ON THE NATURE OF THE BLANDFORD-ZNAJEK MECHANISM

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It is widely accepted in the astrophysical community that the event horizon plays crucial role in the Blandford-Znajek mechanism of extraction of rotational energy of black holes. In fact, this view is a quintessence of the Membrane Paradigm of black holes which suggests that the event horizon, or rather the so-called stretched horizon, is similar to a rotating conducting sphere of finite resistivity. In this paper we argue that this interpretation is rather misleading and the proper explanation of the Blandford-Znajek mechanism has to be sought in the properties of the ergospheric region of black holes.

1 Blandford-Znajek solution and the Membrane Paradigm

We start with describing some basic properties of steady-state force-free magnetospheres of black-holes as discovered in [1] using the 3+1 formulation of electrodynamics. First we introduce vectors of electric and magnetic field as follows

\[ E_i = F_{it}, \quad B^i = \frac{1}{2} e^{ijk} F_{jk}, \]

where \( F_{\mu\nu} \) is the electromagnetic field tensor, and \( e \) is the Levi-Civita alternating tensor. For steady-state axisymmetric solutions \( E_\phi = 0 \). Consider a polar flux tube of magnetic flux

\[ \Phi = \int B^i dS_i \quad (dS_i = e_{ijk} dx^j_{(1)} dx^k_{(2)}) \]

The angular velocity of magnetic field lines defined as

\[ \Omega = \frac{E_r}{\sqrt{-g} B^\theta} = -\frac{E_\theta}{\sqrt{-g} B^r}. \]

is constant along the field lines and, thus, is a function of \( \Phi \). Another constant, \( B_T = \sqrt{-g} F^r\phi \), is loosely called the “azimuthal field” because in the Boyer-Lindquist coordinates

\[ B_T = \alpha g_{\phi\phi} B^\phi \]

where \( \alpha \) is the lapse function. In addition, neither electric charge nor energy or angular momentum can flow across the wall of the flux tube and, thus, the
total poloidal electric current, $I$, energy flux, $E$, and angular momentum flux, $L$, also depend only on $\Phi$. Moreover,

$$\frac{dE}{d\Phi} = -B_T \Omega, \quad \frac{dL}{d\Phi} = -B_T. \quad (4)$$

In the original paper by Blandford and Znajek [1] the so-called “horizon boundary condition” seems to play a rather important role. This condition is a derivative of the usual regularity condition – a free-falling observer crossing the horizon should register a finite electromagnetic field. This leads to

$$B_T = -f(\theta)(\Omega_h - \Omega)\partial_\theta \Phi \quad (5)$$

at the horizon [2], where $\Omega_h = a/(r^2 + a^2)$ is the angular velocity of the black hole and $f(\theta)$ is a positive function of the polar angle. The most important result of [1] is the perturbative solution (we shall refer to it as the BZ solution) for a monopole-like magnetosphere of a slowly rotating black hole that satisfies the “horizon boundary condition” and matches Michel’s flat space wind solution [3] at infinity. In such magnetosphere, all magnetic field lines originate from the event horizon and rotate with $\Omega = 0.5\Omega_h$ ensuring outgoing fluxes of energy and angular momentum.

Although the physical conditions utilized in this model are rather obvious, they are the Kerr metric, negligibly small mass density of matter, and high conductivity of the magnetospheric plasma, no clear explanations of the internal “mechanics” of the BZ process seems to have been given so far. Instead, a surrogate “explanation” was put forward where the so-called “stretched horizon” located just outside of the real event horizon, is considered as an analogue of a magnetized non-perfectly conducting sphere rotating in flat space-time with $\Omega_h$ and, thus, inducing rotation of magnetic field lines originated from its surface. This analogy is often used to describe how the BZ mechanism operates to wider audience not particularly familiar with general relativity. It has been pushed all the way in the Membrane Paradigm [4] where it is regarded as capturing all important features of the real object and even allowing rather accurate analysis without resorting to difficult general relativistic calculations. In fact, this analogy is so popular that in the minds of many the Membrane Paradigm and the BZ mechanism are inseparable. However, nobody can deny that without full understanding of the real thing no analogy can be relied upon with confidence. As far as the electrodynamics of black is concerned, this issue has been addressed in in the critical analysis of the BZ mechanism by Punsly and Coroniti [5] (see also [6]), who showed that the stretched horizon can
not communicate with the outgoing electromagnetic wind by means of Alfvén waves in contrast to a rotating conducting sphere in flat space-time. In [5,6] this causality argument is used to revise the role of the event horizon and even to rise questions about stability of the BZ solution. Recent numerical simulations [13] convincingly show, however, that the BZ solution is asymptotically stable and, thus, leave only one suspect – the Membrane Paradigm.

2 The origin of rotation

In this section we will try to identify what drives the rotation of magnetospheres of black holes. For the sake of argument we assume that magnetic field lines threading the event horizon also thread a massive nonrotating shell located at some distance from the hole and for this reason do not rotate, \( \Omega = 0 \). Without any loss of generality we will consider the case of \( \Omega_h, \partial_\theta \Phi > 0 \). Then eq.[3] requires \( B_T < 0 \) and the second equation in (4) implies \( L > 0 \). Thus, vanishing \( \Omega \) results in torque being applied to the shell due to building up of azimuthal magnetic field! The shell can remain nonrotating only if there is another torque which cancels the torque applied by the black hole. Otherwise both the shell and the magnetic field lines will be forced to rotate in the same sense as the black hole, \( \Omega > 0 \), resulting in extraction of energy from the hole, \( \mathcal{E} > 0 \) (see the first equation in (4).)

This argument applies only to magnetic field lines penetrating the event horizon and makes use of the debatable horizon boundary condition. What can we tell about the field lines which do not penetrate the horizon? Membrane Paradigm gives no reason for such field lines to rotate. Once more let us to consider a static magnetosphere where \( \Omega = 0 \) and, thus, the electric field as defined in (1) vanishes:

\[
E_i = 0, \quad i = \phi, r, \theta.
\]

However, the local “zero angular velocity observer” (ZAMO), which is at rest in the space of the BL space-time foliation, registers not only magnetic field

\[
\dot{B}^i = \sqrt{g_{ii}} B^i
\]

but electric field as well

\[
\dot{E}_\phi = 0;
\]

\[
\dot{E}^\phi = -\sqrt{\frac{g_{\phi\phi}}{g_{rr}}} \Omega_F B^\theta \neq 0,
\]
\[ E^\phi = \sqrt{-g_{\phi\phi}} \Omega r B^r \neq 0, \]

where
\[ \mathcal{L} = g_{tt} g_{\phi\phi} - g_{t\phi} g_{t\phi}. \]

This gives
\[ \hat{B}^2 - \hat{E}^2 = g_{\phi\phi} (B^\phi)^2 - \frac{g_{tt}}{\sin^2 \theta \Delta} \left( g_{\theta\theta} (B^\theta)^2 + g_{rr} (B^r)^2 \right). \quad (6) \]

High conductivity of black hole magnetospheres (e.g.,[1,7]) ensures
\[ \hat{B}^2 - \hat{E}^2 \geq 0. \quad (7) \]

Since \( g_{ii} > 0 \) for all \( i \), \( \Delta = r^2 - 2r + a^2 \) is positive outside of the horizon, and
\[ \begin{cases} g_{tt} < 0 & \text{outside of the ergosphere} \\ g_{tt} > 0 & \text{inside of the ergosphere} \end{cases} \]

there must be a nonvanishing azimuthal component of magnetic field, \( B^\phi \), for all magnetic field lines penetrating the ergosphere irrespective of whether they penetrate the horizon or not! As we have already discussed this leads to torque and ultimately ensures rotation of these field lines.

The key property of the ergospheric region of black holes is the inertial frame dragging effect. Relative to any physical observer the spacial grid of the Boyer-Lindquist coordinates is moving superluminally inside the the ergosphere and subluminally outside. Thus, inside the ergosphere the magnetic field with \( \Omega = 0 \) is static relative to a superluminally moving spacial grid! Moreover, relative to ZAMO the BL spacial grid is rotating in the azimuthal direction and it is the azimuthal component of magnetic field as measured in the BL coordinate basis which is generated due to high conductivity of plasma. This fact suggests that the superluminal motion of the spacial grid is very important. The following flat space-time example confirms this conclusion.

Consider an inertial frame in Minkowskian space-time with pseudo-Cartesian coordinates \( x^\mu \). The corresponding metric form is

\[ ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \]

Now let us introduce a rectangular grid moving in the \( x^1 \)-direction with velocity \( \beta \):
\[ x^\mu = A^\mu_\nu x^\nu \quad \text{where} \quad A^\mu_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \]

Notice that this is not a Lorentz transformation. In these new coordinates the metric form
\[ ds^2 = (-1 + \beta^2)(dx^0)^2 + 2\beta dx^0 dx^1 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \]
has a number of properties similar to the Kerr metric in the BL coordinates. In particular, \( x^0 \) becomes space-like for \( \beta^2 > 1 \). Consider magnetic field static relative to the moving grid, which implies \( E_i = F_{i0} = 0 \). Then, the electromagnetic field registered by our inertial observer is
\[ B^i = B^i, \quad E^1 = 0, \quad E_2 = \beta B^3, \quad E_3 = -\beta B^2 \]
which gives
\[ \dot{B}^2 - \dot{E}^2 = (B^1)^2 + (1 - \beta^2)((B^2)^2 + (B^3)^2). \]

Now we can see that in the case of superluminal motion in the \( x^1 \)-direction, \( \beta^2 > 1 \), the high conductivity condition, \( \dot{B}^2 - \dot{E}^2 > 0 \), requires \( B^1 \neq 0 \). Thus, we observe the same behaviour as in the case of black holes.

## 3 Causality consideration

The “driving force” of the BZ mechanism must be able to communicate with the outgoing wind by means of both fast and Alfvén waves \([5,6,8]\). Therefore, it must be located between the inner and the outer Alfvén surfaces of a black hole magnetosphere \([9]\) and, thus, lay well outside of the horizon (Similar conclusion was reached in \([6,10]\).)

In the limit of force-free degenerate electrodynamics the Alfvén critical surfaces coincide with the light surfaces given by
\[ f(\Omega, r, \theta) = g_{\phi\phi}\Omega^2 + 2g_{\theta\phi}\Omega + g_{tt} = 0, \quad (8) \]
Moreover, for a black hole with positive angular velocity and positive outgoing energy flux one has
\[ 0 < \Omega < \Omega_F \quad (9) \]
for the inner critical surface and
\[ 0 < \Omega_F < \Omega \] (10)
for the outer critical surface, where \( \Omega_F \) is the angular velocity of the local ZAMO. On the surface of the ergosphere
\[ f(\Omega, r, \theta) = g_{\phi\phi}(\Omega - 2\Omega_F). \] (11)
From this one can see that in the limit \( \Omega \to 0 \) the inner light surface coincides with the ergosphere, \( f \) being positive inside and negative outside. Since
\[ \frac{\partial f}{\partial \Omega} = 2g_{\phi\phi}(\Omega - \Omega_F) < 0 \] for \( \Omega = 0, \theta \neq 0 \) (12)
the inner critical surface moves inside the ergosphere as \( \Omega \) increases and must remain inside for all values satisfying (9) (The third factor in (11) vanishes only when the outer critical surface moves inside the ergosphere.)
Thus, for all values of \( \Omega \) consistent with extraction of energy from a black hole, there always exists an outer region of the ergosphere which is causally connected to the outgoing electromagnetic wind. There is no clash between causality and the BZ solution but only between causality and the interpretation of this solution given in the Membrane Paradigm.

4 Numerical simulations

Consider the problem of a Kerr black hole placed in an originally uniform magnetic field aligned along the symmetry axis of the hole. In the vacuum solution by Wald [11] there are magnetic field lines of three kinds: (i) those that never enter the ergosphere, (ii) those that enter the ergosphere but do not thread the event horizon, and (iii) those that thread the event horizon. Should all these three types be present in the corresponding solution of force-free degenerate electrodynamics ([12], “magnetodynamics” seems to be a better name for this system) we would expect the lines of type (i) to remain nonrotating, whereas the lines of both type (ii) and type (iii) to rotate. To test this prediction we carried out time-dependent numerical simulations similar to those described in [13] for the case of monopole magnetospheres. We utilized the Kerr-Schild coordinates and placed the inner boundary \( r = r_{\text{in}} \) of the computational domain inside the event horizon. The outer boundary \( r = r_{\text{out}} \) was placed far away from the hole to ensure no interference. At \( \theta = 0 \) and \( \theta = \pi/2 \) we used relevant symmetry boundary conditions. The initial solution describes
Figure 1: Rotating black hole \((a = 0.9M)\) in originally uniform magnetic field.

the same magnetic field as in [11] and such electric field that 1) the degeneracy conditions

\[
\vec{E} \cdot \vec{B} = 0, \quad B^2 - E^2 > 0
\]

are satisfied everywhere and 2) the magnetic field lines are non-rotating outside of the ergosphere.

Quite soon after the start of simulations, the second degeneracy condition breaks down in the equatorial plane inside the ergosphere where magnetic field lines tend to rotate slower than the minimum angular velocity of massive particles. Even slightly unscreened electric field will create strong equatorial current sheet with Compton drag providing required resistivity. This electric
field will accelerate charged particles to rather high Lorentz factors. These energetic particle will pass their energy to the background photons which will dump it into the black hole. Because for an observer resting at infinity this energy is negative such interactions result in extraction of rotational energy of the black hole. However, any significant deviation from $E^2 = B^2$ is impossible under the typical conditions of astrophysical black holes as it would cause copious pair production. In many respects this model of ergospheric disk is similar to the one described in [6].

The proper description of the ergospheric current sheet appears to be impossible within the framework of magnetodynamics. However, if the net effect of all these processes on the electromagnetic field amounts to ensuring the condition $E^2 \approx B^2$ in the equatorial plane then the following prescription seems to be quite reasonable. If at the end of a normal computational time step we find that in a particular cell $E^2 > B^2$ then the magnitude of $\vec{E}$ is reduced to the magnitude of $\vec{B}$. One can show that this corresponds to a source of energy and momentum in the cell.

Figure 1 shows the numerical solution for a black hole with $a = 0.9M$ by the end of simulations (t=40M) when near the black hole the solution is close to a steady state. $M$ is the black hole mass. As we expected, all magnetic field lines entering the ergospheric region are forced to rotate in the same sense as the black hole irrespective of whether they thread the horizon or not. The magnetic field lines placed outside of the ergosphere remain nonrotating with the exception of those passing very close to the ergosphere. Their negative angular velocity is most certainly a numerical artifact.

5 Conclusions

The results presented in this paper strongly suggest that the Blandford-Znajek mechanism [1] has the same basic “driving force” as the Penrose mechanism [14] which is the ergosphere of a rotating black hole.

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