Rank-based change-point analysis for long-range dependent time series

ANNIKA BETKEN\textsuperscript{1} and MARTIN WENDLER\textsuperscript{2}

\textsuperscript{1}Faculty of Mathematics, Ruhr-Universität Bochum, E-mail: annika.betken@rub.de
\textsuperscript{2}Faculty of Mathematics, Otto-von-Guericke-Universität Magdeburg, E-mail: martin.wendler@ovgu.de

We consider change-point tests based on rank statistics to test for structural changes in long-range dependent observations. Under the hypothesis of stationary time series and under the assumption of a change with decreasing change-point height, the asymptotic distributions of corresponding test statistics are derived. For this, a uniform reduction principle for the sequential empirical process in a two-parameter Skorohod space equipped with a weighted supremum norm is proved. Moreover, we compare the efficiency of rank tests resulting from the consideration of different score functions. Under Gaussianity, the asymptotic relative efficiency of rank-based tests with respect to the CuSum test is 1, irrespective of the score function. Regarding the practical implementation of rank-based change-point tests, we suggest to combine self-normalized rank statistics with subsampling. The theoretical results are accompanied by simulation studies that, in particular, allow for a comparison of rank tests resulting from different score functions. With respect to the finite sample performance of rank-based change-point tests, the Van der Waerden rank test proves to be favorable in a broad range of situations. Finally, we analyze data sets from economy, hydrology, and network traffic monitoring in view of structural changes and compare our results to previous analysis of the data.

MSC 2010 subject classifications: Primary 62G10, 62G30; secondary 62G20 62G35.
Keywords: rank statistic, change-point, long memory, self-normalization, subsampling, empirical process, asymptotic relative efficiency.

Contents

1 Introduction ......................................................... 2
2 Preliminaries ....................................................... 4
   2.1 Weighted Skorohod space .................................. 4
   2.2 Long-range dependence .................................... 6
3 Main Results ....................................................... 8
   3.1 Asymptotic behavior under stationarity .................. 8
   3.2 Asymptotic behavior under local alternatives .......... 10
   3.3 Asymptotic Relative Efficiency for level shifts ....... 13

\textsuperscript{*}Research supported by Collaborative Research Center SFB 823 Statistical modelling of nonlinear dynamic processes.
1. Introduction

Let $X_1, \ldots, X_n$ be random variables and let $F_i, i = 1, \ldots, n$, denote the marginal distribution functions of $X_i, i = 1, \ldots, n$. If $F_k \neq F_{k+1}$ for some $k \in \{1, \ldots, n-1\}$, we say that there is a change-point in $k$ and we refer to $k$ as the time of change. The testing problem

$$H : F_1 = F_2 = \cdots = F_n$$

against

$$A : F_1 = F_2 = \cdots = F_k \neq F_{k+1} = F_{k+2} = \cdots = F_n$$

for some $k \in \{1, \ldots, n-1\}$

is called change-point problem.

The most frequently considered change-point problems relate to the identification of shifts in the mean value of time series. Writing

$$X_n = \mu_n + Y_n$$

for a sequence of unknown constants $\mu_n, n \in \mathbb{N}$, and a mean-zero stochastic process $Y_n, n \in \mathbb{N}$, a change-point in the location of the time series $X_n, n \in \mathbb{N}$, is characterized by the sequence $\mu_n, n \in \mathbb{N}$, satisfying

$$\mu_i = \begin{cases} 
\mu & \text{for } i = 1, \ldots, k, \\
\mu + h_n & \text{for } i = k + 1, \ldots, n
\end{cases}$$

for some $k = \lfloor n \tau \rfloor, 0 < \tau < 1$, denoting the time of change, and a deterministic sequence of shift heights $h_n, n \in \mathbb{N}$, with $h_n \neq 0$ for all $n \in \mathbb{N}$. If the sequence of shift-heights converges to 0, i.e., $\lim_{n \to \infty} h_n = 0$, we refer to local changes and local alternatives, respectively.
Motivated by change-point tests for the change-in-location problem based on the consideration of the partial sums

\[ \sum_{i=1}^{k} (X_i - \bar{X}_n), \quad \bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i, \]

i.e., CuSum-tests, we consider a class of change-point tests based on rank statistics

\[ S_{k,n}(a) := \sum_{i=1}^{k} (a(R_i) - \bar{a}_n), \quad \bar{a}_n = \frac{1}{n} \sum_{i=1}^{n} a(i), \]

where \( a = (a(1), \ldots, a(n)) \) is a vector of scores, and \( R_i = \sum_{j=1}^{n} 1_{\{X_j \leq X_i\}} \) denotes the rank of \( X_i \) among \( X_1, \ldots, X_n \).

Rank statistics for change-point detection have been studied for over 50 years, starting with [13], [53] and [40]. Given independent data-generating random variables with a change in location, the statistical properties of rank-based statistics have been investigated by [49], [27] and [28].

Under the assumption that the time of change is unknown under the alternative, it seems natural to consider the statistics \( |S_{k,n}(a)| \) for every possible time of change \( k \) and to decide in favor of the alternative hypothesis \( A \) if the maximum exceeds a predefined critical value. As a result, change-point tests base test decisions on the values of the statistics

\[ S_n(a) := \max_{1 \leq k < n} |S_{k,n}(a)|. \quad (1) \]

Choosing \( a(i) = i \), a short calculation yields

\[ \sum_{i=1}^{k} (a(R_i) - \bar{a}_n) = \sum_{i=1}^{k} \sum_{j=k+1}^{n} \left( 1_{\{X_j \leq X_i\}} - \frac{1}{2} \right), \]

i.e., this score results in the Wilcoxon-two-sample statistic. [20], [45] and [65] study Wilcoxon-type change-point statistics under the assumption of independent data-generating variables. For long-range dependent time series, [64], [21] and [22] characterize the asymptotic behavior of change-point tests that are based on the two-sample Wilcoxon statistic. A self-normalized version of the Wilcoxon-type change-point test is proposed by [7].

To the best of our knowledge, for dependent data there do not yet exist results for rank-based change-point tests stemming from general score functions. The aim of this paper is to study the (asymptotic and finite sample) behavior of general rank statistics under long-range dependence. This allows for an application of other score functions, including the Median test (choosing \( a(i) = \text{sign}(i - \frac{n+1}{2}) \)) and the Van der Waerden test (choosing \( a(i) = \phi^{-1}(\frac{i}{n+1}) \)). We will use weighted empirical processes to determine the limit distribution of rank statistics following an approach in [50]. For independent data, this techniques are considered in the context of change-point detection by [59].
Section 2 introduces the mathematical framework of weighted Skorohod spaces and subordinated Gaussian processes. The main results on the asymptotic behavior of rank statistics under the hypothesis and under local alternatives follow in Section 3. We discuss self-normalization and subsampling as means of a practical implementation of change-point tests in Section 4. Section 5 contains simulation studies that give insight into the finite sample behavior of rank-based change-point tests. Real life data sets are discussed in Section 6. The proofs of our theoretical results and additional simulation results can be found in the appendix.

2. Preliminaries

Given dependent data, the exact distribution of the statistic \( S_n(a) \) is unknown and, in general, hard to obtain. For this reason, test decisions are based on a comparison of the value of the test statistic with quantiles of its limit distribution. For the determination of the asymptotic distribution of the statistic \( S_n(a) \), it is useful to note that for any function \( h : (0,1) \rightarrow \mathbb{R} \) satisfying \( h \left( \frac{i}{n+1} \right) = a(i) \) we have

\[
S_{k,n}(a) = \sum_{i=1}^{k} a(R_i) - \frac{k}{n} \sum_{i=1}^{n} a(i) = \sum_{i=1}^{k} \left( h \left( \frac{1}{n+1} R_i \right) - \frac{1}{n} \sum_{i=1}^{n} h \left( \frac{1}{n+1} R_i \right) \right) = \int_{0}^{1} h(x) d \left( \hat{G}_k(x) - \frac{k}{n} \hat{G}_n(x) \right),
\]

where \( \hat{G}_k(x) := \sum_{i=1}^{k} 1 \left\{ \frac{i}{n+1} R_i \leq x \right\} \) is the empirical distribution function of the (rescaled) ranks. Under an additional assumption, introduced in Section 2.1, we can use integration by parts (see Lemma B.1 in [11]) to further simplify the above representation, so that

\[
S_{k,n}(a) = \int_{0}^{1} h(x) d \left( \hat{G}_k(x) - \frac{k}{n} \hat{G}_n(x) \right) = - \int_{0}^{1} \left( \hat{G}_k(x-) - \frac{k}{n} \hat{G}_n(x-) \right) dh(x).
\]

2.1. Weighted Skorohod space

In order to derive the asymptotic distribution of the test statistic \( S_n(a) \) defined by (1), we consider the process

\[
\left( \hat{G}_k(x-) - \frac{k}{n} \hat{G}_n(x-) \right), \ x \in [0,1],
\]

as an element of the space \( D[0,1] \), i.e., the set of all functions on \([0,1]\) which are right-continuous and have left limits, and the statistic \( S_{k,n}(a) \) as the image of this process.
under the mapping $g : D[0, 1] \to \mathbb{R}$, $f \mapsto \int_0^1 f(x)dh(x)$. It is important to note that this function is not necessarily continuous with respect to the supremum norm on $D[0, 1]$. In particular, the function $g$ is unbounded for $h = \Phi^{-1}$, i.e., when considering the Van der Waerden test statistic, and, as a linear functional, consequently nowhere continuous. As a result, we must not apply the continuous mapping theorem without further discussion. For this reason, we introduce the weighted supremum norm $\| \cdot \|_\lambda$ on $D[0, 1]$, defined by

$$
\|f\|_\lambda := \sup_{x \in [0, 1]} |(\min\{x, 1-x\})^{-\lambda} f(x)|,
$$

and we consider the space $D_\lambda[0, 1] := \{f \in D[0, 1] : \|f\|_\lambda < \infty\}$.

Note that

$$
\left| \int_0^1 f(x)dh(x) - \int_0^1 g(x)dh(x) \right| \leq \int_0^1 |f(x) - g(x)|d\tilde{h}(x)
\leq \|f - g\|_\lambda \int_0^1 (\min\{x, 1-x\})^\lambda d\tilde{h}(x),
$$

where we define the function $\tilde{h} : [0, 1] \to \mathbb{R}$ by

$$
\tilde{h}(x) := \begin{cases} 
V^x_{1/2}(h) & \text{for } x \geq 1/2 \\
-V^x_{1/2}(h) & \text{for } x < 1/2
\end{cases} \quad (2)
$$

with $V^b_a(f)$ denoting the total variation of a function $f$ over the interval $[a, b]$. For this reason, we impose the following assumption:

**Assumption 1.** We assume that for $\tilde{h} : [0, 1] \to \mathbb{R}$ defined by (2) and some $\lambda \in (0, \frac{1}{3})$

$$
\int_0^1 (\min\{x, 1-x\})^\lambda d\tilde{h}(x) < \infty.
$$

Given Assumption 1, the mapping $f \mapsto \int_0^1 f(x)dh(x)$ is continuous. Moreover, the process $\hat{G}_k(x) - \frac{k}{n} \hat{G}_n(x)$, $x \in [0, 1]$, takes values in $D_\lambda[0, 1]$ almost surely. Due to continuity of $g$ with respect to $\| \cdot \|_\lambda$, convergence in distribution will follow from the continuous mapping theorem and (after rescaling) convergence of $\hat{G}_k(x) - \frac{k}{n} \hat{G}_n(x)$, $x \in [0, 1]$, in $D_\lambda[0, 1]$.

The following example shows that the Van der Waerden score function satisfies Assumption 1:

**Example.** Assumption 1 holds for the score function $\Phi^{-1}$ and for any $\lambda > 0$, since $\Phi^{-1}$
is of bounded variation on compact intervals and since
\[
\int_0^1 (\min\{x, 1-x\})^\lambda d\bar{h}(x)
\]
\[
= \int_0^{1/2} x^\lambda d\left(\Phi^{-1}(x) - \Phi^{-1}\left(\frac{1}{2}\right)\right) + \int_{1/2}^1 (1-x)^\lambda d\left(\Phi^{-1}(x) - \Phi^{-1}\left(\frac{1}{2}\right)\right)
\]
\[
= \int_0^{1/2} x^\lambda d\Phi^{-1}(x) + \int_{1/2}^1 (1-x)^\lambda d\Phi^{-1}(x)
\]
\[
= \int_{-\infty}^\infty \Phi(x)^\lambda dx + \int_0^{\infty} (1-\Phi(x))\lambda dx = \int_{-\infty}^0 \Phi(x)^\lambda dx + \int_0^{\infty} \Phi(-x)^\lambda dx
\]
\[
= 2 \int_{-\infty}^0 (\Phi(x))^\lambda dx < \infty.
\]

2.2. Long-range dependence

In time series analysis, the rate of decay of the autocovariance function is crucial to the characterization of a statistic’s limit distribution. A relatively slow decay of the autocovariances characterizes long-range dependent time series, while a relatively fast decay characterizes short-range dependent processes; see [47], p. 17. We will focus on the consideration of long-range dependent subordinated Gaussian time series, i.e., on random observations generated by transformations of Gaussian processes:

**Model.** Let \( Y_n = G(\xi_n) \), where \( G : \mathbb{R} \rightarrow \mathbb{R} \) is a measurable function and let \( \xi_n, n \in \mathbb{N} \), be a stationary, long-range dependent Gaussian time series with long-range dependence (LRD) parameter \( D \), i.e, \( E \xi_1 = 0 \), \( \text{Var} \xi_1 = 1 \), and

\[
\gamma(k) := \text{Cov}(\xi_1, \xi_{k+1}) \sim k^{-D} L(k), \quad \text{as } k \to \infty,
\]

for some \( D \in (0, 1) \) and a slowly-varying function \( L \).

**Remark 2.1.** For any particular distribution function \( F \), an appropriate choice of the transformation \( G \) yields subordinated Gaussian processes with marginal distribution \( F \). Moreover, there exist algorithms for generating Gaussian processes that, after suitable transformation, yield subordinated Gaussian processes with marginal distribution \( F \) and a predefined covariance structure; see [47]. As a result, subordinated Gaussian processes provide a flexible model for long-range dependent time series.

A very useful tool for studying subordinated Gaussian processes are Hermite polynomials. For \( n \geq 0 \), the **Hermite polynomial** of order \( n \) is defined by

\[
H_n(x) := (-1)^n e^{1/2 x^2} \frac{d^n}{dx^n} e^{-1/2 x^2}, \quad x \in \mathbb{R}.
\]
For any function $G$ with $E[G^2(\xi_1)] < \infty$, the $r$-th Hermite-coefficient is defined by

$$J_r(G) := E[G(\xi_1)H_r(\xi_1)].$$

(3)

Every such $G$ has an expansion in Hermite polynomials, i.e., we have

$$\lim_{n \to \infty} E\left[\left( G(\xi_1) - \sum_{r=0}^{n} \frac{J_r(G)}{r!} H_r(\xi_1) \right)^2 \right] = 0.$$

Given the Hermite expansion, it is possible to characterize the dependence structure of subordinated Gaussian time series $G(\xi_n), n \in \mathbb{N}$: The behavior of the autocorrelations of the transformed process is completely determined by the dependence structure of the underlying process. In fact, it holds that

$$\text{Cov}(G(\xi_1), G(\xi_{k+1})) = \sum_{r=1}^{\infty} \frac{J_r^2(G)}{r!} (\gamma(k))^r.$$

Under the assumption that, as $k$ tends to $\infty$, $\gamma(k)$ converges to 0 with a certain rate, the asymptotically dominating term in this series is the summand corresponding to the smallest integer $r$ for which the Hermite coefficient $J_r(G)$ is non-zero. This index, which decisively depends on $G$, is called Hermite rank.

As, in the following, we will study empirical processes, we do not only consider a single transformation $G$, but the class of transformations $1_{\{G(\xi_1) \leq x\}} - F(x), x \in \mathbb{R}$. For this, we need to define the Hermite rank of this class.

**Definition 2.1.** For $G : \mathbb{R} \to \mathbb{R}$, let $J_r(G; x)$ denote the $r$-th Hermite coefficient in the Hermite expansion of $1_{\{G(\xi_1) \leq x\}} - F(x), x \in \mathbb{R}$, i.e.,

$$J_r(G; x) := E\left(1_{\{G(\xi_1) \leq x\}}H_r(\xi_1)\right),$$

and let $r$ denote the Hermite rank of the class of functions $1_{\{G(\xi_1) \leq x\}} - F(x), x \in \mathbb{R}$, defined by

$$r := \min_{x \in \mathbb{R}} r(x), \quad r(x) := \min \{ q \geq 1 : J_q(G; x) \neq 0 \}.$$

An appropriate scaling for partial sums of a subordinated Gaussian sequence $Y_n = G(\xi_n), n \in \mathbb{N}$, depends on the Hermite rank $r$ of $G$ and the long-range dependence parameter $D$ of $\xi_n, n \in \mathbb{N}$. More precisely, a corresponding scaling sequence $d_{n,r}, n \in \mathbb{N}$, is defined by

$$d^2_{n,r} := \text{Var}\left(\sum_{i=1}^{n} H_r(\xi_i)\right).$$

(4)

Given the previous definitions and notations, we are now in a position to formulate a general assumption on the data-generating process needed for our theoretical results in the following section:
Assumption 2. Let $Y_n = G(\xi_n)$, where $\xi_n, n \in \mathbb{N}$, is a stationary Gaussian time series with mean 0, variance 1, and autocovariance function $\gamma$ satisfying

$$\gamma(k) := \text{Cov}(\xi_1, \xi_{k+1}) \sim k^{-D} L(k),$$

as $k \to \infty$. We assume that $Dr < 1$, where $r$ denotes the Hermite rank of the class of functions $1_{\{G(\xi) \leq x\}} - F(x), x \in \mathbb{R}$. Moreover, we assume that the marginal distribution function $F$ of $Y_n, n \in \mathbb{N}$, is continuous.

Remark 2.2. Without loss of generality, we may assume that $F(x) = x$, because by continuity of $F$, the generalized inverse $F^{-}$ is strictly increasing, $F(X_i)$ is uniformly distributed on $[0,1]$ and rank statistics are therefore not affected by a corresponding transformation.

3. Main Results

Recall that

$$S_{k,n}(a) = -\int_0^1 \left( \hat{G}_k(x-) - \frac{k}{n} \hat{G}_n(x-) \right) dh(x),$$

where $\hat{G}_k(x) := \sum_{i=1}^k 1\{\frac{i}{n} \leq R_i \leq x\}$ with $R_i = \sum_{j=1}^n 1\{X_j \leq X_i\}$ denoting the rank of $X_i$ among observations $X_1, \ldots, X_n$. Given the parametrization

$$S_n(a) = \max_{1 \leq k < n} |S_{k,n}(a)| = \sup_{t \in [0, 1]} \left| \int_0^1 \left( \hat{G}_{\lfloor nt \rfloor}(x-) - \frac{|nt|}{n} \hat{G}_n(x-) \right) dh(x) \right|,$$

the asymptotic distribution of $S_n(a)$ can be derived from an application of the continuous mapping theorem and a limit theorem for the two-parameter process

$$\hat{G}_{\lfloor nt \rfloor}(x-) - \frac{|nt|}{n} \hat{G}_n(x-), \ t \in [0, 1], \ x \in [0, 1].$$

For proofs of corresponding limit theorems, we initially derive reduction principles for the sequential empirical process $F_{\lfloor nt \rfloor}(x) - x, t \in [0, 1], x \in [0, 1]$, where $F_n$ refers to the empirical distribution function of $X_1, \ldots, X_n$, i.e.,

$$F_n(x) := \frac{1}{n} \sum_{i=1}^n 1\{X_i \leq x\}.$$

3.1. Asymptotic behavior under stationarity

The following proposition can be considered as a reduction principle for the empirical process $F_{\lfloor nt \rfloor}(x) - x, t \in [0, 1], x \in [0, 1]$, with respect to the weighted supremum norm and under the assumption of a stationary data-generating process. It makes way for establishing a reduction principle for the two-parameter empirical process of the ranks under the hypothesis of no change; see Theorem 3.1.
Proposition 3.1. Let $X_n = G(\xi_n)$, $n \in \mathbb{N}$, be a subordinated Gaussian sequence satisfying Assumption 2 with marginal distribution $F(x) = x$, $x \in [0,1]$. Moreover, let $d_{n,r}$, $n \in \mathbb{N}$, be the deterministic sequence defined by (4) with $r$ denoting the Hermite rank of the class of functions $1_{\{G(\xi) \leq x\}} - x$, $x \in [0,1]$. Then, there exists a $\vartheta > 0$ such that, as $n \to \infty$, 

$$
\sup_{t \in [0,1], x \in [0,1]} d_{n,r}^{-1}(\min\{x, 1 - x\})^{-\lambda} \left| \frac{|nt|}{n} \hat{G}_n(x) - \frac{|nt|}{n} \hat{G}_n(x^--) \right|
\leq \frac{1}{r!} J_r(F^{-}(x)) \sum_{j=1}^{\lfloor nt \rfloor} H_r(\xi_j)
= O_P(n^{-\vartheta}). \tag{6}
$$

Remark 3.1. Proposition 3.1 is closely related to Theorem 2 in [15] that establishes a reduction principle for the sequential empirical process with respect to another class of weighted norms.

On the basis of Proposition 3.1, we derive a reduction principle for the two-parameter empirical process of the ranks, i.e., for 

$$
\hat{G}_{\lfloor nt \rfloor}(x^--) - \frac{\lfloor nt \rfloor}{n} \hat{G}_n(x^--), \ t \in [0,1], \ x \in [0,1],
$$

with $\hat{G}_k(x) := \sum_{i=1}^{k} \left\{ \frac{i}{n} \cdot R_i \leq x \right\}$.

Theorem 3.1. Let $X_n = G(\xi_n)$, $n \in \mathbb{N}$, be a subordinated Gaussian sequence satisfying Assumption 2 with marginal distribution $F(x) = x$, $x \in [0,1]$. Moreover, let $d_{n,r}$, $n \in \mathbb{N}$, be the deterministic sequence defined by (4) with $r$ denoting the Hermite rank of the class of functions $1_{\{G(\xi) \leq x\}} - x$, $x \in [0,1]$, and consider $\vartheta > 0$ such that (6) holds. For any $\lambda < 1/3$ such that $n^\lambda = o\left(d_{n,r}^{-1}\right)$, $n^{2\lambda} d_{n,r} = o\left(n\right)$ and $d_{n,r}^{\lambda} = o\left(n^{\vartheta}\right)$, we have 

$$
\sup_{t \in [0,1], x \in [0,1]} d_{n,r}^{-1}(\min\{x, 1 - x\})^{-\lambda} \left| \hat{G}_{\lfloor nt \rfloor}(x^--) - \frac{\lfloor nt \rfloor}{n} \hat{G}_n(x^--) \right|
= O_P(1).
$$

According to Theorem 3.1 and (5), it suffices to know the limit of the sequential partial sum process $\sum_{i=1}^{\lfloor nt \rfloor} H_r(\xi_i) \in D[0,1]$, in order to derive the asymptotic distribution of the statistics $S_n(\alpha)$ under the hypothesis of stationarity. In fact, it follows by Theorem 5.6 in [60] that 

$$
\frac{1}{d_{n,r}} \sum_{i=1}^{\lfloor nt \rfloor} H_r(\xi_i) \xrightarrow{D} Z_{r,H}(t), \ t \in [0,1],
$$
where \( Z_{r,H} \) is an \( r \)-th order Hermite process, \( H = 1 - \frac{D^2}{2} \), and \( \xrightarrow{\mathcal{D}} \) denotes convergence in distribution with respect to the \( \sigma \)-field generated by the open balls in \( D[0,1] \), equipped with the supremum norm. As a result, using the representation (5) and applying the continuous mapping theorem yields the asymptotic distribution of the test statistic \( S_n(a) \):

**Corollary 3.1.** Let the assumptions of Theorem 3.1 hold and let \( h : (0,1) \rightarrow \mathbb{R} \) satisfy Assumption 1. Then, we have

\[
\mathbb{D}^{-1} S_n(a) \xrightarrow{\mathcal{D}} \sup_{t \in [0,1]} |Z_{r,H}(t) - tZ_{r,H}(1)| \int_0^1 J_r(f^-(x)) \, dh(x).
\]

In practical applications, the sequence \( d_{n,r} \), the parameters \( r \), \( H \), and the value of the integral on the right-hand side are typically unknown. For this reason, it is difficult to use Corollary 3.1 directly to obtain critical values. In Section 4, we will discuss nonparametric methods to derive critical values.

### 3.2. Asymptotic behavior under local alternatives

In the following, we assume that the considered observations are generated by a triangular array \( X_{n,i}, 1 \leq i \leq n, n \in \mathbb{N} \), with

\[
X_{n,i} = \begin{cases} Y_i & \text{if } i \leq \lfloor n\tau \rfloor, \\ Y_i + h_n & \text{if } i > \lfloor n\tau \rfloor, \end{cases}
\]  

(7)

where \( 0 < \tau < 1 \), \( h_n, n \in \mathbb{N} \), is a non-negative deterministic sequence and \( Y_n = G(\xi_n), n \in \mathbb{N} \), is a subordinated Gaussian sequence according to Model 2 with continuous marginal distribution \( F \) and density \( f \). For convergence of the test statistic \( S_n(a) \) to a non-degenerate limit, we have to assume that \( h_n \to 0 \) (as \( n \to \infty \)) with a certain rate that will be specified later.

In analogy to the asymptotic results in Section 3.1 under the assumption of stationary time series, we first establish a reduction principle for the sequential empirical process with respect to the weighted supremum norm under the assumption of local alternatives:

**Proposition 3.2.** Let \( X_{n,i}, 1 \leq i \leq n, n \in \mathbb{N} \), be a triangular array according to (7) with \( h_n = cn^{-1}d_{n,r} \) for some constant \( c > 0 \) and with \( d_{n,r} \) defined by (4), where \( r \) is the Hermite rank of the class of functions \( 1_{\{G(\xi) \leq x\}} - F(x), x \in [0,1] \). Assume that \( F \) is strictly monotone,

\[
\sup_{x \in [0,1]} (\min\{x, 1-x\})^{-2\lambda} h^{-1} |h^{-1} (x - F(F^-(x) - h)) - f(F^-(x))| = O(h^\rho),
\]

as \( h \to 0 \), for some \( \rho \), \( 0 < \rho < \min\{1, (1 - 2\lambda - \vartheta)^{-1}\vartheta\} \), with \( \vartheta \) and \( \lambda \) as in Proposition 3.1, and

\[
\sup_{x \in [0,1]} (\min\{x, 1-x\})^{-2\lambda} \left| f(F^-(x)) \right| < \infty.
\]  

(9)
Then, if \( n^{\lambda + \rho - 1} = O \left( d_{n,r}^{-1} \right) \), as \( n \to \infty \), and \( 2\lambda + \vartheta < \frac{1}{2} \), we have

\[
\sup_{t \in [0,1], x \in [0,1]} (\min\{x, 1 - x\})^{-\lambda} \left| d_{n,r}^{-1} \left[ nt \right] (F_{nt}(x) - x) - \frac{J_r(F^-(x))}{r! d_{n,r}} \sum_{i=1}^{\lfloor nt \rfloor} H_r(\xi_i) \right| + 1_{\{t > \tau\}} \frac{\lfloor nt \rfloor - \lfloor n\tau \rfloor}{n} (x - F(F^-(x) - h_n)) = O_P \left( h_n^{\min(\rho, \lambda)} \right),
\]

(10)

where \( J_r(F^-(x)) = E \left( I_{\{G(\xi) \leq F^-(x)\}} H_r(\xi) \right) \).

Note that, in comparison to Proposition 3.1, an additional deterministic term is needed to characterize the asymptotic behavior of the empirical process under the alternative.

On the basis of Proposition 3.1, we derive a reduction principle for the two-parameter empirical process of the ranks:

**Theorem 3.2.** Let \( X_{ni}, 1 \leq i \leq n, n \in \mathbb{N}, \) be a triangular according to (7) with \( h_n = cn^{-1}d_{n,r} \) for some constant \( c > 0 \) and with \( d_{n,r} \) defined by (4), where \( r \) is the Hermite rank of the class of functions \( 1_{\{G(\xi) \leq x\}} - F(x), x \in [0,1] \). Assume that for \( F \) and \( f \) the conditions of Proposition 3.2 hold and that, additionally, there is a constant \( C \), such that for \( h \) small enough, there exists an \( \epsilon_1 > 0 \), such that

\[
\left| 1 - \frac{f(F^-(x) + h)}{f(F^-(x))} \right| \leq C (\min\{|x|, |1 - x|\})^{-\epsilon_1} |h|, \quad \text{for } x \in [0,1].
\]

(11)

Then, for any \( \lambda < 1/3 \) such that \( n^{\lambda} = O(d_{n,r}^{-\lambda}) \), \( n^{\lambda}d_{n,r}^{1+\epsilon_1} = o(n) \) and \( d_{n,r}^{\rho + \lambda} = o(n^\rho) \), where \( \rho \) as in Proposition 3.2, we have

\[
\sup_{t \in [0,1], x \in [0,1]} d_{n,r}^{-1} (\min\{x, 1 - x\})^{-(\lambda - \epsilon_1)} \left| \left( \tilde{G}_{nt}(x) - \frac{\lfloor nt \rfloor}{n} \tilde{G}_n(x) \right) - \frac{J_r(F^-(x))}{r!} \sum_{i=1}^{\lfloor nt \rfloor} H_r(\xi_i) - \frac{\lfloor nt \rfloor}{n} \sum_{i=1}^{n} H_r(\xi_i) \right| + 1_{\{t > \tau\}} \frac{\lfloor nt \rfloor - \lfloor n\tau \rfloor}{n} \left( \frac{\lfloor nt \rfloor - \lfloor n\tau \rfloor}{n} \right) d_{n,r}^{-1} n (x - F(F^-(x) - h_n)) = o_P(1),
\]

where \( J_r(F^-(x)) = E \left( I_{\{G(\xi) \leq F^-(x)\}} H_r(\xi) \right) \).

**Example.** It may not be obvious that the conditions (9) and (11) hold for specific distribution functions. Therefore, we discuss the standard normal distribution as an example. Assume that \( G = \text{id} \) such that \( f = \varphi \) and \( F = \Phi \), where \( \varphi \) denotes the standard normal density and \( \Phi \) the standard normal distribution function. It is well-known that

\[
\Phi(x) \approx \frac{1}{|x|} \varphi(x);
\]
see [25]. Consequently \( \varphi(x) \approx |x| \Phi(x) \). As a result, we have

\[
\varphi(\Phi^{-1}(x)) \approx |\Phi^{-1}(x)| \Phi(\Phi^{-1}(x)) = x|\Phi^{-1}(x)| \text{ for } x \to 0.
\]

As \( \Phi(x) \leq e^x \) for \( x \leq 0 \), it holds that \( 0 \geq \Phi^{-1}(x) \geq \log(x) \) for \( x \leq \frac{1}{2} \) and therefore \( |\Phi^{-1}(x)| \leq |\log(x)| \). With similar arguments for \( x \to 1 \), it follows that (9) holds for any \( \lambda < \frac{1}{2} \).

In order to show that (11) holds, one needs a tighter upper bound. For any \( K > 0 \), there exists a constant \( C \), such that \( \Phi(x) \leq Ce^{Kx} \) for \( x \leq 0 \), and we conclude that \( |\Phi^{-1}(x)| \leq \frac{1}{K}|\log(x/C)| \) for \( x \leq \frac{1}{2} \). We focus on the case \( h > 0 \) because for \( h < 0 \), the quotient of densities in (11) is smaller than 1 and thus the difference is bounded. For \( x \leq \frac{1}{2} \) and \( h > 0 \), we have

\[
1 - \frac{\varphi(\Phi^{-1}(x) + h)}{\varphi(\Phi^{-1}(x))} = -x \frac{\varphi'(\Phi^{-1}(x) + h)}{\varphi(\Phi^{-1}(x))} \leq h \frac{|\varphi'(\Phi^{-1}(x))|}{\varphi(\Phi^{-1}(x))} = h \frac{|\varphi'(\Phi^{-1}(x))|}{\varphi(\Phi^{-1}(x))} \leq h \frac{|\varphi'(\Phi^{-1}(x))|}{\varphi(\Phi^{-1}(x))} \leq h |\Phi^{-1}(x) + h| e^{-h\Phi^{-1}(x)} \leq h |\Phi^{-1}(x) + h| e^{-h\Phi^{-1}(x)}
\]
as \( \varphi'(x) = -x \varphi(x) \). Because \( |\Phi^{-1}(x)| \leq \frac{1}{K}|\log(x/C)| \), we arrive at

\[
1 - \frac{\varphi(\Phi^{-1}(x) + h)}{\varphi(\Phi^{-1}(x))} \leq C|\log(x/C)| e^{-h|\log(x/C)|/K} \leq \hat{C} \left( \frac{x}{C} \right)^{1/K}
\]
for any \( h \in (0,1) \) and some constant \( \hat{C} \). As \( K \) can be chosen arbitrarily large, we conclude that (11) holds for any \( \epsilon_1 > 0 \).

**Example.** The conditions (9) and (11) also hold for \( \alpha \)-stable distributions with \( \alpha \in (0,2) \), as the corresponding densities have regularly varying tails, i.e., \( f(x) \sim c_1 |x|^{-(\alpha+1)} \) for \( x \to -\infty \) and \( f(x) \sim c_2 x^{-(\alpha+1)} \) for \( x \to \infty \) and some constants \( c_1, c_2 \). Again, we only study the case \( x \to -\infty \) in detail. From the regular variation of the tail, it follows that \( f(x) \sim \frac{c_3}{|x|^\alpha} F(x) \) for \( x \to -\infty \). Consequently, it holds that

\[
f(F^{-1}(x)) \sim \frac{c_3}{F^{-1}(x)} x, \ x \to 0.
\]

Treating the other tails analogously, it follows that (9) holds for any \( \lambda \leq \frac{1}{2} \) in the same way as for the normal distribution. To show that condition (11) holds, we make again use of the regular variation: For \( x \) close to 0 or 1

\[
1 - \frac{f(F^{-1}(x) + \Delta)}{f(F^{-1}(x))} \sim 1 - \frac{(F^{-1}(x) + \Delta)^{-(\alpha+1)}}{(F^{-1}(x))^{-(\alpha+1)}} \sim 1,
\]
so that the condition holds for any \( \epsilon_1 \geq 0 \).
Example. The conditions (9) and (11) do not hold for all densities. An example, which is excluded, is the uniform distribution, i.e., $f(x) = 1$ if $x \in [0,1]$, and $f(x) = 0$ else. In this case, the left-hand side of (11) is 1 for $\Delta < -x$, so that the condition could only hold for $\epsilon_1 \geq 1$. However, for $\epsilon_1 \geq 1$, the condition $n^{\lambda}d_{n,r}^{\tau} = o(n)$ cannot hold in the long-range dependent case (and neither in the short-range dependent case with $d_{n,r} \approx C \sqrt{n}$).

Based on Theorem 3.2, under local alternatives, the asymptotic distribution of the statistic $S_n(a)$ can be derived by the same arguments as under the assumption of stationarity, i.e., by the representation (5) and the continuous mapping theorem.

**Corollary 3.2.** Let the assumptions of Theorem 3.2 hold and let $h : (0,1) \rightarrow \mathbb{R}$ satisfy Assumption 1. Then, we have

$$d_{n,r}^{-1}S_n(a) \overset{D}{\rightarrow} \sup_{t \in [0,1]} \left| (Z_{r,H}(t) - tZ_{r,H}(1)) \int_0^1 J_r(F^{-}(x))dh(x) \right. \left. + c\delta_{r}(t) \int_0^1 f(F^{-}(x))dh(x) \right|,$$

where $\overset{D}{\rightarrow}$ denotes convergence in distribution in $D_{\lambda}[0,1]$ and

$$\delta_{r}(t) = \begin{cases} t(1 - \tau) & \text{if } t \leq \tau, \\ \tau(1 - t) & \text{if } t > \tau. \end{cases}$$

### 3.3. Asymptotic Relative Efficiency for level shifts

The goal of this section is to calculate the asymptotic relative efficiency of rank tests that are based on two different score functions $a_1$ and $a_2$. For this, we calculate the number of observations needed to detect a level shift of height $h$ at time $\tau$ with a test of predefined asymptotic level $\alpha$ and asymptotic power $\beta$. With $n_1(h)$ and $n_2(h)$ corresponding to these numbers for $a_1$ and $a_2$, we define the asymptotic relative efficiency of the tests by

$$\lim_{h \rightarrow 0} \frac{n_1(h)}{n_2(h)}$$

assuming that this limit exists.

An asymptotic relative efficiency that is smaller than 1 indicates that the change-point test that corresponds to the score function $a_2$ needs on large scale more observations than the change-point test that corresponds to the score function $a_1$ in order to detect a given jump on the same level with the same power. It is therefore called less efficient.

The above definition of asymptotic relative efficiency has as well been considered in [22] for a comparison of change-point tests. [22] show that, when considering the asymptotic
relative efficiency of CuSum and Wilcoxon test, the above limit exists and does not depend on the choice of \( \tau, \alpha, \) or \( \beta. \)

In order to determine the asymptotic relative efficiency of two rank-based testing procedures, we proceed in the same way. For this, we calculate a quantity that is related to the asymptotic relative efficiency, namely the ratio of the sizes of level shifts that can be detected by the two tests, based on the same number of observations \( n \), for given values of \( \tau, \alpha, \) and \( \beta. \) We denote the corresponding level shifts by \( \Delta_1(n) \) and \( \Delta_2(n) \), respectively, assuming that these numbers depend on \( n \) in the following way:

\[
\Delta_1(n) \sim c_1 \frac{d_{n,r}}{n} \quad \text{and} \quad \Delta_2(n) \sim c_2 \frac{d_{n,r}}{n}.
\]

In order to simplify the succeeding argument, we consider a one-sided change-point test, thus rejecting the hypothesis of no change-point for large values of \( \max_{1 \leq k < n} S_{k,n}(a_1) \) and \( \max_{1 \leq k < n} S_{k,n}(a_2) \).

The rank tests reject the null hypothesis when the statistics

\[
\left(\frac{1}{r!} \int_0^1 J_r (F^{-}(x)) \, dh_1(x)\right)^{-1} \max_{1 \leq k < n} S_{k,n}(a_1),
\]

\[
\left(\frac{1}{r!} \int_0^1 J_r (F^{-}(x)) \, dh_2(x)\right)^{-1} \max_{1 \leq k < n} S_{k,n}(a_2)
\]

exceed the upper \( \alpha \) quantile \( q_\alpha \) of the distribution of

\[
\sup_{0 \leq t \leq 1} BB_{r,H}(t), \quad BB_{r,H}(t) := Z_{r,H}(t) - t Z_{r,H}(1).
\]

Thus, if we want the two tests to have identical power, we have to choose \( c_1 \) and \( c_2 \) that

\[
\left(\frac{1}{r!} \int_0^1 J_r (F^{-}(x)) \, dh_1(x)\right)^{-1} c_1 \psi_\tau(t) \int_0^1 f_G(F^{-}(x)) \, dh_1(x)
\]

\[
= \left(\frac{1}{r!} \int_0^1 J_r (F^{-}(x)) \, dh_2(x)\right)^{-1} c_2 \psi_\tau(t) \int_0^1 f_G(F^{-}(x)) \, dh_2(x)
\]

yielding

\[
\frac{\Delta_1(n)}{\Delta_2(n)} = \frac{c_1}{c_2} = \frac{\int_0^1 J_r (F^{-}(x)) \, dh_1(x) \int_0^1 f_G(F^{-}(x)) \, dh_1(x)}{\int_0^1 J_r (F^{-}(x)) \, dh_2(x) \int_0^1 f_G(F^{-}(x)) \, dh_2(x)}.
\]

**Example.** Assume that \( G = \text{id} \) such that \( f = \varphi \) and \( F = \Phi, \) where \( \varphi \) denotes the standard normal density and \( \Phi \) the standard normal distribution function. In this case,
Rank-based change-point analysis

we have

\[ \int_0^1 J_1 (F^{-1}(x)) \, dh_i(x) = \int_0^1 J_1 (\Phi^{-1}(x)) \, dh_i(x) = \int_0^1 E (1_{\xi_1 \leq \Phi^{-1}(x)} \xi_1) \, dh_i(x) \]

\[ = \int_0^1 \int_{-\infty}^{\Phi^{-1}(x)} y \varphi(y) \, dy \, dh_i(x) = - \int_0^1 \varphi(\Phi^{-1}(x)) \, dh_i(x) = - \int_0^1 f(\Phi^{-1}(x)) \, dh_i(x) \]

for \( i = 1, 2 \).

As a result, the asymptotic relative efficiency of rank-based change-point tests is always 1 when considering Gaussian time series. Since [22] have shown that the Wilcoxon-type change-point test has a relative efficiency of 1 with respect to the CuSum change-point test, we may conclude that the asymptotic efficiency of all rank-based change-point tests corresponds to the asymptotic efficiency of the CuSum test under the assumption of Gaussian data. However, for other marginal distributions, this might be different. In particular, the simulation studies considered in Section 5 indicate that rank-based change-point tests have a higher empirical power for heavy-tailed marginal distributions.

4. Practical implementation

In this section, we describe how to meet challenges that go along with an implementation of the established rank-based change-point tests in practice. For this, note that an application of rank tests to a given data set presupposes determination of the scaling factor \( d_{n,r} \), which satisfies \( d_{n,r}^2 \sim c_{r,D} n^{2-rD} L'(n) \), where \( c_{r,D} \) is a constant depending on \( r \) and \( D \); see [21]. In statistical practice, the parameters \( D \), \( r \) and the function \( L \) are usually unknown. With regard to the practical implementation of rank-based change-point tests, we therefore propose to replace the deterministic scaling of rank-based statistics by a data-driven standardization, i.e., by a normalizing sequence that depends on the given realizations only and which is therefore referred to as self-normalization.

Although, by the consideration of self-normalized statistics, we dispose of unknown quantities in the computation of test statistics, we will show that the resulting, self-normalized test statistics converge in distribution to limits that depend on unknown parameters (the Hurst index \( H \) and the Hermite rank \( r \)), as well. To overcome this problem in practice, a subsampling procedure is considered as an alternative to basing test decisions on the limit distributions of test statistics.

Taken by itself, both methods, i.e., self-normalization and subsampling, make applications of change-point tests more feasible. Nonetheless, the particular charm of their practical implementation lies in the combination of the two methods.

4.1. Self-normalized rank tests

The concept of self-normalization has recently been applied to several testing procedures in change-point analysis. Originally established by [39] in another testing context, it
has been adapted to the change-point problem in [56] by definition of a self-normalized
Kolmogorov-Smirnov test statistic. In these papers, short-range dependent processes are
considered. An extension to possibly long-range dependent processes was introduced by
Shao, who established a self-normalized change-point test based on the CuSum statistic;
see [54]. Several inference problems, including a self-normalized cumulative sum test for
the change-point problem and a self-normalization-based wild bootstrap adjusting for
time-dependent variances are considered in [69]. CuSum-based procedures for sequential
monitoring of time series with respect to structural changes are proposed by [23] and [16],
among others. [24] extend the concept of self-normalization to develop a methodology
for testing the null hypothesis of no relevant deviation in functional time series data
against the alternative of relevant changes. [67] propose a self-normalized change-point
test that does not require a priori information on the number of change-points and can
thus be considered as unsupervised. [46] combine self-normalized CuSum-type statistics
and the wild bootstrap, thereby establishing completely data-driven change-point tests.
A self-normalized version of the Wilcoxon change-point test is considered in [7] and [8].
For further, recent discussions and references on self-normalization in different contexts,
we refer to [55].

The definition of self-normalized rank statistics is motivated with reference to an
application of a self-normalization procedure to the CuSum statistic. For this, it is crucial
to note that rank statistics arise from an application of the CuSum statistic to the scores
\(a(R_1), \ldots, a(R_n)\): Given observations \(X_1, \ldots, X_n\), the CuSum test bases test decisions
on the statistic

\[
C_n := \max_{1 \leq k < n} \left| \sum_{i=1}^{k} X_i - \frac{k}{n} \sum_{j=1}^{n} X_j \right|
\]

while rank-based change-point tests decide in favor of a change-point in the data for large
values of the statistic

\[
S_n(a) := \max_{1 \leq k < n} \left| \sum_{i=1}^{k} a(R_i) - \frac{k}{n} \sum_{j=1}^{n} a(R_j) \right|
\]

Therefore, it seems natural to choose a data-driven normalization for rank statistics by
evaluation of the self-normalized CuSum statistic, defined in [54], in \(a(R_1), \ldots, a(R_n)\).
For this reason, we define the self-normalized rank statistic for the change-point problem
by

\[
T_n(a) := \max_{1 \leq k < n} |T_{k,n}(a)|, \tag{12}
\]

\[
T_{k,n}(a) := \frac{S_{k:1,n}(a)}{\left\{ \frac{1}{n} \sum_{i=1}^{k} S_{i:1,k}^2(a) + \frac{1}{n} \sum_{i=k+1}^{n} S_{i:k+1,n}^2(a) \right\}^{1/2}}, \tag{13}
\]
where

\[ S_{t; j,k}(a) = \sum_{h=j}^{t} (a(R_h) - \bar{a}_{j,k}) \quad \text{with} \quad \bar{a}_{j,k} := \frac{1}{k - j + 1} \sum_{t=j}^{k} a(R_t). \]

In order to derive the asymptotic distribution of \( T_n(a) \), recall that

\[ S_{k;1,n}(a) = \sum_{i=1}^{k} a(R_i) - \frac{k}{n} \sum_{i=1}^{n} a(i) = -\int_0^1 \left( \hat{G}_k(x) - \frac{k}{n} \hat{G}_n(x) \right) \, dh(x), \]

where \( \hat{G}_k(x) := \sum_{i=1}^{k} 1\{ \frac{i}{n+1} R_i \leq x \} \), and note that

\[ T_{[nt];n}(a) := G_{S_{[nt];1,n}(a)} + O_P(1), \quad t \in [0,1], \]

where for \( f \in D[0,1] \) the function \( G_f \in D[0,1] \) is defined by

\[ G_f(t) := \frac{f(t)}{V_f(t)}, \]

\[ V_f(t) := \left\{ \int_0^t \left( f(s) - \frac{s}{t} f(t) \right)^2 \, ds + \int_t^1 \left( f(s) - f(t) - \frac{s - t}{1 - t} (f(1) - f(t)) \right)^2 \, ds \right\}^{\frac{1}{2}}. \]

As a result, the limit of the self-normalized process \( T_{[nt];n}(a), t \in [0,1], \) can be derived from the limit distribution of the process \( S_{[nt];1,n}(a), t \in [0,1], \).

Under the hypothesis, i.e., under the assumption of a stationary data-generating process, Theorem 3.1 and the argument that proves Theorem 1 in [7] establish the convergence of the self-normalized rank statistics \( T_n(a) \):

**Corollary 4.1.** Let the assumptions of Theorem 3.1 hold and let \( h : (0,1) \rightarrow \mathbb{R} \) satisfy Assumption 1. Then, we have

\[ T_n(a) \xRightarrow{D} \sup_{t \in [0,1]} \left\{ \frac{|Z_{r,H}(t) - tZ_{r,H}(1)|}{\left\{ \int_0^t V_{r,H}^2(s;0,t) \, ds + \int_t^1 V_{r,H}^2(s;t,1) \, ds \right\}^{\frac{1}{2}}} \right\}. \]

with

\[ V_{r,H}(s; s_1, s_2) = Z_{r,H}(s) - Z_{r,H}(s_1) - \frac{s - s_1}{s_2 - s_1} \left( Z_{r,H}(s_2) - Z_{r,H}(s_1) \right) \]

for \( s \in [s_1, s_2], \) \( 0 < s_1 < s_2 < 1. \)
In practical applications, the parameters \( r \) and \( H \) in the limit of the statistic \( T_n(a) \) are unknown. For this reason, one cannot use Corollary 4.1 directly to obtain critical values. One possibility to arrive at (approximate) asymptotic critical values is to estimate \( r \) and \( H \). For the Hermite rank \( r \), the question of estimation has not been addressed in the literature. Nonetheless, [5] introduce a method for testing the hypothesis that \( r = 1 \) against the alternative that \( r > 1 \). At least in theory, this allows for deciding whether inference based on the usual assumption of \( r = 1 \) is appropriate. From a statistical point of view, however, [3] argue that an Hermite rank \( r > 1 \) easily collapses to \( r = 1 \) when there is a slight perturbation of the data, and is, thereby, unstable, whereas an Hermite \( r = 1 \) is stable. An overview and a comparison of various techniques for estimating the Hurst parameter is given in [61] and [51]. Among these, there are graphical methods, such as the aggregated variance method or the R/S method ([43], [41] and [42]), estimators operating in the frequency domain of time series, such as the Whittle estimator, and semiparametric alternatives such as the GPH estimator ([26]) and the local Whittle estimator ([36], [52]). More recently, an estimation approach based on ordinal-pattern analysis of time series has been considered in [58] and [10].

4.2. Subsampling

The basic idea of resampling procedures is to approximate the distribution function \( F_{T_n} \) of a considered statistic \( T_n := T_n(X_1,\ldots,X_n) \) by the empirical distribution of values of the statistic computed over subsets of the original sample. The so-called sampling-window method, studied by [48], [29], and [57], utilizes evaluations of the test statistic in subsamples of successive observations, i.e., for some blocklength \( l_n < n \), the realizations \( T_{i,n,k} := T_n(X_k,\ldots,X_{k+l_n-1}) \), \( k = 1,\ldots,m_n \), where \( m_n := n-l_n + 1 \), are considered. As a result, multiple (though dependent) realizations of the test statistic \( T_{i,n} \) are obtained. Due to the fact that consecutive observations are chosen, the subsamples retain the dependence structure of the original sample, so that the empirical distribution function of \( T_{i,n,1},\ldots,T_{i,n,m_n} \), defined by

\[
\hat{F}_{m_n,l_n}(t) := \frac{1}{m_n} \sum_{k=1}^{m_n} 1\{T_{i_n,k} \leq t\}, \tag{14}
\]

can be considered as an appropriate estimator for \( F_{T_n} \).

The validity of the subsampling procedure is typically established by proving that the distance between \( \hat{F}_{m_n,l_n} \) and \( F_{T_n} \) vanishes as the number of observations tends to \( \infty \), i.e., by showing that

\[
\left| \hat{F}_{m_n,l_n}(t) - F_{T_n}(t) \right| \xrightarrow{P} 0, \quad \text{as } n \to \infty,
\]

for all points of continuity \( t \) of \( F_T \).

It is shown in [57] that the sampling-window method is consistent for any time series satisfying an \( \alpha \)-mixing condition and for any measurable statistic converging in
distribution to a non-degenerate limit variable. Thereby, consistency of the sampling-window method can be derived for an extensive class of short-range dependent processes under the mildest possible assumptions on the blocklength $l_n$ and the considered statistic $T_n$. In the long-range dependent case, the validity of subsampling has been shown to hold for specific statistics under various model assumptions. [30] prove consistency of the sampling-window method for the sample mean as well as a studentized version of the sample mean under the assumption of subordinated Gaussian processes. [44] attained consistency results with respect to the same statistics for long-range dependent linear processes with possibly non-Gaussian innovations. For this model, an alternative proof for consistency can be found in [6]. [68] generalize these results by proving consistency with respect to the sample mean under the assumption of subordinated long-range dependent linear processes with possibly non-Gaussian innovations. [32] provide a result on the validity of subsampling for a general class of statistics $T_n$ and certain heavy-tailed long-range dependent time series that follow a long memory stochastic volatility model. However, their results are restricted by assumptions that are difficult to check in practice. Moreover, although not explicitly stated in [32], the proof of consistency only holds for statistics $T_n$ that are Lipschitz-continuous (uniformly in $n$); see [33]. Many robust statistics do not satisfy this assumption. In fact, rank-based change-point test statistics can be taken as examples for non-Lipschitz-continuous statistics.

General results on the validity of subsampling for long-range dependent time series are established in [2] and, independently in [9]. Neither of both consistency results makes any restrictive demands concerning the statistic $T_n$ or the considered time series, such as finite moments of the data-generating variables, or continuity of the considered statistics. As a result, both results are applicable to heavy-tailed random variables and rank-based test statistics. In the following, we rigorously state the validity of the subsampling procedure as established in [9]. For this, we have to introduce some assumptions:

**Assumption 3.** $(X_n)_{n \in \mathbb{N}}$ is a stochastic process and $(T_n)_{n \in \mathbb{N}}$ is a sequence of statistics such that $T_n \Rightarrow T$ in distribution as $n \to \infty$ for a random variable $T$ with distribution function $F_T$.

This is a standard assumption for subsampling, see for example [48]. If the distribution does not converge, we cannot expect the distribution of $T_1$ to be close to the distribution of $T_n$.

**Assumption 4.** $X_n = G(\xi_n)$ for a measurable function $G$ and a stationary, Gaussian process $(\xi_n)_{n \in \mathbb{N}}$ with covariance function

$$\gamma(k) := \text{Cov}(\xi_1, \xi_{1+k}) = k^{-D} L_{\gamma}(k)$$

such that

1. $D \in (0, 1]$ and $L_{\gamma}$ is a slowly varying function with

$$\max_{\tilde{k} \in \{k+1, \ldots, k+2^\nu-1\}} \left| \frac{L_{\gamma}(k) - L_{\gamma}(\tilde{k})}{\tilde{k}} \right| \leq K \frac{\nu}{k} \min \{L_{\gamma}(k), 1\}$$
for a constant $K < \infty$ and all $l' \in \{l_k, \ldots, k\}$;

2. $(\xi_n)_{n \in \mathbb{N}}$ has a spectral density $f$ with $f(x) = |x|^{D-1}L_f(x)$ for a slowly varying function $L_f$ bounded away from 0 on $[0, \pi]$ such that $\lim_{x \to 0} L_f(x) \in (0, \infty]$ exists.

We do not impose any conditions on the function $G$: no finite moments or continuity are required, so that our results are applicable for heavy-tailed random variables and robust test statistics. In [9] it is shown that Assumption 4 holds for some standard examples of long range dependent Gaussian processes.

**Assumption 5.** Let $(l_n)_{n \in \mathbb{N}}$ be a non-decreasing sequence of integers such that $l = l_n \to \infty$ as $n \to \infty$ and $l_n = \mathcal{O}(n^{(1+D)/2-\epsilon})$ for some $\epsilon > 0$.

If the dependence of the underlying process $(\xi_n)_{n \in \mathbb{N}}$ gets stronger, the range of possible values for $l$ gets smaller. A popular choice for the block length is $l \approx C\sqrt{n}$ (see for example [30]), which is allowed for all $D \in (0, 1]$. Now, we can state the main result from [9]:

**Theorem 4.1.** Under Assumptions 3, 4 and 5 we have

$$F_{T_n}(t) - \hat{F}_{l,n}(t) \xrightarrow{p} 0$$

as $n \to \infty$ for all points of continuity $t$ of $F_T$.

For this reason, in the following sections, we can formally justify an application of the sampling-window method in simulations and data analysis by referring to the aforementioned results.

5. **Simulations**

In the following, the finite sample performance of self-normalized, rank-based testing procedures is analyzed in the context of testing for changes in the mean of a given set of observations $X_1, \ldots, X_n$. More precisely, we will consider two different rank-based testing procedures, the self-normalized Wilcoxon change-point test and the self-normalized Van der Waerden change-point test, and compare their finite sample performance to that of the self-normalized CuSum change-point test. For this purpose, the rejection rates of all three testing procedures are computed for simulated subordinated Gaussian time series $X_n$, $n \in \mathbb{N}$, $X_n = G(\xi_n)$, where $\xi_n$, $n \in \mathbb{N}$, is a fractional Gaussian noise sequence generated by the function fgnSim from the fArma package in R.

We consider the following choices of marginal distributions that, for subordinated Gaussian time series, are determined by the function $G$:

1. Normal margins: We choose $G(t) = t$. In this case, the Hermite coefficient $J_1(G; x)$ is not equal to 0 for all $x \in \mathbb{R}$ (see [21]), so that $r = 1$, where $r$ denotes the Hermite rank of $1_{G(\xi) \leq x} - F(x), x \in \mathbb{R}$.
2. Pareto margins: In order to get standardized Pareto-distributed data which has a representation as a functional of a Gaussian process, we consider the transformation

\[ G(t) = \left( \frac{a k^2}{(\alpha - 1)^2(\alpha - 2)} \right)^{-\frac{1}{2}} \left( k(\Phi(t))^{-\frac{\alpha}{2}} - \frac{a k}{\alpha - 1} \right) \]

with parameters \( k, \alpha > 0 \) and with \( \Phi \) denoting the standard normal distribution function. Since \( G \) is a strictly decreasing function, it follows by Theorem 2 in [21] that \( r = 1 \), where \( r \) denotes the Hermite rank of \( 1_{\{G(\xi_i) \leq x\}} - F(x), x \in \mathbb{R} \).

3. Cauchy margins: In order to get standardized Pareto-distributed data which has a representation as a functional of a Gaussian process, we consider the transformation

\[ G(t) = \tan \left( \pi \left( \Phi(t) - \frac{1}{2} \right) \right) \]

with \( \Phi \) denoting the standard normal distribution function. Since \( G \) is a strictly increasing function, it follows by Theorem 2 in [21] that \( r = 1 \), where \( r \) denotes the Hermite rank of \( 1_{\{G(\xi_i) \leq x\}} - F(x), x \in \mathbb{R} \).

4. \( \chi^2(1) \) margins: In order to get standardized \( \chi^2(1) \)-distributed data which has a representation as a functional of a Gaussian process, we consider the transformation

\[ G(t) = \frac{1}{2} \left( t^2 - 1 \right) \]

In this case, the Hermite coefficient \( J_1(G; x) \) equals 0 for all \( x \in \mathbb{R} \), while \( J_2(G; x) \) is not equal to 0. It follows that \( r = 2 \), where \( r \) denotes the Hermite rank of \( 1_{\{G(\xi_i) \leq x\}} - F(x), x \in \mathbb{R} \).

All calculations are based on 5,000 realizations of time series and test decisions are based on an application of the sampling-window method for a significance level of 5%, meaning that the values of the test statistics are compared to the 95%-quantile of the empirical distribution function \( \hat{F}_{m_n, t_n} \) defined by (14). The empirical rejection frequencies for all three testing procedures and a sample size of \( n = 500 \) can be found in Figure 1.

As expected, the empirical power increases for greater values of the level shift \( h \). Furthermore, the empirical power is higher for breakpoints located in the middle of the sample \( (\tau = 0.5) \) than for change-point locations that lie close to the boundary of the testing region \( (\tau = 0.25) \). In accordance with the asymptotic considerations in Example 3.3, the three tests behave very similar under normality (upper panels). For the heavy-tailed Pareto and Cauchy distribution (middle panels), the rank-based test clearly outperform the CuSum test. In particular, the empirical power of the CuSum test is not much greater than 5% when considering Cauchy distributed data. Moreover, the rank-based testing procedures yield a better power than the CuSum test when considering \( \chi^2 \)-distributed time series (lower panel), i.e., subordinated Gaussian time series with an Hermite rank \( r = 2 \).

Detailed simulation results can be found in Tables 1, 2, 3, and 4 in the appendix. These display results for sample sizes \( n = 300 \) and \( n = 500 \) and for different block
lengths \( (l_n = \lfloor n^\gamma \rfloor \text{ with } \gamma \in \{0.4, 0.5, 0.6\}) \). Not surprisingly, an increasing sample size goes along with an improvement of the finite sample performance, i.e., the empirical size approaches the level of significance and the empirical power increases. All three testing procedures have an empirical size that is relatively close to the level of significance, an observation that seems to be typical of self-normalized testing procedures as it corresponds to the so-called better size but less power phenomenon for self-normalized tests, which has also been observed in [54], [56], and [7]. The block length \( l_n = \sqrt{n} \) seems to give the best overall performance, although it does not yield the best results for each and every scenario.
Figure 1. Rejection rates of the self-normalized CuSum, the self-normalized Wilcoxon and the self-normalized Van der Waerden change-point tests obtained by subsampling with block length $l_n = 22$ for transformed fractional Gaussian noise time series of length $n = 500$ with Hurst parameter $H = 0.7$, marginal standard normal, Pareto, and Cauchy distribution and a change in location of height $h$ after a proportion $\tau$ of the simulated data.
Given normally distributed observations, there do not seem to be large deviations in the rejection rates of the three testing procedures. Nonetheless, the Van der Waerden test tends to be less conservative, but more efficient than the other testing procedures. At least for independent data, this observations corresponds to the fact that considering normal scores (yielding the Van der Waerden statistic) is known to result in a more efficient testing procedure; see [31]. For Pareto(3, 1)-distributed observations and Cauchy-distributed observations the two rank-based testing procedures clearly outperform the self-normalized CuSum test in that they yield considerably higher empirical rejection rates under the alternative. This observation specifically applies to the Cauchy-distributed time series as for these the self-normalized CuSum test has an extremely low power (which seems to be independent of the height and the location of the change-point). When considering subordinated Gaussian time series with Hermite rank \( r = 2 \), that is \( \chi^2 \)-distributed observations, the rank-based testing procedures also perform better than the CuSum test. Comparing Wilcoxon and Van der Waerden test with respect to these time series, again the Van der Waerden test shows a slight tendency of being more efficient than the Wilcoxon test.

6. Data example

In the following, observations from finance, an area that typically gives rise to long-range dependent time series, is analyzed with regard to structural changes by an application of rank-based testing procedures and the methodologies described in Section 4.

As a relatively recent data set, the considered financial time series, which describes the performance of the British stock market against the background of the United Kingdom European Union membership referendum in 2016, has not yet been considered in the context of change-point analysis. With our consideration of this data, we hope to pave the way for new discussions in applied change-point analysis. Additionally, we aim at a comparison of rank-based change-point tests resulting from different score functions, as well as a comparison of rank-based change-point tests to CuSum-based testing procedures in practice. Two more data sets from hydrology and network traffic monitoring are analyzed in the appendix.

6.1. FTSE 100 Index

The data set corresponds to the closing values of the Financial Times Stock Exchange 100 Index (FTSE 100), a share index of the 100 companies with the highest market capitalisation listed on the London Stock Exchange, recorded daily over a time period of one year from March 2016 to March 2017.

Since in general stock prices do not follow a stationary process, whereas their log-returns display features of stationarity, we analyze the log-returns instead of considering
the closing index itself; see Figure 2. Formally, the log-returns are defined by

\[ L_t := \log R_t, \quad R_t := \frac{P_t}{P_{t-1}}, \]

where \( P_t \) denotes the value of the index on day \( t \).

The plot in Figure 2 does not indicate a change in the location of the time series, but rather a change in its volatility. For this reason, we intend to apply the change-point tests to the absolute log-returns, i.e., the absolute values of the log-returns. A phenomenon that is typically encountered in the context of financial data is the following: the log-returns of stock market indices appear to be uncorrelated, whereas the absolute log-returns tend to be highly correlated; see, e.g., [8]. With respect to the closing index of the FTSE 100, empirical evidence for this assertion is given by the behavior of the sample autocorrelations of the log-returns and absolute log-returns of the index in Figure 3. Moreover, estimation of the Hurst parameter of the absolute log-returns by the local Whittle procedure with bandwidth parameter \( b_n = \lfloor n^{2/3} \rfloor \) yields an estimate \( H = 0.854 \) indicating long-range dependence. Again, we base our analysis of the data on the self-normalized Wilcoxon and the self-normalized Van der Waerden test and compare their performances to that resulting from the change-point test based on the self-normalized CuSum statistic as defined in [34].

![Figure 2](image-url). Log-returns of the daily closing index of the FTSE 100 from March 2016 to March 2017.
Figure 3. Sample autocorrelation function (acf) of the FTSE 100 index’ log-returns and absolute log-returns.

Figure 4 depicts the values of the two-sample statistics $T_{k,n}(a_i)$, $i = 1, 2$, as defined in formula (13), with $a_1(i) = (n + 1)^{-1}i$ (the score function that yields the self-normalized two-sample Wilcoxon statistic) and $a_2(i) = \Phi^{-1}((n + 1)^{-1}i)$ (the score function that yields the self-normalized two-sample Van der Waerden statistic) and the self-normalized two-sample CuSum statistic as defined in [54].

All three line plots achieve their maximum in the autumn of 2016, thereby indicating that, if there is a change in the volatility of the time series, it is most likely located around that point in time. The occurrence of a structural change in financial time series in the year 2016 seems highly plausible due to the outcome of the United Kingdom European Union membership referendum on 23 June 2016. An explanation for a decrease of the volatility around November may refer to the Autumn Statement of the same year, a financial report on the state of the economy, published by the British government on
23 November 2016, possibly soothing markets and thereby resulting in a change of the FTSE 100's volatility.

Approximating the distribution of the self-normalized test statistics by the sampling window method with block size $l = \lfloor \sqrt{n} \rfloor = 15$ yields $p$-values of $0.0118$ for the self-normalized Wilcoxon test, $0.0216$ for the self-normalized Van der Waerden test and $0.2071$ for the self-normalized CuSum test, i.e., at a level of significance of $5\%$ the self-normalized Wilcoxon and the self-normalized Van der Waerden test reject the hypothesis, while the self-normalized CuSum test decides in favor of the hypothesis of no change. This seems plausible insofar the normal quantile plot (see Figure 5) does not substantiate the assumption of a normal distribution, as the tails of the empirical distribution are too heavy. In this case, the more robust Wilcoxon test should be more reliable.

![Figure 5](image_url)  

**Figure 5.** Normal quantile plot of the absolute log-returns of the daily closing index of the FTSE 100 from January 2015 to December 2017.

For a data set where the self-normalized van der Waerden test concurs more with the self-normalized CuSum test see the appendix: section B.1.
Appendix A: Proofs

Proofs under stationarity

Proof of Proposition 3.1. Our goal is to prove a reduction principle for the sequential empirical process \( F_{nt} (x) - x, t \in [0,1], x \in [0,1], \) where

\[
F_n (x) := \frac{1}{n} \sum_{i=1}^{n} 1_{\{X_i \leq x\}}
\]

with respect to the weighted supremum norm. For this, we consider the transformed random variables \( Z_n, n \in \mathbb{N}, \) with

\[
Z_n := \begin{cases} 
\frac{1}{X_n} & \text{if } X_n \leq \frac{1}{2}, \\
\frac{1}{1-X_n} & \text{if } X_n > \frac{1}{2}.
\end{cases}
\]

It follows that for \( x > 2 \)

\[
P (Z_n > x) = \frac{1}{x} \quad \text{and} \quad P (Z_n < -x) = \frac{1}{x}.
\]

Hence, \( Z_n \) has finite \( 3\lambda \)-moment for \( \lambda \in (0,1/3). \) According to Theorem 2 in [15], there exists a constant \( \kappa > 0 \) such that

\[
\sup_{t \in [0,1], x \in [-\infty,\infty]} d_{n,r}^{-1} (1 + |x|)^{\lambda} \left| \sum_{j=1}^{\lfloor nt \rfloor} \left( 1_{\{Z_j \leq x\}} - F_Z (x) - \frac{1}{r!} J_{Z,r} (x) \right) \right| = O_p \left( n^{-\kappa/3} \right),
\]

where \( J_{Z,r} (x) := E (1_{\{Z_1 \leq x\}} H_r (\xi_1)) \) and \( F_Z (x) := P (Z_1 \leq x). \) For \( x \leq 1/2, \) we have \( X_n \leq x \) if and only if \( Z_n \geq 1/x, \) i.e., for \( x \leq 1/2, \) we have

\[
1_{\{X_j \leq x\}} - x - \frac{1}{r!} J_r (x) \left( H_r (\xi_j) \right)
= \left( 1_{\{Z_j \leq x\}} - (1 - F_Z (x^{-1})) \right) - \frac{1}{r!} E \left( 1_{\{Z_j \geq x^{-1}\}} H_r (\xi_j) \right) H_r (\xi_j)
= - \left( 1_{\{Z_j \leq x^{-1}\}} - F_Z (x^{-1}) - \frac{1}{r!} J_{Z,r} (x^{-1}) \right).
\]

Using analogous arguments, we arrive at the same estimation in the case \( x > 1/2. \)

Moreover, we have \( x^{-\lambda} \leq (1 + |1/x|)^{\lambda}. \) As a result, we obtain

\[
\sup_{t \in [0,1], x \in [0,1]} d_{n,r}^{-1} \left( \min \{x, 1-x\} \right)^{-\lambda} \left| \sum_{j=1}^{\lfloor nt \rfloor} \left( 1_{\{Z_j \leq x\}} - F_Z (x) - \frac{1}{r!} J_{Z,r} (x) \right) \right|
\leq \sup_{t \in [0,1], x \in [-\infty,\infty]} d_{n,r}^{-1} (1 + |x|)^{\lambda} \left| \sum_{j=1}^{\lfloor nt \rfloor} \left( 1_{\{Z_j \leq x\}} - F_Z (x) - \frac{1}{r!} J_{Z,r} (x) \right) \right|.
\]

This completes the proof. \( \square \)
Rank-based change-point analysis

Proof of Theorem 3.1. Our goal is to derive a reduction principle for the two-parameter empirical process of the ranks, i.e., for

\[ \hat{G}_{\lfloor nt \rfloor}(x-) - \frac{\lfloor nt \rfloor}{n} \hat{G}_n(x-), \ t \in [0, 1], \ x \in [0, 1], \]

with \( \hat{G}_k(x) := \sum_{i=1}^{k} 1\{ \frac{i}{n+1} R_i \leq x \} \) and \( R_i = \sum_{j=1}^{n} 1\{ X_j \leq X_i \} \).

Recall that \( F_n^{-} \) denotes the generalized inverse of \( F_n \). It then follows that

\[ \hat{G}_{\lfloor nt \rfloor}(x-) - \frac{\lfloor nt \rfloor}{n} \hat{G}_n(x-) = \lfloor nt \rfloor \sum_{i=1}^{\lfloor nt \rfloor} \left( 1\{ \frac{i}{n+1} R_i \leq x \} - \frac{1}{n} \sum_{i=1}^{n} 1\{ \frac{i}{n+1} R_i \leq x \} \right) \]

\[ = \sum_{i=1}^{\lfloor nt \rfloor} \left( 1\{ F_n(X_i) \leq \frac{i}{n+1} x \} - \frac{1}{n} \sum_{i=1}^{n} 1\{ F_n(X_i) \leq \frac{i}{n+1} x \} \right) \]

Unfortunately, the generalized inverse \( F_n^{-} \) is not continuous, which may cause difficulties when considering the weighted supremum norm of the above expression. Therefore, we will consider a continuous modification of \( F_n^{-} \). For this, let \( X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)} \) denote the order statistic of \( X_1, X_2, \ldots, X_n \) and define a continuous modification of \( F_n^{-} \) by \( \tilde{F}_n^{-} : [0, 1] \to [0, 1] \) by

\[ \tilde{F}_n^{-}(x) = \begin{cases} 0 & \text{for } x = 0 \\ X_{(i)} & \text{for } x = \frac{i}{n+1} \\ 1 & \text{for } x = 1 \\ \text{linear interpolated in between.} \end{cases} \quad (15) \]

Because \( F_n \) and \( F_{\lfloor nt \rfloor} \) are constant on the intervals \([X_{(i)}, X_{(i+1)})\), we have

\[ \hat{G}_{\lfloor nt \rfloor}(x-) - \frac{\lfloor nt \rfloor}{n} \hat{G}_n(x-) = [\lfloor nt \rfloor] F_{\lfloor nt \rfloor}(\tilde{F}_n^{-}(x)-) - [\lfloor nt \rfloor] F_n(\tilde{F}_n^{-}(x)-). \]

For the proof of Theorem 3.1, we eliminate the expression \( \tilde{F}_n^{-}(x) \) on the right-hand side of the equality and then apply Proposition 3.1. For this, we have to replace \( J_r(x) \) by \( J_r \left( \tilde{F}_n^{-}(x) \right) \), i.e., we have to show that

\[ \sup_{x \in [0, 1]} (\min\{x, 1-x\})^{-\lambda} \left( J_r \left( \tilde{F}_n^{-}(x) \right) - J_r(x) \right) = o_P(1). \]

Observe that for \( x < y \)

\[ |J_r(x) - J_r(y)| \leq C \sqrt{y-x} \]
for some constant $C$. Therefore, it suffices to show that

$$
\sup_{x \in [0, 1]} (\min\{x, 1 - x\})^{-\lambda} \sqrt{\left| \hat{F}_n^-(x) - x \right|} = \sup_{x \in [0, 1]} (\min\{x, 1 - x\})^{-2\lambda} \left| \hat{F}_n^-(x) - x \right| = o_P(1).
$$

Because $\hat{F}_n^-$ is piecewise linear, we have

$$
\frac{\hat{F}_n^-(x)}{x} = (n + 1) \hat{F}_n^-(\frac{1}{n + 1}) \quad \text{for } x \leq \frac{1}{n + 1},
$$

$$
\frac{1 - \hat{F}_n^-(x)}{1 - x} = (n + 1) \left(1 - \hat{F}_n^-(\frac{n}{n + 1})\right) \quad \text{for } x \geq \frac{n}{n + 1}.
$$

The function $\hat{F}_n^-(x) - x, x \in [0, 1]$, takes its maximum for some $x \in \{\frac{1}{n+1}, \frac{2}{n+1}, \ldots, \frac{n}{n+1}\}$, as it is linear between these points. As a result, we may conclude that

$$
\sup_{x \in [0, 1]} (\min\{x, 1 - x\})^{-2\lambda} \left| \hat{F}_n^-(x) - x \right| = (n + 1)^{2\lambda} \sup_{x \in [0, 1]} \left| \hat{F}_n^-(x) - x \right|.
$$

Letting $X(1) \leq X(2) \leq \ldots \leq X(n)$ denote the order statistics of $X_1, X_2, \ldots, X_n$, such that $nF_n(X(i)) = i$ by definition of $X(i)$, it follows that

$$
\sup_{x \in [0, 1]} |\hat{F}_n^-(x) - x| = \max_{i=1, \ldots, n} \left| \hat{F}_n^-(\frac{i}{n + 1}) - \frac{i}{n + 1} \right| = \max_{i=1, \ldots, n} \left| X(i) - \frac{i}{n + 1} \right|
$$

$$
\leq \max_{i=1, \ldots, n} \left| \frac{i}{n} - X(i) \right| + \frac{1}{n + 1} \leq \sup_{x \in [0, 1]} |F_n(x) - x| + \frac{1}{n + 1}.
$$

By Proposition 3.1, it therefore holds that

$$
\sup_{x \in [0, 1]} |\hat{F}_n(x) - x| = O_P \left( n^{-1}d_{n,r} \right).
$$

Hence, we finally arrive at

$$
\sup_{x \in [0, 1]} (\min\{x, 1 - x\})^{-2\lambda} \left| \hat{F}_n^-(x) - x \right| = O_P \left( n^{2\lambda - 1}d_{n,r} \right) = o_P(1). \tag{16}
$$

Since $J_r \left( \hat{F}_n^-(x) \right)$ converges in probability to $J_r(x)$ with respect to the weighted supremum
Because \( \tilde{\lambda} \) norm, it remains to show that

\[
\sup_{t \in [0,1], x \in [0,1]} d_{n,r}^{-1}(\min\{x, 1-x\})^{-\lambda} \left| \left( \hat{G}_{\lfloor nt \rfloor} (x) - \frac{|nt|}{n} \hat{G}_n (x) \right) - \frac{1}{r!} J_r \left( \hat{F}_n (x) \right) \left( \sum_{i=1}^{\lfloor nt \rfloor} H_r (\xi_i) - \frac{|nt|}{n} \sum_{i=1}^{n} H_r (\xi_i) \right) \right| = o_P(1).
\]

For this, note that

\[
\sup_{t \in [0,1], x \in [0,1]} d_{n,r}^{-1}(\min\{x, 1-x\})^{-\lambda} \left| \left( \hat{G}_{\lfloor nt \rfloor} (x) - \frac{|nt|}{n} \hat{G}_n (x) \right) - \frac{1}{r!} J_r \left( \hat{F}_n (x) \right) \left( \sum_{i=1}^{\lfloor nt \rfloor} H_r (\xi_i) - \frac{|nt|}{n} \sum_{i=1}^{n} H_r (\xi_i) \right) \right| \leq \sup_{x \in [0,1]} \left( \frac{\min\{\hat{F}_n (x), 1-\hat{F}_n (x)\}}{\min\{x, 1-x\}} \right)^\lambda \times \sup_{t \in [0,1], x \in [0,1]} d_{n,r}^{-1}(\min\{x, 1-x\})^{-\lambda} \left| \left( |nt| F_{\lfloor nt \rfloor} (x) - |nt| F_n (x) \right) - \frac{1}{r!} J_r \left( \hat{F}_n (x) \right) \left( \sum_{i=1}^{\lfloor nt \rfloor} H_r (\xi_i) - \frac{|nt|}{n} \sum_{i=1}^{n} H_r (\xi_i) \right) \right|.
\]

We will treat the two factors on the right-hand side of the above formula separately. For the first factor, we have

\[
\sup_{x \in [0,1]} \left| \left( \min\{x, 1-x\} \right)^{-\lambda} \min\{\hat{F}_n (x), 1-\hat{F}_n (x)\} \right| \leq \sup_{x \in [0,1]} \left| \frac{\hat{F}_n (x)}{x} \right|^{\lambda} + \sup_{x \in [\frac{1}{2}, 1]} \left| \frac{1-\hat{F}_n (x)}{1-x} \right|^{\lambda} \leq \sup_{x \in [0,\frac{1}{2}]} \left| \frac{\hat{F}_n (x) - x}{x} \right|^{\lambda} + \sup_{x \in [\frac{1}{2}, 1]} \left| \frac{\hat{F}_n (x) - x}{1-x} \right|^{\lambda} + 2.
\]

Because \( \hat{F}_n \) is piecewise linear, it holds that

\[
\frac{\hat{F}_n (x)}{x} = (n+1) \hat{F}_n \left( \frac{1}{n+1} \right) \quad \text{for } x \leq \frac{1}{n+1},
\]

\[
\frac{1-\hat{F}_n (x)}{1-x} = (n+1) \left( 1 - \hat{F}_n \left( \frac{n}{n+1} \right) \right) \quad \text{for } x \geq \frac{n}{n+1}.
\]
such that
\[
\sup_{x \in [0, 1]} \left| (\min\{x, 1 - x\})^{-\lambda} (\min\{\hat{F}_n^-(x), 1 - \hat{F}_n^-(x)\})^\lambda \right| \\
\leq 2(n + 1)^\lambda \left( \sup_{x \in [0, 1]} \left| \hat{F}_n^-(x) - x \right| \right)^\lambda + 2.
\]

By (16), it therefore follows that
\[
\sup_{x \in [0, 1]} (\min\{x, 1 - x\})^{-\lambda} (\min\{\hat{F}_n^-(x), 1 - \hat{F}_n^-(x)\})^\lambda = O_P (d_{n,r}^\lambda).
\]

So as to determine the order of the second factor, we split it into two summands:
\[
\sup_{t \in [0, 1], x \in [0, 1]} d_{n,r}^{-1} (\min\{x, 1 - x\})^{-\lambda} \left( [nt] F_{[nt]}(x-) - [nt] F_n(x-) \right) \\
- \frac{J_r(x)}{r!} \left( \sum_{i=1}^{[nt]} H_r(\xi_i) - \frac{[nt]}{n} \sum_{i=1}^{n} H_r(\xi_i) \right) \\
\leq \sup_{t \in [0, 1], x \in [0, 1]} d_{n,r}^{-1} (\min\{x, 1 - x\})^{-\lambda} \left( [nt] F_{[nt]}(x-) - F(x-) \right) - \frac{J_r(x)}{r!} \sum_{i=1}^{[nt]} H_r(\xi_i) \\
+ \sup_{t \in [0, 1], x \in [0, 1]} d_{n,r}^{-1} (\min\{x, 1 - x\})^{-\lambda} \left( n F_n(x-) - F(x-) \right) - \frac{J_r(x)}{r!} \sum_{i=1}^{n} H_r(\xi_i).
\]

The second summand is smaller than the first summand. Therefore, it suffices to deal with the first summand. Due to continuity of $J_r$ and $F$, we have
\[
\sup_{t \in [0, 1], x \in [0, 1]} d_{n,r}^{-1} (\min\{x, 1 - x\})^{-\lambda} \left( [nt] F_{[nt]}(x-) - F(x-) \right) - \frac{1}{r!} J_r(x) \sum_{i=1}^{[nt]} H_r(\xi_i) \\
= \sup_{t \in [0, 1], x \in [0, 1]} d_{n,r}^{-1} (\min\{x, 1 - x\})^{-\lambda} \left( [nt] F_{[nt]}(x) - F(x) \right) - \frac{1}{r!} J_r(x) \sum_{i=1}^{[nt]} H_r(\xi_i).
\]

The right-hand side of the above equation is $O_P(n^{-\theta})$ due to Proposition 3.1.
All in all, we arrive at

\[
\sup_{x \in [0, 1]} \frac{(\min \{ \hat{F}_n^-(x), 1 - \hat{F}_n^-(x) \})^\lambda}{(\min \{ x, 1 - x \})^\lambda} \times \sup_{t \in [0, 1], x \in [0, 1]} d_{n,t}^{-1} (\min \{ x, 1 - x \})^{-\lambda} \left| \left( \left\lfloor nt \right\rfloor F_{\lfloor nt \rfloor} (x) - \left\lfloor nt \right\rfloor F_n (x) \right) - J_r (x) \right| \\
= O_p \left( d_{n,d}^\lambda (n^{-d}) + d_{n,r}^{-1} n^\lambda \right) = o_p(1),
\]

since by assumption \( n^\lambda = o \left( d_{n,d}^\lambda (n^{-d}) \right) \), \( n^\lambda d_{n,r} = o(n) \) and \( d_{n,r}^\lambda = o(n^d) \) for any \( \lambda < 1/3 \). This completes the proof of Theorem 3.1.

**Proofs under local alternatives**

Recall that, under the alternative of a change in the mean, the observations are generated by a triangular array \( X_{n,i} \), \( 1 \leq i \leq n \), \( n \in \mathbb{N} \), defined by

\[
X_{n,i} = \begin{cases} 
Y_i & \text{if } i \leq \lfloor n\tau \rfloor, \\
Y_i + h_n & \text{if } i > \lfloor n\tau \rfloor,
\end{cases}
\]

where \( 0 < \tau < 1 \), \( h_n, n \in \mathbb{N} \), is a non-negative deterministic sequence and \( Y_n = G(\xi_n), n \in \mathbb{N} \), is a subordinated Gaussian sequence.

Let \( F \) denote the marginal distribution function of \( Y_n, n \in \mathbb{N} \), and let \( Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)} \) denote the order statistics of \( Y_1, Y_2, \ldots, Y_n \).

Consider the following (stochastic) transformation:

\[
H_n(x) := \begin{cases} 
F(x) & \text{if } x < Y_n - h_n, \\
F(x - h_n) & \text{if } x > Y_n + h_n,
\end{cases}
\]

linear interpolated in between.

Its inverse is given by

\[
H_n^-(x) = \begin{cases} 
F^-(x) & \text{if } x < F \left( Y_n - h_n \right), \\
F^-(x) + h_n & \text{if } x > F \left( Y_n \right),
\end{cases}
\]

linear interpolated in between.

Let \( Y_{n,i} := H_n \left( X_{n,i} \right), 1 \leq i \leq n, n \in \mathbb{N} \), denote the transformed observations and note that \( H_n \) is a strictly monotone function. As a consequence, we have

\[
R_i = \sum_{j=1}^{n} 1 \{ X_{n,j} \leq X_{n,i} \} = \sum_{j=1}^{n} 1 \{ Y_{n,j} \leq Y_{n,i} \},
\]
i.e., the rank statistics are not affected by the transformation $H_n$. Instead of considering the triangular array $X_{n,i}$, $1 \leq i \leq n$, $n \in \mathbb{N}$, we may therefore as well consider the transformed observations $Y_{n,i}$, $1 \leq i \leq n$, $n \in \mathbb{N}$.

In the following, $F_{k,l}$ refers to the empirical distribution function of $Y_{n,k}, \ldots, Y_{n,l}$, i.e.,

$$F_{k,l}(x) := \frac{1}{l-k+1} \sum_{i=k}^{l} 1\{Y_{n,i} \leq x\}.$$ 

For notational convenience, we write $F_l$ instead of $F_{1,l}$.

**Proof of Proposition 3.2.** Our goal is to prove a reduction principle for the sequential empirical process $F_{\lfloor nt \rfloor}(x) - x$, $t \in [0,1], x \in [0,1]$, where

$$F_n(x) := \frac{1}{n} \sum_{i=1}^{n} 1\{Y_{n,i} \leq x\}, \quad Y_{n,i} := H_n(X_{n,i}),$$

with respect to the weighted supremum norm.

For this, we split the weighted supremum norm.

For notational convenience, we write $F_l$ instead of $F_{1,l}$.

Convergence of the first summand follows directly from Proposition 3.1. Therefore, it remains to show that the second summand is $\mathcal{O}_P(h^\rho_n)$.

Noting that $\lfloor nt \rfloor F_{\lfloor nt \rfloor}(x) = \lfloor nt \rfloor F_{\lfloor nt \rfloor}(x) + (\lfloor nt \rfloor - \lfloor nt \rfloor) F_{\lfloor nt \rfloor + 1, \lfloor nt \rfloor}(x)$, we split the
summand as follows:

$$\sup_{t \in [\tau, 1], x \in [0, 1]} (\min\{x, 1 - x\})^{-\lambda} \left| d_{n,r}^{-1} \left| nt \right| \left( F_{[nt]}(x) - x \right) \right| - \frac{1}{r!} J_r(F^-(x)) d_{n,r}^{-1} \sum_{i=1}^{\lceil nt \rceil} H_r(\xi_i) + \left( \frac{\lceil nt \rceil}{n} - \frac{\lceil n\tau \rceil}{n} \right) d_{n,r}^{-1} n \left( x - F^-(x) - h_n \right) \right|$$

$$= \sup_{x \in [0, 1]} (\min\{x, 1 - x\})^{-\lambda} \left| d_{n,r}^{-1} \left| n\tau \right| \left( F_{[n\tau]}(x) - x \right) - \frac{1}{r!} J_r(F^-(x)) d_{n,r}^{-1} \sum_{i=1}^{\lceil n\tau \rceil} H_r(\xi_i) \left| + \sup_{t \in [\tau, 1], x \in [0, 1]} (\min\{x, 1 - x\})^{-\lambda} \left| d_{n,r}^{-1} \left( \left\lfloor nt \right\rfloor - \left\lfloor n\tau \right\rfloor \right) \left( F_{\left\lfloor nt \right\rfloor+\left\lfloor nt \right\rfloor}(x) - x \right) \right| - \frac{1}{r!} J_r(F^-(x)) d_{n,r}^{-1} \sum_{i=\left\lfloor n\tau \right\rfloor+1}^{\lceil nt \rceil} H_r(\xi_i) + \left( \frac{\left\lfloor nt \right\rfloor}{n} - \frac{\left\lfloor n\tau \right\rfloor}{n} \right) d_{n,r}^{-1} n \left( x - F^-(x) - h_n \right) \right|.$$ 

The first summand on the right-hand side of the above inequality is of order $O_P \left( h_n^p \right)$ according to Proposition 3.1. Therefore, we restrict our considerations to the second summand, which can be written as

$$\sup_{t \in [\tau, 1], x \in [0, 1]} (\min\{x, 1 - x\})^{-\lambda} \left| d_{n,r}^{-1} \left( \left\lfloor nt \right\rfloor - \left\lfloor n\tau \right\rfloor \right) \left( F_{\left\lfloor nt \right\rfloor+\left\lfloor nt \right\rfloor}(x) - x \right) \right| - \frac{1}{r!} J_r(F^-(x)) d_{n,r}^{-1} \sum_{i=\left\lfloor n\tau \right\rfloor+1}^{\lceil nt \rceil} H_r(\xi_i) + \left( \frac{\left\lfloor nt \right\rfloor}{n} - \frac{\left\lfloor n\tau \right\rfloor}{n} \right) d_{n,r}^{-1} n \left( x - F^-(x) - h_n \right) \right|$$

$$= \sup_{t \in [\tau, 1], x \in [0, 1]} (\min\{x, 1 - x\})^{-\lambda} \left| d_{n,r}^{-1} \sum_{i=\left\lfloor n\tau \right\rfloor+1}^{\lceil nt \rceil} \left(1_{\{y_i \leq h_n^{-1}(x) - h_n\}} - x \right) \right| - \frac{1}{r!} J_r(F^-(x)) d_{n,r}^{-1} \sum_{i=\left\lfloor n\tau \right\rfloor+1}^{\lceil nt \rceil} H_r(\xi_i) + \left( \frac{\left\lfloor nt \right\rfloor}{n} - \frac{\left\lfloor n\tau \right\rfloor}{n} \right) d_{n,r}^{-1} n \left( x - F^-(x) - h_n \right) \right|.$$
Repeated application of the triangular inequality yields

\[
\begin{align*}
\sup_{t \in [\tau, 1], x \in [0, 1]} \left( \min\{x, 1 - x\} \right)^{-\lambda} & \left| d_{n,r}^{-1} \sum_{i = \lfloor n\tau \rfloor + 1}^{\lfloor nt \rfloor} \left( 1 \{ Y_i \leq H_n^-(x) - h_n \} - x \right) 
- \frac{J_r(F^-(x))}{r!} d_{n,r}^{-1} \sum_{i = \lfloor n\tau \rfloor + 1}^{\lfloor nt \rfloor} H_r(\xi_i) + \left( \frac{|nt|}{n} - \frac{|n\tau|}{n} \right) d_{n,r}^{-1} n \left( x - F \left( F^-(x) - h_n \right) \right) \right| \\
\leq & \sup_{t \in [\tau, 1], x \in [0, 1]} \left( \min\{x, 1 - x\} \right)^{-\lambda} \left| d_{n,r}^{-1} \sum_{i = \lfloor n\tau \rfloor + 1}^{\lfloor nt \rfloor} \left( 1 \{ Y_i \leq H_n^-(x) - h_n \} - F \left( H_n^-(x) - h_n \right) \right) 
- \frac{1}{r!} J_r(H_n^-(x) - h_n) d_{n,r}^{-1} \sum_{i = \lfloor n\tau \rfloor + 1}^{\lfloor nt \rfloor} H_r(\xi_i) \right| \\
+ & \sup_{t \in [\tau, 1], x \in [0, 1]} \left( \min\{x, 1 - x\} \right)^{-\lambda} \left| d_{n,r}^{-1} \left( |nt| - |n\tau| \right) \left( F \left( H_n^-(x) - h_n \right) - x \right) 
+ \left( \frac{|nt|}{n} - \frac{|n\tau|}{n} \right) d_{n,r}^{-1} n \left( x - F \left( F^-(x) - h_n \right) \right) \right| \\
+ & \frac{1}{r!} \sup_{t \in [\tau, 1], x \in [0, 1]} \left( \min\{x, 1 - x\} \right)^{-\lambda} \left| J_r(H_n^-(x) - h_n) - J_r(F^-(x)) \right|
\times \sup_{t \in [\tau, 1]} \left| d_{n,r}^{-1} \sum_{i = \lfloor n\tau \rfloor + 1}^{\lfloor nt \rfloor} H_r(\xi_i) \right| .
\end{align*}
\]
For the first summand (17) on the right-hand side of the above inequality, we have

\[
\sup_{t \in [r,1], \; x \in [0,1]} \left( \min \{x, 1-x\} \right)^{-\lambda} d_{n,r}^{-1} \sum_{i=\lceil n\tau \rceil + 1}^{\lceil nt \rceil} \left( \mathbb{1}_{\{Y_i \leq H_n^{-1}(x) - h_n\}} - F \left( H_n^{-1}(x) - h_n \right) \right)
\]

\[
- \frac{1}{r!} J_r(H_n^{-1}(x) - h_n) d_{n,r}^{-1} \sum_{i=\lceil n\tau \rceil + 1}^{\lceil nt \rceil} H_r(\xi_i) \bigg|_{x \to 0}.
\]

\[
= \sup_{t \in [r,1], \; x \in [0,1]} \left( \min \{x, 1-x\} \right)^{-\lambda} d_{n,r}^{-1} \sum_{i=\lceil n\tau \rceil + 1}^{\lceil nt \rceil} \left( \mathbb{1}_{\{Y_i \leq F(H_n^{-1}(x) - h_n)\}} - F \left( H_n^{-1}(x) - h_n \right) \right)
\]

\[
- \frac{1}{r!} J_r(H_n^{-1}(x) - h_n) d_{n,r}^{-1} \sum_{i=\lceil n\tau \rceil + 1}^{\lceil nt \rceil} H_r(\xi_i) \bigg|_{x \to 0}.
\]

\[
\leq \sup_{x \in [0,1]} \left( \min \{F(H_n^{-1}(x) - h_n), 1 - F(H_n^{-1}(x) - h_n)\} \right)^{-\lambda} \left( \min \{x, 1-x\} \right)
\]

\[
\sup_{t \in [r,1], \; x \in [0,1]} \left( \min \{x, 1-x\} \right)^{-\lambda} d_{n,r}^{-1} \sum_{i=\lceil n\tau \rceil + 1}^{\lceil nt \rceil} \left( \mathbb{1}_{\{Y_i \leq x\}} - x \right) - \frac{J_r(F^{-}(x))}{r!} \sum_{i=\lceil n\tau \rceil + 1}^{\lceil nt \rceil} H_r(\xi_i). \]

The second factor on the right-hand side of the above inequality is \( \mathcal{O}_P(n^{-\theta}) \) according to Proposition 3.1.

For the second summand (18), we have

\[
\sup_{t \in [r,1], \; x \in [0,1]} \left( \min \{x, 1-x\} \right)^{-\lambda} \left( \left\lceil \frac{|nt| - |n\tau|}{n} \right\rceil \right) \left( F \left( H_n^{-1}(x) - h_n \right) - x \right)
\]

\[
+ \left( \frac{|nt|}{n} - \frac{|n\tau|}{n} \right) d_{n,r}^{-1} n \left( x - F \left( F^{-}(x) - h_n \right) \right) \bigg|_{x \to 0}.
\]

\[
\leq \sup_{x \in [0,1]} \left( \min \{x, 1-x\} \right)^{-\lambda} \frac{n}{d_{n,r}} \left| F \left( F^{-}(x) - h_n \right) - F \left( H_n^{-1}(x) - h_n \right) \right| \bigg|_{x \to 0}.
\]

All in all, it therefore remains to show that

\[
\sup_{x \in [0,1]} \left( \min \{F(H_n^{-1}(x) - h_n), 1 - F(H_n^{-1}(x) - h_n)\} \right)^{-\lambda} \left( \min \{x, 1-x\} \right) = \mathcal{O}_P(1), \tag{19}
\]

\[
\sup_{x \in [0,1]} \left( \min \{x, 1-x\} \right)^{-\lambda} \frac{n}{d_{n,r}} \left| F \left( F^{-}(x) - h_n \right) - F \left( H_n^{-1}(x) - h_n \right) \right| = \mathcal{O}_P \left( h_n^{\min(\rho,\lambda)} \right), \tag{20}
\]

and

\[
\sup_{x \in [0,1]} \left( \min \{x, 1-x\} \right)^{-\lambda} \left| J_r(H_n^{-1}(x) - h_n) - J_r(F^{-}(x)) \right| = \mathcal{O}_P \left( h_n^{\min(\rho,\lambda)} \right). \tag{21}
\]
In order to show (19), note that
\[
\sup_{x \in [0,1]} \left( \frac{\min\{F(H_n^- (x) - h_n), 1 - F(H_n^- (x) - h_n)\}}{\min\{x, 1 - x\}} \right)^\lambda \\
\leq \sup_{x \in [0,1]} \left( \frac{F(H_n^- (x) - h_n)}{x} \right)^\lambda + \sup_{x \in [0,1]} \left( \frac{1 - F(H_n^- (x) - h_n)}{1 - x} \right)^\lambda \\
\leq \sup_{x \in [0,1]} \left( \frac{F(F^- (x))}{x} \right)^\lambda + \sup_{x \in [0,1]} \left( \frac{1 - F(H_n^- (x) - h_n)}{1 - x} \right)^\lambda \\
= 1 + \sup_{x \in [0,1]} \left( \frac{1 - F(H_n^- (x) - h_n)}{1 - x} \right)^\lambda.
\]
For the second summand on the right-hand side, we have
\[
\sup_{x \in [F(Y_n), 1]} \left( \frac{1 - F(H_n^- (x) - h_n)}{1 - x} \right)^\lambda = 1.
\]
Moreover, using the fact that \(H_n^- (x) \geq F^- (x)\), it follows that
\[
\sup_{x \in [0, F(Y_n)]} \left( \frac{1 - F(H_n^- (x) - h_n)}{1 - x} \right)^\lambda \leq \sup_{x \in [0, F(Y_n)]} \left( \frac{1 - F(F^- (x) - h_n)}{1 - x} \right)^\lambda \\
= 1 + \sup_{x \in [0, F(Y_n)]} \left( \frac{x - F(F^- (x) - h_n)}{1 - x} \right)^\lambda.
\]
Note that, since \(\sup_{x \in [0, F(Y_n)]}(1 - x)^{2\lambda - 1} = (1 - F(Y_n))^{2\lambda - 1} = O_p(n^{2\lambda - 1})\),
\[
\sup_{x \in [0, F(Y_n)]} \left( \frac{x - F(F^- (x) - h_n)}{1 - x} \right)^\lambda \\
\leq h_n n^{\lambda(1-2\lambda)} \sup_{x \in [0,1]} \left( \frac{\min\{x, 1 - x\}^{-2\lambda} h_n^{-1} (x - F(F^- (x) - h_n))}{x} \right)^\lambda \\
= O \left( h_n n^{\lambda(1-2\lambda)} \right).
\]
The right-hand side of the above inequality is \(O(1)\) due to the fact that \(n^{\lambda + \rho - 1} = O \left( d_n^{-1} \right)\) by assumption. All in all, (19) follows.

In order to show (21), recall that for \(x < y\)
\[
|J_r(x) - J_r(y)| \leq C \sqrt{F(y) - F(x)}.
\]
As a result, we have
\[
\sup_{x \in [0,1]} (\min\{x, 1-x\})^{-\lambda} |J_r(H_n^- (x) - h_n) - J_r(F^- (x))| \leq \sqrt{\sup_{x \in [0,1]} (\min\{x, 1-x\})^{-2\lambda} (x - F(H_n^- (x) - h_n))}.
\]

Note that
\[
\sup_{x \in [0,1]} (\min\{x, 1-x\})^{-2\lambda} (x - F(H_n^- (x) - h_n)) \\
\leq h_n \sup_{x \in [0,1]} (\min\{x, 1-x\})^{-2\lambda} d_{n,r,n}^{-1} (x - F(F^- (x) - h_n)) \\
\leq h_n \sup_{x \in [0,1]} (\min\{x, 1-x\})^{-2\lambda} |d_{n,r,n}^{-1} (x - F(F^- (x) - h_n))| - f(F^- (x))| \\
+ h_n \sup_{x \in [0,1]} (\min\{x, 1-x\})^{-2\lambda} f(F^- (x)).
\]

Due to Assumptions (8) and (9), the right-hand side of the above inequality is \(O_P(h_n)\) and (21) follows, so that it remains to show (20). For this, it is enough to consider the interval \([F(Y_n) - h_n, 1]\). To see this, note that \(H_n^- (x) = F^- (x)\) for \(x \leq F(Y_n) - h_n\), so that
\[
\sup_{x \in [0,F(Y_n)-h_n]} (\min\{x, 1-x\})^{-\lambda} |d_{n,r,n}^{-1} (F(F^- (x) - h_n) - F(H_n^- (x) - h_n))| = 0.
\]

On the other hand, we have \(|H_n^- (x) - F^- (x)| \leq h_n\) for \(x \geq F(Y_n) - h_n\). As \(1 = F_n(Y_n)\) and consequently
\[
1 - F(Y_n) = F_n(Y_n) - F(Y_n) \leq \frac{d_n}{n} \sup_x \frac{n}{d_n} |F_n(x) - x| = O_P(h_n),
\]
it follows that for \(x \geq Y_n - h_n\)
\[
(\min\{x, 1-x\})^{-\lambda} \leq O_P(h_n^\lambda)(\min\{x, 1-x\})^{-2\lambda}
\]
for \(x \geq F(Y_n) - h_n\). As a result, we obtain
\[
\sup_{x \in [F(Y_n) - h_n, 1]} (\min\{x, 1-x\})^{-\lambda} |d_{n,r,n}^{-1} (F(F^- (x) - h_n) - F(H_n^- (x) - h_n))| \\
= \sup_{x \in [F(Y_n) - h_n, 1]} (\min\{x, 1-x\})^{-\lambda} |d_{n,r,n}^{-1} (x - F(F^- (x) - h_n)) - f(F^- (x))| \\
+ \sup_{x \in [F(Y_n) - h_n, 1]} (\min\{x, 1-x\})^{-\lambda} f(F^- (x)) \leq O_P(h_n^\lambda) + O_P(h_n^\lambda) \sup_{x \in [0,1]} (\min\{x, 1-x\})^{-2\lambda} f(F^- (x)) \\
= O_P(h_n^{\min\{p,\lambda\}}),
\]
using Assumptions (8) and (9). Therefore, (20) holds and the proof is complete.

Before proving Theorem 3.2, we establish a number of auxiliary results:

**Lemma A.1.** Given the assumptions of Theorem 3.2, it holds that

\[
\sup_{x \in [0, 1]} \left| \tilde{F}_n^-(x) - x \right| = O_P(h_n)
\]

with \( \tilde{F}_n^- \) defined by

\[
\tilde{F}_n^-(x) = \begin{cases} 
0 & \text{if } x = 0, \\
\frac{Y_{n,(i)}}{n} & \text{if } x = \frac{i}{n+1}, \\
1 & \text{if } x = 1, \\
\text{linear interpolated in between,}
\end{cases}
\]

and

\[
\sup_{x \in [0, 1]} (\min\{x, 1-x\})^{-\lambda} \left| \tilde{F}_n^-(x) - x \right| = O_P(n^\lambda h_n).
\]

**Proof.** It is clear that the function \( \tilde{F}_n^-(x) - x, x \in [0, 1] \), takes its maximum for some \( x \in \{1/(n+1), 2/(n+1), \ldots, n/(n+1)\} \), because it is linear between these points. As a result, we may conclude that

\[
\sup_{x \in [0, 1]} (\min\{x, 1-x\})^{-\lambda} \left| \tilde{F}_n^-(x) - x \right| = \text{max}_{i=1, \ldots, n} \left| \tilde{F}_n^-(\frac{i}{n+1}) - \frac{i}{n+1} \right| = \text{max}_{i=1, \ldots, n} \left| Y_{n,(i)} - \frac{i}{n+1} \right|
\]

By definition, \( F_n(Y_{n,(i)}) = i/n \). It then follows that

\[
\sup_{x \in [0, 1]} \left| \tilde{F}_n^-(x) - x \right| = \max_{i=1, \ldots, n} \left| \tilde{F}_n^-(\frac{i}{n+1}) - \frac{i}{n+1} \right| = \max_{i=1, \ldots, n} \left| Y_{n,(i)} - \frac{i}{n+1} \right|
\]

\[
\leq \text{max}_{i=1, \ldots, n} \left| \frac{i}{n} - Y_{n,(i)} \right| + \frac{1}{n+1} = \sup_{x \in [0, 1]} |F_n(x) - x| + \frac{1}{n+1}.
\]

Proposition 3.2 yields

\[
\sup_{x \in [0, 1]} |F_n(x) - x| = O_P\left(\frac{d_{n,r}}{n}\right).
\]

This completes the proof.
Lemma A.2. Given the assumptions of Theorem 3.2, it holds that
\[
\sup_{x \in [0,1]} \left( \frac{\min\{\hat{F}_n^-(x), 1 - \hat{F}_n^-(x)\}}{\min\{x, 1 - x\}} \right)^\lambda \leq O_P(n^{2\epsilon_1})
\]
with \(\hat{F}_n^-\) defined in (15) and
\[
\sup_{x \in [0,1]} \left( \frac{\min\{x, 1 - x\}}{\min\{\hat{F}_n^-(x), 1 - \hat{F}_n^-(x)\}} \right)^{\epsilon_1} = O_P(n^{2\epsilon_1}).
\]

Proof. Because \(\hat{F}_n^-\) is piecewise linear, we have
\[
\frac{\hat{F}_n^-(x)}{x} = (n + 1)\hat{F}_n^- \left( \frac{1}{n + 1} \right) \quad \text{for } x \leq \frac{1}{n + 1},
\]
\[
\frac{1 - \hat{F}_n^-(x)}{1 - x} = (n + 1) \left( 1 - \hat{F}_n^- \left( \frac{n}{n + 1} \right) \right) \quad \text{for } x \geq \frac{n}{n + 1}.
\]

Using this and the inequality \(x^\lambda \leq 1 + |x - 1|^\lambda\), we get
\[
\sup_{x \in [0,1]} \left( \frac{\min\{\hat{F}_n^-(x), 1 - \hat{F}_n^-(x)\}}{\min\{x, 1 - x\}} \right)^\lambda
\leq \sup_{x \in [1/(n+1), 1/2]} \left( \frac{\hat{F}_n^-(x)}{x} \right)^\lambda + \sup_{x \in [1/2, n/(n+1)]} \left( \frac{1 - \hat{F}_n^-(x)}{1 - x} \right)^\lambda
\leq \sup_{x \in [1/(n+1), 1/2]} \left( \frac{\hat{F}_n^-(x) - x}{x} \right)^\lambda + \sup_{x \in [1/2, n/(n+1)]} \left( \frac{|1 - \hat{F}_n^-(x) - (1 - x)|}{(1 - x)^\lambda} \right)^\lambda + 2
\leq (n + 1)^\lambda \sup_{x \in [0,1]} |\hat{F}_n^-(x) - x|^\lambda + 2 = O_P(d_{n,r}^\lambda).
\]

Similar arguments lead to
\[
\sup_{x \in [0,1]} \left( \frac{\min\{x, 1 - x\}}{\min\{\hat{F}_n^-(x), 1 - \hat{F}_n^-(x)\}} \right)^{\epsilon_1}
\leq \left( \max \left\{ \frac{1}{\hat{F}_n^-(1/(n+1))}, \frac{1}{1 - \hat{F}_n^- (n/(n+1))} \right\} \right)^{\epsilon_1} \sup_{x \in [0,1]} |\hat{F}_n^-(x) - x|^\epsilon_1 + 2.
\]

It remains to show that the first factor on the right-hand side of the above inequality is of order \(O_P(n^{\epsilon_1})\). For any constant \(C > 0\), it holds that
\[
P \left( \frac{1}{\hat{F}_n^-(1/(n+1))} \geq Cn \right) = P \left( Y_{n,(1)} \leq \frac{1}{Cn} \right) \leq \sum_{i=1}^n P \left( Y_i \leq \frac{1}{Cn} \right) \leq n \frac{1}{C} \leq \frac{1}{C}.
\]
and consequently \((F_n^{-1}(1/(n+1)))^{-1} = O_P(n)\). The same holds for \((1 - F_n^{-1}(n/(n+1)))^{-1}\).

This completes the proof. 

**Lemma A.3.** Given the assumptions of Theorem 3.2, it holds that

\[
\sup_{x \in [0,1]} \left( \min\{x, 1-x\}\right) \lambda \left| J_r \left( F^- \left( F_n^{-1}(x) \right) \right) - J_r \left( F^{-} (x) \right) \right| = O_P \left( n^\lambda \sqrt{h_n} \right).
\]

**Proof.** Recall that for \(x < y\)

\[
|J_r(x) - J_r(y)| \leq C \sqrt{F(y) - F(x)}.
\]

With Lemma A.1 it then follows that

\[
\begin{align*}
\sup_{x \in [0,1]} \left( \min\{x, 1-x\}\right) \lambda \left| J_r \left( F^- \left( F_n^{-1}(x) \right) \right) - J_r \left( F^{-} (x) \right) \right| \\
\leq \sqrt{\sup_{x \in [0,1]} \left( \min\{x, 1-x\}\right) - 2\lambda \left| F_n^{-1}(x) - x \right|} \\
= O_P \left( n^\lambda \sqrt{h_n} \right)
\end{align*}
\]

\[\square\]

**Lemma A.4.** Given the assumptions of Theorem 3.2, it holds that

\[
\sup_{x \in [0,1]} \left( \min\{x, 1-x\}\right) - (\lambda - \epsilon) \left| d_{n,r}^{-1} n \left( F_n^{-1}(x) - F \left( F_n^{-1}(x) - h_n \right) \right) \\
- d_{n,r}^{-1} n \left( x - F \left( F_n^{-1}(x) - h_n \right) \right) \right| = o_P(1).
\]

**Proof.** We rewrite the difference as

\[
\left( F_n^{-1}(x) - F \left( F_n^{-1}(x) - h_n \right) \right) - \left( x - F \left( F_n^{-1}(x) - h_n \right) \right) = g(F_n^{-1}(x)) - g(x)
\]

with \(g(x) := x - F \left( F_n^{-1}(x) - h_n \right)\). The function \(g\) has the derivative

\[
g'(x) = 1 - \frac{f(F^{-}(x) - h_n)}{f(F^{-}(x))},
\]

which yields

\[
g(F_n^{-1}(x)) - g(x) = \left( 1 - \frac{f(F^{-}(\xi) - h_n)}{f(F^{-}(\xi))} \right) \left( \tilde{F}_n^{-1}(x) - x \right)
\]
for some $\zeta_x \in \left( \min\{\hat{F}_n^-(x), x\}, \max\{\hat{F}_n^-(x), x\} \right)$. We conclude that

$$\sup_{x \in [0, 1]} \left( \min\{x, 1 - x\}\right)^{-\lambda(\lambda^*)} \left| d_{n,r}^{-1} n \left( \hat{F}_n^-(x) - F^\left( F_n^-(x) - h_n \right) \right) - \left. \frac{d_{n,r}^{-1} n (x - F^\left( F_n^-(x) - h_n \right))}{x} \right| \right.$$ 

$$\leq d_{n,r}^{-1} \sup_{x \in [0, 1]} \left( \min\{\zeta_x, 1 - \zeta_x\}\right)^{-\epsilon_1} \sup_{x \in [0, 1]} \left( \min\{\zeta_x, 1 - \zeta_x\}\right)^{\epsilon_1} \left( 1 - \frac{f(F^\left( \zeta_x \right) - h_n)}{f(F^\left( \zeta_x \right))} \right) \sup_{x \in [0, 1]} \left( \min\{x, 1 - x\}\right)^{-\lambda} \left| \hat{F}_n^-(x) - x \right| .$$

By condition (11), Lemma A.1, and Lemma A.2 it follows that the right-hand side of the above inequality is of the order $O_P \left( nd_{n,r}^{-1} \epsilon_1 h_n n^\lambda h_n \right) = O_P \left( n^{\lambda-1} d_{n,r}^{1+\epsilon_1} \right) = o_P \left( 1 \right)$.

**Proof of Theorem 3.2.** Our goal is to derive a reduction principle for the two-parameter empirical process of the ranks, i.e., for

$$\hat{G}_{[nt]}(x) - \frac{\lfloor nt \rfloor}{n} \hat{G}_n(x) \quad t \in [0, 1], \quad x \in [0, 1]$$

with $\hat{G}_k(x) := \sum_{i=1}^k \{ \frac{i}{nt} \leq R_i \leq x \}$ and $R_i = \sum_{j=1}^n 1\{ Y_{n,j} \leq Y_{n,i} \}$. Recall that

$$\hat{G}_{[nt]}(x) - \frac{\lfloor nt \rfloor}{n} \hat{G}_n(x) = \lfloor nt \rfloor F_{[nt]}(\hat{F}_n^-(x)) - \lfloor nt \rfloor F_n(\hat{F}_n^-(x)),$$

i.e., we have to show that

$$\sup_{t \in [0, 1], x \in [0, 1]} d_{n,r}^{-1} (\min\{x, 1 - x\})^{-\lambda} \left| \lfloor nt \rfloor F_{[nt]}(\hat{F}_n^-(x)) - \lfloor nt \rfloor F_n(\hat{F}_n^-(x)) \right|$$

$$\leq \frac{1}{\epsilon_1} \sup_{x \in [0, 1]} \left| \frac{\lfloor nt \rfloor - \lfloor nt \rfloor}{n} \right| \left( 1 - \frac{\lfloor nt \rfloor}{n} \right) \frac{n}{d_{n,r} x (F(x) - h_n)} = o_P(1).$$
Moreover, Lemma A.2 yields

\[
\sup_{t \in [0,1], x \in [0,1]} d_{n,r}^{-1}(\min \{x, 1 - x\})^{-\lambda} \left| [nt] F_{[nt]} (\hat{F}_n (x) - ) - [nt] F_n (\hat{F}_n (x) - ) \right|
\]

\[
- \frac{1}{r!} J_r (F^- (x)) \left( \sum_{i=1}^{[nt]} H_r (\xi_i) - \frac{[nt]}{n} \sum_{i=1}^{n} H_r (\xi_i) \right)
\]

\[
+ \left( 1_{\{t > \tau\}} \left( \frac{|nt|}{n} - \frac{|n\tau|}{n} \right) - \frac{|nt|}{n} \left( 1 - \frac{|n\tau|}{n} \right) \right) \frac{n}{d_{n,r}} (x - F^- (F^n (x) - h_n))
\]

\[
= \sup_{t \in [0,1], x \in [0,1]} d_{n,r}^{-1}(\min \{x, 1 - x\})^{-\lambda} \left| [nt] F_{[nt]} (\hat{F}_n (x) - ) - [nt] F_n (\hat{F}_n (x) - ) \right|
\]

\[
- \frac{1}{r!} J_r (F^- (\hat{F}_n^-(x))) \left( \sum_{i=1}^{[nt]} H_r (\xi_i) - \frac{[nt]}{n} \sum_{i=1}^{n} H_r (\xi_i) \right)
\]

\[
+ \left( 1_{\{t > \tau\}} \left( \frac{|nt|}{n} - \frac{|n\tau|}{n} \right) - \frac{|nt|}{n} \left( 1 - \frac{|n\tau|}{n} \right) \right) \frac{n}{d_{n,r}} (\hat{F}_n^- (x) - F^- (\hat{F}_n^- (x) - h_n))
\]

\[
+ o_P (1)
\]

Moreover, Lemma A.2 yields

\[
\sup_{t \in [0,1], x \in [0,1]} d_{n,r}^{-1}(\min \{x, 1 - x\})^{-\lambda} \left| [nt] F_{[nt]} (\hat{F}_n (x) - ) - [nt] F_n (\hat{F}_n (x) - ) \right|
\]

\[
- \frac{1}{r!} J_r (F^- (\hat{F}_n^-(x))) \left( \sum_{i=1}^{[nt]} H_r (\xi_i) - \frac{[nt]}{n} \sum_{i=1}^{n} H_r (\xi_i) \right)
\]

\[
+ \left( 1_{\{t > \tau\}} \left( \frac{|nt|}{n} - \frac{|n\tau|}{n} \right) - \frac{|nt|}{n} \left( 1 - \frac{|n\tau|}{n} \right) \right) d_{n,r}^{-1} (x - F^- (F^n (x) - h_n))
\]

\[
= \mathcal{O}_P (n^{-\lambda} h_n^\lambda)
\]

\[
\sup_{t \in [0,1], x \in [0,1]} d_{n,r}^{-1}(\min \{x, 1 - x\})^{-\lambda} \left| [nt] F_{[nt]} (x - ) - [nt] F_n (x - ) \right|
\]

\[
- \frac{J_r (F^- (x))}{r!} \left( \sum_{i=1}^{[nt]} H_r (\xi_i) - \frac{[nt]}{n} \sum_{i=1}^{n} H_r (\xi_i) \right)
\]

\[
+ \left( 1_{\{t > \tau\}} \left( \frac{|nt|}{n} - \frac{|n\tau|}{n} \right) - \frac{|nt|}{n} \left( 1 - \frac{|n\tau|}{n} \right) \right) \frac{n}{d_{n,r}} (\hat{F}_n^- (x) - F^- (\hat{F}_n^- (x) - h_n))
\].
Due to continuity of $J_r$ and $F$, it remains to show that

\[
\sup_{t \in [0,1], x \in [0,1]} d_{n,r}^{-1} (\min \{x, 1-x\})^{-\lambda} \left| \frac{\lfloor nt \rfloor}{n} F_{\lfloor nt \rfloor} (x) - \frac{\lfloor nt \rfloor}{n} F_n (x) \right|
\]

\[
- \frac{1}{r!} J_r (F^- (x)) \left( \sum_{i=1}^{\lfloor nt \rfloor} H_r (\xi_i) - \frac{\lfloor nt \rfloor}{n} \sum_{i=1}^{n} H_r (\xi_i) \right)
\]

\[
+ \left( 1_{\{t > \tau\}} \frac{\lfloor nt \rfloor - \lfloor n\tau \rfloor}{n} - \frac{\lfloor nt \rfloor}{n} \left(1 - \frac{\lfloor n\tau \rfloor}{n} \right) \right) \frac{n}{d_{n,r}} (x - F (F^- (x) - h_n)) \right| = o_P \left( d_{n,r}^{-\lambda} \right).
\]

It follows by Proposition 3.2 that the above expression is $O_P (h_n^p)$ with $\rho$ as in that proposition. As $d_{n,r}^{p+\lambda} = o (n^p)$, this completes the proof.

\[\square\]

**Appendix B: Additional simulation results**

This section provides a detailed description of the finite sample performance of rank-based testing procedures. More precisely, Tables 1, 2, and 3 report the frequencies of rejections of the self-normalized Wilcoxon change-point test, the self-normalized Van der Waerden change-point test, and the self-normalized CuSum test for normal margins, Pareto margins, and Cauchy margins. All calculations are based on 5,000 realizations of time series with sample sizes $n = 300$ and $n = 500$. Test decisions are based on an application of the sampling-window method for a significance level of 5%, meaning that the values of the test statistics are compared to the 95%-quantile of the empirical distribution function $\hat{F}_{m,n,l_n}$ defined by (14). Moreover, block lengths $l_n = \lfloor n^\gamma \rfloor$ with $\gamma \in \{0.4, 0.5, 0.6\}$ are considered. Under the alternative $A$ of a change in the mean, the power of the testing procedures is analyzed by considering different choices for the height of the level shift, denoted by $h$, and the location of the change-point, denoted by $\tau$. In the tables, the columns that are superscribed by $h = 0$ correspond to the frequency of a type 1 error, i.e. the rejection rate under the hypothesis.
Table 1. Rejection rates of the self-normalized CuSum (C), the self-normalized Wilcoxon (W) and the self-normalized Van der Waerden (V) change-point tests obtained by subsampling with block length \( l_n = \lfloor n^\gamma \rfloor \), \( \gamma \in \{0.4, 0.5, 0.6\} \), for transformed fractional Gaussian noise time series of length \( n \) with Hurst parameter \( H \), marginal standard normal distribution and a change in location of height \( h \) after a proportion \( \tau \) of the simulated data.

| \( n \) | \( l_n \) | \( h = 0 \) | \( h = 0.5 \) | \( h = 1 \) |
|-------|--------|-----------|-----------|-----------|
|       |        | W         | V         | C         | W         | V         | C         | W         | V         | C         |
| 300   | 9      | 0.043     | 0.062     | 0.061     | 0.266     | 0.316     | 0.304     | 0.701     | 0.762     | 0.752     |
| 300   | 17     | 0.063     | 0.071     | 0.072     | 0.316     | 0.335     | 0.338     | 0.734     | 0.770     | 0.773     |
| 300   | 30     | 0.070     | 0.075     | 0.077     | 0.319     | 0.330     | 0.334     | 0.699     | 0.725     | 0.728     |
| 500   | 12     | 0.055     | 0.059     | 0.060     | 0.407     | 0.444     | 0.442     | 0.855     | 0.881     | 0.881     |
| 500   | 22     | 0.064     | 0.066     | 0.065     | 0.429     | 0.448     | 0.445     | 0.853     | 0.876     | 0.878     |
| 500   | 41     | 0.069     | 0.070     | 0.068     | 0.422     | 0.428     | 0.430     | 0.824     | 0.840     | 0.843     |

\( \tau = 0.25 \)

| \( n \) | \( l_n \) | \( h = 0 \) | \( h = 0.5 \) | \( h = 1 \) |
|-------|--------|-----------|-----------|-----------|
| 300   | 9      | 0.050     | 0.056     | 0.047     | 0.500     | 0.565     | 0.547     | 0.955     | 0.973     | 0.967     |
| 300   | 17     | 0.056     | 0.058     | 0.060     | 0.566     | 0.585     | 0.580     | 0.966     | 0.969     | 0.969     |
| 300   | 30     | 0.056     | 0.057     | 0.057     | 0.556     | 0.571     | 0.571     | 0.952     | 0.952     | 0.950     |
| 500   | 12     | 0.060     | 0.072     | 0.071     | 0.690     | 0.726     | 0.718     | 0.993     | 0.994     | 0.994     |
| 500   | 22     | 0.064     | 0.066     | 0.065     | 0.708     | 0.721     | 0.720     | 0.992     | 0.992     | 0.993     |
| 500   | 41     | 0.070     | 0.070     | 0.068     | 0.700     | 0.709     | 0.708     | 0.985     | 0.986     | 0.985     |

\( \tau = 0.5 \)

| \( n \) | \( l_n \) | \( h = 0 \) | \( h = 0.5 \) | \( h = 1 \) |
|-------|--------|-----------|-----------|-----------|
| 300   | 9      | 0.073     | 0.083     | 0.069     | 0.293     | 0.345     | 0.307     | 0.757     | 0.800     | 0.759     |
| 300   | 17     | 0.064     | 0.073     | 0.070     | 0.318     | 0.333     | 0.328     | 0.753     | 0.765     | 0.756     |
| 300   | 30     | 0.073     | 0.077     | 0.077     | 0.319     | 0.322     | 0.321     | 0.730     | 0.732     | 0.727     |
| 500   | 12     | 0.063     | 0.075     | 0.069     | 0.377     | 0.415     | 0.406     | 0.855     | 0.874     | 0.862     |
| 500   | 22     | 0.068     | 0.072     | 0.070     | 0.386     | 0.403     | 0.400     | 0.855     | 0.860     | 0.853     |
| 500   | 41     | 0.074     | 0.077     | 0.077     | 0.389     | 0.386     | 0.385     | 0.821     | 0.820     | 0.821     |

\( H = 0.6 \)

\( H = 0.7 \)

\( H = 0.8 \)

\( H = 0.9 \)
Table 2. Rejection rates of the self-normalized CuSum (C), the self-normalized Wilcoxon (W) and the self-normalized Van der Waerden (V) change-point tests obtained by subsampling with block length $l_n = \lfloor n^\gamma \rfloor$, $\gamma \in \{0.4, 0.5, 0.6\}$, for transformed fractional Gaussian noise time series of length $n$ with Hurst parameter $H$, marginal Pareto(3)-distribution and a change in location of height $h$ after a proportion $\tau$ of the simulated data.

| $h$ | $\tau = 0.25$ | $\tau = 0.5$ |
|-----|----------------|---------------|
|     | $h = 0$        | $h = 0.5$     | $h = 1$ |
|     | $h = 0.5$        | $h = 1$ |
| $n$ | $l_n$ | W  | V  | C  | W  | V  | C  | W  | V  | C  |
| 300 | 9    | 0.045 | 0.063 | 0.017 | 0.843 | 0.919 | 0.214 | 0.974 | 0.989 | 0.688 |
| 300 | 17   | 0.066 | 0.072 | 0.045 | 0.869 | 0.919 | 0.341 | 0.975 | 0.986 | 0.785 |
| 300 | 30   | 0.082 | 0.084 | 0.061 | 0.828 | 0.883 | 0.350 | 0.936 | 0.959 | 0.733 |
| 500 | 12   | 0.053 | 0.062 | 0.026 | 0.944 | 0.975 | 0.436 | 0.995 | 0.999 | 0.882 |
| 500 | 22   | 0.058 | 0.062 | 0.042 | 0.948 | 0.972 | 0.497 | 0.993 | 0.997 | 0.902 |
| 500 | 41   | 0.066 | 0.068 | 0.052 | 0.923 | 0.958 | 0.494 | 0.980 | 0.988 | 0.865 |
| 300 | 9    | 0.061 | 0.077 | 0.026 | 0.574 | 0.691 | 0.130 | 0.809 | 0.879 | 0.471 |
| 300 | 17   | 0.070 | 0.076 | 0.050 | 0.569 | 0.654 | 0.195 | 0.803 | 0.848 | 0.553 |
| 300 | 30   | 0.075 | 0.080 | 0.064 | 0.528 | 0.612 | 0.201 | 0.726 | 0.777 | 0.508 |
| 500 | 12   | 0.067 | 0.076 | 0.038 | 0.691 | 0.785 | 0.224 | 0.898 | 0.932 | 0.658 |
| 500 | 22   | 0.071 | 0.075 | 0.051 | 0.691 | 0.769 | 0.263 | 0.888 | 0.921 | 0.683 |
| 500 | 41   | 0.074 | 0.074 | 0.058 | 0.644 | 0.718 | 0.261 | 0.826 | 0.867 | 0.635 |
| 300 | 9    | 0.078 | 0.098 | 0.039 | 0.350 | 0.434 | 0.084 | 0.574 | 0.664 | 0.352 |
| 300 | 17   | 0.075 | 0.085 | 0.060 | 0.311 | 0.372 | 0.114 | 0.525 | 0.585 | 0.378 |
| 300 | 30   | 0.077 | 0.084 | 0.073 | 0.285 | 0.340 | 0.121 | 0.447 | 0.507 | 0.324 |
| 500 | 12   | 0.066 | 0.083 | 0.044 | 0.397 | 0.495 | 0.131 | 0.620 | 0.697 | 0.427 |
| 500 | 22   | 0.069 | 0.073 | 0.054 | 0.379 | 0.448 | 0.145 | 0.587 | 0.645 | 0.430 |
| 500 | 41   | 0.067 | 0.072 | 0.060 | 0.337 | 0.398 | 0.146 | 0.520 | 0.578 | 0.381 |
| 300 | 9    | 0.097 | 0.121 | 0.052 | 0.268 | 0.331 | 0.138 | 0.395 | 0.470 | 0.380 |
| 300 | 17   | 0.073 | 0.089 | 0.064 | 0.208 | 0.254 | 0.141 | 0.320 | 0.375 | 0.372 |
| 300 | 30   | 0.072 | 0.081 | 0.070 | 0.174 | 0.213 | 0.123 | 0.264 | 0.316 | 0.285 |
| 500 | 12   | 0.078 | 0.097 | 0.063 | 0.249 | 0.309 | 0.152 | 0.386 | 0.457 | 0.411 |
| 500 | 22   | 0.067 | 0.077 | 0.065 | 0.212 | 0.255 | 0.147 | 0.334 | 0.383 | 0.388 |
| 500 | 41   | 0.071 | 0.077 | 0.064 | 0.186 | 0.225 | 0.126 | 0.288 | 0.335 | 0.309 |
\[
\begin{align*}
\tau &= 0.25 \\
\tau &= 0.5
\end{align*}
\]

| \(n\) | \(l_n\) | \(n\) | \(l_n\) | \(n\) | \(l_n\) | \(n\) | \(l_n\) | \(n\) | \(l_n\) |
|---|---|---|---|---|---|---|---|---|---|
| 300 | 9 | 0.034 | 0.044 | 0.013 | 0.202 | 0.262 | 0.010 | 0.464 | 0.488 | 0.010 |
| 300 | 17 | 0.059 | 0.063 | 0.048 | 0.279 | 0.311 | 0.042 | 0.553 | 0.541 | 0.043 |
| 300 | 30 | 0.070 | 0.072 | 0.063 | 0.286 | 0.319 | 0.053 | 0.544 | 0.530 | 0.057 |
| 500 | 12 | 0.051 | 0.055 | 0.021 | 0.377 | 0.447 | 0.024 | 0.723 | 0.729 | 0.026 |
| 500 | 22 | 0.062 | 0.062 | 0.043 | 0.414 | 0.471 | 0.048 | 0.759 | 0.743 | 0.047 |
| 500 | 41 | 0.066 | 0.071 | 0.050 | 0.417 | 0.460 | 0.057 | 0.745 | 0.722 | 0.055 |
| 300 | 9 | 0.034 | 0.044 | 0.014 | 0.155 | 0.205 | 0.013 | 0.373 | 0.402 | 0.010 |
| 300 | 17 | 0.057 | 0.059 | 0.044 | 0.215 | 0.250 | 0.046 | 0.457 | 0.447 | 0.042 |
| 300 | 30 | 0.069 | 0.070 | 0.051 | 0.232 | 0.260 | 0.057 | 0.454 | 0.437 | 0.056 |
| 500 | 12 | 0.038 | 0.039 | 0.020 | 0.281 | 0.334 | 0.019 | 0.588 | 0.589 | 0.022 |
| 500 | 22 | 0.048 | 0.049 | 0.037 | 0.315 | 0.357 | 0.040 | 0.618 | 0.605 | 0.046 |
| 500 | 41 | 0.056 | 0.057 | 0.046 | 0.323 | 0.369 | 0.049 | 0.603 | 0.584 | 0.054 |
| 300 | 9 | 0.042 | 0.057 | 0.018 | 0.121 | 0.164 | 0.015 | 0.243 | 0.276 | 0.015 |
| 300 | 17 | 0.062 | 0.069 | 0.044 | 0.156 | 0.178 | 0.044 | 0.286 | 0.293 | 0.050 |
| 300 | 30 | 0.067 | 0.072 | 0.056 | 0.171 | 0.190 | 0.055 | 0.295 | 0.293 | 0.062 |
| 500 | 12 | 0.046 | 0.056 | 0.025 | 0.176 | 0.218 | 0.029 | 0.364 | 0.377 | 0.027 |
| 500 | 22 | 0.045 | 0.059 | 0.039 | 0.200 | 0.225 | 0.049 | 0.388 | 0.385 | 0.045 |
| 500 | 41 | 0.060 | 0.063 | 0.045 | 0.208 | 0.226 | 0.058 | 0.392 | 0.374 | 0.053 |
| 300 | 9 | 0.044 | 0.057 | 0.018 | 0.171 | 0.214 | 0.015 | 0.243 | 0.276 | 0.015 |
| 300 | 17 | 0.062 | 0.069 | 0.044 | 0.156 | 0.178 | 0.044 | 0.286 | 0.293 | 0.050 |
| 300 | 30 | 0.067 | 0.072 | 0.056 | 0.171 | 0.190 | 0.055 | 0.295 | 0.293 | 0.062 |
| 500 | 12 | 0.046 | 0.056 | 0.025 | 0.176 | 0.218 | 0.029 | 0.364 | 0.377 | 0.027 |
| 500 | 22 | 0.045 | 0.059 | 0.039 | 0.200 | 0.225 | 0.049 | 0.388 | 0.385 | 0.045 |
| 500 | 41 | 0.060 | 0.063 | 0.045 | 0.208 | 0.226 | 0.058 | 0.392 | 0.374 | 0.053 |
| 300 | 9 | 0.044 | 0.057 | 0.018 | 0.121 | 0.164 | 0.015 | 0.243 | 0.276 | 0.015 |
| 300 | 17 | 0.062 | 0.069 | 0.044 | 0.156 | 0.178 | 0.044 | 0.286 | 0.293 | 0.050 |
| 300 | 30 | 0.067 | 0.072 | 0.056 | 0.171 | 0.190 | 0.055 | 0.295 | 0.293 | 0.062 |
| 500 | 12 | 0.046 | 0.056 | 0.025 | 0.176 | 0.218 | 0.029 | 0.364 | 0.377 | 0.027 |
| 500 | 22 | 0.045 | 0.059 | 0.039 | 0.200 | 0.225 | 0.049 | 0.388 | 0.385 | 0.045 |
| 500 | 41 | 0.060 | 0.063 | 0.045 | 0.208 | 0.226 | 0.058 | 0.392 | 0.374 | 0.053 |

Table 3. Rejection rates of the self-normalized CuSum (C), the self-normalized Wilcoxon (W) and the self-normalized Van der Waerden (V) change-point tests obtained by subsampling with block length \(l_n = \lfloor n^\gamma \rfloor\), \(\gamma \in \{0.4, 0.5, 0.6\}\), for transformed fractional Gaussian noise time series of length \(n\) with Hurst parameter \(H\), marginal Cauchy-distribution and a change in location of height \(h\) after a proportion \(\tau\) of the simulated data.
Table 4. Rejection rates of the self-normalized CuSum (C), the self-normalized Wilcoxon (W) and the self-normalized Van der Waerden (V) change-point tests obtained by subsampling with block length $l_n = \lfloor n^\gamma \rfloor$, $\gamma \in \{0.4, 0.5, 0.6\}$, for transformed fractional Gaussian noise time series of length $n$ with Hurst parameter $H$, marginal $\chi^2$-distribution and a change in location of height $h$ after a proportion $\tau$ of the simulated data.
Additional data examples

In the following two more data sets are analyzed with regard to structural changes by an application of rank-based testing procedures and the methodologies described in Section 4. The observations stem from hydrology and network traffic monitoring, areas that typically give rise to long-range dependent time series. These examples have been well-studied in the literature, such that we will embed our analysis into the context of existing results.

B.1. Argentina rainfall

The first data set consists of 113 measurements of yearly rainfall volume in the Argentinian province of Tucumán from 1884 to 1996; see Figure 6. The data was monitored by the Agricultural Experiment Station Obispo Colombres (EEAOC). It was provided by Dr. César M. Lamelas, a meteorologist at EEAOC, and reported in [66]. The construction of a dam on the Salí river, one of the main running waters in the province of Tucumán, between 1952 and 1962 may serve as an explanation for an abrupt change in the data.

![Figure 6](image_url). Annual rainfall volume (in millimeters) in the Argentinian province of Tucumán from 1884 to 1996.

We base our analysis of the data on the 83 measurements of yearly rainfall from 1914 to 1996. As examples of rank-based change-point tests we choose the self-normalized Wilcoxon and the self-normalized Van der Waerden test and compare the performances of these tests to that resulting from the change-point test based on the self-normalized CuSum statistic defined in [54].
Figure 7. Values of the self-normalized two-sample CuSum, Wilcoxon, and Van der Waerden statistics between 1940 and 1970 computed on the basis of the annual rainfall volume in the Argentinian province of Tucumán from 1914 to 1996.

Figure 7 depicts the values of the two-sample statistics $T_{k,n}(a_i), i = 1, 2$, as defined in formula (13), with $a_1(i) = (n+1)^{-1}i$ (the score function that yields the self-normalized two-sample Wilcoxon statistic) and $a_2(i) = \Phi^{-1}\left((n+1)^{-1}i\right)$ (the score function that yields the self-normalized two-sample Van der Waerden statistic) and the self-normalized two-sample CuSum statistic as defined in [54], between 1940 and 1970. All three line plots achieve their maximum in the year 1957, thereby indicating a potential change-point location that corresponds to this year. In this regard, our findings agree with the results of previous analysis and the expectation of a change occurring as a consequence of the construction of a dam on the Salí river between 1952 and 1962.

However, approximating the distribution of the self-normalized test statistics by the sampling window method with block size $l = \lfloor \sqrt{n} \rfloor = 9$ yields $p$-values of 0 for the self-normalized CuSum test, 0.0225 for the self-normalized Van der Waerden test and 0.0858 for the self-normalized Wilcoxon test, i.e., at a level of significance of 5% the self-normalized Van der Waerden and the self-normalized CuSum test reject the hypothesis, while the self-normalized Wilcoxon test decides in favor of the hypothesis of no change. In order to decide which test decision is more plausible, we compare our findings with previous analysis on that same data set:

[66] base a change-point test on isotonic regression and consider additionally a trend detection test for stationary time series proposed in [14]. The isotonic regression method of [66] strongly favors a location shift in the data between 1955 and 1956, whereas Brillinger’s test does not show any evidence of a change. [17] examine the possibility of changes in mean and variance of the observations. For this, they apply two different information criteria, both suggesting a change-point occurring around 1954. Since, according to their analysis, their is no sufficient indication of a change in the variability of the time series, the change-point is attributed to an increase in the mean precipitation.

[1] provide statistically significant evidence for a change-point by carrying out Bayesian inference. [34] argue that assuming independence of the data-generating random variables cannot be justified with regard to the precipitation data, indicating that valid change-
point analysis has to account for serial correlations among the observations. By adjusting for dependence, they base their analysis of the data on a Bayes-type statistic developed in [35] and a likelihood ratio statistic studied in [19]. Both procedures provide statistical evidence for a change in the mean of the data around 1956. Incorporating the prior knowledge about a potential change-point location (construction of the dam on the Salí river) by restricting the search area for the change-point accordingly, [56] identify a change in the data on the basis of a self-normalized CuSum statistic.

Since CuSum-type hypothesis tests for a change in mean may be susceptible to outliers in the data, [56] additionally note that the self-normalized median test, as a robust alternative to CuSum-based testing procedures, rejects the hypothesis of stationarity at the 5% significance level, as well. [63] provide further evidence for a change in location by pointing out that Hodges-Lehmann and CuSum-type tests, resulting as special cases from the consideration of U-quantile-based change-point tests, both reject the null hypothesis of no change at the 5% level of significance.

As noted by [63], the normal quantile plot (see Figure 8) supports the assumption of normally distributed data. The Van der Waerden test is known to be more efficient when the normality assumption is satisfied, yielding an explanation for contradicting outcomes of the two rank-based testing procedures and a justification for choosing the Van der Waerden test over the Wilcoxon test when normally distributed data is considered.

B.2. Ethernet traffic

The second data set consists of the arrival rate of Ethernet data (bytes per 10 milliseconds) from a local area network (LAN) measured at Bellcore Research and Engineering Center in 1989. The data has been taken from the longmemo package in R. For more information on the LAN traffic monitoring see [37] and [4].

Figure 9 reveals that the observations are strongly right-skewed. As the Wilcoxon and the Van der Waerden statistics are computed from ranks, this is not expected to affect tests and estimators that are based on these statistics. Moreover, estimation of the Hurst

Figure 8. Normal quantile plot of the annual rainfall volume in the Argentinian province of Tucumán from 1914 to 1996.
parameter by the local Whittle procedure with bandwidth parameter $b_n = \lfloor n^{2/3} \rfloor$ yields an estimate $\hat{H} = 0.845$ indicating long-range dependence. This is consistent with the results of [38] and [62].

![Ethernet traffic in bytes per 10 milliseconds from a LAN measured at Bellcore Research Engineering Center.](image)

**Figure 9.** Ethernet traffic in bytes per 10 milliseconds from a LAN measured at Bellcore Research Engineering Center.

Again, we base our analysis of the data on the two self-normalized rank-based test statistics $T_n(a_i), i = 1, 2$, as defined in formula (12), and the self-normalized CuSum statistic as defined in [54]. If compared to their asymptotic critical values, none of the test statistics values can be interpreted as an indication of a structural change in the data for any value of $H \in \left(\frac{1}{2}, 1\right)$. Furthermore, approximating the distribution of the self-normalized test statistics by the sampling window method with block size $l = 40$ yields $p$-values of 0.7159 for the self-normalized Wilcoxon test, 0.7164 for the self-normalized Van der Waerden test and 0.7972 for the self-normalized CuSum test, i.e., at any common choice of a level of significance all three change-point tests reject the hypothesis. In addition, as shown in [9], even when accounting for ties or multiple changes in the data, an application of the self-normalized Wilcoxon change-point test does not provide evidence for a change-point in the mean.

Let us compare our findings to previous analysis by other authors: [18] examined this data set in view of change-points under the assumption that a FARIMA model holds for segments of the data. The number of different sections and the location of potential change-points are chosen by a model selection criterion. The algorithm proposed by [18] detects multiple changes in the parameters of the corresponding FARIMA time series. However, the change-point estimation algorithm proposed in that paper is not robust to skewness or heavy-tailed distributions and decisively relies on the assumption of FARIMA time series. This seems to contradict observations made by [12] as well as [62] who stress that the Ethernet traffic data is very unlikely to be generated by FARIMA processes.

All in all, we analyzed three data sets from different domains of application in change-point analysis. Even though the self-normalized CuSum and Wilcoxon change-point tests
have been studied before, our theoretical results facilitate the consideration of other rank-based statistics. In particular, we find that test decisions that are based on the self-normalized Van der Waerden test in some cases concur with those of the self-normalized CuSum change-point test while in others they coincide with conclusions drawn from an application of the self-normalized Wilcoxon change-point test.

Acknowledgments

The authors would like to thank Prof. Marie Hušková for encouraging research on the considered topic.

References

[1] Alvérez, E. E. and Dey, D. K. (2009). Bayesian isotonic changepoint analysis. *Annals of the Institute of Statistical Mathematics* **61** 355 – 370.
[2] Bai, S. and Taqqu, M. S. (2017). On the validity of resampling methods under long memory. *The Annals of Statistics* **45** 2365 – 2399.
[3] Bai, S. and Taqqu, M. S. (2019). Sensitivity of the Hermite rank. *Stochastic Processes and their Applications* **129** 822 – 840.
[4] Beran, J. (1994). *Statistics for Long-Memory Processes*. Chapman & Hall.
[5] Beran, J., Möhrle, S. and Ghosh, S. (2016). Testing for Hermite rank in Gaussian subordination processes. *Journal of Computational and Graphical Statistics* **25** 917 – 934.
[6] Beran, J., Feng, Y., Ghosh, S. and Kulik, R. (2013). *Long-Memory Processes*. Springer-Verlag Berlin Heidelberg.
[7] Betken, A. (2016). Testing for Change-Points in Long-Range Dependent Time Series by Means of a Self-Normalized Wilcoxon Test. *Journal of Time Series Analysis* **37** 785 – 809.
[8] Betken, A. and Kulik, R. (2019). Testing for change in long-memory stochastic volatility time series. *Journal of Time Series Analysis* **40** 707 – 738.
[9] Betken, A. and Wendler, M. (2018). Subsampling for general statistics under long range dependence with application to change point analysis. *Statistica Sinica* **28** 1199–1224.
[10] Betken, A., Buchsteiner, J., Dehling, H., Münker, I., Schnurr, A. and Woerner, J. H. (2020). Ordinal patterns in long-range dependent time series. *Scandinavian Journal of Statistics*.
[11] Beutner, E., Zähle, H. et al. (2012). Deriving the asymptotic distribution of U- and V-statistics of dependent data using weighted empirical processes. *Bernoulli* **18** 803 – 822.
[12] Bhansali, R. J. and Kokoszka, P. S. (2001). Estimation of the Long-Memory Parameter: A Review of Recent Developments and an Extension. *Lecture Notes-Monograph Series* **37** 125 – 150.
[13] BHATTACHARYYA, G. K. and JOHNSON, R. A. (1968). Nonparametric tests for shift at an unknown time point. *The Annals of Mathematical Statistics* **39** 1731 – 1743.

[14] BRILLINGER, D. R. (1989). Consistent detection of a monotonic trend superposed on a stationary time series. *Biometrika* **76** 23 – 30.

[15] BUCHSTEINER, J. (2015). Weak convergence of the weighted sequential empirical process of some long-range dependent data. *Statistics & Probability Letters* **96** 170 – 179.

[16] CHAN, N. H., NG, W. L. and YAU, C. Y. (2018). A Self-Normalized Approach to Sequential Change-point Detection for Time Series. *Statistica Sinica*.

[17] CHEN, J., GUPTA, A. K. and PAN, J. (2006). Information criterion and change point problem for regular models. *Sankhyā: The Indian Journal of Statistics, Series A* 252 – 282.

[18] COULON, M., CHABERT, M. and SWAMI, A. (2009). Detection of Multiple Changes in Fractional Integrated ARMA Processes. *IEEE Transactions on Signal Processing* **57** 48 – 61.

[19] Csörgő, M. and HORVÁTH, L. (1997). *Limit theorems in change-point analysis*. Wiley Chichester; New York.

[20] DARKHOVSKH, B. S. (1976). A Nonparametric Method for the a Posteriori Detection of the “Disorder” Time of a Sequence of Independent Random Variables. *Theory of Probability and Its Applications* **21** 178 – 183.

[21] DEHLING, H., ROOCH, A. and TAQUÉ, M. S. (2013). Non-Parametric Change-Point Tests for Long-Range Dependent Data. *Scandinavian Journal of Statistics* **40** 153 – 173.

[22] DEHLING, H., ROOCH, A. and TAQUÉ, M. S. (2017). Power of change-point tests for long-range dependent data. *Electronic Journal of Statistics* **11** 2168 – 2198.

[23] DETTE, H. and GÖSMANN, J. (2019). A likelihood ratio approach to sequential change point detection for a general class of parameters. *Journal of the American Statistical Association* 1 – 17.

[24] DETTE, H., KOCOT, K. and VOLGUSHEV, S. (2020). Testing relevant hypotheses in functional time series via self-normalization. *arXiv:1809.06092*.

[25] FELLER, W. (1968). *An Introduction to Probability Theory and its Applications*. 1 41 – 45.

[26] GEWEKE, J. and PORTER-HUDAK, S. (1983). The estimation and application of long memory time series models. *Journal of Time Series Analysis* **4** 221 – 238.

[27] GOMBAY, E. (1994). Testing for change-points with rank and sign statistics. *Statistics & Probability Letters* **20** 49 – 55.

[28] GOMBAY, E. and HUŠKOVÁ, M. (1998). Rank based estimators of the change-point. *Journal of Statistical Planning and Inference* **67** 137 – 154.

[29] HALL, P. and JING, B. (1996). On Sample Reuse Methods for Dependent Data. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* **58** 727 – 737.

[30] HALL, P., JING, B.-Y. and LAHIRI, S. N. (1998). On the sampling window method for long-range dependent data. *Statistica Sinica* **8** 1189 – 1204.

[31] HODGES JR., J. L. and LEHMANN, E. L. (1961). Comparison of the normal
scores and Wilcoxon tests. In *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability* 1 307 – 317.

[32] Jach, A., McElroy, T. and Politis, D. N. (2012). Subsampling inference for the mean of heavy-tailed long-memory time series. *Journal of Time Series Analysis* 33 96 – 111.

[33] Jach, A., McElroy, T. and Politis, D. N. (2016). Corrigendum to ‘Subsampling Inference for the Mean of Heavy-Tailed Long-Memory Time Series’. *Journal of Time Series Analysis* 37 713 – 720.

[34] Jandhyala, V. K., Fotopoulos, S. B. and You, J. (2010). Change-point analysis of mean annual rainfall data from Tucumán, Argentina. *Environmetrics* 21 687 – 697.

[35] Jandhyala, V. K. and MacNeill, I. B. (1991). Tests for parameter changes at unknown times in linear regression models. *Journal of Statistical Planning and Inference* 27 291 – 316.

[36] Künsch, H. R. (1987). Statistical aspects of self-similar processes. In *Proceedings of the first World Congress of the Bernoulli Society* 1 67 – 74. VNU Science Press Utrecht, The Netherlands.

[37] Leland, W. E. and Wilson, D. V. (1991). High time-resolution measurement and analysis of LAN traffic: Implications for LAN interconnection. In *INFOCOM’91. Proceedings*. 1360 – 1366. IEEE.

[38] Leland, W. E., Taqqu, M. S., Willinger, W. and Wilson, D. V. (1994). On the self-similar nature of Ethernet traffic (extended version). *IEEE/ACM Transactions on Networking* 2 1 – 15.

[39] Lobato, I. N. (2001). Testing That a Dependent Process Is Uncorrelated. *Journal of the American Statistical Association* 96 1066 – 1076.

[40] Lombard, F. (1987). Rank tests for changepoint problems. *Biometrika* 74 615 – 624.

[41] Mandelbrot, B. B. (1975). Limit theorems on the self-normalized range for weakly and strongly dependent processes. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete* 31 271 – 285.

[42] Mandelbrot, B. B. and Taqqu, M. S. (1979). Robust R/S analysis of long run serial correlation, paper presented at the 42nd Session of the International Statistical Institute. *Int. Stat. Inst., Manila* 4 – 14.

[43] Mandelbrot, B. B. and Wallis, J. R. (1969). Computer experiments with fractional Gaussian noises: Part 1, averages and variances. *Water resources research* 5 228–241.

[44] Nordman, D. J. and Lahiri, S. N. (2005). Validity of the Sampling Window Method for Long-Range Dependent Linear Processes. *Econometric Theory* 21 1087 – 1111.

[45] Pettitt, A. N. (1979). Two-Sample Cramér-Von Mises Type Rank Statistics. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* 41 46 – 53.

[46] Pešta, M. and Wendler, M. (2018). Nuisance-parameter-free changepoint detection in non-stationary series. *TEST* 1 – 30.
[47] Pipiras, V. and Taqqu, M. S. (2017). Long-Range Dependence and Self-Similarity. Cambridge University Press.

[48] Politis, D. N. and Romano, J. P. (1994). Large Sample Confidence Regions Based on Subsamples under Minimal Assumptions. The Annals of Statistics 22, 2031 – 2050.

[49] Praagman, J. (1988). Bahadur Efficiency of Rank Tests for the Change-Point Problem. The Annals of Statistics 198 – 217.

[50] Pyke, R. and Shorack, G. R. (1968). Weak Convergence of a Two-sample Empirical Process and a New Approach to Chernoff-Savage Theorems. The Annals of Mathematical Statistics 39, 755 – 771.

[51] Rea, W., Oxley, L., Reale, M. and Brown, J. (2009). Estimators for long range dependence: an empirical study. arXiv preprint arXiv:0901.0762.

[52] Robinson, P. M. (1995). Gaussian Semiparametric Estimation of Long Range Dependence. The Annals of Statistics 23, 1630 – 1661.

[53] Sen, P. K. (1978). Invariance principles for linear rank statistics revisited. Sankhyā: The Indian Journal of Statistics, Series A 215 – 236.

[54] Shao, X. (2011). A simple test of changes in mean in the possible presence of long-range dependence. Journal of Time Series Analysis 32, 598 – 606.

[55] Shao, X. (2015). Self-normalization for time series: a review of recent developments. Journal of the American Statistical Association 110, 1797 – 1817.

[56] Shao, X. and Zhang, X. (2010). Testing for change points in time series. Journal of the American Statistical Association 105, 1228 – 1240.

[57] Sherman, M. and Carlstein, E. (1996). Replicate Histograms. Journal of the American Statistical Association 91, 566 – 576.

[58] Sinn, M. and Keller, K. (2011). Estimation of ordinal pattern probabilities in Gaussian processes with stationary increments. Computational Statistics & Data Analysis 55, 1781 – 1790.

[59] Szyszkoowicz, B. (1994). Weak convergence of weighted empirical type processes under contiguous and changepoint alternatives. Stochastic Processes and their Applications 50, 281 – 313.

[60] Taqqu, M. S. (1979). Convergence of Integrated Processes of Arbitrary Hermite Rank. Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete 50, 53 – 83.

[61] Taqqu, M. S., Teverovsky, V. and Willinger, W. (1995). Estimators for long-range dependence: an empirical study. Fractals 3, 785 – 798.

[62] Taqqu, M. S. and Teverovsky, V. (1997). Robustness of Whittle-type Estimators for Time Series with Long-Range Dependence. Communications in Statistics. Stochastic Models 13, 723 – 757.

[63] Vogel, D. and Wendler, M. (2017). Studentized U-quantile processes under dependence with applications to change-point analysis. Bernoulli 23, 3114 – 3144.

[64] Wang, L. (2008). Change-point detection with rank statistics in long-memory time-series models. Australian & New Zealand Journal of Statistics 50, 241 – 256.

[65] Wolfe, D. A. and Schechtman, E. (1984). Nonparametric statistical procedures for the changepoint problem. Journal of Statistical Planning and Inference 9, 389 – 396.
[66] Wu, W. B., Woodroofe, M. and Mentz, G. (2001). Isotonic regression: Another look at the changepoint problem. *Biometrika* **88** 793 – 804.

[67] Zhang, T. and Lavitas, L. (2018). Unsupervised self-normalized change-point testing for time series. *Journal of the American Statistical Association* **113** 637 – 648.

[68] Zhang, T., Ho, H.-C., Wendler, M. and Wu, W. B. (2013). Block sampling under strong dependence. *Stochastic Processes and their Applications* **123** 2323 – 2339.

[69] Zhao, Z. and Li, X. (2013). Inference for modulated stationary processes. *Bernoulli* **19** 205 – 227.