ASYMPTOTIC BEHAVIOR OF SPHERICALLY OR CYLINDRICALLY SYMMETRIC SOLUTIONS TO THE COMPRESSIBLE NAVIER-STOKES EQUATIONS WITH LARGE INITIAL DATA

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ABSTRACT. In this paper, we study the asymptotic behavior of global spherically or cylindrically symmetric solutions to the compressible Navier-Stokes equations for the viscous heat conducting ideal polytropic gas flow with large initial data in \( H^1 \), when the heat conductivity coefficient depends on the temperature, practically, \( \kappa(\theta) = \tilde{\kappa}_1 + \tilde{\kappa}_2 \theta^q \) where constants \( \tilde{\kappa}_1 > 0, \tilde{\kappa}_2 > 0 \) and \( q > 0 \) (as to the case of \( \tilde{\kappa}_1 = 0 \), please refer to the Appendix). In addition, the exponential decay rate of solutions toward to the constant state as time tends to infinity for the initial boundary value problem in bounded domain is obtained. The mass density and temperature are proved to be pointwise bounded from below and above, independent of time although strong nonlinearity in heat diffusion. The analysis is based on some delicate uniform energy estimates independent of time.

1. Introduction. In this paper, we consider the exponential stability of spherically or cylindrically symmetric solutions to the compressible Navier-Stokes equations with the heat conductivity coefficient depending on temperature in Eulerian coordinates in \( \{ x \in \mathbb{R}^N : 0 < a \leq |x| \leq b \} \), cf., for instance, [5, 6, 16]:

\[
\begin{aligned}
\rho_t + (\rho u)_r + \frac{m \rho u}{r} &= 0, \\
\rho u_t + \rho u u_r - \rho \frac{u^2}{r} + P_r &= \beta(u_r + \frac{m u}{r})_r, \\
\rho v_t + \rho u v_r + \frac{\rho u v}{r} &= \mu(v_r + \frac{m v}{r})_r, \\
\rho w_t + \rho u w_r &= \mu(w_{rr} + \frac{m w_r}{r}), \\
C_v \rho \theta_t + C_v \rho u \theta_r + P(u_r + \frac{m u}{r}) &= (\kappa(\theta) \theta_r)_r + \frac{m \kappa \theta}{r} + \varphi, \\
\varphi &= \lambda(u_r + \frac{m u}{r})^2 + \mu \left( w^2 + 2 u^2 + (v_r - \frac{m v}{r})^2 + \frac{2 m u^2}{r^2} \right).
\end{aligned}
\]

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We consider the initial and boundary conditions:

\[
(\rho, u, v, w, \theta)|_{t=0} = (\rho_0, u_0, v_0, w_0, \theta_0)(r) \quad \text{in} \ [a, b],
\]

and

\[
(u, v, w, \theta_r)|_{r=a,b} = 0, \quad t \geq 0.
\]

Here \( t \geq 0 \) is the time, and \( a \leq r \leq b \) is the radial space variable. \( \rho = \rho(r, t) > 0 \), \( \theta = \theta(r, t) \) are the mass density and temperature, respectively. In the spherically symmetric case, \( m = N - 1, \ r = |x| \), the velocity field \( \mathbf{u}(x, t) = u(r, t)\hat{r} \), and \( v = w = 0 \). In the cylindrically symmetric case, \( m = 1, r = \sqrt{x_1^2 + x_2^2} \), and the velocity field

\[
\mathbf{u}(x, t) = u(r, t)\left(\frac{x_1, 0}{r}\right) + v(r, t)\left(\frac{-x_2, x_1, 0}{r}\right) + w(r, t)(0, 0, 1).
\]

We consider ideal polytropic gas, then the pressure

\[
P(\rho, \theta) = R\rho \theta
\]

with positive constant \( R \); The viscosity coefficients \( \mu \) and \( \lambda \), satisfy \( \mu > 0, 2\mu + N\lambda \geq 0 \), and \( \beta = 2\mu + \lambda \). For perfect gas with constant specific heat, the statistical thermodynamics predicts a constant ratio of the heat conductivity coefficient. At high temperature, however, the heat conductivity coefficient becomes a rather sensitive function of temperature due to radiation, cf. [1, 30]. Generally, the heat conductivity coefficient \( \kappa(\theta) \) satisfies

\[
\kappa(\theta) = \tilde{\kappa}_1 + \tilde{\kappa}_2 \theta^q,
\]

where the constants \( \tilde{\kappa}_1 \geq 0, \tilde{\kappa}_2 \geq 0, q \geq 0 \).

There are many classical literatures on the well-posedness and large-time behavior of solutions to (1)-(3) for constant viscosity and constant heat conductivity coefficients \( (\tilde{\kappa}_1 > 0, \tilde{\kappa}_2 = 0) \). Kazhikhov and Shelukhin [15] firstly obtained the global existence and uniqueness of classical solutions in one-dimensional bounded domain with arbitrarily large initial data. Then, significant progress has been made on the mathematical aspect of the initial and initial boundary value problems. For initial boundary value problem in bounded domains, the global existence and the large-time behavior of solutions have been established in [20, 21]. In fact, the global solutions converge exponentially to constant states as time tends to infinity. This argument has been applied to the case of spherically symmetric solutions and cylindrically symmetric solutions with large initial data [10, 25, 26]. Recently, Cui and Yao [2] got the exponential decay for the global spherically or cylindrically symmetric solutions with large initial data for the compressible \( p \)-th power Newtonian fluid. For initial boundary value problem in unbounded domains and Cauchy problem, Jiang [12] got the asymptotic behavior of solutions to the compressible viscous ideal gas in one dimension. Recently, Li and Liang [18] studied the large-time behavior of solutions to the initial and initial boundary value problems in one-dimensional unbounded domains. They proved the temperature was bounded from below and above uniformly in both time and space. Wan and Wang [27] studied the cylindrically symmetric solutions of (1)-(3) in three-dimensional exterior domains.

For the cases of heat conductivity coefficient depending on temperature\( (\tilde{\kappa}_1 \geq 0, \tilde{\kappa}_2 > 0) \), Jenssen and Karper [8] proved the global existence of a weak solution in one dimension with \( \mu = \tilde{\mu} \) (constant), \( \kappa(\theta) = \tilde{\kappa}_2 \theta^b, b \in [0, \frac{3}{2}] \). Later, Pan and Zhang [22] considered the existence and uniqueness of global strong solutions in one dimension under the milder assumption \( \mu = \tilde{\mu}, \kappa = \tilde{\kappa}_2 \theta^b, b \geq 0 \). Hsiao and Luo [7] considered the large-time behavior of solutions for viscous one-dimensional real
gas in one-dimensional bounded domain, when \( \mu = \tilde{\mu} \) and the heat conductivity coefficient \( \kappa(\rho, \theta) \) satisfies
\[
\kappa_0(1 + \theta^q) \leq \kappa(\rho, \theta) \leq C_1(1 + \theta^q),
\]
\[
|\kappa(\rho, \theta)| + |\kappa(\rho, \theta)| \leq C_2(1 + \theta^q). 
\]
(5)

In the case of the viscosity coefficients \( \mu \) depending on \( \rho \) and \( \kappa(\rho, \theta) \) satisfying (5), the global existence of solutions to the one-dimensional free boundary problem was studied by Kawohl [14]. For more about this case, see [4, 9, 11, 17, 23]. Then, Liu, Yang, Zhao and Zou [19] studied the existence and uniqueness of a global smooth non-vacuum solutions for one-dimensional Cauchy problem with \( \mu, \kappa \) depending on temperature, provided that \( \gamma - 1 \) is sufficiently small. Next, when the viscosity and heat conductivity coefficients satisfy \( \mu(\rho, \theta) = \tilde{\mu} h(\rho) \theta^q, \kappa(\rho, \theta) = \tilde{\kappa} h(\rho) \theta^q \) with \( |\alpha| \leq \varepsilon_0 \) for a positive constant \( \varepsilon_0 \), Wang and Zhao [29] obtained the existence of global non-vacuum solutions to the Cauchy problem in one dimension. This result was generalized to the symmetric flows [28]. For other studies related to this topic such as the boundary layers, as well as other related models, we can refer to [13, 24] and the references therein.

In this paper, we will consider spherically or cylindrically symmetric Navier-Stokes equations (1)-(3) under the assumptions that the viscosity coefficient \( \mu, \lambda \) being constants and the heat conductivity coefficient \( \kappa(\theta) \) satisfying
\[
\kappa(\theta) = \tilde{\kappa}_1 + \tilde{\kappa}_2 \theta^q, \quad \text{with} \quad \tilde{\kappa}_1 > 0, \tilde{\kappa}_2 > 0, q > 0. \tag{6}
\]
We obtain the exponential decay of global smooth solutions without any restrictions on the size of initial data.

To state the main result, let us introduce the notations.

(i) For \( 1 \leq p \leq \infty \), \( L^p \) denotes the \( L^p \) space on \( [0, L] \) with the norm \( \| \cdot \|_{L^p} \). Specially, we put \( \| \cdot \| = \| \cdot \|_{L^2} \). For \( m \in \mathbb{N}, W^{m,p} \) denotes the Sobolev spaces on \( [0, L], \) whose norm is \( \| \cdot \|_{W^{m,p}} \), and \( H^m = W^{m,2} \).

(ii) \( Q_T = [0, L] \times [0, T] \) for \( T > 0 \).

As in [2, 10], it is convenient to transfer the problem (1)-(3) into the equations in Lagrangian coordinates. Hence we consider the following initial-boundary problem in the Lagrangian coordinates:
\[
\begin{align*}
&\eta_t = (r^m u_x)_x, \quad 0 < x < L, \ t > 0, \\
&u_t = r^m \left[ -\frac{R \theta}{\eta} + \beta \frac{(r^m u)_x}{\eta} \right] + u^2 r, \\
&v_t = \mu r^m \left[ \frac{(r^m v)_x}{\eta} \right] - u v r, \\
&w_t = \mu r^m \left[ \frac{(r^m w)_x}{\eta} \right] + \frac{\mu \eta \eta v}{r^2}, \\
&C_v \theta_t = -\left[ \frac{R \theta(r^m u)_x}{\eta} + \frac{\kappa(\theta) r^{2m} \theta x}{\eta} \right] + \eta \varphi.
\end{align*}
\]
(7)

Here the new variables are \( (x, t) \), with \( x \in [0, L], \ t \geq 0 \). \( \eta(x, t), u(x, t), v(x, t), w(x, t), \) and \( \theta(x, t) \) are the specific volume, radial velocity, angular velocity, axial velocity and temperature of the flows, respectively. Moreover, the radius \( r(x, t) \) also depends on the Lagrangian mass coordinates \( (x, t) \), and
\[
\varphi = \frac{\beta(r^m u)_x^2 + \mu(r^m v)_x^2 + \mu r^{2m} w_x^2}{\eta^2} - \frac{2 \mu (r^{m-1} u_x^2 + r^{m-1} v_x^2)}{\eta}.
\]
(8)
The initial conditions are
\[ (\eta, u, v, w, \theta) \big|_{t=0} = (\eta_0, u_0, v_0, w_0, \theta_0). \]  
(9)

The boundary conditions are
\[ (\eta, u, v, w, \theta_x) \big|_{x=0,L} = 0. \]  
(10)

It follows from the Lagrangian transformation that
\[ r_x = \frac{\eta}{r^m}, \quad r_t = u, \]  
(11)

and
\[ r(0, t) = a, \quad r(L, t) = b, \quad r_0(x) = \left( a^{m+1} + (m+1) \int_0^x \eta y \, dy \right)^{\frac{1}{m+1}}. \]  
(12)

Also we denote
\[ \sigma = -R\theta \eta + \beta (r^m u)_x. \]  
(13)

In what follows, we use \( C \) (and \( C_i \) or \( c \)) to denote a generic positive constant depending only on the parameters of the system and the bounds of the initial data, but independent of \( t \) and \( T \). For simplicity, \( \epsilon, \epsilon_1, \epsilon_2, \delta \) denote any small constants.

The following is the main result of this paper:

**Theorem 1.1.** Let the initial data satisfy
\[ 0 < c_0 < \eta_0 < C_0, \quad \theta_0 > 0, \]  
and \( \eta_0(x), u_0(x), v_0(x), w_0(x), \theta_0(x) \in H^1([0, L]) \). Assume that the initial data are compatible with boundary conditions, and the condition (6) holds. Then there exists a constant \( C > 0 \), such that the problem (7)-(10) admits a unique solution \( (\eta, u, v, w, \theta) \) satisfying
\[ C^{-1} \leq \eta(x, t) \leq C, \quad C^{-1} \leq \theta(x, t) \leq C, \]  
and for any \( t > 0 \),
\[ \int_0^L (\eta_x^2 + u_x^2 + v_x^2 + w_x^2 + \theta_x^2) \, dx \leq C e^{-C t}, \]
\[ \int_0^L |r_{xx} - \tilde{r}_{xx}|^2 \, dx \leq C e^{-C t}. \]

Moreover,
\[ \max_{x \in [0, L]} \{ |\eta(x, t) - \tilde{\eta}|, |u(x, t)|, |v(x, t)|, |w(x, t)|, |\theta(x, t) - \tilde{\theta}| \} \leq C e^{-C t}, \]
\[ \max_{x \in [0, L]} \{|r(x, t) - \tilde{r}| + |r_x - \tilde{r}_x|\} \leq C e^{-C t}, \]
where we define
\[ \tilde{\eta} = \frac{1}{L} \int_0^L \eta_0 \, dx, \quad \tilde{\theta} = \frac{1}{2C_v L} \int_0^L \left( 2C_v \theta_0 + v_0^2 + v_0^2 + w_0^2 \right) \, dx, \]
\[ \tilde{r} = (a^{m+1} + (m+1) \tilde{\eta} x)^{\frac{1}{m+1}}. \]  
(14)

Obviously, \( \tilde{\eta}, \tilde{\theta} > 0 \).

**Remark 1.1.** When \( \tilde{k}_1 > 0, \tilde{k}_2 = 0 \), i.e. the constant heat conductivity, the similar result as Theorem 1.1 has been proved in [2].
Remark 1.2. When \( \tilde{\kappa}_1 = 0, \tilde{\kappa}_2 > 0 \), i.e., the degenerate heat conductivity, we can get the similar result as this paper for \( \frac{1}{2} < q < 1 \). The proof can be founded in the Appendix.

Remark 1.3. In fact, Theorem 1.1 implies that \( \lim_{t \to \infty} (\| \eta(x,t) - \tilde{\eta} \|_{H^1([0,L])} + \| u(x,t) \|_{H^1([0,L])} + \| v \|_{H^1([0,L])} + \| w \|_{H^1([0,L])} + \| \theta(x,t) - \tilde{\theta} \|_{H^1([0,L])}) = 0 \), and \( \lim_{t \to \infty} \| r(x,t) - \tilde{r}(x) \|_{H^2([0,L])} = 0 \).

Remark 1.4. The similar results holds when the initial data is in \( H^2([0,L]) \) and \( H^4([0,L]) \) as in [26]; The same conclusions as in Theorem 1.1 hold for the \( p \)-th power Newtonian fluid where the pressure \( P = R\rho \theta \), see [2].

Compared with the previous result, there are two differences in this paper. On the one hand, we don’t need the smallness of the initial data, that is, all initial data can be large. On the other hand, for the heat conductivity coefficient \( \kappa(\theta) \) satisfying (6), in order to obtain the large-time behavior of solutions to the initial boundary problem (7)-(10), all the estimates should be independent of any length of time.

This will result in some mathematical difficulties. The first difficulty encountered here is to establish uniform point-wise positive upper bound of the temperature \( \theta \). To overcome this difficulty, we use the iterative method of [3, 22]. So we can mutually control two functionals \( Y \) and \( Z \) by choosing suitable \( \alpha \). The second difficulty is to establish uniform lower bound of the temperature. Compared with the constant heat conductivity coefficient, it is difficult to get \( \left| \frac{d}{dt} \int_0^L \theta^2 dx \right| \in L^1([0,t]) \) for \( q > 0 \). Fortunately, we can obtain \( \left| \frac{d}{dt} \int_0^L (1 + \theta^q)2\theta^2 dx \right| \in L^1([0,t]) \) for any \( q > 0 \), see Lemma 2.17.

2. Proof of Theorem 1.1. The global existence and uniqueness of solutions \((\eta, u, v, w, \theta)\) to the problem (7)-(10) in \( H^1 \) space has been obtained in [5, 15, 21]. Therefore, we are only concern about the large-time behavior of solutions in \( H^1 \).

For simplicity of presentation, we will take \( \tilde{\kappa}_1 = \tilde{\kappa}_2 = 1 \).

As the constant heat conductivity coefficient, it is important to get the uniform upper bounds and lower bounds of the volume \( \eta(x,t) \) and temperature \( \theta(x,t) \). In fact, the basic energy estimate, the uniform upper bound and lower bound of the volume \( \eta(x,t) \) could be obtained by the same method as the case of constant heat conductivity coefficient. Therefore, the proofs of Lemmas 2.1, 2.2 and 2.3 are similar to the arguments in [2, 25], just with some slightly modifications, we omit the proofs here.

Lemma 2.1. Under the conditions of Theorem 2.1, it holds that

\[
\int_0^L (u^2 + v^2 + w^2) dx + \int_0^t \int_0^L \left( \frac{\eta \phi}{\theta} + \frac{\kappa r^{2m} \eta^2 \theta}{\eta \theta^2} \right) dx ds \leq C, \tag{15}
\]

\[
\int_0^t \int_0^L \left( \frac{r^{2m} u^2}{\eta \theta} + \frac{w^2}{\eta \theta} \right) dx ds \leq C, \tag{16}
\]

\[0 < a \leq r(x,t) \leq b, \tag{17}\]

\[0 < C_1 \leq \int_0^L \theta^l dx \leq C_2, \quad \forall \ l \in [0,1]. \tag{18}\]
and
\[ \int_0^L \eta dx = \int_0^L \eta_0 dx = \frac{b^{m+1} - a^{m+1}}{m + 1} = \eta^*, \quad (19) \]
for any \( t \in [0, T]. \) Moreover, there exists a function \( x_0(t) \in [0, L], \) such that
\[ C_1 \leq \theta(x_0(t), t) \leq C_2, \quad (20) \]

**Lemma 2.2.** Under the conditions of Theorem 2.1, it holds that
\[ \int_0^t \int_0^L v^2 dx ds \leq C, \]
for any \( t \in [0, T]. \)

**Lemma 2.3.** Under the conditions of Theorem 2.1, it holds that \( c \leq \eta(x, t) \leq C. \)

**Corollary 2.1.** Under the conditions of Theorem 2.1, it holds that
\[ \int_0^t \| u \|_{L^\infty}^2 dx \leq C, \quad (21) \]
for any \( t \in [0, T]. \)

Next, by some simple calculations, we can get the lower bound of the temperature \( \theta \) which depend on the fixed time \( T. \)

**Lemma 2.4.** Under the conditions of Theorem 2.1, there exists two constants \( C_1 > 0 \) and \( C_2 > 0, \) independent of \( T, \) such that
\[ \theta(x, t) \geq \frac{1}{C_1 + C_2 T}, \quad \text{for any } (x, t) \in [0, L] \times [0, T]. \]

**Proof.** Let \( h = \frac{1}{T} \). Choosing \( \gamma > 0 \) to satisfy \( 0 < \frac{2\mu}{\gamma^2} < \gamma < 1. \) Then multiplying (7), by \( \frac{1}{\eta^*} \), we have
\[ C_0 h_1 = \left( \frac{\kappa r^{2m} h^2}{\eta} \right)_x - \left\{ \frac{2\kappa r^{2m} \theta}{\eta} h^2_x + \frac{h^2}{\eta} \beta (r^m u^2)_x - \frac{h^2}{\eta} R \theta (r^m u)_x + \frac{\mu h^2 r^{2m} u^2}{\eta} \right\} 
+ \frac{\mu h^2}{\eta} \left[ (r^m v^2)_x - 2m \eta (r^{m-1} v^2)_x \right] - 2m \mu h^2 (r^{m-1} u^2)_x 
= \left( \frac{\kappa r^{2m} h^2}{\eta} \right)_x - \left\{ \frac{2\kappa r^{2m} \theta}{\eta} h^2_x + \frac{h^2}{\eta} \beta (r^m u^2)_x + \frac{h^2}{\eta} (1-\gamma) \beta (r^m u)_x - \frac{h^2}{\eta} R \theta (r^m u)_x + \frac{\mu h^2 r^{2m} u^2}{\eta} \right\} 
+ \frac{\mu h^2}{\eta} \left[ (r^m v^2)_x - 2m \eta (r^{m-1} v^2)_x \right] - 2m \mu h^2 (r^{m-1} u^2)_x 
= \left( \frac{\kappa r^{2m} h^2}{\eta} \right)_x - \left\{ \frac{2\kappa r^{2m} \theta}{\eta} h^2_x + \frac{h^2}{\eta} \gamma \beta \left[ m (\gamma \beta - 2 \mu) + 2 \mu \right] \left( r^{m-1} u \right)_x 
+ \frac{2(\gamma \beta - 2 \mu)}{\gamma \beta} \frac{m r^{m-1} \eta uu_x}{r} \right\} + \frac{h^2}{\eta} \gamma \beta \left[ (r^m u)_x - \frac{R \theta}{2(1-\gamma) \beta} \right] + \frac{2(1-\gamma) \beta}{R} \right\} + \frac{R^2}{4(1-\gamma) \beta \eta} 
= \left( \frac{\kappa r^{2m} h^2}{\eta} \right)_x - \left\{ \frac{2\kappa r^{2m} \theta}{\eta} h^2_x + \frac{h^2}{\eta} m (\gamma \beta - 2 \mu) + 2 \mu \right\}.
\[
\left[ m r^{-1} \eta u + \frac{m(\gamma \beta - 2\mu)}{m(\gamma \beta - 2\mu) + 2\mu} r^m u_x \right]^2 + \frac{2h^2}{\eta} \frac{m(\gamma \beta - 2\mu)}{m(\gamma \beta - 2\mu) + 2\mu} (r^m u_x)^2 + \frac{\mu h^2}{\eta} (r^m v_x - \frac{mv}{r})^2 + \frac{\mu h^2 r^2 w_x^2}{\eta} \\
+ \frac{h^2}{\eta} (1 - \gamma) \beta \left( (r^m u_x - \frac{R \theta}{2(1 - \gamma) \beta})^2 \right) \right] \leq \left( \frac{\kappa r^{2m} h_x}{\eta} \right)_x + \frac{R^2}{4(1 - \gamma) \beta \eta},
\]
where we note that all the terms in the bracket \{·\} on the right hand side of the above resulting equation are nonnegative. Now multiplying the above inequality by \(2ph^2\eta^{-1}\) with \(p \geq 1\) being an arbitrary integer, and integrating the result over \([0, L]\) with respect to \(x\), one yields
\[
C_v \frac{d}{dt} \| h(t) \|_{L^2p}^2 = 2pC_v \left( \| h(t) \|_{L^2p} \right) \frac{d}{dt} \| h(t) \|_{L^2p}
\leq -2p(2p - 1) \int_0^L h^{2p-2} \frac{\kappa r^{2m} h_x^2}{\eta} dx + \frac{pR^2}{2(1 - \gamma) \beta} \int_0^L h^{2p-1} \frac{1}{\eta} dx \\
\leq \frac{pR^2}{2(1 - \gamma) \beta} \| h \|_{L^2p}^2 \left( \| h \|_{L^2p} \right) \frac{d}{dt} \| h \|_{L^2p}.
\]
Then, by the assumption of the initial data \(\theta_0\), we have
\[
\| h \|_{L^2p} \leq \frac{1}{\min_{x \in [0, L]} \theta_0(x)} + \frac{R^2}{4C_v(1 - \gamma) \beta} \int_0^t \left( \| h \|_{L^2p}^2 \right) ds \leq C_1 + C_2 t, \tag{22}
\]
where \(C_1 > 0\) and \(C_2 > 0\) are independent of \(p, T\).
Letting \(p \to +\infty\), we get
\[
\theta(x, t) \geq \frac{1}{C_1 + C_2 T}, \text{ for any } (x, t) \in [0, L] \times [0, T].
\]
This completes the proof. \(\square\)

Next two lemmas give \(H^1\) estimates of \(v\) and \(w\).

**Lemma 2.5.** Under the conditions of Theorem 2.1, we have
\[
\| v \|_{L^2^\infty}^2 + \| v \|_{H^1}^2 + \int_0^t (\| v \|_{L^2^\infty}^2 + \| v \|_{H^1}^2 + \| \nu_i \|^2) ds \leq C,
\]
for any \(t \in [0, T]\).

**Proof.** Multiplying (7)_3 by \(v\), and integrating the resulting equality with respect to \(x\) over \([0, L]\), we have
\[
\frac{1}{2} \frac{d}{dt} \int_0^L v^2 dx + \mu \int_0^L \frac{(r^m v^2)^2}{\eta} dx = -\int_0^L \frac{w v^2}{r} dx \leq C \int_0^L |u| v^2 dx \\
\leq C \| u \|_{L^\infty} \int_0^L v^2 dx + C \int_0^L v^2 dx.
\]
It follows from Gronwall inequality, Lemma 2.2 and (21) that
\[
\| v \|^2 + \int_0^t \int_0^L (r^m v)^2 dx ds \leq C,
\]
this together with \( \frac{1}{2}(r^m v_x)^2 \leq (r^m v)^2 + \left( \frac{mv}{r^m} \right)^2 \), Sobolev inequality and Lemma 2.2 again, imply

\[
\int_0^t (\| v \|^2_{L^\infty} + \| v \|^2_{H^1}) ds \leq C.
\]

Next, multiplying (7)_3 by \( v_t \), integrating the resulting equality with respect to \( x \) over \([0, L]\) and using (7)_1, (11), one gets

\[
\begin{align*}
\frac{\mu}{2} \frac{d}{dt} \int_0^L \frac{(r^m v)^2}{\eta} dx + \| v_t \|^2 &
= \frac{\mu}{2} \int_0^L \left( \frac{(r^m v)^2}{\eta} \right)_x dx - \mu \int_0^L \left( m r^{m-1} v u \right) \left( \frac{(r^m v)_x}{\eta} \right)_x dx - \int_0^L \frac{uvv_t}{r} dx \\
&= \mu \int_0^L \frac{r^m u \left( \frac{(r^m v)^2}{\eta} \right)_x}{\eta} dx - \int_0^L \left( m r^{m-1} v u \right) \left( \frac{v_t}{r^m} + \frac{uv}{r^m+1} \right) dx - \int_0^L \frac{uvv_t}{r} dx \\
&\leq \frac{1}{2} \| v_t \|^2 + C \int_0^L u^2 v^2 dx + C \int_0^L \frac{u^2 (r^m v)^2}{\eta^2} dx \\
&\leq \frac{1}{2} \| v_t \|^2 + C \| u \|^2_{L^\infty} \| r^m v \|^2 + C \| u \|^2_{L^\infty} \int_0^L \frac{(r^m v)^2}{\eta} dx.
\end{align*}
\]

By Gronwall inequality, Poincaré inequality, and (21), we have

\[
\| v_x \|^2 + \| v \|^2_{L^\infty} + \int_0^t \| v_t \|^2 ds \leq C.
\]

This completes the proof of Lemma 2.5.

Analogously, we can also use the same method as that in Lemma 2.5 to prove the following lemma for the axial velocity \( w \).

**Lemma 2.6.** Under the conditions of Theorem 2.1, we have

\[
\| w \|^2_{L^\infty} + \| w \|^2_{H^1} + \int_0^t (\| w \|^2 + \| w \|^2_{H^1} + \| w \|^2_{L^\infty}) ds \leq C,
\]

for any \( t \in [0, T] \).

**Lemma 2.7.** Under the conditions of Theorem 2.1, for any \( \alpha \in (0, 1] \), it holds that

\[
\int_0^t \int_0^L \left( \frac{k r^m \theta^2}{\eta \theta^{1+\alpha}} + \frac{u_x^2 + w_x^2}{\theta^\alpha} \right) dx ds \leq C,
\]

for any \( t \in [0, T] \).

**Proof.** We only prove \( \alpha \in (0, 1) \), the case of \( \alpha = 1 \) is the direct result of Lemma 2.1. We use the idea as in [25]. Multiplying (7)_3 by \( \theta^{-\alpha} \left( \int_0^L \theta^{-1-\alpha} dx \right)^{-1} \), then integrating the result with respect to \( x \) over \([0, L]\), using (7)_1, we find that

\[
\begin{align*}
&\left( \int_0^L \theta^{-1-\alpha} dx \right)^{-1} \int_0^L \frac{\alpha k r^m \theta^2}{\eta \theta^{1+\alpha}} + \frac{\eta v}{\theta^\alpha} dx \\
&= \left( \int_0^L \theta^{1-\alpha} dx \right)^{-1} C_v \left( \int_0^L \theta^{-\alpha} \theta_t dx \right) + R \left( \int_0^L \theta^{-1-\alpha} dx \right)^{-1} \int_0^L \frac{(r^m u)_x}{\eta} \theta^{1-\alpha} dx
\end{align*}
\]
Thus, using (18), and integrating the result inequality with respect to $s$ over $[0, t]$, we have for any $\epsilon > 0$

\[
\begin{align*}
&\int_0^t \int_0^L \left( \frac{\alpha k r^2 m \theta^2}{\eta \theta^{1+\alpha}} + \frac{\eta \theta^4}{\theta^\alpha} \right) dxds \\
&\leq C \log \int_0^L \theta^{1-\alpha} dx - C \log \int_0^L \theta_0^{1-\alpha} dx + C \int_0^L \log \eta dx - C \int_0^L \log \eta_0 dx \\
&+ R \int_0^t \left( \int_0^L \frac{(r^m u)^2}{\eta \theta} dx \right)^{\frac{1}{2}} \left( \int_0^L \theta dx \right)^{\frac{1}{2}} \left\| \frac{\theta^{1-\alpha} - L^{-1} \int_0^L \theta^{1-\alpha} dx}{L} \right\|_{L^\infty} ds \\
&\leq C + C\epsilon \int_0^t \int_0^L \frac{(r^m u)^2}{\eta \theta} dxds + \epsilon \int_0^t \int_0^L \theta^{-\alpha} \theta_x dx \right\|^2 ds \\
&\leq C + C\epsilon \int_0^t \int_0^L \frac{(r^m u)^2}{\eta \theta} dxds + C\epsilon \int_0^t \int_0^L \frac{r^{2m} \theta^2}{\theta^{1+\alpha}} dxds \\
&\leq C + C\epsilon \int_0^t \int_0^L \frac{r^{2m} \theta^2}{\theta^{1+\alpha}} dxds,
\end{align*}
\]

where we have used Lemma 2.1 and the following inequality,

\[
\int_0^t \left\| \theta^{1-\alpha} - L^{-1} \int_0^L \theta^{1-\alpha} dx \right\|^2_{L^\infty} ds \leq C \int_0^t \left\| \int_0^L \theta^{-\alpha} \theta_x dx \right\|^2 ds \\
\leq C \int_0^t \int_0^L \frac{r^{2m} \theta^2}{\theta^{1+\alpha}} dxds \int_0^L \theta^{1-\alpha} dxds.
\]

Finally, as (16), we find

\[
\int_0^t \int_0^L \left( \frac{kr^2 m \theta^2}{\eta \theta^{1+\alpha}} + \frac{u_x^2 + w_x^2}{\theta^\alpha} \right) dxds \leq C.
\]

This completes the proof of Lemma 2.7. \qed

As in [3, 22], in order to get the first-order derivative estimate of the radial velocity $u$, and the uniform estimates of the temperature, we define the following two functionals

\[
Z(T) = \sup_{0 \leq \tau \leq T} \int_0^L u_{xx}^2 dx, \quad \text{and} \quad Y(T) = \sup_{0 \leq \tau \leq T} \int_0^L (1 + \theta^\alpha)^2 \theta_x^2 dx.
\]

Lemma 2.8. Under the conditions of Theorem 2.1, it holds that

\[
\max_{Q_T} | u | \leq C(1 + Z^{\frac{1}{2}}), \quad (23)
\]
\[
\max_{Q_T} |u_x| \leq C(1 + Z^{\frac{3}{8}}),
\]
and
\[
\max_{Q_T} \theta \leq C(1 + Y^{\frac{1}{2m+2}}).
\]

Proof. Applying (10) and Sobolev inequality, we have
\[
u^2(x,t) \leq 2 \int_0^L |u| |u_x| \, dx,
\]
and
\[
u_x^2(y,t) \leq C \int_0^L u^2(x,t) \, dx + C \int_0^L |u_x| \, |u_{xx}| \, dx.
\]
By interpolation inequality, we get
\[
\int_0^L u_x^2 \, dx \leq C \int_0^L u^2 \, dx + C \left( \int_0^L u_{xx}^2 \, dx \right)^{\frac{1}{2}} \left( \int_0^L u_x^2 \, dx \right)^{\frac{1}{2}}.
\]
Therefore
\[
\max_{0 \leq t \leq T} \int_0^L u_x^2 \, dx \leq C + CZ^{\frac{1}{4}},
\]
and
\[
\max_{Q_T} |u| \leq C + CZ^{\frac{1}{8}}, \quad \max_{Q_T} |u_x| \leq C + CZ^{\frac{3}{8}}.
\]
Finally, for (25), it follows from (20) that
\[
\max_{Q_T} \theta^{3+2q} \leq C + \max_{Q_T} \left( \theta^{3+2q} - (\theta(x_0(t),t))^{3+2q} \right)^2
\]
\[
\leq C + C \sup_{0 \leq t \leq T} \left( \int_{x_0(t)}^L \theta^{1+2q} \theta_y \, dy \right)^2
\]
\[
\leq C + C \sup_{0 \leq t \leq T} \left( \int_0^L \theta^{2q} \theta_y^2 \, dx \right) \left( \int_0^L \theta \, dx \right)
\]
\[
\leq C + CY.
\]
Thus, the proof of Lemma 2.8 is finished. \(\square\)

Using the above lemma, we will get the relations between the first-order derivatives estimates of \(\eta, v, w, u\) and \(Z, Y\). And the following four lemmas are key to obtain the large time behavior of the solution and the uniform upper bound of \(\theta\).

**Lemma 2.9.** Under the conditions of Theorem 2.1, there exists a constant \(\alpha\) such that
\[
\| \eta_x \|^2 + \int_0^t \int_0^L \theta \eta_x^2 \, dx \, ds \leq C(1 + Y^{\frac{1}{2m+2}}),
\]
for any \(t \in [0, T]\), and \(\alpha\) will be determined later.

Proof. It follows from (7) that
\[
\left( \frac{\beta \eta_x}{\eta} \right)_t = \left( \frac{R \theta}{\eta} \right)_t + \left( \frac{u}{r^m} \right)_t + \frac{mu^2 - v^2}{r^{m+1}}.
\]
Multiplying it by \(\frac{\eta_x}{\eta}\) and integrating the result over \([0, L]\), we have
\[
\frac{d}{dt} \int_0^L \frac{\beta \eta_x^2}{\eta^2} \, dx + \int_0^L \frac{R \theta \eta_x^2}{\eta^3} \, dx
\]
\[
\int_0^L R \theta_x \eta x \, dx + \int_0^L \left( \frac{u}{r^m} \right) \eta x \, dx + \int_0^L \frac{mu^2 - v^2}{r^{m+1}} \eta \, dx.
\] (26)

Hence by means of Cauchy inequality, Lemma 2.3 and Lemmas 2.7-2.8, we have
\[
\int_0^t \int_0^L R \theta_x \eta x \, dxds \leq \frac{R}{2} \int_0^t \int_0^L \frac{\theta^2 x^2}{\eta^3} \, dxds + C \max_{Q_T} \theta^\alpha \int_0^t \int_0^L \frac{\theta^2 x}{\eta^2} \, dxds
\]
\[
\leq C + \frac{R}{2} \int_0^t \int_0^L \frac{\theta^2 x^2}{\eta^3} \, dxds + \frac{C}{1 + Y}.
\] (27)

Utilizing the boundary condition of \( u \), it easy to see that
\[
\int_0^L \left( \frac{u}{r^m} \right) \eta x \, dx = \frac{d}{dt} \int_0^L \left( \frac{u}{r^m} \right) \eta x \, dx - \int_0^L \frac{u}{r^m} \left( \eta u \right)_{x} \, dx
\]
\[
= \frac{d}{dt} \int_0^L \left( \frac{u}{r^m} \right) \eta x \, dx + \int_0^L \left( \frac{r^m u}{\eta} \right)_{x} \, dx.
\] (28)

On the one hand, it follows form Lemma 2.1 that
\[
\int_0^L \frac{u}{r^m} \eta x \, dx \leq \frac{1}{2} \int_0^L \frac{\eta^2 x}{\eta^2} \, dx + C \int_0^L u^2 \, dx \leq \frac{1}{2} \int_0^L \frac{\eta^2 x}{\eta^2} \, dx + C.
\] (29)

On the other hand, using Lemmas 2.7, 2.8 and (21), one knows
\[
\int_0^t \int_0^L \left( \frac{u}{r^m} \right) \left( \frac{r^m u}{\eta} \right)_{x} \, dx \leq \int_0^t \int_0^L \frac{2mu}{r^{2m+1}} \left( r^m u \right)_{x} \, dxds
\]
\[
\leq C \max_{Q_T} \theta^\alpha \int_0^t \int_0^L \frac{u^2}{\theta^2} \, dxds + C \int_0^t \| u \|_L^2 \, ds \leq C(1 + Y^{\frac{\alpha}{1 + \alpha}}).
\] (30)

where \( \alpha \in (0, 1) \) will be determined later.

Finally, it is easy to see that
\[
\int_0^t \int_0^L \frac{mu^2 - v^2}{r^{m+1}} \frac{\eta}{\eta} \, dxds
\]
\[
\leq C \int_0^t \int_0^L \frac{(u^2 + v^2)\eta^2}{\eta^2} \, dxds + C \int_0^t \int_0^L \left( u^2 + v^2 \right) \, dxds
\]
\[
\leq C \int_0^t \left( \| u \|_{L^\infty}^2 + \| v \|_{L^\infty}^2 \right) \int_0^L \frac{\eta^2}{\eta^2} \, dxds + C.
\] (31)

Substituting (27)-(31) into (26) and using (21), Lemma 2.5, Gronwall inequality, we have
\[
\int_0^L \frac{\eta^2}{\eta^2} \, dx + \int_0^L \int_0^L \frac{\theta^2 x}{\eta^3} \, dxds \leq C(1 + Y^{\frac{\alpha}{1 + \alpha}}).
\]

The proof of Lemma 2.9 is completed. \( \square \)

**Lemma 2.10.** Under the conditions of Theorem 2.1, we have
\[
\int_0^t \| v_{xx} \|^2 \, ds \leq C(1 + Y^{\frac{\alpha}{1 + \alpha}}),
\]
for any \( t \in [0, T] \).
Proof. From (7) and the integration by parts, we have
\[
\frac{d}{dt} \int_0^L v_x^2 dx = 2 \int_0^L v_x v_{xt} dx = -2 \int_0^L v_x v_t dx
\]
\[
= -2 \int_0^L v_{xx} \left[ \mu \frac{(r^m v)_x}{\eta} - \frac{uv}{r} \right] dx
\]
\[
\leq - \int_0^L \frac{\mu^2 m v_x^2}{\eta} dx + C \int_0^L (v^2 x^2 + v_x^2 x^2 + 2 v^2 + u^2 v^2) dx.
\]
After integrating the above inequality with respect to \(s\) over \([0,t]\), we have
\[
\int_0^L v_x^2 dx + \int_0^t \int_0^L v_{xx}^2 dx ds \leq C + C \int_0^t \int_0^L (v^2 x^2 + v_x^2 x^2 + 2 v^2 + u^2 v^2) dx ds.
\] (32)
Then utilizing (21), the boundary condition of \(v\) and Lemmas 2.5, 2.9, one gets
\[
\int_0^t \int_0^L (v^2 + v_x^2)v_x^2 dx ds
\]
\[
\leq C \sup_{0 \leq t \leq T} \int_0^L \eta^2 dx \int_0^t (\|v\|_{L^\infty}^2 + \|v_x\|_{L^\infty}^2) ds
\]
\[
\leq C(1 + Y^{\frac{2\alpha}{3+2q}}) \left[ 1 + \left( \int_0^t \int_0^L v_x^2 dx ds \right)^{\frac{1}{2}} \left( \int_0^t \int_0^L v_{xx}^2 dx ds \right)^{\frac{1}{2}} \right]
\]
\[
\leq \frac{1}{2} \int_0^t \int_0^L v_{xx}^2 dx ds + C(1 + Y^{\frac{2\alpha}{3+2q}}). \tag{33}
\]
Substituting (33) into (32) and applying Lemma 2.5, we derive
\[
\|v_x\|^2 + \int_0^t \|v_{xx}\|^2 ds \leq C(1 + Y^{\frac{2\alpha}{3+2q}}).
\]
This completes the proof of Lemma 2.10. \(\square\)

Next, we can also obtain the similar estimate for the axial velocity \(w\).

Lemma 2.11. Under the conditions of Theorem 2.1, we have
\[
\int_0^t \|w_{xx}\|^2 ds \leq C(1 + Y^{\frac{2\alpha}{3+2q}}),
\]
for any \(t \in [0,T]\).

Lemma 2.12. Under the conditions of Theorem 2.1, we have
\[
\|u_x\|^2 + \int_0^t \|u_{xx}\|^2 ds \leq C(1 + Y^{\frac{2\alpha}{3+2q}}),
\]
for any \(t \in [0,T]\).

Proof. We deduce from the integration by parts and (7) that
\[
\frac{d}{dt} \int_0^L u_x^2 dx = 2 \int_0^L u_x u_{xt} dx = -2 \int_0^L u_{xx} u_t dx
\]
\[
= -2 \int_0^L u_{xx} \left[ r^m \left( -\frac{R}{\eta} + \frac{\beta (r^m u)_x}{\eta} \right) \frac{v}{r} \right] dx
\]
\[
\leq -\int_0^L \frac{\beta_3^{2m} u_x^2}{\eta} \, dx + C \int_0^L (u^2 \eta_x^2 + u_x^2 \eta_x^2 + u_x^2 + u^2 + \theta^2 \eta_x^2 + \theta_x^2 + v^4) \, dx. \tag{34}
\]

Then, from Hölder inequality, \(W^{1,1}([0,L]) \hookrightarrow L^\infty([0,L])\), the boundary condition of \(u\), Lemmas 2.5, 2.7, 2.8, 2.9, and (21), one has

\[
C \int_0^t \left( u^2 \eta_x^2 + u_x^2 \eta_x^2 \right) \, dx \, ds
\]

\[
\leq C \sup_{0 \leq t \leq T} \int_0^L \eta_x^2 \, dx \left( \| u \|_{L^\infty}^2 + \| u_x \|_{L^\infty}^2 \right) \, ds
\]

\[
\leq C (1 + Y^{\frac{\alpha}{1+\alpha}}) \left[ 1 + \left( \int_0^t \int_0^L u_x^2 \, dx \, ds \right)^\alpha \left( \int_0^t \int_0^L u^2 \, dx \, ds \right)^\beta \right] \tag{35}
\]

\[
\leq C (1 + Y^{\frac{\alpha}{1+\alpha}}) \left( \frac{1}{2} \right) \int_0^t \int_0^L u_x^2 \, dx \, ds + C(1 + Y^{\frac{\alpha}{1+\alpha}}) \int_0^t \int_0^L u^2 \, dx \, ds \tag{36}
\]

and

\[
\int_0^t \int_0^L \left( \theta^2 \eta_x^2 + \theta_x^2 \right) \, dx \, ds
\]

\[
\leq C \max_{Q_T} \theta \int_0^t \int_0^L \theta \eta_x^2 \, dx \, ds + C \max_{Q_T} \theta^{1+\alpha} \int_0^t \int_0^L \frac{\theta^2}{\theta^{1+\alpha}} \, dx \, ds \leq C (1 + Y^{\frac{\alpha}{1+\alpha}}).
\]

After integrating the resulting inequality with respect to \(s\) over \([0,t]\), and taking \(0 < \alpha < \frac{1}{2}\), one gets

\[
\int_0^L u_x^2 \, dx + \int_0^t \int_0^L u_x^2 \, dx \, ds \leq C (1 + Y^{\frac{\alpha}{1+\alpha}}).
\]

We completed the proof of Lemma 2.12.

Next, we give the relation between \(Y\) and \(Z\) in the following lemma.

**Lemma 2.13.** Under the conditions of Theorem 2.1, for any constant \(\delta > 0\), there exists \(C_\delta > 0\) such that

\[
Y + \int_0^t \int_0^L (1 + \theta^q) \theta_x^2 \, dx \, ds \leq \delta Z + C_\delta.
\]

**Proof.** Multiplying \((T)_5\) by \((1 + \theta^q) \theta_t\) and integrating it over \([0,L]\), yields

\[
\int_0^L C_v (1 + \theta^q) \theta_t^2 \, dx + \int_0^L \frac{r^{2m}(1 + \theta^q) \theta_x}{\eta} \left( (1 + \theta^q) \theta_t \right)_x \, dx
\]

\[
= - \int_0^L \frac{R \theta (r^{2m} u_x)}{\eta} (1 + \theta^q) \theta_t \, dx + \int_0^L \eta \phi (1 + \theta^q) \theta_t \, dx. \tag{37}
\]

It is easy to see that

\[
\int_0^L \frac{r^{2m}(1 + \theta^q) \theta_x}{\eta} \left( (1 + \theta^q) \theta_t \right)_x \, dx
\]
Substituting (38) into (37), and integrating the result over \([0,t]\), we obtain

\[
\int_0^t \frac{2m(1 + \theta^q) \theta^2_x}{\eta} dx + \int_0^t (1 + \theta^q) \theta^2_x dx ds
\]

\[
\leq C + C \int_0^t \int_0^L \frac{\theta(r^m u)_x}{\eta} (1 + \theta^q)|\theta_s|dxds + C \int_0^t \int_0^L \frac{(r^m u)_x^2}{\eta} (1 + \theta^q)|\theta_s|dxds
\]

\[
+ C \int_0^t \int_0^L \frac{(r^m v)_x^2}{\eta} (1 + \theta^q)|\theta_s|dxds + C \int_0^t \int_0^L \frac{r^m u_x^2}{\eta} (1 + \theta^q)|\theta_s|dxds
\]

\[
\leq C + \frac{1}{2} \int_0^t \int_0^L (1 + \theta^q) \theta^2_x dx ds + C \int_0^t \int_0^L (1 + \theta^q) \theta^2 (r^m u)_x^2 dx ds
\]

\[
+ C \int_0^t \int_0^L (1 + \theta^q) (r^m u)_x^4 dx ds + C \int_0^t \int_0^L (1 + \theta^q) (r^m v)_x^4 dx ds
\]

\[
+ C \int_0^t \int_0^L (1 + \theta^q) r^m u_x^4 dx ds
\]

\[
+ C \int_0^t \left( \frac{(m-1)\eta}{r^2} u_x^2 + 2r^m u u_x + \frac{(m-1)\eta}{r^2} v^2 + 2r^m v v_x \right)^2 dx ds
\]

\[
+ C \int_0^t \left( \| u \|_{L^\infty} + \| (r^m u)_x \|_{L^\infty} \right) \int_0^L (1 + \theta^q) \theta^2_x dx ds.
\]

Firstly, for \(I_1\), by using Cauchy inequality, Lemmas 2.7, 2.8, and (21), we have

\[
I_1 \leq C \int_0^t \int_0^L (1 + \theta^q) \theta^2 u_x^2 dx ds + C \int_0^t \int_0^L (1 + \theta^q) \theta^2 u^2 dx ds
\]

\[
\leq C(1 + \max_{Q_r} \theta^q) \max_{Q_r} \theta^{2+\alpha} \int_0^t \int_0^L \frac{u_x^2}{\theta^\alpha} dx ds + C(1 + \max_{Q_r} \theta^{1+\eta}) \int_0^t \| u \|_{L^\infty}^2 ds
\]

\[
\leq C + Cy^{\frac{2+\alpha}{2+\alpha}}.
\]
Next, for $I_2$, from Hölder inequality, (21) and Lemmas 2.1, 2.8, 2.12, we infer

$$I_2 \leq C(1 + \max_{Q_T} \theta^q) \int_0^t \int_0^L (u^4 + u_x^4) \, dx \, ds$$

$$\leq C(1 + Y^{\frac{n}{n+2}}) \left( \int_0^t \|u\|_{L^\infty}^2 \, ds + \sup_{0 \leq t \leq T} \int_0^L u_x^2 \, dx + \int_0^t \|u_x\|_{L^\infty}^2 \, ds \right)$$

$$\leq C(1 + Y^{\frac{n}{n+2}}) \left( 1 + \sup_{0 \leq t \leq T} \int_0^L u_x^2 \, dx + \int_0^t \|u_x\|_{L^\infty}^2 \, ds \right)$$

$$\leq C(1 + Y^{\frac{n}{n+2}})(1 + Y^{\frac{2+2q}{n+2}})$$

$$\leq C + CY^{\frac{2+2q}{n+2}}.$$

For $I_3 + I_4$, applying Lemmas 2.5, 2.6, 2.8-2.11, we infer that

$$I_3 + I_4 \leq C(1 + \max_{Q_T} \theta^q) \left( \int_0^t \|v\|_{L^\infty}^2 \, ds + \sup_{0 \leq t \leq T} \int_0^L v_x^2 \, dx + \int_0^t \|v_x\|_{L^\infty}^2 \, ds \right)$$

$$+ C(1 + \max_{Q_T} \theta^q) \sup_{0 \leq t \leq T} \int_0^L u^2 \, dx + \int_0^t \|u_x\|_{L^\infty}^2 \, ds$$

$$\leq C(1 + Y^{\frac{n}{n+2}}) \left( 1 + \int_0^t \|v_x\|^2 \, ds + \int_0^t \|w_x\|^2 \, ds \right)$$

$$\leq C + CY^{\frac{2+2q}{n+2}}.$$

For $I_5$, from (21) and Lemmas 2.5, 2.8, 2.12, we get

$$I_5 \leq C(1 + \max_{Q_T} \theta^q) \left( \sup_{0 \leq t \leq T} \int_0^L u_x^2 \, dx + \int_0^t \|u\|_{L^\infty}^2 \, ds \right)$$

$$+ C \left( \sup_{0 \leq t \leq T} \int_0^L v_x^2 \, dx + \int_0^t \|v\|_{L^\infty}^2 \, ds \right)$$

$$\leq C(1 + Y^{\frac{n}{n+2}})(1 + Y^{\frac{2+2q}{n+2}})$$

$$\leq C + CY^{\frac{2+2q}{n+2}}.$$

Finally, for $I_6$, using Poincaré inequality, Lemmas 2.7, 2.8, we obtain

$$I_6 \leq \int_0^t \left( (\|u\|_{L^\infty} + \|r^m u_x\|_{L^\infty}) \int_0^L (1 + \theta^q)^2 \theta_x^2 \, dx \right) \, ds$$

$$\leq (\max_{Q_T} |u| + \max_{Q_T} |u_x|)(1 + \max_{Q_T} \theta^{1+\alpha+q}) \int_0^t \int_0^L \frac{(1 + \theta^q) \theta_x^2 \, dx \, ds}{\theta^{1+\alpha}}$$

$$\leq C(1 + Z^\frac{1}{2})(1 + Y^{\frac{1+\alpha+q}{n+2}})$$

$$\leq C + CY^{\frac{1+\alpha+q}{n+2}}Z^\frac{1}{2} + CY^{\frac{1+\alpha+q}{n+2}} + CZ^\frac{1}{2}.$$

Thus taking $0 < \alpha < \min \left\{ \frac{1}{2}, \frac{7+2q}{8} \right\}$, we have

$$\int_0^T \int_0^L (1 + \theta^q) \theta_x^2 \, dx \, ds + Y$$

$$\leq C + \frac{1}{2} \int_0^T \int_0^L (1 + \theta^q) \theta_x^2 \, dx \, ds + CY^{\frac{1+\alpha+q}{n+2}} Z^\frac{1}{2} + CY^{\frac{2+2q}{n+2}} + CZ^\frac{1}{2}.$$
\[
\leq C_\delta + \frac{1}{2} \int_0^t \int_0^L (1 + \theta^2) \theta^2 dxds + \delta Z.
\]

This completes the proof of the Lemma 2.13.

\[\square\]

**Lemma 2.14.** Under the conditions of Theorem 2.1, for any constant \(\delta > 0\), there exists \(C_\delta > 0\), such that

\[
\int_0^L u_t^2 dx + \int_0^t \int_0^L u_{xx}^2 dxds \leq \delta Z + C_\delta.
\]

**Proof.** Differentiating (7)\(2\) with respect to \(t\), multiplying the resulting equation by \(u_t\) and integrating it over \([0, L] \times [0, t]\), we get

\[
\frac{1}{2} \int_0^L u_t^2 dx = \frac{1}{2} \int_0^L u_t^2(x, 0) dx + \int_0^t \int_0^L mr^{m-1}u u_s \left( - \frac{R\theta}{\eta} + \beta \frac{(r^m u)_x}{\eta} \right) dxds
\]

\[
- \int_0^t \int_0^L \left( (r^m u)_{xx} - (mr^{m-1}u^2)_x \right) \left[ \frac{\beta (r^m u)_x}{\eta} - \beta \frac{(r^m u)^2}{\eta^2} \right] dxds
\]

\[
+ \int_0^t \int_0^L \left( 2v u s u_x - \frac{v^2 u s_x}{r^2} \right) dxds.
\]

For \(II_1\), using Cauchy inequality, Lemmas 2.7, 2.8, and 2.9, we have

\[
II_1 \leq C \int_0^t \int_0^L u^2 u_s^2 dxds + C \int_0^t \int_0^L \theta^2 dxds
\]

\[
+ C \int_0^t \int_0^L \theta^2 \eta^2 dxds + C \int_0^t \int_0^L \left( \frac{(r^m u)_x}{\eta} \right)^2 dxds
\]

\[
\leq C \int_0^t \| u \|^2_{L^\infty} \int_0^L u_s^2 dxds + C \max \theta^{1+\alpha} \int_0^t \int_0^L \frac{\theta^2}{\eta^{1+\alpha}} dxds
\]

\[
+ C \max \theta \int_0^t \int_0^L \theta \eta^2 dxds + C \int_0^t \int_0^L \left( (r^m u)_x^2 u_x^2 + (r^m u)^2 \right) dxds
\]

\[
\leq C \int_0^t \| u \|^2_{L^\infty} \int_0^L u_s^2 dxds + C(1 + Y \frac{1+\alpha}{4\pi})
\]

\[
+ C(1 + Y \frac{1+\alpha}{4\pi})(1 + Y \frac{a}{\pi^2}) + C(1 + Y \frac{1+\alpha}{4\pi})
\]

\[
\leq C \int_0^t \| u \|^2_{L^\infty} \int_0^L u_s^2 dxds + CY \frac{1+\alpha}{4\pi},
\]

where we have used

\[
\int_0^t \int_0^L ( (r^m u)_x^2 u_x^2 + (r^m u)_x u_{xx} ) dxds
\]

\[
\leq C \int_0^t \int_0^L (\eta^2 u_x^2 + \eta^2 u^2 + u_{xx}^2 + u^2 + u_x^2) dxds
\]
\[
\leq C \left( 1 + \sup_{0 \leq t \leq T} \int_0^L \eta_x^2 dx \right) \int_0^t (\|u\|_{L^\infty}^2 + \|u_x\|_{L^\infty}^2) \, ds + \int_0^t \int_0^L u_{xx}^2 \, dx \, ds \\
\leq C(1 + Y^{\frac{2}{r+2}})(1 + \int_0^t \int_0^L u_{xx}^2 \, dx \, ds) \\
\leq C + CY^{\frac{1+\alpha}{r+2}}.
\]

For $I_{I_2}$, using Cauchy inequality, Lemmas 2.7, 2.8, and 2.12, we have
\[
I_{I_2} = - \int_0^t \int_0^L \frac{\beta (r^m u)^2}{\eta} \, dx \, ds + \int_0^t \int_0^L \frac{\beta (r^m u)_{xx}}{\eta^2} \, dx \, ds \\
+ \int_0^t \int_0^L m \beta (r^{m-1} u^2)_x \frac{(r^m u)_{xx}}{\eta} \, dx \, ds - \int_0^t \int_0^L m \beta (r^{m-1} u^2)_x \frac{(r^m u)^2}{\eta^2} \, dx \, ds \\
\leq - \int_0^t \int_0^L \frac{\beta (r^m u)^2}{\eta} \, dx \, ds + \frac{1}{8} \int_0^t \int_0^L \frac{\beta (r^m u)^2}{\eta^2} \, dx \, ds \\
+ C \int_0^t \int_0^L \left( (r^m u)_x^4 + (r^{m-1} u^2)_x^2 \right) \, dx \, ds \\
\leq - \frac{7}{8} \int_0^t \int_0^L \frac{\beta (r^m u)^2}{\eta} \, dx \, ds + C \int_0^t \int_0^L (u_x^4 + u_x^6 + u^4) \, dx \, ds \\
\leq - \frac{7}{8} \int_0^t \int_0^L \frac{\beta (r^m u)^2}{\eta} \, dx \, ds + C(1 + \sup_{0 \leq t \leq r} \int_0^L u_x^2 \, dx) \int_0^t \|u_x\|_{L^\infty}^2 \, ds + C \\
\leq - \frac{7}{8} \int_0^t \int_0^L \frac{\beta (r^m u)^2}{\eta} \, dx \, ds + CY^{\frac{2+\alpha}{r+2}}.
\]

For $I_{I_3}$, using Cauchy inequality, Lemmas 2.7, 2.8, 2.13, we have
\[
I_{I_3} \leq \int_0^t \int_0^L (r^m u)^2 \, dx \, ds + C \int_0^t \int_0^L \theta^2 \, dx \, ds + C \int_0^t \int_0^L \theta^2 (r^m u)_x^2 \, dx \, ds \\
+ C \int_0^t \int_0^L (r^{m-1} u^2)_x^2 \, dx \, ds \\
\leq \epsilon \int_0^t \int_0^L (r^m u)^2 \, dx \, ds + C + C_{\max} \theta^{2+\alpha} \int_0^t \int_0^L \frac{(r^m u)^2}{\theta^\alpha} \, dx \, ds \\
+ \int_0^t \int_0^L (u_x^2 + u^4) \, dx \, ds \\
\leq \epsilon \int_0^t \int_0^L (r^m u)^2 \, dx \, ds + C + C_{\max} Y^{\frac{2+\alpha}{r+2}}.
\]

Finally, for $I_{I_4}$, using Cauchy inequality, Lemma 2.5, we have
\[
I_{I_4} \leq C \int_0^t \int_0^L v^2 \, dx \, ds + C \int_0^t \int_0^L v_x^2 \, dx \, ds \\
+ C \int_0^t \int_0^L u_x^2 \, dx \, ds + C \int_0^t \int_0^L v^4 \, dx \, ds \\
\leq C \int_0^t (\|u\|_{L^\infty}^2 + \|v\|_{L^\infty}^2) \int_0^L u_x^2 \, dx \, ds + C.
\]
This deduces
\[
\int_0^L u_x^2 \, dx + \int_0^t \int_0^L \frac{(r^m u)_x^2}{\eta} \, dx \, ds \\
\leq C \int_0^t (\| u \|_{L^\infty}^2 + \| v \|_{L^\infty}^2) \int_0^L u_x^2 \, dx \, ds + C + CY^{\frac{3+2\alpha}{4}} + \delta Z.
\]

Utilizing Gronwall inequality, Lemmas 2.5, 2.13, and (21), for any \( q > 0 \), we have
\[
\int_0^L u_x^2 \, dx + \int_0^t \int_0^L \frac{(r^m u)_x^2}{\eta} \, dx \, ds \\
\leq \exp \left\{ \int_0^t (\| u \|_{L^\infty}^2 + \| v \|_{L^\infty}^2) \, ds \right\} \left( C + CY^{\frac{3+2\alpha}{4}} + \delta Z \right) \\
\leq C + C(\delta Z)^{\frac{3+2\alpha}{4}} + \delta Z \\
\leq C + \delta Z.
\]

This completes the proof of Lemma 2.14.

By means of these lemmas, we can obtain the estimates of \( Y \) and \( Z \).

**Lemma 2.15.** Under the conditions of Theorem 2.1, we have
\[
Z = \sup_{0 \leq t \leq T} \int_0^L u_x^2 \, dx \leq C, \quad Y = \sup_{0 \leq t \leq T} \int_0^L (1 + \theta^2)^2 \theta_x^2 \, dx \leq C,
\]
and
\[
\max_{Q_T} \theta \leq C.
\]

**Proof.** By (7)_2, Cauchy inequality, Poincaré inequality, and Lemmas 2.1, 2.5, 2.8, 2.12, 2.13, 2.14, we obtain
\[
\sup_{0 \leq t \leq T} \int_0^L u_x^2 \, dx \leq C \sup_{0 \leq t \leq T} \int_0^L \left( u_x^2 + \theta_x^2 + \theta^2 \eta_x^2 + u^2 \eta_x^2 + u_x^2 \eta_x^2 + u^2 + u_x^2 + v^4 \right) \, dx \\
\leq \delta Z + C_\delta + C(\max_{Q_T} \theta^2 + \max_{Q_T} u^2 + \max_{Q_T} | u_x |^2) \sup_{0 \leq t \leq T} \int_0^L \eta_x^2 \, dx \\
+ C + CY^{\frac{3+2\alpha}{4}} \\
\leq \delta Z + C_\delta + C(1 + Y^{\frac{3+2\alpha}{4}} + Z^{\frac{7+2\alpha}{8}})(1 + Y^{\frac{3+2\alpha}{4}}) + C_\delta(1 + (\delta Z)^{\frac{3+2\alpha}{4}}) \\
\leq 2\delta Z + C_\delta,
\]
where we take \( 0 < \alpha < \min\{ \frac{7+2\alpha}{8}, \frac{3+2\alpha}{4}, \frac{1}{2} \} \).

Thus we get
\[
Z \leq C.
\]

This togethers with Lemma 2.13, (25), we finished the proof of the Lemma 2.15.

Using Lemma 2.15, we can get the following Corollary 2.2.

**Corollary 2.2.** Under the conditions of Theorem 2.1, it holds that
\[
\int_0^t \int_0^L \theta_x^2 \, dx \, ds \leq C,
\]
for any \( t \in [0, T] \).
The following lemma is useful for proving $u \in C([0,T], H^1[0,L])$.

**Lemma 2.16.** Under the conditions of Theorem 2.1, we have

$$\| u_x \|_{L^2}^2 + \| u_x \|_{L^\infty}^2 + \int_0^t \| u_t \|_{L^2}^2 \, ds \leq C,$$

for any $t \in [0,T]$.

**Proof.** Multiplying (7) by $u_t$, then integrating the resulting equality with respect to $x$ over $[0,L]$, we have

$$\frac{\beta}{2} \int_0^L \frac{(r^m u_x)^2}{\eta} \, dx + \int_0^t \int_0^L u_t^2 \, dx \, ds$$

$$= \frac{\beta}{2} \int_0^L \frac{(r^2 u_x)^2}{\eta_0} \, dx - \frac{\beta}{2} \int_0^t \int_0^L \frac{(r^m u_x)^2}{\eta^2} \eta \, dx \, ds - \int_0^t \int_0^L r^m u_t \left( \frac{R\theta}{\eta} \right)_x \, dx \, ds$$

$$- \beta \int_0^t \int_0^L m r^{m-1} u^2 \left( \frac{(r^m u_x)}{\eta} \right)_x \, dx \, ds + \int_0^t \int_0^L \frac{v^2 u_t}{r} \, dx \, ds$$

$$= : \sum_{j=0}^4 \text{III}_j. \quad (40)$$

Since $u_0 \in H^1$, we have

$$\text{III}_0 \leq C.$$  

Using integration by parts, (7)$_1$, Lemmas 2.5, 2.9, 2.15 and (39), we know

$$\text{III}_1 = \beta \int_0^t \int_0^L \frac{(r^m u_x)(r^m u)_x}{\eta} \left( \frac{r^m u_x}{\eta} \right)_x \, dx \, ds$$

$$= \int_0^t \int_0^L \frac{(r^m u_x)(r^m u)_x}{\eta} \left[ \frac{u_t}{r^m} + \left( \frac{R\theta}{\eta} \right)_x \right] \, dx \, ds$$

$$\leq \frac{1}{6} \int_0^t \int_0^L u_t^2 \, dx \, ds + C \int_0^t \| u \|_{L^\infty}^2 \int_0^L \frac{(r^m u_x)^2}{\eta} \, dx \, ds$$

$$+ C \int_0^t \int_0^L (\theta^2 \eta_x^2 + \theta_x^2 + v^4) \, dx \, ds$$

$$\leq \frac{1}{6} \int_0^t \int_0^L u_t^2 \, dx \, ds + C \int_0^t \| u \|_{L^\infty}^2 \int_0^L \frac{(r^m u_x)^2}{\eta} \, dx \, ds$$

$$+ C \max_{Q_T} \int_0^t \int_0^L \theta \eta_x^2 \, dx \, ds + C$$

$$\leq \frac{1}{6} \int_0^t \int_0^L u_t^2 \, dx \, ds + C \int_0^t \| u \|_{L^\infty}^2 \int_0^L \frac{(r^m u_x)^2}{\eta} \, dx \, ds + C.$$  

It follows form (39) that

$$\text{III}_2 \leq \frac{1}{6} \int_0^t \int_0^L u_t^2 \, dx \, ds + C \int_0^t \int_0^L (\theta^2 \eta_x^2 + \theta_x^2) \, dx \, ds \leq \frac{1}{6} \int_0^t \int_0^L u_t^2 \, dx \, ds + C.$$  

It is easy to see that

$$\text{III}_3 = - \int_0^t \int_0^L m r^{m-1} u^2 \left[ \frac{u_t}{r^m} + \left( \frac{R\theta}{\eta} \right)_x \right] \, dx \, ds$$
Under the conditions of Theorem 2.1, we have

$$\theta$$

Lemma 2.17.

Then using Gronwall inequality, the above estimates and (21), we get

$$\int_0^L \frac{(r^m u)^2}{\eta} dx + \int_0^t \int_0^L \theta_t^2 dxds \leq C \exp \left\{ \int_0^t \| u \|_{L_\infty}^2 ds \right\} \leq C.$$  

This together with Lemma 2.15 prove the Lemma 2.16.

The following lemma gives the estimate of $d_{\pm}$.

Theorem 2.17. Under the conditions of Theorem 2.1, we have

$$\int_0^L (1 + \theta^q) \theta_t^2 dx + \int_0^t \int_0^L (1 + \theta^q) \theta_t^2 dxds \leq C,$$

for any $t \in [0, T]$.

Proof. Multiplying (7) by $(1 + \theta^q) \theta_t$ and using Lemma 2.15, we deduce that

$$\frac{1}{2} \frac{d}{dt} \int_0^L \frac{(1 + \theta^q)^2 r^{2m}}{\eta} \theta_t^2 dx + C \int_0^L (1 + \theta^q) \theta_t^2 dx$$

$$= \int_0^L \frac{m r^{2m-1} (1 + \theta^q)^2 u \theta_t^2}{\eta} dx - \int_0^L \frac{r^{2m} (1 + \theta^q)^2 (r^m u)_x \theta_t^2}{\eta} dx$$

$$+ \int_0^L (1 + \theta^q) \theta_t \left\{ - \frac{R \theta}{\eta} + \frac{(r^m u)_x}{\eta} \right\} (r^m u)_x$$

$$+ \mu \frac{(r^m u)^2}{\eta} + \mu \frac{r^{2m} u^2}{\eta} - 2\mu m (r^{m-1} u^2 + r^{m-2} v^2)_x \right\} dx. \quad (41)$$

Then, after integrating the above inequality with respect to $s$ over $[0, t]$, we have

$$\int_0^t \int_0^L (1 + \theta^q)^2 \theta_t^2 dxds$$

$$\leq C + C \int_0^t \int_0^L (1 + \theta^q)^2 u \theta_t^2 dxds + C \int_0^t \int_0^L (1 + \theta^q)^2 (r^m u)_x \theta_t^2 dxds$$

$$+ C \int_0^t \int_0^L (1 + \theta^q) \theta_t \left\{ - \frac{R \theta}{\eta} + \frac{(r^m u)_x}{\eta} \right\} (r^m u)_x dxds$$

$$- \mu \int_0^t \int_0^L (1 + \theta^q) \theta_t \left( \frac{(r^m u)^2}{\eta} + \frac{r^{2m} u^2}{\eta} \right) dxds \quad (IV_1, IV_2, IV_3, IV_4)$$
ASYMPTOTIC BEHAVIOR OF SYMMETRIC SOLUTION TO N-S EQUATIONS

From Cauchy inequality, Hölder inequality, Lemmas 2.5-2.12, 2.15, and (21), (39), we have

\[ IV_1 + IV_2 \leq C \int_0^t \int_0^L \mu_m \| u \|_{H^2}^2 \, ds + C \leq C, \]

\[ IV_3 \leq C \int_0^t \int_0^L \theta_x^2 \, dxds + C \max_{0 \leq t \leq T} \| \theta \|_{L^\infty} \int_0^t \int_0^L \frac{(r_m u)_x^2}{\theta} \, dxds \]

\[ \leq C \int_0^t \int_0^L \theta_x^2 \, dxds + C \max_{0 \leq t \leq T} \| \theta \|_{L^\infty} \int_0^t \int_0^L \| u_x \|_{L^\infty}^2 \, ds + C \]

\[ \leq C \int_0^t \int_0^L \theta_x^2 \, dxds + C \max_{0 \leq t \leq T} \| \theta \|_{L^\infty} \int_0^t \int_0^L \| u_x \|_{L^\infty}^2 \, ds + C \]

\[ IV_4 \leq C \int_0^t \int_0^L \theta_x^2 \, dxds + C \int_0^t \int_0^L \left( (r_m v)_x^4 + r_m w_x^4 \right) \, dxds \]

\[ \leq C \int_0^t \int_0^L \theta_x^2 \, dxds + C \sup_{0 \leq t \leq T} \int_0^t \int_0^L v_x^2 \, dx \int_0^t \int_0^L \| v_x \|_{L^\infty}^2 \, ds + C \]

\[ \leq C \int_0^t \int_0^L \theta_x^2 \, dxds + C \int_0^t \int_0^L \| v_x \|_{L^2}^2 \, ds + C \int_0^t \int_0^L \| w_x \|_{L^2}^2 \, ds + C \leq C, \]

and

\[ IV_5 \leq C \int_0^t \int_0^L \theta_x^2 \, dxds + C \int_0^t \int_0^L \left( (r_m u^2 + r_m v^2)_x^2 \right) \, dxds \leq C. \]

Therefore, we get

\[ \int_0^t | d | \int_0^L \frac{\mu_m (1 + \theta^q) \theta_t (r_m^m - 1 u^2 + r_m^m - 1 v^2)_x \, dxds. \]

This completes the proof of the Lemma 2.17.

Recalling (41), and using previous lemmas, we have the following useful result.

**Corollary 2.3.** Under the conditions of Theorem 2.1, it holds that

\[ \int_0^t \left( 1 + \theta^q \right)^2 \theta_x^2 \, dx + \int_0^t \int_0^L \left( 1 + \theta^q \right) \theta_x^2 \, dxds \leq C. \]

for any \( t \in [0, T] \).

With all the \textit{a priori} estimates in Lemmas 2.1-2.17 at hand, we will establish the asymptotic behavior and the convergence rates of \( \eta, u, v, w, \theta \) and \( r \).
Lemma 2.18. Under the conditions of Theorem 2.1, we have

\[
\lim_{t \to \infty} \int_0^L (\eta_x^2 + u_x^2 + v_x^2 + w_x^2 + \theta_x^2) dx = 0,
\]

\[
\lim_{t \to \infty} \int_0^L |r_{xx} - \tilde{r}_{xx}|^2 dx = 0,
\]

\[
\lim_{t \to \infty} \max_{x \in [0,L]} (|\eta(x,t) - \tilde{\eta}|, |u(x,t)|, |v(x,t)|, |w(x,t)|, |\theta(x,t) - \tilde{\theta}|) = 0,
\]

and

\[
\lim_{t \to \infty} \max_{x \in [0,L]} (|r(x,t) - \tilde{r}| + |r_x(x,t) - \tilde{r}_x|) = 0,
\]

where \(\tilde{\eta}, \tilde{\theta}\) and \(\tilde{r}(x)\) have been defined in (14).

Proof. It follows from Lemmas 2.5-2.8, 2.10-2.12, 2.15, Corollary 2.3 that

\[
\int_0^t \int_0^L \left( u_x^2 + v_x^2 + w_x^2 + (1 + \theta^q)^2 \theta_x^2 \right) dx ds \leq C,
\]

and

\[
\int_0^t \left( \frac{d}{ds} \int_0^L u_x^2 dx \right) + \frac{d}{ds} \int_0^L v_x^2 dx + \frac{d}{ds} \int_0^L w_x^2 dx + \frac{d}{ds} \int_0^L (1 + \theta^q)^2 \theta_x^2 dx \right) ds \leq C.
\]

Thus we have

\[
\lim_{t \to \infty} \int_0^L \left( u_x^2 + v_x^2 + w_x^2 + (1 + \theta^q)^2 \theta_x^2 \right) dx = 0. \tag{42}
\]

Then by Sobolev inequality, we have

\[
\lim_{t \to \infty} \max_{x \in [0,L]} |u(x,t)| \leq \lim_{t \to \infty} \left( \int_0^L u_x^2 dx \right)^{\frac{1}{2}} = 0. \tag{43}
\]

Similarly, we can also get the estimates of \(v(x,t)\) and \(w(x,t)\). It follows from Poincaré inequality, (7), (14) and (42) that

\[
|\theta - \tilde{\theta}| \leq |\theta - \frac{1}{L} \int_0^L \theta dx| + |\frac{1}{L} \int_0^L \theta dx - \frac{1}{2C_L} \int_0^L \left( 2C \theta_0 + u_0^2 + v_0^2 + w_0^2 \right) dx | \leq C \left( \int_0^L \theta_x^2 dx \right)^{\frac{1}{2}} \rightarrow 0, \text{ as } t \to \infty.
\]

Hence, there exists some time \(T_0 > 0\) such that as \(t \geq T_0 > 0\),

\[
0 < \frac{\dot{\theta}}{2} \leq \theta \leq \frac{3\dot{\theta}}{2},
\]

which, along with Lemma 2.4, yields

\[
\theta \geq \min \left( \frac{\theta}{2}, \frac{1}{C + CT_0} \right), \text{ for any } (x,t) \in [0,L] \times [0,\infty). \tag{44}
\]

By Lemmas 2.9, 2.15 and (44), we have

\[
\lim_{t \to \infty} \int_0^L \eta_x^2 dx = 0. \tag{45}
\]
It follows from (45) that
\[ |\eta(x, t) - \tilde{\eta}| = |\eta(x, t) - \frac{1}{L} \int_0^L \eta dx| \leq C \left( \int_0^L \eta_x^2 dx \right)^{\frac{1}{2}} \to 0, \text{ as } t \to \infty. \] (46)

On the other hand, we have
\[ r^{m+1}(x, t) = r^{m+1}(0, t) + (m+1) \int_0^x \eta dy = a^{m+1} + (m+1) \eta x + (m+1) \int_0^x (\eta - \tilde{\eta}) dy. \]

Hence, we have
\[ \max_{x \in [0, L]} |r^{m+1}(x, t) - (a^{m+1} + (m+1) \eta x)| \leq (m+1) \int_0^x (\eta - \tilde{\eta}) dy \leq C \max_{x \in [0, L]} |\eta(x, t) - \tilde{\eta}|. \]

By (46), we obtain
\[ \lim_{t \to \infty} \max_{x \in [0, L]} |r(x, t) - \tilde{r}| = 0. \] (47)

Next, using (46) and (47), we have
\[ |r_x(x, t) - \tilde{r}_x| = |\eta \frac{\eta}{r^m} - \frac{\tilde{\eta}}{\tilde{r}^m}| \leq |\eta(x, t) - \tilde{\eta}| \frac{r^{m+1} - r^m - r^m \tilde{\eta}}{r^m \tilde{r}^m}. \]

Then we get
\[ \lim_{t \to \infty} \max_{x \in [0, L]} |r_x(x, t) - \tilde{r}_x| = 0. \]

Finally, we have
\[ \int_0^L |r_{xx}(x, t) - \tilde{r}_{xx}|^2 dx \]
\[ = \int_0^L \left| \eta \frac{\eta}{r^m} - \frac{m \eta^2}{r^{2m+1}} + \frac{m \tilde{\eta}^2}{\tilde{r}^{2m+1}} \right|^2 dx \]
\[ \leq C \int_0^L \eta_x^2 dx + C \int_0^L \left| \frac{\eta^2}{r^{2m+1}} - \frac{\tilde{\eta}^2}{\tilde{r}^{2m+1}} \right|^2 dx \]
\[ \leq C \int_0^L \eta_x^2 dx + C \max_{x \in [0, L]} \left( \frac{|\eta - \tilde{\eta}|^2}{r^{2m+1}} + \frac{|r^m - \tilde{r}^m|}{r^{2m+1}} \right)^2 dx. \]

By (46), (47), we complete the proof of Lemma 2.18. \hfill \square

**Proof of Theorem 1.1**

Define
\[ K(t) = \int_0^L \eta_x^2 dx, \]
\[ V(t) = \int_0^L (u_x^2 + v_x^2 + w_x^2 + \theta_x^2) dx, \]
\[ Z(t) = \int_0^L \left( C_v \theta + \frac{1}{2} (u^2 + v^2 + w^2) - \bar{\theta} S + \nu \right) dx, \text{ for } S = C_v \log \theta + R \log \eta, \]
and
\[ \nu = C_v \bar{\theta} (\log \bar{\theta} - 1) + R \bar{\theta} \log \tilde{\eta}. \]

It follows from (7) that
\[ \left( C_v \theta + \frac{1}{2} (u^2 + v^2 + w^2) - \bar{\theta} S + \nu \right) + \frac{\eta \nu \bar{\theta}}{\theta} + \frac{\kappa r^{2m} \theta^2 \bar{\theta}}{\eta \theta^2}. \]
\[
\begin{align*}
&= \left( r^m u + \frac{\mu r^m v (r^m v)}{\eta} + \frac{\mu r^{2m} w w_x}{\eta} - 2 \mu m (r^m u^2 + r^{m-1} v^2) \\
&\quad + \left( 1 - \frac{\theta}{\bar{\theta}} \right) \nu r^{2m} \theta_x \right)_x.
\end{align*}
\]

Hence, we obtain
\[
\frac{d}{dt} \tilde{Z}(t) + cV(t) \leq 0. \tag{48}
\]

It follows from Taylor expansion and Lemma 2.18 that
\[
C_v(\theta - \bar{\theta} \log \theta) + C_v \tilde{\theta} (\log \bar{\theta} - 1) \leq C(\theta - \bar{\theta})^2,
\]
and
\[
\int_0^L \log \frac{\eta}{\tilde{\eta}} dx \leq C \int_0^L (\eta - \tilde{\eta})^2 dx.
\]

Then by Poincaré inequality and Cauchy inequality, we infer
\[
\tilde{Z}(t) = \int_0^L \left( C_v \theta + \frac{1}{2} (u^2 + v^2 + w^2) - \tilde{\theta} S + \nu \right) dx
\]
\[
= \int_0^L \left[ C_v (\theta - \bar{\theta} \log \theta) + C_v \tilde{\theta} (\log \bar{\theta} - 1) + R \tilde{\theta} \log \frac{\eta}{\tilde{\eta}} + \frac{1}{2} (u^2 + v^2 + w^2) \right] dx
\]
\[
\leq C \int_0^L [(\theta - \bar{\theta})^2 + (\eta - \tilde{\eta})^2 + u^2 + v^2 + w^2] dx
\]
\[
\leq C \int_0^L (\theta_x^2 + \eta_x^2 + u_x^2 + v_x^2 + w_x^2) dx
\]
\[
\leq C(\tilde{K}(t) + V(t)). \tag{49}
\]

On the other hand, it follows from (26) and Lemma 2.16 that
\[
\frac{d}{dt} K(t) + K(t) \leq \frac{d}{dt} K(t) + C \int_0^L \theta \eta_x^2 dx
\]
\[
\leq C \int_0^L \theta \eta_x^2 dx + \int_0^L \left( \frac{u}{r^m} \right)_t \eta_x dx + \int_0^L \frac{m u^2 - v^2}{r^{m+1}} \eta_x dx
\]
\[
\leq C(\tilde{K}(t) + C \| u \|^2_{L^1} = \int_0^L \frac{(r^m u)^2}{\eta} dx + C \int_0^L (\eta_x^2 \theta_x^2 + \theta_x^4 + v^4 + u^4) dx
\]
\[
\leq C(\tilde{K}(t) + C \int_0^L (u_x^2 + v_x^2 + w_x^2 + \theta_x^2) dx
\]
\[
\leq CV(t). \tag{50}
\]

Finally, by Lemmas 2.10-2.12, 2.17, we also have
\[
\frac{d}{dt} V(t) + c \int_0^L (\theta_{xx}^2 + u_{xx}^2 + v_{xx}^2 + w_{xx}^2) dx
\]
\[
\leq C \int_0^L (u_x^2 + v_x^2 + w_x^2 + \theta_x^2 + u^2 \eta_x^2 + \theta_x^2 + u^2 \eta_x^2 + w^2 \eta_x^2 + w^2 \eta_x^2 + u^2 \eta_x^2 + v^2 \eta_x^2 + v^2 \eta_x^2)
\]
\[
+ w^2 \eta_x^2 + \theta_x^2 \eta_x^2 + \theta^2 \eta_x^2 + \theta^2 \eta_x^2 + \theta^2 \eta_x^2 + \theta^2 \eta_x^2 + v^4 + v^4 + w^4 + u^4 + v^4 + w^4) dx
\]
\begin{align*}
\leq \frac{c}{2} \int_0^L (\theta_{xx}^2 + u_{xx}^2 + v_{xx}^2 + w_{xx}^2) dx + C \int_0^L (u_x^2 + v_x^2 + w_x^2 + \theta_x^2) dx + C \int_0^L \eta_{xx}^2 dx.
\end{align*}

Thus
\[
\frac{d}{dt} V(t) \leq C(V(t) + K(t)).
\]

Adding (48) multiplied by \(\epsilon\) to (50), one has
\[
\frac{d}{dt}(\bar{Z}(t) + \epsilon K(t)) + C(V(t) + K(t)) \leq 0.
\]

Adding (52) multiplied by \(\epsilon\) to (51), using (49), we obtain
\[
\frac{d}{dt}(\bar{Z}(t) + \epsilon K(t) + \epsilon V(t)) + C(\bar{Z}(t) + \epsilon K(t) + \epsilon V(t)) \leq 0.
\]

Hence we can get the asymptotic behavior of \(r(x, t)\). This completes the proof of Theorem 1.1.

3. Appendix. Proof of Remark 1.2. For the case of \(\tilde{\kappa}_1 = 0\), the proof is similar to that of \(\kappa_1 > 0\) with some slight modifications. We only give some key steps of proof.

\textbf{Proof.} Under the conditions of Theorem 1.1, for any \(t \in [0, T]\) and any constant \(\delta > 0\), there exists \(C_\delta > 0\), such that

\begin{itemize}
\item[(a)]
\[
\| \eta_x \|^2 + \int_0^t \int_0^L \theta \eta_{xx}^2 dx ds \leq C + Y^{\frac{3}{2+q}};
\]
\item[(b)]
\[
\int_0^L (v_x^2 + w_x^2) dx + \int_0^t \int_0^L (u_x^2 + w_{xx}^2) dx ds \leq C + CY^{\frac{3}{2+q}};
\]
\item[(c)]
\[
\int_0^L u_x^2 dx + \int_0^t \int_0^L u_{xx}^2 dx ds \leq C + CY^{\frac{3}{2+q}};
\]
\item[(d)]
\[
Y + \int_0^t \int_0^L (1 + \theta^q) \theta_x^2 dx ds \leq \delta Z + C_\delta.
\]
\end{itemize}

As the same as the proof of Lemma 2.9, for the inequality (27), we deal with as follow:
\[
\int_0^t \int_0^L \frac{R \theta_x \eta_x}{\eta^2} dx ds \leq \frac{R}{2} \int_0^t \int_0^L \theta \eta_{xx}^2 dx ds + C \max_{Q_T} \theta^{1-q} \int_0^t \int_0^L \theta \eta_x^2 dx ds
\leq \frac{R}{2} \int_0^t \int_0^L \theta \eta_{xx}^2 dx ds + CY^{\frac{3}{2+q}},
\]

where \(q < 1\). Thus we get the estimate of (a).

For the estimate of (b), by (33) and (a), we have
\[
\int_0^t \int_0^L (v_x^2 + w_x^2) \eta_{xx}^2 dx ds \leq C \sup_{0 \leq t \leq T} \int_0^L \eta_{xx}^2 dx \int_0^t \big( \| v \|_{L_\infty}^2 + \| v_x \|_{L_\infty}^2 \big) ds
\leq C(1 + Y^{\frac{3}{2+q}}) \left(1 + \left( \int_0^t \int_0^L v_{xx}^2 dx ds \right)^\frac{1}{2} \left( \int_0^t \int_0^L v_{xx}^2 dx ds \right)^\frac{1}{2} \right)
\leq \frac{1}{2} \int_0^t \int_0^L v_{xx}^2 dx ds + C(1 + Y^{\frac{3}{2+q}}).
\]
Next, we can also obtain the similar estimate for \( w \).

By (53), (36) and (a), it follows
\[
C \int_0^t \int_0^L (u^2 \eta_x^2 + u_x^2 \eta_x^2 + \theta \eta_x^2 + \theta_x^2) dx ds
\leq C \sup_{0 \leq t \leq T} \int_0^L \eta_x^2 dx \int_0^t (\| u \|_{L^\infty}^2 + \| u_x \|_{L^\infty}^2) ds + C \max_{Q_T} \theta \int_0^t \int_0^L \theta_x^2 dx ds
\leq C (1 + Y^{\frac{2-q}{2+q}}) \left( 1 + (\int_0^t \int_0^L u_x^2 dx ds)^{\frac{1}{2}} \left( \int_0^t \int_0^L u_{xx}^2 dx ds \right)^{\frac{1}{2}} \right) + C (1 + Y^{\frac{2-q}{2+q}})
\leq \frac{1}{2} \int_0^t \int_0^L u_{xx}^2 dx ds + C (1 + Y^{\frac{2-q}{2+q}}) \max_{Q_T} \theta \int_0^t \int_0^L u_x^2 dx ds + C (1 + Y^{\frac{2-q}{2+q}})
\leq \frac{1}{2} \int_0^t \int_0^L u_{xx}^2 dx ds + C (1 + Y^{\frac{3-2q}{2+2q}}),
\] where taking \( q < 1 \).

Finally, as for the relation between \( Y \) and \( Z \), the proof is similar to Lemma 2.13 with some slight modifications. For convenient reading, we give a simplified proof in the following.

For \( I_1 \),
\[
I_1 \leq C (1 + \max_{Q_T} \theta^q) \max_{Q_T} \theta^3 \int_0^t \int_0^L \frac{(r^m u)^2_x}{\theta} dx ds \leq C + CY^{\frac{-q}{2+q}}.
\]

For \( I_2 \),
\[
I_2 \leq C (1 + \max_{Q_T} \theta^q) \int_0^t \| r^m u \|_{L^\infty}^2 \| (r^m u)_{xx} \|_{L^2}^2 ds
\leq C (1 + Y^{\frac{q}{2+q}}) \int_0^t \int_0^L r^m u \| (r^m u)_x \| dx \| (r^m u)_{xx} \|_{L^2}^2 ds
\leq C (1 + Y^{\frac{q}{2+q}}) \left( \int_0^L u^2 dx \right)^{\frac{1}{2}} \left( \int_0^L (r^m u)_x^2 dx \right)^{\frac{1}{2}} \| (r^m u)_{xx} \|_{L^2}^2 ds
\leq C (1 + Y^{\frac{q}{2+q}}) (1 + Y^{\frac{3-2q}{2+2q}})(1 + Y^{\frac{3-2q}{2+2q}})
\leq C + CY^{\frac{-q}{2+2q}}.
\]

For \( I_3 + I_4 \),
\[
I_3 + I_4 \leq C (1 + \max_{Q_T} \theta^q) \left( \int_0^t \| v \|_{L^\infty}^2 ds + \sup_{0 \leq t \leq T} \int_0^L v_x^2 dx \int_0^t \| v_x \|_{L^\infty}^2 ds \right)
+ C (1 + \max_{Q_T} \theta^q) \sup_{0 \leq t \leq T} \int_0^L u_x^2 dx \int_0^t \| w_x \|_{L^\infty}^2 ds
\leq C (1 + Y^{\frac{q}{2+q}}) \left( 1 + \int_0^t \| v_x \|^2 ds + \int_0^t \| w_x \|^2 ds \right)
\leq C + CY^{\frac{2-q}{2+2q}}.
\]
For $I_5$,

$$I_5 \leq C(1 + \max Q_T) \sup_{0 \leq t \leq T} \int_0^L u_x^2 \, dx \int_0^t \| u \|_{L^\infty}^2 \, ds + C \int_0^t \| v \|_{L^\infty}^2 \int_0^L v_x^2 \, dx \, ds$$

$$\leq C(1 + Y^{\frac{2}{q+2}})(1 + Y^{\frac{2}{q+2}})$$

$$\leq C + CY^{\frac{2}{q+2}}.$$

For $I_6$, using Poincaré inequality, Lemmas 2.1, 2.8, we obtain

$$I_6 \leq \int_0^t (\| u \|_{L^\infty} + \| (r^m u)_x \|_{L^\infty}) \int_0^L (1 + \theta^q)^{2 \theta^2} \, dx \, ds$$

$$\leq \max_{Q_T} u_x \mid (1 + \max_{Q_T} \theta^{2+q}) \int_0^t \int_0^L (1 + \theta^q)^{2 \theta^2} \, dx \, ds$$

$$\leq C(1 + Z^{\frac{2}{q}})(1 + Y^{\frac{2}{q+2}})$$

$$\leq C + CY^{\frac{2}{q+2}} Z^{\frac{2}{q}} + C Y^{\frac{2}{q+2}} + C Z^{\frac{2}{q}}.$$

Thus for $q > \frac{1}{2}$, we have

$$\int_0^t \int_0^L (1 + \theta^q)^{2 \theta^2} \, dx \, ds + Y$$

$$\leq C + \epsilon \int_0^t \int_0^L (1 + \theta^q)^{2 \theta^2} \, dx \, ds + CY^{\frac{2}{q+2}} Z^{\frac{2}{q}} + CY^{\frac{3+q}{2}} + CY^{\frac{2+q}{2+q}} + CZ^{\frac{2}{q}}$$

$$\leq C + \epsilon \int_0^t \int_0^L (1 + \theta^q)^{2 \theta^2} \, dx \, ds + \delta Z.$$

Using the same method as the case of the $\tilde{k}_1 > 0$, for $\frac{1}{2} < q < 1$, we can get Lemma 2.14 and Lemma 2.15. We omit the proof here.

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