Stochastic Consensus over Multi-Channel Networks of MIMO Linear Symmetric Agents*

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Multi-agent systems over noisy networks with multi-input/multi-output linear symmetric agents are considered. The information is assumed to be sent from an agent to its neighbors via multi-channels. The communication graph of each channel is allowed to be time varying and a different topology. The entire network which is defined by the union of all graph is assumed to be connected in an undirected graph case or weakly connected and balanced in a directed one. The aim of this study is to establish a stopping rule of a stochastic averaging consensus under a noisy and time-varying network. The convergence analysis reveals an explicit relation between the number of iterations and the closeness of the consensus. The results are illustrated through numerical examples.

1. Introduction

An averaging consensus is a basic distributed algorithm which works over a multi-agent networked systems and gives the average value of initial states of the agents [1]. In this algorithm, each agent communicates only with its neighbors and exchanges information iteratively. Since communication noise is unavoidable and usually interferes with the information exchange, several stochastic consensus algorithms have been proposed under noisy environment [2,3]. Stopping rules, which provide a relation between the explicit number of iterations and the quality of the consensus, have also been established for stochastic noisy environment, where the agents are assumed to be first order systems [4] or higher order systems [5]. The latter employs a linear symmetric system [6].

In terms of opinion dynamics, it is significant to consider how to reach a consensus [7]. Multi-input/multi-output (MIMO) agents can represent more complex opinion formation rather than single-input/single-output (SISO) ones. Furthermore, it is also important to think the multi-channel communication networks. We receive much information to form our opinions from many different channels, for example, newspapers or televisions. Even we can easily influence the other persons’ opinion via social networks. These situation can be represented as the multi-channel.

In this paper, we tackle a multi-agent system over noisy networks, where each agent is a MIMO linear symmetric system with multi-channels. We give a communication gain to achieve the stochastic averaging consensus and the stopping rule under an appropriate assumption for the symmetric system. These results include the existing ones [4], [5], and [8] as special cases if the agents are first order systems, SISO symmetric systems, or MIMO symmetric systems with a single channel.

This paper is organized as follows. We define multi-agent systems to be considered in Section 2. We analyze the convergence of the consensus which gives the stopping rule for directed and undirected topology cases in Section 3. We show numerical examples in Section 4. Finally, we conclude this paper in Section 5. The preliminary versions of this study was presented at a conference [8].

Notations: Let \( I_h \in \mathbb{R}^{h \times h} \) and \( 1_N \in \mathbb{R}^N \) be the identity matrix and the vector whose elements are all 1. The expectation and the covariance of a random variable are denoted by \( \mathbb{E}[\cdot] \) and \( \text{Cov}[\cdot] \). The Kronecker product is denoted by \( \otimes \). The notation \( \| \cdot \| \) represents the Euclidean norm for a vector and the spectral norm for a matrix. The trace of a matrix is denoted by \( \text{Tr}(\cdot) \).

2. Multi-Agent Systems

Let us consider MIMO linear symmetric agents having the same dynamics
\begin{equation}
  x_i[k+1] = Ax_i[k] + \sum_{h=1}^{M} B_h u_{hi}[k],
  \quad y_{hi}[k] = C_h x_i[k],
  \quad A = A^\top, \quad B_h = C_h^\top, \quad h \in \mathcal{H}, \quad i \in \mathcal{V},
\end{equation}

where $x_i[k] \in \mathbb{R}^n$ is the state of agent $i$, $u_{hi}[k] \in \mathbb{R}^{n_a}$ is the $h$-th channel input, $y_{hi}[k] \in \mathbb{R}^{n_y}$ is the $h$-th channel output, $\mathcal{H} = \{1, 2, \ldots, M\}$ is the set of channels, $\mathcal{V} = \{1, 2, \ldots, N\}$ is the set of agents, $A \in \mathbb{R}^{n \times n}$, $B_h \in \mathbb{R}^{n_y \times n_a}$, and $C_h \in \mathbb{R}^{n_a \times n}$ are the coefficient matrices, $M \in \mathbb{N}$ is the number of channels, $N \in \mathbb{N}$ is the number of agents, and $k \in \mathbb{N}$ is the discrete time.

We use the following condition on the MIMO linear symmetric agents.

**Assumption 1** There exist a positive definite matrix $R_h = R_h^\top \in \mathbb{R}^{m_h \times m_h}$ and $\eta \in (0, 1]$ such that

\begin{equation}
  0 < R_h \leq I_{m_h}, \quad 0 \leq A \leq I_n, \quad 0 \leq A - B_h R_h C_h \leq (1 - \eta) I_n
\end{equation}

for all $h \in \mathcal{H}$.

This assumption means that each agent can be stabilized by any one channel output feedback with an arbitrary low positive-definite gain since it is equivalent to the following conditions: there exists a positive definite matrix $R_h = R_h^\top \in \mathbb{R}^{m_h \times m_h}$ such that $0 < R_h \leq I_{m_h}$ and $0 \leq A - \kappa B_h R_h C_h < I_n$ for each $h \in \mathcal{H}$ and $\kappa \in (0, 1]$.

We remark that the MIMO linear symmetric agent (1) with multi-channels includes one with a single channel investigated in the conference version [8] of this paper when $M = 1$. It can be the SISO one investigated in the authors’ previous work [5] when $M = 1$, $B$ and $C$ are vectors, and $R = 1$. It can also be the standard first order agent investigated in [4] when $A = B = C = R = 1$ with $M = \eta = 1$.

For the set (1) of the symmetric agents, we introduce a different interaction for each channel $h$ as

\begin{align}
  u_{hi}[k] &= r[k] R_h \sum_{j \in \mathcal{N}_h[i]} (z_{hij}[k] - y_{hi}[k]),
  
  z_{hij}[k] &= y_{hi}[k] + w_{hij}[k],
  
  h \in \mathcal{H}, \quad i \in \mathcal{V}, \quad j \in \mathcal{N}_h[i],
\end{align}

where $r[k] \in \mathbb{R}$ is the communication gain to be determined later, $z_{hij}[k] \in \mathbb{R}^{m_h}$ is the $h$-th channel information which agent $i$ receives from agent $j$ ($i \neq j$), $w_{hij}[k] \in \mathbb{R}^{m_h}$ is the communication noise, and $\mathcal{N}_h[i] \subseteq \mathcal{V} \setminus \{i\}$ is the set of neighbors that can communicate with agent $i$ for $h$-th channel, which introduces a time-varying graph $G_h[k] = (\mathcal{V}, \mathcal{E}_h[k])$ for $h$-th channel at time $k$, where $\mathcal{E}_h[k] \subseteq \mathcal{V} \times \mathcal{V}$ denotes the set of edges for $h$-th channel at time $k$. With this representation, $\mathcal{N}_h[i] = \{j \in \mathcal{V} \mid (i, j) \in \mathcal{E}_h[k]\}$. Note that the edge $(i, j) \in \mathcal{E}_h[k]$ is the ordered pair in the directed graph, i.e., $i$ and $j$ are the tail and the head of the edge. We should also notice that $\mathcal{N}_h[i]$ is the set of in-neighbors for agent $i$ in the directed graph. It means that agent $i$ can receive the output from agent $j$ if $j \in \mathcal{N}_h[i]$ at time $k$.

We assume that the random variables $w_{hij}[k]$ are i.i.d. with respect to $h$, $i$, $j$, and $k$. We also assume that the communication noise vector $w_{hij}[k]$ satisfies

\begin{equation}
  E[w_{hij}[k]] = 0, \quad \text{Cov}[w_{hij}[k]] \leq V_h
\end{equation}

for any $h$, $i$, $j$, and $k$, where $V_h = V_h^\top \geq 0$.

In this paper, we consider the stochastic averaging consensus of the multi-agent system (1) with the noisy interactions (3). It is defined by satisfying

\begin{equation}
  \lim_{k \to \infty} \mathbb{P} \left( \exists i \in \mathcal{V}, \text{s.t.} \left| x_i[k] - \frac{1}{N} \sum_{j=1}^{N} x_j[k] \right| \geq \epsilon \right) = 0
\end{equation}

for any $\epsilon > 0$, where $\mathbb{P}$ is the probability measure on the noise sequence. The goal of this paper is to construct a stopping rule which provides a relation between the number of iterations and the quality of the consensus by giving an appropriate communication gain $r[k]$ for undirected and directed noisy networks.

In order to represent (1) and (3) as a compact form, we introduce the graph Laplacian $L_h[k]$ of $G_h[k]$ as

\begin{equation}
  [L_h[k]]_{ij} = \begin{cases} 
  -1 & \text{if } (j, i) \in \mathcal{E}_h[k], \\
  d_h^{(i)}[k] & \text{if } j = i, \\
  0 & \text{otherwise},
\end{cases}
\end{equation}

where $d_h^{(i)}[k]$ is the number of the neighbors of agent $i$ at time $k$ according to $G_h[k]$. Note that

\begin{equation}
  \sum_{h=1}^{M} L_h[k] \mathbf{1}_N = 0
\end{equation}

always holds by the definition of the graph Laplacian $L_h[k]$.

By using the graph Laplacian $L_h[k]$, the agents (1) and the interaction (3) can be rewritten as

\begin{align}
  x[k+1] &= (I_N \otimes A) + \sum_{h=1}^{M} (I_N \otimes B_h) u_h[k],
  
  y_{hi}[k] &= (I_N \otimes C_h) x[k],
  
  h \in \mathcal{H},
\end{align}

and

\begin{align}
  u_h[k] &= -r[k] (I_N \otimes R_h) (L_h[k] \otimes I_{m_h}) y_{hi}[k] \\
  &+ r[k] (I_N \otimes R_h) W_h[k] \mathbf{1}_N, \quad h \in \mathcal{H},
\end{align}

where

\begin{equation}
  x[k] = [x_1^\top[k] \ x_2^\top[k] \ \cdots \ x_N^\top[k]]^\top \in \mathbb{R}^{Nn},
  
  y_{hi}[k] = [y_{h_{11}}[k] \ y_{h_{12}}[k] \ \cdots \ y_{h_{1N}}[k]]^\top \in \mathbb{R}^{N_m n},
  
  u_h[k] = [u_{h_{11}}[k] \ u_{h_{12}}[k] \ \cdots \ u_{h_{1N}}[k]]^\top \in \mathbb{R}^{N_m n},
  
  W_h[k] = \begin{bmatrix}
  \vdots & \cdots & \vdots \\
  w_{h_{11}}[k] & \cdots & w_{h_{1N}}[k] \\
  \vdots & \cdots & \vdots \\
  w_{h_{N1}}[k] & \cdots & w_{h_{NN}}[k]
  \end{bmatrix} \in \mathbb{R}^{N m_n \times N}.
\end{equation}

Here we state the following assumption for the
Employing a state coordinate transformation

\[ \xi_1[k] = \begin{bmatrix} \frac{1}{N} \otimes I_n \end{bmatrix} x[k], \]
\[ \xi_2[k] = \begin{bmatrix} \frac{1}{\sqrt{N}} \otimes I_n \end{bmatrix} x[k], \]
\[ x[k] = \begin{bmatrix} S \otimes I_n & \frac{1}{\sqrt{N}} \otimes I_n \end{bmatrix} \begin{bmatrix} \xi_1[k] \\ \xi_2[k] \end{bmatrix}, \]

with an orthogonal complement \( S \in \mathbb{R}^{N \times (N-1)} \) of \( \frac{1}{N} \sqrt{N} \) and Assumption 2, we obtain

\[ \xi_1[k+1] = (I_{N-1} \otimes A) - r[k] \sum_{h=1}^{M} (L_h[k] \otimes B_h R_h C_h) \xi_1[k], \]
\[ \xi_2[k+1] = A \xi_2[k] + r[k] \sum_{h=1}^{M} B_h R_h \bar{w}_h[k], \]

where

\[ \bar{w}_h[k] = \begin{bmatrix} (S \otimes I_m) \end{bmatrix} W_h[k] 1_N \in \mathbb{R}^{(N-1)m_1}, \]
\[ \bar{w}_h[k] = \begin{bmatrix} \frac{1}{\sqrt{N}} \otimes I_m \end{bmatrix} W_h[k] 1_N \in \mathbb{R}^{m_1}. \]

Note that \( \bar{w}_h[k] \) and \( \bar{w}_h[k] \) satisfy

\[ \mathbb{E}[\bar{w}_h[k]] = 0, \quad \text{Cov}[\bar{w}_h[k]] \leq N (I_{N-1} \otimes V_h), \]
\[ E[\bar{w}_h[k]] = 0, \quad \text{Cov}[\bar{w}_h[k]] \leq NV_h \]

for any \( h \in H \).

We now define the average and the deviation of the states of all agents at \( k \) as

\[ \bar{x}[k] = \left( \frac{1}{N} \otimes I_n \right) x[k], \]
\[ \bar{x}[k] = x[k] - (1_N \otimes I_n) \bar{x}[k] \]
\[ = \left( I_N - \frac{1}{N} 1_N \otimes I_n \right) x[k]. \]

Then we see that

\[ \bar{x}[k] = \frac{1}{\sqrt{N}} \xi_2[k], \]
\[ \bar{x}[k] = (S \otimes I_n) \xi_1[k], \]

which implies that \( ||\bar{x}[k]|| = ||\xi_1[k]|| \). That is, we can consider the convergence of \( \bar{x}[k] \) as that of \( \xi_1[k] \).

3. Stochastic Consensus

3.1 Undirected Graph Topology Case

In this section, we suppose that each graph \( G_h[k] \) is undirected for any \( k \). Here we state the following assumption.

**Assumption 3** There exist \( \lambda_a \in \mathbb{R} \) and \( \lambda_b \in \mathbb{R} \) such that

\[ 0 < \lambda_a I_N - 1 \leq \sum_{h=1}^{M} S^T L_h[k] S \leq \lambda_b I_{N-1} \]

for all \( k \).

This assumption means that the multi-graph \( G[k] \) must be connected for any \( k \).

The following lemma is the key tool for the proof.

**Lemma 1** Let Assumptions 1 and 3 hold. Then the inequality

\[ \left\| (I_{N-1} \otimes A) - \frac{1}{\lambda_b} \sum_{h=1}^{M} (S^T L_h[k] S \otimes B_h R_h C_h) \right\| \leq 1 - \frac{\eta \lambda_a}{\lambda_b} \]

holds for all \( k \).

(Proof) From the inequality (2), and the lower bound of the inequality (7), we have

\[ (I_{N-1} \otimes A) - \frac{1}{\lambda_b} \sum_{h=1}^{M} (S^T L_h[k] S \otimes B_h R_h C_h) \]
\[ \leq (I_{N-1} \otimes A) \]
\[ - \frac{1}{\lambda_b} \sum_{h=1}^{M} (S^T L_h[k] S \otimes (A - (1 - \eta I_n))) \]
\[ \leq (I_{N-1} \otimes A) - \frac{\lambda_a}{\lambda_b} (I_{N-1} \otimes A - (1 - \eta) I_n) \]
\[ = \left( 1 - \frac{\lambda_a}{\lambda_b} \right) (I_{N-1} \otimes A) + \frac{\lambda_a}{\lambda_b} (1 - \eta) I_{N-1} \]
\[ \leq \left( 1 - \frac{\lambda_a}{\lambda_b} \right) I_{N-1} + \frac{\lambda_a}{\lambda_b} (1 - \eta) I_{N-1} \]
\[ = \left( 1 - \frac{\eta \lambda_a}{\lambda_b} \right) I_{N-1} \]

for all \( h \in H \). Similarly, using the inequality (2) and the upper bound of the inequality (7), we also have
\[(I_{N-1} \otimes A) - \frac{1}{\lambda h} \sum_{h=1}^{M} (S^T L_{h} [k] S \otimes B_{h} R_{h} C_{h})\]

\[\geq (I_{N-1} \otimes A) - \frac{1}{\lambda h} \sum_{h=1}^{M} (S^T L_{h} [k] S \otimes A)\]

\[\geq (I_{N-1} \otimes A) - (I_{N-1} \otimes A) = 0\]

for all \(h \in \mathcal{H}\). Thus we have Lemma 1. \(\square\)

Now, let us select the communication gain as

\[r[k] = \frac{1}{\eta \lambda_h} (k_0 + k), \quad k_0 \geq \frac{\lambda_h}{\eta \lambda_a} - 1, \quad (8)\]

where \(k_0 \in \mathbb{N}\). Then we have the following result.

**Theorem 1** Let Assumptions 1 and 3 hold. For given constants \(\alpha \in (0, \infty), \beta \in (0, \infty), \) and \(\gamma \in (0, 1), \) select \(k_f \in \mathbb{N}\) which satisfies

\[k_f \geq \max(\tau_1, \tau_2), \quad \tau_1 = \left(\frac{1}{\alpha} - 1\right) k_0 + 1,\]

\[\tau_2 = \frac{N(N - 1)}{\beta^2 \gamma^2 \lambda_a^2} \sum_{h=1}^{M} \text{Tr}(V_h) - k_0 + 1.\]

Then the deviation \(\tilde{x}(k_f)\) satisfies

\[P(\|\tilde{x}(k_f)\| \leq \alpha \|\tilde{x}[1]\| + \beta) \geq 1 - \gamma \quad (9)\]

for any initial state \(x[1]\). Furthermore, for each \(k \in \mathbb{N}\), the average \(\bar{x}[k]\) satisfies

\[E[\|\bar{x}[k]\|] = A^{k-1} \bar{x}[1],\]

\[E[\|\|\bar{x}[k] - E[\bar{x}[k]]\|\|^2] = \text{Tr}(\text{Cov}[\bar{x}[k]])\]

\[\leq \frac{n^2}{6 \eta^2 \lambda_a^2} \sum_{h=1}^{M} \text{Tr}(V_h)\]

for any initial state \(x[1]\).

(Proof) Let us introduce \(\Gamma(k)\) and \(\Phi(k, \ell)\) as

\[\Gamma(k) = (I_{N-1} \otimes A) - r[k] \sum_{h=1}^{M} (S^T L_{h} [k] S \otimes B_{h} R_{h} C_{h}),\]

\[\Phi(k, \ell) = \begin{cases} \Gamma(k-1) \Gamma(k-2) \cdots \Gamma(\ell) & \text{if } k > \ell \\ I_{(N-1)n} & \text{otherwise} \end{cases}\]

where Lemma 1 says that \(\|\Gamma(k)\| \leq (k_0 + k - 1)/(k_0 + k)\) and thus \(\|\Phi(k, \ell + 1)\| \leq (k_0 + \ell)/(k_0 + k - 1)\). \[5\]

Then \(\xi_1[k]\) can be expressed as

\[\xi_1[k] = \Phi(k, 1) \xi_1[1] + \sum_{\ell=1}^{k-1} r[\ell] \Phi(k, \ell + 1) \sum_{h=1}^{M} (I_{N-1} \otimes B_{h} R_{h} \bar{W}_{h1}[\ell]).\]

Since \(E[\xi_1[k]] = \Phi(k, 1) \xi_1[1]\), we have

\[\|E[\xi_1[k]]\| \leq \|\Phi(k, 1)\| \|\xi_1[1]\|\]

\[\leq \frac{k_0}{k_0 + k - 1} \|\xi_1[1]\|\]

\[\leq \frac{k_0}{k_0 + \tau_1 - 1} \|\xi_1[1]\| = \alpha \|\xi_1[1]\|\]

if \(k \geq \tau_1\). We also have

\[\text{Tr}(\text{Cov}[\xi_1[k]])\]

\[= \text{Tr} \left( \sum_{h=1}^{M} (I_{N-1} \otimes B_{h} R_{h} \bar{W}_{h1}[\ell]) \text{Tr}(\Phi(k, \ell + 1)^{T}) \right)\]

\[\leq N \cdot \text{Tr} \left( \sum_{h=1}^{M} \sum_{\ell=1}^{k-1} r[\ell] \|\Phi(k, \ell + 1)\|^2 \sum_{h=1}^{M} \text{Tr}(V_h) \right)\]

\[\leq N(N - 1) \sum_{\ell=1}^{k-1} r[\ell] \|\Phi(k, \ell + 1)\|^2 \sum_{h=1}^{M} \text{Tr}(V_h)\]

\[\leq N(N - 1) \sum_{\ell=1}^{k-1} \frac{1}{\eta^2 \lambda_a^2} \frac{\lambda_a}{(k_0 + k - 1)^2} \sum_{h=1}^{M} \text{Tr}(V_h)\]

\[\leq N(N - 1) \frac{1}{\eta^2 \lambda_a^2} \frac{k_0}{k_0 + k - 1} \sum_{h=1}^{M} \text{Tr}(V_h) \leq \beta^2 \gamma\]

if \(k \geq \tau_2\), where we used \(\|B_{h} R_{h} \bar{W}_{h1}\|^2 \leq 1\) for all \(h \in \mathcal{H}\) because \(B_{h} R_{h} R_{h} B_{h}^\top \leq B_{h} R_{h} B_{h}^\top \leq A \leq I_{n}\). Since Markov’s inequality says that

\[P(\|\xi_1[k]\| \leq \|E[\xi_1[k]]\| + \sqrt{\frac{\text{Tr}(\text{Cov}[\xi_1[k]])}{\gamma}}) \geq 1 - \gamma\]

holds, we see that the first part of Theorem 1 has been proved.

Furthermore, for each \(k \in \mathbb{N}\), we have

\[\xi_2[k] = A^{k-1} \xi_2[1] + \sum_{\ell=1}^{k-1} r[\ell] A^{k-1-\ell} \sum_{h=1}^{M} B_{h} R_{h} \bar{W}_{h2}[\ell].\]

Since \(E[\bar{W}_{h2}[k]] = 0\), we obtain

\[E[\bar{x}[k]] = \frac{1}{\sqrt{N}} E[\xi_2[k]] = \frac{1}{\sqrt{N}} A^{k-1} \xi_2[1] = A^{k-1} \bar{x}[1].\]

We also obtain

\[\text{Tr}(\text{Cov}[\bar{x}[k]])\]

\[= \frac{1}{N} \text{Tr} \left( \sum_{h=1}^{M} B_{h} R_{h} \bar{W}_{h2}[\ell] \right) \text{Tr}(A^{k-1-\ell}) \]

\[\leq \text{Tr} \left( \sum_{h=1}^{M} B_{h} R_{h} \bar{W}_{h2}[\ell] \cdot \sum_{h=1}^{M} B_{h} R_{h} B_{h}^\top (A^{k-1-\ell})^\top \right)\]
\[ \begin{align*}
\leq & \sum_{\ell=1}^{k-1} r^{2}[\ell] \| A^{k-1-\ell} \| \sum_{h=1}^{M} \| B_{h} R_{h} \|^{2} \text{Tr}(V_{h}) \\
\leq & \sum_{\ell=1}^{k-1} \frac{1}{\eta^{2} \lambda_{\eta}^{2}(k_{0} + \ell)} \sum_{h=1}^{M} \text{Tr}(V_{h}) \\
\leq & \frac{\pi^{2}}{6\eta^{2} \lambda_{\eta}^{2}} \sum_{h=1}^{M} \text{Tr}(V_{h}),
\end{align*}\]

which concludes the second part of Theorem 1. □

**Theorem 1** gives a stopping rule of the stochastic averaging consensus over a noisy undirected network. If we stop the iteration at \( k_{f} \), the probability such that the deviation \( ||\hat{x}[k_{f}]|| \) is less than or equal to a desired accuracy is with \( 1 - \gamma \), where the desired accuracy is defined by \( \alpha ||\hat{x}[1]|| + \beta \). The topology of the given network is significantly related to the stopping rule since the stopping criteria \( k_{f} \) is calculated by \( \lambda_{a} \) and \( \lambda_{b} \) of (7). Also, Theorem 1 includes the existing ones \([4]\) and \([5]\) as special cases if the agents are first order systems or SISO symmetric systems.

### 3.2 Directed Graph Topology Case

In this section, we suppose that each graph \( G_{h}[k] \) is directed for any \( k \). Here we state the following assumption.

**Assumption 4** Let a multi-graph \( \tilde{G}[k] \) is balanced for all \( k \). Then, there exist \( \lambda_{a} \in \mathbb{R} \) and \( \sigma_{a} \in \mathbb{R} \) such that

\[ 0 < \lambda_{a} I - \delta \leq \sum_{h=1}^{M} S^{T} (L_{h}[k] + L_{h}^{T}[k]) S / 2, \tag{10} \]

\[ M \sum_{h=1}^{M} S^{T} L_{h}[k] S S^{T} L_{h}[k] S \leq \sigma_{a}^{2} I_{N-1} \tag{11} \]

for all \( k \).

This assumption indicates that the balanced multi-graph \( \tilde{G}[k] \) must be weakly connected for any \( k \). The upper bound \( \sigma_{a}^{2} \) always exists.

We further assume the following.

**Assumption 5** The matrix \( A \) is invertible, i.e.,

\[ \det A \neq 0. \tag{12} \]

The following lemma is the key tool for the proof.

**Lemma 2** Let Assumptions 1, 2, 4, and 5 hold. Then the inequality

\[ \| (I_{N-1} \otimes A) - \delta \sum_{h=1}^{M} (\hat{L}_{h}[k] \otimes E_{h}) \|
\leq 1 - \frac{(\eta \lambda_{a})^{2}}{2\sigma_{a}^{2}} \]

holds for all \( k \).

(Proof) For simplicity, we introduce \( \hat{L}_{h}[k] = S^{T} L_{h}[k] S, \ E_{h} = B_{h} R_{h} C_{h} \), and \( \delta = \eta \lambda_{a} / \sigma_{a}^{2} \). We will prove that

\[ \| (I_{N-1} \otimes A) - \delta \sum_{h=1}^{M} (\hat{L}_{h}[k] \otimes E_{h}) \| \leq 1 - \frac{\delta \eta \lambda_{a}}{2}. \]

From (2) and (12), we have

\[ \| (I_{N-1} \otimes A) - \delta \sum_{h=1}^{M} (\hat{L}_{h}[k] \otimes E_{h}) \|
\leq \| (I_{N-1} \otimes A^{1/2}) \|
\leq \| (I_{N-1} \otimes A^{1/2}) - \delta \sum_{h=1}^{M} (\hat{L}_{h}[k] \otimes A^{-1/2} E_{h}) \|
\leq \| (I_{N-1} \otimes A^{1/2}) - \delta \sum_{h=1}^{M} (\hat{L}_{h}[k] \otimes A^{-1/2} E_{h}) \|
\leq \lambda_{\max} \left( (I_{N-1} \otimes A) - \delta \sum_{h=1}^{M} ((\hat{L}_{h}[k] + \hat{L}_{h}^{T}[k]) \otimes E_{h}) + \delta^{2} \sum_{h=1}^{M} (\hat{L}_{h}[k] \otimes E_{h}^{T} A^{-1/2}) \right), \tag{13} \]

where \( \lambda_{\max}(\cdot) \) is the largest eigenvalue of a matrix.

From (2) and (10), the sum of the first and second terms in (13) can be bounded by

\[ \| (I_{N-1} \otimes A) - \delta \sum_{h=1}^{M} (\hat{L}_{h}[k] + \hat{L}_{h}^{T}[k]) \otimes E_{h} \|
\leq \| (I_{N-1} \otimes A) \|
\leq \| (I_{N-1} \otimes A^{1/2}) \|
\leq \| (I_{N-1} \otimes A^{1/2}) - \delta \sum_{h=1}^{M} (\hat{L}_{h}[k] \otimes A^{-1/2} E_{h}) \|
\leq \lambda_{\max} \left( (I_{N-1} \otimes A) - \delta \sum_{h=1}^{M} ((\hat{L}_{h}[k] + \hat{L}_{h}^{T}[k]) \otimes (A - (1 - \eta) I_{N}^{-1})) \right)
\leq (1 - 2\delta \lambda_{a} (I_{N-1} \otimes A + 2\delta \lambda_{a} (1 - \eta) I_{N-1}))
\leq (1 - 2\delta \lambda_{a} (1 - \eta) I_{(N-1)n})
\leq \lambda_{\max} \left( (I_{N-1} \otimes A) - \delta \sum_{h=1}^{M} (\hat{L}_{h}[k] \otimes E_{h} A^{-1/2}) \right), \tag{13} \]

This concludes the second part of Theorem 1. □
from (2).
Applying these bounds to (13), we see that
\[
\left\| (I_{N-1} \otimes A) - \delta \sum_{h=1}^{M} (L_h[k] \otimes E_h) \right\|^2 \\
\leq 1 - 2\delta \eta \lambda_a + \delta^2 \sigma_a^2 = 1 - \delta \eta \lambda_a \\
\leq \left(1 - \frac{\delta \eta \lambda_a}{2}\right)^2,
\]
which establishes Lemma 2. \(\square\)

Now, let us select the communication gain as
\[
r[k] = \frac{2}{\eta \lambda_a (k_0 + k)}, \quad k_0 \geq \frac{2\sigma_a^2}{\eta^2 \lambda_a^2} - 1 \tag{14}
\]
where \(k_0 \in \mathbb{N}\). Then we have the following result.

**Theorem 2** Let Assumptions 1, 2, 4, and 5 hold. For given constants \(c \in (0, \infty)\), \(\beta \in (0, \infty)\), and \(\gamma \in (0, 1)\), select \(k_f \in \mathbb{N}\) which satisfies
\[
k_f \geq \max\{\tau_1, \tau_2\},
\]
\[
\tau_1 = \left(1 - \frac{1}{\alpha} \right) k_0 + 1, \quad \tau_2 = 4N(N-1) k_f \sum_{h=1}^{M} \text{Tr}(V_h) - k_0 + 1.
\]
Then the \(k_f\)-th deviation \(\bar{x}(k_f)\) satisfies
\[
P(\|\bar{x}(k_f)\| \leq \alpha \|\bar{x}(1)\| + \beta) \geq 1 - \gamma \tag{15}
\]
for any initial state \(x[1]\). Furthermore, the average \(\bar{x}[k]\) satisfies
\[
E[\bar{x}[k]] = A^{k-1} \bar{x}[1], \\
E[\|\bar{x}[k] - E[\bar{x}[k]]\|^2] = \text{Tr}(\text{Cov}[\bar{x}[k]]) \\
\leq \frac{2\sigma^2}{3\eta^2 \lambda_a^2} \sum_{h=1}^{M} \text{Tr}(V_h)
\]
for any initial state \(x[1]\) and \(k \in \mathbb{N}\).

(Proof) The theorem can be proved in a similar manner by the proof of Theorem 1 in the previous section with the communication gain (14) and Lemma 2 instead of (8) and Lemma 1. \(\square\)

**Theorem 2** gives a stopping rule of the stochastic averaging consensus over a noisy directed networks, which is similar to Theorem 1. In fact, Theorem 2 includes the existing ones [4] and [5] as special cases.

4. Numerical Example

4.1 Undirected Graph Case

Let us consider a multi-agent system with \(M = 2\) and \(N = 4\). We set \(m_1 = 2, m_2 = 1, n = 5,\) and
\[
A = \begin{bmatrix}
 0.667 & 0.071 & -0.132 & -0.007 & 0.068 \\
 0.071 & 0.667 & 0.101 & -0.108 & 0.077 \\
 -0.132 & 0.101 & 0.667 & 0.034 & -0.169 \\
 -0.007 & -0.108 & 0.034 & 0.667 & -0.188 \\
 0.068 & 0.077 & -0.169 & -0.188 & 0.667
\end{bmatrix},
\]
\[
B_1 = C_1^T = \begin{bmatrix}
 0.480 & 0.002 \\
 0.710 & 0.010 \\
 0.010 & 0.790 \\
 0.000 & 0.000 \\
 0.000 & 0.000
\end{bmatrix}, \\
B_2 = C_2^T = \begin{bmatrix}
 0.950 & 0.020 \\
 0.020 & 0.900 \\
 0.000 & 0.001 \\
 0.000 & 0.000 \\
 0.000 & 0.003 \\
 0.792 & 0.000
\end{bmatrix},
\]
where the inequality (2) holds with \(0 < \eta \leq 0.096\). The initial state was chosen as
\[
x_1[1] = [1 2 3 4 5]^T, \\
x_2[1] = [6 7 8 9 10]^T, \\
x_3[1] = [11 12 13 14 15]^T, \\
x_4[1] = [16 17 18 19 20]^T.
\]
We used the following time-varying undirected graph Laplacians \(L_1[k]\) and \(L_2[k]\): if \(k\) is an odd number,
\[
L_1[k] = L_1, \quad L_2[k] = L_2,
\]
on otherwise
\[
L_1[k] = L_2, \quad L_2[k] = L_1,
\]
where
\[
L_1 = \begin{bmatrix}
 1 & 0 & 0 & -1 \\
 0 & 1 & -1 & 0 \\
 0 & -1 & 1 & 0 \\
 -1 & 0 & 0 & 1
\end{bmatrix}, \\
L_2 = \begin{bmatrix}
 1 & -1 & 0 & 0 \\
 0 & 1 & 1 & 0 \\
 0 & 0 & 1 & 1 \\
 0 & 0 & -1 & 1
\end{bmatrix}.
\]
Note that the lower bound \(\lambda_a\) and the upper bound \(\lambda_b\) of the matrix \(\sum_{h=1}^{M} L_h[k]\) are 2.0 and 4.0. The upper bound \(V_1\) of \(\text{Cov}[\bar{x}[1]]\) and \(V_2\) of \(\text{Cov}[\bar{x}[2]]\) were set as
\[
V_1 = \begin{bmatrix}
 1.000 & 0.300 \\
 0.300 & 0.500
\end{bmatrix}, \quad V_2 = 0.700.
\]
The other parameters were chosen as \(\eta = 0.096, \alpha = 0.002, \beta = 0.660\) and \(\gamma = 0.200\).

We executed 1,000 times with different noise sequences. Figs. 1, 2, and 3 show that the behavior of the state \(x[k]\), average \(\bar{x}[k]\), and the deviation \(\tilde{x}[k]\) at a certain trial, respectively. Then, the worst value of \(\|\bar{x}[9,981]\|\) in the trials was 0.103, i.e., the inequality \(\|\bar{x}[9,981]\| \leq 0.710\) is always satisfied for all trials. Theorem 1 also says that the expectation of the average converges to
\[
\lim_{k \to \infty} E[\bar{x}[k]] = \lim_{k \to \infty} A^{k-1} \bar{x}[1] \\
= \begin{bmatrix}
 1.112 & 0.757 & -1.339 & -1.486 & 1.919
\end{bmatrix}^T.
\]
According to Fig. 2, each element of the average moves around each element of the above vector when the number of iterations \(k\) is sufficiently large. These re-
4.2 Directed Graph Case

We used the same agent dynamics (1) including the initial state in Section 4.1. On the other hand, we used

$$L_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

for constructing the time-varying directed graph Laplacians via the same manner. Note that the lower bound $\lambda_a$ and the upper bound $\sigma_b$ are $\lambda_a = 1.0$ and $\sigma_b = 8.0$.

The upper bounds $V_1$ and $V_2$ were set as

$$V_1 = \begin{bmatrix} 0.500 \\ 0.200 \\ 0.200 \\ 0.100 \end{bmatrix}, \quad V_2 = 0.300.$$ 

The other parameters were chosen as $\eta = 0.096$, $\alpha = 0.025$, $\beta = 0.880$, and $\gamma = 0.200$.

When we set $k_0 = 1,750$, Theorem 2 says that $P(\|x[68,251]\| \leq 1.505) \geq 0.800$ holds. We executed 1,000 times with different noise sequences. Figs. 4, 5, and 6 show that the behavior of the state $x[k]$, average $\bar{x}[k]$, and the deviation $\tilde{x}[k]$ at a certain trial.

Then, the worst value of $\|\tilde{x}[68,251]\|$ in the trials was 0.0460, i.e., the inequality $\|\tilde{x}[68,251]\| \leq 1.505$ is always satisfied for all trials, which is consistent with Theorem 2.

5. Conclusions

We have proposed the stopping rule for the multi-agent system over noisy networks of MIMO multi-agent systems with multi-channels. The theorem establishes the relation between the closeness of the consensus and the number of iterations explicitly with a probabilistic guarantee, which gives a stopping rule for the averaging consensus.

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