Sequential Voting with Confirmation Network

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July 12, 2018

Abstract
We discuss voting scenarios in which the set of voters (agents) and the set of alternatives are the same; that is, voters select a single representative from among themselves. Such a scenario happens, for instance, when a committee selects a chairperson, or when peer researchers select a prize winner. Our model assumes that each voter either renders worthy (confirms) or unworthy any other agent. We further assume that the prime goal of any agent is to be selected himself. Only if that is not feasible, will he try to get one of those he confirms selected. In this paper we investigate the open-sequential ballot system in the above model. We consider both plurality (where each voter has one vote) and approval (where a voter may vote for any subset). Our results show that it is possible to find scenarios in which the selected agent is much less popular than the optimal (most popular) agent. We prove, however, that in the case of approval voting, the ratio between their popularity is always bounded from above by 2. In the case of plurality voting, we show that there are cases in which some of the equilibria give an unbounded ratio, but there always exists at least one equilibrium with ratio 2 at most.

1 Introduction
Consider a committee voting to select a chairperson. Each committee member would like the honour of serving as chairperson himself. As second best option he prefers one of several other members to win the position. The committee members’ preference profile can be represented by a simple directed graph, as for example in Figure 1, which we refer to as a confirmation network.

In the confirmation network of Figure 1 member #5 has the most political power being the most popular member—he is supported by three other members, while everyone else is supported by one at most.

Assume that the confirmation network is known to all, and that the ballot is conducted sequentially in a prefixed order.1 There are two possible voting methods: plurality, in which each voter selects only one other member, and approval, in which members vote for any subset.

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1This project has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement n 740435).

1E.g., by sitting order.
of the other members. In either voting method, a member is not allowed to vote for himself, but he is allowed to abstain. The member with the most votes wins and is elected. Ties are broken by a predetermined and publically known preference order.

Game-theoretically, we have a multi-stage game, describable as an extensive-form game — a tree with all possible voting-sequences, and an outcome at every leaf. The standard solution for this kind of game is a subgame perfect equilibrium (SPE). To find an SPE, we start with the last voter, and for any possible voting history, we assume this voter will choose a ballot which gives him a best outcome (notice that there may be more than one ‘best outcome’). Moving to the next-to-last voter, we know, for any voting history, how the last voter will respond to any of his ballots. Thus, we can find his best votes for any sequence of voting history. We can continue this backward reasoning until we find the best votes for all voters in any scenario. A subgame perfect equilibrium outcome is a result which can be achieved when everyone chooses one of their best votes.

We exemplify the model and its complexity in two scenarios. Example 1.1 shows a case in which the most popular member is not elected in the unique SPE of plurality voting. The same scenario with approval voting leads to two different SPEs—in one of them the most popular member is the winner. In the network of Example 1.2 each member confirms at most one other member. Later (Proposition 2.2) we will see that under this condition the outcome is always ‘almost-optimal’. Nevertheless, Example 1.2 shows that the outcome is not trivial: not only is the outcome different between plurality and approval voting, but one of the members manages to get a better result in approval by voting for someone he does not confirm.

**Example 1.1.** Assume that in the network of Figure 1 the voting order is lexicographic, and so is the tie-breaking order. We will show that in this case we have a unique SPE for plurality voting, and a different unique SPE for approval voting. *If the voting method is plurality*, we claim that member #1’s best vote is to abstain. This will place member #2 in a dilemma: voting for #5 will allow #3 to abstain and get elected by the tie-breaking rule. Member #2 will then opt to vote for #1, resulting ultimately in the election of #1, even though #5, is the most popular member having the most confirmations. Now, *if approval is the voting method and #1 abstains, member #2 may vote just for #1 and get him elected as before. He may also vote for both #1 and #5, in which case member #3 still has no chance of being elected and will vote for #5, and #5, the most popular member, will win and be elected. We see that both ballots of #2 lead to an outcome which he confirms, hence both are ‘best-votes’. Later, when we formalize the model (1.2), we add a ‘truth-bias’ assumption which states that each member prefers the*
vote which is closest to his true confirmation set. Under this assumption member #2 favours the vote \{#1, #5\} over just \{#1\}. In this case, #1 does not gain from abstaining; thus, using the ‘truth-bias’ assumption once more, we get that #1’s best vote is to be truthful (i.e. vote for #5). Everyone else will be truthful as well, and #5 will be elected.

**Example 1.2.** Figure 2 shows a network with four voters and at most one confirmation (out-going edge) for each voter. The voting order and the tie-breaking order are both lexicographic. In plurality voting, both #1 and #2 are truthful, and #3 is elected after abstaining. However, in approval, #1 can achieve a better result by voting for both #4 and #2. Since #2 precedes #3 in the tie-breaking order, #3 cannot be elected and will now vote for #4. Thus, in approval voting, #4 is elected.

![Figure 2: Four committee members and their confirmations.](image)

1.1 Related work

Voting systems and their limitations have been long studied as part of the broader field of computational social choice (see the recent handbook, [5]). The classical voting model assumes that the sets of voters and alternatives are disjoint, and that each voter has a totally ordered preference over the alternatives. Sequential voting with this model has been studied before and showed to contain surprising ‘paradoxes’. In [7], Desmedt and Elkind considered both simultaneous and sequential plurality voting. They showed that a sequential voting system with at least three alternatives is prone to strategic voting, which might lead to an unexpected outcome, such as a Condorcet winner who does not win the election. Conitzer and Xia ([6]) further exemplified this phenomenon in a wide range of sequential voting systems, characterized by their domination index.

A confirmation network as an underlying model for simultaneous voting has also been studied. Holzman and Moulin ([9]) took an axiomatic approach to show the possibilities and limitations of such electoral systems. The main requirement of such systems, in their paper, is that no voter will be able to manipulate the system to select him by delivering a dishonest, strategic ballot. Alon et al. ([1]) investigated the same model, and showed the impossibility of incentive-compatible (that is, ‘non-manipulable’), deterministic voting systems. They suggested a probabilistic system with a bounded ratio between the maximal in-degree and the expected in-degree of the elected agent. Further works with the same theme can be found in [3, 8, 2].

In this paper, we discuss for the first time sequential voting with the underlying model of a confirmation network.
1.2 The model

Let $A = [n]$ be a set of agents. Their preferences are described by a directed graph $G(A, E)$ where the interpretation of $(x, y) \in E$ is that agent $x$ confirms agent $2y$. The agents vote sequentially in lexicographic order. We consider two voting rules: plurality, where each agent is allowed to vote for at most one other agent, and approval, in which each agent may vote for any subset of the other agents (abstentions are allowed). The winner of the ballot is the one who receives the most votes; ties are broken according to some prefixed ordering, which is also known to the voters. To ease the notation, whenever we refer to a graph, we assume it includes a voting order and a tie-breaking order. The utility of agent $x$ from the result $y$ is

$$U_x(y) = \begin{cases} 
1, & y = x \\
1/2, & (x, y) \in E \\
0, & \text{otherwise.}
\end{cases}$$

We are interested in voting strategies that form a subgame perfect equilibrium (SPE). At least one SPE is guaranteed to exist [12]. However, if in some subgame more than one ‘best vote’ option is available to some agents, multiple SPEs exist, possibly with different outcomes. Such a situation can occur, for instance, when an agent does not confirm anyone and is also not confirmed by any other agent (i.e., the agent is an ‘isolated’ node). If many agents are isolated, and so indifferent to the outcome, they will each make an arbitrary vote and anyone may be elected.

To avoid this problem, we take the same approach as in [11] by assuming the agents are truth-biased. Namely, when an agent faces several best-votes, he will select the one which best reflects his true confirmations. In order to quantify the truth-bias assumption, we add the following bonus utility. Let $f(x)$ be the number of agents that $x$ confirms and actually votes for, and let $g(x)$ be the number of agents he does not confirm and nevertheless votes for. Then his bonus utility is

$$B_x = \epsilon^2 \cdot f(x) - \epsilon \cdot g(x),$$

where $\epsilon < 1/2n$ is arbitrary. When the result is $y$, the actual utility of agent $x$ is given by $U_x(y) + B_x$.

1.3 Definitions and notations

We will use the following notations from graph theory. For $a \in A$ let $d(a) = d_{in}(a) := \#\{b \in A : (b, a) \in E\}$ be the popularity of $a$. We denote by $\Delta_{in}(G) := \max_{a \in V} d(a)$ the maximum in-degree. Similarly, $\Delta_{out}(G)$ is the maximum out-degree. An agent $m$ is most popular if $d(m) = \Delta_{in}(G)$.

Our goal in this paper is to find conditions in which sequential voting with a confirmation network gives a ‘reasonable’ outcome and to exemplify cases in which strategic voting leads to an

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2 General social choice settings assume that each agent has ordinal preferences over all possible alternatives (i.e., an elected agent in our case). In such general settings it is arguable how to measure the quality of the elected agent. Thus, we restrict attention to a simplified setting where the ordinal preference of each agent has only three levels: each agent prefers himself, those he confirms are second, and those he does not confirm are last. In such a simplified setting we have a natural measure for the quality of the elected agent: the number of incoming edges. As we saw in Examples 1.1 and 1.2, and will see in the results, even in this simplified setting the strategic analysis is quite involved.

3 There is nothing particular about this function; any three-level function will do. Actually, $U$ will not be explicitly used in the remainder of the paper.
unexpected result. An optimal outcome is when the most popular agent is elected. Let \( W \subseteq A \) be the set of all winners in any SPE. We define the following measures to appraise an outcome \( w \in W \). Let \( G_{-E_w} \) be the graph we get from \( G \) after removing all the out-edges of \( w \). Let

\[
D(w) = \Delta_{in}(G_{-E_w}) - d(w), \quad R(w) = \frac{\Delta_{in}(G_{-E_w})}{d(w)}
\]

be the additive gap and the multiplicative ratio, respectively, between the most popular agent and \( w \). We have two justifications for defining these measures on \( G_{-E_w} \) and not directly on \( G \). The first is philosophical: we do not want \( w \)'s own confirmations to influence the way he is measured\(^4\). The second is mathematical elegance. We pay a small price in the definitions in order to get clearer theorems. It is obvious, though, that \( \Delta_{in}(G_{-E_w}) \geq \Delta_{in}(G) - 1 \); thus it makes little difference, especially for large values of \( \Delta_{in} \).

With a slight abuse of notation, we define for any graph \( G \) with plurality/approval voting, \( D(G) = \min_{w \in W} D(w) \); for either plurality/approval let\(^5\) \( D = \sup G D(G) \). We shall promptly see that \( D \) is unbounded for both plurality and approval voting. In order to give a better description of the limitations of the two voting methods, and to differentiate between them, we would like to chart the asymptotic bounds of the multiplicative ratio when\(^6\)\(^7\) \( D \to \infty \). To that end, we define for a graph \( G \),

\[
\underline{\mathcal{R}}(G) = \min_{w \in W} \mathcal{R}(w), \quad \bar{\mathcal{R}}(G) = \max_{w \in W} \mathcal{R}(w),
\]

the maximal/minimal multiplicative ratio between the most popular agent and the winners. For any positive integer \( k \), we denote by \( G_k \) the family of graphs with \( D(G) \geq k \), and define\(^8\)\(^9\)

\[
\underline{\mathcal{R}} = \lim_{k \to \infty} \sup_{G \in G_k} \mathcal{R}(G), \quad \bar{\mathcal{R}} = \lim_{k \to \infty} \sup_{G \in G_k} \bar{\mathcal{R}}(G).
\]

### 1.4 Our results

Although the confirmation-network model has been previously presented and discussed (see related work above), we analyse it here for the first time as groundwork for sequential voting. We examine both plurality and approval ballots. Our goal is to show the limitations, as well as to find bounds for the ‘price’ of these systems. In Section 2, we show a sharp transition of the additive gap. In networks where each agent confirms at most one other agent (i.e., the maximum out-degree is one) there is a unique outcome, and \( D(G) \) is always zero. However, already for networks where agents confirm at most two other agents (i.e., \( \Delta_{out} = 2 \)), \( D \) is unbounded. In Section 3, we prove bounds on \( \mathcal{R} \) and \( \bar{\mathcal{R}} \). For approval voting we show that \( 1.5 \leq \mathcal{R} \leq \bar{\mathcal{R}} \leq 2 \). Whereas, for plurality voting we show that \( \mathcal{R} \leq 2 \) and \( \bar{\mathcal{R}} = \infty \). These results indicate that in

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\(^4\)This argument relates to the notion of ‘incentive compatibility’ which is central in \([1]\) and \([9]\).

\(^5\)For ease of notation we did not add a subscript to distinguish between \( D \) of plurality and \( D \) of approval. The results are the same for both anyway.

\(^6\)Note that if we define \( \mathcal{R} \) without this asymptotic then it will be predominated by graphs with a constant number of nodes and low values of \( D \).

\(^7\)Alternatively, we can take the asymptotic with respect to \( \Delta_{in} \to \infty \). The results will be the same.

\(^8\)Again we do not have different notations for plurality and approval. It will be clear from the context to which of the two we refer.

\(^9\)\( \underline{\mathcal{R}} \) and \( \bar{\mathcal{R}} \) are analogue to the ‘price of stability’ and the ‘price of anarchy’, respectively (see \([10]\) Section 17.1.3).
worst-case scenarios approval voting succeeds in selecting more popular agents than plurality voting. In Section 4 we sketch a generalization of our results to a $k$-approval voting method. We wrap up with a discussion and open problems in Section 5.

2 Bounds on the additive gap

We start our discussion with the special case of graphs with a maximum out-degree of one (i.e. each agent confirms at most one other agent). In this case, we show that both plurality and approval voting have a unique winner in any SPE, and the winner is a most-popular agent or almost most-popular agent. In approval voting, a vote of an agent to all of his confirmed agents is called truthful. In plurality voting, a vote of an agent to one of his confirmed agents (or abstention if he does not confirm anyone) is called truthful. Our proof is based on the following observation, which is a consequence of our truth-bias assumption.

Observation 2.1. For every SPE with the winner $w$, any agent who does not confirm $w$ is truthful in the SPE path.

The reason is that the election of $w$ is a worse outcome for any agent who does not confirm $w$; hence, being truthful is the only best vote for such an agent.

Proposition 2.2. For the class of graphs with a maximum out-degree of one, both plurality and approval have a unique outcome, and for both plurality and approval $D = 0$.

Before we prove this proposition, let us exemplify it in the scenario of Example 1.2. The network in that example has for every node at most one out-edge, so the proposition applies. Indeed, we showed there that both plurality and approval have a unique SPE. In addition, in approval voting agent #4 is elected, and he is the most popular. In plurality, agent #3 is elected; notice, that if we remove his out-edge to #4, then he becomes one of the most popular agents.

Proof. We start by showing that the outcome is unique using backward induction. Given a subgame (i.e., a history of votes) if an agent has a vote which gets him elected, then this will be the outcome. Moreover, if he cannot get elected but he can get the one he confirms elected, then this would be the outcome (here we use the assumption that he only confirms one agent). If he cannot get elected and cannot get the one he confirms elected, by Observation 2.1 his unique best action is to be truthful, and by induction the outcome is determined uniquely.

Now fix an SPE. Let $w$ be the winner of this SPE and let $m \neq w$ be one of the most popular agents. We denote by

$$C_W := \{v \in V : (v, w) \in E\}; \; C_M := \{v \in V : (v, m) \in E\};$$

the set of agents which confirm $w$ and those which confirm $m$, respectively. By our assumption on the out-degree, $C_M \cap C_W = \emptyset$. Thus, by Observation 2.1 the agents in $C_M \setminus \{w\}$ are truthful. So $m$ gets the votes of all those who confirm him, except perhaps the vote of $w$. Again by Observation 2.1, no agent in $V \setminus C_W$ votes for $w$, which means that $w$ cannot get more votes than his in-degree. Since $w$ is elected, we reach the conclusion that

$$|C_W| \geq |C_M| - 1_{(w, m) \in E},$$

\[\text{\textsuperscript{10}}\text{Meaning, that if we ignore his own confirmations, the elected agent is most popular.}\]

\[\text{\textsuperscript{11}}\text{In fact, this simple observation holds even for a wider solution concept of Nash equilibria.}\]

\[\text{\textsuperscript{12}}\text{Though it might be different between the two, see Example 1.2.}\]
and that is exactly the same as $D = 0$. 

The proof of Proposition 2.2 can be generalized to subgames in which the remaining voters confirm at most one agent. Suppose we are in the middle of a voting process with graph $G$. Let $U \subset V$ be the voters who have not yet voted and let $G'$ be the graph we get from $G$ after removing the out-edges of vertices in $V \setminus U$. Let $\bar{s} = (s_1, \ldots, s_n)$ be the current scoring vector\textsuperscript{13}. We define the potential of a vertex $v \in V$ in this subgame to be $\rho(v) = d_{in}(v, G') + s(v)$, where $d_{in}(v, G')$ is the in-degree of $v$ in $G'$\textsuperscript{14}. Let $P = \max_{v \in V} \rho(v)$.

**Proposition 2.3.** Using the definitions above, if $\Delta_{out}(G') \leq 1$ then there is a unique SPE for the remaining voting process; if $w$ is the winner of this process and $m$ is any agent with $\rho(m) = P$, then

$$ P - \rho(w) \leq 1_{(w, m) \in E(G')}.$$ 

We omit the proof which is very similar to that of Proposition 2.2. Proposition 2.3 will be used in the proof of Proposition 2.4.

In contrast to Proposition 2.2, we will now show that even for graphs with a maximum out-degree of two, $D$ is no longer bounded. In the proof, we will show a voting scenario in which voters who confirm both a very popular agent and a much less popular one are forced to vote only for the less popular.

**Proposition 2.4.** For the class of graphs with a maximum out-degree of at least two, $D$ is unbounded, for both plurality and approval.

**Proof.** We will build a series of graphs, $\{G_k\}$, such that $\forall k, \Delta_{out}(G_k) = 2$, and there is a voting order and a tie-breaking order under which both plurality and approval voting have a unique outcome $w$, with $d(w) = \Delta_{in}(G_k) - k$. Figure 3 depicts the graph $G_k$. The agents in $B$ and $D$ are classified by their types (the number of agents in each type is denoted below its circle). The voting starts with the agents in $D$ by lexicographic order of their type, then agents in $C$ by reverse lexicographic order and finally the agents in $B$. The tie-breaking rule orders the agents in $C$ in reverse lexicographic order, $c_{k+1} \succ \ldots \succ c_1$.

Notice that by Observation 2.1, the winner in any SPE must be from $C$; otherwise we will have a winner which got no votes and which is not ranked first in the tie-breaking order. Suppose we are in the subgame which starts right after the votes of all the voters in $D$. Since all the remaining voters have at most one out-edge, according to Proposition 2.3 the winner must be an agent which will have the highest potential if he abstains. Since any agents in $C$ do not have in-edges from agents in $C$ which vote after them, we can conclude that the winner will be the first agent which will have the highest potential if he abstains.

Now, the agents in $D$ confirm both $c_{k+1}$ and one other agent. The point will be that the only best vote for these agents is to vote only for $c_{k+1}$. Before proving the general case, we demonstrate this phenomenon in the simplest case, when $k = 2$ (Figure 4). Here, if the agents of type $d_1$ give $c_1$ at least one vote (e.g. if one votes for $c_3$ and the other for $\{c_1, c_3\}$), then $c_3$ cannot be elected (since after abstaining his total votes will be two and $c_1$’s potential is at least three). Therefore $c_3$ is truthful and $c_2$ abstains and wins (he will have two votes from $B$ and one from $c_3$; agent $c_1$ will have three votes as well but $c_2$ precedes him in the tie-breaking order).

\textsuperscript{13}That is, $s_i$ is the number of votes agent $i$ received from the voters in $V \setminus U$.

\textsuperscript{14}In other words, $\rho(v)$ is the maximum number of votes $v$ can hope to reach when the voting is done.
Figure 3: The graph $G_k$. Agent $c_1$ has popularity $\frac{1}{2}k(k + 3) - 1$ while the winner, $c_{k+1}$, has popularity $\frac{1}{2}k(k + 1) - 1$.

This result is unfavourable for the agents of $d_1$. However, if the agents $d_1$ vote only for $c_3$, then $c_3$ can now abstain; having the same potential as $c_1$ and $c_2$, agent $c_3$ wins by a tie-breaking.

Figure 4: The graph $G_2$.

Turning to the general case, assume first that all the agents in $d_i$, $1 \leq i \leq k - 1$ vote for $c_i$, and perhaps also for $c_{k+1}$ (in case of approval voting). Since after the votings of $D$, the agents $c_3, \ldots, c_{k+1}$ all have a lower potential than $c_1$, while $c_2$ can abstain and have the highest potential (remember that the tie-breaking rule prefers $c_2$ over $c_1$), $c_2$ must be the winner. This outcome is unfavourable for the voters of $d_1$. We claim that a better vote for them is to vote only for $c_{k+1}$, since that leads to the election of $c_{k+1}$. Indeed, if now all the voters of $d_i$,

\footnote{To be more precise: each voter in $d_1$ considers the situation in his turn. If all the voters before him voted}
$2 \leq i \leq k - 1$ vote for $c_i$, then now $c_1, c_4, \ldots, c_{k+1}$ cannot be elected since $c_2$ will have a higher potential than theirs. However $c_3$ can abstain and win by tie-breaking. Thus, the agents of $d_2$ are now dissatisfied. If they now all vote just for $c_{k+1}$ the same reasoning continues and shows that now $c_4$ will be the winner unless $d_3$ all vote just for $c_{k+1}$. Eventually, if all the voters of $D$ vote just for $c_{k+1}$ and $c_{k+1}$ abstains, he will get elected. The agents of $D$ are all satisfied with this result, which shows that this is an equilibrium. Indeed, our reasoning shows that this is the only equilibrium for both plurality and approval. The difference between the popularity of the winner, $c_{k+1}$, and the most popular, $c_1$, is $k$, which implies the claim of the proposition.  

3 \quad \text{Bounds on the multiplicative ratio}

In Section 2 we showed that, as much as we can tell from the additive gap measure, both plurality and approval voting systems perform poorly: the popularity of the elected candidate may be unboundedly far from that of the most popular. This raises the question whether a constant fraction of popularity can be achieved in sequential voting. We shall see in Theorems 3.1 and 3.2, that the bounds of the multiplicative ratio are non-trivial and are quite different between plurality and approval voting.

**Theorem 3.1.** In plurality voting, $\mathcal{R} \leq 2$ and $\overline{\mathcal{R}}$ is unbounded.

**Proof.** We shall first prove that $\overline{\mathcal{R}}$ is unbounded. We show a series of graphs and SPEs, such that the ratio between the most popular agent and the winner goes to infinity. To this end, consider the graph in Figure 5. Suppose the voting order is: $d_1, d_2, d_3$ and then the rest vote in arbitrary order. In addition, suppose the tie-breaking order is $c_3, c_2, c_1$ followed by the rest. We claim that the following profile of strategies is an SPE\textsuperscript{16}.

- Agent $d_1$: always vote for $c_1$.
- Agent $d_2$: always vote for $c_2$.
- Agent $d_3$: if $d_1$ voted for $c_1$, then vote for $c_2$. Otherwise, vote for $c_3$.
- The rest of the agents: be truthful (abstain).

![Figure 5: There is an SPE in which $c_3$ is elected.](image)

To see that all the agents always act rationally, we start from the last voters and proceed backwards to the first. Agents $c_1, c_2, c_3, b_1$ confirm no one, thus, abstaining is always a best vote

\textsuperscript{16}Note that we only want to show that it is an SPE; we do not claim uniqueness here.
for them. Agent \(d_3\) always gets an agent which he confirms elected, so his votes are best possible as well. Moving on to agent \(d_2\), he will get \(c_2\) elected when \(d_1\) votes for \(c_1\) and that is a best outcome for him. On the other hand, if \(d_1\) votes for \(b_1\), then \(d_2\) knows that \(d_3\) is about to vote for \(c_3\), so the result will be bad for him no matter how he votes. The best thing he can do is to vote for someone he confirms (like \(c_2\)). Lastly, agent \(d_1\) is indifferent between voting for \(b_1\) and \(c_1\) because anyhow the elected will be someone he does not confirm (\(c_3\) in the former case and \(c_2\) in the latter). Thus, assuming that he votes for \(b_1\) is legitimate.

This proves the existence of a graph and an SPE with a multiplicative ratio 3. Figure 6 shows the general case. Here, there is an SPE in which \(d_1, \ldots, d_{k-1}\) vote for \(b_1, \ldots, b_{k-1}\), respectively; \(d_k\) then votes for \(c_k\), who is elected. If \(d_i\) decides to vote for any of \(c_1, \ldots, c_i\), then \(d_{i+1}, \ldots, d_k\) all vote for \(c_{i+1}\), hence \(d_i\) gains nothing. This is an SPE with a ratio of \(k\), which shows that \(\overline{R}\) is unbounded.

\[
\begin{align*}
C_1 & \rightarrow d_1 \rightarrow b_1 \\
C_2 & \rightarrow d_2 \rightarrow b_2 \\
& \vdots \\
C_{k-1} & \rightarrow d_{k-1} \rightarrow b_{k-1} \\
C_k & \rightarrow d_k
\end{align*}
\]

Figure 6: There is an SPE in which \(c_k\) is elected.

In order to prove that \(R \leq 2\), we need to show that there is always an SPE in which the winner’s in-degree is at least half of \(\Delta_{in}\). Let \(G\) be any graph, and let \(m\) be a most popular agent. Assume that every agent who confirms \(m\) would vote for him whenever it is his best vote. Fix an SPE with this condition, and let \(w \neq m\) be the winner. Notice first, that by Observation 2.1 \(w\) cannot get more than \(d(w)\) votes, since anyone who does not confirm him would not vote for him. Let \(C_{m,w}\) be the set of agents who confirm both \(m\) and \(w\), and let \(C_{m,\overline{w}}\) be the set of agents that confirm \(m\) and do not confirm \(w\). By Observation 2.1 and our assumption on the SPE, all the agents in \(C_{m,\overline{w}}\) vote for \(m\), which means that \(|C_{m,\overline{w}}\| \leq d(w)\). In addition, \(|C_{m,w}| \leq d(w)\). Hence, we get that \(m\)’s popularity in \(G_{-w}\) is at most \(d(m) = |C_{m,w}| + |C_{m,\overline{w}}\| \leq 2d(w)\), and the claim follows. \(\square\)

In the next theorem, we prove finite bounds for both \(\overline{R}\) and \(\overline{R}\) in the approval voting.

**Theorem 3.2.** In approval voting, \(\frac{3}{2} \leq R \leq \overline{R} \leq 2\).

**Proof.** The proof of the upper bound on \(\overline{R}\) is similar to the proof of the upper bound on \(R\) in Theorem 3.1. This time we do not need to make any assumption on the SPE. Any voter who confirms the most popular agent and does not confirm the winner is voting for the most popular agent, by Observation 2.1, and the claim follows in a similar way.

To show the lower bound, we construct a series of graphs \(\{H_k\}_{k \geq 2}\) where \(\Delta_{in}(H_k) = \Theta(k)\) and
\( R(H_k) = 3/2 \). In the graph \( H_k \) the agent \( m \) is the most popular, and there are four sets of additional agents:

- The \( k \) agents in \( C = \{c_1, \ldots, c_k\} \) are the only agents, besides \( m \), with a positive in-degree. They all have precisely \( k \) in-edges less than \( m \), and all confirm only \( m \). We will show that \( c_1 \) is the winner in the unique SPE.
- The \( k - 1 \) agents in \( D = \{d_1, \ldots, d_{k-1}\} \) are those who confirm \( m \) but are forced not to vote for him. For any \( 1 \leq i \leq k - 1 \), agent \( d_i \) confirms \( m \) and all the agents \( \{c_j : j \leq i\} \).
- The \( k - 1 \) agents in \( E = \{e_1, \ldots, e_{k-1}\} \) provide the threat which prevents agents of \( D \) from voting for \( m \). Agent \( e_i, 1 \leq i \leq k - 1 \), confirms all the agents \( \{c_j : j \leq i\} \).
- Finally, the set \( B \) contains agents of \( k \) different types which serve as ‘bias’ for \( m \) and the agents in \( C \). For \( 2 \leq i \leq k \) there are \( 2i - 3 \) agents of type \( b_i \) and they only confirm \( c_i \). In addition, there are \( k - 1 \) agents of type \( b_m \) who confirm \( m \).

![Figure 7: \( H_k \). Agent \( m \) has a popularity lead of \( k \) over all other agents, yet agent \( c_1 \) is elected.](image)

The general graph is represented in Figure 7. The voting starts with the agents in \( C, D \) and \( E \) who vote in lexicographic order alternately: \( c_1, d_1, e_1, c_2, d_2, e_2, \ldots, c_{k-1}, d_{k-1}, e_{k-1}, c_k \); then the agents in \( B \cup \{m\} \) vote in arbitrary order. The tie-breaking rule places the agents in \( C \) in lexicographic order and \( m \) below all of them. We will prove by induction on \( k \) that the winner in the unique SPE is agent \( c_1 \). Notice that the popularity of \( m \) in \( G_{-E_{c_1}} \) is \( 3(k - 1) \) while the popularity of \( c_1 \) is \( 2(k - 1) \), which implies the claimed ratio. Our induction base is \( k = 2 \) (Figure 8). Here is a sketch of the unique SPE in this scenario. The voting starts with \( c_1 \) who
abstains. If $d_1$ and $e_1$ are both truthful, then $c_2$ will be truthful as well (since $c_1$ beats him anyhow), and $m$ will be elected. As $e_1$ does not confirm this result, he will vote only for $c_2$, who can now abstain and get elected. Agent $d_1$, foreseeing this possibility, must vote only for $c_1$. Everyone after $d_1$ will now be truthful and $c_1, c_2$ and $m$ all end up with two votes, leading to the election of $c_1$ by tie-breaking. For a general $k$ it is enough to show that the following holds after $c_1$ abstains:

1. If $d_1, e_1$ are both truthful, then the winner is $m$.
2. If $d_1$ is truthful and $e_1$ votes for $c_2$, then $c_2$ is the winner.
3. If $d_1$ votes for $c_1$ and $e_1$ is truthful, then $c_1$ wins.

Notice that after any of the above voting sequences we get a graph which is equivalent to the graph $H_{k-1} \cup c_1$, by which we mean the graph we get after adding the agent $c_1$ and all his in-edges to $H_{k-1}$. By ‘equivalent’, we mean that the graphs are isomorphic except that now all the agents have two more votes; but that does not influence the remaining votes or the outcome.

We prove the three claims above using our induction hypothesis, that $c_2$ wins in $H_{k-1}$ after abstaining.

**Proof of (1.)** If we think of $b_1$ as a ‘sure vote’ for $c_2$ (since this voter cannot get elected and only confirms $c_2$), then both $c_1$ and $c_2$ have two votes and all the remaining voters in $D \cup E$ confirm both of them. Moreover, they are adjacent in the tie-breaking order with $c_1$ preceding $c_2$. It is not hard to see that by our truth-bias assumption, any best vote will either include both $c_1$ and $c_2$ or neither. Hence $c_2$ cannot possibly win, and he will be truthful and vote for $m$. Now if $d_2$ is truthful, then $m$ has a lead of at least $k-1$ over all other agents, and there are only $k-3$ agents in $D$ left to vote, so there are not enough voters to ‘turn the tables’.

**Proof of (2.)** This time $c_1$ has missed a vote and so he is behind all the agents in $C$. This means that we may ignore $c_1$ and our situation is exactly equivalent to $H_{k-1}$, so $c_2$ wins.
Proof of (3.) Now $m$ has missed a vote. While $c_2$ still cannot win (by the same argument as in (1.)), after $c_2$ is truthful, it is just the same situation as if he abstained in $H_{k-1}$ (remember that $m$ already missed a vote), but now $c_1$ is taking the place of $c_2$ and by induction, $c_1$ is elected. \[\Box\]

4 Generalizing to $k$-approval

The two voting methods we discussed (namely, plurality and approval) can be generalized to a $k$-approval voting method in which every voter is allowed to vote for at most $17$ $k$ other agents. So plurality is no more than 1-approval, and approval is the same as $n - 1$-approval. The two propositions of the additive gap (Propositions 2.2 and 2.4) had a single proof for both plurality and approval, and it is not hard to see that it can be generalized for any $k$-approval. We will now extend Theorem 3.1 to any $k$-approval with $k = o(n)$. The bound on $R$ is proven in a similar manner, and we’ll only show a series of graphs in which $R$ is unbounded.

![Figure 9](image)

Figure 9: There is an SPE in which $c_3$ is elected.

In the graph in Figure 9 agent $d_1$ confirms $c_1$ and $k$ additional agents, denoted $b_1$. Likewise, $d_2, d_3$ confirm $c_2, c_3$, respectively, and $k - 1$ additional agents. The voting order is $d_1, d_2, d_3$ and then the rest, and the tie-breaking order is $c_3, c_2, c_1$ and then the rest. We describe an SPE in which $c_3$ is elected. In this SPE, if $d_1$ votes for any subset which includes $c_1$, then $d_2$ and $d_3$ will both vote for a subset which includes $c_2$, and $c_2$ is elected. Since the outcome is the worst $d_1$ can get, voting only for the $k$ agents of $b_1$ is a best vote for him. In this case, $d_3$ decides to vote for $c_3$ and the $k - 1$ agents of $b_3$, and $c_3$ is elected no matter how $d_2$ votes. The multiplicative ratio in this SPE is 3, but it is obvious how to extend it to get any ratio.

5 Discussion and open problems

Additive gap vs. multiplicative ratio

We have seen (Proposition 2.2) that in the special case where every agent confirms at most one other agent, the elected agent will be most-popular or almost most-popular. However, as soon as the maximum out-degree of the graph is higher than one, this is no longer the case. In fact, we have shown (Proposition 2.4 and the discussion in Section 4) that $D$ is unbounded even

\[\text{17}\text{We allow a voter to vote for less than } k \text{ other agents and even abstain. Though this is not the standard definition of } k \text{-approval, this is the correct generalization to the two voting methods we considered.}\]
when $\Delta_{in}(G) = 2$, for any $k$-approval voting. We, therefore, come to the conclusion that the additive gap is not a sufficient measure for the quality of a voting method in this model. Thus we turned to a multiplicative ratio for a finer evaluation.

Indeed the multiplicative ratio gave us different bounds for plurality and approval voting. In the case of plurality, we have seen (Theorem 3.1) that even though there might be an SPE with a ‘bad’ outcome ($\overline{R}$ is unbounded), for every graph, we are guaranteed to have an SPE with a ratio of 2, at most. Moreover, our proof explains how to distinguish this SPE from other SPEs: you just give a small extra incentive for those who confirm the most popular to actually vote for him. The case of approval voting is clearer. Here (Theorem 3.2) we have proved finite bounds for $R$ and $\overline{R}$.

Plurality vs. approval

In [4], Brams demonstrated the superiority of approval voting over plurality voting in simultaneous voting systems. We conclude from our results, that in our setting, plurality voting (and even $k$-approval voting for any $k = o(n)$) allows SPEs with unbounded multiplicative ratio, while in approval, this ratio, in any SPE, will be between 1.5 and 2. We cannot draw a comparison from our results of the ‘best outcome’. To achieve that, we need to bound $\overline{R}$ from below for plurality voting.

Question 5.1. In plurality voting, is it possible to find a series of graphs, $\{G_k\}$, with $\Delta_{in}(G_k) \to \infty$ and $\overline{R}(G_k) \geq \alpha$ for some $\alpha > 1$ and all $k$?

Notice that in the series of graphs of Proposition 2.4, when $D(G_k) = k$, we have $\Delta_{in}(G_k) = \Theta(k^2)$. So in this particular example, $\overline{R}(G_k) = \overline{R}(G_k) = 1 + O(1/k)$. If there is a non-trivial (i.e. different than 1) bound for $\overline{R}$ in plurality voting, then there is a series of graphs, $\{G_k\}$ such that $D(G_k) = k$ and $\Delta_{in}(G_k) = \Theta(k)$ (this is exactly what we have shown for approval voting, when we proved the lower bound in Theorem 3.2). So a rephrase of the above question would be:

Question 5.2. In plurality voting, is it possible to construct a series of graphs, $\{G_k\}$, with $\Delta_{in}(G_k) = \Theta(k)$ and such that $D(G_k) \to \infty$?

It is worthwhile to note here that the example giving the lower bound of Theorem 3.2 does not work for plurality. To see that, consider the graph $H_2$ (Figure 8). If $c_1$ abstains, then even if $d_1$ votes for $c_1$, $e_1$ might opt to vote for $c_2$, and as a result, $c_2$ will be elected. So, for the case of plurality voting, the graph $H_2$ has an SPE in which $m$ is elected, and the proof fails.

For the approval voting method, we have proved both a lower and upper bound on $\overline{R}, \overline{R}$. Still, it could be nice to further narrow these bounds or even find the exact asymptotic values of $\overline{R}, \overline{R}$.

Question 5.3. Can the bounds of Theorem 3.2 be narrowed down?

$k$-approval and a threshold between plurality and approval

Finally, we have seen that $k$-approval has the same bounds as plurality for any $k = o(n)$. When $k = n – 1$ this voting method is precisely approval; and so a natural question is what can be said about the threshold function which separates $k$-approval from plurality.
Question 5.4. Find a minimal function, $f(n)$, such that the voting method $f(n)$-approval has a finite bound for $\bar{\mathcal{R}}$.

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