Maximum temperature for an Ideal Gas
of $\hat{U}(1)$ Kac-Moody Fermions\textsuperscript{1}

Belal E. Baaquie\textsuperscript{2}

Department of Physics, Faculty of Science,
National University of Singapore, Kent Ridge,
Singapore 0511.

Abstract

A lagrangian for gauge fields coupled to fermions with the Kac-Moody group as its gauge group yields, for the pure fermions sector, an ideal gas of Kac-Moody fermions. The canonical partition function for the $\hat{U}(1)$ case is shown to have a maximum temperature $kT_M = |\lambda|/\pi$, where $\lambda$ is the coupling of the super charge operator $G_0$ to the fermions. This result is similar to the case of strings but unlike strings the result is obtained from a well-defined lagrangian.

\textsuperscript{1}To appear in Physical Review, Dec 15, 1995.

\textsuperscript{2}E-mail address: phybeb@leonis.nus.sg
Introduction

The existence of a maximum temperature is widely supposed to hold in string theory. In this paper we discuss another case where the same phenomenon occurs, and the result is shown to hold for a model with an exact lagrangian.

Gauge fields coupled to fermions having an arbitrary Kac-Moody group for its gauge symmetry has been derived in Ref 1, and has a number of new features. The pure gauge sector is nonlinear even for $\hat{U}(1)$ case. The fermion sector has a new mass-like coupling to the super-charge operator $G_0$ due to the necessity of attenuating the high mass states inside Feynman loop integrations (Ref 1).

In this paper, we examine the pure fermion sector. The simplest case of $\hat{U}(1)$ Kac-Moody fermions is studied and we derive the existence of a maximum temperature for the free energy.

Consider a $d$-dimensional Euclidean space time $M_d$. Let $Q_n$ and $h_n$ be the generators of $\hat{U}(1)$ super Kac-Moody algebra with

$$[Q_n, Q_m] = ik\delta_{n+m}$$  \hspace{1cm} (1)

$$[h_n, h_m] = k\delta_{n+m}$$  \hspace{1cm} (2)

We consider only the Ramond sector (integer modes $h_n$). Hence we have the Virasoro generator

$$L_0 = \frac{1}{2k}Q_0^2 + \frac{1}{k} \sum_{n=1}^{\infty} (Q_{-n}Q_n + nh_n h_n) + \frac{1}{16}$$  \hspace{1cm} (3)

and the super-charge operator

$$G_0 = \frac{1}{k}Q_0 h_0 + \frac{1}{k} \sum_{n=1}^{\infty} (Q_{-n} h_n + h_{-n} Q_n)$$  \hspace{1cm} (4)
with

\[ G_0^2 = L_0 - \frac{c}{24}. \] (5)

These generators, together with \( L_n \) & \( G_n \), form the \( N = 1 \) super Virasoro algebra with \( c = 1 + 1/2 = 3/2 \).

### Lagrangian

The free Kac-Moody fermion lagrangian is given by [Ref 1]

\[ L = \bar{\psi}(x)(\partial_\mu \gamma^\mu + \lambda G_0 + m + \mu \gamma_0)\psi(x) \] (6)

where \( \lambda \) and \( m \) have dimension of mass and \( \mu \) is the chemical potential.

Under gauge transformations we have [Ref 1]

\[ \psi \rightarrow \Phi \psi, \quad \bar{\psi} \rightarrow \bar{\psi} \Phi^\dagger \]

where

\[ \Phi(x) = \exp\{i \sum_{n=-\infty}^{+\infty} \phi_n(x)Q_n\} \]

is an element of the Kac-Moody group and \( \phi_n(x) \) is an infinite collection of gauge functions.

The partition function is given by \((\beta = \frac{1}{kT})\)

\[ Z = \text{tr} \exp\{-\beta (\mathcal{H} - \mu N)\} \] (7)

\[ = \int D\bar{\psi} D\psi \exp\{\int_0^\beta dt \int d^{d-1}x \mathcal{L}\}. \] (8)
\(\psi, \bar{\psi}\) are arbitrary elements of an irrep of the super Virasoro algebra. We have commuting operators \(h_0, Q_0, G_0\) and \(L_0\). A vacuum state is specified by \(|q, h\rangle\) which is annihilated by all \(Q_n, h_n, n > 0\) and where \(Q_0|q, h\rangle = q|q, h\rangle\) and \(L_0|q, h\rangle = h|q, h\rangle\). We choose \(q = 0\), and since we are in the Ramond sector we have \(h = 1/16\); we can consequently choose a chiral representation with a supersymmetric vacuum given by \(G_0|q, h\rangle = 0\), and this is possible since for our case \(c/24 = 1/16\) [Ref 2].

In effect, all the states of the representation space are created by \(Q_{-n}, h_{-n}, n > 0\), acting on \(|q, h\rangle = |0, 1/16\rangle\) and the operators \(Q_0, h_0\) are set to zero everywhere.

Hence we have the expansion for the fermion field

\[
\psi(x) = \psi(x_0) + \psi_n(x)h_{-n} + \psi_{nm}(x)h_{-n}h_{-m} + \psi_{nml}(x)h_{-n}h_{-m}h_{-l} + \cdots \tag{9}
\]

where \(\psi_0, \psi_n, \cdots\) are fermionic degrees of freedom which carry an irrep of the bosonic Kac-Moody algebra with \(q = 0\). From eqn (9) we see that \(\psi(x)\) is similar to a superstring field with the \(\psi_0, \psi_n\)'s etc being the excited string modes.

The expansion given in eqn (19) is not suitable for performing the path integration in eqn (8). We instead choose a basis for \(\psi, \bar{\psi}\) to diagonalize \(L\) in eqn (6) and hence evaluate \(Z\) exactly. Let \(|n\rangle\) be an eigenstate of \(L_0\) with \(L_0|n\rangle = (n + \frac{1}{16})|n\rangle\), \(n = 0, 1, 2, \ldots \infty\).

The degeneracy of each state \(|n\rangle\) is given by \(d_n\) where [Ref 3]

\[
\prod_{n=1}^{\infty} \frac{(1 + q^n)}{(1 - q^n)} = \sum_{n=0}^{\infty} d_n q^n = \text{tr} \quad q^{L_0 - \frac{c}{24}}. \tag{10}
\]

The numerator on the left hand side comes from the fermionic states at level \(n\) and the denominator from the bosonic ones.

Define, for \(n \neq 0\),

\[
|n, \pm \rangle = (\sqrt{n} \pm G_0)|n\rangle \tag{11a}
\]
with \( G_0 | n, \pm > = \pm \sqrt{n} | n, \pm > \) (11b)

We have the expansion

\[
\psi(x) = \psi_0(x) | 0 > + \sum_{n=1}^{\infty} \sum_{\pm} \psi_n^\pm(x) | n, \pm >
\]

and similarly for \( \bar{\psi}(x) \).

Then, from eqns (6) and (12)

\[
L = \bar{\psi}_0(x) (\partial + m + \mu \gamma_0) \psi_0(x)
+ \sum_{n=1}^{\infty} \sum_{\pm} \bar{\psi}_n(x) (\partial \pm \lambda \sqrt{n} + m + \mu \gamma_0) \psi_n(x).
\]

(13)

We see from eqn(13) that the zero mode’s \( \bar{\psi}_0(x), \psi_0(x) \) reproduce the ordinary Dirac lagrangian, and the higher modes give an infinite collection of Dirac particles with masses increasing as \( | m \pm \lambda \sqrt{n} | \).

**Partition Function, Maximum Temperature**

Since all the degrees of freedom are decoupled, the path integral in eqn (8), for each mode \( \psi_n^\pm(x) \), is an ideal gas of fermions with mass \( | m \pm \lambda n | \) and can be evaluated exactly using standard techniques [Ref 4]. Performing the path integrations yields the free energy

\[
F = \ln \frac{Z}{V} = 2^{d/2-1} \sum_{n=0}^{\infty} d_n \sum_{\pm} \int_{p} \left\{ \ln(1 + e^{\mu \beta} e^{-\beta \sqrt{p^2 + (m \pm \lambda \sqrt{n})^2}}) + \ln(1 + e^{-\mu \beta} e^{-\beta \sqrt{p^2 + (m \pm \lambda \sqrt{n})^2}}) \right\}
\]

(14)

where \( V = \int d^{d-1}x, \int_{p} \equiv \int \frac{d^{d-1}p}{(2\pi)^{d-1}}, \) and \( d_0 \to d_0/2. \)
We expect $T_M$ to be determined by very high mass states, and hence to leading order we can set $m = 0$; we will consider the case where $kT_M >> \mu$ and hence to leading order we can also set $\mu = 0$. We further consider the case of $\lambda$ such that $\beta \lambda >> 1$ and this will in effect make $T_M$ small; we consequently study this temperature regime of $F$ to ascertain the existence of a maximum temperature.

Doing a low temperature expansion for $F$, ignoring $m$ and $\mu$, yields [Ref 4]

$$F = 2^{d/2+1} \sum_{n=0}^{\infty} d_n \sum_{\pm} \int_p e^{-\beta \sqrt{p^2 + \lambda^2 n^2}} + \ldots$$

$$= \frac{2}{\pi d/2} \left( \frac{\lambda}{\beta} \right)^{d-2} \sum_{n=0}^{\infty} d_n n^{d-2} K_{(d-2)/2}(\beta |\sqrt{n}) + \ldots$$

(15)

where $K_{(d-2)/2}$ is the associated Bessel function and its asymptotic expansion yields ($a = \text{const}$)

$$F = a \sum_{n=1}^{\infty} d_n n^{(d-3)/4} e^{-\beta |\lambda| \sqrt{n}}.$$  

(16)

Clearly the value for $d_n$ as $n \to \infty$ will determine if $F$ is finite for all $\beta$. To determine $d_n$, note, for $q \to 1$ [Ref 5, 6]

$$\prod_{n=1}^{\infty} \frac{1 + q^n}{1 - q^n} = \theta_4^{-1}(0|q) \simeq (1 - q)^{1/2} \exp\{-\frac{\pi^2}{4(1 - q)}\}$$

(17)

where $\theta_4(0 \mid q)$ is the Jacobi theta function. We perform a contour integral using eqn(10) to evaluate $d_n$; there is a sharp maxima of the integrand at $q = 1$, and using the saddle-point method with eqn (17) yields

$$d_n \simeq \frac{1}{n} e^{\pi \sqrt{n}}.$$  

(18)

Hence

$$F \simeq \sum_{n=1}^{\infty} n^{(d-7)/4} e^{\pi \sqrt{n}} e^{-\beta |\lambda| \sqrt{n}}.$$  

(19)
We see that $F$ diverges for all $\beta \mid \lambda \mid < \pi$; hence $F$ is finite for all $kT < kT_M$, where the maximum temperature is $kT_M = \mid \lambda \mid / \pi$.

As $T \to T_M$ from below, we have from eq.(19)

$$F \propto \begin{cases} 
\frac{1}{(T_M - T)^{d-3}}, & d > 3 \\
\ln(T_M - T), & d = 3 \\
\text{finite}, & d < 3 
\end{cases}$$

(20)

Discussion

An obvious similarity of Kac-Moody fermions and strings is the exponentially growing density of states, which in both cases destabilizes the system at high enough temperature. There are however a number of dissimilarities between the two cases.

Firstly a finite maximum temperature exist for Kac-Moody fermions because of the competition between the square root of relativistic energy $\sqrt{p^2 + \lambda^2 n}$ and the square root in entropy $\pi \sqrt{n}$. If for example we had used $L_0$ instead of $G_0$ in the lagrangian $L$, the free energy would have been convergent for all temperature.

Secondly, Kac-Moody fermions have a well defined path integral for all temperature and hence we can study the singularity at $T_M$ in detail using $L$ of eqn (6), whereas in the case of strings short distance effects due to gravity are supposed to invalidate concepts such as maximum temperature, phase transitions etc [Ref 6]. Nevertheless, studies of an ideal gas of strings have been carried out and lead to results that differ significantly from the point particle case [Ref 7].

$T_M$ in general is a complicated function of $\lambda$, and we obtained the leading behaviour of $T_M$ for the region $\mu / \lambda << 1 << \beta \lambda$. Given the exponential growth of the density of states, one expects that the free energy $F$ will have a singularity at $T = T_M$ as in eq.(20); a detailed study of $F$ near $T_M$ including the nature of the phase transition at $T_M$ can be obtained from eqn (14). It can also be determined, using the lagrangian $L$ in eqn (10),
whether the canonical and microcanonical ensembles are equivalent for this system which has an exponentially growing density of states.

In conclusion, an ideal gas of Kac-Moody fermions provides an exact model to study the physics of a system with a finite maximum temperature.

References

1. B. E. Baaquie, Phys. Lett. 271B (1991) 343.
2. C. Itzykson & J-M Drouffe, *Statistical Field Theory Vol II* (Cambridge Univ Press 1989).
3. S. Ketov, *Conformal Field Theory* (World Scientific, 1995).
4. J I. Kaputsa, *Finite Temperature Field Theory* (Cambridge Univ Press, 1989).
5. K. Chandrasekharan, *Elliptic Functions* (Springer Verlag, 1985).
6. M. B. Green, J. H. Schwarz & E. Witten *Superstring Theory Vol I & Vol II* (Cambridge Univ Press, 1987).
7. N. Deo, S. Jain & C-I Tan, Phys. Lett. 220B (1989) 125, Phys.Rev. D40 (1989) 2626.