Semisolid sets and topological measures

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Abstract

This paper is one in a series that investigates topological measures on locally compact spaces. A topological measure is a set function which is finitely additive on the collection of open and compact sets, inner regular on open sets, and outer regular on closed sets. We examine semisolid sets and give a way of constructing topological measures from solid-set functions on locally compact, connected, locally connected spaces. For compact spaces our approach produces a simpler method than the current one. We give examples of finite and infinite topological measures on locally compact spaces and present an easy way to generate topological measures on spaces whose one-point compactification has genus 0. Results of this paper are necessary for various methods for constructing topological measures, give additional properties of topological measures, and provide a tool for determining whether two topological measures or quasi-linear functionals are the same.

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1 Introduction

This paper deal with the theory of topological measures and quasi-linear functionals. The origins of the theory are connected to quantum physics and have a fascinating history. Mathematical interpretations of quantum physics by G. W. Mackey and R. V. Kadison (34, 35, 31) led to very interesting mathematical problems. Let $R$ be a von Neumann algebra, and let $P$ be the lattice of orthogonal projections in $R$. The extension problem asks whether given a measure $\mu$ on $P$ there exists a positive state $\rho$ on $R$ such that $\rho|P = \mu$. A. Gleason (26) obtained the first important affirmative answer when $R$ is the family of all bounded linear operators on a separable Hilbert space $H$ with $\dim H \geq 3$. The extension problem can be viewed as a special case of the following linearity problem for quasi-states (see 1, 2). Let $\mathcal{A}$ be a $C^*$-algebra with identity 1. A quasi-state is a function $\rho : \mathcal{A} \to \mathbb{C}$ which is a state on each $C^*$-subalgebra of $\mathcal{A}$ generated by a single self-adjoint $a \in \mathcal{A}$ and 1, and which satisfies $\rho(a + ib) = \rho(a) + i\rho(b)$ for self-adjoint $a, b \in \mathcal{A}$. The problem is to determine whether $\rho$ is linear. In 1 Theorem 1] J. Aarnes claimed that any positive quasi-linear functional $\rho$ on an abelian $C^*$-algebra is linear. However, C. Akemann and M. Newberger found a gap in the proof (see 8). It turned out that the gap was unbridgeable, which J. Aarnes demonstrated almost twenty years later. Any abelian unital $C^*$-algebra is isomorphic to $C(X)$ for some compact Hausdorff space $X$, and in his seminal paper 3 Aarnes introduced set functions generalizing measures (initially called quasi-measures, now topological measures) and corresponding quasi-linear functionals on $C(X)$ for a compact Hausdorff space. Quasi-linear functionals (also called quasi-integrals) are functionals that are linear on singly generated subalgebras, but in general not linear. On locally compact spaces, there

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is an order-preserving bijection between quasi-linear functionals and compact-finite topological measures, which is also "isometric" when topological measures are finite \([13, 14, 12]\).

Interestingly, quasi-linear functionals are also related to the mathematical model of quantum mechanics of von Neumann \([38]\). Let \(\mathcal{A}\) be the algebra of observables in quantum mechanics. In a simple form, \(\mathcal{A}\) is a space of Hermitian operators on a finite dimensional Hilbert space. In von Neumann’s definition, a quantum state is a linear positive normalized functional on \(\mathcal{A}\). A number of physicists disagreed with the additivity axiom \(\rho(A + B) = \rho(A) + \rho(B)\) if \(A, B\) belong to a singly generated subalgebra of \(\mathcal{A}\). A positive homogeneous functional with additivity as in \((\ast)\) is a positive quasi-linear functional introduced by Aarnes, while additivity as in \((\ast\ast)\) leads to the notion of a Lie quasi-state (see \([21, 36]\) Sect. 5.6)). For more information about the physical interpretation of quasi-linear functionals see \([1, 2, 3, 18, 19, 21, 25, 36]\).

M. Entov and L. Polterovich first linked the theory of quasi-linear functionals and topological measures to symplectic geometry. Their seminal paper \([19]\) was followed by extensive research in which topological measures and quasi-linear functionals are used in connection with rigidity phenomena in symplectic geometry. Topological measures and quasi-linear functionals became an indispensable part of function theory on symplectic manifolds, which is the subject of an excellent monograph \([36]\).

Entov and Polterovich introduced symplectic quasi-states and partial symplectic quasi-states, which are subclasses of quasi-linear functionals (see \([19, 18, 36]\)). Symplectic quasi-states exist on a variety of manifolds, including \(CP^n\), complex Grassmanian, \(S^2\), \(S^2 \times S^2\) (see \([18, 19, 20, 36]\) Ch. 5)). On a closed oriented surface any positive quasi-linear functional is a symplectic quasi-state (see \([36]\) Ch. 5)). Symplectic quasi-states and topological measures are closely related to and can be obtained by homogeneous quasi-morphisms. One can also determine that a quasi-morphism is not a homomorphism by showing that a certain quasi-state is not linear, i.e. a certain topological measure is not a measure (see \([33]\), for example). Topological measures can be used to distinguish Lagrangian knots that have identical classical invariants (\([36]\) Sect. 6.2, Sect. 12.6)). Symplectic quasi-states help to determine how well a pair of functions can be approximated by a pair of Poisson commuting functions, and, more generally, provide bounds for the profile function (\([9, 18]\) Sect. 4.3, \([21, 36]\) Sect. 8.3)). Symplectic quasi-states, unlike Langrangian Floer theory, allow one to prove results about nondisplaceability for singular sets (\([18]\) Sect. 4.1, Sect. 4.5)).

A nice account of applications of topological measures and symplectic quasi-states to symplectic geometry can be found in \([36]\) Ch.6).

For a (symplectic) quasi-state \(\rho\) we may define \(\pi(f, g) = |\rho(f + g) - \rho(f) - \rho(g)|\). The functional \(\pi\) (which is nontrivial if \(\rho\) is not linear, i.e. the corresponding topological measure is not a measure) is important in a number of interesting results in symplectic geometry. The Poisson bracket \(\{f, g\}\) of functions \(f, g\) involves first derivatives of the functions, and at first glance there is no restriction on change in the uniform norm of \(\{f, g\}\) resulting from perturbations in \(f, g\). For a symplectic quasi-state \(\rho\) there is an estimate \(\pi(f, g) \leq c\|\{f, g\}\|\), and, therefore, there is such a restriction (see \([23, 24]\)). For this remarkable phenomenon as well as for a discussion of how \(\pi(f, g)\) appears in the context of simultaneous measurement of noncommuting observables \(f, g\) and provides a lower bound for the error, see \([24]\).

Function theory on symplectic manifolds would not be possible without (a) the theory of topological measures and quasi-linear functionals and (b) rigidity phenomena. The \(C^0\)-rigidity property holds for open or closed manifolds, and it allows one to extend the notion of Poisson commutativity from smooth to continuous functions (see \([22, 36]\) Sect. 2.1)). At the moment, function theory on symplectic manifolds is mostly developed for closed manifolds, perhaps because until very recently the theory of topological measures and quasi-linear functionals dealt almost exclusively with the compact case. We believe that results of this paper (together with other recent papers devoted to the theory of topological measures and quasi-linear functionals on locally
compact spaces) may allow extension of the fascinating function theory on symplectic manifolds to nonclosed manifolds and lead the way to new contributions.

The theory of quasi-linear functionals is connected with the theory of Choquet integrals. If \( \mu \) is a topological measure, the quasi-linear functional \( \int_X f \, d\mu \) is a symmetric Choquet integral (see \[14\], \[12\] and \[15\, Ch. 7\]). Many results about Choquet integrals are for a supermodular (also called 2-monotone) or totally monotone set function, and/or for a set function whose domain is a \( \sigma \)-algebra, an algebra, or is closed under intersection and union. None of this is applicable for topological measures, so the results of Choquet theory do not automatically translate for quasi-linear functionals. Nevertheless, we do have some of the typical results of Choquet integrals, such as properties of being monotone, homogeneous, and additive on cocomonotonic functions, and they hold sometimes under weaker conditions. Some results for quasi-linear functionals are stronger than those for Choquet integrals; others show that some assumptions can not be weakened in Choquet theory results. The interconnection of the two theories deserves more investigation. This will require deeper understanding of topological measures, of which this paper is a part.

Topological measures and deficient topological measures are defined on open and closed subsets of a topological space, which means that there is no algebraic structure on the domain. They lack subadditivity and other properties typical for measures, and many standard techniques of measure theory and functional analysis do not apply to them. Nevertheless, many classical results of probability theory hold for topological measures. These include Aleksandrov’s Theorem for equivalent definitions of weak convergence of topological and deficient topological measures. There is also a version of Prokhorov’s Theorem, which relates the existence of a weakly convergent subsequence in any sequence in a family of topological measures to the characteristics of being a uniformly bounded in variation and uniformly tight family. It is also possible to define Prokhorov and Kantorovich-Rubenstein metrics and show that convergence in either of them implies weak convergence of topological measures. See \[13\].

There are connections of the theory of quasi-linear functionals and topological measures with symplectic geometry, probability theory, Choquet theory, fractals, etc. Current results are just scratching the surface of interesting and fruitful investigations in different directions. Until very recently, other than paper \[37\] and a couple of preprints, including \[6\], there were no works devoted to quasi-linear functionals and topological measures on locally compact spaces. This impeded both the development of the field and its connection with other areas of mathematics. To remedy the situation, the author has written several papers extending the theory to the locally compact setting. The current paper, devoted to semisolid sets and construction of topological measures on locally compact spaces, is a key part of that series. This paper is for anyone interested in (1) learning about quasi-linear functionals on locally compact noncompact spaces or on compact spaces; (2) further study of quasi-linear functionals, signed quasi-linear functionals, and other related nonlinear functionals; (3) applying topological measures and quasi-linear functionals in other areas of mathematics.

Aarnes’s fundamental paper \[5\], devoted to construction of topological measures on compact spaces, is perhaps the most technically difficult of all papers on topological measures in the compact case. The construction technique employed in \[5\] was later nicely simplified by D. Grubb, but, unfortunately, the simplification was never published. Transitioning to the locally compact situation is not mechanical. (For one thing, on a compact space we work with open sets and closed sets, while on a locally compact space the focus is on compact sets and open sets, which are no longer complements of each other. One also has to deal with unbounded sets.) A preprint by Aarnes \[6\] is devoted to extension of the results from \[5\] to the locally compact case. While this work contains many excellent ideas, it is not entirely satisfactory. It is very technical, long, sometimes proves only parts of stated theorems, at times asks the reader to adapt lengthy proofs from other papers to its subject matter, and does not quite do what is needed (see, for instance, \[6\, Sect. 6\] where examples are obtained by a method which in general can produce a trivial topological measure from a non-trivial initial set function). It has also never been published in a mainstream journal and has remained a hard to obtain and even harder to understand preprint.

In this paper we (I) introduce a concept of semisolid sets on locally compact spaces and study the structure of solid and semisolid sets (II) develop an approach for constructing topological
measures on Hausdorff locally compact spaces. Our results allows us to extend a solid-set function to a topological measure on \( X \) when \( X \) is a Hausdorff, locally compact, connected, and locally connected space. The restriction of a topological measure to solid sets with compact closure is a solid-set function that uniquely determines the topological measure. We obtain an easy way to construct topological measures on noncompact locally compact spaces whose one-point compactification has genus 0. Thus, we are able to produce a variety of finite and infinite topological measures on \( \mathbb{R}^n \), half-spaces, punctured balls, and so on. When \( X \) is compact our approach produces a simpler method for constructing a topological measure from a solid-set function than the one in \([5]\) (the only method currently available). Results of this paper are at the core of obtaining various methods for constructing topological measures and quasi-linear functionals; they are essential for studying their properties and also immediately provide some properties of topological measures; they give an effective method for determining whether two topological measures or quasi-linear functionals are the same.

The paper is organized as follows. In Section 2 we give necessary topological preliminaries. In Section 3 we define semisolid and solid sets and study solid hulls of connected sets. In Section 4 we study the structure of solid and semisolid sets. In Section 5 we give a definition and basic properties of topological measures on locally compact spaces, and in Section 6 we do the same for solid-set functions. In Section 7 on a locally compact, connected, and locally connected space we extend a solid-set function from bounded solid sets to compact connected and bounded semisolid sets. In Section 8 the extension is done to the finite unions of disjoint compact connected sets, and in Section 9 to open and closed sets. In Section 10 we show that extension produces a topological measure that is uniquely defined by a solid-set function (see Theorem 10.7 and Theorem 10.10). In Section 11 we discuss irreducible partitions and genus of a space. In Section 12 we define a solid-set function on a compact space in a different way and in Section 13 we extend it to a topological measure. In Section 14 we give examples of topological measures on compact spaces. In Section 15 we give examples and present an easy way (Theorem 15.8) to generate topological measures on Hausdorff, locally compact, connected, and locally connected spaces whose one-point compactification has genus 0.

The spaces we consider in this paper are Hausdorff.

In this paper by a component of a set we always mean a connected component. We denote by \( \overline{E} \) the closure of a set \( E \) and by \( \partial E \) the boundary of \( E \). We denote by \( \sqcup \) a union of disjoint sets, and by \( |J| \) the cardinality of a finite set \( J \). A set \( E \) is co-connected if its complement is connected. A set \( A \subseteq X \) is called bounded if \( \overline{A} \) is compact.

Several collections of sets will be used often. They include: \( \mathcal{O}(X) \), the collection of open subsets of \( X \); \( \mathcal{C}(X) \), the collection of closed subsets of \( X \); and \( \mathcal{K}(X) \), the collection of compact subsets of \( X \). Let \( \mathcal{A}(X) = \mathcal{C}(X) \cup \mathcal{O}(X) \). If \( X \) is compact, of course, \( \mathcal{C}(X) = \mathcal{K}(X) \).

Often we will work with open, compact or closed sets with particular properties. We use subscripts \( c, s \) or \( ss \) to indicate (open, compact, closed) sets that are, respectively, connected, solid, or semisolid. For example, \( \mathcal{O}_c(X) \) is the collection of open connected subsets of \( X \), and \( \mathcal{K}_s(X) \) is the collection of compact solid subsets of \( X \).

Given any collection \( \mathcal{E} \) of subsets of \( X \) we denote by \( \mathcal{E}^* \) the subcollection of all bounded sets belonging to \( \mathcal{E} \). For example, \( \mathcal{A}^*(X) = \mathcal{K}(X) \cup \mathcal{O}^*(X) \) is the collection of compact and bounded open sets, and \( \mathcal{A}_s^*(X) = \mathcal{K}_s(X) \cup \mathcal{O}_s^*(X) \) is the collection of compact semisolid and bounded open semisolid sets. By \( \mathcal{K}_0(X) \) we denote the collection of finite unions of disjoint compact connected sets.

**Definition 1.1.** A nonnegative set function \( \mu \) on a family of sets that includes compact sets is called compact-finite if \( \mu(K) < \infty \) for each compact \( K \). A set function \( \mu \) is monotone on a collection of sets \( \mathcal{E} \) if \( \mu(A) \leq \mu(B) \) whenever \( A \subseteq B, A, B \in \mathcal{E} \). A nonnegative set function \( \mu \) is called simple if it only assumes values 0 and 1; \( \mu \) on \( X \) is finite if \( \mu(X) < \infty \).

We consider set functions that are not identically \( \infty \).
2 Preliminaries

This section contains necessary topological preliminaries. Some results in this section are known, but sometimes we give proofs for the reader’s convenience.

Remark 2.1. An application of compactness (see, for example, [17, Corollary 3.1.5]) shows that
(i) If \( K_\alpha \backslash K, K \subseteq U \), where \( U \in \mathcal{O}(X) \), \( K, K_\alpha \in \mathcal{C}(X) \), and \( K \) and at least one of \( K_\alpha \) are compact, then there exists \( \alpha_0 \) such that \( K_\alpha \subseteq U \) for all \( \alpha \geq \alpha_0 \).
(ii) If \( U_\alpha \not\supseteq U, K \subseteq U \), where \( K \in \mathcal{K}(X) \), \( U, U_\alpha \in \mathcal{O}(X) \) then there exists \( \alpha_0 \) such that \( K \subseteq U_\alpha \) for all \( \alpha \geq \alpha_0 \).

Remark 2.2. (a) Suppose \( X \) is connected, \( U \in \mathcal{O}_c(X) \) and \( F \in \mathcal{C}_c(X) \) are disjoint sets. If \( U \cap F \neq \emptyset \) then \( U \cup F \) is connected.
(b) If \( X \) is locally compact and locally connected, for each \( x \in X \) and each open set \( U \) containing \( x \), there is a connected open set \( V \) such that \( x \in V \subseteq U \) and \( V \) is compact.
(c) If \( V = \bigsqcup_{t \in T} V_t \) where \( V \) and \( V_t \) are open sets, then \( V_t \cap V_r = \emptyset \) for \( t \neq r \). In particular, \( V \) is a decomposition of an open set \( V \) into connected components, then all components \( V_t \) are open, and \( V_t \cap V_r = \emptyset \) for \( t \neq r \).
(d) If \( \{ E_t : t \in T \} \) is a family of connected sets such that any two of the sets have a nonempty intersection then \( \bigcup_{t \in T} E_t \) is connected.

Lemma 2.3. Let \( U \) be an open connected subset of a locally compact and locally connected set \( X \). Then for any \( x, y \in U \) there is \( V_{xy} \in \mathcal{O}_c(X) \) such that \( x, y \in V_{xy} \subseteq V \subseteq U \).

Proof. Fix \( x \in U \). Let \( A = \{ y \in U : \exists V_{xy} \in \mathcal{O}_c(X) \text{ such that } x, y \in V_{xy} \subseteq V \subseteq U \} \). Clearly, \( A \) is open. The set \( U \setminus A \) is also open: if \( y \in U \setminus A \) pick by Remark 2.2 \( V \in \mathcal{O}_c(X) \) such that \( y \in V \subseteq U \) and \( V \subseteq U \setminus A \). Since \( A, U \setminus A \) are open and \( x \in A \), we have \( A = U \).

Lemma 2.4. Let \( K \subseteq U, K \subseteq \mathcal{K}(X), U \in \mathcal{O}(X) \) in a locally compact space \( X \). Then there exists a set \( V \in \mathcal{O}_c(X) \) such that \( K \subseteq V \subseteq U \). (See, for example, [16, Chapter XI, 6.2])

Lemma 2.5. Let \( X \) be a locally compact, locally connected space, \( K \subseteq U, K \subseteq \mathcal{K}(X), U \in \mathcal{O}(X) \). If either \( K \) or \( U \) is connected there exist \( V \in \mathcal{O}_c(X) \) and \( C \in \mathcal{K}_c(X) \) such that \( K \subseteq V \subseteq C \subseteq U \).

One may take \( C = V \).

Proof. Case 1: \( K \in \mathcal{K}_c(X) \). For each \( x \in K \) by Remark 2.2 there is \( V_x \in \mathcal{O}_c(X) \) such that \( x \in V_x \subseteq V_x \subseteq U \). By compactness of \( K \), we write \( K \subseteq V_{x_1} \cup \ldots \cup V_{x_n} \). Since \( x_i \in K \cap V_{x_i} \), \( K \cup V_{x_i} \) is connected for each \( 1 \leq i \leq n \). Hence, \( V = \bigcup_{i=1}^n V_{x_i} = \bigcup_{i=1}^n (K \cup V_{x_i}) \) is a bounded open connected set for which \( K \subseteq V \subseteq V \subseteq \bigcup_{i=1}^n V_{x_i} \subseteq U \). Take \( C = V \).

Case 2: \( U \in \mathcal{O}_c(X) \). As in Case 1 we may find \( V_1, \ldots, V_n \in \mathcal{O}_c(X) \) such that \( K \subseteq V_1 \cup \ldots \cup V_n \subseteq U \). Pick \( x_i \in V_i \) for \( 1 \leq i \leq n \). By Lemma 2.3 choose \( w_i \in \mathcal{O}_c(X) \) with \( x_i \in W_i \subseteq U \) for \( 1 \leq i \leq n \). Then \( V = \bigcup_{i=1}^n V_i \cup \bigcup_{i=1}^n W_j = \bigcup_{i=1}^n (V_i \cup W_i) \) is open connected and \( K \subseteq \bigcup_{i=1}^n V_i \subseteq V \subseteq U \). Again, let \( C = V \).

Lemma 2.6. Let \( X \) be a locally compact, locally connected space. Suppose \( K \subseteq U, K \subseteq \mathcal{K}(X), U \in \mathcal{O}(X) \). Then there exists \( C \in \mathcal{K}_0(X) \) such that \( K \subseteq C \subseteq U \).

Proof. Let \( U = \bigsqcup_{i \in I} U_i \) be the decomposition into connected components. Since \( X \) is locally connected, each \( U_i \) is open, and by compactness of \( K \) there exists a finite set \( I \subseteq I' \) such that \( K \subseteq \bigsqcup_{i \in I} U_i \). Then \( K \cap U_i = K \setminus \bigsqcup_{j \neq i} U_j \) is a compact set. For each \( i \in I \) by Lemma 2.3 choose \( C_i \in \mathcal{K}_c(X) \) such that \( K \cap U_i \subseteq C_i \subseteq U_i \). The set \( C = \bigcup_{i \in I} C_i \) is the desired set.

Lemma 2.7. Let \( X \) be a connected, locally connected space. Let \( A \in \mathcal{A}(X) \) and let \( B \) be a component of \( X \setminus A \). Then

(i) If \( A \) is open then \( B \) is closed and \( \overline{A} \cap B = \emptyset \).
(ii) If $A$ is closed then $B$ is open and $A \cap \overline{B} \neq \emptyset$.

(iii) $A \cup \bigcup_{s \in S} B_s$ is connected for any family $\{B_s\}_{s \in S}$ of components of $X \setminus A$.

(iv) $B$ is connected and co-connected.

Proof. The proof of (i) and (ii) is not hard. (iii) Apply parts (a) and (d) of Remark 2.2. (iv) Let $X \setminus A = \bigcup_{s \in S} B_s$ be a decomposition into connected components. For each $t \in S$ the set $X \setminus B_t = A \cup \bigcup_{s \neq t} B_s$ is a connected set by the previous part. □

Lemma 2.8. Let $X$ be a connected, locally connected space. Let $K \in \mathcal{X}(X)$, $K \subseteq U \in \mathcal{O}_{*}^*(X)$. Then at most a finite number of connected components of $X \setminus K$ are not contained in $U$.

Proof. Let $X \setminus K = \bigcup_{s \in S} W_s$ be the decomposition of $X \setminus K$ into connected components. Each component $W_s$ intersects $U$ since otherwise we would have $W_s \subseteq X \setminus U$, so $W_s \subseteq X \setminus U$, and $W_s \cap K = \emptyset$, which contradicts Lemma 2.7. Assume that there are infinitely many components of $X \setminus K$ that are not contained in $U$. Then we may choose components $W_i$, $i = 1, 2, \ldots$, such that $W_i \cap U \neq \emptyset$ and $W_i \cap (X \setminus U) \neq \emptyset$ for each $i$. Connectedness of $W_i$ implies that $W_i \cap \partial U \neq \emptyset$ for each $i$. Let $x_i \in W_i \cap \partial U$. By compactness of $\partial U$, let $x_0 \in \partial U$ be the limit point of $(x_i)$. Then $x_0 \in X \setminus U \subseteq X \setminus K = \bigcup_{s \in S} W_s$, i.e. $x_0 \in W_i$ for some $t \in S$. But then all but finitely many $x_i$ must also be in $W_i$, which is impossible, since $W_i \cap W_t = \emptyset$ for $t \neq i$. □

Corollary 2.9. Let $X$ be a connected, locally connected space. Let $K \in \mathcal{X}(X)$ and let $W$ be the union of bounded components of $X \setminus K$. Then $W \in \mathcal{O}_{*}^*(X)$.

Proof. By Lemma 2.5 pick $V \in \mathcal{O}_{*}^*(X)$ such that $K \subseteq V$. From Lemma 2.8 it follows that $W$ is bounded. By Lemma 2.7 $W$ is open. □

Remark 2.10. If $A \subseteq B$, $A, B \in \mathcal{O}(X)$ then each unbounded component of $X \setminus B$ is contained in an unbounded component of $X \setminus A$.

Lemma 2.11. Let $X$ be a connected, locally connected space. Assume $A \subseteq B$, $A, B \in \mathcal{O}_{*}^*(X)$. Then each unbounded component of $X \setminus B$ is contained in an unbounded component of $X \setminus A$ and each unbounded component of $X \setminus A$ contains an unbounded component of $X \setminus B$.

Proof. Suppose first that $A \subseteq K$, $K \in \mathcal{X}(X)$. The first assertion is Remark 2.10. Now suppose to the contrary that $E$ is an unbounded component of $X \setminus A$ which contains no unbounded components of $X \setminus K$. Then $E$ is contained in the union of $K$ and all bounded components of $X \setminus K$. By Corollary 2.7 this union is a bounded set, and so is $E$, which gives a contradiction.

Now suppose $A \subseteq U$, $U \in \mathcal{O}_{*}^*(X)$, so $K = U$ is compact. Let $E$ be an unbounded component of $X \setminus A$. By the previous part, $E$ contains an unbounded component $Y$ of $X \setminus K$. But $Y \subseteq G$ for some unbounded component $G$ of $X \setminus U$. Then $G \subseteq E$. □

Lemma 2.12. Let $X$ be locally compact, connected, locally connected. Let $A \in \mathcal{O}_{*}^*(X)$. Then the number of unbounded components of $X \setminus A$ is finite.

Proof. Suppose first that $A \in \mathcal{X}(X)$. By Lemma 2.5 let $U \subseteq \mathcal{O}_{*}^*(X)$ be such that $A \subseteq U$. Then the assertion follows from Lemma 2.8. Now suppose that $A \in \mathcal{O}_{*}^*(X)$. Then $A \in \mathcal{X}(X)$, so the number of unbounded components of $X \setminus A$ is finite. From Lemma 2.11 it follows that the number of unbounded components of $X \setminus A$ is also finite. □

Lemma 2.13. Let $X$ be locally compact, connected, locally connected. Suppose $D \subseteq U$ where $D \in \mathcal{X}(X)$, $U \in \mathcal{O}_{*}^*(X)$. Let $C$ be the intersection of the union of bounded components of $X \setminus D$ with the union of bounded components of $X \setminus U$. Then $C$ is compact and $U \cup C$ is open.
Remark 3.6. Let \( \tilde{X} \) be a noncompact, connected set. By Lemma 2.7, \( \tilde{X} \) is bounded and \( \tilde{X} \) is unbounded if and only if it contains an unbounded component. If \( \tilde{X} \) is bounded, then \( \tilde{X} \) is solid by Remark 2.10. The set \( \tilde{X} \) is open by Lemma 2.7. Now \( \tilde{X} \) is the union of bounded components of \( \tilde{X} \), and \( \tilde{X} \) is the union of unbounded components of \( \tilde{X} \). By Lemma 2.12, it follows that a bounded set \( \tilde{X} \) is solid, and \( \tilde{X} \) is solid if and only if the number of bounded components of \( \tilde{X} \) is finite. For a bounded solid set \( \tilde{X} \), \( \tilde{X} \) is a bounded solid set, and \( \tilde{X} \) is connected by Lemma 2.8.

Proof. Write \( X \setminus D = V \cup W \), where \( V \) is the union of bounded components of \( X \setminus D \), and \( W \) is the union of unbounded components of \( X \setminus D \). Also write \( X \setminus U = B \cup F \), where \( B \) is the union of bounded components of \( X \setminus U \), and \( F \) is the union of unbounded components of \( X \setminus U \). By Lemma 2.12, \( F \) is a closed set. Let \( C = V \cap B \). Clearly, \( C \) and \( U \) are disjoint. Note that \( U \cup B = X \setminus F \) is open, so \( U \cup C = U \cup (V \cap B) = (U \cup V) \cap (U \cup B) \) is also open. Now we shall show that \( X \setminus C \) is open.

Remark 2.14. Lemma 2.8 is stated without proof in [6, Lemma 3.4]. Lemma 2.12 is close to [6, Lemma 3.5], and Lemma 2.13 is related to a part in the proof of Lemma 5.9 in [6].

3 Solid Hulls

Definition 3.1. A set \( A \) is semisolid if \( A \) is connected, and \( X \setminus A \) has only finitely many components. If \( X \) is locally compact, noncompact, a set \( A \) is solid if \( A \) is connected, and \( X \setminus A \) has only unbounded components. If \( X \) is compact, a set \( A \) is solid if \( A \) and \( X \setminus A \) are connected.

Example 3.2. If \( X = [0, 1]^2 \) is the unit square, the smaller square \( A = [1/4, 3/4]^2 \) is solid, and its boundary is not solid, but is semisolid.

Remark 3.3. Let \( X \) be noncompact locally compact, locally connected, connected. From Lemma 2.12 it follows that a bounded set \( B \) is semisolid if and only if only the number of bounded components of \( X \setminus B \) is finite. For a bounded solid set \( A \), \( A = \bigcup_{i=1}^{n} B_i \), where \( n \in \mathbb{N} \) and \( B_i \)’s are unbounded connected components.

Lemma 3.4. Let \( X \) be locally compact, locally connected, connected. If \( A \in \mathcal{A}_c^+(X) \) then each bounded component of \( X \setminus A \) is a solid bounded set.

Proof. Let \( X \setminus A = \bigcup_{i \in I} B_i \cup \bigcup_{j \in J} D_j \) be the decomposition of \( X \setminus A \) into components, where \( B_i \)’s are bounded components, \( D_j \)’s are unbounded ones (and \( J = \emptyset \) when \( X \) is compact). Pick a bounded component \( B_k \). Then \( X \setminus B_k = A \cup \bigcup_{i \neq k} B_i \cup \bigcup_{j \in J} D_j \). The set on the right hand side is connected by Lemma 2.7, it is also unbounded if \( X \) is noncompact. Hence, \( B_k \) is solid.

A set \( A \in \mathcal{A}_c^+(X) \) may not be solid. But we may make it solid by filling in the "holes" by adding to \( A \) all bounded components of \( X \setminus A \). More precisely, we have

Definition 3.5. Let \( X \) be locally compact, locally connected, connected. For \( A \in \mathcal{A}_c^+(X) \) let \( \{ A_i \}_{i=1}^{n} \) be the unbounded components of \( X \setminus A \), and \( \{ B_i \}_{i \in T} \) be the bounded components of \( X \setminus A \). We say that \( \tilde{A} = A \cup \bigcup_{i \in T} B_i = X \setminus \bigcup_{i=1}^{n} A_i \) is a solid hull of \( A \).

Remark 3.6. The set \( \tilde{A} \) is connected by Lemma 2.7. If \( X \) is noncompact, \( X \setminus \tilde{A} \) has only unbounded components, so \( \tilde{A} \) is solid. If \( X \) is compact then \( \tilde{A} = X \) for any connected closed or connected open set \( A \).

The next lemma gives some properties of solid hulls.

Lemma 3.7. Let \( X \) be noncompact locally compact, connected, locally connected. Let \( A, B \in \mathcal{A}_c^+(X) \).

(a1) If \( A \subseteq B \) then \( \tilde{A} \subseteq \tilde{B} \).

(a2) \( \tilde{A} \) is a bounded solid set, \( A \subseteq \tilde{A} \), and \( A \) is solid iff \( A = \tilde{A} \).

(a3) \( \bar{\tilde{A}} = \tilde{A} \).

(a4) If \( A \) is open, then so is \( \tilde{A} \). If \( A \) is compact, then so is \( \tilde{A} \).
(a5) If \( A, B \) are disjoint bounded connected sets, then their solid hulls \( \widetilde{A}, \widetilde{B} \) are either disjoint or one is properly contained in the other.

Proof. Part \([a1]\) follows since each unbounded component of \( X \setminus B \) is contained in an unbounded component of \( X \setminus A \). If \( A \) is compact, choose by Lemma \( 2.3 \) a set \( U \in \mathscr{E}^*_c(X) \) that contains \( A \). Since \( \widetilde{A} \) is a union of \( A \) and bounded components of \( X \setminus A \), applying Lemma \( 2.8 \) we see that \( \widetilde{A} \) is bounded. The rest of parts \([a2]\) and \([a5]\) is immediate. For part \([a4]\) note that if \( A \) is open (closed) then each of finitely many (by Lemma \( 2.12 \)) unbounded components of \( X \setminus A \) is closed (open) by Lemma \( 2.7 \). To prove part \([a5]\) let \( A, B \in \mathscr{E}^*_c(X) \) be disjoint. If \( A \subseteq B \) then \( \widetilde{A} \subseteq \widetilde{B} \) by parts \([a1]\) and \([a3]\). To prove that the inclusion is proper, assume to the contrary that \( \widetilde{A} = \widetilde{B} \). If one of the sets \( A, B \) is open and the other is closed, this equality means that \( \widetilde{A} \) is a proper clopen subset of \( X \), which contradicts the connectedness of \( X \). Suppose \( A \) and \( B \) are both closed (both open). Then it is easy to see that \( A = E \), where \( E \) is a bounded component of \( X \setminus B \), an open (closed) set. Thus, \( A \) is a proper clopen subset of \( X \), which contradicts the connectedness of \( X \). Therefore, \( \widetilde{A} \) is properly contained in \( \widetilde{B} \). Similarly, if \( B \subseteq \widetilde{A} \) then \( \widetilde{B} \subseteq \widetilde{A} \), and the inclusion is proper. Suppose neither of the above discussed cases \( A \subseteq \widetilde{B} \) or \( B \subseteq \widetilde{A} \) occurs. Then by connectedness we have \( A \subseteq G, B \subseteq E \), where \( G \) is an unbounded component of \( X \setminus B \) and \( E \) is an unbounded component of \( X \setminus A \). Then \( B \subseteq \widetilde{B} \subseteq X \setminus G \subseteq X \setminus A \), i.e. \( \widetilde{B} \) is contained in a component of \( X \setminus A \). Since \( \widetilde{B} \) is connected and \( B \subseteq E \) we must have \( \widetilde{B} \subseteq E \subseteq X \setminus \widetilde{A} \).

Remark 3.8. Suppose disjoint sets \( A_1, A_2, \ldots, A_n \in \mathscr{E}^*_c(X) \). On \( \{A_1, A_2, \ldots, A_n\} \) consider a partial order where \( A_i \leq A_j \) iff \( \widetilde{A}_i \subseteq \widetilde{A}_j \). (See Lemma \( 3.7 \)) Let \( A_1, \ldots, A_p \) where \( p \leq n \) be maximal elements in \( \{A_1, A_2, \ldots, A_n\} \) with respect to this partial order. Notice that \( A_1, \ldots, A_p \) are all disjoint by part \([a5]\) of Lemma \( 3.7 \). For a maximal element \( A_k, k \in \{1, \ldots, p\} \) let

\[ I_k = \{i \in \{p+1, \ldots, n\} : A_i \text{ is contained in a bounded component of } X \setminus A_k \} \]

The sets \( I_k, k = 1, \ldots, p \) are disjoint (otherwise, if \( i \in I_k \cap I_m, 1 \leq k, m \leq p \) then \( \widetilde{A}_k \cap \widetilde{A}_m \neq \emptyset \)). Then \( \{1, \ldots, n\} = \{1, \ldots, p\} \cup \bigcup_{k=1}^{p} I_k. \) Indeed, if \( i \in \{1, \ldots, n\} \setminus \{1, \ldots, p\} \) we must have \( A_i \subseteq \widetilde{A}_k \) for some maximal element \( A_k \) (where \( k \in \{1, \ldots, p\} \)). For maximal elements \( A_k \) and \( A_k \) are disjoint, \( A_i \) must be contained in a bounded component of \( A_k \), i.e. \( i \in I_k \). Note that \( A_1 \cup \ldots \cup A_n \subseteq \bigcup_{i=1}^{n} \widetilde{A}_i \).

Remark 3.9. The closure of a solid set need not be solid. For example, in the infinite strip \( X = \mathbb{R} \times [0, 4] \) the open set \( U = ((1, 3) \times (0, 4)) \cup ((5, 7) \times (0, 4)) \cup ((2, 6) \times (1, 3)) \) is solid, while its closure is not. However, we have the following result.

Lemma 3.10. Let \( X \) be locally compact, connected, locally connected.

1. Suppose \( X \) is noncompact. If \( K \subseteq U, K \in \mathscr{K}(X), U \in \mathscr{G}^*_c(X) \) then there exist \( W \in \mathscr{G}^*_c(X) \) and \( C \in \mathscr{K}(X) \) such that \( K \subseteq W \subseteq C \subseteq U \).

2. Suppose \( X \) is noncompact. If \( K \subseteq V, K \in \mathscr{K}(X), V \in \mathscr{G}(X) \) then there exists \( W \in \mathscr{G}^*_c(X) \) such that \( K \subseteq W \subseteq \overline{V} \subseteq V \).

3. Suppose \( X \) is compact. If \( K \subseteq U, K \in \mathscr{K}(X), U \in \mathscr{G}(X) \) or \( K \in \mathscr{G}(X), U \in \mathscr{G}_c(X) \) then there exists \( V \in \mathscr{G}_c(X) \) and \( C \in \mathscr{K}(X) \) such that \( K \subseteq V \subseteq C \subseteq U \).

Proof. 1.) One may take \( W \) to be the solid hull of the set \( V \) and \( C \) to be the solid hull of the set \( \overline{V} \), where \( V \) is from Lemma \( 2.5 \). Then \( K \subseteq W \subseteq C \subseteq U \) by Lemma \( 3.7 \).

2.) By Lemma \( 2.7 \) we may choose \( U \in \mathscr{G}^*_c(X) \) such that

\[ K \subseteq U \subseteq \overline{U} \subseteq V. \quad (3.1) \]

Since \( K \in \mathscr{K}(X) \) let \( X \setminus K = \bigcup_{j=1}^{p} V_j \) be the decomposition into connected components. Each \( V_j \) is an unbounded open connected set. Since \( X \setminus U \subseteq X \setminus K \), for \( j = 1, \ldots, n \) let \( E_j \) be the
union of all bounded components of $X \setminus U$ contained in $V_j$, and let $F_j$ be the union of (finitely many by Lemma 2.12) unbounded components of $X \setminus U$ contained in $V_j$. By Lemma 2.7 each $F_j$ is closed. By Lemma 2.11 each $F_j$ is nonempty. Then by Lemma 2.7 nonempty set $F_j \cup \overline{U} \subseteq V_j$. Now, $E_j \subseteq \overline{U}$, so $E_j$ is bounded. Note that $X = K \cup \bigcup_{i=1}^{n} V_j$, and a limit point $x$ of $E_j$ can not be in $V_i$ for $i \neq j$; and it can not be in $K$, since in this case a neighborhood $U$ of $x$ contains no points of $E_j$. Thus, $(F_j \cup \overline{U}) \cup E_j$ is a compact set contained in $V_j$. By Lemma 2.5 there exists $D_j \in \mathcal{K}_c(X)$ such that
\[(F_j \cup \overline{U}) \cup E_j \subseteq D_j \subseteq V_j.\] (3.2)

Let $B_j = D_j \cup F_j$. Then $B_j$ is connected because from (3.2) one sees that $D_j$ intersects every component comprising $F_j$. Thus, each $B_j$ is an unbounded closed connected set, $B_j \cap K = \emptyset$. Set $B = \bigcup_{j=1}^{n} B_j$. Then $X \setminus U \subseteq B$ and $B \cap K = \emptyset$. Since $K \subseteq X \setminus B$, let $O$ be the connected component of $X \setminus B$ such that $K \subseteq O \subseteq X \setminus B$. Since $B = \bigcup_{j=1}^{n} B_j \subseteq X \setminus O$, $B$ is contained in the union of unbounded components of $X \setminus O$. Hence, each bounded component of $X \setminus O$ is disjoint from $B$, and so $\overline{O} \subseteq X \setminus B$. Thus $K \subseteq O \subseteq \overline{O} \subseteq X \setminus B \subseteq U$. By (3.1) we see that $K \subseteq \overline{O} \subseteq U \subseteq \overline{U} \subseteq V$, and we may take $W = \overline{O}$.

3.) We will prove the statement for the first case $K \subseteq \mathcal{K}_c(X)$, $U \in \mathcal{O}(X)$, and the second case will follow immediately by considering complements of sets. $K$ is connected, so by taking a component of $U$ containing $K$ we may assume that $U \in \mathcal{O}_c(X)$. We have $X \setminus U \subseteq X \setminus K$, where $X \setminus K \in \mathcal{O}_c(X)$. By Lemma 2.5 there exists $W \in \mathcal{O}_c(X)$ such that $X \setminus U \subseteq W \subseteq X \setminus K$, so $K \subseteq X \setminus W \subseteq X \setminus U$. Since $W$ is connected, by Lemma 2.7 the components of $X \setminus W$ are solid. Let $V \in \mathcal{O}_c(X)$ be the component of $X \setminus W$ that contains $K$. Note that $V \subseteq X \setminus W \subseteq X \setminus W' \in \mathcal{O}(X)$, so $V' \subseteq X \setminus W$. Then $K \subseteq V \subseteq V' \subseteq X \setminus W \subseteq U$. Similarly we can get $W_1 \in \mathcal{O}_c(X)$ such that $K \subseteq W_1 \subseteq W_1 \subseteq V \subseteq V'$. Then $X \setminus W_1 \subseteq X \setminus W_1 \subseteq V \subseteq V'$. As above, we find $E \in \mathcal{O}_c(X)$ such that $X \setminus V \subseteq E \subseteq X \setminus W_1$. Then $K \subseteq W_1 \subseteq W_1 \subseteq E \subseteq V \subseteq V' \subseteq U$. Since $W_1 \in \mathcal{O}_c(X)$ and $X \setminus E \subseteq \mathcal{O}_c(X)$, this finishes the proof. 

In the spirit of Lemma 3.10 and Lemma 2.5 we have

**Lemma 3.11.** Let $X$ be locally compact, connected, locally connected. Suppose $K \subseteq W$, $K \in \mathcal{K}_c(X)$, $W \in \mathcal{O}_c(X)$. Then there exist $V \in \mathcal{O}_c(X)$ and $D \in \mathcal{K}_c(X)$ such that $K \subseteq V \subseteq D \subseteq W$.

**Proof.** Suppose first that $X$ is compact. By Lemma 2.5 choose $U \in \mathcal{O}_c(X)$ and $C \in \mathcal{K}_c(X)$ such that $K \subseteq U \subseteq C \subseteq W$. Let $X \setminus W = \bigcup_{i=1}^{n} E_i$, $X \setminus C = \bigcup_{i \in T} V_i$, $X \setminus U = \bigcup_{s \in S} D_s$ be decompositions into connected components of $X \setminus W$, $X \setminus C$, $X \setminus U$ respectively. Then $\bigcup_{i=1}^{n} E_i \subseteq \bigcup_{i \in T} V_i \subseteq \bigcup_{s \in S} D_s$. Let $T_0 = \{ t \in T : V_i \text{ is unbounded} \}$. Let us index by $T'$ the family of all bounded components of $X \setminus C$ each of which contains a component of $X \setminus W$. So $T'$ is finite, and $\bigcup_{i=1}^{n} E_i \subseteq \bigcup_{i \in T} V_i \subseteq \bigcup_{i \in T} V_i$. Now let us index by $S'$ the family of all bounded components of $X \setminus U$ each of which contains a component $V_i$ for some $t \in T'$. Note that $S'$ is a finite index set and $\bigcup_{t \in T'} V_i \subseteq \bigcup_{s \in S'} D_s$. Then $V$ is bounded. Also, $V$ is open. By Lemma 2.7 $V$ is connected. Since $X \setminus V = (X \setminus \overline{U}) \cup \bigcup_{s \in S'} D_s \subseteq \bigcup_{s \in S} D_s = X \setminus U$ we see that $V \in \mathcal{O}_c(X)$ (as the first equality indicates that $X \setminus V$ has finitely many components), and that $U \subseteq V$. Now consider $D = \overline{C} \setminus \bigcup_{i \in T'} V_i$. Then $D$ is compact. By Lemma 2.7 $D$ is connected. We have $X \setminus D = (X \setminus \overline{C}) \cup \bigcup_{i \in T} V_i \subseteq (X \setminus \overline{U}) \cup \bigcup_{s \in S'} D_s = X \setminus V$, so $X \setminus D$ has finitely many components, and $V \subseteq D$. Thus, $D \in \mathcal{K}_c(X)$. Also, $X \setminus W = \bigcup_{i=1}^{n} E_i \subseteq \bigcup_{i \in T} V_i \cup \bigcup_{i \in T'} V_i = (X \setminus \overline{C}) \cup \bigcup_{i \in T} V_i = X \setminus D$. Therefore, $D \subseteq W$. Then we have: $K \subseteq U \subseteq V \subseteq D \subseteq W$, where $V \in \mathcal{O}_c(X)$ and $D \in \mathcal{K}_c(X)$.

If $X$ is compact use $\overline{U} = \overline{C} = X$, $T_0 = \emptyset$ and the same (but simplified) argument. 

Let $V$ be an open subset of $X$ endowed with the subspace topology. Let $D \subseteq V$. By $\overline{D'}$ we denote the closure of $D$ in $V$ with the subspace topology. As before, $\overline{D}$ stands for the closure of $D$ in $X$.

**Lemma 3.12.** Let $V \in \mathcal{O}_c(X)$, $D \subseteq V$. Suppose $V$ is endowed with the subspace topology.
a) If $D$ is bounded in $V$ with the subspace topology then $\overline{D}^V = \overline{D}$ and $\overline{D} \subseteq V$.

b) If $D$ is bounded in $X$ and $\overline{D} \subseteq V$ then $D$ is bounded in $V$.

Proof. (a) Clearly, $\overline{D}^V \subseteq \overline{D}$, $\overline{D}^V \subseteq \overline{D}$, and $D \subseteq \overline{D}^V$, so we have $\overline{D} \subseteq \overline{D}^V \subseteq \overline{D}$. Thus, $\overline{D} = \overline{D}^V \subseteq V$. (b) Let $\overline{D} \subseteq V$. Again, $\overline{D} = \overline{D}^V$. Since $\overline{D}$ is compact in $X$, $\overline{D}^V$ is compact in $V$.  

Remark 3.13. Let $V \in \mathcal{O}(X)$ be endowed with the subspace topology. 

(1) From Lemma 3.12 we see that $D$ is bounded in $V$ iff $\overline{D} \subseteq V$. Hence, $D$ is unbounded in $V$ iff $\overline{D} \cap (X \setminus V) \neq \emptyset$. If $X$ is compact and $V = X$ this criteria shows (as expected) that there are no unbounded components in $X$.

(2) If $E$ is connected in $V$ endowed with the subspace topology then $E$ is connected in $X$.

The next two results give relations between being a solid set in a subspace of $X$ and being a solid set in $X$.

**Lemma 3.14.** Let $X$ be locally compact, locally connected, connected. Let $C \subseteq V$, $C \in \mathcal{C}_s(X)$, $V \in \mathcal{O}(X)$. Suppose $V \setminus C = \bigcup_{i \in T} V_i$ is the decomposition into disjoint open connected sets. Then $\overline{V_i} \cap (X \setminus V) \neq \emptyset$ for each $i$, and $C \in \mathcal{C}_s(V)$.

Proof. If $X$ is compact and $V = X$ then $V \setminus C$ has only one component $U = X \setminus C$, and $\overline{V_i} \cap C \neq \emptyset$ because otherwise $X$ is disconnected. We assume now that $X$ is noncompact or $X$ is compact and $V \neq X$. Write $X \setminus C = (X \setminus V) \bigcup_{i \in T} V_i$. Assume to the contrary that there exists $r \in T$ such that $\overline{V_r} \cap (X \setminus V) = \emptyset$. By Remark 2.12 $\overline{V_r} \cap V_i = \emptyset$ for each $i \neq r$. Thus, $\overline{V_r} \subseteq C \cup V_r$. Since $V_r \subseteq X \setminus C$ and $V_r$ is connected in $X$, assume that $V_r$ is contained in a component $U$ of $X \setminus C$. (If $X$ is compact then $U = X \setminus C$.) Then $V_r \subseteq U \cap \overline{V_r} \subseteq U \cap (C \cup V_r) = V_r$, so $U \cap \overline{V_r} = V_r$. Thus, $U = V_r$ is a component of $X \setminus C$. If $X$ is noncompact this is impossible, since $V_r$ is bounded and $C$ is solid; if $X$ is compact this is impossible since $V_r = U = X \setminus C$ implies $X \setminus V = \emptyset$. Thus, $\overline{V_r} \cap (X \setminus V) \neq \emptyset$ for each $i$. If we take $V \setminus C = \bigcup_{i \in T} V_i$ to be the decomposition into connected components in $V$ endowed with the subspace topology, then by Remark 3.13 each $V_i$ is unbounded in $V$, i.e. $C \in \mathcal{C}_s(V)$.  

**Lemma 3.15.** Suppose $X$ is locally compact, locally connected, connected. Let $A \subseteq V$, $V \in \mathcal{O}_s(X)$. If $A \in \mathcal{A}_s(V)$ then $A \in \mathcal{A}_s(X)$.

Proof. First suppose $X$ is noncompact. If $A \in \mathcal{A}_s(V)$ then $A$ is connected in $X$ and bounded in $X$. Since $V \in \mathcal{O}_s(X)$, write $X \setminus V = \bigcup_{i \in I} F_i$ where $F_i$ are unbounded connected components. Let $X \setminus A = \bigcup_{i \in I} E_i$ be the decomposition into connected components in $V$. Each $E_i$ is unbounded in $V$, i.e. $\overline{E_i} \cap (X \setminus V) \neq \emptyset$, hence, $\overline{E_i} \cap F_i \neq \emptyset$ for some $i \in I$. Let $I' = \{i \in I : F_i \cap \overline{E_i} \neq \emptyset \}$, and for $i \in I'$ let $T_i = \{t \in T : \overline{E_i} \cap F_t \neq \emptyset \}$. For $i \in I'$ the set $F_i \cup \bigcup_{t \in T_i} E_t$ is unbounded and connected. Since $X \setminus A = (X \setminus V) \cup (V \setminus A) = \bigcup_{i \in I'} (F_i \cup \bigcup_{t \in T_i} E_t) \cup \bigcup_{i \in I', t \in T_i} F_i$ is a disjoint union of unbounded connected sets, the statement follows.

For $X$ compact we use a simplified version of the same argument: write $X \setminus V = F$ where $F$ is a closed connected set, and note that $X \setminus A = F \cup \bigcup_{i \in I} E_i$ is connected.  

**Example 3.16.** Let $X = \{z \in \mathbb{C} : 1 \leq |z| \leq 4\}$, $V = \{z \in \mathbb{C} : 1 \leq |z| < 3\}$, $B = \{z \in \mathbb{C} : 1 \leq |z| \leq 2\}$, $W = \{z \in \mathbb{C} : 2 < |z| < 3\}$. Then $V \in \mathcal{O}_s(X)$, $B \in \mathcal{C}_s(V)$ and $B \in \mathcal{X}_s(X)$, while $W \notin \mathcal{A}_s(V)$ and is not solid in $X$. (Note that $W \in \mathcal{O}_s(X)$, which will follow from Lemma 4.3 below.)

**4 Structure of solid and semisolid sets**

Now we shall take a closer look at the structure of open solid or semisolid sets that contain closed solid or closed connected sets.
Lemma 4.1. Let $X$ be locally compact, connected, locally connected. Let $C \subseteq V, C \in \mathcal{K}_s(X)$.

(i) Suppose $V \in \mathcal{O}_s^*(X)$. If $V \setminus C$ is connected then $V = C \sqcup W$ where $W \in \mathcal{O}_s^*(X)$. If $V \setminus C$ is disconnected then $V = C \sqcup \bigsqcup_{i=1}^n V_i$ where $V_i \in \mathcal{O}_s^*(X), i = 1, \ldots, n, n \in \mathbb{N}$.

(ii) Suppose $V \in \mathcal{O}_{ss}^*(X)$. Then $V = C \sqcup \bigsqcup_{i=1}^n V_i$ where $V_i \in \mathcal{O}_{ss}^*(X), i = 1, \ldots, n, n \in \mathbb{N}$.

Proof. (i) Suppose $V \in \mathcal{O}_s^*(X)$ and let $X \setminus V = \bigsqcup_{j \in J} F_j$ be the decomposition into connected components, so $J$ is a finite index set and each $F_j$ is unbounded (respectively, $X \setminus V = F$ where $F$ is connected if $X$ is compact). If $V \setminus C$ is connected then taking $W = V \setminus C$ we see that $X \setminus V = (X \setminus V) \cup C = C \sqcup \bigsqcup_{j \in J} F_j$ has finitely many components, i.e. $W \in \mathcal{O}_s^*(X)$.

Now assume that $V \setminus C$ is not connected. By Lemma 3.14 $C \in \mathcal{K}_s(V)$. The set $V \setminus C$ is also disconnected in $V$, so using Remark 3.3 let $V \setminus C = \bigsqcup_{i=1}^m V_i, m \geq 2$ be the decomposition into connected (unbounded in $V$) components in $V$. Each $V_i$ is connected in $X$. To show that each $V_i \in \mathcal{O}_s^*(X)$ we only need to check that the components of $X \setminus V_i$ are unbounded (respectively, that $X \setminus V_i$ is connected if $X$ is compact). For notational simplicity, we shall show it for $V_1$. For $2 \leq i \leq n$ by Lemma 3.14 $V_i$ intersects $X \setminus V_i$, hence, intersects some $F_j$. Let $J_1 = \{j \in J : F_j \cap V_i \neq \emptyset\}$ for some $2 \leq i \leq n$. By Lemma 2.7 the set $C \sqcup \bigsqcup_{i=1}^m V_i$ is connected in $V$, so by Remark 3.13 and Remark 2.2 the set $\bigcup_{j \in J_1} F_j \sqcup C \sqcup \bigsqcup_{i=1}^m V_i$ is connected. It is also unbounded. Now

$$X \setminus V_1 = (X \setminus V) \cup (V \setminus V_1) = \bigcup_{j \in J} F_j \cup C \sqcup \bigsqcup_{i=1}^n V_i = \left( \bigcup_{j \in J_1} F_j \cup C \sqcup \bigsqcup_{i=2}^n V_i \right) \cup \left( \bigcup_{j \in J \setminus J_1} F_j \right).$$

Since $X \setminus V_1$ is the disjoint union of connected unbounded sets, it follows that $V_1$ is solid. (If $X$ is compact $X \setminus V_1 = F \cup C \sqcup \bigsqcup_{i=1}^m V_i$ is connected.)

(ii) Suppose $V \in \mathcal{O}_{ss}^*(X)$ and let $F_1, \ldots, F_k$ be the components of $X \setminus V$. Since $C$ is compact, by Lemma 3.14 $C \in \mathcal{K}_s(V)$. Let $V \setminus C = \bigsqcup_{i=1}^m V_i, m \geq 1$ be the decomposition into connected components in $V$ according to Remark 3.3. Each $V_i$ is connected in $X$, and to show that each $V_i \in \mathcal{O}_{ss}^*(X)$ we only need to check that $X \setminus V_i$ has finitely many components. For simplicity, we shall show it for $V_1$. We have $X \setminus V_2 = (X \setminus V) \cup (V \setminus V_1) = \bigsqcup_{i=1}^m F_j \sqcup C \sqcup \bigsqcup_{i \neq 1} V_i$. Since $X \setminus V_1$ is a finite disjoint union of connected sets, the number of components of $X \setminus V_1$ is finite, so $V_i \in \mathcal{O}_{ss}^*(X)$.

Lemma 4.2. Let $X$ be locally compact, connected, locally connected. Suppose that $V = \bigsqcup_{i \in T} C_i \sqcup \bigsqcup_{i \in T} U_i$, where $V \in \mathcal{O}_{ss}^*(X), C_i \in \mathcal{K}_s(X), U_i \in \mathcal{O}_s^*(X)$. Then $T$ is finite, and each $U_i \in \mathcal{O}_{ss}^*(X)$.

Proof. The proof works for $X$ compact and noncompact and uses induction on $m$. Let $m = 1$. Using Lemma 4.1 we have $V \setminus C_1 = \bigsqcup_{i=1}^m V_i = \bigsqcup_{i \in T} U_i$. Since sets $V_i$’s and $U_i$’s are connected, $T$ must be finite. Now let $V = \bigsqcup_{i=1}^m C_i \sqcup \bigsqcup_{i \in T} U_i$ and assume that the result holds for any bounded open semisolids set which contains less than $m$ compact solid sets. Using Lemma 4.1 we see that $V = C_1 \sqcup \bigsqcup_{i=1}^m V_i = C_1 \cup \bigsqcup_{j=2}^m C_j \cup \bigsqcup_{i \in T} U_i$, where $V_i \in \mathcal{O}_{ss}^*(X)$. All involved sets are connected, so each set $V_i$ is a disjoint union of sets from the collection $\{C_2, \ldots, C_m, U_i, t \in T\}$. By the induction hypothesis each $V_i$ contains finitely many sets, and it follows that $T$ is finite. Since $T$ is finite, we see that for each $t \in T$ the set $X \setminus U_t = (X \setminus V) \cup \bigsqcup_{i=1}^m C_i \cup \bigsqcup_{t \neq i} U_t$ has finitely many components, so $U_t \in \mathcal{O}_{ss}^*(X)$.

Lemma 4.3. Let $X$ be locally compact, connected, locally connected. If $A = \bigsqcup_{t \in T} A_t, A, A_t \in \mathcal{K}_s^*(X)$ with at most finitely many $A_t \in \mathcal{K}_s(X)$ then $T$ is finite.

Proof. Assume first that $A \in \mathcal{O}_s^*(X)$. If the cardinality $|T| > 1$ then there must be a compact solid set among $A_t$, and the result follows from Lemma 3.3. Assume now that $A \in \mathcal{K}_s(X)$ and write $A = \bigsqcup_{i=1}^m C_j \sqcup \bigsqcup_{i \in T} U_i$, where $C_i \in \mathcal{K}_s(X), U_i \in \mathcal{O}_s^*(X)$. By Lemma 3.10 choose $V \in \mathcal{O}_s^*(X)$ such that $A \subseteq V$. By Lemma 4.1 we may write $V \setminus A = \bigsqcup_{i=1}^m V_i$, where $V_i \in \mathcal{O}_{ss}^*(X)$. Then $V = \bigsqcup_{j=1}^m C_j \sqcup \bigsqcup_{i \in T} U_i \sqcup \bigsqcup_{i=1}^m V_i$, and by Lemma 4.2 $T$ is finite.
Lemma 4.4. Let $X$ be non-compact, locally compact, connected, locally connected. Suppose $C \subseteq U$, $C \in \mathcal{K}(X)$, $U \in \mathcal{O}^*_s(X)$. If $U \setminus \bar{C}$ is disconnected then $U = C \cup \bigcup_{S \in S} V_s$ where $V_s \in \mathcal{O}^*_s(X)$. If $U \setminus \bar{C}$ is connected then $U = C \cup \bigcup_{S \in S} V_s \cup W$ where $V_s \in \mathcal{O}^*_s(X)$, $W \in \mathcal{O}^*_s(X)$.

Proof. Note that $\bar{C} \in \mathcal{K}(X)$ and $\bar{C} \subseteq U$ by Lemma 3.7. Assume that $U \setminus \bar{C}$ is disconnected. By Lemma 3.11 we may write $U = \bar{C} \cup \bigcap_{i=1}^n U_i, U_i \in \mathcal{O}^*_s(X)$. But $\bar{C} = C \cup \bigcup_{S \in S} V_s$, where $V_s$ are bounded components of $X \setminus C$, so by Lemma 3.12 each $V_s \in \mathcal{O}^*_s(X)$. After reindexing, one may write $U = C \cup \bigcup_{S \in S} V_s, V_s \in \mathcal{O}^*_s(X)$. The proof for the case when $U \setminus \bar{C}$ is connected follows similarly from Lemma 4.1.

Lemma 4.5. Let $X$ be locally compact, connected, locally connected. Suppose $\bigcup_{j=1}^n C_j \subseteq V$, $C_1, \ldots, C_n \in \mathcal{K}(X)$, $V \in \mathcal{O}^*_ss(X)$. Then $V = \bigcup_{j=1}^n C_j \cup \bigcap_{t \in T} U_t$, where each $U_t \in \mathcal{O}^*_s(X)$ and all but finitely many are solid.

Proof. The proof is by induction on $n$. Suppose $n = 1$, i.e. $C \subseteq V, C \in \mathcal{K}(V), V \in \mathcal{O}^*_ss(X)$. Then $C \in \mathcal{K}(V)$, where $V$ is equipped with the subspace topology. Being open, $V$ is a locally compact, locally connected subspace. Let $V \setminus \bar{C} = \bigcup_{t \in T} U_t$ be the decomposition into components in $X$, i.e. each $U_t \in \mathcal{O}^*_ss(V)$, and so $U_t \in \mathcal{O}^*_s(X)$. By Lemma 3.7 $C \cap \bigcap_{t \in T} U_t \neq \emptyset$, so also $C \cap \bigcap_{t \in T} U_t \neq \emptyset$ for each $t$. Then $X \setminus U_t = (X \setminus V) \cup (C \cup \bigcup_{t \in T, t \neq U_t} U_t)$, where the set in the last parenthesis is connected by Remark 3.2. Since $V$ is semisolid, $V \setminus U_t$ has finitely many components, so $U_t \in \mathcal{O}^*_s(X)$ for each $t$. By Lemma 2.12 all but finitely many of $U_t$’s are bounded in $V$, hence, by Lemma 5.4 belong to $\mathcal{O}^*_s(V)$, hence, by Lemma 5.15 belong to $\mathcal{O}^*_s(X)$.

Assume the result for less than $n$ compact connected sets. For $n$ compact connected sets we have $V \setminus \bigcup_{j=1}^n C_j = (V \setminus C_1) \setminus (C_2 \cup \ldots \cup C_n) = \bigcup_{j \in T} U_j \setminus (C_2 \cup \ldots \cup C_n)$, where by the first part finitely many of the sets $U_j$ are in $\mathcal{O}^*_ss(X)$ and the rest are in $\mathcal{O}^*_s(X)$. $C_2, \ldots, C_n$ are contained in finitely many of $U_j$’s, each of which contains less than $n$ disjoint compact connected sets, and the result follows by induction.

Lemma 4.6. Let $X$ be locally compact, noncompact, connected, locally connected. Suppose that $\bigcup_{i=1}^n F_i \subseteq W$, where $F_1, \ldots, F_n \in \mathcal{K}(X), W \in \mathcal{O}^*_s(X)$, and $W \setminus (F_1 \cup \ldots \cup F_n)$ is disconnected. Then there is a solid decomposition $W = \bigcup_{i \in I} F_i \cup \bigcap_{j=1}^m W_j$, where $\emptyset \neq I \subseteq \{1, \ldots, n\}, p \in N, U_j \in \mathcal{O}^*_s(X)$.

Proof. By Lemma 3.2 write $W = \bigcup_{i=1}^n F_i \cup \bigcap_{j=1}^m W_j$, where $m \geq 2$, and $W_j \in \mathcal{O}^*_s(X)$. Let $W_1, \ldots, W_p (p \leq m)$ be maximal elements among $W_1, \ldots, W_n$ with respect to the partial order given by $W_j \leq W_t$ if $W_j \subseteq W_t$. By Lemma 3.7 $W_1 \cap W_t = \emptyset$ for any two maximal elements $W_j, W_t$. Let $I$ be the index set for $F_i$’s each of which is not contained in $\bigcup_{j=1}^m W_j$. Sets $W_j \subseteq W$, so $\bigcup_{j=1}^m W_j \cup \bigcup_{i \in I} F_i \subseteq W$. For a maximal component $B$ of $X \setminus W_j$, since $B \subseteq W$ we have $B \subseteq \bigcup_{i=1}^n F_i \cup \bigcup_{k \neq j} W_k \subseteq X \setminus W_j$. Each set from the family $\{F_1, \ldots, F_n, W_k, k \neq j\}$ intersects only one component of $X \setminus W_j$, so $B$ is a union of some sets from this family. Then $W_j$ is a union of some sets from the family $\{F_1, \ldots, F_n, W_1, \ldots, W_m\}$. Together with Remark 3.8 it gives $W = \bigcup_{j=1}^m W_j \cup \bigcup_{i \in I} F_i$. If $I = \emptyset$, the statement of the lemma follows with $U_j = W_j$.

Assume that $I \neq \emptyset$. By connectedness, $W$ is the single maximal element among $W_1, \ldots, W_m$. Without loss of generality, $W = W_m$, and so $W_i \nsubseteq W_m$ for $i = 1, \ldots, m-1$. Among sets $W_1, \ldots, W_{m-1}$ again find maximal elements $W_1, \ldots, W_{q}, q \leq m-1$. As above, $W = W_m = \bigcup_{j=1}^q W_j \cup \bigcup_{i \in I} F_i$. If there is only one maximal element (say, $W_1$) then $I' = \emptyset$ because $W_1 \nsubseteq W_m$. If there are at least two maximal elements then $I' \neq \emptyset$ because of connectedness of $W$.

Remark 4.7. Part (a5) of Lemma 3.7, part 2 of Lemma 3.10 and the last statement in Lemma 4.1.14 are close to [6 Lemmas 3.8, 3.9, 4.2]. The existence of $V$ in the last part of Lemma 4.10 was first proved in [5, Lemma 3.3]. The case “$V \setminus C$ is disconnected” in the first part of Lemma 4.11 is [6 Lemma 4.3], and Lemma 4.3 is an expanded (to compact sets as well) version of [6 Lemma 4.4]. Our proofs are modified, expanded, or different.
Remark 4.8. For a compact space, solid sets are connected and co-connected; they are essential in the construction of a topological measure from a solid-set function. It is tempting to employ sets that are connected and co-connected in the locally compact setting, but this is not the right collection of sets for various reasons. For one, one may end up with topological measure $\mu = 0$ starting from a non-trivial set function on sets that are connected and co-connected (see [6, Example 6.2]). The key role in the locally compact noncompact case will be played by semisolid sets. For a compact space the definition of a semisolid set as a set whose complement has finitely many components was present in [27], but it is not clear, who coined the term. The same definition of semisolid set in locally compact spaces works well for our goals. The definition of solid sets is different for compact spaces and locally compact noncompact spaces ([5], [6]), but as we shall see, it is what allows us to prove many results simultaneously for compact and noncompact spaces. Also, with the current definition of a solid set in the locally compact case, a bounded solid set in a locally compact space is solid in its one-point compactification (Lemma [15.1] below).

5 Definition and basic properties of topological measures

Definition 5.1. A topological measure on a locally compact space $X$ is a set function $\mu : \mathcal{C}(X) \cup \mathcal{O}(X) \to [0, \infty]$ satisfying the following conditions:

(TM1) if $A, B, A \cup B \in \mathcal{K}(X) \cup \mathcal{O}(X)$ then $\mu(A \cup B) = \mu(A) + \mu(B)$;

(TM2) $\mu(U) = \sup\{\mu(K) : K \in \mathcal{K}(X), K \subseteq U\}$ for $U \in \mathcal{O}(X)$;

(TM3) $\mu(F) = \inf\{\mu(U) : U \in \mathcal{O}(X), F \subseteq U\}$ for $F \in \mathcal{C}(X)$.

Remark 5.2. It is important that in Definition 5.1 condition (TM1) holds for sets from $\mathcal{K}(X) \cup \mathcal{O}(X)$. In fact, (TM1) fails on $\mathcal{C}(X) \cup \mathcal{O}(X)$. See Example 15.6 or Example 15.7 below.

We have the following immediate properties of topological measures on locally compact spaces.

Lemma 5.3. The following is true for a topological measure:

(t1) $\mu$ is monotone, i.e. if $A \subseteq B, A, B \in \mathcal{C}(X) \cup \mathcal{O}(X)$ then $\mu(A) \leq \mu(B)$.

(t2) If an increasing net $U_t \nearrow U$, where $U_t, U \in \mathcal{O}(X)$ then $\mu(U_t) \nearrow \mu(U)$. In particular, $\mu$ is additive on $\mathcal{O}(X)$.

(t3) $\mu(\emptyset) = 0$.

(t4) If $V \cap K \subseteq U$, where $V, U \in \mathcal{O}(X), K \in \mathcal{K}(X)$ then $\mu(V) + \mu(K) \leq \mu(U)$.

(t5) If $\mu$ is compact-finite then $\mu(A) < \infty$ for each $A \in \mathcal{A}^s(X)$. $\mu$ is finite iff $\mu$ is real-valued.

(t6) If $X$ is locally compact, locally connected then for any $U \in \mathcal{O}(X)$

$$\mu(U) = \sup\{\mu(C) : C \in \mathcal{K}_0(X), C \subseteq U\}.$$  

(t7) If $X$ is locally compact, connected, locally connected then $\mu(X) = \sup\{\mu(K) : K \in \mathcal{K}_c(X)\} = \sup\{\mu(K) : K \in \mathcal{K}_s(X)\}.$

Proof. (t1) Immediate from Definition 5.1. (t2) Suppose $U_t \nearrow U, U_t \in \mathcal{O}(X).$ Let compact $K \subseteq U$. By Remark 2.1 there is $t'$ such that $K \subseteq U_s$ for all $t \geq t'.$ Then $\mu(K) \leq \mu(U_t) \leq \mu(U)$ for all $t \geq t'$, and we see from the inner regularity condition (TM2) of Definition 5.1 (whether $\mu(U) < \infty$ or $\mu(U) = \infty$) that $\mu(U_t) \nearrow \mu(U)$. (t3) Easy to see since $\mu$ is not identically $\infty$. (t4) Easy to see from part (TM2) of Definition 5.1. (t5) If $U$ is an open bounded set then $\mu(U) \leq \mu(U) < \infty$. (t6) By Lemma 2.6 for arbitrary $K \subseteq U, K \in \mathcal{K}(X), U \in \mathcal{O}(X)$ there is $C \in \mathcal{K}_0(X)$ with $K \subseteq C \subseteq U$. By monotonicity $\mu(U) = \sup\{\mu(K) : K \in \mathcal{K}, K \subseteq U\} \leq \sup\{\mu(C) : C \in \mathcal{K}_0(X), K \subseteq C \subseteq U\} \leq \mu(U)$. (t7) Follows from Lemma 2.6 and Lemma 5.1. □
Proposition 5.4. Let $X$ be locally compact. Consider the following conditions:

(c1) $\mu(U) = \mu(K) + \mu(U \setminus K)$ whenever $K \subseteq U$, $K \in \mathcal{K}(X)$, $U \in \mathcal{O}(X)$.

(c2) $\mu(K \cup C) = \mu(K) + \mu(C)$ for any disjoint compact sets $K, C$.

(c3) $\mu(U \cup V) = \mu(U) + \mu(V)$ for any disjoint open sets $U, V$.

For a set function $\mu : \mathcal{O}(X) \cup \mathcal{C}(X) \to [0, \infty]$ we have:

1. If $\mu$ satisfies (TM2) of Definition 5.1 then (c2) $\Rightarrow$ (c3).
2. If $\mu$ is monotone on open sets and satisfies (TM3) then (c3) $\Rightarrow$ (c2).
3. If $\mu$ satisfies (TM2) and (TM3) then (c3) and (c2) are equivalent.
4. If $\mu$ satisfies (TM2), (TM3), (c1) and one of (c2), (c3) then (TM1) holds for $\mu$; hence, $\mu$ is a topological measure.

Proof. Note that (TM2) implies monotonicity on $\mathcal{O}(X)$ and (TM3) implies monotonicity on $\mathcal{C}(X)$.

(1.) For a compact $D \subseteq U \cup V$ write $D = K \cup C$, where $K \subseteq U, C \subseteq V$. Then by (c2) and (TM2), $\mu(D) = \mu(K) + \mu(C) \leq \mu(U) + \mu(V)$, so taking supremum over all $D \subseteq U \cup V$ we have $\mu(U \cup V) \leq \mu(U) + \mu(V)$. If $K \subseteq U, C \subseteq V$ then $\mu(K) + \mu(C) = \mu(K \cup C) \leq \mu(U \cup V)$, so by (TM2), $\mu(U) + \mu(V) \leq \mu(U \cup V)$.

(2.) Suppose $K, C$ are disjoint compact sets, $K \cup C \subseteq W$, $W \in \mathcal{O}(X)$. $X$ is completely regular, so choose disjoint open sets $U, V$ such that $K \subseteq U, C \subseteq V, U \cup V \subseteq W$. Then using (TM3) and monotonicity on open sets we have: $\mu(K) + \mu(C) \leq \mu(U) + \mu(V) = \mu(U \cup V) \leq \mu(W)$. Taking infimum over all $W$ containing $K \cup C$ we see that $\mu(K) + \mu(C) \leq \mu(K \cup C)$. Since $\mu(K \cup C) \leq \mu(U \cup V) = \mu(U) + \mu(V)$, by (TM3) we have $\mu(K \cup C) \leq \mu(K) + \mu(C)$.

(3.) Follows from previous parts.

(4.) Our proof basically follows the proof of Proposition 2.2 where the result first appeared for compact-finite topological measures. By part 3 we only need to check (TM1) in the situation when $A \in \mathcal{K}(X)$, $B \in \mathcal{O}(X)$, and $A \cup B$ is either compact or open. If $A \cup B$ is open then using condition (c1) we get $\mu(A \cup B) = \mu((A \cup B) \setminus A) + \mu(A) = \mu(B) + \mu(A)$. Now suppose $A \cup B \in \mathcal{K}(X)$. Let $C \in \mathcal{K}(X)$, $C \subseteq B$. Then finite additivity and monotonicity of $\mu$ on $\mathcal{K}(X)$ gives $\mu(A) + \mu(C) = \mu(A \cup C) \leq \mu(A \cup B)$. By (TM2), $\mu(A) + \mu(B) \leq \mu(A \cup B)$. The opposite inequality is obvious if $\mu(A) = \infty$, so let $\mu(A) < \infty$, and for $\epsilon > 0$ pick $U \in \mathcal{O}(X)$ such that $A \subseteq U$ and $\mu(U) < \mu(A) + \epsilon$. Then compact set $A \cup B$ is contained in the open set $B \setminus U$. Also, the compact set $(A \cup B) \setminus U = B \setminus U$ is contained in $B \setminus U$, and $(B \cup U) \setminus (B \setminus U) = U$. Applying (TM2) and then (c1) we have: $\mu(A \cup B) \leq \mu(B \cup U) = \mu((B \cup U) \setminus (B \setminus U)) + \mu(B \setminus U) = \mu(U) + \mu(B \setminus U) \leq \mu(U) + \mu(B) \leq \mu(A) + \mu(B) + \epsilon$. Thus, $\mu(A \cup B) \leq \mu(A) + \mu(B)$. This finishes the proof.

Remark 5.5. Of course, any topological measure satisfies (c1) of Proposition 5.4. It is interesting to note that a similar condition regarding a bounded open subset of a closed set fails for topological measures, i.e. $\mu(F) = \mu(U) + \mu(F \setminus U)$ where $F$ is closed and $U$ is open bounded, in general is not true, as Example 5.5.7 below shows.

Using complements of sets, we see that when $X$ is compact, the definition of a real-valued topological measure is equivalent to:

Definition 5.6. A real-valued topological measure on a compact space $X$ is a set function $\mu : \mathcal{O}(X) \cup \mathcal{C}(X) \to [0, \infty)$ that satisfies the following conditions:

(T1) $\mu(A \cup B) = \mu(A) + \mu(B)$ if $A, B, A \cup B \in \mathcal{O}(X) \cup \mathcal{C}(X)$.

(T2) $\mu(U) = \sup \{\mu(C) : C \in \mathcal{C}(X), C \subseteq U\}$ for $U \in \mathcal{O}(X)$.

Remark 5.7. By Lemma 5.3 Definition 5.6 is equivalent to the definition of a (real-valued) topological measure given in all previous papers that use topological measures on compact spaces.
Remark 5.10. Let $X$ be any closed or open set $C$. Then (TM1) holds for $\mu$, so $\mu$ is a topological measure.

Proof. As in the proof of Proposition 5.4, (k2) and (k3) are equivalent. The condition (k1) means that $\mu(X) = \mu(A) + \mu(X \setminus A)$ for any closed or open set $A$. Suppose $C \cup U = D$ is closed. Then $X \setminus D$ is open, disjoint from $U$, so by (k3) and (k1) we have $\mu(U \cup (X \setminus D)) = \mu(U) + \mu(X \setminus D) = \mu(U) + \mu(X) - \mu(D)$. The complement of $U \cup (X \setminus D)$ is $C$, so by (k1) $\mu(U \cup (X \setminus D)) = \mu(X) - \mu(C)$. Then $\mu(U) + \mu(C) = \mu(D)$. The case where $C \cup U$ is open can be proved similarly.

Sometimes it is convenient to use the definition of a topological measure given through closed sets. The following result is from [39, Sect. 2].

Lemma 5.9. $\mu$ is a topological measure on $X$ iff $\mu$ is a real-valued, finite, nonnegative set function on $\mathcal{C}(X)$ satisfying the following properties:

1. if $C \subseteq K$, $C, K \in \mathcal{C}(X)$ then $\mu(C) \leq \mu(K)$.
2. if $C, K \in \mathcal{C}(X)$ are disjoint then $\mu(C) + \mu(K) = \mu(C \cup K)$.
3. if $C \in \mathcal{C}(X)$ and $\epsilon > 0$ then $\exists K \in \mathcal{C}(X)$ such that $C \cap K = \emptyset$ and $\mu(C) + \mu(K) > \mu(X) - \epsilon$.

4. $\mu(U) = \mu(X) - \mu(X \setminus U)$ for every $U \in \mathcal{O}(X)$.

Thus, we can define a topological measure on $X$ by giving a set function on $\mathcal{C}(X)$ satisfying the first three conditions of Lemma 5.9 and then extend it to open sets using the last condition.

Remark 5.10. If $X$ is compact and $\mu$ is finite, condition (T1) of Definition 5.1 is equivalent, as was noticed in [28], to the following three conditions:

(i) $\mu(U \cup V) = \mu(U) + \mu(V)$ for any two disjoint open sets $U, V$.
(ii) If $X = U \cup V$, $U, V \in \mathcal{O}(X)$ then $\mu(U) + \mu(V) = \mu(X) + \mu(U \cap V)$.
(iii) $\mu(X \setminus U) = \mu(X) - \mu(U)$ for any open set $U$.

The same equivalence holds if we replace open sets by closed sets. Thus, when $X$ is compact, a finite topological measure (and more generally, a bounded signed topological measure) can be defined by its actions on open (respectively, on closed) sets. The idea of determining a topological measure on a closed manifold by its values on closed submanifolds with boundary is in [40, Sect. 2].

Remark 5.11. A measure on $X$ is a countably additive set function on a $\sigma$-algebra of subsets of $X$ with values in $[0, \infty]$. A Borel measure on $X$ is a measure on the Borel $\sigma$-algebra on $X$. Let $X$ be locally compact, and let $\mathcal{M}$ be the collection of all Borel measures on $X$ that are inner regular on open sets and outer regular on all Borel sets (i.e. $m(U) = \sup\{m(K) : K \subseteq U, K$ is compact$\}$ for every open set $U$, and $m(E) = \inf\{m(U) : E \subseteq U, U$ is open$\}$ for every Borel set $E$). Thus, $\mathcal{M}$ includes regular Borel measures and Radon measures. We denote by $M(X)$ the restrictions
to $\mathcal{O}(X) \cup \mathcal{G}(X)$ of measures from $\mathcal{M}$, and by $TM(X)$ the set of all topological measures on $X$. Then

$$M(X) \subseteq TM(X).$$

The inclusions follow from the definitions. When $X$ is compact, there are examples of topological measures that are not measures in numerous papers, beginning with [3]. In Sections 14 and 15 we give examples of topological measures that are not measures on compact and locally compact noncompact spaces.

The next result tells when a topological measure is a Borel measure ([11]).

**Theorem 5.12.** Let $\mu$ be a topological measure (or more generally, a deficient topological measure) on a locally compact space $X$. The following are equivalent:

(a) If $C, K$ are compact subsets of $X$, then $\mu(C \cup K) \leq \mu(C) + \mu(K)$.

(b) If $U, V$ are open subsets of $X$, then $\mu(U \cup V) \leq \mu(U) + \mu(V)$.

(c) $\mu$ admits a unique extension to an inner regular on open sets, outer regular Borel measure $m$ on the Borel $\sigma$-algebra of subsets of $X$. $m$ is a Radon measure iff $\mu$ is compact-finite. If $\mu$ is finite then $m$ is an outer regular and inner closed regular Borel measure.

### 6 Solid-set functions

Our goal now is to extend a set function defined on a smaller collection of subsets of $X$ than $\mathcal{O}(X) \cup \mathcal{G}(X)$ to a topological measure on $X$. One such collection is the collection of solid bounded open and solid compact sets, and the corresponding set function is a solid-set function.

**Definition 6.1.** A function $\lambda : \mathcal{A}_s(X) \rightarrow [0, \infty)$ is a solid-set function on $X$ if

(s1) $\sum_{i=1}^{n} \lambda(C_i) \leq \lambda(C)$ whenever $\bigsqcup_{i=1}^{n} C_i \subseteq C, \ C, C_i \in \mathcal{X}_s(X)$;

(s2) $\lambda(U) = \sup\{\lambda(K) : K \subseteq U, \ K \in \mathcal{X}_s(X)\}$ for $U \in \mathcal{O}_s(X)$;

(s3) $\lambda(K) = \inf\{\lambda(U) : K \subseteq U, \ U \in \mathcal{O}_s(X)\}$ for $K \in \mathcal{X}_s(X)$;

(s4) $\lambda(A) = \sum_{i=1}^{n} \lambda(A_i)$ whenever $A = \bigsqcup_{i=1}^{n} A_i, \ A, A_i \in \mathcal{A}_s(X)$.

**Lemma 6.2.** Let $X$ be locally compact, connected, locally connected. Suppose $\lambda$ is a solid-set function on $X$. Then

(i) $\lambda(\emptyset) = 0$.

(ii) $\lambda$ is a superadditive set function, i.e. if $\bigsqcup_{t \in T} A_t \subseteq A, \ A_t, A \in \mathcal{A}_s(X)$, then $\sum_{t \in T} \lambda(A_t) \leq \lambda(A)$.

**Proof.** From Definition 6.1 we see that $\lambda(\emptyset) = 0$. Now let $\bigsqcup_{t \in T} A_t \subseteq A, \ A_t, A \in \mathcal{A}_s(X)$. Since $\sum_{t \in T} \lambda(A_t) = \sup\{\sum_{t \in T'} \lambda(A_t) : T' \subseteq T, \ T' \text{ finite}\}$, it is enough to assume that $T$ is finite. By [s2] in Definition 6.1 we may take all sets $A_t$ to be disjoint compact solid. If also $A \in \mathcal{X}_s(X)$, the assertion is just part [s1] of Definition 6.1 If $A \in \mathcal{O}_s(X)$ then by Lemma 3.10 choose $C \in \mathcal{X}_s(X)$ such that $\bigsqcup_{t \in T} A_t \subseteq C \subseteq A$. Now the assertion follows from parts [s1] and [s2] of Definition 6.1. 

\[\square\]
7 Extension to $\mathcal{A}_{ss}^*(X) \cup \mathcal{K}_c(X)$

Now we assume that $X$ is locally compact, connected, locally connected. We start with a solid-set function $\lambda : \mathcal{A}_{ss}^*(X) \rightarrow [0, \infty)$ on a locally compact, connected, locally connected space $X$. Our goal is to extend $\lambda$ to a topological measure on $X$. We shall do this in steps, each time extending the current set function to a new set function defined on a larger collection of sets.

Definition 7.1. For $A \in \mathcal{A}_{ss}^*(X) \cup \mathcal{K}_c(X)$ define

$$\lambda_1(A) = \lambda(A) - \sum_{i \in I} \lambda(B_i)$$

where $\{B_i : i \in I\}$ is the family of bounded components of $X \setminus A$. In particular, when $X$ is compact

$$\lambda_1(A) = \lambda(X) - \sum_{i \in I} \lambda(B_i). \quad (7.1)$$

Remark 7.2. By Lemma 6.2 each $B_i \in \mathcal{A}_{ss}^*(X)$. If $A \in \mathcal{A}_{ss}^*(X) \cup \mathcal{K}_c(X)$ then $\bigcup_{i \in I} B_i \subseteq \bar{A}$ and by Lemma 6.2 $\sum_{i \in I} \lambda(B_i) \leq \lambda(\bar{A})$. If $A \in \mathcal{A}_{ss}^*(X)$ then in Definition 7.1 we subtract finitely many terms. If $A \in \mathcal{K}_c(X)$ we use additivity on open solid sets (part (ii) in Lemma 5.3).

Lemma 7.3. The set function $\lambda_1 : \mathcal{A}_{ss}^*(X) \cup \mathcal{K}_c(X) \rightarrow [0, \infty)$ defined in Definition 7.2 satisfies the following properties:

(i) $\lambda_1$ is real-valued and $\lambda_1 = \lambda$ on $\mathcal{A}_{ss}^*(X)$.

(ii) Suppose $\bigcup_{i=1}^n A_i \cup \bigcup_{s \in S} B_s \subseteq A$, where $A, A_i \in \mathcal{A}_{ss}^*(X) \cup \mathcal{K}_c(X)$ and $B_s \in \mathcal{A}_{ss}^*(X)$. Then

$$\sum_{i=1}^n \lambda_1(A_i) + \sum_{s \in S} \lambda_1(B_s) \leq \lambda_1(A).$$

In particular, if $\bigcup_{i=1}^n C_i \subseteq C$ where $C_i, C \in \mathcal{K}_c(X)$ then $\sum_{i=1}^n \lambda_1(C_i) \leq \lambda_1(C)$ and if $A \subseteq B$, $A, B \in \mathcal{A}_{ss}^*(X) \cup \mathcal{K}_c(X)$ then $\lambda_1(A) \leq \lambda_1(B)$.

(iii) Suppose that $\bigcup_{i=1}^n A_i \cup \bigcup_{s \in S} B_s = A$, where $A, A_i \in \mathcal{A}_{ss}^*(X) \cup \mathcal{K}_c(X)$ and $B_s \in \mathcal{A}_{ss}^*(X)$ with at most finitely many of $B_s \in \mathcal{K}_c(X)$. Then $\sum_{i=1}^n \lambda_1(A_i) + \sum_{s \in S} \lambda_1(B_s) = \lambda_1(A)$.

Proof. (i) Easy to see from Lemma 5.7, Lemma 6.2 and Remark 7.2.

(ii) First we assume that $X$ is noncompact. Suppose that $\bigcup_{i=1}^n A_i \cup \bigcup_{s \in S} B_s \subseteq A$, where $A, A_i \in \mathcal{A}_{ss}^*(X) \cup \mathcal{K}_c(X)$ and $B_s \in \mathcal{A}_{ss}^*(X)$. We may assume that $A \in \mathcal{A}_{ss}^*(X)$, since the inequality

$$\sum_{i=1}^n \lambda_1(A_i) + \sum_{s \in S} \lambda_1(B_s) \leq \lambda_1(A) \quad (7.2)$$

is equivalent to

$$\sum_{i=1}^n \lambda_1(A_i) + \sum_{s \in S} \lambda_1(B_s) + \sum_{t \in T} \lambda_1(D_t) \leq \lambda_1(\bar{A}), \quad (7.3)$$

where $\{D_t : t \in T\}$ is the disjoint family of bounded components of $X \setminus A$, and by Lemma 5.4 each $D_t \in \mathcal{A}_{ss}^*(X)$.

The proof is by induction on $n$. For $n = 0$ the statement is Lemma 6.2. Suppose now $n \geq 1$ and assume the result is true for any disjoint collection (contained in a bounded solid set) of bounded semisolid or compact connected sets among which there are less than $n$ non-solid sets. Assume
now that we have \(n\) disjoint sets \(A_1, \ldots, A_n\) from the collection \(\mathscr{A}'_s(X) \cup \mathcal{C}_s(X)\). Consider a partial order on \(\{A_1, A_2, \ldots, A_n\}\) where \(A_i \leq A_j\) if \(\overline{A_i} \subseteq \overline{A_j}\). (See Lemma 7.7) Let \(A_1, \ldots, A_p\) where \(p \leq n\) be maximal elements in \(\{A_1, A_2, \ldots, A_n\}\) with respect to this partial order. For a maximal element \(A_k\), \(k \in \{1, \ldots, p\}\) define the following index sets:

\[
I_k = \{i \in \{p+1, \ldots, n\} : A_i \text{ is contained in a bounded component of } X \setminus A_k\},
\]

\[
S_k = \{s \in S : B_s \text{ is contained in a bounded component of } X \setminus A_k\}.
\]

Let \(\{E_\alpha\}_{\alpha \in H}\) be the disjoint family of bounded components of \(X \setminus A_k\). Then \(I_k = \bigcup_{\alpha \in H} I_{k, \alpha}\), \(S_k = \bigcup_{\alpha \in H} S_{k, \alpha}\) where \(I_{k, \alpha} = \{i \in \{p+1, \ldots, n\} : A_i \subseteq E_\alpha\}\), \(S_{k, \alpha} = \{s \in S : B_s \subseteq E_\alpha\}\). The set \(I_k\) and each set \(I_{k, \alpha}\) has cardinality \(< n\). The set \(E_\alpha\) is solid by Lemma 6.4 and

\[
\bigcup_{i \in I_{k, \alpha}} A_i \cup \bigcup_{s \in S_{k, \alpha}} B_s \subseteq E_\alpha. \tag{7.4}
\]

By induction hypothesis \(\sum_{i \in I_{k, \alpha}} \lambda_1(A_i) + \sum_{s \in S_{k, \alpha}} \lambda_1(B_s) \leq \lambda_1(E_\alpha)\). It follows that

\[
\sum_{i \in I_k} \lambda_1(A_i) + \sum_{s \in S_k} \lambda_1(B_s) = \sum_{\alpha \in H} \left(\sum_{i \in I_{k, \alpha}} \lambda_1(A_i) + \sum_{s \in S_{k, \alpha}} \lambda_1(B_s)\right) \leq \sum_{\alpha \in H} \lambda_1(E_\alpha).
\]

Then using part (i) and Definition 7.1 we have:

\[
\lambda_1(A_k) + \sum_{i \in I_k} \lambda_1(A_i) + \sum_{s \in S_k} \lambda_1(B_s) \leq \lambda_1(A_k) + \sum_{\alpha \in H} \lambda_1(E_\alpha) = \lambda_1(\widetilde{A}_k). \tag{7.5}
\]

From Remark 6.3 \(\{1, \ldots, n\} = \{1, \ldots, p\} \cup \bigcup_{k=1}^p I_k\). Similarly, the sets \(S_k, k = 1, \ldots, p\) are also disjoint. Consider the index set \(S' = S \setminus \bigcup_{k=1}^p I_k\). Since \(\{A_k\}_{k=1}^p \bigcup \{B_s\}_{s \in S'}\) is a collection of disjoint solid sets contained in the solid set \(A\), applying (7.4) and Lemma 6.2 we have:

\[
\sum_{i=1}^n \lambda_1(A_i) + \sum_{s \in S} \lambda_1(B_s) = \sum_{k=1}^p \left(\lambda_1(A_k) + \sum_{i \in I_k} \lambda_1(A_i) + \sum_{s \in S_k} \lambda_1(B_s)\right) \leq \sum_{k=1}^p \lambda(\widetilde{A}_k) + \sum_{s \in S'} \lambda(B_s) \leq \lambda(A). \tag{7.6}
\]

Now suppose \(X\) is compact. As in (7.2) and (7.3), the inequality \(\sum_{i=1}^n \lambda_1(A_i) + \sum_{s \in S} \lambda_1(B_s) \leq \lambda_1(A)\) is equivalent to \(\sum_{i=1}^n \lambda_1(A_i) + \sum_{s \in S} \lambda_1(B_s) + \sum_{t \in T} \lambda_1(D_t) \leq \lambda_1(X)\), where \(\{D_t : t \in T\}\) is the disjoint family of components of \(X \setminus A\), and each \(D_t\) is a solid set. We need to show that

\[
\sum_{i=1}^n \lambda_1(A_i) + \sum_{s \in S} \lambda_1(B_s) \leq \lambda_1(X) \text{ if } \bigcup_{i=1}^n A_i \cup \bigcup_{s \in S} B_s \subseteq X,
\]

where each \(B_s\) is solid and each \(A_i \in \mathscr{A}'_s(X) \cup \mathcal{C}_s(X)\).

The proof is by induction on \(n\). For \(n = 0\) use Lemma 6.2. Suppose now \(n \geq 1\) and assume the result is true for any disjoint family from \(\mathscr{A}'_s(X) \cup \mathcal{C}_s(X) \cup \mathscr{A}'(X)\) that contains less than \(n\) non-solid sets. Let \(X \setminus A = \bigcup_{j \in J} E_j\) be the decomposition into components; each \(E_j\) is a solid set by Lemma 3.3. Each \(A_i, i \geq 2\) and each \(B_s\) is contained in a component of \(X \setminus A_1\), so for each \(j \in J\) define index sets \(I_j = \{i \in \{2, \ldots, n\} : A_i \subseteq E_j\}\), \(S_j = \{s \in S : B_s \subseteq E_j\}\). Since \(X \setminus E_j\) is solid, \(|I_j| \leq n - 1\) and

\[
(X \setminus E_j) \cup \bigcup_{i \in I_j} A_i \cup \bigcup_{s \in S_j} B_s \subseteq X. \tag{7.7}
\]
by induction hypothesis we have \( \lambda_1(X \setminus E_j) + \sum_{i \in I_j} \lambda_1(A_i) + \sum_{s \in S_j} \lambda_1(B_s) \leq \lambda_1(X) \), i.e. by Definition 7.1, \( \sum_{i \in I_j} \lambda_1(A_i) + \sum_{s \in S_j} \lambda_1(B_s) \leq \lambda_1(E_j) \). Then we have:

\[
\sum_{i=1}^{n} \lambda_1(A_i) + \sum_{s \in S} \lambda_1(B_s) = \lambda_1(A_1) + \sum_{j \in J} \left( \sum_{i \in I_j} \lambda_1(A_i) + \sum_{s \in S_j} \lambda_1(B_s) \right)
\leq \lambda_1(A_1) + \sum_{j \in J} \lambda_1(E_j) = \lambda_1(X),
\]

where the last equality is by definition of \( \lambda_1 \).

(iii) The proof is almost identical to the proof of the previous part, and we keep the same notations. When \( X \) is noncompact, we may assume again that \( A \in \mathcal{O}_s(X) \), since the inequalities (7.2) and (7.3) become equalities. The proof is by induction on \( n \), and the case \( n = 0 \) is given by Lemma 4.3 and part (s4) of Definition 6.1. The inequalities in the induction step become equalities once one observes that \( \bigcup_{k=1}^{p} A_k \cup \bigcup_{s \in S} B_s = A \) (see Remark 3.8) and that (7.4) above becomes \( \bigcup_{i \in I_k, s \in S} A_i \cup \bigcup_{s \in S, n} B_s = E_n \). The last inequality in (7.6) becomes an equality by Lemma 4.3 and part (s4) of Definition 6.1. When \( X \) is compact, the inclusion in (7.7) becomes the equality, and then all subsequent inequalities in the proof of the previous part become equalities as well. \( \square \)

8 Extension to \( \mathcal{X}_0(X) \)

Our goal now is to extend the set function \( \lambda_1 \) to a set function \( \lambda_2 \) defined on \( \mathcal{X}_0(X) \). Recall that \( K \in \mathcal{X}_0(X) \) if \( K = \bigcup_{i=1}^{n} K_i \) where \( n \in \mathbb{N} \) and \( K_i \in \mathcal{X}(X) \) for \( i = 1, \ldots, n \).

**Definition 8.1.** For \( K = \bigcup_{i=1}^{n} K_i \), where \( K_i \in \mathcal{X}(X) \), let \( \lambda_2(K) = \sum_{i=1}^{n} \lambda_1(K_i) \).

**Lemma 8.2.** Suppose \( C = C_1 \cup \ldots \cup C_n \subseteq U, C_i \in \mathcal{X}(X), U \in \mathcal{O}_s(X) \). Given \( \epsilon > 0 \), there exists \( B \subseteq U \setminus C, B \in \mathcal{X}_0(X) \) such that \( \lambda(U) - \lambda_2(C) - \lambda_2(B) < \epsilon \).

**Proof.** Let \( \epsilon > 0 \). The proof is by induction on \( n \). Let \( n = 1 \), so \( C = C_1 \). Suppose \( U \setminus C \) is disconnected. By Lemma 4.1, \( U = C \cup \bigcup_{i=1}^{m} W_i \), where \( W_i \in \mathcal{O}_s(X) \). Pick \( K_i \in \mathcal{X}(X) \) such that \( \sum_{i=1}^{m} (\lambda(W_i) - \lambda(K_i)) < \epsilon \) and let \( B = K_1 \cup \ldots \cup K_m \). Then \( \lambda(U) - \lambda(C) - \lambda_2(B) = \sum_{i=1}^{m} (\lambda(W_i) - \lambda(K_i)) < \epsilon \). Now suppose \( U \setminus C \) is connected. Using complete regularity of \( X \) and Lemma 3.10 for \( \epsilon' = \epsilon / 2 \) pick \( K,F \in \mathcal{X}_0(X) \) and \( W,V,O \in \mathcal{O}_s(X) \) such that \( K \subseteq W \subseteq \overline{W} \subseteq U, C \subseteq O \subseteq F \subseteq V, V \subseteq K, \lambda(U) - \lambda(K) < \epsilon' \), and \( \lambda(V) - \lambda(C) < \epsilon' \). Then also

\[
\lambda(U) - \lambda(W) < \epsilon', \quad \lambda(F) - \lambda(C) < \epsilon',
\]

so we approximate \( U \) and \( C \) by \( W \) and \( F \). By the above argument it is enough to assume \( W \setminus F \) is connected. Since \( W \) and \( F \) are solid sets and \( X \setminus (W \setminus F) = (X \setminus W) \cup F \), we see that \( W \setminus F \) is a bounded open semisolid set. If \( X \) is compact, then \( \lambda_1(W \setminus F) = \lambda(X) - \lambda(X \setminus W) - \lambda(F) = \lambda(W) - \lambda(F) \). If \( X \) is noncompact, then \( \lambda_1(W \setminus F) = \lambda(W \setminus F) \). In any case,

\[
\lambda_1(W \setminus F) = \lambda(W) - \lambda(F).
\]

Since \( W \setminus F \) is connected, so is \( B = \overline{W \setminus F} \). Thus, \( B \in \mathcal{X}_0(X) \), \( B \subseteq \overline{W \setminus O} \subseteq U \setminus C \) and using part (iii) of Lemma 7.3

\[
\lambda(U) - \lambda_2(C) - \lambda_1(B) \leq \lambda(U) - \lambda_2(C) - \lambda_1(W \setminus F) = \lambda(U) - \lambda(C) - \lambda(W) + \lambda(F) < 2\epsilon' = \epsilon.
\]

Now suppose that lemma holds for any open bounded open solid set which contains less than \( n \) disjoint compact solid sets. We shall show that the statement holds for \( U \in \mathcal{O}_s(X) \) containing \( C = C_1 \cup \ldots \cup C_n, C_i \in \mathcal{X}(X) \). Suppose that \( U \setminus C \) is disconnected. By Lemma 4.6 write \( U = \bigcup_{i \in I} C_i \cup \bigcup_{j=1}^{p} U_j \), where \( p \in \mathbb{N} \) and \( I \neq \emptyset \). We can write \( \{1, \ldots, n\} = I \cup \bigcup_{j=1}^{p} I_j \), where
Proof. Part (i) easily follows from the definition of (iii) Let $C$ Definition 9.1.

The set function Lemma 9.2. the following properties:

\[
\sum_{i=1}^{n} \lambda(C_i) - \lambda_2(B) = \sum_{j=1}^{p} \left( \lambda(U_i) - \sum_{j \in I_j} \lambda(C_j) - \lambda_2(B_j) \right) < \epsilon.
\]

Now suppose $U \setminus C$ is connected. As in the induction step $n = 1$, choose $K, F_i \in \mathcal{K}_s(X)$ and $W, V_i, O_i \in \mathcal{O}_s(X)$ such that $K \subseteq W \subseteq U, C_i \subseteq O_i \subseteq F_i \subseteq V_i, V_i \subseteq K$ for $i = 1, \ldots, n,$ and $\lambda(U) - \lambda(K) < \epsilon, \sum_{i=1}^{n} (\lambda(V_i) - \lambda(C_i)) < \epsilon$. Again, it is enough to assume $W \setminus \bigcup_{i=1}^{n} F_i$ is connected. Then $B = W \setminus \bigcup_{i=1}^{n} F_i$ is a compact connected set for which the argument similar to one in step $n = 1$ shows that $B \subseteq U \setminus C$ and $\lambda(U) - \lambda_2(C) - \lambda_1(B) < \epsilon$. \[\square\]

Lemma 8.3. The set function $\lambda_2$ from Definition 7.1 satisfies the following properties:

(i) $\lambda_2$ is real-valued, $\lambda_2 = \lambda_1$ on $\mathcal{K}_s(X)$ and $\lambda_2 = \lambda$ on $\mathcal{K}_s(X)$.

(ii) $\lambda_2$ is finitely additive on $\mathcal{K}_o(X)$.

(iii) $\lambda_2$ is monotone on $\mathcal{K}_o(X)$.

(iv) $\lambda_1(U) = \sup \{ \lambda_2(K) : K \subseteq U, K \in \mathcal{K}_o(X) \}$ for $U \in \mathcal{O}_s^*(X)$.

Proof. Part (i) easily follows from the definition of $\lambda_2$ and Lemma 7.3. Part (ii) is obvious.

(iii) Let $C \subseteq K$, where $C, K \in \mathcal{K}_o(X)$. Write $C = \bigcup_{i=1}^{n} C_i, K = \bigcup_{j=1}^{m} K_j$, where the sets $C_i (i = 1, \ldots, n)$ and $K_j (j = 1, \ldots, m)$ are compact connected. By connectedness, each $C_i$ is contained in one of the sets $K_j$. Consider index sets $I_j = \{ i : C_i \subseteq K_j \}$ for $j = 1, \ldots, m$. By Lemma 7.3, we have $\sum_{i \in I_j} \lambda_1(C_i) \leq \lambda_1(K_j)$. Then $\lambda_2(C) = \sum_{i=1}^{n} \lambda_1(C_i) = \sum_{j=1}^{m} \sum_{i \in I_j} \lambda_1(C_i) \leq \sum_{j=1}^{m} \lambda_1(K_j) = \lambda_2(K)$.

(iv) If $K \subseteq U, K \in \mathcal{K}_o(X), U \in \mathcal{O}_s^*(X)$ then by part (iii) of Lemma 7.3 $\lambda_2(K) \leq \lambda_1(U)$. Let $\tilde{U} = U \cup \bigcup_{i=1}^{n} C_i$, where $C_i$ are bounded components of $X \setminus U$. By Lemma 8.2 find $B \in U, B \in \mathcal{K}_o(X)$ such that $\lambda(\tilde{U}) - \sum_{i=1}^{n} \lambda_1(C_i) - \lambda_2(B) = \lambda_1(U) - \lambda_2(B) < \epsilon$. \[\square\]

9 Extension to $\mathcal{O}(X) \cup \mathcal{C}(X)$

We are now ready to extend the set function $\lambda_2$ to a set function $\mu$ defined on $\mathcal{O}(X) \cup \mathcal{C}(X)$.

Definition 9.1. For an open set $U$ and a closed set $F$ we define

\[
\mu(U) = \sup \{ \lambda_2(K) : K \subseteq U, K \in \mathcal{K}_o(X) \},
\]

\[
\mu(F) = \inf \{ \mu(U) : F \subseteq U, U \in \mathcal{O}(X) \}.
\]

Note that $\mu$ may assume $\infty$.

Lemma 9.2. The set function $\mu$ in Definition 9.1 satisfies the following properties:

(p1) $\mu$ is monotone, i.e. if $A \subseteq B, A, B \in \mathcal{O}(X) \cup \mathcal{C}(X)$ then $\mu(A) \leq \mu(B)$.

(p2) $\mu(A) < \infty$ for each $A \in \mathcal{A}^*(X)$. In particular, $\mu$ is compact-finite.

(p3) $\mu \geq \lambda_2$ on $\mathcal{K}_o(X)$.

(p4) Let $K \subseteq V, K \in \mathcal{K}_o(X), V \in \mathcal{O}(X)$. Then for any positive $\epsilon$ there exists $K_1 \in \mathcal{K}_o(X)$ such that $K \subseteq K_1 \subseteq V$ and $\mu(K_1) - \mu(K) < \epsilon$.

(p5) $\mu = \lambda$ on $\mathcal{A}_s^*(X)$.

(p6) $\mu$ is finitely additive on open sets.
(p7) If \( G = F \cup K \), where \( G, F \in \mathcal{C}(X) \), \( K \in \mathcal{K}(X) \) then \( \mu(G) = \mu(F) + \mu(K) \). In particular, \( \mu \) is finitely additive on compact sets.

(p8) \( \mu \) is additive on \( \mathcal{O}(X) \), i.e. if \( V = \bigcup_{i \in I} V_i \), where \( V, V_i \in \mathcal{O}(X) \) for all \( i \in I \), then 
\[
\mu(V) = \sum_{i \in I} \mu(V_i).
\]

(p9) If \( G \cup V = F \) where \( G, F \in \mathcal{C}(X) \), \( V \in \mathcal{O}(X) \) then \( \mu(G) + \mu(V) \leq \mu(F) \).

(p10) If \( G \cup V \subseteq U \) where \( G \in \mathcal{C}(X) \), \( V, U \in \mathcal{O}(X) \) then \( \mu(G) + \mu(V) \leq \mu(U) \).

(p11) \( \mu = \lambda_1 \) on \( \mathcal{K}_c(X) \cup \mathcal{O}^*_s(X) \) and \( \mu = \lambda_2 \) on \( \mathcal{K}_0(X) \).

(p12) \( \mu(U) = \sup\{\mu(C) : C \subseteq U, C \in \mathcal{K}(X)\} \) for \( U \in \mathcal{O}(X) \).

Proof. (p1) Clearly, \( \mu \) is monotone on open sets and on closed sets. Let \( V \in \mathcal{O}(X), F \in \mathcal{C}(X) \). The monotonicity in the case \( F \subseteq V \) is obvious. Suppose \( V \subseteq F \). For any open set \( U \) with \( F \subseteq U \) we have \( V \subseteq U \), so \( \mu(V) \leq \mu(U) \). Taking infimum over sets \( U \) we obtain \( \mu(V) \leq \mu(F) \).

(p2) Let \( K \in \mathcal{K}(X) \). By Lemma 2.20 choose \( V \in \mathcal{O}^*_s(X) \) and \( C \in \mathcal{K}_c(X) \) such that \( K \subseteq V \subseteq C \subseteq X \). For any \( D \in \mathcal{K}_0(X) \), \( D \subseteq V \) by Lemma 5.3 we have \( \lambda_2(D) \leq \lambda_2(C) \), and \( \lambda_2(C) \leq \infty \). By Definition 9.1 \( \mu(V) \leq \lambda_2(C) \), and then \( \mu(K) \leq \mu(V) \leq \lambda_2(C) \). Thus, \( \mu \) is compact-finite. If \( U \) is an open bounded set then \( \mu(U) \leq \mu(U) < \infty \).

(p3) Let \( K \in \mathcal{K}_0(X) \). For any open set \( U \) containing \( K \) we have \( \mu(U) \geq \lambda_2(K) \) by the definition of \( \mu \). Taking infimum over sets \( U \) we obtain \( \mu(K) \geq \lambda_2(K) \).

(p4) \( \mu(K) \leq \infty \), so by Definition 9.1 find \( U \in \mathcal{O}(X) \) such that \( U \subseteq V \), \( \mu(U) - \mu(K) < \epsilon \). Let \( U_1, \ldots, U_n \) be finitely many connected components of \( U \) that cover \( K \). By Lemma 2.20 pick \( V_i \in \mathcal{O}^*_s(X) \) such that \( K \cap U_i \subseteq V_i \subseteq V_i \subseteq U_i \) for \( i = 1, \ldots, n \). We may take \( K_1 = \bigcup_{i=1}^n V_i \), for \( K_2 \subseteq V \) and \( \mu(K_1) - \mu(K) \leq \mu(\bigcup_{i=1}^n U_i) - \mu(K) \leq \mu(U) - \mu(K) \). (p5) We shall show that \( \mu = \lambda \) on \( \mathcal{O}^*_s(X) \). Let \( U \in \mathcal{O}^*_s(X) \), so by part (p2) \( \mu(U) < \infty \). By Definition 9.1 given \( \epsilon > 0 \), choose \( K \in \mathcal{K}_0(X) \) such that \( K \subseteq U \) and \( \mu(U) - \epsilon < \lambda_2(K) \). By Lemma 5.10 there exists \( C \in \mathcal{K}_c(X) \) such that \( K \subseteq C \subseteq U \). Now using Lemma 5.3 and Definition 9.1 we have: \( \mu(U) - \epsilon < \lambda_2(K) \leq \lambda_2(C) = \mu(C) \leq \sup\{\lambda_2(C) : C \subseteq U, C \in \mathcal{K}_c(X)\} = \lambda(U) \). Hence, \( \mu(U) = \lambda(U) \).

\[
\lambda(U) = \sup\{\lambda(C) : C \subseteq U, C \in \mathcal{K}_c(X)\} = \sup\{\lambda_2(C) : C \subseteq U, C \in \mathcal{K}_c(X)\}
\leq \sup\{\lambda_2(C) : C \subseteq U, C \in \mathcal{K}_0(X)\} = \mu(U).
\]

Therefore, \( \mu(U) = \lambda(U) \) for any \( U \in \mathcal{O}^*_s(X) \). Now we shall show that \( \mu = \lambda \) on \( \mathcal{K}_c(X) \). From part (p3) above and Lemma 8.3 we have \( \mu \geq \lambda_2 = \lambda \) on \( \mathcal{K}_c(X) \). Since \( \mu = \lambda \) on \( \mathcal{O}^*_s(X) \), for \( C \in \mathcal{K}_c(X) \) we have by Definition 9.1 and Definition 9.1

\[
\lambda(C) = \inf\{\lambda(U) : U \in \mathcal{O}^*_s(X), C \subseteq U\} = \inf\{\mu(U) : U \in \mathcal{O}^*_s(X), C \subseteq U\}
\geq \inf\{\mu(U) : U \in \mathcal{O}(X), C \subseteq U\} = \mu(C).
\]

Therefore, \( \mu = \lambda \) on \( \mathcal{K}_c(X) \).

(p6) Let \( U_1, U_2 \in \mathcal{O}^*_s(X) \) be disjoint. For any \( C_1, C_2 \in \mathcal{K}_0(X) \) with \( C_i \subseteq U_i \), \( i = 1, 2 \) by Lemma 8.3 and Definition 9.1 we obtain \( \mu(U_1) + \mu(U_2) \leq \mu(U_1 \cup U_2) \). For the converse inequality, note that given \( C \subseteq U_1 \cup U_2, C \in \mathcal{K}_0(X) \) we have \( C = C_1 \cup C_2 \), \( C_i = C \cap U_i \in \mathcal{K}_0(X), i = 1, 2 \) (since each connected component of \( C \) must be contained either in \( U_1 \) or \( U_2 \)). Then \( \lambda_2(C) = \lambda_2(C_1) + \lambda_2(C_2) \leq \mu(U_1) + \mu(U_2) \), giving \( \mu(U_1 + U_2) \leq \mu(U_1) + \mu(U_2) \).

(p7) A compact and a closed set that are disjoint can be separated by disjoint open sets, and we can use the argument from part 2 of Proposition 6.3.

(p8) Let \( V = \bigcup_{i \in I} V_i \) with \( V_i \in \mathcal{O}(X) \) for all \( i \in I \). By parts (p6) and (p1) for any finite \( I' \subseteq I \) we have \( \sum_{i \in I'} \mu(V_i) = \mu(\bigcup_{i \in I'} V_i) \leq \mu(V) \), so \( \sum_{i \in I} \mu(V_i) \leq \mu(V) \). To prove the opposite inequality, first assume that \( \mu(V) < \infty \). For \( \epsilon > 0 \) find a compact \( C \in \mathcal{K}_0(X) \) contained in \( V \) such that \( \mu(V) - \epsilon < \lambda_2(C) \). By compactness, \( C \subseteq \bigcup_{i \in I'} V_i \) for some finite subset \( I' \) of \( I \). Then by
connectedness $C = \bigsqcup_{i \in I'} C_i$ where $C_i = C \cap V_i \subseteq V_i$, and $C_i \in \mathcal{K}_0(X)$ for each $i \in I'$. By Lemma 8.3 and part (p3) we have:

$$\mu(V) - \epsilon < \lambda_2(C) = \lambda_2\left(\bigcup_{i \in I'} C_i\right) = \sum_{i \in I'} \lambda_2(C_i) \leq \sum_{i \in I'} \mu(C_i) \leq \sum_{i \in I'} \mu(V_i) \leq \sum_{i \in I} \mu(V_i).$$

This shows that $\mu(V) = \sum_{i \in I} \mu(V_i)$ when $\mu(V) < \infty$.

Now suppose $\mu(V) = \infty$. For $n \in \mathbb{N}$ find a compact $K \subseteq V$ such that $\mu(K) > n$. Choose a finite index set $I_n \subseteq I$ such that $K \subseteq \bigsqcup_{i \in I_n} V_i$. Then $\sum_{i \in I} \mu(V_i) \geq \sum_{i \in I_n} \mu(V_i) = \mu\left(\bigsqcup_{i \in I_n} V_i\right) \geq \mu(K) > n$. Thus $\sum_{i \in I} \mu(V_i) = \infty = \mu(V)$.

(p9) It is enough to show the statement for the case $\mu(F) < \infty$. If $K \subseteq F \subseteq \mathcal{K}_0(X)$ then $G \sqcup K \subseteq F$. By parts (p3), (p7) and (p1) $\mu(G) + \lambda_2(K) \leq \mu(G) + \mu(K) \leq \mu(F)$. Then $\mu(G) + \mu(V) \leq \mu(F)$.

(p10) It is enough to show the statement for the case $\mu(U) < \infty$. If $K \subseteq F \subseteq \mathcal{K}_0(X)$ then $F = G \sqcup K \subseteq U$. By parts (p3), (p7) and Definition 9.1 $\mu(G) + \lambda_2(K) \leq \mu(G) + \mu(K) = \mu(F) \leq \mu(U)$. Then $\mu(G) + \mu(V) \leq \mu(U)$.

(p11) Let $C \in \mathcal{K}_0(X)$. According to Lemma 3.4 and Definition 5.5 we write $\tilde{C} \in \mathcal{K}_0(X)$ as $\tilde{C} = C \cup \bigsqcup_{i \in I} U_i$, where $U_i \in \mathcal{S}_s^*(X)$ are the bounded components of $X \setminus C$. Given $\epsilon > 0$ choose by Definition 6.1 $V \in \mathcal{S}_s^*(X)$ such that $\tilde{C} \subseteq V$ and $\lambda(V) - \lambda(\tilde{C}) < \epsilon$. By parts (p8), (p9) and (p1)

$$\mu(C) + \sum_{i \in I} \mu(U_i) = \mu(C) + \mu\left(\bigsqcup_{i \in I} U_i\right) \leq \mu(\tilde{C}) \leq \mu(V).$$

Then using part (p5) and Definition 7.1 we have:

$$\mu(C) \leq \mu(V) - \sum_{i \in I} \mu(U_i) = \lambda(V) - \sum_{i \in I} \lambda(U_i) \leq \lambda(\tilde{C}) - \sum_{i \in I} \lambda(U_i) + \epsilon = \lambda_1(C) + \epsilon.$$ Thus, $\mu(C) \leq \lambda_1(C)$. By part (p3) and Lemma 8.3 $\mu(C) \geq \lambda_2(C) = \lambda_1(C)$. So $\mu = \lambda_1$ on $\mathcal{K}_0(X)$.

By part (iv) of Lemma 8.3 $\mu = \lambda_1$ on $\mathcal{S}_s^*(X)$. From part (p7) and Definition 8.1 we have $\mu = \lambda_2$ on $\mathcal{S}_s^*(X)$.

(p12) Using part (p3)

$$\mu(U) = \sup\{\lambda_2(C) : C \subseteq U, C \in \mathcal{K}_0(X)\} \leq \sup\{\mu(C) : C \subseteq U, C \in \mathcal{K}_0(X)\} \leq \sup\{\mu(C) : C \subseteq U, C \in \mathcal{K}(X)\}.$$ For the converse inequality, given $C \subseteq U$, $U \in \mathcal{O}(X)$, $C \in \mathcal{K}(X)$ choose by Lemma 2.6 $K \in \mathcal{K}_0(X)$ with $C \subseteq K \subseteq U$. Then by parts (p1) and (p11) $\mu(C) \leq \mu(K) = \lambda_2(K)$, so

$$\sup\{\mu(C) : C \subseteq U, C \in \mathcal{K}(X)\} \leq \sup\{\lambda_2(K) : K \subseteq U, K \in \mathcal{K}_0(X)\} = \mu(U).$$

\[\square\]

10 Finite additivity on $\mathcal{O}(X) \cup \mathcal{K}(X)$

Finite additivity of $\mu$ (defined in Definition 9.1) on $\mathcal{O}(X) \cup \mathcal{K}(X)$ will be established in a series of lemmas.

Lemma 10.1. If $C \subseteq U$, $C \in \mathcal{K}_0(X)$, $U \in \mathcal{O}(X)$ then $\mu(U) = \mu(C) + \mu(U \setminus C)$.

Proof. Let $C = C_1 \cup C_2 \cup \ldots \cup C_n$. By Lemma 1.5 $U = \bigsqcup_{j=1}^n C_j \cup \bigsqcup_{t \in T} U_t$, where sets $U_t \in \mathcal{S}_s^*(X)$, and all but finitely many are in $\mathcal{S}_s^*(X)$. By part (iii) of Lemma 7.3 $\lambda(U) = \sum_{t \in T} \lambda_1(C_t) + \sum_{t \in T} \lambda_1(U_t)$, so by parts (p11) and (p8) of Lemma 9.2 $\mu(U) - \mu(C) = \sum_{t \in T} \mu(U_t) = \mu(U \setminus C)$.

Lemma 10.2. If $X$ is compact, $K \subseteq X$, $K \in \mathcal{C}(X)$, $U \in \mathcal{O}(X)$ then $\mu(U) = \mu(K) + \mu(X \setminus K)$.
Proof. Since $X$ is solid, by part [p5] of Lemma 9.2, $\mu$ is finite. Let $K \in \mathcal{C}(X)$. Given $\epsilon > 0$ by part [p4] choose $K_1 \in \mathcal{H}_0(X), K \subseteq K_1$ such that $\mu(K) > \mu(K_1) - \epsilon$. Using Lemma 10.1 we have: $\mu(K) + \mu(X \setminus K) > \mu(K_1) - \epsilon + \mu(X \setminus K) \geq \mu(K_1) + \mu(X \setminus K_1) - \epsilon = \mu(X) - \epsilon$. Thus, $\mu(K) + \mu(X \setminus K) \geq \mu(X)$. The opposite inequality is by part [p9] of Lemma 9.2. \hfill \Box

Lemma 10.3. If $X$ is noncompact, $K \subseteq U, K \in \mathcal{H}(X), U \in \mathcal{O}^\ast(X)$ then $\mu(U) = \mu(K) + \mu(U \setminus K)$.

Proof. Using part [p12] of Lemma 9.2 and Lemma 2.5 choose sets $K, \ldots$ such that

$$K \subseteq W \subseteq D \subseteq U \text{ and } \mu(U) - \mu(W) < \epsilon. \quad (10.1)$$

Let $B$ be the union of bounded components of $X \setminus U$ and let the open set $V$ be the union of bounded components of $X \setminus D$. Set $C = B \cap V$. By Lemma 2.13 $C$ is compact and $U \cup C$ is open. The solid hull $\bar{D} = D \cup V$. Then by part [p9] of Lemma 9.2 $\mu(D) + \mu(V) \leq \mu(\bar{D})$. By Lemma 5.7 $V \subseteq \bar{D} \subseteq \bar{U} = U \cup B$, so $V \subseteq U \cup (B \cap V) = U \cup C$. It follows that $K \cup C \subseteq D \cup V = \bar{D} \subseteq U \cup C$. Since $U \cup C$ is open, by Lemma 3.10 we may find $W' \in \mathcal{O}^\ast(X)$ such that

$$K \cup C \subseteq \bar{D} \subseteq W' \subseteq U \cup C. \quad (10.2)$$

Then

$$W' \setminus (K \cup C) \subseteq U \setminus K. \quad (10.3)$$

According to part [p4] of Lemma 9.2 pick $K_1 \in \mathcal{H}_0(X)$ such that

$$K \cup C \subseteq K_1 \subseteq W' \text{ and } \mu(K_1) \leq \mu(K \cup C) + \epsilon. \quad (10.4)$$

By Lemma 10.1, $\mu(W') = \mu(W' \setminus K_1) + \mu(K_1)$. Now using (10.1), Definition 7.1, (10.2), (10.4), (10.3), additivity on $\mathcal{O}(X)$ and finite additivity of $\mu$ on $\mathcal{H}(X)$ in Lemma 9.2 we have:

$$\mu(U) - \epsilon < \mu(W) \leq \mu(D) = \mu(\bar{D}) - \mu(V) \leq \mu(W') - \mu(C) = \mu(W' \setminus K_1) + \mu(K_1) - \mu(C) \leq \mu(W' \setminus (K \cup C)) + \mu(K \cup C) + \epsilon - \mu(C) \leq \mu(U \setminus K) + \mu(K \cup C) + \epsilon = \mu(U \setminus K) + \mu(K) + \epsilon.$$

Thus, $\mu(U) \leq \mu(U \setminus K) + \mu(K)$. The opposite inequality is by part [p10] of Lemma 9.2. \hfill \Box

Lemma 10.4. If $K \subseteq U, K \in \mathcal{H}(X), U \in \mathcal{O}^\ast(X)$ then $\mu(U) = \mu(K) + \mu(U \setminus K)$.

Proof. Let $U = \bigcup_{i \in I} U_i$ be the decomposition of $U$ into connected components, and let $I'$ be a finite subset of $I$ such that $K \subseteq \bigcup_{i \in I'} U_i$. For $i \in I'$ let $K_i = K \cap U_i \in \mathcal{H}(X)$ and let $K = \bigcup_{i \in I'} K_i$. By Lemma 10.3 we know that

$$\mu(K_i) + \mu(U_i \setminus K_i) = \mu(U_i) \quad \text{for each } i \in I'. \quad (10.5)$$

Then using finite additivity of $\mu$ on $\mathcal{H}(X)$ and additivity of $\mu$ on $\mathcal{O}(X)$ in Lemma 9.2 and (10.5) we have:

$$\mu(K) + \mu(U \setminus K) = \sum_{i \in I'} \mu(K_i) + \mu(U \setminus \bigcup_{i \in I'} K_i) = \sum_{i \in I'} \mu(K_i) + \sum_{i \in I'} \mu(U_i \setminus K_i) + \sum_{i \in I \setminus I'} \mu(U_i)$$

$$= \sum_{i \in I'} \mu(U_i) + \sum_{i \in I \setminus I'} \mu(U_i) = \sum_{i \in I} \mu(U_i) = \mu(U). \quad \Box$$

Lemma 10.5. If $K \subseteq U, K \in \mathcal{H}(X), U \in \mathcal{O}(X)$ then $\mu(U) = \mu(K) + \mu(U \setminus K)$.
Proof. First assume that $\mu(U) < \infty$. Given $\epsilon > 0$ by Definition 9.1 find $C \in \mathcal{A}(X)$ such that $K \subseteq C$ and $\mu(U) - \mu(C) < \epsilon$. Using Lemma 2.3 find $V \in \mathcal{A}^*(X)$ such that $K \subseteq C \subseteq V \subseteq U$. By Lemma 10.4 $\mu(V) = \mu(V \setminus K) + \mu(K)$. Then using monotonicity of $\mu$ in Lemma 9.2 we see that

$$\mu(U) - \epsilon < \mu(C) \leq \mu(V) = \mu(V \setminus K) + \mu(K) \leq \mu(U \setminus K) + \mu(K).$$

(10.6)

Therefore, $\mu(U) \leq \mu(U \setminus K) + \mu(K)$. The opposite inequality is part (p10) of Lemma 9.2. Therefore, $\mu(U) = \mu(U \setminus K) + \mu(K)$ if $\mu(U) < \infty$. Now assume $\mu(U) = \infty$. For $n \in \mathbb{N}$ choose $C \in \mathcal{A}(X)$ such that $K \subseteq C$ and $\mu(C) > n$. By Lemma 2.3 find $V \in \mathcal{A}^*(X)$ such that $K \subseteq C \subseteq V \subseteq U$. Using again (10.6) we have: $n < \mu(C) \leq \mu(V \setminus K) + \mu(K) \leq \mu(U \setminus K) + \mu(K)$, i.e. $n - \mu(K) \leq \mu(U \setminus K)$. Since $\mu(K) \in \mathbb{R}$ by part (p2) of Lemma 9.2 it follows that $\mu(U \setminus K) = \infty$, and $\mu(U \setminus K) + \mu(K) = \mu(U)$.

**Remark 10.6.** Our proof of Lemma 10.3 is close to that of [6] Lemma 5.9. We would like to point out that the part related to Lemma 10.1 is the lengthiest and technically most difficult in the entire [6] as it involves [6] Lemma 5.8, Proposition 4.1, Lemma 4.3 as well as lengthy adaptations of arguments from [5].

**Theorem 10.7.** Let $X$ be locally compact, connected, locally connected. A solid-set function on $X$ extends uniquely to a compact-finite topological measure on $X$.

**Proof.** Definitions 7.1 8.1 and 9.1 extend solid-set function $\lambda$ to a set function $\mu$. We shall show that $\mu$ is a topological measure. Definition 9.1 and part (p12) of Lemma 9.2 show that $\mu$ satisfies (TM2) and (TM3) of Definition 5.1. Proposition 5.8, part (p6) of Lemma 9.2 and Lemma 10.5 show that $\mu$ is a topological measure if $X$ is noncompact. Proposition 5.8, part (p6) of Lemma 9.2 and Lemma 10.2 show that $\mu$ is a topological measure if $X$ is compact. By part (p2) of Lemma 9.2 $\mu$ is compact-finite. To show the uniqueness of the extension suppose $\nu$ is a topological measure on $X$ such that $\mu = \nu = \lambda$ on $\mathcal{A}^*_\nu(X)$. If $A \in \mathcal{A}_c(X)$ then by Definition 5.5 $A = \tilde{A} \setminus \bigcup_{s \in S} B_s$, where $\tilde{A}, B_s \in \mathcal{A}^*_\nu(X)$, so from Definition 7.1 it follows that $\mu = \nu$ on $\mathcal{A}_c(X)$, and, hence, on $\mathcal{A}_0(X)$. From part (16) of Lemma 5.3 it then follows that $\mu = \nu$ on $\mathcal{A}(X)$, so $\mu = \nu$.

**Remark 10.8.** We will summarize the extension procedure for obtaining a topological measure $\mu$ from a solid-set function $\lambda$ on a locally compact, connected, locally connected space. First, for a compact connected set $C$ we have:

$$\mu(C) = \lambda(\tilde{C}) - \sum_{i=1}^n \lambda(B_i),$$

where $\tilde{C}$ is the solid hull of $C$ and $B_i$ (open solid sets) are bounded components of $X \setminus C$. Hence, when $X$ is compact and $C$ is a closed connected set

$$\mu(C) = \lambda(X) - \sum_{i=1}^n \lambda(B_i).$$

For $C \in \mathcal{A}_0(X)$ of the form $C = \bigcup_{i=1}^n C_i$, $C_i \in \mathcal{A}^*_c(X)$ we have:

$$\mu(C) = \sum_{i=1}^n \mu(C_i).$$

Finally,

$$\mu(U) = \sup \{\mu(K) : K \subseteq U, K \in \mathcal{A}_0(X)\} \quad \text{for an open set } U,$$

$$\mu(F) = \inf \{\mu(U) : F \subseteq U, U \in \mathcal{A}(X)\} \quad \text{for a closed set } F.$$

**Theorem 10.9.** If a solid-set function $\lambda$ is extended to a topological measure $\mu$ then the following holds: if $\lambda : \mathcal{A}^*_\nu(X) \to \{0,1\}$ then $\mu$ is also simple; if $\sup \{\lambda(K) : K \in \mathcal{A}_c(X)\} = M < \infty$ then $\mu$ is finite and $\mu(X) = M$. 

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Proof. Follows from Remark 10.8 part (p5) of Lemma 9.2 and part (17) of Lemma 5.3.

Theorem 10.10. The restriction $\lambda$ of a compact-finite topological measure $\mu$ to $A_s^*(X)$ is a solid-set function, and $\mu$ is uniquely determined by $\lambda$.

Proof. Let $\lambda$ be the restriction of $\mu$ to $A_s^*(X)$. Monotonicity of a topological measure (see Lemma 5.3 and (TM1) of Definition 5.1) show that $\lambda$ satisfies conditions (s1) and (s4) of Definition 6.1. For $U \in \mathcal{O}_s^*(X)$ by (TM2) let $K \in \mathcal{X}(X)$ be such that $\mu(U) - \mu(K) < \epsilon$ and by Lemma 3.10 we may assume that $K \in \mathcal{X}_s(X)$. Part (s2) of Definition 6.1 follows. Part (s3) of Definition 6.1 follows from (TM3) and Lemma 5.10. Since $\mu$ is compact-finite, $\lambda$ is real-valued. Therefore, $\lambda$ is a solid-set function.

Remark 10.11. When $X$ is compact our method of extending a solid-set function to a topological measure is different and simpler than the original method in [5].

Remark 10.12. Comparing values on semisolid or solid sets one may determine whether two topological measures (or two quasi-integrals) coincide. Lemma 6.2, Lemma 7.3, and Lemma 9.2 give us some additional properties of topological measures. For example, by part (p7) of Lemma 6.2, Lemma 7.3, and Lemma 9.2 respectively, (TM1) and Lemma 5.3 show that $\mathcal{A}_s(X)$ is uniquely determined by a solid-set function, and $(\mathcal{A}_s(X) \cup \mathcal{G}_s(X))$.

11 Irreducible partitions and genus of a space

Let $X$ be a compact Hausdorff connected locally connected space. $A_s(X) = \mathcal{O}_s(X) \cup \mathcal{G}_s(X)$.

Definition 11.1. A solid partition of a solid set $A$ is a finite disjoint collection of sets $\{A_i : i \in I, A_i \in A_s(X)\}$ such that $A = \bigcup_{i \in I} A_i$.

Remark 11.2. In [5] Def. 2.1 a solid partition of $X$ is a disjoint collection of solid sets $\{A_i : i \in I \subseteq A_s(X)\}$ such that $X = \bigcup_{i \in I} A_i$ and only finitely many of $A_i$’s are closed. By Lemma 4.3 this definition and Definition 11.1 are consistent.

If we take any solid set $A$ we immediately get a solid partition $\{A, A \setminus A\}$ of $X$. Such partitions are called trivial.

Definition 11.3. Let $\{A_i : i \in I\}$ be a solid partition of $X$. Let $I' = \{i \in I : A_i \in \mathcal{X}_s(X)\}$. We say that $\{A_i : i \in I'\}$ is an irreducible partition if $X \setminus \bigcup_{i \in J} A_i$ is connected for any proper subset $J \subset I'$.

Example 11.4. Let $X = \{z \in \mathbb{C} : |z| \leq 2\}, C_1 = \{z \in X : 1 \leq Re z \leq 2, Im z = 0\}, C_2 = \{z \in X : -2 \leq Re z \leq -1, Im z = 0\}, C_3 = \{z \in X : 1 \leq Im z \leq 2, Re z = 0\}, C_4 = \{z \in X : -2 \leq Im z \leq -1, Re z = 0\}$. Then $C_1, C_2, C_3, C_4$ and the four components of $X \setminus (C_1 \cup \ldots \cup C_4)$ constitute a non-trivial partition of $X$, which is not irreducible. The partition consisting of $C_1, C_2$ and the two components of $X \setminus (C_1 \cup C_2)$ is non-trivial irreducible.

Definition 11.5. We define the genus of $X$, $g(X)$, to be the maximal value of $n - 1$, where $n$ is the number of closed solid sets in an irreducible partition of $X$, where the max is taken over irreducible partitions of $X$.

Remark 11.6. (a1) If there is only one open (closed) solid set in a solid partition, then there is only one closed (open) solid set in this partition, and the partition is trivial. (a2) If $\{A_i : i \in I\}$ is a solid partition, and $|I'| = 2$ then $\{A_i : i \in I'\}$ is an irreducible non-trivial partition. (a3) If $\{A_i : i \in I\}$ is a non-trivial irreducible partition then $|I \setminus I'| \geq 2$. (a4) $g(X) = 0$ if and only if every irreducible partition of $X$ is trivial. (a5) By [29] Cor. 16 $g(X) + 1$ is equal to the maximum number of components of a set of the form $V \cap W$ where $V$ and $W$ are open solid sets with $V \cup W = X$. (a6) By [29] Cor. 15 in Definition 11.5 we may take open solid sets instead of closed solid sets.
When \( X \) is compact, connected, locally path connected \( g(X) \leq n(X) \) where \( n(X) \) is the largest integer \( n \) so that there is a surjective map from \( \pi_1(X) \) onto \( FG(n) \). If \( \pi_1(X) \) is abelian, then \( g(X) \leq 1 \). In particular, if \( X \) is a topological group, \( g(X) \leq 1 \). See [30, 29].

**Remark 11.7.** We are particularly interested in the case \( g(X) = 0 \), which can also be described as follows: if the union of two open solid sets in \( X \) is the whole space, their intersection must be connected. Intuitively, \( X \) does not have holes or loops. In the case where \( X \) is locally path connected, \( g(X) = 0 \) if the fundamental group \( \pi_1(X) \) is finite (in particular, if \( X \) is simply connected). Knudsen [32] was able to show that if \( H^1(X) = 0 \) then \( g(X) = 0 \), and in the case of CW-complexes the converse also holds.

**Definition 11.8.** We say that a finite disjoint family \( \{C_j\}_{j \in J} \subseteq \mathcal{G}(X) \) generates a non-trivial irreducible partition if there is \( J' \subseteq J \) and a finite disjoint family \( \{U_i\}_{i \in I} \subseteq \mathcal{O}(X) \) such that \( \bigsqcup_{j \in J'} C_j \cup \bigsqcup_{i \in I} U_i = X \) is a non-trivial irreducible partition of \( X \).

The next result is [5, Lemma 2.1].

**Lemma 11.9.** Suppose \( X \) is disconnected where \( \{C_j\}_{j \in J} \subseteq \mathcal{G}(X) \) is a finite disjoint family. Then \( \{C_j\}_{j \in J} \) generates a non-trivial irreducible partition of \( X \).

**Example 11.10.** The same finite family \( \{C_j\}_{j \in J} \subseteq \mathcal{G}(X) \) may generate different irreducible partitions; moreover, the number of closed solid set in these partitions may be different. Consider \( X = \{z \in C : |z| \leq 7\} \setminus \left( \{z \in C : |z| < 1\} \cup \{z \in C : |z - 5| < 1\} \right) \). Let \( C_1 = \{z \in X : \text{Re} z = 0, 1 \leq \text{Im} z \leq 7\} \), \( C_2 = \{z \in X : \text{Im} z = 0, 1 \leq \text{Re} z \leq 7\} \), \( C_3 = \{z \in X : \text{Im} z = 0, 6 \leq \text{Re} z \leq 7\} \), and \( C_4 = \{z \in X : 6 \leq \text{Re} z \leq 7\} \).

**Corollary 11.11.** \( g(X) = 0 \) if and only if \( X \) is connected for any finite disjoint family \( \{C_i\}_{i=1}^n \) of closed solid sets.

**Proof.** (\( \Rightarrow \)) Suppose \( g(X) \geq 1 \). Then there exists a non-trivial irreducible partition \( \{A_i\}_{i \in I} \) of \( X \). By part (a1) of Remark 11.6 \( X \) is disconnected, where \( I' = \{i \in I : A_i \in \mathcal{G}(X)\} \).

(\( \Leftarrow \)) Suppose that \( X \) is disconnected. By Lemma 11.9 there is a non-trivial irreducible partition. Then \( g(X) \geq 1 \) by part (a4) of Remark 11.6. \( \square \)

### 12 Solid-set function: a different definition

When \( X \) is compact we have another definition of a solid-set function:

**Definition 12.1.** A set function \( \lambda : \mathcal{A}(X) \to \mathbb{R} \) is a solid-set function if

1. \( \sum_{i \in I} \lambda(A_i) \leq \lambda(X) \) if \( \{A_i\}_{i \in I} \subseteq \mathcal{A}(X) \) is a disjoint finite collection.
2. \( \lambda(U) = \sup\{\lambda(C) : C \in \mathcal{G}(X), C \subseteq U\} \) for \( U \in \mathcal{O}(X) \).
3. \( \sum_{i \in I} \lambda(A_i) = \lambda(X) \) for an irreducible partition \( \bigsqcup_{i \in I} A_i = X \).

(In particular, \( \lambda(A) + \lambda(X \setminus A) = \lambda(X) \) for any \( A \in \mathcal{A}(X) \).)

**Remark 12.2.** An equivalent definition of a solid-set function may be obtained if one replaces the first condition in Definition 12.1 by the following condition: \( \sum_{i=1}^n \lambda(C_i) \leq \lambda(C) \) if \( \bigsqcup_{i=1}^n C_i \subseteq C, C, C_i \in \mathcal{G}(X) \). This gives the original definition of a solid-set function from [5].

Part 2 in Definition 12.1 is equivalent to the following: \( \lambda(C) = \inf\{\lambda(U) : C \subseteq U, U \in \mathcal{O}(X)\} \) for \( C \in \mathcal{G}(X) \).

If \( g(X) = 0 \) then by Remark 11.6 condition 3 in Definition 12.1 becomes: \( \lambda(A) + \lambda(X \setminus A) = \lambda(X) \) for any \( A \in \mathcal{A}(X) \).
Lemma 12.3. For the set function \( \lambda \) given by Definition 12.1 we have:

1. \( \lambda(\emptyset) = 0 \).
2. \( \lambda(C) = \inf \{ \lambda(U) : C \subseteq U, U \in \mathcal{C}(X) \} \) for any \( C \in \mathcal{C}(X) \).
3. If \( \bigcup_{s \in S} A_s \subseteq A \) where \( A, A_s \in \mathcal{A}_s(X) \) then \( \sum_{s \in S} \lambda(A_s) \leq \lambda(A) \).
4. If \( \bigcup_{s \in S} A_s = A \) is a solid partition of \( A \) then \( \sum_{s \in S} \lambda(A_s) = \lambda(A) \).

Proof. Parts (1) and (2) are easy. In part (3) it is enough to assume that \( S \) is finite, since \( \sum_{s \in S} \lambda(A_s) = \sup \{ \sum_{s' \in S'} \lambda(A_s) : S' \subseteq S, S' \text{ finite} \} \). Then \( \bigcup_{s \in S} A_s \subseteq A \) is a finite disjoint collection of solid sets. By Definition 12.1 \( \sum_{s \in S} \lambda(A_s) + \lambda(X \setminus A) \leq \lambda(X) = \lambda(A) - \lambda(X \setminus A) \), and we obtain the result.

(4). Since \( \bigcup_{s \in S} A_s = A \) is a solid partition of \( A \) iff \( (X \setminus A) \cup \bigcup_{s \in S} A_s = X \) is a solid partition of \( X \), it is enough to prove the statement for solid partitions of \( X \). Let \( \bigcup_{j \in J} C_j \cup \bigcup_{s \in S} V_s = X \), \( C_j \in \mathcal{C}(X), V_s \in \mathcal{C}(X) \) be a solid partition of \( X \). \( J \) is finite, and the proof is by induction on \( |J| \). For \( |J| = 1 \) or \( |J| = 2 \) the result holds by Remark 11.6 and Definition 12.1. Assume that \( |J| \geq 3 \) and that the result holds for any solid partition of \( X \) with less then \( |J| \) closed solid sets. Note that \( |S| \geq 2 \) by Remark 11.6, i.e. \( X \setminus \bigcup_{j \in J} C_j \) is disconnected. Obtain by Lemma 11.9 a non-trivial irreducible partition of \( X \)

\[
X = \bigcup_{j \in J'} C_j \cup \bigcup_{i \in I} U_i. \tag{12.1}
\]

Then \( \bigcup_{i \in I} U_i = X \setminus \bigcup_{j \in J'} C_j = \bigcup_{s \in S} V_s \cup \bigcup_{j \in J \setminus J'} C_j \). By connectedness, for each \( i \in I \)

\[
U_i = \bigcup_{j \in J_i} C_j \cup \bigcup_{s \in S_i} V_s \text{ where } J_i = \{ j \in J \setminus J' : C_j \in U_i \}, \quad S_i = \{ s \in S : V_s \in U_i \}.
\]

The partition in (12.1) is non-trivial, so \( |J'| \geq 2 \), and then \( |J_i| \leq |J \setminus J'| \leq n - 2 \). Then \( (X \setminus U_i) \cup \bigcup_{j \in J_i} C_j \cup \bigcup_{s \in S_i} V_s = X \) is a solid partition of \( X \) with at most \( n - 1 \) closed solid sets, so by induction hypothesis \( \lambda(X \setminus U_i) + \sum_{j \in J_i} \lambda(C_j) + \sum_{s \in S_i} \lambda(V_s) = \lambda(X) \). By part 3 of Definition 12.1

\[
\lambda(U_i) = \sum_{j \in J_i} \lambda(C_j) + \sum_{s \in S_i} \lambda(V_s). \tag{12.2}
\]

Applying (12.2) and then Definition 12.1 to the irreducible partition in (12.1) we have: \( \sum_{j \in J} \lambda(C_j) + \sum_{s \in S} \lambda(V_s) = \sum_{j \in J'} \lambda(C_j) + \sum_{i \in I} \left( \sum_{j \in J_i} \lambda(C_j) + \sum_{s \in S_i} \lambda(V_s) \right) = \sum_{j \in J'} \lambda(C_j) + \sum_{i \in I} \lambda(U_i) = \lambda(X) \).

Remark 12.4. Lemma 12.3 establishes the equivalence of Definition 12.1 and Definition 6.1 when \( X \) compact.

13 Extension to a topological measure

We shall describe a way to extend a solid-set function \( \lambda \) from Definition 12.1 to a topological measure. This extension procedure is related to one given by D. Grubb 27, which significantly simplified the original one presented in 9.

STEP 1. We extend \( \lambda \) to \( \lambda_1 \) on \( \mathcal{C}(X) \cup \mathcal{A}_s(X) \) exactly as in formula (12.1). Note that Lemma 7.3 holds for \( \lambda_1 \).
STEP 2. Let $\mathcal{O}_0(X) = \{X \setminus C : C \in \mathcal{C}_0(X)\}$. We extend the set function $\lambda_1$ to a set function $\lambda_2$ defined on $\mathcal{O}_0(X) = \mathcal{C}_0(X) \cup \mathcal{O}_0(X)$. On $\mathcal{C}_0(X)$ the function $\lambda_2$ is defined as in Definition 9.1 and for $U \in \mathcal{O}_0(X)$ let

$$\lambda_2(U) = \lambda(X) - \lambda_2(X \setminus U).$$

(13.1)

We still have Lemma 8.3 and we also have:

$$\lambda_2(U) = \sup\{\lambda_2(C) : C \subseteq U, C \in \mathcal{C}_0(X)\} \quad \text{for} \quad U \in \mathcal{O}_0(X).$$

(13.2)

**Proof.** Let $U = \bigsqcup_{i \in I} U_i$ be the decomposition into components of $U \in \mathcal{O}_0(X)$. By Lemma 4.3 components $U_i \in \mathcal{O}_{ss}(X)$, and all but finitely many are in $\mathcal{O}_s(X)$. For $\epsilon > 0$, choose a finite set $T \subseteq I$ for which $\sum_{i \in I \setminus T} \lambda_1(U_i) < \epsilon/2$. By part (iv) of Lemma 8.3 for each $i \in T$ choose a set $C_i \subseteq U_i$ such that $C_i \subseteq U_i$ and $\sum_{i \in T} \lambda_1(U_i) - \sum_{i \in T} \lambda_2(C_i) < \epsilon/2$. Let $C = \bigsqcup_{i \in T} C_i \in \mathcal{C}_0(X)$. Then $C \subseteq U$ and

$$\lambda_2(C) = \sum_{i \in T} \lambda_2(C_i) > \sum_{i \in T} \lambda_1(U_i) - \frac{\epsilon}{2} \geq \sum_{i \in I} \lambda_1(U_i) - \epsilon.$$

(13.3)

Now writing $X \setminus U = \bigsqcup_{j=1}^m F_j$, $F_j \in \mathcal{C}_s(X)$ by part (iii) of Lemma 7.3 we have

$$\lambda_1(X) = \sum_{j=1}^m \lambda_1(F_j) + \sum_{i \in I} \lambda_1(U_i).$$

(13.4)

Then by (13.3), (13.4) and (13.1) we have $\lambda_2(C) + \epsilon \geq \sum_{i \in I} \lambda_1(U_i) = \lambda_1(X) - \sum_{j=1}^m \lambda_1(F_j) = \lambda_2(U)$. 

We extend the set function $\lambda_2$ to a set function $\mu$ defined on $\mathcal{A}(X)$: for an open set $U$ define $\mu(U)$ as in Definition 9.1 and for a closed set $C$ let

$$\mu(C) = \mu(X) - \mu(X \setminus C).$$

(13.5)

**Lemma 13.1.** The set function $\mu$ satisfies the following properties:

1. $\mu = \lambda_2$ on $\mathcal{O}_0(X) \cup \mathcal{C}_0(X)$, $\mu = \lambda$ on $\mathcal{A}(X)$, and $\mu$ is finite.

2. $\mu$ is monotone on open sets, $\mu$ is monotone on closed sets.

3. $\mu(U) = \sup\{\mu(C) : C \subseteq U, C \in \mathcal{C}(X)\}$, $U \in \mathcal{O}(X)$.

4. $\mu(C) = \inf\{\mu(U) : C \subseteq U, U \in \mathcal{O}(X)\}$, $C \in \mathcal{C}(X)$.

5. $\mu$ is finitely additive on open sets.

6. $\mu$ is finitely additive on closed sets.

**Proof.** (1.) Note that $\mu(U) = \lambda_2(U)$ for any $U \in \mathcal{O}_0(X)$ by (13.2). For $C \in \mathcal{C}_0(X)$ then $\mu(C) = \mu(X) - \mu(X \setminus C) = \lambda_2(X) - \lambda_2(X \setminus C) = \lambda_2(C)$. Thus, $\mu = \lambda_2$ on $\mathcal{O}_0(X) \cup \mathcal{C}_0(X)$. In particular, $\mu = \lambda = \lambda_1$ on $\mathcal{C}(X)$, and hence, $\mu = \lambda$ on solid sets; $\mu(X) = \lambda(X) < \infty$. (2.) Obvious. (3.) As in the proof of part (i) of Lemma 8.2 we have $\mu(U) = \sup\{\mu(K) : K \subseteq U, K \in \mathcal{C}(X)\}$. The second statement follows immediately from the first by definition of $\mu$. (4.) and (5.) are proved as in Lemma 9.2. (6.) Suppose $A, B \in \mathcal{A}(X)$ are disjoint. By parts (i) and (ii) we may assume that $A, B$ are not both closed or both open. We may also assume that $A$ and $A \cup B$ are not both closed or both open. Then the sets $A$ and $X \setminus (A \cup B)$ are both closed or both open, and disjoint. Hence by part (i) or part (ii) $\mu(A) + \mu(X \setminus (A \cup B)) = \mu(A \cup X \setminus (A \cup B)) = \mu(X \setminus B)$, which by (13.3) gives $\mu(A) + \mu(X) - \mu(A \cup B) = \mu(X) - \mu(B)$, i.e. $\mu(A) + \mu(B) = \mu(A \cup B)$. 

**Theorem 13.2.** Any solid-set function on $X$ extends uniquely to a topological measure on $X$.

**Proof.** Parts (i), (ii) and (iii) of Lemma 13.1 show that that the set function $\mu$ is a topological measure that is an extension of a given solid-set function $\lambda$. The proof of uniqueness is similar to the one in Theorem 10.7. 

\[\square\]
14 Examples for a compact space

In the following examples the underlying compact space has genus 0. This means by Remark [12.2] that checking the last condition of a solid-set function in Definition [12.1] is greatly simplified.

Example 14.1 (Aarnes circle measure). Let $X$ be the unit circle and $B$ be the boundary of $X$. Fix a point $p$ in $X \setminus B$. Define $\mu$ on solid sets as follows: $\mu(A) = 1$ if i) $B \subset A$ or ii) $p \in A$ and $A \cap B \neq \emptyset$. Otherwise, we let $\mu(A) = 0$. Then $\mu$ is a solid-set function and hence extends to a topological measure on $X$. Note that $\mu$ is not a point mass. To demonstrate that $\mu$ is not a measure we shall show that $\mu$ is not subadditive. Let $A_1$ be a closed solid set which is an arc that is a proper subset of $B$, $A_2$ be a closed solid set that is the closure of $B \setminus A_1$, and $A_3 = X \setminus B$ be an open solid subset of $X$. Then $X = A_1 \cup A_2 \cup A_3$, $\mu(X) = 1$, but $\mu(A_1) + \mu(A_2) + \mu(A_3) = 0$.

Example 14.2. Another solid-set function (hence, a topological measure) is obtained if in the Example [14.1] we take $B$ to be any set of points of $X$, and $p \in X \setminus B$.

Example 14.3. Let $X$ be a sphere (or a square.) Fix points $p_1, p_2, p_3$ in $X$. Define $\mu$ on solid sets as follows: $\mu(A) = 1$ if $A$ contains the majority of the three points, otherwise, $\mu(A) = 0$. The resulting topological measure is non-subadditive, since $\mu(X) = 1$, and it is easy to represent $X$ as a union of three overlapping solid sets each of which contains exactly one of the points $x, y, z$ and, hence, has measure 0. Notice also that $\mu(A) = 1$ for any connected set $A$ that contains at least 2 points, since any component of $X \setminus A$ (a solid set by Lemma [5.4]) contains at most 1 point.

Example 14.4. Let $X$ be a sphere and let $P = \{p_1, \ldots, p_n\}$ be a set of $n$ distinct points in $X$, with $n$ an odd number. We define a solid-set function $\mu$ on $X$ by letting $\mu(A) = 1$ if $A \cap P$ contains a majority of points of $P$; otherwise let $\mu(A) = 0$.

Example 14.5. Let $X$ be the unit sphere and $P = \{p_1, \ldots, p_{2n+1}\}$ an odd-numbered subset of $X$. If $A \in \mathcal{A}_s(X)$ let $\sharp A$ the number of points in $A \cap P$. For $k = 0, \ldots, n$ let $\mu(A) = k/n$ if $\sharp A \in \{2k, 2k+1\}$. Thus defined, $\mu$ is a solid-set function in $X$.

Remark 14.6. Example [14.1] Example [14.3] Example [14.5] were the first examples of topological measures that are not (restrictions to $\mathcal{O}(X)$ or $\mathcal{K}(X)$) of regular Borel measures. They were presented by J. Aarnes in [3], [5] and [11]. For more examples of topological measures on compact spaces see, for instance, [7] and [10].

15 Examples for a locally compact space

When $X$ is locally compact, the hardest condition to verify is the condition [84] that deals with solid partitions. But, as we shall see in this section, it turns out that this condition holds trivially for spaces whose one-point compactification has genus 0. We denote by $\hat{X}$ the one-point compactification of $X$.

Lemma 15.1. Let $X$ be locally compact noncompact and $\hat{X}$ be its one-point compactification. If $A \in \mathcal{A}_s^*(X)$ then $A$ is solid in $\hat{X}$.

Proof. Since $A$ is connected in $X$, it is also connected in $\hat{X}$. Let $X \setminus A = \bigcup^{n}_{i=1} B_i$ be the decomposition into connected components. Each $B_i$ is an unbounded subset of $X$. We can write $\hat{X} \setminus A = \bigcup^{n}_{i=1} E_i$ where each $E_i = B_i \cup \{\infty\}$. It is easy to see that each $E_i$ is connected in $\hat{X}$. Thus, $\hat{X} \setminus A$ is connected, and so $A$ is solid in $\hat{X}$. \qed

Lemma 15.2. Let $X$ be a locally compact noncompact space whose one-point compactification $\hat{X}$ has genus 0. If $A \in \mathcal{A}_s^*(X)$ then any solid partition of $A$ is the set $\hat{A}$ itself.

Proof. Suppose first that $V \in \mathcal{O}_s^*(X)$ and its solid partition is given by $V = \bigcup^{n}_{i=1} C_i \cup \bigcup^{m}_{j=1} U_j$, where each $C_i \in \mathcal{K}_s(X)$ and each $U_j \in \mathcal{O}_s^*(X)$. From Lemma [15.1] it follows that $\hat{X} \setminus V$ and each $C_i$
are closed solid sets in $\hat{X}$. Since $\hat{X}$ has genus 0, by Remark 11.1. $\hat{X}\setminus((\hat{X}\setminus V)\cup\bigcup_{j=1}^{n} C_{i}) = \bigcup_{j=1}^{m} U_{j}$ must be connected in $\hat{X}$. It follows that $m = 1$ and we may write $V = \bigcup_{j=1}^{m} C_{i} \cup U_{1}$. Then $\{U_{1}, \hat{X}\setminus V, C_{1}, \ldots, C_{n}\}$ is a solid partition of $\hat{X}$, and it has only one open set. By Remark 11.6 this solid partition also has only one closed set in it, and it must be $\hat{X}\setminus V$. So each $C_{i} = \emptyset$, and the solid partition of $V$ is $V = U_{1}$, i.e. the set itself.

Now suppose that $C \in \mathcal{K}_{s}(X)$ and its solid partition is given by $C = \bigcup_{i=1}^{n} C_{i} \cup \bigcup_{j=1}^{m} U_{j}$, where each $C_{i} \in \mathcal{K}_{s}(X)$ and each $U_{j} \in \mathcal{O}_{s}(X)$. Then $\{\hat{X}\setminus C, U_{1}, \ldots, U_{m}, C_{1}, \ldots, C_{n}\}$ is a solid partition of $\hat{X}$. Again by Remark 11.1. $\hat{X}\setminus\bigcup_{j=1}^{m} C_{i} = (\hat{X}\setminus C) \cup U_{1} \ldots \cup U_{m}$ must be connected in $\hat{X}$. It follows that $U_{j} = \emptyset$ for $j = 1, \ldots, m$. Then by connectedness of $C$ we see that the solid partition of $C$ must be the set itself. 

**Remark 15.3.** From Lemma 16.2 it follows that for any locally compact noncompact space whose one-point compactification has genus 0 the last condition of Definition 6.1 holds trivially. This is true, for example, for $X = \mathbb{R}^{n}$, half-plane in $\mathbb{R}^{n}$ with $n \geq 2$, or for a punctured ball in $\mathbb{R}^{n}$ with the relative topology.

**Example 15.4.** Lemma 15.2 may not be true for spaces whose one-point compactification has genus greater than 0. For example, let $X$ be an infinite strip $\mathbb{R} \times [0, 1]$ without the ball of radius $1/4$ centered at $(-1/2, 1/2)$, so $\hat{X}$ has genus greater than 0. It is easy to give an example of a solid partition of a bounded solid set (say, rectangle $[0, n] \times [0, 1]$ or $(0, n) \times [0, 1]$) which consists of $n$ solid sets (rectangles of the type $(i, i+1) \times [0, 1]$ or $[i, i+1] \times [0, 1]$) for any given odd $n \in \mathbb{N}$, $n > 1$.

We are ready to give examples of topological measures on locally compact spaces.

**Example 15.5.** Let $X$ be a locally compact space whose one-point compactification has genus 0. Let $\lambda$ be a real-valued topological measure on $X$ (or, more generally, a real-valued deficient topological measure on $X$; for definition and properties of deficient topological measures on locally compact spaces see [11]). Let $P$ be a set of two distinct points. For each $A \in \mathcal{A}^{s}_{\infty}(X)$ let $\nu(A) = 0$ if $\sharp A = 0$, $\nu(A) = \lambda(A)$ if $\sharp A = 1$, and $\nu(A) = 2\lambda(X)$ if $\sharp A = 2$, where $\sharp A$ is the number of points in $A \cap P$. We claim that $\nu$ is a solid-set function. By Remark 15.3 we only need to check the first three conditions of Definition 6.1. The first one is easy to see. Using Lemma 3.10 it is easy to verify conditions (s2) and (s3) of Definition 6.1. The solid-set function $\nu$ extends to a unique compact-finite topological measure on $X$. Suppose, for example, that $\lambda$ is the Lebesgue measure on $X = \mathbb{R}^{2}$, the set $P$ consists of two points $p_{1} = (0, 0)$ and $p_{2} = (2, 0)$. Let $K_{i}$ be the closed ball of radius 1 centered at $p_{i}$ for $i = 1, 2$. Then $K_{1}, K_{2}$ and $C = K_{1} \cup K_{2}$ are compact solid sets, $\nu(K_{1}) = \nu(K_{2}) = \pi$, $\nu(C) = \pi$. Since $\nu$ is not subadditive, $\nu$ is a topological measure that is not a measure. Note that $\nu(X) = \infty$.

The next two examples are adapted from [6] Example 2.2 and are related to Example 14.1.

**Example 15.6.** Let $X$ be the unit disk on the plane with removed origin. $X$ is a locally compact Hausdorff space with respect to the relative topology. Any subset of $X$ whose closure in $\mathbb{R}^{2}$ contains the origin is unbounded in $X$. For $A \in \mathcal{A}^{s}_{\infty}(X)$ (since $X$ is also a solid subset of the unit disk by Lemma 15.1 we define $\mu^{\prime}(A) = \mu(A)$ where $\mu$ is the solid-set function on the unit disk from Example 14.1. From Remark 15.3 Lemma 3.10 and the fact that $\mu$ is a solid-set function on $X$ we see that $\mu^{\prime}$ is a solid-set function on $X$. By Theorem 10.7 $\mu^{\prime}$ extends uniquely to a topological measure on $X$, which we also call $\mu^{\prime}$. Note that $\mu^{\prime}$ is simple. We claim that $\mu^{\prime}$ is not a measure. Let $U_{1} = \{z \in X : \text{Im } z > 0\}$, $U_{2} = \{z \in X : \text{Im } z < 0\}$ and $F = \{z \in X : \text{Im } z = 0\}$. Then $U_{1}, U_{2}$ are open (unbounded) in $X$ and $F$ is a closed (unbounded) set in $X$ consisting of two disjoint segments. Note that $X = F \cup U_{1} \cup U_{2}$. Using Remark 10.3 we calculate $\mu^{\prime}(F) = \mu^{\prime}(U_{1}) = \mu^{\prime}(U_{2}) = 0$. The boundary of the disk, $C$, is a compact connected set, $X \setminus C$ is unbounded in $X$, so $C \in \mathcal{K}_{s}(X)$. Since $\mu^{\prime}(C) = 1$, we have $\mu^{\prime}(X) = 1$. Thus, $\mu^{\prime}$ is not subadditive, so it is not a measure.

This example also shows that on the unit disk without the origin (a locally compact space) finite additivity of topological measures holds on $\mathcal{K}(X) \cup \mathcal{O}(X)$ by Definition 5.1 but fails on
$\mathcal{C}(X) \cup \mathcal{O}(X)$. This is in contrast to topological measures on the unit disk (and on all compact spaces), where finite additivity holds on $\mathcal{C}(X) \cup \mathcal{O}(X)$.

**Example 15.7.** Let $X = \mathbb{R}^2$, $l$ be a straight line and $p$ a point of $X$ not on the line $l$. For $A \in \mathcal{A}_s^\ast(X)$ define $\mu(A) = 1$ if $A \cap l \neq \emptyset$ and $p \in A$; otherwise, let $\mu(A) = 0$. Using Lemma 3.10 it is easy to verify the first three conditions of Definition 6.1. From Remark 15.3 it follows that $\mu$ is a solid-set function on $X$. By Theorem 10.7 $\mu$ extends uniquely to a topological measure on $X$, which we also call $\mu$. Note that $\mu$ is not subadditive, for we may cover a compact ball with Lebesgue measure greater than 1 by finitely many balls of Lebesgue measure less than 1. Hence, $\mu(X) = 1$. Failure of subadditivity shows that $\mu$ is not a measure.

The sets $F$ and $X \setminus F$ are both unbounded. Now take a bounded open disk $V$ around $p$ that does not intersect $l$. Then $X = V \cup (X \setminus V)$, where $V \in \mathcal{O}(X)$, $\mu(V) = \mu(X \setminus V) = 0$, while $\mu(X) = 1$.

This example also shows that on a locally compact space finite additivity of topological measures holds on $\mathcal{X}(X) \cup \mathcal{O}(X)$ by Definition 5.1 but fails on $\mathcal{C}(X) \cup \mathcal{O}(X)$. It fails even in the situation $X = V \cup F$, where $V$ is a bounded open set, and $F$ is a closed set.

The last two examples suggest the following result.

**Theorem 15.8.** Let $X$ be a noncompact locally compact, connected, locally connected space whose one-point compactification $\hat{X}$ has genus 0. Suppose $\nu$ is a solid-set function on $\hat{X}$. For $A \in \mathcal{A}_s^\ast(X)$ define $\mu(A) = \nu(A)$. Then $\mu$ is a solid-set function on $X$ and, thus, extends uniquely to a topological measure on $X$.

**Proof.** Let $A \in \mathcal{A}_s^\ast(X)$. By Lemma 15.1 $A$ is a solid set in $\hat{X}$. Using Lemma 3.10 the fact that $\nu$ is a solid-set function on $\hat{X}$, and that a bounded solid set does not contain $\infty$, it is easy to verify the first three conditions of Definition 6.1. By Remark 15.3 $\mu$ is a solid-set function on $X$.

**Theorem 15.8** allows us to obtain a large variety of topological measures on a locally compact space from examples of topological measures on compact spaces.

**Example 15.9.** Let $X$ be a locally compact space whose one-point compactification has genus 0. Let $n$ be a natural number. Let $P$ be the set of distinct $2n + 1$ points. For each $A \in \mathcal{A}_s^\ast(X)$ let $\nu(A) = i/n$ if $|A| = 2i + 1$, where $|A|$ is the number of points in $A \cap P$. By Example 14.5 and Theorem 15.8 $\nu$ is a solid-set function on $X$; it extends to a unique topological measure on $X$ that assumes values $0, 1/n, \ldots, 1$.

We conclude with an example of another infinite topological measure.

**Example 15.10.** Let $X = \mathbb{R}^n$ for any $n \geq 2$, and $\lambda$ be the Lebesgue measure on $X$. For $U \in \mathcal{O}_s^\ast(X)$ define $\mu(U) = 0$ if $0 \leq \lambda(U) \leq 1$ and $\mu(U) = \lambda(U)$ if $\lambda(U) > 1$. For $C \in \mathcal{X}_s(X)$ define $\mu(C) = 0$ if $0 \leq \lambda(C) < 1$ and $\mu(C) = \lambda(C)$ if $\lambda(C) \geq 1$. It is not hard to check the first three conditions of Definition 6.1. From Remark 15.3 it follows that $\mu$ is a solid-set function on $X$. By Theorem 10.7 $\mu$ extends uniquely to a topological measure on $X$, which we also call $\mu$. Note that $\mu(X) = \infty$. $\mu$ is not subadditive, for we may cover a compact ball with Lebesgue measure greater than 1 by finitely many balls of Lebesgue measure less than 1. Hence, $\mu$ is not a measure.

The vast majority of papers dealing with quasi-linear functionals and topological measures (including papers in symplectic geometry) consider a compact underlying space, finite topological measures and bounded quasi-linear functionals. In symplectic geometry one often sees quasi-linear functionals corresponding to simple topological measures. This section shows how to obtain a variety of finite and infinite topological measures on locally compact spaces.

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