On Universal Point Sets for Planar Graphs

Jean Cardinal¹,∗ Michael Hoffmann²,† Vincent Kusters²,†

¹Département d’Informatique, Université Libre de Bruxelles, jcardin@ulb.ac.be
²Institute of Theoretical Computer Science, ETH Zürich, {hoffmann,kustersv}@inf.ethz.ch

Abstract

A set $P$ of points in $\mathbb{R}^2$ is $n$-universal, if every planar graph on $n$ vertices admits a plane straight-line embedding on $P$. Answering a question by Kobourov, we show that there is no $n$-universal point set of size $n$, for any $n \geq 15$.

Introduction

We consider plane, straight-line embeddings of graphs in $\mathbb{R}^2$. In those embeddings, every vertex is represented by a point, every edge is represented by a line segment connecting its endpoints, and no two edges intersect except at a common endpoint.

An $n$-universal (or short universal) point set for planar graphs can accommodate every planar graph on $n$ vertices. A longstanding open problem is to give precise bounds on the minimum number of points in an $n$-universal point set. The currently known asymptotic bounds are apart by a linear factor: On one hand, it is known that every planar graph can be embedded on a grid of size $n - 1 \times n - 1$ [5, 10]. On the lower bound side, it was shown by Kurowski [8] that at least $1.235n$ points are necessary, improving earlier bounds of $1.206n$ [4] and $n + \sqrt{n}$ [5].

The following, somewhat simpler question was asked ten years ago by Kobourov [6]: what is the largest value of $n$ for which a universal point set of size $n$ exists? We prove the following.

Theorem 1 There is no $n$-universal point set of size $n$, for any $n \geq 15$.

According to the Open Problem Project page [6], Kobourov proved there exist 14-universal point sets of size 14. Hence our bound is tight, and the answer to the above question is $n = 14$. Like all existing lower bound arguments, our proof is based on counting planar 3-trees (also called stacked triangulations sometimes), which we combine with a known lower bound on the rectilinear crossing number [1 9].

Proof

A planar 3-tree is a triangulation obtained by iteratively splitting a triangular face into three new triangles with a degree-3 vertex, starting from a single triangle.

For every integer $n \geq 4$, we define a family $\mathcal{T}_n$ of labeled planar 3-trees on $[n] := \{1, 2, \ldots, n\}$ as follows: (i) $\mathcal{T}_1$ contains only the complete graph, (ii) $\mathcal{T}_n$ contains every graph that can be constructed by making a new vertex $n$ adjacent to the three vertices of one of the $2n - 4$ facial triangles of some $T \in \mathcal{T}_{n-1}$.

Lemma 2 For $n \geq 5$, we have $|\mathcal{T}_n| = 4 \times 6 \times \ldots \times (2n - 6)$.
Theorem 3 ([1] [9]) Given a set \( P \subset \mathbb{R}^2 \) of \( n \geq 4 \) points in general position, more than \( \frac{3}{8} \binom{n}{4} \) 4-tuples of \( P \) are in convex position.

Lemma 4 Less than \( \frac{3}{8} n! \) graphs from \( \mathcal{T}_n \) can be embedded on any set of \( n \geq 4 \) points in \( \mathbb{R}^2 \).

Proof. Let \( P \subset \mathbb{R}^2 \) be a set of \( n \) points and denote by \( \mathcal{F}_n \subseteq \mathcal{T}_n \) the set of trees that admit a planar straight-line embedding onto \( P \). Consider a labeled planar 3-tree \( T \in \mathcal{F}_n \) and let \( \phi : |n| \to P \) denote a planar straight-line embedding of \( T \). In this way we can associate to each tree \( T \in \mathcal{F}_n \) a permutation \( \phi_T = \phi(1), \ldots, \phi(n) \) of the points in \( P \).

Consider now a permutation \( \pi = p_1, p_2, \ldots, p_n \) of the points in \( P \). With \( \pi \) we can associate a unique tree \( T_\pi \in \mathcal{F}_n \) as follows: The first three points form a triangle \( p_1 p_2 p_3 \); each subsequent point \( p_i \), for \( 4 \leq i \leq n \), is located in a unique face of the triangulation constructed so far, and so \( p_i \) is inserted as a degree three vertex, connected to the three vertices of \( f \). If at some point during the construction the point \( p_i \) together with the vertices from \( f \) forms a convex quadrilateral, we cannot connect \( p_i \) to all vertices of \( f \) without crossing an edge of \( f \). In such a case, the permutation \( \pi \) does not have an associated graph in \( \mathcal{F}_n \). Otherwise, the above construction yields a plane drawing of some labeled planar 3-tree \( T_\pi \in \mathcal{F}_n \).

Combining the two arguments, we obtain a bijective mapping between \( \mathcal{F}_n \) and some subset \( \Pi \) of the symmetric group \( S_n \). It follows that \( |\mathcal{F}_n| = |\Pi| \leq |S_n| = n! \).

Next we can quantify the difference between \( |\Pi| \) and \( |S_n| \) using Theorem 3. Note that the general position assumption is no restriction: In case of collinearities, a slight perturbation of the point set yields a new point set that still admits all planar straight-line drawings of the original point set. Consider a permutation \( \pi = p_1, \ldots, p_n \) such that \( p_1, p_2, p_3, p_4 \) form a convex quadrilateral. For any tree in \( \mathcal{F}_n \), these first four vertices must form a complete graph \( K_4 \), but there is no way obtain a plane straight line drawing of \( K_4 \) on four points in convex position. It follows that \( \pi \in S_n \setminus \Pi \). We know from Theorem 3 that more than a fraction of \( 3/8 \) of the 4-tuples of \( P \) are in convex position and therefore \( |\Pi| < \frac{5}{8} n! \). \( \square \)

Proof of Theorem 1 Using Lemma 2 one can check that \( |T_{15}| > \frac{1}{2} 15! \) (the values are 980'995'276'800 and 817'296'480'000, respectively). Since the latter by Lemma 4 is an upper bound on the number of graphs of \( \mathcal{T}_n \) that we can embed, not all graphs of \( \mathcal{T}_n \) have an embedding. Hence not every planar 3-tree on 15 vertices can be embedded on any given set of 15 points. \( \square \)

The number of non-isomorphic planar 3-trees on \( n \) vertices was computed by Beineke and Pippert [2], and appears as sequence A027610 on Sloane’s Encyclopedia of Integer Sequences. For \( n = 15 \), this number is 321’776. Hence we can also phrase our result in the language of simultaneous embeddings [3]. For a collection \( \mathcal{G} = \{G_1, \ldots, G_k\} \) of planar graphs on \( n \) vertices, a simultaneous embedding without mapping for \( \mathcal{G} \) is a collection of planar straight line embeddings \( \phi_i : G_i \to P \) onto the same set \( P \subset \mathbb{R}^2 \) of \( n \) points.

Corollary 5 There is a collection of 321’776 planar graphs that do not admit a simultaneous (plane straight-line) embedding without mapping.

Denote by \( \sigma \) the maximum natural number such that every collection of \( \sigma \) planar graphs on the same number of vertices admit a simultaneous embedding without mapping. We have shown that \( \sigma < 321’776 \) and from Fáry-Wagner’s Theorem [7] [11] we know that \( \sigma \geq 1 \). To our knowledge these are the best bounds currently known. It is a very interesting and probably challenging open problem to determine the exact value of \( \sigma \).

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