Arithmetic of triangles

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Abstract

In this paper we consider a set of similar triangles with parallel sides together with a set of points of the plane. It turns out that the set $\mathbb{R}_2 = \{\pm(x) = \pm(x^2, x, 1); x \in \mathbb{R}\}$ describes this set of triangles quite well. The set $\mathbb{R}_2$ is the subset of the ring $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z); x, y, z \in \mathbb{R}\}$ with addition and multiplication by coordinates. The set $\mathbb{R}_2$ has two operations. The multiplication is the same as in the ring $\mathbb{R}^3$ while the addition is three-argument and describes a homothety and a translation of elements of the set $\mathbb{R}_2$. Depending on the size of the arguments of addition, we have six types of its geometric interpretations. The implemented addition, however, has its limitations. It turns out that what is feasible in an algebraic sense is not necessarily closed in a set of triangles.

In the section 4 we use the construction of adding to describe the dissection of the triangle into 15 triangles of different sides.

In the next section we show how to make sense of the limit $\lim_{n \to \infty} \left( \frac{3}{2} \right)^n = 0$.

In the last two sections we give two geometric interpretations of the set $\mathbb{R}_1 = \{\pm(x) = \pm(x, 1); x \in \mathbb{R}\}$.

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Introduction

Let us take the ring $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x,y,z); x, y, z \in \mathbb{R}\}$ with addition and multiplication

\[
(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2), \quad (1)
\]

\[
(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) = (x_1 \cdot x_2, y_1 \cdot y_2, z_1 \cdot z_2). \quad (2)
\]

Let us consider the subset of the ring $\mathbb{R}^3$, the set $\mathbb{R}_2 = \{±\langle x \rangle = ±(x^2, x, 1); x \in \mathbb{R}\}$. It is closed under multiplication (2) but not under addition (1).

The set $\mathbb{R}_2$ is closed under the following kind of addition

\[
\forall x, y, z, t \in \mathbb{R} \quad \langle x+y+z+t \rangle = \langle x+y+t \rangle + \langle x+z+t \rangle + \langle y+z+t \rangle
- \langle x+t \rangle - \langle y+t \rangle - \langle z+t \rangle + \langle t \rangle \quad (3)
\]

because the equations

\[
\forall x, y, z, t \in \mathbb{R} \quad \forall i = 2, 1, 0 \quad \langle x+y+z+t \rangle^i = \langle x+y+t \rangle^i + \langle x+z+t \rangle^i + \langle y+z+t \rangle^i
- \langle x+t \rangle^i - \langle y+t \rangle^i - \langle z+t \rangle^i + \langle t \rangle^i
\]

are true.

If we multiply the Eq. (3) by $-1$, we get the definition of addition for the elements $-\langle x \rangle$ of the set $\mathbb{R}_2$.

It can be proved by mathematical induction that the following equation follows from Eq. (3).

\[
\forall n \in \mathbb{N} \quad \langle n \rangle = \frac{n(n+1)}{2} \langle 1 \rangle - (n-1)(n+1)\langle 0 \rangle + \frac{n(n-1)}{2} \langle -1 \rangle. \quad (4)
\]

1. For $n = 0, \quad n = 1$ and $n = 2$ we have respectively

\[
\langle 0 \rangle = 0\langle 1 \rangle + 1\langle 0 \rangle + 0\langle -1 \rangle,
\langle 1 \rangle = 1\langle 1 \rangle - 0\langle 0 \rangle + 0\langle -1 \rangle,
\langle 2 \rangle = \langle 1 + 1 + 1 - 1 \rangle = 3\langle 1 \rangle - 3\langle 0 \rangle + 1\langle -1 \rangle.
\]
2. Let us assume Eq. (4) holds for $n = k - 2$ and $n = k - 1$ for each $k > 2$, that is

$$\langle k-2 \rangle = \frac{(k-2)(k-1)}{2} \langle 1 \rangle - (k-3)(k-1) \langle 0 \rangle + \frac{(k-3)(k-2)}{2} \langle -1 \rangle,$$

$$\langle k-1 \rangle = \frac{(k-1)(k)}{2} \langle 1 \rangle - (k-2)k \langle 0 \rangle + \frac{(k-2)(k-1)}{2} \langle -1 \rangle.$$ 

3. Using Eq. (3) we have

$$\langle k \rangle = \langle (k-1) + 1 + 1 - 1 \rangle = 2(k-1) + \langle 1 \rangle - \langle k-2 \rangle - 2\langle 0 \rangle + \langle -1 \rangle$$

$$= 2 \left[ \frac{(k-1)(k)}{2} \langle 1 \rangle - (k-2)k \langle 0 \rangle + \frac{(k-2)(k-1)}{2} \langle -1 \rangle \right] + \langle 1 \rangle$$

$$- \left[ \frac{(k-2)(k-1)}{2} \langle 1 \rangle - (k-3)(k-1) \langle 0 \rangle + \frac{(k-3)(k-2)}{2} \langle -1 \rangle \right]$$

$$- 2\langle 0 \rangle + \langle -1 \rangle$$

$$= \frac{k(k+1)}{2} \langle 1 \rangle - (k-1)(k+1) \langle 0 \rangle + \frac{k(k-1)}{2} \langle -1 \rangle.$$ 

Similarly, we can prove Eq. (4) for $n < 0$.

But you can check purely for accounting that Eq. (4) holds for each $x \in \mathbb{R}$.

$$\forall x \in \mathbb{R} \quad \langle x \rangle = \frac{x(x+1)}{2} \langle 1 \rangle - (x-1)(x+1) \langle 0 \rangle + \frac{x(x-1)}{2} \langle -1 \rangle. \quad (5)$$

Let us transform Eq. (5).

$$\langle x \rangle = \frac{x^2+x}{2} \langle 1 \rangle - (x^2-1) \langle 0 \rangle + \frac{x^2-x}{2} \langle -1 \rangle$$

$$= x^2 \langle 1 \rangle - \langle 0 \rangle + \langle -1 \rangle$$

$$+ x \langle 1 \rangle - \langle -1 \rangle + \langle 0 \rangle.$$

It is easy to check that the elements $\frac{\langle 1 \rangle - 2\langle 0 \rangle + \langle -1 \rangle}{2} = A_2$, $\frac{\langle 1 \rangle - \langle -1 \rangle}{2} = A_1$, $\langle 0 \rangle = A_0$ are orthogonal. So we can put $A_2 = (1, 0, 0)$, $A_1 = (0, 1, 0)$, $A_0 = (0, 0, 1)$ and now we know why $\langle x \rangle = (x^2, x, 1)$.

1 **Geometric interpretation of the set $\mathbb{R}_2$.**

Let us set any closed triangle on the plane (the triangle does not have to be equilateral) and let us denote it by the symbol $\langle 1 \rangle$. Each triangle obtained from the
fixed triangle \(\langle 1 \rangle\) by any translation will be denoted by \(\langle 1 \rangle\) too. The triangle obtained from the triangle \(\langle 1 \rangle\) by any homothety of ratio \(x > 0\) will be denoted by \(\langle x \rangle\).
Each point of the plane will be denoted by \(\langle 0 \rangle\).
The symbol \(-\langle x \rangle\) denotes the triangle, which lying on the triangle \(\langle x \rangle\) gives an empty set denoted by an element \((0,0,0) \in \mathbb{R}^3\).
An element \(\langle x \rangle\) for each \(x \in \mathbb{R}, x > 0\) will be interpreted as a closed triangle and marked with a black color \(\blacktriangle\). The interior of this triangle should be black but we would like to mark that sides and vertexes belong to this triangle, so we coloured it in gray. An element \(-\langle x \rangle\) will be interpreted as a closed triangle (we will call it sometimes antytriangle) and marked with a red color \(\redtriangle\). An element \(\langle 0 \rangle\) will be marked as a black point and an element \(-\langle 0 \rangle\) will be marked as a red point.
To see the result of putting the black triangle on the red one, we will denote the empty set of green.
An element \(\langle -x \rangle\) for each \(x \in \mathbb{R}, x > 0\) will be interpreted as a open triangle symmetrical to the triangle \(\langle x \rangle\) in relation to any side and will be marked with a black color \(\blacktriangle\).
At the end an element \(-\langle -x \rangle\) for each \(x \in \mathbb{R}, x > 0\) will be interpreted as a open triangle symmetrical to the triangle \(-\langle x \rangle\) in relation to any side and will be marked with a red color \(\redtriangle\).

Why the triangles \(\langle -x \rangle\) and \(-\langle -x \rangle\) for \(x > 0\) are open we will see after introducing the geometric interpretation of the operation (3).

The geometric interpretation of the individual components of Eq. (4) is as follows (Fig. 1). The triangle \(\langle n \rangle\) has \(n^2\) of all triangles \(\langle 1 \rangle\) and \(-\langle 1 \rangle\).
If we assume that the length of each side of the triangle \(\langle 1 \rangle\) is equal to 1 (even when the triangle is not equilateral) then \(n\) is the length of each side of the triangle \(\langle n \rangle\).
The number 1 in \((n^2,n,1)\) means one triangle. (Each point \(\langle 0 \rangle\) is a triangle with sides equal to 0. Each triangle \(\langle -n \rangle\) has sides equal to \(-n\).)
The triangle \( \langle n \rangle \) has \( \frac{n(n+1)}{2} \) of triangles \( \langle 1 \rangle \) and \( \frac{n(n-1)}{2} \) of triangles \( \langle -1 \rangle \).

Since the vertices of the closed triangles \( \langle 1 \rangle \) overlap, these vertices as points \( \langle 0 \rangle = (0,0,1) \) must be subtracted in the description \( [4] \) of the triangle \( \langle n \rangle \).

We have \( 3(n-1) \) double vertices lying on the 3 sides of the triangle and \( \frac{(n-2)(n-1)}{2} \) triple vertices lying inside the triangle \( \langle n \rangle \). So we must substract

\[
3(n-1) + 2 \frac{(n-2)(n-1)}{2} = (n-1)(n+1)
\]

vertices \( \langle 0 \rangle \) lying in the description \( [4] \) of the triangle \( \langle n \rangle \).

Using linear combinations of elements of the set \( \mathbb{R}_2 \) we can describe different geometric figures (as elements of the ring \( \mathbb{R}_3 \)) and thus better understand the meaning of the number \( x \) in the element \( \langle x^2, x, 1 \rangle \). (The colors white and green mean the same, ie. empty set) (Fig. [2] [3].

_Geometric interpretation of the operation \( [3] \)._

Let us fix the ordered successive components \( x, y, z \) of the sum \( \langle x + y + z + t \rangle \). This components extend or shorten the triangle \( \langle t \rangle \) in the directions I, II, III, or I’, II’, III’, depending on whether the numbers \( x, y, z \) are positive or negative (Fig. [4]).
\[ \langle a \rangle - \langle b \rangle = (a^2 - b^2, a - b, 0), \quad a > b > 0. \]

\[ (-a) - (-b) = (a^2 - b^2, -a + b, 0), \quad a > b > 0. \]

Fig. 2:

\[ \langle a + b \rangle - \langle a \rangle - \langle b \rangle = (2ab, 0, -1), \quad a, b > 0. \]

\[ (a + b + c + t) - (a) - (b) - (c) = (x, t, -2), \quad a, b, c > 0. \]

Fig. 3:
Therefore Eq. (3) should be properly written as
\[
\forall x, y, z, t \in \mathbb{R} \\
\langle x + y + z + t \rangle = \langle x + y + 0 + t \rangle + \langle x + 0 + z + t \rangle + \langle 0 + y + z + t \rangle \\
- \langle x + 0 + 0 + t \rangle - \langle 0 + y + 0 + t \rangle - \langle 0 + 0 + z + t \rangle \\
+ \langle 0 + 0 + 0 + t \rangle.
\]

Since the above record is long and inconvenient, we will replace it with the following one
\[
\forall x, y, z, t \in \mathbb{R} \\
\langle x, y, z, t \rangle = \langle x, y, 0, t \rangle + \langle x, 0, z, t \rangle + \langle 0, y, z, t \rangle \\
- \langle x, 0, 0, t \rangle - \langle 0, y, 0, t \rangle - \langle 0, 0, z, t \rangle + \langle 0, 0, 0, t \rangle,
\]
where we can write \(\langle 0, 0, 0, t \rangle\) as \(\langle t \rangle\).

From now on, we will replace Eq. (3) by Eq. (6).

\[
\langle x, y, z, t \rangle = \langle x, y, 0, t \rangle + \langle x, 0, z, t \rangle + \langle 0, y, z, t \rangle \\
- \langle x, 0, 0, t \rangle - \langle 0, y, 0, t \rangle - \langle 0, 0, z, t \rangle + \langle 0, 0, 0, t \rangle,
\]

where we can write \(\langle 0, 0, 0, t \rangle\) as \(\langle t \rangle\).

Below we are further examples of the creation of new triangles from the triangle \(\langle t \rangle\) (Fig. 5-7).

Because some components of Eq. (6) are positive (i.e. black) and others are negative (i.e. red), for simplicity we omit these colors in Fig. 5-7.

The triangle \(\langle t \rangle\) has vertices \(ABC\), and the triangle \(\langle x, y, z, t \rangle\) has vertices \(A'B'C'\).

The side \(B'C'\) of the triangle \(\langle x, 0, 0, t \rangle\) is created by a moving the side \(BC\) in the direction I or I’ while \(A = A'\) and the sides \(A'B'\) and \(A'C'\) are lying on the lines
containing respectively the sides $AB$ and $AC$ (Fig. 5).
The sides $B'C'$ and $A'C'$ of the triangle $\langle x, y, 0, t \rangle$ are created by a moving respectively the sides $BC$ and $AC$ while the side $A'B'$ is lying on the line containing the side $AB$ (Fig. 6).
The sides created by moving the sides $BC$, $AC$ and $AB$ by values $x$, $y$ and $z$ are blue, yellow and brown respectively.

![Fig. 5: The triangle $\triangle A'B'C' = \langle x, 0, 0, t \rangle$ for different $x$.](image)

![Fig. 6: The triangle $\triangle A'B'C' = \langle x, y, 0, t \rangle$ for different $x$, $y$.](image)
Let us consider Eq. (6) for following numbers.

\[
\langle -1 \rangle = \langle 2, -2, -4, 3 \rangle = \langle 2, -2, 0, 3 \rangle + \langle 2, 0, -4, 3 \rangle + \langle 0, -2, -4, 3 \rangle \\
- \langle 2, 0, 0, 3 \rangle - \langle 0, -2, 0, 3 \rangle - \langle 0, 0, -4, 3 \rangle \\
+ \langle 0, 0, 0, 3 \rangle.
\]

(7)

In Fig. 8 we can see individual components of Eq. (7). The first (black) components are the sum \(\langle 2, -2, 0, 3 \rangle + \langle 2, 0, -4, 3 \rangle + \langle 0, -2, -4, 3 \rangle + \langle 0, 0, 0, 3 \rangle\), the second (red) components are the sum \(-\langle 2, 0, 0, 3 \rangle - \langle 0, -2, 0, 3 \rangle - \langle 0, 0, -4, 3 \rangle\) and the right side of the equation is the result \(\langle 2, -2, -4, 3 \rangle = \langle -1 \rangle\).

Numbers 2 staying on the triangles means an overlapping two triangles.
A simplified diagram of Eq. (8) can be seen in Fig. 9.

From the below equation
\[
\langle -1 \rangle = \langle -1, -1, -1, 2 \rangle = 3\langle 0 \rangle - 3\langle 1 \rangle + \langle 2 \rangle
\]
we can get the way of building of the triangle \(\langle -1 \rangle\). Fig. 10 shows each step of the construction of \(\langle -1 \rangle\).

Similarly, from the relation
\[
\langle -x \rangle = \langle -x, -x, -x, 2x \rangle = 3\langle 0 \rangle - 3\langle x \rangle + \langle 2x \rangle
\]
we can see that for each \(x \in \mathbb{R}, x > 0\) the triangle \(\langle -x \rangle\) is the opened triangle.
2 The equation true in the geometric sense.

Let us consider Eq. (6) for concrete numbers

\[ \langle 4 \rangle = \langle 1, 1, 2, 0 \rangle = \langle 1, 1, 0, 0 \rangle + \langle 1, 0, 2, 0 \rangle + \langle 0, 1, 2, 0 \rangle \]
\[ - \langle 1, 0, 0, 0 \rangle - \langle 0, 1, 0, 0 \rangle - \langle 0, 0, 2, 0 \rangle + \langle 0, 0, 0, 0 \rangle \]
\[ = \langle 2 \rangle + 2 \langle 3 \rangle - 2 \langle 1 \rangle - 2 \langle 2 \rangle + \langle 0 \rangle. \]  

(8)

After reduction we will get

\[ \langle 4 \rangle = \langle 1, 1, 2, 0 \rangle = 2 \langle 3 \rangle - 2 \langle 1 \rangle + \langle 0 \rangle. \]  

(9)

From the arithmetic point of view Eq. (9) is true. But it easy to see that we can not build the triangle \( \langle 4 \rangle \) using only two triangles \( \langle 3 \rangle \), two triangles \( -\langle 1 \rangle \) and point \( \langle 0 \rangle \). We need the triangles \( \langle 2 \rangle \) and \( -\langle 2 \rangle \) too. They are not reducible to the empty set because they do not lie on one another (Fig. 11).

![Fig. 11: A geometric interpretation of Eq. (8) and Eq. (9).](image)

But if we put Eq. (8) in the following equation

\[ \langle 6 \rangle = \langle 2, 2, 2, 0 \rangle = 3 \langle 4 \rangle - 3 \langle 2 \rangle + \langle 0 \rangle, \]  

(10)

we get

\[ \langle 6 \rangle = 2 \langle 4 \rangle + 2 \langle 3 \rangle + \langle 2 \rangle - 4 \langle 2 \rangle - 2 \langle 1 \rangle + 2 \langle 0 \rangle \]
\[ = 2 \langle 4 \rangle + 2 \langle 3 \rangle - 3 \langle 2 \rangle - 2 \langle 1 \rangle + 2 \langle 0 \rangle. \]  

(11)

We could reduce \( \langle 2 \rangle - \langle 2 \rangle \) in Eq. (11) because after reduction we can still build the triangle \( \langle 6 \rangle \) from the remaining elements (Fig. 12).
Let us note that $\langle 2 \rangle$ and $-\langle 2 \rangle$ lie one on the other but $\langle 2 \rangle$ is from equation Eq. (8) and $-\langle 2 \rangle$ is from equation Eq. (10).

Fig. 12: The elements of Eq. (11) which build the triangle $\langle 6 \rangle$.

So it makes sense to introduce the following definitions.

**Definition 2.1.** The equation $\langle x \rangle = \sum_j \alpha_j \langle x_j \rangle$, where $\alpha_j \in \{-1, 1\}$ is true in the geometric sense if we can build the triangle $\langle x \rangle$ from the elements $\alpha_j \langle x_j \rangle$.

**Definition 2.2.** The equation $\langle x \rangle = \sum_j \alpha_j \langle x_j \rangle$, where $\alpha_j \in \{-1, 1\}$ is true in the arithmetic sense if the equations $x_i = \sum_j \alpha_j x_{ij}$, where $\alpha_j \in \{-1, 1\}$, $i = 0, 1, 2$ hold.

**Corollary 2.3.** If the equation is true in the geometric sense, it is also true in the arithmetic sense.

You can see that Eq. (8) is true in the geometric sense, while Eq. (9) only in the arithmetic sense.

Fig. 13 shows that for $x, y, z, t > 0$ Eq. (6) is true in the geometric sense.

Fig. 13: Interpretation of Eq. (6) for $x, y, z, t > 0$. 12
We want to prove that Eq. (6) is true in the geometric sense $\forall x, y, z, t \in \mathbb{R}$.

Let’s transform Eq. (6).

\[
\langle x, y, 0, t \rangle = \langle x, y, z, t \rangle + \langle x, 0, 0, t \rangle + \langle 0, y, 0, t \rangle \\
- \langle x, 0, z, t \rangle - \langle 0, y, z, t \rangle - \langle 0, 0, 0, t \rangle + \langle 0, 0, z, t \rangle.
\] (12)

Eq. (12) does not describe the transition from the triangle $\langle t \rangle$ to the triangle $\langle x+y+t \rangle$ because it contains terms with the variable $z$. Eq. (12) is equivalent to the following Eq. (13) and describes the transition from the triangle $\langle z+t \rangle$ to the triangle $\langle x+y+t \rangle$.

\[
\langle x, y, -z, z+t \rangle = \langle x, y, 0, z+t \rangle + \langle x, 0, -z, z+t \rangle + \langle 0, y, -z, z+t \rangle \\
- \langle x, 0, 0, z+t \rangle - \langle 0, y, 0, z+t \rangle - \langle 0, 0, -z, z+t \rangle \\
+ \langle 0, 0, 0, z+t \rangle.
\] (13)

Just in Eq. (6), the triangle $\langle x, y, 0, t \rangle$ becomes the triangle $\langle x, y, -z, z+t \rangle$.

Similarly Eq. (14) describes the transition from the triangle $\langle y+z+t \rangle$ to the triangle $\langle x+t \rangle$ and Eq. (15) describes the transition from the triangle $\langle x+y+z+t \rangle$ to the triangle $\langle t \rangle$.

\[
\langle x+t \rangle = \langle x, -y, -z, y+z+t \rangle \\
= \langle x, -y, 0, y+z+t \rangle + \langle x, 0, -z, y+z+t \rangle + \langle 0, y, -z, y+z+t \rangle \\
- \langle x, 0, 0, y+z+t \rangle - \langle 0, y, 0, y+z+t \rangle - \langle 0, 0, -z, y+z+t \rangle \\
+ \langle 0, 0, 0, y+z+t \rangle.
\] (14)

\[
\langle t \rangle = \langle -x, -y, -z, x+y+z+t \rangle \\
= \langle -x, -y, 0, x+y+z+t \rangle + \langle -x, 0, -z, x+y+z+t \rangle \\
+ \langle 0, -y, -z, x+y+z+t \rangle - \langle -x, 0, 0, x+y+z+t \rangle \\
- \langle 0, -y, 0, x+y+z+t \rangle - \langle 0, 0, -z, x+y+z+t \rangle \\
+ \langle 0, 0, 0, y+z+t \rangle.
\] (15)

**Theorem 2.4.** $\forall x, y, z, t \in \mathbb{R}$ *Equation (6) is true in the geometric sense.*

**Proof.** If in Eq. (6) one element, for example, $z$ is negative we can replace Eq. (6) with Eq. (13).

If in Eq. (6) two elements $y, z$ are negative we can replace Eq. (6) with Eq. (14).

If Eq. (6) has three elements $x, y, z$ negative we will take Eq. (15).

So it is sufficient to consider the cases $x > 0$, $y > 0$, $z > 0$ and any $t$. We have 10 following cases.
(1) $t > 0$, (Fig. 13)
   In next cases $t < 0$.

(2) $x + t > 0$, $y + t > 0$, $z + t > 0$, (Fig. 14).

(3) $x + t > 0$, $y + t > 0$, $z + t < 0$, (Fig. 15).

(4) $x + t > 0$, $y + t < 0$, $z + t < 0$, $y + z + t > 0$, (Fig. 16).

(5) $x + t > 0$, $y + z + t < 0$, (Fig. 17).

(6) $x + t < 0$, $y + t < 0$, $z + t < 0$, $x + y + t > 0$, $y + z + t > 0$, $x + z + t > 0$, (Fig. 18).

(7) $x + t < 0$, $x + y + t > 0$, $y + z + t < 0$, $x + z + t > 0$, (Fig. 19).

(8) $x + y + t > 0$, $y + z + t < 0$, $x + z + t < 0$, (Fig. 20).

(9) $x + y + t < 0$, $y + z + t < 0$, $x + z + t < 0$, $x + y + z + t > 0$, (Fig. 21).

(10) $x + y + z + t < 0$, (Fig. 22).

The proof is based on reviewing each figure and founding that by using the components of Eq. (6), we always get the triangle $\langle x + y + z + t \rangle$ from the triangle $\langle t \rangle$. In Figs. 14-22 the triangle $\langle x + y + z + t \rangle$ is denoted by $\langle t' \rangle$.

Fig. 14: $x + t > 0$, $y + t > 0$, $z + t > 0$.          Fig. 15: $x + t > 0$, $y + t > 0$, $z + t < 0$.
Fig. 16: \( x + t > 0, \ y + t < 0, \ z + t < 0, \ y + z + t > 0 \).

Fig. 17: \( x + t > 0, \ y + z + t < 0 \).

Fig. 18: \( x + t < 0, \ y + t < 0, \ z + t < 0, x + y + t > 0, y + z + t > 0, x + z + t > 0 \).

Fig. 19: \( x + t < 0, x + y + t > 0, y + z + t < 0, x + z + t > 0 \).

Fig. 20: \( x + y + t > 0, y + z + t < 0, x + z + t < 0 \).

Fig. 21: \( x + y + t < 0, y + z + t < 0, x + z + t < 0, x + y + z + t > 0 \).
One can show that the case (8) can be replaced by the case (3). Indeed, if we replace Eq. (6) by equivalent Eq. (13) then ⟨t⟩ acts as ⟨t′⟩ = ⟨x + y + z + t⟩ and ⟨z + t⟩ = ⟨−x, −y, 0, (x + y + z + t)⟩ acts as ⟨x + y + t⟩. In the case (3) only t′ and x + y + t are positive and in the case (8) only t and l + t are negative. By changing the sign in all components of the case (8) we will receive the case (3).

Similarly we can ignore cases (7), (9) and (10), which are equivalent to cases (4), (2) and (1) respectively. And so we have 6 different cases represented by 6 different figures.

In Figs. 23-28 we can see individual terms of Eq. (6) for cases (1)-(6). The first (black) component is the sum ⟨x + y + t⟩ + ⟨x + z + t⟩ + ⟨y + z + t⟩, the second (red) component is the sum −⟨x + t⟩ − ⟨y + t⟩ − ⟨z + t⟩, the third component is ⟨t⟩ and the right side of the equation is the triangle ⟨x + y + z + t⟩. Numbers 2 or 3 staying on the triangles mean an overlapping two or three triangles.

Fig. 22: x + y + z + t < 0.

Fig. 23: Equation (6) for x > 0, y > 0, z > 0, t > 0.
Fig. 24: Equation (6) for $x + t > 0$, $y + t > 0$, $z + t > 0$, $t < 0$.

Fig. 25: Equation (6) for $x + t > 0$, $y + t > 0$, $z + t < 0$, $t < 0$.

Fig. 26: Equation (6) for $x + t > 0$, $y + t < 0$, $z + t < 0$, $t < 0$. 
Fig. 27: Equation (6) for $x + t > 0$, $y + z + t < 0$, $t < 0$.

Fig. 28: Equation (6) for $x + t < 0$, $y + t < 0$, $z + t < 0$, $x + y + t > 0$, $y + z + t > 0$, $x + z + t > 0$, $t < 0$.

So the result of the equation (6) is always a triangle.

**Remark 2.5.** Let us note that only in the case (6) no triangles of the same sign do not overlap (Fig. 28).

**Corollary 2.6.** $\forall x, y, z, t \in \mathbb{R}$
If $x + y + z = 0$, then the triangles $\langle t \rangle$ and $\langle x, y, z, t \rangle$ from Eq. (6) are congruent.

We can see that for $x + y + z = 0$ the expression $\langle x, y, z, . \rangle$ can be treated as a translation vector.

**Theorem 2.7.** For every two triangles $ABC$ and $A'B'C'$ with respective parallel sides there exist numbers $x, y, z, t \in \mathbb{R}$ such that
$\triangle ABC = \langle t \rangle$, $\triangle A'B'C' = \langle x, y, z, t \rangle$.

**Proof.** The proof follows from the geometric construction of Eq. (6).

18
Remark 2.8. To get the triangle \( \langle x, y, z, t \rangle \) from the triangle \( \langle 1 \rangle \) we can successively build triangles \( \langle 0, 0, 0, t \rangle, \langle 0, y, 0, (x + t) \rangle, \langle 0, 0, z, (x + y + t) \rangle \).

Let us note that the Eq. (6) is also true in the geometric sense. For \( n = \frac{1}{2} \) Eq. (6) will take the following form.

\[
\langle \frac{1}{2} \rangle = \frac{3}{8} \langle 1 \rangle + \frac{3}{4} \langle 0 \rangle - \frac{1}{8} \langle -1 \rangle.
\]

We will not have to look for an interpretation of the number \( \langle \frac{1}{2} \rangle \) if we find that

\[
\langle 1 \rangle = \frac{3}{8} \langle 2 \rangle + \frac{3}{4} \langle 0 \rangle - \frac{1}{8} \langle -2 \rangle.
\]

Just the number \( \langle \frac{1}{2} \rangle \) is a triangle with a side \( \frac{1}{2} \).

Similarly for each real number \( x \in \mathbb{R} \) the notation

\[
\langle x \rangle = \frac{x(x + 1)}{2} \langle 1 \rangle - (x^2 - 1) \langle 0 \rangle + \frac{x(x - 1)}{2} \langle -1 \rangle \tag{16}
\]

is another form of the triangle \( \langle x \rangle = (x^2, x, 1) \) with a side \( x \).

3 Generalization of the equation (6).

**Theorem 3.1.** \( \forall n \in \mathbb{Z} \ \forall t \in \mathbb{R} \) the following equation is true in the geometric sense.

\[
\langle n + t \rangle = \frac{n(n + 1)}{2} \langle 1 + t \rangle - (n - 1)(n + 1) \langle t \rangle + \frac{n(n - 1)}{2} \langle -1 + t \rangle \tag{17}
\]

**Proof.** First, we will prove the theorem by mathematical induction for \( n > -2 \).

1. The theorem is true for \( n = -1, 0, 1 \).

2. Let us assume it holds for \( n = k - 3, \ n = k - 2, \ n = k - 1 \) for each \( k > 1 \), that is

\[
\langle k - 3 + t \rangle = \frac{(k - 3)(k - 2)}{2} \langle 1 + t \rangle - (k - 4)(k - 2) \langle t \rangle + \frac{(k - 3)(k - 4)}{2} \langle -1 + t \rangle \\
\langle k - 2 + t \rangle = \frac{(k - 2)(k - 1)}{2} \langle 1 + t \rangle - (k - 3)(k - 1) \langle t \rangle + \frac{(k - 2)(k - 3)}{2} \langle -1 + t \rangle \\
\langle k - 1 + t \rangle = \frac{(k - 1)k}{2} \langle 1 + t \rangle - (k - 2)k \langle t \rangle + \frac{(k - 1)(k - 2)}{2} \langle -1 + t \rangle
\]
3. Using Eq. (17) for \( n = k \) we have

\[
\langle k + t \rangle = \langle 1, 1, 1, k - 3 + t \rangle
\]

\[
= 3\langle k - 1 + t \rangle - 3\langle k - 2 + t \rangle + \langle k - 1 + t \rangle. \tag{18}
\]

If \( k - 2 + t > 0 \) then the triangle \( \langle k - 1 + t \rangle \) contains the triangle \( \langle k - 2 + t \rangle \) and we can reduce the components of both triangles, that is

\[
\langle k + t \rangle = 3\langle k - 1 + t \rangle - 3\langle k - 2 + t \rangle + \langle k - 1 + t \rangle
\]

\[
= 3 \left[ \frac{1}{2} (k - 1) \langle 1 + t \rangle - (k - 2)k\langle t \rangle + \frac{1}{2} (k - 1)(k - 2) \langle -1 + t \rangle \right]
\]

\[
- 3 \left[ \frac{1}{2} (k - 2)(k - 1) \langle 1 + t \rangle - (k - 3)(k - 1)\langle t \rangle + \frac{1}{2} (k - 2)(k - 3) \langle -1 + t \rangle \right]
\]

\[
+ \left[ \frac{1}{2} (k - 3)(k - 2) \langle 1 + t \rangle - (k - 4)(k - 2)\langle t \rangle + \frac{1}{2} (k - 3)(k - 4) \langle -1 + t \rangle \right]
\]

\[
= \frac{k(k + 1)}{2} \langle 1 + t \rangle - (k - 1)(k + 1)\langle t \rangle + \frac{k(k - 1)}{2} \langle -1 + t \rangle.
\]

The case \( k - 2 + t < 0 \) requires more detailed treatment. We have to consider two cases.

a) \( k - 2 + t < 0 \) and \( k - 1 + t < 0 \), Fig. 29

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig29.png}
\caption{\( \langle k + t \rangle = 3\langle k - 1 + t \rangle - 3\langle k - 2 + t \rangle + \langle k - 3 + t \rangle \).}
\end{figure}

We have to show that the triangle \(-\langle k - 2 + t \rangle\) from Eq. (18) coincides with the triangle \( \langle k - 2 + t \rangle \), where

\[
\langle k - 1 + t \rangle = \langle 1, 1, 1, k - 4 + t \rangle = 3\langle k - 2 + t \rangle - 3\langle k - 3 + t \rangle + \langle k - 4 + t \rangle. \tag{19}
\]
Fig. 30: $\langle k - 1 + t \rangle = 3\langle k - 2 + t \rangle - 3\langle k - 3 + t \rangle + \langle k - 4 + t \rangle$.

Fig. 31: Reduction of triangles $\langle k - 2 + t \rangle$ and $-\langle k - 2 + t \rangle$.

Figures 30 and 31 show that two components $\langle k - 2 + t \rangle$ of Eq. (19) coincide with two components $-\langle k - 2 + t \rangle$ of equation (18). Since we have three components $\langle k - 1 + t \rangle$ in Eq. (18), all components $-\langle k - 2 + t \rangle$ of equation (18) coincide with $\langle k - 2 + t \rangle$ of three equations (19).

Similarly, we can reduce triangles $\pm \langle k - 3 + t \rangle$, $\pm \langle k - 4 + t \rangle$, $\ldots$, until we go back to an equation

$$3\langle 1 + t \rangle - 3\langle t \rangle + \langle -1 + t \rangle = \langle 2 + t \rangle$$

that is true in the geometric sense.

b) $k - 2 + t < 0$ and $k - 1 + t > 0$. For example, for $k = 4$, $t = -\frac{5}{2}$. Fig. 32
Fig. 32: \( \langle k + t \rangle = 3 \langle k - 1 + t \rangle - 3 \langle k - 2 + t \rangle + \langle k - 3 + t \rangle \).

Similarly, as in the case a) we have to show that the triangle \(-\langle k - 2 + t \rangle\) from Eq. (18) coincides with the triangle \(\langle k - 2 + t \rangle\) from Eq. (19).

Fig. 33: \( \langle k - 1 + t \rangle = 3 \langle k - 2 + t \rangle - 3 \langle k - 3 + t \rangle + \langle k - 4 + t \rangle \).

Fig. 34: Reduction of triangles \(\langle k - 2 + t \rangle\) and \(-\langle k - 2 + t \rangle\).

Figures 33 and 34 show that two components \(\langle k - 2 + t \rangle\) of Eq. (19) coincide with two components \(-\langle k - 2 + t \rangle\) of equation (18). Further reasoning as in the case a).
Similarly we can proof this theorem for \( n < -1 \). \( \square \)

If we replace 1 with any number \( a \in \mathbb{R} \) we get

**Corollary 3.2.** \( \forall n \in \mathbb{Z} \forall a, t \in \mathbb{R} \)

the following equation is true in the geometric sense.

\[
\langle na + t \rangle = \langle \underbrace{a + a + \cdots + a}_n \rangle + t + \frac{n(n-1)}{2}(-a + t)
\]

Now we can proof more general theorem.

**Theorem 3.3.** \( \forall n, k \in \mathbb{Z} \forall a, t \in \mathbb{R} \)

the following equation is true in the geometric sense.

\[
\langle na + t \rangle = \frac{(n-k)(n-k+1)}{2} \langle (k+1)a + t \rangle - (n-k-1)(n-k+1) \langle ka + t \rangle + \frac{(n-k)(n-k-1)}{2} \langle (k-1)a + t \rangle.
\]

*Proof. * We have to consider a few cases.

The first two follow from Corollary 3.2.

1. For \( n - k > 2 \), that is \( n > k + 2 \)

\[
\langle na + t \rangle = \langle \underbrace{a + a + \cdots + a}_{n-k} + (ka + t) \rangle
\]

\[
= \frac{(n-k)(n-k+1)}{2} \langle a + (ka + t) \rangle - (n-k-1)(n-k+1) \langle ka + t \rangle + \frac{(n-k)(n-k-1)}{2} \langle -a + (ka + t) \rangle
\]

\[
= \frac{(n-k)(n-k+1)}{2} \langle (k+1)a + t \rangle - (n-k-1)(n-k+1) \langle ka + t \rangle + \frac{(n-k)(n-k-1)}{2} \langle (k-1)a + t \rangle.
\]
2. For $k - n > 2$, that is $n < k - 2$

$$\langle na + t \rangle = \langle -a - a - \cdots - a + (ka + t) \rangle$$

$$= \frac{(k - n)(k - n + 1)}{2} \langle -a + (ka + t) - (k - n - 1)(k - n + 1)\rangle$$

$$+ \frac{(k - n)(k - n - 1)}{2} \langle -(a) + (ka + t) \rangle$$

$$= \frac{(n - k)(n - k + 1)}{2} \langle (k + 1)a + t \rangle - (n - k - 1)(n - k + 1)\langle ka + t \rangle$$

$$+ \frac{(n - k)(n - k - 1)}{2} \langle (k - 1)a + t \rangle.$$ 

The next two follow from Theorem 2.4.

3. For $n = k + 2$ we have

$$\langle (k + 2)a + t \rangle = \langle a + a + a + (k - 1)a + t \rangle$$

$$= 3\langle (k + 1)a + t \rangle - 3\langle ka + t \rangle + \langle (k - 1)a + t \rangle.$$ 

4. For $n = k - 2$ we get

$$\langle (k - 2)a + t \rangle = \langle -a - a - a + (k + 1)a + t \rangle$$

$$= 3\langle (k - 1)a + t \rangle - 3\langle ka + t \rangle + \langle (k + 1)a + t \rangle.$$ 

The last cases for $n = k + 1, k - 1$ are trivial.

An example of Eq. (20) for $k = 3, a = 1, t = 0$.

$$\langle n \rangle = \frac{(n - 3)(n - 2)}{2} \langle 4 \rangle - (n - 4)(n - 2)\langle 3 \rangle + \frac{(n - 4)(n - 3)}{2} \langle 2 \rangle.$$ 

The reader can try to build triangles $\langle 5 \rangle$ and $\langle 1 \rangle$ using triangles $\langle 4 \rangle, \langle 3 \rangle$ and $\langle 2 \rangle$.

Substituting $k - 1$ for $k$, Eq. (21) will take the form

$$\langle na + t \rangle = \frac{(n - k + 1)(n - k + 2)}{2} \langle ka + t \rangle - (n - k)(n - k + 2)\langle (k - 1)a + t \rangle$$

$$+ \frac{(n - k)(n - k + 1)}{2} \langle (k - 2)a + t \rangle.$$ 

(22)
The triangle \( \langle na + t \rangle \) can also be written in the following form.

\[
\langle na + t \rangle = \frac{(n-k+1)(n-k+2)}{2} \langle ka + t \rangle \\
+ \frac{(n-k)(n-k+1)}{2} \left( \langle (k+1)a + t \rangle - 3\langle ka + t \rangle \right) \\
+ \frac{(n-k-1)(n-k)}{2} \left( \langle (k+2)a + t \rangle - 3\langle (k+1)a + t \rangle + 3\langle ka + t \rangle \right) \\
= \frac{(n-k+1)(n-k+2)}{2} \langle ka + t \rangle \\
+ \frac{(n-k)(n-k+1)}{2} \left( \langle (k+1)a + t \rangle - 3\langle ka + t \rangle \right) \\
+ \frac{(n-k-1)(n-k)}{2} \langle (k-1)a + t \rangle.
\]  

(23)

Let us note that

\[
\langle ka + t \rangle + \left( \langle (k+1)a + t \rangle - 3\langle ka + t \rangle \right) + \langle (k-1)a + t \rangle = (2,0,0).
\]

For \( k = a = 1, \ t = 0 \) we have (Fig. 35)

\[
\langle 1 \rangle + \left( \langle 2 \rangle - 3\langle 1 \rangle \right) + \langle 0 \rangle = (2,0,0).
\]

Fig. 35:

The expression

\[
\langle b_{k,a,t} \rangle = \langle (k+1)a + t \rangle - 3\langle ka + t \rangle
\]

not belongs to \( \mathbb{R}_2 \) but it is useful for the demonstration of the geometric interpretation of the triangles.

For this we have to assume that (Fig. 36)

\[
(k+1)a + t - 2(ka + t) \geq 0 \quad \text{and} \quad ka + t \geq 0.
\]
Hence
\[-t/t \geq k \geq -t/a\]
The above assumptions give all components of the Eq. (23) \( \geq 0 \) (at most except the vertices of the figure \( \langle b_{k,a,t} \rangle \)).

Fig. 36:

For \( a = 3, \ t = -1 \) we have \( k = 1 \) and Eq. (23) will take the form (Fig. 37).
\[
\langle n \rangle = \frac{n(n+1)}{2} \langle 2 \rangle + \frac{(n-1)n}{2} \langle 5 \rangle - \frac{3}{2} \langle 2 \rangle + \frac{(n-2)(n-1)}{2} \langle -1 \rangle.  \tag{24}
\]

Fig. 37: Eq. (24) for \( n = 3 \) and \( n = -2 \).

Eqs. (6), (17), (20) and (21) are special cases of the following equation
\[
\forall n, a, b, c \in \mathbb{Z}
\langle n \rangle = \frac{(n-b)(n-c)}{(a-b)(a-c)} \langle a \rangle - \frac{(n-a)(n-c)}{(b-a)(b-c)} \langle b \rangle + \frac{(n-a)(n-b)}{(c-a)(c-b)} \langle c \rangle,  \tag{25}
\]
where coefficients are the Lagrange interpolating polynomials.
Eq. (25) is not true in the geometric sense for all \( a, b, c \in \mathbb{Z} \).
For \( a = 2, \ b = 0, \ c = -1 \) we have
\[
\langle n \rangle = \frac{n(n+1)}{6} \langle 2 \rangle - \frac{(n-2)(n+1)}{2} \langle 0 \rangle + \frac{(n-2)n}{3} \langle -1 \rangle.  \tag{26}
\]
Can be proved that the coefficients of Eq. (26) are integers for \( n \in \{6k, 6k+2, 6k+3, 6k+5\} \), where \( k \in \mathbb{Z} \). But for example an equation

\[
\langle 3 \rangle = 2\langle 2 \rangle - 2\langle 0 \rangle + \langle -1 \rangle.
\]

is not true in the geometric sense.

Probably Eq. (26) is not true in the geometric sense for any \( n \in \mathbb{Z} \).

It is supposed that Eq. (25) is not true in the geometric sense when \( a - b \neq b - c \).

In general, the criterion of whether a given equation is true in the geometric sense is unknown.

If we knew such a criterion, we could prove the theorem on dissection of any triangle into at least 15 similar triangles of different sides.

### 4 Dissection of triangles into triangles

It is known \([2, 3]\) that a square can be dissected into at least 21 squares of different sides. It is also known \([4]-[6]\) that a triangle can be dissected into at least 15 similar triangles of different sides. At the same, it is considered that the triangles, one of which is a mirror image of the other are different.

To describe the dissection of triangle, we will use the simplified version of the set \( \mathbb{R}_2 \). Let us take the set \( \mathbb{R}_{02} = \{ \pm \langle x \rangle = \pm (x^2, x); x \in \mathbb{R} \} \) with the multiplication

\[
(x_1^2, x_1) \cdot (x_2^2, x_2) = (x_1x_2)^2, (x_1, x_2)
\]

and addition \((3)\). The elements of \( \mathbb{R}_{02} \) are interpreted in the same way as the elements of set \( \mathbb{R}_2 \), as similar triangles. But elements \((x^2, x)\) describes triangles without an edge.

In Fig. 38 we have one of two known possible optimum dissection of the triangle.
This dissection can be written by using 7 times Eq. (6).

\[ \langle 39 \rangle = \langle 19, 12, 20, -12 \rangle \]
\[ = \langle 19, 12, 0, -12 \rangle + \langle 0, 12, 20, 12 \rangle + \langle 19, 0, 20, -12 \rangle \]
\[ - \langle 19, 0, 0, -12 \rangle - \langle 0, 12, 0, -12 \rangle - \langle 0, 0, 20, -12 \rangle + \langle 0, 0, 0, -12 \rangle \]
\[ = \langle 19 \rangle + \langle 20 \rangle + \langle 27 \rangle - \langle 7 \rangle_1 - \langle 8 \rangle_2 + \langle -12 \rangle, \tag{27} \]

where

\[ \langle 27 \rangle = \langle 11, 16, 11, -11 \rangle = \langle 16_1 \rangle + \langle 11 \rangle + \langle 16_2 \rangle - \langle 5 \rangle_3 + \langle -11 \rangle, \tag{28} \]

where

\[ \langle 16_1 \rangle = \langle 7, 7, 9, -7 \rangle = \langle 7 \rangle_1 + \langle 9 \rangle_1 + \langle 9 \rangle_2 - \langle 2 \rangle_4 + \langle -7 \rangle, \tag{29} \]
where
\[ \langle 9_2 \rangle = \langle 2, 7, 2, -2 \rangle = \langle 7_1 \rangle + \langle 2\rangle + \langle 7_2 \rangle - \langle 5\rangle + \langle -2 \rangle, \] (30)

where
\[ \langle 7_2 \rangle = \langle 5, 5, 2, -5 \rangle = \langle 5\rangle + \langle 2\rangle + \langle 2\rangle - \langle -3\rangle + \langle -5 \rangle, \] (31)

where
\[ \langle 16_2 \rangle = \langle 8, 8, 8, -8 \rangle = \langle 8_1 \rangle + \langle 8_2 \rangle + \langle 8\rangle + \langle -8 \rangle, \] (32)

where
\[ \langle 8_1 \rangle = \langle 3, 5, 3, -3 \rangle = \langle 5\rangle + \langle 5\rangle + \langle 3\rangle - \langle 2\rangle + \langle -3\rangle. \] (33)

It should be noted that the triangles in order to be reduced have to be of different signs and lie one on the other. The strikethrough triangle \(-\langle 7\rangle\) in Eq. (27) means that it reduces with the strikethrough triangle \(\langle 7\rangle\) in Eq. (29). The other strikethrough triangles mean the same.

We can see Eqs. (27)-(33) on Fig. [39].

The blue triangle \(-\langle 7\rangle\) in the black triangle \(\langle 39\rangle\) means that it reduces with the black triangle \(\langle 7\rangle\) in the blue triangle \(\langle 16\rangle\). Similarly reduce other triangles.

The gray number \(\langle 27 \rangle\) means that the triangle \(\langle 27 \rangle\) will be replaced with a distribution into smaller triangles \(\langle 27 \rangle = \langle 11, 16, 11, -11 \rangle\). It is similar with the other gray numbers.
Fig. 39: Scheme of a reduction of triangles
5 Geometric series in \( \mathbb{R}_2 \).

Let us fill the triangle \( \langle 1 \rangle \) with the triangles \( \langle -\frac{1}{2^n} \rangle \) (Fig. 40).

![Fig. 40:](image)

Because the triangles \( \langle -\frac{1}{2^n} \rangle \) are open let us add vertices of these triangles and the vertices of the triangle \( \langle 1 \rangle \) (Fig. 41).

![Fig. 41:](image)

Now let’s sum up the triangles \( \langle -\frac{1}{2^n} \rangle \) and added vertices.

We must find the sum

\[
\langle -\frac{1}{2} \rangle + 3\langle -\frac{1}{2^2} \rangle + 3^2\langle -\frac{1}{2^3} \rangle + \cdots
\]

Each triangle \( \langle -\frac{1}{2^n} \rangle = \frac{1}{2^n}A_2 - \frac{1}{2^n}A_1 + 1A_0 \).

The sum of the coefficients of \( A_2 \) is equal to

\[
\frac{1}{4} + \frac{3}{4^2} + \frac{3^2}{4^3} + \cdots = \frac{1}{1 - \frac{3}{4}} = 1.
\]
The sum of the coefficients of $A_1$ is equal to
\[
-\frac{1}{2} - \frac{3}{2^2} - \frac{3^2}{2^3} + \cdots = \lim_{n \to \infty} \left( -\frac{1}{2} \right) \cdot \frac{1 - \left(\frac{3}{2}\right)^n}{1 - \frac{3}{2}}. \tag{34}
\]

If we assume that $\lim_{n \to \infty} \left(\frac{3}{2}\right)^n = 0$, then the sum in Eq. (34) equals to $1$.

We must note that the parallel sides of the triangles $\langle -\frac{1}{2} \rangle$ do not lie on one number line, so they should be summed up according to some other rules. It seems that summing should also take into account the coefficients $A_2$. The sum of the coefficients of $A_0$ is equal to
\[
1 + 3 + 3^2 + \cdots = \lim_{n \to \infty} \frac{1 - 3^n}{1 - 3}. \tag{35}
\]

If we assume that $\lim_{n \to \infty} 3^n = 0$, then the sum in Eq. (35) equals to $-\frac{1}{2}$.

Let’s sum up the added black vertices of the triangles $\langle -\frac{1}{2^n} \rangle$.
\[
3 + 3^2 + 3^3 + \cdots = \lim_{n \to \infty} 3 \cdot \frac{1 - 3^n}{1 - 3}. \tag{36}
\]

If we assume that $\lim_{n \to \infty} 3^n = 0$, then the sum in Eq. (36) equals to $-\frac{3}{2}$.

If we add up the sums received from the Eqs. (35), (36) and three vertices of the triangle $\langle 1 \rangle$, we get $-\frac{1}{2} - \frac{3}{2} + 3 = 1$.

Thus, the sums of the coefficients $A_2$, $A_1$, $A_0$ are equal to $1$.

We could say that it is possible to build the triangle $\langle 1 \rangle$ from the triangles $\langle -\frac{1}{2^n} \rangle$.

### 6 The set $\mathbb{R}_1$.

The set $\mathbb{R}_2$ has its counterpart on the number line. Let us take the ring $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y); x, y \in \mathbb{R}\}$ with addition and multiplication
\[
(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2), \tag{37}
\]
\[
(x_1, y_1) \cdot (x_2, y_2) = (x_1 \cdot x_2, y_1 \cdot y_2). \tag{38}
\]

Let us consider the subset of the ring $\mathbb{R}^2$, the set $\mathbb{R}_1 = \{\pm (x) = \pm (x, 1); x \in \mathbb{R}\}$.

It is closed under multiplication (38) but not under addition (37).

The set $\mathbb{R}_1$ is closed under the following kind of addition
\[
\forall x, y, t \in \mathbb{R} \quad \langle x + y + t \rangle = \langle x + t \rangle + \langle y + t \rangle - \langle t \rangle. \tag{39}
\]
because the equations
\[
\forall x, y, t \in \mathbb{R} \quad \forall i = 1, 0
\]
\[
(x + y + t)^i = (x + y + t)^i + (x + z + t)^i + (y + z + t)^i
\]
\[
- (x + t)^i - (y + t)^i - (z + t)^i + (t)^i
\]
are true.
Later in this article, an element \( \langle x \rangle \) can be replaced with \(-\langle x \rangle \) and vice versa.

We can check that for each \( x \in \mathbb{R} \) the following equation holds.
\[
\forall x \in \mathbb{R} \quad \langle x \rangle = x\langle 1 \rangle - (x - 1)\langle 0 \rangle. \tag{40}
\]
Let us transform Eq. (40).
\[
\langle x \rangle = x\langle 1 \rangle - (x - 1)\langle 0 \rangle = x\left(\langle 1 \rangle - \langle 0 \rangle\right) + 1\langle 0 \rangle.
\]
It is easy to check that the elements \( \langle 1 \rangle - \langle 0 \rangle = A_1, \langle 0 \rangle = A_0 \) are orthogonal. So \( A_1 = (1, 0), \ A_0 = (0, 1) \).

An element \( \langle x \rangle \) for each \( x \in \mathbb{R}, x > 0 \) will be interpreted as a one-dimensional vector and marked with a black color .
An element \(-\langle x \rangle\) will be interpreted as a one-dimensional vector with a direction opposite to the direction of vector \(-\langle x \rangle\) and marked with a red color .
An element \(\langle 0 \rangle\) will be marked as a black point and an element \(-\langle 0 \rangle\) will be marked as a red point.
To see the result of putting the black triangle on the red one, we will denote the empty set of green.
An element \(\langle -x \rangle\) for each \( x \in \mathbb{R}, x > 0 \) will be interpreted as a one-dimensional vector but without endpoints and marked with a black color with green endpoints .
An element \(-\langle -x \rangle\) for each \( x \in \mathbb{R}, x > 0 \) will be interpreted as a one-dimensional vector without endpoints and marked with a red color with green endpoints .

Why the vectors \(\langle -x \rangle\) and \(-\langle -x \rangle\) for \( x > 0 \) are without the endpoints we will see after introducing the geometric interpretation of the operation (39).
We will create a geometric construction of the addition (39).

The sum \( \langle x + t \rangle \) is obtained by placing the tail of the vector \( \langle x \rangle \) at the head of the vector \( \langle t \rangle \) (Fig. 42). The sum \( \langle y + t \rangle \) is obtained by placing the tail of the vector \( \langle y \rangle \) at the head of the vector \( \langle t \rangle \).

However, it should be emphasized that each of the elements \( \langle x + t \rangle \) and \( \langle y + t \rangle \) should be treated as one vector. The division of the vector \( \langle x + t \rangle \) into the two vectors \( \langle t \rangle \) and \( \langle x \rangle \) shown in the Fig. 42 is only to illustrate where the vector \( \langle x + t \rangle \) lies versus the vector \( \langle t \rangle \).

\[
\begin{align*}
\langle t \rangle, & \quad t > 0 \quad \longrightarrow \quad \langle t \rangle, \quad t < 0 \quad \longrightarrow \quad \langle t \rangle, \quad t < 0 \\
\langle x \rangle, & \quad x > 0 \quad \longrightarrow \quad \langle x \rangle, \quad x > 0 \quad \longrightarrow \quad \langle x \rangle, \quad x > 0 \\
\langle x + t \rangle & \quad \longrightarrow \quad \langle x + t \rangle \quad \longrightarrow \quad \langle x + t \rangle
\end{align*}
\]

Fig. 42: The sum \( \langle x + t \rangle \) for different \( t \).

Figure (43) illustrates the geometric interpretation of Eq. (39) for positive real numbers \( x, y, t \).

\[
\begin{align*}
\langle t \rangle & \quad \longrightarrow \quad \langle t \rangle \\
\langle x + t \rangle & \quad \longrightarrow \quad \langle x + t \rangle \\
\langle y + t \rangle & \quad \longrightarrow \quad -\langle t \rangle \\
\langle x + y + t \rangle & \quad \longrightarrow \quad \langle x + y + t \rangle
\end{align*}
\]

Fig. 43: The geometric interpretation of Eq. (39) for positive real numbers \( x, y, t \).

From the below equation

\[
\langle -1 \rangle = \langle -1 - 1 + 1 \rangle = \langle -1 + 0 + 1 \rangle + \langle 0 - 1 + 1 \rangle - \langle 1 \rangle = 2\langle 0 \rangle - \langle 1 \rangle
\]

we can get the way of building of the triangle \( \langle -1 \rangle \). Fig. 44 shows each step of the construction of \( \langle -1 \rangle \).
The definitions that an equation is true in geometric or algebraic sense are the same as for the elements of the set $\mathbb{R}_2$.
Because elements of the set $\mathbb{R}_1$ are vectors so Eq. (39) is true in the geometric sense $\forall x, y, t \in \mathbb{R}$.
According to proof of Theorem 2.4, to show the geometric interpretation of Eq. (39) for any real numbers $x, y, t$ it is enough to show it for positive $x, y$ and any $t$.
We have the following cases.

(1) $t > 0$, (Fig. 43).

In next cases $t < 0$.

(2) $x + t > 0, y + t > 0$, (Fig. 45).

(3) $x + t > 0, y + t < 0$, (Fig. 45).

(4) $x + t < 0, y + t < 0, x + y + t > 0$.

(5) $x + t < 0, y + t < 0, x + y + t < 0$.
But if we replace Eq. (39) by equivalent Eq. (41)

\[ \langle t \rangle = \langle -y + (x + y + t) \rangle + \langle -x + (x + y + t) \rangle - \langle x + y + t \rangle \]  

(41)

then \( \langle t \rangle \) acts as \( \langle t' \rangle = \langle x + y + t \rangle \), \( \langle x + t \rangle = \langle -y + (x + y + t) \rangle \) acts as \( \langle y + t \rangle \) and \( \langle y + t \rangle = \langle -x + (x + y + t) \rangle \) acts as \( \langle x + t \rangle \).

In the case (4) only \( t' \) is positive and in the case (2) only \( t \) is negative. By changing the sign in all components of the case (4) we will receive the case (2).

Similarly, in the case (5) all components are negative and in the case (1) all components are positive. By changing the sign in all components of the case (5) we will receive the case (1).

So we have 3 different cases of adding in the set \( \mathbb{R}_1 \).

It can be proved by mathematical induction that the Eq. (40) follows from Eq. (39) for natural \( n \) (Fig. 46).

\[ \forall n \in \mathbb{N} \quad \langle n \rangle = n\langle 1 \rangle - (n - 1)(\langle 0 \rangle) = n\langle 1 \rangle + (n - 1)(-\langle 0 \rangle). \]  

(42)

Fig. 46: Interpretation of the equation (42).

7 Other geometric interpretation of the set \( \mathbb{R}_1 \).

The set \( \mathbb{R}_1 \) has another richer geometric interpretation.

An element \( \langle 1 \rangle \) can be interpreted as a closed triangle without one vertex and marked with a black color .

An element \( \langle 0 \rangle \) can be interpreted as a triangle .

The Figures 47-49 show the interpretation of \( \langle x \rangle \) for different \( x \).
In Figures 50-55 we can see the components of the Eq. (39). To make it easier to understand the rules of addition, we gave direction to the bottom edge of each element. We can see that the rules of adding are the same as in the section 6.
We must show that the Eq. (39) is true in the geometric sense for all $x, y, t$.

**Theorem 7.1.** $\forall x, y, z, t \in \mathbb{R}$ Equation (39) is true in the geometric sense.

**Proof.** We need to consider three cases from section 6 for positive $x, y$ and any $t$.

(1) $t > 0$, (Fig. 56).

In next cases $t < 0$.

(2) $x + t > 0, y + t > 0$, (Fig. 57).
(3) $x + t > 0, y + t < 0$, (Fig. 58).

Fig. 56: $\langle x + y + t \rangle$, for $x > 0, y > 0, t > 0$.

Fig. 57: $\langle x + y + t \rangle$, for $x + t > 0, y + t > 0, t < 0$.

Fig. 58: $\langle x + y + t \rangle$, for $x + t > 0, y + t < 0, t < 0$.

The cases for $x = 0 \lor y = 0 \lor t = 0$ we leave for consideration to the reader. □

Using Eq. (39) we can check the interpretation of the element $\langle \frac{1}{3} + \frac{1}{3} + 0 \rangle = \langle \frac{2}{3} \rangle$ (Fig. 59). (The Figure is to a scale enlarged twice).
Fig. 59: $\langle \frac{1}{3} \rangle + \langle \frac{1}{3} \rangle + ( - \langle 0 \rangle ) = \langle \frac{2}{3} \rangle$.

The Figure 60 shows that the interpretation of the element $\langle \frac{1}{3} \rangle$ is well defined.

Fig. 60: $\langle \frac{1}{3} \rangle + \langle \frac{2}{3} \rangle + ( - \langle 0 \rangle ) = \langle 1 \rangle$.

Let us assume that $\langle 1 \rangle_1 = \langle 1 \rangle_2 - \langle 0 \rangle_2$, where $\langle x \rangle_1 = \langle x \rangle \in \mathbb{R}_1$, $\langle x \rangle_2 = \langle x \rangle \in \mathbb{R}_2$.

Then

$\forall x \in \mathbb{R} \; \langle x \rangle_1 = \langle x \rangle_2 - \langle x - 1 \rangle_2$.

But only in the geometric sense because

$\langle x \rangle_1 = \langle x, 1 \rangle$ and $\langle x \rangle_2 - \langle x - 1 \rangle = (2x - 1, 1, 0)$.

The above interpretation of the set $\mathbb{R}_1$ is not the only one. We can use, for example the following interpretation.

\[ \triangle \] as the element $\langle 1 \rangle$ and \[ \triangle \] as the element $\langle 0 \rangle$.

Geometric interpretation of the set $\mathbb{R}_1$ in the section 7 shows that the elements of the set $\mathbb{R}_2$ can be interpreted in the form of three-dimensional figures. But that is material for another article.

**References**

[1] Wolfram MathWorld.
[2] A.J.W Duijvestijn, *Simple perfect squared square of lowest order* J. Combin. Theory, Ser. B, 25, pp. 240-243, (1978)

[3] I. Gambini, *A method for cutting squares into distinct squares*, Discrete Applied Mathematics, Volume 98, Issues 1–2, pp. 65-80, (1999)

[4] W. T. Tutte, *Dissections into Equilateral Triangles*, The Mathematical Gardner, edited by David Klarner. Published by Van Nostrand Reinhold (1981).

[5] W. T. Tutte, *The Dissection of Equilateral Triangles into Equilateral Triangles*, Proc. Cambridge Phil. Soc 44, pp. 463-482, (1948)

[6] A. Drápal, C. Hämäläinen, *An enumeration of equilateral triangle dissections*, Discrete Applied Mathematics, Volume 158, Issue 14 (2010)