An $A_5$ inverse seesaw model with perturbed golden ratio mixing

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(Dated: June 15, 2022)

We propose a theory based on the $A_5$ discrete group which successfully accounts for lepton masses and mixings. In the proposed model, the neutrino masses arise from an inverse seesaw mechanism. The leptonic mixing consists in a perturbed golden ratio mixing pattern with a strong correlation arising between the leptonic mixing observables, which can be expressed as $\theta_{13}$ predicting the other mixing angles and the Dirac CP phase.

I. INTRODUCTION

The problem of fermion masses and mixing is one of the theoretical issues that afflicts the Standard Model. The typical solution to the problem is the introduction of extra symmetries - often referred to as family symmetries - which justify the replication of the fermions in 3 generations. Through the mechanism of breaking the family symmetry it is possible to explain the different and hierarchical masses of the generations and the mixing between fermions.

The family symmetry introduced can be Abelian or non-Abelian, discrete or continuous. Non-Abelian discrete symmetries gained favour as they are better suited to explain leptonic mixing patterns. Within the more popular non-Abelian discrete family symmetries in the literature are $S_3$ [1–6], $A_4$ [7–10], $S_4$ [11–18], $\Delta(27)$ [19–24] and $A_5$ [25–33]. The largest of these, $A_5$, can be broken into subgroups which lead to what is referred to as golden ratio mixing patterns. The original golden ratio mixing pattern with vanishing $\theta_{13}$ is clearly excluded, with the more recent $A_5$ models [26–33] obtaining deviations via corrections in the golden mixing pattern arising from the vacuum alignments of the $A_5$ triplets and $A_5$ quintuplet [26] as well as corrections in the neutrino [27, 30–33] or charged lepton [27, 29] sectors. In this paper we explore controlled deviations of the golden ratio mixing patterns, in order to have an excellent fit to leptonic mixing data. We consider an extended fermion sector leading to an inverse seesaw mechanism, such that, on the family symmetry basis, the neutrino mixing is the Golden Ratio, but on the same basis the charged lepton mass is not diagonal. The leptonic mixing therefore is a perturbed Golden Ratio pattern.

In Section II we describe the model in terms of symmetries and the respective assignments of the fields. Section III details the breaking of the family symmetry and the associated mass terms that arise, leading to the leptonic masses and mixing. We conclude in Section IV.

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The model under consideration is an $A_5$ flavor supersymmetric theory for leptons where the inverse seesaw mechanism is implemented to generate the tiny masses of the light active neutrinos and lepton mixings features a perturbed golden ratio pattern. The $A_5$ family symmetry of the model is supplemented by the $Z_9 \times Z_N$ discrete group. The successful and viable implementation of the inverse seesaw mechanism requires the inclusion of six right handed Majorana neutrinos $\nu_{iR}$ and $N_{iR}$ ($i = 1, 2, 3$), which are grouped into an $A_5$ leptonic triplet. Furthermore, in the supersymmetric model under consideration, the scalar sector of the MSSM is extended by the inclusion of several gauge singlet scalar fields. Such gauge singlet scalar fields are needed to generate the required Yukawa terms in the supersymmetric model under consideration, the scalar sector of the MSSM is extended by the inclusion of several scalar triplets and the $5$ scalar quintuplet can be obtained from the following superpotential:

$$W = M_0 \xi_1 \xi_2 + \kappa_1 \xi_1 \phi_N^2 + \kappa_2 \left( \phi_e^3 \right) + \kappa_3 \left( \phi_\nu^3 \right) + \frac{\kappa_4}{3} \xi_1^3 + \frac{\kappa_5}{3} \xi_2^3 + M_1 \phi_e^2 + M_2 \phi_\nu \phi_e + \kappa \phi_e \phi_\nu \phi_e \phi_e + \cdots$$

where $\xi_1$ and $\xi_2$ are driving fields. Notice that the $\phi_e^3$ and $\phi_e \phi_e \phi_e \phi_e \phi_e$ terms are not present in the superpotential since they yield vanishing results as follows from the $A_5$ multiplications rules. Finally, to close this section, we briefly compare our model with the one proposed in [26]. In the model of [26], the light active neutrino masses are generated from a type I seesaw mechanism and the reactor mixing angle arises from a perturbation of the VEV configuration of the $A_5$ triplets and $A_5$ quintuplet. On the other hand, in our proposed model, the masses of the light active neutrinos are produced via an inverse seesaw mechanism and the required perturbation of the golden mixing pattern that generates a non vanishing reactor mixing angle arises from the charged lepton sector.

| $l_L$ | $l_{1R}$ | $l_{2R}$ | $l_{3R}$ | $\nu_R$ | $N_R$ | $H_u$ | $H_d$ | $\phi_e$ | $\phi_\nu$ | $\varphi_e$ | $\varphi_\nu$ | $\varphi_e \varphi_\nu$ | $\theta$ | $\rho$ | $\phi_N$ | $\xi_1$ | $\xi_2$ |
|-------|---------|---------|---------|--------|-------|-------|-------|--------|--------|--------|--------|---------------|--------|-------|--------|--------|--------|
| 3     | 1       | 1       | 1       | 3      | 3     | 1     | 1     | 3      | 3      | 3      | 1      | 1               | 5      | 1     | 1      |
| $Z_9$ | -1      | -1      | -1      | -3     | 0     | 1     | 0     | 0      | 0      | 0      | 1      | 3               | 3      | -3    |
| $Z_N$ | 0       | 0       | -x      | x      | 0     | 0     | 0     | 0      | 0      | 0      | 0      | 0               | 0      | 0     |

Table I: Lepton and scalar assignments under the $A_5 \times Z_9 \times Z_N$ discrete group.
III. LEPTON MASSES AND MIXINGS

After the SM gauge symmetry and the $A_5 \times Z_3 \times Z_N$ discrete group are spontaneously broken, the following mass matrix for SM charged leptons is obtained:

$$
M_l = \begin{pmatrix}
\frac{y_1 v_e v^2}{\lambda^2} & 0 & \frac{z_3 v_e v_d}{\lambda^2} \\
\frac{z_1 y_1 v_e v^2}{\lambda^2} & \frac{y_2 v_e v^2}{\lambda^2} & 0 \\
0 & 0 & \frac{y_3 v_e v_d}{\lambda^2}
\end{pmatrix},
$$

where we assume that the entries of $M_l$ fulfill the following hierarchy:

$$
|(M_l)_{11}| \sim |(M_l)_{22}| \ll |(M_l)_{13}| < |(M_l)_{33}|.
$$

This hierarchy is justified by the insertions of the fields (such as $\rho$). The matrix $M_l M_l^\dagger$ is diagonalized by a rotation matrix according to:

$$
R_l^\dagger M_l M_l^\dagger R_l = \begin{pmatrix}
0 & 0 & m_e \\
0 & m_\mu & 0 \\
0 & 0 & m_\tau
\end{pmatrix},
\quad
R_l = \begin{pmatrix}
\cos \theta & 0 & \sin \theta e^{i\theta} \\
0 & 1 & 0 \\
-\sin \theta e^{-i\theta} & 0 & \cos \theta
\end{pmatrix},
\quad
\tan \theta \approx \frac{|(M_l)_{13}|}{|(M_l)_{33}|}.
$$

Furthermore, the neutrino Yukawa terms yield the following neutrino mass terms:

$$
-\mathcal{L}_{\text{mass}}^{(\nu)} = \frac{1}{2} \left( \overline{\nu_L} v_R N_R \right) M_\nu \begin{pmatrix}
\nu_L \\
\nu_R \\
N_R
\end{pmatrix} + \sum_{n=1}^2 \sum_{m=1}^2 \left( m_\Omega \right)_{nm} \overline{\Omega_{nR}} \Omega_{mR}^C + \text{H.c.},
$$

where the neutrino mass matrix is given by:

$$
M_\nu = \begin{pmatrix}
0_{3 \times 3} & m_{\nu D} & 0_{3 \times 3} \\
0_{3 \times 3} & m_{\nu D}^T & M \\
0_{3 \times 3} & M^T & \mu
\end{pmatrix},
$$

and the submatrices $m_{\nu D}$, $M$ and $\mu$ have the following structure:

$$
m_{\nu D} = \frac{y_\nu v_e}{\sqrt{2}} \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix},
\quad
M = y_\nu^N v_\nu \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix},
\quad
\mu = \frac{v_e v_\nu v_\nu^N}{\sqrt{6} \Lambda^2} \begin{pmatrix}
\frac{2}{3} (p + q) & \frac{p}{\sqrt{2}} & \frac{p}{\sqrt{2}} \\
\frac{p}{\sqrt{2}} & q - \frac{1}{3} (p + q) & q \\
\frac{p}{\sqrt{2}} & q & q
\end{pmatrix}.
$$

The active light neutrino masses are generated from an inverse seesaw mechanism, and the physical neutrino mass matrices are given by:

$$
\tilde{M}_\nu = m_{\nu D} (M^T)^{-1} \mu M^{-1} m_{\nu D}^T,
$$

$$
M_\nu^{(-)} = -\frac{1}{2} (M + M^T) + \frac{1}{2} \mu,
$$

$$
M_\nu^{(+)} = \frac{1}{2} (M + M^T) + \frac{1}{2} \mu.
$$

here $\tilde{M}_\nu$ is the mass matrix for the active light neutrinos ($\nu_a$), whereas $M_\nu^{(-)}$ and $M_\nu^{(+)}$ are the mass matrices for sterile neutrinos. In the limit $\mu \rightarrow 0$, which corresponds to unbroken lepton number, the active light neutrinos become massless. The smallness of the $\mu$ parameter yields a small mass splitting for the two pairs of sterile neutrinos, thus implying that the sterile neutrinos form pseudo-Dirac pairs.
Then, we obtain that the mass matrix for the light active neutrinos has the form:

\[
\widetilde{M}_\nu = \frac{y_{\nu L}^2 v^2 v_N}{2\sqrt{2} g_{\nu N}^2 v^2 A^2} \begin{pmatrix}
\frac{2}{3} (p + q) & \frac{p}{\sqrt{2}} & \frac{p}{\sqrt{2}} \\
\frac{2}{3} q & -\frac{1}{3} (p + q) & q \\
-\frac{1}{3} (p + q) & 0 & 0
\end{pmatrix} = \begin{pmatrix}
\frac{2}{3} (A + B) & \frac{A}{\sqrt{2}} & \frac{A}{\sqrt{2}} \\
\frac{A}{\sqrt{2}} & B & -\frac{1}{3} (A + B) \\
\frac{A}{\sqrt{2}} & -\frac{1}{3} (A + B) & B
\end{pmatrix},
\]  

The light active neutrino mass matrix given above can be diagonalized by a rotation matrix:

\[
R_\nu = \begin{pmatrix}
\frac{\sqrt{3}}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\
\frac{1}{\sqrt{2}} & \frac{\sqrt{3}}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix} = \begin{pmatrix}
\frac{\sqrt{1 + \sqrt{5}}}{\sqrt{2} \sqrt{3}} & -\frac{\sqrt{1 + \sqrt{5}}}{\sqrt{2} \sqrt{3}} & 0 \\
\frac{\sqrt{1 + \sqrt{5}}}{\sqrt{2} \sqrt{3}} & \frac{\sqrt{1 + \sqrt{5}}}{\sqrt{2} \sqrt{3}} & -\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2} \sqrt{3}} & \frac{1}{\sqrt{2} \sqrt{3}} & \frac{1}{\sqrt{2}}
\end{pmatrix}, \quad \phi_g = \frac{1 + \sqrt{5}}{2},
\]  

according to the following relation:

\[
R_\nu^T M_\nu R_\nu = \begin{pmatrix}
\frac{(3\sqrt{5} + 1) A + 4B}{\sqrt{6}} & 0 & 0 \\
0 & \frac{(3\sqrt{5} - 1) A - 4B}{\sqrt{6}} & 0 \\
0 & 0 & -\frac{2}{3} (A + 4B)
\end{pmatrix}.
\]

Then, the PMNS leptonic mixing matrix has the form:

\[
U = R_\nu^T R_\nu \simeq \begin{pmatrix}
\cos \theta & 0 & -\sin \theta e^{i\phi} \\
0 & 1 & 0 \\
\sin \theta e^{-i\phi} & 0 & \cos \theta
\end{pmatrix} \begin{pmatrix}
\frac{\sqrt{\phi_g}}{\sqrt{5}} & \frac{\sqrt{\phi_g}}{\sqrt{5}} & 0 \\
\frac{\sqrt{\phi_g}}{\sqrt{5}} & \frac{\sqrt{\phi_g}}{\sqrt{5}} & -\frac{1}{\sqrt{2}} \\
\frac{\sqrt{\phi_g}}{\sqrt{5}} & \frac{\sqrt{\phi_g}}{\sqrt{5}} & \frac{1}{\sqrt{2}}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\frac{1}{5} \frac{\sqrt{2} \phi_g \cos \theta - \frac{2}{15} \sqrt{2} e^{i\phi} (\sin \theta)}{\sqrt{\phi_g}} & \frac{1}{5} \frac{\sqrt{2} \phi_g \cos \theta - \frac{2}{15} \sqrt{2} e^{i\phi} \sin \theta}{\phi_g} & -\frac{1}{5} \frac{2 \phi_g \cos \theta - \frac{2}{15} \sqrt{2} e^{i\phi} \sin \theta}{\phi_g} \\
\frac{5}{10} \frac{\sqrt{2} \phi_g \cos \theta + \frac{2}{15} \sqrt{2} e^{-i\phi} \sin \theta}{\sqrt{\phi_g}} & \frac{5}{10} \frac{\sqrt{2} \phi_g \cos \theta + \frac{2}{15} \sqrt{2} e^{-i\phi} (\sin \theta)}{\phi_g} & \frac{1}{2} \sqrt{2}
\end{pmatrix}
\]

The experimental values of the SM charged lepton masses:

\[m_e(m_Z) = 0.4883266 \pm 0.0000017\text{MeV}, \quad m_\mu(m_Z) = 102.87267 \pm 0.00021\text{MeV}, \quad m_\tau(m_Z) = 1747.43 \pm 0.12\text{MeV},\]

neutrino mass squared splittings, leptonic mixing angles and leptonic Dirac CP violating phase, as shown in Table II, can be very well reproduced for the following benchmark point:

\[
M_l = \begin{pmatrix}
0.000485825 & 0 & 0.0343002 - 0.366171i \\
0.0000470777 & 0.101891 & 0 \\
0 & 0 & 1.70203
\end{pmatrix} \text{GeV}, \quad A = 0.236\text{meV}, \quad B = 17.82\text{meV}
\]  

As shown in Figure 1, there is a linear correlation between the different lepton sector observables. As shown in Table II and Figure 1, our model is consistent with the neutrino oscillation experimental data.
Table II: Model and experimental values of the neutrino mass squared splittings, leptonic mixing angles, and $CP$-violating phase. The experimental values are taken from Refs. [34, 35].

| Observable                  | Model value | Neutrino oscillation global fit values (NH) |
|-----------------------------|-------------|-------------------------------------------|
|                             |             | Best fit ±1σ [34] | Best fit ±1σ [35] | 3σ range [34] | 3σ range [35] |
| $\Delta m_{21}^2$ [10$^{-3}$eV$^2$] | 7.50        | 7.50$^{+0.22}_{-0.20}$ | 7.42$^{+0.21}_{-0.20}$ | 6.94 – 8.14 | 6.82 – 8.04 |
| $\Delta m_{32}^2$ [10$^{-3}$eV$^2$] | 2.56        | 2.56$^{+0.03}_{-0.04}$ | 2.517$^{+0.026}_{-0.026}$ | 2.46 – 2.65 | 2.435 – 2.598 |
| $\theta_{12}$ (°)           | 33.15       | 34.3 ± 1.0           | 33.44$^{+0.77}_{-0.74}$ | 31.4 – 37.4 | 31.27 – 35.86 |
| $\theta_{13}$ (°)           | 8.59        | 8.58$^{+0.11}_{-0.10}$ | 8.57 ± 0.12           | 8.16 – 8.94 | 8.20 – 8.93  |
| $\theta_{23}$ (°)           | 45.65       | 48.79$^{+0.93}_{-1.03}$ | 49.2$^{+0.9}_{-1.0}$   | 41.63 – 51.32 | 40.1 – 51.7  |
| $\delta_{CP}$ (°)           | −76.54      | 216$^{+41}_{-35}$ | 197$^{+27}_{-24}$ | 144 – 360 | 120 – 369 |

IV. CONCLUSIONS

We have built a supersymmetric model where leptonic mixing consists in a perturbed golden ratio mixing pattern. The model is based on an $A_5$ family symmetry, supplemented by the $Z_9 \times Z_N$ discrete group. The tiny masses of the light active neutrinos are generated by an inverse seesaw mechanism mediated by right handed Majorana neutrinos. In this framework, renormalizable and non-renormalizable Majorana mass terms generated after the spontaneous breaking of the discrete symmetries. The inverse seesaw mechanism is guaranteed through the smallness of the $\mu$ parameter, which arises from non-renormalizable Yukawa terms involving gauge singlet Majorana neutrinos. The resulting mass matrix for the light active neutrinos features a golden ratio mixing pattern in the family symmetry basis, in this same basis the charged lepton mass matrix is not diagonal. Therefore, the charged lepton sector provides the required perturbation, leading to a leptonic mixing pattern in excellent agreement with the experimental data. Furthermore, our model predicts linear correlations among the leptonic mixing angles and the Dirac leptonic $CP$ violating phase.

Acknowledgments

A.E.C.H is supported by ANID-Chile FONDECYT 1210378, ANID PIA/APOYO AFB180002 and ANID- Programa Milenio - code ICN2019_044. IdMV acknowledges funding from Fundação para a Ciência e a Tecnologia (FCT) through the contract UID/FIS/00777/2020 and was supported in part by FCT through projects CFTP-FCT Unit 777 (UID/FIS/00777/2019), PTDC/FIS-PAR/29436/2017, CERN/FIS-PAR/0004/2019 and CERN/FIS-PAR/0008/2019 which are partially funded through POCTI (FEDER), COMPETE, QREN and EU. AECH thanks the Instituto Superior Técnico, Universidade de Lisboa for hospitality, where part of this work was done.

Appendix A. $A_5$ tensor product rules

In this appendix we show the tensor product rules of the $A_5$ discrete group. We assign $a = (a_1, a_2, a_3)^T$ and $b = (b_1, b_2, b_3)^T$ to the $3$ representation, while $a' = (a'_1, a'_2, a'_3)^T$ and $b' = (b'_1, b'_2, b'_3)^T$ belong to the $3'$ representation. $c = (c_1, c_2, c_3, c_4, c_5)^T$ and $d = (d_1, d_2, d_3, d_4, d_5)^T$ are pentaplets; $f = (f_1, f_2, f_3, f_4)^T$ and $g = (g_1, g_2, g_3, g_4)^T$ are tetraplets.
\[ \mathbf{3} \otimes \mathbf{3} = \mathbf{3}_a + (1 + 5)_s \]

\[ \begin{align*}
1 &= a_1 b_1 + a_2 b_3 + a_3 b_2 \\
3 &= (a_2 b_3 - a_3 b_2, a_1 b_2 - a_2 b_1, a_3 b_1 - a_1 b_3)^T \\
5 &= (a_1 b_1 - \frac{a_2 b_3}{2} - \frac{a_3 b_2}{2}, \sqrt{3} (a_1 b_2 + a_2 b_1), -\sqrt{3} a_2 b_2, -\sqrt{3} a_3 b_3, -\sqrt{3} (a_1 b_3 + a_3 b_1))^T
\end{align*} \]

\[ \mathbf{3}' \otimes \mathbf{3}' = \mathbf{3}'_a + (1 + 5)_s \]

\[ \begin{align*}
1 &= a'_1 b'_1 + a'_2 b'_3 + a'_3 b'_2 \\
3' &= (a'_2 b'_3 - a'_3 b'_2, a'_1 b'_2 - a'_2 b'_1, a'_3 b'_1 - a'_1 b'_3)^T \\
5 &= (a'_1 b'_1 - \frac{a'_2 b'_3}{2} - \frac{a'_3 b'_2}{2}, \sqrt{3} a'_2 b'_3, -\sqrt{3} (a'_1 b'_2 + a'_2 b'_1), -\sqrt{3} a'_3 b'_3, -\sqrt{3} (a'_1 b'_3 + a'_3 b'_1))^T
\end{align*} \]

\[ \mathbf{3} \otimes \mathbf{3}' = 4 + 5 \]

\[ \begin{align*}
4 &= \left( a_2 b'_1 - \frac{a_3 b'_2}{\sqrt{2}}, -a_1 b'_2 + \frac{a_3 b'_1}{\sqrt{2}}, a_1 b'_3 - \frac{a_2 b'_3}{\sqrt{2}}, -a_3 b'_1 + \frac{a_2 b'_3}{\sqrt{2}} \right)^T \\
5 &= \left( a_1 b'_1 - \frac{a_2 b'_2}{\sqrt{3}}, a_1 b'_2 + \frac{\sqrt{2} a_3 b'_1}{\sqrt{3}}, a_1 b'_3 + \frac{\sqrt{2} a_2 b'_3}{\sqrt{3}}, a_3 b'_1 + \frac{\sqrt{2} a_2 b'_3}{\sqrt{3}} \right)^T
\end{align*} \]
\[3 \otimes 4 = 3' + 4 + 5\]

\[3' = (a_2g_4 - a_3g_1, \frac{1}{\sqrt{2}}(\sqrt{2}a_1g_2 + a_2g_1 + a_3g_3), -\frac{1}{\sqrt{2}}(\sqrt{2}a_1g_3 + a_2g_2 + a_3g_4))^T\]

\[4 = (a_1g_1 + \sqrt{2}a_3g_2, -a_1g_2 + \sqrt{2}a_2g_1, a_1g_3 - \sqrt{2}a_3g_4, -a_1g_4 - \sqrt{2}a_2g_3))^T\]

\[5 = (a_3g_1 + a_2g_4, \frac{3}{\sqrt{3}}(\sqrt{2}a_1g_1 - a_3g_2), \frac{1}{\sqrt{6}}(\sqrt{2}a_1g_2 - 3a_3g_3 + a_2g_1),\]

\[\frac{1}{\sqrt{6}}(\sqrt{2}a_1g_3 - 3a_2g_2 + a_3g_4), \frac{2}{\sqrt{3}}(-\sqrt{2}a_1g_4 + a_3g_2))^T\]

\[3' \otimes 4 = 3 + 4 + 5\]

\[3 = (a'_2g_3 - a'_3g_2, \frac{1}{\sqrt{2}}(\sqrt{2}a'_1g_1 + a'_2g_4 - a'_3g_3), \frac{1}{\sqrt{2}}(-\sqrt{2}a'_1g_4 + a'_2g_2 - a'_3g_1))^T\]

\[4 = (a'_1g_1 + \sqrt{2}a'_3g_2, a'_1g_2 - \sqrt{2}a'_4g_4, -a'_1g_3 + \sqrt{2}a'_2g_1, -a'_1g_4 - \sqrt{2}a'_2g_3))^T\]

\[5 = (a'_3g_1 + a'_2g_4, \frac{1}{\sqrt{6}}(\sqrt{2}a'_1g_1 - 3a'_3g_4 - a'_3g_3), -\frac{2}{\sqrt{3}}(\sqrt{2}a'_1g_2 + a'_3g_4),\]

\[-\frac{1}{\sqrt{3}}(-\sqrt{2}a'_1g_4 + 3a'_3g_1 + a'_2g_2))^T\]

\[3 \otimes 5 = 3 + 3' + 4 + 5\]

\[3 = \left(\frac{2a_1c_1}{\sqrt{3}} + a_3c_2 - a_2c_5, \frac{a_2c_1}{\sqrt{3}}, \frac{a_1c_2}{\sqrt{3}} - \sqrt{2}a_3c_3, -\frac{a_3c_1}{\sqrt{3}} - a_1c_5 - \sqrt{2}a_2c_4\right)^T\]

\[3' = \left(\frac{a_1c_1}{\sqrt{3}} + \frac{a_2c_5 - a_3c_2}{\sqrt{3}}, \frac{a_1c_3 + \sqrt{2}(a_3c_4 - a_2c_5)}{\sqrt{3}}, \frac{a_1c_4 + \sqrt{2}(a_2c_3 + a_3c_5)}{\sqrt{3}}\right)^T\]

\[4 = \left(4a_1c_2 + 2\sqrt{3}a_2c_1 + \sqrt{2}a_3c_3, 2a_1c_3 - 2\sqrt{2}a_3c_2 - 3\sqrt{2}a_3c_4, \right.\]

\[2a_1c_4 - 3\sqrt{2}a_2c_3 + 2\sqrt{2}a_3c_5, -4a_1c_5 + \sqrt{2}a_2c_4 + 2\sqrt{3}a_3c_1)^T\]

\[5 = \left(\frac{a_2c_3 + a_3c_2, a_2c_1 - \frac{a_1c_2 + \sqrt{2}a_3c_3}{\sqrt{3}}, \frac{a_1c_4 + \sqrt{2}a_2c_2}{\sqrt{3}}, \frac{2a_1c_4 - \sqrt{2}a_3c_5}{\sqrt{3}}, -\frac{a_1c_5 - \sqrt{2}a_2c_4}{\sqrt{3}}\right)^T\]

\[3' \otimes 5 = 3 + 3' + 4 + 5\]

\[3 = \left(a'_1c_1 + \frac{a'_3c_3 + a'_2c_4}{\sqrt{3}}, \frac{-a'_1c_2 + \sqrt{2}(a'_3c_4 + a'_2c_5)}{\sqrt{3}}, \frac{a'_1c_5 + \sqrt{2}(a'_2c_3 - a'_3c_2)}{\sqrt{3}}\right)^T\]

\[3' = \left(\frac{2a'_1c_1}{\sqrt{3}} - a'_3c_3 - a'_2c_4, \frac{-a'_1c_2 + \sqrt{2}a'_3c_3 - a'_3c_5}{\sqrt{3}}, -\frac{a'_3c_1}{\sqrt{3}} - a'_1c_4 + \sqrt{2}a'_2c_2\right)^T\]

\[4 = \left(2a'_1c_2 + 3\sqrt{2}a'_2c_1 - 2\sqrt{2}a'_3c_3, -4a'_1c_3 + 2\sqrt{3}a'_2c_1 + \sqrt{2}a'_3c_5, \right.\]

\[-4a'_1c_4 - \sqrt{2}a'_2c_2 + 2\sqrt{3}a'_3c_1, -2a'_1c_5 - 2\sqrt{2}a'_2c_3 - 3\sqrt{2}a'_3c_2)^T\]

\[5 = \left(\frac{a'_2c_4 - a'_3c_3, 2a'_1c_2 + \sqrt{2}a'_3c_4}{\sqrt{3}}, -\frac{a'_2c_1 - a'_1c_4 - \sqrt{2}a'_3c_5}{\sqrt{3}}, \right.\]

\[a'_3c_1 + \frac{a'_1c_4 + \sqrt{2}a'_2c_2}{\sqrt{3}}, -2a'_1c_5 + \sqrt{2}a'_3c_3\right)^T\]
\[ 4 \otimes 4 = (3 + 3')_a + (1 + 4 + 5)_s \]

\[
\begin{align*}
1 &= f_{1g4} + f_{2g5} + f_{3g2} + f_{4g1} \\
3 &= (f_{1g4} - f_{4g1} + f_{3g2} - f_{2g3})\sqrt{2}(f_{2g4} - f_{4g2}), \sqrt{2}(f_{1g3} - f_{3g1})^T \\
3' &= (f_{1g4} - f_{4g1} + f_{2g3} - f_{3g2})\sqrt{2}(f_{4g1} - f_{4g3}), \sqrt{2}(f_{1g2} - f_{2g1})^T \\
4 &= (f_{3g3} - f_{4g2} - f_{2g4} + f_{1g9} + f_{3g4} + f_{4g3} - f_{1g4} - f_{1g2} - f_{2g3} - f_{3g2} + f_{1g3} + f_{3g1})^T \\
5 &= (f_{1g4} + f_{4g1} - f_{3g2} - f_{2g3} - \frac{2}{3}(2f_{3g3} + f_{2g4} + f_{4g2}), \frac{2}{3}(-2f_{1g1} + f_{3g4} + f_{4g3}), \\
&\quad \sqrt{2}\left(\frac{2}{3}(-2f_{4g4} + f_{2g1} + f_{1g2}), \frac{2}{3}(2f_{2g2} + f_{1g3} + f_{3g1})\right)^T \\
\end{align*}
\]

\[ 4 \otimes 5 = 3 + 3' + 4 + 5 + 5 \]

\[
\begin{align*}
3 &= (4f_{1c5} - 4f_{4c4} - 2f_{3c3} - 2f_{2c4}, -2\sqrt{3}f_{1c1}^T - \sqrt{2}(2f_{2c5} - 3f_{3c4} + f_{4c3}), \\
&\quad \sqrt{2}(-f_{1c4} + 3f_{2c3} + 2f_{3c2} - 2\sqrt{3}f_{1c1})^T \\
3' &= (2f_{1c5} - 2f_{4c2} + 4f_{3c4} + 4f_{2c4}, -2\sqrt{3}f_{1c1} + \sqrt{2}(2f_{4c4} + 3f_{1c2} - f_{3c5}), \\
&\quad \sqrt{2}(-f_{2c2} - 3f_{4c5} + 2f_{3c1} - 2\sqrt{3}f_{3c1})^T \\
4 &= (3f_{1c1} + \sqrt{6}(f_{2c5} + f_{3c4} - 2f_{4c3}, -3f_{2c1} + \sqrt{6}(f_{4c4} - f_{1c2} + 2f_{4c3}), \\
&\quad -3f_{3c1} + \sqrt{6}(f_{1c3} + f_{1c5} - 2f_{2c2}, 3f_{4c1} + \sqrt{6}(f_{2c3} - f_{3c2} - 2f_{1c4}))^T \\
5_1 &= (f_{1c5} + 2f_{2c4} - 2f_{3c3} + f_{4c2}, -2f_{1c1} + \sqrt{6}f_{2c5}, f_{2c1} + \sqrt{3}(2f_{1c1} - f_{3c5} + 2f_{4c4}), \\
&\quad -f_{3c1} - \sqrt{2}(f_{1c2} + f_{4c5} + 2f_{1c3}), -2f_{1c1} - \sqrt{6}f_{3c2})^T \\
5_2 &= (f_{2c4} - f_{3c3}, -f_{1c1} + \frac{2f_{2c5} - f_{3c4} - f_{4c3}}{\sqrt{6}}, -\frac{2}{3}(f_{1c2} + f_{3c5} - f_{4c4}), \\
&\quad -\sqrt{2}(f_{1c3} + f_{2c2} + f_{4c5}), -f_{4c1} - \frac{2f_{3c2} + f_{1c4} + f_{2c3}}{\sqrt{6}})^T \\
\end{align*}
\]
\[ 5 \otimes 5 = (3 + 3') + (1 + 4 + 5 + 5) \]

\[ 3 = (2(c_4 d_5 - c_3 d_2) + c_2 d_5 - c_5 d_2, \sqrt{3}(c_2 d_1 - c_1 d_2) + \sqrt{2}(c_3 d_5 - c_5 d_3), \]
\[ \sqrt{3}(c_5 d_1 - c_1 d_5) + \sqrt{2}(c_3 d_2 - c_2 d_4))^T \]

\[ 3' = (2(c_2 d_5 - c_3 d_2) + c_3 d_4 - c_4 d_3, \sqrt{3}(c_3 d_1 - c_1 d_3) + \sqrt{2}(c_4 d_5 - c_5 d_4), \]
\[ \sqrt{3}(c_1 d_4 - c_4 d_1) + \sqrt{2}(c_4 d_2 - c_2 d_3))^T \]

\[ 4_a = ((c_1 d_2 + c_2 d_1) - \frac{(c_3 d_5 + c_5 d_3) - 4c_4 d_4}{\sqrt{6}}, -(c_1 d_3 + c_3 d_1) - \frac{(c_4 d_5 + c_5 d_4) - 4c_2 d_2}{\sqrt{6}}, \]
\[ (c_1 d_4 + c_4 d_1) - \frac{(c_2 d_3 + c_3 d_2) + 4c_5 d_5}{\sqrt{6}}, (c_1 d_5 + c_5 d_1) - \frac{(c_2 d_4 + c_4 d_2) + 4c_3 d_3}{\sqrt{6}})^T \]

\[ 4_a = ((c_1 d_2 - c_2 d_1) + \frac{3}{2}(c_3 d_5 - c_5 d_3), (c_1 d_3 - c_3 d_1) + \frac{3}{2}(c_4 d_5 - c_5 d_4), \]
\[ (c_1 d_4 - c_4 d_1) + \frac{3}{2}(c_2 d_2 - c_3 d_3), (c_1 d_5 - c_5 d_1) + \frac{3}{2}(c_2 d_4 - c_4 d_2))^T \]

\[ 5_1 = (c_1 d_1 + c_2 d_5 + c_3 d_2 + \frac{c_4 d_4 + c_4 d_4}{2}, -(c_1 d_2 + c_2 d_1) + \frac{3}{2}c_4 d_4, \frac{1}{2}(c_1 d_3 + c_3 d_1 - \sqrt{6}(c_4 d_5 + c_5 d_4)), \]
\[ \frac{1}{2}(c_1 d_4 + c_4 d_1 + \sqrt{6}(c_2 d_3 + c_3 d_2)), -(c_1 d_5 + c_5 d_1) - \frac{3}{2}c_3 d_3)^T \]

\[ 5_2 = \left( \frac{2c_1 d_1 + c_2 d_5 + c_3 d_2}{2}, -3(c_1 d_2 + c_2 d_1) + \sqrt{6}(2c_4 d_4 + c_3 d_5 + c_5 d_3), -\frac{2c_4 d_5 + 2c_3 d_4 + c_2 d_2}{2}, \right. \]
\[ \left. \frac{2c_2 d_2 + 2c_3 d_2 - c_5 d_5}{2}, -3(c_1 d_5 + c_5 d_1) + \sqrt{6}(-2c_3 d_4 + c_2 d_4 + c_4 d_2) \right)^T \]

\[ 1 = c_1 d_1 + c_3 d_4 + c_4 d_3 - c_2 d_5 - c_5 d_2 \]
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