ROUGH POTENTIAL RECOVERY IN THE PLANE

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Dedicated to the memory of Tuulikki

Abstract. We consider the problem of reconstructing a potential $V$ from the scattering amplitude at a fixed positive energy. More specifically, we consider the method of Bukgheim to recover $V$ from the Dirichlet-to-Neuman map on the boundary of a bounded planar domain which contains the support of the potential. We draw a connection between this and a question of Carleson regarding the convergence to initial data of solutions to time-dependent Schrödinger equations. As a consequence we are able to recover the potentials with only half a derivative in $L^2$. We also show that our result is sharp.

1. Introduction

We consider the Schrödinger equation $\Delta u = Vu$ on a bounded domain $\Omega$ in the plane. For each solution $u$, we are given the value of both $u$ and $\nabla u \cdot n$ on the boundary $\partial \Omega$, where $n$ is the exterior unit normal on $\partial \Omega$. The goal is then to recover the potential $V$ from this information.

We suppose throughout that $V \in L^2(\Omega)$ and that $0$ is not a Dirichlet eigenvalue for the Hamiltonian $-\Delta + V$. Then for each $f \in H^{1/2}(\partial \Omega)$, there is a unique solution $u$ in $H^1(\Omega)$ to the Dirichlet problem

$$\begin{cases}
\Delta u = Vu \\
u|_{\partial \Omega} = f,
\end{cases}$$

and the Dirichlet-to-Neumann (DN) map $\Lambda_V$ can be formally defined by

$$\Lambda_V : f \mapsto \nabla u \cdot n|_{\partial \Omega}.$$

Then a restatement of our goal is to recover $V$ from knowledge of $\Lambda_V$.

We come to this problem via a question of Calderón regarding impedance tomography [12], however the DN map $\Lambda_{V-\kappa^2}$ and the scattering amplitude at energy $\kappa^2$ are uniquely determined by each other, and indeed the DN map can be recovered from the scattering amplitude (see the appendix for explicit formulae). Thus we are also addressing the question of whether it is
possible to recover the potential from the scattering data at a fixed positive energy.

In higher dimensions, Sylvester and Uhlmann proved that smooth potentials are uniquely determined by the DN map [50] (see [37, 38, 14] for non-smooth potentials and [9, 40, 22] for the conductivity problem). The uniqueness result was extended to a reconstruction procedure by Nachman [32, 33]. The planar case is quite different mathematically as it is not overdetermined. Here the first uniqueness and reconstruction algorithm was proved by Nachman [34] via $\partial$-methods for potentials of conductivity type (see also [10] for uniqueness with less regularity). Sun and Uhlmann [46, 48] proved uniqueness for potentials satisfying nearness conditions to each other. Isakov and Nachman [25] then reconstructed the real valued $L^p$-potentials, $p > 1$, in the case that their eigenvalues are strictly positive. The $\partial$-method in combination with the theory of quasiconformal maps gave the uniqueness result for the conductivity equation with measurable coefficients [2]. The problem for the general Schrödinger equation was solved only in 2008 by Bukhgeim [11] for $C^1$-potentials. Bukhgeim’s result has since been improved and extended to treat related inverse problems (see for example [7, 39, 24, 23]).

The aim of this article is to emphasise a surprising connection between the pioneering work of Bukhgeim [11] and Carleson’s question [13] regarding the convergence to initial data of solutions to time-dependent Schrödinger equations. Elaborating on this new point of view we obtain a reconstruction theorem for general planar potentials with only half a derivative in $L^2$, which is sharp with respect to the regularity. The precise statements are given in the forthcoming Corollary 1.3 and Theorem 1.4.

To describe the results in more detail, we recall that the starting point in [11] was to consider solutions to $\Delta u = Vu$ of the form $u = e^{i\psi}(1 + w)$, where from now on

$$\psi(z) \equiv \psi_{k,x}(z) = \frac{k}{8}(z - x)^2, \quad z \in \mathbb{C}, \quad x \in \Omega.$$ 

Solutions of this type have a long history (see for example [21, 50, 27, 16]), and in this form they were considered first by Bukhgeim. We will recover the potential with a countable number of measurements on the boundary, so we take $k \in \mathbb{N}$. We will require the homogeneous Sobolev spaces with norm given by $\|f\|_{H^s} = \|(-\Delta)^{s/2}f\|_{L^2}$, where $(-\Delta)^{s/2}$ is defined via the Fourier transform as usual. In Section 2, we prove that if the zero extension of the potential $V$ is contained in $H^s$ with $0 < s < 1$, and $k$ is sufficiently large, then we can take $w \equiv w_{k,x} \in H^s$ with a bound for the norms which is decreasing to zero in $k$. We write $u_{k,x} = e^{i\psi}(1 + w)$ for these $w \in H^s$.

The definition of the DN map yields the basic integral formula in inverse problems; Alessandrini’s identity. Indeed, if $u, v \in H^1(\Omega)$ satisfy $\Delta u = Vu$ and $\Delta v = 0$, then the formula states that

$$\langle (\Lambda V - \Lambda_0)[u], v \rangle := \int_{\partial\Omega} (\Lambda V - \Lambda_0)[u] v = \int_{\Omega} V u v.$$
Taking $u = u_{k,x}$, which is also in $H^1(\Omega)$, and $v = e^{i\psi}$ this yields

$$\langle (\Lambda V - \Lambda_0)[u_{k,x}], e^{i\psi} \rangle = \int_{\partial\Omega} e^{i(\psi + \overline{\psi})} V(1 + w),$$

and so the integral over $\Omega$ can be obtained from information on the boundary.

The bulk of the article is concerned with recovering the potential from the integral on the right-hand side of (2). However, in order to calculate the value of the integral, without knowing the value of the potential $V$ inside $\Omega$, we need to calculate the value of the left-hand side of (2). That is to say, we must determine the values of $u_{k,x}$ on the boundary from the DN map. In the case of linear phase, this was achieved by Nachman [34] for $L^p$-potentials $V$, with $p > 1$, and Lipschitz boundary. For $C^1$-potentials, with $C^2$-boundary, the result was extended by Novikov and Santacesaria to quadratic phases [39]. Here we show that for quadratic phases almost no regularity is needed. We consider potentials in the inhomogeneous $L^2$-Sobolev space $H^s$, defined as before with $(-\Delta)^{s/2}$ replaced by $(I - \Delta)^{s/2}$. Our starting point is similar to [34] but we give a shorter argument, avoiding single layer potentials.

**Theorem 1.1.** Let $V \in H^s$ with $s > 0$ and suppose that $\Omega$ is Lipschitz. Then we can identify bounded operators $R_k : H^{1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$, depending only on $k$ and $\Lambda V - \Lambda_0$, such that

$$u_{k,x}|_{\partial\Omega} = R_k[e^{i\psi}|_{\partial\Omega}].$$

For $C^1$-potentials, Bukhgeim [11] proved that the right-hand side of (2), multiplied by $(4\pi i)^{-1}k$, converges to $V(x)$ for all $x \in \Omega$, when $k$ tends to infinity. In Section 4 we obtain this convergence for potentials in $H^s$ with $s > 1$. For discontinuous potentials we are no longer able to recover at each point. Instead we bound the fractal dimension of the sets where the recovery fails. As Sobolev spaces are only defined modulo sets of zero Lebesgue measure, we consider first the potential spaces $L^{s,2} = (-\Delta)^{-s/2}L^2(\mathbb{R}^2)$, and bound the Hausdorff dimension of the points where the recovery fails.

**Theorem 1.2.** Let $V \in L^{s,2}$ with $1/2 \leq s < 1$. Then

$$\dim_H \left\{ x \in \Omega : \frac{k}{4\pi i} \langle (\Lambda V - \Lambda_0)[u_{k,x}], e^{i\psi} \rangle \not\to V(x) \text{ as } k \to \infty \right\} \leq 2 - s.$$ 

As the members of $H^s$ coincide almost everywhere with members of $L^{s,2}$, we see that rough and unbounded potentials can be recovered almost everywhere from information on the boundary. Note that these results are stable in the sense that $k \in \mathbb{N}$ can be replaced by any sequence $\{n_k\}_{k \in \mathbb{N}}$ such that $n_k$ tends to infinity as $k$ tends to infinity.

**Corollary 1.3.** Let $V \in H^{1/2}$. Then

$$\lim_{k \to \infty} \frac{k}{4\pi i} \langle (\Lambda V - \Lambda_0)[u_{k,x}], e^{i\psi} \rangle = V(x), \quad \text{a.e. } x \in \Omega.$$
In Section 5, we will prove that this is sharp in the sense of the following theorem. Note that even though there is divergence on a set of full Hausdorff dimension when \( s < \frac{1}{2} \), the dimension of the divergence set is bounded above by \( \frac{3}{2} \) when \( s \geq \frac{1}{2} \).

**Theorem 1.4.** Let \( s < \frac{1}{2} \). Then there exists a potential \( V \in H^s \), supported in \( \Omega \), for which

\[
\left\{ x \in \Omega : \frac{k}{4\pi i} \left( (\Lambda_V - \Lambda_0)[u_{k,x}], e^{i\psi} \right) \not\rightarrow V(x) \text{ as } k \to \infty \right\} \neq \emptyset.
\]

Blåsten \([7]\) proved that potentials in \( H^s \) with \( s > 0 \) are uniquely determined by the DN map. It is a curious phenomenon that, within the Bukhgeim approach, uniqueness and reconstruction have different smoothness barriers.

The DN map \( \Lambda_{V - \kappa^2} \) can be recovered from the scattering amplitude at a fixed energy \( \kappa^2 > 0 \) (see the appendix), from which we are able to recover the potential \( V - \kappa^2 \chi_\Omega \) rather than \( V \). We are free to choose the domain \( \Omega \).

Taking \( \Omega \) to be a square, we obtain the following recovery formula. Here \( U_{k,x} \) are Bukhgeim solutions which solve \( \Delta u = (V - \kappa^2)u \) in \( \Omega \).

**Theorem 1.5.** Let \( V \in H^{1/2} \) be supported in a square \( \Omega \). Then

\[
\lim_{k \to \infty} \frac{k}{4\pi i} \left( (\Lambda_{V - \kappa^2} - \Lambda_0)[U_{k,x}], e^{i\psi} \right) + \kappa^2 = V(x), \quad \text{a.e. } x \in \Omega.
\]

Interpreting the problem acoustically, it is unsurprising that we are unable to recover potentials in \( H^s \) with \( s < 1/2 \). Taking

\[
V(x) = \kappa^2(1 - c^{-2}(x)),
\]

where \( c(x) \) denotes the speed of sound at \( x \), the scattering solutions \( u \) also satisfy \( c^2 \Delta u + \kappa^2 u = 0 \). Now there are potentials in \( H^s \), with \( s < 1/2 \), which are singular on closed curves (see for example \([52]\)). Thus the speed of sound is zero on the curve and so a continuous solution \( u \) would be zero. That is to say, the continuous incident waves cannot pass through the curve, and we should not expect to be able to detect a modification of the potential in the interior. From this point of view, the uniqueness result of Blåsten \([7]\) reflects tunneling phenomena in quantum mechanics.

2. **The Bukhgeim solutions**

Writing \( \partial_z = \frac{1}{2}(\partial_x - i\partial_y) \) and \( \partial_\nu = \frac{1}{2}(\partial_x + i\partial_y) \), we consider the complex analytic interpretation of the Schrödinger equation \( 4\partial_\nu \partial_\nu u = Vu \). When looking for solutions of the form \( u = e^{i\psi}(1 + w) \), the equation is equivalent to the system

\[
2\partial_\nu w = e^{-i(\psi + \bar{\psi})}v, \quad 2\partial_x v = e^{i(\psi + \bar{\psi})}V(1 + w),
\]

which is solved in \( \Omega \) whenever

\[
w = \frac{1}{4} \partial_x^{-1} \left[ e^{-i(\psi + \bar{\psi})} \chi_\Omega \partial_x^{-1} \left[ e^{i(\psi + \bar{\psi})} V(1 + w) \right] \right].
\]
Here, we take $Q$ to be a fixed, auxiliary, axis-parallel square which properly contains $\Omega$. Thus, defining the operator $S^{k}_V \equiv S^{k,x}_V$ by
\[
S^{k}_V[F] = \frac{1}{4\pi^2} \left[ e^{-i(\psi + \overline{\psi})} \chi_Q \partial_z^{-1} \left[ e^{i(\psi + \overline{\psi})} V F \right] \right],
\]
we see that as soon as $\|S^{k}_V\|_{\dot{H}^s \to \dot{H}^s} < 1$, we can treat $(I - S^{k}_V)^{-1}$ by Neumann series to deduce that it is a bounded operator on $\dot{H}^s$. This yields a solution
\[
u_{k,x} = e^{i\psi}(1 + w)
\]
where
\[
w = w_{k,x} = (I - S^{k}_V)^{-1} S^{k}_V[1] \in \dot{H}^s.
\]

In what remains of this section, we prove that $S^{k}_V$ is contractive for sufficiently large $k$. This property will be crucial in the proof of Theorem 1.1 as well as in Section 4. We write $S^{k}_V[f] = \frac{1}{4} S^{k}_V[V f]$, where
\[
S^{k}_V = \partial_z^{-1} \circ M^{-k} \circ \partial_z^{-1} \circ M^k
\]
and the multiplier operators $M^{\pm k}$ are defined by
\[
M^{\pm k}[F] = e^{\pm i(\psi + \overline{\psi})} \chi_Q F.
\]
The key ingredient in the proof of the following estimate, is the classical lemma of van der Corput [13].

**Lemma 2.1.** Let $0 < \lambda < s_1, s_2 < 1$. Then
\[
\|M^{\pm k}[F](\cdot, x)\|_{\dot{H}^{-s_2}} \leq C k^{-\lambda} \|F(\cdot, x)\|_{\dot{H}^{s_1}}, \quad x \in \Omega.
\]

**Proof.** We will first prove that
\[
\|M^{\pm k} F\|_{B^{s_1}_{2,\infty}} \leq C k^{-\lambda} \|F\|_{B^{s_1}_{2,1}},
\]
where the Besov norms are defined as usual by
\[
\|f\|_{B^{s_1}_{2,\infty}} = \sup_{j \in \mathbb{Z}} 2^{-j} \|P_j f\|_{L^2} \quad \text{and} \quad \|f\|_{B^{s_1}_{2,1}} = \sum_{j \in \mathbb{Z}} 2^j \|P_j f\|_{L^2}.
\]
Here, $\hat{P}_j f = \hat{\vartheta}(2^{-j} | \cdot |) \hat{f}$ with $\vartheta$ satisfying $\supp \vartheta \subset (1/2, 2)$ and
\[
\sum_{j \in \mathbb{Z}} \vartheta(2^{-j} \cdot) = 1.
\]
As $\|F\|_{B^{s_1}_{2,\infty}} \leq C \|\hat{F}\|_{\infty}$ and $\|\hat{F}\|_{1} \leq C \|F\|_{B^{s_1}_{2,1}}$, the estimate (4) would follow from
\[
\|M^{\pm k} F\|_{\infty} \leq C k^{-\lambda} \|\hat{F}\|_{1}.
\]
Now, by the Fourier inversion formula and Fubini’s theorem,
\[
|M^{\pm k} F(\xi)| = \frac{1}{(2\pi)^2} \left| \int_Q e^{\pm i (\psi(z) + \overline{\psi}(z))} \int \hat{F}(\omega) e^{iz \cdot \omega} d\omega e^{-iz \cdot \xi} dz \right|
\]
\[
\leq \int \left| \int_Q e^{\pm i (\xi_1 x_1^2 + (x_2 - x_2)^2)} e^{iz \cdot \omega} d\omega \right| \|\hat{F}(\omega)\| d\omega
\]
so that (5) follows by two applications of van der Corput’s lemma [13] (factorising the integral into the product of two integrals).
Now, by a trivial decomposition, for \( j = 1, 2 \), it will suffice to prove
\[
\| M_j \pm \mathcal{F} \|_{\dot{H}^{-s}} \leq C k^{-\lambda} \| F \|_{\dot{H}^{s_1}}, \quad 0 < \lambda < s_1, s_2 < 1,
\]
whenever \( F \) is frequency supported in \( \mathbb{D} \) or \( \mathbb{R}^2 \setminus \mathbb{D} \), where
\[
M_j^{\pm k} F = \left( \hat{M}_j^{\pm k} F \chi_{\mathbb{D}} \right) \vee \quad \text{and} \quad M_2^{\pm k} F = \left( \hat{M}_2^{\pm k} F \chi_{\mathbb{R}^2 \setminus \mathbb{D}} \right) \vee,
\]
so we have four cases. It is clear that (5) and thus (4) also hold for these slightly modified operators.

By summing geometric series in (4), we obtain
\[
\| M_j^{\pm k} \|_{\dot{H}^{-s}} \leq C k^{-\lambda} \| F \|_{\dot{H}^{s_3}},
\]
with \( s_4 \) less than or greater than one depending on whether we are in the case \( j = 1 \) or \( 2 \), and similarly for \( s_3 \). On the other hand, by the Hardy–Littlewood–Sobolev inequality, Hölder’s inequality and interpolation, one has
\[
\| M_j^{\pm k} \|_{\dot{H}^{-s}} \leq C \| F \|_{\dot{H}^{s_5}}, \quad 0 \leq s_5, s_6 < 1.
\]
Interpolating between the two estimates yields (6), and so we are done. \( \Box \)

In the following lemma, we optimise the decay in \( k \), which will be important in Section 4.

**Lemma 2.2.** Let \( 0 < s, \lambda < 1 \). Then
\[
\| S_k^1 \|_{\dot{H}^s} \leq C k^{-\lambda} \| F(\cdot, x) \|_{\dot{H}^s}, \quad x \in \Omega.
\]

**Proof.** By two applications of Lemma 2.1,
\[
\| S_k^1 \|_{\dot{H}^s} \leq \| M_k \circ \partial_z^{-1} \circ M_k \|_{\dot{H}^{s} \to \dot{H}^{s_1}} \leq C k^{-\lambda_1} \| \partial_z^{-1} \circ M_k \|_{\dot{H}^{s} \to \dot{H}^{s_2}} \leq C k^{-\lambda_1} \| M_k \|_{\dot{H}^{s} \to \dot{H}^{s_3}} \leq C k^{-\lambda_1 - \lambda_2},
\]
where \( 0 < \lambda_1 < 1 - s \) and \( 0 < \lambda_2 < s \), and so we are done. \( \Box \)

In the following lemma, we use Lemma 2.1 only once, and gain some integrability using the Hardy–Littlewood–Sobolev theorem. By taking \( k \) sufficiently large, we obtain our contraction and thus our Bukhgeim solution \( u = u_{k,x} \) as described above.

**Lemma 2.3.** Let \( 0 < \lambda < 2s, 1 - s \). Then
\[
\| S_k^1 \|_{\dot{H}^s} \leq C k^{-\lambda} \| V \|_{\dot{H}^s} \| F(\cdot, x) \|_{\dot{H}^s}, \quad x \in \Omega.
\]

**Proof.** By the Cauchy–Schwarz and Hardy–Littlewood–Sobolev inequalities,
\[
\| V F \|_q \leq \| V \|_{2q} \| F \|_{2q} \leq C \| V \|_{\dot{H}^s} \| F \|_{\dot{H}^s},
\]
where \( q = \frac{1}{1 - s} \). Thus, as \( S_k^1 = S_k^1 \| F \| = S_k^1 \| V F \| \), it will suffice to prove that
\[
\| S_k^1 \|_{L_q \to \dot{H}^s} \leq C k^{-\lambda}.
\]
When $0 < s < 1/3$, by Lemma 2.1 we have
\[
\|S^k_1\|_{L^q \to H^s} \leq \|M^{-k} \circ \partial_z^{-1} \circ M^k\|_{L^q \to H^{s-1}} \leq C k^{-\lambda} \|\partial_z^{-1} \circ M^k\|_{L^q \to H^{2s}} \\
\leq C k^{-\lambda} \|M^k\|_{L^q \to H^{2s-1}} \\
\leq C k^{-\lambda} \|M^k\|_{L^q \to H^s} \\
\leq C k^{-\lambda} \|M^k\|_{L^q \to L^s},
\]
where $0 < \lambda < 2s, 1 - s$. When $s \geq 1/3$, we also use Hölder’s inequality:
\[
\|S^k_1\|_{L^q \to H^s} \leq \|M^{-k} \circ \partial_z^{-1} \circ M^k\|_{L^q \to H^{s-1}} \leq C k^{-\lambda} \|\partial_z^{-1} \circ M^k\|_{L^q \to H^{1-s}} \\
\leq C k^{-\lambda} \|M^k\|_{L^q \to H^{-s}} \\
\leq C k^{-\lambda} \|M^k\|_{L^q \to L^s},
\]
where $q^* = \frac{2}{s+1}$, and so we are done. \(\square\)

### 3. Proof of Theorem 1.1

In this section we show that the boundary values of our Bukhgeim solution $u_{k,x}$ can be determined from knowledge of $\Lambda_V$. The argument is inspired by [34, Theorem 5] but we replace the Faddeev green function $G_k$ by its analogue in terms of the operator $S^k_V$ and avoid the use of single layer potentials.

Indeed, considering the kernel representation of $S^k_1$, we can write $S^k_V[F]$ in the form
\[
S^k_V[F](z) = \int_{\Omega} g_\psi(z, \eta)V(\eta)F(\eta)\,d\eta.
\]
where $g_\psi$, the kernel of $S^k_1$, is given by
\[
g_\psi(z, \eta) = \chi_Q(\eta)\frac{e^{i\left(\psi(\eta) + \overline{\psi(\eta)}\right)}}{4\pi^2} \int_Q \frac{1}{(\omega - \eta)(z - \omega)}e^{-i\left(\psi(\omega) + \overline{\psi(\omega)}\right)}\,d\omega.
\]
In order to work directly with exponentially growing solutions we conjugate $g_\psi$ with the exponential factors, so that
\[
\int_{\Omega} G_\psi(z, \eta)V(\eta)F(\eta)\,d\eta = e^{i\psi(z)}S^k_V[e^{-i\psi}F](z),
\]
where $G_\psi(z, \eta) = e^{i\psi(z)}g_\psi(z, \eta)e^{-i\psi(\eta)}$. Notice also that when $z \in Q \setminus \Omega$ and $\eta \in \Omega$, we have that
\[
\Delta_\eta G_\psi(z, \eta) = 0.
\]
Thus, if we take (7) with $F = P_V(f)$, where $P_V(f)$ solves $\Delta u = Vu$ with $u|_{\partial \Omega} = f$, using Alessandrini’s identity we obtain that, for each $z \in Q \setminus \Omega$,
\[
\left<(\Lambda_V - \Lambda_0)[f], G_\psi(z, \cdot)|_{\partial \Omega}\right> = e^{i\psi(z)}S^k_V[e^{-i\psi}P_V(f)](z).
\]
In particular the right-hand side belongs to $H^1(Q \setminus \Omega)$ and hence we can define the operator $\Gamma_\psi : H^{1/2} \to H^{1/2}$ by
\[
\Gamma_\psi[f] = T_r \circ \left<(\Lambda_V - \Lambda_0)[f], G_\psi|_{\partial \Omega}\right>,
\]
where $T_r : H^1(Q \setminus \Omega) \to H^{1/2}(\partial \Omega)$ is the trace operator. Now, by the definitions of $u_{k,x}$ and $w$, we also deduce from (7) and (8) that

$$\int_\Omega G_\psi(\cdot, \eta)V(\eta) u_{k,x}(\eta) \, d\eta = e^{i\psi} S^k_V \left[ 1 + w \right] = e^{i\psi} w = u_{k,x} - e^{i\psi}. $$

Combining (7), (8), and (9) we obtain the integral identity

$$(I - \Gamma_\psi)[u_{k,x}|_{\partial \Omega}] = e^{i\psi}|_{\partial \Omega}. $$

Thus, we can determine $u_{k,x}$ on the boundary if we can invert $(I - \Gamma_\psi)$. By the Fredholm alternative it will suffice to show that $\Gamma_\psi$ is compact and that $(I - \Gamma_\psi)$ has a trivial kernel on $H^{1/2}(\partial \Omega)$.

**Theorem 3.1.** Let $V \in \dot{H}^s$ with $0 < s < 1$. Then

(i) $\Gamma_\psi$ is compact

(ii) $\Gamma_\psi[f] = f \Rightarrow f = 0$.

**Proof of (i).** We have that

$$\Gamma_\psi[f] = T_r[e^{i\psi} S^k_V \left[ e^{-i\psi} P_V(f) \right]].$$

As the set of compact operators is a left and right ideal, we consider the boundedness properties of each component of the composition. Firstly, $P_V : H^{1/2}(\partial \Omega) \to H^1(\Omega)$ is bounded. Secondly, $H^1(\Omega) \hookrightarrow L^p(\Omega)$ compactly for all $2 < p < \infty$. Now taking $p$ sufficiently large and $\frac{1}{2} = \frac{1}{q} + \frac{1}{p}$, by the boundedness of the Cauchy transform followed by the Hardy–Littlewood–Sobolev inequality,

$$\|S^k_V[e^{-i\psi} G]\|_{H^1(Q \setminus \Omega)} \leq C\|V G\|_{L^2(\Omega)} \leq C\|V\|_{L^q(\Omega)}\|G\|_{L^p(\Omega)} \leq C\|V\|_{H^s} \|G\|_{L^p(\Omega)}.$$

Finally, $T_r : H^1(Q \setminus \Omega) \to H^{1/2}(\partial \Omega)$ is bounded. Since the embedding $H^1(\Omega) \hookrightarrow L^p(\Omega)$ is compact, it follows that $\Gamma_\psi$ is compact.

**Proof of (ii).** Letting $\rho = S^k_V[e^{-i\psi} P_V(f)]$, we have that

$$\partial z^i e^{i\psi} \rho = \frac{1}{4} e^{-i\psi} \chi_Q \partial z^i [e^{i\psi} V P_V(f)],$$

so that

$$4\partial z \partial z^i [e^{i\psi} \rho] = V P_V(f) \quad \text{on } \Omega.$$

This can be rewritten as $\Delta [e^{i\psi} \rho - P_V(f)] = 0$ on $\Omega$. Now by hypothesis $\Gamma_\psi[f] = f$, so that by (8) we have $e^{i\psi} \rho = f$ on $\partial \Omega$. Combining the two, we see that

$$e^{i\psi} \rho = P_V(f) \quad \text{on } \Omega.$$

From the definition of $\rho$ we see that $\rho = S^k_V[\rho]$, and as soon as $S^k_V$ is strictly contractive, that $\rho = 0$. This of course follows from Lemma 2.3 for large enough $k$. Thus, $f = e^{i\psi} \rho = 0$, so that $I - \Gamma_\psi$ is injective as desired. \(\square\)
4. Potential recovery

In order to recover the potential at $x \in \Omega$, it remains to show that the right-hand side of Alessandrini’s identity (2) converges to $V(x)$. That is to say $T^k_{1+w}V(x)$ converges to $V(x)$ as $k$ tends to infinity, where

$$T^k_{1+w}[F](x) = \frac{k}{4\pi i} \int_{\mathbb{R}^2} e^{i(\psi(z)+\psi(z))} F(z)(1+w(z)) \, dz.$$  

First we show that $T^k_{w}V$ can be considered to be a remainder term.

**Theorem 4.1.** Let $V \in \dot{H}^s$ with $0 < s < 1$. Then

$$\lim_{k \to \infty} T^k_{w}[V](x) = 0, \quad x \in \Omega.$$  

Moreover, if $k \geq (1 + c\|V\|_{\dot{H}^s}^2)^{\max\{\frac{1}{6}, \frac{1}{2}\}}$ and $0 < \lambda < s$, then

$$\sup_{x \in \Omega} |T^k_{w}[V](x)| \leq Ck^{-\lambda}\|V\|_{\dot{H}^s}^2.$$  

**Proof.** By Lemma 2.1,

$$|T^k_{w}[V](x)| \leq Ck\|M^k[V]\|_{\dot{H}^{s-1}}\|w\|_{\dot{H}^s},$$  

$$\leq Ck^{1-\lambda}\|V\|_{\dot{H}^s}\|(I - S^k_v)^{-1}S^k_v[1]\|_{\dot{H}^s},$$  

where $0 < \lambda < s$. By Lemma 2.3, we can treat $(I - S^k_v)^{-1}$ by Neumann series to deduce that it is a bounded operator on $\dot{H}^s$. In particular, when $s > 1/3$, this is true if $k \geq 1$ and $Ck^{-\frac{1}{3s}}\|V\|_{\dot{H}^s} \leq \frac{1}{2}$. Then

$$|T^k_{w}[V](x)| \leq Ck^{1-\lambda}\|V\|_{\dot{H}^s}\|S^k_v[1]\|_{\dot{H}^s},$$  

$$\leq Ck^{-\lambda+i\epsilon}\|V\|_{\dot{H}^s}^2, \quad \epsilon > 0,$$

by an application of Lemma 2.2, which is the desired estimate. \(\square\)

Noting that $e^{i(\psi(z)+\psi(z))} = \exp\left(ik\frac{(z_1-x_1)^2-(z_2-x_2)^2}{4}\right)$, it remains to prove

$$(10) \quad \lim_{k \to \infty} T^k_{1}[V](x) = V(x),$$

where $T^k_{1}$ is defined by

$$T^k_{1}[F](x) = \frac{k}{4\pi i} \int_{\mathbb{R}^2} \exp\left(ik\frac{(z_1-x_1)^2-(z_2-x_2)^2}{4}\right) F(z) \, dz.$$  

Now when $F$ is a Schwartz function, this is equal to $e^{i\frac{1}{k}\square}F(x)$, where

$$e^{i\frac{1}{k}\square}[F](x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{ix\xi} e^{-i\frac{1}{k}(\xi^2_1-\xi^2_2)} \hat{F}(\xi) \, d\xi.$$  

This follows easily, making use of the distributional formula

$$\frac{k}{4\pi i} \int e^{ik\frac{z^2_1-z^2_2}{4}} \phi(z) \, dz = \int e^{-i\frac{1}{k}(\xi^2_1-\xi^2_2)} \hat{\phi}(\xi) \, d\xi,$$
which holds for Schwartz functions \( \phi \). We see that when \( V \) is a Schwartz function, \( T^k_1V \) solves the time-dependent nonelliptic Schrödinger equation,

\[
i\partial_t u + \Box u = 0,
\]

where \( \Box = \partial_{x_1}^2 - \partial_{x_2}^2 \), with initial data \( V \) at time \( 1/k \). When \( V \in H^s \) with \( s > 1 \), both \( V \) and its Fourier transform are integrable, and so both \( T^k_1V \) and \( e^{i \frac{k}{2} \Box} V \) are continuous functions which are again equal pointwise. Thus, in the following lemma we obtain the convergence \([10]\) and therefore complete the reconstruction for potentials in \( H^s \) with \( s > 1 \).

**Lemma 4.2.** Let \( V \in H^s \) with \( 1 < s < 3 \). Then

\[
|e^{i \frac{k}{2} \Box} V(x) - V(x)| \leq Ck^{\frac{s}{2}}\|V\|_{H^s}, \quad x \in \Omega.
\]

**Proof.** By the Fourier inversion formula and the Cauchy–Schwarz inequality,

\[
|e^{i \frac{k}{2} \Box} V(x) - V(x)| = \frac{1}{(2\pi)^2} \left| \int \hat{V}(\xi) e^{i \xi \cdot x} \left( e^{-i \frac{k}{2}(\xi_1^2 - \xi_2^2)} - 1 \right) d\xi \right|
\]

\[
\leq \|V\|_{H^s} \left( \int \left| e^{-i \frac{k}{2}(\xi_1^2 - \xi_2^2)} - 1 \right|^2 \frac{d\xi}{|\xi|^{2s}} \right)^{1/2}
\]

\[
= \|V\|_{H^s} \left( \int 2 - 2 \cos \left( \frac{k}{2}(\xi_1^2 - \xi_2^2) \right) \frac{d\xi}{|\xi|^{2s}} \right)^{1/2}
\]

\[
= 2k^{\frac{s-2}{2}}\|V\|_{H^s} \left( \int \frac{\sin^2 \left( \frac{k}{2}(\xi_1^2 - \xi_2^2) \right)}{|\xi|^{2s}} d\xi + \int_{\mathbb{R}^2 \setminus \{0\}} \frac{1}{|\xi|^{2s}} d\xi \right)^{1/2},
\]

where we have used the trigonometric identity \( 2\sin^2 \theta = 1 - \cos 2\theta \) and the fact that \( \sin \theta \leq |\theta| \).

 Altogether we see that \( |T^k_1V(x) - V(x)| \leq Ck^{\frac{s-2}{2}} \) for all \( x \in \Omega \) and \( V \in H^s \) with \( 1 < s < 3 \), which improves upon the decay rate of \([39]\) where they recovered \( C^2 \) potentials. Note that there can be no decay rates, at least for the main term, for the potentials of \( H^s \) with \( s \leq 1 \) as they would then be uniform limits of continuous functions and thus continuous.

For discontinuous potentials we are no longer able to recover at each point. Instead we bound the fractal dimension of the sets where the recovery fails. This point of view has its origins in the work of Beurling who bounded the capacity of the divergence sets of Fourier series \([6]\) (see also \([3]\)). Now Sobolev spaces are only defined modulo sets of zero Lebesgue measure, and so we consider first the potential spaces

\[
L^{s,2} = \{ I_s * g : g \in L^2(\mathbb{R}^2) \},
\]

where \( I_s \) is the Riesz potential \( |\cdot|^{s-2} \). As \( \hat{I_s}(\xi) = C_s |\xi|^{-s} \), we have that \( I_s * g \) is also a member of (an equivalence class of) \( H^s \).
To bound the dimension of the sets where the recovery fails, we will prove maximal estimates with respect to fractal measures. We say that a positive Borel measure $\mu$ is $\alpha$-dimensional if
\begin{equation}
\begin{aligned}
c_\alpha(\mu) := \sup_{x \in \mathbb{R}^2, r > 0} \frac{\mu(B(x, r))}{r^\alpha} < \infty, \quad 0 \leq \alpha \leq 2,
\end{aligned}
\end{equation}
and denote by $\mathcal{M}^\alpha(\Omega)$ the $\alpha$-dimensional probability measures which are supported in $\Omega$. For $0 < s < 1$, we will require the elementary inequality
\begin{equation}
\begin{aligned}
\|I_s * g\|_{L^1(d\mu)} \lesssim c_\alpha(\mu) \|g\|_{L^2(\mathbb{R}^2)}, \quad \alpha > 2 - 2s,
\end{aligned}
\end{equation}
which holds whenever $\mu \in \mathcal{M}^\alpha(\Omega)$ and $g \in L^2(\mathbb{R}^2)$. To see this, we note that by Fubini’s theorem and the Cauchy–Schwarz inequality,
\begin{equation}
\|I_s * g\|_{L^1(d\mu)} \leq \|I_s * \mu\|_{L^2} \|g\|_{L^2},
\end{equation}
so that (12) follows by proving
\begin{equation}
\|I_s * \mu\|_{L^2}^2 \lesssim c_\alpha(\mu), \quad \alpha > 2 - 2s.
\end{equation}
Now by Plancherel’s theorem,
\begin{equation}
\|I_s * \mu\|_{L^2}^2 = (2\pi)^{-2} \|\hat{I}_s \hat{\mu}\|_{L^2}^2 \lesssim \int \hat{\mu}(\xi) \overline{\hat{\mu}(\xi)} \hat{I}_{2s}(\xi) \, d\xi \lesssim \int \mu * I_{2s}(y) \, d\mu(y)
\end{equation}
\begin{equation}
= \int \int \frac{d\mu(x) \, d\mu(y)}{|x - y|^{2-2s}},
\end{equation}
which is nothing more than the $(2 - 2s)$-energy. Then, by an appropriate dyadic decomposition,
\begin{equation}
\int \int \frac{d\mu(x) \, d\mu(y)}{|x - y|^{2-2s}} \lesssim \sum_{j=0}^{\infty} c_\alpha(\mu) 2^{-j\alpha} 2^j (2 - 2s) \, d\mu(y) \lesssim c_\alpha(\mu)
\end{equation}
whenever $\alpha > 2 - 2s$ and $\mu \in \mathcal{M}^\alpha(\Omega)$.

The Fourier transform of less regular potentials $V$ is not necessarily integrable, and so in that case $e^{i\xi \cdot \Box} V$ is not even well-defined. Instead we make do with the pointwise limit
\begin{equation}
T_k[V](x) = \lim_{N \to \infty} G_N * T_k[V](x) = \lim_{N \to \infty} e^{i\xi \cdot \Box} [G_N * V](x), \quad x \in \Omega,
\end{equation}
where $G_N = N^2 G(N \cdot)$ and $G$ is the Gaussian $e^{-|\cdot|^2}$. This formula holds as $V$ is compactly supported and integrable; conditions which the initial data in the time-dependent theory does not normally satisfy. We will also require the following lemma due, in this form, to Sjölin [43].

**Lemma 4.3.** [43] Let $x, t \in \mathbb{R}$, $\gamma \in [1/2, 1)$ and $N \geq 1$. Then
\begin{equation}
\left| \int_{\mathbb{R}} \frac{\eta(N^{-1} \xi) e^{i(x \xi - t \xi^2)}}{|\xi|^{\gamma}} \, d\xi \right| \lesssim \frac{1}{|x|^{1-\gamma}},
\end{equation}
where the constant implied by the symbol $\lesssim$ depends only on $\gamma$ and the Schwartz function $\eta$. 

**ROUGH POTENTIAL RECOVERY IN THE PLANE**

11
In the following theorem, we employ the Kolmogorov–Seliverstov–Plessner method, as used by Carleson [13] for the one-dimensional Schrödinger equation. Dahlberg and Kenig [19] proved that the result of Carleson is sharp and noted that his argument could be applied to the higher dimensional problem (for which the argument is no longer sharp for the elliptic equation, see [8]). We refine their argument, which extends to the nonelliptic case, by proving estimates which hold uniformly with respect to fractal measures.

**Theorem 4.4.** Let $1/2 \leq s < 1$. Then

$$\left\| \sup_{k \geq 1} N \geq 1 \left| e^{i \frac{t}{k} \Box} [G_N * I_s * g] \right| \right\|_{L^1(d\mu)} \lesssim \sqrt{c_0(\mu)} \| g \|_{L^2(\mathbb{R}^2)}, \quad \alpha > 2 - s,$$

whenever $\mu \in \mathcal{M}^\alpha(\Omega)$ and $g \in L^2$.

**Proof.** By linearising, it will suffice to prove

$$\left( 14 \right) \int_\Omega e^{it|x| \Box} [G_N(x) * I_s * g] w(x) \, d\mu(x) \right| \left( 2 \right) \lesssim c_0(\mu) \| g \|_{L^2}, \quad \alpha > 2 - s,$$

uniformly in measurable functions $t : \Omega \to \mathbb{R}$, $N : \Omega \to \mathbb{N}$ and $w : \Omega \to \mathbb{D}$. By Fubini’s theorem and the Cauchy–Schwarz inequality, the left-hand side of (14) is bounded by

$$\int |\hat{g}(\xi)|^2 d\xi \int \left| \int G\left( \frac{\xi}{N(x)} \right) e^{i t(x) | \xi|^2} e^{i x \cdot \xi} w(x) \, d\mu(x) \right|^2 \frac{d\xi}{|\xi|^{2s}}.$$

Writing the squared integral as a double integral, and applying Fubini’s theorem again, it will suffice to show that

$$\int \int G\left( \frac{\xi_j}{N(x)} \right) G\left( \frac{\xi_j}{N(y)} \right) e^{i t(x) - (t(y)) | \xi_j|^2} e^{i x \cdot \xi_j} \frac{d\xi_j}{|\xi_j|^{2s}} \times$$

$$w(x) w(y) \, d\mu(x) d\mu(y) \lesssim c_0(\mu)$$

uniformly in the functions $t$, $N$ and $w$. Now, as $|\xi|^{2s} \gtrsim |\xi_1|^s |\xi_2|^s$, the left-hand side of (15) is bounded by

$$\prod_{j=1}^2 \left| \int G\left( \frac{\xi_j}{N(x)} \right) G\left( \frac{\xi_j}{N(y)} \right) e^{i (t(x) - t(y)) \xi_j^2} e^{i x \cdot \xi_j} \frac{d\xi_j}{|\xi_j|^s} \right| \times$$

$$w(x) w(y) \, d\mu(x) d\mu(y),$$

and by Lemma 4.3, we have

$$\left| \int G\left( \frac{\xi_j}{N(x)} \right) G\left( \frac{\xi_j}{N(y)} \right) e^{i (t(x) - t(y)) \xi_j^2} e^{i x \cdot \xi_j} \frac{d\xi_j}{|\xi_j|^s} \right| \lesssim \frac{1}{|x_j - y_j|^{1-s}}.$$

Substituting in, we see that the left-hand side of (15) is bounded by

$$\left( 16 \right) C \int \int \frac{|w(x) w(y)| \, d\mu(x) d\mu(y)}{|x_1 - y_1|^{1-s} |x_2 - y_2|^{1-s}} \leq C \int \frac{d\mu(x) d\mu(y)}{|x_1 - y_1|^{1-s} |x_2 - y_2|^{1-s}}.$$
To complete the proof, we are required to bound (16) by $c_\alpha(\mu)$. This will require a dyadic decomposition which lends itself to the singularities along the axis-parallel lines $A_y$ defined by

$$A_y = \{ x \in \Omega : x_1 = y_1 \text{ or } x_2 = y_2 \}, \quad y \in \Omega.$$  

Covering $A_y$ by balls $\{B_j\}_{j \geq 1}$ of radius $r_j$ and using the definition (11) of $c_\alpha(\mu)$, we have

$$\mu(A_y) \leq \sum_{j \geq 1} \mu(B_j) \leq c_\alpha(\mu) \sum_{j \geq 1} r_j^\alpha.$$  

Taking the infimum over all such coverings and using the fact that the $\alpha$-Hausdorff measure of $A_y$ is zero when $\alpha > 1$, we see that $\mu(A_y) = 0$ for all $\mu \in M^\alpha(\Omega)$. Thus we can ignore the sets $A_y$ when decomposing the inner integral of (16).

For each $j, \ell \in \mathbb{Z}$ we break up $Q \supset \Omega$ into dyadic rectangles of dimensions $2^{-j} \times 2^{-\ell}$ and consider the unique rectangle $R_{j,\ell}$ which contains $y$. We call the unique rectangles $R_{j-1,\ell-1}$, $R_{j-1,\ell}$, and $R_{j,\ell-1}$ that contain $R_{j,\ell}$, the mother, the father, and the stepfather respectively. We write $R_n \sim R_{j,\ell}$ if their mothers touch, but their fathers and stepfathers do not. As $\mu(A_y) = 0$, we can write

$$\int F(x, y) d\mu(x) = \sum_{j, \ell \geq 0} \sum_{n: R_n \sim R_{j,\ell}} \int_{R_n} F(x, y) d\mu(x),$$  

which yields

$$\text{(16)} \leq C \int \sum_{j, \ell \geq 0} \sum_{n: R_n \sim R_{j,\ell}} 2^{j(1-s)} 2^{\ell(1-s)} \mu(R_n) \mu(y).$$  

Without loss of generality, we can suppose that

$$\sum_{\ell \geq j \geq 0} \sum_{n: R_n \sim R_{j,\ell}} 2^{j(1-s)} 2^{\ell(1-s)} \mu(R_n) \mu(y) \leq \sum_{j \geq \ell \geq 0} \sum_{n: R_n \sim R_{j,\ell}} 2^{j(1-s)} 2^{\ell(1-s)} \mu(R_n),$$  

where $y$ is associated with a single point.
so that
\[
\sum_{\ell \geq 0} \int \sum_{R_{j,\ell}^n \sim R_{j,\ell}} 2^{j(1-s)}2^{\ell(1-s)} \mu(R_{j,\ell}^n) \, d\mu(y).
\]

Now by covering each rectangle by discs of radius $2^{-j}$, and using the definition (11) of $c_\alpha(\mu)$, we see that
\[
\mu(R_{j,\ell}^n) \lesssim 2^{j-\ell} c_\alpha(\mu) 2^{-j\alpha},
\]
and for each rectangle $R_{j,\ell}$ there are exactly nine rectangles $R_{j,\ell}^n$ which satisfy $R_{j,\ell}^n \sim R_{j,\ell}$. Thus
\[
(16) \lesssim c_\alpha(\mu) \sum_{j \geq \ell \geq 0} 2^{j(2-s-\alpha)} 2^{-\ell s} \lesssim c_\alpha(\mu),
\]
when $\alpha > 2 - s$, and so we are done. \(\square\)

Proof of Theorem 1.2. By Alessandrini’s identity (2) and Frostman’s lemma (see for example [31]), it will suffice to prove that
\[
\mu\left\{ x : \limsup_{k \to \infty} \limsup_{N \to \infty} |e^{i\frac{k}{N}}[G_N * V](x) - V(x)| \neq 0 \right\} = 0
\]
whenever $\mu \in M^\alpha(\Omega)$ and $V \in L^s,2(\Omega)$ with $\alpha > 2 - s$. By Theorem 4.1 and (13), this would follow from
\[
\mu\left\{ x : \limsup_{k \to \infty} \limsup_{N \to \infty} |e^{i\frac{k}{N}}[G_N * V](x) - V(x)| \neq 0 \right\} = 0.
\]

Writing $V = I_s * g$, where $g \in L^2$, we take a Schwartz function $h$ so that $\|g - h\|_{L^2} < \epsilon$. Then
\[
\mu\left\{ x : \limsup_{k \to \infty} \limsup_{N \to \infty} |e^{i\frac{k}{N}}[G_N * V](x) - V(x)| > \lambda \right\}
\]
\[
\leq \mu\left\{ x : \sup_{k \geq 1} \sup_{N \geq 1} |e^{i\frac{k}{N}}[G_N * (g - h)](x)| > \lambda/3 \right\} +
\]
\[
\mu\left\{ x : \limsup_{k \to \infty} \limsup_{N \to \infty} |e^{i\frac{k}{N}}[G_N * I_s * h](x) - I_s * h(x)| > \lambda/3 \right\} +
\]
\[
\mu\left\{ x : |I_s * (h - g)(x)| > \lambda/3 \right\}.
\]

As the terms involving $h$ are continuous in all parameters, the second set of the three is empty, so by the elementary inequality (12) and Theorem 4.4, we see that
\[
\mu\left\{ x : \limsup_{k \to \infty} |T_k^{1+w} V(x) - V(x)| > \lambda \right\} \lesssim \lambda^{-1} \sqrt{c_\alpha(\mu)} \|g - h\|_{L^2}
\]
\[
\lesssim \lambda^{-1} \sqrt{c_\alpha(\mu)} \epsilon,
\]
for all $\epsilon > 0$, which yields (17), and so we are done. \(\square\)

Proof of Theorem 1.5. For $V \in H^{1/2}$, the potentials $q = V - \kappa^2 \chi_\Omega$ are contained in $H^s$ for $0 < s < 1/2$ (see for example [20]) and so we find
Bukhgiem solutions $U_{k,x}$, associated to $q$, and recover their value on the boundary as before. Corollary 1.3 requires the potential $q$ to be contained in $H^{1/2}$ which is not satisfied for any domain. However, it is clear from the proof of Theorem 4.4 that we can relax this additional condition further to

$$\| (i \frac{\partial}{\partial x_1})^{1/4} (i \frac{\partial}{\partial x_2})^{1/4} q \|_{L^2(\mathbb{R}^2)} < \infty,$$

which is satisfied when $\Omega$ is a square, but not when it is a disc. \hfill \Box

Finally we remark that the uniqueness result of Blåsten [7] can be observed using the connection with the time-dependent Schrödinger equation. Indeed if the scattering data or boundary measurements are the same for two potentials $V_1$ and $V_2$, then by Alessandrini’s identity (2),

$$\| V_2 - V_1 \|_{L^2} = \| V_2 - T_k V_1 - V_1 \|_{L^2},$$

so that by the triangle inequality and Lemma 4.1, it suffices to prove

$$\| V - T_k V \|_{L^2} \to 0 \quad \text{as} \quad k \to \infty,$$

which is a well-known property of the Schrödinger flow.

5. Proof of Theorem 1.4

First we construct a real potential $V$, supported in $\Omega$, and contained in $H^s$ with $s < 1/2$, for which

$$\left\{ x \in \Omega : \lim_{k \to \infty} e^{it\square}[V](x) \neq V(x) \right\} \neq 0.$$

Throughout this section we work with a different set of coordinates from the previous sections. Indeed, for Schwartz functions $F$, we now write

$$e^{it\square}[F](x) = \frac{1}{(2\pi)^2} \int e^{ix \cdot \xi} e^{-2it \xi_1 \xi_2} \hat{F}(\xi) \, d\xi.$$

Let $\phi_0$ be a positive Schwartz function, compactly supported in $[-1/4, 1/4]$, and consider $\phi = \phi_0 \ast \phi_0$, which is supported in $[-1/2, 1/2]$. Note that $\hat{\phi} = (\hat{\phi_0})^2 \geq 0$. We consider the potential $V$ defined by

$$V(x) = \sum_{j \geq 2} V_j(x) = \sum_{j \geq 2} 2^{(1-\beta)j+1} \cos(2^j x_2) \phi(2^j x_1) \phi(x_2)
= \sum_{j \geq 2} 2^{(1-\beta)j} e^{i2^j x_2} \phi(2^j x_1) \phi(x_2) + \sum_{j \geq 2} 2^{(1-\beta)j} e^{-i2^j x_2} \phi(2^j x_1) \phi(x_2)
= \sum_{j \geq 2} V_j^+(x) + \sum_{j \geq 2} V_j^-(x),$$

which is supported in $[-1/8, 1/8] \times [-1/2, 1/2]$. If $\beta \in (1/2 + s, 1)$, by changes of variables,

$$\| V \|_{H^s} \leq C \sum_{j \geq 2} 2^{(1-2\beta+2s)j} \int |\hat{\phi}(\xi_1)\hat{\phi}(\xi_2)|^2 (1 + |\xi|^2)^s d\xi < \infty.$$
Thus $V$ is finite almost everywhere, and we will show that $e^{\frac{i\pi}{16} \square} V$ diverges on $\left[ \frac{1}{16}, \frac{1}{4} \right] \times \left[ \frac{1}{16}, \frac{1}{4} \right]$.

This potential is an adaptation of an initial datum for the time-dependent nonelliptic Schrödinger equation considered in [11]. The initial datum there was not real, the diverging sequence of time was allowed to depend on the point $x$, and more crucially, the initial datum was not compactly supported. Thus our arguments will have a different flavour, working on the frequency and spatial side simultaneously.

By changes of variables and the Fourier inversion formula,

$$e^{i\frac{\pi}{16} \square} [V_+]^j (x) = \frac{2^{(1-\beta)j} e^{i2^j x_2}}{(2\pi)^2} \int \widehat{\phi}(\xi_1) \widehat{\phi}(\xi_2) e^{-i2^{j+1} \xi_1 \xi_2} e^{i(2^j \xi_1 (x_1 - 2^{j+1} t) + 2^j x_2)} d\xi_2$$

$$= \frac{2^{(1-\beta)j} e^{i2^j x_2}}{2\pi} \int \phi(2^j (x_1 - 2^{j+1} t - 2^j t \xi_2)) \widehat{\phi}(\xi_2) e^{i\xi_2 x_2} d\xi_2,$$

Taking $t = 1/k$ with $k$ the nearest natural number to $2^{j+1}/x_1$,

$$e^{i\frac{\pi}{16} \square} [V_+]^j (x) = \frac{2^{(1-\beta)j} e^{i2^j x_2}}{2\pi} \int \phi(\zeta(x_1, j) - \frac{2^{j+1}}{k} \xi_2) \widehat{\phi}(\xi_2) e^{i\xi_2 x_2} d\xi_2,$$

where $|\zeta(x_1, j)| \leq \frac{1}{4}$ when $x_1 \in \left[ \frac{1}{16}, \frac{1}{4} \right]$, so that, using the compact support of $\phi$, we see that

$$|e^{i\frac{\pi}{16} \square} [V_+]^j (x)| \geq \frac{2^{(1-\beta)j} \cos(1)}{2\pi} \int_{-16}^{16} \phi(\zeta(x_1, j) - \frac{2^{j+1}}{k} \xi_2) \widehat{\phi}(\xi_2) \cos(\xi_2 x_2) d\xi_2$$

$$\geq C \frac{2^{(1-\beta)j}}{2\pi} \int_{-16}^{16} \phi(\zeta(x_1, j) - \frac{2^{j+1}}{k} \xi_2) \widehat{\phi}(\xi_2) d\xi_2.$$

Now when $x_2 \in \left[ -\frac{1}{16}, \frac{1}{16} \right]$, we have $|\xi_2 x_2| \leq 1$, so that $|\cos(\xi_2 x_2)| > \cos(1)$. Using the fact that $\phi$ and $\widehat{\phi}$ are nonnegative, we obtain

$$|e^{i\frac{\pi}{16} \square} [V_+]^j (x)| \geq C \frac{2^{(1-\beta)j}}{2\pi} \int_{-16}^{16} \phi(\zeta(x_1, j) - \frac{2^{j+1}}{k} \xi_2) \widehat{\phi}(\xi_2) d\xi_2$$

It remains to bound from above the solution associated to the other pieces of the potential. Again, by the Fourier inversion formula,

$$|e^{i\frac{\pi}{16} \square} [V_\ell]^\pm (x)| = \frac{2^{(1-\beta)j} (2\pi)^2}{(2\pi)^2} \int \widehat{\phi}(\xi_1) \widehat{\phi}(\xi_2) e^{-i\frac{\ell}{2} \xi_1 \xi_2} e^{i(2^\ell \xi_1 (x_1 + \frac{2^{j+1}}{k} + 2^j x_2))} d\xi_2$$

$$= \frac{2^{(1-\beta)j} (2\pi)^2}{2\pi} \int \phi(2^\ell (x_1 + \frac{2^{j+1}}{k} + \frac{2^j \xi_2}) \widehat{\phi}(\xi_2) e^{i\xi_2 x_2} d\xi_2.$$

Using the fact that $\phi(y) \leq C |y|^{-1/2}$, we obtain

$$|e^{i\frac{\pi}{16} \square} [V_\ell]^\pm (x)| \leq C 2^{(1/2-\beta)\ell} \int \frac{|\widehat{\phi}(\xi_2)|}{|x_1 + \frac{2^{j+1}}{k} + \frac{\ell \xi_2}{2}|^{1/2}} d\xi_2.$$
Taking $0 < \epsilon < \min\{1/4, 1 - \beta\}$, and using the rapid decay of $\hat{\phi}$, we see that
\[
|e^{i\frac{k}{k}\Delta} [V^\pm_k](x)| \leq C2^{(1/2 - \beta)k} \left( \int_{|\xi_2| < 2^j} \frac{1}{x_1 + \frac{2^{j+1}}{k} - \frac{2}{k} \xi_2 |1/2} d\xi_2 + C2^{-j} \right).
\]

Now one can check that when $\ell \neq j$ or $j = \ell$ and $\mp$ is an addition,
\[
\left| \frac{2}{k} \xi_2 \right| \leq \frac{3}{4} |x_1 + \frac{2^{j+1}}{k}|
\]
when $|\xi_2| \leq 2^j$. Indeed, when $j > \ell$, the left-hand side is less than $\frac{1}{4} |x_1|$ which is less than the right-hand side. On the other hand, when $j < \ell$ or $j = \ell$ and $\mp$ is an addition, the left-hand side is less than $\frac{1}{2} |x_1|$ which is less than the right-hand side. Thus, the integrand of the final integral is nonsingular so that the integral is bounded by $C|x_1|^{-1/2}2^j \leq C2^j$.

By summing a geometric series in $\ell$, we obtain
\[
\left| \sum_{\ell \neq j} e^{i\frac{k}{k}\Delta} [V^\pm_k](x) + e^{i\frac{k}{k}\Delta} [V^-_j](x) \right| \leq C2^j,
\]
and we can conclude that on $[\frac{1}{10}, \frac{1}{4}] \times [-\frac{1}{10}, \frac{1}{10}],$
\[
|e^{i\frac{k}{k}\Delta} [V]| \geq |e^{i\frac{k}{k}\Delta} [V^+_j]| - \left| \sum_{\ell \neq j} e^{i\frac{k}{k}\Delta} [V^+_k] + e^{i\frac{k}{k}\Delta} [V^-_j](x) \right| \geq C1 2^{j(1 - \beta)} - C2^j,
\]
which diverges as $j$ tends to infinity. Considering forty-five degree rotations of the $V_j$, which are Schwartz functions, via the pointwise equality, this yields
\[
|T^k[V]| \geq |T^k[V^+_j]| - \left| \sum_{\ell \neq j} T^k[V^+_k] + T^k[V^-_j] \right| \geq C1 2^{j(1 - \beta)} - C2^j
\]
on a forty-five degree rotation of $[\frac{1}{10}, \frac{1}{4}] \times [-\frac{1}{10}, \frac{1}{10}]$, so that $|T^k[V]|$ diverges as $k$ tends to infinity. Thus, by Theorem 4.1 combined with Alessandrinis’s identity (2),
\[
\left\{ x : \frac{1}{4\pi} \left( (\Lambda V - \Lambda_0)[u_{k,x} |\partial\Omega], e^{i\frac{k}{k}\Delta} |\partial\Omega \right) \not\rightarrow V(x) \text{ as } k \to \infty \right\}
\]
contains a forty-five degree rotation of $[\frac{1}{10}, \frac{1}{4}] \times [-\frac{1}{10}, \frac{1}{10}]$, which has nonzero Lebesgue measure. \[\square\]

Note that this result is stable in the sense that $k \in \mathbb{N}$ can be replaced by any sequence $\{n_k\}_{k \in \mathbb{N}}$ satisfying $n_k \in [k, k + 1)$.

**Remark 5.1.** In [44], Sjölin asked for which values of $s$ is it true that
\[
\lim_{k \to \infty} e^{i\frac{k}{k}\Delta} f(x) = 0, \quad \text{a.e. } x \in \mathbb{R}^d \setminus (\text{supp} f),
\]
for all $f \in H^s$. In principle, this question could have stronger positive results and weaker negative results than Carleson’s question: for which values of $s$ is it true that
\[
\lim_{k \to \infty} e^{i\frac{k}{k}\Delta} f(x) = f(x), \quad \text{a.e. } x \in \mathbb{R}^d,
\]

Sjölin’s question is open. In principle, this question could have stronger positive results and weaker negative results than Carleson’s question: for which values of $s$ is it true that
for all $f \in H^s$? Indeed, before Bourgain’s recent breakthrough [8], Sjölin proved a stronger positive result for his question than what was known for Carleson’s question in three dimensions. Here we solve Sjölin’s question completely for the nonelliptic equation in two dimensions. That is to say,

$$\lim_{k \to \infty} e^{i k \Box} f(x) = 0, \quad \text{a.e. } x \in \mathbb{R}^2 \setminus (\text{supp} f),$$

for all $f \in H^s$ if and only if $s \geq 1/2$.

**Appendix A. The DN map from the scattering amplitude**

It is well–known that in the absence of zero Dirichlet eigenvalues there is a unique weak solution to the Dirichlet problem (1) that satisfies

$$\|u\|_{H^1(\Omega)} \leq C\|f\|_{H^{1/2}(\partial \Omega)}$$

(see for example [17]– in two dimensions $L^{n/2}(\mathbb{R}^n)$ can be replaced by $L^2(\mathbb{R}^2)$). Here $H^{1/2}(\partial \Omega) := H^1(\Omega)/H^0_0(\Omega)$, where $H^0_0(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$. The DN map $\Lambda_V$ is then defined by

$$\langle \Lambda_V[f], \psi \rangle = \int_{\partial \Omega} \Lambda_V[f] \psi = \int_{\Omega} Vu \Psi + \nabla u \cdot \nabla \Psi,$$

for all $\Psi \in H^1(\Omega)$ with $\psi = \Psi + H^0_0(\Omega)$. When the solution and boundary are sufficiently regular, this definition coincides with that of the introduction by Green’s formula. To see that $\Lambda_V$ maps from $H^{1/2}(\partial \Omega)$ to $H^{-1/2}(\partial \Omega)$, the dual of $H^{1/2}(\partial \Omega)$, we note that by Hölder’s inequality and the Hardy–Littlewood–Sobolev inequality,

$$\left| \langle \Lambda_V[f], \psi \rangle \right| \leq \|u\|_{H^1(\Omega)} \|\Psi\|_{H^1(\Omega)} + \|V\|_2 \|u\|_{L^4(\Omega)} \|\Psi\|_{L^4(\Omega)} \leq (1 + C\|V\|_2) \|u\|_{H^1(\Omega)} \|\Psi\|_{H^1(\Omega)}$$

whenever $\Psi \in H^1(\Omega)$, so that by (18), we obtain

$$\left| \langle \Lambda_V[f], \psi \rangle \right| \leq C(1 + \|V\|_2) \|f\|_{H^{1/2}(\partial \Omega)} \|\psi\|_{H^{1/2}(\partial \Omega)}.$$
and let $S_V$ and $S_0$ be the corresponding near-field operators defined via single layer potentials:

$$S_V[f](x) = \int_{\partial \Omega} G_V(x, y) f(y) dy, \quad S_0[f](x) = \int_{\partial \Omega} G_0(x, y) f(y) dy.$$  

These are bounded and invertible, mapping $H^{-1/2}(\partial \Omega)$ to $H^{1/2}(\partial \Omega)$ (the two–dimensional proof can be found in [25, Proposition A.1]). Then Nachman’s formula [32],

$$\Lambda_{V-\kappa^2} = \Lambda - \kappa^2 + S_V^{-1} - S_0^{-1},$$

allows us to recover the DN map.

Thus it remains to recover the single layer potential $S_V$ from the scattering amplitude $A_V$ at energy $\kappa^2$. For $\omega \in S^1$, the outgoing scattering solution $v(\cdot, \omega, \kappa)$ is the unique solution to the Lippmann–Schwinger equation

$$v(y, \omega, \kappa) = e^{i\kappa y \cdot \omega} - \int_{\mathbb{R}^2} G_0(y, z) V(z) v(z, \omega, \kappa) dz. \quad (19)$$

For $(\sigma, \omega) \in S^1 \times S^1$, the scattering amplitude then satisfies

$$A_V(\sigma, \omega, \kappa) = \int_{\mathbb{R}^2} e^{-i\kappa \sigma \cdot z} V(z) v(z, \omega, \kappa) dz. \quad (20)$$

When $\Omega$ is a disc, Nachman recovers $S_V$ via formulae given by expansions in spherical harmonics as below. Otherwise he uses a density argument (we remark that Sylvester [49] also invokes density in order to recover). Since we have been obliged to work with $\Omega$ a square, at this point we deviate and instead follow an argument of Stefanov [45], obtaining an explicit formula for the Green’s function $G_V$ in terms of $A_V$. Alternatively it seems likely that we could pass to the DN map on the square from that on the disc via the argument in [34, Section 6] for the conductivity problem, but we prefer this more direct approach.

Stefanov worked in three dimensions, with bounded potentials, and a number of details change in two dimensions, so we present the argument. We recover $G_V$ outside of a disc which contains the potential, but which is contained in the domain, so that $S_V$ can be obtained by integrating along the sides of our square $\Omega$. First we require the following asymptotics.

**Lemma A.1.**

$$G_V(x, y) - G_0(x, y) = \frac{-i}{8\pi \kappa} \frac{e^{i\kappa |x|}}{|x|^{3/2}} \frac{e^{i\kappa |y|}}{|y|^{3/2}} A_V \left( \frac{x}{|x|}, -\frac{y}{|y|}, \kappa \right) + o\left( \frac{1}{|x|^{1/2} |y|^{1/2}} \right).$$

**Proof.** It is well–known (see for example (3.66) in [35]) that $G_V$ satisfies

$$G_V(x, z) = \frac{e^{i\pi}}{(8\pi)^{3/2}} \frac{e^{i\kappa |x|}}{\kappa \pi |x|^{3/2}} v(z, -\frac{x}{|x|}, \kappa) + o\left( \frac{1}{|x|^{1/2}} \right), \quad (21)$$
and, in particular,

\begin{equation}
G_0(y, z) = \frac{e^{i\frac{\pi}{4}}}{(8\pi)^{\frac{1}{2}} \kappa^\frac{3}{2} |y|^\frac{3}{2}} e^{-i\kappa \frac{|y|}{2}} e^{i\pi |y|^\frac{1}{2}} + o\left(\frac{1}{|y|^\frac{1}{2}}\right).
\end{equation}

From the definitions, it easy to verify that

\begin{equation}
G_V(x, y) - G_0(x, y) = -\int_{\mathbb{R}^2} G_0(x, z) V(z) G_0(y, z) \, dz.
\end{equation}

Substituting in (21) and (22), see that

\begin{equation}
G_V(x, y) - G_0(x, y) = \frac{-i}{8\pi \kappa} e^{i\kappa |x|} e^{i\kappa |y|} \int_{\mathbb{R}^2} e^{-i\kappa \frac{|y|}{2}} V(z) v\left(z, -\frac{x}{|x|}, \kappa\right) \, dz + o\left(\frac{1}{|x|^\frac{1}{2} |y|^\frac{1}{2}}\right),
\end{equation}

so that (20) and the symmetries of $A_V$ yield the result. \(\square\)

In the following, $J_n$ and $H_n^{(1)}$ denote the Bessel and Hankel functions of the first kind of $n$th order, respectively (see for example [29]). We also write $x$ in polar coordinates as $(|x|, \phi_x)$.

**Theorem A.2.** Let $V \in H^s$ with $s > 0$ be supported in the disc of radius $\rho$, centred at the origin, and consider the Fourier series

\[ A_V(\sigma, \omega, \kappa) = \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} a_{n,m} e^{in\phi_x} e^{im\phi_y}. \]

Then

\[ G_V(x, y) - G_0(x, y) = \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \frac{(-1)^m}{16} e^{in+m\kappa} a_{n,m} H_n^{(1)}(\kappa|x|) H_m^{(1)}(\kappa|y|) e^{in\phi_x} e^{im\phi_y}, \]

where the series is uniformly, absolutely convergent for $|x| > |y| > R > \rho$.

**Proof.** We can expand $G_0(x, y) = \frac{1}{4} H_0^{(1)}(\kappa|x| - \kappa|y|)$ as

\[ G_0(x, y) = H_0^{(1)}(\kappa|x|) J_0(\kappa|y|) + 2 \sum_{n \geq 1} H_n^{(1)}(\kappa|x|) J_n(\kappa|y|) \cos(\phi_x - \phi_y), \]

(see for example [15, Section 3.4] or [42, Theorem 3.4]). As $H_n^{(1)} = (-1)^n H_n^{(1)}$ and $J_n = (-1)^n J_n$, in order to separate variables it will be convenient to write this as

\[ G_0(x, y) = \sum_{n \in \mathbb{Z}} H_n^{(1)}(\kappa|x|) J_n(\kappa|y|) e^{in\phi_x} e^{-in\phi_y}. \]

As before, it is easy to check that

\[ G_V(x, y) - G_0(x, y) = -\int_{\mathbb{R}^2} G_0(x, z) V(z) G_V(z, y) \, dz, \]
and so substituting (23) into this we obtain $G_V - G_0 = -I_1 + I_2$, where

$$I_1 = \int G_0(x, z)V(z)G_0(z, y)\,dz$$
$$I_2 = \int G_0(x, z_1)V(z_1)\int G_V(z_1, z_2)V(z_2)G_0(y, z_2)\,dz_2\,dz_1.$$

Now in both integrals we introduce the expansion of $G_0$ (note that $G_0(x, y) = G_0(y, x)$), extracting the terms independent of $z, z_1, z_2$. In this way we get

(24) $I_1 = \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \alpha_{n,m} H_n^{(1)}(\kappa |x|) H_m^{(1)}(\kappa |y|) e^{im\phi_x} e^{im\phi_y},$

(25) $I_2 = \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \beta_{n,m} H_n^{(1)}(\kappa |x|) H_m^{(1)}(\kappa |y|) e^{im\phi_x} e^{im\phi_y},$

where

$$\alpha_{n,m} = \int_{\mathbb{R}^2} V(z)J_n(\kappa |z|)J_m(\kappa |z|)e^{-i(n+m)\phi_z} \,dz,$$

$$\beta_{n,m} = \int_{\mathbb{R}^4} J_n(\kappa |z_1|)V(z_1)G_V(z_1, z_2)V(z_2)J_m(\kappa |z_2|)e^{-i\phi_{z_1} - im\phi_{z_2}} \,dz_1\,dz_2.$$

It remains to show that the sums (24) and (25) converge uniformly and absolutely for $|x| > |y| > R > \rho$. Once we know that this is the case, we can take limits and use the asymptotics of the Hankel functions for large $r$;

$$H_n^{(1)}(r) = e^{-i(n\pi/4 + \pi/4)}\left(\frac{2}{\pi r}\right)^{1/2} r + o\left(\frac{1}{r^{3/4}}\right)$$

(see for example [29, Section 5.16]), and then Proposition [A.1] tells us that

$$(-i)^{n+m} 2\pi (\beta_{n,m} - \alpha_{n,m}) = \frac{(-1)^m}{8\pi} a_{n,m}.$$

To see that the sums converge note that, by Hölder’s inequality, we have

$$|\alpha_{n,m}| \leq C_p \|V\|_{L^2} \|J_n(\kappa \cdot)\|_{L^\infty(B_\rho)} \|J_m(\kappa \cdot)\|_{L^\infty(B_\rho)},$$

$$|\beta_{n,m}| \leq \|G_V\|_{L^2(B_\rho \times B_\rho)} \|V\|_{L^2} \|J_n(\kappa \cdot)\|_{L^\infty(B_\rho)} \|J_m(\kappa \cdot)\|_{L^\infty(B_\rho)}.$$

At this point we deviate from [45] as there seems to be less local knowledge regarding $G_V$ in two dimensions. Instead we can rewrite (23) as

$$G_V(\cdot, y) = G_0(\cdot, y) - (-\Delta + V - \kappa^2 - i0)^{-1}[VG_0(\cdot, y)],$$

and use that the resolvent is bounded from $L^2((1+|\cdot|^2)^\delta)$ to $L^2((1+|\cdot|^2)^{-\delta})$ with $\delta > 1/2$ (see [11, Theorem 4.2]). Thus, using that $V$ is compactly supported, and taking $1/2 = 1/p + 1/q$ with large $p$ so that $1 - \frac{2}{q} = s$,

$$||G_V(\cdot, y)||_{L^2(B_\rho)} \leq ||G_0(\cdot, y)||_{L^2(B_\rho)} + C_p \|V G_0(\cdot, y)\|_{L^2(B_\rho)}$$

$$\leq ||G_0(\cdot, y)||_{L^2(B_\rho)} + C_p \|V\|_q ||G_0(\cdot, y)||_{L^p(B_\rho)}$$

$$\leq ||G_0(\cdot, y)||_{L^2(B_\rho)} + C_p \|V\|_{H^s} ||G_0(\cdot, y)||_{L^p(B_\rho)}.$$
we see that \( \| G_V \|_{L^2(B_\rho \times B_\rho)} \leq C \). Then, using the Taylor series expansion for the Bessel function,

\[
|J_n(r)| = \sum_{j \geq 0} \frac{(-1)^j}{j!(|n| + j)!} \left( \frac{r}{2} \right)^{2j+|n|} \leq C_{\rho} \frac{1}{|n|!} \left( \frac{\rho}{2} \right)^{|n|}, \quad 0 \leq r \leq \rho,
\]

we see that

\[
|\alpha_{n,m}| \leq C_{\rho} \| V \|_{L^2} \frac{1}{|n|!} \left( \frac{\rho}{2} \right)^{|n|} \frac{1}{|m|!} \left( \frac{\rho}{2} \right)^{|m|},
\]
\[
|\beta_{n,m}| \leq C_{\rho} \| V \|_{H^s}^3 \frac{1}{|n|!} \left( \frac{\rho}{2} \right)^{|n|} \frac{1}{|m|!} \left( \frac{\rho}{2} \right)^{|m|}.
\]

Finally we require the Hankel function estimate,

\[
|H^{(1)}_n(r)| \leq C_R |n|! \left( \frac{2}{R} \right)^{|n|}, \quad R \leq r,
\]

which again follows from the power series expansion given that the function decays at infinity (see for example [29, Section 5.5] or [4, Lemma 1]). The sums (24) and (25) are then bounded by a constant multiple of

\[
\sum_{n \geq 0} \sum_{m \geq 0} \left( \frac{\rho}{R} \right)^n \left( \frac{\rho}{R} \right)^m
\]

provided that \(|x| > |y| > R > \rho\), which is convergent, and so we are done. \(\square\)

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