MORSE THEORY FOR LOOP-FREE CATEGORIES

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Abstract. We extend discrete Morse-Bott theory to the setting of loop-free (or acyclic) categories. First of all, we state a homological version of Quillen’s Theorem A in this context and introduce the notion of cellular categories. Second, we present a notion of vector field for loop-free categories. Third, we prove a homological collapsing theorem in the absence of critical objects in order to obtain the Morse inequalities. Examples are provided through the exposition. This answers partially a question by T. John: whether there is a Morse theory for loop-free (or acyclic) categories? [14].

1. Introduction

Recently, some topological concepts were extended to small categories. This is the case of the Euler characteristic [3, 18], the Lusternik-Schnirelman category [33], the topological complexity [20] and the Euler Calculus [31, 32]. Among small categories, the loop-free (or acyclic) ones are of particular importance since they stand as one of the main settings for working in Combinatorial Algebraic Topology [16, 34, 35]. Their relation to other common objects in Combinatorial Algebraic Topology is illustrated by the following diagram:

\[
\begin{array}{ccc}
\text{Loop-free Categories} & \xrightarrow{\mathcal{K}} & \Delta - \text{Complexes} \\
\xrightarrow{\text{sd}} & \uparrow i & \xleftarrow{\text{sd}} \\
\text{Posets} & \xrightarrow{\mathcal{K}} & \text{Simplicial Complexes} \\
\xleftarrow{\mathcal{X}} & & \xrightarrow{i}
\end{array}
\]

where sd stands for the subdivision functors [16, 22], \( \mathcal{K} \) is the order complex functor [16, 35] and \( \mathcal{X} \) is the face poset functor [22].
Morse Theory is an active field of research with manifestations in diverse areas of Mathematics. Despite it began in the smooth setting [23, 25], it rapidly extended to other contexts, leading to PL versions [1, 2, 4], a purely combinatorial approach on simplicial and regular cell complexes [10, 11, 27], an algebraic version [17, 30] and a theory for posets [7, 8, 24]. The purpose of this work is to extend Morse theory for posets to Morse-Bott theory for loop-free categories.

The organization of the paper is as follows:

In Section 2 we present some necessary preliminaries on small categories and a homological version of Quillen’s Theorem A. Section 3 is devoted to developing homology for loop-free categories and introducing the notion of cellular categories. In Section 4 we present the notion of vector field for graded loop-free categories and its dynamical interpretation. In Section 5 we prove a homological collapsing theorem and the Morse inequalities.

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2. Homological Theorem A for categories

In this section we state a homological version of Quillen Theorem A for small categories. For a detailed presentation the reader is referred to [16, 21, 28, 29, 33, 36].

2.1. Preliminaries on small categories. Recall that a category is said to be small if its arrows form a set. Given a small category $\mathcal{C}$, we denote by $\text{Ob}(\mathcal{C})$ its set of objects, by $\text{Arr}(\mathcal{C})$ its set of arrows and by $\mathcal{C}(c, c')$ the set of arrows between the objects $c$ and $c'$. Moreover, we define two maps $t, s: \text{Arr}(\mathcal{C}) \to \text{Ob}(\mathcal{C})$ which send an arrow to its target (codomain) and source (domain), respectively.

In order to define homology for small categories we briefly recall the definition of the nerve functor $N$ from small categories to simplicial sets. Given the small category $\mathcal{C}$, its nerve $N\mathcal{C}$ is a simplicial set whose $m$-simplices are the composable $m$-tuples of arrows in $\mathcal{C}$:

$$c_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_m} c_m.$$ 

The face maps are obtained by composing or deleting arrows and the degeneracy maps are obtained by inserting identities. An $m$-simplex of $N\mathcal{C}$ is called non-degenerate if it includes no identity. Given a functor
Let $F : \mathcal{C} \to \mathcal{D}$ between small categories, we define $NF : \mathcal{NC} \to \mathcal{NcD}$ as follows: if $c_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_m} c_m$ is an $m$-simplex in $\mathcal{NC}$, then

$$NF(c_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_m} c_m) = F(c_0) \xrightarrow{F(\alpha_1)} \cdots \xrightarrow{F(\alpha_m)} F(c_m).$$

We define the homology (with coefficients in a principal ideal domain) of small categories as the homology of the associated objects by the nerve functor.

2.2. Homological Quillen’s Theorem A. We recall the notions of left and right homotopy fibers due to Quillen.

**Definition 2.1.** Let $\mathcal{C}$ and $\mathcal{D}$ be small categories, let $F : \mathcal{C} \to \mathcal{D}$ be a functor and let $d$ be an object of $\mathcal{D}$. The **left homotopy fiber** $F/d$ of $F$ is the small category whose objects are:

$$\text{Ob}(F/d) = \{(c, g) \in \text{Ob}(\mathcal{C}) \times \mathcal{D}(F(c), d)\}$$

and whose arrows are:

$$F/d((c, g), (c', g')) = \{f \in C(c, c') : g' \circ F(f) = g\}.$$  

Dually, the **right homotopy fiber** $d/F$ of $F$ is the small category whose objects are:

$$\text{Ob}(d/F) = \{(c, g) \in \text{Ob}(\mathcal{C}) \times \mathcal{D}(d, F(c))\}$$

and whose arrows are:

$$d/F((c, g), (c', g')) = \{f \in C(c, c') : F(f) \circ g = g'\}.$$  

We now state a homological version of Quillen Theorem A. For the proof we refer the reader to [36]:

**Theorem 2.2** (Homological Theorem A). Let $\mathcal{C}$ and $\mathcal{D}$ be small categories and let $F : \mathcal{C} \to \mathcal{D}$ be a functor. If all the left homotopy fibers or all the right homotopy fibers are homologically trivial, then $F$ induces an isomorphism $H_*(F) : H_*(\mathcal{C}) \to H_*(\mathcal{D})$ in homology.

3. Homology of loop-free categories

In this section we introduce the notion of cellular categories.

3.1. Homology of loop-free categories. We begin by recalling the concept of loop-free category.

**Definition 3.1.** A small category $\mathcal{C}$ is **loop-free** or **acyclic** if it satisfies the following two conditions:

1. Only the identity arrows have inverses.
2. Any arrow from an object to itself is an identity.
From now on, we will assume that all loop-free categories are finite, that is, their set of arrows are finite.

We recall a construction for loop-free categories which simplifies the computation of homology. We refer the reader to [34], [16] and [13, Appendix] for the notion of (regular) Δ-complexes or regular trisps.

**Definition 3.2** ([16, 33]). Let $\mathcal{C}$ be an loop-free category. Its order complex $K(\mathcal{C})$ is a Δ-complex (or regular trisp) whose $m$-simplices are the composable $m$-tuples of arrows in $\mathcal{C}$ not including identities. For an object $c$, the face map $d_c$ is given by composing arrows at $c$ or by deleting the arrows starting or ending at $c$.

Observe that for an loop-free category $\mathcal{C}$, it is equivalent to compute the homology as $H \circ K(\mathcal{C})$ or as $H \circ N(\mathcal{C})$.

3.2. **Cellular categories.** We begin by introducing a grading for our loop-free categories. In order to do so, we will add an extra assumption (to make our work easier) to the notion of graded loop-free category which appears in [16]. Then, we will introduce the notion of cellular categories. It can be seen as an extension of the concepts of cellular posets defined in [6] and [24].

**Definition 3.3.** Let $\mathcal{C}$ be a loop-free category and $c \in \text{Ob}(\mathcal{C})$. We say that $c$ is a minimal object of $\mathcal{C}$ (or that $c$ is minimal, for short) if $c$ is not the target of any non identity arrow of $\mathcal{C}$, that is, $t^{-1}(c) = \{\text{id}_c\}$. Dually, we say that $c$ is a maximal object of $\mathcal{C}$ (or that $c$ is maximal, for short) if $c$ is not the source of any non identity arrow of $\mathcal{C}$, that is, $s^{-1}(c) = \{\text{id}_c\}$.

**Definition 3.4.** We call an arrow indecomposable if it can not be represented as a composition of two non identity arrows. A loop-free category $\mathcal{C}$ is called graded if there is a map $r: \text{Ob}(\mathcal{C}) \to \mathbb{Z}$ such that:

1. whenever $m: c \to c'$ is a non identity indecomposable arrow, we have $r(c') = r(c) + 1$.
2. Moreover, if $c$ is a minimal object of $\mathcal{C}$, then $r(c) = 0$.

For an object $c$, the integer $r(c)$ will be referred as its degree. Then we will write $c^{(r(c))}$.

We introduce some definitions that will be necessary later.

**Definition 3.5.** Let $\mathcal{C}$ be a loop-free category and let $c$ be an object of $\mathcal{C}$. Let $\mathcal{C} - \{c\}$ denote the full subcategory of $\mathcal{C}$ with $\text{Ob}(\mathcal{C} - \{c\}) = \text{Ob}(\mathcal{C}) - \{c\}$.
Definition 3.6. Let $\mathcal{C}$ be a loop-free category and let $c$ be an object of $\mathcal{C}$. Then $U_c$ is the full subcategory of $\mathcal{C}$ whose objects are $s(t^{-1}(c))$. Moreover, we define $\hat{U}_c = U_c - \{c\}$.

Definition 3.7. Let $\mathcal{C}$ be a graded loop-free category. It is said to be cellular if for each $c \in \text{Ob}(\mathcal{C})$, $\hat{U}_c$ has the homology of a wedge of $n_c$ $(r(c) - 1)$-spheres for some $n_c \geq 1$.

Example 3.8. We provide an example of a cellular category in Figure 3.1. We show only indecomposable arrows and we do not include the identities in the picture.

Figure 3.1. Example of a cellular category.

4. Vector fields and Morse theory

4.1. Vector fields. We begin with the definition of vector field for loop-free categories. Given an arrow $f \in \text{Arr}(\mathcal{C})$, we denote $\text{Arr}(f) := \mathcal{C}(s(f), t(f))$. If two arrows $f, g$ have the same source and target, or $\text{Arr}(f) = \text{Arr}(g)$, we say that they have the same type.

Definition 4.1. Let $\mathcal{C}$ be a graded loop-free category. A vector field $\mathcal{V}$ on $\mathcal{C}$ is a subset of the non identity indecomposable arrows of $\mathcal{C}$ satisfying the following conditions:

1. If $f, g \in \mathcal{V}$, then $s(f) \neq t(g)$.
2. If $f \in \mathcal{V}$ and $\#\text{Arr}(f) = 1$, then $s^{-1}(s(f)) \cap \mathcal{V} = \{f\}$ and $t^{-1}(t(f)) \cap \mathcal{V} = \{f\}$.
3. If $f \in \mathcal{V}$ and $\#\text{Arr}(f) > 1$, then $\#(\mathcal{V} \cap \text{Arr}(f)) \leq \#\text{Arr}(f) - 1$.

Elements of $\mathcal{V}$ are called vectors and elements of the set $\text{Ob}(\mathcal{C}) \setminus (s(\mathcal{V}) \cup t(\mathcal{V}))$ are called critical.

Remark 1. Intuitively, the conditions of Definition 4.1 say that every object can be a source or a target of at most a single vector unless these vectors of the same type. Note that conditions (1) and (2) rephrase the conditions that Forman [9] use to define his combinatorial vector;
while new condition \([3]\) lets us to deal with multiple arrows of the same type.

We need to recall some notions from graph theory.

**Definition 4.2.** Let \(v\) and \(v'\) be nodes in a directed multigraph (multiple arrows with same source and target are allowed) \(G\). A sequence \(v = v_0, v_1, \ldots, v_j = v'\) is a *path* (of length \(j\)) from \(v\) to \(v'\) if \(v_{i-1}v_i\) is an edge in \(G\) for every \(i\). A path is *non-trivial* if it has length greater than zero.

Given a loop-free category \(C\), let us denote by \(H(C)\) the directed multigraph defined as follows. The elements of \(H(C)\) are the objects of \(C\) while the set of edges of \(H(C)\) consists of the indecomposable non-identity arrows of \(C\). If \(V\) is a vector field on \(C\), we write \(H_V(C)\) for the directed multigraph obtained from \(H(C)\) by reversing the orientations of the edges which are not in \(V\) and adding the identity arrows of critical elements.

**Definition 4.3.** Let \(V\) be a vector field on a loop-free graded category \(C\) and let \(c^{(k)}, \tilde{c}^{(k)} \in \text{Ob}(C)\) be two objects of \(C\). A *\(V\)-path*, \(\gamma\), of index \(k\) from \(c^{(k)}\) to \(\tilde{c}^{(k)}\) is a sequence:

\[
(\tilde{c}^{(k)} = x_{k}^{(k)}, y_{0}^{(k+1)}, x_{1}^{(k+1)}, y_{1}^{(k+1)}, \ldots, y_{r-1}^{(k+1)}, x_{r}^{(k)} = \tilde{c}^{(k)})
\]

with \(r \geq 1\) such that for each \(i = 0, 1, \ldots, r - 1\):

1. There is a \(f_i \in V\) such that \(f_i : x_i \to y_i\),
2. There is a \(g_i \in \text{Arr}(C) - V\) such that \(g_i : x_{i+1} \to y_i\).

A *\(V\)-cycle* \(\gamma\) is a *\(V\)-path* such that \(\tilde{c}^{(k)} = c^{(k)}\). A *\(V\)-cycle* also can be interpreted as a non-trivial closed path in the directed multigraph \(H_V(C)\).

**4.2. Critical subcategories.** We now present the notion of chain recurrent set, which generalizes its homonymous concept appearing in \([8, 9]\).

**Definition 4.4.** Let \(V\) be a vector field on a graded loop-free category \(C\). We say that \(c^{(k)} \in C\) is an object of the *chain recurrent set* \(R\) if one of the following conditions holds:

- \(c\) is a critical element of \(V\).
- There is a \(V\)-cycle \(\gamma\) such that \(c \in \gamma\).

The chain recurrent set decomposes into disjoint subsets \(\Lambda_i\) by means of the equivalence relation defined as follows:

1. If \(c\) is a critical element, then it is only related to itself.
(2) Given \( c, c' \in \mathcal{R} \) not critical, \( c \neq c' \), \( c \sim c' \) if there is a \( \mathcal{V} \)-cycle \( \gamma \) such that \( c, c' \in \gamma \).

Let \( \Lambda_1, \ldots, \Lambda_k \) be the equivalence classes of \( \mathcal{R} \). The \( \Lambda_i \)'s are called basic sets. Each \( \Lambda_i \) consists either of a single critical element of \( \mathcal{V} \) or a union of cycles, each of which has the same index. We write \( \Lambda_i^{(k)} \) and say that \( \Lambda_i \) has index \( k \) if \( \Lambda_i \) consists of a critical point of index \( k \) or a union of closed paths of index \( k \).

**Definition 4.5.** Let \( \mathcal{G} \) be a multigraph. A strongly path-connected component \( \mathcal{G}' \) of \( \mathcal{G} \) is a maximal subgraph such that for \( v, v' \in \mathcal{G}' \) there exist non-trivial paths from \( v \) to \( v' \) and vice versa. In particular, a singleton \( \{v\} \) is a strongly path-connected component if and only if there is a self-loop attached to \( v \).

**Remark 2.** Note that with the multigraph interpretation of \( \mathcal{V} \), we can retrieve basic sets by computing the family of all strongly path-connected components of \( \mathcal{H}_V(\mathcal{C}) \) following the ideas of [15, 19, 26].

The set

\[
\{ f \in \mathcal{V} : \#\text{Arr}(f) = 1 \text{ and } s(f), t(f) \notin \mathcal{R} \}
\]

will be referred as the gradient like part of the vector field.

**Definition 4.6.** Let \( \mathcal{C} \) be a graded loop-free category and \( \mathcal{V} \) a vector field on \( \mathcal{C} \). The vector field \( \mathcal{V} \) is homologically admissible if for every arrow \( f \) in the gradient like part of \( \mathcal{V} \), the subcategory \( \hat{\mathcal{U}}_{t(f)} - \{s(f)\} \) is homologically trivial.

**Example 4.7.** In Figure 4.1 we provide a representation of \( \mathcal{H}_V(\mathcal{C}) \) for a vector field \( \mathcal{V} \) on the cellular category of Example 3.8. The orange arrow is the gradient like part. The other colors represent different basic sets.

![Figure 4.1. Example of a vector field.](image)
4.3. Filtration induced by a vector field. Let \( \mathcal{C} \) be a finite graded loop-free category and \( \mathcal{V} \) a vector field on \( \mathcal{C} \). We show how this vector field induces a filtration on \( \mathcal{C} \):

\[
C_0 \hookrightarrow C_1 \hookrightarrow \cdots \hookrightarrow C_i \hookrightarrow C_{i+1} \hookrightarrow \cdots \hookrightarrow C_n = \mathcal{C}
\]

where each \( C_i \) is a full subcategory of \( \mathcal{C} \). We define \( C_0 \) as the empty category. Now we apply the following iterative process. For \( C_i \), a full subcategory of \( \mathcal{C} \), we denote by \( M_i \) the set of minimal elements of \( \mathcal{C} \) in \( \mathcal{C} - C_i \). If there is a critical element \( c \in M_i \), then we define \( C_i \) as the full subcategory with the object \( c \). If there are no critical elements in \( M_i \) and there is a \( \mathcal{V} \)-cycle \( \gamma \) such that \( \text{Ob}(\bigcup_{c \in \gamma} \mathring{U}_c) - \gamma \subset \text{Ob}(C_i) \), then we define \( C_{i+1} \) as the full subcategory adding the objects in \( \gamma \). If there are no critical elements nor \( \mathcal{V} \)-cycles satisfying the stated conditions in \( M_i \), then we proceed as follows. Among all the arrows \( f \) in the gradient part of \( \mathcal{V} \) such that \( s(f) \in M_i \), we pick one satisfying that \( \text{Ob}(\mathring{U}_{t(f)} - s(f)) \subset \text{Ob}(C_i) \). Then we define \( C_{i+1} \) as the full subcategory with the objects \( \text{Ob}(C_i) \cup \{ s(f), t(f) \} \).

Example 4.8. We illustrate in Figure 4.2 the procedure presented above for the vector field and category of Example 4.7.

![Figure 4.2. Filtration induced by a vector field.](image)

5. The Morse inequalities

In this section we will assume that loop-free small category \( \mathcal{C} \) is cellular and vector field \( \mathcal{V} \) is homologically admissible.
5.1. A homological collapsing theorem. In order to prove the Morse inequalities, we need the following homological collapsing theorem:

**Theorem 5.1** (Homological collapsing theorem). Let $\mathcal{C}$ be a finite cellular category and $\mathcal{V}$ a homologically admissible vector field on $\mathcal{C}$. Consider a filtration

$$C_0 \hookrightarrow C_1 \hookrightarrow \cdots \hookrightarrow C_i \hookrightarrow C_{i+1} \hookrightarrow \cdots \hookrightarrow C_n = C$$

as constructed in Subsection 4.3. If there is an $f$ in the gradient part of $\mathcal{V}$ such that $\text{Ob}(C_{i+1}) - \text{Ob}(C_i) = \{s(f), t(f)\}$, then $C_i \hookrightarrow C_{i+1}$ induces an isomorphism in homology.

**Proof.** First of all, each right homotopy fiber of the inclusion functor $i: C_{i+1} - \{s(f)\} \hookrightarrow C_{i+1}$ has an initial object, so it is homologically trivial (it is, in fact, contractible [20]). Therefore, by the Homological Theorem A (Theorem 2.2), $i: C_{i+1} - \{s(f)\} \hookrightarrow C_{i+1}$ induces an isomorphism in homology. Now, consider the inclusion: $i: C_{i+1} - \{s(f), t(f)\} \hookrightarrow C_{i+1} - \{s(f)\}$. The result would follow if we proved that $H_*(C_{i+1} - \{s(f)\}, C_{i+1} - \{s(f), t(f)\}) \cong 0$. Apply the Excision theorem to the subcomplexes $\mathcal{K}(U_t(f))$ and $\mathcal{K}(C_{i+1} - \{s(f), t(f)\})$ to obtain the isomorphism:

$$H_*(U_t(f), \tilde{U}_t(f) - \{s(f)\}) \cong H_*(C_{i+1} - \{s(f)\}, C_{i+1} - \{s(f), t(f)\}).$$

Observe that $U_t(f)$ has a terminal object $t(f)$, so $\mathcal{K}(U_t(f))$ is homologically trivial. Since the vector field $\mathcal{V}$ is homologically admissible, then $\tilde{U}_t(f) - \{s(f)\}$ is homologically trivial. By the homology long exact sequence of the pair $(\mathcal{K}(U_t(f)), \mathcal{K}(\tilde{U}_t(f) - \{s(f)\}))$, it follows that $H_*(\mathcal{K}(U_t(f)), \mathcal{K}(\tilde{U}_t(f) - \{s(f)\})) \cong 0$ and we obtain the desired result. \hfill $\square$

5.2. The Morse inequalities. We generalize Morse inequalities from the context of regular CW-complexes [9, Theorem 3.1] and posets [7, 8, 24] to loop-free categories. This result can be seen a combinatorial analogue of a theorem due to Conley [12, Theorem 1.2], [5].

Given a subcategory $\mathcal{D} \hookrightarrow \mathcal{C}$ we denote by $\mathcal{D}$ the full subcategory with objects $\text{Ob}(\mathcal{D}) = \cup_{d \in \mathcal{D}} U_d$ and by $\hat{\mathcal{D}}$ the full subcategory with objects $\text{Ob}(\hat{\mathcal{D}}) = \text{Ob}(\mathcal{D}) - \text{Ob}(\mathcal{D})$.

**Definition 5.2.** For each $k \geq 0$, we define

$$m_k = \sum_{\text{basic sets } \Lambda_i} \text{rank } H_k(\bar{\Lambda}_i, \hat{\Lambda}_i).$$

The following result is proved by means of analogous techniques as [8, Lemma 4.2.2].
Lemma 5.3. If the index of a basic set $\Lambda$ is $k$, then $H_i(\bar{\Lambda}, \dot{\Lambda}) = 0$ unless $i = k, k + 1$. Moreover, if $\Lambda$ is just a critical point $x^{(k)}$, then $H_i(\bar{\Lambda}, \dot{\Lambda}) = 0$ for $i \neq k$.

Theorem 5.4 (Strong Morse-Bott inequalities). Let $\mathcal{C}$ be a finite cellular category and let $\mathcal{V}$ be a homologically admissible vector field on $\mathcal{C}$. Then, for every $k \geq 0$:

$$m_k - m_{k-1} + \cdots + (-1)^k m_0 \geq b_k - b_{k-1} + \cdots + (-1)^k b_0$$

where $b_k$ stands for the Betti number of $\mathcal{C}$ of dimension $k$ with coefficients in a principal ideal domain.

Proof. Consider a filtration of $\mathcal{C}$ as described in Subsection 4.3:

$$\mathcal{C}_0 \hookrightarrow \mathcal{C}_1 \hookrightarrow \cdots \hookrightarrow \mathcal{C}_i \hookrightarrow \cdots \hookrightarrow \mathcal{C}_n = \mathcal{C}.$$ 

We will check that the inequalities hold for every $i$, that is:

$$m_k - m_{k-1} + \cdots + (-1)^k m_0 \geq b_k - b_{k-1} + \cdots + (-1)^k b_0$$

for every $\mathcal{C}_i$. We argue by induction on $i$. For $\mathcal{C}_0$ it holds trivially. Assume it holds for $\mathcal{C}_i$ and let us show that then it also holds for $\mathcal{C}_{i+1}$.

There are two cases to consider:

1. There is an arrow $f$ in the gradient part of $\mathcal{V}$ such that $\mathcal{C}_{i+1}$ is the full subcategory with the objects $\text{Ob}(\mathcal{C}_i) \cup \{s(f), t(f)\}$. Then, by the Homological Collapsing Theorem (Theorem 5.1), $\mathcal{C}_i \hookrightarrow \mathcal{C}_{i+1}$ induces an isomorphism in homology. Therefore $b_k(\mathcal{C}_i) = b_k(\mathcal{C}_{i+1})$ for all $k$. Moreover, $m_k(\mathcal{C}_i) = m_k(\mathcal{C}_{i+1})$ by Definition 5.2.

2. The subcategory $\mathcal{C}_{i+1}$ is the full subcategory whose objects are the union of the objects of $\mathcal{C}_i$ and the elements of a basic set $\Lambda$. Then $m_k(\mathcal{C}_{i+1}) - m_k(\mathcal{C}_i) = \text{rank } H_k(\bar{\Lambda}, \dot{\Lambda})$. By excision, it follows that: $H_k(\mathcal{C}_{i+1}, \mathcal{C}_i) \cong H_k(\bar{\Lambda}, \dot{\Lambda})$. Now the result is obtained by standard arguments (see [23, p. 28-31]).

$\square$

Corollary 5.5 (Weak Morse-Bott inequalities). Let $\mathcal{C}$ be a finite cellular category and let $\mathcal{V}$ be a homologically admissible vector field on $\mathcal{C}$. Then:

1. For every $k \geq 0$, $m_k \geq b_k$;
2. $\chi(\mathcal{C}) = \sum_{i=0}^\infty (-1)^k b_k = \sum_{i=0}^\infty (-1)^k m_k$.

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