Trajectory attractors for the Sun–Liu model for nematic liquid crystals in 3D

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Abstract
In this paper we prove the existence of a trajectory attractor (in the sense of Chepyzhov and Vishik) for a nonlinear PDE system obtained from a 3D liquid crystal model accounting for stretching effects. The system couples a nonlinear evolution equation for the director \( d \) (introduced in order to describe the preferred orientation of the molecules) with an incompressible Navier–Stokes equation for the evolution of the velocity field \( u \). The technique is based on the introduction of a suitable trajectory space and of a metric accounting for the double-well type nonlinearity contained in the director equation. Finally, a dissipative estimate is obtained by using a proper integrated energy inequality. Both the cases of (homogeneous) Neumann and (non-homogeneous) Dirichlet boundary conditions for \( d \) are considered.

Mathematics Subject Classification: 35D30, 35B41, 35K45, 35Q30, 76A15

1. Introduction

In this paper we prove the existence of a trajectory attractor for the following PDE system

\[
\begin{align*}
\frac{\partial u}{\partial t} + \text{div}(u \otimes u) + \nabla p &= \text{div}(\nu(\nabla u + \nabla^T u)) - \text{div}(\nabla d \otimes \nabla d) \\
-\text{div}(\alpha(\Delta d - \nabla_d W(d)) \otimes d - (1 - \alpha)d \otimes (\Delta d - \nabla_d W(d))) + h, \\
\frac{\partial d}{\partial t} + u \cdot \nabla d - ad \cdot \nabla u + (1 - \alpha)d \cdot \nabla^T u &= (\Delta d - \nabla_d W(d)), \\
\text{div}(u) &= 0,
\end{align*}
\]

in \( \Omega \times (0, T) \), where \( \Omega \) is a bounded smooth domain in \( \mathbb{R}^3 \).

The first equation is a momentum balance ruling the evolution of the velocity field \( u \) (\( p \) denotes the pressure of the system and \( h \) is an external body force), relation (1.3) represents the incompressibility constraint, while (1.2) describes the dynamics of the director field \( d \),
which represents here a vector pointing in the preferred direction from the molecules at a neighbourhood of any point of our domain. The nonlinear function \( W \) stands for a relaxation of a constraint that should be imposed on the unitary vector \( d \), whose modulus should be equal to 1. In order to relax this non-convex constraint, we introduce the double-well potential \( W \), which is a regular potential with some coercivity properties (see the following section 2.1 for the precise assumptions on \( W \)). For example the classical double well potential \( W(d) = ((|d|^2 - 1)^2 \) is included in our analysis, but also a more general growth is admitted. The constant \( \nu \) is a positive viscosity coefficient and \( \alpha \in [0, 1] \) is a parameter related to the shape of the liquid crystal molecules. For instance, the spherical, rod-like and disc-like liquid crystal molecules correspond to the cases \( \alpha = \frac{1}{2}, 1 \) and 0, respectively.

Concerning the notation, \( \nabla_d \) represents the gradient with respect to the variable \( d \). \( \nabla_d \nabla d \) denotes the \( 3 \times 3 \) matrix whose \((i, j)\)-th entry is given by \( \nabla_i d \cdot \nabla_j d \), for \( i \leq j \leq 3 \), and \( \otimes \) stands for the usual Kronecker product, i.e., \( (u \otimes w)_{ij} := u_i w_j \), for \( i, j = 1, 2, 3 \). Finally, \( \nabla^T \) indicates the transpose of the gradient.

Equations (1.1)–(1.3) come from a model introduced in [30] in order to describe the evolution of a liquid crystal substance under constant temperature (i.e. in the isothermal case). This system is obtained as a correction to a simplification of the celebrated Leslie–Ericksen evolution of a liquid crystal substance under constant temperature (i.e. in the isothermal case).

Concerning the notation, \( \nabla_d \) represents the gradient with respect to the variable \( d \). \( \nabla_d \nabla d \) denotes the \( 3 \times 3 \) matrix whose \((i, j)\)-th entry is given by \( \nabla_i d \cdot \nabla_j d \), for \( i \leq j \leq 3 \), and \( \otimes \) stands for the usual Kronecker product, i.e., \( (u \otimes w)_{ij} := u_i w_j \), for \( i, j = 1, 2, 3 \). Finally, \( \nabla^T \) indicates the transpose of the gradient.

The resulting model (1.1)–(1.3) has been subsequently analysed both from the point of view of existence of strong solutions and of their long-time behaviour in the paper [31], where (as in [30]) the authors explicitly manifest the impossibility of proving the existence of solutions for a standard weak formulation of the problem due to the nonlinearity of the stretching term and of the lack of maximum principle for equation (1.2), and so of an \( L^\infty \) -estimate for \( d \).

In [4], properly choosing the space of the test functions in the weak momentum equation, the existence of well-defined weak solutions for system (1.1)–(1.3) is rigorously derived and an integrated energy inequality is obtained. It is worth noting that the uniqueness of such solutions in the 3D case, but also the proof of regularizing effects even in the 2D case, are not known yet, while the existence of weak solutions for the corresponding non-isothermal system has been recently proved in [13].

These results contained in the paper [4] are our starting point in order to perform the analysis on the long-time behaviour of solutions. As in [4] we consider both Neumann boundary conditions for \( d \) (see, e.g., [24] where it is pointed out that the Neumann boundary conditions for \( d \) are also suitable for the implementation of a numerical scheme) and non-homogeneous time-dependent Dirichlet ones (see also [2, 8, 9, 17, 18] for other contributions concerning liquid crystal systems without stretching terms endowed with a non-homogeneous time-dependent Dirichlet boundary condition), while for \( u \) we take into account only homogeneous Dirichlet boundary conditions. For the resulting Cauchy boundary value problem we prove the existence of a trajectory attractor in the sense of Chepyzhov and Vishik (see [5, 6]). The main components in the proofs are the dissipative estimates (see propositions 1
and 3) which are based on the use of a proper integral Gronwall lemma (see lemma 2), which was not needed, e.g. in [4], where the existence of solutions for the Sun–Liu model was proved in finite time intervals \([0, T]\).

We point out that, due to the lack of uniqueness of solutions, the choice of the notion of attractor is essential. Indeed, there are two main approaches when one deals with dissipative systems without uniqueness (see also [11] for a nonstandard analysis method). The first one is based on the theory of global attractors for semigroups of multi-valued maps (see [3, 25, 26] and also, for 3D incompressible Navier–Stokes, [1, 7, 19, 23] and references therein). The second more geometric approach consists of working in a phase space made of trajectories with the translation semigroup acting on them. Since the translation semigroup is single-valued, one can then rely on the results from the classical theory of attractors (see [5, 6] and also [14, 29]). In this paper we apply the second approach which seems more effective when the external forces are time dependent.

We essentially prove two types of results. The first one leads to a ‘weaker’ definition of trajectory attractor, but it holds true for quite general potentials \(W\). The second one leads to the standard definition of the trajectory attractor in the sense of Chepyzhov and Vishik, but it holds true only for polynomially fast growing potentials \(W\). We point out that the consideration of more general potentials, representing the configuration energy of the crystal, with respect to the standard Ginzburg–Landau double well \(W(d) = (|d|^2 - 1)^2\) can be useful as a possible better approximation of the constraint \(|d| = 1\).

In the first case, in order to prove the existence of the trajectory attractor under quite general assumptions on the potential \(W\) (a \(C^2\) function which is the sum of a convex and ‘coercive’ part and of a possibly non-convex part with Lipschitz continuous derivative), we generalize the result [6, theorem 3.1] and we show that it is not necessary to prove the closure of the space of trajectories in the local topology in order to obtain the existence of the trajectory attractor. This is indeed one of the main contribution of this paper (see theorem 3) and it could also be useful for the analysis of the long-time behaviour of other nonlinear PDE systems arising in different contexts. Moreover, we show in theorem 3 that the closure property not only better characterizes the trajectory attractor, but we can prove it for our system under more restrictive assumptions on the potential, which, however, are still satisfied by the classical double-well potential \(W(d) = (|d|^2 - 1)^2\).

In the second case, the trajectory space is instead defined in order to take into account the polynomial growth assumed on the potential \(W\). In this setting, we can immediately prove the closure of the trajectory space, leading to the standard definition of the trajectory attractor in the sense of Chepyzhov and Vishik, without any adjoint request on \(W\) (see theorem 4). We note that in both cases the metric introduced on the subset of the trajectory space (suggested by the energy estimate) explicitly depends on the potential \(W\). This turns out to be meaningful in other nonlinear models, such as for Penrose–Fife phase transition systems (see, e.g., [28] where the phase space explicitly depended on the nonlinearities of the problem as well).

Regarding other contributions in the literature on the long-time behaviour of solutions for this system accounting for stretching terms, we can quote two recent papers: [16], where the authors prove the existence of a finite-dimensional global attractor in the 2D case and [27] in which the authors prove—via Łojasiewicz–Simon techniques—the convergence of any weak solution to a single stationary state under suitable conditions on the data, which are different in the 2D and 3D cases.

**Structure of the paper.** We split the rest of the paper into three parts: in sections 2 and 3 we prove the existence of the trajectory attractor for (1.1)–(1.3) in the case of homogeneous Neumann and non-homogeneous Dirichlet boundary conditions for \(d\), respectively. Finally,
in the last section 4, some further properties of the trajectory attractor are studied. More specifically, in section 2.1, we introduce some notation and recall the main results concerning system (1.1)–(1.3), which are proved in [4] and regarding the general theory of trajectory attractors introduced in [5]. Sections 2.2, 2.3 and 3 are devoted to the main results of the paper (theorems 3, 4 and 5), where the existence of the trajectory attractor for (1.1)–(1.3) is established under different functional settings, assumptions on the potential and boundary conditions for \( d \) (homogeneous Neumann or non-homogeneous Dirichlet).

2. The case of homogeneous Neumann boundary conditions for \( d \)

In this section we deal with a suitable weak formulation of the PDE system (1.1)–(1.3) coupled with Neumann homogeneous boundary conditions for \( u \). First, in section 2.1, we introduce some notation and preliminary results which we recall for reader’s convenience, then, in sections 2.2 and 2.3, we state and prove our main results: the existence of the trajectory attractor under two different assumptions on the potential \( W \) in (1.2).

2.1. Notation and preliminaries

Let us introduce the classical Hilbert spaces for the Navier–Stokes equation

\[
G_{\text{div}} := \{ u \in C_0^\infty(\Omega)^3 : \text{div}(u) = 0 \}\] and

\[
V_{\text{div}} := \{ u \in H_0^1(\Omega)^3 : \text{div}(u) = 0 \}.
\]

We denote by \( (\cdot, \cdot) \) and \( \| \cdot \| \) the scalar product and the norm, respectively, both in \( L^2(\Omega) \) and in \( L^2(\Omega)^3 \). We also set \( V := H^1(\Omega)^3 \) and the duality between a Banach space \( X \) and its dual \( X' \) will be denoted by \( \langle \cdot, \cdot \rangle \). The space \( V_{\text{div}} \) is endowed with the scalar product

\[
(u, v)_{V_{\text{div}}} := (\nabla u, \nabla v), \quad \forall u, v \in V_{\text{div}}.
\]

We shall also use the first eigenvalue of the Stokes operator \( A \) with a no-slip boundary condition. Recall that \( A : D(A) \subset G_{\text{div}} \rightarrow G_{\text{div}} \) is defined as \( A := -\Delta \) with domain \( D(A) = H^2(\Omega)^3 \cap \text{div} \), where \( P : L^2(\Omega)^3 \rightarrow G_{\text{div}} \) is the Leray projector. Notice that we have \( (Au, v) = (u, v)_{V_{\text{div}}} \) for all \( u \in D(A) \) and for all \( v \in V_{\text{div}} \), that \( A^{-1} : G_{\text{div}} \rightarrow G_{\text{div}} \) is a self-adjoint compact operator in \( G_{\text{div}} \). Thus, according with classical spectral theorems, it possesses a sequence \( \{ \lambda_j \} \) of eigenvalues satisfying \( 0 < \lambda_1 \leq \lambda_2 \leq \cdots \) and \( \lambda_j \rightarrow \infty \), and a family \( \{ u_j \} \subset D(A) \) of eigenfunctions which is orthonormal in \( G_{\text{div}} \).

Let \( X \) be a Banach space and \( 1 \leq p < \infty \). We shall denote by \( L^p_{\text{loc}}(0, \infty; X) \) the space of translation bounded functions in \( L^p_{\text{loc}}((0, \infty); X) \). We recall that \( f \in L^p_{\text{loc}}(0, \infty; X) \) iff

\[
\| f \|_{L^p_{\text{loc}}(0, \infty; X)} := \sup_{t \geq 0} \| f(t) \|_X < \infty.
\]

Furthermore, \( L^p_{\text{loc}, w}(0, \infty; X) \) will stand for the space of functions in \( L^p_{\text{loc}}((0, \infty); X) \) endowed with the local weak convergence topology, i.e. a sequence \( \{ f_n \} \) converges to \( f \) in \( L^p_{\text{loc}, w}(0, \infty; X) \) iff \( f_n \rightharpoonup f \) weakly in \( L^p(0, M; X) \), for every \( M > 0 \).

We are ready now to recall from [4] the weak formulation of the PDE system (1.1)–(1.3) which we complement with the following boundary and initial conditions

\[
\begin{align*}
\Delta u &= 0, &\text{on } \Gamma \times (0, T), \\
\partial_n d &= 0, &\text{on } \Gamma \times (0, T), \\
u(0) &= u_0, &d(0) = d_0, \quad \text{in } \Omega.
\end{align*}
\]
**Definition 1.** A couple \( w = [u, d] \) is a weak solution to system (1.1)–(2.5) corresponding to the initial data \( u_0, d_0 \) if \( u, d \) are such that

\[
\begin{align*}
\mathbf{u} & \in L^\infty_{\text{loc}}([0, \infty); G_{\text{div}}) \cap L^2_{\text{loc}}([0, \infty); V_{\text{div}}), \\
\mathbf{u}_t & \in L^2_{\text{loc}}([0, \infty); W^{-1.3/2}(\Omega)^3), \\
\mathbf{d} & \in L^2_{\text{loc}}([0, \infty); V) \cap L^2_{\text{loc}}([0, \infty); H^2(\Omega)^3), \\
W(\mathbf{d}) & \in L^\infty_{\text{loc}}([0, \infty); L^1(\Omega)), \\
\mathbf{d}_t & \in L^2_{\text{loc}}([0, \infty); L^{3/2}(\Omega)^3),
\end{align*}
\]  
(2.7) \hspace{1cm} \text{for} \hspace{1cm} u, \hspace{1cm} d \hspace{1cm} \text{satisfy the boundary and initial conditions (2.5), (2.6), the equation (1.2) is satisfied a.e. in} \hspace{1cm} \Omega \times (0, T) \hspace{1cm} \text{and we have}

\[
\begin{align*}
\langle u_t, \psi \rangle - \int_\Omega u \otimes u : \nabla \psi + \int_\Omega v(\nabla u + \nabla^T u) : \nabla \psi &= \int_\Omega (\nabla d \otimes \nabla d) : \nabla \psi + \alpha \int_\Omega (\Delta d - \nabla d W(\mathbf{d})) \otimes d : \nabla \psi \\
&\quad - (1 - \alpha) \int_\Omega d \otimes (\Delta d - \nabla d W(\mathbf{d})) : \nabla \psi + \langle h, \phi \rangle,
\end{align*}
\]  
(2.8) \hspace{1cm} \text{for a.e.} \hspace{1cm} t > 0 \hspace{1cm} \text{and for every} \hspace{1cm} \psi \in W^{1,3}_0(\Omega)^3 \hspace{1cm} \text{with} \hspace{1cm} \text{div} (\mathbf{r}) = 0.

In [4] the existence of a global in time weak solution is proved under the following assumptions on the potential \( W \)

\[
\begin{align*}
W & \in C^2(\mathbb{R}^3), \quad W \geq 0, \\
W & = W_1 + W_2, \quad \text{with} \ W_1 \text{ convex and} \ W_2 \in C^{1,1}(\mathbb{R}^3), \\
\mathbf{h} & \in L^2_{\text{loc}}([0, \infty); V_{\text{div}}),
\end{align*}
\]  
(2.12) \hspace{1cm} \text{and on the external force} \hspace{1cm} \mathbf{h}

\[
\mathbf{h} \in L^2_{\text{loc}}([0, \infty); V_{\text{div}}).
\]  
(2.13)

Namely, from [4] we recall the following.

**Theorem 1.** Suppose that (2.12)–(2.14) are satisfied and let the initial data be such that

\[
\mathbf{u}_0 \in G_{\text{div}}, \quad d_0 \in V, \quad W(d_0) \in L^1(\Omega).
\]  
(2.15)

Then, problem (1.1)–(1.3), (2.4)–(2.6) admits a global in time weak solution \( w := [u, d] \) on \([0, \infty)\) corresponding to \( u_0, d_0 \) and satisfying the following energy inequality

\[
\mathcal{E}(w(t)) + \int_t^s \left( \| \Delta d - \nabla d W(\mathbf{d}) \|^2 + \nu \| \nabla u \|^2 \right) dt \leq \mathcal{E}(w(s)) + \int_s^t \langle h, u \rangle d\tau,
\]  
(2.16)

for all \( t \geq s \), for a.e. \( s \in (0, \infty) \), including \( s = 0 \), where

\[
\mathcal{E}(w) := \frac{1}{2} \| u \|^2 + \frac{1}{2} \| \nabla d \|^2 + \int_\Omega W(\mathbf{d}), \quad w = [u, d].
\]  
(2.17)

**Remark 1.** The regularity of the test function \( \phi \) can be justified by the regularity properties of the solution which imply that

\[
\mathbf{u} \otimes u, \quad \nabla d \otimes \nabla d, \quad (\Delta d - \nabla d W(\mathbf{d})) \otimes d \in L^2_{\text{loc}}([0, \infty); L^{3/2}(\Omega)^{3\times 3}).
\]

Their divergence is therefore in \( L^3_{\text{loc}}([0, \infty); W^{-1.3/2}(\Omega)^3) \).

Let us resume some basic definitions and results from the theory of trajectory attractors for non-autonomous evolution equations due to Chepyzhov and Vishik (see [5, chapters XI and XIV] and [6] for details).
Consider an abstract nonlinear non-autonomous evolution equation with the symbol $\sigma$ in a set $\Sigma$. The symbol $\sigma$ is a functional parameter which represents all time-dependent terms (like external forces) and coefficients of the equation.

The solutions are sought in a topological (usually Banach) space $\mathcal{W}_M$ which consists of vector-valued functions $w : [0, M] \to E$, where $E$ is a given Banach space. The space $\mathcal{W}_M$ is endowed with a given topology $\Theta_M$, such that $(\mathcal{W}_M, \Theta_M)$ is a Hausdorff topological space with a countable base. By means of $\mathcal{W}_M$ the space $\mathcal{W}_M^\sigma$ is defined as $\mathcal{W}_M^\sigma := \{ w : [0, \infty) \to E : \Pi_{[0,M]} w \in \mathcal{W}_M, \text{for all } M > 0 \}$, where $\Pi_{[0,M]}$ is the restriction operator on the interval $[0, M]$. The space $\mathcal{W}_M^\sigma$ is endowed with a local convergence topology $\Theta^\sigma_{loc}$, i.e., the topology that induces the following definition of convergence for a sequence $\{w_n\} \subset \mathcal{W}_M^\sigma$ to $w \in \mathcal{W}_M^\sigma$

$$w_n \to w \quad \text{in } \Theta^\sigma_{loc} \quad \text{if } \Pi_{[0,M]} w_n \to \Pi_{[0,M]} w \quad \text{in } \Theta_M,$$

for every $M > 0$. It can be seen that the space $(\mathcal{W}_M^\sigma, \Theta^\sigma_{loc})$ is a Hausdorff topological space with a countable base. On the space $\mathcal{W}_M^\sigma$ the translation semigroup $\{ T(t) \}_{t \geq 0}$ is defined, for every $w \in \mathcal{W}_M^\sigma$, as

$$T(t) w := w(\cdot + t), \quad \forall t \geq 0.$$

The semigroup $\{ T(t) \}$ is continuous in the topology $\Theta^\sigma_{loc}$ (see, e.g., [6, proposition 2.2]).

For each $\sigma \in \Sigma$ let us denote by $K^\sigma_\Theta$ the set of some solutions from $\mathcal{W}_M$ and by $\mathcal{K}^\sigma_\sigma$ the set of some solutions from $\mathcal{W}_M^\sigma$. The set $\mathcal{K}^\sigma_{\sigma}$ is said to be a trajectory space of the evolution equation corresponding to the symbol $\sigma \in \Sigma$.

Now, let $\mathcal{W}_M^\sigma$ be a subspace of $\mathcal{W}_M^\sigma$ and assume that a metric $\rho_{\mathcal{W}^\sigma}$ is defined on $\mathcal{W}_M^\sigma$. Assume also that $\mathcal{K}^\sigma_{\sigma} \subset \mathcal{W}_M^\sigma$, for every $\sigma \in \Sigma$.

Recall that the family of trajectory spaces $\{ \mathcal{K}^\sigma_\sigma \}_{\sigma \in \Sigma}$ is said to be translation-coordinated (tr.-coord.) if for any $\sigma \in \Sigma$ and any $w \in \mathcal{K}^\sigma_{\sigma}$ we have $T(t) w \in \mathcal{K}^\sigma_{\sigma}$, for every $t \geq 0$. The symbol space $\Sigma$ is assumed to be invariant with respect to the translation semigroup $\{ T(t) \}$, i.e., $T(t) \Sigma \subset \Sigma$, for all $t \geq 0$.

Consider the united trajectory space $\mathcal{K}^\sigma_{\Sigma} := \cup_{\sigma \in \Sigma} \mathcal{K}^\sigma_{\sigma}$ of the family $\{ \mathcal{K}^\sigma_{\sigma} \}_{\sigma \in \Sigma}$. We have $\mathcal{K}^\sigma_{\Sigma} \subset \mathcal{W}_M^\sigma$ and if the family $\{ \mathcal{K}^\sigma_{\sigma} \}_{\sigma \in \Sigma}$ is tr.-coord. then we have $T(t) \mathcal{K}^\sigma_{\Sigma} \subset \mathcal{K}^\sigma_{\Sigma}$, for every $t \geq 0$, i.e., the translation semigroup $\{ T(t) \}$ acts on $\mathcal{K}^\sigma_{\Sigma}$.

Introduce now the family

$$\mathcal{B}^\sigma_{\Sigma} := \{ B \subset \mathcal{K}^\sigma_{\Sigma} : B \text{ bounded in } \mathcal{W}_M^\sigma \text{ w.r.t. the metric } \rho_{\mathcal{W}^\sigma} \}.$$

**Definition 2.** A set $P \subset \mathcal{W}_M^\sigma$ is said to be a uniformly (w.r.t. $\sigma \in \Sigma$) attracting set for the family $\{ \mathcal{K}^\sigma_{\sigma} \}_{\sigma \in \Sigma}$ in the topology $\Theta^\sigma_{loc}$ if $P$ is uniformly (w.r.t. $\sigma \in \Sigma$) attracting for the family $\mathcal{B}^\sigma_{\Sigma}$, i.e., for any $B \in \mathcal{B}^\sigma_{\Sigma}$ and for any neighbourhood $O(P)$ in $\Theta^\sigma_{loc}$ there exists $\epsilon > 0$ such that $T(t) B \subset O(\mathcal{P})$, for every $t \geq \epsilon$.

**Definition 3.** A set $A_{\Sigma} \subset \mathcal{W}_M^\sigma$ is said to be a uniform (w.r.t. $\sigma \in \Sigma$) trajectory attractor of the translation semigroup $\{ T(t) \}$ in the topology $\Theta^\sigma_{loc}$ if $A_{\Sigma}$ is compact in $\Theta^\sigma_{loc}$, $A_{\Sigma}$ is a uniformly (w.r.t. $\sigma \in \Sigma$) attracting set for $\{ \mathcal{K}^\sigma_{\sigma} \}_{\sigma \in \Sigma}$ in the topology $\Theta^\sigma_{loc}$, and $A_{\Sigma}$ is the minimal compact and uniformly (w.r.t. $\sigma \in \Sigma$) attracting set for the family $\{ \mathcal{K}^\sigma_{\sigma} \}_{\sigma \in \Sigma}$ in the topology $\Theta^\sigma_{loc}$, i.e., if $P$ is any compact uniformly (w.r.t. $\sigma \in \Sigma$) attracting set for the family $\{ \mathcal{K}^\sigma_{\sigma} \}_{\sigma \in \Sigma}$, then $A_{\Sigma} \subset P$.

From the definition it follows that, if the trajectory attractor exists, then it is unique.

In order to prove some properties of the trajectory attractor we need the set $\mathcal{K}^\sigma_{\Sigma}$ to be closed in $\Theta^\sigma_{loc}$. Assume that $\Sigma$ is a complete metric space. Recall that the family $\{ \mathcal{K}^\sigma_{\Sigma} \}_{\sigma \in \Sigma}$ is called (see, e.g., $\Theta^\sigma_{loc}$)-closed if the graph set $\cup_{\sigma \in \Sigma} \mathcal{K}^\sigma_{\Sigma} \times \{ \sigma \}$ is closed in the topological space $\Theta^\sigma_{loc} \times \Sigma$. If $\{ \mathcal{K}^\sigma_{\Sigma} \}_{\sigma \in \Sigma}$ is (see, e.g., $\Theta^\sigma_{loc}$)-closed and $\Sigma$ is compact, then $\mathcal{K}^\sigma_{\Sigma}$ is closed in $\Theta^\sigma_{loc}$. 
By applying [5, chapter XI, theorem 2.1] to the topological space \( W^{\mu}_{r,loc} \), to the family \( B^\mu_{\Sigma} \) and to the family

\[
B^\mu_{\omega}(\Sigma) := \{ B \subseteq K^\mu_{\omega}(\Sigma) : B \text{ bounded in } W^{\mu}_{r,loc} \text{ w.r.t. the metric } \rho^{\mu}_{W^{\mu}} \},
\]

where \( K^\mu_{\omega}(\Sigma) := \bigcup_{t \in (0,\infty)} K^\mu_{\omega} \sigma_t \) and where \( \omega(\Sigma) \) is the \( \omega \)-limit set of \( \Sigma \), we can state the following (see also [6, theorem 3.1] and [5, chapter XIV, theorem 3.1]).

**Theorem 2.** Let the spaces \( (W^{\mu}_{r,loc}, \Theta^{\mu}_{loc}) \) and \( (W^{\mu}_{b}, \rho^{\mu}_{W^{\mu}}) \) be as above, and the family of trajectory spaces \( \{ K^\mu_{\omega} \}_{\sigma \in \Sigma} \) corresponding to the evolution equation with symbols \( \sigma \in \Sigma \) be such that \( K^\mu_{\omega} \sigma \subseteq W^{\mu}_{b} \), for every \( \sigma \in \Sigma \). Assume there exists a subset \( P \subseteq W^{\mu}_{b} \) which is compact in \( \Theta^{\mu}_{loc} \) and uniformly (w.r.t. \( \sigma \in \Sigma \)) attracting in \( \Theta^{\mu}_{loc} \) for the family \( \{ K^\mu_{\omega} \}_{\sigma \in \Sigma} \) in the topology \( \Theta^{\mu}_{loc} \). Then, the translation semigroup \( \{ T(t) \}_{t \geq 0} \) (acting on \( K^\mu_{\omega} \sigma \)) if the family \( \{ K^\mu_{\omega} \}_{\sigma \in \Sigma} \) is tr.-coord.) possesses a (unique) uniform (w.r.t. \( \sigma \in \Sigma \)) trajectory attractor \( A_{\Sigma} \subseteq P \) which is strictly invariant

\[
T(t) A_{\Sigma} = A_{\Sigma}, \quad \forall t \geq 0.
\]

In addition, if the family \( \{ K^\mu_{\omega} \}_{\sigma \in \Sigma} \) is tr.-coord. and \( (\Theta^{\mu}_{loc}, \Sigma) \)-closed, with \( \Sigma \) a compact metric space, then \( A_{\Sigma} \subseteq K^\mu_{\omega} \sigma \) and

\[
A_{\Sigma} = A_{\omega(\Sigma)},
\]

where \( A_{\omega(\Sigma)} \) is the uniform (w.r.t. \( \sigma \in \omega(\Sigma) \)) trajectory attractor for the family \( B^\mu_{\omega}(\Sigma) \), and \( A_{\omega(\Sigma)} \subseteq K^\mu_{\omega}(\Sigma) \).

Now, let us suppose that a dissipative estimate of the following form holds

\[
\rho^{\mu}_{W^{\mu}}(T(t)w, w_0) \leq \Lambda_0 \left( \rho^{\mu}_{W^{\mu}}(w, w_0) \right) e^{-kt} + \Lambda_1, \quad \forall t \geq t_0, \tag{2.18}
\]

for every \( w \in K^\mu_{\Sigma} \), for some fixed \( w_0 \in W^{\mu}_{b} \) and for some \( \Lambda_0 : [0, \infty) \to [0, \infty) \) locally bounded and some constants \( \Lambda_1 \geq 0, k > 0 \), where both \( \Lambda_0 \) and \( \Lambda_1 \) are independent of \( w \). Furthermore, suppose that the ball

\[
B^{\mu}_{W^{\mu}}(w_0, 2\Lambda_1) := \{ w \in W^{\mu}_{b} : \rho^{\mu}_{W^{\mu}}(w, w_0) \leq 2\Lambda_1 \}
\]

is compact in \( \Theta^{\mu}_{loc} \). By virtue of (2.18) such ball is a uniformly (w.r.t. \( \sigma \in \Sigma \)) attracting set for the family \( \{ K^\mu_{\omega} \}_{\sigma \in \Sigma} \) in the topology \( \Theta^{\mu}_{loc} \) (actually, \( B^{\mu}_{W^{\mu}}(w_0, 2\Lambda_1) \) is uniformly (w.r.t. \( \sigma \in \Sigma \)) absorbing for the family \( K^\mu_{\omega} \)). Theorem 2 therefore entails that the translation semigroup \( \{ T(t) \}_{t \geq 0} \) possesses a (unique) uniform (w.r.t. \( \sigma \in \Sigma \)) trajectory attractor \( A_{\Sigma} \subseteq B^{\mu}_{W^{\mu}}(w_0, 2\Lambda_1) \).

2.2. The trajectory attractor for a general smooth potential \( W \)

We now apply the scheme described in section 2.1 to system (1.1)–(1.3) coupled with boundary conditions (2.4)–(2.5) in order to prove the existence of a trajectory attractor for that system.

For \( M > 0 \) introduce the space

\[
\mathcal{W}_M := \left[ \{ \{ u, d \} \in L^\infty(0, M; G_{div} \times V) \cap L^2(0, M; V_{div} \times H^2(\Omega)^3) : \begin{align*}
\{ u, d \} & \in L^2(0, M; W^{-1,3/2}(\Omega)^3), \\
\{ d \} & \in L^2(0, M; L^{3/2}(\Omega)^3) \end{align*} \right], \tag{2.19}
\]

endowed with the weak topology \( \Theta_M \) which induces the following notion of weak convergence: a sequence \( \{ u_m, d_m \} \subset \mathcal{W}_M \) is said to converge to \( \{ u, d \} \in \mathcal{W}_M \) in \( \Theta_M \) if

\[
\begin{align*}
\{ u_m \} & \rightharpoonup u \quad \text{weakly* in } L^\infty(0, M; G_{div}) \text{ and weakly in } L^2(0, M; V_{div}), \tag{2.20} \\
\{ u_m \} & \rightharpoonup u \quad \text{weakly in } L^2(0, M; W^{-1,3/2}(\Omega)^3), \tag{2.21} \\
\{ d_m \} & \rightharpoonup d \quad \text{weakly* in } L^\infty(0, M; V) \text{ and weakly in } L^2(0, M; H^2(\Omega)^3), \tag{2.22} \\
\{ d_m \} & \rightharpoonup d \quad \text{weakly in } L^2(0, M; L^{3/2}(\Omega)^3). \tag{2.23}
\end{align*}
\]
Then the space
\[ \mathcal{W}^+_t := \left\{ [u, d] \in L^\infty_{loc}([0, \infty); G_{\div} \times V) \cap L^2_{loc}([0, \infty); V_{\div} \times H^2(\Omega)^3) : \right. \]
\[ u_t \in L^2_{loc}([0, \infty); W^{-1,3/2}(\Omega)^3), \quad d_t \in L^2_{loc}([0, \infty); L^{3/2}(\Omega)^3) \right\} \] (2.24)
is defined, as well as the inductive limit weak topology \( \Theta^+_t \).

In \( \mathcal{W}^+_t \) we consider the following subspace
\[ \mathcal{K}_{h}^+ := \left\{ [u, d] \in L^\infty_{loc}([0, \infty); G_{\div} \times V) \cap L^2_{loc}([0, \infty); V_{\div} \times H^2(\Omega)^3) : \right. \]
\[ u_t \in L^2_{loc}([0, \infty); W^{-1,3/2}(\Omega)^3), \quad d_t \in L^2_{loc}([0, \infty); L^{3/2}(\Omega)^3), \]
\[ W(d) \in L^\infty([0, \infty); L^1(\Omega)) \right\}, \] (2.25)
and on \( \mathcal{W}^+_t \) we define the following metric
\[ \rho_{\mathcal{W}^+_t}(w_1, w_2) := \|w_1 - w_2\|_{L^\infty_{loc}([0, \infty); G_{\div} \times V)} + \|w_1 - w_2\|_{L^2_{loc}([0, \infty); V_{\div} \times H^2(\Omega)^3)} \]
\[ + \|u_t - u_t\|_{L^2_{loc}([0, \infty); W^{-1,3/2}(\Omega)^3)} + \|d_t - d_t\|_{L^2_{loc}([0, \infty); L^{3/2}(\Omega)^3)} \]
\[ + \left\| \int_\Omega W(d_t) - \int_\Omega W(d_2) \right\|_{L^\infty([0, \infty; \mathbb{R}^3)} \] (2.26)

for every \( w_1 = [u_1, d_1], w_2 = [u_2, d_2] \in \mathcal{W}^+_t \).

**Definition 4.** For every \( h \in L^2_{loc}([0, \infty); V_{\div}^w) \) the trajectory space \( \mathcal{K}_{h}^+ \) of system (1.1)–(1.3), (2.4)–(2.5) with external force \( h \) is the set of all weak solutions \( w = [u, d] \) of this system with the regularity properties (2.7)–(2.10) for \( u, d, \) and satisfying the energy inequality (2.16) for all \( t \geq s \) and for a.a. \( s \in [0, \infty) \).

The trajectory space \( \mathcal{K}_h^+ \) on the bounded interval \([0, M]\) can be defined similarly, for every \( M > 0 \).

**Remark 2.** Notice that in the definition of the trajectory space \( \mathcal{K}_h^+ \) we do not assume that the energy inequality (2.16) is satisfied also for \( s = 0 \). In this way the family \( \{\mathcal{K}_h^+\}_{h \in \Sigma} \) (\( \Sigma \) may be a generic symbol space included in \( L^2_{loc}([0, \infty); V_{\div}^w) \)) is tr.-coord. and therefore the translation semigroup \( \{T(t)\} \) acts on \( \mathcal{K}_h^+ \).

According to theorem 1, if (2.12) and (2.13) hold, then, for every \( w_0 = [u_0, d_0] \) such that
\[ u_0 \in G_{\div}, \quad d_0 \in V, \quad W(d_0) \in L^1(\Omega), \] (2.29)
and every \( h \) such that
\[ h \in L^2_{loc}([0, \infty); V^w) \] (2.30)
there exists a trajectory \( w \in \mathcal{K}_{h}^+ \) for which \( w(0) = w_0 \).

Consider now
\[ h_0 \in L^2_{loc}([0, \infty); V^w) \]
so that \( h_0 \) is translation compact (tr.-c.) in \( L^2_{loc,w}([0, \infty); V^w) \) (see, e.g., [6, proposition 6.8]). For the symbol space \( \Sigma \) we choose the hull of \( h_0 \) in \( L^2_{loc,w}([0, \infty); V^w) \)
\[ \Sigma = \mathcal{H}_s(h_0) := \left\{ [T(t)h_0, t \geq 0] \right\}_{L^2_{loc,w}([0, \infty); V^w)} \] (2.31)
which is a compact metric space. Recall (see [6, proposition 6.9]) that every \( h \in \mathcal{H}_s(h_0) \) is also tr.-c. in \( L^2_{loc,w}([0, \infty); V^w) \) and
\[ \|h\|_{L^2_{loc,w}([0, \infty); V^w)} \leq \|h_0\|_{L^2_{loc,w}([0, \infty); V^w)}, \quad \forall h \in \mathcal{H}_s(h_0). \] (2.32)
In order to prove the closure of the space of the trajectory attractor, we shall also assume that $h_0$ is tr.-c. in $L^2_{loc}([0, \infty); V'_{\text{div}})$ or tr.-c. in $L^2_{loc,u}([0, \infty); G_{\text{div}})$. The latter condition is equivalent to the assumption that $h_0 \in L^2_{\text{div}}(0, \infty; G_{\text{div}})$. It is not difficult to prove that the hull of $h_0$ with one of these assumptions (defined as in (2.31) with the closure in the above spaces) coincides with the hull $H_{\epsilon}(h_0)$ defined as in (2.31).

In order to state our first result on the existence of the trajectory attractor, we shall make the following assumption on the potential $W$:

(W1) $W$ satisfies (2.12), (2.13) and there exist $c_0 \geq 0$, $c_1 > 0$, $c_2 \in \mathbb{R}$ and $\delta > 0$ such that

$$W_1(d) \leq c_0(1 + |\nabla d W_1(d)|^2),$$  \hspace{0.5cm} (2.33)

$$W_1(d) \geq c_1|d|^{2+\delta} - c_2,$$  \hspace{0.5cm} (2.34)

for every $d \in \mathbb{R}^3$.

Let us now state the following lemma which will be useful in order to prove our next main theorem 3.

**Lemma 1.** Take assumption (W1) on $W$, then there exist $\kappa, \eta, l > 0$ (independent of $d$) such that

$$\| - \Delta d + \nabla d W_1(d) \|^2 \geq \kappa \| \nabla d \|^2 + \eta \int_\Omega W(d) - l,$$  \hspace{0.5cm} (2.35)

for all $d \in H^2(\Omega)^3$, with $\partial_\alpha d = 0$ on $\partial \Omega$.

**Proof.** Using (W1), we have

$$\| - \Delta d + \nabla d W_1(d) \|^2 = \| - \Delta d + d \|^2 + |d|^2 - 2(\Delta d, -\Delta d + d)$$

$$\quad + \| \nabla d W_1(d) \|^2 + 2(-\Delta d, \nabla d W_1(d)).$$  \hspace{0.5cm} (2.36)

By means of (2.33) we obtain

$$\| \nabla d W_1(d) \|^2 \geq \frac{1}{c_0} \int_\Omega W_1(d) - |\Omega|.$$  \hspace{0.5cm} (2.37)

Moreover, using the convexity of $W_1$ (which implies that $(-\Delta d, \nabla W_1(d)) \geq 0$) and the fact that (2.34) implies that $W_1(d) \geq c_3|d|^2 - c_4$, from (2.36) we get

$$\| - \Delta d + \nabla d W_1(d) \|^2 \geq \epsilon \| - \Delta d + d \|^2 - \frac{\epsilon}{1 - \epsilon} |d|^2 + \frac{1}{c_0} \int_\Omega W_1(d) - |\Omega|$$

$$\geq \epsilon c_1 \| \nabla d \|^2 + \epsilon \left( c_1 - \frac{1}{1 - \epsilon} \right) |d|^2 + \frac{1}{c_0} \int_\Omega W_1(d) - |\Omega|$$

$$\geq \epsilon c_1 \| \nabla d \|^2 + \left( \frac{c_1}{2c_0} - \epsilon \left( c_1 - \frac{1}{1 - \epsilon} \right) \right) |d|^2 + \frac{1}{2c_0} \int_\Omega W_1(d) - c_5$$

$$\geq \epsilon c_1 \| \nabla d \|^2 + \frac{1}{2c_0} \int_\Omega W_1(d) - c_5,$$  \hspace{0.5cm} (2.38)

provided $\epsilon$ is chosen small enough. In (2.38) the positive constant $c_1$ is such that

$$c_1 |d|_{V} \leq \| - \Delta d + d \|,$$

for every $d \in H^2(\Omega)^3$, with $\partial_\alpha d = 0$ on $\partial \Omega$. Now, from (2.38) we have

$$\| - \Delta d + \nabla d W(d) \|^2 \geq \frac{1}{2} \| - \Delta d + \nabla d W_1(d) \|^2 - \| \nabla d W_2(d) \|^2$$

$$\geq \frac{\epsilon c_1}{2} \| \nabla d \|^2 + \frac{1}{4c_0} \int_\Omega W_1(d) - \| \nabla d W_2(d) \|^2 - \frac{c_5}{2},$$  \hspace{0.5cm} (2.39)
and observe that, due to (2.34) and to the assumption (2.13) on $W_2$, we can choose $\eta > 0$ such that
\[
\frac{1}{4c_0} W_1(d) - |\nabla d W_2(d)|^2 \geq \eta W(d) - c_\eta, \quad \forall d \in \mathbb{R}^3.
\]
We therefore get (2.35) with $\kappa = \epsilon c_1/2$ and $l$ depending on $\Omega$, $W$ and with $\eta$ depending on $W$ only.

In order to prove that the united trajectory space $\mathcal{K}^+_{H(t_0)}$ is closed in $\Theta^+_{loc}$ we shall also need the following growth assumption on $W$

(W2) There exists $b > 0$ such that
\[
W(d) \leq b(1 + |d|^6), \quad \forall d \in \mathbb{R}^3.
\]

**Remark 3.** Notice that both assumptions (W1) and (W2) are satisfied in the case of the physically interesting double-well potential
\[
W(d) = (|d|^2 - 1)^2.
\]
This function is usually assumed as a good smooth approximation for a potential penalizing the deviation of the length $|d|$ from the value 1, which is due to liquid crystal molecules being of similar size.

We can now state our main result.

**Theorem 3.** Let (W1) holds and that $h_0 \in L^2_{loc}(0, \infty; V^1_{div})$. Then, the semigroup $\{T(t)\}$ acting on $\mathcal{K}^+_{H(t_0)}$ possesses the uniform (w.r.t. $h \in \mathcal{H}_{\epsilon}(h_0)$) trajectory attractor $\mathcal{A}_{H(t_0)}$. This set is strictly invariant, bounded in $W^1_0$ and compact in $\Theta^+_{loc}$. In addition, if (W2) holds and $h_0$ is tr-$c.$ in $L^2_{loc}([0, \infty); V^1_{div})$ or $h_0 \in L^2_{tb}(0, \infty; G_{div})$, then $\mathcal{K}^+_{H(t_0)}$ is closed in $\Theta^+_{loc}$, $\mathcal{A}_{H(t_0)} \subset \mathcal{K}^+_{H(t_0)}$ and
\[
\mathcal{A}_{H(t_0)} = \mathcal{A}_{\Theta(H(t_0))}.
\]

For the proof of theorem 3 we need two propositions. The first proposition establishes a dissipative estimate of the form (2.18) for our problem.

For the reader’s convenience we first recall an integral Gronwall lemma which is a modification of a lemma by to Ball (see [1, lemma 7.2]) and which is crucial for proving proposition 1. For its proof we refer to [15].

**Lemma 2.** Let $\theta \in L^1(0, T)$ for every $T > 0$ and suppose that
\[
\theta(t) + k \int_0^t \theta(\tau)d\tau \leq \int_0^t f(\tau)d\tau + \theta(s) + k \int_0^s \theta(\tau)d\tau \quad (2.40)
\]
holds for a.e. $t, s \in (0, \infty)$, with $t \geq s$, where $f \in L^1(0, T)$ for every $T > 0$ and the constant $k \geq 0$ are given. Then we have
\[
\theta(t) \leq \theta(s)e^{-k(t-s)} + \int_s^t e^{-k(t-\tau)} f(\tau)d\tau, \quad (2.41)
\]
for a.e. $t, s \in (0, \infty)$, with $t \geq s$. Furthermore, suppose $\theta : [0, \infty) \to \mathbb{R}$ is a l.s.c. representative satisfying (2.40) for a.e. $t, s \in (0, \infty)$, with $t \geq s$. Then (2.40) and (2.41) also hold for every $t \geq s$ and for a.e. $s > 0$, and if, in addition, (2.40) holds for $s = 0$, then we have
\[
\theta(t) \leq \theta(0)e^{-kT} + \int_0^t e^{-k(t-\tau)} f(\tau)d\tau, \quad (2.42)
\]
for all \( t \in [0, \infty) \). In particular, suppose \( f(t) = l + g(t) \), where \( l \in \mathbb{R} \) is a given constant and \( g \in L^1_{loc}([0, \infty)) \), i.e., \( g \) belongs to \( L^1_{loc}([0, \infty)) \) and is translation bounded, that is,
\[
\|g\|_{L^1_{\infty}} := \sup_{t \geq 0} \int_{t}^{t+1} |g(\tau)| \, d\tau < \infty.
\]

Then we have
\[
\theta(t) \leq \theta(0) e^{-kt} + \frac{l}{k} + \frac{\|g\|_{L_{\infty}^1}}{1 - e^{-k}},
\]
for all \( t \in [0, \infty) \).

**Proposition 1.** Assume (W1) holds and that \( h_0 \in L^2_{\infty}(0, \infty; V'_{div}) \). Then, for all \( h \in \mathcal{H}_v(h_0) \) we have \( \mathcal{K}_h \subset W^v_t \) and the following dissipative estimate holds
\[
\rho_{\mathcal{K}_h}^v(T(t)w, 0) \leq c \rho_{\mathcal{K}_h}^v(w, 0) e^{-\frac{t}{2}} + \Lambda_0, \quad \forall t \geq 1,
\]
for all \( w \in \mathcal{K}_h \). Here \( \Lambda_0, k \) and \( c \) are positive constants (independent of \( w \)) that depend on \( W, \Omega, v \) with only \( \Lambda_0 \) depending on the norm of \( h_0 \) in \( L^2_{\infty}(0, \infty; V'_{div}) \). In particular \( k \) can be given by \( k = \min(\eta, 2\kappa, \nu\lambda_1) \), where \( \lambda_1 \) is the first eigenvalue of the Stokes operator and \( \eta, \kappa \) are such that (2.35) holds.

**Proof.** Take now \( w \in \mathcal{K}_h \), with \( h \in \mathcal{H}_v(h_0) \). Recalling the definition of the energy \( \mathcal{E}(2.17) \), using (2.35) and Poincaré inequality we have
\[
\| - \Delta d + \nabla dW(d) \| + \frac{v}{2} \| \nabla u \|^2 \geq k \mathcal{E}(w) - l, \quad w = [u, d]
\]
where \( k = \min(\eta, 2\kappa, \nu\lambda_1) \), \( \lambda_1 \) is the first eigenvalue of the Stokes operator, and \( l \) (depending on \( \Omega, W \) only) is the same as in (2.35).

Therefore, by combining (2.45) with the energy inequality (2.16) we deduce that \( w \) satisfies the integral inequality
\[
\mathcal{E}(w(t)) + k \int_0^t \mathcal{E}(w(\tau)) \, d\tau \leq l(t - s) + \frac{1}{2v} \int_s^t \| h(\tau) \|_{V'_{div}}^2 \, d\tau + \mathcal{E}(w(s)) + k \int_0^t \mathcal{E}(w(\tau)) \, d\tau,
\]
for all \( t \geq s \) and for a.e. \( s \in (0, \infty) \). We can now apply lemma 2 and deduce that
\[
\mathcal{E}(w(t)) \leq \mathcal{E}(w(s)) e^{-kt(t-s)} + \frac{1}{2v} \int_s^t e^{-k(t-\tau)} \left( \| h(\tau) \|_{V'_{div}}^2 + 2vl \right) \, d\tau,
\]
for all \( t \geq s \) and for a.e. \( s \in (0, \infty) \). Notice that, due to the regularity properties of the solution, which imply that \( u \in C_\omega([0, \infty); G_{div}), \quad d \in C_\omega([0, \infty); V) \) (and hence \( d \in C([0, \infty); L^2(\Omega)^3) \)), and due to the fact that, thanks to (2.13), \( W \) is a quadratic perturbation of a convex function, then \( \mathcal{E}(w(\cdot)) : [0, \infty) \to \mathbb{R} \) is lower semicontinuous.

Hence
\[
\mathcal{E}(w(t)) \leq e^t \sup_{s \in (0,1)} \mathcal{E}(w(s)) e^{-kt} + \frac{1}{2v} \int_0^t e^{-k(t-\tau)} \left( \| h(\tau) \|_{V'_{div}}^2 + 2vl \right) \, d\tau
\]
\[
\leq e^t \sup_{s \in (0,1)} \mathcal{E}(w(s)) e^{-kt} + K^2, \quad \forall t \geq 1,
\]
where
\[
K^2 = \frac{l}{k} + \frac{1}{2v(1 - e^{-k})} \| h_0 \|_{L^2_{\infty}(0, \infty; V'_{div})}^2.
\]
Now, observe that due to (2.34) we have
\[ \mathcal{E}(w) \geq c_6 \left( \| u \|^2 + \| d \|_{V'}^2 + \int_\Omega W(d) \right) - c_7, \tag{2.49} \]
and
\[ \sup_{s \in (0, 1)} \mathcal{E}(w(s)) \leq \frac{1}{2} \| u \|^2_{L^\infty(0, 1; G_{av})} + \frac{1}{2} \| \nabla d \|^2_{L^\infty(0, 1; L^1(\Omega)^{3'})} + \sup_{s \in (0, 1)} \int_\Omega W(d(s)) \]
\[ \leq c \rho_\text{av}(w, 0), \quad \forall w = [u, d] \in V_0^N. \tag{2.50} \]
Henceforth in this proof we shall denote by \( c \) a nonnegative constant, which may vary even within the same line, that possibly depends on \( W \), \( \Omega \) and \( v \), but is independent of \( w \) and \( h_0 \).

By combining (2.49) and (2.50) with (2.48) we get
\[ \| u(t) \| + \| d(t) \|_{V'} + \left( \int_\Omega W(d(t)) \right)^{1/2} \leq c \rho_\text{av}(w, 0) e^{-\frac{2t}{\nu}} + cK + c, \quad \forall t \geq 1, \tag{2.51} \]
and hence
\[ \| T(t)u \|_{L^\infty(0, \infty; G_{av})} + \| T(t)d \|_{L^\infty(0, \infty; V')} + \int_\Omega W(T(t)d) \] \[ \leq c \rho_\text{av}(w, 0) e^{-\frac{2t}{\nu}} + cK + c, \quad \forall t \geq 1. \tag{2.52} \]
From the energy inequality (2.16) we have
\[ \int_t^{t+1} \left( \| -\Delta d + \nabla dW(d) \|^2 + \frac{\nu}{2} \| \nabla u \|^2 \right) \, d\tau \]
\[ \leq \mathcal{E}(w(t)) - \mathcal{E}(w(t + 1)) + \frac{1}{2\nu} \int_t^{t+1} \| h(\tau) \|^2_{V_{loc}} \, d\tau, \tag{2.53} \]
for a.e. \( t > 0 \).

Notice that, thanks to the convexity of \( W_1 \) and to the assumption (2.13) on \( W_2 \), we have
\[ \| -\Delta d + \nabla dW(d) \|^2 \geq \frac{1}{2} \| -\Delta d + \nabla dW_1(d) \|^2 - \| \nabla dW_2(d) \|^2 \]
\[ \geq \frac{1}{4} \| -\Delta d + d \|^2 - \frac{1}{2} \| d \|^2 - \| \nabla dW_2(d) \|^2 \]
\[ \geq \frac{1}{4} \| -\Delta d + d \|^2 - c \| d \|^2 - c, \tag{2.54} \]
and therefore (2.53) and (2.51) entail
\[ \int_t^{t+1} \left( \frac{1}{4} \| d(\tau) \|^2_{H^1(\Omega)} + \frac{\nu}{2} \| \nabla u(\tau) \|^2 \right) \, d\tau \leq c \rho_\text{av}(w, 0) e^{-\frac{2t}{\nu}} + cK^2 + c, \quad \forall t \geq 1, \tag{2.55} \]
which implies that
\[ \| T(t)d \|_{L^\infty(0, \infty; H^1(\Omega)^3)} + \| T(t)u \|_{L^\infty(0, \infty; V_{loc})} \leq c \rho_\text{av}(w, 0) e^{-\frac{2t}{\nu}} + cK + c, \quad \forall t \geq 1. \tag{2.56} \]

Now, recall that, due to the interpolation inequality
\[ L^\infty(0, M; L^2(\Omega)) \cap L^2(0, M; L^3(\Omega)) \subset L^3(0, M; L^3(\Omega)) \]
and to the regularity property of the solution, we have that
\[ u \cdot \nabla d, \quad d \cdot \nabla u \in L^2_{loc}([0, \infty); L^{3/2}(\Omega)^3), \]
and so
\[
\| \mathbf{u} \cdot \nabla \mathbf{d} \|_{L^1(0,t+1;L^3(\Omega^i)')} \leq \| \mathbf{u} \|_{L^1(0,t+1;L^3(\Omega^i)')} \| \nabla \mathbf{d} \|_{L^1(0,t+1;L^3(\Omega^i)')} \\
\leq c (\| \mathbf{u} \|_{L^\infty(0,t+1;\mathcal{L}(\Omega^i))} + \| \mathbf{u} \|_{L^2(0,t+1;V_{\text{div}})}) (\| \mathbf{d} \|_{L^\infty(0,t+1;V)} + \| \mathbf{d} \|_{L^2(0,t+1;H^1(\Omega^i))}).
\]
and
\[
\| \mathbf{d} \cdot \nabla \mathbf{u} \|_{L^3(0,t+1;L^3(\Omega^i)')} \leq c \| \mathbf{d} \|_{L^\infty(0,t+1;V)} \| \mathbf{u} \|_{L^2(0,t+1;V_{\text{div}})}.
\]
By using (2.51) and (2.55) we hence get
\[
\| \mathbf{u} \cdot \nabla \mathbf{d} \|_{L^1(0,t+1;L^3(\Omega^i)')} + \| \mathbf{d} \cdot \nabla \mathbf{u} \|_{L^1(0,t+1;L^3(\Omega^i)')} \leq \rho_{V^i}^2 (\mathbf{w}, \mathbf{0}) e^{-2t} + c K^2 + c,
\]
for all \( t \geq 1 \). Furthermore, from (2.53) and (2.51) we have
\[
\| - \Delta \mathbf{d} + \nabla W(\mathbf{d}) \|_{L^1(0,t+1;L^3(\Omega^i)')} \leq \rho_{V^i}^2 (\mathbf{w}, \mathbf{0}) e^{-3t} + c K + c, \quad \forall t \geq 1.
\]
Therefore, by using (1.2), (2.57) and (2.58) we obtain
\[
\| \mathbf{d} \|_{L^3(0,t+1;L^3(\Omega^i)')} \leq \rho_{V^i}^2 (\mathbf{w}, \mathbf{0}) e^{-\frac{3}{2}t} + c K^2 + c, \quad \forall t \geq 1,
\]
and from this last inequality
\[
\| T(t) \|_{L^1(0,t;L^3(\Omega^i)')} \leq \rho_{V^i}^2 (\mathbf{w}, \mathbf{0}) e^{-\frac{3}{2}t} + c K^2 + c, \quad \forall t \geq 1.
\]
Finally, observe that the regularity properties of the solution also entail
\[
\mathbf{u} \otimes \mathbf{u}, \quad \nabla \mathbf{d} \otimes \nabla \mathbf{d}, \quad (\Delta \mathbf{d} - \nabla W(\mathbf{d})) \otimes \mathbf{d} \in L^2_{t,w}(0, \infty; L^3(\Omega^i)^{3 \times 3}),
\]
and we have
\[
\| \mathbf{u} \otimes \mathbf{u} \|_{L^1(0,t+1;L^3(\Omega^i)')} \leq \| \mathbf{u} \|_{L^\infty(0,t+1;L^3(\Omega^i)')}^2 + \| \mathbf{u} \|_{L^2(0,t+1;V_{\text{div}})}^2,
\]
and
\[
\| \nabla \mathbf{d} \otimes \nabla \mathbf{d} \|_{L^1(0,t+1;L^3(\Omega^i)')} \leq \| \nabla \mathbf{d} \|_{L^3(0,t+1;L^3(\Omega^i)')}^2 + \| \mathbf{d} \|_{L^2(0,t+1;V_{\text{div}})}^2 + \| \mathbf{d} \|_{L^3(0,t+1;L^3(\Omega^i)')}^2.
\]
Therefore, from the variational formulation (2.11) for the equation of the velocity we get
\[
\| \mathbf{u} \|_{L^1(0,t+1;W^{-1,3}(\Omega^i))} \leq \| \mathbf{u} \otimes \mathbf{u} \|_{L^1(0,t+1;L^3(\Omega^i)')} + c \| \mathbf{u} \|_{L^2(0,t+1;V_{\text{div}})}^2 + \| \nabla \mathbf{d} \otimes \nabla \mathbf{d} \|_{L^1(0,t+1;L^3(\Omega^i)')} + \| \Delta \mathbf{d} - \nabla W(\mathbf{d}) \|_{L^1(0,t+1;L^3(\Omega^i)')} + c \| \mathbf{h} \|_{L^1(0,t+1;V_{\text{div}})}^2.
\]
By combining (2.62) with the previous estimates and with (2.51), (2.55) and with (2.58), we easily obtain
\[
\| \mathbf{u} \|_{L^1(0,t+1;W^{-1,3}(\Omega^i))} \leq \rho_{V^i}^2 (\mathbf{w}, \mathbf{0}) e^{-\frac{3}{2}t} + c K^2 + c, \quad \forall t \geq 1,
\]
whence
\[
\| T(t) \mathbf{u} \|_{L^1(0,t;W^{-1,3}(\Omega^i))} \leq \rho_{V^i}^2 (\mathbf{w}, \mathbf{0}) e^{-\frac{3}{2}t} + c K^2 + c, \quad \forall t \geq 1.
\]
Collecting now (2.52), (2.56), (2.59) and (2.63) we deduce that \( K^+_{\mathbf{h}} \subset V^+_{\mathbf{y}} \) and that (2.44) holds with \( \Lambda_0 = c K^2 + c \). \( \square \)

The next proposition states that \( \{ \mathbf{k}^M_{\mathbf{h}} \}_{\mathbf{h}} \in L^2(0,M;V_{\text{div}}) \) is \( (\Theta_M, L^2(0,M;V_{\text{div}})) \)-closed and \( \{ \mathbf{k}^M_{\mathbf{h}} \}_{\mathbf{h}} \in L^2(0,M;G_{\text{div}}) \) is \( (\Theta_M, L^2(0,M;G_{\text{div}})) \)-closed, for every \( M > 0 \), under the further assumption (W2) on \( W \).
Proposition 2. Suppose that (W1) and (W2) hold. Let \( w_m := [u_m, d_m] \in \mathcal{K}_M^M \) be such that \( \{w_m\} \) converges to \( w := [u, d] \) in \( \Theta_M \) and \( \{h_m\} \) be such that one of the following convergence assumptions holds

(a) \( h_m \in L^2(0, M; V_{\text{div}}^\varepsilon) \) and \( h_m \rightharpoonup h \), strongly in \( L^2(0, M; V_{\text{div}}^\varepsilon) \).

(b) \( h_m \in L^2(0, M; G_{\text{div}}) \) and \( h_m \rightarrow h \), weakly in \( L^2(0, M; G_{\text{div}}) \).

Then \( w \in \mathcal{K}_M^M \).

Proof. Since \( w_m = [u_m, d_m] \in \mathcal{K}_M^M \), then, every weak solution \( w_m \) is such that: (i) the regularity properties (2.7)–(2.10) hold for each solution \( w_m \), (ii) the weak formulation (2.11) for \( u_m \) corresponding to the external force \( h_m \) and (1.2), (2.5) for \( d_m \) are satisfied, and (iii) the energy inequality

\[
\mathcal{E}(w_m(t)) + \int_t^s \left( \| - \Delta d_m + \nabla_x W(d_m) \| + v \| \nabla u_m \| \right) \, dt \leq \mathcal{E}(w_m(s)) + \int_s^t \langle h_m, u_m \rangle \, dt
\]  

holds for every \( m \in \mathbb{N} \), for all \( t \) and a.e. \( s \) with \( t \geq s \) and \( t \in [0, M] \). The weak convergences (2.20)–(2.23) imply that the sequence \( \{u_m\} \) is bounded in \( L^\infty(0, M; G_{\text{div}}) \), the sequence \( \{d_m\} \) is bounded in \( L^\infty(0, M; V) \) and hence also in \( L^\infty(0, M; L^6(\Omega)^3) \). The growth assumption (W2) then entails

\[
\| \mathcal{E}(w_m(s)) \| \leq c,
\]

for every \( m \) and a.e. \( s \in [0, M] \). Therefore, by using (2.64) and the convergence assumption for the sequence \( \{h_m\} \) we deduce that

\[
\| - \Delta d_m + \nabla_x W(d_m) \|_{L^2(0, M; L^2(\Omega)^3)} \leq c.
\]  

Since, by (2.22) the sequence \( \{-\Delta d_m\} \) is bounded in \( L^2(0, M; L^2(\Omega)^3) \), then we infer that \( \{\nabla_x W(d_m)\} \) is bounded in \( L^2(0, M; L^2(\Omega)^3) \) as well and therefore, up to a subsequence, \( \nabla_x W(d_m) \rightharpoonup G \) weakly in \( L^2(0, M; L^2(\Omega)^3) \). Since we also have, as a consequence of the convergences (2.22), (2.23) and of the Aubin–Lions lemma, that \( d_m \rightharpoonup d \) strongly in \( L^2(0, M; V) \), we deduce that \( G = \nabla_x W(d) \).

It is easy to check that \( w = [u, d] \) is a weak solution corresponding to the external force \( h \). Indeed, we can take \( \varphi \in \mathcal{D}(\Omega)^3 \) with \( \text{div}(\varphi) = 0 \), write the variational formulation (2.11) for \( u_m \) and equation (1.2) for \( d_m \) and pass to the limit as \( m \to \infty \). Then we can use the weak convergences (2.20)–(2.23) which imply the strong convergences \( u_m \rightharpoonup u \) in \( L^2(0, M; G_{\text{div}}) \) (and hence \( u_m \otimes u_m \rightharpoonup u \otimes u \) in \( L^1(0, M; L^1(\Omega)^{3 \times 3}) \)) and the assumed convergence for the sequence \( \{h_m\} \) to conclude that \( u \) satisfies the variational formulation (2.11) with external force \( h \) for every test function \( \varphi \in \mathcal{D}(\Omega)^3 \) with \( \text{div}(\varphi) = 0 \), and that \( d \) satisfies (1.2). By density and (2.60), the weak formulation for \( u \) is satisfied also for every \( \varphi \in W_0^{1,3}(\Omega) \) with \( \text{div}(\varphi) = 0 \).

It remains to prove that \( w \) satisfies the energy inequality (2.16) with external force \( h \) on \( [0, M] \). Let us first assume the convergence condition (a) for the sequence \( \{h_m\} \). We then consider (2.64), pass to the limit as \( m \to \infty \), use the strong and weak convergences for the sequences \( \{u_m\}, \{d_m\} \) and, on the left hand side of the inequality, the lower semicontinuity of the \( L^2(0, M; L^2(\Omega)) \)—norm and Fatou’s lemma for the nonlinear integral term. On the right hand side we use the fact that, since, by the Aubin–Lions lemma, we have the compact and continuous embeddings

\[
L^2(0, M; H^2(\Omega)^3) \cap H^1(0, M; L^{3/2}(\Omega)^3) \hookrightarrow \hookrightarrow L^2(0, M; H^{2-\varepsilon}(\Omega)^3),
\]  

\[
\hookrightarrow L^2(0, M; C(\Omega)^3),
\]  

and
for \(0 < \delta < 1/2\), then, up to a subsequence, we have \(d_m(s) \to d(s)\) in \(C(\bar{\Omega})^3\) for a.e. \(s \in [0, M]\) and therefore
\[
\int_{\Omega} W(d_m(s)) \to \int_{\Omega} W(d(s)), \quad \text{a.e. } s \in [0, M].
\]
We hence recover (2.16) for \(w\) with the forcing term \(h\).

On the other hand, if (b) holds, we can argue as in [5, chapter XV, proposition 1.1] and exploit the strong convergence \(u_m \to u\) in \(L^2(0, M; G_{div})\) which implies
\[
\int_s^t (h_m(\tau), u_m(\tau))d\tau \to \int_s^t (h(\tau), u(\tau))d\tau, \quad \text{as } m \to \infty.
\]

In both cases (a) and (b) we therefore conclude that \(w \in K_h^M\).

**Proof of theorem 3.** By proposition 1 the ball
\[
B_{W}^+(0, 2\Lambda_0) := \{w \in W^+_h: \rho_{W}^+(w, 0) \leq 2\Lambda_0\}
\]
is a uniformly (w.r.t. \(h \in \mathcal{H}(h_0)\)) absorbing set for the family \(\{K^+_h\}h \in \mathcal{H}(h_0)\). Such ball is also precompact in \(\Theta^+_{loc}\). The first part of theorem 2 entails the existence of the uniform (w.r.t. \(h \in \mathcal{H}(h_0)\)) trajectory attractor \(A^+_h(h_0) \subset B_{W}^+(0, 2\Lambda_0)\). This set is compact in \(\Theta^+_{loc}\) and, since \((t)\) is obviously continuous in \(\Theta^+_{loc}\), \(A^+_h(h_0)\) is also strictly invariant.

Furthermore, assuming also (W2) and that \(h_0\) is tr.-c. in \(L^2_{loc}(0, \infty; G_{div})\) or \(h_0 \in L^2_{loc}(0, \infty; G_{div})\), then proposition 2 implies that \(\{K^+_h\}h \in \mathcal{H}(h_0)\) is \((\Theta^+_{loc}, \mathcal{H}_h(h_0))\)-closed when (W2) holds true. Since \(\mathcal{H}_h(h_0)\) is a compact metric space, then \(K^+_h(h_0)\) is closed in \(\Theta^+_{loc}\) and the second part of theorem 2 allows the conclusion of the proof.

2.3. The trajectory attractor for a potential \(W\) with polynomial growth

The results on the existence of the trajectory attractor and on the closedness of the trajectory space can be recovered under alternative functional setting and assumptions on the potential. Indeed, let \(p \geq 2\) and for every \(M > 0\) introduce the space
\[
W_{p,M} := \left\{[u, d] \in L^\infty(0, M; G_{div} \times (V \cap L^p(\Omega)^3)) \cap L^2(0, M; V_{div} \times H^2(\Omega)^3): \begin{array}{l}
u, \in L^2(0, M; W^{-1,3/2}(\Omega)^3), d, \in L^2(0, M; L^{3/2}(\Omega)^3)\end{array}\right\}.
\]
The topology \(\Theta_{p,M}\) on \(W_{p,M}\) is now chosen to induce the following notion of weak convergence: a sequence \([u_m, d_m]\) \(\subset W_{p,M}\) is said to converge to \([u, d]\) \(\in W_M\) in \(\Theta_{p,M}\) if (2.20)–(2.23) hold and if in addition
\[
d_m \rightharpoonup^* d \quad \text{weakly}^* \text{ in } L^\infty(0, M; L^p(\Omega)^3).
\]
Then define
\[
W_{p,loc}^+ := \left\{[u, d] \in L^\infty_{loc}(0, \infty; G_{div} \times (V \cap L^p(\Omega)^3)) \cap L^2_{loc}(0, \infty; V_{div} \times H^2(\Omega)^3): \begin{array}{l}
u, \in L^2_{loc}(0, \infty; W^{-1,3/2}(\Omega)^3), d, \in L^2_{loc}(0, \infty; L^{3/2}(\Omega)^3)\end{array}\right\},
\]
endowed with its inductive limit topology \(\Theta_{p,loc}^+\), and
\[
W_{p,b}^+ := \left\{[u, d] \in L^\infty(0, \infty; G_{div} \times (V \cap L^p(\Omega)^3)) \cap L^2_{p,b}(0, \infty; V_{div} \times H^2(\Omega)^3): \begin{array}{l}
u, \in L^2_{p,b}(0, \infty; W^{-1,3/2}(\Omega)^3), d, \in L^2_{p,b}(0, \infty; L^{3/2}(\Omega)^3)\end{array}\right\},
\]

(2.70)
which is now a Banach space with the norm

\[ \|w_1 - w_2\|_{W_p} := \|w_1 - w_2\|_{L^2(0,\infty;G_{\text{div}}, (V\cap L^p(\Omega)))} + \|w_1 - w_2\|_{L^2(0,\infty;V_{\text{div}}, H^1(\Omega))} \]

\[ + \|u_1 - (u_2)\|_{L^2(0,\infty;W^{1,3/2}(\Omega))} + \|d_1 - (d_2)\|_{L^2(0,\infty;L^2(\Omega))}, \]  

for every \( w_1 = [u_1, d_1], w_2 = [u_2, d_2] \in W_p^+ \). On the potential \( W \) we now need the following assumption

(W3) There exist two positive constants \( C_1, C_2 \) and \( p \in (2, +\infty) \) such that

\[ C_1(|d|^p - 1) \leq W(d) \leq C_2(1 + |d|^p), \quad \forall d \in \mathbb{R}^3, \]

Remark 4. Let us note that assumption (W3) is satisfied with \( p = 4 \) in the standard double-well potential case \( W(d) = (|d|^2 - 1)^2 \).

For every \( h \in L^2_{\text{loc}}([0, \infty); V_{\text{div}}') \), the trajectory space \( K^{h, p} \) can be defined exactly as in definition 4 with the additional requirement that

\[ d \in L^2_{\text{loc}}([0, \infty); L^p(\Omega)^3). \]

Thanks to (W3), then, if assumptions (2.12), (2.13) are satisfied, theorem 1 ensures that for every \( w_0 = [u_0, d_0] \) such that \( u_0 \in G_{\text{div}}, d_0 \in V \cap L^p(\Omega)^3 \) and every \( h \in L^2_{\text{loc}}([0, \infty); V_{\text{div}}') \) there exists a trajectory \( w \in K^{h, p} \) such that \( w(0) = w_0 \). Furthermore, the space \( K^{h, p} \) of trajectories on the interval \([0, M]\) can be defined in an obvious way, as well as the united trajectory space

\[ K^{h, p} := \bigcup_{h \in H_0} K^{h, p}. \]

In place of (W1) on the potential \( W \) we shall therefore make the following assumption

(W1)* \( W \) satisfies (2.12), (2.13), and (2.33).

Instead of theorem 3 we can now prove the following.

Theorem 4. Assume that (W1)* and (W3) hold and that \( h_0 \in L^2_{\text{loc}}(0, \infty; V_{\text{div}}') \). Then, \( \{T(t)\} \) acting on \( K^{h, p}(h_0) \) possesses the uniform (w.r.t. \( h \in H_0) \) trajectory attractor \( A_{p,H_0}(h_0) \). This set is strictly invariant, bounded in \( W_{p,b}^+ \), compact in \( \Theta_{p,\text{loc}}^+ \). In addition, if \( h_0 \) is tr.-c. in \( L^2_{\text{loc}}(0, \infty; V_{\text{div}}') \) or \( h_0 \in L^2(0, \infty; G_{\text{div}}, \text{ then } K^{h, p}(h_0) \) is closed in \( \Theta_{p,\text{loc}}^+ \), \( A_{p,H_0}(h_0) \subset K^{h, p}(h_0) \) and

\[ A_{p,H_0}(h_0) = A_{p,\text{loc}}(H_0(h_0)). \]

Similarly to theorem 3, theorem 4 is a consequence of two propositions. The first one concerns with a dissipative estimate, and the second one establishes the closure property of the space of trajectories. 

Proposition 3. Let (W1)* and (W3) be satisfied and assume that \( h_0 \in L^2_{\text{loc}}(0, \infty; V_{\text{div}}') \). Then, for every \( h \in H_0(h_0) \) we have \( K^{h, p} \subset W_{p,b}^+ \) and

\[ \|T(t)w\|_{W_{p}^+} \leq \Gamma(\|w\|_{W_{p}^+})e^{-\sigma t} + \Gamma_0, \quad \forall t \geq 1, \]

for every \( w \in K^{h, p} \). Here, \( \Gamma_0, \sigma \) and \( \Gamma \) are two positive constants and a monotone positive increasing function, respectively, (independent of \( w \)) that depend on \( W, \Omega, v \) and \( p \), with only \( \Gamma_0 \), depending on the \( L^2_{\text{loc}}(0, \infty; V_{\text{div}}') \)-norm of \( h_0 \).
On the other hand, (2.49) and (2.50) will now be replaced by it is easy to check that estimate (2.35) still holds and also that (2.45)–(2.48) can be rewritten. On the other hand, (2.49) and (2.50) will now be replaced by

\[ E(w) \geq c_6 \left( \|u\|^2 + \|d\|_V^2 + \|d\|_{L^p(\Omega)}^p \right) - c_7, \]  

(2.76)

\[ \sup_{s \in (0,1)} E(w(s)) \leq \frac{1}{2} \|u\|_{L^\infty(0,1; G_{\text{opt}})}^2 + \frac{1}{2} \|\nabla d\|_{L^\infty(0,1; L^1(\Omega)^{3\times3})}^2 + c_8 \|d\|_{L^\infty(0,1; L^1(\Omega)^{3\times3})}^p + c_9, \]  

(2.77)

respectively. Here all nonnegative constants \( c_i \) depend possibly on \( W, \Omega, v \) and \( p \), but do not depend either on the solution \( w \), or on \( h_0 \). Hence, in place of (2.51) we get

\[ \|u\|^2 + \|d\|_V^2 + \|d\|_{L^p(\Omega)}^p \leq c_{10} \left( \|u\|_{L^\infty(0,1; G_{\text{opt}})}^2 + \|\nabla d\|_{L^\infty(0,1; L^1(\Omega)^{3\times3})}^2 + \|d\|_{L^\infty(0,1; L^1(\Omega)^{3\times3})}^p \right), \]

\[ \times e^{-\gamma t} + K^2 + c_{11}, \quad \forall t \geq 1, \]  

(2.78)

the constant \( K \) being given as in the proof of proposition 1. Hence we have

\[ \|T(t)u\|_{L^\infty(0,\infty; G_{\text{opt}})} + \|T(t)d\|_{L^\infty(0,\infty; V_1)} + \|T(t)d\|_{L^\infty(0,\infty; L^p(\Omega)^{3\times3})} \]

\[ \leq c_{12} \left( \|u\|_{L^\infty(0,1; G_{\text{opt}})} + \|\nabla d\|_{L^\infty(0,1; L^1(\Omega)^{3\times3})} + \|d\|_{L^\infty(0,1; L^1(\Omega)^{3\times3})}^{p/2} \right), \]

\[ \times e^{-\frac{\gamma t}{2}} + K + c_{13}, \quad \forall t \geq 1. \]  

(2.79)

Once we have (2.79), for the remaining part of the proof we can argue as in the proof of proposition 1. At the end we arrive at (2.75) with \( \sigma = k/p, \Gamma(R) = c_{14} R^p \) and \( \Gamma_0 = c_{15} K^2 + c_{16} \).

\[ \square \]

Proposition 4. Assume that (W1)* and (W3) are satisfied. Let \( w_m := [u_m, d_m] \in K_{p, h_m}^M \) be such that \( \{w_m\} \) converges to \( w := [u, d] \) in \( \Theta_{p,H} \) and assume that \( \{h_m\} \) and \( h \) satisfy (a) or (b) from proposition 2. Then \( w \in K_{p,H}^M \).

\[ \square \]

3. The case of non-homogeneous Dirichlet boundary conditions for \( d \)

Let us now consider the physically relevant case of the non-homogeneous Dirichlet boundary condition for \( d \)

\[ d|_\Gamma = g. \]  

(3.80)

where the boundary datum \( g \) is supposed to be at least such that

\[ g \in H^1_{\text{loc}}(0, \infty); H^{-1/2}(\Gamma)^3 \cap L^2_{\text{loc}}(0, \infty); H^{3/2}(\Gamma)^3. \]

This condition, together with (2.29), (2.30) and with the compatibility condition

\[ d_0|_\Gamma = g(0), \]
ensure the existence of a global in time weak solution on \([0, \infty)\) corresponding to \(u_0, d_0\) and 
\(g, h\) with the regularity properties (2.7)–(2.10) and satisfying the following energy inequality 
(see [4])

\[
\mathcal{E}(w(t)) + \int_s^t \left( \|\Delta d + \nabla u\|^2 + v \|\nabla u\|^2 \right) dt \leq \mathcal{E}(w(s)) + \int_s^t (g, \partial_t d)_{H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)} dt + \int_s^t (h, u) dt, \quad w := [u, d] 
\]

(3.81)

for all \(t \geq s\), for a.e. \(s \in (0, \infty)\), including \(s = 0\), where the energy functional \(\mathcal{E}\) is the same as for the case of homogeneous Neumann boundary condition for \(d\) (see theorem 1).

We can recover the result on the existence of the trajectory attractor also for the case of the non-homogeneous boundary condition for \(d\), assuming that either (W1) or (W1)\(^*\) and (W3) holds for the potential \(W\). Indeed, if (W1) holds, introducing the spaces \(\mathcal{W}_{p,M, \gamma, \alpha}^*\) and \(\mathcal{V}_{h}^*\) defined as in (2.19), (2.24) and (2.25), respectively, we can define the trajectory space \(K_{g,h}^{*}\) of system (1.1)–(2.4), (3.80) with boundary datum \(g\) and external force \(h\) as the set of all weak solutions \(w = [u, d]\) on the time interval \([0, \infty)\) to this system satisfying the energy inequality (3.81) for all \(t \geq s\) and for a.a. \(s \in (0, \infty)\). On the other hand, if (W1)\(^*\) and (W3) hold, the spaces \(\mathcal{W}_{p,M, \gamma, \alpha}^*\), \(\mathcal{V}_{p,loc, \gamma, \alpha, \alpha}^*\), \(\mathcal{V}_{p,b, \gamma, \alpha, \alpha}^*\), defined as in (2.67), (2.69) and (2.70), respectively, can be introduced and the trajectory space \(K_{p, h}^{*}\) can be defined in a similar way. The definition of the trajectory spaces \(K_{g,h}^{*}\) and \(K_{p, h}^{*}\) on the bounded time interval \([0, M]\) is obvious.

Let us now introduce the symbol spaces for the Dirichlet datum \(g\)

\[
\mathcal{E}_M := \{g \in C([0, M]; H^{3/2}(\Gamma)^3) : g_0 \in L^2(0, M; H^{-1/2}(\Gamma)^3)\}, \\
\mathcal{E}^\gamma_{loc} := \{g \in C([0, \infty); H^{3/2}(\Gamma)^3) : g_0 \in L^2_{loc}(0, \infty; H^{-1/2}(\Gamma)^3)\}, \\
\mathcal{E}_{loc, w} := \{g \in C([0, \infty); H^{3/2}(\Gamma)^3) : g_0 \in L^2_{loc,w}(0, \infty; H^{-1/2}(\Gamma)^3)\}. 
\]

and also

\[
\mathcal{E}^\gamma_{loc, w} := \{g \in C([0, \infty); H^{3/2}(\Gamma)^3) : g_0 \in L^2_{loc,w}(0, \infty; H^{-1/2}(\Gamma)^3)\}. 
\]

Take then

\[
g_0 \text{ tr.-c. in } C([0, \infty); H^{3/2}(\Gamma)^3), \quad (g_0) \in L^2_{loc}(0, \infty; H^{-1/2}(\Gamma)^3), \quad (3.82) 
\]

so that \(g_0\) is tr.-c. in \(\mathcal{E}^\gamma_{loc, w}\) and set

\[
\mathcal{H}_+(g_0) := \{T(t)g_0, t \geq 0\} |_{\mathcal{E}^\gamma_{loc, w}}. 
\]

(3.83)

We shall also assume that \(g_0\) is tr.-c. in \(\mathcal{E}^\gamma_{loc, w}\). It can be proved that in this case the hull of \(g_0\) (defined as in (3.83) with the closure in \(\mathcal{E}^\gamma_{loc, w}\)) coincides with the hull \(\mathcal{H}_+(g_0)\) defined as in (2.31). The united trajectory spaces are now given by

\[
K_{\mathcal{H}+(g_0) \times \mathcal{H}_+(h_0)}^{\mathcal{H}_+(g_0) \times \mathcal{H}_+(h_0)} := \bigcup_{g \in \mathcal{H}+(g_0), h \in \mathcal{H}_+(h_0)} K_{g,h}^{\mathcal{H}_+(g_0) \times \mathcal{H}_+(h_0)} 
\]

or by

\[
K_{\mathcal{H}+(g_0) \times \mathcal{H}_+(h_0)}^{\mathcal{H}_+(g_0) \times \mathcal{H}_+(h_0)} := \bigcup_{g \in \mathcal{H}+(g_0), h \in \mathcal{H}_+(h_0)} K_{p, h}^{\mathcal{H}_+(g_0) \times \mathcal{H}_+(h_0)} 
\]

We therefore can state the following.

**Theorem 5.** Assume that (W1) ((W1)* and (W3)) holds and that \(g_0\) is tr.-c. in \(C([0, \infty); H^{3/2}(\Gamma)^3)\) with \((g_0)_t \in L^2_{loc}(0, \infty; H^{-1/2}(\Gamma)^3)\), and \(h_0 \in L^2_{loc}(0, \infty; \mathcal{V}_{h1}^\gamma)\). Then, \([T(t)]\) acting on \(K_{\mathcal{H}+(g_0) \times \mathcal{H}_+(h_0)}^{\mathcal{H}_+(g_0) \times \mathcal{H}_+(h_0)} (K_{\mathcal{H}+(g_0) \times \mathcal{H}_+(h_0)})\) possesses the uniform (w.r.t. \([g, h] \in \mathcal{H}_+(g_0) \times \mathcal{H}_+(h_0)\)) trajectory attractor \(A_{\mathcal{H}+(g_0) \times \mathcal{H}_+(h_0)} (A_{\mathcal{H}+(g_0) \times \mathcal{H}_+(h_0)})\). This set is strictly
invariant, bounded in \(W^p_b(V^p,b)\) and compact in \(\Theta^p_{loc}(\Theta^p_{loc})\). In addition, if (W2) holds (or if (W1) holds and (W3) hold), if \(g_0\) is tr.-c. in \(\mathbb{R}^p\) and if \(h_0\) is tr.-c. in \(L^2_{loc}(0, \infty; V^p_{div})\) or \(h_0 \in L^2_{\mu}(0, \infty; G^{\infty}_{div})\), then \(K^p_{\kappa\ell}(g_0) \times \Gamma_{\kappa\ell}(h_0)\) is closed in \(\Theta^p_{loc}(\Theta^p_{loc})\).

Proof. We can easily recover a dissipative estimate of the form (2.44) (or (2.75)) and also the closure property of the space of trajectories. Let us start by proving the dissipative inequality, taking, e.g., assumption (W1). First observe that, due to the convexity of \(W_1\) and to (2.33), we have

\[
\| - \Delta d + \nabla dW(d) \|^2 \geq \frac{1}{2} \| \Delta d \|^2 + \frac{1}{2c_0} \int_\Omega W_1(d) - \frac{1}{2} |\Omega|
\]

and compact in \(\Theta^p_{loc}(\Theta^p_{loc})\) and

\[
\mathcal{A}_{\kappa\ell}(g_0) \times \Gamma_{\kappa\ell}(h_0) = \mathcal{A}_{\kappa\ell}(g_0) \times \Gamma_{\kappa\ell}(h_0)
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\]
for all \( t \geq s \) and for a.a. \( s \in (0, \infty) \), where \( k' = \min(\alpha_S/2, \alpha_T, \alpha_V) \), \( l' = \alpha_b\|\Omega\| \) and
\[
m(t) := 2\alpha^2\|g(t)\|_{H^{-1/2}(\Gamma)} + \alpha_b\|g(t)\|_{H^{-1/2}(\Gamma)} + \alpha_d\|\nabla_dW_1(g(t))\|_{L^2(\Gamma)}^2 + \frac{1}{2\nu}\|h(t)\|_{V_d}^2.
\]
(3.88)

Once we have (3.87), it is not difficult to argue as in the proof of proposition 1 (notice in particular that an estimate similar to (2.55) can be obtained in this case by exploiting once again the elliptic regularity estimates for \( d \) already used above). In particular, it is easy to check that, due to the assumption (3.82) on \( g_0 \), for every \( g \in \mathcal{H}_s(g_0) \) we have
\[
\|g\|_{L^2(0,\infty;H^{1/2}(\Gamma))} + \|g_0\|_{L^2(0,\infty;H^{1/2}(\Gamma))}, \quad \|g(t)\|_{L^2(0,\infty;H^{1/2}(\Gamma))} \leq \|\nabla_dW_1(g(t))\|_{L^2(\Gamma)} + \|h(t)\|_{V_d}.
\]

Therefore, we can first prove that \( K_{g,b}^s \subset V_d^3 \) and then recover an inequality of the form (2.44), with \( k' \) in place of \( k \) and with \( \Lambda_0 \) depending on the constants \( \alpha_i \) and on the norms \( \|g_0\|_{L^2(0,\infty;H^{1/2}(\Gamma))} \), \( \|g_0\|_{L^2(0,\infty;H^{1/2}(\Gamma))} \), \( \|\nabla_dW_1(g(t))\|_{L^2(\Gamma)} \) and \( \|h_0\|_{L^2(0,\infty;V_d)} \). If assumption (W1)* and (W3) hold in place of (W1) we can argue as above and as in proposition 3 and recover a dissipative estimate in the form (2.75), where the constant \( \Gamma_0 \) now depends also on the above norms of \( g_0 \), \( g_0 \), and \( \nabla_dW_1(g_0) \). We omit the details.

Let us now prove the closure property of the space of trajectories, assuming first, e.g., assumptions (W1) and (W2). Let \( g_m \in \mathcal{M}_M \), \( h_m \in L^2(0,M;V_\text{div}) \) and \( w_m \in K_{g,b}^m \) with \( w_m := [u_m, d_m] \) such that \( w_m \to w \) in \( \Theta_M \), \( h_m \to h \) in \( L^2(0,M;V_\text{div}) \) and \( g_m \to g \) in \( \mathcal{M}_M \).

Then, we claim that \( w \in K_{g,b}^* \). Indeed, we can argue as in the proof of proposition 2. In particular, in the energy inequality (3.81), written for each \( w_m \) and corresponding to \( g_m \) and \( h_m \), the second term on the right hand side can be estimated as
\[
\left| \int_s^t \langle (g_m)_t, \partial_\nu d_m \rangle d\tau \right| \leq c\|g_m\|_{L^2(0,M;H^{1/2}(\Gamma))} \|d_m\|_{L^2(0,M;H^{1/2}(\Gamma))} \leq c,
\]
for all \( t \in [0,M] \) and a.a. \( s \in [0,M] \) with \( t \geq s \), due to the convergence assumption on the sequence \( \{g_m\} \) and to (2.22). Hence, using (W2) and the convergence assumption on \( \{h_m\} \) we again infer that the right hand side of (3.81) is bounded and recover the control (2.65). In order to prove that \( w \) is a weak solution corresponding to \( g \) and \( h \) satisfying the energy inequality (3.81), we notice in particular that the convergence assumption on \( \{g_m\} \) and (2.22) imply that \( d_1 = g \) and furthermore that
\[
\int_s^t \langle (g_m)_t, \partial_\nu d_m \rangle d\tau \to \int_s^t \langle g_t, \partial_\nu d \rangle d\tau.
\]
Hence \( \{K_{g,b}^M(g_m)_t \}_{g_m \in \mathcal{M}_M} \subset \{(\Theta_M, \mathcal{M}_M \times L^2(0,M;V_\text{div})\} \)-closed.

Furthermore, if the convergence assumption on \( \{h_m\} \) is replaced by \( h_m \to h \), weakly in \( L^2(0,M;G_\text{div}) \), then, arguing as in [5, chapter XV, proposition 1.1] (see the end of the proof of proposition 2), we can still conclude that \( \{K_{g,b}^M \}_{g \in \mathcal{M}_M} \subset \{(\Theta_M, \mathcal{M}_M \times L^2(0,M;G_\text{div})\} \)-closed.

Finally, if (W1)* and (W3) hold, it is easy to check that \( \{K_{g,b}^M \}_{g \in \mathcal{M}_M} \subset \{(\Theta_{p,M}, \mathcal{M}_M \times L^2(0,M;V_\text{div})\} \)-closed and that \( \{K_{p,g,b}^M \}_{g \in \mathcal{M}_M} \subset \{(\Theta_{p,M}, \mathcal{M}_M \times L^2(0,M;G_\text{div})\} \)-closed, by the same argument as above and proposition 4. This concludes the proof of theorem 5.

\[ \square \]

**Remark 5.** If (W1)* and (W3) hold the same conclusions of theorem 5 still hold under a different assumption on the Dirichlet datum \( g_0 \). Indeed, suppose that \( W_1 \) satisfies the additional assumption
\[
|\nabla_dW_1(d)| \leq C_5(1 + |d|^{p-1}), \quad \forall d \in \mathbb{R}^3, \quad p > 2,
\]
(3.89)
(compare with (W3)), and introduce the symbol space
\[ \mathbb{S}^+_\text{loc} := \{ g \in L^2_{\text{loc}}([0, \infty); H^{3/2}(\Gamma))^3 : g_t \in L^2_{\text{loc}}([0, \infty); H^{-1/2}(\Gamma)^3) \}, \]
and also
\[ \mathbb{S}^+_{\text{loc}, w} := \{ g \in L^2_{\text{loc}, w}([0, \infty); H^{3/2}(\Gamma)^3 : g_t \in L^2_{\text{loc}, w}([0, \infty); H^{-1/2}(\Gamma)^3) \} . \]
Assume that
\[ g_0 \in L^2_{\text{loc}}(0, \infty; H^{3/2}(\Gamma)^3) , \quad g_t \in L^2_{\text{loc}}(0, \infty; H^{-1/2}(\Gamma)^3) . \]
Then, if we define the hull \( \mathcal{H}_+(g_0) \) as in (3.83) with now the closure in \( \mathbb{S}^+_{\text{loc}, w} \), by arguing as in the proof above and using assumption (3.89) in (3.86), we can still get a dissipative estimate in the form (2.75) with the constant \( \Gamma_0 \) now depending on the norms \( \| g_0 \|_{L^2_{\text{loc}}(0, \infty; H^{3/2}(\Gamma)^3)} \), \( \| (g_0)_t \|_{L^2_{\text{loc}}(0, \infty; H^{-1/2}(\Gamma)^3)} \), and \( \| \nabla w(t) \|_{L^2(0, \infty; L^2(\Gamma))} \). Finally, assuming in addition that \( (g_0)_t \) is tr.-c. in \( L^2_{\text{loc}}([0, \infty); H^{-1/2}(\Gamma)^3) \), then the closure property of the space of trajectories can be recovered as well.

**Remark 6.** Notice that if \( w_m \to w \) in \( \Theta_M \), then due to the compact embedding (2.66), we have
\[ \partial_t u_m \to \partial_t u , \text{ strongly in } L^2(0, M; H^{1/2}(\Gamma)^3) , \]
for every \( 0 < \delta < 1/2 \). Therefore, as far as the closure property of the space of trajectories is concerned, the assumption on \( (g_0)_t \) in theorem 5 could be replaced by
\[ (g_0)_t \in L^2_{\text{loc}}(0, \infty; H^{-1/2}(\Gamma)^3) , \quad 0 < \delta < 1/2 . \]

### 4. Further properties of the trajectory attractor

Let us consider only the case of a homogeneous Neumann boundary condition for \( d \) and refer to both functional settings and assumptions on the potential introduced in the previous section. The results of this section can be reproduced also for the case of non-homogeneous Dirichlet boundary condition for \( d \) without any difficulty (see theorem 5).

We start to discuss some structural properties of the trajectory attractor.

Denote by \( Z(h_0) := Z(\mathcal{H}_+(h_0)) \) the set of all complete symbols in \( \omega(\mathcal{H}_+(h_0)) \). Recall that a function \( \zeta \in L^2_{\text{loc}}(\mathbb{R}; V^+_{\text{loc}}) \) is a complete symbol in \( \omega(\mathcal{H}_+(h_0)) \) if \( \Pi_{-2}(\Pi_{0, \infty}) \zeta \in \omega(\mathcal{H}_+(h_0)) \) for all \( t \in \mathbb{R} \), where \( \Pi_{-2} = \Pi_{0, \infty} \). It can be proved (see [6, section 4] or [5, chapter XIV, section 2]) that, due to the strict invariance of \( \omega(\mathcal{H}_+(h_0)) \), given a symbol \( h \in \omega(\mathcal{H}_+(h_0)) \) there exists at least one complete symbol \( \tilde{h} \) (not necessarily unique) which is an extension of \( h \) on \( (-\infty, 0] \) and such that \( \Pi_{-2}(\Pi_{0, \infty}) \zeta \in \omega(\mathcal{H}_+(h_0)) \) for all \( t \in \mathbb{R} \). Note that we have \( \Pi_{-2}Z(h_0) = \omega(\mathcal{H}_+(h_0)) \).

Let us refer first to the functional setting introduced in theorem 3.

To every complete symbol \( \zeta \in Z(h_0) \) there corresponds by [5, chapter XIV, definition 2.5] (see also [6, definition 4.4]) the kernel \( K_{\zeta} \) in \( W_b \) which consists of the union of all complete trajectories which belong to \( W_b \), i.e., all weak solutions \( w = [u, d] : \mathbb{R} \to G_{\text{dv}} \times V \) with external force \( \zeta \in Z(h_0) \) (in the sense of definition 1 with the interval \([0, \infty)\) replaced by \( \mathbb{R} \)) satisfying (2.16) on \( \mathbb{R} \) (i.e., for all \( t \geq s \) and for a.a. \( s \in \mathbb{R} \) that belong to \( W_b \). We recall that the space \( (W_b, \rho_{W_b}) \) is defined as the space \( (W^0_{\text{loc}}, \rho_{W^0_{\text{loc}}}) \) (see (2.25) and (2.28)) with the time interval \((0, \infty)\) replaced by \( \mathbb{R} \). The space \( (W_{\text{loc}}, \Theta_{\text{loc}}) \) can be defined in the same way.

Set
\[ K_{Z(h_0)} := \bigcup_{\zeta \in Z(h_0)} K_{\zeta} . \]
Then, if the assumptions of theorem 3 hold we also have (see, e.g., [6, theorem 4.1])
\[ \mathcal{A}_{\mathcal{H}_+(h_0)} = \mathcal{A}_{\omega(\mathcal{H}_+(h_0))} = \Pi_+ K_{\mathcal{Z}(h_0)}, \]
and the set \( K_{\mathcal{Z}(h_0)} \) is compact in \( \Theta_{\text{loc}} \) and bounded in \( \mathcal{W}_0 \).

On the other hand, it is not difficult to see that, under the assumptions of theorem 3, \( K_\zeta \neq \emptyset \) for all \( \zeta \in \mathcal{Z}(h_0) \). Indeed, by virtue of [6, theorem 4.1] (see also [5, chapter XIV, theorem 2.1]), this is a consequence of the fact that the family \( \{ K_h \}_{h \in \mathcal{H}_+(h_0)} \) of trajectory spaces satisfies the following condition: there exists \( R > 0 \) such that \( B_{\mathcal{W}_0^s}(0, R) \cap K_h \neq \emptyset \) for all \( h \in \mathcal{H}_+(h_0) \). In order to check this condition fix an initial datum \( w_0^h = [u_0^h, d_0^h] \), with \( u_0^h, d_0^h \) taken as in theorem 1. We know that for every \( h \in \mathcal{H}_+(h_0) \) there exists a trajectory \( w_0^h \in K_h \) such that \( w_0^h(0) = w_0^h \) and such that the energy inequality (2.16) holds for all \( t \geq s \) and for a.a. \( s \in (0, \infty) \), including \( s = 0 \). Arguing as in proposition 1 (see (2.47) written for \( s = 0 \) and all \( t \geq 0 \) we get an estimate of the form \( \rho_{w_0^h}(w_0^h, h_0) \) (see also (2.32)), where the positive constant \( \Lambda \) depends on \( \mathcal{E}(w_0^h) \) and on the norm \( ||h_0||_{L^1(0, \infty; \mathcal{V}_{\text{loc}})} \). The above condition is thus fulfilled by choosing \( R = \Lambda (w_0^h, h_0) \).

In the case we consider the functional setting of theorem 4, we can similarly introduce the kernel \( K_{p, \zeta} \) in \( \mathcal{W}_{p,b} \), with the Banach space \( (\mathcal{W}_{p,b}, \rho_{\mathcal{W}_{p,b}}) \) always defined as the space \( (\mathcal{W}_{p,b}, \rho_{\mathcal{W}_{p,b}}) \) (see (2.70) and (2.72)) with the time interval \( (0, \infty) \) replaced by \( \mathbb{R} \). The space \( (\mathcal{W}_{p,\text{loc}}, \Theta_{p,\text{loc}}) \) can be defined in the same way from the space \( (\mathcal{W}_{p,b}, \Theta_{p,b}) \). Hence, in this case, if the assumptions of theorem 4 hold, we have
\[ K_{p,\zeta} \subset \subset \mathcal{W}_{p,b}, \]
and the set \( K_{p,\zeta}(h_0) \) is compact in \( \Theta_{p,\text{loc}} \) and bounded in \( \mathcal{W}_{p,b} \). The proof that \( K_{p,\zeta} \neq \emptyset \) for all \( \zeta \in \mathcal{Z}(h_0) \) is exactly the same as above.

As far as the attraction properties are concerned, we observe that, due to compactness results, the trajectory attractor attracts the subsets of the family \( B_{\mathcal{H}_+(h_0)}^+ \) (if we refer to theorem 3), or the subsets of \( K_{p,\mathcal{H}_+(h_0)}^+ \) bounded in \( \mathcal{W}_{p,b}^+ \) (if we refer to theorem 4), in some strong topologies. Indeed, set
\[
\mathcal{X}_{\delta_1, \delta_2} := H^1(\Omega)^3 \times H^{1+\delta_1} (\Omega)^3, \quad \mathcal{Y}_{\delta_1, \delta_2} := H^{-\delta_1} (\Omega)^3 \times H^{\delta_2} (\Omega)^3,
\]
where \( 0 \leq \delta_1, \delta_2 < 1 \) and \( 2 \leq s < p \). Then, by using the compact embeddings
\[
L^2(0, M; \mathcal{V}_{\text{div}} \times H^2(\Omega)^3) \hookrightarrow H^1(0, M; W^{-1,3/2}(\Omega)^3 \times L^{3/2}(\Omega)^3) \hookrightarrow \hookrightarrow L^2(0, M; \mathcal{X}_{\delta_1, \delta_2}),
\]
\[
L^\infty(0, M; \mathcal{G}_{\text{div}} \times \mathcal{V}) \hookrightarrow L^1(0, M; W^{-1,3/2}(\Omega)^3 \times L^{3/2}(\Omega)^3) \hookrightarrow C([0, M]; \mathcal{Y}_{\delta_1, \delta_2}),
\]
\[
L^\infty(0, M; \mathcal{G}_{\text{div}} \times (\mathcal{V} \cap L^p(\Omega)^3)) \hookrightarrow H^1(0, M; W^{-1,3/2}(\Omega)^3 \times L^{3/2}(\Omega)^3) \hookrightarrow C([0, M]; \mathcal{Y}_{\delta_1, \delta_2})
\]
then theorem 3 and theorem 4 imply the following two corollaries (see [5, chapter XIV, theorem 2.2]).

**Corollary 1.** Let \((W1)\) and \((W2)\) hold and assume \( h_0 \in L^2_{\text{loc}}(0, \infty; \mathcal{V}_{\text{div}}^0) \). Then, for every \( 0 \leq \delta_1, \delta_2 < 1 \) the trajectory attractor \( \mathcal{A}_{\mathcal{H}_+(h_0)} \) from theorem 3 is compact in \( L^2_{\text{loc}}([0, \infty); \mathcal{X}_{\delta_1, \delta_2}) \cap C([0, \infty); \mathcal{Y}_{\delta_1, \delta_2}) \), bounded in \( L^2_{\text{loc}}(0, \infty; \mathcal{X}_{\delta_1, \delta_2}) \cap C_b([0, \infty); \mathcal{Y}_{\delta_1, \delta_2}) \), and for every \( B \in B_{\mathcal{H}_+(h_0)}^+ \) and every \( M > 0 \) we have, for \( t \to +\infty \)
\[
dist_{\mathcal{L}^2_{\text{loc}}([0, M]; \mathcal{X}_{\delta_1, \delta_2})} (\Pi_{[0, M]} T(t) B, \Pi_{[0, M]} \mathcal{A}_{\mathcal{H}_+(h_0)}) \to 0,
\]
\[
dist_{C([0, M]; \mathcal{Y}_{\delta_1, \delta_2})} (\Pi_{[0, M]} T(t) B, \Pi_{[0, M]} \mathcal{A}_{\mathcal{H}_+(h_0)}) \to 0.
\]
Corollary 2. Let \((W1)^*\) and \((W3)\) hold and assume \(h_0 \in L^2_{loc}(0, \infty; V^l_{div})\). Then, for every \(0 < \delta_1, \delta_2 < 1\) and every \(2 < s < p\) the trajectory attractor \(A_{p,\mathcal{H},(h_0)}\) from theorem 4 is compact in \(L^2_{loc}(0, \infty; X_{\delta_1,\delta_2}) \cap \mathcal{C}_h((0, \infty); Y^l_{\delta_1,\delta_2})\), bounded in \(L^2(0, \infty; X_{\delta_1,\delta_2}) \cap C^0((0, \infty); Y^l_{\delta_1,\delta_2})\), and for every \(B \subset K^+_{p,\mathcal{H},(h_0)}\) bounded in \(Y^l_{p,b}\) and every \(M > 0\) we have, for \(t \to +\infty\)

\[
\begin{align*}
\text{dist}_{L^2(0, \infty; X_{\delta_1,\delta_2})} \left( \Pi_{[0,M]} T(t) B, \Pi_{[0,M]} A_{p,\mathcal{H},(h_0)} \right) & \to 0, \\
\text{dist}_{C^0((0, \infty); Y^l_{\delta_1,\delta_2})} \left( \Pi_{[0,M]} T(t) B, \Pi_{[0,M]} A_{p,\mathcal{H},(h_0)} \right) & \to 0.
\end{align*}
\]

In corollary 1 and corollary 2 we have denoted by \(\text{dist}_X(A, B)\) the Hausdorff semidistance in the Banach space \(X\) between \(A, B \subset X\).

Let us now define, for every \(B \subset K^+_{\mathcal{H},(h_0)}\) and every \(B_p \subset K^+_{p,\mathcal{H},(h_0)}\), the sections

\[
B(t) := \left\{ [u(t), \varphi(t)] : [u, \varphi] \in B \right\} \subset Y^l_{\delta_1,\delta_2}, \quad t \geq 0,
\]

\[
B_p(t) := \left\{ [u(t), \varphi(t)] : [u, \varphi] \in B_p \right\} \subset Y^l_{\delta_1,\delta_2}, \quad t \geq 0. \tag{4.92}
\]

Similarly we set

\[
\begin{align*}
\mathcal{A}_{\mathcal{H},(h_0)}(t) := & \left\{ [u(t), \varphi(t)] : [u, \varphi] \in \mathcal{A}_{\mathcal{H},(h_0)} \right\} \subset Y^l_{\delta_1,\delta_2}, \quad t \geq 0, \\
\mathcal{K}_Z(h_0) := & \left\{ [u(t), \varphi(t)] : [u, \varphi] \in \mathcal{K}_Z(h_0) \right\} \subset Y^l_{\delta_1,\delta_2}, \quad t \in \mathbb{R}, \\
\mathcal{A}_{p,\mathcal{H},(h_0)}(t) := & \left\{ [u(t), \varphi(t)] : [u, \varphi] \in \mathcal{A}_{p,\mathcal{H},(h_0)} \right\} \subset Y^l_{\delta_1,\delta_2}, \quad t \geq 0, \\
\mathcal{K}_{p,Z}(h_0)(t) := & \left\{ [u(t), \varphi(t)] : [u, \varphi] \in \mathcal{K}_{p,Z}(h_0) \right\} \subset Y^l_{\delta_1,\delta_2}, \quad t \in \mathbb{R}.
\end{align*}
\]

Then, as a further consequence of theorem 3 and theorem 4 we have (see [5, chapter XIV, definition 2.6, corollary 2.2]) the following two corollaries.

Corollary 3. Let \((W1)\) and \((W2)\) hold and assume that \(h_0\) is tr.-c. in \(L^2_{loc}(0, \infty; V^l_{div})\) or \(h_0 \in L^2_{\mathcal{H}}(0, \infty; \mathcal{G}_{div})\). Then the bounded subset

\[ A_{gl} := \mathcal{A}_{\mathcal{H},(h_0)}(0) = \mathcal{K}_Z(h_0)(0) \]

is the uniform (w.r.t. \(h \in \mathcal{H}_s(h_0)\)) global attractor in \(Y^l_{\delta_1,\delta_2}\), \(0 < \delta_1, \delta_2 < 1\), of system \((1.1)-(2.5)\), namely (i) \(A_{gl}\) is compact in \(Y^l_{\delta_1,\delta_2}\), (ii) \(A_{gl}\) satisfies the attracting property

\[
\text{dist}_{Y^l_{\delta_1,\delta_2}} (B(t), A_{gl}) \to 0, \quad t \to +\infty,
\]

for every \(B \subset \mathcal{B}_{\mathcal{H},(h_0)^+}\), and (iii) \(A_{gl}\) is the minimal set satisfying (i) and (ii).

Corollary 4. Let \((W1)^*\) and \((W3)\) hold and assume that \(h_0\) is tr.-c. in \(L^2_{\mathcal{H}}(0, \infty; V^l_{div})\) or \(h_0 \in L^2_{\mathcal{H}}(0, \infty; \mathcal{G}_{div})\). Then the bounded subset

\[ A_{p,gl} := \mathcal{A}_{p,\mathcal{H},(h_0)}(0) = \mathcal{K}_{p,Z}(h_0)(0) \]

is the uniform (w.r.t. \(h \in \mathcal{H}_s(h_0)\)) global attractor in \(Y^l_{\delta_1,\delta_2}\), \(0 \leq \delta_1, \delta_2 < 1\) and \(2 < s < p\), of system \((1.1)-(2.5)\), namely (i) \(A_{p,gl}\) is compact in \(Y^l_{\delta_1,\delta_2}\), (ii) \(A_{p,gl}\) satisfies the attracting property

\[
\text{dist}_{Y^l_{\delta_1,\delta_2}} (B(t), A_{p,gl}) \to 0, \quad t \to +\infty,
\]

for every \(B_p \subset \mathcal{K}_{p,\mathcal{H},(h_0)}\), bounded in \(Y^l_{p,b}\), and (iii) \(A_{p,gl}\) is the minimal set satisfying (i) and (ii).
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