Linear Filters and Hereditary Torsion Theories in Functor Categories

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Abstract

We introduce the notion of a Gabriel filter for a preadditive category \( C \) and we show that there is a bijective correspondence between Gabriel filters of \( C \) and hereditary torsion theories in the category of additive functors \((C, \text{Ab})\). Thus, a generalization of the theorem given by Gabriel [6] and Maranda [7]; this establishes a bijective correspondence between Gabriel filters for a ring and hereditary torsion theories in the corresponding category of modules.

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1 Introduction and Basic Results

In this paper $\mathcal{C}$ will be an arbitrary skeletally small preadditive category and $\text{Mod}(\mathcal{C})$ will denote the category of contravariant functors from $\mathcal{C}$ to the category of Abelian groups $\text{Ab}$. Following the approach by Mitchell [8], we can think of $\mathcal{C}$ as a ring “with several objects” and $\text{Mod}(\mathcal{C})$ as a category of $\mathcal{C}$ -modules. The aim of the paper is to show that the notions of a linear filter can be extended to preadditive categories obtaining generalizations of the theorem explained by Gabriel that establishes a bijective correspondence hereditary torsion theories and linear filters. The notion of torsion theory (or torsion pair) was introduced by S. E. Dickson [5] in the 1960s in the setting of Abelian categories, by generalizing the classical notion for Abelian groups. Since then it has received a lot of attention in various contexts, such as non commutative localization or representation theory of Artin algebras. Of particular importance are the hereditary torsion pairs, because of their role in localization theory, notably their characterization in terms of Gabriel filters of ideals and Gabriel topologies.

1.1 Torsion Theories

In this part we recall some basic concepts about Torsion theories. The content of this subsection is taken from [10].

Let $\mathcal{A}$ be an Abelian category.

Definition 1.1. A Torsion theory for $\mathcal{A}$ is a pair $(\mathcal{T}, \mathcal{F})$ of classes of objects in $\mathcal{A}$ such that

(i) $\text{Hom}(T, F) = 0$ for all $T \in \mathcal{T}, F \in \mathcal{F}$;

(ii) If $\text{Hom}(C, F) = 0$ for all $F \in \mathcal{F}$, then $C \in \mathcal{T}$;

(iii) If $\text{Hom}(T, C) = 0$ for all $T \in \mathcal{T}$, then $C \in \mathcal{F}$.

$\mathcal{T}$ is called a Torsion class and its objects are Torsion objects, while $\mathcal{F}$ is a Torsion free class consisting of Torsion free objects.

Any given class $\mathcal{B}$ of objects generates a Torsion theory in the following way:

$\mathcal{F} = \{ F | \text{Hom}(C, F) = 0 \text{ for all } C \in \mathcal{B} \}$,

$\mathcal{T} = \{ T | \text{Hom}(T, F) = 0 \text{ for all } F \in \mathcal{F} \}$.

Clearly this pair $(\mathcal{T}, \mathcal{F})$ is a Torsion theory, and $\mathcal{T}$ is the smallest Torsion class containing $\mathcal{B}$. Dually, the class $\mathcal{B}$ cogenerates a Torsion theory $(\mathcal{T}, \mathcal{F})$ such that $\mathcal{F}$ is the smallest Torsion free theory.
A Torsion pair \((\mathcal{T}, \mathcal{F})\) is \textit{hereditary} if \(\mathcal{T}\) is closed under subobjects. In case \(\mathcal{A}\) is a module category, the hereditary torsion pairs are in bijective correspondence with the Gabriel filters of the ring; see [[10]. Theorem 5.1, Ch. VI].

A class \(\mathcal{B}\) of objects is called a pretorsion class if it is closed under quotients objects and coproducts and it is a pretorsion-free class if it is closed under subobjects and products. A pretorsion class is hereditary if it is closed under subobjects.

In the first section, we fix the notation and recall some notions from functor categories that will be used throughout the paper. In the second section, we define the concept of linear filter for a preadditive category \(\mathcal{C}\), and we generalize the theorem given in [[10]. Theorem 5.1, Ch. VI] and we establish a one to one correspondence between Gabriel filters in a preadditive category and hereditary torsion theories in the category of \(\mathcal{C}\)-modules. In the third section, we see that the linear filters induce a topology for \(\mathcal{C}(A, B)\) for every pair of objects \(A, B\) in \(\mathcal{C}\) that makes that the composition

\[
\mathcal{C}(A, B) \times \mathcal{C}(B, C) \to \mathcal{C}(A, C)
\]

be continuous, by obtaining similar results as those given in [[10], Ch. VI]. Finally, in the fourth section, we explore some examples of linear filters for path categories, and we show how some natural examples of hereditary torsion theories appear in the category of \(\mathcal{C}\)-modules.

### 1.2 The Category of \(\mathcal{C}\)-modules

Let \(\mathcal{C}\) be a preadditive skeletally small category. By \(\text{Mod}(\mathcal{C})\) we denote the category of additive contravariant functors from \(\mathcal{C}\) to the category of Abelian groups. \(\text{Mod}(\mathcal{C})\) is then an Abelian category with arbitrary sums and products; in fact it has arbitrary limits and colimits, and the filtered limits are exact (Ab5 in Grothendieck terminology). Furthermore, \(\text{Mod}(\mathcal{C})\) has enough projective and injective objects. For any object \(C \in \mathcal{C}\), the representable functor \(\mathcal{C}(\cdot, C)\) is projective, the arbitrary sums of representable functors are projective, and any object \(M \in \text{Mod}(\mathcal{C})\) is covered by an epimorphism

\[
\prod_{i \in I} \mathcal{C}(\cdot, C_i) \to M \to 0;
\]

see [3].

We will indistinctly say that \(M\) is an object of \(\text{Mod}(\mathcal{C})\) or that \(M\) is a \(\mathcal{C}\) -module. A representable functor \(\mathcal{C}(\cdot, C)\) will sometimes be denoted by \((\cdot, C)\).
Let $M$ be a $C$-module and $C$ and object in $\mathcal{C}$. Thus, by Yoneda’s Lemma there exists a one to one correspondence

$$\theta = \theta_{C,M} : \mathcal{C}(-,C), M \to M$$

where $(\mathcal{C}(-,C), M)$ is the class of natural transformations from the functor $\mathcal{C}(-,C)$ to the functor $M$, given by $\theta(\eta) = \eta_C(1_C)$.

Since $\mathcal{C}$ is a skeletally small category, then the class $(\mathcal{C}(-,C), M)$ is a set. We can then index it as $(\mathcal{C}(-,C), M) = \{\eta^x : \mathcal{C}(-,C) \to M\}_{x \in M(C)}$; furthermore, it induces a morphism of $\mathcal{C}$-modules

$$\eta^C = (\eta^x)_{x \in M(C)} : \coprod_{x \in M(C)} \mathcal{C}(-,C) \to M.$$ 

Clearly

$$\eta^C_C = (\eta^x_C)_{x \in M(C)} : \coprod_{x \in M(C)} \mathcal{C}(C,C) \to M(C)$$

is an epimorphism, and we have an epimorphism of $\mathcal{C}$-modules

$$\eta = (\eta^C_C)_{C \in \mathcal{C}} : \coprod_{C \in \mathcal{C}} \coprod_{x \in M(C)} \mathcal{C}(-,C) \to M.$$ 

As above, let $x \in M(C)$ and $\eta^x : \mathcal{C}(-,C) \xrightarrow{p} \text{Im}(\eta^x) \xrightarrow{j^x} M$ be the canonical factorization of $\eta^x$. The family $\{j^x : \text{Im}(\eta^x) \to M\}$ induces a morphism of $\mathcal{C}$-modules $j^C = (j^x)_{x \in M(C)} : \coprod \text{Im}(\eta^x) \to M$. Then $j^C_C = (\tilde{j}^x_C)_{x \in M(C)} : \coprod \text{Im}(\eta^x_C) \to M(C)$ is an isomorphism, and we have an isomorphism of $\mathcal{C}$-modules

$$j = (j^C_C)_{C \in \mathcal{C}} : \coprod_{C \in \mathcal{C}} \coprod_{x \in M(C)} \text{Im}(\eta^x) \xrightarrow{\cong} M.$$ 

Finally, we can define the above isomorphism as

$$\coprod_{C \in \mathcal{C}} \coprod_{x \in M(C)} \mathcal{C}(-,C)/\text{Ker}(\eta^x) \xrightarrow{\cong} M. \quad (1)$$

1.3 The Functor $D : \text{Mod}(\mathcal{C}) \to \text{Mod}(\mathcal{C}^{\text{op}})$

We consider the functor

$$D : \text{Mod}(\mathcal{C}) \to \text{Mod}(\mathcal{C}^{\text{op}})$$

defined by

$$(DM)(C) = \text{Hom}_{\text{Ab}}(M(C), \mathbb{Q}/\mathbb{Z})$$

for all pairs $M \in \text{Mod}(\mathcal{C})$ and $C \in \mathcal{C}$.

We leave to the reader the following.
Proposition 1.2. Let \( M \in \text{Mod}(\mathcal{C}) \) and \( N \in \text{Mod}(\mathcal{C}^{op}) \). Then we have an isomorphism
\[
\Phi : \text{Hom}_\mathcal{C}(M, DN) \to \text{Hom}_{\mathcal{C}^{op}}(N, DM)
\]
defined by \( [\Phi(\eta)](Y)(X) = \eta_Y(X) \) for each pair \( X \in M(\mathcal{C}) \) and \( Y \in N(\mathcal{C}) \).

Now we can prove the following.

Proposition 1.3. Let \( C \in \mathcal{C} \). Then the \( \mathcal{C} \)-module \( D(\mathcal{C}(C, -)) \) is injective.

Proof. It follows from Yoneda’s Lemma and Proposition 1.2.

1.4 Ideals

A right ideal in an additive category \( \mathcal{C} \) is a subfunctor of \( \mathcal{C}(-, C) \) for some \( C \in \mathcal{C} \), and a left ideal is a subfunctor of \( \mathcal{C}(C, -) \); see [8] and [4]. A two-sided ideal is a subfunctor of the two variable functor \( \mathcal{C}(-, ?) \). Given a two-sided ideal \( I \) in \( \mathcal{C} \) we can form the quotient category \( \mathcal{C}/I \) which has the same objects as \( \mathcal{C} \) and
\[
(\mathcal{C}/I)(A, B) = \mathcal{C}(A, B)/I(A, B).
\]

Given a family \( \mathcal{B} \) of objects in \( \mathcal{C} \) we can define a two-sided ideal \( I_\mathcal{B} \) of \( \mathcal{C} \) generated by \( \mathcal{B} \) setting \( f \in I_\mathcal{B}(A, B) \) if and only if \( f \) is a finite sum of maps of the form \( A \xrightarrow{h} C \xrightarrow{g} B \) with \( C \) in \( \mathcal{B} \).

If \( I \) is a two-sided ideal in \( \mathcal{C} \) and \( M \) is a \( \mathcal{C} \)-module the \( \mathcal{C} \)-submodule \( IM \) is defined as
\[
IM(A) = \sum_{f \in I(A, C), C \in \mathcal{C}} \text{Im}(f).
\]

Therefore, \( IM \) is a subfunctor of \( M \). Moreover, any epimorphism \( M \to M' \) induces an epimorphism \( IM \to IM' \).

Note that for every morphism of \( \mathcal{C} \)-modules \( \eta : \mathcal{C}(-, C) \to M \), the functor kernel \( \text{Ker}(\eta) \subset \mathcal{C}(-, C) \) is a right ideal.

2 Linear Filters and Hereditary Torsion Theories

In this section we introduce the notions of linear filter and Gabriel filter for an arbitrary preadditive category \( \mathcal{C} \). In addition, we show that there is a one-to-one correspondence between linear filters in \( \mathcal{C} \) and classes of pretorsion theories in \( \text{Mod}(\mathcal{C}) \), as well as a correspondence between Gabriel filters and classes of hereditary torsion theories.
2.1 Linear Filters

Let be $N$ a $\mathcal{C}$-module, $K$ be a $\mathcal{C}$-submodule of $N$ and $C$ an object in $\mathcal{C}$. For each $C' \in \mathcal{C}$ and each $x \in N(C)$, we consider the set

$$(K(C') : x) = \{ f \in \mathcal{C}(C', C)|N(f)(x) \in K(C') \}.$$

Thus, we can define a $\mathcal{C}$-module $(K(\cdot) : x) : \mathcal{C} \to \text{Ab}$ as $(K(\cdot) : x)(C') = (K(C') : x)$ for all $C' \in \mathcal{C}$. Clearly $(K(\cdot) : x)$ is an ideal of $\mathcal{C}(\cdot, C)$.

Let $0$ be the trivial $\mathcal{C}$-module, $0(C) = 0_{\text{Ab}}$, for all $C \in \mathcal{C}$. We denote by $\text{Ann}(\cdot, \cdot)$ the functor $(0(\cdot) : \cdot) : \mathcal{C} \to \text{Ab}$ as $\text{Ann}(\cdot, \cdot)(C') = \text{Ann}(\cdot, C') = (0(C') : x) = \{ f \in \mathcal{C}(C', C)|N(f)(x) \in 0(C') \} = \{ f \in \mathcal{C}(C', C)|N(f)(x) = 0 \}$.

We have the following lemma.

**Lemma 2.1.** (i) Let $N$ and $K$ be $\mathcal{C}$-modules such that $K \subset N$. Thus, $\text{Ann}(x + K(C), \cdot) = (K(\cdot) : x)$ for all $C \in \mathcal{C}$ and $x \in N(C)$.

(ii) Let $C \in \mathcal{C}$ and $I$ be an ideal of $\mathcal{C}(\cdot, C)$. Then $(I(\cdot) : 1_C) = I$, for the identity morphism $1_C \in \mathcal{C}(C, C)$.

**Proof.** We leave the proof for the reader. \hfill \qed

Now we can define the concept of a linear filter for a preadditive category. This definition generalizes the definition of linear filter for rings.

**Definition 2.2.** 1) A family $\mathcal{F}$ of ideals of $\mathcal{C}(\cdot, C)$ is a filter for $\mathcal{C}(\cdot, C)$ if the following conditions hold:

$(T_1)$ If $I \in \mathcal{F}$ and $I \subset J$, then $J \in \mathcal{F}$;

$(T_2)$ If $I$ and $J$ belong to $\mathcal{F}$, then $I \cap J \in \mathcal{F}$.

2) A collection $\{\mathcal{F}_C\}_{C \in \mathcal{C}}$ is a linear filter for the category $\mathcal{C}$ if each $\mathcal{F}_C$ is a filter for $\mathcal{C}(\cdot, C)$ and

$(T_3)$ For all $I \in \mathcal{F}_C$, $B \in \mathcal{C}$ and each $h \in \mathcal{C}(B, C)$, we have $(I(\cdot) : h) \in \mathcal{F}_B$, where $(I(C') : h) = \{ f : C' \to B|\mathcal{C}(f, A)(h) = hf \in I(C') \}$ for all $C' \in \mathcal{C}$.

3) A collection $\{\mathcal{F}\}_{C \in \mathcal{C}}$ is a Gabriel filter for the category $\mathcal{C}$ if the following holds:

$(T_4)$ If $J \in \mathcal{F}_C$, assume that $I \subset \mathcal{C}(\cdot, C)$ is an ideal such that $(I(\cdot) : h) \in \mathcal{F}_B$ for all $h \in J(B)$ for all $B \in \mathcal{C}$, then $I \in \mathcal{F}_C$. 
Now we show that there is a bijection between hereditary pretorsion classes and linear filters on \( \mathcal{C} \). To achieve this, we need the following two lemmas.

**Lemma 2.3.** Let \( \mathcal{F} = \{ \mathcal{F}_C \} \) be a linear filter on \( \mathcal{C} \). Then the class \( \mathcal{T}_\mathcal{F} \) consisting of \( \mathcal{C} \)-modules \( M \) for which \( \text{Ann}(x, -) \in \mathcal{F}_C \), for all \( x \in M(C) \) and \( C \in \mathcal{C} \), is a hereditary pretorsion class.

*Proof.* 1) \( \mathcal{T}_\mathcal{F} \) is closed under subobjects. Let \( N \in \mathcal{T}_\mathcal{F} \) and \( K \) be a \( \mathcal{C} \)-submodule of \( N \). It follows that for all \( C \in \mathcal{C} \) and \( x \in N(C) \) the ideal \( \text{Ann}(x, -) \in \mathcal{F}_C \). Then \( \text{Ann}(x, -) \in \mathcal{F}_C \) for all \( x \in K(C) \), since \( K(C) \subset N(C) \).

2) \( \mathcal{T}_\mathcal{F} \) is closed under quotients. Let \( N \in \mathcal{T}_\mathcal{F} \) and \( K \) be a \( \mathcal{C} \)-submodule of \( N \). Then for all \( C \in \mathcal{C} \) and \( x \), \( K(C) \in N(C) \), the ideal \( \text{Ann}(x, -) \in \mathcal{F}_C \). It follows that for all \( x \), \( K(C) \in N(C) \).

3) \( \mathcal{T}_\mathcal{F} \) is closed under coproducts. Let \( \{ N_\lambda \}_{\lambda \in \Lambda} \) be a family of \( \mathcal{C} \)-modules, \( N_\lambda \in \mathcal{T}_\mathcal{F} \), and \( \overline{X} = (x_\lambda)_{\lambda \in \Lambda} \subseteq \prod_{\lambda} N_\lambda(C) = \prod_{\lambda} N_\lambda(C) \). If we consider the set \( \{ x_\lambda, \ldots, x_\lambda \} \) consisting of all the nonzero coordinates of \( \overline{X} \), then we have \( \text{Ann}(\overline{X}, -) = \cap_{i=1}^n \text{Ann}(x_\lambda, -) \in \mathcal{F}_C \); this implies that \( \prod_{\lambda} N_\lambda \in \mathcal{T}_\mathcal{F} \). \( \square \)

**Lemma 2.4.** Let \( \mathcal{T} \) be a hereditary pretorsion class. For each \( C \in \mathcal{C} \), let \( \mathcal{F}_C = \{ I \subset \mathcal{C}(-, C) \} \) be the family of right ideals for which \( \frac{\mathcal{C}(\cdot, C)}{I} \in \mathcal{T} \). Then the class \( \mathcal{F}_\mathcal{T} = \{ \mathcal{F}_C \}_{C \in \mathcal{C}} \) is a linear filter on \( \mathcal{C} \).

*Proof.* \((T_1)\) Let \( I \in \mathcal{F}_C \) and \( J \subset \mathcal{C}(-, C) \) a right ideal such that \( I \subset J \). The monomorphism

\[
\varphi : \frac{\mathcal{C}(-, C)}{J} \rightarrow \frac{\mathcal{C}(-, C)}{I} \quad \varphi_C(f + J(C')) = f + I(C')
\]

for all \( f \in \mathcal{C}(C', C) \) and \( C' \in \mathcal{C} \), implies that \( \frac{\mathcal{C}(-, C)}{J} \in \mathcal{T} \), since \( \mathcal{T} \) is closed under subobjects. Hence \( J \in \mathcal{F}_C \).

\((T_2)\) If \( I, J \in \mathcal{F}_C \), then \( I \cap J \in \mathcal{F}_C \) because we have the monomorphism

\[
\varphi : \frac{\mathcal{C}(-, C)}{I \cap J} \rightarrow \frac{\mathcal{C}(-, C)}{I} \prod \frac{\mathcal{C}(-, C)}{J}
\]

given by \( \varphi_C(f + (I \cap J)(C')) = (f + I(C'), f + J(C')) \), \( C' \in \mathcal{C} \), \( f \in \mathcal{C}(C', C) \), since \( \mathcal{T} \) is a closed under coproducts.

\((T_3)\) Let \( I \in \mathcal{F}_C \). Then \( \frac{\mathcal{C}(-, C)}{I} \in \mathcal{T} \). We will prove that

\[
\frac{\mathcal{C}(-, C')}{(I(-) : h)} \subset \frac{\mathcal{C}(-, C)}{I}
\]
for all $h \in \mathcal{C}(C', C)$ and $C' \in \mathcal{C}$. Indeed, let $h \in \mathcal{C}(C', C)$, then the natural morphisms $(-, h) : \mathcal{C}(-, C') \to \mathcal{C}(-, C)$ and $\pi : \mathcal{C}(-, C) \to \frac{\mathcal{C}(-, C)}{I}$ induce the exact sequence

$$0 \to (I(-) : h) \to \mathcal{C}(-, C') \xrightarrow{\pi h} \frac{\mathcal{C}(-, C)}{I}$$

which implies $\frac{\mathcal{C}(-, C')}{(I(-) : h)} \subset \frac{\mathcal{C}(-, C)}{I}$. Hence $\frac{\mathcal{C}(-, C')}{(I(-) : h)} \in \mathcal{T}$, since it is closed under subobjects, and finally $(I(-) : h) \in \mathcal{F}_{C'}$.

Now we are ready for the first part of the main theorem for this section.

**Theorem 2.5.** The maps $\mathcal{F} \to \mathcal{T}_F$, $\mathcal{T} \to \mathcal{F}_T$ induce a bijection between heredity pretorsion classes and linear filters on $\mathcal{C}$.

**Proof.** Starting with a linear filter $\mathcal{F} = \{, \mathcal{F}_C\}$ we get $\mathcal{T} = \{M|\text{Ann}(x, -) \in \mathcal{F}_C\}$ for all $x \in M(C)$ and each $C \in \mathcal{C}$; furthermore, we obtain $\mathcal{I} = \{\mathcal{I}_C\}_{C \in \mathcal{C}}$ such that $\mathcal{I}_C = \{I \subset \mathcal{C}(-, C)\frac{\mathcal{C}(-, C)}{I} \in \mathcal{T}\}$.

We claim that $\mathcal{I}_C = \mathcal{F}_C$ for all $C \in \mathcal{C}$. Indeed, let $I \in \mathcal{I}_C$, then $\frac{\mathcal{C}(-, C)}{I} \in \mathcal{T}$. It follows that $\text{Ann}(h + I(C), -) \in \mathcal{F}_C$ for all $h \in \mathcal{C}(C, C)$ and in particular $\text{Ann}(1_C + I(C), -) \in \mathcal{F}_C$, but $\text{Ann}(1_C + I(h), -) = I$ by Lemma 2.1. Hence $I \in \mathcal{F}_C$.

Conversely, let $F \in \mathcal{F}_C$. Then $\frac{\mathcal{C}(-, C)}{F} \in \mathcal{T}$. Since $\text{Ann}(g + F(B), -) = (F(-) : g) \in \mathcal{F}_B$ for all $B \in \mathcal{C}$ and each $g \in \mathcal{C}(B, C)$, it follows that $F \in \mathcal{I}_C$.

On the other hand, if we start with the class $\mathcal{T} = \{M\}$ we first get $\mathcal{F} = \{\mathcal{F}_C\}_{C \in \mathcal{C}}$ such that $\mathcal{F}_C = \{I \subset \mathcal{C}(-, C)\frac{\mathcal{C}(-, C)}{I} \in \mathcal{T}\}$; we obtain $\mathcal{T}' = \{N|\text{Ann}(x, -) \in \mathcal{F}_C\}$ for all $x \in N(C)$ and each $C \in \mathcal{C}$.

We will prove $\mathcal{T} = \mathcal{T}'$. Let $M \in \mathcal{T}$, then we put

$$M \cong \prod_{C \in \mathcal{C}} \prod_{x \in M(C)} \frac{\mathcal{C}(-, C)}{I_x}.$$

By (1), it follows that $\frac{\mathcal{C}(-, C)}{I_x} \in \mathcal{T}$ for all $x \in F(C)$ since $\mathcal{T}$ is closed under subobjects. This implies that $\{I^x\}_{x \in M(C)} \subset \mathcal{F}_C$. Now, let $B \in \mathcal{C}$, and assume that $\bar{X} = (\bar{X}_\lambda)_{\lambda \in N} \in M(B)$ is nonzero. Let $\{\bar{X}_{\lambda_1}, \ldots, \bar{X}_{\lambda_n}\}$ be the set of all nonzero coordinates, and we can include for $i = 1, \ldots, n$ the following:

$$\bar{X}_{\lambda_i} = x_i + I^{x_i}(B), x_i \in \mathcal{C}(B, C_i)$$

for some $C_i \in \mathcal{C}$.
Thus, \( \text{Ann}(X, -) = \cap_{i=1}^n \text{Ann}(X_{x_i}, -) = \cap_{i=1}^n \text{Ann}(x_i + I^{x_i}(B), -) = \cap_{i=1}^n (I^{x_i}(-) : x_i) \in \mathcal{F}_B \) by \( T_2 \) and by Lemma 2.1. Hence, \( M \in \mathcal{T}' \).

Now, assume that \( M \in \{ M | \text{Ann}(x, -) \in \mathcal{F}_C \) for all \( x \in M(C), C \in \mathcal{C} \} \) and let \( X \in M(C) \) whose the unique nonzero coordinate is \( 1_C + I^{x_C}(C) \). Thus

\[
\text{Ann}(X) = \text{Ann}(1_C + I^{x_C}(C), -) = I^{x_C} \in \mathcal{F}_C,
\]

and this implies \( \mathcal{C}(-, C)_{I^{x_C}} \in \mathcal{T} \) since \( \mathcal{T} \) is closed under subjects. Finally \( M \cong \coprod_{C \in \mathcal{C}} \prod_{x \in M(C)} \frac{\mathcal{C}(-, C)}{I^{x}} \) lies in \( \mathcal{T} \), since \( \mathcal{T} \) is closed under coproducts. \( \square \)

Now we prove the main theorem of this section.

**Theorem 2.6.** The maps \( \mathcal{F} \rightarrow \mathcal{T}_g, \mathcal{T} \rightarrow \mathcal{F}_\mathcal{T} \) induce a bijection between heredity torsion classes and Gabriel filters on \( \mathcal{C} \).

**Proof.** We will only prove that if \( \mathcal{F} \) is a hereditary torsion class, the corresponding topology \( \mathcal{F} = \{ \mathcal{F}_C \}, \) where \( \mathcal{F}_C = \{ I \subset \mathcal{C}(-, C)| \frac{\mathcal{C}(-, C)}{I} \in \mathcal{T} \} \), satisfies \( T_1 \). Indeed, let \( I \subset \mathcal{C}(-, C) \) be an ideal such that \( (I(-) : h) \in \mathcal{F}_{C'}, \) for all \( h \in J(C') \) and \( C' \in \mathcal{C} \) for some \( J \in \mathcal{F}_C \). Consider the exact sequence

\[
0 \rightarrow \frac{J}{I} \rightarrow \frac{\mathcal{C}(-, C)}{I} \rightarrow \frac{\mathcal{C}(-, C)}{I + J} \rightarrow 0,
\]

where \( \frac{\mathcal{C}(-, C)}{I + J} \in \mathcal{T}, \) since it is a quotient \( \mathcal{C} \)-module of \( \frac{\mathcal{C}(-, C)}{I} \in \mathcal{T}. \) Now, we have \( (I(-) : h) = ((I \cap J)(-) : h) \) for all \( h \in J(C') \), but this implies that

\[
((I \cap J)(-)) : h) \in \mathcal{F}_{C'}, \text{ and } \frac{\mathcal{C}(-, C)}{((I \cap J)(-)) : h) \in \mathcal{F}.
\]

In this way we have a map of \( \mathcal{C} \)-modules for all \( h \in J(C') \):

\[
\frac{\mathcal{C}(-, C)}{I \cap J} \xrightarrow{\varphi_h} \frac{J}{I \cap J}
\]

given by \( (\varphi_h)_{C'}(g + ((I \cap J)(C') : h)) = hg + (I \cap J)(C') \). An epimorphism of \( \mathcal{C} \)-modules is then induced:

\[
\varphi = (\varphi_h) : \prod_{h \in J(C')} \frac{\mathcal{C}(-, C')}{(I \cap J : h) \rightarrow \frac{J}{I \cap J}},
\]

given by \( \varphi_{C'}(g + ((I \cap J)(C') : h)) \) for all \( g \in \mathcal{C}(C', C') \).
It follows that $\prod_{h \in J(C')} \frac{C(-, C')}{(I \cap J)(-)} : h \in T$, since $T$ is closed under coproducts. Finally $\frac{J}{I \cap J} \in T$ because $F$ is closed under quotients. Since $T$ is closed under extensions, it follows that $\frac{C(-, C)}{I} \in T$ and therefore $I \in T$.

Assume that $F = \{F_C\}_{C \in C}$ is a Gabriel filter for $C$, and let $0 \rightarrow K \rightarrow N \rightarrow \frac{N}{K} \rightarrow 0$ be an exact sequence of $C$-modules for which $K$ and $\frac{N}{K}$ in $T$.

Let $C \in C$ and $x \in N(C)$. Then $x + K(C) \in \frac{N}{K}(C)$ and $\text{Ann}(x + K(C), -) \in F_C$ since $\frac{N}{K} \in T$. Let $C' \in C$ and $h \in \text{Ann}(x + K(C), C') \subset C(C', C)$; we will then prove the following:

1) Since $h \in \text{Ann}(x + K(C), C') \subset C(C', C)$ we have

$$0 = \left(\frac{N}{K}\right)(h)(x + K(C)) = N(h)(x) + K(C').$$

It follows that $N(h)(x) \in K(C')$. In addition $\text{Ann}(N(h)(x)) \in F_{C'}$, because $K \in T$.

2) Let $C'' \in C$. Then $f \in (\text{Ann}(x, -) : h)(C'') \subset C(C'', C')$ if and only if $hf \in \text{Ann}(x, C'')$ if and only if $N(hf)(x) = N(f)(N(h)(x)) = 0$ if and only if $f \in \text{Ann}(N(h)(x), C'')$. It follows that $(\text{Ann}(x, -) : h) = \text{Ann}(N(h)(x), -)$ and (2) holds.

We proved that $(\text{Ann}(x, -) : h) = \text{Ann}(N(h)(x), -) \in F_{C'}$, for all $h \in \text{Ann}(x + K(C), C')$, $C' \in C$. It follows that $\text{Ann}(x, -) \in F_C$, since $\text{Ann}(x + K(C), -) \in F_{C'}$, i.e. $N \in T$. □

3 Linear Topologies

We have noticed that a hereditary torsion theory $T$ is characterized by the class $F_T = \{F_C\}_{C \in C}$ such that $F_C = \{I_{\lambda}\}_{\lambda \in \Lambda}$ is a family of right ideals for which $\frac{C(-, C)}{I_{\lambda}} \in T$. It turns out that for such a family $F_C = \{I_{\lambda}\}_{\lambda \in \Lambda}$, $C \in C$ of right ideal $I_{\lambda} \subset C(-, C)$ we have that $F_C(C') = \{I_{\lambda}(C')\}$ is the family of neighborhoods of the zero map $0 : C' \rightarrow C$ for certain topology on $C(C', C)$. For this reason we start with a general review of topological groups.
Remember that an Abelian group $G$ is a topological group if it is equipped with a topology for the group operations $(a, b) \mapsto a + b$ and $a \mapsto -a$ are continuous functions $G \times G \to G$ and $G \to G$.

We have the following well known proposition.

**Proposition 3.1.** Let $G$ be a topological group. For an $a \in G$ the translation map

$$
\mathcal{L}_a : G \to G \\
g \mapsto a + g
$$

is a homeomorphism.

Therefore, $U$ is a neighborhood of $a \in G$ if and only if $U - a$ is a neighborhood of 0. Thus the topology of $G$ is completely determined by a neighbourhood basis of 0.

**Proposition 3.2.** Let $\mathcal{C}$ be a preadditive category and $\mathcal{F} = \{\mathcal{F}_C\}_{C \in \mathcal{C}}$ be a linear filter for $\mathcal{C}$. Then there exists a topology $T_{(A,C)}$ for $\mathcal{C}(A,C)$, for each pair of objects $A, C \in \mathcal{C}$, for which the composition $\mathcal{C}(A,B) \times \mathcal{C}(A,B) \to \mathcal{C}(A,C)$ and the sum $\mathcal{C}(A,B) \times \mathcal{C}(A,B) \mapsto \mathcal{C}(A,B)$, $(f, g) \mapsto (f + g)$, are continuous. Moreover $\mathcal{F}_C(A)$ is a basis of neighborhoods for $0 \in \mathcal{C}(A,C)$.

**Proof.** We consider a family of subsets $T_{(A,C)} \subset 2^{\mathcal{C}(A,C)}$ defined as follows: $U \in T_{(A,C)}$ if $U$ is the empty set or if for each $x \in U$ there exist $I \in \mathcal{F}_C$ such that $x + I(A) \subset U$.

1. $T_{(A,C)}$ is a topology for $\mathcal{C}(A,C)$.
2. Clearly $\emptyset, \mathcal{C}(A,C)$ are in $T_{(A,C)}$.
3. Let $U, V$ be in $T_{(A,B)}$ and $x \in U \cap V$. Then there are $I$ and $J$ in $\mathcal{F}_C$ such that $x + I(A) \subset U$ and $x + J(A) \in V$. We have that $I \cap J \in \mathcal{F}_C$ since $\mathcal{F}$ is a linear filter of $\mathcal{C}$. If $y \in x + (I \cap J)(A) = x + I(A) \cap J(A)$, then there exist $r \in I(A) \cap J(A)$ such that $y = x + r \subset U \cap V$. It follows that $x + (I \cap J)(A) \subset U \cap V$, and hence $U \cap V \in T_{(A,B)}$.
4. Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be a family of sets such that $U_\lambda \in T$, and $x \in \bigcup U_{\lambda \in \Lambda}$. Then $x \in U_\lambda$ for some $\lambda \in \Lambda$, and there exists $I \in \mathcal{F}_C$ such that $x + I(A) \subset U_\lambda \subset \bigcup U_{\lambda \in \Lambda}$.

Now, if $U$ is a neighborhood of $0 \in \mathcal{C}(A,C)$, then there exists $U' \in T_{(A,C)}$ for which $0 \in U' \subset U$. In addition exist $I \in \mathcal{F}_C$ such that $I(A) = 0 + I(A) \subset U' \subset U$; it follows that $\mathcal{F}_C(A)$ is a basis of neighborhoods for 0.

b) For each pair $A, B \in \mathcal{C}$, the map $\mathcal{C}(A,B) \times \mathcal{C}(A,B) \mapsto \mathcal{C}(A,B)$, $(f, g) \mapsto (f + g)$ is continuous. Let $f, g \in \mathcal{C}(A,B)$ and $U \subset \mathcal{C}(A,B)$ open such that $f + g \in U$. Then there exists $I \in \mathcal{F}_B$ for $(f + g) + I(A) \subset U$. 


Let \((r_1, r_2) \in (f + I(A), g + I(A))\); if we put \(r_1 = f + t_1, r_2 = g + t_2, t_1, t_2 \in f + g + I(A)\), it follows that \(r_1 + r_2 = f + g + t_1 + t_2 \in f + g + I(A)\).

c) \(\mathcal{C}(A, B) \times \mathcal{C}(B, C) \to \mathcal{C}(A, C), (f, g) \mapsto (gf)\) is continuous. Indeed, let \(f \in \mathcal{C}(A, B), g \in \mathcal{C}(B, C)\) and \(U \subset \mathcal{C}(A, C)\) be an open set such that \(gf \in U\). Then there exits \(I \in \mathcal{F}_C\), such that \(gf + I(A) \subset U\). Now, we have \((I(-) : g) \in \mathcal{F}_B\), and if we take \((r_1, r_2) \in (f + (I(A) : g)) \times (g + I(A)) \subset \mathcal{C}(A, B) \times \mathcal{C}(B, C)\), then we can put \(r_1 = f + t_1, t_1 \in (I(A) : g)\), and \(r_2 = g + t_2, t_2 \in I(A)\). It follows that

\[
(r_2 r_1) = (g + t_2)(f + t_1)
= gf + gt_1 + t_2 f + t_2 t_1 \in gf + I(A) \subset \mathcal{C}(A, C)
\]

because \(gt_1 + t_2 f + t_2 t_1 \in I(A)\).

\(\square\)

4 Examples

In this section we show two examples, one of hereditary pretorsion theory and the other one of hereditary torsion theory, both in the category of \(\mathcal{C}\)-modules.

Definition 4.1. Let \(\mathcal{U}\) be a subclass of objects of \(\text{Mod}(\mathcal{C})\). We say that a \(\mathcal{C}\)-module \(N\) is \(\mathcal{U}\)-subgenerated if \(N\) is a subobject of another \(\mathcal{U}\)-generated object. We denote by \(\sigma[\mathcal{U}]\) to the full subcategory of \(\text{Mod}(\mathcal{C})\) consistent with all the \(\mathcal{U}\)-subgenerated objects.

We see that \(\sigma[\mathcal{U}]\) is a pretorsion class for \(\text{Mod}(\mathcal{C})\); moreover, we have the following.

Proposition 4.2. Let \(\mathcal{T}\) be a pretorsion class of \(\text{Mod}(\mathcal{C})\). Then \(\mathcal{T}\) is the category \(\sigma[M]\), where

\[
M = \coprod_{C \in \mathcal{C}} \{ \frac{\mathcal{C}(-, C)}{I} : \frac{\mathcal{C}(-, C)}{I} \text{ is an object of } \mathcal{T} \}.
\]

Proof. Since \(\mathcal{T}\) is closed under submodules, coproducts and quotients, it follows that \(\sigma[M]\) is a full subcategory of \(\mathcal{T}\).

Conversely, for every \(\mathcal{C}\)-module \(N\) we have

\[
N \cong \coprod_{C \in \mathcal{C}} \prod_{x \in N(C)} (-, C)/K^x.
\]

by (1). If \(N\) is an object of \(\mathcal{T}\), then we have that each \(\mathcal{C}(-, C)/K^x\) is a subobject of \(N\). It follows \(\mathcal{C}(-, C)/K^x \in \mathcal{T}\), since \(\mathcal{T}\) is closed under subobjects, and therefore, \(N\) is an object in \(\sigma[M]\). \(\square\)
Theorem 4.3. Let $I(-,?)$ be a bilateral ideal of $\mathcal{C}$ and

$$F = \prod_{C \in \mathcal{C}} \mathcal{C}(-,C)/I(-,C),$$

then $\sigma[F]$ is the full subcategory of $\text{Mod}(\mathcal{C})$ consisting of all objects $N$ such that $IN = 0$.

Proof. Let $M$ be a $F$-generated $\mathcal{C}$-module. Then there exists an epimorphism $F^A \to M$. Then we have an epimorphism $I(F^A) \to IM$, but it is easy to see that $I(F^A) = 0$, then $IM = 0$. Clearly, $IN = 0$ for all $\mathcal{C}$ submodules $N \subset M$.

Conversely, let $N$ be a $\mathcal{C}$-module for which $IN = 0$. Let $\eta = \{\eta_A\}_{A \in \mathcal{C}} : \mathcal{C}(-,C) \to N$ be a natural transformation, then we can define a natural transformation $\hat{\eta} = \{\hat{\eta}_B\}_{B \in \mathcal{C}} : \mathcal{C}(-,C)/I \to N$ as

$$\hat{\eta}_B : \mathcal{C}(B,C)/I(B) \to M(B)$$

$$f + I(B) \mapsto \eta_B(f).$$

We claim that the natural transformation $\hat{\eta}$ is well defined. Indeed, if $f + I(B) = g + I(B)$, then we have $f - g \in I(B) \subset \mathcal{C}(B,C)$. Therefore there is a commutative diagram

\[
\begin{array}{ccc}
\mathcal{C}(C,C) & \xrightarrow{\eta_C} & N(C) \\
\downarrow \mathcal{C}(f-g,C) & & \downarrow N(f-g) \\
\mathcal{C}(B,C) & \xrightarrow{\eta_B} & N(B)
\end{array}
\]

It follows that $\eta_B(f) - \eta_B(g) = \eta_B(f - g) = \eta_B\mathcal{C}(f - g,C)(1_C) = N(f - g)(\eta_C(1_C)) = 0$ because $IN = 0$. In this way if we proceed as in (1) we have a well defined epimorphism

$$\prod_{C \in \mathcal{C}} \prod_{x \in N(C)} \frac{\mathcal{C}(-,C)}{I} \to N;$$

therefore, $N$ is $F$-generated and also lies in $\sigma[F]$.

Now, we show an example of hereditary torsion theory in the category of $\mathcal{C}$-modules.

Let $\mathcal{C} = \{C_{\lambda}\}_{\lambda \in \Lambda}$ be a family of objects in $\mathcal{C}$. The class of $\mathcal{C}$-modules $\mathcal{T} = \{M|M(C_{\lambda}) = 0\}$ is a hereditary torsion class. Indeed, it is closed under subobjects, epimorphic images, and coproducts. In addition if $0 \to M \to L \to N \to 0$ is an exact sequence with $M,N \in \mathcal{T}$, then $0 \to M(C_{\lambda}) \to \ldots$
L(C_\lambda) \rightarrow N(C_\lambda) \rightarrow 0 is an exact sequence of Abelian groups for all C_\lambda \in \mathcal{C}. M(C_\lambda) = M(C_\lambda) = 0, however, for all \lambda \in \Lambda implies L(C_\lambda) = 0 for all \lambda \in \Lambda. Thus, \mathcal{T} is closed under extensions and it is a hereditary torsion class.

We observe that M \in \mathcal{T} if and only if \boxed{M} = \boxed{M} = 0, however, for all \lambda \in \Lambda implies \boxed{L} = 0 for all \lambda \in \Lambda.

Thus, \mathcal{T} is closed under extensions and it is a hereditary torsion class.

We will prove that hereditary torsion classes are cogenerated by injective objects.

By [3] we have that in the category Mod(\mathcal{C}) injective envelopes exist. Thus following the proof given in [10], the following proposition holds in the category of \mathcal{C}-modules.

**Proposition 4.4** ([10], Prop. 3.2). A torsion theory (\mathcal{T}, \mathcal{F}) is hereditary if and only if \mathcal{F} is closed under injective envelopes.

Now, we can prove the following.

**Theorem 4.5.** A torsion theory is hereditary if and only if it can be cogenerated by an injective module.

*Proof.* The proof is similar to that given in [10] but we include it here for readers’ benefit. Let E be an injective module and \mathcal{T} = \{M | \text{Hom}(M, E) = 0\}. If M \in \mathcal{T} and L is a \mathcal{C}-module of M with a nonzero homomorphism \alpha : L \rightarrow E, then extends to a homomorphism M \rightarrow E, which is impossible. Hence L \in \mathcal{T}, and the torsion theory cogenerated by E is hereditary.

Conversely, assume that (\mathcal{T}, \mathcal{F}) is a hereditary torsion theory. Set

\[ E = \prod_{\lambda \in \Lambda} E(\mathcal{C}(\lambda, -)/I) \]

with the product taken over the injective envelopes of all right ideals I \subset \mathcal{C}(\lambda, C) such that \mathcal{C}(\lambda, C)/I \in \mathcal{F}, for all C \in \mathcal{C}. Then E is a torsion free module because is a product of injective envelopes, so Hom(M, E) = 0 for every M \in \mathcal{T}. On the other hand, let M be a \mathcal{C}-module and put it as \boxed{M} = \boxed{\prod_{C \in \mathcal{C}} \prod_{x \in M(C)} \mathcal{C}(\lambda, C)/I^x}, if M \notin \mathcal{T}; there exist a direct summand \mathcal{C}(\lambda, C_0)/I^x with a nonzero morphism \alpha : \mathcal{C}(\lambda, C_0)/I^x \rightarrow F for some F \in \mathcal{F}. Otherwise, if Hom(\mathcal{C}(\lambda, C)/I^x, \mathcal{F}) = 0 for all pair C, x we should have \mathcal{C}(\lambda, C)/I^x \in \mathcal{T} for all C, x; M would then in \mathcal{T}, since it is closed under coproducts.
Now, if we put $F = \bigsqcup_{B \in \mathcal{C}} \bigsqcup_{y \in F(B)} \mathcal{C}(-, B)/I^y$ as in (1), then all the direct summands $\mathcal{C}(-, B)/I^y$ lie in $\mathcal{F}$. Therefore, there is a monomorphism $F \to E$, so $\alpha$ induces a morphism $\mathcal{C}(-, C_0)/I^{\alpha_0} \to E$ which can be extend to a nonzero map $M \to E$. It follows that $M \in \mathcal{T}$ if and only if $\text{Hom}(M, E) = 0$, $E$ cogenerates the torsion theory.

### 4.1 Dense Ideals in Path Categories

In this subsection we show an example of linear filters certains categories that appear in the study of representations of dimensional finite algebras.

**Definition 4.6.** An ideal $I(-, C) \subseteq \mathcal{C}(-, C)$ is dense in $\mathcal{C}(-, C)$ if for all $B \in \mathcal{C}$ and $g \in \mathcal{C}(B, C)$, there exist $D \in \mathcal{C}$ and $h \in \mathcal{C}(D, B)$ such that $gh \in I(D, C)$.

Therefore, we have examples of linear filters for categories as the following proposition says.

**Proposition 4.7.** Let $\mathcal{F}_C$ be the family of all ideals in $\mathcal{C}(-, C)$ that are dense. The collection $\mathcal{F} = \{\mathcal{F}_C\}_{C \in \mathcal{C}}$ is a linear filter for $\mathcal{C}$.

**Proof.**

a) Assume that $I(-, C) \subseteq \mathcal{C}(-, C)$ is dense, and $I \subseteq J$. Let $f \in \mathcal{C}(C', C)$, then there exists a morphism $g : C'' \to C'$ such that $fg \in I(C'', C) \subseteq J(C'', C)$, and $J$ is dense in $\mathcal{C}$.

b) Assume that $I(-, C), J(-, C) \subseteq \mathcal{C}(-, C)$ are dense. Let $f \in \mathcal{C}(C', C)$.

Since $I$ is dense in $\mathcal{C}$, then there is a morphism $g : C'' \to C'$ such that $fg \in I(C'', C) \subseteq \mathcal{C}(C'', C)$. Again, since $J$ is dense there exists $h : C'' \to C''$ for $(fg)h \in J(C'', C)$. Therefore, if $f \in \mathcal{C}(C', C)$ then the existence of $gh : C'' \to C'$ implies that $f(gh) \in (I \cap J)(C'', C)$.

c) Let $I \subseteq \mathcal{C}(-, A)$ be a dense ideal and $h \in \mathcal{C}(C, A)$. We will show that $(I(-) : h) \subseteq \mathcal{C}(-, C)$ is dense. Let $f \in \mathcal{C}(C', C)$, then $hf \in \mathcal{C}(C', A)$. Since $I \subseteq \mathcal{C}(-, A)$ is dense, there exist $C'' \in \mathcal{C}$ and $g : C'' \to C'$ for $(hf)g \in I(C'')$. It follows that $fg \in (I(C'') : h)$.

Now, we show two examples where dense ideals appear.

**Example 1.** If $K$ is a field, let $K(\mathbb{Z}A_\infty, \sigma)$ be the mesh category, see [[11] Sec. 2.1]. Indeed, we can think of $K(\mathbb{Z}A_\infty, \sigma)$ as an additive category whose indecomposable objects the vertices and the set of morphisms between two indecomposable objects is the $K$-vectorial space generated by the paths between them for which the squares commute. Let $f : A \to C$ be a morphism. Then the ideal $(f)$ generated by $f$, defined as $(f)(D) = \{fg|g \in \mathcal{C}(D, A)\}$, is dense in $\mathcal{C}(-, C)$. For this, let $g : B \to C$ a map and consider the set $\text{Supp}(f) = \{X \in \mathcal{C}|\text{there exists a path } t : X \to A\}$, then clearly there exists
an object \( D \in \text{Supp}(f) \) and a path \( h : D \to B \). Since \( D \in \text{Supp}(f) \) there exists a path \( t' : D \to A \). Then, by the mesh relations we have \( gh = ft' \in (f)(D) \). This is, \((f)\) is dense in \( C(−, C) \).

**Example 2.** Let \((\mathcal{X}, \tau) \cong \mathbb{Z}A_\infty/(\tau^r)\) a stable tube of rank \( r \) and \( B \) a complete set of representatives of the space of orbits of \((\mathbb{Z}A_\infty, \tau)\), see [[11]] . If \( B \) is the full subcategory of \( K((\mathcal{X}, \tau)) \) which objects are finite direct sums of objects in \( B \). Then the ideal \( I_B(−, C) \) is dense in \( \mathbb{Z}A_\infty/(\tau^r) \). Indeed, if \( f : X \to C \) is a map in \( \mathbb{Z}A_\infty/(\tau^r) \) the we can write \( f = (f_i)_{i=1}^n : \bigoplus X_i \to C \). Then, there exists a path \( g_i : B_i \to X_i \), for \( i = 1, \ldots, n \). In this way the map \( f(g_i) = (f_i g_i) : B = \bigoplus B_i \to C \) lies in \( I_B(B, C) \), i.e \( I_B(−, C) \) is dense in \( \mathbb{Z}A_\infty/(\tau^r) \).
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