On First Integrals for Holomorphic Vector Fields

Jonny Ardila

1 Introduction.

One of the key stones in the theory of holomorphic foliations is the article [8], where is presented the following important result about the existence of holomorphic first integrals.

**Theorem.** Let $F$ be a germ in $0 \in \mathbb{C}^2$ of holomorphic foliation of codimension 1. Suppose that:

1. $\text{Sing}(F) = \{0\}$.
2. There are only finite separatrices $S_k$.
3. The leaves that do not accumulate in $0$ are close.

Then, exist $V$ a neighborhood of $0$, such that $F|_V$ has a holomorphic first integral.

Years latter in [9], one of their authors revisited this result in order to create a new proof, a simpler and more geometric one.

The present work is motivated by that demonstration and is divided in two parts:

In the first part, we adapt the technique used in [9] to a vector fields in $\mathbb{C}^3$ under the conditions exposed in [2] giving a new proof for the existence theorem presented there:

**Theorem I.** Suppose that $X \in \text{Gen}(\mathfrak{X}(\mathbb{C}^n, 0))$ satisfies condition $(\star)$ and let $S_X$ be the axis associated to the separable eigenvalue of $X$. Then, $F(X)$ has a holomorphic first integral if and only if, the leaves of $F(X)$ are closed off $\text{sing}(F(X))$ and transversely stable with respect to $S_X$.

In the second part, we return to dimension 2 and modified the proof in [9] obtaining:

**Theorem II.** Let $F$ be a holomorphic Morse type foliation of codimension 1 in a neighborhood $U$ of $B^4 \subset \mathbb{C}^2$. Suppose that:

1. $\text{Sing}(F) = \{0\}$.
2. There are only finite separatrices.
3. The leaves that not accumulated in $0$ are close in $U$.

Then, $F|_B$ has a holomorphic first integral.
that solves in some how a limitation that \cite{9} has, as the author himself comments in it:

"Cette preuve(ou encore la conclusion du théorème) n’est pas entièrement satisfaisante. En effet, nous n’avons montré l’existence d’une intégrale première \( p_\alpha \) que sur un voisinage \( V \) des séparatrices strictement contenu dans la boule \( B \). Notre méthode, l’étude de l’espace des feuilles \( V/\mathcal{F}_V \) nécessite la transversalité des feuilles de \( \mathcal{F}_V \) à la sphère a \( B \). Elle ne permet pas d’étendre \( p_\alpha \) en une intégrale première sur \( B \)."

Throughout this work we identify \( \mathbb{C}^n \) with \( \mathbb{R}^{2n} \) (together with the euclidean norm \( \| \| \)) and use the notation:

- \( B_m^r \) for the open ball in 0 of radius \( r \) in \( \mathbb{R}^m \), \( \partial B_m^r \) the sphere of radius \( r \) and \( B_m^r \) the closure of \( B_m^r \).
- \( \mathcal{F}_r \), \( \partial \mathcal{F}_r \) and \( \mathcal{F}_r \) for the foliations induced by \( \mathcal{F} \) in \( B_m^r \), \( \partial B_m^r \) and \( B_m^r \) respectively.

We will omit \( m \) and \( r \), if \( m = 2n \) and \( r \) does not play a relevant role.

2 Generic Vector Fields in dimension 3.

This section is dedicated to prove Theorem 1 but, is necessary to introduce first some definitions and notation.

Denote the ring of germs of holomorphic functions on \( (\mathbb{C}^n, 0) \) by \( \mathcal{O}_n \), the ring of formal series on \( (\mathbb{C}^n, 0) \) by \( \hat{\mathcal{O}}_n \), the group of formal diffeomorphisms of \( (\mathbb{C}^n, 0) \) by \( \hat{\text{Diff}}(\mathbb{C}^n, 0) \) and \( \text{Diff}(\mathbb{C}^n, 0) \) the subgroup of analytic diffeomorphisms (or just diffeomorphisms) of \( \hat{\text{Diff}}(\mathbb{C}^n, 0) \). Given a germ of a holomorphic vector field \( \mathcal{X} \in \mathfrak{X}(\mathbb{C}^n, 0) \) we shall denote by \( \mathcal{F}(\mathcal{X}) \) the germ of a one-dimensional holomorphic foliation on \( (\mathbb{C}^n, 0) \) induced by \( \mathcal{X} \).

Definition 1. We shall say that \( \mathcal{F}(\mathcal{X}) \) is non-degenerate generic if \( d\mathcal{X}(0) \) is non-singular, diagonalizable, and after some suitable change of coordinates \( \mathcal{X} \) leaves invariant the coordinate planes. Denote the set of germs of non-degenerate generic vector fields on \( (\mathbb{C}^n, 0) \) by \( \text{Gen}(\mathfrak{X}(\mathbb{C}^n, 0)) \).

Definition 2. We say that a germ of a holomorphic foliation \( \mathcal{F}(\mathcal{X}) \) has a holomorphic first integral, if there is a germ of a holomorphic map \( F : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n-1}, 0) \) such that:

(a) \( F \) is a submersion off some proper analytic subset. Equivalently if we write \( F = (f_1, \ldots, f_{n-1}) \) in coordinate functions, then the \( (n-1) \)-form \( df_1 \wedge \cdots \wedge df_{n-1} \) is non-identically zero.

(b) The leaves of \( \mathcal{F}(\mathcal{X}) \) are contained in level curves of \( F \).

Further, a germ \( f \) of a meromorphic function at the origin \( 0 \in \mathbb{C}^n \) is called \( \mathcal{F}(\mathcal{X}) \)-invariant if the leaves of \( \mathcal{F}(\mathcal{X}) \) are contained in the level sets of \( f \). This can be precisely stated in terms of representatives for \( \mathcal{F}(\mathcal{X}) \) and \( f \), but can also be written as \( i_\mathcal{X}(df) = \mathcal{X}(f) \equiv 0 \).
Definition 3 (condition (⋆)). Let $\mathcal{X}$ be a germ of a holomorphic vector field at the origin such that the origin $0 \in \mathbb{C}^m, m \geq 3$ is a nondegenerate singularity of $\mathcal{X}$ (i.e. $d\mathcal{X}(0)$ is non-singular). We say that $\mathcal{X}$ satisfies condition (⋆) if there is a real line $L \subset \mathbb{C}$ through the origin, separating a certain eigenvalue $\lambda$ from the others.

2.1 Proof of Theorem I.

The proof is divided in two parts:
A. Construction a neighborhood $V$ of the origin.
B. Study of the quotient space $V/F_V$.

Lemma A. There exists open sets $V, U$ with $V \subset U \subset \overline{B}$ such that, $V$ is a neighborhood $\mathcal{F}$-invariant of $S$ (the union of separatrices), and the leaves in $V$ cut $\partial U$ transversally.

Let’s set $V^* = V \setminus S$, $F_{V^*}$ the foliation induced by $\mathcal{F}$ in $V^*$, $V^*/F_{V^*}$ the quotient space and $q_{V^*}$ the quotient map ($q_{V^*} : V^* \rightarrow V^*/F_{V^*}$).

Lemma B. Exist a homeomorphic map $h : V^*/F_{V^*} \rightarrow B^*$, such that $h \circ q_{V^*} = p_{V^*}$ is a submersion.

Proof of Theorem I. Therefore, $p_{V^*}$ is holomorphic and bounded in $V^*$, and $S$ is an analytic set that do not disconnect $V$ (see [5]) then, $p_{V^*}$ extents as a holomorphic first integral in $V$.

2.2 Proof of the Lemmas.

First of all, fix a small enough ball $B = B_2^n$ centered in $0 \in \mathbb{C}^n (\cong \mathbb{R}^{2n})$.

Though we only need the transversality of the coordinate axes with the sphere, remember that we are working with germs of vector fields and we have this property after a coordinate change so, what we mean with small enough is that is contained in a neighborhood of the origin where the germ is defined and has this property.

Proof of Lemma A. The proof is divided in the following affirmations:

Affirmation 4. If $L$ is a leaf of $\mathcal{F}$ transverse to $\partial B$. Then, it exist a fundamental systems of neighborhoods $\mathcal{F}$-invariant of $L$ in $\overline{B}$.

-If $L$ is transverse to $\partial B$ using the fact that leaves in $\mathcal{F}$ are compact and closed off the origin then, the holonomy of $L$ is finite. We can use Reeb’s Theorem in $(\partial S_X, \partial \mathcal{F})$ showing the affirmation.

Remark 5. Is important to observe that this affirmation is actually true for any open set $U \subset B$.

Now, we have that the curve $S_X \cap \partial B$ posses a neighborhood where $\partial \mathcal{F}$ is a transversally holomorphic foliation without singularities. $S_X \cap \partial B$ is compact and, using the Lemma 2 in [2] (is here where we need the transversely stable hypothesis of the leaves respect to $S_X$), its holonomy is periodic then, applying Reeb’s in $(\partial S_X, \partial \mathcal{F})$ we have:
Affirmation 6. The leaf $\partial S_X$ of $\partial F$ possesses a tubular neighborhood $T_1(\epsilon)$ in $\partial B$

$$J_1 : B^{2n-2}_\epsilon \times S^1 \to T_1(\epsilon),$$

such that $J_1^{-1}(\partial F)$ is the suspension of a periodic diffeomorphism in $B^{2n-2}_\epsilon$.

$T_1(\epsilon)$ is $\partial F$-invariant and $T_1(\epsilon') = J_1(B^{2n-2}_\epsilon \times S^1)$ with $0 < \epsilon' < \epsilon$ forms a fundamental system of neighborhoods of $\partial S_X$ in $\partial B$. In addition $T_1(\epsilon)$ is transverse to $F$.

Next, remember that the vector fields in $\text{Gen}(X(\mathbb{C}^n, 0))$ satisfying condition $(\star)$ and having a holomorphic first integral can be written as:

$$X(x) = \sum_{i=1}^{n-1} k_i x_i (1 + a_i(x)) \frac{\partial}{\partial x_i} - k_n x_n (1 + a_n(x)) \frac{\partial}{\partial x_n},$$

where $k_1, \ldots, k_n \in \mathbb{Z}^+$, in this case $S_X$ is the $x_n$ axis. Now taking $x_n = 0$ we have that $X_0(x) = X(x_1, \ldots, x_{n-1}, 0)$ is a hyperbolic vector field (i.e., $k_j > 1$ for all $j$) in Poincaré’s domain then is smoothly linearizable. In order to see the hyperbolicity, suppose that $k_1 = 1$ and $k_2 > 1$ then in the plane $\{x_1, x_2\}$ restricted to the sphere we have that only one leaf is closed and the others are open spiraling to it (see [1] Cap. 3) in contradiction to our hypothesis. Also, we are omitting the tangent to the identity case using that generically $X$ has $n$ different eigenvalues. Consider the linear vector field,

$$j^1X_0(x) = k_1 x_1 \frac{\partial}{\partial x_1} + \cdots + k_{n-1} x_{n-1} \frac{\partial}{\partial x_{n-1}},$$

we are interested in the transversality of this vector field with the sphere of radius $\delta$ in $\{x_n = 0\}$, in order to see this we calculate the tangent points of the field with it i.e., the solutions of the following equation where $\langle \cdot \rangle$ is the Hermitian product:

$$\langle j^1X_0(x), x \rangle = k_1 |x_1|^2 + \cdots + k_{n-1} |x_{n-1}|^2 = 0.$$ 

Hence this vector field is always transverse to the spheres, if $\varphi$ is the biholomorphism that conjugates $X_0$ with is linear part, i.e., $X_0 = \varphi^* (j^1X_0)$, we can take the inverse image of a ball $B^{2n-2}_\delta$ in the complex hyperplane $\{x_n = 0\}$ and, the fact that the vector field $j^1X_0$ is transverse to this ball guarantee us that $X_0$ is
transverse to \( \varphi^{-1}(\partial B_3^{2n-2}) \). Define now the following open set which is a solid cylinder intercepted with \( B \),

\[
U = B \cap \{ x \in \mathbb{C}^n \mid (x_1, \ldots, x_{n-1}, 0) \in \varphi^{-1}(B_3^{2n-2}) \}.
\]

Now, for each leaf in \( F \cap \varphi^{-1}(\partial B_3^{2n-2}) \) we can apply the Affirmation 6 and obtain a neighborhood similar to \( T_1(\epsilon) \) (in the sense that the holonomy is finite but we cannot guarantee its periodicity) and the compactness of \( \varphi^{-1}(\partial B_3^{2n-2}) \) allows to take a finite covering of it formed by a finite union of those neighborhoods, we note this covering by \( T_2(\epsilon) \).

**Affirmation 7.** It exist \( 0 < \epsilon' < \epsilon \) such that the intersection of \( \partial U \) with the \( F \)-saturated \( V(\epsilon') \) of \( T_1(\epsilon') \) is contained in \( T(\epsilon) = T_1(\epsilon) \cup T_2(\epsilon) \).

- By contradiction, take a sequence \( \{a_k\}_k \) of points in \( T_1(\epsilon) \) such that \( a_k \to a \in \partial S_X \) and satisfying \( L_{a_k} \cap \partial U \not\subset T(\epsilon) \) where \( L_{a_k} \) is the leaf in \( F \) passing by \( a_k \). Take \( b_k \) a point in \( (L_{a_k} \cap \partial U) \setminus T(\epsilon) \), then \( \{b_k\}_k \) is a sequence in a compact thus \( b_k \to b \) (using the same notation for a subsequence), if \( L_b \) is transverse to \( \partial U \) then, by Affirmation 4 it exist a saturated neighborhood of \( L_b \) by transverse leaves to \( U \) then, it does not contain separatrices meaning that \( L_b \) is far from \( S_1 \), contradiction. If \( L_b \) is not transverse to \( \partial U \) we can take small \( U \) and apply Affirmation 4 again, a contradiction.

**Remark 8.** In the previous affirmation (Aff. 7) we are using implicitly that the only separatrices are \( S_X \) and the ones contained in \( \{x_n = 0\} \). This is easy to see by taking the vector field in the form \( \epsilon \) and supposing that there exits a integral curve \( c(T) = (x_1(T), \ldots, x_n(T)) \) such that \( c(T) \to 0 \) if \( |T| \to \infty \). Without lost of generality take real time \( t \) and \( \lambda_n = -1 \), in this case the condition \((*)\) implies that \( \text{Re}(\lambda_i) > 0 \) for \( i = 1, \ldots, n-1 \), and this can be used to show that the equations in \( \mathcal{A}(c(t)) = c'(t), \) i.e., \( x'_i(t) = \lambda_i x_i(t)(1 + a_i(c(t))) \), can not go to 0 simultaneously.

**Affirmation 9.** It exist \( 0 < \epsilon_1 < \epsilon' \) such that \( V(\epsilon_1) = V \), the \( F \)-saturated of \( T_1(\epsilon_1) \), is a neighborhood of \( 0 \) in \( U \).

- The pseudo-group of holonomy is generated by an enumerable set of biholomorphisms with finitely many non trivial fixed points. The set of leaves of \( F \) with non-trivial holonomy is meager (see [4] proposition 2.7, pag. 96) so we can choose \( \epsilon_1 \) such that \( 0 < \epsilon_1 < \epsilon' \) and the leaves cutting \( J_1(\partial B_3^{2n-2} \times 1) = C_{\epsilon_1} \) have trivial holonomy. Again, the compactness of the leaves allows to apply Reeb stability theorem. For all \( a \in C_{\epsilon_1} \) the leaf \( L_a \) in \( F \) through \( a \) possess a \( F \)-saturated tubular neighborhood (see Fig. 2):

\[
J_a : \tau_a \times L_a \to T(L_a),
\]

such that \( J_a^{-1}(F) \) is foliated by fibers \( z \times L_a \), where \( \tau_a \) is a small curve transverse to \( F \) through \( a \) contained in \( T_1(\epsilon) \). In particular the \( F \)-saturated of \( \nu_a = \tau_a \cap C_{\epsilon_1} \) is \( C^\infty \)-diffeomorphic to the product \( \nu_a \times L_a \) and the saturated of \( C_{\epsilon_1} \) is a \( C^\infty \)-hypersurface (whose boundary is contained in \( \partial U \) fibered over \( S^1 \). By construction, is the boundary of \( V = V(\epsilon_1) \) the \( F \)-saturated of \( T_1(\epsilon_1) \).

**Proof of Lemma 2** We are going show that \( \tilde{\Delta} = V^*/FV^* \) and \( (B_3^{2n-2})* \) (the punctured ball) are biholomorphic. Note that \( V^* \) is the saturation of \( T^*_1(\epsilon_1) = T_1(\epsilon_1) \setminus \partial S_X \) which is the same that the saturation of \( J_1(\mathbb{D}_{\epsilon_1} \times \{1\}) =: \Delta^*_1 \), and,
\[ \Delta_{\epsilon_1}^* \text{ is transverse to } \mathcal{F}_{V^\ast} \text{ and biholomorphic to } (B^{2n-2})^* \text{ by a biholomorphism } \Phi : \Delta_{\epsilon_1}^* \to (B^{2n-2})^* \text{ that conjugates the action of a diagonal diffeomorphism } G \text{ centered in } 0 \in \mathbb{C}^n \text{ and the periodic holonomy } H \text{ in } \Delta_{\epsilon_1}^*. \]

\[ \begin{array}{ccc}
\Delta_{\epsilon_1}^* & \xrightarrow{\Phi} & (B^{2n-2})^* \\
\downarrow q & & \downarrow q \\
\Delta & \xrightarrow{\Psi} & (B^{2n-2})^*/G
\end{array} \]

The action of the groups \( H \) and \( G \), in their respective spaces, is free and proper then, the quotient spaces \( \Delta \) and \((B^{2n-2})^*/G\) are manifolds with a unique smooth structure such that \( q \) is a smooth submersion (see [7] Chap. 21), and they are biholomorphic too by a biholomorphism \( \Psi \). We also know that \( q : (B^{2n-2})^* \to (B^{2n-2})^*/G \) is a finite covering now, remember that the holonomy \( H \) is periodic then \( G \) is also periodic and suppose that \( G(x_1, \ldots, x_{n-1}) = (\lambda_1 x_1, \ldots, \lambda_{n-1} x_{n-1}) \) where \( \lambda_j = e^{\alpha_j i} \) with \( \alpha_j \in [0, 2\pi) \) and \( j = 1, \ldots, n-1 \), and define

\[ S(\alpha_j) = \{ x \in \mathbb{C}^n \mid 0 \leq \arg(x_j) < \alpha_j \text{ and } x_k = 0, \text{ for } k \neq j \}. \]

Thus the map

\[ \begin{array}{ccc}
S(\alpha) & \xrightarrow{q} & \mathbb{D}^*/G
\end{array} \]

Figure 3: \( S(\alpha) \) in dimension 2.
\[ q : (B^{2n-2})^* \cap S(\alpha_1) \times \cdots \times S(\alpha_{n-1}) \to (B^{2n-2})^*/G \]

is bijective, hence \( (B^{2n-2})^*/G \) is biholomorphic to \( (B^{2n-2})^* \cap S(\alpha_1) \times \cdots \times S(\alpha_{n-1}) \) which is biholomorphic to \( (B^{2n-2})^* \). Therefore, it exist a biholomorphism \( h : \tilde{\Delta} \to (B^{2n-2})^* \) as we wanted and \( h \circ q \) is a submersion due to the smooth structure we are considering in \( \tilde{\Delta} \).

3 Vector Fields in dimension 2.

In this part, we make a small modification to the technique used in [9] (but, we preserve the same philosophy) and, that allow us to obtain an interesting result (Theorem II). The difference between our result and the one in [9] is that we managed to get a holomorphic first integral defined in the whole ball.

Something important is that here we need a small sphere \( B \). Also, in what follows \( M \) will always mean the variety of contacts restricted to \( B \).

Let \( B^* \) notes \( B \) without the union of the separatrices, \( F_{B^*} \) the foliation induced by \( F \) in \( B^* \), \( B^*/F_{B^*} \) the quotient space and \( q_{B^*} \) the quotient map \( (q_{B^*} : B^* \to B^*/F_{B^*}) \). The Theorem II is consequence of the following lemma which is the analogous of Lemma B,

**Lemma B’**. Exist a continuous map

\[ h : B^*/F_{B^*} \to \mathbb{D}^* \]

such that \( h \circ q_{B^*} = p_{B^*} \) is holomorphic.

**Proof of Theorem II**. Therefore, \( p_{B^*} \) is holomorphic and bounded in \( B^* \), and the union of the separatrices is an analytic set that does not disconnect \( B \) then, \( p_{B^*} \) extents as a holomorphic first integral in \( B \).

We need the following definitions and results taken from [4]:

**Definition 10.** The foliation \( F \) of \( X \) is of Morse type if the singularities of \( r_X(x) = -\langle X(x), x \rangle X(x) \) on each leaf are nondegenerate. \( X \) is of Morse type if \( F \) is.

**Theorem 11.** Let \( X \) be a holomorphic vector field in \( U \) with a unique zero at 0. If \( X \) is a field of Morse type, then either we have \( M = \{0\} \) or \( M \setminus \{0\} \) is a smooth manifold of real codimension two. In the latter case, each connected component of \( M \setminus \{0\} \) consists entirely of either minimal points in the leaves or saddle points. The foliation \( F \) (defined by \( X \)) is transversal to \( M \setminus \{0\} \) (and therefore \( M \setminus \{0\} \) can be given a complex structure).

**Proposition 12.** Let \( \text{sat}(M) \) be the saturation of \( M \) in \( B \) and let \( S = B \setminus \text{sat}(M) \). Then the \( \omega \)-limit set of \( r_X(x) \) restricted to \( S \) consists of \( \{0\} \) alone. For each \( x \in \text{sat}(M) \), its \( \omega \)-limit set consists of a single point, which is in \( M \).
Theorem 13. The radial flow $r_{\mathcal{X}}$ endows $S$ with the structure of a foliated cone, with base $S \cap \partial B_{r}$ foliated by $\mathcal{F} \cap \partial B_{r}$ and (deleted) top at 0. Every leaf of $\mathcal{F}$ contained in $S$ is an immersed copy of $\mathbb{R}^2$ or $S^1 \times \mathbb{R}$, depending on whether it intersects $\partial B_{r}$ in a line or in a circle (i.e., a closed orbit). Each closed orbit in $S$ corresponds to a separatrix of $\mathcal{F}$.

Proof of Lemma $[\mathcal{L}]$. First, if $M = \{0\}$ this means that $\mathcal{X}$ is transverse to the spheres $\partial B_{r}$ then, by $[\mathcal{L}]$, it exhibits only one singularity at $0 \in B$ which is accumulated by each orbit of $\mathcal{X}$ (in addition it belongs to the Poincaré domain), which contradicts the finite many separatrices hypothesis. Therefore $M \neq \{0\}$ and we are in the second case of Theorem $[\mathcal{T}]$.

The proof is an adaptation of the proof of Lemma 2 in $[\mathcal{G}]$ to our case.

We are going to endow $\tilde{\Delta} = B^*/\mathcal{F}_{B^*}$ (the leaves space) with a Riemannian surface structure and show that each connected component of $\tilde{\Delta}$ is biholomorphic to a closed punctured disk. Note first that $\tilde{\Delta}$ is Hausdorff, with the quotient topology $q_{B^*} : B^* \to B^*/\mathcal{F}_{B^*}$ (just take two different leaves and apply local stability). Note also that Proposition $[\mathcal{P}]$ and Theorems $[\mathcal{T}]$ and $[\mathcal{T}]$ implies that, $S$ is the union of the separatrices and $B^*$ is the saturation of $M^* = M \setminus \{0\}$ which is transverse to $\mathcal{F}_{B^*}$ and diffeomorphic to the cone with base $M \cap \partial B$ and deleted top $\{0\}$. With this in mind we can identify

$$B^*/\mathcal{F}_{B^*} = q_{B^*}(B^*) = q_{B^*}(M^*)$$

$$\tilde{\Delta} := q(M^*).$$

Suppose that the finite many connected components of $M$ are $M_1, \ldots, M_k$ and sat($M$) = sat($M_1 \cup \cdots \cup M_k$) with $r \leq k$, consider $\tilde{\Delta}_1$ a connected component of $\tilde{\Delta}$ and suppose that $q^{-1}(\tilde{\Delta}_1) = M_1^* \cup \cdots \cup M_k^*$ with $s \leq r$. Let $a \in q^{-1}(\tilde{\Delta}_1)$ and $L_a$ be the leaf in $\mathcal{F}_{B^*}$ passing by $a$. Observe that $L_a$ is compact and with finite holonomy. This holonomy is a subgroup of the group of rotations centered at $0 \in \mathbb{C}$, and is isomorphic to $\mathbb{Z}/n(a)\mathbb{Z}$, with $n(a) \in \mathbb{Z}$. Applying Reeb’s to $(L_a, \mathcal{F}_{B^*})$ we find a neighborhood $\mathcal{F}_{B^*}$-invariant of $L_a$ that can be thought as the $\mathcal{F}_{B^*}$-saturated of $\Delta_a$, a neighborhood of $a$ in $M$, where $\Delta_a$ is biholomorphic to $\mathbb{D}_1$ (in $\mathbb{D}_1$ we consider the induced topology), the former neighborhood is biholomorphically conjugated to $\mathbb{D}_1 \times L_a$, having a first integral $z \to z^{n(a)}$.

Therefore, there exist a biholomorphism $\varphi_a : \mathbb{D}_1 \to \Delta_a$ that conjugates the action of a periodic rotation in $\mathbb{D}_1$ with the holonomy of $a$ in $\Delta_a$ and, a homeomorphism $g_a : q(\Delta_a) \to \mathbb{D}_1$ that makes the next diagram commutative.

$$\begin{array}{ccc}
\mathbb{D}_1 & \xrightarrow{\varphi_a} & \Delta_a \\
g_a & & \downarrow q \\
 & & q(\Delta_a)
\end{array}$$

Where $g_a$ (see Fig. $[\mathcal{I}]$) can be defined by

$$g_a \circ q \circ \varphi_a(z) = z^{n(a)}$$

$$\left(\varphi_a^{-1} \circ q_a^{-1} \circ q \circ \varphi_a(z)\right)^{n(a)} = z^{n(a)},$$

we are notating by $q_a$ the restriction of $q$ to $\Delta_a$. So $g_a(\cdot) = (\varphi_a^{-1} \circ q_a^{-1}(\cdot))^{n(a)}$ is a homeomorphism ($g_a$) composed with a biholomorphism ($\varphi_a$) and, $g_a \circ q$ is
holomorphic over the $F_V$-saturate of $\Delta_a$. In order to see that \{g_a \mid a \in q^{-1}(\tilde{\Delta}_1)\} is an atlas that define a differentiable structure in $\tilde{\Delta}_1$ (and therefore, $\tilde{\Delta}_1$ is a real manifold of dimension two i.e., locally a surface) we need to show:

- $\tilde{\Delta}_1 = \bigcup q(\Delta_a)$.
- If $q(\Delta_a) \cap q(\Delta_b) \neq \emptyset$, $g_a \circ q^{-1}_a(q(\Delta_a) \cap q(\Delta_b))$ and $g_b(q(\Delta_a) \cap q(\Delta_b))$ are open sets and $g_b \circ g^{-1}_a : g_b(q(\Delta_a) \cap q(\Delta_b)) \to g_b(q(\Delta_a) \cap q(\Delta_b))$ is a biholomorphism.

To see the second one, note that $q(\Delta_a) \cap q(\Delta_b)$ is intersection of two open sets and by continuity $g_b(q(\Delta_a) \cap q(\Delta_b)) = (\varphi^{-1}_b \circ q^{-1}_a(q(\Delta_a) \cap q(\Delta_b)))^{n(a)}$ is open. Finally,

$$g_b \circ g^{-1}_a(\cdot) = (\varphi^{-1}_b \circ q^{-1}_a(g_a \circ \varphi_a((\cdot)^{1/n(a)})))^{n(b)} = (\cdot)^{n(b)/n(a)}.$$

**Remark 14.** In the previous construction we can take points in the intersection of $q^{-1}(\Delta_1)$ with the sphere and obtain that $\tilde{\Delta}_1$ is a manifold with boundary. In order to do this, we need $B$ to be contained in an open set where the hypothesis of the previous theorems and proposition remains valid or take a small ball $B' \subset B$.

Then by construction $q$ (which is $q_B$) is holomorphic (because $g_a \circ q$ is holomorphic, thinking $\tilde{\Delta}$ as a manifold). Observe that $q : q^{-1}(\Delta_1) \to \tilde{\Delta}_1$ is proper and a finite covering.

We have that $\tilde{\Delta}_1$ can not be simply connected because, in that case it would be biholomorphic to $\mathbb{D}_1$ or $\mathbb{C}$ and the preimage of its boundary $S^1$ (hyperbolic) or $\{\infty\}$ (parabolic) necessarily has to be $\partial \mathbb{D}_1$ and 0, which are of different kind.

Furthermore, $\pi_1(\tilde{\Delta}_1)$ is generated by one element, take a point $q(a)$ and two different elements $\alpha, \beta \in \pi_1(\tilde{\Delta}_1, q(a))$ and due to the fact that $q$ is a finite
covering, there exist \( l, t \in \mathbb{Z} \) such that \( \alpha^l \) and \( \beta^t \) are lifted to closed curves in \( a \) which are homotopic. Then, \( \pi_1(\overline{\Delta}_1) \) is generated by one element and, \( \overline{\Delta}_1 \) and \( \mathbb{D}^* \) are homeomorphic. Finally, if \( B_1, B_2 \) are the boundaries of \( \overline{\Delta}_1 \) we have that \( q^{-1}(B_i) \) is a boundary of \( M^*_i \cup \cdots \cup M^*_s \) of the same class that \( B_i \) then \( \overline{\Delta}_1 \) and \( \mathbb{D}^* \) are biholomorphic.

We complete the prove observing that this biholomorphism can be extended up to the edge, in fact \( \overline{\Delta}_1 \) can be considered as a manifold with boundary biholomorphic to \( \mathbb{D}^* \). Hence each connected component of \( \overline{\Delta} \) is biholomorphic to a one closed punctured discs \( \mathbb{D}^* \).

![Figure 5: \( h : \overline{\Delta} \to \mathbb{D}^* \)](image)

Now, it is possible to construct a holomorphic map between a finite union of closed punctured discs and one closed punctured discs, just by fixing one as a center and, forming rings enlarging the others (Figure 5). Thus there exist a holomorphic map \( h : \overline{\Delta} \to \mathbb{D}^* \).

\[\square\]

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