Accelerating Local Search for the
Maximum Independent Set Problem

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Abstract. Computing high-quality independent sets quickly is an important problem in combinatorial optimization. Several recent algorithms have shown that kernelization techniques can be used to find exact maximum independent sets in medium-sized sparse graphs, as well as high-quality independent sets in huge sparse graphs that are intractable for exact (exponential-time) algorithms. However, a major drawback of these algorithms is that they require significant preprocessing overhead, and therefore cannot be used to find a high-quality independent set quickly.
In this paper, we show that performing simple kernelization techniques in an online fashion significantly boosts the performance of local search, and is much faster than pre-computing a kernel using advanced techniques. In addition, we show that cutting high-degree vertices can boost local search performance even further, especially on huge (sparse) complex networks. Our experiments show that we can drastically speed up the computation of large independent sets compared to other state-of-the-art algorithms, while also producing results that are very close to the best known solutions.

1 Introduction

The maximum independent set problem is a classic NP-hard problem [14] with applications spanning many fields, such as classification theory, information retrieval, computer vision [12], computer graphics [31], map labeling [15] and routing in road networks [21]. Given a graph $G = (V, E)$, our goal is to compute a maximum cardinality set of vertices $I \subseteq V$ such that no vertices in $I$ are adjacent to one another. Such a set is called a maximum independent set (MIS).

1.1 Previous Work

Since the MIS problem is NP-hard, all known exact algorithms for these problems take exponential time, making large graphs infeasible to solve in
practice. Instead, heuristic algorithms such as local search are used to efficiently compute high-quality independent sets. For many practical instances, some local search algorithms even quickly find exact solutions [3,17].

**Exact Algorithms.** Much research has been devoted to reducing the base of the exponent for exact branch-and-bound algorithms. One main technique is to apply reductions, which remove or modify subgraphs that can be solved simply, reducing the graph to a smaller instance. Reductions have consistently been used to reduce the running time of exact MIS algorithms [32], with the current best polynomial-space algorithm having running time $O(1.2114^n)$ [7]. These algorithms apply reductions during recursion, only branching when the graph can no longer be reduced [13].

Relatively simple reduction techniques are known to be effective at reducing graph size in practice [18]. Recently, Akiba and Iwata [2] showed that more advanced reduction rules are also highly effective, finding an exact minimum vertex cover (and by extension, an exact maximum independent set) on a corpus of large social networks with up to 3.2 million vertices in less than a second. However, their algorithm still requires $O(1.2210^n)$ time in the worst case, and its running time has exponential dependence on the kernel size. Since much larger graph instances have consistently large kernels, they remain intractable in practice [25]. Even though small benchmark graphs with up to thousands of vertices have been solved exactly with branch-and-bound algorithms [29,30,33], many similarly-sized instances remain unsolved [8]. Even a graph on 4,000 vertices was only recently solved exactly, and it required hundreds of machines in a MapReduce cluster [34]. Heuristic algorithms are clearly still needed in practice, even for small graphs.

**Heuristic Approaches.** There are a wide range of heuristics and local search algorithms for the complementary maximum clique problem (see for example [6,18,16,20,28,17]). These algorithms work by maintaining a single solution and attempt to improve it through node deletions, insertions, swaps, and plateau search. Plateau search only accepts moves that do not change the objective function, which is typically achieved through node swaps—replacing a node by one of its neighbors. Note that a node swap cannot directly increase the size of the independent set. A very successful approach for the maximum clique problem has been presented by Grosso et al. [17]. In addition to plateau search, it applies various diversification operations and restart rules. The iterated local search algorithm of Andrade et al. [3] is one of the most successful local search algorithms in practice. On small
benchmark graphs requiring hours of computation to solve with exact algorithms, their algorithm often finds optimal solutions in milliseconds. However, for huge complex networks such as social networks and web graphs, it is consistently outperformed by other methods \[24,25\]. We give further details of this algorithm in Section \[2.1\].

To solve these largest—and intractable—graphs, Lamm et al. \[25\] proposed \textbf{ReduMIS}, an algorithm that uses reduction techniques combined with an evolutionary approach. It finds the exact MIS for many of the benchmarks used by Akiba et al. \[2\], and consistently finds larger independent sets than other heuristics. Its major drawback is the significant preprocessing time it takes to apply reductions and initialize its evolutionary component, especially on larger instances. Thus, though \textbf{ReduMIS} finds high-quality independent sets faster than existing methods, it is still slow in practice on huge complex networks. However, for many of the applications mentioned above, a near-optimal independent set is not needed in practice. The main goal then is to quickly compute an independent set of sufficient quality. Hence, to find high-quality independent sets faster, we need a different approach.

1.2 Our Results

We develop an advanced local search algorithm that quickly computes large independent sets by combining iterated local search with reduction rules that reduce the size of the search space without losing solution quality. By running local search on the kernel, we significantly boost its performance, especially on huge sparse networks. In addition to exact kernelization techniques, we also apply inexact reductions that remove high-degree vertices from the graph. In particular, we show that cutting a small percentage of high-degree vertices from the graph minimizes performance bottlenecks of local search, while maintaining high solution quality. Experiments indicate that our algorithm finds large independent sets much faster than existing state-of-the-art algorithms, while still remaining competitive with the best solutions reported in literature.

2 Preliminaries

Let \( G = (V = \{0, \ldots, n - 1\}, E) \) be an undirected graph with \( n = |V| \) nodes and \( m = |E| \) edges. The set \( N(v) = \{u : \{v, u\} \in E\} \) denotes the open neighborhood of \( v \). We further define the open neighborhood of a set of nodes \( U \subseteq V \) to be \( N(U) = \cup_{v \in U} N(v) \setminus U \). We similarly define the
closed neighborhood as $N[v] = N(v) \cup \{v\}$ and $N[U] = N(U) \cup U$. A graph $H = (V_H, E_H)$ is said to be a subgraph of $G = (V, E)$ if $V_H \subseteq V$ and $E_H \subseteq E$. We call $H$ an induced subgraph when $E_H = \{\{u, v\} \in E : u, v \in V_H\}$.

For a set of nodes $U \subseteq V$, $G[U]$ denotes the subgraph induced by $U$.

An independent set is a set $I \subseteq V$, such that all nodes in $I$ are pairwise nonadjacent. An independent set is maximal if it is not a subset of any larger independent set. The maximum independent set problem is that of finding the maximum cardinality independent set among all possible independent sets. Such a set is called a maximum independent set (MIS).

Finally, we note the maximum independent set problem is equivalent to the maximum clique and minimum vertex cover problems. We see this equivalence as follows: Given a graph $G = (V, E)$ and an independent set $I \in V$, $V \setminus I$ is a vertex cover and $I$ is a clique in the complement graph (the graph containing all edges missing in $G$). Thus, algorithms for any of these problems can also solve the maximum independent set problem.

2.1 The ARW Algorithm

We now review the local search algorithm by Andrade et al. [3] (ARW) in more detail, since we use this algorithm in our work. For the independent set problem, Andrade et al. [3] extended the notion of swaps to $(j, k)$-swaps, which remove $j$ nodes from the current solution and insert $k$ nodes. The authors present a fast linear-time implementation that, given a maximal solution, can find a $(1, 2)$-swap or prove that no $(1, 2)$-swap exists. One iteration of the ARW algorithm consists of a perturbation and a local search step. The ARW local search algorithm uses $(1, 2)$-swaps to gradually improve a single current solution. The simple version of the local search iterates over all nodes of the graph and looks for a $(1, 2)$-swap. By using a data structure that allows insertion and removal operations on nodes in time proportional to their degree, this procedure can find a valid $(1, 2)$-swap in $O(m)$ time, if it exists.

A perturbation step, used for diversification, forces nodes into the solution and removes neighboring nodes as necessary. In most cases a single node is forced into the solution; with a small probability the number of forced nodes $f$ is set to a higher value ($f$ is set to $i + 1$ with probability $1/2^i$). Nodes to be forced into a solution are picked from a set of random candidates, with priority given to those that have been outside the solution for the longest time. An even faster incremental version of the algorithm (which we use here) maintains a list of candidates, which are nodes that may be involved in $(1, 2)$-swaps. It ensures a node is not examined twice unless there is some change in its neighborhood. Furthermore, an external memory version of
this algorithm by Liu et al. [26] runs on graphs that do not fit into memory on a standard machine. The ARW algorithm is efficient in practice, finding the exact maximum independent sets orders of magnitude faster than exact algorithms on many benchmark graphs.

3 Techniques for Accelerating Local Search

First, we note that while local search techniques such as ARW perform well on huge uniformly sparse mesh-like graphs, they perform poorly on complex networks, which are typically scale-free. We first discuss why local search performs poorly on huge complex networks, then introduce the techniques we use to address these shortcomings.

The first performance issue is related to vertex selection for perturbation. Many vertices are always in some MIS. These include, for example, vertices with degree 1. However, ARW treats such vertices like any other. During a perturbation step, these vertices may be forced out of the current solution, causing extra searching that may not improve the solution.

The second issue is that high-degree vertices may slow ARW down significantly. Most internal operations of ARW (including (1,2)-swaps) require traversing the adjacency lists of multiple vertices, which takes time proportional to their degree. Although high-degree vertices are only scanned if they have at most one solution neighbor (or belong to the solution themselves), this happens often in complex networks.

A third issue is caused by the particular implementation. When performing an (1,2)-swap involving the insertion of a vertex v, the original ARW implementation (as tested by Andrade et al. [3]) picks a pair of neighbors u, w of v at random among all valid ones. Although this technically violates that $O(m)$ worst-case bound (which requires the first such pair to be taken), the effect is minimal on the small-degree networks. On large complex networks, this can become a significant bottleneck.

To deal with the third issue, we simply modified the ARW code to limit the number of valid pairs considered to a small constant (100). Addressing the first two issues requires more involved techniques (kernelization and high-degree vertex cutting, respectively), as we discuss next.

3.1 Exact Kernelization

First, we investigate kernelization, a technique known to be effective in practice for finding an exact minimum vertex cover (and hence, a maximum independent set) [12]. In kernelization, we repeatedly apply reductions to
the input graph $G$ until it cannot be reduced further, producing a kernel $\mathcal{K}$. Even simple reduction rules can significantly reduce the graph size, and in some cases $\mathcal{K}$ is empty—giving an exact solution without requiring any additional steps. We note that this is the case for many of the graphs in the experiments by Akiba and Iwata [2]. Furthermore, any solution of $\mathcal{K}$ can be extended to a solution of the input.

The size of the kernel depends entirely on the structure of the input graph. In many cases, the kernel can be too large, making it intractable to find an exact maximum independent set in practice (see Section 4). In this case “too large” can mean a few thousand vertices. However, for many graphs, the kernel is still significantly smaller than the input graph, and even though it is intractable for exact algorithms, local search algorithms such as ARW have been shown to find the exact MIS quickly on small benchmark graphs. It therefore stands to reason that ARW would perform better on a small kernel.

**Reductions.** We now briefly mention the reduction rules that we consider. Each of these exact reductions allow us to choose vertices that are in some MIS by following simple rules. If an MIS is found on the kernel graph $\mathcal{K}$, then each reduction may be undone, producing an MIS in the original graph.

Akiba and Iwata [2] use a full suite of reduction rules, which they show can efficiently solve the minimum vertex cover problem exactly for a wide variety of instances. We consider all of their reductions here. These include simple rules typically used in practice such as pendant vertex removal and vertex folding [9], and more advanced (and time-consuming) rules such as a linear programming relaxation with a half-integral solution [19,27], unconfined [35], twin [35], alternative [35], and packing [2] reductions. Since details on these reductions are not necessary for understanding our results, we defer them to Appendix A. (Refer to Akiba and Iwata [2] for a more thorough discussion, including implementation details.)

The most relevant reduction for our purposes is the isolated vertex removal. If a vertex $v$ forms a single clique $C$ with all its neighbors, then $v$ is called isolated and is always contained in some maximum independent set. To see this, at most one vertex from $C$ may be in an MIS. If a neighbor of $v$ is in an MIS, then so is $v$. Otherwise, $v$ is in the MIS. This reduction was not included in the
exact algorithm by Akiba and Iwata [2]; however, Butenko et al. [8] show that it is highly effective on graphs derived from error-correcting codes.

3.2 Inexact Reductions: Cutting High-Degree Vertices

To further boost local search, we investigate removing (cutting) high-degree vertices outright. This is a natural strategy: intuitively, vertices with very high degree are unlikely to be in a large independent set (consider a maximum independent set of graphs with few high-degree vertices, such as a star graph, or scale-free networks). In particular, many reduction rules show that low-degree vertices are in some MIS, and applying them results in a small kernel [25]. Thus, high-degree vertices are left behind. This is especially true for huge complex networks considered here, which generally have few high-degree vertices.

Besides intuition, there is much additional evidence to support this strategy. In particular, the natural greedy algorithm that repeatedly selects low-degree vertices to construct an independent set is typically within 1%–10% of the maximum independent set size for sparse graphs [3]. Moreover, several successful algorithms make choices that favor low-degree vertices. **ReduMIS** [25] forces low-degree vertices into an independent set in a multi-level algorithm, giving high-quality independent sets as a result. Exact branch-and-bound algorithms order vertices so that vertices of high-degree are considered first during search. This reduces the search space size initially, at the cost of finding poor initial independent sets. In particular, optimal and near-optimal independent sets are typically found after high-degree vertices have been evaluated and excluded from search; however, it is then much slower to find the remaining solutions, since only low-degree vertices remain in the search. This slowness can be observed in the experiments of Batsyn et al. [5], where better initial solutions from local search significantly speed up exact search.

We considered two strategies for removing high-degree vertices from the graph. When we cut by **absolute degree**, we remove the vertices with degree higher than a threshold. In **relative degree** cutting, we iteratively remove high-degree vertices and their incident edges from the graph. This is the mirror image of the greedy algorithm that repeatedly selects smallest-degree vertices in the graph to be in an independent set until the graph is empty. We stop removing until a fixed fraction of all vertices is eliminated. Unlike absolute cutting, this better ensures that clusters of high-degree vertices are removed, leaving high-degree vertices that are isolated from one another, which are more likely to be in large independent sets.
3.3 Putting Things Together

We use reductions and cutting in two ways. First, we define an algorithm that applies the standard technique of producing the kernel in advance, and then run ARW on the kernel. Second, we investigate applying reductions online as ARW runs.

Preprocessing. Our first algorithm (KerMIS) uses exact reductions in combination with relative degree cutting. It uses the full set of reductions from Akiba and Iwata [2], as described in Section 3. Note that we do not include isolated vertex removal, as it was not included in their reductions. After computing a kernel, we then cut 1% of the highest-degree vertices using relative cutting, breaking ties randomly. We then run ARW on the resulting graph.

Online. Our second approach (OnlineMIS) applies a set of simple reductions on the fly. For this algorithm, we use only the isolated vertex removal reduction (for degree zero, one and two), since it does not require the graph to be modified—we can just mark isolated vertices and their neighbors as removed during local search. In more detail, we first perform a quick single pass when computing the initial solution for ARW. We force isolated vertices into the initial solution, and mark them and their neighbors as removed. Note that this does not result in a kernel, as this pass may create more isolated vertices. We further mark the top 1% of high-degree vertices as removed during this pass. As the local search continues, whenever we check if a vertex can be inserted into the solution, we check if it is isolated and update the solution and graph similarly to the single pass. Thus, OnlineMIS kernelizes the graph in an online fashion as local search proceeds.

4 Experimental Evaluation

4.1 Methodology

We implemented our algorithms (OnlineMIS, KerMIS) including the kernelization techniques using C++ and compiled all code using gcc 4.6.3 with full optimizations turned on (-O3 flag). We further compiled the original implementations of ARW and ReduMIS using the same settings. For ReduMIS, we use the same parameters as Lamm et al. [25] (convergence parameter $\mu = 1,000,000$, reduction parameter $\lambda = 0.1 \cdot |Z|$, and cutting percentage $\eta = 0.1 \cdot |K|$). For all instances, we perform three independent runs of each algorithm. For small instances, we run each algorithm sequentially with a 5-hour wall-clock time limit to compute its best solution. For huge graphs,
with tens of millions of vertices and at least one billion edges, we enforce a time limit of ten hours.

Each run was performed on a machine that is equipped with four Octa-Core Intel Xeon E5-4640 processors running at 2.4 GHz. It has 512 GB local memory, \( 4 \times 20 \text{ MB L3-Cache} \) and \( 4 \times 8 \times 256 \text{ KB L2-Cache} \).

We consider social networks, autonomous systems graphs, and Web graphs taken from the 10th DIMACS Implementation Challenge \[4]\, and two additional large Web graphs, \text{webbase-2001} \[23]\, and \text{wikilinks} \[22]\,. We also include road networks from Andrade et al. \[3]\, and meshes from Sander et al. \[31]\,. The graphs \text{europe} and \text{USA-road} are large road networks of Europe \[10]\, and the USA \[11]\,. The instances \text{as-Skitter-big}, \text{web-Stanford} and \text{libimseti} are the hardest instances from Akiba and Iwata \[2]\,. We further perform experiments on huge instances with billions of edges taken from the Laboratory of Web Algorithmics \[23]\,: \text{it-2004}, \text{sk-2005}, \text{and uk-2007}.

### 4.2 Accelerated Solutions

We now illustrate the speed improvement over existing heuristic algorithms. First, we measure the speedup of OnlineMIS over other high-quality heuristic search algorithms. In particular, in Table 1 we report the maximum speedup that OnlineMIS compared with the state-of-the-art competitors. We compute the maximum speedup for an instance as follows. For each solution size \( i \), we compute the speedup \( s_{\text{Alg}}^i = t_{\text{Alg}}^i / t_{\text{OnlineMIS}}^i \) of OnlineMIS over algorithm Alg for that solution size. We then report the maximum speedup \( s_{\text{Alg}}^{\text{max}} = \max_i s_{\text{Alg}}^i \) for the instance.

As can be seen in Table 1, OnlineMIS always has a maximum speedup greater than 1 over every other algorithm. We first note that OnlineMIS is significantly faster than ReduMIS and KerMIS. In particular, in 14 instances, OnlineMIS achieves a maximum speedup of over 100 over ReduMIS. KerMIS performs only slightly better than ReduMIS in this regard, with OnlineMIS achieving similar speedups on 12 instances. Though, on meshes, KerMIS fairs especially poorly. On these instances, OnlineMIS always finds a better solution than KerMIS (instances marked with an *), and on the bunny and feline instances, OnlineMIS achieves a maximum speedup of over 10,000 against KerMIS. Furthermore, on the venus mesh graph, KerMIS never matches the quality of a single solution from OnlineMIS, giving infinite speedup. ARW is the closest competitor, where OnlineMIS only has 2 maximum speedups greater than 100. However, on a further 6 instances, OnlineMIS achieves a maximum speedup over 10, and on 11 instances ARW
Table 1. For each graph instance, we give the number of vertices $n$ and the number of edges $m$. We further give the maximum speedup for $\text{OnlineMIS}$ over other heuristic search algorithms. For each solution size $i$, we compute the speedup $s_{\text{Alg}} = t_{\text{Alg}} / t_{\text{OnlineMIS}}$ of $\text{OnlineMIS}$ over algorithm $\text{Alg}$ for that solution size. We then report the maximum speedup $s_{\text{max}}^{\text{Alg}} = \max_i s_{\text{Alg}}$ for the instance. When an algorithm never matches the final solution quality of $\text{OnlineMIS}$, we give the highest non-infinite speedup and give an *. A ‘$\infty$’ indicates that all speedups are infinite.

| Name   | Graph          | $n$  | $m$  | $s_{\text{max}}^{\text{ARW}}$ | $s_{\text{max}}^{\text{KerMIS}}$ | $s_{\text{max}}^{\text{ReduMIS}}$ |
|--------|----------------|------|------|------------------------------|---------------------------------|----------------------------------|
| Huge instances: |                |      |      |                              |                                 |                                  |
| it-2004 |                | 41291594 | 1027474947 | 4.51                      | 221.26                          | 266.30                           |
| sk-2005 |                | 50636154 | 1810063330 | 356.87*                   | 201.68                          | 302.64                           |
| uk-2007 |                | 105896555 | 1154392916 | 11.63*                    | 108.13                          | 122.50                           |
| Social networks and Web graphs: |                |      |      |                              |                                 |                                  |
| amazon-2008 |            | 735323   | 3523472    | 43.39*                    | 13.75                           | 50.75                            |
| as-Skitter-big |          | 1696415  | 11905298   | 356.06*                   | 2.68                            | 7.62                             |
| deviki-2013 |            | 1532354  | 33093029   | 36.22*                    | 632.94                          | 1726.28                          |
| enwiki-2013 |            | 4206785  | 91939728   | 51.01*                    | 146.58                          | 244.64                           |
| eu-2005 |                | 862664   | 22217686   | 5.52                       | 62.37                           | 217.39                           |
| hollywood-2011 |          | 2180759  | 114492816  | 4.35                       | 5.51                            | 11.24                            |
| libimseti |              | 220970   | 17233144   | 15.16*                    | 218.30                          | 1116.65                          |
| ljournal-2008 |           | 5363260  | 49514271   | 2.51                       | 3.00                            | 5.33                             |
| orkut |                | 3072441  | 117185082  | 1.82*                     | 478.94*                         | 8751.62*                         |
| web-Stanford |            | 281903   | 1992636    | 50.70*                    | 29.83                           | 59.31                            |
| webbase-2001 |           | 11814215 | 854809761  | 4.38                      | 33.54                           | 36.18                            |
| wikilinks |              | 25890800 | 543159884  | 3.88                      | 11.54                           | 11.89                            |
| youtube |                | 1134890  | 543159884  | 6.83                       | 1.83                            | 7.29                             |
| Road networks: |            |      |      |                              |                                 |                                  |
| europe |                | 18029721 | 22217686   | 5.57                       | 12.79                           | 14.20                            |
| USA-road |              | 23947347 | 28854312   | 7.17                       | 24.41                           | 27.84                            |
| Meshes: |                |      |      |                              |                                 |                                  |
| buddha |                | 1087716  | 1631574    | 1.16                       | 154.04*                         | 976.10*                          |
| bunny |                | 68790   | 103017     | 3.26                       | 16616.83*                       | 526.14                           |
| dragon |                | 1500800 | 2250000    | 2.22*                     | 567.39*                         | 692.60*                          |
| feline |                | 41262   | 61893      | 2.00*                     | 13777.42*                       | 315.48                           |
| gameguy |               | 42623   | 63850      | 3.23                       | 98.82*                          | 102.03                           |
| venus |                | 5672    | 8508       | 1.17                       | $\infty$                       | 157.78*                          |

fails to match the final solution quality of $\text{OnlineMIS}$, giving an effective infinite maximum speedup.

We now give several representative convergence plots in Fig. 2, which illustrate the early solution quality of $\text{OnlineMIS}$ compared to $\text{ARW}$, the closest competitor. We construct these plots as follows. Whenever an algorithm finds a new large independent set $I$ at time $t$, it reports a tuple $(t, |I|)$; the convergence plots show average values over all three runs. In the non-mesh instances, $\text{OnlineMIS}$ takes a early lead over $\text{ARW}$, though solution quality converges over time. Lastly, we give the convergence plot for the bunny mesh graph. Reductions and high-degree cutting aren’t effective on meshes, thus $\text{ARW}$ and $\text{OnlineMIS}$ have similar initial solution sizes.
4.3 Time to High-Quality Solutions

We now look at the time it takes an algorithm to find a high-quality solution. We first determine the largest independent set found by any of the four algorithms, which represent the best-known solutions [25], and compute how long it takes each algorithm to find an independent set within $99.5\%$ of this size. The results are shown in Table 2. With a single exception, OnlineMIS is the fastest algorithm to be within $99.5\%$ of the target solution. In fact, OnlineMIS finds such a solution at least twice as fast as ARW in 14 instances; it is almost 10 times faster on the largest instance, uk-2007. Further, OnlineMIS is orders of magnitude faster than ReduMIS (by a factor of at least 100 in 7 cases). We also see that KerMIS is faster than ReduMIS in 19 cases, but much slower than OnlineMIS for all instances. It does eventually find the largest independent set (among all algorithms) for 10 instances. This shows that the full set of reductions is not always necessary, especially when the goal is to get a high-quality solution quickly. It also justifies our choice of cutting: the solution quality of KerMIS rivals (and sometimes even improves) that of ReduMIS. Further information about overall solution quality can be found in Appendix B.
Table 2. For each algorithm, we give the average time $t_{avg}$ to reach 99.5% of the best solution found by any algorithm. The fastest such time for each instance is marked in bold. We also give the size of the largest solution found by any algorithm and list the algorithms (abbreviated by first letter) that found this largest solution in the time limit. A '-' indicates that the algorithm did not find a solution of sufficient quality.

| Graph       | OnlineMIS $t_{avg}$ | ARW $t_{avg}$ | KerMIS $t_{avg}$ | ReduMIS $t_{avg}$ | Best IS Size | Best IS Algorithms |
|-------------|---------------------|--------------|-----------------|-----------------|--------------|-------------------|
| Huge instances:                          |                   |               |                 |                 |              |                   |
| it-2004     | 86.01               | 327.35       | 7 892.04        | 9 448.18        | 25 620 285   | R                 |
| sk-2005     | 152.12              | -             | 10 854.46       | 16 316.59       | 30 086 766   | K                 |
| uk-2007     | 403.36              | 3 789.74     | 23 022.26       | 26 081.36       | 67 282 659   | K                 |
| Social networks and Web graphs:          |                   |               |                 |                 |              |                   |
| amazon-2008 | 0.76                | 1.26          | 5.81            | 15.23           | 309 794      | K, R              |
| as-Skitter-big | 1.26             | 2.70          | 2.82            | 8.00            | 1 170 580    | K, R              |
| deviki-2013 | 4.10                | 7.88          | 898.77          | 2 589.32        | 697 923      | K                 |
| enviki-2013 | 10.49               | 19.26         | 850.01          | 1 428.71        | 2 178 457    | K                 |
| eu-2005     | 1.32                | 3.11          | 29.01           | 95.65           | 452 353      | R                 |
| hollywood-2011 | 1.28           | 1.46          | 7.06            | 14.38           | 523 402      | O, A, K, R       |
| libimseti   | 0.44                | 0.45          | 50.21           | 257.29          | 127 293      | R                 |
| ljournal-2008 | 3.79            | 8.30          | 10.20           | 18.14           | 2 970 397    | K, R              |
| orkut       | 42.19               | 49.18         | 2 024.36        | -               | 839 086      | K                 |
| web-Stanford | 1.58             | 8.19          | 3.57            | 7.12            | 163 390      | R                 |
| webbase-2001 | 144.51           | 343.86        | 2 920.14        | 3 150.05        | 80 009 826   | R                 |
| wikipedias  | 34.40               | 85.54         | 348.63          | 358.98          | 19 418 724   | R                 |
| youtube     | 0.26                | 0.81          | 0.48            | 1.90            | 857 945      | A, K, R           |
| Road networks:                               |                   |               |                 |                 |              |                   |
| europe      | 28.22               | 75.67         | 91.21           | 101.21          | 9 267 811    | R                 |
| USA-road    | 44.21               | 112.67        | 259.33          | 295.70          | 12 428 105   | R                 |
| Meshes:                                             |                   |               |                 |                 |              |                   |
| buddha      | 26.23               | 26.72         | 119.05          | 1 699.19        | 480 853      | A                 |
| bunny       | 3.21                | 9.22          | -               | 70.40           | 32 349       | R                 |
| dragon      | 3.32                | 4.90          | 5.18            | 97.88           | 66 502       | A                 |
| feline      | 1.24                | 1.27          | -               | 39.18           | 18 853       | R                 |
| gameguy     | 15.13               | 10.60         | 60.77           | 12.22           | 20 727       | R                 |
| venus       | 0.32                | 0.36          | -               | 6.52            | 2 684        | O, A, R           |

5 Conclusion and Future Work

We have shown that applying reductions on the fly during local search leads to high-quality independent sets quickly. Furthermore, cutting few high-degree vertices has little effect on the quality of independent sets found during local search. Lastly, by kernelizing with advanced reduction rules, we can further speed up local search for high-quality independent sets, in the long-run—rivaling the current best heuristic algorithms for complex networks. Determining which reductions give a desirable balance between high-quality results and speed is an interesting topic for future research. While we believe that OnlineMIS gives a nice balance, it is possible that further reductions may achieve higher-quality results even faster.
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A Reductions

We briefly describe the reduction rules from Akiba and Iwata [2]. Each of these exact reductions allow us to choose vertices that are in some MIS by following simple rules. If an MIS is found on the kernel graph K, then each reduction may be undone, producing an MIS in the original graph. Refer to Akiba and Iwata [2] for a more thorough discussion, including implementation details.

Pendant vertices: Any vertex $v$ of degree one, called a pendant, is in some MIS; therefore $v$ and its neighbor $u$ can be removed from $G$.

Vertex folding [9]: For a vertex $v$ with degree 2 whose neighbors $u$ and $w$ are not adjacent, either $v$ is in some MIS, or both $u$ and $w$ are in some MIS. Therefore, we can contract $u$, $v$, and $w$ to a single vertex $v'$ and decide which vertices are in the MIS later.

Linear Programming: A well-known [27] linear programming relaxation for the MIS problem with a half-integral solution (i.e., using only values 0, 1/2, and 1) can be solved using bipartite matching: maximize $\sum_{v \in V} x_v$ such that $\forall (u, v) \in E$, $x_u + x_v \leq 1$ and $\forall v \in V$, $x_v \geq 0$. Vertices with value 1 must be in the MIS and can thus be removed from $G$ along with their neighbors. We use an improved version [19] that computes a solution whose half-integral part is minimal.

Unconfined [35]: Though there are several definitions of unconfined vertex in the literature, we use the simple one from Akiba and Iwata [2]. A vertex $v$ is unconfined when determined by the following simple algorithm. First, initialize $S = \{v\}$. Then find a $u \in N(S)$ such that $|N(u) \cap S| = 1$ and $|N(u) \setminus N[S]|$ is minimized. If there is no such vertex, then $v$ is confined. If $N(u) \setminus N[S] = \emptyset$, then $v$ is unconfined. If $N(u) \setminus N[S]$ is a single vertex $w$, then add $w$ to $S$ and repeat the algorithm. Otherwise, $v$ is confined. Unconfined vertices can be removed from the graph, since there always exists an MIS $\mathcal{I}$ that contains no unconfined vertices.

Twin [35]: Let $u$ and $v$ be vertices of degree 3 with $N(u) = N(v)$. If $G[N(u)]$ has edges, then add $u$ and $v$ to $\mathcal{I}$ and remove $u$, $v$, $N(u)$, $N(v)$ from $G$. Otherwise, some vertices in $N(u)$ may belong to some MIS $\mathcal{I}$. We still remove $u$, $v$, $N(u)$ and $N(v)$ from $G$, and add a new gadget vertex $w$ to $G$ with edges to $u$’s two-neighborhood (vertices at a distance 2 from $u$). If $w$ is in the computed MIS, then none of $u$’s two-neighbors are $\mathcal{I}$, and therefore $N(u) \subseteq \mathcal{I}$. Otherwise, if $w$ is not in the computed MIS, then some of $u$’s two-neighbors are in $\mathcal{I}$, and therefore $u$ and $v$ are added to $\mathcal{I}$.

Alternative: Two sets of vertices $A$ and $B$ are set to be alternatives if $|A| = |B| \geq 1$ and there exists an MIS $\mathcal{I}$ such that $\mathcal{I} \cap (A \cup B)$ is either
A or B. Then we remove A and B and \( C = N(A) \cap N(B) \) from \( G \) and add edges from each \( a \in N(A) \setminus C \) to each \( b \in N(B) \setminus C \). Then we add either A or B to \( \mathcal{I} \), depending on which neighborhood has vertices in \( \mathcal{I} \).

Two structures are detected as alternatives. First, if \( N(v) \setminus \{ u \} \) induces a complete graph, then \( \{ u \} \) and \( \{ v \} \) are alternatives (a funnel). Next, if there is a cordless 4-cycle \( a_1b_1a_2b_2 \) where each vertex has at least degree 3. Then sets \( A = \{ a_1, a_2 \} \) and \( B = \{ b_1, b_2 \} \) are alternatives when \( |N(A) \setminus B| \leq 2 \), \( |N(A) \setminus B| \leq 2 \), and \( N(A) \cap N(B) = \emptyset \).

**Packing**: Given a non-empty set of vertices \( S \), we may specify a packing constraint \( \sum_{v \in S} x_v \leq k \), where \( x_v \) is 0 when \( v \) is in some MIS \( \mathcal{I} \) and 1 otherwise. Whenever a vertex \( v \) is excluded from \( \mathcal{I} \) (i.e., in the unconfined reduction), we remove \( x_v \) from the packing constraint and decrease the upper bound of the constraint by one. Initially, packing constraints are created whenever a vertex \( v \) is excluded or included into the MIS. The simplest case for the packing reduction is when \( k \) is zero: all vertices must be in \( \mathcal{I} \) to satisfy the constraint. Thus, if there is no edge in \( G[S] \), \( S \) may be added to \( \mathcal{I} \), and \( S \) and \( N(S) \) are removed from \( G \). Other cases are much more complex. Whenever packing reductions are applied, existing packing constraints are updated and new ones are added.

### B Overall Solution Quality

Next, we show that OnlineMIS has high solution quality when given a time limit for searching (5 hours for normal graphs, 10 hours for huge graphs). Table 3 shows the average solution size over the three runs. Although long-run quality is not the goal of the OnlineMIS algorithm, in 11 instances OnlineMIS finds a larger independent set than ARW, and in 4 instances OnlineMIS finds the largest solution in the time limit. As seen in Table 3, OnlineMIS also finds a solution within 0.1% of the best solution found by any algorithm for all graphs. However, in general OnlineMIS finds lower-quality solutions than ReduMIS, which we believe is from high-degree cutting removing vertices in large independent sets. Nonetheless, as this shows, even when cutting out 1% of the vertices, the solution quality remains high.

We further test KerMIS, which first kernelizes the graph using the advanced reductions from ReduMIS, removes 1% of the highest-degree vertices, and then runs ARW on the remaining graph. On 8 instances, KerMIS finds a better solution than ReduMIS. However, kernelization and cutting take a long time (over 3 hours for sk-2005, 10 hours for uk-2007), and therefore KerMIS is much slower to get to a high-quality solution than OnlineMIS. Thus, our experiments show that the full set of reductions is
Table 3. For each algorithm, we include average solution size and average time $t_{avg}$ to reach it within a time limit (5 hours for normal graphs, 10 hours for huge graphs). Solutions in italics indicate the larger solution between ARW and OnlineMIS local search, bold marks the largest overall solution. A '-' in our indicates that the algorithm did not find a solution in the time limit.

| Graph Name | OnlineMIS | ARW | KerMIS | ReduMIS |
|------------|-----------|-----|--------|---------|
| Name       | Avg. t$_{avg}$ | Avg. t$_{avg}$ | Avg. t$_{avg}$ | Avg. t$_{avg}$ |
| Huge instances: |          |      |        |         |
| it-2004    | 25,610,697 35.324 25,612,993 33.407 | 25,619,988 35.751 | **25,620,246** 35.645 |
| sk-2005    | 30,680,869 34.480 30,373,880 11.387 | 30,686,684 34.923 | 30,684,867 35.837 |
| uk-2007    | 67,265,560 35.982 67,101,065 8.702 | **67,282,347** 35.663 | 67,278,359 35.782 |
| Social networks and Web graphs: |          |      |        |         |
| amazon-2008 | 309,792 6.154 309,791 12.195 | 309,793 818 | **309,794** 153 |
| as-Skitter-big | 1,170,560 7.163 1,170,548 14.017 | **1,170,580** 4 | 1,170,580 9 |
| deviki-2013 | 697,789 17.481 697,669 16.030 | **697,921** 14.070 | 697,798 17.283 |
| enviki-2013 | 2,178,255 13.612 2,177,965 17.336 | **2,178,436** 17.408 | 2,178,327 17.697 |
| eu-2005    | 452,296 11.995 452,311 22.968 | 452,342 5.512 | **452,353** 2.332 |
| hollywood-2011 | 523,402 33 | 523,402 101 | **523,402** 9 |
| libimseti  | 157,288 8.250 127,284 9.308 | **127,292** 102 | 127,292 16.747 |
| ljournal-2008 | 2,970,236 428 2,970,887 16.571 | **2,970,937** 36 | 2,970,937 41 |
| orkut      | 839,073 17.764 839,001 17.933 | 839,004 19.765 | 804,244 34.197 |
| web-Stanford | 163,384 5.938 163,382 10.924 | 163,388 35 | **163,390** 12 |
| webbase-2001 | 79,998,332 35.240 80,002,845 35.922 | 80,009,041 30.960 | **80,009,820** 31.954 |
| wikilinks | 19,404,530 21.069 19,416,213 34.085 | 19,418,693 23.133 | **19,418,724** 854 |
| youtube   | 857,914 < 1 | 857,945 93 | **857,945** < 1 | **857,945** 2 |
| Road networks: |          |      |        |         |
| europe     | 9,267,573 15.622 | 9,267,587 28.450 | 9,267,804 27.039 | **9,267,809** 115 |
| USA-road | 12,426,557 10.490 | 12,426,582 31.583 | 12,427,819 32.490 | **12,428,099** 4799 |
| Meshes:    |          |      |        |         |
| buddha     | 480,795 17.895 | **480,808** 17.906 | 480,592 16.695 | 479,905 17.782 |
| bunny      | 32,283 13.258 | 32,287 13.486 | 32,110 14.185 | **32,344** 1.309 |
| dragon     | 66,501 15.203 | 66,496 14.775 | 66,386 16.577 | 66,447 3.456 |
| feline     | 18,846 15.193 | 18,844 10.547 | 18,732 15.055 | **18,851** 706 |
| gameguy    | 20,662 6.868 | 20,674 12.119 | 20,655 7.467 | **20,727** 191 |
| venus      | **2 684** 507 | **2 684** 528 | 2 664 9 | 2 683 74 |

Not always necessary, especially when the goal is to get a high-quality solution quickly. This also further justifies our choice of cutting, as the solution quality of KerMIS remains high. On the other hand, instances as-Skitter-big, ljournal-2008, and youtube are solved quickly with advanced reduction rules.