The analysis of solutions behaviour of Van der Pol Duffing equation describing local brain hemodynamics

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Abstract. This article proposes the generalized model of Van der Pol — Duffing equation for describing the relaxation oscillations in local brain hemodynamics. This equation connects the velocity and pressure of blood flow in cerebral vessels. The equation is individual for each patient, since the coefficients are unique. Each set of coefficients is built based on clinical data obtained during neurosurgical operation in Siberian Federal Biomedical Research Center named after Academician E. N. Meshalkin. The equation has solutions of different structure defined by the coefficients and right side. We investigate the equations for different patients considering peculiarities of their vessel systems. The properties of approximate analytical solutions are studied. Amplitude-frequency and phase-frequency characteristics are built for the small-dimensional solution approximations.

1. Introduction

Different pathologies of cerebral blood vessel network are very dangerous for human health and life. For example, arterial aneurysm\cite{1,2} rupture may lead to cerebral hemorrhage, blood stroke and very often death. That is why studying of bloodstream behavior in the cerebral blood vessel network is a problem of a great importance.

There are different approaches to modelling of bloodstream behavior in the surroundings of anomalies. One of them is to construct empirical model based on experimental data obtained during clinical measurements. This approach is described in this paper\cite{3} and proved to be effective in different applications.

The equation of a nonlinear oscillator of Van der Pol — Duffing (VPD) is a simple model for description of bloodstream behaviour. While building the model velocity $u = u(t)$ is chosen as the control function, i.e. the right side of the equation, and pressure $q = q(t)$ is considered as sought quantity. Such equations have large and complex sets of solutions with the structure determined by different values of coefficients\cite{4}.

Being quite simple this model describes the behaviour of hemodynamic parameters near pathologies quite well.
2. Choosing mathematical model
2.1. Obtaining clinical data
The hemodynamic parameters are measured during the neurosurgical operations with the help of ComboMap device with ComboWire sensor. Blood pressure and velocity are measured simultaneously inside the vessel. The measurements are made both in arterial and venous pool in different areas of vasculature. The obtained pressure and velocity are written into the file with ADC device [5, 6] at frequency \( \approx 200\text{Hz} \). The obtained data are cleaned by using software noise wavelet filters to get rid of high-frequency noise, vibrations and the low-frequency oscillations corresponding to the respiratory rhythm. Gabor wavelet decomposition is used for this purpose [7].

2.2. Justification of the mathematical model choice
As the structure of arterial vessel is complex and multilayer, the model should consider how active vessel walls and vessel surroundings influence the bloodstream. Under such conditions the processes should be described by the multidimensional system. But we can measure and study only the projection of this system on the "velocity - pressure" plane.

So the question about such simplification correctness arises. The answer is positive in the following sense: pressure and velocity can be represented as the combination of fast and slow bloodstream changes (1)

\[
p(t) = p_m(t) + a_p(t)q(t); v(t) = v_m(t) + a_v(t)u(t),
\]

where \( p_m(t), v_m(t) \) are mean values, \( a_p(t), a_v(t) \) are amplitude values and \( q(t), u(t) \) are dimensionless normalized values of pressure and velocity.

The values \( p_m(t), v_m(t), a_p(t), a_v(t) \) are "slow variables" and \( q(t), u(t) \) are "fast variables". Slow changes are caused by the reaction to the medical intervention, whereas fast occur during one cardiac cycle. The behaviour of slow variables is regulated by central nervous system, while the behaviour of fast dimensionless variables is regulated by local vessel reaction to the pulse wave. Thus, being quite simple for investigation, fast variables describe hemodynamic parameters quite good. Our model helps to predict the fast variables behaviour.

3. VDP model
The model of generalized Van der Pol–Duffing equation [8] was suggested to identify the characteristic behaviour of "fast" hemodynamic parameters in the surroundings of vascular pathologies

\[
\varepsilon q''(t) + P_2(q)q' + P_3(q) = ku(t).
\]

The functions \( q = q(t) \) and \( u = u(t) \) in (2) are dimensionless and normalized values of blood pressure and velocity, such that \(|q| \leq 1, |u| \leq 1\). Moreover, in case of arteries velocity is the control function and pressure is found as the solution of equation. Polynomials \( P_2(q) = a_0 + a_1q + a_2q^2 \), \( P_3(q) = b_1q + b_2q^2 + b_3q^3 \) characterize friction and elasticity of the system. Real coefficients \( \{a_i, b_j, k\} \ (i = 0, 1, 2; j = 1, 2, 3) \) are built based on clinical data with the methods of inverse problem theory [9]. Finally, coefficient \( \varepsilon = 10^{-3} \) corresponds to the relaxation oscillation character.

The model was confirmed experimentally with the large amount of clinical data. The pressure values obtained in the experiment coincide well with the pressure values predicted by the model.

Later the behaviour of model solutions is studied in case when the right side of the model is replaced by the harmonic function. The function has the form of \( B \cos(\omega t) \) depending from two parameters which are frequency \( \omega \) and amplitude \( B \). These parameters are chosen from physiologically significant range of values. Such investigation allows to identify the important
properties of the model. Though due to nonlinear character of the model full research of solution properties could not be conducted.

Numerical study of amplitude-phase-frequency solution characteristics shows the system having complex behaviour in concerned range of right side frequencies and amplitudes [10].

4. Existing theorem

Let apply Wong theorem [11] to the equation (2). Theorem satisfaction guarantees the existence of bounded solution and freedom from oscillation stop. The application of Wong theorem to the equation (2) gives the necessary conditions on the coefficients (3)

\[
\begin{align*}
a_3 &> 0 \\
b_3 &> 0 \\
a_1 - \frac{a_2^2}{4a_3} &> 0.
\end{align*}
\]

(3)

There is an example of theorem accomplishment (Figure 1: a-b), the form of solution is close to linear. The left picture on the figure is solution plot in polar coordinates and the right picture is the graph of solution in yellow and right side in blue. When the theorem is not fulfilled, the nonlinear effects appear on solution (Figure 1: c-d) and the oscillation stop occurs.

Figure 1. Examples of solution behaviour depending on Wong theorem. a-b: Theorem is fulfilled, the form of solution is close to linear. c-d: Theorem is not fulfilled, the form of solution is nonlinear.
5. Analytical study by Galerkin method

Galerkin method is often used to find approximate solutions of systems with essential nonlinearities [11]. For this sought solution is represented as the linear combination of linearly independent functions.

The solution is approximated with trigonometric polynomials. The approximate solution of initial equation can be represented in the following form (4) in case of units period $T = 2\pi$.

$$u_m(t) = c_0 + \sum_{i=1}^{m} (c_{2i-1} \sin it + c_{2i} \cos it), \quad (4)$$

where $c_0, c_1, \ldots, c_{2m}$ are constants.

Compute the residual projection on the solution expansion subspace $\{1, \sin it, \cos it, i = 1, \ldots, m\}$. And than equating residual to zero, we obtain the equations for identification of the coefficients $c_0, c_1, \ldots, c_{2m}$.

5.1. One-dimensional approximation

Consider the one-dimensional Galerkin approximation. The approximate solution is of the form $q = A \cos(\omega t)$. The equation (5) connects amplitude and frequency

$$3A_3b_3 - 4Bk + 4A(b_1 - \varepsilon \omega^2) = 0. \quad (5)$$

Build the amplitude-frequency characteristic, where $\omega$ is the right side frequency, $A$ is the amplitude of approximate solution and $B$ is the right side amplitude. Different lines on the plot (Figure 2:a) correspond to different values of $B$. Coefficient $a_i$ specifying damping and $b_2$ do not affect the behaviour of approximate solution.

5.2. Two-dimensional approximation

Consider the two-dimensional case. Solution representation is of the form $q = A \cos(\omega t - \varphi)$. The equations for amplitude (6) and frequency (7) split:

$$16B^2k^2 - A^2(4b_1 + 3A^2b_3)^2 + ((4a_1 + A^2a_3)^2 -$$

$$- 8(4b_1 + 3A^2b_3)\varepsilon)\omega^2 + 16\varepsilon^2\omega^4 = 0, \quad (6)$$

$$B^3k^3(a_3\omega \cos(\varphi) - 3b_3 \sin(\varphi))^3 +$$

$$+ 4Bk\omega^2(-a_3b_1 + 3a_1b_3 + a_3\varepsilon\omega^2)^2 *$$

$$(a_1\omega \cos \varphi + (-b_1 + \varepsilon \omega^2) \sin \varphi) = 0. \quad (7)$$

Build the amplitude-frequency characteristic (Figure 2:b). Coefficients $a_2$ and $b_2$ do not influence the amplitude behaviour. Build the phase-frequency characteristic (Figure 2:c). Coefficients $a_2, b_2$ do not influence the phase behaviour.

5.3. Three-dimensional approximation

Consider three-dimensional approximation. Solution is of the form $q = A_0 + A_1 \cos(\omega t - \varphi_1)$. Obtain equations for two amplitudes (8, 9)

$$2A_0b_1 + 2A_0^2b_2 + A_1^2b_2 + 2A_0^3b_3 + 3A_0A_1^2b_3 = 0, \quad (8)$$
\begin{align}
16B^2k^2 - A_1^6(9b_3^2 + a_3^2\omega^2) + \\
+ 8A_1^4(-3b_1b_3 + (2b_2 + 3A_0b_3)^2 - a_1a_3\omega^2 - (A_0a_3(a_2 + A_0a_3) - 3b_3\varepsilon)\omega^2) - \\
- 16A_1^2(b_1^2 + 2A_0^2b_2b_3 + 3A_0b_3^2 + ((a_1 + A_0(a_2 + A_0a_3))^2 - \\
- 2(b_1 + A_0(2b_2 + 3A_0b_3))\varepsilon)\omega^2 + \varepsilon^2\omega^4) = 0, 
\end{align}

where amplitudes do not depend on phase. Build the amplitude-frequency characteristic (Figure 2: d). Amplitudes $A_0$ and $A_1$ also do not depend from right side parameters and damping coefficients.

Phase equation (10) does not split off and phase depends on $A_0$ and $A_1$

\[
\tan \phi_1 = \frac{(4a_1 + 4A_0a_2 + 4A_0^2a_3 + A_1^2a_3)\omega}{4b_1 + 8A_0b_2 + 12A_0^2b_3 + 3A_1^2b_3 - 4\varepsilon\omega^2}.
\]

Build the phase-frequency characteristic (Figure 2: e). Phase does not depend explicitly from right side amplitude of initial equation.

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\textbf{Figure 2.} Examples of amplitude-frequency and phase-frequency characteristics for one patient. a: Amplitude-frequency characteristic in case of one-dimensional approximation. b-c: Amplitude-frequency and phase-frequency characteristics in case of two-dimensional approximation. d-e: Amplitude-frequency and phase-frequency characteristics in case of three-dimensional approximation.
For all considered amplitude- and phase-frequency characteristics, the ambiguous dependence of solution amplitude and phase on the right side frequency near a physiologically significant frequency range is characteristic. Under unfavourable circumstances this can result to a dangerous from the medical point of view switching of oscillatory regimes.

6. Conclusion
The study of generalized Van der Pol–Duffing equations built on the clinical data of real patients is presented in this paper. The equation has solutions of different structure defined by the coefficients and right side. Conditions on the coefficients guaranteeing the absence of oscillation stop are found out. Amplitude-frequency and phase-frequency characteristics are built for Galerkin small-dimensional solution approximations in case of one patient. The dependencies of these characteristics from equation coefficients are found out.

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