Abstract

We construct local $M$-operators for an integrable discrete time version of the classical Heisenberg magnet by convolution of the twisted quantum trigonometric $4 \times 4$ R-matrix with certain vectors in its "quantum" space. Components of the vectors are identified with $\tau$-functions of the model. Hirota’s bilinear formalism is extensively used. The construction generalizes the known representation of $M$-operators in continuous time models in terms of Lax operators and classical $r$-matrix.

1 Introduction

The unified treatment [1] of non-linear soliton equations as hamiltonian systems having enough number of conserved quantities in involution is based on the (classical) $r$-matrix. Its role is to provide a universal form of Poisson brackets for elements of Lax operators. An alternative though less popular point of view on the $r$-matrix (which we are going to follow) comes from the Zakharov-Shabat approach [1], [2] which consists in representing soliton equations as 2D zero curvature conditions for a pair of matrix functions (called $L$ and $M$ operators) depending on a spectral parameter. The $r$-matrix works there as a machine producing $M$-operators from $L$-operators.

In the paper [3], for the simplest example of the lattice sine-Gordon (SG) model, we have found a similar machine in the completely discrete set-up, i.e., in lattice integrable models with discrete time. Remarkably enough, it appears to be a quantum $R$-matrix with the "quantum" parameter $q$ related to the time lattice spacing. The formula for local $M$-operators in the discrete case has as simple structure as in the continuous one (see (1.5) below) with the $R$-matrix in place of the $r$-matrix. This comes as the result of a computation. At the moment we do not try to "explain" why a typical quantum $R$-matrix takes part in a purely classical problem.

In this paper we show that the construction still works for a much more general model – partially anisotropic lattice Heisenberg magnet (HM) in discrete time. (This model contains discrete versions of SG, KdV and other equations as special cases.) Although the final result looks like a straightforward generalization of the one for the lattice SG model, the derivation gets much more involved and some important modifications are necessary.

Let us recall the $r$-matrix construction of $M$-operators for continuous flows. Let $\mathcal{L}_l(z)$ be a classical $2 \times 2$ $L$-operator on 1D lattice with the periodic boundary condition $\mathcal{L}_{l+N}(z) = \mathcal{L}_l(z); z$ is the spectral parameter. For the moment we assume the ultralocality – Poisson brackets between elements of the $\mathcal{L}_l(z)$ at different
sites are zero. The monodromy matrix $T_l(z)$ is ordered product of $L$-operators along the lattice from the site $l$ to $l+N-1$:

$$T_l(z) = L_{l+N-1}(z) \ldots L_{l+1}(z)L_l(z).$$ (1.1)

Hamiltonians of commuting flows are obtained by expanding $\log T(z)$ in $z$, where $T(z) = \text{tr} T_l(z)$ does not depend on $l$ due to the periodic boundary condition. All these flows admit a zero curvature representation with the generating function of $M$-operators given by $\tilde{M}$.

$$M_l(z; \omega) = T^{-1}(\omega) \text{tr}_1 \left[ r(z/\omega)(T_l(\omega) \otimes I) \right].$$ (1.2)

Here $r(z)$ is a (classical) $4 \times 4$ $r$-matrix, acting in the tensor product of two 2-dimensional spaces, $\text{tr}_1$ means trace in the first space, $I$ is the unity matrix. Expanding the r.h.s. of eq. (1.2) in $w$, one gets $M$-operators depending on the spectral parameter $z$. From the hamiltonian point of view, the zero curvature condition follows from Poisson brackets for elements of the $L$-operator. A similar $r$-matrix formula exists for $M$-operators in non-ultralocal models, though, in this case the $r$-matrix is not skew-symmetric. In general, $M_l(z; \omega)$ is non-local.

It is well known [5], [1] how to construct local $M$-operators from the generating function. Suppose there exists $z_0$ such that $\det L_l(z_0) = 0$ for any $l$. This means that $L_l(z_0)$ is a 1-dimensional projector:

$$L_l(z_0) = \frac{\langle \alpha(l) \rangle \langle \beta(l) \rangle}{P(l)},$$ (1.3)

where

$$|\alpha\rangle = \left( \begin{array}{c} \alpha_1 \\ \alpha_2 \end{array} \right), \quad \langle \beta| = (\beta_1, \beta_2)$$ (1.4)

are two-component vectors and $P(l)$ is a scalar normalization factor. Components of the vectors as well as the $P(l)$ depend on dynamical variables. It is easy to see that $M_l(z; \omega)$ is a local quantity:

$$M_l(z) \equiv M_l(z; z_0) = \frac{\langle \beta(l) \rangle r(z/z_0) \langle \alpha(l-1) \rangle}{\langle \beta(l) | \alpha(l-1) \rangle}$$ (1.5)

(note that the normalization factor cancels). The scalar product is taken in the first space only, so the result is a $2 \times 2$ matrix with the spectral parameter $z$. It obeys the zero curvature condition

$$\partial_t L_l(z) = M_{l+1}(z)L_l(z) - L_l(z)M_l(z).$$ (1.6)

Let us outline the results of this work. We consider the completely discretized partially anisotropic ($XXZ$) HM and represent $M$ and $L$ operators $M_l(z)$, $L_l(z)$ by formulas of the type (1.5):

$$M_l(z) = \frac{\langle \beta(l) | R(z/z_0; q, \xi) | \tilde{\beta}(l-1) \rangle}{\langle \beta(l) | \alpha(l-1) \rangle},$$ (1.7)

$$L_l(z) = \frac{\langle \beta(l) | R(z/z_0; q, \xi') \alpha(l) \rangle}{\langle \beta(l) | \alpha(l-1) \rangle}.$$ (1.8)

In the r.h.s., $R(z; q, \xi), R(z; q, \xi')$ are quantum $4 \times 4$ $R$-matrices (to be specified below) with the ”quantum” parameter $q$ and the Drinfeld twist parameters $\xi, \xi'$ related to the space-time lattice. The vectors $|\alpha(l)\rangle$, $|\beta(l)\rangle$ are the same as in eq. (1.3).

$$|\tilde{\beta}(l)\rangle \equiv \left( \begin{array}{cc} 0 & (-\xi \xi')^{\frac{l}{2}} \\ (-\xi \xi')^{-\frac{l}{2}} & 0 \end{array} \right) |\beta(l)\rangle.$$ (1.9)

It is worth noting here that the continuous and discrete time models have a common $L$-operator; formula (1.8) gives its $R$-matrix representation.

To visualize these formulas, it is convenient to use the language of the algebraic Bethe ansatz [7], [8]. The scalar product is taken in the ”quantum” (vertical) space, so one gets a $2 \times 2$ matrix in the ”auxiliary” (horizontal) space:
The $M$-operator (1.7) generates shifts of a discrete time variable $m$. The discrete zero curvature condition
\[
M_{l+1,m}(z) L_{l,m}(z) = L_{l,m+1}(z) M_{l,m}(z)
\]
gives rise to the HM equations of motion in discrete space-time.

The key ingredient of the construction is to pass from the initial dynamical variables to the pair of vectors $|\alpha(l)\rangle$, $|\beta(l)\rangle$. In the papers [3] on exact lattice regularization of integrable models components of these vectors were expressed in terms of canonical variables of the model. Those formulas looked quite complicated and were hardly considered as something instructive. Here we shed some light on their meaning. Using equations of motion of the completely discretized model (derived from the discrete zero curvature condition (1.9)), we show that (suitably normalized) components of the vectors $|\alpha(l)\rangle$, $|\beta(l)\rangle$ are $\tau$-functions.

The $\tau$-function is one of the most fundamental objects of the theory (see e.g. [3], [4]). It is worth stressing that quantum $R$-matrices in the classical discrete problems can hardly be noticed until one reformulates the non-linear equations and elements of the $L$-$M$ pair entirely in terms the $\tau$-functions. That is why we make a long excursion into Hirota’s bilinear formalism [10]-[12] (Sect. 2). Our guiding principle is Miwa’s interpretation [13] of the discrete time flows, in which discretized integrable equations are treated as members of the same infinite hierarchy as their continuous counterparts. A general method to produce discrete soliton equations was developed in [14], where, in particular, the discrete isotropic ($XXX$) HM model was proposed. Following these ideas, we deal with bilinear form of the discrete $XXZ$ HM model. The matrix $L$-$M$-pair for the latter is derived in Sect. 3 from basic linear problems for a scalar wave function [5], [15].

Like in the SG model, there are two lattice versions of the classical HM: the discrete HM model on a space-time lattice [4] and the model on a space lattice with continuous time introduced in [4] for the more general $XYZ$-case. It is interesting to note that both of them were proposed back in 1982 but the ideas underlying one and another seemed to be "orthogonal" and did not intersect until the very recent time [9]. A motivation of this work was just an attempt to understand one of the models in terms of the other one.

The manner of exposition here is different from the one in [3], where we started with either Faddeev-Volkov or Izergin-Korepin $L$-operator for the lattice SG model in terms of lattice fields and passed to the $\tau$-function by means of a special substitution. In this paper we start directly from the $\tau$-function (and bilinear equations it obeys) rather than from lattice spin variables. The reason is twofold. First, the equations of motion in terms of lattice spin variables are too complicated and, anyway, are of no practical use for us. Second, in the discrete case there is no canonical way to introduce spin variables whereas Hirota’s bilinear formalism is gauge-invariant and free of artifacts. Having this in mind, let us remark that by Heisenberg model we mean here nothing more than a convenient name for the properly reduced 2D Toda lattice hierarchy [22]. (Actually we deal with a model which is slightly more general than the $XXZ$ HM itself.) In this sense a name like partially anisotropic chiral field model would be also appropriate. Needless to say that in the discrete set-up the specific features of each of the two models become irrelevant [8]. What is relevant is the type of reduction of the 2D Toda hierarchy that is the same in both cases.

Treated as a hierarchy, the bilinear equations imply infinitely many discrete variables (commuting flows) from the very beginning. Generally speaking, any two of them could be chosen as space-time coordinates. However, for our purpose we need a distinguished space-time lattice. In Sect. 4 we introduce such a lattice with coordinates denoted by $l$, $m$ and derive the matrix $L$-$M$-pair for translations $l \to l+1$, $m \to m+1$. These

\[\langle\beta|R(z)\alpha\rangle = \langle\beta| \]
\[|\alpha\rangle\]

1 On the quantum level, these ideas were partially linked together in [17]-[20], where quantum SG model in discrete space-time (a quantization of Hirota’s discrete SG equation) was constructed; a generalization to affine Toda field theories on the lattice was suggested in [24].
and $M$ operators are represented in the form \((1.8), \(1.7)\) in Sect. 5. In Sect. 6 we discuss the continuous time limit of the construction and show how eq. \((1.5)\) is reproduced. As it was already mentioned, the $r$-matrix in that formula turns out to be not necessarily skew-symmetric. This fact is a signal of non-ultralocality of the model in general case. Sect. 7 is a conclusion where we point out a few unsolved problems motivated by our results.

2 Hirota’s bilinear formalism

In this section we present the results of the papers \([13], [14]\) in the form convenient for our purpose. We illustrate formulas by the graphical representation of discrete flows suggested in the review \([23]\) to which we refer for more details.

2.1 General form of 3-term bilinear equations

The key object of Hirota’s approach is $\tau$-function. The $\tau$-function $\tau = \tau(a, b, c, \ldots)$ (as a function of the discrete variables $a, b, c, \ldots$) obeys a number of bilinear partial difference equations. To each discrete variable $a$ a complex parameter $\lambda_a \in \mathbb{C}$ ("Miwa’s variable") is associated. One may think of the $\lambda_a$ as a continuous "number" which marks the discrete flow. For any two discrete flows $a, b$ we put

$$\lambda_{ab} \equiv \lambda_a - \lambda_b, \quad \lambda_{ab} = -\lambda_{ba}.$$  

Let us recall how to compose bilinear equations. Each triplet \(\{a, b, c\}\) of discrete flows gives rise to a 3-term bilinear equation \([12], [13]\) for the $\tau$:

$$\lambda_{bc} \tau(a + 1, b, c)\tau(a, b + 1, c + 1) + \lambda_{ca} \tau(a, b + 1, c)\tau(a + 1, b, c + 1) + \lambda_{ab} \tau(a, b, c + 1)\tau(a + 1, b + 1, c) = 0. \quad (2.1)$$

All other variables which the $\tau$ may depend on enter this equation as parameters. Each quadruplet $\{a, b, c, d\}$ gives rise to another 3-term bilinear equation:

$$\lambda_{ad} \lambda_{bc} \tau(a + 1, b, c, d + 1)\tau(a, b + 1, c + 1, d) + \lambda_{bd} \lambda_{ca} \tau(a, b + 1, c, d + 1)\tau(a + 1, b, c + 1, d) + \lambda_{cd} \lambda_{ab} \tau(a, b, c + 1, d + 1)\tau(a + 1, b + 1, c, d) = 0. \quad (2.2)$$

Remark Links between eqs. \((2.1), (2.2)\) exist in both directions. On the one hand, eq. \((2.1)\) is a particular case of \((2.2)\) when $\lambda_d \to \infty$. (According to \([13]\), this limit means that the dependence on $d$ in the $\tau$ disappears.) On the other hand, eq. \((2.2)\), though linearly independent of eqs. \((2.1)\), is an algebraic consequence of equations of the type \((2.1)\) written for the triplets $\{a, b, c\}, \{a, b, d\}$ and $\{a, c, d\}$. In this sense all what we are going to derive in the sequel follows already from eqs. \((2.1)\).

2.2 Basic bilinear equations for the discrete HM model

Let us consider five discrete flows and denote the corresponding variables by $p, p', n, u, v$. We denote the $\tau$-function by $\tau(p, p', n, u, v) \equiv \tau_{n}^{p, p'}(u, v)$. In the latter notation the $u, v$ are separated from the others because they will play the role of chiral space-time coordinates. We call them chiral variables.
Bilinear equations for the $\tau$-function of the $XXZ$ HM model are obtained from (2.1), (2.2) by imposing the reduction

$$
\tau_{n+1, p'+1}^p(u, v) = \tau_{n+1}^{p, p'}(u, v),
$$

(2.3)

so we are left with four independent variables. Let us choose them to be $p, n, u, v$.

**Remark** The discrete KdV equation is a particular case $\lambda_n = \infty$, so the dependence on $n$ disappears. The discrete SG model corresponds to further specification $\lambda_{p'} = \lambda_p$.

Using the graphical representation introduced in [23], we can display the above configuration of discrete flows in the figure (the graph of flows):

![Graphical representation](image)

The dashed line is drown here to indicate that the reduction (2.3) is the same as in the case of the 1D Toda chain in discrete time (a detailed discussion of this point see in [23]). Indeed, the vector field $\partial_{\bar{p}} \equiv -\partial_n + \partial_p$ defines the flow $\bar{p}$ corresponding to the dashed line; in these terms the reduction acquires the more familiar Toda-like form $\tau^{p+1, \bar{p}+1} = \tau^{p, \bar{p}}$. This turns into a differential condition as $\lambda_p \to \infty, \lambda_{p'} \to \lambda_n$. In this case one gets the isotropic ($XXX$) HM model in discrete time.

**Remark** Looking at the graph makes it easier to deal with different discrete flows when there are many of them. We refer to [23] for the exact meaning of the graphical elements. Here we only note that keeping in mind solutions of the finite-gap type (see e.g. [2], [24]), the reader may think of this figure as drown on a patch of a Riemann surface with local coordinate $\lambda^{-1}$. Miwa’s variables are coordinates of punctures on the surface. The lines are then cuts between the punctures, which give rise to discrete commuting flows on the Jacobian via the Abel map.

The condition (2.3) allows one to get rid of $p'$ if necessary. Let us give a list of bilinear equations which are obtained in this way from eqs. (2.1), (2.2). In front of each equation we indicate the triplet or quadruplet which it comes from.

The simplest equation involves the variables $p, n$ only:

$$
\{pp'\} : \quad \lambda_{p'n} \tau_{n+1}^{p-1}(u, v) \tau_{n-1}^{p+1}(u, v) - \lambda_{pn} \tau_{n+1}^{p-1}(u, v) \tau_{n}^{p+1}(u, v) = \lambda_{p'p} \left( \tau_{n}^{p}(u, v) \right)^2.
$$

(2.4)

It is the discrete time 1D Toda chain equation in bilinear form [12]. In our case it plays the role of a constraint on the dynamical variables since the space-time coordinates $u, v$ enter as parameters.

Equations of the next group contain shifts of $p, n, u$ only:

$$
\{pp'u\} : \quad \lambda_{p'u} \tau_{n+1}^{p+1}(u, v) \tau_{n+1}^{-1}(u+1, v) - \lambda_{pu} \tau_{n+1}^{p-1}(u, v) \tau_{n+1}^{p+1}(u+1, v) = \lambda_{p'p} \tau_{n+1}^{p}(u+1, v),
$$

(2.5)

$$
\{pnu\} : \quad \lambda_{un} \tau_{n+1}^{p+1}(u, v) \tau_{n+1}^{p}(u+1, v) + \lambda_{pu} \tau_{n+1}^{p}(u, v) \tau_{n+1}^{p+1}(u+1, v) = \lambda_{pn} \tau_{n+1}^{p}(u+1, v),
$$

(2.6)

For the general theory of finite-gap solutions to Hirota’s difference equation see [23].
\[ \{p'n\} : \quad \lambda_{p'n} \tau_{n+1}^p(u, v) \tau_{n+1}^{p+1}(u + 1, v) + \lambda_{p'u} \tau_{n}^p(u, v) \tau_{n}^{p+1}(u + 1, v) = \lambda_{p'n} \tau_{n+1}^p(u, v) \tau_{n+1}^{p+1}(u + 1, v) , \]  
(2.7)

\[ \{pp'n\} : \quad \lambda_{pn} \lambda_{p'u} \tau_{n+1}^{p+1}(u, v) \tau_{n+1}^{p-1}(u + 1, v) - \lambda_{p'n} \lambda_{p'u} \tau_{n+1}^{p-1}(u, v) \tau_{n+1}^{p+1}(u + 1, v) = \lambda_{p'n} \lambda_{p'u} \tau_{n+1}^p(u, v) \tau_{n+1}^p(u + 1, v) . \]  
(2.8)

Similar equations can be written for \( p, n, v \) – it is enough to replace \( u \) by \( v \) everywhere.

**Remark** Combining eqs. (2.7), (2.6),

\[ \text{eq. (2.5)} \quad \frac{\lambda_{p'u} \tau_{n+1}^{p-1}(u + 1, v)}{\lambda_{p'u} \tau_{n+1}^p(u + 1, v)} = \frac{\text{eq. (2.6)}}{\lambda_{p'n} \tau_{n+1}^p(u + 1, v)} , \]

and plugging (2.7) into the l.h.s., we obtain eq. (2.8).

Equations involving both coordinates \( u, v \) read:

\[ \{nuv\} : \quad \lambda_{un} \tau_{n+1}^p(u, v + 1) \tau_{n+1}^p(u + 1, v) - \lambda_{uv} \tau_{n+1}^p(u + 1, v) \tau_{n+1}^p(u + 1, v) \]
\[ = \lambda_{uv} \tau_{n+1}^p(u + 1, v + 1) \tau_{n+1}^p(u, v) , \]  
(2.9)

\[ \{pp'uv\} : \quad \lambda_{pu} \lambda_{pv} \tau_{n+1}^{p+1}(u, v + 1) \tau_{n+1}^{p-1}(u + 1, v) - \lambda_{pu} \lambda_{pv} \tau_{n+1}^{p-1}(u, v + 1) \tau_{n+1}^{p+1}(u + 1, v) \]
\[ = \lambda_{pu} \lambda_{pv} \tau_{n+1}^p(u + 1, v + 1) \tau_{n+1}^p(u, v) , \]  
(2.10)

\[ \{p'uv\} : \quad \lambda_{pu} \lambda_{pv} \tau_{n+1}^{p+1}(u, v + 1) \tau_{n+1}^{p-1}(u + 1, v) - \lambda_{pu} \lambda_{pv} \tau_{n+1}^{p-1}(u, v + 1) \tau_{n+1}^{p+1}(u + 1, v) \]
\[ = \lambda_{pu} \lambda_{pv} \tau_{n+1}^p(u + 1, v + 1) \tau_{n+1}^p(u, v) , \]  
(2.11)

\[ \{p'nuv\} : \quad -\lambda_{p'u} \lambda_{vn} \tau_{n+1}^{p+1}(u, v + 1) \tau_{n+1}^p(u + 1, v) + \lambda_{p'u} \lambda_{vn} \tau_{n+1}^{p+1}(u + 1, v) \tau_{n+1}^p(u, v + 1) \]
\[ = \lambda_{p'u} \lambda_{vn} \tau_{n+1}^p(u + 1, v + 1) \tau_{n+1}^p(u, v) , \]  
(2.12)

\[ \{p'uv\} : \quad \lambda_{pu} \lambda_{pv} \tau_{n+1}^{p+1}(u, v + 1) \tau_{n+1}^p(u + 1, v) - \lambda_{pu} \lambda_{pv} \tau_{n+1}^p(u + 1, v + 1) \tau_{n+1}^{p+1}(u, v + 1) \]
\[ = \lambda_{pu} \lambda_{pv} \tau_{n+1}^p(u + 1, v + 1) \tau_{n+1}^p(u, v) , \]  
(2.13)

\[ \{p'nuv\} : \quad -\lambda_{p'u} \lambda_{vn} \tau_{n+1}^p(u, v + 1) \tau_{n+1}^{p+1}(u + 1, v) + \lambda_{p'u} \lambda_{vn} \tau_{n+1}^p(u + 1, v) \tau_{n+1}^{p+1}(u, v + 1) \]
\[ = \lambda_{p'u} \lambda_{vn} \tau_{n+1}^p(u + 1, v + 1) \tau_{n+1}^p(u, v) , \]  
(2.14)

The above list of linearly independent 3-term bilinear equations is by no means complete. The full list contains many other equations which either follow from the already written ones or can be derived from "higher" analogues of eqs. (2.1), (2.2) for more than 4 variables (which, in their turn, are algebraic corollaries of eqs. (2.1)). Some of them are given in the next subsection.
2.3 Some useful corollaries of the basic equations

Note that the pair of equations (2.11), (2.12) (respectively (2.13), (2.14)) can be considered as a linear system for \( \tau^{p+1}(u, v+1) \tau^{p}_n(u + 1, v), \tau^{p+1}_n(u, v+1) \tau^{p}_n(u + 1, v) \) (respectively, for \( \tau^{p+1}_n(u + 1, v) \tau^{p+1}_n(u, v+1) \)). Resolving these systems, we get

\[
\lambda_p \lambda_{un} \tau^{p+1}_n(u + 1, v) + \lambda_p \lambda_{un} \tau^{p+1}_n(u + 1, v)
= \Lambda \tau^{p+1}_n(u + 1, v),
\]

(2.15)

\[
\lambda_p \lambda_{vn} \tau^{p+1}_n(u + 1, v) + \lambda_p \lambda_{vn} \tau^{p+1}_n(u + 1, v)
= \Lambda \tau^{p+1}_n(u + 1, v),
\]

(2.16)

\[
\lambda_p \lambda_{un} \tau^{p+1}_n(u + 1, v) + \lambda_p \lambda_{un} \tau^{p+1}_n(u + 1, v)
= \Lambda \tau^{p+1}_n(u + 1, v),
\]

(2.17)

\[
\lambda_p \lambda_{vn} \tau^{p+1}_n(u + 1, v) + \lambda_p \lambda_{vn} \tau^{p+1}_n(u + 1, v)
= \Lambda \tau^{p+1}_n(u + 1, v),
\]

(2.18)

where

\[
\Lambda \equiv \lambda_{un} \lambda_{pn} - \lambda_{un} \lambda_{vn}.
\]

(2.19)

These equations are equally useful in what follows. They are even more informative than eqs. (2.11)-(2.14) since allow one to "fuse" the variables, i.e. to put \( \lambda_v = \lambda_u \). (In (2.11)-(2.14) this leads to the identity \( 0 = 0 \).) In this case \( \tau(u + 1, v + 1) \) converts into \( \tau(u + 2) \) and we get:

\[
\lambda_{un} \lambda_{pn} \tau^{p+1}_n(u - 1, v) \tau^{p}_n(u + 1, v) + \lambda_{un} \lambda_{pn} \tau^{p+1}_n(u - 1, v) \tau^{p}_n(u + 1, v)
= (\lambda_{pn} - \lambda_{un}^2) \tau^{p}_n(u, v) \tau^{p+1}_n(u, v),
\]

(2.20)

\[
\lambda_{vn} \lambda_{pn} \tau^{p+1}_n(u - 1, v) \tau^{p}_n(u + 1, v) + \lambda_{vn} \lambda_{pn} \tau^{p+1}_n(u - 1, v) \tau^{p}_n(u + 1, v)
= (\lambda_{pn} - \lambda_{vn}^2) \tau^{p+1}_n(u, v) \tau^{p+1}_n(u, v).
\]

(2.21)

The same equations can be written down for \( v \) in place of \( u \).

At last, we present two other useful equations which are obtained from (2.11), (2.12), (2.13), (2.14) and (2.20), (2.21) (together with their \( v- \)counterparts) by a procedure similar to the one explained in the remark after eq. (2.8):

\[
\lambda_{pn} \lambda_{pn} \tau^{p+1}_n(u + 1, v) \tau^{p+1}_n(u + 1, v) - \lambda_{vn} \lambda_{vn} \tau^{p+1}_n(u + 1, v) \tau^{p+1}_n(u, v)
= \Lambda \tau^{p}_n(u + 1, v) \tau^{p}_n(u + 1, v),
\]

(2.22)
3.2 Vector linear problem

Therefore, skipped. These equations are basic tools for deriving matrix direction in the matrix form:

\[
\lambda_{p'v}\lambda_{p'u}\lambda_{p+1}\tau_{n}^{u}+(u+1,v)\tau_{n}^{u+1}(u,v) - \lambda_{p+1}\lambda_{p+1}\tau_{n}^{u+1}(u+1,v)\tau_{n+1}^{u+1}(u,v) = \lambda_{p'v}\lambda_{p'u}\tau_{n+1}^{u}(u+1,v)\tau_{n+1}^{u+1}(u,v).
\]

(2.23)

To conclude the section, let us mention that the complete list of linearly independent equations for the \(\tau\)-function (even restricted by the 3-term ones) is somewhat longer. Here we have selected those used in the sequel.

3 Linearization of the discrete HM model

The bilinear equations from the previous section can be represented as compatibility conditions for an overdetermined system of linear problems for a "wave function" \(\Psi\). This is what we mean by the linearization. For us this is a systematic way to find \(L-M\)-pairs.

3.1 Scalar linear problems

The bilinear equations \ref{2.4}-\ref{2.14} follow from compatibility of a system of linear equations for a "wave function" \(\Psi = \Psi_{n}^{p'}(u,v)\). The prototype of linear equations for \(\Psi\) is \ref{2.4}, \ref{2.14}, \ref{2.15} (see also \ref{2.23} for a review):

\[
\Psi(a+1,b) = \Psi(a,b+1) - \lambda_{a,b}\tau(u(a+1,b+1))\Psi(a,b),
\]

(3.1)

where \(a, b\) stand for any two elementary discrete variables.

The reduction condition \ref{2.3} for the \(\Psi\)-function reads

\[
\Psi_{n}^{P+1,p+1}(u,v) = z^{2}\Psi_{n}^{P,p}(u,v),
\]

(3.2)

where \(z\) is a spectral parameter. Hiding the variable \(p'\) with the help of this condition, we have:

\[
\{nu\} : \quad \Psi_{n}^{P}(u+1) = \Psi_{n+1}(u) - \lambda_{a,n}\tau_{n}^{P}(u+1)\Psi_{n+1}(u),
\]

(3.3)

\[
\{pu\} : \quad \Psi_{n}^{p+1}(u) = \Psi_{n+1}(u+1) - \lambda_{p,n}\tau_{n+1}^{p}(u+1)\Psi_{n+1}(u),
\]

(3.4)

\[
\{p'u\} : \quad z^{2}\Psi_{n+1}^{p+1}(u) = \Psi_{n+1}(u+1) - \lambda_{p'u,n}\tau_{n+1}^{p+1}(u+1)\Psi_{n+1}(u),
\]

(3.5)

(and similar equations for \(v\) in place of \(u\)). In \ref{3.3}-\ref{3.5}, \(v\) is supposed to be the same everywhere and, therefore, skipped. These equations are basic tools for deriving matrix \(L-M\)-pairs.

3.2 Vector linear problem

Combining equations \ref{3.3}-\ref{3.5}, one can represent translation of the vector \(\begin{pmatrix} \Psi_{n}^{p}(u) \\ \Psi_{n+1}^{p+1}(u) \end{pmatrix}\) along the \(u\)-direction in the matrix form:

\[
\begin{pmatrix} \Psi_{n}^{p+1}(u+1) \\ \Psi_{n+1}^{p+1}(u+1) \end{pmatrix} = \begin{pmatrix} \lambda_{p,u} \tau_{n}^{p}(u)\tau_{n+1}^{p+1}(u) \\ \lambda_{p+1,u} \tau_{n}^{p+1}(u+1)\tau_{n+1}^{p+1}(u+1) \end{pmatrix} \begin{pmatrix} 1 \\ z^{2} + \lambda_{p'u} \tau_{n}^{p+1}(u)\tau_{n+1}^{p+1}(u+1) \end{pmatrix} \begin{pmatrix} \Psi_{n}^{p+1}(u) \\ \Psi_{n+1}^{p+1}(u) \end{pmatrix}.
\]

(3.6)

8
Let us change the gauge passing to the wave function

\[
\begin{pmatrix}
\Phi_1(u) \\
\Phi_2(u)
\end{pmatrix} = D(u) \begin{pmatrix}
\Psi_n^p(u) \\
\Psi_n^{p+1}(u)
\end{pmatrix}
\]

(3.7)

where \(D(u)\) is the diagonal matrix

\[
D(u) = \left(\tau_{n+1}^p(u)\tau_n^{p+1}(u)\right)^{-\frac{1}{2}} \begin{pmatrix}
\lambda_{pn}^\frac{1}{z} \tau_p^n(u) & 0 \\
0 & \lambda_{pn}^{-\frac{1}{z}} \tau_n^{p+1}(u)
\end{pmatrix}.
\]

Then the linear problem (3.6) acquires the form

\[
\begin{pmatrix}
\Phi_1(u+1) \\
\Phi_2(u+1)
\end{pmatrix} = L^{(u)}(z) \begin{pmatrix}
\Phi_1(u) \\
\Phi_2(u)
\end{pmatrix}.
\]

(3.8)

The \(L\)-operator \(L^{(u)}(z)\) can be compactly written in terms of the three fields

\[
\psi^0(u,v) = \frac{\tau_n^{p+1}(u,v)}{\tau_n^{p+1}(u,v)}, \quad \psi^-(u,v) = \frac{\tau_p^p(u,v)}{\tau_{n+1}^p(u,v)}, \quad \psi^+(u,v) = \frac{\tau_{n+1}^{p+1}(u,v)}{\tau_n^{p+1}(u,v)}.
\]

(3.9)

For brevity we also use the notation \(\phi(u,v) \equiv \left[\psi^0(u,v)\right]^\frac{1}{2}\). The \(L\)-operator reads

\[
L^{(u)}(z) = \begin{pmatrix}
\lambda_{pn} \frac{\phi(u+1)}{\phi(u)} & z\lambda_{pn}^\frac{1}{z} \frac{\psi^-(u+1)}{\phi(u)\phi(u+1)} \\
\lambda_{pn}^\frac{1}{z} \frac{\phi(u)\phi(u+1)\psi^+(u)}{\phi(u+1)} & z\lambda_{pn}^{-\frac{1}{z}} \frac{\phi(u) + z^2 \phi(u+1)}{\phi(u)}
\end{pmatrix}.
\]

(3.10)

Note that

\[
\det L^{(u)}(z) = \lambda_{pn} \lambda_p' - \lambda_{un} z^2.
\]

(3.11)

We call the \(L^{(u)}(z)\) chiral \(L\)-operator because it shifts the chiral variable of the wave function via (3.8). In the next subsection we study the discrete zero curvature condition with matrices of the type (3.10).

### 3.3 The zero curvature condition for chiral \(L\)-operators

Let

\[
\begin{array}{c|c|c|c|c}
\hline
& C=(u,v+1) & D=(u+1,v+1) \\
C=(u,v) & A=(u,v) & B=(u+1,v) \\
\hline
\end{array}
\]

be an elementary cell of the \(u,v\)-lattice. In this notation (borrowed from [17]-[19]) the \(L\)-operator (3.10) reads:

\[
^3\text{In general the coordinate axes are not orthogonal to each other; in particular, at } \lambda_u = \lambda_v \text{ the lattice collapses to a 1D one.}
\]
\[ L_{B^{\leftarrow}A}(z) = \begin{pmatrix} \lambda_{pa} \phi(B) \phi(A) & z\lambda_{pn} \frac{\psi^{-}(B)}{\phi(A)\phi(B)} \\ z\lambda_{pn} \phi(A)\phi(B) \psi^{+}(A) & \lambda_{p'u} \phi(A) + z\lambda_{pn} \phi(B) \phi(A) \end{pmatrix}. \] (3.12)

Similarly, we introduce another chiral \( L \)-operator, \( L_{C^{\leftarrow}A}(z) \), which is given by the same formula with \( \phi(C) \) in place of \( \phi(B) \) and \( \lambda_{pa}, \lambda_{p'u} \) in place of \( \lambda_{pn}, \lambda_{p'u} \), respectively.

The discrete zero curvature condition
\[ L_{D^{\leftarrow}B}(z)L_{B^{\leftarrow}A}(z) = L_{D^{\leftarrow}C}(z)L_{C^{\leftarrow}A}(z) \] (3.13)
is equivalent to the following non-linear equations of motion for the fields \( \psi^{0}, \psi^{\pm} \):
\[
\begin{align*}
(\lambda_{p'u}\psi^{0}(C) - \lambda_{p'v}\psi^{0}(B)) & (\psi^{0}(A)\psi^{0}(A) - \psi^{0}(B)\psi^{0}(C)) \\
&= \lambda_{pn}\psi^{0}(B)\psi^{0}(C)\psi^{0}(D) \left[ \psi^{+}(C)\psi^{-}(C) - \psi^{+}(B)\psi^{-}(B) \right], \tag{3.14}
\end{align*}
\]
\[
\begin{align*}
\lambda_{pn}\psi^{0}(D)\psi^{0}(C)\psi^{-}(B) + \lambda_{p'u}\psi^{0}(A)\psi^{0}(C)\psi^{-}(D) \\
&= \lambda_{pn}\psi^{0}(D)\psi^{0}(B)\psi^{-}(C) + \lambda_{p'v}\psi^{0}(A)\psi^{0}(B)\psi^{-}(D), \tag{3.15}
\end{align*}
\]
\[
\begin{align*}
\lambda_{pn}\psi^{0}(D)\psi^{0}(C)\psi^{+}(C) + \lambda_{p'u}\psi^{0}(A)\psi^{0}(C)\psi^{+}(A) \\
&= \lambda_{pn}\psi^{0}(D)\psi^{0}(B)\psi^{+}(B) + \lambda_{p'v}\psi^{0}(A)\psi^{0}(B)\psi^{+}(A). \tag{3.16}
\end{align*}
\]

Equations of motion for the discrete HM model of XXX type given in [14] are written for a different choice of dynamical variables. It would be useful to establish a direct correspondence between them.

**Remark** The discrete KdV equation in the Faddeev-Volkov form [17], [18] is reproduced from (3.13) or (3.16) in the limit \( \lambda_{n} \to \infty \) when the \( n \)-dependence disappears, so that \( \psi^{-} = \psi^{+} = 1 \) and we are left with the single field \( \psi^{0} \). The Faddeev-Volkov chiral \( L \)-operators for the discrete KdV equation are reproduced from (3.12) in the same limit provided the renormalized spectral parameter \( \zeta = z\lambda_{pn}^{\dagger} \) is finite.

### 3.4 Antichiral \( L \)-operators

Here we introduce another type of \( L \)-operators which will be called *antichiral*.

Let us put \( \lambda_{u} = \lambda_{n} \) in the chiral \( L \)-operator (3.12) and, correspondingly, identify \( u \) with \( n \). In this way we get the operator which generates the translation
\[
A = (u, v, n) \longrightarrow A^{\dagger} = (u, v, n + 1)
\]
in the 3D lattice spanned by \( u, v, n \):
\[
L_{A^{\dagger} \leftarrow A}^{(n)}(z) = \begin{pmatrix} \lambda_{pn} \frac{\phi(A^{\dagger})}{\phi(A)} & z\lambda_{pn} \frac{\psi^{-}(A^{\dagger})}{\phi(A)\phi(A^{\dagger})} \\ z\lambda_{pn} \phi(A)\phi(A^{\dagger}) \psi^{+}(A) & \lambda_{p'u} \phi(A) + z\lambda_{pn} \phi(B) \phi(A^{\dagger}) \end{pmatrix}. \]
\[
\begin{pmatrix}
\frac{r_{n+1}(A)r_{n+2}(A)}{r_n(A)} & \frac{z\lambda_{pn}}{r_{n+1}(A)} \\
\frac{z\lambda_{pn}}{r_{n+1}(A)} & z^2 + \lambda_{pn} \frac{r_{n+1}(A)r_{n+2}(A)}{r_{n+1}(A)}
\end{pmatrix}.
\] (3.17)

The similar notation \((B^\uparrow = (u+1,v,n+1), B^\downarrow = (u+1,v,n-1))\) etc will be used for the vertices in the upper and lower layers of the 3D lattice.

Now we define the antichiral \(L\)-operator to be

\[
\bar{L}^{(\bar{v})}_{C\uparrow-A}(z) = -\lambda_{pn}\lambda_{p'n} \left[ L^{(n)}_{C\uparrow-C\downarrow}(z) \right]^{-1} L^{(n)}_{C\uparrow-C\downarrow}(z).
\] (3.18)

It generates a discrete flow \(\bar{v}\) (which we call antichiral) defined by the vector field \(\partial\bar{v} \equiv -\partial_n + \partial_u\) in the space of variables. We can display the flow \(\bar{v}\) on the graph of flows from Sect. 2.2:

\[\begin{array}{c}
\lambda_u \\
\lambda_v \\
\lambda_n \\
\lambda_p
\end{array}\]

\[\begin{array}{c}
\lambda_u \\
\lambda_v \\
\lambda_n \\
\lambda_p
\end{array}\]

\[
\bar{L}^{(\bar{v})}_{C\uparrow-A}(z) = \begin{pmatrix}
\lambda_{mn} z^2 \frac{\phi(A)}{\phi(C^\downarrow)} - \lambda_{p'n} \lambda_{pn} \frac{\phi(C^\downarrow)}{\phi(A)} & -z\lambda_{pn} \frac{\psi^-(A)}{\phi(A)\phi(C^\downarrow)} \\
-z\lambda_{pn} \phi(A)\phi(C^\downarrow) \psi^+(C^\downarrow) & -\lambda_{pn} \frac{\phi(A)\phi(C^\downarrow)}{\phi(C^\downarrow)}
\end{pmatrix}.
\] (3.19)

\((v\text{-counterparts of eqs. (2.6), (2.7) have been used}).

**Remark** Although we do not need the antichiral flows themselves, we found it instructive to mention them here in order to motivate the choice of the \(L\)-operator in the next section. Let us stress that antichiral flows have “equal rights” with the chiral ones. The symmetry can be seen from the figure and from the explicit form of the \(L\)-operators. Note also that in the case of the discrete KdV or SG models there is no difference between chiral and antichiral flows.

## 4 ”Composite” \(L\) and \(M\) operators

The chiral and antichiral \(L\)-operators introduced in the previous section are building blocks for more complicated ones obtained as their ordered products. Not all of them have an \(R\)-matrix representation of the desired form. Our next task is, therefore, to find a special pair of flows \(l, m\) such that the corresponding \(L\) and \(M\) operators meet the requirement. The lattice spanned by \(l, m\) will be called the space-time lattice. It is embedded into the 3D lattice with coordinates \(u, v, n\).

Our experience in the lattice SG model suggests that the good candidates are ”composite” \(L\) and \(M\) operators which generate translations along diagonals of the chiral space-time lattice. In the discrete HM
model this prescription works literally for the M-operator only. That is why, contrary to the tradition, we
discuss the M-operator first. The L-operator requires an important modification coming from the fact that
in general the space-time lattice is embedded into the 3D lattice in a different way than the chiral one.

4.1 The ”composite” M-operator

Consider the ”composite” operator which transfers the wave function along the diagonal \( C \to B \):

\[
\hat{M}_{B\to C}(z) \equiv z^{-1}(\lambda_{uv}z^2 - \lambda_{u}\lambda_{p}v)L_{B\to A}^{(u)}(z) \left[ L_{C\to A}^{(v)}(z) \right]^{-1}.
\] (4.1)

From [3,12] we have:

\[
\hat{M}_{B\to C}(z) = \begin{pmatrix}
z\lambda_{uv} \frac{\phi(C)}{\phi(B)} - z^{-1}\lambda_{u}\lambda_{p}v\frac{\phi(B)}{\phi(C)} & \lambda_{u}^{-\frac{1}{2}}\frac{\psi-(D)}{\phi(D)}\phi(M)(C, B) \\
\lambda_{p}^{\frac{1}{2}}\phi(A)^2\phi(M)(C, B) & z\lambda_{uv} \frac{\phi(B)}{\phi(C)} - z^{-1}\lambda_{u}\lambda_{p}v\frac{\phi(C)}{\phi(B)}
\end{pmatrix},
\] (4.2)

\[\phi(M)(C, B) = \lambda_{u}^{-\frac{1}{2}}\frac{\phi(C)}{\phi(B)} - \lambda_{p}^{\frac{1}{2}}\phi(B) - \lambda_{p}^{\frac{1}{2}}\phi(C).\]

The diagonal elements are brought into this form using eqs. (2.6) and (2.7). The non-diagonal elements are
obtained with the help of eqs. (3.17), (3.18).

Let us express the matrix elements in terms of the \( \tau \)-function. For any two vertices \( X, Y \) of the lattice
we set

\[W(X, Y) \equiv \left[ \tau_n^{p+1}(X)_{n+1}(X)\tau_n^{p+1}(Y)\tau_n^{p+1}(Y) \right]^{\frac{1}{2}}.
\]

Using eqs. (2.11), (2.13), we find:

\[
\hat{M}_{B\to C}(z) = \frac{1}{W(C, B)} \begin{pmatrix}
z\lambda_{uv} \tau_{n+1}^{p+1}(C)\tau_{n}^{p+1}(B) & -\lambda_{uv} \lambda_{p}^{\frac{1}{2}}\tau_{n+1}^{p+1}(A)\tau_{n}^{p+1}(D) \\
-\lambda_{u}^{\frac{1}{2}}\lambda_{p}^{\frac{1}{2}}\tau_{n+1}^{p+1}(A)\tau_{n}^{p+1}(D) & z\lambda_{uv} \tau_{n+1}^{p+1}(B)\tau_{n+1}^{p+1}(C) - z^{-1}\lambda_{u}^{\frac{1}{2}}\lambda_{p}^{\frac{1}{2}}\tau_{n+1}^{p+1}(C)\tau_{n+1}^{p+1}(B)
\end{pmatrix}.
\] (4.3)

Moreover, with the help of eqs. (2.17), (2.18) it is possible to express the matrix elements through the
\( \tau \)-functions at the points \( A \) and \( D \) only, while the dependence on the points \( C, B \) is confined in the prefactor
\( W^{-1}(C, B) \). Indeed, the r.h.s. of eqs. (2.17), (2.18) are just the two terms on the diagonal of the matrix in
(4.3). Substituting them by the left hand sides, we see that the matrix depends on \( A \) and \( D \) only. This is
used in Sect. 5.

4.2 The ”composite” L-operator – first version

The first candidate is suggested by our experience in the discrete SG model [3]:

\[
L_{D\to A}^{(uv)}(z) = z^{-1}L_{D\to B}^{(v)}(z)L_{B\to A}^{(u)}(z).
\] (4.4)

Using the bilinear equations from Sect. 2 many times, we can express its matrix elements through the \( \tau \)-function:

\[
L_{D\to A}^{(uv)}(z) = \frac{1}{\lambda_{uv}W(A, D)} \begin{pmatrix}
L_{11}^{(uv)} & L_{12}^{(uv)} \\
L_{21}^{(uv)} & L_{22}^{(uv)}
\end{pmatrix}.
\] (4.5)
with

\[
L_{11}^{(uv)} = \lambda_{pu} \zeta_u(z) \tau^p_{n+1}(B) \tau^{p+1}_{n+1}(C) - \lambda_{pu} \zeta_v(z) \tau^p_{n+1}(C) \tau^{p+1}_{n+1}(B),
\]

\[
L_{12}^{(uv)} = \lambda_{pn} z \left( \zeta_u(z) \tau^p_{n+1}(B) \tau^{p+1}_{n+1}(C) - \zeta_v(z) \tau^p_{n+1}(C) \tau^{p+1}_{n+1}(B) \right),
\]

\[
L_{21}^{(uv)} = \lambda_{pn} z \left( \zeta_u(z) \tau^p_{n+1}(B) \tau^{p+1}_{n+1}(C) - \zeta_v(z) \tau^p_{n+1}(C) \tau^{p+1}_{n+1}(B) \right),
\]

\[
L_{22}^{(uv)} = \lambda_{pn} \zeta_u(z) \frac{\zeta_v(z)}{\lambda_{un} \lambda_{vn}} \left( \lambda_{p'v} \tau^p_{n+1}(B) \tau^{p+1}_{n+1}(C) - \lambda_{p'u} \tau^p_{n+1}(C) \tau^{p+1}_{n+1}(B) \right) + \frac{\lambda_{pn}}{\lambda_{un} \lambda_{vn}} \left( \lambda_{p'v} \lambda_{un} \zeta_u(z) \tau^p_{n+1}(B) - \lambda_{p'u} \lambda_{vn} \zeta_v(z) \tau^p_{n+1}(C) \tau^{p+1}_{n+1}(B) \right). \tag{4.9}
\]

Here

\[
\zeta_u(z) = \lambda_{un} z^2 - \lambda_{pu} \lambda_{p'u}, \quad \zeta_v(z) = \lambda_{vn} z^2 - \lambda_{pv} \lambda_{p'v}. \tag{4.10}
\]

As it will be clear later, the desired \( R \)-matrix representation for this operator does not exist since the matrix elements have a "wrong" dependence on \( z \). We should seek for an \( L \)-operator which would depend on \( z \) as in (4.3).

### 4.3 The "composite" \( L \)-operator – improved version

The above \( L \)-operator can be "improved" by an additional translation in \( n \). The right choice is the operator which generates the translation

\[
A = (u, v, n) \rightarrow D^\downarrow = (u + 1, v + 1, n - 1)
\]

in the 3D lattice spanned by \( u, v, n \). Its elementary cell is shown in the figure:

\[
\begin{align*}
A^{\downarrow} = (u, v, n - 1) & \quad \rightarrow \quad B^{\downarrow} = (u + 1, v, n - 1) \\
C = (u, v + 1, n) & \quad \rightarrow \quad D = (u + 1, v + 1, n) \\
B = (u + 1, v, n) & \quad \rightarrow \quad C^{\downarrow} = (u, v + 1, n - 1) \\
A = (u, v, n) & \quad \rightarrow \quad \text{elementary cell}
\end{align*}
\]

One of possible ways to represent such an \( L \)-operator in terms of already mentioned ones is:

\[
\hat{L}_{D^\downarrow \leftarrow A}(z) = -\lambda_{pn} \lambda_{p'n} \left[ L^{(n)}_{D^\downarrow \leftarrow D^\downarrow}(z) \right]^{-1} L^{(uv)}_{D^\downarrow \leftarrow A}(z) \tag{4.11}
\]
with $L_{D_{+}-D_{-}}^{(n)}(z)$ as in (3.17) and $\mathcal{L}_{D_{+}^{(u)}}^{(w)}(z)$ as in (1.3). Matrix elements of $\hat{L}_{D_{+}^{(u)}-A}^{(w)}(z)$ are Laurent polynomials in $z$. From (4.11), (3.12) and (4.6)-(4.10) we see that the diagonal elements are of at most 3-d degree, the right upper (resp., left lower) element is of at most 4-th (resp., 2-n d) degree.

It turns out that this $L$-operator actually has a surprisingly simple structure – much simpler than one could expect from (4.11). In particular, using the bilinear equations, one can show that the highest degrees of $z$ cancel leaving us with at most $z$ and $z^{-1}$-terms in the diagonal elements and $z$-independent terms in the non-diagonal ones.

This cancellation could be expected from the following equivalent representation of the $L$-operator (4.11) which makes use of the antichiral flow introduced in Sect. 3.4:

$$\hat{L}_{D_{+}^{(u)}-A}^{(w)}(z) = L_{D_{+}^{(u)}-C_{+}}^{(w)}(z) L_{C_{+}^{(v)}-A}^{(w)}(z).$$

This means that the flow generated by the $L$-operator (4.11) is the "superposition" of the chiral and antichiral ones.

Computing the $L$-operator by means of any one of the two formulas, we find:

$$\hat{L}_{D_{+}^{(u)}-A}^{(w)}(z) = \begin{pmatrix} z\lambda_{un}\lambda_{vn} \phi(A) \phi(D) + z^{-1}\lambda_{pu}\lambda_{pv}\lambda_{p'n} \phi(D^1) & \lambda_{pn}^\frac{1}{2} \psi^-(B) \phi(B)^2 \varphi_L(A, D^1) \\ \lambda_{pn}^\frac{1}{2} \psi^+(C^1) \phi(C^1) \varphi_L(A, D^1) & \frac{1}{z\lambda_{pn}\lambda_{p'n} \phi(A) \phi(D^1) + z^{-1}\lambda_{pn}\lambda_{p'u}\lambda_{p'v} \phi(A) \phi(D^1)} \end{pmatrix},$$

(4.12)

In terms of the $\tau$-function we have:

$$\hat{L}_{D_{+}^{(u)}-A}^{(w)}(z) = \frac{1}{W(A, D^1)} \begin{pmatrix} z\lambda_{un}\lambda_{vn} \tau_n^p(D) \tau_{n+1}^p(A) + z^{-1}\lambda_{pu}\lambda_{pv}\lambda_{p'n} \tau_{n-1}^p(D) \tau_{n+1}^p(A) & \lambda_{pn} \tau_{n}^p(B) \tau_{n}^p(C) \\ \lambda_{pn}^\frac{1}{2} \tau_{n}^p(B) \tau_{n}^p(C) & \frac{1}{z\lambda_{pn}\lambda_{p'n} \tau_{n-1}^p(D) \tau_{n+1}^p(A) + z^{-1}\lambda_{pn}\lambda_{p'u}\lambda_{p'v} \tau_{n}^p(D) \tau_{n}^p(A)} \end{pmatrix},$$

(4.13)

($\Lambda$ is given in (2.19)).

### 4.4 The space-time lattice

Let us consider the 2D lattice spanned by coordinates $l, m$ which are flows generated by the $L$ and $M$ introduced above. The vector fields

$$\partial_l = \partial_u + \partial_v - \partial_n, \quad \partial_m = \partial_u - \partial_v$$

are then "coordinate axes" of this lattice. (Note the equivalent representations of these vector fields in terms of antichiral flows: $\partial_l = \partial_u + \partial_v, \partial_m = \partial_u - \partial_v$.) Shifting the origin of coordinates, we always can embed this lattice into the 3D one as the 2D plane defined by the homogeneous linear equation

$$u + v + 2n = 0$$

(4.14)

provided $(u, v, n) \in \mathbb{Z}^3$. In the sequel this 2D lattice is referred to as the space-time lattice (STL). Note that the chiral lattice is the plane $n = 0$, i.e. it is the linear projection of the STL along the $n$-direction.
The space-time coordinates \( l, m \) are introduced by the formulas

\[
l = \frac{1}{2} (u + v), \quad m = \frac{1}{2} (u - v)
\]  

(4.15)

provided the point \((u, v, n)\) belongs to the STL. (Under the same condition \( l \) could be equivalently defined as \( l = -n \).)

**Remark** After an obvious matching of the notation, the zero curvature condition on the elementary cell of the STL for the constructed "composite" \( L-M \)-pair takes the form (1.9). It gives rise to a system of non-linear equations for fields at the vertices of eight neighbouring cubes of the 3D lattice. This system is a direct consequence of the basic equations of motion (3.14)-(3.16).

## 5 \( R \)-matrix representation of the \( L-M \)-pair

Our goal is to represent the \( L \) and \( M \) operators (4.3), (4.13) constructed in the previous section as convolutions of a quantum \( R \)-matrix with some vectors in its "quantum" space.

### 5.1 The quantum \( R \)-matrix

Consider the following quantum \( R \)-matrix with the spectral parameter \( z \):

\[
R(z) = \frac{1}{4} [2a(z) + b_+(z) + b_-(z)] I \otimes I + \frac{1}{4} [2a(z) - b_+(z) - b_-(z)] \sigma_3 \otimes \sigma_3 \\
+ \frac{1}{4} [b_+(z) - b_-(z)] (I \otimes \sigma_3 - \sigma_3 \otimes I) + \frac{1}{2} c(z) (\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2)
\]  

(5.1)

\( (\sigma_i \) are Pauli matrices), where

\[
a(z) = qz - q^{-1}z^{-1}, \quad b_{\pm}(z) = \xi^{\pm1} (z - z^{-1}), \quad c(z) = q - q^{-1}.
\]  

(5.2)

When necessary, we write \( R(z) = R(z; q, \xi) \); \( q \) is the "quantum" parameter and \( \xi \) is the parameter of Drinfeld’s twist [20], [27]:

\[
R(z; q, \xi) = F(\xi) R(z; q, 1) F(\xi),
\]

\[
F(\xi) = \xi^{\frac{1}{2} (I \otimes \sigma_3 - \sigma_3 \otimes I)} = \text{diag} \left( 1, \xi^{\frac{1}{2}}, \xi^{-\frac{1}{2}}, 1 \right).
\]

The \( R \)-matrix (5.1) satisfies the quantum Yang-Baxter equation for any \( q, \xi \):

\[
R_{12}(z_1/z_2) R_{13}(z_1/z_3) R_{23}(z_2/z_3) = R_{23}(z_2/z_3) R_{13}(z_1/z_3) R_{12}(z_1/z_2),
\]  

(5.3)

where \( R_{12}(z) \) acts in \( \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \) as the \( R(z) \) in the first and the second spaces and as the \( I \) in the third one (similarly for \( R_{13}(z) \) and \( R_{23}(z) \)).

The first (resp., second) space of tensor products in (5.1) is called "quantum" (resp., "auxiliary") space. It is convenient to represent the \( R \)-matrix as a 2×2 block matrix in the "auxiliary" space. Let \( i, i' \) number
block rows and columns and let \( j, j' \) number rows and columns inside each block (i.e., in the "quantum" space). Then matrix elements of the \( R \)-matrix (5.3) are denoted by \( R(z)_{j j'}^{i i'} \).

Let \( |\alpha\rangle, |\beta\rangle \) be two vectors in the quantum space (see (1.4)). Each block of the \( R \)-matrix is an operator in the quantum space. Consider its action to \( |\alpha\rangle \) and subsequent scalar product with \( \langle \beta | \). The result is a 2×2 matrix in the auxiliary space:

\[
\langle \beta | R(z) | \alpha \rangle_{ii'} = \sum_{jj'} R(z)_{j j'}^{i i'} \alpha_j \beta_j.
\]

Substituting the matrix (5.1), we find:

\[
\langle \beta | R(z) | \alpha \rangle = \begin{pmatrix}
\beta_1 \alpha_1 a(z) + \beta_2 \alpha_2 b_+(z) & \beta_2 \alpha_1 b_-(z) + \beta_2 \alpha_2 c(z) \\
\beta_1 \alpha_2 c(z) & \beta_1 \alpha_1 b_-(z) + \beta_2 \alpha_2 a(z)
\end{pmatrix}
\]

\[
= \begin{pmatrix}
z(q\beta_1 \alpha_1 + \xi \beta_2 \alpha_2) & (q - q^{-1})\beta_2 \alpha_1 \\
-z^{-1}(q^{-1}\beta_1 \alpha_1 + \xi \beta_2 \alpha_2) & (q - q^{-1})\beta_1 \alpha_2
\end{pmatrix}
\] (5.4)

Hereafter all scalar products of the type \( \langle \beta | R(z) | \alpha \rangle \) are understood as to be taken in the first ("quantum") space.

### 5.2 The \( L-M \)-pair in terms of the \( R \)-matrix

We are going to bring the \( L \) and \( M \) operators (4.11), (4.2) into the form (5.4). The best result is achieved after the diagonal gauge transformation

\[
\hat{\mathcal{M}}_{B+C}(z) \rightarrow \mathcal{M}_{B+C}(z) = \left( \frac{\tau_n^{p+1}(B)\tau_n^{p+1}(B)}{\tau_n^{p+1}(C)\tau_n^{p+1}(C)} \right)^{\frac{1}{2}} \hat{\mathcal{M}}_{B+C}(z),
\] (5.5)

\[
\hat{\mathcal{L}}_{D i+A}(z) \rightarrow \mathcal{L}_{D i+A}(z) = \left( \frac{\tau_n^{p+1}(D)\tau_n^{p}(D)}{\tau_n^{p+1}(A)\tau_n^{p}(A)} \right)^{\frac{1}{2}} \hat{\mathcal{L}}_{D i+A}(z)
\] (5.6)

and shift of the spectral parameter

\[
z \rightarrow kz, \quad k = \left( \frac{\lambda_{pu}\lambda_{nu}}{\lambda_{un}} \right)^{\frac{1}{2}}.
\] (5.7)

The matrices \( \mathcal{M}_{B+C}(kz) \), \( \mathcal{L}_{D i+A}(kz) \) are to be expressed through the \( \tau \)-function as is explained at the end of Sect. 4.1.

In order to present the result, let us introduce the vectors

\[
|\alpha(u, v, n)\rangle = \begin{pmatrix}
\mu \tau_n^p(u, v) \\
\mu^{-1} \tau_n^{p+1}(u, v)
\end{pmatrix}, \quad |\beta(u, v, n)\rangle = \begin{pmatrix}
\mu \tau_n^{p+1}(u, v) \\
\mu^{-1} \tau_n^p(u, v)
\end{pmatrix},
\] (5.8)

where

\[
\mu = \left( \frac{\lambda_{un}\lambda_{pu}}{\lambda_{pu}\lambda_{nu}} \right)^{\frac{1}{2}}
\] (5.9)
and identify the parameters as follows:

\[ q = \left( \frac{\lambda_{vu} \lambda_{pu} \lambda_{p'v}}{\lambda_{un} \lambda_{pu} \lambda_{p'u}} \right)^{\frac{1}{2}}, \quad \xi = \left( \frac{\lambda_{vu} \lambda_{pv} \lambda_{p'v}}{\lambda_{un} \lambda_{pv} \lambda_{p'u}} \right)^{\frac{1}{2}}, \quad \xi' = -\left( \frac{\lambda_{uv} \lambda_{pu} \lambda_{pv}}{\lambda_{un} \lambda_{pu} \lambda_{p'u}} \right)^{\frac{1}{2}} \]  

(5.10)

(note that \( \xi' = -\mu^4 \)). For brevity we also set

\[ \rho \equiv [\lambda_{pu} \lambda_{uv} \lambda_{pv} \lambda_{p'u} \lambda_{p'v}]^{\frac{1}{2}}, \]

then \( q - q^{-1} = -\lambda_{pu} \lambda_{uv} \Lambda \rho^{-1} \) (recall that \( \Lambda \) is given by (2.19)).

After these preparations, the comparison with (5.4) immediately yields the formulas

\[ \langle \beta(B) | R(z; q, \xi') | \alpha(C) \rangle = -\lambda_{uv} \rho^{-1} \tau_{n+1}^p (A) \tau_{n+1}^p (A) \mathcal{L}_{D_i \leftarrow A}(kz), \]  

(5.11)

\[ \langle \beta(D) | R(z; q, \xi) | \alpha(A^\dagger) \rangle = \Lambda \rho^{-1} \tau_{n+1}^p (C) \tau_{n+1}^p (C) \mathcal{M}_{B \leftarrow C}(kz). \]  

(5.12)

To recast them into the form (5.5), (5.7) respectively, we, first of all, fix the constant time slice \( m = 0 \) of the STL. It is embedded into the 3D lattice as the 1D lattice with vertices at the points \( A_l = (l, l, -l) \), \( l \in \mathbb{Z} \). Let us introduce the similar notation for other points: \( B_l = (l + 1, l, -l) \), \( B_l = (l + 1, l - 1, -l) \), \( D_l^+ = A_{l+1} = (l + 1, l + 1, -l - 1) \) etc and put

\[ \mathcal{L}_l(z) = \mathcal{L}_{D_l^+ \leftarrow A_l}(z), \quad \mathcal{M}_l(z) = \mathcal{M}_{B_l \leftarrow A_l}(z). \]  

(5.13)

Now we notice that

\[ \tau_{n+1}^p (u, v) \tau_{n+1}^p (u, v) = \frac{\lambda_{pu} \lambda_{vu} \lambda_{pv} \lambda_{p'u}}{\lambda_{un} \lambda_{pu} \lambda_{p'n} - \lambda_{un}^2} \langle \beta(u + 1, v, n) | \alpha(u - 1, v, n + 1) \rangle \]  

due to eq. (2.20). This relation allows us to rewrite eqs. (5.11), (5.12) in the form

\[ \mathcal{L}_l(kz) = -\gamma \frac{\langle \beta(B_l) | R(z; q, \xi') | \alpha(C_l) \rangle}{\langle \beta(B_l) | \alpha(C_{l-1}) \rangle}, \]  

(5.15)

\[ \mathcal{M}_l(kz) = \gamma \frac{\langle \beta(B_l) | R(z; q, \xi) | \beta(B_{l-1}) \rangle}{\langle \beta(B_l) | \alpha(C_{l-1}) \rangle}, \]  

(5.16)

where

\[ | \bar{\beta}(B_l) \rangle = \begin{pmatrix} \mu \tau_{l+1}^p (l + 1, l) \\ -\mu^{-1} \tau_{l+1}^p (l + 1, l) \end{pmatrix} = \begin{pmatrix} 0 & \mu^2 \\ \mu^{-2} & 0 \end{pmatrix} | \beta(B_l) \rangle \]

and \( \gamma \) is a constant:

\[ \gamma = [\lambda_{vu} \lambda_{pv} \lambda_{p'v}]^{\frac{1}{2}} (\lambda_{pu} \lambda_{pv} \lambda_{p'u} - \lambda_{un}^2). \]

These formulas differ from the ones announced in the Introduction merely by the irrelevant prefactors. To match the notation, let us rename the vectors: \( | \alpha(C_l) \rangle \rightarrow | \alpha(l) \rangle, \quad | \beta(B_l) \rangle \rightarrow | \beta(l) \rangle \), so from now on we set

\[ | \alpha(l) \rangle = \begin{pmatrix} \mu \tau_{l+1}^p (l + 1, l) \\ -\mu^{-1} \tau_{l+1}^p (l + 1, l) \end{pmatrix}, \quad | \beta(l) \rangle = \begin{pmatrix} \mu \tau_{l+1}^p (l + 1, l) \\ -\mu^{-1} \tau_{l+1}^p (l + 1, l) \end{pmatrix}. \]  

(5.17)

Their location in the 3D lattice is shown in the figure:
The constant time slice \( m = 0 \) is the chain of vertices \((\ldots, A_{l-1}, A_l, A_{l+1}, \ldots)\). The dashed line shows the translation generated by the \( M \)-operator (5.16).

**Remark** The \( L \)-operator \( L_l(z) \) (5.15) has two degeneracy points: \( z = k \) and \( z = kq^{-1} \). At the first one the \( R \)-matrix is proportional to the permutation operator \( P \) in \( \mathbb{C}^2 \otimes \mathbb{C}^2 \): \( R(1; q, \xi') = (q - q^{-1})P \). Therefore,

\[
L_l(k) = \lambda_{uv}^2 \Lambda \frac{|\tilde{\alpha}(l)\rangle \langle \tilde{\beta}(l)|}{\tau_{l+1}^{p+1}(l, l)\tau_{l-1}^{p}(l, l)},
\]

(cf. (1.3)). At the second one we get:

\[
L_l(kq^{-1}) = \lambda_{uv}^2 \Lambda \frac{|\tilde{\beta}(l)\rangle \langle \tilde{\alpha}(l)|}{\tau_{l+1}^{p+1}(l, l)\tau_{l-1}^{p}(l, l)},
\]

\[
|\tilde{\beta}(l)\rangle = \begin{pmatrix} 0 & -\xi' \\ 1 & 0 \end{pmatrix} |\beta(l)\rangle, \quad |\tilde{\alpha}(l)\rangle = \begin{pmatrix} 0 & 1 \\ -\xi' - 1 & 0 \end{pmatrix} |\alpha(l)\rangle.
\]

Formulas of the type (5.15), (5.16) for the same \( L-M \)-pair can also be written in terms of the vectors \( |\tilde{\alpha}(l)\rangle \), \( |\tilde{\beta}(l)\rangle \).

### 6 Continuous time limit

In this section we show that the \( r \)-matrix formula for local \( M \)-operators (1.5) is a degenerate case of our \( R \)-matrix formula. It follows from Sect. 2 that \( \lambda_{uv} \) plays the role of lattice spacing for the discrete time variable \( m \). At the first glance, the continuous time limit would then imply \( \lambda_u \rightarrow \lambda_u \), i.e. \( q \rightarrow 1, \xi \rightarrow 1 \) in the \( R(z; q, \xi) \), so, in agreement with eq. (1.5), we do get a classical \( r \)-matrix. However, this would imply

\[
\lim_{q \rightarrow 1} |\tilde{\alpha}(l)\rangle = |\alpha(l)\rangle \quad \text{that is certainly wrong in general.}
\]

The correct limit to a continuous time coordinate is more subtle and needs some clarification.

#### 6.1 Limiting form of the \( M \)-operator

We need a continuous time limit such that the \( L \)-operator (6.15) remains the same. The naive limit does not work because varying \( \lambda_u \) one changes the STL and, therefore, the \( L \)-operator itself. In the correct limiting procedure, the time lattice spacing should approach zero independently of \( \lambda_u \) and \( \lambda_{uv} \).
To achieve the goal, we exploit the advantage of Miwa’s approach \([13]\) and introduce \(w\) – another "copy" of the chiral flow \(v\) with Miwa’s variable \(\lambda_v\). The bilinear equations from Sect. 2 apparently hold true for \(w\) and \(\lambda_w\) in place of \(v\) and \(\lambda_v\) respectively. Now we can safely tend \(\lambda_w \to \lambda_u\) thus eventually getting a continuous flow, leaving \(\lambda_v\) and all the other parameters unchanged. Set

\[
\hat{q} = \left( \frac{\lambda_{un} \lambda_{pu} \lambda_{p'u}}{\lambda_{un} \lambda_{pu} \lambda_{p'u}} \right)^{\frac{1}{2}}, \quad \hat{\xi} = \left( \frac{\lambda_{un} \lambda_{pu} \lambda_{p'u}}{\lambda_{un} \lambda_{pu} \lambda_{p'u}} \right) \hat{q}, \quad -\lambda_{uw} \equiv \varepsilon
\]

and consider the limit \(\varepsilon \to 0\). We have:

\[
\hat{q} = 1 + \frac{\lambda_{p'u} \lambda_{un} + \lambda_{pu} \lambda_{p'n}}{2\lambda_{un} \lambda_{pu} \lambda_{p'u}} \varepsilon + O(\varepsilon^2), \quad \hat{\xi} = 1 + \frac{\lambda_{p'u} \lambda_{un} - \lambda_{pu} \lambda_{p'n}}{2\lambda_{un} \lambda_{pu} + \lambda_{p'u}} \varepsilon + O(\varepsilon^2).
\]

Discrete \(M\)-operators are defined up to multiplication by a scalar function of \(z\) independent of dynamical variables. To pursue the continuous time limit, it is convenient to normalize the \(M\)-operators by the condition \(M^0_l(z) = I\) at \(\varepsilon = 0\). Then the next term (of order \(\varepsilon\)) yields local \(M\)-operator of a continuous time flow.

To implement this project, consider the 4D lattice with coordinates \((u, v, w, n)\). The \(m = 0\) slice of the former STL is embedded into this lattice as the 1D sublattice with vertices \(A_l = (l, l, 0, -l), l \in \mathbb{Z}\). The continuous time \(M\)-operator at the site \(A_l\) is obtained by expansion of the discrete \(M\)-operator \(M^0_{B_l+A_l}(z) = \lambda_{un} k(z - z^{-1}) I + O(\varepsilon)\) which generates the translation \(A_l = (l, l, 0, -l) \to \bar{B}^l = (l + 1, l, -1, -l)\). Passing to the normalized \(M\)-operator, we have:

\[
M^0_{\bar{B}^l+A_l}(z) = I + \varepsilon \tilde{M}_l(z) + O(\varepsilon^2).
\]

For clarity, the \(u, w\)-section of the 4D lattice (\(v = \text{const}, n = \text{const}\)) is displayed in the figure:

![Diagram](image)

The point \(\bar{B}^l\) tends to the point \(A_l\) as \(\varepsilon \to 0\), so the parallelogram collapses to the \(u\)-axis. Using eqs. (13), (2.13), we obtain:

\[
\tilde{M}_l(kz) = \frac{1}{2\lambda_{p'u}(z - z^{-1})} \left( \begin{array}{c}
\frac{1}{z} - z + (z + \frac{1}{z}) U_l \\
2\mu \frac{\tau_{p+1}^p(l + 1, l) \tau_{l-1}^{p+1} (l - 1, l)}{\tau_{p}^{p+1} (l, l) \tau_{l}^{p+1} (l, l)}
\end{array} \right),
\]

where

\[
U_l \equiv \frac{\tau_{l+1}^{p+1} (l + 1, l) \tau_{l}^{p} (l - 1, l)}{\tau_{l+1}^{p} (l, l) \tau_{l}^{p+1} (l, l)}.
\]

Note the redundant freedom which allows one to add to the \(\tilde{M}_l(z)\) a term \(h(z) I\) with a scalar function \(h(z)\) independent of dynamical variables. This obviously does not affect the zero curvature condition [L,B]. In the next subsection we shall use this freedom to redefine the \(M\)-operator.
6.2 Comparison with the $r$-matrix formula

Rather than to compute the limit of the r.h.s. of eq. (5.16) directly, it is easier to compare eq. (1.5) (for some $r$) and look at the "discrepancy" (if any). As a "first approximation" let us try the standard classical $r$-matrix

$$r^{(0)}(z) = \frac{1}{2(z - z^{-1})} \left[ (z + z^{-1})I \otimes I + 2\sigma_1 \otimes \sigma_1 + 2\sigma_2 \otimes \sigma_2 + (z + z^{-1})\sigma_3 \otimes \sigma_3 \right]$$

which works in the lattice SG model. It is skew-symmetric, i.e. for $z_1 \neq z_2$ it enjoys the property $r^{(0)}_{12}(z_1/z_2) = -r^{(0)}_{21}(z_2/z_1)$ where we use the notation from Sect. 5.1.

Plugging the vectors (5.17) into (1.5) with this $r$-matrix and using eq. (2.21), we compute the "discrepancy":

$$\frac{\lambda_{pn}\lambda_{p'n} - \lambda_{un}^2}{2\lambda_{un}\lambda_{pu}\lambda_{p'u'}} \frac{\langle \beta(l)|r^{(0)}(z)|\alpha(l - 1) \rangle}{\langle \beta(l)|\alpha(l - 1) \rangle} - \tilde{M}_I(kz) = \frac{1}{2} \begin{pmatrix} \lambda_{p'u}^{-1} & 0 \\ 0 & \lambda_{pu}^{-1} - \lambda_{un}^{-1} \end{pmatrix}$$

which stands in the right hand side. The result means that our "first approximation" is not so bad. To get zero in the r.h.s., we need two corrections – one in the $\tilde{M}_I(z)$ and one in the $r^{(0)}$. The first one is rather a matter of definition. Using the redundant freedom mentioned above, we are free to redefine the $M$-operator:

$$M_I(z) \equiv \tilde{M}_I(z) + \frac{1}{4} \left( \lambda_{p'u}^{-1} + \lambda_{pu}^{-1} - \lambda_{un}^{-1} \right) I.$$  

(6.7)

The second one is more important. Let $r^{(\kappa)}(z)$ be the modified $r$-matrix:

$$r^{(\kappa)}(z) = r^{(0)}(z) + \kappa I \otimes \sigma_3,$$  

(6.8)

$$\kappa = \frac{\lambda_{p'u} \lambda_{un} - \lambda_{pu} \lambda_{p'u}}{2\lambda_{un} \lambda_{pu} \lambda_{p'u}}.$$  

(6.9)

The $\kappa$-term violates the skew-symmetry; nevertheless, the $r^{(\kappa)}(z)$ obeys the classical Yang-Baxter equation written in the form modified for not necessarily skew-symmetric $r$-matrices [28]:

$$[r^{(\kappa)}_{12}(z_1/z_2), r^{(\kappa)}_{13}(z_1/z_3)] + [r^{(\kappa)}_{12}(z_1/z_2), r^{(\kappa)}_{23}(z_2/z_3)] + [r^{(\kappa)}_{32}(z_3/z_2), r^{(\kappa)}_{13}(z_1/z_3)] = 0.$$  

(6.10)

Incorporating the corrections into eq. (6.6), we arrive at the $r$-matrix formula

$$M_I(kz) = \frac{\lambda_{pn}\lambda_{p'u} - \lambda_{un}^2}{2\lambda_{un}\lambda_{pu}\lambda_{p'u}} \frac{\langle \beta(l)|r^{(\kappa)}(z)|\alpha(l - 1) \rangle}{\langle \beta(l)|\alpha(l - 1) \rangle}.$$  

(6.11)

At $\kappa = 0$ (i.e. $\lambda_{p'u} \lambda_{un} = \lambda_{pu} \lambda_{p'u}$) one gets the standard $r$-matrix. (This just happens for the lattice SG model.) Note that in this case $\xi = 1 + O(\varepsilon^2)$ (see (6.2)), so the right classical $r$-matrix is obtained as a "germ" of the quantum one:

$$r^{(0)}(z) = \lim_{\varepsilon \to 0} \frac{R(z; \hat{q}, \hat{\xi}) - (z - z^{-1})I \otimes I}{(\hat{q} - 1)(z - z^{-1})}$$

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(provided $\xi = 1 + O(\varepsilon^2)$!).

**Remark** The non-skew-symmetric $r$-matrix (6.8) signifies that the $L$-operator at $\kappa \neq 0$ is not ultralocal. In this case Poisson brackets between elements of the monodromy matrix $T$ are given by a general quadratic Poisson bracket algebra

$$\{T_1, T_2\} = r^+_{12}T_1T_2 + T_1s^+_{12}T_2 - T_2s^-_{12}T_1 - T_1T_2r^-_{12}$$

studied (together with its quantum version) in [22], [30], [31]. The matrices $r^\pm, s^\pm$ meet a number of consistency conditions which ensure antisymmetry and Jacobi identity for the Poisson bracket. Besides, in integrable systems the constraint $r^+ + s^+ = s^- + r^-$ holds true. Under these conditions, the (non-skew-symmetric) $r$-matrix $r^+ - s^-$ satisfies the Yang-Baxter equation of the form (6.10). In our case the quadruplet $r^+, s^+, s^-, r^-$ is as follows: $r^+ = r^{(0)}, s^+ = -\kappa I \otimes I, s^- = -\kappa I \otimes \sigma_3, r^- = r^{(0)} + \kappa(I \otimes \sigma_3 - \sigma_3 \otimes I)$, so $r^+ - s^- = r^{(a)}$.

7 Conclusion

The main result of this work is the $R$-matrix representation (5.15), (5.16) of the local $L$-$M$ pair for the classical discrete HM model. In our opinion, the very fact that typical quantum $R$-matrices naturally arise in a purely classical problem is important and interesting by itself. To recover them inside classical discrete problems, one has to pass to Hirota’s bilinear formalism and make use of Miwa’s interpretation of discrete flows. As a by-product, we have shown that components of the vectors $|\alpha\rangle, |\beta\rangle$ (representing the $L$-operator at the degeneracy point) are $\tau$-functions.

Let us recall that the quantum Yang-Baxter equation already appeared in connection with purely classical problems, though in a different context [22], [31]. However, the class of solutions relevant to classical problems is most likely very far from $R$-matrices of known quantum integrable models. In our construction, the role of the quantum Yang-Baxter equation remains obscure; instead, the most popular $4 \times 4$ trigonometric quantum $R$-matrix is shown to take part in the zero curvature representation of a classical discrete model. We believe that a conceptual explanation of this phenomenon nevertheless relies on the quantum Yang-Baxter equation.

We should stress that the ”quantum deformation parameter” $q$ of the trigonometric $R$-matrix in our context seems to have nothing to do with any kind of quantization. This fact suggests to extend this hidden $R$-matrix structure to the quantum level. In this case the vectors $|\alpha\rangle, |\beta\rangle$ will have operator components. If such an extension does exist, this would mean that there are two different $R$-matrices in a quantum integrable model rather than one. At the same time this would be an intelligent explanation of our $R$-matrix formulas – in the classical limit one of the $R$-matrices degenerates to a classical one while the other one survives.

As far as other possible projects are concerned, there are many problems apparently arising from this work. To mention only few, we point out first of all a detailed comparison with the hamiltonian approach, which is missing in our exposition. More specifically, it would be very instructive to identify the Poisson bracket algebra for elements of the $L$-operator (4.12) (in general non-ultralocal) and, starting from this algebra, to give an alternative derivation of the $r$-matrix formula (6.11) for the $M$-operator. Another interesting question is to find a discrete time analogue of the non-local generating function (1.2) of $M$-operators. At last, we point out the challenging problem to pursue all the program for the ”master model” with $2 \times 2 L$-$M$-pair – a discrete analogue of the Landau-Lifshitz model ($XYZ$ magnet). In this case the spectral parameter lives on an elliptic curve and the proper extension of the bilinear formalism is not completely clear. (In particular, how the reduction condition (6.2) looks like?)

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References

[1] L.Faddeev and L.Takhtadjan, *Hamiltonian methods in the theory of solitons*, Springer, 1987.

[2] V.Zakharov, S.Manakov, S.Novikov and L.Pitaevskiy, *Theory of solitons. The inverse problem method*. Nauka, Moscow, 1980.

[3] A.Zabrodin, *Zero curvature representation for classical lattice sine-Gordon model via quantum R-matrix*, preprint ITEP-TH-47/97, [hep-th/9709168](http://arxiv.org/abs/hep-th/9709168), to be published in ZhETP Lett.

[4] E.K.Sklyanin, *Quantum version of the inverse scattering method*, Zap. Nauchn. Semin. LOMI 95 (1980) 55-128.

[5] A.G.Izergin and V.E.Korepin, *The lattice quantum sine-Gordon model*, Lett. Math. Phys. 5 (1981) 199-205; *Lattice versions of quantum field theory models in two dimensions*, Nucl. Phys. B205 (1982) 401-413.

[6] L.Faddeev and L.Takhtadjan, *Quantum inverse scattering method and XYZ Heisenberg model*, Uspekhi Mat. Nauk, 34:5 (1979) 13-63.

[7] V.E.Korepin, N.M.Bogoliubov and A.G.Izergin, *Quantum inverse scattering method and correlation functions*, Cambridge University Press, 1993.

[8] M.Jimbo and T.Miwa, *Solitons and infinite dimensional Lie algebras*, Publ. RIMS, Kyoto Univ. 19 (1983) 943-1001.

[9] G.Segal and G.Wilson, *Loop groups and equations of KdV type*, Publ. IHES 61 (1985) 5-65.

[10] R.Hirot, *Nonlinear partial difference equations I*, Journ. Phys. Soc. Japan 43 (1977) 1424-1433.

[11] R.Hirot, *Nonlinear partial difference equations II; Discrete time Toda equations*, Journ. Phys. Soc. Japan 43 (1977) 2074-2078.

[12] R.Hirot, *Discrete analogue of a generalized Toda equation*, Journ. Phys. Soc. Japan 50 (1981) 3785-3791.

[13] T.Miwa, *On Hirota’s difference equations*, Proc. Japan Acad. 58 Ser.A (1982) 9-12.

[14] E.Date, M.Jimbo and T.Miwa, *Method for generating discrete soliton equations I, II*, Journ. Phys. Soc. Japan 51 (1982) 4116-4131.

[15] S.Saito and N.Saito, *Linearization of bilinear difference equations*, Phys. Lett. A120 (1987) 322-326; *Gauge and dual symmetries and linearization of Hirota’s bilinear equations*, Journ. Math. Phys. 28 (1987) 1052-1055.

[16] E.Sklyanin, *On some algebraic structures connected with the Yang-Baxter equation*, Funk. Anal. i ego Pril., 16:4 (1982) 27-34.

[17] L.Faddeev and A.Volkov, *Hirota equation as an example of integrable symplectic map*, Lett. Math. Phys. 32 (1994) 125-133.

[18] A.Volkov, *Quantum lattice KdV equation*, University of Uppsala preprint (1995), [hep-th/9509024](http://arxiv.org/abs/hep-th/9509024).

[19] L.D.Faddeev, *Current-like variables in massive and massless integrable models*, Lectures at E.Fermi Summer School, Varenna 1994, [hep-th/9406191](http://arxiv.org/abs/hep-th/9406191).

[20] V.Bazhanov, A.Bobenko and N.Reshetikhin, *Quantum discrete sine-Gordon model at roots of 1: integrable system on the integrable classical background*, Commun. Math. Phys. 175 (1996) 377-400.

[21] R.Kashaev and N.Reshetikhin, *Affine Toda field theory as a 3-dimensional integrable system*, Commun. Math. Phys. 188 (1997) 251-266.

22
[22] K. Ueno and K. Takasaki, *Toda lattice hierarchy*, Adv. Studies in Pure Math. **4** (1984) 1-95.

[23] A. Zabrodin, *A survey of Hirota’s difference equations*, preprint ITEP-TH-10/97, [solv-int/9704001](https://arxiv.org/abs/solv-int/9704001), to be published in Teor. Mat. Fyz.

[24] I. Krichever, *Algebraic curves and non-linear difference equations*, Uspekhi Mat. Nauk, **33:4**, 215-216.

[25] I. Krichever, P. Wiegmann and A. Zabrodin, *Elliptic solutions to difference non-linear equations and related many-body problems*, preprint ITEP-TH-13/97, [hep-th/9704090](https://arxiv.org/abs/hep-th/9704090), to be published in Commun. Math. Phys.

[26] V. G. Drinfeld, *Quasi-Hopf algebras*, Leningrad Math. J. **1:6** (1990) 1419-1457.

[27] N. Reshetikhin, *Multiparameter quantum groups and twisted quasitriangular Hopf algebras*, Lett. Math. Phys. **20** (1990) 331-335.

[28] O. Babelon and C.-M. Viallet, *Hamiltonian structures and Lax equations*, Phys. Lett. **B237** (1989) 411-416.

[29] L. Freidel and J. M. Maillet, *Quadratic algebras and integrable systems*, Phys. Lett. **B262** (1991) 278-284.

[30] F. W. Nijhoff, H. W. Capel and V. G. Papageorgiou, *Integrable quantum mappings*, Phys. Rev. **A46** (1992) 2155-2158; F. W. Nijhoff and H. W. Capel, *Integrable quantum mappings and quantization aspects of integrable discrete-time systems*, [hep-th/9212083](https://arxiv.org/abs/hep-th/9212083).

[31] M. Semenov-Tian-Shansky, *Monodromy map and classical r-matrices*, Zap. Nauchn. Semin. LOMI **200** (1993) 156-166, [hep-th/9402054](https://arxiv.org/abs/hep-th/9402054); M. Semenov-Tian-Shansky and A. Sevostyanov, *Classical and quantum nonultralocal systems on the lattice*, [hep-th/9509029](https://arxiv.org/abs/hep-th/9509029).

[32] E. K. Sklyanin, On classical limits of $SU(2)$-invariant solutions of the Yang-Baxter equation, Zap. Nauchn. Semin. LOMI **146** (1985) 119-136.

[33] A. Weinstein and P. Xu, *Classical solutions of the quantum Yang-Baxter equation*, Commun. Math. Phys. **148** (1992) 309-343.