Asymptotic Behavior of the $T^3 \times R$ Gowdy Spacetimes

Boro Grubišić and Vincent Moncrief

Department of Physics, Yale University
217 Prospect St.
New Haven, CT 06511

Abstract

We present new evidence in support of the Penrose’s strong cosmic censorship conjecture in the class of Gowdy spacetimes with $T^3$ spatial topology. Solving Einstein’s equations perturbatively to all orders we show that asymptotically close to the boundary of the maximal Cauchy development the dominant term in the expansion gives rise to curvature singularity for almost all initial data. The dominant term, which we call the “geodesic loop solution”, is a solution of the Einstein’s equations with all space derivatives dropped. We also describe the extent to which our perturbative results can be rigorously justified.

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1On leave from the Institute of Theoretical Physics, University of Zagreb, Croatia
1 Introduction

It is well known [1] that in the class of globally hyperbolic spacetimes arbitrary Cauchy data for the vacuum Einstein equations have unique maximal developments (maximal Cauchy developments). We would like to know under which conditions a similar statement could be true in some larger class of spacetimes, possibly all vacuum Einstein spacetimes satisfying certain differentiability conditions (see [2] for a nice review). The Taub-NUT spacetime [3] is prototype of a spacetime where the globally hyperbolic part (Taub) can be extended into at least two non-isometric Taub-NUT spacetimes [4], thus violating the uniqueness of the maximal development. Penrose’s Strong Cosmic Censorship (SCC) conjecture states that for almost all (generic) initial data the maximal development is, indeed, globally hyperbolic, not larger, hence restoring the uniqueness of the maximal development. The spacetimes like Taub-NUT obviously necessitate the exclusion of some spacetimes as special (nongeneric). In the case of Taub-NUT it is the high symmetry (its isometry group is four dimensional) that makes it nongeneric.

To prove the SCC we would have to specify arbitrary Cauchy data, find the maximal globally hyperbolic development, and show that for almost all initial data that development is inextendible by proving, for example, that:

1. The maximal globally hyperbolic development is geodesically complete

or

2. For every incomplete geodesics some curvature scalar blows up when approaching the incomplete end or ends (the singularity).

The Hawking-Penrose singularity theorems [5] tell us that, generically, the first case is not true, but they do not tell us anything about the behavior of the curvature when we approach the incomplete end of a geodesic (the singularity). This is not surprising since these theorems use Einstein’s equations to a very limited extent. To find the behavior of the curvature when approaching the singularity we’ll have to find the detailed asymptotic behavior of the solutions of Einstein’s equations close to the singularity.

The proof of the SCC for the general case is presently out of our reach due to the complexity of Einstein’s equations, but, to gain some insight into the general case, we can try to prove the SCC in the restricted class of spacetimes having an $n$-dimensional isometry group, with least possible $n$.

Following that strategy in this paper we analyse the asymptotic behavior of the solutions of Einstein’s equations in the class of Gowdy spacetimes on
$T^3 \times R$; spacetimes that have 2-dimensional $U(1) \times U(1)$ isometry group $\mathbb{R}$. We develop a new method for studying the asymptotic, singular behavior of large classes of cosmological solutions of Einstein’s equations. The method consists principally in showing that one can solve the $n$-th order perturbed Einstein’s equations for all $n$ using the “variation of constants” method, and then use the freedom in choice of the $n$-th order solution to obtain an asymptotic (when approaching the singularity) sequence of functions which is presumably the asymptotic expansion of some exact solution of Einstein’s equations. This sequence is uniquely determined by the zeroth order solution which we call the “geodesic loop solution”. The zeroth order equations are just the Einstein’s field equations with “space” derivatives dropped and the method is applicable only to the spacetimes whose dynamics is “velocity-dominated” in the sense of Eardley, Liang and Sachs [8, 9].

For the first two orders our method gives the same results as the method suggested by Cosgrove, based on the “multiple scales” technique of applied analysis [10]. The full extent of applicability of our method is not yet known. In the present paper we develop and apply it to the case of Gowdy metrics on $T^3 \times R$ whose asymptotic behavior has been analysed before by Mansfield, using a different perturbative method [11]. In a separate work we shall apply it to $U(1)$ symmetric spacetimes. We believe, but this has not yet been explicitly demonstrated, that our basic method should be applicable to general, non-symmetric solutions of Einstein’s equations.

An advantage to starting with with the Gowdy metrics on $T^3 \times R$ is that much is known about their behavior. In particular one can prove a “global existence theorem” to the effect that, in a suitably rigid coordinate system (in which the time coordinate $\tau$ measures the geometrical area of the two-tori which arise as orbits of the isometry group), all sufficiently smooth Gowdy solutions extend globally to all $\tau \in (0, \infty)$ and that this interval of existence exhausts the maximal globally hyperbolic development [12]. The same reference shows that the $\tau = \text{const.}$ hypersurfaces always approach a “crushing singularity” of uniformly diverging mean curvature as $\tau \to 0^+$ but approach a boundary of infinite three volume as $\tau \to \infty$. The former boundary always arises after a finite lapse of proper time (as measured, say, along the normal trajectories to the chosen foliation from some non-singular reference Cauchy surface) making the spacetime past geodesically incomplete, whereas the latter occurs only after an infinite lapse of proper time. Our main interest here is with the “crushing singular” boundary which occurs as $\tau \to 0^+$, we would like to show that the spacetime is inextendible.
beyond that boundary. It is expected to be curvature singular generically, but may in some special cases instead correspond to Cauchy horizon across which (analytic) extensions are possible \cite{13, 14, 15}.

The mathematical methods used in proving the main results of reference \cite{12} give only rather weak information about the behaviors of the solutions near their crushing boundaries at $\tau = 0$. For the special case of polarized Gowdy metrics (which are governed by a linear hyperbolic equation) one can strengthen the arguments of Ref.\cite{12} to derive genuinely sharp estimates for the behaviors of the metric functions and their derivatives near the crushing boundaries \cite{9}. In this case one can rigorously characterize the asymptotic behavior of every solution and even classify the solutions in a natural way by studying the asymptotic behaviors of their curvature tensors. While we have considerable hope that similar arguments can be developed for the general, non-linear Gowdy metrics, this has not yet been accomplished.

Lacking a direct (i.e., non-perturbative) means for establishing the asymptotic, singular behavior of the generic Gowdy spacetime, we have developed the perturbative approach to be presented here. This perturbative approach has (as we have mentioned) the advantage of applicability far beyond the rather limited class of Gowdy metrics. When restricted to the Gowdy class however, it yields results which may eventually be provable by direct, non-perturbative methods. For the larger classes of spacetimes to which the perturbative method definitely applies (e.g., those having one spacelike Killing field), the possibility of a direct (non-perturbative) proof of the corresponding results seems currently to be rather remote. Thus the Gowdy metrics provide a natural test-case for the perturbative method. The perturbative results provide natural conjectures which one can reasonably hope to prove by direct methods and the direct methods, if successful, can be expected to justify the (less rigorously founded) perturbative approach.

The outline of the rest of the paper is as follows: In section 2 we define the Gowdy spacetimes and briefly review the field equations. In section 3.1 we formulate the perturbative method, in section 3.2 we solve the zeroth and first order equations and find the asymptotic behavior of the corresponding solutions. In section 3.3 we show that, as $\tau \to 0^+$, all higher order solutions decay faster than the zeroth order solution, called the “geodesic loop solution”. In section 3.4 we show that for almost all initial data the “geodesic loop spacetimes”, which presumably capture the asymptotic behaviors of the exact solutions of the Gowdy field equations, are curvature-singular when, $\tau \to 0^+$, and hence inextendible beyond their maximal Cauchy developments. In section 4 we describe known rigorous results, which all support
our perturbative analysis.

2 Gowdy $T^3 \times R$ Spacetimes

Consider a spacetime that can be foliated by a family of compact, connected and orientable spacelike hypersurfaces. If the maximal isometry group of the spacetime is two-dimensional and if it acts invariantly and effectively on the foliation than the isometry group must be $U(1) \times U(1)$. Moreover, the foliation surfaces must be homeomorphic to $T^3$, $S^1 \times S^2$, $S^3$ or to a manifold covered by one of these, and the action of the $U(1) \times U(1)$ is unique up to a diffeomorphism. In such a spacetime the Killing vector fields $K_a$, $a = 1, 2$ associated with the isometry group must commute, and two scalar functions $c_a = \epsilon_{\alpha\beta\gamma\delta} K_\alpha^a K_\beta^b \nabla^\gamma K_\delta^d$ (1) must be constant [16]. The spacetimes which satisfy the above mentioned symmetry requirements, and in which both constants $c_a$ vanish, will be called Gowdy spacetimes [7].

For the simplest case of $T^3$ spatial topology, Gowdy [7] showed that in such a spacetime a coordinate system can be found such that the metric can be expressed in the form

$$ds^2 = e^{2A}[-e^{-2t} dt^2 + d\theta^2] + e^{-t}[(\cosh W + \cos \Phi \sinh W)(dx^1)^2 + 2e^{-t} \sin \Phi \sinh W \, dx^1 \, dx^2 + e^{-t}[(\cosh W - \cos \Phi \sinh W)(dx^2)^2]$$

where $\partial/\partial x^1$ and $\partial/\partial x^2$ are commuting Killing vector fields on $T^3$ in the $S^1 \times S^1$ directions, and the functions $W$, $\Phi$ and $A$ depend only on the two remaining coordinates: “time” $t$ and “angle” $\theta$ parametrizing the third $S^1$ factor in $T^3$. For convenience, instead of the time $\tau$ that measures the geometric area of the two-dimensional group orbits we use the time $t = -\ln \tau$. The singularity at $\tau = 0$, mentioned in the introduction, corresponds to $t \to \infty$.

Einstein’s vacuum field equations give the equations of motion for the metric functions $W$, $\Phi$ and $A$

$$W_{tt} - \frac{1}{2} \sinh W \Phi_t^2 = e^{-2t}[W_{,\theta\theta} - \frac{1}{2} \sinh 2W \Phi_\theta^2]$$

$$\Phi_{tt} + 2 \coth W \, W_t \Phi_t = e^{-2t}[\Phi_{,\theta\theta} + 2 \coth W \, W_{,\theta} \Phi_\theta]$$

$$A_t = \frac{1}{4}[1 - W_{,t}^2 - \sinh^2 W \, \Phi_x^2 \Phi_t - e^{-2t}(W_{,\theta}^2 + \sin^2 W \Phi_\theta^2)]$$

$$A_\theta = \frac{1}{2}[W_t W_{,\theta} + \Phi_t \Phi_{,\theta} \sinh^2 W]$$

4
which have to be solved on the \( t\theta \) cylinder.

To solve this system of equations we proceed in two steps. We first solve equations (3) and (4) for \( W \) and \( \Phi \). These two equations can be put into Hamiltonian form with the Hamiltonian

\[
H[X, \Pi] = \int_0^{2\pi} \mathcal{H} d\theta
\]

\[
\mathcal{H} = \frac{1}{2} [g^{AB} \Pi_A \Pi_B + e^{-2t} g_{AB} X^A X^B],
\]

where \( X^1 \) and \( X^2 \) are \( W \) and \( \Phi \) respectively; prime denotes differentiation with respect to \( \theta \) and \( g_{AB} \) is the metric of two-dimensional hyperbolic space

\[
g = dW^2 + \sinh^2 W d\Phi^2.
\]

From Hamilton’s equations it follows that imposing periodicity in \( \theta \) (with period \( 2\pi \)) on initial conditions is enough to obtain \( W \) and \( \Phi \) that are periodic at all times. A solution \( W(t, \theta) \) and \( \Phi(t, \theta) \) can be conveniently pictured as a closed loop moving in the two-dimensional hyperbolic space with speed

\[
u = \sqrt{g^{AB} \Pi_A \Pi_B}.
\]

After obtaining \( W \) and \( \Phi \) we can find \( A \) by evaluating the line integral

\[
A(t, \theta) = a_0 + \int_{\Gamma} \alpha
\]

over some path \( \Gamma \) on the \( t\theta \) cylinder with the end points at \( (t_0, \theta_0) \) and \( (t, \theta) \), and with components \( \alpha_t \) and \( \alpha_\theta \) of the two dimensional one-form \( \alpha \) equal to the right-hand sides of the equations (3) and (4) respectively. Expressed in Hamiltonian variables

\[
\alpha_t = \frac{1}{2} \left[ \frac{1}{2} - \mathcal{H} \right],
\]

\[
\alpha_\theta = -\frac{1}{2} X^{A'} \Pi_A.
\]

As a consequence of the form of the Hamiltonian the function \( \alpha_\theta \) satisfies the identity

\[
\frac{\partial \alpha_\theta}{\partial X^{A'}} \left( \frac{\delta H}{\delta \Pi_A} \right)' - \frac{\partial \alpha_\theta}{\partial \Pi_A} \frac{\delta H}{\delta X^A} = -\frac{1}{2} \mathcal{H}' = \alpha_{t, \theta}
\]
for arbitrary functions \( X^A \) and \( \Pi_A \). For \( X^A \) and \( \Pi_A \) that satisfy the Hamilton’s equations of motion the left-hand side of (14) becomes the time derivative of \( \alpha_\theta \) and we obtain

\[
\alpha_{\theta,t} = \alpha_{t,\theta} \quad \text{or} \quad d\alpha = 0, \tag{15}
\]

which guarantees that the line integral does not depend on the path. To have the function \( A \) globally defined on the \( t\theta \) cylinder we must impose one more constraint on \( W \) and \( \Phi \), namely that \( A \) must also be periodic in \( \theta \) with period \( 2\pi \), which leads to the constraint

\[
\int_0^{2\pi} \alpha_\theta(t, \theta) d\theta = 0. \tag{16}
\]

This constraint has to be imposed only at one time \( t_0 \) because it is conserved during the time evolution due to equation (15).

Two important subclasses of Gowdy spacetimes are obtained by imposing special conditions that are conserved by the evolution equations. We shall refer to the spacetimes satisfying the condition \( \Phi = 0 \) as the “polarized” Gowdy spacetimes and to the spacetimes satisfying \( W = W(t) \) and \( \Phi = n\theta \), for some \( n \in \mathbb{Z} \), as the “circular loop” spacetimes. For the polarized case the evolution equations for \( W \) and \( \Phi \) reduce to one linear partial differential equation, and for the circular loop case to one non-linear ordinary differential equation.

To facilitate further analysis we will introduce new coordinates \( x \) and \( y \) in the \( W\Phi \) hyperbolic space, such that

\[
\tanh W = 2\sqrt{\frac{x^2 + (y - 1)^2}{x^2 + (y + 1)^2}}, \tag{17}
\]

\[
\tan \Phi = \frac{x^2 + y^2 - 1}{2x}. \tag{18}
\]

These are just the Poincaré half-plane coordinates in which the hyperbolic metric becomes

\[
g = \frac{dx^2 + dy^2}{y^2}, \quad x \in \mathbb{R}, \quad y > 0. \tag{19}
\]

The equations of motion for \( x \) and \( y \) are also given by Hamilton’s equations where now, however, the metric in (8) is replaced by expression (19) and \( \{X^1, X^2\} \) now stand for \( \{x, y\} \). With this modification the Hamiltonian density becomes a rational function of its arguments. The functions appearing
in the Gowdy metric (3), expressed in terms of the \( x \) and \( y \), are

\[
\cosh W + \cos \Phi \sinh W = \frac{(x + y)^2 + 1}{2y}
\]

\[
\cosh W - \cos \Phi \sinh W = \frac{(x - y)^2 + 1}{2y}
\]

\[
\sin \Phi \sinh W = \frac{x^2 + y^2 - 1}{2y}.
\]

Note that the \( y > 0 \) restriction is necessary to obtain a metric (3) that is of Lorentzian signature and that the transformation \( x \to -x \) gives an isometric metric (it amounts to exchange of \( x^1 \) and \( x^2 \)).

3 The Perturbative Expansion

3.1 The Perturbative Method

As already mentioned, in general we do not know how to solve the evolution equations for \( x \) and \( y \). What we can do, however, is try to solve the equations perturbatively to all orders and analyse the asymptotic behavior of the expansion to learn about the behavior of the Gowdy space-times close to the singularity at \( t = \infty \). To facilitate this analysis we will first introduce a seemingly even more complicated problem. We will try to solve the evolution equations for the \( \epsilon \)-dependent Hamiltonian density

\[
\mathcal{H}_\epsilon = \frac{1}{2}[g^{AB} \Pi_A \Pi_B + \epsilon e^{-2t} g_{AB} X^A X^B].
\]

The case of real interest to us is \( \epsilon = 1 \), but it is easy to see that any solution of the \( \epsilon \)-dependent equations for some \( \epsilon_1 > 0 \) is also a solution of the equations for any other \( \epsilon_2 > 0 \), provided we shift the time \( t \) by an appropriate constant. Precisely stated, if \( X_{\epsilon_1}(t) \) is a solution of the equations of motion for \( \epsilon = \epsilon_1 > 0 \) then

\[
X_{\epsilon_2}(t) = X_{\epsilon_1}(t - \frac{1}{2} \ln \epsilon_2/\epsilon_1)
\]

is a solution of the equations for \( \epsilon = \epsilon_2 > 0 \). So, with the introduction of \( \epsilon \) we have not changed any significant property of the problem, but the \( \epsilon \) in the Hamiltonian will help us obtain simple recursive form for the perturbative calculations.
Assume that there exists an \( \epsilon \)-dependent family \( X(\epsilon, t, \theta) \) of solutions to the modified equations which depends smoothly upon the parameter \( \epsilon \), in particular around \( \epsilon = 0 \), and expand it into a power series in \( \epsilon \) to find evolution equations for each term in the expansion:

\[
X(\epsilon, t, \theta) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} X^{(n)}(t, \theta) \quad (23)
\]

\[
X^{(n)}(t, \theta) = \frac{d^n}{d\epsilon^n} X(\epsilon, t, \theta) \bigg|_{\epsilon=0} \quad (24)
\]

\[
X = \begin{pmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{pmatrix} \quad (25)
\]

Whether the family actually exists depends on the convergence properties of the “formal” perturbative expansion \((23)\) which can be changed using the freedom in choosing appropriate \( X^{(n)} \) in every order since the \( n \)-th order perturbation equations don’t have unique solutions; but as we shall see, irrespective of the convergence of the expansion, the recursive calculations of the higher order terms are always well defined in terms of the lower order terms. It could easily happen that the expansion could not be made convergent but could nevertheless be made asymptotic as \( t \to \infty \), in which case we would still be able to study the behavior of Gowdy spacetime close to the singularity. The question still remains whether we can study all, or almost all, Gowdy spacetimes using this perturbative method; i.e. can we for any Gowdy solution find an \( \epsilon \)-dependent family which for \( \epsilon = 1 \) reduces to that particular Gowdy solution and is smooth around \( \epsilon = 0 \)? We will return to this question later, for now let us just say that a function counting argument, together with the properties of our expansion, suggests that the expansion is possible for an open subset of too strong all Gowdy solutions on \( T^3 \times R \).

To clarify the analysis let us for a moment consider an \( s \)-dimensional system of nonlinear partial differential equations on \( R \times M \), with “time” coordinate in \( R \) and “space” coordinates in an arbitrary finite dimensional manifold \( M \). In shorthand notation we write

\[
\dot{X} = h(\epsilon, X, Y) = g(X) + \epsilon f(Y), \quad (26)
\]

where \( X \) is an \( s \)-component function and \( Y \) is a multicomponent function that contains all the components of \( X \) and some number of partial derivatives
of $X$ with respect to the “spatial” coordinates. For $\epsilon = 0$ this system reduces to a system of ordinary differential equations. By differentiating (26) with respect to $\epsilon$ and then putting $\epsilon = 0$ we obtain the equations that have to be satisfied by $X^{(n)}$ if a family $X(\epsilon)$ exists:

$$\dot{X}^{(n)} = \left. \frac{d^n}{d\epsilon^n} h(\epsilon, X, Y) \right|_{\epsilon=0} \equiv h^{(n)} \implies (27)$$

$$n = 0 \quad \dot{X}^{(0)} = g(X^{(0)}) \quad (28)$$

$$n > 0 \quad \dot{X}^{(n)} = g^{(n)} + n f^{(n-1)} \quad (29)$$

Now we will analyse the above perturbative equations without worrying about the existence of an $\epsilon$-dependent family of solutions of the equations (26). The derivatives of $g(X)$ and $f(Y)$ can be calculated using the chain rule, and with all the indices explicitly written, the expression for $g(X)$ is

$$g^{(n)}_j \equiv \left. \frac{d^n}{d\epsilon^n} g_j(X) \right|_{\epsilon=0} = \sum_{k=1}^{n} \sum_{i_1 \cdots i_k} \partial^k g_{j i_1 \cdots i_k} (X^{(0)}) \left[ \sum_{m_1 \cdots m_k} X^{(m_1)}_{i_1} \cdots X^{(m_k)}_{i_k} \right] \quad (30)$$

$$\partial^k g_{j i_1 \cdots i_k} (X^{(0)}) \equiv \left. \frac{\partial^k g_j(X)}{\partial X_{i_1} \cdots \partial X_{i_k}} \right|_{\epsilon=0} \quad . \quad (31)$$

From now on we will suppress the vector indices, like $ji_1 \cdots i_k$, and arguments of $\partial^n g$ and $\partial^n f$ to simplify the notation. For the same reason we will put $g(X^{(0)}) \equiv g^{(0)}$ and $f(Y^{(0)}) \equiv f^{(0)}$. In this shorthand notation the evolution equations for $X^{(n)}$ for any $n \geq 1$ are

$$\dot{X}^{(n)} = \partial g X^{(n)} + \sum_{k=2}^{n} \partial^k g \left[ \sum_{m_1 \cdots m_k} X^{(m_1)} \cdots X^{(m_k)} \right]$$

$$+ \delta_{1n} f^{(0)} + \sum_{k=1}^{n-1} \partial^k f \left[ \sum_{m_1 \cdots m_k} Y^{(m_1)} \cdots Y^{(m_k)} \right] \quad (32)$$

The first three equations including the zeroth are

$$n = 0 \quad \dot{X}^{(0)} = g(X^{(0)}) \quad (33)$$
We see that to all orders, except the zeroth, the perturbative equations are ordinary inhomogeneous linear differential equations of the form

\[ \dot{X}^{(n)} = \partial g X^{(n)} + S^{(n)}, \]  

where the matrix that multiplies \( X^{(n)} \) is the same for all orders and the source \( S^{(n)} \) depends only on the lower order functions \( X^{(k)} \) and \( Y^{(l)} \), i.e. \( k, l < n \). Once we know the general solution for the zeroth order equation (33), which we shall refer to as the seed solution, we can recursively solve all the higher order equations using the “variation of constants” method for ordinary linear differential equations. The general solution of the \( n \)-th order equation (36) is

\[ X^{(n)}(t) = X^{(n)}_{\text{hom}}(t) + \int_{t_0}^{t} G(t, t') S^{(n)}(t') dt', \]  

where \( X^{(n)}_{\text{hom}} \) is a solution of the homogeneous part of Eq. (36) and \( G \) is its Green’s function. Both can be calculated from the seed solution. By differentiating the seed solution with respect to the constants \( c_i \) it depends on, and linearly combining these derivatives, we obtain the general homogeneous solution

\[ X_{\text{hom}} = \sum_{i=1}^{s} \alpha_i \frac{\partial X^{(0)}}{\partial c_i}, \]  

where \( \alpha_i \) are arbitrary constants. The Green’s function \( G(t, t') \) for (37) is the Jacobian matrix for the transformation

\[ \frac{\partial X^{(0)}(t)}{\partial X^{(0)}(t')} \]  

and can be calculated from the seed solution expressed in terms of the initial values. It is easy to check that

\[ \dot{X}_{\text{hom}} = \partial g X_{\text{hom}} \]  

and

\[ \frac{\partial}{\partial t} G(t, t') = \partial g(t) G(t, t') \quad \forall t', \]
and confirm that (37) satisfies (36). The lower limit of integration in (37) could have been chosen to be different at every order as well, but a change in its value corresponds just to adding a solution of the homogeneous equation.

Using any finite number of the solutions $X^{(n)}$ of the perturbative equations (27) we can construct an $\epsilon$-dependent function

$$X_N(\epsilon) = \sum_{n=0}^{N} \frac{\epsilon^n}{n!} X^{(n)},$$

which we shall refer to as the formal expansion of $X$. For any formal expansion we can define a function

$$\delta_N(\epsilon) = \dot{X}_N(\epsilon) - h(\epsilon, X_N),$$

which, as a consequence of perturbative equations, satisfies

$$\frac{d^n}{d\epsilon^n} \delta_N(\epsilon) \big|_{\epsilon=0} \equiv \delta^{(n)}_N = 0, \quad \forall n \leq N.$$ (44)

To simplify the notation we have assumed that $h$ depends only on $X$. Given a formal expansion $X_N$ of $X$, we can generate a formal expansion $F_N$ of any smooth function $F$ by

$$F_N(\epsilon) = \sum_{n=0}^{N} \frac{\epsilon^n}{n!} F^{(n)}, \quad F^{(n)} = \left. \frac{d^n}{d\epsilon^n} F(X_N(\epsilon)) \right|_{\epsilon=0},$$

where $F^{(n)}$ in general depends on all $X^{(k)}$, with $k \leq n$, and is explicitly given by (30) with $g$ replaced by $F$.

An interesting consequence of the above definition and Eq. (44) is that if we have a constant of motion for the evolution governed by (26), i.e. a function $C(X)$ such that identically for any $X$ and $\epsilon$

$$\frac{\partial C(X)}{\partial X} h(\epsilon, X) = 0,$$ (46)

all the terms $C^{(n)}$ in the formal expansion of $C$ will be constant in time also. To see that, consider a formal expansion $C_N$ of $C$. Using (46) we can evaluate

$$\dot{C}(X_N) = \frac{\partial C(X_N)}{\partial X} \dot{X}_N = \frac{\partial C(X_N)}{\partial X} \left[ h(\epsilon, X_N) + \delta_N(\epsilon) \right]$$

$$= \frac{\partial C(X_N)}{\partial X} \delta_N(\epsilon),$$ (47)

11
differentiating $n$ times with respect to $\epsilon$ and using (44) we obtain

$$\dot{C}^{(n)} = \frac{d^n}{d\epsilon^n} \dot{C}(X_N(\epsilon)) \bigg|_{\epsilon=0} = 0. \quad (48)$$

The general idea of using $\delta^{(n)}_N = 0$ could be employed to find other properties of the perturbative equations that are inherited from the “full” equations.

To conclude this general discussion we once more note that one has considerable freedom in choosing the formal expansion apart from the freedom in the choice of the seed solution. One can use this freedom to try to obtain an expansion in which, when time goes to some $t_0$, all terms in the expansion form an asymptotic sequence of functions i.e.

$$\frac{X^{(n+1)}(t)}{X^{(n)}(t)} \to 0 \quad t \to t_0, \quad (49)$$

for all $n$; or perhaps even to obtain a convergent expansion.

Since we are interested in finding the asymptotic behavior of the solutions of the Gowdy equations when $t \to \infty$, we shall chose the lower limit of integration in (37) to be $t_0 = \infty$ and let $X^{(n)}_{hom} = 0$ for all $n$. This choice leads uniquely to the fastest possible decay of the higher order terms, because the homogeneous solutions of the perturbative Gowdy equations do not decay faster than the seed solution when $t \to \infty$. The $X^{(n)}$ in (37) after shifting the integration variable $t' \to t' - t$ becomes

$$X^{(n)}(t) = - \int_0^\infty G(t, t + t') \ S^{(n)}(t + t') dt', \quad (50)$$

and depends only on the seed solution. As we shall see, this choice leads to a uniformly behaved asymptotic sequence for a large range of seed solutions, which suggests that the sequence might be an asymptotic expansion of an exact Gowdy solution.
3.2 The Geodesic Loop Solution and the First Order Correction

Returning to the Gowdy equations we note that Hamilton’s equations for the Hamiltonian (21) are of the form (26) with

\[
X = \begin{pmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{pmatrix}, \quad Y = \begin{pmatrix} x \\ y \\ x' \\ y' \\ x'' \\ y'' \end{pmatrix}
\]

(51)

and

\[
g(X) = \begin{pmatrix} X_3 \\ X_4 \\ A \\ C - B \end{pmatrix}, \quad f(Y) = e^{-2t} \begin{pmatrix} 0 \\ 0 \\ D - E \\ F + G - H \end{pmatrix}
\]

(52)

where

\[
A = \frac{2}{X_2} X_3 X_4, \quad D = Y_5 \\
B = \frac{1}{X_2} X_3^2, \quad E = \frac{2}{Y_2} Y_3 Y_4 \\
C = \frac{1}{X_2} X_4^2, \quad F = Y_6 \\
G = \frac{1}{Y_2} Y_3^2, \quad H = \frac{1}{Y_2} Y_4^2
\]

(53)

The zeroth order equations are, as is obvious from the Hamiltonian (21), just the geodesic equations in the Poincaré half–plane whose general solution, using our notation, is

\[
X_1^{(0)} = a + b \tanh(ct + d) \\
X_2^{(0)} = b \frac{1}{\cosh(ct + d)} \\
X_3^{(0)} = bc \frac{1}{\cosh^2(ct + d)} \\
X_4^{(0)} = -bc \frac{\sinh(ct + d)}{\cosh^2(ct + d)}
\]

(54-57)
where $a, b > 0$, $c > 0$ and $d$ are time independent but otherwise arbitrary smooth functions of $\theta$. For convenience we shall refer to these as the “geodesic loop solutions”, since each curve $X^{(0)}(t, \theta_0)$, for fixed $\theta_0$, is a geodesic. The speed of these geodesic loops, evaluated using Eq. (10), is equal to the function $c(\theta)$ and is, as it should be, constant in time. The geodesic loop solutions have the same number of free initial data as the general Gowdy solutions. Note that when $a, b, c$ and $d$ do not depend on $\theta$, the geodesic loop, in this case degenerated to just one point, is a solution to the full Gowdy equations. The geodesic loop solutions give rise to the “geodesic loop spacetimes” which, we hope to prove, are asymptotically (as $t \to \infty$) approached by Gowdy spacetimes.

Differentiating the geodesic loop solutions with respect to $a, b, c$ and $d$ we obtain the general homogeneous solution for the perturbative Gowdy equations. It is easy to see that it does not decay faster than the seed solution when $t \to \infty$, which justifies our choice of $X^{(0)}$ in (54) as unique for a (potentially) well behaved asymptotic expansion.

Since we are interested in the asymptotic behavior when $t \to \infty$ we will expand the above functions in power series in $1/\xi$, where $\xi = e^{(c t + d)}$, which are convergent for all $\xi > 1$ i.e. for all $t$ and $\theta$ for which $c t + d > 0$. To exhibit the asymptotic behavior we will factor out the lowest power of $\xi$. For example:

\begin{align}
X_1^{(0)} &= a + b + \sum_{k=1}^{\infty} 2b(-1)^k \xi^{-2k} \\
X_2^{(0)} &= \xi^{-1} \sum_{k=0}^{\infty} 2b(-1)^k \xi^{-2k}.
\end{align}

(58)

(59)

In fact all the the zeroth order functions have a similar form. The other two components of $X^{(0)}$ are time derivatives of the first two, and $Y^{(0)}$ includes also $\theta$ derivatives of $X^{(0)}$.

**Definition 1** Let $F: \mathbb{R} \to \mathbb{R}$ be a function that can be expanded into a uniformly convergent series

$$F(t) = \xi^{-N} \sum_{k=0}^{\infty} a_k \xi^{-2k},$$

(60)

where $\xi = e^{(c t + d)}$, $N \in \mathbb{R}$, all $a_k$ are polynomials in $t$ of degree less than or equal to some fixed integer $p \geq 0$ and $a_0 \neq 0$. Then, for easy reference, we
call $F$ a $\xi$–expandable function, $N(F)$ the decay exponent of $F$, $p(F)$ the polynomial degree of $F$ and $a_0(F)$ the dominant coefficient of $F$. If $F(t) = 0$ for all $t$ we put $N(F) = \infty$.

This definition will be important in the inductive proof of the decay of all higher order terms since all the functions we will be dealing with in this paper satisfy the conditions of the above definition, i.e. they are $\xi$–expandable. For any two such functions $F$ and $G$ the product $FG$ is obviously a $\xi$–expandable function, with $p(FG) = p(F) + p(G)$ and with $N(FG) = N(F) + N(G)$. Moreover, if $N(F) - N(G) = 2k$ for some integer $k > 0$ the sum $F + G$ will also be $\xi$–expandable and $N(F + G) = N(G)$. In the special case when $N(F) = N(G)$ we have $N(F + G) \geq N(F)$, because of the possible cancellation of the dominant coefficients. If $k$ dominant coefficients cancel we have $N(F + G) = N(F) + 2k$. So, in general, for any two $\xi$-expandable functions $F$ and $G$ whose sum is also $\xi$-expandable, we have

$$N(F + G) \geq \min\{N(F), N(G)\}$$  \hspace{1cm} (61)

Owing to the uniform convergence of expansion (60) the integral

$$\int_0^\infty F(t)dt = \int_0^\infty \sum_{k=0}^\infty a_k \xi^{-2k-N}dt = \sum_{k=0}^\infty \int_0^\infty a_k \xi^{-2k-N}dt,$$  \hspace{1cm} (62)

for all $F$ for which $N(F) \geq 0$. Therefore, to evaluate (50) we need to integrate only simple integrals of the form

$$\int_0^\infty t^re^{-(2k+N)t}dt = \frac{r!}{[(2k + N)c]^r+1},$$  \hspace{1cm} (63)

where the integer $k \geq 0$ and the integer $r \leq p(F)$, the polynomial degree of $F$. Note that all the zeroth order functions $X^{(0)}$ have $p(X^{(0)}) = 0$, and that each differentiation with respect to $\theta$ brings one $t$ down from the exponent and therefore increases the polynomial degree by 1.

In the generic case, i.e. when no special conditions are imposed on the functions $a$, $b$, $c$ and $d$, the decay exponents for the zeroth order functions are

$$N(X^{(0)}) = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \quad N(Y^{(0)}) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix},$$  \hspace{1cm} (64)
from which follows that

\[
\begin{align*}
N(A) &= 2 & N(D) &= 0 \\
N(B) &= 3 & N(E) &= 0 \\
N(C) &= 1 & N(F) &= 1 \\
N(G) &= -1 & N(H) &= 1.
\end{align*}
\]  

(65)

From (34) and (52) we can calculate the first order source vector \( S^{(1)}(t) \); its decay exponents are

\[
N(S^{(1)}) = N(f^{(0)}) = \begin{pmatrix} \infty \\ \infty \\ \lambda + 2 \\ \lambda + 1 \end{pmatrix},
\]

(66)

where \( \lambda = 2(1-c)/c \), and its polynomial degree \( p(S^{(1)}) = 2 \).

From the form of Gowdy’s \( g \) and \( f \) we see that the first two components of the 4-dimensional source vector \( S^{(k)} \) are zero for all \( k \), which implies that we need just the last two columns of the Green’s function to evaluate integrals (50). Explicit calculation of the Green’s function \( 4 \times 4 \) matrix \( G(t, t + t') \) shows that it depends on \( \theta \) only through the functions \( c \) and \( d \), and that all its components are of the form (60) with the coefficients \( a_k \) polynomials of the first order in \( t' \) multiplied by \( e^{nt'} \), with \( n \) ranging from -1 to 2. The time \( t \) appears only in the exponents, therefore the polynomial degree of \( G \) is zero. Asymptotically for \( t \to \infty \) we obtain

\[
G \sim \begin{pmatrix}
* & * & (1 - \zeta^2)/2c & \xi^{-1}(\zeta^{-1} - \zeta + 2\zeta c t')/c \\
* & * & \xi^{-1}(1 - \zeta^2 + 2c t')/c & -\zeta t' \\
* & * & \xi^2 & -\xi^{-1}4\zeta c t' \\
* & * & \xi^{-1}(3\zeta^2 - 2c t' - 3) & \zeta(c t' + 1)
\end{pmatrix}.
\]

(67)

Here we have put asterisks for uninteresting components and we have written \( \zeta \) for \( e^{ct'} \). The highest power of \( \zeta \) in the components of the Green’s function will be important for the convergence properties of the integral (50), and it is convenient to introduce another simple definition to keep track of the exponential functions of the integration variable \( t' \). Let \( N'(F) \) denote the function defined like \( N(F) \) in Definition 1, with \( t \) changed to \( t' \) and with \( \zeta = e^{ct'} \) written in place of \( \xi = e^{(ct+d)} \). All the functions we use obviously satisfy the conditions stated in Definition 1, with respect to both \( t \) and \( t' \).
Pairing the exponents $N$ and $N'$ in parentheses for brevity, from the exact expression for the Green’s function we obtain

$$(N, N')(G) = \begin{pmatrix} * & * & (0, -2) & (1, -1) \\ * & * & (1, -2) & (0, -1) \\ * & * & (0, -2) & (1, -1) \\ * & * & (1, -2) & (0, -1) \end{pmatrix}. \quad (68)$$

Because the identity $N(F(t + t')) = N'(F(t + t'))$ is valid for any function satisfying the conditions in Definition 1, using (66), we obtain

$$(N, N')(S) = \begin{pmatrix} (\infty, \infty) \\ (\infty, \infty) \\ (\lambda + 2, \lambda + 2) \\ (\lambda + 1, \lambda + 1) \end{pmatrix}. \quad (69)$$

Multiplying the Green’s function and the source we find the integrand whose exponents are

$$(N, N')(GS) = \begin{pmatrix} (\lambda + 2, \lambda) \\ (\lambda + 1, \lambda) \\ (\lambda + 2, \lambda) \\ (\lambda + 1, \lambda) \end{pmatrix}. \quad (70)$$

The integral will converge for all $\theta$ for which

$$\lambda(\theta) > 0 \iff c(\theta) \in (0, 1). \quad (71)$$

Finally, the integration gives the first order functions which are, because of (62), also $\xi$-expandable. We were able to calculate the first order integral explicitly, without restoring to the $\xi$-expansion, and thus obtain the exact expressions for the first order functions, whose asymptotic form agreed with the result obtained by $\xi$-expansion. The polynomial degree of $X^{(1)}$ is 2, the same as $p(S^{(1)})$, since $p(G) = 0$, but the dominant coefficient of all $X^{(1)}$-s are just first degree polynomials in $t$. The differences of the decay exponents of the first and zeroth order functions are

$$N(X^{(1)}) - N(X^{(0)}) = \begin{pmatrix} \lambda + 2 \\ \lambda \\ \lambda \end{pmatrix}, \quad N(Y^{(1)}) - N(Y^{(0)}) = \begin{pmatrix} \lambda + 2 \\ \lambda \\ \lambda + 2 \end{pmatrix}. \quad (72)$$
and, again, \( \lambda > 0 \) is the condition that has to be satisfied if we want that

\[
\frac{X^{(1)}}{X^{(0)}}_i \sim t e^{-2(1-c)t} \to 0 \quad \text{when} \quad t \to \infty,
\]

(73)

for \( i = 1, 2, 3, \) (\( X^{(1)}_1 \) decays even faster).

The condition \( c(\theta) < 1 \) can be relaxed for \( \theta \) in some \( I_0 \subset [0, 2\pi) \), if \( a(\theta) + b(\theta) = \text{const.} \) for \( \theta \in I_0 \). Explicit calculation in this case shows that the decay exponents of the first order source increase by 2 and the first order integral converges for all \( c(\theta) > 0 \) , \( \theta \in I_0 \) giving

\[
N(X^{(1)}) - N(X^{(0)}) = \begin{pmatrix}
\lambda + 4 \\
\lambda + 2 \\
\lambda + 2 \\
\lambda + 2
\end{pmatrix}, \quad N(Y^{(1)}) - N(Y^{(0)}) = \begin{pmatrix}
\lambda + 4 \\
\lambda + 2 \\
\lambda + 2 \\
\lambda + 2
\end{pmatrix},
\]

(74)

Now when we know the decay exponents of the zeroth and first order functions we can go on to calculate the decay exponents of the higher order functions using (50), but all we actually need to know is whether they increase with order.

### 3.3 Higher Order Terms

The lower bounds for the decay exponents of \( X^{(m)} \) can be easily found for any order \( m \) following a rather simple inductive argument.

**Theorem 1** Let \( X^{(m)} \) be a solution of the perturbative Gowdy equations given by (50), and let its geodesic loop parameters \( a, b, c \) and \( d \) be arbitrary smooth functions of \( \theta \) except that \( c(\theta) \in (0, 1) \), then for any \( m \geq 1 \), the decay exponent of \( X^{(m)} \) satisfies the inequality

\[
N(X^{(m)}) \geq N(X^{(1)}) + (m - 1)\lambda.
\]

(75)

**Proof:** The statement of the theorem is trivially true for \( m = 1 \).

Inductive hypothesis: Assume that the statement (73) is true for all \( 1 \leq m \leq n - 1 \). Using the result (72) from the previous section we obtain

\[
N(X^{(m)}) \geq N(X^{(0)}) + m\lambda,
\]

(76)
for all \(1 \leq m \leq n - 1\). Now, all we need to prove the theorem is show that the decay exponents of the \(n\)-th order source \(S^{(n)}\) satisfy the inequality

\[
N(S^{(n)}) \geq N(S^{(1)}) + (n - 1)\lambda
\]

which will, because of the inequality (61), after multiplication with the Green’s function and integration over \(t'\) give the desired result (75) for \(m = n\).

The components of the source vector \(S^{(n)}\) for \(n > 1\) are sums of parts of the form

\[
\begin{align*}
S^{(n)}(g) & = S^{(n)}(f) \\
S^{(n)}(g) & = \partial^k g X^{(m_1)} \cdots X^{(m_k)} \bigg|_{\sum m_i = n} \\
S^{(n)}(f) & = \partial^k f Y^{(m_1)} \cdots Y^{(m_k)} \bigg|_{\sum m_i = n - 1}
\end{align*}
\]

where for \(g\) we can substitute \(A, B\) or \(C\) and for \(f\) \(D, E, F, G\) and \(H\). Again, because of the inequality (61), we need to find only the minimal decay exponent of any term in the sum to obtain a lower bound to the decay exponent of \(S^{(n)}\). To do this we need to take a closer look at the form of the terms above.

All the functions \(A, B, C, D, E, F, G\) and \(H\) are of the form \(P = \prod_j Z_j^{k_j}\), where \(Z\) stands for \(X\) and \(Y\), e.g. \(A = 2X_2^{-1}X_3X_4\). Derivatives of such functions are easy to evaluate and we have

\[
\frac{\partial P}{\partial X_j} = k_j P \Rightarrow N(\frac{\partial P}{\partial X_j}) = N(P) - N(X_j)
\]

if \(k_j \neq 0\), otherwise \(N = \infty\). Using the above property we find

\[
\begin{align*}
N(\partial^k f Y^{(m_1)} \cdots Y^{(m_k)}) & = N(f^{(0)}) \\
& + \sum_{i=1}^k [N(Y^{(m_i)}) - N(Y^{(0)})] \\
N(\partial^k g X^{(m_1)} \cdots X^{(m_k)}) & = N(g^{(0)}) \\
& + \sum_{i=1}^k [N(X^{(m_i)}) - N(X^{(0)})]
\end{align*}
\]
In the sums above only $X$-s and $Y$-s of the order $n-1$ or lower appear and the inequality (76) together with the conditions $\sum m_i = n-1$ and $\sum m_i = n$, for $Y$ and $X$ sums respectively, enable us to conclude that the sums satisfy the following inequalities

$$\sum_{i=1}^{k} [N(Y^{(m_i)}) - N(Y^{(0)})] \geq (n-1)\lambda, \quad (84)$$

$$\sum_{i=1}^{k} [N(X^{(m_i)}) - N(X^{(0)})] \geq n\lambda. \quad (85)$$

Recalling that $S^{(1)} = f^{(0)}$ we obtain

$$N(S^{(n)}(f)) \geq N(S^{(1)}) + (n-1)\lambda, \quad (86)$$

and using $N(g^{(0)}) = N(f^{(0)}) - \lambda$ and (83) we get

$$N(S^{(n)}(g)) \geq N(S^{(1)}) + (n-1)\lambda. \quad (87)$$

Therefore, the source $S^{(n)}$ has the desired behavior, which concludes the proof.

When no cancellations in the dominant coefficients are present in the course of calculations, the statement of the theorem becomes an equality, and for $n \geq 1$ and for $t \to \infty$ we obtain

$$x^{(n)} = X_1^{(n)} \sim p_{2n} e^{-2n(1-c)t} e^{-\Delta t} \quad (88)$$

$$y^{(n)} = X_2^{(n)} \sim q_{2n} e^{-2n(1-c)t} e^{-ct}. \quad (89)$$

The $p_{2n}$ and $q_{2n}$ are polynomials in $t$ of degree $2n$ or possibly lower, which can be easily shown using an argument similar to the one used for the decay exponents. We see that the terms decay faster and faster with order and that we have obtained an asymptotic sequence. When cancellations are present, the decay is even faster and we can group the terms to obtain an asymptotic sequence again.

A statement similar to the Theorem 1. is true for the case $a+b=\text{const.}$ and $c$ arbitrarily large. Then the higher order terms decay even faster, and their decay exponents satisfy the inequality

$$N(X^{(n)}) \geq N(X^{(1)}) + (n-1)(\lambda + 2) \quad (90)$$
3.4 Expansion of the Metric Functions and Curvature

Having obtained the asymptotic sequence of solutions to the perturbative equations for the Gowdy functions \(x\) and \(y\) we can generate the expansion for any smooth function of \(x\) and \(y\) using (45). For example, let \(F\) be a function of \(X\) of the form

\[
F(X) = \frac{P(X)}{Q(X)},
\]

(91)

where \(P\) and \(Q\) are polynomial functions of \(X\). To find the expansion

\[
F_M = F(X(0)) + \sum_{n=1}^{M} \frac{\epsilon^n}{n!} F^{(n)}
\]

(92)

to arbitrary finite order \(M\), we have to calculate the \(F^{(n)}\)-s. They are given in terms of \(\partial^{k}F\) by the equation (30), with \(g\) replaced by \(F\). The first partial derivative of \(F\),

\[
\partial F = \frac{P [\partial P / P - \partial Q / Q]}{Q}
\]

(93)

is again a function of the form \(\tilde{P}/\tilde{Q}\), with \(\tilde{P}\) and \(\tilde{Q}\) polynomials in \(X\). For each monomial \(T\) in, say, \(P\) we have \(N(\partial T) = N(T) - N(X)\), which using (61) gives \(N(\partial P) \geq N(P) - N(X)\). Since \(N(\partial P/P) = N(\partial P) - N(P) \geq -N(X)\), using (93) we obtain

\[
N(\partial F) \geq N(F) - N(X),
\]

(94)

and repeating the same reasoning \(k\) times,

\[
N(\partial^{k}F) \geq N(F) - kN(X).
\]

(95)

Using the definition of \(F^{(N)}\) and Theorem 1 together with (93) we obtain that higher order terms in the expansion of \(F\) decay faster than \(F^{(0)}\), and that their decay exponents satisfy the inequality

\[
N(F^{(n)}) \geq N(F^{(0)}) + n\lambda \quad n \geq 0.
\]

(96)

The functions (20) that appear in the Gowdy metric (3), expressed in terms of \(x = X_1\) and \(y = X_2\), are of the form (93), or in the case of the function \(A\), which is defined by integral (11), the integrand is of the form (93). For the first three functions (20) the expansion is straightforward and
gives higher order corrections that are globally defined functions on the \( t\theta \) cylinder, whose decay exponents satisfy \( (96) \).

The terms in the expansion of the function \( A \) are

\[
A^{(n)}(t, \theta) = A_0^{(n)} + \int_{t_0}^{t} \alpha_t^{(n)}(t', \theta_0) dt' + \int_{\theta_0}^{\theta} \alpha_\theta^{(n)}(t, \theta') d\theta',
\]

where \( \alpha_t^{(n)} \) and \( \alpha_\theta^{(n)} \) are terms in the expansion of the functions \( \alpha_t \) and \( \alpha_\theta \) defined in \((11)\) and \((12)\), and \( A_0^{(n)} \) a suitable constant.

To determine the asymptotic behavior of the integrals note that, since both \( \alpha_t^{(n)} \) and \( \alpha_\theta^{(n)} \) are \( \xi \)-expandable functions we can integrate term by term. The time integral, when evaluated at the upper limit \( t \), gives a \( \xi \)-expandable function with the decay exponent equal to the decay exponent of \( \alpha_t^{(n)} \). When evaluated at the lower limit \( t_0 \) it gives a constant \( k(t_0, \theta_0) \) that could spoil the decay properties of \( A^{(n)} \). To prevent that we choose \( A_0^{(n)} = -k(t_0, \theta_0) \).

Using \( (96) \) the theta integral can be shown to be

\[
\left| \int_{\theta_0}^{\theta} \alpha_\theta^{(n)}(t, \theta') d\theta' \right| \leq K^{(n)} \exp[-2 \inf(1 - c(\theta))nt],
\]

for some constant \( K^{(n)} > 0 \).

We have chosen a special path of integration for \( A^{(n)} \), but it is easy to show that \( A^{(n)} \) is path independent. Using the functional relation \( (14) \) between \( \alpha_\theta \) and \( \alpha_t \) and reasoning similar to that that led to \( (48) \) we obtain

\[
\alpha_{\theta,t}^{(n)} = \alpha_{t,\theta}^{(n)}
\]

which guarantees the path independence. To have each \( A^{(n)} \) defined globally on the \( t\theta \) cylinder they must be periodic in \( \theta \), with period \( 2\pi \), which is equivalent to

\[
C^{(n)} = \int_{0}^{2\pi} \alpha_\theta^{(n)}(t, \theta) d\theta = 0.
\]

Equation \( (99) \) guarantees that \( \dot{C}^{(n)} = 0 \), and we have to impose the constraint at one time only.

The decay exponent of \( \alpha_\theta^{(0)} \) is zero, which means that \( C^{(0)} = 0 \) will impose a constraint on the geodesic loop functions \( a, b, c \) and \( d \). Evaluating \( C^{(0)} \) and then going to the limit \( t \to \infty \) we obtain

\[
\int_{0}^{2\pi} c(\theta) \left[ \frac{a'(\theta)}{b(\theta)} + d'(\theta) \right] d\theta = 0.
\]
For \( n \geq 1 \) the decay exponents of \( a^{(n)}_{\theta} \) are, according to (96), greater than 1, which together with the constancy of \( C^{(n)} \) implies that \( C^{(n)} = 0 \), without the need to impose any new constraints on \( a, b, c \) and \( d \).

So, we conclude that the expansion of the function \( A \) is globally defined provided the condition (101) is satisfied by the seed solution. Then, asymptotically for \( t \to \infty \), the zeroth order term in the expansion of \( A \) is

\[
A^{(0)}(t, \theta) \sim \frac{1}{4} \left[ 1 - c^2(\theta) \right] t - \frac{1}{2} \int_{\theta_0}^{\theta} c(\vartheta) a'(\vartheta) + b(\vartheta) d'(\vartheta) d\vartheta,
\]

(102)
and the higher order terms decay exponentially faster.

Therefore, we have shown that for any geodesic loop spacetime with metric \( g^{(0)} \) whose geodesic loop functions \( a, b > 0, c > 0 \) and \( d \) satisfy the constraint (101) and

\[
c(\theta) < 1 \quad \text{or} \quad (103)
\]
\[
c(\theta) \text{arbitrary and } a(\theta) + b(\theta) = \text{const}., \quad (104)
\]
there exists a sequence \( g^{(n)} \) of solutions of the perturbative Gowdy equations, to arbitrarily high order \( N \), such that, asymptotically as \( t \to \infty \),

\[
\sum_{n=0}^{N} g^{(n)} \sim g^{(0)}
\]
(105)

The functions \( a, b, c, \) and \( d \) satisfying (101) and (103) form an open subset in the set of all Gowdy initial data. This gives us a hope that the exact Gowdy solutions from an open subset \( \mathcal{G} \) of all Gowdy solutions asymptotically approach geodesic loop solutions. The geodesic loop approximation may not be valid for all Gowdy spacetimes however. Consider the speed of the exact Gowdy loops (11): there may exist Gowdy spacetimes with asymptotic speed \( \lim_{t \to \infty} u = c > 1 \), and for such spacetimes our perturbative expansion is not asymptotically dominated by the zeroth order (the geodesic loop) term. In the case of \( c > 1 \), unless the condition (104) is satisfied, the higher order terms in the expansion exponentially increase instead of decrease compared to the zeroth order term, as \( t \to \infty \).

All the rigorous results on Gowdy spacetimes obtained so far show that the geodesic loop approximation is indeed asymptotically valid. In the special case of the circular loop spacetimes, which because of circular symmetry can’t satisfy the condition (104), it was rigorously proven [4] that the geodesic loop approximation is asymptotically valid and that the asymptotic speed \( c \) is always less than unity with all values between zero and
unity attained by some spacetime. Special condition \( (104) \) includes polarized Gowdy metrics which are rigorously known to asymptotically approach geodesic loop spacetimes with arbitrary large speed \( c \). We will discuss these and some other exact results in the last section.

For now, assume that our perturbative results are rigorously true, i.e. that the geodesic loop approximation is valid for all spacetimes with asymptotic speed \( c < 1 \). The asymptotic behavior of curvature, as \( t \to \infty \), in Gowdy spacetimes can be calculated using their corresponding geodesic loop asymptotes. The asymptotic behavior of the curvature in the geodesic loop spacetimes was analysed by Mansfield \([11]\). He showed that for \( c \neq 1 \) the geodesic loop spacetime is curvature singular, as \( t \to \infty \), and

\[
R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} \sim \frac{1}{4} e^{-4f(\theta)}(c(\theta)^2 - 1)^2(c(\theta)^2 + 3)e^{(c(\theta)^2+3)t}, \tag{106}
\]

where

\[
f(\theta) = \int_{\theta_0}^{\theta} c(\vartheta) \frac{d(\vartheta) + b(\vartheta)d(\vartheta)}{b(\vartheta)} d\vartheta. \tag{107}
\]

This means that the corresponding Gowdy spacetime with the asymptotic speed \( c < 1 \) is curvature singular, as \( t \to \infty \), and therefore inextendible beyond its maximal globally hyperbolic development, in agreement with the SCC. Mansfield also showed that for the special case of \( c = 1 \) and \( a + b = \text{const.} \) the spacetime is not curvature singular and is extendible.

4 Concluding Remarks

Chruściel and Moncrief have derived some “light cone” and “higher-order-energy” estimates for Gowdy equations with a view towards rigorously proving the geodesic loop asymptotic behavior suggested by the perturbation calculations presented here \([18]\). Several complications make such a proof significantly more difficult here than for polarized case treated in Refs.\([4]\) and \([17]\). Aside from the fact that now the basic equations are non-linear there is also the complication that not every geodesic loop can be realized as the asymptote of some exact solution. The reason for this is that our perturbative calculations require \( c(\theta) < 1 \) to be applicable, unless some special non-generic condition such as \( a + b = \text{const.} \) is satisfied.

Nevertheless, it is quite conceivable that exact field equations always force the asymptotic speed \( c \) below unity (unless the aforementioned special condition is satisfied) and thus force the solutions to achieve the geodesic
loop asymptotic behavior. An example of this phenomenon was found by Chruściel and Moncrief who studied exact circular loop solutions, the motion of which is governed by a non-linear, second order *ordinary* differential equation (for the “radius” of the loop). The circularity is preserved by the exact field equations and such solutions are prevented, by their circular symmetry, from achieving the special condition which permits an asymptotic speed greater than or equal to unity (unless the geodesic loop behavior is violated). It was found, however, that every such circular loop solution does indeed asymptotically approach a geodesic loop (of speed less than unity) and that every value of asymptotic speed (strictly less than unity) is in fact achieved by some exact circular solution [2].

The polarized solutions considered by Isenberg and Moncrief [9] satisfy the special condition mentioned above \((a + b = \text{const.})\) and can achieve arbitrarily large asymptotic speeds while still realizing the asymptotic geodesic loop behavior. Not only is this asymptotic behavior universally satisfied by the polarized solutions but also one can rigorously justify computing the asymptotic behavior of the Riemann tensor or (for sufficiently smooth solutions) its covariant derivatives by means of computations made purely within the geodesic loop approximation.

A further example of exact results which realize the geodesic loop asymptotic behavior (this time with \(c = 1\)) has been found by Mansfield [11] who transformed a class of exact analytic solutions, derived using the Ernst formalism, back to the Gowdy representation. This family of solutions (which is infinite dimensional and generically non-polarized) turns out to be none other than the set of “generalized Taub-NUT” spacetimes (restricted to the Gowdy symmetry class) defined on \(T^3 \times R\) which develop compact Cauchy horizons instead of curvature singularity at the boundaries of their maximal Cauchy development [14, 15, 19]. Much larger families of such generalized Taub-NUT spacetimes (admitting generically only one spacelike Killing field) are defined in Refs. [14] and [15] and have been used in Ref. [20] as backgrounds for perturbative expansion. An infinite dimensional family of analytic, curvature singular solutions having only one Killing field (and defined on \(S^2 \times S^1 \times R\)) was constructed in [19] by applying suitably chosen Geroch transformation to the generalized Taub-NUT solutions defined on \(S^3 \times R\). One hopes that these rigorously derived singular solutions will help one to understand the asymptotic behaviors of the perturbative solutions discussed in [20]. At our present level of understanding it seems not unreasonable to hope that the higher order perturbation methods discussed here can be successfully applied to the study of completely general
(non-symmetric) cosmological solutions of Einstein’s equations near their singular boundaries.

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