THE CAUCHY PROBLEMS FOR THE 2D COMPRESSIBLE EULER EQUATIONS AND IDEAL MHD SYSTEM ARE ILL-POSED IN $H^\frac{7}{4}(\mathbb{R}^2)$

XINLIANG AN$^1$, HAOYANG CHEN$^2$, AND SILU YIN$^3$

Abstract. In a fractional Sobolev space $H^\frac{7}{4}(\mathbb{R}^2)$, we prove the desired low-regularity ill-posedness results for the 2D compressible Euler equations and the 2D ideal compressible MHD system. For the Euler equations, it is sharp with respect to the regularity of the fluid velocity and density. The mechanism behind the obtained ill-posedness is the instantaneous shock formation.

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1. Introduction and Main results

We prove the $H^\frac{7}{4}$ ill-posedness for Cauchy problems of the 2D compressible Euler equations and the 2D ideal compressible MHD system. Both of them are physical systems with multiple
wave-speeds. The 2D ideal compressible MHD system takes the form:

\[
\begin{cases}
\partial_t \varrho + \nabla \cdot (\varrho u) = 0, \\
\varrho \{\partial_t + (u \cdot \nabla)\} u + \nabla p + \frac{\mu_0}{2} \nabla |H|^2 - \mu_0 H \cdot \nabla H = 0, \\
\partial_t H + (u \cdot \nabla) H - H \cdot \nabla u + H \nabla \cdot u = 0, \\
\partial_t S + (u \cdot \nabla) S = 0, \\
\nabla \cdot H = 0.
\end{cases}
\]

(1.1)

Here, \(\mu_0\) is the magnetic permeability constant, \(\varrho\) is the fluid density\(^1\), \(u = (u_1, u_2) \in \mathbb{R}^2\) is the fluid velocity, \(H = (H_1, H_2) \in \mathbb{R}^2\) is the magnetic field intensity, \(S\) is the entropy and \(p\) is the pressure satisfying the equation of state \(p = p(\varrho, S)\). We consider the polytropic gas. The pressure \(p(\varrho, S)\) obeys the following equation of state

\[p = A \varrho^\gamma,\]

where \(A\) is a positive constant and \(\gamma > 1\) is the adiabatic gas constant. If the magnetic field vanishes, i.e., \(H_1 = H_2 = 0\), the above system (1.1) reduces to the following 2D compressible Euler equations allowing non-trivial entropy and vorticity

\[
\begin{cases}
\partial_t \varrho + \nabla \cdot (\varrho u) = 0, \\
\varrho \{\partial_t + (u \cdot \nabla)\} u + \nabla p = 0, \\
\partial_t S + (u \cdot \nabla) S = 0.
\end{cases}
\]

(1.2)

In this paper, we first show that the Cauchy problems of the 2D ideal compressible MHD equations (1.1) are ill-posed in the fractional Sobolev space \(H^{\frac{7}{4}}(\mathbb{R}^2)\), in contrast to the \(H^2\) ill-posedness in 3D that we obtained in [4]. Furthermore, this ill-posedness is driven by the instantaneous shock formation. To our best knowledge, this is the first nonlinear result of low regularity ill-posedness for the 2D ideal compressible MHD. In the absence of the magnetic field, the ideal MHD system reduces to the compressible Euler equations. Our work thus also serves as the first low-regularity ill-posedness result for the 2D compressible Euler equations (1.2). For these 2D Euler equations, this \(H^{\frac{7}{4}}\) ill-posedness is the desired result according to the corresponding low-regularity local well-posedness result by Zhang [39], where she proved the local well-posedness in \(H^s\) for \(s > \frac{7}{4}\) with respect to the regularity of fluid velocity, vorticity and density. See Remark 1.4 for a more detailed discussion.

1.1. Background and history. Our research is motivated by a series of classic works on low-regularity ill-posedness and shock formation. We first review studies on low-regularity ill-posedness. Under planar symmetry, Lindblad [22, 24] constructed sharp counterexamples to the local well-posedness for semilinear and quasilinear wave equations in three dimensions. Low-regularity local well-posedness for the quasilinear wave equation was proven by Tataru-Smith [34]. They showed that for \(n\) dimensional quasilinear wave equations, the Cauchy problems are locally well-posed in \(H^s(\mathbb{R}^n)\) with \(s > n/2 + 3/4\) for \(n = 2\) and \(s > (n + 1)/2\) for \(n = 3, 4, 5\). For \(n = 3\), via the vector field method, Wang [35] gave a different proof and re-obtained Smith-Tataru’s conclusion. See also Zhou-Lei [42] for radially symmetric global solutions in low regularity. In [17], Ettinger-Lindblad generalized the above results to the Einstein’s equations

\[1\] In our analysis, the value of \(\varrho\) is close to 1.
and constructed a sharp counterexample for local well-posedness of Einstein vacuum equations in wave coordinates. An exploration by Granowski [18] later showed that Lindblad’s ill-posedness in [24] is stable under perturbations out of planar symmetry. In [3–5], we generalized Lindblad’s work on a scalar wave equation in [24] by showing that the Cauchy problems for 3D elastic waves and 3D ideal compressible MHD system are ill-posed in $H^3(\mathbb{R}^3)$ and $H^2(\mathbb{R}^3)$, respectively. In the absence of magnetic fields, the $H^2$ ill-posedness in [4] also holds for 3D compressible Euler equations. According to the low-regularity local well-posedness result in [36] by Wang, this $H^2$ ill-posedness is sharp with respect to the regularity of the fluid velocity $u$ and density $\rho$. Unlike the 3D case, in 2D, as suggested by Smith-Tataru [34], the desired sharp result would be in the fractional Sobolev space $H^{7/4}(\mathbb{R}^2)$. Recently, Ohlmann [29] generalized Lindblad’s result [24] to the 2D case and proved the ill-posedness for a 2D quasilinear wave equation in the logarithmic Sobolev space $H^{7/4}(\ln H)^{-\beta}$ with $\beta > \frac{1}{2}$. This space is slightly more singular than $H^{7/4}$. For 2D Euler equations, the low-regularity local well-posedness was proved by Zhang [39] in $H^s$ with $s > \frac{7}{4}$ for initial fluid velocity, vorticity and density. In this paper, we will derive the desired $H^{7/4}$ ill-posedness for 2D compressible Euler equations and for the more general 2D ideal MHD system. We employ the wave-decomposition approach introduced by John [20] to decompose our system. Compared with the 3D case, the decomposed system here is equipped with a different structure, which is owing to the correct selection of the eigenvectors for the coefficient matrix in 2D. Using the initial-data construction adopted in this paper, in [6] we also report the $H^{11/4}$ ill-posedness for the 2D elastic wave system.

For the incompressible case, the corresponding low-regularity ill-posedness in Sobolev spaces was proved by Bourgain-Li [9] in 2D and [10] in 3D. We also refer to a different proof by Elgindi-Jeong [16] for 2D, and refer to Bourgain-Li [8] for ill-posedness in $C^m$ spaces.

As mentioned above, our $H^{7/4}$ ill-posedness is driven by the instantaneous shock formation. The studies on shock formation for Euler equations and other hyperbolic systems have a long history. In 1D case, the classical method is to analyze along the characteristics. For compressible Euler equations, Riemann [30] proved the finite-time shock formation via using Riemann invariants. This was later extended to $2 \times 2$ genuinely nonlinear strictly hyperbolic system by Lax [21]. For larger $n \times n$ hyperbolic systems, John [20] developed the decomposition–of–waves method to prove shock formation for genuinely nonlinear strictly-hyperbolic system. See also [25] by Liu and [41] by Zhou for generalizations of John’s approach to linearly degenerate hyperbolic systems. In [14], Christodoulou-Perez also studied the shock formation for electromagnetic plane waves in nonlinear crystals. The equations of the electromagnetic plane waves satisfy a first-order strictly hyperbolic and genuinely nonlinear system. In more than one spatial dimensions, with no symmetry assumption, Alinhac [12] proved singularity formation for solutions to quasilinear wave equations via a Nash-Moser iteration scheme. Based on this approach, Yin-Zheng-Jin [38] constructed the blow-up solution for 2D irrotational compressible Euler equations. This approach does not reveal information beyond the first blow-up point. In [7], Christodoulou developed a geometric approach and provided a detailed understanding

\[\text{See also [15] by Disconzi-Luo-Mazzzone-Speck and [40] by Zhang-Andersson}
\]

\[\text{Ohlmann studied the model equation } \Box u = (Du)D^2u, \text{ with } D = \partial_{x_1} - \partial_t.
\]

\[\text{There is a logarithmic loss between Ohlmann’s result and the desired } H^{7/4} \text{ ill-posedness.}\]
and a complete description of shock formation for 3D irrotational compressible Euler equations. This work was later extended to a large class of equations, seeing [13, 26–28, 31–33]. In particular, applying Christodoulou’s geometric approach, Luk-Speck [26, 27] constructed shock formation for 2D and 3D compressible Euler equations, allowing the presence of non-trivial vorticity. Buckmaster-Shkoller-Vicol [11, 12] also gave a different approach via using self-similar variables. For 2D compressible MHD, very little is known about its shock formation and the corresponding low-regularity solutions. For its small data global well-posedness results, we also refer to Hu [19], Wu-Wu [37] and references therein.

1.2. Main results. We construct counterexamples to $H^\frac{7}{4}$ local well-posedness for the 2D ideal compressible MHD system (1.1). The ill-posedness is driven by the instantaneous shock formation. We work under plane symmetry. By Gauss’s Law, the first component $H^1_1$ of the magnetic field stays to be a constant. We proceed with the $H^1_1 \neq 0$ case and the $H^1_1 = 0$ case, respectively. In the former case, the system is strictly hyperbolic. While for $H^1_1 = 0$, it is not.

Besides $H^1_1$, the remaining unknowns $\Phi = (u_1, u_2, \varrho - 1, H_2, S)^T$ then satisfy a quasilinear hyperbolic system stated in (2.2). Our main theorem is as below.

**Theorem 1.1.** The Cauchy problems of the 2D ideal compressible MHD equations (1.1) are ill-posed in $H^\frac{7}{4}(\mathbb{R}^2)$ in the following sense:

There exists a family of compactly supported, smooth initial data $\Phi_0(\eta)$ satisfying

$$
\|\Phi_0(\eta)\|_{H^\frac{7}{4}(\mathbb{R}^2)} \to 0 \text{ as } \eta \to 0
$$

with $\eta > 0$ being a small parameter. For each initial datum $\Phi_0(\eta)$, there is a finite $T^*_\eta > 0$ such that the corresponding Cauchy problem of the 2D ideal MHD system (1.1) admits a local-in-time regular solution $\Phi_\eta \in C^\infty(\mathbb{R}^2 \times [0, T^*_\eta])$. Moreover, the following statements hold:

i) (Instantaneous shock formation) For each solution $\Phi_\eta$, a shock forms at $T^*_\eta$. More precisely, we have that $|\Phi_\eta|$ remains small, while $|\nabla \Phi_\eta|$ blows up at $T^*_\eta$. Moreover, the lifespan vanishes as $\eta \to 0$, i.e., $T^*_\eta \to 0$, which means that the constructed initial data $\Phi_0(\eta)$ give rise to the instantaneous shock formation.

ii) (Blow-up of $H^1$-norm) The $H^1$-norm of the solution $\Phi_\eta$ blows up at $T^*_\eta$. In particular, we have

$$
\|\nabla u_1(\cdot, T^*_\eta)\|_{L^2(\Omega_{T^*_\eta})} = +\infty, \quad \|\nabla \varrho(\cdot, T^*_\eta)\|_{L^2(\Omega_{T^*_\eta})} = +\infty,
$$

where $\Omega_{T^*_\eta}$ is a spatial neighborhood of the first (shock) singularity.

**Remark 1.1.** In our proof, we algebraically reformulate the MHD system (1.1) via a wave-decomposition approach. The success of our analysis crucially relies on a key fact that all the coefficients of the decomposed hyperbolic system are uniformly bounded. For a real physical system without any imposed mathematical condition on its structure, it is far from trivial that the coefficients $c_{im}, \gamma_{im}, \gamma_{km}$ of the decomposed system defined in (2.19)-(2.21) are bounded. Our carefully designed eigenvectors (2.4)-(2.6) ensure the required boundedness. Instead of directly verifying around 300 coefficients as for 3D in [4], in this paper, for the 2D case, we find a concise

\footnote{For $H_1 = 0$ case, see Section 6 for the details.}
proof via analyzing the leading-order terms of the potential problematic coefficients. See Section 2.3 for the discussion.

Remark 1.2. To construct the suitable initial data, for here we get inspired by the ideas in Lindblad [23, 24], Ohlmann [29] and An-Chen-Yin [4] and we extend them. Our initial data take the form

\[ \theta \big| \ln(x) \big|^{\alpha} \chi_\eta(x) \psi \left( \frac{|\ln(x)|^m x^2}{\sqrt{2}} \right) \]

Ohlmann in [29] set the initial data to be \( f_{\varepsilon}(x) \); \( \phi(x) \); \( \chi_\eta \) for \( x > 0 \), \( 0 \) for \( x \leq 0 \).

Remark 1.3. Besides the just-mentioned constructed initial data, compared with Ohlmann’s approach in [29], our result also exhibits the following differences. First, we investigate the physical MHD system (1.1) and the Euler equations (1.2). The specific structures of these two systems are crucial for our analysis. Ohlmann [29] studied a quasilinear equation

\[ 2u = (Du) D^2 u, \quad \text{with } D = \partial_x - \partial_t. \]

Second, in [29] by Ohlmann, there is a logarithmic loss for the regularity, i.e., the ill-posedness in [29] holds in the logarithmic Sobolev space \( H^\frac{7}{4} (\ln H)^{-\beta} \) with \( \beta > \frac{1}{2} \). For the 2D ideal MHD system and the Euler equations, we prove the desired \( H^\frac{7}{4} \) ill-posedness. Third, in our work, the underlying mechanism behind our ill-posedness is explored. And we demonstrate that the ill-posedness results from the instantaneous shock formation. In [29], the mechanism driving the \( H^\frac{7}{4} (\ln H)^{-\beta} \) ill-posedness, seems to be further explored.

As a corollary of the above theorem, in the absence of magnetic fields, i.e., \( H_1 = H_2 = 0 \), we have the following \( H^\frac{7}{4} \) ill-posedness for the 2D compressible Euler equations (1.2) allowing non-trivial entropy and vorticity.

Theorem 1.2. For the 2D compressible Euler equations (1.2), we have that its Cauchy problem is \( H^\frac{7}{4} \) ill-posed with respect to the fluid velocity and density. More precisely, there exits a family of compactly supported smooth initial data for (1.2) satisfying

\[ \| \theta_0(n) \|_{H^\frac{7}{4}(\mathbb{R}^2)} + \| u_0(n) \|_{H^\frac{7}{4}(\mathbb{R}^2)} + \| s_0(n) \|_{H^\frac{7}{4}(\mathbb{R}^2)} \to 0 \quad \text{as} \quad \eta \to 0. \]

After a finite time \( T^*_\eta \), a shock forms and the solution ceases to be smooth. Tied to this family of initial data, the shock formation is instantaneous, i.e., \( T^*_\eta \to 0 \) as \( \eta \to 0 \). Furthermore, the \( H^1 \)-norm of the velocity \( u_1 \) and density \( \phi \) blow up at the shock formation time \( T^*_\eta \):

\[ \| u_1(n)(\cdot, T^*_\eta) \|_{H^1(\Omega_T)} = +\infty, \quad \| \phi(n)(\cdot, T^*_\eta) \|_{H^1(\Omega_T)} = +\infty \]

with \( \Omega_{T^*_\eta} \) being a spatial region around the first (shock) singularity.
Remark 1.4. For the 2D compressible Euler equations, the above $H^{7/4}$ ill-posedness is the desired result. It is sharp with respect to the regularity of the fluid velocity $u$ and density $\varrho$. As is well-known, for the 2D compressible Euler equations the classical local well-posedness result holds for initial data in $H^s$ with $s > 2$. Recently, Zhang [39] improved the classical result and proved the low regularity local well-posedness in $H^s$ space with $s > 7/4$ for fluid velocity, vorticity and density. Zhang’s result left a room to be improved for the regularity of fluid vorticity. See the following two pictures:

![Diagram of regularity for the density and vorticity](image)

For 2D incompressible Euler equations, referring to Bourgain-Li [9] and Elgindi-Jeong [16], they proved $H^1$ ill-posedness with respect to vorticity. Our ill-posedness result can serve as a sharp counterpart with respect to the fluid velocity and density.

1.3. Organization of the paper. As in [4,23,24], we investigate the ill-posedness under plane symmetry. Suppose that $U(x,t) = (u_1, u_2, \varrho, H_1, H_2, S)(x,t)$ is a planar symmetric solution to (1.1). Then, by Gauss’s Law, we have that $H_1 \equiv \text{const.}$ The ideal MHD system (1.1) is then reduced to a $5 \times 5$ first-order hyperbolic system:

$$\partial_t \Phi + A(\Phi) \partial_x \Phi = 0$$

with $\Phi = (u_1, u_2, \varrho - 1, H_2, S)^T$. Then we algebraically decompose the system via employing John’s wave-decomposition method in [20]. The decomposed system satisfies (2.16)-(2.18), which forms a transport system along different characteristics. In Section 2, we carefully explore the structures of the corresponding system.

In Section 3, we construct vanishing $H^{7/4}$ initial data to deduce the ill-posedness for the Cauchy problems of (1.1). In contrast with the 3D case in [4], our construction of $H^{7/4}$ initial data here is more subtle. And it is inspired by Lindblad [23,24], Ohlmann [29] and An-Chen-Yin [4]. We modify and extend the construction in [4,23,24,29].

We first consider the $H_1 \neq 0$ case. Based on the decomposition of waves, in Section 4 we prove the (first) shock formation at time $T^*_{\eta} < +\infty$. And we exhibit the detailed quantitative information all the way up to the shock formation time $T^*_{\eta}$. The key point here is to trace the evolution of $\rho_i$ ($i = 1, \cdots, 5$): the so-called inverse density of the $i$th characteristics. We derive a positive uniform lower bound for $\{\rho_i\}_{i=2,\cdots,5}$. Whereas, for the first family of characteristics,
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its inverse density $\rho_1(z_0, t) \to 0$ as $t \to T^*_\eta$, which indicates the compression of characteristics and whence the shock formation.

Prescribing the aforementioned initial data, we have $T^*_\eta \to 0$ as $\eta \to 0$. In Section 5, we then estimate the $H^1(\mathbb{R}^2)$ norm of the solution to (1.1) at time $T^*_\eta$. In a suitable constructed spatial region $\Omega_{T^*_\eta}$, we deduce that $\|\Phi(\cdot, T^*_\eta)\|_{H^1(\Omega_{T^*_\eta})} = +\infty$. This further implies that the top order $\dot{H}^{7/4}$ norm also blows up within $\Omega_{T^*_\eta}$, and the desired strong ill-posedness result. Furthermore, both the inflation of $H^1$ norm and the vanishing of $T^*_\eta$ are driven by the instantaneous shock formation: $\rho_1(z_0, t) \to 0$ as $t \to T^*_\eta$.

In a similar manner, we treat the non-strictly hyperbolic $H_1 \equiv 0$ case. The details are in Section 6. Finally, in Section 7, we apply the above method and results for the 2D compressible Euler equations and obtain the desired $H^{7/4}$ ill-posedness.

1.4. Acknowledgements. XA is supported by NUS startup grant R-146-000-269-133, and MOE Tier 1 grants R-146-000-321-144 and R-146-000-335-114. HC acknowledges the support of MOE Tier 1 grant R-146-000-335-114 and NSFC (Grant No. 12171097). SY is supported by NSFC under Grant No. 12001149 and Zhejiang Provincial Natural Science Foundation of China under Grant No. LQ19A010006.

2. Preliminaries

In this section, we apply the wave-decomposition method to the 2D ideal compressible MHD system (1.1) and analyze its structure.

2.1. The corresponding hyperbolic system. We establish the ill-posedness under planar symmetry. The planar symmetric solution of (1.1) satisfies

$$U(x_1, x_2, t) = (u_1, u_2, \varrho, H_1, H_2, S)(x_1, x_2, t) = (u_1, u_2, \varrho, H_1, H_2, S)(x_1, t).$$

For notational simplicity, we denote $x_1$ by $x$ from now on. Using Gauss’s Law, i.e., the last equation in (1.1), we have

$$\nabla \cdot H = \frac{\partial H_1}{\partial x_1} + \frac{\partial H_2}{\partial x_2} = \frac{\partial H_1}{\partial x} = 0, \text{ hence } H_1(x, t) = H_1(t).$$

Moreover, following from the third equation of (1.1), it holds that $\partial_t H_1 = 0$. Thus, $H_1(x, t)$ remains to be a constant during the evolution. In this paper, we assume that the constant $H_1$ is sufficiently small such that

$$H_1^2 \ll \min\{\mu_0^{-1} A_\gamma, 1\},$$

where $\mu_0$ denotes the magnetic permeability constant, $A$ is a positive constant and $\gamma > 1$ is the adiabatic gas constant. Via (1.1), the remaining unknowns $(u_1, u_2, \varrho, H_2, S)^T$ then satisfy the
following system:

\[
\begin{aligned}
\frac{\partial}{\partial t} u_1 + u_1 \partial_x u_1 + \frac{c^2}{\rho} \partial_x \rho + \frac{\mu_0 H_2}{\rho} \partial_x H_2 + \frac{c^2}{\gamma} \partial_x S &= 0, \\
\frac{\partial}{\partial t} u_2 + u_1 \partial_x u_2 - \frac{\mu_0 H_1}{\rho} \partial_x H_2 &= 0, \\
\frac{\partial}{\partial t} \rho + \rho \partial_x u_1 + u_1 \partial_x \rho &= 0, \\
\frac{\partial}{\partial t} H_2 + u_1 \partial_x H_2 + H_2 \partial_x u_1 - H_1 \partial_x u_2 &= 0, \\
\frac{\partial}{\partial t} S + u_1 \partial_x S &= 0,
\end{aligned}
\]  

(2.1)

where \( c = \sqrt{\frac{\partial_p \rho}{\rho}} \) is the sound speed. Let \( \Phi = (u_1, u_2, \rho-1, H_2, S)^T \). The quasilinear hyperbolic system (2.1) then takes the form below

\[
\frac{\partial}{\partial t} \Phi + A(\Phi) \partial_x \Phi = 0,
\]  

(2.2)

where

\[
A(\Phi) = \begin{pmatrix}
  u_1 & 0 & \frac{c^2}{\rho} & \frac{\mu_0 H_2}{\rho} & \frac{c^2}{\gamma} \\
  0 & u_1 & 0 & -\frac{\mu_0 H_1}{\rho} & 0 \\
  \rho & 0 & u_1 & 0 & 0 \\
  H_2 & -H_1 & 0 & u_1 & 0 \\
  0 & 0 & 0 & 0 & u_1
\end{pmatrix}.
\]

Via calculations, we have that the eigenvalues of the coefficient matrix \( A(\Phi) \) are

\[
\lambda_1 = u_1 + C_f, \quad \lambda_2 = u_1 + C_s, \quad \lambda_3 = u_1, \quad \lambda_4 = u_1 - C_s, \quad \lambda_5 = u_1 - C_f
\]  

(2.3)

with

\[
C_f = \left\{ \frac{\mu_0}{2 \rho} (H_1^2 + H_2^2) + \frac{c^2}{2} + \frac{1}{2} \sqrt{\left[\frac{\mu_0}{\rho} (H_1^2 + H_2^2) + c^2\right]^2 - \frac{4 \mu_0}{\rho} H_1^2 c^2} \right\}^{1/2},
\]

\[
C_s = \left\{ \frac{\mu_0}{2 \rho} (H_1^2 + H_2^2) + \frac{c^2}{2} - \frac{1}{2} \sqrt{\left[\frac{\mu_0}{\rho} (H_1^2 + H_2^2) + c^2\right]^2 - \frac{4 \mu_0}{\rho} H_1^2 c^2} \right\}^{1/2}.
\]

A quasilinear hyperbolic system is called strictly hyperbolic if the eigenvalues of its coefficient matrix are all real and distinct. We first consider the case of \( H_1 \neq 0 \). For this case, provided that \( |\Phi| < 2 \delta \) with \( \delta \) being sufficiently small, the ideal MHD system (2.2) is strictly hyperbolic, and the above five eigenvalues are completely distinct,

\[
\lambda_5(\Phi) < \lambda_4(\Phi) < \lambda_3(\Phi) < \lambda_2(\Phi) < \lambda_1(\Phi).
\]
We then choose the right eigenvectors of \( A(\Phi) \) as follows:

\[
 r_1 = \begin{pmatrix} 
 1 \\
 \frac{(C_f^2-c^2)H_1}{C_f^2H_2} \\
 \frac{\rho H_2}{C_s} \\
 \frac{\rho(C_f^2-c^2)}{\mu_0C_s} \\
 0 
 \end{pmatrix}, \\
 r_2 = \begin{pmatrix} 
 \frac{H_2}{(C_f^2-c^2)H_1} \\
 \frac{\rho H_2}{C_s} \\
 \frac{\rho(C_f^2-c^2)}{\mu_0C_s} \\
 0 
 \end{pmatrix}, \\
 r_3 = \begin{pmatrix} 
 0 \\
 0 \\
 -\frac{\rho}{\gamma} \\
 1 
 \end{pmatrix}, \\
 r_4 = \begin{pmatrix} 
 \frac{H_2}{(C_f^2-c^2)H_1} \\
 \frac{\rho H_2}{C_s} \\
 \frac{\rho(C_f^2-c^2)}{\mu_0C_s} \\
 0 
 \end{pmatrix}, \\
 r_5 = \begin{pmatrix} 
 \frac{1}{(C_f^2-c^2)H_1} \\
 \frac{\rho H_2}{C_s} \\
 \frac{\rho(C_f^2-c^2)}{\mu_0C_s} \\
 0 
 \end{pmatrix}.
\]

The corresponding left eigenvectors are set to be dual to the right ones satisfying:

\[
l_i r_j = \delta_{ij} \quad \text{for} \quad i,j = 1, \cdots, 5.
\] (2.5)

Here \( \delta_{ij} \) is the Kronecker symbol. Specifically, we list these \( l_i \):

\[
l_1 = \frac{C_f^4H_2}{2H_2^2C_f^2 + 2H_1^2(C_f^2-c^2)^2}(H_2, -\frac{(C_f^2-c^2)H_1}{C_f^2}, \frac{c^2H_2}{C_f}, \frac{C_f^2 - c^2}{\gamma C_f}),
\]

\[
l_2 = \frac{C_f^4H_2}{2H_2^2C_f^2 + 2H_1^2(C_f^2-c^2)^2}(1, -\frac{(C_f^2-c^2)H_1}{C_f^2C_sH_2}, \frac{c^2}{C_sH_2}, \frac{C_f^2 - c^2}{\gamma C_f}),
\]

\[
l_3 = (0, 0, 0, 0, 1),
\]

\[
l_4 = \frac{C_f^4H_2}{2H_2^2C_f^2 + 2H_1^2(C_f^2-c^2)^2}(1, -\frac{(C_f^2-c^2)H_1}{C_f^2C_sH_2}, -\frac{C_f^2 - c^2}{\gamma C_f}),
\]

\[
l_5 = \frac{C_f^4H_2}{2H_2^2C_f^2 + 2H_1^2(C_f^2-c^2)^2}(H_2, -\frac{(C_f^2-c^2)H_1}{C_f^2}, -\frac{c^2H_2}{C_f}, \frac{C_f^2 - c^2}{\gamma C_f}).
\]

**Remark 2.1.** Designing proper right and left eigenvectors is crucial in our approach. As what we will explain below, the above selection of eigenvectors guarantees that the equivalent form of (1.1) satisfies the genuinely nonlinear condition, as well as the boundedness condition for the corresponding coefficients in the decomposed system.

2.2. Decomposition of waves. Employing the left and right eigenvectors designed in the above subsection, we introduce

\[
w_i := l_i \partial_x \Phi \quad \text{for} \quad i = 1, \cdots, 5.
\] (2.7)

By (2.5) and (2.7), there holds the following formula

\[
\partial_x \Phi = \sum_{k=1}^{5} w_k r_k.
\] (2.8)
Before writing down the explicit equations for \( w_i \), we first introduce the characteristic and bi-characteristic coordinates. We define the \( i^{th} \) characteristic originating from \( z_i \) to be \( C_i(z_i) = \left\{ (X_i(z_i,t), t) : 0 \leq t \leq T \right\} \) with \( X_i(z_i,t) \) solving
\[
\begin{align*}
\partial_t X_i(z_i,t) &= \lambda_i \Phi(X_i(z_i,t), t), \quad t \in [0,T], \\
\rho_i(z_i,0) &= z_i.
\end{align*}
\] (2.9)

For any \((x,t) \in \mathbb{R} \times [0,T] \), there is a unique \((z_i, s_i) \in \mathbb{R} \times [0,T] \) such that \((x,t) = (X_i(z_i, s_i), s_i) \). Hence, we call \((z_i, s_i) \) the characteristic coordinate. In addition, we can also locate the points using the intersection of two transversal characteristics \( C_i(y_i) \) and \( C_j(y_j) \) when \( i \neq j \). For any \((x,t) \in \mathbb{R} \times [0,T] \), there is a unique pair of \((y_i, y_j) \) such that the characteristics \( C_i \) and \( C_j \) intersect at \((x,t) \). We define \((y_i, y_j) \) to be the bi-characteristic coordinate. In particular, these coordinates satisfy
\[
(x,t) = (X_i(y_i, t', (y_i, y_j)), t'(y_i, y_j)) = (X_j(y_j, t'((y_i, y_j), t'(y_i, y_j)).
\]

A direct calculation yields the following rules of coordinate transformations:
\[
\begin{align*}
\partial_{z_i} &= \rho_i \partial_x, \quad \partial_{s_i} = \lambda_i \partial_x + \partial_t, \quad \partial_{y_i} = \frac{\rho_i}{\lambda_j - \lambda_i}, \quad \partial_{y_j} = \frac{\rho_j}{\lambda_i - \lambda_j}, \\
\partial_{y_i} &= \frac{\rho_i}{\lambda_j - \lambda_i} \partial_{s_j} = \partial_{z_i} + \frac{\rho_i}{\lambda_i - \lambda_j} \partial_{s_i},
\end{align*}
\] (2.10)
and
\[
\begin{align*}
dx &= \rho_i dz_i + \lambda_i ds_i, \quad dt = ds_i, \quad dz_i = dy_i, \quad dz_j = dy_j, \\
dx &= \frac{\rho_i \lambda_j}{\lambda_j - \lambda_i} dy_i + \frac{\rho_j \lambda_i}{\lambda_i - \lambda_j} dy_j, \quad dt = \frac{\rho_i}{\lambda_j - \lambda_i} dy_i + \frac{\rho_j}{\lambda_i - \lambda_j} dy_j.
\end{align*}
\] (2.12)

In order to characterize the shock formation, we employ a geometric quantity \( \rho_i \) (the inverse density of the \( i^{th} \) characteristics)
\[
\rho_i := \partial_{z_i} X_i.
\]

This geometric quantity describes the compression among the characteristics of the \( i^{th} \) family. And it follows from (2.9) that
\[
\rho_i(z_i,0) = 1.
\] (2.14)

We further define
\[
v_i := \rho_i w_i.
\] (2.15)

Then, via the wave decomposition as we did in [4], we get the following system for \((\rho_i, w_i, v_i)\):
\[
\begin{align*}
\partial_{s_i} \rho_i &= c_i^i v_i + \left( \sum_{m \neq i} c_{im} w_m \right) \rho_i, \\
\partial_{s_i} w_i &= - c_i^i w_i^2 + \left( \sum_{m \neq i} (-c_{im} + \gamma_{im}^i) w_m \right) w_i + \sum_{m \neq i, k \neq i} \gamma_{km}^i w_k w_m, \\
\partial_{s_i} v_i &= \left( \sum_{m \neq i} \gamma_{im}^i w_m \right) v_i + \sum_{m \neq i, k \neq i} \gamma_{km}^i w_k w_m \rho_i,
\end{align*}
\] (2.16) (2.17) (2.18)

where \( \partial_{s_i} = \lambda_i \partial_x + \partial_t \) and
\[
c_{im}^i = \nabla \Phi \lambda_i \cdot r_m,
\] (2.19)
\[
\gamma_{im}^i = -(\lambda_i - \lambda_m)l_i \cdot (\nabla \Phi r_i \cdot r_m - \nabla \Phi r_m \cdot r_i), \quad \text{when } m \neq i, \quad (2.20)
\]
\[
\gamma_{km}^i = -(\lambda_k - \lambda_m)l_i \cdot (\nabla \Phi r_k \cdot r_m), \quad \text{when } k \neq i, \ m \neq i. \quad (2.21)
\]
Noting that in (2.17), if \( \epsilon_{ii}^i \neq 0 \) always holds for certain \( i \), then, the \( i \)th characteristics are genuinely nonlinear in the sense of P. D. Lax and genuinely nonlinear. Notice that along the first characteristics, in (2.16) and (2.17), we have \( c_{11}^i(\Phi) \) being away from zero. This is owing to

\[
c_{11}^i(0)\bigg|_{H_1=0} = \nabla \Phi \lambda_1(0)r_1(0) = (1, 0, \frac{\sqrt{A_1}(\gamma - 1)}{2}, 0, \frac{\sqrt{A_2}(\gamma - 1)}{2}) \cdot (1, 0, \frac{1}{\sqrt{A_1}}, 0, 0)^T = \frac{\gamma + 1}{2} > 0.
\]
Regarding \( H_1 \) as a parameter, when \( H_1 \) and \( |\Phi| \) are small enough, we have that \( c_{11}^i(\Phi) \approx \frac{\gamma + 1}{2} \).
This implies that the genuine nonlinearity, i.e., \( c_{11}^i(\Phi) > 0 \).

2.3. Boundedness of the coefficients. Consider the coefficients \( c_{im}^i, \gamma_{im}^i, \gamma_{km}^i \) given in (2.19)-(2.21). Note that \( C_f^2 - c^2 = 0 \) when \( H_2 = 0 \). Furthermore, it also holds \( \partial_{H_2}(C_f^2 - c^2) = O(H_2) \) and \( \partial_{H_2 H_2}(C_f^2 - c^2) = O(1) \). Hence, the leading order of \( C_f^2 - c^2 \) is \( O(H_2^2) \). One can then verify that \( l_i, \nabla \Phi r_i \) and \( r_i \) are all regular provided that \( |\Phi| \) is small enough. Hence, the coefficients \( |c_{ii}^i, \gamma_{im}^i, \gamma_{km}^i| \) are all uniformly bounded from above under the assumption \( \Phi \in B_{2d}(0) \).

2.4. Characteristic strips. At the end of this section, we introduce the characteristic strips in which our estimates would be carried out. For initial data supported in \( I_0 = [a, b] \), we define the \( i \)th characteristic strip \( R_i \) by

\[
R_i := \cup_{z_i \in I_0} C_i(z_i).
\]
Via deriving estimates for the decomposed system (2.16)-(2.18), we can prove the formation of shock, which is characterized by the vanishing of \( \rho_1 \) in \( R_1 \). See Section 2 for the details.

Corresponding to five different characteristic speeds \( \lambda_i \), there are five characteristic strips starting from \( I_0 = [a, b] \). Now we show that these characteristic strips \( R_1, \cdots, R_5 \) totally separate with each other after a time \( t_0 \). We take the supremum and infimum of the eigenvalues

\[
\bar{\lambda}_i := \sup_{\Phi \in B_{2d}^2(0)} \lambda_i(\Phi), \quad \underline{\lambda}_i := \inf_{\Phi \in B_{2d}^2(0)} \lambda_i(\Phi), \quad \text{for } i = 1, \cdots, 5
\]
and define

\[
\sigma := \min_{\alpha, \beta \in \{1, \cdots, 5\}} (\Delta_\alpha - \bar{\lambda}_\beta).
\]
It holds that \( \sigma \) has a uniform positive lower bound provided \( \delta \) being sufficiently small. For \( \alpha \in \{1, \cdots, 5\} \) and \( z \in I_0 \), by (2.9), we obtain that

\[
z + \Delta_\alpha t \leq X_\alpha(z, t) \leq z + \bar{\lambda}_\alpha t.
\]
Then, for all \( \alpha < \beta \), with \( \alpha, \beta \in \{1, \cdots, 5\} \), we have

\[
X_\alpha(a, t) - X_\beta(b, t) \geq (a + \lambda_\alpha t) - (b + \lambda_\beta t) = -(b - a) + (\lambda_\alpha - \lambda_\beta)t \geq -(b - a) + \sigma t.
\]
Note that the above inequality is strictly positive if
\[ t > t_0 := \frac{b - a}{\sigma} > 0. \] (2.22)

Hence the above \( t_0 \) in (2.22) is the separating time, after which the five characteristic strips are well separated.

3. Constructions of initial data

In this section, we construct the \( H^{\frac{7}{4}} \) initial data for the 2D ideal compressible MHD system (1.1), with which we prove the desired ill-posedness. Our construction of initial data is inspired by Lindblad [23, 24], Ohlmann [29] and An-Chen-Yin [4]. In [23, 24], Lindblad employed the function \( |\ln(x)|^\alpha \) to construct initial data with low regularity. The corresponding Cauchy problems of the 3D quasilinear wave equations are proved to be ill-posed in suitable Sobolev spaces. In [29], Ohlmann generalized Lindblad’s result to 2D quasilinear wave equations. He introduced a cut-off function \( \phi(|\ln(x)|^\delta x^2) \) such that the initial data belong to the logarithmic Sobolev space \( H^{\frac{7}{4}}(\ln H)^{-\beta} \), which is slightly more singular than the fractional Sobolev space \( H^{\frac{7}{4}} \). In this paper, we generalize Lindblad’s and Ohlmann’s approach and also extend our approach in [4]. We don’t have the logarithmic loss here.

We first construct the \( H^{\frac{3}{4}} \) initial data for the decomposed system (2.16)-(2.18). Assume \( \alpha, \delta, m, \theta \) to be positive constants and require \( \theta \) to be sufficiently small, we then prescribe
\[ \hat{w}_1^{(n)}(x, x_2) = w_1^{(n)}(x, x_2, 0) = -\theta |\ln(x)|^\alpha \mathcal{X}_q(x)\psi\left(\frac{|\ln(x)|^\delta x^{2m}}{x^m}\right) \] (3.1)
with
\[ \mathcal{X}(x) = \begin{cases} 1, & x \in \left[\frac{6}{5}, \frac{9}{5}\right], \\ 0, & x \in (-\infty, 1] \cup [2, +\infty). \end{cases} \]
\[ \psi(x) = \begin{cases} 1, & |x| \leq \frac{1}{3}, \\ 2, & \frac{2}{3} < |x| < \frac{1}{2}, \\ 0, & |x| \geq \frac{2}{3}, \end{cases} \]
and \( \mathcal{X}_q(x) = \mathcal{X}(\frac{x}{q}) \). We further assign
\[ \hat{w}_k^{(n)}(x, x_2) = w_k^{(n)}(x, x_2, 0) = -\theta^2 \mathcal{X}_q(x)\psi\left(\frac{|\ln(x)|^\delta x^{2m}}{x^m}\right) \quad \text{for} \quad k = 2, \cdots, 5. \] (3.2)

With the above initial data, we prove the following lemma.

**Lemma 3.1.** Assume that
\[ 2\alpha + \delta(1 - \frac{1}{m}) < 0 \quad \text{and} \quad \frac{1}{4} < m \leq \frac{1}{2}. \]

Then for \( \hat{w}_1^{(n)}(x, x_2) \) defined in (3.1), we have \( \hat{w}_1^{(n)} \in \dot{H}^{\frac{3}{4}}(\mathbb{R}^2) \).

**Proof.** Recall that the \( \dot{H}^{\frac{3}{4}}(\mathbb{R}^2) \) norm of \( \hat{w}_1^{(n)} \) is defined as
\[ \|\hat{w}_1^{(n)}\|^2_{\dot{H}^{\frac{3}{4}}(\mathbb{R}^2)} := \iint_{\mathbb{R}^2} |\xi|^\frac{1}{2}[\mathcal{F}(\hat{w}_1^{(n)})(\xi_1, \xi_2)]^2 d\xi_1 d\xi_2. \] (3.3)
We first calculate the Fourier transform of \( \hat{w}_1^{(\eta)} \) as below:

\[
\mathcal{F}(\hat{w}_1^{(\eta)})(\xi_1, \xi_2)
= -\int_x \int_{x^2} e^{-2\pi i x_1 \xi_1} e^{-2\pi i x_2 \xi_2} \theta |\ln(x)|^\alpha \mathcal{X}_\eta(x) \psi \left( \frac{|\ln(x)|^\delta x_2^{2m}}{x^m} \right) dx dx_2
= -\int_x e^{-2\pi i x_1 \xi_1} \theta |\ln(x)|^\alpha \mathcal{X}_\eta(x) dx \int_{x^2} e^{-2\pi i x_2 \xi_2} \psi \left( \frac{|\ln(x)|^\delta x_2^{2m}}{x^m} \right) dx_2
= -\frac{1}{2m} \int_\eta e^{-2\pi i x_1 \xi_1} \theta |\ln(x)|^\alpha \mathcal{X}_\eta(x) \sqrt{\frac{x}{|\ln(x)|^{2m}}} dx \int_{|y| \leq \frac{1}{2}} \exp \left\{ -2\pi i y \frac{1}{2m} \sqrt{\frac{x}{|\ln(x)|^{2m}}} \xi_2 \right\} \psi(y) y^{\frac{1}{2m}-1} dy.
\]

To estimate the \( \dot{H}^\frac{3}{2}(\mathbb{R}^2) \) norm of \( \hat{w}_1^{(\eta)} \), we divide the domain of integration in (3.3) into four regions:

\[
D_1 = \{ (\xi_1, \xi_2) : |\xi_1| \leq \frac{1}{\eta}, |\xi_2| \leq \frac{|\ln(\eta)|^\delta}{\sqrt{\eta}} \},
\]
\[
D_2 = \{ (\xi_1, \xi_2) : |\xi_1| > \frac{1}{\eta}, |\xi_2| \leq \frac{|\ln(\eta)|^\delta}{\sqrt{\eta}} \},
\]
\[
D_3 = \{ (\xi_1, \xi_2) : |\xi_1| \leq \frac{1}{\eta}, |\xi_2| > \frac{|\ln(\eta)|^\delta}{\sqrt{\eta}} \},
\]
\[
D_4 = \{ (\xi_1, \xi_2) : |\xi_1| > \frac{1}{\eta}, |\xi_2| > \frac{|\ln(\eta)|^\delta}{\sqrt{\eta}} \}.
\]

- **Estimate in \( D_1 \).** In the compact region \( D_1 \), the Fourier transform part is controlled via

\[
|\mathcal{F}(\hat{w}_1^{(\eta)})(\xi_1, \xi_2)| \lesssim \eta \theta |\ln(\eta)|^\alpha \frac{\sqrt{\eta}}{|\ln(\eta)|^{\frac{2}{2m}}} = \theta \eta^\frac{3}{2} |\ln(\eta)|^{\alpha - \frac{3}{2m}}.
\]

Hence, there holds

\[
\int_{D_1} |\xi_1|^\frac{3}{2} |\mathcal{F}(\hat{w}_1^{(\eta)})(\xi_1, \xi_2)|^2 d\xi_1 d\xi_2 \lesssim \eta^{-\frac{3}{2}} |\ln(\eta)|^{\delta - \frac{3}{2}} \theta^2 \eta^\frac{3}{2} |\ln(\eta)|^{2\alpha - \frac{3}{2m}}.
\]

(3.4)

- **Estimate in \( D_2 \).** In the non-compact domain \( D_2 \), we claim that

\[
|\mathcal{F}(\hat{w}_1^{(\eta)})(\xi_1, \xi_2)| \lesssim \frac{1}{|\xi_1|^q} \theta \eta^\frac{3}{2} - q |\ln(\eta)|^{\alpha - \frac{3}{2m}}.
\]

(3.5)

for any positive integer \( q \).
For the case of \( q = 1 \): inequality (3.5) is achieved via using integration by parts with respect to \( x \):

\[
\mathcal{F}(\hat{u}_1^{(q)})(\xi_1, \xi_2) = -\frac{1}{2m} \int_0^{2\eta} e^{-2\pi ix \xi_1 \theta} |\ln(x)|^\alpha \mathcal{X}_\eta(x) \frac{\sqrt{x}}{|\ln x|^{2m}} \, dx \int_{|y| \leq \frac{1}{2}} \exp \left\{ -2\pi iy \frac{2m}{|\ln x|^{2m}} \xi_2 \right\} \psi(y) y^{\frac{1}{2m} - 1} \, dy
\]

\[
= \frac{C}{\xi_1} \int_0^{2\eta} \theta |\ln(x)|^\alpha \mathcal{X}_\eta(x) \frac{\sqrt{x}}{|\ln x|^{2m}} \, dx \int_{|y| \leq \frac{1}{2}} \exp \left\{ -2\pi iy \frac{2m}{|\ln x|^{2m}} \xi_2 \right\} \psi(y) y^{\frac{1}{2m} - 1} \, dy
\]

\[
= \frac{C}{\xi_1} \int_0^{2\eta} e^{-2\pi ix \xi_1 \theta} |\ln(x)|^\alpha \mathcal{X}_\eta(x) \frac{\sqrt{x}}{|\ln x|^{2m}} \, dx \int_{|y| \leq \frac{1}{2}} \exp \left\{ -2\pi iy \frac{2m}{|\ln x|^{2m}} \xi_2 \right\} \psi(y) y^{\frac{1}{2m} - 1} \, dy
\]

\[
+ \frac{C}{\xi_1} \int_0^{2\eta} -e^{-2\pi ix \xi_1 \theta} |\ln(x)|^\alpha \mathcal{X}_\eta(x) \frac{\sqrt{x}}{|\ln x|^{2m}} \, dx \int_{|y| \leq \frac{1}{2}} \exp \left\{ -2\pi iy \frac{2m}{|\ln x|^{2m}} \xi_2 \right\} \psi(y) y^{\frac{1}{2m} - 1} \, dy
\]

\[
+ \frac{C}{\xi_1} \int_0^{2\eta} e^{-2\pi ix \xi_1 \theta} |\ln(x)|^\alpha \mathcal{X}_\eta(x) \frac{1}{|\ln x|^{2m+1}} \, dx \int_{|y| \leq \frac{1}{2}} \exp \left\{ -2\pi iy \frac{2m}{|\ln x|^{2m}} \xi_2 \right\} \psi(y) y^{\frac{1}{2m} - 1} \, dy
\]

\[
+ \frac{C}{\xi_1} \int_0^{2\eta} e^{-2\pi ix \xi_1 \theta} |\ln(x)|^\alpha \mathcal{X}_\eta(x) \frac{1}{|\ln x|^{2m+1}} \, dx \int_{|y| \leq \frac{1}{2}} \exp \left\{ -2\pi iy \frac{2m}{|\ln x|^{2m}} \xi_2 \right\} \psi(y) y^{\frac{1}{2m} - 1} \, dy
\]

\[
+ \frac{C}{\xi_1} \int_0^{2\eta} e^{-2\pi ix \xi_1 \theta} |\ln(x)|^\alpha \mathcal{X}_\eta(x) \frac{1}{|\ln x|^{2m+1}} \, dx \int_{|y| \leq \frac{1}{2}} \exp \left\{ -2\pi iy \frac{2m}{|\ln x|^{2m}} \xi_2 \right\} \psi(y) y^{\frac{1}{2m} - 1} \, dy
\]

(3.6)

In (3.6) and hereafter, we abuse the notation \( C \) to denote a uniform constant and the value of \( C \) may change in different lines. Straightforward calculation yields

\[
|I_1| \lesssim \eta \theta \frac{1}{\eta} |\ln \eta|^{\alpha - 1} \frac{\sqrt{\eta}}{|\ln \eta|^{2m}} \leq \theta \sqrt{\eta} |\ln \eta|^{\alpha - \frac{4}{2m}},
\]

\[
|I_2| \lesssim \eta \theta \frac{1}{\eta} |\ln \eta|^{\alpha} \frac{\sqrt{\eta}}{|\ln \eta|^{2m}} = \theta \sqrt{\eta} |\ln \eta|^{\alpha - \frac{4}{2m}},
\]

\[
|I_3| \lesssim \eta \theta \frac{1}{\sqrt{\eta} |\ln \eta|^{2m}} \frac{1}{\eta} \leq \theta \sqrt{\eta} |\ln \eta|^{\alpha - \frac{4}{2m}},
\]
\(|I_4| \lesssim \eta \theta \ln \eta^{\alpha} \frac{1}{\sqrt{\eta}} \ln \eta^{\alpha - \frac{\delta}{2m}} \),
\(|I_5| \lesssim \eta \theta \ln \eta^{\alpha} \frac{1}{\sqrt{\eta}} \ln \eta^{\alpha - \frac{\delta}{2m}} = \theta \sqrt{\eta} \ln \eta^{\alpha - \frac{\delta}{2m}}
\(|I_6| \lesssim \eta \theta \ln \eta^{\alpha} \frac{1}{\sqrt{\eta}} \ln \eta^{\alpha - \frac{\delta}{2m}} = \theta \sqrt{\eta} \ln \eta^{\alpha - \frac{\delta}{2m} - 1}.

Since \(m \leq \frac{1}{2}\), it holds that \(1 - \frac{1}{m} \leq -\frac{1}{2m}\). This implies that \(|I_5|\) and \(|I_6|\) are bounded by \(\theta \sqrt{\eta} \ln \eta^{\alpha - \frac{\delta}{2m}}\). Inserting the above estimates of \(I_j\) into \((3.6)\), we hence deduce

\[|\mathcal{F}(\hat{\omega}_1^{(\eta)})(\xi_1, \xi_2)| \lesssim \frac{1}{|\xi_1|} \theta \sqrt{\eta} \ln \eta^{\alpha - \frac{\delta}{2m}}.\]

For the case of \(q > 1\): inequality \((3.5)\) can be obtained via using integration by parts for \(q\) times as we proceed in \((3.6)\). Then, taking \(q \geq 2\), by \((3.5)\) within \(D_2\) we derive the following estimate

\[
\iint_{D_2} |\xi|^3 |\mathcal{F}(\hat{\omega}_1^{(\eta)})(\xi_1, \xi_2)|^2 d\xi_1 d\xi_2 \lesssim \iint_{D_2} |\xi|^3 \theta^2 \frac{1}{\xi_1^2} \eta^{3 - 2q} |\ln \eta^{\alpha - \frac{\delta}{m}}| d\xi_1 d\xi_2
\]
\[
\lesssim \theta^2 \eta^{3 - 2q} |\ln \eta^{\alpha - \frac{\delta}{m}}| \sqrt{\eta} \int_{|\xi_1| > \eta^{-1}} |\xi_1|^3 |\eta^{3 - 2q} d\xi_1
\]
\[
\lesssim \theta^2 |\ln \eta^{\alpha + \delta(1 - \frac{1}{2m})}|.
\]

Here, we impose the requirement \(q \geq 2\) to guarantee that the integral on the second line of \((3.7)\) is integrable.

- **Estimate in \(D_3\).** In this domain, employing integration by parts for \(\mathcal{F}(\hat{\omega}_1^{(\eta)})\) with respect to \(y\), we obtain

\[
\mathcal{F}(\hat{\omega}_1^{(\eta)})(\xi_1, \xi_2)
\]
\[
= -\frac{1}{2m} \int_{|y| \leq \frac{1}{2}} e^{-2\pi i x \xi_1} \theta |\ln(x)|^\alpha X_\eta(x) \frac{\sqrt{x}}{|\ln x|^\frac{3}{2m}} dx \int_{|y| \leq \frac{1}{2}} \exp \left\{ -2\pi iy \frac{1}{\sqrt{x}} \sqrt{\frac{x}{\ln x}} |\xi_2| \right\} \psi(y) y^{-\frac{1}{2m}} d\psi(y)
\]
\[
= -\frac{C}{\xi_2} \int_{|y| \leq \frac{1}{2}} e^{-2\pi i x \xi_1} \theta |\ln(x)|^\alpha X_\eta(x) \frac{\sqrt{x}}{|\ln x|^\frac{3}{2m}} y^{-\frac{1}{2m} - 1} \frac{|\ln x|^\frac{\delta}{2m}}{\sqrt{x}} d\psi(y) \exp \left\{ -2\pi iy \frac{1}{\sqrt{x}} \sqrt{\frac{x}{\ln x}} |\xi_2| \right\} dy
\]
\[
= -\frac{C}{\xi_2} \int_{|y| \leq \frac{1}{2}} e^{-2\pi i x \xi_1} \theta |\ln(x)|^\alpha X_\eta(x) \psi(y) \exp \left\{ -2\pi iy \frac{1}{\sqrt{x}} \sqrt{\frac{x}{\ln x}} |\xi_2| \right\} dy dx
\]
\[
\lesssim \frac{1}{|\xi_2|} \theta |\ln \eta^{\alpha}|.
\]

\[(3.8)\]
Applying integration by parts twice, we further get

$$\mathcal{F}(\hat{w}_1^{(\eta)})(\xi_1, \xi_2) = C_2^{-\frac{1}{2}} \int_0^{2\eta} \int_{|y| \leq \frac{1}{2}} e^{-2\pi i y \xi_1} \theta |\ln(x)|^\alpha \chi(x) \left[ \frac{|\ln x|^{\frac{\delta}{2}}}{\sqrt{x}} \psi''(y) y^{1-\frac{1}{2m}} \exp \left\{ -2\pi i y \frac{1}{2m} \sqrt{|\ln x| \frac{1}{2m}} \xi_2 \right\} \right] dy dx \lesssim \frac{1}{\xi_2^2} \theta \sqrt{|\eta|} \ln|\eta|^\alpha \frac{1}{2m}.$$ (3.9)

In (3.9), noting the fact that $m > \frac{1}{4}$ implies $1 - \frac{1}{2m} > -1$, we have that the integrand $y^{1-\frac{1}{2m}}$ is integrable with respect to $y$ for $|y| \leq \frac{1}{2}$. Analogously, via integration by parts, for any positive integer $q'$, we derive that

$$|\mathcal{F}(\hat{w}_1^{(\eta)})(\xi_1, \xi_2)| \lesssim \frac{1}{|\xi_2|^p} \theta |\eta|^{1 - \frac{1}{2}(q'-1)} |\ln|\eta|^\alpha \frac{1}{2m} (q'-1).$$ (3.10)

In particular, taking $q' = 2$, integrating (3.10) in $D_3$, we hence deduce

$$\int \int_{D_3} |\xi_1|^\frac{3}{2} |\mathcal{F}(\hat{w}_1^{(\eta)})(\xi_1, \xi_2)|^2 d\xi_1 d\xi_2 \lesssim \int_{|\xi_1| \leq \sqrt{\eta}} \int_{|\xi_2| > \frac{|\ln|\eta|^\delta}{\sqrt{|\eta|}} } |\xi_1|^\frac{3}{2} \theta_2 \frac{1}{|\xi_2|} \eta |\ln|\eta|^\alpha \frac{2}{m} + \frac{1}{|\xi_2|} \eta |\ln|\eta|^\alpha \frac{2}{m} d\xi_1 d\xi_2 \lesssim \int \int_{D_3} \int_{|\xi_1| < |\xi_2|} |\xi_1|^\frac{3}{2} \theta_2 \frac{1}{|\xi_2|} \eta |\ln|\eta|^\alpha \frac{2}{m} + \frac{1}{|\xi_2|} \eta |\ln|\eta|^\alpha \frac{2}{m} d\xi_1 d\xi_2$$

$$\lesssim \theta_2 |\ln|\eta|^\alpha \frac{2}{m} + \frac{1}{|\xi_2|} \eta |\ln|\eta|^\alpha \frac{2}{m} + \frac{1}{|\xi_2|} \eta |\ln|\eta|^\alpha \frac{2}{m}.$$ (3.11)

Note that $\frac{1}{m} - 3 \leq 1 - \frac{1}{2m}$ because $m \leq \frac{1}{2}$. Thus, we derive

$$\int \int_{D_3} |\xi_1|^\frac{3}{2} |\mathcal{F}(\hat{w}_1^{(\eta)})(\xi_1, \xi_2)|^2 d\xi_1 d\xi_2 \lesssim \theta_2 |\ln|\eta|^\alpha \frac{2}{m} + \frac{1}{|\xi_2|} |\ln|\eta|^\alpha \frac{2}{m} + \theta_2 |\ln|\eta|^\alpha \frac{2}{m}.$$ (3.11)

- **Estimate in $D_4$.** Within $D_4$, combining (3.5) and (3.10), for any positive integers $p > 1$ and $p' > 1$, we have

$$|\mathcal{F}(\hat{w}_1^{(\eta)})(\xi_1, \xi_2)| \lesssim \frac{1}{|\xi_1|^p |\xi_2|^{p'}} \theta |\eta|^{\frac{1}{2} - \frac{1}{2}} |\ln|\eta|^\alpha \frac{2}{m} (p'-1).$$ (3.12)
Here, we set \( p \geq 2 \) and \( p' = 2 \). Then it follows from (3.12) that

\[
\int \int_{D_4} |\xi|^{\frac{3}{2}} \left| F(w_1^{(n)})(\xi_1, \xi_2) \right|^2 d\xi_1 d\xi_2 \\
\lesssim \int_{|\xi_1| > \eta^{-1}} \int_{|\xi_2| > \frac{1}{|\xi_1|^{2p}|\xi_2|^{4}}^{\frac{1}{|\xi_1|^{2p}|\xi_2|^{4}}} |\xi|^{\frac{3}{2}} \theta^2 \frac{1}{|\xi_1|^{2p}|\xi_2|^{4}} \eta^1 - 2p \ln \eta |^2 + \delta \frac{d}{\eta^p} d\xi_1 d\xi_2
\]

\[
\leq \int \int_{D_4 \cap \{|\xi_1| \leq |\xi_2|\}} |\xi_2|^{2\alpha + \frac{3}{2}} \theta^2 \frac{1}{|\xi_1|^{2p}|\xi_2|^{4}} \eta^1 - 2p \ln \eta |^2 + \delta \frac{d}{\eta^p} d\xi_1 d\xi_2
\]

\[
+ \int \int_{D_4 \cap \{|\xi_1| > |\xi_2|\}} |\xi_1|^{\frac{3}{2}} \theta^2 \frac{1}{|\xi_1|^{2p}|\xi_2|^{4}} \eta^1 - 2p \ln \eta |^2 + \delta \frac{d}{\eta^p} d\xi_1 d\xi_2
\]

\[
\lesssim \theta^2 |\ln \eta|^{2\alpha + \frac{3}{2}} \left( \frac{|\ln \eta|^{\frac{2}{\alpha m}}}{\sqrt{\eta}} \right)^{\frac{3}{2}} + \theta^2 |\ln \eta|^{2\alpha + \frac{\delta}{m} + \delta \frac{d}{\eta^p}} |\ln \eta|^{2\alpha + \delta (1 - \frac{1}{m})}.
\]

In summary, from (3.4), (3.7), (3.11) and (3.13), we deduce that

\[
\|w_1^{(n)}\|_{H^{\frac{3}{2}}(\mathbb{R}^2)}^2 \lesssim \theta^2 |\ln \eta|^{2\alpha + \frac{\delta}{m} - \frac{3}{2} \delta} + \theta^2 |\ln \eta|^{2\alpha + \delta (1 - \frac{1}{m})}.
\]

As required in the assumption, for \( \eta > 0 \) being small and \( 2\alpha + \delta (1 - \frac{1}{m}) < 0 \), the right hand side of (3.14) is bounded. This completes the proof of this lemma.

By far we have constructed the \( H^{\frac{3}{2}} \) initial data for the decomposed system. Based on this, we now go back to the original system (2.1) by reversing the process of wave decomposition. By the formula of decomposition of waves (2.8), we have

\[
\partial_2 \Phi(x, x, 0) = \sum_{k=1}^{5} w_i^{(n)}(x, x, 0) r_i(\Phi(x, x, 0)).
\]

Noting that the right eigen vectors \( r_i \) chosen in Section 2 is Lipschitz continuous in \( \Phi \), by the ODE argument, one can solve (3.15) with the initial data \( \Phi^{(n)}(x, x, 0) \). Moreover, since \( r_i(\Phi) \in L^\infty(B_{2\delta}^5(0)) \) and \( w_i^{(n)}(x, x, 0) \in H^{\frac{1}{2}}(\mathbb{R}^2) \), we obtain

\[
\Phi(x, x, 0) = (u_1, u_2, \varrho - 1, H_2, S)^{(n)}(x, x, 0) \in H^{\frac{1}{2}}(\mathbb{R}^2).
\]

These \( \Phi(x, x, 0) \) constitute the \( H^{\frac{1}{2}}(\mathbb{R}^2) \) initial data for the 2D ideal MHD system (2.1). Furthermore, when \( j \neq 1 \), we have \( \|w_j^{(n)}\|_{H^{\frac{1}{2}}(\mathbb{R}^2)} \leq \|w_1^{(n)}\|_{H^{\frac{1}{2}}(\mathbb{R}^2)} \). Together with (3.14), this implies the vanishing of \( \|\Phi_0^{(n)}\|_{H^{\frac{1}{2}}(\mathbb{R}^2)} \) as \( \eta \to 0 \).

4. Shock formation

In this section, we prove the (first) shock forms at a finite time \( T^* \) within \( \mathcal{R}_1 \). To demonstrate this, we carry out a priori estimates in \( L^\infty \) norm.
4.1. $L^\infty$ estimates. Consider the decomposed diagonal system (2.16)-(2.18) under planar symmetry. For the initial data designed in Section 3, we have

$$W_0^{(\eta)} := \max_i \sup_{z_i} |w_i^{(\eta)}(z_i, 0)| = -w_1^{(\eta)}(z_0, 0) > 0$$

for some $z_0 \in (\eta, 2\eta)$. To study the Cauchy problem for system (2.16)-(2.18), we estimate the following quantities:

$$S(t) := \max_i \sup_{(z_i', s_i') \in [\eta, 2\eta]} \rho_i(z_i', s_i'),$$

$$J(t) := \max_i \sup_{(z_i', s_i') \in [\eta, 2\eta]} |v_i(z_i', s_i')|,$$

$$V(t) := \max_i \sup_{(x', t') \in [\eta, \infty]} |w_i(x', t')|,$$

$$\bar{U}(t) := \sup_{(x', t') \in [\eta, \infty]} |\Phi(x', t')|.$$

Let $\Phi \in C^2(\mathbb{R} \times [0, T], B_{2\eta}^0(0))$ be a solution to (1.1) for $t \in [0, T]$ with $T > 0$. Our aim is to show that, for $t \in [0, T^*_\eta]$ it holds

$$S(t) = O(1), \quad J(t) = O(W_0^{(\eta)}), \quad V(t) = O\left(\eta W_0^{(\eta)}\right), \quad \bar{U}(t) = O(\eta W_0^{(\eta)}),$$

and

$$T^*_\eta \leq O\left(\frac{1}{W_0^{(\eta)}}\right).$$

The main strategy is to derive the following a priori estimates for $t < T^*_\eta$

$$S(t) = O(1 + tJ + tVS),$$

$$J(t) = O(W_0^{(\eta)} + tVJ + tV^2S),$$

$$V(t) = O\left(\eta W_0^{(\eta)} + \theta^2V + \eta V^2\right),$$

$$\bar{U}(t) = O(\eta J + \eta V + \eta V^2).$$

Once these a priori estimates are obtained, we then set up a bootstrap argument to deduce (4.1). In particular, supposing that

$$tV \leq \theta^2, \quad J \leq \theta^2,$$

via (4.3)-(4.5), we have that

$$S(t) = O(1 + tJ + \theta^2S) \Rightarrow S(t) = O(1 + tJ),$$

$$V(t) = O\left(\eta W_0^{(\eta)} + \theta^2V + \eta \theta^2V\right) \Rightarrow V(t) = O\left(\eta W_0^{(\eta)}\right),$$

$$J(t) = O(W_0^{(\eta)} + \theta^2J + \theta^2VS) \Rightarrow J(t) = O(W_0^{(\eta)} + \theta^2V(1 + tJ)),$$

$$J(t) = O(W_0^{(\eta)} + \theta^2V) = O(W_0^{(\eta)}).$$
Then for $t < O\left(\frac{1}{W_0^{(\eta)}}\right)$, we get $S(t) = O(1)$. These estimates improve the bootstrap assumptions in (4.7). Moreover, with (4.6) we also obtain $\bar{U}(t) = O(\eta W_0^{(\eta)})$.

Now we derive the a priori estimates stated in (4.3)-(4.6). Recall that for any fixed $x_2$, the initial data are supported in $[\eta, 2\eta]$. And from (2.22), we also know that the characteristic strips are well separated after time

$$t_0^{(\eta)} := \frac{\eta}{\sigma}.$$

According to Section 2.3, all the coefficients in equations (2.16)-(2.18) are proved to be bounded. Here we denote this uniform upper bound to be $\Gamma$. The a priori estimates are deduced in two regions: the non-separating region $[0, t_0^{(\eta)}]$ and the separating region $[t_0^{(\eta)}, T]$ as in Figure 1.

**Non-separating region** $[0, t_0^{(\eta)}]$. During this time period, all the characteristic strips may overlap with each other.

- **Estimate of $W(t)$**: We first bound

$$W(t) = \max_i \sup_{0 \leq t' \leq t} |w_i(x', t')|.$$

Using (2.17), we have

$$\frac{\partial}{\partial s_i} |w_i| \leq \Gamma W^2.$$

By comparing $w_i$ with the solution $Y$ to the following ODE

$$\begin{cases} \frac{d}{dt} Y = \Gamma Y^2, \\ Y(0) = W_0^{(\eta)}, \end{cases}$$

we then deduce

$$|w_i| \leq Y(t) = \frac{W_0^{(\eta)}}{1 - \Gamma W_0^{(\eta)}} t$$

for $t < \min\left\{ \frac{1}{\Gamma W_0^{(\eta)}}, t_0^{(\eta)} \right\}$. (4.8)
Noticing that $t_0^{(η)} = O(η)$, it yields
\[ ΓW_0^{(η)t_0^{(η)}} = O(ηW_0^{(η)}) = O(θ). \] (4.9)

Invoking \(4.9\) in \(4.8\), for a small parameter $ε \in (0, \frac{1}{100}]$ we obtain
\[ |u_ι(x, t)| ≤ (1 + ε)W_0^{(η)} \quad \text{for any } x ∈ ℝ \text{ and } t ∈ [0, t_0^{(η)}]. \]

This further implies that
\[ |W(t)| ≤ (1 + ε)W_0^{(η)} \quad \text{for } t ∈ [0, t_0^{(η)}]. \] (4.10)

- **Estimate of $V(t)$:** Consider the exterior of the characteristic strips $R_i$. Employing characteristic coordinates, we can label any point $(x', t')$ by $(z_i^*, s_i^*)$ satisfying $z_i^* \notin [η, 2η]$ and $w_i^{(η)}(z_i^*, 0) = 0$. Integrating \(2.16\) along the characteristic $C_i$, we then obtain
\[ V(t) = O\left(∫_0^{t_0^{(η)}} w_i w_j dσ_i\right) = O(η[W(t)]^2) = O(η[W_0^{(η)}]^2). \] (4.11)

- **Estimate of $S(t)$:** By \(2.16\), the inverse density $ρ_i$ satisfies
\[ \frac{∂ρ_i}{∂s_i} = O(ρ_i W). \] (4.12)

Integrating \(4.12\) along the characteristic $C_i$, we obtain
\[ ρ_i(z_i, t) = ρ_i(z_i, 0) \exp\left(O(tW(t))\right). \]

Together with \(2.14\) and \(4.10\), we then derive that
\[ ρ_i(z_i, t) = \exp\left(O(ηW_0^{(η)})\right) \quad \text{for } t ∈ [0, t_0^{(η)}]. \]

We can choose $θ$ to be small enough such that
\[ 1 - ε ≤ ρ_i(z_i, t) ≤ 1 + ε \quad \text{for } t ∈ [0, t_0^{(η)}]. \]

Here, $θ$ is a small parameter, that measures the amplitude of the initial data. Therefore, it holds
\[ S(t) = O(1) \quad \text{for } t ∈ [0, t_0^{(η)}]. \] (4.13)

- **Estimate of $J(t)$:** Employing \(2.18\), we have
\[ \frac{∂v_i}{∂s_i} = O(S(t)[W(t)]^2). \]

Integrating the above equation along $C_i$, together with \(4.10\) and \(4.13\), for $t ∈ [0, t_0^{(η)}]$ we obtain
\[ J(t) = O(W_0^{(η)} + t[W(t)]^2) = O(W_0^{(η)} + η[W^{(η)}]^2) = O(W_0^{(η)}). \] (4.14)

- **Estimate of $\overline{U}(t)$:** Note that
\[ \overline{U}(x, t) = ∫_X S_{5(η)} \frac{∂Φ(x', t)}{∂x} dx' = ∫_X S_{5(η)} \sum_k w_k r_k(x', t) dx'. \] (4.15)
Using the estimate (4.10), we hence obtain the following estimate of $\bar{U}$,

$$\bar{U}(t) = \sup_{(x', t') \in \{(x', t') \mid 0 \leq t' \leq t \}} |\Phi(x', t')| = O\left(W(t)(\eta + (\bar{\lambda}_1 - \bar{\lambda}_5)t)\right) = O(\eta W_0^{(\eta)}), \quad \forall \ t \in [0, t_0^{(\eta)}].$$  \hspace{1cm} (4.16)

**Separating region** $[t_0^{(\eta)}, T]$. Based on the obtained estimates for $t \in [0, t_0^{(\eta)}]$, we next consider the time interval $t > t_0^{(\eta)}$. And the five characteristic strips $R_i$ are all separated.

- **Estimate of $S(t)$**: For $(x, t) \in R_i$, by (2.16), we have

$$\frac{\partial \rho_i}{\partial s_i} = O(J + V S). \hspace{1cm} (4.17)$$

Integrating (4.17) along the characteristic $C_i$, we obtain

$$\rho_i(z_i, t) = \rho_0(z_i, 0) + \int_0^t O(J + V S) dt'.$$

We hence conclude that

$$S(t) = O(1 + tJ + tV S). \hspace{1cm} (4.18)$$

- **Estimate of $J(t)$**: For $(x, t) \in R_i$, using (2.18), $v_i$ satisfies

$$\frac{\partial v_i}{\partial s_i} = O(V J + V^2 S). \hspace{1cm} (4.19)$$

We then integrate (4.19) and derive that

$$J(t) = O(W_0^{(\eta)} + tV J + tV^2 S). \hspace{1cm} (4.20)$$

- **Estimate of $V(t)$**: Now we investigate the dynamics of $w_i$ outside the characteristic strip $R_i$. Via (2.17), we have that

$$\frac{\partial w_i}{\partial s_i} = O(V^2) + O\left(\sum_{k \neq i} w_k\right)V + O\left(\sum_{m \neq i, k \neq i} w_m w_k\right). \hspace{1cm} (4.21)$$

Note that $C_i$ issues from $z_i \notin [\eta, 2\eta]$ and terminates at $(x, t) \notin R_i$. If $t' \geq t_0^{(\eta)}$ and $(X_i(z_i, t'), t') \in C_i$, it holds either $(X_i(z_i, t'), t') \in (\mathbb{R} \times [t_0^{(\eta)}, t]) \setminus \bigcup_k R_k$ or $(X_i(z_i, t'), t') \in R_k$ for some $k \neq i$. When $(x, t) \notin R_i$, for $m \neq i, k \neq i, m \neq k$, there are only three scenarios: $(x, t)$ stays in $R_m$, or $(x, t)$ stays in $R_k$, or $(x, t)$ stays out of all the characteristics. In all of these three cases, the third term $O\left(\sum_{m \neq i, k \neq i} w_m w_k\right)$ on the right hand side of (4.21)
can be absorbed by the second term \(O\left(\sum_{k \neq i} w_k\right)V\). To estimate \(O\left(\sum_{k \neq i} w_k\right)V\) for \((x, t) \notin \mathcal{R}_i\), we further integrate \((4.21)\) along \(C_i\). Noticing that \(w_i^{(n)}(z, 0) = 0\), we then obtain

\[
w_i(x, t) = O\left(tV^2 + V \sum_{k \neq i} \int_0^t w_k(X_i(z, t'), t') dt'\right)
\]

\[
= O\left(tV^2 + V \sum_{k \neq i} \int_{t_0}^{t} w_k(X_i(z, t'), t') dt'\right) + O\left(V \sum_{k \neq i} \int_{t_0}^{t} w_k(X_i(z, t'), t') dt'\right)
\]

\[
= O\left(tV^2 + [W_0^{(n)}]^2 + V \sum_{k \neq i} \int_{I_k} w_k(X_i(z, t'), t') dt'\right),
\]

where we denote \(I_k := \{t' \in [t_0^{(n)}, t] : (x, t') \in C_i \cap \mathcal{R}_k\}\) for \(k \neq i\). Here, we use the estimate \(V(t) \leq W(t) = O(W_0^{(n)})\) for \(t \leq t_0^{(n)}\) in the second equality. The bi-characteristic coordinates are then employed to control \(M\) and we have

\[
\int_{I_k} w_k(X_i(z, t'), t') dt' = O\left(\int_{y_k \in [\eta, 2\eta]} \frac{\rho_k(y_k, t'(y_i, y_k))}{\lambda_i - \lambda_k} w_k(y_k, t'(y_i, y_k)) dy_k\right) = O(\eta J).
\]

Together with \((4.22)\) and \((4.23)\), we thus obtain

\[
V(t) = O(tV^2 + [W_0^{(n)}]^2 + \eta V J).
\]

\[
\Phi(x, t) = O((\eta + (\lambda_1 - \lambda_2) t) V).
\]

\[
\Phi(x, t) = O(\eta J + \eta V + \eta t V).
\]

Here we use the characteristic coordinates. Together with \((4.25)\) and \((4.26)\), we derive

\[
\Phi(x, t) = O(\eta J + \eta V + \eta t V).
\]
We now finish the proof of the a priori estimates in (4.3)-(4.6) and have obtained the $L^\infty$ estimates in (4.1) via the aforementioned bootstrap argument.

4.2. Finite-time blow-up of $w_1$. We further derive an accurate lower bound and upper bound for $T^*_\eta$. When $t$ goes to $T^*_\eta$, the first family of characteristics within $\mathcal{R}_1$ collapse to a (first) shock. And the quantity $w_1$ blows up at the shock formation time.

We first impose a bootstrap assumption $v_1 < 0$, and we will later improve this bound. Recall that the equation for $\rho_1$ is

$$\frac{\partial \rho_1}{\partial s_1} = c_{11}^1(\Phi) v_1 + O\left(\sum_{k \neq 1} w_k\right) \rho_1.$$ 

Then noting the fact $c_{11}^1(\Phi) > 0$, we have

$$-c_{11}^1 |v_1| - |O\left(\sum_{k \neq 1} w_k\right)|\rho_1 \leq \frac{\partial \rho_1}{\partial s_1} \leq -c_{11}^1 |v_1| + |O\left(\sum_{k \neq 1} w_k\right)|\rho_1. \quad (4.28)$$

Since $|\Phi| = O(\eta W_0^{(q)}) \leq \delta$, we can choose $\theta$ to be sufficiently small such that

$$(1 - \varepsilon) c_{11}^1 (0) \leq c_{11}^1(\Phi) \leq (1 + \varepsilon) c_{11}^1(0). \quad (4.29)$$

Employing bi-characteristic coordinates, we then deduce

$$\int_0^t \sum_{k \neq 1} w_k(X_1(z_1, t'), t') dt' = O(\eta W_0^{(q)} + \eta J) = O(\eta W_0^{(q)}),$$

which implies

$$1 - \varepsilon \leq \exp\left(\int_0^t O\left(\sum_{k \neq 1} w_k(X_1(z_1, t'), t')\right) dt'\right) \leq 1 + \varepsilon, \quad (4.30)$$

and

$$1 - \varepsilon \leq \exp\left(-\int_0^t O\left(\sum_{k \neq 1} w_k(X_1(z_1, t'), t')\right) dt'\right) \leq 1 + \varepsilon. \quad (4.31)$$

Applying Grönwall’s inequality for (4.28) and combining with (4.29)-(4.31), we get

$$(1 - \varepsilon)\left(1 - (1 + \varepsilon)^2 c_{11}^1(0) \int_0^t |v_1(z, t')| dt'\right) \leq \rho_1(z, t)$$

$$\leq (1 + \varepsilon)\left(1 - (1 - \varepsilon)^2 c_{11}^1(0) \int_0^t |v_1(z, t')| dt'\right). \quad (4.32)$$

Next, for $v_1$ we have

$$\frac{\partial v_1}{\partial s_1} = O\left(\sum_{m \neq 1} w_m\right) v_1 + O\left(\sum_{m, k \neq 1} w_m w_k\right) \rho_1. \quad (4.33)$$

Integrating (4.33) along $C_1$, we obtain

$$v_1(z, t) = w_1^{(q)}(z, 0) + O(t VJ + t V^2 S) = w_1^{(q)}(z, 0) + O(\eta |W_0^{(q)}|^2).$$

Since $W_0^{(q)} = -w_1^{(q)}(z_0, 0)$, the above inequality implies

$$-(1 + \varepsilon) W_0^{(q)} \leq v_1(z_0, t) \leq -(1 - \varepsilon) W_0^{(q)}.$$
This improves the aforementioned bootstrap assumption $v_1 < 0$ and it implies
\[
(1 - \varepsilon)W_0^{(\eta)} \leq |v_1(z_0, t)| \leq (1 + \varepsilon)W_0^{(\eta)}.
\] (4.34)
Employing (4.32), we further obtain that
\[
\rho_1(z_0, t) \geq 1 - (1 + \varepsilon)\delta_{11}(0)W_0^{(\eta)}
\] (4.35)
and
\[
\rho_1(z_0, t) \leq 1 - (1 - \varepsilon)\delta_{11}(0)W_0^{(\eta)}.
\] (4.36)
From (4.35), we have that $\rho_1(z_0, t) > 0$ when
\[
t < \frac{1}{(1 + \varepsilon)\delta_{11}(0)W_0^{(\eta)}}.
\]
Meanwhile, we conclude that there exists $T_\eta^*$ (shock formation time) such that
\[
\lim_{t \to T_\eta^*} \rho_1(z_0, t) = 0.
\]
And $T_\eta^*$ obeys
\[
\frac{1}{(1 + \varepsilon)\delta_{11}(0)W_0^{(\eta)}} \leq T_\eta^* \leq \frac{1}{(1 - \varepsilon)\delta_{11}(0)W_0^{(\eta)}}.
\] (4.37)
Finally, since
\[
w_1 = \frac{v_1}{\rho_1},
\]
as defined in (2.15), by (4.34) and the positivity of $W_0^{(\eta)}$ we deduce that $w_1 \to \infty$ when $t \to T_\eta^*$ in $\mathcal{R}_1$.

4.3. Upper bound estimates for other $w_i$ ($i \neq 1$). For $w_i$ with $i \neq 1$, even though its evolution equation may be of the Riccati-type, its initial datum is designed to be much smaller compared with $w_1$’s datum. This guarantees that for $i \neq 1$ the evolution of $w_i(z_i, t)$ will not influence the blow-up of $w_1$ as $t \to T_\eta^*$. In particular, when $t \leq T_\eta^*$, we show below $\{\bar{W}_i = \sup_{z_i \in [s_i, s_i]} w_i(z_i, s_i)\}_{i=2,3,4,5}$ are all bounded.

Invoking the estimate of $V$ into the equations of $w_i$ with $i \neq 1$, for $(x, t) \in \mathcal{R}_i$, we have that
\[
\frac{\partial w_i}{\partial s_i} = -c_{ii}^{\delta}w_i^2 + O\left(\sum_{k \neq i} w_k\right)w_i + O\left(\sum_{m \neq i, k \neq i} w_m w_k\right)
\]
\[
= -c_{ii}^{\delta}w_i^2 + O(V)w_i + O(V^2)
\]
\[
= -c_{ii}^{\delta}w_i^2 + O(\eta W_0^{(\eta)}w_i) + O(\eta^2 |W_0^{(\eta)}|^4).
\]
Equivalently, we rewrite the above equation as
\[
\frac{d}{ds} \left[\exp\left(O(\eta |W_0^{(\eta)}|^2)s\right)w_i\right] = -\exp\left(O(\eta |W_0^{(\eta)}|^2)s\right)c_{ii}^{\delta}w_i^2 + \exp\left(O(\eta |W_0^{(\eta)}|^2)s\right)O(\eta^2 |W_0^{(\eta)}|^4).\] (4.38)
Using (4.37), we then have
\[
\exp\left(O(\eta |W_0^{(\eta)}|^2)s\right) \leq \exp\left(O(\eta |W_0^{(\eta)}|^2)T_\eta^*\right) \leq \exp\left(O(\eta W_0^{(\eta)})\right) = O(\epsilon^\theta).
\]
Here, $\theta$ and $\eta$ are small parameters, which characterize the magnitude of the initial data and the size of their supports, respectively. Then, by choosing $\theta$ to be sufficiently small, we further obtain

$$1 - \varepsilon \leq \exp\left( O\left( \eta [W_0^{(n)}]^2 \right) s \right) \leq 1 + \varepsilon.$$ 

Thus, back to $(4.39)$, it holds that

$$\frac{d}{ds} \left[ \exp\left( O\left( \eta [W_0^{(n)}]^2 \right) s \right) w_i \right] \leq C w_i^2 + O\left( \eta^2 [W_0^{(n)}]^4 \right).$$

Integrating $(4.39)$ along $C_i$, we get

$$\bar{W}_i \leq O\left( w_i(z, 0) + t \bar{W}_i^2 + t \eta^2 [W_0^{(n)}]^4 \right)$$

$$\leq O\left( [W_0^{(n)}]^2 + t \bar{W}_i^2 + t \eta^2 [W_0^{(n)}]^3 \right)$$

$$\leq O\left( [W_0^{(n)}]^2 + t \bar{W}_i^2 \right).$$

Now we introduce an additional bootstrap assumption

$$t \bar{W}_i \leq \theta^\frac{1}{2}.$$ 

Then by $(4.40)$, it holds

$$\bar{W}_i \leq O\left( [W_0^{(n)}]^2 \right).$$

And by $(4.37)$, we get

$$t \bar{W}_i \leq O\left( W_0^{(n)} \right) = O(\theta) < \theta^\frac{1}{2}.$$ 

This improves the bootstrap assumption $(4.41)$ and gives the desired upper bound for $|w_i|$ with $i \neq 1$.

5. Proof of the ill-posedness

We are ready to prove the $H^\frac{7}{4}$ ill-posedness stated in Theorem 1.1. Our ill-posedness result is twofold. The first part is the instantaneous shock formation. This immediately follows from our construction $(3.1)(3.2)$ for initial data and our estimate for the shock formation time $(4.37)$. In particular, we obtain that, as $\eta \to 0$, the shock formation time $T_\eta^* \to 0$ and the instantaneous shock forms.

In this section, we also prove the second part of the ill-posedness, which is the norm inflation. More precisely, we show that the $H^1$ norm of the solutions to 2D ideal compressible MHD system $(1.1)$ blows up at the shock formation time $T_\eta^*$.

Without loss of generality, we restrict our focus to the following region

$$\Omega_0 = \{(x, y_2) : (x - \frac{3\eta}{2})^2 + (y_2)^2 \leq \left( \frac{\eta}{2} \right)^2 \}.$$ 

Considering the initial data constructed in Section 3 we note that

$$\Omega_0 \subseteq \{(x, y_2) : \psi\left( \frac{|\ln(x)|^d y_2^{2m}}{x^m} \right) = 1\}$$

with $\psi\left( \frac{|\ln(x)|^d y_2^{2m}}{x^m} \right)$ a cut-off function given in $(3.1)$. Therefore, the initial data are planar symmetric within $\Omega_0$, i.e., $w_i^{(n)}(x, y_2)|_{\Omega_0} = \tilde{w}_i^{(n)}(x)$ for $i = 1, 2, 3, 4, 5$. Now we consider the
domain of future dependence for $\Omega_0$ with respect to the maximal speed $\lambda_1$. We also denote its $T^*_\eta$-slice as $\Omega^*_{T^*_\eta}$. Then, for $\Phi \in B^2_\varepsilon(0)$, $\Omega^*_{T^*_\eta}$ is uniformly close to the following 2D-disk

$$\Omega^*_{T^*_\eta} \approx \{(x, y_2, T^*_\eta) : x = X_1(z, T^*_\eta), \ (x - \lambda_1 T^*_\eta - \frac{3\eta}{2})^2 + y_2^2 \leq \frac{\eta^2}{4}, \ \text{for } \eta \leq z \leq 2\eta\}.$$  \hfill (5.1)

This implies that

$$\int_{(x,y_2)\in\Omega^*_{T^*_\eta}} dy_2 \approx (1 + O(\varepsilon)) \left(z - \eta + O(\varepsilon)\eta\right) (2\eta - z + O(\varepsilon)\eta).$$  \hfill (5.2)

Before showing the blow-up of the solution’s $H^1$-norm, we first derive an upper bound estimate for $|\partial_{z_1} \rho_1(z_1, s_1)|$.

**Proposition 5.1.** There exists a uniform constant $C$ depending only on $\varepsilon, \theta$ and $\eta$, such that for any $s_1 \leq T^*_\eta$, there holds $|\partial_{z_1} \rho_1(z_1, s_1)| \leq C$.

**Proof.** This proposition can be proven in the same fashion as we did for the 3D case in [3, 4]. We outline the main steps here. Since the coordinate transformation obeys (2.11), it holds

$$\partial_{z_1} \rho_1 = \partial_{y_1} \rho_1 + \frac{\rho_1}{2C_f} \partial_{s_1} \rho_1.$$  \hfill (5.3)

To control $\partial_{z_1} \rho_1$, we first estimate $\partial_{y_1} \rho_1$. Denote

$$\tau^{(5)}_1 := \partial_{y_1} \rho_1, \ \ \pi^{(5)}_1 := \partial_{y_1} v_1.$$  \hfill (5.4)

Since by (2.11), it holds that

$$\partial_{y_5} = \frac{\rho_5}{\lambda_1 - \lambda_5} \partial_{s_5} = \frac{\rho_5}{2C_f} \partial_{s_1},$$

we have that $\tau^{(5)}_1$ obeys

$$\partial_{y_5} \tau^{(5)}_1 := \partial_{y_1} \partial_{y_5} \rho_1 = \partial_{y_1} \left(\frac{\rho_5}{2C_f} \partial_{s_1} \rho_1\right) = \partial_{y_1} \left(\frac{\rho_1 \rho_5}{2C_f} \sum_m c_{1m} w_m\right)$$

$$= \frac{\rho_5}{2C_f} \left(c_{11} \partial_{y_1} v_1 + \sum_{m \neq 1} c_{1m} w_m \partial_{y_1} \rho_1\right) + \frac{\rho_1}{2C_f} \left(\sum_{m \neq 1} c_{1m} w_m \partial_{y_1} \rho_5 + c_{15} \partial_{y_1} v_5\right)$$

$$+ \frac{\rho_1 \rho_5}{2C_f} \left(\sum_{m=2,3,4} c_{1m} \frac{1}{\rho_m} \partial_{y_1} v_m - \sum_{m=2,3,4} c_{1m} \frac{w_m}{\rho_m} \partial_{y_1} \rho_m\right)$$

$$- \frac{\rho_1 \rho_5}{2C_f} \rho_1 \rho_5 \sum_m c_{1m} w_m + \frac{\rho_1 \rho_5}{2C_f} \sum_m \partial_{y_1} c_{1m} w_m.$$  \hfill (5.5)

In (5.5), the quantities remaining to be estimated are: $\partial_{y_1} \lambda_1, \partial_{y_1} c_{1m}, \partial_{y_1} \rho_m$ and $\partial_{y_1} v_m$. Firstly, for $\partial_{y_1} \lambda_1$, using bi-characteristic coordinates $(y_1, y_i)$ $(i \neq 1)$, it satisfies

$$\partial_{y_1} \lambda_1 = \nabla \Phi \lambda_1 \cdot \partial_{y_1} \Phi = \nabla \Phi \lambda_1 \cdot \left[\partial_{s_1} X_i \partial_{y_i} t' \partial_x \Phi + \partial_{y_i} t' \partial_t \Phi\right]$$

$$= \nabla \Phi \lambda_1 \cdot \left[\lambda_i \frac{\rho_1}{\lambda_i - \lambda_1} \sum_k w_k r_k + \frac{\rho_1}{\lambda_i - \lambda_1} (\lambda_i - \Phi) \sum_k w_k r_k\right]$$

$$= O(v_1 + \rho_1 \sum_{k \neq 1} w_k).$$
Similarly, we get
\[ \partial_y c_{1m}^1 = \nabla \Phi c_{1m}^1 \cdot \partial_y \Phi = \nabla \Phi c_{1m}^1 \cdot [\partial_x X_t \partial_y t' \partial_x \Phi + \partial_y t' \partial_t \Phi] = \nabla \Phi c_{1m}^1 \cdot \left[ \lambda_i \frac{\rho_1}{\lambda_i - \lambda_1} \sum_k w_k r_k + \frac{\rho_1}{\lambda_i - \lambda_1} (-A(\Phi) \sum_k w_k r_k) \right] = O\left(v_1 + \rho_1 \sum_{k \neq 1} w_k \right). \]
\[ \text{(5.6)} \]

Next, by (2.16)-(2.18), we have
\[ \partial_y \rho_m = \frac{\rho_1}{\lambda_m - \lambda_1} \partial_s \rho_m = O\left(\rho_m v_1 + \rho_1 \rho_m \sum_{k \neq 1} w_k \right) \text{ when } m \neq 1, \]
\[ \text{(5.7)} \]
and
\[ \partial_y v_m = \frac{\rho_1}{\lambda_m - \lambda_1} \partial_s v_m = O\left(\rho_m v_1 \sum_{k \neq 1} w_k + \rho_1 \rho_m \sum_{j \neq 1, k \neq 1} w_j w_k \right) \text{ when } m \neq 1. \]
\[ \text{(5.8)} \]

Therefore, by (5.5)-(5.8), one can rewrite (5.4) as
\[ \partial_y \tau_{1}^{(5)} := B_{11} \tau_{1}^{(5)} + B_{12} \pi_{1}^{(5)} + B_{13} \]
\[ \text{(5.9)} \]
with $B_{11}, B_{12}, B_{13}$ being uniform constants depending on $\eta$.

Next, we deduce the following evolution equation for $\pi_{1}^{(5)}$ in the same manner
\[ \partial_y \pi_{1}^{(5)} = \frac{\rho_5}{2C_f} \left( \sum_{p 
eq 1, q 
eq 1, p \neq q} \gamma_{1p}^1 w_p w_q \pi_{1}^{(5)} \right) + \sum_{p \neq 1} \gamma_{1p}^1 w_p \pi_{1}^{(5)} \]
\[ - \frac{\partial_y C_f}{2C_f^2} \left( \sum_{p 
eq 1} \gamma_{1p}^1 w_p v_1 \rho_5 + \sum_{p \neq 1, q \neq 1, p \neq q} \gamma_{pq}^1 w_p w_q \rho_5 \rho_1 \right) \]
\[ + \frac{\rho_5 \rho_1}{2C_f} \left( \sum_{p \neq 1} \partial_y \gamma_{1p}^1 w_p w_m + \sum_{p \neq 1, q \neq 1, p \neq q} \partial_y \gamma_{pq}^1 w_p w_q \right) \]
\[ + \frac{\rho_1}{2C_f} \left( \sum_{p \neq 1} \gamma_{1p}^1 w_p w_1 + \sum_{p \neq 1, q \neq 1, p \neq q} \gamma_{pq}^1 w_p w_q \right) \partial_y \rho_5 \]
\[ + \frac{\rho_5 \rho_1}{2C_f} \left( \sum_{p = 2, 3, 4} \gamma_{1p}^1 w_1 \rho_p + \sum_{p \neq 1, q \neq 1, p \neq q} \gamma_{pq}^1 w_q \rho_p \right) (\partial_y v_p - w_p \partial_y \rho_5) \]
\[ + \frac{\rho_5 \rho_1}{2C_f} \sum_{p \neq 1, q \neq 1, p \neq q} \gamma_{pq}^1 \rho_p (\partial_y v_q - w_q \partial_y \rho_5) + \frac{\rho_1}{2C_f} \gamma_{15}^1 w_1 \partial_y v_5 \]
\[ := B_{21} \tau_{1}^{(5)} + B_{22} \pi_{1}^{(5)} + B_{23}. \]
\[ \text{Similarly to (5.9), the constants } B_{21}, B_{22}, B_{23} \text{ here are also uniformly bounded.} \]
Then, we bound the initial data of $\tau_1^{(5)}$ and $\pi_1^{(5)}$. It follows from (2.11) and (2.14) that

$$
\tau_1^{(5)}(z_1, 0) = \partial_{z_1} \rho_1(z_1, 0) - \frac{\rho_1(z_1, 0)}{2C_f} \partial_{s_1} \rho_1(z_1, 0)
$$

$$
= - \frac{1}{2C_f} \sum_k c_{1k} w_k(z_1, 0) = O(W_0^{(\eta)}) < +\infty.
$$

And since $v_1^{(n)}(z_1, 0) = w_1^{(n)}(z_1, 0)$, we similarly obtain

$$
\pi_1^{(5)}(z_1, 0) = \partial_{z_1} v_1(z_1, 0) - \frac{\rho_1(z_1, 0)}{2C_f} \partial_{s_1} v_1(z_1, 0)
$$

$$
= \partial_{z_1} w_1(z_1, 0) - \frac{1}{2C_f} \sum_{q\neq 1, q\neq p} \gamma_{pq} w_p(z_1, 0) w_q(z_1, 0) \rho_1(z_1, 0)
$$

$$
= O(\partial_{z_1} w_1(z_1, 0) + [W_0^{(n)}]^2) < +\infty.
$$

Now by applying Grönwall’s inequality to (5.9) and (5.10), for $s_1 \leq \tau^*$, we deduce that $\tau_1^{(5)} := \partial_{y_1} \rho_1(z_1, s_1)$ is bounded on $z_1 \in [\eta, 2\eta]$.

Back to (5.3), invoking all these estimates, we hence prove

$$
\partial_{z_1} \rho_1 = \partial_{y_1} \rho_1 + O(v_1 + \sum_{m \neq 1} w_m \rho_1).
$$

Employing the estimates for $J(t)$, $S(t)$ and $V(t)$ obtained in Section 4 consequently, we conclude that $|\partial_{z_1} \rho_1|$ is uniformly bounded by a uniform constant $C$. \hfill \Box

Next, we begin to estimate the $H^1$ norm of the solution. The integration is taken in a subinterval $(z_0, z^*_1) \subseteq [\eta, 2\eta]$ where $|w_1^{(n)}(z, 0)| > \frac{1}{2} W_0^{(n)}$ for $z \in (z_0, z^*_1)$. In this subinterval, it follows from (1.34) that $|v_1(z, t)|$ admits a lower bound $|v_1(z, t)| \geq \frac{1}{2} W_0^{(n)}$. Then, together with the shock formation at $(z_0, \tau^*_1)$, i.e., $\rho_1(z_0, \tau^*_1) = 0$ and Proposition 5.1, we obtain

$$
||w_1(\cdot, \tau^*_1)||^2_{L^2(\Omega_{\tau^*_1})} 
$$

$$
\geq C \int_{\eta}^{2\eta} \frac{|v_1(z, T^*_\eta)|^2}{\rho_1(z, T^*_\eta)} \rho_1(z, T^*_\eta) \sqrt{(z - \eta + O(\varepsilon) \eta)(2\eta - z + O(\varepsilon) \eta)} \, dz
$$

$$
\geq C(1 - \varepsilon)^2(1 - 2\varepsilon)^2 [W_0^{(n)}]^2 \int_{z_0}^{z^*_1} \frac{1}{\rho_1(z, T^*_\eta)} \sqrt{(z - \eta + O(\varepsilon) \eta)(2\eta - z + O(\varepsilon) \eta)} \, dz
$$

$$
\geq C \sqrt{(z_0 - \eta)(2\eta - z^*_1)} [W_0^{(n)}]^2 \int_{z_0}^{z^*_1} \rho_1(z, T^*_\eta) - \rho_1(z_0, T^*_\eta) \, dz
$$

$$
\geq C \sqrt{(z_0 - \eta)(2\eta - z^*_1)} [W_0^{(n)}]^2 \int_{z_0}^{z^*_1} \frac{1}{(\sup_{z \in (z_0, z^*_1)} |\partial_z \rho_1|)(z - z_0, \tau^*_\eta)} \, dz
$$

$$
\geq C \eta \int_{z_0}^{z^*_1} \frac{1}{z - z_0} \, dz = +\infty.
$$
Here we crucially use the property \( \rho_1(z_0, T^*_n) = 0 \) of shock formation and the boundedness of \( \partial_{z_1} \rho_1 \) in Proposition 5.1. Thus, the proved ill-posedness is driven by instantaneous shock formation.

Finally, we go back to the original 2D ideal MHD system (1.1). Noting that since the region \( \Omega_{T^n} \) stays in \( \mathbb{R}_1 \), therefore \( w_i|_{\Omega_{T^n}} \) are controlled by \( V(t) \) for \( i = 2, 3, 4, 5 \). Combining this with the fact that \( |r_k| = O(1) \), we hence obtain

\[
\| \partial_x u_1 \|_{L^2(\Omega_{T^n})} = \| \sum_{k=1}^5 w_k r_k \|_{L^2(\Omega_{T^n})} \geq C \left( \| w_1 \|_{L^2(\Omega_{T^n})} - \sum_{k=2,4,5} \| w_k \|_{L^2(\Omega_{T^n})} \right)
\]

\[
\geq C \left( \| w_1 \|_{L^2(\Omega_{T^n})} - 3V(T^n)|\Omega_{T^n}|^{\frac{1}{2}} \right)
\]

\[
\geq C \left( \| w_1 \|_{L^2(\Omega_{T^n})} - 3\eta^2 W_0^{(n)} \right) = +\infty
\]

and

\[
\| \partial_x \rho \|_{L^2(\Omega_{T^n})} = \| \sum_{k=1}^5 w_k r_{k3} \|_{L^2(\Omega_{T^n})} \geq C \left( \| w_1 \|_{L^2(\Omega_{T^n})} - \sum_{k=2,3,4,5} \| w_k \|_{L^2(\Omega_{T^n})} \right)
\]

\[
\geq C \left( \| w_1 \|_{L^2(\Omega_{T^n})} - 4\eta^2 W_0^{(n)} \right) = +\infty.
\]

This concludes the proof of Theorem 1.1 when \( H_1 \neq 0 \).

6. PROOF FOR \( H_1 = 0 \) CASE

In this section, we deal with the case of \( H_1 = 0 \). We first explore the detailed structures of the decomposed system for this scenario. Then we prove the shock formation and the desired ill-posedness. Compared with the \( H_1 \neq 0 \) case, the system here is non-strictly hyperbolic. Nonetheless, as will be explained below, since the non-separating characteristic speeds are identical, we can treat the corresponding characteristics as in the same family. The arguments in Section 4 and Section 5 then apply to the \( H_1 = 0 \) case.

When \( H_1 = 0 \), under planer symmetry, \( \Phi = (u_1, u_2, \rho - 1, H_2, S)^T \) satisfies:

\[
\partial_t \Phi + A(\Phi)\partial_x \Phi = 0
\]

with the coefficient matrix being

\[
A(\Phi) = \begin{pmatrix}
u_1 & 0 & c^2/q & \mu_0 H_2/q & c^2/\gamma \\
0 & u_1 & 0 & 0 & 0 \\
\nu & 0 & u_1 & 0 & 0 \\
H_2 & 0 & 0 & u_1 & 0 \\
0 & 0 & 0 & 0 & u_1
\end{pmatrix}
\]

The eigenvalues of (6.2) are

\[
\lambda_1 = u_1 + C_f, \quad \lambda_2 = \lambda_3 = \lambda_4 = u_1, \quad \lambda_5 = u_1 - C_f,
\]

where \( C_f \) is defined by

\[
C_f = \sqrt{\frac{\mu_0 H_2^2}{\gamma} + c^2}.
\]
And the eigenvalues satisfy

$$\lambda_5 < \lambda_4 = \lambda_3 = \lambda_2 < \lambda_1.$$  

Since $A(\Phi)$ admits an eigenvalue of triple multiplicity, the system is not strictly hyperbolic. We choose the right eigenvectors as:

$$r_1 = \begin{pmatrix} C_f \varepsilon \\ 0 \\ 0 \\ H_2 \varepsilon \\ 0 \end{pmatrix}, r_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, r_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -\frac{\gamma}{\varepsilon} \end{pmatrix}, r_4 = \begin{pmatrix} \frac{-C_f}{\varepsilon} \\ 0 \\ 0 \\ 1 \\ -\gamma \mu_0 H_2 \frac{\varepsilon}{\rho c^2} \end{pmatrix}, r_5 = \begin{pmatrix} 0 \\ -\frac{C_f}{\varepsilon} \\ 0 \\ 0 \\ 1 \end{pmatrix}. \quad (6.4)$$

And the left eigenvectors are set to be the dual of the right ones:

$$l_i r_j = \delta_{ij}. \quad (6.5)$$

Via calculations, we construct the following left eigenvectors

$$l_1 = \left( \frac{\rho}{2 C_f}, \frac{\rho c^2}{2 C_f}, \frac{\mu_0 H_2}{2 C_f}, \frac{\rho c^2}{\gamma C_f} \right),$$

$$l_2 = (0, 1, 0, 0, 0),$$

$$l_3 = (0, 0, \frac{\mu_0 H_2}{\rho c^2}, \frac{\mu_0 H_2}{C_f}, \frac{-\rho c^2}{\gamma C_f}),$$

$$l_4 = (0, 0, -\frac{c^2 H_2}{\rho c^2}, \frac{c^2}{C_f}, \frac{-c^2 H_2}{\gamma C_f})$$

$$l_5 = \left( -\frac{\rho}{2 C_f}, \frac{\rho c^2}{2 C_f}, \frac{\mu_0 H_2}{2 C_f}, \frac{\rho c^2}{\gamma C_f} \right), \quad (6.6)$$

which satisfies (6.5). With these eigenvectors, we further decompose $\partial_s \Phi$ as in Section 2.2 and obtain:

$$\partial_s \rho_i = c_{i \rho}^i v_i + \left( \sum_{m \neq i} c_{im}^i w_m \right) \rho_i, \quad (6.7)$$

$$\partial_s w_i = -c_{i \rho}^i w_i^2 + \left( \sum_{m \neq i} (-c_{im}^i + \gamma_{im}^i) w_m \right) w_i + \sum_{m \neq i, k \neq i} \gamma_{km}^i w_k w_m, \quad (6.8)$$

$$\partial_s v_i = \left( \sum_{m \neq i} \gamma_{im}^i w_m \right) v_i + \sum_{m \neq i, k \neq i} \gamma_{km}^i w_k w_m \rho_i, \quad (6.9)$$

where

$$c_{im}^i = \nabla_\Phi \lambda_i \cdot r_m,$$

$$\gamma_{im}^i = -(\lambda_i - \lambda_m) l_i \cdot (\nabla_\Phi r_i \cdot r_m - \nabla_\Phi r_m \cdot r_i), \quad m \neq i,$$

$$\gamma_{km}^i = -(\lambda_k - \lambda_m) l_i \cdot (\nabla_\Phi r_k \cdot r_m), \quad k \neq i, m \neq i.$$
Since $H_2$ is small around zero, one can check that $C_f \approx c$. Thus $r_i$, $\nabla \Phi r_i$ and $l_i$ are of order $O(1)$. Hence the coefficients $c_{im}^i, \gamma_{im}^i, \gamma_{km}^i$ are all uniformly bounded by $O(1)$. Especially, we have
\[
c_{11}^1(0) = \nabla \Phi \lambda_1(0) \cdot r_1(0) = \left(1, 0, \frac{(\gamma - 1)A_7}{2}, 0, A_7\right) \cdot \left(\frac{\sqrt{A_7}}{2}, 0, 1, 0, 0\right)^T = \left(\frac{(\gamma + 1)\sqrt{A_7}}{2}\right) > 0.
\]

The first family of characteristic is now genuinely nonlinear. The main difference between the case of $H_1 = 0$ and $H_1 \neq 0$ is the non-strict hyperbolicity. For $\lambda_2 = \lambda_3 = \lambda_4$, the corresponding characteristic strips $\mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4$ are the same. Our strategy is to consider characteristic waves that propagate in and out of the following three characteristic strips: $\{\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3\}$, where $\mathcal{R}_2 := \mathcal{R}_2 = \mathcal{R}_3 = \mathcal{R}_4$. These three strips will be completely separated when $t > t_0^{(\eta)}$. See the following figure.

![Figure 2. Separation of three characteristic strips.](image-url)

Fortunately, according to the decomposed system (6.7)-(6.9), the interaction terms of the $i^{th}$ and the $m^{th}$ characteristic waves vanish for $i, m \in \{2, 3, 4\}$. In particular, since $\nabla \Phi \lambda_i = (1, 0, 0, 0, 0)$ and $r_{m1} = 0$, one can see that all $c_{im}^i = \nabla \Phi \lambda_i r_m = 0$ for $i, m \in \{2, 3, 4\}$. Since there is the factor $\lambda_i - \lambda_m$ in $\gamma_{im}^i$, we further get $\gamma_{im}^i = 0$ for $i, m \in \{2, 3, 4\}$. And for $i \in \{1, \cdots, 5\}$ and $k, m \in \{2, 3, 4\}$, similarly we have $\gamma_{km}^i = 0$.

Proceeding in the same fashion as in (3.4), we introduce the following quantity:
\[
V(t) = \max \left\{ \sup_{(x', t') \notin \mathcal{R}_1, 0 \leq t' \leq t} |w_1(x', t')|, \sup_{(x', t') \notin \mathcal{R}_5, 0 \leq t' \leq t} |w_5(x', t')|, V_2(t) \right\}, \quad (6.10)
\]

where
\[
V_2(t) := \sup_{(x', t') \notin \mathcal{R}_2, 0 \leq t' \leq t} \{|w_2(x', t')|, |w_3(x', t')|, |w_4(x', t')|\}.
\]

With the same initial data designed in (3.1) and (3.2), for $t \in [0, T^\eta)$, we get the following estimates in a similar manner as in Section 4
\[
S(t) = O(1), \quad J(t) = O(W_0^{(\eta)}), \quad V(t) = O\left(\eta[W_0^{(\eta)}]^2\right), \quad \tilde{U}(t) = O(\eta W_0^{(\eta)}), \quad (6.11)
\]
and
\[
\frac{1}{(1 + \varepsilon)^4 c_1(0) W_0(\eta)} \leq T^*_\eta \leq \frac{1}{(1 - \varepsilon)^4 c_1(0) W_0(\eta)}. \tag{6.12}
\]
Here \(T^*_\eta\) is the shock formation time. In particular, a shock forms as \(t \to T^*_\eta\), i.e.,
\[
\lim_{t \to T^*_\eta} \rho_1(z_0, t) = 0.
\]
Via the same argument as in Section 5, we have that the \(H^1\) norm of the solution to (6.2) blows up at \(T^*_\eta\). Hence the \(H^\frac{7}{4}\) ill-posedness stated in Theorem 1.1 is also true when \(H_1 = 0\).

7. \(H^\frac{7}{4}\) ill-posedness for 2D compressible Euler equations

When the magnetic field vanishes, i.e., \(H_1 = H_2 = 0\), the MHD system (1.1) reduces to the compressible Euler system:
\[
\begin{aligned}
\partial_t \rho + \nabla \cdot (\rho u) &= 0, \\
\rho \{\partial_t + (u \cdot \nabla)\} u + \nabla p &= 0, \\
\partial_t S + (u \cdot \nabla) S &= 0.
\end{aligned} \tag{7.1}
\]
Now define
\[
\tilde{U}(x_1, x_2, t) = (u_1, u_2, \rho, S)^T(x_1, x_2, t).
\]
And we denote its planar symmetric solution to be
\[
\tilde{\Phi} := (u_1, u_2, \rho - 1, S)^T(x_1, t) = (u_1, u_2, \rho - 1, S)^T(x, t).
\]
Under plane symmetry, the 2D compressible Euler system (7.1) then reads
\[
\partial_t \tilde{\Phi} + A(\tilde{\Phi}) \partial_x \tilde{\Phi} = 0 \tag{7.2}
\]
with
\[
A(\tilde{\Phi}) = \begin{pmatrix}
    u_1 & 0 & c^2/\rho & c^2/\gamma \\
    0 & u_1 & 0 & 0 \\
    \rho & 0 & u_1 & 0 \\
    0 & 0 & 0 & u_1
\end{pmatrix}. \tag{7.3}
\]
By a direct calculation, we have the eigenvalues of (7.3) are
\[
\lambda_1 = u_1 + c, \quad \lambda_2 = \lambda_3 = u_1, \quad \lambda_4 = u_1 - c. \tag{7.4}
\]
This indicates that the Euler system under planar symmetry is not strictly hyperbolic. We then choose the right eigenvectors as:
\[
\begin{align*}
    r_1 &= \begin{pmatrix} 1 \\ 0 \\ \rho/c \\ 0 \end{pmatrix}, \\
    r_2 &= \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \\
    r_3 &= \begin{pmatrix} 0 \\ 0 \\ 1 \\ -\gamma/\rho \end{pmatrix}, \\
    r_5 &= \begin{pmatrix} 1 \\ 0 \\ -\rho/c \\ 0 \end{pmatrix}.
\end{align*} \tag{7.5}
\]
The left eigenvectors are set dual to the right ones, i.e.,
\[
\begin{align*}
    l_1 &= \left(\frac{1}{2}, 0, \frac{c}{2\rho}, -\frac{c}{2\gamma}\right), \\
    l_2 &= (0, 1, 0, 0), \\
    l_3 &= (0, 0, 0, -\frac{\rho}{\gamma}), \\
    l_4 &= \left(\frac{1}{2}, 0, -\frac{c}{2\rho}, -\frac{c}{2\gamma}\right). \tag{7.6}
\end{align*}
\]
Then we derive the decomposed system of $w_i, \rho_i, v_i$ as in (6.7)-(6.9) with bounded coefficients $c_{im}^i, \gamma_{im}^i, \gamma_{km}^i$. In particular, the first family of characteristics is genuinely nonlinear because

$$c_{11}^i(0) = \nabla \Phi \lambda_1(0) \cdot r_1(0) = \left(1, 0, (\gamma - 1)\sqrt{A\gamma}, 1\right) \cdot \left(1, 0, \frac{1}{\sqrt{A\gamma}}, 0\right)^T = \gamma > 0.$$ 

Note that the interaction terms of the second and third characteristic waves vanish. The desired $H_{\frac{7}{4}}$ ill-posedness follows from the argument as in Section 6.

**Remark 7.1.** In terms of the low-regularity local well-posedness result in [39] by Zhang, one can see that our $H_{\frac{7}{4}}$ ill-posedness for 2D compressible Euler equations is sharp with respect to fluid velocity and density. Moreover, we introduce the vorticity $\omega = -\partial_x u_2 + \partial_x u_2$. Under plane symmetry, the vorticity $\omega = \partial_x u_2$ is regular since

$$\partial_x u_2 = \sum_{k=1}^4 w_k r_{k2} = w_2.$$ 

The system (7.2) therefore allows non-trivial vorticity and entropy.

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