Erratum: Reply to comment on ‘Poynting flux in the neighbourhood of a point charge in arbitrary motion and the radiative power losses’ (2018 *Eur. J. Phys.* 39 018002)

Ashok K Singal

Astronomy and Astrophysics Division, Physical Research Laboratory, Navrangpura, Ahmedabad, 380 009, India

E-mail: ashokkumar.singal@gmail.com

The following error was introduced during the production process. On pages 2, 3 and 6, the $\mathcal{P}$ symbol was incorrectly typeset as a $\rho$. This applies to the left-hand side of equations (3), (8), (23) and (24). The corrected equations are

\[ P_2 - P_1 = \frac{2e^2}{3c^3} \dot{\mathbf{v}} \cdot \mathbf{v} - \frac{2e^2}{3c^3} \mathbf{v} \cdot \ddot{\mathbf{v}} = -\frac{2e^2}{3c^3} \frac{d(\mathbf{v} \cdot \mathbf{v})}{dt}. \]  

(3)

\[ P = \frac{2e^2}{3r^2c} v_0^2. \]  

(8)

\[ P = \frac{2e^2}{3r^2c} (\mathbf{v} + \mathbf{v}_r/c)^2. \]  

(23)

\[ P = \frac{2e^2(\mathbf{v}_0 - \mathbf{v}_0r^2/2c^2)^2}{3r^2c} = \frac{2e^2v_0^2}{3r^2c} - \frac{2e^2v_0 \cdot \ddot{v}_0}{3c^3}. \]  

(24)
Reply to comment on ‘Poynting flux in the neighbourhood of a point charge in arbitrary motion and the radiative power losses’

Ashok K Singal

Astronomy and Astrophysics Division, Physical Research Laboratory, Navrangpura, Ahmedabad—380 009, India

E-mail: ashokkumar.singal@gmail.com

Received 25 August 2017
Accepted for publication 6 October 2017
Published 21 December 2017

Abstract
Doubts have been expressed in a comment about the tenability of the formulation for radiative losses in our recent published work (Singal 2016 Eur. J. Phys. 37 045210). We provide our reply to the comment.

This is a reply to comment by Rowland on my recently published work (Singal 2016 Eur. J. Phys. 37 045210). All points raised by Rowland are discussed below, though not in same order. Like Rowland, we shall also confine our discussion largely to non-relativistic motion, unless otherwise specified.

1. A difference in the physical interpretation of two power formulas

We begin by pointing out the difference in the physical interpretation of the two power formulas in question. First is the well-known Larmor’s formula, representing the power going into electromagnetic radiation from an accelerated charge

\[ P_1 = \frac{2e^2}{3c^3}v^2_{\text{rel}}. \]  

(1)

This formula has a text-book derivation [2–4] where Poynting flux through a spherical surface of large enough radius is computed from acceleration fields, assuming any contribution of velocity fields could be neglected. The latter condition is met in almost all cases, a notable exception being where the velocity of the accelerated charge may be a monotonic function of time, e.g., in the case of a uniformly accelerated charge [5]. Leaving apart any such peculiar cases, Larmor’s formula does represent electromagnetic radiation power which will be received by a set of distant observers stationed on a spherical surface of radius \( r \) around the
retarded position of the charge. Thereafter, in the standard text-book approach, one equates the Poynting flux at time $t$ to the kinetic energy loss rate of the charge at a retarded time $t - r/c$, purportedly using Poynting’s theorem of energy conservation. However, a fallacy lies in this particular step. Poynting’s theorem does not relate Poynting flux through a surface at some time $t$ to energy loss rate by the enclosed charge at a retarded time $t - r/c$. In fact most of the confusion in this hundred years old controversy has arisen due to this oversight. In Poynting’s theorem all quantities need to be calculated for the same instant of time [2–4]. Applying Poynting’s theorem correctly in terms of real time values of the charge motion [1], one gets the instantaneous rate of loss of the mechanical energy by the charge as

$$\mathcal{P}_2 = -\frac{2e^2}{3c^3} \mathbf{v} \cdot \mathbf{v}. \quad (2)$$

One should clearly distinguish between the electromagnetic power received by a set of far-off observers and the instantaneous loss of mechanical power by the charge. In literature both power rates are treated as not only equal but almost synonymous. However, the two need not be the same as seen from equations (1) and (2). The difference in the two power formulas is

$$\rho_2 - \rho_1 = -\frac{2e^2}{3c^3} \dot{\mathbf{v}} \cdot \mathbf{v} - \frac{2e^2}{3c^3} \ddot{\mathbf{v}} \cdot \mathbf{v} = \frac{2e^2}{3c^3} \frac{d(\mathbf{v} \cdot \mathbf{v})}{dt}. \quad (3)$$

The last term on the right-hand side in equation (3) is known as the Schott term, after Schott [6] who first pointed it out, and is thought in literature to arise from an acceleration-dependent energy in electromagnetic fields. The meaning of this elusive, century-old term is still being debated [7–11] and it does not seem to make an appearance elsewhere in physics. We shall later demonstrate that the Schott term arises as a consequence of not keeping a proper distinction between ‘real’ and ‘retarded’ times while calculating power losses for a radiating charge.

The rate of momentum being carried away by the electromagnetic radiation (due to its $\sin^2 \theta$ pattern) in the instantaneous rest frame of the charge is zero. However in the frame where the charge moves with a velocity $\mathbf{v}$, one gets [12, 13]

$$\dot{\mathbf{p}}_1 = \frac{\mathcal{P}_1}{c^2} \mathbf{v}. \quad (4)$$

Now contrary to the view expressed by Rowland in his comment (his equation (7) and the discussion following that), the negative of $\dot{\mathbf{p}}_1$ cannot be the radiation reaction on the charge, as an application of equation (4) along with equation (1) to a radiating synchrotron source case leads to results which are mutually inconsistent when compared in two different inertial reference frames [1].

On the other hand, the momentum conservation theorem, using Maxwell’s stress tensor, directly leads to a rate of change of momentum of the charge [14]

$$\dot{\mathbf{p}}_2 = \frac{2e^2}{3c^3} \mathbf{v}. \quad (5)$$

The result in equation (5), known as the Abraham–Lorentz radiation reaction formula, has been obtained earlier from the self-force of the charge, calculated albeit in a rather cumbersome manner [2, 3, 6, 15, 16].

It should be noted that equation (2), representing the rate of change in the mechanical energy of a charge, was not derived using the radiation drag force (equation (5)) as mentioned in the comment by Rowland after his equation (8), but was instead calculated directly from
Poynting flux when written in terms of real time values of the charge motion [1]. The result though does turn out to be consistent with the work being done against the drag force \( p_2 \) (equation (5)).

2. Absence of radiation from a uniformly accelerated charge

Using the vector identity \( \mathbf{v} = \mathbf{n} \times (\mathbf{n} \times \mathbf{v}) - \mathbf{n} \times \{ \mathbf{n} \times \mathbf{v} \} \), the transverse component of the electric field of an accelerated charge, moving with a non-relativistic velocity \( \mathbf{v} \) and an acceleration \( \mathbf{v} \) at the retarded time, can be written as

\[
\mathbf{E}_T = \frac{\mathbf{en} \times (\mathbf{n} \times \mathbf{v})}{c^2r^2} + \frac{\mathbf{en} \times (\mathbf{n} \times \mathbf{v})}{c^2r}.
\]

The conventional wisdom is that the acceleration fields \((a1/r)\) solely represent the radiation from a charge, since the contribution of the velocity fields \((\propto1/r^2)\) appears to be negligible for a large enough value of \( r \). However, in the case of a uniform acceleration, the *retarded value* of the velocity will be \( \mathbf{v} = \mathbf{v}_0 - \dot{\mathbf{v}}_r/c \), where \( \mathbf{v}_0 \) is the *present* velocity of the charge. Then equation (6) for the transverse component of the electric field becomes

\[
\mathbf{E}_T = \frac{\mathbf{en} \times (\mathbf{n} \times \mathbf{v})}{c^2r^2} - \frac{\mathbf{en} \times (\mathbf{n} \times \mathbf{v})}{c^2r} + \frac{\mathbf{en} \times (\mathbf{n} \times \mathbf{v})}{c^2r} = \frac{\mathbf{en} \times (\mathbf{n} \times \mathbf{v})}{c^2r},
\]

which for the Poynting flux begets

\[
\rho = \frac{2e^2}{3c^3} \mathbf{v}_0^2.
\]

The transverse component of the electric field here (equation (7)) is the same as would be that of a charge moving with a uniform velocity \( \mathbf{v}_0 \), equal to the 'present' velocity of the accelerated charge. It is clear that in the case of a uniformly accelerated charge, there is no term proportional to \( \dot{\mathbf{v}}^2 \), independent of \( r \), which is commonly called the radiated power. Nor is there any term proportional to \( \mathbf{v} \cdot \dot{\mathbf{v}} \), the Schott energy term, as suggested in equation (10) of Rowland. Instead, the Poynting flux in equation (8) is merely what would be for a charge moving with a uniform velocity \( \mathbf{v}_0 \), justifying its usage in equation (17) of [1].

3. No power losses from an instantaneously stationary charge

It is already evident from equations (7) that the transverse component of electric field in the instantaneous rest frame \( (\mathbf{v}_0 = 0) \) of a uniformly accelerated charge is nil, as the acceleration fields there get cancelled *neatly* by the transverse term of the velocity fields *at all distances*. Consequently there is a nil Poynting flux through any surface around such a charge (equations (8)), which is consistent with the assertion made in [1] that there is no radiation from an accelerated charge that is instantaneously stationary.

Now from Poynting’s theorem we shall explicitly demonstrate that there are no radiative losses from a uniformly accelerated charge in its instantaneous rest-frame. Let a charge moving with a uniform acceleration \( a \) along the +z axis, starting from \( z = -\infty \) at time \( t = -\infty \), momentarily comes to rest at a point \( z = \eta \) at time \( t = 0 \), and then onwards moves with an increasing velocity along the +z axis. Choosing the origin of the coordinate system so that \( \eta = c^2/a \), the position and velocity of the charge at any time \( t \) are given by \( z_0 = (\eta^2 + c^2t^2)^{1/2} \) and \( \mathbf{v} = c^2t/z_0 \). The electromagnetic fields in cylindrical coordinates \((\rho, \phi, z)\) at any instant \( t \) are given by [17]
where \( \xi = ([z_0^2 - z^2 - \rho^2]^2 + 4\eta^2\rho^2)^{1/2} \). All other field components are zero. We shall restrict all discussions to a region \( z + ct > 0 \) because it is only within this region that the light signals from the retarded positions of the charge could have reached [17].

The first thing we notice from equation (9) is that at time \( t = 0 \), when the charge has come to rest momentarily at \( z_0 = \eta \), the magnetic field is zero throughout. Therefore the Poynting flux from any closed surface \( \Sigma \) surrounding the charge will be zero.

\[
\int_{\Sigma} d\Sigma \left( \mathbf{n} \cdot \mathbf{S} \right) = 0.
\]

Further, the field energy density, \( (E^2 + B^2)/8\pi \), equal at times \( t \) and \( -t \), is an even function of \( t \). Therefore the electromagnetic field energy in the volume enclosed by \( \Sigma \)

\[
\mathcal{E}_{\text{em}} = \int \frac{E^2 + B^2}{8\pi} \, dv,
\]

is also an even function of \( t \), implying

\[
\frac{d\mathcal{E}_{\text{em}}}{dt} = 0,
\]

at \( t = 0 \). Then from the Poynting’s theorem, the rate of change of the mechanical energy \( (\mathcal{E}_{\text{me}}) \) of the charge at \( t = 0 \) given by

\[
\frac{d\mathcal{E}_{\text{me}}}{dt} = -\frac{d\mathcal{E}_{\text{em}}}{dt} - \int_{\Sigma} d\Sigma \left( \mathbf{n} \cdot \mathbf{S} \right) = 0,
\]

where all quantities are evaluated at the same time, \( t = 0 \). This immediately implies no power losses by the instantly stationary charge. Here we see no sign of the acceleration-dependent Schott term \( \propto d(\mathbf{v} \cdot \mathbf{v})/dt \), mentioned in the comment by Rowland which is supposed to make the net power loss from a uniformly accelerated charge zero even when there is radiated power as per Larmor’s formula (the latter is also not seen here).

4. Schott energy term—a difference in the self-field energy of an accelerated charge between retarded and present time

In order to better apprehend the difference between equations (1) and (2) and their relations with retarded and real times, we consider the effect of the self-force of an accelerated charge on itself. For this we take the charge to be a spherical shell of a small radius \( \epsilon \), in order to avoid divergence of fields at the centre of a point charge, though the final results turn out to be independent of the radius assumed for the sphere. Force on each infinitesimal element of the spherical shell is calculated due to the time-retarded fields from the remainder parts of the charged shell and then total force on the charge is calculated by integrating over the whole shell.

It has been shown [18] that for an accelerated charge, there is a net self-force proportional to the acceleration which is as if due to the time-retarded fields of a co-moving, equivalent point charge at the centre [19]. Thus the charged spherical shell experiences a force proportional to the acceleration it had at a time interval \( \tau = \epsilon/c \) earlier, because of fields from the centre delayed due to the finite speed \( c \) of propagation. Effectively the self-force on the charge at time \( t \) is then proportional to the acceleration it had at a retarded time \( t_0 = t - \epsilon/c \)
\[ f_i = -\frac{2e^2}{3\epsilon c^2} [\dot{v}]_{ret}, \]  
(14)

where a square bracket denotes a retarded-time value \[ [\cdot] \]. Accordingly, for an accelerated charge, the power loss during work done against self-force of the charge is given by

\[ P_i = -f_i \cdot v = \frac{2e^2}{3\epsilon c^2} [\dot{v}]_{ret} \cdot v. \]  
(15)

Now if we write the velocity too in terms of its value at the retarded time \[ t_0 \] (to a first order in \( \epsilon/c \))

\[ v = [v]_{ret} + \epsilon/c \]  
(16)

we get

\[ P_i = \frac{2e^2}{3\epsilon c^2} [\dot{v}]_{ret} + \frac{2e^2}{3\epsilon c^3} [\ddot{v}]_{ret}. \]  
(17)

On the right-hand side, the first term shows the rate of change of self-field energy of the accelerated charge due to its changing momentum \textit{at the retarded time}, and the second term comprises Larmor’s formula, again evaluated at the retarded time.

However if we instead express the acceleration itself in terms of its real-time value at \( t = t_0 + \tau \)

\[ [\dot{v}]_{ret} = \dot{v} - \dot{v}\tau + \cdots, \]  
(18)

then the self-force (equation (14)) can be written in terms of real-time values as

\[ f_i = -\frac{2e^2}{3\epsilon c^2} \dot{v} + \frac{2e^2}{3\epsilon c^3} \ddot{v}, \]  
(19)

and the corresponding formula for power (equation (15)), in terms of real time, is written as

\[ P_i = \frac{2e^2}{3\epsilon c^2} (\dot{v} \cdot v)_{ret} - \frac{2e^2}{3\epsilon c^3} (\ddot{v} \cdot v)_{ret}. \]  
(20)

Now the rate of change of energy in self-fields between retarded and real times (the leading terms on the right-hand sides of equations (17) and (20)), differs by

\[ \frac{2e^2}{3\epsilon c^2} [\dot{v} \cdot v]_{ret} - \frac{2e^2}{3\epsilon c^2} (\dot{v} \cdot v)_{ret} = -\frac{2e^2}{3\epsilon c^3} \frac{d(\dot{v} \cdot v)}{dt} \tau = -\frac{2e^2}{3\epsilon c^3} \frac{d(\dot{v} \cdot v)}{dt}, \]  
(21)

a result independent of \( \epsilon \). This explains the genesis of the mysterious Schott term and makes it obvious that this term, hitherto thought to arise from some acceleration-dependent energy in fields, is actually nothing but the difference in rate of change of energy in self-fields of the charge between retarded and present times.

5. Energy being radiated at a negative rate?

Rowland in his comment has shown through a specific example, where acceleration of a charge is increasing in its direction of motion, equation (2) yields a negative value, apparently implying energy being radiated at a negative rate. Actually in such cases it is not that the Poynting flux is negative (or inward) but that the rate of change of the kinetic energy of the charge is more than what should be expected from the Poynting flux calculated from its motion at the retarded time. We can elucidate it in the following manner.
From (equation (6)), we can write the transverse component of the electric field as

\[ E_T = \frac{en \times [n \times (v + vr/c)]}{cr^2}, \]  

with \( v \) and \( \dot{v} \) being the (non-relativistic) velocity and acceleration at the retarded time. Then for the Poynting flux we get

\[ \rho = \frac{2e^2}{3r^2c} (v + vr/c)^2. \]  

The expression \((v + vr/c)\) represents the extrapolated ‘present’ velocity \(v_0\) of the charge, assuming the acceleration \(\dot{v}\) remained constant during the time interval \(\tau = r/c\). The Poynting flux through a spherical surface centred on the retarded position of the charge is always given in accordance with (23), i.e., \(\propto v_0^2/r^2\). This is true for all \(r\), including in the neighbourhood of the charge.

On the other hand, the Poynting flux through a spherical surface in the neighbourhood of the charge is also written as [1]

\[ \rho = \frac{2e^2(v_0 - \dot{v}r^2/2c^2)^2}{3r^2c} = \frac{2e^2v_0^2}{3r^2c} - \frac{2e^2v_0^2 \cdot \dot{v}_0}{3c^3}. \]  

A comparison of equations (23) and (24) shows that \(\mathcal{P}_2\) (equations (2)) will be a negative quantity when \(v_0^2 > v_\infty^2\), as in the example by Rowland. On the other hand \(\mathcal{P}_2 > 0\) when \(v_0^2 < v_\infty^2\), as happens in vast majority of cases, for instance in a circular motion or in an oscillatory motion where displace of charge from its equilibrium position gives rise to a restoring force proportional to the displacement. In no case is the Poynting flux through the spherical surface centred on the retarded position of the charge negative or inward (see equations (23) and (24)).

It may be worth pointing out here that the Poynting flux through the spherical surface centred on the ‘present’ position of the charge could surprisingly be negative even though it is always positive through the surface centred on the retarded position of the charge. For example, in the case of a uniform acceleration, there is always an outward flow of Poynting flux through a spherical surface centred on the retarded position of the charge (equation (8)). However, in our chosen coordinate system in section 3, the charge occupies the same position \(z_0\) at times \(-t\) and \(t\). We can choose a fixed finite closed surface, say, a sphere centred at \(z_0\). The Poynting vector, at any point on the surface, at time \(t\) is exactly equal in magnitude but opposite in direction to its value at time \(-t\). This is easily seen from equation (9), where the electric field vector is independent of sign of \(t\) while the magnetic field changes sign with \(t\). Thus if there is an outward flow of Poynting flux through the surface at time \(t\), it immediately follows that there was an equal inward flow of Poynting flux through that surface at time \(-t\). Of course an even simpler example is that of a charge moving with a uniform velocity \(v_0\), where there is always an outward flow of Poynting flux through a spherical surface centred on the retarded position of the uniformly moving charge (equation (8)), but at the same time we know that the electric field of a charge moving with a uniform velocity is radial with respect to its ‘present’ position and therefore there is a nil Poynting flux through a surface centred on the present position of the charge.

**ORCID iDs**

Ashok K Singal @ https://orcid.org/0000-0002-8479-7656
References

[1] Singal A K 2016 Poynting flux in the neighbourhood of a point charge in arbitrary motion and radiative power losses Eur. J. Phys. 37 045210
[2] Jackson J D 1975 Classical Electrodynamics 2nd edn (New York: Wiley)
[3] Panofsky W K H and Phillips M 1962 Classical Electricity and Magnetism 2nd edn (Boston, MA: Addison-Wesley)
[4] Griffiths D J 1999 Introduction to electrodynamics 3rd edn (Upper Saddle River, NJ: Prentice)
[5] Singal A K 1997 The equivalence principle and an electric charge in a gravitational field: II. A uniformly accelerated charge does not radiate Gen. Relativ. Gravit. 29 1371–90
[6] Schott G A 1912 Electromagnetic Radiation (Cambridge: Cambridge University Press)
[7] Teitelboim C 1970 Splitting of the Maxwell tensor: radiation reaction without advanced fields Phys. Rev. D 1 1572–82
[8] Heras J A and O’Connell R F 2006 Generalization of the Schott energy in electrodynamic radiation theory Am. J. Phys. 74 150–3
[9] Grøn Ø 2011 The significance of the Schott energy for energy-momentum conservation of a radiating charge obeying the Lorentz–Abraham–Dirac equation Am. J. Phys. 79 115–22
[10] Hammond R T 2010 Relativistic particle motion and radiation reaction in electrodynamics El. J. Theor. Phys. 23 221–58
[11] Rowland D R 2010 Physical interpretation of the Schott energy of an accelerating point charge and the question of whether a uniformly accelerated charge radiates Eur. J. Phys. 31 1037–51
[12] Rohrlich F 1997 The dynamics of a charged sphere and the electron Am. J. Phys. 65 1051–6
[13] Hartemann F V and Luhmann N C Jr 1995 Classical electrodynamical derivation of the radiation damping force Phy. Rev. Lett. 74 1107–10
[14] Singal A K 2017 Radiation reaction from electromagnetic fields in the neighborhood of a point charge Am. J. Phys. 85 202–6
[15] Lorentz H A 1909 The Theory of Electron (Leipzig: Teubner) reprinted 1952 2nd edn (New York: Dover)
[16] Yaghjian A D 2006 Relativistic Dynamics of a Charged Sphere 2nd edn (New York: Springer)
[17] Fulton T and Rohrlich F 1960 Classical radiation from a uniformly accelerated charge Ann. Phys. 9 499–517
[18] Singal A K 2016 Compatibility of Larmor’s formula with radiation reaction for an accelerated charge Found. Phys. 46 554–74
[19] Templin J D 1998 An approximate method for the direct calculation of radiation reaction Am. J. Phys. 66 403–9