THE SESHADRI CONSTANTS OF TANGENT SHEAVES ON TORIC VARIETIES

CHIH-WEI CHANG

ABSTRACT. In this paper, we investigate the Seshadri constant $\varepsilon(X, T_X; p)$ of the tangent sheaf $T_X$ on a complete $\mathbb{Q}$-factorial toric variety $X$. We show that $\varepsilon(X, T_X; 1) > 0$ if and only if the following statement holds true: if $a_1v_1 + \cdots + a_kv_k = 0$ where $a_i$'s are positive real numbers and $v_i$'s are primitive generators of some rays in the fan $\Delta$ that defines $X$, then $k \geq \dim X + 1$. Based on the result, we show that a smooth projective toric variety $X$ with $\varepsilon(X, T_X; p) > 0$ for some $p \in X$ is isomorphic to the projective space, confirming a special case of the conjecture proposed by M. Fulger and T. Murayama.

1. Introduction

The classical Seshadri constant of a nef invertible sheaf $\mathcal{L}$ on a projective variety $X$ at a closed point $p$, introduced by Demailly [Dem92], is defined to be

$$\varepsilon(\mathcal{L}, p) = \sup \{ t \in \mathbb{R}_{\geq 0} | \pi^*c_1(\mathcal{L}) - tE \text{ is nef, } \pi : \text{Bl}_p(X) \to X, \ E = \text{exc}(\pi) \}$$

$$= \inf \{ \frac{c_1(\mathcal{L}) \cdot C}{\text{mult}_p(C)} | C \text{ is an irreducible curve passing through } p \} \in \mathbb{R}_{\geq 0}.$$ 

The Seshadri constants of nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors are defined in the same way. The constants measure the local positivity of $\mathcal{L}$ and have a closed relation with the separation of jets of $\mathcal{L}$ and of the adjoint bundle $\omega_X \otimes_{\mathcal{O}_X} \mathcal{L}$. More precisely, the Seshadri criterion [Laz04, Theorem 1.4.12] asserts that a Cartier divisor $D$ on a projective variety $X$ is ample if and only if $\inf_{p \in X} \{ \varepsilon(D, p) \} > 0$, and, if $X$ is smooth projective, $\omega_X \otimes_{\mathcal{O}_X} \mathcal{L}$ separates $\ell$-jets at $p$ if $\varepsilon(\mathcal{L}, p) \geq \dim X + \ell + 1$ [Dem92]. See [Laz04, Chapter 5] or [BDRH+08] for more detailed introductions. Despite the fact that the classical Seshadri constants are fundamental and richly structured, there are few studies on the possible generalizations. The most notable works in this direction, among others, are Beltrametti–Schneider–Sommese [BSS93, BSS96] and Hacon [Hac00] on the Seshadri constants of ample vector bundles. Recently, Fulger and Murayama defined in [FM21] the Seshadri constants of not necessarily nef coherent sheaves on projective varieties. To state their definition, recall that for any coherent sheaf $\mathcal{G}$ on a smooth projective curve $C$, the minimal slope of $\mathcal{G}$ is defined to be

$$\mu_{\text{min}}(\mathcal{G}) := \min \{ \mu(\mathcal{H}) | \mathcal{H} \text{ is a quotient of } \mathcal{G}, \ \mu(\mathcal{H}) := \frac{\deg \mathcal{H}}{\text{rk } \mathcal{H}}, \ \deg(\mathcal{H}) := \chi(\mathcal{H}) - \text{rk } \mathcal{H} \cdot \chi(\mathcal{O}_C) \}$$

with the convention that the slope of a torsion sheaf is $\infty$. Following [FM21], we define

$$\overline{\mu}_{\text{min}}(\mathcal{G}) := \left\{ \begin{array}{ll}
\mu_{\text{min}}(\mathcal{G}) & , \text{if } \text{char}(k) = 0 \\
\lim_{n \to \infty} \frac{\mu_{\text{min}}((F^n)^* \mathcal{G})}{p^n} & , \text{if } \text{char}(k) = p
\end{array} \right.$$ 

where $k$ is the underlying field and $F : C \to C$ is the absolute Frobenius morphism. The Seshadri constant of a coherent sheaf $\mathcal{F}$ on a projective variety $X$ at a closed point $p \in X$ is defined to be
\[ \varepsilon(X, \mathcal{F}; p) := \inf_{\nu C \subseteq X} \left\{ \frac{\Pi_{\min}(\nu^* \mathcal{F})}{\mult_p(C)} \mid C \text{ is an irreducible curve, } \nu : \tilde{C} \to C \text{ is the normalization} \right\} \]

\[ \in \mathbb{R} \cup \{ \pm \infty \}. \]

We usually write \( \varepsilon(\mathcal{F}, p) \) if no confusion arises. These constants agree with the classical Seshadri constants when \( \mathcal{F} \) is a nef invertible sheaf, and it has been shown in [FM21] that they behave similarly to their classical counterparts.

\begin{itemize}
  \item \( \inf \{ \varepsilon(\mathcal{F}, p) \mid p \in X \} > 0 \) if and only if \( \mathcal{F} \) is ample [FM21 Theorem 3.11].
  \item Suppose \( X \) is smooth of dimension \( n \) and \( \mathcal{F} \) is an ample locally free sheaf of rank \( r \). If \( \varepsilon(\mathcal{F}, p) > \frac{r}{n+1} \), then \( \omega_X \otimes S^i \mathcal{F} \otimes \det \mathcal{F} \) separates \( s \)-jets at \( p \) [FM21 Proposition 5.7].
\end{itemize}

Besides being useful in studying the local positivity of coherent sheaves, Seshadri constants can also be employed to characterize the projective spaces. Bauer and Szemberg showed in [BS09 Theorem 1.7] that a complex Fano manifold is isomorphic to the projective space if the Seshadri constant of \(-K_X\) is larger than \( \dim X \) at some point. Later, Liu and Zhuang generalized the theorem to complex \( \mathbb{Q} \)-Fano varieties [LZ18 Theorem 2]. Fulger and Murayama asked a question in higher rank similar to these observations:

**Conjecture 1.1** ([FM21 Conjecture 4.9]). Let \( X \) be a smooth projective variety over an algebraically closed field. If there exists \( p \in X \) such that \( \varepsilon(T_X, p) > 0 \), then \( X \simeq \mathbb{P}^n \).

In other words, it is expected that the Seshadri constants of tangent bundles can not strictly lie between 0 and 1. The conjecture has been verified when \( X \) is a smooth projective surface or a Fano manifold [FM21 Proposition 4.8 and Corollary 4.12]. The purpose of this paper is to study Conjecture [1.1] when \( X \) is toric. Our main results are the following:

**Theorem 1.2.** Suppose \( X \) is a smooth projective toric variety. If \( \varepsilon(T_X, p) > 0 \) for some \( p \in X \), then \( X \simeq \mathbb{P}^n \).

**Corollary 1.3** (= Proposition [1.1]). Suppose \( X \) is a smooth projective toric variety. Then

\[ \varepsilon(T_X, p) = \min_{\rho \in \Delta(1)} \{ \varepsilon(D_\rho, p) \}. \]

If \( X \) is a projective \( \mathbb{Q} \)-factorial toric variety which is possibly singular, we still have \( \varepsilon(T_X, p) \geq \min_{\rho \in \Delta(1)} \{ \varepsilon(D_\rho, p) \} \) (see Lemma [2.1] and the proof of Proposition [2.3]). The author does not know any such \( X \) violating Corollary [1.3]. It is worth mentioning that an explicit formula is given in [HMPT10 Proposition 3.2] for the Seshadri constants of nef toric vector bundles at the torus invariant points. On the other hand, when \( p \) is not a torus invariant point, it’s not easy to calculate the Seshadri constants even for torus invariant divisors. For example, if \( H \) is the ample generator of the Picard group of \( X = \mathbb{P}(a, b, c) \) and \( 1 \in X \) is the identity of its dense torus, then \( \varepsilon(H, 1) = 1/\sqrt{abc} \) is equivalent to Nagata’s conjecture for \( abc \) points ([CK11 Proposition 5.2]). Somewhat, Corollary [1.3] is only of theoretical interest.

The proof of Theorem [1.2] is divided into two parts: \( p \in T \) or \( p \in X \setminus T \). If \( p \in X \setminus T \), we first show that the normal sequence of a \( T \)-invariant prime divisor \( Y \) inside the smooth projective toric variety \( X \) splits, and then apply an induction argument. To deal with the case \( p \in T \) (may assume \( p = 1 \)), we establish a combinatorial criterion of when \( \varepsilon(X, T_X; 1) > 0 \) (Theorem [2.10]). Combining with [Bat91 Proposition 3.2] on the existence of certain primitive collection, we show that the only smooth projective toric variety with \( \varepsilon(X, T_X; 1) > 0 \) is the projective space. For this part, if char \( k = 0 \), we remark that one can simply apply [CMSB02 Corollary 0.4(11)] to get the same result (see [FM21 Proposition 4.8(2)] for details). The advantage of our argument is that it does not depend on the characteristic of \( k \).
2. A combinatorial criterion

In the remaining of the paper, all varieties are defined over an algebraic closed field $k$ of any characteristic. The main result of this section is Theorem 2.10. Instead of repeating [FM21, Section 3], we just recall some of the lemmas that are necessary in our calculation.

**Lemma 2.1.** Let $X$ be a projective variety.

1. If $\mathcal{V} \to \mathcal{Q}$ is a surjective morphism of coherent sheaves on $X$, then $\varepsilon(\mathcal{Q}, p) \geq \varepsilon(\mathcal{V}, p)$ for all $p \in X$. More generally, if $\mathcal{V}|_U \to \mathcal{Q}|_U$ is surjective for some open subset $U$ of $X$ containing $p$, then $\varepsilon(\mathcal{Q}, p) \geq \varepsilon(\mathcal{V}, p)$.

2. ([FM21, Lemma 3.31]) If
   \[ \mathcal{G} \to \mathcal{F} \to \mathcal{H} \to 0 \]
   is an exact sequence of coherent sheaves on $X$, then for all $p \in X$
   \[ \varepsilon(\mathcal{F}, p) \geq \min\{\varepsilon(\mathcal{G}, p), \varepsilon(\mathcal{H}, p)\}. \]
   If $\mathcal{F} = \mathcal{G} \oplus \mathcal{H}$, then "=" holds.

3. Suppose $f : Y \to X$ is a birational morphism such that $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is an isomorphism for some open subset $U$ containing $p$. Then $\varepsilon(X, \mathcal{F}; p) = \varepsilon(Y, f^*\mathcal{F}; p)$ for any coherent sheaf $\mathcal{F}$ on $X$.

**Proof.** (1) The first part is just [FM21, Lemma 3.18]. For the second part, note that the pullback of $\text{coker}(\mathcal{V} \to \mathcal{Q})$ via the normalization of any irreducible curve passing through $p$ is torsion. The statement now follows from (2) (it's true even when $\infty$ occurs).

(3) Let $C$ be an irreducible curve on $X$ passing through $p$, and let $C'$ be its strict transform on $Y$. Note that if $\nu : \tilde{C} \to C'$ is the normalization, then so is the composition $\tilde{C} \xrightarrow{\nu} C' \xrightarrow{f} C$ and
   \[ \frac{\mu_{\text{min}}((f \circ \nu)^* \mathcal{F})}{\text{mult}_p(C)} = \frac{\mu_{\text{min}}(\nu^*(f^* \mathcal{F}))}{\text{mult}_p(C')}. \]

We refer to [Ful93] and [CLS11] for basic knowledge of toric geometry.

**Notation 2.2.** (1) $N \simeq \mathbb{Z}^n$ is a lattice and $M := \text{Hom}(N, \mathbb{Z})$ is the dual lattice. The convex cone in $N_\mathbb{R} := N \otimes \mathbb{R}$ generated by $\{v_1, \ldots, v_k\} \subseteq N$ is denoted by $\langle v_1, \ldots, v_k \rangle$. A strongly convex rational polyhedral cone $\sigma \subseteq N_\mathbb{R}$ is a cone of the form $\langle v_1, \ldots, v_k \rangle$ for some $\{v_1, \ldots, v_k\} \subseteq N$ such that $\sigma$ does not contain any non-trivial linear subspace of $N_\mathbb{R}$.

(2) A fan $\Delta$ in $N$ is a set of strongly convex rational polyhedral cones in $N_\mathbb{R}$ such that (i) for all $\sigma \in \Delta$, each face of $\sigma$ is also in $\Delta$, and (ii) if $\sigma, \sigma' \in \Delta$, then $\sigma \cap \sigma'$ is a face of each. It gives rise to a toric variety $X(\Delta)$.

(3) $\Delta(k) := \{\sigma \in \Delta \mid \dim \sigma = k\}$. If $\rho \in \Delta(1)$, denote the primitive generator of $\rho$ by $v_\rho \in N$ and write $G(\Delta) := \{v_\rho \mid \rho \in \Delta(1)\}$.

(4) The dense torus $N \otimes k^*$ is denoted by $T$. Let $\sigma \in \Delta$.
   - $O_\sigma$ is the $T$-orbit corresponding to $\sigma$. Write $V_\rho := \overline{O_\sigma} = \bigcup_{\sigma \leq \rho} O_\sigma$. In the case that $\rho \in \Delta(1)$, we write $D_\rho$ instead of $V_\rho$.
   - $U_\sigma := \text{maxSpec} k[\lambda^m \mid m \in \sigma^\vee \cap M] = \bigcup_{\tau \leq \sigma} O_\tau$.

(5) We say $\rho, \rho' \in \Delta(1)$ are adjacent if they are contained in the same maximal cone in $\Delta$.

(6) For any $O_X$-module $\mathcal{F}$, we write $\tilde{\mathcal{F}} := \mathcal{F}^{\vee \vee}$.

(7) We say that a morphism $f : \mathcal{F} \to \mathcal{G}$ of sheaves on a toric variety $X(\Delta)$ is essentially surjective if $f|_T : \mathcal{F}|_T \to \mathcal{G}|_T$ is surjective.

**Proposition 2.3.** Let $X = X(\Delta)$ be a complete $\mathbb{Q}$-factorial toric variety. Then $\varepsilon(T_X, 1) \geq 0$. 

Proof. From [CLST11, Theorem 8.1.6] there is a short exact sequence
\[ 0 \to \tilde{\Omega}_X^1 \to \bigoplus_{\rho \in \Delta(1)} \mathcal{O}_X(-D_\rho) \to \text{Cl}(X) \otimes_{\mathbb{Z}} \mathcal{O}_X \to 0. \] (2.1)

The dual of (2.1) gives
\[ \bigoplus_{\rho \in \Delta(1)} \mathcal{O}_X(D_\rho) \to T_X. \] (2.2)

Let \( \sigma \in \Delta(n) \) be a maximal cone and consider the sequence (2.1) for \( U_\sigma \):
\[ 0 \to \tilde{\Omega}_{U_\sigma}^1 \to \bigoplus_{\rho \leq \sigma} \mathcal{O}_{U_\sigma}(-D_\rho|_{U_\sigma}) \to \text{Cl}(U_\sigma) \otimes_{\mathbb{Z}} \mathcal{O}_{U_\sigma} \to 0. \]

Taking the dual and using the fact that \( \text{Cl}(U_\sigma) \) is torsion, we have \( T_{U_\sigma} \cong \bigoplus_{\rho \leq \sigma} \mathcal{O}_{U_\sigma}(D_\rho|_{U_\sigma}) \).

Now the restriction of (2.2) to \( U_\sigma \) is given by the projection
\[ \bigoplus_{\rho \leq \sigma} \mathcal{O}_X(D_\rho|_{U_\sigma}) \to \bigoplus_{\rho \leq \sigma} \mathcal{O}_X(D_\rho|_{U_\sigma}) \cong T_{U_\sigma}, \]

which is surjective, and thus so is (2.2) as \( \{ U_\sigma \mid \sigma \in \Delta(n) \} \) covers \( X \). The proposition now follows from Lemma 2.1(1), (2), and the fact that \( D_\rho \cdot C \geq 0 \) for any \( \rho \in \Delta(1) \) and any irreducible curve \( C \) passing through 1.

Definition 2.4 (the condition (\( \dagger \))). Let \( \Delta \) be a complete simplicial fan in \( N \) of rank \( n \). The condition (\( \dagger \)) is defined as follows: if \( a_1 v_1 + \cdots + a_r v_r = 0 \) for some \( \{ v_1, \cdots, v_r \} \subseteq G(\Delta) \) and \( a_i \in \mathbb{R}_{>0} \), then \( k \geq n + 1 \). See Proposition 2.8(1) for the geometric meaning.

Proposition 2.5. \( \varepsilon(T_X, 1) = 0 \) if (\( \dagger \)) is not satisfied.

Proof. We may assume that
\[ n_1 v_1 + \cdots + n_r v_r = 0, \quad n_i \in \mathbb{N} \text{ for all } i, \ r \leq n. \]

Then \( \{ v_1, \cdots, v_r \} \) spans a subspace \( V \subseteq N_\mathbb{R} \) of dimension \( \ell < n \). Let \( \Delta_0 \) be the noncomplete fan in \( N \cap V \) consisting of \( 0, \rho_1, \cdots, \rho_r \), where \( \rho_i \) is the ray generated by \( v_i \). Then
\[ Y(\Delta_0) \hookrightarrow Y(\Delta_0) \times (k^*)^{n-\ell} \hookrightarrow X(\Delta). \] (2.3)

The following construction is taken from the proof of [Pay06, Proposition 2]. Let \( \phi_i : k^* \to (N \cap V) \otimes k^* \) be the one-parameter subgroup corresponding to \( v_i \) for \( 1 \leq i \leq r \), and consider the following one-parameter subgroup:
\[ \phi : k^* \to (N \cap V) \otimes k^*, \quad \phi(t) = \prod_{i=1}^{r} \phi_i(t - \lambda_i)^{n_i}, \quad \lambda_i \text{'s are distinct elements in } k^*. \]

Consider \( k^* \) as the subset \( \{(a, 1) \mid a \in k^*\} \subseteq \mathbb{P}^1 \). Then \( \phi \) can be completed into a morphism \( \bar{\phi} : \mathbb{P}^1 \to X(\Delta) \) via (2.3) such that
- \( \bar{\phi}([1, 0]) = 1 \),
- \( \bar{\phi}([a, 1]) = \phi(a) \in Y(\Delta_0) \) for \( a \in k \setminus \{ \lambda_1, \cdots, \lambda_r \} \), and
- \( \bar{\phi}([\lambda_k, 1]) = \left( \prod_{1 \leq i \leq r, i \neq k} \phi_i(\lambda_k - \lambda_i)^{n_i} \right) \cdot x_{\rho_k} \in \mathcal{O}_{\rho_k} \subseteq Y(\Delta_0) \) where \( x_{\rho_k} \) is the distinguished point corresponding to \( \rho_k \).

Now let \( C := \bar{\phi}(\mathbb{P}^1) \). Then \( C \) lies in the fiber of \( 1 \in (k^*)^{n-\ell} \) of \( Y(\Delta_0) \times (k^*)^{n-\ell} \to (k^*)^{n-\ell} \), which is a trivial family of smooth varieties. We thus have \( \nu^* T_X \cong \mathcal{O}_{\mathbb{P}^1}^{n-\ell} \oplus E \) where \( \nu : \mathbb{P}^1 \to C \) is the normalization and \( E \) is a vector bundle on \( \mathbb{P}^1 \). Hence \( \varepsilon(T_X, 1) \leq 0 \) by Lemma 2.1(2) and the definition of Seshadri constants (\( \dagger \)). Finally, by Proposition 2.3 we must have \( \varepsilon(T_X, 1) = 0 \). \( \square \)
If \( \pi : Y \to X \) is the blow-up along a smooth subvariety \( Z \subseteq X \setminus \text{Sing}(X) \) and \( E \subseteq Y \) is the exceptional divisor, there is a naturally defined morphism \( \pi^*T_X(-E) \to T_Y \) \([FM21, 4.10.2]\). Such a morphism can be used to estimate the Seshadri constants of \( T_Y \) in terms of that of \( T_X \) and \( E \). The following is the toric analog.

**Lemma 2.6.** Suppose

\[
Y = Y(\Delta) \xrightarrow{\pi} X = X(\Delta)
\]

is a birational toric morphism between complete \( \mathbb{Q} \)-factorial toric varieties. Let \( E \) be an effective \( T \)-invariant divisor on \( Y \) such that \( \text{exc}(\pi) \subseteq \text{Supp}(E) \). Then for any sufficiently divisible \( m \in \mathbb{N} \), there is a naturally defined morphism \( \pi^*T_X(-mE) \to T_Y \). In particular, it is essentially surjective in the sense of Notation 2.2(7).

**Proof.** \([CLS11, \text{Theorem 8.1.4(b)}]\) gives the following short exact sequence:

\[
0 \to \Omega^1_Y \to M \otimes_{\mathbb{Z}} \mathcal{O}_X \to \bigoplus_{\rho \in \Delta(1)} \mathcal{O}_{D_\rho} \to 0.
\]

Applying \( \pi^* \), we obtain

\[
\pi^*\Omega^1_Y \to M \otimes_{\mathbb{Z}} \mathcal{O}_Y \to \bigoplus_{\rho \in \Delta(1)} \pi^*\mathcal{O}_{D_\rho} \to 0. \tag{2.4}
\]

Consider the short exact sequence \( 0 \to \mathcal{O}_X(-D_\rho) \to \mathcal{O}_X \to \mathcal{O}_{D_\rho} \to 0 \) and the induced long exact sequence of the functor \( \pi^* \). We have \( L^1\pi^*\mathcal{O}_{D_\rho} = 0 \) as \( \pi^*\mathcal{O}_X(-D_\rho) = \mathcal{O}_Y(-\pi^*D_\rho) \to \mathcal{O}_Y \) is injective and hence \((2.4)\) is a short exact sequence. For sufficiently divisible \( m \in \mathbb{N} \), \( -\pi^*D_\rho + mE = -\pi^{-1}D_\rho + E_\rho \) for some effective \( \mathbb{Q} \)-divisor \( E_\rho \) on \( Y \). Now we have the following commutative diagram

\[
\begin{array}{ccc}
\mathcal{O}_Y(-\pi^{-1}D_\rho) & \to & \mathcal{O}_Y \\
\downarrow & & \downarrow \\
\mathcal{O}_Y(-\pi^*D_\rho + mE) & \to & \mathcal{O}_Y(mE)
\end{array}
\]

which induces a morphism \( \mathcal{O}_{\pi^{-1}D_\rho} \to \pi^*\mathcal{O}_{D_\rho}(mE) \). Fix an \( m \) that works for every \( \rho \in \Delta(1) \). Then we have the following commutative diagram

\[
\begin{array}{ccc}
0 & \to & \hat{\Omega}^1_Y \\
\downarrow & & \downarrow \\
\pi^*\hat{\Omega}^1_Y(mE) & \to & \bigoplus_{\rho \in \Delta(1)} \mathcal{O}_{D_\rho}(mE) \to 0 \\
\downarrow \alpha & & \downarrow \\
0 & \to & \pi^*\hat{\Omega}^1_Y(mE) \to M \otimes_{\mathbb{Z}} \mathcal{O}_Y(mE) \to \bigoplus_{\rho \in \Delta(1)} \pi^*\mathcal{O}_{D_\rho}(mE) \to 0
\end{array}
\]

where the map \( \mathcal{O}_{D_\rho} \to \pi^*\mathcal{O}_{D_\rho}(mE) \) appearing in \( \alpha \) is the map just defined if \( \pi^{-1}D_\rho = D_{\tilde{\rho}} \) and is the zero map otherwise. Hence there is a morphism \( \hat{\Omega}^1_Y \to \pi^*\hat{\Omega}^1_Y(mE) \), and the required morphism is just the composition \( \pi^*T_X(-mE) \to (\pi^*\hat{\Omega}^1_Y)(Y)(-mE) \to T_Y \). \( \square \)

**Lemma 2.7.** Let \( f : X \dashrightarrow Y \) be a toric birational map between complete \( \mathbb{Q} \)-factorial toric varieties. Suppose there exist an effective, \( T \)-invariant Cartier divisor \( D \) on \( X \) and a sequence of irreducible curves \( \{1 \subseteq C_k \subseteq X\}_{k \in \mathbb{N}} \) such that

1. \( \text{exc}(f) \subseteq \text{Supp}(D) \) and
2. \( \lim_{k \to \infty} \frac{C_k \cdot D}{\text{mult}_1(C_k)} = \lim_{k \to \infty} \frac{\text{mult}_{\text{min}}(\nu_k^*T_X)}{\text{mult}_1(C_k)} = 0 \), where \( \nu_k : \bar{C}_k \to C_k \) is the normalization.

Then \( \varepsilon(T_Y, 1) = 0 \).
Note that (2) is stronger than merely assuming \(\varepsilon(D, 1) = \varepsilon(T_X, 1) = 0\); in (2) we are assuming that the Seshadri constants are zero and they can be achieved by the same sequence of irreducible curves.

Proof. Let \(W\) be the complete \(\mathbb{Q}\)-factorial toric variety given by a common simplicial refinement of the fans of \(X\) and \(Y\). Then it gives the following commutative diagram of toric maps

\[
\begin{array}{ccc}
W & \xleftarrow{\alpha} & X \\
\downarrow{\beta} & & \downarrow{f} \\
& Y
\end{array}
\]

Let \(E = \alpha^*D\). We have \(\varepsilon(T_Y, 1) = \varepsilon(\beta^*T_Y, 1)\) by Lemma 2.1(3), and Lemma 2.6 gives the essentially surjective morphism \(\beta^*T_Y \to T_W(mE)\) which implies \(\varepsilon(T_W(mE), 1) \geq \varepsilon(\beta^*T_Y, 1)\) by Lemma 2.1(1). From the other essentially surjective morphism \(T_W(mE) \to \alpha^*T_X(mE)\) we get \(\varepsilon(\alpha^*T_X(mE), 1) \geq \varepsilon(T_W(mE), 1)\). Decompose \(\nu_k : \check{C}_k \to C_k\) into \(\check{C}_k \to \alpha_k^*C_k \to C_k\) and note that, because of assumption (2),

\[
\frac{\mu_{\min}(\check{\nu}_k^*\alpha^*T_X(mE))}{\mu(\alpha_k^{-1}C_k)} = \frac{\mu_{\min}(\nu_k^*T_X(mD))}{\mu(\alpha_k^{-1}C_k)} \to 0 \text{ as } k \to \infty.
\]

Putting all these together, we conclude \(\varepsilon(T_Y, 1) \leq 0\) and hence \(\varepsilon(T_Y, 1) = 0\) by Proposition 2.3.

Proposition 2.8. Let \(X = X(\Delta)\) be a complete \(\mathbb{Q}\)-factorial toric variety of dimension \(n\) such that \(\Delta\) satisfies (\(\dag\)).

1. There is no surjective toric morphism from \(X\) to a toric variety \(Y(\Delta')\) such that \(\dim X > \dim Y(\Delta') > 0\).
2. Suppose \(\varepsilon(D_\rho, 1) = 0\). Then we can contract \(D_\rho\) by running \(D_\rho\)-MMP.

Proof. (1) Suppose we have such a morphism \(f : X(\Delta) \to Y(\Delta')\). Let \(\Delta\) and \(\Delta'\) be fans in \(N\) and \(N'\), respectively. Then \(f\) corresponds to a surjective map between lattices \(\phi : N \to N'\) such that the induced surjective map \(\phi : N_\mathbb{R} \to N'_\mathbb{R}\) satisfies the following: for any \(\sigma \in \Delta\), we have \(\phi(\sigma) \subseteq \sigma'\) for some \(\sigma' \in \Delta'\). If there exists \(\sigma \in \Delta\) such that \(\sigma \not\subseteq \ker(\phi)\) and \(\ker(\phi) \cap \text{reint}(\sigma) \neq \emptyset\), then \(\phi(\sigma)\) is not strongly convex and can not be contained in any cone in \(\Delta'\). Hence \(\{\sigma \in \Delta \mid \sigma \not\subseteq \ker(\phi)\}\) is a complete fan in \(N \cap \ker(\phi)\). By assumption \(\ker(\phi)\) is not 0 nor \(N_\mathbb{R}\), which violates (\(\dag\)).

(2) By (1) we can not have a fiber-type contraction in \(D_\rho\)-MMP. To prove the statement, we need to exclude the existence of the complete simplicial fan \(\hat{\Delta}\) in \(N\) such that \(\Delta(1) = \hat{\Delta}(1)\), \(\text{exc}(f : X(\Delta) \dasharrow \hat{X}(\hat{\Delta})) \subseteq \text{Supp}(D_\rho)\), and \(\check{D}_\rho = f_!D_\rho\) is nef. Suppose on the contrary that we do have one. First note that in this case \(\kappa(\hat{X}(\hat{\Delta}), \check{D}_\rho) = \dim Z\) where \(g : \hat{X}(\hat{\Delta}) \to Z\) is the toric morphism defined by \(\check{D}_\rho\). \(\kappa(\hat{X}(\hat{\Delta}), \check{D}_\rho) = 0\) is impossible since \(\check{D}_\rho \not\sim 0\). Hence \(\kappa(\hat{X}(\hat{\Delta}), \check{D}_\rho) = n\) again by (1), and \(\check{D}_\rho = g^*A\) for some ample divisor on \(Z\). By Lemma 2.1(3), \(\varepsilon(\hat{X}(\hat{\Delta}), \check{D}_\rho, 1) = \varepsilon(Z, A; 1) > 0\). Taking a common simplicial refinement of \(\Delta\) and \(\hat{\Delta}\), we get the following commutative diagram

\[
\begin{array}{ccc}
W & \xleftarrow{\alpha} & X \\
\downarrow{\beta} & & \downarrow{f} \\
& \hat{X}
\end{array}
\]

Now there exists \(m \in \mathbb{N}\) such that \(\alpha^*mD_\rho - \beta^*\check{D}_\rho\) is an effective torus invariant divisor. We conclude that, again by Lemma 2.1(3),

\[
0 = \varepsilon(mD_\rho, 1) = \varepsilon(\alpha^*mD_\rho, 1) \geq \varepsilon(\beta^*\check{D}_\rho, 1) = \varepsilon(\check{D}_\rho, 1),
\]
which is a contradiction.

\[ \text{Proposition 2.9.} \; \varepsilon(T_X, 1) > 0 \; \text{if} \; (\dagger) \; \text{is satisfied.} \]

**Proof.** Suppose on the contrary that the fan \( \Delta \) of \( X \) satisfies (\( \dagger \)) and \( \varepsilon(T_X, 1) = 0 \). From the surjective morphism \([2.2]\) and Lemma \([2.1](2)\), we can choose a sequence of irreducible curves \( \{1 \in C_k \subseteq X\} \) such that, for some \( \rho \in \Delta(1) \),

\[
\lim_{k \to \infty} \frac{C_k \cdot D_\rho}{\text{mult}_1(C_k)} = \lim_{k \to \infty} \frac{\overline{P}_{\text{min}}(\nu_k^* T_X)}{\text{mult}_1(C_k)} = 0,
\]

where \( \nu_k : \tilde{C}_k \to C_k \) is the normalization. By Proposition \([2.8]\) we have a toric birational contraction \( f : X \to X' \) such that \( \text{exc}(f) \subseteq \text{Supp}(D_\rho) \) and \( \rho(X) > \rho(X') \). Then the fan of \( X' \) also satisfies (\( \dagger \)) and \( \varepsilon(T_{X'}, 1) = 0 \) by Lemma \([2.7]\). We get a contradiction since this process can not be carried out for infinitely many times. \( \square \)

**Theorem 2.10.** Suppose \( X = X(\Delta) \) is a complete \( \mathbb{Q} \)-factorial toric variety. Then \( \varepsilon(T_X, 1) > 0 \) if and only if \( \Delta \) satisfies (\( \dagger \)) in Definition \([2.4]\).

**Proof.** It’s just the combination of Proposition \([2.5]\) and Proposition \([2.9]\). \( \square \)

3. Proof of Theorem \([1.2]\)

**Proposition 3.1.** If \( X = X(\Delta) \) is a smooth projective toric variety of dimension \( n \) and \( Y \subseteq X \) is a \( T \)-invariant prime divisor, then the normal sequence of \( Y \) in \( X \)

\[
0 \to T_Y \to T_X|_Y \to N_{Y/X} \to 0
\]

splits.

**Proof.** We have \( Y = D_\rho \) for some \( \rho \in \Delta(1) \). It is enough to show that \( T_X|_Y \) contains a line sub-bundle \( L \) such that \( L \nsubseteq T_{Y,y} \) for any \( y \in Y \).

Let \( \mathcal{F} \subseteq T_X|_T \) be the \( T \)-invariant line sub-bundle given by \( k \nu_\rho \subseteq N \otimes k = T_{X,1} \), and let \( L := j_* \mathcal{F} \cap T_X \), \( j \) being the inclusion \( j : T \hookrightarrow X \). If \( \sigma \in \Delta(n) \) is a maximal cone containing \( \rho \), then \( L|_{U_\sigma} \) is just the line sub-bundle \( \mathcal{O}_{U_\sigma}(D_\rho|_{U_\sigma}) \) of \( T_{U_\sigma} \) via \( T_{U_\sigma} \simeq \bigoplus_{\rho \in \sigma} \mathcal{O}_{U_\sigma}(D_\rho|_{U_\sigma}) \) (see the proof of Proposition \([2.3]\)), as \( L|_{U_\sigma} \) contains \( \mathcal{O}_{U_\sigma}(D_\rho|_{U_\sigma}) \) and is torsion over it. Hence \( L|_Y \) is a line sub-bundle of \( T_X|_Y \) since \( Y \subseteq U = \bigcup_{\rho \in \sigma \in \Delta(n)} U_\sigma \). By local computation, it’s easy to see that \( L \nsubseteq T_{Y,y} \) for any \( y \in Y \). \( \square \)

**Definition 3.2.** A non-empty subset \( \mathcal{B} = \{v_1, \cdots, v_k\} \subseteq G(\Delta) \) is called a primitive collection if for each \( i, \mathcal{B} \setminus \{v_i\} \) generates a \((k-1)\)-dimensional cone in \( \Delta \), while \( \mathcal{B} \) does not generate a \( k \)-dimensional cone in \( \Delta \).

**Proposition 3.3** (\([\text{Bar91}] \) Proposition 3.2). Suppose \( X = X(\Delta) \) is a smooth projective toric variety. Then there exists a primitive collection \( \mathcal{B} = \{v_1, \cdots, v_k\} \) such that \( v_1 + \cdots + v_k = 0 \).

**Proof of Theorem \([1.2]\)** Suppose \( p \in T \). We may assume \( p = 1 \). Let \( \mathcal{B} = \{v_1, \cdots, v_k\} \subseteq G(\Delta) \) be a primitive collection such that \( v_1 + \cdots + v_k = 0 \), whose existence is guaranteed by Proposition \([3.3]\).

If \( k \leq n \), then \( \Delta \) does not satisfy (\( \dagger \)) and by Theorem \([2.10]\) we must have \( \varepsilon(T_X, 1) = 0 \). Hence \( k = n + 1 \) and \( X \simeq \mathbb{P}^n \).

Suppose \( p \in X \setminus T \) and thus \( p \in D_\rho \) for some \( \rho \in \Delta(1) \). Then the splitting short exact sequence from Proposition \([3.1]\)

\[
0 \to T_{D_\rho} \to T_X|_{D_\rho} \to N_{D_\rho/X} \to 0
\]

tells us by Lemma \([2.12]\)

\[
\varepsilon(T_X, p) \leq \varepsilon(T_X|_{D_\rho}, p) = \min \{\varepsilon(T_{D_\rho}, p), \varepsilon(N_{D_\rho/X}, p)\}.
\]
By induction on \( \dim X \), we may assume \( D_\rho \simeq \mathbb{P}^{n-1} \) and \( N_{D_\rho/X} \) is ample.

Let \( \sigma = \langle v_1, \ldots, v_n = v_\rho \rangle \in \Delta(n) \). Then all the rays adjacent to \( \rho \) other than \( \rho \) itself are \( \langle v_1, \ldots, v_n-1, v_n \rangle \) and \( \langle v_1, \ldots, v_n+1 \rangle \), where \( v_{n+1} = -v_1 - \cdots - v_{n-1} - \ell v_n \) for some \( \ell \in \mathbb{Z} \). Since \( N_{D_\rho/X} \simeq \mathcal{O}_{D_\rho}(D_\rho|\rho) \) is ample, we must have \( \ell > 0 \) and thus all the other rays in \( \Delta \) are contained in \( \langle v_1, \ldots, v_n-1, v_{n+1} \rangle \) (see Figure 1). Let \( \mathcal{B} = \{w_1, \ldots, w_k\} \subseteq G(\Delta) \) be a primitive collection as in Proposition 2.10. The condition \( w_1 + \cdots + w_k = 0 \) implies that \( v_n \in \mathcal{B} \).

**Case 1:** \( k \geq 3 \). From the definition of primitive collections, \( \langle w_i \rangle \) is adjacent to \( \rho \) for each \( i \) and hence \( \mathcal{B} \subseteq \{v_1, \ldots, v_{n+1}\} \). Since \( w_1 + \cdots + w_k = 0 \) in \( N_\mathbb{R}/\langle \pm \rho \rangle \), we conclude that \( \mathcal{B} = \langle v_1, \ldots, v_{n+1} \rangle \) and hence \( X \simeq \mathbb{P}^n \).

**Case 2:** \( k = 2 \). We have \( -v_n \in \mathcal{B} \subseteq G(\Delta) \) and hence \( -\rho \in \Delta \). By using the argument similar to that in Proposition 2.3 we conclude that \( p \notin \mathcal{O}_\rho \). Consequently, \( p \in D_\rho' \) for some \( \rho' = \langle v_i \rangle, i \neq n \). But the same argument implies \( v_i \in \mathcal{B} \), which is absurd.

The following example demonstrates that the conjecture fails when the variety is not smooth, even in the toric case:

**Example 3.4.** Let \( N = \mathbb{Z}^3 \) and consider the fan \( \Delta \) generated by \((1, 0, 0), (0, 1, 0), (0, 0, 1) \) and \((-1, -2, -3) \). Then \( X(\Delta) \) is isomorphic to the weighted projective space \( \mathbb{P}(1, 1, 2, 3) \), which has terminal singularities. We have \( \varepsilon(T_X, 1) > 0 \) by Theorem 2.10 while \( X \not\simeq \mathbb{P}^3 \).

### 4. Some Formulas for Seshadri Constants

**Proposition 4.1.** Suppose \( X = X(\Delta) \) is a smooth projective toric variety. Then for any \( p \in X \),

\[
\varepsilon(T_X, p) = \min_{\rho \in \Delta(1)} \{\varepsilon(D_\rho, p)\}.
\]

**Proof.** If \( X \simeq \mathbb{P}^n \), then \( \varepsilon(T_{\mathbb{P}^n}, p) = 1 = \varepsilon(D, p) \) for any \( p \in \mathbb{P}^n \) and any \( T \)-invariant divisor \( D \) on \( \mathbb{P}^n \). Suppose \( X \) is not isomorphic to the projective space. For any \( p \in X \), the surjective morphism \( \bigoplus_{\rho \in \Delta(1)} \mathcal{O}_X(D_\rho) \to T_X \) from the proof of Proposition 2.3 implies \( \varepsilon(T_X, p) \geq \min_{\rho \in \Delta(1)} \{\varepsilon(D_\rho, p)\} \). If \( p \in D_\rho \), the proof of Theorem 1.2 gives \( \varepsilon(T_X, p) \leq \min \{\varepsilon(T_{D_\rho}, p), \varepsilon(N_{D_\rho/X}, p)\} \). Using \( N_{D_\rho/X} \simeq \mathcal{O}_X(D_\rho) \), we obtain

\[
\min_{p \in D_\rho} \{\varepsilon(D_\rho, p)\} \geq \varepsilon(T_X, p) \geq \min_{\rho \in \Delta(1)} \{\varepsilon(D_\rho, p)\}.
\]

Here we set \( \min_{p \in D_\rho} \{\varepsilon(D_\rho, p)\} = \infty \) if \( p \in T \). By Theorem 1.2 we must have \( \varepsilon(T_X, p) \leq 0 \) for all \( p \in X \). If \( \varepsilon(T_X, p) < 0 \), then \( \min_{\rho \in \Delta(1)} \{\varepsilon(D_\rho, p)\} < 0 \) and thus \( \min_{\rho \in \Delta(1)} \{\varepsilon(D_\rho, p)\} = \infty \).
min_{p \in D_\rho}\{\varepsilon(D_\rho, p)\}. Hence in this case \(\varepsilon(T_X, p) = \min_{p \in D_\rho}\{\varepsilon(D_\rho, p)\} = \min_{\rho \in \Delta(1)}\{\varepsilon(D_\rho, p)\}\) if \(\varepsilon(T_X, p) = 0\), then \(\min_{p \in D_\rho}\{\varepsilon(D_\rho, p)\} \geq 0\) and thus \(\min_{\rho \in \Delta(1)}\{\varepsilon(D_\rho, p)\} \geq 0\). As a result \(\varepsilon(T_X, p) = 0 = \min_{\rho \in \Delta(1)}\{\varepsilon(D_\rho, p)\}\).

**Corollary 4.2.** Assume as above. Then \(\varepsilon(T_X, p)\) is lower semicontinuous in \(p\).

*Proof.* If \(X \cong \mathbb{P}^n\), then \(\varepsilon(T_X, p) = 1\) is a constant. Suppose \(X \not\cong \mathbb{P}^n\). Proposition 4.1 and Theorem 1.1 imply \(\varepsilon(T_X, p) = \min\{0, \min_{p \in D_\rho}\{\varepsilon(D_\rho, p)\}\}\) and the statement follows. \(\square\)

Note that the corollary is true no matter \(T_X\) is nef or not. It would be interesting to see whether Corollary 4.2 is true for every smooth projective variety \(X\).

**Example 4.3.** In the proof of Proposition 4.1, we see that \(\varepsilon(T_X, p) = \min_{p \in D_\rho}\{\varepsilon(D_\rho, p)\}\) if \(\varepsilon(T_X, p) < 0\). In general, to apply Proposition 4.1, we need to calculate \(\varepsilon(D_\rho, p)\) for every \(\rho \in \Delta(1)\) no matter \(D_\rho\) contains \(p\) or not. Let \(N = \mathbb{Z}^2\), and let \(\Delta\) be the fan generated by \(v_1 = (1, 0), v_2 = (0, 1), v_3 = (0, -1),\) and \(v_4 = (-1, r)\). Then \(X(\Delta) \cong \Sigma_r\), the \(r\)-th Hirzebruch surface. If \(p\) is a general point on \(D_2\), then \(\varepsilon(D_2, p) = -r\) and hence \(\varepsilon(T_X, p) < 0\). As a result \(\varepsilon(T_X, p) = \min_{p \in D_\rho}\{\varepsilon(D_\rho, p)\} = -r\), since \(D_2\) is the only \(T\)-invariant divisor containing \(p\).

Now let \(p\) be a general point on \(D_3\). We have \(\varepsilon(D_1, p) = \varepsilon(D_2, p) = \varepsilon(D_4, p) = 0\). Note that \(\varepsilon(D_3, p)\) can be calculated on \(Y(\Delta')\), where \(\Delta'\) is the fan generated by \(v_1, v_2\) and \(v_4\). Then \(\varepsilon(D_3, p) > 0\) since \(D_3\) on \(Y(\Delta')\) is ample. Hence \(\varepsilon(T_X, p) = 0 \neq \varepsilon(D_3, p) = \min_{p \in D_\rho}\{\varepsilon(D_\rho, p)\}\).

Although the above example shows that we can actually have \(0 > \varepsilon(T_X, p) > -\infty\), it is quite rare in the sense of the following proposition.

**Proposition 4.4.** Suppose \(X\) is a smooth projective toric variety. If \(\dim(\text{orb}(p)) \geq 2\), then \(\varepsilon(T_X, p) = 0\) or \(-\infty\). In other words, the Seshadri constant is "non-trivial" (meaning that \(0 > \varepsilon(T_X, p) > -\infty\) only if \(\dim(\text{orb}(p)) \leq 1\)).

Note that \(\dim(\text{orb}(p)) \leq 1\) does not guarantee \(\varepsilon(T_X, p) > -\infty\). For example, let \(X = Bl_p\mathbb{P}^3 \to \mathbb{P}^3\) and let \(E = \text{exc}(\pi)\). We have \(\varepsilon(E, p) = -\infty\) for any \(p \in E\) since \(E \cong \mathbb{P}^2\) and \(E|_E\) is anti-ample. Then \(\varepsilon(T_X, p) = -\infty\) for any \(p \in E\) by Proposition 4.1.

**Lemma 4.5.** Let \(X = X(\Delta)\) be a complete \(\mathbb{Q}\)-factorial toric variety such that \(\dim X \geq 2\), and let \(D = \sum_{\rho \in \Delta(1)} a_\rho D_\rho\) be a \(T\)-invariant \(\mathbb{Q}\)-divisor. If \(\varepsilon(D, 1) < 0\), then \(\varepsilon(D, 1) = -\infty\).

*Proof.* By assumption, there exists an irreducible curve \(C\) passing through \(1 \in T\) such that \(C \cdot D < 0\). Let \(m_\rho = C \cdot D_\rho\). Then we have \(N := \sum_{\rho \in \Delta(1)} m_\rho a_\rho < 0\). For any \(k > 0\), by [Pay06, Section 3] we have an irreducible curve \(C_k\) such that \(C_k \cap T \neq \emptyset\) and \(C_k \cdot D_\rho = km_\rho\). After applying suitable automorphism \(t \in T\), we may assume \(1 \in C_k\) and \(\text{mult}(C_k) = 1\). Thus \(\frac{C_k \cdot D}{\text{mult}(C_k)} = kN \to -\infty\) as \(k \to \infty\). \(\square\)

*Proof of Proposition 4.4.* Suppose \(\varepsilon(T_X, p) < 0\). Then by Proposition 4.1 there exists an irreducible curve \(C\) containing \(p\) such that \(C \cdot D_\rho < 0\) for some \(\rho \in \Delta(1)\). Let \(T\) be the unique cone such that \(\mathcal{O}_T \cap C\) is dense in \(C\). Note that \(V_T\) contains \(C\) and \(p\), and we may assume \(1_T \in C\) after applying suitable automorphism \(t \in T\). We can write \(D_\rho|_{V_T} \sim E\) for some \(\mathcal{O}_T\)-invariant divisor \(E\) on \(V_T\) and thus \(C \cdot E < 0\). If \(\dim(\text{orb}(p)) \geq 2\), then \(\dim V_T > 2\) and \(\varepsilon(E, 1_T \in \mathcal{O}_T) = -\infty\) by Lemma 4.5. Therefore \(\varepsilon(T_X, 1_T) = -\infty\) again by Proposition 4.1 and we have \(\varepsilon(T_X, p) = -\infty\) by semi-continuity Corollary 4.2. \(\square\)

Finally, we raise a question on the necessity of the projectiveness assumption.

**Question 4.6.** Is there any smooth complete fan \(\Delta\), other than the fan of the projective space, satisfying the condition (†) in Definition 2.1?
In view of Proposition 3.3, $X(\Delta)$ must be non-projective. If such an example does exist, then it will serve as a counterexample of Conjecture 1.1 without assuming that $X$ is projective. If the answer to Question 4.6 is no, then the proof of Theorem 1.2 can be carried over and we can generalize the theorem to smooth complete toric varieties.

**Acknowledgement.** The author would like to thank Jungkai Chen for his warm encouragement during the preparation of the paper. This work is supported by NSTC (National Science and Technology Council) and the National Taiwan University. Part of the work was done during the author’s stay in NCTS (National Center for Theoretical Sciences).

**References**

[Bat91] Victor V. Batyrev. On the classification of smooth projective toric varieties. *Tohoku Math. J. (2)*, 43(4):569–585, 1991.

[BDRH+08] Thomas Bauer, Sandra Di Rocco, Brian Harbourne, Michal Kapustka, Andreas Leopold Knutsen, Wioletta Syzdek, and Tomasz Szemberg. A primer on seshadri constants. 2008.

[BS09] Thomas Bauer and Tomasz Szemberg. Seshadri constants and the generation of jets. *J. Pure Appl. Algebra*, 213(11):2134–2140, 2009.

[BSS93] Mauro C. Beltrametti, Michael Schneider, and Andrew J. Sommese. Applications of the Ein-Lazarsfeld criterion for spannedness of adjoint bundles. *Math. Z.*, 214(4):593–599, 1993.

[BSS96] Mauro C. Beltrametti, Michael Schneider, and Andrew J. Sommese. Chern inequalities and spannedness of adjoint bundles. In *Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan, 1993)*, volume 9 of *Israel Math. Conf. Proc.*, pages 97–107. Bar-Ilan Univ., Ramat Gan, 1996.

[CK11] Steven Dale Cutkosky and Kazuhiko Kurano. Asymptotic regularity of powers of ideals of points in a weighted projective plane. *Kyoto J. Math.*, 51(1):25–45, 2011.

[CLS11] David A. Cox, John B. Little, and Henry K. Schenck. *Toric varieties*, volume 124 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2011.

[CMSB02] Koji Cho, Yoichi Miyaoka, and N. I. Shepherd-Barron. Characterizations of projective space and applications to complex symplectic manifolds. In *Higher dimensional birational geometry (Kyoto, 1997)*, volume 35 of *Adv. Stud. Pure Math.*, pages 1–88. Math. Soc. Japan, Tokyo, 2002.

[Dem92] Jean-Pierre Demailly. Singular Hermitian metrics on positive line bundles. In *Complex algebraic varieties (Bayreuth, 1990)*, volume 1507 of *Lecture Notes in Math.*, pages 87–104. Springer, Berlin, 1992.

[FM21] Mihai Fulger and Takumi Murayama. Seshadri constants for vector bundles. *J. Pure Appl. Algebra*, 225(4):Paper No. 106559, 35, 2021.

[Ful93] William Fulton. *Introduction to toric varieties*, volume 131 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1993. The William H. Roever Lectures in Geometry.

[Hac00] Christopher D. Hacon. Remarks on Seshadri constants of vector bundles. *Ann. Inst. Fourier (Grenoble)*, 50(3):767–780, 2000.

[HMP10] Milena Hering, Mircea Mustaţă, and Sam Payne. Positivity properties of toric vector bundles. *Ann. Inst. Fourier (Grenoble)*, 60(2):607–640, 2010.

[Laz04] Robert Lazarsfeld. *Positivity in algebraic geometry. I*, volume 48 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics*. Springer-Verlag, Berlin, 2004. Classical setting: line bundles and linear series.

[LZ18] Yuchen Liu and Ziquan Zhuang. Characterization of projective spaces by Seshadri constants. *Math. Z.*, 289(1-2):25–38, 2018.

[Pay06] Sam Payne. Stable base loci, movable curves, and small modifications, for toric varieties. *Math. Z.*, 253(2):421–431, 2006.

**Department of Mathematics, National Taiwan University, Taiwan**

**Email address:** cwchang@ncts.ntu.edu.tw