ON THE INTERSECTION DENSITY OF PRIMITIVE GROUPS OF DEGREE A PRODUCT OF TWO ODD PRIMES

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Abstract. A subset $F$ of a finite transitive group $G \leq \text{Sym}(\Omega)$ is intersecting if for any $g, h \in F$ there exists $\omega \in \Omega$ such that $\omega^g = \omega^h$. The intersection density $\rho(G)$ of $G$ is the maximum of $\left\{ \frac{|F|}{|G_\omega|} : F \subset G \text{ is intersecting} \right\}$, where $G_\omega$ is the stabilizer of $\omega$ in $G$. In this paper, it is proved that if $G$ is an imprimitive group of degree $pq$, where $p$ and $q$ are distinct odd primes, with at least two systems of imprimitivity then $\rho(G) = 1$. Moreover, if $G$ is primitive of degree $pq$, where $p$ and $q$ are distinct odd primes, then it is proved that $\rho(G) = 1$, whenever the socle of $G$ admits an imprimitive subgroup.

1. Introduction

Let $\Omega$ be a finite set and $G \leq \text{Sym}(\Omega)$ be a finite transitive and faithful group. A subset $F \subset G$ is intersecting if any two permutations of $F$ agree on at least one element of $\Omega$. That is, for all $\sigma, \pi \in F$, there exists $\omega \in \Omega$ such that $\omega^\sigma = \omega^\pi$. If $F \subset G$ is an intersecting family, then its intersection density is the rational number

$$\rho(F) := \frac{|F|}{|G_\omega|},$$

where $\omega \in \Omega$. The intersection density of the group $G$ is the rational number $\rho(G) := \max \{ \rho(F) : F \subset G \text{ is intersecting} \}$. The intersection density of groups was first introduced in [24]. Note that $\rho(G) \geq 1$ since a point-stabilizer of $G$ is an intersecting subset of $G$. We say that the transitive group $G \leq \text{Sym}(\Omega)$ has the Erdős-Ko-Rado property or EKR property if $\rho(G) = 1$. Moreover, we say that $G$ has the strict-EKR property if it has the EKR property and the only intersecting families with intersection density equal to 1 are cosets of point-stabilizers.

The study of transitive groups having the EKR property started with the 1977 paper of Deza and Frankl [10]. It was proved in this paper that $\text{Sym}(n)$ has the EKR property. In 2004, Cameron and Ku [9], independently Larose and Malvenuto [23], proved that $\text{Sym}(n)$ has the strict-EKR property. Since then, many works on the EKR property of transitive groups have appeared in the literature [2, 3, 4, 7, 12, 13, 15, 24, 27, 30, 34]. Examples of groups having the EKR property are finite doubly transitive groups [30] and transitive groups admitting sharply transitive sets. Examples of groups that do not have the EKR property are given in [19, 28, 31]. The
following conjecture on the intersection density of transitive groups of certain degrees was posed in [29].

**Conjecture 1.1.** Let $G \leq \text{Sym}(\Omega)$ be a transitive group.

(a) If $|\Omega|$ is a prime power, then $\rho(G) = 1$.

(b) If $|\Omega| = 2p$, where $p$ is an odd prime, then $\rho(G) \leq 2$. Moreover, this upper bound is tight for any odd prime $p$.

(c) If $|\Omega| = pq$, where $p$ and $q$ are distinct odd primes, then $\rho(G) = 1$.

Conjecture 1.1 (a) was recently proved by Li et al. [24], and independently, Hujdurović et al. [20].

**Theorem 1.2 ([20, 24]).** If $G$ is a transitive group of prime power degree, then $\rho(G) = 1$.

In this paper, we are interested in the intersection density of transitive groups of degree $pq$, where $p$ and $q$ are distinct primes with $p > q$. In [31], it was proved that when $q = 2$, the intersection density of transitive groups of degree $2p$ is in the set $[1, 2] \cap \mathbb{Q}$. In [29, 31], it was proved that there exists a transitive group of degree $2\ell$, for any odd $\ell$, with intersection density equal to 2. These results settle Conjecture 1.1(b).

A question raised in [31, Question 6.1] is whether a transitive group $G$ of degree $2p$ always has integral intersection density, that is, $\rho(G) \in \{1, 2\}$. This question was recently answered by Hujdurović et al. in [20].

**Theorem 1.3 ([20]).** If $G$ is a transitive group of degree $2p$, where $p$ is an odd prime, then $\rho(G) \in \{1, 2\}$.

For the rest of this paper, we suppose that $G$ is a transitive group of degree $pq$, where $p$ and $q$ are distinct primes with $p > q$. Hujdurović et al. [18] recently proved that Conjecture 1.1(c) fails for certain imprimitive groups of degree $pq$. An example of groups for which the conjecture fails is TransitiveGroup(33,18). Further, it was proved in [18] that if an imprimitive group of degree $pq$ has a block of size $p$, then it has the EKR property. In this paper, we will see in fact that the imprimitive groups violating Conjecture 1.1(c) have exactly one system of imprimitivity, with blocks of size $q$.

In this paper, we prove that Conjecture 1.1(c) holds for imprimitive groups of degree $pq$ with at least two systems of imprimitivity. Our first main result is stated as follows.

**Theorem 1.4.** If $G \leq \text{Sym}(\Omega)$ is an imprimitive group of degree $pq$, where $p$ and $q$ are distinct odd primes, with at least two different systems of imprimitivity then $\rho(G) = 1$.

Next, we consider the primitive cases (see [18, Section 2] for a comprehensive description of these groups). If $G$ is doubly transitive of degree $pq$, then by the main result of [30], $\rho(G) = 1$. Therefore, we assume that $G$ is simply primitive (i.e., a primitive group which is not doubly transitive). Recall that if $G$ is a group, then the socle of $G$, denoted $\text{Soc}(G)$, is the subgroup generated by all the minimal normal subgroups of $G$. If $G \leq \text{Sym}(\Omega)$ is primitive, then its socle $\text{Soc}(G)$ must be transitive.
since it is normal in $G$ and $G$ is faithful. Using [29, Lemma 6.5], we deduce that $ho(G) \leq \rho(Soc(G))$. Therefore, in order to prove Conjecture 1.1(c) for the primitive cases, it is enough to prove that the possible socles of primitive groups of degree $pq$ have the EKR property. The socle of primitive groups of degree $pq$ have been classified by Marušič and Scapellato [26]. This classification is given in Table 1. Among the possible socles of $G$, there are seven families. Our second main result concerns the intersection density of primitive groups of degree $pq$.

**Theorem 1.5.** If $G$ is primitive of degree $pq$ and $Soc(G)$ is one of the groups in lines $1-11, 14, 16, 17$ of Table 1, then $\rho(G) = 1$. In particular, if $Soc(G)$ contains an imprimitive group, then $\rho(G) = 1$.

This paper is organized as follows. In Section 2, we recall some preliminary results on derangement graphs and the conjugacy class scheme. In Section 3, we prove that the groups in lines 1-8, 14, and 16 have the EKR property. In Section 6, we prove that if $G \leq Sym(\Omega)$ has degree $pq$ with socle equal to the group in line 9 – 10, then $G$ has the EKR property. To prove this, we recall some results on the representation theory of the alternating group in Section 4 and refine the result on the EKR property for $Sym(n)$ acting on the 2-subsets [28] in Section 5. In Section 7, we prove that the group in line 11 has the EKR property. The proof of Theorem 1.4 is given in Section 8. Finally, we end this paper by given some interesting directions for further research on the EKR property of primitive groups of degree $pq$ in Section 9.

### Table 1. Socle of simply primitive groups of degree $pq$.  

| Line | $S$ | $(p, q)$ | action | Information |
|------|-----|----------|---------|-------------|
| 1    | $Alt(7)$ | $(7, 5)$ | triples |  |
| 2    | $PSL(4, 2)$ | $(5, 5)$ | 2-spaces |  |
| 3    | $PSL(5, 2)$ | $(31, 5)$ | 2-spaces |  |
| 4    | $PSL(2, 23)$ | $(23, 11)$ | cosets of Sym(4) |  |
| 5    | $PSL(2, 11)$ | $(11, 5)$ | cosets of $Alt(4)$ |  |
| 6    | $M_{11}$ | $(11, 5)$ |  |
| 7    | $M_{23}$ | $(11, 7)$ |  |
| 8    | $PSL(2, p)$ | $(p, p = \frac{p+1}{2})$ | pairs | $p \geq 5$ |
| 9    | $Alt(p)$ | $(p, p+1)$ | pairs | $p \geq 5$ |
| 10   | $Alt(p+1)$ | $(p, p+1)$ | pairs | $p \geq 5$ |
| 11   | $PSp(4, k)$ | $(k^2 + 1, k + 1)$ | 1-spaces | $p, q$ are Fermat primes |
| 12   | $PSL(2p, 2)$ | $(2^d - 1, 2^d - 1 + 1)$ | singular 1-spaces | $d = 1$ and $d$ is a Fermat prime |
| 13   | $PSL(2, p)$ | $(p, \frac{p+1}{2})$ | cosets of $D_{p-1}$ | $p \geq 13$ and $p \equiv 1(mod 4)$ |
| 14   | $PSL(2, p)$ | $(p, \frac{p+1}{2})$ | cosets of $D_{p+1}$ | $p \geq 13$ and $p \equiv 3(mod 4)$ |
| 15   | $PSL(2, d)$ | $(d^2 + 1, q)$ | cosets of $PGL(2, q)$ |  |
| 16   | $PSL(2, p)$ | $(19, 3), (29, 7), (59, 29)$ | cosets of $Alt(5)$ |  |
| 17   | $PSL(2, 61)$ | $(61, 31)$ | cosets of $Alt(5)$ |  |

2. **Background results**

In this section, we give a brief review of the EKR theory of transitive permutation groups. We let $G \leq Sym(\Omega)$ be a finite transitive group throughout this section.
2.1. Derangement graphs. A derangement of $G$ is a permutation without a fixed point. A well-known result due to Camille Jordan \cite{Jordan1872} asserts that a finite transitive group always has a derangement. Let $\text{Der}(G)$ be the set of all derangements of $G$. The derangement graph $\Gamma_G$ of $G$ is the graph whose vertex-set is $G$ and two group elements $g, h \in G$ are adjacent if $hg^{-1} \in G$. It is not hard to see that $\Gamma_G$ is the Cayley graph of $G$ with connection set equal to $\text{Der}(G)$. Consequently, $\Gamma_G$ is regular of valency $|\text{Der}(G)|$ and is vertex transitive. Moreover, $\Gamma_G$ is a normal Cayley graph since $\text{Der}(G)$ is the union of conjugacy classes of derangements.

If $\mathcal{F} \subset G$ is intersecting, then for any $g, h \in \mathcal{F}$, $\omega^g = \omega^h \iff \omega = \omega^{hg^{-1}}$ for all $\omega \in \Omega$. In other words, $hg^{-1} \notin \text{Der}(G)$, meaning that $g$ and $h$ are non-adjacent in $\Gamma_G$. Consequently, $\mathcal{F}$ is intersecting if and only if it is a coclique or an independent set of $\Gamma_G$. Therefore, the problem of studying the maximum intersecting families of $G \leq \text{Sym}(\Omega)$ reduces to the study of the maximum cocliques of $\Gamma_G$.

2.2. Maximum cocliques. Let $X = (V, E)$ be an undirected graph. A clique in $X$ is a subgraph of $X$ such that every pair of vertices are adjacent; that is, a complete subgraph of $X$. A coclique of $X$ is a subgraph of $X$ in which every pair of vertices are non-adjacent; that is, an empty subgraph of $X$. The size of the maximum clique and coclique of $X$ are $\omega(X)$ and $\alpha(X)$, respectively.

If $X = (V, E)$ is a vertex-transitive graph (i.e., its full automorphism group acts transitively on $V(X)$), then one can obtain bounds on the size of the maximum cocliques of $X$ using the maximum cliques. This bound is given in the next lemma.

**Lemma 2.1** (Clique-coclique Bound \cite{Godsil2001}). If $X = (V, E)$ is a vertex-transitive graph, then $\omega(X)\alpha(X) \leq |V(X)|$. Equality holds if and only if every clique of $X$ intersects every coclique of $X$ at a unique vertex.

**Corollary 2.2.** Let $G \leq \text{Sym}(\Omega)$. If $G$ admits a regular subgroup, then $\rho(G) = 1$.

**Proof.** Let $H$ be a regular subgroup of $G$ and assume that $\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$. First, we note that if $S$ is a clique of $\Gamma_G$, then $|S| \leq |\Omega|$. Since there is at most one permutation that maps $\omega_i$ to $\omega_j$, for any $i \in \{1, 2, \ldots, n\}$. For any $\omega, \omega' \in \Omega$, there is exactly one element of $H$ that maps $\omega$ to $\omega'$. Consequently, no two permutations of $H$ agree on an element of $\Omega$. In other words, $H$ is a clique in the derangement graph $\Gamma_G$ of size $|H| = |\Omega|$. By the Clique-coclique bound, we have $\alpha(\Gamma_G) \leq \frac{|G|}{|H|}$ and so $\rho(G) = 1$. \hfill $\square$

The next result gives an upper bound on the size of the maximum coclique using an algebraic method. If $S \subset V(X)$, then the characteristic vector of $S$ is the $\{0, 1\}$-vector of $\mathbb{Z}^{|V(X)|}$ indexed by $V(X)$ whose $s$-entry is equal to 1 if $s \in S$, and 0 otherwise. Let $\mathbf{1}$ be the vector whose entries are all equal to 1 (its dimension should be clear from the context).

**Lemma 2.3** (Hoffman bound - Ratio Bound \cite{Hoffman1952}). Let $X = (V, E)$ be a regular graph with degree equal to $d$ and minimum eigenvalue $\tau$. Then,
\[
\alpha(X) \leq \frac{\tau}{\tau - d}|V(X)|.
\]
Moreover, if equality holds and $C$ is a maximum coclique of $X$, then the translated characteristic vector $v_C - \frac{|C|}{|V(X)|}1$ is an eigenvector with eigenvalue $\tau$.

A weighted adjacency matrix corresponding to a graph $X = (V, E)$ is a $|V(X)| \times |V(X)|$ real symmetric matrix $A$ whose row sum is constant and the entry $A_{u,v} = 0$ if $u \not\sim_X v$. The following result is a refinement of the Ratio Bound.

**Lemma 2.4** (Weighted Ratio Bound [16, 17]). Let $X = (V, E)$ be a regular graph and let $A$ be a weighted adjacency matrix of $X$. Suppose that $d$ and $\tau$ are respectively the row sum and the minimum eigenvalue of $A$. Then,

$$\alpha(X) \leq \frac{\tau}{\tau - d}|V(X)|.$$

### 2.3. Conjugacy class schemes.

Throughout this subsection we let $G$ be an abstract group and $C$ be the set of all conjugacy classes of $G$. We say that a conjugacy class $C$ of $G$ is inverse-closed if $x^{-1} \in C$, for any $x \in C$.

For any conjugacy class $C \in C$, define the $\{0, 1\}$-matrix $A_C$ whose rows and columns are indexed by group elements and $A_{g,h} = 1$ if and only if $hg^{-1} \in C$. The conjugacy class scheme of $G$ is the association scheme obtained from the set of matrices $\mathcal{A}(G) = \{A_C \mid C \in C\}$.

If the conjugacy classes of $G$ are all inverse-closed, then every matrix in $\mathcal{A}(G)$ is symmetric, that is, the conjugacy class association scheme of $G$ is symmetric. Moreover, the matrices in $\mathcal{A}(G)$ commute with each other, so the matrices in $\mathcal{A}(G)$ are simultaneously diagonalizable. Therefore, an eigenvalue of a linear combination of matrices of $\mathcal{A}(G)$ is a linear combination of eigenvalues of matrices of $\mathcal{A}(G)$.

If $G$ has a conjugacy class which is not inverse-closed, then one of the matrices in $\mathcal{A}(G)$ is not symmetric. It is still possible to prove that the eigenvalues of a linear combination of matrices of $\mathcal{A}(G)$ are linear combinations of eigenvalues of the matrices in $\mathcal{A}(G)$ (see [16, Section 3.4]). Due to the existence of the idempotents of $\mathcal{A}(G)$ (see [16, Theorem 3.4.4]), the matrices in $\mathcal{A}(G)$ are still diagonalizable. Since the matrices of $\mathcal{A}(G)$ are pairwise commuting, they are simultaneously diagonalizable. Hence, we obtain similar properties as when the matrices of $\mathcal{A}(G)$ are symmetric.

Therefore, the matrices of $\mathcal{A}(G)$ admit an orthogonal basis $\mathcal{B}$ of eigenvectors. Let $v \in \mathcal{B}$. For any $C \in C$, let $\lambda_C$ be the eigenvalue of $A_C$ corresponding to the eigenvector $v$. The discussion in the two previous paragraphs leads to the following straightforward result.

**Lemma 2.5.** Let $v \in \mathcal{B}$. If $A = \sum_{C \in C} k_C A_C$ is a linear combination of $\mathcal{A}(G)$, then $v$ is an eigenvector of $A$ with eigenvalue

$$\sum_{C \in C} k_C \lambda_C.$$
graph can be determined by a result of Babai in [6]. From this, the spectrum of the derangement graph of a transitive group $G$ can be found. Recall that a Cayley graph is normal if its connection set is invariant under conjugation. The following result gives the eigenvalues of normal Cayley graphs.

**Lemma 2.6** ([6]). Let $X = \text{Cay}(G, D)$ be a Cayley graph such that $D$ is invariant by conjugation. Let $(X_1, V_1), (X_2, V_2), \ldots, (X_k, V_k)$ be a complete list of distinct irreducible representations of $G$ and let $\chi_i$ be the character afforded by $X_i$, for any $i \in \{1, 2, \ldots, k\}$. Then,

$$CG \cong \bigoplus_{i=1}^{k} U_i,$$

where $U_i$ is the sum of all submodules isomorphic to $V_i$ in the regular $CG$-module. Moreover, $U_i$ is an eigenspace of $A$ with eigenvalue

$$\xi_{x_i} = \frac{1}{\dim X_i} \sum_{g \in D} \chi_i(g),$$

for any $i \in \{1, 2, \ldots, k\}$. The dimension of each eigenspace $U_i$ is equal to

$$\sum_{\{j | \xi_{x_j} = \xi_{x_i}\}} \chi_j(1)^2.$$

Using Lemma 2.6, we obtain the following.

**Corollary 2.7.** Let $C \in \mathcal{C}$ and $g$ be a representative of $C$. The matrix $A_C$ is the adjacency matrix of the Cayley graph $\text{Cay}(G, C)$ and its eigenvalues are of the form

$$\xi_{\chi} = \frac{1}{\chi(1)} |C| \chi(g),$$

for any irreducible character $\chi$ of $G$.

From this corollary, we get an expression of the eigenvalues of any weighted adjacency matrix of the derangement graph of a transitive group $G \leq \text{Sym}(\Omega)$. Let $\mathcal{D} = \{D_1, D_2, \ldots, D_k\}$ be the set of all conjugacy classes of derangements of $G$. For any $i \in \{1, 2, \ldots, k\}$, let $g_i$ be an arbitrary element of $D_i$. Consider the weighted adjacency matrix

$$A = \sum_{i=1}^{k} \omega_i A_{D_i}.$$

The eigenvalues of $A$ are given in the following lemma.

**Lemma 2.8.** The eigenvalues of $A$ are of the form

$$\frac{1}{\chi(id)} \sum_{i=1}^{k} \omega_i \chi(g_i)|D_i|,$$

where $\chi$ is an irreducible character of $G$. 
3. Primitive cases

Let $G \leq \text{Sym}(\Omega)$ be a primitive group of degree $pq$. If $G$ is doubly transitive, then $G$ has the EKR property (see [30]). Hence, we may assume that $G$ is simply primitive (primitive but not doubly transitive). Let $S := \text{Soc}(G)$ be the socle of $G$. Since $S$ is the subgroup generated by the minimal normal subgroups of $G$, it is easy to see that $S \triangleleft G$. As $G$ is primitive, $S$ is transitive. A consequence of the No-Homomorphism Lemma [5] (see also [29, Lemma 6.5]) is that if $S$ has the EKR property, then so does $G$. Therefore, to prove the primitive case of Conjecture 1.1(c), it is enough to prove that the socle of $G$ has the EKR property.

3.1. Simply primitive. We use the classification of the socles of simply primitive groups of degree $pq$ from [26] to determine their EKR property. The classification is given in Table 1.

The straightforward cases. We use Sagemath [35] to verify that the groups in lines 1-8 and 16 have the EKR property.

The group $\text{Alt}(7)$ of degree 35 in line 1 has its point-stabilizers of size 72 which is equal to the upper bound given by the Ratio Bound. Therefore, $\text{Alt}(7)$ of degree 35 has the EKR property. Similarly, the Ratio Bound on the derangement graph of the groups in line 2 and line 16 with $p = 19$, are equal to the size of their respective point-stabilizers.

The group $\text{PSL}(5, 2)$ in line 3 has a regular subgroup isomorphic to $C_{31} \rtimes C_5$. Therefore, it has the EKR property. Similarly, the groups in lines 4-6, 8 and 16 when $p \in \{29, 59\}$ have regular subgroups. Therefore, they have the EKR property.

Infinite families. Now, we consider the groups in line 14. When $p \equiv 3 \pmod{4}$, the group $K = C_p \rtimes C_{(p-1)/2}$ is a regular subgroup of $\text{PSL}(2, p)$ with the action described in line 14. Therefore, the group in line 14 has the EKR property (see [24] for details).

For the case of the groups in lines 9-10, we prove a more general statement. We prove that for any $n \geq 3$, $\text{Alt}(n)$ acting on the 2-subsets of $\Omega = \{1, 2, \ldots, n\}$ has the EKR property. This can be further reformulated in terms of the natural action of $\text{Sym}(n)$ on $\Omega$. We say that $F \subset \text{Sym}(n)$ is 2-setwise intersecting if any two permutations of $F$ agree on a 2-subset of $\Omega$. It is not hard to see that proving that the group $\text{Alt}(n)$ acting on the 2-subsets of $\Omega$ has the EKR property is equivalent to proving that for any $n \geq 3$, if $F \subset \text{Alt}(n)$ is 2-setwise intersecting, then $|F| \leq \frac{|\text{Alt}(n)|}{(n-2)!}$. We prove the following.

Theorem 3.1. For $n \geq 19$, if $F \subset \text{Alt}(n)$ is 2-setwise intersecting, then $|F| \leq (n-2)!$.

The proof of this theorem is given in Section 6.

For $(p, \frac{p+1}{2}) \in \{(7, 3), (11, 5)\}$, it is easy to see that $\text{Alt}(p)$ acting on the 2-subsets of $\{1, 2, \ldots, p\}$ contains a regular subgroup, thus, $\rho(\text{Alt}(p)) = 1$. For $(p, \frac{p+1}{2}) \in \{(5, 3), (13, 7)\}$, we verified on Sagemath that the action of $\text{Alt}(p+1)$ on the 2-subsets
of \{1, 2, \ldots, p + 1\} has the EKR property; thus \(\rho(\text{Alt}(p + 1)) = 1\). This settles the EKR property for the groups in lines 9 and 10.

For the group in line 11, it was shown in [26] that \(\text{PSp}(4, k)\) admits an imprimitive subgroup which is isomorphic to \(\text{PSL}(2, k^2)\). In Section 7, we prove that this subgroup has the EKR property. Consequently, the group in line 11 has the EKR property.

4. Representation Theory of the Symmetric Group and the Alternating Group

In this section, we give a brief review on the representation theory of the symmetric group and the alternating group. For more detailed results on the representation theory of the symmetric group and the alternating group, we refer the reader to [14] and [33].

Recall that the irreducible \(\mathbb{C}\text{Sym}(n)\)-modules are the Specht modules. These irreducible representations of \(\text{Sym}(n)\) are indexed by the partitions of \(n\). For more details on this, see [33, Section 2.3].

Given a partition \(\lambda \vdash n\), the irreducible character of \(\text{Sym}(n)\) corresponding to the Specht module of \(\lambda\) is denoted by \(\chi^\lambda\). The dimension \(f^\lambda\) of the Specht module corresponding to \(\lambda\) is \(\chi^\lambda(\text{id})\), where \(\text{id}\) is the identity permutation of \(\text{Sym}(n)\). This number is also the dimension of the character \(\chi^\lambda\). The dimension of \(\chi^\lambda\) can be computed using the Hook Length Formula [33, Section 3.10]. In the next subsection, we present a recursive method to compute the irreducible characters values of \(\text{Sym}(n)\).

4.1. The Murnaghan-Nakayama Rule. For any \(\sigma \in \text{Sym}(n)\) with cycle type \(\rho = (a_1, a_2, \ldots, a_k)\), we let \(\chi^\lambda_\rho := \chi^\lambda(\sigma)\). The Murnaghan-Nakayama Rule is a combinatorial tool with which the irreducible characters of \(\text{Sym}(n)\) can be computed. To state the Murnaghan-Nakayama rule, we need to introduce some new definitions.

Let \(\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_k] \vdash n\) and \(\mu = [\mu_1, \mu_2, \ldots, \mu_l] \vdash m\), where \(m < n\). We say that \(\lambda\) contains \(\mu\), and write \(\mu \subset \lambda\) if \(\mu_i \leq \lambda_i\), for all \(i \in \{1, 2, \ldots, l\}\). When \(\mu \subset \lambda\), the corresponding skew diagram \(\lambda/\mu\) is the set of cells of \(\lambda\) that are not in \(\mu\). That is,

\[\lambda/\mu = \{c \in \lambda \mid c \notin \mu\}.\]

A rim hook \(\zeta\) of a Young diagram \(\lambda \vdash n\) is a skew diagram of \(\lambda\) whose cells are on a path with upward and rightward steps. The leg length \(\ell(\zeta)\) of the rim hook \(\zeta\) of \(\lambda\) is the number of rows it spans minus 1 and the length \(|\zeta|\) of \(\zeta\) is the number of cells it has.

Given a partition \(\lambda\) and a rim hook \(\zeta\) of \(\lambda\), we define \(\lambda \setminus \zeta\) to be the set of cells of \(\lambda\) that are not in \(\zeta\). Since \(\zeta\) is a skew diagram, \(\lambda \setminus \zeta\) is a Young diagram of \(n - |\zeta|\).

Given a composition \(\rho = (a, \rho_2, \ldots, \rho_k)\) of \(n\), we define the operation \(\rho \setminus a\) to be the composition of \(n - a\) obtained by deletion of the first entry (which is \(a\)) of \(\rho\).

**Lemma 4.1** (Murnaghan-Nakayama Rule [33]). Let \(\lambda \vdash n\) and \(\rho = (\rho_1, \rho_2, \ldots, \rho_k)\) be a composition of \(n\). Then,

\[\chi^\lambda_\rho = \sum_{\zeta \in RH_{\rho_1}(\lambda)} (-1)^{\ell(\zeta)} \chi^\lambda_{\rho \setminus \zeta}.\]
where \( RH_{\rho_1}(\lambda) \) is the set of all rim hooks of length \( \rho_1 \) of \( \lambda \).

4.2. Representation theory of the alternating group. The conjugacy classes of \( \text{Alt}(n) \) are intertwined with the conjugacy classes of \( \text{Sym}(n) \). In fact, if \( C \) is a conjugacy class consisting of even permutations of \( \text{Sym}(n) \), then either \( C \) is a conjugacy class of \( \text{Alt}(n) \) or it splits into two conjugacy classes in \( \text{Alt}(n) \). This result is given without proof in the next proposition.

Proposition 4.2. Let \( C \) be a conjugacy class (of even permutations) of \( \text{Sym}(n) \) with cycle type \( \lambda \vdash n \). Then, \( C \) splits if and only if \( \lambda \) contains no even parts and all parts are distinct.

Next, we describe the irreducible characters of \( \text{Alt}(n) \). The irreducible characters of \( \text{Alt}(n) \) are also similar to the irreducible characters of \( \text{Sym}(n) \). The value of the characters of \( \text{Alt}(n) \) depend on the symmetry of the Young diagrams of \( n \). Recall that if \( \lambda \vdash n \) is a Young diagram of \( n \), then its transpose \( \lambda' \) is the Young diagram obtained by interchanging the rows and columns of \( \lambda \). If \( \lambda = \lambda' \), then we say that \( \lambda \) is self-transpose.

For any group \( G \), we let \( \text{Irr}(G) \) be the complete set of irreducible characters of \( G \). If \( \chi^\lambda \in \text{Irr}(\text{Sym}(n)) \), then the restriction of \( \chi^\lambda \) to \( \text{Alt}(n) \) is either an irreducible character of \( \text{Alt}(n) \) or is the sum of two irreducible characters of \( \text{Alt}(n) \). This result is presented in the next lemma, which is derived from [14, Proposition 5.1] (see also [3, Theorem 3.2]).

Lemma 4.3. Let \( \lambda \vdash n \). If \( \chi_\lambda \) and \( \chi_{\lambda'} \) are the restrictions of the characters \( \chi^\lambda \) and \( \chi^{\lambda'} \) to \( \text{Alt}(n) \) respectively, then one of the following happens.

(i) If \( \lambda \) is not self-transpose, then \( \chi_\lambda = \chi_{\lambda'} \) is an irreducible representation of \( \text{Alt}(n) \).

(ii) If \( \lambda \) is self-transpose, then there exist \( \phi_1, \phi_2 \in \text{Irr}(\text{Alt}(n)) \) such that \( \phi_1 \) and \( \phi_2 \) are constituents of \( \chi_\lambda \) and \( \chi_\lambda = \phi_1 + \phi_2 \).

Moreover, each irreducible character of \( \text{Alt}(n) \) arises as one of these two cases.

For any \( \lambda \vdash n \), we let \( \chi_\lambda \) be the restriction of the irreducible character \( \chi^\lambda \) to \( \text{Alt}(n) \). We let \( f_\lambda := \chi_\lambda(id) \) be the dimension of the character \( \chi_\lambda \). For the self-transpose partitions of \( n \), we recall the following lemmas (see [14, Proposition 5.3] and [21, Theorem 2.5.13]).

Lemma 4.4. Let \( \lambda \vdash n \) such that \( \lambda = \lambda' \) and let \( \chi'_\lambda \) and \( \chi''_\lambda \) be, respectively, the irreducible characters corresponding to the two irreducible constituents of \( \chi_\lambda \). Suppose that \( c \) is a non-split conjugacy class of \( \text{Sym}(n) \) with cycle type \( \mu \). Then, \( \chi'_\lambda(\sigma) = \chi''_\lambda(\sigma) = \frac{1}{2} \chi^\lambda(\sigma) \).

Lemma 4.5. Let \( \lambda \vdash n \) such that \( \lambda = \lambda' \) and let \( \chi'_\lambda \) and \( \chi''_\lambda \) be, respectively, the irreducible characters corresponding to the two irreducible constituents of \( \chi_\lambda \). Suppose that \( c \) is a conjugacy class of \( \text{Sym}(n) \) with cycle type \( \mu \), which splits as \( c = c' \cup c'' \) in \( \text{Alt}(n) \). Let \( \sigma \in c, \sigma' \in c', \) and \( \sigma'' \in c'' \).

- If \( \mu \neq \lambda \), then

\[
\chi'_\lambda(\sigma') = \chi''_\lambda(\sigma'') = \chi''_\lambda(\sigma') = \chi'_\lambda(\sigma'') = \frac{1}{2} \chi^\lambda(\sigma).
\]
• If \( \mu = \lambda \), then \( \chi^{\lambda}(\sigma) = 2\chi^{\lambda}_{\lambda}(\sigma) = 2\chi^{\prime}_{\lambda}(\sigma) \). Moreover,
\[
\chi^{\prime}_{\lambda}(\sigma') = \chi^{\prime\prime}_{\lambda}(\sigma'') = x \quad \text{and} \quad \chi^{\prime}_{\lambda}(\sigma') = \chi^{\prime\prime}_{\lambda}(\sigma'') = y,
\]
such that the values \( x \) and \( y \) are the two numbers \( \frac{1}{2} \left( -1 \frac{n-5}{2} \pm \sqrt{\left( -1 \frac{n-5}{2} \right) q_1 q_2 \ldots q_r} \right) \), where \( q_1, q_2, \ldots, q_r \) are the lengths of the odd cycles in the decomposition of an element of \( c' \cup c'' \).

4.3. Character values. In this subsection, we compute the maximum values that the irreducible characters can take on the conjugacy classes with cycle type \( (n) \), \( (n-1,1) \), \( (n-3,3) \), \( (n-4,3,1) \), \( (n-6,3^2) \), and \( (n-5,4,1) \).

First, we recall the following result, which is proved in [8].

**Lemma 4.6.** Let \( k, n \in \mathbb{N} \) such that \( 3k + 1 \leq n \) and \( \lambda \vdash n \). Then, the Young diagram \( \lambda \) has at most one rim hook of length \( n - k \).

**Lemma 4.7.** Let \( n \geq 19 \). For any \( \lambda \vdash n \) and \( \sigma \in \text{Sym}(n) \) with cycle type that is one of \( (n) \), \( (n-1,1) \), \( (n-3,3) \), \( (n-4,3,1) \), and \( (n-5,4,1) \), \( \chi^{\lambda}(\sigma) \in \{-1, 0, 1\} \). If \( \sigma \in \text{Sym}(n) \) has cycle type \( (n-6,3^2) \), then \( \chi^{\lambda}(\sigma) \in \{0, \pm 1, \pm 2\} \).

**Proof.** Suppose that the cycle type of \( \sigma \) is one of \( (n) \), \( (n-1,1) \), \( (n-3,3) \), \( (n-4,3,1) \), and \( (n-5,4,1) \). Since \( n \geq 19 \), there is at most one rim hook of length at least \( n - 6 \) in any Young diagram \( \lambda \vdash n \). Therefore, we have
\[
\left| \chi^{\lambda}_{(n)} \right| \leq \max_{\nu=0} \left| \chi^{\nu} \right| = 1
\]
\[
\left| \chi^{\lambda}_{(n-1,1)} \right| \leq \max_{\nu=1} \left| \chi^{\nu}_{(1)} \right| = 1
\]
\[
\left| \chi^{\lambda}_{(n-3,3)} \right| \leq \max_{\nu=3} \left| \chi^{\nu}_{(3)} \right| = 1
\]
\[
\left| \chi^{\lambda}_{(n-4,3,1)} \right| \leq \max_{\nu=4} \left| \chi^{\nu}_{(3,1)} \right| = 1
\]
\[
\left| \chi^{\lambda}_{(n-5,4,1)} \right| \leq \max_{\nu=5} \left| \chi^{\nu}_{(4,1)} \right| = 1
\]
\[
\left| \chi^{\lambda}_{(n-6,3^2)} \right| \leq \max_{\nu=6} \left| \chi^{\nu}_{(3^2)} \right| = 2.
\]

The proof follows since the symmetric group has integral character values. \( \square \)

5. EKR property of \( \text{Sym}(n) \) acting on pairs

In this section, we refine the result of Meagher and the author in [28] on the 2-setwise intersecting permutation of \( \text{Sym}(n) \).

Let \( G \leq \text{Sym}(\Omega) \). We say that \( \mathcal{F} \subseteq G \) is \( t \)-setwise intersecting if for any \( \sigma, \tau \in \mathcal{F} \), there exists a \( t \)-subset of \( \Omega \) such that \( S^\sigma = S^\tau \). We shall prove that if \( \mathcal{F} \subseteq \text{Sym}(\Omega) \), where \( \Omega := \{1, 2, \ldots, n\} \), is 2-setwise intersecting, then \( |\mathcal{F}| \leq 2(n-2)! \). We will prove this result using the weighted Ratio Bound.
5.1. A weighted adjacency matrix for the symmetric group. In this subsection, we find a weighted adjacency matrix with which we can prove that if $F \subset \text{Sym}(n)$ is 2-setwise intersecting, then $|F| \leq 2(n - 2)!$. This result is already known to be true [28]. However, the weighted adjacency matrix that we give in this paper enables us to prove the same property for the alternating group acting on the 2-subsets.

If $\rho \vdash n$, then we let $A_\rho$ be the matrix of the conjugacy class scheme $A(\text{Sym}(n))$ that corresponds to the conjugacy class of $\text{Sym}(n)$ with cycle type $\rho$. Consider the weighted adjacency matrix.

$$A = x_1A_{(n)} + x_2A_{(n-1,1)} + x_3A_{(n-3,3)} + x_4A_{(n-4,3,1)} + x_5A_{(n-6,3^2)} + x_6A_{(n-5,4,1)}.$$  \hspace{1cm} (2)

The irreducible constituents of the permutation character of $\text{Sym}(n)$ acting on the 2-subsets of $\Omega$ are $\chi^{[n]}$, $\chi^{[n-1,1]}$ and $\chi^{[n-2,2]}$ (see [28]). The main idea for our proof is to find real numbers $x_1, x_2, x_3, x_4, x_5$ and $x_6$ so that

(i) the largest eigenvalue of $A$ is $\binom{n}{2} - 1$, which is afforded by the irreducible character $\chi^{[n]}$,

(ii) the smallest eigenvalue of $A$ is $-1$, which is afforded by the irreducible characters $\chi^{[n-1,1]}$ and $\chi^{[n-2,2]}$,

(iii) all other eigenvalues of $A$ are in the interval $[-1, \binom{n}{2} - 1]$.

**Remark 5.1.** If (i), (ii), and (iii) are satisfied, then by the weighted Ratio bound the spanning subgraph $Y$ of $\Gamma_{\text{Sym}(n)}$ (here $\Gamma_{\text{Sym}(n)}$ is the derangement graph of $\text{Sym}(n)$ acting on the 2-subsets of $\Omega$) which corresponds to the weighted adjacency matrix $A$ will be such that

$$\alpha(Y) \leq \frac{n!}{1 - \binom{n}{2} - 1} = 2(n - 2)!.$$  

Since $Y$ is a spanning subgraph, we will have $\alpha(\Gamma_{\text{Sym}(n)}) \leq \alpha(Y) \leq 2(n - 2)!$.

The values of the irreducible characters that are constituent of the permutation character is given in the following table.

| Representation | $A_{(n)}$ | $A_{(n-1,1)}$ | $A_{(n-3,3)}$ | $A_{(n-4,3,1)}$ | $A_{(n-6,3^2)}$ | $A_{(n-5,4,1)}$ |
|---------------|----------|--------------|--------------|----------------|----------------|----------------|
| $\chi^{[n]}$  | 1        | 1            | 1            | 1              | 1              | 1              |
| $\chi^{[n-1,1]}$ | -1     | 0            | -1           | 0              | -1             | 0              |
| $\chi^{[n-2,2]}$ | 0       | -1           | 0            | -1             | 0              | -1             |

**Table 2.** Values of $\chi^{[n]}$, $\chi^{[n-1,1]}$ and $\chi^{[n-2,2]}$ on the conjugacy classes corresponding to $A_{(n)}$, $A_{(n-1,1)}$, $A_{(n-3,3)}$, $A_{(n-4,3,1)}$, $A_{(n-6,3^2)}$ and $A_{(n-5,4,1)}$.

The sizes of the conjugacy classes $A_{(n)}$, $A_{(n-1,1)}$, $A_{(n-3,3)}$, $A_{(n-4,3,1)}$, $A_{(n-6,3^2)}$, and $A_{(n-5,4,1)}$ are respectively $(n - 1)!$, $n(n - 2)!$, $2\binom{n}{3}(n - 4)!$, $8\binom{n}{4}(n - 5)!$, $40\binom{n}{6}(n - 7)!$, and $30\binom{n}{5}(n - 6)!$. Define
\[
\begin{align*}
\omega_1 = (n - 1)!x_1 \\
\omega_2 = n(n - 2)!x_2 \\
\omega_3 = 2\binom{n}{3}(n - 4)! \\
\omega_4 = 8\binom{n}{4}(n - 5)!x_4 \\
\omega_5 = 40\binom{n}{5}(n - 7)!x_5 \\
\omega_6 = 30\binom{n}{6}(n - 6)!x_6.
\end{align*}
\]

By Lemma 2.5 and Corollary 2.7, the eigenvalue of \( A \) afforded by the irreducible character corresponding to \( \lambda \vdash n \) is

\[
\xi^\lambda = \frac{1}{f^\lambda} \left( \omega_1 \chi_{\lambda(n)} + \omega_2 \chi_{\lambda(n-1,1)} + \omega_3 \chi_{\lambda(n-3,3)} + \omega_4 \chi_{\lambda(n-4,3,1)} + \omega_5 \chi_{\lambda(n-6,3^2)} + \omega_6 \chi_{\lambda(n-5,4,1)} \right).
\]

(3)

If (i) and (ii) are satisfied, then we have

\[
\begin{cases}
\omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5 + \omega_6 = \alpha \\
-\omega_1 - \omega_3 - \omega_5 = -\beta \\
-\omega_2 - \omega_4 - \omega_6 = -\gamma
\end{cases}
\]

(4)

where \( \alpha = \binom{n}{2} - 1, \beta = n - 1, \) and \( \gamma = \binom{n}{2} - n \). It is straightforward to see that this system of linear equation has infinitely many solutions and has four free variables. A general solution to (4) is as follows

\[
\begin{cases}
\omega_1 = \beta - t_1 - t_3 \\
\omega_2 = \alpha - \beta - t_2 - t_4 \\
\omega_3 = t_1 \\
\omega_4 = t_2 \\
\omega_5 = t_3 \\
\omega_6 = t_4
\end{cases}
\]

(5)

where \( t_1, t_2, t_3, t_4 \in \mathbb{R} \). We use these values in (5) to derive the weights \( x_1, x_2, \ldots, x_6 \) on the matrix \( A \) defined in (2). The eigenvalues corresponding to (2) are in function of the parameters \( t_1, t_2, t_3, t_4 \).

Using [28, Lemma 3.2], the only irreducible characters of \( \text{Sym}(n) \) of dimension less than \( 2\binom{n+1}{2} \) are those corresponding to the partitions \([n], [1^n], [n-1, 1], [2, 1^{n-1}], [n-2, 2], [3, 1^{n-3}], [n-2, 1^2], \) and \([2^2, 1^{n-4}]\). We first show that condition (i), (ii) and (iii) are satisfied for these characters. The eigenvalues of the irreducible constituents of the permutation character are \( \xi^{[n]} = \alpha \) and \( \xi^{[n-1,1]} = \xi^{[n-2,2]} = -1 \). Using the character values in Table 7, the eigenvalues \( A \) corresponding to the remaining characters with
low dimensions are
\[
\xi^{[1^n]} = (-1)^{n-1} (2\beta - \alpha - 2t_1 + 2t_2 + 2t_4)
\]
\[
\xi^{[2,1^{n-2}]} = \frac{(-1)^n}{n-1} (\beta - 2t_1)
\]
\[
\xi^{[2^2,1^{n-4}]} = \frac{(-1)^{n-1}}{\left(\frac{n}{2}\right)} (\alpha - \beta - 2t_2 - 2t_4)
\]
\[
\xi^{[n-2,1^2]} = \frac{\beta}{\left(\frac{n-1}{2}\right)}
\]
\[
\xi^{[3,1^{n-3}]} = \frac{(-1)^{n-1}}{\left(\frac{n-1}{2}\right)} (\beta - 2t_1).
\]

We would like to find the parameters \((t_i)_{i=1,2,3,4,5,6}\) so that the above eigenvalues \((6)\) are all in the interval \([-1, \infty)\). We will prove later that the eigenvalues of \(A\) obtained from these values of \((t_i)_{i=1,2,3,4,5,6}\) are at most \(\alpha\). We consider two cases depending on the parity of \(n\).

5.2. **Even case.** When the eigenvalues in \((6)\) are greater than \(-1\), we obtain the polytope

\[
(Q) \begin{cases}
2\beta - \alpha - 2t_1 + 2t_2 + 2t_4 \leq 1 \\
\frac{1}{2} \left( \beta - \left( \frac{n-1}{2} \right) \right) \leq t_1 \leq \beta \\
0 \leq t_2 + t_4.
\end{cases}
\]

Consider the polytope of \(\mathbb{R}^4\) defined such that

\[
(P) \begin{cases}
2\beta - \alpha - 2t_1 + 2t_2 + 2t_4 \leq 1 \\
0 \leq t_1 \leq \beta \\
0 \leq t_2 + t_4 \leq \gamma \\
-\frac{(n+2)}{3} < t_3 \leq 0, \\
t_2, t_4 \geq 0.
\end{cases}
\]

We note that \((P)\) is nonempty. It is easy to check that the eigenvalues in \((6)\) are still greater than \(-1\) for any \(t := (t_1, t_2, t_3, t_4) \in (P)\), since \((P)\) is contained in the polytope \((Q)\). Let us prove that every eigenvalue of \(A\) that is afforded by an irreducible character of dimension larger than \(2\left(\frac{n+1}{2}\right)\) is in the interval \([-1, 1]\), whenever \(t \in (P)\).
First, we show that the weights $\omega_i(t) \geq 0$, for any $t \in (P)$ and $i \in \{1, 2, 3, 4, 6\}$, whereas $\omega_5(t) \leq 0$. We have
\[
\begin{align*}
\omega_1(t) &= \beta - t_1 - t_3 \geq -t_3 \geq 0 \\
\omega_2(t) &= \alpha - \beta - t_2 - t_4 = \gamma - t_2 - t_4 \geq 0 \\
\omega_3(t) &= t_1 \geq 0 \\
\omega_4(t) &= t_2 \geq 0 \\
\omega_5(t) &= t_3 \leq 0 \\
\omega_6(t) &= t_4 \geq 0.
\end{align*}
\]
Together with Lemma 4.7, we will use the above properties of the weights on the eigenvalues in (3). For any $\lambda \vdash n$ such that $\ell^\lambda \geq 2^{(n+1)/2}$, the eigenvalue $\xi^\lambda$ in (3) is such that
\[
\begin{align*}
|\xi^\lambda| &\leq \frac{1}{\ell^\lambda} \left( |\omega_1(t)\chi^\lambda_{(1)}| + |\omega_2(t)\chi^\lambda_{(n-1,1)}| + |\omega_3(t)\chi^\lambda_{(n-3,3)}| + |\omega_4(t)\chi^\lambda_{(n-4,3,1)}| + |\omega_5(t)\chi^\lambda_{(n-6,3^3)}| \right. \\
&\quad \left. + |\omega_6(t)\chi^\lambda_{(n-5,4,1)}| \right) \\
&\leq \frac{1}{2^{(n+1)/2}} (\omega_1(t) + \omega_2(t) + \omega_3(t) + \omega_4(t) - 2\omega_5(t) + \omega_6(t)) \\
&= \frac{1}{2^{(n+1)/2}} (\alpha - 3t_3) < 1 \quad \text{(because of the fourth line in the definition of (P)).}
\end{align*}
\]
In other words, the eigenvalues from irreducible characters of dimension larger than $2^{(n+1)/2}$ are all strictly greater than $-1$. We let the reader verify that the eigenvalues in (6) are all less than $\alpha$, whenever $t \in (P)$.

5.3. **Odd case.** When the eigenvalues in (6) are greater than $-1$, they define the polytope
\[
\begin{align*}
(Q') \quad \left\{ \begin{array}{l}
2\beta - \alpha - 2t_1 + 2t_2 + 2t_4 \geq -1 \\
0 \leq t_1 \leq \frac{1}{2} \left( \beta + \binom{n-1}{2} \right) \\
t_2 + t_4 \leq \gamma.
\end{array} \right.
\end{align*}
\]
Consider the polytope of $\mathbb{R}^4$ defined such that
\[
\begin{align*}
(P') \quad \left\{ \begin{array}{l}
2\beta - \alpha - 2t_1 + 2t_2 + 2t_4 \geq -1 \\
0 \leq t_1 \leq \frac{1}{2} \beta \\
t_2 + t_4 \leq \gamma \\
0 \leq t_3 \leq \frac{1}{2} \beta \\
t_2, t_4 \geq 0.
\end{array} \right.
\end{align*}
\]
Similar to the even case, the polytope \((P')\) is nonempty and the eigenvalues in (6) are strictly larger than \(-1\), for any \(t \in (P')\). For any \(t \in (P')\), the weights of \(A\) are such that

\[
\begin{align*}
\omega_1(t) &= \beta - t_1 - t_3 \geq \beta - \beta = 0 \\
\omega_2(t) &= \alpha - \beta - t_2 - t_4 \geq \gamma - \gamma = 0 \\
\omega_3(t) &= t_1 \geq 0 \\
\omega_4(t) &= t_2 \geq 0 \\
\omega_5(t) &= t_3 \geq 0 \\
\omega_6(t) &= t_4 \geq 0.
\end{align*}
\]

The eigenvalues of the irreducible characters of large dimensions are also bounded by 1 in absolute value, in this case. For any \(\lambda \vdash n\) such that \(f^\lambda \geq 2\binom{n+1}{2}\), we have

\[
\begin{align*}
|\chi^\lambda| &\leq \frac{1}{f^\lambda} \left( |\omega_1(t)\chi^\lambda_{n(n-1)}| + |\omega_2(t)\chi^\lambda_{(n-3,1)}| + |\omega_3(t)\chi^\lambda_{(n-4,3,1)}| + |\omega_4(t)\chi^\lambda_{(n-6,3,2)}| + |\omega_5(t)\chi^\lambda_{(n-5,4,1)}| \right) \\
&\leq \frac{1}{2\binom{n+1}{2}} (\omega_1(t) + \omega_2(t) + \omega_3(t) + \omega_4(t) + 2\omega_5(t) + \omega_6(t)) \\
&= \frac{1}{2\binom{n+1}{2}} (\alpha + t_3) \leq 1 \quad \text{(because of the fourth line in the definition of \((P')\))}.
\end{align*}
\]

From this, we conclude that all eigenvalues of \(A\) are at least \(-1\). Again, we let the reader verify that any eigenvalue of \(A\) is also bounded from above by \(\alpha\).

Therefore, we found a weighted adjacency matrix of a spanning subgraph of the derangement graph of \(\Gamma_G\) for which (i), (ii) and (iii), for any \(n \geq 19\). We conclude that any 2-setwise intersecting family of \(\text{Sym}(n)\) is at most of size \(2(n-2)!\) (see Remark 5.1).

6. EKR property of \(\text{Alt}(n)\) acting on pairs

In this section, we use the weighted adjacency matrix that we found in Section 5 to prove that if \(\mathcal{F} \subset \text{Alt}(n)\) is a 2-setwise intersecting family, then \(|\mathcal{F}| \leq (n-2)!\). This is equivalent to saying that \(\text{Alt}(n)\) acting on the 2-subsets of \(\Omega\) has the EKR property.

Remark 6.1. We want to find weights in the polytope \((P)\) and \((P')\), defined in the previous section, such that the weights \((\omega_i)\) can take the value 0, whenever the corresponding conjugacy class consists of odd permutations. We will see that this is crucial to the proof that \(\text{Alt}(n)\) acting on the pairs has the EKR property. Thus, we want the weights \((\omega_i)\) to satisfy the following table (the symbol \(\checkmark\) indicates that the weights can be non-zero).
If $\rho$ is the cycle type of a permutation $\sigma \in \text{Sym}(n)$, corresponding to a split conjugacy class $C_{\rho}$, then there is a pair of conjugacy classes in $\text{Alt}(n)$, denoted $C'_\rho$ and $C''_\rho$, such that $C_{\rho} = C'_\rho \cup C''_\rho$. The matrices of these conjugacy classes in the conjugacy class scheme $\mathcal{A}(\text{Alt}(n))$ are denoted by $B'_\rho$ and $B''_\rho$, respectively. If the conjugacy class $C_{\rho}$ does not split in $\text{Alt}(n)$, then its conjugacy class in $\text{Alt}(n)$ is denoted by $C_{\rho}$ and its matrix in $\mathcal{A}(\text{Alt}(n))$ is denoted by $B_{\rho}$.

In this section, we will prove that we can easily construct a weighted adjacency matrix of $\text{Alt}(n)$ from the ones used in Section 5.

6.1. Even case. In this subsection, we let $t_2 = t_3 = t_4 = 0$ and $t_1 = \beta$. Let $t = (t_1, t_2, t_3, t_4)$. First, we note that $t \in (P)$, which is the polytope defined in (7). We use these same weights on the conjugacy classes (split and non-split) corresponding to the ones used in the previous section.

We note first that the weights $t = (t_1, 0, 0, 0)$ follows what is desired in Table 3. For any $i \in \{1, 2, 3, 4, 5, 6\}$, we let $\omega_i$ be the same as the ones defined in (5). Consider the weighted adjacency matrix

$$
B = \frac{1}{2} \omega_2(t) \left( B'_{(n-1,1)} + B''_{(n-1,1)} \right) + \frac{1}{2} \omega_3(t) \left( B'_{(n-3,3)} + B''_{(n-3,3)} \right)
= \frac{1}{2} \gamma \left( B'_{(n-1,1)} + B''_{(n-1,1)} \right) + \frac{1}{2} \beta \left( B'_{(n-3,3)} + B''_{(n-3,3)} \right).
$$

(9)

Recall that if $\chi \vdash n$, then $\chi_\lambda$ denotes the restriction of $\chi^\lambda$ to $\text{Alt}(n)$. If $C$ is a conjugacy class of $\text{Alt}(n)$ and $\lambda \vdash n$, then $\chi_\lambda(C)$ is equal to $\chi_\lambda(\sigma)$, where $\sigma \in C$. If $C = \{id\}$, then recall that $f_\lambda = \chi_\lambda(C)$ is the dimension of the character $\chi_\lambda$.

Suppose that $\chi_\lambda$ is an irreducible character of $\text{Alt}(n)$. That is, $\lambda \neq \lambda'$. For any $\chi \in \text{Irr}(\text{Alt}(n))$, we let $\xi_\chi$ be the eigenvalue of $B$ afforded by $\chi$.

Using Lemma 4.3, the eigenvalue of $B$ afforded by $\chi_\lambda$ is

$$
\xi_{\chi_\lambda}(t) = \frac{1}{2f_\lambda} \left( \gamma \chi_\lambda \left( C'_{(n-1,1)} \right) + \gamma \chi_\lambda \left( C''_{(n-1,1)} \right) + \beta \chi_\lambda \left( C'_{(n-3,3)} \right) + \beta \chi_\lambda \left( C''_{(n-3,3)} \right) \right)
= \frac{1}{2f_\lambda} \left( \gamma \chi_\lambda^\lambda_{(n-1,1)} + \gamma \chi_\lambda^\lambda_{(n-1,1)} + \beta \chi_\lambda^\lambda_{(n-3,3)} + \beta \chi_\lambda^\lambda_{(n-3,3)} \right)
= \frac{1}{f_\lambda} \left( \gamma \chi^\lambda_{(n-1,1)} + \beta \chi^\lambda_{(n-3,3)} \right)
= \frac{1}{f_\lambda} \left( \omega_2(t) \chi^\lambda_{(n-1,1)} + \omega_3(t) \chi^\lambda_{(n-3,3)} \right).
$$

(see (3)).

Since $t \in (P)$, we conclude that $\xi_{\chi_\lambda}(t) = \xi^\lambda(t) \in [-1, \alpha]$.
Suppose that \( \chi \) is an irreducible character of \( \text{Alt}(n) \) which cannot be obtained by restricting an irreducible character of \( \text{Sym}(n) \). Therefore, there exist \( \lambda \vdash n \) and \( \chi' \in \text{Irr}(\text{Alt}(n)) \) such that \( \chi_{\lambda} = \chi + \chi' \). We note that \( \lambda = \lambda' \), therefore, it cannot correspond to the compositions \( (n-1,1) \) and \( (n-3,3) \), which are the cycle types of the conjugacy classes considered in this subsection (see Lemma 4.5 for details). Moreover, \( f_\lambda = 2f_{\lambda} \). The eigenvalue of \( B \) corresponding to \( \chi \) is

\[
\xi_{\chi}(t) = \frac{1}{\chi(id)} \left( \gamma \chi(C'_{(n-1,1)}) + \gamma \chi(C''_{(n-1,1)}) + \beta \chi(C'_{(n-3,3)}) + \beta \chi(C''_{(n-3,3)}) \right)
\]

\[
= \frac{1}{2f_\lambda} \left( \gamma \frac{\chi_{(n-1,1)}}{2} + \gamma \frac{\chi_{(n-1,1)}}{2} + \beta \frac{\chi_{(n-3,3)}}{2} + \beta \frac{\chi_{(n-3,3)}}{2} \right)
\]

\[
= \frac{1}{f_\lambda} \left( \omega_2(t)\chi_{(n-1,1)} + \omega_3(t)\chi_{(n-3,3)} \right)
= \xi_\lambda(t).
\]

As \( t \in (P) \), we also have \( \xi_{\chi}(t) = \xi_{\lambda}(t) \in [-1, \alpha] \).

6.2. Odd case. Similar to the previous subsection, we consider a weighted adjacency matrix obtained from the weights in (2). In this subsection, we let \( t = (t_1, t_2, t_3, t_4) = (0, t_2, t_3, \gamma - t_2) \), where \( t_2 \geq 0 \) and \( 0 \leq t_3 \leq \frac{1}{2} \beta \). It is easy to verify that \( t \in (P') \) (see (8)). Moreover,

\[
\begin{align*}
\omega_1(t) & = \beta - t_1 - t_3 = \beta - t_3 \\
\omega_2(t) & = \alpha - \beta - t_2 - t_4 = 0 \\
\omega_3(t) & = t_1 = 0 \\
\omega_4(t) & = t_2 \\
\omega_5(t) & = t_3 \\
\omega_6(t) & = t_4 = \gamma - t_2,
\end{align*}
\]

Consider the matrix

\[
B = \frac{1}{5} \omega_1(t) \left( B_{(n)} + B''_{(n)} \right) + \frac{1}{2} \omega_4(t) \left( B'_{(n-4,3,1)} + B''_{(n-4,3,1)} \right) + \omega_5(t)B_{(n-6,3^2)} + \omega_6(t)B_{(n-5,4,1)}
\]

\[
= \frac{1}{2}(\beta - t_3) \left( B_{(n)} + B''_{(n)} \right) + \frac{1}{2} t_2 \left( B'_{(n-4,3,1)} + B''_{(n-4,3,1)} \right) + t_3 B_{(n-6,3^2)} + (\gamma - t_2)B_{(n-5,4,1)}.
\]
Suppose that $\chi_\lambda$ is an irreducible character of $\text{Alt}(n)$. The eigenvalue of $B$ afforded by $\chi_\lambda$ is

$$
\xi_{\chi_\lambda}(t) = \frac{1}{f_\lambda} \left( \frac{1}{2} (\beta - t_3) \chi_\lambda \left( C'(n) \right) + \frac{1}{2} (\beta - t_3) \chi_\lambda \left( C''(n) \right) + \frac{1}{2} t_2 \chi_\lambda \left( C'(n,4,3,1) \right) + \frac{1}{2} t_2 \chi_\lambda \left( C''(n,4,3,1) \right) + t_3 \chi_\lambda \left( C(n-6,3^2) \right) + (\gamma - t_2) \chi_\lambda \left( C(n-5,4,1) \right) \right)
$$

$$
= \frac{1}{f_\lambda} \left( \frac{1}{4} (\beta - t_3) \chi_\lambda^3 \left( C(n) \right) + \frac{1}{4} (\beta - t_3) \chi_\lambda \left( C'(n) \right) + \frac{1}{4} t_2 \chi_\lambda \left( C(n,4,3,1) \right) + \frac{1}{4} t_2 \chi_\lambda \left( C''(n,4,3,1) \right) + \frac{1}{2} t_3 \chi_\lambda \left( C(n-6,3^2) \right) + \frac{1}{2} (\gamma - t_2) \chi_\lambda \left( C(n-5,4,1) \right) \right)
$$

$$
= \frac{1}{f_\lambda} \left( (\beta - t_3) \chi_\lambda^3 \left( n \right) + t_2 \chi_\lambda \left( n-4,3,1 \right) + t_3 \chi_\lambda \left( n-6,3^2 \right) + (\gamma - t_2) \chi_\lambda \left( n-5,4,1 \right) \right)
$$

$$
= \frac{1}{f_\lambda} \left( \omega_1(t) \chi_\lambda^3 \left( n \right) + \omega_2(t) \chi_\lambda \left( n-4,3,1 \right) + \omega_5(t) \chi_\lambda \left( n-6,3^2 \right) + \omega_6(t) \chi_\lambda \left( n-5,4,1 \right) \right)
$$

$$
= \xi^\lambda(t).
$$

Therefore, $\xi_{\chi_\lambda}(t) = \xi^\lambda(t) \in [-1, \alpha]$ since $t \in (P')$.

Suppose that $\chi$ is an irreducible character of $\text{Alt}(n)$ which does not correspond to a restriction of an irreducible character of $\text{Sym}(n)$. There exist $\lambda \vdash n$ and $\chi' \in \text{Irr}(\text{Alt}(n))$ such that $\chi_\lambda = \chi + \chi'$. We note that $\lambda = \lambda'$ holds in this case. Therefore, $\lambda$ cannot correspond to $(n)$ and $(n-4,3,1)$, which are the cycle types of the split conjugacy classes considered in this subsection. The eigenvalue of $B$ corresponding to $\chi$ is

$$
\xi_{\chi}(t) = \frac{1}{f_\chi} \left( \frac{1}{2} \beta \chi \left( C'(n) \right) + \frac{1}{2} \beta \chi \left( C''(n) \right) + \frac{1}{2} t_2 \chi \left( C'(n,4,3,1) \right) + \frac{1}{2} t_2 \chi \left( C''(n,4,3,1) \right) + t_3 \chi \left( C(n-6,3^2) \right) + (\gamma - t_2) \chi \left( C(n-5,4,1) \right) \right)
$$

$$
= \frac{1}{f_\chi} \left( \frac{1}{4} \beta \chi \left( C'(n) \right) + \frac{1}{4} \beta \chi \left( C''(n) \right) + \frac{1}{4} t_2 \chi \left( C'(n,4,3,1) \right) + \frac{1}{4} t_2 \chi \left( C''(n,4,3,1) \right) + \frac{1}{2} t_3 \chi \left( C(n-6,3^2) \right) + \frac{1}{2} (\gamma - t_2) \chi \left( C(n-5,4,1) \right) \right)
$$

$$
= \frac{1}{f_\chi} \left( \beta \chi \left( C(n) \right) + t_2 \chi \left( n-4,3,1 \right) + t_3 \chi \left( n-6,3^2 \right) + (\gamma - t_2) \chi \left( n-5,4,1 \right) \right)
$$

$$
= \xi^\chi(t).
$$

We conclude that $\xi_{\chi}(t) \in [-1, \alpha]$.

6.3. Summary. The content of the previous two subsections can be summarized as follows. For any $n \geq 19$, one can find a weighted adjacency matrix $B$, depending on the parity of $n$, such that:

- the eigenvalue afforded by $\chi_{[n]}$ is $\alpha = \left( \begin{array}{c} n \\ 2 \end{array} \right) - 1$, which is the maximum eigenvalue of $B$;
- the smallest eigenvalue of $B$ is $-1$, which is afforded by $\chi_{[n-1,1]}$ and $\chi_{[n-2,2]}$;
- all other eigenvalues of $B$ are in the interval $[-1, \alpha]$.

Consequently, if $F \subset \text{Alt}(n)$ is 2-setwise intersecting, then $|F| \leq \frac{n}{1 - \frac{1}{n}} = (n - 2)!$. This is equivalent to $\rho(\text{Alt}(n)) = 1$. 

7. Line 11

In this section, we prove that the group in line 11 of Table 1 has the EKR property. The group PSp(4, k) acts on the 1-spaces (i.e., the 1-dimensional subspaces) of \( \mathbb{F}_k^4 \), that is, PG(1, \( \mathbb{F}_k^4 \)). This group is simply primitive of degree \( \frac{k^4 - 1}{k - 1} = (k + 1)(k^2 + 1) \). The condition in line 11 that \( p = k^2 + 1 \) and \( q = k + 1 \) are Fermat primes forces \( k \) to be an even power of 2.

The character table of the group PSp(4, k) is readily available but is rather complicated. Fortunately, this group contains a transitive subgroup which is imprimitive. The subgroup in question is PSL(2, \( k^2 \)) \( \leq \) PSp(4, k) and details about it can be found in [26]. In this section, we prove that the subgroup PSL(2, \( k^2 \)) acting on PG(1, \( \mathbb{F}_k^4 \)) has the EKR property, implying that the group in line 11 also has the EKR property.

7.1. Conjugacy classes. The conjugacy classes of PSL(2, \( k^2 \)) \( \cong \) SL(2, \( k^2 \)) is available in [1]. We will need some information on how the group PSL(2, \( k^2 \)) is embedded in PSp(4, k), which consists of 4 \( \times \) 4 matrices, before describing the conjugacy classes.

Let us describe the embedding of PSL(2, \( k^2 \)) into PSp(4, k). Let \( f(t) = t^2 + rt + s \) be an irreducible polynomial over \( \mathbb{F}_k \). Let \( M = \) the companion matrix of \( f(t) \) and \( \alpha \) be a root of \( f(t) \). By definition, \( f(t) \) is the characteristic polynomial of \( M \) and by the Cayley-Hamilton theorem, we have \( f(M) = 0 \). Now, we identify the field \( \mathbb{F}_k \) with the field \( K_1 \) of \( \mathbb{F}_k \)-multiples of the 2 \( \times \) 2 identity matrix over \( \mathbb{F}_k \). With this identification, it is easy to see that \( \mathbb{F}_{k^2} = \mathbb{F}_k(\alpha) \) can also be identified to the field \( K_2 := K_1(M) = \{ A + MB \mid A, B \in K_1 \} \). In [26], it has been proved that the group PSL(2, \( k^2 \)) can be identified with the set of all matrices \( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \), such that \( A, B, C, D \in K_2 \) and \( AD - CB = I_4 \), where \( I_n \) is the \( n \times n \) identity matrix. It is easy to see that the map \( a_1 + a_2 \alpha \mapsto a_1 I_2 + a_2 M \) from \( \mathbb{F}_k(\alpha) \) to \( K_1(M) \) gives the embedding of PSL(2, \( k^2 \)) in PSp(4, k).

Next, let us describe the matrices of PSL(2, \( k^2 \)) given by the above embedding. There are four families of conjugacy classes in PSL(2, \( k^2 \)), which are given below.

(i) The conjugacy class consisting of the identity matrix \( I_4 = I_2 \otimes I_2 \).

(ii) The class consisting of matrices conjugate to the matrix \( \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \otimes I_2 \).

(iii) The conjugacy classes consisting of matrices conjugate to a matrix of the form \( \begin{bmatrix} A & 0 \\ 0 & A^{-1} \end{bmatrix} \), where \( A \in K_2 \setminus \{ I_2 \} \).

(iv) Consider the finite field \( F \cong \mathbb{F}_{k^4} \) which is an extension of degree 2 of the field \( K_2 \). That is, there exists a matrix \( M' \) which is the root of an irreducible polynomial of degree 2 over \( K_2 \) such that \( F = K_2(M') \). Since \( M' \in F \) and \( F \) has characteristic 2, there exists \( U \in F \) such that \( M' = U^2 \) (this follows from the Frobenius automorphism). Let \( N : F^* \to K_2^* \) such that \( N(z) = z^{k^2} \). The map \( N \) is a homomorphism of groups and \( E = \ker N \) is a cyclic subgroup of \( F^* \) of order \( k^2 + 1 \). The last type of conjugacy classes of PSL(2, \( k^2 \)) are obtained from the matrices in \( E \). A conjugacy class of this type consists of matrices conjugate to
a matrix of the form

\[ A_Z := \begin{bmatrix} X & M'Y \\ Y & X \end{bmatrix}, \]

where \( Z = X + UY \in E \). Note that the determinant of \( A_Z \) is equal to 1 since

\[
\det(A_Z) = X^2 - M'Y^2 = (X - UY)(X + UY) = Z^{k^2}Z = I_2, \text{ since } Z \in E.
\]

We observe the following straightforward fact.

**Proposition 7.1.** Let \( A = cI_2 + dM \in K_2 \) and \( B = \begin{bmatrix} a & 0 \\ 0 & a' \end{bmatrix} \), where \( a, a' \in F_k \). If \( A \) and \( B \) are conjugate, then \( a = a' \).

**Proof.** Since \( A \) and \( B \) are conjugate, there exists an invertible matrix \( P \) such that \( B = PAP^{-1} = cI_2 + dPMP^{-1} \). Since \( B \) is a diagonal matrix, either \( d = 0 \) or \( PMP^{-1} \) is a diagonal matrix. The latter is impossible otherwise \( M \) would be equal to an element of \( K_1 \). Therefore, \( d = 0 \) and so \( a = c = a' \). \( \square \)

Using the information above, we can easily describe the conjugacy classes of derangements of \( \text{PSL}(2, k^2) \). There are two types of conjugacy classes of derangements in \( \text{PSL}(2, k^2) \), which are described as follows.

- **Type 1:** The conjugacy classes of matrices conjugate to a matrix the form

  \[ B = \begin{bmatrix} A & 0 \\ 0 & A^{-1} \end{bmatrix}, \text{ where } A \in K_2. \]

  Let us find the matrices of this type (iii) that fix a 1-space of \( F^4_k \). If \( B = \begin{bmatrix} A & 0 \\ 0 & A^{-1} \end{bmatrix} \) fixes an element of \( \text{PG}(1, F^4_k) \), then there exist \( u, v \in F^2_k \) and \( a \in F^*_k \) such that \( Au = au \) and \( A^{-1}v = av \).

  (i) First, suppose that \( u, v \neq 0 \). Then, \( Au = au \) and \( Av = a^{-1}v \). Assume that \( a = 1 \) and \( A \) is similar to \( \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \). Since \( A \in K_2 \), we may write \( A = cI + dM \), for some \( c, d \in F_k \). By similarity, there exists an invertible matrix \( P \) over \( F_k \) such that \( \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = PAP^{-1} = cI + dPMP^{-1} \). Hence, \( d = 0 \) or \( PMP^{-1} \) is a upper triangular matrix. However, the latter is impossible since \( M \) would have eigenvalues in \( F_k \) in this case and so \( f(t) \) would have roots in \( F_k \). Therefore, \( d = 0 \) and we get a contradiction.

  If \( A \) is diagonalizable and \( a = 1 \), then \( A = I_2 \) and \( B = I_4 \) (this is conjugacy class (i)).

  Consequently, we may assume that \( a \neq 1 \). In this case, \( a \neq a^{-1} \) because \( F_k \) has characteristic 2, and so \( A \) must be similar to the matrix \( \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \).

  However, \( \text{Diag}(a, a^{-1}) \) cannot be an element of \( K_2 \) by Proposition 7.1. We conclude that the case that \( u \neq 0 \) and \( v \neq 0 \) is impossible.
(ii) Next, assume without loss of generality that \( v = 0 \). Then, we have \( Au = au \). Since \( a \in \mathbb{F}_k \) is an eigenvalue of \( A \), \( A \) must have another eigenvalue, say \( a' \), in \( \mathbb{F}_k \). If \( a \neq a' \), then \( A \) is diagonalizable, and we can use the same argument as above (Proposition 7.1) to conclude that this case cannot happen since \( A \) would not be in \( K_2 \). If \( a = a' \) and \( A \) is not diagonalizable, then \( A \) is similar to the matrix \( C = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \). Similar to the argument for \( \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \), we can prove that we get a contradiction in this situation.

The only case left to examine is when \( a = a' \) and \( A \) is diagonalizable. In other words, \( A = aI_2 \). In this case, \( B = \begin{bmatrix} aI_2 & 0 \\ 0 & a^{-1}I_2 \end{bmatrix} \). It is easy to see that \( B \) cannot be a derangement since the 1-space generated by the vector

\[
\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}
\]

is fixed by \( B \). Therefore, we proved that if \( B \) has a fixed point, then it has to be of this form. There are \( \frac{k^2-2}{2} \) elements \( a \) of \( K_1 \) that give such matrices (fixing a 1-space). Therefore, there are \( \frac{k^2-2}{2} - \frac{k-2}{2} = \binom{k}{2} \) conjugacy classes of derangements of this type. Each conjugacy class of this type has size \( k^2(k^2 + 1) \).

• Type 2: The conjugacy class of matrices of \( \text{PSL}(2, k^2) \) that are described in (iv). There are \( \frac{k^2}{2} \) classes of this type, each of size \( k^2(k^2 - 1) \). We will show that these classes consist of derangements.

Let us prove that \( \mathcal{A}_Z = \begin{bmatrix} X & M'Y \\ Y & X \end{bmatrix} \), for \( Z = X + UY \in E \), is a derangement for the action of \( \text{PSL}(2, k^2) \) on the 1-spaces of \( \mathbb{F}_k^2 \). If \( \mathcal{A}_Z \) fixes a 1-space, then the \( 4 \times 4 \) matrix \( \mathcal{A}_Z \) has an eigenvalue in \( \mathbb{F}_k \). The characteristic polynomial of \( \mathcal{A}_Z \) (over \( \mathbb{F}_k \)) is

\[
\begin{vmatrix} X - tI_2 & M'Y \\ Y & X - tI_2 \end{vmatrix} = \det \left((X - tI_2)^2 - U^2Y^2\right) \text{ (since } Y \text{ and } X - tI_2 \text{ commute)} \\
= \det(X - tI_2 - UY) \det(X - tI_2 + UY) \\
= \det(Z^{-1} - tI_2) \det(Z - tI_2).
\]

Since \( ZZk^2 = I_2 \), which means that the order of \( Z \) divides \( k^2 + 1 \), one can conclude that \( Z \) does not have an eigenvalue in \( \mathbb{F}_k \) (see \[32\, \text{Lemma 4.2}\]). Therefore, \( \mathcal{A}_Z \) does not have an eigenvalue in \( \mathbb{F}_k \). In other words, \( \mathcal{A}_Z \) is a derangement.
We will denote the representatives of the conjugacy classes of this type by $A_{Z_1}, A_{Z_2}, \ldots, A_{Z_k}$

7.2. **The irreducible characters.** In this subsection, we describe the irreducible characters and certain (not all) constituents of the permutation character of the action of $\text{PSL}(2, k^2)$ on the 1-spaces of $\mathbb{F}_k^4$.

The irreducible characters of $\text{PSL}(2, k^2)$ are available in [1]. We refer to [1, Page 5,10] for the notations and the properties of the irreducible representations. There are four families of irreducible characters of $\text{PSL}(2, k^2)$, when $k$ is power of 2. These characters are:

- the trivial character $\rho'(1)$,
- the character $\rho(1)$,
- the character $\rho(\alpha)$, where $\alpha$ is an irreducible representation of $K_2^*$,
- the character $\pi(\chi)$, where $\chi$ is a non-trivial irreducible representation of the kernel $E$ (a cyclic group of order $k^2 + 1$) of the norm map $N$ defined in the previous subsection.

Recall that the permutation character of a permutation group $G$ is the character $\text{fix}$ which gives the number of fixed points of a permutation of $G$.

**Proposition 7.2.** The irreducible characters $\rho'(1)$ and $\rho(1)$ are constituents of the permutation character of the action of $\text{PSL}(2, k^2)$ on the 1-spaces of $\mathbb{F}_k^4$.

**Proof.** Since a character of a group is a non-negative linear combination of irreducible characters, it is enough to prove that the coefficients of $\rho'(1)$ and $\rho(1)$ in this combination are both nonzero. It is obvious that the trivial character $\rho'(1)$ is a constituent of the permutation character since $\langle \rho'(1), \text{fix} \rangle = 1$ ($\text{PSL}(2, k^2)$ acts transitively on the 1-spaces).

For $\rho(1)$, we also prove that $\langle \text{fix}, \rho(1) \rangle = 1$. Note that a matrix in family of conjugacy classes (iii) which is not a derangement fixes $2(k+1)$ (each diagonal element fixes $q + 1$ 1-spaces). Similar to the case of the trivial character,

$$\langle \rho(1), \text{fix} \rangle = \frac{1}{|\text{PSL}(2, k^2)|} \sum_{A \in \text{PSL}(2, k^2)} \rho(1)(A) \frac{\text{fix}(A)}{2}$$

$$= \frac{1}{k^2(k^4 - 1)} \left( k^2 \times 1 \times (k^2 + 1)(k + 1) + 0 \times \frac{(k - 2)}{2} k^2(k^2 + 1) \times 2(k + 1) +1 \times \frac{k}{2} k^2(k^2 + 1) \times 0 + (-1) \times \frac{k^4(k^2 - 1)}{2} \times 0 \right)$$

$$= \frac{1}{k^2(k^4 - 1)} (k^2(k^2 + 1)(k + 1) + k^2(k + 1)(k^2 + 1)(k - 2))$$

$$= \frac{1}{k^2(k^4 - 1)} (k^2(k^2 + 1)(k + 1)(k - 1))$$

$$= \frac{1}{k^2(k^4 - 1)} (k^2(k^4 - 1))$$

$$= 1.$$
This completes the proof. □

Next, we describe the values of the irreducible characters \( \rho(\alpha) \) and \( \pi(\chi) \) where \( \alpha \) is an irreducible representation of \( K_2^* \) and \( \chi \) is an irreducible representation of \( E \). Let \( \chi \) be an irreducible representation of \( E \) and \( \mathcal{A}_{Z_1}, \mathcal{A}_{Z_2}, \ldots, \mathcal{A}_{Z_{k^2}} \) be representatives from the \( \frac{k^2}{2} \) conjugacy classes of Type 2 of \( \text{PSL}(2, k^2) \). Then using the character table \[1\], we have

\[
\frac{k^2}{2} \sum_{i=1}^{\frac{k^2}{2}} \pi(\chi)(\mathcal{A}_{Z_i}) = -\frac{k^2}{2} \sum_{i=1}^{\frac{k^2}{2}} (\chi(Z_i) + \chi(Z_i^{-1})) = -\sum_{\zeta \in \{z \in \mathbb{C} \mid z^{k^2}=1, \ z \neq 1\}} \zeta = 1. \tag{10}
\]

**Remark 7.3.** The second equality in (10) follows due to the fact that each non-identity element of \( E \) determines a conjugacy class of Type 2.

Let \( 1_{K_1^*} \) be the trivial character of \( K_1^* \). Let \( \alpha \) be an irreducible representation of \( K_2^* \) and assume that the restriction \( \alpha|_{K_1^*} \) of \( \alpha \) on \( K_1^* \) is equal to the trivial representation \( 1_{K_1^*} \) of \( K_1^* \). By the property of \( k^2 \)-th roots of unity, we have

\[
\sum_{A \in K_2^* \setminus K_1^*} \rho(\alpha)(A) + \sum_{A \in K_1^*} \rho(\alpha)(A) = \sum_{A \in K_2^*} \alpha(A) = 0.
\]

Therefore,

\[
\sum_{A \in K_2^* \setminus K_1^*} \rho(\alpha)(A) = -\sum_{A \in K_1^*} \rho(\alpha)(A) = -\sum_{A \in K_1^*} 1 = -(k - 1).
\]

If \( \alpha|_{K_1^*} \neq 1_{K_1^*} \), then by the orthogonality of characters and the fact that the dimension of \( \alpha \) is 1, we have

\[
\langle 1_{K_1^*}, \alpha|_{K_1^*} \rangle_{K_1^*} = \frac{1}{|K_1^*|} \sum_{x \in K_1^*} \alpha|_{K_1^*}(x) = 0. \tag{11}
\]

Using (11), it is straightforward that if the restriction of \( \alpha \) on \( K_1^* \) is not equal to the trivial character of \( K_1^* \), then \( \sum_{A \in K_1^*} \rho(\alpha)(A) = 0 \). Hence,

\[
\sum_{A \in K_2^* \setminus K_1^*} \rho(\alpha)(A) = 0.
\]

To summarize, the value of the character \( \rho(\alpha) \) on all conjugacy classes of derangement of Type 1 is

\[
\sum_{A \in K_2^* \setminus K_1^*} \rho(\alpha)(A) = \begin{cases} -(k - 1), & \text{if } \alpha|_{K_1^*} \text{ is the trivial character of } K_1^*; \\ 0, & \text{otherwise}. \end{cases} \tag{12}
\]
7.3. Maximum cocliques. In this subsection, we prove that PSL(2, k^2) acting on the 1-spaces of F^4_k has the EKR property. To do this, we will use a weighted adjacency matrix for which the Ratio Bound yields the order of a point-stabilizer.

If \[
\begin{bmatrix}
A & 0 \\
0 & A^{-1}
\end{bmatrix}
\] is a matrix of PSL(2, k^2) in the conjugacy class of Type 1, then we let \(T_A^{(1)}\) be the matrix in the conjugacy class scheme \(A(PSL(2, k^2))\) that corresponds to it. For a conjugacy class of Type 2, we let \(T_Z^{(2)}\) be the matrix in \(A(PSL(2, k^2))\) that is obtained from the conjugacy class of the matrix \(A_Z = \begin{bmatrix} X & M'Y \\ Y & X \end{bmatrix}\), where \(Z = X + UY \in E\).

Let us uniformly assign the same weights on the conjugacy classes of the same type. That is, let \(\omega_1\) and \(\omega_2\) be two real numbers and define the weighted adjacency matrix

\[
T(\omega_1, \omega_2) = \omega_1 \sum_{A \in K_2 \setminus K_1} T_A^{(1)} + \omega_2 \sum_{i=1}^{k^2} T_{Z_i}^{(2)}. \tag{13}
\]

Now, we use Lemma 2.5 to find the eigenvalues of \(T(\omega_1, \omega_2)\) as a function of \(\omega_1\) and \(\omega_2\). Similar to what we saw in Section 5 and Section 6, we would like to find values of \(\omega_1\) and \(\omega_2\) so that the largest eigenvalue (afforded by the trivial character) of \(T = T(\omega_1, \omega_2)\) is \((k^2 + 1)(k + 1) - 1\) and its smallest eigenvalue is \(-1\). However, our strategy is not quite the same as what was presented in Section 6, in the sense that we will only set one non-trivial irreducible character of the permutation character to afford the eigenvalue \(-1\) and prove that all the other eigenvalues of \(T\) are in the interval \([-1, (k^2 + 1)(k + 1) - 1]\). To summarize, we would like to find \(\omega_1\) and \(\omega_2\) such that

- The eigenvalue of \(T(\omega_1, \omega_2)\) afforded by \(\rho'(1)\) is \((k^2 + 1)(k + 1) - 1\).
- The eigenvalue of \(T(\omega_1, \omega_2)\) afforded by \(\rho(1)\) is \(-1\).
- The eigenvalues of \(T(\omega_1, \omega_2)\) afforded by all other irreducible characters are in the interval \([-1, (k^2 + 1)(k + 1) - 1]\).

The following table gives the values of the characters \(\rho'(1)\) and \(\rho(1)\) on the all conjugacy classes of the same type (Type 1 and Type 2).

| Representation | Dimension | Type 1: \(\sum_{A \in K_2 \setminus K_1} T_A^{(1)}\) | Type 2: \(\sum_{i=1}^{k^2} T_{Z_i}^{(2)}\) |
|---------------|-----------|---------------------------------|---------------------------------|
| \(\rho'(1)\)  | 1         | \(\binom{k}{2} k^2 (k^2 + 1)\) | \(k^2 (k^2 - 1)\) |
| \(\rho(1)\)   | \(k^2\)   | \(\binom{k}{2} (k^2 + 1)\)     | \(-\frac{k^2}{2} (k^2 - 1)\)   |

Table 4. Sum of character values afforded by \(\rho'(1)\) and \(\rho(1)\) on the conjugacy classes of the same type.
Using Table 4, the eigenvalues of $T(\omega_1, \omega_2)$ afforded by $\rho'(1)$ and $\overline{\rho}(1)$ are

$$
\begin{cases}
\left(\frac{k}{2}\right)k^2(k^2 + 1)\omega_1 + \frac{k^4}{2}(k^2 - 1)\omega_2 = (k^2 + 1)(k + 1) - 1 \\
\frac{1}{k^2}\left(\frac{k}{2}k^2(k^2 + 1)\omega_1 - \frac{k^4}{2}(k^2 - 1)\omega_2\right) = -1.
\end{cases}
$$

The system of linear equations (14) with indeterminate $\omega_1$ and $\omega_2$ has a unique solution which is

$$
\begin{align*}
\omega_1 &= \frac{k^3 + k}{2\left(\frac{k}{2}\right)k^2(k^2 + 1)} \\
\omega_2 &= \frac{k^3 + 2k^2 + k}{k^4(k^2 - 1)}.
\end{align*}
$$

Now, let us apply the values in (15) to the weighted adjacency matrix given in (13). The eigenvalues of $T(\omega_1, \omega_2)$ afforded by the irreducible character $\pi(\chi)$, where $\chi$ is an irreducible character of $E$, is

$$
\frac{k^3 + 2k^2 + k}{k^4(k^2 - 1)} \times \frac{1}{k^2 - 1} \sum_{i=1}^{k^2} k^2(k^2 - 1)\pi(\chi)(A_{Z_i}) = \frac{k + 1}{k(k - 1)}.
$$

The eigenvalue of $A(\omega_1, \omega_2)$ afforded by the irreducible character $\rho(\alpha)$, where $\alpha$ is an irreducible character of $K_2^*$, is either equal to 0 or

$$
\frac{k^3 + k}{2\left(\frac{k}{2}\right)k^2(k^2 + 1)} \times \frac{1}{k^2 + 1} \left(k^2(k^2 + 1) \times ((k^2 - 1))\right) = -1,
$$

depending on the restriction of $\alpha$ on $K_1^*$ (see (12)). Therefore, the eigenvalues of $T(\omega_1, \omega_2)$ are

$$(k^2 + 1)(k + 1) - 1, \frac{k + 1}{k(k - 1)}, 0, -1.$$ 

By Lemma 2.4, we have

$$
\alpha(\Gamma_{PSL(2, k^2)}) \leq \frac{|PSL(2, k^2)|}{1 - \frac{(k^2 + 1)(k+1)-1}{(k^2 + 1)(k + 1)}} = \frac{|PSL(2, k^2)|}{(k^2 + 1)(k + 1)}.
$$

In other words, $\rho(PSL(2, k^2)) = 1$. We conclude that $\rho(PSp(4, k)) = 1$.

8. IMPRIMITIVE CASE

In this section, we prove Theorem 1.4. Suppose that $G \leq \text{Sym}(\Omega)$ is imprimitive of degree $pq$, where $p > q$ are odd prime and has at least two systems of imprimivity. We will use a classification result with respect to the number of systems of imprimitivity, which is due to Lucchini [25].

**Theorem 8.1** (Lucchini [25]). Let $G \leq \text{Sym}(\Omega)$ be an imprimitive group of degree $pq$, where $p > q$ are odd primes. Let $m$ be the number of systems of imprimitivity of $G$. If $m \geq 2$, then either $m = 2$ or $m = p + 1$. In particular, we have the following:
(i) there is at most one system of imprimitivity with blocks of size $p$;
(ii) if $G$ has a block of imprimitivity of size $p$ and one of size $q$, then $G \leq \text{Sym}(p) \times \text{Sym}(q)$,
(iii) if $G$ has at least two systems of imprimitivity of size $q$, then $q \mid (p - 1)$ and one of the following holds:
(a) $G$ is a non-abelian group of order $pq$ having $p$ systems of imprimitivity with blocks of size $q$ and one with blocks of size $p$;
(b) $p = 7$, $q = 3$, and $G = \text{PSL}(2,7)$ such that $G$ has exactly two systems of imprimitivity with blocks of size 3,
(c) $p = 11$, $q = 5$, and $G = \text{PSL}(2,11)$ such that $G$ has exactly two systems of imprimitivity with blocks of size 5.

We use Theorem 8.1 to prove that $G$ has the EKR property. First, we note that Hujdurović et al. [18, Proposition 2.1] recently proved that if $G$ has blocks of size $p$ then $\rho(G) = 1$. In this section, we show in more details (in term of their algebraic structures) why these groups have the EKR property.

Suppose that $G$ has at least two systems of imprimitivity with blocks of size $q$. If we are in the case (iii), then $G$ is a non-abelian group of order $pq$. Since $|G| = pq$, there exists a cyclic group $H \leq G$ with $|H| = q$. If there exists any $g \in G \setminus H$ such that $gHg^{-1} \cap H \neq \{id\}$, then there exists a common non-identity element $x$ of order $q$ in $H$ and $gHg^{-1}$. By primality of $q$, $H = \langle x \rangle = gHg^{-1}$. Note that $g \in G \setminus H$ necessarily has order $p$ (it cannot have order $pq$ since $G$ would be cyclic) and so the conjugation by $g$ has order $p$ in $\text{Aut}(G)$. By the Sylow theorems, it is easy to see that $G$ has $p$ Sylow $q$-subgroups, which we denote $H_1 = H, H_2, \ldots, H_p$. Since $G$ acts transitively by conjugation on $\{H_1, H_2, \ldots, H_p\}$ and the order of the automorphism which conjugates by $g$ is $p$, we conclude that $gHg^{-1} \neq H$. Thus, we have a contradiction. We conclude that $H \neq gHg^{-1}$, which means $H \cap gHg^{-1} = \{id\}$, for any $g \in G \setminus H$. In other words, $G$ is a Frobenius group. Therefore, $G$ has the EKR property [2].

Next, we assume that $G$ has one system of imprimitivity with blocks of size $p$ and one with blocks of size $q$. Hujdurović et al. [18, Proposition 2.1] recently proved that such a group $G$ has the EKR property. Here, we give an alternative proof by exhibiting a regular subgroup of $G$. By (ii), $G \leq \text{Sym}(p) \times \text{Sym}(q)$. We prove that $G$ always has a regular subgroup.

To prove this result, we need to recall some results from [25]. We derive the following lemma from Lemma 1.1 and Lemma 1.2 in [25].

**Lemma 8.2.** Let $G$ be an imprimitive group of degree $pq$, where $p$ and $q$ are distinct odd primes. Assume that $\mathcal{P} = \{P_1, P_2, \ldots, P_q\}$ and $\mathcal{Q} = \{Q_1, Q_2, \ldots, Q_p\}$ are two systems of imprimitivity of $G$. Then, $|P_i \cap Q_j| = 1$ for any $1 \leq i \leq q$ and $1 \leq j \leq p$.

The following argument is also in [25]; we include it here for completeness. Let $\mathcal{P} = \{P_1, P_2, \ldots, P_q\}$ and $\mathcal{Q} = \{Q_1, Q_2, \ldots, Q_p\}$ be the systems of imprimitivity of $G$, where $|P_i| = p$, for any $i \in \{1,2,\ldots,q\}$ and $|Q_i| = q$ for any $i \in \{1,2,\ldots,p\}$. Consider the homomorphisms $\pi_\mathcal{P} : G \to \text{Sym}(\mathcal{P})$ and $\pi_\mathcal{Q} : G \to \text{Sym}(\mathcal{Q})$ of the induced action of $G$ on $\mathcal{P}$ and $\mathcal{Q}$, respectively. It was proved in [25] that the map $\pi_\mathcal{P}$:
$G \to \text{Sym}(Q) \times \text{Sym}(P)$ such that $\pi (g) = (\pi_Q(g), \pi_P(g))$ is a group homomorphism which is injective. In other words, $G$ can be embedded in $\text{Sym}(Q) \times \text{Sym}(P)$.

For any $1 \leq i \leq p$ and $1 \leq j \leq q$, define $\omega_{i,j} := P_i \cap Q_j$ (see Lemma 8.2). Let $A$ be the $p \times q$ matrix whose $(i, j)$-entry is $\omega_{i,j}$. Note that since $G$ is transitive on the entries of $A$, $G$ cannot be a proper subgroup of $\text{Sym}(Q)$ or $\text{Sym}(P)$; otherwise the rows or columns, respectively, will be orbits of $G$. This is also equivalent to $\pi_Q(G)$ or $\pi_P(G)$ being trivial. Therefore, $G$ contains a permutation of $\text{Sym}(Q)$ of order $p$ and a permutation of $\text{Sym}(P)$ of order $q$. Let $\sigma_P$ and $\sigma_Q$ be permutations of $G$ that are derangements of $\text{Sym}(Q)$ and $\text{Sym}(P)$, respectively. Set $H := \langle \sigma_P \sigma_Q \rangle$. We claim that $H$ is a regular subgroup of $G$.

First, we note that $\sigma_P \sigma_Q = \sigma_Q \sigma_P$ since $G \leq \text{Sym}(Q) \times \text{Sym}(P)$. Hence, the order of $\sigma_P \sigma_Q$ is equal to $pq$. Consider the entries $\omega_{i,j}$ and $\omega_{u,v}$ of the matrix $A$. There exist $s, t \geq 0$ such that $\omega_{i,j}^s = \omega_{u,j}$ and $\omega_{u,j}^t = \omega_{u,v}$. Therefore, the permutation $\sigma = \sigma_Q^s \sigma_P^t \in \langle \sigma_Q \rangle \times \langle \sigma_P \rangle$, is such that $\omega_{i,j}^\sigma = \omega_{u,v}$. Hence, $H$ is a transitive subgroup of order $pq$, thus acting regularly on $\Omega$. It follows from the clique-coclique bound that $G$ has the EKR property.

If (iiiib) holds, then $G$ is the transitive group $\text{TransitiveGroup}(21,14)$ of the library of transitive groups of SageMath. Using SageMath, we can easily verify that $G = \text{PSL}(2, 7)$ has the EKR property. If (iiiic) holds, then $G = \text{PSL}(2, 11)$ has a maximal subgroup of order 55 which is regular. Therefore, $G = \text{PSL}(2, 11)$ has the EKR property.

We conclude that any imprimitive group of degree $pq$ with at least two systems of imprimitivity has the EKR property. Therefore, the remaining imprimitive groups of degree $pq$ to consider are those with exactly one system of imprimitivity. It was also recently proved by Hujdurović et al. [18] that there are transitive groups of degree $pq$ with blocks of size $q$ that do not have the EKR property.

9. Future works

9.1. Summary of the results. In this paper it is proved that if $G \leq \text{Sym}(\Omega)$ is imprimitive of degree $pq$ with at least two systems of imprimitivity, then $G$ has the EKR property. Moreover, if $G \leq \text{Sym}(\Omega)$ is primitive of degree $pq$, with socle equal to one of the groups lines 1-11, 14, 16 and 17 in Table 1, then $G$ has the EKR property. In order to prove the latter, we used a consequence of the No-Homomorphism Lemma (see [29, Lemma 6.5]) which says that if $\text{Soc}(G)$ (which is transitive) has the EKR property, then so does $G$. Therefore, to prove the primitive case of Conjecture 1.1(c), it is enough to show that the groups in lines 12-13, 15 and 18 have the EKR property.

The result of this paper for the primitive case is summarized with the following table (the label “Yes” in the table means that the group has the EKR property).
The remaining families of primitive groups of degree $pq$ that are left to check are the groups from [11, Table 3]. These groups are the socles of primitive groups of degree $pq$ that do not admit imprimitive subgroups. We make the following conjecture about these groups.

**Conjecture 9.1.** The groups in lines 12, 13, 15 and 18 of Table 5 have the EKR property.

We note that the technique used to prove the main results in this work (i.e., the Ratio Bound) does not work for the groups in Conjecture 9.1.

**9.2. PSL(2, q) acting on 2-subsets of PG(1, q).** In this subsection, we consider the action of the group PSL(2, q), $q$ a prime power, acting on 2-subset of PG(1, q). Note that when $q = p = 3 \mod 4$ is a prime, then this group action is permutation isomorphic to the group PSL(2, p) acting on the cosets of $D_{p-1}$ in Table 1. To see this, let $\{u, v\}$ be a 2-subset of PG(1, q), where $u$ and $v$ are respectively the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Note that if a matrix $A \in$ PSL(2, p) leaves $\{u, v\}$ invariant, then $Au = u$ and $Av = v$, or $Au = v$ and $Av = u$. It is easy to see that the subgroup generated by $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ swaps $u$ and $v$, thus leaving $\{u, v\}$ invariant. Moreover, the subgroup $K$ of matrices of the form $\text{Diag}(\alpha, \alpha^{-1})$, where $\alpha \in \mathbb{F}_q^*$, fixes $\{u, v\}$ pointwise. We let the reader verify that $K \times \langle A \rangle$ is isomorphic to $D_{p-1}$. Therefore, the two actions are equivalent.

| Line | $S$ | $(p, q)$ | action | Information | EKR |
|------|-----|---------|--------|-------------|-----|
| 1    | Alt(7) | $(7, 3)$ | triples |            | Yes |
| 2    | PSL(4, 2) | $(3, 5)$ | 2-spaces |            | Yes |
| 3    | PSL(5, 2) | $(31, 5)$ | 2-spaces |            | Yes |
| 4    | PSL(2, 23) | $(23, 11)$ | cosets of Sym(4) |            | Yes |
| 5    | PSL(2, 11) | $(11, 5)$ | cosets of Alt(4) |            | Yes |
| 6    | $M_{11}$ | $(11, 9)$ |        |            | Yes |
| 7    | $M_{23}$ | $(11, 7)$ |        |            | Yes |
| 8    | $M_{23}$ | $(23, 11)$ |        |            | Yes |
| 9    | Alt$(p)$ | $\left(p, \frac{p+1}{2}\right)$ | pairs | $p \geq 5$ | Yes |
| 10   | Alt$(p + 1)$ | $\left(p, \frac{p+1}{2}\right)$ | pairs | $p \geq 5$ | Yes |
| 11   | PSp(4, k) | $(k^2 + 1, k + 1)$ | 1-spaces | $p, q$ are Fermat primes | Yes |
| 12   | PSp*(2d, 2) | $\left(2^d - \varepsilon, 2^d + 1 + \varepsilon\right)$ | singular 1-spaces | $\varepsilon = 1$ and $d$ is a Fermat prime | Yes |
| 13   | PSL(2, p) | $\left(p, \frac{p+1}{2}\right)$ | cosets of $D_{p-1}$ | $p \geq 13$ and $p \equiv 1 (mod 4)$ | - |
| 14   | PSL(2, p) | $\left(\frac{p+1}{2}, q\right)$ | cosets of $D_{p-1}$ | $p \geq 13$ and $p \equiv 3 (mod 4)$ | Yes |
| 15   | PSL(2, $q^2$) | $\left(\frac{p+1}{2}, q\right)$ | cosets of $\text{PGL}(2, q)$ |         | - |
| 16   | PSL(2, p) | $(19, 3), (29, 7), (59, 29)$ | cosets of Alt(5) |         | Yes |
| 17   | PSL(2, 11) | $(11, 5)$ | cosets of Alt(4) |         | Yes |
| 18   | PSL(2, 61) | $(61, 31)$ | cosets of Alt(5) |         | - |

**Table 5.** Status of the EKR property for the socle of simply primitive groups of degree $pq$. 
9.2.1. When $q$ is even. When $q$ is an even prime power, then $\text{PSL}(2, q) = \text{SL}(2, q)$. The action of $\text{PSL}(2, q)$ considered in this subsection is the one on the 2-subsets of $\text{PG}(1, q)$. We prove that this group does not have the EKR property.

**Theorem 9.2.** Let $q = 2^k$, for some $k \geq 1$. If $\mathcal{F} \subset \text{PSL}(2, q)$ is intersecting, then $|\mathcal{F}| \leq q(q-1)$. Moreover, if $\rho(\text{PSL}(2, q)) = q^2$.

Again, we use the character table of $\text{PSL}(2, q)$ in [1]. We use the notation in [1] for the irreducible characters and the conjugacy classes. Using Lemma 2.6 and the character table in [1], it is not hard to see that the eigenvalues (with the multiplicities) of $\text{PSL}(2, q)$ acting on the 2-subsets of $\text{PG}(1, q)$ are

$$\frac{q^2(q-1)}{2}, q \left(\frac{q(q-1)^2}{2}\right), 0 \left(\frac{(q+1)^2(q-2)}{2}\right), -q \left(\frac{q-1}{2}\right).$$

Using the Ratio Bound, we get that

$$\alpha(\Gamma_{\text{PSL}(2, q)}) \leq -\frac{q(q-1)}{2} - \frac{q^2(q-1)}{2} q(q-1)(q+1) = q(q-1).$$

In other words, if $\mathcal{F} \subset \text{PSL}(2, q)$ acting on the 2-subsets of $\text{PG}(1, q)$ is intersecting, then $|\mathcal{F}| \leq q(q-1)$.

Now, let us prove that this bound is tight. Let $f(t) \in \mathbb{F}_2[t]$ be an irreducible polynomial of degree $k$ and let $\alpha$ be a root of $f(t)$. Recall that $\mathbb{F}_q$ is an $\mathbb{F}_2$-vector space spanned by $1, \alpha, \alpha^2, \ldots, \alpha^{k-1}$. For any $i \in \{0, 1, 2, \ldots, k-1\}$, let

$$A_i = \begin{bmatrix} 1 & \alpha^i \\ 0 & 1 \end{bmatrix}.$$ 

Let $\beta \in \mathbb{F}_q$ be a primitive element. We define

$$B = \begin{bmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{bmatrix}.$$ 

Let $N = \langle A_0, A_1, A_2, \ldots, A_{k-1} \rangle$ and $M = \langle B \rangle$. Note that $N$ and $M$ are subgroups of $\text{PSL}(2, q)$. Moreover, $N \cong \mathbb{F}_q$ as subgroups. For any $i \in \{0, 1, 2, \ldots, k-1\}$, we have

$$BA_iB^{-1} = \begin{bmatrix} 1 & \alpha^i \beta^2 \\ 0 & 1 \end{bmatrix} \in N.$$ 

In other words, $M$ normalizes $N$. Since $N \cap M$ is trivial, we conclude that $H := \langle N, M \rangle = N \rtimes M \cong \mathbb{F}_q \rtimes C_{q-1}$.

Next, we prove that $H$ is intersecting. It is enough to prove that $H$ does not have a derangement (see [29, Lemma 2.3]). A typical element of $H$ is of the form

$$\begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \beta^t & 0 \\ 0 & \beta^{-t} \end{bmatrix} = \begin{bmatrix} \beta^t y & \beta^{-t} \\ 0 & \beta^{-1} \end{bmatrix},$$

where $y \in \mathbb{F}_q$ and $t \in \{0, 1, \ldots, q-1\}$. Hence, a typical element of $H$ has two eigenvalues in $\mathbb{F}_q$, which implies that it fixes a 2-subsets of $\text{PG}(1, q)$. We conclude that $H$ is a coclique of size $q(q-1)$ of $\text{PSL}(2, q)$.
9.2.2. When $q = 1 \mod 4$ or $q = 3 \mod 4$. Using Sagemath, we were able to prove that $\text{PSL}(2,13)$ acting on the cosets of $D_{12}$ has the EKR property. However, none of the known techniques worked in general for the groups in line 13. We believe that new techniques are needed to prove that the group in line 13 has the EKR property. We have compiled in the following table the status of the EKR property for the action of $\text{PSL}(2,q)$ acting on the 2-subsets of $\text{PG}(1,q)$.

| $q$ mod 4 | 3 | 4 | 5 | 7 | 8 | 9 | 11 | 13 |
|-----------|---|---|---|---|---|---|----|----|
| Order of point-stabilizers | 2 | 6 | 4 | 6 | 14 | 8 | 10 | 12 |
| Maximum cocliques | 4 | 12 | 4 | 12 | 56 | 8 | 17 | 12 |

Table 6. EKR for the group in line 13 for small values.

We conjecture the following, which is more general than what we need for the group in line 13 of Table 1.

**Conjecture 9.3.** Let $q$ be a prime power and consider the action of $\text{PSL}(2,q)$ on the 2-subsets of the projective line $\text{PG}(1,q)$.

1. If $q \equiv 1 \mod 4$, then $\rho(\text{PSL}(2,q)) = 1$.
2. If $q \equiv 3 \mod 4$, then $\text{PSL}(2,q)$ does not have the EKR property.

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# Appendix A. Character values

| Representation | Cycle type | Dimension | (n) | (n – 1, 1) | (n – 3, 3) | (n – 4, 3, 1) | (n – 6, 3²) | (n – 5, 4, 1) |
|----------------|------------|-----------|-----|------------|------------|-------------|-------------|-------------|
| $\chi^{[n]}$   | 1          | 1         | 1   | 1          | 1          | 1           | 1           | 1           |
| $\chi^{[n-1,1]}$ | n – 1     | –1        | 0   | –1         | 0          | –1          | 0           | 0           |
| $\chi^{[n-2,2]}$ | $\binom{n}{2} - n$ | 0         | –1 | 0          | –1         | 0           | –1          | 0           |
| $\chi^{[n-2,1^*]}$ | $\binom{n-1}{2}$ | 1         | 0   | 1          | 0          | 1           | 0           | 0           |
| $\chi^{[3,1^n-1]}$ | $\binom{n-1}{2}$ | $(-1)^{n-1}$ | 0   | $(-1)^n$  | 0          | $(-1)^{n-3}$ | 0           | 0           |
| $\chi^{[2^2,1^{n-4}]}$ | $\binom{n}{2} - n$ | 0         | $(-1)^{n-1}$ | 0          | $(-1)^n$  | 0           | $(-1)^{n-2}$ | 0           |
| $\chi^{[2,1^{n-2}]}$ | n – 1     | $(-1)^n$ | 0   | $(-1)^{n-1}$ | 0          | $(-1)^n$  | 0           | 0           |
| $\chi^{[1^4]}$ | 1         | $(-1)^{n-1}$ | $(-1)^n$ | $(-1)^{n-1}$ | $(-1)^n$ | $(-1)^{n-1}$ | $(-1)^{n-2}$ | $(-1)^{n-1}$ |

Table 7. Character values

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