§1. Introduction and statements

In what follows a tensor category is understood to be an ACU $\otimes$-category in the sense of Saavedra Rivano [5, Ch. I, 2.4.1]. We denote the unit object by $\mathbb{1}$, the commutativity constraint by $\psi$, and the tensor structure by $\otimes$. There is also an associativity constraint that we omit and all these constraints are subject to natural compatibility conditions (loc. cit. I, 2.4). Recall (Deligne [2, 2.1.2]) that an object $X$ of a tensor category is said to be dualizable if there is an object $X^\vee$ and morphisms $\delta_X : \mathbb{1} \to X \otimes X^\vee$ and $\text{ev}_X : X^\vee \otimes X \to \mathbb{1}$ such that the diagrams

\[
\begin{array}{ccc}
\mathbb{1} \otimes X & \xrightarrow{\delta_X \otimes \text{id}_X} & X \otimes X^\vee \otimes X \\
\psi_{1, X} & & \text{id}_X \otimes \text{ev}_X \\
& \xleftarrow{X \otimes \mathbb{1}} X \otimes\mathbb{1}
\end{array} \quad \quad \begin{array}{ccc}
X^\vee \otimes X \otimes X^\vee & \xleftarrow{\text{id}_X \otimes \delta_X} & X^\vee \otimes \mathbb{1} \\
\text{ev}_X \otimes \text{id}_X & & \psi_{X^\vee, X} \\
& \xrightarrow{X \otimes \mathbb{1}} \mathbb{1} \otimes X^\vee
\end{array}
\]

are commutative. For example for the tensor category of modules over a commutative ring, dualizability is (e.g. loc. cit. 2.6) the same as being finitely generated and projective. With an appropriate interpretation, the morphism $\text{ev}_X$ gives the trace. More concretely, let $X$ be a dualizable object and $f : X \to X$ an endomorphism. The trace of $f$, here denoted by $\text{tr}(f ; X)$, is defined to be the composite

\[
\mathbb{1} \xrightarrow{\delta_X} X \otimes X^\vee \xrightarrow{f \otimes \text{id}_X} X \otimes X^\vee \xrightarrow{\text{id}_X \otimes \text{ev}_X} X \otimes X \xrightarrow{\psi_{X^\vee, X}} X^\vee \otimes \mathbb{1}.
\]

This is an element of $\text{End}(\mathbb{1})$. The resulting map $\text{tr} : \text{End}(X) \to \text{End}(\mathbb{1})$ is linear. Moreover, when defined, the trace $\text{tr}(f \otimes g ; X \otimes Y)$ is the product of $\text{tr}(f ; X)$ and $\text{tr}(g ; Y)$. For the proofs of these and other properties see any of the references cited above.

We clarify some terminologies. A tensor category as above is (Mac Lane [3]) also called an (additive) symmetric monoidal category. A symmetric monoidal category in which each functor $Z \mapsto Z \otimes X$ has a right adjoint is (Eilenberg-Kelly [1]) said to be closed. Recall the following result.

**Theorem 1.1** (May [4, 0.1]).— For any distinguished triangle $\Delta : X \to Z \to Y \to X[1]$ of dualizable objects in a closed symmetric monoidal category with a compatible triangulation we have

\[
\text{tr}(\text{id}; Z) = \text{tr}(\text{id}; X) + \text{tr}(\text{id}; Y).
\]

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In what follows we let $D$ be a $k$-linear Karoubian (i.e. pseudo-abelian) rigid tensor triangulated category where $k = \bar{k}$ is an algebraically closed field of characteristic zero. Note that linearity means ([5, Ch. I, 0.1.2]) that $\text{End}(D)$ is a $k$-algebra. Here the term rigid tensor triangulated means a closed symmetric monoidal category with a compatible triangulation in the sense of [4] and in which every object is dualizable.

An endomorphism $f = (f_X, f_Z, f_Y)$ of a distinguished triangle $\Delta$ in $D$ is a commutative diagram

$$
\begin{array}{cccc}
X & \longrightarrow & Z & \longrightarrow Y & \longrightarrow X[1] \\
\downarrow f_X & & \downarrow f_Z & & \downarrow f_Y & & \downarrow f_{X[1]} \\
X & \longrightarrow & Z & \longrightarrow Y & \longrightarrow X[1]
\end{array}
$$

with both rows being the given triangle $\Delta$. For example $\text{id} = (\text{id}_X, \text{id}_Z, \text{id}_Y)$ is an endomorphism of $\Delta$. The compositions of endomorphisms of triangles are defined in an obvious manner and is associative. We prove the following result.

**Proposition 1.2.** Let $f$ be an endomorphism of a distinguished triangle $X \rightarrow Z \rightarrow Y \rightarrow X[1]$ in $D$ with $f^n = \text{id}$ for an integer $n > 0$. Then

$$
\text{tr}(f_Z; Z) = \text{tr}(f_X; X) + \text{tr}(f_Y; Y).
$$

§2. Proof

Let $D$ and $k$ be as above. We prove a more general result than 1.2. Let $G$ be a group. A $G$-object in $D$ is a pair $(X, \rho)$ consisting of an object $X$ of $D$ and a $k$-algebra homomorphism $\rho : kG \rightarrow \text{End}_D(X)$ where $kG$ is the group algebra of $G$. We may denote $\rho(a)$ by $a_X$ or simply $a$. Let $Y$ be another $G$-object. An $G$-morphism or $G$-equivariant morphism from $X$ to $Y$ is a morphism $f : X \rightarrow Y$ with $a_Y f = f a_X$ for all $a \in kG$. If $X$ is an $G$-object define the central function

$$
\chi_X : G \rightarrow \text{End}_D(\mathbb{1}), \quad g \mapsto \text{tr}(g; X).
$$

We say that the distinguished triangle $\Delta$ is $G$-equivariant, if $X$, $Y$, and $Z$ are equipped with actions $\rho_X : G \rightarrow \text{Aut}_D(Z)$ (similarly for $X$ and $Y$) and such that all morphisms (including the differential) are $G$-equivariant.

**Theorem 2.1.** If $G$ is torsion and $X \rightarrow Z \rightarrow Y \rightarrow X[1]$ is $G$-equivariant, then as functions $G \rightarrow \text{End}_D(\mathbb{1})$ we have

$$
\chi_Z = \chi_X + \chi_Y.
$$

**Proof.** We may assume that $G$ is finite. Let $\text{Irr} kG$ be the set of isomorphism classes of irreducible $k-$representations of $G$. In $D$ we have a natural $G$-equivariant isomorphism

$$
X \simeq \prod_{V \in \text{Irr} kG} V \otimes_k S_V(X)
$$

where $S_V(X) = \text{Hom}_{kG}(V, X)$ are certain objects and on which $G$ acts trivially. To see this, consider the contravariant functor $D \rightarrow (k-\text{mod})$ given by

$$\text{Obj}(D) \ni Y \mapsto \text{Hom}_{kG}(V, \text{Hom}_D(Y, X)).$$
This is representable. Indeed if in the above definition we replace $V$ by any finitely generated free $kG$-module $M$ and consider the corresponding functor, we see immediately that the functor is representable by an object $S_M(X) = \text{a finite direct sum of } X$. The general case follows from this and the fact that $V$ is a finitely generated projective $kG$-module and hence the kernel (i.e. image) of a projector $\pi$ on a free $kG$-module $M$. Since $D$ is Karoubian, we can define $S_V(X) = \text{coker}(\pi^*)$ where $\pi^*: S_M(X) \to S_M(X)$ is induced by $\pi$. This is easily seen to represent $S_V(X)$. Once we have these objects, the decomposition of $X$ follows from the corresponding one for $kG$. It follows that the sequence
\[
S_V(X) \to S_V(Z) \to S_V(Y) \to S_V(X[1])
\]
being a direct summand of the original distinguished triangle is distinguished in $D$. Finally we note that by the above decomposition and $k$-linearity of trace we have
\[
\text{tr}(g,X) = \sum \chi_V(g) \text{tr}(\text{id}; S_V(X))
\]
where $\chi_V: G \to k$ is the usual character of $V$. Similarly for $Z$ and $Y$. The result follows from this and 1.1. \hfill $\square$

**Proof of 1.2.** Apply the result 2.1 with $G = \mathbb{Z}/n\mathbb{Z}$ and the action $m \mapsto f_Z^m$ (resp. $m \mapsto f_X^m$, $m \mapsto f_Y^m$) on $Z$ (resp. $X, Y$). \hfill $\square$

§3. Remark

We conclude this short note by indicating a corollary of the proof of 2.1. We let $\mathfrak{A}$ a Karoubian tensor category with $k \subseteq \text{End}_\mathfrak{A}(1)$ where $k$ is an algebraic closure of $\mathbb{Q}$. Define $\mathbb{Z}_\mathfrak{A}$ to be the subring (=subgroup) of $\text{End}_\mathfrak{A}(1)$ generated by all $\text{tr}(\text{id}; X)$ with $X$ being dualizable in $\mathfrak{A}$.

**Corollary 3.1.** Let $f: X \to X$ be an endomorphism of a dualizable object in $\mathfrak{A}$ with $f^n = \text{id}$ for an integer $n > 0$. Then $\text{tr}(f; X) \in \text{End}_\mathfrak{A}(1)$ is integral over $\mathbb{Z}_\mathfrak{A}$.

**Proof.** Similar to the proof of 1.2 consider $X$ with an action of $G = \mathbb{Z}/n\mathbb{Z}$. Note that in the category $\mathfrak{A}$ the decomposition (2) and the formula (3) hold with exactly the same proof. Since the element $\chi_V(g) \in k$ is integral over $\mathbb{Z}$, the result follows from (3). \hfill $\square$

**References**

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