Maximum logarithmic derivative bound on quantum state estimation as a dual of the Holevo bound

Koichi Yamagata*
The University of Electro-Communications Department of Informatics,
1-5-1, Chofugaoka, Chofu, Tokyo 182-8585, Japan

Abstract
In quantum estimation theory, the Holevo bound is known as a lower bound of weighed traces of covariances of unbiased estimators. The Holevo bound is defined by a solution of a minimization problem, and in general, explicit solution is not known. When the dimension of Hilbert space is two and the number of parameters is two, a explicit form of the Holevo bound was given by Suzuki. In this paper, we focus on a logarithmic derivative lies between the symmetric logarithmic derivative (SLD) and the right logarithmic derivative (RLD) parameterized by \( \beta \in [0, 1] \) to obtain lower bounds of weighted trace of covariance of unbiased estimator. We introduce the maximum logarithmic derivative bound as the maximum of bounds with respect to \( \beta \). We show that all monotone metrics induce lower bounds, and the maximum logarithmic derivative bound is the largest bound among them. We show that the maximum logarithmic derivative bound has explicit solution when the \( d \) dimensional model has \( d+1 \) dimensional \( D \) invariant extension of the SLD tangent space. Furthermore, when \( d = 2 \), we show that the maximization problem to define the maximum logarithmic derivative bound is the Lagrangian duality of the minimization problem to define Holevo bound, and is the same as the Holevo bound. This explicit solution is a generalization of the solution for a two dimensional Hilbert space given by Suzuki. We give also examples of families of quantum states to which our theory can be applied not only for two dimensional Hilbert spaces.

1 Introduction
Let \( \mathcal{X} = \{ \rho_\theta; \theta \in \Theta \subset \mathbb{R}^d \} \) be a smooth parametric family of density operators on a Hilbert space \( \mathcal{H} \). An estimator is represented by a pair \( (M, \hat{\theta}) \) of a POVM \( M \) taking values on any finite set \( \mathcal{X} \) and a map \( \hat{\theta} : \mathcal{X} \to \Theta \). An estimator \( (M, \hat{\theta}) \) is called unbiased if

\[
E_\theta[M, \hat{\theta}] = \sum_{x \in \mathcal{X}} \hat{\theta}(x)\text{Tr} \rho_\theta M_x = \theta
\]

is satisfied for all \( \theta \in \Theta \). An estimator \( (M, \hat{\theta}) \) is called locally unbiased\(^{[1]}\) at a given point \( \theta_0 \in \Theta \) if the condition \((1.1)\) is satisfied around \( \theta_0 \) up to the first order of the Taylor expansion, i.e.,

\[
\sum_{x \in \mathcal{X}} \hat{\theta}^i(x)\text{Tr} \rho_{\theta_0} M_x = \theta_0^i \quad (i = 1, \ldots, d),
\]

\[
\sum_{x \in \mathcal{X}} \hat{\theta}^i(x)\text{Tr} \frac{\partial}{\partial \theta^j} \rho_{\theta_0} M_x = \delta^i_j \quad (i, j = 1, \ldots, d),
\]

where \( \frac{\partial}{\partial \theta^0} \rho_{\theta_0} \bigg|_{\theta = \theta_0} \). It is well-known that the covariance matrix \( V_{\theta_0}[M, \hat{\theta}] \) of an locally unbiased estimator \( (M, \hat{\theta}) \) at \( \theta_0 \) satisfies the following inequalities:

\[
V_{\theta_0}[M, \hat{\theta}] \geq J_{\theta_0}^{(S^{-1})},
\]

*koichi.yamagata@uec.ac.jp
where $J_{\theta_0}^{(S)} := \left[\text{Re}(\text{Tr}\, \rho_{\theta_0} L_i^{(S)} L_j^{(S)})\right]_{ij}$ is the symmetric logarithmic derivative (SLD) Fisher information matrix at $\theta_0$ with SLDs $L_i^{(S)}$ ($1 \leq i \leq d$) defined by
\[
\partial_i \rho_{\theta_0} = \frac{1}{2} \left( \rho_{\theta_0} L_i^{(S)} + L_i^{(S)} \rho_{\theta_0} \right),
\]
and $J_{\theta_0}^{(R)} := \left[\text{Tr}\, L_i^{(R)} \rho_{\theta_0} L_j^{(R)}\right]_{ij}$ is the right logarithmic derivative (RLD) Fisher information matrix at $\theta_0$ with RLDs $L_i^{(R)}$ ($1 \leq i \leq d$) defined by
\[
\partial_i \rho_{\theta_0} = \rho_{\theta_0} L_i^{(R)}.
\]
These matrix inequalities imply
\[
\text{Tr} \, GV_{\theta_0}[M, \hat{\theta}] \geq \text{Tr} \, GJ_{\theta_0}^{(S)} =: C_{\theta_0, G}^{(S)} \quad \text{(1.8)}
\]
\[
\text{Tr} \, GV_{\theta_0}[M, \hat{\theta}] \geq \text{Tr} \, GJ_{\theta_0}^{(R)} + \text{Tr} \, \sqrt{G} \text{Im} J_{\theta_0}^{(R)} \sqrt{G} =: C_{\theta_0, G}^{(R)} \quad \text{(1.9)}
\]
for any $d \times d$ real positive matrix $G$, because
\[
\min_V \{\text{Tr} \, GV; V \geq J, V \text{ is a real matrix}\} = \text{Tr} \, GJ + \text{Tr} \, \sqrt{G} \text{Im} J \sqrt{G} \quad \text{(1.10)}
\]
for any positive complex matrix $J$ (see Appendix A for the proof).

A tighter lower bound of $\text{Tr} \, GV_{\theta_0}[M, \hat{\theta}]$ than the SLD bound $C_{\theta_0, G}^{(S)}$ and the RLD bound $C_{\theta_0, G}^{(R)}$ is known as the Holevo bound [1] defined by
\[
C_{\theta_0, G}^{(H)} := \min_{Z, B} \{\text{Tr} \, GV; V \text{ is a real matrix such that } V \geq Z(B), Z_{ij}(B) = \text{Tr} \, \rho_{\theta_0} B_j B_i,\} \quad \text{(1.11)}
\]
\[
B_1, \ldots, B_d \text{ are Hermitian operators on } \mathcal{H} \text{ such that } \text{Tr} \, \partial_i \rho_{\theta_0} B_j = \delta_{ij},\}
\]
(see Appendix B for the derivation and the proof) and it satisfies
\[
\text{Tr} \, GV_{\theta_0}[M, \hat{\theta}] \geq C_{\theta_0, G}^{(H)} \geq \max(C_{\theta_0, G}^{(S)}, C_{\theta_0, G}^{(R)}).
\]
(1.13)

It is known that the Holevo bound is asymptotically achievable in theory of quantum local asymptotically normality [2, 3, 4]. Note that the minimization problem over $V$ in (1.12) is explicitly solved by using (1.10), and
\[
C_{\theta_0, G}^{(H)} = \min_B \left\{\text{Tr} \, GZ(B) + \text{Tr} \, \sqrt{G} \text{Im} Z(B) \sqrt{G}; Z_{ij}(B) = \text{Tr} \, \rho_{\theta_0} B_j B_i,\right. \quad \text{(1.14)}
\]
\[
B_1, \ldots, B_d \text{ are Hermitian operators on } \mathcal{H} \text{ such that } \text{Tr} \, \partial_i \rho_{\theta_0} B_j = \delta_{ij}.\}
\]
However, the minimization problem over $B$ in (1.14) is not trivial in general. Suzuki [5] showed that, when dim $\mathcal{H} = 2$ and $d = 2$, the Holevo bound can be represented explicitly by using the SLD bound and the RLD bound as
\[
C_{\theta_0, G}^{(H)} = \begin{cases} C_{\theta_0, G}^{(R)} & \text{if } C_{\theta_0, G}^{(R)} \geq \frac{C_{\theta_0, G}^{(S)} + C_{\theta_0, G}^{(R)}}{2}, \\ C_{\theta_0, G}^{(R)} + S_{\theta_0, G} & \text{otherwise}, \end{cases} \quad \text{(1.15)}
\]
where $C_{\theta_0, G}^{(S)}$ and $S_{\theta_0, G}$ are positive values defined by
\[
C_{\theta_0, G}^{(S)} := \text{Tr} \, GZ(L^{(S)}) + \text{Tr} \, \sqrt{G} \text{Im} Z(L^{(S)}) \sqrt{G}, \quad \text{(1.16)}
\]
and we show that the inequalities

\[ S_{\theta_0, G} := \left[ \frac{1}{2} (c_{\theta_0, G}^{(Z)} + c_{\theta_0, G}^{(S)}) - c_{\theta_0, G}^{(R)} \right]^2. \]  

(1.17)

In this paper, we focus on a logarithmic derivative \( L_i^{(\beta)} \) lies between SLD \( L_i^{(S)} \) and RLD \( L_i^{(R)} \), that defined by

\[ \partial_i \rho_{\theta_0} = \frac{(1 + \beta)}{2} \rho_{\theta_0} L_i^{(\beta)} + \frac{(1 - \beta)}{2} L_i^{(\beta)} \rho_{\theta_0} \]  

(1.18)

with \( \beta \in [0, 1] \). When \( \beta = 0 \), \( L_i^{(\beta)} \) coincides with SLD \( L_i^{(S)} \), and when \( \beta = 1 \), \( L_i^{(\beta)} \) coincides with RLD \( L_i^{(R)} \). The Fisher information matrix with respect to \( \{ L_i^{(\beta)} \}_{i=1}^d \) is

\[ J_{\theta_0}^{(\beta)} := \left[ \text{Tr} \partial_i \rho_{\theta_0} L_j^{(\beta)} \right]_{ij}, \]  

(1.19)

and we show that the inequalities

\[ V_{\theta_0}[M, \hat{\theta}] \geq J_{\theta_0}^{(\beta)^{-1}} \]  

(1.20)

and

\[ \text{Tr} GV_{\theta_0}[M, \hat{\theta}] \geq \text{Tr} G J_{\theta_0}^{(\beta)^{-1}} + \text{Tr} \left[ \sqrt{\text{Im} J_{\theta_0}^{(\beta)^{-1}}} \sqrt{G} \right] =: \mathcal{C}_{\theta_0, G}^{(\beta)}, \]  

(1.21)

for the covariance matrix \( V_{\theta_0}[M, \hat{\theta}] \) of any locally unbiased estimator \( (M, \hat{\theta}) \) at \( \theta_0 \). We call

\[ \max_{0 \leq \beta \leq 1} \mathcal{C}_{\theta_0, G}^{(\beta)} \]  

(1.22)

a maximum logarithmic derivative bound. More generally, monotone metrics introduced by Petz can also induce Fisher information matrices and lower bounds of \( \text{Tr} GV_{\theta_0}[M, \hat{\theta}] \) \([6, 7]\). We show that the maximum logarithmic derivative bound is the largest bound among them.

The maximization problem \((1.22)\) is also not trivial in general. However, when the model \( \mathcal{F} = \{ \rho_\theta; \theta \in \Theta \subset \mathbb{R}^d \} \) has \( d + 1 \) dimensional real space \( \mathcal{F} \supset \text{span} \{ L_i^{(S)} \}_{i=1}^d \) such that \( \mathcal{D}_{\rho_\theta_0}(\mathcal{F}) \subset \mathcal{F} \) at \( \theta_0 \in \Theta \), we show that \( \max_{0 \leq \beta \leq 1} \mathcal{C}_{\theta_0, G}^{(\beta)} \) has explicit solution:

\[ \max_{0 \leq \beta \leq 1} \mathcal{C}_{\theta_0, G}^{(\beta)} = \begin{cases} c_{\theta_0, G}^{(1)} & \text{if } \hat{\beta} \geq 1, \\ c_{\theta_0, G}^{(\beta)} & \text{otherwise}, \end{cases} \]  

(1.23)

with

\[ \hat{\beta} = \begin{cases} \frac{\text{Tr} \left[ \sqrt{\text{Im} J_{\theta_0}^{(S)^{-1}}} \sqrt{J_{\theta_0}^{(S)}} \right]}{2 \text{Tr} G \left( J_{\theta_0}^{(S)^{-1}} - \text{Re}(J_{\theta_0}^{(R)}) \right)} & \text{if } J_{\theta_0}^{(S)^{-1}} \neq \text{Re}(J_{\theta_0}^{(R)}), \\ \infty & \text{otherwise}, \end{cases} \]  

(1.24)

where \( \mathcal{D}_{\rho_\theta_0} \) is the commutation operator (see Section \[3\]). Furthermore, when \( d = 2 \), we show that the maximization problem \((1.22)\) is the Lagrangian duality of the minimization problem to define Holevo bound, thus

\[ \max_{0 \leq \beta \leq 1} \mathcal{C}_{\theta_0, G}^{(\beta)} = \mathcal{C}_{\theta_0, G}^{(H)}. \]  

(1.25)

Actually, the explicit solution \((1.23)\) is a generalization of the solution \((1.15)\) for \( \mathcal{H} = 2 \) (see Appendix \[3\]).

This paper is organized as follows. In Section \[2\] we introduce logarithmic derivatives and Fisher information matrices induced by monotone metrics, and we derive the maximum logarithmic derivative bound. In Section \[3\] we introduce a commutation operator \( \mathcal{D} \), and the Holevo bound is rewritten in simpler form by using a \( \mathcal{D} \) invariant space of Hermitian operators. In Section \[4\] we show that \( \max_{0 \leq \beta \leq 1} \mathcal{C}_{\theta_0, G}^{(\beta)} \) has explicit
Thus the minimization of $\Tr GV$ solution (1.23) when the $d$ dimensional model has $d + 1$ dimensional $\mathcal{D}$ invariant extension of SLD tangent space. Further, we show the maximum logarithmic derivative bound is the same as the Holevo bound if $d = 2$. At the end of the section, we give examples of families of quantum states to which our theory can be applied not only for dim $\mathcal{H} = 2$. Section 5 is the conclusion. For the reader’s convenience, some additional material is presented in the Appendix. In Appendix A a proof of (1.10) is given. In Appendix B a brief proof and derivation of Holevo bound is presented. In Appendix C the Schur complement, which plays an important role, is introduced. In Appendix D details of the commutation operator $\mathcal{D}$ as a tool for multiple inner products is described. In Appendix E it is shown that the explicit form (1.15) for dim $\mathcal{H} = 2$ and $d = 2$ can be derived from (1.23).

2 Maximum logarithmic derivative bound

Let $\mathcal{F} = \{ \rho_\theta; \theta \in \Theta \subseteq \mathbb{R}^d \}$ be a smooth parametric family of density operators on a finite dimensional Hilbert space $\mathcal{H}$. The covariance matrix $V_{0\theta} [M, \hat{\theta}]$ of an locally unbiased estimator $(M, \hat{\theta})$ at $\theta_0$ satisfies the classical Cramér Rao inequality,

$$V_{0\theta} [M, \hat{\theta}] \geq J_{0\theta}^{(M)^{-1}}$$

(2.1)

where

$$J_{0\theta}^{(M)} = \left[ \sum_{r \in \mathcal{F}} \frac{(\Tr \partial_r \rho_\theta M_r) (\Tr \partial_j \rho_\theta M_j)}{\Tr \rho_\theta M_r} \right]_{ij}$$

(2.2)

is the classical Fisher information matrix with respect to the POVM $M$. The equality is achieved when

$$\hat{\theta}(x) = \theta_0 + \sum_{j=1}^d \left[ J_{0\theta}^{(M)^{-1}} \right]_{ij} \frac{\Tr \partial_j \rho_\theta M_j}{\Tr \rho_\theta M_j} x_j$$

(2.3)

Thus the minimization of $\Tr GV_{0\theta}[M, \hat{\theta}]$ is reduced to the minimization of $\Tr GJ_{0\theta}^{(M)^{-1}}$ for any $d \times d$ real positive matrix $G$. In this section, we consider lower bounds of $\Tr GJ_{0\theta}^{(M)^{-1}}$ directly induced by monotone metrics.

Let $P : (0, \infty) \rightarrow (0, \infty)$ be an operator monotone function such that $P(1) = 1$. Let $\mathcal{B}(\mathcal{H})$ be the set of all linear operators on $\mathcal{H}$. A monotone metric at $\theta_0 \in \Theta$ is an inner product $K_{\theta_0}^{(P)}(\cdot, \cdot)$ on $\mathcal{B}(\mathcal{H})$ defined by

$$K_{\theta_0}^{(P)}(X, Y) = \Tr X^* \left[ (R_{\rho_\theta_0} P(L_{\rho_\theta_0} R_{\rho_\theta_0}^{-1}))^{-1} Y \right]$$

(2.4)

where $L_{\rho_\theta_0}$ and $R_{\rho_\theta_0}$ are super operators on $\mathcal{B}(\mathcal{H})$ defined by

$$L_{\rho_\theta_0}(X) = \rho_\theta_0 X, \quad R_{\rho_\theta_0}(X) = X \rho_\theta_0,$$

(2.5)

(2.6)

with a strictly positive operator $\rho_\theta_0 \in \mathcal{F}$. Note that $L_{\rho_\theta_0}$ and $R_{\rho_\theta_0}$ are commutative so the super operator $(R_{\rho_\theta_0} P(L_{\rho_\theta_0} R_{\rho_\theta_0}^{-1}))^{-1}$ is well-defined. The monotone metric $K_{\theta_0}^{(P)}(\cdot, \cdot)$ has the monotonicity

$$K_{\theta_0}^{(P)}(X, X) \geq K_{\theta_0}^{(P)}(T(X), T(X))$$

(2.7)

under any channel $T : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}')$ mapping to another Hilbert space $\mathcal{H}'$.

The logarithmic derivative $L_{\theta_0}^{(P)}$ and the Fisher information matrix $J_{\theta_0}^{(P)}$ with respect to $P$ are

$$L_{\theta_0}^{(P)} = (R_{\rho_\theta_0} P(L_{\rho_\theta_0} R_{\rho_\theta_0}^{-1}))^{-1} \partial_\theta \rho_\theta_0 \quad (i = 1, \ldots, d),$$

$$J_{\theta_0}^{(P)} = \left[ K_{\theta_0}^{(P)}(\partial_i \rho_\theta_0, \partial_j \rho_\theta_0) \right]_{ij} = \left[ \Tr \partial_i \rho_\theta_0 L_{\theta_0}^{(P)} \right]_{ij}.$$  

(2.8)

(2.9)
Because a linear map $T^{(M)} : X \mapsto \text{Diag} (\text{Tr} X M_1, \text{Tr} X M_2, \ldots, \text{Tr} X M_{|\mathcal{X}|})$ ($X \in \mathcal{B} (\mathcal{H})$) is a quantum channel, for any POVM $M$ taking values on $\mathcal{X} = \{1, 2, \ldots, |\mathcal{X}|\}$, the monotonicity $[2, 7]$ of $K_{\rho_0}^{(P)} (\cdot, \cdot)$ induces a matrix inequality

$$J_{\beta_0}^{(P)} \geq \left[ K_{T^{(M)} (\rho_0)}^{(P)} (T^{(M)} (\partial_{i} \rho_0), T^{(M)} (\partial_{j} \rho_0)) \right]_{ij} = J_{\theta_0}^{(M)} ,$$

where Diag($\cdots$) indicates a diagonal matrix. The inequality (2.10) implies

$$\text{Tr} G J_{\beta_0}^{(M)} = \text{Tr} G J_{\beta_0}^{(P)} \geq \text{Tr} G J_{\beta_0}^{(P)} \geq \text{Tr} G J_{\beta_0}^{(P)} = : C_{\beta_0G}^{(P)},$$

due to (1.10). To obtain a tighter bound, we consider maximizing $C_{\beta_0G}^{(P)}$ with respect to $P$. In existing studies, such maximization was considered in several models, and SLD or RLD bounds were derived $[9]$. In this section, we consider maximization of $C_{\beta_0G}^{(P)}$ in general models.

In quantum state estimation, a family of linear functions

$$\mathcal{F} = \left\{ P^{(\beta)} (x) = \frac{1 + \beta}{2} x + \frac{1 - \beta}{2} ; \beta \in [-1, 1] \right\},$$

is particularly important for monotone metrics among operator monotone functions because the operator monotone function $P$ which maximize the lower bound $C_{\beta_0G}^{(P)}$ is always in the family $\mathcal{F}$ of functions as we will show in Theorem 2.1. We write $K_{\rho_0}^{(\beta)}, L_{\beta_0}, J_{\beta_0}$, and $C_{\beta_0G}^{(\beta)}$ instead of $K_{\rho_0}^{(P^{(\beta)} (x))}, L_{\beta_0}^{(P^{(\beta)} (x))}, J_{\beta_0}^{(P^{(\beta)} (x))}$, and $C_{\beta_0G}^{(P^{(\beta)} (x))}$. The $\beta$ logarithmic derivative $L_{\beta_0}^{(\beta)}$ is defined by

$$\partial_{i} \rho_{0} = \frac{1 + \beta}{2} \rho_{0} L_{\beta_0}^{(\beta)} + \frac{1 - \beta}{2} L_{\beta_0}^{(\beta)} \rho_{0}.$$

Let $(\cdot, \cdot)^{(\beta)}$ be an inner product on $\mathcal{B} (\mathcal{H})$ defined by

$$\langle X, Y \rangle^{(\beta)} = \frac{1}{2} \text{Tr} X^{*} \left\{ (1 + \beta) \rho_{0} Y + (1 - \beta) Y \rho_{0} \right\}.$$

By using this inner product, the $\beta$ Fisher information matrix can be written as

$$J_{\beta_0}^{(\beta)} = \left[ K_{\rho_0}^{(\beta)} (\partial_{i} \rho_0, \partial_{j} \rho_0) \right]_{ij} = \left[ \langle L_{\beta_0}^{(\beta)} (x), L_{\beta_0}^{(\beta)} (y) \rangle^{(\beta)} \right]_{ij}.$$

Note that $L_{\beta_0}^{(0)}$ coincides with SLD $L_{\beta_0}^{(S)}$, and $L_{\beta_0}^{(1)}$ coincides with RLD $L_{\beta_0}^{(R)}$.

Let us prove the optimality of the family $\mathcal{F}$ of functions. Because any operator monotone function $P : (0, \infty) \to (0, \infty)$ is differentiable and concave $[8]$, there always exists $\beta \in [-1, 1]$ such that $\frac{d}{dx} P (x) \big|_{x=1} = \frac{1 + \beta}{2}$ if $P (1) = 1$. Since the line $P^{(\beta)} (x)$ is a tangent to the concave function $P (x)$ at $x = 1$,

$$P (x) \leq P^{(\beta)} (x)$$

for any $x \in (0, \infty)$, and this implies

$$J_{\beta_0}^{(P)} \geq J_{\beta_0}^{(\beta)}$$

and

$$C_{\beta_0G}^{(\beta)} \geq C_{\beta_0G}^{(P)}.$$

Further, $C_{\beta_0G}^{(\beta)} = C_{\beta_0G}^{(-\beta)}$ because $L_{\beta_0}^{(\beta)}$ is conjugate transpose of $L_{\beta_0}^{(-\beta)}$. Therefore, we do not need to consider operator monotone functions other than $P^{(\beta)}$ for $\beta \in [0, 1]$.

Collecting these results, we have the following theorem.
Theorem 2.1. For any locally unbiased estimator $(M, \hat{\theta})$ at $\theta_0$, a $d \times d$ real positive matrix $G$, and an operator monotone function $P : (0, \infty) \to (0, \infty)$ such that $P(1) = 1$ and $\frac{d}{dx}P(x)|_{x=1} = \frac{1 + \beta}{2}$ with $\beta \in [-1, 1]$.

$$\text{Tr} G V_{\theta_0}[M, \hat{\theta}] \geq \text{Tr} G J^{(M)}_{\theta_0}^{-1} \geq C_{0, G}^{(\beta)} \geq C_{0, G},$$

(2.20)

From this theorem, we have an inequality

$$\text{Tr} G V_{\theta_0}[M, \hat{\theta}] \geq \max_{0 \leq \beta \leq 1} C_{0, G}^{(\beta)},$$

(2.21)

and we call the RHS of this inequality the maximum logarithmic derivative bound.

3 Equivalent expressions of Holevo bound

In this section, we give a simpler form of the Holevo bound by using a commutation operator. Let $\mathcal{H} = \{ \rho_\theta; \theta \in \Theta \subset \mathbb{R}^d \}$ be a smooth parametric family of density operators on a finite dimensional Hilbert space $\mathcal{H}$. Let $\mathcal{D}_{\rho_\theta} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be the commutation operator with respect to a faithful state $\rho_\theta \in \mathcal{H}$ on the set of linear operators $\mathcal{B}(\mathcal{H})$ on $\mathcal{H}$ defined by

$$\mathcal{D}_{\rho_\theta}(X)\rho_\theta + \rho_\theta \mathcal{D}_{\rho_\theta}(X) = \sqrt{-1} (X \rho_\theta - \rho_\theta X),$$

(3.1)

for $X \in \mathcal{B}(\mathcal{H})$. The commutation operator can also be defined by

$$\mathcal{D}_{\rho_\theta} = \frac{1}{\sqrt{-1}} (L_{\rho_\theta} - R_{\rho_\theta}) (L_{\rho_\theta} + R_{\rho_\theta})^{-1}.$$  

(3.2)

When $X$ is a Hermitian operator, $\mathcal{D}_{\rho_\theta}(X)$ is also a Hermitian operator. Through the commutation operator, the $\beta$ logarithmic derivatives $\{L_i^{(\beta)}\}_{i=1}^d$ and the corresponding inner product are linked by the following relations:

$$L_i^{(\beta)} = (I + \beta \sqrt{-1} \mathcal{D}_{\rho_\theta})^{-1} L_i^{(0)} \quad (i = 1, \ldots, d),$$

(3.3)

$$\langle A, B \rangle^{(\beta)} = \langle A, (I + \beta \sqrt{-1} \mathcal{D}_{\rho_\theta})(B) \rangle^{(0)} \quad (A, B \in \mathcal{B}(\mathcal{H}))$$

(3.4)

for $\beta \in [0, 1]$. Note that $I + \beta \sqrt{-1} \mathcal{D}_{\rho_\theta}$ is invertible for $\rho_\theta > 0$ since the operator norm of $\mathcal{D}_{\rho_\theta}$ is\[\max \left\{ \frac{\lambda}{2 \beta}; \lambda, \mu \text{ are eigenvalues of } \rho_\theta \right\} < 1.\] For details about the commutation operator $\mathcal{D}_{\rho_\theta}$, see Appendix D. By considering a $\mathcal{D}_{\rho_\theta}$-invariant extension $\tilde{\mathcal{F}} \supset \mathcal{F}$ of the SLD tangent space $\mathcal{F} := \text{span}_{\mathbb{R}} \{ L_i^{(S)} \}_{i=1}^d$, the minimization problem to define the Holevo bound is simplified as follows.

Theorem 3.1. Suppose that a quantum statistical model $\mathcal{F} = \{ \rho_\theta; \theta \in \Theta \subset \mathbb{R}^d \}$ on $\mathcal{H}$ has a $\mathcal{D}_{\rho_\theta}$-invariant extension $\tilde{\mathcal{F}}$ of the SLD tangent space of $\mathcal{F}$ at $\theta = \theta_0$. Let $\{ D_j^{(S)} \}_{j=1}^d$ be a basis of $\tilde{\mathcal{F}}$. The Holevo bound defined by (1.72) is rewritten as

$$C_{0, G}^{(\tilde{H})} = \min_F \left\{ \text{Tr} G Z + \text{Tr} \left| \sqrt{\text{Im} Z \sqrt{G}} \right| ; Z = F^\top \Sigma F, \right. \text{ } \left. F \text{ is an } r \times d \text{ real matrix satisfying } F^\top \text{Re}(\tau) = I, \right\},$$

(3.5)

(3.6)

where $\Sigma$ and $\tau$ are $r \times r$ and $r \times d$ complex matrix whose $(i, j)$th entries are given by $\Sigma_{ij} = \text{Tr} \rho_\theta D_j^{(S)} D_i^{(S)}$ and $\tau_{ij} = \text{Tr} \rho_\theta (D_j^{(S)} D_i^{(S)}).$
Proof. Let \( \mathcal{F}^\perp \) be the orthogonal complement of \( \mathcal{F} \) in the set \( B(H) \) of Hermitian operators with respect to the inner product \( \langle \cdot, \cdot \rangle \), and let \( \mathcal{F} : B(H) \rightarrow \mathcal{F} \) and \( \mathcal{F}^\perp : B(H) \rightarrow \mathcal{F}^\perp \) be the projections associated with the decomposition \( B(H) = \mathcal{F} \oplus \mathcal{F}^\perp \). For \( X \in \mathcal{F}^\perp \) and \( Y \in \mathcal{F} \),

\[
\text{Tr} X \rho_{th} Y = \langle X, Y \rangle^{(0)} = \left( X, (I + \sqrt{-1} D_{\rho_{th}})(Y) \right)^{(0)} = 0.
\]  

(3.7)

Let \( \{B_j\}_{j=1}^d \) be observables achieving the minimum in (1.12), \( \{\mathcal{P}(B_j)\}_{j=1}^d \) also satisfies the local unbiasedness condition

\[
\text{Tr} \partial \rho_{th} \mathcal{P}(B_j) = \left( L_i^{(S)}, \mathcal{P}(B_j) \right)^{(0)} = \left( L_i^{(S)}, B_j \right)^{(0)} = \delta_{ij}.
\]

(3.8)

Further, because of (3.7),

\[
Z_{ij}(B) = \text{Tr} B_j \rho_{th} B_j = \text{Tr} \left\{ \mathcal{P}(B_j) + \mathcal{P}^\perp(B_j) \right\} \rho_{th} \left\{ \mathcal{P}(B_j) + \mathcal{P}^\perp(B_j) \right\}
\]

(3.9)

\[
= \text{Tr} \mathcal{P}(B_j) \rho_{th} \mathcal{P}(B_j) + \text{Tr} \mathcal{P}^\perp(B_j) \rho_{th} \mathcal{P}^\perp(B_j) = Z_{ij}(\mathcal{P}(B)) + Z_{ij}(\mathcal{P}^\perp(B)).
\]

(3.10)

This decomposition implies \( Z(B) \geq Z(\mathcal{P}(B)) \), thus \( \{B_j\}_{j=1}^d \subset \mathcal{F} \).

Observables \( \{B_j\}_{j=1}^d \subset \mathcal{F} \) can be expressed by \( B_j = \sum_k f_k D_k^{(S)} \) with an \( r \times d \) real matrix \( F \). By using \( F \), the local unbiasedness condition in (1.12) is written as

\[
\left( L_i^{(S)}, B_j \right)^{(0)} = F_k \left( L_i^{(S)}, D_k^{(S)} \right)^{(0)} = F_k (\Re \tau)_k = \delta_{ij},
\]

(3.11)

and \( Z(B) \) is written as \( Z(B) = F^\dagger \Sigma F \).

Due to this theorem, we can easily see \( C_{\theta_0, G}^{(H)}(\rho_{\theta_0}^\otimes n) = \frac{1}{n} C_{\theta_0, G}^{(H)}(\rho_\theta) \). In this paper, we use further rewrite of the Holevo bound as follows.

**Corollary 3.2.** Suppose \( D_i^{(S)} = L_i^{(S)} \) for \( 1 \leq i \leq d \) in Theorem 3.1 and let \( R = (\Re \Sigma)^{-1} \Sigma (\Re \Sigma)^{-1} = \begin{pmatrix} R_1 & R_2^* \cr R_2 & R_3 \end{pmatrix} \) with \( d \times d \), \( (r-d) \times d \), and \( (r-d) \times (d-d) \) block matrices \( R_1, R_2, \) and \( R_3 \). The Holevo bound is rewritten as

\[
C_{\theta_0, G}^{(H)} = \min_f \left\{ \text{Tr} GZ(f) + \text{Tr} \sqrt{\text{Im} Z(f)} \sqrt{G} \right\},
\]

(3.12)

where

\[
Z(f) = (I, f^\dagger) R \begin{pmatrix} I \\ f \end{pmatrix} = R_1 + R_2 f + f^* R_2 + f^* R_3 f.
\]

(3.13)

**Proof.** Let \( \Sigma = \begin{pmatrix} \Sigma_1 & \Sigma_2 \\ \Sigma_2 & \Sigma_3 \end{pmatrix} \) be partitioned in the same manner as \( R \). For an \( r \times d \) real matrix \( F \), the condition \( F^\dagger \Re(\tau) = I \) implies

\[
F \Re \Sigma = \begin{pmatrix} I \\ f \end{pmatrix}
\]

(3.15)

with an \( (r-d) \times d \) real matrix \( f \) because \( \tau = \begin{pmatrix} \Sigma_1 \\ \Sigma_2 \end{pmatrix} \). By using \( f, F^\dagger \Sigma F \) in Theorem 3.1 can be written as

\[
F^\dagger \Sigma F = (I, f^\dagger) (\Re \Sigma)^{-1} \Sigma (\Re \Sigma)^{-1} \begin{pmatrix} I \\ f \end{pmatrix} = (I, f^\dagger) R \begin{pmatrix} I \\ f \end{pmatrix}
\]

(3.16)

\( \square \)
Actually, $R$ in Corollary 3.2 coincides with the inverse RLD Fisher information matrix of a supermodel $\mathcal{J} \supset \mathcal{J}$ of $\mathcal{J}$ that have SLDs $\{D^{(i)}\}_{i=1}^r$ due to Lemma D.2 in Appendix D. Furthermore, $\beta$ Fisher information matrix can be calculated directly by Schur complement of $\text{Re} R + \beta \sqrt{-1} \text{Im} R$ as follows.

**Lemma 3.3.** Let $D_i^{(\beta)} = (I + \beta \sqrt{-1} \mathcal{D}_{\rho_0})^{-1}(D_i^{(S)})$ be the $\beta$ logarithmic derivative with respect to the extended SLD $D_i$ ($1 \leq i \leq r$) at $\rho_0$ given in Corollary 3.2 and let

$$f^{(\beta)}_{\rho_0} = \left[ \left\langle D_i^{(\beta)}, D_j^{(\beta)} \right\rangle \right]_{1 \leq i, j \leq r} \quad (3.17)$$

be the extended $\beta$ Fisher information matrix. Then $R$ given in Corollary 3.2 satisfies

$$f^{(\beta)-1}_{\rho_0} = \text{Re} R + \beta \sqrt{-1} \text{Im} R. \quad (3.18)$$

Further, the inverse $\beta$ Fisher information matrix $f^{(\beta)-1}_{\rho_0}$ can be represented by $R$ as

$$f^{(\beta)-1}_{\rho_0} = R_1^{(\beta)} - R_2^{(\beta)} R_3^{(\beta)-1} R_2^{(\beta)}.$$ \quad (3.19)

where $R_1^{(\beta)} = \text{Re} R_1 + \beta \sqrt{-1} \text{Im} R_1$, $R_2^{(\beta)} = \text{Re} R_2 + \beta \sqrt{-1} \text{Im} R_2$, $R_3^{(\beta)} = \text{Re} R_3 + \beta \sqrt{-1} \text{Im} R_3$.

**Proof.** The proof of (3.18) is given in Lemma D.2. The proof of (3.19) is immediate, because $f^{(\beta)}_{\rho_0}$ is the $(1,1)$ block of

$$f^{(\beta)}_{\rho_0} = \left( \begin{array}{cc} R_1^{(\beta)} & R_2^{(\beta)} \\ R_2^{(\beta)} & R_3^{(\beta)} \end{array} \right)^{-1}, \quad (3.20)$$

and it is the same as the inverse of $f^{(\beta)-1}_{\rho_0} / R_3^{(\beta)} = R_1^{(\beta)} - R_2^{(\beta)} R_3^{(\beta)-1} R_2^{(\beta)}$, where $f^{(\beta)-1}_{\rho_0} / R_3^{(\beta)}$ is the Schur complement given in Appendix C.

From this lemma, a relation between the Holevo bound and $C^{(\beta)}_{0, G}$ can be obtained directly.

**Lemma 3.4.** For any $\beta \in [0, 1]$,

$$C^{(H)}_{0, G} \geq C^{(\beta)}_{0, G}. \quad (3.21)$$

**Proof.** Let $Z^{(\beta)}(f) := (I, f^T)(\text{Re} R + \beta \sqrt{-1} \text{Im} R) \left( \begin{array}{c} I \\ f \end{array} \right)$. Then we see

$$C^{(H)}_{0, G} = \min_f \left\{ \text{Tr} G Z(f) + \text{Tr} \left[ \sqrt{G} \text{Im} Z(f) \sqrt{G} \right] \right\}; \quad (3.22)$$

if $f$ is an $(r-d) \times d$ real matrix,

$$\geq \min_f \left\{ \text{Tr} G Z^{(\beta)}(f) + \text{Tr} \left[ \sqrt{G} \text{Im} Z^{(\beta)}(f) \sqrt{G} \right] \right\}; \quad (3.23)$$

if $f$ is an $(r-d) \times d$ real matrix

$$\geq \min_f \left\{ \text{Tr} G Z^{(\beta)}(f) + \text{Tr} \left[ \sqrt{G} \text{Im} Z^{(\beta)}(f) \sqrt{G} \right] \right\}; \quad (3.24)$$

if $f$ is an $(r-d) \times d$ complex matrix

$$= C^{(\beta)}_{0, G}. \quad (3.25)$$

The last equality is obtained from

$$Z^{(\beta)}(f) = R_1^{(\beta)} - R_2^{(\beta)} R_3^{(\beta)-1} R_2^{(\beta)} + (f + R_2^{(\beta)} R_3^{(\beta)-1}) R_3^{(\beta)} (f + R_2^{(\beta)} R_3^{(\beta)-1}) \\quad (3.26)$$

and the minimum is achieved when $f = -R_2^{(\beta)} R_3^{(\beta)-1} R_2^{(\beta)}$. \hfill $\square$
For a numerical computation of the Holevo bound, it was proposed to apply a linear semi-definite program\cite{10}. The minimization problem given in Corollary 3.2 can be rewritten to a linear semi-definite program:

\[ \text{minimize}_{f, V} \text{Tr} G V \]

subject to \[ \begin{pmatrix} V & (I_d \cdot f^T) \sqrt{R} \\ \sqrt{R} & I_r \end{pmatrix} \geq 0, \]

where \( V \) is a \( d \times d \) real matrix, \( I_d \) and \( I_r \) are the identity matrices of size \( d \) and \( r \), since we can see that the inequality \( V \geq Z(f) = (I_d \cdot f^T) R \left( \begin{array}{c} I_d \\ f \end{array} \right) \) is equivalent to (3.31) by considering the Schur complement of (3.31).

The relationship between the bounds introduced in this paper is

\[ 2C^{(S)}_{\theta_0, G} \geq C^{(H)}_{\theta_0, G} \geq \max_{0 \leq \beta \leq 1} C^{(R)}_{\theta_0, G} \geq \max \{ C^{(S)}_{\theta_0, G}, C^{(R)}_{\theta_0, G} \}. \]

In the first inequality, \( 2C^{(S)}_{\theta_0, G} \) is known as an upper bound\cite{11, 12} of the Holevo bound. This inequality can be shown as follows. In Corollary 3.2, it can be seen that

\[ \text{Tr} \ G \text{Re} Z(f^{(S)}) \leq C^{(H)}_{\theta_0, G} \leq \text{Tr} \ G \text{Re} Z(f^{(S)}) + \text{Tr} \left| \sqrt{G} \text{Im} Z(f^{(S)}) \sqrt{G} \right|, \]

where \( f^{(S)} := -\left( \text{Re} R_3 \right)^{-1} (\text{Re} R_2). \) Because

\[ \text{Re} Z(f^{(S)}) = J^{(S)^{-1}} \geq \sqrt{-\text{Im} Z(f^{(S)})}, \]

we obtain the inequality

\[ 2C^{(S)}_{\theta_0, G} \geq C^{(H)}_{\theta_0, G}. \]

For any sufficiently smooth model \( \{ \rho_0^h; \theta \in \Theta \subset \mathbb{R}^d \} \), it is known that a sequence of i.i.d. extension models \( \{ \rho_{0^h+\sqrt{h}/\sqrt{\pi}}; h \in \mathbb{R}^d \} \) with a local parameter \( h \in \mathbb{R}^d \) has a sequence of estimators that achieves the Holevo bound \( C^{(H)}_{\theta_0, G} \) asymptotically by using the theory of the quantum local asymptotic normality\cite{2, 3, 4}. On the other hand, \( C^{(S)}_{\theta_0, G}, C^{(R)}_{\theta_0, G} \) and \( \max_{0 \leq \beta \leq 1} C^{(\beta)}_{\theta_0, G} \) can not be always achieved by considering i.i.d. extension. Therefore, these bounds are informative only when they are consistent with the Holevo bound. It can be obviously seen from Theorem 3.1 that \( C^{(H)}_{\theta_0, G} = C^{(R)}_{\theta_0, G} \) if SLDs are \( \mathcal{D}_{\theta_0} \) invariant. It can be also seen from (3.33) that \( C^{(H)}_{\theta_0, G} = C^{(S)}_{\theta_0, G} \) if and only if \( \text{Im} Z(f^{(S)}) = 0 \). Since the maximum logarithmic derivative bound \( \max_{0 \leq \beta \leq 1} C^{(\beta)}_{\theta_0, G} \) is larger than the SLD bound and the RLD bound, if the Holevo bound \( C^{(H)}_{\theta_0, G} \) is equal to the SLD bound or RLD bound, \( C^{(H)}_{\theta_0, G} = \max_{0 \leq \beta \leq 1} C^{(\beta)}_{\theta_0, G} \) also holds. In Section 4, we provide another case of satisfying \( C^{(H)}_{\theta_0, G} = \max_{0 \leq \beta \leq 1} C^{(\beta)}_{\theta_0, G} \) that is different from SLD or RLD bounds and has an explicit solution. Further, we give examples of models that can achieve \( \max_{0 \leq \beta \leq 1} C^{(\beta)}_{\theta_0, G} \) (see Example 4.5 and 4.6).

When \( \rho_{0^h} \) is not strictly positive, the \( \beta \) logarithmic derivatives \( \{ L_{\beta}^{(1)} \}_{\beta=1} \) that satisfy (2.14) for \( -1 < \beta < 1 \) can be defined on the quotient space \( \mathcal{D}(\mathcal{H})/\sim_{\rho_{0^h}} \) with respect to an equivalence relation defined by

\[ A \sim_{\rho_{0^h}} B \iff A - B \in \ker L_{\rho_{0^h}} \cap \ker R_{\rho_{0^h}}. \]

The inner product \( \langle \cdot, \cdot \rangle^{(1)} \) on \( \mathcal{D}(\mathcal{H})/\sim_{\rho_{0^h}} \) and the \( \beta \) Fisher information matrix \( J_{\beta}^{(1)} \) can be also defined by (2.15) and (2.16). The commutation operator \( \mathcal{D}_{\rho_{0^h}} \) is defined by (3.1) as a super operator on \( \mathcal{D}(\mathcal{H})/\sim_{\rho_{0^h}} \).

For \( \beta = 1 \), RLDs \( \{ L_{1}^{(R)} \}_{r=1} \) cannot be defined, however \( \langle \cdot, \cdot \rangle^{(1)} \) can be defined as a pre-inner product on \( \mathcal{D}(\mathcal{H})/\sim_{\rho_{0^h}} \) and (3.4) is valid. Theorem 3.1 and Corollary 3.2 also holds in a similar way by dealing with \( \mathcal{D}(\mathcal{H})/\sim_{\rho_{0^h}} \) instead of \( \mathcal{D}(\mathcal{H}) \).
4 $\mathcal{D}_{\rho_0}$ invariant extension with one dimension

In Corollary 3.2 if $\{D_i\}_{i=d+1}$ are orthogonal to $\mathcal{T} = \text{span}_\mathbb{R} \{D_i\}_{i=1}$ with respect to the inner product $\langle \cdot, \cdot \rangle^{(0)}$, $\text{Re} R = 0$. Further, if $r = d + 1$ and $\langle D_r, D_j \rangle^{(0)} = 1$, $R$ can take form of

$$R = \begin{pmatrix} A & \sqrt{-1} \langle b, \cdot \rangle \\ -\sqrt{-1} \langle \cdot, b \rangle & 1 \end{pmatrix}$$

with a real vector $|b\rangle \in \mathbb{R}^d$. In this case,$$
J^{(\beta)}_{\theta_0}^{-1} = \text{Re} A + \beta \sqrt{-1} \text{Im} A - \beta^2 |b\rangle \langle b|$$
due to (3.19), and

$$C^{(\beta)}_{\theta_0, G} = \text{Tr} G \text{Re} A + \beta \text{Tr} \sqrt{G \text{Im} A \sqrt{G}} - \beta^2 |b\rangle \langle G | b|.$$ 

Therefore $A$ and $|b\rangle \langle b|$ can be expressed by $J^{(R)}_{\theta_0}^{-1}$ and $J^{(S)}_{\theta_0}$ as

$$A = J^{(S)}_{\theta_0}^{-1} + \sqrt{-1} \text{Im}(J^{(R)}_{\theta_0}^{-1})$$

$$|b\rangle \langle b| = J^{(S)}_{\theta_0}^{-1} - \text{Re}(J^{(R)}_{\theta_0}^{-1}).$$

Let us calculate the maximum logarithmic derivative bound

$$\max_{0 \leq \beta \leq 1} C^{(\beta)}_{\theta_0, G} = \max_{0 \leq \beta \leq 1} \text{Tr} G \text{Re} A + \beta \text{Tr} \sqrt{G \text{Im} A \sqrt{G}} - \beta^2 |b\rangle \langle G | b|.$$ 

If $|b\rangle \neq 0$, the quadratic function

$$g_1 : \beta \mapsto \text{Tr} G \text{Re} A + \beta \text{Tr} \sqrt{G \text{Im} A \sqrt{G}} - \beta^2 |b\rangle \langle G | b|$$

is maximized at

$$\beta = \frac{\text{Tr} \sqrt{G \text{Im} A \sqrt{G}}}{2 |b\rangle \langle G | b|} > 0.$$ 

If $\frac{\text{Tr} \sqrt{G \text{Im} A \sqrt{G}}}{2 |b\rangle \langle G | b|} \geq 1$, $C^{(\beta)}_{\theta_0, G}$ is maximized at $\beta = 1$, thus

$$\max_{0 \leq \beta \leq 1} C^{(\beta)}_{\theta_0, G} = g_1(1) = \text{Tr} G \text{Re} A + \text{Tr} \sqrt{G \text{Im} A \sqrt{G}} - |b\rangle \langle G | b|.$$ 

If $\frac{\text{Tr} \sqrt{G \text{Im} A \sqrt{G}}}{2 |b\rangle \langle G | b|} < 1$, $C^{(\beta)}_{\theta_0, G}$ is maximized at $\beta = \frac{\text{Tr} \sqrt{G \text{Im} A \sqrt{G}}}{2 |b\rangle \langle G | b|}$, thus

$$\max_{0 \leq \beta \leq 1} C^{(\beta)}_{\theta_0, G} = g_1 \left( \frac{\text{Tr} \sqrt{G \text{Im} A \sqrt{G}}}{2 |b\rangle \langle G | b|} \right) = \text{Tr} G \text{Re} A + \frac{1}{4} \left( \frac{\text{Tr} \sqrt{G \text{Im} A \sqrt{G}}}{|b\rangle \langle G | b|} \right)^2.$$ 

When $|b\rangle = 0$, $g_1$ is a linear function and $C^{(\beta)}_{\theta_0, G}$ is maximized at $\beta = 1$, so

$$\max_{0 \leq \beta \leq 1} C^{(\beta)}_{\theta_0, G} = g_1(1) = \text{Tr} G \text{Re} A + \text{Tr} \sqrt{G \text{Im} A \sqrt{G}}.$$ 

Collecting these result, we have the following theorem.
Letting \( \epsilon = \epsilon \), the Holevo bound is
\[
C_{\theta_0, G}^{(\epsilon)} = \begin{cases} 
C_{\theta_0, G}^{(1)} & \text{if } \epsilon \geq 1, \\
C_{\theta_0, G}^{(\epsilon)} & \text{otherwise},
\end{cases}
\]
where
\[
\hat{\epsilon} = \begin{cases} 
\frac{\operatorname{Tr} \left[ \sqrt{\operatorname{Im} \langle \theta_0, G^* \rangle} \right]}{2 \operatorname{Tr} \left[ \sqrt{\langle \theta_0, G^* \rangle} \right]} & \text{if } J^{(S)}_{\theta_0} \neq \operatorname{Re}(\langle \theta_0, G^* \rangle) , \\
\infty & \text{otherwise}.
\end{cases}
\]

In general, even when \( R \) can take form of \([4.1]\), the equality of
\[
C_{\theta_0, G}^{(H)} = \max_{0 \leq \beta \leq 1} C_{\theta_0, G}^{(\beta)}
\]
is not always achieved. However, when \( d = 2 \), these two bounds are consistent.

**Theorem 4.2.** When \( d = 2 \) and the model has three dimensional \( \mathcal{D}_{\theta_0} \) invariant extended SLD tangent space, the Holevo bound \( C_{\theta_0, G}^{(H)} \) is the same as \( \max_{0 \leq \beta \leq 1} C_{\theta_0, G}^{(\beta)} \), which is given explicitly in Theorem 4.1.

**Proof.** By using Corollary 3.2, the Holevo bound is
\[
C_{\theta_0, G}^{(H)} = \min_{f} \left\{ \operatorname{Tr} G Z(f) + \operatorname{Tr} \left[ \sqrt{\operatorname{Im} Z(f)} \sqrt{G} \right] \right\},
\]
where
\[
Z(f) = \operatorname{Re} A + |f \rangle \langle f| + \sqrt{-1} (\operatorname{Im} A + |b \rangle \langle f| - |f \rangle \langle b|).
\]

Letting
\[
|\hat{b} \rangle := \sqrt{G} |b \rangle ,
\]
\[
|\hat{f} \rangle := \sqrt{G} |f \rangle ,
\]
and
\[
\hat{A} := \sqrt{\operatorname{Im} A} \sqrt{G} = a \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]
with \( a \in \mathbb{R} \), then
\[
C_{\theta_0, G}^{(H)} = \min_{f} \left\{ \operatorname{Tr} G \operatorname{Re} A + \langle \hat{f} | \hat{f} \rangle + \operatorname{Tr} \left[ \hat{A} + |\hat{b} \rangle \langle \hat{f}| - |\hat{f} \rangle \langle \hat{b}| \right] \right\}
\]
\[
= \min_{f} \left\{ \operatorname{Tr} G \operatorname{Re} A + \langle \hat{f} | \hat{f} \rangle + \operatorname{Tr} \left[ \left( a + b_2 f_1 - \hat{b} f_2 \right) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right] \right\}
\]
\[
= \min_{f} \left\{ \operatorname{Tr} G \operatorname{Re} A + \langle \hat{f} | \hat{f} \rangle + 2 |a + b_2 f_1 - \hat{b} f_2| \right\}.
\]

By representing \( |\hat{f} \rangle \) as \( |\hat{f} \rangle = s \begin{pmatrix} \hat{b}_1 \\ \hat{b}_2 \end{pmatrix} + t \begin{pmatrix} \hat{b}_2 \\ -\hat{b}_1 \end{pmatrix} \) with \( s, t \in \mathbb{R} \), we have
\[
C_{\theta_0, G}^{(H)} = \min_{s,t} \left\{ \operatorname{Tr} G \operatorname{Re} A + \langle \hat{b} | \hat{b} \rangle s^2 + (\langle \hat{b} | \hat{b} \rangle t^2 + 2 |a + \langle \hat{b} | \hat{b} \rangle t| \right\}
\]
\[
= \min_{t} \left\{ \operatorname{Tr} G \operatorname{Re} A + (\langle \hat{b} | \hat{b} \rangle t^2 + 2 |a + \langle \hat{b} | \hat{b} \rangle t| \right\}
\]
\[
= \min_{\epsilon} \left\{ \operatorname{Tr} G \operatorname{Re} A + (\langle \hat{b} | \hat{b} \rangle t^2 + 2 |a + \langle \hat{b} | \hat{b} \rangle t| \right\}
\]
\[
\leq \min_{\epsilon>0} \left\{ \operatorname{Tr} G \operatorname{Re} A + (\langle \hat{b} | \hat{b} \rangle t^2 + 2 |a + \langle \hat{b} | \hat{b} \rangle t| \right\}.
\]
Let
\[ g_2(t) = \text{Tr} G \Re\text{a} + 2|a| + 2\langle \hat{b} | \hat{b} \rangle t. \] (4.27)

Because of (4.9), (4.10) and \( C^{(\beta)}_{0, G} \geq \max_{0 \leq \beta \leq 1} C^{(\beta)}_{0, G} \), we have
\[ g_2(t) \geq \begin{cases} 
\text{Tr} G \Re\text{a} + 2|a| - \langle \hat{b} | \hat{b} \rangle, & \text{if } |a| \geq \langle \hat{b} | \hat{b} \rangle, \\
\text{Tr} G \Re\text{a} + \frac{\epsilon^2}{(2\eta)}, & \text{otherwise},
\end{cases} \] (4.28)

where the equality is achieved at \( t = \max\{-|\langle \hat{b} | \hat{b} \rangle|, -1\} \).

Let us consider the Lagrangian duality of the quadratic programming (4.26). The Lagrangian function is
\[ \mathcal{L}(t, \lambda) = \text{Tr} G \Re\text{a} + 2|a| + 2\langle \hat{b} | \hat{b} \rangle t + \langle \hat{b} | \hat{b} \rangle t^2 - \lambda (|a| + \langle \hat{b} | \hat{b} \rangle t), \] (4.29)
\[ = \text{Tr} G \Re\text{a} + 2|a| - \lambda |a| + \langle \hat{b} | \hat{b} \rangle ((2 - \lambda)t + t^2). \] (4.30)

For any fixed \( \lambda \in \mathbb{R}, \mathcal{L}(t, \lambda) \) is minimized at \( t = \frac{1 - \lambda}{2}, \) and the Lagrangian dual function is
\[ g(\lambda) = \min_t \mathcal{L}(t, \lambda) = \mathcal{L}\left(\frac{\lambda - 2}{2}, \lambda\right) = \text{Tr} G \Re\text{a} - (\lambda - 2)|a| - \langle \hat{b} | \hat{b} \rangle \left(\frac{\lambda - 2}{2}\right)^2. \] (4.31)
\[ = g_1\left(\frac{2 - \lambda}{2}\right). \] (4.32)

Hence the Lagrangian dual programming is
\[ \max_{\lambda \geq 0} g_1\left(\frac{2 - \lambda}{2}\right). \] (4.33)

The solution of this maximization is the same as (4.6). It is known that in quadratic programming the Lagrangian duality problem has the same solution. This is the reason why the two bounds coincide.

The optimal observables \( B_1, B_2 \) for the minimization of (1.14) to define Holevo bound can be described by \( \beta^* := \arg\max_{0 \leq \beta \leq 1} C^{(\beta)}_{0, G} \) given in Theorem 4.1 as follows. From the proof of Theorem 4.2, we see that the minimization of (4.15) is achieved when
\[ |f\rangle = |f^{(\beta^*)}\rangle := \beta^*\sqrt{G^{-1}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \sqrt{G}|\hat{b}\rangle. \] (4.34)

This means the minimization of (3.6) is achieved when \( F = F^{(\beta^*)} := \left(\frac{1}{|f^{(\beta^*)}\rangle}\right) (\Re\Sigma)^{-1}, \) and the minimization of (1.14) is achieved when
\[ B_i = B_i^{(\beta^*)} := \sum_{j=1}^3 F_{ji}^{(\beta^*)} D_j \quad (i = 1, 2). \] (4.35)

When \( \dim \mathcal{H} = 2 \) and \( d = 2 \), any model \( \mathcal{N} \) has two SLDs \( L_1 \) and \( L_2 \) at any point \( \theta_0, \) and \( \hat{\mathcal{N}} = \{ X \in \mathcal{B}_b(\mathcal{H}) \mid \text{Tr} \rho_0 X = 0 \} \supset \mathcal{X} = \{ L_1, L_2 \} \) is \( \mathcal{D}_b \) invariant three dimensional space. Therefore, \( R \) takes the form of (4.14), thus Theorem 4.1 and Theorem 4.2 are applicable. This is the essential reason why the Holevo bound can be expressed by (1.15).

More generally, if a two-dimensional smooth parametric family \( \{ \sigma_\xi; \xi \in \Xi \subset \mathbb{R}^2 \} \) of density operators on a Hilbert space \( \mathcal{H} \) with an open set \( \Xi \subset \mathbb{R}^2 \) is \( \mathcal{D}_e \) invariant, a three-dimensional smooth parametric family \( \{ \rho_\xi; \xi \in \Xi \subset \mathbb{R}^2, 0 < \eta < 1 \} \) is also \( \mathcal{D}_e(\hat{\xi}, \eta) \) invariant. Therefore, Theorem 4.1 and 4.2 are applicable for any two-dimensional submodel of \( \{ \rho_\xi; \xi \in \Xi \subset \mathbb{R}^2 \} \), and the maximum logarithmic derivative bound and the Holevo bound can be calculated explicitly. We show examples below.
Example 4.3. Let
\[
\rho_\theta = a(1 - |\theta|) \left\{ \frac{1}{2} (\theta^1 \sigma_1 + \theta^2 \sigma_2 + \sqrt{1 - |\theta|^2} \sigma_3) + (1 - a(1 - |\theta|)) \right\} \frac{I}{2}, |\theta| < 1
\] (4.36)
be a family of density operators on \( \mathcal{H} = \mathbb{C}^2 \) parameterized by \( \theta = (\theta^1, \theta^2) \) with fixed \( 0 < a < 1 \), where \( \sigma_1, \sigma_2, \sigma_3 \) are Pauli matrices. Let \( D_1 = \partial_1 \rho_\theta, D_2 = \partial_2 \rho_\theta, D_3 = \rho_\theta - \frac{I}{2} \). A linear space of observables \( \text{span}_\mathcal{R} \{ \{ D_i \}_{i=1}^d \} = \{ X \in \mathcal{B}_R(\mathcal{H}) : \text{Tr} X = 0 \} \) is \( \mathcal{D}_0 \) invariant at any \( \theta \). The extended RLD Fisher information matrix is calculated by \( J_{\theta \theta} = \text{Tr} D_i \rho_{\theta}^{-1} D_j \) and its inverse is
\[
J_{\theta \theta}^{-1} = \frac{1}{a^2 (1 - r)} \begin{pmatrix}
\frac{1 + r}{\sqrt{1 - a^2 (1 - r)^2}} & -\sqrt{a^2 (1 - r)^2} \\
\frac{1 - r}{\sqrt{1 - a^2 (1 - r)^2}} & \frac{1 + r}{\sqrt{1 - a^2 (1 + r)^2}}
\end{pmatrix},
\] (4.37)
at \( \theta = (r, 0) \) with \( 0 \leq r < 1 \). The inverse \( \beta \) Fisher information matrix \( J_{\theta \theta}^{-1} \) is the Schur complement of \( \text{Re} J_{\theta \theta}^{-1} + \beta \text{Im} J_{\theta \theta}^{-1} \) due to (3.19). Let us consider lower bounds of \( \text{Tr} G \rho_{\theta} | M, \hat{\theta} \) with a SLD weight \( G = J_{\theta \theta}^{(S)} \). The \( \beta \) bound is
\[
C_{\theta_0, G}^{(\beta)} = 2 + \frac{2a \sqrt{(1 - a^2 (1 - r)^2)(2 - a^2 (1 - r)^3)(1 - r)^3}}{2 - a^2 (1 - r)^3} \beta - \frac{a^2 (1 - r)^2 (1 + r)}{2 - a^2 (1 - r)^3} \beta^2.
\] (4.38)
By using Theorem 4.1 we see that the maximum of \( C_{\theta_0, G}^{(\beta)} \) is achieved by
\[
\beta = \min \left\{ 1, \frac{1}{a(1 + r)} \sqrt{\frac{(1 - a^2 (1 - r)^2)(2 - a^2 (1 - r)^3)}{1 - r}} \right\}.
\] (4.39)
In Fig. 1(left), the behavior of the optimal \( \beta \) is plotted as a function of \( r \) when \( a = 0.95 \). Due to Theorem 4.2, \( \max_{0 \leq \beta \leq 1} C_{\theta_0, G}^{(\beta)} \) is the same as the Holevo bound \( C_{\theta_0, G}^{(H)} \). This result illustrates a principle behind the explicit expression of the Holevo bound (1.15).

Example 4.4. Here we show an example of the case when \( \text{dim} \mathcal{H} > 2 \). Let
\[
\rho_\theta = a(1 - |\theta|) \left\{ \frac{1}{2} (\theta^1 \sigma_1 + \theta^2 \sigma_2 + \sqrt{1 - |\theta|^2} \sigma_3)^{\otimes 2} + (1 - a(1 - |\theta|)) \right\} \frac{I}{2}, |\theta| < 1
\] (4.40)
be a family of density operators on \( \mathcal{H} = \mathbb{C}^4 \) parameterized by \( \theta = (\theta^1, \theta^2) \) with fixed \( 0 < a < 1 \). Let \( D_1 = \partial_1 \rho_\theta, D_2 = \partial_2 \rho_\theta, D_3 = \rho_\theta - \frac{I}{2} \). A linear space of observables \( \text{span}_\mathcal{R} \{ \{ D_i \}_{i=1}^d \} = \{ X \in \mathcal{B}_R(\mathcal{H}) : \text{Tr} X = 0 \} \) is \( \mathcal{D}_0 \) invariant at any \( \theta \). The extended RLD Fisher information matrix is calculated by \( J_{\theta \theta} = \text{Tr} D_i \rho_{\theta}^{-1} D_j \). By the similar calculation as in Example 4.3, we see that the maximum of \( C_{\theta_0, G}^{(\beta)} \) is achieved by
\[
\beta = \min \left\{ 1, \frac{1}{3a(1 + r)} \sqrt{\frac{(1 - a(1 - r))(1 + 3a(1 - r))(7 - r - a(1 - r)(-11 + 12a(1 - r)^2 + 5r))}{1 - r}} \right\}
\] (4.41)
for a SLD weight \( G = J_{\theta \theta}^{(S)} \) at \( \theta = (r, 0) \) with \( 0 \leq r < 1 \). In Fig. 1(right), the behavior of the optimal \( \beta \) is plotted as a function of \( r \) when \( a = 0.95 \). Due to Theorem 4.2, \( \max_{0 \leq \beta \leq 1} C_{\theta_0, G}^{(\beta)} \) is the same as the Holevo bound \( C_{\theta_0, G}^{(H)} \). This example is not included in the result of (1.15) for \( \text{dim} \mathcal{H} = 2 \).
Next, let us show examples of models that can achieve the maximum logarithmic derivative bounds. It is known that the Holevo bounds can be achieved for quantum Gaussian shift models and pure states models. The Holevo bounds can also be achieved as SLD bounds for models that have commutative SLDs. We can derive the similar property by combining the above models that have the achievable Holevo bounds. The following examples show models of a tensor product of a one-dimensional model and quantum Gaussian shift models or pure states model that have achievable maximum logarithmic derivative bounds.

Example 4.5. Let
\[
\{\sigma^{(i)}_\eta; \eta_1 < \eta < \eta_2\} \tag{4.42}
\]
be any one-dimensional family of density operators on a Hilbert space \(\mathcal{H}_1\) parameterized by \(\eta \in \mathbb{R}\), and let
\[
\{\sigma^{(2)}_\xi; \xi \in \mathbb{R}^2\} \tag{4.43}
\]
be a two-dimensional family of quantum Gaussian states, where \(\sigma^{(2)}_\xi\) is a quantum Gaussian state represented on a Hilbert space \(\mathcal{H}_2\) defined by a characteristic function
\[
\varphi^{(2)}_\xi(\zeta) = \text{Tr}\sigma^{(2)}_\xi e^{\sqrt{-1}\zeta^T\xi} = e^{-\frac{1}{4}(\xi^2 + \sqrt{-1}\zeta^T\xi)} \quad (\zeta \in \mathbb{R}^2) \tag{4.44}
\]
with \(s \geq 1\) and canonical observables \(X_1, X_2\) such that
\[
[X_1, X_2] = 2\sqrt{-1}I. \tag{4.45}
\]
Let us consider a three-dimensional quantum statistical model
\[
\{\hat{\rho}(\eta, \xi) = \sigma^{(1)}_\eta \otimes \sigma^{(2)}_\xi; \eta_1 < \eta < \eta_2, \xi \in \mathbb{R}^2\}. \tag{4.46}
\]
Since it is known that \(\tilde{D}_1^{(2)} = \frac{1}{\sqrt{2}}X_i (i = 1, 2)\) are the SLDs of \(\sigma^{(2)}_\xi\) and their tangent space is \(\mathcal{D}_\xi\) invariant, the SLD tangent space of this three-dimensional model is \(\mathcal{D}_{(\eta, \xi)}\) invariant at every \((\eta, \xi)\). Therefore, Theorem 4.1 and 4.2 are applicable for any two-dimensional submodel \(\{\hat{\rho}_\theta; \theta \in \Theta \subset \mathbb{R}^2\}\) of \(\{\hat{\rho}(\eta, \xi)\}_{(\eta, \xi)}\) and the maximum logarithmic derivative bound and the Holevo bound can be calculated explicitly. Furthermore, we can show that the maximum logarithmic derivative bound can be achieved. Let \(D_1, D_2\) be SLDs of \(\rho_\theta\) at \(\theta = \Theta\), let \(D_3\) and \(R\) be an observable and a 3 \times 3 matrix obtained in the same way as (4.1). Note that \(D_1, D_2, D_3\) are in \(\text{span}_\mathbb{R} \{\tilde{D}_1^{(1)} \otimes I, I \otimes D_1^{(2)}, I \otimes D_2^{(2)}\}\), where \(\tilde{D}_1^{(1)}\) is the SLD of \(\sigma^{(1)}_\eta\) and \(\tilde{D}_i^{(2)} = \frac{1}{\sqrt{2}}X_i (i = 1, 2)\) are the SLDs of \(\sigma^{(2)}_\xi\). Due to Theorem 4.1 the maximum of \(C^{(\beta)}_{\theta, \xi}\) is achieved when \(\beta = \beta^* := \min \{1, \frac{\text{Tr}|V_{\sqrt{-1}\xi}|}{2\text{Tr}|G(\beta)|}\}\).
for any weight matrix $G$. By using (4.35), it can be seen that the minimization of (4.14) is achieved when $B_i = F_{j_i}^{(\beta)} := \sum_{j=1}^{3} F_{ij}^{(\beta)} D_j$ $(i = 1, 2)$. Note that $B_1, B_2$ satisfy a commutation relation

$$[B_1, B_2] = -2\sqrt{-1}\text{Im}Z(B)I_2.$$

Let $\sigma^{(3)}$ be another ancilla Gaussian states defined by a characteristic function

$$\phi^{(3)}(\xi) = \text{Tr} \sigma^{(3)} e^{-\sqrt{-1}Y_1} = e^{-\frac{1}{2} \xi^T \gamma(3) \xi} \quad (\xi \in \mathbb{R}^2)$$

with canonical observables $Y_1, Y_2$ such that

$$[Y_1, Y_2] = 2\sqrt{-1}\text{Im}Z(B)I_2$$

and a real positive matrix

$$V^{(3)} = \sqrt{G}^{-1} \left| \sqrt{G} \text{Im}Z(B) \sqrt{G} \right| \sqrt{G}^{-1}.$$ (4.50)

It can be seen that two observables $\hat{B}_i := \theta_i^* + B_i \otimes I + I \otimes Y_i$ $(i = 1, 2)$ can be measured simultaneously because they are commutative. Further, these observables satisfy locally unbiased conditions and achieve the Holevo bound, i.e.,

$$\begin{align*}
\text{Tr} (\rho_\theta \otimes \sigma^{(3)}) \hat{B}_i & = \theta_i^* \quad (1 \leq i \leq 2) \quad (4.51) \\
\text{Tr} (\partial \rho_\theta \otimes \sigma^{(3)}) \hat{B}_i & = \delta_{ij} \quad (1 \leq i, j \leq 2) \quad (4.52) \\
\text{Tr} (\rho_\theta \otimes \sigma^{(3)}) (\hat{B}_i - \theta_i^*) (\hat{B}_j - \theta_j^*) & = (\text{Re} Z(B) + V^{(3)})_{ij} \quad (1 \leq i, j \leq 2). \quad (4.53)
\end{align*}$$

**Example 4.6.** Let

$$\left\{ \sigma^{(1)}_\eta : \eta_1 < \eta < \eta_2 \right\}$$

be any one-dimensional family of density operators on a Hilbert space $\mathcal{H}_1$ parameterized by $\eta \in \mathbb{R}$, and let

$$\left\{ \sigma^{(2)}_\xi : |\psi_\xi\rangle \langle \psi_\xi| : \xi \in \Xi \subset \mathbb{R}^2 \right\}$$

be a two-dimensional family of pure states on a Hilbert space $\mathcal{H}_2$ with an open set $\Xi \subset \mathbb{R}^2$. Let us consider a three-dimensional quantum statistical model

$$\left\{ \tilde{\rho}_{(\eta, \xi)} = \sigma^{(1)}_\eta \otimes \sigma^{(2)}_\xi : \eta_1 < \eta < \eta_2, \xi \in \Xi \subset \mathbb{R}^2 \right\}.$$ (4.56)

Suppose $\text{span}\mathcal{D}_{\eta_0} \left\{ \tilde{\rho}_{(\eta, \xi)} = |\psi_\xi\rangle \langle \psi_\xi| + |\tilde{\psi}_{\eta_0}\rangle \langle \tilde{\psi}_{\eta_0}| \right\}_{\xi = 1}^2$ is $\mathcal{D}_{\eta_0}$ invariant at a fixed point $\xi_0$. It can be seen that $\mathcal{D}_{\eta_0}$ invariance for $\left\{ \sigma^{(2)}_\xi \right\}_{\xi = 1}^2$ is equivalent to $|\tilde{\psi}_{\eta_0}\rangle \in \text{span}\mathcal{R} \left\{ |\partial \psi_{\eta_0}\rangle, \sqrt{-1} |\partial \psi_{\eta_0}\rangle \right\}$. Since this three-dimensional model is also $\mathcal{D}_{(\eta_0, \xi_0)}$ invariant at a fixed point $(\eta_0, \xi_0)$, Theorems 4.1 and 4.2 are applicable for two-dimensional submodel $\left\{ \rho_\theta : \theta \in \mathcal{Q} \subset \mathbb{R}^2 \right\}$ of $\left\{ \tilde{\rho}_{(\eta, \xi)} : (\eta, \xi) \right\}$ at $\theta_0$ such that $\rho_{\theta_0} = \tilde{\rho}_{(\eta_0, \xi_0)}$ and the maximum logarithmic derivative bound and the Holevo bound can be calculated explicitly. Furthermore, we can show that the maximum logarithmic derivative bound can be achieved. Let $D_1, D_2$ be SLDs of $\rho_{\theta_0}$ at $\theta = \theta_0$, let $D_3$ and $R$ be an observable and a $3 \times 3$ matrix obtained in the same way as (4.1). Due to Theorem 4.1, the maximum of $C_{B, G}^{(\beta)}$ is achieved when $\beta = \beta^* := \min \left\{ i, \frac{\text{Tr} \left| \sqrt{G} \text{Im} \sqrt{G} \right|}{2\sqrt{\text{Im}G_{\beta G}}} \right\}$ for any weight matrix $G$. By using (4.35), it can be seen that the minimization of (4.14) is achieved when $B_i = B_i^{(\beta)} := \sum_{j=1}^{3} F_{ij}^{(\beta)} D_j$ $(i = 1, 2)$. Because $D_1, D_2, D_3$ are in $\text{span}\mathcal{D}_{\eta_0} \left\{ \tilde{D}_{(1)} \otimes I, I \otimes \tilde{D}_{(2)}, I \otimes \tilde{D}_{(2)} \right\}$, where $\tilde{D}_{(1)}$ is the SLD of $\sigma^{(1)}_\eta$ and $\tilde{D}_{(2)} = 2 \partial \sigma^{(2)}_\xi$ $(i = 1, 2)$ are the SLDs of $\sigma^{(2)}_\xi$, there exist $1 \times 2$ and $2 \times 2$ real
matrix $F^{(1)}$ and $F^{(2)}$ such that $B_i^{(β)} = F_{ii}^{(1)} \hat{D}^{(1)} \otimes I + \sum_{j=1}^{2} F_{ij}^{(2)} \otimes \hat{D}^{(2)}$. Let $B_i^{(1)} := F_{ii}^{(1)} \hat{D}^{(1)} \in \mathcal{B}(\mathcal{H}_1)\otimes \mathcal{B}(\mathcal{H}_2)$ and $B_i^{(2)} := \sum_{j=1}^{2} F_{ij}^{(2)} \otimes \hat{D}^{(2)} \in \mathcal{B}(\mathcal{H}_2)$, and let $Z_i^{(1)} = \text{Tr} \sigma_0^{(1)} B_j^{(1)} B_i^{(1)}$ and $Z_i^{(2)} = \text{Tr} \sigma_0^{(2)} B_j^{(2)} B_i^{(2)}$. Because
\[ \{B_i^{(1)} \otimes I\}_{i=1}^{2} \text{ and } \{I \otimes B_i^{(2)}\}_{i=1}^{2} \text{ are independent}, \]
$Z(B^{(β)}) = Z^{(1)} + Z^{(2)}$. Note that $Z^{(1)}$ is a real matrix. Let
\[ \{\psi^{(3)}\}, \{\hat{I}_1^{(3)}\}, \{\hat{I}_2^{(3)}\} \in \mathcal{H}_3 \] be vectors in a Hilbert space $\mathcal{H}_3 = \mathbb{C}^2$ such that
\[ \langle \psi^{(3)} | \hat{I}_1^{(3)} \rangle = \langle \psi^{(3)} | \hat{I}_2^{(3)} \rangle = 0, \]
and $\|\psi^{(3)}\| = 1$ with a positive real matrix $V^{(3)} = \sqrt{G^{-1}} |\sqrt{G} \text{Im} Z^{(2)} | \sqrt{G}^{-1}$. Because
\[ \hat{I}_i^{(3)} := |\psi^{(3)}\rangle \langle \psi^{(3)}| \] (4.58)
\[ \langle \hat{I}_i^{(3)} | \hat{I}_j^{(3)} \rangle = 0 \quad (i \neq j), \quad \langle \hat{I}_i^{(3)} | \hat{I}_i^{(3)} \rangle = \left( \text{Re} Z^{(2)} + V^{(3)} \right)_{ij} \in \mathbb{R}, \]
satisfy $\langle \psi^{(3)} | \hat{I}_i^{(3)} \rangle = \langle \theta_i | \hat{I}_i^{(3)} \rangle = 0$ and $\langle \hat{I}_j^{(3)} | \hat{I}_i^{(3)} \rangle = \left( \text{Re} Z^{(2)} + V^{(3)} \right)_{ij} \in \mathbb{R}$, there exist an orthonormal basis $\{ |k\rangle \}_{k=1}^{\dim \mathcal{H}_2 \otimes \mathcal{H}_3}$ of $\mathcal{H}_2 \otimes \mathcal{H}_3$ such that $\langle k | \psi^{(3)} \rangle, \langle k | \hat{I}_1^{(3)} \rangle, \langle k | \hat{I}_2^{(3)} \rangle$ are real numbers and $\langle k | \psi^{(3)} \rangle \neq 0$ for $1 \leq k \leq \dim \mathcal{H}_2 \otimes \mathcal{H}_3$. It can be seen that two observables
\[ \hat{B}_i = \theta_i^{(1)} + B_i^{(1)} \otimes I + I \otimes \sum_{k=1}^{\dim \mathcal{H}_2 \otimes \mathcal{H}_3} \frac{\langle k | \hat{I}_1^{(3)} \rangle}{\langle k | \psi^{(3)} \rangle} |k\rangle \langle k| \]
(4.60)
(i = 1, 2) can be measured simultaneously, and they satisfy locally unbiased conditions and achieve the Holevo bound, i.e.,
\[ \text{Tr} (\rho_{\theta_i} \otimes \sigma^{(3)}) \hat{B}_i = \theta_i^{(1)} \quad (1 \leq i \leq 2) \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ endangered.

5 Conclusion

In this paper, we focused on a logarithmic derivative $L^{(β)}_i$ lies between SLD $L^{(S)}_i$ and RLD $L^{(R)}_i$ with $β \in [0, 1]$ to obtain lower bounds of weighted trace of covariance $\text{Tr} GV_{\theta} [M, \hat{θ}]$ of a locally unbiased estimator $(M, \hat{θ})$ at $θ_0$ of a parametric family of quantum states. We showed that all monotone metrics induce lower bounds of $\text{Tr} GV_{\theta} [M, \hat{θ}]$, and the maximum logarithmic derivative bound $\max_{0 \leq β \leq 1} C_{\theta_0,G}^{(β)}$ is the largest bound among them. We showed that $\max_{0 \leq β \leq 1} C_{\theta_0,G}^{(β)}$ has explicit solution when the $d$ dimensional model has $d + 1$ dimensional real space $\mathcal{F} \supset \text{span}_R \{L^{(S)}_i\}_{i=1}^d$ such that $\mathcal{F}_{\theta_0} \subset \mathcal{F}$ at $θ_0 \in Θ$. Furthermore, when $d = 2$, we showed that the maximization problem $\max_{0 \leq β \leq 1} C_{\theta_0,G}^{(β)}$ is the Lagrangian duality of the minimization problem to define Holevo bound, and is the same as the Holevo bound. This explicit solution is the generalization of the solution [115] given for a two dimensional Hilbert space.

Acknowledgment

The author is grateful to Prof. A. Fujiwara for valuable comments.
A Proof of (1.10)

In this appendix, we give a proof of (1.10).

Lemma A.1. For a $d \times d$ positive complex matrix $J$ and a real positive matrix $G$,

$$\min \{ \text{Tr} GV; V \text{ is a } d \times d \text{ real matrix such that } V \geq J \} = \text{Tr} GJ + \text{Tr} \sqrt{G} \text{Im} J \sqrt{G}. \quad (A.1)$$

Proof. Let $\{|i\rangle\}_{i=1}^{d}$ be normalized eigenvectors of $\sqrt{G} \text{Im} J \sqrt{G}$. For a $d \times d$ real matrix $V$ such that $V \geq J$, because

$$\sqrt{G}V \sqrt{G} \geq \sqrt{G} \text{Re} J \sqrt{G} \pm \sqrt{-1} \sqrt{G} \text{Im} J \sqrt{G}, \quad (A.2)$$

we have

$$\langle i | \sqrt{G}V \sqrt{G} | i \rangle \geq \langle i | \sqrt{G} \text{Re} J \sqrt{G} | i \rangle + \left| \langle i | \sqrt{G} \text{Im} J \sqrt{G} | i \rangle \right|. \quad (A.3)$$

Therefore we obtain the inequality

$$\sum_{i=1}^{d} \langle i | \sqrt{G}V \sqrt{G} | i \rangle = \text{Tr} \sqrt{G}V \sqrt{G} \quad (A.4)$$

$$\geq \sum_{i=1}^{d} \left\{ \langle i | \sqrt{G} \text{Re} J \sqrt{G} | i \rangle + \left| \langle i | \sqrt{G} \text{Im} J \sqrt{G} | i \rangle \right| \right\} \quad (A.5)$$

$$= \text{Tr} \sqrt{G} \text{Re} J \sqrt{G} + \text{Tr} \sqrt{G} \text{Im} J \sqrt{G}. \quad (A.6)$$

The equality is achieved when

$$V = \text{Re} J + \sqrt{G^{-1}} \sqrt{G} \text{Im} J \sqrt{G} \sqrt{G^{-1}}. \quad (A.7)$$

B Derivation of Holevo bound

In this appendix, it is proved briefly that the Holevo bound is lower than the weighted trace of the covariance of any unbiased estimator.

Theorem B.1. Let $\mathcal{S} = \{ \rho_\theta; \theta \in \Theta \subset \mathbb{R}^d \}$ be a smooth parametric family of density operators on a finite dimensional Hilbert space $H$. For a locally unbiased estimator $(M, \hat{\theta})$ at $\theta_0$ and a $d \times d$ positive real matrix $G$,

$$\text{Tr} GV_0[M, \hat{\theta}] \geq C^{(H)}_{\theta_0, G}, \quad (B.1)$$

where $C^{(H)}_{\theta_0, G}$ is the Holevo bound defined by (1.12).

Proof. Let

$$X_{M,i} := \sum_{x \in \mathcal{X}} (\hat{\theta}^i(x) - \theta_{0,i}) M_x. \quad (B.2)$$

It follows that

$$V_0[M, \hat{\theta}]_{ij} = Z(X_{M})_{ij} = \sum_{x \in \mathcal{X}} \text{Tr} \left[ (\hat{\theta}^i(x) - \theta_{0,i} - X_{M,i}) \rho_0 \left( (\hat{\theta}^j(x) - \theta_{0,j} - X_{M,j}) M_x \right) \right], \quad (B.3)$$

thus

$$V_0[M, \hat{\theta}] \geq Z(X_{M}). \quad (B.4)$$

Further, $X_{M,i}$ satisfies $\text{Tr} \partial_i \rho_0 X_{M,i} = \delta_i^j$. Then (B.1) is proved. \[\square\]
C Schur complement

In this paper, we utilize Schur complement. Suppose $A_1, A_2, A_3$ are $p \times p, p \times q, q \times q$ complex matrices such that $A_3$ is invertible. The Schur complement of the block $A_3$ of $A = \begin{pmatrix} A_1 & A_2 \\ A_2 & A_3 \end{pmatrix}$ is defined by

$$A/A_3 := A_1 - A_2^* A_3^{-1} A_2. \quad (C.1)$$

The matrix $A$ can be decomposed as

$$A = \begin{pmatrix} A_1 & A_2 \\ A_2 & A_3 \end{pmatrix} = \begin{pmatrix} I & A_2 A_3^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} A/A_3 & 0 \\ 0 & A_3 \end{pmatrix} \begin{pmatrix} I & 0 \\ A_3^{-1} A_2 & I \end{pmatrix}. \quad (C.2)$$

Therefore $\text{rank} A = \text{rank} A_3$ if and only if $A/A_3 = 0$. When $A$ is invertible, $A/A_3$ is also invertible and

$$A^{-1} = \begin{pmatrix} (A/A_3)^{-1} & -(A/A_3)^{-1} A_2 A_3^{-1} \\ -A_3^{-1} A_2 (A/A_3)^{-1} & A_3^{-1} + A_3^{-1} A_2 (A/A_3)^{-1} A_2 A_3^{-1} \end{pmatrix}. \quad (C.2)$$

D Multiple inner products and $\mathcal{D}$ invariant space

In quantum statistics, multiple inner products are used. The commutation operator $\mathcal{D}$ can link the multiple monotone metrics. In general, any inner product $(\cdot, \cdot)$ on a Hilbert space $\mathcal{H}$ with a fixed inner product $(\cdot, \cdot)$ has a positive operator $S$ uniquely such that $(v, w) = (v, Sw)$ for $v, w \in \mathcal{H}$. For a strictly positive operator $S$ on the Hilbert space $\mathcal{H}$, the following Lemma holds.

**Lemma D.1.** Let $\mathcal{K}_0 \subset \mathcal{K}$ be a linear subspace of $\mathcal{H}$, and let $\{e_i\}_{i=1}^d$ be a basis of $\mathcal{K}_0$. The following conditions are equivalent:

(i) $S(\mathcal{K}_0) = \mathcal{K}_0$.

(ii) $K^{(3)} = K^{(2)} K^{(1)} K^{(2)}$-1, where $K^{(1)}, K^{(2)}, K^{(3)}$ are $d \times d$ matrix defined by $K_{ij}^{(1)} = \langle e_i, S e_j \rangle, K_{ij}^{(2)} = \langle e_i, e_j \rangle, K_{ij}^{(3)} = \langle e_i, S^{-1} e_j \rangle$.

**Proof.** The Gram matrix of $\{S e_i\}_{i=1}^d \cup \{S^{-1} e_i\}_{i=1}^d$ with respect to the inner product $\langle \cdot, \cdot \rangle$ is

$$K = \begin{pmatrix} K_{ij}^{(1)} & K_{ij}^{(2)} \\ K_{ij}^{(2)} & K_{ij}^{(3)} \end{pmatrix}. \quad (D.1)$$

The condition (i) is equivalent to $\text{rank} K = d$ because $\dim \left\{ \text{span} \{S e_i\}_{i=1}^d \cup \{S^{-1} e_i\}_{i=1}^d \right\} = d$, and is equivalent to

$$K/K^{(3)} = K^{(1)} - K^{(2)} K^{(3)}^{-1} K^{(2)} = 0, \quad (D.2)$$

where $K/K^{(3)}$ is the Schur complement given in Appendix C.

By using Lemma D.1, we can obtain a useful property of $\mathcal{D}$ invariant space. Let $\mathcal{S} = \{\rho_\theta; \theta \in \Theta \subset \mathbb{R}^d\}$ be a smooth parametric family of density operators on a finite dimensional Hilbert space $\mathcal{H}$. Let $\mathcal{D}_{\rho_0} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be the commutation operator with respect to a faithful state $\rho_0 \in \mathcal{S}$. Through a positive super operator $I + \sqrt{-1} \beta \mathcal{D}_{\rho_0}$, the $\beta$ logarithmic derivatives $\{L^{(\beta)}_{i=1}^d\}$ and the corresponding inner product $\langle \cdot, \cdot \rangle^{(\beta)}_{\rho_0}$ are linked by (3.3) and (3.4). From these relations, we have the following lemma.

**Lemma D.2.** The following conditions are equivalent:
Therefore (1.23).

In this appendix, we show that the explicit form (1.15) given for $\dim \mathcal{H}$ can be transformed into an inequality (1.23), for any $\beta \in [0, 1]$.

Proof. At first, let us prove (i)$\iff$(ii). Because super operators $\mathcal{D}_{\rho_0}$ and $(\frac{L_{\rho_0} + R_{\rho_0}}{2})^{-1}$ are commutative,

$$
\mathcal{D}_{\rho_0} \left( \partial \rho_0 \right) = \mathcal{D}_{\rho_0} \left( \partial \rho_0 \right) \circ \mathcal{D}_{\rho_0} \left( \frac{L_{\rho_0} + R_{\rho_0}}{2} \right)^{-1} \left( \partial \rho_0 \right) = \mathcal{D}_{\rho_0} \left( L_i^{(S)} \right).
$$

Therefore $\mathcal{D}_{\rho_0} \left( \partial \rho_0 \right) \in \text{span}_R \left\{ \partial \rho_0 \right\}_{i=1}$ if and only if $\mathcal{D}_{\rho_0} \left( L_i^{(S)} \right) \in \text{span}_R \left\{ L_i^{(S)} \right\}_{i=1}$.

The proof of (ii)$\iff$(ii)$'$ is trivial because $\mathcal{D}_{\rho_0} \left( X \right)$ is self-adjoint for any self-adjoint operator $X$.

The proof of (ii)$'$ is also trivial because of (3.3).

The proof of (iii)$\iff$(iv) is given by Lemma 9.1 with a positive operator $I + \sqrt{-1} \beta \mathcal{D}_{\rho_0}$ because

$$
\text{Re} Z_{ij} + \beta \sqrt{-1} \text{Im} Z_{ij} = \langle L_i^{(S)}, (I + \sqrt{-1} \beta \mathcal{D}_{\rho_0}) L_j^{(S)} \rangle^{(0)}.
$$

$$
J_{\theta_0}^{(S)} = \langle L_i^{(S)}, L_j^{(S)} \rangle^{(0)}.
$$

E. Relation between the bounds (1.15) and (1.23)

In this appendix, we show that the explicit form (1.15) given for $\dim \mathcal{H} = 2$ and $d = 2$ can be derived from (1.23).

The inequality condition in (1.23)

$$
\hat{\beta} = \frac{\text{Tr} \left| \sqrt{\text{Im} J_{\theta_0}^{(R)^{-1}} \sqrt{G}} \right|}{2 \text{Tr} G \left\{ J_{\theta_0}^{(S)^{-1}} - \text{Re} (J_{\theta_0}^{(R)^{-1}}) \right\}} \geq 1
$$

can be transformed into an inequality

$$
2 \text{Tr} G \text{Re} (J_{\theta_0}^{(R)^{-1}}) + 2 \text{Tr} \left| \sqrt{\text{Im} J_{\theta_0}^{(R)^{-1}} \sqrt{G}} \right| \geq 2 \text{Tr} G J_{\theta_0}^{(S)^{-1}} + \left| \sqrt{\text{Im} J_{\theta_0}^{(R)^{-1}} \sqrt{G}} \right|.
$$

The left hand side of (E.2) is equal to $2 \tilde{c}_{\theta_0,G}^{(R)}$. The right hand side of (E.2) is equal to $c_{\theta_0,G}^{(S)} + c_{\theta_0,G}^{(Z)}$ because

$$
\text{Tr} G J_{\theta_0}^{(S)^{-1}} + \left| \sqrt{\text{Im} J_{\theta_0}^{(R)^{-1}} \sqrt{G}} \right| = \text{Tr} G Z(L^{(S)}) + \text{Tr} \left| \sqrt{\text{Im} Z(L^{(S)}) \sqrt{G}} \right| = c_{\theta_0,G}^{(Z)}.
$$
where

$$Z(L^{(S)}) = A = J_{b_0}^{(S)^{-1}} + \sqrt{-1} \text{Im}(J_{b_0}^{(R)^{-1}})$$

(E.4)
given in (4.4) is used. Therefore the inequality (E.2) is equivalent to the inequality condition

$$C_{\theta_b,G}^{(R)} \geq \frac{C_{\theta_b,G}^{(Z)} + C_{\theta_b,G}^{(S)}}{2}$$

(E.5)
in (1.15).

Further, by using (4.3), (4.4), and (4.5), we have

$$C_{\theta_b,G}^{(b)} = \text{Tr} G \text{Re} A + \frac{\beta \text{Tr} \left| \sqrt{G} \text{Im} A \sqrt{G} \right|}{\beta} \langle b | G | b \rangle$$

(E.6)

$$= \text{Tr} G \text{Re} A + \frac{\text{Tr} \left| \sqrt{G} \text{Im} A \sqrt{G} \right|}{2 \langle b | G | b \rangle} \text{Re} A \left| \sqrt{G} \text{Im} A \sqrt{G} \right| - \text{Tr} \left( \frac{\text{Tr} \left| \sqrt{G} \text{Im} A \sqrt{G} \right|}{2 \langle b | G | b \rangle} \right)^2 \langle b | G | b \rangle$$

(E.7)

$$= \text{Tr} G \text{Re} A + \frac{\left( \text{Tr} \left| \sqrt{G} \text{Im} A \sqrt{G} \right| \right)^2}{4 \langle b | G | b \rangle}$$

(E.8)

$$= \text{Tr} G \text{Re} A - \langle b | G | b \rangle + \text{Tr} \left| \sqrt{G} \text{Im} A \sqrt{G} \right| + \frac{\left( \text{Tr} \left| \sqrt{G} \text{Im} A \sqrt{G} \right| \right)^2}{4 \langle b | G | b \rangle} - \text{Tr} \left| \sqrt{G} \text{Im} A \sqrt{G} \right| + \langle b | G | b \rangle$$

(E.9)

$$= C_{\theta_0,G}^{(R)} + \frac{\left( \text{Tr} \left| \sqrt{G} \text{Im} A \sqrt{G} \right| - 2 \langle b | G | b \rangle \right)^2}{4 \langle b | G | b \rangle}$$

(E.10)

$$= C_{\theta_0,G}^{(R)} + \frac{\left( \frac{1}{2} \beta \text{Tr} G \text{Re} A + \left| \sqrt{G} \text{Im} A \sqrt{G} \right| \right) - \left( \text{Tr} G \text{Re} A + \langle b | G | b \rangle \right)}{\langle b | G | b \rangle}$$

(E.11)

$$= C_{\theta_0,G}^{(R)} + \frac{\left( \frac{1}{2} \left( C_{\theta_0,G}^{(Z)} + C_{\theta_0,G}^{(S)} \right) - C_{\theta_0,G}^{(R)} \right)^2}{C_{\theta_0,G}^{(Z)} - C_{\theta_0,G}^{(R)}} = C_{\theta_0,G}^{(R)} + S_{\theta_0,G}.$$  

(E.12)

Thus, it is confirmed that (1.15) can be derived from (1.23).

References

[1] A. S. Holevo, Probabilistic and Statistical Aspects of Quantum Theory, 2nd English ed., Edizioni della Normale, Pisa (2011).

[2] K. Yamagata, A. Fujiwara, and R. D. Gill, “Asymptotic Normality based on a new Quantum Likelihood Ratio,” Ann. Stat. 41 (4), 2197–2217 (2013).

[3] A. Fujiwara and K. Yamagata, “Noncommutative Lebesgue decomposition and contiguity with application to quantum local asymptotic normality,” Bernoulli 26 (3), 2105–2142 (2020).

[4] M. Guţă and J. Kahn, “Local asymptotic normality for qubit states,” Phys. Rev. A 73 (5), 052108, 15. MR2229156 (2006).

[5] J. Suzuki, “Explicit formula for the Holevo bound for two-parameter qubit-state estimation problem,” J. Math. Phys. 57, 042201 (2016).

[6] D. Petz, “Monotone metrics on matrix spaces,” Linear Algebra Appl. 244, 81–96 (1996).

[7] K. Yamagata, “Quantum monotone metrics induced from trace non-increasing maps and additive noise,” J. Math. Phys. 61,052202, (2020).
[8] R. Bhatia, Matrix Analysis, Graduate Texts in Mathematics 169, Springer, New York (1997).

[9] P. J. D. Crowley, A. Datta, M. Barbieri, and I. A. Walmsley, “A tradeoff in simultaneous quantum-limited phase and loss estimation in interferometry,” Phys. Rev. A 89, 023845 (2014).

[10] F. Albarelli, J. F. Friel, and A. Datta, “Evaluating the Holevo Cramér-Rao Bound for Multiparameter Quantum Metrology,” Phys. Rev. Lett. 123, 200503 (2019).

[11] F. Albarelli and A. Datta, “Upper bounds on the Holevo Cramér-Rao bound for multiparameter quantum parametric and semiparametric estimation,” arXiv:1911.11036 (2019).

[12] A. Carollo, B. Spagnolo, A. A. Dubkov, and D. Valenti, “On quantumness in multi-parameter quantum estimation,” J. Stat. Mech.: Theory Exp. 2019 (9), 094010, (2019).

[13] K. Matsumoto, “A new approach to the Cramér-Rao-type bound of the pure-state model,” J. Phys. A: Gen. Phys. 35, 3111–3123, (2002).