OPTIMAL SELECTION FOR GOOD POLYNOMIALS OF DEGREE UP TO FIVE

AUSTIN DUKES, ANDREA FERRAGUTI, AND GIAOCe MICHELI

Abstract. Good polynomials are the fundamental objects in the Tamo-Barg constructions of Locally Recoverable Codes (LRC). In this paper we classify all good polynomials up to degree 5, providing explicit bounds on the maximal number $\ell$ of sets of size $r+1$ where a polynomial of degree $r+1$ is constant, up to $r = 4$. This directly provides an explicit estimate (up to an error term of $O(\sqrt{q})$, with explicit constant) for the maximal length and dimension of a Tamo-Barg LRC. Moreover, we explain how to construct good polynomials achieving these bounds. Finally, we provide computational examples to show how close our estimates are to the actual values of $\ell$, and we explain how to obtain the best possible good polynomials in degree 5.

1. Introduction

A code $C$ with length $n$ and dimension $k$ is called a locally recoverable code (LRC) with locality $r$, or a $(n, k, r)$-LRC, if, for any $v = (v_1, \ldots, v_n) \in C$ and any $1 \leq i \leq n$, the coordinate $v_i$ is a function of at most $r$ other coordinates $v_{i_1}, v_{i_2}, \ldots, v_{i_r}$ of $v$. In other words, the value of any symbol in a particular codeword can be recovered by accessing at most $r$ other symbols of the codeword.

Given the linear $(n, k, r)$-LRC $C$, Gopalan et al. [12] and Papailiopoulos and Dimakis [18] proved that the minimum distance $d(C)$ of $C$ satisfies the upper bound $d(C) \leq n - k - \lceil k/r \rceil + 2$. As in the literature, we will say $C$ is an optimal LRC if the minimum distance $d(C)$ of $C$ achieves this bound, that is, if $d(C) = n - k - \lceil k/r \rceil + 2$.

A powerful approach to constructing LRCs was given by Tamo and Barg in [20], and it can be accomplished by constructing polynomials of degree $r + 1$ which are constant on pairwise disjoint subsets of $\mathbb{F}_q$ of size $r + 1$. Such polynomials are called good polynomials.

More formally, for a nonnegative integer $\ell$ the polynomial $f \in \mathbb{F}_q[X]$ is said to be $(r, \ell)$-good if

- the degree of $f$ is $r + 1$, and
- there are pairwise disjoint sets $A_1, \ldots, A_\ell \subseteq \mathbb{F}_q$, each of cardinality $r + 1$, such that $f(A_i) = \{t_i\}$ for some $t_i \in \mathbb{F}_q$, i.e., $f$ is constant on $A_i$.

Given a good polynomial, one can construct an optimal linear LRC as follows (we use the notation of [13], which is the most convenient for our purposes). Fix $r \geq 1$,
and let $f(X) \in \mathbb{F}_q$ be a good polynomial. Write $n = (r+1)\ell$ and $k = rt$, where $t \leq \ell$. For $a = (a_{ij} | i = 0, \ldots, r-1; j = 0, \ldots, t-1) \in (\mathbb{F}_q)^k$, define encoding polynomials

$$f_a(X) = \sum_{i=1}^{r-1} \sum_{j=0}^{t-1} a_{ij} f(X)^j X^i.$$ 

Let $A = \bigcup_{i=1}^{\ell} A_i$ and define

$$C = \left\{ (f_a(x), x \in A) | a \in (\mathbb{F}_q)^k \right\}.$$ 

Then $C$ is an optimal linear $(n, k, r)$-LRC code over $\mathbb{F}_q$.

In the rest of the paper we classify all $(r, \ell)$-good polynomials up to $r = 4$ as follows: for any fixed prime power $q$ (even or odd) and a fixed $r$ up to 4, we provide an explicit estimate (of the form $cq + O(\sqrt{q})$, where $c \in [0, 1)$, and the implied constant in the error term is explicitly computable) on the maximal $\ell$ such that a polynomial of degree $r+1$ is $(r, \ell)$-good. Moreover, we provide examples of polynomials achieving these values for $\ell$, showing that the estimate is the best possible.

The machinery we use involves Galois theory, the classification of transitive subgroups of the symmetric group $S_n$ up to $n = 5$, and the theory of function fields, using the results and techniques of [15, 16], then further developed in [1, 2, 4, 5, 8, 9].

### 2. Monodromy groups and totally split places

Let $q$ be a prime power and $f \in \mathbb{F}_q[X]$ with $f \notin \mathbb{F}_q[X^p]$, where $p$ is the characteristic of $\mathbb{F}_q$. Throughout the paper, we will call such polynomials separable, for short. Let $t$ be transcendental over $\mathbb{F}_q$.

**Definition 2.1.** The *arithmetic monodromy group* of $f$, denoted by $A(f)$, is the Galois group of $f(X) - t$ seen as a polynomial over $\mathbb{F}_q(t)$. The *geometric monodromy group* of $f$, denoted by $G(f)$, is the Galois group of $f(X) - t$ seen as a polynomial over $\overline{\mathbb{F}}_q(t)$.

It is easy to see for any non-constant $f$, the polynomial $f - t$ is geometrically irreducible. Hence both the arithmetic and the geometric monodromy groups are isomorphic to transitive subgroups of the symmetric group of degree $\deg f$, after choosing a labeling of the roots of $f - t$ in $\overline{\mathbb{F}}_q(t)$. Different labelings yield conjugate embeddings.

Recall that $G(f)$ is a normal subgroup of $A(f)$, and the quotient is isomorphic to the Galois group of the extension $(M \cap \overline{\mathbb{F}}_q)/\mathbb{F}_q$, where $M$ is the splitting field of $f - t$.

For any fixed $n$, it is possible to construct a polynomial $f$ of degree $n$ having $A(f) = G(f) = S_n$. Hence one can define a function

$$G_n(\cdot) : \{\text{prime powers}\} \to \mathbb{N}$$
that assigns to every prime power $q$ the least positive integer such that there exists a separable $f \in \mathbb{F}_q[X]$ of degree $n$ with $|G(f)| = |A(f)| = G_n(q)$. Notice that $G_n(q) \geq n$ for every $q$, because a transitive group of degree $n$ must have at least order $n$.

Thanks to the techniques introduced in [16], given a separable polynomial $f \in \mathbb{F}_q[X]$ such that $A(f) = G(f)$, one can obtain an explicit estimate on cardinality of the set

$$T^1_{\text{split}}(f) := \{t_0 \in \mathbb{F}_q: f(X) - t_0 \text{ splits into } \deg f \text{ distinct linear factors}\}.$$

This is done via the following result:

**Proposition 2.2.** [16, Proposition 3.1] Let $f \in \mathbb{F}_q[X]$ be a separable polynomial of degree $n$ with $G(f) = A(f)$ and let $g_f$ be the genus of the splitting field of $f(X) - t$. Then

$$\frac{q + 1 - 2g_f\sqrt{q}}{|G(f)|} - \frac{n}{2} \leq |T^1_{\text{split}}(f)| \leq \frac{q + 1 + 2g_f\sqrt{q}}{|G(f)|}.$$

The genus $g_f$ can be bounded solely in terms of $\deg f$, using for example Castelnuovo’s inequality. As noticed in [16, Proposition 3.3], if $\text{char } \mathbb{F}_q \nmid |G(f)|$ we have $g_f \leq \frac{(n-2)|G(f)|+2}{2}$.

It is clear from the above proposition that for a fixed $n$, minimizing $|G(f)|$ maximizes the expected number of totally split places, which in turn maximises the dimension of the Tamo-Barg code.

In this paper, we compute the function $G_n$ for every $n \in \{2, \ldots, 5\}$. The simpler cases $n = 2, 3, 4$ are completely treated in Section 3. When $n = 5$ the problem becomes more difficult as, up to conjugation, there are 5 transitive subgroups of the symmetric group $S_5$:

- The cyclic group $C_5$, generated by a 5-cycle;
- The dihedral group $D_5$, generated by a 5-cycle and a product of two disjoint transpositions;
- The affine general linear group $AGL_1(\mathbb{F}_5)$, isomorphic to $C_5 \rtimes C_4$, generated by a 5-cycle and a 4-cycle.
- The alternating group $A_5$;
- The symmetric group $S_5$.

Nevertheless, we are able to prove the following theorem for good polynomials of degree 5.

**Theorem 2.3.** Let $q$ be a prime power. Then:

$$G_5(q) = \begin{cases} 5 & \text{if } 5 \mid q(q - 1) \\ 10 & \text{if } 5 \mid q + 1 \\ 120 & \text{otherwise} \end{cases}.$$
As we mentioned above, the estimate of Proposition 2.2 can be made explicit, leading to a formula for the maximal dimension of a Tamo-Barg code of locality 4. When \( G(f) = A(f) \cong D_5, C_5 \) the genus of the splitting field of \( f \) is necessarily zero (by Hurwitz formula, for example), and the error term is therefore \( O(1) \). In the next theorem we give an explicit estimate for the remaining cases, restricting for simplicity to the case \( 2, 3, 5 \nmid q \).

**Theorem 2.4.** Let \( q \) be a prime power with \( 2, 3, 5 \nmid q \). Let \( f \in \mathbb{F}_q[X] \) of degree 5 with \( G(f) = A(f) \cong S_5 \). Then:

\[
\frac{q + 1 - 72\sqrt{q}}{120} - \frac{5}{2} \leq \left| T_{\text{split}}(f) \right| \leq \frac{q + 1 + 72\sqrt{q}}{120}.
\]

**Proof.** Let \( x \) be a root of \( f(X) - t \). Let \( F := \overline{\mathbb{F}_q}(x) \) and \( M \) be the splitting field of \( f - t \) over \( \overline{\mathbb{F}_q}(t) \). By Proposition 2.2 all we have to do is bound the genus \( g_M \) of \( M \). We will do it via Riemann-Hurwitz applied to the degree 24 extension \( M/F \). Notice that \( F \) has genus \( g_F \) equal to zero. We have that:

\[
2g_M - 2 = 24(2g_F - 2) + \sum_P \sum_{Q|P} (e_{Q|P} - 1),
\]

because thanks to our assumptions on \( q \) the ramification is tame. Here the external sum is over all places \( P \) of \( F \), while the internal one is over all places \( Q \) of \( M \) dividing \( P \), and \( e_{Q|P} \) is the ramification index. Since \( f \) has degree 5, its derivative has degree 4 and therefore there are at most 5 places of \( F \) that can ramify in \( M \) (notice that a place of \( \overline{\mathbb{F}_q}(t) \) ramifies in \( M \) if and only if it ramifies in \( F \)). Now since \( M/F \) is a Galois extension the ramification index \( e_{Q|P} \) depends only on \( P \), and it is at most 24. On the other hand, there are at least two places of \( F \) that ramify in \( M \), since there are at least two places of \( \overline{\mathbb{F}_q}(t) \) that ramify in \( F \): the infinite place and a finite one, since the derivative of \( f \) has positive degree. All in all, we have that

\[
\sum_P \sum_{Q|P} (e_{Q|P} - 1) = \sum_P e_{Q|P} - \sum_P \sum_{Q|P} 1 \leq 5 \cdot 24 - 2 = 118,
\]

and substituting in the above equation yields \( g_M \leq 36 \). \( \square \)

### 3. Degrees up to 4

In this section we compute \( G_2, G_3 \) and \( G_4 \). We start with two general lemmas.

**Lemma 3.1.** Let \( p \) be a prime and \( q = p^m \) for some \( m \geq 1 \). Let \( f = X^q - X \in \mathbb{F}_q[X] \). Then \( A(f) = G(f) \cong (C_p)^m \).

**Proof.** Let \( x \) be a root of \( f \) and let \( F := \mathbb{F}_q(x) \). Then \( F \) is a Galois extension of \( \mathbb{F}_q(t) \), because \( x + \alpha \) is a root of \( f(X) - t \) for every \( \alpha \in \mathbb{F}_q \), and therefore \( |A(f)| = q \). Since \( f(X) - t \) is absolutely irreducible, both \( A(f) \) and \( G(f) \) act transitively on the set of roots of \( f(X) - t \) and therefore it must be that \( G(f) = A(f) \). If \( \sigma \in A(f) \) and \( r = x + \alpha \) is a root of \( f - t \) for some \( \alpha \in \mathbb{F}_q \), then \( \sigma(r) = r + \beta \) for some
\(\beta \in \mathbb{F}_q\), and therefore \(\sigma^p(r) = r\). It follows that \(\sigma^p\) is the identity, and therefore \(A(f) = (C_p)^m\).

**Lemma 3.2.** Let \(\ell\) be a prime and \(q\) a prime power with \(\ell \nmid q\). Let \(f \in \mathbb{F}_q(X)\) be a degree \(\ell\) polynomial. Then \(G(f) = A(f) \cong C_\ell\) if and only if \(\ell \mid q - 1\) and \(f = (X - a)^\ell + b\) for some \(a, b \in \mathbb{F}_q\).

**Proof.** Sufficiency is obvious.

Conversely, suppose that \(G(f) = A(f) \cong C_\ell\). Let \(x\) be a root of \((X - t)\) and \(F := \mathbb{F}_q(x)\). Then the ramification in \(F\) is always tame, and hence the Riemann-Hurwitz formula implies that there must be a finite place of \(\mathbb{F}_q(t)\) that ramifies in \(F\). Let this place correspond to \(b \in \mathbb{F}_q\). Then \((X - b)\) must factor as \((X - a)^\ell\) for some \(a \in \mathbb{F}_q\). Comparing the coefficients of the linear terms, it follows immediately that \(a \in \mathbb{F}_q\), and hence \(b \in \mathbb{F}_q\). But then \(f = (X - a)^\ell + b\), and in order to have \(G(f) = A(f)\) the field of constants \(F \cap \mathbb{F}_q\) must be \(\mathbb{F}_q\), implying immediately that \(\ell \mid q - 1\) because certainly \(F\) contains a primitive \(\ell\)-th root of unity.

**Theorem 3.3.** The following holds:

1. \(G_2(q) = 2\) for every \(q\).
2. \(G_3(q) = \begin{cases} 3 & \text{if } 3 \mid q(q-1) \\ 6 & \text{otherwise} \end{cases}\).

**Proof.** When \(n = 2\) and \(q\) is odd, every quadratic \(f \in \mathbb{F}_q[X]\) has \(G(f) = A(f) \cong C_2\). When \(q\) is even, by Lemma 3.1 if \(f = X^2 + X \in \mathbb{F}_q(X)\) we have \(G(f) = A(f) = 2\).

When \(n = 3\) and \(3 \mid q\), by Lemma 3.1 for \(f = X^3 - X\) we have \(G(f) = A(f) \cong C_3\). When \(3 \nmid q(q-1)\) by Lemma 3.2 we cannot have \(G(f) = A(f) = C_3\). The only other transitive group inside \(S_3\) is \(S_3\) itself, and hence \(G_3(q) = 6\).

**Theorem 3.4.**

\(G_4(q) = \begin{cases} 24 & \text{if } q = 2 \\ 4 & \text{if } 4 \mid q - 1 \text{ or } q = 2^m \text{ for some } m > 1 \\ 8 & \text{otherwise} \end{cases}\)

**Proof.** Recall that the transitive groups of degree 4 are: \(C_4, C_2 \times C_2, D_4, A_4\) and \(S_4\). Here the non-trivial elements of \(C_2 \times C_2\) are products of two disjoint transpositions, and therefore this copy of \(C_2 \times C_2\) is contained in \(A_4\).

If \(q\) is even and greater than 2, then there exist distinct elements \(\alpha_1, \alpha_2, \alpha_3 \in \mathbb{F}_q^*\) such that \(\alpha_1 + \alpha_2 + \alpha_3 = 0\). Now let \(f = X(X + \alpha_1)(X + \alpha_2)(X + \alpha_3)\); this is of the form \(X^4 + aX^2 + bX\) for some \(a, b\) with \(b \neq 0\). Therefore if \(x\) is a root of \(f(X) - t\), then all other roots are of the form \(x + \alpha_i\) for some \(i \in \{1, 2, 3\}\). It follows that \(\mathbb{F}_q(x)/\mathbb{F}_q(t)\) is a Galois extension of degree 4, and therefore \(G(f) = A(f)\) and \(G_2(q) = 4\). In fact it is easy to see that these monodromy groups are isomorphic to
$C_2 \times C_2$: if $\sigma$ is any element of $G(f)$ and $r$ is any root of $f - t$, then $\sigma(r) = r + \alpha$ for some $\alpha \in \mathbb{F}_q$, and hence $\sigma^2(r) = r$, showing that non-trivial elements have order 2.

If $q = 2$, a quick search shows that the only two separable polynomials $f$ with $A(f) \neq S_4$ are $X^4 + X$ and $X^4 + X^2 + X$. However, the first one has $A(f) \cong D_8$ and $G(f) \cong C_2 \times C_2$, while the second one has $A(f) \cong A_4$ and $G(f) \cong C_2 \times C_2$. Hence $G_4(2) = 24$.

If $4 \mid q - 1$, then for $f = X^4$ we have $G(f) = A(f) \cong C_4$, and therefore $G_4(q) = 4$.

Finally, suppose $q$ is odd and $4 \nmid q - 1$. If $A(f) \leq A_4$, then the discriminant of $f - t$ is a square in $\mathbb{F}_q(t)$. However, such discriminant is always a polynomial of degree 3 in $t$, so $A(f) \leq A_4$ can never happen (and therefore, in particular, we cannot have $A(f) \cong C_2 \times C_2$). If it was $A(f) = G(f) \cong C_4$, then since the ramification in the splitting field $F$ of $f - t$ is tame, by Riemann-Hurwitz there is a finite place of $\mathbb{F}_q(t)$ that ramifies in $F$. Let $R$ be a place of $F$ lying over it. Then the decomposition group $D(R|P)$ is either $C_4$ or $C_2$. In the former case, for some $t_0 \in \mathbb{F}_q$ the polynomial $f - t_0$ factors as $(X - a)^4$, and this leads to a contradiction as in the proof of Lemma 3.2. In the latter case, up to translations we have $f - t_0 = X^2(X - a)^2$ for some $a, t_0 \in \mathbb{F}_q$ with $a \neq 0$, because the element of order 2 in $C_4$ is a product of two disjoint transpositions. This implies that $f$ equals the composition $g \circ h$, where $g = X^2$ and $h = X(X - a)$, and it is a well-known fact (see for example [10]) that for the Galois group of $g \circ h - t$ to be smaller than $D_4$ one would need $t$ and $a^2 / 4 - t$ to be linearly dependent in the $\mathbb{F}_2$-vector space $\mathbb{F}_q(t)^*/(\mathbb{F}_q(t)^*)^2$, which clearly does not hold since $a \neq 0$. Hence $G_4(q) \geq 8$. On the other hand, again for the same well-known reasons one has that if $f = X^4 + bX^2$ for some $b \neq 0$ then $G(f) = A(f) = D_4$. Hence $G_4(q) = 8$. \hfill\qed

4. Degree 5: AGL$_1(\mathbb{F}_5)$ never occurs

In this section we will prove that if $5 \nmid q$ then there exists no degree 5 polynomial $f$ with $G(f) \cong AGL_1(\mathbb{F}_5)$. From now on, we let $M$ be the splitting field of $f - t$ over $\mathbb{F}_q(t)$. If $P$ is a place of $\mathbb{F}_q(t)$ and $R$ is a place of $M$ lying above it, we denote by $D(R|P)$ the corresponding decomposition group. For every $t_0 \in \mathbb{F}_q$, we denote by $P_{t_0}$ the corresponding place of $\mathbb{F}_q(t)$.

We start with a preliminary lemma.

**Lemma 4.1.** Let $q$ be a prime power with $5 \nmid q$. Let $f \in \mathbb{F}_q[X]$ be a degree 5 polynomial and assume that $G(f) \cong AGL_1(\mathbb{F}_5)$. Then there are $a, b, t_0 \in \mathbb{F}_q$ with $a \neq b$ such that $f - t_0 = (X - a)^4(X - b)$.

**Proof.** We start by showing that $M = \mathbb{F}_q(x, x') = \mathbb{F}_q(x)\mathbb{F}_q(x')$ for any two roots $x \neq x'$ of $f(X) - t$ in $M$. Observe that $\mathbb{F}_q(t)(x) = \mathbb{F}_q(x)$ since $t = f(x) \in \mathbb{F}_q(x)$ (and similarly for $x'$) and write $F = \mathbb{F}_q(x)$ and $F' = \mathbb{F}_q(x')$. Clearly we have $[F : \mathbb{F}_q(t)] = [F' : \mathbb{F}_q(t)] = \deg f = 5$. Because $G(f) \cong AGL_1(\mathbb{F}_5)$ is 2-transitive, the stabilizer $G_x \subseteq G$ of $x$ acts transitively on the four other roots of $f - t$. In particular,
since $G_x$ is the Galois group of $(f(X) - t)/(X - x)$ over $F(t)$, and since the orbit of $x'$ under the action of $G_x$ is a set of size 4, we have $[FF' : F] = [G : G_x] = 4$. By definition $M \supseteq FF'$, so since $|G| = 20$ we have $M = FF' = F_q(x, x')$.

Now, let $P_\infty$ be the place at infinity of $F_q(t)$ and let $R_\infty$ be a place of $M$ lying over $P_\infty$. Let $Q_\infty = R_\infty \cap F$ and consider the ramification index $e(Q_\infty|P_\infty)$ of $Q_\infty$ over $P_\infty$. Recalling that $f(x) = t$ in $F$, we have

$$v_{Q_\infty}(f(x)) = v_{Q_\infty}(t) = v_{P_\infty}(t) \cdot e(Q_\infty|P_\infty) = -e(Q_\infty|P_\infty),$$

since $P_\infty$ is a pole of order 1 of $t$. On the other hand, we also have $v_{Q_\infty}(f(x)) = \deg f \cdot v_{Q_\infty}(x) = -5$ by the strict triangle inequality. This yields $e(Q_\infty|P_\infty) = 5$, and an identical argument applied to $Q_\infty' = R_\infty \cap F'$ yields $e(Q_\infty'|P_\infty) = 5$. Since we have seen that $M$ is the compositum of the fields $F$ and $F'$ (both of which are tame extensions of $F_q(t)$ as we are working in characteristic $\neq 5$), it now follows from Abhyankar’s Lemma [19, Theorem 3.9.1] that $e(R_\infty|P_\infty) = \lcm\{e(Q_\infty|P_\infty), e(Q_\infty'|P_\infty)\} = 5$. Thus the decomposition group $D(R_\infty|P_\infty)$ is a group of order 5, and hence it is isomorphic to $C_5$.

Next, we claim that there must be some $t_0 \in F_q$ such that for any place $R$ of $M$ lying over $P_{t_0}$, the decomposition group $D(R|P_{t_0})$ is isomorphic to $C_4$. To prove it, start by noticing that there must be some $t_0 \in F_q$ such that the decomposition group of any place of $M$ lying above it contains a cycle of order 4. In fact, consider the subset of $G(f)$ of elements of even order that belong to some decomposition group: this contains no transpositions because the transitive copy of $AGL_1(F_5)$ inside $S_5$ contains no transpositions, and on the other hand if all such elements had order 2 then they would all be products of two transpositions. However the decomposition groups generate $G(f)$ and in this latter case it would follow that $G(f) \leq A_5$, which is false once again. So let $t_0 \in F_q$ be such that for some place $R$ of $M$ lying above $P_{t_0}$, the decomposition group contains a cycle of order 4. If we had $C_4 \subseteq D(R|P_{t_0}) \neq C_4$, it would follow $D(R|P_{t_0}) \cong G(f)$ by the maximality of $C_4$ in $G(f)$. But then $R|P_{t_0}$ would be totally ramified, and hence $f(X) - t_0 = (X - a_0)^5$ for some $a_0 \in F_q$. Since the field of constants of $M/F_q(t)$ is trivial, this factorization implies $5 \mid q - 1$. But then $G \cong C_5$, an immediate contradiction. Thus $D(R|P_{t_0}) \cong C_4$.

To conclude the proof, notice that specializing at the place $P_{t_0}$ and applying the Dedekind-Kummer Theorem [19, Theorem 3.3.7] allows us to write $f(X) - t_0 = (X - a)^4(X - b)$ for $a, b \in F_q$ with $a \neq b$.

We are now ready to prove that $AGL_1(F_5)$ cannot occur as a geometric monodromy group. The proof will require separate arguments for even and odd characteristics.

**Theorem 4.2.** Let $q$ be a prime power with $5 \mid q$ and $f \in F_q[X]$ a polynomial of degree 5. Then $G(f) \not\cong AGL_1(F_5)$.

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1This is because if $G$ is the subgroup of $G(f)$ generated by all the decomposition subgroups, then $M^G$ is an unramified extension of $F_q(t)$, and there are no non-trivial such extensions.
Proof. Assume by contradiction that \( G(f) \cong \text{AGL}_1(\mathbb{F}_5) \). By Lemma \ref{lemma:1}, there are \( t_0, a, b \in \mathbb{F}_q \) such that \( f - t_0 = (X - a)^4(X - b) \). Since \( \text{Gal}(f(X) - t \mid \mathbb{F}_q(t)) \cong \text{Gal}(f(X - c) - (t - d) \mid \mathbb{F}_q(t - d)) \) for every \( c, d \in \mathbb{F}_q \), we can assume without loss of generality that \( t_0 = a = 0 \) and \( b \neq 0 \).

**Odd characteristic.** Computing \( f'(X) = X^3(5X - 4b) \) shows that \( f'(4b/5) = 0 \), so for \( t_1 = f(4b/5) \in \mathbb{F}_q \), we see that \( f(X) - t_1 \) is divisible by \((X - 4b/5)^2\). Furthermore, since \( X = 4b/5 \) is not a root of \( f''(X) = 4X^2(5X - 3b) \), it follows that \( X = 4b/5 \) is precisely a double root of \( f(X) - t_1 \). Notice that \( f(4b/5) \neq 0 \) and hence \( t_1 \neq 0 \), so \( X = 0 \) is not a root of \( f(X) - t_1 \). This implies that the only repeated root of \( f(X) - t_1 \) is \( X = 4b/5 \) since the only roots of \( f'(X) \) are \( X = 0 \) and \( X = 4b/5 \). In other words, we can write \( f(X) - t_1 = (X - 4b/5)^2(X - x_1)(X - x_2)(X - x_3) \) for pairwise distinct elements \( 4b/5, x_1, x_2, x_3 \in \mathbb{F}_q \). Finally, let \( R \) be any place of \( M \) lying over \( P_1 \). Then the previous factorization shows that there is a transposition in \( D(R|P_1) \leq G(f) \). But \( G(f) \) contains no transpositions, and we have a contradiction.

**Even characteristic.** From now on, we let \( x \) be a root of \( f(X) - t \) in \( M \) and \( f := \mathbb{F}_q(x) \). Fix a place \( R \) of \( M \) lying above the place \( P_0 \) of \( \mathbb{F}_q(t) \). The natural action of \( C_4 \cong D(R|P_0) \subseteq G \) on the set of roots of \( f(X) - t \) yields orbits of sizes 4 and 1, so there must be two places \( Q_0 \) and \( Q_1 \) of \( F \) lying over \( P_0 \) with ramification indices \( e(Q_0|P_0) = 4 \) and \( e(Q_1|P_0) = 1 \) by \cite[Lemma 2.1]{2}. Let \( R_0 \) be a place of \( M \) lying over \( Q_0 \). We have just seen that \( 4 = [D(R_0|P_0)] = e(R_0|P_0) = e(R_0|Q_0) \cdot e(Q_0|P_0) \), so it follows that \( e(R_0|Q_0) = 1 \).

Before proceeding, we introduce the following notation: given a function field \( K \) and a place \( P \) of \( K \), we will write \( \hat{K}_P \) to denote the completion of \( K \) at \( P \) with respect to the \( P \)-adic metric. In particular, \( \hat{F}_{Q_0} = \mathbb{F}_q((x)) \) and \( \mathbb{F}_q(t)|P_0 = \mathbb{F}_q((t)) \).

Using a well-known number theoretical fact (see for example \cite[Proposition II.9.6]{1}), we have \( \text{Gal}(\hat{M}_{R_0} \mid \mathbb{F}_q((t))) \cong D(R_0|P_0) \cong C_4 \). Observe that \( \hat{M}_{R_0} \geq \hat{F}_{Q_0} \supseteq \mathbb{F}_q((t)) \), so because \( [\hat{F}_{Q_0} : \mathbb{F}_q((t))] = e(Q_0|P) = 4 \) and \( [\hat{M}_{R_0} : \mathbb{F}_q((t))] = e(R_0|P_0) = 4 \) we have \( \hat{M}_{R_0} = \hat{F}_{Q_0} \). In particular, \( \hat{F}_{Q_0}/\mathbb{F}_q((t)) \) is a Galois extension. Denoting the local Galois group \( \text{Gal}(\hat{F}_{Q_0} \mid \mathbb{F}_q((t))) \) by \( \hat{G} \), we have \( \hat{G} \cong C_4 \).

The above shows that every root of \( f(X) - t \) in \( \hat{M}_{R_0} \) can be expressed as an element of \( \hat{F}_{Q_0} = \mathbb{F}_q((x)) \), that is, as a Laurent series in \( x \). We proceed by showing that if \( z \neq x \) is any other root of \( f(X) - t \), then we can write \( z = x + ux^i \) for some \( i \geq 2 \) and \( u \in \mathbb{F}_q[x] \). First, recall that \( f(X) - t = X^4(X - b) - t \) so that \( b \) is a simple root of \( f(X) \). Then by Hensel’s lifting lemma there is some \( \tilde{b} \in \mathbb{F}_q((t)) \) such that we can write \( f(X) - t = \tilde{f}(X)(X - \tilde{b}) \) over \( \mathbb{F}_q((t)) \), where \( \tilde{f}(X) \in \mathbb{F}_q((t))[X] \) and \( \deg \tilde{f} = 4 \). Further, the polynomial \( \tilde{f} \) must be irreducible over \( \mathbb{F}_q((t)) \) since otherwise we could write \( \tilde{f}(X) = \tilde{g}(X)(X - r) \) for an irreducible \( \tilde{g} \in \mathbb{F}_q((t))[X] \) and some \( r \in \mathbb{F}_q((t)) \), or we could write \( \tilde{f}(X) = \tilde{h}_1(X)\tilde{h}_2(X) \) for two irreducible quadratic polynomials \( \tilde{h}_1, \tilde{h}_2 \in \mathbb{F}_q((t))[X] \) having distinct roots in...
The second assertion follows immediately from the fact that the transitive copy of $\bar{\mathbb{G}}_1$ and $\bar{\mathbb{G}}_2$ be the splitting fields of $\bar{\mathcal{h}}_1(X)$ and $\bar{\mathcal{h}}_2(X)$, respectively, in $\mathbb{F}_q$. Then $\mathbb{F}_q = H_1H_2$ so that $C_4 \cong \bar{\mathbb{G}} = \text{Gal}(H_1H_2 \mid \mathbb{F}_q((t))) \subseteq \text{Gal}(H_1 | \mathbb{F}_q((t))) \times \text{Gal}(H_2 | \mathbb{F}_q((t))) \cong C_2 \times C_2$, another contradiction. Thus we conclude that $\bar{\mathbb{G}}$ is the Galois group of $\bar{f}(X)$ over $\bar{\mathbb{F}}_q((t))$; in particular, we have that $\bar{\mathbb{G}}$ acts transitively on the roots of $\bar{f}(X)$ in $\mathbb{F}_q$ and hence there is some automorphism $\tau \in \bar{\mathbb{G}}$ of $\mathbb{F}$ satisfying $\tau(x) = z$.

Observe that $\langle x \rangle$ is the unique maximal ideal of the ring $\mathbb{F}_q[[x]]$, so since $\tau$ is an automorphism of $\mathbb{F}_q((x))$ (and hence $\tau$ preserves maximal ideals) we must have $\langle \tau(x) \rangle = \tau(\langle x \rangle) = \langle x \rangle$. Then $z \equiv 0 \mod \langle x \rangle$ if and only if $\langle \tau(x) \rangle \equiv 0 \mod \langle \tau(x) \rangle$, and the latter clearly holds. This allows us to write $z = cx + ux^i$ for some $c \in \mathbb{F}_q$ and for some $i \geq 1$, and we can assume further that $i \geq 2$ since otherwise we could replace $cx$ by $c'x$ for an appropriate $c' \in \mathbb{F}_q$ so that this holds. Now computing $\tau^4(x)$ by using $\tau(x) = cx + ux^i$ and comparing coefficients with $\tau^4(x) = x$ yields $c^4 = 1$ and hence $c = 1$ as we are working over a field with characteristic 2. Thus we can write $z = x + ux^i$ for some $i \geq 2$ and some $u \in \mathbb{F}_q[[x]]^*$.

Observe the following:

$$f(z) - t = z^4(z - b) - t$$

$$= (x + ux^i)^4 (x + ux^i - b) - t$$

$$= (x^4 + u^4x^{4i}) (x - b + ux^i) - t$$

$$= x^4(x - b) - t + ux^{4+i} + u^4x^{4i}(x - b) + u^5x^{5i}$$

$$= ux^{4+i} + u^4x^{4i}(x - b) + u^5x^{5i},$$

where the last equality holds since $x$ is a root of $f(X) - t$. Let $A = ux^{4+i} + u^4x^{4i}(x - b) + u^5x^{5i} = f(z) - t$. Since $z$ was chosen to be another root of $f(X) - t$, we must have $A = 0$. But $v_{\mathbb{F}_q}(A) = v_{\mathbb{F}_q}(x) - \min\{4+i, 4i, 5i\} = 4+i \neq \infty = v_{\mathbb{F}_q}(0)$, a contradiction since $u \neq 0$. Thus our initial assumption was false, so $G(f) \not\cong \text{AGL}_1(\mathbb{F}_5)$. $\square$

5. Degree 5: if $D_5$ or $A_5$ occurs, then $5 \mid q^2 - 1$

Assume $q$ is a prime power and $f \in \mathbb{F}_q[X]$ is a separable polynomial of degree 5. Let $t$ be transcendental over $\mathbb{F}_q$ and let $M$ be the splitting field of $f(X) - t$. Let $A(f)$ and $G(f)$ be the arithmetic and geometric monodromy groups of $f$, respectively.

**Theorem 5.1.** Suppose that $5 \mid q$ and $A(f) \leq A_5$. Then $5 \mid q^2 - 1$. In particular, if $A(f) \leq D_5$, then $q^2 - 1$.

**Proof.** The second assertion follows immediately from the fact that the transitive copy of $D_5$ inside $S_5$ lies inside $A_5$.

If $q$ is odd, just use the fact that $A(f) \leq A_5$ if and only if the discriminant of $f(X) - t$ is a square in $\mathbb{F}_q(t)$. When $f(X)$ is monic of degree 5, the discriminant has the form $5^2t^4 + \sum_{i=0}^3 a_it^i$. Hence 5 needs to be a square in $\mathbb{F}_q$; this implies that either
$q$ is an even power of a prime $p$ or, by quadratic reciprocity, that $q \equiv \pm 1 \mod 5$. In any case, $5 \mid q^2 - 1$.

If $q = 2^n$ for some $n \geq 1$, one needs to use the characteristic two analogue of the discriminant, called Berlekamp discriminant (see \[5\]). If $k$ is a field of characteristic 2 and $g \in k[x]$ has degree $n$, the Berlekamp discriminant of $g$ is an element $\Delta \in k$, that can be effectively computed using the coefficients of $g$, that has the property that $\text{Gal}(g) \leq A_n$ if and only if the polynomial $X^2 + X + \Delta$ has a root in $k$.

Now let $f = X^5 + aX^4 + bX^3 + cX^2 + dX \in \mathbb{F}_{2^n}[X]$. We will show that if $A(f) \leq A_5$ then $2 \mid n$, and consequently $5 \mid q^2 - 1$ once again. One can compute the Berlekamp discriminant $\Delta_f$ of $f - t$, seen as a polynomial over $\mathbb{F}_{2^n}(t)$, and see that this is given by an expression of the form $r(t)/s(t)^2$, where $r, s \in \mathbb{F}_{2^n}(t)$ are two monic polynomials with $\deg r = 4$ and $\deg s = 2$. Suppose that $A(f) \leq A_5$, and hence $X^2 + X + \Delta_f$ has a root in $\mathbb{F}_{2^n}(t)$. Then there are coprime polynomials $u(t), v(t) \in \mathbb{F}_{2^n}[t]$, with $v(t)$ monic, such that $\Delta_f = (u(t)^2 + u(t)v(t))/v(t)^2$. Therefore if $r, s$ share a common factor, this can only have degree 2 or 4, and if it has degree 2 then it must be of the form $(t + \alpha)^2$ for some $\alpha$. Clearly if they share a factor of degree 4 then $\Delta_f = r(t)/s(t)^2 = 1$, so that $X^2 + X + 1$ has a root in $\mathbb{F}_{2^n}(t)$ and consequently $2 \mid n$. Otherwise looking at degrees one sees that it must be $\deg u = \deg v$ and $\deg(u^2 + uv) = 2 \deg v$. If the leading coefficient of $u$ is $\delta \in \mathbb{F}_{2^n}$, these two conditions, together with the fact that $r(t)$ is monic of degree 4, imply that $\delta^2 + \delta + 1 = 0$ and consequently that $2 \mid n$.

**Proof of Theorem 2.3**: First, suppose that $5 \mid q(q - 1)$. Then by Lemmas 3.1 and 4.2, $G_5(q) = 5$.

Now suppose that $5 \nmid q(q - 1)$. Then by Lemma 3.2 we have $G_5(q) > 5$. If $5 \mid q + 1$, it is known (see for example \[17\] Section 3) that degree 5 Dickson polynomials of the first kind, e.g. $f = X^5 - 5X^3 + 5X$, satisfy $G(f) = A(f) = D_5$. Hence $G_5(q) = 10$.

Finally, suppose that $5 \nmid q(q^2 - 1)$. Then by Lemma 3.2 and Theorems 4.2 and 5.1 we cannot have $G_5(q) = 5, 10, 20$ or 60. Hence $G_5(q) = 120$.

**Remark 5.2.** Notice that the $q/10$ asymptotic when $5|q + 1$ was in fact obtained in \[14\] using Dickson Polynomials and an independent approach.

6. **Computational examples**

Let us show with a couple of explicit examples how the number of totally split places compares to the theoretical estimate given by Proposition 2.2. We pick examples with $G_5(q) = 120$; for each of these values of $q$ we pick polynomials $f$ with $G(f) = A(f) \cong S_5$ for $5 \nmid q(q^2 - 1)$. As proved in Theorem 2.3 it is not possible to do better for these $q$’s.

In order to construct polynomials with geometric monodromy (and therefore also arithmetic monodromy) $S_5$, one can use the following well-known group theoretical fact (see \[11\]): if $G \leq S_5$ is a transitive subgroup containing a transposition and a
cycle of prime length $\ell > 2$, then $G = S_5$. In order to force the geometric monodromy group to contain two such elements it is enough, by ramification arguments, to pick $g(X) \in \mathbb{F}_q[X]$ irreducible of degree 3, and set $f = X^2 g(X)$.

| $q$  | $T_{\text{split}}^1(f)$ | $\lfloor \frac{q}{120} \rfloor$ |
|------|--------------------------|---------------------------------|
| $2^{13}$ | 78 | 68 |
| $2^{15}$ | 278 | 273 |
| $2^{17}$ | 1088 | 1092 |
| $2^{19}$ | 4332 | 4369 |

(a) $f = X^2(X^3 + X + 1)$

| $q$  | $T_{\text{split}}^1(f)$ | $\lfloor \frac{q}{120} \rfloor$ |
|------|--------------------------|---------------------------------|
| $3^7$ | 21 | 18 |
| $3^{11}$ | 1474 | 1476 |
| $3^{13}$ | 13338 | 13286 |

(b) $f = X^2(X^3 - X + 1)$

| $q$  | $T_{\text{split}}^1(f)$ | $\lfloor \frac{q}{120} \rfloor$ |
|------|--------------------------|---------------------------------|
| 19583 | 156 | 163 |
| 19597 | 163 | 163 |
| 19687 | 155 | 164 |
| 19753 | 194 | 164 |
| 19793 | 179 | 164 |
| 19913 | 189 | 165 |
| 19927 | 160 | 166 |
| 19963 | 162 | 166 |
| 19993 | 156 | 166 |
| 19997 | 161 | 166 |

(c) $f = X^2(X^3 + X + 3)$

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University of South Florida, 4202 E Fowler Ave, 33620 Tampa, US.
Email address: austindukes@usf.edu

Scuola Normale Superiore, Piazza dei Cavalieri 7, 56126 Pisa
Email address: andrea.ferraguti@sns.it

University of South Florida, 4202 E Fowler Ave, 33620 Tampa, US.
Email address: gmicheli@usf.edu