GELFAND–TSETLIN BASES FOR REPRESENTATIONS OF FINITE W-ALGEBRAS AND SHIFTED YANGIANS

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Abstract. Remarkable subalgebras of the Yangian for $\mathfrak{gl}_n$ called the shifted Yangians were introduced in a recent work by Brundan and Kleshchev in relation to their study of finite $W$-algebras. In particular, in that work a classification of finite-dimensional irreducible representations of the shifted Yangians and the associated finite $W$-algebras was given. We construct a class of these representations in an explicit form via bases of Gelfand–Tsetlin type.

1. Introduction

A striking relationship between the Yangians and finite $W$-algebras was first discovered by Ragoucy and Sorba [14]; see also Briot and Ragoucy [1]. This relationship was developed in full generality by Brundan and Kleshchev [4]. The finite $W$-algebras associated to nilpotent orbits in the Lie algebra $\mathfrak{gl}_N$ turned out to be isomorphic to quotients of certain subalgebras of the Yangian $Y(\mathfrak{gl}_n)$. These subalgebras, called the shifted Yangians in [4], admit a description in terms of generators and relations. This leads to respective presentations of the finite $W$-algebras and thus provides new tools to study their structure and representations. The representation theory of the shifted Yangians and associated $W$-algebras was developed in a subsequent paper by Brundan and Kleshchev [5] where deep connections of the shifted Yangian representation theory were established. In particular, a classification of the finite-dimensional irreducible representations of the shifted Yangians and the finite $W$-algebras was given in terms of their highest weights. Moreover, in the case of the shifted Yangian associated to $\mathfrak{gl}_2$ all such representations were explicitly constructed.

Our aim in this paper is to construct in an explicit form a family of representations of the shifted Yangians and finite $W$-algebras via bases of Gelfand–Tsetlin type. Such bases for certain classes of representations of the Yangian $Y(\mathfrak{gl}_n)$ were constructed in different ways by Nazarov and Tarasov [12, 13] and Molev [10]. We mainly employ the approach of [12, 13] which turns out to be more suitable for the generalization to the case of the shifted Yangians. In more detail, following
2. Shifted Yangians and finite $W$-algebras

As in [4], given a pyramid $\pi = (p_1, \ldots, p_n)$ with $p_1 \leq \cdots \leq p_n$, introduce the corresponding shifted Yangian $Y_\pi(\mathfrak{gl}_n)$ as the associative algebra defined by generators

\begin{align*}
&d_i^{(r)}, \quad i = 1, \ldots, n, \quad r \geq 1, \\
&f_i^{(r)}, \quad i = 1, \ldots, n - 1, \quad r \geq 1, \\
&e_i^{(r)}, \quad i = 1, \ldots, n - 1, \quad r \geq p_{i+1} - p_i + 1,
\end{align*}

subject to the following relations:

\begin{align*}
[d_i^{(r)}, d_j^{(s)}] &= 0, \\
[e_i^{(r)}, f_j^{(s)}] &= -\delta_{ij} \sum_{t=0}^{r+s-1} d_i^{(t)} d_{i+1}^{(r+s-t-1)}, \\
[d_i^{(r)}, e_j^{(s)}] &= (\delta_{ij} - \delta_{i, j+1}) \sum_{t=0}^{r-1} d_i^{(t)} e_j^{(r+s-t-1)}, \\
[d_i^{(r)}, f_j^{(s)}] &= (\delta_{i, j+1} - \delta_{ij}) \sum_{t=0}^{r-1} f_j^{(r+s-t-1)} d_i^{(t)},
\end{align*}
natural subalgebra of \( Y(\mathfrak{g}l_{n}) \) providing a proof of the original result of Drinfeld \([6]\). Moreover, for an algebra \( Y(\pi) \) be defined for more general types of pyramids. However, in accordance to \([4]\), each of these algebras is isomorphic to \( Y(\pi) \). Note that the algebra \( Y(\pi) \) depends only on the differences \( p_{i+1} - p_{i} \). In the particular case of a rectangular pyramid \( \pi \) with \( p_{1} = \cdots = p_{n} \), the algebra \( Y_{\pi}(\mathfrak{g}l_{n}) \) is isomorphic to the Yangian \( Y(\mathfrak{g}l_{n}) \); see e.g. \([11]\) for the description of its structure and representations. The isomorphism with the RTT presentation of \( Y(\mathfrak{g}l_{n}) \) was constructed in \([3]\) providing a proof of the original result of Drinfeld \([6]\). Moreover, for an arbitrary pyramid \( \pi \), the shifted Yangian \( Y_{\pi}(\mathfrak{g}l_{n}) \) can be regarded as a natural subalgebra of \( Y(\mathfrak{g}l_{n}) \). Note also that the shifted Yangians can be found for more general types of pyramids. However, in accordance to \([4]\), each of these algebras is isomorphic to \( Y_{\pi}(\mathfrak{g}l_{n}) \) for an appropriate left-justified pyramid \( \pi \).

Introduce formal generating series in \( u^{-1} \) by

\[
[r]! \sum_{t=0}^{r} d_{i}^{(r)}(t) d_{i}^{(r-t)} = \delta_{r0}, \quad r = 0, 1, \ldots
\]

for all admissible \( i, j, r, s, t \), where \( d_{i}^{(0)} = 1 \) and the elements \( d_{i}^{(r)} \) are found from the relations

\[
[e_{i}^{(r)}, e_{j}^{(s+1)}] = e_{i}^{(r)} e_{j}^{(s)} + \text{if } |i - j| > 1,
\]

\[
[e_{i}^{(r)}, f_{j}^{(s)}] = 0 \quad \text{if } |i - j| > 1,
\]

\[
[e_{i}^{(r)}, [e_{j}^{(s)}, f_{k}^{(t)}]] = 0 \quad \text{if } |i - j| = 1,
\]

\[
[f_{i}^{(r)}, [f_{j}^{(s)}, f_{k}^{(t)}]] = 0 \quad \text{if } |i - j| = 1,
\]

and set

\[
a_{i}(u) = d_{1}(u) d_{2}(u - 1) \ldots d_{i}(u - i + 1)
\]

for \( i = 1, \ldots, n, \) and

\[
b_{i}(u) = a_{i}(u) e_{i}(u - i + 1), \quad c_{i}(u) = f_{i}(u - i + 1) a_{i}(u)
\]
for \( i = 1,\ldots,n - 1 \). It is clear that the coefficients of the series \( a_i(u) \), \( b_i(u) \) and \( c_i(u) \) generate the algebra \( Y_\pi(\mathfrak{gl}_n) \). It is not difficult to rewrite the defining relations in terms of these coefficients. We point out a few of these relations here which will be frequently used later on; see also [3]. We have

\begin{align*}
[&a_i(u), c_j(v)] = 0, \quad [b_i(u), c_j(v)] = 0, \quad \text{if } i \neq j, \\
&[c_i(u), c_j(v)] = 0, \quad \text{if } |i - j| \neq 1, \\
&(u - v)[a_i(u), c_i(v)] = c_i(u)a_i(v) - c_i(v)a_i(u).
\end{align*}

Let \( N \) be the number of bricks in the pyramid \( \pi \). Due to the main result of [4], the finite \( W \)-algebra \( W(\pi) \), associated to \( \mathfrak{gl}_N \) and the pyramid \( \pi \), can be defined as the quotient of \( Y_\pi(\mathfrak{gl}_n) \) by the two-sided ideal generated by all elements \( d_i^{(r)} \) with \( r \geq p_1 + 1 \). We refer the reader to [4, 5] for a discussion of the origins of the finite \( W \)-algebras and more references. Note that in the case of a rectangular pyramid of height \( p \), the algebra \( W(\pi) \) is isomorphic to the Yangian of level \( p \); this relationship was originally observed in [1] and [14].

We will use the same notation for the images of the elements of \( Y_\pi(\mathfrak{gl}_n) \) in the quotient algebra \( W(\pi) \). Set

\[ A_i(u) = u^{p_1} (u - 1)^{p_2} \cdots (u - i + 1)^{p_i} a_i(u) \]

for \( i = 1,\ldots,n \), and

\[ B_i(u) = u^{p_1} (u - 1)^{p_2} \cdots (u - i + 2)^{p_{i-1}} (u - i + 1)^{p_{i+1}} b_i(u), \]

\[ C_i(u) = u^{p_1} (u - 1)^{p_2} \cdots (u - i + 1)^{p_{i+1}} c_i(u) \]

for \( i = 1,\ldots,n - 1 \). The following lemma is immediate from the results of Brown and Brundan [2]. Here we regard \( A_i(u) \), \( B_i(u) \), and \( C_i(u) \) as series with coefficients in \( W(\pi) \).

**Lemma 2.1.** All series \( A_i(u) \), \( B_i(u) \), and \( C_i(u) \) are polynomials in \( u \).

**Proof.** In terms of the RTT presentation of the Yangian, each of the series \( a_i(u) \in Y(\mathfrak{gl}_n)[[u^{-1}]] \) coincides with a quantum minor of the matrix of the generators; see [3, Theorem 8.6]. Therefore the statement for the \( A_i(u) \) follows from the results of [2, Section 3]. Note that the polynomial \( A_i(u) \) in \( u \) is monic of degree \( p_1 + \cdots + p_i \). Furthermore, the defining relations of \( Y_\pi(\mathfrak{gl}_n) \) imply \([f_i^{(1)}, a_i(u)] = c_i(u)\), and so \( C_i(u) = [f_i^{(1)}, A_i(u)] \) is a polynomial in \( u \) of degree \( p_1 + \cdots + p_{i-1} + 1 \). Similarly,

\[ b_i(u) (u - i + 1)^{p_{i+1} - p_i} = [a_i(u), e_i^{(p_{i+1} - p_i + 1)}], \]

which gives

\[ B_i(u) = [A_i(u), e_i^{(p_{i+1} - p_i + 1)}], \]
so that $B_i(u)$ is a polynomial in $u$ of degree $p_1 + \cdots + p_i - 1$. □

Note that by [5, Theorem 6.10], all coefficients of the polynomial $A_n(u)$ belong to the center of $W(\pi)$ and these coefficients (excluding the leading one) are algebraically independent generators of the center.

For $i = 1, \ldots, n - 1$ define the elements $h_i^{(r)} \in Y_\pi(\mathfrak{gl}_n)$ by the expansion

$$1 + \sum_{r=1}^{\infty} h_i^{(r)} u^{-r} = d_i(u)^{-1} d_{i+1}(u)$$

and set

$$H_i^{(r)}(u) = u^r + u^{r-1} h_i^{(1)} + \cdots + h_i^{(r)}.$$

**Lemma 2.2.** For $i = 1, \ldots, n - 1$ in the algebra $W(\pi)$ we have

$$(u - v) [B_i(u), C_i(v)] = A_i^{(i+1)}(u) A_i(v) - A_i^{(i+1)}(v) A_i(u),$$

where $A_i^{(i+1)}(u)$ is the polynomial in $u$ with coefficients in $W(\pi)$ given by

$$A_i^{(i+1)}(u) = u^{p_1} (u - 1)^{p_2} \cdots (u - i + 2)^{p_{i-1}} (u - i + 1)^{p_i+1}$$

$$(u - 1) a_i(u+1)^{-1} (a_i(u+1) a_{i-1}(u) + c_i(u+1) b_i(u))$$

$- H_i^{(p_i+1-p_1)}(u - i) A_i(u).$$

Moreover,

$$B_i(u) C_i(u-1) = A_i^{(i+1)}(u) A_i(u-1) - A_i^{(i+1)}(u) A_{i-1}(u-1)$$

$$+ H_i^{(p_i+1-p_1)}(u - i) A_i(u) A_i(u-1).$$

**Proof.** Observe that for any fixed $i \in \{1, \ldots, n - 1\}$ the elements $d_i^{(r)}, d_i^{(r+1)}, e_i^{(r)}$ and $f_i^{(r)}$ of $Y_\pi(\mathfrak{gl}_n)$ satisfy the defining relations of the shifted Yangian $Y_{\pi_i}(\mathfrak{gl}_2)$, where $\pi_i = (p_i, p_{i+1})$. Therefore, it suffices to prove the first relation in the case $i = 1$; the proof for the remaining values of $i$ will then easily follow. Working in the Yangian $Y(\mathfrak{gl}_2)$, we can derive the relation

$$(u - v - 1) [d_1(u), e_1(v)] = (e_1(v) - e_1(u)) d_1(u);$$

see e.g. [3]. This allows us to calculate the commutators $[d_1(u), e_1^{(r)}]$ and leads to an equivalent expression for $b_1(u)$ in the subalgebra $Y_{\pi}(\mathfrak{gl}_2)$:

$$b_1(u) = d_1(u) e_1(u) = (1 - u^{-1})^{p_2-p_1} e_1(u-1) d_1(u).$$

Furthermore, starting from the relations

$$[e_1^{(r)}, f_1^{(s)}] = - \sum_{t=0}^{r+s-1} d_1^{(t)} d_2^{(r+s-t-1)}$$

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in \( Y_\pi(\mathfrak{gl}_2) \), it is now straightforward to derive that

\[
(u - v) [b_1(u), c_1(v)] = a_1(u + 1)^{-1}(a_2(u + 1) + c_1(u + 1) b_1(u)) a_1(v)
- (u^{-1} v)^{p_2 - p_1} a_1(v + 1)^{-1}(a_2(v + 1) + c_1(v + 1) b_1(v)) a_1(u)
- u^{p_1 - p_2}(H_1^{(p_2 - p_1)}(u) - H_1^{(p_2 - p_1)}(v)) a_1(u) a_1(v).
\]

The desired relation in \( W(\pi) \) is then obtained by multiplying both sides by the product \( u^{p_2} v^{p_1} \). Furthermore, by the defining relations,

\[
u^{p_2} a_1(u + 1)^{-1}(a_2(u + 1) + c_1(u + 1) b_1(u))
= u^{p_2} (d_2(u) + f_1(u) d_1(u) c_1(u)).
\]

This is a polynomial in \( u \) due to \([5, \text{Theorem 3.5}]\). Hence, by Lemma 2.1, \( A'_2(u) \) is a polynomial in \( u \) too.

The second part of the lemma is implied by the first part by taking into account the relations in the shifted Yangian \( Y_\pi(\mathfrak{gl}_n) \),

\[
a_i(u)^{-1} c_i(u) = c_i(u - 1) a_i(u - 1)^{-1}
\]

and

\[
(u - i)^{p_i + 1 - p_i} a_i(u - 1)^{-1} b_i(u - 1) = (u - i + 1)^{p_i + 1 - p_i} b_i(u) a_i(u)^{-1},
\]

which are implied by the defining relations. \( \square \)

3. CONSTRUCTION OF BASIS VECTORS

Using the canonical homomorphism \( Y_\pi(\mathfrak{gl}_n) \to W(\pi) \) we can extend every representation of the finite \( W \)-algebra \( W(\pi) \) to the shifted Yangian \( Y_\pi(\mathfrak{gl}_n) \). In what follows we work with representations of \( W(\pi) \), and the results can be easily interpreted in the shifted Yangian context.

Let us recall some definitions and results from \([3] \) regarding representations of \( W(\pi) \). Fix an \( n \)-tuple \( \lambda(u) = (\lambda_1(u), \ldots, \lambda_n(u)) \) of monic polynomials in \( u \) with coefficients in \( \mathbb{C} \), where \( \lambda_i(u) \) has degree \( p_i \). We let \( L(\lambda(u)) \) denote the irreducible highest weight representation of \( W(\pi) \) with the highest weight \( \lambda(u) \). Then \( L(\lambda(u)) \) is generated by a nonzero vector \( \zeta \) (the highest vector) such that

\[
B_i(u) \zeta = 0 \quad \text{for} \quad i = 1, \ldots, n - 1, \quad \text{and}
\]

\[
u^{p_i} d_i(u) \zeta = \lambda_i(u) \zeta \quad \text{for} \quad i = 1, \ldots, n.
\]

Write

\[
\lambda_i(u) = (u + \lambda_i^{(1)}) (u + \lambda_i^{(2)}) \ldots (u + \lambda_i^{(p_i)}), \quad i = 1, \ldots, n.
\]
We will assume that the parameters $\lambda_i^{(k)}$ satisfy the conditions: for any value $k \in \{1, \ldots, p_i\}$ we have

$$\lambda_i^{(k)} - \lambda_{i+1}^{(k)} \in \mathbb{Z}_+, \quad i = 1, \ldots, n - 1,$$

where $\mathbb{Z}_+$ denotes the set of nonnegative integers. In this case the representation $L(\lambda(u))$ of $W(\pi)$ is finite-dimensional.

Denote by $q_k$ the number of bricks in the column $k$ of the pyramid $\pi$. We have $q_1 \geq \cdots \geq q_l > 0$, where $l = p_n$ is the number of the columns in $\pi$. If $p_{i-1} < k \leq p_i$ for some $i \in \{1, \ldots, n\}$ (taking $p_0 = 0$), then we set $\lambda^{(k)} = (\lambda_i^{(k)}, \ldots, \lambda_n^{(k)})$. Then $q_k = n - i + 1$. Let $L(\lambda^{(k)})$ denote the finite-dimensional irreducible representation of the Lie algebra $\mathfrak{g}_{q_k}$ with the highest weight $\lambda^{(k)}$. The vector space

$$(6) \quad L(\lambda^{(1)}) \otimes \cdots \otimes L(\lambda^{(l)})$$

can be equipped with an action of the algebra $W(\pi)$, and $L(\lambda(u))$ is isomorphic to a subquotient of the module (6). In particular,

$$(7) \quad \dim L(\lambda(u)) \leq \prod_{k=1}^l \dim L(\lambda^{(k)}).$$

In what follows we will only consider a certain family of representations of $W(\pi)$ by imposing a \textit{generality condition} on the highest weights of the representations $L(\lambda(u))$. We will assume that

$$\lambda_i^{(k)} - \lambda_j^{(m)} \notin \mathbb{Z}, \quad \text{for all } i, j \text{ and all } k \neq m.$$

The \textit{Gelfand–Tsetlin pattern} $\Lambda(u)$ (associated with $\lambda(u)$) is an array of monic polynomials in $u$ of the form

$$\begin{align*}
\lambda_{n1}(u) & \quad \lambda_{n2}(u) & \quad \cdots & \quad \lambda_{nn}(u) \\
\lambda_{n-1,1}(u) & \quad \lambda_{n-1,n-1}(u) \\
\cdots & \quad \cdots \\
\lambda_{21}(u) & \quad \lambda_{22}(u) \\
\lambda_{11}(u)
\end{align*}$$

where

$$\lambda_{ri}(u) = (u + \lambda_{ri}^{(1)}) \cdots (u + \lambda_{ri}^{(p_i)}), \quad 1 \leq i \leq r \leq n,$$

with $\lambda_{ri}^{(k)} = \lambda_i^{(k)}$ and the following conditions hold

$$\lambda_{r+1,i}^{(k)} - \lambda_{ri}^{(k)} \in \mathbb{Z}_+ \quad \text{and} \quad \lambda_{ri}^{(k)} - \lambda_{r+1,i+1}^{(k)} \in \mathbb{Z}_+.$$
for $k = 1, \ldots, p_i$ and $1 \leq i \leq r \leq n - 1$. We have $\lambda_{n_i}(u) = \lambda_i(u)$ for $i = 1, \ldots, n$, so that the top row coincides with $\lambda(u)$.

Most arguments in the rest of the paper will not be essentially different from [13, Section 3], so we only sketch the main steps in the construction of the basis. Given a pattern $\Lambda(u)$, introduce the corresponding element $\zeta_\Lambda$ of $L(\lambda(u))$ by the formula

$$\zeta_\Lambda = \prod_{i=1}^{n-1} \left\{ \prod_{k=1}^{p_i} \left( C_{n-1}(-l_{n-1,i}^{(k)} - 1) \cdots C_{n-1}(-l_i^{(k)}) \right) \right\} \times \prod_{k=1}^{p_i} \left( C_{n-2}(-l_{n-2,i}^{(k)} - 1) \cdots C_{n-2}(-l_i^{(k)}) + 1 \right) \times \cdots \times \prod_{k=1}^{p_i} \left( C_i(-l_{i,i}^{(k)} - 1) \cdots C_i(-l_i^{(k)}) + 1 \right),$$

where we have used the notation

$$l_i^{(k)} = \lambda_i^{(k)} - i + 1 \quad \text{and} \quad l_{ri}^{(k)} = \lambda_{ri}^{(k)} - i + 1.$$

Note that by (3) we have $[C_i(u), C_i(v)] = 0$, so that the order of the factors in the products over $k$ is irrelevant.

**Lemma 3.1.** We have

$$A_r(u) \zeta_\Lambda = \lambda_{r1}(u) \cdots \lambda_{rr}(u - r + 1) \zeta_\Lambda,$$

for $r = 1, \ldots, n$.

**Proof.** When applying $A_r(u)$ to $\zeta_\Lambda$, separating the first factor, we need to calculate $A_r(u) C_s(v) \eta$ for the respective value of $v$. By (3), the operator $A_r(u)$ commutes with $C_s(v)$ for $s \neq r$. Furthermore, by (3),

$$A_r(u) C_r(v) \eta = \frac{1}{u - v} C_r(u) A_r(v) \eta + \frac{u - v - 1}{u - v} C_r(v) A_r(u) \eta.$$

The calculation is completed by induction on the number of factors $C_i(v)$ in the expression for $\zeta_\Lambda$, taking into account that $A_r(v) \eta = 0$. \qed

**Lemma 3.2.** For any $1 \leq i \leq r \leq n - 1$ and $k = 1, \ldots, p_i$ we have

$$B_r(-l_{ri}^{(k)}) \zeta_\Lambda = -\lambda_1(-l_{ri}^{(k)}) \cdots \lambda_i(-l_{ri}^{(k)} - i + 1)$$

$$\times \lambda_{r+1,i+1}(-l_{r+1,i+1}^{(k)} - i) \cdots \lambda_{r+1,r+1}(-l_{r+1}^{(k)} - r)$$

$$\times \lambda_1(-l_{r+1}^{(k)} - 1) \cdots \lambda_i(-l_{r}^{(k)} - i + 1)$$

$$\times \lambda_{r-1,i}(-l_{r-1,i}^{(k)} - i) \cdots \lambda_{r-1,r-1}(-l_{r-1}^{(k)} - r + 1) \zeta_{\Lambda + \delta_{ri}^{(k)}},$$
where $\zeta_{\Lambda^+}^{(k)}$ corresponds to the pattern obtained from $\Lambda(u)$ by replacing $\lambda_{ri}^{(k)}$ by $\lambda_{ri}^{(k)} + 1$, and the vector $\zeta_{\Lambda}$ is considered to be zero, if $\Lambda(u)$ is not a pattern.

Proof. The argument is based on Lemma 2.2. As in the proof of Lemma 3.1, separating the first factor, we need to calculate the image $B_r(-l_{ri}^{(k)}) C_s(v) \eta$ for the respective value of $v$. By (3), the operator $B_r(u)$ commutes with $C_s(v)$ for $s \neq r$. If $s = r$ then we consider two cases. If $-l_{ri}^{(k)} - v \neq 1$, then applying the first relation of Lemma 2.2 together with Lemma 3.1, we find that

$$B_r(-l_{ri}^{(k)}) C_r(v) \eta = C_r(v) B_r(-l_{ri}^{(k)}) \eta$$

and proceed by induction. If $v = -l_{ri}^{(k)} - 1$, then we apply the second relation of Lemma 2.2 together with Lemma 3.1 to get

$$B_r(-l_{ri}^{(k)}) C_r(-l_{ri}^{(k)} - 1) \eta = -A_{r+1}(-l_{ri}^{(k)} + 1) A_{r-1}(-l_{ri}^{(k)} - 1) \eta.$$

One more application of Lemma 3.1 leads to the desired formula. □

The following theorem provides a basis of the Gelfand–Tsetlin type for the representation $L(\lambda(u))$.

**Theorem 3.3.** The vectors $\zeta_{\Lambda}$ parameterized by all patterns $\Lambda(u)$ associated with the highest weight $\lambda(u)$, form a basis of the representation $L(\lambda(u))$ of the algebra $W(\pi)$.

Proof. It is easy to verify that if the array of monic polynomials obtained from $\Lambda(u)$ by increasing the entry $\lambda_{ri}^{(k)}$ by 1 is a pattern, then the coefficient of the vector $\zeta_{\Lambda+\delta_{ri}^{(k)}}$ in the formula of Lemma 3.2 is nonzero. This implies that each vector $\zeta_{\Lambda} \in L(\lambda(u))$ associated with a pattern $\Lambda(u)$ is nonzero.

Furthermore, by Lemma 3.1, $\zeta_{\Lambda}$ is an eigenvector for all operators $A_r(u)$ with distinct sets of eigenvalues. This shows that the vectors $\zeta_{\Lambda}$ are linearly independent.

Finally, for each $i \in \{1, \ldots, n\}$ and $p_{i-1} < k \leq p_i$ the set of parameters $(\lambda_{rj}^{(k)})$ with $i \leq j \leq r \leq n$ forms a Gelfand–Tsetlin pattern associated with the highest weight $\lambda^{(k)}$ of the irreducible representation $L(\lambda^{(k)})$ of the Lie algebra $\mathfrak{gl}_{qk}$. Hence, the number of patterns $\Lambda(u)$ coincides with the product of dimensions $\dim L(\lambda^{(k)})$ for $k = 1, \ldots, l$. Comparing this with (7), we conclude that the number of patterns coincides with $\dim L(\lambda(u))$. □
Note that by Theorem 3.3, we have the equality in (7), and thus we recover a result from [5] that the representation (6) of $W(\pi)$ is irreducible.

4. ACTION OF THE GENERATORS

We will calculate the action of the generators of $W(\pi)$ in a normalized basis of $L(\lambda(u))$. For any pattern $\Lambda(u)$ associated to $\lambda(u)$ set

$$N_{\Lambda} = \prod_{r=1}^{n} \prod_{i=1}^{m} \prod_{p_i} \prod_{j=1}^{n} (l_{ij}^{(m)} - l_{ij}^{(k)})(l_{ij}^{(m)} - l_{ij}^{(k)} + 1) \cdots (l_{ij}^{(m)} - l_{ij}^{(k)} - 1)$$

where the pairs $(r, i)$ run over the set of indices satisfying the inequalities $1 \leq i \leq r \leq n - 1$. This constant is clearly nonzero for any pattern $\Lambda(u)$. Introduce normalized vectors $\xi_{\Lambda} \in L(\lambda(u))$ by

$$\xi_{\Lambda} = N_{\Lambda}^{-1} \zeta_{\Lambda}.$$ 

By Theorem 3.3, the vectors $\xi_{\Lambda}$ form a basis of the representation $L(\lambda(u))$. The algebra $W(\pi)$ is generated by the coefficients of the polynomials $A_r(u)$ with $r = 1, \ldots, n$ and the coefficients of the polynomials $B_r(u)$ and $C_r(u)$ with $r = 1, \ldots, n - 1$. Since $B_r(u)$ and $C_r(u)$ are polynomials in $u$ of degree less than $p_1 + \cdots + p_r$, it suffices to find the values of these polynomials at $p_1 + \cdots + p_r$ different values of $u$. The polynomial can then be calculated by the Lagrange interpolation formula. For these values we take the numbers $-l_{ri}^{(k)}$ with $i = 1, \ldots, r$ and $k = 1, \ldots, p_i$.

**Theorem 4.1.** We have

(8) $$A_r(u) \xi_{\Lambda} = \lambda_{r+1}(u) \cdots \lambda_{r+r}(u - r + 1) \xi_{\Lambda},$$

for $r = 1, \ldots, n$, and

(9) $$B_r(-l_{ri}^{(k)}) \xi_{\Lambda} = -\lambda_{r+1,1}(-l_{ri}^{(k)}) \cdots \lambda_{r+1,r+1}(-l_{ri}^{(k)} - r) \xi_{\Lambda + \delta_{ri}^{(k)}},$$

$$C_r(-l_{ri}^{(k)}) \xi_{\Lambda} = \lambda_{r-1,1}(-l_{ri}^{(k)}) \cdots \lambda_{r-1,r-1}(-l_{ri}^{(k)} - r + 2) \xi_{\Lambda - \delta_{ri}^{(k)}},$$

for $r = 1, \ldots, n - 1$, where $\xi_{\Lambda + \delta_{ri}^{(k)}}$ corresponds to the pattern obtained from $\Lambda(u)$ by replacing $\lambda_{ri}^{(k)}$ by $\lambda_{ri}^{(k)} \pm 1$. 
Proof. The formulas for the action of $A_r(u)$ and $B_r(-l^{(k)}_{ri})$ follow respectively from Lemmas 3.1 and 3.2 by taking into account the normalization constant. Now consider the vector $C_r(-l^{(k)}_{ri})\xi_\Lambda$. Arguing as in the proof of Lemma 3.1 and using (8), we find that

$$A_s(u) C_r(-l^{(k)}_{ri})\xi_\Lambda = C_r(-l^{(k)}_{ri}) A_s(u)\xi_\Lambda$$

$$= \lambda_{s1}(u) \ldots \lambda_{ss}(u - s + 1) C_r(-l^{(k)}_{ri})\xi_\Lambda$$

for $s \neq r$, while

$$A_r(u) C_r(-l^{(k)}_{ri})\xi_\Lambda = \frac{u + l^{(k)}_{ri} - 1}{u + l^{(k)}_{ri}} C_r(-l^{(k)}_{ri}) A_r(u)\xi_\Lambda$$

$$= \frac{u + l^{(k)}_{ri} - 1}{u + l^{(k)}_{ri}} \lambda_r(u) \ldots \lambda_r(u - r + 1) C_r(-l^{(k)}_{ri})\xi_\Lambda.$$

If $\lambda^{(k)}_{ri} = \lambda^{(k)}_{r+1,i+1}$, then the vector $\xi_{\Lambda-\delta^{(k)}_{ri}}$ is zero and we need to show that $C_r(-l^{(k)}_{ri})\xi_\Lambda = 0$. Indeed, otherwise the vector $C_r(-l^{(k)}_{ri})\xi_\Lambda$ must be proportional to a certain basis vector of $L(\lambda(u))$. However, this is impossible because none of the basis vectors has the same set of eigenvalues as $C_r(-l^{(k)}_{ri})\xi_\Lambda$.

If $\lambda^{(k)}_{ri} - \lambda^{(k)}_{r+1,i+1} \geq 1$, then by the same argument we have

$$C_r(-l^{(k)}_{ri})\xi_\Lambda = \alpha \xi_{\Lambda-\delta^{(k)}_{ri}}$$

for a certain constant $\alpha$. Its value is found by the application of the operator $B_r(-l^{(k)}_{ri} + 1)$ to the vectors on both sides with the use of (8), (9) and the second relation in Lemma 2.2. □

Note that in the particular case of a rectangular pyramid $\pi$ the normalized basis $\{\xi_\Lambda\}$ coincides with the basis of [10] constructed in a different way.

Let us denote by $\pi'$ the pyramid with the rows $p_1, \ldots, p_{n-1}$. Then the finite $W$-algebra $W(\pi')$ may be identified with the subalgebra of $W(\pi)$ generated by the elements (2), excluding all $h^{(r)}_n, e^{(r)}_n$ and $f^{(r)}_n$. Theorem 4.1 implies the following branching rule for the reduction $W(\pi) \downarrow W(\pi')$ and thus shows that the basis $\{\xi_\Lambda\}$ is consistent with the chain of subalgebras (1). 

**Corollary 4.2.** The restriction of the $W(\pi)$-module $L(\lambda(u))$ to the subalgebra $W(\pi')$ is isomorphic to the direct sum of irreducible highest weight $W(\pi')$-modules $L'(\mu(u))$,

$$L(\lambda(u)) \downarrow_{W(\pi')} \cong \bigoplus_{\mu(u)} L'(\mu(u)),$$
where $\mu(u)$ runs over all $(n - 1)$-tuples of monic polynomials in $u$ of the form $\mu(u) = (\mu_1(u), \ldots, \mu_{n-1}(u))$, such that

$$\mu_i(u) = (u + \mu_i^{(1)})(u + \mu_i^{(2)}) \ldots (u + \mu_i^{(p_i)}), \quad i = 1, \ldots, n - 1,$$

and the following conditions hold:

$$\lambda_i^{(k)} - \mu_i^{(k)} \in \mathbb{Z}_+ \quad \text{and} \quad \mu_i^{(k)} - \lambda_{i+1}^{(k)} \in \mathbb{Z}_+$$

for $k = 1, \ldots, p_i$ and $1 \leq i \leq r \leq n - 1$. □

For each $i = 1, \ldots, n - 1$ introduce the polynomials $\tau_{ni}(u)$ and $\tau_{in}(u)$ with coefficients in $W(\pi)$ by the formulas

$$\begin{align*}
\tau_{ni}(u) &= C_{n-1}(u) C_{n-2}(u) \ldots C_i(u), \\
\tau_{in}(u) &= B_i(u) B_{i+1}(u) \ldots B_{n-1}(u).
\end{align*}$$

(10)

Define the vector $\zeta_\mu \in L(\lambda(u))$ corresponding to the $(n - 1)$-tuple of polynomials $\mu(u) = (\mu_1(u), \ldots, \mu_{n-1}(u))$ by the formula

$$\zeta_\mu = \prod_{i=1}^{n-1} \prod_{k=1}^{p_i} \left( \tau_{ni}(-m_i^{(k)} - 1) \ldots \tau_{ni}(-l_i^{(k)} + 1) \tau_{ni}(-\lambda_i^{(k)}) \right) \zeta,$$

where the ordering of the factors corresponds to increasing indices $i$ and $k$, and we used the notation

$$m_i^{(k)} = \mu_i^{(k)} - i + 1 \quad \text{and} \quad l_i^{(k)} = \lambda_i^{(k)} - i + 1.$$

Due to Theorem 4.1, the vector $\zeta_\mu$ generates a $W(\pi')$-submodule of $L(\lambda(u))$, isomorphic to $L'(\mu(u))$. Moreover, the operators $\tau_{ni}(-m_i^{(k)})$ and $\tau_{ni}(-l_i^{(k)})$ take $\zeta_\mu$ to the vectors proportional to $\zeta_{\mu-\delta_i^{(k)}}$ and $\zeta_{\mu+\delta_i^{(k)}}$, respectively. So, the polynomials (10) valued at appropriate points can be regarded as the lowering and raising operators for the reduction $W(\pi) \downarrow W(\pi')$; cf. [11, Chapter 5].

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