$hp$-version collocation method for a class of nonlinear Volterra integral equations of the first kind

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Abstract

In this paper, we present a collocation method for nonlinear Volterra integral equation of the first kind. This method benefits from the idea of $hp$-version projection methods. We provide an approximation based on the Legendre polynomial interpolation. The convergence of the proposed method is completely studied and an error estimate under the $L^2$-norm is provided. Finally, several numerical experiments are presented in order to verify the obtained theoretical results.

Keywords: nonlinear operator, first kind Volterra integral equation, $hp$-collocation method, error analysis.

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1 Introduction

Volterra integral equations of first kind are important in theory and application, for example solving an exterior homogeneous wave equation with Dirichlet boundary condition leads to a time dependent single layer boundary integral equation which can be seen as a Volterra integral equation of the first kind [15]. They could be categorized according to their kernels in two types. The first equations are those with well-behaved kernels and the second types have unbounded kernels at $s = t$, like Abel’s integral equation [21].

This paper deals with the numerical solution of nonlinear Volterra integral equation of first kind

$$Ku(t) := \int_0^t \kappa(s, t)\psi(s, u(s))ds = f(s), \quad 0 \leq t \leq T < \infty.$$  \hspace{1cm} (1)

A general form of Eq. (1) can be expressed as follows:

$$Ku(t) := \int_0^t \kappa(s, t, u(s))ds = f(t), \quad t \in I := [0, T].$$ \hspace{1cm} (2)

A lot of theoretical and numerical researches have been devoted to the second kind Volterra integral equations. The theoretical study of them is given in [16] and a comprehensive numerical investigation based on collocation method is presented in [6]. But a few numbers of them deal with the numerical solution of the first kind integral equations, especially the nonlinear integral equations.

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Eggermont has studied the numerical solution of the Volterra integral equation of the first kind in a series of papers [11, 12, 13]. For the linear case, the collocation scheme has been analyzed as a projection method using piecewise polynomials [11]. In addition, the super-convergence property of the collocation projection method is studied in [13]. The quadrature method for nonlinear Volterra integral equation has been studied in [12] by considering it as a collocation-projection method. Furthermore, an asymptotic optimal error estimation and a comprehensive study of the zero stability of the method are presented.

Brunner et al. have given a comprehensive convergence analysis for collocation, (quadrature) discontinuous Galerkin and full discontinuous Galerkin methods for linear Volterra integral equation of the first kind extensively [3, 19, 20]. Furthermore, the global order of convergence for the collocation method in the space of piecewise polynomials of degree \( m \geq 0 \), with jump discontinuities on the set of knots is studied in [4, 5]. Recent studies for the linear Abel’s integral equations based on finite element Galerkin method are analyzed in [14, 29]. In mentioned works, to obtain an efficient approximation, one should increase the number of mesh points (\( h \)-version) or the degree of polynomials in the expansion (\( p \)-version).

In order to have the advantages of both \( h \)- and \( p \)-versions, it is possible to vary time steps and approximation orders simultaneously which is called \( hp \)-version methods. Kaushen and Brunner by following similar ideas have studied the convergence analysis of two-step collocation method based on a Runge-Kutta approach for the first kind Volterra integral equation [19]. A multi-step collocation method for the second kind Volterra integral equations and its linear stability properties have been studied by Conte and Paternoster [7]. Recently, Zhang and Liang [34] have modified these approaches and introduced a type of multi-step collocation method by using the Lagrange polynomials in each subinterval for the first kind linear Volterra equation. In such methods, the resulting system can be solved efficiently, in addition the flexibility of method makes it more suitable for large \( T \) [31].

Due to the efficiency and accuracy, the \( hp \)-version Galerkin and collocation methods have received considerable attentions. For example, the \( hp \)-version of discontinuous Galerkin and Petrov-Galerkin have been studied for integro-differential equations of Volterra types, for more details see [24, 32]. Sheng et al. have introduced a multi-step Legendre-Gauss spectral collocation method and given a full analysis of convergence in \( L^2 \)-norm for the nonlinear Volterra integral equations of the second kind [26]. This approach has been extended to the Volterra integral and integro-differential equations with vanishing delays [31, 27]. Locally varying time steps makes these methods popular for investigating the numerical solution of integral equations with weakly singular kernels, this idea is completely studied in [30].

The aim of this paper is to analyze the \( hp \)-version Legendre collocation method for the first kind nonlinear Volterra integral equations. An important aspect of this method is its flexibility with respect to the step-size and the order of polynomials in each sub-interval. As we will see in the numerical experiments, the proposed collocation method works for the long-time integration intervals and also it gives remarkable results for the approximation of the equations with non-smooth solutions. The notations used in this paper are borrowed from [17, 20].

This paper is organized in the following way. In Section 2 we give some regularity results for the nonlinear Volterra integral equation of the first kind. Section 3 is devoted to the description of the \( hp \)-version collocation method for the first kind nonlinear Volterra integral equation. In Section 4 an error analysis of the proposed method is provided in \( L^2 \) spaces. Finally, in order to show the applicability and efficiency of the method and compare with other methods, several examples with smooth and non-smooth solutions are illustrated in Section 5.

### 2 Well-posedness of the problem

In order to introduce a numerical scheme for the solution of the Eq. [10], knowledge of the smoothness properties of the exact solution is necessary. The existence of the solution for this kind of equation is investigated by Banach and Schauder’s fixed point theorems, for more detail see [6, 15] and references
Proof. At the first step, we differentiate both sides of Eq. (1) in the sense of distributional derivative, hence from the assumptions (i-iii), it reads that

\[ u \] and (iii), we deduce that the function \( u \) and \( \psi \) are continuous w.r.t. \( u \). Therefore, without loss of generality for \( m > n \), the function \( u_n \) is well-defined and belongs to \( H^{m-1}(I) \). Now using induction hypothesis, \( u_n \) is well-defined and belongs to \( H^{m-1}(I) \). From the assumptions (i), (ii), and (iii), we deduce that the function

\[ \int_0^t \frac{\kappa(s, t)}{\kappa(t, t)} \psi(s, u_n(s))ds, \quad n \geq 0. \]

By the assumptions (iii) and (iv), \( \psi(t, u(t)) \) is strictly monotonic continuous function with respect to \( u \). So by considering the Inverse Theorem [8, p. 68], \( u_0 \) is well-defined and belongs to \( H^{m-1}(I) \). Now, assuming the hypothesis, \( u_n \) is well-defined and belongs to \( H^{m-1}(I) \). From the assumptions (i), (ii), and (iii), we deduce that the function

\[ \int_0^t \frac{\kappa(s, t)}{\kappa(t, t)} \psi(s, u_n(s))ds, \]

belongs to \( H^{m-1}(I) \). Hence by the Inverse Theorem, \( u_{n+1} \in H^{m-1}(I) \).

Using the assumptions (iv) and (v) one can conclude that

\[ |u_{n+1} - u_n| \leq \left( \frac{dL}{M} \right)^n n! \max_{s \in I} |u_1(s) - u_0(s)|, \]

where \( L \) is the Lipschitz constant in the assumption (v) and \( d := \max \left\{ \frac{1}{\kappa(t,s)} \right\} | (t, s) \in I \times I \} \). Therefore, without loss of generality for \( m > n \),

\[ |u_m - u_n| \leq \sum_{i=n}^{m-1} |u_{i+1} - u_i| \leq \|u_1(s) - u_0(s)\| \infty \sum_{i=n}^{m-1} \left( \frac{dL}{M} \right)^i \frac{1}{i!}. \]
The term $$\sum_{i=0}^{\infty} \left( \frac{dLT}{M} \right)^i \frac{1}{i!}$$ is convergent, so the Cauchy sequence $$\{u_n\}$$ is convergent uniformly to $$\lim_{n \to \infty} u_n(t) = u(t),$$

where $$u(t)$$ belongs to $$H^{m-1}(I)$$. This result follows from the fact that $$u_n(t) \in H^{m-1}(I)$$. ■

3 Numerical scheme

In this section, we propose an hp-version Legendre collocation method for Volterra integral equation of the first kind. To make the paper self-contained, some basic properties of the shifted Legendre polynomial interpolation are introduced in the following subsection.

3.1 Preliminaries

The Legendre-Gauss interpolation operator. We denote $$\{t_i, w_i\}_{i=0}^{M}$$ as the Legendre-Gauss quadrature nodes in $$(-1, 1)$$ and their corresponding weights. Set $$\Lambda := (-1, 1]$$ and let $$P_M(\Lambda)$$ be the set of polynomials of degree at most $$M$$. For any function $$\phi \in P_{2M+1}(\Lambda)$$, the following equality could be obtained from the main property of Gauss quadrature,

$$\int_{\Lambda} \phi(t) dt = \sum_{j=0}^{M} w_j \phi(t_j).$$

Thanks to the above equation, for any $$\phi \psi \in P_{(2M+1)}(\Lambda)$$ and $$\phi \in P_M(\Lambda),$$

$$(\phi, \psi) = \langle \phi, \psi \rangle_M,$$  

where $$(., .)$$ denotes the inner product of $$L^2(\Lambda)$$ and the discrete inner product

$$\langle u, v \rangle_M := \sum_{i=0}^{M} w_j u(t_j)v(t_j), \quad \|v\|_M = \langle v, v \rangle_M^{\frac{1}{2}}.$$  

Let define $$I_M^t : C(\Lambda) \to P_M(\Lambda)$$ as the Legendre-Gauss interpolation operator in the $$t$$-direction with the following property

$$I_M^t v(t_j) = v(t_j), \quad 0 \leq j \leq M.$$  

Regarding the relation (5), for any $$\phi \in P_{M+1}(\Lambda),$$

$$\langle I_M^t v, \phi \rangle = \langle I_M^t v, \phi \rangle_M = \langle v, \phi \rangle_M.$$  

Let $$L_i(t)$$ be the Legendre polynomial of degree $$i$$. Since the set of Legendre polynomials form an orthogonal complete set in $$L^2(\Lambda)$$, namely, a function $$v \in L^2(\Lambda)$$ can be represented as

$$v(t) = \sum_{i=0}^{\infty} c_i L_i(t),$$

so $$I_M^t v(t)$$ may expand as

$$I_M^t v(t) = \sum_{i=0}^{M} \hat{v}_i L_i(t),$$  

by using the orthogonality condition of the Legendre polynomials

$$\hat{v}_i = \frac{2i+1}{2} \langle I_M^t v, L_i \rangle = \frac{2i+1}{2} \langle I_M^t v, L_i \rangle_M, \quad i = 0, 1, \ldots, M.$$
3.2 Description of the numerical scheme

For a fixed integer \( N \), let \( I_k := \{ t_n : 0 = t_0 < t_1 < \cdots < t_N = T \} \) be as a mesh on \( I \), \( h_n := t_n - t_{n-1} \) and \( h_{\max} = \max_{1 \leq n \leq N} h_n \). Moreover, denote \( u^n(t) \) the solution of Eq. (1) on the \( n \)-th subinterval of \( I \), namely,

\[
u^n(t) = u(t), \quad t \in I_n := (t_{n-1}, t_n], \quad n = 1, 2, \cdots, N.
\]

By above mesh, we rewrite the Eq. (1) as

\[
\int_0^{t_{n-1}} \kappa(s, t) \psi(s, u(s)) \, ds + \int_{t_{n-1}}^t \kappa(s, t) \psi(s, u(s)) \, ds = f(t),
\]

then for any \( t \in I_n \), this equation can be written as

\[
\int_{t_{n-1}}^t \kappa(s, t) \psi(s, u^n(s)) \, ds = f(t) - \sum_{k=1}^{n-1} \int_{t_{k-1}}^{t_k} \kappa(\xi, t) \psi(\xi, u^k(\xi)) \, d\xi.
\]

The problem (7) is converted into an equivalent problem in \( \Lambda := (-1, 1) \). For this aim, we transfer \( t \in I_n \) to \( x \in \Lambda \) by

\[
t = \frac{h_n x + t_{n-1} + t_n}{2},
\]

in other words, we have

\[
\int_{t_{n-1}}^{h_n x + t_{n-1} + t_n} \kappa\left(s, \frac{h_n x + t_{n-1} + t_n}{2}\right) \psi\left(s, u^n(s)\right) \, ds = f\left(h_n x + t_{n-1} + t_n\right)
\]

\[
- \sum_{k=1}^{n-1} \int_{t_{k-1}}^{t_k} \kappa\left(\xi, \frac{h_n x + t_{n-1} + t_n}{2}\right) \psi\left(\xi, u^k(\xi)\right) \, d\xi.
\]

Furthermore, the integral intervals \( I_k \) and \( \left(t_{n-1}, \frac{h_n x + t_{n-1} + t_n}{2}\right) \) can convert to \( \Lambda \) and \((-1, x]\), respectively under the following transformation

\[
\xi = \frac{h_n \eta + t_{k-1} + t_k}{2}, \quad s = \frac{h_n \tau + t_{n-1} + t_n}{2}.
\]

Hence, Eq. (8) becomes

\[
\frac{h_n}{2} \int_{-1}^x \kappa\left(\frac{h_n \tau + t_{n-1} + t_n}{2}, \frac{h_n x + t_{n-1} + t_n}{2}\right) \psi\left(\frac{h_n \tau + t_{n-1} + t_n}{2}, u^n\left(\frac{h_n \tau + t_{n-1} + t_n}{2}\right)\right) \, d\tau
\]

\[
= f\left(\frac{h_n x + t_{n-1} + t_n}{2}\right)
\]

\[
- \sum_{k=1}^{n-1} \frac{h_k}{2} \int_{\Lambda} \kappa\left(\frac{h_k \eta + t_{k-1} + t_k}{2}, \frac{h_n x + t_{n-1} + t_n}{2}\right) \psi\left(\frac{h_k \eta + t_{k-1} + t_k}{2}, u^k\left(\frac{h_k \eta + t_{k-1} + t_k}{2}\right)\right) \, d\eta.
\]

Finally, using the linear transform

\[
\tau = \sigma(x, \theta) := \frac{1 + x}{2} + \theta \frac{1 - x}{2},
\]

Eq. (9) reads

\[
\frac{h_n}{4} (1 + x) \int_{\Lambda} \tilde{\kappa}^n(\sigma(x, \theta), x) \psi\left(\sigma(x, \theta), \tilde{u}^n(\sigma(x, \theta))\right) \, d\theta = \]

\[
\hat{f}(x) - \sum_{k=1}^{n-1} \frac{h_k}{2} \int_{\Lambda} \tilde{\kappa}^k(\eta, x) \psi(\eta, \tilde{u}^k(\eta)) \, d\eta, \quad x \in \Lambda,
\]

(10)
where

\[
\tilde{u}^k(x) = u^k \left( \frac{h_k x + t_{k-1} + t_k}{2} \right),
\]
\[
\tilde{f}(x) = f \left( \frac{h_n x + t_{n-1} + t_n}{2} \right),
\]
\[
\tilde{\kappa}^k(\eta, x) = \kappa \left( \frac{h_k \eta + t_{k-1} + t_k}{2}, \frac{h_n x + t_{n-1} + t_n}{2} \right).
\]

### 3.3 The hp-version of Legendre-Gauss collocation method

In order to seek a solution \( \tilde{u}_{M_n}^M(x) \in \mathcal{P}_{M_n}(\Lambda) \) of Eq. (11) by hp-collocation method, at first step this equation will be fully discretized as

\[
\mathcal{T}_{M_n}^x \left( \frac{h_n}{2} (1 + x) \int_\Lambda \mathcal{T}_{M_n}^\eta \left( \tilde{\kappa}^n(\sigma(x, \theta), x) \psi(\sigma(x, \theta), \tilde{u}_{M_n}^n(\sigma(x, \theta))) \right) d\theta \right)
\]
\[
= \mathcal{T}_{M_n}^x \left( \tilde{f}(x) - \sum_{k=1}^{n-1} \frac{h_k}{2} \int_\Lambda \mathcal{T}_{M_k}^\eta \left( \tilde{\kappa}^k(\eta, x) \psi(\eta, \tilde{u}_{M_k}^k(\eta)) \right) d\eta \right), \quad x \in \Lambda,
\]

where

\[
\tilde{u}_{M_n}^M(x) = \sum_{p=0}^{M_n} \tilde{u}_{p}^n L_p(x),
\]

\[
\mathcal{T}_{M_n}^x \mathcal{T}_{M_n}^\eta \left( (1 + x) \tilde{\kappa}^n(\sigma(x, \theta), x) \psi(\sigma(x, \theta), \tilde{u}_{M_n}^n(\sigma(x, \theta))) \right) = \sum_{p, q=0}^{M_n, M_k} a_{pq}^n L_p(x) L_q(\theta), \quad (13)
\]

\[
\mathcal{T}_{M_n}^x \mathcal{T}_{M_n}^\eta \left( \tilde{\kappa}^k(\eta, x) \psi(\eta, \tilde{u}_{M_k}^k(\eta)) \right) = \sum_{p, q=0}^{M_n, M_k} b_{pq}^k L_p(x) L_q(\eta),
\]

and

\[
\mathcal{T}_{M_n}^M \tilde{f}(x) = \sum_{p=0}^{M_n} c_p^n L_p(x). \quad (14)
\]

Then, we have

\[
\int_\Lambda \mathcal{T}_{M_n}^x \mathcal{T}_{M_n}^\eta \left( (1 + x) \tilde{\kappa}^n(\sigma(x, \theta), x) \psi(\sigma(x, \theta), \tilde{u}_{M_n}^n(\sigma(x, \theta))) \right) d\theta = \sum_{p, q=0}^{M_n, M_k} a_{pq}^n L_p(x) \int_\Lambda L_q(\theta) d\theta \]
\[
= 2 \sum_{p=0}^{M_n} a_{p0}^n L_p(x), \quad (15)
\]

and similarly,

\[
\int_\Lambda \mathcal{T}_{M_n}^x \mathcal{T}_{M_n}^\eta \left( \tilde{\kappa}^k(\eta, x) \psi(\eta, \tilde{u}_{M_k}^k(\eta)) \right) d\eta = \sum_{p, q=0}^{M_n, M_k} b_{pq}^k L_p(x) \int_\Lambda L_q(\eta) d\eta \]
\[
= 2 \sum_{p=0}^{M_n} b_{p0}^k L_p(x). \quad (16)
\]
We denote the Legendre-Gauss quadrature nodes in $(-1, 1)$ and the corresponding weights by $\{x_{k,i}, w_{k,i}\}_{i=0}^{M_k}$ which are related to $k$-th subinterval. It can be determined from Eqs. (14)-(16) that

$$
\tilde{u}_p^n = \frac{2p+1}{2} \sum_{i=0}^{M_n} \tilde{u}_n^{n, M_n}(x_{n,i}) L_p(x_{n,i}) w_{n,i},
$$

$$
a_{p0}^n = \frac{2p+1}{4} \sum_{i,j=0}^{M_n} (1+x_{n,i}) K^n(\sigma(x_{n,i}, x_{n,j}), x_{n,i}) \psi(\sigma(x_{n,i}, x_{n,j}), \tilde{u}_n^{n, M_n}(\sigma(x_{n,i}, x_{n,j}))) L_p(x_{n,i}) w_{n,i} w_{n,j},
$$

$$
b_k^p = \frac{2p+1}{4} \sum_{i=0}^{M_n} \sum_{j=0}^{M_n} K^k(x_{n,i}, x_{n,j}) \psi(x_{n,i}, \tilde{u}_n^{n, M_n}(x_{n,i})) L_p(x_{n,i}) w_{n,i} w_{n,j},
$$

$$
c_p^n = \frac{2p+1}{2} \sum_{i=0}^{M_n} \tilde{f}_n^{n, M_n}(x_{n,i}) L_p(x_{n,i}) w_{n,i}.
$$

With equations (14) - (16) and (12) reads

$$
0 = \sum_{p=0}^{M_n} c_p^n L_p(x) + \sum_{p=0}^{M_n} \tilde{a}_p^n L_p(x) + \sum_{p=0}^{M_n} \tilde{b}_p^n L_p(x),
$$

where

$$
\tilde{a}_p^n = \frac{1}{2} h_n a_{p0}, \quad \tilde{b}_p^n = \sum_{k=1}^{n-1} h_k b_{p0}.
$$

Consequently, we compare the expansion coefficient to obtain

$$
0 = \tilde{a}_p^n + \tilde{b}_p^n - c_p^n, \quad 0 \leq p \leq M_n.
$$

(18)

To evaluate the unknown coefficients $\tilde{a}_p^n$ for any given $n$, we solve the nonlinear system (18) with the Newton’s iteration method. Finally the approximate solution obtain as

$$
u_n^N(x) = \sum_{k=1}^{N} \sum_{p=0}^{M_k} \tilde{u}_p^k L_p \left( \frac{2x - t_{k-1} - t_k}{h_k} \right).
$$

(19)

### 4 Error analysis

In this section, we should give functional framework and for this aim some weighted Sobolev spaces are defined.

Let us define the weight function $\chi^{(\alpha, \beta)}(x) = (1-x)^\alpha(1+x)^\beta$ for $\alpha, \beta > -1$. For $r \in \mathbb{N}$, $H_{\chi^{(\alpha, \beta)}}^r(\Lambda)$ is a weighted Sobolev space defined by

$$
H_{\chi^{(\alpha, \beta)}}^r(\Lambda) = \left\{ v \mid v \text{ is measurable and } \|v\|_{r, \chi^{(\alpha, \beta)}} < \infty \right\},
$$

equipped with the following norm

$$
\|v\|_{r, \chi^{(\alpha, \beta)}} = \left( \sum_{k=0}^{r} \|\partial_x^k v\|_{\chi^{(\alpha+k, \beta+k)}}^2 \right)^{\frac{1}{2}},
$$

and semi-norm

$$
|v|_{r, \chi^{(\alpha, \beta)}} = \|\partial_x^r v\|_{\chi^{(\alpha+r, \beta+r)}},
$$

where $\|v\|_{\chi^{(\alpha, \beta)}}$ is an appropriate norm for the space $L_{\chi^{(\alpha, \beta)}}^2(\Lambda)$.  

Lemma 2 (\cite{18}) (Grönwall inequality) Assume that there are numbers $\alpha$, $\beta_l \geq 0$ ($l = 0, 1, \ldots, n - 1$) and $0 \leq M_0 < 1$ such that

$$0 \leq \varepsilon_n \leq \alpha + \sum_{l=0}^{n-1} \beta_l \varepsilon_l + M_0 \varepsilon_n, \quad n \geq 1.$$  

Then the quantities $\varepsilon_n$ fulfill the following estimate for $n \geq 0$

$$\varepsilon_n \leq \frac{\alpha}{1 - M_0} + \exp\left(\sum_{l=0}^{n-1} \frac{\beta_l}{1 - M_0}\right).$$

In the following, some theoretical results regarding the convergent of the method is stated.

Lemma 3 For any $\tilde{v} \in H^m(\Lambda)$ with integer $1 \leq m \leq M_n + 1$ and $1 \leq n \leq N$, 

$$\|\tilde{v} - I_{\tilde{v}}\|_{\Lambda}^2 \leq cM^{-2m} \|\partial_x^m \tilde{v}\|_{H^m(\Lambda)}^2 \leq cM^{-2m} (M_n - 2m) \|\partial_x^m \tilde{v}\|_{L^2(I_n)},$$

where $\tilde{v}(x) = v(t)|_{t=h_n+1/n+1}$. 

Proof. First inequality is proved in \cite{17}. For the second inequality, we have 

$$\|\tilde{v} - I_{\tilde{v}}\|_{\Lambda}^2 \leq cM^{-2m} \int_{\Lambda} (\partial_x^m \tilde{v})^2 (1 - x^2)^m dx$$

$$\leq cM^{-2m} \int_{I_n} (\partial_x^m \tilde{v})^2 (t - t_{n-1})^m (t_n - t)^m dt$$

$$\leq cM^{-2m} (M_n - 2m) \|\partial_x^m \tilde{v}\|_{L^2(I_n)}.$$

Theorem 4 Let $\tilde{u}_n$ be the solution Eq. (10) and $\tilde{u}_{M_n}^n$ be the solution of Eq. (13). Assume that $\kappa(s,t) \in C^m(I \times I)$, $f(t)|_{I_n} \in H^m(I_n)$, $u(t)|_{I_n} \in H^m(I_n)$ and $\psi : I_n \times H^m(I_n) \to H^m(I_n)$ for $n = 1, 2, \ldots, N$. and $\psi(., u)$ fulfills the Lipschitz condition with respect to the second variable, i.e.,

$$|\psi(., u_1) - \psi(., u_2)| \leq \gamma|u_1 - u_2|, \quad \gamma \geq 0.$$  

Then, for any $1 \leq n \leq N$, $B_1 = B_2 + B_3$, 

$$\|B_1\|^2 \leq cM^{-2m} \|\partial_t^m f\|_{L^2(\Lambda)}^2 + cT \sum_{k=1}^{n-1} \left( \gamma^2 h_k \|e_k\|^2 + \sum_{k=1}^{n-1} \gamma^2 h_k \|e_k\|^2 \right) + cT \sum_{k=1}^{n-1} \|\psi(., u)\|^2_{H^m(I_k)},$$

where

$$B_1 = \frac{h_n}{2} (1 + x) \kappa^n(\sigma(x, .), x, \psi(\sigma(x, .), \tilde{u}_n^n(\sigma(x, .)))) - \frac{h_n}{2} I_{\tilde{u}_n}^n ((1 + x) \kappa^n(\sigma(x, .), x, \psi(\sigma(x, .), \tilde{u}_n^n(\sigma(x, .)))) \|M_n)$$

$$B_2 = \tilde{f}(x) - I_{\tilde{f}} \tilde{f}(x),$$

$$B_3 = -\sum_{k=1}^{n-1} \frac{h_k}{2} (\tilde{k}^k(., x), \psi(., \tilde{u}_k^k(., x))) + \sum_{k=1}^{n-1} \frac{h_k}{2} I_{\tilde{u}_k}^k (\tilde{k}^k(., x), \psi(., \tilde{u}_k^k(., x))) \|M_k)$$

and $e_k = \tilde{u}_k - \tilde{u}_{M_k}^k$, $1 \leq k \leq N$. 

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Proof. The main problem (11) can be converted into the interval Λ as Eq. (10). In the present scheme, we approximate it by Eq. (18). Regarding Eq. (5), we have from Eqs. (10) and (18) that

\[ \frac{h}{4}((1 + x)\tilde{r}^n(\sigma(x, .), x), \psi(\sigma(x, .), \tilde{u}^n(\sigma(x, .))) = \tilde{f}(x) - \sum_{k=1}^{n-1} \frac{h}{4}((\tilde{r}^k(., x), \psi(., \tilde{u}^k(.))), \quad x \in \Lambda, \]  

(24)

and

\[ \frac{h}{4}I_{M_n}(1 + x)\tilde{r}^n(\sigma(x, .), x)\psi(\sigma(x, .), \tilde{u}_{M_n}^n(\sigma(x, .)))_{M_n} = I_{M_n}\tilde{f}(x) - \sum_{k=1}^{n-1} \frac{h}{4}I_{M_n}(\tilde{r}^k(., x)\psi(., \tilde{u}_{M_n}^k(.)\rangle_{M_k}. \]  

(25)

By subtracting (25) from (24), we have

\[ B_1 = B_2 + B_3, \]  

(26)

where

\[ B_1 = \frac{h}{4}((1 + x)\tilde{r}^n(\sigma(x, .), x), \psi(\sigma(x, .), \tilde{u}^n(\sigma(x, .))) - \frac{h}{4}I_{M_n}(1 + x)\tilde{r}^n(\sigma(x, .), x)\psi(\sigma(x, .), \tilde{u}_{M_n}^n(\sigma(x, .)))_{M_n}, \]

\[ B_2 = \tilde{f}(x) - I_{M_n}\tilde{f}(x), \]

\[ B_3 = -\sum_{k=1}^{n-1} \frac{h}{4}(\tilde{r}^k(., x), \psi(., \tilde{u}^k(.) + \sum_{k=1}^{n-1} \frac{h}{4}I_{M_n}(\tilde{r}^k(., x), \psi(., \tilde{u}_{M_n}^k(.)\rangle_{M_k}. \]  

(27)

In order to obtain an estimate error for the term \( B_1 \), we need error bounds for \( \|B_i\|, \ i = 2, 3. \) First using Lemma 3 we infer that

\[ \|B_2\|^2 = \|\tilde{f}(x) - I_{M_n}\tilde{f}(x)\|^2 \leq ch_n^{2m-1}M_n^{-2m}\|\nabla^m f(t)\|^2_{L^2(Ln)}. \]  

(28)

With the same argument in (26) about \( \|B_3\| \), one can conclude that

\[ \|B_3\|^2 \leq cT\sum_{k=1}^{n-1} \left( \gamma^2 h_k^2\|e_k\|^2 + ch_k^{2m}M_k^{-2m}(\gamma^2\|\nabla^m u\|^2_{L^2(Lk)}) \right. \]

\[ + \|\psi(., u)\|^2_{H^m(I_k)} \right) + cTh_n^{2m}M_n^{-2m}\sum_{k=1}^{n-1} \|\psi(., u)\|^2_{H^m(I_k)}. \]  

(29)

So the desire result follows from \( \|B_1\|^2 \leq 2(\|B_2\|^2 + \|B_3\|^2). \)  

Theorem 5 Assume that the Fréchet derivative of the operator \( K\) with respect to \( u \) satisfies at \( ||(K'u)(t)|| \geq \alpha > 0 \), then under the hypothesis of the Theorem 4, for sufficiently small \( h_{\text{max}} \) the following error estimate is obtained

\[ \|e_n\|^2 = \|\tilde{u}^n - \tilde{u}_{M_n}^n\|^2 \leq \frac{c}{\beta^2} \exp(c\gamma^2T^2) \left( h_n^{2m-1}M_n^{-2m}\|\nabla^m f\|^2_{L^2(Ln)} + h_n^{2m-3}M_n^{-2m}(\gamma^2 h_n^2\|\nabla^m u\|^2_{L^2(Ln)}) \right. \]

\[ + \|\psi(., u)\|^2_{H^m(I_n)} \left. + h_n^{2m-1}M_n^{-2m}\|\psi(., u_{M_n}^N)\|^2_{H^m(I_n)} \right) + T\sum_{k=1}^{n-1} \left( ch_k^{2m}M_k^{-2m}(\gamma^2\|\nabla^m u\|^2_{L^2(Lk)} + \|\psi(., u)\|^2_{H^m(I_k)}) \right. \]

\[ + h_n^{2m}M_n^{-2m}\sum_{k=1}^{n-1} \|\psi(., u)\|^2_{H^m(I_k)} \right). \]  

(30)
Proof. For convenience, let
\[ F(x, \tau, u(\tau)) := \frac{h_n}{2} \hat{\kappa}(x, \tau)\psi(\tau, u(\tau)), \quad \tau \in (-1, x]. \tag{31} \]
and \(G(u) := \int_{-1}^{x} F(x, \tau, \bar{u})d\tau\). Under the mean value theorem \(\text{[11 p. 229]}\), we have
\[ \int_{-1}^{x} F(x, \tau, \bar{u}^n(\tau))d\tau - \int_{-1}^{x} F(x, \tau, \bar{u}_{M_n}^n(\tau))d\tau = G'(\xi)(\bar{u}^n(x) - \bar{u}_{M_n}^n(x)), \tag{32} \]
where \(\xi \in (\min\{\bar{u}^n, \bar{u}_{M_n}^n\}, \max\{\bar{u}^n, \bar{u}_{M_n}^n\})\) and \(G'\) denotes the Fréchet derivative, namely,
\[ G'(u)h(x) = \int_{-1}^{x} \hat{\kappa}(x, \tau) \frac{\partial \psi(\tau, u(\tau))}{\partial u} h(\tau)d\tau. \]
Under the assumption \(\|K^tu(t)\| \gg 0\), we deduce that \(\delta := |G'(u)h(t)| \gg 0\). Therefore,
\[ \left| \bar{u}^n(x) - \bar{u}_{M_n}^n(x) \right| \leq \frac{1}{\delta} \left| \int_{-1}^{x} F(x, \tau, \bar{u}^n(\tau))d\tau - \int_{-1}^{x} F(x, \tau, \bar{u}_{M_n}^n(\tau))d\tau \right|. \tag{33} \]
Since
\[ \int_{-1}^{x} F(x, \tau, \bar{u}^n(\tau))d\tau = \frac{h_n}{4} \int_{\Lambda} (1 + x) \hat{\kappa}(x, \sigma(x, \eta))\psi(\sigma(x, \eta), \bar{u}^n(\sigma(x, \eta)))d\eta, \tag{34} \]
then from (33), we deduce that
\[ \left| e_n(x) \right| = \left| \bar{u}^n(x) - \bar{u}_{M_n}^n(x) \right| \leq \frac{1}{\delta} \frac{h_n}{4} (1 + x) \int_{\Lambda} \hat{\kappa}(x, \sigma(x, \eta))\psi(\sigma(x, \eta), \bar{u}^n(\sigma(x, \eta)))d\eta 
- \hat{\kappa}^n(x, \sigma(x, \eta))\psi(\sigma(x, \eta), \bar{u}_{M_n}^n(\sigma(x, \eta)))d\eta \right| \leq \frac{1}{\delta} (\|B_1(x)\| + E_1(x) + E_2(x)), \tag{35} \]
where \(B_1(x)\) is defined in (31) and
\[ E_1(x) = \left| \frac{h_n}{4} \left( \int_{\Lambda} \hat{\kappa}(x, \sigma(x, \eta))\psi(\sigma(x, \eta), \bar{u}_{M_n}^n(\sigma(x, \eta)))d\eta 
- (1 + x) \int_{\Lambda} \hat{\kappa}(x, \sigma(x, \eta))\psi(\sigma(x, \eta), \bar{u}_{M_n}^n(\sigma(x, \eta)))d\eta \right) \right|, \tag{36} \]
\[ E_2(x) = \left| \frac{h_n}{4} (1 + x) \left( \int_{\Lambda} \hat{\kappa}(x, \sigma(x, \eta))\psi(\sigma(x, \eta), \bar{u}_{M_n}^n(\sigma(x, \eta)))d\eta 
- \int_{\Lambda} \hat{\kappa}(x, \sigma(x, \eta))\psi(\sigma(x, \eta), \bar{u}_{M_n}^n(\sigma(x, \eta)))d\eta \right) \right|. \]
Thus
\[ \|e_n\|^2 \leq \frac{3}{2^2} (\|B_1\|^2 + \|E_1\|^2 + \|E_2\|^2). \tag{37} \]
In order to estimate the term \(\|e_n\|^2\), we need the error bound for \(\|E_i\|\), for \(i = 1, 2\). Owing to Lemma \(\text{[20]}\) and the same discussion about the error of the terms \(D_4\) and \(D_2\) in (20), we have
\[ \|E_1\|^2 = \frac{h_n^2}{16} \left\| (I_{M_n} - I)(1 + x) \int_{\Lambda} \hat{\kappa}(x, \sigma(x, \eta))\psi(\sigma(x, \eta), \bar{u}_{M_n}^n(\sigma(x, \eta))) \right\|^2 
- \hat{\kappa}(x, \sigma(x, \eta))\psi(\sigma(x, \eta), \bar{u}_{M_n}^n(\sigma(x, \eta))) \right\|^2 + \left\| \psi(\cdot, u) \right\|^2_{H^m(I_n)}, \tag{38} \]
10
and
\[ \| E_2 \| = \frac{h_n^2}{16} \left( 1 + x \right) \left( \int_I I_{M_n} \left( \tilde{k}^n(x, \sigma(x, \eta)) \psi(\sigma(x, \eta), \tilde{a}_{M_n}^n(\sigma(x, \eta))) - \tilde{k}^n(x, \sigma(x, \eta)) \psi(\sigma(x, \eta), \tilde{a}_{M_n}^n(\sigma(x, \eta))) \psi d\eta \right) \right) \]
\[ \leq c h_n^{2m+1} M_n^{-2m} \left( \psi(., u_N^m) \right)_{H^m(I_n)}, \]
where the constant \( d \) depends on the term \( \max_{(s,t) \in \Omega} |\kappa(s,t)| \) and \( \gamma \) is the Lipschitz constant. Consequently,
\[ (1 - d \gamma^2 h_n^2) \| e_n \|^2 \leq \frac{1}{\delta^2} \left( c h_n^{2m-1} M_n^{-2m} \| \partial_t^m f \|_{L^2(I_n)}^2 + c T \sum_{k=1}^{n-1} \left( \gamma^2 h_k^2 \| e_k \|^2 + c h_k^{2m} M_k^{-2m} \| \partial_t^m u \|_{L^2(I_k)}^2 \right) \]
\[ + \| \psi(., u) \|_{H^m(I_n)}^2 \left( \right) + c T h_n^{2m} M_n^{-2m} \sum_{k=1}^{n-1} \| \psi(., u) \|_{H^2(I_k)}^2 + c h_n^{2m-1} M_n^{-2m} \left( \gamma^2 h_n^2 \| \partial_t^m u \|_{L^2(I_n)}^2 \right) \]
\[ + \| \psi(., u) \|_{H^m(I_n)}^2 \left( \right) + c h_n^{2m+1} M_n^{-2m} \left( \| \psi(., u_N^m) \|_{H^m(I_n)} \right). \]

We assume that \( h_{\max} \) is sufficiently small such that
\[ d \gamma^2 h_{\max}^2 \leq \beta < 1, \]
now using Lemma 2 we have
\[ \| e_n \|^2 \leq \frac{c}{\delta^2} \exp(c \gamma^2 T^2 \left( h_n^{2m-1} M_n^{-2m} \| \partial_t^m f \|_{L^2(I_n)}^2 + h_n^{2m-1} M_n^{-2m} \left( \gamma^2 h_n^2 \| \partial_t^m u \|_{L^2(I_n)}^2 \right) \]
\[ + \| \psi(., u) \|_{H^m(I_n)}^2 \left( \right) + h_n^{2m-1} M_n^{-2m} \| \psi(., u_N^m) \|_{H^2(I_n)}^2 + T \sum_{k=1}^{n-1} \left( h_k^{2m} M_k^{-2m} \left( \gamma^2 \| \partial_t^m u \|_{L^2(I_k)}^2 \right) \]
\[ + \| \psi(., u) \|_{H^m(I_k)}^2 \left( \right) + h_n^{2m-1} M_n^{-2m} \sum_{k=1}^{n-1} \| \psi(., u) \|_{H^2(I_k)}^2 \right), \]

hence the desired result is obtained. \( \blacksquare \)

**Theorem 6** Assume that \( u(t) \) be the exact solution of Eq. 11 and \( u_N^m(t) \) be the global approximate solution obtained from Eq. 19. Under the hypothesis of Theorem 5, the following error estimate can be derived as
\[ \| u - u_N^m \|_{L^2(I)} \leq \frac{c}{\delta^2} \exp(c \gamma^2 T^2) h_{\max}^m M_{\min} \left( \| \partial_t^m f \|_{L^2(I)} + \gamma (1 + T) \| \partial_t^m u \|_{L^2(I)} \right) \]
\[ + T \| \psi(., u) \|_{H^m(I)} + \| \psi(., u_N^m) \|_{H^m(I)} \right). \]

**Proof.** The global convergence error of the approximate solution \( u_N^m(t) \) which is given by
\[ u_N^m(t) \big|_{t \in I_n} = \tilde{u}_{M_n}^n(x) 2^{-i_n-1-i_n}, \quad 1 \leq n \leq N, \]
and the exact solution \( u(t) \) which is fulfilled in
\[ u(t) \big|_{t \in I_n} = \tilde{u}^n(x) 2^{-i_n-1-i_n}, \quad 1 \leq n \leq N, \]

can be easily obtained using Theorem 5 and the following formula
\[ \| u - u_N^m \|_{L^2(I)} = \frac{1}{2} \sum_{k=1}^{N} h_k \| e_k \|^2. \]
Therefore,

\[ \| u - u^N_m \|^2 \leq \frac{C}{\delta^2} \exp(\epsilon T^2) \sum_{n=1}^{N} \left( h_n^{2m} M_n^{-2m} \| \partial_t^m f \|_{L^2(I_n)}^2 + h_n^{2m} M_n^{-2m} \left( \gamma^2 h_n^2 \| \partial_t^m u \|_{L^2(I_n)}^2 \right) \right) + \| \psi(., u) \|_{H^m(I_n)}^2 + T h_n \sum_{k=1}^{n-1} \left( h_k^{2m} M_k^{-2m} \left( \gamma^2 \| \partial_t^m u \|_{L^2(I_k)}^2 \right) \right) + \| \psi(., u) \|_{H^m(I_k)}^2 \).

All terms of the above error bound could be simplified using \( h_{\text{max}} \) and \( M_{\text{min}} \) as follows

\[ \sum_{n=1}^{N} h_n^{2m} M_n^{-2m} \| \partial_t^m f \|_{L^2(I_n)}^2 \leq h_{\text{max}}^{2m} M_{\text{min}}^{-2m} \| \partial_t^m f \|_{L^2(I)}, \]

similarly, and

\[ \sum_{n=1}^{N} h_n^{2m} M_n^{-2m} \| \psi(., u^N_m) \|_{H^m(I_n)}^2 \leq h_{\text{max}}^{2m} M_{\text{min}}^{-2m} \| \psi(., u^N_m) \|_{H^m(I)}. \]

Furthermore,

\[ \sum_{n=1}^{N} T h_n \sum_{k=1}^{n-1} h_k^{2m} M_k^{-2m} \gamma^2 \| \partial_t^m u \|_{L^2(I_k)}^2 \leq T h_{\text{max}}^{2m} M_{\text{min}}^{-2m} \gamma^2 \sum_{n=1}^{N} h_n \sum_{k=1}^{N} \| \partial_t^m u \|_{L^2(I_k)}^2 \leq \gamma^2 T^2 h_{\text{max}}^{2m} M_{\text{min}}^{-2m} \| \partial_t^m u \|_{L^2(I)}, \]

and

\[ \sum_{n=1}^{N} T h_n \sum_{k=1}^{n-1} h_k^{2m} M_k^{-2m} \| \psi(., u) \|_{H^m(I_k)}^2 \leq T h_{\text{max}}^{2m} M_{\text{min}}^{-2m} \sum_{n=1}^{N} h_n \sum_{k=1}^{N} \| \psi(., u) \|_{H^m(I_k)}^2 \leq T^2 h_{\text{max}}^{2m} M_{\text{min}}^{-2m} \| \psi(., u) \|_{H^m(I)}^2. \]

Moreover, the last term can be bounded as

\[ \sum_{n=1}^{N} h_n^{2m+1} M_n^{-2m} \sum_{k=1}^{n-1} \| \psi(., u) \|_{H^m(I_k)}^2 \leq h_{\text{max}}^{2m} M_{\text{min}}^{-2m} \sum_{n=1}^{N} h_n \sum_{k=1}^{N} \| \psi(., u) \|_{H^m(I_k)}^2 \leq T h_{\text{max}}^{2m} M_{\text{min}}^{-2m} \| \psi(., u) \|_{H^m(I)}^2. \]

Consequently, the combination of the above error bounds for Eq. (43) lead to the desired result.

\[ \Box \]

5 Numerical results

The numerical experiments are used to illustrate the efficiency of the \( hp \)-collocation method for the first kind Hammerstein integral equations. The experiments are implemented in Mathematica® software platform. The programs are executed on a PC with 3.50 GHz Intel(R) Core(TM) i5-4690K processor. In order to analyze the method, the following notations are introduced:

\[ E_1^N(T) = \left( \sum_{k=1}^{M_k} \sum_{j=0}^{M} \frac{h_k}{2} w_{k,j} (u^k(x_{k,j}) - u^k_{M_k}(x_{k,j}))^2 \right)^{\frac{1}{2}}, \]
The discrete $L^2$-norm error is denoted by $E_N^2(T)$, also the maximum of absolute error at the mesh knots is shown by $E_N(T)$ and finally $E_N^N(T)$ indicates the infinite norm. Furthermore, the order of convergence $\rho_N$ is defined by $\log_2\left(\frac{E_{2N}^N(T)}{E_N^N(T)}\right)$. Let $L$ denotes the number of unknown coefficients, in this regard we have $L = M \times N$ for the $hp$-collocation method.

For the solution of the nonlinear systems which arise in the formulation of the method, one may use the Newton’s iteration method which needs an initial guess. In these examples, the initial points are selected by a algorithm based on the steepest descent method.

**Remark 7** In [25], an optimal control problem is solved numerically using a mesh refinement method based on collocation at Legendre-Gauss-Radau points. A relative error estimate is defined and then it is used to choose increasing the degree of polynomials or refinement of the mesh-size. The described scheme is called “adaptive $hp$-collocation method”. If we provide some facilities to modify the degree of polynomials in each subinterval or change the mesh-size during the approximation procedure then the desired error could be fulfilled.

**Example 1** ([19, 33]) The following linear Volterra integral equation of first kind is considered

$$\int_0^t \exp(-ts)u(s)ds = \frac{\exp(-t(t + 1))\sin(t) - (t + 1)\cos(t)\exp(-t(t + 1)) + t + 1}{1 + (t + 1)^2}, \quad t \in [0, 1],$$

with the exact solution $u(t) = \exp(-t)\cos(t)$.

This equation has been solved by piecewise polynomial collocation method [19] and a type of multi-step collocation method [33]. Table 1 reports the obtained error results for $hp$- and multi-step collocation methods. By comparing the results, we can conclude that $hp$-collocation gives better results. In addition, the best results reported in [19] with $M = 4$ and $N = 256$ has the absolute error around $10^{-10}$ while the present scheme achieves the error $10^{-14}$. Figure 4 shows the superiority of $hp$-version against $h$- and $p$-versions. The figure on the left with fixed $m = 4$ and different values of $N$ depicts $h$-version. Also, the figure on the right demonstrates $p$-version for each fixed $N = 1, 2, 4, 8$ and various values $M$ which can be seen as an $hp$-version method.

Table 1: A comparison between multi-step collocation [33] and $hp$-collocation methods for Example 1 in terms of $E_N^3(1)$.

| $N$ | multi-step method       | $hp$-collocation method | $\rho_N$ |
|-----|------------------------|-------------------------|----------|
| 2   | 9.5385e - 04           | 2.4708e - 05            |          |
| 2   | 4.8403e - 05           | 1.7028e - 06            | 3.8589   |
| 3   | 2.8037e - 06           | 1.1137e - 07            | 3.9345   |
| 4   | 1.6961e - 07           | 7.1140e - 09            | 3.9685   |
| 5   | 1.0443e - 08           | 4.4938e - 10            | 3.9846   |
| 6   | 6.4788e - 10           | 2.8104e - 11            | 3.9990   |
| 7   | 4.2577e - 11           | 1.7397e - 12            | 4.0138   |

**Example 2** ([22, 28]) In this example, we apply the methods to the following nonlinear Volterra integral equation of the first kind

$$\int_0^t (\sin(t - s) + 1)\cos(u(s))ds = f(t), \quad t \in [0, 1],$$
whose \( f(t) \) is chosen such that \( u(t) = t \) be the exact solution. Due to the invertibility of the kernel, this equation can be converted into a second kind integral equation. Using this idea, two numerical schemes based on Sinc Nyström and Haar wavelet methods are discussed in \cite{22} and \cite{28}, respectively. Table 2 and 3 demonstrates the results with comparisons. In Table 3 and in the column for Sinc method, the number of the basis functions can be obtained from the parameter \( N \) as \( L = 2N + 1 \). In addition, in the column of Haar wavelet method, \( J \) denotes the level of wavelets. The number of the basis functions are derived by \( L = 2^J \).

Table 2: The \( E_3^N(1) \) error of the \( hp \)-collocation method for various \( M \) and \( N \) for Example 2

| \( M \) \( \setminus \) \( N \) | 1      | 2      | 4      | 6      | 8      |
|----------------|--------|--------|--------|--------|--------|
| 2              | 2.48e−02 | 6.61e−04 | 1.69e−07 | 7.60e−04 | 4.29e−04 |
| 4              | 1.04e−04 | 7.08e−05 | 4.55e−09 | 9.06e−08 | 2.87e−08 |
| 6              | 1.88e−07 | 3.20e−09 | 5.14e−11 | 4.55e−12 | 8.07e−13 |
| 8              | 1.87e−10 | 7.88e−13 | 9.32e−14 | 3.55e−15 | 2.22e−15 |
| 10             | 1.44e−13 | 4.21e−15 | 1.59e−14 | 8.88e−16 | 1.44e−15 |

Table 3: The comparison of Haar wavelet \cite{28} and Sinc Nyström \cite{22} methods with \( hp \)-collocation method for Example 2 in the sense of \( E_3^N(1) \) error.

| \( J \) | \( L \) | Haar wavelet | \( N \) | \( L \) | SE Sinc | DE Sinc | \( L \) | \( hp \)-collocation |
|---------|--------|--------------|--------|--------|---------|---------|--------|---------------------|
| 2       | 4      | 1.2e−03      | 4      | 9      | 5.68e−02 | 9.25e−02 | 4      | 1.04e−04           |
| 3       | 8      | 3.1e−04      | 4      | 9      | 5.68e−02 | 9.25e−02 | 8      | 1.87e−10           |
| 4       | 16     | 8.0e−05      | 8      | 17     | 4.44e−03 | 3.95e−03 | 16     | 7.88e−13           |
| 5       | 32     | 2.0e−05      | 16     | 33     | 2.01e−04 | 8.88e−05 | 32     | 9.32e−15           |
| 6       | 64     | 5.0e−06      | 32     | 65     | 1.21e−06 | 4.20e−08 | 64     | 8.88e−16           |
| 7       | 128    | 1.2e−06      | 64     | 127    | 6.33e−10 | 8.29e−15 | 128    | 8.88e−16           |
| 8       | 256    | 3.1e−07      | 64     | 127    | 6.33e−10 | 8.29e−15 | 256    | 8.88e−16           |
| 9       | 512    | 7.9e−08      | 64     | 127    | 6.33e−10 | 8.29e−15 | 512    | 8.88e−16           |
Calculation for long $T$.

**Example 3** ([26]) In the following example, we consider solving the equation

$$\int_0^t \sin(t - su(s))\,ds = f(t), \quad t \in [0, T],$$

with the exact solution $u(t) = 1$. The Figure 2 shows considerable results for various $T$.

![Figure 2: Plots of the error in logarithmic scale in different time $T$ for Example 3](image)

**Steepest gradient solution**

**Example 4** ([26]) Consider the nonlinear Volterra integral equation

$$\int_0^t u^2(s)\,ds = f(t), \quad t \in [0, 10],$$

where $f(t) = \sqrt{\pi} \left( \text{erf}(10) + \text{erf}(2(t - 5)) \right)$. The exact solution is $u(t) = \exp(-2(t - 5)^2)$. The Figure 3 depicts the results for $N = 10, 20, 30$ and different $M$.

![Figure 3: Plots of $E_1^N(10)$ in logarithmic scale for Example 4](image)

**5.1 Special cases**

In this part, we present some examples of the first kind integral equations which do not fulfill the assumptions in [11, 19, 22, 28, 33] and the theorems in Sections 2 and 4. Due to the lack of enough smoothness properties for kernel $\kappa(s, t)$ and right-hand function $f(t)$, these equations could not be converted to the second kind ones. In all following examples, the advantage and efficiency of $hp$-collocation method to approximate the non-smooth solutions vs. $p$- and $h$-version methods are shown.
Non-differentiable kernel

**Example 5** As a test problem, consider the following first kind Volterra-Hammerstein integral equation
\[ \int_0^t \kappa(s,t) \left( \frac{3u^2(s)}{s+1} + \sin(u(s)) \right) ds = f(t), \quad t \in [0,1], \]
where
\[ \kappa(s,t) = \begin{cases} s^2 - t + 5, & 0 \leq t < 0.5, \quad 0 \leq s \leq 1, \\ 1, & 0.5 \leq t \leq 1, \quad 0 \leq s \leq 1, \end{cases} \]
and the exact solution is \( u(t) = t^3 \). Figure 4 describes the \( E_{N}^{2}(1) \) error for \( N = 1, 2, 4 \).

![Figure 4: Plots of the \( E_{N}^{2}(1) \) error in logarithmic scale for Example 5.](image)

Discontinuous solution. In this part, we focus on the nonlinear examples with discontinuous solutions.

**Example 6** As another test problem consider the following integral equation
\[ \int_0^t (t + 4s) \left( u(s) - \frac{s}{2} \right)^2 ds = f(t), \quad t \in [0,1], \]
where
\[ u(t) = \begin{cases} \frac{t-1}{2}, & t < 0.5, \\ 2e^{-t}, & t \geq 0.5, \end{cases} \]
is the exact solution. In previous examples, we take the degree of polynomial for each \( I_k \) with \( M_k = M \) or \( L = MN \). Here, we take \( M_k = 1, \ k = 1, \ldots, \frac{N}{2} \) and \( M_k = M_0, \ k = \frac{N}{2} + 1, \ldots, N \). The applicability of this scheme is verified by using less basis functions due to the behavior of the solution. In Figure 6, we observe the \( E_{N}^{3}(1) \) error for different values of \( N \) and \( M \). As expected from the theoretical achievements, by decreasing \( h \) we get better numerical results. The function \( f(t) \) is discontinuous and for equations with discontinuous right-hand side function or its corresponding discontinuous solution, all \( p \)-version schemes are incapable to approximate the solution. On the other hand, some recent numerical methods are based on hybrid functions \([23]\) which could be categorized into \( hp \)-version methods since they approximate functions locally, but they solve the final system globally. The superiority of \( hp \)-collocation method against hybrid functions method is shown in Figure 6 for various \( N \) and \( M \), even if we choose \( M_k = M \) for the whole interval.

**Example 7** In this example, we solve the following integral equation
\[ \int_0^t u(s) ds = f(t), \quad t \in [0,10], \]
Figure 5: Plots of the $E^N_3(1)$ error in logarithmic scale for Example 6.

where $f(t)$ can is determined such that

$$u(t) = \begin{cases} 
  t, & t < 5, \\
  \frac{1}{t}, & t \geq 5,
\end{cases}$$

be the exact solution. Figure 6 displays the error $E^N_1(10)$ for different $h$. As the theoretical results predict by increasing $M$, the error is reduced.

Figure 6: Comparison between $hp$-collocation method and the hybrid method in [23] with the $E^N_3(1)$ error in logarithmic scale for Example 6.

Function $\kappa(t, t) = 0$.

Example 8 Consider the following integral equation

$$\int_0^t (s - t)e^{u(s)}\,ds = f(t), \quad t \in [0, 1],$$
where \( u(t) = |t - 0.5| \) is the exact solution. Figure 8 shows the results in terms of different \( E_i^N(1) \), \( i = 1, 2, 3 \). As expected, their behavior are almost the same. The results are reported for \( N = 2 \) and various \( M \). Note that the solution has finite regularity.

![Figure 8: Plots of the \( E_i^N(1) \) errors in logarithmic scale for Example 8.](image)

**Singular solution**

In the following test problem, we consider an equation which has a weakly singular solution.

**Example 9** (20) The final example is the following integral equation

\[
\int_0^t u^2(s)ds = f(t), \quad t \in [0, 1],
\]

where \( u(t) = t^r \) with non-integer \( r > \frac{1}{2} \) is the exact solution. Note that this solution has finite regularity. Figure 9 depicts the \( E_i^N(1) \) error for different \( r = 0.51, 1.51, 2.51 \). As we expect for bigger \( r \), the error is reduced significantly.

![Figure 9: Plots of the \( E_i^N(1) \) error in logarithmic scale with \( r = 0.51, 1.51, 2.51 \), for Example 9.](image)

**Conclusion**

Integral equations of the first kind and their approximations have a long history and many researchers have worked on them. In this paper, the idea of \( hp \)-version projection methods has been studied and a prior error analysis for the \( hp \)-version collocation method for the Volterra integral equations of the first kind developed. The existence and uniqueness of the solution have been investigated in the suitable Sobolev spaces under some reasonable assumptions on the nonlinearity. Numerical treatments indicate that the proposed scheme is effective and powerful to deal with smooth and non-smooth solutions, especially for long-time integration.
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