Topologically free partial actions and faithful representations of partial crossed products

A.V. Lebedev

Belarus State University / University of Bialystok

In this paper we investigate the interrelation between the topological freedom of partial actions of discrete groups and faithful representations of partial crossed products

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1 Introduction

The notion of a partial crossed product of a $C^*$-algebra by an action of the group $\mathbb{Z}$ by partial automorphisms was introduced by R. Exel [1]. It was then generalized further by K. McClanachan [2] up to partial crossed products by partial actions of discrete groups and by N. Sieben [3] up to partial crossed products by actions of inverse semigroups. A fruitful discussion of these and related objects one can also find in [4]. Partial crossed product is a natural generalization of the crossed product of a $C^*$-algebra by a group of automorphisms. To investigate this universal object it is important to have its faithful representations. The description of the characteristic properties of such representations is the theme of this article. Among the main properties in the presence of which one can obtain these representations is the existence of a contractive conditional expectation onto the 'coefficient' algebra (in this
paper we call this property — property (*)) and the topological freedom of the partial action. It is shown that the topological freedom implies property (*) and therefore gives us a powerful instrument to construct faithful representations of partial crossed products.

In this introductory section we gather the known necessary notions and facts on the partial actions and partial crossed products. In the next Section 2 we introduce the notion of the topologically free partial action and prove the principle result of the article (Theorem 2.8) linking topological freedom of the action and property (*). Finally in Section 3 on the base of this result we describe the existence of faithful representations of partial crossed products and reduced partial crossed products.

Let $A$ be a $C^*$-algebra and $G$ be a discrete group. We recall the definition of a partial $C^*$-dynamical systems and the corresponding partial crossed products for discrete groups (see, for example, [2]).

**Definition.** A partial action of $G$ on $A$ (denoted by $\alpha$) is a collection $\{D_g\}_{g \in G}$ of closed two-sided ideals of $A$ and a collection $\{\alpha_g\}_{g \in G}$ of isomorphisms $\alpha_g: D_{g^{-1}} \to D_g$ such that

1. $\alpha_g(D_{g^{-1}} \cap D_h) \subset D_{gh}$ for $g, h \in G$
2. $\alpha_{hg}(d) = \alpha_h(\alpha_g(d))$ for $d \in D_{g^{-1}} \cap D_{g^{-1}h^{-1}}$
3. $D_e = A$, $\alpha_e = \text{Id}_A$

We shall say that $(A, G, \alpha)$ is a partial dynamical system.

Let

$L = \{a \in l^1(G, A) : a(g) \in D_g\}$

with the usual norm $\|a\|_1 = \sum \|a(g)\|$. Define a convolution multiplication and involution on $L$ as follows.

$$(a * b)(g) = \sum_{h \in G} \alpha_h \left[ \alpha_{h^{-1}}(a(h)) b(h^{-1}g) \right]$$

$$a^*(g) = \alpha_g(a(g^{-1})^*)$$

With these operations $L$ becomes a Banach *-algebra.

**Definition.** The partial crossed product of $A$ and $G$ is the universal enveloping $C^*$-algebra of $L$. We denote the partial crossed product by $A \times_\alpha G$.

**Definition.** A covariant representation of $(A, G, \alpha)$ is a triple $(\pi, u, H)$ where $\pi: A \to B(H)$ is a representation of $A$ on a Hilbert space $H$ (here
\( B(H) \) is the algebra of all linear bounded operators on \( H \), \( u : G \to B(H) \)
is a function \( g \mapsto u_g \) with \( u_g \) being a partial isometry on \( H \) with the initial
subspace \([\pi(D_g^{-1}H)]\) and the final subspace \([\pi(D_gH)]\) such that

1. \( u_g\pi(d)u_g^{-1} = \pi(\alpha_g(d)) \quad d \in D_g^{-1} \)
2. \( \pi(d)[u_gu_h - u_{gh}] = 0 \quad d \in D_g \cap D_h \)
3. \( u_g^* = u_{g^{-1}} \).

**Definition.** Let \( (\pi, u, H) \) be a covariant representation of \((A, G, \alpha)\). We define the representation \( \pi \times u : L \to B(H) \) by

\[
(\pi \times u)(a) = \sum \pi(a(g))u_g.
\]

By the definition of \( A \times \alpha G \) \( (\pi \times u) \) extends up to a \(*\)-representation of \( A \times \alpha G \).

**1.1 Reduced partial crossed product.** A special and important particular covariant representation of \((A, G, \alpha)\) is the so-called *reduced partial crossed product* which is defined in the following way (see [2], Section 3).

First we associate with any representation \( \pi : A \to B(H) \) a certain representation \( \tilde{\pi} \) (the regular representation) of \( A \) on \( l^2(G, H) \). Let

\[ \pi_g : D_g \to B(H) \]

be defined by

\[ \pi_g(d) = \pi(\alpha_{g^{-1}}(d)). \]

By [3], 2.10.4. there exists a unique extension \( \pi'_g \) of \( \pi_g \) to \( A \) which annihilates \([\pi_g(D_g)H]^\perp \). This extension is given by

\[ \pi'_g(d) = \lim_{\lambda} \pi_g(v_{\lambda}d) \]

where \( \{v_{\lambda}\}_\lambda \) is an approximate identity for \( D_g \). Now we define

\[ \tilde{\pi} : A \to B(l^2(G, H)) \]

by

\[ \tilde{\pi}(d)\xi(g) = \pi'_g(d)\xi(g) \quad \xi \in l^2(G, H), \quad g \in G, \quad d \in A. \]

For the regular representation \( \lambda \) of \( G \) \( \lambda : G \to B(l^2(G, H)) \)
\( (\lambda_g\xi)(h) = \xi(g^{-1}h) \) we have ([2], Proposition 3.1.)

\[ \lambda_g \tilde{\pi}(d)\lambda_{g^{-1}} = \tilde{\pi}(\alpha_g(d)) \quad g \in G, \quad d \in D_g^{-1}. \]
If we let $\tilde{\lambda}_g = \lambda_g P_g^{-1}$ where $P_g$ is the orthogonal projection onto the Hilbert space $[\tilde{\pi}(D_g)l^2(G, H)]$ then $([\tilde{\pi}, \tilde{\lambda}, l^2(G, H))$ is a covariant representation of $(A, G, \alpha)$.

Let $\| \cdot \|_r$ be the norm on $l^1(G, A)$ defined by

$$\| a \|_r = \sup \{ \| (\tilde{\pi} \times \lambda)(a) \| (\pi, H) \in \text{Rep}(A) \}$$

where $\text{Rep}(A)$ is the set of all representations of $A$.

The reduced partial crossed product $A \times \alpha_r G$ of $A$ by $G$ is the completion of $l^1(G, A)$ with respect to the norm $\| \cdot \|_r$.

In fact there is no need to use all representations of $A$ to define the reduced crossed product. As the next result tells it is enough to exploit any its faithful one.

**Theorem 1.2** (\[2\], Proposition 3.4.) Let $\pi : A \to B(H)$ be a representation. Then $\tilde{\pi}$ is faithful iff $\tilde{\pi} \times \lambda$ is faithful on $A \times \alpha_r G$.

## 2 Property (*) and topologically free action

**Definition.** Let $(\pi, u, H)$ be a covariant representation of $(A, G, \alpha)$. We shall say that $(\pi \times u)$ possesses property (*) if for any finite sum

$$\sum_{g \in F} \pi(a(g))u_g, \ \ F \subset G, \ |F| < \infty$$

we have

$$\left\| \sum_{g \in F} \pi(a(g))u_g \right\| \geq \| a(e) \|$$

**Remark 2.1** It follows from \[2\], Proposition 3.5 that $A \times \alpha_r G$ and $A \times \alpha G$ possess property (*).

2.2 If $(\pi \times u)$ possesses property (*) then the mapping

$$\mathcal{N} \left( \sum_{g \in F} \pi(a(g))u_g \right) = a(e)$$

is uniquely extended up to the mapping (positive, contractive, conditional expectation)

$$\mathcal{N} : (\pi \times u) \to A.$$
Remark 2.3 R. Exel [6, Theorem 3.3. proved that one can formulate property (*) in somewhat weaker but anyway equivalent way. In fact he proved a more general statement concerning graded \(C^*\)-algebras. Hereafter we formulate his result (its simplification) in terms of the objects considered in this paper.

Let \((\pi \times u)\) be such that \(\pi\) is a faithful representation of \(A\) and

\[
E: (\pi \times u) \rightarrow A
\]

be a bounded linear map such that

(a) \(E(\pi(a)I) = a, \quad a \in A,\)
(b) \(E(\pi(a(g))u_g) = 0, \quad g \neq 0.\)

Then \((\pi \times u)\) possesses property (*) and \(E = \mathcal{N}\) where \(\mathcal{N}\) is that mentioned in 2.2.

2.4 Now we proceed to one of the main notions of the article: topologically free action. To start with we note that partial action defines in a natural way a partial dynamical system (the action of a group by partial homeomorphisms) on the primitive ideal space \(\text{Prim} A\) and the spectrum \(\hat{A}\) of \(A\). Here we give the description of this partial dynamical system.

For any ideal \(J \subset A\) we set \(\text{supp} J = \{x \in \text{Prim} A : x \not\supset J\}\). It is known (see [5, 3.2.1.]) that the mapping \(x \rightarrow x \cap J\) establishes a homeomorphism \(\text{supp} J \leftrightarrow \text{Prim} J\) (with respect to the Jacobson topology) and \(\text{supp} J\) is an open set in \(\text{Prim} A\). Set also \(\hat{A}' = \{\pi \in \hat{A} : \pi(J) \neq 0\}\) (here \(\hat{A}\) is the spectrum of \(A\)). Then the mapping \(\pi \rightarrow \pi|_J\) establishes a homeomorphism \(\hat{A}' \leftrightarrow \hat{J}\) (with respect to the Jacobson topology) and \(\hat{A}'\) is an open set in \(\hat{A}\) (see [5, 3.2.1.]).

Let us define the mapping \(\tau_g : \hat{A}^{D_g^{-1}} \rightarrow \hat{A}^{D_g}\) in the following way: for any \(\pi \in \hat{A}^{D_g^{-1}}\) we set

\[
\tau_g(\pi)(j) = \pi(\alpha_g^{-1}(j)), \quad j \in D_g.
\]

The foregoing observations tell us that \(\tau_g\) is a homeomorphism.

Let us also define the mapping \(t_g : \text{supp} D_g^{-1} \rightarrow \text{supp} D_g\) in the following way: for any point \(x \in \text{supp} D_g^{-1}\) such that \(x = \ker \pi\) where \(\pi \in \hat{A}^{D_g^{-1}}\) we set

\[
t_g(x) = \ker \tau_g(\pi).
\]
Clearly $t_g$ is a homeomorphism.

For $\tau_g$ and $t_g$ defined in the above described way we have that $\{\tau_g\}_{g \in G}$ defines an action of $G$ by partial homeomorphisms of $\hat{A}$ and $\{t_g\}_{g \in G}$ defines an action of $G$ by partial homeomorphisms of $\text{Prim} \ A$.

2.5 We say that the action $\{\alpha_g\}_{g \in G}$ is topologically free iff for any finite set $\{g_1,...g_k\} \subset G$ and any nonempty open set $U \subset \text{supp} \ D_{g_1^{-1}} \cap ... \cap \text{supp} \ D_{g_k^{-1}}$ there exists a point $x \in U$ such that all the points $t_{g_i}(x)$, $i = 1, k$ are distinct.

This condition can be also formulated in the following way: for any finite set $\{g_1,...g_k\} \subset G$ and any nonempty open set $U$ there exists a point $x \in U$ such that all the points $t_{g_i}(x)$, $i = 1, k$ that are defined ($\Leftrightarrow x \in \text{supp} \ D_{g_i^{-1}}$) are distinct.

If we denote by $X_g$ the set

$$X_g = \{x \in \text{supp} \ D_{g^{-1}} : t_g(x) = x\}$$

then the foregoing condition can be also written in the next way: for any finite set $\{g_1,...g_n\}$, $g_i \neq e$ the interior of the set $[\cup_{i=1}^n X_{g_i}]$ is empty.

The main statement of this section is Theorem 2.8 and the most important technical result is Lemma 2.7. Among the technical instruments of the proof of this lemma is the next Lemma 2.6 which is useful in its own right.

**Lemma 2.6** ([10], Lemma 12.15). Let $B$ be a $C^*$-subalgebra of the algebra $L(H)$ of linear bounded operators in a Hilbert space $H$. If $P_1$, $P_2 \in B'$ are two orthogonal projections such that the restrictions

$$B|_{HP_1} \text{ and } B|_{HP_2}$$

(where $HP_1 = P_1(H)$, $HP_2 = P_2(H)$) are both irreducible and these restrictions are distinct representations then

$$H_{P_1} \perp H_{P_2}.$$ 

**Lemma 2.7** Let the action $\{\alpha_g\}_{g \in G}$ be topologically free and $(\pi \times u)$ is such that $\pi$ is a faithful representation of $A$. Let $F$ be a finite subset of $G$, and
$a \in L$ be any function such that $a(g) = 0, \ g \notin F$, and $c \in (\pi \times u)$ be the operator of the form

$$c = \sum_{g \in F} \pi(a(g))u_g \quad (2.1)$$

Then for every $\varepsilon > 0$ there exists an irreducible representation $\pi'$ of $\pi(A)$ such that for any irreducible representation $\nu$ of $(\pi \times u)$ which is an extension of $\pi'$ we have

(i) \quad $\|\pi'[\pi(a(e))]\| \geq \|a(e)\| - \varepsilon$,

(ii) \quad $P_{\pi'} \pi'[\pi(a(e))] P_{\pi'} = P_{\pi'} \nu(c) P_{\pi'}$

where $P_{\pi'}$ is the orthogonal projection onto $H_{\pi'}$ in $H_{\nu}$.

**Proof.** As $\pi(A) \cong A$ we shall identify throughout the proof $\pi(A)$ and $A$ (in order to shorten the notation).

For any $d \in A$ and $x \in \text{Prim} A$ we denote by $\check{d}(x)$ the number

$$\check{d}(x) = \inf_{j \in x} \|d + j\| \quad (2.2)$$

For every $d \in A$ the function $\check{d}(x)$ is lower semicontinuous on $\text{Prim} A$ and attains its upper bound equal to $\|d\|$ (see [5], 3.3.2. and 3.3.6.).

Let $x_0 \in \text{Prim} A$ be a point at which $\check{a}(e)(x_0) = \|a(e)\|$ and $\pi_0$ be an irreducible representation of $A$ such that $x_0 = \ker \pi_0$ (thus $\|\pi_0(a(e))\| = \|a(e)\|$). Since the function $\check{a}(e)(x)$ is lower semicontinuous it follows that for any $\varepsilon > 0$ there exists an open set $U \subset \text{Prim} A$ such that

$$\check{a}(e)(x) > \|a(e)\| - \varepsilon \quad \text{for every } x \in U. \quad (2.3)$$

As the action $\{\alpha_g\}_{g \in G}$ is topologically free there exists a point $x' \in U$ such that all the points $t_g(x'), \ i = 1, k$ are distinct (if they are defined $\iff x' \notin \text{supp } D_{\pi_0}^{-1}$).

Let $\pi'$ be an irreducible representation of $A$ such that $\ker \pi' = x'$ and let $\nu$ be any extension of $\pi'$ up to an irreducible representation of $(\pi \times u)(L)$. We shall denote by the same letter $\nu$ an extension of the mentioned representation up to an irreducible representation of the $C^*$-algebra $C$ generated by $(\pi \times u)(L)$ and $\{u_g\}_{g \in G}$ (see [5], 2.10.2.). For this representation $\nu$ we have

$$H_{\pi'} \subset H_{\nu}$$

where $H_{\pi'}$ is the representation space for $\pi'$ and $H_{\nu}$ is that for $\nu$. 

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By the choice of $\pi'$ and (2.3) we conclude that there exists a vector $\xi \in H_{\pi'}$ such that $\|\xi\| = 1$ and

$$\|\pi'(a(e))\xi\| > \|a(e)\| - \varepsilon.$$  \hspace{1cm} (2.4)

Thus (i) is proved.

To prove (ii) let us observe first that for any vectors $\xi, \eta \in H_{\pi'}$ we have

$$\langle \pi'(d_1)\eta, \nu(d_2u_g)\xi \rangle = 0, \quad d_1 \in A, \quad d_2 \in D_g, \quad g \in F, \quad g \neq e.$$  \hspace{1cm} (2.5)

Which in turn will imply

$$P_{\pi'} \nu(d_2u_g) P_{\pi'} = 0, \quad g \in F, \quad d_2 \in D_g, \quad g \neq e$$  \hspace{1cm} (2.6)

To prove (2.5) we consider the following possible positions of $x'$.

$x' \notin \text{supp} \, D_g$. In this case we have $\pi'(d_2^*) = 0$ and

$$\langle \pi'(d_1)\eta, \nu(d_2)\nu(u_g)\xi \rangle = \langle \nu(d_2)\pi'(d_1)\eta, \nu(u_g)\xi \rangle =$$

$$\langle \pi'(d_2)\pi'(d_1)\eta, \nu(u_g)\xi \rangle = 0.$$

$x' \notin \text{supp} \, D_g^{-1}$. Observing that $\nu(u_g^*u_g)$ is the projection onto the essential space of $\nu(D_g^{-1})$ we conclude that $\nu(u_g^*u_g)\xi = 0$ and therefore we have

$$\langle \pi'(d_1)\eta, \nu(d_2)\nu(u_g)\xi \rangle = \langle \pi'(d_1)\eta, \nu(d_2)\nu(u_g^*u_g)\xi \rangle =$$

$$\langle \pi'(d_1)\eta, \nu(d_2)\nu(u_g^*u_g)\xi \rangle = 0.$$

Finally let $x' \in [\text{supp} \, D_g \cap \text{supp} \, D_g^{-1}]$.

In this case $\pi'$ is an irreducible representation as for $D_g$ so also for $D_g^{-1}$ and $t_g(x') \in \text{supp} \, D_g$ (according to the definition of $t_g$ [2.4]). Moreover we have

$$\nu(u_g^*u_g)\eta = \eta, \quad \nu(u_g^*u_g)\eta = \eta \quad \text{for any} \quad \eta \in H_{\pi'}.$$  \hspace{1cm} (2.7)

Since $\nu(u_g)$ is a partial isometry the observation (2.7) implies that $H_{\pi'}$ belongs as to the initial and final subspaces of $\nu(u_g)$ so also to the initial and final subspaces of $\nu(u_g^*)$ and the mappings

$$\nu(u_g) : H_{\pi'} \to \nu(u_g) \left[ H_{\pi'} \right] \quad \text{and} \quad \nu(u_g^*) : H_{\pi'} \to \nu(u_g^*) \left[ H_{\pi'} \right]$$  \hspace{1cm} (2.8)
are isomorphisms.

Let \( P_1 \) be the orthogonal projection of \( H_\nu \) onto \( H_\pi' \). By the definition of \( \nu \) we have that \( P_1 \in \nu(A)' \) and \( (2.7) \) means that
\[
P_1 = P_1 \nu(u_g^* u_g) = P_1 \nu(u_g u_g^*). \tag{2.9}
\]

Set \( P_2 = \nu(u_g) P_1 \nu(u_g^*) \). The foregoing observations imply that
\[
\nu(u_g) : P_1(H_\pi') \to P_2(H_\pi')
\]
is an isomorphism. Observe also that
\[
P_2 \in [\nu(D_g)]'. \tag{2.10}
\]
Indeed. For any \( d \in D_g \) we have
\[
\nu(d) = \nu(u_g u_g^*) \nu(d) = \nu(d) \nu(u_g u_g^*)
\]
and
\[
\nu(\alpha_{g^{-1}}(d)) = \nu(u_g^* u_g) \nu(\alpha_{g^{-1}}(d)) = \nu(\alpha_{g^{-1}}(d)) \nu(u_g^* u_g),
\]
and
\[
\nu(u_g^*) \nu(d) \nu(u_g) = \nu(\alpha_{g^{-1}}(d)),
\]
and
\[
\nu(\alpha_g \circ \alpha_{g^{-1}}(d)) = \nu(a).
\]
Using this we obtain for any \( d \in D_g \)
\[
P_2 \nu(d) = \nu(u_g) P_1 \nu(u_g^*) \nu(d) = \nu(u_g) P_1 \nu(u_g^*) \nu(u_g u_g^*) \nu(d) =
\nu(u_g) P_1 \left[ \nu(u_g^*) \nu(d) \nu(u_g) \right] \nu(u_g^*) = \nu(u_g) P_1 \nu(\alpha_{g^{-1}}(d)) \nu(u_g^*) =
\nu(u_g) \nu(\alpha_{g^{-1}}(d)) \nu(u_g^*) P_1 \nu(u_g^*) = \nu(\alpha_g \circ \alpha_{g^{-1}}(d)) \nu(u_g) P_1 \nu(u_g^*) =
\nu(d) P_2
\]
Thus \( (2.10) \) is true.

In addition the irreducibility of \( \nu(D_g)|_{H_{P_1}} \) implies the irreducibility of \( \nu(D_g)|_{H_{P_2}} \) (here \( H_{P_1} = P_1(H_\nu) = H_{\pi'} \) and \( H_{P_2} = P_2(H_\nu) \)).

Now observe that for \( d \in D_g \) we have
\[
P_1 \nu(d) = 0 \iff \tilde{d}(x') = 0
\]
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and

\[ P_2 \nu(d) = 0 \iff \nu(u_g) P_1 \nu(u_g^*) \nu(u_g) = 0 \iff \nu(u_g) P_1 \nu(u_g^*) = 0 \iff \nu(u_g) P_1 \nu(u_g^*) = 0 \iff P_1 \nu(\alpha_{g^{-1}}(d)) = 0 \iff \alpha_{g^{-1}}(d)(x') = 0 \iff \tilde{d}(t_g(x')) = 0. \]

So (since the points \( x' \) and \( t_g(x') \) are distinct) we conclude that the representations \( \nu(D_g)|_{H_{P_1}} \) and \( \nu(D_g)|_{H_{P_2}} \) are distinct. Applying Lemma 2.6 we find that

\[ P_1 \cdot P_2 = 0. \quad (2.11) \]

By applying (2.11), (2.7), (2.9) and (2.10) we have for any \( \eta, \xi \in H_{P_1} = H_\pi, \ d_1 \in A, \ d_2 \in D_g \)

\[ \langle \pi'(d_1) \eta, \nu(d_2 u_g) \xi \rangle = \langle P_1 \nu(d_1) \eta, \nu(d_2 u_g) \xi \rangle = \langle P_1 \nu(d_1) \eta, \nu(d_2) \nu(u_g) \xi \rangle = \langle P_2 \cdot P_1 \nu(d_1) \eta, \nu(d_2) \nu(u_g) \xi \rangle = 0 \]

which finishes the proof of (2.5) (and therefore the proof of (2.6) as well).

Now returning to the operator (2.1) (recall that we are identifying \( A \) and \( \pi(A) \)) and using (2.6) we have that

\[ P_{\pi'} \nu \left[ \sum_{g \in F} a(g) u_g \right] P_{\pi'} = P_{\pi'} \nu(a(e)) P_{\pi'} = P_{\pi'} \pi'(a(e)) P_{\pi'} \]

so (ii) is true and the proof of the lemma is complete.

As an immediate corollary of this lemma we obtain the next

**Theorem 2.8** Let the action \( \{ \alpha_g \}_{g \in G} \) be topologically free. If \( (\pi \times u) \) is such that \( \pi \) is a faithful representation of \( A \) then \( (\pi \times u) \) possesses property \((^*)\).

**Proof.** Let \( c \) be the operator (2.1). Take \( \pi' \) mentioned in the statement of Lemma 2.7. Then we have by (ii) and (i)

\[ \|c\| \geq \|\nu(c)\| \geq \|P_{\pi'} \nu(c) P_{\pi'}\| \geq \|a(e)\| - \varepsilon \]

In view of the arbitrariness of \( \varepsilon \) this implies property \((^*)\).

One more simple corollary of Lemma 2.7 and Theorem 2.8 is the following
Lemma 2.9 Let the action \( \{\alpha_g\}_{g \in G} \) be topologically free and \((\pi \times u)\) is such that \(\pi\) is a faithful representation of \(A\). Then for any \(c \in (\pi \times u)\) and every \(\varepsilon > 0\) there exists an irreducible representation \(\pi'\) of \(\pi(A)\) such that for any irreducible representation \(\nu\) of \((\pi \times u)\) which is an extension of \(\pi'\) we have

(i) \(\|\pi'[N(c)]\| \geq \|N(c)\| - \varepsilon\),

(ii) \(\|P_{\pi'}[N(c)] P_{\pi'} - P_{\pi'} \nu(c) P_{\pi'}\| \leq \varepsilon\).

Proof. Follows from the standard approximation argument in view of the density of finite sums of the form (2.1) in \((\pi \times u)\) and the fact that \((\pi \times u)\) possesses property (*)

3 Property (*), topologically free action, partial crossed products and partial reduced crossed products

It is reasonable to consider \(A \times_\alpha G\) as the maximal \(C^*\)-algebra possessing property (*) (it follows from the construction of \(A \times_\alpha G\) and Remark 2.1). On the other hand it has been shown by R. Exel that \(A \times_{ar} G\) is the minimal \(C^*\)-algebra possessing this property. The exact meaning of 'minimality' is given in the next statement which is a reformulation (in fact simplification) of [6], Theorem 3.3. (we recall at this point that according to Theorem 1.2 for any faithful representation \(\pi\) of \(A\) \(\bar{\pi} \times \lambda\) is a faithful representation of \(A \times_{ar} G\)).

Theorem 3.1 Let \((\pi \times u)\) be such that \(\pi\) is a faithful representation of \(A\). If \((\pi \times u)\) possesses property (*) then the mapping

\[
(\pi \times u) \ni \sum \pi(a(g)) u_g \mapsto \sum \tilde{\pi}(a(g)) \tilde{\lambda}_g \in \tilde{\pi} \times \lambda
\]

can be extended up to a \(C^*\)-algebra epimorphism (here \(\tilde{\pi} \times \lambda\) is that mentioned in Theorem 1.2).

Remark 3.2 It is also known that if \(G\) is an amenable group then the canonical surjection \(\Lambda : A \times_\alpha G \rightarrow A \times_{ar} G\) is an isomorphism (see, for example, [2], Proposition 4.2).

This observation along with Theorem 3.1 leads to the next result
Theorem 3.3 Let $G$ be an amenable group and $(\pi^i, u^i, H^i), \ i = 1, 2$ be two covariant representations of $(A, G, \alpha)$ such that both $\pi^i \times u^i, \ i = 1, 2$ possess property (*) then the mapping
\[
\sum \pi^1(a(g)) u^1_g \mapsto \sum \pi^2(a(g)) u^2_g
\]
give rise to the isomorphism of the algebras $\pi^1 \times u^1$ and $\pi^2 \times u^2$.

Remark 3.4 The importance of property (**) for the first time (probably) was clarified by O’Donovan [8] in connection with the description of C*-algebras generated by weighted shifts. The most general result (of Theorem 3.3 type) establishing the crucial role of this property in the theory of crossed products of C*-algebras by discrete groups of automorphisms was obtained in [9] for an arbitrary C*-algebra and amenable discrete group (see also [10], Chapters 2,3 for complete proofs and various applications). The relation of the corresponding property to the faithful representations of crossed products by endomorphisms generated by isometries was investigated in [11, 12].

It is worth mentioning that in [10], Theorem 12.8 (an analogue to Theorem 3.3) was proved in a direct way not exploiting the reduced crossed product so in particular the isomorphism of $\Lambda : A \times_{\alpha} G \rightarrow A \times_{\alpha r} G$ for amenable groups can also be derived from this result (the proof of [10], Theorem 12.8 can be easily extended up to a partial crossed product situation).

Theorem 2.8 gives us a possibility to verify property (*) in an automatic way by means of the property of the underlying partial dynamical system. This theorem along with the foregoing results leads to the following Theorems 3.5, 3.6.

Theorem 3.5 Let the action $\{\alpha_g\}_{g \in G}$ be topologically free and $(\pi \times u)$ is such that $\pi$ is a faithful representation of $A$. Then the mapping
\[
(\pi \times u) \ni \sum \pi(a(g)) u_g \mapsto \sum \tilde{\pi}(a(g)) \tilde{\lambda}_g \in \tilde{\pi} \times \lambda
\]
can be extended up to a C*-algebra epimorphism (here $\tilde{\pi} \times \lambda$ is that mentioned in Theorem 1.2).

Proof. Apply Theorem 2.8 and Theorem 3.1

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Theorem 3.6 Let $G$ be an amenable group and the action $\{\alpha_g\}_{g \in G}$ be topologically free. If $(\pi^i, u^i, H^i), \ i = 1, 2$ be two covariant representations of $(A, G, \alpha)$ such that both $\pi^i, \ i = 1, 2$ are faithful representations of $A$ then the mapping
$$\sum \pi^1(a(g))u^1_g \mapsto \sum \pi^2(a(g))u^2_g$$
give rise to the isomorphism of the algebras $\pi^1 \times u^1$ and $\pi^2 \times u^2$.

**Proof.** Apply Theorem 2.8 and Theorem 3.3.

The next Theorem 3.7 and Corollary 3.8 are in a way opposite to Theorem 3.1. They form a generalization of [7], Theorem 2.6. (where $A = C_0(X)$).

**Theorem 3.7** Let the action $\{\alpha_g\}_{g \in G}$ be topologically free. If $I$ is an ideal in $A \times_{\alpha r} G$ then $I = \{0\}$ iff $I \cap A = \{0\}$.

**Proof.** Let $I \cap A = \{0\}$. Denote by $\pi : A \times_{\alpha r} G \to (A \times_{\alpha r} G)/I$ the quotient map and let $c \in I$ be an element such that $c \geq 0$ and $\pi(c) = 0$. To prove that $I = \{0\}$ we have to verify that
$$c = 0. \tag{3.1}$$

Since the mapping
$$\mathcal{N} : A \times_{\alpha r} G \to A$$
defined in 2.2 is faithful (see, for example, [6, Proposition 2.12]) (3.1) will be proved if we prove that
$$\mathcal{N}(c) = 0. \tag{3.2}$$

So let us verify the latter property.

Since $I \cap A = \{0\}$ it follows that $\pi(A) \cong A$. Given $\varepsilon > 0$ take $\pi'$ form the statement of Lemma 2.9 (we can refer to this representation either as to the representation of $\pi(A)$ so also as to the representation of $A$) and extend it up to an irreducible representation $\nu$ of $\pi(A \times_{\alpha r} G)$ (here we consider $\pi(A \times_{\alpha r} G)$ as $(\pi \times u)$ in the statement of Lemma 2.9). Evidently $\nu \circ \pi$ is an irreducible representation of $A \times_{\alpha r} G$.

Now the condition $\pi(c) = 0$ and property (ii) of the statement of Lemma 2.9 imply
$$\varepsilon \geq \|P_{\pi'} \nu'[\mathcal{N}(\pi(c))] P_{\pi'} - P_{\pi'} \nu(\pi(c)) P_{\pi'}\| = \|\pi'[\mathcal{N}(\pi(c))]\| = \|\pi'[\mathcal{N}(c)]\|.$$ 

This and (i) implies
$$\|\mathcal{N}(c)\| \leq 2\varepsilon.$$

Which proves (3.2) by the arbitrariness of $\varepsilon$. 

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Corollary 3.8 Let the action \( \{ \alpha_g \}_{g \in G} \) be topologically free. A representation \( \pi \) of the reduced partial crossed product \( A \times_{\alpha_r} G \) is faithful iff it is faithful on \( A \).

Proof. Take in the statement of Theorem 3.7 \( I = \ker \pi \).

Remark 3.9 The interrelation between the topological freedom of the action and property (\( \ast \)) and application of these properties to various crossed product type results have been intensively exploited by many authors. The treatment of the topological freedom as an instrument of investigation of ideals in the crossed products was started (probably) by D.P. O’Donovan in [8], Theorem 1.2.1. Theorem 3.6 in the case of a commutative algebra \( A \) and the action of the group \( \mathbb{Z} \) by automorphisms was proved in [13, 14]. The development of this field and its numerous (not purely \( C^* \)-algebraic) applications such as, for example, the construction of symbolic calculus and the solvability theory of functional differential equations one can find in [15, 10, 16, 17]. For the general automorphism situation Theorem 3.6 was obtained in [9] (see also in this connection [10], Chapters 2,3). Among the already mentioned ‘purely’ \( C^* \)-algebraic sources we have to emphasize an outstanding contribution to the theme made in [6]. A deep and versatile study of the topological freedom (in the situation \( A = C_0(X) \)) and its application to a series of structural problems in partial crossed product theory is implemented in [7].

In the Lebesgue space situation the topological freedom corresponds to the so-called metrical freedom. The interrelation between this property, property (\( \ast \)) and the corresponding crossed product results (in the automorphisms situation) were investigated and applied to the solution of the problem of classification of measure preserving automorphisms by W.B Arveson and K.B. Josephson in [18, 19].

In the endomorphisms situation namely in the case when a \( C^* \)-algebra endomorphism is generated by a single isometry the interrelations between the topological freedom of the action and property (\( \ast \)) have been investigated in [20, 21, 22] where in particular the analogues to Theorems 2.8, 3.3, 3.6 for the situation considered were obtained. In fact this research has been inspired by the pioneering work by V.A. Arzumanian and A.M. Vershik [22, 23, 24, 25] where the corresponding Lebesgue space objects have been introduced and studied. Recently this theme has got a new development in the work by R. Exel [26, 27], and R. Exel and A.M. Vershik [28].
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