On the pseudo-nullity of fine Selmer groups

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Abstract

In this paper, we will study the pseudo-nullity of the fine Selmer group and its related question. We investigate certain situations, where one can deduce the pseudo-nullity of the dual fine Selmer groups of a general Galois module over a admissible $p$-adic Lie extension $F_\infty$ from the knowledge that pseudo-nullity of the Galois group of the maximal abelian unramified pro-$p$ extension of $F_\infty$ at which every prime of $F_\infty$ above $p$ splits completely. In particular, this gives us a way to construct examples of the pseudo-nullity of the dual fine Selmer group of a Galois module that is unramified outside $p$. We will illustrate our results with many examples.

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1 Introduction

Throughout the paper, $p$ will denote a fixed odd prime. Let $F$ be a number field. Let $S$ be a finite set of primes of $F$ which contains the primes above $p$ and the infinite primes. We then denote by $F_S$ the maximal algebraic extension of $F$ which is unramified outside $S$. For any algebraic (possibly infinite) extension $\mathcal{L}$ of $F$ contained in $F_S$, we write $G_S(\mathcal{L}) = \text{Gal}(F_S/\mathcal{L})$. Let $R$ be a commutative complete regular local ring with maximal ideal $m$ and residue field $k$, where $k$ is finite of characteristic $p$. We denote by $T$ a finitely generated free $R$-module with a continuous $R$-linear $G_S(F)$-action. Here $T$ is endowed with the canonical topology arising from its filtration by powers of $m$.

Let $v$ be a prime in $S$. For every finite extension $L$ of $F$ contained in $F_S$, we define

$$K_v^2(T/L) = \bigoplus_{w|v} H^2(L_w, T),$$

where $w$ runs over the (finite) set of primes of $L$ above $v$. If $\mathcal{L}$ is an infinite extension of $F$ contained in $F_S$, we define

$$\mathcal{K}_v^2(T/\mathcal{L}) = \lim_{\rightarrow} K_v^2(T/L),$$

where
where the inverse limit is taken over all finite extensions $L$ of $F$ contained in $\mathcal{L}$. For any algebraic (possibly infinite) extension $\mathcal{L}$ of $F$ contained in $F_S$, the *dual fine Selmer group* of $T$ over $\mathcal{L}$ (with respect to $S$) is defined to be

$$Y_S(T/\mathcal{L}) = \ker\left( H^2_S(\mathcal{L}/F, T) \rightarrow \bigoplus_{v \in S} K^2_v(T/\mathcal{L}) \right),$$

where we write $H^2_S(\mathcal{L}/F, T) = \lim \leftarrow \mathcal{L} H^2(G_S(\mathcal{L}), T)$.

To facilitate further discussion, we now recall the following conjecture which was studied in [CS, JhS, Lim].

**Conjecture A**: For any number field $F$, $Y_S(T/F^{\text{cyc}})$ is a finitely generated $R$-module, where $F^{\text{cyc}}$ is the cyclotomic $\mathbb{Z}_p$-extension.

We say that $F^\infty$ is an $S$-admissible $p$-adic Lie extension of $F$ if (i) $\text{Gal}(F^\infty/F)$ is a compact pro-$p$ $p$-adic Lie group, (ii) $F^\infty$ contains $F^{\text{cyc}}$ and (iii) $F^\infty$ is contained in $F_S$. We say that $F^\infty$ is a strongly $S$-admissible $p$-adic Lie extension of $F$ if $\text{Gal}(F^\infty/F)$ is a compact pro-$p$ $p$-adic Lie group without $p$-torsion. Write $G = \text{Gal}(F^\infty/F)$ and $H = \text{Gal}(F^\infty/F^{\text{cyc}})$. We recall that a finitely generated $R[G]$-module $M$ is said to be torsion (resp., pseudo-null) if $\text{Ext}^i_{R[G]}(M, R[G]) = 0$ for $i = 0$ (resp., $i = 0, 1$). If the $R[G]$-module $M$ in question is finitely generated over $R[H]$, then it follows from a well-known result of Venjakob (cf. [V]) that it is a pseudo-null $R[H]$-module if and only if it is a torsion $R[H]$-module.

Now under the assumption that Conjecture A is valid, it follows from the argument of [CS, Lemma 3.2] that $Y_S(T/F^\infty)$ is finitely generated over $R[H]$. We can now state the following question, noting the above result of Venjakob.

**Question B**: Let $F^\infty$ be an $S$-admissible $p$-adic Lie extension of $F$ of dimension $> 1$, and suppose that $Y_S(T/F^{\text{cyc}})$ is a finitely generated $R$-module. Is $Y_S(T/F^\infty)$ a pseudo-null $R[G]$-module, or equivalently a torsion $R[H]$-module?

When $T$ is the Tate module of all the $p$-power roots of unity, the dual fine Selmer group is precisely $\text{Gal}(K(F^\infty)/F^\infty)$, where $K(F^\infty)$ is the maximal abelian unramified pro-$p$ extension of $F^\infty$ at which every prime of $F^\infty$ above $p$ splits completely. In this context, Hachimori and Sharifi [HSh] has constructed a class of admissible $p$-adic Lie extensions $F^\infty$ of $F$ of dimension $> 1$ such that $\text{Gal}(K(F^\infty)/F^\infty)$ is not pseudo-null. Despite so, they have speculated that the pseudo-nullity statement should hold for admissible $p$-adic extensions that “come from algebraic geometry” (see [HSh, Question 1.3] for details, and see also [Sh2, Conjecture 7.6] for a related assertion and [Sh3] for positive results in this direction).

When $T$ is the Tate module of an elliptic curve, this is precisely [CS, Conjecture B]. In this context, the conjecture has also been studied and verified in certain numerical examples.
When $T$ is the $R(1)$-dual of the Galois representation attached to a normalized eigenform ordinary at $p$, this is [Jh] Conjecture B. In the case when $T$ is the $R(1)$-dual of the Galois representation coming from a $\Lambda$-adic form, this is [Jh] Conjecture 2. We should mention that in the formulation of their conjectures in [CS, Jh], they did not have any restriction of the admissible $p$-adic Lie extensions.

The following question is the theme of this paper.

**Question B’**: Let $F_\infty$ be an $S$-admissible $p$-adic Lie extension of a number field $F$ of dimension $> 1$ with the property that $G_S(F_\infty)$ acts trivially on $T/mT$. Suppose that $\text{Gal}(K(F_\infty)/F_\infty)$ is a finitely generated $R[[H]]$-module. Can one deduce that $Y_S(T/F_\infty)$ is a pseudo-null $R[[G]]$-module from the knowledge that $\text{Gal}(K(F_\infty)/F_\infty)$ is a pseudo-null $\mathbb{Z}_p[[G]]$-module?

We note that the assumptions in the above question imply that $Y_S(T/F_\infty)$ is a finitely generated $R[[H]]$-module (see [Lim, Theorem 3.5, Lemma 5.2]). In fact, one can think of Question B’ as whether the pseudo-null analogue of [Lim, Theorem 3.5] is valid. In this paper, we will be studying Question B’. Namely, we will give some partial (positive) results in support of the question B’ (see Theorems 2.3 and 2.4). We then apply Theorem 2.3 to prove the pseudo-nullity of the dual fine Selmer group of a certain class of Galois modules that are unramified outside $p$, and therefore, this gives a positive answer the question B for this particular class of Galois modules. We then give several examples of these pseudo-null dual fine Selmer groups which come from elliptic curves, modular forms and abelian varieties.

We end the introduction by mentioning that in view of the counterexamples of Hachimori-Sharifi [HSh] and the pseudo-nullity conjectures made in [CS, Jh], it seems unlikely that the converse statement of Question B’ will hold. It will be interesting to find a numerical counterexample to the converse statement which we are not able to at the moment.

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## 2 Question B’

We will retain the notion and notation of Section 1. We first record a lemma which gives a relationship between the dual fine Selmer group of $T$ over $F_\infty$ and the Galois group $\text{Gal}(K(F_\infty)/F_\infty)$ under the assumption that $G_S(F_\infty)$ acts trivially on $T$. We denote $M(n)$ to be the $n$th Tate twist of $M$. 


Lemma 2.1. Suppose that $F$ contains $\mu_p$. Let $F_\infty$ be an $S$-admissible $p$-adic Lie extension of $F$ such that $G_S(F_\infty)$ acts trivially on $T$. Then we have an isomorphism

$$Y_S(T/F_\infty) \cong \text{Gal}(K(F_\infty)/F_\infty) \otimes_{Z_p} T(-1).$$

Proof. For a pro-$p$ group or a discrete $p$-primary group $M$, its Pontryagin dual is denoted to be $M^\vee = \text{Hom}_{cts}(M, \mathbb{Q}_p/\mathbb{Z}_p)$. As the group $G_S(F_\infty)$ acts trivially on $T$ and $\mu_p \subseteq F_\infty$, it also acts trivially on $T^\vee(1)$. Therefore, one calculates that

$$Y_S(T/F_\infty)^\vee = \text{Hom}_{Z_p}(\text{Gal}(K(F_\infty)/F_\infty), T^\vee(1)).$$

On the other hand, one has the following adjunction isomorphism

$$\left(\text{Gal}(K(F_\infty)/F_\infty) \otimes_{Z_p} T(-1)\right)^\vee \cong \text{Hom}_{Z_p}(\text{Gal}(K(F_\infty)/F_\infty), T^\vee(1)).$$

Combining this with the above equality and taking dual, we obtain the required isomorphism. \hfill \Box

As we are mostly interested in $S$-admissible $p$-adic Lie extensions $F_\infty$ that satisfy the following condition, we give a name to it.

$$(\text{Dim}_S)$$: For each $v \in S$, the decomposition group of $G$ at $v$ has dimension $\geq 2$.

We record another result (cf. [Lim, Theorem 5.4], see also [Jh, Theorem 10]) which is the main ingredient in proving our main results. In view of the discussion in this paper, we will state a slightly strengthened version of [Lim, Theorem 5.4].

Proposition 2.2. Let $F_\infty$ be an $S$-admissible $p$-adic Lie extension of $F$ which satisfies $(\text{Dim}_S)$. Assume that $Y_S(T/F_\infty)$ is a finitely generated $R[H]$-module. Suppose that there exists a prime ideal $p$ of $R$ such that the ring $R/p$ is also regular local. If $Y_S((T/pT)/F_\infty)$ is a pseudo-null $R/p[G]$-module, then $Y_S(T/F_\infty)$ is a pseudo-null $R[G]$-module.

Proof. Let $F_0$ be a finite extension of $F$ contained in $F_\infty$ such that $G_0 := \text{Gal}(F_\infty/F^t)$ is a compact pro-$p$ $p$-adic Lie group without $p$-torsion. Clearly, the decomposition group of $G_0$ at $v$ for every $v \in S$ has dimension $\geq 2$. Write $H_0 = \text{Gal}(F_\infty/F_0^{\text{cyc}})$. Since $F_0$ is a finite $p$-extension of $F$, it follows that $H_0$ is a subgroup of $H$ with finite index. Thus, we also have that $Y_S(T/F_\infty)$ is a finitely generated $R[H_0]$-module. Now it is easy to verify that, for every $i \geq 0$, one has an isomorphism

$$\text{Ext}^i_{R[G]}(M, R[G]) \cong \text{Ext}^i_{R[G_0]}(M, R[G_0])$$

for any $R[G]$-module $M$. Hence, we are reduced to proving the theorem over $G_0$. But this is precisely the statement of [Lim, Theorem 5.4]. \hfill \Box

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We can now proceed to prove the following two propositions which serve as (positive) evidence in support of Question B'.

**Theorem 2.3.** Suppose that $F$ contains $\mu_p$. Let $F_\infty$ be an $S$-admissible $p$-adic Lie extension of $F$ which satisfies $\text{(Dim}_S\text{)}$. Assume that $G_S(F_\infty)$ acts trivially on $T/mT$, and assume that $\text{Gal}(K(F_\infty)/F_\infty)$ is a finitely generated $Z_p[H]$-module. If $\text{Gal}(K(F_\infty)/F_\infty)/p$ is a pseudo-null $Z/pZ[G]$-module, then $Y_S(T/F_\infty)$ is a pseudo-null $R[G]$-module.

Note that the condition that $\text{Gal}(K(F_\infty)/F_\infty)/p$ is a pseudo-null $Z/pZ[G]$-module implies that $\text{Gal}(K(F_\infty)/F_\infty)$ is a pseudo-null $Z_p[G]$-module by Theorem 2.2.

**Proof of Theorem 2.3.** Combining the assumptions in the Theorem and Lemma 2.1, we have an isomorphism

$$Y_S((T/mT)/F_\infty) \cong \text{Gal}(K(F_\infty)/F_\infty) \otimes_{Z_p} T/mT(-1).$$

Since $p$ kills $T/mT$, the latter is equal to

$$(\text{Gal}(K(F_\infty)/F_\infty)/p) \otimes_{Z_p} T/mT(-1).$$

It then follows from the pseudo-nullity hypothesis of the theorem that $Y_S((T/mT)/F_\infty)$ is a pseudo-null $R/m[\text{Gal}(F_\infty/Q(\mu_p))]$-module. We may now apply Proposition 2.2 to obtain the required conclusion.

Before stating the next result, we introduce another notation. Now if $p$ is a prime ideal of $R$, we then denote the residual representation of $\rho$ mod $p$ by

$$\rho_p : G_S(F) \rightarrow \text{Aut}_{R/p}(T/pT).$$

**Theorem 2.4.** Suppose that $F$ contains $\mu_p$. Let $F_\infty$ be an $S$-admissible $p$-adic Lie extension of $F$ which satisfies $\text{(Dim}_S\text{)}$. Assume that $\text{Gal}(K(F_\infty)/F_\infty)$ is a finitely generated $Z_p[H]$-module. Suppose that there exists a prime ideal $p$ of $R$ such that the ring $R/p$ is a regular local ring that is a free $Z_p$-algebra, and $\text{Gal}(F_S/F_\infty) \subseteq \ker \rho_p$. If $\text{Gal}(K(F_\infty)/F_\infty)$ is a pseudo-null $Z_p[G]$-module, then $Y_S(T/F_\infty)$ is a pseudo-null $R[G]$-module.

**Proof.** By Proposition 2.2 one is reduced to showing that $Y_S((T/pT)/F_\infty)$ is a pseudo-null $R/p[G]$-module. Since $F$ contains $\mu_p$, the field $F_\infty$ necessarily contains $\mu_{p\infty}$, and therefore, the group $G_S(F_{\infty})$ will act trivially on $(T/pT)^{\vee}(1)$. By Lemma 2.1 we have

$$Y_S((T/pT)/F_\infty) \cong \text{Gal}(K(F_\infty)/F_\infty) \otimes_{Z_p} T/pT(-1).$$

The conclusion is now immediate.  

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3 Question B

In this section, we will apply Theorem 2.3 to prove the pseudo-nullity of the dual fine Selmer group of certain classes of Galois modules that are unramified outside \( p \) over an \( p \)-adic admissible \( p \)-adic extension that is also unramified outside \( p \).

From now on, we will say an admissible \( p \)-adic Lie extension to mean a \( S \)-admissible \( p \)-adic extension, where the set \( S \) consists only of the primes above \( p \). We will also assume that \( T \) is unramified outside \( p \).

**Theorem 3.1.** Let \( p \) be a regular prime. Suppose that \( F_\infty \) is an admissible \( p \)-adic Lie extension of \( \mathbb{Q}(\mu_p) \) of dimension > 1. Consider the conditions

(1) \( G_S(F_\infty) \) acts trivially on \( T \),

(2) \( G_S(F_\infty) \) acts trivially on \( T/\mathfrak{m}T \),

(3) For the (unique) prime \( v \) of \( \mathbb{Q}(\mu_p) \) above \( p \), the decomposition group of \( \text{Gal}(F_\infty/\mathbb{Q}(\mu_p)) \) at \( v \) has dimension \( \geq 2 \).

If (1) holds, then \( Y_S(T/F_\infty) = 0 \). If (2) and (3) hold, then \( Y_S(T/F_\infty) \) is a pseudo-null \( R[c][G] \)-module.

The above theorem can be seen as a generalization of the results in [Oc]. The conclusion of the theorem under assumption (1) is probably well-known, but nevertheless, we have included it for completeness. In the case when \( p \) is irregular, we also have a statement in the spirit of the previous theorem, but under more restrictive conditions.

**Theorem 3.2.** Let \( p \) be an irregular prime < 1000. Suppose that \( F_\infty \) is an admissible \( p \)-adic Lie extension of \( \mathbb{Q}(\mu_p) \) (of dimension > 1) satisfying all of the following conditions

(1) \( F_\infty \) is unramified outside \( p \),

(2) \( G_S(F_\infty) \) acts trivially on \( T/\mathfrak{m}T \),

(3) \( F_\infty \) contains \( \mathbb{Q}(\mu_p, p^{-\infty}) \).

Then \( Y_S(T/F_\infty) \) is a pseudo-null \( R[c][G] \)-module.

In preparation for the proofs, we collect some preliminary results on the group \( \text{Gal}(K(F_\infty)/F_\infty) \), where \( K(F_\infty) \) is the maximal abelian unramified pro-\( p \) extension of \( F_\infty \) at which every prime of \( F_\infty \) above \( p \) splits completely. We begin stating the following well-known statement (for instance, see [Oc, Section 4]).
Proposition 3.3. Let $p$ be a regular prime. Let $F_{\infty}$ be an admissible $p$-adic Lie extension of $\mathbb{Q}(\mu_p)$. Then one has $\text{Gal}(K(F_{\infty})/F_{\infty}) = 0$.

The next result concerns the structure of group $\text{Gal}(K(F_{\infty})/F_{\infty})$ for irregular primes $< 1000$ which is an immediate consequence of [Sh2, Corollary 5.9] and [Sh3, Propositions 3.3 and 2.1a]

Proposition 3.4. Let $p$ be an irregular prime $< 1000$. Let $F_{\infty}$ be an admissible $p$-adic extension of $\mathbb{Q}(\mu_p)$ which contains $\mathbb{Q}(\mu_{p^\infty}, p^{-p}\infty)$. Then one has that $\text{Gal}(K(F_{\infty})/F_{\infty})$ is a finitely generated $\mathbb{Z}_p/\llbracket \text{Gal}(F_{\infty}/\mathbb{Q}(\mu_{p^\infty}, p^{-p}\infty)) \rrbracket$-module.

We can now prove our theorems.

Proof of Theorem 3.1. Suppose that condition (1) holds. Then by Lemma 2.1, we have an isomorphism $Y_S(T/F_{\infty}) \cong \text{Gal}(K(F_{\infty})/F_{\infty}) \otimes_{\mathbb{Z}_p} T(-1)$.

By Proposition 3.3, this in turn implies that $Y_S(T/F_{\infty}) = 0$.

Now suppose that conditions (2) and (3) hold. Applying the above argument to $T/mT$, we obtain $Y_S((T/mT)/F_{\infty}) = 0$, and in particular, a pseudo-null $R/m[\text{Gal}(F_{\infty}/\mathbb{Q}(\mu_p))]$-module. The required conclusion is then immediate from an application of Theorem 2.2.

Proof of Theorem 3.2. Combining the assumptions in the Theorem and Lemma 2.1, we have an isomorphism $Y_S((T/mT)/F_{\infty}) \cong \text{Gal}(K(F_{\infty})/F_{\infty}) \otimes_{\mathbb{Z}_p} T/mT(-1)$.

Since $p$ kills $T/mT$, the latter is equal to $(\text{Gal}(K(F_{\infty})/F_{\infty})/p) \otimes_{\mathbb{Z}_p} T/mT(-1)$.

Write $F' = \mathbb{Q}(\mu_{p^{\infty}}, p^{-p^{\infty}})$. It then follows from Proposition 3.4 that $Y_S((T/mT)/F_{\infty})$ is a finitely generated $R/m[\text{Gal}(F_{\infty}/F')]$-module and, in particular, a pseudo-null $R/m[\text{Gal}(F_{\infty}/\mathbb{Q}(\mu_p))]$-module. The required conclusion now follows from an application of Theorem 2.2.

4 Examples

We will now discuss some numerical examples of Theorems 3.1 and 3.2.

(a) (See also [Oc]) Let $E$ be an elliptic curve defined over $\mathbb{Q}$ that has good reduction away from $p$ and possesses a $\mathbb{Q}$-rational isogeny. A complete list (up to isogeny) of such elliptic curves can be found in [RT, Table 2]. By [RT, Proposition 5], we have that $\mathbb{Q}(E[p])$ is a finite $p$-extension of $\mathbb{Q}(\mu_p)$. Now let $T$ be the Tate module of the $p$-division points of such an elliptic
curve. We first consider the case that \( p \) is regular (i.e., \( p = 3, 7, 11, 19, 43, 163 \)). It then follows from Theorem \([3.1]\) that \( Y(T/F_\infty) \) is a pseudo-null (resp., trivial) \( \mathbb{Z}_p[\Gal(F_\infty/Q(\mu_p))] \)-module when \( F_\infty \) is an admissible \( p \)-adic Lie extension containing \( Q(E[p]) \) (resp., \( Q(E[p^{\infty}]) \)) whose decomposition group of \( \Gal(F_\infty/Q(\mu_p)) \) at the unique prime \( v \) of \( F \) has dimension \( \geq 2 \).

Now consider the case when \( p = 67 \) (an irregular prime). By an application of Theorem \([3.2]\) one has that \( Y(T/F_\infty) \) is a pseudo-null \( \mathbb{Z}_p[\Gal(F_\infty/Q(\mu_{67}))] \)-module whenever \( F_\infty \) is an admissible 67-adic Lie extension containing \( Q(E[67], \mu_{67\infty}, 67^{-67\infty}) \). Examples of such admissible 67-adic Lie extensions are

\[
Q(\mu_{67\infty}, 67^{-67\infty}, E[67]), \ Q(E[67\infty], 67^{-67\infty}).
\]

(b) The discussion in (a) can also be applied to the case of a modular form of higher weight with appropriate assumptions. Let \( N \) be a power of the prime \( p \) (we also allow \( N = 1 \)). Let \( f \in S_k(N) \) be a primitive cuspidal modular form of positive weight \( k \geq 2 \), level \( N \) and trivial character for the group \( \Gamma_0(N) \). Fix an embedding \( \mathbb{Q} \) in \( \mathbb{Q}_p \). By the results of Eichler, Shimura and Deligne, there is an associated Galois representation, which we denote by

\[
\rho_f : \Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \text{GL}_2(\mathbb{Q}_p)
\]

which is unramified outside \( p \) (since \( N \) is a power of \( p \)). After conjugation, one may therefore assume that \( \rho_f \) takes value in \( \text{GL}_2(\mathbb{Z}_p) \). Let \( T \) be the Galois module associated to \( \rho_f \) which is a free \( \mathbb{Z}_p \)-module of rank 2. Denote the residual representation to be

\[
\hat{\rho}_f : \Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \text{GL}_2(\mathbb{Z}/p\mathbb{Z}).
\]

Let \( K \) be the finite extension of \( Q \) corresponding to fixed field of ker \( \hat{\rho}_f \). If the reduced representation \( \hat{\rho}_f \) is reducible and contains \( \mu_p \) as a sub-representation, then the field \( Q(\mu_p) \) is contained in \( K \) and the Galois group \( \Gal(K/Q(\mu_p)) \) is a finite \( p \)-group. Therefore, one can obtain examples of psuedo-nullity of fine Selmer group of \( T \) as in (a).

(c) Assume that \( p \geq 5 \). Let \( n \geq 1 \). Let \( a, b, c \) be integers such that \( 1 \leq a, b, c < p^n \), \( a + b + c = p^n \) and at least one of (and hence at least two of) \( a, b, c \) is not divisible by \( p \). Let \( J = J_{a,b,c} \) be the Jacobian variety of the curve \( y^p = x^a(1 - x)^b \). Set \( T \) to be the Tate module of the \( p \)-division points of \( J \). Write \( R = \mathbb{Z}_p[\zeta_{p^n}] \) and \( \pi = 1 - \zeta_{p^n} \). Note that \( \pi \) is a generator of the maximal ideal of \( R \). It follows from \([11] \text{ Chap. II §§4, Chap. II Theorem 5B}] \) (see also \([Gr] \text{ Section 2}] \) for the case \( n = 1 \)) that \( T \) is a free \( R \)-module of rank one and that \( G_S(\mathbb{Q}(\mu_{p^n})) \) acts trivially on \( T/\pi T \). Therefore, it follows from Theorem \([3.1]\) that if the prime \( p \) is regular, then \( Y(T/F_\infty) \) is a pseudo-null \( \mathcal{O}[\Gal(F_\infty/Q(\mu_p))] \)-module when \( F_\infty \) is an admissible \( p \)-adic Lie extension of \( \mathbb{Q}(\mu_p) \) whose decomposition group of \( \Gal(F_\infty/Q(\mu_p)) \) at the unique prime \( v \) of
$F$ has dimension $\geq 2$. Similarly, if $p$ is an irregular prime $< 1000$, one applies Theorem 3.2 to conclude that $Y(T/F_{\infty})$ is a pseudo-null $O[\text{Gal}(F_{\infty}/Q(\mu_p))]$-module for an admissible $p$-adic Lie extension $F_{\infty}$ of $Q(\mu_p)$ which contains $Q(\mu_{p^\infty}, p^{-p^\infty})$.

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