Given the abstract wave equation $\ddot{\phi} - \Delta_\alpha \phi = 0$, where $\Delta_\alpha$ is the Laplace operator with a point interaction of strength $\alpha$, we define and study $\bar{W}_\alpha$, the associated wave generator in the phase space of finite energy states. We prove the existence of the phase flow generated by $\bar{W}_\alpha$, and describe its most relevant properties with particular emphasis on the associated symplectic structure and scattering theory.
I. INTRODUCTION.

To introduce the problem we begin with a well known example. Given the free scalar, zero mass, wave equation

$$\ddot{\phi} - \Delta \phi = 0,$$

the usual attitude in the literature is to search the solutions in the real Sobolev-Hilbert space $H^2(\mathbb{R}^3)$; in order to fix the notations we recall that $H^s(\mathbb{R}^3)$, $s \in \mathbb{R}$, is defined as the set of tempered distributions with a Fourier transform which is square integrable w.r.t. the measure with density $(1 + |k|^2)^s$. This is a standard mathematical choice but not the more natural one. In fact, equation (1) can be written in the first order form

$$\dot{\psi} = W\psi,$$

where the linear operator

$$W : H^2(\mathbb{R}^3) \oplus H^1(\mathbb{R}^3) \to H^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$$

is defined as

$$W \left( \begin{array}{c} \phi \\ \dot{\phi} \end{array} \right) = \left( \begin{array}{cc} 0 & 1 \\ \Delta & 0 \end{array} \right) \left( \begin{array}{c} \phi \\ \dot{\phi} \end{array} \right).$$

Here $\Delta$ is the usual Laplace operator viewed as a self-adjoint operator on $L^2(\mathbb{R}^3)$. It is well known that equation (2) generates a strongly continuous one parameter group of evolution

$$U^t : H^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \to H^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3).$$

This group group is energy preserving, i.e. there exists an energy form

$$\mathcal{E}(\phi, \dot{\phi}) = \frac{1}{2} \left( \|\dot{\phi}\|_2^2 + \|\sqrt{-\Delta} \phi\|_2^2 \right),$$

coinciding with the Hamiltonian of the system, preserved by the flow. Moreover $U^t$ constitutes a group of canonical transformations w.r.t. the symplectic form

$$\omega \left( (\phi, \dot{\phi}), (\varphi, \dot{\varphi}) \right) := \langle \phi, \dot{\varphi} \rangle - \langle \varphi, \dot{\phi} \rangle$$

($\langle \cdot, \cdot \rangle$ denoting the usual scalar product on $L^2(\mathbb{R}^3)$) and $W$ is nothing but the Hamiltonian vector field corresponding, via $\omega$, to $\mathcal{E}$. 

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As the form of Hamiltonian $E$ suggests, a more natural domain for the study of the system described by (3) is the space of the finite energy states, which is larger than the original one, because the first component $\phi$ of such a state is not necessarily square integrable, as instead is implicit in the standard Sobolev environment recalled above. This more suitable description goes as follows.

Let us define (general and more complete definitions will be given in the following section) $\bar{H}^1(\mathbb{R}^3)$ as the completion of the space $C^\infty_0(\mathbb{R}^3)$ in the norm $\|\sqrt{-\Delta} \phi\|_2$. Now it is possible to define the new operator $\bar{W}$ on $\bar{H}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$, the Hilbert space of finite energy states, by

$$\bar{W} : \bar{H}^2(\mathbb{R}^3) \oplus H^1(\mathbb{R}^3) \rightarrow \bar{H}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3), \quad \bar{W}(\phi, \dot{\phi}) := (\dot{\phi}, \Delta \phi). \quad (4)$$

where

$$\bar{H}^2(\mathbb{R}^3) := \{ \phi \in \bar{H}^1(\mathbb{R}^3) : \Delta \phi \in L^2(\mathbb{R}^3) \}.$$ 

It is an easy matter to verify that $\bar{W}$ is a skew-adjoint operator (see e.g. [1, thm. 2.1.2], [2, §XI.10]) so that due to Stone theorem it defines a strongly continuous one parameter group of evolution

$$\bar{U}^t : \bar{H}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \rightarrow \bar{H}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$$

which is trivially energy preserving, just because the energy coincides with the norm of the Hilbert space, and the flow is given by a group of isometric operators. This procedure generalizes to the case in which one considers an abstract wave equation with a positive self-adjoint operator in the place of $-\Delta$ (see [3], [4, §8]).

Here we consider and study in detail the case in which $-\Delta$ is replaced by $-\Delta_\alpha$, the Laplace operator with a point interaction of strength $\alpha$ (see section II for its precise definition), and construct the corresponding wave generator $\bar{W}_\alpha$; since $-\Delta_\alpha$ is not positive when $\alpha < 0$ one can not directly use the results appearing in [3] and [4].

The abstract wave equation corresponding to $\Delta_\alpha$, i.e.

$$\ddot{\phi} - \Delta_\alpha \phi = 0,$$ 

was introduced for the first time in [5]. There, when $\phi$ is vector-valued and when $\alpha = -\frac{me}{c}$ ($m$ the phenomenological mass, $c$ the velocity of light, $e$ the electric charge), it is shown that (5) describes the evolution of the electromagnetic field self-interacting with a point particle in dipole approximation (the
so called linearized Pauli-Fierz model). Another model connected with the
wave equation (5), often studied in the fifties’ and sixties’ literature on exact
models in quantum field theory, is the so called “pair theory” (see ([6]-[8]
and references therein). The classical version of this model is the regularized
version of the one we study here, and many at the time unanswered questions
about its behaviour in the ultraviolet limit find their rigorous collocation in
the present work.

In [5], [9], [10] it is also shown that the Cauchy problem is well posed on
the phase space \( D^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \), \( D^1(\mathbb{R}^3) \simeq H^1(\mathbb{R}^3) \oplus \mathbb{R} \), (refer to section II
for the definition of \( D^1(\mathbb{R}^3) \)) and that the corresponding strongly continuous
one parameter group of evolution

\[
U^t_\alpha : D^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \to D^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)
\]

preserves the energy

\[
\mathcal{E}_\alpha(\phi, \dot{\phi}) := \frac{1}{2} \left( \| \dot{\phi} \|_2^2 + F_\alpha(\phi, \phi) \right),
\]

where \( F_\alpha \) denotes the bilinear form corresponding to the self-adjoint operator
\(-\Delta_\alpha\). Therefore, analogously to the case of the free wave equation, the
problem of defining (5) on the larger space of finite energy states naturally
arises. The theory of delta point interactions was originally developed in
the context of nonrelativistic quantum mechanics (see [11] and ref erences
therein); this made natural to use \( L^2(\mathbb{R}^3) \) as the underlying Hilbert space
and so, in order to define the dynamics on the space of finite energy states,
one has to modify the original definition of \(-\Delta_\alpha\), to allow the elements of
its domain being not square integrable. This is done in section III where
we also show (thm. 3.1) that the operators \( \bar{W}_\alpha \) here constructed generate
an evolution group \( \bar{U}^t_\alpha \), a fact that, in the case \( \alpha \leq 0 \), is not immediately
evident. So, as an aside result, a conserved energy form exists; this form
however is not positive when \( \alpha < 0 \), and therefore it is not suitable to define
the norm of the appropriate phase space.

In section IV we treat the Hamiltonian formulation of the wave equa-tions with delta interactions. Here we solve the problem by giving a complex
structure \( J_\alpha \) commuting with the operator \( \bar{W}_\alpha \). This leads to an equivalent
Schrödinger-like first-order formulation which, also in view of a future quan-
tization of the dynamical system under study, plays a key role. The complex
structure before mentioned is obtained considering separately the case \( \alpha \leq 0 \)
from the other case: in the strictly negative case we obtain an invariant splitting of the phase space and the complex structure in such a way that the Hamiltonian vector field appears separately as a Schrödinger equation both on the stable and unstable part of the phase space; in particular, on the unstable subspace, which is finite dimensional, the Hamiltonian is that of an harmonic repulsor, and the Schrödinger equation is the corresponding ordinary differential equation, as expected.

The last topic treated (see section V) is the scattering theory for the pair of operators $(\hat{W}_\alpha, \hat{W})$. Being the Hilbert phase spaces for $\hat{W}_\alpha$ and $\hat{W}$ different, resort has to be made to the two Hilbert space scattering theory introduced by Kato in [4]. Using then Birman invariance principle and the trace condition of Birman-Kuroda theorem, we are able to prove the existence of the Möller wave operators and their completeness (thms. 5.1 and 5.2). As a consequence of the machinery needed for the definition of wave operators, one obtains a relation (see (12)-(14)) between the evolution group (acting on the real Hilbert space of states with finite energy) generated by $\hat{W}_\alpha$ and the unitary group (acting on the complex Hilbert space $L^2(\mathbb{R}^3)$, the complexification of $L^2(\mathbb{R}^3)$) generated by $\sqrt{-\Delta_\alpha}$ (in the case $\alpha < 0$ one consider only the positive part of the operator). This can be seen as a variation of the procedure applied in section III in the case one uses the standard complex structure on $L^2(\mathbb{R}^3)$: indeed the two structures are related in a simple way (see (10) and (11)). The relations (12)-(14) could also be used to define the group $\hat{U}^t_\alpha$, the generator of which is easily seen to be $\hat{W}_\alpha$, so providing an alternative proof of the existence of the dynamics.

II. PRELIMINARIES.

We start by giving definitions and main properties of the Sobolev type spaces needed in the sequel, and to which we made reference in the introduction. We define the family of pre-Hilbert spaces $\check{H}^s(\mathbb{R}^3)$, $s \in \mathbb{R}$, as the set of tempered distributions with a Fourier transform (denoted by $\hat{\cdot}$ or by $\mathcal{F}$) which is square integrable w.r.t. to the measure with density $|k|^{2s}$. The scalar product is defined as

$$\langle \phi_1, \phi_2 \rangle_s := \int_{\mathbb{R}^3} dk |k|^{2s} \hat{\phi}_1(k) \hat{\phi}_2(k).$$

Note that, when $s > 0$, $H^s(\mathbb{R}^3) \subset \check{H}^s(\mathbb{R}^3)$ and $\check{H}^{-s}(\mathbb{R}^3) \subset H^{-s}(\mathbb{R}^3)$, the embeddings being continuous. Since $|k|^{-2s}$ is locally integrable for any $s <$
\forall s < \frac{3}{2}, \quad L^2(\mathbb{R}^3, |x|^{2s}dx) \subset \mathcal{S}'(\mathbb{R}^3), \quad \tilde{H}^s(\mathbb{R}^3) \equiv \mathcal{F}^{-1}(L^2(\mathbb{R}^3, |k|^{2s}dk))

and thus \(\tilde{H}^s(\mathbb{R}^3)\) is complete for any \(s < 3/2\) and coincides with the usual Riesz potential spaces (see e.g. [12, §7.1.2]).

We can then define the isomorphism \((r - s < 3/2)\)

\[ (-\bar{\Delta})^{s/2} : \tilde{H}^r(\mathbb{R}^3) \to \tilde{H}^{r-s}(\mathbb{R}^3), \quad \mathcal{F}((-\bar{\Delta})^{s/2}\phi)(k) := |k|^s\hat{\phi}(k). \]

Our notation is justified by observing that, in the case \(0 < r = s < 3/2\), \((-\bar{\Delta})^{s/2}\) coincides with the closure of \((-\Delta)^{s/2} : H^s(\mathbb{R}^3) \to L^2(\mathbb{R}^3).\)

Since, contrarily to what happens for the usual Sobolev chain \(H^s(\mathbb{R}^3)\), \(\tilde{H}^r(\mathbb{R}^3)\) is not included in \(\tilde{H}^s(\mathbb{R}^3)\) when \(r > s\), we also define the sequence of spaces

\[ \tilde{H}^n(\mathbb{R}^3) := \bigcap_{k=1}^n \tilde{H}^k(\mathbb{R}^3) \equiv \mathcal{F}^{-1}\left( \bigcap_{k=1}^n L^2(\mathbb{R}^3, |x|^{2k}dx) \right). \]

Obviously \(\tilde{H}^n(\mathbb{R}^3)\) is a Hilbert space with norm

\[ \|\phi\|_{\tilde{H}^n} := \left( \sum_{k=1}^n \|(-\bar{\Delta})^{k/2}\phi\|^2 \right)^{1/2}. \]

We come now to point interactions; for their general theory of we refer to [11]; here we confine ourselves to the essential definitions and results. The operator \(-\Delta_\alpha\) describing a standard point interaction at the origin with strength \(\alpha\) is defined as follows. Let us introduce the dense linear subspace of \(L^2(\mathbb{R}^3)\)

\[ D^2_\alpha(\mathbb{R}^3) := \left\{ \phi \in L^2(\mathbb{R}^3) : \phi = \phi_\lambda + Q_\phi G_\lambda, \phi_\lambda \in H^2(\mathbb{R}^3), \left( \alpha + \frac{\sqrt{\lambda}}{4\pi} \right) Q_\phi = \phi_\lambda(0) \right\}, \]

where \(0 < \lambda \neq -\text{sign}(\alpha) (4\pi \alpha)^2\) and

\[ G_\lambda(x) = \frac{e^{-\sqrt{\lambda}|x|}}{4\pi|x|}. \]

The Laplacian with a point interaction with strength \(\alpha\) is the operator

\[ -\Delta_\alpha : D^2_\alpha(\mathbb{R}^3) \to L^2(\mathbb{R}^3), \quad -\Delta_\alpha \phi := -\Delta \phi_\lambda - \lambda Q_\phi G_\lambda. \]
Its resolvent is given by

\[ (-\Delta_\alpha + \lambda)^{-1} = (-\Delta + \lambda)^{-1} + \left( \alpha + \frac{\sqrt{\lambda}}{4\pi} \right)^{-1} G_\lambda \otimes G_\lambda, \]

where \( G_\lambda \otimes G_\lambda(\phi) := \langle G_\lambda, \phi \rangle G_\lambda. \)

The bilinear form corresponding to \(-\Delta_\alpha\) has domain \( D^1(\mathbb{R}^3) \times D^1(\mathbb{R}^3), \)

\[ D^1(\mathbb{R}^3) := \{ \phi \in L^2(\mathbb{R}^3) : \phi = \phi_\lambda + Q_\phi G_\lambda, \phi_\lambda \in H^1(\mathbb{R}^3), Q_\phi \in \mathbb{R} \}, \]

and is defined by

\[ F_\alpha(\phi, \varphi) := \langle (-\Delta + \lambda)^{1/2} \phi_\lambda, (-\Delta + \lambda)^{1/2} \varphi_\lambda \rangle - \lambda \langle \phi, \varphi \rangle + \left( \alpha + \frac{\sqrt{\lambda}}{4\pi} \right) Q_\phi Q_\varphi \]

(see [13]). Both the expressions for \( F_\alpha \) and \(-\Delta_\alpha\) contain the arbitrary parameter \( \lambda \), but contrarily to the appearance, they do not depend on it. Indeed (following [14, §2]) the operator and form domain can be defined in the following alternative, and more useful, way. Note that, since for any \( \lambda > 0 \)

\[ G_\lambda \in L^2(\mathbb{R}^3), \quad G - G_\lambda \in \bar{H}^2(\mathbb{R}^3), \quad (G - G_\lambda)(0) = \frac{\sqrt{\lambda}}{4\pi}, \]

where

\[ G(x) = \frac{1}{4\pi |x|}, \]

defining

\[ \phi_{\text{reg}} := \phi_\lambda + Q_\phi (G_\lambda - G) \in \bar{H}^2(\mathbb{R}^3) \]

we have equivalently

\[ D^2_\alpha(\mathbb{R}^3) = \left\{ \phi \in L^2(\mathbb{R}^3) : \phi = \phi_{\text{reg}} + Q_\phi G, \phi_{\text{reg}} \in \bar{H}^2(\mathbb{R}^3), Q_\phi \in \mathbb{R}, \alpha Q_\phi = \phi_{\text{reg}}(0) \right\}. \]

Correspondingly, the form domain is

\[ D^1(\mathbb{R}^3) = \left\{ \phi \in L^2(\mathbb{R}^3) : \phi = \phi_{\text{reg}} + Q_\phi G, \phi_{\text{reg}} \in \bar{H}^1(\mathbb{R}^3), Q_\phi \in \mathbb{R} \right\} \]

so that, with this definition, the singular part of the field is exactly Coulombian. However such a singular field \( G \) is not in the configuration space \( D^1(\mathbb{R}^3) \).
The removal of this incongruence will lead, in the following section, to the introduction of the operator $\bar{W}_\alpha$.

With the domains so given we can redefine the operator and the form as

$$-\Delta_\alpha \phi = -\bar{\Delta} \phi_{\text{reg}}$$

and

$$F_\alpha(\phi, \varphi) = \langle (-\bar{\Delta})^{1/2} \phi_{\text{reg}}, (-\bar{\Delta})^{1/2} \varphi_{\text{reg}} \rangle + \alpha Q_\phi Q_\varphi .$$

Now it is well known (see [11, Chap. I.1]) that $-\Delta_\alpha$ is a selfadjoint operator in $L^2(\mathbb{R}^3)$. An important property is that $-\Delta_\alpha$ is positive only for $\alpha \geq 0$, whereas for $\alpha < 0$ it is only bounded from below; more precisely if $\alpha \geq 0$ (repulsive delta interactions) the spectrum of the operator is absolutely continuous and coinciding with $[0, +\infty)$; if $\alpha < 0$ (attractive delta interactions) the spectrum is given by $\{-\lambda_0\} \cup [0, +\infty)$, where $-\lambda_0 = -(4\pi \alpha)^2$ is an isolated negative eigenvalue, and the remaining part of the spectrum is absolutely continuous. In the Schrödinger case this eigenvalue corresponds to a bound state, while in the wave case where one has a second order equation in time, it leads to unstable solutions exponentially running away in the past or in the future (see [5], [9], [10] and reference therein for the meaning of these well known runaway solutions in classical electrodynamics).

We now come to the wave generator associated to the standard delta operator. Its domain and action are given by

$$W_\alpha : D^2_\alpha(\mathbb{R}^3) \oplus D^1(\mathbb{R}^3) \to D^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) ,$$

$$W_\alpha \left( \begin{array}{c} \phi \\ \dot{\phi} \end{array} \right) = \left( \begin{array}{cc} 0 & 1 \\ \Delta_\alpha & 0 \end{array} \right) \left( \begin{array}{c} \phi \\ \dot{\phi} \end{array} \right) .$$

(6)

By considering the Hilbert space structure given by $D^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \simeq H^1(\mathbb{R}^3) \oplus \mathbb{R} \oplus L^2(\mathbb{R}^3)$ this operator is the generator of a strongly continuous group of operators

$$U^t_\alpha : D^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \to D^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) .$$

In the case $\alpha \geq 0$ this is an immediate consequence of the skew-adjointness of $W_\alpha$ with respect to the positive energy scalar product on the phase space given by

$$\langle \langle (\phi, \dot{\phi}), (\varphi, \dot{\varphi}) \rangle \rangle_\alpha := \langle \dot{\phi}, \dot{\varphi} \rangle + F_\alpha(\phi, \varphi) .$$

(7)

More precisely one has the following result (the proof being a straightforward calculation):
Theorem 2.1. For any $\alpha \in \mathbb{R}$, with respect to the scalar product $\langle\langle \cdot, \cdot \rangle\rangle_\beta$, $\beta \geq 0$, one has

$$D(W_\alpha^*) = D(W_\beta)$$

and

$$W_\alpha^*(\phi, \dot{\phi}) = -\left(\dot{\phi}_{\text{reg}} + \frac{\alpha}{\beta}Q_{\phi}G, \Delta_{\text{reg}}\phi\right), \quad \beta > 0,$$

$$W_\alpha^*(\phi, \dot{\phi}) = -\left(\dot{\phi}_{\text{reg}} + Q_{\phi}G, \Delta_{\text{reg}}\phi\right) \equiv W_\beta(\phi, \dot{\phi}), \quad \beta = 0 = \alpha .$$

In the case $\alpha < 0$ the operator $W_\alpha$ is readily proven to be a generator by considering the operator

$$W_{\alpha, \lambda}(\phi, \dot{\phi}) := (\dot{\phi}, (\Delta_\alpha - \lambda)\phi_{\text{reg}}),$$

where $\lambda > \lambda_0$. This, being now $-\Delta_\alpha + \lambda$ positive, is skew-adjoint with respect to the scalar product

$$\langle\langle (\phi, \dot{\phi}), (\varphi, \dot{\varphi}) \rangle\rangle_{\alpha, \lambda} := \langle \dot{\phi}, \dot{\varphi} \rangle + F_\alpha(\phi, \varphi) + \lambda \langle \phi, \varphi \rangle ,$$

and so it generates a group of isometries (w.r.t. the Hilbert structure given by (8)). The original operator $W_\alpha$, being a perturbation of the previous one by a bounded operator, also generates a strongly continuous group of operators on the phase space (which however are no more isometries).

We now describe an alternative way to prove that $W_\alpha$, $\alpha < 0$, is a generator. Such a different method will play a key role in the next sections. As we already said before in the case $\alpha < 0$ the self-adjoint operator $-\Delta_\alpha$ has a negative eigenvalue $-\lambda_0$ (with corresponding normalized eigenvector $4\pi \sqrt{-2\alpha} G_{\lambda_0}$) which gives rise to the runaway solutions of the wave equation associated to $W_\alpha$. Proceeding as in [10, §4] (note that there we worked with the different decomposition $\phi = \phi_{\lambda_0} + Q_{\phi}G_{\lambda_0}$) we consider the linear operator

$$W_\alpha^{\text{nr}} : [D_\alpha^2(\mathbb{R}^3)]_{nr} \oplus [D_\alpha^1(\mathbb{R}^3)]_{nr} \rightarrow [D_\alpha^1(\mathbb{R}^3)]_{nr} \oplus [L^2(\mathbb{R}^3)]_{nr},$$

where, given any vector subspace $\mathcal{V} \subseteq L^2(\mathbb{R}^3)$, we have defined the corresponding “non runaway” subspace $[\mathcal{V}]_{nr}$ by

$$[\mathcal{V}]_{nr} := \{ \phi \in \mathcal{V} : \langle \phi, G_{\lambda_0} \rangle = 0 \} ,$$

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and
\[ \Delta^{nr}_α := (\Delta_α)_{|[D^*_α(R^3)]_{nr}} \equiv P_{nr} \cdot (\Delta_α)_{|[D^*_α(R^3)]_{nr}} ; \]
P_{nr} being the orthogonal projector onto \([L^2(R^3)]_{nr}\). By simple calculations one has (see [10, §4])
\[ [D^1(R^3)]_{nr} = \left\{ \phi \in D^1(R^3) : Q_\phi = -4\pi \sqrt{\lambda_0} \langle \phi_{\text{reg}}, G_{\lambda_0} \rangle \right\} \]
and
\[ [D^2_α(R^3)]_{nr} = \left\{ \phi \in D^2_α(R^3) : \phi_{\text{reg}}(0) = \lambda_0 \langle \phi_{\text{reg}}, G_{\lambda_0} \rangle \right\} \]
and
\[ \Delta^{nr}_α \phi = \bar{\Delta} \phi_{\text{reg}} - 8\pi \sqrt{\lambda_0} \langle \bar{\Delta} \phi_{\text{reg}}, G_{\lambda_0} \rangle G_{\lambda_0} \]
\[ = \Delta \phi_{\text{reg}} + 8\pi \sqrt{\lambda_0} \langle (-\Delta + \lambda_0) \phi_{\text{reg}}, G_{\lambda_0} \rangle G_{\lambda_0} - 8\pi \sqrt{\lambda_0} \lambda_0 \langle \phi_{\text{reg}}, G_{\lambda_0} \rangle G_{\lambda_0} \]
\[ = \bar{\Delta} \phi_{\text{reg}} . \]
The non-negative bilinear form associated to \(-\Delta^{nr}_α\) is then
\[ F^{nr}_α(\phi, \varphi) = \langle (-\Delta)^{1/2} \phi_{\text{reg}}, (-\Delta)^{1/2} \varphi_{\text{reg}} \rangle - 4\pi \lambda_0^{3/2} \langle \phi_{\text{reg}}, G_{\lambda_0} \rangle \langle \varphi_{\text{reg}}, G_{\lambda_0} \rangle \]
and \(W^{nr}_α\) is skew-adjoint w.r.t. the scalar product
\[ \langle \langle (\phi, \dot{\phi}), (\varphi, \dot{\varphi}) \rangle \rangle^{nr}_α := \langle \dot{\phi}, \dot{\varphi} \rangle + F^{nr}_α(\phi, \varphi) . \]
The strongly continuous one parameter group of evolution generated by \(W^{nr}_α\) preserves the non-negative energy
\[ \mathcal{E}^{nr}_α(\phi, \dot{\phi}) := \frac{1}{2} \left( \langle \dot{\phi}, \dot{\phi} \rangle + F^{nr}_α(\phi, \phi) \right) \]
with coincides with the Hamiltonian of the sistem w.r.t. the symplectic form \(\omega\) (see [10, thm. 4.2] for an alternative Hamiltonian picture).

Since \(\Delta_α G_{\lambda_0} = \lambda_0 G_{\lambda_0}\), and
\[ D^2_α(R^3) \simeq [D^2_α(R^3)]_{nr} \oplus \mathbb{R} , \]
\[ D^1(R^3) \simeq [D^1(R^3)]_{nr} \oplus \mathbb{R} , \]
\[ L^2(R^3) \simeq [L^2(R^3)]_{nr} \oplus \mathbb{R} , \]

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we can write
\[ W_\alpha = W_\alpha^{nr} \times \Lambda_0, \]
where
\[ \Lambda_0 : \mathbb{R}^2 \to \mathbb{R}^2, \quad \Lambda_0(x, \dot{x}) := (\dot{x}, \lambda_0 x). \]
Therefore \( W_\alpha, \alpha < 0, \) is a generator and
\[ U_t^\alpha \equiv e^{tW_\alpha} = e^{tW_\alpha^{nr}} \times e^{t\Lambda_0}. \]

Here and below, given two linear operators \( A_1 : D(A_1) \to H_1 \) and \( A_2 : D(A_2) \to H_2, \) \( A_1 \times A_2 : D(A_1) \times D(A_2) \to H_1 \oplus H_2 \) denotes the the linear operator defined by
\[ A_1 \times A_2(\phi_1, \phi_2) := (A_1 \phi_1, A_2 \phi_2). \]

In conclusion, for any \( \alpha \in \mathbb{R}, \) \( U_t^\alpha \) is a group of canonical transformation w.r.t. the symplectic form \( \omega, \) and \( W_\alpha \) is the Hamiltonian vector field corresponding to the energy
\[ \mathcal{E}_\alpha(\phi, \dot{\phi}) = \frac{1}{2} \left( \| \dot{\phi} \|_2^2 + F_\alpha(\phi, \dot{\phi}) \right) \equiv \mathcal{E}(\phi_{\text{reg}}, \dot{\phi}) + \frac{\alpha}{2} Q_\phi^2. \]

Let us remark that the flow \( U_t^\alpha \) can be explicitly calculated (see [5, thm. 3.1]).

III. THE OPERATOR \( \bar{W}_\alpha. \)

Now we would like to mimic the construction of the energy space for the usual wave generator and the extension of the operator itself, to the case of delta point interactions. To this end, let us define the linear operator
\[ \bar{W}_\alpha : \bar{D}_\alpha^2(\mathbb{R}^3) \oplus D^1(\mathbb{R}^3) \to \bar{D}_\alpha^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3), \quad \bar{W}_\alpha(\phi, \dot{\phi}) := (\dot{\phi}, \bar{\Delta} \phi_{\text{reg}}), \]
where
\[ \bar{D}_\alpha^2(\mathbb{R}^3) := \left\{ \phi = \phi_{\text{reg}} + Q_\phi G, \ \phi_{\text{reg}} \in \bar{H}^2(\mathbb{R}^3), \ Q_\phi \in \mathbb{R}, \ \alpha Q_\phi = \phi_{\text{reg}}(0) \right\}, \]
\[ \bar{D}_\alpha^1(\mathbb{R}^3) := \left\{ \phi = \phi_{\text{reg}} + Q_\phi G, \ \phi_{\text{reg}} \in \bar{H}^1(\mathbb{R}^3), \ Q_\phi \in \mathbb{R} \right\}. \]

Analogously to the free case \( \bar{D}_\alpha^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \) describes now the space of finite energy states. Moreover the Coulombian singularity \( G \) is now in the configuration space \( \bar{D}_\alpha^1(\mathbb{R}^3). \)
Introducing the Hilbert space structure given by $\bar{D}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \simeq \bar{H}^1(\mathbb{R}^3) \oplus \mathbb{R} \oplus L^2(\mathbb{R}^3)$ we want now to show that also in this case $\bar{W}_\alpha$ generates a strongly continuous one parameter group of evolution. When $\alpha > 0$, considering, similarly to the case of $W_\alpha$, the scalar product

$$
\langle \langle (\phi, \dot{\phi}), (\varphi, \dot{\varphi}) \rangle \rangle_\alpha := \langle \dot{\phi}, \dot{\varphi} \rangle + \langle (-\Delta)^{1/2} \phi_{\text{reg}}, (-\Delta)^{1/2} \varphi_{\text{reg}} \rangle + \alpha Q\phi Q\varphi ,
$$

one can prove that $\bar{W}_\alpha$ is skew-adjoint and so it is a generator. Note that when $\alpha = 0$, contrarily to situation discussed in the previous section, $\langle \langle \cdot, \cdot \rangle \rangle_\alpha$ is no more a scalar product, being annihilated by the zero energy eigenvector $(G, 0)$ (this fact has to be compared with the presence of a zero energy resonance for $-\Delta_0$). In order to show that also in the case $\alpha \leq 0$ $\bar{W}_\alpha$ is a generator one can not use the same strategy as before consisting in a translation, since the scalar product (8) is now ill-defined, $\bar{D}^1(\mathbb{R}^3)$ being not a subset of $L^2(\mathbb{R}^3)$. So the perturbation argument fails and we are forced to proceed in an alternative way. The decomposition of $W_\alpha$, $\alpha < 0$, introduced at the end of the previous section is our starting point: we simply extend it to the case of $\bar{W}_\alpha$. Therefore we define, when $\alpha < 0$,

$$
[\bar{D}_2^2(\mathbb{R}^3)]_{nr} = \{ \phi \in \bar{D}_2^2(\mathbb{R}^3) : \phi_{\text{reg}}(0) = \lambda_0 \langle \dot{\phi}_{\text{reg}}, G_{\lambda_0} \rangle \}
$$

and

$$
[\bar{D}^1(\mathbb{R}^3)]_{nr} = \{ \phi \in \bar{D}^1(\mathbb{R}^2) : Q\phi = -4\pi \sqrt{\lambda_0} \langle \dot{\phi}_{\text{reg}}, G_{\lambda_0} \rangle \},
$$

and

$$
\bar{W}^nr_\alpha : [\bar{D}_2^2(\mathbb{R}^3)]_{nr} \oplus [\bar{D}^1(\mathbb{R}^3)]_{nr} \to [\bar{D}^1(\mathbb{R}^3)]_{nr} \oplus [L^2(\mathbb{R}^3)]_{nr} ,
$$

$$
\bar{W}^nr_\alpha(\phi, \dot{\phi}) := (\phi, \bar{\Delta}^nr_\alpha \dot{\phi}) ,
$$

where

$$
\bar{\Delta}^nr_\alpha : [\bar{D}_2^2(\mathbb{R}^3)]_{nr} \to [L^2(\mathbb{R}^3)]_{nr} ,
$$

$$
\bar{\Delta}^nr_\alpha \phi := \bar{\Delta}\phi_{\text{reg}} .
$$

With such definitions $\bar{W}^nr_\alpha$ results skew-adjoint with respect to the scalar product on $[\bar{D}^1(\mathbb{R}^3)]_{nr} \oplus [L^2(\mathbb{R}^3)]_{nr}$ given by

$$
\langle \langle (\phi, \dot{\phi}), (\varphi, \dot{\varphi}) \rangle \rangle^nr_\alpha := \langle \dot{\phi}, \dot{\varphi} \rangle + \langle (-\Delta)^{1/2} \phi_{\text{reg}}, (-\Delta)^{1/2} \varphi_{\text{reg}} \rangle - 4\pi \lambda_0^{3/2} \langle \dot{\phi}_{\text{reg}}, G_{\lambda_0} \rangle \langle \varphi_{\text{reg}}, G_{\lambda_0} \rangle .
$$

Moreover, since

$$
\bar{D}_2^2(\mathbb{R}^3) \simeq [\bar{D}_2^2(\mathbb{R}^3)]_{nr} \oplus \mathbb{R} , \quad \bar{D}^1(\mathbb{R}^3) \simeq [\bar{D}^1(\mathbb{R}^3)]_{nr} \oplus \mathbb{R} ,
$$

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similarly to the case of $W_\alpha$ we have

$$\tilde{W}_\alpha = \tilde{W}_\alpha^{nr} \times \Lambda_0.$$ 

For the case $\alpha = 0$ a similar decomposition is possible by using the projection onto the subspace orthogonal to the eigenvector $(G, 0)$. Indeed, defining

$$\tilde{W}(0) : \tilde{H}_0^2(R^3) \oplus D^1(R^3) \to \tilde{H}^1(R^3) \oplus L^2(R^3), \quad \tilde{W}(0)(\phi, \dot{\phi}) := (\dot{\phi}_{reg}, \tilde{\Delta}\phi),$$

where $\tilde{H}_0^2(R^3) := \{\phi \in \tilde{H}^2(R^3) : \phi(0) = 0\}$, the operator $\tilde{W}(0)$ is skew-adjoint with respect to the scalar product

$$\langle\langle (\phi, \dot{\phi}), (\varphi, \dot{\varphi})\rangle\rangle_{(0)} := \langle\dot{\varphi}, \dot{\varphi}\rangle + \langle(-\tilde{\Delta})^{1/2}\phi, (-\tilde{\Delta})^{1/2}\varphi\rangle$$

and, since $\tilde{D}^1(R^3) \simeq \tilde{H}^1(R^3) \oplus \mathbb{R}$, the following decomposition holds:

$$\tilde{W}_0 = \tilde{W}(0) \times 0.$$ 

We can now state our result regarding the existence of dynamics:

**Theorem 3.1.** $\tilde{W}_\alpha$ is a closed operator coinciding with the closure of $W_\alpha$. It generates a strongly continuous group of evolution

$$\tilde{U}^t_\alpha : \tilde{D}^1(R^3) \oplus L^2(R^3) \to \tilde{D}^1(R^3) \oplus L^2(R^3),$$

which can be defined as

$$\tilde{U}^t_\alpha(\phi, \dot{\phi}) = \lim_{n \to \infty} U^t_\alpha(\phi_n, \dot{\phi}),$$

where $\{\phi_n\}_{1}^{\infty} \subset D^1(R^3)$ is any sequence such that $\phi_n \to \phi$ in $\tilde{D}^1(R^3)$.

**Proof.** $\tilde{W}_\alpha$ is a generator since it is skew-adjoint when $\alpha > 0$ and $\tilde{W}_0 = \tilde{W}(0) \times 0$, $\tilde{W}_\alpha = \tilde{W}_\alpha^{nr} \times \Lambda_0$, $\alpha < 0$, where both $\tilde{W}(0)$ and $\tilde{W}_\alpha^{nr}$ are skew-adjoint. Therefore $\tilde{W}_\alpha$ is closed. By its definition $\tilde{W}_\alpha$ is equal to $W_\alpha$ on $D^2_\alpha(R^3) \oplus D^1(R^3)$ and so it coincides with the closure of $W_\alpha$ if $D^2_\alpha(R^3) \oplus D^1(R^3)$ is a core. This is proven as follows:

analogously to the case of $W_\alpha$, any $\phi \in \tilde{D}^2_\alpha(R^3)$ admits the representation

$$\phi = \phi_\lambda + Q_\phi G_\lambda$$

where

$$\phi_\lambda = \phi_{reg} - Q_\phi (G_\lambda - G) \in \tilde{H}^2(R^3)$$

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\[
\left( \alpha + \frac{\sqrt{\lambda}}{4\pi} \right) Q_\phi = \phi(0).
\]

Consider then a sequence \( \phi_n^\alpha \) in \( H^2(\mathbb{R}^3) \) and define
\[
\phi_n := \phi_n^\alpha + Q_n G_\lambda \in D^2_\alpha(\mathbb{R}^3),
\]
where
\[
Q_n := \left( \alpha + \frac{\sqrt{\lambda}}{4\pi} \right)^{-1} \phi_n^\alpha(0).
\]

Now if \( \phi_n^\alpha \) converges in \( \bar{H}^2(\mathbb{R}^3) \) to \( \phi_\lambda \), we have that \( Q_n \) converges to \( Q_\phi \), thanks to the continuous embedding of \( \bar{H}^2(\mathbb{R}^3) \) in \( C^0_c(\mathbb{R}^3) \) (see e.g. [12, §5.6.2]).

Being \( \bar{W}_\alpha \) equal to \( W_\alpha \) on \( D^2_\alpha(\mathbb{R}^3) \oplus D^1(\mathbb{R}^3) \) the same is true for the corresponding groups of evolution. Since \( D^2_\alpha(\mathbb{R}^3) \oplus D^1(\mathbb{R}^3) \) is dense in \( D^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \) which is continuously embedded in \( \bar{D}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \), one has the equality \( \bar{U}_\alpha^t(\phi, \dot{\phi}) = U_\alpha^t(\phi, \dot{\phi}) \) for any \( \phi \in D^1(\mathbb{R}^3) \). The proof is then concluded by the denseness of \( D^1(\mathbb{R}^3) \) in \( \bar{D}^1(\mathbb{R}^3) \).

Let us remark that, since \( \bar{W}_\alpha \) is the closure of \( W_\alpha \), our construction coincides, in the case \( \alpha \geq 0 \), with the abstract one given in [3] (see also [4, §8] for a similar construction). Moreover, since \( W_\alpha = W^nr_\alpha \times \Lambda_0 \) when \( \alpha < 0 \), one has that \( W^nr_\alpha \) is the closure of \( W^nr_\alpha \).

IV. THE SYMPLECTIC STRUCTURE.

The standard symplectic structure recalled in the introduction,
\[
\omega \left( (\phi, \dot{\phi}), (\varphi, \dot{\varphi}) \right) := \langle \phi, \dot{\varphi} \rangle - \langle \varphi, \dot{\phi} \rangle
\]

it is not well defined on the phase space finite energy states, i.e \( \bar{D}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \). This requires a different approach to the Hamiltonian description of the dynamical system described in the previous paragraph. The problem shows up already in the case of the free wave equation, with the phase space \( \bar{H}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \); usually in the standard literature on infinite dimensional Hamiltonian systems (see e.g. [15]) only the easier case of the free field with strictly positive mass is explicitly discussed.

We recall that (see [16], [15]) when the Hilbert space carries a complex structure \( J \), it is possible to complexify the space in such a way that the imaginary part of the complex scalar product turns out to be a symplectic
form, while the real part is the old (real) scalar product, coinciding with the energy. Any skew-adjoint operator $A$ commuting with $J$ remains skew-adjoint within the complex Hilbert space, so that $iA := J \cdot A$ is self-adjoint. Therefore, since $e^{tA} = e^{-it^2A}$, $A$ generates a strongly continuous group of unitary (hence symplectic) transformations. More precisely, collecting the known results on the subject (see e.g. [15, §2.6, §2.7], [16, chap. II]), we state the following:

**Theorem 4.1.** Let $A$ be an injective skew-adjoint operator on the real Hilbert space $H$ with inner product $\langle \cdot, \cdot \rangle$. Then the closure of the densely defined linear operator

$$A \cdot (-A^2)^{-1/2} : \text{Range}(A) \to H$$

defines a complex structure $J$ commuting with $A$. Defining, for any $\psi \in H$, the multiplication by the complex number $i$ as

$$i \psi := J \psi,$$

$H$ becomes a complex Hilbert space with Hermitean inner product

$$[\psi_1, \psi_2] := \langle \psi_1, \psi_2 \rangle + i \langle \psi_1, J \psi_2 \rangle.$$

The strongly continuous one parameter group $U^t := e^{tA}$ is a group of symplectic transformations relatively to the symplectic form

$$\Omega(\psi_1, \psi_2) := \text{Im} [\psi_1, \psi_2]$$

and the linear vector field

$$A : D(A) \to H$$

is Hamiltonian with associated densely defined Hamiltonian function

$$\mathcal{H} : D(Q) \to \mathbb{R}, \quad \mathcal{H}(\psi) := \frac{1}{2} Q(\psi),$$

where $Q$ denotes the quadratic form associated to the self-adjoint operator $J \cdot A$.

A wide class of examples is obtained by the following construction, which is a simple consequence of the above theorem. Let us consider an injective nonnegative self-adjoint operator

$$B : D(B) \to K$$
on the Hilbert space $K$ and let us consider the closure of
\[
\begin{pmatrix}
0 & 1 \\
-B^2 & 0
\end{pmatrix}
\]
on the Hilbert space $H = \overline{D(B)} \oplus K$, where $\overline{D(B)}$ is the completion of $D(B)$ with respect to the norm $\|u\|_B := \|Bu\|_K$. In the case in which this closure is injective, the complex structure $\mathcal{J}$ given by the previous theorem is
\[
\mathcal{J}_B : \overline{D(B)} \oplus K \to \overline{D(B)} \oplus K, \quad \mathcal{J}_B(u, v) = (B^{-1}v, Bu),
\]
where $B$ and $B^{-1}$ are the closures respectively of $B$ and its inverse $B^{-1} : \text{Range}(B) \to \overline{D(B)}$. This allows to endow $H$ with the structure of a complex Hilbert space, which we continue to call $H$; precisely, defined a generic element as $w := (u, v) \in \overline{D(B)} \oplus K$, the Hermitean scalar product in $H$ is
\[
[w_1, w_2]_B := \langle \langle w_1, w_2 \rangle \rangle_B + i \langle \langle w_1, \mathcal{J}_B w_2 \rangle \rangle_B,
\]
where
\[
\langle \langle w_1, w_2 \rangle \rangle_B := \langle Bw_1, Bw_2 \rangle + \langle v_1, v_2 \rangle.
\]
On the product $H \times H$ we have the symplectic form
\[
\Omega_B : \overline{D(B)} \oplus K \times \overline{D(B)} \oplus K \to \mathbb{R} \quad \Omega_B (w_1, w_2) = \langle \langle w_1, \mathcal{J}_B w_2 \rangle \rangle_B.
\]
With respect to the complex variable $w$ the wave equation
\[
\ddot{u} = -B^2 u
\]
assumes the Schrödinger-like form
\[
-i \dot{w} = Bw.
\]
Moreover such an equation is Hamiltonian w.r.t. the symplectic form $\Omega_B$ and the densely defined Hamiltonian function
\[
\mathcal{H}_B : D(B^{3/2}) \times D(B^{1/2}) \to \mathbb{R}, \quad \mathcal{H}_B(w) = \frac{1}{2} \left( \| B^{1/2}v \|^2 + \| B^{3/2}u \|^2 \right),
\]
where the operator $B^s$ is defined as the closure of $B^s$.

The strongly continuous symplectic group of operators obtained by solving the equation (9) preserves the energy $\mathcal{E}_B(u, v) := \frac{1}{2} [w, w]_B$. 

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An immediate example is given by the choice $B = \sqrt{-\Delta} : H^1(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$, corresponding to the standard wave equation and leading to the complex structure

$$\mathcal{J} : \tilde{H}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \to \tilde{H}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3),$$

$$\mathcal{J}(\phi, \dot{\phi}) := \left((-\Delta)^{-1/2}\phi, -(-\Delta)^{1/2}\phi\right).$$

Other concrete examples are obtained when the operator $B^2$ is a point interaction, more precisely $B = \sqrt{-\Delta_\alpha} : D^1(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ with $\alpha > 0$. In this case the corresponding complex structure is given by

$$\mathcal{J}_\alpha : \tilde{D}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \to \tilde{D}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3),$$

$$\mathcal{J}_\alpha(\phi, \dot{\phi}) := \left((-\Delta_\alpha)^{-1/2}\phi, -(-\Delta_\alpha)^{1/2}\phi\right).$$

The same procedure is not directly applicable to the cases $\alpha \leq 0$, due to the lack of skewadjointness and injectivity for the operator $\tilde{W}_\alpha$. A natural way out is to project the operator on the subspace of absolute continuity and to apply the abstract scheme to this projection. This works well for the case $\alpha < 0$, whereas the case $\alpha = 0$ deserves a different treatment. Here are the details of the two constructions.

In the case $\alpha < 0$ we have seen in section III that $\tilde{W}_\alpha^{nr}$ is skew-adjoint, w.r.t. the scalar product $\langle\langle \cdot, \cdot \rangle\rangle_{\alpha}^{nr}$, and one-to-one. Therefore we can apply to it thm. 4.1 (or better the successive example with $B = \sqrt{-\Delta_\alpha^{nr}}$) obtaining the complex structure $\mathcal{J}_\alpha$ commuting with $W_\alpha$, $\alpha < 0$, defined as

$$\mathcal{J}_\alpha := \mathcal{J}_\alpha^{nr} \times j,$$

where

$$\mathcal{J}_\alpha^{nr} : [\tilde{D}^1(\mathbb{R}^3)]_{nr} \oplus [L^2(\mathbb{R}^3)]_{nr} \to [\tilde{D}^1(\mathbb{R}^3)]_{nr} \oplus [L^2(\mathbb{R}^3)]_{nr},$$

$$\mathcal{J}_\alpha^{nr}(\phi, \dot{\phi}) = \left((-\Delta_\alpha^{nr})^{-1/2}\dot{\phi}, -(-\Delta_\alpha^{nr})^{1/2}\phi\right)$$

and

$$j : \mathbb{R}^2 \to \mathbb{R}^2, \quad j(x, \dot{x}) := (\dot{x}, -x).$$

Here, analogously to the case $\alpha > 0$, the linear operators

$$(-\Delta_\alpha^{nr})^{1/2} : [\tilde{D}^1(\mathbb{R}^3)]_{nr} \to [L^2(\mathbb{R}^3)]_{nr},$$

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and
\[ (-\Delta^\alpha_{nr})^{-1/2} : [L^2(R^3)]_{nr} \to [\tilde{D}^1(R^3)]_{nr}, \]
onumber
are defined as the closures of
\[ (-\Delta^\alpha_{nr})^{1/2} : [D^1(R^3)]_{nr} \subset [\tilde{D}^1(R^3)]_{nr} \to [L^2(R^3)]_{nr} \]
and
\[ (-\Delta^\alpha_{nr})^{-1/2} : \text{Range} \left((-\Delta^\alpha_{nr})^{1/2}\right) \subset [L^2(R^3)]_{nr} \to [\tilde{D}^1(R^3)]_{nr} \]
respectively.

We have then the complex Hilbert space of the couples
\[ (\psi, z) := (\phi, \dot{\phi}, (x, \dot{x})) \in [\tilde{D}^1(R^3)]_{nr} \oplus [L^2(R^3)]_{nr} \oplus \mathbb{R}^2, \]
with the Hermitean scalar product
\[ \langle\langle \psi_1, z_1 \rangle\rangle_{\alpha} := \langle\langle \psi_1, \psi_2 \rangle\rangle_{\alpha}^{nr} + i \langle\langle \psi_1, J^nr_{\alpha} \psi_2 \rangle\rangle_{\alpha}^{nr}, \]
and
\[ [z_1, z_2] := (z_1, z_2) + i (\dot{z}_1, j \dot{z}_2), \quad (z_1, z_2) := (x_1 \dot{x}_2 + x_2 \dot{x}_1). \]
The associated symplectic form is
\[ \Omega_{\alpha} : [\tilde{D}^1(R^3)]_{nr} \oplus [L^2(R^3)]_{nr} \oplus \mathbb{R}^2 \to \mathbb{R}, \]
\[ \Omega_{\alpha} ((\psi_1, z_1), (\psi_2, z_2)) = \langle\langle \psi_1, J^nr_{\alpha} \psi_2 \rangle\rangle_{\alpha}^{nr} + (z_1, j \dot{z}_2). \]
With respect to the complex variables \((\psi, z)\) the wave equation corresponding to \(W_{\alpha}\) takes the Schrödinger-like form
\[
\begin{cases}
-\dot{\psi} = (-\Delta^nr_{\alpha})^{1/2} \psi \\
-\dot{z} = L_0 z
\end{cases}
\]
with \(L_0 := \left( \begin{array}{cc} -\lambda_0 & 0 \\ 0 & 1 \end{array} \right) \).
where
\[ Q_{\alpha}^{nr}(\phi, \dot{\phi}) = \frac{1}{2} \left( \|(-\bar{\Delta}_{\alpha}^{nr})^{1/4} \dot{\phi} \|_{2}^{2} + \|(-\bar{\Delta}_{\alpha}^{nr})^{3/4} \phi \|_{2}^{2} \right) \]
is the quadratic form associated to the self-adjoint operator \( J_{\alpha}^{nr} \cdot W_{\alpha}^{nr} \).

If \((\phi, \dot{\phi}) \in \bar{D}^{1}(\mathbb{R}^{3}) \oplus L^{2}(\mathbb{R}^{3})\) has the orthogonal decomposition \((\phi, \dot{\phi}) \equiv \psi + z G\), where \( G \) denotes the normalized eigenvector corresponding to \( \lambda_{0} \), then
\[ E_{\alpha}(\phi, \dot{\phi}) = \frac{1}{2} \left[ \psi, \psi \right]_{\alpha}^{nr} + \frac{1}{2} \left( L_{0} z, z \right). \]

Therefore, being
\[ \bar{U}_{\alpha}^{t} = e^{itW_{\alpha}^{nr}} \times e^{itA_{0}} \]
a strongly continuous group of unitary and symplectic transformations, the energy is conserved by the flow.

We come now to the case \( \alpha = 0 \). In this case, in order to apply thm. 4.1, which requires injectivity, it is necessary to project onto the subspace orthogonal to the eigenvector \((G, 0)\). Being \( \bar{W}_{(0)} \) one-to-one and skew-adjoint w.r.t. the scalar product \( \langle \cdot, \cdot \rangle_{(0)} \), one can then apply thm. 4.1 thus obtaining a one parameter group \( \bar{U}_{(0)}^{t} \) of symplectic transformations such that
\[ \bar{U}_{0}^{t} = \bar{U}_{(0)}^{t} \times 1 \]
and so \( \bar{U}_{0}^{t} \) preserves the energy \( E_{0}(\phi, \dot{\phi}) = E(\phi_{reg}, \dot{\phi}) \).

Since \((-\Delta_{0})^{1/2} \phi = (-\bar{\Delta})^{1/2} \phi_{\text{reg}}\) (note that this equality holds true only in the case \( \alpha = 0 \)) one has
\[ \phi_{\text{reg}} = (-\bar{\Delta})^{-1/2} \cdot (-\Delta_{0})^{1/2} \phi \]
and so, when \((\phi, \dot{\phi}) \in \bar{H}_{0}^{2}(\mathbb{R}^{3}) \oplus D^{1}(\mathbb{R}^{3})\),
\[ \mathcal{J} \cdot \bar{W}_{(0)}(\phi, \dot{\phi}) = \left( (-\bar{\Delta})^{1/2} \phi, (-\Delta_{0})^{1/2} \dot{\phi} \right) \equiv \left( (-\Delta_{0})^{1/2} \phi, (-\Delta_{0})^{1/2} \dot{\phi} \right) \]
Moreover \( \mathcal{J} \) commutes with \( \bar{W}_{(0)} \) (see (15) in the next section) and so \( \mathcal{J} \) coincides with the complex structure associated to \( \bar{W}_{(0)} \) by thm. 4.1. With respect to the complex variable \( \psi = (\phi, \dot{\phi}) \) the wave equation corresponding to \( \bar{W}_{(0)} \) assumes the Schrödinger-like form
\[ -i \dot{\psi} = (-\bar{\Delta}_{0})^{1/2} \psi. \]
We summarize the results obtained in the following

**Theorem 4.2.** For every $\alpha \in \mathbb{R}\setminus\{0\}$ there exists a symplectic form

$$\Omega_{\alpha} : \tilde{D}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \times \tilde{D}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \to \mathbb{R}$$

with respect to which the vector field

$$\bar{W}_{\alpha} : \tilde{D}^2(\mathbb{R}^3) \times D^1(\mathbb{R}^3) \to \tilde{D}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$$

is Hamiltonian. Moreover for $\alpha \leq 0$ the analogous result occurs for the reduced vector fields

$$\bar{W}_{\alpha}^{nr} : [\tilde{D}^2^1(\mathbb{R}^3)]_{nr} \times [\tilde{D}^1(\mathbb{R}^3)]_{nr} \to [\tilde{D}^1(\mathbb{R}^3)]_{nr} \oplus [L^2(\mathbb{R}^3)]_{nr}$$

and

$$\bar{W}_{(0)} : \tilde{H}^2_0(\mathbb{R}^3) \oplus D^1(\mathbb{R}^3) \to \tilde{H}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) .$$

For every $\alpha \in \mathbb{R}$ the evolution group $\tilde{U}_t$ preserves the energy

$$\mathcal{E}_\alpha(\phi, \dot{\phi}) = \frac{1}{2} \left( \|\dot{\phi}\|_2^2 + F_\alpha(\phi, \phi) \right) .$$

**V. SCATTERING THEORY.**

The Hilbert spaces where the operators $\bar{W}_{\alpha}$ and $\bar{W}$ act on, respectively $\tilde{D}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$ and $\tilde{H}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$, are different (also as sets), and so one is forced to use a two Hilbert space formulation to treat scattering theory for the pair $(\bar{W}_{\alpha}, \bar{W})$. We refer to the seminal paper by Kato [4] for the relevant constructions and results in scattering theory with two Hilbert spaces. Our approach will follow the lines of the construction given in [4, §8-9] (also see [17, §3.5]).

From now on, given the real vector space $L^2(\mathbb{R}^3)$, we will denote by $L^2_\mathbb{C}(\mathbb{R}^3)$ the complex vector space

$$L^2_\mathbb{C}(\mathbb{R}^3) := \left\{ \phi_1 + i\phi_2, \; \phi_1, \phi_2 \in L^2(\mathbb{R}^3) \right\} .$$

We begin introducing the isometries

$$C : \tilde{H}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \to L^2_\mathbb{C}(\mathbb{R}^3) ,$$

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\[ C(\phi, \dot{\phi}) \equiv C_0(\phi, \dot{\phi}) := (-\overline{\Delta})^{1/2} \phi - i\dot{\phi}, \]
\[ C_\alpha : \bar{D}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \to L^2_\xi(\mathbb{R}^3), \quad \alpha > 0, \]
\[ C_\alpha(\phi, \dot{\phi}) := (-\overline{\Delta}_\alpha)^{1/2} \phi - i\dot{\phi}, \]
\[ C_\alpha : [\bar{D}^1(\mathbb{R}^3)]_{nr} \oplus [L^2(\mathbb{R}^3)]_{nr} \to [L^2_\xi(\mathbb{R}^3)]_{nr}, \quad \alpha < 0, \]
\[ C_\alpha((\phi, \dot{\phi})) := (-\overline{\Delta}_{\alpha}^{nr})^{1/2} \phi - i\dot{\phi}. \]

These isometries lead to the following relations:
\[ J = C^{-1} \cdot iC, \quad \text{(10)} \]
\[ J_\alpha = C^{-1}_\alpha \cdot iC_\alpha, \quad \alpha > 0, \quad J_{\alpha}^{\text{nr}} = C^{-1}_\alpha \cdot iC_\alpha, \quad \alpha < 0, \quad \text{(11)} \]
\[ \bar{U}^t = C^{-1} \cdot e^{it\sqrt{-\overline{\Delta}}} \cdot C, \quad \bar{U}^{t=0} = C^{-1} \cdot e^{it\sqrt{-\overline{\Delta}_0}} \cdot C \times 1, \quad \text{(12)} \]
\[ \bar{U}^t_\alpha = C^{-1}_\alpha \cdot e^{it\sqrt{-\overline{\Delta}_\alpha}} \cdot C_\alpha, \quad \alpha > 0, \quad \text{(13)} \]
\[ \bar{U}^t_{\alpha} = C^{-1}_\alpha \cdot e^{it\sqrt{-\overline{\Delta}_{\alpha}^{nr}}} \cdot C_\alpha \times e^{t\Lambda_\alpha}, \quad \alpha < 0. \quad \text{(14)} \]

Note that the two equalities
\[ J = C^{-1} \cdot iC, \quad \bar{U}^{t(0)} = C^{-1} \cdot e^{it\sqrt{-\overline{\Delta}_0}} \cdot C \quad \text{(15)} \]

imply, as we stated in the previous section, that \( J \) commutes with \( \bar{W}_{(0)} \).

Moreover the relations (12)-(14) provide an alternative construction of the dynamics generated by \( \bar{W}_\alpha \). In fact one could use such relations as definitions of \( \bar{U}^t_\alpha \) and then check by differentiating with respect to the time parameter that this evolution group is generated by the operator \( \bar{W}_\alpha \).

We introduce now the identification operators
\[ J_\alpha : \bar{D}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \to \bar{H}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3), \]
\[ J_\alpha(\phi, \dot{\phi}) := \begin{cases} 
((-\overline{\Delta})^{-1/2} \cdot (-\overline{\Delta}_\alpha)^{1/2} \phi, \dot{\phi}), & \text{for } \alpha > 0 \\
(\phi_{\text{reg}}, \dot{\phi}) \equiv ((-\overline{\Delta})^{-1/2} \cdot (-\overline{\Delta}_0)^{1/2} \phi, \dot{\phi}), & \text{for } \alpha = 0 \\
((-\overline{\Delta})^{-1/2} \cdot (-\overline{\Delta}_{\alpha}^{nr})^{1/2} \cdot \Pi_{\alpha \text{nr}} \phi, \dot{\phi}), & \text{for } \alpha < 0,
\end{cases} \]
where \( \Pi_{\alpha \text{nr}} \) denotes the projection
\[ \Pi_{\alpha \text{nr}} : \bar{D}^1(\mathbb{R}^3) \to [\bar{D}^1(\mathbb{R}^3)]_{\alpha \text{nr}}, \]

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and
\[ J'_\alpha : \tilde{H}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \to \tilde{D}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3), \]
\[
J'_\alpha(\phi, \dot{\phi}) := \begin{cases} 
(\phi, \dot{\phi}), & \text{for } \alpha > 0 \\
(\phi, \dot{\phi}), & \text{for } \alpha = 0 \\
(\phi, \dot{\phi}), & \text{for } \alpha < 0.
\end{cases}
\]

We can then define the Möller wave operators
\[
\Omega_{\pm}(\tilde{W}, \tilde{W}_\alpha; J_\alpha) := s\lim_{t \to \pm\infty} U^{-t} : J_\alpha \cdot \hat{U}^t \cdot P_{ac}(\tilde{W}_\alpha),
\]
\[
\Omega_{\pm}(\tilde{W}_\alpha, \tilde{W}; J'_\alpha) := s\lim_{t \to \pm\infty} U^{-t} : J'_\alpha \cdot \hat{U}^t,
\]
where
\[
P_{ac}(\tilde{W}_\alpha) : \tilde{D}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \to \tilde{D}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3),
\]
\[
P_{ac}(\tilde{W}_\alpha)(\phi, \dot{\phi}) := \begin{cases} 
(\phi, \dot{\phi}), & \text{for } \alpha > 0 \\
(\phi_{reg}, \dot{\phi}), & \text{for } \alpha = 0 \\
(\Pi_{nr}\phi, P_{nr}\dot{\phi}), & \text{for } \alpha < 0.
\end{cases}
\]

Concerning the existence of such wave operators we have the following

**Theorem 5.1.** *The Möller wave operators*

\[
\Omega_{\pm}(\tilde{W}, \tilde{W}_\alpha; J_\alpha) := s\lim_{t \to \pm\infty} U^{-t} : J_\alpha \cdot \hat{U}^t \cdot P_{ac}(\tilde{W}_\alpha),
\]
\[
\Omega_{\pm}(\tilde{W}_\alpha, \tilde{W}; J'_\alpha) := s\lim_{t \to \pm\infty} U^{-t} : J'_\alpha \cdot \hat{U}^t
\]
exist, are complete and are mutually adjoint isometries, i.e.

\[
\text{Range } \Omega_{+}(\tilde{W}, \tilde{W}_\alpha; J_\alpha) = \text{Range } \Omega_{-}(\tilde{W}, \tilde{W}_\alpha; J_\alpha) = 1_{\tilde{H}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)},
\]
\[
\text{Range } \Omega_{+}(\tilde{W}_\alpha, \tilde{W}; J'_\alpha) = \text{Range } \Omega_{-}(\tilde{W}_\alpha, \tilde{W}; J'_\alpha) = \text{Range } P_{ac}(\tilde{W}_\alpha),
\]
\[
\Omega_{\pm}(\tilde{W}, \tilde{W}_\alpha; J_\alpha) = \Omega_{\pm}(\tilde{W}_\alpha, \tilde{W}; J'_\alpha) = P_{ac}(\tilde{W}_\alpha),
\]
\[
\Omega_{\pm}(\tilde{W}, \tilde{W}_\alpha; J_\alpha)^* \cdot \Omega_{\pm}(\tilde{W}, \tilde{W}_\alpha; J_\alpha) = P_{ac}(\tilde{W}_\alpha),
\]
\[
\Omega_{\pm}(\tilde{W}_\alpha, \tilde{W}; J'_\alpha)^* \cdot \Omega_{\pm}(\tilde{W}_\alpha, \tilde{W}; J'_\alpha) = 1_{\tilde{H}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)},
\]
\[
\Omega_{\pm}(\tilde{W}, \tilde{W}_\alpha; J_\alpha)^* = \Omega_{\pm}(\tilde{W}_\alpha, \tilde{W}; J'_\alpha).
\]

**Proof.** With the above definitions one has

\[
\Omega_{\pm}(\tilde{W}, \tilde{W}_\alpha; J_\alpha) = C^{-1} \cdot \Omega_{\pm}(\sqrt{-\Delta}, \sqrt{H_\alpha}; I_\alpha) \cdot C \cdot P_{ac}(\tilde{W}_\alpha),
\]

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\[ \Omega_{\pm}(\bar{W}_a, \bar{W}; J'_a) = C_{\alpha}^{-1} \cdot \Omega_{\pm}(\sqrt{H_a}, \sqrt{-\Delta}; I'_a) \cdot C \]

where
\[ \Omega_{\pm}(\sqrt{-\Delta}, \sqrt{H_a}; I_a) := \lim_{t \to \pm\infty} e^{-it\sqrt{-\Delta}} \cdot I_a \cdot e^{it\sqrt{H_a}}, \]
\[ \Omega_{\pm}(\sqrt{H_a}, \sqrt{-\Delta}; I'_a) := \lim_{t \to \pm\infty} e^{-it\sqrt{H_a}} \cdot I'_a \cdot e^{it\sqrt{-\Delta}}. \]

Here \( I'_a := P_{ac}(-\Delta_a), \) \( I_a \) is its left inverse, and
\[
H_a := \begin{cases} 
-\Delta_a, & \text{for } \alpha \geq 0 \\
-\Delta_{\alpha}^{nr}, & \text{for } \alpha < 0
\end{cases}
\]

By Birman invariance principle one has
\[ \Omega_{\pm}(\sqrt{-\Delta}, \sqrt{H_a}; I_a) = \Omega_{\pm}(-\Delta, H_a; I_a) \]
and
\[ \Omega_{\pm}(\sqrt{H_a}, \sqrt{-\Delta}; I'_a) = \Omega_{\pm}(H_a, -\Delta; I'_a). \]

Therefore one has the identities
\[
\Omega_{\pm}(\sqrt{-\Delta}, \sqrt{H_a}; I_a) = \lim_{t \to \pm\infty} e^{it\Delta} \cdot I_a \cdot e^{itH_a} = \lim_{t \to \pm\infty} e^{it\Delta} \cdot e^{-it\Delta_a} \cdot P_{ac}(-\Delta_a) = \Omega_{\pm}(-\Delta, -\Delta_a)
\]
and
\[
\Omega_{\pm}(\sqrt{H_a}, \sqrt{-\Delta}; I'_a) = \lim_{t \to \pm\infty} P_{ac}(-\Delta_a) \cdot e^{it\Delta_a} \cdot e^{-it\Delta} = P_{ac}(-\Delta_a) \cdot \Omega_{\pm}(-\Delta_a, -\Delta) = \Omega_{\pm}(-\Delta_a, -\Delta).
\]

In conclusion one obtains the equalities
\[ \Omega_{\pm}(\bar{W}, \bar{W}_a; J_a) = C_{\alpha}^{-1} \cdot \Omega_{\pm}(-\Delta, -\Delta_a) \cdot C \cdot P_{ac}(\bar{W}_a), \]
\[ \Omega_{\pm}(\bar{W}_a, \bar{W}; J'_a) = C_{\alpha}^{-1} \cdot \Omega_{\pm}(-\Delta_a, -\Delta) \cdot C \]
and the proof is concluded since the wave operators \( \Omega_{\pm}(-\Delta, -\Delta_a), \) and \( \Omega_{\pm}(-\Delta_a, -\Delta) \) exist, are complete and are mutually adjoint isometries. This
is proven (see [11, appendix E]) by the Birman-Kuroda theorem being the resolvent difference
\[ (-\Delta_\alpha + z)^{-1} - (-\Delta + z)^{-1} \]
a rank one (hence trace class) operator.

The previous theorem holds true also with the different ($\alpha$-independent and much simpler and natural) couple of identification operators defined by
\[ J : \bar{D}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \to \bar{H}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3), \quad J(\phi, \dot{\phi}) := (\phi_{\text{reg}}, \dot{\phi}), \]
\[ J' : \bar{H}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \to \bar{D}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3), \quad J'(\phi, \dot{\phi}) := (\phi, \dot{\phi}). \]
This is true by [4, thms. 10.3 and 10.5] since the condition 10.1 in [4] is verified with $m = M = 1$. In our situation such a condition simply reads
\[ \forall \phi \in H^1(\mathbb{R}^3) \cap \text{Range} P_{ac}(-\Delta_\alpha), \quad \| \sqrt{H_\alpha} \phi \|_{L^2} = \| \sqrt{-\Delta} \phi \|_{L^2}. \]

In more detail one has the following

**Theorem 5.2.** $J$ is ($\bar{U}_t^\alpha, \pm$)-equivalent to $J_\alpha$, i.e.
\[ \lim_{t \to \pm \infty} (J_\alpha - J) \cdot \bar{U}_t^\alpha \cdot P_{ac}(\bar{W}_\alpha) = 0, \]

Therefore
\[ \Omega_{\pm}(\bar{W}, \bar{W}_\alpha; J) := \lim_{t \to \pm \infty} \bar{U}^{-t} \cdot J \cdot \bar{U}_t^\alpha \cdot P_{ac}(\bar{W}_\alpha) \]
exist and are equal to $\Omega_{\pm}(\bar{W}, \bar{W}_\alpha; J_\alpha)$.

$J'$ is a ($\bar{U}_t^\alpha, \pm$)-asymptotic left-inverse to $J$, i.e.
\[ \lim_{t \to \pm \infty} (J' \cdot J - 1) \cdot \bar{U}_t^\alpha \cdot P_{ac}(\bar{W}_\alpha) = 0, \]

thus
\[ \Omega_{\pm}(\bar{W}_\alpha, \bar{W}; J') := \lim_{t \to \pm \infty} \bar{U}_t^\alpha \cdot J' \cdot \bar{U}^{-t} \]
exist and are equal to $\Omega_{\pm}(\bar{W}_\alpha, \bar{W}; J'_\alpha)^* \equiv \Omega_{\pm}(\bar{W}_\alpha, \bar{W}; J'_\alpha)$.

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