CONSERVATIVE L-SYSTEMS AND THE LIVŠIC FUNCTION

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Abstract. We study the connection between the Livšic class of functions $s(z)$ that are the characteristic functions of densely defined symmetric operators $\hat{A}$ with deficiency indices $(1,1)$, the characteristic functions $S(z)$ (the Möbius transform of $s(z)$) of a maximal dissipative extension $T$ of $\hat{A}$ (determined by the von Neumann parameter $\kappa$ of the extension relative to an appropriate basis in the deficiency subspaces) and the transfer functions $W_\Theta(z)$ of a conservative L-system $\Theta$ with the main operator $T$. It is shown that under a natural hypothesis $S(z)$ and $W_\Theta(z)$ are reciprocal to each other. In particular, when $\kappa = 0$, $W_\Theta(z) = \frac{1}{S(z)} = -\frac{1}{s(z)}$. It is established that the impedance function of a conservative L-system with the main operator $T$ coincides with the function from the Donoghue class if and only if the von Neumann parameter vanishes ($\kappa = 0$). Moreover, we introduce the generalized Donoghue class and obtain the criteria for an impedance function to belong to this class. All results are illustrated by a number of examples.

1. Introduction

Suppose that $T$ is a densely defined closed operator in a Hilbert space $\mathcal{H}$ such that its resolvent set $\rho(T)$ is not empty. We also assume that $\text{Dom}(T) \cap \text{Dom}(T^*)$ is dense and that the restriction $\hat{A} = T|_{\text{Dom}(T) \cap \text{Dom}(T^*)}$ is a closed symmetric operator with finite equal deficiency indices. Let $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$ be the rigged Hilbert space associated with $\hat{A}$ (see Appendix A for a detailed discussion on a concept of rigged Hilbert spaces).

One of the main objectives of the current paper is the study of the L-system

$$
\Theta = \begin{pmatrix}
\hat{A} & K \\
\mathcal{H}_+ & \mathcal{H} & \mathcal{H}_-
\end{pmatrix},
$$

where the state-space operator $\hat{A}$ is a bounded linear operator from $\mathcal{H}_+$ into $\mathcal{H}_-$ such that $\hat{A} \subset T \subset \mathcal{A}$, $\hat{A} \subset T^* \subset \mathcal{A}^*$, $E$ is a finite-dimensional Hilbert space, $K$ is a bounded linear operator from the space $E$ into $\mathcal{H}_-$, $J = J^* = J^{-1}$ is a self-adjoint isometry on $E$ such that $\text{Im} \hat{A} = KJK^*$. Due to the fact that $\mathcal{H}_+$ is dual to $\mathcal{H}_-$, $\text{Im} \hat{A}$ is a well defined bounded operator from $\mathcal{H}_+$ into $\mathcal{H}_-$ (see Appendix A). Note that the main operator $T$ associated with the system $\Theta$ is uniquely determined by the state-space operator $\hat{A}$ as its restriction onto the domain $\text{Dom}(T) = \{f \in \mathcal{H}_+ | \hat{A}f \in \mathcal{H}\}$.

Recall that the operator-valued function given by

$$
W_\Theta(z) = I - 2iK^*(\hat{A} - zI)^{-1}KJ, \quad z \in \rho(T),
$$

2010 Mathematics Subject Classification. Primary: 81Q10, Secondary: 35P20, 47N50.
Key words and phrases. L-system, transfer function, impedance function, Herglotz-Nevanlinna function, Weyl-Titchmarsh function, Livšic function, characteristic function, Donoghue class, symmetric operator, dissipative extension, von Neumann parameter.
is called the \textit{transfer function} of the L-system $\Theta$ and
\[ V_\Theta(z) = i[W_\Theta(z) + I^{-1}]^{-1}W_\Theta(z) - I = K^*(\text{Re} \, zI - 1)^{-1}K, \quad z \in \rho(T) \cap C_, \]
is called the \textit{impedance function} of $\Theta$. Note that if $\varphi_+ = W_\Theta(z)\varphi_-$, with $\varphi_-$ the input and $\varphi_+$ the output, then L-system $[\ref{1}]$ can be associated with the system of equations
\[
\begin{align*}
(\mathcal{A} - zI)x &= KJ\varphi_- \\
\varphi_+ &= \varphi_- - 2iK^*x
\end{align*}
\] (given $\varphi_-$, one needs to find $x$ and then determine $\varphi_+$).

We remark that the concept of L-systems $[\ref{1}, \ref{2}]$ generalizes the one of the Livšic systems in the case of a bounded main operator. It is also worth mentioning that those systems are conservative in the sense that a certain metric conservation law holds (see $[\ref{3}]$ Preface). An overview of the history of the subject and a detailed description of the L-systems can be found in $[\ref{4}]$.

Another important object of interest is the \textit{Livšic function}. Recall that M. Livšic $[\ref{5}]$ introduced a fundamental concept of a characteristic functions of a densely defined symmetric operator $\hat{A}$ with deficiency indices $(1, 1)$ as well as of its maximal non-self-adjoint extension $T$. In $[\ref{6}, \ref{7}]$ two of the authors (K.A.M. and E.T.) suggested to define characteristic functions of a symmetric operator and of its dissipative extension as functions associated with the pairs $(\hat{A}, A)$ and $(T, A)$, rather than with the single operators $\hat{A}$ and $T$, respectively, by introducing an auxiliary self-adjoint (reference) extension $A$ of $\hat{A}$. Following $[\ref{6}, \ref{7}]$ we call the characteristic function associated with the pair $(\hat{A}, A)$ the \textit{Livšic function}. For a detailed treatment of the aforementioned concepts of the Livšic and the characteristic functions we refer to $[\ref{8}]$ (see also $[\ref{9}, \ref{10}, \ref{11}, \ref{12}, \ref{13})$).

The main goal of the present paper is the following.

First, we establish the connection between: (i) the Livšic class of functions $s(z)$ that are the characteristic functions of a densely defined symmetric operators $\hat{A}$ with deficiency indices $(1, 1)$; (ii) the characteristic functions $S(z)$ (the Möbius transform of $s(z)$) of a maximal dissipative extension $T$ of $\hat{A}$ determined by the von Neumann parameter $\kappa$; and (iii) the transfer functions $W_\Theta(z)$ of an L-system $\Theta$ with the main operator $T$. It is shown (see Theorem $[\ref{14}]$) that under some natural hypothesis $S(z)$ and $W_\Theta(z)$ are reciprocal to each other. In particular, when $\kappa = 0$, we have $W_\Theta(z) = \frac{1}{s(z)} = -\frac{1}{\rho(z)}$.

Second, in Theorem $[\ref{15}]$ we show that the impedance function of a conservative L-system with the main operator $T$ coincides with a function from the Donoghue class $\mathcal{M}$ if and only if the von Neumann parameter vanishes ($\kappa = 0$). For $0 \leq \kappa < 1$ we introduce the generalized Donoghue class $\mathcal{M}_\kappa$ and establish a criteria (see Theorem $[\ref{16}]$ in Section $[\ref{17}]$) for an impedance function to belong to $\mathcal{M}_\kappa$. In particular, when $\kappa = 0$ the class $\mathcal{M}_\kappa$ coincides with the Donoghue class $\mathcal{M} = \mathcal{M}_0$.

We conclude with several examples that illustrate the main results and concepts.

2. Preliminaries

For a pair of Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$ we denote by $[\mathcal{H}_1, \mathcal{H}_2]$ the set of all bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. Let $\hat{A}$ be a closed, densely defined, symmetric operator in a Hilbert space $\mathcal{H}$ with inner product $(f, g), f, g \in \mathcal{H}$. Any operator $T$
in $\mathcal{H}$ such that

$$\hat{A} \subset T \subset \hat{A}^*$$

is called a quasi-self-adjoint extension of $\hat{A}$.

Consider the rigged Hilbert space (see [1], [3]) $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$, where $\mathcal{H}_+ = \text{Dom}(\hat{A}^*)$ and

$$(f,g)_+ = (f,g) + (\hat{A}^* f, \hat{A}^* g), \ f, g \in \text{Dom}(\hat{A}^*).$$

Let $\mathcal{R}$ be the Riesz-Berezansky operator $\mathcal{R}$ (see [1], [3]) which maps $\mathcal{H}_-$ onto $\mathcal{H}_+$ such that $(f,g) = (\hat{f}, \mathcal{R}g)_+ \ (\forall f \in \mathcal{H}_+, \ g \in \mathcal{H}_-)$ and $\|\mathcal{R}g\|_+ = \|g\|_-$. Note that identifying the space conjugate to $\mathcal{H}_+$ with $\mathcal{H}_+$, we get that if $\mathcal{A} \in [\mathcal{H}_+, \mathcal{H}_-]$, then $\mathcal{A}^* \in [\mathcal{H}_+, \mathcal{H}_-]$. An operator $\mathcal{A} \in [\mathcal{H}_+, \mathcal{H}_-]$ is called a self-adjoint bi-extension of a symmetric operator $\hat{A}$ if $\mathcal{A} = \mathcal{A}^*$ and $\mathcal{A} \supset \hat{A}$. Let $\mathcal{A}$ be a self-adjoint bi-extension of $\hat{A}$ and let the operator $\hat{A}$ in $\mathcal{H}$ be defined as follows:

$$\text{Dom}(\hat{A}) = \{f \in \mathcal{H}_+ : \hat{A} f \in \mathcal{H}\}, \ \hat{A} = \mathcal{A}| \text{Dom}(\hat{A}).$$

The operator $\hat{A}$ is called a quasi-kernel of a self-adjoint bi-extension $\mathcal{A}$ (see [22], [3, Section 2.1]). According to the von Neumann Theorem (see [3, Theorem 1.3.1]) the domain of $\hat{A}$, a self-adjoint extension of $\hat{A}$, can be expressed as

$$(\text{Dom}(\hat{A})) = \text{Dom}(\hat{A}) \oplus (I + U)\mathfrak{N}_i,$$

where $U$ is a (-) (and (+))-isometric operator from $\mathfrak{N}_i$ into $\mathfrak{N}_{-i}$ and

$$\mathfrak{N}_{\pm i} = \text{Ker} (\hat{A}^* \mp iI)$$

are the deficiency subspaces of $\hat{A}$. A self-adjoint bi-extension $\mathcal{A}$ of a symmetric operator $\hat{A}$ is called t-self-adjoint (see [3, Definition 3.3.5]) if its quasi-kernel $\hat{A}$ is self-adjoint operator in $\mathcal{H}$. An operator $\mathcal{A} \in [\mathcal{H}_+, \mathcal{H}_-]$ is called a quasi-self-adjoint bi-extension of an operator $T$ if $\mathcal{A} \supset T \supset \hat{A}$ and $\mathcal{A}^* \supset T^* \supset \hat{A}$. In what follows we will be mostly interested in the following type of quasi-self-adjoint bi-extensions.

**Definition 1** ([3]). Let $T$ be a quasi-self-adjoint extension of $\hat{A}$ with nonempty resolvent set $\rho(T)$. A quasi-self-adjoint bi-extension $\mathcal{A}$ of an operator $T$ is called a ($\ast$)-extension of $T$ if $\text{Re} \mathcal{A}$ is a t-self-adjoint bi-extension of $\hat{A}$.

In what follows we assume that $\hat{A}$ has equal finite deficiency indices and will say that a quasi-self-adjoint extension $T$ of $\hat{A}$ belongs to the class $\Lambda(\hat{A})$ if $\rho(T) \neq \emptyset$, $\text{Dom}(\hat{A}) = \text{Dom}(T) \cap \text{Dom}(T^*)$, and hence $T$ admits ($\ast$)-extensions. The description of all ($\ast$)-extensions via Riesz-Berezansky operator $\mathcal{R}$ can be found in [3, Section 4.3].

**Definition 2.** A system of equations

\[
\begin{cases}
(\hat{A} - zI)x = KJ\varphi_-
\varphi_+ = \varphi_- - 2iK^*x
\end{cases}
\]

or an array

$$(5) \quad \Theta = \left( \begin{array}{cc} \hat{A} & K \\ \mathcal{H}_+ & \mathcal{H}_- & \mathcal{H}_- & \end{array} \right)$$

is called an L-system if:

1. $\mathcal{A}$ is a ($\ast$)-extension of an operator $T$ of the class $\Lambda(\hat{A})$;
2. $J = J^* = J^{-1} \in [E,E], \ \dim E < \infty;$
3) \( \text{Im} \mathcal{A} = KJK^*, \) where \( K \in [E, \mathcal{H}_-], \) \( K^* \in [\mathcal{H}_+, E], \) and \( \text{Ran}(K) = \text{Ran}(\text{Im} \mathcal{A}). \)

In the definition above \( \varphi_- \in E \) stands for an input vector, \( \varphi_+ \in E \) is an output vector, and \( x \) is a state space vector in \( \mathcal{H}. \) The operator \( \mathcal{A} \) is called the state-space operator of the system \( \Theta, \) \( T \) is the main operator, \( J \) is the direction operator, and \( K \) is the channel operator. A system \( \Theta \) in \( \mathcal{H} \) is called minimal if the operator \( \hat{A} \) is a prime operator in \( \mathcal{H}, \) i.e., there exists no non-trivial reducing invariant subspace of \( \mathcal{H} \) on which it induces a self-adjoint operator.

We associate with an L-system \( \Theta \) the operator-valued function

\[
W_{\Theta}(z) = I - 2iK^* (\mathcal{A} - zI)^{-1} KJ, \quad z \in \rho(\hat{A}),
\]

which is called the transfer function of the L-system \( \Theta. \) We also consider the operator-valued function

\[
W_{\Theta}(z) = K^* (\text{Re} \mathcal{A} - zI)^{-1} K, \quad z \in \rho(\hat{A}).
\]

It was shown in [1, Section 6.3] that both \([1] \) and \([2]\) are well defined. The transfer operator-function \( W_{\Theta}(z) \) of the system \( \Theta \) and an operator-function \( V_{\Theta}(z) \) of the form \([3]\) are connected by the following relations valid for \( \text{Im} z \neq 0, \) \( z \in \rho(\hat{A}), \)

\[
V_{\Theta}(z) = i[W_{\Theta}(z) + I]^{-1} [W_{\Theta}(z) - I] J,
\]

\[
W_{\Theta}(z) = (I + iV_{\Theta}(z))^2 (I - iV_{\Theta}(z) J).
\]

The function \( V_{\Theta}(z) \) defined by \([3]\) is called the impedance function of an L-system \( \Theta \) of the form \([3]\). The class of all Herglotz-Nevanlinna functions in a finite-dimensional Hilbert space \( E, \) that can be realized as impedance functions of an L-system, was described in \([3]\), (see also \([3]\) Definition 6.4.1)).

3. On \((*)\)-extension parametrization

Let \( \hat{A} \) be a densely defined closed symmetric operator with \((n, n) \) (\( n < \infty \)) deficiency indices. Then (see \([3]\) Section 2.3)

\[
\mathcal{H}_+ = \text{Dom}(\hat{A}^*) = \text{Dom}(\hat{A}) \oplus \mathfrak{N}_i \oplus \mathfrak{N}_{-i},
\]

where \( \oplus \) stands for the \((+)-orthogonal\) sum. Moreover, all operators from the class \( \Lambda(\hat{A}) \) are of the form (see \([3]\) Theorem 4.1.9), \([2]\)

\[
\text{Dom}(T) = \text{Dom}(\hat{A}) \oplus (K + I) \mathfrak{N}_i, \quad T = \hat{A}^* \upharpoonright \text{Dom}(T),
\]

\[
\text{Dom}(T^*) = \text{Dom}(\hat{A}) \oplus (K^* + I) \mathfrak{N}_{-i}, \quad T^* = \hat{A}^* \upharpoonright \text{Dom}(T^*),
\]

where \( K \in [\mathfrak{N}_i, \mathfrak{N}_{-i}]. \) Let \( \mathcal{M} = \mathfrak{N}_i \oplus \mathfrak{N}_{-i} \) and \( P^+ \) be a \((+)-orthogonal\) projection onto a corresponding subspace shown in its subscript. In this case (see \([2]\)) all quasi-self-adjoint bi-extensions of \( T \in \Lambda(\hat{A}) \) can be parameterized via an operator \( H \in [\mathfrak{N}_{-i}, \mathfrak{N}_i] \) as follows

\[
\mathcal{A} = \hat{A}^* + \mathcal{R}^{-1}(S - \frac{i}{2^H}) P^+_M, \quad \mathcal{A}^* = \hat{A}^* + \mathcal{R}^{-1}(S^* - \frac{i}{2^H}) P^+_M,
\]

where \( \mathcal{J} = P^+_{\mathfrak{N}_i} - P^+_{\mathfrak{N}_{-i}} \) and \( S : \mathfrak{N}_i \oplus \mathfrak{N}_{-i} \to \mathfrak{N}_i \oplus \mathfrak{N}_{-i}, \) satisfies the condition

\[
S = \left( -\frac{i}{2} I - HK \right) \mathcal{K} \left( \frac{i}{2} I - KH \right).
\]
Introduce \((2n \times 2n)\) block-operator matrices \(S_A\) and \(S_{A^*}\) by

\[
S_A = S - \frac{i}{2} \mathbf{J} = \begin{pmatrix}
-HK & H \\
\mathcal{K}(HK - iI) & iI - KH
\end{pmatrix},
\]

\[
S_{A^*} = S^* - \frac{i}{2} \mathbf{J} = \begin{pmatrix}
-K^*H^* - iI & (K^*H^* - iI)K^* \\
H^* & -H^*K^*
\end{pmatrix}.
\]

By direct calculations one finds that

\[
\frac{1}{2}(S_A + S_{A^*}) = \frac{1}{2} \left( \begin{pmatrix}
-HK - K^*H^* - iI & H + (K^*H^* + iI)K^* \\
\mathcal{K}(HK - iI) + H^* & iI - KH - H^*K^*
\end{pmatrix} \right),
\]

and

\[
\frac{1}{2t}(S_A - S_{A^*}) = \frac{1}{2} \left( \begin{pmatrix}
-HK + K^*H^* + iI & H - (K^*H^* + iI)K^* \\
\mathcal{K}(HK - iI) - H^* & iI - KH + H^*K^*
\end{pmatrix} \right).
\]

In the case when the deficiency indices of \(\dot{A}\) are \((1, 1)\), the block-operator matrices \(S_A\) and \(S_{A^*}\) in \([7]\) become \((2 \times 2)\) matrices with scalar entries. As it was announced in \([2]\), (see also \([3\text{ Section 3.4}]\) and \([24]\)), in this case any quasi-self-adjoint bi-extension \(A\) of \(T\) is of the form

\[
A = \dot{A}^* + [p(\cdot, \varphi) + q(\cdot, \psi)] \varphi + [v(\cdot, \varphi) + w(\cdot, \psi)] \psi,
\]

where \(S_A = \begin{pmatrix} p & q \\ v & w \end{pmatrix}\) is a \((2 \times 2)\) matrix with scalar entries such that \(p = -HK\), \(q = H\), \(v = \mathcal{K}(HK - i)\), and \(w = i - KH\). Also, \(\varphi\) and \(\psi\) are \((-\text{normalized})\) elements in \(\mathcal{R}^{-1}(\mathcal{N}_i)\) and \(\mathcal{R}^{-1}(\mathcal{N}_{-1})\), respectively. Both parameters \(H\) and \(\mathcal{K}\) are complex numbers in this case and \(|\mathcal{K}| < 1\). Similarly we write

\[
A^* = \dot{A}^* + \left[ p^x(\cdot, \varphi) + q^x(\cdot, \psi) \right] \varphi + \left[ v^x(\cdot, \varphi) + w^x(\cdot, \psi) \right] \psi,
\]

where \(S_{A^*} = \begin{pmatrix} p^x & q^x \\ v^x & w^x \end{pmatrix}\) is such that \(p^x = -\bar{K}H - i\), \(q^x = (\bar{K}H - i)\mathcal{K}\), \(v^x = \bar{H}\), and \(w^x = -\bar{H}\mathcal{K}\). A direct check will confirm that \(\dot{A} \subset T \subset A\) and we present it below for the reader’s convenience. Indeed, recall that \(\|\varphi\|_\perp = \|\psi\|_\perp = 1\). Using formulas \([10]\) and \([11]\) from Appendix A we get

\[
1 = (\varphi, \varphi)_\perp = (\mathcal{R}\varphi, \mathcal{R}\varphi)_+ = \|\mathcal{R}\varphi\|_+^2 = 2\|\mathcal{R}\varphi\|^2 = (\sqrt{2}\mathcal{R}\varphi, \sqrt{2}\mathcal{R}\varphi).
\]

Set \(g_+ = \sqrt{2}\mathcal{R}\varphi \in \mathcal{N}_i\) and \(g_- = \sqrt{2}\mathcal{R}\psi \in \mathcal{N}_{-1}\) and note that \(g_+\) and \(g_-\) form orthonormal bases in \(\mathcal{N}_i\) and \(\mathcal{N}_{-1}\), respectively. Now let \(f \in \text{Dom}(T)\), where \(\text{Dom}(T)\) is defined in \([\text{i}]\). Then,

\[
f = f_0 + (\mathcal{K} + 1)f_1 = f_0 + Cg_+ + \mathcal{K}Cg_-,
\]

\(f_0 \in \text{Dom}(\dot{A}),\ f_1 \in \mathcal{N}_i\),

for some choice of the constant \(C\) that is specific to \(f \in \text{Dom}(T)\). Moreover,

\[
A\dot{f} = Tf + [p(f, \varphi) + q(f, \psi)] \varphi + [v(f, \varphi) + w(f, \psi)] \psi, \quad f \in \text{Dom}(T).
\]

Let us show that the last two terms in the sum above vanish. Consider \((f, \varphi)\) where \(f\) is decomposed into the \((+\text{-orthogonal})\) sum \([\text{i}]\). Using \((+\text{-orthogonality})\) of \(\mathcal{N}_i\)
and $\mathfrak{N}_-, i$ we have

\[(f, \varphi) = (f_0 + Cg_+ + K\mathcal{C}g_-, \varphi) = (f_0, \varphi) + (Cg_+, \varphi) + (K\mathcal{C}g_-, \varphi) \]
\[= 0 + (Cg_+, \mathcal{R}\varphi)_+ + (K\mathcal{C}g_-, \mathcal{R}\varphi)_+ \]
\[= (Cg_+, (1/\sqrt{2})g_+)_+ + (K\mathcal{C}g_-, (1/\sqrt{2})g_+)_+ \]
\[= \frac{C}{\sqrt{2}}(g_+, g_+)_+ = \frac{C}{\sqrt{2}}\|g_+\|^2 = \sqrt{2}C. \]

Similarly,

\[(f, \psi) = (f_0 + Cg_+ + K\mathcal{C}g_-, \psi) = (f_0, \psi) + (Cg_+, \psi) + (K\mathcal{C}g_-, \psi) \]
\[= 0 + (Cg_+, \mathcal{R}\psi)_+ + (K\mathcal{C}g_-, \mathcal{R}\psi)_+ \]
\[= (Cg_+, (1/\sqrt{2})g_-)_+ + (K\mathcal{C}g_-, (1/\sqrt{2})g_-)_+ \]
\[= \frac{K\mathcal{C}}{\sqrt{2}}(g_-, g_-)_+ = \frac{K\mathcal{C}}{\sqrt{2}}\|g_-\|^2 = \sqrt{2}KC. \]

Consequently,

\[p(f, \varphi) + q(f, \psi) = -H\mathcal{K}(f, \varphi) + H(f, \psi) = H[-\mathcal{K}\sqrt{2}C + \sqrt{2}KC] = 0. \]

Applying similar argument for the last bracketed term in (14) we show that

\[v(f, \varphi) + w(f, \psi) = 0 \]

as well. Thus, $\hat{\mathcal{A}} \subset T \subset \mathcal{A}$. Likewise, using (10) one shows that $\hat{\mathcal{A}} \subset T^* \subset \hat{\mathcal{A}}^*.$

The following theorem was announced by one of the authors (E.T.) in [22] and we present its proof below for convenience of the reader.

**Theorem 3.** Let $T \in \Lambda(\hat{\mathcal{A}})$ and $A$ be a self-adjoint extension of $\hat{\mathcal{A}}$ such that $U$ defines $\text{Dom}(A)$ via (11) and $\mathcal{K}$ defines $T$ via (11). Then $\mathcal{A}$ is a $(\ast)$-extension of $T$ whose real part $\text{Re} \mathcal{A}$ has the quasi-kernel $\mathcal{A}$ if and only if $UK^* - I$ is a homeomorphism and the operator parameter $H$ in (11)-(12) takes the form

\[H = i(I - \mathcal{K}^*\mathcal{K})^{-1}(I - \mathcal{K}^*U)(I - U^*\mathcal{K})^{-1} - \mathcal{K}^*U\mathcal{K}^*. \]

**Proof.** First, we are going to show that $\text{Re} \mathcal{A}$ has the quasi-kernel $A$ if and only if the system of operator equations

\[X^*(I - \hat{\mathcal{K}}^*) + \hat{\mathcal{K}}X(\hat{\mathcal{K}} - I) = i(\hat{\mathcal{K}} - I) \]

\[\hat{\mathcal{K}}^*X^*(\hat{\mathcal{K}}^* - I) + X(I - \hat{\mathcal{K}}) = i(I - \hat{\mathcal{K}}^*) \]

has a solution. Here $\hat{\mathcal{K}} = U^*\mathcal{K}$. Suppose $\text{Re} \mathcal{A}$ has the quasi-kernel $A$ and $U$ defines $\text{Dom}(A)$ via (11). Then there exists a self-adjoint operator $H \in [\mathfrak{N}_{-i}, \mathfrak{N}_{i}]$ such that $\mathcal{A}$ and $\mathcal{A}^*$ are defined via (11) where $S_{\mathcal{A}}$ and $S_{\mathcal{A}^*}$ are of the form (12). Then $\frac{1}{2}(S_{\mathcal{A}} + S_{\mathcal{A}^*})$ is given by (13). According to [3, Theorem 3.4.10] the entries of the operator matrix (13) are related by the following

\[-H\mathcal{K} - K^*H^* - iI = -(H + (K^*H^* + iI)\mathcal{K}^*)^*U, \]

\[\mathcal{K}(HK - iI) + H^* = -(iI - \mathcal{K}H - H^*\mathcal{K}^*)^*U. \]

Denoting $\hat{\mathcal{K}} = U^*\mathcal{K}$ and $\hat{H} = HU$, we obtain

\[\hat{H}^*(I - \hat{\mathcal{K}}^*) + \hat{\mathcal{K}}\hat{H}(\hat{\mathcal{K}} - I) = i(\hat{\mathcal{K}} - I), \]

\[\hat{\mathcal{K}}^*\hat{H}^*(\hat{\mathcal{K}}^* - I) + \hat{H}(I - \hat{\mathcal{K}}) = i(I - \hat{\mathcal{K}}^*), \]
and hence $\hat{H}$ is the solution to the system (19). To show the converse we simply reverse the argument.

Now assume that $UK^* - I$ is a homeomorphism. We are going to prove that the operator $T$ from the statement of the theorem has a unique ($*$)-extension $\hat{A}$ whose real part $\Re \hat{A}$ has the quasi-kernel $A$ that is a self-adjoint extension of $\hat{A}$ parameterized via $U$. Consider the system (19). If we multiply the first equation of (19) by $\tilde{K}^*$ and add it to the second, we obtain

$$(I - \tilde{K}^* \tilde{K})(I - \tilde{K}) = i(\tilde{K}^*(\tilde{K} - I) + (I - \tilde{K}^*)�).$$

Since $I - \tilde{K}^* \tilde{K} = I - K^* K, I - \tilde{K}^* = I - U^* K,$ and $T \in \Lambda(\hat{A}),$ then the operators $I - \tilde{K}^* \tilde{K}$ and $I - \tilde{K}$ are boundedly invertible. Therefore,

$$(20)\quad \text{X} = -i(I - \tilde{K}^* \tilde{K})^{-1}[(I - \tilde{K}^*)(I - \tilde{K})^{-1} - \tilde{K}^*].$$

By the direct substitution one confirms that the operator $\text{X}$ in (20) is a solution to the system (19). Applying the uniqueness result [3, Theorem 4.4.6] and the above reasoning we conclude that our operator $T$ has a unique ($*$)-extension $\hat{A}$ whose real part $\Re \hat{A}$ has the quasi-kernel $A$.

Combining the two parts of the proof, replacing $\tilde{K}$ with $U^* K$, and $X$ with $\tilde{H} = HU$ in (20) we complete the proof of the theorem. □

Suppose that for the case of deficiency indices $(1, 1)$ we have $K = K^* = \tilde{K} = \kappa$ and $U = 1$. Then formula (18) becomes

$$(21) \quad S_A = \left(\begin{array}{cc}
-\frac{i}{\kappa} & \frac{1}{1+\kappa} \\
\frac{1}{1+\kappa} & -\frac{i}{\kappa}
\end{array}\right), \quad S_{A^*} = \left(\begin{array}{cc}
\frac{i\kappa}{1+\kappa} & -\frac{i}{1+\kappa} + i\kappa \\
-\frac{i}{1+\kappa} + i\kappa & -\frac{i\kappa}{1+\kappa}
\end{array}\right).$$

Performing direct calculations gives

$$(22)\quad \frac{1}{2i}(S_A - S_{A^*}) = \frac{1 - \kappa}{2 + 2\kappa} \left(\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array}\right).$$

Using (22) with (12) one obtains

$$(23) \quad \text{Im } \hat{A} = \left(\frac{1 - \kappa}{2 + 2\kappa}\right) \left((\cdot, \varphi + (\cdot, \psi))\varphi + [(\cdot, \varphi) + (\cdot, \psi)]\psi\right)$$

$$= \left(\frac{1 - \kappa}{2 + 2\kappa}\right) ((\cdot, \varphi + \psi)(\varphi + \psi))$$

$$= (\cdot, g)g,$$

where

$$(24) \quad g = \sqrt{\frac{1 - \kappa}{2 + 2\kappa}}(\varphi + \psi) = \sqrt{\frac{1 - \kappa}{1 + \kappa}} \left(\frac{1}{\sqrt{2}} \varphi + \frac{1}{\sqrt{2}} \psi\right).$$

1Throughout this paper $\kappa$ will be called the von Neumann parameter.
Consider a special case when \( \kappa = 0 \). Then the corresponding \((\ast)\)-extension \( A_0 \) is such that
\[
\text{Im } A_0 = \left( \frac{1}{2} \right) \left( (\cdot, \varphi + \psi)(\varphi + \psi) \right) = (\cdot, g_0) g_0,
\]
where
\[
g_0 = \frac{1}{\sqrt{2}} (\varphi + \psi).
\]

4. The Livšic function

Suppose that \( \dot{A} \) is closed, prime\footnote{A symmetric operator \( \dot{A} \) is prime if there does not exist a subspace invariant under \( \dot{A} \) such that the restriction of \( \dot{A} \) to this subspace is self-adjoint.}, densely defined symmetric operator with deficiency indices \((1, 1)\). In \cite{Livsic1975}, a part of Theorem 13 (for a textbook exposition see \cite{Batty1987}) M. Livšic suggested to call the function
\[
s(z) = \frac{z - i}{z + i} \left( \frac{g_z, g_-}{g_z, g_+} \right), \quad z \in \mathbb{C}_+,
\]
the \textit{characteristic function} of the symmetric operator \( \dot{A} \). Here \( g_\pm \in \ker (\dot{A}^* \mp iI) \) are normalized appropriately chosen deficiency elements and \( g_\pm \neq 0 \) are arbitrary deficiency elements of the symmetric operators \( \dot{A} \). Livšic result identifies the function \( s(z) \) (modulo \( z \)-independent unimodular factor) with a complete unitary invariant of a prime symmetric operator with deficiency indices \((1, 1)\) that determines the operator uniquely up to unitary equivalence. He also gave the following criterion for a contractive analytic mapping from the upper half-plane \( \mathbb{C}_+ \) to the unit disk \( \mathbb{D} \) to be the characteristic function of a densely defined symmetric operator with deficiency indices \((1, 1)\).

**Theorem 4** (\cite{Livsic1975}). For an analytic mapping \( s \) from the upper half-plane to the unit disk to be the characteristic function of a densely defined symmetric operator with deficiency indices \((1, 1)\) it is necessary and sufficient that
\[
s(i) = 0 \quad \text{and} \quad \lim_{z \to \infty} z (s(z) - e^{2i\alpha}) = \infty \quad \text{for all} \quad \alpha \in [0, \pi),
\]
\[
0 < \varepsilon \leq \arg(z) \leq \pi - \varepsilon.
\]

The Livšic class of functions described by Theorem 4 will be denoted by \( \mathcal{L} \).

In the same article, Livšic put forward a concept of a characteristic function of a quasi-self-adjoint dissipative extension of a symmetric operator with deficiency indices \((1, 1)\).

Let us recall Livšic’s construction. Suppose that \( \dot{A} \) is a symmetric operator with deficiency indices \((1, 1)\) and that \( g_\pm \) are its normalized deficiency elements,
\[
g_\pm \in \ker (\dot{A}^* \mp iI), \quad \|g_\pm\| = 1.
\]

Suppose that \( T \neq (T)^* \) is a maximal dissipative extension of \( \dot{A} \),
\[
\text{Im}(T f, f) \geq 0, \quad f \in \text{Dom}(T).
\]

Since \( \dot{A} \) is symmetric, its dissipative extension \( T \) is automatically quasi-self-adjoint \cite{Kato1980, Shtrao1967}, that is,
\[
\dot{A} \subset T \subset \dot{A}^*.
\]
and hence, according to (1) with $K = \kappa$, 

\begin{equation}
    g_+ - \kappa g_- \in \text{Dom}(T) \quad \text{for some } |\kappa| < 1.
\end{equation}

Based on the parametrization (29) of the domain of the extension $T$, Livšic suggested to call the Möbius transformation

\begin{equation}
    S(z) = \frac{s(z) - \kappa}{s(z) - 1}, \quad z \in \mathbb{C}_+,
\end{equation}

where $s$ is given by (27), the characteristic function of the dissipative extension $T_{14}$. All functions that satisfy (30) for some function $s(z) \in \mathcal{L}$ will form the Livšic class $\mathcal{L}_\kappa$. Clearly, $\mathcal{L}_0 = \mathcal{L}$.

A culminating point of Livšic’s considerations was the discovery that the characteristic function $S(z)$ (up to a unimodular factor) of a dissipative (maximal) extension $T$ of a densely defined prime symmetric operator $\hat{\mathcal{A}}$ is a complete unitary invariant of $T$ (see [14, the remaining part of Theorem 13]).

In 1965 Donoghue [10] introduced a concept of the Weyl-Titchmarsh function $M(\hat{\mathcal{A}}, \mathcal{A})$ associated with a pair $(\hat{\mathcal{A}}, \mathcal{A})$ by

\begin{equation}
    M(\hat{\mathcal{A}}, \mathcal{A})(z) = ((Az + I)(A - zI)^{-1}g_+, g_+), \quad z \in \mathbb{C}_+,
\end{equation}

where $\hat{\mathcal{A}}$ is a symmetric operator with deficiency indices $(1, 1)$ and $\mathcal{A}$ is its self-adjoint extension.

Denote by $\mathfrak{M}$ the Donoghue class of all analytic mappings $M$ from $\mathbb{C}_+$ into itself that admits the representation

\begin{equation}
    M(z) = \int_{\mathbb{R}} \left( \frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right) d\mu,
\end{equation}

where $\mu$ is an infinite Borel measure and

\begin{equation}
    \int_{\mathbb{R}} \frac{d\mu(\lambda)}{1 + \lambda^2} = 1, \quad \text{equivalently, } M(i) = i.
\end{equation}

It is known [10, 13, 12, 17] that $M \in \mathfrak{M}$ if and only if $M$ can be realized as the Weyl-Titchmarsh function $M(\hat{\mathcal{A}}, \mathcal{A})$ associated with a pair $(\hat{\mathcal{A}}, \mathcal{A})$.

We will also say that an analytic function $M$ from $\mathbb{C}_+$ into itself belongs to the generalized Donoghue class $\mathfrak{M}_\kappa$, $(0 \leq \kappa < 1)$ if it admits the representation (31) and

\begin{equation}
    \int_{\mathbb{R}} \frac{d\mu(\lambda)}{1 + \lambda^2} = \frac{1 - \kappa}{1 + \kappa}, \quad \text{equivalently, } M(i) = i \frac{1 - \kappa}{1 + \kappa}.
\end{equation}

Clearly, $\mathfrak{M}_0 = \mathfrak{M}$.

The Weyl-Titchmarsh function $M$ is a (complete) unitary invariant of the pair of a symmetric operator with deficiency indices $(1, 1)$ and its self-adjoint extension that determines the pair of operators uniquely up to unitary equivalence.

Livšic’s definition of a characteristic function of a symmetric operator (see eq. (27)) has some ambiguity related to the choice of the deficiency elements $g_{\pm}$. To avoid this ambiguity we proceed as follows. Suppose that $\mathcal{A}$ is a self-adjoint extension of a symmetric operator $\hat{\mathcal{A}}$ with deficiency indices $(1, 1)$. Let $g_{\pm}$ be deficiency elements $g_{\pm} \in \text{Ker}((\hat{\mathcal{A}})^* \mp iI)$, $\|g_{\pm}\| = 1$. Assume, in addition, that

\begin{equation}
    g_+ - g_- \in \text{Dom}(\mathcal{A}).
\end{equation}
Following [17] we introduce the Livšic function $s(\hat{A}, A)$ associated with the pair $(\hat{A}, A)$ by

$$s(z) = \frac{z - i}{z + i} \frac{g_+}{g_+}, \quad z \in \mathbb{C}_+,$$

where $0 \neq g_+ \in \text{Ker}((\hat{A}^*) - zI)$ is an arbitrary (deficiency) element.

A standard relationship between the Weyl-Titchmarsh and the Livšic functions associated with the pair $(\hat{A}, A)$ was described in [17]. In particular, if we denote by $M = M(\hat{A}, A)$ and by $s = s(\hat{A}, A)$ the Weyl-Titchmarsh function and the Livšic function associated with the pair $(\hat{A}, A)$, respectively, then

$$s(z) = M(z) - i \frac{M(z)}{M(z) + i}, \quad z \in \mathbb{C}_+. \tag{34}$$

**Hypothesis 5.** Suppose that $T \neq T^*$ is a maximal dissipative extension of a symmetric operator $\hat{A}$ with deficiency indices $(1, 1)$. Assume, in addition, that $A$ is a self-adjoint (reference) extension of $\hat{A}$. Suppose, that the deficiency elements $g_+ \in \text{Ker}(\hat{A}^* - iI)$ are normalized, $\|g_+\| = 1$, and chosen in such a way that

$$g_+ - g_- \in \text{Dom}(A), \quad g_+ - \kappa g_- \in \text{Dom}(T) \text{ for some } |\kappa| < 1. \tag{35}$$

Under Hypothesis [17], we introduce the characteristic function $S = S(\hat{A}, T, A)$ associated with the triple of operators $(\hat{A}, T, A)$ as the Möbius transformation

$$S(z) = \frac{s(z) - \kappa}{s(z) - 1}, \quad z \in \mathbb{C}_+, \tag{36}$$

of the Livšic function $s = s(\hat{A}, A)$ associated with the pair $(\hat{A}, A)$.

We remark that given a triple $(\hat{A}, T, A)$, one can always find a basis $g_\pm$ in the deficiency subspace $\text{Ker}(\hat{A}^* - iI) + \text{Ker}(\hat{A}^* + iI),$

$$\|g_\pm\| = 1, \quad g_\pm \in (\hat{A}^* \mp iI),$$

such that

$$g_+ - g_- \in \text{Dom}(A) \quad \text{and} \quad g_+ - \kappa g_- \in \text{Dom}(T),$$

and then, in this case,

$$\kappa = S(\hat{A}, T, A)(i). \tag{37}$$

Our next goal is to consider a functional model of a prime dissipative triple\footnote{We call a triple $(\hat{A}, T, A)$ a prime triple if $\hat{A}$ is a prime symmetric operator.} parameterized by the characteristic function and obtained in [17].

Given a contractive analytic map $S$,

$$S(z) = \frac{s(z) - \kappa}{s(z) - 1}, \quad z \in \mathbb{C}_+, \tag{38}$$

where $|\kappa| < 1$ and $s$ is an analytic, contractive function in $\mathbb{C}_+$ satisfying the Livšic criterion \cite{28}, we use \cite{13} to introduce the function

$$M(z) = \frac{1}{i} \frac{s(z) + 1}{s(z) - 1}, \quad z \in \mathbb{C}_+, \tag{39}$$

so that

$$M(z) = \int_{\mathbb{R}} \left( \frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right) d\mu(\lambda), \quad z \in \mathbb{C}_+,$$
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for some infinite Borel measure with
\[ \int_{\mathbb{R}} \frac{d\mu(\lambda)}{1 + \lambda^2} = 1. \]

In the Hilbert space \( L^2(\mathbb{R}; d\mu) \) introduce the multiplication (self-adjoint) operator by the independent variable \( B \) on

\[ \text{Dom}(B) = \left\{ f \in L^2(\mathbb{R}; d\mu) \left| \int_{\mathbb{R}} |\lambda^2 f(\lambda)|^2 d\mu(\lambda) < \infty \right. \right\} , \]

denote by \( \dot{B} \) its restriction on

\[ \text{Dom}(\dot{B}) = \left\{ f \in \text{Dom}(B) \left| \int_{\mathbb{R}} f(\lambda)d\mu(\lambda) = 0 \right. \right\} , \]

and let \( T_B \) be the dissipative restriction of the operator \( (\dot{B})^* \) on

\[ \text{Dom}(T_B) = \text{Dom}(\dot{B}) + \dim \text{span} \left\{ \frac{1}{\cdot - i} - S(i) \frac{1}{\cdot + i} \right\} . \]

We will refer to the triple \( (\dot{B}, T_B, B) \) as the model triple in the Hilbert space \( L^2(\mathbb{R}; d\mu) \).

It was established in [17] that a triple \( (\dot{A}, T, A) \) with the characteristic function \( S \) is unitarily equivalent to the model triple \( (\dot{B}, T_B, B) \) in the Hilbert space \( L^2(\mathbb{R}; d\mu) \) whenever the underlying symmetric operator \( \dot{A} \) is prime. The triple \( (\dot{B}, T_B, B) \) will therefore be called the functional model for \( (\dot{A}, T, A) \).

It was pointed out in [17], if \( \kappa = 0 \), the quasi-self-adjoint extension \( T \) coincides with the restriction of the adjoint operator \( (\dot{A})^* \) on

\[ \text{Dom}(T) = \text{Dom}(\dot{A}) + \text{Ker}(\dot{A}^* - iI) . \]

and the prime triples \( (\dot{A}, T, A) \) with \( \kappa = 0 \) are in one-to-one correspondence with the set of prime symmetric operators. In this case, the characteristic function \( S \) and the Livšč function \( s \) coincide (up to a sign),

\[ S(z) = -s(z), \quad z \in \mathbb{C}^+. \]

For the resolvents of the model dissipative operator \( T_B \) and the self-adjoint (reference) operator \( B \) from the model triple \( (\dot{B}, T_B, B) \) one gets the following resolvent formula.

**Theorem 6 ([1]).** Suppose that \( (\dot{B}, T_B, B) \) is the model triple in the Hilbert space \( L^2(\mathbb{R}; d\mu) \). Then the resolvent of the model dissipative operator \( T_B \) in \( L^2(\mathbb{R}; d\mu) \) has the form

\[ (T_B - zI)^{-1} = (B - zI)^{-1} - p(z)(\cdot, g_z)g_z, \]

with

\[ p(z) = \left( M(\dot{B}, B)(z) + \frac{\kappa + 1}{\kappa - 1} \right)^{-1}, \quad z \in \rho(T_B) \cap \rho(B). \]

Here \( M(\dot{B}, B) \) is the Weyl-Titchmarsh function associated with the pair \( (\dot{B}, B) \) continued to the lower half-plane by the Schwarz reflection principle, and the deficiency elements \( g_z \) are given by

\[ g_z(\lambda) = \frac{1}{\lambda - z}, \quad \mu\text{-a.e.} . \]
The same resolvent formula takes place\(^4\) for a given triple \((\hat{A}, T, \hat{A})\) satisfying Hypothesis 3.

**Remark 7.** Without loss of generality, we can assume that \(\kappa\) is real and \(0 \leq \kappa < 1\). Indeed, if \(\kappa = |\kappa|e^{i\theta}\), then one may consider \(\theta = 0\) by the changing of the basis \(g_-\) to the basis \(e^{i\theta}g_-\) in the deficiency subspace \(\text{Ker}(A^* + iI)\). Thus, for the remainder of this paper we assume that the von Neumann parameter \(\kappa\) is real and \(0 \leq \kappa < 1\).

5. Transfer function vs Livšic function

The theorem below is the principal result of the current paper.

**Theorem 8.** Let

\[
\Theta = \begin{pmatrix}
\hat{A} & K \\
\mathcal{H}_+ \subset \mathcal{H} & 1 \\
\end{pmatrix}
\]

be an L-system whose main operator \(T\) and the quasi-kernel \(\hat{A}\) of \(\text{Re}\hat{A}\) satisfy the conditions of Hypothesis 3 with the reference operator \(A = \hat{A}\) and the von Neumann parameter \(\kappa\). Then the transfer function of \(W_\Theta(z)\) and the characteristic function \(S(z)\) of the triple \((\hat{A}, T, \hat{A})\) are reciprocals of each other, i.e.,

\[
W_\Theta(z) = \frac{1}{S(z)}, \quad z \in \mathbb{C}_+ \cap \rho(T),
\]

and \(\frac{1}{W_\Theta(z)} \in \Sigma_{\kappa}\).

**Proof.** We are going to break the proof into three major steps.

**Step 1.** Let us consider the model triple \((\hat{B}, T_B, \mathcal{B})\) developed in Section 8 and described via formulas (39)-(41) with \(\kappa = 0\). Let \(\mathbb{B}_0 \in [\mathcal{H}_+, \mathcal{H}_-]\) be a \((\ast)\)-extension of \(T_{B_0}\) such that \(\text{Re}\mathbb{B}_0 \supset \mathcal{B} = \mathcal{B}_0^\ast\). Clearly, \(T_{B_0} \in \Lambda(\hat{B})\) and \(\mathcal{B}\) is the quasi-kernel of \(\text{Re}\mathbb{B}_0\). It was shown in [3] Theorem 4.4.6 that \(\mathbb{B}_0\) exists and unique. We also note that by the construction of the model triple the von Neumann parameter \(\mathcal{K} = \kappa\) that parameterizes \(T_{B_0}\) via (3) equals to zero, i.e., \(\mathcal{K} = \kappa = 0\). At the same time the parameter \(U\) that parameterizes the quasi-kernel \(\mathcal{B}\) of \(\text{Re}\mathbb{B}_0\) is equal to 1, i.e., \(U = 1\). Consequently, we can use the derivations of the end of Section 8 on \(\mathbb{B}_0\), use formulas (39), (40), and conclude that

\[
\text{Im}\mathbb{B}_0 = \langle \cdot, g_0 \rangle g_0, \quad g_0 = \frac{1}{\sqrt{2}} (\varphi + \psi) \in \mathcal{H}_-,
\]

where \(\varphi \in \mathcal{H}_-\) and \(\psi \in \mathcal{H}_-\) are basis vectors in \(\mathcal{R}^{-1}(\mathcal{H}_u)\) and \(\mathcal{R}^{-1}(\mathcal{H}_{-i})\), respectively. Now we can construct (see (3)) an L-system of the form

\[
\Theta_0 = \begin{pmatrix}
\mathbb{B}_0 & K_0 \\
\mathcal{H}_+ \subset \mathcal{H} & 1 \\
\end{pmatrix}
\]

where \(K_0c = c \cdot g_0, K_0^*f = (f, g_0), (f \in \mathcal{H}_+).\) The transfer function of this L-system can be written (see (3), (3) and (4)) as

\[
W_{\Theta_0}(z) = 1 - 2i((\mathbb{B}_0 - zI)^{-1}g_0, g_0),
\]

and the impedance function is

\[
V_{\Theta_0}(z) = ((\text{Re}\mathbb{B}_0 - zI)^{-1}g_0, g_0) = ((\mathcal{B} - zI)^{-1}g_0, g_0).
\]

\(^4\)Here and below when we write \((\mathcal{B} - zI)^{-1}g_0\) for \(g_0 \in \mathcal{H}_-\) we mean that the resolvent \((\mathcal{B} - zI)^{-1}\) is considered as extended to \(\mathcal{H}_-\) (see (3)).
At this point we would like to apply Theorem 3 and obtain the following resolvent formula

\[(T_{\mathcal{B}_0} - zI)^{-1} = (\mathcal{B} - zI)^{-1} - \frac{1}{M(\mathcal{B}, \mathcal{B})(z) - t} (\cdot, g_z)g_z,\]

where \(g_z = 1/(t - z)\) and \(M(\mathcal{B}, \mathcal{B})(z)\) is the Weyl-Titchmarsh function associated with the pair \((\mathcal{B}, \mathcal{B})\). Moreover,

\[W_{\mathcal{B}_0}(z) = 1 - 2i((\mathbb{B}_0 - zI)^{-1}g_0, g_0)\]

\[= 1 - 2i(T_{\mathcal{B}_0} - zI)^{-1}g_0, g_0)\]

\[= 1 - 2i \left( (\mathcal{B} - zI)^{-1}g_0, g_0) - \left( \frac{1}{M(\mathcal{B}, \mathcal{B})(z) - i} (g_0, g_z)g_z, g_0) \right) \right].\]

Without loss of generality we can assume that

\[(49) \quad g_z = (\mathcal{B} - zI)^{-1}g_0 = (\Re \mathbb{B}_0 - zI)^{-1}g_0 = \frac{1}{t - z},\]

Indeed, clearly \((\Re \mathbb{B}_0 - zI)^{-1}g_0 \in \mathfrak{N}_z\), where \(\mathfrak{N}_z\) is the deficiency subspace of \(\mathcal{B}\), and thus

\[(\Re \mathbb{B}_0 - zI)^{-1}g_0 = \frac{\xi}{t - z},\]

for some \(\xi \in \mathbb{C}\). Let us show that \(|\xi| = 1\). For the impedance function \(V_{\mathcal{B}_0}(z)\) in (17) we have

\[(50) \quad \Im V_{\mathcal{B}_0}(z) = \frac{1}{2i} \left[ ((\Re \mathbb{B}_0 - zI)^{-1}g_0, g_0) - ((\Re \mathbb{B}_0 - \bar{z}I)^{-1}g_0, g_0) \right]

\[= \frac{1}{2i} \left[ (z - \bar{z})((\Re \mathbb{B}_0 - zI)^{-1}(\Re \mathbb{B}_0 - \bar{z}I)^{-1}g_0, g_0) \right]

\[= \Im z((\Re \mathbb{B}_0 - \bar{z}I)^{-1}g_0, (\Re \mathbb{B}_0 - \bar{z}I)^{-1}g_0)

\[= \Im z \left( \frac{\xi}{t - \bar{z}}, \frac{\xi}{t - \bar{z}} \right)_{L^2(\mathbb{R};d\mu)} = (\Im z)|\xi|^2 \int_{\mathbb{R}} \frac{d\mu}{|t - z|^2}.\]

On the other hand, we know [3] that \(V_{\mathcal{B}_0}(z)\) is a Herglotz-Nevanlinna function that has integral representation

\[V_{\mathcal{B}_0}(z) = Q + \int_{\mathbb{R}} \left( \frac{t}{t - z} - \frac{t}{1 + t^2} \right) d\mu, \quad Q = \bar{Q} \]

Using the above representation, the property \(V_{\mathcal{B}_0}(z) = V_{\mathcal{B}_0}(\bar{z})\), and straightforward calculations we find that

\[(51) \quad \Im V_{\mathcal{B}_0}(z) = (\Im z) \int_{\mathbb{R}} \frac{d\mu}{|t - z|^2}.\]

Considering that \(\int_{\mathbb{R}} \frac{d\mu}{|t - z|^2} > 0\), we compare (3) with (51) and conclude that \(|\xi| = 1\). Since \(|\xi| = 1\), \(\xi\) can be scaled into \(g_0\) and we obtain (3).

Taking into account (49) and denoting \(M_0 = M(\mathcal{B}, \mathcal{B})(z)\) for the sake of simplicity, we continue

\[W_{\mathcal{B}_0}(z) = 1 - 2i \left( V_{\mathcal{B}_0}(z) - \frac{1}{M_0 - i} V_{\mathcal{B}_0}^2(z) \right)

\[= 1 - 2i \left( \frac{W_{\mathcal{B}_0}(z) - 1}{W_{\mathcal{B}_0}(z) + 1} + \frac{1}{M_0 - i} \left( \frac{W_{\mathcal{B}_0}(z) - 1}{W_{\mathcal{B}_0}(z) + 1} \right)^2 \right).\]
Thus,

\[
W_{\theta_0}(z) - 1 = 2 \frac{W_{\theta_0}(z) - 1}{W_{\theta_0}(z) + 1} - \frac{2i}{M_0 - i} \left( \frac{W_{\theta_0}(z) - 1}{W_{\theta_0}(z) + 1} \right)^2,
\]

or

\[
1 = \frac{2}{W_{\theta_0}(z) + 1} - \frac{2i}{M_0 - i} \cdot \frac{W_{\theta_0}(z) - 1}{(W_{\theta_0}(z) + 1)^2}.
\]

Solving this equation for \(W_{\theta_0}(z) + 1\) yields

\[
W_{\theta_0}(z) + 1 = \frac{(M_0 - 2i) \pm M_0}{M_0 - i}.
\]

Assume that \(M_0(z) \neq i\) for \(z \in C_+\) and consider two outcomes for formula (52).

First case leads to \(W_{\theta_0}(z) + 1 = 2\) or \(W_{\theta_0}(z) = 1\) which is impossible because it would lead (via (53)) to \(V_{\theta_0}(z) = 0\) that contradicts (51). The second case is

\[
W_{\theta_0}(z) + 1 = -\frac{2i}{M_0 - i},
\]

leading to (see (54))

\[
W_{\theta_0}(z) = -\frac{2i}{M_0 - i} - 1 = -\frac{M_0 + i}{M_0 - i} = -\frac{1}{s(z)}, \quad z \in C_+ \cap \rho(T_{B_0}),
\]

where \(s(z)\) is the Livšic function associated with the pair \((\hat{B}, B)\). As we mentioned in Section 5, in the case when \(\kappa = 0\) the characteristic function \(S\) and the Livšic function \(s\) coincide (up to a sign), or \(S(z) = -s(z)\). Hence,

\[
W_{\theta_0}(z) = -\frac{1}{s(z)} = \frac{1}{S(z)}, \quad z \in C_+ \cap \rho(T_{B_0}),
\]

where \(S(z)\) is the characteristic function of the model triple \((\hat{B}, T_{B_0}, B)\).

In the case when \(M_0(z) = i\) for all \(z \in C_+\), formula (52) would imply that \(s(z) \equiv 0\) in the upper half-plane. Then, as it was shown in [24, Lemma 5.1], all the points \(z \in C_+\) are eigenvalues for \(T_{B_0}\) and the function \(W_{\theta_0}(z)\) is simply undefined in \(C\) making (52) irrelevant.

As we mentioned above, if \(M_0(z) = i\) for all \(z \in C_+\) the function \(W_{\theta_0}(z)\) is undefined and (52) does not make sense in \(C_+\). One can, however, in this case re-write (52) in \(C_-\). Using the symmetry of \(M_0(z)\) we get that \(M_0(z) = -i\) for all \(z \in C_-\). Then (52) yields that \(W_{\theta_0}(z) = 0\). On the other hand, (54) extended to \(C_-\) in this case implies that \(s(z) = \infty\) for all \(z \in C_-\) and hence (53) still formally holds true here for \(z \in C_+ \cap \rho(T_{B_0})\).

Let us also make one more observation. Using formulas (3) and (53) yields

\[
W_{\theta_0}(z) = \frac{1 - iV_{\theta_0}(z)}{1 + iV_{\theta_0}(z)} = \frac{-V_{\theta_0}(z) + i}{V_{\theta_0}(z) - i} = -\frac{M_0(z) + i}{M_0(z) - i},
\]

and hence

\[
V_{\theta_0}(z) = M_0(z), \quad z \in C_+.
\]
Step 2. Now we are ready to treat the case when \( \kappa = \bar{\kappa} \neq 0 \). Assume Hypothesis 4 and consider the model triple \((\bar{\mathcal{B}}, T_B, \mathcal{B})\) described by formulas (9) and (21) with some \( \kappa, |\kappa| < 1 \). Let \( \mathcal{B} \in [\mathcal{H}_+, \mathcal{H}_-] \) be a \((\ast)\)-extension of \( T_B \) such that Re \( \mathcal{B} \supset \mathcal{B} = B^* \). Below we describe the construction of \( \mathcal{B} \). Equation (35) of Hypothesis 4 implies that
\[
\text{Im} \mathcal{B} = (\cdot, g) g, \quad g = \sqrt{\frac{1 - \kappa}{1 + \kappa}} \left( \frac{1}{\sqrt{2}} \varphi - \frac{1}{\sqrt{2}} \psi \right).
\]

We notice that if we followed the same basis pattern for the \((\ast)\)-extension \( \mathcal{B}_0 \) (when \( \kappa = 0 \)) then (4) would become slightly modified as follows
\[
\text{Im} \mathcal{B}_0 = (\cdot, g_0) g_0, \quad g_0 = \frac{1}{\sqrt{2}} (\varphi - \psi),
\]

As before we use \( \mathcal{B} \) to construct a model L-system of the form
\[
\Theta' = \left( \begin{array}{cc} \mathcal{B} & \mathcal{B}' \\ \mathcal{H}_+ \subset \mathcal{H}_- & \mathcal{K}' \\ 1 & C \end{array} \right),
\]

where \( \mathcal{K}' = c \cdot g, \mathcal{K}' \ast f = (f, g), (f \in \mathcal{H}_+) \). The impedance function of \( \Theta' \) is
\[
V_{\Theta}(z) = ((\text{Re} \mathcal{B} - zI)^{-1}g, g) = ((\mathcal{B} - zI)^{-1}g, g)
\]
\[
= \left( (\mathcal{B} - zI)^{-1} \sqrt{\frac{1 - \kappa}{1 + \kappa}} \left( \frac{1}{\sqrt{2}} \varphi - \frac{1}{\sqrt{2}} \psi \right), \sqrt{\frac{1 - \kappa}{1 + \kappa}} \left( \frac{1}{\sqrt{2}} \varphi - \frac{1}{\sqrt{2}} \psi \right) \right)
\]
\[
= \frac{1 - \kappa}{1 + \kappa} ((\mathcal{B} - zI)^{-1}g_0, g_0) = \left( \frac{1 - \kappa}{1 + \kappa} \right) V_{\Theta_0}(z) = \left( \frac{1 - \kappa}{1 + \kappa} \right) M_0(z).
\]

Here we used relations (6) and (10). On the other hand, using (5), (22), and (14) yields
\[
S(z) = \frac{s(z) - \kappa}{\kappa s(z) - 1} = \frac{M_0 - i}{M_0 + i} = \frac{(1 - \kappa)M_0 - i(\kappa + 1)}{(\kappa - 1)M_0 - (\kappa + 1)i}
\]
\[
= \frac{1 - \kappa}{1 + \kappa} M_0 - i = \frac{V_{\Theta}(z) - i}{V_{\Theta}(z) + i} = \frac{1}{W_{\Theta}(z)}.
\]

Thus,
\[
W_{\Theta}(z) = \frac{1}{S(z)}, \quad z \in \mathbb{C}_+ \cap \rho(T_B),
\]

where \( S(z) \) is the characteristic function of the model triple \((\bar{\mathcal{B}}, T_B, \mathcal{B})\).
Step 3. Now we are ready to treat the general case. Let
\[ \Theta = \begin{pmatrix} \mathcal{A} & K \\ \mathcal{H} & H & \mathcal{C} \end{pmatrix} \]
be an L-system from the statement of our theorem. Without loss of generality we can consider our L-system \( \Theta \) to be minimal. If it is not minimal, we can use its so-called “principal part”, which is an L-system that has the same transfer and impedance functions (see [3, Section 6.6]). We use the von Neumann parameter \( \kappa \) of \( T \) and the conditions of Hypothesis 5 to construct a model system \( \Theta' \) given by (57). By construction \( W_{\Theta}(z) = W_{\Theta'}(z) \) and the characteristic functions of \((\hat{A}, T, \hat{A})\) and the model triple \((\hat{B}, T_{\hat{B}}, \hat{B})\) coincide. The conclusion of the theorem then follows from Step 2 and formula (59). \( \square \)

Corollary 9. If under conditions of Theorem 8 we also have that the von Neumann parameter \( \kappa \) of \( T \) equals zero, then \( W_{\Theta}(z) = -1/s(z) \), where \( s(z) \) is the Livšic function associated with the pair \((\hat{A}, \hat{A})\).

Corollary 10. Let \( \Theta \) be an arbitrary L-system of the form (42). Then the transfer function of \( W_{\Theta}(z) \) and the characteristic function \( S(z) \) of a triple \((\hat{A}, T, \hat{A})\) satisfying Hypothesis 5 with reference operator \( A = \hat{A}_1 \) are related via
\[ W_{\Theta}(z) = \frac{\nu}{S(z)}, \quad z \in \mathbb{C}_+ \cap \rho(T), \]
where \( \nu \in \mathbb{C} \) and \( |\nu| = 1 \).

Proof: The only difference between the L-system \( \Theta \) here and the one described in Theorem 3 is that the set of conditions of Hypothesis 3 is satisfied for the latter. Moreover, there is an L-system \( \Theta_1 \) of the form (42) with the same main operator \( T \) that complies with Hypothesis 3. Then according to the theorem about a constant \( J \)-unitary factor [3, Theorem 8.2.1], \( W_{\Theta}(z) = \nu W_{\Theta_1}(z) \), where \( \nu \) is a unimodular complex number. Applying Theorem 3 to the L-system \( \Theta_1 \) yields \( W_{\Theta_1}(z) = 1/S(z) \), where \( S(z) \) is the characteristic function of the triplet \((\hat{A}, T, \hat{A}_1)\) and \( \hat{A}_1 \) is the quasi-kernel of the real part of the operator \( A_1 \) in \( \Theta_1 \). Consequently,
\[ W_{\Theta}(z) = \nu W_{\Theta_1}(z) = \frac{\nu}{S(z)}, \]
where \( |\nu| = 1 \). \( \square \)

6. Impedance Functions of the Classes \( \mathfrak{M} \) and \( \mathfrak{M}_\infty \)

We begin by stating and proving the following important lemma.

Lemma 11. Let \( \Theta_\kappa \) of the form (42) be an L-system whose main operator \( T \) (with the von Neumann parameter \( \kappa \), \( 0 \leq \kappa < 1 \)) and the quasi-kernel \( \hat{A} \) of \( \text{Re} A \) satisfy the conditions of Hypothesis 3 with the reference operator \( A = \hat{A} \). Then the impedance function \( V_{\Theta_\kappa}(z) \) admits the representation
\[ V_{\Theta_\kappa}(z) = \left( \frac{1 - \kappa}{1 + \kappa} \right) V_{\Theta_0}(z), \]
where \( V_{\Theta_0}(z) \) is the impedance function of an L-system \( \Theta_0 \) with the same set of conditions but with \( \kappa_0 = 0 \), where \( \kappa_0 \) is the von Neumann parameter of the main operator \( T_0 \) of \( \Theta_0 \).
Proof. Once again we rely on our derivations above. We use the von Neumann parameter $\kappa$ of $T$ and the conditions of Hypothesis (5) to construct a model system $\Theta'$ given by (57). By construction $V_{\Theta_0}(z) = V_{\Theta^\prime}(z)$. Similarly, the impedance function $V_{\Theta_0}(z)$ coincides with the impedance function of a model system (15). The conclusion of the lemma then follows from (54) and (58). □

Theorem 12. Let $\Theta$ of the form (42) be an L-system whose main operator $T$ has the von Neumann parameter $\kappa$, $(0 \leq \kappa < 1)$. Then its impedance function $V_{\Theta}(z)$ belongs to the Donoghue class $\mathfrak{M}$ if and only if $\kappa = 0$.

Proof. First of all, we note that in our system $\Theta$ the quasi-kernel $\hat{A}$ of $\text{Re} \ A$ does not necessarily satisfy the conditions of Hypothesis (5). However, if $\Theta_\kappa$ is a system from the statement of Lemma (11) with the same $\kappa$ and Hypothesis (5) requirements, then

$$W_{\Theta}(z) = \nu W_{\Theta_\kappa}(z),$$

where $\nu$ is a complex number such that $|\nu| = 1$. This follows from the theorem about a constant $J$-unitary factor \cite[Theorem 8.2.1]{4}. We know that for system $\Theta_\kappa$ Theorem \cite{3} applies and hence formula (13) takes place. Assume that $\kappa \neq 0$. Combining (12) with (62) and using the normalization condition (37) we obtain

$$W_{\Theta}(i) = \frac{\nu}{\kappa}.$$  

We know that according to \cite[Theorem 6.4.3]{3} the impedance function $V_{\Theta}(z)$ admits the following integral representation

$$V_{\Theta}(z) = Q + \int_\mathbb{R} \left( \frac{1}{\lambda - z} - \frac{1}{1 + \lambda^2} \right) d\mu,$$

where $Q$ is a real number and $\mu$ is an infinite Borel measure such that

$$\int_\mathbb{R} \frac{d\mu(\lambda)}{1 + \lambda^2} = L < \infty.$$  

By direct check $V_{\Theta}(i) = Q + iL$. Therefore, applying (13) directly to $W_{\Theta}(z)$ and using (12) yields

$$W_{\Theta}(i) = \frac{1 - iV_{\Theta}(i)}{1 + iV_{\Theta}(i)} = \frac{(1 - i)Q + L}{(1 + i)Q - L} = \frac{\nu}{\kappa}.$$  

Solving this relation for $Q$ gives us

$$Q = i \frac{\nu(1 - L) - \kappa(1 + L)}{\nu + \kappa}.$$  

Taking into account that since $\nu\bar{\nu} = 1$ and recalling our agreement in Section (4) to consider real $\kappa$ only, we get

$$\bar{Q} = -i \frac{\bar{\nu}(1 - L) - \kappa(1 + L)}{\bar{\nu} + \kappa}.$$  

But $Q = \bar{Q}$ and hence equating (65) and (66) and solving for $L$ yields

$$L = \frac{\nu - \kappa^2 \nu}{(\nu + \kappa)(1 + \kappa \nu)}. $$  

Clearly, $V_{\Theta}(z) \in \mathfrak{M}$ if and only if $Q = 0$ and $L = 1$. Setting the right hand side of (12) to 1 and solving for $\kappa$ gives $\kappa = 0$ or $\kappa = -(\nu^2 + 1)/(2\nu)$, but only $\kappa = 0$
makes \( Q = 0 \) in (66). Thus, \( Q = 0 \) and \( L = 1 \) if and only if \( \kappa = 0 \) which contradicts our assumption. Therefore, \( V_0(z) \in \mathfrak{M} \) if and only if \( \kappa = 0 \). \( \square \)

Consider the L-system \( \Theta \) of the form (12). This L-system does not necessarily comply with the conditions of Hypothesis 8 and hence the quasi-kernel \( \hat{A} \) of \( \text{Re} \, \hat{A} \) is parameterized via (1) by some complex number \( U \), \( |U| = 1 \). Then \( U = e^{2i\beta} \), where \( \beta \in [0, \pi) \). This representation allows us to introduce a one-parametric family of L-systems \( \Theta_0(\beta) \) that all have \( \kappa = 0 \). That is

\[
\Theta_0(\beta) = \left( \begin{array}{ccc}
\hat{A}_{00}(\beta) & \hat{A}_{01}(\beta) & 1 \\
\hat{A}_{10}(\beta) & \hat{A}_{11}(\beta) & 0 \\
\hat{A}_{20}(\beta) & \hat{A}_{21}(\beta) & 0 \\
\end{array} \right).
\]

We note that \( \Theta_0(\beta) \) satisfies the conditions of Hypothesis 8 only for the case when \( \beta = 0 \). Hence, the L-system \( \Theta_0 \) from Lemma 4 can be written as \( \Theta_0 = \Theta_0(0) \) using (13). Moreover, it directly follows from Theorem 2 that all the impedance functions \( V_{\Theta_0(\beta)}(z) \) belong to the Donoghue class \( \mathfrak{M} \) regardless of the value of \( \beta \in [0, \pi) \).

The next theorem gives criteria on when the impedance function of an L-system belongs to the generalized Donoghue class \( \mathfrak{M}_\kappa \).

**Theorem 13.** Let \( \Theta_\kappa, \ 0 \leq \kappa < 1 \), of the form (12) be an L-system with the main operator \( T \). Then its impedance function \( V_{\Theta_\kappa}(z) \) belongs to the generalized Donoghue class \( \mathfrak{M}_\kappa \) and \( (6\bar{1}) \) holds if and only if the triple \( (A, T, A) \) satisfies Hypothesis 8 with \( A = \hat{A} \), the quasi-kernel of \( \text{Re} \, \hat{A} \).

**Proof.** It follows from Theorem 12 and the definition of \( \mathfrak{M}_\kappa \) that our function \( V_{\Theta_\kappa}(z) \) would belong to the generalized Donoghue class \( \mathfrak{M}_\kappa \) if and only if it satisfies equation (61). Then in one direction the proof of the Theorem directly follows from Lemma 4. Let us prove it in the other direction. Suppose \( V_{\Theta_\kappa}(z) \) satisfies equation (61) for some L-system \( \Theta_0 \) with \( \kappa = 0 \). Then according to Theorem 12 \( V_{\Theta_\kappa}(z) \) belongs to the Donoghue class \( \mathfrak{M} \) and hence has its integral representation (65) with \( Q = 0 \) in particular. Clearly, then (62) implies that \( V_{\Theta_\kappa}(z) \) has \( Q = 0 \) in its integral representation (65). Moreover,

\[
V_{\Theta_\kappa}(i) = \left( \begin{array}{c}
1 - \kappa \\
1 + \kappa \\
\end{array} \right), \quad V_{\Theta_0}(i) = i \left( \begin{array}{c}
1 - \kappa \\
1 + \kappa \\
\end{array} \right) = i \int_{\mathbb{R}} d\mu(\lambda) \frac{1}{1 + \lambda^2},
\]

where \( \mu(\lambda) \) is the measure from the integral representation (62) of \( V_{\Theta_0}(z) \). Thus,

\[
L = \int_{\mathbb{R}} d\mu(\lambda) \frac{1}{1 + \lambda^2} = \frac{1 - \kappa}{1 + \kappa}.
\]

Assume the contrary, i.e., the quasi-kernel \( \hat{A} \) of \( \text{Re} \, \hat{A} \) of \( \Theta_\kappa \) does not satisfy the conditions of Hypothesis 8. Then, consider another L-system \( \Theta' \) of the form (12) which is only different from \( \Theta \) by the fact that its quasi-kernel \( \hat{A}' \) of \( \text{Re} \, \hat{A}' \) satisfies the conditions of Hypothesis 8 for the same value of \( \kappa \). Applying the theorem about a constant \( J \)-unitary factor \( \Theta \) (Theorem 8.2.1) then yields

\[
W_{\Theta_\kappa}(z) = \nu W_{\Theta'}(z),
\]

where \( \nu \) is a complex number such that |\( \nu \)| = 1. Our goal is to show that \( \nu = 1 \). Since we know the values of \( Q \) and \( L \) in the integral representation (65) of \( V_{\Theta_\kappa}(z) \),
we can use this information to find \( \nu \) from (65). We have then
\[
0 = i \frac{\nu(1 - L) - \kappa(1 + L)}{\nu + \kappa}, \quad \text{where} \quad L = \frac{1 - \kappa}{1 + \kappa}.
\]
Consequently, \( \nu(1 - L) - \kappa(1 + L) = 0 \) or
\[
\nu = \kappa \frac{1 + L}{1 - L} = \kappa \frac{1 + \frac{1 - \kappa}{1 + \kappa}}{1 - \frac{1 - \kappa}{1 + \kappa}} = \kappa \cdot \frac{2}{2\kappa} = 1.
\]
Thus, \( \nu = 1 \) and hence
\[
W_{\Theta_\kappa}(z) = W_{\Theta'}(z).
\]
Then we can apply the Theorem on bi-unitary equivalence [1 Theorem 6.6.10] for L-systems \( \Theta_\kappa \) and \( \Theta' \) and obtain that \( \hat{A} \) and \( \hat{A}' \) are unitary equivalent and so are the pairs \( (\hat{A}, \hat{A}) \) and \( (\hat{A}, \hat{A}') \). Consequently, the Weyl-Titchmarsh functions \( M(\hat{A}, \hat{A}) \) and \( M(\hat{A}, \hat{A}') \) coincide. At the same time, both \( \hat{A} \) and \( \hat{A}' \) are self-adjoint extensions of the symmetric operator \( \hat{A} \) giving us the following relation between \( M(\hat{A}, \hat{A}) \) and \( M(\hat{A}, \hat{A}') \) (see [18 Subsection 2.2])
\[
(70) \quad M(\hat{A}, \hat{A}') = \frac{\cos \alpha M(\hat{A}, \hat{A}) - \sin \alpha}{\cos \alpha + \sin \alpha M(\hat{A}, \hat{A})}, \quad \text{for some} \ \alpha \in [0, \pi).
\]
Using \( M(\hat{A}, \hat{A}')(z) = M(\hat{A}, \hat{A})(z) \) for \( z \in \mathbb{C}_+ \) on [3] and solving for \( M(\hat{A}, \hat{A})(z) \) gives us that either \( \alpha = 0 \) or \( M(\hat{A}, \hat{A})(z) = i \) for all \( z \in \mathbb{C}_+ \). The former case of \( \alpha = 0 \) gives \( \hat{A} = \hat{A}' \), and thus \( \hat{A} \) satisfies the conditions of Hypothesis [10] which contradicts our assumption. The latter case would imply (via (64)) that \( s(z) = s(\hat{A}, \hat{A}')(z) \equiv 0 \) and consequently \( S(z) = S(\hat{A}, \hat{A}, T)(z) \equiv \kappa \) in the upper half-plane. Then (43) and (62) imply that \( W_{\Theta_\kappa}(z) = \theta/\kappa \) for some \( \theta \) such that \( |\theta| = 1 \) and
\[
V_{\Theta_\kappa}(z) = i \frac{\theta/\kappa - 1}{\theta/\kappa + 1} = i \frac{\theta - \kappa}{\theta + \kappa}, \quad z \in \mathbb{C}_+.
\]
Taking into account that in this case \( V_{\Theta_0}(z) \equiv i \), we see that formula (61) will not hold unless \( \theta = 1 \) which brings us back to the case of \( \alpha = 0 \) and \( \hat{A} = \hat{A}' \) again. Therefore, we have also arrive at a contradiction and the conditions of Hypothesis [10] must hold for \( \hat{A} \).

Using similar reasoning as above we introduce another one parametric family of L-systems
\[
(71) \quad \Theta_\kappa(\beta) = \left( \begin{array}{cc}
A_\kappa(\beta) & K_\kappa(\beta) \\
\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & 1
\end{array} \right),
\]
which is different from the family in [18] by the fact that all the members of the family have the same fixed von Neumann parameter \( \kappa \neq 0 \). It easily follows from Theorem [12] that for all \( \beta \in [0, \pi) \) there is only one impedance function \( V_{\Theta_\kappa(\beta)}(z) \) that belongs to the class \( \mathfrak{M}_\kappa \). This happens when \( \beta = 0 \) and consequently the L-system \( \Theta_\kappa(0) \) complies with the conditions of Hypothesis [10]. The results of Theorems [12 and 13] can be illustrated with the help of Figure 8 describing the parametric region for the family of L-systems \( \Theta(\beta) \). When \( \kappa = 0 \) and \( \beta \) changes from 0 to \( \pi \), every point on the unit circle with cylindrical coordinates \( (1, \beta, 0) \), \( \beta \in [0, \pi] \) describes an L-system \( \Theta_\kappa(\beta) \) and Theorem [12] guarantees that \( V_{\Theta_\kappa(\beta)}(z) \) belongs to the class \( \mathfrak{M} \). On the other hand, for any \( \kappa_0 \) such that \( 0 < \kappa_0 < 1 \) we apply
Theorem 13 to conclude that only the point \((1, 0, \kappa_0)\) on the wall of the cylinder is responsible for an L-system \(\Theta_{\kappa_0}(0)\) such that \(V_{\Theta_{\kappa_0}(0)}(z)\) belongs to the class \(\mathfrak{M}_{\kappa_0}\).

7. Examples

Example 1. Following [1] we consider the prime symmetric operator

\[ \dot{A}x = \frac{i}{t} dx, \quad \text{Dom}(\dot{A}) = \left\{ x(t) \mid x(t) - \text{abs. cont.}, x'(t) \in L^2_{[0, \ell]}, x(0) = x(\ell) = 0 \right\}. \]

Its (normalized) deficiency vectors of \(\dot{A}\) are

\[ g_+ = \frac{\sqrt{2}}{\sqrt{e^{2\tau} - 1}} e^t \in \mathfrak{H}_i, \quad g_- = \frac{\sqrt{2}}{\sqrt{1 - e^{-2\tau}}} e^{-t} \in \mathfrak{H}_{-i}. \]

If we set \(C = \frac{\sqrt{2}}{\sqrt{e^{2\tau} - 1}}\), then (73) can be re-written as

\[ g_+ = Ce^t, \quad g_- = Ce^{-t}. \]

Let

\[ Ax = \frac{i}{t} dx, \quad \text{Dom}(A) = \left\{ x(t) \mid x(t) - \text{abs. cont.}, x'(t) \in L^2_{[0, \ell]}, x(0) = -x(\ell) \right\}. \]

be a self-adjoint extension of \(\dot{A}\). Clearly, \(g_+(0) - g_-(0) = C - Ce^\ell\) and \(g_+(\ell) - g_-(\ell) = Ce^\ell - C\) and hence (32) is satisfied, i.e., \(g_+ - g_- \in \text{Dom}(A)\).

Then the Livšic characteristic function \(s(z)\) for the pair \((\dot{A}, A)\) is defined and equal (see [4])

\[ s(z) = \frac{e^\ell - e^{-iz}}{1 - e^\ell e^{-iz}}. \]
We introduce the operator

$$T x = i \frac{dx}{dt}, \quad \text{Dom}(T) = \left\{ x(t) \mid x(t) - \text{abs. cont.}, x'(t) \in L^2_{[0,\ell]}, x(0) = 0 \right\}. \quad (76)$$

By construction, $T$ is a dissipative extension of $\hat{A}$ parameterized by a von Neumann parameter $\kappa$. To find $\kappa$ we use (73) with (29) to obtain

$$x(t) = Ce^\ell - \kappa Ce^\ell e^{-t} \in \text{Dom}(T), \quad x(0) = 0,$$

yielding

$$\kappa = e^{-\ell}. \quad (78)$$

Obviously, the triple of operators $(\hat{A}, T, A)$ satisfy the conditions of Hypothesis 3 since $|\kappa| = e^{-\ell} < 1$. Therefore, we can use (34) to write out the characteristic function $S(z)$ for the triple $(\hat{A}, T, A)$

$$S(z) = \frac{s(z) - \kappa}{\kappa s(z) - 1} = \frac{e^\ell - \kappa + e^{-iz}(\kappa e^\ell - 1)}{\kappa e^\ell - 1 + e^{-iz}(e^\ell - \kappa)}, \quad (79)$$

and apply the value of $\kappa = e^{-\ell}$ to get

$$S(z) = e^{iz}. \quad (80)$$

Now we shall use the triple $(\hat{A}, T, A)$ for an L-system $\Theta$ that we about to construct. First, we note that by the direct check one gets

$$T^* x = i \frac{dx}{dt}, \quad \text{Dom}(T) = \left\{ x(t) \mid x(t) - \text{abs. cont.}, x'(t) \in L^2_{[0,\ell]}, x(\ell) = 0 \right\}. \quad (81)$$

Following the steps of Example 7.6 of [3], we have

$$\hat{A}^* x = i \frac{dx}{dt}, \quad \text{Dom}(\hat{A}^*) = \left\{ x(t) \mid x(t) - \text{abs. cont.}, x'(t) \in L^2_{[0,\ell]} \right\}. \quad (82)$$

Then $H_+ = \text{Dom}(\hat{A}^*) = W^1_2$ is the Sobolev space with scalar product

$$(x, y)_+ = \int_0^\ell x(t)y(t) dt + \int_0^\ell x'(t)y'(t) dt. \quad (83)$$

Construct rigged Hilbert space $W^1_2 \subset L^2_{[0,\ell]} \subset (W^1_2)_-$ and consider operators

$$\mathbb{A} x = i \frac{dx}{dt} + ix(0) [\delta(t) - \delta(t - \ell)], \quad \mathbb{A}^* x = i \frac{dx}{dt} + ix(\ell) [\delta(t) - \delta(t - \ell)], \quad (84)$$

where $x(t) \in W^1_2$, $\delta(t)$, $\delta(t - \ell)$ are delta-functions and elements of $(W^1_2)_-$ that generate functionals by the formulas $(x, \delta(t)) = x(0)$ and $(x, \delta(t - \ell)) = x(\ell)$. It is easy to see that $\mathbb{A} \supset T \supset \hat{A}$, $\mathbb{A}^* \supset T^* \supset \hat{A}$, and

$$\text{Re} \mathbb{A} x = \frac{i}{2} \frac{dx}{dt} + i(x(0) + x(\ell)) [\delta(t) - \delta(t - \ell)].$$

Clearly, $\text{Re} \mathbb{A}$ has its quasi-kernel equal to $A$ in (74). Moreover,

$$\text{Im} \mathbb{A} x = \left( \cdot, \frac{1}{\sqrt{2}} [\delta(t) - \delta(t - \ell)] \right) = \frac{1}{\sqrt{2}} [\delta(t) - \delta(t - \ell)] = (\cdot, g),$$

where $g = \frac{i}{\sqrt{2}} [\delta(t) - \delta(t - \ell)]$. Now we can build

$$\Theta = \begin{pmatrix} \mathbb{A} & K & 1 \\ W^1_2 \subset L^2_{[0,\ell]} & (W^1_2)_- & \mathbb{C} \end{pmatrix}, \quad (85)$$
that is a minimal L-system with
\[ Kc = c \cdot g = c \cdot \frac{1}{\sqrt{2}} [\delta(t) - \delta(t - l)], \quad (c \in \mathbb{C}), \]
(86)
\[ K^* x = (x, g) = \left( x, \frac{1}{\sqrt{2}} [\delta(t) - \delta(t - l)] \right) = \frac{1}{\sqrt{2}} [x(0) - x(l)], \]
and \( x(t) \in W^2_2 \). In order to find the transfer function of \( \Theta \) we begin by evaluating the resolvent of operator \( T \) in (76). Solving the linear differential equation of the first order with the initial condition from (76) yields
\[ R(T)f = (T - zI)^{-1} f = -ie^{-itz} \int_0^t f(s)e^{izs} \, ds, \quad f \in L^2_{[0, \ell]} . \]
(87)

Similarly, one finds that
\[ R(T^*)f = (T^* - zI)^{-1} f = ie^{-itz} \int_0^\ell f(s)e^{izs} \, ds, \quad f \in L^2_{[0, \ell]} . \]
(88)

We need to extend \( R(T) \) to \( (W^2_2)_{-} \) to apply it to the vector \( g \). We can accomplish this via finding the values of \( \hat{R}(T)\delta(t) \) and \( \hat{R}(T)\delta(t - \ell) \) (here \( \hat{R}(T) \) is the extended resolvent). We have
\[
(\hat{R}(T)\delta(t), f) = (\delta(t), R(T^*)f) = R(T^*)f \bigg|_{t=0} = -i \int_0^\ell e^{-itz} f(s) ds
\]
\[ = (-ie^{-itz}, f), \quad f \in L^2_{[0, \ell]}, \]
and hence \( \hat{R}(T)\delta(t) = -ie^{-itz} \). Similarly, we determine that \( \hat{R}(T)\delta(t - \ell) = 0 \).

Consequently,
\[ \hat{R}(T)g = -\frac{i}{\sqrt{2}} e^{-itz}. \]

Therefore,
\[ W_\Theta(z) = 1 - 2i((T - zI)^{-1} g, g) = 1 - 2i \left( -\frac{i}{\sqrt{2}} e^{-itz}, \frac{1}{\sqrt{2}} [\delta(t) - \delta(t - \ell)] \right) \]
\[ = 1 - (e^{-itz}, \delta(t) - \delta(t - \ell)) = 1 - 1 + e^{-it\ell} = e^{-it\ell}. \]
(89)

This confirms the result of Theorem 8 and formula (53) by showing that \( W_\Theta(z) = 1/S(z) \). The corresponding impedance function is found via (53) and is
\[ V_\Theta(z) = \frac{e^{-it\ell} - 1}{e^{-it\ell} + 1}. \]

Direct substitution yields
\[ V_\Theta(i) = i \frac{e^\ell - 1}{e^\ell + 1} = i \frac{1 - e^{-\ell}}{1 + e^{-\ell}} = i \frac{1 - \kappa}{1 + \kappa}, \]
and thus \( V_\Theta(z) \in \mathbb{M}_\kappa \) with \( \kappa = e^{-\ell} \).
Example 2. In this Example we will rely on the main elements of the construction presented in Example 1 but with some changes. Let \( \hat{A} \) and \( A \) be still defined by formulas (72) and (74), respectively and let \( s(z) \) be the Livšic characteristic function \( s(z) \) for the pair \( (\hat{A},A) \) given by (83). We introduce the operator
\[
T_0 x = i \frac{dx}{dt}, \quad \text{Dom}(T_0) = \left\{ x(t) \mid x(t) - \text{abs. cont.}, x'(t) \in L^2_{[0,t]}, x(t) = e^{\ell} x(0) \right\}.
\]
It turns out that \( T_0 \) is a dissipative extension of \( \hat{A} \) parameterized by a von Neumann parameter \( \kappa = 0 \). Indeed, using (83) with (82) again we obtain
\[
x(t) = C e^t - \kappa C e^t e^{-t} \in \text{Dom}(T), \quad x(t) = e^{\ell} x(0),
\]
yielding \( \kappa = 0 \). Clearly, the triple of operators \( (\hat{A}, T_0, A) \) satisfy the conditions of Hypothesis \( \text{(H)} \) but this time, since \( \kappa = 0 \), we have that \( S(z) = -s(z) \).

Following the steps of Example 1 we are going to use the triple \( (\hat{A}, T_0, A) \) in the construction of an L-system \( \Theta_0 \). By the direct check one gets
\[
T_0^* x = i \frac{dx}{dt}, \quad \text{Dom}(T) = \left\{ x(t) \mid x(t) - \text{abs. cont.}, x'(t) \in L^2_{[0,t]}, x(t) = e^{-\ell} x(0) \right\}.
\]
Once again, we have \( \hat{A}^* \) defined by (82) and \( H_+ = \text{Dom}(\hat{A}^*) = W_2^1 \) is a space with scalar product (83). Consider operators
\[
\begin{align*}
\mathcal{A}_0 x &= i \frac{dx}{dt} + i \frac{2}{e^t - 1} (x(t) - e^{\ell} x(0)) [\delta(t - \ell) - \delta(t)], \\
\mathcal{A}_0^* x &= i \frac{dx}{dt} + i \frac{2}{e^t - 1} (x(0) - e^{\ell} x(t)) [\delta(t - \ell) - \delta(t)],
\end{align*}
\]
where \( x(t) \in W_2^1 \). It is easy to see that \( \mathcal{A} \supset T_0 \supset \mathcal{A}, \mathcal{A}^* \supset T_0^* \supset \hat{A} \), and
\[
\text{Re} \mathcal{A}_0 x = i \frac{dx}{dt} - \frac{i}{2} (x(0) + x(t)) [\delta(t - \ell) - \delta(t)].
\]
Thus \( \text{Re} \mathcal{A}_0 \) has its quasi-kernel equal to \( A \) in (74). Similarly,
\[
\text{Im} \mathcal{A}_0 x = \left( \frac{1}{2} \right) e^{\ell} e^{\ell} - e^{\ell} (x(t) - x(0)) [\delta(t - \ell) - \delta(t)].
\]
Therefore,
\[
\text{Im} \mathcal{A}_0 = \left( \frac{1}{2} \right) \frac{e^{\ell} - 1}{2(e^\ell - 1)} [\delta(t - \ell) - \delta(t)]\sqrt{\frac{e^{\ell} + 1}{2(e^\ell - 1)} [\delta(t - \ell) - \delta(t)]}
\]
\[
= \langle \cdot, g_0 \rangle g_0,
\]
where \( g_0 = \sqrt{\frac{e^{\ell} + 1}{2(e^\ell - 1)}} [\delta(t - \ell) - \delta(t)] \). Now we can build
\[
\Theta_0 = \begin{pmatrix}
\mathcal{A}_0 & K_0 & 1 \\
W_2^1 \subset L^2_{[0,t]} & \subset (W_2^1)^* & \mathbb{C}
\end{pmatrix},
\]
that is a minimal L-system with $K_0 c = c \cdot g_0$, $(c \in \mathbb{C})$, $K^n_0 x = (x, g_0)$ and $x(t) \in W^n_2$.

Following Example 1 we derive

$$R_z(T_0) = (T_0 - zI)^{-1} f$$

(94)

$$= -ie^{-itz} \left( \int_0^t f(s)e^{itz} ds + \frac{e^{-itz}}{e^t - e^{-it}} \int_0^t f(s)e^{itz} ds \right),$$

and

$$R_z(T^n_0) = (T^n_0 - zI)^{-1} f$$

(95)

$$= -ie^{-itz} \left( \int_0^t f(s)e^{itz} ds + \frac{e^{-itz}}{e^t - e^{-it}} \int_0^t f(s)e^{itz} ds \right),$$

for $f \in L^2_{[0, t]}$. Then again

$$(\hat{R}_z(T_0)\delta(t), f) = (\delta(t), R_z(T_0)f) = \frac{i e^{itz}}{e^t - e^{-it}} \int_0^t e^{-izs} f(s) ds$$

$$= \frac{i e^{itz}}{e^{-itz} - e^t} (e^{-itz}, f), \quad f \in L^2_{[0, t]}.$$

Similarly,

$$(\hat{R}_z(T_0)\delta(t - \ell), f) = (\delta(t - \ell), R_z(T_0)f) = \frac{i e^{itz}e^{-\ell}}{e^{-itz} - e^t} \int_0^t e^{-izs} f(s) ds = \frac{i}{e^{-itz} - e^t} (e^{-itz}, f), \quad f \in L^2_{[0, t]},$$

Hence,

$$\hat{R}_z(T_0)\delta(t) = \frac{ie^{itz}}{e^{-itz} - e^t} e^{-itz}, \quad \hat{R}_z(T_0)\delta(t - \ell) = \frac{i}{e^{-itz} - e^t} e^{-itz},$$

and

$$\hat{R}_z(T_0)g_0 = \hat{R}_z(T_0) \sqrt{\frac{e^\ell + 1}{2(e^\ell - 1)}} [\delta(t - \ell) - \delta(t)] = \sqrt{\frac{e^\ell + 1}{2(e^\ell - 1)}} i - ie^{itz}. $$

Using techniques of Example 1 one finds the transfer function of $\Theta_0$ to be

$$W_{\Theta_0}(z) = 1 - 2i(\hat{R}_z(T_0)g_0, g_0)$$

$$= 1 - 2i \left( \sqrt{\frac{e^\ell + 1}{2(e^\ell - 1)}} i - ie^{itz} \sqrt{\frac{e^\ell + 1}{2(e^\ell - 1)}} [\delta(t - \ell) - \delta(t)] \right)$$

$$= 1 + \frac{e^\ell + 1}{e^\ell - 1} \left( \frac{e^\ell - 1}{e^{-itz} - e^t} e^{-itz}, \delta(t - \ell) - \delta(t) \right)$$

$$= 1 - \frac{e^\ell + 1}{e^\ell - 1} \left( \frac{e^\ell - 1}{e^{-itz} - e^t} - \frac{(e^\ell - 1)e^{-itz}}{e^{-itz} - e^t} \right)$$

$$= 1 + (e^\ell + 1) \left( \frac{1 - e^{-itz}}{e^{-itz} - e^t} \right)$$

$$= \frac{e^\ell e^{-itz} - 1}{e^t - e^{-itz}}.$$
This confirms the result of Corollary 9 and formula (53) by showing that $W_{\Theta_0}(z) = -1/s(z)$. The corresponding impedance function is

$$V_{\Theta_0}(z) = i \left( \frac{e^\ell + 1}{e^\ell - 1} \right) \frac{e^{-i\ell z} - 1}{e^{-i\ell z} + 1}.$$  

A quick inspection confirms that $V_{\Theta_0}(i) = i$ and hence $V_{\Theta_0}(z) \in \mathbb{R}$.

**Remark.** We can use Examples 1 and 2 to illustrate Lemma 11 and Theorem 13. As one can easily tell that the impedance function $V_{\Theta_0}(z)$ from Example 2 above and the impedance function $V_{\Theta}(z)$ from Example 1 are related via (61) with $\kappa = e^{-\ell}$, that is $V_{\Theta}(z) = \frac{1 - e^{-\ell}}{1 + e^{-\ell}} V_{\Theta_0}(z)$.

Let $\Theta$ be the L-system of the form (85) described in Example 1 with the transfer function $W_{\Theta}(z)$ given by (89). It was shown in [3, Theorem 8.3.1] that if one takes a function $W(z) = -W_{\Theta}(z)$, then $W(z)$ can be realized as a transfer function of another L-system $\Theta_1$ that shares the same main operator $T$ with $\Theta$ and in this case

$$V_{\Theta_1}(z) = -1/V_{\Theta}(z) = i \frac{e^{-i\ell z} + 1}{e^{-i\ell z} - 1}.$$  

Clearly, $V_{\Theta_1}(z)$ and $V_{\Theta_0}(z)$ are not related via (61) even though $\Theta_1$ has the same operator $T$ with the same parameter $\kappa = e^{-\ell}$ as in $\Theta$. The reason for that is the fact that the quasi-kernel of the real part of $A_1$ of the L-system $\Theta_1$ does not satisfy the conditions of Hypothesis 5 as indicated by Theorem 13.

**Example 3.** In this Example we are going to extend the construction of Example 2 to obtain a family of L-systems $\Theta_0(\beta)$ described in (68). Let $\dot{A}$ be defined by formula (72) but the operator $A$ be an arbitrary self-adjoint extension of $\dot{A}$. It is known then [1] that all such operators $A$ are described with the help of a unimodular parameter $\mu$ as follows

$$Ax = \frac{dx}{dt},$$  

$$(97) \quad \text{Dom}(A) = \left\{ x(t) \mid x(t) \in \text{Dom}(\dot{A}^*), \mu x(\ell) + x(0) = 0, |\mu| = 1 \right\}.$$

In order to establish the connection between the boundary value $\mu$ in (21) and the von Neumann parameter $U$ in [1] we follow the steps similar to Example 1 to guarantee that $g_+ + U g_- \in \text{Dom}(\dot{A})$, where $g_\pm$ are given by (74). Quick set of calculations yields

$$(98) \quad U = \frac{1 + \mu e^\ell}{\mu + e^\ell}.$$  

For this value of $U$ we set the value of $\beta$ so that $U = e^{2i\beta}$, where $\beta \in [0, \pi)$ and thus establish the link between the parameters $\mu$ and $\beta$ that will be used to construct the family $\Theta_0(\beta)$. In particular, we note that $\beta = 0$ if and only if $\mu = -1$. 

Once again, having $\hat{A}^*$ defined by (82) and $\mathcal{H}_+ = \text{Dom}(\hat{A}^*) = W^1_2$ a space with scalar product (83), consider the following operators

\begin{align}
&\mathcal{A}_0(\beta)x = \frac{dx}{dt} + i\frac{\mu}{\mu + e^{-\ell}}(x(0) - e^{-\ell}x(\ell)) [\mu\delta(t - \ell) + \delta(t)], \\
&\mathcal{A}_0^\ast(\beta)x = i\frac{dx}{dt} + i\frac{1}{\mu + e^{-\ell}}(e^{-\ell}x(0) - x(\ell)) [\mu\delta(t - \ell) + \delta(t)],
\end{align}

where $x(t) \in W^1_2$. It is immediate that $\hat{A} \supset T_0 \supset \hat{A}$, $\hat{A}^* \supset T_0^* \supset \hat{A}$, where $T_0$ and $T_0^*$ are given by (80) and (82). Also, as one can easily see, when $\beta = 0$ and consequently $\mu = -1$, the operators $\mathcal{A}_0(0)$ and $\mathcal{A}_0^\ast(0)$ in (82) match the corresponding pair $\mathcal{A}_0$ and $\mathcal{A}_0^\ast$ in (93). By performing direct calculations we obtain

\[
\text{Re } \mathcal{A}_0(\beta)x = i\frac{dx}{dt} + \frac{i}{2}(\nu x(\ell) + x(0)) [\mu\delta(t - \ell) + \delta(t)],
\]

where

\[
\nu = \frac{2\mu e^{-\ell} + e^{-2\ell} + 1}{\mu + 2e^{-\ell} + \mu e^{-2\ell}},
\]

and $|\nu| = 1$. Consequently, $\text{Re } \mathcal{A}_0$ has its quasi-kernel

\[
\mathcal{A}_0(\beta) = i\frac{dx}{dt}, \quad \text{Dom}(A) = \left\{ x(t) \mid x(t) \in \text{Dom}(\hat{A}^*), \nu x(\ell) + x(0) = 0 \right\}.
\]

Moreover,

\[
\text{Im } \mathcal{A}_0(\beta)x = \left( \frac{1}{2} \right) \left( \frac{1 - e^{-2\ell}}{\mu + e^{-2\ell}} \right) (\bar{\mu}x(\ell) + x(0)) [\mu\delta(t - \ell) + \delta(t)].
\]

Therefore,

\[
\text{Im } \mathcal{A}_0(\beta) = \left( \frac{1}{2} \right) \left( \frac{\sqrt{1 - e^{-2\ell}}}{\sqrt{2|\mu + e^{-2\ell}|}} \right) [\mu\delta(t - \ell) + \delta(t)] = (\cdot, g_0(\beta))g_0(\beta),
\]

where $g_0(\beta) = \sqrt{\frac{e^{\ell} + 1}{2(e^{\ell} - 1)}} [\delta(t - \ell) - \delta(t)]$. Now we can compose our one-parametric L-system family

\[
\Theta_0(\beta) = \begin{pmatrix}
\mathcal{A}_0(\beta) & K_0(\beta) & 1 \\
W^1_2 \subset L^2_{[0, \ell]} \subset (W^1_2)^* & \mathbb{C}
\end{pmatrix},
\]

where $K_0(\beta)c = c \cdot g_0(\beta)$, $(c \in \mathbb{C})$, $K^*_0(\beta)x = (x, g_0(\beta))$ and $x(t) \in W^1_2$. Using techniques of Example 2 one finds the transfer function of $\Theta_0(\beta)$ to be

\[
W_{\Theta_0(\beta)}(z) = 1 - 2i(\hat{R}(\hat{T}_0)g_0(\beta), g_0(\beta)) = \left( \frac{e^{\ell} + \mu}{\mu e^{\ell} + 1} \right) \frac{e^{\ell}(e^{-it\beta} - 1)}{e^{\ell} - e^{-it\beta}}.
\]

The corresponding impedance function is again found via (8)

\[
V_{\Theta_0(\beta)}(z) = i\left( \frac{\bar{\mu}e^{-it\beta} - 1}{\mu e^{-it\beta} + 1} \right) \frac{e^{\ell}(e^{-it\beta} - 1) + 2e^{\ell}e^{-it\beta} - 2\bar{\mu}e^{\ell}}{(\mu e^{-it\beta} + 1)(e^{2\ell} - 1)}.
\]

A quick inspection confirms that $V_{\Theta_0(\beta)}(t) = i$ and hence $V_{\Theta_0(\beta)}(z)$ belongs to the Donoghue class $\mathfrak{M}$ for all $\beta \in [0, \pi]$ (equivalently $|\mu| = 1$). Also, one can see that if $\beta = 0$ and consequently $\mu = -1$ the conditions of Hypothesis 3 are satisfied and the
L-system $\Theta_0(0)$ coincides with the L-system $\Theta_0$ of Example 2 and so do its transfer and impedance function.

**Example 4.** In this Example we will generalize the results obtained in Examples 1 and 2. Once again, let $\hat{A}$ and $A$ be defined by formulas (72) and (74), respectively and let $s(z)$ be the Livšic characteristic function $s(z)$ for the pair $(\hat{A}, A)$ given by (75). We introduce a one-parametric family of operators

$$\tag{102} T_\rho x = \frac{dx}{dt}, \quad \text{Dom}(T_\rho) = \left\{ x(t) \mid x(t) - \text{abs. cont.}, x'(t) \in L^2_{[0, \ell]}, x(0) = \rho x(0) \right\}. $$

We are going to select the values of boundary parameter $\rho$ in a way that will make $T_\rho$ compliant with Hypothesis 1. By performing the direct check we conclude that $\text{Im}(T_\rho f, f) \geq 0$ for $f \in \text{Dom}(T_\rho)$ if $|\rho| > 1$. This will guarantee that $T_\rho$ is a dissipative extension of $\hat{A}$ parameterized by a von Neumann parameter $\kappa$. For further convenience we assume that $\rho \in \mathbb{R}$. To find the connection between $\kappa$ and $\rho$ we use (73) again to obtain

$$\tag{103} x(t) = Ce^t - \kappa Ce^\ell e^{-t} \in \text{Dom}(T), \quad x(\ell) = \rho x(0). $$

Solving (103) in two ways yields

$$\tag{104} \kappa = \frac{\rho - e^\ell}{\rho e^\ell - 1} \quad \text{and} \quad \rho = \frac{\kappa - e^\ell}{\kappa e^\ell - 1}. $$

Using the first of relations (104) to find which values of $\rho$ provide us with $0 \leq \kappa < 1$ we obtain

$$\tag{105} \rho \in (-\infty, -1) \cup [e^\ell, +\infty). $$

Now assuming (103) we can acknowledge that the triplet of operators $(\hat{A}, T_\rho, A)$ satisfy the conditions of Hypothesis 1. Following Examples 1 and 2, we are going to use the triplet $(\hat{A}, T_\rho, A)$ in the construction of an L-system $\Theta_\rho$. By the direct check we have

$$\tag{106} T_\rho^* x = \frac{dx}{dt}, \quad \text{Dom}(T_\rho) = \left\{ x(t) \mid x(t) - \text{abs. cont.}, x'(t) \in L^2_{[0, \ell]}, \rho x(\ell) = x(0) \right\}. $$

Once again, we have $\hat{A}^*$ defined by (32) and $\mathcal{H}_+ = \text{Dom}(\hat{A}^*) = W^1_2$ is a space with scalar product (38). Consider operators

$$\tag{107} \mathcal{A}_\rho x = \frac{dx}{dt} + i \frac{x(t) - \rho x(0)}{\rho - 1} \left[ \delta(t - \ell) - \delta(t) \right], $$

$$\mathcal{A}_\rho^* x = \frac{dx}{dt} + i \frac{x(0) - \rho x(\ell)}{\rho - 1} \left[ \delta(t - \ell) - \delta(t) \right], $$

where $x(t) \in W^1_2$. One easily checks that since $\text{Im} \rho = 0$, then $\mathcal{A}_\rho^*$ is the adjoint to $\mathcal{A}_\rho$ operator. Evidently, that $\mathcal{A} \supset T_\rho \supset \hat{A}$, $\mathcal{A}^* \supset T^*_{\rho} \supset \hat{A}$, and

$$\Re \mathcal{A}_\rho x = \frac{1}{2} \frac{dx}{dt} - i \frac{1}{2} (x(0) + x(\ell)) \left[ \delta(t - \ell) - \delta(t) \right]. $$

Thus $\Re \mathcal{A}_\rho$ has its quasi-kernel equal to $A$ defined in (34). Similarly,

$$\Im \mathcal{A}_\rho x = \left( \frac{1}{2} \right) \frac{\rho + 1}{\rho - 1} (x(\ell) - x(0)) \left[ \delta(t - \ell) - \delta(t) \right]. $$
Therefore,
\[
\text{Im} \, \hat{A}_\rho = \left( \sqrt{\frac{\rho + 1}{2(\rho - 1)}} [\delta(t - \ell) - \delta(t)] \right) \sqrt{\frac{\rho + 1}{2(\rho - 1)}} [\delta(t - \ell) - \delta(t)]
\]
\[
\hat{g}_\rho = \sqrt{\frac{\rho + 1}{2(\rho - 1)}} [\delta(t) - \delta(t)].
\]
Now we can build
\[
\Theta_\rho = \begin{pmatrix}
K_\rho & K_\rho \\
W^1_2 \subset L^2_{[0,l]} \subset (W^2_1)^- & \mathbb{C}
\end{pmatrix},
\]
which is a minimal L-system with
\[
K_\rho c = c \cdot g_\rho, \quad (c \in \mathbb{C}),
\]
\[
x(t) \in W^1_2.
\]
Evaluating the transfer function \(W\Theta_\rho(z)\) resembles the steps performed in Example 2. We have
\[
R_z(T_\rho) = (T_\rho - zI)^{-1}f
\]
\[
(108)
\]
This leads to
\[
\hat{R}_z(T_\rho)g_\rho = \hat{R}_z(T_\rho) \sqrt{\frac{\rho + 1}{2(\rho - 1)}} [\delta(t) - \delta(t)] = i \sqrt{\frac{\rho + 1}{2(\rho - 1)}} \left( \frac{1 - \rho}{e^{-i\ell z} - \rho} \right) e^{-i\ell z},
\]
and eventually to
\[
W\Theta_\rho(z) = 1 - 2i(\hat{R}_z(T_\rho)g_\rho, g_\rho) = \frac{\rho e^{-i\ell z} - 1}{\rho - e^{-i\ell z}}.
\]
Evaluating the impedance function \(V\Theta_\rho(z)\) results in
\[
V\Theta_\rho(z) = i \left( \frac{\rho + 1}{\rho - 1} \right) \frac{1 - e^{-i\ell z}}{1 + e^{-i\ell z}}.
\]
Using direct calculations and (104) gives us
\[
\frac{\rho + 1}{\rho - 1} = \frac{1 - \kappa}{1 + \kappa} \frac{e^{\ell} + 1}{e^{\ell} - 1},
\]
and thus
\[
V\Theta_\rho(z) = \left( \frac{1 - \kappa}{1 + \kappa} \right) V\Theta_0(z),
\]
which confirms the result of Lemma 1.

**APPENDIX A. RIGGED HILBERT SPACES**

In this Appendix we are going to explain the construction and basic geometry of rigged Hilbert spaces.

We start with a Hilbert space \( \mathcal{H} \) with inner product \( (x, y) \) and norm \( \| \cdot \| \). Let \( \mathcal{H}_+ \) be a dense in \( \mathcal{H} \) linear set that is a Hilbert space itself with respect to another inner product \( (x, y)_+ \), generating the norm \( \| \cdot \|_+ \). We assume that \( \| x \| \leq \| x \|_+ \), \( (x \in \mathcal{H}_+) \), i.e., the norm \( \| \cdot \|_+ \) generates a stronger than \( \| \cdot \| \) topology in \( \mathcal{H}_+ \). The space \( \mathcal{H}_+ \) is called the space with the positive norm.
Now let $\mathcal{H}_-$ be a space dual to $\mathcal{H}_+$. It means that $\mathcal{H}_-$ is a space of linear functionals defined on $\mathcal{H}_+$ and continuous with respect to $\| \cdot \|_+$. By the $\| \cdot \|_-$ we denote the norm in $\mathcal{H}_-$ that has a form

$$\|h\|_- = \sup_{u \in \mathcal{H}_+} \frac{|\langle h, u \rangle|}{\|u\|_+}, \quad h \in \mathcal{H}.$$  

The value of a functional $f \in \mathcal{H}_-$ on a vector $u \in \mathcal{H}_+$ is denoted by $(u, f)$. The space $\mathcal{H}_-$ is called the space with the negative norm.

Consider an embedding operator $\sigma : \mathcal{H}_+ \to \mathcal{H}$ that embeds $\mathcal{H}_+$ into $\mathcal{H}$. Since $\|\sigma f\| \leq \|f\|_+$ for all $f \in \mathcal{H}_+$, then $\sigma \in [\mathcal{H}_+, \mathcal{H}]$. The adjoint operator $\sigma^*$ maps $\mathcal{H}$ into $\mathcal{H}_-$ and satisfies the condition $\|\sigma^* f\|_- \leq \|f\|$ for all $f \in \mathcal{H}$. Since $\sigma$ is a monomorphism with a $(-)$-dense range, then $\sigma^*$ is a monomorphism with $(-)$-dense range. By identifying $\sigma^* f$ with $f (f \in \mathcal{H})$ we can consider $\mathcal{H}$ embedded in $\mathcal{H}_-$ as a $(-)$-dense set and $\|f\|_- \leq \|f\|$. Also, the relation

$$(\sigma f, h) = (f, \sigma^* h), \quad f \in \mathcal{H}_+, \; h \in \mathcal{H},$$

implies that the value of the functional $\sigma^* h \in \mathcal{H}$ calculated at a vector $f \in \mathcal{H}_+$ as $(f, \sigma^* h)$ corresponds to the value $(f, h)$ in the space $\mathcal{H}$.

It follows from the Riesz representation theorem that there exists an isometric operator $\mathcal{R}$ which maps $\mathcal{H}_-$ onto $\mathcal{H}_+$ such that $(f, g) = (\mathcal{R} f, g)$ for all $f \in \mathcal{H}_+$, $g \in \mathcal{H}_-$ and $\|\mathcal{R} g\|_+ = \|g\|_-$. Now we can turn $\mathcal{H}_-$ into a Hilbert space by introducing $(f, g)_- = (\mathcal{R} f, \mathcal{R} g)_+$. Thus,

$$(f, g)_- = (f, \mathcal{R} g) = (\mathcal{R} f, g) = (\mathcal{R} f, \mathcal{R} g)_+ + (f, g \in \mathcal{H}_-),$$

and

$$(u, v)_+ = (u, \mathcal{R}^{-1} v) = (\mathcal{R}^{-1} u, v) = (\mathcal{R}^{-1} u, \mathcal{R}^{-1} v)_-, \quad (u, v \in \mathcal{H}_+).$$

The operator $\mathcal{R}$ (or $\mathcal{R}^{-1}$) will be called the Riesz-Berezansky operator. Applying the above reasoning, we define a triplet $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$ to be called the rigged Hilbert space.

Now we explain how to construct a rigged Hilbert space using a symmetric operator. Let $A$ be a closed symmetric operator whose domain $\text{Dom}(A)$ is not assumed to be dense in $\mathcal{H}$. Setting $\text{Dom}(\bar{A}) = \mathcal{H}_0$, we can consider $\bar{A}$ as a densely defined operator from $\mathcal{H}_0$ into $\mathcal{H}$. Clearly, $\text{Dom}(\bar{A}^*)$ is dense in $\mathcal{H}$ and $\text{Ran}(\bar{A}^*) \subset \mathcal{H}_0$. We introduce a new Hilbert space $\mathcal{H}_+ = \text{Dom}(\bar{A}^*)$ with inner product

$$(f, g)_+ = (f, g) + (\bar{A}^* f, \bar{A}^* g), \quad (f, g \in \mathcal{H}_+),$$

and then construct the operator generated rigged Hilbert space $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$.  

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