Bandwidth Maximization of Disturbance Observer Based on Experimental Frequency Response Data

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Abstract : A disturbance observer (DOB) has been widely employed in industrial field due to its simplicity and effectiveness in disturbance rejection. This paper focuses on systematic bandwidth-maximized DOB design by frequency response data-based convex optimization. The transformation process from original non-convex optimization to convex optimization has been formulated. Simulation results have verified the feasibility and generality of the proposal and shown that the designed DOB is able to achieve good disturbance rejection performance.

Key Words : disturbance observer, frequency response data, convex optimization.

1. Introduction

Unavoidable disturbances deteriorate the performance of industrial control systems. To reject the effects of disturbance, a disturbance observer (DOB) has been proposed [1] and used in various applications, such as robot manipulators [2], high-speed positioning systems [3], etc.

In ideal disturbance observer configuration as shown in Fig. 1, the difference between \( u \) and \( \hat{u}_p \) (estimation of plant \( P_r \)) input \( (u_p) \) obtained by using nominal plant inversion \( (P_{r}^{-1}) \) creates estimated disturbance \( \hat{d} \), which is fed back to compensate effects of real disturbance \( d \). In practical applications, a low pass filter (\( Q \) filter) is necessary to guarantee the causality of the system, and the high bandwidth of the said filter is desired to ensure satisfactory disturbance rejection performance.

Previous research on the \( Q \) filter employed a parametric model (transfer function or state space representation) as a real plant. Since unknown coefficients of the parametric model are identified from experimentally obtained input-output relationship known as frequency response data (FRD), unmodeled dynamics which limit the shaping of \( Q \) filter response are introduced due to fitting process. Moreover, the \( Q \) filter designed by the aforementioned methods cannot be regarded as optimal. To directly utilize all the information of FRD in shaping \( Q \) filter response and optimize the bandwidth of the DOB, frequency response data-based DOB design has been explored in this paper.

Previous frequency response data-based research mainly focused on designing a linearly parameterized fixed order feedback controller while optimizing specifications of the control system [12]–[15], e.g., integrator gain. Convex optimization was used to compute robust controllers for single-input-single-output systems depicted by frequency response data in [12], which was applied to specifically design a PID controller in [13],[14] and further extended to the multi-input-multi-output case in [15].

In regard to the aforementioned works, we integrated the frequency response data-based method into DOB optimization design, and the preliminary work which designed a second order \( Q \) filter was presented in [16]. In this paper, we confirm the previous findings and enhance its generality and performance.

1. A systematic method of designing the second order \( Q \) filter by FRD is derived. The \( Q \) filter parameters are properly tuned to maximize the bandwidth of the DOB and provide satisfactory disturbance attenuation performance.

2. A general derivation process from non-convex constraints to convex constraints in DOB design has been developed. An iterative convex optimization process is established to solve the problem.

3. The requirement for the nominal plant and the damping factor of the \( Q \) filter have been mitigated compared with [16]. The proposed method is versatile for both minimum phase and non-minimum phase plants. The order of the
nominal plant is arbitrary, but the relative order should be two.

The remaining part of this paper is organized as follows. Section 2 provides mathematical preliminaries of this paper. Problem formulation is developed in Section 3. Non-convex constraints will be derived in this section followed by mathematical transformation to convex constraints in Section 4. Based on the constraints obtained, simulation results are shown in Section 5. Section 6 presents discussions, and this paper ends by giving concluding remarks in Section 7.

2. Preliminaries

A convex optimization problem is one in which the objective and constraint functions are convex, which means they satisfy the inequality [17]:

\[ f(\alpha x + \beta y) \leq \alpha f(x) + \beta f(y) \]  

for all \( x, y \in \mathbb{R}^n \) and all \( \alpha, \beta \in \mathbb{R} \) with \( \alpha + \beta = 1, \alpha \geq 0, \beta \geq 0 \).

A linear matrix inequality (LMI) has the following form:

\[ F(x) = F_0 + \sum_{i=1}^{m} x_i F_i > 0, \]

in which \( x \in \mathbb{R}^m \) is the variable and the symmetric matrices \( F_i = F_i^T \in \mathbb{R}^{n \times n}, i = 0, \ldots, m \) are given [18].

Besides these, linear approximation is extensively employed. The basic concept is to estimate the value of a function, \( f(x) \), near a point \( x_0 = [x_0(1), x_0(2), \ldots, x_0(n)]^T \), using the following formula:

\[ f(x) \approx f(x_0) + \nabla f(x_0)(x-x_0), \]

in which \( \nabla \) denotes the vector differential operator, and

\[ \nabla f(x_0) = [\partial f(x_0)/\partial x(1), \partial f(x_0)/\partial x(2), \ldots, \partial f(x_0)/\partial x(n)]. \]

Additionally, |A| denotes the magnitude of A, and \( j\omega_k \) means sequential frequency points in which \( j \) is the imaginary unit while \( \omega_k \) and \( k \) are frequency point and the index of frequency point, respectively.

3. Problem Formulation

In the disturbance observer system as shown in Fig. 2, \( P_r \) and \( P_n \) denote the real plant and the nominal plant, defined by FRD and a transfer function, respectively. Signals \( d, \bar{d}, u, \) and \( y \) are external disturbance input, estimated disturbance, control input, and output, respectively. The to-be-designed low pass filter is represented by \( Q \).

In this paper, \( Q \) is selected as follows in which \( a = [a_1, a_2]^T \) is the parameter to be decided and the relative order of \( P_n \) is 2:

\[ Q = \frac{1}{a_2 s^2 + a_1 s + 1}. \]

\[ L = P_n^{-1}(1-Q)^{-1} P_s(j\omega_k) = \frac{P_s(j\omega_k)P_n^{-1}}{(a_2 s^2 + a_1 s) + N} \]

\[ S = \frac{1}{1+(1-Q)^{-1} P_s(j\omega_k)P_n(j\omega_k)} = \frac{D + N}{D} \]

\[ \frac{y}{d} = \frac{1+(1-Q)^{-1} P_s(j\omega_k)P_n(j\omega_k)}{1+(1-Q)^{-1} P_s(j\omega_k)^2 P_n(j\omega_k)^2} = \frac{S P_s(j\omega_k)}{D}, \]

\[ S = \frac{1}{1+(1-Q)^{-1} P_s(j\omega_k)P_n(j\omega_k)} = \frac{1-S}{T}, \]

in which \( N = P_s(j\omega_k)P_n^{-1}, D = a_2 s^2 + a_1 s, \) and \( L, S, \) and \( T \) denote the open loop function, sensitivity function, and complementary sensitivity function for Fig. 2, respectively.

Several constraints should be satisfied to obtain satisfactory disturbance rejection performance. First, to guarantee the stability margins (the gain margin and the phase margin), the stability circle condition (14), which is graphically shown in Fig. 3, should be met.

The mathematical representation of stability circle condition is shown as the following inequality:

\[ |\sigma + L(j\omega_k)| - r_m \geq 0, \]

in which \( L \) is the open loop function while \( \sigma \) (the center point of the circle \( C_2 (-\sigma, 0) \) and \( r_m \) (the radius of the circle \( C_2 \)) can be calculated by using the following equations ([14]), where \( g_m \) is the gain margin and \( \phi_m \) represents for phase margin:

\[ \sigma = \frac{g_m - 1}{2 g_m (g_m \cos \phi_m - 1)}, \]

\[ r_m = \frac{(g_m - 1)^2 + 2 g_m (1 - \cos \phi_m)}{2 g_m (g_m \cos \phi_m - 1)}. \]

Second, by selecting weighting functions \( W_p \) and \( W_m \) as follows for \( S \) and \( T \), respectively, the constraints for the sensitivity function and the complementary sensitivity function are established as shown in Figs. 4 and 5. The optimization objective is \( \omega_p \), which represents the bandwidth frequency of the disturbance observer. Variable \( \omega_k \) together with \( a_1 \) and \( a_2 \) is an optimization variable. When an optimal point \( \omega_{p(opt)} \) is found, \( a = [a_1, a_2]^T \) is obtained simultaneously. These constraints are
common in conventional $H_\infty$ theory, but obtainable $Q$ is of high order, whereas $Q$ has fixed form in this paper.

\[ W_p = \frac{\omega_p}{s}, |W_p| \leq 1, \]  
\[ W_m = \frac{s + \omega}{1.25s}, |W_m| \leq 1(M_i = 1.25). \]  

Third, $\omega_p$ and $\omega$ set lower and upper bounds for the $Q$ filter bandwidth frequency, and $a_1$ and $a_2$ should all be positive to guarantee the stability of the $Q$ filter itself. While in our previous study [16], the damping factor of the $Q$ filter ($\frac{a_1}{2\sqrt{a_1^2}}$) was limited to [0.5, 1], we managed to remove this constraint in this paper, which adds more freedom in tuning parameters.

In summary, the optimization problem can be formulated into the following form:

Maximize $\omega_p$ subject to $0 < a_1, 0 < a_2$, $0 < \omega_p \leq \frac{1}{\sqrt{a_2}} \leq \omega_t$, $|U(j\omega) + \sigma| \geq r_m$, $|W_p(j\omega_p, \omega_p)|S(j\omega_p)| \leq 1$, $|W_m(j\omega_m, \omega_m)|T(j\omega_m)| \leq 1$.  

4.  Convex Constraints Derivation

In this section, the above-listed non-convex constraints are all transformed into linear functions or LMI form in terms of variables $\omega_p$, $\omega_t$, $a_1$, $a_2$. The derived constraints are a sufficient condition of original constraints, which means that if the newly-obtained constraints are satisfied, the original constraints hold undoubted.

4.1 Constraint in Eq. (12c)

The left side of Eq. (12c) ($\omega_p \leq \frac{1}{\sqrt{a_2}}$) is changed into the following form by using linear approximation of $\frac{1}{\sqrt{a_2}}$:

\[ \omega_p \leq \frac{1}{\sqrt{a_2(i)}} \Leftrightarrow \omega_p \leq \frac{1}{\sqrt{a_2(i)}} = \frac{a_2(i) - a_2(i-1)}{2}/\text{newly-obtained} \]  

in which $i$ is the iteration index and $a_2(i)$ and $\omega_p$ mean the current values while $a_2(i-1)$ means the previous value in the iterative optimization process.

The Schur complement is used to deal with the right side of Eq. (12c) ($\frac{1}{\sqrt{a_2(i)} - \omega_p(i)}$):

\[ \frac{1}{a_2(i)} \leq \omega_p(i) \Leftrightarrow \frac{1}{a_2(i)} - \omega_p(i) \leq 2\omega_p(i) - \omega_p(i) - \omega_p(i) \]

\[ \Leftrightarrow 2\omega_p(i) - \omega_p(i) - \omega_p(i) \leq 1 \geq 0. \]  

Similarly, $\omega_p(i)$ means the current value, while $\omega_p(i-1)$ means the previous value in the iterative optimization process.

4.2 Constraint in Eq. (12d)

In this subsection, Eq. (12d) is converted to convex constraint in the following way.

\[ |U(j\omega_k, a_1) + \sigma| - r_m = \frac{N(j\omega_k)}{D(j\omega_k, a_1)} + \sigma - r_m \geq 0 \]  

\[ \Leftrightarrow \left| N(j\omega_k) + D(j\omega_k, a_1)\sigma \right| - r_m = |D(j\omega_k, a_1)|. \]

Define $F(j\omega_k, a_1) = |N(j\omega_k) + D(j\omega_k, a_1)\sigma|$. Since $F(j\omega_k, a_1)$ is a convex function of variable $a_1$ and the linear approximation of it would be no larger than the original value, the following equation holds:

\[ F(j\omega_k, a_1) \geq r_m |D(j\omega_k, a_1)| \Leftrightarrow |\Psi - r_m |D(j\omega_k, a_1)|| \geq 0, \]  

where

\[ \Psi = F(j\omega_k, a_1) + \nabla F(j\omega_k, a_1)(a_1 - a_1), \]  

\[ \nabla F(j\omega_k, a_1) = \frac{\partial |N(j\omega_k) + D(j\omega_k, a_1)\sigma|}{\partial a_1(i)} . \]  

4.3 Constraint in Eq. (12e)

For the sensitivity function constraint, by substituting (11a) and (7b) into (12e), the following inequality is obtained:

\[ |W_p(j\omega_k, \omega_p) S(j\omega_k, a_1)| \leq 1 \]

\[ \Leftrightarrow \left| \omega_p \right|D(j\omega_k, a_1) \leq |D(j\omega_k, a_1)|. \]  

Squaring both sides of Eq. (19) and turning this inequality into matrix inequality form by using the Schur complement, we have

\[ \left( \begin{array}{cc} \omega_p(i) & \omega_p \omega_p(i) \\ \omega_p(i) & \omega_p(i) \end{array} \right)^2 |D(j\omega_k, a_1)|^2 \leq |D(j\omega_k, a_1)| + |N(j\omega_k)|^2 \]

\[ \Leftrightarrow \left[ \begin{array}{cc} \omega_p(i) & \omega_p(i) \\ \omega_p(i) & \omega_p(i) \end{array} \right] \left( \begin{array}{cc} \omega_p(i) & \omega_p(i) \\ \omega_p(i) & \omega_p(i) \end{array} \right)^T \left( \begin{array}{cc} D(j\omega_k, a_1) \right) \leq |D(j\omega_k, a_1)| + |N(j\omega_k)|^2 \]

\[ \left[ S_{11} S_{12} \right] \left( \begin{array}{cc} S_{11} & S_{12} \\ S_{12} & S_{22} \end{array} \right) \geq 0. \]
To obtain a sufficient condition of original constraint, lower bounds of $S_{11}$ and $S_{22}$ are required. For $S_{11} = \frac{(\omega_p)^2}{\omega_p}$, a lower bound of $\omega_p^2$ is obtained by using the following technique [19]:

$$\omega_p^2 - \omega_{p-1}^2 \geq 0$$

$$\omega_p^2 - \omega_{p-2}^2 \geq 0$$

$$\omega_p^2 - \omega_{p-3}^2 \geq \phi_1(i) > 0,$$

in which $\phi_1(i)$ is a newly-introduced variable and the constraints for it can be expressed in the following form:

$$\begin{align*}
\frac{2\omega_p^2 - \phi_1(i)\omega_{p-1}^2}{\omega_p(i)} \geq 0, \phi_1(i) > 0.
\end{align*}$$

In conclusion,

$$S_{11} = \frac{(\omega_p)^2}{\omega_p} \geq \omega_p^2 \phi_1(i) > 0.$$  \hspace{1cm} (23)

As for $S_{22}$, since $|D(j\omega_k, a_i) + N(j\omega_k)|^2$ is a convex function of variable $a_i$, the linear approximation is employed to find a lower bound of it:

$$S_{22} = \frac{|D(j\omega_k, a_i) + N(j\omega_k)|^2}{\text{original convex}} \geq (M(j\omega_k, a_i) + \nabla(M(j\omega_k, a_i)))(a_i - a_{i-1}) = \Phi,$$

newly-obtained \hspace{1cm} (24)

in which

$$\nabla(M(j\omega_k, a_i)) = \begin{bmatrix}
\frac{\partial[D(j\omega_k, a_{i-1}) + N(j\omega_k)]}{\partial a_{i-1}} \\
\frac{\partial[D(j\omega_k, a_i) + N(j\omega_k)]}{\partial a_i}
\end{bmatrix}.$$

In summary, the original nonlinear constraint Eq. (12c) is transformed into the following form by combining Eq. (20), Eq. (22), Eq. (23), and Eq. (24):

$$\begin{align*}
\frac{\omega_p^2 \phi_1(i)}{(D(j\omega_k, a_i))} \Phi & \geq 0, \phi_1(i) > 0, \hspace{1cm} (26a)
\end{align*}$$

$$\begin{align*}
\frac{2\omega_p^2 - \phi_1(i)\omega_{p-1}^2}{\omega_p(i)} \geq 0. \hspace{1cm} (26b)
\end{align*}$$

4.4 Constraint in Eq. (12f):

Following a similar process as used in dealing with Eq. (12e), the complementary sensitivity function constraint is changed into the following form:

$$\begin{equation}
W_m(\omega_N)T(j\omega_k, a_i) \leq 1 \Rightarrow \begin{bmatrix} j\omega_k + \omega_N & 1 & N(j\omega_k) \\ 1 & 1.25\omega_N & 1.25 \\ N(j\omega_k) & 1.25 & 1.25 \\ \end{bmatrix} \leq \frac{1 + \frac{N(j\omega_k)D(j\omega_k, a_i)}{N(j\omega_k)D(j\omega_k, a_i)}}{1 + \frac{N(j\omega_k)D(j\omega_k, a_i)}{N(j\omega_k)D(j\omega_k, a_i)}} \hspace{1cm} (27)
\end{equation}$$

$$\Rightarrow \begin{align*}
\frac{j\omega_k + \omega_N}{1.25\omega_N} & \leq \frac{|D(j\omega_k, a_i) + N(j\omega_k)|^2}{N(j\omega_k)(j\omega_k + \omega_N)} \hspace{1cm} (27)
\end{align*}$$

$$\Rightarrow \begin{align*}
\frac{|D(j\omega_k, a_i) + N(j\omega_k)|^2}{N(j\omega_k)(j\omega_k + \omega_N)} & \geq 0. \hspace{1cm} (27)
\end{align*}$$

$$\begin{bmatrix} T_{11} & T_{12} \\ T_{12} & T_{22} \end{bmatrix} \geq 0.$$

As before, $T_{11}$ and $T_{22}$ need transformation. Since $T_{22} = S_{22}$, this part is omitted due to the repetition. For $T_{11}$,

$$\omega_N^2 \geq 2\omega_{N-1}^2 \omega_N - \omega_{N-1}^2 \hspace{1cm} (28)$$

By combining Eq. (24), Eq. (27) and Eq. (28), the original non-convex constraint is changed into

$$\begin{align*}
\frac{2\omega_{N-1}^2 \omega_N - \omega_{N-1}^2}{|N(j\omega_k)(j\omega_k + \omega_N)|} \geq 0. \hspace{1cm} (29)
\end{align*}$$

4.5 Problem Reformulation

After finishing all the processes mentioned above, the original problem is reformulated as follows.

Maximize

$$a_{i+1} > 0, a_{i+2} > 0, \omega_N > \omega_p > 0,$$

Subject to

$$\begin{align*}
\phi_1(i) > 0, |\gamma - r_m D(j\omega_k, a_i)| & \geq 0, \hspace{1cm} (30a)
\end{align*}$$

$$\begin{align*}
\omega_N \geq \frac{1}{2\Phi} \frac{a_{i+1}^2}{a_{i+1}^2 - a_{i+1}^2} \hspace{1cm} (30b)
\end{align*}$$

$$\begin{align*}
\frac{2\omega_{N-1}^2 \omega_N - \omega_{N-1}^2}{1} \geq 0. \hspace{1cm} (30c)
\end{align*}$$

$$\begin{align*}
\frac{2\omega_{N-1}^2 \omega_N - \omega_{N-1}^2}{1} \geq 0. \hspace{1cm} (30d)
\end{align*}$$

$$\begin{align*}
\frac{2\omega_{N-1}^2 \omega_N - \omega_{N-1}^2}{1} \geq 0. \hspace{1cm} (30e)
\end{align*}$$

$$\begin{align*}
\frac{2\omega_{N-1}^2 \omega_N - \omega_{N-1}^2}{1} \geq 0. \hspace{1cm} (30f)
\end{align*}$$

$$\begin{align*}
\frac{2\omega_{N-1}^2 \omega_N - \omega_{N-1}^2}{1} \geq 0. \hspace{1cm} (30g)
\end{align*}$$

The new optimization problem is a convex optimization problem and can be solved by commercial solvers.

5. Simulation Result

5.1 Simulation Plant and Simulation Condition

The simplified model of the simulation plant is shown in Fig. 6. The linear motor (actuator) provides actuation force (input, $F$) and moves the table towards a predefined position. The position of the table (output or control target $x$) is recorded by a table-side linear encoder. As the table-side linear encoder position (vertical direction $l_{sys}$) changes, the plant dynamics (from linear motor actuation force $F$ to table’s position $x$) varies from minimum phase to non-minimum phase [20].

Input disturbances, e.g., the friction of the linear motor, are taken into account. The real plant and the nominal plant are represented by FRD ($P_s(j\omega)$) and the transfer function ($P_s$), respectively. For both minimum phase and non-minimum phase plants, a disturbance observer has been designed by utilizing our proposal.

During all the simulations, the desired gain margin and the phase margin are 6 dB and 30°. According to Eq. (9) and Eq. (10),

$$\sigma = 1.03, \ r_m = 0.525.$$  \hspace{1cm} (31)
5.2 Case 1: Stable Minimum Phase Plant

The system is minimum phase when $l_{enc}$ is 80 mm. The corresponding nominal plant ($P_n$) is fourth order, and the Bode plots of $P_r$ (FRD) and $P_n$ are shown in Fig. 7:

\begin{align}
P_n &= \frac{59.275(s^2 + 8.402s + 6.573 \times 10^4)}{s(s + 2.101)(s^2 + 10.89s + 3.665 \times 10^4)}, \\
L &= \frac{P_r(j\omega_k)P_n^{-1}}{a_2s^2 + a_1s}, \quad S = \frac{a_2s^2 + a_1s}{a_2s^2 + a_1s + P_r(j\omega_k)P_n^{-1}}, \\
T &= \frac{P_r(j\omega_k)P_n^{-1}}{a_2s^2 + a_1s + P_r(j\omega_k)P_n^{-1}}.
\end{align}

After optimization, $\omega_{p(0pt)} = 88.6$ rad/s (14.1 Hz). In the meantime, $a_1 = 0.0071$, $a_2 = 5.79 \times 10^{-5}$, and the damping factor is 0.46. The magnitude plots of optimized $S$, initial $S$ and $W_p$ (optimized) are shown in Fig. 8a.

The stability margin constraint, Eq. (12d), holds successfully, and the bandwidth of the open loop function is maximized as the optimized Nyquist plot becomes tangent to the small dotted circle (Fig. 8b) whose center is located at $(-\sigma, 0)$, i.e., $(-1.03, 0)$ and the radius is $r_v = 0.525$.

The constraints for $S$ and $T$ are satisfied as $|W_pS|$ and $|W_nT|$ are always no larger than 0 dB in Fig. 8c.

5.3 Case 2: Stable Non-Minimum Phase Plant

When the vertical distance from the table side linear encoder to the actuator ($l_{enc}$) is 300 mm, the plant dynamics becomes non-minimum phase (unstable zeros appear). The nominal plant is shown in Eq. (33) as fourth order and contains one unstable zero. Figure 9 depicted the Bode plots of $P_r$ (FRD) and $P_n$:

\begin{equation}
P_n = \frac{-206.68(s - 125.6)(s + 120)}{s(s + 2.101)(s^2 + 10.89s + 3.665 \times 10^4)}.
\end{equation}

Since the inverse of the nominal plant is unstable, an approximate inverse of the nominal plant (zero magnitude error approximation [23]) is used:

\begin{equation}
\tilde{P}_n^{-1} = \frac{s(s + 2.101)(s^2 + 10.89s + 3.665 \times 10^4)}{206.68(s + 125.6)(s + 120)}.
\end{equation}

The corresponding $L$, $S$, and $T$ are obtained as

\begin{align}
L &= \frac{P_r(j\omega_k)\tilde{P}_n^{-1}}{a_2s^2 + a_1s}, \quad S = \frac{a_2s^2 + a_1s}{a_2s^2 + a_1s + P_r(j\omega_k)\tilde{P}_n^{-1}}, \\
T &= \frac{P_r(j\omega_k)\tilde{P}_n^{-1}}{a_2s^2 + a_1s + P_r(j\omega_k)\tilde{P}_n^{-1}}.
\end{align}

After optimization, $\omega_{p(0pt)} = 33.1$ rad/s (5.26 Hz) and $a_1 = 0.0239$, $a_2 = 1.98 \times 10^{-6}$. The tuned $Q$ filter’s damping factor is 8.59. The magnitude plots of optimized $S$, initial $S$, and

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5.2 Case 1: Stable Minimum Phase Plant

The system is minimum phase when $l_{enc}$ is 80 mm. The corresponding nominal plant ($P_n$) is fourth order, and the Bode plots of $P_r$ (FRD) and $P_n$ are shown in Fig. 7:

\begin{align}
P_n &= \frac{59.275(s^2 + 8.402s + 6.573 \times 10^4)}{s(s + 2.101)(s^2 + 10.89s + 3.665 \times 10^4)}, \\
L &= \frac{P_r(j\omega_k)P_n^{-1}}{a_2s^2 + a_1s}, \quad S = \frac{a_2s^2 + a_1s}{a_2s^2 + a_1s + P_r(j\omega_k)P_n^{-1}}, \\
T &= \frac{P_r(j\omega_k)P_n^{-1}}{a_2s^2 + a_1s + P_r(j\omega_k)P_n^{-1}}.
\end{align}

After optimization, $\omega_{p(0pt)} = 88.6$ rad/s (14.1 Hz). In the meantime, $a_1 = 0.0071$, $a_2 = 5.79 \times 10^{-5}$, and the damping factor is 0.46. The magnitude plots of optimized $S$, initial $S$ and $W_p$ (optimized) are shown in Fig. 8a.

The stability margin constraint, Eq. (12d), holds successfully, and the bandwidth of the open loop function is maximized as the optimized Nyquist plot becomes tangent to the small dotted circle (Fig. 8b) whose center is located at $(-\sigma, 0)$, i.e., $(-1.03, 0)$ and the radius is $r_v = 0.525$.

The constraints for $S$ and $T$ are satisfied as $|W_pS|$ and $|W_nT|$ are always no larger than 0 dB in Fig. 8c.
$W_p$ (optimized) are shown in Fig. 10a. Similar to the minimum phase case, the optimized Nyquist plot in this case also becomes almost tangent to the small dotted circle as shown in Fig. 10b, which implies that the stability margin constraint Eq. (12d) is satisfied. Since $|W_pS|$ and $|W_mT|$ are always no larger than 0 dB in Fig. 10c, it is verified that the constraints Eq. (12e) and Eq. (12f) hold successfully.

### 5.4 Disturbance Rejection Performance

Simulations have been conducted to test the disturbance rejection performance of the above-designed $Q$ filters by using Fig. 11, and when the non-minimum phase nominal plant is employed, $P_n^{-1}$ in the figure is replaced by $\bar{P}_n^{-1}$. For both minimum and non-minimum phase plants, $P_r$ is a well-identified 8th order transfer function (tf) (FRD can not be used in this simulation). The feedback controller $C_{fb}$ is chosen as follows:

$$C_{fb} = \frac{1.3 + \frac{3.75}{s} + \frac{0.139}{0.0249s + 1}}{s}.$$  \hspace{1cm} (36)

By making reference input as zero and injecting unit step disturbance to the system, in both cases (minimum phase plant and non-minimum phase plant), three different output responses are obtained for comparison.

1. Only the feedback controller $C_{fb}$ works in the system.
2. The selected initial disturbance observer [16] and the feedback controller $C_{fb}$ work in the system together.
3. The optimized disturbance observer and the feedback controller $C_{fb}$ work in the system together.

#### 5.4.1 Case 1: stable minimum phase plant

The transfer function of $P_r$ is chosen as follows:

$$P_r = \frac{43.834(s^2 + 8.402s + 6.573 \times 10^4)}{s(s + 2.101)(s^2 + 10.89s + 3.665 \times 10^3)} \times \frac{(s^2 + 74.07s + 4.128 \times 10^5)(s^2 + 308.38s + 3.606 \times 10^9)}{(s^2 + 45.4s + 3.139 \times 10^3)(s^2 + 262.2s + 3.507 \times 10^6)}.$$  \hspace{1cm} (37)

The disturbance rejection performance is depicted in Fig. 12. It can be concluded that the maximum deviation from the reference position (0) has been decreased by 53.0% compared to the initial DOB and 94.6% compared to the feedback controller only case, which implies that designed $Q$ filter has improved the disturbance rejection performance.

#### 5.4.2 Case 2: stable non-minimum phase plant

For non-minimum phase plant case, $P_r$ is defined by Eq. (38):

$$P_r = \frac{164.38(s + 1089)(s - 878)(s - 125.6)}{s(s + 2.101)(s^2 + 10.89s + 3.665 \times 10^3)} \times \frac{(s + 120)(s^2 + 185.5s + 1.447 \times 10^9)}{(s^2 + 45.4s + 3.139 \times 10^3)(s^2 + 262.2s + 3.507 \times 10^6)}.$$  \hspace{1cm} (38)

The proposed $Q$ filter design outperforms the initial design, and the disturbance rejection performance has been improved as shown in Fig. 13. The maximum deviation from the reference position has been decreased by 67.7% compared to the
As long as the frequency response data can be obtained for a single plant, the proposed method can provide bandwidth-maximized design for each plant. The only limitation on the frequency response data is that the plant should be stable.

7. Conclusion

This paper has proposed a general design method for maximizing bandwidth of disturbance observer configuration directly based on frequency response data. Moreover, all the non-convex constraints have been transformed into convex form, which is solved by convex optimization. The numerical case studies present the feasibility and generality of the proposed method for both minimum and non-minimum phase plants.

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