On partial regularity for the steady Hall magnetohydrodynamics system

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Abstract

We study partial regularity of suitable weak solutions of the steady Hall magnetohydrodynamics equations in a domain $\Omega \subset \mathbb{R}^3$. In particular we prove that the set of possible singularities of the suitable weak solution has Hausdorff dimension at most one. Moreover, in the case $\Omega = \mathbb{R}^3$, we show that the set of possible singularities is compact.

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1 Introduction

The resistive incompressible Hall magnetohydrodynamics (Hall-MHD) is described by the following equations:

$$\begin{align*}
\frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p &= (\nabla \times B) \times B + \nu \Delta u + f, \\
\frac{\partial B}{\partial t} - \nabla \times (u \times B) + \nabla \times ((\nabla \times B) \times B) &= \mu \Delta B + \nabla \times g, \\
\nabla \cdot u &= 0, \quad \nabla \cdot B = 0,
\end{align*}$$

where 3D vector fields $u = u(x,t), B = B(x,t)$ are the fluid velocity and the magnetic field respectively. The scalar field $p = p(x,t)$ is the pressure, while the positive
constants $\nu$ and $\mu$ represent the viscosity and the magnetic resistivity respectively. The given vector fields $f$ and $\nabla \times g$ are external forces on the magnetically charged fluid flows. Historically, the Hall-MHD system was first considered by Lighthill ([15]). Compared with the usual MHD system, the Hall-MHD system contains the extra term $\nabla \times ((\nabla \times B) \times B)$, called the Hall term. The inclusion of this term is essential in understanding the problem of magnetic reconnection, which corresponds to the change of the topology of magnetic field lines. This phenomena of magnetic reconnection is really observed, for example, in space plasma([10, 12]), star formation([22]) and neutron star([19]). For the other physical features related to the Hall-MHD we refer [20, 21], while for a comprehensive review of the physical aspect of the equations we refer [17]. Since the Hall term involves the second order derivative of the magnetic field, it becomes important when the magnetic shear is very large, and this occurs during the reconnection procedure. In the laminar flows this term is small compared with the other term, and can be neglected, which is the case of the usual MHD.

Since the Hall term is quadratically nonlinear, containing the second order derivative, it causes major difficulties in the mathematical study of the Hall-MHD system, and only recently the rigorous results on the Cauchy problem appeared. In [1] the authors proved the global existence of weak solutions, while the local in time well-posedness as well as the global in time well-posedness for small initial data was proved in [4]. This later result was refined in [5]. In the case of $\mu = 0$ it is proved in [7] that the Cauchy problem is not globally in time well-posed, rigorously verifying the numerical experiment of [8]. For a special axially symmetric initial data the authors of [9] proved the global well-posedness of the system. In [6] the long time behaviors of the solution were also studied. Since the Hall-MHD system has more complicated structure than the usual MHD system and the Navier-Stokes equations, the study of full regularity of weak solutions would be extremely difficult. Therefore, it might be reasonable to begin with the partial regularity, similarly to the case of the Navier-Stokes equations, the partial regularity of which was studied e.g. in [18, 2, 14, 16, 23]. In the time-dependent problem, mainly due to the difficulty of defining the correct localized energy inequality for a suitable weak solutions we concentrate the partial regularity problem of the steady Hall-MHD system. Contrary to the case of the Navier-Stokes equations and the usual MHD system the full regularity of the steady weak solutions is difficult to deduce. Instead, we prove that the set of possible singularity of the steady suitable weak solution of the Hall-MHD system has Hausdorff dimension at most 1(see Remark 5.2 and Theorem 5.3 below). Moreover, for a steady suitable weak solution on $\mathbb{R}^3$ the set of possible singularity is a compact set(see Corollary 7.3 below). The partial regularity of the time dependent problem will be studied elsewhere.
2 Weak solution and higher regularity of $u$

We consider the following steady Hall-MHD system in $\mathbb{R}^3$.

\begin{align}
(u \cdot \nabla)u - \Delta u &= -\nabla p + (\nabla \times B) \times B + f, \\
\nabla \times (B \times u) - \Delta B &= -\nabla \times ((\nabla \times B) \times B) + \nabla \times g, \\
\nabla \cdot u &= 0, \quad \nabla \cdot B = 0.
\end{align}

(2.1) \quad (2.2) \quad (2.3)

We note that we set $\mu = \nu = 1$ for convenience. As $(u \cdot \nabla)u = (\nabla \times u) \times u + \frac{1}{2}|u|^2$, (2.1) turns into

\begin{align}
(\nabla \times u) \times u - \Delta u &= -\nabla \left( p + \frac{|u|^2}{2} \right) + (\nabla \times B) \times B + f \quad \text{in} \quad \mathbb{R}^3.
\end{align}

(2.4)

Applying $\nabla \times$ to the both sides of the above, we get

\begin{align}
\nabla \times (\omega \times u) - \Delta \omega &= \nabla \times ((\nabla \times B) \times B) + \nabla \times f \quad \text{in} \quad \mathbb{R}^3,
\end{align}

(2.5)

where $\omega$ stands for the vorticity $\nabla \times u$. Taking the sum of (2.2) and (2.5), we are led to

\begin{align}
\nabla \times (V \times u) - \Delta V &= \nabla \times (f + g) \quad \text{in} \quad \mathbb{R}^3,
\end{align}

(2.6)

where

\begin{align}
V &= B + \omega.
\end{align}

(2.7)

As $\nabla \cdot V = 0$, there exists a solenoidal potential $v$ such that $\nabla \times v = V$. From (2.6) we deduce that $v$ solves the system in $\mathbb{R}^3$,

\begin{align}
\nabla \cdot v &= 0, \\
(v \cdot \nabla)v - \Delta v &= -\nabla \pi + (\nabla \times v) \times b + f + g,
\end{align}

(2.8) \quad (2.9)

where $b = v - u$. Clearly, $\nabla \times b = B$.

**Definition 2.1.** Let $f \in L^{6/5}$ and $g \in L^2$. We say $(u, p, B) \in \hat{W}^{1,2} \times L^2_{\text{loc}} \times \hat{W}^{1,2}$ is a weak solution to (2.3)–(2.2) if

\begin{align}
\int_{\mathbb{R}^3} \nabla u : \nabla \varphi - u \otimes u : \nabla \varphi dx \\
= \int_{\mathbb{R}^3} p \nabla \cdot \varphi dx + \int_{\mathbb{R}^3} ((\nabla \times B) \times B) \cdot \varphi dx + \int_{\mathbb{R}^3} f \cdot \varphi dx,
\end{align}

(2.10)

\begin{align}
\int_{\mathbb{R}^3} \nabla B : \nabla \varphi + B \times u \cdot \nabla \times \varphi dx \\
= -\int_{\mathbb{R}^3} ((\nabla \times B) \times B) \cdot \nabla \times \varphi dx + \int_{\mathbb{R}^3} g \cdot \nabla \times \varphi dx
\end{align}

(2.11)

for all $\varphi \in C^\infty_c$. Here $\hat{W}^{1,2}$ stands for the homogeneous Sobolev space.
Remark 2.2. Since $B \in \dot{W}^{1,2} \hookrightarrow L^6$, by the Calderón-Zygmund inequality and Sobolev’s embedding theorem we infer $b \in L^\infty$. In particular, having $V \in L^2_{\text{loc}}$ we easily deduce that $v \in W^{1,2}_{\text{loc}}$ and $\pi \in L^2_{\text{loc}}$. Furthermore, $(v, \pi)$ satisfies the following integral identity for all $\varphi \in C^\infty_c$

\[
\int_{\mathbb{R}^3} \nabla v : \nabla \varphi - v \otimes v : \nabla \varphi \, dx = \int_{\mathbb{R}^3} \pi \nabla \cdot \varphi \, dx + \int_{\mathbb{R}^3} ((\nabla \times v) \times b) \cdot \varphi \, dx + \int_{\mathbb{R}^3} (f + g) \cdot \varphi \, dx.
\]

(2.12)

By using standard regularity methods we get the following

Theorem 2.3. Let $f \in L^{6/5}$ and $g \in L^2$. Let $(u, p, B) \in \dot{W}^{1,2} \times L^2_{\text{loc}} \times \dot{W}^{1,2}$ be a weak solution to (2.3). Suppose, $f, g \in L^q_{\text{loc}}(\Omega)$ for some $\frac{6}{5} < q < +\infty$ and for an open set $\Omega \subset \mathbb{R}^3$. Then

\[
V \in W^{1,q}_{\text{loc}}(\Omega), \quad v \in W^{2,q}_{\text{loc}}(\Omega), \quad u \in W^{q,2}_{\text{loc}}(\Omega),
\]

where $q \wedge 2 = \min\{q, 2\}$.

Proof First, assume $\frac{3}{2} \leq q < +\infty$. Since $u \in L^6$ and $V \in L^2_{\text{loc}}$, we see that $-V \times u + f + g \in L^{3/2}_{\text{loc}}(\Omega)$. Observing (2.6), by the aid of Calderón-Zygmund’s inequality and Sobolev’s embedding theorem we find $V \in W^{1,3/2}_{\text{loc}}(\Omega) \subset L^{3}_{\text{loc}}(\Omega)$. As $\omega = V - B$, this implies $\omega \in L^3_{\text{loc}}(\Omega)$. Taking into account $\nabla \cdot u = 0$, we get $u \in W^{1,3}_{\text{loc}}(\Omega)$. Once more appealing to Sobolev’s embedding theorem, we obtain $u \in L^q_{\text{loc}}(\Omega)$ for all $1 \leq s < +\infty$, and thus $-V \times u + f + g \in L^q_{\text{loc}}(\Omega)$. Again applying Calderón-Zygmund’s inequality, we see that $V \in W^{1,q}_{\text{loc}}(\Omega)$. As $\nabla \cdot v = 0$ from the last statement we infer $v \in W^{2,q}_{\text{loc}}(\Omega)$. Finally, recalling $\omega = V - B$ and $B \in W^{1,2}_{\text{loc}}(\Omega)$, we obtain $\omega \in W^{1,q,2}_{\text{loc}}(\Omega)$.

In case $\frac{6}{5} < q < \frac{3}{2}$, we immediately get $-V \times u + f + g \in L^q_{\text{loc}}(\Omega)$, and the assertion can be proved as in the previous case. 

3 Caccioppoli-type inequality for $B$

Suppose $f, g \in L^2_{\text{loc}}(\Omega)$ for some open set $\Omega \subset \mathbb{R}^3$. As it has been proved in Section 2 we get $u \in W^{2,2}_{\text{loc}}(\Omega)$ if $(u, p, B)$ is a weak solution to the steady Hall-MHD system. By means of Sobolev’s embedding theorem this implies $u \in L^\infty_{\text{loc}}(\Omega)$. Accordingly, $-B \times u + g \in L^2_{\text{loc}}(\Omega)$.

Thus, for the sake of generality in the present and next section we study the local regularity for the following general model. Let $\Omega \subset \mathbb{R}^3$ be a domain. We consider the system

\[
-\Delta B = -\nabla \times ((\nabla \times B) \times B) + \nabla \times F \quad \text{in} \quad \Omega.
\]

(3.1)

We start our discussion with the following notion of a weak solution to (3.1).
Definition 3.1. Let \( F \in L^2_{\text{loc}}(\Omega) \). (i) A vector function \( B \in W^{1,2}_{\text{loc}}(\Omega) \) is said to be a weak solution to (3.1) if
\[
\int_\Omega \nabla B : \nabla \varphi \, dx = -\int_\Omega (\nabla \times B) \times B \cdot \nabla \times \varphi \, dx + \int_\Omega F \cdot \nabla \times \varphi \, dx
\]
for all \( \varphi \in C^\infty_c(\Omega) \).

(ii) A weak solution \( B \) to (3.1) is called a suitable weak solution to (3.1) if, in addition, the following local energy inequality holds:
\[
\int_\Omega \phi |\nabla B|^2 \, dx \\
\leq \frac{1}{2} \int_\Omega \Delta \phi |B|^2 \, dx - \int_\Omega (\nabla \times B) \times B \cdot (B \times \nabla \phi) \, dx \\
+ \int_\Omega (\phi F \cdot \nabla \times B + F \cdot B \times \nabla \phi) \, dx
\]
(3.3)
for all non-negative \( \phi \in C^\infty_c(\Omega) \).

Remark 3.2. Let \( B \) be a suitable weak solution to (3.1). Then for every constant vector \( \Lambda \in \mathbb{R}^3 \) there holds
\[
\int_\Omega \phi |\nabla B|^2 \, dx \\
\leq \frac{1}{2} \int_\Omega \Delta \phi |B - \Lambda|^2 \, dx - \int_\Omega (\nabla \times B) \times B \cdot ((B - \Lambda) \times \nabla \phi) \, dx \\
+ \int_\Omega \{ \phi F \cdot \nabla \times B + F \cdot (B - \Lambda) \times \nabla \phi \} \, dx
\]
(3.4)
for all non-negative \( \phi \in C^\infty_c(\Omega) \). This can be readily seen by combining (3.3) and (3.2) with \( \varphi = \phi \Lambda \).

Now, we state the following Caccioppoli-type inequality

Lemma 3.3. Let \( F \in L^2_{\text{loc}}(\Omega) \). Let \( B \in W^{1,2}_{\text{loc}}(\Omega) \) be a suitable weak solution to (3.1). Then for every ball \( B_r = B_r(x_0) \subset \subset \Omega \) and \( 0 < \rho < r \) there holds
\[
\frac{1}{r} \int_{B_\rho} |\nabla B|^2 \, dx \\
\leq \frac{cr^2}{(r-\rho)^2} (1 + |B_{r,x_0}|^2) \int_{B_r} |B - B_{r,x_0}|^2 \, dx + \frac{cr^2}{(r-\rho)^2} \int_{B_r} |B - B_{r,x_0}|^4 \, dx \\
+ \frac{c}{r} \|F\|_{2,B_r}^2
\]
(3.5)
where \( B_{r,x_0} \) stands for the mean value

\[
\int_{B_r} B \, dx := \frac{1}{\text{meas}(B_r)} \int_{B_r(x_0)} B \, dx,
\]

and \( c = \text{const} > 0 \) denotes a universal constant.

**Proof** Let \( B_r = B_r(x_0) \subset \subset \Omega \) be a fixed ball. Given \( \rho \in (0, r) \), we consider a cut-off function \( \zeta \) defined as \( \zeta \in C^\infty_c(B_r) \) such that \( 0 \leq \zeta \leq 1 \) in \( \mathbb{R}^3 \), \( \zeta \equiv 1 \) on \( B_\rho \) and \( |\nabla \zeta|^2 + |\nabla^2 \zeta| \leq c(r - \rho)^{-2} \) in \( \mathbb{R}^3 \). From (3.4) with \( \phi = \zeta^2 \) and \( \Lambda = B_{r,x_0} \) we obtain the following Caccioppoli-type inequality

\[
\int_{B_r} \zeta^2 |\nabla \mathbf{B}|^2 \, dx \leq \frac{c}{(r - \rho)^2} \int_{B_r} |\mathbf{B} - B_{r,x_0}|^2 \, dx + c \int_{B_r} |\mathbf{F}|^2 \, dx
\]

(3.6)

Applying Hölder’s and Young’s inequality, we estimate

\[
J \leq \frac{c}{(r - \rho)^2} \int_{B_r} |\mathbf{B}|^2 |\mathbf{B} - B_{r,x_0}|^2 \, dx + \frac{1}{2} \int_{B_r} \zeta^2 |\nabla \mathbf{B}|^2 \, dx
\]

\[
\leq \frac{c}{(r - \rho)^2} |B_{r,x_0}|^2 \int_{B_r} |\mathbf{B} - B_{r,x_0}|^2 \, dx + \frac{c}{(r - \rho)^2} \int_{B_r} |\mathbf{B} - B_{r,x_0}|^4 \, dx
\]

\[
+ \frac{1}{2} \int_{B_r} \zeta^2 |\nabla \mathbf{B}|^2 \, dx.
\]

Inserting the above estimate of \( J \) into the right-hand side of (3.6), and dividing the resulting estimate by \( r \), we are led to

\[
\frac{1}{r} \int_{B_\rho} |\nabla \mathbf{B}|^2 \, dx \leq \frac{c r^2}{(r - \rho)^2} (1 + |B_{r,x_0}|^2) \int_{B_r} |\mathbf{B} - B_{r,x_0}|^2 \, dx + \frac{c r^2}{(r - \rho)^2} \int_{B_r} |\mathbf{B} - B_{r,x_0}|^4 \, dx
\]

(3.7)

\[+ \frac{c}{r} \|\mathbf{F}\|_{2,B_r}^2.\]

Whence, the claim.\[\blacksquare\]
Remark 3.4. Setting

\[ E(r) = E(r, x_0) = \left( \int_{B_r(x_0)} |B - B_{r,x_0}|^4 dx \right)^{1/4} \quad 0 < r < \text{dist}(x_0, \partial \Omega), \]

(3.5) becomes

\[ \left( \frac{1}{r} \int_{B_r} |\nabla B|^2 dx \right)^{1/2} \leq \frac{cr}{r - \rho} \left\{ (1 + |B_{r,x_0}|)E(r) + E(r)^2 \right\} + \frac{c}{r^{1/2}} \| F \|_{2,B_r}. \]  

\[ (3.8) \]

4 Blow-up

We begin our discussion with the following fundamental estimate for solutions to the model problem, which will be used in the blow-up lemma below.

Lemma 4.1. Let \( \Lambda \in \mathbb{R}^3 \). Let \( W \in L^4(B_1) \cap W^{1,2}_{\text{loc}}(B_1) \) be a weak solution to

\[ -\Delta W = -\nabla \times ((\nabla \times W) \times \Lambda) \quad \text{in} \quad B_1, \]

i.e.

\[ \int_{B_1} \nabla W : \nabla \Phi dx = -\int_{B_1} ((\nabla \times W) \times \Lambda) \cdot \nabla \times \Phi dx \]

for all \( \Phi \in W^{1,2}(B_1) \) with \( \text{supp}(\Phi) \subset B_1 \). Then,

\[ \left( \int_{B_{\tau}} |W - W_{B_{\tau}}|^4 dx \right)^{1/4} \leq C_0 \tau (1 + |\Lambda|^3) \left( \int_{B_1} |W - W_{B_1}|^4 dx \right)^{1/4} \quad \forall \ 0 < \tau < 1, \]

where \( C_0 > 0 \) denotes a universal constant.

Proof Since the assertion is trivial for \( \frac{1}{2} < \tau < 1 \), we may assume that \( 0 < \tau < \frac{1}{2} \). Let \( \zeta \in C_c^\infty(B_1) \) be a suitable cut-off function for \( B_{1/2} \), i.e. \( 0 \leq \zeta \leq 1 \) in \( B_1 \) and \( \zeta \equiv 1 \) on \( B_{1/2} \). In (4.2) inserting the admissible test function \( \Phi = \zeta^{2m}(W - W_{B_1}) \ (m \in \mathbb{N}) \), by using Cauchy-Schwarz’s inequality along with Young’s inequality we obtain

\[ \int_{B_1} \zeta^{2m} |\nabla W|^2 dx \leq c(1 + |\Lambda|^2) \int_{B_1} \zeta^{2m-2} |W - W_{B_1}|^2 dx. \]

If \( W \) is smooth in \( B_1 \), since (4.1) is a linear system, the same inequality holds for \( D^\alpha W \) in place of \( W \) for any multi-index \( \alpha \). By a standard mollifying argument together with Sobolev’s embedding theorem we see that \( W \) is smooth. In particular, in (4.4) putting \( m = 3 \) it follows that

\[ \int_{B_1} \zeta^6 |D^\alpha W|^2 dx \leq c(1 + |\Lambda|^6) \int_{B_1} |W - W_{B_1}|^2 dx \quad \forall |\alpha| \leq 3. \]
By means of Sobolev’s embedding theorem and Jensen’s inequality we get
\begin{equation}
\| \nabla W \|_{4, B_{1/2}}^4 \leq c(1 + |\Lambda|^2) \int_{B_2} |W - W_{B_2}|^4 dx.
\end{equation}

Applying Poincaré’s inequality, we obtain
\begin{equation}
\int_{B_r} |W - W_{B_r}|^4 dx \leq c r^4 \| \nabla W \|_{4, B_{1/2}}^4.
\end{equation}

Combination of (4.6) and (4.7) yields the desired estimate. 

In our discussion below we make use of the notion of the Morrey space. We say \( F \in \mathcal{M}^{p, \lambda}_{\text{loc}}(\Omega) \) if for all \( K \subset \subset \Omega \)
\[
[F]_{\mathcal{M}^{p, \lambda}_{\text{loc}}(K)} = \sup \left\{ r^{-\lambda} \int_{B_r(x_0)} |F|^p dx \mid x_0 \in K, 0 < r \leq \text{dist}(K, \partial \Omega) \right\} < +\infty.
\]

**Lemma 4.2.** Let \( F \in \mathcal{M}^{2, \lambda}_{\text{loc}}(\Omega) \) for some \( 1 < \lambda \leq 3 \). For every \( 0 < \tau < \frac{1}{2}, 0 < M < +\infty, K \subset \subset \Omega \) and \( 0 < \alpha < \frac{\lambda - 1}{2} \), there exist positive numbers \( \varepsilon_0 = \varepsilon_0(\tau, M, K, \alpha) \) and \( R_0 = R_0(\tau, M, K, \alpha) < \text{dist}(K, \partial \Omega) \) such that, if \( B \in W^{1, 2}_{\text{loc}}(\Omega) \) is a suitable weak solution to (3.1), and for \( x_0 \in K \) and \( 0 < R \leq R_0 \) the condition
\begin{equation}
|B_{R, x_0}| \leq M, \quad E(R, x_0) + R^\alpha \leq \varepsilon_0
\end{equation}
is fulfilled, then
\begin{equation}
E(\tau R, x_0) \leq 2\tau C_0(1 + M^3)(E(R, x_0) + R^\alpha),
\end{equation}
where \( C_0 > 0 \) stands for the constant which appears on the right-hand side of (4.3).

**Proof** Assume the assertion of the Lemma is not true. Then there exist \( 0 < \tau < \frac{1}{2}, 0 < M < +\infty, K \subset \subset \Omega \) and \( 0 < \alpha < \frac{\lambda - 1}{2} \) together with a sequence \( B^{(k)} \in W^{1, 2}_{\text{loc}}(\Omega) \) being suitable weak solutions to (3.1) as well as sequences \( x_k \in K, 0 < R_k < \text{dist}(K, \partial \Omega) \) and \( \varepsilon_k \to 0 \) as \( k \to +\infty \) such that
\begin{equation}
|B^{(k)}_{R_k, x_k}| \leq M, \quad E_k(R_k, x_k) + R_k^\alpha = \varepsilon_k
\end{equation}
and
\begin{equation}
E_k(\tau R_k, x_k) > 2\tau C_0(1 + M^3)(E_k(R_k, x_k) + R_k^\alpha).
\end{equation}
Here we have used the notation
\[
E_k(r, x_0) = \left( \int_{B_r(x_0)} |B^{(k)} - B^{(k)}_{r, x_0}|^4 dx \right)^{1/4}, \quad x_0 \in K, 0 < r \leq \text{dist}(K, \partial \Omega).
\]
Note that (4.10) yields \( R_k \to 0 \) as \( k \to +\infty \).
Next, define
\[
W_k(y) = \frac{1}{\varepsilon_k}(B^{(k)}(x_k + R_k y) - B^{(k)}_{R_k,x_k}),
\]
\[
F_k(y) = F(x_k + R_k y), \quad y \in B_1(0)
\]
\((k \in \mathbb{N})\). Furthermore, set
\[
\mathcal{E}_k(\sigma) = \left( \int_{B_{\sigma}} |W_k - (W_k)_{B_{\sigma}}|^4 dy \right)^{1/4}, \quad 0 < \sigma \leq 1.
\]
Then (4.10) and (4.11) turn into
\[
|B^{(k)}_{R_k,x_k}| \leq M, \quad \mathcal{E}_k(1) + \frac{R_k^\alpha}{\varepsilon_k} = 1,
\]
and
\[
\mathcal{E}_k(\tau) > 2\tau C_0(1 + M^3)\left( \mathcal{E}_k(1) + \frac{R_k^\alpha}{\varepsilon_k} \right) = 2\tau C_0(1 + M^3)
\]
respectively.

Using the chain rule, we find that (3.1) transforms into
\[
-\Delta W_k = -\varepsilon_k \nabla \times ((\nabla \times W_k) \times W_k) - \nabla \times ((\nabla \times W_k) \times B^{(k)}_{R_k,x_k})
\]
\[
+ \frac{R_k}{\varepsilon_k} \nabla \times F_k \quad \text{in} \quad B_1.
\]

Thus, \(W_k \in W^{1,2}(B_1)\) is a weak solution to (4.14). Let \(0 < \sigma < 1\). Using the transformation formula, noticing that \(|B^{(k)}_{R_k,x_k}| \leq M\), the Caccioppoli-type inequality (3.8) with \(r = R_k\) and \(\rho = \sigma R_k\) turns into
\[
\|\nabla W_k\|_{2,B_\sigma} \leq c(1 - \sigma)^{-1} \left( (1 + M) \mathcal{E}_k(1) + \varepsilon_k \mathcal{E}_k(1)^2 \right) + \frac{cR_k^{1/2}}{\varepsilon_k} \|F\|_{2,B_{R_k}(x_k)}.
\]
Observing (4.12), and verifying
\[
\frac{R_k^{1/2}}{\varepsilon_k} \|F\|_{B_{R_k}(x_k)} \leq \frac{R_k^{(\lambda - 1)/2}}{\varepsilon_k} [F]_{M^2,\lambda,K} \leq R_k^{(\lambda - 1)/2 - \alpha}[F]_{M^2,\lambda,K}
\]
from (4.15), we get
\[
\|\nabla W_k\|_{2,B_\sigma} \leq c(1 - \sigma)^{-1}(M + 1) + c[F]_{M^2,\lambda,K}.
\]

In addition, in view of (4.12) we estimate
\[
\|W_k\|_{4,B_1} = (\text{mes } B_1)^{1/4} \mathcal{E}_k(1) \leq (\text{mes } B_1)^{1/4}.
\]
From (4.17) and (4.18) it follows that \((W_k)\) is bounded in \(W^{1,2}(B_\sigma)\) for all \(0 < \sigma < 1\) and bounded in \(L^4(B_1)\). Thus, by means of reflexivity, eventually passing to subsequences, we get \(W \in L^4(B_1) \cap W^{1,2}_{\text{loc}}(B_1)\) and \(\Lambda \in \mathbb{R}^3\) such that

\[
W_k \to W \quad \text{weakly in } L^4(B_1) \quad \text{as } \quad k \to +\infty,
\]

\[
W_k \to W \quad \text{weakly in } W^{1,2}(B_\sigma) \quad \text{as } \quad k \to +\infty \quad \forall \ 0 < \sigma < 1,
\]

\[
B^{(k)}_{R_k,x_k} \to \Lambda \quad \text{in } \mathbb{R}^3 \quad \text{as } \quad k \to +\infty.
\]

On the other hand, by the compactness of the embedding \(W^{1,2}(B_\sigma) \hookrightarrow L^4(B_\sigma)\) from (4.19) we infer

\[
W_k \to W \quad \text{strongly in } L^4(B_\sigma) \quad \text{as } \quad k \to +\infty \quad \forall \ 0 < \sigma < 1.
\]

Accordingly,

\[
\lim_{k \to +\infty} \mathcal{E}_k(\sigma) = \mathcal{E}(\sigma) \quad \forall \ 0 < \sigma < 1,
\]

where \(\mathcal{E}(\sigma) = \left( \int_{B_\sigma} |W - W_{B_\sigma}|^4 dy \right)^{1/4}\). In particular, by the aid of (4.23) (with \(\sigma = \tau\)) from (4.13) we get

\[
\mathcal{E}(\tau) \geq 2\tau C_0 (1 + M^3).
\]

In view of (4.16) we have

\[
\frac{R_k}{\varepsilon_k} \|F_k\|_{2,B_1} = \frac{R_k^{-1/2}}{\varepsilon_k} \|F\|_{B_{R_k}(x_k)} \leq R_k^{(\lambda - 1)/2 - \alpha} [F]_{M^2,\lambda,K} \to 0
\]

as \(k \to +\infty\). Therefore, with the help of (4.19), (4.20), (4.21) and (4.22), letting \(k \to +\infty\) in (4.14), we see that \(W \in W^{1,2}_{\text{loc}}(B_1) \cap L^4(B_1)\) is a weak solution to

\[
-\Delta W = -\nabla \times ((\nabla \times W) \times \Lambda) \quad \text{in } \quad B_1.
\]

As \(|\Lambda| \leq M\) appealing to Lemma 4.1, we find

\[
\mathcal{E}(\tau) \leq \tau C_0 (1 + M^3) \mathcal{E}(1).
\]

On the other hand, by virtue of the lower semi continuity of the norm together with (4.13) and (4.23) we get

\[
\mathcal{E}(1) \leq \liminf_{k \to +\infty} \left( \mathcal{E}_k(1) + \frac{R_k^\alpha}{\varepsilon_k} \right) \leq \frac{1}{2\tau C_0 (1 + M^3)} \lim_{k \to +\infty} \mathcal{E}_k(\tau) = \frac{1}{2\tau C_0 (1 + M^3)} \mathcal{E}(\tau).
\]

Estimating the right of (4.26) by the inequality we have just obtained, we see that \(\mathcal{E}(\tau) \leq \frac{1}{2} \mathcal{E}(\tau)\) and hence \(\mathcal{E}(\tau) = 0\), which contradicts to (4.24). Whence, the assumption is not true, and this completes the proof of the Lemma. \(\blacksquare\)
5 Partial regularity

The aim of the present section is to prove the partial regularity of a suitable weak solution $B \in W^{1,2}_{\text{loc}}(\Omega)$ to system (3.1), which will lead to the partial regularity of a suitable weak solution to the steady Hall-MHD system. As we will see below, the set $\Sigma(B)$ of possible singularities is given by means of

$$\Sigma(B) = \left\{ x_0 \in \Omega \left| \liminf_{r \to 0^+} \frac{1}{r} \int_{B_r(x_0)} |\nabla B|^2 \, dx > 0 \right. \right\} \cup \left\{ x_0 \in \Omega \left| \sup_{r>0} |B_{r,x_0}| = +\infty \right. \right\}.$$  \hfill (5.1)

**Theorem 5.1.** Let $F \in M^{2,\lambda}_{\text{loc}}(\Omega)$ for some $1 < \lambda < 3$. Let $B \in W^{1,2}_{\text{loc}}(\Omega)$ be a suitable weak solution to the system (3.1). Then, $\Omega \setminus \Sigma(B)$ is an open set, on which $B$ is $\alpha$-Hölder continuous with respect to any exponent $0 < \alpha < \frac{\lambda-1}{2}$.

**Proof** Let $x_0 \in \Omega \setminus \Sigma(B)$ (cf. (5.1)). Set $d = \text{dist}(x_0, \partial \Omega)$, and define $K = \overline{B_{d/2}(x_0)}$. Using Sobolev Poincaré’s inequality, we have

$$\liminf_{r \to 0^+} \int_{B_r(x_0)} |B - B_{r,x_0}|^4 \, dx \leq c \liminf_{r \to 0^+} \left( \frac{1}{r} \int_{B_r(x_0)} |\nabla B|^2 \, dx \right)^2 = 0.$$  We set

$$M := \sup_{0<r<d/2} |B_{r,x_0}| + 3 < +\infty.$$  

Take $\tau > 0$ such that

$$2\tau^{1-\alpha}C_0(1 + M^3) \leq \frac{1}{2}, \quad \tau^\alpha \leq \frac{1}{2}.$$ \hfill (5.2)

Let $\varepsilon_0 = \varepsilon_0(\tau, M, K, \alpha)$ and $R_0 = R_0(\tau, M, K, \alpha)$ denote the numbers according to Lemma 4.2. Then, we choose $0 < R_1 \leq R_0$ such that

$$E(R_1, x_0) + 2R_1^\alpha < \min\{\varepsilon_0, \varepsilon_1\},$$ \hfill (5.3)

where $\varepsilon_1 > 0$ fulfills

$$2\tau^{-3}\varepsilon_1 \leq 1.$$ \hfill (5.4)

By the absolutely continuity of the Lebesgue integral there exists $0 < \delta < \frac{d}{2}$ such that for all $y \in \overline{B_\delta(x_0)}$

$$E(y, R_1) + 2R_1^\alpha \leq \min\{\varepsilon_0, \varepsilon_1\},$$ \hfill (5.5)

and

$$|B_{R_1,y}| \leq \sup_{0<r<d/2} |B_{r,x_0}| + 1 = M - 2.$$ \hfill (5.6)
Fix \( y \in \overline{B_\delta(x_0)} \). We claim that for every \( j \in \mathbb{N} \cup \{0\} \) there holds
\[
E(\tau^j R_1, y) \leq 2^{-j} \tau^{\alpha j} E(R_1, y) + (1 - 2^{-j}) \tau^{\alpha j} R_1^\alpha, \tag{5.7}
\]
\[
|B_{\tau^j R_1, y}| \leq M - 2^{-j+1}. \tag{5.8}
\]

Clearly, for \( j = 0 \), (5.7) is trivially fulfilled, while (5.8) holds in view of (5.6).

Now, assume (5.7) and (5.8) are fulfilled for \( j \in \mathbb{N} \cup \{0\} \). Then (5.7) along with (5.3) immediately implies
\[
E(\tau^{j+1} R_1, y) + \tau^{\alpha(j+1)} R_1^\alpha \leq \tau^{\alpha j} (E(R_1, y) + 2R_1^\alpha) \leq \tau^{\alpha j} \min\{\varepsilon_0, \varepsilon_1\}. \tag{5.9}
\]

Thus, we are in a position to apply Lemma 4.2 with \( R = \tau^j R_1 \). This together with (5.2) gives
\[
E(\tau^{j+1} R_1, y) + \tau^{\alpha(j+1)} R_1^\alpha \leq \tau^{\alpha j} (E(R_1, y) + 2R_1^\alpha) \leq \tau^{\alpha j} \min\{\varepsilon_0, \varepsilon_1\}. \tag{5.10}
\]

This proves (5.7) for \( j + 1 \).

It remains to show (5.8) for \( j + 1 \). First, from (5.7) along with (5.5) we infer
\[
E(\tau^j R_1, y) \leq \tau^{\alpha j} (E(R_1, y) + R_1^\alpha) \leq \tau^{\alpha j} \varepsilon_1. \tag{5.11}
\]

Using triangle inequality and Jensen’s inequality, we find
\[
|B_{\tau^{j+1} R_1, y}| \leq |B_{\tau^j R_1, y}| + |B_{\tau^{j+1} R_1, y} - B_{\tau^j R_1, y}|
\leq |B_{\tau^j R_1, y}| + 2\tau^{-3} E(\tau^j R_1, y).
\]

Estimating the first member on the right by using (5.8) and the second one by the aid of (5.11) together with (5.3) and (5.4), we obtain
\[
|B_{\tau^{j+1} R_1, y}| \leq M - 2^{-j+1} + 2\tau^{-3} \tau^{\alpha j} \varepsilon_1
\leq M - 2^{-j+1} + 2^{-j} = M - 2^{-j}.
\]

This completes the proof of (5.8) for \( j + 1 \). Whence, the claim.

From (5.7) we get a constant \( C_1 > 0 \) such that
\[
\left( \int_{B_r(y)} |B - B_{r,y}|^4 dx \right)^{1/4} \leq C_1 r^\alpha, \quad \forall 0 < r < \frac{d}{2}, \quad \forall y \in \overline{B_\delta(x_0)}.
\]

Thus, by the well-known equivalence of the Campanato space and the Hölder space (see e.g. [3] or Theorem 1.3 of [11]) we conclude
\[
B_{|B_{B_\delta(x_0)}|} \in C^\alpha(B_\delta(x_0)). \tag{5.12}
\]
Finally, we shall verify that $B_\delta(x_0) \subset \Omega \setminus \Sigma(B)$. Let $y \in B_\delta(x_0)$ be arbitrarily chosen. Firstly, notice that $\sup_{0 < r < d/2} |B_{r,y}| < +\infty$ (see (5.12)). Secondly, from the Caccioppoli-type inequality (3.8) with $0 < r < d/2$ and $\rho = r/2$ replacing $x_0$ by $y$ therein we deduce

$$\left( \frac{1}{r} \int_{B_{r/2}(y)} |\nabla B|^2 dx \right)^{1/2} \leq c \left\{ (1 + |B_{r,y}|) E(r, y) + E(r, y)^2 \right\}^{1/2} + cr^{(\lambda-1)/2}[F]_{M^{2,\lambda}}.$$

As $\lim_{r \to 0^+} E(r, y) = 0$ and $\lambda > 1$ the right-hand side of the above inequality tends to zero as $r \to 0^+$. Hence, $y \in \Omega \setminus \Sigma(B)$. This completes the proof of the theorem.

**Remark 5.2.** By the result of [11, Chap. IV, 2.] there holds

$$H^2(\Sigma(B)) = 0 \quad \forall \beta > 1,$$

which implies that $\Sigma(B)$ has Hausdorff dimension at most one. We don’t know, however, whether the one-dimensional Hausdorff measure of $\Sigma(B)$ is finite.

As a consequence of Theorem 5.1 we get the following partial regularity result for the steady Hall-MHD system.

**Theorem 5.3.** Let $f \in L^{6/5} \cap L^2_{\text{loc}}$ and $g \in L^2 \cap M^{2,\lambda}_{\text{loc}}$ for some $1 < \lambda \leq 2$. Let $(u, p, B)$ be a weak solution to the Hall-MHD system such that $B$ satisfies the local energy inequality (3.3) with $F = -B \times u + g$. Then, for $\Sigma(B)$ defined by (5.1) we have that $\mathbb{R}^3 \setminus \Sigma(B)$ is an open set such that $B$ is $\alpha$-Hölder continuous on $\mathbb{R}^3 \setminus \Sigma(B)$ for any $0 < \alpha < \frac{\lambda - 1}{2}$.

**Proof** Arguing similarly as in the proof of Theorem 2.3, from $f, g \in L^2_{\text{loc}}$ we deduce that $V = B + \omega \in W^{1,2}_{\text{loc}}$. As $B \in W^{1,2}$ we see that $\omega \in W^{1,2}_{\text{loc}}$, and hence $u \in W^{2,2}_{\text{loc}}$. By means of Sobolev’s embedding theorem we find $u \in L^\infty_{\text{loc}}$. This shows that $B \times u \in L^6_{\text{loc}} \subset M^{2,2}_{\text{loc}}$. Consequently, $F = -B \times u + g \in M^{2,\lambda}_{\text{loc}}$. The assertion of the theorem is now an immediate consequence of Theorem 5.1.

6 Higher regularity

In Section 5 we have proved the partial Hölder regularity of a suitable weak solution $(u, p, B)$ of the Hall-MHD system for $f$ and $g$ being sufficiently regular. The aim of the present section is to show that if both $f$ and $g$ are smooth, then $(u, p, B)$ is smooth in $\mathbb{R}^3 \setminus \Sigma(B)$. To prove this we first shall establish a regularity result for the following linearized problem.

Let $\Omega \subset \mathbb{R}^3$ be an open set, and let $B \in C(\Omega)$. We consider the linear system

$$-\Delta A = -\nabla \times ((\nabla \times A) \times B) + \nabla \times F \quad \text{in} \quad \Omega.$$

We have the following regularity result.
Theorem 6.1. Let $F \in M_{\text{loc}}^{2,\lambda}(\Omega)$ for some $1 < \lambda < 3$. Let $A \in W_{\text{loc}}^{1,2}(\Omega)$ be a weak solution to (6.1). Then

\begin{equation}
A \in C^{\alpha}(\Omega) \quad \text{with} \quad \alpha = \frac{\lambda - 1}{2}.
\end{equation}

Proof Let $x_0 \in \Omega$. Set $d = \text{dist}(x_0, \partial \Omega)$. As $B$ is continuous, we get

\begin{align*}
M := \max_{y \in B_{d/2}(x_0)} |B(y)| < +\infty.
\end{align*}

Let $y \in B_{d/4}(x_0)$ be fixed. For the sake of notational simplicity in what follows we use the notation

\begin{align*}
S(R, y) = \left( \frac{1}{R} \int_{B_R(y)} |\nabla A|^2 \, dx \right)^{1/2}, \quad 0 < R \leq \frac{d}{4}.
\end{align*}

Furthermore, by $\text{osc}(f; x_0, R)$ we denote the oscillation of a continuous function $f$ over $B_R(x_0)$, which equals the supremum of $|f(x) - f(y)|$ taken over all $x, y \in B_R(x_0)$. As $B$ is continuous we may choose $0 < R_0 < \frac{d}{4}$ such that

\begin{align*}
\sup_{B_R(y)} |B - B_{R,y}| \leq \text{osc}(B; x_0, 2R) \leq \frac{1}{2} \quad \forall 0 < R \leq R_0.
\end{align*}

Next, for $0 < R \leq R_0$ let $Z \in W^{1,2}(B_R(y))$ denote the unique weak solution to

\begin{align*}
-\Delta Z &= -\nabla \times ((\nabla \times Z) \times B_{R,y}) - \nabla \times ((\nabla \times A) \times (B - B_{R,y})) + \nabla \times F \quad \text{in} \quad B_R(y),
\end{align*}

\begin{align*}
Z = 0 \quad \text{on} \quad \partial B_R(y).
\end{align*}

It can be easily checked that

\begin{align*}
R^{-1/2} \|\nabla Z\|_{2,B_R(y)} &\leq 4 \text{osc}(B; x_0, 2R) R^{-1/2} \|\nabla A\|_{2,B_R(y)} + 2R^{-1/2} \|F\|_{2,B_R(y)} \leq 4 \text{osc}(B; x_0, 2R) S(R, y) + 2R^{(\lambda - 1)/2} [F]_{M^{2,\lambda}, B_{d/2}(x_0)}.
\end{align*}

Setting $W = A - Z$ and $\Lambda = A_{R,y}$, it follows that $W \in W^{1,2}(B_R(y))$ is a weak solution to (4.1) in $B_R(y)$. Analogously as Lemma 4.1 one shows that get

\begin{align*}
\left( \frac{1}{\tau} \int_{B_{\tau R}(y)} |\nabla W|^2 \, dx \right)^{1/2} &\leq C_0 \tau (1 + M^2) \left( \frac{1}{R} \int_{B_R(y)} |\nabla W|^2 \, dx \right)^{1/2}
\end{align*}

for all $0 < \tau < 1$.

Next, let $\frac{\lambda - 1}{2} < \beta < 1$ be fixed. Take $0 < \tau < \frac{1}{2}$ such that

\begin{align*}
C_0 \tau^{1-\beta}(1 + M^2) &\leq \frac{1}{2}.
\end{align*}
Thus, using triangle inequality along with (6.5), (6.6) and (6.7), we get

\[ S(\tau R, y) \leq \left( \frac{1}{\tau R} \int_{B_\tau R(y)} |\nabla W_{\tau R, y}|^2 \, dx \right)^{1/2} + \tau^{-1/2} R^{-1/2} \|\nabla Z\|_{2, R} \]

\[ \leq C_0 \tau (1 + M^2) S(R, y) + (1 + \tau^{-1/2}) R^{-1/2} \|\nabla Z\|_{2, R} \]

\[ \leq \frac{1}{2} \tau^\beta S(R, y) + 4(1 + \tau^{-1/2}) \text{osc}(B; x_0, 2R) S(R, y) + C_1 R^\alpha, \]

where

(6.8) \[ C_1 = 2(1 + \tau^{-1/2})[F]_{M^2, \lambda, B_{d/2}(x_0)}, \quad \alpha = \frac{\lambda - 1}{2}. \]

Take \( 0 < R_1 \leq R_0 \) such that \( 4(1 + \tau^{-1/2}) \text{osc}(B; x_0, 2R_1) \leq \frac{\tau^\beta}{2} \). Then from the estimate above we deduce

(6.9) \[ S(\tau R, y) \leq \tau^\beta S(R, y) + C_1 R^\alpha \quad \forall \ 0 < R \leq R_1. \]

By using a routine iteration argument, we infer from (6.9) that that

\[ S(\tau^k R_1, y) \leq \tau^{k\beta} S(\tau^{k-1} R_1) + C_1 \tau^{ak} R_1^\alpha \]

\[ \leq \tau^{k\beta} S(\tau^{k-2} R_1) + C_1 \tau^{ak} (1 + \tau^{\beta - \alpha} + \ldots + \tau^{(\beta - \alpha)(k-1)}) R_1^\alpha \]

\[ \leq \tau^{k\alpha} \left( 1 + C_1 R_1^\alpha \right) \frac{1}{1 - \tau^{\beta - \alpha}}. \]

Thus, there exists a constant \( C_2 > 0 \) such that

(6.10) \[ S(R, y) \leq C_2 R^\alpha \quad \forall \ 0 < R \leq \frac{d}{4}, \quad y \in B_{d/4}(x_0). \]

By the aid of the Poincaré inequality from (6.10) we obtain

(6.11) \[ \left( \int_{B_R} |A - A_{R, y}|^2 \, dx \right)^{1/2} \leq cR^\alpha \quad \forall \ 0 < R \leq \frac{1}{4}, \quad y \in B_{d/4}(x_0), \]

which leads to the desired Hölder regularity of \( A \).

We are now in a position to prove the higher regularity for a continuous weak solution \((u, p, B)\) to the steady Hall-MHD system. More precisely, we have the following

**Theorem 6.2.** For \( f \in L^{6/5} \) and \( g \in L^2 \) let \((u, p, B)\) be a weak solution to the steady Hall-MHD system. Let \( \Omega \subset \mathbb{R}^3 \) be an open set such that \( B \) is continuous in \( \Omega \) and \( f, g \in C^k(\Omega) \ (k \in \mathbb{N} \cup \{0\}) \). Then \( B, \omega \in C^{k, \alpha}(\Omega) \) for all \( 0 < \alpha < 1 \).

**Proof** First, let us consider the case \( k = 0 \). As \( f, g \in L^\infty(\Omega) \) by virtue of Theorem 2.3 and Sobolev’s embedding theorem we get

(6.12) \[ V \in W^{1,q}_{\text{loc}}(\Omega) \quad \forall \ 1 \leq q < +\infty, \quad V \in C^\alpha_{\text{loc}}(\Omega) \quad \forall \ 0 < \alpha < 1. \]
With help of Sobolev’s embedding theorem we see that \(-B \times u + g \in \mathcal{M}^{2,\lambda}_{\text{loc}}\) for all \(0 < \lambda < 3\). Hence, from Theorem 6.1 with \(A = B\) we immediately get \(B \in C_\text{loc}^{\alpha}(\Omega)\) for all \(0 < \alpha < 1\). With \(A = B\) we infer \(\omega \in C_\text{loc}^{\alpha}(\Omega)\) and since \(\nabla \cdot u = 0\) it follows \(u \in C_\text{loc}^{1,\alpha}(\Omega)\). This completes the proof of the assertion in case \(k = 0\).

Suppose \(f, g \in C_\text{loc}^{k}(\Omega)\) for some \(k \in \mathbb{N}\). From the proof above we immediately get \(B, \omega \in C_\text{loc}^{\alpha}(\Omega)\) for all \(0 < \alpha < 1\). Now, assume that \(B, \omega \in C_\text{loc}^{j-1,\alpha}(\Omega) \cap W_\text{loc}^{1,2}(\Omega)\) for all \(0 < \alpha < 1\) and that \(j \in \{1, \ldots, k-1\}\). Let \(\nu \in \mathbb{N}^3\) be a multi-index with \(|\nu| = j-1\). Define \(A = D^\nu B\) in \(\Omega\). Applying \(D^\nu\) to both sides of \((\ref{eq:2.2})\), we are led to

\[
-\Delta A = -\nabla \times ((\nabla \times A) \times B) + \sum_{|\mu| \leq j-2, \mu \leq \nu} \nabla \times ((\nabla \times D^\mu B) \times D^{\nu-\mu} B) + \nabla \times D^\nu (B \times u + g)
\]

\[(6.13)\]

in \(\Omega\). By our assumption, we have \(G \in C_\text{loc}^{\alpha}(\Omega)\) for all \(0 < \alpha < 1\). Applying the method of differences, we see that \(A \in W^{1+\theta,2}_{\text{loc}}(\Omega)\) for every \(0 < \theta < 1\). Consequently, \(B \in W^{j+\theta,2}_{\text{loc}}(\Omega)\) for all \(0 < \theta < 1\). By virtue of Sobolev’s embedding theorem it follows that

\[(6.14)\]

\[B \in W^{j,q}_{\text{loc}}(\Omega) \quad \forall 1 \leq q < 6.\]

This shows that the \(\nabla \times G \in L^2_{\text{loc}}(\Omega)\). Therefore, we are able to perform the method of difference quotient which yields \(A \in W^{2,2}_{\text{loc}}(\Omega)\). Recalling our assumption, having \(A \in C_\text{loc}^{\alpha}(\Omega)\) for all \(0 < \alpha < 1\) by the interpolation inequality due to Kufner and Wannebo \([13]\), we obtain \(A \in W^{1,q}_{\text{loc}}(\Omega)\) for all \(1 \leq q < +\infty\). This proves that

\[(6.15)\]

\[B \in W^{j+1,2}_{\text{loc}}(\Omega) \cap \bigcap_{1 \leq q < \infty} W^{j,q}_{\text{loc}}(\Omega).\]

Repeating the above argument with \(|\nu| = j\) and \(A = D^\nu B\), we see that \(A \in W^{1,2}_{\text{loc}}(\Omega)\) is a weak solution to

\[(6.16)\]

\[-\Delta A = -\nabla \times ((\nabla \times A) \times B) + \nabla \times G.\]

Thanks to \((6.15)\) we have \(G \in \mathcal{M}^{2,\lambda}_{\text{loc}}(\Omega)\) for all \(0 < \lambda < 3\), so that Theorem 6.1 yields that \(A \in C_\text{loc}^{\alpha}(\Omega)\) for all \(0 < \alpha < 1\), and that implies

\[(6.17)\]

\[B \in C_\text{loc}^{j,\alpha}(\Omega) \quad \forall 0 < \alpha < 1.\]

Furthermore, according to our assumption we have \(V \in W^{j-1,q}_{\text{loc}}(\Omega)\) for all \(1 \leq q < +\infty\). Consequently, \(-V \times u + f + g \in W^{j-1,q}_{\text{loc}}(\Omega)\) for all \(1 \leq q < +\infty\). Hence using the Calderón-Zygmund inequality, from \((\ref{eq:2.6})\) we deduce that \(V \in W^{j,q}_{\text{loc}}(\Omega)\) for all \(1 \leq q < +\infty\). In particular, \(-V \times u + f + g \in W^{j,q}_{\text{loc}}(\Omega)\) for all \(1 \leq q < +\infty\). Once, more employing Calderón-Zygmund’s inequality, we find \(V \in W^{j+1,q}_{\text{loc}}(\Omega)\) for all \(1 \leq q < +\infty\), and with the help of Sobolev’s embedding theorem we get \(V \in C_\text{loc}^{j,\alpha}(\Omega)\).
for all $0 < \alpha < 1$. Finally, recalling $\omega = V - B$ in view of (6.15) and (6.17), we conclude

\begin{equation}
\omega \in W^{j+1,2}_{\text{loc}}(\Omega) \cap \bigcap_{0 < \alpha < 1} C^{j,\alpha}_{\text{loc}}(\Omega).
\end{equation}

The desired regularity now follows from above by induction over $j = 0, \ldots, k$.

\section{Direct method and compactness of the singular set}

In this section we prove that a suitable weak solution becomes regular outside a sufficiently large ball, which is due to the decay property

\begin{equation}
\lim_{R \to +\infty} \int_{\{|x| > R\}} |B|^6 \, dx = 0.
\end{equation}

First let us state an alternative Caccioppoli-type inequality for the system (3.1).

\begin{lemma}
Let $F \in L^2$ and let $B \in \dot{W}^{1,2}$ be a suitable weak solution to the system (3.1) in $\mathbb{R}^3$. Then for every Ball $B_r = B_r(x_0)$ and $0 < \rho < r$ there holds

\begin{equation}
\begin{aligned}
\frac{1}{r} \int_{B_r} |\nabla B|^2 \, dx & \leq \frac{c r^2}{(r - \rho)^2} \left\{ 1 + \left( \int_{B_r} |B|^6 \, dx \right)^{1/3} \right\} \left( \int_{B_r} |B - B_{r,x_0}|^3 \, dx \right)^{2/3} \\
& \quad + \frac{c}{r} \|F\|^2_{2,B_r},
\end{aligned}
\end{equation}

where $c = \text{const} > 0$ denotes a universal constant.
\end{lemma}

\begin{proof}
This can be easily achieved by estimating the integral $J$ on the right-hand side of (3.6) by using Hölder’s inequality and Young’s inequality as follows

\begin{align*}
J & \leq \frac{c}{(r - \rho)^2} \int_{B_r} |B|^2 |B - B_{r,x_0}|^2 \, dx + \frac{1}{2} \int_{B_r} \zeta^2 |\nabla B|^2 \, dx \\
& \leq \frac{cr^3}{(r - \rho)^2} \left( \int_{B_r} |B|^6 \, dx \right)^{1/3} \left( \int_{B_r} |B - B_{r,x_0}|^3 \, dx \right)^{2/3} + \frac{1}{2} \int_{B_r} \zeta^2 |\nabla B|^2 \, dx.
\end{align*}

Using the well-known properties of harmonic functions, one easily verifies the following

\end{proof}
Lemma 7.2. Let $W \in W^{1,3/2}(B_{R/2}(x_0))$ be harmonic in $B_{R/2}(x_0)$. Then, there exists an absolute constant $C_0$ such that for all $0 < \tau < \frac{1}{2}$

\[(7.3) \quad \left( \int_{B_{r}} |W - W_{\tau R, x_0}|^3 dx \right)^{1/3} \leq C_0 \tau \left( \int_{B_{R/2}} |W - W_{R/2, x_0}|^3 dx \right)^{1/3}.
\]

In what follows, let $F \in \mathcal{M}^{2,\lambda}$ for some $1 < \lambda < 3$, i.e.

\[ [F]^{2}_{\mathcal{M}^{2,\lambda}} = \sup \left\{ r^{-\lambda} \int_{B_{r}(x_0)} |F|^2 dx \big| x_0 \in \mathbb{R}^3, 0 < r \leq 1 \right\} < +\infty. \]

Clearly

\[(7.4) \quad R^{-1/2} \|F\|_{2,R} \leq \gamma_0 R^\alpha \quad \forall 0 < R \leq 1, \]

where

\[ \alpha = \frac{\lambda - 1}{2}, \quad \gamma_0 = [F]_{\mathcal{M}^{2,\lambda}}. \]

Furthermore, define

\[
S(r, x_0) = \left( \frac{1}{r} \int_{B_{r}(x_0)} |\nabla B|^2 dx \right)^{1/2},
\]

\[
E(r, x_0) = \left( \int_{B_{r}(x_0)} |B - B_{r, x_0}|^3 dx \right)^{1/3},
\]

\[
M(r, x_0) = \left( \int_{B_{r}(x_0)} |B|^6 dx \right)^{1/6}, \quad 0 < r < +\infty, \ x_0 \in \mathbb{R}^3.
\]

Fix $x_0 \in \mathbb{R}^3$ and $0 < R \leq 1$. Assume that $M(R, x_0) \leq 1$. Then, from (7.2) with $r = R$ and $\rho = \frac{R}{2}$ along with (7.4) we deduce

\[(7.5) \quad S(R/2, x_0) \leq c(E(R, x_0) + \gamma_0 R^\alpha), \]

where $c > 0$ denotes an absolute constant.

Let $\alpha < \beta < 1$ be fixed. Take $0 < \tau < \frac{1}{4}$ such that

\[(7.6) \quad 2C_0 \tau^{1-\beta} \leq \frac{1}{2}, \quad \tau^{\beta-\alpha} \leq \frac{1}{2}, \quad \tau^\alpha \leq \frac{1}{2}.
\]

Let $Z \in W^{1,3/2}(B_{R/2}(x_0))$ denote a weak solution to

\[(7.7) \quad -\Delta Z = -\nabla \times ((\nabla \times B) \times B) + \nabla \times F \quad \text{in} \ B_{R/2}(x_0),
\]

\[(7.8) \quad Z = 0 \quad \text{on} \ \partial B_{R/2}(x_0). \]
By the well-known $L^p$-theory of the Laplace equation we get
\[
R^{-1}\|\nabla Z\|_{3/2,B_{R/2}(x_0)} \leq \frac{cR^{-1}}{B_{R/2}(x_0)}\|\nabla \times B\|_{3/2,B_{R/2}(x_0)} + cR^{-1}\|F\|_{3/2,B_{R}(x_0)}
\]
(7.9)
\[
\leq cM(R, x_0)S(R/2, x_0) + c\gamma_0 R^\alpha.
\]
Esimating the left hand side from below by using Sobolev-Poincaré’s inequality and the right hand side from above by the aid of (7.5), recalling that $M(R, x_0) \leq 1$, we are led to
\[
(\int_{B_{R/2}(x_0)}|Z|^3 dx)^{1/3} \leq C_1(M(R, x_0)E(R, x_0) + \gamma_0 R^\alpha),
\]
(7.10)
where $C_1 > 1$ stands for an absolute constant.

Next, we assume that
\[
3\tau^{-1}C_1M(R, x_0) \leq \frac{1}{2}\tau^\beta.
\]
(7.11)
We note here that (7.11) yields $M(R, x_0) \leq 1$ and thus (7.5) remains true. We make use of triangle inequality, then apply (7.3) (note that $W$ is harmonic). This together with (7.10) and (7.6) gives
\[
E(\tau R, x_0) \leq \left(\int_{B_{R}(x_0)}|W - W_{\tau R,x_0}|^3 dx\right)^{1/3} + 3\tau^{-1}\left(\int_{B_{R/2}(x_0)}|Z|^3 dx\right)^{1/3}
\]
\[
\leq 2C_0\tau E(R, x_0) + 3\tau^{-1}\left(\int_{B_{R/2}(x_0)}|Z|^3 dx\right)^{1/3}
\]
\[
\leq \frac{1}{2}\tau^\beta E(R, x_0) + 3\tau^{-1}C_1M(R, x_0)E(R, x_0) + 3\tau^{-1}C_1\gamma_0 R^\alpha,
\]
and observing (7.11), we therefore obtain
\[
E(\tau R, x_0) \leq \tau^\beta E(R, x_0) + C_2\gamma_0 R^\alpha,
\]
(7.12)
where
\[
C_2 = 3\tau^{-1}C_1.
\]

Next, we shall estimate $|M(\tau R, x_0)|$. By using triangle inequality and Sobolev-Poincaré inequality it follows that
\[
M(\tau R, x_0) \leq |B_{\tau R,x_0}| + c\tau^{-1/2}S(R/2, x_0)
\]
\[
\leq M(R, x_0) + c\tau^{-1/2}S(R/2, x_0) + 2\tau^{-1}E(R, x_0)
\]
(7.13)
\[
\leq M(R, x_0) + C_3(E(R, x_0) + \gamma_0 R^\alpha)
\]
with a constant $C_3 > 0$ depending on $\tau$ only.

Let $0 < M_0 \leq 1$ such that
\[
3\tau^{-1}C_1 M_0 = \frac{1}{2}\tau^\beta.
\]
(7.14)
Let $0 < R_1 \leq 1$ chosen so that

\begin{equation}
2C_3(2C_2\tau^{-\alpha} + 1)\gamma_0 R_1^\alpha \leq \frac{M_0}{2}.
\end{equation}

Since $B \in L^6$, there exists $0 < \rho_0 < +\infty$ such that

\begin{equation}
M(R_1, x_0) + 2C_3 E(R_1, x_0) \leq (1 + 4C_3) M(R_1, x_0) \leq \frac{M_0}{2} \quad \forall |x_0| \geq \rho_0.
\end{equation}

Let $x_0 \in \mathbb{R}^n$, $|x_0| \geq \rho_0$. We claim that for every $k \in \mathbb{N} \cup \{0\}$

\begin{equation}
E(\tau^k R_1, x_0) \leq \tau^k E(R_1, x_0) + 2(1 - 2^{-k})\tau^\alpha R_1^\alpha,
\end{equation}

\begin{equation}
M(\tau^k R_1, x_0) \leq M(R_1, x_0) + 2(1 - 2^{-k})\left\{C_3 E(R_1, x_0) + C_3(2C_2\tau^{-\alpha} + 1)\gamma_0 R_1^\alpha\right\}.
\end{equation}

We prove the claim by using induction over $k \in \mathbb{N} \cup \{0\}$.

Firstly, note that for $k = 0$ both (7.17) and (7.18) are trivially fulfilled. Assume (7.17) and (7.18) hold for $k \in \mathbb{N} \cup \{0\}$. Observing (7.15) and (7.16), the assumption (7.18) implies

\begin{equation}
M(\tau^k R_1, x_0) \leq M_0.
\end{equation}

By the choice of $M_0$ (7.19) yields (7.11) for $R = \tau^k R_1$. Hence from (7.12) with $R = \tau^k R_1$ we infer

\[E(\tau^{k+1} R_1, x_0) \leq \tau^\beta E(\tau^k R_1, x_0) + C_2\gamma_0 \tau^\alpha R_1^\alpha.\]

Now, estimating the first term by the assumption (7.17) taking into account (7.6), we arrive at

\[E(\tau^{k+1} R_1, x_0) \leq \tau^\beta(1 - 2^{-k})\tau^\alpha R_1^\alpha
+ 2(1 - 2^{-k})\tau^\alpha R_1^\alpha
\leq \tau^\beta(1 - 2^{-k})\tau^\alpha R_1^\alpha
= M(R_1, x_0) + 2(1 - 2^{-k})\left\{C_3 E(R_1, x_0) + C_3(2C_2\tau^{-\alpha} + 1)\gamma_0 R_1^\alpha\right\}.\]

whence, (7.18) for $k + 1$.

It remains to verify (7.18) for $k + 1$. In fact, by means of (7.13) with $R = \tau^k R_1$ together with the assumption (7.17) and (7.18) we estimate

\begin{align*}
M(\tau^{k+1} R_1, x_0) \\
&\leq M(\tau^k R_1, x_0) + C_3 E(\tau^k R_1, x_0) + \gamma_0 \tau^\alpha R_1^\alpha) \\
&\leq M(R_1, x_0) + 2(1 - 2^{-k})\left\{C_3 E(R_1, x_0) + C_3(2C_2\tau^{-\alpha} + 1)\gamma_0 R_1^\alpha\right\} \\
&+ C_3(\tau^k E(R_1, x_0) + 2\tau^\alpha(1 - 2^{-k})\tau^\alpha R_1^\alpha + \gamma_0 \tau^\alpha R_1^\alpha) \\
&\leq M(R_1, x_0) + 2(1 - 2^{-k})\left\{C_3 E(R_1, x_0) + C_3(2C_2\tau^{-\alpha} + 1)\gamma_0 R_1^\alpha\right\} \\
&+ 2^{-k}(C_3 E(R_1, x_0) + C_3(2C_2\tau^{-\alpha} + 1)\gamma_0 R_1^\alpha) \\
&= M(R_1, x_0) + 2(1 - 2^{-k})\left\{C_3 E(R_1, x_0) + C_3(2C_2\tau^{-\alpha} + 1)\gamma_0 R_1^\alpha\right\}.\]

This implies the following
Theorem 7.3. Let $F \in M^{2,\lambda}$, $1 < \lambda < 3$. Let $B \in \dot W^{1,2}$ be a suitable weak solution to (3.1). Then there exists $\rho_0 > 0$ such that $\Sigma(B) \subset B_{\rho_0}$.

As a consequence of Theorem 7.3 we get

Corollary 7.4. Let $f \in L^{6/5} \cap L^2$ and $g \in L^2 \cap L^q$ for some $3 < q < +\infty$. Let $(u, p, B)$ be a suitable weak solution to the steady Hall-MHD system. Then there exists $\rho_0 > 0$ such that $B_{\rho_0}$ is Hölder continuous in $\{x : |x| > \rho_0\}$. In particular, $\Sigma(B)$ is a compact set of Hausdorff dimension at most one.

Proof. To prove the corollary we only need to verify that $-B \times u + g \in M^{2,\lambda}$ for some $1 < \lambda < 3$. Then the assertion follows immediately from Theorem 7.3 with $F = -B \times u + g \in M^{2,\lambda}$.

First, using Hölder’s inequality, we find $\|g\|_{2,B_R} \leq c R^{3(q-2)/q} \|g\|_q$ for every ball $B_R \subset \mathbb{R}^3$, which implies $g \in M^{2,3(q-2)/q}$. Owing to $3 < q < +\infty$ we have $1 < \frac{3(q-2)}{q} < 3$.

Next, as $V = B + \omega \in L^6 \cap L^2$ and $u \in L^6$, we see that $-V \times u + (f + g) \in L^{3/2} + L^3 + L^2$.

By Calderón-Zygmund’s inequality it follows that $\nabla V \in W^{1,3/2} + W^{1,3} + W^{1,2}$. By means of Sobolev’s embedding theorem we get

$$
\|\omega\|_{3,B_2(x_0)} \leq \|V\|_{3,B_2(x_0)} + \|B\|_{3,B_2(x_0)} \\
\leq c (\|\omega\|_6 \|u\|_6 + \|B\|_6 \|u\|_6 + \|f + g\|_2)
$$

for all $x_0 \in \mathbb{R}^3$, with an absolute constant $c > 0$. As $\nabla \cdot u = 0$, we obtain

$$
\|u\|_{10,B_1(x_0)} \leq c (\|u\|_6 (1 + \|\omega\|_2 + \|B\|_6) + \|f + g\|_2) \quad \forall x_0 \in \mathbb{R}^3.
$$

Accordingly, $B \times u \in M^{2,7/5}$. This completes the proof of the corollary.

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