EXISTENCE AND NON-EXISTENCE OF BOUNDED PACKING
IN CAT(0) SPACES AND GROMOV HYPERBOLIC SPACES

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ABSTRACT. The main result of this paper is that given a group $G$ acting geometrically by isometries on a CAT(0) space $X$ and a cyclic subgroup $H$ of $G$ generated by a rank-1 isometry of $X$, $H$ has bounded packing in $G$. We give two proofs of this result. The first one is by a clever argument of Mj (Lemma 3.3 of [Mj08]) and the characterization of rank-1 isometries by Hamenstadt ([Ham09]). The second proof follows directly from some results of Dahmani, Guirardel and Osin ([DGO14]) and Sisto ([Sis13]). Then using Mikhailova’s construction, we show the existence of a finitely generated subgroup of the direct product of two free groups $F_2 \times F_2$ without the bounded packing property answering a question of Hruska-Wise ([HW09]). We also prove the existence of finitely presented subgroups of CAT(0) groups without bounded packing using Wise’s modified Rip’s construction ([Wiz98]) and the 1-2-3 theorem of Baumslag, Bridson, Miller and Short ([BBIS00]).

1. Introduction

Bounded packing was defined by Hruska and Wise ([HW09]) motivated by the concept of width of subgroups due to [GMRS97]. Hruska and Wise proved bounded packing of quasi-convex subgroups of Gromov hyperbolic groups and relatively quasi-convex subgroups of relatively hyperbolic groups under mild and natural restrictions. Bounded packing proves to be useful for two reasons. One is that failure of bounded packing of $H$ in $G$ - which is a geometric condition, implies that $H$ is not separable in $G$ - which is an algebraic property. This result is due to Yang ([Yan11]). On the other hand, as commented by Hruska-Wise in [HW09] (Corollary 3.1) the following result follows from the work of Sageev ([Sag97]):

**Theorem:** Suppose $H$ is a finitely generated codimension 1 subgroup of a finitely generated group $G$. If $H$ has bounded packing in $G$, then the corresponding CAT(0) cube complex $C$ is finite dimensional.

However, constructing new examples of groups with subgroups with (or without) bounded packing seems difficult although it is known to hold for quasi-convex subgroups of hyperbolic groups, all subgroups of polycyclic groups and nilpotent groups and so on. (See Example 2.19, Example 2.22 of [HW09], and the main results of [Yan11] and [Sar15] for more examples.) Moreover, many natural questions about bounded packing remain unanswered. For example in [HW09] we have:

**Problem** (Problem 2.24 of [HW09]): Give an example of a cyclic subgroup $Z$ of a finitely generated group $G$ such that $Z$ does not have bounded packing in $G$.

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Recently Wise and Woodhouse ([WW15]) have shown that abelian subgroups of groups acting geometrically on CAT(0) cube complexes have bounded packing. However, the following special case of the above problem is still not answered.

**Problem** (Problem 1.1(1) of [WW15]): Let $G$ act properly and cocompactly on a CAT(0) space. Does each [cyclic] abelian subgroup $A$ of $G$ have bounded packing?

Towards this direction we show that if $G$ is a group acting geometrically a CAT(0) space $X$ and $H$ is a cyclic subgroup generated by a rank-1 isometry then $H$ has bounded packing in $G$. However, we prove that there are finitely generated subgroups of $F_2 \times F_2$ without bounded packing answering Question 2.25 of [HW09]. Also we show that there are finitely presented subgroups of CAT(0) groups without bounded packing using Wise’s modified Rip’s construction ([Wis98]) and the 1-2-3 theorem of Baumslag, Bridson, Miller and Short ([BBIS00]).

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## 2. Bounded Packing

**Definition 2.1.** ([HW09]) Suppose $G$ is a countable group with a proper, left invariant metric $d$. Let $H \leq G$ be a subgroup. We say that $H$ has bounded packing in $G$ (with respect to $d$) if for any $D > 0$ there is a number $n = n(G, H, D)$ such that given any collection of left cosets $I \subset G/H$ of $H$ in $G$ such that $d(g_1H, g_2H) \leq D$ for all $g_1H, g_2H \in I$, we have $|I| \leq n$.

By Lemma 2.2 of [HW09] we know that bounded packing for a subgroup $H$ of a countable group $G$ is independent of the choice of the particular left invariant proper metric on $G$. On the other hand, by the work of Higman, Neumann and Neumann ([HBN49]) we know that any countable group can be embedded in a 2-generated group. Since every finitely generated group admits a proper, invariant metric- namely a word metric, it follows that any countable group admits a left invariant proper metric. Therefore, bounded packing of subgroups makes sense for subgroups of all countable groups.

The following lemma is a summary of some basic results about bounded packing which are proved by Hruska-Wise in the second section of [HW09]. We will make repeated use of it later.

**Lemma 2.2.** ([HW09]) Let $G$ be a countable group with a proper left invariant metric. Then the following are true.

1. Every finite subgroup of $G$ has bounded packing in $G$.
2. Every finite index subgroup of $G$ has bounded packing in $G$.
3. For any sequence of subgroups $K \subset H \subset G$ if $K$ has bounded packing in $H$ and $H$ has bounded packing in $G$, then $K$ has bounded packing in $G$.
4. For any sequence of subgroups $K \subset H \subset G$ if $[H : K] < \infty$ then $K$ has bounded packing in $G$ if and only if $H$ has bounded packing in $G$. 
Lemma 3.2. First and the second conditions of the theorem hold.

For convenience it is broken into the following two lemmas. We assume that the infinite sequence of distinct cosets $G$ on such that each one of them has two members close to identity. Now, since the metric of the metric on $G$ by the following conditions hold:

\[ (1) \quad \text{Limit set property: For any } x \in X \text{ and any infinite sequence of distinct elements } \{h_n\} \text{ in } H \text{ there is a subsequence } \{h_{n_k}\} \text{ such that } \lim_{k \to \infty} h_{n_k} x \text{ exists and it is in } A. \]

\[ (2) \quad \text{Dynamical quasi-convexity: For any sequence of distinct cosets } \{g_i H\} \text{ there is a subsequence } \{g_{i_k} H\} \text{ such that } \lim_{n \to \infty} g_{i_n} A \text{ is a single point in } X. \]

(3) For any pair of distinct cosets $g_1 H, g_2 H$ we have $g_1 A \cap g_2 A = \emptyset$.

Then $H$ has bounded packing in $G$.

The goal of this section is to prove the following theorem:

**Theorem 3.1.** Suppose $G$ is a finitely generated group acting by homeomorphism on a compact space $X$. Suppose $H \leq G$ and $A \subset X$ invariant under $H$ such that the following conditions hold:

1. Limit set property: For any $x \in X$ and any infinite sequence of distinct elements $\{h_n\}$ in $H$ there is a subsequence $\{h_{n_k}\}$ such that $\lim_{k \to \infty} h_{n_k} x$ exists and it is in $A$.
2. Dynamical quasi-convexity: For any sequence of distinct cosets $\{g_i H\}$ there is a subsequence $\{g_{i_k} H\}$ such that $\lim_{n \to \infty} g_{i_n} A$ is a single point in $X$.
3. For any pair of distinct cosets $g_1 H, g_2 H$ we have $g_1 A \cap g_2 A = \emptyset$.

Then $H$ has bounded packing in $G$.

The proof is an adaptation of the arguments in the proof of Lemma 3.3 of [Mj08]. For convenience it is broken into the following two lemmas. We assume that the first and the second conditions of the theorem hold.

**Lemma 3.2.** Suppose $C_i$ is an infinite sequence of sets of cosets of $H$ in $G$ such that for all $i$ and $xH, yH \in C_i$, $d(xH, yH) \leq D$ and $|C_i| \to \infty$. Then there is an infinite sequence of distinct cosets $\{g_n H\}$ such that $d(1, g_1 H) \leq D$, $d(1, g_2 H) \leq D$, $d(1, g_n H) \to \infty$ and $d(g_1 H, g_2 H) \leq D$ for $i = 1, 2$ and $j \geq 3$.

**Proof:** For all $i$ we can find an element $x_i \in G$ such that under left multiplication by $x_i$ two cosets of $C_i$ intersect $B(1; D)$. Thus we get a sequence of sets of cosets of $H$ such that each one of them has two members close to identity. Now, since the metric on $G$ is proper only finitely many cosets of $H$ can intersect $B(1; D)$. Therefore, after passing to a subsequence we may assume that each of $C'_i := x_i C_i$ contains two fixed cosets intersecting $B(1; D)$. Call these cosets $g_1 H, g_2 H$. Now, since $|C_i| \to \infty$ we can pick one coset $g_n H$ from $C'_n$ such that $d(1, g_n H) \to \infty$ because of the properness of the metric on $G$. Also, by construction $d(g_i H, g_j H) \leq D$ for $i = 1, 2$ and $j \geq 3$.

**Lemma 3.3.** Suppose we have an an infinite sequence of distinct cosets $\{g_n H\}$ such that $d(1, g_1 H) \leq D$, $d(1, g_2 H) \leq D$ and $d(g_i H, g_j H) \leq D$ for $i = 1, 2$ and $j \geq 3$. Then $g_1 A \cap g_2 A \neq \emptyset$.

**Proof:** We can have a sequence of distinct cosets $\{g_n H\}$ such that there are points $v_{ij} \in g_i H, w_{ij} \in g_j H$ for $i = 1, 2$, and $j \geq 3$ such that $d(v_{ij}, w_{ij}) \leq D$. Therefore, we can write $w_{ij} = v_{ij}, z_{ij}$ where $d(1, z_{ij}) \leq D$. If necessary, by passing to a subsequence, we may assume that $g_n A$ converges to $p \in X$. Now fix a point $q \in A$
and $i, 1 \leq i \leq 2$. Without loss of generality we can assume that $v_{ij}q$ is a convergent sequence. Passing to a subsequence we can assume that $z_{ij} = z$. Write $v_{ij} = g_i h_{ij}$ where $h_{ij} \in H$. Now consider the sequence $w_{ij}q = (g_i h_{ij}z)q = g_i (h_{ij} (zq))$. By the limit set property of $H$ we know that this sequence has a limit in $g_i A$. Thus $p \in g_i A$ for $i = 1, 2$. □

Proof of the theorem: Suppose $H$ does not have bounded packing in $G$. Then there is a constant $D > 0$ and an infinite sequence of sets $C_i$ of cosets of $H$ in $G$ such that $|C_i| \to \infty$ and for all $i$ and $xH, yH \in C_i$ we have $d(xH, yH) \leq D$.

Now, by the above lemmas then we can find two distinct cosets $g_1 H, g_2 H$ such that $g_1 A \cap g_2 A \neq \emptyset$. This contradicts the third hypothesis of the theorem and we are done. □

In [HW09] the authors raises the question if relatively quasi-convex subgroups of relatively hyperbolic groups have finite width. Since finite width is a consequence of bounded packing (see the proof of Corollary 8.11 in [HW09]), we obtain a partial answer to this question in the form of the following corollary.

**Definition 3.4.** Suppose $G$ is a relatively hyperbolic group and $H$ is a subgroup. Let $\partial G$ be the Bowditch boundary (see [Bow97] for detail) of $G$ and let $A$ be the limit set of $H$ in $\partial G$. We will call $H$ a dynamically malnormal subgroup of $G$ if for all $g \in G \setminus H$ we have $gA \cap A = \emptyset$.

Note that the same notion can be defined for a hyperbolic group $G$ taking the action of it on the Gromov boundary. However, for a quasi convex subgroup of a hyperbolic group dynamical malnormality is equivalent to almost malnormality which means for all $g \in G \setminus H$, $H \cap gHg^{-1}$ is finite (see [Sho91]). However, this is no longer true if we take a relatively quasi convex subgroup of a relatively hyperbolic group.

**Corollary 3.5.** Suppose $G$ is a relatively hyperbolic group and $H$ is a relatively quasi-convex, dynamically malnormal subgroup of $G$. Then $H$ has bounded packing in $G$.

In particular, $H$ has finite width in $G$.

**Proof**: One just needs to check the conditions of the above theorem. A relatively hyperbolic group $G$ acts on its Bowditch boundary $\partial G$ which is a compact space. Now for any relatively quasi-convex subgroup $H$ we can take $A$ to be its limit set. Condition (1) is true for any convergence action on a compact set. We know that $G$ acts on $\partial G$ as a convergence group and hence so does $H$. Condition (2) is proved by Dahmani (see Proposition 1.8 of [Dah03]). Condition (3) holds by definition of dynamical malnormality. □

**Corollary 3.6.** In a relatively hyperbolic group maximal parabolic subgroups have bounded packing.

This follows easily since we know that the maximal parabolic subgroups are dynamically malnormal and their limit sets are singleton sets.

**Remark 3.7.** If we use the fact that quasi-convex subgroup of a hyperbolic group has finite height as proved in [GMRS97] and the Lemma 3.3 of [Mj08] then using the proof of the above theorem we get an alternative proof of the fact that a quasi-convex subgroup of a hyperbolic group has bounded packing. However, this method fails to deal with the relatively hyperbolic situation.
4. Bounded packing of cyclic subgroups generated by rank-1 elements

This section contains two proofs of the following theorem using completely different sets of ideas. The first one is an application of Theorem 3.1, whereas the second proof uses results from Dahmani-Guirardel-Osin ([DGO14]) and Sisto ([Sis13]).

**Theorem 4.1.** (Bounded packing of rank-1 cyclic subgroups) Suppose $G$ is a CAT(0) group and $H$ is a cyclic subgroup generated by a rank-1 element. Then $H$ has bounded packing in $G$.

**Proof:** Let $Y$ be a CAT(0) space on which $G$ acts properly and cocompactly. Let $X = \partial Y$ be the visual boundary of $Y$ with the visual topology. Then $G$ acts on $X$ by homeomorphism. We will check that the pair $(G, H)$ satisfies all the conditions of Theorem 3.1 to finish the proof. For this purpose we shall use the results of Hamenstadt ([Ham09]). Let $A = \{\xi_1, \xi_2\} \subset X$ - the set containing the two fixed points in $X$ of the non-trivial isometries in $H$.

The condition (1) of Theorem 3.1 is verified by the fact that rank-1 isometries have north-south dynamics on $X$ (Lemma 4.4 of [Ham09]).

For condition (2) suppose $\{g_nH\}$ is a sequence of distinct cosets of $H$ in $G$ and suppose $\gamma = [\xi_1, \xi_2]$ is the geodesic line in $Y$ invariant under $H$. Suppose now that $g_n\xi_i$ converges to $\alpha_i$ for $i = 1, 2$ and $\alpha_1 \neq \alpha_2$. Fix a point $p \in Y$ and drop a perpendicular $[p, p_n]$ on to $g_n\gamma$. Then by Lemma 3.2(2) of [Ham09] the points $p_n$ are uniformly close to the geodesics $[p, g_n\xi_1]$ and $[p, g_n\xi_2]$. Since $g_n\xi_i$ converges to $\alpha_i$ and $\alpha_1 \neq \alpha_2$, we have that $\angle_p(g_n\xi_1, g_n\xi_2) \geq \epsilon$ for some $\epsilon > 0$ for all $n$. Now choose points $q_n \in [p, g_n\xi_1]$, $r_n \in [p, g_n\xi_2]$ such that $q_n, r_n$ are both uniformly close to $p_n$. Then $d(q_n, r_n)$ is uniformly small but $d(p, q_n), d(p, r_n)$ are going to infinity. This is so because $\{g_nH\}$ are distinct cosets and hence their distances from $1 \in G$ are going to infinity. This gives a contradiction because suppose $\Delta A_pB_pC_p$ is a comparison triangle in $\mathbb{E}^2$ of the geodesic triangle $\triangle pq_nr_n \subset Y$ where $p$ is mapped to $A_p$. Then $\angle_p(A_pB_p, C_p) \geq \angle P(q_n, r_n) \geq \epsilon$ whereas $d(B_p, C_p)$ is uniformly small and $d(A_p, B_p)$ and $d(A_p, C_p)$ are arbitrarily large. Such a sequence of triangles can not exist in $\mathbb{E}^2$. □

For condition (3) is a direct consequence of Lemma 4.5 of Caprace ([Cap10]) and Proposition 4.3 of [Ham09]. □

**Remark 4.2.** We note that in general a purely rank-1 subgroup (i.e. one whose non-identity elements are all rank-1 isometries) of a CAT(0) group may not have bounded packing. In fact, as in Example 2.22 of [HW09] one can have subgroups of CAT(-1) groups without bounded packing.

**An alternative proof of Theorem 4.1**

Now we sketch an alternative proof of the above theorem using results of Dahmani, Guirardel and Osin ([DGO14]) and Sisto ([Sis13]). We refer the reader to [DGO14] for terminologies. We will need the following theorems as ingredients of the proof.

**Theorem 4.3.** (See Theorem 1.1 and 1.3 of [Sis13]) Suppose $G$ is a group acting properly and cocompactly on a CAT(0) space $X$ and $g \in G$ acts as a rank-1 isometry on $X$. Then there is a virtually cyclic subgroup $E(g)$ of $G$ containing $g$ such that $E(g)$ is hyperbolically embedded in $G$. 


Theorem 4.4. (Corollary 6.36 of [DGO14]) Let \( \{ H_\lambda \}, \lambda \in \Lambda \) be a hyperbolically embedded collection of subgroups of a group \( G \). Then for every \( \alpha > 0 \) there exists finite subsets \( F_\lambda \subset H_\lambda \setminus \{1\} \) such that any collection \( \{ N_\lambda \}, \lambda \in \Lambda \), where \( N_\lambda \leq H_\lambda \) and \( N_\lambda \cap F_\lambda = \emptyset \) for every \( \lambda \in \Lambda \), is \( \alpha \)-rotating.

Theorem 4.5. (Corollary 5.4 of [DGO14]) Let \( H \) be a 200-rotating subgroup of a group \( G \). Then the normal subgroup of \( G \) generated by \( H \) is a free product of a (usually infinite) family of conjugates of \( H \).

Theorem 4.6. Suppose \( H \) is a residually finite hyperbolically embedded subgroup of a group \( G \). Then \( H \) has bounded packing in \( G \).

We first prove the following:

Lemma 4.7. In an infinite free product of groups \( G_1 \ast G_2 \ast \cdots \) each free factor has bounded packing.

Proof of the lemma: Let us show, without loss of generality, that \( G_1 \) has bounded packing in \( G_1 \ast G_2 \ast \cdots \). Suppose \( G_1 \) does not have bounded packing. Then there is \( D > 0 \) and an infinite collection of sets of cosets of \( G_1 \) which are pairwise at a distance \( D \). We know that \( d(xG_1, yG_1) \leq D \) if and only if \( d(1, G_1x^{-1}yG_1) \leq D \) whence the double coset has an element in \( B(1, D) \). Since the metric of the group is proper we conclude that there is a finitely generated subgroup containing \( G_1 \) in which \( G_1 \) fails to have bounded packing. Therefore, we are reduced to the case of a finite free product. This case follows easily by looking at the tree of spaces corresponding to the free product decomposition. This completes the proof of the lemma.

Proof of Theorem 4.6: We apply Theorem 4.4 with \( \alpha = 200 \). Since \( H \) is residually finite there is a normal subgroup \( N \) of finite index in \( H \) satisfying the condition of Theorem 4.4. Now, by Theorem 4.5 the normal subgroup in \( G \) generated by \( N \) is a free product of copies of conjugates of \( N \). It follows that this normal subgroup has bounded packing in \( G \) by Lemma 2.2(7), whereas \( N \) has bounded packing in the free product by Lemma 4.7. Hence, by Lemma 2.2(4) \( N \) has bounded packing in \( G \). Again by Lemma 2.2(4) we have that \( H \) has bounded packing in \( G \).

Proof of Theorem 4.1: Let \( H = \langle g \rangle \). By Theorem 4.3 \( E(g) \) is a virtually cyclic subgroup of \( G \) which is hyperbolically embedded in \( G \). Hence \( E(g) \) has bounded packing in \( G \) by Theorem 4.6. Since \( H \) has finite index in \( E(g) \) it now follows from Lemma 2.2(4) that \( H \) has bounded packing in \( G \).

5. Non-existence of bounded packing

Proposition 5.1. Let \( \mathbb{F}_2 \) denote the free group on two generators. Then there are finitely generated subgroups of \( \mathbb{F}_2 \times \mathbb{F}_2 \) without the bounded packing property.

Proof: Let \( G \) be any countable group and let \( \Delta \) be the diagonal in \( G \times G \). It is clear that any coset of \( \Delta \) in \( G \times G \) can be written uniquely as \( (g, 1)\Delta \) for some \( g \in G \) and any two cosets \( (g, 1)\Delta \) and \( (gh, 1)\Delta \) are close essentially means that there is \( x \in G \) such that \( xhx^{-1} \) is close to identity in \( G \).

We first search for groups \( G \) such that the diagonal \( \Delta \leq G \times G \) does not have bounded packing. Towards that goal, it is enough to furnish a group \( G \) which has a sequence of distinct elements \( g_n \) such that for all \( m, n \), a conjugate of \( (g_m)^{-1}g_n \)
is uniformly close to identity. Bartholdi, Cornulier and Kochloukova( see [BKC15]) constructed these types of examples using generalized wreath products. By their construction we can have examples of the form $G = \mathbb{Z}_x \wr Q$ where the $Q$-action on $X$ is 2-transitive and $X$ is an infinite set. Hence, one can take $g_x$ to be the 1 in the $x$-th copy of $\mathbb{Z}$ in $\bigoplus_{x \in X} \mathbb{Z} \subset G$ for all $x \in X$. It follows that for all $x, y \in X$ $g_x^{-1}g_y \in G$ is a conjugate of $g_x^{-1}g_y$ for some fixed $x_0, y_0 \in X$; this means each $g_x^{-1}g_y \in G$ has a conjugate uniformly close to the identity in $G$. Therefore, for any such group $G$ the diagonal $\Delta \leq G \times G$ does not have bounded packing. Note that in these examples the groups $G$ are finitely presented too.

Next, choose such a group $G$. Since the group $G$ is finitely generated (presented) there is a surjective group homomorphism $\phi: \mathbb{F}_n \to G$. Now we can use Mihailova’s construction. That will give us that $(\phi \times \phi)^{-1}(\Delta) \leq \mathbb{F}_n \times \mathbb{F}_n$ is finitely generated, where $\phi \times \phi$ is the naturally induced map $\mathbb{F}_n \times \mathbb{F}_n \to G \times G$ and $\Delta \leq G \times G$ is the diagonal.

Since there are natural embeddings of $\mathbb{F}_n \times \mathbb{F}_n$ in $\mathbb{F}_2 \times \mathbb{F}_2$, we are done. \hfill \Box

**Remark 5.2.** Any finitely presented subgroup $G$ of $\mathbb{F}_2 \times \mathbb{F}_2$ is virtually a product $H_1 \times H_2$ where $H_i \leq \mathbb{F}_i$ are finitely generated subgroups. We know that finitely generated subgroups of free groups are quasi-convex and by the results of [HW99] that quasi-convex subgroups of hyperbolic groups have bounded packing. Hence $H_i \leq \mathbb{F}_2$ have bounded packing. Therefore, $G$ has bounded packing in $\mathbb{F}_2 \times \mathbb{F}_2$. So one can get only finitely generated counter-examples like above.

**Proposition 5.3.** There is a CAT(0) group which has a finitely presented subgroup without bounded packing.

We shall make use of the following theorems for the proof.

1-2-3 Theorem (Baumslag-Bridson-Miller-Short [BBIS00]) Suppose that $1 \to N \to \Gamma \overset{\nu}{\to} Q \to 1$ is exact, and consider the fibre product

$$P := \{(\gamma_1, \gamma_2)|\nu(\gamma_1) = \nu(\gamma_2)\} \subset \Gamma \times \Gamma.$$  

If $N$ is finitely generated, $\Gamma$ is finitely presented and $Q$ is of type $F$, then $P$ is finitely presented.

**Theorem 5.4.** (Wise [Wis98]) Let $Q$ be a finitely presented group. Then there exists a group $G$ which is the fundamental group of a compact negatively curved 2-complex and a finitely generated normal subgroup $N \leq G$ such that $G/N \simeq Q$.

**Proof of the proposition:** The proof of this proposition also follows the same line of arguments as that of Proposition 5.1 by using the above theorems. First we find a group $Q$ which satisfies $F_3$ using [BKC15] and then apply Wise’s construction to get a CAT(-1) group $G$ with a surjective map $\phi: G \to Q$ whose kernel is finitely generated. Finally we use the 1-2-3 Theorem of Baumslag-Bridson-Miller-Short to conclude that $H = (\phi \times \phi)^{-1}(\Delta) \leq G \times G$ is finitely presented where $\Delta \leq Q \times Q$ -the diagonal subgroup, does not have bounded packing. Thus $H \leq G \times G$ will not have bounded packing. Since $G$ is a CAT(-1) group $G \times G$ is a CAT(0) group. \hfill \Box

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