Coherent Perfect Rotation

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Two classes of conservative, linear, optical rotary effects (optical activity and Faraday rotation) are distinguished by their behavior under time reversal. In analogy with coherent perfect absorption, where counterpropagating light fields are controllably converted into other degrees of freedom, we show that only time-odd (Faraday) rotation is capable of coherent perfect rotation in a linear and conservative medium, by which we mean the complete transfer of counterpropagating coherent light fields into their orthogonal polarization. This highlights the necessity of time reversal odd processes (not just absorption) and coherence in perfect mode conversion and may inform device design.

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Coherent Perfect Absorption (CPA) \cite{1,2} illuminates the role optical coherence plays in the perfect conversion of optical energy into other modes (typically incoherent fluorescence or heat). CPA is a non-conservative linear process, typically modeled using a non-Hermitian Hamiltonian. In its original formulation, this non-Hermitian Hamiltonian included absorption/gain to explicitly break the time reversal invariance of the underlying fundamental processes. This is also the case with the formulation of CPA in PT-invariant theories \cite{3,4}, which has led to a fertile way to explore many subtleties in optical processes.

In this paper we develop theory for Coherent Perfect Rotation (CPR), the conservative transfer of any fixed input polarization state of coherent counterpropagating light fields completely into its orthogonal polarization. CPR highlights the necessity of combining T-odd processes (in, for example, magneto-optics) with optical coherence to achieve this perfect conversion. By contrast T-even conservative processes cannot effect such a transformation. CPR denotes a conservative (thus fully Hermitian Hamiltonian) process that first appears at a particular “threshold” value of the parameter scaling the T-odd process, and there are many phenomenological correspondences between CPA and CPR, illustrated schematically in Fig. 1. Beyond revealing a connection between T-odd processes, Hermiticity, and CPA, CPR may inform the design of novel magnetooptical sensors and devices.

We adopt a $4 \times 4$ transfer matrix approach to describe linear optical transport of a monochromatic ray moving back and forth along the $\hat{z}$-axis,

\[
\mathcal{M} = \begin{pmatrix} M & C \\ B & M' \end{pmatrix} \quad \text{with} \quad \vec{v}_{i+1} = \mathcal{M}_i \vec{v}_i, \quad (1)
\]

where the $M_i$’s, $B$ and $C$ are $2 \times 2$ (in general complex) matrices; here we are working in the basis where the local field (complex) amplitudes are $\vec{v} = (E_x, H_y, E_y, -H_x)$. Note that for birefringent materials $M \neq M'$, but since we are interested in systems that transform any input polarization into the orthogonal polarization, we will focus on the case of non-birefringent materials in which $M = M'$. The more familiar single polarization form of the transport is in terms of the $2 \times 2$ matrix $M = C = 0$.

Throughout we work in units in which the familiar propagation eigenstates of a single polarization in the vacuum are $\vec{e}_R = (E_x, H_y) = (1,1)$ for a rightmoving wave and $\vec{e}_L = (-1,1)$ for a lefturng wave. Thus for review, we represent the coherent scattering from a linear material whose $(2 \times 2)$ transfer matrix is $M$ by $\vec{e}_\text{in} = (1,1) + r(-1,1)$ as the incident fields from the left and $\vec{e}_\text{out} = t(1,1) = M \vec{e}_\text{in}$ being the fields on the right, with $r$ and $t$ denoting the reflection and transmission amplitudes (generally complex numbers).

For reference, solving the transport in this basis gives,

$t = 2(m_{11}m_{22} - m_{12}m_{21})/(m_{11} + m_{22} - m_{12} - m_{21})$ and 

$r = (m_{11} - m_{22} + m_{12} - m_{21})/(m_{11} + m_{22} - m_{12} - m_{21}),$

where the $m_{ij}$ are the matrix elements of $M$ (note differ-

\[\begin{array}{c|c}
\hline
\text{Absorption} & \text{Rotation} \\
\hline
\begin{array}{c}
\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
\end{array} & \begin{array}{ccc}
\mathcal{M} & \Lambda, \alpha \\
\end{array} \\
\hline
\end{array}\]

\[\begin{array}{c|c}
\hline
\text{in} & \text{out} \\
\hline
\begin{array}{c}
\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
\end{array} & \begin{array}{ccc}
\mathcal{M} & \Lambda, \alpha \\
\end{array} \\
\hline
\end{array}\]

FIG. 1. CPA and CPR are distinct from critical coupling and critical rotation. For a fixed value of $\Lambda$, the system’s length in terms of the vacuum wavelength, critical coupling and CPA occur at a particular value of the absorption $\alpha$ and index. Critical half-wave rotation and CPR first occur at “threshold” values for the material’s Verdet-magnetic field product $V$. Only CPA and CPR depend upon the amplitude and relative phase of the counterpropagating beams; these can be used to control the onset of CPA or CPR.
ence in basis to Ref. [10]).

In the 2 × 2 case, T-symmetry indicates that real diagonal elements of $M$ are T-even whereas real off-diagonal elements are T-odd. In general, matrices $C$ and $B$ in $M$ can each be written as a sum of T-even and T-odd parts. Thus in the chosen basis the T-even part is of the form

$$\{C \text{ or } B\}_{T-\text{even}} = \begin{bmatrix} \text{Re} & \text{Im} \\ \text{Im} & \text{Re} \end{bmatrix}$$

where $\text{Im}$ ($\text{Re}$) stand for imaginary (real) matrix elements. Note these elements can all be different from one another.

In contrast, for the T-odd part of the $B$ and $C$ matrices,

$$\{C \text{ or } B\}_{T-\text{odd}} = \begin{bmatrix} \text{Im} & \text{Re} \\ \text{Re} & \text{Im} \end{bmatrix} \cdot$$

where, again, all entries could be different. T-odd pieces in the $2 \times 2$ $M$ are associated with absorption/gain. However, addressing polarization changing processes in the $4 \times 4$ basis there are combinations of these T-odd matrix elements that conserve the total power.

For materials without linear birefringence the resulting $O(2)$ symmetry about the axial direction implies $M = M'$ and $B = -C$, regardless of the T-symmetry of the underlying matrices.

In steady state, the local power flux will be a constant of the transport for a conservative system. A local expression for the power flux in the chosen basis is

$$\sim \vec{v}^\dagger \mathcal{P} \vec{v}$$

where $\mathcal{P} = \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix}$ in which for each polarization $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. The statement that the transport is conservative is thus $M^\dagger \mathcal{P} M = \mathcal{P}$. This leads to important constraints on the matrix elements of $M$ and $B, C$. In the more familiar $2 \times 2$ formulation of transport for a single polarization, a conservative system satisfies $M^\dagger PM = P$, indicating that $m_{11}$ and $m_{22}$ must be purely real, while $m_{12}$ and $m_{21}$ must be purely imaginary (det$(M)$=1 is automatic in 1-d linear transport as it preserves the $E_x, H_y$ commutator) Note that this is consistent with $M$ being T-even, as expected. For example, for normal incidence on a purely dielectric material of thickness $L$, index $n$, the $M = \begin{bmatrix} \cos \delta & \frac{i}{n} \sin \delta \\ \frac{i}{n} \sin \delta & \cos \delta \end{bmatrix}$ where $\delta = nk_0 L$ and $k_0$ is the vacuum wavenumber.

Similarly, for conservative transport in the full $4 \times 4$ system, the $M$, $B$ and $C$ jointly satisfy,

$$M^\dagger PM + C^\dagger PC = M'^\dagger PM' + B'^\dagger PB = P$$

and

$$M^\dagger PC + B'^\dagger PM' = 0$$

The relations Eqs. (1)-(5) indicate that T-symmetry and power conservation are not identical. Solving Eq. (5) in the uniaxial case where $M = M'$ and $B = -C$, if we restrict to T-even, conservative transport, gives

$$m_{12}c_{21} - m_{21}c_{12} = m_{22}c_{11} - m_{11}c_{22}$$

along with $c_{21}/c_{12} = m_{21}/m_{12}$ and $c_{11}/c_{22} = m_{11}/m_{22}$. Combining these equations indicates that $c_{ij} = \alpha m_{ij}$ with $\alpha$ a real constant. Then, Eq. (1) gives $(1 + \alpha^2)M^\dagger PM = P$. Then for transport with rotation in a purely dielectric material, $M = \cos \gamma \begin{bmatrix} \cos \delta & \frac{i}{n} \sin \delta \\ \frac{i}{n} \sin \delta & \cos \delta \end{bmatrix}$ and $\alpha = \tan \gamma$. Thus only a single parameter, $\gamma$, governs the overall rotation of the frame in the T-even case, as would be the case for optical activity in which $\gamma$ is proportional to the product of the concentration of chiral centers and sample length.

Assuming both that the components of $M$ remain T-even and the system is uniaxial, the case of conservative, T-odd $C$ in Eqs. (1)-(5) reduces to

$$\det M - \det C = 1$$

and

$$m_{11}c_{22} + m_{22}c_{11} = m_{21}c_{12} + m_{12}c_{21} \cdot$$

Thus, studying conjugation and scaling symmetry of the above equations, we see that there are three (real) parameters that determine the longitudinal T-odd polarization mixing in a uniaxial material. One of these parameters is the ordinary Faraday rotation parameter (the Verdet constant times the applied longitudinal magnetic field). The other two parameters in a general solution of Eqs. (7) and (8) are less familiar though lead to the same phenomena.

As emphasized in the literature, the adjective “coherent” in CPA and CPR indicates its reliance on the relative phase between the counterpropagating light fields in achieving the mode conversion. Thus CPA and CPR are necessarily two-port processes, in contrast to critical coupling [7-9], itself sometimes referred to as 1-port CPA. Also in contrast to 1-port devices, CPA is only possible using T-odd processes such as Faraday rotation as we now show.

As noted in the original formulation [1], CPA can be understood via $2 \times 2$ transfer matrices, and we review it briefly here to motivate the $4 \times 4$ transfer matrix expressions below that describe CPR. In CPA there are incoming fields only, and in our choice of basis, these are $\vec{v}_\| = (1, 1)$ and $\vec{v}_\perp = f(-1, 1)$ (note $f$ is complex). These fields are related via the transfer matrix as, $\vec{v}_\perp = M \vec{v}_\|$ which in terms of the matrix elements of $M$ indicates that CPA requires the condition $m_{11} + m_{22} + m_{12} + m_{21} = 0$.

In terms of a fixed optical element size, this (complex) equation yields both the wavelength of the CPA pole in
the S-matrix and the critical value of the dissipative coupling (which necessarily has T-odd components in $M$).

It is straightforward to find the location of a CPR resonance using the $4 \times 4$ basis. For fields on the left take $\vec{v}_l = (1, 1, -l, l)$ where $l$ is the amplitude the outgoing rotated wave. On the right, take $\vec{v}_r = (-d, d, s, s)$; this configuration thus consists of only incoming fields of one polarization and outgoing fields of the orthogonal polarization only, the CPR state. In analogy with the CPA state, these boundary conditions lead to a condition on the size, wavelength and rotary power of the system. For uniaxial systems with the $4 \times 4$ form of $M$ as described earlier, we require

$$M \left( \begin{array}{c} 1 \\ 1 \\ -1 \\ 1 \end{array} \right) + C \left( \begin{array}{c} 1 \\ 1 \\ -1 \\ 1 \end{array} \right) l = \left( \begin{array}{c} 1 \\ 1 \\ -1 \\ 1 \end{array} \right) d \quad (9)$$

and

$$- C \left( \begin{array}{c} 1 \\ 1 \\ -1 \\ 1 \end{array} \right) l = \left( \begin{array}{c} 1 \\ 1 \\ -1 \\ 1 \end{array} \right) s \quad (10)$$

Any optically-active, uniaxial, conservative process never solves the above pair, and thus cannot be used to achieve CPR. For this case, as indicated in the preliminaries, $C \sim M$ and thus $M = \left[ \begin{array}{cc} M \cos \gamma & -M \sin \gamma \\ M \sin \gamma & M \cos \gamma \end{array} \right]$, for $\gamma$ proportional to the concentration-length product of the chiral centers. Using this form in Eqs. (9) and (10) and eliminating $l$, $s$ and $d$, we arrive at the single constraint

$$- (m_{11} - m_{22})^2 + (m_{12} - m_{21})^2 = 4 \cos^2 \gamma \quad (11)$$

Power conservation discussed earlier indicates that $m_{11}$ and $m_{22}$ must be purely real in this basis such that $m_{12}$ and $m_{21}$ are purely imaginary; thus Eq. (11) can never be achieved unless both sides are identically zero. If so, then both $m_{11} = m_{22}$ and $m_{12} = m_{21}$. Thus the condition $\det(M) = 1$ would imply that there exists some angle $\phi$ such that $m_{11} = \cos \phi = m_{22}$ and $m_{12} = i \sin \phi = m_{21}$. For $\phi \neq 0$ then this case would correspond to a material that has a net index of refraction of unity. Alternatively plugging the choice $\phi = 0$ into Eqs. (9) and (10) the equations become degenerate, relaxing the requirement on the index although yielding a solution for any inputs ($1 \mathrm{or} d$ in any relation, since $M = 1$) independently. This is not CPR; it is instead the rotation analogue of critical coupling (Fig 1). To reiterate, such a system conservatively rotates the polarization of light from any given polarization state completely into the orthogonal state whether it is illuminated from one side or the other, independent of any phase relationship between the incoming fields. Indeed, a single slab of an optically active material can be tuned in width and chiral concentration to create this analogue of critical coupling for rotation. There are likely to be other ways to achieve this rotational analogue of critical coupling, including one we discuss below, but again, this is not CPR.

The main new idea of this letter is that CPR is achievable with T-odd rotation, as we now show analytically for a slab dielectric Faraday rotator. The $M$ and $C$ in the chosen basis for a slab are

$$M = \frac{1}{2} \left[ \begin{array}{cc} C_1 + C_2 & i(S_1/n_1 + S_2/n_2) \\ i(n_1 S_1 + n_2 S_2) & C_1 + C_2 \end{array} \right] \quad (12)$$

and

$$C = \frac{1}{2} \left[ \begin{array}{cc} i(C_1 - C_2) & -(S_1/n_1 - S_2/n_2) \\ -(n_1 S_1 - n_2 S_2) & i(C_1 - C_2) \end{array} \right] \quad (13)$$

where $C_{1,2}$ ($S_{1,2}$) refer to the cosine (sine) of $\delta_{1,2}$ which is $n_{1,2}k_0L$ in which the $n_{1,2}$ are the indices of refraction of the left- and right- circular polarization in the slab, the $k_0$ refers to the vacuum wavevector and $L$ is the thickness of the slab. For a dielectric slab in an external magnetic field pointing along the direction of propagation, the $\delta n = n_1 - n_2$ is proportional to the product of the Verdet and the magnetic field. Note that this $C$ given by Eq. (13) has the requisite symmetry of Eq. (3) and is conservative, satisfying Eqs. (1)–(5).

The system Eqs. (11) and (13) are 4 (complex) relations for 3 complex quantities ($d, s, l$), so, being overdetermined, demand a condition on the $n_{1,2}, k_0 L$ which may or may not be physically satisfiable. Algebra shows this condition to be

$$\left(n_1 + \frac{1}{n_1}\right) S_1 C_2 - \left(n_2 + \frac{1}{n_2}\right) S_2 C_1 =$$
$$\pm \left[n_1 - \frac{1}{n_1}\right] S_1 - \left[n_2 - \frac{1}{n_2}\right] S_2 \quad (14)$$

Whenever this condition is satisfied, the fields fall into the (external) parity eigenstates $l = \pm s$ and $d = \pm 1$, as expected. Again, these are necessarily two-port resonances, as is CPA, and thus examples of CPR states. A numerical solution is shown in Fig. 2 for terbium-gallium-garnet with $n = (n_1 + n_2)/2 = 1.95$ subject to a coherent 632.8nm light source and $\delta \eta = 2.7 \times 10^{-5}$ produced by a 1T external field. Here, we have plotted the LHS-hand-side squared (LHS)$^2$ of Eq. (14) as a dashed line and the (envelope of the) right-hand-side squared (RHS)$^2$ as a gray line. The first of many CPR states exists under these conditions at $L/\lambda_c \approx 0.603$, where $L$ is the length of the slab and $\lambda_c$ is the critical half-wave rotation length. It is rather easy to understand some general trends in the location of the CPR resonances in $\lambda$. Increasing $\tilde{n} = (n_1 + n_2)/2$ or $\delta \eta$ brings the location of the first CPR resonance into lower $k_0 L$, as would be the case in CPA with $\delta \eta$ playing the role of $\alpha$, the absorption constant. Thus for a fixed $L$ and a given range of $k_0$, there is a threshold $\delta \eta$ at which CPR states first appear, again reminiscent of CPA.

Also in Fig. 2 is a graph of the total power reflected with the same polarization as the input fields for this case, clearly indicating the first CPR resonance near
FIG. 2. (a) The LHS (black dashed line) and the envelope of the fast oscillating RHS (thick gray line) of the CPR condition Eq. [13] plotted as a function of $L/L_c$. The inset is a small portion of the full graph where the fast oscillations of the RHS are shown. (b) Plot of the total reflected intensity in the same polarization as the input fields as a function of the length, $L$, multiplied by the vacuum wave number, $k_0$. The thin line corresponds to the case where the counterpropagating fields have the same relative phase and the thick line to the case where they are 180 degrees out of phase. The inset shows the line splitting of the CPR resonances.

$k_0L ≈ 70,201$, or $L ≈ 7.07\text{mm}$. Notably for this simple slab geometry, all resonances come in pairs of the same parity, and are part of a parity-alternating series of pairs of resonances. As in CPA, these CPR resonances are bound-state like (zero width). Unlike CPA where there is but one resonance, for CPR given $n$, $L$, and a range of $k_0$ there are many, and they occur generically in “doublets.”

Finally, just as one can reach critical coupling in a 1-port version of CPA, one can see that for particular values of $n_1$, $n_2$, and $k_0L$ there can be a degeneracy of the positive and negative parity resonances. For Fig. 2, this occurs for $k_0L ≈ 116,355$. At degeneracy, taking linear combinations of the CPR resonances yields incoherent critical rotation solutions (in detail they are at $S_1 = 0 = S_2$ and $C_1 = -C_2 = ±1$). These are optically indistinguishable from the critical rotator of the optical activity example already discussed.

An experimental verification of CPR is planned using a high Verdet glass. The CPR resonances are thin, indicating that small changes in a substantial magnetic field (or in the material itself) may be readily detectable through changes in the extinction of a reflected polarization. At the level of technological application, note that an optical modulator based on CPA will necessarily have limited dynamic range as the material will always absorb some of the light even when not in CPA. A CPR-based optical modulator may not suffer the same limitations.

In conclusion, we have shown that Faraday rotation has the appropriate symmetries to manifest Coherent Perfect Rotation (CPR) and analytically developed an example of CPR in a dielectric Faraday slab rotator. CPR has deep phenomenologically similarity with CPA, but with a Hermitian Hamiltonian. It appears likely that other types of coherent perfect mode conversion will have similar phenomenology, and necessitate T-odd processes.

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