State observers as a means for estimating derivatives of deterministic signals

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Abstract. To estimate the derivatives of a deterministic signal arriving in real time, a state observer is introduced, built as a replica of the virtual canonical model with an unknown bounded input. The results of a comparative analysis of observers with linear and piecewise linear corrective actions are presented, which are confirmed by the results of numerical modeling for piecewise differentiable composite function. Its derivatives have jump discontinuity points at the moments when the form of the function changes. It is shown that at these moments the observer variables with linear corrective actions with high gains have large surges, which increase by an order of magnitude with an increase in the order of the estimated derivative. Observers with bounded piecewise linear corrective actions do not have these problems and are recommended for use in practical applications.

1. Introduction
At the synthesis of high-precision automatic control systems, there is a problem of obtaining derivatives of external signals arriving from an independent source. For example, the current values of the derivatives of the given actions of not only the first but also of the highest orders are needed for the synthesis of tracking systems with programmed control. To obtain the derivatives of the higher orders, we need either the complete analytical description of the control actions or their real differentiation. The first approach can turn out to be computationally rather laborious if the desired trajectory has a complex, compound form [1], and the second approach can lead to the appearance of a delay and accumulation of errors, which increase with increasing order of differentiation [1, 3].

In this paper, within the framework of the methods of the state observers synthesis of dynamic objects operating under the action of external disturbances, an alternative approach to the estimation of the derivatives of deterministic signals, which is free from the indicated disadvantages, is proposed.

2. Problem Statement
Let the control system receive an unnoisy signal \( g(t) \) from an external uncontrolled source in real time; the signal is assumed to be a continuous, piecewise differentiated time function, the analytical form of which is unknown. Its derivatives up to and including \((n+1)\) order are assumed to be piecewise continuous and bounded by known constants

\[
|g^{(i)}(t)| \leq G_{i+1}, i = 1, n + 1, t \geq 0. \tag{1}
\]

The problem of restoring the current values of the derivatives of a given signal up to the \(n\)-th order inclusive is considered.
To formalize the problem, let us introduce a virtual dynamic model of the \((n+1)\)-th order, which has a canonical form
\[
g_{i+1} = g_i, i = 1, n+1. \tag{2}\]
The output of model (1) is the measured signal \(g_1(t) = g(t)\); its state variables are the signal \(g_1(t)\) and its nonmeasurable derivatives \(g^{(i)}(t) = g_i, i = 1, n\); the input signal, which is a derivative of \((n+1)\)-order, \((g^{(n+1)}(t) = g_{n+2}(t))\) is treated as an external bounded disturbance. Model (2) generates a fairly wide class of functions. Moments of signal \(g_1(t)\) shape change (i.e. possible points of the jump discontinuity of its derivatives) can be interpreted as intermittent changes of initial conditions in model (1).

Model (2) is observable with respect to the output \(g_1(t)\). In this connection, the following solution to the above problem is proposed: to construct a state observer that will perform as a dynamic differentiator for model (2). Namely, the state variables of the observer will be estimates of unmeasured derivatives \(g^{(i)}(t), i = 1, n\) and will be used for control purposes.

When synthesizing a state observer for model (2), we should take into account the following factors. First, virtual model (2) has an unknown input, so the observation problem can be solved only with some (given) accuracy. And to ensure invariance with respect to the "external disturbance", we should use "suppression" methods in the synthesis of the observer. These include high-gain feedback (linear corrective actions with high gains \([2, 4]\)) and sliding mode observers with discontinuous corrective actions \([5, 12]\). The research results showed that in practical applications it is advisable to use – instead of discontinuous corrective actions – their various continuous approximations (smooth sigmoids \([6, 7]\) or nonsmooth piecewise linear functions with saturation \([8-10]\)), which provide better quality (smoothness) of estimated signals. The indicated prelimit realizations of the sign function provide approximately the same results using a state observer and disturbance observer. Considering that in comparison with sigmoids, piecewise linear functions are simpler in microprocessor implementation, they will be considered in this work.

Secondly, the signals being estimated are assumed to be piecewise-continuous, which leads to several surges in the process of observation. As is known the observation process is initially accompanied by rather large observation errors (since the initial conditions of the observed plant are unknown), which rapidly tend to zero. Therefore, in practice, the convergence time of observation errors is often fixed, and the estimated signals (observer variables) are used in the control system after the completion of the observation error transient processes. In this case, the described technique is ineffective, since each point of the jump discontinuity, the moment of appearance of which is not known in advance, will lead to a surge of estimated signals. To protect the control system from the impact of these surges, it will be necessary either to specifically limit control signals everywhere or to find ways to reduce overshoot in the state observer.

Thus, our goal is a comparative analysis of the derivatives’ observers with linear and piecewise linear corrective actions, taking into account both of these factors.

3. Two approaches to the synthesis of derivatives’ observers built based on a virtual dynamic model

To solve the posed problem, a dynamic state observer is introduced as a replica of virtual model (2):
\[
\dot{z}_i = z_{i+1} + v_i, \quad i = 1, n; \quad \dot{z}_{n+1} = v_{n+1}, \tag{3}\]
where \(z = (z_1, ..., z_n)' \in R^{n+1}\) is state vector, \(v_i (i = 1, n + 1)\) are corrective actions of the observer, which are formed by measurements \(g_i(t)\) and \(z_i(t)\) to ensure system stabilization with respect to observation errors \(e_i = g_i - z_i, i = 1, n+1\):
\[
\dot{e}_i = e_{i+1} - v_i, \quad i = 1, n, \quad \dot{e}_{n+1} = g_n(t) - v_{n+1}.
\]  

A feature of system (4) is the presence of an "external disturbance" \( g_{n+1}(t) = g_{n+2}(t) \). First, we will consider a few special cases.

If there is a reason to suppose that at all intervals the function \( g_i(t) \) is described by algebraic polynomials with the maximum degree \( \overline{n} = n \), then \( g_{n+1}(t) \equiv 0 \). In this case, there is no external disturbance in system (4) and a standard linear correction

\[ v_i = a_1 e_1, v_2 = a_2 e_2, \ldots, v_{n+1} = a_{n+1} e_1, \]

can be used to stabilize it. The parameters of this correction are the coefficients of the Hurwitz polynomial

\[ s^{n+1} + a_1 s^n + a_2 s^{n-1} + \ldots + a_n s + a_{n+1}. \]  

This approach will provide asymptotic convergence of the observation errors in closed system (4)-(5):

\[ \lim_{t \to +\infty} e_i(t) = 0 \Rightarrow \lim_{t \to +\infty} z_i(t) = g_i(t), t = \overline{1, n+1}. \]  

If it is known that the maximum degree of the polynomials is \( 1 \leq \overline{n} < n \), then all derivatives starting from \( (n + 1) \)-th will be identically equal to zero. Accordingly, the order of observer (3) should be reduced by \( n - \overline{n} \). If \( n < \overline{n} < \infty \), then, on the contrary, it is necessary to increase the order of the observer by \( \overline{n} - n \) in order to provide asymptotic estimates (6).

If there is a reason to suppose that the "external disturbance" decreases very quickly and \( \lim_{t \to +\infty} g_{n+2}(t) = 0 \), then we can "ignore it", since in closed-loop system (4)-(5) asymptotic estimates (6) are achieved, on which the decreasing parasitic signal is superimposed.

In the general case, when the "external disturbance" \( g_{n+2}(t) \) is bounded (1), but does not decrease over time, it is necessary to use "suppression" methods to suppress it. For this purpose, the standard linear correction is supplemented by a high gain \( l > 1 \) as follows [2, 4]:

\[ v_i = a_1 l^1 e_1, v_2 = a_2 l^2 e_2, \ldots, v_{n+1} = a_{n+1} l^{n+1} e_1, \]

that provides the stabilization of observation errors with a given accuracy in virtual closed-loop system (4), (7)

\[ |e_i(t)| = |g_i(t) - z_i(t)| \leq \delta(1/l), i = \overline{1, n+1}, t \geq (1/l), \lim_{t \to +\infty} T(1/l) = 0. \]  

Note that result (8) is valid if \( g_{n+2}(t) \) is locally Lipschitz [4]. In this case, this condition is not fulfilled everywhere, but it is fulfilled only within each interval. In more detail, let the given signal \( g_i(t) \) for \( t \geq 0 \) be continuous, piecewise differentiable, and change its form \( k \) times at moments

\[ 0 = \tau_0 < \tau_1 < \tau_2 < \ldots < \tau_k < \infty, \quad \tau_{\text{min}} = \min \{ \tau_j - \tau_{j-1} \} j, k \in \mathbb{N}. \]

At \( t = \tau_j, i = \overline{l, k} \), its derivatives can have jump discontinuities, which are treated as changing of initial conditions in virtual model (2) at the specified moments. Thus, in system (4), in general, there will be \( k + 1 \) transients, and when \( 0 < T < \tau_{\text{min}} \) and the setting is adequate, the observation errors \( e_i(t), i = \overline{2, n+1} \) will be in the given neighborhood of zero in the following time intervals:

\[ |e_i(t)| \leq \delta, \quad T + j - \tau < \tau_{j+1}, j = \overline{0, k-1}, \tau_k + T \leq t; \quad |e_i(t)| \leq \delta, t \geq 0. \]
As is known [4], the main disadvantage of high-gain observers is a large overshoot at the beginning of the transition process. When evaluating the derivatives of piecewise differentiated signals, one should expect a large surge of observation errors \( \varepsilon_i(t), i = 2, n + 1 \) and, consequently, the evaluation signals \( z_i(t), i = 2, n + 1 \) at \( t \in [\tau_j ; \tau_j + T], j = 0, k \), which increase with the growth of high gain \( l \), mismatches in the initial conditions \( g_i(0) - z_i(0) \), and values of jumps \( g_i(\tau_j + 0) - g_i(\tau_j - 0), j = 1, k \).

Thus, the use of derivatives observer (3), (7) in practical applications will lead to a significant decrease in the quality of controlled processes if the control system uses standard linear control laws formed based on estimates \( z_i(t), i = 2, n + 1 \). To improve the quality, it will be necessary to limit the control actions specifically, which can complicate the analysis and synthesis of a closed-loop system with dynamic feedback. Also, due to the used hierarchy of high gain (7), surges of estimated signals will increase each time by an order of magnitude with an increase in the order of the estimated derivative. This fact limits the allowable order of virtual model (1) and the possibility of an adequate estimation of higher derivatives with a large number of jump discontinuity points.

In order to overcome this problem, we propose another method to suppress the disturbance. Let us use piecewise linear corrective actions with saturation [8-10] of the following type in derivatives observer (3):

\[
\begin{align*}
v_i &= m_i \text{sat}(l_i \varepsilon_i) = \\
&= \begin{cases} 
    m_i \text{sign} \varepsilon_i, & |\varepsilon_i| > 1/l_i, \\
    m_i l_i \varepsilon_i, & |\varepsilon_i| \leq 1/l_i; 
\end{cases} \\
v_j &= m_j \text{sat}(l_j \varepsilon_j) = \\
&= \begin{cases} 
    m_j \text{sign} \varepsilon_j, & |\varepsilon_j| > 1/l_j, \\
    m_j l_j \varepsilon_j, & |\varepsilon_j| \leq 1/l_j, 
\end{cases}
\end{align*}
\tag{10}
\]

As we can see, each corrective action (10) has two configurable parameters: amplitude \( m_i > 0 \) and high gain \( l_i > 0 \). Bounded and continuous functions (10) are a hybrid of high-gain feedback [2, 4] and discontinuous controls [5, 12]. They bring positive properties of both methods to closed-loop system (4) but they are free from their disadvantages. According to the cascade synthesis procedure [8-10], the selection of amplitudes sequentially, from top to bottom, ensures that the arguments of the corrective actions get into linear zones, and estimation accuracy (9) is ensured by selecting high gains. The setting of these parameters is based on inequalities and, unlike the setting of high-gain observers (3), (7), does not require compiling reference polynomials (5).

In [9] in the framework of synthesis of a single-channel tracking system under the action of external disturbances, an observer similar to (3), (10) was used to estimate mixed variables (functions of state variables, external influences, and their derivatives) by measuring the tracking error. Hierarchical systems of inequalities were obtained for selection of the parameters of piecewise-linear corrective actions, under which, for a given time \( T > 0 \) with a given accuracy \( \delta > 0 \), an estimation of non-measured signals of the canonical system, similar to (1)-(2), is provided. These procedures, without the restriction of generality, can be used to adjust corrective actions (10) of derivatives observer (3).

Let us highlight the main advantage of the observer (3), (10) in the problem of estimating piecewise continuous signals. Corrective actions (10) are bounded everywhere

\[
v_i(t) \leq m_i = \text{const}, i = 1, n + 1,
\]

consequently, surges of the estimated signals of the derivatives at the beginning of all transients, which are caused by the change of the shape of output piecewise-differentiated signal \( g(t) \), will also be significantly bounded. Moreover, the magnitudes of these surges do not depend on the order of the estimated derivative, which allows obtaining adequate estimations of the highest derivatives of any required order.

Thus, the application of observer (3), (10) expands the class of admissible functions significantly, and the estimated signals of its derivatives can be directly used in practical applications without additional restrictions of control signals.
4. Example
To illustrate the presented above comparative analysis of the derivatives observers with linear and piecewise linear corrective actions, let us consider the results of numerical example modeling in a MATLAB-Simulink environment. For the signal

\[ g_i(t) = \begin{cases} 
4 - t^2, & t \in [0; 2); \\
0.5(t-2)^2, & t \in [2; 4); \\
4, & t \in [4; 6); \\
4e^{-(t-6)}, & t \in [6; +\infty), 
\end{cases} \]

whose values are available at any moment \( t \), estimates of the first three derivatives were required to obtain. For this purpose, observers (3) of the 4th order with piecewise-linear (10) and linear (7) corrective actions were constructed, the parameters of which were selected so as to provide approximately equal settling time \( T = 0.1 \text{ [s]} \) and accuracy of estimation \( \delta = 0.02 \) (9):

\[
m_1 = 150, m_2 = 116, m_3 = m_4 = 150; l_1 = l_2 = 30, l_3 = l_4 = 10; \]
\[
a_1 = 8, a_2 = 24, a_3 = 32, a_4 = 16, l = 168. \]

Figures 1-4 present graphs of the signal \( g_i(t) \), its three derivatives \( g_i^{(i)}(t) = g_{i+1}, \ i = 1,2,3 \) and their estimates \( z_i(t) \) (from below), as well as \( \varepsilon_i(t) = g_i(t) - z_i(t), \ i = 1,4 \) (from above). The graphs on the left side are obtained for the observer with piecewise-linear correction with saturation (10), (12), and on the right side – with linear correction (7), (13).

Fig. 1. Graphs of the \( \varepsilon_1(t) = g_1(t) - z_1(t) \).

Fig. 2. Graphs of the \( \varepsilon_2(t) = g_2(t) - z_2(t) \).
For observation errors $e_i, i = 1,4$ on the interval $t \in [2; 4]$, Table 1 presents: settling time $t_s$, $|e_i(t)| \leq 0.02, t \in [t_i; t_f]$; steady-state accuracy $\delta_i$, and overshoot $e_{max,i} \geq |e_i(t)|, t \in [2; t_i]$. 

| Signal  | Observer (3), (10), (11) | Observer (3), (7), (12) |
|--------|--------------------------|--------------------------|
| $e_1 = g_1 - z_1$ | $t_s, s | \delta_i | e_{max,i}$ | $t_s, s | \delta_i | e_{max,i}$ |
| 2.0308  | 1.55 \cdot 10^{-15} | 8.70 \cdot 10^{-4} | 2.0247  | 1.11 \cdot 10^{-15} | 0.0021  |
| $e_2 = g_2 - z_2$ | 2.0342  | 2.25 \cdot 10^{-4} | 4.0003 | 2.0333  | 2.25 \cdot 10^{-4} | 4.0003  |
| $e_3 = g_3 - z_3$ | 2.0570  | 3.00 \cdot 10^{-4} | 3.1754 | 2.0557  | 3.00 \cdot 10^{-4} | 867.8230 |
| $e_4 = g_4 - z_4$ | 2.0743  | 1.24 \cdot 10^{-5} | 3.0000 | 2.0722  | 7.93 \cdot 10^{-9} | 61091.4089 |

As we can see, these characteristics confirm the conclusions about the advantage of an observer with piecewise linear correction when estimating piecewise continuous derivatives of high orders.

It should be noted that given signal (11) is a special case described in the previous section. Namely, in all open time intervals, its fourth derivative is either identically equal to zero, or is infinitely small, i.e., in virtual system (4), the external disturbance is absent or fades out quickly. If this fact is known a priori, then it is possible not to use high gain in linear correction (7) and slightly reduce the overshoot at the break points. However, given signal (11) changes shape quite often, after 2 seconds. Therefore, we still have to use sufficiently large correction factors and, as a consequence, get a large overshoot, the magnitude of which increases with increasing order of the estimated derivative. Otherwise, with a decrease in the correction coefficients in modulus, the time of transient processes increases, which leads to unacceptably large estimation errors at intervals.
5. Conclusion
In the work for the estimation of the derivatives of deterministic signals, standard observers were used, which repeat the structure of estimation objects. Within the framework of systems with piecewise linear feedbacks, the construction of a non-standard observer-differentiator without its own dynamics is promising, which will lead to the appearance of explicitly estimated signals in the system regarding observation errors and will significantly simplify the procedure for adjusting the parameters of corrective actions.

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