Quantization of the External Algebra on a Poisson-Lie Group

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Abstract

We show that the external algebra $\mathcal{M}$ on $GL(N)$ can be equipped with the graded Poisson brackets compatible with the group action. We prove that there are only two graded Poisson-Lie structures (brackets) on $\mathcal{M}$ and we obtain their explicit description. We realize that just these two structures appear as the quasiclassical limit of the bicovariant differential calculi on the quantum linear group $GL_q(N)$.

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1 Introduction

During last few years there was a number of attempts to construct a quantum group (QG) gauge theory [1]-[5]. Such a theory looks rather promising since it permits to go beyond the rigid frames of ordinary gauge theory keeping therewith it’s best features [2, 6]. It is well known that the appropriate description of ordinary gauge theories is in terms of differential geometry and it seems natural and attractive to employ this language for the QG case.

One of the essential components of differential geometry is the calculus of differential forms on a Lie group. The possible algebraic structure of differential calculus on QG-s $SU_q(N), SO_q(N)$ was proposed in [7, 8] and for the $GL_q(N)$ in [9]-[11]. Later it was found that this structure can be put in $R$-matrix notations [12]-[14]. In a rather abstract formulation the notions of the usual differential geometry were generalized to the QG case in [15].

In the theory of QG there is a powerful and instructive look on QG as on a result of quantization (deformation) of an underlying classical structure [16]-[19]. Namely, an algebra of functions on a Lie group can be equipped with the Poisson brackets $\{,\}$ compatible with a group multiplication. This group is defined as a Poisson-Lie group. For simple Lie groups Poisson-Lie structures are defined by a tensor $r$ being a solution of the Classical Yang-Baxter Equation (CYBE). Just this bracket should be quantized to obtain a non-commutative algebra $N_h$ called a function algebra on a QG. The defining relations of $N_h$ are specified by a quantum $R(h)$-matrix being a solution of the Quantum Yang-Baxter Equation (QYBE): $RTT = TTR$ [20] while the classical limit of $R(h)$ (when $h$ tends to zero) is $r$.

From this point of view it seems rather unsatisfactory that despite the existence of a number of papers on quantum differential calculus (see refs. above) the issue of a Poisson-Lie structure on the external algebra of a Lie group was not discussed in the current literature. In this note we are going to fill this gap.

We analyze the external algebra on $G = GL(N)$ equipped with the Poisson-Lie structure defined by the Sklyanin bracket with $r$-matrix being the lift of the canonical $r$-matrix for $SL(N)$. We show that the Poisson-Lie structure on the algebra of ordinary function on $G$ can be extended to the external algebra of forms. As we prove there exists a pair of graded Poisson brackets for this complex and we present their explicit expression. The brackets appear to be defined by the same $r$ matrix as the Sklyanin brackets [17] are.

Next we turn to the issue of quantization. By the explicit calculation we realize that the classical limit of the QG external algebra [21] is nothing but the graded Poisson-Lie structure described above. We observe that some phenomena which were thought to be especially of quantum nature do have classical counterparts in the Poisson-Lie external algebra.

The paper is organized as follows. In Section 2 we give a brief outlook of essential notions of Hopf algebras that are necessary to state the problem. In Section 3 we define the Hopf superalgebra $M$ and formulate the consistency condition for graded Poisson brackets. By solving this condition together with the graded Jacobi identity we arrive at the explicit formula for the Poisson-Lie brackets on $M$. In Section 4
we translate our result from the abstract language of Hopf algebras into the notions of ordinary differential geometry thus obtaining the graded Poisson-Lie structure on the external algebra of $G$. In Section 5 we demonstrate that the classical limit of QG external complex coincides with our structure. In Section 6 we discuss possible generalization of our approach.

2 Basic facts and definitions

The most transparent way to define the Poisson-Lie structure is to use the notion of a Hopf algebra. Hence we start with recalling the basic definitions [20, 19].

An associative unital algebra $A$ is called a Hopf algebra $(A, \cdot, k)$ over a field $k$ if it is equipped with a coproduct $\Delta : A \to A \otimes A$ (algebra homomorphism), a counit $\varepsilon : A \to k$ (algebra homomorphism) and an antipode $S : A \to A$ (algebra antihomomorphism), satisfying the following axioms:

\begin{align*}
(\Delta \otimes id)\Delta(a) &= (id \otimes \Delta)\Delta(a), \quad (\varepsilon \otimes id)\Delta(a) = (id \otimes \varepsilon)\Delta(a) = a, \quad (2.1) \\
(S \otimes id)\Delta(a) &= (id \otimes S)\Delta(a) = 1\varepsilon(a), \quad \Delta(S(a)) = \sigma(S \otimes S)\Delta(a), \quad (2.2) \\
\varepsilon(S(a)) &= \varepsilon(a), \quad \Delta(1) = 1 \otimes 1, \quad S(1) = 1, \quad \varepsilon(1) = 1, \quad (2.3)
\end{align*}

where $\sigma$ is the twist map $\sigma(a \otimes b) = (b \otimes a)$ and $a \in A$. Note, that a linear space supplied only with $\Delta$ and $\varepsilon$ is called a coalgebra, and an algebra being a coalgebra is called a bialgebra.

Let $G$ be a Lie group and $\mathcal{G}$ be its Lie algebra with a basis $\{e_\mu\}$. A classical example of a Hopf algebra is delivered by a commutative algebra $\text{Fun}(G)$ of functions on $G$. The corresponding $\Delta$, $\varepsilon$ and $S$ are given by

\begin{equation}
\Delta f(g_1, g_2) = f(g_1 g_2), \quad (\varepsilon f)(g) = f(e), \quad (S f)(g) = f(g^{-1}),
\end{equation}

where $e$ is the group unity and $\text{Fun}G \otimes \text{Fun}G$ is identified with $\text{Fun}(G \times G)$.

If $G$ is supplied with a Poisson bracket $\{\cdot, \cdot\}_G$

\begin{equation}
\{f, h\}_G(g) = \eta^\mu \nu(g) \partial_\mu f \partial_\nu h,
\end{equation}

where $\eta(g) = \eta^\mu \nu e_\mu \otimes e_\nu$ is a Poisson tensor field and $\partial_\mu$ is a basis of left-invariant vector fields on $G$, the Poisson-Lie group is defined by the following requirement

\begin{equation}
\Delta\{f, h\}_G = \{(\Delta(f), \Delta(h))\}_{G \otimes G},
\end{equation}

for any $f, h \in \text{Fun}(G)$. For simple Lie groups all the solutions of eq.(2.5) are labelled by constant ($g$-independent) elements $r$ with values in $\mathcal{G} \otimes \mathcal{G}$:

\begin{align*}
\eta(g) &= \text{Ad}_g^{-1} r - r, \\
r &= r^\mu \nu e_\mu \otimes e_\nu. \quad (2.6)
\end{align*}

This $r$ can be identified with a classical $r$-matrix.
In the case of a matrix Lie group the appropriate coordinate system in $\text{Fun}(G)$ is given by the matrix elements of a matrix $T = [t^j_i]$ in the fundamental representation of $G$. In other words, $\text{Fun}(G)$ is defined to be an algebra generated by the variables $t^j_i$. Roughly speaking, functions on $G$ are identified with formal power series in $t^j_i$. For simple Lie groups the Poisson-Lie brackets in terms of $t^j_i$-s read:

\[ \{t^j_i, t^l_k\} = r^{mn}_{ik} t^m_k t^n_l - t^m_k r^m_{jn}, \]  

(2.7)

where $r^{mn}_{ik}$ is the $r$-matrix (2.6) taken in the corresponding representation of $G$.

In our paper we shall deal with $\text{GL}(N)$ which is not a simple group and therefore eq.(2.5) has also solutions with a non-trivial $g$-dependence. However, for the purposes of quantization only solutions associated with classical $r$-matrices are relevant. Thus, following [20] we employ (2.7) for the Poisson-Lie structure on $\text{GL}(N)$ with $r$ being a trivial lifting of $r$-matrix for $\text{SL}(N)$. In the following it will be important that the bracket (2.7) is degenerate and the function $\det T$ lies in its center:

\[ \{t^j_i, \det T\} = 0. \]  

(2.8)

Fixing the value of $\det T$ equal to unity we obtain the Poisson-Lie structure on $\text{SL}(N)$.

One more fact from ordinary differential geometry will be necessary. Let $G^*$ be the dual space of $G$. The cotangent bundle $T^*G$ on $G$ is trivialized by means of right (left) action of $G$ on itself: $T^*G \approx G \times G^*$. Let us define the Maurer-Cartan right-invariant form $R_g$ on $G$ which takes value in $G$:

\[ R_g(X_g) = X_e, \]

where $X_g$ is a right-invariant vector field corresponding to the element $X_e \in G$. Under the gauge transformations $g \rightarrow g_1 g$ the form $R_g$ transforms as follows:

\[ R_g \rightarrow R_{g_1 g} = g_1 R_g g_1^{-1} + dg_1 g_1^{-1}. \]  

(2.9)

(This equation is well-known in physical literature as the gauge transformation law.) One can treat the right hand side of eq.(2.9) as the differential form on $G \times G$. With the identification $T^*(G \times G) \approx T^*G \otimes T^*G$ eq.(2.9) can be written as:

\[ R_g \rightarrow R_{g_1 g} = (g_1 \otimes I)(I \otimes R_g)(I \otimes g_1^{-1}) + R_{g_1} \otimes I. \]  

(2.10)

\section{Graded Poisson-Hopf structures associated with $\text{GL}(N)$}

Now we can introduce our basic object $\mathcal{M}$. To describe the external algebra of right-invariant forms we add to the system of coordinates $t^j_i$ new anticommuting variables $\theta^j_i$. Hence, by definition $\mathcal{M}$ is a free associative algebra generated by $t^j_i, \theta^j_i, t$ modulo the relations:

\[ t^j_i t^l_k = t^l_k t^j_i, \quad t \theta^j_i = \theta^j_i t, \quad t \det T = I, \]
\[ t^j_k \theta^l_i = \theta^l_k t^j_i, \quad t \theta^j_i = \theta^j_i t, \quad \theta^j_i \theta^l_k = -\theta^l_k \theta^j_i. \]

The algebra \( \mathcal{M} \) has a natural grading with \( \text{deg} (t^j_i) = 0 \) and \( \text{deg} (\theta^j_i) = 1 \).

Let us define the multiplication law in \( \mathcal{M} \otimes \mathcal{M} \) as follows:

\[ (a \otimes b)(c \otimes d) = (-1)^{\text{deg} b \text{deg} c} (ac \otimes bd) \quad (3.1) \]

for any \( a, b, c, d \in \mathcal{M} \). Now one can endow \( \mathcal{M} \) with the structure of the graded Hopf algebra defining the action of \( \Delta, \varepsilon, S \) on the generators \( t^j_i, \theta^j_i, t \) as follows:

\[ \Delta t^j_i = t^k_i \otimes t^l_k, \quad \varepsilon(t^j_i) = \delta^j_i, \quad S(t^j_i) = tt^j_i \quad (3.2) \]

\[ \Delta \theta^j_i = \theta^j_i \otimes I + t^k_i S(t^j_p) \otimes \theta^p_k, \quad (3.3) \]

\[ \varepsilon(\theta^j_i) = 0, \quad S(\theta^j_i) = -S(t^k_i) \theta^p_k t^j_p. \quad (3.4) \]

\[ \Delta(t) = t \otimes t, \quad \varepsilon(t) = 1, \quad S(t) = \det T. \quad (3.5) \]

in the usual notation one has \( \| t^j_i \|^{-1} = tt^j_i \).

To an arbitrary element of \( \mathcal{M} \) the actions of \( \Delta, \varepsilon \) are extended as to be homomorphisms and \( S \) as to be antihomomorphism. The coproduct law eq.(3.3) mimics the transformation law for the right invariant forms on a Lie group (see eqs.(2.9), (2.10)).

Our main goal is to equip \( \mathcal{M} \) with a Poisson structure consistent with the coproduct on \( \mathcal{M} \). Precisely, it means the following. We introduce a bilinear operation \( \{,\} : \mathcal{M} \otimes \mathcal{M} \to \mathcal{M} \) called brackets. The algebra \( \mathcal{M} \) has odd and even generators, so it is natural to require for this bracket the fulfillment of the super Jacobi identity:

\[ (-1)^{\text{deg} a \text{deg} c} \{\{a,b\},c\} + (-1)^{\text{deg} b \text{deg} c} \{\{c,a\},b\} + (-1)^{\text{deg} a \text{deg} b} \{\{b,c\},a\} = 0, \quad (3.6) \]

the graded Leibniz rule

\[ \{a \cdot b, c\} = a\{b, c\} + (-1)^{\text{deg} b \text{deg} c} \{a, c\} b, \quad (3.7) \]

and the graded symmetry property:

\[ \{a, b\} = (-1)^{\text{deg} a \text{deg} b + 1} \{b, a\}, \quad \text{deg} \{a, b\} = (\text{deg} a + \text{deg} b) \mod 2. \quad (3.8) \]

Let us require now that our algebra \( \mathcal{M} \) supplied with the Poisson structure would be a Poisson-Hopf algebra, i.e. the Poisson brackets would satisfy

\[ \Delta \{a, b\}_M = \{\Delta(a), \Delta(b)\} \cdot \mathcal{M} \otimes \mathcal{M}, \quad (3.9) \]

where the bracket on \( \mathcal{M} \otimes \mathcal{M} \) is defined as

\[ \{a \otimes b, c \otimes d\} \cdot \mathcal{M} \otimes \mathcal{M} = (-1)^{\text{deg} b \text{deg} c} \{a, c\} \otimes bd + (-1)^{\text{deg} b \text{deg} c} ac \otimes \{b, d\} \quad (3.10) \]

for any elements \( a, b, c, d \in \mathcal{M} \). In what follows we will call the Hopf algebra \( \mathcal{M} \) equipped with the brackets defined above as the Poisson-Hopf superalgebra.
To define the brackets on $\mathcal{M}$ it is enough to define them on the set of generators $t_i^j, \theta_i^j$ and then to extend by the Leibniz rule to the whole algebra.

The linear space $\mathcal{N}$ spanned by the generators $t_i^j$ and $t$ forms the Hopf subalgebra that can be identified with the commutative algebra of functions on $GL(N)$. Hence, we equip $\mathcal{N}$ with the bracket (2.7) described in Sec. 2:

$$\{T_1, T_2\} = [r_+, T_1 T_2].$$

(3.11)

Here we use the standard tensor notation: $T_1 = T \otimes I$, $T_2 = I \otimes T$. The $r_+$ matrix satisfies the CYBE and the condition:

$$P r P + r = 2 P$$

(3.12)

where $P_{ik}^{sp} = \delta_i^p \delta_k^s$ is a permutation operator. Now our goal is to extend these brackets on $\mathcal{N}$ to a Poisson-Hopf structure on $\mathcal{M}$.

To find the $\{\theta, T\}$ bracket we shall take it in a most general form and after that find the coefficients by imposing the constraints that follow from eq. (3.9) and Jacobi identity for $\theta$ and two $T$-s. The realization of this program leads to obvious but rather tedious calculations which are sketched in Appendix A. The result we obtain is the following. The brackets are arranged into two families. In the first one the brackets are parametrized by two continuous parameters $\alpha$ and $\beta$ and by the sign $\epsilon$ of $m$:

$$\{\theta_1, T_2\}^{\pm}_{\alpha, \beta} = \alpha \theta_2 T_2 + r_{\pm}^{12} \theta_1 T_2 - \theta_1 r_{\pm}^{12} T_2 + \alpha_2 \text{tr} \theta T_2 + \alpha_3 \text{tr} \theta P^{12} T_2 + \beta \theta_1 T_2,$$

(3.13)

where

$$\alpha_2 = -\frac{\alpha^2}{m + \alpha N}, \quad \alpha_3 = -\frac{\alpha m}{m + \alpha N}, \quad \alpha \neq -\frac{m}{N}, \quad m = \pm 2.$$  

(3.14)

In the second family we have

$$\{\theta_1, T_2\}^{\pm}_{\alpha_2, \beta} = r_{\pm}^{12} \theta_1 T_2 - \theta_1 r_{\pm}^{12} T_2 + \alpha_2 \text{tr} \theta T_2 + \beta \theta_1 T_2,$$

(3.15)

where now $\alpha_2, \beta$ are arbitrary.

Following the same strategy we get for the $\{\theta, \theta\}$ bracket (the calculation is sketched in Appendix B) in tensor notations:

$$\{\theta_1, \theta_2\}^{\pm}_\alpha = \alpha (\theta_1 \theta_1 + \theta_2 \theta_2) + r_+^{12} \theta_1 \theta_2 + \theta_1 \theta_2 r_+^{12} -$$

(3.16)

$$\theta_1 (r_+^{12} - \frac{m + 2}{2} P^{12}) \theta_2 + \theta_2 (r_+^{12} + \frac{m - 2}{2} P^{12}) \theta_1.$$

Let us stress that this bracket prolongs the $\{\theta, t\}$ brackets from the first family only. We find that there are no extensions for the brackets from the second family consistent with the coproduct $\Delta$. Hence, in eq. (3.10) $m = \pm 2$ as it must be for the bracket (3.13). Thus, the bracket for two $\theta$-s is uniquely determined by the bracket $\{\theta, T\}$.

One also has to check the Jacobi identity for the system of brackets given by eqs. (3.13) and (3.16):

$$\sum (-1)^{\text{deg}(1) \text{deg}(3)} \{\{\theta, \theta\}, T\} = 0,$$

(3.17)

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\[ \sum \{ \{ \theta, \theta \}, \theta \} = 0, \quad (3.18) \]

but it is convenient before turning to the explicit calculation to make some preliminary studies.

At this stage our result looks like that we have obtained the infinite family of candidates for brackets given by eqs. (2.7), (3.13), (3.16).

However, one has to take into account an arbitrariness in the choice of generators \( t_i^j \) and \( \theta_i^j \) in the Hopf algebra \( \mathcal{M} \). There exists a nondegenerate change of variables:

\[
\begin{align*}
T & \rightarrow \tilde{T} = T (\det T)^s \\
t & \rightarrow \tilde{t} = (t)^{N_s+1} \\
\theta & \rightarrow \tilde{\theta} = \theta + k \text{ tr } \theta, \quad k, s \neq -1/N
\end{align*}
\]  

(3.19)

that does not affect the form of the coproduct

\[ \Delta \tilde{t}^j_i = \tilde{t}^i_j \otimes \tilde{t}^j_i, \quad \Delta \tilde{\theta}^j_i = \tilde{\theta}^j_i \otimes I + \tilde{t}^k_i S(\tilde{t}^j_p) \otimes \tilde{\theta}^p_k. \]

Hence, the natural question arises how this covariance of the coproduct reflects itself on the level of brackets. First of all, one can check that the \( T \) brackets (3.11) are not affected by the transformation (3.19):

\[ \{ \tilde{T}(T)_1, \tilde{T}(T)_2 \} = \{ T_1, T_2 \}|_{T=T \det(T)^s}. \quad (3.20) \]

This directly follows from the above mentioned fact that \( \det T \) lies in the center of the Sklyanin brackets. To answer this question for the brackets given by eqs. (3.13), (3.16) let us make the transformation (3.19) and calculate the bracket \( \{ \theta, \tilde{T} \}_\alpha^\pm \) for some given \( \alpha \) and \( \beta \). We get:

\[ \{ \tilde{\theta}(\theta)_1, \tilde{T}(T)_2 \}_\alpha^\pm = \{ \theta_1 + k \text{ tr } \theta I_1, T (\det T)^2 \}_\alpha^\pm = \{ \tilde{\theta}_1, \tilde{T}_2 \}_\alpha^\pm |_{\tilde{T}=T \det(T)^s}. \]

The same "covariance" holds for the bracket \( \{ \theta, \theta \}_\alpha^\pm \):

\[ \{ \tilde{\theta}(\theta)_1, \tilde{\theta}(\theta)_2 \}_\alpha^\pm = \{ \theta_1, \tilde{T}_2 \}_\alpha^\pm |_{\tilde{\theta}=\theta + k \text{ tr } \theta} \]

(3.21)

where in both cases

\[ \alpha' = \alpha + k(\alpha N + m), \quad \beta' = \beta + s(\beta N + m). \quad (3.22) \]

So, under the change of generators given by eq. (3.19) the brackets \( \{ , \}_\alpha^\pm \) and \( \{ , \}_\alpha^\pm \) are transformed into the bracket \( \{ , \}_\alpha^\pm, \beta^\pm \) and \( \{ , \}_\alpha^\pm \) with \( \alpha' \) and \( \beta' \) given by eq. (3.22). The inverse assertion is also true: for any two pairs of admissible \( \alpha, \beta \) and \( \alpha', \beta' \) \( (\alpha, \beta, \alpha', \beta' \neq -m/N) \) there exist such \( k \) and \( s \) that under the transformation (3.19) the \( \alpha, \beta \) brackets convert into the \( \alpha', \beta' \) brackets.

Hence, we have proved that by the appropriate change of variables any bracket from the family can be put into the "canonical" form with \( \alpha = 0 = \beta \):

\[
\begin{align*}
\{ \theta_1, T_2 \}_\alpha^\pm &= r^2_1 \theta_1 T_2 - \theta_1 r^2_2 T_2 \\
\{ \theta_1, \theta_2 \}_\alpha^\pm &= r^2_1 \theta_1 \theta_2 + \theta_1 \theta_2 r^2_2 - \theta_1 r^2_2 \theta_2 + \theta_2 r^2_2 \theta_1.
\end{align*}
\]

(3.23)
Now it is the place to turn to the Jacobi identity (3.17), (3.18). Due to the covariance described above it is enough to check these identities for an arbitrary fixed values of $\alpha$ and $\beta$ and it is convenient to take the brackets in the canonical form (3.23). The direct calculation leads to

$$\sum (-1)^{\delta} \{\{\theta_1, \theta_2\}^\pm, T_3\}^\pm =$$

$$= C(r_+)\theta_1\theta_2T_3 + \theta_1\theta_2C(r_+)T_3 + \theta_2C(r_+)\theta_1T_3 - \theta_1C(r_+)\theta_2T_3,$$

(3.24)

where $\delta =\deg(1)\deg(3)$ and

$$\sum \{\{\theta_1, \theta_2\}^\pm, \theta_3\}^\pm =$$

$$= C(r_+)\theta_1\theta_2\theta_3 - \theta_1\theta_2\theta_3C(r_+) + \sum \text{cycl perm} (\theta_1\theta_2C(r_+)\theta_3 - \theta_1C(r_+)\theta_2\theta_3).$$

(3.25)

Hence, due to the CYBE: $C(r_+)$ the Jacobi identities are satisfied.

To summarize, all possible Poisson-Hopf structures on $\mathcal{M}$ are arranged into two-parameter infinite family. However, these parameters $\alpha$ and $\beta$ seem to be redundant and may be removed by the appropriate change of generators of the algebra $\mathcal{M}$, that leads to the canonical form given by eqs. (3.23).

4 Geometric interpretation

Up to now we have treated the problem in the pure algebraic framework and now we have to make a contact with the usual differential geometry on $G$. As it is clear from the previous Section, one needs to specify a coordinate system on $G$ and a basis of right-invariant differential forms which can be identified with abstract generators of the Hopf algebra $\mathcal{M}$.

Let $T(g)$ be a fundamental representation of $G$ (a homomorphism of $G$ onto itself). Note, that $T$ is not uniquely defined. Clearly, the representation

$$\tilde{T} = (\det T)^s T, \quad s \neq -1/N$$

(4.1)

is not equivalent to $T$ for they differ by the value of determinant: $\det \tilde{T} = (\det T)^{sN+1}$. The point $s = -1/N$ is forbidden because the corresponding representation becomes non-exact and cannot serve as a coordinate system on $G$. Thus, if we want to identify the matrix elements $t^i_j$ of $T(g)$ with the set of coordinates on $G$ we see that there exists the infinite number of non-equivalent coordinate systems labeled by the function $\xi(g) = \det T(g)$. In other words, fixing a one-dimensional representation $\xi(g)$ of $G$ one chooses the coordinate system $\{t^i_j\}$.

The first order differential forms, i.e. the sections of $\mathcal{T}^*G$ form a bimodule $\Gamma$ over $Fun(G)$. The linear bases in $\Gamma$ can be chosen from right(left) invariant forms in the following way. Any representation $T$ of $G$ gives rise to the representation of the Lie algebra $\mathcal{G}$. For a given representation $T$ one can define the Lie-valued right-invariant Maurer-Cartan form by the expression $\theta^i_j = (dT)^i_k (T^{-1})^k_j$. The value of $\theta^i_j$ on a right-invariant vector field $v$ is a constant matrix $\theta^i_j(v)$ which canonically defines the element of $\mathcal{G}$ (right-invariant vector field) as $(\theta(v))^i_k \frac{\partial}{\partial t^j_k}$. The matrix elements
\( \theta^j_i \) form a bases in \( \Gamma \). When one changes a coordinate system on \( G \) according to eq.(4.1), i.e. one goes to some other representation \( \tilde{T} \), one also changes the matrix elements \( \theta^j_i \). To obtain \( \tilde{\theta} \) - Maurer-Cartan form in the new basis one can use the following formal derivation as a hint:

\[
\tilde{\theta} = d \ln \tilde{T} = d \ln T + d \ln(\det T)^s = d \ln T + s \text{tr} \ln T = d \ln T + s \text{tr} d \ln T.
\]

That gives

\[
\tilde{\theta} = d \tilde{T} \tilde{T}^{-1} = \theta + s \text{tr} \theta I. \tag{4.2}
\]

We refer to eq.(4.3) as to the consistency criteria between a bases in \( \text{Fun}(G) \) and in \( \Gamma \).

Coming back to the Hopf algebra \( \mathcal{M} \) we see that it is natural to identify the subalgebra \( \mathcal{N} \) with \( \text{Fun}(G) \) and the bimodule \( \Gamma \in \mathcal{M} \) generated by \( \theta^j_i \) with the bimodule of the first order differential forms on \( G \). In this case eqs.(4.1) and (4.2) literally coincide with the formulas for the change of variables in \( \mathcal{M} \) with the only difference: two parameters \((k, s)\) in the algebra are replaced by the single parameter \( s \) in the group. The discussion above eliminates the geometric roots of transformations \((3.19)\) in the Hopf superalgebra \( \mathcal{M} \). Now we recognize that the first line in \((3.19)\) reflects the existence of non-equivalent fundamental representations of \( G \). The insensitivity of the Hopf algebra \( \mathcal{M} \) to the shift given by the last line of \((3.19)\) can be interpreted as a manifestation of reducibility of the adjoint representation of \( GL \).

Each orbit of the adjoint representation may be identified with the set of matrices with fixed trace, while the value of this trace labels these orbits. However, there is no link in \( \mathcal{M} \) between the generators \( t^j_i \) and \( \theta^j_i \). In other words, the generators \( \theta \) can belong to a representation of \( G \) which does not correspond (see eq.(4.3) to \( T(g) \)). That is why the changes of variables in \( \mathcal{M} \) are labeled by two parameters \( k \) and \( s \).

It seems that without fixing this redundant freedom by some extra constraint we can not give a geometric interpretation of our pure algebraic bracket.

We think that the constraint we choose in the sequel can be justified in the following way. Let \( \mathcal{M} \) be a graded Hopf algebra equipped with the Poisson brackets given by eqs.(3.13) and (3.16) with some fixed values of \( \alpha, \beta \). Let us define the operator \( d^\pm_{\alpha, \beta} \equiv \{ \frac{1}{\alpha N + m} \text{tr} \theta, \}^\pm_{\alpha, \beta} \) acting on generators \( t^j_i, \theta^j_i \) as

\[
d^\pm_{\alpha, \beta} T = \left\{ \frac{1}{\alpha N + m} \text{tr} \theta, T \right\}^\pm_{\alpha, \beta}, \quad d^\pm_{\alpha, \beta} \theta = \left\{ \frac{1}{\alpha N + m} \text{tr} \theta, \theta \right\}^\pm_{\alpha}.
\]

The operator \( d^\pm_{\alpha, \beta} \) satisfies the Leibniz rule because the brackets \( \{, \}^\pm_{\alpha, \beta} \) and \( \{, \}^\pm_{\alpha} \) do. The property \( d^2_{\alpha, \beta} = 0 \) is due to the Jacobi identity and \( \{ \text{tr} \theta, \text{tr} \theta \}^\pm_{\alpha} = 0 \). Hence we find \( d^\pm_{\alpha, \beta} \) to be good candidates for the operator of exterior derivative on \( \mathcal{M} \).
However, there are well known conditions for \( d_{\alpha,\beta}^\pm \) to be a real \( d \). Namely, \( d_{\alpha,\beta}^\pm \) must act properly on coordinate functions \( t_i \) and satisfy the Maurer-Cartan equation. We have:

\[
d_{\alpha,\beta}^\pm T = \left( \theta - \frac{\alpha - \beta}{\alpha N + m} \text{tr} \theta \right) T \quad (4.5)
\]
or in more familiar form

\[
d_{\alpha,\beta}^\pm TT^{-1} = \theta - \frac{\alpha - \beta}{\alpha N + m} \text{tr} \theta \quad (4.6)
\]

So, we realize that if and only if \( \alpha = \beta \) one has the proper action of \( d_{\alpha,\beta}^\pm \) on \( t \)-s, while the Maurer-Cartan equation is satisfied automatically as

\[
d_{\alpha,\beta}^\pm \theta = \theta \theta.
\]

In other words, the Poisson-Hopf superalgebras with \( \alpha = \beta \) can be equipped with the operation of exterior derivation \( d = d_{\alpha,\alpha}^\pm \). We postulate that only these algebras are admissible. The only transformations of coordinates in admissible algebras are those that keep \( \alpha' = \beta' \), i.e. transformations with \( k = s \). Thus, we can state the one to one correspondence between admissible Poisson-Hopf superalgebras and the external algebra on the Poisson-Lie group \( G \). As for consistency criteria eq.\((4.3)\) for general \( \alpha \) and \( \beta \) it reads

\[
d_{\alpha,\beta}^\pm (\ln \det T) = \frac{\beta N + m}{\alpha N + m} \text{tr} \theta
\]

and is obviously satisfied if \( \alpha = \beta \). Now one can say that \( \theta \)-s in admissible algebras \( (\alpha = \beta) \) take their value in the representation of the Lie algebra \( G \) that corresponds to the given representation \( T(g) \).

To summarize, we prove that there exist only two different Poisson- Lie structures on the external algebra on the Poisson-Lie group \( G \). In the proper coordinate system the brackets take the canonical form given by eqs.\((3.11)\) and \((3.23)\). The appearance of two Poisson structures is a direct consequence of the fact that the Sklyanin bracket can be defined both by the \( r_+ \) and \( r_- \) matrices. The graded extensions of the Sklyanin bracket to the whole algebra \( \mathcal{M} \) are more sensitive and do depend on the choice of \( r \)-matrix.

5 Connection with the bicovariant differential calculus on \( G \)

The aim of this section is to establish a connection of our classical construction with the bicovariant differential calculus on \( G \) proposed in [9] - [11]. Namely, we will show that the classical limit of this calculus reproduce the Poisson-Lie structure on the external algebra described above.

We start with a review of basic definitions. Let \( R \) be a quantum \( R \)-matrix, i.e. an invertible \( N^2 \times N^2 \)-matrix solution of the QYBE depending on a parameter \( q \) (\( q = \)}
exp h). The associated bialgebra $\mathcal{N}_h$ is a noncommutative algebra generated by $1$ and $N^2$ generators $t^j_i$ modulo the relations:

$$RT_1T_2 = T_2T_1R.$$  

(5.1)

The action of the coproduct $\Delta$ and the counit $\varepsilon$ on the generators is

$$\Delta(T) = T \otimes T, \quad \varepsilon(T) = I.$$  

(5.2)

The algebra of regular functions $Fun(G_q)$ on a quantum group $G_q$ is obtained by the choice of the corresponding $R$-matrix and further factorizing $\mathcal{N}_h$ by some additional relations [20].

Let us consider the bicovariant differential calculus on the quantum group $G_q$ in the matrix form [12]-[14] (from now on $G_q$ denotes the quantum linear group $GL_q(N)$). It can be defined as the free associative algebra $M_h$ generated by the symbols $T$ and $dT$ modulo the quadratic relations:

$$R^\pm T_1T_2 = T_2T_1R^\pm,$$  

(5.3)

$$R^\pm (dT)_1T_2 = T_2(dT)_1R^\mp,$$  

(5.4)

$$R^\pm (dT)_1(dT)_2 = -(dT)_2(dT)_1R^\mp.$$  

(5.5)

Here we use the notation $R^+ = R, R^- = \sigma(R^{-1})$ where $\sigma$ is a permutation map. We also suppose that $R$-matrix satisfies the Hecke relation:

$$R^+ = R^{-1} + \lambda P_{12}, \quad \lambda = q - 1/q.$$  

(5.6)

Note, that the sign in eq.(5.3) is irrelevant while two possible signs in (5.4) and (5.5) reflect the fact that there are two different bicovariant differential calculi on $QG$ (following [7] they will be referred as “+” and “−” calculi respectively). As we will see this fact strictly corresponds to the existence of two Poisson-Lie structures on the external algebra of $G$.

In terms of quantum right-invariant forms $\theta = dTT^{-1}$ the defining relations (5.4) and (5.5) take the form:

$$T_2\theta_1 = R^\pm \theta_1 R^{-1^\pm}T_2$$  

(5.7)

$$R^\pm \theta_1 R^{-1^\mp} = -\theta_2 R^\pm \theta_1 R^{-1^\pm}.$$  

(5.8)

The algebra defined by eqs.(5.3) – (5.5) is a Hopf algebra. In terms of $\theta$-s $\Delta$ and $\varepsilon$ have the form:

$$\Delta(\theta) = \theta \otimes I + (T \otimes \theta)(S(T) \otimes I), \quad \varepsilon(\theta) = 0.$$  

(5.9)

Let $\mathcal{M}$ be a commutative Poisson-Hopf algebra. A noncommutative Hopf algebra $\mathcal{M}_h$ is defined to be a quantization of $\mathcal{M}$ if: 1) $\mathcal{M}_h$ is a free module over the ring $C[[h]]$, where $h$ is a parameter of quantization, 2) as a Hopf algebra $\mathcal{M}_h/h\mathcal{M}_h$ is isomorphic to $\mathcal{M}$ and 3) one can define on $\mathcal{M}$ the Poisson bracket:

$$\{a, b\} = \lim_{h \to 0} \left( \frac{1}{h} \left[ a, b \right] \right)$$

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which should coincide with the original bracket on $\mathcal{M}$ [13]. Let us suppose that $R_\pm$ are quasi-classical, i.e.:

$$R_\pm = 1 + hr_\pm + o(h).$$

Then the QYBE for $R_\pm$ implies that $r_\pm$ satisfy the CYBE: $C(r_\pm) = 0$ and eq.(5.6) gives eq.(3.12) for $r_+$. The quasi-classical expansion of the multiplication law in $\mathcal{M}_h$ has the following form

$$T_2\theta_1 = \theta_1T_2 + h(r_{12}^\pm \theta_1 T_2 - \theta_1 r_{12}^\pm T_2) + O(h^2),$$

(5.10)

$$\theta_1\theta_2 = -\theta_2\theta_1 - h(r_{12}^\pm \theta_1 \theta_2 + \theta_1 r_{12}^\pm \theta_2 - \theta_1 r_{12}^\pm \theta_2 + \theta_2 r_{12}^\pm \theta_1) + O(h^2).$$

(5.11)

Now let us define on the algebra $\mathcal{M}$ being a "classical" limit ($h \to 0$) of $\mathcal{M}_h$ the Poisson brackets:

$$\{\theta_1, T_2\} = -\lim_{h \to 0} \frac{1}{h} [\theta_1, T_2]^\pm$$

and $$\{\theta_1, \theta_2\} = -\lim_{h \to 0} \frac{1}{h} [\theta_1, \theta_2]^\pm.$$  

(5.12)

Here we use the square brackets $[,]$ for the graded commutator. Using eqs.(5.10), (5.11) we see at once that the r.h.s. of eqs.(5.12) coincides with the canonical brackets given by eqs.(3.23). Thus, if $\mathcal{M}$ is identified with the external algebra supplied with the Poisson-Lie structure then the quantization of this structure gives the quantum algebra $\mathcal{M}_h$ describing the bicovariant differential calculus on the corresponding quantum group.

It can be seen that some of the constructions [8], [21] proposed for quantum algebras have the "classical" origin. We leave the detailed comparison of "classical" and "quantum" results to the reader and note only the connection of the "classical" and the "quantum" operator of exterior derivative.

The general discussion of quantum differential calculus was given in [22]. It was shown that in a noncommutative graded algebra the operator $d$ of exterior derivative can be defined as: $da = [\xi, a]$, where $\xi$ is an element of degree one satisfying $[\xi, \xi] = 0$. For the case of $SU_q(N)$ this idea was used in [7] and the explicit formula for $\xi$ via the quantum trace was proposed in [8]. The same operator $\xi$ occurred in the construction of the bicovariant differential calculus on $G_q$ [21]. Now we are going to consider the quasi-classical limit of this differential.

Following [8], [21] let us define in $\mathcal{M}_h$ the "right-left invariant" element:

$$\xi = \frac{1}{q^{2N-1}} \text{tr} (D\theta),$$

where $D$ is the numerical matrix $D = \text{diag}(1, q^2, \ldots, q^{2(N-1)})$ [22]. An element $\text{tr} (D\theta)$ is a quantum analog of the left-right invariant form $\text{tr} \theta$ on $G$ as well as the quantum determinant $\text{det}_q T$ is an analog of $\text{det} T$. Let us define for the " + " calculus

$$df = -\frac{1}{\lambda} [\xi, f],$$

(5.13)

and for the " - " one

$$df = \frac{q^{2N}}{\lambda} [\xi, f],$$

(5.14)
where \( f \in \mathcal{M}_h \). From eqs. (5.3), (5.7) and (5.8) for the “+” calculus one has (see [21] for details)

\[
dT = \theta T, \quad d\theta = \theta^2
\]

and the same for the “-” one. It is trivial to find that in both cases the “classical” limit \((h \to 0)\) reproduces the action of the exterior derivative \(d\) which we have in our differential algebra \(\mathcal{M}\). For example,

\[
dT = -\left[\frac{1}{\lambda} \xi, T \right] \approx \left\{ \frac{1}{m} \text{tr} \theta, T \right\}^\pm,
\]

where \(m = +2\). Thus, the possibility of quantizing the differential algebra \(\mathcal{M}\) lies in the existence on \(\mathcal{M}\) the Poisson-Lie structure in which the operator \(d\) of exterior derivative can be expressed as in Section 4.

What are the lessons we learn from the above discussion? At first, we see that the representation of \(d\) via a commutator is not a privilege of the non-commutative geometry. This representation is an attribute of a graded Lie algebra without any reference to an underlying geometry. At second, we see that the quantum operator \(d\) is really a quantization of the classical external derivative. At third, in general one can find in \(\mathcal{M}_h\) some other elements \(\xi\) to define the \(d\) operator, but their limit when \(h \to 0\) should coincide with the element \(\text{tr} \theta\) in \(\mathcal{M}\). This follows from the fact that for the Poisson-Lie structure on \(\mathcal{M}\) described above there is only one element, namely \(\text{tr} \theta\), which represents the external derivative. Clearly, suppose that there exists one more element, say \(\nu\) that also generates \(d\). Then, for any function \(f \in \mathcal{M}\) one has:

\[
\left\{ \frac{1}{m} \text{tr} \theta - \nu, f \right\}^\pm = 0.
\]

while the difference \(\omega = \text{tr} \theta /m - \nu\) is a one form. Taking \(\omega = c_i^j(T)\theta_{ji}\) and choosing \(f = \det T\) one arrives at:

\[
0 = \left\{ c_i^j\theta_{ji}^\pm, \det T \right\}^\pm = \left\{ c_i^j, \det T \right\}^\pm \theta_{ji}^\pm + c_i^j\left\{ \theta_{ji}, \det T \right\}^\pm = mc_i^j\theta_{ji}^\pm \det T
\]

for \(\det T\) is a central element of the Sklyanin bracket. Thus, we see that \(\omega = 0\) and \(\nu = \text{tr} \theta /m\).

One additional comment is necessary. Comparing eq. (5.9) with (3.3) one can see that the coproduct law in \(\mathcal{M}\) is not deformed under quantization. Hence, it is natural to search for the quantum counterpart of the covariance (3.19). Let us show that the transformations

\[
\tilde{T} = T(\det_q T)^*, \quad \tilde{\theta} = \theta + k \text{tr} (D\theta) \cdot I
\]

do not affect the form of the coproduct (5.2) and (5.9). Using the definition of \(D\)-matrix and eq. (5.3) one can find the identities [21]

\[
(D^{-1})^t D^t S(T)^t = S(T)^t (D^{-1})^t D^t D^t = 1.
\]

Applying \(\Delta\) to \(\tilde{\theta}\) one has

\[
\Delta(\tilde{\theta}_{ji}^\pm) = (\theta_{ji} + k \text{tr} (D\theta) \delta_{ji}) \otimes I + (t_i^r S(t_{ji}^r) + k D_s^m t_m^r S(t_{p}^r) \delta_{ji}^p) \otimes \theta_{ij}^p =
\]
\[ \tilde{\theta}_i^j \otimes I + (t_i^r S(t_p^j) + k(T^t D^t S(T)^t_r \delta_i^j) \otimes \theta_r^p \]

and by virtue of eq.(5.16) and the centrality of \( \det_q T \) with respect to \( T \) one obtains

\[ \Delta(\tilde{\theta}_i^j) = \tilde{\theta}_i^j \otimes I + \tilde{t}_i^r S(\tilde{t}_p^j) \otimes \tilde{\theta}_r^p. \]

Note that the form of the defining relations (5.7),(5.8) do depend on the choice of generators \( T \) and \( \theta \) just as the form of the brackets (3.13) and (3.16) depends on a coordinate system.

6 Conclusion

In this note we have endeavored to develop the idea that the bicovariant differential calculus on a quantum group can be considered as the result of quantization of some underlying graded Poisson-Lie structure just in the same sense as a quantum group itself is a quantization of a Poisson-Lie group. The graded Poisson-Lie structures in question are defined on the external algebra of the Poisson-Lie group \( G \) and specifying the properties of \( r \)-matrix we obtain their complete description. It turns out that there exist only two different graded Poisson-Lie structures the quantization of which gives \( \pm \) bicovariant differential calculi on the quantum linear group \( GL_q(N) \).

Now we will briefly discuss the possible applications of the proposed construction. Recall that describing the Poisson-Lie structure on \( G \) we specify \( r \)-matrix as the trivial lift of the canonical \( r \)-matrix for \( SL(N) \). Here the natural question arises if it is possible to present the bicovariant differential calculi on \( SL_q(N) \) as the quantization of an appropriate classical counterpart. To answer this question one has to describe the Poisson-Lie structure on the external algebra of \( SL(N) \) and we conjecture that our brackets will be a suitable tool for doing this. Another interesting question is whether it is possible to build a Poisson-Lie structure on the external algebra of \( G \) induced by the canonical \( r \)-matrices for other classical groups. Seemingly, these brackets would be of importance for studying Poisson-Lie structures on their external algebras. This will be the subject of subsequent publications.

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APPENDIX

A

The requirement of grading for the coordinate tensor \( \{ \theta^i_k, t^j_i \} \) suggests that it has the following decomposition over the basis \( \theta^i_j \):

\[
\{ \theta^j_k, t^i_k \} = C_{0}^{' ij} s^i m^j s^m + C_{1}^{' ij} j_{i} j_{j} j_{k} \theta^i_j \theta^j_k \theta^k_i + \ldots .
\]  \tag{A.1}

Due to the anticommutativity of \( \theta \) the series has a finite numbers of terms and the structure tensors \( C_0, C_1, \ldots \) considered as unknown functions of even variables \( t^i_j \) are antisymmetric with respect to the simultaneous permutation \( i_k \leftrightarrow i_m, j_k \leftrightarrow j_m \) for any \( k, m \). Applying the coproduct to both sides of eq.\( (3.9) \) we derive from eq.\( (A.1) \) the set of conditions for tensors \( C \). Firstly, all \( C_i \) when \( i \neq 0 \) should be equal to zero. Secondly, \( C_{0}^{' ij} m^j (t) \) should be linear in \( t^i_j \), i.e. of the form \( C_{ik}^{' j} m^j (t) = C_{ik sp}^{' jr} m^j r^p (C_0 \equiv C) \)

\[
C_{ik sp}^{' jr} m^j r^p (t^i_j) t^j_k \otimes \theta_{m^t_s}^b = \left( \begin{array}{l}
\end{array} \right)
\tag{A.2}
\]

where the tensor \( D \) is defined to be

\[
D_{ir k}^{' j} n = \left( \begin{array}{l}
\end{array} \right) = (r_+)_{ik} m^t s^t t^i_j n S(t^j_k) - t^i_j k (r_+)_{ms} m^s S(t^j_k) - (A.3)
\]

Equating of terms in \( (A.2) \) containing one generator \( t^i_j \) in the right multiple of tensor product shows that

\[
C_{ik sp}^{' j} m r^p \sim C_{ik s}^{' j} m r^p \delta^l_p .
\]

Taking into account this expression for \( C \) one can equate now the terms in \( (A.2) \) containing \( t^i_j \theta \) in the right multiple of tensor product. The result is

\[
t^i_j k (C_{ip}^{' j} s^i m^l S(t^j_k) = (r_+)_{ip} m^l S(t^j_k) = (A.4)
\]

Such form of \( (A.4) \) implies that one can define a tensor \( \Phi \):

\[
\Phi_{m^l s^i}^r = C_{ip}^{' j} s^i m^l S(t^j_k) = (A.5)
\]

for which eq.\( (A.4) \) reads

\[
t^i_j k (\Phi_{m^l s^i}^r S(t^j_k) = S(t^j_k) i_{t^i_j k} t^j_k (A.6)
\]

The solutions of eq.\( (A.6) \) can be easily enumerated. In fact, due to the property of the antipode \( t^i_k S(t^j_k) = S(t^j_k) t^j_k = \delta^i_j \), they correspond to different possibilities of
contracting indexes on each side of (A.6). Thus, the general solution for \( \Phi \) has the form:

\[
\Phi_{pm s}^{rl} = \sum_{\text{perm}} \alpha_{(pm)} \delta_p^r \delta_m^l \delta_s^l
\]  

(A.7)

where \( \alpha_{(pm)} \in C \) is an arbitrary coefficient corresponding to a given permutation of indexes \( r, m, l \) and the sum is over all permutations of these indexes. So, for the bracket \( \{ \theta_i^j, t_k^l \} \) consistent with the coproduct we have

\[
\{ \theta_i^j, t_k^l \} = ((r_+)^s_p + \alpha_4 P^s_p) \theta_j^l - \theta_i^m ((r_+)^j_p + \alpha_6 P^j_p) t_p^l + \alpha_1 \delta_i^j \theta_k^l - \theta_i^m (\alpha_2 t_k^l + \alpha_3 t_k^l + \alpha_5 t_k^l)
\]  

(A.8)

or in tensor notations:

\[
\{\theta_1, T_2\} = r_A^{12} \theta_1 T_2 - \theta_1 r_B^{12} T_2 + \alpha_1 \theta_2 T_2 + \alpha_2 \text{tr} \theta T_2 + \alpha_3 \text{tr} \theta P^{12} T_2 + \alpha_5 \theta_1 T_2. \quad (A.9)
\]

Here \( r_A = r_+ + \alpha_4 P \) and \( r_B = r_+ + \alpha_6 P \).

The further fixing of coefficients in (A.3) is accomplished by imposing the Jacobi identity (3.3) for \( \theta \) and two \( T \)-s, that reads:

\[
\{\{\theta_1, T_2\}, T_3\} - \{\{\theta_1, T_3\}, T_2\} - \{\theta_1, \{T_2, T_3\}\} = 0. \quad (A.10)
\]

To calculate (A.11) one needs a formula

\[
\{ \text{tr} \theta, T \} = \gamma_1 \theta T + \gamma_2 \text{tr} \theta T,
\]

(A.11)

which is derived from (A.9). Here \( \gamma_1 = \alpha_1 N + \alpha_4 - \alpha_6 \) and \( \gamma_2 = \alpha_2 N + \alpha_3 + \alpha_5 \). By straightforward calculations one can find that

\[
\sum_{\text{perm}} \{\{\theta_1, T_2\}, T_3\} = C(r_A, r_+) \theta_1 T_2 T_3 - \theta_1 C(r_B, r_+) T_2 T_3 + K_1 + \text{tr} \theta K_2, \quad (A.12)
\]

where two factors:

\[
K_1 \equiv \alpha_1 (r_A^{12} - r_B^{12}) \theta_3 T_3 T_2 + (\alpha_1^2 + \gamma_1 \alpha_2) \theta_3 T_3 T_2 - \alpha_1 (P^{23} r_A^{23} P^{23} + r_B^{23}) \theta_3 T_3 T_2 + \quad (A.13)
\]

\[
\alpha_1 \theta_3 (P^{23} r_B^{23} P^{23} + r_B^{23}) T_2 T_3 + \gamma_1 \alpha_3 P^{12} \theta_3 T_3 T_2 + \alpha_1 (r_B^{13} - r_A^{13}) \theta_2 T_2 T_3 -
\]

\[
(\alpha_1^2 + \gamma_1 \alpha_2) \theta_2 T_2 T_3 + \alpha_1 (r_A^{23} - r_B^{23}) \theta_2 T_2 T_3 + \alpha_1 \theta_2 (r_B^{23} - r_B^{23}) T_2 T_3 - \gamma_1 \alpha_3 P^{13} \theta_2 T_2 T_3,
\]

\[
K_2 \equiv \alpha_2 (r_B^{13} - r_A^{13}) + \alpha_3 \alpha_5 P^{13} + \alpha_3 P^{12} r_B^{13} - \alpha_3 r_A^{13} P^{12} - \alpha_3 \gamma_2 P^{13} + \quad (A.14)
\]

\[
+ \alpha_3 P^{13} r_B^{23} - \alpha_3 r_B^{23} P^{13} + \alpha_2 (r_A^{12} - r_B^{12}) + \alpha_3 r_A^{13} P^{13} - \alpha_3 P^{13} r_B^{12} + \alpha_3 \gamma_2 P^{12} +
\]

\[
\alpha_3 P^{12} r_B^{23} - \alpha_3 r_B^{23} P^{12} - \alpha_3 \alpha_5 P^{12}
\]

and tensors \( C(r_A, r_+) \) (\( C(r_B, r_+) \)):

\[
C(r_A, r_+) = [r_A^{12}, r_A^{13}] + [r_A^{12}, r_B^{23}] + [r_A^{13}, r_B^{23}], \quad (A.15)
\]

were introduced.
A simple analysis shows that all terms in (A.12) are linear independent. Therefore to obey the Jacobi identity one has to choose the coefficients $\alpha_i$ ($i = 1, \ldots, 6$) in such a way as to obtain

$$C(r_A, r_+) = C(r_B, r_+) = K_1 = K_2 = 0.$$  

Note that $C(r_+) \equiv C(r_-, r_+) = 0$ is the CYBE.

We start with the first two equations:

$$C(r_A, r_+ + r_B) = C(r_+, r_+ + r_B) = K_1 = K_2 = 0.$$  

The two obvious solutions are $\alpha_4 = 0, -2$ or, in other words, $r_A = r_\pm$, where $r_- = r_+ - 2P$ and the same pair for $\alpha_6$.

The requirement $K_1 = K_2 = 0$ gives the system of equations for the remaining coefficients $\alpha$:

$$m\alpha_1 + \gamma_1\alpha_3 = 0,$$

$$\alpha_1^2 + \gamma_1\alpha_2 = 0,$$

$$\alpha_3(\alpha_4 + \alpha_6 + 2) = 0,$$

$$m\alpha_2 - \alpha_2\alpha_5 + \gamma_2\alpha_3 = 0,$$

where we put for simplicity $\alpha_4 - \alpha_6 = m$. In fact the last line in (A.17) directly follows from the first and the second ones. The third and the fourth lines show that except for the point $\alpha_1 = \alpha_3 = 0$ one can not choose $r_A = r_B$ and that we have only two admissible combinations: $(r_A = r_+, r_B = r_-)$ or $(r_A = r_-, r_B = r_+)$. The general solution of the system (A.17) reads:

$$\alpha_2 = -\frac{\alpha_4^2}{m + \alpha N}, \quad \alpha_3 = -\frac{\alpha m}{m + \alpha N}, \quad m = \pm 2.$$  

$$\alpha = \alpha_3 = 0, \quad \alpha_4 = \alpha_6 = 0, -2, \quad m = 0$$

where $\alpha = \alpha_1$ remains an arbitrary parameter in eq. (A.18). Thus, the brackets are arranged into two families. In the first one the brackets are parametrized by two continuous parameters $\alpha$ and $\beta = \alpha_5$ and by the sign $\epsilon$ of $m$:

$$\{\theta_1, T_2\}_{\alpha, \beta}^\pm = \alpha\theta_2 T_2 + r_+^{12}\theta_1 T_2 - \theta_1 r_+^{12} T_2 + \alpha_2 \text{tr } \theta T_2 + \alpha_3 \text{tr } \theta P^{12} T_2 + \beta\theta_1 T_2,$$  

where $\alpha_2$ and $\alpha_3$ are expressed via $\alpha$ as in (A.18). In the second family we have

$$\{\theta_1, T_2\}_{\alpha, \beta}^\pm = r_+^{12}\theta_1 T_2 - \theta_1 r_+^{12} T_2 + \alpha_2 \text{tr } \theta T_2 + \beta\theta_1 T_2,$$  

where now $\alpha_2, \beta$ are arbitrary.
Following the same steps as in Appendix A let us define the bracket \( \{ \theta^j, \theta^l \} \). From the same arguments as above we have to take the brackets in the form:

\[
\{ \theta^j, \theta^l \} = W^j_{ik} \frac{mn}{sp}(t) \theta^m_s \theta^p_n,
\]

where \( W^j_{ik} \frac{mn}{sp}(t) \) is the unknown structure tensor as the function of even variables \( t^j_i \). Applying \( \Delta \) to both sides of (B.1) and using eq. (3.9):

\[
\{ \Delta \theta^j, \Delta \theta^l \} = \Delta W^j_{ik} \frac{mn}{sp}(t) \Delta \theta^m_s \Delta \theta^p_n,
\]

we find that \( W \) should be independent of \( t \)-s. Let us stress, that calculating the l.h.s. of eq. (B.2) we use \( \{ \theta, T \} \) brackets in the general form (A.9). Equating the terms containing a one \( \theta \) generator in the right multiple of tensor product we obtain the condition on the antisymmetric under the permutation \( s \leftrightarrow p, m \leftrightarrow n \) part of \( W \):

\[
W^j_{ik} \frac{ps}{sp} - W^j_{ik} \frac{mn}{sp} = (r_A)_{ki} \frac{mn}{ps} \delta^j_p \delta^s_i + (r_A)_{ki} \frac{mn}{sp} \delta^j_i \delta^s_p - \delta^m_k (r_B)_{ps} \frac{ln}{pi} \delta^s_j - \delta^m_k (r_B)_{ps} \frac{ln}{pi} \delta^s_j
\]

(8)

\[
\delta^m_k (r_A)_{ks} \frac{mj}{p} \delta^l_t + \delta^m_k (r_A)_{ks} \frac{nj}{p} \delta^l_t - \delta^m_k (r_B)_{ps} \frac{lm}{pi} \delta^s_j - \delta^m_k (r_B)_{ps} \frac{ln}{pi} \delta^s_j
\]

(9)

\[
\delta^m_k \delta^i_l (r_B)_{ps} \frac{jl}{p} + \alpha_1 \delta^m_k \delta^i_l (r_B)_{ps} \frac{jm}{ps} \delta^s_j - \delta^m_k \delta^j_i (r_B)_{ps} \frac{lm}{pi} \delta^s_l - \delta^m_k \delta^j_i (r_B)_{ps} \frac{ln}{pi} \delta^s_l
\]

(10)

Let us consider now the terms in (B.2) containing two \( \theta \)-generators in the right multiples of tensor product. They produce the equation for the tensor \( \hat{\Phi} \):

\[
t^j_{ik} \frac{nm}{ps} \hat{\Phi}^r \frac{ac}{bd} S(t^j_r) S(t^l_r) = S(t^r_b) S(t^p_b) \hat{\Phi}^j_{ik} \frac{mn}{sp} \hat{t}^l_{nm} \hat{a}^c
\]

(11)

where \( \hat{\Phi} \) is defined by

\[
\hat{\Phi}^r \frac{ac}{bd} = W^r \frac{ac}{bd} - (r+)_{mn} \frac{bd}{d} \delta^r_b + \delta^c_n (r+)_{mn} \frac{ac}{d} \delta^r_b + (r+)_{mn} \frac{pc}{d} \delta^r_d \delta^c_n
\]

(12)

Equation (B.11) has the general solution:

\[
\hat{\Phi}^j_{ik} \frac{mn}{sp} = \sum_{perm} \mu_{ijklmn} \delta^l_k \delta^m_s \delta^p_n
\]

(13)

where the sum is extended over all permutations of indexes \( j, l, m, n \) and \( \mu_{ijklmn} \) is an arbitrary coefficient corresponding to the given permutation of the indexes. Now substituting (B.13) in (B.12) we obtain the \( W \)-tensor. After taking into account the symmetry properties of the brackets one can go further and require this general expression for the \( W \)-tensor to be consistent with (B.12). At this step it appears to be relevant to distinguish between the \( \{ \theta, t \} \) brackets from the first and from the second families. We find that for the second family \( (r_A = r_B, \alpha_1 = 0) \) there is no solution for eq. (B.12), i.e. the \( \{ \theta, t \} \) brackets given by eq. (A.20) can not be prolonged upto the \( \{ \theta, \theta \} \) brackets consistent with the coproduct \( \Delta \). For the first family we can write down the general expression for the bracket \( \{ \theta, \theta \} \). In tensor notations we can present this bracket as:

\[
\{ \theta_1, \theta_2 \}^\alpha_\alpha = \alpha (\theta_1 \theta_1 + \theta_2 \theta_2) + \theta_1 \theta_2 \theta_1 + \theta_1 \theta_2 \theta_2 - \theta_1 \left( r^2 \frac{+}{m+2} P^2 \right) + \theta_2 \left( r^2 \frac{+}{m-2} P^2 \right) \theta_1
\]

(14)
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