Random Coxeter Groups

Angelica Deibel

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Abstract

Much is known about random right-angled Coxeter groups (i.e., right-angled Coxeter groups whose defining graphs are random graphs under the Erdős-Rényi model). In this paper, we extend this model to study random general Coxeter groups and give some results about random Coxeter groups, including some information about the homology of the nerve of a random Coxeter group and results about when random Coxeter groups are $\delta$-hyperbolic and when they have the FC-type property.

1 Introduction

Let $\Gamma$ be a graph with vertices $v_1, \ldots, v_k$ in which each edge is labelled with an integer at least 2. The Coxeter group associated to $\Gamma$ is $W_\Gamma = \langle v_1, \ldots, v_k | (v_i, v_j)^{m_{i,j}} \rangle$, where $m_{i,j}$ is the label of the edge connecting $v_i$ and $v_j$. If $m_{i,j} = 2$ whenever $v_i$ and $v_j$ are adjacent in $\Gamma$, then $W_\Gamma$ is a right-angled Coxeter group, and usually in this case we just think of $\Gamma$ as a graph without the edge labels.

A lot of previous work has been done on random right-angled Coxeter groups; i.e., Coxeter groups associated to a random graph under the Erdős-Rényi model. In the Erdős-Rényi model, a random graph on $n$ vertices is chosen by independently including each possible edge with probability $p(n)$. Typically, we are interested in the asymptotic behavior as $n \to \infty$. The study of random graphs is itself a large field with many results (see, e.g., [2]), and when an interesting group property can be identified in the defining graph of a right-angled Coxeter group, the results and methods of random graph theory can be applied to give results about the groups (e.g. [1],[3]).

In the same way, properties of Coxeter groups which can be identified from the defining graph can be studied via random edge-labelled graphs under an extended version of the Erdős-Rényi model. In this model, a random edge-labelled graph on $n$ vertices is chosen by independently choosing each possible edge to appear labelled $m$ with probability $p_m(n)$ such that for all $n$, $\sum_{m=2}^{\infty} p_m(n) \leq 1$. We will sometimes denote $\sum_{m=2}^{\infty} p_m(n)$ by $p(n)$. In this paper, we use this model to give some results about random Coxeter groups which extend previous work on random right-angled Coxeter groups, as well as some results about general random Coxeter groups which were not interesting in the right-angled case.

One area of interest in the study of random right-angled Coxeter groups has been topological properties of the nerve of a random right-angled Coxeter group. The nerve of a
Coxeter group $W_{\Gamma}$ is the simplicial complex whose vertices are the vertices of the defining graph $\Gamma$, and in which a simplex appears if the vertices in the simplex generate a finite subgroup of $W_{\Gamma}$. In \cite{4}, Davis shows that the cohomology of $W_{\Gamma}$ with coefficients in $\mathbb{Z}W_{\Gamma}$ can be computed from the homology of the nerve and certain subcomplexes of the nerve. In the right-angled case, the nerve is just the flag complex whose 1-skeleton is the defining graph of $\Gamma$. In \cite{7}, Kahle gives conditions on $p$ under which a random flag complex asymptotically almost surely has trivial $H_k$ or non-trivial $H_k$ (among other results). Then in \cite{6}, Davis and Kahle use these results along with additional work to determine the cohomology of a random right-angled Coxeter group. In particular, they show that random right-angled Coxeter groups are asymptotically almost surely rational duality groups; i.e., that for a random right-angled Coxeter group $W_{\Gamma}$, $H^k(W_{\Gamma},\mathbb{Q}W_{\Gamma})$ is asymptotically almost surely trivial in all but one dimension. In section 4, we show that if $n^kp_2^{(k)}\to\infty$ and $\frac{p^2-p_2}{p^2}\not\to\infty$, then the nerve of $W_{\Gamma}$ asymptotically almost surely has trivial $H_i$ for all $i \geq k+1$. Also, in section 5 we show that if $n^kp_2^{(k)}\to\infty$ and $np_3\not\to0$ and $np^{k+1}\to0$, then the nerve of $W_{\Gamma}$ asymptotically almost surely has non-trivial $H_k$. This partially extends the work of Kahle in \cite{7} and is the first step towards understanding the cohomology of a random Coxeter group in the general setting.

Another of the various properties that have been studied in random right-angled Coxeter groups is hyperbolicity. A group is hyperbolic if its Cayley graph has the property that for some $\delta$, all triangles are “$\delta$-slim”; i.e., the $\delta$-neighborhood of the union of two sides of a geodesic triangle in the Cayley graph contains the third side. (This notion of hyperbolicity is different and not equivalent to a Coxeter group being a hyperbolic reflection group.) In \cite{3}, Charney and Farber give conditions under which right-angled Coxeter groups are asymptotically almost surely hyperbolic or not. In section 6, we give conditions under which $W_{\Gamma}$ is asymptotically almost surely hyperbolic, and we also give conditions under which $W_{\Gamma}$ is asymptotically almost surely not hyperbolic.

The study of random general Coxeter groups also produces some questions which were not interesting in the right-angled case. One example of this is the FC-type property. We say a Coxeter group $W_{\Gamma}$ is of FC type if every complete subgraph of $\Gamma$ generates a finite subgroup of $W_{\Gamma}$. In the right-angled case, this is always true since any complete subgraph generates a subgroup isomorphic to a product of $\mathbb{Z}/2\mathbb{Z}$'s, but general Coxeter groups may or may not have this property. In section 7 we give conditions under which a random Coxeter group is asymptotically almost surely of FC type and conditions under which it asymptotically almost surely is not.

## 2 Subgraphs of Random Edge-Labelled Graphs

$\Gamma$ is a random edge-labelled graph where edges are labelled $m$ with probability $p_m(n)$ for every $m$, and $W_{\Gamma}$ is the Coxeter group associated to $\Gamma$. For each $m$, $\Gamma_m$ is the subgraph of $\Gamma$ whose edges are labelled $m$ in $\Gamma$.

The following proofs rely on computing the expected number of subgraphs of $\Gamma$ which are isomorphic to a particular graph $\Gamma'$. To simplify the proofs, we will give the general procedure for computing these expected values here. First, pick an ordering of the vertices $w_1, \ldots, w_k$.
of $\Gamma'$, so that we can consider a $k$-tuple $(v_1, \ldots, v_k)$ of vertices in $\Gamma$ to be an instance of $\Gamma'$ if the edge between every pair $v_i, v_j$ has the same label as the edge between $w_i, w_j$. Then for each $k$-tuple $\alpha = (v_1, \ldots, v_k)$ of vertices of $\Gamma$, let $X_\alpha$ be the random variable which takes the value 1 if $\alpha$ is an instance of $\Gamma'$ and takes the value 0 otherwise, and let $X = \sum_\alpha X_\alpha$. Then the expected number of subgraphs of $\Gamma$ isomorphic to $\Gamma'$ is $\frac{1}{b}E(X)$, where $b$ is the number of permutations of the vertices of an instance of $\Gamma'$ which give another instance of $\Gamma'$ (for any particular $\Gamma'$, this is easy to compute, though we typically won’t need that information). So, the expected number of subgraphs isomorphic to $\Gamma'$ is:

$$
\frac{1}{b}E(X) = \frac{1}{b} \sum_\alpha E(X_\alpha) = \frac{1}{b} n(n-1) \cdots (n-k) E(X_\alpha) \sim \frac{1}{b} n^k E(X_\alpha),
$$

(since $E(X_\alpha)$ does not actually depend on $\alpha$). Now, $X_\alpha$ will take the value 1 precisely when each edge between vertices in $\alpha$ has the “correct” label – i.e., the one that matches the label on the corresponding edge in $\Gamma'$. Since the choices of labels for different edges are independent, if edge $e_i$ in $\Gamma'$ is labelled $m_i$, $E(X_\alpha)$ is given by the product $\prod_i p_{m_i}(n)$. So,

$$
\frac{1}{b}E(X) \sim \frac{1}{b} n^k \prod_i p_{m_i}(n).
$$

For a particular graph $\Gamma'$, we can find the expected number of subgraphs isomorphic to $\Gamma'$ just by knowing (1) the number $b$ of symmetries of $\Gamma'$, (2) the number $k$ of vertices in $\Gamma'$, and (3) the number of edges in $\Gamma'$ which are labelled with each label $m$, all of which are easy to identify.

It will also be important to be able to compute the expected number of subgraphs of $\Gamma$ which are isomorphic to two copies of $\Gamma'$ (possibly sharing some vertices and edges). Such subgraphs can be characterized by the number of vertices and the number and labels of edges shared by the two copies of $\Gamma$. We can consider a $(2k-\ell)$-tuple $(s_1, \ldots, s_{\ell}, v_1, \ldots, v_{k-\ell}, w_1, \ldots, w_{k-\ell})$ of vertices in $\Gamma$ to be an instance of two copies of $\Gamma'$ sharing $\ell$ vertices if there is some choice of $1 \leq i_1 < i_2 < \cdots < i_{\ell} \leq k$ and $1 \leq j_1 < j_2 < \cdots < j_{\ell} \leq k$ such that the $k$-tuple whose $i_q$'th entry is $s_q$ and whose $r$'th entry for $r \neq i_q$ is $v_{r-\#(q;i_q<r)}$ is an instance of $\Gamma'$, and also similarly the $k$-tuple whose $j_q$'th entry is $s_q$ and whose $r$'th entry for $r \neq j_q$ is $w_{r-\#(q;j_q<r)}$ is an instance of $\Gamma'$.

For each $(2k-\ell)$-tuple $\alpha$, let $Y_{\ell,\alpha}$ be the random variable that takes the value 1 if $\alpha$ is an instance of two copies of $\Gamma'$ sharing $\ell$ vertices and takes the value 0 otherwise. Then the expected number of pairs of subgraphs isomorphic to $\Gamma'$ which share $\ell$ vertices is

$$
\sum_\alpha E(Y_{\ell,\alpha}) = \sum_{a_2 + a_3 + \cdots + a_\infty = (\ell)} b_{\ell,a_2,a_3,\ldots,a_\infty} n^{k-\ell} p_2^{c_2-a_2} p_3^{c_3-a_3} \cdots p_\infty^{c_\infty-a_\infty},
$$

where $c_i$ is the number of edges of $\Gamma'$ labelled $i$, and $b_{\ell,a_2,a_3,\ldots,a_\infty}$ is the number of choices of two subgraphs of $\Gamma'$, each of which has $\ell$ vertices and $a_i$ edges labelled $i$ for each $i$. (I.e., the number of choices for the overlapping part of the two copies of $\Gamma'$). Note that for any
particular choice of $\Gamma'$ and $\ell$, only finitely many of the $a_i$ will be non-zero; in the proofs to follow, we will simplify notation a bit by dropping those which are zero.

So, the expected number of subgraphs of $\Gamma$ isomorphic to two copies of $\Gamma'$ is

$$\mathbb{E}(X^2) = \sum_{\ell=0}^{k} \sum_{(2k-\ell)-\text{tuples } \alpha} \mathbb{E}(Y_{\ell,\alpha})$$

$$= \sum_{\ell=0}^{k} \sum_{a_2+a_3+\cdots+a_{\infty}\leq(\ell\choose 2)} b_{\ell,a_2,a_3,\ldots,a_{\infty}} n^{k-\ell} p_2^{a_2} p_3^{a_3} \cdots p_{\infty}^{a_{\infty}}.$$

This looks complicated, but it turns out we won't actually need to know what $b_{\ell,a_2,a_3,\ldots,a_{\infty}}$ is unless it is zero (i.e., unless there is no way to have two copies of $\Gamma'$ share $\ell$ vertices and $a_i$ $i$-labelled edges for each $i$), except for $b_{0,0,\ldots,0}$ (which will always equal 1). So, to compute as much of this expected value as will be required, we only need to know the number of vertices of $\Gamma'$, the number of edges of $\Gamma'$ with each label, and, for each choice of $(\ell, a_2, \ldots, a_{\infty})$, whether or not there is a subgraph of $\Gamma'$ with $\ell$ vertices and $a_i$ $i$-labelled edges for each $i$. Generally, $\Gamma'$ won't be too complicated and this will be pretty simple.

3 The nerve of a Coxeter group

The nerve of a Coxeter group is a simplicial complex associated to the group whose homology gives some information about the cohomology of the group. It is defined as follows:

**Definition 1.** For an edge-labelled graph $\Gamma$, the nerve of the Coxeter groups $W_\Gamma$ is the simplicial complex whose vertices are the vertices of $\Gamma$, and whose simplices are the sets of vertices which generate a finite subgroup of $W_\Gamma$. We will denote the nerve by $N(\Gamma)$.

It’s easy to tell which simplices are contained in $N(\Gamma)$ because the subgroup of $W_\Gamma$ generated by any subset of the vertices is also a Coxeter group and the finite Coxeter groups are well-understood and classified by Dynkin diagrams (see, e.g. [5]). For reference, the diagrams for the irreducible finite Coxeter groups are given in Figure 1 in the Appendix.

The nerve of a right-angled Coxeter group is just the flag complex on its defining graph (since every clique in the defining graph of a right-angled Coxeter group generates a product of $\mathbb{Z}/2\mathbb{Z}$’s, which is finite). In [7], Kahle obtains the following result about the homology of a random flag complex:

**Theorem 1** (Kahle). For $k \geq 0$,

- If $p^k n \to \infty$ and $p^{k+1} n \to 0$, then the random flag complex asymptotically almost surely has nontrivial $H_k$.

- If $p^k n \to 0$ or if $p^{2k+1} n \to \infty$, then the random flag complex asymptotically almost surely has trivial $H_k$.

In the next two sections, we will study the nerve of a random Coxeter group in the general setting. In section 4 we give some results about the dimension of the nerve of a random Coxeter group and draw conclusions about conditions on the $p_m$ under which $N(\Gamma)$ asymptotically almost surely has trivial $H_k$. In section 5 we give some conditions on the $p_m$ under which $N(\Gamma)$ asymptotically almost surely has non-trivial homology.
4 Dimension of the nerve

The dimension of the nerve of $\Gamma$ is equal to the size of the largest induced subgraph of $\Gamma$ corresponding to a finite subgroup of $W_{\Gamma}$.

For this section, let $p_B = \sum_{m=3}^{\infty} p_m$.

**Theorem 2.** (i) If $n^kp_2^{(k)} \rightarrow 0$ and $\frac{p_B}{p_2} \not\rightarrow \infty$, then asymptotically almost surely $\dim(N(\Gamma)) < k$.

(ii) If $n^kp_2^{(k)} \rightarrow \infty$, then asymptotically almost surely $\dim(N(\Gamma)) \geq k$.

(iii) If $n^kp_2^{(k)} \not\rightarrow 0$ and $\frac{p_m}{p_2} \rightarrow \infty$ for some $m$, then asymptotically almost surely $\dim(N(\Gamma)) \geq k$.

**Proof.** To prove (i), we will show that $\Gamma$ asymptotically almost surely does not contain any induced subgraph on at least $k$ vertices which corresponds to a finite subgroup of $W_{\Gamma}$. Any such subgraph has at most $k - 1$ edges labelled with a number other than 2, so it suffices to show that $\Gamma$ asymptotically almost surely does not contain any subgraph on $k$ vertices with fewer than $k$ non-2 edges. But the expected number of such subgraphs is

$$\sum_{i=0}^{k-1} \binom{n}{k} c_i p_B^{i} p_2^{(k) - i} \sim \sum_{i=0}^{k-1} c_i n^k p_B^{i} p_2^{(k) - i} = \sum_{i=0}^{k-1} c_i n^k p_2^{(k)} \left(\frac{p_B}{p_2}\right)^i \rightarrow 0,$$

since $n^kp_2^{(k)} \rightarrow 0$ and $\frac{p_B}{p_2} \not\rightarrow \infty$. ($c_i$ is the sum over all graphs $\Gamma'$ with $k$ vertices and $i$ edges of the number of symmetries of $\Gamma'$) So, every induced subgraph of $\Gamma$ which induces a finite subgroup of $W_{\Gamma}$ is on fewer than $k$ vertices; hence $\dim(N(\Gamma)) < k$.

For (ii), we’ll show that $\Gamma$ asymptotically almost surely contains a $k$-clique whose edges are all 2’s. Let $\Gamma_2$ be the (unlabelled) graph whose vertices are the vertices of $\Gamma$ and whose edges are those which are labelled 2 in $\Gamma$. Then $\Gamma_2$ can be interpreted as an Erdős-Rényi random graph with edge probability $p(n) = p_2(n)$. Also, a set of vertices in $\Gamma$ spans a $k$-clique whose edges are all 2’s if and only if the corresponding subgraph in $\Gamma_2$ is a $k$-clique, so it suffices to show that a random graph with $n^kp_2^{(k)}$ asymptotically almost surely contains a $k$-clique. This is a standard result (see [2]).

For (iii), we’ll show that $\Gamma$ asymptotically almost surely contains a $k$-clique with one edge labelled $m$ and the rest labelled 2. For a $k$-tuple of vertices $\alpha = (v_1, \ldots, v_k)$, let $X_\alpha$ be the random variable which takes the value 1 if edge $(v_1, v_2)$ is labelled $m$ and all other edges are labelled 2 and takes the value 0 otherwise, and let $X = \sum_\alpha X_\alpha$. Then

$$\frac{\mathbb{E}(X^2)}{\mathbb{E}(X)^2} = 1 + \frac{b_{1,0}}{n} + \sum_{j=1}^{k} \left[ \frac{b_{j,1}}{n^j p_m p_2^{(j) - 1}} + \frac{b_{j,0}}{n^j p_2^{(j)}} \right],$$
where \( b_{j,i} \) is the number of ways two \( k \)-tuples of vertices can share \( i \) \( m \)-edges and \( \binom{j}{2} - i \) \( 2 \)-edges. Since \( n^k p_2^{\binom{j}{2}} \not\to 0 \), we have \( n^j p_2^{\binom{j}{2}} \to \infty \) for \( j < k \), and since additionally \( \frac{p_m}{p_2} \to \infty \), we have \( n^j p_m^\binom{j}{2} - 1 \) \( \binom{p_m}{p_2} \to \infty \); hence, \( \frac{p(x^2)}{E(x^2)} \to 1 \). So, \( \Gamma \) asymptotically almost surely contains a graph on \( k \) vertices corresponding to a finite subgroup of \( \Gamma \); hence \( N(\Gamma) \) asymptotically almost surely contains a \( k \)-simplex and \( \dim(N(\Gamma)) \geq k \). \( \square \)

**Theorem 3.** If \( n^k p_2^{\binom{j}{2}} \to \infty \) and \( n^{k+1} p_2^{\binom{j}{2}+1} \to 0 \), and \( \frac{p_m}{p_2} \not\to \infty \), then asymptotically almost surely \( \dim(N(\Gamma)) = k \)

**Proof.** Since \( n^{k+1} p_2^{\binom{j}{2}+1} \to \infty \) and \( \frac{p_m}{p_2} \not\to \infty \), asymptotically almost surely \( \dim(N(\Gamma)) \leq k \) by part (i) of Theorem 2. But since \( n^k p_2^{\binom{j}{2}} \to \infty \), \( \dim(N(\Gamma)) \geq k \) by part (ii) of Theorem 2. So, \( \dim(N(\Gamma)) = k \). \( \square \)

**Corollary 1.** If \( n^k p_2^{\binom{j}{2}} \to \infty \) and \( \frac{p_m}{p_2} \not\to \infty \), then the nerve of \( W_\Gamma \) asymptotically almost surely has trivial \( H_i \) for all \( i \geq k + 1 \).

**Proof.** Since \( N(\Gamma) \) has dimension at most \( k \), it has trivial \( H_i \) for \( i \geq k + 1 \). \( \square \)

## 5 Non-trivial homology of the nerve

For \( k \geq 1 \), denote by \( Z_k \) the edge-labelled graph on the \( 2k+2 \) vertices \( \{x_0^+, \ldots, x_k^+, x_0^-, \ldots, x_k^-\} \) in which the edges \( (x_i^+, x_{i+1}^-) \), \( (x_i^+, x_{i+1}^+) \), \( (x_k^+, x_0^-) \), and \( (x_k^-, x_0^+) \) are labelled 3, the edges \( (x_i^+, x_i^-) \) are labelled \( \infty \), and all other edges are labelled 2 (in the picture, solid lines indicate edges labelled 3, dashed lines indicate edges labelled \( \infty \), and edges not shown are labelled 2):

![Diagram of edge-labelled graph](image)

**Lemma 1.** The nerve of \( Z_k \) has non-trivial \( H_k \).

**Proof.** Any set of \( k+1 \) vertices of \( Z_k \) gives a simplex in \( Z_k \) iff it does not contain any pair \( x_i^+, x_i^- \), so each \( k \)-simplex of \( N(Z_k) \) is given by \( [x_0^+, \ldots, x_k^+] \) for some choice of \( \pm \) for each \( i \). For each map \( \sigma : \{0, \ldots, k\} \to \{\pm\} \), let \( d_\sigma \) be the number of things that map to \( - \). Consider \( \sum_{\sigma} (-1)^{d_\sigma} [x_0^{\sigma(0)}, \ldots, x_k^{\sigma(k)}] \). In the image under \( \partial_k \), each \( [x_0^{\sigma(0)}, \ldots, x_i^{\sigma(i)}, \ldots, x_k^{\sigma(k)}] \) will appear twice—once from the term from which \( x_i^+ \) was removed and once from the term from which \( x_i^- \) was removed, and these two terms will have opposite sign (since the term in the pre-image which contained \( x_i^- \) had one more negative sign than the term which contained \( x_i^+ \)). So, \( \sum_{\sigma} (-1)^{d_\sigma} [x_0^{\sigma(0)}, \ldots, x_k^{\sigma(k)}] \in \ker(\partial_k) = H_k(N(Z_k)) \) (since \( N(Z_k) \) has no \( k + 1 \)-simplices). \( \square \)
Lemma 2. If an edge-labelled graph $\Gamma$ contains a subgraph isomorphic to $Z_k$ in which the $\{x_i^+\}$ do not have a common neighbor in $\Gamma$, then the nerve of $\Gamma$ retracts onto the nerve of $Z_k$.

Proof. Define the retraction $r$ on the vertices as follows: first, send each of the $x_i^+$ to itself. Then, for any other vertex $v$, there will be some $x_i^+$ which is not adjacent to $v$. Send $v$ to $x_i^-$. To show that $r$ extends simplicially to a retraction $N(\Gamma) \rightarrow N(Z_k)$, it suffices to show that whenever $[v_1, \ldots, v_j]$ is a simplex in $N(\Gamma)$, $[r(v_1), \ldots, r(v_j)]$ is also a simplex in $N(Z_k)$. This is equivalent to showing that $\{r(v_1), \ldots, r(v_j)\}$ induce a complete subgraph in $Z_k$, since every complete subgraph of $Z_k$ generates a finite subgroup of $W_{\Gamma}$. But suppose this is not the case; i.e., suppose there exist $a$ and $b$ such that $r(v_a)$ and $r(v_b)$ are not adjacent in $Z_k$. Then $\{r(v_a), r(v_b)\} = \{x_i^+, x_i^-\}$ for some $i$; without loss of generality, say $r(v_a) = x_i^+$ and $r(v_b) = x_i^-$. Then $v_a = x_i^+$, and $v_b$ is not adjacent to $x_i^+$, a contradiction since $\{v_1, \ldots, v_j\}$ induces a complete subgraph in $\Gamma$. So, $r : N(\Gamma) \rightarrow N(Z_k)$ is a retraction. $\square$

Lemma 3. Let $\Gamma$ be a random edge-labelled graph whose edges are labelled $m$ with probability $p_m(n)$. Then if $n^k p_2^{(k)} \rightarrow \infty$ and $np_3 \not\rightarrow 0$ and $n(1 - p_{\infty})^{k+1} \rightarrow 0$, $\Gamma$ asymptotically almost surely contains a subgraph isomorphic to $Z_k$ in which the $\{x_i^+\}$ do not have a common neighbor in $\Gamma$.

Proof. First, we’ll show that $\Gamma$ asymptotically almost surely contains a subgraph isomorphic to $Z_k$. For each $k$-tuple $\alpha$, let $X_\alpha$ be the random variable which takes the value 1 if $\alpha$ is an instance of $Z_k$ and 0 otherwise. Let $X_k = \sum_\alpha X_\alpha$. Then

$$\frac{\mathbb{E}(X_k^2)}{\mathbb{E}(X_k)^2} = 1 + \sum_{i=1}^{2k+2} \sum_{j=0}^{i/2} \sum_{0}^{i} \frac{b_{i,j-\ell,j}}{n^i p_2^{(i)} - j - \ell} p_3^j p_\infty^\ell,$$

where $b_{i,j-\ell,j}$ is the number of ways two $(2k+2)$-tuples can share $i$ vertices, $\binom{i}{2} - j - \ell$ 2-edges, $j$ 3-edges, and $\ell$ $\infty$-edges. Since $n^i p_2^{(i)} \rightarrow \infty$, also $n^i p_2^{(i)} \rightarrow \infty$ for $i < k$, and since also $\frac{p_2}{p_3} \not\rightarrow 0$ and $n(1 - p_{\infty})^{k+1} \rightarrow 0 \implies p_\infty \not\rightarrow 0$, for each $i$ $n^i p_2^{(i)} - (\frac{p_3}{p_2})^\ell p_\infty^\ell \rightarrow \infty$. So, $\frac{\mathbb{E}(X_k^2)}{\mathbb{E}(X_k)^2} \rightarrow 1$, and hence $\Gamma$ asymptotically almost surely contains a subgraph isomorphic to $Z_k$.

Now, the probability that a particular instance of $Z_k$ is such that the $\{x_i^+\}$ have a common neighbor is $n(1 - p_{\infty})^{k+1} \rightarrow 0$; hence the probability that in a particular instance of $Z_k$ the $\{x_i^+\}$ have no common neighbor $\rightarrow 1$. So, the probability that $\Gamma$ contains a $Z_k$ with no common neighbor for $\{x_i^+\}$ is

$$P(\exists Z_k \text{ with no common neighbor}) = P(\exists Z_k) P(\exists Z_k \text{ with no common neighbor} | \exists Z_k) \geq P(\exists Z_k) P(\text{a particular } Z_k \text{ has no common neighbor}) \rightarrow 1.$$ $\square$

Theorem 4. Let $\Gamma$ be a random edge-labelled graph whose edges are labelled $m$ with probability $p_m(n)$. Then if $n^k p_2^{(k)} \rightarrow \infty$ and $np_3 \not\rightarrow 0$ and $n(1 - p_{\infty})^{k+1} \rightarrow 0$, the nerve of $W_{\Gamma}$ asymptotically almost surely has non-trivial $H_k$. 

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Proof. By Lemma 3, \( \Gamma \) asymptotically almost surely contains a subgraph which is an instance of \( Z_k \) such that the \( \{x_i^+\} \) have no common neighbor. So, by Lemma 2, \( N(\Gamma) \) retracts onto \( N(Z_k) \), which has nontrivial \( H_k \).

\[ \square \]

6 Hyperbolicity

Another property which is of interest for Coxeter groups is hyperbolicity, defined as follows:

Definition 2.  
- For \( \delta \geq 0 \), a metric space \( X \) is \( \delta \)-hyperbolic if every geodesic triangle in \( X \) is “\( \delta \)-slim”; i.e., if the \( \delta \)-neighborhood of the union of two sides of the triangle contains the third side.
- A metric space \( X \) is hyperbolic if it is \( \delta \)-hyperbolic for some \( \delta \).
- A group \( G \) is hyperbolic if its Cayley graph with respect to some (and hence any) finite generating set is a hyperbolic metric space.

In [8], Moussong gives the following condition for when a Coxeter group is hyperbolic in this sense:

Theorem 5 (Moussong). A Coxeter group \( W_\Gamma \) is hyperbolic iff the following hold:

- \( \Gamma \) does not contain a subgraph on 3 or more vertices which generates a Euclidean reflection group in \( W_\Gamma \).
- \( \Gamma \) does not contain two disjoint subgraphs \( S \) and \( T \) such that each of \( S \) and \( T \) generates an infinite subgroup of \( \Gamma \) and every pair of vertices \( s \in S \) and \( t \in T \) is adjacent via an edge labelled 2.

These conditions are easy to check because the finite Coxeter groups and Euclidean reflection groups are well understood. The tables of both are given in the Appendix for reference (the finite Coxeter groups in Table 1 and the Euclidean reflection groups in Table 2).

Lemma 4. If \( n^4 p_2^2 p_\infty^2 \to \infty \), then \( \Gamma \) asymptotically almost surely contains the following subgraph: \[
\begin{array}{c}
\backslash / \\
\backslash / \\
\backslash / \\
\end{array}
\]
where the solid edges are labelled 2 and the dashed edges are labeled \( \infty \).

Proof. If \( X \) is the number of such subgraphs, we have:

\[
\frac{\mathbb{E}(X^2)}{\mathbb{E}(X)^2} = 1 + \frac{b_{1,0}}{n} + \frac{b_{2,1}}{n^2 p_2} + \frac{b_{2,0}}{n^2 p_\infty} + \frac{b_{3,2}}{n^3 p_2^2 p_\infty} + \frac{b_{4,4}}{n^4 p_2^4 p_\infty^2},
\]

where \( b_{i,j} \) is the number of ways two instances of the subgraph in question can share \( i \) vertices and \( j \) 2-edges. To show this \( \to 1 \), we need to show that the denominator of each term besides the first \( \to \infty \). Since \( n^4 p_2^4 p_\infty^2 \to \infty \):

- \( n^2 p_2 \to \infty \) since \( n^4 p_2^4 p_\infty^2 = (n^2 p_2)^2 (p_2^2 p_\infty^2) \)
\[ n^2p_\infty \to \infty \text{ since } n^4p_2^4p_\infty^2 = (n^2p_\infty)^2(p_2^4) \]
\[ n^3p_2^2p_\infty \to \infty \text{ since } n^4p_2^4p_\infty^2 = (n^3p_2^2p_\infty)(n^2p_\infty^2). \]

**Theorem 6.** If \( n^4p_2^4p_\infty^2 \to \infty \), then \( W_\Gamma \) is asymptotically almost surely not hyperbolic.

**Proof.** By Lemma 4, \( \Gamma \) asymptotically almost surely has a subgraph isomorphic to a square with edges labelled 2 whose diagonals are labeled \( \infty \). This corresponds to a direct product of two infinite subgroups, which by Moussong’s condition means that \( W_\Gamma \) is asymptotically almost surely not hyperbolic. \( \square \)

**Theorem 7.** If \( np_3 \to \infty \), then \( W_\Gamma \) is asymptotically almost surely not hyperbolic.

**Proof.** If \( np_3 \to \infty \), then \( \Gamma \) asymptotically almost surely contains a 3-labelled triangle: Let \( \Gamma_3 \) be the (unlabelled) graph whose vertices are the vertices of \( \Gamma \) and whose edges are those which are labelled 3 in \( \Gamma \). Then \( \Gamma_3 \) can be interpreted as an Erdős-Rényi random graph with edge probability \( p(n) = p_3(n) \), and \( \Gamma \) contains a 3-labelled triangle if and only if \( \Gamma_3 \) contains a triangle. So, it suffices to show that a random graph with \( np \to \infty \) contains a triangle, which is a standard result (see [2]). By Moussong’s condition, this means \( W_\Gamma \) is asymptotically almost surely not hyperbolic. \( \square \)

**Theorem 8.** If \( np_4 \to \infty \) and \( np_2 \not\to 0 \), then \( W_\Gamma \) is asymptotically almost surely not hyperbolic.

**Proof.** For each 3-tuple of vertices \( \alpha = (v_1, v_2, v_3) \), let \( X_\alpha \) be the random variable which takes the value 1 if edges \((v_1, v_2)\) and \((v_2, v_3)\) are labelled 4 and edge \((v_1, v_3)\) is labelled 2. Let \( X = \sum_\alpha X_\alpha \). Then
\[
\mathbb{E}(X^2) / \mathbb{E}(X)^2 = 1 + \frac{b_{1,0}}{n} + \frac{b_{2,0}}{n^2p_2} + \frac{b_{2,1}}{n^2p_4} + \frac{b_{3,2}}{n^3p_2^2p_2},
\]
where \( b_{i,j} \) is the number of ways two 3-tuples can share \( i \) vertices and \( j \) 4-edges. This \( \to 1 \) since \( n \to \infty \), \( n^2p_2 = n(np_2) \to \infty \), \( n^2p_4 \to \infty \), and \( n^3p_2^2p_2 = (np_4)^2(np_2) \). So, \( \Gamma \) asymptotically almost surely contains a triangle whose edges are labelled 4, 4, 2; hence it is not hyperbolic by Moussong’s condition. \( \square \)

**Theorem 9.** If \( np_6 \to \infty \) and \( np_3 \not\to 0 \) and \( np_2 \not\to 0 \), then \( W_\Gamma \) is asymptotically almost surely not hyperbolic.

**Proof.** For each 3-tuple of vertices \( \alpha = (v_1, v_2, v_3) \), let \( X_\alpha \) be the random variable which takes the value 1 if edge \((v_1, v_2)\) is labelled 6, edge \((v_2, v_3)\) is labelled 3, and edge \((v_1, v_3)\) is labelled 2. Let \( X = \sum_\alpha X_\alpha \). Then
\[
\mathbb{E}(X^2) / \mathbb{E}(X)^2 = 1 + \frac{b_{1,0,0}}{np_6} + \frac{b_{2,1,0}}{n^2p_6} + \frac{b_{1,0,1}}{n^2p_3} + \frac{b_{2,0,0}}{n^2p_2} + \frac{b_{3,1,1}}{n^3p_6p_3p_2},
\]
where \( b_{v,v_6,e_3} \) is the number of ways two 3-tuples can share \( v \) vertices, \( e_6 \) 6-edges, and \( e_3 \) 3-edges. This \( \to 1 \) since \( n \to \infty \), \( n^2p_6 \to \infty \), \( n^2p_3 \to \infty \), \( n^2p_2 \to \infty \), and \( n^3p_6p_3p_2 = (np_6)(np_3)(np_2) \). So, \( \Gamma \) asymptotically almost surely contains a triangle whose edges are labelled 6, 3, 2; hence it is not hyperbolic by Moussong’s condition. \( \square \)
Lemma 5. If \( np_3 \to 0 \), \( np_2 \not\to \infty \), \( np_4 \not\to \infty \), and \( np_6 \not\to \infty \), and at least one of \( np_2 \) or \( np_4 \to 0 \), then \( W_\Gamma \) asymptotically almost surely contains no subgraph on at least three vertices corresponding to a Euclidean reflection group.

Proof. We will show that each of the Euclidean reflection groups asymptotically almost surely does not appear in \( \Gamma \):

- The expected number of \( \tilde{A}_{k-1} \) is \( n^k p_3^k p_2^{(k-1)} \), so the total number over all \( k \) is
  \[
  \sum_{k=3}^{\infty} n^k p_3^k p_2^{(k-1)} \leq \sum_{k=3}^{\infty} (np_3)^k = \frac{np_3}{1 - np_3} - np_3 - (np_3)^2 \to 0
  \]
since \( np_3 \to 0 \).

- The expected number of \( \tilde{B}_{k-1} \) is \( n^k p_4 p_3^{k-2} p_2^{(k-1)} \), so the total number over all \( k \) is
  \[
  \sum_{k=4}^{\infty} n^k p_4 p_3^{k-2} p_2^{(k-1)} \leq (np_2)^2 \sum_{k=4}^{\infty} (np_3)^{k-2} = (np_2)^2 \left( \frac{np_3}{1 - np_3} - np_3 \right) \to 0
  \]
since \( np_3 \to 0 \) and \( np_2 \not\to \infty \).

- The expected number of \( \tilde{C}_{k-1} \) is \( n^k p_4^2 p_3^{k-2} p_2^{(k-1)} \), so the total number over all \( k \) is
  \[
  \sum_{k=4}^{\infty} n^k p_4^2 p_3^{k-2} p_2^{(k-1)} \leq (np_2)^3 \sum_{k=4}^{\infty} (np_3)^k - 3 = (np_2)^3 \left( \frac{np_3}{1 - np_3} \right) \to 0
  \]
since \( np_3 \to 0 \) and \( np_2 \not\to \infty \).

- The expected number of \( \tilde{D}_{k-1} \) is \( n^k p_3^{k-1} p_2^{(k-1)} \), so the total number over all \( k \) is
  \[
  \sum_{k=6}^{\infty} n^k p_3^{k-1} p_2^{(k-1)} \leq (np_2) \sum_{k=6}^{\infty} (np_3)^{k-1}
  = (np_2) \left( \frac{np_3}{1 - np_3} - np_3 - (np_3)^2 - (np_3)^3 - (np_3)^4 \right) \to 0
  \]
since \( np_3 \to 0 \) and \( np_2 \not\to \infty \).

- The expected number of \( \tilde{B}_2 \) is \( n^3 p_2^2 p_2 = (np_4)^2 (np_2) \to 0 \) since \( np_4 \not\to \infty \) and \( np_2 \not\to \infty \) and one of them \( \to 0 \).

- The expected number of \( \tilde{G}_2 \) is \( n^3 p_6 p_3 p_2 = (np_6)(np_3)(np_2) \to 0 \) since \( np_3 \to 0 \), \( np_6 \not\to \infty \) and \( np_2 \not\to \infty \).

- The expected number of \( \tilde{F}_4 \) is \( n^5 p_4^3 p_2^6 \leq (np_3)^3 (np_2)^2 \to 0 \) since \( np_3 \to 0 \) and \( np_2 \not\to \infty \).
• The expected number of $\tilde{E}_{k-1}$ for $k = 7, 8, 9$ is $n^k p_3^{k-1} p_2^{(k)-k+1} \leq (np_2)(np_3)^{k-1} \to 0$ since $np_3 \to 0$ and $np_2 \not\to \infty$.

The expected number of subgraphs on at least three vertices corresponding to an irreducible affine Coxeter group is the sum of the above, and hence $\to 0$. So, $\Gamma$ asymptotically almost surely contains no such subgraph, and hence asymptotically almost surely contains no subgraph on at least three vertices corresponding to any affine Coxeter group.

**Lemma 6.** If $np_2 \to 0$, then $\Gamma$ asymptotically almost surely does not contain any subgraph corresponding to a direct product of infinite parabolic subgroups of $W_\Gamma$.

**Proof.** Any infinite parabolic subgroup of $W_\Gamma$ must have at least two generators, so it suffices to show $\Gamma$ asymptotically almost surely does not contain any direct product of two edges with any labels $m_1$ and $m_2$. The expected number of such subgraphs is $n^4 p_2^4 p_{m_1} p_{m_2} \leq (np_2)^4 \to 0$ since $np_2 \to 0$.

**Theorem 10.** If $np_3 \to 0$, $np_2 \to 0$, $np_4 \not\to \infty$, and $np_6 \not\to \infty$, then $W_\Gamma$ is asymptotically almost surely hyperbolic.

**Proof.** By Lemmas 5 and 6, $W_\Gamma$ asymptotically almost surely satisfies Moussong’s conditions; hence is hyperbolic.

## 7 The FC-type property

**Definition 3.** A Coxeter group $W_\Gamma$ is said to be of FC-type if every clique in $\Gamma$ generates a finite subgroup of $W_\Gamma$.

Denote by $\Gamma_3$ the (unlabelled) graph whose vertices are the vertices of $\Gamma$ and whose edges are those edges which are labelled 3 in $\Gamma$. Note that $\Gamma_3$ can be interpreted as an Erdős-Rényi random graph with edge probability $p(n) = p_3(n)$. We will say a graph $\Gamma'$ is a 3-labelled cycle if $\Gamma_3$ is a cycle and we will say $\Gamma'$ is a 3-labelled tree if $\Gamma_3$ is a tree. A connected 3-labelled subgraph of $\Gamma$ is a subgraph of $\Gamma$ which is connected in $\Gamma_3$.

**Lemma 7.** If $n^5 p_3^4 \to 0$, then $\Gamma_3$ asymptotically almost surely does not contain a tree on 5 vertices.

**Proof.** There are four trees on 5 vertices: (a) •——•——•——•, (b) •——•——•——•, (c) •——•——•——•, and (d) •——•——•——•. The expected number of graphs isomorphic to (a) is $\frac{1}{2} n^5 p_3^4$, the expected number of graphs isomorphic to (b) is $\frac{1}{2} n^5 p_3^4$, the number of graphs isomorphic to (c) is $\frac{1}{24} n^5 p_3^4$, and the expected number of graphs isomorphic to (d) is $n^5 p_3^4$. So, the total expected number of trees on 5 vertices is $\frac{49}{24} n^5 p_3^4 \to 0$. 

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Lemma 8. If \( np_3 \to 0 \), \( \Gamma_3 \) asymptotically almost surely does not contain any cycle.

Proof. The expected number of \( k \)-cycles is \( \frac{1}{2k} n^k p_3^k \), so the total expected number of cycles is
\[
\sum_{k=3}^{\infty} \frac{1}{2k} n^k p_3^k = \frac{1}{2} \left( \sum_{k=1}^{\infty} \frac{(np_3)^k}{k} - np_3 - \frac{(np_3)^2}{2} \right) = \frac{1}{2} \left( -\ln(1 - np_3) - np_3 - \frac{(np_3)^2}{2} \right) \to 0. \]

\[\square\]

Lemma 9. If \( np_3 \to 0 \) and \( np_2 \not\to \infty \), or if \( np_3 \not\to \infty \) and \( np_2^6 \to 0 \), then \( \Gamma \) asymptotically almost surely does not contain a 3-labelled tree on 5 vertices.

Proof. The expected number of 3-labelled trees on 5 vertices in \( \Gamma \) is \[
\frac{49}{24} n^5 p_3^4 p_2^6 = \frac{49}{24} (np_3)^4 (np_2^6) \to 0. \]

Let \( p_B = \sum_{m=4}^{\infty} p_m \). A “\( B \)-labelled” edge of \( \Gamma \) is one labelled with any number greater than 3.

Theorem 11. If \( n^3 p_B^2 \to 0 \), \( n^3 p_B p_3 \to 0 \), and \( n^5 p_3^4 \to 0 \), then \( A_\Gamma \) is asymptotically almost surely of FC-type.

Proof. Since \( n^5 p_3^4 \to 0 \), the connected 3-labelled subgraphs of \( \Gamma \) are all associated to finite Coxeter groups by the previous theorem. The expected number of adjacent pairs of \( B \)-labelled edges is \( \frac{1}{2} n^3 p_B^2 \to 0 \), and the expected number of adjacent pairs of edges with one labelled \( B \) and the other labelled 3 is \( \frac{1}{2} n^3 p_B p_3 \to 0 \). So, \( B \)-labelled edges asymptotically almost surely do not appear adjacent to any edge labelled \( m \) with \( m \geq 3 \). Hence, the only connected 3-and-\( B \)-labelled subgraphs which appear with positive probability are 3-labelled trees on at most 4 vertices whose other edges in \( \Gamma \) are all labelled 2, as well as \( B \)-labelled edges. Since each of these is associated to a finite Coxeter group, \( A_\Gamma \) is asymptotically almost surely of FC-type.

\[\square\]

Theorem 12. If \( np_B \to 0 \) and \( np_2 \not\to \infty \) and \( np_3 \to 0 \), then \( A_\Gamma \) is asymptotically almost surely of FC-type.

Proof. The expected number of triangles with edges labelled \( B, B, B \) is \( \frac{1}{6} n^3 p_B^3 = (np_B)^3 \to 0 \). The expected number of triangles with edges labelled \( B, B, 3 \) is \( \frac{1}{2} n^3 p_B p_3 = \frac{1}{2} (np_B)^2 (np_3) \to 0 \). The expected number of triangles with edges labelled \( B, 2, 2 \) is \( \frac{1}{2} n^3 p_B p_2^2 = \frac{1}{2} (np_B)^2 (np_2) \to 0 \). So, \( \Gamma \) asymptotically almost surely does not contain any adjacent pair of \( B \)-labelled edges. The expected number of triangles whose edges are labelled \( B, 3, 3 \) is \( \frac{1}{2} n^3 p_B p_3^2 = \frac{1}{2} (np_B)(np_3)^2 \to 0 \). The expected number of triangles whose edges are labelled \( B, 3, 2 \) is \( n^3 p_B p_3 p_2 = (np_B)(np_3)(np_2) \to 0 \). So, \( \Gamma \) asymptotically almost surely does not contain any adjacent pairs of edges with one labelled \( B \) and the other labelled 3. So, the only connected 3-and-\( B \)-labelled subgraphs which appear with positive probability in \( \Gamma \) are those listed in the previous theorem. Since each of these corresponds to a finite Coxeter group, \( A_\Gamma \) is asymptotically almost surely of FC-type.

\[\square\]

Theorem 13. If \( np_3 \to \infty \), \( A_\Gamma \) is asymptotically almost surely not of FC-type.

Proof. For every 3-tuple \( \alpha = (v_1, v_2, v_3) \) of vertices in \( \Gamma \), let \( X_\alpha \) be the random variable which takes the value 1 if \( \alpha \) spans a 3-labelled triangle, and takes the value 0 otherwise. Let \( X =
\[ \sum X_\alpha \] (so the expected number of 3-labelled triangles in \( \Gamma \) is \( \frac{1}{6} \mathbb{E}(X) \)). Then \( \mathbb{E}(X) = n^3 p_3^3 \), so \( \mathbb{E}(X)^2 = n^6 p_3^6 \). \( \mathbb{E}(X^2) = b_{0,0} n^6 p_3^6 + b_{1,0} n^5 p_3^5 + b_{2,1} n^4 p_3^4 + b_{3,3} n^3 p_3^3 \), where \( b_{i,j} \) is the number of ways two 3-tuples can share \( i \) vertices and \( j \) 3-edges. So, \( \frac{\mathbb{E}(X^2)}{\mathbb{E}(X)^2} = \frac{b_{1,0}}{n} + \frac{b_{2,1}}{n^2 p_3} + \frac{b_{3,3}}{n^3 p_3^3} \).

This \( \to 1 \) since \( b_{0,0} = 1 \), \( n \to \infty \), \( n^2 p_3 = n(np_3) \to \infty \), and \( n^3 p_3^3 = (np_3)^3 \to \infty \). \( \square \)

**Theorem 14.** If for \( k = 2 \) or 3, \( np_B \) and \( np_k \) both \( \not\to 0 \) and one of them \( \to \infty \), then \( A_\Gamma \) is asymptotically almost surely not of FC-type.

**Proof.** For every 3-tuple \( \alpha = (v_1, v_2, v_3) \), let \( X_\alpha \) be the random variable which takes the value 1 if edges \( (v_1, v_2) \) and \( (v_2, v_3) \) are labelled \( B \) and edge \( (v_3, v_1) \) is labelled \( k \), and takes the value 0 otherwise. Let \( X = \sum X_\alpha \) (so the expected number of triangles with two \( B \)-labelled edges and one \( k \)-labelled edge is \( \frac{1}{2} \mathbb{E}(X) \)). Then \( \mathbb{E}(X) = n^3 p_B^2 p_k \), so \( \mathbb{E}(X)^2 = n^6 p_B^4 p_k^2 \). \( \mathbb{E}(X^2) = b_{0,0,0} n^6 p_B^4 p_k^2 + b_{1,0,0} n^5 p_B^4 p_k^2 + b_{2,1,0} n^4 p_B^4 p_k^2 + b_{3,2,1} n^3 p_B^4 p_k^2 + b_{3,3} n^3 p_B^2 p_k^2 \), where \( b_{i,j,m} \) is the number of ways two of these triangles can share \( i \) vertices, \( j B \)-edges, and \( m k \)-edges. So, \( \frac{\mathbb{E}(X^2)}{\mathbb{E}(X)^2} = b_{0,0,0} + b_{1,0,0} + b_{2,1,0} + b_{3,2,1} + b_{3,3} \to 1 \) since \( b_{0,0,0} = 1 \), \( n \to \infty \), \( n^2 p_B = n(np_B) \to \infty \), \( n^2 p_k = n(np_k) \to \infty \), and \( n^3 p_B^2 p_k = (np_B)^2(np_k) \to \infty \). \( \square \)

### 8 Appendix: Coxeter diagrams of irreducible finite and Euclidean reflection groups

In the following tables, we use the Dynkin diagram convention, so unlabelled edges should be interpreted as 3-labelled in the defining graph and missing edges should be interpreted as 2-labelled in the defining graph.

- **A\(_n\)**
  - \( \bullet - \bullet - \cdots - \bullet \)
- **B\(_n\)**
  - \( 4 \)
- **D\(_n\)**
  - \( \bullet - \bullet - \cdots - \bullet \)
- **I\(_2(m)\)**
  - \( m \)
- **H\(_3\)**
  - \( 5 \)
- **H\(_4\)**
  - \( 5 \)

- **F\(_4\)**
  - \( \bullet - \bullet - \bullet - \bullet \)
- **E\(_6\)**
  - \( \bullet \)
- **E\(_7\)**
  - \( \bullet \)
- **E\(_8\)**
  - \( \bullet \)

**Figure 1:** Coxeter diagrams for irreducible finite Coxeter groups
Figure 2: Coxeter diagrams for Euclidean reflection groups

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