Study the spheroidal wave functions by SUSYQM *

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Abstract

The perturbation method in supersymmetric quantum mechanics (SUSYQM) is used to study the spheroidal wave functions’ eigenvalue problem. Expanding the super-potential in series of the parameter $\alpha$, the first order term of ground eigen-value and the eigen-function are gotten. In the paper, the very excellent results are that all the first two terms approximation on eigenfunctions obtained are in closed form. They give useful information for the involved physical problems in application of spheroidal wave functions.

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Introduction

Since 1930s, spheroidal wave functions have made strong contributions to extensively theoretical and practical applications in pure mathematics, applied mathematics, physics and engineering. They appear in the fields, such as wavelet, random matrix, non-commute geometry, gravitational wave detection, quantum field theory in curved space-time, black hole stable problem; 3G mobile and broad band satellite telecommunication; steady flow of a viscous fluid and so on[1]-[5]. Nevertheless, they perhaps are one of the hardest work for researchers. They welcome the new thought and methods to deal with them. Since the appearance of the supersymmetric quantum mechanics (SUSYQM), its great power of solving the differential equation attracts tremendous attention.

In this paper, we first use SUSYQM to study The spheroidal differential equations. First, brief introduction to the spheroidal problems and the ordinary methods to treat them. The spheroidal differential equations are

$$\left[ \frac{d}{dx} \left( 1 - x^2 \right) \frac{d}{dx} \right] + E + \alpha x^2 - \frac{m^2}{1 - x^2} \right] \Theta = 0, \; x \in (-1, +1).$$

(1)

With the condition $\Theta$ is finite at the boundaries $x = \pm 1$, they consist of the Sturm-Liouville eigenvalue problem. The parameter $E$ can only takes the values $E_0, E_1, \ldots, E_n, \ldots$, which are

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called the eigenvalues of the problem, and the corresponding solutions (the eigenfunctions) \( \Theta_0, \Theta_1, \ldots, \Theta_n, \ldots \) are called the spheroidal wave functions [1]-[3].

The equations (1) have two parameters: \( m, \alpha \). When \( \alpha = 0 \), the spheroidal wave functions reduce to the spherical wave functions, that is, the associated Legendre-functions \( P^m_l(x) \). Though the spheroidal wave equations are extension of the ordinary spherical wave functions equations, the difference between this two kinds of wave equations are far greater than their similarity[1]. The spherical wave equations belong to the case of the confluent super-geometrical equations with one regular and one irregular singularities, whereas the spheroidal wave equations are the confluent Heun equations containing two regular and one irregular singularities. The extra singularity makes it extremely difficult to solve them[1]-[3], so very little information have been obtained concerning the analytically exact solutions. Therefore approximate and numerical methods are two main resources to rely on for the problem.

Traditionally, the three-term recurrence relation methods are used to evaluate the eigenvalues and eigenfunctions of spheroidal wave functions: one could solve the transcendental equation in continued fraction form or its equivalent or by power series expansion etc[1]-[3],[6]-[13]. For the details of these methods and their advantage and disadvantage, one could see the reference [13]. These methods mainly work for the numerical purpose, and also rely heavily on the numerical method. However, all previous works concentrate largely on the calculations of the eigenvalues, and particularly emphasize the small parameter approximation and large parameter limits form of the eigenvalues. Little effort has been devoted to the related eigenfunctions due to the difficulty and complexity.

Here, we give brief review on the eigenfunctions in small parameter approximation. No matter what method may be used, all eigenfunctions come into this kind of form in the end

\[
\Theta_n(x) = P^m_n + \sum_{q=1}^{\infty} \Theta_{nq}(x)\alpha^q. \tag{2}
\]

Though there are many excellent works on the eigenvalues’ approximation of small parameter, no good works exist for the eigenfunctions approximation \( \Theta_{nq}(x) \). Even for the first eigenfunctions approximation, the existing works are only in the series’ form

\[
\Theta_{n1}(x) = \sum_{r=0, r \neq n}^{\infty} B_r P^m_{n+r}(x) \tag{3}
\]

with infinite numbers \( B_r \) needed to evaluate. Obviously, this series form does not reveal much information about the eigen-functions, even about its 1st order approximation eigenfunctions \( \Theta_{n1}(x) \) itself.

In recent years, supersymmetric quantum mechanics have attracted tremendous attention for solvable potential problems. They not only provide clear insight into the factorization method of Infeld and Hull [16], but also greatly improve the methods to solve the differential equations. See reference [14] for review on its development.

The spheroidal eigenvalue problem is treated by the method of SUSYQM. This is the first time for researchers to use SUSYQM to study the ground eigenvalue and eigenfunction (that is \( \Theta_0 \)) of spheroidal wave functions in the small parameter \( \alpha \) approximation.

In usual small parameter approximation method, the key concept is the eigenfunctions and they are expanded in the form (2). On the contrary, the super-potential is the central concept in SUSYQM, and is expanded in the series form of the parameter \( \alpha \). This new method
is applied to study the spheroidal equations and unexpected results are obtained: the ground eigenfunction of the first order is in closed form, this in turn gives useful information on the eigenfunction and is helpful for their application. We also get the ground function for higher order terms in parameter $\alpha$.

**The ground eigenvalue and eigenfunction in the first order**

In the following, we will use the new perturbation method in supersymmetry quantum to resolve the spheroidal eigenfunctions’ problem.

Though the form (1) is more familiar for research, the problem is easier to solve in the original differential equation than in the equation (1). The original form is obtained from the eq. (4) by the transformation

$$x = \cos \theta,$$

that is,

$$\left[ \frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d}{d\theta} \right) + \alpha \cos^2 \theta - \frac{m^2}{\sin^2 \theta} \right] \Theta = -E \Theta \quad (5)$$

the corresponding boundary conditions become $\Theta$ is finite at $\theta = 0, \pi$.

One writes the eqn. (5) in the form of the Schrödinger equation by the transformation

$$\Theta = \frac{\Psi}{\sin \frac{\theta}{2}} \quad (6)$$

the differential equations turn out to be

$$\frac{d^2 \Psi}{d\theta^2} + \left[ \frac{1}{4} + \alpha \cos^2 \theta - \frac{m^2 - \frac{1}{4}}{\sin^2 \theta} + E \right] \Psi = 0 \quad (7)$$

and the boundary conditions become

$$\Psi_{\theta=0} = \Psi_{\theta=\pi} = 0 \quad (8)$$

From the equation (7), one knows the potential is

$$V(\theta, \alpha, m) = -\frac{1}{4} - \alpha \cos^2 \theta + \frac{m^2 - \frac{1}{4}}{\sin^2 \theta} \quad (9)$$

The super-potential $W$ is determined by the potential $V(\theta, \alpha, m)$ through the Reccita’s equation

$$W^2 - W' = V(\theta, \alpha) - E_0 \quad (10)$$

where the substraction of the ground energy just makes the eqn. (7) factorable. Actually this equation is the same hard to treat as that in the original form (7). The approximate method naturally comes to one’s mind. Hence, when the absolute value of $\alpha$ is small, it is the the super-potential $W$ that could be expanded as series of the parameter $\alpha$, that is,

$$W = W_0 + \alpha W_1 + \alpha^2 W_2 + \alpha^3 W_3 + \ldots \quad (11)$$
\[ W^2 - W' = W_0^2 - W_0' + \alpha (2W_0W_1 - W_1') + \alpha^2 (2W_0W_2 + W_1^2 - W_2') + \alpha^3 (2W_0W_3 + 2W_1W_2 - W_3') + \alpha^4 (2W_0W_4 + 2W_1W_3 + W_2^2 - W_4') + \ldots \] \hspace{1em} (12)

One can write the perturbation equation as

\[ W^2 - W' = V(\theta, \alpha, m) - \sum_{n=0}^{\infty} 2E_{0n} \alpha^n = -\frac{1}{4} + \frac{m^2 - 1}{4\sin^2 \theta} - \alpha \cos^2 \theta - \sum_{n=0}^{\infty} 2E_{0n} \alpha^n \] \hspace{1em} (13)

There are two lower indices in the parameter \( E_{0n} \), with the index 0 referring to the ground state and the other index \( n \) meaning the \( n \)th term in parameter \( \alpha \). The last term \( \sum_{n=0}^{\infty} 2E_{0n} \alpha^n \) is subtracted from the above equation in order to make the ground state energy actually zero for the application of the theory of SUSYQM. Later, one must add the term to our calculated eigen-energy. Comparing the equations (12), (9), and (13), one could get

\[ W_0^2 - W_0' = -\frac{1}{4} + \frac{m^2 - 1}{4\sin^2 \theta} - 2E_{00} \] \hspace{1em} (14)
\[ 2W_0W_1 - W_1' = -\alpha \cos^2 \theta - 2E_{01} \] \hspace{1em} (15)
\[ 2W_0W_2 + W_1^2 - W_2' = -2E_{02} \] \hspace{1em} (16)
\[ 2W_0W_3 + 2W_1W_2 - W_3' = -2E_{03} \] \hspace{1em} (17)
\[ 2W_0W_4 + 2W_1W_3 + W_2^2 - W_4' = -2E_{04} \] \hspace{1em} (18)
\[ \vdots \] \hspace{1em} (19)

From the eq.(14), we get

\[ W_0 = -\left( m + \frac{1}{2} \right) \cot \theta, \ 2E_{00} = m(m + 1). \] \hspace{1em} (20)

Then, we can write the other equations more concisely

\[ W_1' + (2m + 1) \cot \theta W_1 = \cos^2 \theta + 2E_{01} \] \hspace{1em} (21)
\[ W_2' + (2m + 1) \cot \theta W_2 = W_1^2 + 2E_{02} \] \hspace{1em} (22)
\[ W_3' + (2m + 1) \cot \theta W_3 = 2W_1W_2 + 2E_{03} \] \hspace{1em} (23)
\[ W_4' + (2m + 1) \cot \theta W_4 = 2W_1W_3 + W_2^2 + 2E_{04} \] \hspace{1em} (24)
\[ W_5' + (2m + 1) \cot \theta W_5 = 2W_1W_4 + 2W_2W_3 + 2E_{05} \] \hspace{1em} (25)
\[ \vdots \] \hspace{1em} (26)

After obtaining the zero term \( W_0 \) for the super-potential \( W \), the first order \( W_1 \) can be gotten as

\[ W_1 = \frac{\tilde{A}_1}{\sin^{2m+1} \theta} \] \hspace{1em} (27)

with

\[ \frac{d\tilde{A}_1}{d\theta} = \sin^{2m+1} \theta \left( \cos^2 \theta + 2E_{01} \right) \] \hspace{1em} (28)
\[ \tilde{A}_1 = \int \left( \sin^{2m} \theta \cos^2 \theta + 2E_{01} \sin^{2m} \theta \right) \sin \theta \, d\theta \] \hspace{1em} (29)
Suitably changing the independent variable to \( x = \cos \theta \) and expanding the term as following

\[
\sin^{2m} \theta = \left( 1 - \cos^2 \theta \right)^m = \left( 1 - x^2 \right)^m = \sum_{k=0}^{m} (-1)^k \binom{m}{k} x^{2k}
\]  

(30)

where \( \binom{m}{k} = \frac{m!}{k! \times (m-k)!} \), then it reaches

\[
A_1 = \sum_{k=0}^{m} (-1)^{k+1} \binom{m}{k} \left( \frac{\cos^{2k+3} \theta}{2k+3} + \frac{2E_{01} \cos^{2k+1} \theta}{2k+1} \right).
\]  

(31)

So,

\[
W_1 = \frac{A_1}{\sin^{2m+1} \theta} = \frac{\sum_{k=0}^{m} (-1)^{k+1} \binom{m}{k} \left( \frac{\cos^{2k+3} \theta}{2k+3} + \frac{2E_{01} \cos^{2k+1} \theta}{2k+1} \right)}{\sin^{2m+1} \theta}.
\]  

(32)

The quantity \( E_{01} \) needs to be determined by the boundary conditions. This in turn require to calculate the ground eigenfunction upon to the first order by

\[
\Psi_0 = N \exp \left[ - \int W \, d\theta \right]
\]  

(33)

\[
= N \exp \left[ - \int W_0 \, d\theta - \alpha \int W_1 \, d\theta \right] * \exp O(\alpha^2)
\]  

(34)

\[
= N \sin^{m+\frac{1}{2}} \theta \exp \left[ -\alpha \int W_1 \, d\theta \right] * \exp O(\alpha^2).
\]  

(35)

Whenever the eigenfunction is obtained, the boundary conditions \( \Psi|_{\theta=0} = \Psi|_{\theta=\pi} = 0 \) would choose the proper \( E_{01} \). This sounds very easy, but is a tough task in reality. The complete calculating process is left to the appendix 1. The results are

\[
2E_{01} = - \frac{1}{2m+3}
\]  

(36)

\[
W_1 = \frac{\sin \theta \cos \theta}{2m+3}.
\]  

(37)

With the first order term of the super-potential \( W_1 \), one could compute the second term \( W_2 \) by the same process. This is not easy either. The appendix 2 gives the detail of the calculation. Nevertheless, the results are elegant:

\[
2E_{02} = - \frac{2m+2}{(2m+3)^3(2m+5)}
\]  

(38)

\[
W_2 = \left[ \frac{-\sin \theta \cos \theta}{(2m+3)^3(2m+5)} + \frac{\sin^3 \theta \cos \theta}{(2m+3)^2(2m+5)} \right].
\]  

(39)

The ground eigenfunction upon to the second order becomes

\[
\Psi_0 = N \exp \left[ - \int W \, d\theta \right]
\]  

(40)

\[
= N \exp \left[ - \int W_0 \, d\theta - \alpha \int W_1 \, d\theta - \alpha^2 \int W_2 \, d\theta \right] * \exp O(\alpha^3)
\]  

(41)

\[
= (\sin \theta)^{m+\frac{1}{2}} \exp \left[ \frac{\alpha \sin^2 \theta}{4m+6} \right]
\]  

\[
* \exp \left[ \frac{\alpha^2 \sin^2 \theta}{2(2m+3)^3(2m+5)} - \frac{\alpha^2 \sin^4 \theta}{4(2m+3)^2(2m+5)} \right] * \exp O(\alpha^3).
\]  

(42)
When \( m = 0 \), these results \((36), (37), (38), (39)\) reduce respectively to

\[
2E_{01} = -\frac{1}{3} \quad W_1 = \frac{1}{3} \sin \theta \cos \theta; \tag{43}
\]

\[
2E_{02} = -\frac{2}{135} \quad W_2 = -\frac{1}{135} \sin \theta \cos \theta + \frac{1}{45} \sin^3 \theta \cos \theta. \tag{44}
\]

They are in complete accordance with that in reference \([18]\) where the spheroidal wave functions are treated by SUSYQM in the case \( m = 0 \).

Back to equation \((1)\) with their relationship \((6)\), we could obtain

\[
\Theta_0 = \left(1 - x^2\right)^\frac{m}{2} \exp \left(\frac{-\alpha (1 - x^2)}{4m + 6}\right) \times \exp \left[\frac{\alpha^2 (1 - x^2)^2}{2(2m + 3)^2(2m + 5)} - \frac{\alpha^2 (1 - x^2)^2}{4(2m + 3)^2(2m + 5)}\right] \exp O(\alpha^3). \tag{45}
\]

Expanding the exponential functions in the above equation, the results elegantly turn out as

\[
\Theta_0 = \left(1 - x^2\right)^\frac{m}{2} \left[1 - \alpha \frac{1 - x^2}{4m + 6} + \alpha^2 \left(\frac{1 - x^2}{2(2m + 3)^2(2m + 5)} + \frac{(1 - x^2)^2}{8(2m + 3)(2m + 5)}\right)\right] + O(\alpha^3). \tag{46}
\]

The above equation clearly shows how the ground state function changes as the function of variable \( x \) when \( \alpha \) is small and could be compared to the non-perturbation case \( P_m^m = (1 - x^2)^\frac{m}{2} \). The result is much better than the usual result of the series form

\[
\Theta_0(x) = P_m^m + \alpha \sum_{q=1}^{\infty} B_q P_m^{m+q}(x) + \alpha^2 \sum_{q=1}^{\infty} C_q P_m^{m+q}(x) + O(\alpha^3) \tag{47}
\]

with infinite numbers \( B_q, \ q = 1, 2, \ldots \) need to be determined for the first order term and \( C_q, \ q = 1, 2, \ldots \) need to be determined for the second order term. From another point view, the results in the eqns. \((45), (46)\) give the method to determine the infinite numbers \( B_q, \ C_q, \ q = 1, 2, \ldots \).

In conclusion, SUSYQM provides a new opportunity to treat the spheroidal wave functions and indeed they give new results in the eqns. \((45), (46)\). Further calculations can be done by the same way, nevertheless, the higher order term \( W_n \) is more complex than the lower one. The maximus of \( W_1 \) and \( W_2 \) satisfy

\[
\max W_1 = \frac{1}{6}, \quad \max W_2 = \frac{\sqrt{11}}{2160} < \frac{1}{540} < \max W_1. \tag{48}
\]

Though the further calculation is not processed here, the reasonable guess is that the higher term \( W_n \) is, the smaller its maximus. The guess mainly comes from the calculation of the quantity \( W_1, \ W_2 \), see the appendix 1 and appendix 2 for details. If the guess is right,

\[
W(\theta) = \sum_{n=0}^{\infty} W_n \alpha^n \tag{49}
\]

might be analytic function in all complex plane \( \alpha \), the only singularity of the function \( W \) as the variable \( \alpha = \infty \). Only the calculation in the appendixes can process on and on, could the guess be tested. Might some day the computer can do the work, this is the reason that the calculations in appendixes are extremely detailed.
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Appendix1: Simplification of the super potential of the first term $W_1$

The calculations are very complex, we rewrite the eq.(32) here again for convenience

$$W_1 = \frac{\tilde{A}_1}{\sin^{2m+1} \theta} = \sum_{k=0}^{m} (-1)^{k+1} \binom{m}{k} \left[ \frac{\cos^{2k+3} \theta}{2k+3} + \frac{2E_{01} \cos^{2k+1} \theta}{2k+1} \right].$$

As state before, the quantity $E_{01}$ is determined by the requirement that the eigenfunction is zero at the boundaries $\theta = 0, \pi$; this in turn demands $\int W_1 d\theta$ finite at the boundary. Therefore, the calculation of the term $\int W_1 d\theta$ is first processed. By transformation

$$\tau = \sin \theta$$

and denoting $\int W_1 d\theta$ by $I$, it reads

$$I = \int W_1 d\theta = \int \sum_{k=0}^{m} (-1)^{k+1} \binom{m}{k} \left[ \frac{(1-\tau^2)^k}{2k+3} + \frac{2E_{01}(1-\tau^2)^k}{2k+1} \right] d\tau$$

Using formula

$$(1-\tau^2)^k = \sum_{l=0}^{k} (-1)^l \binom{k}{l} \tau^{2l}$$

and

$$(1-\tau^2)^{k+1} = \sum_{l=0}^{k+1} (-1)^l \binom{k+1}{l} \tau^{2l+1},$$

it becomes

$$I = \int \sum_{k=0}^{m} (-1)^{k+1} \binom{m}{k} \left[ \frac{\sum_{l=0}^{k+1} (-1)^l \binom{k+1}{l} \tau^{2l-2m-1}}{2k+3} + \frac{2E_{01} \sum_{l=0}^{k} (-1)^l \binom{k}{l} \tau^{2l-2m-1}}{2k+1} \right] d\tau$$

$$= \sum_{k=0}^{m} \sum_{l=0}^{k+1} \frac{(-1)^{k+l+1} \binom{m}{k+1} \binom{k+1}{l} \tau^{2l-2m}}{(2k+3)(2l-2m)} + \sum_{k=0}^{m} \sum_{l=0}^{k} \frac{2E_{01} (-1)^{k+l} \binom{m}{k} \binom{k}{l} \tau^{2l-2m}}{(2k+1)(2l-2m)}$$

In order to exchange the sums order in the above equations, one must notice the fact that

$$l \leq k + 1 \Rightarrow k \geq l - 1$$

in the first term $\sum_{k=0}^{m} \sum_{l=0}^{k+1} \frac{(-1)^{k+l+1} \binom{m}{k+1} \binom{k+1}{l} \tau^{2l-2m}}{(2k+3)(2l-2m)}$ and $l \leq k \Rightarrow k \geq l$
in the second term $\sum_{k=0}^{m} \sum_{l=0}^{k} \frac{2E_{01}(-1)^{k+l+1}(m)_{l}(k)_{l}}{(2k+1)(2l-2m)} \tau^{2l-2m}$. So

$$I = \sum_{l=0}^{m} \sum_{k=0}^{l} \frac{(-1)^{k+l+1}(m)_{l}(k)_{l}}{(2k+1)(2l-2m)} \tau^{2l-2m} + \sum_{l=0}^{m} \sum_{k=0}^{l} \frac{2E_{01}(-1)^{k+l+1}(m)_{l}(k)_{l}}{(2k+1)(2l-2m)} \tau^{2l-2m} \qquad (56)$$

$$= \sum_{l=0}^{m} \sum_{k=0}^{l} \frac{(-1)^{k+l+1}(m)_{l}(k)_{l}}{(2k+1)(2l-2m)} + 2E_{01} \sum_{k=0}^{l} \frac{(-1)^{k+l+1}(m)_{l}(k)_{l}}{(2k+1)(2l-2m)} \right] \tau^{2l-2m} + \frac{\tau^2}{4m+6} \qquad (57)$$

where the first sum under the condition $l = m + 1$ becomes $\frac{\tau^2}{4m+6}$.

Defining

$$N_1 \ t = \sum_{k=0}^{m} \frac{(-1)^{k+l+1}(m)_{l}(k)_{l}}{(2k+1)(2l-2m)} \ , \ l = 0, 1, 2, \ldots \qquad (58)$$

$$N_2 \ l = \sum_{k=0}^{m} \frac{(-1)^{k+l+1}(m)_{l}(k)_{l}}{(2k+1)(2l-2m)} \ , \ l = 0, 1, 2, \ldots \qquad (59)$$

it gets

$$I = \sum_{l=0}^{m} [N_1 \ l + 2E_{01}N_2 \ l] \tau^{2l-2m} + \frac{\tau^2}{4m+6} \qquad (60)$$

There are terms $\tau^{2l-2m}$ in the above equation. With the eq.\(51\) and the fact $l < m$, these terms as $\sin^{2l-2m} \theta$ become infinite as $\theta \to 0, \pi$. By eq.\(35\) and the eigenfunction’s boundary condition at $\theta = 0, \pi$, the coefficients of those terms must be zero. There is only one quantity $E_{01}$ unfixed, could one choose proper $E_{01}$ to make the eigenfunction finite at the boundaries? Actually, one only has one choice to select $E_{01}$ by

$$N_1 \ 0 + 2E_{01}N_2 \ 0 = 0, \ \ N_1 \ 0 = \frac{1}{2m} \sum_{k=0}^{m} \frac{(-1)^{k+2}(m)_{k}}{(2k+1)} \ , \ N_2 \ 0 = \frac{1}{2m} \sum_{k=0}^{m} \frac{(-1)^{k+2}(m)_{k}}{(2k+1)}, \qquad (61)$$

do the other terms in eqn.\(60\) automatically become zero under the condition \(61\).

Fortunately, this can be done and the following is the proof. The inductive reasoning is used to give the proof. In order to determine the quantity $E_{01}$ under the condition \(61\), one must first simplify $N_1 \ 0, \ N_2 \ 0$. From the formula (see reference [19] on page 389)

$$\int_{0}^{1} (1 - \tau^2)^m d\tau = \int_{0}^{\frac{\pi}{2}} \cos^{2m+1} \theta d\theta = \frac{(2m)!}{(2m+1)!!} \qquad (62)$$

and the similarly one

$$\int_{0}^{1} \tau^2 (1 - \tau^2)^m d\tau = \int_{0}^{1} (1 - \tau^2)^m d\tau - \int_{0}^{1} (1 - \tau^2)^{m+1} d\tau \qquad (63)$$

$$= \left[ \frac{(2m)!!}{(2m+1)!!} - \frac{(2m+2)!!}{(2m+3)!!} \right] \qquad (64)$$
and also with the formula (53), it is easy to obtain

\[
\int_0^1 \tau^2(1 - \tau^2)^m d\tau = \int_0^1 \sum_{k=0}^{m} \binom{m}{k} (-1)^k \tau^{2k+2} d\tau = \sum_{k=0}^{m} \frac{(-1)^{k+2} \binom{m}{k}}{2k+3}
\]  \hspace{1cm} (65)

\[
\int_0^1 (1 - \tau^2)^m d\tau = \int_0^1 \sum_{k=0}^{m} \binom{m}{k} (-1)^k \tau^{2k} d\tau = \sum_{k=0}^{m} \frac{(-1)^{k+2} \binom{m}{k}}{2k+1}
\]  \hspace{1cm} (66)

Comparing these equations with that of (61), one reaches

\[
N_{1\ 0} = \frac{1}{2m} \sum_{k=0}^{m} \frac{(-1)^{k+2} \binom{m}{k}}{2k+3} = \frac{1}{2m} \left[ \frac{(2m)!!}{(2m+3)!!} - \frac{(2m+2)!!}{(2m+1)!!} \right]
\]

\[
N_{2\ 0} = \frac{1}{2m} \sum_{k=0}^{m} \frac{(-1)^{k+1} \binom{m}{k}}{2k+1} = \frac{1}{2m} \frac{(2m)!!}{(2m+1)!!}.
\]  \hspace{1cm} (67)

Consequently, the quantity \(2E_{01}\) is simplified as

\[
2E_{01} = -\frac{N_{1\ 0}}{N_{2\ 0}} = -\frac{1}{2m+3}.
\]  \hspace{1cm} (68)

\(N_{1\ 0} + 2E_{01}N_{2\ 0} = 0\) is guaranteed by the choice in eqn. (68). According to inductive reasoning, one needs to prove

\[
M_{1\ l+1} = N_{1\ l+1} + 2E_{01}N_{2\ l+1} = 0
\]  \hspace{1cm} (69)

under the assumption that

\[
M_{1\ l} = N_{1\ l} + 2E_{01}N_{2\ l} = 0.
\]  \hspace{1cm} (70)

The key idea is to find the connection between the terms \(M_{1\ l} = N_{1\ l} + 2E_{01}N_{2\ l}\) and \(M_{1\ l+1} = N_{1\ l+1} + 2E_{01}N_{2\ l+1}\). By the definitions of \(N_{1\ l}, N_{2\ l}, l = 0, 1, 2, \ldots\) in eqns. (58)-(59), one has

\[
N_{1\ l+1} = \sum_{k=l}^{m} \frac{(-1)^{k+l+2} \binom{m}{k} \binom{k+1}{l+1}}{(2k+3)(2l+2-2m)}
\]  \hspace{1cm} (71)

\[
N_{2\ l+1} = \sum_{k=l+1}^{m} \frac{(-1)^{k+l+2} \binom{m}{k} \binom{k}{l+1}}{(2k+1)(2l+2-2m)}.
\]  \hspace{1cm} (72)

Due to the following relation

\[
\frac{1}{2k+3} \binom{k+1}{l+1} = \frac{(k+1)!}{(2k+3)(k-l)!(l+1)!} = \frac{(k+1)!}{(2k+3)(k+1-l)!l!} \ast \frac{k+1-l}{l+1} = \frac{(k+1)!}{(k+1-l)!} \ast \frac{k+\frac{3}{2}}{l+\frac{1}{2}} \ast \frac{l+\frac{1}{2}}{(l+1)(2k+3)} = -\frac{1}{l+1} \times \frac{1}{2k+3} \binom{k+1}{l} + \frac{1}{2l+2} \binom{k+1}{l},
\]  \hspace{1cm} (73)
it is easy to get

$$ (2l + 2 - 2m)N_{1 \ l+1} = \sum_{k=l}^{m} \frac{(-1)^{k+l} (m)_{(k+1)}}{2k + 3} \quad (74) $$

$$ = \sum_{k=l}^{m} (-1)^{k+l} \binom{m}{k} \left[ -\frac{k}{2k + 3} \binom{k + 1}{l} + \frac{k + 1}{2l + 2} \binom{k + 1}{l} \right] \quad (75) $$

$$ = \sum_{k=l+1}^{m} \left[ -\frac{l + \frac{1}{2}}{l + 1} \times \frac{1}{2k + 3} \right] (-1)^{k+l+1} \binom{m}{k} \binom{k + 1}{l} - \frac{(m-1)}{2(l + 1)} \quad (76) $$

Now, the calculation becomes

$$ (2l + 2 - 2m)N_{1 \ l+1} = \frac{l + \frac{1}{2}}{l + 1} \binom{2l - 2m}{l + 1} - \frac{(m-1)}{2(l + 1)} \binom{m}{k} \binom{k + 1}{l} \quad (77) $$

under the help of eqn. (58). Definition of the quantity

$$ Q_{1 \ l} = -\frac{(m-1)}{2l + 2} + \sum_{k=l}^{m} (-1)^{k+l} \binom{m}{k} \binom{k + 1}{l} \quad (78) $$

may make the equation (77) simple as

$$ (2l + 2 - 2m)N_{1 \ l+1} = \frac{l + \frac{1}{2}}{l + 1} \binom{2l - 2m}{l + 1} + Q_{1 \ l} \quad l = 0, 1, 2, \ldots, m - 1. \quad (79) $$

In completely similar way, one could get the relation between $N_{2 \ l}$ and $N_{2 \ l+1}$. the formula

$$ \frac{1}{2k + 1} \left( \binom{k}{l} \right) = \frac{1}{2k + 1} \frac{k!}{(k - l - 1)!(l + 1)!} = \frac{1}{2k + 1} \frac{k!}{(k - l)!(l)!} \binom{k - l}{l + 1} $$

$$ = \frac{k!}{(k - l)!(l)!} \binom{k}{l + 1} \frac{1}{2k + 1} \binom{k - l}{l + 1} $$

$$ = -\frac{l + \frac{1}{2}}{l + 1} \times \frac{1}{2k + 1} \binom{k}{l} + \frac{1}{2l + 2} \binom{k}{l} \quad (80) $$

helps to simplify the quantity $N_{2 \ l+1}$

$$ (2l + 2 - 2m)N_{2 \ l+1} = \sum_{k=l+1}^{m} \frac{(-1)^{k+l} \binom{m}{k} \binom{k}{l+1}}{2k + 1} \quad (81) $$

$$ = \sum_{k=l+1}^{m} (-1)^{k+l} \binom{m}{k} \left[ -\frac{l + \frac{1}{2}}{l + 1} \times \frac{1}{2k + 1} \binom{k}{l} + \frac{1}{2l + 2} \binom{k}{l} \right] \quad (82) $$
The calculation may also go by different way:

\[ f \text{ under the special case of } Q \text{ makes the equation (84) simply become} \]

\[ (2l + 2 - 2m)N_{2 \ l+1} = \frac{(l + \frac{3}{2})(2l - 2m)}{(l + 1)} N_{2 \ l} + \frac{(m)}{2l + 2} \sum_{k=1}^{m} (-1)^{k+l+1} \binom{m}{k} \binom{k}{l}. \quad (83) \]

Using the eqn. (83) and the fact \(-\frac{l + \frac{3}{2}}{2l + 2} \binom{m}{l} = \frac{1}{2l + 2} \binom{m}{l}\), it is easy to obtain

\[ (2l + 2 - 2m)N_{2 \ l+1} = \frac{(l + \frac{3}{2})(2l - 2m)}{(l + 1)} N_{2 \ l} + Q_{2 \ l}, \ l = 0, 1, 2, \ldots, m - 1. \quad (84) \]

The similar definition of the quantity

\[ Q_{2 \ l} = \frac{(m)}{2l + 2} \sum_{k=1}^{m} (-1)^{k+l} \binom{m}{k} \binom{k}{l} \quad (85) \]

makes the equation (84) simply become

\[ (2l + 2 - 2m)N_{2 \ l+1} = \frac{(l + \frac{3}{2})(2l - 2m)}{(l + 1)} N_{2 \ l} + Q_{2 \ l}, \ l = 0, 1, 2, \ldots, m - 1. \quad (86) \]

The eqns. (79), (80) tell that

\[ M_{1 \ l+1} = N_{1 \ l+1} + 2E_{01} N_{2 \ l+1} \]

\[ = \frac{(l + \frac{3}{2})(2l - 2m)}{(l + 1)(2l - 2m + 2)} M_{1 \ l} + [Q_{1 \ l} + 2E_{01} Q_{2 \ l}], \ l = 0, 1, 2, \ldots, m - 1. \quad (88) \]

the relation between \(M_{1 \ l}, M_{1 \ l+1}\) would be the desired result if one could prove that \(Q_{1 \ l} = Q_{2 \ l} = 0, \ l = 0, 1, 2, \ldots, m - 1. \) This is the following work.

Using the formula

\[ (fg)^{(m)} = \sum_{k=0}^{m} \binom{m}{k} f^{(m-k)} g^{(k)}, \quad (89) \]

under the special case of \( f = \tau, \ g = (1 - \tau)^m, \) one gets

\[ [\tau(1 - \tau)^m]^{(l)} = \tau [(1 - \tau)^m]^{(l)} + \binom{l}{1} [(1 - \tau)^m]^{(l-1)} \]

\[ = (-1)^l m(m-1) \ldots (m-l+1) \tau (1 - \tau)^{m-l} \]

\[ + (-1)^{l-1} l \times m(m-1) \ldots (m-l+2) (1 - \tau)^{m-l+1}. \quad (91) \]

The calculation may also go by different way:

\[ [\tau(1 - \tau)^m]^{(l)} = \left[ \sum_{k=0}^{m} (-1)^k \binom{m}{k} \tau^{k+1} \right]^{(l)} \]
\[
\sum_{k=l-1}^{m} (-1)^k \binom{m}{k} (k+1)k(k-1)\ldots(k-l+2)\tau^{k-l+1}
= \frac{1}{(l-1)!} \sum_{k=l-1}^{m} (-1)^k \binom{m}{k} \binom{k+1}{l} \tau^{k-l+1}
\]

Comparing the results of eqns. (91), (94) and taking the function \([\tau(1-\tau)^m]^{(l)}\) at \(\tau = 1\) under the condition \(l < m\), the good result reaches

\[
\left[ \sum_{k=l-1}^{m} (-1)^k \binom{m}{k} \binom{k+1}{l} \tau^{k-l+1} \right]_{\tau=1} = \sum_{k=l-1}^{m} (-1)^k \binom{m}{k} \binom{k+1}{l} = 0.
\]

By eqn. (78), it is easy to get

\[Q_{\ell l} = 0, \ l = 0, 1, 2, \ldots, m - 1.\]

Similarly, one also could prove that

\[Q_{2l} = 0, \ l = 0, 1, 2, \ldots, m - 1.\]

Therefore, the eqns. (88), (96), (97) imply that

\[M_{1 l+1} = 0, \ l = 0, 1, 2, \ldots, m - 1\]

under the condition

\[M_l = 0;\]

by induction one gets

\[M_n = 0, \ n = 1, 2, \ldots, m.\]

Hence, the boundary conditions could be satisfied by just selecting the only one quantity

\[2E_{01} = -\frac{1}{2m+3}.\]

With the good results \(M_{1 l} = N_{1 l} + 2E_{01}N_{2 l} = 0, l = 0, 1, 2, \ldots, m\), one can greatly simplify the first order super-potential \(W_1\) in the eqn. (50). By similar method as before, rewrite \(W_1\) by changing the independent variable to

\[\tau = \sin \theta\]

and expanding terms \((1-\tau^2)^{k+1}\), \((1-\tau^2)^k\), that is

\[
W_1 = \frac{\sum_{k=0}^{m}(-1)^{k+1}\binom{m}{k}}{\tau^{2m+1}} \left[ \frac{(1-\tau^2)^{k+1}}{2k+3} + \frac{2E_{01}(1-\tau^2)^k}{2k+1} \right]
= (1-\tau^2)^{\frac{1}{2}} \sum_{k=0}^{m}(-1)^{k+1}\binom{m}{k} \left[ \frac{(1-\tau^2)^{k+1}}{2k+3} + \frac{2E_{01}(1-\tau^2)^k}{2k+1} \right] \cdot \tau^{2l-2m-1}
\]
then changing the order of the sums just as before and dividing \(W_1\) by \((1 - \tau^2)^{1/2}\) for good looking in the formula, it reads

\[
\frac{W_1}{(1 - \tau^2)^{1/2}} = \sum_{l=0}^{m+1} \sum_{k=l-1 \geq 0}^{m} \frac{(-1)^{k+l+1}(m)^{(k+1)}}{(2k+3)} \tau^{2l-2m-1} + \sum_{l=0}^{m} \sum_{k=l}^{m} \frac{2E_{0l}(m)^{(k)}}{(2k+1)} \tau^{2l-2m-1} \\
= \sum_{l=0}^{m} \sum_{k=l-1 \geq 0}^{m} \frac{(-1)^{k+l+1}(m)^{(k+1)}}{(2k+3)} \tau^{2l-2m-1} + 2E_{0l} \sum_{k=l}^{m} \frac{(-1)^{k+l+1}(m)^{(k)}}{(2k+1)} \tau^{2l-2m-1} \\
+ \frac{\tau}{2m+3} \\
= \left[ N_1 I + 2E_{0l} N_2 \right] \tau^{2l-2m-1} + \frac{\tau}{2m+3} \\
= \tau^{2l-2m-1} + \frac{\tau}{2m+3} \\
\] (106)

So the quantity \(W_1\) could be written tidily as

\[
W_1 = \frac{\tau (1 - \tau^2)^{1/2}}{2m+3} = \frac{\sin \theta \cos \theta}{2m+3}. \\
\] (109)

**Appendix 2: Simplification of the super potential of the first term \(W_2\)**

Though the process of the calculation of \(W_2\) repeats that of \(W_1\) in the appendix 1, there still are some needs to write it down. Perhaps this may be used for further computation, even by computer.

\[
W_2 = \frac{\bar{A}_2}{\sin^{2m+1} \theta} \\
\] (110)

where

\[
\bar{A}_2 = \int (W_1^2 + 2E_{02}) \sin^{2m+1} \theta d\theta \\
= \int \left[ \frac{1}{(2m+3)^2} \sin^{2m+3} \theta \cos^2 \theta + 2E_{02} \sin^{2m+1} \theta \right] d\theta \\
= - \int \left[ \frac{1}{(2m+3)^2} (1 - \theta^2)^{m+1} \theta^2 + 2E_{02} (1 - \theta^2)^m \right] d\theta \\
\] (113)
The other important term is

$$ II = \int W_2 d\theta $$

$$ = \int \left[ \frac{1}{(2m+3)^2} \sum_{k=-1}^{m+1} (-1)^{k+1} \binom{m+1}{k} x^{2k+2} + 2E_{02} \sum_{k=0}^{m} (-1)^{k+1} \binom{m}{k} x^{2k} \right] dx $$

$$ = \sum_{k=0}^{m+1} \frac{(-1)^{k+1} \binom{m+1}{k} x^{2k+3}}{(2m+3)^2(2k+3)} + 2E_{02} \sum_{k=0}^{m} \frac{(-1)^{k+1} \binom{m}{k} x^{2k+1}}{2k+1} $$

$$ = \sum_{k=0}^{m+1} \frac{(-1)^{k+1} \binom{m+1}{k} \cos^{2k+3} \theta}{(2m+3)^2(2k+3)} + 2E_{02} \sum_{k=0}^{m} \frac{(-1)^{k+1} \binom{m}{k} \cos^{2k+1} \theta}{2k+1} $$

By the transformation $\tau = \sin \theta$, one may have

$$ \int W_2 d\theta = \int \left[ \frac{(-1)^{l+1} \binom{m+1}{k} (1 - \tau^2)^{k+1}}{(2m+3)^2(2k+3)\tau^{2m+1}} + 2E_{02} \sum_{k=0}^{m} \frac{(-1)^{k+1} \binom{m}{k} (1 - \tau^2)^{k}}{(2k+1)\tau^{2m+1}} \right] d\tau $$

$$ = \sum_{k=0}^{m+1} \sum_{l=0}^{k+1} \frac{(-1)^{k+1} \binom{m+1}{k} \binom{k+1}{l} \tau^{2l-2m}}{(2m+3)^2(2k+3)(2l-2m)} + 2E_{02} \sum_{k=0}^{m} \sum_{l=0}^{k} \frac{(-1)^{k+1} \binom{m}{k} \binom{k}{l} \tau^{2l-2m}}{(2k+1)(2l-2m)} $$

Exchanging the sum order, it reads

$$ II = \sum_{l=0}^{m+2} \sum_{k=-1}^{l+1} \frac{(-1)^{k+l+1} \binom{m+1}{k} \binom{k+1}{l} \tau^{2l-2m}}{(2m+3)^2(2k+3)(2l-2m)} + 2E_{02} \sum_{l=0}^{m} \sum_{k=0}^{l} \frac{(-1)^{k+l+1} \binom{m}{k} \binom{k}{l} \tau^{2l-2m}}{(2k+1)(2l-2m)} $$

$$ = \sum_{l=0}^{m+1} \sum_{k=-1}^{l+1} \frac{(-1)^{k+l+1} \binom{m+1}{k} \binom{k+1}{l} \tau^{2l-2m}}{(2m+3)^2(2k+3)(2l-2m)} + 2E_{02} \sum_{k=0}^{m} \sum_{l=0}^{k} \frac{(-1)^{k+l+1} \binom{m}{k} \binom{k}{l} \tau^{2l-2m}}{(2k+1)(2l-2m)} $$

$$ = \sum_{l=0}^{m+1} \sum_{k=-1}^{l+1} \frac{(-1)^{k+l+1} \binom{m+1}{k} \binom{k+1}{l} \tau^{2l-2m}}{(2m+3)^2(2k+3)(2l-2m)} + 2E_{02} \sum_{k=0}^{m} \sum_{l=0}^{k} \frac{(-1)^{k+l+1} \binom{m}{k} \binom{k}{l} \tau^{2l-2m}}{(2k+1)(2l-2m)} $$

$$ + \sum_{l=m+1}^{m+2} \sum_{k=-1}^{l+1} \frac{(-1)^{k+l+1} \binom{m+1}{k} \binom{k+1}{l} \tau^{2l-2m}}{(2m+3)^2(2k+3)(2l-2m)} $$

It is better to define the following quantities

$$ N_3, l = \sum_{k=-1}^{m+1} \frac{(-1)^{k+l+1} \binom{m+1}{k} \binom{k+1}{l}}{(2m+3)^2(2k+3)(2l-2m)}, \quad l = 0, 1, 2, \ldots, m $$

$$ N_4, l = \sum_{k=l}^{m} \frac{(-1)^{k+l+1} \binom{m}{k} \binom{k}{l}}{(2k+1)(2l-2m)}, \quad l = 0, 1, 2, \ldots, m $$
and simplify the last term

$$\sum_{l=m+1}^{m+2} \sum_{k=l+1}^{m+1} \frac{(-1)^{k+l+1}(m+1)(k+1)}{k} r^{2l-2m}$$

$$= \frac{(2m+6)r}{(2m+3)^3(2m+5)} + \frac{\tau^3}{3(2m+3)^2(2m+7)}. \quad (129)$$

Then the express becomes

$$II = \sum_{l=0}^{m} \left[ N_{3,l} + 2E_{02}N_{4,l} \right] r^{2l-2m}$$

$$= \frac{(2m+6)r}{(2m+3)^3(2m+5)} + \frac{\tau^3}{3(2m+3)^2(2m+7)}. \quad (130)$$

Just as done before, one needs to select proper $E_{02}$ to make all $N_{3,l} + 2E_{02}N_{4,l}$ zero. By comparing the quantities $N_{1,l}, N_{2,l}, N_{3,l}, N_{4,l}$, the good relations among them could be revealed:

$$N_{4,l} = N_{2,l}, \quad l = 0, 1, 2, \ldots, m \quad (131)$$

and the great similarity between $N_{1,l}, N_{3,l}$. The transformation forms between $N_{1,l+1}, N_{1,l}, N_{3,l+1}, N_{3,l}$ are similar.

$$N_{3,l+1} = \sum_{k=l+1}^{m+1} \frac{(-1)^{k+l+1}(m+1)(k+1)}{(2m+3)^2(2k+3)(2l+2m+2)} \quad (132)$$

By the relation (73)

$$\frac{1}{2k+3} \binom{k+1}{l+1} = -\frac{l+\frac{1}{2}}{l+1} \times \frac{1}{2k+3} \binom{k+1}{l} + \frac{1}{2l+2} \binom{k+1}{l} \quad (133)$$

one could get

$$\sum_{k=l+1}^{m+1} \frac{(-1)^{k+l}(m+1)}{2k+3} \binom{k+1}{l+1} \quad (134)$$

$$= \sum_{k=l}^{m+1} (-1)^{k+l} \binom{m+1}{k} \left[ -\frac{l+\frac{1}{2}}{l+1} \times \frac{1}{2k+3} \binom{k+1}{l} + \frac{1}{2l+2} \binom{k+1}{l} \right]$$

$$= \sum_{k=l+1}^{m+1} \left[ \frac{l+\frac{1}{2}}{l+1} \times \frac{1}{2k+3} \right] (-1)^{k+l+1} \binom{m+1}{k} \binom{k+1}{l} - \frac{(m+1)}{2(l+1)} \quad (135)$$

$$+ \sum_{k=l}^{m+1} (-1)^{k+l} \binom{m+1}{k} \binom{k+1}{l}; \quad (136)$$

with the help of eqn. (126), the calculation becomes

$$N_{3,l+1} = \frac{(l+\frac{1}{2})(2l-2m)}{(l+1)(2l+2-2m)} N_{3,l} + \frac{\sum_{k=l}^{m+1} (-1)^{k+l} \binom{m+1}{k} \binom{k+1}{l}}{(2m+3)^2(2l+2-2m)}. \quad (137)$$
The definition of the quantity

\[ Q_3 l = \frac{(m+1)}{2l+2} + \sum_{k=l}^{m+1} \frac{(-1)^{k+l}(m+1)^{k+1}}{2l+2} \]

\[ = \frac{1}{2l+2} \sum_{k=l-1}^{m+1} (-1)^{k+l} \binom{m+1}{k} \binom{k+1}{l} \] (138)

makes the above equation (137) become

\[ N_{3, l+1} = \frac{(l+\frac{1}{2})(2l-2m)}{(l+1)(2l+2-2m)} N_{3, l} + \frac{Q_3 l}{(2m+3)^2(2l+2-2m)}, \quad l = 0, 1, 2, \ldots, m-1. \] (139)

Another useful relation is

\[ N_{4, l+1} = N_{2, l+1} = \frac{(l+\frac{1}{2})(2l-2m)}{(l+1)(2l+2-2m)} N_{2, l} \] (140)

\[ = \frac{(l+\frac{1}{2})(2l-2m)}{(l+1)(2l+2-2m)} N_{4, l} \] (141)

by the use of \( Q_{2, l} = 0 \). Now the quantities \( M_2, l \) is defined as

\[ M_2, l = N_3, l + 2E_{02} N_4, l, \quad l = 0, 1, 2, \ldots, m \] (142)

then the relation between \( M_2, l \) and \( M_2, l+1 \) is

\[ M_{2, l+1} = \frac{(l+\frac{1}{2})(2l-2m)}{(l+1)(2l-2m+2)} M_{2, l} + \frac{Q_3 l}{(2m+3)^2(2l+2-2m)} M_{2, l}, \quad l = 0, 1, \ldots, m-1. \] (143)

In the following, it will be proven that \( Q_{3, l} = 0 \). With the help

\[ (fg)^{(m)} = \sum_{k=0}^{m} \binom{m}{k} f^{(m-k)} g^{(k)} \] (144)

under the special case \( f = \tau, \ g = (1-\tau)^{m+1} \), one gets

\[ \left[ \tau (1-\tau)^{m+1} \right]^{(l)} = \tau \left[ (1-\tau)^{m+1} \right]^{(l)} + \binom{l}{1} \left[ (1-\tau)^{m+1} \right]^{(l-1)} \]

\[ = (-1)^l m(m+1) \ldots (m-l+2) \tau (1-\tau)^{m-l+1} \]

\[ + (-1)^{l-1} l \times m(m+1) \ldots (m-l+2)(1-\tau)^{m-l+2}. \] (146)

The alternative way to compute is

\[ \left[ \tau (1-\tau)^{m+1} \right]^{(l)} = \left[ \sum_{k=0}^{m+1} (-1)^k \binom{m+1}{k} \tau^{k+1} \right]^{(l)} \]

\[ = \sum_{k=l-1}^{m+1} (-1)^k \binom{m+1}{k} (k+1)(k+1) \ldots (k-l+2) \tau^{k-l+1} \]

\[ = \frac{1}{(l-1)!} \sum_{k=l-1}^{m+1} (-1)^k \binom{m+1}{k} \binom{k+1}{l} \tau^{k-l+1}. \] (149)
The valuation of \([\tau(1 - \tau)^{m+1}]^{(l)}\) at \(\tau = 1\) under the condition \(l < m + 1\) is

\[
\left[ \sum_{k=l-1}^{m+1} (-1)^k \binom{m+1}{k} \binom{k+1}{l} \tau^{k-l+1} \right]_{\tau=1} = \sum_{k=l-1}^{m+1} (-1)^k \binom{m+1}{k} \binom{k+1}{l} = 0. \tag{150}
\]

From the eqn. (138), it is easy to get

\[
Q_{3 \ l} = 0 \ l = 0, 1, 2, \ldots, m - 1. \tag{151}
\]

Finally, one gets the good result

\[
M_{2 \ l+1} = \frac{(l + \frac{1}{2})(2l - 2m)}{(l + 1)(2l - 2m + 2)} M_{2 \ l}, \ l = 0, 1, 2, \ldots, m - 1. \tag{152}
\]

So the choice of \(M_{2 \ 0} = 0\) guarantees \(M_{2 \ l} = 1, 2, \ldots, m\). The quantity \(E_{02}\) is obtained by

\[
E_{02} = - \frac{N_{3 \ 0}}{N_{4 \ 0}}. \tag{153}
\]

As in the case \(N_{1 \ 0}\),

\[
N_{3 \ 0} = \frac{1}{2m} \sum_{k=0}^{m+1} (-1)^{k+2} \frac{(m+1)}{k} \frac{(2m + 2)!}{(2m + 3)^2(2k + 3)} = \frac{1}{2m(2m + 3)^2} \left[ \frac{(2m + 2)!}{(2m + 3)!} - \frac{(2m + 4)!}{(2m + 5)!} \right]
\]

\[
N_{4 \ 0} = N_{2 \ 0} = \frac{1}{2m} \sum_{k=0}^{m} (-1)^{k+1} \frac{(m)}{k} \frac{1}{2k + 1} = \frac{1}{2m} \frac{(2m)!}{(2m + 1)!} \cdot \tag{154}
\]

Hence

\[
E_{02} = - \frac{N_{3 \ 0}}{N_{4 \ 0}} = - \frac{2m + 2}{(2m + 3)^2(2m + 5)}. \tag{155}
\]

The expression

\[
W_2 = \frac{\sum_{k=0}^{m+1} (-1)^{k+1} \frac{(m+1)}{k} \cos^{2k+3} \theta}{(2m + 3)^2(2k + 3)} + 2E_{02} \sum_{k=0}^{m} (-1)^{k+1} \frac{(m)}{k} \cos^{2k+1} \theta \tag{156}
\]

could greatly simplified by \(\tau = \sin \theta\) and the use of the elegant formula \(M_{2 \ l} = 0\):

\[
\frac{W_2}{\cos \theta} = \sum_{k=0}^{m+1} (-1)^{k+1} \frac{(m+1)}{k} \frac{(1 - \tau^2)^{k+1}}{(2m + 3)^2(2k + 3)\tau^{2m+1}} + 2E_{02} \sum_{k=0}^{m} (-1)^{k+1} \frac{(m)}{k} \frac{(1 - \tau^2)^{k+1}}{(2k + 1)\tau^{2m+1}} \tag{157}
\]

\[
= \sum_{k=0}^{m+1} \sum_{l=0}^{k+1} (-1)^{k+l+1} \frac{(m+1)}{k} \frac{(k+1)}{l} \frac{(1 - \tau^2)^{k+1}}{(2m + 3)^2(2k + 3)\tau^{2m-2l+1}} + 2E_{02} \sum_{k=0}^{m} \sum_{l=0}^{k} (-1)^{k+l+1} \frac{(m)}{k} \frac{(k+1)}{l} \frac{(1 - \tau^2)^{k+1}}{(2k + 1)\tau^{2m-2l+1}}. \tag{158}
\]

Exchanging the sums order, it is easy to get

\[
\frac{W_2}{\cos \theta} = \sum_{l=0}^{m+2} \sum_{k=l-1}^{m+1} (-1)^{k+l+1} \frac{(m+1)}{k} \frac{(k+1)}{l} \frac{(1 - \tau^2)^{k+1}}{(2m + 3)^2(2k + 3)\tau^{2m-2l+1}} + 2E_{02} \sum_{l=0}^{m} \sum_{k=l}^{m} (-1)^{k+l+1} \frac{(m)}{k} \frac{(k+1)}{l} \frac{(1 - \tau^2)^{k+1}}{(2k + 1)\tau^{2m-2l+1}}. \tag{159}
\]
comparing the eqns. (126) and (127), one may have

\[
W_2 = \sum_{l=0}^{m} \left[ N_3 l + 2E_{02} N_4 l \right] + \sum_{l=m+1}^{m+2} \sum_{k=l-1 \geq 0}^{m+1} \frac{(-1)^{k+l+1} (m+1) (l+1)}{(2m+3)^2 (2k+3) \tau^{2m-2l+1}}
\]

(160)

\[
= \sum_{l=m+1}^{m+2} \sum_{k=l-1 \geq 0}^{m+1} \frac{(-1)^{k+l+1} (m+1) (l+1)}{(2m+3)^2 (2k+3) \tau^{2m-2l+1}}
\]

(161)

\[
= \frac{-\tau}{(2m+3)^3 (2m+5) + (2m+3)^2 (2m+5)} \tau^3
\]

(162)

by the use of \( M_{2 l} = N_{3 l} + 2E_{02} N_{4 l} = 0 \). The elegant form of \( W_2 \) is

\[
W_2 = \left[ \frac{-1}{(2m+3)^3 (2m+5)} + \frac{\sin^2 \theta}{(2m+3)^2 (2m+5)} \right] \sin \theta \cos \theta.
\]

(163)

This ends the appendix 2.

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