Abstract

Using the middle convolution functor $MC_\chi$ which was introduced by N. Katz we prove the existence of rigid local systems whose monodromy is dense in the simple algebraic group $G_2$. We derive the existence of motives for motivated cycles which have a motivic Galois group of type $G_2$. Granting Grothendieck’s standard conjectures, the existence of motives with motivic Galois group of type $G_2$ can be deduced, giving a partial answer to a question of Serre.

Introduction

The method of rigidity was first used by B. Riemann [27] in his study of Gauß’ hypergeometric differential equations $2F_1 = 2F_1(a, b, c)$: Consider the monodromy representation

$$\rho : \pi^\text{top}_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, s) \to \text{GL}(V_s)$$

which arises from analytic continuation of the vector space $V_s \simeq \mathbb{C}^2$ of local solutions of $2F_1$ at $s$, along paths in $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ which are based at $s$. Let $\gamma_i, i = 0, 1, \infty$, be simple loops around the points $0, 1, \infty$ (resp.) which are based at $s$. Then the monodromy representation $\rho$ is rigid in the sense that it is determined up to isomorphism by the Jordan canonical forms of $\rho(\gamma_i), i = 0, 1, \infty$.

One can translate the notion of rigidity into the language of local systems by saying that the local system $\mathcal{L}(2F_1)$ on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ which is given by the holomorphic solutions of $2F_1$ is rigid in the following sense: The monodromy representation of $\mathcal{L}(2F_1)$ (as defined in [8]) is determined up to isomorphism by the local monodromy representations at the missing points. This definition of rigidity extends in the obvious way to other local systems. Since Riemann’s work, the concept of a rigid local system has proven to be a very fruitful and has appeared in many different branches of mathematics and physics (see e.g. [5], [20]).
A key observation turned out to be the following: The local sections of the rank-two local system $\mathcal{L}(2F_1)$ can be written as linear combinations of convolutions $f * g$, where $f$ and $g$ are solutions of two related Fuchsian systems of rank one (see to [23], Introduction). By interpreting the convolution as higher direct image and using a transition to étale sheaves, N. Katz [23] proved a vast generalization of the above observation: Let $\mathcal{F}$ be any irreducible étale rigid local system on the punctured affine line in the sense specified below. Then $\mathcal{F}$ can be transformed to a rank one sheaf by a suitable iterative application of middle convolutions $\text{MC}_\chi$ and tensor products with rank one objects to it (loc. cit., Chap. 5). (The definition of the middle convolution $\text{MC}_\chi$ and its main properties are recalled in Section 1.) This yields Katz Existence Algorithm for irreducible rigid local systems, which tests, whether a given set of local representations comes from an irreducible and rigid local system (loc. cit., Chap. 6). This algorithm works simultaneously in the tame étale case and in the classical case of rigid local systems mentioned above (loc. cit., Section 6.2 and 6.3).

Let $\mathcal{F}$ be a lisse constructible $\bar{\mathbb{Q}}_\ell$-sheaf on a non-empty Zariski open subset $j : U \to \mathbb{P}_k^1$ which is tamely ramified at the missing points $\mathbb{P}_k^1 \setminus U$ (compare to [18]). Call $\mathcal{F}$ rigid, if the monodromy representation

$$\rho_\mathcal{F} : \pi_1^\text{tame}(U, \bar{\eta}) \to \text{GL}(\mathcal{F}_\eta)$$

of $\mathcal{F}$ is determined up to isomorphism by the conjugacy classes of the induced representations of tame inertia groups $I_s^\text{tame}$, where $s \in D := \mathbb{P}_k^1 \setminus U$. Sometimes, we call such a sheaf an étale rigid local system. If $\mathcal{F}$ is irreducible, then $\mathcal{F}$ is rigid if and only if the following formula holds:

$$\chi(\mathbb{P}_k^1, j_*\text{End}(\mathcal{F})) = (2 - \text{Card}(D))\text{rk}(\mathcal{F})^2 + \sum_{s \in D} \dim(\text{Centralizer}_{\text{GL}(\mathcal{F}_\eta)}(I_s^\text{tame})) = 2,$$

see [23], Chap. 2 and 6.

In preparation to Thm. 1 below, recall that there exist only finitely many exceptional simple linear algebraic groups over an algebraically closed field which are not isomorphic to a classical group, see [6]. The smallest of them is the group $G_2$ which admits an embedding into the group $\text{GL}_7$. Let us also fix some notation: Let $\mathbf{1}$ (resp. $-\mathbf{1}$, resp. $U(n)$) denote the trivial $\bar{\mathbb{Q}}_\ell$-valued representation (resp. the unique quadratic $\bar{\mathbb{Q}}_\ell$-valued character, resp. the standard indecomposable unipotent $\bar{\mathbb{Q}}_\ell$-valued representation of degree $n$) of the tame fundamental group $\pi_1^\text{tame}(\mathbb{G}_m, k)$, where $k$ is an algebraically closed field of characteristic $\neq 2, \ell$. The group $\pi_1^\text{tame}(\mathbb{G}_m, k)$ is isomorphic to the tame inertia group $I_s^\text{tame}$. This can be used to view representations of $I_s^\text{tame}$ as representations of $\pi_1^\text{tame}(\mathbb{G}_m, k)$. We prove the
following result:

**Theorem 1:** Let \( \ell \) be a prime number and let \( k \) be an algebraically closed field of characteristic \( \neq 2, \ell \). Let \( \varphi, \eta : \pi_1^{\text{tame}}(\mathbb{G}_m, k) \to \hat{\mathbb{Q}}_\ell^* \) be continuous characters such that

\[
\varphi, \eta, \varphi \eta, \eta \varphi^2, \varphi \eta^2 \neq -1.
\]

Then there exists an étale rigid local system \( \mathcal{H}(\varphi, \eta) \) of rank 7 on \( \mathbb{P}_k^1 \setminus \{0, 1, \infty\} \) whose monodromy group is Zariski dense in \( G_2(\hat{\mathbb{Q}}_\ell) \) and whose local monodromy is as follows:

- **The local monodromy at 0 is of type**
  \[
  -1 \oplus -1 \oplus -1 \oplus -1 \oplus 1 \oplus 1 \oplus 1.
  \]

- **The local monodromy at 1 is of type**
  \[
  \text{U}(2) \oplus \text{U}(2) \oplus \text{U}(3).
  \]

- **The local monodromy at } \infty \text{ is of the following form:**

| Local monodromy at } \infty | conditions on } \varphi \text{ and } \eta |
|-----------------------------|----------------------------------|
| U(7)                        | } \varphi = \eta = 1 \ |
| U(3, } \varphi \oplus \text{U}(3, } \varphi \oplus 1 | } \varphi = \eta \neq 1, \ \varphi^3 = 1 |
| U(2, } \varphi \oplus \text{U}(2, } \varphi \oplus \text{U}(1, } \varphi^2 \oplus \text{U}(1, } \varphi^2 \oplus 1 | } \varphi = \eta, \ \varphi^4 \neq 1 \neq \varphi^6 |
| U(2, } \varphi \oplus \text{U}(2, } \varphi \oplus \text{U}(3) | } \varphi = \bar{\eta}, \ \varphi^4 \neq 1 |
| } \varphi \oplus } \eta \oplus } \varphi } \eta \oplus } \bar{\varphi } \bar{\eta } \oplus } \bar{\varphi } \oplus 1 | } \varphi, } \eta, } \varphi } \eta, } \bar{\eta }, } \bar{\varphi }; } \bar{\varphi }, 1 |

Thm. 1 is proved in a slightly more general form in Thm. 1.3.1 below, where it is also proved that these are the only étale rigid local systems of rank 7 whose monodromy is dense in \( G_2 \). The proof of Thm. 1.3.1 relies heavily on Katz’ Existence Algorithm. Using the canonical homomorphism

\[
\pi_1^{\text{top}}(\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}) \to \pi_1^{\text{et}}(\mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\}),
\]

the existence of a rigid local system in the classical sense (corresponding to a representation of the topological fundamental group \( \pi_1^{\text{top}}(\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}) \)) whose monodromy group is Zariski dense in \( G_2 \) can easily be derived.
Suppose that one has given several local representations

\[
I^\text{tame}_s \longrightarrow \text{GL}(V) \quad \text{with} \quad s \in D \cup \{\infty\}
\]

which are assumed to come from an irreducible rigid local system \(\mathcal{F}\) on \(\mathbb{P}^1 \setminus D \cup \{\infty\}\). It is an empirical observation that the rigidity condition \(\chi(\mathbb{P}^1, j_\ast \text{End}(\mathcal{F})) = 2\) and the (necessary) irreducibility condition

\[
\chi(\mathbb{P}^1, j_\ast \mathcal{F}) = (1 - \text{Card}(D)) \text{rk}(\mathcal{F}) + \sum_{s \in D \cup \{\infty\}} \dim(\mathcal{F}_s^\text{tame}) \leq 0
\]

contradict each other in many cases. This is especially often the case if the Zariski closure of the monodromy group of \(\mathcal{F}\) is supposed to be small in the underlying general linear group. It is thus astonishing that the above mentioned irreducible and rigid \(G_2\)-sheaves exists at all. In fact, the local systems given by Thm. 1 and Thm. 1.3.1 are the first – and maybe the only – examples of tamely ramified rigid sheaves such that the Zariski closure of the monodromy group is an exceptional simple algebraic group.

We remark that in positive characteristic, wildly ramified lisse sheaves on \(\mathbb{G}_m\) with \(G_2\)-monodromy were previously found by N. Katz (see [21], [22]). Also, the conjugacy classes in \(G_2(\mathbb{F}_\ell)\) which correspond to the local monodromy of the above rigid local system \(\mathcal{H}(1,1)\) already occur in the work of Feit, Fong and Thompson on the inverse Galois problem (see [16], [31]); but only the situation in \(G_2(\mathbb{F}_\ell)\) was considered and the transition to rigid local systems was not made.

We then apply the above results to give a partial answer to a question of Serre on the existence of motives with exceptional motivic Galois groups. Recall that a motive in the Grothendieck sense is a triple \(M = (X, P, n)\), \(n \in \mathbb{Z}\), where \(X\) is a smooth projective variety over a field \(K\) and \(P\) is an idempotent correspondence, see e.g. [28]. Motives appear in many branches of mathematics (see [15]) and play a central role in the Langlands program [25]. Granting Grothendieck’s standard conjectures, the category of Grothendieck motives has the structure of a Tannakian category. Thus, by the Tannakian formalism, every Grothendieck motive \(M\) has conjecturally an algebraic group attached to it, called the motivic Galois group of \(M\) (see [28] and [12]).

An unconditional theory of motives for motivated cycles was developed by André [2], who formally adjoins a certain homological cycle (the Lefschetz involution) to the algebraic cycles in order to obtain the Tannakian category of motives for motivated cycles. Let us also mention the Tannakian category of motives for absolute Hodge cycles, introduced by Deligne [13] (for a definition of an absolute Hodge cycle, take a homological cycle which satisfies the most visible properties
of an algebraic cycle). In both categories, one has the notion of a motivic Galois group, given by the Tannakian formalism. It can be shown that any motivated cycle is an absolute Hodge cycle, so every motive for motivated cycles is also a motive for absolute Hodge cycles, see [2]. Since the category of motives for motivated cycles is the minimal extension of Grothendieck’s category which is unconditionally Tannakian, we will work and state our results mainly in this category.

The motivic Galois group is expected to encode essential properties of a motive. Many open conjectures on motivic Galois groups and related Galois representations are considered in the article of J.-P. Serre [30]. Under the general assumption of Grothendieck’s standard conjectures, Serre (loc. cit, 8.8) asks the following question “plus hasardeuse”: Do there exist motives whose motivic Galois group is an exceptional simple algebraic group of type $G_2$ (or $E_8$)? It follows from Deligne’s work on Shimura varieties that such motives cannot be submotives of abelian varieties or the motives parametrized by Shimura varieties, see [10]. Thus, motives with motivic Galois group of type $G_2$ or $E_8$ are presumably hard to construct.

There is the notion of a family of motives for motivated cycles, cf. [2] and Section 3.2. Using this, we prove the following result (see 3.3.1):

**Theorem 2:** There is a family of motives $M_s$ parametrized by $S = \mathbb{P}^1 \setminus \{0, 1, \infty\}$, such that for any $s \in S(\mathbb{Q})$ outside a thin set, the motive $M_s$ has a motivic Galois group of type $G_2$.

Since the complement of a thin subset of $\mathbb{Q}$ is infinite (see [29]), Theorem 2 implies the existence of infinitely many motives for motivated cycles whose motivic Galois group is of type $G_2$. A proof of Thm. 2 will be given in Section 3. It can be shown that under the assumption of the standard conjectures, the motives $M_s$ are Grothendieck motives with motivic Galois group of type $G_2$ (see Rem. 3.3.2). In this sense, we obtain a positive answer to Serre’s question in the $G_2$-case.

The method of the construction of the motives $M_s$ is the motivic interpretation of rigid local systems with quasi-unipotent local monodromy, introduced by N. Katz in [23], Chap. 8. It follows from Katz’ work that the sheaf $\mathcal{H}(1, 1)$ in Thm. 1 arises from the cohomology of a smooth affine morphism $\pi : \text{Hyp} \to \mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$ which occurs during the convolution process (see Thm. 2.4.1 and Cor. 2.4.2). Then a desingularization of the relative projective closure of Hyp and the work of Y. André [2] on families of motives imply that a suitable compactification and specialization of $\pi$ gives motives over $\mathbb{Q}$ whose motivic Galois groups are of type $G_2$. 

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In the Appendix to this paper, which is a joint work with N. Katz, the Galois representations associated with the above motives $M_s$ are studied. It follows from Thm. 1 of the Appendix., that for two coprime integers $a$ and $b$ which each have at least one odd prime divisor, the motive $M_s, s = 1 + \frac{a}{b}$, gives rise to $\ell$-adic Galois representations whose image is Zariski dense in the group $G_2$. This implies that the motivic Galois group of $M_s$ is of type $G_2$. By letting $a$ and $b$ vary among the squarefree coprime odd integers $> 2$, one obtains infinitely many non-isomorphic motives $M_{1+\frac{a}{b}}$ with motivic Galois group of type $G_2$ (Appendix, Cor. 2 (ii)).

We remark that Gross and Savin [17] propose a completely different way to construct motives with motivic Galois group $G_2$ by looking at the cohomology of Shimura varieties of type $G_2$ with nontrivial coefficients. The connection between these approaches has yet to be explored. Due to an observation of Serre, at least the underlying Hodge types coincide.

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1 Middle convolution and $G_2$-local systems

Throughout the section we fix an algebraically closed field $k$ and a prime number $\ell \neq \text{char}(k)$.

1.1 The middle convolution Let $G$ be an algebraic group over $k$ and let $\pi : G \times G \to G$ the multiplication map. Let $D^b_c(G, \bar{\mathbb{Q}}_\ell)$ denote the bounded derived category of constructible $\bar{\mathbb{Q}}_\ell$-sheaves on $G$ (compare to [11], Section 1, and [23], Section 2.2). Given two objects $K, L \in D^b_c(G, \bar{\mathbb{Q}}_\ell)$, define their $!$-convolution as

$$K \ast_! L := R\pi_!(K \boxtimes L) \in D^b_c(G, \bar{\mathbb{Q}}_\ell)$$

and their $*$-convolution as

$$K \ast_* L := R\pi_*(K \boxtimes L) \in D^b_c(G, \bar{\mathbb{Q}}_\ell).$$
And element $K \in D^b_c(G, \overline{\mathbb{Q}}_\ell)$ is called a \textit{perverse sheaf} (compare to [1]), if $K$ and its dual $D(K)$ satisfy
\[
\dim \left( \text{Supp}(H^i(K)) \right) \leq -i, \quad \text{resp.} \quad \dim \left( \text{Supp}(H^i(D(K))) \right) \leq -i.
\]
Suppose that $K$ is a perverse sheaf with the property that for any other perverse sheaf $L$ on $G$, the sheaves $K \ast_1 L$ and $K \ast_* L$ are again perverse. Then one can define the \textit{middle convolution} $K \ast_{\text{mid}} L$ of $K$ and $L$ as the image of $L \ast_1 K$ in $L \ast_* K$ under the “forget supports map” in the abelian category of perverse sheaves.

Let us now consider the situation, where $G = \mathbb{A}^1_k$: For any nontrivial continuous character $\chi : \pi_{\text{tame}}^1(G_m, k) \rightarrow \overline{\mathbb{Q}}_\ell \times \mathbb{L}$, let $L_\chi$ denote the corresponding lisse sheaf of rank one on $G_m, k$. Let $j : G_m \rightarrow \mathbb{A}^1$ denote the inclusion. From $j_* \mathcal{L}_\chi$ one obtains a perverse sheaf $j_* \mathcal{L}_\chi[1]$ on $\mathbb{A}^1$ by placing the sheaf in degree $-1$. Since $\ast$-convolution (resp. $\ast_1$-convolution) with $j_* \mathcal{L}_\chi[1]$ preserves perversity (see [23], Chap. 2), the middle convolution $K \ast_{\text{mid}} j_* \mathcal{L}_\chi[1]$ is defined for any perverse sheaf $K$ on $\mathbb{A}^1$.

The following notation will be used below: For any scheme $W$ and any map $f : W \rightarrow G_m$, define
\[
(1.1.1) \quad \mathcal{L}_{\chi(f)} := f^* \mathcal{L}_\chi.
\]
The identity character will be denoted by $1$ and $-1$ denotes the unique quadratic character of $\pi_{\text{tame}}^1(G_m)$. The \textit{inverse character} of $\chi$ will be denoted by $\overline{\chi}$ (by definition, $\chi \otimes \overline{\chi} = 1$).

The following category will be of importance below:

\textbf{1.1.1 Definition.} Let $\mathcal{T}_\ell = \mathcal{T}_\ell(k)$ denote the full subcategory of constructible $\overline{\mathbb{Q}}_\ell$-sheaves $\mathcal{F}$ on $\mathbb{A}^1_k$ which satisfy the following conditions:

- There exists a dense open subset $j : U \rightarrow \mathbb{A}^1$ such that $j^* \mathcal{F}$ is lisse and irreducible on $U$, and such that $\mathcal{F} \simeq j_* j^* \mathcal{F}$.

- The lisse sheaf $j^* \mathcal{F}$ is tamely ramified at every point of $\mathbb{P}^1 \setminus U$.

- There are at least two distinct points of $\mathbb{A}^1$ at which $\mathcal{F}$ fails to be lisse.

The properties of $\mathcal{T}_\ell$ imply that $\mathcal{F}[1] \ast_{\text{mid}} \mathcal{L}_\chi[1]$ is a single sheaf placed in degree $-1$ ([23], Chap. 5), leading to the \textit{middle convolution functor}
\[
\text{MC}_\chi : \mathcal{T}_\ell \rightarrow \mathcal{T}_\ell, \quad \mathcal{F} \mapsto (\mathcal{F}[1] \ast_{\text{mid}} \mathcal{L}_\chi[1])[-1],
\]
An important property of $MC_{\chi}$ is the following:

\[(1.1.2) \quad MC_{\chi} \circ MC_{\rho} = MC_{\chi \rho} \quad \text{if} \quad \chi \rho \neq 1, \quad \text{and} \quad MC_{\chi} \circ MC_{\chi} = \text{Id.}\]

Let $U \subseteq \mathbb{A}^1$ be an open subset of $\mathbb{A}^1$ such that $\mathcal{F}|_U$ is lisse and let $\iota : U \to \mathbb{P}^1$ be the canonical inclusion. The sheaf $\mathcal{F} \in \mathcal{T}_\ell$ is called cohomologically rigid, if the index of rigidity

\[\operatorname{rig}(\mathcal{F}) = \chi(\mathbb{P}^1, \iota_*(\text{End}(\mathcal{F}|_U)))\]

is equal to 2. Then $MC_{\chi}$ carries rigid elements in $\mathcal{T}_\ell$ to rigid elements in $\mathcal{T}_\ell$ by the following result, see [23], 6.0.17:

\[(1.1.3) \quad \operatorname{rig}(\mathcal{F}) = \operatorname{rig}(MC_{\chi}(\mathcal{F})).\]

1.2 The numerology of the middle convolution

We recall the effect of the middle convolution on the Jordan canonical forms of the local monodromy, given by Katz in [23], Chap. 6:

Let $\mathcal{F} \in \mathcal{T}_\ell$ and let $j : U \to \mathbb{A}^1_x$ denote an open subset such that $j^* \mathcal{F}$ is lisse. Let $D := \mathbb{A}^1 \setminus U$. Then, for any point $s \in D \cup \{\infty\} = \mathbb{P}^1 \setminus U$, the sheaf $\mathcal{F}$ gives rise to the local monodromy representation $\mathcal{F}(s)$ of the tame inertia subgroup $I(s)^{\text{tame}}$ (of the absolute Galois group of the generic point of $\mathbb{A}^1$) at $s$. The representation $\mathcal{F}(s)$ decomposes as follows as a direct sum (character)$\otimes$(unipotent representation), where the sum is over the set of continuous $\mathbb{Q}_\ell$-characters $\rho$ of $\pi_1^{\text{tame}}(\mathbb{G}_{m,k}) \simeq I(s)^{\text{tame}}$:

$$\mathcal{F}(s) = \bigoplus_{\rho} \mathcal{L}_{\rho(x-s)} \otimes \text{Unip}(s, \rho, \mathcal{F}) \quad \text{for all} \quad s \in D$$

and

$$\mathcal{F}(\infty) = \bigoplus_{\rho} \mathcal{L}_{\rho(x)} \otimes \text{Unip}(\infty, \rho, \mathcal{F}).$$

Here, the following convention is used: If one starts with a rank one object $\mathcal{F}$, which at $s \in D$ gives locally rise to a character $\chi_s$ of $\pi_1^{\text{tame}}(\mathbb{G}_m)$ then

$$\chi_\infty = \prod_{s \in D} \chi_s.$$

For $s \in D \cup \{\infty\}$, write $\text{Unip}(s, \rho, \mathcal{F})$ as a direct sum of Jordan blocks of lengths $\{n_i(s, \rho, \mathcal{F})\}_i$. This leads to a decreasing sequence of non-negative integers

$$e_1(s, \rho, \mathcal{F}) \geq e_2(s, \rho, \mathcal{F}) \geq \ldots \geq e_k(s, \rho, \mathcal{F}) = 0 \quad \text{for} \quad k \gg 0,$$
where the number \( e_j(s, \rho, \mathcal{F}) \) is defined to be the number of Jordan blocks in \( \text{Unip}(s, \rho, \mathcal{F}) \) whose length is \( \geq j \).

1.2.1 Proposition. Let \( \mathcal{F} \in \mathcal{T}_\ell \) be of generic rank \( n \). Then the following holds:

(i) \[
\text{rk}(\text{MC}_\chi(\mathcal{F})) = \sum_{s \in D} \text{rk}(\mathcal{F}(s)/(\mathcal{F}(s)^{(s)})) - \text{rk}((\mathcal{F}(\infty) \otimes \mathcal{L}_\chi)^{(\infty)})
\]
\[
= \sum_{s \in D} (n - e_1(s, 1, \mathcal{F})) - e_1(\infty, \chi, \mathcal{F}).
\]

(ii) For \( s \in D \) and \( i \geq 1 \), the following holds:

\[
e_i(s, \rho \chi, \text{MC}_\chi(\mathcal{F})) = e_i(s, \rho, \mathcal{F}) \quad \text{if} \quad \rho \neq 1 \quad \text{and} \quad \rho \chi \neq 1,
\]
\[
e_{i+1}(s, 1, \text{MC}_\chi(\mathcal{F})) = e_i(s, \chi, \mathcal{F}),
\]
\[
e_i(s, \chi, \text{MC}_\chi(\mathcal{F})) = e_{i+1}(s, 1, \mathcal{F}).
\]

Moreover,
\[
e_1(s, 1, \text{MC}_\chi(\mathcal{F})) = \text{rk}(\text{MC}_\chi(\mathcal{F})) - n + e_1(s, 1, \mathcal{F}).
\]

(iii) For \( s = \infty \) and \( i \geq 1 \), the following holds:

\[
e_i(\infty, \rho \chi, \text{MC}_\chi(\mathcal{F})) = e_i(\infty, \rho, \mathcal{F}) \quad \text{if} \quad \rho \neq 1 \quad \text{and} \quad \rho \chi \neq 1,
\]
\[
e_{i+1}(\infty, \chi, \text{MC}_\chi(\mathcal{F})) = e_i(\infty, 1, \mathcal{F}),
\]
\[
e_i(\infty, 1, \text{MC}_\chi(\mathcal{F})) = e_{i+1}(\infty, \chi, \mathcal{F}).
\]

Moreover,
\[
e_1(\infty, \chi, \text{MC}_\chi(\mathcal{F})) = \sum_{s \in D} (\text{rk}(\mathcal{F}) - e_1(s, 1, \mathcal{F})) - \text{rk}(\mathcal{F}).
\]

Proof: Claim (i) is [23], Cor. 3.3.7. The first three equalities in (ii) are loc. cit., 6.0.13. The last equality in (ii) follows from loc. cit., 6.0.14. To deduce (iii), we argue as follows: From loc. cit., 3.3.6, and 6.0.5, for any \( \mathcal{F} \in \mathcal{T}_\ell \) there exists an \( I(\infty)^{\text{tame}} \)-representation \( M(\infty, \mathcal{F}) \) of rank \( \sum_{s \in D} (n - e_1(s, 1, \mathcal{F})) \) with the following properties:

\[
E_i(\infty, \rho, \mathcal{F}) = e_i(\infty, \rho, \mathcal{F}) \quad \text{if} \quad \rho \neq 1,
\]
\[
E_{i+1}(\infty, 1, \mathcal{F}) = e_i(\infty, 1, \mathcal{F}) \quad \text{for} \quad i \geq 1,
\]
\[
E_1(\infty, 1, \mathcal{F}) = \text{rk}(M(\infty, \mathcal{F})) - \text{rk}(\mathcal{F}).
\]
where the numbers $E_i(\infty, \rho, \mathcal{F})$ denote the invariants associated to $M(\infty, \mathcal{F})$ which are defined analogously as the invariants $e_i(s, \rho, \mathcal{F})$ for $\mathcal{F}(s)$. Moreover, by loc. cit., 6.0.11, the following holds:

$$E_i(\infty, \rho\chi, MC_{\chi}(\mathcal{F})) = E_i(\infty, \rho, \mathcal{F}), \quad \text{for all } i \geq 1 \text{ and } \rho.$$  

By combining the last equations, it follows that if $\rho\chi \neq 1$ and $\rho \neq 1$, then

$$e_i(\infty, \rho\chi, MC_{\chi}(\mathcal{F})) = E_i(\infty, \rho, \mathcal{F}) = e_i(\infty, \rho, \mathcal{F}).$$

If $\rho = 1$, since $\chi$ is nontrivial, the following holds

$$e_{i+1}(\infty, \chi, MC_{\chi}(\mathcal{F})) = E_{i+1}(\infty, \chi, MC_{\chi}(\mathcal{F})) = E_{i+1}(\infty, 1, \mathcal{F}) = e_i(\infty, \rho, \mathcal{F}).$$

Moreover,

$$e_i(\infty, 1, MC_{\chi}(\mathcal{F})) = E_{i+1}(\infty, 1, MC_{\chi}(\mathcal{F})) = E_{i+1}(\infty, \chi, \mathcal{F}) = e_{i+1}(\infty, \chi, \mathcal{F}),$$

since $\chi$ and thus $\chi$ are nontrivial. Finally,

$$e_1(\infty, \chi, MC_{\chi}(\mathcal{F})) = E_1(\infty, \chi, MC_{\chi}(\mathcal{F})) = E_1(\infty, 1, \mathcal{F}) = \text{rk}(M(\infty, \mathcal{F})) - \text{rk}(\mathcal{F}) = \sum_{s \in D} (\text{rk}(\mathcal{F}) - e_1(s, 1, \mathcal{F})) - \text{rk}(\mathcal{F}),$$

where the last equality follows from loc. cit., 6.0.6. 

Let $\mathcal{F} \in \mathcal{T}_l$ and let $\mathcal{L}$ be a middle extension sheaf on $\mathbb{A}^1$ (i.e., there exists an open subset $j : U \to \mathbb{A}^1$ such that $j^* \mathcal{L}$ is lisse and such that $\mathcal{L} \simeq j^* j^* \mathcal{L}$). Assume that $\mathcal{F}_{| U}$ is also lisse. Then the middle tensor product of $\mathcal{F}$ and $\mathcal{L}$ is defined as

$$\text{MT}_\mathcal{L}(\mathcal{F}) = j_*(\mathcal{F}_{| U} \otimes \mathcal{L}_{| U}),$$

compare to loc. cit., 5.1.9. Obviously, the generic rank of $\text{MT}_\mathcal{L}(\mathcal{F})$ is the same as the generic rank of $\mathcal{F}$. For any $s \in D \cup \{ \infty \}$, denote by $\chi_{s, \mathcal{L}}$ the unique character $\rho$ with $e_1(s, \rho, \mathcal{L}) = 1$. Then the following holds (loc. cit., 6.0.10):

$$(1.2.1) \quad e_i(s, \rho \chi_{s, \mathcal{L}}, \text{MT}_\mathcal{L}(\mathcal{F})) = e_i(s, \rho, \mathcal{F}).$$
1.3 Classification of irreducible rigid local systems with $G_2$-monodromy

In this section, we give a complete classification of rank 7 rigid sheaves $H \in \mathcal{T}_\ell$ whose associated monodromy group is Zariski dense in the group $G_2(\bar{\mathbb{Q}}_\ell)$.

Let us first collect the information on the conjugacy classes of the simple algebraic group $G_2$ which are needed below. In Table 1 below, we lists the possible Jordan canonical forms of elements of the group $G_2(\bar{\mathbb{Q}}_\ell) \leq \text{GL}_7(\bar{\mathbb{Q}}_\ell)$ together with the dimensions of the centralizers in the group $G_2$ and in the group $\text{GL}_7(\bar{\mathbb{Q}}_\ell)$. That the list exhausts all possible cases can be seen (using Jordan decomposition) from the fact that a semisimple element in $G_2(\bar{\mathbb{Q}}_\ell) \leq \text{GL}_7(\bar{\mathbb{Q}}_\ell)$ is of the form $\text{diag}(x, y, xy, 1, (xy)^{-1}, y^{-1}, x^{-1})$, and from the same arguments as in [26] for the classification of the unipotent classes. The computation of the centralizer dimension in $G_2(\bar{\mathbb{Q}}_\ell)$ follows using the same arguments as in [7], the centralizer dimension in the group $\text{GL}_7$ follows e.g. from [23], 3.1.15.

We use the following conventions: $E_n \in \bar{\mathbb{Q}}_{\ell}^{n \times n}$ denotes the identity matrix, $J(n)$ denotes a unipotent Jordan block of length $n$, $\epsilon \in \bar{\mathbb{Q}}_{\ell}^\times$ denotes a primitive 3-rd root of unity, and $i \in \bar{\mathbb{Q}}_{\ell}^\times$ denotes a primitive 4-th root of unity. Moreover, an expression like $(xJ(2), x^{-1}J(2), x^2, x^{-2}, 1)$ denotes a matrix in Jordan canonical form in $\text{GL}_7(\bar{\mathbb{Q}}_\ell)$ with one Jordan block of length 2 having eigenvalue $x$, one Jordan block of length 2 having eigenvalue $x^{-1}$, and three Jordan blocks of length 1 having eigenvalues $x^2, x^{-2}, 1$ (resp.).

| Jordan form                  | Centralizer dimension in $G_2$ | Centralizer dimension in $\text{GL}_7$ | Conditions |
|------------------------------|---------------------------------|---------------------------------------|------------|
| $E_7$                        | 14                              | 49                                    |            |
| $(J(2), J(2), E_3)$          | 8                               | 29                                    |            |
| $(J(3), J(2), J(2))$         | 6                               | 19                                    |            |
| $(J(3), J(3), 1)$            | 4                               | 17                                    |            |
| $J(7)$                       | 2                               | 7                                     |            |
| $(-E_4, E_3)$                | 6                               | 25                                    |            |
| $(-J(2), -J(2), E_3)$        | 4                               | 17                                    |            |
| $(-J(2), -J(2), J(3))$       | 4                               | 11                                    |            |
| $(-J(3), -1, J(3))$          | 2                               | 9                                     |            |
Jordan form | Centralizer dimension in \( G_2 \) | \( \text{GL}_7 \) | Conditions  
--- | --- | --- | ---  
\((\epsilon E_3, 1, \epsilon^{-1} E_3)\) | 8 | 19 |  
\((\epsilon J(2), \epsilon^{-1} J(2), \epsilon, \epsilon^{-1}, 1)\) | 4 | 11 |  
\((\epsilon J(3), \epsilon^{-1} J(3), 1)\) | 2 | 7 |  
\((i, i, -1, 1, i^{-1}, -1, -1)\) | 4 | 13 |  
\((i J(2), i^{-1} J(2), -1, -1, 1)\) | 2 | 9 |  
\((x, x, x^{-1}, x^{-1}, 1, 1, 1)\) | 4 | 17 | \(x^2 \neq 1\)  
\((x, x, x^{-2}, 1, x^{-1}, x^{-1}, x^{-2})\) | 4 | 11 | \(x^4 \neq 1 \neq x^3\)  
\((x, -1, -x, 1, -x^{-1}, -1, x^{-1})\) | 2 | 9 | \(x^4 \neq 1\)  
\((x J(2), x^{-1} J(2), x^2, x^{-2}, 1)\) | 2 | 7 | \(x^4 \neq 1\)  
\((x J(2), x^{-1} J(2), J(3))\) | 2 | 7 | \(x^2 \neq 1\)  
\((x, y, xy, 1, (xy)^{-1}, y^{-1}, x^{-1})\) | 2 | 7 | \(x, y, xy, 1, (xy)^{-1}, y^{-1}, x^{-1}\) pairwise different  

**Table 1:** The \( \text{GL}_7 \) conjugacy classes of \( G_2 \)  

We will also use the following notation in Thm. 1.3.1 below: Let \( U(i) \) denote the \( \mathbb{Q}_\ell \)-valued representation of \( \pi_1^\text{tame}(\mathbb{G}_m) \) which sends a generator of \( \pi_1^\text{tame}(\mathbb{G}_m) \) to the Jordan block \( J(i) \). For any character \( \chi \) of \( \pi_1^\text{tame}(\mathbb{G}_m) \) let  

\[
U(i, \chi) := \chi \otimes U(i),
\]

let  

\[-U(i) := -1 \otimes U(i)\]

\((-1\) denoting the unique quadratic character of \( \pi_1^\text{tame}(\mathbb{G}_m)\)), and let \( U(i, \chi)^j \) denote the \( j \)-fold direct sum of the representation \( U(i, \chi) \).  

**1.3.1 Theorem.** Let \( \ell \) be a prime number and let \( k \) be an algebraically closed field with \( \text{char}(k) \neq 2, \ell \). Then the following holds:
(i) Let \( \alpha_1, \alpha_2 \in \mathbb{A}^1(k) \) be two disjoint points and let \( \varphi, \eta : \pi_1^{\text{tame}}(\mathbb{G}_m, k) \to \overline{\mathbb{Q}}_\ell^\times \) be continuous characters such that

\[
\varphi, \eta, \varphi \eta^2, \eta \varphi^2, \varphi \eta \neq -1.
\]

Then there exists an irreducible cohomologically rigid sheaf \( \mathcal{H} = \mathcal{H}(\varphi, \eta) \in \mathcal{T}_\ell(k) \) of generic rank 7 whose local monodromy is as follows:

- The local monodromy at \( \alpha_1 \) is \( -1^4 \oplus 1^3 \).
- The local monodromy at \( \alpha_2 \) is \( U(2)^2 \oplus U(3) \).
- The local monodromy at \( \infty \) is of the following form:

| Local monodromy at \( \infty \) | conditions on \( \varphi \) and \( \eta \) |
|----------------------------------|------------------------------------------|
| \( U(7) \)                      | \( \varphi = \eta = 1 \)                |
| \( U(3, \varphi) \oplus U(3, \varphi^2) \oplus 1 \) | \( \varphi = \eta \neq 1, \quad \varphi^3 = 1 \) |
| \( U(2, \varphi) \oplus U(1, \varphi) \oplus U(2, \varphi^2) \oplus U(1, \varphi^2) \oplus 1 \) | \( \varphi = \eta, \quad \varphi^4 \neq 1 \neq \varphi^6 \) |
| \( U(2, \varphi) \oplus U(2, \varphi^2) \oplus U(3) \) | \( \varphi = \eta^2, \quad \varphi^4 \neq 1 \) |
| \( \varphi \oplus \eta \oplus \varphi \eta \oplus \varphi \eta^2 \oplus \eta \oplus \varphi \oplus 1 \) | \( \varphi, \eta, \varphi \eta, \varphi \eta^2, \eta, \varphi, 1 \) pairwise different |

Moreover, the restriction \( \mathcal{H}|_{\mathbb{A}^1(k) \setminus \{\alpha_1, \alpha_2\}} \) is lisse and the monodromy group associated to \( \mathcal{H} \) is a Zariski dense subgroup of the simple exceptional algebraic group \( G_2(\overline{\mathbb{Q}}_\ell) \).

(ii) Assume that \( \mathcal{H} \in \mathcal{T}_\ell \) is a cohomologically rigid \( \overline{\mathbb{Q}}_\ell \)-sheaf of generic rank 7 which fails to be lisse at \( \infty \) and such that the monodromy group associated to \( \mathcal{H} \) is Zariski dense in the group \( G_2(\overline{\mathbb{Q}}_\ell) \). Then \( \mathcal{H} \) fails to be lisse at exactly two disjoint points \( \alpha_1, \alpha_2 \in \mathbb{A}^1(k) \) and, up to a permutation of the points \( \alpha_1, \alpha_2, \infty \), the above list exhaust all the possible local monodromies of \( \mathcal{H} \).

**Proof:** Let us introduce the following notation: Let \( j : U := \mathbb{A}^1_k \setminus \{\alpha_1, \alpha_2\} \to \mathbb{P}^1 \) denote the tautological inclusion. Let \( x - \alpha_i, i = 1, 2 \), denote the morphism \( U \to \mathbb{G}_m \) which is induced by sending \( x \in U \) to \( x - \alpha_i \). For any pair of continuous characters \( \chi_1, \chi_2 : \pi_1^{\text{tame}}(\mathbb{G}_m) \to \overline{\mathbb{Q}}_\ell^\times \), let

\[
\mathcal{L}(\chi_1, \chi_2) := j_*(\mathcal{L}_{\chi_1(x-\alpha_1)} \otimes \mathcal{L}_{\chi_2(x-\alpha_2)})
\]

(using the notation of Formula 1.1.1). Let

\[
\mathcal{F}_1 = \mathcal{L}(-1, -\varphi \eta) \in \mathcal{T}_\ell.
\]

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Define inductively a sequence of sheaves $\mathcal{H}_0, \ldots, \mathcal{H}_6$ in $\mathcal{T}_\ell$, by setting

$$\mathcal{H}_0 := \mathcal{F}_1,$$
and
$$\mathcal{H}_i := M_{\mathcal{F}_{i+1}}(MC_{\rho_i}(\mathcal{H}_{i-1})), \quad \text{for} \quad i = 1, \ldots, 6,$$

where the $\mathcal{F}_i$ and $\rho_i$ are as follows:

- $\mathcal{F}_3 = \mathcal{F}_5 = \mathcal{F}_7 = L(-1, 1)$,
- $\mathcal{F}_2 = L(1, -\varphi)$,
- $\mathcal{F}_4 = L(1, -\varphi \eta)$,
- $\mathcal{F}_6 = L(1, -\varphi)$

and

- $\rho_1 := -\varphi \eta^2$,
- $\rho_2 := -\varphi \eta^2$,
- $\rho_3 := -\varphi \eta$,
- $\rho_4 := -\varphi \eta$,
- $\rho_5 := -\varphi$,
- $\rho_6 := -\varphi$.

We now distinguish 5 cases, which correspond to the different types of local monodromy at $\infty$ listed above:

**Case 1:** Let $\varphi = \eta = 1$. The following table lists the local monodromies of the sheaves $\mathcal{H}_0, \ldots, \mathcal{H}_6 = \mathcal{H}$ at the points $\alpha_1, \alpha_2, \infty$ (the proof is a direct computation, using Prop. 1.2.1 and Equation (1.2.1)):

|   | at $\alpha_1$ | at $\alpha_2$ | at $\infty$ |
|---|---------------|---------------|-------------|
| $\mathcal{H}_0$ | $-1$          | $-1$          | $1$         |
| $\mathcal{H}_1$ | $U(2)$        | $-U(2)$       | $U(2)$      |
| $\mathcal{H}_2$ | $-1^2 \oplus 1$ | $U(3)$       | $U(3)$      |
| $\mathcal{H}_3$ | $U(2)^2$      | $U(2) \oplus -1^2$ | $U(4)$      |
| $\mathcal{H}_4$ | $1^2 \oplus -1^3$ | $U(2)^2 \oplus -1$ | $U(5)$      |
| $\mathcal{H}_5$ | $U(2)^3$      | $-U(2) \oplus -1^2 \oplus 1^2$ | $U(6)$      |
| $\mathcal{H}_6$ | $-1^4 \oplus 1^3$ | $U(2)^2 \oplus U(3)$ | $U(7)$      |

By the results of Section 1.1 and Prop. 1.2.1, the sheaf $\mathcal{H} = \mathcal{H}_6$ is a cohomologically rigid irreducible sheaf of rank 7 in $\mathcal{T}_\ell$ which is lisse on the open subset $U = A^1_k \setminus \{\alpha_1, \alpha_2\} \subseteq A^1_k$. The lisse sheaf $\mathcal{H}|_U$ corresponds to a representation

$$\rho : \pi_1^{\text{tame}}(A^1 \setminus \{\alpha_1, \alpha_2\}) \to \text{GL}(V),$$

where $V$ is a $\mathbb{Q}_\ell$-vector space of dimension 7. Let $G$ be the image of $\rho$. Note that $G$ is an irreducible subgroup of $\text{GL}(V)$ since $\mathcal{H}$ is irreducible. In the following, we fix an isomorphism $V \simeq \mathbb{Q}_\ell^7$. This induces an isomorphism $\text{GL}(V) \simeq \text{GL}_7(\mathbb{Q}_\ell)$, so we can view $G$ as a subgroup of $\text{GL}_7(\mathbb{Q}_\ell)$.
We want to show that $G$ is contained in a conjugate of the group $G_2(\bar{\mathbb{Q}}_\ell) \leq \text{GL}_7(\bar{\mathbb{Q}}_\ell)$. For this we argue as in [22], Chap. 4.1: First note that the local monodromy at $s \in \{\alpha_1, \alpha_2, \infty\}$ can be (locally) conjugated in $\text{GL}_7(\bar{\mathbb{Q}}_\ell)$ into the orthogonal group $O_7(\bar{\mathbb{Q}}_\ell)$. It follows thus from the rigidity of the representation $\rho$ that there exists an element $x \in \text{GL}(V)$ such that

$$\text{Transpose}(\rho(g)^{-1}) = \rho(g)^x \quad \forall g \in \pi_{tame}^1(A^1 \setminus \{\alpha_1, \alpha_2\}).$$

In other words, the group $G$ respects the nondegenerate bilinear form given by the element $x^{-1}$. Since $G$ is irreducible and the dimension of $V$ is 7, this form has to be symmetric. Thus we can assume that $G$ is contained in the orthogonal group $O_7(\bar{\mathbb{Q}}_\ell)$. By the results of Aschbacher [4], Thm. 5 (2) and (5), an irreducible subgroup $G$ of $O_7(K)$ ($K$ denoting an algebraically closed field or a finite field) lies inside an $O_7(K)$-conjugate of $G_2(K)$, if and only if $G$ has a nonzero invariant in the third exterior power $\Lambda^3(V)$ of $V = K^7$. In our case, this is equivalent to

$$H^0(U, \Lambda^3(\bar{\mathcal{H}})) \simeq H^0(\mathbb{P}^1, j_*\Lambda^3(\bar{\mathcal{H}})) \neq \{0\},$$

where $\bar{\mathcal{H}} = \mathcal{H}|_U$. Poincaré duality implies that the last formula is equivalent to

$$H^2_c(U, \Lambda^3(\bar{\mathcal{H}})) \simeq H^2(\mathbb{P}^1, j_*\Lambda^3(\bar{\mathcal{H}})) \neq \{0\}.$$

The Euler-Poincaré Formula implies that

$$(1.3.3) \chi(\mathbb{P}^1, j_*\Lambda^3(\bar{\mathcal{H}})) = h^0(\mathbb{P}^1, j_*\Lambda^3(\bar{\mathcal{H}})) - h^1(\mathbb{P}^1, j_*\Lambda^3(\bar{\mathcal{H}})) + h^2(\mathbb{P}^1, j_*\Lambda^3(\bar{\mathcal{H}})) = \chi(U) \cdot \text{rk}(\Lambda^3(\bar{\mathcal{H}})) + \sum_{s \in \{\alpha_1, \alpha_2, \infty\}} \dim(\Lambda^3(\bar{\mathcal{H}})^I(s))$$

$$= -35 + 19 + 13 + 5 = 2$$

(note that $\chi(U) = h^0(U) - h^1(U) + h^2(U) = -1$, $\text{rk}(\Lambda^3(\bar{\mathcal{H}})) = 35$, and that for $s = \alpha_1, \alpha_2, \infty$, the dimension of the local invariants $\dim(\Lambda^3(\bar{\mathcal{H}})^I(s))$ can be computed to be equal to 19, 13, 5, resp.). It follows thus from the equivalence of (1.3.1) and (1.3.2) that $h^0(U, \Lambda^3(\bar{\mathcal{H}})) \geq 1$. Therefore, the monodromy group $G$ can be assumed to be contained in $G_2(\bar{\mathbb{Q}}_\ell)$. Let $\overline{G}$ denote the Zariski closure of $G$ in $G_2(\bar{\mathbb{Q}}_\ell)$. By [4], Cor. 12, a Zariski closed proper maximal subgroup of $G_2(\bar{\mathbb{Q}}_\ell)$ is either reducible or $G$ is isomorphic to the group $\text{SL}_2(\bar{\mathbb{Q}}_\ell)$ acting on the vector space of homogeneous polynomials of degree 6. In the latter case, the unipotent elements of the image of $\text{SL}_2(\bar{\mathbb{Q}}_\ell)$ are equal to the identity matrix, or they are conjugate in $\text{GL}_7(\bar{\mathbb{Q}}_\ell)$ to a Jordan block of length 7. Since the local monodromy of $\mathcal{H}$ at $\alpha_2$ is not of this form, $G$ must coincide with $G_2(\bar{\mathbb{Q}}_\ell)$. This finishes the proof of Claim (i) in the Case 1.
In the following, we only list the local monodromy of the sheaves \( H_0, \ldots, H_6 = H \) in the remaining Cases 2-5. In each case, rigidity implies that the image of \( \pi_1(U) \) is contained in an orthogonal group, and one can compute that an analogue of Formula (1.3.3) holds. Thus the image of \( \pi_1(U) \) is Zariski dense in \( G_2 \) by the same arguments as in Case 1.

**Case 2**: \( \varphi = \eta \) and \( \varphi \) is nontrivial of order 3. Then

| \( \mathcal{H} \) | at \( \alpha_1 \) | at \( \alpha_2 \) | at \( \infty \) |
|-----------------|-----------|-----------|-----------|
| \( H_0 \)       | \(-1\)    | \(-\varphi\) | \(\varphi\) |
| \( H_1 \)       | \(U(2)\)  | \(-\varphi \oplus -\varphi\) | \(\varphi \oplus \varphi\) |
| \( H_2 \)       | \(-1^2 \oplus 1\) | \(\varphi \oplus \varphi \oplus 1\) | \(\varphi \oplus \varphi \oplus 1\) |
| \( H_3 \)       | \(U(1,\varphi)^2 \oplus 1^2\) | \(\varphi \oplus 1 \oplus U(1,-1)^2\) | \(U(2,\varphi) \oplus 1 \oplus \varphi\) |
| \( H_4 \)       | \(1^2 \oplus -1^3\) | \(-\varphi \oplus U(1,\varphi)^2 \oplus 1^2\) | \(U(2) \oplus U(2,\varphi) \oplus \varphi\) |
| \( H_5 \)       | \(U(1,\varphi)^3 \oplus 1^3\) | \(U(2,-\varphi) \oplus 1^2 \oplus U(1,-\varphi)^2\) | \(U(3,\varphi) \oplus U(2) \oplus \varphi\) |
| \( H_6 \)       | \(-1^4 \oplus 1^3\) | \(U(2)^2 \oplus U(3)\) | \(U(3,\varphi) \oplus U(3,\varphi) \oplus 1\) |

**Case 3**: \( \varphi = \eta \) and \( \varphi^4 \neq 1 \neq \varphi^6 \). Then

| \( \mathcal{H} \) | at \( \alpha_1 \) | at \( \alpha_2 \) | at \( \infty \) |
|-----------------|-----------|-----------|-----------|
| \( H_0 \)       | \(-1\)    | \(-\varphi^2\) | \(\varphi^2\) |
| \( H_1 \)       | \(\varphi^3 \oplus 1\) | \(-\varphi^2 \oplus -\varphi\) | \(\varphi^4 \oplus \varphi^2\) |
| \( H_2 \)       | \(-1^2 \oplus 1\) | \(\varphi \oplus \varphi^2 \oplus 1\) | \(\varphi^3 \oplus \varphi \oplus \varphi\) |
| \( H_3 \)       | \(U(1,\varphi)^2 \oplus 1^2\) | \(\varphi \oplus 1 \oplus U(1,-1)^2\) | \(\varphi^2 \oplus \varphi \oplus \varphi^3 \oplus \varphi\) |
| \( H_4 \)       | \(1^2 \oplus -1^3\) | \(-\varphi \oplus U(1,\varphi)^2 \oplus 1^2\) | \(\varphi^2 \oplus 1 \oplus \varphi^3 \oplus \varphi \oplus \varphi\) |
| \( H_5 \)       | \(U(1,\varphi)^3 \oplus 1^3\) | \(U(2,-\varphi) \oplus 1^2 \oplus U(1,-\varphi)^2\) | \(1 \oplus U(2,\varphi^2) \oplus \varphi \oplus \varphi^3 \oplus \varphi\) |
| \( H_6 \)       | \(-1^4 \oplus 1^3\) | \(U(2)^2 \oplus U(3)\) | \(U(2,\varphi) \oplus U(2,\varphi) \oplus \varphi^2 \oplus \varphi^2 \oplus 1\) |
Case 4: $\varphi = \eta$ and $\varphi^4 \neq 1$. Then


division 
| $\mathcal{H}_0$ | at $\alpha_1$ | at $\alpha_2$ | at $\infty$ |
|----------------|--------------|--------------|-------------|
|                | $-1$         | $-1$         | $1$         |
| $\mathcal{H}_1$ | $\varphi \oplus 1$ | $-1 \oplus -\varphi$ | $U(2)$ |
| $\mathcal{H}_2$ | $-1^2 \oplus 1$ | $\varphi \oplus \varphi^2 \oplus 1$ | $U(3, \varphi)$ |
| $\mathcal{H}_3$ | $U(2)^2$ | $\varphi \oplus 1 \oplus U(1, -\varphi)^2$ | $\varphi^2 \oplus U(3, \varphi)$ |
| $\mathcal{H}_4$ | $1^2 \oplus -1^3$ | $-\varphi \oplus U(1, \varphi)^2 \oplus 1^2$ | $\varphi^2 \oplus 1 \oplus U(3, \varphi)$ |
| $\mathcal{H}_5$ | $U(1, \varphi)^3 \oplus 1^3$ | $U(2, \varphi) \oplus 1^2 \oplus U(1, -\varphi)^2$ | $1 \oplus U(2, \varphi^2) \oplus U(3, \varphi)$ |
| $\mathcal{H}_6$ | $-1^4 \oplus 1^3$ | $U(2)^2 \oplus U(3)$ | $U(2, \varphi) \oplus U(2, \varphi) \oplus U(3)$ |

Case 5: The characters

$\varphi, \eta, \varphi \eta, \varphi \eta, \eta, \varphi, 1$

are pairwise disjoint and

$\varphi \eta \neq -1 \neq \varphi \eta^2, \varphi^2 \eta.$

Then


division 
| $\mathcal{H}_0$ | at $\alpha_1$ | at $\alpha_2$ | at $\infty$ |
|----------------|--------------|--------------|-------------|
|                | $-1$         | $-\varphi \eta$ | $\varphi \eta$ |
| $\mathcal{H}_1$ | $\varphi \eta^2 \oplus 1$ | $-\varphi \eta \oplus -\varphi$ | $\varphi \eta \oplus \varphi^2 \eta^2$ |
| $\mathcal{H}_2$ | $1 \oplus -1^2$ | $\eta \oplus \eta^2 \oplus 1$ | $\eta \oplus \varphi \oplus \varphi \eta^2$ |
| $\mathcal{H}_3$ | $U(1, \varphi \eta)^2 \oplus 1^2$ | $-\eta \oplus 1 \oplus U(1, -\varphi \eta)^2$ | $\varphi \eta \oplus \varphi^2 \eta^2 \oplus \varphi \oplus \eta^2$ |
| $\mathcal{H}_4$ | $U(1, -1)^3 \oplus 1^2$ | $-\varphi \oplus U(1, \varphi)^2 \oplus 1^2$ | $\varphi \oplus \varphi^2 \eta \oplus \varphi \eta \oplus \varphi \eta^2$ |
| $\mathcal{H}_5$ | $U(1, \varphi)^3 \oplus 1^3$ | $U(2, -\varphi) \oplus U(1, -\varphi)^2 \oplus 1^2$ | $\varphi \oplus \eta \varphi^2 \oplus \eta \oplus \varphi \eta \oplus \varphi \eta \oplus \varphi \eta^2$ |
| $\mathcal{H}_6$ | $-1^4 \oplus 1^3$ | $U(2)^2 \oplus U(3)$ | $1 \oplus \varphi \eta \oplus \eta \varphi \oplus \eta \varphi \oplus \eta \varphi \oplus \varphi$ |

By what was said above, this finishes the proof of Claim (i).

Let us prove Claim (ii): Let $D$ be a reduced effective divisor on $\mathbb{A}^1$, let $U := \mathbb{A}^1 \setminus D$, and let $j : U \to \mathbb{P}^1$ denote the obvious inclusion. A sheaf $\mathcal{H} \in \mathcal{T}_k(k)$ which is lisse
on $U$ is cohomologically rigid, if and only if
\[
\text{rig}(\mathcal{H}) = (1 - \text{Card}(D))\text{rk}(\mathcal{H}|_U)^2 + \sum_{s \in D \cup \{\infty\}} \sum_{i, \chi} e_i(s, \chi, \mathcal{H})^2 = 2,
\]
compare to [23], 6.0.15. (Note that the sum $\sum_{i, \chi} e_i(s, \chi, \mathcal{H})^2$ gives the dimension of the centralizer of the local monodromy in the group $\text{GL}_{\text{rk}(\mathcal{H}|_U)}(\overline{\mathbb{Q}}_\ell)$, see loc. cit. 3.1.15.) Another necessary condition for $\mathcal{H}$ to be contained in $\mathcal{T}_\ell$ is that $\mathcal{H}$ is irreducible. This implies that
\[
\chi(\mathbb{P}^1, j_*(\mathcal{H}|_U)) = (1 - \text{Card}(D))\text{rk}(\mathcal{H}) + \sum_{s \in D \cup \{\infty\}} e_1(s, 1, \mathcal{H}) \leq 0,
\]
since the same arguments as on Formula (1.3.3) apply. Assume first that $\text{Card}(D) > 2$ and that $\mathcal{H}$ fails to be lisse at all points of $D$. Then, by (1.3.4),
\[
\sum_{s \in D \cup \{\infty\}} \sum_{i, \chi} e_i(s, \chi, \mathcal{H})^2 = 2 + (\text{Card}(D) - 1)7^2.
\]
Since $\sum_{i, \chi} e_i(s, \chi, \mathcal{H})^2 \leq 29$ by Table 1, one immediately concludes that the cardinality of $D$ is $\leq 3$. If $\text{Card}(D) = 3$ then, by Table 1, the following combinations of the centralizer dimensions can occur:
\[
(25, 25, 25, 25), \quad (29, 29, 29, 13), \quad (29, 29, 25, 17).
\]
In each case one obtains a contradiction to (1.3.5) (using a quadratic twist at each local monodromy in the case $(25, 25, 25, 25)$).

Thus $D = \{\alpha_1, \alpha_2\}$, where $\alpha_1, \alpha_2$ are two disjoint points of $\mathbb{A}^1(k)$, and
\[
\sum_{s \in \{\alpha_1, \alpha_2, \infty\}} \sum_{i, \chi} e_i(s, \chi, \mathcal{H})^2 = 7^2 + 2 = 51.
\]
This leaves one with 7 possible cases $P_1, \ldots, P_7$, which are listed in Table 2.

Using Table 1 and the inequality (1.3.5), one can exclude $P_1, P_4$ and $P_7$ by possibly twisting the local monodromy at $\alpha_1, \alpha_2, \infty$ by three suitable (at most quadratic) characters, whose product is 1. The possible case $P_5$ can be excluded using the inequality in (1.3.5) and a twist by suitable characters of order at most 4 whose product is 1.

Since the monodromy representation of $\mathcal{H}$ is dense in the group $G_2(\overline{\mathbb{Q}}_\ell)$, one obtains an associated sheaf $\text{Ad}(\mathcal{H}) \in \mathcal{T}_\ell$ of generic rank 14, given by the adjoint representation of $G_2$. This is again irreducible, which implies that
\[
\chi(\mathbb{P}^1, j_*(\text{Ad}(\mathcal{H})|_U)) = (1 - \text{Card}(D)) \cdot 14 + \sum_{s \in D \cup \{\infty\}} \dim(C_{G_2}(\mathcal{H}(s))) \leq 0.
\]

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The possible centralizer dimensions

| case | $\sum_{i,\chi} e_i(\alpha_1, \chi, \mathcal{H})^2$ | $\sum_{i,\chi} e_i(\alpha_2, \chi, \mathcal{H})^2$ | $\sum_{i,\chi} e_i(\infty, \chi, \mathcal{H})^2$ |
|------|---------------------------------|---------------------------------|---------------------------------|
| $P_1$ | 29                              | 13                              | 9                               |
| $P_2$ | 29                              | 11                              | 11                              |
| $P_3$ | 25                              | 19                              | 7                               |
| $P_4$ | 25                              | 17                              | 9                               |
| $P_5$ | 25                              | 13                              | 13                              |
| $P_6$ | 19                              | 19                              | 13                              |
| $P_7$ | 17                              | 17                              | 17                              |

**Table 2:** The possible centralizer dimensions

This can be used to exclude the case $P_2$ and $P_6$, because in these cases one has

$$\chi(\mathbb{P}^1, j_* \textnormal{End}_{\sigma_2}(\mathcal{H}|_U)) = -14 + 8 + 4 + 4 > 0$$

and

$$\chi(\mathbb{P}^1, j_* \textnormal{End}_{\sigma_2}(\mathcal{H}|_U)) = -14 + 8 + 8 + 4 > 0 \quad (\text{resp.}).$$

By the same argument, in case $P_3$, one can exclude that the centralizer dimension 19 comes from a non-unipotent character.

Thus, up to a permutation of the points $\alpha_1, \alpha_2, \infty$, we are left with the following possibility for the local monodromy: The local monodromy at $\alpha_1$ is an involution, the local monodromy at $\alpha_2$ is unipotent of the form $U(2)^2 \oplus U(3)$ and the local monodromy at $\infty$ is regular, i.e., the dimension of the centralizer is 7. By Table 1, Thm. 1.3.1 (i) above lists all the possibilities for the local monodromy at $\infty$ to be regular, except in Case 3, when $\varphi$ has order 6, or in Case 4, when $\varphi$ is a character of order 4, or in Case 5, when

$$\varphi \overline{\eta} = -1, \quad \text{or} \quad \varphi \eta^2 = -1, \quad \text{or} \quad \varphi^2 \eta = -1.$$

In all these cases one can show that no such local system exists by inverting the construction of $\mathcal{H}$ (using Formula (1.1.2) and middle tensor products with the dual sheaves $\mathcal{F}^\vee_i$) and deriving a contradiction to (1.3.4) or (1.3.5). $\Box$

Let us assume that $\alpha_1 = 0, \alpha_2 = 1$ and that $k = \overline{\mathbb{Q}}$. Let $\iota : \mathbb{A}^1_{\mathbb{Q}} \to \mathbb{A}^1_{\overline{\mathbb{Q}}}$ be the basechange map. We call a sheaf $\mathcal{H}$ as in Thm. 1.3.1 to be defined over $\mathbb{Q}$, if $\mathcal{H}|_{\mathbb{A}^1_{\overline{\mathbb{Q}}} \setminus \{0,1\}}$ is of the form $\iota^*(\mathfrak{f})$ where $\mathfrak{f}$ is lisse on $\mathbb{A}^1_{\overline{\mathbb{Q}}} \setminus \{0,1\}$. 19
1.3.2 Theorem. If a sheaf $\mathcal{H}$ as in Thm. 1.3.1 (i) is defined over $\mathbb{Q}$, then the trace of the local monodromy at $\infty$ is contained in $\mathbb{Q}$. This is the case, if and only if the local monodromy at $\infty$ is of the following form:

| $\mathbf{U}(7)$ | $\mathbf{U}(3, \varphi) \oplus \mathbf{U}(3, \overline{\varphi}) \oplus 1$ | $\varphi$ of order 3 |
|----------------|---------------------------------|----------------------|
| $\mathbf{U}(2, \varphi) \oplus \mathbf{U}(2, \overline{\varphi}) \oplus \mathbf{U}(3)$ | $\varphi$ of order 3 or 6 |
| $\varphi \oplus \eta \oplus \varphi \eta \oplus \overline{\varphi} \eta \oplus \overline{\varphi} \oplus 1$ | $\varphi$ of order 8 and $\eta = \varphi^2$ |
| | $\varphi$ of order 7 or 14 and $\eta = \varphi^2$ |
| | $\varphi$ of order 12 and $\eta = -\varphi$ |

Proof: The first claim follows from the structure of the local fundamental group at $\infty$. Using the local monodromy of the sheaves $\mathcal{H}$ listed in Thm. 1.3.1 (i), one obtains the result by an explicit computation. $\square$

2 The motivic interpretation of the rigid $G_2$-sheaves

In this section we recall the motivic interpretation of the middle convolution in the universal setup of [23], Chap. 8. This leads to an explicit geometric construction of the rigid $G_2$-sheaves found in the previous section.

2.1 Basic definitions Let us recall the setup of [23], Chap. 8: Let $k$ denote an algebraically closed field and $\ell$ a prime number which is invertible in $k$. Let further $\alpha_1, \ldots, \alpha_n$ be pairwise disjoint points of $\mathbb{A}^1(k)$ and $\zeta$ a primitive root of unity in $k$. Fix an integer $N \geq 1$ such that $\text{char}(k)$ does not divide $N$ and let

$$R := R_{N, \ell} := \mathbb{Z}[\zeta_N, \frac{1}{N\ell}];$$

where $\zeta_N$ denotes a primitive $N$-th root of unity. Set

$$S_{N,n,\ell} := R_{N,\ell}[T_1, \ldots, T_n][1/\Delta], \quad \Delta := \prod_{i \neq j} (T_i - T_j).$$

Fix an embedding $R \to \overline{\mathbb{Q}}_\ell$ and let $E$ denote the fraction field of $R$. For a place $\lambda$ of $E$, let $E_\lambda$ denote the $\lambda$-adic completion of $E$. Let $\phi : S_{N,n,\ell} \to k$ denote the unique ring homomorphism for which $\phi(\zeta_N) = \zeta$ and for which

$$\phi(T_i) = \alpha_i, \quad i = 1, \ldots, n.$$
Let $\mathbb{A}^1_{\mathbb{A}^n_{S,n,\ell}} \setminus \{T_1, \ldots, T_n\}$ denote the affine line over $S$ with the $n$ sections $T_1, \ldots, T_n$ deleted. Consider more generally the spaces

$$\mathbb{A}(n, r + 1)_R := \text{Spec } (R[T_1, \ldots, T_n, X_1, \ldots, X_{r+1}][\frac{1}{\Delta_{n,r}}]),$$

where

$$\Delta_{n,r} := \left(\prod_{i \neq j} (T_i - T_j)\right) \left(\prod_{a,j} (X_a - T_j)\right) \left(\prod_{k} (X_{k+1} - X_k)\right)$$

(here the indices $i,j$ run through $\{1, \ldots, n\}$, the index $a$ through $\{1, \ldots, r+1\}$ and the index $k$ runs through $\{1, \ldots, r\}$; when $r = 0$ the empty product $\prod_k (X_{k+1} - X_k)$ is understood to be 1).

Let

$$\text{pr}_i : \mathbb{A}(n, r + 1)_R \to \mathbb{A}^1_{\mathbb{A}^n_{S,n,\ell}} \setminus \{T_1, \ldots, T_n\},$$

$$(T_1, \ldots, T_n, X_1, \ldots, X_{r+1}) \mapsto (T_1, \ldots, T_n, X_i).$$

On $(\mathbb{G}_m)_R$ with coordinate $Z$, one has the Kummer covering of degree $N$, of equation $Y^N = Z$. This is a connected $\mu_N(R)$-torsor whose existence defines a surjective homomorphism $\pi_1((\mathbb{G}_m)_R) \to \mu_N(R)$. The chosen embedding $R \to \mathbb{Q}_\ell$ defines a faithful character

$$\chi_N : \mu_N(R) \to \mathbb{Q}_\ell^\times$$

and the composite homomorphism

$$\pi_1((\mathbb{G}_m)_R) \to \mu_N(R) \to \mathbb{Q}_\ell^\times$$

defines the Kummer sheaf $\mathcal{L}_{\chi_N}$ on $(\mathbb{G}_m)_R$. For any scheme $W$ and any map $f : W \to (\mathbb{G}_m)_R$, define

$$\mathcal{L}_{\chi(f)} := f^* \mathcal{L}_\chi.$$

### 2.2 The middle convolution of local systems

Denote by $\text{Lisse}(N, n, \ell)$ the category of lisse $\mathbb{Q}_\ell$-sheaves on

$$\mathbb{A}(n, 1)_R = (\mathbb{A}^1 - (T_1, \ldots, T_n))_{S,n,\ell}.$$ 

For each nontrivial $\mathbb{Q}_\ell$-valued character $\chi$ of the group $\mu_N(R)$, Katz [23] defines a left exact middle convolution functor

$$\text{MC}_\chi : \text{Lisse}(N, n, \ell) \to \text{Lisse}(N, n, \ell)$$

as follows:
2.2.1 Definition. View the space \( \mathbb{A}(n, 2)_R \) with its second projection \( \text{pr}_2 \) to \( \mathbb{A}(n, 1)_R \), as a relative \( \mathbb{A}^1 \) with coordinate \( X_1 \), minus the \( n+1 \) sections \( T_1, \ldots, T_n, X_2 \). Compactify the morphism \( \text{pr}_2 \) into the relative \( \mathbb{P}^1 \)

\[
\overline{\text{pr}}_2 : \mathbb{P}^1 \times \mathbb{A}(n, 1)_R \to \mathbb{A}(n, 1)_R,
\]

by filling in the sections \( T_1, \ldots, T_n, X_2, \infty \). Moreover, let \( j : \mathbb{A}(n, 2)_R \to \mathbb{P}^1 \times \mathbb{A}(n, 1)_R \) denote the natural inclusion. The middle convolution of \( F \in \text{Lisse}(N, n, \ell) \) is defined as follows

\[
\text{MC}_\chi(F) := R^1(\overline{\text{pr}}_2)_J(j_*(\text{pr}_1^*(F) \otimes \mathcal{L}_\chi(X_2 - X_1))) \in \text{Lisse}(N, n, \ell),
\]

see loc. cit. Section 8.3.

For any \( F \in \text{Lisse}(N, n, \ell) \), and any nontrivial character \( \chi \) as above, let \( F_k \) denote the restriction of \( F \) to the geometric fibre \( U_k = \mathbb{A}^1_k \setminus \{\alpha_1, \ldots, \alpha_n\} \) of \( (\mathbb{A}^1 - (T_1, \ldots, T_n))_{S, n, \ell} \) which is defined by the homomorphism \( \phi : S \to k \). Define \( \chi_k \) as the restriction of \( \chi \) to \( \mathbb{G}_{m,k} \) and let \( j : U_k \to \mathbb{P}^1_k \) denote the inclusion. Then the following holds:

\[
(2.2.1) \quad \text{MC}_{\chi_k}(j_*F_k)|_{U_k} = \text{MC}_\chi(F)_k,
\]

where on the left, the middle convolution \( \text{MC}_{\chi_k}(F_k) \) is defined as in Section 1.1 and on the right, the middle convolution is defined as in Def. 2.2.1 above (see [23], Lemma 8.3.2).

2.3 The motivic interpretation of the middle convolution In [23], Thm. 8.3.5 and Thm. 8.4.1, the following result is proved:

2.3.1 Theorem. Fix an integer \( r \geq 0 \). For a choice of \( n(r+1) \) characters

\[
\chi_{a,i} : \mu_N(R) \to \widehat{\mathbb{Q}}_\ell^\times, \quad i = 1, \ldots, n, \quad a = 1, \ldots, r+1,
\]

and a choice of \( r \) nontrivial characters

\[
\rho_k : \mu_N(R) \to \widehat{\mathbb{Q}}_\ell^\times, \quad k = 1, \ldots, r,
\]

define a rank one sheaf \( \mathcal{L} \) on \( \mathbb{A}(n, r+1)_R \) by setting

\[
\mathcal{L} := \bigotimes_{a,i} \mathcal{L}_{\chi_{a,i}(X_a - T_i)} \bigotimes_k \mathcal{L}_{\rho_k(X_{k+1} - X_k)}.
\]

Then the following holds:
(i) The sheaf $\mathcal{K} := R^p (pr_{r+1})_!(\mathcal{L})$ is mixed of integral weights in $[0, r]$. There exists a short exact sequence of lisse sheaves on $\mathbb{A}^1_{S, N, n, \ell} \setminus \{T_1, \ldots, T_n\}$:

$$
0 \to \mathcal{K}_{\leq r-1} \to \mathcal{K} \to \mathcal{K}_{= r} \to 0,
$$

such that $\mathcal{K}_{\leq r-1}$ is mixed of integral weights $\leq r - 1$ and where $\mathcal{K}_{= r}$ is punctually pure of weight $r$.

(ii) Let $\chi = \chi_N : \mu_N(R) \to \overline{\mathbb{Q}}^\times$ be the faithful character defined in the last section and let $e(a, i), i = 1, \ldots, n, a = 1, \ldots, r + 1$, and $f(k), k = 1, \ldots, r$ be integers with $\chi_{a, i} = \chi^e(a, i)$, and $\rho_k = \chi^f(k)$.

In the product space $\mathbb{G}_{m, R} \times \mathbb{A}(n, r + 1)$, consider the hypersurface $\text{Hyp}$ given by the equation

$$
Y^N = \left( \prod_{a, i} (X_a - T_i)^{e(a, i)} \right) \left( \prod_{k=1, \ldots, r} (X_{k+1} - X_k)^{f(k)} \right)
$$

and let

$$
\pi : \text{Hyp} \to \mathbb{A}^1_{S, N, n, \ell} \setminus \{T_1, \ldots, T_n\},
$$

$$(Y, T_1, \ldots, T_n, X_1, \ldots, X_{r+1}) \mapsto (T_1, \ldots, T_n, X_{r+1}).$$

The group $\mu_N(R)$ acts on $\text{Hyp}$ by permuting $Y$ alone, inducing an action of $\mu_N(R)$ on $R^p \pi_!(\overline{\mathbb{Q}}^\ell)$. Then the sheaf $\mathcal{K}$ is isomorphic to the $\chi$-component $(R^p \pi_!(\overline{\mathbb{Q}}^\ell))^\chi$ of $R^p \pi_!(\overline{\mathbb{Q}}^\ell)$.

(iii) For $a = 1, \ldots, r + 1$, let

$$
\mathcal{F}_a = \mathcal{F}_a(X_a) := \bigotimes_{i=1, \ldots, n} \mathcal{L}_{\chi_{a, i}(X_a - T_i)} \in \text{Lisse}(N, n, \ell).
$$

Let

$$
\mathcal{H}_0 := \mathcal{F}_1,
$$

$$
\mathcal{H}_1 := \mathcal{F}_2 \otimes \text{MC}_{\rho_2}(\mathcal{H}_0),
$$

$$
\vdots
$$

$$
\mathcal{H}_r := \mathcal{F}_{r+1} \otimes \text{MC}_{\rho_r}(\mathcal{H}_{r-1}).
$$

Then $\mathcal{K}_{= r} = \mathcal{H}_r$. 

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2.4 Motivic interpretation for rigid $G_2$-sheaves

Let $\epsilon : \pi_1(\mathbb{G}_{m,R}) \to \mu_N(R)$ be the surjective homomorphism of Section 2.1. By composition with $\epsilon$, every character $\chi : \mu_N(R) \to \bar{\mathbb{Q}}_{\ell}^\times$ gives rise to a character of $\pi_1(\mathbb{G}_{m,R})$, again denoted by $\chi$. For a sheaf $\mathcal{K} \in \text{Lisse}(N, n, \ell)$ which is mixed of integral weights in $[0, r]$ let $W^r(\mathcal{K})$ denote the weight-$r$-quotient of $\mathcal{K}$.

2.4.1 Theorem. Let $\varphi, \eta$ and $\mathcal{H}(\varphi, \eta)$ be as in Thm. 1.3.1. Let $N$ denote the least common multiple of 2 and the orders of $\varphi, \eta$. Let further $\chi = \chi_N : \mu_N(R) \to \bar{\mathbb{Q}}_{\ell}^\times$ be the character of order $N$ which is defined in Section 2.1, and let $n_1, n_2$ be integers with

$$\varphi = \chi_{k_1}^{n_1} \quad \text{and} \quad \eta = \chi_{k_2}^{n_2}, \quad n_1, n_2 \in \mathbb{Z},$$

where $\chi_k$ is the restriction of $\chi$ to $\mathbb{G}_{m,k}$. Let $\text{Hyp} = \text{Hyp}(n_1, n_2)$ denote the hypersurface in $\mathbb{G}_{m,R} \times \mathbb{A}(2, 6 + 1)_{\mathbb{R}}$, given by the following equation:

$$Y^N = \left( \prod_{1 \leq a \leq 7; 1 \leq i \leq 2} (X_a - T_i)^{e(a,i)} \right) \left( \prod_{1 \leq k \leq 6} (X_{k+1} - X_k)^{f(k)} \right),$$

where the numbers $e(a, i)$ and the $f(k)$ are as follows:

| $e(1, 1)$ | $e(2, 1)$ | $e(3, 1)$ | $e(4, 1)$ | $e(5, 1)$ | $e(6, 1)$ | $e(7, 1)$ |
|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| $\frac{N}{2}$ | 0 | $\frac{N}{2}$ | 0 | $\frac{N}{2}$ | 0 | $\frac{N}{2}$ |

| $e(1, 2)$ | $e(2, 2)$ | $e(3, 2)$ | $e(4, 2)$ | $e(5, 2)$ | $e(6, 2)$ | $e(7, 2)$ |
|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| $\frac{N}{2} + n_1 + n_2$ | $\frac{N}{2} - n_1$ | 0 | $\frac{N}{2} + n_1 - n_2$ | 0 | $\frac{N}{2} - n_1$ | 0 |

| $f(1)$ | $f(2)$ | $f(3)$ | $f(4)$ | $f(5)$ | $f(6)$ |
|---------|--------|--------|--------|--------|--------|
| $\frac{N}{2} - n_1 - 2n_2$ | $\frac{N}{2} + n_1 + 2n_2$ | $\frac{N}{2} - n_1 - n_2$ | $\frac{N}{2} + n_1 + n_2$ | $\frac{N}{2} - n_1$ | $\frac{N}{2} + n_1$ |

Let $\pi = \pi(n_1, n_2) : \text{Hyp}(n_1, n_2) \to \mathbb{A}_{S_{N,n,\ell}}^1 \setminus \{T_1, T_2\}$, given by $(Y, T_1, T_2, X_1, \ldots, X_7) \mapsto (T_1, T_2, X_7)$. Then the higher direct image sheaf $W^6 \left[ (R^6_{\pi_1(\mathbb{Q}_{\ell})})^X \right]$ is contained in $\text{Lisse}(N, n, \ell)$. Moreover, for any algebraically closed field $k$ whose characteristic does not divide $\ell N$, one has an isomorphism

$$\mathcal{H}(\varphi, \eta) |_{\mathbb{A}_k^1 \setminus \{\alpha_1, \alpha_2\}} = (W^6 \left[ (R^6_{\pi_1(\mathbb{Q}_{\ell})})^X \right]) |_{\mathbb{A}_k^1 \setminus \{\alpha_1, \alpha_2\}}.$$
Proof: This is just a restatement of Thm. 2.3.1 in the situation of Thm. 1.3.1. The last formula follows from Formula (2.2.1) and Thm. 2.3.1 (iii).

We now turn to the special case, where \( n_1 = n_2 = 0, N = 2, \) and \( \alpha_1 = 0, \alpha_2 = 1. \) In this case, the higher direct image sheaves which occur in Thm. 2.4.1 can be expressed in terms of the cohomology of a smooth and proper map of schemes over \( \mathbb{Q}. \) This will be crucial in the next section.

2.4.2 Corollary. Let \( N = 2, n_1 = n_2 = 0, \) and let

\[
\text{Hyp} = \text{Hyp}(0, 0) \subseteq \mathbb{G}_{m,R} \times \mathbb{A}(2, 6 + 1)_R
\]

be the associated hypersurface equipped with the structural morphism \( \pi = \pi(0, 0) : \text{Hyp} \to \mathbb{A}^1_{\mathbb{S}} \setminus \{T_1, T_2\}. \) Let \( \pi_Q : \text{Hyp}_Q \to \mathbb{A}^1_{\mathbb{Q}} \setminus \{0, 1\}, \) denote the basechange of \( \pi \) induced by \( T_1 \mapsto 0 \) and \( T_2 \mapsto 1. \) Then the following holds:

(i) There exists a smooth and projective scheme \( X \) over \( \mathbb{A}^1_{\mathbb{Q}} \setminus \{0, 1\} \) and an open embedding of \( j : \text{Hyp}_Q \to X \) such that

\[
D = X \setminus \text{Hyp}_Q = \bigcup_{i \in I} D_i
\]

is a strict normal crossings divisor over \( \mathbb{A}^1_{\mathbb{Q}} \setminus \{0, 1\}. \) The involutory automorphism \( \sigma \) of \( \text{Hyp} \) (given by \( Y \mapsto -Y \)) extends to an automorphism \( \sigma \) of \( X. \)

(ii) Let \( \bigsqcup_{i \in I} D_i \) denote the disjoint union of the components of \( D \) and let

\[
\pi_X : X \to \mathbb{A}^1_{\mathbb{Q}} \setminus \{0, 1\} \quad \text{and} \quad \pi_{\bigsqcup_{i \in I} D_i} : \bigsqcup_{i \in I} D_i \to \mathbb{A}^1_{\mathbb{Q}} \setminus \{0, 1\}
\]

denote the structural morphisms. Let \( G := W^6 \left[ (R^6 \pi_{\mathbb{Q}})^X \right] |_{\mathbb{A}^1_{\mathbb{Q}} \setminus \{0, 1\}}. \) Then

\[
G \simeq \Pi \left[ \ker (R^6 (\pi_X)_*(\mathbb{Q})) \to R^6 (\pi_{\bigsqcup_{i \in I} D_i})_*(\mathbb{Q})) \right],
\]

where \( \Pi \) denotes the formal sum \( \frac{1}{2}(\sigma - 1). \)

Proof: Let \( \Delta \subseteq \mathbb{A}^7 \) be the divisor defined by the vanishing of

\[
(2.4.1) \quad \prod_{i=1}^6 (X_{i+1} - X_i) \prod_{i=1}^7 X_i \prod_{i=1}^7 (X_i - 1).
\]
By Thm. 2.4.1, the hypersurface Hyp is an unramified double cover of an open subset of $\mathbb{A}^7 \setminus \Delta$ defined by

$$Y^2 = \prod_{i=1}^{6} (X_{i+1} - X_i) \prod_{i=1,3,5,7} X_i \prod_{i=1,2,4,6} (X_i - 1).$$

This defines a ramified double cover $\alpha : \overline{X} \to \mathbb{P}_S^6 = \mathbb{P}^6 \times S$, where $S := \mathbb{A}^1_{X_7} \setminus \{0,1\}$. The image of the complement $\overline{X} \setminus \text{Hyp}$ under $\alpha$ is a relative divisor $L$ over $S$ on $\mathbb{P}_S^6$. The divisor $L$ is the union of the relative hyperplane at infinity $L_0 = \mathbb{P}_S^6 \setminus (\mathbb{A}^6_{X_1,\ldots,X_6} \times S)$ with the 20 linear hyperplanes $L_i = L_{T_i}$, $i = 1,\ldots,20$, which are defined by the vanishing of the partial projection of the irreducible factors $T_i$ of the right hand side of Equation (2.4.1). The singularities of $\overline{X}$ are situated over the singularities of the ramification locus $R$ of $\alpha : \overline{X} \to \mathbb{P}_S^6$ which is a subdi visor of $L$ by Equation (2.4.2).

There is a standard resolution of any linear hyperplane arrangement $L = \bigcup_i L_i \subseteq \mathbb{P}^n$ given in [14], Section 2. By this we mean a birational map $\tau : \mathbb{P}^n \to \mathbb{P}^n$ which factors into several blow ups and which has the following properties: The inverse image of $L$ under $\tau$ is a strict normal crossings divisor in $\mathbb{P}^n$ and the strict transform of $L$ is nonsingular (see [14], Claim in Section 2). The standard resolution depends only on the combinatorial intersection behaviour of the irreducible components $L_i$ of $L$, therefore it can be defined for locally trivial families of hyperplane arrangements.

In our case, we obtain a birational map $\tau : \tilde{\mathbb{P}}_S^6 \to \mathbb{P}_S^6$ such that $\tilde{L} := \tau^{-1}(L)$ is a relative strict normal crossings divisor over $S$ and such that the strict transform of $L$ is smooth over $S$. Let $\hat{\alpha} : X \to \hat{\mathbb{P}}_S^6$ denote the pullback of the double cover $\alpha$ along $\tau$ and let $\hat{R}$ be the ramification divisor of $\hat{\alpha}$. Then $\hat{R}$ is a relative strict normal crossings divisor since it is contained in $\tilde{L}$. Write $\hat{R}$ as a union $\bigcup_k \hat{R}_k$ of irreducible components. By successively blowing up the (strict transforms of the) intersection loci $\hat{R}_{k_1} \cap \hat{R}_{k_2}$, $k_1 < k_2$, one ends up with a birational map $f : \hat{\mathbb{P}}_S^6 \to \hat{\mathbb{P}}_S^6$. Let $\hat{\alpha} : X \to \hat{\mathbb{P}}_S^6$ denote the pullback of the double cover $\hat{\alpha}$ along $f$. Then the strict transform of $\hat{R}$ in $\hat{\mathbb{P}}_S^6$ is a disjoint union of smooth components. Moreover, since $\hat{R}$ is a normal crossings divisor, the exceptional divisor of the map $f$ has no components in common with the ramification locus of $\hat{\alpha}$. It follows that the double cover $\hat{\alpha} : X \to \hat{\mathbb{P}}_S^6$ is smooth over $S$ and that $D = X \setminus \text{Hyp}$ is a strict normal crossings divisor over $S$. This desingularization is obviously equivariant with respect to $\sigma$ which finishes the proof of Claim (i).

Let $\pi_X : X \to S$ denote the structural map (the composition of $\hat{\alpha} : X \to \hat{\mathbb{P}}_S^6$ with the natural map $\hat{\mathbb{P}}_S^6 \to S$). There exists an $n \in \mathbb{N}$ such that the morphism $\pi_X$ extends to a morphism $X_A \to \mathbb{A}^4_A \setminus \{0,1\}$ of schemes over $A := \mathbb{Z}[\frac{1}{2n}]$. We
assume that \( n \) is big enough that \( D_A := X_A \setminus \text{Hyp}_A \) is a normal crossings divisor over \( \mathbb{A}_A \setminus \{0,1\} \). In the following, we mostly omit the subscript \( A \) but we will tacitly work in the category of schemes over \( A \) (making use of the fact that \( A \) is finitely generated over \( \mathbb{Z} \), in order to be able to apply Deligne’s results on the Weil conjectures). Let \( \pi_D : D \to \mathbb{A}^1 \setminus \{0,1\} \) (resp. \( \pi_{\text{Hyp}} : \text{Hyp} \to \mathbb{A}^1 \setminus \{0,1\} \)) be the structural morphisms. The excision sequence gives an exact sequence of sheaves
\[
\chi(\pi_D)_* (\bar{Q}_\ell) \to \chi(\pi_{\text{Hyp}})_! (\bar{Q}_\ell) \to \chi(\pi_X)_* (\bar{Q}_\ell) \to \chi(\pi_D)_* (\bar{Q}_\ell) \to R^7(\pi_{\text{Hyp}})_! (\bar{Q}_\ell).
\]
By exactness and the work of Deligne (Weil II, [11]), the kernel of the map \( R^6(\pi_{\text{Hyp}})_!(\bar{Q}_\ell) \to R^6(\pi_X)_*(\bar{Q}_\ell) \) is an integral constructible sheaf which is mixed of weights \( \leq 5 \). Thus (2.4.3) implies an isomorphism
\[
W^6(\chi(\pi_{\text{Hyp}})_!(\bar{Q}_\ell)) \to \ker(\chi(\pi_X)_*(\bar{Q}_\ell)) \to \ker(\chi(\pi_D)_*(\bar{Q}_\ell)).
\]
By the exactness of (2.4.3) and by functoriality, one thus obtains the following chain of isomorphisms
\[
W^6(\chi(\pi_{\text{Hyp}})_!(\bar{Q}_\ell)) \simeq \ker(\chi(\pi_X)_*(\bar{Q}_\ell)) \to \ker(\chi(\pi_D)_*(\bar{Q}_\ell)).
\]
where the superscript \( \chi \) stands for the \( \chi \)-component of the higher direct image in the sense of 2.3.1 (the notion extends in an obvious way to \( X \) and to \( D \)).

We claim that the natural map
\[
\ker(\chi(\pi_X)_*(\bar{Q}_\ell)) \to \ker(\chi(\pi_D)_*(\bar{Q}_\ell))
\]
is an isomorphism. For this we argue as follows: Since the sheaf \( W^6(\chi(\pi_{\text{Hyp}})_!(\bar{Q}_\ell)) \) is lisse (see Thm.2.4.1), the isomorphisms given in (2.4.5) imply that
\[
\ker(\chi(\pi_X)_*(\bar{Q}_\ell)) \to \ker(\chi(\pi_D)_*(\bar{Q}_\ell)).
\]
is lisse. It follows from proper base change that
\[
\ker(\chi(\pi_X)_*(\bar{Q}_\ell)) \to \ker(\chi(\pi_{\text{Hyp}, D_A,i})_*(\bar{Q}_\ell)).
\]
is lisse. Thus, by the specialization theorem (see [22], 8.18.2), in order to prove that the map in (2.4.6) is an isomorphism, it suffices to show this for any \( \text{closed} \) geometric point \( s \) of Hyp. In view of (2.4.5), we have thus to show that
\[
W^6(H^6_{\ell}(\text{Hyp}_s, \bar{Q}_\ell)) \simeq \ker(H^6(X_s, \bar{Q}_\ell) \to H^6(\prod_i D_{s,i}, \bar{Q}_\ell)).
\]
Let $X^0_\delta = X_\delta$, and for positive natural numbers $i$, let $X^i_\delta$ denote the disjoint union of the irreducible components of the locus, where $i$ pairwise different components of $D_\delta$ meet. It follows from the Weil conjectures [9] that the spectral sequence $E_1 = H^j(X^0_\delta, \mathbb{Q}_\ell) \Rightarrow H^{i+j}(U_\delta, \mathbb{Q}_\ell)$ degenerates at $E_2$. Consequently,

$$W^6(H^6_c(Hyp, \mathbb{Q}_\ell)) \cong \ker(H^6(X_\delta, \mathbb{Q}_\ell) \to H^6(\coprod_i D_{s,i}, \mathbb{Q}_\ell)).$$

This implies (2.4.7) and thus proves that the map in (2.4.6) is an isomorphism as claimed. So,

$$W^6(R^6\pi_{Hyp,*}(\mathbb{Q}_\ell)) \cong \ker \left( R^6(\pi_{X_A})_* (\mathbb{Q}_\ell) \to R^6(\pi_{\coprod_i D_{A,i}})_*(\mathbb{Q}_\ell) \right).$$

where the last equality is a tautology using the representation theory of finite (cyclic) groups. It follows that

$$\mathcal{G} = W^6(R^6(\pi_{Hyp})_!(\mathbb{Q}_\ell))_{|_{h_\delta \setminus \{0,1\}}} \cong \Pi \left( \ker(R^6(\pi_X)_* (\mathbb{Q}_\ell) \to R^6(\pi_{\coprod_i D_i})_* (\mathbb{Q}_\ell)) \right),$$

as claimed. □

### 3 Relative motives with motivic Galois group $G_2$

#### 3.1 Preliminaries on motives

For an introduction to the theory of motives, as well as basic properties and definitions, we refer the reader to the book of Y. André [3]. Let $K$ and $E$ denote a fields of characteristic zero. Let $V_K$ denote the category of smooth and projective varieties over $K$. If $X \in V_K$ is purely $d$-dimensional, denote by $\text{Corr}^0(X,X)_E$ the $E$-algebra of codimension-$d$-cycles in $X \times X$ modulo homological equivalence (the multiplication is given by the usual composition of correspondences). This notion extends by additivity to an arbitrary object $X \in V_K$. A Grothendieck motive with values in $E$ is then a triple $M = (X,p,m)$, where $X \in V_K$, $m \in \mathbb{Z}$, and where $p \in \text{Corr}^0(X,X)_E$ is idempotent. For any $X \in V_K$ one has associated a motive $h(X) = (X,\Delta(X),0)$ (called the motive of $X$), where $\Delta(X) \subseteq X \times X$ denotes the diagonal.

One also has the theory of motives for motivated cycles due to Y. André [2], where the ring of correspondences $\text{Corr}^0(X,X)_E$ is replaced by a larger ring $\text{Corr}^0_{\text{mot}}(X,X)_E$ of motivated cycles by adjoining a certain homological cycle (the Lefschetz involution) to $\text{Corr}^0(X,X)_E$ (see [2] and [3]). The formal definition of a motivated cycle is as follows: For $X,Y \in V_K$, let $\text{pr}^{XY}_X$ denote the projection...
A motivated cycle is an element \((pr^X_Y)*((\alpha \cup *_{XY}(\beta)) \in H^*(X))\), where \(\alpha, \beta\) are \(E\)-linear combinations of algebraic cycles on \(X \times Y\) and \(*_{XY}\) is the Lefschetz involution on \(H^*(X \times Y)\) relatively to the line bundle \(\eta_{X \times Y} = [X] \otimes \eta_Y + \eta_X \otimes [Y]\) (with \(\eta_X\), resp. \(\eta_Y\), arbitrary ample line bundles on \(X\), resp. \(Y\)). Define \(\text{Corr}^0_{\text{mot}}(X, X)_E\) as the ring of the motivated codimension-\(d\)-cycles in analogy to \(\text{Corr}^0(X, X)_E\). A motive for motivated cycles with values in \(E\) is then a triple \(M = (X, p, m)\), where \(X \in \mathbb{V}_K\), \(m \in \mathbb{Z}\), and where \(p \in \text{Corr}^0_{\text{mot}}(X, X)_E\) is idempotent with respect to the composition of motivated cycles.

The category of motivated cycles is a neutral Tannakian category ([2], Section 4). Thus, by the Tannakian formalism (see [12]), every motive for motivated cycles \(M\) with values in \(E\) has attached an algebraic group \(G_M\) over \(E\) to it, called the motivic Galois group of \(M\). Similarly, granting Grothendieck’s standard conjectures, the category of motives has the structure of a Tannakian category. Thus, by the Tannakian formalism and by assuming the standard conjectures, every motive in the Grothendieck sense \(M\) has attached an algebraic group \(\tilde{G}_M\) to it, called the motivic Galois group of \(M\). The following lemma and the remark following it were communicated to the authors by Y. André:

3.1.1 Lemma. Let \(M = (X, p, n)\) be a motive for motivated cycles with motivic Galois group \(G_M\). Assume that that Grothendieck’s standard conjectures hold. Then the motive \(M\) is defined by algebraic cycles and the motivic Galois group \(\tilde{G}_M\) in the Grothendieck sense coincides with the motivic Galois group \(G_M\) of motives for motivated cycles.

Proof: The first claim follows from the fact that the standard conjectures predict the algebraicity of the Lefschetz involution in the auxiliary spaces \(X \times X \times Y\) which are used to define the projector \(p\) (see [28]). The last claim follows from the following interpretation of \(G_M\) (resp. \(\tilde{G}_M\)): The motivic Galois group for motivated cycles \(G_M\) is the stabilizer of all motivated cycles which appear in the realizations of all submotives of the mixed tensors \(M^\otimes n \otimes (M^*)^\otimes n\), where \(M^*\) denotes the dual of \(M\) (this can be seen using the arguments in [3], Chap. 6.3). Similarly, under the assumption of the standard conjectures, the motivic Galois group \(\tilde{G}_M\) is the stabilizer of all algebraic cycles which appear in the realizations of submotives of the mixed tensors of \(M\), see [3], Chap. 6.3. Under the standard conjectures these spaces coincide, so \(G_M = \tilde{G}_M\). \(\square\)

3.1.2 Remark. The above lemma can be strengthened or expanded as follows. It is possible to define unconditionally and purely in terms of algebraic cycles a group which, under the standard conjectures, will indeed be the motivic Galois group of the motive \(X = (X, \text{Id}, 0)\), where \(X\) is a smooth projective variety.
Namely, let $G_X^{\text{alg}}$ be the closed subgroup of $\prod_i \text{GL}(H^i(X)) \times \mathbb{G}_m$ which fixes the classes of algebraic cycles on powers of $X$ (viewed as elements of $H(X)^{\otimes n} \otimes \mathbb{Q}(r)$, the factor $\mathbb{G}_m$ acting on $\mathbb{Q}(1)$ by homotheties). Then the motivic Galois group $G_X$ is related to $G_X^{\text{alg}}$ as follows (cf. [3], 9.1.3): $G_X = \text{im}(G_X^{\text{alg}} \times Y \rightarrow G_X^{\text{alg}})$ for a suitable projective smooth variety $Y$. Under the standard conjectures, one may take $Y$ to be a point.

3.2 Results on families of motives

It is often useful to consider variations of motives over a base. Suppose one has given the following data:

(i) A smooth and geometrically connected variety $S$ over a field $K \subseteq \mathbb{C}$.

(ii) Smooth and projective $S$-schemes $X$ and $Y$ of relative dimensions $d_X$ and $d_Y$, equipped with invertible ample line bundles $L_X$ and $L_Y$.

(iii) Two $\mathbb{Q}$-linear combinations $Z_1$ and $Z_2$ of integral codimension-$d_X + d_Y$-subvarieties in $X \times_S X \times_S Y$ which are flat over $S$ such that the following holds for one (and thus for all) $s \in S(\mathbb{C})$: The class $q_s := (\text{pr}_{X_s \times X_s}^*(Z_1) \cup \ast (X_2)_s) \in H^{2d_X}(X_s \times X_s)(d_x) \subseteq \text{End}(H^*(X_s))$

satisfies $q_s \circ q_s = q_s$, where $\ast$ denotes the Lefschetz involution relative to $[(L_X/S)_s] \otimes [Y_s] + [X_s] \otimes [(L_Y/S)_s]$.

(iv) an integer $j$.

Then the assignment $s \mapsto (X_s, p_s, j)$, $s \in S(\mathbb{C})$, defines a family of motives in the sense of [2], Section 5.2. The following result is due to Y. André (see [2], Thm. 5.2 and Section 5.3):

3.2.1 Theorem. Let $s \mapsto (X_s, p_s, j)$, $s \in S(\mathbb{C})$, be a family of motives with coefficients in $E$ and let $H_E(M_s) := p_s(H_B^*(X_s, E))$ denote the $E$-realization of $M_s$, where $H_B^*(X_s, E)$ denotes the singular cohomology ring of $X_s(\mathbb{C})$. Then there exists a meager subset $\text{Exc} \subseteq S(\mathbb{C})$ and a local system of algebraic groups $G_s \leq \text{Aut}(H_E(M_s))$ on $S(\mathbb{C})$, such that the following holds:

(i) $G_{M_s} \subseteq G_s$ for all $s \in S(\mathbb{C})$.

(ii) $G_{M_s} = G_s$, if and only if $s \notin \text{Exc}$.

(iii) $G_s$ contains the image of a subgroup of finite index of $\pi_1^{\text{top}}(S(\mathbb{C}), s)$. 

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Let \( g'_s \) denote the Lie algebra of \( G_s \), and let \( h_s \) denote the Lie algebra of the Zariski closure of the image of \( \pi_1^{top}(S(\mathbb{C}), s) \). Then the Lie algebra \( h_s \) is an ideal in \( g'_s \).

Moreover, if \( S \) is an open subscheme of \( \mathbb{P}^n \) which is defined over a number field \( K \), then \( \text{Exc} \cap \mathbb{P}^n(K) \) is a thin subset of \( \mathbb{P}^n(K) \) (thin in the sense of [29]).

### 3.3 Motives with motivic Galois group \( G_2 \)

Let us call an algebraic group \( G \) which is defined over a subfield of \( \overline{\mathbb{Q}} \) to be of type \( G_2 \) if the group of \( \overline{\mathbb{Q}} \)-points \( G(\overline{\mathbb{Q}}) \) is isomorphic to the simple exceptional algebraic group \( G_2(\overline{\mathbb{Q}}) \) (see [6] for the definition of the algebraic group \( G_2 \)). It is the aim of this section to prove the existence of motives for motivated cycles having a motivic Galois group of type \( G_2 \).

We start in the situation of Cor. 2.4.2: Let \( S := \mathbb{A}^1_{\mathbb{Q}} \setminus \{0, 1\} \), let \( \pi_Q : \text{Hyp} \rightarrow S \) be as in Cor. 2.4.2, and let \( \pi_X : X \rightarrow S \) be the strict normal crossings compactification of Hyp given by Cor. 2.4.2. Let

\[
\begin{equation}
\mathcal{G} \simeq \Pi \left[ \ker \left( R^6(\pi_X)_*(\mathbb{Q}_\ell) \rightarrow R^6(\pi_{\bigoplus D_i})_* (\mathbb{Q}_\ell) \right) \right]
\end{equation}
\]

be as in Cor. 2.4.2, where the \( D_i \) are the components of the normal crossings divisor \( D = X \setminus \text{Hyp} \) over \( S \). We want to use the right hand side of this isomorphism to define a family of motives \((N_s)_{s \in S(\mathbb{C})}\) for motivated cycles such that the \( \overline{\mathbb{Q}} \)-realization of \( N_s \) coincides naturally with the stalk \( \mathcal{G}_s \) of \( \mathcal{G} \). This is done in three steps:

- Let
  \[
  \psi^*_s : H^*(X_s, \mathbb{Q}_\ell) \rightarrow H^*(\bigoplus_i D_{s,i}, \mathbb{Q}_\ell)
  \]
  be the map which is induced by the tautological map \( \psi_s := \bigoplus_i D_{s,i} \rightarrow X_s \). Let \( \Gamma_{\psi_s} \in \text{Corr}^0_{\text{mot}}(X_s, \bigoplus_i D_{s,i})_{\mathbb{Q}} \) be the graph of \( \psi_s \). Note that \( \Gamma_{\psi_s} \) can be seen as a morphism of motives
  \[
  \Gamma_{\psi_s} = \psi^*_s : h(X_s) \rightarrow h(\bigoplus_i D_{s,i}).
  \]

Since the category of motives for motivated cycles is abelian (see [2], Section 4), there exists a kernel motive

\[
K_s = (X_s, p_s, 0), \quad p_s \in \text{Corr}^0_{\text{mot}}(X_s, X_s)_{\mathbb{Q}},
\]

of the morphism \( \psi^*_s \) such that

\[
p_s(H^*(X_s, \mathbb{Q}_\ell)) = \ker \left( H^*(X_s, \mathbb{Q}_\ell) \rightarrow H^*(\bigoplus_i D_{s,i}, \mathbb{Q}_\ell) \right).
\]
• The Künneth projector \( \pi_6^{X_s} : H^*(X_s) \to H^i(X_s) \) is also contained in \( \text{Corr}^0_{\text{mot}}(X_s, X_s)_Q \) (see [2], Prop. 2.2).

• Let \( \Pi_s \) denote the following projector in \( \text{Corr}^0_{\text{mot}}(X_s, X_s)_Q \):

\[
\Pi_s := \frac{1}{2}(\Delta(X_s) - \Gamma_{\sigma_s}),
\]

where \( \Delta(X_s) \subseteq X_s \times X_s \) denotes the diagonal of \( X_s \) and \( \Gamma_{\sigma_s} \leq X_s \times X_s \) denotes the graph of \( \sigma_s \). By construction, the action of \( \Pi_s \) on \( H^6(X_s) \) is the same as the action (induced by) the idempotent \( \Pi = \frac{1}{2}(1 - \sigma) \) which occurs in Cor. 2.4.2.

By (3.3.1) one has

\[
\mathcal{G}_s = \Pi \left[ \ker \left( H^6(X_s, \mathbb{Q}_\ell) \to H^6(\prod D_{s,i}, \mathbb{Q}_\ell) \right) \right], \quad \forall s \in S(\mathbb{C}).
\]

Thus, by combining the above arguments, one sees that the stalk \( \mathcal{G}_s \) is the \( \mathbb{Q}_\ell \)-realization \( H_{\mathbb{Q}_\ell}(N_s) \) of the motives

\[
N_s := (X_s, \Pi_s \cdot p_s \cdot \pi_6^{X_s}, 0) \quad \text{with} \quad \Pi_s \cdot p_s \cdot \pi_6^{X_s} \in \text{Corr}^0_{\text{mot}}(X_s, X_s)_Q.
\]

We set

\[
M_s := N_s(3) = (X_s, \Pi_s \cdot p_s \cdot \pi_6^{X_s}, 3).
\]

3.3.1 Theorem. The motives \( M_s, s \in S(\mathbb{C}) \), form a family of motives such that for any \( s \in S(\mathbb{Q}) \) outside a thin set, the motive \( M_s \) has a motivic Galois group of type \( G_2 \).

Proof: That the motives \( (N_s)_{s \in S(\mathbb{C})} \) form a family of motives (in the sense of Section 3.2) can be seen by the following arguments: Let \( \Gamma_\sigma \subseteq X \times S X \) be the graph of the automorphism \( \sigma \) and let \( \Delta(X) \subseteq X \times S X \) be the diagonal. By Cor. 2.4.2, the projectors \( \Pi_s \) arise from the \( \mathbb{Q} \)-linear combination of schemes \( \frac{1}{2}(\Delta(X) - \Gamma_\sigma) \) over \( S \) via base change to \( s \). The Künneth projector \( \pi_6^{X_s} \in \text{Corr}^0_{\text{mot}}(X_s, X_s) \) is invariant under the action of \( \pi_1(S) \). It follows thus from the theorem of the fixed part as in [2], Section 5.1, that \( \pi_6^{X_s} \) arises from the restriction of the Künneth projector \( \pi_6^{\hat{X}} \), where \( \hat{X} \) denotes a normal crossings compactification over \( \mathbb{Q} \) of the morphism \( \pi_X : X \to S \) (which exists by Hironaka [19]). By [2], Prop. 2.2., the projector \( \pi_6^{\hat{X}} \) is a motivated cycle. Since this cycle gives rise to the Künneth projector \( \pi_6^{X_s} \) on one fibre via restriction, one can use the local triviality of the family \( X/S \) to show that the restriction of \( \pi_6^{\hat{X}} \in \text{Corr}^0(\hat{X} \times \hat{X}) \) to \( X \times S X \) gives
rise to a family of motives \((X_s, \pi^6_{X_s}, 0)\). A similar argument applies to the projectors \(p_s\). Therefore the motives \(M_s = N_s(3), \ s \in S(\mathbb{C})\), form indeed a family of motives.

Let \(G_s^\text{an}\) be the local system on \(S(\mathbb{C})\) which is defined by the composition of the natural map \(\pi^{1\text{top}}_1(S(\mathbb{C}), s) \to \pi_1(S, s)\) with the monodromy representation of \(G\). By the comparison isomorphism between singular- and étale cohomology, the local system \(G_s^\text{an}\) coincides with the local system which is defined by the singular \(\mathbb{Q}_l\)-realizations \(H_{\mathbb{Q}_l}(N_s)\) of the above family \((N_s)_{s \in S(\mathbb{C})}\). It follows from Thm. 2.4.1 that \(\mathcal{G}_{|A_1 \setminus \{0, 1\}} \simeq \mathcal{H}(\mathbf{1}, \mathbf{1})|_{A_1 \setminus \{0, 1\}}\), where \(\mathcal{H}(\mathbf{1}, \mathbf{1})\) is as in Thm. 1.3.1. It follows thus from Thm. 1.3.1 (i) that the image of \(\pi^{1\text{top}}_1(S(\mathbb{C}), s) \to \pi_1(S, s)\) in \(\text{Aut}(H_{\mathbb{Q}_l}(N_s)) \simeq \text{GL}_7(\mathbb{Q}_l)\) under the monodromy map is Zariski dense in the group \(G_2(\mathbb{Q}_l)\).

By Thm. 3.2.1, (i) and (ii), and since \(S\) is open in \(\mathbb{P}^1\), there exists a local system \((G_s)_{s \in S(\mathbb{C})}\) of algebraic groups with \(G_s \leq \text{Aut}(H_{\mathbb{Q}_l}(M_s))\) such that the following holds: The motivic Galois group \(G_{M_s}\) is contained in \(G_s\), and there exists a thin subset \(\text{Exc} \subseteq \mathbb{Q}\) such that if \(s \in \mathbb{Q} \setminus \text{Exc}\), then \(G_s = G_{M_s}\). By Thm. 3.2.1 (iii), \(G_s\) contains a subgroup of finite index of the image of \(\pi_1(S(\mathbb{C}), s)\). Thus, by what was said above, the group \(G_s\) contains the group \(G_2\) for all \(s \in S(\mathbb{C})\). Let \(g_2\) denote the Lie algebra of the group \(G_2\). Let \(g_s^\prime\) denote the Lie algebra of the group \(G_s\). By Thm. 3.2.1 (iv), the Lie algebra \(g_2\) is an ideal of \(g_s^\prime\). It follows from this and from \(N_{\text{GL}_7}(G_2) = \mathbb{G}_m \times G_2\) (where \(\mathbb{G}_m\) denotes the subgroup of scalars of \(\text{GL}_7\)) that \(G_{N_s} \leq \mathbb{G}_m \times G_2\) for all \(s \in S(\mathbb{C})\). The representation \(\rho_{N_s}\) of \(G_{N_s}\), which belongs to the motive \(N_s\) under the Tannaka correspondence, is therefore a tensor product \(\chi \otimes \rho\), where \(\chi : G_{N_s} \to \mathbb{G}_m\) is a character and \(\rho : G_{N_s} \to \text{GL}_7\) has values in \(G_2 \leq \text{GL}_7\). Let \(A_s\) denote the dual of the motive which belongs to \(\chi\) under the Tannaka correspondence. Then \(G_{N_s \otimes A_s} = G_2\) for all \(s \in \mathbb{Q} \setminus \text{Exc}\).

We claim that for \(s \in \mathbb{Q} \setminus \text{Exc}\), the motive \(A_s\) is the motive \((\text{Spec}(s), \text{Id}, 3)\) : The Galois representation which is associated to the motive \(N_s\) is equivalent to that of the stalk of \(G\) at \(s\) (viewed as \(\mathbb{Q}\)-point) and is therefore pure of weight 6. By [24], Thm. 3.1, any rank-one motive over \(\mathbb{Q}\) is a Tate twist of an Artin motive. Therefore, the \(\ell\)-adic realization of any rank-one motive over \(\mathbb{Q}\) is a power of the cyclotomic character with a finite character. In order that \(G_{N_s \otimes A_s}\) is contained in \(G_2\), the \(\ell\)-adic realization of \(A_s\) has to be of the form \(\epsilon \otimes \chi^2\), where \(\epsilon\) is of order \(\leq 2\). But if the order of \(\epsilon\) is equal to 2, we derive a contradiction to Thm. 1 of the Appendix. It follows that \(A_s\) is the motive \((\text{Spec}(s), \text{Id}, 3)\) and that the motivic Galois group of \(M_s = N_s \otimes A_s = N_s(3)\) is of type \(G_2\). \(\square\)
3.3.2 Remark. Under the hypothesis of the standard conjectures, Thm. 3.3.1 and Lemma 3.1.1 imply the existence of Grothendieck motives whose motivic Galois group is of type $G_2$. Moreover, it follows from Rem. 3.1.2 that, independently from the standard conjectures, there is a projective smooth variety $X$ over $\mathbb{Q}$ such that the group $G^\text{alg}_X$ (which is defined in Rem. 3.1.2) has a quotient $G_2$.

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