Unitary Fermi Gas: Scaling Symmetries and Exact Map

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The weakly bound system of dimers formed in a cold Fermi gas at infinite scattering length reveals an underlying SU(1,1) symmetry, which facilitates an exact map from the interacting to the non-interacting system. A systematic treatment of this symmetry for the ground state, is shown to result in a shift of energy proportional to the scaling exponent. For the excited states, novel breathing modes at integral values of the harmonic frequency \( \omega \) are predicted in one-dimension and consequences for the realization of exclusion statistics are pointed out.

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Universality of the dilute Fermi gas at unitarity has garnered considerable attention in the recent literature [1]. The unitarity regime characterized by a diverging scattering length \( a \), is the cross section reaching a maximum value \( 4\pi/k^2 \), where \( k \) is the relative momentum of the scattering particles. Using Feshbach resonance, the infinite scattering length is achieved when the energy of a pair of scattering atoms is tuned close to that of a molecular bound state by an external magnetic field, leading to a substantial resonance scattering. A hallmark feature of this system is that the pairing correlations have a range shorter than the de Broglie wavelength, leading to significantly novel behavior [2, 3]. The unitary regime is described by weakly interacting particles, which provides an ideal ground to study physics at the interface of weak coupling BCS superfluidity and BEC. However, the lack of a small parameter poses a challenging problem in understanding the structure of many-body wave functions and the associated bound states. The only available length scale in the ground state is inter particle spacing, \( a^{-1/3} \), where \( n \) is the density, since, \( a \) must drop out from all physical observables. Thus, universality requires that for both bosons and fermions, the only relevant energy scale is the Fermi energy \( E_f \). The energy \( E_0 \) of the unitary fermions has to then scale as \( \xi E_f \), where the parameter \( \xi \) does not depend on the specific form of the short range potential and is same for all potentials which have a zero energy S-wave bound state. Determining this scaling parameter \( \xi \) is central to an understanding of the degenerate Fermi gas and also its superfluid transition temperature. From the theoretical view point, a careful inspection [4] of the short distance behavior of the two-body wave function \( \psi \), in the unitary regime, shows a scaling behavior [5],

\[
\sum_{i=1}^{N} \vec{r}_i \cdot \nabla \psi = \gamma \psi, \tag{1}
\]

where \( \gamma \) is the scaling exponent. Amusingly, this leads to a shift in the ground state energy by a factor

\[
E = (\gamma + dN/2)\hbar \omega. \tag{2}
\]

To study this scaling regime, it is instructive to consider the problem of short-range interacting particles in d-dimensions and the general Hamiltonian

\[
H = \sum_{i=1}^{N} \frac{P_i^2}{2m} + \sum_{i<j} V(\vec{r}_i - \vec{r}_j), \tag{3}
\]

which satisfies the condition [1] and the Schrödinger equation \( -\frac{\hbar^2}{2m} \sum_{i=1}^{N} \nabla_i^2 \Psi + \sum_{i<j} V(\vec{r}_i - \vec{r}_j) \Psi = E \Psi \), without an external potential. Note that the two-body potential \( V \) depends only on the inter particle distance vector \( \vec{r}_i - \vec{r}_j \) and is characterized by the s-wave scattering length \( a \). The precise form of this potential is unimportant in the unitary regime. The scaling properties of this system are as follows

\[
\begin{align*}
\vec{r} &\to \lambda \vec{r}, & \Psi(\vec{r}) &\to \lambda^{d/2} \Psi(\lambda \vec{r}) \tag{4} \\
H &\to \frac{1}{\lambda^2} \left( \sum_{i=1}^{N} \frac{P_i^2}{2m} + \sum_{i<j} V(\vec{r}_i - \vec{r}_j) \right),
\end{align*}
\]

where we concentrate on potentials which have the scaling law \( V(\lambda \vec{r}) = V(\vec{r})/\lambda^2 \). As pointed out in Ref. [10], in additional to a \( 1/r^2 \) type potential in d-dimensions, it is also possible to have \( V(\vec{r} - \vec{r}') = \delta(\vec{r} - \vec{r}') \) in two dimensions. It should be mentioned that a distinctive role of \( d = 4 \) has emerged in recent literature due to the \( 1/r^2 \) behavior of the two-body wave function, leading to a logarithmic divergence in the normalization [6]. Both numerical [3] and \( \epsilon \)-expansion around \( d = 4 \) indicate, that the unitary regime can be well described by weakly interacting fermions and bosons [6]. Here, the potential \( V \) possesses a degree \( -2 \) characteristic of \( 1/r^2 \), as the manifestation of scale invariance can only allow a \( 1/r^2 \) type interaction between pairs.

In a harmonic trap, the weakly interacting particles can show several new features. The interactions of the
type given in [3], can capture the dimerization effect accurately. The formation of molecules, which subsequently condense to a BEC, necessitates the search of a zero-energy bound state, leading to a dimerization of the fermionic gas. If we consider the trapping Hamiltonian to be

\[ H_{\text{trap}} = \sum_i \frac{1}{2} m \omega_i^2 \rho_i^2, \]  

then there can be additional breathing modes with universal frequency \(2\omega\) [8, 9]. Remarkably, there exists a hidden SO(2,1) Lorentz symmetry [10], the continuous representations of which, can explain exactly the frequency of breathing modes. The starting point for such an SO(2,1) algebra is a suitable combination of the generators given in Eq. (11), (3) and (5) [10].

In this letter, we point out that in the unitary regime and for a particular class of wave-functions, this underlying SO(2,1) symmetry goes much deeper in establishing an exact map from the interacting to a non-interacting system and also naturally explains the shift in the ground state energy [5]. The SO(2,1) algebra in the scaling regime can have interesting consequences for the existence of fractional exclusion statistics [11]. There have been several works in recent literature which suggest a number of universal features connecting interacting and non-interacting regimes [12, 13, 14]. We wish to see which of these universal features are captured by the SO(2,1) symmetry algebra.

To illustrate our proposal, let us start with a general Hamiltonian of the type \( H + H_{\text{trap}} \) having a ground state wave function \( \Psi_0 \) such that \( V(\rho) = \epsilon_0 + \frac{1}{\lambda^{2m}} \sum_{i=1}^{N} \nabla_i^2 \Psi_0 \), where, \( \epsilon_0 \) is the ground-state energy [15, 16]. Let us now focus on a certain class of wave functions of the product type \( \Psi_0 = \psi G \), where \( \psi \) captures the pairing correlations associated with the scaling properties of the dimer potential given in Eq. (1) and \( G \) governs the dynamics of the trap. This form of the wave function is quite suitable in the dimerization regime, where we are interested in extracting out the features universal to the interacting and the non-interacting systems. It is worth mentioning that \( \sum_i \rho_i^2 \) completely decouples from all the degrees of freedom [10]. Without loss of generality, we consider

\[ \psi = \prod_i f(\rho_{ij}), \]  

to satisfy the scaling behavior advocated in Eq. (1), where \( f(\rho_{ij}) \) denotes a homogeneous polynomial of definite degree. For instance, if \( f \) is of the form \( \prod_{i<j} |\rho_i - \rho_j|^\beta \), where \( i, j = 1, \cdots, N \), then the degree of the polynomial is \( \beta N(N - 1)/2 \).

In order to identify the \( SO(2,1) \) or \( SU(1,1) \) generators, it is useful to make a similarity transformation with respect to \( \psi \) and obtain

\[ \tilde{H} = \psi^{-1}(H) \psi = -\hat{A} + \epsilon_0, \]  

where \( \hat{A} \equiv \frac{1}{2} \sum_{i=1}^{N} \nabla_i^2 + \sum_{i<j} \nabla_i (\ln \psi) \nabla_j \). Let us note that there are two different ways of forming a \( SU(1,1) \) algebra. An inter particle potential of the type \( V(\rho_i - \rho_j) = \frac{g^2}{2m} \sum_{i,j} (\rho_i - \rho_j)/|\rho_i - \rho_j|^3 \), leads to \( \tilde{H} = -\frac{\hbar^2}{2m} \sum_{i,j} \nabla_i^2 - \alpha \sum_{i,j=1} (\rho_i - \rho_j)/|\rho_i - \rho_j|^2 \nabla_i \epsilon_0/2 \), where \( \alpha = (1 + \sqrt{1 + 4g^2})/2 \). Going to the Cartan basis, the two generators which are common to these two \( SU(1,1) \) algebras are

\[ T_0 = -\frac{1}{2} \sum_i (\hat{r}_i \cdot \nabla_i + \epsilon_0), \]  

\[ T_+ = \frac{1}{2} \sum_i \hat{r}_i^2, \]  

where \( T_0 \) is the generator of scale transformations. The choice of the third generator can be either \( T_+ = \frac{\hbar^2}{2m} \sum_{i,j} \nabla_i^2 \) or formed from \( \tilde{H} \) as \( T_+ = \frac{1}{2} \sum_i \nabla_i^2 + \alpha \sum_{i,j=1} (\rho_i - \rho_j)/|\rho_i - \rho_j|^2 \nabla_i \) where the \( f \) and \( i \) indices stand for “free” and “interacting”, respectively. Both of these satisfy the usual \( SU(1,1) \) algebra:

\[ [T_+, T_-] = -2T_0, \quad [T_0, T_{\pm}] = \pm T_{\pm}. \]  

The above generators obtained, after performing certain similarity transformations, are tailor made for the class of wave functions of Eq. (1) in the scaling regime and are slightly different from [10]. In particular, our algebra is valid in the infinite scattering limit, where the scaling law of Eq. (1) comes into effect. The reason for the existence of two different ways of forming the algebra is the special property of the Euler operator \( Q = \sum_i \hat{r}_i \cdot \nabla_i \), in any dimension \( d \),

\[ [Q, O^d] = dO^d, \]  

which brings out the scaling dimension of a general operator \( O \). Thus, any two operators scaling with same degree can be combined to form a new algebra. For the case at hand, the two generators \( T_+^d \) and \( T_-^d \), both have the same degree with respect to \( T_0 \). For instance, when the two body interaction is of the \( 1/\rho^2 \) type, scaling \( \tilde{r}_{ij} \rightarrow \lambda \tilde{r}_{ij} \), one can check that \( [T_0, \tilde{A}] = 1/\lambda^2 \tilde{A} \), showing the same behavior as \( 1/\lambda^2 \sum_{i=1}^{N} \nabla_i^2 \). The fact that there exists a well defined Hamiltonian with good scaling properties as in [3], confirms that there are two different choices for \( SU(1,1) \).

A group theory view point to the zero energy bound states can be obtained by analyzing the ground state of the system, using the freedom of choosing a third generator to perform a series of \( SU(1,1) \) transformations. In this map, particular attention should be paid to the way in which the ground state and excited states transform under symmetry rotations. After an \( SU(1,1) \) rotation we obtain,

\[ e^{-T_-^d \tilde{H}} e^{T_-^d} \equiv \tilde{H} \equiv \sum_{i=1}^{N} \hat{r}_i \cdot \nabla_i - \hat{A} + \epsilon_0. \]
where \( \epsilon_0 = \frac{1}{2}N \frac{1}{2}N(N-1)\alpha \). Putting the normalizations back, remarkably, there is shift in the ground state energy, by \( \gamma \omega_0 \). Thus, for an \( N \) particle system, we see that the scaling exponent takes a universal formula independent of any details of the system as \( \gamma = \alpha \frac{N(N-1)}{2} \). The above analysis is in conformity with the suggestions in \([5]\), where \( \gamma \) appears exactly in the actual shift of ground state energy of the \( N \) particle system, connecting up well with the recent correspondence between non-relativistic conformal systems and their gravity duals \([12,13]\). Each value of the scaling exponent is in general related to the dimension of primary operators \([12,18]\). In this regard, the relation between the dimension of the operators and the scaling exponent in our case is \( \Delta = \alpha \frac{N(N-1)}{2} + 2\Delta \). For two particles of species spin up and spin down, \( \gamma \) can take values 0 or \(-1\) \([5]\), which means \( \alpha \) takes exactly the same values. For the case of three particles \( \gamma \approx 0.22728 \) signifies the value is 0.07576 for \( \alpha \). Since, both the \( SU(1,1) \) algebras in Eq. (3) can be embedded in the Schrödinger algebra, the exact map we constructed, confirms the duality between free and unitary fermions envisaged in \([12]\). The duality was originally argued based on insights from the AdS/CFT correspondence \([19]\) regarding the existence of a pair of non relativistic conformal field theories with operators of different dimensions. These features continue to hold for the Calogero type of models, as emphasized in \([20]\). The duality between interacting and non-interacting systems is not restricted to the \( 1/r^2 \) potential, but to any general short-range potential which satisfies the scaling law in Eq. (1).

Now, owing to the reduction of \([\hat{H}, \exp\{-\hat{A}/2\}]\) to the commutator with respect to the Euler operator as \( [\sum_i \hat{r}_i \cdot \hat{\nabla}_i, \exp\{-\hat{A}/2\}] = \hat{A} \exp\{-\hat{A}/2\} \), the operator \( \hat{T} \equiv \Phi_0 \exp\{-\hat{A}/2\} \) diagonalizes the original Hamiltonian \( \hat{H} \), giving \( \hat{H}_D = \sum_i \hat{r}_i \cdot \hat{\nabla}_i + E_0 \). \( \tag{12} \)

Finally, making another \( SU(1,1) \) rotation with respect to the generator \( T_+ \) in Eq. (6), one arrives at decoupled oscillators \( \hat{H}_{\text{decoupled}} = -\frac{1}{2} \sum_i \hat{\nabla}_i^2 + \frac{1}{2} \sum_i \hat{r}_i^2 + (E_0 - \frac{1}{2}N) \). \( \tag{13} \)

This shows an exact map from the interacting model to the free model by a series of \( SU(1,1) \) transformations. Notice that only the ground state energy gets shifted and the excited states are same as the system of decoupled oscillators. In fact, in the absence of trap, the \( SU(1,1) \) algebra can be used to make a unitary transformation, which takes particles interacting via the potential in (3) to free particles \([21]\). Thus, \( U^{-1} \hat{H} U = -\frac{\hbar^2}{2m} \sum_i \hat{\nabla}_i^2 \) where one takes \( U = e^{-(T_+^i + T_-^i)} e^{H + T_-} \). It is known that the free system obtained in this way obeys fractional statistics \([22]\). A careful scrutiny of the energy reveals the possibility of exclusion statistics of the fermions near unitarity, which also points towards a mutual \( 1/r^2 \) type short-range interaction \([11]\).

For example, in the three-body problem in a trap, in the unitarity limit, the interaction potential can actually be replaced by the following contact conditions \( \psi(r_{1i}, r_{2i}, r_{3i}) = \left( \frac{1}{r_{ij}} - \frac{1}{a} \right) A(R_{ij}, r_k) + O(r_{ij}), \tag{14} \) in the limit \( r_{ij} \equiv |r_i - r_j| \to 0 \) taken for fixed positions of the other particle \( k \) and of the center of mass \( R_{ij} \) of \( i \) and \( j \). In the molecule formation domain, \( A \neq 0 \) and the wave function is singular, implying high probability. However, for the class of wave functions in Eq. (10), which are of the Laughlin type, \( A = 0 \). Consequently, in the \( r_{ij} \to 0 \) limit, the wave function has a dependence as in Eq. (11). The density of states in these cases is high, and the wave function actually vanishes when two particles come close to each other, much like the non-interacting system, signifying a dimerization, where the particles are weakly interacting. In the condensation regime the wave function should be singular, but yet normalizable, pointing towards molecule formation. In this domain, the information about the inter particle correlations is present in the Jastrow factor, and hence the statistics can come into play. Universality of the degenerate gas on the other hand, suggests that the scaling exponent \( \gamma \) does not depend on the specific statistics of the system. Nevertheless, at unitarity, it might be useful to model the interaction as coming from fractional exclusion statistics and intriguingly, the estimates for energy per particle are in good agreement with Monte Carlo simulations \([11]\). Together with the fact that these particles are interacting via a \( 1/r^2 \) potential, points towards a possible realization of exclusion statistics in this weakly interacting system of fermions \([11]\). As is widely known, exclusion statistics is realized in one-dimension \([22]\) by the Calogero model \([24]\) and in two dimensions, by the kinetic and potential energy densities of fermions interacting with a zero-range potential in the mean-field approximation \([25]\). There is evidence emerging that such statistics can also be realized for cold atoms in three dimensions at unitarity \([26]\).

The eigen functions of \( \hat{H}_D \) are homogeneous and symmetric functions, due to the presence of the Euler operator. The unitary irreducible representations of this \( SU(1,1) \) algebra, with \( T_0 \) diagonal can be constructed as well and point towards the existence of analytic scaling solutions, similar to the coherent states found in \([27]\), in a different context. The \( n \)-th energy level eigen function of \( \hat{H} \) takes the form \( \Psi_n = \Phi_0 P_n \), where, \( P_n \equiv \exp\{-\hat{A}/2\} S_n \) are totally symmetric, inhomogeneous polynomials, with \( S_n \) denoting a choice of symmetric polynomials. One particular choice can be made
by knowing that, $\vec{r}_i$ and $\vec{\nabla}_i$ serve as the creation and annihilation operators respectively. The ground-state can be chosen: $\vec{\nabla}_i \phi_0 = 0 \, \text{ for } i = 1, 2, \cdots, N$. In two and higher dimensions, the excited states are built out of monomials depending on the square of the particle coordinates as $\prod_i^n (r_i^2)^{n_i}$, where $n_i = 0, 1, 2, \cdots$. Here, $r_i$ is to be read as the hyperspherical coordinate which decouples from the rest of the system. The corresponding energy is $E = 2 \sum_i n_i + E_0$ leading to equally spaced excited states with universal frequency $2\omega$; the factor of 2 ensuing from the typical $1/r^2$ behavior. More over, these modes are independent of the initial N-particle system, with regard to the linear and non-linear response of atom-atom interactions in a trap $E_i$. The picture changes appreciably in one-dimension, where, the symmetric polynomials can be constructed without recourse to the hyperspherical coordinates. In particular, the excited states can be formed out of the particle coordinates themselves as $\prod_i^n (x_i)^{n_i}$, which are understood to be symmetric with respect to the exchange of particle coordinates. Thus, the spectrum of excited states turns out to be integrally spaced with frequency $\sum_i n_i \omega$, for $n_i = 0, 1, 2, \cdots$. This leads us to predict that as opposed to higher dimensions, the resulting shift in the energies of the breathing modes in one-dimension are integral. This though mystifies the universality of the unitary fermi gas, suggesting a dependence of the breathing modes on the dimensionality of the trap, by singling out the special role of one-dimension.

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