Complementary Schur Asymptotics for Partitions

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Abstract

We deduce from the strong form of the Hardy–Ramanujan asymptotics for the partition function $p(n)$ an asymptotics for $p_{-S}(n)$, the number of partitions of $n$ that do not use parts from a finite set $S$ of positive integers. We apply this to construct highly oscillating partition ideals.

1 Introduction

Let $n \in \mathbb{N}$. We say that $\lambda = (\lambda_1, \ldots, \lambda_k)$, where $\lambda_1 \geq \cdots \geq \lambda_k > 0$ are positive integers, is a partition of $n$ if $\sum_{i=1}^{k} \lambda_i = n$. We say that $k$ is the length of $\lambda$ and $\lambda_1, \ldots, \lambda_k$ are the parts of the partition $\lambda$.

Let $p(n)$ be the number of partitions of $n$. The famous result of Hardy and Ramanujan [5] gives the asymptotics of $p(n)$ as

$$p(n) \sim e^{C\sqrt{n}}$$

where $C = \pi \sqrt{2/3} \approx 2.656$. \hspace{1cm} (1)

As usual, $f(n) \sim g(n)$ means that $\lim_{n \to \infty} \frac{f(n)}{g(n)} = 1$. For more information on partitions refer to the book [1] of G. Andrews. In [6] F. Johansson presents an almost optimum numerical algorithm for evaluating $p(n)$.

One can be also interested in the number $p_{S}(n)$ of partitions of $n$ with parts from a given (finite or infinite) set $S \subset \mathbb{N}$. Such partitions are often called “restricted partitions”. I. Schur [9] proved that for finite $S$ with cardinality $|S| = t$ (and such that the gcd($S$) = 1) one has the asymptotics

$$p_{S}(n) \sim \frac{n^{t-1}}{(t-1)! \prod_{s \in S} s} = \frac{1}{n(t-1)! \prod_{s \in S} \frac{n}{s}}.$$ 

Note that adding a new element $s$ to $S$ increases $p_{S}(n)$ by the factor $\frac{n}{s|S|}$.

Motivated by these two partition asymptotics, we obtain what we call a complementary Schur asymptotic. Nicolas and Sárközy [7] found an asymptotics of the number of partitions using only parts $\geq m$ for a wide range of parameter $m$. We extend this result. For a finite set of integers $S \subset \mathbb{N}$, $|S| = t$, let $p_{-S}(n)$
be the number of partitions of \( n \) not using any part from \( S \). Our main result is that

\[
p_{-S}(n) \sim p(n) \left( \prod_{s \in S} \frac{C_s}{2^{\sqrt{n}}} \right)^t = p(n) \prod_{s \in S} \frac{C_s}{2^{\sqrt{n}}}. \]

Now each element \( s \) in \( S \) decreases \( p(n) \) by the factor \( \frac{C_s}{2^{\sqrt{n}}} \). We apply this to construct highly oscillating partition ideals. This theorem is an extension of result [7] of Nicolas and Sárközy who considered the case \( S = [m] \).

The paper is organized as follows. In Section 2 we state the strong Hardy–Ramanujan asymptotics of \( p(n) \) and restate our main result. Section 3 presents some useful lemmas. Section 4 is devoted to the proof of the main theorem. In the last Section 5 we present an application of the main result to oscillations of growth functions of partition ideals.

## 2 Two asymptotics for partitions

Let \( n \in \mathbb{N} \),

\[
\lambda_n := \sqrt{n - \frac{1}{24}}, \quad C := \pi \sqrt{\frac{2}{3}} \quad \text{and} \quad D \in (C/2, C)
\]

be a constant.

As we mentioned earlier, the classical asymptotics for \( p(n) \) proved by Hardy and Ramanujan [5] is

\[
p(n) \sim e^{C\sqrt{n}} / 4\sqrt{n}. \quad \text{In [5 formula (1.55)] they gave much stronger asymptotics which we give in the next theorem. They state it as a value of a derivative which we compute explicitly.}
\]

**Theorem 1** (Hardy–Ramanujan). For \( n = 1, 2, \ldots \), the partition function \( p(n) \) satisfies

\[
p(n) = e^{C\lambda_n} \left( C - \frac{1}{\lambda_n} \right) + O \left( e^{D\sqrt{n}} \right).
\]

Note that now the error is only about a square root of the main term. This resembles strong asymptotic relations for coefficients of power series with unique dominant singularity (see P. Flajolet and R. Sedgewick [3] Chapter V or think of the Fibonacci numbers). We use this result with small error term to deduce our asymptotic relation for \( p_{-S}(n) \). Our main theorem says:

**Theorem 2.** Let \( S = \{s_1, s_2, \ldots, s_t\} \subset \mathbb{N} \) be a finite set of integers with \( |S| = t \). Then the number \( p_{-S}(n) \) of partitions of \( n \) with parts in \( \mathbb{N} \setminus S \) satisfies

\[
p_{-S}(n) \sim p(n) \left( \prod_{s \in S} \frac{C_s}{2^{\sqrt{n}}} \right)^t = p(n) \prod_{s \in S} \frac{C_s}{2^{\sqrt{n}}}. \]

We base our proof on manipulating the strong asymptotics of \( p(n) \). First we estimate \( p(n - s) \) for a fixed \( s \in \mathbb{N} \), and then express \( p_{-S}(n) \) as a sum of values \( p(n - s) \) for various numbers \( s \).
3 Auxiliary results

Let $S \subset \mathbb{N}$ be a finite set with $|S| = t$ and let $s \in [t]$. First we determine the asymptotics of $p(n - s)$ and then we prove an algebraic identity needed later in the proof of the main result.

We set

$$q(n) := e^{C_n \lambda_n} \left( C - \frac{1}{\lambda_n} \right).$$

Lemma 1. Let $t \in \mathbb{N}$. Then for all $n, s \in \mathbb{N}$ with $n > s$ we have

$$p(n - s) = e^{C \sqrt{n}} \sum_{z=0}^{t} g(z, s)n^{-z/2} + O \left( e^{C \sqrt{n}} n^{-\frac{3}{4}} \right)$$

(2)

where for $z \in \{0, 1, \ldots, t\}$ we denote by $g(z, s)$ a real polynomial in $s$ with degree $z$ and leading term

$$g(z, s) = \frac{(-1)^z C^{z+1}}{2^z z!} s^z + h(z, s)$$

(3)

where $h(z, s)$ is a real polynomial in $s$ with degree at most $z - 1$.

Proof. Let $t \in \mathbb{N}$. We expand the main term in Theorem 1 in powers of $n$. Since $e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!} = \sum_{i=0}^{t} \frac{x^i}{i!} + O(x^{t+1})$ for $|x| < c$ and $(c_i$ are some real constants, not necessarily always the same)

$$e^{C \lambda_n} = e^{C \sqrt{n}(1-1/24n)^{1/2}} = e^{C \sqrt{n} + c_1 n^{-1/2} + c_2 n^{-2/2} + \cdots + c_l n^{-l/2} + O(n^{-l+1/2})}$$

$$\frac{1}{\lambda_n^2} = \frac{1}{n(1-1/24n)} = c_1 n^{-1} + \cdots + c_l n^{-l} + O(n^{-l-1})$$

$$C - \frac{1}{\lambda_n} = C - \frac{1}{\sqrt{n}(1-1/24n)^{1/2}}$$

$$= C + c_1 n^{-1/2} + c_2 n^{-2/2} + \cdots + c_l n^{-l/2} + O(n^{-l+1/2})$$

we get, for every $n > n_0(t)$ and some coefficients $a_k$,

$$q(n) = e^{C \lambda_n} \lambda_n^2 \left( C - \frac{1}{\lambda_n} \right) = e^{C \sqrt{n}} \left( \sum_{k=0}^{t} a_k n^{-k/2 - 1} + O(n^{-l+1/2}) \right)$$

(4)

(note that $a_0 = C$). For all integers $n > s > 0$ we have, expanding again $(n-s)^{-k/2-1} = n^{-k/2-1}(1-s/n)^{-k/2-1} = \ldots$ by I. Newton’s binomial theorem, that

$$\sum_{k=0}^{t} a_k (n-s)^{-k/2-1}$$

$$= \sum_{k=0}^{t} a_k \left( \sum_{l=0}^{t} \binom{-1-k/2}{l} (-s)^l n^{-1-k/2-l} + O(n^{-k/2-l}) \right)$$

$$= \sum_{w=0}^{2t} f(w, s)n^{-w/2-1} + O(n^{-l-2})$$

(5)
where
\[
f(w, s) = \sum_{k,l \geq 0, k+2l = w} (-1)^l a_k \left( -1 - \frac{k}{2} \right) s^l
\]  
(6)
is a real polynomial in \(s\) with \(\deg_s f(w, s) = \lfloor w/2 \rfloor\).

Next we combine expansions of the square root and of the exponential function, as above, and denote \(x = n^{-1/2}\). For all integers \(n > s > 0\), where \(s\) is fixed, we get
\[
e^{C \sqrt{n-s}} = e^{C \sqrt{n(1-s/n)^{1/2}}}
\]
\[
e^{C \sqrt{1+O(n^{-1})}}(1-\ldots)(1+\ldots)(1+O(s/n))
\]
\[
e^{C \sqrt{e^{C x^2}}(1-\ldots)(1+\ldots)+O((s/n)^{1/2})}
\]
\[
e^{C \sqrt{e^{C x^2}}}(\sum d(i, s)n^{-i/2} + O(n^{-i/2}))
\]  
(7)
where \(d(i, s)\) is a polynomial in \(s\) with \(\deg_s d(i, s) = i\) and
\[
d(i, s) = \frac{(-1)^i}{2i!} c_i s^i + c_{i-1} s^{i-1} + \ldots
\]  
(8)
\((c_{i-1} s^{i-1} + \ldots\) are the remaining terms with degree less than \(i\).

Combining results (6), (7) and (8) we have
\[
q(n-s) = e^{C \sqrt{\pi n}} \left( \sum_{i=0}^{t} d(i, s)n^{-i/2} + O(n^{-i/2}) \right) \left( \sum_{w=0}^{2t} f(w, s)n^{-w/2} + O(n^{-w/2}) \right)
\]
\[
e^{C \sqrt{\pi n}} \sum_{z=0}^{t} g(z, s)n^{-z/2} + O(e^{C \sqrt{\pi n^{-1/2}}})
\]
where
\[
g(z, s) = \sum_{i, w \geq 0, i+w = z} d(i, s)f(w, s)
\]
is a real polynomial in \(s\) with \(\deg_s g(z, s) = z\). Indeed, \(\max_{i, w \geq 0, i+w = z}(i + \lfloor w/2 \rfloor) = z\), attained uniquely for \(i = z\) and \(w = 0\). Moreover, (6) and (7) imply
\[
g(z, s) = \frac{(-1)^z C^{z+1}}{2^z z!} s^z + h(z, s)
\]
where \(h(z, s)\) are the remaining terms with \(\deg_s h(z, s) \leq z - 1\). The exponent of \(C\) increased by one since \(a_0 = C\). Therefore for all integers \(n > s > 0\),
\[
p(n-s) = \frac{q(n-s)}{4\pi \sqrt{2}} + O(e^{D \sqrt{n}})
\]
\[
= \frac{e^{C \sqrt{\pi n}}}{4\pi \sqrt{2}} \sum_{z=0}^{t} g(z, s)n^{-z/2} + O(e^{C \sqrt{\pi n^{-1/2}}}).
\]
We state and prove an identity needed in the main proof. Recall that for \( t \in \mathbb{N}_0 \), \([t]=\{1,2,\ldots,t\}\).

**Lemma 2.** Let \( t \geq z \geq 0 \) be integers and \( s_1, s_2, \ldots, s_t \) be variables. Then

\[
\sum_{J \subset [t]} (-1)^{|J|} \left( \sum_{i \in J} s_i \right)^z = \begin{cases} 0 & \text{for } z < t \\ (-1)^t t! \prod_{i=1}^t s_i & \text{for } z = t. \end{cases}
\]

The first case holds in fact more generally for any polynomial in \( \sum_{i \in J} s_i \) with degree at most \( t - 1 \).

**Proof.** Let \( k \leq t \) be positive integers, \( j_i \) with \( 1 \leq j_1 < j_2 < \cdots < j_k \leq t \) be some indices and \( \alpha_1, \ldots, \alpha_k \) be positive integers with \( \sum \alpha_i = z \), so \( k \leq z \). We denote the polynomial on the left side as \( f = f(s_1, \ldots, s_t) \) and examine the coefficient \( [s_{j_1}^{\alpha_1} \cdots s_{j_k}^{\alpha_k}]f \). Clearly, only the sets \( J \) containing \( \{j_1, \ldots, j_k\} \) contribute to it. Each such \( J \) contributes \(( -1)^{|J|} \binom{z}{\alpha_1, \ldots, \alpha_k}\) to the coefficient. Summing over all \( J \) containing \( \{j_1, \ldots, j_k\} \) we obtain

\[
[s_{j_1}^{\alpha_1} \cdots s_{j_k}^{\alpha_k}]f = (-1)^k \binom{z}{\alpha_1, \ldots, \alpha_k} \sum_{l=0}^{t-k} (-1)^l \binom{t-k}{l}
\]

(here \( l = |J\setminus\{j_1, \ldots, j_k\}| \)). If \( z < t \) then \( k < t \) and the sum is \(( 1-1)^{t-k} = 0 \) by the binomial theorem. If \( z = t \) then only \( k = z = t \) yields non-zero contribution, for \( \alpha_1 = \cdots = \alpha_k = 1 \), and we get the coefficient

\[
[s_1 \cdots s_t]f = (-1)^t \binom{z}{1, \ldots, 1} = (-1)^t t! = (-1)^t t!,
\]

which proves the theorem. \( \square \)

**4 Proof of the Theorem**

**Proof of Theorem** Let \( J \subset [t] \). Note that the partitions of \( n \) using each part \( s_j, j \in J \), at least once are in bijection with the partitions of \( n - \sum_{j \in J} s_j \). Thus by the principle of inclusion and exclusion,

\[
p_-(n) = \sum_{J \subset [t]} (-1)^{|J|} p(n - \sum_{j \in J} s_j).
\]

By Lemma 2 we have

\[
p_-(n) = \sum_{J \subset [t]} (-1)^{|J|} \left( \frac{e^{C\sqrt{n}}}{4\pi n^{3/2}} \sum_{z=0}^t g(z, \sum_{j \in J} s_j) n^{-z/2} + O \left( e^{C\sqrt{n}} n^{-\frac{t+1}{2}} \right) \right)
\]

\[
= \frac{e^{C\sqrt{n}}}{4\pi n^{3/2}} \sum_{z=0}^t \sum_{J \subset [t]} (-1)^{|J|} g(z, \sum_{j \in J} s_j) n^{-z/2} + O \left( e^{C\sqrt{n}} n^{-\frac{t+1}{2}} \right).
\]
We apply Lemma 2 to the polynomial $g(z, \sum_{j \in J} s_j)$ when $z \in [t-1]$—first we understand $s_j$ as variables and only at the end we substitute for them the numbers $s_j$—and obtain, by the first case,

$$\sum_{J \subset [t]} (-1)^{|J|} g(z, \sum_{j \in J} s_j) = 0.$$ 

Hence only term $z = t$ remains yielding

$$p_{-S}(n) = \frac{e^{C\sqrt{n}}}{4\pi n \sqrt{2}} \sum_{J \subset [t]} (-1)^{|J|} g\left(t, \sum_{j \in J} s_j\right) n^{-t/2} + O\left(e^{C\sqrt{n}n^{-\frac{t+1}{2}}}\right).$$

Finally, we expand $g(t, \sum s_j)$ by equation (3) and use Lemma 2. By the first case of Lemma 2 the contributions of $h(z, \sum s_j)$ sum up to zero. By the second case the contribution of the leading term in $g(z, \sum s_j)$ is

$$\sum_{J \subset [t]} (-1)^{|J|} \frac{(-1)^t C^{t+1}}{2^t t!} \prod_{j \in J} s_j = \frac{C^{t+1}}{2^t} \prod_{i=1}^t s_i.$$ 

Thus

$$p_{-S}(n) = \frac{e^{C\sqrt{n}}}{4\pi n \sqrt{2}} \left[ \left( \frac{C^{t+1}}{2^t} \prod_{j=1}^t s_j \right) n^{-t/2} \right] + O\left(e^{C\sqrt{n}n^{-\frac{t+1}{2}}}\right),$$

which in view that $C = \pi \sqrt{2/3}$ and $p(n) \sim e^{C\sqrt{n}}/4\pi \sqrt{3}$ gives the desired asymptotics

$$p_{-S}(n) \sim \frac{C e^{C\sqrt{n}}}{4\pi n \sqrt{2}} \left( \frac{C}{2} \right)^t \prod_{j=1}^t s_j = p(n) \left( \frac{C}{2\sqrt{n}} \right)^t \prod_{j=1}^t s_j.$$

5 Partition functions of ideals

Let $X$ be a set of partitions and $\lambda, \gamma \in X$. We say that $\lambda$ is a subpartition of $\gamma$ if no part from $\lambda$ has more occurrences in $\lambda$ then in $\gamma$. We denote this relation $\lambda < \gamma$. A set of partitions $X$ is a partition ideal, if $\lambda < \gamma$ and $\gamma \in X$ always implies $\lambda \in X$. By $p(n, X)$ we denote the number of partitions of $n$ lying in $X$. Let $Z$ be a set of partitions such that no two partitions in $Z$ are comparable by $<$. We denote by $F_Z$ the set of all partitions that do not contain any element of $Z$. Clearly, $F_Z$ is a partition ideal. We call $Z$ a basis of the ideal $F_Z$. We denote its counting function by

$$p_{-Z}(n) = p(n, F_Z).$$
Recall that the notation $p_{-S}(n)$ and $p_{-Z}(n)$ where $S$ is a set of positive integers and $Z$ is a set of partitions. Finally, two partitions are independent if their parts are pairwise distinct.

We make use of the Cohen–Remmel theorem [2, 8] that gives sufficient condition for equality of counting functions of two partition ideals in terms of their bases. In the theorem we use the following notation. For a partition $\lambda$, $|\lambda|$ is the sum of all parts of $\lambda$, and for several partitions $\lambda_i$ their union is the partition $\lambda$ such that the multiplicity of any part equals to the maximum multiplicity attained over all $\lambda_i$.

**Theorem 3** (Cohen, 1981; Remmel, 1982). Let $\Lambda = \{\lambda_1, \lambda_2, \ldots\}$ and $\Gamma = \{\gamma_1, \gamma_2, \ldots\}$ be two finite or infinite sequences of partitions (of the same length), such that for every finite set $I \subset \mathbb{N}$,

$$\left| \bigcup_{i \in I} \lambda_i \right| = \left| \bigcup_{i \in I} \gamma_i \right|.$$  

Then $p_{-\Lambda}(n) = p_{-\Gamma}(n)$ for every positive integer $n$.

Now we state three applications of our Theorem 2. We are inspired by similar results of Hančl [4], but use a different approach.

**Theorem 4.** Let $F_Z$ be a partition ideal with basis $Z$, where $Z$ contains infinitely many pairwise independent partitions, and let $k$ be any positive integer. Then

$$p(n, F_Z) < K e^{C\sqrt{n}} n^{-k}$$

for any sufficiently large $n$, where $K = K(k, Z)$ is a fixed constant (and $C = \pi \sqrt{2/3}$).

**Proof.** Let $k \in \mathbb{N}$ and $t = 2k - 1$. Let $\lambda^1, \lambda^2, \ldots, \lambda^t$ be mutually independent partitions from $Z$ such that $|\lambda^1| < |\lambda^2| < \cdots < |\lambda^t|$. By Theorem 3 applied to $\Lambda = \{\lambda^1, \lambda^2, \ldots, \lambda^t\}$ and $\Gamma = \{|\lambda^1|, |\lambda^2|, \ldots, |\lambda^t|\}$,

$$p(n, F_Z) \leq p_{-\Lambda}(n) = p_{-\Gamma}(n).$$

From Theorem 2 we have

$$p_{-\Gamma}(n) \sim K e^{C\sqrt{n}} n^{-1/2} < K e^{C\sqrt{n}} n^{-k}$$

where $K = K(k, Z)$ is a constant and the last inequality holds for any sufficiently large $n$. \hfill \Box

**Conjecture 1.** Let $X = F_Z$ be a partition ideal with finite basis $Z$. Then the asymptotics for the counting function $p(n, X)$ is of the form

$$p(n, X) \sim K e^{C\sqrt{n}} n^{-1-k},$$

where $K$ is a constant and $k = m/2$ for some $m \in \mathbb{N}$.
Let \( \varepsilon \in (0, 1) \). For the forthcoming theorem we set
\[
f(n) = \left(1 - \frac{\log^{1+\varepsilon} n}{\sqrt{n}}\right)^2.
\]
Thus \( f(n) \) goes to 1 as fast as \( n^{-1/2} \log^{1+\varepsilon} n \) goes to 0.

**Theorem 5.** Let \( \varepsilon > 0 \) and \( f(n) \) be as above. Then there is a partition ideal \( X \) such that both

1. \( p(n, X) = 0 \)
2. \( p(n, X) > p(n f(n)) \)

holds for infinitely many positive integers \( n \).

**Proof.** We define the sequences \((s_i)_{i=1}^\infty\) and \((t_i)_{i=1}^\infty\) of positive integers such that
\[
s_1 = 2,
\]
\[
s_{i+1} = t_i^3 + 2
\]
and, given \( s_i \), we set
\[
t_i = \max \left\{ s_i, \exp \left( \left( \frac{3s_i + 10}{2C^*} \right)^{1/\varepsilon} \right), 2n_0 \right\},
\]
where \( n_0 = n_0(s_i) \) is such that for any \( n \geq n_0 \) we have \( f(n) \in \left(\frac{1}{2}, 1\right) \) and both
\[
2 \cdot \frac{e^{C^*}}{4n^{\sqrt{3}}} > p(n) > \frac{1}{2} \cdot \frac{e^{C^*}}{4n^{\sqrt{3}}} \quad \text{and} \quad p_{-[s_i]}(n) > \frac{1}{2} p(n) \prod_{s=1}^{s_i} C_s \frac{2}{2\sqrt{n}}.
\]

Existence of \( n_0 \) follows from the Hardy–Ramanujan asymptotics \([1]\) and Theorem \([2]\).

Let \( I_i = [s_i, t_i] \cap \mathbb{N} \). Let \( X \) be the partition ideal consisting of the partitions that use parts from any of the intervals \( I_i \) with multiplicities at most \( t_i \), and do not use other parts. Our aim is to prove that the first condition is satisfied for \( n = s_{i+1} - 1 \) and the second condition is satisfied for \( n = t_{i+1} \).

Any partition of \( s_{i+1} - 1 \) lying in \( X \) may use only parts \( \leq t_i \) but, as the multiplicities are restricted, sum of all the parts \( \leq t_i \) equals
\[
\sum_{i=1}^{s_i} \sum_{j=1}^{t_i} j \leq t_i \sum_{j=1}^{t_i} j = \frac{t_i^2(t_i + 1)}{2} < s_{i+1} - 1.
\]
Hence easily \( p(s_{i+1} - 1, X) = 0 \) for any positive integer \( i \).

Let \( K = (8\sqrt{3})^{-1} \). To prove the second property we first show that for a fixed positive integer \( i \) and any \( n > \max\{t_i, n_0\} \) we have
\[
p(n)^{1 - \sqrt{f(n)}} > n^{3s_i/2}.
\]

\[8\]
Indeed, the definition of \( f(n) \) implies \( \sqrt{n}(1 - \sqrt{f(n)}) = \log^{1+\varepsilon} n \) and \( 0 < f(n) < 1 \), which, combined with the definition of \( t_i \) and \( n_0 \), yields for any \( n \geq \max\{t_i, n_0\} \) bound

\[
p(n)^{1-\sqrt{f(n)}} n^{-3s_i/2} > K^{1-\sqrt{f(n)}} e^{C\sqrt{n}(1-\sqrt{f(n)})} n^{-1+\sqrt{f(n)}-3s_i/2}
\]

\[
> K e^{C\log^{1+\varepsilon} n - n^{-1-3s_i/2}}
\]

\[
= K n^{C\log^* n - n^{-1-3s_i/2}} \geq K n^4 > 1,
\]

by (9) and the bounds \( K > 1/16 \) and \( n \geq 2 \). Now Theorem 2 and (10) implies

\[
p(t_i, X) \geq p_{-[s_i]}(t_i) > \frac{1}{2} p(t_i) \left( \frac{C}{2\sqrt{t_i}} \right)^{s_i} s_i! > \frac{1}{2} p(t_i)^{\sqrt{f(t_i)}} \left( \frac{C t_i}{2} \right)^{s_i} s_i!.
\]

We apply again asymptotics (1) for \( p(t_i) \) and \( p(t_i f(t_i)) \) and have

\[
p(t_i)^{\sqrt{f(t_i)}} > \left( \frac{e C \sqrt{t_i}}{8\sqrt{3} t_i} \right)^{\sqrt{f(t_i)}} \left( \frac{1}{8\sqrt{3} t_i} \right)^{2t_i f(t_i) \sqrt{3p(t_i f(t_i))}}
\]

\[
> \frac{1}{8} (8t_i \sqrt{3})^{1-\sqrt{f(t_i)}} p(t_i f(t_i)) > \frac{1}{8} p(t_i f(t_i)).
\]

Putting these results together we get that

\[
p(t_i, X) > \frac{1}{16} p(t_i f(t_i)) \left( \frac{C t_i}{2} \right)^{s_i} s_i! > p(t_i f(t_i))
\]

since \( s_i \geq 2, C t_i \geq 6 \). That completes the proof. \( \square \)

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References

[1] G.E. Andrews. *The Theory of partitions*. Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam, 1976. Encyclopedia of Mathematics and its Applications, Vol. 2.

[2] D. I. A. Cohen. *Pie-sums: a combinatorial tool for partition theory*. Combin. Theory, Ser A, 31:223–236, 1981.

[3] P. Flajolet and R. Sedgewick. *Analytic Combinatorics*. Cambridge University Press, 2009.
[4] J. Jr. Hančl. General enumeration of integer partitions. Diploma thesis, 2011.

[5] G. H. Hardy and S. Ramanujan. Asymptotic formulae in combinatorial analysis. Proc. Lond. Math. Soc., 17:75–115, 1918

[6] F. Johansson. Efficient implementation of the Hardy-Ramanujan-Rademacher formula. LMS J. Comput. Math., 15:341–359, 2012

[7] J. L. Nicolas and A. Sárközy. On partitions without small parts. Journal de théorie des nombres de Bordeaux, 12:227–254, 2000

[8] J. B. Remmel. Bijective proofs of some classical partition identities. J. Combin. Theory, Ser A, 33:273–286, 1982.

[9] I. Schur. Zur Additiven Zahlentheorie. S.-B. Preuss. Akad. Wiss. Phys. Math. Klasse, 488–495, 1926.