SUBMANIFOLDS WITH BIHARMONIC GAUSS MAP

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Abstract. We generalize the Ruh-Vilms problem by characterizing the submanifolds in Euclidean spaces with proper biharmonic Gauss map and we construct examples of such hypersurfaces.

1. Introduction

As it is classically known, most of the extrinsic geometry of an oriented submanifold \( M^m \) in the Euclidean space \( \mathbb{R}^{m+n} \) can be described by its Gauss map \( \gamma : M \to G(m, n) \) which assigns to every point \( p \in M \) the tangent space \( T_p M \), thought of as a point of the Grassmannian of oriented \( m \)-dimensional subspaces of \( \mathbb{R}^{m+n} \). A splendid example is the celebrated Ruh-Vilms Theorem which asserts that the Gauss map \( \gamma : M \to G(m, n) \) is a harmonic map if and only if the mean curvature vector field of \( M \) in \( \mathbb{R}^{m+n} \) is parallel. Here we say that a smooth map \( \phi : (M, g) \to (N, h) \) between Riemannian manifolds is harmonic if it is a critical point of the energy functional \( E(\phi) = \frac{1}{2} \int_M |d\phi|^2 \, v_g \), i.e. \( \phi \) is a solution of the corresponding Euler-Lagrange equation which is given by the vanishing of the tension field \( \tau(\phi) = \text{trace} \nabla d\phi \).

A natural extension of harmonic maps is provided by biharmonic maps (as suggested by J. Eells and J.H. Sampson in [7]) which are the critical points of the bienergy functional \( E_2(\phi) = \frac{1}{2} \int_M |\tau(\phi)|^2 \, v_g \). In [10] G.Y. Jiang derived the first variation formula of the bienergy showing that the Euler-Lagrange equation for \( E_2 \) is

\[
\tau_2(\phi) = -J(\tau(\phi)) = -\Delta \tau(\phi) - \text{trace} \nabla^N (d\phi, \tau(\phi)) d\phi \tag{1.1}
\]

where \( J \) is (formally) the Jacobi operator of \( \phi \), \( \Delta \) is the rough Laplacian defined on sections of \( \phi^{-1}(TN) \) and \( R^N(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \) is the curvature operator on \( (N, h) \).

In this paper we propose to study the biharmonic equation \( (\tau_2(\phi) = 0) \) for the Gauss map of submanifolds in the Euclidean space, in the intent to generalize the Ruh-Vilms Theorem to the case of biharmonicity. To pursue our intent we first derive the equation that characterizes the submanifolds in the Euclidean space with biharmonic Gauss map (Theorem 3.1). Although the condition that ensures the biharmonicity of the Gauss map is rather technical, in the case of hypersurfaces it simplifies and gives the following: the Gauss map of an orientable hypersurface \( M^m \) in \( \mathbb{R}^{m+1} \) is proper biharmonic if and only if \( \text{grad} \, f \neq 0 \) and

\[
\Delta \text{grad} \, f + A^2(\text{grad} \, f) - |A|^2 \, \text{grad} \, f = 0,
\]

where \( \Delta \) denotes the rough Laplacian on \( C(TM) \) while \( f \) and \( A \) denote the mean curvature function and the shape operator, respectively (see also [2]).

2000 Mathematics Subject Classification. 58E20.

Key words and phrases. Biharmonic maps, Gauss map, hypercones, isoparametric hypersurfaces.
The last part of the paper is devoted to the construction of examples of hypersurfaces with biharmonic Gauss map. We study the biharmonicity of the Gauss map for hypercones generated by constant mean curvature hypersurfaces in spheres (Theorem 4.3) and, in particular, by isoparametric hypersurfaces, obtaining explicit examples. Non-existence results for hypercones in $\mathbb{R}^3$ and $\mathbb{R}^4$ with proper biharmonic Gauss map are obtained (Theorem 4.4 and Theorem 4.5).

2. Preliminaries

2.1. Biharmonic maps between Riemannian manifolds. We recall the following facts on biharmonic maps:

(i) the equation $\tau_2(\phi) = 0$ is called the biharmonic equation and a map $\phi$ is biharmonic if and only if its tension field is in the kernel of the Jacobi operator;

(ii) a harmonic map is obviously a biharmonic map. We call proper biharmonic the biharmonic non-harmonic maps;

(iii) a harmonic map is an absolute minimum of the bienergy;

(iv) if $M$ is compact and $\text{Riem}^N \leq 0$, i.e. the sectional curvature of $(N,h)$ is non-positive, then $\phi : M \rightarrow N$ is biharmonic if and only if it is harmonic;

(v) if $\phi : M \rightarrow N$ is a Riemannian immersion with $|\tau(\phi)| = \text{constant}$ and $\text{Riem}^N \leq 0$, then $\phi$ is biharmonic if and only if it is harmonic (minimal).

The first three remarks are immediate consequences of the definition of the bienergy and of (1.1). The non-existence results (iv) and (v) are proved in [9] and in [16], respectively.

On the other hand, in Euclidean spheres we do have examples of proper biharmonic submanifolds, i.e. non-minimal submanifolds for which the inclusion map is biharmonic. It was conjectured in [3] that the only proper biharmonic hypersurfaces in $S^{m+1}$ are the open parts of the hypersphere $S^m(\frac{1}{\sqrt{2}})$ and of the generalized Clifford torus $S^{m_1}(\frac{1}{\sqrt{2}}) \times S^{m_2}(\frac{1}{\sqrt{2}})$, $m_1 + m_2 = m$ and $m_1 \neq m_2$.

For a general account on biharmonic maps see [12].

2.2. The Gauss map. Consider $M^m$ to be a $m$-dimensional oriented submanifold in $\mathbb{R}^{m+n}$. The map which assigns to every point $p \in M$ the oriented tangent space $T_pM$, thought of as a point of the Grassmannian of oriented $m$-dimensional subspaces of $\mathbb{R}^{m+n}$,

$$\gamma : M \longrightarrow G(m,n)$$

$$p \longmapsto T_pM,$$

is called the Gauss map associated to $M$.

As usually, the Riemannian structure on $G(m,n)$ is defined by considering the Euclidean metric on $\mathbb{R}^{m+n}$ and by identifying the tangent space to $G(m,n)$ at a point $P \in G(m,n)$ as follows:

$$T_PG(m,n) = \text{Hom}(P,P^\perp) = P^* \otimes P^\perp.$$ 

Thus, if we fix a positive oriented orthonormal basis $e_1, \ldots, e_m$ of $P$ and complete it to an orthonormal basis of $\mathbb{R}^{m+n}$ with $e_{m+1}, \ldots, e_{m+n}$, spanning $P^\perp$, then a basis of $T_PG(m,n)$ will be given by

$$\{e_i^* \otimes e_{m+a}\}_{i=1, m \atop a=1, n}$$
which can be also written as

\[ \{ e_1 \wedge \ldots \wedge e_{i-1} \wedge e_{m+a} \wedge e_{i+1} \wedge \ldots \wedge e_m \}_{i=1,m}^{a=1,m}. \]

The Riemannian metric on the Grassmannian \( G(m, n) \) is given by requesting that the basis \( \{ e_i^* \otimes e_{m+a} \} \) is an orthonormal basis.

The curvature tensor field can be determined by identifying the Grassmannian as a symmetric space (see, for example, [17 p.219]) and in our formalism

\[
R_P(\rho_1, \rho_2)\rho_3 = \langle X_1, X_2 \rangle \langle \eta_2, \eta_3 \rangle X_3^* \otimes \eta_1 - \langle X_1, X_2 \rangle \langle \eta_1, \eta_3 \rangle X_3^* \otimes \eta_2
\]

\[
+ \langle X_2, X_3 \rangle \langle \eta_1, \eta_2 \rangle X_1^* \otimes \eta_3 - \langle X_1, X_3 \rangle \langle \eta_1, \eta_2 \rangle X_2^* \otimes \eta_3,
\]

where \( P \in G(m, n) \) and \( \rho_i = X_i^* \otimes \eta_i, \ X_i \in P, \ \eta_i \in P^\perp, \ i = 1, 2, 3. \)

We recall that for the pull-back bundle of the tangent bundle induced by \( \gamma \) we have the isometric identification

\[
\gamma^{-1}(TG(m, n)) = \bigcup_{p \in M} T_{\gamma(p)}G(m, n) = \bigcup_{p \in M} (T_p^* M \otimes N_p M) = T^* M \otimes NM,
\]

where \( NM \) denotes the normal bundle of \( M \) in \( \mathbb{R}^{m+n} \).

2.3. The tension field of the Gauss map. In the following we intend to recall the fundamental technical steps needed for the characterization of the harmonicity of the Gauss map.

Consider \( v \in T_p M \). In order to compute \( d\gamma(v) \) consider \( \sigma : I \rightarrow M \) to be a curve with \( \sigma(0) = p \) and \( \dot{\sigma}(0) = v \). Let now \( \{ e_i \}_{i=1}^m \) be a positive oriented orthonormal basis in \( T_p M \). By parallel transporting it along \( \sigma \) we obtain a positive oriented orthonormal basis \( \{ e_i(t) \}_{i=1}^m \) in \( T_{\sigma(t)} M \), for all \( t \). Since \( e_i(t) \) are obtained by parallel transport along \( \sigma \), we have

\[
\dot{e}_i(t) = \nabla^M_{\dot{\sigma}} e_i = \nabla^M_{\dot{\sigma}} e_i + B(\dot{\sigma}, e_i(t)) = B(\dot{\sigma}, e_i(t)),
\]

where \( B \) is the second fundamental form of \( M \) in \( \mathbb{R}^{m+n} \). This implies that

\[
d\gamma_p(v) = \frac{d}{dt}\bigg|_{t=0} (\gamma \circ \sigma)(t) = \frac{d}{dt}\bigg|_{t=0} (e_i(t) \wedge \ldots \wedge e_m(t))
\]

\[
= \sum_{i=1}^m e_i^* \otimes B(v, e_i),
\]

by using the standard identifications.

The fundamental result concerning the harmonicity of the Gauss map was obtained in [18]. We shall present here a computation that follows [6]. By using (2.2) one can compute the tension field of the Gauss map in terms of the mean curvature of \( M \). Since the bundles \( \gamma^{-1}(TG(m, n)) \) and \( T^* M \otimes NM \) are isometric, we can write

\[
\nabla^\gamma \rho = \nabla \omega \otimes \eta + \omega \otimes \nabla^\perp \eta,
\]

where the section \( \rho \in C(\gamma^{-1}(TG(m, n))) \) in the pull-back bundle is such that it can naturally be identified with \( \omega \otimes \eta \in C(T^* M \otimes NM) \).

Consider \( \{ E_i \}_{i=1}^m \) to be a local positive oriented orthonormal frame field, geodesic at \( p \in M \). By using the expression (2.2) for the differential of the Gauss map and the consequence of the Codazzi equation, \( \nabla_{E_i} B(E_j, E_k) = \nabla_{E_j} B(E_i, E_k) \), for all \( i, j, k \), we get at \( p \),

\[
\tau(\gamma) = \sum_{i=1}^m \nabla d\gamma(E_i, E_i) = m \sum_{j=1}^m E_j^* \otimes \nabla^\perp_{E_j} H,
\]
where $H = \frac{1}{m} \text{trace } B$ is the mean curvature vector field of $M$ in $\mathbb{R}^{m+n}$. We note that $E_j^*$ coincides with $E_j^0$ obtained by the musical isomorphism $\flat$.

Then, the Ruh-Vilms Theorem is an immediate consequence of (2.3).

### 3. The biharmonic equation for the Gauss map

Inspired by the expression for the tension field given in the previous section, we now characterize the biharmonicity of the Gauss map in terms of the second fundamental form of the submanifold. We obtain

**Theorem 3.1.** The Gauss map associated to a $m$-dimensional orientable submanifold $M$ of $\mathbb{R}^{m+n}$ is biharmonic if and only if

$$
\nabla_X^\perp \Delta^\perp H - m\nabla_{A_X}^\perp H + \text{trace } B(2A^\perp
\nabla_X^\perp H) - A\nabla_X^\perp H = 0,
$$

(3.1)

for all $X \in C(TM)$, where $A$ denotes the Weingarten operator and $H$ the mean curvature vector field of $M$ in $\mathbb{R}^{m+n}$.

**Proof.** We fix an orientation on $M$ and consider $\{E_i\}_{i=1}^m$ to be a local positive oriented orthonormal frame field, geodesic at $p \in M$. In order to determine the bitension field of the Gauss map, by using (2.1), (2.2) and (2.3) we obtain

$$
\text{trace } R(d\gamma, \tau(\gamma))d\gamma = \sum_{i=1}^m R(d\gamma(E_i), \tau(\gamma))d\gamma(E_i)
$$

$$
= m \sum_{h,i,j,k=1}^m R(E_j^* \otimes B(E_i, E_j), E_k^* \otimes \nabla_{E_h}^\perp H) E_i^* \otimes B(E_i, E_k)
$$

$$
= m \sum_{h,i,j,k=1}^m \left\{ \langle \nabla_{E_h}^\perp H, B(E_i, E_k) \rangle E_i^* \otimes B(E_i, E_j)
$$

$$
- \delta_{jh} \langle B(E_i, E_j), B(E_i, E_k) \rangle E_i^* \otimes \nabla_{E_h}^\perp H
$$

$$
+ \delta_{hk} \langle \nabla_{E_h}^\perp H, B(E_i, E_j) \rangle E_i^* \otimes B(E_i, E_k)
$$

$$
- \delta_{jk} \langle \nabla_{E_j}^\perp H, B(E_i, E_j) \rangle E_i^* \otimes B(E_i, E_k) \right\}
$$

$$
= m \sum_{i,j,k=1}^m \left\{ \langle \nabla_{E_k}^\perp H, B(E_i, E_k) \rangle E_k^* \otimes B(E_i, E_j)
$$

$$
- \langle B(E_i, E_j), B(E_i, E_k) \rangle E_k^* \otimes \nabla_{E_j}^\perp H
$$

$$
+ \langle \nabla_{E_k}^\perp H, B(E_i, E_j) \rangle E_j^* \otimes B(E_i, E_k)
$$

$$
- \langle \nabla_{E_k}^\perp H, B(E_i, E_j) \rangle E_j^* \otimes B(E_i, E_k) \right\}.
$$

Further computations lead to

$$
\text{trace } R(d\gamma, \tau(\gamma))d\gamma = m \sum_{k=1}^m \left\{ E_k^* \sum_{i,j=1}^m \left( (2\langle \nabla_{E_j}^\perp H, B(E_i, E_k) \rangle
$$

$$
- \langle B(E_i, E_j), \nabla_{E_k}^\perp H \rangle) B(E_i, E_j)
$$

$$
- \langle B(E_i, E_j), B(E_i, E_k) \rangle \nabla_{E_j}^\perp H \right) \right\}.
$$
By using the Weingarten operator we can express
\[
2\langle \nabla_{E_j} \parallel H, B(E_i, E_k) \rangle - \langle B(E_i, E_j), \nabla_{E_k} \parallel H \rangle = \langle 2A\nabla_{\parallel E_j}^{-1}(E_k) - A\nabla_{\parallel E_k}^{-1}(E_j), E_i \rangle,
\]
and from the Gauss equation of \( M \) in \( \mathbb{R}^{m+n} \),
\[
\langle B(Y, T), B(X, Z) \rangle = \langle B(X, T), B(Y, Z) \rangle - \langle R^M(X, Y)Z, T \rangle, \ \forall X, Y, Z, T \in C(TM),
\]
for \( Y = Z = E_i, X = E_k, T = E_j \), we get
\[
\sum_{i,j=1}^{m} \langle B(E_i, E_j), B(E_i, E_k) \rangle E_j = \sum_{i,j=1}^{m} \left\{ \langle B(E_k, E_j), B(E_i, E_i) \rangle - \langle R^M(E_k, E_i)E_i, E_j \rangle \right\} E_j
\]
\[
= \sum_{i=1}^{m} \left\{ m\langle A_H(E_k), E_i \rangle E_i - R^M(E_k, E_i)E_i \right\}
\]
\[
= mA_H(E_k) - \text{Ricci}^M(E_k),
\]
where \( \text{Ricci}^M \) denotes the Ricci tensor field of \( M \).

By summing up all of the above we obtain
\[
\text{trace } R(d\gamma, \tau(\gamma))d\gamma = m \sum_{k=1}^{m} E_k^* \otimes \left( \sum_{j=1}^{m} B(2A\nabla_{\parallel E_j}^{-1}H(E_k) - A\nabla_{\parallel E_k}^{-1}H(E_j), E_j) \right)
\]
\[
(3.2)
\]
\[-m\nabla_{A_H(E_k)H}^\perp + \nabla_{\text{Ricci}^M(E_k)H}^\perp.\]

In order to compute \(-\Delta^H \tau(\gamma)\) we recall that, since \( \{E_i\}_{i=1}^{m} \) is geodesic at \( p \), \((\nabla_{E_i}E_k)_p = 0 \) and \((\nabla_{E_i}\nabla_{E_k}E_i)_p = 0 \), for all \( i, k = 1, \ldots, m \). Thus, at \( p \) we have
\[
\text{trace } \nabla^2 \tau(\gamma) = \sum_{i=1}^{m} \nabla^\gamma_{E_i} \nabla^\gamma_{E_i} \tau(\gamma) = \sum_{i,k=1}^{m} \nabla^\gamma_{E_i} \nabla^\gamma_{E_i} (E_k^* \otimes \nabla_{E_k}^\perp H)
\]
\[
= \sum_{i,k=1}^{m} \left( \nabla_{E_i} \nabla_{E_i} E_k^* \otimes \nabla_{E_k}^\perp H + E_k^* \otimes \nabla_{E_i}^\perp \nabla_{E_k}^\perp H \right)
\]
\[
= \sum_{i,k=1}^{m} \left( \nabla_{E_i} \nabla_{E_i} E_k^* \otimes \nabla_{E_k}^\perp H + E_k^* \otimes \nabla_{E_i}^\perp \nabla_{E_i}^\perp \nabla_{E_k}^\perp H \right)
\]
\[
(3.3)
\]
\[
\sum_{i,k=1}^{m} E_k^* \otimes \nabla_{E_i}^\perp \nabla_{E_i}^\perp \nabla_{E_k}^\perp H.
\]

Moreover, by using the curvature tensor fields \( R^M \) and \( R^\perp \), for \( \nabla \) and \( \nabla^\perp \), respectively, at \( p \) we obtain
\[
\nabla^\perp_{E_i} \nabla^\perp_{E_i} \nabla_{E_k}^\perp H = \nabla^\perp_{E_i} \left( R^\perp(E_i, E_k)H + \nabla_{E_k}^\perp \nabla_{E_i}^\perp H + \nabla_{[E_i,E_k]}^\perp H \right)
\]
\[
= \langle \nabla_{E_i}^\perp R^\perp(E_i, E_k)H + R^\perp(E_i, E_k) \nabla_{E_i}^\perp H + \nabla_{E_k}^\perp \nabla_{[E_i,E_k]}^\perp H \rangle
\]
\[
= \langle \nabla_{E_i}^\perp R^\perp(E_i, E_k)H + 2R^\perp(E_i, E_k) \nabla_{E_i}^\perp H + \nabla_{E_k}^\perp \left( \nabla_{E_i}^\perp \nabla_{E_i}^\perp H - \nabla_{E_i}^\perp \nabla_{E_i}^\perp H \right) + \nabla_{R^M(E_k,E_i)E_i}^\perp H,\rangle
\]
thus

\[
\text{trace } \nabla^2 \tau(\gamma) = m \sum_{k=1}^{m} E_k \otimes \left( \text{trace } \left\{ \left( \nabla^\perp R^\perp \right)(\cdot, E_k)H + 2R^\perp(\cdot, E_k)\nabla^\perp H \right\} \right) - \nabla^\perp_{E_k} \Delta^\perp H + \nabla^\perp_{\text{Ricci}^M(E_k)} H.
\]

(3.4)

We finally substitute (3.2) and (3.4) into the biharmonic equation and conclude. □

3.1. The case of hypersurfaces. Let \( M \) be a nowhere zero mean curvature hypersurface in \( \mathbb{R}^{m+1} \). We obtain

**Theorem 3.2.** The Gauss map of a non-minimal hypersurface \( M^m \) in \( \mathbb{R}^{m+1} \) is proper biharmonic if and only if \( \text{grad } f \neq 0 \) and

\[
\Delta \text{grad } f + A^2(\text{grad } f) - |A|^2 \text{grad } f = 0.
\]

(3.5)

where \( \Delta \) denotes the rough Laplacian on \( C(TM) \) and \( f \) and \( A \) denote the mean curvature function and, respectively, the shape operator of \( M \) in \( \mathbb{R}^{m+1} \).

**Proof.** In this case we can consider the expression of the mean curvature vector field as \( H = f \eta \), where \( f = |H| \) is the mean curvature function of \( M \) in \( \mathbb{R}^{m+1} \) and \( \eta = \frac{H}{|H|} \) is a unit section in the normal bundle \( NM \).

In the following we shall use the general biharmonic equation (3.1) and express it for the case of hypersurfaces.

Since \( H = f \eta \) and \( \nabla^\perp \eta = 0 \), we get that

\[
\text{trace } R^\perp(\cdot, X)\nabla^\perp H = 0
\]

(3.6) and

\[
\text{trace}(\nabla^\perp R^\perp)(\cdot, X)H = 0.
\]

(3.7)

By considering now \( \{E_i\}_{i=1}^{m} \) to be a local orthonormal frame field on \( M \) we obtain

\[
\text{trace } B\left( 2A\nabla^\perp_{(\cdot)}H(X) - A\nabla^\perp_{\cdot}H(\cdot, X), \cdot \right) = \sum_{i=1}^{m} \left\{ 2B(A(X), E_i(f)E_i) - X(f)B(A(E_i), E_i) \right\}
\]

(3.8)

and

\[
\nabla^\perp_{A\eta(X)}H = \langle X, fA(\text{grad } f) \rangle \eta.
\]

(3.9)

For the final term, we recall that \( \text{grad } \Delta f = \Delta \text{grad } f + \text{Ricci}^M(\text{grad } f) \). Also, as a consequence of the Gauss equation for \( M \) in \( \mathbb{R}^{m+1} \), we have

\[
\text{Ricci}^M(\text{grad } f) = mfA(\text{grad } f) - A^2(\text{grad } f).
\]

Thus, it follows that

\[
\nabla^\perp_{\Delta^\perp H} = \langle X, \Delta \text{grad } f \rangle \eta
\]

(3.10)

Finally, by replacing the expressions (3.6), (3.7), (3.8), (3.9), (3.10) in (3.1), we deduce that the Gauss map of the hypersurface is biharmonic if and only if

\[
\langle X, \Delta \text{grad } f + A^2(\text{grad } f) - |A|^2 \text{grad } f \rangle = 0, \quad \forall X \in C(TM),
\]

and this completes the proof. □
Example 3.3. The first example of hypersurface with proper biharmonic Gauss map is obtained by analyzing the right cylinders in $\mathbb{R}^3$. A right cylinder in $\mathbb{R}^3$ determined by a curve parametrized by arc length has proper biharmonic Gauss map if and only if the curve is a clothoid or a Cornu spiral. Indeed, consider a right cylinder in $\mathbb{R}^3$, determined by a curve $\sigma: I \rightarrow \mathbb{R}^2$, parametrized by arc length. Denote by $s$ the parameter along $\sigma$ and by $t$ the parameter along the generatrix. If $k$ denotes the signed curvature of $\sigma$, then the mean curvature function is $f(s, t) = \pm \frac{1}{2}k(s)$. Moreover, since $A^2(\nabla f) = |A|^2 \nabla f = k^2(s) \nabla f$, equation (3.5) becomes $k = 0$, hence $k$ is a second degree polynomial in $s$ and, by using the fundamental theorem of plane curves, we deduce that $\sigma$ is a clothoid.

By a straightforward computation we obtain

Proposition 3.4. Let $M^m$ be a submanifold in $\mathbb{R}^{m+n}$. Then the generalized cylinder $\mathbb{R} \times M$ in $\mathbb{R}^{m+n+1}$ has biharmonic Gauss map if and only if $M$ has biharmonic Gauss map.

4. Hypercones with biharmonic Gauss map

In order to obtain some examples of hypersurfaces with biharmonic Gauss map, in the following we shall study the hypercones generated by hypersurfaces of the unit Euclidean sphere.

Let us first consider $M$ to be an arbitrary $r$-dimensional submanifold of the unit Euclidean sphere $S^{m+1}$. The cone in $\mathbb{R}^{m+n+2}$ generated by $M$ is defined by the immersion $\phi: (0, \infty) \times M \rightarrow \mathbb{R}^{m+n+2}$

\[
\phi: (0, \infty) \times M \rightarrow \mathbb{R}^{m+n+2} \\
(t, p) \mapsto t \cdot p.
\]

The differential of $\phi$ is determined by

\[
d\phi_{(t, p)}(\partial/\partial t_{(t, p)}) = p = x^\alpha(p)e_\alpha(\phi(p)),
\]

\[
d\phi_{(t, p)}(X_{(t, p)}) = tX_p = t\xi^\alpha(p)e_\alpha(\phi(p)),
\]

where $\{e_\alpha\}_{\alpha=1}^{m+2}$ is the canonical orthonormal frame field on $\mathbb{R}^{m+2}$, $X \in C(TM)$ with $X(p) = \xi^\alpha(p)e_\alpha(p) \in \mathbb{R}^{m+2}$, for all $p \in M$, and, typically, we use the same notation for a vector field and for its lift to the product manifold.

If we denote by $\overline{g}$ the metric on $M$, then the immersion $\phi: (0, \infty) \times M \rightarrow \mathbb{R}^{m+n+2}$ induces on the product $(0, \infty) \times M$ the warped metric $g = dt^2 + t^2\overline{g}$. Thus the cone can be seen as the warped product $M = (0, \infty) \times_{\phi^2} M$.

Denote by $\nabla$ and $\overline{\nabla}$ the Levi-Civita connections on $M$ and on $\overline{M}$, respectively, and recall that (see [14, p.206]) $\overline{\nabla}$ is completely determined by

\[
\begin{cases}
\nabla_{\partial/\partial t}\partial/\partial t = 0 \\
\nabla_{\partial/\partial t}X = \nabla_X\partial/\partial t = \frac{1}{t}X \\
\nabla_XY = \overline{\nabla}_XY - \frac{1}{t}\langle X, Y \rangle \partial/\partial t,
\end{cases}
\]

where $X, Y \in C(TM)$. The second fundamental form of the cone in $\mathbb{R}^{m+n+2}$, obtained by using (4.1), is given by

\[
B(\partial/\partial t, \partial/\partial t) = 0, \quad B(X, \partial/\partial t) = 0, \quad B(X, Y) = t\overline{B}(X, Y),
\]

for all $X, Y \in C(TM)$, where $\overline{B}$ denotes the second fundamental form of $M$ in $S^{m+1}$. 

Thus, if we denote by $A$ the Weingarten operator of $\overline{M}$ in $S^{m+1}$ with respect to an arbitrary fixed unit section $\overline{\eta}$ in the normal bundle of $\overline{M}$ in $S^{m+1}$, we obtain the expression for the Weingarten operator $A$ of the cone with respect to the unit section $\eta(t, p) = \overline{\eta}(p)$, $(t, p) \in M$, in the normal bundle of $M$ in $\mathbb{R}^{m+2}$,

\begin{equation}
A(\partial/\partial t) = 0 \quad \text{and} \quad A(X) = \frac{1}{t} \overline{\overline{A}}(X),
\end{equation}

for all $X \in C(T \overline{M})$, and, consequently, $|A|^2 = \frac{1}{t^2} |\overline{A}|^2$.

Moreover, for a smooth function $f \in C^\infty(M)$, we have

\begin{equation}
\nabla f = \frac{\partial f}{\partial t} \partial/\partial t + \frac{1}{t^2} \nabla f,
\end{equation}

and

\begin{equation}
\Delta f = -\frac{\partial^2 f}{\partial t^2} - \frac{n}{t} \frac{\partial f}{\partial t} + \frac{1}{t^2} \Delta f,
\end{equation}

where $f_t \in C^\infty(\overline{M})$, $f_t(p) = f(t, p)$, for all $p \in \overline{M}$ and $t \in (0, \infty)$.

We are now ready to write down the conditions for the biharmonicity of the Gauss map associated to a hypercone.

\textbf{Theorem 4.1.} Let $\overline{M}$ be a non-minimal hypersurface of $S^{m+1}$. The Gauss map associated to the hypercone $(0, \infty) \times_{t^2} \overline{M}$ is proper biharmonic if and only if

\begin{equation}
\begin{cases}
\nabla \overline{\overline{f}} + \overline{A}^2(\overline{\nabla} \overline{f}) + (2m - 3 - |\overline{A}|^2) \overline{\nabla} \overline{f} = 0 \\
3\Delta \overline{f} + (3m - 6 - |\overline{A}|^2) \overline{f} = 0,
\end{cases}
\end{equation}

where $\overline{A}$ and $\overline{f} \in C^\infty(\overline{M})$ are the shape operator and the mean curvature function of $\overline{M}$ in $S^{m+1}$, respectively.

\textbf{Proof.} Consider $\partial/\partial t$ and $\{E_i\}_{i=1}^m$ a local orthonormal frame field on $\overline{M}$, geodesic at $p$. Then $\{\partial/\partial t, \frac{1}{t} E_i\}_{i=1}^m$ constitutes a local orthonormal frame field on $(0, \infty) \times_{t^2} \overline{M}$.

Denoting by $\overline{f}$ is the mean curvature function of $\overline{M}$ in $S^{m+1}$ and using (1.2), we get the mean curvature function $f$ of the hypercone,

\[ f = \frac{m}{(m+1)} \frac{1}{t} \overline{f}. \]

Using (1.4), we get

\[ \nabla f = \frac{m}{(m+1)} \left( -\frac{1}{t^2} \overline{f} \partial/\partial t + \frac{1}{t^2} \nabla \overline{f} \right), \]

and this, together with (4.3), implies

\begin{equation}
A^2(\nabla f) = \frac{m}{(m+1)} \frac{1}{t^5} \overline{A}^2(\nabla \overline{f}).
\end{equation}

Also,

\begin{equation}
-|A|^2 \nabla f = \frac{m}{(m+1)} \frac{1}{t^4} |\overline{A}|^2 \left( \overline{f} \partial/\partial t - \frac{1}{t} \nabla \overline{f} \right).
\end{equation}
In order to compute $\Delta(\text{grad } f) = -\text{ trace } \nabla^2(\text{grad } f)$ we shall use (4.1). Thus,
\[
\text{trace } \nabla^2 \left( \frac{1}{t^2} \partial \partial t \right) = \nabla \partial \partial t \nabla \partial \partial t \left( \frac{1}{t^2} \partial \partial t \right) + \frac{1}{t^2} \sum_{i=1}^{m} \left\{ \frac{1}{t^2} \nabla E_i \nabla E_i \left( \frac{1}{t^2} f \partial \partial t \right) \right\} \\
= \frac{6}{t^4} \partial \partial t + \frac{1}{t^2} \sum_{i=1}^{m} \left\{ E_i (E_i (f)) \partial \partial t + \frac{2}{t} E_i (f) E_i - 3 f \partial \partial t \right\} \\
= \frac{1}{t^4} (6 - 3 m) \partial \partial t + \frac{2}{t^5} \text{grad } f,
\]
and
\[
\text{trace } \nabla^2 \left( \frac{1}{t^3} \text{grad } f \right) = \nabla \partial \partial t \nabla \partial \partial t \left( \frac{1}{t^3} \text{grad } f \right) + \frac{1}{t^2} \sum_{i=1}^{m} \left\{ \frac{1}{t^3} \nabla E_i \nabla E_i \text{grad } f - \nabla \nabla E_i \left( \frac{1}{t^3} \text{grad } f \right) \right\} \\
= \frac{6}{t^5} \text{grad } f + \frac{1}{t^2} \sum_{i=1}^{m} \left\{ \frac{1}{t^3} \nabla E_i (\nabla E_i \text{grad } f - t \langle E_i, \text{grad } f \rangle \partial \partial t) \right\} \\
= \frac{2}{t^4} \Delta \partial \partial t + \frac{1}{t^5} (5 - 2 m) \text{grad } f - \Delta \text{grad } f).
\]

Using the two expressions above we obtain
\[
\Delta(\text{grad } f) = \frac{m}{m+1} \left\{ \frac{1}{t} ((6 - 3 m) \partial \partial t - 3 \Delta \partial \partial t) + \frac{1}{t^5} ((2 m - 3) \text{grad } f + 3 \text{grad } f) \right\}.
\]

By substituting (4.7), (4.8) and (4.9) in (3.5), we obtain the desired result. □

**Corollary 4.2.** Let $\overline{M}$ be a non-minimal hypersurface of $S^{m+1}$ with constant norm of the shape operator. We have

(i) if the Gauss map associated to the hypercone $(0, \infty) \times \overline{M}$ is proper biharmonic, then
\[
2 \overline{\mathbf{A}}^2 (\text{grad } \overline{f}) - m \overline{\mathbf{A}} (\text{grad } \overline{f}) - \frac{2}{3} |\overline{\mathbf{A}}|^2 \text{grad } \overline{f} = 0.
\]

(ii) if $\overline{M}$ is compact, then the Gauss map associated to the hypercone is proper biharmonic if and only if $\overline{M}$ has constant mean curvature in $S^{m+1}$, $m > 2$ and $|\overline{\mathbf{A}}|^2 = 3(m - 2)$.

**Proof.** The second equation of (4.6) implies
\[
3 \text{grad } \overline{\Delta} \overline{f} + (3 m - 6 - |\overline{\mathbf{A}}|^2) \text{grad } \overline{f} = 0,
\]
and, since $\text{grad } \overline{\Delta} \overline{f} = \overline{\Delta} \text{grad } \overline{f} + \text{Ricci}^{\overline{M}} (\text{grad } \overline{f})$, we obtain
\[
\overline{\Delta} \text{grad } \overline{f} = \left( 2 - m + \frac{1}{3} |\overline{\mathbf{A}}|^2 \right) \text{grad } \overline{f} - \text{Ricci}^{\overline{M}} (\text{grad } \overline{f}).
\]

We substitute this expression in the first equation of (4.6) and it follows that
\[
\overline{A}^2 (\text{grad } \overline{f}) - \text{Ricci}^{\overline{M}} (\text{grad } \overline{f}) + \left( m - 1 - \frac{2}{3} |\overline{\mathbf{A}}|^2 \right) \text{grad } \overline{f} = 0.
\]
Finally, from the Gauss equation for $\overline{M}$ in $S^{m+1}$ we deduce that
\[
\text{Ricci}^{\overline{M}}(X) = (m - 1)X + m\bar{f}A(X) - \bar{A}^2(X), \quad \forall X \in C(T\overline{M}),
\]
and we conclude.

In order to prove (ii), we integrate the second equation of (4.6) and we get
\[
3m - 6 - |A|^2 = 0,
\]
and then $\bar{f}$ is constant. \(\square\)

As a consequence of Theorem 4.1 we obtain,

**Theorem 4.3.** Let $\overline{M}$ be a constant non-zero mean curvature hypersurface of $S^{m+1}$. The Gauss map associated to the hypercone $(0, \infty) \times_t \overline{M}$ is proper biharmonic if and only if $m > 2$ and $|A|^2 = 3(m - 2)$, where $A$ is the shape operator of $\overline{M}$ in $S^{m+1}$.

**Proof.** If $\bar{f}$ is constant, then the first condition of (4.6) is identically satisfied and the second one implies $|A|^2 = 3(m - 2)$. The converse is immediate. \(\square\)

For the case of hypercones in $\mathbb{R}^3$ and $\mathbb{R}^4$, i.e. $m = 1$ and $m = 2$, we have the following non-existence results.

**Theorem 4.4.** There exist no cones in $\mathbb{R}^3$ with proper biharmonic Gauss map.

**Proof.** Consider a cone in $\mathbb{R}^3$ generated by a curve $\sigma : I \to S^2$, parametrized by arc length. Denote by $s$ the parameter on the curve and by $T = \dot{\sigma}$ the tangent vector field along $\sigma$. Since $\nabla^S_T T = kN$, with $N$ the unit normal vector field along $\sigma$, the mean curvature function is given by $\bar{f} = \pm k$,
\[
A(\partial/\partial s) = k\partial/\partial s \quad \text{and} \quad |A|^2 = k^2,
\]
\[
\text{grad} \bar{f} = \pm \dot{k}\partial/\partial s \quad \text{and} \quad \Delta \text{grad} \bar{f} = \mp \ddot{k}\partial/\partial s.
\]
Thus, condition (4.6) becomes
\[
\begin{cases}
\ddot{k} + \dot{k} = 0 \\
k(3 + k^2) + 3\ddot{k} = 0.
\end{cases}
\]
This implies that $\ddot{k}k^2 = 0$, hence $k = 0$, i.e. the Gauss map of the cone is harmonic, and we conclude. \(\square\)

**Theorem 4.5.** There exist no hypercones in $\mathbb{R}^4$, over compact non-minimal surfaces $\overline{M}^2 \subset S^3$, with proper biharmonic Gauss map.

**Proof.** Suppose that the Gauss map of the hypercone over $\overline{M}$ is proper biharmonic. Since $m = 2$, the second equation of (4.6) leads to
\[
3\Delta \bar{f} - |A|^2 \bar{f} = 0.
\]
By integrating condition (4.12) on $\overline{M}$ and by using the fact that $\bar{f}$ is positive, we conclude that $|A|^2 = 0$ and we have a contradiction. \(\square\)

When $m > 2$, we have examples of hypercones with proper biharmonic Gauss map. Recall that if $\overline{M}$ is a hypersurface in $S^{m+1}$, then the cone over $\overline{M}$ has harmonic Gauss map if and only if $\overline{M}$ is minimal in $S^{m+1}$ (see [3, 19]). This does not hold in the case of the biharmonicity. Indeed, by considering $\overline{M}$ to be a constant mean curvature proper biharmonic hypersurface of $S^{m+1}$ and by using the fact that the squared norm of the shape operator of such a submanifold is equal to $m$ (see [4]) we get
Theorem 4.6. Let $\overline{M}$ be a constant mean curvature proper biharmonic hypersurface of $S^{m+1}$. Then the hypercone $(0, \infty) \times_{\ell} \overline{M}$ has proper biharmonic associated Gauss map if and only if $m = 3$.

Remark 4.7. Theorem 4.6 can be deduced, for the particular case of the hypersphere of radius equal to $\frac{1}{\sqrt{2}}$, in a more geometrical manner. The argument is the following. In [11], the authors proved that if $\psi : N \to S^m(\frac{1}{\sqrt{2}})$ is a harmonic map and $i : S^m(\frac{1}{\sqrt{2}}) \to S^{m+1}$ denotes the inclusion map, then the tension and bitension fields of the composition are given by

$$\tau(i \circ \psi) = -2e(\psi)\overline{n}$$
$$\frac{1}{2} \tau_2(i \circ \psi) = (\Delta e(\psi))\overline{n} - 2d\psi(\text{grad } e(\psi)),$$

where $e(\psi)$ denotes the energy density of the map $\psi$ and $\overline{n}$ the unit section of the normal bundle of $S^m(\frac{1}{\sqrt{2}})$ in $S^{m+1}$. The Gauss map associated to the hypercone $(0, \infty) \times_{\ell} S^m(\frac{1}{\sqrt{2}})$ are given by

$$\gamma : (0, \infty) \times_{\ell} S^m(\frac{1}{\sqrt{2}}) \to S^{m+1}$$
$$\gamma(t, p) = \overline{n}(p),$$

i.e. $\gamma(t, x^1, \ldots, x^{m+1}, \frac{1}{\sqrt{2}}) = (x^1, \ldots, x^{m+1}, -\frac{1}{\sqrt{2}})$. We can thus think of $\gamma$, up to an isometry, as the composition $i \circ \psi$, where

$$\psi : (0, \infty) \times_{\ell} S^m(\frac{1}{\sqrt{2}}) \to S^m(\frac{1}{\sqrt{2}})$$
$$\psi(t, p) = p.$$

The map $\psi$ is the projection onto the second factor of a warped product, so it is a harmonic map. Now, since $d\psi(\partial/\partial t) = 0$ and $d\psi(X) = X$, for all $X \in C(TS^m(\frac{1}{\sqrt{2}}))$, the energy density of $\psi$ is $e(\psi) = \frac{m}{2t^2}$ and $\text{grad } e(\psi) = -\frac{m}{t^2} \partial/\partial t$. Thus, we deduce that

$$\Delta e(\psi) = \frac{m(m-3)}{t^4}$$
and
$$d\psi(\text{grad } e(\psi)) = 0.$$

Finally, by using (4.13) and (4.14), we conclude that the Gauss map associated to the hypercone $(0, \infty) \times_{\ell} S^m(\frac{1}{\sqrt{2}})$ in $\mathbb{R}^{m+2}$ is proper biharmonic if and only if $m = 3$, in accordance with Theorem 4.6.

5. Hypercones generated by isoparametric hypersurfaces in spheres

We recall that a hypersurface $\overline{M}^m$ in $S^{m+1}$ is said to be isoparametric of type $\ell$ if it has constant principal curvatures $k_1 > \ldots > k_\ell$ with respective constant multiplicities $m_1, \ldots, m_\ell$, $m = m_1 + m_2 + \ldots + m_\ell$. E. Cartan classified in [5] the isoparametric hypersurfaces with $\ell = 1, 2, 3$. For $\ell > 3$ a full classification of isoparametric hypersurfaces is not yet known. Nevertheless, it is known that the number $\ell$ is either 1, 2, 3, 4 or 6 (see [13]) and the following information on the principal curvatures and their multiplicities is available.

(i) If $\ell = 1$, then $\overline{M}$ is totally umbilical.
(ii) If $\ell = 2$, then $\overline{M} = S^{m_1}(r_1) \times S^{m_2}(r_2)$, $r_1^2 + r_2^2 = 1$.
(iii) If $\ell = 3$, then $m_1 = m_2 = m_3 = 2^q$, $q = 0, 1, 2, 3$.
(iv) If $\ell = 4$, then $m_1 = m_3$ and $m_2 = m_4$. Moreover, $(m_1, m_2) = (2, 2)$ or $(4, 5)$, or $m_1 + m_2 + 1$ is a multiple of $2^\rho(s)$, where $\rho(s)$ is the number of integers $r$ with $1 \leq r \leq s$, $r \equiv 0, 1, 2, 4$ (mod 8) and $m_\rho = \min\{m_1, m_2\}$.
(v) If $\ell = 6$, then $m_1 = m_2 = \ldots = m_6 = 1$ or 2.
Moreover, there exists an angle $\theta$, $0 < \theta < \frac{\pi}{4}$, such that

$$(5.1) \quad k_\alpha = \cot \left( \theta + \frac{(\alpha - 1)\pi}{\ell} \right), \quad \alpha = 1, \ldots, \ell.$$  

We now study the biharmonicity of the Gauss map of the hypercones generated by isoparametric hypersurfaces in spheres. We shall detail this study according to Proposition 5.3.

Isoparametric hypersurface with $\ell = 1$. In this case $\overline{M}$ is a hypersphere $S^m(a)$, $a \in (0, 1)$, in $S^{m+1}$. Since $|\overline{A}|^2 = \frac{1 - a^2}{a^2}$, by using Theorem 4.3 we obtain

**Proposition 5.1.** Consider the hypercone $(0, \infty)^2 \times S^m(a)$, $a \in (0, 1)$, in $\mathbb{R}^{m+2}$. Its associated Gauss map is proper biharmonic if and only if $m > 2$ and $a = \sqrt{\frac{m}{4m-6}}$.

**Remark 5.2.** We underline the fact that Proposition 5.1 provides examples of hypersurfaces with proper biharmonic associated Gauss map in any $(m+2)$-dimensional Euclidean space, with $m > 2$.

Isoparametric hypersurface with $\ell = 2$. In this case $\overline{M}$ is a generalized torus $S^{m1}(r_1) \times S^{m2}(r_2) \subset S^{m+1}$, $m_1 + m_2 = m$, $r_1^2 + r_2^2 = 1$. The squared norm of the shape operator is $|\overline{A}|^2 = \left( \frac{r_1}{m_1} \right)^2 m_1 + \left( \frac{r_2}{m_2} \right)^2 m_2$ and, by using Theorem 4.3 we get

**Proposition 5.3.** Consider the hypercone $(0, \infty)^2 \times (S^{m1}(r_1) \times S^{m2}(r_2))$ in $\mathbb{R}^{m+2}$. Its associated Gauss map is proper biharmonic if and only if $m > 3$, $\frac{m_1}{r_1} \neq \frac{m_2}{r_2}$ and

$$(5.2) \quad \frac{m_1}{r_1} + \frac{m_2}{r_2} = 4m - 6.$$  

**Remark 5.4.** In order to obtain an example, consider $m > 3$, $m_1 = 1$, and $m_2 = m - 1$. Then

$$r_1^2 = \frac{3m - 4 \pm \sqrt{9m^2 - 40m + 40}}{2(4m - 6)} \quad \text{and} \quad r_2^2 = \frac{5m - 8 \mp \sqrt{9m^2 - 40m + 40}}{2(4m - 6)}$$

are solutions for (5.2).

Isoparametric hypersurface with $\ell = 3$. In this case, taking into account (5.1), there exists $\theta \in (0, \frac{\pi}{3})$ such that

$$k_1 = \cot \theta, \quad k_2 = \cot \left( \theta + \frac{\pi}{3} \right) = \frac{k_1 - \sqrt{3}}{1 + \sqrt{3}k_1}, \quad k_3 = \cot \left( \theta + \frac{2\pi}{3} \right) = \frac{k_1 + \sqrt{3}}{1 - \sqrt{3}k_1}.$$  

Thus, the square of the norm of the shape operator is

$$(5.3) \quad |\overline{A}|^2 = 2^q(\overline{k}_1^2 + \overline{k}_2^2 + \overline{k}_3^2) = 2^q \frac{9k_1^2 + 45k_2^2 + 6}{(1 - 3k_1^2)^2}$$

and $m = 3 \cdot 2^q$, $q = 0, 1, 2, 3$.

On the other hand, from Theorem 4.3 the hypercone generated by $\overline{M}$ has proper biharmonic Gauss map if and only if

$$|\overline{A}|^2 = 3(m - 2) = 3(3 \cdot 2^q - 2).$$

The last equation, together with (5.3), implies that $k_1$ is a solution of

$$(5.4) \quad 3 \cdot 2^q x^6 + (18 - 27 \cdot 2^q) x^4 + (-12 + 33 \cdot 2^q) x^2 + 2 - 2^q = 0.$$  

If $q = 0$, equation (5.4) becomes $3x^6 - 9x^4 + 11x^2 + 1 = 0$ and it has no real roots.
If $q = 1$, equation (5.4) becomes $x^2(x^2 - 3)^2 = 0$, which has one root $x = \sqrt{3}$ in $(\sqrt{3}/3, \infty)$. Notice that when $k_1 = \sqrt{3}$, $M$ is minimal.

If $q = 2$, equation (5.4) becomes $6x^6 - 45x^4 + 60x^2 - 1 = 0$ and it has two distinct roots in $(\sqrt{3}/3, \infty)$, different from $\sqrt{3}$.

If $q = 3$, equation (5.4) becomes $4x^6 - 33x^4 + 42x^2 - 1 = 0$ and it has two distinct roots in $(\sqrt{3}/3, \infty)$, different from $\sqrt{3}$.

We can conclude,

**Proposition 5.5.** Consider an isoparametric hypersurface of type 3 in $S^{m+1}$. The Gauss map of its hypercone is proper biharmonic if and only if

(i) $q = 2$ and the first principal curvature $k_1$ is one of the two roots of the equation $6x^6 - 45x^4 + 60x^2 - 1 = 0$ in $(\sqrt{3}/3, \infty)$, or

(ii) $q = 3$ and the first principal curvature $k_1$ is one of the two roots of the equation $4x^6 - 33x^4 + 42x^2 - 1 = 0$ in $(\sqrt{3}/3, \infty)$.

**Isoparametric hypersurface with $\ell = 4$.** In this case, taking into account (5.1), there exists $\theta \in (0, \pi/4)$ such that

\[
\begin{align*}
k_1 &= \cot \theta, \\
k_2 &= \cot (\theta + \pi/4) = \frac{k_1 - 1}{k_1 + 1}, \\
k_3 &= \cot (\theta + \pi/2) = -\frac{1}{k_1}, \\
k_4 &= \cot (\theta + 3\pi/4) = -\frac{k_1 + 1}{k_1 - 1}.
\end{align*}
\]

The square of the norm of the shape operator is

\[
|A|^2 = m_1 \left( k_1^2 + \frac{1}{k_1^2} \right) + m_2 \left[ \left( \frac{k_1 - 1}{k_1 + 1} \right)^2 + \left( \frac{k_1 + 1}{k_1 - 1} \right)^2 \right] \\
= m_1 \lambda + 16m_2 \frac{1}{\lambda} + 2(m_1 + m_2), \tag{5.5}
\]

where $\lambda = \left( k_1 - \frac{1}{k_1} \right)^2$.

Since in this case $m = 2(m_1 + m_2)$, from Theorem 4.3, the hypercone generated by $M$ has proper biharmonic Gauss map if and only if

\[
|A|^2 = 3(m - 2) = 6(m_1 + m_2 - 1).
\]

The last equation, together with (5.5), implies that $\lambda$ is a solution of

\[
m_1 \lambda^2 - (4(m_1 + m_2) - 6)\lambda + 16m_2 = 0. \tag{5.6}
\]

Notice that if $(m_1, m_2) = (2, 2)$ or $(4, 5)$, equation (5.6) has no real roots. Consequently, we obtain

**Proposition 5.6.** Consider an isoparametric hypersurface of type 4 in $S^{m+1}$. The Gauss map of its hypercone is proper biharmonic if and only if its first principal curvature $k_1$ is given by the condition that $\lambda = \left( k_1 - \frac{1}{k_1} \right)^2$ is the positive solution of the equation

\[
m_1 \lambda^2 - (4(m_1 + m_2) - 6)\lambda + 16m_2 = 0,
\]

and $m_1 + m_2 + 1$ is a multiple of $2^{\rho(s)-1}$, where $\rho(s)$ is the number of integers $r$ with $1 \leq r \leq s$ and $r \equiv 0, 1, 2, 4 \pmod{8}$. 

Example 5.7. In order to obtain an explicit example for this case we shall consider from the Takagi list (see [20]) the following homogeneous hypersurfaces with four principal curvatures,
\[(5.7) \quad M = S(U(k) \times U(2))/(T^2 \times SU(k-2)) \subset S^{2n+1},\]
where \(n = 2k + 1, \ n \geq 5.\)

Since \(m_1 = n - 2\) and \(m_2 = 2,\) from Proposition [5.6] we deduce that the Gauss map of the hypercone over \(M\) is biharmonic if and only if
\[(5.8) \quad (n - 2)\lambda^2 - (4n - 6)\lambda + 32 = 0.\]

By denoting \(\sin^2 2\theta = x \in (0, 1)\) we have \(\lambda = 4\frac{1-x}{x},\) and equation (5.8) becomes
\[(5.9) \quad (4n - 3)x^2 - (6n - 11)x + 2(n - 2) = 0.\]

Since \(n\) is odd, (5.9) has real roots if and only if \(n \geq 9.\) It is easy to verify that, for \(n \geq 9,\) the two real roots of (5.9) are in \((0, 1)\) and we obtain
\[(5.10) \quad \sin^2 2\theta = \frac{6n - 11 \pm \sqrt{4n^2 - 44n + 73}}{2(4n - 3)}.\]

Notice that \(M\) is minimal if and only if
\[(5.11) \quad \cot^2 \theta = \frac{\sqrt{n} \pm \sqrt{2}}{\sqrt{n} + \sqrt{2}}.\]

Since \(n \geq 9,\) from (5.10) and (5.11) we get that \((0, \infty) \times \mathbb{E}^2 M\) has proper biharmonic Gauss map if and only if \(n \geq 9\) and
\[\theta = \frac{1}{2} \arcsin \sqrt{\frac{6n - 11 \pm \sqrt{4n^2 - 44n + 73}}{2(4n - 3)}}.\]

Isoparametric hypersurface with \(\ell = 6.\) In this case, taking into account (5.11), there exists \(\theta \in (0, \pi/6)\) such that

\[k_1 = \cot \theta, \quad k_2 = \cot (\theta + \frac{\pi}{6}) = \frac{\sqrt{3}k_1 - 1}{k_1 + \sqrt{3}},\]

\[k_3 = \cot (\theta + \frac{\pi}{3}) = \frac{k_1 - \sqrt{3}}{1 + \sqrt{3}k_1}, \quad k_4 = \cot (\theta + \frac{\pi}{2}) = -\frac{1}{k_1},\]

\[k_5 = \cot (\theta + \frac{2\pi}{3}) = \frac{k_1 + \sqrt{3}}{1 - \sqrt{3}k_1}, \quad k_6 = \cot (\theta + \frac{5\pi}{6}) = \frac{1 + \sqrt{3}k_1}{\sqrt{3} - k_1}.\]

If \(m_i = 1, i = 1, \ldots, 6,\) then the square of the norm of the shape operator is
\[(5.12) \quad |A|^2 = \frac{9k_1^{12} + 495k_1^8 - 528k_1^6 + 495k_1^4 + 9}{k_1^4(3k_1^2 - 10k_1^2 + 3)^2},\]

and since \(m = 6\) the hypercone generated by \(M\) has proper biharmonic Gauss map if and only if
\[|A|^2 = 12.\]

The last equation, together with (5.12), implies that \(k_1\) is a solution of the equation
\[x^{12} - 12x^{10} + 135x^8 - 216x^6 + 135x^4 - 12x^2 + 1 = 0,\]

which has no real roots.
If \( m_i = 2, \ i = 1, \ldots, 6 \), a similar computation leads to the conclusion that \( k_1 \) is a solution of the equation
\[
3x^{12} - 45x^{10} + 465x^8 - 766x^6 + 465x^4 - 45x^2 + 3 = 0,
\]
which has no real roots.

Conclusively, we have

**Proposition 5.8.** There exist no isoparametric hypersurface of type 6 whose associated hypercone has proper biharmonic Gauss map.

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