Soliton Spectrum of Integrable Models with Local Symmetries

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ABSTRACT

The soliton spectrum (massive and massless) of a family of integrable models with local $U(1)$ and $U(1) \otimes U(1)$ symmetries is studied. These models represent relevant integrable deformations of $SL(2,R) \otimes U(1)^{n-1}$ - WZW and $SL(2,R) \otimes SL(2,R) \otimes U(1)^{n-2}$ - WZW models. Their massless solitons appear as specific topological solutions of the $U(1)$ (or $U(1) \otimes U(1)$) - CFTs. The nonconformal analog of the GKO-coset formula is derived and used in the construction of the composite massive solitons of the ungauged integrable models.

\textsuperscript{1}to appear in JHEP (2002)
1 Introduction

The $G$-WZW models and their gauged $G/H$-versions (for appropriated choice of $G$ and $H$) are known to describe string theories on curved backgrounds [1], [2]. The simplest examples are the $SL(2, R) -$ WZW model, representing string $AdS_3$ target space time [3] and $SL(2)/U(1) -$ WZW, giving rise to 2-d black hole geometry [1], [2]. In this paper we consider specific relevant perturbations ($\hat{G}, G_0, \mu_{ab}$) \[4\] \[5\] \[6\] of certain physically interesting (gauged) WZW models

$$
\mathcal{L}^{pert}(\hat{G}, G_0) = \mathcal{L}^{WZW}_{G_0}(g_0) - \frac{k}{2\pi} \mu_{ab} \text{Tr}(T_a g_0 T_b g_0^{-1}) \tag{1.1}
$$

where $G_0 \subset \hat{G}$ is finite dimensional subgroup of the affine group $\hat{G}$ with generators $T_a, g_0 \in G_0$ and $\mu_{ab}$ are real parameters. They are expected to describe the nonconformal counterparts of the Maldacena string/gauge theory correspondence \[1\] (i.e., deformations of $AdS_d/CFT_{d-1}$ \[8\]). The first question to be addressed is about the classical (and quantum) integrability of the models (1.1), i.e., for a given affine group $G$, how to chose $G_0 \subset \hat{G}$ and $\mu_{ab}$ such that (1.1) is exactly integrable? Another important question concerns the nonperturbative topologically stable solutions of the model (1.1) and their particle or strings interpretation.

The main purpose of the present paper is the description of the semiclassical spectrum of the finite energy topological solutions (solitons and solitonic strings) of a class of integrable perturbations that preserve (a) one local $U(1)$($p = 1$) or (b) two $U(1) \otimes U(1)$($p = 2$) local symmetries. The simplest representative of type (a) integrable models (IMs) studied in this paper is given by the Lagrangian [3]

$$
\mathcal{L}^u_{p=1} = \frac{1}{2} \eta_{ij} \partial \varphi_i \partial \varphi_j + \frac{n}{2(n+1)} \partial R_u \partial R_u + \partial \chi_u \partial \psi_u e^{\beta(R_u - \varphi_1)} - V_u \tag{1.2}
$$

with potential

$$
V_u = \frac{m^2}{\beta^2} \left( \sum_{i=1}^{n-1} e^{\beta(\varphi_{i-1} + \varphi_{i+1} - 2\varphi_i)} + e^{\beta(\varphi_1 + \varphi_{n-1})} \left( 1 + \beta^2 \psi_u \chi_u e^{\beta(R_u - \varphi_1)} - n \right) \right)
$$

where $\varphi_0 = \varphi_n = 0, i, j = 1, 2, \cdots n - 1, \beta^2 = -\frac{2\pi}{\kappa}$ and $\eta_{ij} = 2\delta_{ij} - \delta_{i,j-1} - \delta_{i,j+1}$ is the $A_{n-1}$ Killing form. The above $\mathcal{L}^u_{p=1}$ is indeed a special case of the $\mathcal{L}^{pert}(\hat{G}, G_0)$ (1.1) obtained by taking $\hat{G} = A_n^{(1)}$, \[ G_0 = SL(2) \otimes U(1)^{n-1} \], $\mu_{ab} = \mu_a \mu_b$ and

$$
\mu_a T_a = \epsilon_+ = m \left( \sum_{i=2}^{n} E^{(0)}_{\alpha_i} + E^{(1)}_{(\alpha_2 + \cdots + \alpha_n)} \right), \quad \bar{\mu}_b T_b = \epsilon_- = m \left( \sum_{i=2}^{n} E^{(0)}_{-\alpha_i} + E^{(-1)}_{(\alpha_2 + \cdots + \alpha_n)} \right) \tag{1.3}
$$

The fields $\varphi_i, R_u, \psi_u, \chi_u$ that appear in (1.2) parametrize the $g_0 \in G_0$ group element as follows

$$
g_0 = \exp \left( \beta \chi_u E^{(0)}_{-\alpha_1} \right) \exp \left( \beta \lambda_1 \cdot H^{(0)} R_u + \beta \sum_{i=2}^{n} \varphi_{i-1} h^{(0)}_i \right) \exp \left( \beta \psi_u E^{(0)}_{\alpha_1} \right)
$$
An important property of these IMs is their invariance under local $U(1)$ transformations ($\beta = i\beta_0$)

$$R'_u = R_u + \beta_0 \left( w(z) + \bar{w}(\bar{z}) \right), \quad \psi'_u = \psi_u e^{-i\beta_0 \bar{w}(\bar{z})}, \quad \chi'_u = \chi_u e^{-i\beta_0 w(z)}, \quad \varphi'_j = \varphi_j$$

(1.4)

where $w, \bar{w}$ are arbitrary chiral functions. The IM (1.2) represents $A_{n-1}$-abelian affine Toda theory interacting with the thermal perturbation (with $\Phi_{ab}^{(j=1)}$) of the $SL(2, R)$-WZW model. Two particular cases should be mentioned: (1) $n = 1$ (no $\varphi_i$ at all) and the IM (1.2) just coincides with deformed $SL(2, R)$-WZW model. They have again the form (1.5), but with $R^{\text{vac}}$ replaced by $R^{CFT}$, i.e.,

$$R^{CFT} = \beta_0 \left( w(z) + \bar{w}(\bar{z}) \right)$$

(1.6)

The spectrum of the left-moving solitons of this chiral $U(1)$-CFT (with appropriate periodic b.c.'s for $w, \bar{w}$), derived in Sect. 3.4 has the form:

$$E_{L-sol}^{CFT} = \frac{4n}{(n+1)\beta_0} \left( s + \frac{j_w}{n} \right)^2 |a_0|, \quad j_w = j_\varphi - 2j_q, \quad j_\varphi, j_q = 0, \pm 1, \cdots, \pm (n-1) \text{ mod } n$$

$$Q_0 = \frac{4\pi n}{n+1} \left( s + \frac{j_w}{n} \right), \quad s = 0, \pm 1, \cdots$$

(1.7)

where $|a_0|$ is an arbitrary infrared mass scale and $Q_0$ denotes the left-$U(1)$ charge. The conformal sector of the theory provides each one of the $n$-vacua states ($E^v = 0$) with a tower of conformal states $(j_\varphi, j_w, s)$ with $E^{CFT} \neq 0$. The massless solitons are topologically stable solutions that interpolate between the vacua $(0, 0, 0)$ and an arbitrary conformal state $(0, j_w, s)$. Together with such massless solutions, it is natural to expect the existence of two types of massive solitons:

1. interpolating between different vacua, called $g$-solitons: $(0, 0, 0) \rightarrow (j_\varphi, 0, 0)$

2. interpolating between vacua and arbitrary exited conformal states, called $u$-solitons: $(0, 0, 0) \rightarrow (j_\varphi, j_w, s)$
as it is shown by the diagram

\[
\begin{array}{c}
\text{vacua} \\
(0,0,0) \\
\downarrow M_g \neq 0 \\
\text{gauged IM} \\
(j_\varphi,0,0) \\
\downarrow M^\text{CFT} = 0 \\
\text{u(1)-CFT} \\
(0,j_\omega,s) \\
\downarrow M_g = 0 \\
\text{ungauged IM} \\
(j_\varphi,j_\omega,s) \\
\end{array}
\]

In order to construct such soliton solutions and to derive their semi-classical spectrum, it is crucial to observe that the type (1) g-solitons in fact coincide with the \(U(1)\)-charged topological solitons of the gauged dyonic IM \([9], [10]\) (with one global \(U(1)\) symmetry) obtained from (1.2) by axial gauging of the local \(U(1)\):

\[
\mathcal{L}^g_{\alpha x}(p = 1) = \frac{1}{2} \sum_{i,j=1}^{n-1} \eta_{ij} \partial \varphi_i \bar{\partial} \varphi_j + \frac{\partial \chi_g \bar{\psi}_g e^{-\beta \varphi_1}}{1 + \beta^2 \frac{n+1}{2n} \psi_g \chi_g e^{-\beta \varphi_1}} - V_g,
\]

\[
V_g = \frac{m^2}{\beta^2} \left( \sum_{i=1}^{n-1} e^{-\beta \eta_{ij} \varphi_j} + e^{\beta (\varphi_1 + \varphi_{n-1}) (1 + \beta^2 \psi_g \chi_g e^{-\beta \varphi_1})} - n \right)
\]

The semiclassical spectrum of the 1-solitons of this gauged IM has been derived in our recent paper \([9]\).

The type (2) u-solitons turns out to be the conformal dressing of the g-solitons, i.e. by performing specific \(U(1)\) gauge transformations (1.4) (with \(w\) and \(\bar{w}\) given by eqn. (3.40)) of the already known g-solitons \([9], [10]\). The spectrum of these new u-solitons can be obtained by applying the following nonconformal version of the Goddard-Kent-Olive (GKO) coset construction \([14]\),

\[
T^u_{G_0} = T^g_{G_0/G_0} + T^CFT_{G_0}\]

establishing the relation between the stress-tensors (and energies) of the ungauged IM (1.2), the gauged IMs (1.8) and the \(U(1)\)-CFT. The spectrum of the left-moving u-solitons (\(w \neq 0, \bar{w} = 0\)), derived in Sect. 3.5, has the form,

\[
M^2_u = M_g \left( M_g + \frac{8nm_0}{(n+1)\beta_0^2} (s + j_w/n)^2 \right),
\]

\[
M_g = \frac{4nm}{\beta_0^2} \sin(\frac{4\pi}{4n} j_e - \frac{\beta_0}{4n} j_e), \quad Q_{el} = \beta_0 j_e, \quad j_e = 0, \pm 1, \ldots
\]

\[
Q^u_{R} = \frac{n + 1}{2n} \left( Q_{el} + \frac{2\pi n}{n+1} (s + j_w/n) \right), \quad Q^u_{\theta} = \frac{\pi}{2} (s + j_w/n)
\]
where \( m_0 = |a_0| e^{bL} \) and g-soliton velocity \( v_g \) coincides with the left rapidity \( b_L \), i.e., \( v_g = b_L \). The stability of these strong coupling particles is ensured by their nontrivial topological charges: \( j_{\varphi}, s, j_{\psi} \).

The paper is organized as follows. In Sect. 2 we present a brief introduction to the path integral version of the Hamiltonian reduction method. The Lagrangians of the different IMs (with one local \( U(1) \), two local \( U(1) \otimes U(1) \) and one local and one global \( U(1)'s \) ) considered in this paper are derived together with the proof of their integrability. Sect. 2.3 is devoted to the identification of the above IMs as relevant integrable deformations of the conformal minimal models of certain extended conformal algebras \( W_{n+1}^{(n)} \) and \( V_{n+1}^{(1,1)} \). The discussion of Sect. 3 is concentrated on the symmetries (continuous \( U(1) \) and discrete \( Z_n \otimes Z_2 \)), their currents and especially on the allowed b.c.'s for the fields of IMs (1.2) and (1.8). The main result of Sect. 3.2 is the derivation of the nonconformal GKO coset construction. In Sect. 3.4, we study the properties of the different \( U(1)-CFT \) solitons: left, right and left-right ones. The spectrum of the composite u-solitons is obtained in Sect. 3.5. The spectral flow of the \( U(1)-\)charges \( Q_0, \bar{Q}_0 \) induced from certain topological \( \theta \)-terms added to the original Lagrangian (1.2) is derived in Sect. 3.6. Sect. 3.7 is devoted to the nontopological solitons of the deformed \( SL(2,R) \)-WZW model. The constructions of different soliton solutions of the ungauged IM (2.12) with \( U(1) \otimes U(1) \) local symmetries as well as of the so called intermediate IM are presented in Sect. 4. Sect. 5 contains our conclusions and few remarks concerning the use of the charge spectrum of our soliton and string solutions in the off-critical (i.e. nonconformal) \( AdS_3/CFT_2 \) correspondence. The IMs with local \( SL(2,R) \otimes U(1) \) symmetry and an example of nonrelativistic IM with local \( U(1) \) symmetry are also discussed.

## 2 Integrable Perturbations of Gauged \( A^{(1)}_n \)-WZW Model

### 2.1 Effective Lagrangians from Hamiltonian reduction

Different integrable perturbations \((\hat{G}, G_0, \mu_{ab})\) of the gauged WZW model \((\mathbb{1}, \mathbb{1}, \mathbb{1}, \mathbb{1})\) are known to be (at least classically) in one to one correspondence to the admissible graded structures \((\hat{G}, Q, \epsilon_{\pm})\) of the affine algebra \( \hat{G} \)

\[
[Q, G_k] = kG_k, \quad [G_k, G_l] \subset G_{k+l}, \quad k, l \in \mathbb{Z}, \quad G_k \subset \hat{G}, \quad [Q, \epsilon_{\pm}] = \pm \epsilon_{\pm}
\]

introduced by an appropriate grading operator \( \lambda \)

\[
Q = \tilde{h}d + \sum_{i=1}^{\text{rank} \hat{G} = n} s_i \lambda_i \cdot H^{(0)}, \quad [d, E_{\alpha}^{(a)}] = a E_{\alpha}^{(a)}, \quad a, s_i, \tilde{h} \in \mathbb{Z}
\]

such that the zero graded subalgebra \( G_0 \), i.e., \([Q, G_0] = 0\) is finite dimensional. The functional integral version of the Hamiltonian reduction method \((\mathbb{1}, \mathbb{1}, \mathbb{1}, \mathbb{1})\) consists in considering the two-loop (i.e., affine) \( \hat{G} \)-WZW model \((\mathbb{1}, \mathbb{1}, \mathbb{1}, \mathbb{1})\) and imposing certain constraints on the \( \hat{G} \)-currents \( J_{\alpha}^{(a)} = Tr \left( g^{-1} \partial g E_{\alpha}^{(a)} \right) \) and \( \bar{J}_{\alpha}^{(a)} = Tr \left( \bar{\partial} gg^{-1} E_{-\alpha}^{(a)} \right) \) from the positive and negative \({^2\lambda}_i\) are the fundamental weights and \( \alpha_i \cdot H^{(a)}, E_{\alpha}^{(a)} \) the generators of \( \hat{G} \). The operator \( d \) is the derivation operator.
grades \(|q| \geq 1\) subalgebras \(\mathcal{H}_\pm = \oplus_{l \in \mathbb{Z}_\pm} \hat{G}_l\). More precisely, we introduce the \(\mathcal{H}_\pm\)-invariant gauged \(\mathcal{H}_-\backslash \hat{G}/\mathcal{H}_+\)-WZW model

\[
S(g, A_+, \bar{A}_-) = S_{\text{WZW}}(g) - \frac{k}{2\pi} \int d^2x Tr \left( A_- (\partial g g^{-1} - \epsilon_+) + \bar{A}_+ (g^{-1} \partial g - \epsilon_-) + A_- g \bar{A}_+ g^{-1} \right)
\]  

(2.2)

where \(\bar{A}_+(z, \bar{z}) \in \mathcal{H}_+, \ A_-(z, \bar{z}) \in \mathcal{H}_-, \ g(z, \bar{z}) \in \hat{G}\) and \(\epsilon_\pm\) are specific constant linear combinations of grade \(\pm 1\) generators of \(\hat{G}\), say,

\[
\epsilon_+ = \mu_a T_a = \sum_i \mu_i E^{(0)}_{\alpha_i} + \mu_0 E^{(1)}_{\sigma}, \quad \epsilon_- = \bar{\mu}_i T_{\bar{\alpha}_i} = \sum_i \bar{\mu}_i E^{(0)}_{-\bar{\alpha}_i} + \bar{\mu}_0 E^{(-1)}_{\sigma},
\]

where \(\mu_{ab} = \mu_a \bar{\mu}_b\) and \(\sigma\) is a fixed composite root, such that \([Q, E^{(\pm 1)}_{\pm \sigma}] = \pm E^{(\pm 1)}_{\pm \sigma}, \ \alpha_i\) denote the simple roots of \(\hat{G}\). Then the effective Lagrangian \((1.1)\) appears as a result of performing the Gaussian integral on \(A_-\) and \(\bar{A}_+\) in the partition function

\[
Z = \int Dg D\bar{g} A_- D\bar{A}_+ \exp(-S(g, A_-, \bar{A}_+)) \sim \int Dg_0 \exp(-S_{\text{pert}}(g_0))
\]

(2.3)

of the gauged two loop WZW model, where \(g_0(z, \bar{z}) \in G_0 \subset \hat{G}\). By construction for each admissible graded structure \((2.1)\) of \(\hat{G}\) and for any choice of \(\epsilon_\pm\) the corresponding \(\mathcal{L}_{\text{pert}}(1.1)\) represents an integrable model, as we shall demonstrate in the next Subsection 2.2. Therefore the individual properties of such IMs are determined by \(G_0\) and \(\epsilon_\pm\), containing all the information about the physical fields \(g_0 = g_0(\varphi_l, \psi_a, \chi_a, \cdots)\) and their nonconformal interactions. Depending on the grading operator \(Q\) the zero grade subgroup \(G_0 \subset \hat{G}\) can be abelian \((G_0 = U(1)^L, 0 \leq l \leq n)\), or nonabelian, say \(G_0 = SL(2)^p \otimes U(1)^{n-p}\). For example for \(\hat{G} = A_n^{(1)}\), taking \(Q\) in the form (principal gradation) \(Q = (n + 1) d + \sum_{i=1}^{n} \lambda_i \cdot H^{(0)}\) one conclude that \(G_0 = U(1)^n\). We next choose the most general \(\epsilon_\pm\), such that

\[
\epsilon_\pm = \sum_{i=1}^{n} \epsilon^{(\pm)}_{\alpha_i} E^{(0)}_{\alpha_i} + \mu^{(\pm)}_{0} E^{(\pm)}_{\sigma(\alpha_1 + \cdots + \alpha_n)}
\]

(2.4)

and observe that the corresponding \(\mathcal{L}_{\text{pert}}(1.1)\) takes the form of the well known abelian affine Toda model \((1.3)\). It represents an integrable perturbation of the \(W_{n+1}\)-minimal models \((20)\) which describes marginal \((\mu^{(\pm)}_{\alpha})\) and relevant \((\mu^{(\pm)}_{0})\) perturbations of a string on flat background with certain tachyonic potential. An interesting nonflat strings backgrounds are provided by the non-abelian (NA) affine Toda models \((19)\), i.e., when \(G_0 \subset \hat{G}\) is non-abelian. The simplest example of such model is defined by the following algebraic data:

\[
\hat{G} = A_n^{(1)}, \quad G_0 = SL(2) \otimes U(1)^{n-1}, \quad Q = nd + \sum_{i=2}^{n} \lambda_i \cdot H^{(0)}, \quad \epsilon_\pm = m \sum_{i=2}^{n} \epsilon^{(0)}_{\pm \alpha_i} + \epsilon^{(\pm)}_{\sigma(\alpha_2 + \cdots + \alpha_n)}
\]

(2.5)

\(^{a}\)these are the simplest grade \(|q| = 1\) IMs. Imposing constraints on \(J^{(a)}_n, \bar{J}^{(a)}_{-\alpha}\) of grades \(|q| \geq s\), one can construct in this way the so called higher grades IMs \([18]\).
and physical fields parametrizing the zero grade group element

\[ g_0 = e^{\beta \chi_u E^{(0)}_{-\alpha_1}} e^{\beta \lambda_1 \cdot H R_u + \beta \sum_{i=1}^{n-1} \varphi_i h_{i+1}} e^{\beta \psi_u E^{(0)}_{\alpha_1}} \]

Its Lagrangian derived from eqns. (2.2) and (2.3) has the form (1.2) and represents an integrable perturbation of \( G_0 \)-WZW model. The main new feature of this \( A_n^{(1)}(p = 1) \) affine NA-Toda model in comparison with the abelian one defined by eqn. (2.4), is the existence of nontrivial invariant subalgebra \( \mathcal{G}^0_0 \subset \mathcal{G}_0 \), such that \( [\mathcal{G}^0_0, \epsilon_{\pm}] = 0 \), i.e. \( \mathcal{G}^0_0 = U(1) = \{ \lambda_1 \cdot H^{(0)} \} \). It contains all the information about the continuous symmetries of the model. For the abelian affine Toda we have \( \mathcal{G}^0_0 = \emptyset \), i.e., no continuous symmetries. In the NA-Toda case, (i.e., eqn. (1.1) and (2.4) leading to (1.2)) one can easily verify that the Lagrangian (1.1) (and therefore (1.2)) is invariant under chiral \( \mathcal{G}^0_0 = U(1) \) transformations:

\[ g'_0 = e^{\beta \omega(z) \lambda_1 \cdot H} g_0 e^{\beta \omega(z) \lambda_1 \cdot H} \] (2.6)

which is the compact form of the field transformation (1.4).

Similarly to the conformal (i.e. unperturbed) WZW models one can further gauge fix the above chiral \( \mathcal{G}^0_0 \)-symmetry. The standard procedure of gauge fixing [21] consists in considering the following “improved” action by the addition of auxiliary \( \mathcal{G}^0_0 \)-fields \( A_0(z, \bar{z}), \bar{A}_0(z, \bar{z}) \), playing the role of Lagrange multipliers:

\[
S(g_0, A_0, \bar{A}_0) = S_u(g_0) - \frac{k}{2\pi} \int d^2 x Tr \left( \pm A_0 \partial g_0 g_0^{-1} + \bar{A}_0 g_0^{-1} \partial g_0 \pm A_0 g_0 \bar{A}_0 g_0^{-1} + A_0 \bar{A}_0 \right)
\] (2.7)

where the signs \( \pm \) takes place for axial/vector gaugings of the \( U(1) \), respectively. The new \( A_0, \bar{A}_0 \)-dependent terms added to the action \( S_u(g_0) \) are responsible for imposing the additional constraints \( J_{\lambda_1 \cdot H} = Tr \left( \lambda_1 \cdot H g_0 \partial g_0 \right) = \bar{J}_{\lambda_1 \cdot H} = Tr \left( \lambda_1 \cdot H \partial g_0 g_0^{-1} \right) = 0 \). They promote the chiral \( U(1) \) symmetry (2.4) to the following local \( U(1) \) symmetry \( \alpha_0 = \alpha_0(z, \bar{z}) \in U(1) \)

\[ g'_0 = \alpha_0 g_0 \alpha'_0, \quad \bar{A}'_0 = \bar{A}_0 - \bar{\alpha}_0 \alpha'_0, \quad A'_0 = A_0 - \alpha_0^{-1} \partial \alpha_0 \] (2.8)

where \( \alpha'_0 = \alpha_0 \) in the case of axial gauging and \( \alpha'_0 = \alpha_0^{-1} \) for the vector gauging. Again, one can take the Gaussian integral on \( A_0, \bar{A}_0 \) as in eqn. (2.3) and the result is the effective Lagrangian (1.8) of the axial singular affine NA-Toda denoted as dyonic \( A_n^{(1)}(p = 1) \) IMs (\( p = 1 \) is the number of the gauged fixed \( U(1)'s) \).

According to refs. [4], [22] the CPT invariant vector gauged Lagrangian has the following form:

\[
\mathcal{L}_0^v(p = 1) = \frac{1}{2} \sum_{l=1}^{n-1} \left( \partial \phi_i \bar{\partial} (\phi_l + \phi_{l+1} \cdots + \phi_{n-1}) + \bar{\partial} \phi_i \partial (\phi_l + \phi_{l+1} \cdots + \phi_{n-1}) \right) - \frac{1}{2} \partial A \bar{\partial} B + \bar{\partial} A \partial B - V_v,
\]
This algebraic data was used in deriving the Lagrangian (2.2) for the gauged fixed loop WZW model. Again, by integrating the auxiliary fields $A$, $B$ and $\phi_i$ as follows

$$g_0^\nu = \begin{pmatrix} d_2 & 0 \\ 0 & d_{n-1} \end{pmatrix}, \quad d_{n-1} = \text{diag}(e^{\beta \phi_1}, \ldots, e^{\beta \phi_{n-1}}), \quad d_2 = \begin{pmatrix} A \\ \frac{A}{ue^{\beta \phi_1 + \cdots + \phi_{n-1}}} \end{pmatrix}, \quad B e^{-\beta (\phi_1 + \cdots + \phi_{n-1})} \bigg| \begin{array}{c} u \\ \frac{A}{ue^{\beta \phi_1 + \cdots + \phi_{n-1}}} \end{array} \bigg)$$

The nonlocal field $u$ (i.e., the vector analog of $R_g$) is defined by the following first order equations [3]:

$$\partial \ln \left( u e^{\beta (\phi_1 + \cdots + \phi_{n-1})} \right) = -\beta^2 \left( A \partial B \right) \frac{1}{1 - \beta^2 AB}, \quad \bar{\partial} nu = -\beta^2 \left( B \bar{\partial} A \right) \frac{1}{1 - \beta^2 AB}.$$ 

The main difference between “ungauged” IM $\mathcal{L}^u(p = 1)$ (1.2) and its gauged, axial (1.8) and vector (2.3) versions,is that the latters are invariant only under global $U(1)$ transformations, say $(\beta = i \beta_0)$

$$A' = e^{ia\beta_0} A, \quad B' = e^{-ia\beta_0^2} B, \quad \phi'_i = \phi_i - \frac{a}{n} \beta_0 \quad (2.10)$$

They contain one less field (no $R_u$ or $u$) and represent more complicated target space metric $g_{AB} \sim \frac{1}{1 - \beta^2 AB}$, than the one of the ungauged model (1.2). The reason we are considering them once more in the present paper is due to the crucial role they are going to play (see Sect. 3.3) in the description of different 1-soliton solutions of the ungauged IM (1.2).

Together with the IMs (1.2) with one local $U(1)$ symmetry we shall study the soliton spectrum of another class of multicharged $A_{n}^{(1)}(p = 2)$ IMs [23], [24] with two local $U(1) \otimes U(1)$ symmetries and also IMs with one local and one global $U(1)$ described by Lagrangian (2.16) below. The starting point in the description of their effective Lagrangians is the following specific graded structure of $\hat{G} = G_{n}^{(1)}$:

$$Q = (n - 1) + \sum_{i=2}^{n-1} \lambda_i \cdot H^{(0)}, \quad G_0 = SL(2) \otimes SL(2) \otimes U(1)^{n-2}, \quad G_0^0 = U(1) \otimes U(1),$$

$$\epsilon_{\pm} = m \left( \sum_{i=2}^{n-1} E^{(0)}_{\pm \alpha_i} + E^{(\pm 1)}_{+ (\alpha_2 + \cdots + \alpha_{n-1})} \right),$$

$$g_0 = e^{\beta} \sum_{a=1,n} \lambda_a \cdot H R_{g}^{a} + \beta \sum_{i=1}^{n-2} \phi_{i+1} e^{\beta} \sum_{a=1,n} \psi_{a}^{\alpha} E_{a}^{(0)} \quad (2.11)$$

This algebraic data was used in deriving the Lagrangian (2.2) for the gauged fixed $\mathcal{H}_\pm$ two-loop WZW model. Again, by integrating the auxiliary fields $A_+$, $\tilde{A}_-$, one gets (1.1). More
explicitly the ungauged multicharged $A_n^{(1)}(p = 2)$ IM is given by [23]

$$L^u_{p=2} = \frac{1}{2} \sum_{i=1}^{n-2} \eta_{ij} \partial \varphi_i \partial \varphi_j + \partial \chi^u_1 \partial \psi^u_1 e^{\beta (R^a_1 - \varphi_1)} + \partial \chi^u_n \partial \psi^u_n e^{\beta (R^a_n - \varphi_{n-2})}$$

$$+ \frac{1}{2(n+1)} \left( n \partial R^u_1 \partial R^a_1 + n \partial R^u_n \partial R^a_n + \partial R^u_1 \partial R^a_n + \partial R^u_n \partial R^a_1 \right) - V_u^{p=2} \quad (2.12)$$

with potential

$$V_u^{p=2} = \frac{m^2}{\beta^2} \left( \sum_{i=1}^{n-2} e^{-\beta \eta_{ij} \varphi_j} + e^{\beta (\varphi_1 + \varphi_{n-2})} (1 + \beta^2 \psi^u_1 \chi^u_1 e^{\beta (R^a_1 - \varphi_1)}) \right)$$

$$\times \left( 1 + \beta^2 \psi^u_1 \chi^u_1 e^{\beta (R^a_n - \varphi_{n-2})} - n + 1 \right), \quad (2.13)$$

where $\varphi_0 = \varphi_{n-1} = 0$. For the specific choice of constant grade $\pm 1$ elements $\epsilon_{\pm} \quad (2.11)$, the form of the invariant subalgebra $G_0^0 = \{\lambda_1 \cdot H, \lambda_n \cdot H\}$ is an indication that the IM \quad (2.12) is invariant under the following chiral $U(1) \otimes U(1)$ transformations

$$g_0 = e^{\beta \sum_a \lambda_a H \bar{w}_a(z)} g_0 e^{\beta \sum_a \lambda_a H \bar{w}_a(z)} \quad (2.14)$$

with $\partial \bar{w}_a = \partial \bar{w}_a = 0$. The field transformation encoded in eq. \quad (2.14) have the form

$$(R^u_a)' = R^u_a + \beta_0 (w_a + \bar{w}_a), \quad \varphi_i' = \varphi_i, \quad (\psi^u_a)' = e^{-i \beta \bar{w}_a \psi^u_a}, \quad (\chi^u_a)' = e^{-i \beta \bar{w}_a \chi^u_a} \quad (2.15)$$

By axial or vector gauge fixing one of the local symmetries (say, the one generated by $(\lambda_1 + \lambda_n) \cdot H$) one can derive an interesting “intermediate” IM with one local and one global $U(1)$ symmetries. We take eqn. \quad (2.7) with $g_0 \in G_0 = SL(2) \otimes SL(2) \otimes U(1)^{n-2}$ and $A_0 = a_0(z, \bar{z}) (\lambda_1 + \lambda_n) \cdot H, \bar{A}_0 = \bar{a}_0(z, \bar{z}) (\lambda_1 + \lambda_n) \cdot H, (a_0, \bar{a}_0$ are arbitrary functions) and by performing the Gaussian integration over $a_0, \bar{a}_0$ we obtain the effective Lagrangian for the intermediate axial IM \quad [23]

$$L^{\text{interm}}_{p=2} = \frac{n-1}{n+1} \partial R \partial \bar{R} + \frac{1}{2} \sum_{i=1}^{n-2} \eta_{ij} \partial \varphi_i \partial \varphi_j$$

$$+ \frac{1}{\Delta_0} \left( 1 + \frac{\beta^2}{4} \bar{\psi}_n \bar{\chi}_n e^{-\beta (\varphi_{n-2} + R)} \bar{\partial} \bar{\psi}_1 \partial \chi_1 e^{\beta (R - \varphi_1)} \right)$$

$$+ \frac{1}{\Delta_0} \left( 1 + \frac{\beta^2}{4} \bar{\psi}_1 \bar{\chi}_1 e^{-\beta (\varphi_1 + R)} \bar{\partial} \bar{\psi}_n \partial \chi_n e^{-\beta (R + \varphi_{n-2})} - \frac{\beta^2}{4} (\bar{\psi}_n \bar{\chi}_1 \bar{\partial} \bar{\psi}_1 \partial \chi_n \right.$$ (2.16)

$$+ \bar{\psi}_1 \bar{\chi}_n \bar{\partial} \bar{\psi}_n \partial \chi_n) e^{-\beta (\varphi_1 + \varphi_{n-2})} \right) - V_{\text{interm}}$$

with potential

$$V_{\text{interm}} = \frac{m^2}{\beta^2} \left( \sum_{i=1}^{n-2} e^{-\beta \eta_{ij} \varphi_j} + e^{\beta (\varphi_1 + \varphi_{n-2})} (1 + \beta^2 \bar{\psi}_1 \bar{\chi}_1 e^{\beta (R - \varphi_1)}) (1 + \beta^2 \bar{\psi}_n \bar{\chi}_n e^{-\beta (R + \varphi_{n-2})} - n + 1 \right)$$

and the denominator $\Delta_0$ is quadratic in $\bar{\psi}_a, \bar{\chi}_a$:

$$\Delta_0 = 1 + \frac{\beta^2}{4} \left( \bar{\psi}_1 \bar{\chi}_1 e^{\beta (R - \varphi_1)} + \bar{\psi}_n \bar{\chi}_n e^{-\beta (R + \varphi_{n-2})} \right)$$

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The relations between the fields of the ungauged (2.12) and the above axial gauged IM (2.16) has the form

\[ \tilde{\psi}_a = \psi_a e^{\frac{\beta}{2} (R_i^a + R_n^a)}, \quad \tilde{\lambda}_a = \lambda_a e^{\frac{\beta}{2} (R_i^a + R_n^a)}, \quad \tilde{R} = \frac{1}{2} (R_i^a - R_n^a) \]  

as one can see by comparing the group elements \( g_0 \in G_0 = SL(2) \otimes SL(2) \otimes U(1)^{n-2} \) from eqn. (2.11) with \( g_0^f \in G_0/U(1) \) (for axial gauging), i.e.,

\[ g_0 = e^{\frac{\beta}{2} (\lambda_a \cdot H R_i^a + \lambda_1 \cdot H R_n^a)} g_0^f \text{, interm}_1 \]

where \( g_0^f \text{, interm} = e^{\frac{\beta}{2} \sum_{a=1}^{n} \chi_a E_1^{(a)} \beta (\lambda_1 - \lambda_a) \cdot H \tilde{R} + \beta \sum_{i=1}^{n-2} \psi_i \tilde{h}_{i+1} e^{\beta \sum_{a=1}^{n} \psi_a E_1^{(a)}} \}. \) Note that the \( p = 2 \)

Lagrangian (2.19) is quite similar to (2.8) of the \( p = 1 \) gauged IM. Both are invariant under one global \( U(1) \) symmetry. The denominators of both are quadratic in \( \psi_a, \chi_a, \) but (2.16) involves an extra pair \( \psi_n, \chi_n \) of charged fields and it is invariant under the following chiral \( U(1) \) transformation:

\[ \tilde{\psi}_n = e^{-i \beta_0^2 \bar{w}} \psi_n, \quad \tilde{\lambda}_n = e^{-i \beta_0^2 \bar{w}} \lambda_n, \quad \tilde{\chi}_n = e^{i \beta_0^2 \bar{w}} \chi_n, \quad \tilde{R}_n = \tilde{R} + \beta_0 (w + \bar{w}), \quad \varphi_n = \varphi_l. \]  

The description of the 1-soliton spectrum of this IM (2.16) is one of the main purposes of the present paper. As we shall show in the next Sect. 4.3, the structure of its soliton solutions is quite similar to the ones of \( p = 1 \) ungauged IM (2.3). Due to the common local \( U(1) \) symmetry, they share the same \( \beta (\lambda_1 - \lambda_a) \cdot H \tilde{R} + \beta \sum_{i=1}^{n-2} \psi_i \tilde{h}_{i+1} e^{\beta \sum_{a=1}^{n} \psi_a E_1^{(a)}} \} \}

The result of the matrix Gaussian integration (over \( a_{0a} \), \( a_{0a} \), \( a = 1, n \)) is the following effective Lagrangian of the completely axial gauged IM [23] (2.20)

\[ \mathcal{L}_{n}^{p=2} = \frac{1}{2} \sum_{i=1}^{n-2} \eta_{ij} \partial \varphi_i \partial \varphi_j + \frac{1}{2} \left( 1 + \beta^2 \right) \frac{n}{2(n-1)} \psi_n \chi_n e^{-\beta \varphi_n - 2} \partial \psi_1 \partial \chi_1 e^{-\beta \varphi_1} \]

\[ + \left( 1 + \beta^2 \right) \frac{n}{2(n-1)} \psi_1 \chi_1 e^{-\beta \varphi_1} \partial \psi_n \partial \chi_n e^{-\beta \varphi_n - 2} \]

\[ + \frac{\beta^2}{2(n-1)} (\chi_1 \psi_n \partial \psi_1 \partial \chi_n + \chi_n \psi_1 \partial \psi_1 \partial \chi_1) e^{-\beta (\varphi_1 + \varphi_n - 2)} \right) - V_n^{p=2} \]

with potential

\[ V_n^{p=2} = \frac{\mu^2}{\beta^2} \left( \sum_{i=1}^{n-2} e^{-\beta \eta_{ij} \varphi_j} + e^{\beta (\varphi_1 + \varphi_n - 2)} (1 + \beta^2 \psi_n \chi_n e^{-\beta \varphi_n - 2}) (1 + \beta^2 \psi_1 \chi_1 e^{-\beta \varphi_1}) - n + 1 \right) \]  

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where \( \varphi_0 = \varphi_{n-1} = 0 \) and
\[
\Delta = 1 + \frac{\beta^2 n}{2(n-1)}(\psi_1 \chi_1 e^{-\beta \varphi_1} + \psi_n \chi_n e^{-\beta \varphi_n}) + \frac{\beta^4(n+1)}{4(n-1)}\psi_1 \psi_n \chi_1 \chi_n e^{-\beta (\varphi_1 + \varphi_n)}.
\]
The fields \( \psi_a, \chi_a \) are related to the charged fields \( \psi_a^u, \chi_a^u \) of the ungauged IM as follows:
\[
g_0 = e^{-i\frac{2}{3} \lambda_a \cdot R_a} g_0^f, \text{axial} e^{-i\frac{2}{3} \lambda_a \cdot R_a} \quad (2.21)
\]
i.e., \( \psi_a = \psi_a^u e^{\frac{i}{2} R_a} \), \( \chi_a = \chi_a^u e^{\frac{i}{2} R_a} \). The IM \((2.20)\) is invariant under global \( U(1) \otimes U(1) \) transformations only \( (\epsilon_a = \text{const}) \),
\[
\psi_a' = \psi_a e^{i\beta \epsilon_a}, \quad \chi_a' = \chi_a^u e^{-i\beta \epsilon_a}, \quad \varphi_l' = \varphi_l \quad (2.22)
\]
Its charged \((Q_1, Q_n)\) 1-solitons \( M_g(p = 2) \) \([23]\) appears as the basic ingredient in the construction of the 1-solitons of both the ungauged \( p = 2 \) IM \((2.12)\) and the intermediate IM \((2.16)\).

### 2.2 Zero curvature representation and classical integrability

The proof of the (classical) integrability of the 2-d models \((1.2)\), \((1.8)\), \((2.12)\), \((2.16)\) and \((2.20)\) introduced in Sect. 2.1, is based on the following two basic ingredients \([25]\)

- **zero curvature representation**
  \[
  \partial \bar{A} - \bar{\partial} A - [A, \bar{A}] = 0, \quad A, \bar{A} \in \bigoplus_{i=0,\pm 1} G_i
  \quad (2.23)
  \]
of their equation of motion.

- **fundamental Poisson bracket (FPR) relation**
  \[
  \{A_x(x_1, t) \otimes A_x(x_2, t)\}_{PB} = [r_{cd}, A_x(x_1, t) \otimes I + I \otimes A_x(x_2, t)] \delta(x_1 - x_2),
  \quad (2.24)
  \]
  where \( r_{cd} \) denotes the classical \( r \)-matrix and \( A_x = \frac{1}{2}(A - \bar{A}) \).

The Leznov-Saveliev matrix form \([19]\) of the equations of motion derived from the corresponding Lagrangians \((1.8), (2.12), (2.16)\) and \((2.20)\) is
\[
\bar{\partial}(g_0^{-1} \partial g_0) + [\epsilon_- - g_0^{-1} \epsilon_+ g_0] = 0, \quad \partial(\bar{\partial} g_0 g_0^{-1}) - [\epsilon_+ g_0^{-1} \epsilon_-] = 0,
\quad (2.25)
\]
with \( g_0(\psi_a, \chi_a, \varphi_l, R_a) \in G_0 \) given by eqns. \((2.3)\), \((2.11)\) and \((2.18)\). It allows us to deduce the explicit form of the 2-d pure gauge potentials, namely,
\[
A = -g_0 \epsilon_- g_0^{-1}, \quad \bar{A} = \epsilon_+ + \bar{\partial} g_0 g_0^{-1}
\quad (2.26)
\]
They indeed satisfy eqns. \((2.24)\) if \( g_0 \) is a solution of eqns. \((2.25)\). Their explicit form in terms of the fields \( \varphi_l, \psi_a, \chi_a, R_a \), etc is given in refs. \([9], [11], [23], [24]\).
The derivation of \( r_{cl} \) and the proof of eqn. (2.24) is based on the explicit realization of \( (A_x)_{ij} \) in terms of fields and their canonical momenta. We next calculate the equal time matrix PBs:

\[
\{A_x(x_1, t) \otimes A_x(x_2, t)\}_{ij,kl} = \{(A_x)_{ik}, (A_x)_{ji}\}
\]

making use of the basic canonical PBs, say \( \{\phi_j(x,t), \Pi_{\phi_k}(y,t)\} = \delta_{jk}\delta(x-y) \). The result for the IM (1.2) has the form \[26\]

\[ r_{cl} = \beta^2 \left( C^+ - C^- \right) \]  

(2.27)

where

\[ C^+ = \sum_{m=1}^{\infty} \left( \sum_{a,b=1}^{n} (K^{-1})_{ab} h_a^{(m)} \otimes h_b^{(-m)} + \sum_{\alpha > 0} \left( E_{\alpha}^{(m)} \otimes E_{-\alpha}^{(-m)} + E_{-\alpha}^{(m)} \otimes E_{\alpha}^{(-m)} \right) \right) \]

\[ + \frac{1}{2} \sum_{\alpha > 0} E_{\alpha}^{(0)} \otimes E_{-\alpha}^{(0)} \]  

(2.28)

and \( C^- = \sigma(C^+) \), \( \sigma(A \otimes B) = B \otimes A \); \( K_{ab}^{-1} \) denotes the inverse of the Cartan matrix for \( GA_n \). It turns out that \( r_{cl} \) has an universal form (2.27), (2.28) for each given algebra, say, \( A_n^{(1)} \) and does not depend on the choice of the specific graded structure and of \( \epsilon_\pm \), i.e., the abelian and nonabelian affine \( A_n \)-Toda model provide different representations of the same FPR algebra (2.24).

The complete proof of the integrability of the considered models require two more steps. The first is the construction of the infinite set of conserved charges. Their existence and explicit form is well known consequence of the zero curvature representation (2.23) \[25\], namely

\[ Tr(T(\tau)^m) = P_m(\tau), \quad \partial_\tau P_m = 0 \quad m = 0, \pm 1, \cdots \]

\[ T(\tau) = \lim_{L \rightarrow \infty} P \exp \int_{-L}^{L} A_x(\tau, x) dx \]  

(2.29)

i.e., \( P_m(0) \) are the conserved charges we seek. The last step is to prove that these conserved charges are in involution, i.e.

\[ \{P_{m_1}, P_{m_2}\}_{PB} = 0 \]

which is ensured by the specific form of the FPR (2.24) and the explicit form of the charges \( P_m \) (2.29).

2.3 Identification of Dyonic IMs as perturbed CFTs

The Hamiltonian reduction [15] and the integrable relevant perturbations of certain conformal minimal models [4],[5],[6], are known to be two equivalent methods for constructing and

\[4\] It is not however true for the quantum R-matrices, which acquire specific form for each fixed graded structure and \( \epsilon_\pm \).
solving a large class of IMs. As we have shown in Sect. 2.1, the first method is based on
an appropriate graded structure \( (Q, G_0, \epsilon_\pm) \) of the defining affine algebra \( \hat{G} \) (say, \( A_1^{(1)} \)) and
each choice of \( G, Q \) and \( \epsilon_\pm \) determines one IMs. The second method introduced by Zamolod-
chikov [4], [6], consists in considering certain quantum CFT (i.e. conformal minimal model
of one of the extended Virasoro algebras: conformal current algebra, \( W_n, Z_n \)-parafermionic
algebra, [27], [28], etc.) and adding to its Lagrangian a linear combination \( \sum_{a=1}^l g_a V_a \bar{V}_a \)
of certain relevant vertex operators of dimension \( \Delta_a + \bar{\Delta}_a < 2 \) representing highest weight
representations of the correseponding extended conformal algebra, i.e.,

\[
\mathcal{L}_{\text{pert}} = \mathcal{L}_{\text{conf}} + \sum_{a=1}^l g_a V_a(z)\bar{V}_a(\bar{z})
\]

(2.30)

The relation between these two methods has been demonstrated on the examples of sine-
Gordon (SG) model as \( \phi_{1,2}\phi_{1,2} \) perturbation of the Virasoro minimal models [30], abelian
affine Toda as \( W_{n+1} \) minimal model perturbations [20], the Lund-Regge model as \( \phi_{0,0}^{(2)} \) per-
turbation of the \( Z_N \) parafermions [30], [24], etc. (see ref. [33] for review). The problem
we address in this Section is to recognize the IMs derived in Sect.2.1 as perturbed CFTs.
Therefore we have to answer the following two questions:

- which are the extended conformal algebras behind the conformal limits \( V = V_{\text{conf}} +
V_{\text{pert}} \rightarrow V_{\text{conf}} \) of the dyonic IMs (1.2), (1.8), (2.12), (2.16) and (2.20)?

- how to identify the nonconformal part of their potentials \( V_{\text{pert}} \), say for the IM (1.8)

\[
V_{\text{pert}}^{(g)} = e^{\beta(\varphi_1 + \varphi_{n-1})} \left( 1 + \beta^2 \psi_g \chi_g e^{-\beta \varphi_1} \right)
\]

(2.31)

with certain linear combinations of vertex operators of the underlying conformal models
(of the \( V_{n+1} \)-algebra [14], [15], [31] for the gauged IM (1.8))?

One should consider separately the \( U(1) \)-IMs (1.2) (and (1.8)) with \( n = 1 \) and \( n \geq 2 \), due
to the fact that they are based on different type of extended conformal algebras, namely:

- the \( n = 1 \) ungauged IM (1.2) is governed by the chiral \( SL(2, R) \) conformal current
algebra and its gauged version \( SL(2, R)/U(1) \) - by the parafermionic algebra [27].

- the \( n \geq 2 \) ungauged IMs (1.2) appear to be certain perturbations of the minimal
models of the \( W_{n+1}^{(n)} \)-Bershadsky-Polyakov algebra [13], [12]. The conformal limit of
the gauged IMs (1.8) with \( n \geq 2 \) is characterized by the nonlocal \( V_{n+1}^{(1)} \)-algebra of
mixed PF-\( W_n \)-type [16], [35], [34].

Similar separation takes place in the case of \( U(1) \otimes U(1) \) multicharged IMs (2.12), (2.16)
and (2.20). The simplest case \( n = 2 \) represents relevant perturbations of the \( SL(2, R) \otimes
SL(2, R) \) WZW models, while the \( n \geq 3 \) case is related to specific quadratic (non-Lie)
\( W_{n+1}^{(n,2)} \)-algebra spanned by two sets \( J_{a\pm}^s, a = 1, n \) of spin \( s = \frac{n}{2} \) currents, two spin \( s = 1 \)
currents \( J_{a, H}^\pm \) and \( n - 2 \) currents \( T_s \) of spin \( s = 2, 3, \ldots n - 1 \). Its algebraic structure (i.e.
the OPE’s of these currents) is quite similar to the one of \( W_{n+1}^{(n)} \) algebra, but including an
extra set of currents \( J_{a\pm}^\pm \), \( J_{a, H}^\pm \).
We first consider few different integrable perturbations of the $SL(2,R)$-WZW model [36, 37, 38]. For $n=1$ the Lagrangian (1.2) takes the form

$$L_{u}^{p=1}(n=1) = \partial R \bar{\partial} R + \partial \chi u \bar{\partial} u e^{2\beta R} - V_u, \quad V_u = m^2 \chi u \bar{\partial} u e^{2\beta R}, \quad R = \frac{1}{2} R_u \quad (2.32)$$

Its conformal limit $V_u \to 0$, (i.e. $m^2 \to 0$) coincides with the $SL(2,R)$-WZW model. We next remember that the vertex operators, representing the primary fields of the discrete series of highest weight representations [11], [39], [40] are given by

$$V_{m,\bar{m}}^{j}(z,\bar{z}) = \psi^{j+m} \chi^j \bar{\psi}^{j} \chi^{\bar{m}} e^{2\beta j R} \equiv \Phi_{j}^{m}(z) \Phi_{j}^{m}(\bar{z}), \quad m, \bar{m} = -j, -j+1, \ldots, j \quad (2.33)$$

of conformal dimensions $\Delta_{\Phi} = \bar{\Delta}_{\Phi} = \frac{(j+1)k}{k-2}$, i.e. $\Delta_{\Phi} = 2\Delta_{\Phi}$. Therefore the perturbation of $SL(2,R)$ WZW model by $V_{0}^{(1)}$, can be represented by $L_{u}^{p=1}(n=1)$ of eqn. (2.32). Since $V_{u} = Tr \epsilon_{+} \epsilon_{-} - V_{0,0}^{(1)}$ for $\epsilon_{\pm} = mh^{(\pm 1)}$ of grade $\pm 1$ with respect to the homogeneous grading $Q = d$ and $g_{0} \in SL(2,R)$ we conclude that the IM $(n=1)$ defined by the above grading structure $Q_{j}, \epsilon_{\pm}$ of $G_{0} = SL(2,R)$ is identical with the $V_{0,0}^{(1)}$ perturbation of the $SL(2,R)$ WZW model. Note that in the classical limit $k \to \infty$ and $\Delta_{\Phi} \to 0$ and therefore the constant $m$ has dimension 1 in mass units. If we take the most general grade $\pm 1$ elements

$$\epsilon_{\pm} = m \left( a_{1}^{(\pm)} h^{(\pm 1)} + a_{2}^{(\pm)} E_{\alpha}^{(\pm 1)} + a_{3}^{(\pm)} E_{-\alpha}^{(\pm 1)} \right) \quad (2.34)$$

instead of $\epsilon_{\pm} \sim h^{(\pm 1)}$, we get a six parameter family of integrable perturbations of the $SL(2,R)$ WZW model

$$V_{u} = \left( \frac{1}{2} a_{1}^{+} + a_{3}^{+} \psi \right) \left( \frac{1}{2} a_{1}^{-} + a_{2}^{-} \chi \right) + a_{2}^{+} a_{3}^{+} e^{-2\beta R}$$

$$+ \left( a_{3}^{+} \psi + a_{3}^{+} \bar{\psi} - a_{3}^{+} \right) \left( a_{1}^{-} \chi + a_{2}^{-} \bar{\chi} - a_{2}^{-} \right) e^{2\beta R} \quad (2.35)$$

considered in ref. [37]. For arbitrary $a_{i}^{\pm}$ both $SL(2,R)_{left}$ and $SL(2,R)_{right}$ are broken. Moreover, when $a_{i}^{+} = a_{i}^{-} = a_{i}$, we have

$$\left[ a_{1}h^{(0)} + a_{2}E_{\alpha}^{(0)} + a_{3}E_{-\alpha}^{(0)}, \epsilon_{\pm} \right] = 0$$

and therefore such perturbations has chiral $U(1)_{left} \otimes U(1)_{right}$ symmetries. Particular examples of chiral integrable perturbations by

$$V_{u}^{\pm} = \Phi_{j}^{\pm j}(z), \quad V_{u}^{(0)} = \Phi_{j}^{0}(z)$$

have been introduced and studied in ref. [38]. It is important to mention that exhausting the possible gradings $(Q = d)$ and $\epsilon_{\pm}$’s (see eqn. (2.34)) one can classify all the integrable perturbations of a given WZW model, i.e. listing the admissible graded structures $(\tilde{G}, Q, \epsilon_{\pm})$ one separates few integrable linear combinations of the vertices $V_{m,\bar{m}}^{j}$ among the large set of combinations with $j \leq \frac{k}{2}$.  

In the cases when the perturbation preserves the chiral $U(1)$ symmetry (i.e. $a_{i}^{+} = a_{i}^{-}$) one can further gauge fix this symmetry as it was explained in Sect. 2.1 (see eqn. (2.7)). The
corresponding $n = 1$ gauged IMs \((1.8)\) (or more general for $\epsilon_\pm$ given by eqn. \((2.34)\)) give rise to different integrable perturbations of the gauged $SL(2, R)/U(1)$ - WZW (i.e. noncompact parafermions \([31]\)) and $SU(2)/U(1)$ - (i.e. compact parafermions) studied in ref. \([32, 36]\).

The conformal limits of IMs \((1.2)\) with $n \geq 2$ represent certain conformal $A_n$-non-abelian Toda models introduced in ref. \([16, 35]\) (see Sect. 2 and 8 of ref. \([16]\)). They can be defined as conformal gauged $G_0 = H_- \backslash A_n / H_+$-WZW model based on the finite dimensional Lie algebra $A_n$, with $H_\pm \in A_n$ being the of positive and negative graded nilpotent subalgebras according to the following grading operator

$$Q^{\text{conf}} = \sum_{i=2}^{n} \lambda_i \cdot H^{(0)}, \quad \epsilon^{\text{conf}}_\pm = \sum_{i=2}^{n} E^{(0)}_{\pm \alpha_i}, \quad (2.36)$$

Similar to the conformal abelian Toda theory, the nontrivial conformal part $V^{\text{conf}} = Tr(\epsilon_+^{\text{conf}} g_0 \epsilon_+^{\text{conf}} g_0^{-1})$ of potential $V_n(n \geq 2)$ is originated from the specific set of constraints on the $A_n$-WZW currents:

$$J_{-\alpha_i} = J_{\alpha_i} = 1, \quad i = 2, \cdots, n, \quad J_{-[\alpha]} = J_{[\alpha]} = 0, \quad [\alpha] = \text{non simple root} \quad (2.37)$$

encoded in $\epsilon_\pm^{\text{conf}}$. The vertices $V_j = e^{-\beta \sum \eta_j \phi_j}$ of dimension \((1,1)\) represent the screening operators of the $W_{n+1}^{(n)}$-algebra. As it is well known these constraints reduce the original $A_n$-chiral conformal current algebra to specific higher spin quadratic algebra of $W_{n+1}^{(n+1)}$-type studied by Polyakov \([12]\) and Bershadsky \([13]\). For example the $n = 2$ Bershadsky-Polyakov (BP) algebra $W_3^{(2)}$ is generated by four chiral currents $T_W, G^\pm$ and $J$ of spins $s = 2, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}$. It has the following OPE form \([13]\)

$$J(z_1)J(z_2) = \frac{2k + 3}{3z_{12}^2} + O(z_{12}), \quad G^\pm(z_1)G^\pm(z_2) = O(z_{12})$$

$$J(z_1)G^\pm(z_2) = \pm \frac{1}{z_{12}} G^\pm(z_2) + O(z_{12})$$

$$T_W(z_1)T_W(z_2) = \frac{c_W}{2z_{12}^2} + \frac{2}{z_{12}^2} T_W(z_2) + \frac{1}{z_{12}} \partial T_W(z_2) + O(z_{12})$$

$$T_W(z_1)G^\pm(z_2) = \frac{3}{2z_{12}^2} G^\pm(z_2) + \frac{1}{z_{12}} \partial G^\pm(z_2) + O(z_{12})$$

$$T_W(z_1)J(z_2) = \frac{1}{z_{12}^2} J(z_2) + \frac{1}{z_{12}} \partial J(z_2) + O(z_{12})$$

$$G^+(z_1)G^-(z_2) = (k + 1) \left( \frac{2k + 3}{z_{12}^3} + \frac{3}{2} \frac{(k + 1)}{z_{12}} J(z_2) \right) + \frac{1}{z_{12}} \left( 3 : J^2 : -(k + 3) T(z_2) + \frac{3}{2} (k + 1) \partial J \right) + O(z_{12}) \quad (2.38)$$

where $c_W = \frac{8k}{k+3} - 6k - 1$ is its central charge. In fact the original constraints of the $n = 2$ BP model

$$J_{-\alpha_2} = J_{\alpha_2} = 0, \quad J_{-\alpha_1 - \alpha_2} = J_{\alpha_1 + \alpha_2} = 1$$

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are the image of the \((2.37)\) under the action of particular \(A_2\)-Weyl reflection \(w_{\alpha_1}\), i.e.,
\[
w_{\alpha_1}(\alpha) = \alpha - (\alpha \cdot \alpha_1)\alpha_1, \quad w_{\alpha_1}^2 = 1.
\]
as it has been shown in Sect 8 of ref. [16]. According to the analysis of the h.w. representations of the \(W_3^{(2)}\) algebra \((2.38)\) one can construct a specific class of degenerate vertex operators, that realize in the \(\psi_u, \chi_u, R_u, \varphi\) variables of our model \((1.2)\) acquire the form:
\[
V^j_{\lambda m, \lambda n}(z, \bar{z}) = \psi_u^{j+m} \chi_u^{j+m} e^{\beta(\beta R_u + \alpha_{r,s} \varphi)}
\]
(2.39)
where \(\alpha_{r,s}\) and \(\beta_{r,s}\) are certain charges defined in Sect. 9 of ref. [16] (see also [13]). For \(j = 1, m = \bar{m} = 0, \alpha_{r,s} = \beta_{r,s} = 1\) and for \(j = m = \bar{m} = \beta_{r,s} = 0, \alpha_{r,s} = 2\) one recognize the two vertex operators that form the nonconformal part (i.e., the integrable perturbation) \(V_{\text{pert}}\) of \(V_u\), i.e.,
\[
V_{\text{pert}} = e^{2\beta \varphi} + \beta^2 \psi_u \chi_u e^{\beta(R + \varphi)} = V_{0,0}^{0,0} + \beta^2 V_{0,0}^{1,1}
\]
Similar identifications takes place for the gauged version \((1.8)\) of the \(n = 2\) IM \((1.2)\). In this case the role of the \(W_3^{(2)}\) algebra and its vertex operators \((2.39)\) is played by the \(V_3^{(1,1)}\) algebra of mixed PF-\(W_2\)-type \([10], [36]\) (spanned by the stress tensor \(T\) and two PF currents \(V^\pm\) of spin \(s^\pm = 3/2(1 - \frac{1}{2k+3})\)) and the corresponding h.w. vertex operators. The conformal extended algebras \(V_{n+1}^{(n+1)}\) of the classical symmetries of the conformal limits \(V_{\text{pert}} \to 0\) for generic \(n \geq 2\) gauged IMs \((1.8)\) have been constructed in ref. [16]. Their quantum versions, the h.w. representations and the vertex operators are known, however only in the particular case of \(n = 2\), (i.e. \(V_3^{(1,1)}\)). Therefore, the problem of identification of \(n \geq 3\) gauged (and ungauged) dyonic IMs as perturbed CFT m.m.’s of the \(V_{n+1}^{(n+1)}\) (and \(W_{n+1}^{(n)}\)) algebras remains open.

3 Dyonic IMs with \(U(1)\) local symmetry

3.1 Gauged vs. ungauged IMs: conserved charges relations

The invariance of the ungauged IM Lagrangian \((1.2)\) under local \(U(1)\) transformations \((1.4)\) (see eqn. \((2.6)\) for their matrix form) gives rise to the following chiral \(U(1)\) conserved currents:
\[
J(z) = Tr(g_0^{-1} \partial g_0 \lambda_1 \cdot H^{(0)}) = \frac{2n}{n+1} \beta_0 \partial R_u - 2i \beta_0 \psi u \partial \chi u e^{i\beta_0(R_u - \varphi_1)},
\]
\[
\bar{J}(z) = Tr(\bar{g}_0 g_0^{-1} \lambda_1 \cdot H^{(0)}) = \frac{2n}{n+1} \beta_0 \partial \bar{R}_u - 2i \beta_0 \chi u \bar{\psi} u e^{i\beta_0(R_u - \varphi_1)}
\]
(3.1)
i.e., \(\bar{\partial} J = \partial \bar{J} = 0\) as one can verify by taking traces of eqns. \((2.23)\) with \(\lambda_1 \cdot H^{(0)}\) (remembering that \([\lambda_1 \cdot H^{(0)}, \epsilon_\pm] = 0\)). Hence the solutions of eqns. \((2.23)\) are characterised by two infinite sets of conserved charges \((m \in \mathbb{Z})\),
\[
Q_{m+1} = \oint J(z) z^{-m-1} dz, \quad \bar{Q}_{m+1} = \oint \bar{J}(\bar{z}) \bar{z}^{-m-1} d\bar{z}
\]
(3.2)
Since the conservation of the chiral \( U(1) \) currents \( J \) and \( \bar{J} \) is a consequence of the conservation of both vector \( J_\mu^{vec} = J_\mu \) and axial \( J_\mu^{axial} = \epsilon_{\mu\nu}J^\nu \) currents (i.e., \( \partial^\mu J_\mu^{vec,axial} = 0 \) and therefore \( J_\mu = \partial_\mu \Phi, \quad \Phi = \Phi(z) + \bar{\Phi}(\bar{z}) \)) we realize that

\[
J = J_0 + J_1 = \partial\Phi(z), \quad \bar{J} = J_0 - J_1 = \partial\bar{\Phi}(\bar{z}) \tag{3.3}
\]

It implies the following relations between axial and vector \( U(1) \)-charges \( Q_{vec}, Q_{axial} \) with the \( J, \bar{J} \)-zero modes \( Q_0, \bar{Q}_0 \):

\[
Q_{vec} = \int_{-\infty}^{\infty} J_0 dx = \frac{1}{2} \int_{-\infty}^{\infty} (J + \bar{J}) dx = \frac{1}{2}(Q_0 + \bar{Q}_0), \\
Q_{ax} = \int_{-\infty}^{\infty} J_1 dx = \frac{1}{2} \int_{-\infty}^{\infty} (J - \bar{J}) dx = \frac{1}{2}(Q_0 - \bar{Q}_0) \tag{3.4}
\]

Taking into account the “chirality” of the free fields \( \Phi(z), \bar{\Phi}(\bar{z}) \), i.e., \( \partial\Phi(z) = \partial\bar{\Phi}(\bar{z}) = 0 \), we have \( \partial\Phi = 2\partial_x \Phi \) and \( \bar{\partial}\Phi = -2\partial_{\bar{z}} \bar{\Phi} \). Therefore all charges \( Q_{vec}, Q_{axial}, Q_0, \bar{Q}_0 \) are defined in terms of the asymptotics of \( \Phi \) and \( \bar{\Phi} \) at \( x \to \pm\infty \) (and fixed \( t \), say \( t = 0 \)):

\[
Q_0 = 2 \int_{-\infty}^{\infty} \partial_x \Phi dx = 2 (\Phi(\infty) - \Phi(-\infty)), \\
\bar{Q}_0 = -2 \int_{-\infty}^{\infty} \partial_x \bar{\Phi} dx = -2 (\bar{\Phi}(\infty) - \bar{\Phi}(-\infty)) \tag{3.5}
\]

We next observe that the second terms of \( J \) and \( \bar{J} \) given by eqn. \( \text{(3.1)} \) are in fact components of the global \( U(1) \)-current \( I_\mu^u = (I, \bar{I}) \), i.e.,

\[
I^u = 2i\beta_0^2 \bar{\psi}_u \partial X_u e^{i\beta_0 (R_u - \varphi_1)}, \\
\bar{I}^u = -2i\beta_0^2 \bar{\psi}_u \partial \bar{X}_u e^{i\beta_0 (R_u - \varphi_1)}, \quad \bar{\partial}I^u + \partial\bar{I}^u = 0, \tag{3.6}
\]

generated by the following global \( U(1) \)-transformations (\( \epsilon = \text{const} \)):

\[
\psi'_u = \psi_u e^{i\beta_0 \epsilon}, \quad \chi'_u = \chi_u e^{-i\beta_0 \epsilon}, \quad R'_u = R_u, \quad \varphi'_1 = \varphi_1 \tag{3.7}
\]

As one might expect, its charge \( Q_{al,clus}^u = \frac{1}{2} \int_{-\infty}^{\infty} (I + \bar{I}) dx \) is not independent of \( Q_0 \) and \( \bar{Q}_0 \):

\[
Q_{ul}^u = \frac{2n}{n + 1} Q_R^u - \frac{1}{2}(Q_0 - \bar{Q}_0), \quad Q_R^u = \beta_0 \int_{-\infty}^{\infty} \partial_x R_u dx \tag{3.8}
\]

due to the relation

\[
\frac{1}{2} (J - \bar{J}) = \frac{2n\beta_0}{n + 1} \partial_x R_u - \frac{1}{2}(I + \bar{I}) \tag{3.9}
\]

between the currents \( J, \bar{J}, I, \bar{I} \) and the topological current \( J_\mu^{Ru} = \beta_0 \epsilon_{\mu\nu} \partial_x R_u \), encoded in eqn. \( \text{(3.1)} \): It reflects the fact that the transformation \( \text{(3.7)} \) is a particular case \( w = \bar{w} = -\epsilon \) of eqn. \( \text{(1.4)} \).

As we have shown in Sect. 2.1, to each ungauged \( G_0 \)-IM with local \( G_0^0 \subset G_0 \) symmetry, one can make in correspondence a new \( G_0/G_0^0 \)-gauged IM, by gauge fixing \( G_0^0 \). The procedure
of elimination of the extra $G_0^0$-field degrees of freedom ($R_u$ for $G_0^0 = U(1)$) consists in imposing
constraints on the $G_0^0$-chiral currents ($J(z)$ and $\bar{J}(z)$):

\[ J_g = Tr((g_0^f)^{-1} \partial g_0^f \lambda \cdot H) \]
\[ = \frac{2n}{n+1} \beta_0 \partial R_g (1 + \beta_0 \frac{n+1}{2n} \psi_g \chi_g e^{-i\beta_0 \varphi_1}) - 2i \beta_0^2 \psi_g \partial \chi_g e^{-i\beta_0 \varphi_1} = 0 \]
\[ \bar{J}_g = Tr(\bar{\partial} g_0^f (g_0^f)^{-1} \lambda \cdot H) \]
\[ = \frac{2n}{n+1} \beta_0 \bar{\partial} R_g (1 + \beta_0 \frac{n+1}{2n} \psi_g \chi_g e^{-i\beta_0 \varphi_1}) - 2i \beta_0^2 \chi_g \bar{\partial} \psi_g e^{-i\beta_0 \varphi_1} = 0 \] (3.10)

The relation between the field variables $g_0^u(\psi_u, \chi_u, R_u, \varphi_l) \in G_0$ of the ungauged IM (1.2)
\[ g_0^u = e^{\chi_u E_{-\alpha_1}(0)} e^{\beta_0(1) \cdot H(0) R_u + \sum_{i=1}^{n-1} \psi_i E_{-1}(0)} e^{\psi_i E_{-1}(0)} \]
and of the gauged IM (1.8) $g_0^f(\psi_g, \chi_g, \varphi_l) \in G_0/G_0^0$
\[ g_0^f = e^{\chi_g E_{-\alpha_1}(0)} e^{\beta \sum_{i=1}^{n-1} \varphi_i E_{-1}(0)} e^{\psi_i E_{-1}(0)} \]
is given by
\[ g_0^u = e^{\beta \bar{\psi}(z) \lambda \cdot H(0) \alpha_0 g_0^f} e^{\beta \psi w(z) \lambda \cdot H(0)}, \quad \alpha_0 = e^{\frac{1}{2} \beta_0(1) \cdot H(0) R_g} \]
(3.11)

In components we find the following relations:
\[ R_u = R_g + \beta_0 (w + \bar{w}), \quad \varphi_l^u = \varphi_l^0 = \varphi_l, \]
\[ \psi_u = \psi_g e^{\frac{1}{2} \beta_0 R_g - i\beta_0 \bar{w}}, \quad \chi_u = \chi_g e^{\frac{1}{2} \beta_0 R_g - i\beta_0 \bar{w}}, \quad \theta_g = \frac{1}{2i \beta_0} \ln(\chi_g/\psi_g) \]
\[ \theta_u = \theta_g \frac{1}{2} \beta_0 (w - \bar{w}), \quad \theta_u = \frac{1}{2i \beta_0} \ln(\chi_u/\psi_u) \] (3.12)

They reflect (a) the axial gauge fixing of chiral $U(1)_L \otimes U(1)_R$ symmetry and (b) the specific
choice of the representative $g_0^f$ of the coset $G_0/G_0^0$ (i.e. of the nonlocal field $R_g$) such that
the $L_{p=1}^g$ (1.8) is local in $\psi_g, \chi_g$, i.e., independent of $R_g$. It is instructive to verify that by
gauge transforming $R_g, \psi_g$ and $\chi_g$ according to (3.12) with
\[ w = \frac{1}{\beta_0^2} \left( \frac{1}{2n} \right) \Phi, \quad \bar{w} = \frac{1}{\beta_0^2} \left( \frac{1}{2n} \right) \bar{\Phi} \]
one recover eqn. (3.11) starting from the constraints (3.10) and vice versa.

The form of the global $U(1)$ current $I^g_\mu = (I^g, I^g)$ of the gauged IM (1.8)
\[ I^g = 2i \beta_0^2 \frac{\psi_g \partial \chi_g e^{-i\beta_0 \varphi_1}}{1 + \beta_0 \frac{n+1}{2n} \psi_g \chi_g e^{-i\beta_0 \varphi_1}}, \quad \bar{I}^g = -2i \beta_0^2 \frac{\chi_g \bar{\partial} \psi_g e^{-i\beta_0 \varphi_1}}{1 + \beta_0 \frac{n+1}{2n} \psi_g \chi_g e^{-i\beta_0 \varphi_1}} \] (3.13)
i.e., $\partial I^g + \bar{\partial} \bar{I}^g = 0$, suggests that the constraints (3.10) (i.e. the gauge fixing conditions) can
be considered as a requirement of the proportionality of the electric current $I^g_\mu$ (3.13) and
the $R_g$-topological current $J^R_g = \beta_0 \epsilon_{\mu \nu} \partial^\nu R_g$, i.e.,
\[ I^g_\mu = \frac{2n}{n+1} J^R_g \] (3.14)
Note that in the ungauged IM (1.2) this relation is replaced by the more general one (3.9)

\[ I^u_\mu = \epsilon_{\mu\nu} \left( \frac{2n}{n+1} \partial^\nu R_u - J^\nu \right) \] (3.15)

and therefore \( I^u_\mu \) and \( J^R_u \) are independent due to the contribution of \( J^\nu \). The relation (3.12) established between the fields of the gauged and ungauged IMs allows us to relate the corresponding \( U(1) \) and topological currents and charges of both models:

\[ Q^u_R = Q^g_R + \frac{n+1}{4n} (Q_0 - \bar{Q}_0) = Q^g_R + \frac{n+1}{2n} Q_{ax} \]

\[ Q^u_\theta = Q^g_\theta + \frac{n+1}{8n} (Q_0 + \bar{Q}_0) = Q^g_\theta + \frac{n+1}{4n} Q_{vec} \] (3.16)

where we have introduced the topological charge \( Q_\theta \) related to the field \( \theta_g = \frac{1}{2i\beta_0} \ln(\chi_g/\psi_g) \).

Replacing \( Q^u_R \) in eqn. (3.8) we find that

\[ Q^{el}_u = \frac{2n}{n+1} Q^g_R = Q^{el}_g \] (3.17)

i.e., the global \( U(1) \) charges (of the currents \( I^u_\mu \) and \( I^g_\mu \)) do coincide. This leads to the conclusion that one can determine the charge spectrum of the ungauged IM in terms of the charges \( Q^g_R, Q^g_\theta, Q^g_{ax} \) (and \( Q_{top} = \beta_0 \int_{-\infty}^{\infty} \partial_x \varphi_0 dx \)) of the gauged IM and the \( J, \bar{J} \) charges \( (Q_{vec}, Q_{ax} \text{ or } Q_0, \bar{Q}_0) \), i.e., the asymptotics of \( \Phi, \bar{\Phi} \).

### 3.2 Nonconformal GKO coset construction

The relation between the fields \( (g^u_0, g^g_0 \text{ and } \Phi, \bar{\Phi}) \), currents and charges (3.15), (3.16) and (3.17) of the ungauged \( G_0 \)-IM, the gauged \( G_0/G^0_0 \)-IM and the chiral \( U(1) \)-CFT address the question whether the stress-tensors (and the energies) of these theories are related in similar manner. Having in mind that for the conformal limits \( 5 \) of the considered IMs (1.2) and (1.8) according to Goddard-Kent-Olive (GKO) coset construction \( 14 \), we have

\[ T^{CFT}_{G_0} = T^{CFT}_{G_0/G_0} + T^{CFT}_{G^0_0} \] (3.18)

one might expect that an appropriate nonconformal extension of the GKO formula (3.18) to take place.

We start the derivation of the integrable models analog of eqn. (3.18) by calculating the ungauged IM stress-tensor

\[ T^u = \frac{1}{2} \eta_{ij} \partial \varphi_i \partial \varphi_j + \frac{n}{2(n+1)} (\partial R_u)^2 + \partial \chi_u \partial \psi_u e^{i\beta_0 (R_u - \varphi_1)} + V^u \] (3.19)

\[ T^u = T^u(\partial \rightarrow \bar{\partial}), \quad T^u_{00} = \frac{1}{2}(T^u + \bar{T}^u), \quad T^u_{01} = \frac{1}{2}(T^u - \bar{T}^u) \]

We next substitute \( R_u, \psi_u \) and \( \chi_u \) in \( T^u \), taking into account eqns. (3.12) and the constraints (3.10) as well. By straightforward simplifications we realize that \( T^u \) can be written in terms

\[ V^u_{\text{pert}} = \frac{m^2}{\beta_0} \left( e^{\beta(\varphi_1 + \varphi_{n-1})} (1 + \beta^2 \psi_u \chi_u e^{\beta(R_u - \varphi_1)} - n) \right) \text{ and } V^g_{\text{pert}} \]
of the gauged IM fields $\psi_g, \chi_g$ and $\varphi_l$ together with the $U(1)$-CFT currents $J = \beta_0^2 \frac{2n}{n+1} \partial w$ and $\bar{J} = \beta_0^2 \frac{2n}{n+1} \partial \bar{w}$ only, i.e.,

$$T^u = \frac{1}{2} n_j \partial \varphi_i \partial \varphi_j + \frac{\partial \chi_g \partial \psi_g}{\Delta} e^{-\beta \varphi_1} + V_g + \frac{\beta_0^2 n}{2(n+1)} (\partial w)^2$$

(3.20)

where $\Delta = 1 + \beta_0^2 \frac{2n+1}{2n} \psi_g \chi_g e^{-\beta \varphi_1}$. Finally, we remind that the stress-tensor of the gauged IM, derived from its Lagrangian (1.8) has the form

$$T^g = \frac{1}{2} n_j \partial \varphi_i \partial \varphi_j + \frac{\partial \chi_g \partial \psi_g}{\Delta} e^{-\beta \varphi_1} + V_g$$

Therefore the nonconformal version of the GKO formula (3.18) is given by

$$T^u_{G_0} = T^g_{G_0/G_0^a} + T^{CFT}_{G_0}, \quad \bar{T}^u_{G_0} = \bar{T}^g_{G_0/G_0^a} + \bar{T}^{CFT}_{G_0}$$

(3.21)

or equivalently

$$T^u_{00} = T^g_{00} + T^{CFT}_{00}, \quad \bar{T}^u_{00} = \bar{T}^g_{00} + \bar{T}^{CFT}_{00}$$

(3.22)

where $T^{CFT}, \bar{T}^{CFT}$ are the $U(1)$-CFT stress-tensors:

$$T^{CFT} = \frac{\beta_0^2 n}{2(n+1)} (\partial w)^2, \quad \bar{T}^{CFT} = \frac{\beta_0^2 n}{2(n+1)} (\partial \bar{w})^2$$

(3.23)

Note that the $T^{CFT}, \bar{T}^{CFT}$ are chiral, i.e., $\partial T^{CFT} = \partial \bar{T}^{CFT} = 0$ but $T^u, T^g, \bar{T}^u, \bar{T}^g$ are not, since the corresponding gauged and ungauged IMs are not conformal invariant.

The formal splitting of the $G_0$-ungauged IM fields, currents and stress-tensor in $U(1)$-CFT and gauged $G_0/G_0^a$-IM parts cannot be considered as an indication of a direct sum, since certain properties of the $U(1)$-CFT (b.c.’s of $w, \bar{w}$) depend on the gauged IM b.c.’s in the way indicated by the interaction terms of the ungauged IM, as we shall show in Subsect. 3.3.

### 3.3 Vacua, boundary conditions and discrete symmetries

The potential $V_u$ of the ungauged IM (1.8) for imaginary coupling $\beta = i \beta_0$ manifest $n$-distinct zeroes:

$$\varphi_l^{(N)} = \frac{2 \pi l N}{\beta_0} n, \quad \psi_u \chi_u = 0, \quad R_u = \beta_0 a_R, \quad \theta_u = \beta_0 a_\theta$$

(3.24)

where $N = 0, \pm 1, \cdots \pm (n - 1) \mod n$, $l = 1, 2, \cdots n - 1$ and $a_R, a_\theta$ are real parameters. They represent the constant vacua solutions (i.e. $E_{\text{vac}} = 0$, all charges $Q_{\text{all}}^{\varphi, \psi} = 0$) of the eqn. of motion (2.25). The vacua values and the boundary conditions of the massless fields $R_u$ and $\theta_u$ remain undefined by the $V_u = 0$ condition [4]. As usually the global symmetries of the model ($Z_2 \otimes Z_n$ and $U(1)_{\text{vector}} \otimes U(1)_{\text{axial}}$ in our case (1.2)) determine the complete vacua

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6 due to the fact that $\partial \bar{V}_u / \partial R_u = \partial \bar{V}_u / \partial \theta_u = 0$, (i.e. $R_u$ and $\theta_u$ are flat directions) as one can see by suitable change of variables $\psi_u = \psi_u e^{-\frac{2}{\beta} R_u}, \chi_u = \chi_u e^{-\frac{2}{\beta} R_u}$ and $V_u \to \bar{V}_u$
structure and allowed b.c.’s of all the fields. The $Z_n$-group that leave $\mathcal{L}_{p=1}^{u}$ invariant acts as follows:

$$
\varphi'_l = \varphi_l + \frac{2\pi l N}{\beta_0 n}, \quad R'_u = R_u + \frac{2\pi}{\beta_0}(s_R + \frac{N - q - \bar{q}}{n})
$$

$$
\psi'_u = \psi_u e^{\pi i (\frac{\bar{q}}{n} + s_1)}, \quad \chi'_u = \chi_u e^{\pi i (\frac{\bar{q}}{n} + s_2)}, \quad \theta'_u = \theta_u + \frac{\pi}{\beta_0}(\bar{q} - q + s_0),
$$

where $q, \bar{q} = 0, \pm 1, \cdots \pm (n - 1)$ and $s_1, s_2, 2s_R = s_1 + s_2, s_R, s_0 = s_2 - s_1$ are integers. The $Z_n$-charges of fields $\psi_u, \chi_u$ and $e^{\beta \varphi_l}$ are given by $q, \bar{q}$ and $N$ mod $n$ respectively. One can further combine the above $Z_n$ with the CP-transformation ($P x = -x, \quad P \partial = \partial, \quad P^2 = 1$):}

$$
\varphi''_l = \varphi'_l, \quad R''_u = R_u, \quad \psi''_u = \chi_u, \quad \chi''_u = \psi_u
$$

into the larger dihedral group $D_n$, requiring that $\psi_u$ and $\chi_u$ are conjugated, i.e.,

$$
\bar{q} = n - q
$$

and $w(t - x) \leftrightarrow \bar{w}(t + x)$. Indeed by considering only chiral (say, left) $U(1)$ transformations ($w \neq 0, \bar{w} = 0$) one breaks the CP-invariance ($\bar{q} = 0$, but $q \neq n$) and eqn. (3.26) does not take place in this case. Completing the discussion about the relation between global symmetries and the vacua solutions we have also to mention the following global symmetries of $\mathcal{L}_{p=1}^{u}$:\

- **vector $U(1)$**

  $$
  \psi'_u = e^{-i \beta \theta \bar{q}} \psi_u, \quad \chi'_u = e^{i \beta \theta \bar{q}} \chi_u,
  \quad R'_u = R_u, \quad \varphi'_l = \varphi_l, \quad \theta'_u = \theta_u + \beta \theta \bar{q}
  $$

- **axial $U(1)$**

  $$
  \psi''_u = e^{-i \frac{\beta}{2} \theta \bar{q} \beta} \psi_u, \quad \chi''_u = e^{-i \frac{\beta}{2} \theta \bar{q} \beta} \chi_u,
  \quad R''_u = R_u + \beta \theta \beta, \quad \varphi''_l = \varphi_l, \quad \theta''_u = \theta_u
  $$

Therefore the allowed b.c.’s for the fields at $x \rightarrow \pm \infty$ are given by

$$
\varphi_l^{(N)}(\pm \infty) = \frac{2\pi l N}{\beta_0 n}, \quad R_u(\pm \infty) = \beta_0 a^\pm_R + \frac{2\pi}{\beta_0}(s_R^\pm + \frac{N \pm q \pm \bar{q}^\pm}{n}),
$$

$$
\psi_u \chi_u(\pm \infty) = 0, \quad \theta_u(\pm \infty) = \beta_0 a^\pm_\theta + \frac{\pi}{\beta_0}(s_\theta^\pm + \frac{\bar{q}^\pm - q^\pm}{n})
$$

Together with the vacua sector (defined by $D_n \otimes U(1)_{\text{vector}} \otimes U(1)_{\text{axial}}$), the ungauged IM (1.2) (in contrast to the gauged (1.8)) admits a new conformal sector. Due to chiral $U(1)$ symmetry (1.4) its equations of motion (2.25) have conformal (1-D string-like) solutions

$$
\varphi_l^{(N)} = \frac{2\pi l N}{\beta_0 n}, \quad \psi_u \chi_u = 0, \quad \theta_u^{CFT} = \frac{\beta_0}{2}(\bar{w} - w), \quad R_u^{CFT} = \beta_0 (w + \bar{w})
$$
with nonvanishing energy \( E^{CFT} = \pm P^{CFT} \neq 0 \) and charges \( (Q_0, \bar{Q}_0 \neq 0) \). The complete description of the \( U(1) \) CFT representing the conformal sector of the IM (1.2), requires the knowledge of the b.c.’s for the free fields \( \Phi = \frac{2n\beta^2}{n+1} w \) and \( \bar{\Phi} = \frac{2n\beta^2}{n+1} \bar{w} \). The question to be answered is whether one can uniquely determine the \( \Phi, \bar{\Phi} \) b.c.’s from the relation (3.12) in terms of the b.c.’s of the fields of the ungauged IM (\( \psi_u, \chi_u, R_u, \varphi_l \) given by eqn. (3.31)) and of the gauged ones, \( \psi_g, \chi_g, \varphi_l \) (see ref. [9]). We first consider the \( Z_n \)-transformations of \( w \) and \( \bar{w} \) (and \( \Phi, \bar{\Phi} \)). According to eqn. (3.12) we have,

\[
e^{i\beta^2 w} = \frac{\psi_u}{\psi_l} e^{-i\frac{1}{2} \beta_0 R_g}, \quad e^{i\beta^2 \bar{w}} = \frac{x_g}{x_u} e^{-i\frac{1}{2} \beta_0 R_g}
\]

and therefore the \( w, \bar{w} \) properties are a consequence of the \( \psi_u, \chi_u \)-transformation (3.25) and of the \( \psi_g, \chi_g, R_g \) ones [9]:

\[
\psi'_g = \psi_g e^{i\pi (\frac{s}{n} + \tilde{s}_1)}, \quad \chi'_g = \chi_g e^{i\pi (\frac{s}{n} + \tilde{s}_2)}, \quad R'_g = R_g, \quad \tilde{s}_1 + \tilde{s}_2 = 2\tilde{s}
\]

where \( \tilde{s}_1, \tilde{s}_2 \) and \( \tilde{s} \) are integers (\( R_g = \frac{n}{n+1} R \) in the notation of ref. [9]). The result

\[
w' = w + \frac{\pi}{\beta^2} (s + \frac{N - 2q}{n}), \quad \bar{w}' = \bar{w} + \frac{\pi}{\beta^2} (\bar{s} + \frac{N - 2\bar{q}}{n}), \quad s = \tilde{s}_1 - s_1, \quad \bar{s} = \tilde{s}_2 - s_2
\]

is consistent with the \( R_u \) and \( \theta_u \) transformations (3.25) under the identification

\[
s = s_R + s_\theta, \quad \bar{s} = s_R - s_\theta
\]

Note that the particular case of left movers, i.e., \( \bar{w} = 0 \) and \( w \neq 0 \) takes place when the \( Z_n \)-charge \( \bar{q} \) of \( \chi_u \) is half of the \( \varphi_1 \)-charge: \( \bar{q} = \frac{N}{2} \) and \( \tilde{s}_2 = s_2 \).

The transformation properties (3.33) of \( w, \bar{w} \) allow us to single out an important class of b.c.’s (at \( x \rightarrow \pm \infty \)) for the \( U(1) \)-CFT, namely,

\[
\Phi(\pm \infty) = \frac{2\pi}{n+1} (ns_\pm + N_\pm - 2q_\pm), \quad \bar{\Phi}(\pm \infty) = \frac{2\pi}{n+1} (n\bar{s}_\pm + N_\pm - 2\bar{q}_\pm)
\]

They give rise to topological CFT solitons of vortex type constructed in the next Sect. 3.4. Observe that for these CFT solutions with b.c.’s (3.33) (i.e. interpolating between two coformal vacua \( s_+, q_+, N_+ \rightarrow s_-, q_-, N_- \)), the charges \( Q_0, \bar{Q}_0 \) (and \( Q_{ax}, Q_{vee} \) (3.5)) of the \( U(1) \) CFT take the form:

\[
Q_0 = \frac{4\pi n}{n+1} (s + \frac{j_w}{n}), \quad \bar{Q}_0 = -\frac{4\pi n}{n+1} (\bar{s} + \frac{j_\bar{w}}{n}), \quad j_w = j_\varphi - 2j_q, \quad j_\varphi = N_+ - N_-, \quad j_q = q_+ - q_-,
\]

\[
s, \bar{s} \in Z, \quad j_\varphi, j_q, j_\bar{q} = 0, \pm 1, \cdots \pm (n-1) \text{ mod } n
\]

The integers \( s, \bar{s} \) denote the winding numbers of \( w \) and \( \bar{w} \), i.e. the number of times \( w \) winds \( S^1 \) of radius \( r_0 = \frac{1}{2\beta_0} \) when \( x \) is running from \( -\infty \) to \( \infty \). The integers \( j_w \) and \( j_\bar{w} \) are the \( Z_n \)-charges of the vertices \( V_w^{s_j_\varphi} = e^{2i\beta_0^2 w} \) (or \( V_{\bar{w}}^{s_j_\varphi} = e^{2i\beta_0^2 \bar{w}} \)). It is worthwhile to mention the
relation of $Q_0$ and $\bar{Q}_0$ with the topological charges of the conformal fields $R^{CFT} = \beta_0 (w + \bar{w})$ and $\theta^{CFT} = \frac{\beta_0}{2} (w - \bar{w})$, namely:

$$Q^{CFT}_R = \beta_0 \int_{-\infty}^{\infty} \partial_x R^{CFT} dx = \frac{n + 1}{4n} (Q_0 - \bar{Q}_0), \quad Q^{CFT}_\theta = \beta_0 \int_{-\infty}^{\infty} \partial_x \theta^{CFT} dx = \frac{n + 1}{8n} (Q_0 + \bar{Q}_0)$$

We next remind the vacua structure of the gauged IM (1.8) [9]:

$$\varphi_l^{(N)}(\pm \infty) = \frac{2\pi l N_{\pm}}{\beta_0}, \quad \psi_g(\pm \infty) = 0, \quad \theta_g^{(L)}(\pm \infty) = \frac{\pi L_{\pm}}{2\beta_0}.$$  

The corresponding $U(1)$ charged topological g-solitons (≡ gauged solitons) are characterized by their topological $(j_{\varphi}, j_{\theta})$ and electric $j_{el}$ charges,

$$Q^{el} = \frac{2n}{n + 1} \beta_0 \int_{-\infty}^{\infty} \partial_x R_{g} dx = \beta_0^2 j_{el}, \quad Q^\theta = \beta_0 \int_{-\infty}^{\infty} \partial_x \theta_{g} dx = \frac{\pi j_\theta}{2}, \quad j_\theta = L_+ - L_-,$$

$$Q^{top}_{\varphi_l} = \frac{2n}{\beta_0} \int_{-\infty}^{\infty} \partial_x \varphi_{l} dx = \frac{4\pi l}{\beta_0^2} j_{\varphi}, \quad j_{\varphi} = N_+ - N_- \mod n$$  \hspace{1cm} (3.37)

where $j_\theta = 0$ for 1-solitons and $j_\theta \neq 0$ for charged breathers [9], [10]. The CFT dressing of such g-solitons according to eqn. (3.12) (with $w$ and $\bar{w}$ having specific b.c.’s (3.35)) maps each g-soliton to topological soliton (or string) of the ungauged IM (1.2) (called u-soliton), carrying the charges of both CFT and g-solitons:

$$(j_{el}, j_\varphi, j_\theta | s, \bar{s}, j_{w}, \bar{j}_{\bar{w}})$$

The one u-solitons are finite energy topological solutions of the IM (1.2) that interpolate between two different ungauged vacua

$$(j_{\varphi} | s, \bar{s}, j_{w}, \bar{j}_{\bar{w}})$$  \hspace{1cm} (3.38)

($j_\theta = 0$ for 1-solitons). It becomes clear that the $U(1)$-CFT provides each g-vacua $(j_{\varphi}, j_\theta)$ with an infinite tower of conformal “states” (3.38) called u-vacua. The origin of its structure is in the allowed b.c.’s (3.30) for the IM (1.2).

### 3.4 $U(1)$ CFT solitons

We are interested in a specific class of topological solutions of $U(1)$ CFT with finite energy (real and positive) and having eqns. (3.35) as b.c.’s for $\Phi, \bar{\Phi}$ (and $w, \bar{w}$). The angular nature of $w_0 = 2\beta_0^2 w$ (and $\bar{w}_0 = 2\beta_0^2 \bar{w}$), i.e., of $R^{CFT}_0 = 2\beta_0 R^{CFT}$, is an indication that $w_0, \bar{w}_0$ we seek represent the map of 2-D Minkowski space $M_2$ to the torus $T_2$:

$$M_2 \rightarrow S^1_r \otimes S^1_r, \quad r_0 = \frac{1}{2\beta_0^2}$$
with certain discontinuities (branch cuts) allowed. Hence they should satisfy 2-D Poisson equation

$$\partial_\rho \partial_{\bar{\rho}} w_0 = \sum \alpha_i \delta^{(2)}(\rho - \rho_i), \quad \sum \alpha_i = 0 \quad (3.39)$$

where $\alpha_i$ are static charges localized at $\rho_i$ and $\rho = e^{a_0 z}$, $\bar{\rho} = e^{a_0 \bar{z}}$ denote the new coordinates. The problem is quite similar to the vortex solutions of 2-D Euclidean $U(1)$ CFT with “magnetic operators” $V_{w_0}(z_i) = e^{2i\beta_0^2 w(z_i)}$ creating discontinuity $2\pi s$ at $\rho = \rho_i$. It turns out that the simplest solution with all the required properties is given by the (twisted) Cayley transform:

$$w_{\text{top}} = -i \delta \ln \left( \frac{e^{a_0 z} + i}{e^{a_0 z} - i} \right), \quad \bar{w}_{\text{top}} = -i \bar{\delta} \ln \left( \frac{e^{a_0 \bar{z}} + i}{e^{a_0 \bar{z}} - i} \right) \quad (3.40)$$

with

$$\delta = \frac{1}{\beta_0^2} (s + \frac{j_w}{n}), \quad \bar{\delta} = \frac{1}{\beta_0^2} (\bar{s} + \frac{\bar{j}_w}{n}) \quad (3.41)$$

and $a_0$ is an arbitrary infrared (IR) scale. In the pure winding sector ($j_w = 0$, i.e., $j_q = \frac{j_w}{2}$) we have

$$w_{\text{top}}(\infty) = 0, \quad w_{\text{top}}(-\infty) = 2\pi s$$

which confirms the fact that $w_{\text{top}}$ maps the infinite interval $(-\infty, \infty)$ to a circle. The energy of the left-moving solitons ($\bar{w}_0 = 0$) takes the form (see eqn. (3.23)):

$$E_{\text{CFT}}^{L-sol} = \int_{-\infty}^{\infty} T_{\text{CFT}}(z) dx = \frac{2n\beta_0^2}{n+1} \int_{-\infty}^{\infty} (\partial_x w_{\text{top}})^2 dx = \frac{4n}{(n+1)\beta_0^2} (s + \frac{j_w}{n})^2 |a_0| \quad (3.42)$$

The dimensionless quantity $E_{\text{CFT}}^{L-sol}/|a_0| = \Delta$ coincides with the conformal dimension of the vortex operator $V_{w_0}^{s,j_w}$ with topological charge

$$Q_0 = \frac{4\pi n}{n+1} (s + \frac{j_w}{n}) \quad (3.43)$$

i.e. $\Delta = \frac{(n+1)}{(2\pi\beta_0)^n} Q_0^2$. The general solution of eqn. (3.39) (with b.c.’s (3.35)) includes also an arbitrary holomorphic (i.e. string oscillators) part $w_{\text{str}}$, i.e.,

$$w = w_{\text{top}} + w_{\text{str}}, \quad w_{\text{str}}(\pm \infty) = 0 \quad (3.44)$$

It has the same charge (3.43) as $w_{\text{top}}$ but its energy acquires “string” contributions from $w_{\text{str}}$:

$$E_{L-\text{string}} = \frac{2n}{n+1} \left( \frac{2}{\beta_0^2} (s + \frac{j_w}{n})^2 |a_0| + \beta_0^2 \sum_{l \neq 0} Q_l Q_{-l} \right) \quad (3.45)$$

where $\partial w_{\text{str}} = \sum_{l \neq 0} Q_l z^l$. 

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We should mention that \( w_{\text{top}} \) given by eqn. (3.40) is not the most general real non-homogeneous solution of eqn. (3.39). One can construct one parameter family \( w_{\text{top}}^\alpha \) of such solutions with the same b.c.’s (3.35) but with arbitrary position of the branch cut:

\[
w_{\text{top}}^\alpha = -i\delta_\alpha \ln \left( \frac{e^{\alpha a z} + e^{i\alpha}}{e^{a z} + e^{-i\alpha}} \right), \quad \delta_\alpha = \frac{(s + j_w n)}{\beta_0^2} \left( \frac{\pi}{2\alpha} \right)
\]

which for \( \alpha = \frac{\pi}{2} \) coincides with (3.40). By construction they carry the same topological charge (3.43), but their energy is \( \alpha \)-dependent:

\[
E_{CFT}^{L-\text{sol}}(\alpha) = \frac{4n}{(n+1)\beta_0^2} (s + j_w)^2 |a_0| (\frac{\pi}{2\alpha})^2 (1 - \alpha \frac{\cos(\alpha)}{\sin(\alpha)})
\]

and positive for, say, \( \frac{\pi}{4} < \alpha < \pi \).

The left (and right) solitons are massless by construction, since \( E_L = P_L \) (or \( E_R = -P_R \) for the right ones). One could have however a nontrivial scattering of left and right solitons of rapidities \( b_L - b_R \approx 0 \),

\[
E_L = P_L = \frac{1}{2} M_0 e^{b_L}, \quad E_R = -P_R = \frac{1}{2} M_0 e^{-b_R}
\]

leading to massive poles at intermediate (crossover) energy scale \( \sim M_0^2 \) that spoils the infrared scale (and conformal) invariance [42]. Kinematically such possibility indeed exists,

\[
M_0^2 = 4E_{CFT}^L E_{CFT}^R, \quad b_L - b_R \approx 0
\]

and it has been studied in the context of the marginal perturbation of \( SU(2) \) WZW model in ref. [12]. We assume that such phenomenon (existence of preferable scale, breaking conformal symmetry at intermediate energies) takes place in the IM under consideration, i.e. together with massless solitons we also have the massive left-right solitons (for \( \alpha = \frac{\pi}{2} \))

\[
M_0 = \frac{8n}{(n+1)\beta_0^2} (s + j_w) (\bar{s} + j_{\bar{w}}) |a_0|,
Q_0 = \frac{4\pi n}{n+1} (s + j_w), \quad \bar{Q}_0 = -\frac{4\pi n}{n+1} (\bar{s} + j_{\bar{w}})
\]

We remind that the CP-invariance (3.27) imposes

\[
s = -\bar{s}, \quad j_w + j_{\bar{w}} = 2(j_\psi - n), \quad j_q + j_{\bar{q}} = n
\]

The proof of the conjecture of appearence of massive solitons at certain intermediate scale \( M_0 \) requires the construction of the corresponding left-right S-matrices which is out of the scope of this paper.

### 3.5 Spectrum of the ungauged 1-solitons

The explicit construction of the u-solitons is based on eqn. (3.12), replacing \( w \) and \( \bar{w} \) by the \( U(1) \)-CFT solitons (3.40) and \( \psi_g, \chi_g, R_g = \frac{n+1}{n} R \) - by the corresponding 1-soliton solutions
of the gauged IM \cite{18} (see eqns. (4.3)-(4.10) of ref. \cite{9}). The nonconformal GKO formula (3.21) allows us to calculate the energies (and masses) of the ungauged IM 1-solitons in terms of the energies of the constituent gauged and CFT solitons. For the left-u-solitons we find

\[ E_u = E_g + E_{L-sol}^{CFT}, \quad P_u = P_g + P_{L-sol}^{CFT}, \quad E_L^{CFT} = E_{L-sol}^{CFT}, \]

\[ E_g = M_g \cosh b, \quad P_g = -M_g \sinh b, \quad M_u^2 = M_g \left( M_g + 2E_{L-sol}^{CFT} \right) \] (3.49)

where \( b \) is the g-soliton velocity and \( M_g \) its mass \cite{9}.

Due to the arbitrariness of the \(|a_0|\) scale, we can introduce a new scale parameter \( m_0 \), such that \(|a_0| = m_0 e^{-b}\). Then the mass \( M_u \) of the ungauged 1-soliton takes the form

\[ \left( \frac{M_u}{m_0} \right)^2 = \left( \frac{M_g}{m_0} \right) + \frac{8n}{(n+1)\beta_0^2} \left( s + \frac{j_w}{n} \right)^2 \] (3.51)

which in fact determines the dimensionless mass-ratios \( \frac{M_u}{m_0} \) only. According to eqns. (3.16), (3.17) and (3.43) the charge spectrum of the left u-soliton is given by:

\[ Q^u_\theta = \frac{n+1}{8n} Q_0 = \frac{\pi}{2} \left( s + \frac{j_w}{n} \right), \quad Q^g_\theta = 0, \]

\[ Q^u_R = \frac{n+1}{n} \left( \beta_0^2 j_{el} + \frac{2\pi n}{n+1} \left( s + \frac{j_w}{n} \right) \right), \]

\[ Q^u_{el} = Q^g_{el} = \beta_0^2 j_{el}, \quad j_{el} = 0, \pm 1, \pm 2, \cdots \] (3.52)

Similar formulae take place for the right-u-soliton \((w = 0, \bar{w} \neq 0)\).

The spectrum of the left-right u-soliton combines the energies, masses and charges of the g-soliton \cite{9} with the left-right soliton (3.48):

\[ E_{L-R}^u = E_g + E_{L-sol}^{CFT} + E_{R-sol}^{CFT}, \quad M_{L-R}^u = M_g + M_0 \]

\[ Q^u_\theta = \frac{n+1}{8n} (Q_0 + Q_0), \quad Q^u_R = \frac{n+1}{2n} \left( Q^u_{el} + \frac{1}{2} (Q_0 - Q_0) \right) \] (3.53)

Together with charged topological u-solitons representing stable strong coupling particles one can also have u-strings represented by eqns. (3.12), but with \( w_{top}, \bar{w}_{top} \) of eqn. (3.40) replaced by \( w = w_{top} + w_{str} \) (3.44). They have the same charge spectrum as the u-solitons (but with infinite set of charges \( Q_k \) and \( \bar{Q}_k \) added) and the energy spectrum including the string oscillator part:

\[ E_{u}^{\text{string}} = E_g + E_{L-string} + E_{R-string} \] (3.54)

where \( E_{L-string}, (E_{R-string}) \) are given by (3.43).
3.6 \( \theta \)-terms, dyonic effects and spectral flows

Similarly to the gauged IM \([9]\), one can add to the ungauged IM Lagrangian (1.2) certain topological \( \theta \)-terms, i.e.,

\[
L_{\text{impr}} = L_u + \delta L_{\text{top}}^u
\]

with \( \delta L_{\text{top}}^u \) given by

\[
\delta L_{\text{top}}^u = \beta \frac{\nu^2}{8\pi^2} \sum_{k=1}^{n-1} \nu_k^2 \epsilon_{\mu\nu} \partial_\mu \varphi_k \partial_\nu \ln \left( \frac{X_u}{\psi_u} \right) + \pi \nu R \epsilon_{\mu\nu} \partial_\mu R_u \partial_\nu \ln \left( \frac{X_u}{\psi_u} \right),
\]

(3.55)

\( (\nu_k^2, \tilde{\nu}_k, \nu^R \) are real parameters). They do not change the equations of motion, but contribute to the charges \( Q_{el}, Q_0, \bar{Q}_0 \rightarrow Q_{el}^{\text{impr}}, Q_0^{\text{impr}}, \bar{Q}_0^{\text{impr}} \). For example, the electric current \( I_\mu^{\text{impr}} \) (generated by the global \( U(1) \) transformations (3.7)) acquires extra terms,

\[
I_\mu^{\text{impr}} = I_\mu - \beta_0^3 \frac{4\pi^2}{\epsilon_{\mu\nu}} \left( \sum_{k=1}^{n-1} \nu_k^2 \partial_\nu \varphi_k + 2\pi \nu R \partial_\nu R_u \right)
\]

(3.56)

Then, say for left-\( u \)-solitons we have

\[
Q_{el}^{\text{impr}} = Q_{el} - \frac{\nu \beta_0^2}{2\pi} j_{\varphi} - \frac{\nu^R}{\pi} Q_R^u, \quad Q_0^{\text{impr}} = Q_0 - \frac{\tilde{\nu} \beta_0^2}{2\pi} j_{\varphi} - \frac{\nu R \beta_0^2}{\pi} Q_\theta^u,
\]

\[
Q_R^u = \frac{n+1}{2n} (Q_{el} + \frac{1}{2} Q_0), \quad Q_\theta^u = \frac{n+1}{8n} Q_0
\]

(3.57)

where we have introduced \( \nu \) and \( \tilde{\nu} \) as follows

\[
\nu = \frac{1}{n} \sum_{k=1}^{n-1} k \nu_k^2, \quad \tilde{\nu} = \frac{1}{n} \sum_{k=1}^{n-1} k \tilde{\nu}_k^2
\]

Taking into account eqn. (3.43) and that \( Q_{el}^{\text{impr}} = \beta_0^2 j_{el} \) is quantized semiclassically \([9]\), we derive the improved charge spectrum

\[
Q_0^{\text{impr}} = \frac{4\pi n}{n+1} (1 - \frac{\beta_0^2 \nu R (n+1)}{8\pi n}) (s + \frac{j_\varphi}{n}) - \frac{\beta_0^2}{2} \tilde{\nu} j_{\varphi} - \frac{\beta_0^2 \nu R}{2} j_\theta,
\]

\[
Q_{el} = \frac{\beta_0^2 \left( j_{el} + \frac{\nu}{\pi} j_{\varphi} + \frac{\nu R}{2} (s + \frac{j_\varphi}{n}) \right)}{1 - \frac{\beta_0 \nu R}{4\pi n} (n+1)}
\]

(3.58)

The interpretation of the parameters \( \nu_k^2, \tilde{\nu}_k \) and \( \nu^R \) as external constant magnetic fields is similar to one already presented in refs. \([9], [23]\).

Note that the shift in \( Q_0 \) by \( \tilde{\nu} j_{\varphi} \) and \( \nu R j_\theta \) that gives \( Q_0^{\text{impr}} \) (and similar for \( \bar{Q}_0^{\text{impr}} \)) appears to be the nonconformal analog of the spectral flow \([18]\)

\[
\tilde{J}_3^0 = J_3^0 + \frac{k}{4} \tilde{\omega}
\]

(\( \tilde{\omega} \) is an integer) playing an important role in the description of \( AdS_3 \)-CFT (i.e. the representations of \( SL(2, R) \) current algebra).
3.7 Solitons of deformed $SL(2,R)$-WZW model

As we have mentioned in Sect. 2.3, the particular case $n = 1$ and $\beta$ real of the IM\(^{(1.2)}\) represents integrable deformation of the $SL(2,R)$-WZW model described by the Lagrangian\(^{(2.32)}\). The problem of the vacua structure and soliton solutions of this IM should be considered separately by the following three reasons:

- its potential $V_{n=1}^u = m^2\psi_u\chi_u e^{\beta R_u}$ has not distinct zeroes and therefore one cannot expect to have topological solitons.
- for real $\beta$ the $R^{CFT}$ (and $w, \overline{w}$) b.c.’s are not periodic and as a consequence the $U(1)$-CFT solitons are not topological too.
- contrary to the $n \geq 2$ IMs the g-solitons of the $n = 1$ gauged IM (i.e. Lund-Regge\(^{(14)}\), \(^{(13)}\)) which are an important ingredient of the u-solitons are also nontopological.

Therefore we have to make certain modifications in the arguments of Sect. 3.3 - 3.5 in order to derive the spectrum of such nontopological u-solitons. All the relations between currents, charges and stress-tensors of the gauged and ungauged IM of Sect. 3.1 and 3.2 are still valid for $n = 1$ and $\beta$ real. For example, the field relation \(^{(3.12)}\) now reads

$$ R_u = R_g + \beta (w + \overline{w}), \quad \theta_u = \theta_g - \frac{\beta}{2} (w - \overline{w}), $$

$$ \psi_u = \psi_g e^{-\frac{2}{\beta} R_g - \beta^2 w}, \quad \chi_u = \chi_g e^{-\frac{2}{\beta} R_g - \beta^2 \overline{w}} \quad (3.60) $$

Due to the fact that this case ($n = 1$, $\beta$ real)\(^{7}\) we have no analog of the $Z_n$ discrete symmetry \(^{(3.23)}\), the b.c.’s of the $R_u$ (and $w, \overline{w}$) are determined by the global $U(1)_{vector} \otimes U(1)_{axial}$ transformations \(^{(3.28)}\) and \(^{(3.29)}\) only, i.e.,

$$ \psi'_u = \psi_u e^{-\frac{2}{\beta} a_R}, \quad \chi'_u = \chi_u e^{-\frac{2}{\beta} a_R}, \quad R'_u = R_u + \beta a_R, \quad \theta'_u = \theta_u, \quad (3.61) $$

Therefore the allowed b.c.’s for $w$ and $\overline{w}$ have the form

$$ w' = w + a_w, \quad \overline{w}' = \overline{w} + \overline{a}_w, \quad a_w + \overline{a}_w = a_R, \quad \overline{a}_w - a_w = 2a_\theta \quad (3.62) $$

where $a_w, \overline{a}_w, a_R, a_\theta$ are arbitrary real constants. As a consequence of \(^{(3.62)}\) we can chose

$$ w(\pm \infty) = a_w^\pm, \quad \overline{w}(\pm \infty) = \overline{a}_w^\pm \quad (3.63) $$

which determine the spectrum of the $U(1)$-charges $Q_0, \overline{Q}_0$, namely

$$ Q_0 = 2\beta^2 (a_w^+ - a_w^-) \equiv \frac{2\pi a}{\beta^2}, \quad \overline{Q}_0 = -2\beta^2 (\overline{a}_w^+ - \overline{a}_w^-) \equiv -\frac{2\pi \overline{a}}{\beta^2} \quad (3.64) $$

\(^7\)for imaginary $\beta = i\beta_0$ and $n = 1$, i.e., $SU(2,R)$-WZW, however the following $Z$-transformation $\psi'_u = \psi_u e^{i\pi s}, \quad \chi'_u = \chi_u e^{-i\pi s}, \quad R'_u = R_u - \frac{2}{\beta_0} s_R$ takes place ( $\psi_u = \chi_u$ in SU(2) case).
Since $2\beta R_{\text{CFT}}$ and $2\beta^2 w$ are not angular variables in the $SL(2, R)$ case (and the charge spectrum is continuous) the $w_{\text{top}}$ and $\bar{w}_{\text{top}}$ in the form (3.40) have no topological meaning. They map $M_2$ into a rectangle (with finite area $\pi \delta \bar{\delta}$) instead the torus as it is in the case of imaginary $\beta = i\beta_0$. Our choice of $w_{\text{top}}$ and $\bar{w}_{\text{top}}$ is again in the form (3.40), but with $\delta, \bar{\delta}$-continuous, i.e.,

$$\delta = \frac{2a}{\beta^2}, \quad \bar{\delta} = -\frac{2\bar{a}}{\beta^2} \quad (3.65)$$

It is dictated by eqns. (3.39) and indeed ensures the required discontinuities. The energy of such solutions (“nontopological” CFT-solitons) is finite and positive:

$$E_{\text{CFT}}^{L-\text{sol}} = \frac{8a^2}{\beta^2} |a_0|, \quad E_{\text{CFT}}^{R-\text{sol}} = \frac{8\bar{a}^2}{\beta^2} |a_0| \quad (3.66)$$

According to eqn. (3.60) (and similarly to the generic $n \geq 2$ case of Sect. 3.5) the 1-solitons of this deformed $SL(2, R)$-WZW are a composition of the above “CFT-solitons” and the 1-solitons (nontopological) of the gauged $n = 1$ IM (i.e. Lund-Regge model [44]). Their semi classical spectrum [45] is given by

$$M_g = \frac{4m}{\beta^2} \sin\left(\frac{\beta^2 j_{el}}{4}\right), \quad Q_{el}^g = \beta^2 j_{el}, \quad j_{el} = 0, \pm 1, \pm 2, \ldots \quad (3.67)$$

Hence the spectrum of the nontopological left 1-solitons of the IM (2.32) has the form

$$Q_R^u = \beta^2 j_{el} + \frac{a}{\beta^2}, \quad Q_0^u = \frac{a}{2\beta^2}, \quad M^2_u = M_g (M_g + \frac{16a^2}{\beta^2} m_0) \quad (3.68)$$

The energy of the corresponding $n = 1$ string solutions is quite similar to the generic $n$ formula (3.45), (3.54) with $E_{L-\text{sol}}$ replaced by (3.66) and $E_g$ - with the Lund-Regge soliton energy.

It is worthwhile to mention that in the $SU(2)$ case ($\beta = i\beta_0, R_u$ with periodic b.c.’s and $\psi^* = \chi_u$) the charges $Q_0, \bar{Q}_0$ are quantized, i.e. $a = s \in \mathbb{Z}$ and the corresponding CFT-solitons are topological and stable. Similarly to the $n \geq 2$ IMs (1.2) the deformed $SU(2)$ WZW admits four kinds of 1-soliton solutions:

- massless topological $U(1)$ CFT solitons and strings
- massive solitons of the gauged (Lund-Regge) IM
- massive composite left (and right) u-solitons
- massive left-right u-solitons

Whether the composed left-u-solitons of deformed $SL(2, R)$ WZW with spectrum (3.68) represent stable strong coupling particles is an open question. It is clear however that the corresponding u-solitons of the $SU(2)$ model are indeed topologically stable.
4 Multicharged IMs with local and global symmetries

Among the vast family of dyonic IMs introduced in Sect. 2, we have chosen the simplest $A_n^{(1)}(p = 1)$ IM (1.12) (with one local $U(1)$ symmetry) in order to demonstrate how the spectrum of its $u$-solitons can be realized in terms of the charges, energies, etc. of the $g$-solitons of the gauged IM (1.8) and certain $U(1)$ CFT solitons (3.40). It is natural to address the question of whether the established reduction of the ungauged IM properties to the corresponding gauged IM combined with $U(1)$ CFT (with specific b.c.’s) takes place for generic multicharged IMs, i.e., $G_0^0 = U(1)^l$. Our main attention in the present section is concentrated on the $G_0^0 = U(1) \otimes U(1)$ multicharged IM (2.12) and particularly on the intermediate IM (2.16) with one local and one global $U(1)$ symmetries.

4.1 Conserved charges and GKO energy splitting

The chiral $U(1) \otimes U(1)$ conserved currents

$$ J_a = Tr \left( g_0^{-1} \partial g_0 \lambda_a \cdot H^{(0)} \right), \quad \bar{J}_a = Tr \left( \bar{\partial} g_0^{-1} \lambda_a \cdot H^{(0)} \right), \quad a = 1, n $$

($g_0 \in G_0$ is given by eqn. (2.11)) of the ungauged IM (2.12), generated by the transformations (2.14), (2.15), take the following explicit form:

$$ J_1 = \frac{2\beta_0}{n+1} (n \partial R_1^u + \partial R_n^u) - i \beta_0^2 \psi_1^u \partial \chi_1^u e^{i\beta_0 (R_1^u - \varphi_1)}, $$

$$ J_n = \frac{2\beta_0}{n+1} (\partial R_1^u + n \partial R_n^u) - i \beta_0^2 \psi_n^u \partial \chi_n^u e^{i\beta_0 (R_n^u - \varphi_n - 2)}, $$

$$ \bar{J}_1 = \frac{2\bar{\beta}_0}{n+1} (n \bar{\partial} R_1^u + \bar{\partial} R_n^u) - i \beta_0^2 \bar{\psi}_1^u \bar{\partial} \bar{\chi}_1^u e^{i\beta_0 (R_1^u - \varphi_1)}, $$

$$ \bar{J}_n = \frac{2\bar{\beta}_0}{n+1} (\bar{\partial} R_1^u + n \bar{\partial} R_n^u) - i \beta_0^2 \bar{\psi}_n^u \bar{\partial} \bar{\chi}_n^u e^{i\beta_0 (R_n^u - \varphi_n - 2)} \quad (4.1) $$

We denote their charges by

$$ Q_{m+1}^a = \oint J_a(z) z^{-m-1} dz, \quad \bar{Q}_{m+1}^a = \oint \bar{J}_a(\bar{z}) \bar{z}^{-m-1} d\bar{z}, \quad m \in \mathbb{Z} \quad (4.2) $$

Similarly to the $U(1)$ case (3.8), the topological charges

$$ Q_{R_a}^a = \beta_0 \int_{-\infty}^{\infty} \partial_x R_a^u dx, \quad a = 1, n, \quad (4.3) $$

the global $U(1) \otimes U(1)$ electric charges $Q_{e^{1,2}}^a$.

$$ Q_{e^1}^a = i \beta_0^2 \int_{-\infty}^{\infty} (\psi_1^u \partial \chi_1^u - \chi_1^u \partial \psi_1^u) e^{i\beta_0 (R_1^u - \varphi_1)} dx $$

$$ Q_{e^n}^a = i \beta_0^2 \int_{-\infty}^{\infty} (\psi_n^u \partial \chi_n^u - \chi_n^u \partial \psi_n^u) e^{i\beta_0 (R_n^u - \varphi_n - 2)} dx \quad (4.4) $$
and zero modes $Q^a_0$, $\bar{Q}^a_0$ of the chiral currents $J^a$, $\bar{J}^a$ are related as follows

\[
Q^e_{1,n} = \frac{2}{n+1}(nQ^{(1)}_{Ra} + Q^{(n)}_{Ra}) - \frac{1}{2}(Q^{(1)}_0 - \bar{Q}^{(1)}_0) \\
Q^e_{n,n} = \frac{2}{n+1}(Q^{(1)}_{Ra} + nQ^{(n)}_{Ra}) - \frac{1}{2}(Q^{(n)}_0 - \bar{Q}^{(n)}_0)
\] (4.5)

The fact that by construction the gauged IM (2.20) is a result of the gauge fixing of the local $U(1) \otimes U(1)$ symmetries (2.15) (in the way that $J_a = \bar{J}_a = 0$) and of the consequent elimination of two degrees of freedom $R^a_a$, $a = 1, n$ lead to the following relation between the gauged and ungauged fields appearing in (2.20) and in (2.12) respectively,

\[
R^a_a = R^a_a + \beta_0(w_a + \bar{w}_a), \quad \varphi^u_a = \varphi^g_a = \varphi^1_a, \\
\psi^u_a = \psi^g_a e^{-\frac{1}{2}i\beta_0 R^a_a - i\beta^a w_a}, \quad \chi^u_a = \chi^g_a e^{-\frac{1}{2}i\beta_0 R^a_a - i\beta^a \bar{w}_a}, \quad a = 1, n \\
\theta^u_a = \theta^g_a - \frac{1}{2}(w_a - \bar{w}_a), \quad \theta^u_a = \frac{1}{2i\beta_0} ln(\frac{\psi^u_a}{\chi^u_a})
\] (4.6)

The matrix form of these relations is given by eqn. (2.21). As in the case of the $U(1)$-models of Sect. 3, eqns. (1.6) are crucial in the construction of the solutions of the ungauged IM (2.12) in terms of the known solutions \textsuperscript{[23]} of the gauged IM (2.20) and certain $U(1) \otimes U(1)$ - CFT solutions (i.e. specific $w_a, \bar{w}_a$). An important byproduct of eqns. (1.6) are the following relations between the charges (topological and Noether) of both models (2.12) and (2.20)

\[
Q^{(1)}_{Ra} = Q^{(1)}_{R^a} + \frac{n}{4(n-1)} \left( Q^{(1)}_0 - \bar{Q}^{(1)}_0 - \frac{1}{n}(Q^{(n)}_0 - \bar{Q}^{(n)}_0) \right) \\
Q^{(n)}_{Ra} = Q^{(n)}_{R^a} + \frac{n}{4(n-1)} \left( Q^{(n)}_0 - \bar{Q}^{(n)}_0 - \frac{1}{n}(Q^{(1)}_0 - \bar{Q}^{(1)}_0) \right) \\
Q^{(1)}_{\theta^a} = Q^{(1)}_{\theta^a} + \frac{n}{8(n-1)} \left( Q^{(1)}_0 + \bar{Q}^{(1)}_0 + \frac{1}{n}(Q^{(n)}_0 + \bar{Q}^{(n)}_0) \right) \\
Q^{(n)}_{\theta^a} = Q^{(n)}_{\theta^a} + \frac{n}{8(n-1)} \left( Q^{(n)}_0 + \bar{Q}^{(n)}_0 + \frac{1}{n}(Q^{(1)}_0 + \bar{Q}^{(1)}_0) \right)
\] (4.7)

where the charges $Q^a_0, \bar{Q}^a_0$ have been realized in terms of the asymptotics of the free fields $\Phi^a(z), \bar{\Phi}^a(z)$ (i.e., $J^a = \partial \Phi^a, \bar{J}^a = \partial \bar{\Phi}^a$):

\[
Q^a_0 = 2 \int_{-\infty}^{\infty} \partial_z \Phi^a dx = 2(\Phi^a(\infty) - \Phi^a(-\infty)), \\
\bar{Q}^a_0 = -2 \int_{-\infty}^{\infty} \partial_x \Phi^a dx = -2(\Phi^a(\infty) - \Phi^a(-\infty))
\] (4.8)

which are related to $w_a$ and $\bar{w}_a$ as follows

\[
\Phi^{(1)} = \frac{2\beta^2_0}{n+1}(nw_1 + w_n), \quad \bar{\Phi}^{(1)} = \frac{2\beta^2_0}{n+1}(n\bar{w}_1 + \bar{w}_n), \\
\Phi^{(n)} = \frac{2\beta^2_0}{n+1}(w_1 + nw_n), \quad \bar{\Phi}^{(n)} = \frac{2\beta^2_0}{n+1}(\bar{w}_1 + n\bar{w}_n)
\] (4.9)
Again, as in the $U(1)$ case (3.17), the electric charges $Q_{a}^{\ell,u}$ and $Q_{a}^{\ell,g}$ of the ungauged and gauged IMs do coincide:

$$Q_{1}^{\ell,u} = \frac{2}{n+1} \left( nQ_{Rv}^{(1)} + Q_{Rv}^{(n)} \right) = Q_{1}^{\ell,g},$$
$$Q_{n}^{\ell,u} = \frac{2}{n+1} \left( Q_{Rv}^{(1)} + nQ_{Rv}^{(n)} \right) = Q_{n}^{\ell,g} \hspace{1cm} (4.10)$$
as one can verify by substituting (4.7) in (4.5).

The generalization of the nonconformal GKO formulae (3.21) and (3.22) to the case of multicharged IMs (2.12) and (2.20) is straightforward. Substituting eqns. (4.6) in the ungauged IM stress-tensor:

$$T_{p=2}^{u} = \frac{1}{2} \eta_{ij} \partial \varphi_{i} \partial \varphi_{j} + \partial \psi_{1}^{u} \partial \lambda_{1}^{u} e^{\beta(R_{v}^{u} - \varphi_{1})} + \partial \psi_{n}^{u} \partial \lambda_{n}^{u} e^{\beta(R_{v}^{u} - \varphi_{n-2})}$$
$$+ \frac{n}{2(n+1)} \left( (\partial R_{v}^{u})^{2} + (\partial R_{n}^{u})^{2} + \frac{2}{n} \partial R_{1}^{u} \partial R_{n}^{u} \right) + V_{u} \hspace{1cm} (4.11)$$

and $T_{p=2}^{u} = T_{p=2}^{u}(\partial \to \bar{\partial})$ and taking into account the constraints [23]

$$J(z) = Tr((g_{0}^{f})^{-1} \partial g_{0}^{f} \lambda_{a} \cdot H^{(0)}) = \bar{J}(z) = Tr(\bar{\partial} g_{0}^{f} (g_{0}^{f})^{-1} \lambda_{a} \cdot H^{(0)}) = 0,$$

we realize that the following GKO-splitting

$$T_{p=2}^{u} = T_{p=2}^{g} + \beta_{0}^{2} \frac{n}{2(n+1)} \left( (\partial w_{1})^{2} + (\partial w_{n})^{2} + \frac{2}{n} \partial w_{1}(\partial w_{n}) \right) \hspace{1cm} (4.12)$$
takes place. We have denoted by $T_{p=2}^{g}$ the canonical stress-tensor of the gauged IM derived from its Lagrangian (2.20),

$$T_{p=2}^{g} = \frac{1}{2} \eta_{ij} \partial \varphi_{i} \partial \varphi_{j} + \frac{1}{\Delta} \left( \partial \psi_{1}^{g} \partial \lambda_{1}^{g} e^{-\beta \varphi_{1}}(1 + \beta_{0}^{2} \frac{n}{2(n-1)} \psi_{n}^{g} \lambda_{n}^{g} e^{-\beta \varphi_{n-2}}) \right. \right.$$\left. \left. + \partial \psi_{n}^{g} \partial \lambda_{n}^{g} e^{-\beta \varphi_{n-2}}(1 + \beta_{0}^{2} \frac{n}{2(n-1)} \psi_{1}^{g} \lambda_{1}^{g} e^{-\beta \varphi_{1}}) \right. \right.$$\left. \left. + \beta_{0}^{2} \frac{1}{2(n-1)} (\lambda_{1}^{g} \psi_{1}^{g} \partial \varphi_{1} \partial \lambda_{1}^{g} + \lambda_{n}^{g} \psi_{n}^{g} \partial w_{n} \partial w_{n}) e^{-\beta(\varphi_{1} + \varphi_{n-2})} \right) + V_{g} \hspace{1cm} (4.13)$$

One can further diagonalize the $U(1) \otimes U(1)$-CFT stress-tensor

$$T_{p=2}^{CFT} = \beta_{0}^{2} \frac{n}{2(n+1)} \left( (\partial w_{1})^{2} + (\partial w_{n})^{2} + \frac{2}{n} \partial w_{1} \partial w_{n} \right), \hspace{1cm} \partial T_{p=2}^{CFT} = 0$$

by introducing new fields $w_{\pm}$:

$$w_{\pm} = \frac{1}{2}(w_{1} \pm w_{n}), \hspace{1cm} \bar{w}_{\pm} = \frac{1}{2}(\bar{w}_{1} \pm \bar{w}_{n}) \hspace{1cm} (4.14)$$

The result is:

$$T_{p=2}^{CFT} = \beta_{0}^{2} \left( (\partial w_{+})^{2} + \frac{n-1}{n+1}(\partial w_{-})^{2} \right) = T_{+}^{CFT} + T_{-}^{CFT}, \hspace{1cm} (4.15)$$
and the same for $\tilde{T}_{p=2}^{\text{CFT}} = T_{p=2}^{\text{CFT}}(\partial \to \tilde{\partial})$.

The intermediate IM (2.16), introduced in Sect. 2, is a result of the gauge fixing of one of the local $U(1)$ symmetries, say,

$$J_+ = \frac{1}{2} Tr \left( \left( g_0^{\text{int}} \right)^{-1} \partial g_0^{\text{int}} (\lambda_1 + \lambda_n) \cdot H^{(0)} \right) = 0,$$

$$J_- = \frac{1}{2} Tr \left( 1 \left( g_0^{\text{int}} \right)^{-1} (\lambda_1 + \lambda_n) \cdot H^{(0)} \right) = 0$$

(4.16)

of the ungauged IM (2.12). Its Lagrangian is invariant under chiral $U(1)$ (spanned by $1/2(\lambda_1 - \lambda_n) \cdot H^{(0)}$) transformation (2.19) with conserved current $\partial J_+ = \partial J_- = 0$

$$J_- = \frac{1}{2} Tr \left( \left( g_0^{\text{int}} \right)^{-1} \partial g_0^{\text{int}} (\lambda_1 - \lambda_n) \cdot H^{(0)} \right) = 2 \beta_0 \left( \frac{n-1}{n+1} \right) \Delta_0 \partial \bar{R}_u$$

$$-\bar{J}_- = J_-(\partial \to \tilde{\partial}, \psi_a \to \chi_a) = \frac{1}{2} Tr \left( \tilde{\partial} g_0^{\text{int}} (\lambda_1 - \lambda_n) \cdot H^{(0)} \right)$$

(4.17)

where

$$J_- = \partial \Phi_-, \quad \bar{J} = \partial \Phi_-, \quad \Phi_- = 2 \beta_0 \frac{n-1}{n+1} w_- \quad \bar{\Phi}_- = 2 \beta_0 \frac{n-1}{n+1} \bar{w}_-.$$

They are also invariant under global $U(1)$ transformation

$$\bar{\psi}_a' = \bar{\psi}_a e^{i\beta_0^2 \epsilon_-}, \quad \bar{\chi}_a' = \bar{\chi}_a e^{-i\beta_0^2 \epsilon_-}, \quad \bar{R}_u' = \bar{R}_u, \quad \varphi_l' = \varphi_l$$

(4.18)

We denote the charges of the $J_-$ and $\bar{J}_-$ currents by

$$Q_{m+1}^- = \oint J_-(z) z^{-m-1} dz, \quad \bar{Q}_{m+1}^- = \oint \bar{J}_-(\bar{z}) \bar{z}^{-m-1} d\bar{z},$$

i.e., we have for their zero modes:

$$Q_0^- = 2 \int_{-\infty}^{\infty} \partial_x \Phi_- dx, \quad \bar{Q}_0^- = -2 \int_{-\infty}^{\infty} \partial_x \bar{\Phi}_- dx$$

(4.19)

The global $U(1)$-charges are now given by

$$Q_{\pm}^{el,u} = \frac{1}{2} (Q_{1}^{el,u} \pm Q_{n}^{el,u})$$

(4.20)

Taking into account eqn. (4.10), we derive the following relation

$$Q_+^{el,u} = 2 \bar{Q}_0 = Q_+^{el,g}, \quad Q_-^{el,u} = 2 \frac{n-1}{n+1} \bar{Q}_0 = Q_-^{el,g}$$

(4.21)
between \(Q_{\pm}^{el,a}\) and the topological charges

\[
Q_{\tilde{R}_g} = \beta_0 \int_{-\infty}^{\infty} \partial_x \tilde{R}_g dx, \quad \tilde{R}_g = \frac{1}{2}(R_1^q + R_2^q),
\]

\[
Q_{\tilde{R}_g} = \beta_0 \int_{-\infty}^{\infty} \partial_x \tilde{R}_g dx, \quad \tilde{R}_g = \frac{1}{2}(R_1^q - R_2^q)
\]

(4.22)

Note that the topological charge \(Q_{\tilde{R}_a}\), i.e.,

\[
Q_{\tilde{R}_a} = \beta_0 \int_{-\infty}^{\infty} \partial_x \tilde{R}_a dx
\]

is not proportional to the electric charge \(Q_{\pm}^{el,a}\), due to the more general relation

\[
\frac{1}{2}(Q_0 - \tilde{Q}_0) = 2\left(\frac{n-1}{n+1}\right)Q_{\tilde{R}_a} - Q_{\pm}^{el,a}
\]

(4.23)

which follows from eqn. (4.17).

According to eqn. (2.18) the fields of the intermediate model \(\tilde{\psi}_a, \tilde{\chi}_a, \tilde{R}\) can be realized in terms of the fields of the ungauged IM \(\psi_a^u, \chi_a^u, R_a^u\) and the free fields \(w_+\) and \(\bar{w}_+\):

\[
\tilde{R}_a = R_a^{int} + \beta_0(w_+ + \bar{w}_+), \quad \varphi_a^{int} = \varphi_a^{int}, \quad \tilde{R}_a = R_a^{int} = \bar{R},
\]

\[
\psi_a^{\bar{u}} = \psi_a^{\bar{u}} e^{-i\beta_a^0 w_+ + \frac{1}{2}\beta_a^0 \tilde{R}_a}, \quad \chi_a^u = \chi_a^u e^{-i\beta_a^0 w_- + \frac{1}{2}\beta_a^0 \tilde{R}_a}
\]

(4.24)

Similarly we can relate the intermediate IM fields with those of the gauged IM (1.8) \(\psi_a^g, \chi_a^g, R_a^g\), taking into account eqns. (4.24), (1.6) and (4.14):

\[
R = \tilde{R}_g + \beta_0(w_+ + \bar{w}_+), \quad \varphi_a^{int} = \varphi_a^{g}, \quad \tilde{\psi}_1 = \psi_1^{g} e^{-i\beta_a^0 w_- - \frac{1}{2}\beta_0 \tilde{R}_a},
\]

\[
\bar{\psi}_n = \psi_n^{g} e^{i\beta_a^0 w_+ + \frac{1}{2}\beta_0 \tilde{R}_a}, \quad \tilde{\chi}_1 = \chi_1^{g} e^{-i\beta_a^0 w_- - \frac{1}{2}\beta_0 \tilde{R}_a},
\]

\[
\bar{\chi}_n = \chi_n^{g} e^{i\beta_a^0 w_+ + \frac{1}{2}\beta_0 \tilde{R}_a}, \quad \bar{\theta}_a = \theta_a^g + \frac{\beta_0}{2}(w_+ - \bar{w}_+), \quad a = 1, n
\]

(4.25)

Therefore the topological charges of the intermediate IM can be realized in terms of the corresponding gauged IM charges and the \(J_-\) and \(J_+\) zero modes:

\[
Q_{\tilde{R}_a} = Q_{\tilde{R}_a} + \frac{n+1}{4(n-1)}(Q_0^0 - \tilde{Q}_0^0),
\]

\[
Q_{\tilde{\theta}_a}^{(1)} = Q_{\tilde{\theta}_a}^{(1)} - \frac{n+1}{8(n-1)}(Q_0^0 + \tilde{Q}_0^0),
\]

\[
Q_{\tilde{\theta}_a}^{(n)} = Q_{\tilde{\theta}_a}^{(n)} + \frac{n+1}{8(n-1)}(Q_0^0 + \tilde{Q}_0^0)
\]

(4.26)

Finally, the GKO energy splitting for the intermediate IM takes the form

\[
T_{p=2}^{u} = T_{p=2}^{int} + \beta_0^2(\partial w_+)^2, \quad T_{p=2}^{int} = T_{p=2}^{g} + \beta_0^2\left(\frac{n-1}{n+1}\right)(\partial w_-)^2
\]

(4.27)
The stress-tensor of the intermediate IM (2.12) is given by

\[
T_{p=2}^{\text{int}} = \frac{1}{2} \eta_{ij} \partial \phi_i \partial \phi_j + \left( \frac{n - 1}{n + 1} \right) (\partial R)^2 + \frac{1}{\Delta_0} \left( 1 + \frac{\beta_0^2}{4} \bar{\psi}_n \bar{\chi}_n e^{-i\beta_0(R+\varphi_n-2)} e^{i\beta_0(R+\varphi_1)} \partial \bar{\psi}_1 \partial \bar{\chi}_1 \right) + \left( 1 + \frac{\beta_0^2}{4} \bar{\psi}_1 \bar{\chi}_1 e^{-i\beta_0(R-\varphi_1)} e^{i\beta_0(R+\varphi_n-2)} \partial \bar{\psi}_n \partial \bar{\chi}_n \right) - \frac{\beta_0^2}{4} e^{-i\beta_0(\varphi_1+\varphi_n-2)} (\bar{\psi}_1 \bar{\chi}_1 \partial \bar{\psi}_n \partial \bar{\psi}_1) + V_{\text{int}}
\]

and \( T_{p=2}^u \) and \( T_{p=2}^q \) by eqns. (4.11) and (4.13) respectively.

### 4.2 Vacuas Structure and 1-solitons of the \( U(1) \otimes U(1) \) ungauged IM

Together with the local (and global) \( U(1) \otimes U(1) \) symmetries (described in Sect. 4.1) the ungauged IM Lagrangian (2.12) is also invariant under discrete \( Z_2 \otimes Z_2 \otimes Z_{n-1} \) transformations (for imaginary coupling \( \beta = i\beta_0 \)). The action of the \( Z_{n-1} \) group on the fields \( \varphi_a, \psi_a^u, \chi_a^u \) and \( R_a^u \) is quite similar to the \( Z_n \) transformations (3.23) for the \( U(1) \) model (i.e. \( p = 1 \) (1.2)). It is easy to check that \( \mathcal{L}_p^u (2.12) \) remains invariant under the following \( Z_{n-1} \) transformation:

\[
\varphi'_l = \varphi_l + \frac{2\pi}{\beta_0} \frac{ln}{n-1}, \quad l = 1, 2, \ldots, n - 2,
\]

\[(R_a^u)' = R_a^u + \frac{2\pi}{\beta_0} (s_R^{(1)} + \frac{N - q_1 - \bar{q}_1}{r - 1}),\]

\[(R_a^u)' = R_a^u + \frac{2\pi}{\beta_0} (s_R^{(n)} - \frac{N - q_n - \bar{q}_n}{r - 1}),\]

\[(\psi_a^u)' = e^{\pi i (\frac{2q_1}{n-1} + s_1)} \psi_1^u, \quad (\chi_1^u)' = e^{\pi i (\frac{2q_1}{n-1} + s_1)} \chi_1^u,\]

\[(\psi_n^u)' = e^{-\pi i (\frac{2q_n}{n-1} + s_n)} \psi_n^u, \quad (\chi_n^u)' = e^{-\pi i (\frac{2q_n}{n-1} + s_n)} \chi_n^u,\]

where \( s_R^{(a)}, s_a, \tilde{s}_a, (s_a \pm \tilde{s}_a = 2s^{(a)}_a) \) are integers and \( q_a, \bar{q}_a, N = 0, 1, \ldots, n - 2 \mod (n - 1) \) are the \( Z_{n-1} \) charges of the fields \( \psi_a^u, \chi_a^u \) and \( e^{i\beta_0 \varphi_1} \) respectively. The first \( Z_2 \) acts as field reflections:

\[
\varphi''_l = \varphi_{n-l-1}, \quad (R_1^u)'' = R_1^u, \quad (R_n^u)'' = R_n^u, \quad (\psi_1^u)'' = \psi_n^u, \quad (\psi_n^u)'' = \psi_1^u,\]

\[(\chi_1^u)'' = \chi_n^u, \quad (\chi_n^u)'' = \chi_1^u,\]

i.e. interchanging the \( Z_{n-1} \) charges of \( \varphi_1 \) and \( \varphi_{n-2}, \psi_1^u \) and \( \psi_n^u, R_1^u \) and \( R_n^u \), etc.

\[N \to n - 1 - N, \quad q_a \to n - 1 - q_a, \quad \bar{q}_a \to n - 1 - \bar{q}_a\]

This extends the \( Z_{n-1} \) to the diedral group \( D_{n-1} \). The other \( Z_2 \) group represents the following CP-transformations:

\[
\varphi''''_l = \varphi_l, \quad (R_a^u)''' = R_a^u, \quad (\psi_a^u)''' = \chi_a^u, \quad (\chi_a^u)''' = \psi_a^u,\]

\[P_x = -x, \quad P \vartheta = \tilde{\vartheta}, \quad P^2 = 1\]

(4.31)
The vacuum solutions for the ungauged IM (2.12) are given by

$$\varphi_i^{(N)} = \left(\frac{2\pi}{\beta_0}\right) \frac{ln N}{n-1}, \quad \psi_a \chi_a = 0, \quad R_{a}^u = \beta_0 a_R^a, \quad \theta_a^u = \beta_0 a_{\theta}^a$$ (4.32)

For such field configuration the potential $V_{p=2}$ (2.13) shows $(n-1)$-distinct zeroes. By chiral $U(1) \otimes U(1)$ transformations (2.15) one map these constant vacua solutions into a special class of conformal invariant solutions

$$\varphi_i^{(N)} = \frac{2\pi}{\beta_0} \frac{ln N}{n-1}, \quad \psi_a \chi_a = 0$$

$$R_{a}^{u,CFT} = \beta_0 (w_a + \bar{w}_a), \quad \theta_a^{u,CFT} = \frac{\beta_0}{2} (w_a - \bar{w}_a), \quad a = 1, n$$ (4.33)

representing string (2-d free fields $w_a, \bar{w}_a$) in flat 2-d target space, i.e., $U(1) \otimes U(1)$-CFT. Among all possible string-like and particle-like finite energy ($E^{CFT} = \pm P^{CFT}$) solutions of this CFT, we seek for a special family of charged topological massless solitons. As in the case of $U(1)$-CFT (see Sect. 3.3 and 3.4) crucial for the existence of such solutions are the nontrivial b.c. for $w_a, \bar{w}_a$, supported by certain discrete symmetries ($Z_{n-1}$ in our case). It is not necessary to impose an appropriate b.c., since they are already encoded in eqns. (4.10) and (4.29). Taking into account the $Z_{n-1}$ transformations of the gauged IM (2.12) fields $[23]$

$$\begin{align*}
(\psi_1^g)' &= e^{\pi i (\frac{N}{n-1} + \frac{s_1^g}{n})} \psi_1^g, \quad (\chi_1^g)' &= e^{\pi i (\frac{N}{n-1} + \frac{\bar{s}_1^g}{n})} \chi_1^g, \\
(\psi_n^g)' &= e^{-\pi i (\frac{N}{n-1} + \frac{s_n^g}{n})} \psi_n^g, \quad (\chi_n^g)' &= e^{-\pi i (\frac{N}{n-1} + \frac{\bar{s}_n^g}{n})} \chi_n^g, \\
(R_{a}^g)' &= R_{a}^g, \quad \varphi_i' = \varphi_i
\end{align*}$$ (4.34)

we derive the following discrete transformations for $w_a, \bar{w}_a$,

$$w'_a = w_a + \frac{\pi}{\beta_0^2} (s_a + \epsilon_a \frac{N - 2q_a}{n - 1}), \quad \bar{w}'_a = \bar{w}_a + \frac{\pi}{\beta_0^2} (\bar{s}_a + \epsilon_a \frac{N - 2\bar{q}_a}{n - 1}),$$ (4.35)

where $\epsilon_1 = -\epsilon_n = 1$. The $Z_{n-1}$ properties (4.34) of $w_a, \bar{w}_a$ (and of the diagonal fields (4.14) $w^\pm$ and $\bar{w}^\pm$) allow us to distinguish an important class of b.c. for the $U(1) \otimes U(1)$-CFTs free fields $\Phi^a(z)$ and $\Phi^a(\bar{z})$ of eqns. (4.19):

$$\Phi_+(\pm \infty) = \frac{\pi}{n+1} (s_1^+ + s_n^+ + \frac{2(q_1^+ - q_n^+)}{n - 1}),$$

$$\Phi_- (\pm \infty) = \frac{\pi}{n+1} (s_1^- + s_n^- + \frac{2(N_1 - q_1^+ - q_n^+)}{n - 1})$$ (4.36)

where $\Phi_\pm$ are defined as follows

$$\begin{align*}
\Phi_+ &= \frac{1}{2} (\Phi^{(1)} + \Phi^{(n)}) = 2\beta_0^2 w_+, \\
\Phi_- &= \frac{1}{2} (\Phi^{(1)} - \Phi^{(n)}) = 2\beta_0^2 w_-, \\
J_\pm &= \partial \Phi_\pm, \quad J_\pm = \frac{1}{2} (J_1 \pm J_n)
\end{align*}$$ (4.37)

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Therefore the CFT solutions interpolating between two vacua

\[(\Phi_+(\infty), \Phi_-(\infty)) \rightarrow (\Phi_+(-\infty), \Phi_-(-\infty))\]

carry nontrivial topological charges \(Q^\pm_0\) and \(\bar{Q}^\pm_0\)

\[
Q^+_0 = 2 \int_{-\infty}^{\infty} \partial_x \Phi_+ \, dx = 4\pi (s_+ + \frac{j^+_w}{n-1}), \\
Q^-_0 = 2 \int_{-\infty}^{\infty} \partial_x \Phi_- \, dx = 4\pi (n-1) (s_- + \frac{j^-_w}{n-1}), \\
\bar{Q}^+_0 = -4\pi (s_+ + \frac{j^+_\bar{w}}{n-1}), \quad \bar{Q}^-_0 = -4\pi (n-1) (s_- + \frac{j^-_\bar{w}}{n-1})
\]

where \(s_1 \pm s_n = 2s_\pm\), \(2j^+_w = j^{(1)}_w \pm j^{(n)}_w\), \(j^a_w = j^\varphi - 2j^a_q\), \(j^\varphi = N_+ - N_-\), \(j^a_q = q^+_a - q^-_a\).

It becomes clear that the conformal vacua of the \(U(1) \otimes U(1)-\)CFT are characterized by the winding numbers \(s_\pm\) and \(\bar{s}_\pm\) of the fields \(2\beta^2 w_\pm\) and \(2\beta^2_0 \bar{w}_\pm\) and by the \(Z_{n-1}\) charges \(j^\pm_w, j^\pm_\bar{w}\) of the vertices \(V^\pm_{s_\pm j^\pm_w} = e^{2\beta^2 w_\pm}\) and \(\bar{V}^\pm_{s_\pm j^\pm_\bar{w}} = e^{2\beta^2_0 \bar{w}_\pm}\). Similarly to the \((1,1)-\)case of Sect. 3.4, the topological solitons of the \(U(1) \otimes U(1)-\)CFT with all the above properties are nothing but the map of 2-d dimension space \(M_2\) to 4-torus \(T_4 = (S^1)^4\), \(r_0 = \frac{1}{2\beta^2_0}\). Their explicit form is a generalization of the 1-soliton \(w_{\text{top}}\) (4.40) of \((1,1)-\)CFT:

\[
w^\text{top} = -i\delta_\pm \ln \left( \frac{e^{\alpha x+i}}{e^{\alpha x-i}} \right), \quad \bar{w}^\text{top} = -i\bar{\delta}_\pm \ln \left( \frac{e^{\alpha x+i}}{e^{\alpha x-i}} \right),
\]

\[
\delta_\pm = \frac{1}{\beta^2} (s_\pm + \frac{j^\pm}{n-1}), \quad \bar{\delta}_\pm = \frac{1}{\beta^2_0} (s_\pm + \frac{j^\pm_\bar{w}}{n-1})
\]

We next calculate the energy \(E_{L-\text{sol}}^{p=2}\) of the left moving massless solitons (see eqn. (1.15)):

\[
E_{L-\text{sol}}^{p=2} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} T^{CFT} \, dx = 4\beta^2 \int_{-\infty}^{\infty} \left( (\partial_x w_+)^2 + \frac{n-1}{n+1} (\partial_x w_-)^2 \right) \, dx
\]

or explicitly:

\[
E_{L-\text{sol}}^{p=2} = \frac{8\beta^2}{a_0} \left( (s_+ + \frac{j^-}{n-1})^2 + \frac{n-1}{n+1} (s_- + \frac{j^+}{n-1})^2 \right)
\]

One can also have finite energy string-like solutions with the same topological charges (4.38), but admiting arbitrary holomorphic parts \(w_{\text{str}}^\pm\) of \(w^\pm\), i.e.,

\[
w^\pm = w_{\text{top}}^\pm + w_{\text{str}}^\pm, \quad w_{\text{str}}^\pm(\pm\infty) = 0
\]

The energy of such string solutions gets contributions from \(\partial w_{\text{str}}^\pm = \sum_{k \neq 0} Q^\pm_{k}\cdot k\):

\[
E_{L-\text{string}}^{p=2} = \frac{8\beta^2}{a_0} \sum_{k \neq 0} \left( Q^+_k Q^-_k + \frac{n-1}{n+1} Q^-_k Q^-_k \right)
\]
We should mention that the most general particle-like 1-soliton is represented by the following two parameter family of solutions:

\[ w_{\text{top}}^\pm(\alpha_{\pm}) = -i\delta_{\pm}(\alpha_{\pm})\ln \left( e^{\alpha_{\pm}z} + e^{-\alpha_{\pm}z} \right), \quad \delta_{\pm}(\alpha_{\pm}) = \frac{\pi}{2\alpha_{\pm}} \delta_{\pm} \]  

(4.42)

They have the same topological charges (4.38) as \( w_{\text{top}}^\pm(\alpha_{\pm} = \pi/2) = w_{\text{top}}^\pm \), but different energies (for \( \alpha_{\pm} \neq \pi/2, \pi/4 < \alpha_{\pm} < \pi \))

\[ E_{L=\text{sol}}^{p=2}(\alpha_{+}, \alpha_{-}) = \frac{2|a_0|\pi^2}{\beta_0^2} \left( \frac{1}{\alpha_+^2}(s_+ + \frac{j\_w}{n-1})^2(1 - \alpha_+\cotg(\alpha_+)) \right. 
\]

\[ + \left. \frac{1}{\alpha_-^2}(n-1)(s_- + \frac{j\_w}{n-1})^2(1 - \alpha_-\cotg(\alpha_-)) \right) \]  

(4.43)

which coincides with (4.40) for \( \alpha_{\pm} = \pi/2 \).

Given the 1-solitons [23] of the gauged IM (2.20) and the \( U(1) \otimes U(1)\)-CFT topological massless solitons (4.39) and (4.42) we can construct the ungauged 1-solitons of the IM (2.12) according to eqns. (4.6). Then the GKO formula (4.12) allows us to calculate the energy of say, left-u-solitons (i.e., \( w_+^\pm \neq 0, \tilde{w}_+^\pm = 0 \)):

\[ E_{u}^{p=2} = E_{u}^{p=2} + E_{L=\text{sol}}^{CFT}(p = 2), \]

\[ P_{u}^{p=2} = P_{u}^{p=2} + P_{L=\text{sol}}^{CFT}(p = 2), \quad E_{L}^{CFT} = P_{L}^{CFT}, \]

\[ E_{g}^{p=2} = M_{g}^{p=2}\cosh(b), \quad P_{g}^{p=2} = -M_{g}^{p=2}\sinh(b) \]  

(4.44)

where \( b \) is the velocity of the gauged IM 1-solitons and \( M_g \) their mass [23]:

\[ M_{g}^{p=2} = \frac{4m(n-1)}{\beta_0^2}\sin\left(\frac{4\pi j\_\varphi - \beta_0^2 j\_e}{4(n-1)}\right) \]

For \( |a_0| = m_0e^{-b} \), the mass of u-solitons is given by

\[ \left( \frac{M_{g}^{p=2}}{m_0} \right)^2 = \left( \frac{M_{g}^{p=2}}{m_0} \right) + 2 \left( \frac{E_{L=\text{sol}}^{CFT}(p = 2)}{m_0} \right) \]  

(4.45)

Taking into account the charge relations (4.15), (4.7) and (1.10) we have established in Sect. 4.1, we derive the charge spectrum of the left-u-soliton:

\[ Q_0^+ = 4\pi(s_+ + \frac{j\_w}{n-1}), \quad Q_0^- = 4\pi\frac{n-1}{n+1}(s_- + \frac{j\_w}{n-1}), \]

\[ Q_{el}^+ = \frac{1}{2}(Q_{el}^1 - Q_{el}^1) = \frac{\beta_0^2}{2}j\_el, \quad Q_{el}^+ = \frac{1}{2}(Q_{el}^1 + Q_{el}^1) = \frac{\beta_0^2}{2}j_0 \]

\[ Q_{Ru}^+ = \frac{1}{2}(Q_{Ru}^{(1)} - Q_{Ru}^{(1)}) = \frac{n+1}{4(n-1)}(\beta_0^2j\_el + Q_0^-), \]

\[ Q_{Ru}^+ = \frac{1}{2}(Q_{Ru}^{(1)} + Q_{Ru}^{(1)}) = \frac{1}{4}(2\beta_0^2j_0 + Q_0^+), \]

\[ Q_{Ru}^- = \frac{n+1}{4(n-1)}Q_0^+, \quad Q_{Ru}^- = \frac{1}{4}Q_0^- \]  

(4.46)

where \( Q_{el}^+ \) (i.e. \( j\_el \in \mathbb{Z} \) and \( \delta_0 \in \mathbb{R} \)) represent the semiclassical electric charges of the gauged 1-solitons [23].
4.3 Soliton spectrum of the intermediate IM

The intermediate model (2.16) arises from the ungauged IM (2.12) by axial gauge fixing one of the chiral $U(1)$ symmetries, namely by imposing the constraints $J_+ = J_- = 0$ (see eqn. (1.10)) and keeping $J_-$ and $\bar{J}_-$ unconstrained. The same way we have constructed u-solitons as conformal dressing by $w_{\text{top}}^a$ (4.39) of the 1-solitons [23] of the gauged IM (2.20), the intermediate 1-solitons can be constructed by $U(1)$ CFT-dressing of the same g-solitons by $\bar{w}_{\text{top}}$ and $\bar{w}^{-}_{\text{top}}$ according to eqns. (4.23). The relations between the solitons of the gauged, intermediate and ungauged IMs, established in Sect. 4.1 provide an alternative construction of these int-solitons as a reduction of the u-solitons by requiring that $w^+ = \bar{w}^+ = 0$, i.e.,

$$q_1 = q_\alpha = q, \quad \bar{q}_1 = \bar{q}_\alpha = \bar{q}, \quad s_+ = \bar{s}_- = 0$$

In order to make transparent the properties of these solitons (and the origin of the b.c.'s for $w^-, \bar{w}^-$) we shall derive the vacua structure of the IM (2.16) and its discrete symmetries independently of the relations with the ungauged IM (2.12). The $Z_{n-1}$ transformations that leave invariant the Lagrangian (2.16) have the form:

$$\psi'_a = \psi_a e^{i\pi \epsilon_a \left( \frac{2q_\alpha + s_\alpha}{n-1} \right)}, \quad \bar{X}_a = \bar{X}_a e^{i\pi \epsilon_a \left( \frac{2\bar{q}_\alpha + \bar{s}_\alpha}{n-1} \right)}, \quad \epsilon_1 = -\epsilon_n = 1, \quad s_\alpha \pm \bar{s}_\alpha = 2L_\alpha,$$

$$\bar{R}' = \bar{R} + \frac{2\pi}{\beta_0} \left( s_R + \frac{N - q - \bar{q}}{n-1} \right), \quad \varphi' = \varphi + \frac{2\pi}{\beta_0} \frac{lN}{n-1},$$

(4.47)

where $s_\alpha, \bar{s}_\alpha$ and $L_\alpha$ are integers and $q, \bar{q}, N = 0, 1, \cdots n - 2 \mod (n-1)$ are the $Z_{n-1}$ charges of the corresponding fields. Repeating the arguments of Sect. 4.2. we find that

$$w'_- = w_- + \frac{\pi}{\beta_0} \left( s_- + \frac{N - 2q_-}{n-1} \right), \quad \bar{w}'_- = \bar{w}_- + \frac{\pi}{\beta_0} \left( \bar{s}_- + \frac{N - 2\bar{q}_-}{n-1} \right),$$

(4.48)

and therefore the b.c.'s for the free field $\Phi_- = 2\beta_0^2 (n-1) w_-$ (i.e., $J_- = \partial \Phi_-$) are given by

$$\Phi_-(\pm \infty) = \pi \left( s_- + \frac{N - 2q_-}{n-1} \right)$$

(4.49)

As a consequence the charges $Q^-_0$ and $\bar{Q}^-_0$ (4.19) for the solutions having the above asymptotics take the form,

$$Q^-_0 = 4\pi n - 1 \left( s_- + \frac{j_w}{n-1} \right), \quad \bar{Q}^-_0 = -4\pi n - 1 \left( \bar{s}_- + \frac{j_{\bar{w}}}{n-1} \right),$$

$$j_w = j_{\varphi} - 2j_q, \quad j_{\varphi} = N_+ - N_-, \quad s_- = s_+ - s_-, \quad j_q = q_+ - q_-$$

(4.50)

The explicit form $w_{\text{top}}^-$ of the $U(1)$-CFT topological solitons (carrying $Q^-_0, \bar{Q}^-_0$ topological charges) is again in the standard form (1.39) with $\delta_- = \frac{1}{\beta_0} \left( s_- + \frac{j_{\varphi} - 2j_q}{n-1} \right)$. Similarly to the u-soliton case (1.44), (1.45) and (1.46) but now according to the charges and energy relations (1.23), (1.26) and (1.27), we derive the following semiclassical spectrum of the left moving 1-soliton of the intermediate IM (2.16):

$$M^2_{\text{int}} = M_{g=2}^2 \left( M_{g=2}^2 + \frac{16m_0}{\beta_0^2} \right) \left( \frac{n-1}{n+1} \right) \left( s_- + \frac{j_{\varphi} - 2j_q}{n-1} \right)^2,$$

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\[ Q_{el}^- = \frac{\beta_0^2}{2} j_{el}, \quad Q_{el}^+ = \beta_0^2 \delta_0, \]
\[ Q_{\bar{R}} = 4 \frac{n+1}{n-1} \beta_0^2 j_{el} + \pi (s_+ + j_\varphi - \frac{2j_q}{n-1}), \]
\[ Q_{\phi}^{(1)} = -\frac{n+1}{8(n-1)} Q_0^- = -Q_{\phi}^{(n)} \quad (4.51) \]

where \( Q_0^- \) is given by eqn. (4.50). The energy (and the mass \( M_{int}(\alpha) \)) for more general \( \alpha \)-dependent \( w_{top}(\alpha) \) (see eqn. (4.42)) are given by
\[
E_{int}(\alpha) = E_g^{\alpha=2} + E_{L-sol}^{int}(\alpha),
\]
\[
E_{L-sol}^{int}(\alpha) = \frac{2|a_0| n - 1}{\beta_0^2} \frac{\pi}{n+1} \left( 1 - \alpha \coth(\alpha) \right) (s_+ + j_\varphi - \frac{2j_q}{n-1})^2,
\]
\[
M_{int}^2(\alpha) = M_g^{\alpha=2} \left( M_g^{\alpha=2} + 2m_0 \frac{E_{L-sol}^{int}(\alpha)}{|a_0|} \right) \quad (4.52)
\]

As in the case of the \( U(1) \) IM (1.2) (see Sect 3.5), the intermediate IM (2.16) admits together with the charged topological int-solitons of spectrum (4.51) (or (4.52)) string-like solutions given by eqn. (4.25), but with \( w_{top}(\bar{w}_{top}) \) (ref.4.37) replaced by
\[ w_{string} = w_{top} + w_{osc} \]

They carry the same topological and electric charges (and new \( Q_{\bar{k}}, \bar{Q}_{\bar{k}}, k \in Z \) (4.51), but their energy gets contributions from the oscillator part, i.e.,
\[ E_{int}^{string} = E_g + E_{L}^{string} + E_{R}^{string} \]

The question of whether such classical string solutions are topologically stable (and remain stable under quantization) is still open. The fact that they have the same topological numbers but higher energies, is however an indication for their topological instability.

5 Discussion and further developments

Among the vast variety of integrable perturbations of the gauged \( A_n \)-WZW models we have chosen to study the soliton solutions of a specific class of perturbations that preserve one or two chiral \( U(1) \) symmetries and whose potentials have \( n \)-distinct zeros. Our main tool in the construction of the 1-solitons of these models is the nonconformal version of the GKO coset construction [14], that allows us to compose these \( u \)-solitons in terms of massless solitons of the \( U(1) \) (or \( U(1) \otimes U(1) \))-CFT and already known \( g \)-solitons of the gauged \( G_0/U(1) \) (or \( G_0/U(1) \otimes U(1) \)) integrable models [3], [11], [23]. The new ingredients of this construction are the solitons and solitonic strings (left, right and left-right ) of the corresponding CFTs presented in Sect. 3 and 4. The simplest example \( (n=1) \) of \( U(1) \) IMs (1.2) written in the “free field” form
\[
L_{\alpha}^{n=1} = \partial \Phi \bar{\partial} \Phi + \beta \partial \gamma + \bar{\beta} \bar{\partial} \gamma - \beta \bar{\beta} e^{-2\Phi} - m^2 \gamma \bar{\gamma} e^{2\Phi} + 2a_0 \Phi R^{(2)} \sqrt{-g} \quad (5.1)
\]
(where $R^{(2)}$ is the worldsheet curvature of the 2-d metric $g_{\mu\nu}$, $g = \det g_{\mu\nu}$), is known to represent an integrable perturbation of string on $AdS_3$ target space. One expects that $AdS_3/CFT_2$ Maldacena correspondence \cite{7} takes place for the relevant deformed theories, i.e. the $AdS_3$ with perturbation $m^2 \gamma e^{2\Phi}$ to be equivalent to the deformed $CFT_2$ (i.e. 2-d integrable model) living on the border of the $AdS_3$ space. Since the energy $\tilde{E}$ and angular momenta $\tilde{l}$ of the latter theory are related to the charge $Q_0$ and $\tilde{Q}_0$ of the $AdS_3$ model (\ref{5.1}) \cite{43}, \cite{4},

\[ Q_0 = \frac{1}{2} (\tilde{E} + \tilde{l}), \quad \tilde{Q}_0 = \frac{1}{2} (\tilde{E} - \tilde{l}) \quad (5.2) \]

the charge spectrum of (\ref{5.1}) we have derived in Sect. 3 determines the energy spectrum of the border deformed $CFT_2$. One can consider the $n \geq 2$ IMs (\ref{1.2}) as specific integrable deformations of the bosonic string on $AdS_3 \otimes T_{n-1}$ target space which provides its charge spectrum (\ref{3.36}) and (\ref{3.52}) with certain string (and border $CFT_2$) meaning. More interesting examples are given by the IMs (\ref{2.12}) studied in Sect. 4. They can be considered as integrable relevant perturbations of the $AdS_3 \otimes S_3 \otimes T_{n-1}$ string model (taking $R_n \rightarrow iR_n$ and $\psi_n^* = \chi_n$). It is important to mention that the spectral flow (\ref{3.59}) of charges $Q_0, \tilde{Q}_0$ is realized by adding topological $\theta$-terms (\ref{3.55}) to the original deformed string Lagrangians (\ref{1.2}).

As it well known the standard procedure of restoring the conformal invariance of the deformed IM (\ref{5.1}) (and (\ref{1.2}) in general) is to consider their conformal affine versions \cite{17}, \cite{46}. By introducing a new pair of fields $(\nu, \eta)$ one can map the nonconformal IM (\ref{5.1}) to the following conformal integrable model

\[ \mathcal{L}_{CAT}^{n=1} = \partial \Phi \bar{\partial} \Phi + \beta \bar{\partial} \gamma + \beta \partial \bar{\gamma} + \partial \nu \bar{\partial} \eta + \partial \eta \bar{\partial} \nu - \beta \bar{\beta} e^{-2\Phi} - m^2 \gamma e^{2\Phi - \eta} + 2(\alpha_0 \Phi + \gamma_0 \nu) R^{(2)} \sqrt{-g} \quad (5.3) \]

where $\alpha_0$ and $\gamma_0$ are static background charges. An important feature of such IM is that the free field $\eta$ plays the role of renormalization group parameter that interpolate between different conformal backgrounds (i.e. the zeros of the corresponding $\sigma$-model $\beta$-functions):

- $\eta \rightarrow \infty$ leads back to the original conformal $SL(2, R)$-WZW model
- $\eta \rightarrow 0$ is reproducing the relevant perturbation (\ref{5.1})

The model (\ref{5.3}) admits new massless solitons characterized now by the topological charge

\[ Q_\eta = \beta_0 \int_{-\infty}^{\infty} \partial_x \eta dx \]

It is interesting to derive their complete energy and charge spectrum as well as to understand the string meaning of the CAT versions of the solitonic string solutions of (\ref{5.1}).

An interesting example of an off critical bosonic string on curved target space of black hole type \cite{1} is given by the $n = 2$ intermediate IM (\ref{2.16}) (with local $U(1)$ symmetry):

\[ \mathcal{L}_{n=2}^{int} = \frac{1}{3} \partial \bar{R} \partial R + \frac{1}{4} \sum_{a,b=1}^{2} f_{ab} \left( (1 + \frac{\beta^2}{4} \bar{\psi}_a \chi_a e^{\beta \bar{\sigma}_a R}) \partial \bar{X}_b \bar{\partial} \bar{\psi}_b e^{\beta \bar{\sigma}_b R} - \bar{\psi}_a \chi_a \bar{\partial} \bar{\psi}_b \partial X_b \right) \]

\[ \quad - \frac{1}{4} \left( \bar{\psi}_1 \chi_1 e^{\beta R} + \bar{\psi}_2 \chi_2 e^{-\beta R} + \beta^2 \bar{\psi}_1 \chi_1 \bar{\psi}_2 \chi_2 \right) \quad (5.4) \]
where $f_{aa} = 0$, $f_{12} = f_{21} = 1$, $\epsilon_1 = -\epsilon_2 = 1$. For $m^2 = 0$, the $\sigma$-model (5.4) is conformal invariant. Since corresponding target space metric admits both horizons and singularities, it represents a specific 5-d generalization of the 3-d black string model [4]. For $m^2 \neq 0$ the above nonconformal IM has $U(1)$-charged massive and massless solitons (and strings), constructed in Sect.4.3, that can be interpreted as charged strong coupling (stable) particles (and strings). Its CAT version (i.e. with $\nu$ and $\eta$ as in eqns. (5.3)) might play an important role in the understanding of the nonperturbative properties of the IM (5.4) (and of the corresponding string models) due to the fact that the massless solitons introduces new nonperturbative string states.

Another direction of extending the results of the present paper is to consider IMs with nonabelian local symmetries. The simplest example of integrable perturbation of the $SL(3, R)$-WZW model that preserve chiral $SL(2, R) \otimes U(1)$ symmetry was derived in our recent paper [23]

$$\mathcal{L}_{SL(3,R) \text{ pert}} = \mathcal{L}_{WZW}^{SL(3,R)}(g_0) - \frac{m^2}{\beta^2} \left( \frac{2}{3} + \psi_1 \chi_1 e^{\beta(\varphi_2 - \varphi_1)} + \psi_2 \chi_2 e^{\beta(\varphi_1 + \varphi_2)} \right)$$

(5.5)

where the $SL(3, R)$ group element is parametrized as follows:

$$g_0 = e^{\chi_3 E_{-a_1} e^{\chi_1 E_{-a_2} + \chi_2 E_{-a_1 - a_2} e^{\varphi_1 h_1 + \varphi_2 h_2} e^{\psi_1 E_{a_2} + \psi_2 E_{a_1 + a_2} e^{\psi_3 E_{a_1}}}}$$

By the methods of Sect. 2 one can construct integrable perturbations of $SL(n + 1)$-WZW model keeping unbroken certain subgroup $G_0^0 = SU(k), k \leq n$. We expect that the nonconformal GKO coset construction of Sect. 3 (for the particular cases $G_0^0 = U(1)$ or $G_0^0 = U(1) \otimes U(1)$) to take place for arbitrary nonabelian $G_0^0$ as well. As in the $U(1)$ case, the $G_0^0$-CFT massless solitons should be the main ingredient in the construction of the 1-solitons of such IM, as (5.5) for example. The derivation of their energy and charge spectrum (say for $G_0^0 = SL(2)$) is an interesting open problem.

Our last comment concerns the nonrelativistic analogs of IM (1.2). The question is whether one can construct nonrelativistic IMs with local $U(1)$ symmetry. The well known relation between Lund-Regge (L-R) IM ($n = 1$ of eqn. (1.3)) and the nonlinear Schroedinger model [14] and the fact that L-R model is the gauged version of the deformed $SL(2, R)$ WZW (5.1) address the question about the nonrelativistic counterpart (in the sense of ref. [24], [23]) of the IM (5.1). The answer to this question is given by the following integrable system of second order differential equations for the field variables $r(x, t), q(x, t)$ and $u(x, t)$:

$$\partial_t r - \frac{1}{2} \partial_x^2 r + 2u \partial_x r + r \partial_x^2 u - 2r(u^2 - rq) = 0,$$

$$\partial_t q + \frac{1}{2} \partial_x^2 q + 2u \partial_x q + q \partial_x^2 u + 2q(u^2 - rq) = 0,$$

$$\partial_t u + \frac{1}{2} \partial_x(rq) = 0$$

(5.6)

which is invariant under local $U(1)$ transformations:

$$u = u' + \frac{\beta}{2} \partial_x w(x), \quad r = e^{\beta w(x)} r', \quad q = e^{-\beta w(x)} q'$$

(5.7)
We expect that its solitons are related to the Non linear Schroedinger (NLS) solitons in the same way the u-solitons of the IM \([5, 1]\) are related to the L-R nontopological solitons. The generalization of the IM \([5, 6]\) to the family of nonrelativistic IMs with local \(U(1)\) symmetry related to the relativistic IMs \([1, 2]\) is straightforward.

**Acknowledgements** We are grateful to R. Paunov for discussions concerning integrable perturbations of \(SL(2,R)\) WZW model. We thank Fapesp, CNPq and Unesp for partial financial support.

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