An initial value problem arising in mechanics

Abstract We study an initial value problem for a system consisting of an integer order and a distributed-order fractional differential equation describing forced oscillations of a body attached to a free end of a light viscoelastic rod. An explicit form of a solution for a class of linear viscoelastic solids is given in terms of a convolution integral. Restrictions on storage and loss moduli following from the second law of thermodynamics play the crucial role in establishing the form of the solution. Some previous results are shown to be special cases of the present analysis.

Keywords Distributed-order fractional differential equation · Fractional viscoelastic material · Forced oscillations of a body

1 Introduction

Let us consider the motion of a system shown in Fig. 1 consisting of a light viscoelastic rod clamped at one side with a body, moving translatory, attached at the rod’s free end.

Related to the system shown, we study (in the dimensionless form) the initial value problem given by

\[ \int_0^1 \phi_\sigma (\gamma) \, 0D_\gamma^\alpha \sigma (t) \, d\gamma = \int_0^1 \phi_\varepsilon (\gamma) \, 0D_\gamma^\alpha \varepsilon (t) \, d\gamma, \quad t > 0, \] (1)

\[-\sigma (t) + F (t) = \frac{d^2}{dt^2} \varepsilon (t), \quad t > 0, \] (2)

\[ \sigma (0) = 0, \quad \varepsilon (0) = 0, \quad \frac{d}{dt} \varepsilon (0) = 0. \] (3)
In (1)–(3), we use $\sigma$ to denote the stress, $\varepsilon$ denotes the strain, and $F$ denotes the force acting at the free end of a rod. The left Riemann–Liouville fractional derivative of order $\gamma \in (0, 1)$ is defined by

$$0D^\gamma_t y(t) := \frac{d}{dt} \left( \frac{t^{-\gamma}}{\Gamma(1-\gamma)} * y(t) \right), \quad t > 0,$$

see [12]. The Euler gamma function is denoted by $\Gamma$, and $*$ denotes the convolution: $(f * g)(t) := \int_0^t f(\tau) g(t-\tau) d\tau, t \in \mathbb{R}$, if $f, g \in L^1_{\text{loc}}(\mathbb{R})$ and supp $f, g \subset [0, \infty)$. The displacement of an arbitrary point of a rod that is at the initial moment at the position $x$ is

$$u(x, t) = x \varepsilon(t), \quad t > 0, \quad x \in [0, 1],$$

see [1]. As the matter of fact, we used notation $\sigma(t) = \sigma(1, t)$ and $\varepsilon(t) = \varepsilon(1, t), t > 0$.

Equation (1) represents a constitutive equation of a viscoelastic rod, (2) is the equation of motion of a body attached to a free end of a rod, and (3) represents the initial conditions.

Constitutive models containing fractional derivatives may be successfully used to describe real materials in which the memory effects are important. The early studies of application of fractional calculus in viscoelasticity are by Gerasimov [7] and Caputo [5]. We refer to [4], where the thermodynamical restrictions on the parameters appearing in the fractional Zener model were presented. For the study of waves in viscoelastic materials of fractional type, see [9]. The extensive review of the application of the fractional calculus in viscoelasticity is by Rossikhin and Shitikova [11]. Example of the derivation of the fractional constitutive equations in the framework of the rheology can be found in, for example, [13], while for the determination of the fractional model parameters using experimental data, see for example [8,10].

The derivation of the system (1)–(3) is given in [1], where the special case of (1) is considered when the constitutive functions $\phi_\sigma$ and $\phi_\varepsilon$ have the form

$$\phi_\sigma(\gamma) = a\gamma \quad \text{and} \quad \phi_\varepsilon(\gamma) = c\delta(\gamma) + b\gamma, \quad 0 < a \leq b, \quad c > 0,$$

where $\delta$ is the Dirac delta distribution. In the present work, we allow constitutive functions (or distributions) $\phi_\sigma$ and $\phi_\varepsilon$ to be arbitrary satisfying Condition 1 and Assumption 1 that will be specified later in Sects. 2 and 3, respectively. To be physically admissible (1) must satisfy two conditions: one being mathematical, the other being physical. The first condition that is essential on the level of generality treated in the work requires that to real forcing ($\sigma$ is a real-valued function of real variable), there corresponds a real response $\varepsilon$ ($\varepsilon$ is a real-valued function of real variable). This mathematical condition that we believe is new is precisely formulated as (i) of Condition 1. The second condition is physical and requires that in any closed deformation cycle, there is a dissipation of energy. This is the formulation of the second law of thermodynamics for isothermal processes. This condition is stated in its equivalent form as (ii) of Condition 1.

2 Analysis of the problem

In the following, we use the Laplace transform method. The Laplace transform is defined by

$$\tilde{f}(s) = \mathcal{L}[f(t)](s) := \int_0^\infty f(t) e^{-st} dt, \quad \text{Re} s > k,$$