Vortex lattice melting in a boson-ladder in artificial gauge field

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We consider a two-leg boson ladder in an artificial U(1) gauge field and show that, in the presence of interleg attractive interaction, the flux induced Vortex state can be melted by dislocations. For increasing flux, instead of the Meissner to Vortex transition in the commensurate-incommensurate universality class, first an Ising transition from the Meissner state to a charge density wave takes place, then, at higher flux, the melted Vortex phase is established via a disorder point where incommensurability develops in the rung current correlation function and in momentum distribution. Finally, the quasi-long range ordered Vortex phase is recovered for sufficiently small interaction. Our predictions for the observables, such as the spin current and the static structure factor, could be tested in current experiments with cold atoms in bosonic ladders.

Recently, artificial gauge fields1 and artificial spin-orbit coupling2 have been achieved in cold atomic gases using Raman coupling, allowing to probe the effect of external gauge fields on interacting bosons. The analog of the Meissner to vortex (M-to-V) phase transition for superconductors3 was predicted for the bosonic two-leg ladder in Refs. 4. The original proposal was made in the context of Josephson junction ladders, where dissipation spoils quantum coherence5, affecting their use for superconducting qubits based circuits6. In the ultracold atomic gas a simple but already nontrivial realization is the bosonic two-leg ladder in artificial flux7, where the M-to-V transition was observed in non-interacting bosonic ladders at fixed flux $\pi/2$ per plaquette and varying interleg hopping.

From the theoretical point of view4, for bosons with interchain repulsive interactions, the M-to-V transition falls in the commensurate-incommensurate (C-IC) universality class8. Recently it was investigated by Density Matrix Renormalization Group (DMRG) and bosonization approach for hard-core bosons on a two-leg ladder as a function of flux4,10 showing that the region of stability of the M phase over the V one is largely enhanced with respect to the non-interacting case10. Moreover, besides the incommensuration already predicted for low-flux4 at fluxes of the order of $\pi n$, with $n$ the number of particles per rung, a second incommensuration (2-IC) in the correlation functions is induced by the interchain hopping4,11. However, in statistical mechanics, it is known that transitions in C-IC universality class can be turned into different universality classes by various relevant perturbations12,13, thus for a bosonic ladder in an artificial gauge field it remains a relevant question to investigate the robustness of the M-to-V phase transition. In this Letter, we consider the effect of an interchain interaction ($U_{\perp}$) and show that it can spoil the M-to-V transition, leading to the appearance of an intermediate charge density wave phase (CDW) that can be interpreted as a melted vortex phase. The melting of vortices is accompanied by a disorder point that gives rise to incommensurability of correlation functions when the density of dislocations becomes large enough to permit a greater energy gain from the applied flux than from the pinning potential of the vortices. We recall that while in two dimensions dislocations appear at finite temperature15,16, in one dimension even at zero temperature their formation can be driven by quantum fluctuations only.

We consider a two-leg hard-core boson ladder in a flux $\lambda$, with Hamiltonian:

$$H_\lambda = \sum_{j,\sigma} -t \left( b_{j,\sigma}^\dagger e^{i\lambda} b_{j+1,\sigma} + \text{h.c.} \right) + \sum_{j} \Omega \left( b_{j,\downarrow}^\dagger b_{j,\downarrow} + \text{h.c.} \right) + U_{\perp} n_{j,\uparrow} n_{j,\downarrow}, \quad (1)$$

where $b_{j,\sigma}$ annihilates a boson on chain $\sigma = \pm 1/2$ at site $j$, $n_{j,\sigma} = b_{j,\sigma}^\dagger b_{j,\sigma}$ is the associated number operator, $t e^{i\lambda}$ the hopping amplitude along the chains, $\Omega$ the rung hopping, and $U_{\perp}$ the interchain interaction. This Hamiltonian can be mapped onto a system of spin-1/2 bosons with spin-orbit coupling in a transverse magnetic field with each spinor state corresponding to one leg of the ladder. In the rest of this Letter, we will consider the attractive case ($U_{\perp} < 0$).

When $\Omega$ and $U_{\perp} = 0$, the two chains are decoupled and their low energy properties are described by the Tomonaga-Luttinger liquid model17. For weak interchain couplings a low-energy description can be
FIG. 1. Phase diagram at $n = 0.5$ for a fixed value of interchain hopping $\Omega/t = 0.5$ as a function of the applied flux $\lambda$ and as function of the strength of the interchain interaction as from DMRG simulations for $L = 64$ in PBC. The area under the dashed black line shows the stability region for the Meissner phase, the dark-red region between the solid and the dashed black lines shows the region of stability for Meissner-CDW phase. The light-blue area under the dotted line displays the stability region for the Vortex phase and shaded blue area is the one where the second-incommensurability occurs. The light-green region above the solid and dotted lines is where the melted or "Boatling" Vortex phase is stable. The dot-dashed line is the Lifshitz curve where the occurrence of the melted-Vortex phase is detected in the correlation function of rung-current (see text for explanations). Color online.
Meissner-CDW (M-CDW) phase. In the statistical mechanics context, the commensurate phase is our Meissner phase, and the incommensurate phase is the Vortex phase and the liquid phase is the M-CDW phase. This last phase can be detected via the (in-phase) charge density structure factor

$$ S^c(k) = \sum_{j,\sigma,\sigma'} \langle n_{j,\sigma} n_{0,\sigma'} \rangle e^{-ikj} $$

which develops peaks at $k = \pm \pi n$ whose heights don’t scale with $L$, since the Luttinger exponent $K_c > 1$ (see \cite{24}). In the lower panels $A$ and $B$ of Fig. 2 we follow this transition: in the Meissner phase (panel $A$) $S^c(k)$ is smooth and $n(k)$ shows a power-law divergence, while in the M-CDW phase (panel $B$) $S^c(k)$ acquires the above mentioned peaks and $n(k)$ shows a Lorentzian-like peak at $k = 0$. Meanwhile, $C(k)$ and $S^c(k)$ retain a Lorentzian shape on both sides of the transition. For sufficiently large interchain interaction the Meissner phase is replaced by the M-CDW even at zero flux, as shown in Fig. 1. Let us note that the opening of a gap in the spin-1/2 ladders. The corresponding Majorana fermion Hamiltonian \cite{28} has dispersion

$$ E_{\pm}(k)^2 = (u_{\pm}k)^2 + m^2 + \hbar^2 + \Delta^2 $$

where $m = -2\pi \Omega A^2 a$, $\Delta = -2\pi V_1 (B_1 a)^2$ and $h = -\Delta v_0 K_x$. For $h_{\text{eff}} = v_0/\sqrt{\Omega}$, the $-\text{branch}$ of Majorana fermions dispersion becomes gapless showing that the M-CDW quantum phase transition falls in the Ising universality class \cite{30,31}. For $h_{\text{eff}} < |m|$ $e^{i\theta_c}/\sqrt{\pi}$ is long range ordered, while it is short range ordered for $h_{\text{eff}} > |m|$. At the Ising transition, the derivative of the spin current $J_s$, $\partial_\lambda J_s \sim -\ln[\max(|\lambda - \lambda_c|, 1/L)]$ diverges logarithmically like the specific heat of the two-dimensional Ising model \cite{31}. The absence of the square root threshold singularity\cite{8} characteristic of the C-IC transition, $\langle J_s(\lambda) - J_s(\lambda_c) \rangle \sim \sqrt{\lambda - \lambda_c} \theta(\lambda - \lambda_c)$, can be noted in the upper panel in Fig. 2. By plotting the numerical derivative of $\langle J_s \rangle$ a narrow peak can be spotted at $\lambda \approx 0.15\pi$, past the maximum in the spin current (see supplemental material\cite{28}). Another indicator of the nature of the transition is the von Neumann entropy computed with DMRG in PBC for $L = 64, 48$ and $32$ retaining $m = 600$ states.

In Fig. 3. In the main panel, red and black solid dots data show the central charge $c$ as a function of $\lambda$ while red and black solid lines are the numerical derivative of $J_s$ with respect to $\lambda$, respectively for $U_\perp/t = -1.5$ and $U_\perp/t = -2.0$ at $n = 0.5$ and $\Omega/t = 0.5$. In the inset we show the central charge $c$ for different sizes namely $L = 64, 48$ and $32$, fitted with Lorentzians curves whose spread reduces on increasing the size and whose maximum is at $\lambda = 0.15\pi$. Data shown are extracted from Von Neumann entropy computed with PBC in PBC for $L = 64, 48$ and $32$ retaining $m = 600$ states. Despite
the size effects, we can observe a bell shaped curve centered around the critical value \( \lambda_{\text{cdw}} \), the width of which gets smaller with increasing system size. The height of this peak extrapolates to \( c = c_e + c_s = 3/2 \) as size increases, indicating that the critical point belongs to the Ising universality class (see supplementary[28]), while far from \( \lambda_{\text{cdw}} \), \( c = c_e \) extrapolates to unity. Finally, the Ising nature of the transition can also be spotted looking at the value of the peaks in the \( S_c(k = \pi/2) \) that not too close to transition should be proportional to \( (\lambda - \lambda_{\text{cdw}})^{1/4} \). We have verified this behavior for the case reported in Fig. 3 (see supplemental material[28]).

Besides the Ising transition, the fermionized Hamiltonian also predicts[13] the existence of a disorder point[33] in the crossover to the melted-Vortex phase. Indeed, beyond a critical value \( \lambda = \lambda_d \), real space correlations function \( \langle J_x(t)J_x(0) \rangle \) and \( \langle b_x^\dagger b_y \rangle \) both acquire a periodic modulation[29]. Since the wavevector of the modulation is varying with \( \lambda \), the disorder point is of the first kind[32]. In reciprocal space, the modulation gives rise to a superposition of two Lorentzian-like peaks in \( C(k) \) and \( n(k) \) that are remnants of the divergent peaks[2, 21] previously obtained in the QLRO vortex state when \( U_\perp = 0 \) (see panel C in Fig 2 and panels A and B in Fig. 4). The values of \( \lambda \) for which the two peaks in \( C(k) \) can be resolved are at \( \lambda > \lambda_{\text{L},C} \) with \( \lambda_{\text{L},C} \) the Lifshitz point. One has \( \lambda_{\text{L},C} > \lambda_d \) since resolving the peaks requires that the distance between their maxima exceeds their width. This effect is less evident in the spin resolved \( n_s(k) \) where a single Lorentzian-like peaks located at finite \( k \) develops (black solid line in Fig. 1). Similar disorder and Lifshitz points have been found in one-dimensional spin-1/2[30] and spin-1[37] chains as well as frustrated Ising chains in transverse field[32]. In this phase the spin and charge response functions retain respectively the Lorentzian shape centered around \( k = 0 \) and the peaks at \( k = \pm \pi n \). Finally, for even higher flux and sufficiently small interaction, \( K^*_s \) becomes greater than 1 and the operator \( \cos \sqrt{\delta \phi} \) becomes irrelevant, allowing a vortex phase[3] with QLRO. This takes place through a Berezinskii-Kosterlitz-Thouless (BKT) transition[16] at the point \( K^*_s(\lambda_{\text{BKT}}) = 1 \). In Fig. 1 we show the recover of the QLRO vortex phase (panel C from Fig 1) from the melted one by decreasing the strength of interchain interaction \( U_\perp/t \) for fixed applied flux \( \lambda = 0.625\pi \).

To conclude, using bosonization and DMRG we have found (see Fig. 1 that, with an interchain attractive interaction, the commensurate Meissner phase and the incommensurate QLRO vortex phase leave space to a Meissner-CDW and to a melted vortex phase with SRO. Instead of having a single flux-driven M-to-V transition we are left with an Ising transition to the commensurate Meissner-CDW. On increasing the flux an exponentially damped sinusoidal modulation, incommensurate with the ladder, develops in the momentum distribution. At the Lifshitz point \( \lambda = \lambda_{\text{L},C} \) double peaks appear in the rung current structure factor. This indicates the existence of a disorder point[34, 35] where the bosonic Green’s functions and the rung current correlation function develop exponentially damped oscillations in real space. The Meissner-CDW is then crossing over into a melted vortex phase where a SRO with proliferation of dislocations takes over. At higher flux, a BKT transition takes place, and the quasi-long range vortex lattice is recovered.

Our predictions on the melting of vortices in Bose-Einstein condensates in optical lattices can be traced in current experiments, where static structure factors[39] and momentum distributions can be measured, together with the spin current[33, 40]. Using dipolar interactions, tunable by orienting the dipoles with a field[41], or Feshbach resonances, the interaction \( U_\perp \) can be rendered attractive.

The detection of a melting transition as well as the non-trivial effects due to interactions can be relevant for atomtronic ring ladders which have been proposed for readout and gate implementation in quantum technologies[42], analogously to the superconducting qubits in multi-junction circuits.

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Supplementary material for “Vortex lattice melting in a boson-ladder in artificial gauge field”

MAJORANA FERMIONS REPRESENTATION

Fermionization leads to a detailed picture of the transition between the Meissner state and the density wave states. Rescaling \( \theta_s = \sqrt{2} \theta \) and \( \phi_s = \phi / \sqrt{2} \), the Hamiltonian (Insert Eq. number from manuscript) is fermionized using the identities:

\[
\cos \frac{2 \theta}{\pi a} = -i(\psi_R^\dagger \psi_L - \psi_L^\dagger \psi_R), \quad \cos \frac{2 \phi}{\pi a} = i(\psi_R^\dagger \psi_L^\dagger - \psi_L \psi_R), \quad -\frac{1}{\pi} \partial_s \theta = \psi_R^\dagger \psi_R + \psi_L^\dagger \psi_L,
\]

yielding:

\[
H_s = -iu_s \int dx(\psi_R^\dagger \partial_x \psi_R - \psi_L^\dagger \partial_x \psi_L) - i \int dx(\psi_R^\dagger \psi_L - \psi_L^\dagger \psi_R) - i \Delta \int dx(\psi_R^\dagger \psi_L^\dagger - \psi_L \psi_R)
\]

\[
- h \int dx(\psi_R^\dagger \psi_R + \psi_L^\dagger \psi_L) + \int dx \frac{\hbar^2}{2 \pi u_s},
\]

where:

\[
h = -\lambda u_s K_s, \quad m = -2\pi \Omega A_0 a, \quad \Delta = -\frac{\pi}{2} \nu_\perp (B_1 a)^2.
\]

Once we introduce the Majorana fermion operators \( \zeta_{\nu,\sigma}(x) = \frac{1}{\sqrt{2}}(\psi_{\nu} + \sigma \psi_{\nu}^\dagger)(x) \) with \( \nu = R, L, \sigma = \pm \), the Hamiltonian is rewritten:

\[
H_s = -\frac{u_s}{2} \int dx \sum_{j=1}^2 (\zeta_{R,j} \partial_x \zeta_{R,j} - \zeta_{L,j} \partial_x \zeta_{L,j}) - i(m + \Delta) \int dx \zeta_{R,1} \zeta_{L,1} - i(m - \Delta) \int dx \zeta_{R,2} \zeta_{L,2}
\]

\[
- i h \int dx (\zeta_{R,1} \zeta_{R,2} + \zeta_{L,1} \zeta_{L,2}) + \int dx \frac{\hbar^2}{2 \pi u_s}
\]

Hamiltonians of the form have previously been studied in the context of spin-1 chains in magnetic field or spin-1/2 ladders with anisotropic interactions. The quadratic Hamiltonian can be diagonalized in momentum space to obtain the following energy eigenvalues:

\[
E_{\pm}(k)^2 = (u_s k)^2 + m^2 + h^2 + \Delta^2 \pm 2 \sqrt{h^2 (u_s k)^2 + h^2 m^2 + \Delta^2 m^2}.
\]

Since observables such as the rung current are bilinear in the Majorana fermions, obtaining their correlation functions requires knowledge of the Matsubara Green’s function for the Majorana fermions defined as:

\[
G_{\alpha,\beta}(x, \tau) = -\langle T_{\tau} \zeta_{\alpha}(x, \tau) \zeta_{\beta}(0, 0) \rangle = \frac{1}{\beta} \sum_{k, i\nu_n} G_{\alpha,\beta}(q, i\nu_n) e^{i(q - \nu_n \tau)},
\]

where \( \alpha = (\nu, j) \) with \( \nu = R, L \) and \( j = 1, 2 \). In Eq. (S5), the Fourier space Green’s function is the 4 x 4 matrix:

\[
G_{\alpha,\beta}(q, i\nu_n) = (i\nu_n - \mathcal{H}(q))^{-1}_{\alpha,\beta}.
\]

An explicit expression in terms of Pauli matrices is:

\[
G(k, z) = (z - u_s k \mathbb{1} \otimes \tau_3 - h \sigma_2 \otimes \mathbb{1} - m \mathbb{1} \otimes \tau_2 - \Delta \sigma_3 \otimes \tau_2)^{-1}
\]

\[
= \frac{(\sigma_1 \otimes \mathbb{1})(z^2 - \mathcal{H}^2(k))(\sigma_1 \otimes \mathbb{1})(z + \mathcal{H}(k))}{(z^2 - E_+(k)^2)(z^2 - E_-(k)^2)}
\]

where \( E_{\pm}(k) \) have been defined above in (S4).

The real space Green’s function, at equal time and zero temperature, is defined by:

\[
\int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{ikx} \int_{-\infty}^{+\infty} \frac{d\nu}{2\pi} G(k, \nu) = -\int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{ikx} \left[ \frac{H(k)}{2[E_+(k) + E_-(k)]} + \frac{(\sigma_1 \otimes \mathbb{1})H(k)^2(\sigma_1 \otimes \mathbb{1})H(k)}{2E_+(k)E_-(k)[E_+(k) + E_-(k)]} \right].
\]
Thus, it can be obtained from just two integrals:

\[
I_1(x) = \int \frac{dk}{2\pi} \frac{e^{ikx}}{E_+(k)E_-(k)(E_+(k) + E_-(k))},
\]

\[
I_2(x) = \int \frac{dk}{2\pi} \frac{e^{ikx}}{(E_+(k) + E_-(k))},
\]

by taking the appropriate number of derivatives with respect to \(x\).

**Ising transition**

For \(m = \sqrt{\hbar^2 + \Delta^2}\), \(E_-(k) = v_N\frac{\lambda}{m}|k| + O(k^2)\) a single Majorana fermion mode becomes massless at the transition\(^{[S1]}\) between the Meissner and the density wave state. This transition belongs to the Ising\(^{[S3]}\) universality class. As a consequence, at the transition, the Von Neumann entanglement entropy \(S_{vN} = \frac{1}{4} c \ln L = \frac{1}{4}(c_c + c_s) \ln L = \frac{1}{4}(1 + 1/2) \ln L\), where \(c\) is the central charge of the gapless modes, while away from the transition it is \(S_{vN} = \frac{1}{4} c_c \ln L = \frac{1}{3} \ln L\) since the total density mode \(\phi_c\) is gapless. In Fig. \(^{[S1]}\) we show size extrapolation for the central charge of two cases, at \(n = 0.5\), \(U_\perp/t = -1.5\), \(\Omega/t = 0.5\), namely at \(\lambda = 0.15\pi\) where we expect the Ising transition for which we get \(c = 1.6(1)\) and at \(\lambda = 0.75\pi\) in the melted Vortex phase where \(c = 1.0(1)\). Since bosonization predicts that the charge mode is gapless and it is described at low energy by a free boson with \(c_c = 1\), the charge mode exhausts the central charge when \(\lambda \neq \lambda_{cdu}\). At \(\lambda_{cdu}\), the presence of the peak indicates the appearance of an extra gapless field with \(c_s < 1\), i.e., a critical point. Conformal Field Theories with \(c_s < 1\) form a discrete series\(^{[S0]}\) with \(c_s = 1 - \frac{6}{m(m+1)}\) with the integer \(m \geq 3\). The size-extrapolated value of central charge of the critical theory appears to be under \(7/10\), leaving us with \(m = 3\) and \(c_s = 1/2\) as the only possible value, indicating that the critical point belongs to the Ising universality class.

In Fig. \(^{[S1]}\) we show size extrapolation for the central charge of two cases, at \(n = 0.5\), \(U_\perp/t = -1.5\), \(\Omega/t = 0.5\), namely at \(\lambda = 0.15\pi\) where we expect the Ising transition for which we get \(c = 1.6(1)\) and at \(\lambda = 0.75\pi\) in the melted Vortex phase where \(c = 1.0(1)\).
Current

A signature of the Ising transition can be observed also in the spin current, which is defined by

$$\langle J_s \rangle = \frac{1}{L} \int \frac{\partial H}{\partial \lambda} = \frac{1}{L} \frac{\partial E_{GS}}{\partial \lambda} = \frac{1}{L} \frac{\partial}{\partial \lambda} \left( - \sum_{0<s<k} E_+(k) + E_-(k) + L \frac{h^2}{2\pi u_s} \right),$$

from which the $\langle J_s \rangle$ in units of $\frac{\hbar}{2\pi u_s}$ is found:

$$\langle J_s \rangle = \frac{h}{\pi u_s} - \sum_{r=\pm 1} \int_0^\Lambda \frac{dk}{2\pi} \frac{h + r \frac{h(u_k + m^2)}{\sqrt{h^2(u_k^2 + m^2) + m^2}}} {\sqrt{(u_k)^2 + m^2 + \Delta^2 + h^2 + 2r \sqrt{h^2(u_k^2 + m^2) + m^2 \Delta^2}}}. \tag{S11}$$

The integral (S11) is convergent in the limit $\Lambda \to +\infty$.

In the limit $\Lambda \ll 1$, or $h \ll 1$,

$$\langle J_s \rangle = \frac{h}{2\pi u_s} \left[ 3 - \frac{m^2 - \Delta^2}{2m\Delta} \ln \left| \frac{m - \Delta}{m + \Delta} \right| \right],$$

As $\Delta/m$ increases, the proportionality constant between the flux and the current decreases but remains positive, indicating that the Meissner effect is reduced by interchain repulsion. Right at $h = m$, we can obtain the exact expression of $\partial_{\hbar} E_{GS}$ as:

$$\frac{1}{L} \left( \frac{\partial E_{GS}}{\partial \hbar} \right)_{h=m} = \frac{m}{\pi u_s} \left[ 1 - \frac{\Delta}{m} \arctan \left( \frac{m}{\Delta} \right) \right],$$

which shows that the current at $h = m$, i.e., where the commensurate-incommensurate transition would take place in the absence of repulsion, is always reduced by interchain interaction.

In the vicinity of $h = \sqrt{m^2 - \Delta^2}$, the dominant singularity in $\langle J_s \rangle$ is:

$$\langle J_s \rangle \text{sing.} = -h \left( 1 - \frac{m}{\sqrt{h^2 + \Delta^2}} \right) \int_0^{\frac{\pi}{2}} \frac{dk}{2\pi} \frac{1}{\sqrt{(m - \sqrt{h^2 + \Delta^2})^2 + \frac{\Delta^2}{m^2}(u_k)^2}} \approx -h \left( \frac{\sqrt{h^2 + \Delta^2} - m}{\Delta} \right) \ln \left( \frac{2m\Delta}{h \sqrt{h^2 + \Delta^2} - m} \right).$$

Thus, while $\langle J_s \rangle$ is continuous for $h \to \sqrt{m^2 - \Delta^2}$, its derivative instead:

$$\frac{2a}{u_s} \frac{\partial (\langle J_s \rangle)}{\partial h} = \frac{1}{\pi u_s} - \int_0^\Lambda \frac{dk}{2\pi} \sum_{r=\pm 1} \frac{\Delta^2}{\left[ (u_k)^2 + m^2 + \Delta^2 + h^2 + 2r \sqrt{h^2((u_k)^2 + m^2) + m^2 \Delta^2} \right]^2} \times \left[ 1 + \frac{3m^2[(u_k)^2 + m^2]}{h^2((u_k)^2 + m^2) + m^2 \Delta^2} + r \frac{m^2}{h^2((u_k)^2 + m^2) + m^2 \Delta^2} \left( 2 + \frac{(u_k)^2 + m^2}{h^2((u_k)^2 + m^2) + m^2 \Delta^2} \right) \right]. \tag{S12}$$

diverges as $\frac{\ln |h - h_c|}{h}$ for $h^2 = m^2 + \Delta^2$. $\langle J_s \rangle$ is a decreasing function of $h$, and its plot as a function of $h$ presents a vertical tangent at $h_c$. As $\langle J_s \rangle$ is increasing at small $h$ and decreasing for $h = \sqrt{m^2 - \Delta^2}$, a maximum of the current must exist in the range $0 < h_{\text{max}} < \sqrt{m^2 - \Delta^2}$ i.e. inside the Meissner phase.

Charge density wave order parameter

The CDW order is related to the $2k_F$ wave-vector component of the density operator that, expressed using bosonization, is $O_{\text{CDW}}(x) = A e^{i\sqrt{2} \phi_c} \cos \sqrt{2} \phi_s$, where $A$ is a non-universal constant dependent on the microscopic Hamiltonian. On the Meissner side, $\phi_c$ is long range ordered, so the CDW correlations decay exponentially while, on the density wave side and for $U < 0$, $\langle \phi_s \rangle = 0$ and $O_{\text{CDW}}(x) \simeq \tilde{A} (\cos \sqrt{2} \phi_c) e^{i\sqrt{2} \phi_s}$ where $\langle \cos \sqrt{2} \phi_s \rangle \sim (a/\xi)\xi^{1/8}$ with $\xi_s \sim |\lambda - \lambda_{\text{cdw}}|$ the critical correlation length of the Meissner-CDW transition. The exponent $1/8$ is the result of $\cos \sqrt{2} \phi_s$ being
proportional to the order parameter of the Ising transition. For finite size and periodic boundary conditions the CDW correlations takes the form:

\[ \langle \cos \sqrt{2} \phi_s(x) \cos \sqrt{2} \phi_s(0) \rangle \propto \left( \frac{1}{\xi_s} \right)^{1/4} \frac{\cos(\pi n x)}{[L \sin \left( \frac{\pi x}{L} \right)]^{K_c}} \]  
(S13)

and right at the Ising transition it becomes \( \propto \left( \frac{\pi \alpha}{L \sin \left( \frac{\pi x}{L} \right)} \right)^{1/4} \) and hence that \( K_c \to K_c + 1/4 \) in Eq. S13.

Taking the Fourier transform with \( 2 > K_c > 1 \) at \( k = 2\pi \rho_0 \) and at finite \( q \) nearby, we get:

\[ S_c(2\pi \rho_0) - S_c(2\pi \rho_0 + q) \propto \left( \frac{1}{\xi_s} \right)^{1/4} \int_0^L \left( \frac{\pi \alpha}{L \sin \left( \frac{\pi x}{L} \right)} \right)^{K_c} \left[ 1 - \cos(qx) \right] dx \]

\[ \propto \frac{\pi \alpha}{\left| \cos \left( \frac{\pi K_c}{2} \right) \right|} \left( \frac{2\pi \alpha}{L} \right)^{K_c-1} \left[ \frac{\Gamma \left( \frac{K_s}{2} + \frac{L}{2\pi} \right)}{\Gamma \left( 1 - \frac{K_s}{2} + \frac{L}{2\pi} \right)} - \frac{\Gamma \left( \frac{K_s}{2} \right)}{\Gamma \left( 1 - \frac{K_s}{2} \right)} \right]. \]  
(S14)

When \( qL \gg 1 \), one has \( S_c(2\pi \rho_0) - S_c(2\pi \rho_0 + q) \sim |q\alpha|^{K_c-1} \). In the limit \( L \to +\infty \), if \( K_c > 1 \), \( S_c(k) \) is finite for \( k \to 2\pi \rho_0 \) but presents a cusp at that point. This is consistent with the low energy predictions derived from bosonization where \( K_c = [1 + U_{\perp}/(2\pi t \sin(\pi n/2))]^{-1/2} \).

In the transition region between the CDW and the Ising critical point, where the correlation length is finite but comparable to \( L \), we can write:

\[ \langle \cos \sqrt{2} \phi_s(x) \cos \sqrt{2} \phi_s(0) \rangle = \left( \frac{\alpha}{\xi_s} \right)^{1/4} \Phi \left( \frac{x}{L}, \frac{L}{\xi_s} \right). \]  
(S15)

For given \( L \) and \( \xi_s \), we have to distinguish two regimes in \( q \) when we are close to the transition. For \( q\xi_s \ll 1 \), we recover the regime S14. For \( q\xi_s \gg 1 \), the integral in (S15) becomes sensitive to the short distance physics, and we expect: \( S_c(2\pi \rho_0) - S_c(q + 2\pi \rho_0) \sim (q\alpha)^{K_c-3/4} \). To sum up, far from the transition, we can treat the Ising order parameter as a constant and obtain a scaling function that depends only on \( qL \) and \( K_c \) while near the transition (with \( \xi_s/L \) non-negligible) the scaling function depends on \( K_c \) and both \( qL \) and \( q\xi_s \).

In Fig. S2 we show \( S_c(2\pi \rho_0) \) as a function of the applied flux for the case \( U_{\perp}/t = -1.5 \) at \( n = 0.5 \) and \( \Omega/t = 0.5 \), far from transition we can detect a region on both sides of the Ising transition where this quantity is \( \propto \xi_s^{-1/4} \). A fit using \( f(\lambda) = a/(\lambda - \lambda_{cdw})^{1/4} + b \) on both sides of the transition gives us as results for \( \lambda_{cdw} = 0.150(2) \) and \( 0.16(2) \), in agreement with the estimated extracted from the central central charge or from the derivative of the spin current.

**Lifshitz and disorder point**

From the spectrum of the Majorana fermion representation S14 we can deduce the existence of a disorder and a Lifshitz point in some correlation functions. At a disorder point, a real space correlation function acquires a periodic modulation as a function of \( x \). In reciprocal space, its Fourier transform is a the sum of two Lorentzian-shaped peaks. The value of \( \lambda \) where the two peaks are resolved, i. e. where the modulation wavevector is of the order of the peak width, is the Lifshitz point. It does not coincide with the disorder point because of the short range order. This definition of disorder and Lifshitz points applies to all correlation functions, but the existence of these points can be inferred by considering the single particle Green’s function of the Majorana Fermions.

The energy \( E_+(k) \) is always an increasing function of \( k \), while for \( 2h^2 > 2h_L^2 = m^2 + \sqrt{m^4 + 4m^2\Delta^2} \), \( E_-(k) \) has two degenerate minima S2 for \( k = \pm k_L \) with \( k_L^{(\pm)} = \pm \sqrt{\frac{m^2 - \Delta^2 - m^2}{\alpha}} \). For \( h > h_L \), \( E_-(k) \) can be Taylor expanded near \( k = \pm k_L \) as:

\[ E_-(k)^2 \simeq \Delta^2 \left( 1 - \frac{m^2}{k_L^2} \right) + \frac{v^4}{4k^2} (k^2 - k_L^2)^2, \]  
(S16)

and inserting in the expression of \( I_1(x) \), a sinusoidal modulation of the Green’s function of the Majorana fermions is expected S2 at least when \( h > h_L \). A more detailed calculation would show that the sinusoidal modulation of \( I_1(x) \) appears when \( E_-(k) \) develops two disconnected branch cuts symmetric with respect to the imaginary axis in complex \( k \) plane. Now, the correlation function \( \langle J_\perp(x)J_\perp(0) \rangle \) is the trace of a product of two Majorana Fermion
Green’s function and Pauli matrices, and thus also exhibits a sinusoidal modulation of wavevector $2k_L$. So a disorder point is present in the correlator $\langle J_{\perp}(x)J_{\perp}(0) \rangle$, and a Lifshitz point is expected in the Fourier transform $C(k)$. In the case of the real space Green’s function of the original bosons, it is known that it depends on a factor coming from the charge modes and a factor coming from Green’s functions of Ising order and disorder operators $S^2$ associated with the Majorana fermions. The latter factor $S^2, S^7$ can be expressed in terms of block Toeplitz determinants the elements of which are Majorana fermion Green’s functions. Numerical calculations $S^2, S^7$ show that a sinusoidal modulation appears also in the Ising order and disorder parameter correlations. This indicates that a disorder point is also present in the correlators $\langle b_{\sigma}^\dagger b_{\sigma} \rangle$. Considering the Fourier transform, a Lifshitz point is expected in $n(k)$. Since the disorder point is associated with the appearance of a modulation in the Green’s function of the Majorana fermions, it is expected that $\langle b_{\sigma}^\dagger b_{\sigma} \rangle$ and $\langle J_{\perp}(x)J_{\perp}(0) \rangle$ have a disorder point at the same value of $\lambda$. However, in general, their Lifshitz points are not expected to coincide since $\langle b_{\sigma}^\dagger b_{\tau} \rangle$ depends on both “charge” and “spin” modes and since the correlation lengths of the two “spin” parts can differ.

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