R^{2-\alpha} & \text{if } \alpha \in (0, 2) \\ \log R & \text{if } \alpha = 2. \end{cases}$$

**Proof.** This follows by inspection. \qed

The next lemma follows by application of Proposition 2.3. However, in this lemma, we make a significant hypothesis on $\nu_\alpha^{(n)}(e)$. 

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Lemma 5.2. Set \( r_\alpha(t) = t^{1/\alpha} \) if \( \alpha \in (0, 2) \), \( r_2(t) = (t \log t)^{1/2} \). Referring the situation described above, assume that \( V \) is regularly varying of positive index and
\[
\nu_\alpha^{(n)}(e) \leq CV (r_\alpha(n))^{-1}.
\] (5.2)
Then \( \nu_\alpha \) is \((\| \cdot \|, r_\alpha)\)-controlled.

The next theorem provides a basic class of examples when the hypothesis (5.2) regarding \( \nu_\alpha \) can indeed be verified. Note that the result is restricted to the case \( \alpha \in (0, 2) \).

Theorem 5.3. Referring the situation described above, assume that \( V \) is regularly varying of positive index and \( \alpha \in (0, 2) \). Then
\[
\nu_\alpha^{(n)}(e) \simeq V(n^{1/\alpha})^{-1}
\]
and \( \nu_\alpha \) is \((\| \cdot \|, t \mapsto t^{1/\alpha})\) controlled.

Proof. It suffices to prove the upper bound \( \nu_\alpha^{(n)}(e) \leq CV(n^{1/\alpha})^{-1} \). Start by checking that
\[
\nu_\alpha(x) \simeq \sum_{0}^{\infty} \frac{1}{(1 + m)^{1+\alpha}} \frac{1_{B(m)}(x)}{V(m)},
\]
where \( B(m) = \{ x \in G : \| x \| \leq m \} \). Then apply the elementary technique of [3, Section 4.2] to derive the desired upper bound on \( \nu_\alpha^{(n)}(e) \).

Remark 5.4. In the context of Theorem 5.3, we do not know if \( \| \cdot \| \) is well-connected and we also do not know if \( \nu_\alpha \) satisfies a pointwise \((\| \cdot \|, r_\alpha)\)-pseudo-Poincaré inequality. Hence, the techniques used in this paper do not suffice to obtain strong control. However, if \( \| \cdot \| \) is well-connected and \( \nu_\alpha \) satisfies a pointwise \((\| \cdot \|, r_\alpha)\)-pseudo-Poincaré inequality then the strong \((\| \cdot \|, r_\alpha)\)-control follows by Proposition 3.5. This proves the second statement in Theorem 1.6.

5.2 Word-length radial measures

As noticed above, the study of \( \nu_\alpha \) in the case \( \alpha = 2 \) is significantly more difficult than in the case \( \alpha \in (0, 2) \). In fact, we do not know how to treat this case in the generality described in the previous subsection. The following theorem treats the case when \( \| \cdot \| \) is the usual word-length function \( \| \cdot \| = | \cdot | \) on \( G \).

Theorem 5.5. Assume that \( G \) is a group of polynomial volume growth equipped with generating \( k \)-tuple \( S = (s_1, \ldots, s_k) \) and the associated word-length \( | \cdot | \) and volume function \( V \). Let \( D(G) \) be the degree of polynomial volume growth of \( G \). Let \( \nu_2 \) be a symmetric probability measure such that
\[
\nu_2(g) \simeq ((1 + |g|)^2 V(|g|))^{-1}.
\]
Then the have
\[
\nu_2^{(n)}(e) \simeq (n \log n)^{-D(G)/2}.
\]
Further, \( \nu_2 \) is strongly \((| \cdot |, t \mapsto (t \log t)^{1/2})\)-controlled.
Proof. We apply Lemma 5.2 and Proposition 3.5. When $G$ is nilpotent, the upper bound $\nu_2^{(n)}(e) \leq (n \log n)^{-D(G)/2}$ follows from Theorems 4.8 and 5.7 of [18]. Namely, [18, Theorem 5.7] shows that $\nu_2^{(n)}(e) \leq C\mu_{S,2}^{(Kn)}(e)$ and [18, Theorem 4.8] gives $\mu_{S,2}^{(n)}(e) \leq C(n \log n)^{-D(G)/2}$. Further, [18, Theorem 5.7] shows that $\nu_2$ satisfies a pointwise $(| \cdot |, t \mapsto (t \log t)^{1/2})$-pseudo-Poincaré inequality.

Since any group of polynomial volume growth of degree $D(G)$ contains a nilpotent subgroup of finite index (hence, with the same degree of polynomial volume growth) the upper bound $\nu_2^{(n)}(e) \leq (n \log n)^{-D(G)/2}$ follows from the comparison theorem [16, Theorem 2.3]. By direct inspection, the desired pseudo-Poincaré inequality also follows. \qed

Note that Theorem 1.5 includes the result stated in Theorem 5.5 as a special case and provides a very satisfactory result covering the behaviors of word-length radial measures across the second moment threshold.

Proof of Theorem 1.5. The same technique of proof as for Theorem 5.5 gives the much more complete and subtle result stated in the introduction as Theorem 1.5. Namely, let $\phi : [0, \infty) \to [1, \infty)$ be a continuous increasing regularly function of index 2 and let $\nu_\phi$ be as in (1.3), that is, assume that $\nu_\phi$ is symmetric and satisfies $\nu_\phi(g) \asymp \phi(|g|)V(|g|)^{-1}$. First, assume that $G$ is nilpotent and let $\mu_{S,\phi}$ be the measure given by

$$\mu_{S,\phi}(g) = \frac{1}{k} \sum_{1}^{k} \frac{\kappa}{(1+|n|)\phi(n)} 1_s^n(g).$$

Let $r$ be the inverse function of $t \mapsto t^2/\int_0^t \frac{ds}{\phi(s)}$. By [18, Lemma 4.4], the measure $\mu_{S,\phi}$ satisfies the pointwise $(| \cdot |, r)$-pseudo-Poincaré inequality. By [18, Theorem 4.1], it follows that $\mu_{S,\phi}^{(n)}(e) \leq V(r(n))^{-1}$. By [18, Theorem 5.7], we have the Dirichlet form comparison $E_{\mu_{S,\phi}} \leq CE_{\nu_\phi}$.

Now, if $G$ has polynomial volume growth then it contains a nilpotent subgroup with finite index, $G_0$. By inspection, quasi-isometry and comparison of Dirichlet forms (see [16]), it is easy to transfer both the pointwise $(| \cdot |, r)$-pseudo-Poincaré inequality and the decay $\nu_\phi^{(n)}(e) \leq V(r(n))^{-1}$ from $G_0$ to $G$. Further, one checks that the functions $\delta_R$ and $\mathcal{G}(R)$ satisfy $\delta_R \simeq 1/\phi(R)$ and $\mathcal{G}(R) \simeq \int_0^R \frac{dt}{\phi(t)}$. Proposition 2.3 with $r(t) = R(t)$ equals to the inverse function of $s \mapsto s^2/\int_0^s \frac{dt}{\phi(t)}$ shows that $\nu_\phi$ is $(| \cdot |, r)$-controlled. By Proposition 3.5, $\nu_\phi$ is strongly $(| \cdot |, r)$-controlled. \qed

5.3 Assorted further applications

The approach presented here is applicable even in cases where we are not able to obtain sharp results and we illustrate this by an example. Let $G$ be a nilpotent
group equipped with a generating $k$-tuple $S = (s_1, \ldots, s_k)$. Fix $a \in (0, 2)^k$ and set $\alpha_* = \max\{\alpha_i, 1 \leq i \leq k\} \leq 2$. Consider the norm $\| \cdot \|_{S,a}$ defined at (1.5). Let $\nu_*$ be any symmetric probability measure such that

$$\nu_*(g) \simeq \frac{1}{(1 + \|g\|_{S,a})^2} V(\|g\|_{S,a}), \quad V(r) = \#\{g : \|g\|_{S,a} \leq r\}.$$ 

Theorems 3.2, 4.8 and 5.7 of [18] gives the following information. There exists two reals $D = D_{S,a}$ and $d = d_{S,a}$ and a constant $C_1 \in (0, \infty)$ such that

$$\nu_*^{(n)}(e) \leq C_1 n^{-\alpha_* D/2} (\log n)^{-d} \quad (5.3)$$

$$V(r) \simeq r^{\alpha_* D}. \quad (5.4)$$

**Theorem 5.6.** For the probability measure $\nu_*$ on a finitely generated nilpotent group as described above, we have

$$c(\log \log n)^{-\alpha_* D}(n \log n)^{-\alpha_* D/2} \leq \nu_*^{(n)}(e) \leq Cn^{-\alpha_* D/2} (\log n)^{-d}.$$ 

**Proof.** The volume estimate (5.3) and Lemma 5.1 gives $\delta_R \simeq R^{-2}$ and $G(R) \simeq \log R$. In order to apply Proposition 2.3, we set $R(t) \simeq (t \log t)^{1/2}$. Further, we use (5.3) to verify that the choice $r(t) = 6AR(t) \log \log t$ with $A$ large enough satisfies the condition of Proposition 2.3. Indeed, we have

$$m(t) \simeq t^{-\alpha_* D/2}(\log t)^{-d}, \quad V(r) \simeq r^{\alpha_* D}.$$ 

Clearly, for $A$ large enough, the right-hand side is bounded above by a constant as required by Proposition 2.3 which now gives the stated lower bound on $\nu_*^{(n)}(e)$. \hfill $\Box$

5.4 Complementary off-diagonal upper bounds

In contrast with the case (1.4) of measures supported on powers of generators, for norm-radial kernels of type (5.1), we can use Meyer’s construction to derive good off-diagonal bounds for $p(t, e, x)$.

**Proposition 5.7.** Let $G$ be a finitely generated group equipped with a norm $\| \cdot \|$. For $\alpha \in (0, 2)$, let $\nu_\alpha$ be a symmetric probability measure on $G$ satisfying (5.1). Assume that there exist a positive slowly varying function $\ell$ and a real $D > 0$ such that:

1. $\forall r > 1, \quad V(r) \simeq r^D \ell(r)$;

2. $\forall t > 0, \quad x \in G, \quad p(t, x, x) \leq m(t) \simeq [(1 + t)^D \ell(t^{1/\alpha})]^{-1}$.

Then there exists $C$ such that, for all $t > 1$ and $x \in G$, we have

$$p(t, e, x) \leq C m(t) \min \left\{ \left( \frac{t}{\|x\|^\alpha} \right)^{1+D/\alpha} \frac{\ell_1(t^{1/\alpha})}{\ell_1(\|x\|^\alpha)}, 1 \right\}.$$
Remark 5.8. This proposition is stated here mostly for comparison with the next proposition. In fact, for the measure $\nu_\alpha$ with $\alpha \in (0, 2)$, the hypothesis (1) implies automatically that (2) is satisfied as well. See [1, 2, 11, 14]. See [14] for a complete study of this case including two-sided discrete time estimates.

Proposition 5.9. Let $G$ be a finitely generated group equipped with a norm $\| \cdot \|$ with volume $V$. Let $\nu_2$ be a symmetric probability measure on $G$ satisfying (5.1) with $\alpha = 2$. Assume that:

1. $\forall r > 1, V(r) \simeq r^D$,
2. $\forall t > 1, x \in G, \ p(t, x, x) \leq m(t) \simeq (t \log t)^{-D/2}$.

Then there exists $C$ such that, for all $t > 1$ and $x \in G$, we have

$$p(t, e, x) \leq Cm(t) \min \left\{ \left( \frac{t \log \|x\|}{\|x\|^2} \right)^{1+D/2}, 1 \right\}$$

Further, for any $\gamma \in (0, 2)$, there exist $C_\gamma$ such that if $1 \leq t \leq \|x\|^\gamma$ then

$$p(t, e, x) \leq \frac{C_\gamma}{t^{D/2}} \left( \frac{t}{\|x\|^2} \right)^{1+D/2}.$$

Proof of Proposition 5.9. Under the stated hypothesis, we have $\delta_R \simeq R^{-\alpha}$ and $G(R) \simeq R^{2-\alpha}$ and, for $1 \leq t \leq \eta R^{-\alpha}$ (with $\eta$ to be fixed later, small enough), Proposition 2.1 gives

$$p_R(t, e, x) \leq Cm(t) \left( \frac{t}{\|x\|^\alpha} \right) \|x\|/3R.$$

By Meyer’s construction, we have

$$p(t, x, y) \leq p_R(t, x, y) + t \|v'_{R}\|_\infty + \frac{1}{t^{D/\alpha}} \left( \frac{t}{R^{2\alpha}} \right) \|x\|/3R + \frac{t}{R^{2\alpha(1+D/\alpha)} \ell_1(R)}.$$

Choose $R = R(x, t)$ such that the two terms of the sum on the left-hand side are essentially equal, namely, set

$$\left( \log \frac{R^\alpha}{t} \right) \|x\|/3R = \left( \log \frac{R^\alpha}{t} \right) \left( 1 + \frac{D}{\alpha} \right) + \log \frac{\ell_1(R)}{\ell_1(1^{1/\alpha})}.$$

As long as $\eta$ is small enough, this choice of $R$ gives $\|x\| \simeq R$ and

$$p(t, x, y) \leq \frac{2t}{\|x\|^\alpha(1+D/\alpha) \ell_1(\|x\|)} \simeq \frac{1}{t^{D/\alpha} \ell_1(1^{1/\alpha})} \left( \frac{t}{\|x\|^\alpha} \right)^{1+D/\alpha} \frac{\ell_1(1^{1/\alpha})}{\ell_1(\|x\|)}.$$
For any \( t \) (in particular, \( t \geq \eta R^\gamma \)) we can also use \( m(t) \) for an easy upper bound. This gives

\[
p(t, e, x) \leq C \min \left\{ \frac{t}{\|x\|^\alpha \ell_1(t^1/\alpha)} \ell_1(\|x\|), 1 \right\}
\]

or, equivalently,

\[
p(t, e, x) \leq C \min \left\{ t \nu_\alpha(\|x\|), m(t) \right\}.
\]

**Proof of Proposition 5.9.** In the context of proposition 5.9, we have \( \delta_R \simeq R^2 \) and \( G(R) \simeq \log R \). For \( 1 < t \leq \eta R^2 \), \( \eta > 0 \) small enough, Proposition 2.1 and Meyer’s decomposition gives

\[
p(t, x, y) \leq p_R(t, x, y) + t \|v'_R\|_\infty \leq C(t \log t)^{-D/2} \left( \frac{t \log R}{R^2} \right) \frac{\|x\|/3R}{R^{2+D}}.
\]

If \( R^2/\log R \leq t \leq R^2 \) then this bound is not better than the easy bound \( p(t, x, y) \leq m(t) \). By taking \( R \) such that \( \|x\| = 3R(1 + D/2) \), we obtain

\[
p(t, x, y) \leq C m(t) \min \left\{ \left( \frac{t \log \|x\|}{\|x\|^2} \right)^{1+D/2}, 1 \right\}.
\]

However, if \( 1 \leq t \leq \eta \|x\|^\gamma \) with \( \gamma \in (0, 2) \) and \( \eta \) small enough, then we can choose \( R \simeq \|x\| \) so that

\[
(t \log t)^{-D/2} \left( \frac{t \log R}{R^2} \right) \frac{\|x\|/3R}{R^{2+D}} = \frac{t}{R^{2+D}},
\]

equivalently,

\[
\left( \frac{R^2}{t \log R} \right) \frac{\|x\|/3R}{R^{2+D}} = \left( \frac{R^2}{t \log R} \right)^{1+D/2} \frac{(\log R)^{1+D/2}}{(\log t)^{D/2}}.
\]

In the region \( t \leq \|x\|^\gamma \), this yields

\[
p(t, e, x) \leq \frac{2t}{\|x\|^{2+D}} \simeq t^{-D/2} \left( \frac{t}{\|x\|^2} \right)^{1+D/2}.
\]

\[\square\]

### 6 Random walks on wreath products

In this subsection, we illustrate how to use Proposition 1.4 to derive a lower bound for return probability of certain classes of random walks on wreath products.
First we briefly review the definition of wreath products and a special type of random walks on them. Our notation follows [15]. Let $H, K$ be two finitely generated groups. Denote the identity element of $K$ by $e_K$ and identity element of $H$ by $e_H$. Let $K_H$ denote the direct sum:

$$K_H = \sum_{h \in H} K_h.$$ 

The elements of $K_H$ are functions $f : H \to K$, $h \mapsto f(h) = k_h$, which have finite support in the sense that $\{h \in H : f(h) = k_h \neq e_K\}$ is finite. Multiplication on $K_H$ is simply coordinate-wise multiplication. The identity element of $K_H$ is the constant function $e_K : h \mapsto e_K$ which, abusing notation, we denote by $e_K$. The group $H$ acts on $K_H$ by translation:

$$\tau_h f(h') = f(h^{-1}h'), \; h, h' \in H.$$ 

The wreath product $K \wr H$ is defined to be semidirect product

$$K \wr H = K_H \rtimes \tau H,$$

$$\langle f, h \rangle \langle f', h' \rangle = \langle f \tau_h f', hh' \rangle.$$ 

In the lamplighter interpretation of wreath products, $H$ corresponds to the base on which the lamplighter lives and $K$ corresponds to the lamp. We embed $K$ and $H$ naturally in $K \wr H$ via the injective homomorphisms

$$k \mapsto (k_{e_H}, e_H), \; k_{e_H}(e_H) = k, \; k_{e_H}(h) = e_K \text{ if } h \neq e_H$$

$$h \mapsto (e_K, h).$$

Let $\mu$ and $\eta$ be probability measures on $H$ and $K$ respectively. Through the embedding, $\mu$ and $\eta$ can be viewed as probability measures on $K \wr H$. Consider the measure

$$q = \eta * \mu * \eta$$

on $K \wr H$. This is called the switch-walk-switch measure on $K \wr H$ with switch-measure $\eta$ and walk-measure $\mu$.

Let $(X_i)$ be the random walk on $H$ driven by $\mu$, and let $l(n, h)$ denote the number of visits to $h$ in the first $n$ steps:

$$l(n, h) = \# \{i : 0 \leq i \leq n, \; X_i = h\}.$$ 

Set also

$$l^q(n, h) = \begin{cases} l(n, h) & \text{if } h \notin \{e_H, g\} \\ l(n, e_H) - 1/2 & \text{if } h = g \\ l(n, e_H) - 1 & \text{if } h = e_H. \end{cases}$$

From [15], probability that the random walk on $K \wr H$ driven by $q$ is at $(h, g) \in K \wr H$ at time $n$ is given by

$$q^{(n)}((f, g)) = \mathbb{E} \left( \prod_{h \in H} \eta^{l^q(n, h)}(f(h)) \mathbf{1}_{\{X_n = g\}} \right).$$
Note that $E$ stands for expectation with respect to the random walk $(X_i)_{i=0}^\infty$ on $H$ started at $e_H$.

From now on we assume that $\eta$ satisfies $\eta(e_K) = \epsilon > 0$ so that

$$\epsilon_1^{(n-1)}(e_K) \leq \eta^{(n)}(e_K) \leq \epsilon^{-1}\eta^{(n-1)}(e_K).$$

Write $f \lesssim g$ if $C^{-1} f \leq g \leq Cf$. Under these circumstances, we have

$$q^{(n)}(e_K, g) \lesssim \epsilon^3 \mathbb{E} \left( \prod_{h \in H} \eta^{(2(n,h))}(e_K) \mathbf{1}_{\{X_n = g\}} \right)$$

so that we can essentially ignore the difference between $l$ and $l^\ast$.

Set $F_K(n) := -\log \eta^{(2n)}(e_K)$

so that, for any $g \in H$,

$$q^{(n)}((e_K, g)) \sim \mathbb{E} \left( e^{-\sum_{h \in H} F_K(l(n,h))} \mathbf{1}_{\{X_n = g\}} \right). \quad (6.5)$$

**Proposition 6.1.** Let $H$ be a finitely generated group equipped with a symmetric measure $\mu$ with $\mu(e_H) > 0$. Let $K$ be a finitely generated group equipped with a symmetric measure $\eta$ with $\eta(e_K) > 0$. Let $\| \cdot \|$ be a norm with volume function $V$. Let $r$ be a positive continuous increasing function. Assume that:

1. The measure $\mu$ is strongly ($\| \cdot \|, r$)-controlled and $V$ satisfies $V(t) \sim t^D$.

2. The function $r$ satisfies $r(t) = t^{1/\beta} \ell_1(t)$ where $\ell_1$ is a positive continuous slowly varying function.

3. The function $F_K(n) = -\log \eta^{(2n)}(e_K) \sim n^\gamma \ell_2(n)$ where $\gamma \in [0, 1)$ and $\ell_2$ is a positive continuous slowly varying function.

Assume also that the slowly varying functions $\ell_i$, $i = 1, 2$, are such that $\ell_i(t^n) \sim \ell_i(t)$ for all $a > 0$. Then the switch-walk-switch measure $q$ on $K \wr H$ associated with the pair $\eta, \mu$ satisfies

$$q^{(n)}(e) \geq \exp \left( -Cn^{\frac{D(1-\gamma)+\beta}{D(1-\gamma)+\beta}} \ell_1(n)^{\frac{D(1-\gamma)}{D(1-\gamma)+\beta}} \ell_2(n)^{\frac{\beta}{D(1-\gamma)+\beta}} \right).$$

**Proof.** Let $m$ be the spectral measure of $\eta^{(2)}$ in the sense that

$$\int_{[0,1]} t^n \, dm(t) = \nu^{(2n)}(a).$$

For $x \in [0, \infty)$, set

$$F(x) := -\log \int_{[0,1]} t^x \, dm(t).$$

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Observe that $F_K(n) = F(n)$ and that $F$ is a concave function. For $\tau > 0$, let $B(\tau) = \{ h \in H : \| h \| \leq \tau \}$. Since $q^{(2n)}(e) \geq q^{(2n)}(x)$ for any $x \in K \cap H$, (6.3) yields

$$
q^{(n)}(e) \geq \frac{1}{\#B(\tau)} \mathbf{E} \left( e^{-\sum_{h} F_{K}(l(n,h))} 1_{\{ \| X_n \| \leq \tau \}} \right) 
\geq \frac{1}{\#B(\tau)} \mathbf{E} \left( e^{-\sum_{h} F_{K}(l(n,h))} 1_{\{ \max_{1 \leq k \leq n} \{ \| X_k \| \} \leq \tau \}} \right).
$$

Using the concavity of $F$ and the confinement of the walk $(X_n)$ on $H$ in the ball $B(\tau)$ in the last expression, this yields

$$
q^{(n)}(e) \geq \frac{1}{V(\tau)} e^{-V(\tau) F(n/V(\tau))} P_e \left( \max_{1 \leq k \leq n} \{ \| X_k \| \} \leq \tau \right).
$$

Let $\tau_n$ be such that

$$
V(\tau_n) F(n/V(\tau_n)) = n/\rho(\tau_n)
$$

where $\rho$ is the inverse of $r$. By our various assumption, this means

$$
(\tau_n D/n)^{1-\gamma} \ell_2(n/\tau_n^D) = \tau_n^{-\beta} \ell_1(\tau_n)\beta.
$$

Hence $n \to \tau_n$ is a regularly varying function of order $(1-\gamma)/(\beta + D(1-\gamma)) < 1/\beta$. This shows that $r(n) \gg \tau_n$ and, since $\mu$ is strongly ($\| \cdot \|, r)$-controlled, Proposition [1.4] yields

$$
q^{(n)}(e) \geq \frac{1}{V(\tau_n)} e^{-V(\tau_n) F(n/V(\tau_n))} e^{-C n/\rho(\tau_n)} \geq e^{-C n/\rho(\tau_n)}
$$

and

$$
\frac{n}{\rho(\tau_n)} = n^{\frac{D(1-\gamma)+\gamma\beta}{D(1-\gamma)+\beta}} \ell_1(n)^{\frac{\beta D(1-\gamma)}{D(1-\gamma)+\beta}} \ell_2(n)^{-\frac{\beta}{D(1-\gamma)+\beta}}.
$$

This gives the stated lower bound on $q^{(n)}(e)$. \hfill \Box

**Remark 6.2.** The case $\gamma = 1$ is excluded from this computations. It can be treated by the same method but $\tau_n$ become a slowly varying function of $n$.

**Remark 6.3.** In the setting of Proposition [6.1], suppose in addition we have $H = \mathbb{Z}^d$ and $\mu$ is in the domain of attraction of an operator-stable law $\nu$ on $\mathbb{R}^d$, that is there exists a normalizing sequence $B_n \in GL_d(\mathbb{R})$ such that $B_n^{-1} \mu^{*n} \Rightarrow \nu$. Then the lower bound in Proposition [6.1] is sharp and agrees with [17] Theorem 4.2]. Note that in this case, $\det B_n \simeq V(r(n))$, the scaling relation in [17] Theorem 4.2] reads

$$
a_n \frac{\det B_{an} F_K \left( \frac{n}{\det B_{an}} \right)}{n} \simeq 1
$$

and it agrees with

$$
V(\tau_n) F(n/V(\tau_n)) = n/\rho(\tau_n),
$$

with $a_n \simeq \rho(\tau_n)$. 

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Example 6.1. Consider the symmetric probability measure $\mu$ on $\mathbb{Z}$ of the form

$$\mu(n) = \sum_{m \in \mathbb{Z}} \frac{\kappa \ell_1(|n|)}{(1 + |n|)^{1+\alpha}}$$

where and $\alpha \in (0, 2)$ and $\ell_1$ is a positive continuous slowly varying function satisfying $\ell_1(t^b) \simeq \ell_1(t)$ for all $b > 0$. We have

$$\delta_R := \sum_{|n| > R} \mu(n) \sim \frac{\kappa \ell_1(R)}{\alpha R^\alpha} \quad \text{and} \quad \mathcal{G}(R) = \sum_{|n| \leq R} |n|^2 \mu(n) \sim \frac{\kappa}{2 - \alpha} R^{2-\alpha} \ell_1(R).$$

Therefore $R^2 \delta_R / \mathcal{G}(R) \to (2 - \alpha)/\alpha$. By a classical result (see [9]), $\mu$ is in the domain of attraction of an $\alpha$-stable law on $\mathbb{R}$. The normalizing sequence $b_n$ can be chosen as the solution to the equation $n b_n^2 \mathcal{G}(b_n) = 1$, that is $b_n \sim \left(\frac{\kappa}{2-\alpha} \right)^{1/\alpha} n \ell_1(n)$. Let $K$ be a finitely generated group equipped with a symmetric measure $\eta$ with $\eta(e_K) > 0$. Suppose that the function $F_K(n) = -\log \eta^{(2n)}(e_K) \simeq n^\gamma \ell_2(n)$ where $\gamma \in [0, 1)$ and $\ell_2$ is a positive continuous slowly varying function. Assume also that $\ell_2(t^a) \simeq \ell_2(t)$ for all $a > 0$. Then [17, Theorem 4.2] (and the remark following that statement in [17]) implies that the switch-walk-switch measure $q$ on $K \wr H$ associated with the pair $\eta, \mu$ satisfies

$$-\log q(n)(e) \simeq n/a_n \sim n^{\frac{(1-\gamma) + \gamma b}{1 - b} + \frac{\gamma - \gamma}{1 - b} + \frac{\gamma b}{1 - b}} \ell_1(n)^{\frac{(1-\gamma) + \gamma b}{1 - b}} \ell_2(n)^{\frac{\gamma - \gamma}{1 - b}} \ell_2(t^a) \simeq \frac{a_n b_n}{n} F_K \left( \frac{n}{b_n} \right) \simeq 1.$$

This agrees with the lower bound in Proposition 6.1. Note that, by Proposition 3.5, the measure $\mu$ is strongly $\langle |\cdot|, r \rangle$-controlled where $r(n) = \langle n \ell_1(n) \rangle^{1/\alpha}$.

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