SPACE-FILLING CURVES OF SELF-SIMILAR SETS (III): SKELETONS

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Abstract. To study the space-filling curve (SFC) of a self-similar set, Dai, Rao and Zhang [2] introduce a new notion skeleton of a self-similar set which is the basis to get a substitution rule to construct SFC. In this paper, we will study this concept in detail and give an algorithm to find the skeleton of a connected self-similar set which satisfies the finite type condition. Precisely, we prove that a self-similar set with finite type condition has a skeleton.

1. Introduction

The topic of space-filling curve (SFC) has been studied over a century. The idea of constructing the classical SFC was generated to construct SFC’s for a connected self-similar sets by Rao and Zhang [11] and Dai, Rao and Zhang [2]. To produce a SFC, [2] introduce a new concept skeleton of self-similar set which is critical for their construction and also the main concern of present paper.

1.1. Skeletons of self-similar sets. Let \( S = \{S_1, \ldots, S_N\} \) be a finite set contraction similitudes on \( \mathbb{R}^d \). Then there exists a unique non-empty compact set \( K \) satisfying

\[
K = \bigcup_{i=1}^{N} S_i(K).
\]

We call such set \( K \) a self-similar set and the family \( \{S_1, \ldots, S_N\} \) is called an iterated function system, or IFS in short; \( K \) is also called the invariant set of the IFS. (See for instance, [5][3]). Denote \( \Sigma = \{1, 2, \ldots, N\} \). The mapping \( \pi : \Sigma^N \to K \) defined as

\[
\{\pi(\omega)\} = \bigcap_{m=1}^{\infty} S_{\omega|m}(K)
\]

is called a projection. (See Section 2 for details.)

Date: September 16, 2018.

2010 Mathematics Subject Classification. Primary: 28A80. Secondary: 52C20, 20H15.

Key words and phrases. Self-similar set, Skeleton, Finite type condition.

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Supported by project I1136 and by the doctoral program W1230 granted by the Austrian Science Fund (FWF).

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To study the connectivity of self-similar set, Hata introduced the following undirected graph.

**Definition 1.1.** Let \( \{ S_j \}_{j=1}^N \) be an IFS with invariant set \( K \). For any subset \( A \) of \( K \), we define an undirected graph \( H(A) \) as follows.

- The vertex set is \( V = \{ S_1, S_2, \ldots, S_N \} \);
- There is an edge between two vertices \( S_i \) and \( S_j \) if and only if \( S_i(A) \cap S_j(A) \neq \emptyset \).

We call \( H(A) \) the *Hata graph* of \( A \) with respect to \( \{ S_1, \ldots, S_N \} \).

**Remark 1.2.** Hata proved that a self-similar set \( K \) is connected if and only if the graph \( H(K) \) is connected.

![Hata graphs](image)

**Figure 1.** The Hata graph of subset \( A \) of the Sierpiński carpet w.r.t. IFS \( \{ S_i(x) = \frac{x+d_i}{3} \}_{i=1}^8 \), \( \{ d_1, \ldots, d_8 \} = \{ 0, i, 2i, 2i + 1, 2i + 2, i + 2, 2, 1 \} \).

By the Hata graph, we define a skeleton of an IFS as follows.

**Definition 1.3.** Let \( \{ S_j \}_{j=1}^N \) be an IFS such that the invariant set \( K \) is connected. We call a finite subset \( A \) of \( K \) a **skeleton** of \( \{ S_j \}_{j=1}^N \) (or \( K \)), if the following two conditions are satisfied.

1. \( A \) is stable under iteration, that is, \( A \subset \bigcup_{j=1}^N S_j(A) \);
2. The Hata graph \( H(A) \) is connected.

**Remark 1.4.** Kigami and Morán have studied the ‘boundary’ (also called ‘vertices’ if it is finite) of a fractal. A skeleton is usually chosen to be a subset of the ‘boundary’ of a self-similar set since we want a small skeleton; for a so-called p.c.f. self-similar set, the set of vertices, is a skeleton. Indeed, the choices of skeletons are much more arbitrary.

This aim of this paper is to give an algorithm to find a skeleton of an given IFS. To this end, we will give a necessary and sufficient condition of the existence of skeletons at first.
1.2. **Main results.** Let $K$ be the invariant set of IFS $\{S_j\}_{j=1}^N$. Denote Hata graph of $K$ by $H = H(K)$. We shall denote by $\{S_i, S_j\}$ the edge in $H$ connecting $S_i$ and $S_j$. For $e = \{S_i, S_j\} \in H$, a pair $\omega, \gamma \in \{1, \ldots, N\}^\infty$ is called *bifurcation pair* of $e$, if both sequences are eventually periodic, and $\omega_1 = i, \gamma_1 = j, \pi(\omega) = \pi(\gamma)$.

Let $R$ be a sub-graph of the Hata graph $H$. We call $R$ the *spanning subgraph* of $H$ if $R$ is connected, and $R$ has the same vertex set as $H$.

**Theorem 1.5.** Let $K$ be a connected self-similar set. Then $K$ possesses a skeleton if and only if there exists a spanning subgraph $R \subset H(K)$ such that every edge $e \in R$ admits a bifurcation pair.

The self-similar sets satisfying *finite type condition* form an important class of fractals. (See for instance [10, 9, 1].) The exact definition of finite type will be given in Section 4. Roughly speaking, a self-similar set $K$ of finite type is that there are finitely many kinds of neighbors of $K$. We will show that

**Theorem 1.6.** Let $K$ be a self-similar set satisfying the finite type condition. Let $H = H(K)$ be the Hata graph. Then every edge $e \in H$ admits a bifurcation pair.

Combining Theorem 1.5 and Theorem 1.6, we obtain the following result immediately.

**Theorem 1.7.** Let $\{S_j\}_{j=1}^N$ be an IFS satisfying the finite type condition, then its invariant set $K$ has a finite skeleton.

There do exist self-similar sets without skeletons. We will prove the following example in Section 3.

**Example 1.** Let $D = \{0, 1, 2, i, 1+i, 2+i, \varepsilon + 2i, \varepsilon + 1 + 2i, \varepsilon + 2 + 2i\}$ where $\varepsilon$ is a small irrational number, for example, $\varepsilon = \sqrt{2}/4$. Kenyon [6] shows that the self-similar set $K$ generated by the IFS $\{(z+d)/3\}_{d \in D}$ is a reptile. We will show that $K$ has no skeleton.
This paper is organized as follows. We will give some notations and show that the skeleton of \( S \) is still the skeleton of iterations of \( S \) in Section 2. Section 3 will be devoted to prove Theorem 1.5. The finite type condition will be discussed in Section 4. The last Section is devoted to prove Theorem 1.6 and an examples are given to explain our algorithm.

2. Preliminaries

2.1. Symbolic space and the projection. Let \( S = \{S_j\}_{j=1}^{N} \) be an IFS and \( K \) be the invariant set. Denote \( \Sigma = \{1, 2, \ldots, N\} \). We define \( \Sigma^N = \{1, 2 \cdots N\}^N \) and call it the symbolic space with \( N \)-symbols.

For \( m \geq 1 \), define \( \Sigma^m = \{1, 2 \cdots N\}^m \) (we set \( \Sigma^0 = \{\emptyset\} \) for convention), and set \( \Sigma^* = \bigcup_{m=0}^{\infty} \Sigma^m \). For \( \omega = \omega_1 \omega_2 \cdots \omega_m \in \Sigma^N \), set \( \omega|_n = \omega_1 \omega_2 \cdots \omega_n \) be the prefix of \( \omega \) of length \( n \). For \( i_1 i_2 \cdots i_m \in \Sigma^* \), we call

\[
[i_1 i_2 \cdots i_m] = \{\omega \in \Sigma^N; \omega|_m = i_1 i_2 \cdots i_m\}
\]

a cylinder of the symbolic space. We denote \( S_I = S_{i_1} \circ S_{i_2} \circ \cdots \circ S_{i_m} \) if \( I = i_1 i_2 \cdots i_m \in \Sigma^m \).

Define \( \pi : \Sigma^N \to K \) as

\[
\{\pi(\omega)\} = \bigcap_{1}^{\infty} S_{\omega|_m}(K).
\]

Then \( \pi \) is a surjection. We call \( \omega \) a coding of \( x \) if \( \pi(\omega) = x \).

The shift map \( \sigma : \Sigma^N \to \Sigma^N \) is defined as \( \sigma(\omega_1 \omega_2 \omega_3 \cdots) = \omega_2 \omega_3 \cdots \).

For \( k \in \{1, 2, \ldots, N\} \), we define \( \sigma_k : \Sigma^N \to \Sigma^N \) as

\[
\sigma_k(\omega_1 \omega_2 \omega_3 \cdots) = k \omega_1 \omega_2 \omega_3 \cdots.
\]

Then it holds that \( \pi \circ \sigma_i = S_i \circ \pi \) (See [5] [7]).
2.2. Skeleton of iteration of IFS. We define the $n$-th iteration of $S$ to be the IFS
\[ S^n = \{ S_I; I \in \Sigma^n \}. \]

It is well-known that the invariant set $K$ of the IFS $S$ is also the invariant set of $S^n$ (see Falconer [3]). Similarly, we have

**Proposition 2.1.** If $A$ is a skeleton of $S$, then $A$ is also a skeleton of $S^n$.

**Proof.** Denote the Hata graph of $A$ related to $S^n$ by $H(S^n, A)$. We prove the proposition by induction on $n$. Suppose the proposition holds for $n - 1$. That is, $A \subset \bigcup_{I \in \Sigma^{n-1}} S_I(A)$ and $H(S^{n-1}, A)$ is connected.

First, we observe that
\[ A \subset \bigcup_{i=1}^{N} S_i(A) \subset \bigcup_{i=1}^{N} S_i \left( \bigcup_{I \in \Sigma^{n-1}} S_I(A) \right) = \bigcup_{I \in \Sigma^n} S_I(A). \]

Next, we show that $H(S^n, A)$ is connected. Take two words $I = i_1 \ldots i_n$, $J = j_1 \ldots j_n \in \Sigma^n$. We show that there is a trail in $H(S^n, A)$ connecting $S_I$ and $S_J$. Denote $I' = i_2 \ldots i_n$, $J' = j_2 \ldots j_n$.

**Claim 1.** The restriction of $H(S^n, A)$ on $\{i\} \times \Sigma^{n-1}$ is connected.

By induction hypothesis, there exist $L_1, \ldots, L_k \in \Sigma^{n-1}$ such that $S_{I'}, S_{L_1}, \ldots, S_{L_k}, S_{J'}$ is a trail in $H(S^{n-1}, A)$, precisely, there is an edge between every two consecutive vertices. Hence
\[ S_{i_1I'}, S_{i_1L_1}, \ldots, S_{i_1L_k}, S_{i_1J'} \]

is a walk from $S_I$ to $S_{i_1J'}$ in $H(S^n, A)$.

**Claim 2.** $S_I$ and $S_J$ are connected in $H(S^n, A)$ if $\{S_{i_1}, S_{j_1}\}$ is an edge in $H(S, A)$.

The assumption implies that $S_{i_1}(A) \cap S_{j_1}(A) \neq \emptyset$, so
\[ \left( \bigcup_{I'' \in \Sigma^{n-1}} S_{i_1I''}(A) \right) \cap \left( \bigcup_{J'' \in \Sigma^{n-1}} S_{j_1J''}(A) \right) \neq \emptyset, \]
so $S_{i_1I''}(A) \cap S_{j_1J''}(A) \neq \emptyset$ for some $I'', J''$ with length $n - 1$; namely, there is an edge connecting $S_{i_1I''}$ and $S_{j_1J''}$. Therefore, the walk from $S_I$ to $S_{i_1I''}$ (by Claim 1), the edge above, and the walk from $S_{j_1J''}$ to $S_J$ (again by Claim 1) form a walk from $S_I$ to $S_J$.

Now we deal with the general case. Since $H(S, A)$ is connected, there exists a walk connected the vertices $S_{i_1}$ and $S_{j_1}$. Using Claim 2 repeatedly, we obtain that $H(S^n, A)$ is connected. \qed
Proposition 2.2. If a self-similar set $K$ has a skeleton $A = \{a\}$, then $K = \{a\}$ is a singleton.

Proof. Since the connecting graph $H(A)$ is connected, $S_i(a) = S_j(a)$ if there is an edge between the vertices $S_i$ and $S_j$ in $H(A)$, so $S_1(a) = \cdots = S_N(a)$. This together with $A \subseteq \bigcup_{j=1}^{N} S_j(A)$ imply that $K = \{a\}$. \qed

3. Proof of Theorem 1.5

Let $S = \{S_i\}_{i=1}^{N}$ be an IFS and $K$ be the invariant set. In this section, we will characterize the skeleton of $S$ in detail. Recall that a Hata graph $H(A)$ of a subset $A$ of $K$ is defined as follows.

- The vertex set is $V = \{S_1, S_2, \ldots, S_N\}$;
- There is an edge between two vertices $S_i$ and $S_j$ if and only if $S_i(A) \cap S_j(A) \neq \emptyset$.

Recall that a finite subset $A$ of $K$ is called a skeleton if $A \subseteq \bigcup_{j=1}^{N} S_j(A)$ and the Hata graph $H(A)$ is connected.

Let $H = H(K)$ be the Hata graph of $K$. Recall that for an edge $e = \{S_i, S_j\}$ of $H$, a pair $\omega, \gamma \in \{1, \ldots, N\}^\infty$ is called bifurcation pair of $e$, if both sequences are eventually periodic, and $\omega_1 = i, \gamma_1 = j, \pi(\omega) = \pi(\gamma)$. And $R$ is the spanning subgraph of $H$.

We will show the following lemma at first, actually it leads to the sufficient part of Theorem 1.5.

Lemma 3.1. Let $A$ be a skeleton of $K$, then points in $A$ have eventually periodic codings. Moreover, if $S_i(A) \cap S_j(A) \neq \emptyset$, then their common points have two eventually codings.

Proof. Denote $\#A = m$. \forall \ x \in A, \exists \ i_1 \in \Sigma$ such that $x \in S_{i_1}(A)$ since $A \subseteq \bigcup_{i=1}^{N} S_i(A)$. Then there exists $y_1 \in A$ such that $x = S_{i_1}(y_1)$. Thus there exist $i_2 \in \Sigma$ and $y_2 \in A$ such that $y_1 = S_{i_2}(y_2)$. By induction, there exist a sequence $\{i_k\}_{k \geq 1}$ in $\Sigma^N$ and $\{y_k\}_{k \geq 1}$ in $A$ such that $y_k-1 = S_{i_k}(y_k)$. Since $A$ is finite, there must exist $1 \leq \ell \leq m$ such that $y_{m+1} = y_\ell$. So we obtain that $y_\ell$ is the fixed point of $S_{i_{\ell+1}} \circ S_{i_{\ell+2}} \circ \cdots \circ S_{i_{m+1}}$ since $y_\ell = S_{i_{\ell+1}} \circ S_{i_{\ell+2}} \circ \cdots \circ S_{i_{m+1}}(y_{m+1})$. Denote $\omega = (i_{\ell+1}i_{\ell+2}i_{\ell+3}i_{\ell+4} \cdots i_{m+1})^\infty$, then $y_\ell = \pi(\omega)$. Hence $x = S_{i_1} \circ S_{i_2} \circ \cdots \circ S_{i_\ell}(y_\ell) = \pi(i_1i_2i_3 \cdots i_\ell)$. Thus $x$ has eventually periodic coding.

For $z \in S_i(A) \cap S_j(A)$, then there exist $x, y \in A$ such that $z = S_i(x) = S_j(y)$. Since $x, y \in A$, there exist eventually periodic $\omega, \gamma \in \Sigma^\infty$ such that $x = \pi(\omega), y = \pi(\gamma)$. Then $z = S_i(\pi(\omega)) = S_j(\pi(\gamma))$. Hence $z = \pi(i_\omega) = \pi(j_\gamma)$. So $z$ has two eventually periodic codings. \qed
and so that $x$ are eventually periodic, we have that

$$
\sum (\sigma^k(\omega_e); \ k \in \mathbb{N}) \cup (\sigma^k(\gamma_e); \ k \in \mathbb{N}),
$$

the union of orbits of $\omega$ and $\gamma$ under $\sigma$, is a finite set. We claim that

$$
B = \bigcup_{e \in R} \pi(B_e).
$$

is a skeleton of $K$.

First, we show that the Hata graph $H(B)$ contains $R$ as a subgraph, and hence $H(B)$ is connected since $R$ is spanning. For $e = \{S_i, S_j\} \in R$, denote $x_e = \pi(\omega) = \pi(\gamma)$. Then

$$
x_e = S_i \circ \pi(\sigma(\omega)) = S_j \circ \pi(\sigma(\gamma)) \in S_i(B) \cap S_j(B),
$$

which implies that $e \in H(B)$. Therefore, $R$ is a subgraph of $H(B)$.

Take $x \in B$. Then $x$ can be written as $x = \pi(\sigma^k(\omega_e))$ for some $e \in R$. Let $\ell$ be the $(k + 1)$-th entry of the sequence $\omega_e$, then

$$
x = \pi(\ell \sigma^{k+1}(\omega_e)) = S_\ell \circ \pi(\sigma^{k+1}(\omega_e)) \in S_\ell(B),
$$

which verifies that $B \subset \bigcup_{j=1}^N S_j(B)$.

On the other hand, suppose $A$ is a skeleton of $K$. Then $R = H(A)$ is a spanning subgraph of $H(K)$. Then by Lemma 3.1 every edge of $R$ admits a bifurcation pair. □

**Proof of Example 1.** Suppose that $K$ has a skeleton $A$. Then there exist $i \in \{1, 2, 3\}$, $j \in \{4, 5, 6\}$ such that $S_i(A) \cap S_j(A) \neq \emptyset$ (otherwise, the Hata graph $H(A)$ is not connected). Without loss of generality, let us assume that $i = 1$ and $j = 4$. Take a point $(x, y)^t$ from the intersection. By Theorem 1.5 $(x, y)^t$ has two eventually codings with initial letter 1 and 4 respectively. It follows that

$$
\begin{pmatrix}
x \\
y
\end{pmatrix} = \sum_{k=1}^{\infty} 3^{-k} \begin{pmatrix}
x_k \\
y_k
\end{pmatrix} = \sum_{k=1}^{\infty} 3^{-k} \begin{pmatrix}
x_k \\
y_k
\end{pmatrix},
$$

where $(x_1, y_1)^t = (0, 0)^t$, $(\tilde{x}_1, \tilde{y}_1)^t = (0, 1)^t$ and $(x_4, y_4)^t, (\tilde{x}_4, \tilde{y}_4)^t \in D$. From $\sum 3^{-k}y_k = \sum 3^{-k}\tilde{y}_k$ we deduce that $y_k = 2$, $\tilde{y}_k = 0$ for $k \geq 2$, since they are two ternary expansions starting with 0 and 1, respectively. It follows that

$$
(x_k, y_k)^t \in \{d_7, d_8, d_9\} \text{ and } (\tilde{x}_k, \tilde{y}_k)^t \in \{d_1, d_2, d_3\}, \quad k \geq 2,
$$

and so that $x_k \in \{\varepsilon, \varepsilon + 1, \varepsilon + 2\}$ and $\tilde{x}_k \in \{0, 1, 2\}$ for $k \geq 2$. Since $(x_k)_{k \geq 1}$ and $(\tilde{x}_k)_{k \geq 1}$ are eventually periodic, we have that $\sum 3^{-k}x_k$ is irrational and $\sum 3^{-k}\tilde{x}_k$ is rational, which contradicts that $\sum 3^{-k}x_k = \sum 3^{-k}\tilde{x}_k$. 
4. Finite type condition

In this section, we will give the definition finite type condition in detail and state some basic facts. Recall that $\Sigma = \{1, \ldots, N\}$.

Let $\{S_i\}_{i \in \Sigma}$ be an IFS. Let $K$ be the invariant set; without loss of generality, we assume that $0 \in K$. Let us denote $r_I$ the contraction ratio of the map $S_I$, and set

$$r_* = \min\{|r_j|; j = 1, \ldots, N\}, \quad r^* = \max\{|r_j|; j \in \Sigma\}.$$ 

A map $f(z) = S^{-1}_J \circ S_I(z)$ is called a neighbor map if

$$r_* < r^{-1}_J r_I \leq 1/r_*.$$ 

A neighbor map describes the relative position of two cylinders of about the same size. Let

$$\mathcal{N}_0 = \{S^{-1}_J S_I; I, J \in \{1, \ldots, N\}^*, r_* < r^{-1}_J r_I \leq 1/r_*\},$$

be the set of all neighbor maps. Let $\epsilon$ be the empty word, and denote $S_\epsilon = \text{id}$ for convention. For $f, g \in \mathcal{N}_0$, we define there is an edge from $f$ to $g$, if there exist $i$ such that

$$f \circ S_i = g \text{ or } S^{-1}_i \circ f = g;$$

in the first case, we name the edge by $(\epsilon, i)$, or $(f, \epsilon, i, g)$, where in the second case, we name the edge by $(i, \epsilon)$, or $(f, i, \epsilon, g)$. Hence we obtain a directed graph with vertex set $\mathcal{N}_0$, which we call the complete neighbor graph, and denote by $\Delta_0$.

We call $S^{-1}_J \circ S_I$ the basic neighbor maps, where $i \neq j \in \Sigma$.

Lemma 4.1. For any $f \in \mathcal{N}_0$, there is a path starting at some basic neighbor map and terminating at $f$.

Proof. Suppose $f = S^{-1}_J \circ S_I \in \mathcal{N}_0$, we prove the lemma by induction on $|I| + |J|$. The lemma is clearly true if $|I| + |J| = 2$. Let $r$ be the contraction ratio of $f$. If $|r| > 1$, denote $J^*$ obtained by deleting the last symbol of $J$, then $h = S^{-1}_{J^*} \circ S_I$ is a neighbor map, and there is an edge from $h$ to $f$. By induction, there is a path starting at some basic neighbor map and terminating at $h$, and it can be extended to $f$. If $|r| \leq 1$, let $h = S^{-1}_J \circ S_I$, and the above discussion still holds. \qed 

Especially, if $r_j$ have the same values, the neighbor graph can be simplified. First, the set $\mathcal{N}_0$ reduces to

$$\mathcal{N}_0 = \bigcup_{n=1}^{\infty} \{S^{-1}_J S_I; I, J \in \Sigma^n, i_1 \neq j_1\},$$
and there is an edge from $f$ to $g$, if there exist $i$ such that

$$S_j^{-1} \circ f \circ S_i = g;$$

in this case, we name the edge by $(j, i)$, or $(f, j, i, g)$. (Now, there may be several edges from $f$ to $g$.)

**Definition 4.2.** We say $\{S_j\}_{j=1}^N$ satisfies the **finite type condition** if the set

$$\mathcal{N} = \{f \in \mathcal{N}_0; \ f(K) \cap K \neq \emptyset\}$$

is a finite set. We say $\{S_j\}_{j=1}^N$ satisfies the **strong finite type condition** if $\mathcal{N}_T$ are finite for all $T > 0$, where

$$\mathcal{N}_T = \{f \in \mathcal{N}_0; \ |f(0)| < T\} < \infty.$$ 

Let

$$r_K = \max\{|S_i(0)|; \ i = 1, \ldots, N\} \cdot \frac{1}{1 - r^*},$$

then $\bigcup_{i=1}^N S_i(B(0, r_K)) \subset B(0, r_K)$, and hence $K$ is contained in the ball with center 0 and radius $r_K$. Let

$$L = (1 + r^{-1}_*)r_K,$$

then $\mathcal{N} \subset \mathcal{N}_L$. (Indeed, if $|f(0)| > L$, then $B(0, r_K)$ and $B(f(0), r_K/c^*)$ are disjoint and hence $K$ and $f(K)$ are disjoint.) We denote by $\Delta_L$ the restriction of $\Delta_0$ on $\mathcal{N}_L$.

**Lemma 4.3.** $f \in \mathcal{N}$ if and only if $f \in \mathcal{N}_L$ and there is an eventually periodic path in $\Delta_L$ emanating from $f$.

**Proof.** Take any $f \in \mathcal{N}$. Notice that at least one follower of $f$, denoted by $f_1$, is still in $\mathcal{N}$. Repeating this argument, we obtain an infinite path in $\Delta$. This path is an eventually loop since $\mathcal{N}$ is a finite set.

On the other hand, suppose $(j_k, i_k)_{k=1}^\infty$ is an eventually loop emanating from $f$. Let $x = \pi(I(i_k)_{k=1}^\infty)$ and $y = \pi(J(j_k)_{k=1}^\infty)$. The neighbor map $S_{j_n\ldots j_1}^{-1} \circ S_{i_1\ldots i_n}$ are nodes of the eventually loop, and hence $S_{j_n\ldots j_1}(K) \cap S_{i_1\ldots i_n}(K) \neq \emptyset$, for all $n \geq 1$. Therefore, the distance between $x$ and $y$ can be arbitrarily small, namely, $x = y$. It follows that $S_f(K) \cap S_f(K) \neq \emptyset$, namely, $f(K) \cap K \neq \emptyset$. \qed

If the IFS satisfies the strong finite type condition, then the above lemma provides an algorithm to determine the set $\mathcal{N}$.

We shall call the restriction of $\Delta_0$ on $\mathcal{N}$ the **essential neighbor graph** or **neighbor graph** in short, and denote it by $\Delta$. By Lemma 4.3, for any $f \in \mathcal{N}$, there is an eventually periodic path emanating from $f$ in $\Delta$. 

Example 2. IFS with uniform contraction ratios. Let us consider the IFS of the form

\[ S_i(z) = rM(z + d_i), \quad i = 1, \ldots, N, \]

where \( M \) is an orthogonal matrix. Let us denote \( D = \{d_1, \ldots, d_N\} \), and define

\[ D^* = \bigcup_{n=1}^{\infty} \left\{ \sum_{k=0}^{n-1} (rM)^{-k} x_k; \ x_k \in D - D \right\}. \]

By induction, one can show that

\[ N_0 = \{z \mapsto z + b; \ b \in D^*\}. \]

In particular, if \( D^* \) is uniformly discrete, then the set \( \{|b| < T; \ b \in D^*\} \) is a finite set for any \( T > 0 \), and hence the IFS satisfies the strong finite type condition.

Let us consider the ter-dragon. The IFS is of the form (4.1) with \( rM = \exp(i\pi/6)/\sqrt{3} \), and

\[ D = \exp(-i\pi/6)\sqrt{3}\{1, \omega, \omega^2\}. \]

Since \((rM)^{-1} = 1 - \omega\), we have that \( b = \sum_{k=0}^{n-1} (rM)^{-k} x_k \) is an integral combination of \( 1, \omega \), i.e., \( D^* \) is a subset of the lattice \( \mathbb{Z} + \mathbb{Z}\omega \) and hence it is uniformly discrete. By the same argument, the four star tile also satisfies the strong finite type condition.

5. Algorithm of skeleton

In this section, we formulate an algorithm to obtain skeletons for self-similar set of finite type. We will prove Theorem 1.6 at first.

Proof of Theorem 1.6 Suppose \( e = (S_i, S_j) \in H(K) \), then \( S_j(K) \cap S_i(K) \neq \emptyset \). Thus \( f = S_j^{-1} \circ S_i \in \mathcal{N} \). So by Lemma 4.3 there is an eventually cycle

\( (j_1, i_1), (j_2, i_2), \ldots, (j_n, i_n), \ldots \)

in \( \Delta \) emanating from \( f \). Let \( \omega = i(i_k)_{k=1}^{\infty} \) and \( \gamma = j(j_k)_{k=1}^{\infty} \). Clearly, \( \pi(\omega) = \pi(\gamma) \), which implies that \( \{\omega, \gamma\} \) is a bifurcation pair of \( e \). \( \square \)

5.1. Algorithm. For an IFS \( S \) satisfying the finite type condition, we can construct skeletons by the following algorithm:

(i) Compute the neighbor map set \( \mathcal{N} \) and the neighbor graph \( \Delta \);
(ii) Compute the Hata graph \( H \), and choose a spanning subgraph \( R \);
(iii) For each \( e \in R \), find a bifurcation pairs of \( e \) by Theorem 1.6;
(iv) Construct a skeleton according to (3.1).
5.2. Examples.

**Example 3. Skeleton of Terdragon.** Recall the IFS of Terdragon is

\[ S_1(z) = \lambda z + 1, \quad S_2(z) = \lambda z + \omega, \quad S_3(z) = \lambda z + \omega^2, \]

where \( \lambda = \exp(\pi i/6)/\sqrt{3} \) and \( \omega = \exp(2\pi i/3) \).

(i) The neighbor map set is

\[ \mathcal{N} = \{ f_1, f_2, \ldots, f_6 \} = \{ S_3^{-1} \circ S_2^{-1} \circ S_1^{-1} \circ S_2, S_3^{-1} \circ S_1^{-1} \circ S_1, S_3^{-1} \circ S_2 \}. \]

The neighbor graph \( \Delta \) (see Figure 4) is

\[
\begin{align*}
&f_1 \xrightarrow{(3,1)} f_1, \quad f_1 \xrightarrow{(3,2)} f_2, \quad f_2 \xrightarrow{(2,1)} f_2, \quad f_2 \xrightarrow{(3,1)} f_3; \\
&f_3 \xrightarrow{(2,3)} f_3, \quad f_3 \xrightarrow{(2,1)} f_6, \quad f_6 \xrightarrow{(1,3)} f_6, \quad f_6 \xrightarrow{(2,3)} f_5; \\
&f_5 \xrightarrow{(1,2)} f_5, \quad f_5 \xrightarrow{(1,3)} f_4, \quad f_4 \xrightarrow{(3,2)} f_4, \quad f_4 \xrightarrow{(1,2)} f_1.
\end{align*}
\]

![Figure 4](Image)

Figure 4. (a): Neighbor graph of Terdragon. (b): Hata graph \( H \) of Terdragon.

(ii) We choose \( R = H \).

1. The first skeleton. Notice that \( \pi(12^{\infty}) = \pi(23^{\infty}) = \pi(31^{\infty}) = 0 \). For the three edges of \( H \), we choose the bifurcation pairs \( \{12^{\infty}, 23^{\infty}\}, \{23^{\infty}, 31^{\infty}\}, \{31^{\infty}, 12^{\infty}\} \). The resulted skeleton is

\[ \{a_1, a_2, a_3\} = \pi\{1^{\infty}, 2^{\infty}, 3^{\infty}\} = \{-\omega^2, -1, -\omega\}/\lambda, \]

which is the skeleton used in [2], and it is used widely in literature.

2. The second skeleton. For \( S_2, S_3 \in H \), we choose

\[ ((3,2)(3,1)(2,1)(2,3)(1,3)(1,2))^{\infty} \]
Figure 5. Another skeleton of Terdragon as a cycle in $\Delta$ passing all the vertices. Then the bifurcation pair associated to $\{S_2, S_3\}$ is $\{2(332211)\infty, 3(211332)\infty\}$. Similarly, the other two bifurcation pairs associated to $\{S_1, S_2\}$ and $\{S_3, S_1\}$ are $\{1(221133)\infty, 2(133221)\infty\}$ and $\{3(113322)\infty, 1(322113)\infty\}$, respectively. Denote $\omega = (332211)^\infty$, then the resulted skeleton is (see Figure 5)

$$\{a_1, a_2, a_3, a_4, a_5, a_6\} = \pi\{\sigma^k(\omega); k = 0, \ldots, 5\}$$

$$= \{\frac{6}{7} - \frac{4\sqrt{3}}{7}i, \frac{12}{7} - \frac{\sqrt{3}}{7}i, \frac{3}{7} + \frac{5\sqrt{3}}{7}i, -\frac{9}{14} + \frac{13\sqrt{3}}{14}i, -\frac{9}{7} - \frac{\sqrt{3}}{7}i, -\frac{15}{14} - \frac{11\sqrt{3}}{14}i\}.$$

Figure 6. Neighbor graph of Four tile star

Example 4. Skeleton of the four-tile star. The IFS is

$$S_1(x) = -\frac{1}{2}x, \quad S_2(x) = -\frac{1}{2}x - i, \quad S_3(x) = -\frac{1}{2}x + \exp\left(\frac{5\pi i}{6}\right), \quad S_4(x) = -\frac{1}{2}x + \exp\left(\frac{\pi i}{6}\right).$$
The neighbor graph is given by Figure 6, where the basic neighbor maps are marked yellow:

\[ f_1 = f_4^{-1} = S_1^{-1} \circ S_2, \quad f_2 = f_5^{-1} = S_1^{-1} \circ S_3, \quad f_3 = f_6^{-1} = S_1^{-1} \circ S_4. \]

(ii) The Hata graph \( H \) is Figure 7(a), and the spanning subgraph we choose is

\[ R = \{ \{S_1, S_2\}, \{S_1, S_3\}, \{S_1, S_4\}\}. \]

(iii) The bifurcation pairs of the edges of \( \{S_1, S_2\}, \{S_1, S_3\} \) and \( \{S_1, S_4\} \) are chosen to be \( \{1(12)\infty, 2(21)\infty\}, \{1(13)\infty, 3(31)\infty\}, \{1(14)\infty, 4(41)\infty\} \), respectively.

(iv) The resulted skeleton is

\[ \{a_1, a_2, a_3, a_4, a_5, a_6\} = \pi \{ (31)\infty, (14)\infty, (21)\infty, (13)\infty, (41)\infty, (12)\infty \}
\[ \quad = \{-2\sqrt{3}+2i, 2i, 2\sqrt{3}+2i, \sqrt{3}-1, -4i, -\sqrt{3}-1\}. \]

Figure 7. (a): Four-tile star. (b): Hata graph \( H \) and a spanning subgraph \( R \).

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