Chiral Perturbation Theory and
\(U(3)_L \times U(3)_R\) Chiral Theory of Mesons

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Abstract

We examine low energy limit of \(U(3)_L \times U(3)_R\) chiral theory of mesons through integrating out fields of vector and axial-vector mesons. The effective lagrangian for pseudoscalar mesons at \(O(p^4)\) has been obtained, and five low energy coupling constants \(L_i (i = 1, 2, 3, 9, 10)\) have been revealed. They are in good agreement with the results of \(\chi PT\)’s at \(\mu \sim m_\rho\).

1. Introduction

It is well known that, at low energies (e.g.,the energy scale is \(\mu \sim m_\rho\)), the strong, electromagnetic and weak interactions of pseudoscalar mesons can be successfully described by the chiral perturbation theory(\(\chi PT\))\cite{1,2}. This effective theory depends on a number of low-energy coupling constants which cannot be determined from the symmetries of the fundamental theory only. They are in principle determined by the underlying QCD dynamics in terms of renormalization group invariant scale \(\Lambda_{QCD}\) and the heavy quark masses. The foundations of \(\chi PT\) has been discussed in ref.\cite{3}. Actually, \(\chi PT\) is the effective theory of

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QCD at low-energies, and any eligible low-energy theories of mesons have to be consistent with it. By using this claim as a criterion, the authors of ref.[4, 5] have examined, commented on or criticized several chiral theoretical models of mesons. In particular, it was pointed in ref.[4, 6] that a reliable low energy effective theory which includes meson resonances must return to $\chi$PT when the freedom of meson resonance is frozen, or “integrated out”. The remainder effects of these meson resonances (or effects of virtual mesons) are absorbed by the low-energy coupling constances (or chiral coupling constances) of $\chi$PT. In other words, the low energy limit of effective meson theories with excited meson resonances has to be consistent with $\chi$PT.

Recently, based on $U(3)_L \times U(3)_R$ chiral symmetry and an ansatz that mesons are composite fields of quarks, Li[8]-[12] have proposed an effective meson theory with spin 1 meson resonances (hereafter we shall call this theory as the Li model). Distinguishing from other chiral theories, a cutoff $\Lambda \sim 1.6 GeV^2$ (which is higher than $m_\rho = 0.77 GeV$) has been introduced as an intrinsic parameter of this truncated field theory of mesons, and all the masses of the mesons in the model are below this cutoff $\Lambda$. This fact indicates that Li model is a self-consistent meson theory at intermediate energy scale. In this present paper we try to study the low energy limit of Li model and to see whether it is consistent with $\chi$PT or not. Specifically, we will calculate the chiral coupling constants $L_1, L_2, L_3, L_9, L_{10}$ and $H_1$ at $\mu \sim m_\rho$ by using the path integration to freeze the field freedoms of vector and axial-vector meson resonances (or ”integrate out” those meson fields) in the Li model. Because $L_1, L_2, L_3, L_9$ and $L_{10}$ are related to five conditions abstracted from QCD[6], our calculations

\[ g^2 = \frac{8}{3} \frac{N_c}{(4\pi)^2} \frac{D}{4} \Gamma(2 - \frac{D}{2})(\frac{\mu^2}{m^2})^{\frac{D}{2}} \]

This constant $g$ can be determined from decay $\rho \to \pi\pi$. As $g = 0.35$, the cut-off $\Lambda \sim 1.6 GeV$.
in this paper could be thought of as a test of the Li model from the QCD points of view. Therefore, even though the phenomenological predictions have been extensively studied and met with remarkable success\cite{8, 9, 10, 11, 12}, the examination to the model’s low energy limit is still necessary.

In ref.\cite{6}, the effects of virtual meson resonances in several models have been calculated. In those calculations only the terms with one meson resonance in interactions were taken into account. Under this approximation, it is enough for the purpose there to construct the meson resonance fields by simply using the propagators. The corresponding virtual meson resonance effects are due to the exchanges of these virtual particles. However, this method fails to the models in which the terms with higher power of resonances have contributions to the interactions at $O(p^4)$. In Li model, the vector($V$) and the axial vector($A$) mesons are ordinary 4-component vectors. The terms with more than one resonance fields, such as $VV$, $VVV$, $VVVV$, $VVAA$, $AAAA$, $AAA\partial\phi$ ($\phi$ is pseudoscalar meson) and so on, will emerge in the model’s interactions at $O(p^4)$. Thus, in order to catch at the effects at $O(p^4)$ of all virtual meson resonances in Li model, we have to carry out the corresponding non-Gaussian type path integrations to $V$ and $A$ under appropriate approximations. Practically, we will use the perturbation expansions around the saddle point of the model’s Lagrangian to perform the path integrations for our purpose. In this manner, the effects of virtual meson resonances are divided into two parts: effects of virtual resonance exchanges (EVRE) and effects of virtual resonance vertices (EVRV). Actually, the corresponding calculations in ref.\cite{6} are equivalent to performing Gaussian type path integrations, and one of the things done in this paper is to extend the ref.\cite{6} to the non-Gaussian type case. The former evaluates EVRE only, and the latter evaluates both. In addition, we will also show (see section 3)
the effects of quark constitutes of pseudoscalar mesons (EQCPM) in Li model contributes to $L_i(i = 1, 2, 3, 9, 10)$ at $O(p^4)$ besides EVRE and EVRV. All EVRE, EVRV and EQCPM will be taken into consideration in the calculations of $L_i$ presented in this paper.

In sections 2 and 3, the notations and definitions in $\chi$PT and Li model are presented respectively. The section 4 devotes to calculate the low energy coupling constants. Finally, we discuss the results of this paper in section 5.

2. Chiral Perturbation Theory

The effective lagrangian of chiral perturbation theory can be found in ref. [2]

$$L_{\chi PT} = L_2 + L_4 + \ldots$$

$L_2$ is the nonlinear $\sigma$ model lagrangian coupled to external fields $v, a, s$ and $p$

$$L_2 = \frac{f_\pi^2}{16} \langle \nabla_\mu U \nabla^\mu U^\dagger + \chi U^\dagger + \chi^\dagger U \rangle,$$

where

$$\nabla_\mu U = \partial_\mu U - i(v_\mu + a_\mu)U + iU(v_\mu - a_\mu);$$

$$\chi = 2B_0(s + ip) \sim 2m_qB_0;$$

and $< M >$ stands for the trace of the matrix $M$. The external fields $v_\mu, a_\mu, s$ and $p$ are Hermitian $3 \times 3$ matrices in flavour space and are vector, axial-vector, scalar and pseudoscalar external fields respectively. $U$ is a unitary $3 \times 3$ matrix

$$U = exp(-i \frac{2\Phi}{f_\pi}), \quad \Phi = \sum_{i=1}^{8} \lambda_i \phi^i.$$

where $f_\pi = 186 MeV$. The lagrangian (1) exhibits a local $SU(3)_L \otimes SU(3)_R$ symmetry

$$U \rightarrow g_L U g_R^\dagger.$$
\[ v_\mu \pm a_\mu \to g_{L,R}(v_\mu \pm a_\mu)g^\dagger_{L,R} + ig_{L,R}\partial_\mu g^\dagger_{L,R}, \]
\[ s + ip \to g_L(s + ip)g^\dagger_R, \]
where \( g_L, g_R \in SU(3)_L \otimes SU(3)_R \). The general lagrangian \( \mathcal{L}_4 \) is to be used only at tree-level in \( \chi \text{PT} \), and all the states to which it is applied obey the equation of motion
\[ \nabla_\mu (U \nabla^\mu U^\dagger) + \frac{1}{2}(\chi^\dagger U - U^\dagger \chi) = 0. \quad (5) \]
At chiral limit \( \chi \to 0 \), eq. (5) becomes
\[ \nabla_\mu U \nabla^\mu U^\dagger + U \nabla_\mu \nabla^\mu U^\dagger = 0. \quad (6) \]
In this paper we are concerned with the effective lagrangian at order \( p^4 \),
\[ \mathcal{L}_4 = L_1 < \nabla_\mu U \nabla^\mu U^\dagger >^2 + L_2 < \nabla_\mu U \nabla_\nu U^\dagger > < \nabla^\mu U \nabla^\nu U^\dagger > + L_3 < \nabla_\mu U \nabla^\mu U^\dagger \nabla_\nu U \nabla^\nu U^\dagger > + L_4 < \nabla_\mu U \nabla^\mu U^\dagger > < \chi^\dagger U + \chi U^\dagger > + L_5 < \nabla_\mu U \nabla^\mu (\chi^\dagger U + \chi U^\dagger) > + L_6 < \chi^\dagger U + \chi U^\dagger >^2 + L_7 < \chi^\dagger U - \chi U^\dagger >^2 + L_8 < \chi^\dagger U \chi^\dagger U + \chi U^\dagger \chi U^\dagger > - iL_9 < F^\mu_{\nu} \nabla_\mu U^\dagger \nabla_\nu U + F^\mu_{\nu} \nabla_\mu U \nabla_\nu U^\dagger > + L_{10} < UF^\mu_{\nu} U^\dagger F_{L\mu\nu} > + H_1 < F_{R\mu\nu} F^\mu_{L\nu} + F_{L\mu\nu} F^\mu_{R\nu} > + H_2 < \chi^\dagger \chi >, \quad (7) \]
where
\[ F^\mu_{L,R} = \partial^\mu (\nu^\nu \mp a^\nu) - \partial^\nu (\nu^\mu \mp a^\mu) - i[\nu^\mu \mp a^\mu, \nu^\nu \mp a^\nu]. \quad (8) \]
\( L_1, \ldots, L_{10} \) are ten real low-energy coupling constants at order \( p^4 \) which together with \( f_\pi \) and \( B_0 \) determined completely the low-energy behavior of pseudoscalar meson interaction to \( O(p^4) \). It is easy to see that \( L_1, L_2, L_3, L_9, L_{10} \) and \( H_1 \) are independent of quark-masses. Therefore they could receive contributions from the chiral symmetric models beyond \( \chi \text{PT} \).
3. $U(3)_L \times U(3)_R$ Chiral Theory of Mesons

The lagrangian of Li model with 3-flavor reads

$$L = \bar{\psi}(x)(i\gamma \cdot \partial + \gamma \cdot V + e_0 Q \gamma \cdot A + \gamma \cdot A \gamma_5 - mu(x))\psi(x) + \frac{1}{2}m_0^2(V^\mu_\mu V^\mu_\mu + A^i_\mu A^i_\mu) + \bar{\psi}(x)L \gamma \cdot W \psi(x)_L,$$

where $\psi$ are quark fields $u, d$ and $s$, $V$ and $A$ are vector and axial-vector mesons fields respectively, $A$ is the photon field, $Q$ is the electric charge operator of $u, d, s$ quarks, $W^i_\mu$ is $W$ boson. More explicitly, $V, A,$ and $u$ can be written as follows

$$A^i_\mu = \lambda_i a^a_\mu + \lambda_\mu K^{a\mu}_1 + \frac{2}{3} + \frac{1}{\sqrt{3}} \lambda_8 f_\mu + \frac{2}{3} - \frac{1}{\sqrt{3}} \lambda_8 f_{s\mu},$$

$$V^i_\mu = \lambda_i V^i_\mu = \lambda_\mu K^{s\mu}_1 + \frac{2}{3} + \frac{1}{\sqrt{3}} \lambda_8 \omega_\mu + \frac{2}{3} - \frac{1}{\sqrt{3}} \lambda_8 \sqrt{2} \phi_\mu,$$

$$u = \frac{1}{2}(1 + \gamma_5) U + \frac{1}{2}(1 - \gamma_5) U^\dagger,$$

$$m_0^2(V^\mu_\mu V^\mu_\mu + A^i_\mu A^i_\mu) = m_1^2(\rho_\mu^i \rho_\mu^i + \omega_\mu^i \omega_\mu^i + a^a_\mu a_{i\mu} + f^i_\mu f^i_\mu) + m_2^2(K^{s\mu}_1 K^{s\mu}_1 + K^{s\mu}_1 K^{s\mu}_1)$$

$$+ m_3^2(\phi_\mu f^{i \mu} + f_s f_{s\mu}),$$

where $i = 1, \ldots 8; n = 1, 2, 3; a = 4, 5, 6, 7$ and $U$ is the same as in eq. The $U_A(1)$ problem is not taken into consideration in this paper. All meson fields $V, A$ and $u(x)$ emerge in the Lagrangian of eq.(9) as auxiliary fields, because there are no kinetic terms to them. By solving the equations of motion of the fields, one can find that these auxiliary fields are composed fields of quark fields[]. This reflects the basic physics fact that all mesons are the bound states of quarks.

By means of path integral, the quark fields can be integrated out, the effective lagrangian of meson is obtained

$$\mathcal{L}(U, V, A) = \mathcal{L}_{Re} + \mathcal{L}_{Im}$$
The lagrangian $L_{nm}$ describes the physical processes with abnormal parity which will not be discussed in this paper. To the fourth order in the covariant derivatives in Minkowski space, the lagrangian $L_{Re}$ which describes the physical processes with normal parity takes the form

$$L_{Re} = \frac{F^2}{16} < D_\mu U D^\mu U^\dagger > - \frac{g^2}{8} < \mathbf{V}_{\mu\nu} \mathbf{V}^{\mu\nu} + \mathbf{A}_{\mu\nu} \mathbf{A}^{\mu\nu}>$$

$$- \frac{3i}{2} \gamma < (D_\mu U D_\nu U^\dagger + D_\nu U^\dagger D_\mu U) \mathbf{V}^{\mu\nu} >$$

$$- \frac{3i}{2} \gamma < (D_\mu U^\dagger D_\nu U - D_\nu U D_\mu U^\dagger) \mathbf{A}^{\mu\nu} >$$

$$+ \frac{\gamma}{2} < D_\mu D_\nu U D^\mu D^\nu U^\dagger > + \frac{1}{4} m_0^2 < \mathbf{V}_\mu \mathbf{V}_\mu + \mathbf{A}_\mu \mathbf{A}_\mu >$$

$$+ \frac{\gamma}{4} < D_\mu U D_\nu U^\dagger D_\nu D_\mu U^\dagger > - 2D_\mu U D^\mu U^\dagger D_\nu U D^\nu U^\dagger >$$

(10)

with

$$\gamma = \frac{N_c}{3(4\pi)^2}$$

and

$$D_\mu U = \nabla_\mu U - i(\mathbf{V}_\mu - \mathbf{A}_\mu)U + iU(\mathbf{V}_\mu + \mathbf{A}_\mu),$$

$$\mathbf{V}_{\mu\nu} = \partial_\mu (\mathbf{V}_\nu + \nu_\nu) - \partial_\nu (\mathbf{V}_\mu + \nu_\mu) - i[\mathbf{V}_\mu + \nu_\mu, \mathbf{V}_\nu + \nu_\nu] - i[\mathbf{A}_\mu + a_\mu, \mathbf{A}_\nu + a_\nu],$$

(11)

$$\mathbf{A}_{\mu\nu} = \partial_\mu (\mathbf{A}_\nu + a_\nu) - \partial_\nu (\mathbf{A}_\mu + a_\mu) - i[\mathbf{A}_\mu + a_\mu, \mathbf{V}_\nu + \nu_\nu] - i[\mathbf{V}_\mu + \nu_\mu, \mathbf{A}_\nu + a_\nu],$$

$$D_\mu D_\nu U = \nabla_\mu (\nabla_\nu U) - i(\mathbf{V}_\mu - \mathbf{A}_\mu)D_\nu U + iD_\nu U(\mathbf{V}_\mu + \mathbf{A}_\mu),$$

in this form, $\nabla_\mu U$ is the same as eq. (3), there are only three independent parameters in this theory and they are $g, f_\pi, m_\rho$ ($m_0 = gm_\rho$). It is should be noted $F \neq f_\pi$ in lagrangian (10) because of mixing between field $\mathbf{A}_\mu(x)$ and $\partial_\mu \pi(x)$, which should be diagonalized via field redefinition as,

$$\mathbf{A}_\mu(x) \rightarrow \mathbf{A}_\mu(x) + c\partial_\mu \pi(x)$$

the following equations has been given in Ref. [3],

$$\frac{F^2}{f_\pi^2}(1 - \frac{c}{g}) = 1$$

7
\[ c = \frac{f_\pi^2}{2g_m^2} \]

Eq.(10) is the meson Lagrangian of Li model at energy scale \( \Lambda \sim 1.6 GeV \). Besides the pseudoscalar mesons, the vector \((V)\) and axial vector \((A)\) meson resonances are excited in eq.(10). Our object is to reduce it to an effective Lagrangian containing pseudoscalar mesons only by integrating out \( V \) and \( A \), to reveal the values of \( L_1, L_2, L_3, L_9, L_{10} \) and \( H_1 \). The interactions of \( V \) and \( A \) in eq.(10) will contribute to these low energy coupling constants. Because the terms with higher powers of \( V \) and \( A \), such as \( VV, VVV, VVVV, VVAA, AA, AAAAA, AAAAA\partial\phi \) (\( \phi \) is pseudoscalar meson) and so on, emerge in the interactions of eq.(10). As stated in section 1, we have to calculate both EVRE and EVRV. In addition, in eq.(10), there are terms with 4 derivatives to pseudoscalar fields and independent of \( V \) and \( A \). The coupling coefficients of these terms contribute to \( L_i, (i = 1, 2, 3, 9, 10) \) also. They reflect the direct contributions due to \( L_i \) of the effects of quark constitutes of pseudoscalar mesons (EQCPM) in the Li model. Consequently, the low-energy coupling constants at \( O(p^4) \) for Li model receive contributions from three effects: EVRE, EVRV and EQCPM. Hereafter, we will take all effects into consideration.

4. Calculations of Low Energy Coupling Constants

In this section, we will use path integration manner to integrate out field freedoms of \( V \) and \( A \) in \( \mathcal{L}_{Re}(10) \), and to derive the chiral effective Lagrangian for pseudoscalar mesons, which is the low energy limit of the Li model. Comparing it with the standard \( \chi PT \) Lagrangian of eq.(7), we then work out the low-energy coupling constants at order \( p^4 \) for Li model.

8
For the sake of convenience we define
\[ L_\mu = \frac{1}{\sqrt{2}} (V_\mu - A_\mu), \quad R_\mu = \frac{1}{\sqrt{2}} (V_\mu + A_\mu), \]
\[ l_\mu = \frac{1}{\sqrt{2}} (v_\mu - a_\mu), \quad r_\mu = \frac{1}{\sqrt{2}} (v_\mu + a_\mu), \]
\[ L_{\mu\nu} = \frac{1}{\sqrt{2}} (V_{\mu\nu} - A_{\mu\nu}) = \nabla^L_{\mu} L_\nu - \nabla^L_{\nu} L_\mu - \sqrt{2} i [L_\mu, L_\nu], \]
\[ R_{\mu\nu} = \frac{1}{\sqrt{2}} (V_{\mu\nu} + A_{\mu\nu}) = \nabla^R_{\mu} R_\nu - \nabla^R_{\nu} R_\mu - \sqrt{2} i [R_\mu, R_\nu], \]

where
\[ \nabla^L_{\mu} L_\nu = \partial_{\mu} L_\nu - \frac{1}{\sqrt{2}} i [l_\mu, L_\nu], \]
\[ \nabla^R_{\mu} R_\nu = \partial_{\mu} R_\nu - \frac{1}{\sqrt{2}} i [r_\mu, R_\nu]. \]

Then we have
\[ F^L_{\mu\nu} = \sqrt{2} (\partial_\mu l_\nu - \partial_\nu l_\mu) - 2 i [l_\mu, l_\nu], \]
\[ F^R_{\mu\nu} = \sqrt{2} (\partial_\mu r_\nu - \partial_\nu r_\mu) - 2 i [r_\mu, r_\nu], \]
\[ (12) \]

and
\[ [\nabla_\mu, \nabla_\nu] U = i (U F^R_{\mu\nu} - F^L_{\mu\nu} U), \]
\[ [\nabla_\mu, \nabla_\nu] U^\dagger = i (U^\dagger F^L_{\mu\nu} - F^R_{\mu\nu} U^\dagger). \]
\[ (13) \]

With the above notations, the lagrangian \([10]\) can be written as follows
\[
\mathcal{L}_{Re} = \frac{F^2}{16} <D_\mu U D^\mu U^\dagger> - \frac{g^2}{16} <2 L_{\mu\nu} L^{\mu\nu} + 2 R_{\mu\nu} R^{\mu\nu} + F^L_{\mu\nu} F^L_{\mu\nu} + F^R_{\mu\nu} F^R_{\mu\nu} >
- \frac{g^2}{4\sqrt{2}} <L_{\mu\nu} F^{L}_{\mu\nu} + R_{\mu\nu} F^{R}_{\mu\nu} >
- \frac{3i}{\sqrt{2}} \gamma <D_\mu U D_\nu U^\dagger L^{\mu\nu} + D_\mu U^\dagger D_\nu U R^{\mu\nu} >
- \frac{3i}{2} \gamma <D_\mu U^\dagger D_\nu U F^{L\mu\nu} + D_\mu U D_\nu U^\dagger F^{R\mu\nu} >
\]
\[ \begin{align*}
+ \frac{\gamma}{2} < D_\mu D_\nu U D^\mu D^\nu U \uparrow > & + \frac{1}{4} m_0^2 < L_\mu L^\mu + R_\mu R^\mu > \\
+ \frac{\gamma}{4} < D_\mu U D_\nu U \uparrow D^\mu D^\nu U \uparrow - 2D_\mu U D^\mu U \uparrow D_\nu U D^\nu U \uparrow >,
\end{align*} \] (14)

where

\[ \nabla_\mu U = \partial_\mu U - \sqrt{2} i_\mu U + \sqrt{2} i R_\mu, \]

\[ D_\mu U = \nabla_\mu U - \sqrt{2} i L_\mu U + \sqrt{2} i R_\mu. \]

All vector and axial-vector meson resonances in Li model join meson dynamics via \( L_\mu, R_\mu \) in \( \mathcal{L}_{Re} \) (eq. (14)). To order \( p^4 \) in chiral expansion, the virtual particle effects of these spin-1 mesons will induce a local lagrangian of type eq.( 7) with their contributions to \( L_i \). Our objective is to derive this lagrangian by completing path integration over \( L_\mu \) and \( R_\mu \) in \( \mathcal{L}_{Re} \). For this purpose, we divide \( \mathcal{L}_{Re} \) into two parts,

\[ \mathcal{L}_{Re} = (\mathcal{L}_{Re})_2 + (\mathcal{L}_{Re})_{3,4}, \]

where \((\mathcal{L}_{Re})_2\) contains the terms up to the second power of \( L_\mu \) and \( R_\mu \), and \((\mathcal{L}_{Re})_{3,4}\), the terms with third and fourth power of them. \((\mathcal{L}_{Re})_2\) can be rewritten as

\[ \begin{align*}
(\mathcal{L}_{Re})_2 &= \mathcal{L}^{(0)}(U) + < J^L_\mu L^\mu > + < J^R_\mu R^\mu > + < S^L_\mu(U, R)L^\mu > \\
&\quad - \frac{1}{2} < L_\mu M^{\mu\nu}_L L_\nu + R_\mu M^{\mu\nu}_R R_\nu >, 
\end{align*} \] (15)

where

\[ \mathcal{L}^{(0)}(U) = \frac{F^2}{16} < \nabla_\mu U \nabla_\mu U \uparrow >, \]

\[ J^L_\mu = -\frac{\sqrt{2}}{8} F^2 i U \nabla_\mu U \uparrow, \]

\[ J^R_\mu = -\frac{\sqrt{2}}{8} F^2 i U \uparrow \nabla_\mu U, \]

\[ S^L_\mu(U, R) = -\frac{F^2}{4} U R_\mu U \uparrow + 2\gamma \nabla^L_\nu(U \nabla_\nu R_\mu U \uparrow), \]

\[ M^{\mu\nu}_{L.R.} = -(\frac{g^2}{2} - 2\gamma) \nabla^L_{L.R.} g^{\mu\nu} - (\frac{F^2}{4} + \frac{m_0^2}{2}) g^{\mu\nu} + \frac{g^2}{2} \nabla^L_{L.R.} \nabla^L_{L.R.}. \]
Noting the symmetry of $L_\mu$ and $R_\mu$ in $\mathcal{L}_{Re}$, we have
\[ < S_\mu^L(U, R) L^\mu > = < S_\mu^R(U, L) R^\mu >, \]
where
\[ S_\mu^R(U, L) = -\frac{F^2}{4} U^\dagger L_\mu U + 2\gamma \nabla_\nu (U^\dagger \nabla_\nu L_\mu U). \]
The terms with the third and the fourth power are
\[ (\mathcal{L}_{Re})_{3,4} = \mathcal{L}_{Re} - (\mathcal{L}_{Re})_2 = (\mathcal{L}_{Re})_{3,4}(U, L, R). \]
We now evaluate both the exchange effects and the vertex effects of virtual spin-1 meson resonances (or $L$ and $R$) in $\mathcal{L}_{Re}$ and work out the desired effective lagrangian $\mathcal{L}_{eff}$ which contains pseudoscalar mesons only. The functional integral expression for $\mathcal{L}_{eff}$ is
\[ \exp \left\{ i \int d^4x \mathcal{L}_{eff}(U) \right\} = \frac{1}{\det(\delta J_\mu^L, \delta J_\mu^R)} \int [dL][dR] \exp \left\{ i \int d^4x (\mathcal{L}_{Re})_2 \right\}. \]
Owing to Gaussian integral formula, the functional integrations over $L$ and $R$ in above equation can be performed exactly, and $(\mathcal{L}_{Re})_{3,4}$ can be treated perturbatively.

From $(\mathcal{L}_{Re})_2$, the classical field equations of $L$ and $R$ read
\[ M_{\mu\nu}^L L_\nu = J_\mu^L + S_{\mu}^L(U, R), \]
\[ M_{\mu\nu}^R R_\nu = J_\mu^R + S_{\mu}^R(U, L). \]
(17)
Substituting eq. (16) into eq. (17), at $O(p^1)$ the solutions of eq. (17) are given as
\[ L^{(1)c}_\mu = k J_\mu^L = \frac{i}{\sqrt{2} g} \beta U \nabla_\mu U^\dagger, \]
\[ R^{(1)c}_\mu = k R_\mu^R = \frac{i}{\sqrt{2} g} U^\dagger \nabla_\mu U. \]
(18)
where
\[ k = -\frac{2}{m_0^2 + F^2}. \]  

For comparing with ref. [8] (similar to the quantity \( c \) in [8]) we define
\[ \beta = \frac{gF^2}{2(m_0^2 + F^2)}. \]

Furthermore, the solution of eq. (17) at \( O(p^3) \) are
\[ L_{\mu}^{(3)c} = L_{\mu}^{(1)c} + \alpha \Theta_{\mu}, \]
\[ R_{\mu}^{(3)c} = R_{\mu}^{(1)c} + \alpha \Omega_{\mu}, \]

where
\[ \alpha = -\frac{2}{m_0^2} \left( 1 - \frac{F^2}{m_0^2} \right), \]

and index \( c \) denote classical solution of eq. (17)
\[ \Theta_{\mu} = \frac{g^2}{2} - 2\gamma) \nabla^2 L_{\mu}^{(1)c} - \frac{g^2}{2} \nabla_{\nu} L_{\mu}^{L(1)c} + 2\gamma \nabla_{\nu} (U \nabla_{\nu} R_{\mu}^{(1)c} U^\dagger), \]
\[ \Omega_{\mu} = \frac{g^2}{2} - 2\gamma) \nabla^2 R_{\mu}^{(1)c} - \frac{g^2}{2} \nabla_{\nu} R_{\mu}^{R(1)c} + 2\gamma \nabla_{\nu} (U \nabla_{\nu} L_{\mu}^{L(1)c} U^\dagger). \]

Thus, with \( L^c \) and \( R^c \), we obtain the \( \mathcal{L}_{\text{eff}} \) to order \( p^4 \)
\[ \mathcal{L}_{\text{eff}} = (\mathcal{L}_{\text{eff}})_2 + (\mathcal{L}_{\text{eff}})_4 + O(p^6) \]
\[ = \mathcal{L}^{(0)}(U) + \frac{1}{2} < J_{\mu}^{L} L_{\mu}^{(3)c} + J_{\mu}^{R} R_{\mu}^{(3)c} > \]
\[ + \frac{1}{4} < S_{\mu}^{L} (U, R^c) L_{\mu}^{c} > + (\mathcal{L})_{3,4}(U, \frac{R_{\mu}^{c} U^\dagger}{2}, \frac{L_{\mu}^{c}}{2}) + O(p^6). \]  

Substituting eq. (18), (21) into the lagrangian (23), the contributions of lagrangian \( \mathcal{L}_{Re} \) (eq. (10)) to the coupling constants at the order \( p^2 \) can be found from the following expression,
\[ (\mathcal{L}_{\text{eff}})_2 = \frac{F^2}{16} \left( 1 - \frac{2\beta}{g} \left( 1 - \frac{\beta}{4g} \right) \right) < \nabla_{\mu} U \nabla^{\mu} U^\dagger >. \]
This is the lagrangian of kinetic energies for pseudoscalar meson after integrating out $\mathcal{V}_\mu$ and $\mathcal{A}_\mu$ in the Li model. The first two terms are similar to the original Li model (before the integration), however the third one $\frac{F^2}{16 \beta 2g^2} < \nabla_\mu U \nabla_\mu U^\dagger >$ is new. It comes from the mixed term of left-hand and right-hand fields coupling $< S^L_{\mu}(U, R)L^\mu >$ of $\mathcal{L}_{Re}(\text{eq.}(15))$. In the original Li model\cite{8}, $< S^L_{\mu}(U, R)L^\mu >$ contributes to the three-point vertex of pseudoscalar, vector and axial-vector mesons instead of a kinetic term. However, when the vector and axial-vector mesonic fields were integrated out, this interaction contributes to $(\mathcal{L}_{eff})_2$. In general, the form of the lagrangian should be different, since the lagrangian of Li model contains the vector and axial vector resonance mesons dynamical fields which mimic these dynamics in the low energy region.

To normalize the kinetic energy term of pseudoscalar mesons, we have

$$\frac{F^2}{f_\pi^2}(1 - \frac{2\beta}{g}(1 - \frac{\beta}{4g})) = 1. \quad (25)$$

Parameter $F$ can be fixed by eqs.(20) and (25) as long as $g$ and $f_\pi$ are known.

The following equations can be easily proved

$$\nabla^L_\mu(U \nabla_\nu U^\dagger) = \nabla_\mu U \nabla_\nu U^\dagger + U \nabla_\mu \nabla_\nu U^\dagger,$$

$$\nabla^R_\mu(U^\dagger \nabla_\nu U) = \nabla_\mu U^\dagger \nabla_\nu U + U^\dagger \nabla_\mu \nabla_\nu U.$$

Then, from eq.(23), we have

$$(\mathcal{L}_{eff})_4 = f_1 < \nabla_\mu U^\dagger \nabla_\nu U \nabla_\mu U^\dagger \nabla_\nu U > + f_2 < \nabla_\mu U^\dagger \nabla_\nu U \nabla_\mu U^\dagger \nabla_\nu U > + f_3 < \nabla_\mu \nabla_\nu U \nabla_\mu U^\dagger \nabla_\nu U > + f_4 < \nabla_\mu \nabla_\nu U \nabla_\mu U^\dagger \nabla_\nu U > + if_5 < F^{\mu\nu}_L \nabla_\mu U \nabla_\nu U^\dagger + F^{\mu\nu}_R \nabla_\mu U^\dagger \nabla_\nu U > + f_6 < F_{L\mu\nu} U F^{\mu\nu}_R U^\dagger > + f_7 < F_{R\mu\nu} F^{\mu\nu}_R + F_{L\mu\nu} F^{\mu\nu}_L >, \quad (26)$$

where $f_i(i = 1, 2, ..., 7)$ are coupling constants of $(\mathcal{L}_{eff})_4$, and they can be calculated explicitly from eq.(23)(18) and (21)(see below). At $O(p^4)$, EVRE, EVRV and EQCPM of Li model
have been absorbed by \( f_i (i = 1, 2, \ldots, 7) \). In order to transform eq. (26) into the standard lagrangian (4), the following \( SU(3) \) relation (2) should be used,

\[
< \nabla_\mu U^\dagger \nabla_\nu U \nabla^\mu U^\dagger \nabla^\nu U > \\
= -2 < \nabla_\mu U^\dagger \nabla^\mu U \nabla_\nu U^\dagger \nabla^\nu U > + \frac{1}{2} < \nabla_\mu U^\dagger \nabla^\mu U >^2 \\
+ < \nabla_\mu U^\dagger \nabla_\nu U > < \nabla^\mu U^\dagger \nabla^\nu U > .
\] (27)

In addition, using eq. (6) we obtain

\[
< \nabla_\mu \nabla_\nu U \nabla^\mu \nabla^\nu U > \\
= < \nabla_\mu U^\dagger \nabla^\mu U \nabla_\nu U^\dagger \nabla^\nu U > + i < F^\mu_\nu \nabla_\mu U \nabla_\nu U^\dagger + F^{\mu}_\nu \nabla^\mu U \nabla^\nu U > \\
- < F_{\mu\nu} U F^\mu_\nu U > + \frac{1}{2} < F_{\mu\nu} F^\mu_\nu > .
\] (28)

Inserting eq. (13)(26)(27) into eq. (23) we obtain the desired coupling constants \( L_i \) and \( H_1 \) as follows,

\[
L_1 = \frac{f_1}{2} \\
= \left( \frac{1}{k} - \frac{F^2}{8} \right) \left( \frac{1}{2} - \frac{2\gamma}{g^2} \alpha^2 \beta^2 \beta^3 \frac{1}{32g^2} + \frac{\beta^4}{128g^2} \right) \\
+ \frac{\gamma \beta}{2g} (1 - \frac{\beta}{g})^2 (1 - \frac{\beta}{2g}) + \frac{\gamma}{8} (1 - \frac{\beta}{g})^4,
\] (29)

\[
L_2 = 2L_1,
\] (30)

\[
L_3 = -2f_1 + f_2 + f_3 + f_4 \\
= -\left( \frac{1}{k} - \frac{F^2}{8} \right) \left( 1 - \frac{3\gamma}{g^2} \alpha^2 \beta^2 \beta^3 \frac{1}{4g^2} + \frac{\gamma \beta^2}{4g^2} + \frac{3\beta^3}{16g^2} - \frac{3\beta^4}{64g^2} \right) \\
+ \frac{\gamma \beta}{2g} (1 - \frac{\beta}{g})^2 - \frac{4\gamma \beta}{g} (1 - \frac{\beta}{g})^2 (1 - \frac{\beta}{2g}) - \frac{\gamma (1 - \frac{\beta}{g})^4}{4g^2},
\] (31)

\[
L_9 = -f_5 - f_3 - f_4 \\
= -\left( \frac{1}{k} - \frac{F^2}{8} \right) \frac{2\alpha \gamma \beta^2}{g^2} + \frac{\gamma \beta^2}{4g^2} + \frac{\gamma (1 - \frac{\beta}{g})^2 + g\beta}{8} (1 - \frac{\beta}{2g}),
\] (32)

\[
L_{10} = f_6 - f_3
\]
\[
\begin{align*}
\frac{1}{k} - \frac{F^2}{8} &\left(1 - \frac{2\gamma}{g^2}\right) \alpha \beta^2 + \frac{\gamma \beta^2}{4g^2} - \frac{g \beta}{2} - \frac{\gamma}{2} (1 - \frac{\beta}{g})^2, \\
H_1 &\equiv f_7 + \frac{f_3}{2} = -\frac{L_{10}}{2} - \frac{g^2}{16},
\end{align*}
\]

(33)  

(34)

| \(\chi PT[L_i^i(m_\rho)]\) | Li Model |  
|---|---|---|---|---|
|  | \(L_i^E\) | \(L_i^V\) | \(L_i^Q\) | Total |
| \(L_1\) | 0.7 ± 0.3 | 0.84 | -0.76 | 0.79 | 0.87 |
| \(L_2\) | 1.3 ± 0.7 | 1.56 | -1.49 | 1.58 | 1.75 |
| \(L_3\) | -4.4 ± 2.5 | -3.54 | 2.59 | -3.17 | -4.13 |
| \(L_9\) | 6.9 ± 0.7 | 3.2 | -2.8 | 6.3 | 6.7 |
| \(L_{10}\) | -5.2 ± 0.3 | -3.4 | 1.57 | -3.17 | -5.0 |
| \(H_1\) | | | | | -5.2 |

Table 1: \(L_i^i\) in units of \(10^{-3}\), \(\mu = m_\rho\). The \(L_i^E\), \(L_i^V\) and \(L_i^Q\) denote effects of spin -1 meson resonances exchange, effects of spin -1 meson resonances vertices and effects of quark constituent of pseudoscalar meson contribute to \(L_i^i(i = 1, 2, 3, 9, 10)\) respectively.

Above analytical expressions of \(L_i(i = 1, 2, 3, 9, 10)\) and \(H_1\) are main results of this paper. Remember that the Li model’s inputs are \(f_\pi = 186\,MeV, g = 0.35\) and \(m_\rho = 770\,MeV\). Then due to eq.(19)(20)(22)(23), \(F, \beta, \alpha, k\) are fixed and \(\gamma = \frac{N_c}{3(4\pi)^2}, m_0^2 = m_\rho^2 g^2\). The numerical results of \(L_i(i = 1, 2, 3, 9, 10)\) and \(H_1\) are shown in Table 1, which are in good agreement with their values in \(\chi PT\) with renomalization scale parameter \(\mu \sim m_\rho\). Thus, we have successfully derived \(\chi PT\) Lagrangian from Li model without any additional assumptions. This indicates that the low energy limit of Li model is equivalent to \(\chi PT\).

In the rest of this section, we shall analyse contribution of the EQCPM, EVRE and EVRV to chiral coupling constants \(L_i(i = 1, 2, 3, 9, 10)\) respectively. Consequently, \(L_i\) of Li model are expressed as

\[
L_i = L_i^Q + L_i^E + L_i^V
\]

where \(L_i^Q, L_i^E\) and \(L_i^V\) stand for contribution of EQCPM, EVRE and EVRV respectively.
For this purpose, the lagrangian (14) is separated into three parts,

\[ \mathcal{L}_{Re} = \mathcal{L}_{Re}^Q + \mathcal{L}_{Re}^E + \mathcal{L}_{Re}^V \]

where

\[ \mathcal{L}_{Re}^E = \frac{F^2}{16} \left< D_\mu U D_\mu U^\dagger - \nabla_\mu U \nabla^\mu U^\dagger \right> - \frac{g^2}{16} \left< 2L_{\mu\nu} L^{\mu\nu} + 2R_{\mu\nu} R^{\mu\nu} \right> - \frac{3i}{\sqrt{2}} \gamma \left< \nabla_\mu U \nabla_\nu U^\dagger L^{\mu\nu} + \nabla_\mu U^\dagger \nabla_\nu U R^{\mu\nu} \right> \]

\[- \frac{g^2}{4\sqrt{2}} \left< L_{\mu\nu} F_{\mu\nu}^{L} + 2R_{\mu\nu} F_{\mu\nu}^{R} \right> + \frac{1}{4} m_0^2 < L_\mu L^\mu + R_\mu R^\mu >, \]

\[ \mathcal{L}_{Re}^Q = \frac{F^2}{16} \left< \nabla_\mu U \nabla^\mu U^\dagger \right> + \frac{\gamma}{2} \left< \nabla_\mu U \nabla^\mu U \nabla_\nu U^\dagger \right> - \frac{3i}{2} \gamma \left< \nabla_\mu U^\dagger \nabla_\nu U F_{\mu\nu}^{L} + \nabla_\mu U \nabla_\nu U^\dagger F_{\mu\nu}^{R} \right> \]

\[- \frac{\gamma}{4} \left< \nabla_\mu U \nabla_\nu U^\dagger \nabla^\mu U \nabla^\nu U^\dagger - 2\nabla_\mu U \nabla^\mu U^\dagger \nabla_\nu U \nabla^\nu U^\dagger \right> >, \]

\[ \mathcal{L}_{Re}^V = \mathcal{L}_{Re} - \mathcal{L}_{Re}^Q - \mathcal{L}_{Re}^E. \]

There are no meson resonance fields in \( \mathcal{L}_{Re}^Q \) which describe the effects of quark constituent of pseudoscalar meson. But there are one or two spin-1 meson resonances in \( \mathcal{L}_{Re}^E \) which describe the effects of virtual resonances exchange (it is similar with Ref. [4]). The other terms have been included in \( \mathcal{L}_{Re}^V \) which describe the effects of virtual resonances vertices. Three parts are all invariant under global \( U(3)_L \times U(3)_R \) chiral transformation consequently. The EQCPM contributes to chiral coupling constants \( L_{Q}^i (i = 1, 2, 3, 9, 10) \) and EVRE contribute to chiral coupling constants \( L_{E}^i (i = 1, 2, 3, 9, 10) \) can be expressed as (see eq. (29)-(33)),

\[ L_1^Q = \frac{\gamma}{8}; \quad L_2^Q = \frac{\gamma}{4}; \quad L_3^Q = -\frac{\gamma}{2}; \]

\[ L_9^Q = \gamma; \quad L_{10}^Q = -\frac{\gamma}{2}, \]

and

\[ L_1^E = \frac{1}{k} - \frac{F^2}{8} \left( \frac{1}{2} - \frac{2\gamma}{g^2} \frac{\alpha\beta^2}{4} + \frac{\gamma^2}{8g^2} + \frac{\gamma \beta}{2g} \left( 1 - \frac{\beta}{2g} \right) \right). \]

16
\[ L_2^E = 2L_1^E, \]
\[ L_3^E = -(\frac{1}{k} - \frac{F^2}{8})(1 - \frac{3\gamma}{g^2}) \frac{3\alpha\beta^2}{4} - \frac{3\gamma\beta^2}{4g^2} + \frac{\gamma}{2} - \frac{4\gamma\beta}{g}(1 - \frac{\beta}{2g}), \]
\[ L_9^E = -(\frac{1}{k} - \frac{F^2}{8}) \frac{2\alpha\gamma\beta^2}{g^2} + \frac{\gamma\beta^2}{4g^2} + \frac{g\beta}{8}(1 - \frac{\beta}{2g}), \]
\[ L_{10}^E = \left(\frac{1}{k} - \frac{F^2}{8}\right)(\frac{1}{4} - \frac{2\gamma}{g^2}) \alpha\beta^2 + \frac{\gamma\beta^2}{4g^2} - \frac{g\beta}{8}. \]

The numerical results of \( L_i^Q, L_i^E \) and \( L_i^V (i = 1, 2, 3, 9, 10) \) have been shown in table 1. As emphasisation in Introduction and section 2, both the EQCPM and EVRV also contribute to \( L_i (i = 1, 2, 3, 9, 10) \) besides the EVRE when spin -1 meson resonances were integrated out. Numerically, the contribution of \( L_i^Q + L_i^V \) to \( L_i (i = 1, 2, 3) \) are small, and \( L_1^E, L_2^E, L_3^E \) are dominant. But \( L_9^Q + L_9^V \) and \( L_{10}^Q + L_{10}^V \) are close to \( L_9^E \) and \( L_{10} \) respectively.

5. Discussion

It is necessary for meson theories in intermediate energy scale to examine if its low energy limit is equivalent to \( \chi PT \) or not, because \( \chi PT \) is the rigorous theory of QCD at low energies. In this paper, we have done so for Li model, and proved that Li model is satisfied of this necessary condition. In the derivation of \( L_i (i = 1, 2, 3, 9, 10) \) and \( H_1 \), all EVRE, EVRV and EQCPM at \( O(p^4) \) are taken into account. The calculations presented in the paper are analytical and manifest without additional assumptions. They are beyond the similar calculations in refs.[4, 6]. The authors of ref.[6] addressed that there are five conditions abstracted from QCD, which should be satisfied for eligible chiral effective theories with meson resonances. Since we have shown that the five constants \( L_i (i = 1, 2, 3, 9, 10) \) of Li model are consistent with \( \chi PT \), we conclude that Li model satisfies these conditions. In refs.[8, 9, 10, 11, 12], various aspects of the meson physics have been extensive studied. Our investigations in this paper have proved further that the dynamics of pseudoscalar, vector and
axial-vector mesons in Li model is eligible. Consequently the successes of the phenomenology of Li model become understandable more manifestly from QCD point of view. Furthermore, there are some other models [13] in which the $1^-$ and $1^+$ meson resonance fields are treated as ordinary vector and axial-vector fields too (instead of as antisymmetric tensor fields like in Ref[4]). Therefore the low energy limit of those models can be derived and examined by our method presented in this paper. The studies on them will be presented elsewhere.

In the electromagnetic and weak interaction of mesons, the vector and axial-vector mesons play essential role through VMD and AVMD. This was originally illustrated by Sakurai[14], and enjoys considerable phenomenological support. In the Li model, this is no longer an input, but a natural consequence of the model. Thus, our success in this paper indicates that VMD and AVMD are finely compatible with $\chi$PT.

Finally, we like to argue that in the Li model, besides the exchange effects of virtual spin 1 meson resonances, the vertex effects of them and the effects of quark constitutes of pseudoscalar mesons contribute to $L_i(i = 1, 2, 3, 9, 10)$ at $O(p^4)$ too. Therefore, in principle, the compatibility of VMD, AVMD with $\chi$PT in the chiral theories do not require that the values of $L_i(i = 1, 2, 3, 9, 10)$ must be dominated by $V + A$ exchange effects only (like the case of [4]). In the view of the model presented in [4], $L_i(i = 1, 2, 3, 9, 10)$ are dominated by spin 1 meson resonance exchanges. The analysis presented in this paper indicates that the result of [4] is model dependent.

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References

[1] R.Daschen and M.Weinstein, Phys. Rev., 183, 1261(1969). S.Weinberg, Physica, A 96, 327(1979).

[2] J.Gasser and H.Leutwyler, Ann. Phys., 158, 142(1984). J.Gasser and H.Leutwyler, Nucl. Phys., B 250, 465(1985); ibid., 517; ibid., 539.

[3] H.Leutwyler, Ann. Phys., 235, 165(1994).

[4] G.Ecker, J.Gasser, A.Pich and E.de Rafel, Nucl. Phys., B 321, 311(1989).

[5] J.F.Donoghue, C.Ramirez and G.Valencia, Phys. Rev. D 39, 1947(1989).

[6] G.Ecker, J.Gasser, H.Leutwyler, A.Pich and E.de Rafel, Phys. Lett., B223, 425(1989).

[7] J.F.Donoghue and B.R.Holstein, Phys. Rev., D 40, 2378(1989).

[8] B.A.Li, Phys. Rev., D 52, 5165(1995).

[9] B.A.Li, Phys. Rev., D 52, 5184 (1995).

[10] B.A.Li, Phys. Rev., D 55, 1425(1997).

[11] B.A.Li, Phys. Rev., D 55, 1436(1997).

[12] D.N.Gao, B.A.Li and M.L.Yan, Phys.Rev., D56, 4115(1997).
[13] Y. Nambu and G. Jona-Lasinio, Phys. Rev., 122, 345 (1961). M. Bando et al., Phys. Rep., 164, 217 (1988), U.-G. Meissner, Phys. Rep., 161, 213 (1988), J. Bijnens, Nucl. Phys., B390 501 (1993) J. Bijnens et al., Phys. Rep., 265, 369 (1996).

[14] J.J. Sakurai, *Currents and mesons*, University of Chicago press, Chicago, (1969).