CONCENTRATION OF 1-LIPSCHITZ MAPS INTO AN INFINITE DIMENSIONAL $\ell^p$-BALL WITH THE $\ell^q$-DISTANCE FUNCTION

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Abstract. In this paper, we study the Lévy-Milman concentration phenomenon of 1-Lipschitz maps into infinite dimensional metric spaces. Our main theorem asserts that the concentration to an infinite dimensional $\ell^p$-ball with the $\ell^q$-distance function for $1 \leq p < q \leq +\infty$ is equivalent to the concentration to the real line.

1. Introduction

This paper is devoted to investigating the Lévy-Milman concentration phenomenon of 1-Lipschitz maps from mm-spaces (metric measure spaces) to infinite dimensional metric spaces. Here, an mm-space is a triple $(X, d_X, \mu_X)$, where $d_X$ is a complete separable metric on a set $X$ and $\mu_X$ is a finite Borel measure on $(X, d_X)$. The theory of concentration of 1-Lipschitz functions was first introduced by V. D. Milman in his investigation of asymptotic geometric analysis ([17], [18], [19]). Nowadays, the theory blends with various areas of mathematics, such as geometry, functional analysis and infinite dimensional integration, discrete mathematics and complexity theory, probability theory, and so on (see [16], [21], [22], [24] and the references therein for further information).

The theory of concentration of maps into general metric spaces was first studied by M. Gromov ([11], [12], [13]). He established the theory by introducing the observable diameter $\text{ObsDiam}_Y(X; -\kappa)$ for an mm-space $X$, a metric space $Y$, and $\kappa > 0$ in [13] (see Section 2 for the definition of the observable diameter). Given a sequence $\{X_n\}_{n=1}^\infty$ of mm-spaces and a metric space $Y$, we note that $\lim_{n \to \infty} \text{ObsDiam}_Y(X_n; -\kappa) = 0$ for any $\kappa > 0$ if and only if for any sequence $\{f_n : X_n \to Y\}_{n=1}^\infty$ of 1-Lipschitz maps, there exists a sequence $\{m_{f_n}\}_{n=1}^\infty$ of points in $Y$ such that

$$\lim_{n \to \infty} \mu_{X_n}(\{x_n \in X_n \mid d_Y(f_n(x_n), m_{f_n}) \geq \varepsilon\}) = 0$$
for any $\varepsilon > 0$. If $\lim_{n \to \infty} \text{ObsDiam}_{g_r}(X_n; -\kappa) = 0$ for any $\kappa > 0$, then the sequence
\[ \{X_n\}_{n=1}^\infty \]
of mm-spaces is called a Lévy family. The Lévy families were first introduced and analyzed by Gromov and Milman in [10]. In previous works [2], [3], [4], [5], the author proved that if a metric space $Y$ is either an $\mathbb{R}$-tree, a doubling space, a metric graph, or a Hadamard manifold, then $\lim_{n \to \infty} \text{ObsDiam}_{Y}(X_n; -\kappa) = 0$ holds for any $\kappa > 0$ and any Lévy family $\{X_n\}_{n=1}^\infty$. To prove these results, we needed to assume the finiteness of the dimension of the target metric spaces.

In this paper, we treat the case where the dimension of the target metric space $Y$ is infinite. The author has proved in [1] that if the target space is infinite. The author has proved in [1] that if the target metric space $Y$ is an mm-space with some homogeneity property can isometrically be embedded into $Y$, then its observable diameter $\text{ObsDiam}_Y(X; -\kappa)$ is not close to zero. It seems from this result that the concentration to an infinite dimensional metric space cannot happen easily.

A main theorem of this paper is the following. For $1 \leq p \leq +\infty$, we denote by $B^p$ an infinite dimensional $\ell^p$-ball $\{(x_n)_{n=1}^\infty \in \mathbb{R}^\infty \mid \sum_{n=1}^\infty |x_n|^p \leq 1\}$ and by $d_{\ell^p}$ the $\ell^p$-distance function.

**Theorem 1.1.** Let $\{X_n\}_{n=1}^\infty$ be a sequence of mm-spaces and $1 \leq p < q \leq +\infty$. Then, the sequence $\{X_n\}_{n=1}^\infty$ is a Lévy family if and only if
\[ \lim_{n \to \infty} \text{ObsDiam}_{(B^p, d_{\ell^p})}(X_n; -\kappa) = 0 \text{ for any } \kappa > 0. \]

As a result, we obtain the example of the infinite dimensional target metric space such that the concentration to the space happens as often as the concentration to the real line.

The proof of the sufficiency of Theorem 1.1 is easy. The observations of A. Gournay and M. Tsukamoto play important roles for the proof of the converse ([9], [28]). Answering a question of Gromov in [14, Section 1.1.4], Tsukamoto proved in [28] that the “macroscopic” dimension of the space $(B^p, d_{\ell^p})$ for $1 \leq p < q \leq +\infty$ is finite. Gournay independently proved it in [9] in the case of $q = +\infty$. For any $(p, q) \in \{(k, l) \in \mathbb{N} \times \mathbb{N} \mid l \leq k\} \setminus \{(k, 1) \mid k \geq 2\}$, we have an example of a Lévy family which does not satisfy (1.1) (see Proposition 3).

As applications of Theorem 1.1 by virtue of [3] Propositions 4.3 and 4.4, we obtain the following corollaries of a Lévy group action. Lévy groups were first introduced by Gromov and Milman in [10]. Let a topological group $G$ act on a metric space $X$. The action is called bounded if for any $\varepsilon > 0$ there exists a neighborhood $U$ of the identity element $e_G \in G$ such that $d_X(x, gx) < \varepsilon$ for any $g \in U$ and $x \in X$. Note that every bounded action is continuous. We say that the topological group $G$ acts on $X$ by uniform isomorphisms if for each $g \in G$, the map $X \ni x \mapsto gx \in X$ is uniformly continuous. The action is said to be uniformly equicontinuous if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $d_X(gx, gy) < \varepsilon$ for every $g \in G$ and $x, y \in X$ with $d_X(x, y) < \delta$. Given a subset $S \subseteq G$ and $x \in X$, we put $Sx := \{gx \mid g \in S\}$.

**Corollary 1.2.** Let $1 \leq p < q \leq +\infty$ and assume that a Lévy group $G$ boundedly acts on the metric space $(B^p, d_{\ell^p})$ by uniform isomorphisms. Then for any compact subset $K \subseteq G$ and any $\varepsilon > 0$, there exists a point $x_{\varepsilon, K} \in B^p$ such that $\text{diam}(K_{x_{\varepsilon, K}}) \leq \varepsilon$.

**Corollary 1.3.** There are no non-trivial bounded uniformly equicontinuous actions of a Lévy group to the metric space $(B^p, d_{\ell^p})$ for $1 \leq p < q \leq +\infty$.
Gromov and Milman pointed out in [10] that the unitary group $U(\ell^2)$ of the separable Hilbert space $\ell^2$ with the strong topology is a Lévy group. Many concrete examples of Lévy groups are known by the works of S. Glasner [8], H. Furstenberg and B. Weiss (unpublished), T. Giordano and V. Pestov [6, 7], and Pestov [25]. For examples, groups of measurable maps from the standard Lebesgue measure space, full groups of amenable equivalence relations, and the isometry groups of some von Neumann algebras, groups of measure and measure-class preserving automorphisms of the standard Lebesgue measure space, full groups of amenable equivalence relations, and the isometry groups of the universal Urysohn metric spaces are Lévy groups (see the recent monograph [23] for precise statements).

2. Preliminaries

Let $Y$ be a metric space and $\nu$ a Borel measure on $Y$ such that $m := \nu(Y) < +\infty$. We define for any $\kappa > 0$,
\[
\text{diam}(\nu, m - \kappa) := \inf\{\text{diam } Y_0 \mid Y_0 \subseteq Y \text{ is a Borel subset with } \nu(Y_0) \geq m - \kappa\}
\]
and call it the partial diameter of $\nu$.

**Definition 2.1** (Observable diameter). Let $(X, d_X, \mu_X)$ be an mm-space with $m_X := \mu_X(X)$ and $Y$ a metric space. For any $\kappa > 0$ we define the observable diameter of $X$ by
\[
\text{ObsDiam}_Y(X; -\kappa) := \sup\{\text{diam}(f_*(\mu_X), m_X - \kappa) \mid f : X \to Y \text{ is a 1-Lipschitz map}\},
\]
where $f_*(\mu_X)$ stands for the push-forward measure of $\mu_X$ by $f$.

The idea of an observable diameter comes from quantum and statistical mechanics; that is, we think of $\mu_X$ as a state on a configuration space $X$ and $f$ is interpreted as an observable.

Let $(X, d_X, \mu_X)$ be an mm-space. For any $\kappa_1, \kappa_2 \geq 0$, we define the separation distance $\text{Sep}(X; \kappa_1, \kappa_2) = \text{Sep}(\mu_X; \kappa_1, \kappa_2)$ of $X$ as the supremum of the distance $d_X(A, B) := \inf\{d_X(a, b) \mid a \in A \text{ and } b \in B\}$, where $A$ and $B$ are Borel subsets of $X$ satisfying that $\mu_X(A) \geq \kappa_1$ and $\mu_X(B) \geq \kappa_2$.

**Lemma 2.2** (cf. [13] Section 3). Let $X$ and $Y$ be two mm-spaces and $\alpha > 0$. Assume that an $\alpha$-Lipschitz map $f : X \to Y$ satisfies $f_*(\mu_X) = \mu_Y$. Then we have
\[
\text{Sep}(Y; \kappa_1, \kappa_2) \leq \alpha \text{Sep}(X; \kappa_1, \kappa_2).
\]

Relationships between the observable diameter and the separation distance are the following. We refer to [4] Subsection 2.2 for precise proofs.

**Lemma 2.3** (cf. [13] Section 3). Let $X$ be an mm-space and $\kappa, \kappa' > 0$ with $\kappa > \kappa'$. Then we have
\[
\text{ObsDiam}_X(X; -\kappa') \geq \text{Sep}(X; \kappa, \kappa).
\]

**Remark 2.4.** In [13] Section 3, Lemma 2.3 is stated as $\kappa = \kappa'$, but that is not true in general. For example, let $X := \{x_1, x_2\}$, $d_X(x_1, x_2) := 1$, and $\mu_X(\{x_1\}) = \mu_X(\{x_2\}) := 1/2$. Putting $\kappa = \kappa' = 1/2$, we have $\text{ObsDiam}_X(X; -1/2) = 0$ and $\text{Sep}(X; 1/2, 1/2) = 1$. This issue can be removed if in the definition of $\text{diam}(\nu, m - \kappa)$, we ask for $\nu(Y_0) > m - \kappa$ instead of $\nu(Y_0) \geq m - \kappa$.
Lemma 2.5 (cf. [13] Section 3.2.33). Let $\nu$ be a Borel measure on $\mathbb{R}$ with $m := \nu(\mathbb{R}) < +\infty$. Then, for any $\kappa > 0$ we have
\[ \text{diam}(\nu, m - 2\kappa) \leq \text{Sep}(\nu; \kappa, \kappa). \]
In particular, for any $\kappa > 0$ we have
\[ \text{ObsDiam}_{\mathbb{R}}(X; -2\kappa) \leq \text{Sep}(X; \kappa, \kappa). \]

Combining Lemma 2.3 with Lemma 2.5, we obtain the following corollary:

Corollary 2.6 (cf. [13] Section 3.2.33). A sequence $\{X_n\}_{n=1}^{\infty}$ of mm-spaces is a Lévy family if and only if $\lim_{n \to \infty} \text{Sep}(X_n; \kappa, \kappa) = 0$ for any $\kappa > 0$.

Lemma 2.7. Let $\nu$ be a finite Borel measure on $(\mathbb{R}^k, d_{ev})$ with $m := \nu(\mathbb{R}^k)$. Then for any $\kappa > 0$ we have
\[ \text{diam}(\nu, m - \kappa) \leq k^{1/p} \text{Sep}\left(\nu; \frac{\kappa}{2k}, \frac{\kappa}{2k}\right). \]
Proof. For $i = 1, 2, \ldots, k$, let $\text{proj}_i : \mathbb{R}^k \ni (x_i)_{i=1}^k \mapsto x_i \in \mathbb{R}$ be the projection. For any Borel subsets $A_1, A_2, \ldots, A_k \subseteq \mathbb{R}$ with $(\text{proj}_i)_*(\nu)(A_i) \geq m - \kappa/k$, we have
\[ \nu(A_1 \times A_2 \times \cdots \times A_k) = \nu\left(\bigcap_{i=1}^k (\text{proj}_i)^{-1}(A_i)\right) \geq m - \kappa, \]
which leads to
\[ \text{diam}(\nu, m - \kappa) \leq \text{diam}(A_1 \times A_2 \times \cdots \times A_k) \leq k^{1/p} \max_{1 \leq i \leq k} \text{diam} A_i. \]

We therefore get
\[ \text{diam}(\nu, m - \kappa) \leq k^{1/p} \max_{1 \leq i \leq k} \text{diam}(\text{proj}_i)_*(\nu), m - \frac{\kappa}{k}. \]

Combining this with Lemmas 2.2 and 2.5, we obtain
\[ \text{diam}(\nu, m - \kappa) \leq k^{1/p} \max_{1 \leq i \leq k} \text{Sep}(\text{proj}_i)_*(\nu; \frac{\kappa}{2k}, \frac{\kappa}{2k}) \leq k^{1/p} \text{Sep}(\nu; \frac{\kappa}{2k}, \frac{\kappa}{2k}). \]

This completes the proof. \qed

Lemma 2.8. Let $a, b$ be two real numbers with $a < b$. Then, a sequence $\{X_n\}_{n=1}^{\infty}$ of mm-spaces is a Lévy family if and only if
\[ (2.1) \quad \lim_{n \to \infty} \text{ObsDiam}_{[a,b]}(X_n; -\kappa) = 0 \text{ for any } \kappa > 0. \]
Proof. The necessity is obvious. We shall prove the converse. Suppose that the sequence $\{X_n\}_{n=1}^{\infty}$ with the property (2.1) is not a Lévy family. Then, by Corollary 2.6 there exists $\kappa > 0$ and Borel subsets $A_n, B_n \subseteq X_n$ such that $\mu_{X_n}(A_n) \geq \kappa$, $\mu_{X_n}(B_n) \geq \kappa$, and $\limsup_{n \to \infty} d_{X_n}(A_n, B_n) > 0$. Define a function $f_n : X_n \to \mathbb{R}$ by $f_n(x) := \max\{d_{X_n}(x, A_n) + a, b\}$. Since $\mu_{X_n}(B_n) \geq \kappa$ and $\limsup_{n \to \infty} d_{X_n}(A_n, B_n) > 0$, we have
\[ \limsup_{n \to \infty} \text{diam}(f_n)_*(\mu_{X_n}), m_{X_n} - \kappa' > 0 \]
for any $0 < \kappa' < \kappa$. Since each $f_n$ is a 1-Lipschitz function, this contradicts the assumption (2.1). This completes the proof. \qed
3. Proof of the main theorem

To prove the main theorem, we extract arguments from Gournay’s paper [9] and Tsukamoto’s paper [28].

For $k \in \mathbb{N}$, we identify $\mathbb{R}^k$ with the subset \{(x_1, x_2, \ldots, x_k, 0, 0, \ldots) \in \mathbb{R}^\infty \mid x_i \in \mathbb{R}$ for all $i\} \subseteq \mathbb{R}^\infty$. Given $k \in \mathbb{N} \cup \{\infty\}$, let $\mathcal{G}_k$ be the $k$-th symmetric group. We consider the group $G_k := \{\pm1\}^k \rtimes \mathcal{G}_k$. The multiplication in $G_k$ is given by

\[
((\varepsilon_{n})_{n=1}^{k}, \sigma) \cdot (\varepsilon'_{n})_{n=1}^{k}, \sigma') := ((\varepsilon_{n} \varepsilon'_{\sigma^{-1}(n)})_{n=1}^{k}, \sigma \sigma').
\]

The group $G_k$ acts on the space $\mathbb{R}^k$ by

\[
((\varepsilon_{n})_{n=1}^{k}, \sigma) \cdot (x_{n})_{n=1}^{k} := (\varepsilon_n x_{\sigma^{-1}(n)})_{n=1}^{k}.
\]

Note that this action preserves the $k$-dimensional $\ell^p$-ball $B_{\ell^p}^k \subseteq \mathbb{R}^\infty$ and the $\ell^q$-distance function $d_{\ell^q}$. Define a subset $A_k \subseteq B_{\ell^p}^k$ by

\[
A_k := \{x \in B_{\ell^p}^k \mid x_i \geq 0 \text{ for all } i\}.
\]

Given an arbitrary $\varepsilon > 0$, we put $k(\varepsilon) := \lceil(2/\varepsilon)^{pq/(q-p)}\rceil - 1$, where $\lceil(2/\varepsilon)^{pq/(q-p)}\rceil$ denotes the smallest integer which is not less than $(2/\varepsilon)^{pq/(q-p)}$. For $k \geq k(\varepsilon) + 1$, we define a continuous map $f_{k,\varepsilon} : A_k \to \mathbb{R}^{k(\varepsilon)}$ by

\[
f_{k,\varepsilon}(x) := (x_1 - x_{k(\varepsilon)+1}, x_2 - x_{k(\varepsilon)+1}, \ldots, x_k - x_{k(\varepsilon)+1}, 0, 0, \ldots).
\]

For any $x \in B_{\ell^p}^k$, taking $g \in G_k$ such that $gx \in A_k$, we define

\[
F_{k,\varepsilon}(x) := g^{-1}f_{k,\varepsilon}(gx).
\]

This definition of the map $F_{k,\varepsilon} : B_{\ell^p}^k \to B_{\ell^p}^k$ is well-defined (see [28] Section 2) for details. Given $k \in \mathbb{N}$, we put $A_k := \bigcup_{g \in G_k} g\mathbb{R}^k \subseteq \mathbb{R}^\infty$.

**Theorem 3.1** (cf. [9] Proposition 1.3 and [28] Section 2). The map $F_{k,\varepsilon} : B_{\ell^p}^k \to B_{\ell^p}^k$ satisfies that $F_{k,\varepsilon}(B_{\ell^p}^k) \subseteq A_{k(\varepsilon)}$ and

\[
d_{\ell^q}(x, F_{k,\varepsilon}(x)) \leq \frac{\varepsilon}{2}
\]

for any $x \in B_{\ell^p}^k$.

**Lemma 3.2.** The map $F_{k,\varepsilon} : (B_{\ell^p}^k, d_{\ell^q}) \to (A_{k(\varepsilon)}, d_{\ell^q})$ is a $(1 + k(\varepsilon)^{1/q})$-Lipschitz map.

**Proof.** By the definition of the map $F_{k,\varepsilon}$, it suffices to prove that the map $F := F_{2k(\varepsilon)+2,\varepsilon} : (B_{\ell^p}^{2k(\varepsilon)+2}, d_{\ell^q}) \to (B_{\ell^p}^{2k(\varepsilon)+2}, d_{\ell^q})$ is $(1 + k(\varepsilon)^{1/q})$-Lipschitz. Recall that

\[
F(x) := (x_1 - x_{k(\varepsilon)+1}, x_2 - x_{k(\varepsilon)+1}, \ldots, x_k - x_{k(\varepsilon)+1}, 0, 0, \ldots, 0)
\]

for any $x \in A_{2k(\varepsilon)+2}$. We hence get

\[
d_{\ell^q}(F(x), F(y)) \leq d_{\ell^q}(x, y) + k(\varepsilon)^{1/q}|x_{k(\varepsilon)+1} - y_{k(\varepsilon)+1}| \leq (1 + k(\varepsilon)^{1/q})d_{\ell^q}(x, y)
\]

for any $x, y \in A_{2k(\varepsilon)+2}$. Since each $g \in G_{2k(\varepsilon)+2}$ preserves the distance function $d_{\ell^q}$, the map $F$ is $(1 + k(\varepsilon)^{1/q})$-Lipschitz on each $gA_{2k(\varepsilon)+2}$.

Let $x, y \in B_{\ell^p}^{2k(\varepsilon)+2}$ be arbitrary points. Observe that there exist a division $t_0 := 0 \leq t_1 \leq t_2 \leq \cdots \leq t_i = 1 =: t_i$ and $g_1, g_2, \ldots, g_i \in G_{2k(\varepsilon)+2}$ such that

\[
g_{i+1} \cdots g_2 g_1 x_{k(\varepsilon)+1} = y_{k(\varepsilon)+1}.
\]
Claim 3.4. For any $C \in \mathcal{C}_n$, there exists $g \in G_\infty$ such that

$$C = g\{(x_1, \cdots, x_k) \in \mathbb{R}^k \mid x_i \geq a_n \text{ for each } i\}.$$  

Claim 3.5. There exists $C_n \in \mathcal{C}_n$ such that $\nu_{n,k}(C_n) \geq m_n/6$.

Proof. If $\nu_{n,k}(C) < m_n/6$ for all $C \in \mathcal{C}_n$, then there exists $C'_n \subseteq \mathcal{C}_n$ such that

$$\frac{m_n}{6} \leq \nu_{n,k}\left( \bigcup_{C' \in C'_n} C' \right) < \frac{m_n}{3}.$$
Hence (3.4) \( \nu \) get \( D \) ma 2.7 and Claim 3.6, we therefore obtain

\[
2^{1/q} a_n \leq d_{\ell^q} \left( \bigcup_{C' \in \mathcal{C}_n} C', \bigcup_{C'' \in \mathcal{C}_n''} C'' \right) \leq \text{Sep} \left( \nu_{n,k}; \frac{m_n}{6} \right) < a_n,
\]

which is a contradiction. This proves that Tending

\[
\text{Sep} \left( \nu_{n,k}; \frac{m_n}{6} \right) < d_{\ell^q} (C_n, A_k \setminus (D_n)_\delta) \leq \text{Sep} \left( \nu_{n,k}; \frac{m_n}{6}, \frac{\kappa}{2} \right),
\]

which is a contradiction. This completes the proof of the claim. \( \square \)

Claim 3.6. Putting \( D_n := A_k \cap (C_n \setminus \text{Sep}(\nu_{n,k}; m_n/6, \kappa/2)) \), we have \( \nu_{n,k}(D_n) \geq m_n - \kappa/2 \).

Proof. Take any \( \delta > 0 \). Supposing that \( \nu_{n,k}((D_n)_\delta) < m_n - \kappa/2 \), by Claim 3.5 we get

By using Claim 3.3 there exists \( g_n \in G_{\infty} \) such that

\[
C_n = g_n \{ (x_1, \ldots, x_k) \in \mathbb{R}^k \mid x_i \geq a_n \text{ for each } i \},
\]

Since \( a_n \geq \text{Sep}(\nu_{n,k}, m_n/6, \kappa/2) \), we observe that

\[
D_n \subseteq A_k \cap \{ (x_1, \ldots, x_k) \in \mathbb{R}^k \mid x_i \geq a_n \text{ for each } i \}\)

\[
= A_k \cap g_n \left( \bigcup_{j=1}^{k} \{ (x_1, \ldots, x_k) \in \mathbb{R}^k \mid x_j \geq 0 \text{ and } x_i \geq a_n \text{ for each } i \neq j \} \right)
\]

\[
\cup \bigcup_{j=k+1}^{\infty} \{ (x_1, \ldots, x_k, 0, \ldots, 0, x_j, 0, 0, \ldots) \in \mathbb{R}^\infty \mid |x_j| \leq a_n \text{ and } x_i \geq a_n \text{ for each } i = 1, \ldots, k \}\)
\]

\[
\subseteq g_n \mathbb{R}^k.
\]

Hence \( D_n \) is isometrically embedded into the \( \ell^q \)-space \( (\mathbb{R}^k, d_{\ell^q}) \). Combining Lemma 2.7 and Claim 3.6 we therefore obtain

\[
diam(\nu_{n,k}, m_n - \kappa) \leq diam(\nu_{n,k}|_{D_n}, m_n - \kappa)
\]

\[
\leq \text{diam} \left( \nu_{n,k}|_{D_n}, \nu_{n,k}(D_n) - \frac{\kappa}{2} \right)
\]

\[
\leq k^{1/4} \text{Sep} \left( \nu_{n,k}|_{D_n}; \frac{\kappa}{4k}, \frac{\kappa}{4k} \right)
\]

\[
\leq k^{1/4} \text{Sep} \left( \nu_{n,k}; \frac{\kappa}{4k}, \frac{\kappa}{4k} \right) \rightarrow 0 \text{ as } n \rightarrow \infty.
\]

This implies (3.2).

We next consider the case (1). Putting \( b_n := a_n + \text{Sep}(\nu_{n,k}; m_n/2, \kappa/2) \), as in the proof of Claim 3.6 we get

\[
\nu_{n,k}((A_{k-1})_{b_n} \cap A_k) = \nu_{n,k}((B_{n,1})_{\text{Sep}(\nu_{n,k}; m_n/2, \kappa/2)} \cap A_k) \geq m_n - \frac{\kappa}{2}.
\]

Note that there exists a Borel measurable map \( f_n : (A_{k-1})_{b_n} \cap A_k \rightarrow A_{k-1} \) such that

\[
\text{d} e_{\ell^q}(x, f_n(x)) = \min \{ \text{d} e_{\ell^q}(x, y) \mid y \in A_{k-1} \} \leq b_n
\]
Lemma 4.1 (cf. [3, Corollary 2.6])

For any $x \in (A_{k-1})_{b_n} \cap A_k$, put $\nu_{n,k-1} := (f_n)_* (\nu_{n,k}|_{(A_{k-1})_{b_n} \cap A_k})$. An easy calculation proves that

$$\text{Sep} (\nu_{n,k-1}; \kappa_1, \kappa_2) \leq \text{Sep} (\nu_{n,k}; \kappa_1, \kappa_2) + 2b_n$$

for any $\kappa_1, \kappa_2 > 0$. By this and the property (3.2) for $\nu_{n,k}$, the measures $\nu_{n,k-1}$ on $A_{k-1}$ satisfy that

$$\lim_{n \to \infty} \text{Sep} (\nu_{n,k-1}; \kappa_1, \kappa_2) = 0$$

for any $\kappa_1, \kappa_2 > 0$. By the assumption of the induction, we therefore get

$$\lim_{n \to \infty} \text{diam} \left( \nu_{n,k-1}, \nu_{n,k-1}(A_{k-1}) - \frac{\kappa}{2} \right) = 0$$

for any $\kappa > 0$. By using (3.4), we finally obtain

$$\text{diam} (\nu_{n,k}, m_n - \kappa) \leq \text{diam} (\nu_{n,k-1}, m_n - \kappa) + 2b_n$$

$$\leq \text{diam} (\nu_{n,k-1}, \nu_{n,k-1}(A_{k-1}) - \frac{\kappa}{2}) + 2b_n \to 0 \text{ as } n \to \infty.$$

This completes the proof of the lemma. \qed

**Proof of Theorem 1.1**

Lemma 2.8 directly implies the sufficiency of Theorem 1.1. We shall prove the converse. Let $\{f_n : X_n \to (B^\infty_{\ell^p}, d_{\ell^p})\}_{n=1}^\infty$ be any sequence of $1$-Lipschitz maps. Given an arbitrary $\varepsilon > 0$, we shall prove that

$$\text{diam} ((f_n)_*(\mu_{X_n}), m_{X_n} - \kappa) \leq 2 \varepsilon$$

for any $\kappa > 0$ and any sufficiently large $n \in \mathbb{N}$. Put $k := k(\varepsilon)$ and $\nu_{n,k} = (F_{\infty,\varepsilon} \circ f_n)_*(\mu_{X_n})$. Since

$$\text{diam} ((f_n)_*(\mu_{X_n}), m_{X_n} - \kappa) \leq \text{diam} (\nu_{n,k}, m_{X_n} - \kappa) + \varepsilon$$

by (3.1), it suffices to prove that

$$\lim_{n \to \infty} \text{diam} (\nu_{n,k}, m_{X_n} - \kappa) = 0.$$

Since Lemma 2.2 together with Corollary 2.6 and Lemma 3.2 implies that

$$\text{Sep} (\nu_{n,k}; \kappa_1, \kappa_2) \leq (1 + k(\varepsilon)^{1/q}) \text{Sep} (X_n; \kappa_1, \kappa_2) \to 0 \text{ as } n \to \infty$$

for any $\kappa_1, \kappa_2 > 0$, by virtue of Lemma 3.3, we obtain (3.5). This completes the proof. \qed

4. Case of $1 \leq q \leq p \leq +\infty$

For an mm-space $X$, we define the *concentration function* $\alpha_X : (0, +\infty) \to \mathbb{R}$ as the supremum of $\mu_X (X \setminus A_{+r})$, where $A$ runs over all Borel subsets of $X$ with $\mu_X (A) \geq m_X / 2$ and $A_{+r}$ is an open $r$-neighborhood of $A$.

**Lemma 4.1** (cf. [3, Corollary 2.6]). A sequence $\{X_n\}_{n=1}^\infty$ of mm-spaces is a Lévy family if and only if $\lim_{n \to \infty} \alpha_X (r) = 0$ for any $r > 0$.

Let $p \geq 1$. We shall consider the $\ell^p_n$-sphere $S^n_{\ell^p} := \{ (x_i)_{i=1}^n \in \mathbb{R}^n \mid \sum_{i=1}^n |x_i|^p = 1 \}$. We denote by $\mu_{n,p}$ the cone measure and by $\nu_{n,p}$ the surface measure on $S^n_{\ell^p}$ normalized as $\mu_{n,p}(S^n_{\ell^p}) = \nu_{n,p}(S^n_{\ell^p}) = 1$. In other words, for any Borel subset $A \subseteq S^n_{\ell^p}$, we put

$$\mu_{n,p}(A) := \frac{1}{L(B^n_{\ell^p})} \cdot \mathcal{L} (\{ tx \mid x \in A \text{ and } 0 \leq t \leq 1 \}),$$

where $\mathcal{L}$ is the Lebesgue measure on $\mathbb{R}^n$. 

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By the works of G. Schechtman and J. Zinn [27, Theorems 3.1 and 4.1] and R. Latała and J. O. Wojtaszczyk [15, Theorem 5.31], we obtain (4.1)

\[
\alpha(\mathcal{S}_{p,p}^n, d_{\mathcal{L}_p}, \mu_n, \nu) (r) \leq C \exp(-cnr^{\min\{2,p\}}).
\]

This inequality for \( p \geq 2 \) is also mentioned by A. Naor in [23, Introduction] (see also [15, Proposition 5.21]).

**Lemma 4.2.** Let \( 1 \leq q \leq p \leq +\infty \). Then, we have

\[
\alpha(\mathcal{S}_{p,p}^n, d_{\mathcal{L}_p}, \mu_n, \nu) (r) \leq C \exp(-cn^{1+(1/2-1/q)\min\{2,p\}} r^{\min\{2,p\}}) \quad \text{if} \quad q < 2
\]

and

\[
\alpha(\mathcal{S}_{p,p}^n, d_{\mathcal{L}_p}, \mu_n, \nu) (r) \leq C \exp(-cn^{1+(1/2-1/q)\min\{2,p\}} r^{\min\{2,p\}}) \quad \text{if} \quad q \geq 2.
\]

**Proof.** If \( q < 2 \), by \( d_{\mathcal{L}}(x, y) \leq n^{1/q-1/2} d_{\mathcal{L}_p}(x, y) \), we then have

\[
\alpha(\mathcal{S}_{p,p}^n, d_{\mathcal{L}_p}, \mu_n, \nu) (r) \leq \alpha(\mathcal{S}_{p,p}^n, d_{\mathcal{L}_p}, \mu_n, \nu) (n^{1/2-1/q} r)
\]

\[
\leq C \exp(-cn^{1+(1/2-1/q)\min\{2,p\}} r^{\min\{2,p\}}).
\]

If \( q \geq 2 \), by \( d_{\mathcal{L}}(x, y) \leq d_{\mathcal{L}_p}(x, y) \), we then obtain

\[
\alpha(\mathcal{S}_{p,p}^n, d_{\mathcal{L}_p}, \mu_n, \nu) (r) \leq \alpha(\mathcal{S}_{p,p}^n, d_{\mathcal{L}_p}, \mu_n, \nu) (r) \leq C \exp(-cn^{\min\{2,p\}}).
\]

This completes the proof. \( \square \)

**Corollary 4.3.** The sequences \( \{\mathcal{S}_{p,p}^n, d_{\mathcal{L}_p}, \mu_n, \nu\}_{n=1}^\infty \) and \( \{\mathcal{S}_{p,p}^n, d_{\mathcal{L}_p}, \nu_n, \nu\}_{n=1}^\infty \) are both Lévy families for \( (p, q) \in \mathbb{N} \times \mathbb{N} \setminus \{(k, 1) \mid k \geq 2\} \).

**Proof.** Since \( 1 + (1/2 - 1/q)\min\{2,p\} > 0 \), by Lemmas 4.1 and 4.2, the sequence \( \{\mathcal{S}_{p,p}^n, d_{\mathcal{L}_p}, \mu_n, \nu\}_{n=1}^\infty \) is a Lévy family. By virtue of [23, Theorem 6], the sequence \( \{\mathcal{S}_{p,p}^n, d_{\mathcal{L}_p}, \nu_n, \nu\}_{n=1}^\infty \) is also a Lévy family. This completes the proof. \( \square \)

**Proposition 4.4.** Let \( 1 \leq q \leq p \leq +\infty \) and let \( \mu \) be one of the two measures \( \mu_n, \nu \). Then, for any \( \kappa \) with \( 0 < \kappa < 1/2 \), we have

\[
\text{ObsDiam}(B_{L_p}^{-\kappa}) (\mathcal{S}_{p,p}^n, d_{\mathcal{L}_p}, \mu, -\kappa) \geq 2.
\]

**Proof.** Let \( A \subseteq \mathcal{S}_{p,p}^n \) be a Borel subset such that \( \mu(A) \geq 1 - \kappa \). Since \( \mu(A) = \mu(-A) > 1/2 \), we have \( \mu(A \cap (-A)) > 0 \). Hence, there exists \( x \in A \) such that \( -x \in A \). Since \( \text{diam} A \geq d_{\mathcal{L}_p}(x, -x) \geq d_{\mathcal{L}_p}(x, x) = 2 \), we obtain \( \text{diam}(\mu, 1 - \kappa) = 2 \). Since the inclusion map from the space \( \mathcal{S}_{p,p}^n, d_{\mathcal{L}_p} \) to the space \( B_{L_p}^{-\kappa} \) is 1-Lipschitz, we obtain the conclusion. This completes the proof. \( \square \)

Combining Corollary 4.3 with Proposition 4.4, we obtain an example of a Lévy family which does not satisfy (1.1) in the case of \( (p, q) \in \{(k, l) \in \mathbb{N} \times \mathbb{N} \mid l \leq k\} \setminus \{(k, 1) \mid k \geq 2\} \).

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