On Imprimitive Representations of Finite Reductive Groups in Non-defining Characteristic

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Abstract

In this paper, we begin with the classification of Harish-Chandra imprimitive representations in non-defining characteristic. We recall the connection of this problem to certain generalizations of Iwahori-Hecke algebras and show that Harish-Chandra induction is compatible with the Morita equivalence by Bonnafé and Rouquier, thus reducing the classification problem to quasi-isolated blocks. Afterwards, we consider imprimitivity of unipotent representations of certain classical groups. In the case of general linear and unitary groups, our reduction methods then lead to results for arbitrary Lusztig series.

Keywords: modular representation theory, finite reductive groups, Harish-Chandra induction, imprimitive representation

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1 Introduction

In [8] and [9], the imprimitive representations of finite quasi-simple groups in characteristic 0 were classified and some results were obtained for arbitrary characteristic. Focusing now on positive characteristic, one big part of the classification problem of imprimitive representations revolves around Harish-Chandra imprimitive representations of finite reductive groups.

Let $G$ be a connected reductive group over the algebraic closure $\mathbb{F}$ of a field $\mathbb{F}_q$ with $q$ elements and suppose that $G$ is defined over $\mathbb{F}_q$ via a Frobenius morphism $F : G \to G$. We consider a prime $\ell$ not dividing $q$ and an $\ell$-modular system $(K, R, k)$ which is split for $G^F$ and all its subgroups.

For an $F$-stable parabolic subgroup $P \subset G$ with unipotent radical $V$ and $F$-stable Levi complement $L$ and $\Lambda \in \{K, R, k\}$ we consider Harish-Chandra induction

$$R^{G^F}_{\Lambda L^F} : \Lambda L^F - \text{mod} \to \Lambda G^F - \text{mod}, \ X \mapsto \Lambda G^F / V^F \otimes_{\Lambda L^F} X$$
and Harish-Chandra restriction

\[ \ast R^G_{L \subseteq P} : AG^F\text{-mod} \rightarrow AL^F\text{-mod}, X \mapsto AV^F \backslash G^F \otimes_{AG^F} X. \]

These two functors are left and right adjoints of one another and fixing the split Levi subgroup \( L \) while changing the parabolic subgroup gives rise to naturally equivalent functors.

An imprimitive representation is one which is induced from a proper subgroup. When only considering Harish-Chandra induced representations, we get the notion of Harish-Chandra imprimitivity.

**Definition 1.1.** A simple \( kG^F \)-module \( X \) is called (Harish-Chandra) imprimitive if \( R^G_L X' \cong X \) for some \( kL^F \)-module \( X' \) where \( L \subseteq G \) is a proper split Levi subgroup of \( G \). If \( X \) is not imprimitive, it is said to be primitive.

Since we are only considering the concept of Harish-Chandra imprimitivity in the following, there will be no confusion in calling it simply imprimitivity. It is also noteworthy that by [8, Prop. 7.1], the notion of Harish-Chandra imprimitivity coincides with the more general notion of imprimitivity for quasi-simple groups of Lie type.

It takes multiple steps to reach a classification of Harish-Chandra imprimitive representations in non-defining characteristic. It makes sense to look at unipotent representations of groups with connected center first with the idea of reducing the general problem to this case or similar cases. For most of the classical groups, one can apply a result by Christoph Schönnenbeck on Iwahori-Hecke algebras ([11]) to the endomorphism algebras of the Harish-Chandra induction of cuspidal modules to find that imprimitivity is quite rare for unipotent representations but actually does occur in contrast to the analogous situation in characteristic 0 ([8, Corollary 8.5]).

To get from unipotent representations to arbitrary ones is more difficult in positive characteristic than in characteristic 0. For instance, we do not have an analogue to the Jordan decomposition of characters. However, we can at least make use of the Morita equivalence by Bonnafe and Rouquier from [2] to reduce our problem to the study of representations in (quasi-)isolated Lusztig series. For this, we shall prove that Harish-Chandra induction commutes with this kind of Morita equivalences. As a consequence, we will be able to extend our results on unipotent representations to arbitrary Lusztig series for the general linear and unitary groups.
2 Imprimitivity and Hecke Algebras

In characteristic 0, the property of a representation of $G^F$ to be imprimitive turns out to be a property of the Harish-Chandra series it belongs to. In fact [8, Thm. 8.3] implies that if there is one imprimitive representation in a Harish-Chandra series, then all representations in this series are imprimitive. The proof of this theorem relies heavily on the knowledge of the algebras $\text{End}_{kG^F}(R_{L,0}^G X_0)$ where $X_0$ is a cuspidal $kL_0^F$-module. In this section, we shall review the properties of corresponding algebras in characteristic $\ell$.

Let $(L_0, X_0)$ be a cuspidal pair of $(G, F)$, that is, $L_0 \subseteq G$ is a split Levi subgroup of $G$ and $X_0$ is a simple cuspidal $kL_0^F$-module. We consider the full subcategory $kG^F-\text{mod}_{(L_0, X_0)}$ of $kG^F-\text{mod}$ whose objects $X$ admit a monomorphism $X \rightarrow (R_{L_0}^G X_0)^m$ and an epimorphism $(R_{L_0}^G X_0)^n \rightarrow X$ for some positive integers $m, n \in \mathbb{N}$. In particular, every simple $kG^F$-module belonging to the Harish-Chandra series of $(L_0, X_0)$ is an object in $kG^F-\text{mod}_{(L_0, X_0)}$ by [4, Thm. 1.28]. As an important consequence of this, we note that the simple objects in $kG^F-\text{mod}_{(L_0, X_0)}$ that is, the non-zero objects whose only proper subobjects are zero-objects, are precisely the simple $kG^F$-modules belonging to the Harish-Chandra series of $(L_0, X_0)$.

The functor

$$\text{Hom}_{kG^F}(R_{L_0}^G X_0, -) : kG^F-\text{mod}_{(L_0, X_0)} \rightarrow \text{End}_{kG^F}(R_{L_0}^G X_0)^{\circ-}\text{mod}$$

is an equivalence of categories by [4, Thms. 1.20, 1.25].

Recall that the algebra $\text{End}_{kG^F}(R_{L_0}^G X_0)^{\circ}$ has a $k$-basis $\{B_w\}_w$ indexed by $w \in N_{G^F}(L_0, X_0)/L_0^F$, where $N_{G^F}(L_0, X_0) = \{g \in N_G(L)^F \mid \varphi X_0 \cong X_0\}$. This algebra is akin to Iwahori-Hecke algebras in view of [7, Thm. 3.12].

Lemma 2.1. Given any split Levi subgroup $L \subseteq G$ with $L_0 \subseteq L$, the algebra morphism $R_L^G : \text{End}_{kL^F}(R_{L,0}^L X_0) \rightarrow \text{End}_{kG^F}(R_{L_0}^G X_0)$ is injective with $R_L^G(B_w) = B_w$ for all $w \in N_{L^F}(L_0, X_0)/L_0^F$. Moreover, the diagram

$$\begin{array}{ccc}
kG^F-\text{mod}_{(L_0, X_0)} & \xrightarrow{\text{Hom}_{kG^F}(R_{L_0}^G X_0, -)} & \text{End}_{kG^F}(R_{L_0}^G X_0)^{\circ-}\text{mod} \\
R_L^G & \downarrow & \text{Ind}_{kG^F} \\
kL^F-\text{mod}_{(L_0, X_0)} & \xrightarrow{\text{Hom}_{kL^F}(R_{L_0}^L X_0, -)} & \text{End}_{kL^F}(R_{L_0}^L X_0)^{\circ-}\text{mod}
\end{array}$$

is commutative up to natural isomorphism.

Proof. Let us set $X = R_{L_0}^L X_0$. Since Harish-Chandra induction is faithful,
the morphism $R^G_L : \text{Hom}_{kL^F}(X,X) \to \text{Hom}_{kG^F}(R^G_L X, R^G_L X)$ is injective. The identity $R^G_L(B_w) = B_w$ for all $w \in N_{L^F}(L_0, X_0)/L_0^F$ follows with a simple calculation from the definition of the $B_w$ [7, (3.5)] and the transitivity of Harish-Chandra induction.

For $kL^F$-modules $Y$ and $Z$, we consider the natural map

$$\text{Hom}_{kL^F}(Z, X) \otimes \text{End}_{kL^F}(X) \to \text{Hom}_{kL^F}(Z, Y),$$

$\varphi \otimes f \mapsto f \circ \varphi.$

This is an isomorphism for $Z = X$ and thus also for $Z$ being a direct sum of copies of $X$. In particular, by the Mackey formula [5, Thm. 5.1],

$$\text{Hom}_{kL^F}(R^G_L X, X) \otimes \text{End}_{kL^F}(X) \to \text{Hom}_{kG^F}(R^G_L X, R^G_L X),$$

$\varphi \otimes f \mapsto f \circ \varphi$

is an isomorphism natural in $Y \in kL^F_{-}\text{-mod}(_{L_0,X_0})$ and by adjointness, the map

$$\text{End}_{G^F}(R^G_L X) \otimes \text{End}_{kL^F}(X) \to \text{Hom}_{kG^F}(R^G_L X, R^G_L Y),$$

$\varphi \otimes f \mapsto R^G_L(f) \circ \varphi$

is an isomorphism, too.

This result tells us that finding the imprimitive representations in the Harish-Chandra series of $(L_0, X_0)$ amounts to finding the simple $\text{End}_{G^F}(R^G_L X)$-modules which are of the form $\text{Ind}_{R^G_L}^G(M)$ for some simple $\text{End}_{kL^F}(R^L_{L_0}X_0)$-module $M$. The easiest case is the following.

**Corollary 2.2.** Let $L \subseteq G$ be a proper split Levi subgroup of $G$ containing $L_0$. If $N_{G^F}(L_0, X_0) = N_{L^F}(L_0, X_0)$, then every simple $kG^F$-module belonging to the Harish-Chandra series of $(L_0, X_0)$ is imprimitive.

**Proof.** If $N_{G^F}(L_0, X_0) = N_{L^F}(L_0, X_0)$, then

$$R^G_L : \text{End}_{kL^F}(R^L_{L_0}X_0) \to \text{End}_{kG^F}(R^G_L X_0)$$

is a monomorphism between $k$-vector spaces of the same dimension and thus an isomorphism. By Lemma (2.1), this implies that every simple $kG^F$-module belonging to the Harish-Chandra series of $(L_0, X_0)$ is Harish-Chandra induced.

It is conjectured that the converse of this corollary also holds true if the center of $G$ is connected. As in characteristic 0, imprimitivity of a $kG^F$-module
would then imply the imprimitivity of every other module in the same Harish-Chandra series.

3 The Bonnafé – Rouquier Morita equivalence

In characteristic 0, as was mentioned before, imprimitivity can be viewed as a property of Harish-Chandra series. Moreover, it was proven in [8, Thm. 7.3, Thm 8.4] that imprimitivity can also be viewed as a property of Lusztig series in characteristic 0.

Lusztig series are compatible with modular representation theory as was shown by Broué and Michel in [3]. In particular, certain unions of Lusztig series turn out to be unions of ℓ-blocks.

In [2], Bonnafé and Rouquier showed that every ℓ-block is Morita equivalent to some quasi-isolated ℓ-block. In this section, we shall show that this Morita equivalence is compatible with Harish-Chandra induction which also implies that it preserves and reflects the property of being imprimitive. To do so we shall need a result by Bonnafé, Dat and Rouquier from [1] which gives a sufficient condition for Lusztig induction to depend only on the Levi subgroup (and not on the parabolic subgroup).

So let \((G^*, F^*)\) be a group in duality with \((G, F)\). Recall that we have a decomposition

\[
\Lambda G^F = \bigoplus_{[s]} \Lambda G^F e_s^{G^F}
\]

into sums of blocks corresponding to the decomposition

\[
\text{IBr}(G^F) = \bigcup_{[s]} \mathcal{E}_l(G, F; [s])
\]

into ℓ-modular Lusztig series. Both decompositions are indexed by the conjugacy classes of semisimple ℓ’-elements in \(G^*\).

Let us fix a semisimple element \(s \in (G^*)^{F^*}\) and let \(G^*_s \subseteq G^*\) be a rational Levi subgroup containing \(C_{G^*}(s)\). Let \(G_s \subseteq G\) correspond to \(G_s\) under duality. If \(P_s = G_s V_s\) denotes a parabolic subgroup of \(G\) with Levi complement \(G_s\), then the Deligne-Lusztig variety

\[
Y_{G, [s]}^{G_s \subseteq P_s} = \{gV_s \in G/V_s \mid g^{-1}F(g) \in V_s F(V_s)\}
\]

has the property \(H^i_c(Y_{G, [s]}^{G_s \subseteq P_s}, \Lambda) = 0\) except for \(i = d := \dim(Y_{G, [s]}^{G_s \subseteq P_s})\), and \(H^d_c(Y_{G, [s]}^{G_s \subseteq P_s}, \Lambda)\) induces a Morita equivalence between the sum of blocks \(\Lambda G_s^{P_s} e_s^{G^F}\) and \(\Lambda G e_s^{G^F}\) where \(e_s^{G^F}\) and \(e_s^{G^F}\) denote the central idempotents corresponding
Thus, the assumptions of Lusztig induction from $G$ to $G^*$ a dual correspondent and suppose that $s \in L^*$. We set $L_s = L \cap G_s$ and $L_s^* = L^* \cap G_s^*$. Then $L_s \subseteq G_s$ is a split Levi subgroup of $G_s$ and $L_s^*$ is a dual correspondent. Since we have $C_G^*(s) \subseteq G^*_s$, we also have $C_{L^*}(s) = L^* \cap C_G^*(s) \subseteq L^* \cap G^*_s = L^*_s$. The group $P_s \cap L = L_s(V_s \cap L)$ is a parabolic subgroup of $L$ with Levi complement $L_s$.

As above, the associated Deligne-Lusztig variety $Y_{L,\cap P,\cap L_s}^L$ has non-vanishing cohomology only in degree $d' = \dim(Y_{L,\cap P,\cap L_s}^L)$ and the $\Lambda L^F e_s^{L^F} \otimes (\Lambda L^F e_s^{L^F})^*$-module $H_\cdot^d(Y_{L,\cap P,\cap L_s}^L, \Lambda)$ induces a Morita equivalence between $\Lambda L^F e_s^{L^F}$ and $\Lambda L^F e_s^{L^F}$.

We want to show that the two Morita equivalences just obtained turn Harish-Chandra induction from $L_s^F$ to $G_s^F$ into Harish-Chandra induction from $L^F$ to $G^F$.

Let $P = LV$ be a rational parabolic subgroup of $G$ having $L$ as Levi complement. Then $P_1 = (P \cap P_s)V$ and $P_2 = (P \cap P_s)V_s$ are both parabolic subgroups of $G$ having $L_s$ as Levi complement. Their unipotent radicals are given by $V_1 = (V_s \cap L)V$ and $V_2 = (G_s \cap V)V_s$, respectively.

We consider their dual correspondents $V_1^*$ and $V_2^*$ and find that $C_G^*(s) \subseteq G^*_s$ implies

$$C_{V_1^*}(s) = C_{(V_1 \cap L^*)} V^* \cap G^*_1(s) = C_G^* V^*(s)$$

as well as

$$C_{V_2^*}(s) = C_{(G^* \cap V^*)} V^* \cap G^*_1(s) = C_G^* V^*(s).$$

Thus, the assumptions of [1, Cor. 6.5] are satisfied and we conclude that Lusztig inductions with respect to $P_1$ and $P_2$ are equivalent up to shifting (and a Tate twist). Using the transitivity of Lusztig induction (cf. [4, Thm. 7.9] and [2, 3.3]), we find that the diagram

$$\begin{array}{ccc}
D(\Lambda G^F e_s^{G^F} - \text{mod}) & \xrightarrow{H_\cdot^d(Y_{L,\cap P_s}^L, \Lambda) \otimes -} & D(\Lambda G^F e_s^{G^F} - \text{mod}) \\
R_{L_s}^{G^F} \downarrow & & \downarrow R_{L_s}^{G^F} \\
D(\Lambda L^F e_s^{L^F} - \text{mod}) & \xrightarrow{H_\cdot^d(Y_{G_s \cap P_s}^L, \Lambda) \otimes -} & D(\Lambda L^F e_s^{L^F} - \text{mod})
\end{array}$$

is commutative up to shifting and equivalence. However, since all the functors are exact, with the vertical functors being Harish-Chandra induction, they commute with homology which implies that no shifting is required for the diagram.
to commute and so we actually obtain the following result.

**Lemma 3.1.** Given the notation and assumptions of this section, the diagram

\[
\begin{array}{ccc}
\Lambda G^F G^r & \xrightarrow{\text{mod} H^d(Y_G, \mathbb{F}, L)} & \Lambda G^F G^r \\
\Lambda F^G & \xrightarrow{\text{mod} H^d(Y_G, \mathbb{F}, L)} & \Lambda F^G
\end{array}
\]

is commutative up to natural equivalence.

4 **Imprimitivity for unipotent representations of classical groups**

In this section we are going to use the results from Section 4 of [7]. Accordingly, we let \((G, F)\) be such that \(G^F = G_n(q)\) is one of the groups

(a) \(GL_n(q)\) (any \(q, n \geq 0\))
(b) \(GU_n(q)\) (any \(q, n \geq 0\))
(c) \(Sp_n(q)\) (\(q\) a power of 2, \(n \geq 0\) even)
(d) \(CSp_n(q)\) (\(q\) odd, \(n \geq 0\) even)
(e) \(SO_n(q)\) (\(q\) odd, \(n \geq 0\) odd)

The reason for restricting to this list of groups is the following result which is, for instance, not known for the even orthogonal groups.

**Proposition 4.1.** Let \((L_0, X_0)\) be a cuspidal pair of \((G, F)\). If \(X_0\) is unipotent, then we have

\[N_{G^F}(L_0, X_0) = N_{G^r}(L_0)\]

and \(X_0\) is extendible to \(N_{G^r}(L_0)\).

Moreover, the algebra \(\text{End}_{G^r}(R_{L_0}^F X_0)\) is an Iwahori-Hecke algebra associated with the relative Weyl group \(N_{G^r}(L_0)/L_0^F\).

**Proof.** This was proven in [7, 4.3 and 4.4] for the cases (b)–(e). All but the last statements can be proven for case (a) by analogous arguments. The last statement follows in case (a) from [4, 19.20].
We can now obtain a converse of Corollary (2.2) for the unipotent representations of the classical groups we consider.

**Theorem 4.2.** Let \((X_0, L_0)\) be a cuspidal pair of \((G, F)\) where \(X_0\) is unipotent. Then the following statements are equivalent:

(i) There exists a \(kG^F\)-module in the Harish-Chandra series of \((L_0, X_0)\) which is primitive.

(ii) Every \(kG^F\)-module in the Harish-Chandra series of \((L_0, X_0)\) is primitive.

(iii) We have \(N_{G^F}(L_0) \neq N_{L^F}(L_0)\) for every proper split Levi subgroup \(L \subseteq G\) containing \(L_0\).

(iv) We are in one of the cases (b)–(e) or we are in case (a) and \(L_0^F \cong GL_1(q)^n\) or \(L_0^F \cong GL_{e\ell}(q)^m\) where \(e\) is the order of \(q\) modulo \(\ell\) and \(n = me\ell^i\).

**Proof.** The algebra \(\text{End}_{kG^F}(R_{L_0}^G X_0)\) is an Iwahori-Hecke algebra by Proposition (4.1) and the embedding

\[ R_L^G : \text{End}_{kL^F}(R_{L_0}^L X_0) \rightarrow \text{End}_{kG^F}(R_{L_0}^G X_0) \]

identifies the domain with a parabolic subalgebra of this Iwahori-Hecke algebra for any split Levi subgroup \(L_0 \subseteq L \subseteq G\).

If this embedding identifies the domain with a proper parabolic subalgebra of \(\text{End}_{kG^F}(R_{L_0}^G X_0)\), then it follows from [11, Thm. 1.1] that the induced module \(\text{Ind}_{R_L^G} M\) is reducible for every \(\text{End}_{kL^F}(R_{L_0}^L X_0)\)-module \(M\). On the other hand, if the above embedding is an isomorphism, then clearly \(\text{Ind}_{R_L^G} M\) is simple for every simple \(\text{End}_{kL^F}(R_{L_0}^L X_0)\)-module \(M\).

In view of Lemma (2.1), this implies the equivalence of (i) and (ii).

Comparing dimensions we also find that either of these statements is equivalent to \(N_{L^F}(L_0, X_0) \neq N_{G^F}(L_0, X_0)\) for all proper split Levi subgroups \(L_0 \subseteq L \subseteq G\). By Proposition (4.1), this is equivalent to (iii).

In the cases (b)–(e), the structure of the normalizers of Levi subgroups admitting cuspidal unipotent representations has been analyzed in the proof of [7, Prop. 4.3]. It is easy to see that (iii) is always satisfied in these cases.

In case (a), \(L_0^F\) is conjugate in \(G^F\) to a group of the form

\[ GL_1(q)^{m_{-1}} \times \prod_{i=1}^{r} GL_{e\ell^i}(q)^{m_i} \]

with \(e\) the order of \(q\) modulo \(\ell\) and \(m_{-1}, m_0, \ldots, m_r \geq 0\) as well as \(n - m_{-1} = e \sum_{i=1}^{r} m_i \ell^i\) (cf. [6, (7.9)]).
The group of rational points of the smallest split Levi subgroup containing $N_{G^F}(L_0)$ can now easily be seen to be conjugate in $G^F$ to a group of the form

$$GL_{m-1}(q) \times \prod_{i=1}^{r} GL_{e_m(i)}(q).$$

Thus, in case (a), condition (iii) is equivalent to $L^F_0$ being isomorphic to $GL_1(q)^n$ or to $GL_{e_F}(q)^m$ where $n = me^{l_F}$. This completes the proof.

5 Imprimitivity for $GL_n(q)$ and $GU_n(q)$

In this section we let $G = GL_n(F)$ and $F$ be either the standard Frobenius morphism $F_q$ defined by $F_q((a_{i,j})) = (a_{i,j}^q)$ or the twisted Frobenius $F'_q$ defined by $F'_q((a_{i,j})) = (a_{i,j}^q)^{-tr}$ for all $(a_{i,j}) \in GL_n(F)$.

For these groups, we can actually use our results on the Morita equivalence by Bonnafé and Rouquier together with Theorem (4.2) to obtain the converse of Corollary (2.2) for arbitrary Lusztig series.

**Corollary 5.1.** Let $M$ be a simple $kG^F$-module which belongs to the Harish-Chandra series of $(L_0, X_0)$. Then $M = R^G_{s} M'$ for some split Levi subgroup $L_0 \subseteq L \subseteq G$ if and only if $N_{G^F}(L_0, X_0) = N_{L^F}(L_0, X_0)$.

**Proof.** If $X_0$ is unipotent, then the claim holds by Theorem (4.2). There exists a semisimple element $s \in (L_0^F)_{tr}$ such that $X_0$ is an object of $kL_0^F e_{s}^{L_0^F} - \text{mod}$. The groups $G_s = C_G(s)$, $L_s = C_L(s)$ and $L_{0,s} = C_{L_0}(s)$ are rational Levi subgroups of $G$, $L$ and $L_0$, respectively, and the diagram

$$
\begin{array}{ccc}
 kG^F e_s^{G^F} - \text{mod} & \rightarrow & kG_s^F e_s^{G_s^F} - \text{mod} \\
 R^F_s & \rightarrow & R^F_{s,s} \\
 kL^F e_s^{L^F} - \text{mod} & \rightarrow & kL_s^F e_s^{L_s^F} - \text{mod} \\
 R^F_{s,s} & \rightarrow & R^F_{s,s,s} \\
 kL_0^F e_s^{L_0^F} - \text{mod} & \rightarrow & kL_{0,s}^F e_s^{L_{0,s}^F} - \text{mod} \\
 R^F_{s,s} & \rightarrow & R^F_{s,s,s}
\end{array}
$$

commutes by Lemma (3.1) up to natural equivalence. Note that $s$ is central in $G_s$, so we have isomorphisms

$$kG_s^F e_s^{G_s^F} \cong kG_s^F e_1^{G_s^F}$$

and

$$kL_s^F e_s^{L_s^F} \cong kL_s^F e_1^{L_s^F}.$$
as well as
\[ kL_{0,s}^F e_{s}^{L_{0,s}, s} \cong kL_{0,s}^F e_{1}^{L_{0,s}, s} \]
induced by a linear character \( \lambda_s \) of \( G_s^F \). As tensoring with linear characters commutes with Harish-Chandra induction, the diagram

\[ \begin{array}{ccc}
  kG_s^F e_{s}^{G_s^F} - \text{mod} & \xrightarrow{\lambda_s \otimes} & kG_s^F e_{1}^{G_s^F} - \text{mod} \\
  kL_s^F e_{s}^{L_s^F} - \text{mod} & \xrightarrow{\lambda_s \otimes} & kL_s^F e_{1}^{L_s^F} - \text{mod} \\
  kL_{0,s}^F e_{s}^{L_{0,s}, s} - \text{mod} & \xrightarrow{\lambda_s \otimes} & kL_{0,s}^F e_{1}^{L_{0,s}, s} - \text{mod}
\end{array} \]

is commutative.

Combining these diagrams we obtain a unipotent \( kG_s^F \)-module \( M_u \) such that \( \lambda_s \otimes M_u \) corresponds to \( M \) under the Morita equivalence between \( kG_s^F e_{s}^{G_s^F} \) and \( kG_s^F e_{s}^{G_s^F} \). In the same way, we obtain a unipotent cuspidal \( kL_{0,s}^F \)-module \( X_u \) with \( \lambda_s \otimes X_u \) corresponding to \( X_0 \) under the analogous Morita equivalence.

Suppose now that \( M = R_G^F M' \) for some \( kL_s^F \)-module \( M' \). We let \( M'_u \) be the unipotent \( kL_s^F \)-module that corresponds to \( M' \). We thus have \( M_u \cong R_L^F R_G^F M'_u \). Since \( G_s^F \) is a direct product of general linear groups and general unitary groups we have \( N_{G_s^F}(L_{0,s}, X_u) = N_{L_s^F}(L_{0,s}, X_u) \) by Theorem (4.2).

Using

\[ \text{End}_{kL_s^F}(R_{L_{0,s}}^F X_0) \cong \text{End}_{kL_s^F}(R_{L_{0,s}}^F X_u) \]

and

\[ \text{End}_{kG_s^F}(R_{L_{0,s}}^F X_0) \cong \text{End}_{kG_s^F}(R_{L_{0,s}}^F X_u) \]

and comparing dimensions, we obtain \( N_{G}(L_0, X_0) = N_{L_s^F}(L_0, X_0) \) as desired.

\[ \square \]

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