Abstract
We study exact recovery conditions for the linear programming relaxation of the $k$-median problem in the stochastic ball model (SBM). In Awasthi et al. (Relax, no need to round: integrality of clustering formulations. arXiv:1408.4045, 2015; in: Proceedings of the 2015 conference on innovations in theoretical computer science, pp 191–200, 2015), the authors give a tight result for the $k$-median LP in the SBM, saying that exact recovery can be achieved as long as the balls are pairwise disjoint. We give a counterexample to their result, thereby showing that the $k$-median LP is not tight in low dimension. Instead, we give a near optimal result showing that the $k$-median LP in the SBM is tight in high dimension. We also show that, if the probability measure satisfies some concentration assumptions, then the $k$-median LP in the SBM is tight in every dimension. Furthermore, we propose a new model of data called extended stochastic ball model (ESBM), which significantly generalizes the well-known SBM. We then show that exact recovery can still be achieved in the ESBM.

Keywords $k$-median · Stochastic ball model · Linear programming relaxation · Recovery guarantee

Mathematics Subject Classification 90C05 · 90C10 · 68Q87
1 Introduction

Clustering problems form a fundamental class of problems in data science with a wide range of applications in computational biology, social science, and engineering. Although clustering problems are often NP-hard in general, recent results in the literature show that we may be able to solve these problems efficiently if the data exhibits a good structure. More specifically, we may be able to solve these problems in polynomial time if the problem data is generated according to some reasonable model of data. These models of data are defined in such a way that there is a ground-truth that reveals which cluster a data point comes from. In this way, for each instance of the clustering problem generated according to such model of data, it is clear which optimal solution our algorithm should return. If the algorithm returns the correct solution, we say that the algorithm “achieves exact recovery”. Examples of models of data include the stochastic block model and the stochastic ball model.

One of the most successful types of algorithms to achieve exact recovery in polynomial time are convex relaxation techniques, including linear programming (LP) relaxations and semidefinite programming (SDP) relaxations. When these algorithms achieve exact recovery, the optimal solution to the convex relaxation is an integer vector which is also the optimal solution to the underlying integer programming problem which models the clustering problem. Recently, much work has been done to understand the phenomenon of exact recovery for convex relaxation methods, with diverse clustering problems and models. Recent LP relaxations that achieve exact recovery for clustering problems include [9, 16, 17, 31], while some SDP relaxations that achieve exact recovery are [1–6, 9, 14, 15, 19–21, 24, 25, 27, 32].

In this paper we study the $k$-median problem, which is one of the most well-known and studied clustering problems. We are given a set $P$ of $n$ different points in a metric space $(X, d)$ and a positive integer $k \leq n$, and our goal is to partition these $n$ points into $k$ different sets $A_1, A_2, \ldots, A_k$, also known as clusters. Each cluster $A_i$ has a center $a_i \in P$, which satisfies $\sum_{p \in A_i} d(a_i, p) = \min \{ \sum_{p \in A_i} d(q, p) \mid q \in A_i \}$, and each point in $P$ is assigned to the cluster with the closest center. Formally, the $k$-median problem is defined as the following optimization problem:

$$\min \sum_{p \in P} \min_{i \in [k]} d(p, a_i)$$

s.t. $a_1, \ldots, a_k \in P$.

The $k$-median problem is NP-hard even in some very restrictive settings, like the Euclidean $k$-median problem on the plane [28], and only few very special cases of the $k$-median problem are known to be solvable in polynomial time, like the $k$-median problem on trees [22, 33]. Several papers study approximation algorithms for the $k$-median problem, including [7, 8, 12, 13, 23, 26].

The model of data that we consider in this paper, and that is arguably the one used the most in the study of the $k$-median problem, is the stochastic ball model (SBM), formally introduced in Definition 2. In the SBM, we consider $k$ probability measures, each one supported on a unit ball in $\mathbb{R}^m$, and $n$ data points are sampled...
Table 1  Exact recovery results for clustering problems in the SBM

| Problem          | Method | Sufficient condition | References |
|------------------|--------|----------------------|------------|
| $k$-means/$k$-median | Thresholding | $\Delta > 4$ | Simple algorithm |
| $k$-means      | SDP    | $\Delta > 2\sqrt{2/(1+1/\sqrt{m})}$ | Theorem 3 in [9] |
|                 | SDP    | $\Delta > 2 + k^2/m$ | Theorem 9 in [21] |
|                 | SDP    | $\Delta > 2 + O(\sqrt{k/m})$ | Corollary 2 in [25] |
|                 | SDP    | $\Delta > O(\sqrt{\log n/m})$ | Corollary of Theorem 3 in [19] |
| $k$-means      | LP     | $\Delta > 4$ | Theorem 9 in [9] |
|                 | LP     | $\Delta > 1 + \sqrt{3}$ | Theorem 4 in [16] |
| $k$-median     | LP     | $\Delta > 3.75$ | Theorem 6 in [31] |
|                 | LP     | $\Delta > 2$ | Theorem 7 in [9] |
|                 | LP     | $\Delta > 3.29$ | Theorem 6 |
|                 | LP     | $\Delta > 2 + O(\sqrt{k \log m/m})$ | Theorem 7 |

from each of them. In this paper we study the effectiveness of the LP relaxation to achieve exact recovery. The main goal of this paper is then to seek for the minimum pairwise distance $\Delta$ between the ball centers which is needed for the LP relaxation to achieve exact recovery with high probability when the number of the input data points $n$ is large enough. To the best of our knowledge, the only known result in this direction is Theorem 7 in [9] (or Theorem 6 in the conference version of the paper [10]). Unfortunately, as will be discussed later, this result is false.

The SBM has also been used as a model of data for other closely related clustering problems, including $k$-means and $k$-medoids clustering. In Table 1 we summarize the known exact recovery results for clustering problems in the SBM, including some of our results that will discuss later. For more details about the results in the table, including the additional assumptions required, we refer the reader to the corresponding cited paper. We remark that the problem considered in [31] differs from the $k$-median defined in this paper because in the objective function the sum of the squared distances is considered.

1.1 Our contribution

In [9], the authors study the $k$-median problem in the SBM. In the model of data considered in the paper, there are $k$ unit balls and $n$ points are sampled from each ball. The probability measures, supported on each ball, are translations of each other. Moreover, each probability measure is invariant under rotations centered in the ball center and every neighborhood of each ball center has positive probability measure. In Theorem 7 in [9], the authors claim that, if the unit balls are pairwise disjoint, then the LP-relaxation of the $k$-median problem achieves exact recovery with high probability. Unfortunately this result is false. In Example 2 in “Appendix B”, we present an example in $\mathbb{R}^2$ where the balls are pairwise disjoint and the probability measures satisfy the assumption of Theorem 7 in [9], but when $n$ is large enough,
with high probability the LP relaxation does not achieve exact recovery. Our example implies that to achieve exact recovery, a significant distance between the ball centers is needed. In “Appendix C” we also point out the key problem in the proof of Theorem 7 in [9]. Furthermore, we notice that the techniques used in [9] highly depend on the assumptions that we draw the same number of points from each ball, and that the balls have the same radius and the same probability measure. These observations naturally lead to two questions, which are at the heart of this paper.

**Question 1** What is the minimum pairwise distance $\Delta$ between the ball centers which guarantees that the $k$-median LP relaxation in the SBM achieves exact recovery with high probability?

**Question 2** If we relax some of the assumptions in the model of data, will exact recovery still happen for the $k$-median LP relaxation?

In this paper, we provide the first answers to Question 1 and Question 2. We propose a more general version of the SBM called ESBM, which is a natural model for Question 2 formally defined in Definition 3. In the ESBM, the number of points drawn from each ball can be different, the balls can have different radii and different probability measures. We study exact recovery for the $k$-median problem in the ESBM. Informally, we obtain the following results, where we denote by $c_i$ the center and by $r_i$ the radius of ball $i$.

- **Theorem 5:** In the ESBM, if for every $i \neq j$ we have $d(c_i, c_j) > (1+\beta)R + \max\{r_i, r_j\} + O(\sqrt{k \log m/m})$, then the $k$-median LP achieves exact recovery with high probability. Here, $R := \max_{i \in [k]} r_i$ and $\beta$ is a parameter that measures the difference between the numbers of points sampled from the balls.

- **Theorem 6:** In the SBM, if $\Delta > 3.29$, then the $k$-median LP achieves exact recovery with high probability.

- **Theorem 7:** In the SBM, if $\Delta > 2 + O(\sqrt{k \log m/m})$, then the $k$-median LP achieves exact recovery with high probability.

- **Theorem 8:** In the SBM, if $\Delta > 2$ and the density function decreases as we increase the distance from the center, then the $k$-median LP achieves exact recovery with high probability.

We remark that exact recovery can only be considered when the balls are pairwise disjoint. Moreover, we need to assume that $d(c_i, c_j) > 2 \max\{r_i, r_j\}$ for every $i \neq j$, otherwise the ground-truth solution may not be optimal to the $k$-median problem. In particular, in the SBM we need to have $\Delta > 2$.

For the ESBM, Theorem 5 provides sufficient conditions for exact recovery. For the SBM, Theorems 6 and 7 provide the condition $\Delta > \min\{3.29, 2 + O(\sqrt{k \log m/m})\}$ to guarantee exact recovery. This result implies that the $k$-median LP is tight in high dimension. Furthermore, Theorem 8 implies that, if we add strong assumptions on the probability measures, then $\Delta > 2$ also guarantees exact recovery.

The rest of the paper is organized as follows. In Sect. 2 we introduce the integer programming formulation (IP) of the $k$-median problem and the corresponding linear programming relaxation (LP). We then provide deterministic necessary and sufficient conditions which guarantee that a feasible solution to (IP) is optimal to (LP) (Theorem 1). In Sect. 3 we introduce the definition of SBM, ESBM, and exact recovery.
In Sect. 4, we introduce a very general sufficient condition which ensures that exact recovery happens with high probability (Theorem 2). In Sect. 5, we will present our main theorems for exact recovery (Theorems 3–8). Finally, in Sect. 6, we perform numerical experiments to illustrate the empirical performance of (LP) under the SBM and the ESBM.

We conclude this section with what we believe is an interesting open question. As we already mentioned, in the SBM, exact recovery can only be considered when the balls are pairwise disjoint, otherwise the ground-truth solution may not be optimal to the $k$-median problem. In this case, we can set aside the concept of exact recovery and focus instead simply on seeking an optimal solution to the $k$-median problem. A natural question is whether, in this scenario, we are still able to find an optimal solution to the $k$-median problem by simply solving the LP relaxation. Interesting models of data that can be considered for this question are the SBM and ESBM with intersecting balls, as well as subgaussian mixture models (SGMMs), where data points are drawn from a mixture of $k$ subgaussian distributions and certain overlaps are allowed. In fact, SBM can be viewed as a special case of SGMMs. Previous work such as [19, 30] show that SDP relaxations still have desirable theoretical guarantees for clustering data points under SGMMs. On the contrary, to the best of our knowledge, there is no theoretical understanding of the performance of LP relaxations under these more general models of data, where certain overlaps are allowed.

## 2 The $k$-median problem via linear programming

The $k$-median problem can be formulated as an integer linear program as follows.

$$\min \sum_{p,q \in P} d(p, q)z_{pq}$$

subject to

$$\sum_{p \in P} z_{pq} = 1 \quad \forall q \in P$$

$$z_{pq} \leq y_p \quad \forall p, q \in P$$

$$\sum_{p \in P} y_p = k$$

$$y_p, z_{pq} \in \{0, 1\} \quad \forall p, q \in P.$$ (IP)

Here, $y_p = 1$ if and only if $p$ is a center, and $z_{pq} = 1$ if and only if $p$ is the center of $q$. The first constraint says that each point is assigned to exactly one center. The second constraint says that $z_{pq} = 1$ can happen only if $p$ is a center. The third constraint says that there are exactly $k$ centers. It is simple to check that an optimal solution to (IP) provides an optimal solution to the $k$-median problem.

In this paper we consider the linear programming relaxation of (IP) obtained from (IP) by replacing the constraints $y_p, z_{pq} \in \{0, 1\}$ with $y_p, z_{pq} \geq 0$. Such a linear
program, which is given below, has been used in other works in the literature including [13].

$$\begin{align*}
\min & \sum_{p,q \in P} d(p,q)z_{pq} \\
\text{s.t.} & \sum_{p \in P} z_{pq} = 1 \quad \forall q \in P \\
& z_{pq} \leq y_p \quad \forall p, q \in P \\
& \sum_{p \in P} y_p = k \\
& y_p, z_{pq} \geq 0 \quad \forall p, q \in P.
\end{align*}$$

(LP)

The linear program (LP) is called a linear programming relaxation of (IP) because each feasible solution to (IP) is also feasible to (LP). The main advantage of (LP) over (IP) is that the first can be solved in polynomial time, while the second is NP-hard.

### 2.1 Conditions for the integrality of (LP)

Let $\bar{y}, \bar{z}$ be a feasible solution to (IP). The main goals of this section are twofold. First, we provide necessary and sufficient conditions for $(\bar{y}, \bar{z})$ to be an optimal solution to (LP). Second, we give sufficient conditions for $(\bar{y}, \bar{z})$ to be the unique optimal solution to (LP). In particular, under these sufficient conditions the $k$-median problem is polynomially solvable.

We start by writing down the the dual linear program of (LP). To do so, we associate the dual variables $\alpha_q \forall q \in P$, to the first block of constraints, the dual variables $\beta_{pq} \forall p, q \in P$, to the second block of constraints, and the dual variable $\omega$ to the single constraint $\sum_{p \in P} y_p = k$. We obtain the dual linear program

$$\begin{align*}
\max & \sum_{q \in P} \alpha_q - k\omega \\
\text{s.t.} & \alpha_q \leq \beta_{pq} + d(p,q) \quad \forall p, q \in P \\
& \sum_{q \in P} \beta_{pq} \leq \omega \quad \forall p \in P \\
& \beta_{pq} \geq 0 \quad \forall p, q \in P.
\end{align*}$$

(DLP)

It is simple to see that (LP) always has a finite optimum, thus by the Strong Duality Theorem, so does (DLP). In particular, (DLP) is always feasible.

Let $(y, z)$ be a feasible solution to (LP), and let $(\alpha, \beta, \omega)$ be a feasible solution to (DLP). The Complementary Slackness Theorem (see, e.g., Theorem 4.5 in [11]), says that the vector $(y, z)$ is optimal to (LP) and $(\alpha, \beta, \omega)$ is optimal to (DLP) if and only if

$$\beta_{pq} (z_{pq} - y_p) = 0 \quad \forall p, q \in P$$

(1)
\[ z_{pq} (\alpha_q - \beta_{pq} - d(p, q)) = 0 \quad \forall p, q \in P \quad (2) \]

\[ y_p \left( \sum_{q \in P} \beta_{pq} - \omega \right) = 0 \quad \forall p \in P. \quad (3) \]

Now let \((\tilde{y}, \tilde{z})\) be a feasible solution to (IP). Clearly, the vector \((\tilde{y}, \tilde{z})\) is feasible to (LP). Furthermore, let \((\alpha, \beta, \omega)\) be a feasible solution to (DLP). From complementary slackness, the vector \((\tilde{y}, \tilde{z})\) is optimal to (LP) and \((\alpha, \beta, \omega)\) is optimal to (DLP) if and only if

\[ \beta_{pq} = 0 \quad \forall p, q \in P \text{ such that } \tilde{y}_p = 1, \tilde{z}_{pq} = 0 \quad (4) \]

\[ \beta_{pq} = \alpha_q - d(p, q) \quad \forall p, q \in P \text{ such that } \tilde{z}_{pq} = 1 \quad (5) \]

\[ \sum_{q \in P} \beta_{pq} = \omega \quad \forall p \in P \text{ such that } \tilde{y}_p = 1. \quad (6) \]

Next, we provide an interpretation of the dual variables. We can interpret \(\alpha_q\) as the maximum distance a point \(q\) can “see”. We can then interpret \(\beta_{pq}\) as the “contribution” from \(q\) to \(p\). The above conditions (4)–(6), together with (DLP) feasibility, can then be interpreted as follows. When \(q\) is not assigned to a center \(p\), condition (4) says that \(q\) does not contribute to \(p\), and the first constraint in (DLP) implies that \(q\) cannot see \(p\). Vice versa, when \(q\) is assigned to a center \(p\), condition (5) and the third constraint in (DLP), imply that \(q\) can see \(p\), and that \(q\) contributes to \(p\). Hence, a center \(p\) is seen exactly by the points in its cluster, which are also the points that contribute to \(p\). Finally, condition (6) says that the centers of the clusters all get the same contribution \(\omega\).

In the remainder of the paper, we denote by \(t_+\) the positive part of a number \(t\), i.e., \(t_+ := \max\{t, 0\}\). We obtain the following observation regarding (DLP).

**Observation 1** Suppose \((\alpha, \beta, \omega)\) is a feasible solution to (DLP). For each \(p, q \in P\), let \(\beta'_{pq} := (\alpha_q - d(p, q))_+\). Then \((\alpha, \beta', \omega)\) is a feasible solution to (DLP) with the same objective value.

In particular, Observation 1 implies that there is always an optimal solution to (DLP) where \(\beta_{pq} = (\alpha_q - d(p, q))_+\). Next, we define the contribution function.

**Definition 1** (Contribution function) Given \(\alpha \in \mathbb{R}^P\), the contribution function \(C^\alpha(z) : \mathbb{R}^m \rightarrow \mathbb{R}\) is defined by

\[ C^\alpha(z) := \sum_{q \in P} (\alpha_q - d(z, q))_+. \]

According to Observation 1, the contribution function can be seen as the contribution that a point \(p \in P\) gets from all points in \(P\). We are now ready to present our main deterministic result.
Theorem 1 Let \((\bar{y}, \bar{z})\) be a feasible solution to \((\text{IP})\). Let \(a_i, i \in [k]\), be the \(k\) points in \(P\) such that \(\bar{y}_{a_i} = 1\). For every \(i \in [k]\), let \(A_i := \{q \in P\mid \bar{z}_{a_iq} = 1\}\). Then \((\bar{y}, \bar{z})\) is optimal to \((\text{LP})\) if and only if there exists \(\alpha \in \mathbb{R}^P\) such that

\[
C^\alpha(a_1) = \cdots = C^\alpha(a_k)
\]

(7)

\[
C^\alpha(q) \leq C^\alpha(a_1) \quad \forall q \in P \setminus \{a_i\}_{i \in [k]}
\]

(8)

\[
\alpha_q \geq d(a_i, q) \quad \forall i \in [k], \forall q \in A_i
\]

(9)

\[
\alpha_q \leq d(a_i, q) \quad \forall i \in [k], \forall q \in P \setminus A_i.
\]

(10)

Furthermore, if there exists \(\alpha \in \mathbb{R}^P\) such that (7), (9) hold, and (8), (10) are satisfied strictly, then \((\bar{y}, \bar{z})\) is the unique optimal solution to \((\text{LP})\).

Proof In the first part of the proof we show the ‘if and only if’ in the statement. After that, we will show the ‘uniqueness’.

First, we show the implication from left to right. Assume that \((\bar{y}, \bar{z})\) is an optimal solution to \((\text{LP})\). Then by Strong Duality (DLP) also has an optimal solution, which we denote by \((\alpha, \beta, \omega)\). For each \(p, q \in P\), let \(\beta_{pq} := (\alpha_q - d(p, q))_+\). According to Observation 1, \((\alpha, \beta, \omega)\) is also optimal to (DLP). Complementary slackness implies that \((\bar{x}, \bar{y})\) and \((\alpha, \beta', \omega)\) satisfy the complementary slackness conditions (4)–(6). Note that for every \(p \in P\), we have \(\sum_{q \in P} \beta_{pq} = \sum_{q \in P} (\alpha_q - d(p, q))_+ = C^\alpha(p)\). Constraints (7) are then implied by (6), since \(C^\alpha(a_i) = \omega\) for every \(i \in [k]\). Constraints (8) are implied by (6) and the second constraint in (DLP). Constraints (9) are implied by (5) and the third constraint in (DLP). Finally, constraints (10) are implied by (4) and the first constraint in (DLP).

Next, we show the implication from right to left. Let \(\alpha \in \mathbb{R}^P\) such that (7)–(10) are satisfied. For every \(p, q \in P\), we define \(\beta_{pq} := (\alpha_q - d(p, q))_+\) and we let \(\omega := C^\alpha(a_1)\). From (8), we know that \((\alpha, \beta, \omega)\) is feasible to (DLP). We can then check that \((\bar{y}, \bar{z})\) and \((\alpha, \beta, \omega)\) satisfy the complementary slackness conditions (4), (5), and (6) due to (10), (9), and (7), respectively. We conclude that \((\bar{x}, \bar{y})\) is optimal to \((\text{LP})\).

To show the ‘uniqueness’ part of the statement, we continue the previous proof (of the implication from right to left) with the additional assumption that (8), (10) are satisfied strictly.

From complementary slackness we also obtain that \((\alpha, \beta, \omega)\) is an optimal solution to (DLP). Let \((y', z')\) be a feasible solution to \((\text{LP})\). Applying complementary slackness to \((y', z')\) and \((\alpha, \beta, \omega)\), we obtain that \((y', z')\) is an optimal solution to \((\text{LP})\) if and only if these two vectors satisfy conditions (1)–(3). Thus, to prove that \((\bar{y}, \bar{z})\) is the unique optimal solution to \((\text{LP})\), we only need to show that if \((y', z')\) and \((\alpha, \beta, \omega)\) satisfy (1)–(3), then \((y', z') = (\bar{y}, \bar{z})\).

Since for every \(p \in P \setminus \{a_i\}_{i \in [k]}\), we have \(C^\alpha(p) = \sum_{q \in P} \beta_{pq} < \omega\), (3) implies that \(y'_p = 0\) for every \(p \in P \setminus \{a_i\}_{i \in [k]}\). From the primal constraints \(z'_{pq} \leq y'_p\), \(\forall p, q \in P\), we obtain \(z'_{pq} = 0 = \forall p \in P \setminus \{a_i\}_{i \in [k]}, \forall q \in P\). Since for every \(i \in [k]\) and for every \(q \in P \setminus A_i\), we have \(\alpha_q < d(a_i, q)\), we know from (2) that \(z'_{a_iq} = 0\) for every \(i \in [k]\) and for every \(q \in P \setminus A_i\). From the primal constraint \(\sum_{p \in P} z_{pq} = 1\), \(\forall q \in P\) we then obtain \(z'_{a_iq} = 1\) for every \(i \in [k]\) and for every \(q \in A_i\). Primal constraints \(z'_{pq} \leq y'_p\)
\( \forall p, q \in P \) and \( \sum_{p \in P} y_p' = k \) imply \( y_{a_i}' = 1 \) for every \( i \in [k] \). We have thereby shown \( (y', z') = (y, z) \). \( \square \)

We remark that deterministic sufficient conditions which guarantee that an integer solution to \((\text{IP})\) is an optimal solution to \((\text{LP})\) have also been presented in [9, 31]. The main difference with respect to these known results is that Theorem 1 provides necessary and sufficient conditions. In this paper, we do not only use Theorem 1 to prove that \((\text{LP})\) can achieve exact recovery, but we also use it to construct examples where \((\text{LP})\) does not achieve exact recovery.

## 3 Models of data and exact recovery

In Sect. 2, we considered the \(k\)-median problem in a deterministic setting. In the remainder of the paper we will instead consider a probabilistic setting. Furthermore, our discussion of the \(k\)-median problem so far is very general, as it applies to any given input consisting of \(n\) points in a metric space. In the remainder of the paper, we will only consider the Euclidean space. Thus we use \(d(\cdot, \cdot)\) to denote the Euclidean distance and we use \(\|\cdot\|\) to denote the Euclidean norm. We also denote by \(B^m_r(c)\) the closed ball of radius \(r\) and center \(c\) in \(\mathbb{R}^m\) and by \(S^{m-1}_r(c)\) the sphere of radius \(r\) and center \(c\) in \(\mathbb{R}^m\). In this paper, unless otherwise stated, we always assume that the radius \(r\) of balls is positive, i.e., \(r \in \mathbb{R}_+\), where \(\mathbb{R}_+ := \{x \in \mathbb{R} \mid x > 0\}\). On the other hand we allow the radius of spheres to be nonnegative, i.e., \(r \in \{x \in \mathbb{R} \mid x \geq 0\}\). In particular, \(S^{m-1}_0(c)\) is the set containing only the vector \(c\).

In this paper we will consider two models of data for the \(k\)-median problem, which are called the stochastic ball model and the extended stochastic ball model. Before defining these two models of data, we first introduce our notation for basic probability theory, in particular, our notation follows [18]. Let \((\mu, \Omega, \mathcal{F})\) be a probability space, where \(\Omega\) is a set of “outcomes”, \(\mathcal{F}\) is a set of “events”, and \(\mu\) is a probability measure. The set \(\mathcal{F}\) is a \(\sigma\)-algebra on \(\Omega\), and in this paper we will always let \(\mathcal{F}\) be the \(\sigma\)-algebra generated by \(\Omega\). Therefore we will refer to the probability space \((\mu, \Omega, \mathcal{F})\) by simply writing \((\mu, \Omega)\). If \(A \in \mathcal{F}\) is an event, we use \(\bar{A}\) to denote its complementary event. We say \(X\) is an \(m\)-dimensional random vector if \(X\) is a measurable map from \((\Omega, \mathcal{F})\) to \((\mathbb{R}^m, \mathcal{B}^m)\), where \(\mathcal{B}^m\) is the \(\sigma\)-algebra generated by \(\mathbb{R}^m\). If \(m = 1\), we call \(X\) a random variable. In particular, if \((\mu, \Omega, \mathcal{F})\) is a probability space, \(\Omega \subseteq \mathbb{R}^m\) and \(X\) is the identity map, we say that \(X\) is a random vector drawn according to \(\mu\). If \(X\) is a random variable, we define its expected value to be \(\mathbb{E}X = \int_{\Omega} X(x) d\mu(x)\).

We are now ready to define the stochastic ball model.

**Definition 2** (Stochastic ball model (SBM)) For every \(i \in [k]\), let \((\mu, B^{m}_i(0))\) be a probability space. For each \(i \in [k]\), draw \(n\) i.i.d. random vectors \(v^{(i)}_{\ell}\), for \(\ell \in [n]\), according to \(\mu\). The points from cluster \(i\) are then taken to be \(x^{(i)}_{\ell} := c_i + v^{(i)}_{\ell}\), for \(\ell \in [n]\).

Variants of the SBM have been considered in the literature, with different assumptions on the properties that the probability space \((\mu, B^{m}_i(0))\) should satisfy. We refer the reader for example to [21].
In this paper we will also consider a more general model of data, which we call the extended stochastic ball model. The extended stochastic ball model is more general than the SBM in the following ways: (i) we do not require the balls to have the same radius, (ii) we do not require the probability measure on the balls to coincide, and (iii) we allow to draw different numbers of data points from different balls.

**Definition 3** (Extended stochastic ball model (ESBM)) For every \( i \in [k] \), let \((\mu_i, B^m_{ri}(c_i))\) be a probability space. For each \( i \in [k]\), let \( \beta_i \geq 1 \) and draw \( n_i := \beta_i n \) i.i.d. random vectors \( x^{(i)}_\ell \), for \( \ell \in [n_i] \), according to \( \mu_i \). The points from cluster \( i \) are then taken to be \( x^{(i)}_\ell \), for \( \ell \in [n_i] \).

In this paper we will consider three different assumptions on the probability spaces of the form \((\mu_i, B^m_{ri}(c_i))\) that we consider, namely:

(a1) The probability measure \( \mu_i \) is invariant under rotations centered in \( c_i \);

(a2) Every open subset of \( B^m_{ri}(c_i) \) containing \( c_i \) has positive probability measure;

(a3) Every subset of \( B^m_{ri}(c_i) \) with zero Lebesgue measure has zero probability measure.

In this paper we will see that in the ESBM, the linear program (LP) can perform very well in solving the \( k \)-median problem. To formalize this notion we define next the concept of exact recovery.

**Definition 4** (Exact recovery) We say that (LP) achieves exact recovery if it has a unique optimal solution, such solution is also feasible (thus optimal) to (LP), and it assigns each point to the ball from which it is drawn.

The reader might wonder why in the definition of the ESBM we assume that \( n_i = \beta_i n \) for \( i \in [k] \), effectively requiring the \( n_i \) to be of the same order. In Example 1 in “Appendix A” we show that this assumption is needed in order to obtain exact recovery.

### 4 Sufficient conditions for exact recovery in the ESBM

In this section, we introduce general sufficient conditions which guarantee that (LP) achieves exact recovery with high probability in the ESBM. To state our results we first introduce the contribution function in the ESBM.

In the original definition (Definition 1), we assumed that \( \alpha \) is a vector in \( \mathbb{R}^P \). When we will consider the contribution function in the ESBM, we will always assume that for every \( i \in [k] \) there exists \( \alpha'_i \in \mathbb{R} \) such that \( \forall \ell \in [n_i] \) we have \( \alpha_{x^{(i)}_\ell} = \alpha'_i \). For ease of notation, we then define the contribution function in the ESBM.

**Definition 5** (Contribution function in the ESBM) Given \( \alpha \in \mathbb{R}^k \), the contribution function in the ESBM \( C^\alpha(z) : \mathbb{R}^m \to \mathbb{R} \) is defined by

\[
C^\alpha(z) := \sum_{i \in [k]} \sum_{\ell \in [n_i]} (\alpha_i - d(z, x^{(i)}_\ell))^+.
\]

Clearly, given \( \alpha \in \mathbb{R}^P \) and \( \alpha' \in \mathbb{R}^k \) such that \( \forall i \in [k], \ell \in [n_i] \) we have \( \alpha_{x^{(i)}_\ell} = \alpha'_i \), the two definitions are equivalent, i.e., \( C^\alpha(z) = C^{\alpha'}(z) \) for every \( z \in \mathbb{R}^m \). Since each
$x^{(i)}_\ell$ is a random vector drawn according to $\mu_i$, we define $\hat{\Omega} := \prod_{\ell \in [n_k]} B_{c_k}^{m}(0) \times \cdots \times \prod_{\ell \in [n_1]} B_{r_1}^{m}(c_k)$ and we let $\hat{\mu}$ be the corresponding joint probability measure for $x^{(1)}_1, \ldots, x^{(1)}_{n_1}, \ldots, x^{(k)}_1, \ldots, x^{(k)}_{n_k}$. Then for every $z \in \mathbb{R}^m$ and for every $\alpha \in \mathbb{R}^k$, $C^\alpha(z)$ is a random variable on the probability space $(\hat{\mu}, \hat{\Omega})$.

Next, we define the function $G^\alpha(z)$, which plays a fundamental role in our sufficient conditions.

**Definition 6** Given $\alpha \in \mathbb{R}^k$, in the ESBM we define the function $G^\alpha(z) : \mathbb{R}^m \to \mathbb{R}$ as

$$G^\alpha(z) := \frac{1}{n} \mathbb{E} C^\alpha(z).$$

**Observation 2** In the ESBM, we obtain

$$G^\alpha(z) = \sum_{i \in [k]} \beta_i \int_{x \in B_{r_i}^{m}(c_i)} (\alpha_i - d(z, x))_+ d\mu_i(x)$$

$$= \sum_{i \in [k]} \beta_i \int_{B_{r_i}^{m}(z) \cap B_{r_i}^{m}(c_i)} (\alpha_i - d(z, x))d\mu_i(x).$$

**Proof** The expected value of the contribution function is

$$\mathbb{E} C^\alpha(z) = \sum_{i \in [k]} n_i \int_{x \in B_{r_i}^{m}(c_i)} (\alpha_i - d(z, x))_+ d\mu_i(x).$$

Using $n_i = \beta_i n$, for $i \in [k]$, we obtain

$$G^\alpha(z) = \frac{1}{n} \mathbb{E} C^\alpha(z) = \sum_{i \in [k]} \beta_i \int_{x \in B_{r_i}^{m}(c_i)} (\alpha_i - d(z, x))_+ d\mu_i(x)$$

$$= \sum_{i \in [k]} \beta_i \int_{B_{r_i}^{m}(z) \cap B_{r_i}^{m}(c_i)} (\alpha_i - d(z, x))d\mu_i(x),$$

where the last equality holds because $\alpha_i - d(z, x) \geq 0$ if and only if $x \in B_{r_i}^{m}(c_i) \cap B_{r_i}^{m}(z)$.

**Observation 3** In the ESBM, the function from $\mathbb{R}^{k+m}$ to $\mathbb{R}$ defined by $(\alpha, z) \mapsto G^\alpha(z)$ is continuous.

**Proof** To prove this observation, it suffices to show that for every compact set $B \subseteq \mathbb{R}^{k+m}$, the function from $B$ to $\mathbb{R}$ defined by $(\alpha, z) \mapsto G^\alpha(z)$ is continuous. Therefore, let $B \subseteq \mathbb{R}^{k+m}$ be an arbitrary compact set. From Observation 2, $G^\alpha(z)$ can be written in the form

$$G^\alpha(z) = \sum_{i \in [k]} \beta_i \int_{x \in B_{r_i}^{m}(c_i)} (\alpha_i - d(z, x))_+ d\mu_i(x).$$
Hence, it suffices to show that, for every $i \in [k]$, the function from $B$ to $\mathbb{R}$ defined by $(\alpha, z) \mapsto \int_{x \in B_{r_i}^{m}(c_i)} (\alpha_i - d(z, x))_+ d\mu_i(x)$ is continuous.

We know that the function from $B \times B_{r_i}^{m}(c_i)$ to $\mathbb{R}$ defined by $(\alpha, z, x) \mapsto (\alpha_i - d(z, x))_+$ is continuous. Since $B \times B_{r_i}^{m}(c_i)$ is a compact set, the Heine–Cantor theorem implies that $(\alpha_i - d(z, x))_+$ is uniformly continuous over $B \times B_{r_i}^{m}(c_i)$. This implies that for every $\epsilon > 0$, there is some $\delta > 0$, such that for every $x \in B_{r_i}^{m}(c_i)$ and for every $(\alpha^1, z^1), (\alpha^2, z^2) \in B$, when $\| (\alpha^1, z^1) - (\alpha^2, z^2) \| < \delta$, we have $\left| (\alpha^1_i - d(z^1, x))_+ - (\alpha^2_i - d(z^2, x))_+ \right| < \epsilon$. We obtain that

$$\int_{x \in B_{r_i}^{m}(c_i)} (\alpha_i - d(z^1, x))_+ - (\alpha_i - d(z^2, x))_+ d\mu_i(x) \leq \epsilon \mathbb{P}(x \in B_{r_i}^{m}(c_i)) \leq \epsilon.$$

This concludes the proof that, for every $i \in [k]$, the function from $B$ to $\mathbb{R}$ defined by $(\alpha, z) \mapsto \int_{x \in B_{r_i}^{m}(c_i)} (\alpha_i - d(z, x))_+ d\mu_i(x)$ is continuous. \hfill \Box

For ease of notation, given $k$ balls $B_{r_i}^{m}(c_i) \subseteq \mathbb{R}^m$, for $i \in [k]$, throughout the paper we denote by

$$D_i := \min\{d(c_i, c_j) - r_i \mid j \in [k], \ j \neq i \}.$$

We also give the following definition in order to simplify the language in this paper.

**Definition 7** Let $(\mu(n), \Omega(n), \mathcal{F}(n))$ be a probability space, which depends on a parameter $n$, and let $A_n \in \mathcal{F}(n)$ be an event which depends on $n$. We say that $A_n$ happens with high probability, if for every $\delta \in (0, 1)$, there exists $N > 0$ such that when $n > N$, $\mathbb{P}(A_n) > 1 - \delta$.

Note that, when we say with high probability, we always mean with respect to the parameter called $n$ in the probability space. In this paper we use several times the well-known fact that, if a constant number of events happen with high probability, then they also happen together with high probability.

We are now ready to state the main result of this section.

**Theorem 2** Consider the ESBM. For every $i \in [k]$, assume that the probability space $(\mu_i, B_{r_i}^{m}(c_i))$ satisfies (a1), (a2), (a3). For every $i \in [k]$, denote by $E_i := \mathbb{E}d(x, c_i)$, where $x$ is a random vector drawn according to $\mu_i$. Assume that there exists some $\gamma \in \mathbb{R}$ that satisfies $\max_{i \in [k]} \beta_i(r_i - E_i) < \gamma < \min_{i \in [k]} \beta_i(D_i - E_i)$. For every $i \in [k]$, let $\alpha_i := E_i + \frac{\gamma}{\beta_i}$ and assume that $c_i$ is the unique point that achieves $\max \{G^m(z) \mid z \in B_{r_i}^{m}(c_i) \}$. Then (LP) achieves exact recovery with high probability.

Next, we present a corollary of Theorem 2 for the ESBM with some special structure.
Corollary 1  Consider the ESBM. For every $i \in [k]$, assume that the probability space $(\mu_i, B_{r_i}(c_i))$ satisfies (a1), (a2), (a3). For every $i \in [k]$, assume $n_i = n$, $r_i = 1$, and denote by $E_i := \mathbb{E}d(x, c_i)$, where $x$ is a random vector drawn according to $\mu_i$. We further assume $E_1 = \cdots = E_k$. Assume that there exists some $\alpha' \in \mathbb{R}$ that satisfies $1 < \alpha' < \min_{i \neq j} d(c_i, c_j) - 1$. For every $i \in [k]$, let $\alpha_i := \alpha'$ and assume that $c_i$ is the unique point that achieves $\max\{G^\alpha(z) \mid z \in B_{r_i}(c_i)\}$. Then (LP) achieves exact recovery with high probability.

In this paper we often use the concept of median. Let $P$ be a finite set of points in $\mathbb{R}^m$. We say that $x^* \in P$ is a median of $P$ if $x^* \in \arg\min\{\sum_{s \in P} d(x, s) \mid x \in P\}$.

Next, we give an overview of the proof of Theorem 2. We first study points that are drawn from a single ball. The key observation is that when $n$ is large enough, the median of the points drawn from a single ball is very close to the ball center. This allows us to characterize the solution corresponding to the ground-truth. Then, using Hoeffding’s inequality [34], we prove that with high probability, when we add any small perturbation to $\alpha$, the medians still get most contribution, which in turn implies (8). The condition $\max_{i \in [k]} \beta_i (r_i - E_i) < \gamma < \min_{i \in [k]} \beta_i (D_i - E_i)$ guarantees that with high probability, when we add a very small perturbation to $\alpha$, the resulting $\alpha$ satisfies (9) and (10). Finally, using Hoeffding’s inequality, we can guarantee that with high probability the choice of $\alpha$ that satisfies (7) is very close to the parameter $\alpha$ in the statement. As a consequence, (LP) achieves exact recovery with high probability according to Theorem 1.

A careful reader may find that, if we assume that all $E_i$ are the same and all $\beta_i$ equal one, then our proof is similar to Steps 2–4 in the proof of Theorem 7 in [9]. However, we point out here an important difference. In Step 2, the authors show that, after adding a small perturbation to $\alpha$, the points that get most contribution in expectation will be the ball centers. Then in Step 3, they show that with high probability a special choice of $\alpha$ can be seen as the $\alpha$ in Step 2 plus a small perturbation. Finally in Step 4, they use Hoeffding’s inequality to show that with high probability the $\alpha$ in Step 3 can make the median in each ball obtain most contribution. However, we notice that since Step 4 is conditioned on Step 3, the probability spaces considered in Step 3 and Step 4 are different, so in Step 4 the sequence of random variables considered in Hoeffding’s inequality are not independent, and Hoeffding’s inequality cannot be used directly. In our proof this problem is not present.

In Sects. 4.1 and 4.2 we prove some lemmas that will be used in the proof of Theorem 2, which is given in Sect. 4.3. Then, in Sect. 4.4, we prove Corollary 1.

In this paper we use the standard notation $[a, b]$ for closed segments and $(a, b)$ for open segments in $\mathbb{R}$. Throughout the paper this notation is used only when these segments are nonempty. Therefore, each time we write $[a, b]$ or $(a, b)$ we are also implicitly assuming $a \leq b$ and $a < b$, respectively.

4.1 Lemmas about a single ball

In this section we present some lemmas that consider a single probability space of the form $(\mu, B_{r_i}^m(0))$. 
Lemma 1 Let \((\mu, B_r^m(0))\) be a probability space that satisfies (a3). Let \(x_1, \ldots, x_n\) be random vectors drawn i.i.d. according to \(\mu\), where \(n \geq 3\), and let \(\mathcal{M}\) the set of medians of \(\{x_\ell\}_{\ell \in [n]}\). Then \(|\mathcal{M}| = 1\) with probability one.

Proof Since \(\mathcal{M}\) is always non empty, in order to show that \(|\mathcal{M}| = 1\) with probability one, it suffices to show that we have \(|\mathcal{M}| \geq 2\) with probability zero.

Let \(\bar{x}_1, \ldots, \bar{x}_{n-1} \in B_r^m(0)\). Then we have

\[
P(|\mathcal{M}| \geq 2) = \int_{B_r^m(0)} \cdots \int_{B_r^m(0)} P(|\mathcal{M}| \geq 2 \mid x_1 = \bar{x}_1, \ldots, x_{n-1} = \bar{x}_{n-1}) d\mu(\bar{x}_1) \cdots d\mu(\bar{x}_{n-1}).
\]

Hence, to prove the lemma it suffices to show that for every \(\bar{x}_1, \ldots, \bar{x}_{n-1} \in \mathbb{R}^m\) we have

\[
P(|\mathcal{M}| \geq 2 \mid x_1 = \bar{x}_1, \ldots, x_{n-1} = \bar{x}_{n-1}) = 0.
\] (11)

From (a3), we know that \(x_1, \ldots, x_{n-1}\) are different points with probability one. So it is sufficient to show that (11) holds when \(\bar{x}_1, \ldots, \bar{x}_{n-1}\) are all different.

Note that \(|\mathcal{M}| \geq 2\) implies that there exist \(u, v \in [n]\) with \(u \neq v\) such that \(\sum_{\ell \in [n]} d(x_u, x_\ell) = \sum_{\ell \in [n]} d(x_v, x_\ell)\). So we have

\[
P(|\mathcal{M}| \geq 2 \mid x_1 = \bar{x}_1, \ldots, x_{n-1} = \bar{x}_{n-1}) \leq \sum_{u, v \in [n], u \neq v} P\left( \sum_{\ell \in [n]} d(x_u, x_\ell) = \sum_{\ell \in [n]} d(x_v, x_\ell) \mid x_1 = \bar{x}_1, \ldots, x_{n-1} = \bar{x}_{n-1} \right).
\]

Thus, to prove the lemma, it suffices to show that, for every \(\bar{x}_1, \ldots, \bar{x}_{n-1} \in \mathbb{R}^m\) all different, and for every \(u, v \in [n]\) with \(u \neq v\), we have

\[
P\left( \sum_{\ell \in [n]} d(x_u, x_\ell) = \sum_{\ell \in [n]} d(x_v, x_\ell) \mid x_1 = \bar{x}_1, \ldots, x_{n-1} = \bar{x}_{n-1} \right) = 0.
\]

Notice that the above event only depends on the choice of \(x_n\), since \(x_1, \ldots, x_{n-1}\) are fixed to \(\bar{x}_1, \ldots, \bar{x}_{n-1}\) respectively. Thus we define

\[
S := \left\{ x_n \in \mathbb{R}^m \mid \sum_{\ell \in [n]} d(x_u, x_\ell) = \sum_{\ell \in [n]} d(x_v, x_\ell), x_1 = \bar{x}_1, \ldots, x_{n-1} = \bar{x}_{n-1} \right\}.
\]

To prove the lemma, it suffices to show that the Lebesgue measure of \(S\) is zero. In fact, (a3) then implies that \(S\) has zero probability measure. Hence, in the remainder of the proof we show that the Lebesgue measure of \(S\) is zero.
We consider separately two cases. In the first case we assume \( u \neq n \) and \( v \neq n \). Then

\[
S = \left\{ x_n \in \mathbb{R}^m \mid d(\bar{x}_u, x_n) - d(\bar{x}_v, x_n) = \sum_{\ell \in [n] \setminus [n]} d(\bar{x}_v, \bar{x}_\ell) - \sum_{\ell \in [n] \setminus [n]} d(\bar{x}_u, \bar{x}_\ell) \right\}.
\]

We define the function \( f : \mathbb{R}^m \to \mathbb{R} \) defined by

\[
f(x_n) := d(\bar{x}_u, x_n) - d(\bar{x}_v, x_n) - \sum_{\ell \in [n] \setminus [n]} d(\bar{x}_v, \bar{x}_\ell) + \sum_{\ell \in [n] \setminus [n]} d(\bar{x}_u, \bar{x}_\ell).
\]

Note that \( S \) is the zero set of \( f \). The function \( f(x_n) \) is a real analytic function on the connected open domain \( \mathbb{R}^m \setminus \{\bar{x}_u, \bar{x}_v\} \) since the distance function can be written as a composition of exponential functions, logarithms and polynomials. Furthermore, \( f(x_n) \) is not identically zero, since it increases as \( x_n \) moves on the segment from \( \bar{x}_v \) to \( \bar{x}_u \). From Proposition 1 in [29], we obtain that \( S \) has zero Lebesgue measure.

In the second case we assume \( u = n \) and \( v \neq n \). Then

\[
S = \left\{ x_n \in \mathbb{R}^m \mid \sum_{\ell \in [n] \setminus [n] \setminus \{v\}} d(x_n, \bar{x}_\ell) = \sum_{\ell \in [n] \setminus [n]} d(\bar{x}_v, \bar{x}_\ell) \right\}.
\]

We define the function \( f : \mathbb{R}^m \to \mathbb{R} \) defined by

\[
f(x_n) := \sum_{\ell \in [n] \setminus [n] \setminus \{v\}} d(x_n, \bar{x}_\ell) - \sum_{\ell \in [n] \setminus [n]} d(\bar{x}_v, \bar{x}_\ell).
\]

Also in this case \( S \) is the zero set of \( f \). As in the previous case, the function \( f(x_n) \) is a real analytic function on the connected open domain \( \mathbb{R}^m \setminus \{\bar{x}_1, \ldots, \bar{x}_{n-1}, \bar{x}_{n+1}, \ldots, \bar{x}_{n-1}\} \). Furthermore, it is not identically zero, as it increases as the norm of \( x_n \) goes to infinity. Again from Proposition 1 in [29], we obtain that \( S \) has zero Lebesgue measure. So in both cases we have shown that the Lebesgue measure of \( S \) is zero.

The next two lemmas state that, under some assumptions on the probability space \((\mu, B^m_r(0))\), the vector \( z = 0 \) is the unique point that achieves \( \min\{\mathbb{E}d(z, y) \mid z \in B^m_r(0)\} \), where \( y \) be a random vector drawn according to \( \mu \). In Lemma 2 we consider the case \( m = 1 \) and in Lemma 3 we study the case \( m \geq 2 \).

**Lemma 2** Let \((\mu, B^1_r(0))\) be a probability space that satisfies (a1), (a2). Let \( y \) be a random vector drawn according to \( \mu \). Then \( z = 0 \) is the unique point that achieves \( \min\{\mathbb{E}d(z, y) \mid z \in B^1_r(0)\} \).  

**Proof** We show that for every \( z \neq 0 \), we have \( \mathbb{E}d(z, y) > \mathbb{E}d(0, y) \). Let \( z \in [-r, r] \setminus \{0\} \). Then we have

\[
\mathbb{E}d(z, y) = \int_{-r}^{z} (z - y) d\mu(y) + \int_{z}^{r} (y - z) d\mu(y).
\]
Without loss of generality, we assume that \( z > 0 \). We then have
\[
\mathbb{E}d(z, y) - \mathbb{E}d(0, y) = \int_{-r}^{-z} zd\mu(y) + \int_{-z}^{0} zd\mu(y) + \int_{0}^{r} (z - 2y) d\mu(y) + \int_{z}^{r} zd\mu(y).
\]
Since \( \mu \) satisfies (a1), we have
\[
\int_{-r}^{-z} zd\mu(y) = \int_{z}^{r} zd\mu(y) \quad \text{and} \quad \int_{0}^{r} zd\mu(y) = \int_{0}^{r} zd\mu(y).
\]
So we obtain
\[
\mathbb{E}d(z, y) - \mathbb{E}d(0, y) = \int_{-z}^{0} zd\mu(y) + \int_{0}^{z} (z - 2y) d\mu(y) = 2 \int_{0}^{z} (z - y) d\mu(y) > 0,
\]
where the inequality holds due to (a2).

Lemma 3 Let \((\mu, B^m(B_r^m(0)))\) be a probability space with \( m \geq 2 \) that satisfies (a1). Let \( y \) be a random vector drawn according to \( \mu \). Then \( z = 0 \) is the unique point that achieves \( \min \{ \mathbb{E}d(z, y) \mid z \in B^m(0) \} \).

Proof Note that we can write any \( z \in B^m(0) \) as \( z = tv \), for a unit vector \( v \) and a scalar \( t \in [0, r] \). Since \( \mu \) is invariant under rotations centered in the origin, to prove the lemma it suffices to show that for any fixed unit vector \( v \), \( t = 0 \) is the unique point that achieves \( \min \{ \mathbb{E}d(tv, y) \mid t \in [0, r] \} \). To prove the lemma it is sufficient to show that
\[
\frac{\partial}{\partial t} \mathbb{E}d(tv, y) > 0 \quad \forall t \in (0, r). \tag{12}
\]
In fact, we notice that \( \mathbb{E}d(tv, y) \) is a continuous function in \( t \in [0, r] \), since for every \( \epsilon > 0 \) and for every \( t, t' \in [0, r] \) with \( |t - t'| < \epsilon \), we have
\[
|\mathbb{E}d(tv, y) - \mathbb{E}d(t'v, y)| = |\mathbb{E}(d(tv, y) - d(t'v, y))| \leq |t - t'| = |t - t'| < \epsilon.
\]
Hence, if (12) holds, then by the Newton-Leibniz formula, we have
\[
\mathbb{E}d(sv, y) - \mathbb{E}d(0, y) = \int_{0}^{s} \frac{\partial}{\partial t} \mathbb{E}d(tv, y) dt > 0 \quad \forall s > 0.
\]
Thus, in the remainder of the proof we show (12).

We know that
\[
\mathbb{E}d(tv, y) = \int_{B^m(0)} d(tv, y) d\mu(y),
\]
thus we obtain
\[
\frac{\partial}{\partial t} \mathbb{E}d(tv, y) = \frac{\partial}{\partial t} \int_{B^m(0)} d(tv, y) d\mu(y) = \int_{B^m(0)} \frac{\partial}{\partial t} d(tv, y) d\mu(y).
\]
where $\langle \cdot, \cdot \rangle$ denotes the scalar product. In the remainder of the proof, for $s \geq 0$, we denote by $\mu^s$ the uniform probability measure with support $S^{m-1}(0)$. Since $\mu$ is invariant under rotations centered in the origin, we know that a vector $y$ with $\|y\| = s$, $s \in [0, r]$, is drawn according to $\mu^s$.

We evaluate (13) in a fixed $i \in (0, r)$. Let $\hat{\mu}$ be the probability measure of the random variable $\|y\|$ and let $x$ be a random vector drawn according to $\mu^s$. We have

$$
\frac{\partial}{\partial t} \mathbb{E}_{t} d(t, v) \bigg|_{t=i} = \int_{B^m(0)} \frac{\langle t v - y, v \rangle}{d(tv, y)} d\mu(y) = \int_{0}^{r} d\hat{\mu}(s) \int_{S^{m-1}(0)} \frac{\langle t v - x, v \rangle}{d(tv, x)} d\mu^s(x). 
$$

Next, we study the inner integral in (14) and consider two subcases. In the first subcase we have $s \in [0, i]$, and obtain

$$
\langle t v - x, v \rangle = i - \langle x, v \rangle \geq i - \|x\| \geq 0,
$$

where the chain of inequalities holds at equality if and only $x = i v$. So we obtain that the inner integral in (14) is strictly positive when $s \in (0, i]$.

In the second subcase we have $s \in (i, r]$. We define the random variable $\theta \in [0, \pi]$ to be the angle between $x$ and $v$ and we let $\hat{\mu}$ be its probability measure. We also define the random variable $\psi \in [0, \pi]$ to be the angle between $v$ and $i v - x$, and the random variable $\phi \in [0, \pi]$ to be the angle between $x$ and $x - i v$. See Fig. 1 for a depiction of the angles $\theta$, $\psi$, $\phi$. Note that once $\theta$ is determined, $\psi$ and $\phi$ are also determined. Therefore, we can consider the functions $\psi, \phi : [0, \pi] \rightarrow [0, \pi]$ that associate to each angle $\theta$, the corresponding angles $\psi(\theta)$ and $\phi(\theta)$. Then we know that for every $\theta \in [0, \pi]$, $\psi(\theta) = \pi - \phi(\theta) - \theta \leq \pi - \theta$, and, when $\theta \in (0, \pi)$, $\psi(\theta) < \pi - \theta$.

We then have

$$
\int_{S^{m-1}(0)} \frac{\langle t v - x, v \rangle}{d(tv, x)} d\mu^s(x) = \int_{0}^{\pi} \cos \psi(\theta) d\hat{\mu}(\theta) = \int_{0}^{\pi} \cos \psi(\theta) + \cos \psi(\pi - \theta) d\hat{\mu}(\theta) > 0.
$$

In the above formula, the second equality use the fact that $\hat{\mu}$ is symmetric with respect to $\theta = \pi/2$, and the inequality follows because, when $m \geq 2$ and $\theta \in (0, \pi)$, we have $\psi(\theta) < \pi - \theta$ and $\psi(\pi - \theta) < \theta$, which implies $\cos \psi(\theta) + \cos \psi(\pi - \theta) > \cos(\pi - \theta) + \cos \theta = 0$, when $\theta \in (0, \pi/2)$. We obtain that the inner integral in (14) is strictly positive when $s \in (i, r]$. Thus we conclude that (14) is positive. \hfill \square

In the next lemma we make use of Lemmas 2 and 3.
The angles $\theta, \psi, \phi$ in the proof of Lemma 3. The dotted vector is $\bar{t}v - x$ applied to $\bar{t}v$.

**Fig. 1**

**Lemma 4** Let $(\mu, B^m_\tau(0))$ be a probability space that satisfies (a1), (a2). Let $x_1, \ldots, x_n$ be random vectors drawn i.i.d. according to $\mu$, and let $x_*$ be a median of $\{x_\ell \}_{\ell \in [n]}$. Then $\forall \epsilon > 0$, with high probability, we have $\|x_*\| < \epsilon$.

**Proof** Let $y$ be a random vector drawn according to $\mu$. We know from Lemmas 2 and 3 that $x = 0$ is the unique point that achieves $\min\{\mathbb{E}d(x, y) \mid x \in B^m_r(0)\}$. Furthermore, since the function from $B^m_r(0)$ to $\mathbb{R}$ defined by $x \mapsto \mathbb{E}d(x, y)$ is continuous, we know that for every $\epsilon \in (0, r)$, there is some $\tau$ with $0 < \tau < \epsilon < r$ and some $\xi > 0$ such that for each $x \in B^m_r(0) \setminus B^m_\epsilon(0)$ and for each $x' \in B^m_\epsilon(0)$, we have $\mathbb{E}d(x, y) - \mathbb{E}d(x', y) > \xi$. Let $x_{\text{min}} := \text{argmin}\{\|x\| \mid x \in \{x_\ell \}_{\ell \in [n]}\}$ and notice that $x_* = \text{argmin}\{\sum_{\ell \in [n]} d(x, x_\ell)/n \mid x \in \{x_\ell \}_{\ell \in [n]}\}$.

We observe that to prove the lemma it suffices to show that $\|x_{\text{min}}\| \leq \tau$, that $\sum_{\ell \in [n]} d(x_u, x_\ell)/n - \mathbb{E}d(x_u, y) \geq -\xi/2$ for every $u \in [n]$ with $x_u \in B^m_r(0) \setminus B^m_\epsilon(0)$, and that $\sum_{\ell \in [n]} d(x_v, x_\ell)/n - \mathbb{E}d(x_v, y) \leq \xi/2$ for every $v \in [n]$ with $x_v \in B^m_\tau(0)$.

In fact, under these assumptions we obtain that for every $u \in [n]$ with $x_u \in B^m_r(0) \setminus B^m_\epsilon(0)$ and for every $v \in [n]$ with $x_v \in B^m_\tau(0)$, we have

$$
\frac{\sum_{\ell \in [n]} d(x_v, x_\ell)}{n} - \frac{\sum_{\ell \in [n]} d(x_u, x_\ell)}{n}
\leq \left( \frac{\sum_{\ell \in [n]} d(x_v, x_\ell)}{n} - \mathbb{E}d(x_v, y) \right) - \left( \frac{\sum_{\ell \in [n]} d(x_u, x_\ell)}{n} - \mathbb{E}d(x_u, y) \right)
- \left( \mathbb{E}d(x_u, y) - \mathbb{E}d(x_v, y) \right)
< \frac{\xi}{2} + \frac{\xi}{2} - \xi = 0.
$$

Since $\|x_{\text{min}}\| \leq \tau$, the above expression implies that $\|x_*\| \leq \epsilon$. 

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Inspired by the above observation, we define the following events. We denote by $A$ the event that $||x_u|| \leq \epsilon$ and we denote by $T$ the event that $||x_{\min}|| \leq \tau$. For every $w \in [n]$, we denote by $M_w$ the event that at least one of the following events happens:

- $x_w \in B^m_r(0)$ and $\sum_{\ell \in [n]} d(x_w, x_\ell)/n - \mathbb{E}d(x_w, y) \leq \xi/2$;
- $x_w \in B^\epsilon_r(0) \setminus B^m_r(0)$;
- $x_w \in B^m_r(0) \setminus B^\epsilon_r(0)$ and $\sum_{\ell \in [n]} d(x_w, x_\ell)/n - \mathbb{E}d(x_w, y) \geq -\xi/2$.

In the remainder of the proof, we denote by $\bar{E}$ the complement of an event $E$. We know that if $T$ and $M_w$, for all $w \in [n]$, are true then $A$ is true. So we get

$$\mathbb{P}(A) \geq \mathbb{P}
\left(T \cap \bigcap_{w \in [n]} M_w \right) = 1 - \mathbb{P}
\left(\bar{T} \cup \bigcup_{w \in [n]} \bar{M}_w \right) = 1 - \mathbb{P}(\bar{T}) - \sum_{w \in [n]} \mathbb{P}(\bar{M}_w).
\tag{15}$$

We next upper bound $\mathbb{P}(\bar{T})$ and $\mathbb{P}(\bar{M}_w)$.

We define $p := \mathbb{P}(y \in B^m_r(0) \setminus B^\epsilon_r(0))$ and obtain

$$\mathbb{P}(\bar{T}) = p^n. \tag{16}$$

Since $(\mu, B^m_r(0))$ satisfies (a2), we know that $p < 1$.

For $w \in [n]$, we know that $\bar{M}_w$ is true if and only if at least one of the following event is true:

$P_w$: $x_w \in B^m_r(0)$ and $\sum_{\ell \in [n]} d(x_w, x_\ell)/n - \mathbb{E}d(x_w, y) > \xi/2$;
$Q_w$: $x_w \in B^m_r(0) \setminus B^\epsilon_r(0)$ and $\sum_{\ell \in [n]} d(x_w, x_\ell)/n - \mathbb{E}d(x_w, y) < -\xi/2$.

Hence in the following we will upper bound separately $\mathbb{P}(P_w)$ and $\mathbb{P}(Q_w)$.

We start by analyzing $P_w$. For every $z \in B^m_r(0)$, we have

$$\mathbb{P}\left(\sum_{\ell \in [n]} \frac{d(z, x_\ell)}{n} - \mathbb{E}d(z, y) > \frac{\xi}{2} \mid x_w = z\right) = \mathbb{P}\left(\sum_{\ell \neq w} \frac{d(z, x_\ell)}{n} - \mathbb{E}d(z, y) > \frac{\xi}{2} \mid x_w = z\right) = \mathbb{P}\left(\sum_{\ell \neq w} \frac{d(z, x_\ell)}{n} - \mathbb{E}d(z, y) > \frac{\xi}{2}\right) = \mathbb{P}\left(\sum_{\ell \neq w} d(z, x_\ell) - (n - 1)\mathbb{E}d(z, y) > \frac{n\xi}{2} + \mathbb{E}d(z, y)\right) \leq \exp\left(- \frac{2(n\xi/2 + \mathbb{E}d(z, y))^2}{(n - 1)r^2}\right) \leq \exp\left(- \frac{n\xi^2}{2r^2}\right).$$

Here, the second equality holds because $x_\ell$, for $\ell \in [n]$ are independent. In the first inequality, we use the Hoeffding’s inequality and the fact that $d(z, x_\ell) \in [0, r]$. The
The last inequality follows because \( \mathbb{E}d(z, y) \geq 0 \). So we get

\[
\mathbb{P}(P_w) = \int_{B^m_r(0)} \mathbb{P}\left( \frac{\sum_{\ell \in [n]} d(z, x_\ell)}{n} - \mathbb{E}d(z, y) > \frac{\xi}{2} \mid x_w = z \right) d\mu(z)
\]

\[
\leq \sup \left\{ \mathbb{P}\left( \frac{\sum_{\ell \in [n]} d(z, x_\ell)}{n} - \mathbb{E}d(z, y) > \frac{\xi}{2} \mid x_w = z \right) \mid z \in B^m_r(0) \right\}
\]

\[
\leq \exp\left( -\frac{n \xi^2}{2r^2} \right).
\]

Next, we analyze in a similar way \( Q_w \). We have

\[
\mathbb{P}(Q_w) = \int_{B^m_r(0) \setminus B^m_\epsilon(0)} \mathbb{P}\left( \frac{\sum_{\ell \in [n]} d(z, x_\ell)}{n} - \mathbb{E}d(z, y) < -\frac{\xi}{2} \mid x_w = z \right) d\mu(z)
\]

\[
\leq \exp\left( -\frac{n \xi^2}{4r^2} \right),
\]

because for every \( z \in B^m_r(0) \setminus B^m_\epsilon(0) \), we have

\[
\mathbb{P}\left( \frac{\sum_{\ell \in [n]} d(z, x_\ell)}{n} - \mathbb{E}d(z, y) < -\frac{\xi}{2} \mid x_w = z \right)
\]

\[
= \mathbb{P}\left( \frac{\sum_{\ell \neq w} d(z, x_\ell)}{n} - \mathbb{E}d(z, y) < -\frac{\xi}{2} \mid x_w = z \right)
\]

\[
= \mathbb{P}\left( \frac{\sum_{\ell \neq w} d(z, x_\ell)}{n} - \mathbb{E}d(z, y) < -\frac{\xi}{2} \right)
\]

\[
= \mathbb{P}\left( \sum_{\ell \neq w} d(z, x_\ell) - (n - 1)\mathbb{E}d(z, y) < -\left( \frac{n \xi}{2} - \mathbb{E}d(z, y) \right) \right)
\]

\[
\leq \exp\left( -\frac{2(n \xi/2 - \mathbb{E}d(z, y))^2}{(n - 1)r^2} \right) \leq \exp\left( -\frac{2(n \xi/2 - r)^2}{nr^2} \right) \leq \exp\left( -\frac{n \xi^2}{4r^2} \right),
\]

where the last inequality holds when \( n > 4r/((2 - \sqrt{2})\xi) \).

In the rest of the proof, we assume that \( n > 4r/((2 - \sqrt{2})\xi) \). Using the union bound, we have

\[
\mathbb{P}(\tilde{M}_w) \leq \mathbb{P}(P_w) + \mathbb{P}(Q_w) \leq \exp\left( -\frac{n \xi^2}{2r^2} \right) + \exp\left( -\frac{n \xi^2}{4r^2} \right) \leq 2 \exp\left( -\frac{n \xi^2}{4r^2} \right).
\]

Using (15) and (16), we obtain

\[
\mathbb{P}(A) \geq 1 - \mathbb{P}(\tilde{T}) - \sum_{w \in [n]} \mathbb{P}(\tilde{M}_w) \geq 1 - p^n - 2n \exp\left( -\frac{n \xi^2}{4r^2} \right).
\]
The latter quantity goes to 1 as \( n \) goes to infinity because \( p < 1 \) and \( p, \xi \) and \( r \) are all parameters that do not depend on \( n \). So with high probability we have \( \| x^* \| \leq \epsilon \). \( \square \)

In the next lemma we use Lemma 4.

**Lemma 5** Let \((\mu, B^m_r(0))\) be a probability space that satisfies (a1), (a2). Let \( x_1, \ldots, x_n \) be random vectors drawn i.i.d. according to \( \mu \), and let \( x^* \) be a median of \( \{ x_\ell \}_{\ell \in [n]} \). Let \( E := \mathbb{E} \| x \| \), where \( x \) is a random vector drawn according to \( \mu \). Let \( \text{OPT} := \sum_{\ell \in [n]} d(x^*_\ell, x_\ell) \). Then for each \( \epsilon > 0 \), with high probability we have \( \left| \text{OPT} / n - E \right| < \epsilon / 2 \).

**Proof** Let \( \epsilon > 0 \). We apply Lemma 4 and we know that with high probability, we have \( \| x^* \| < \epsilon / 2 \). This implies that, with high probability, we have \( |d(x_\ell, x^*_\ell) - d(x_\ell, 0)| < \epsilon / 2 \) for each \( \ell \in [n] \). Summing the latter \( n \) inequalities, we obtain that with high probability we have

\[
\left| \sum_{\ell \in [n]} \| x^* \| n - E \right| < \epsilon / 2. \tag{17}
\]

On the other hand, according to Hoeffding’s inequality,

\[
\mathbb{P} \left( \left| \sum_{\ell \in [n]} \| x_\ell \| n - E \right| < \frac{\epsilon}{2} \right) > 1 - 2 \exp \left( -\frac{n\epsilon^2}{2r^2} \right).
\]

Since \( \exp(-n\epsilon^2/(2r^2)) \) goes to zero as \( n \) goes to \( +\infty \), with high probability we have

\[
\left| \sum_{\ell \in [n]} \| x_\ell \| n - E \right| < \epsilon / 2. \tag{18}
\]

From (17) and (18), with high probability we have

\[
\left| \frac{\text{OPT}}{n} - E \right| \leq \left| \frac{\text{OPT}}{n} - \frac{\sum_{\ell \in [n]} \| x_\ell \| n}{n} \right| + \left| \frac{\sum_{\ell \in [n]} \| x_\ell \| n}{n} - E \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \]

\( \square \)

### 4.2 Lemmas about several balls

While in Sect. 4.1 we only considered one ball \( B^m_r(0) \), in the three lemmas presented in this section we will consider \( k \) balls \( B^m_{r_i}(c_i) \), for \( i \in [k] \).

**Lemma 6** Let \( B^m_{r_i}(c_i) \), for \( i \in [k] \), be \( k \) balls in \( \mathbb{R}^m \). For every \( i \in [k] \), assume \( r_i < D_i \) and let \( [a_i, b_i] \subset (r_i, D_i) \). Then there exist \( \tau_i > 0 \), \( \forall i \in [k] \), such that \( \forall i, j \in [k], \forall z \in \text{int} B^m_{r_i}(c_i), \text{and} \forall \alpha_j \in [a_j, b_j] \), we have

\[
B^m_{\alpha_j}(z) \cap B^m_{r_j}(c_j) = \begin{cases} B^m_{r_i}(c_i) & \text{if } j = i \\ \emptyset & \text{otherwise.} \end{cases} \tag{19}
\]
Proof We first show the following claim, obtained from the statement of the lemma by fixing some \( \alpha_i \in (r_i, D_i) \), for \( i \in [k] \). Let \( B^m_{\alpha_i}(c_i) \), for \( i \in [k] \), be \( k \) balls in \( \mathbb{R}^m \), assume \( r_i < D_i \), and let \( \alpha_j \in (r_i, D_i) \). Then, for every \( i \in [k] \), there exists \( \tau_i(\alpha) > 0 \) such that \( \forall z \in \text{int} B^m_{\tau_i(\alpha)}(c_i) \), we have (19).

To prove the claim we choose, for every \( i \in [k] \), \( \tau_i(\alpha) := \min\{\alpha_i - r_i, D_1 - \alpha_1, \ldots, D_k - \alpha_k\} > 0 \). Let \( z \in \text{int} B^m_{\tau_i(\alpha)}(c_i) \). We first show that \( B^m_{\tau_i(\alpha)}(z) \cap B^m_{\tau_j(\alpha)}(c_j) = B^m_{\tau_j}(c_j) \). We only need to prove that for each \( x \in B^m_{\tau_j}(c_j) \), we have \( x \in B^m_{\tau_j}(z) \), and this holds because

\[
d(x, z) \leq d(x, c_i) + d(c_i, z) < r_i + \tau_i(\alpha) \leq r_i + (\alpha_i - r_i) = \alpha_i.
\]

Next we show that we have \( B^m_{\tau_i(\alpha)}(z) \cap B^m_{\tau_j}(c_j) = \emptyset \) for \( j \neq i \). We only need to prove that for each \( x \in B^m_{\tau_j}(c_j) \), we have \( x \notin B^m_{\tau_j}(z) \). We have

\[
d(x, c_i) \geq d(c_i, c_j) - d(x, c_j) = d(c_i, c_j) - r_j \geq D_j,
\]

thus

\[
d(x, z) \geq d(x, c_i) - d(c_i, z) \geq D_j - d(c_i, z) > D_j - \tau_i(\alpha) \geq \alpha_j,
\]

where the last inequality follows from the definition of \( \tau_i(\alpha) \). We have shown \( d(x, z) > \alpha_j \), thus \( x \notin B^m_{\tau_j}(z) \). This concludes the proof of the claim.

To prove the lemma, we define the set \( S := \prod_{i \in [k]} [a_i, b_i] \) and take \( \tau_i := \inf\{\tau_i(\alpha) \mid \alpha \in S\} = \min\{\tau_i(\alpha) \mid \alpha \in S\} > 0 \), where the equality follows from the extreme value theorem, since \( S \) is compact and \( \tau_i(\alpha) \) is a continuous function over \( S \), for every \( i \in [k] \).

Lemma 7 Consider the ESBM. Let \( \alpha \in \mathbb{R}^k \) and let \( s_i \in \mathbb{R}^m \) for every \( i \in [k] \). Assume that we have

\[
B^m_{\alpha_j}(s_i) \cap B^m_{\tau_j}(c_j) = \begin{cases} B^m_{\tau_j}(c_j) & \text{if } j = i \\ \emptyset & \text{otherwise} \end{cases} \quad \forall i, j \in [k]. \quad (20)
\]

Then, we have

\[
\begin{align*}
\alpha_i & \geq d(s_i, x^{(i)}_\ell) & \forall i \in [k], \forall \ell \in [n_i] \\
\alpha_j & < d(s_i, x^{(j)}_\ell) & \forall i, j \in [k], i \neq j, \forall \ell \in [n_j] \\
C^\alpha(s_i) & = n_i \alpha_i - \sum_{\ell \in [n_i]} d(s_i, x^{(i)}_\ell) & \forall i \in [k].
\end{align*}
\]

Proof From (20) with \( j = i \) we obtain that for every \( i \in [k] \) we have \( B^m_{\alpha_j}(s_i) \cap B^m_{\tau_i}(c_i) = B^m_{\tau_i}(c_i) \) thus \( B^m_{\tau_i}(c_i) \subseteq B^m_{\alpha_i}(s_i) \). Since \( x^{(i)}_\ell \in B^m_{\tau_i}(c_i) \) for every \( i \in [k], \ell \in [n_i] \), we obtain

\[
\alpha_i \geq d(s_i, x^{(i)}_\ell) \quad \forall i \in [k], \forall \ell \in [n_i].
\]
From (20) with \( i \neq j \), we obtain that for every \( i, j \in [k] \) with \( i \neq j \), we have \( B^m_{\alpha_j}(s_i) \cap B^m_{\alpha_j}(c_j) = \emptyset \). Since \( x_{\ell}^{(j)} \in B^m_{\alpha_j}(c_j) \) for every \( j \in [k], \ell \in [n_j] \), we obtain

\[
a_j < d(s_i, x_{\ell}^{(j)}) \quad \forall i, j \in [k], i \neq j, \forall \ell \in [n_j].
\]

We obtain that for every \( i \in [k] \),

\[
C^\alpha(s_i) = \sum_{j \in [k]} \sum_{\ell \in [n_j]} (\alpha_j - d(s_i, x_{\ell}^{(j)}))^+ = \sum_{\ell \in [n_i]} (\alpha_i - d(s_i, x_{\ell}^{(i)}))^+ + \sum_{j \in [k], j \neq i} \sum_{\ell \in [n_j]} (\alpha_j - d(s_i, x_{\ell}^{(j)}))^+ = \sum_{\ell \in [n_i]} (\alpha_i - d(s_i, x_{\ell}^{(i)})) = n_i \alpha_i - \sum_{\ell \in [n_i]} d(s_i, x_{\ell}^{(i)}).
\]

\[\Box\]

**Lemma 8** Consider the ESBM. For every \( i \in [k] \), assume that the probability space \((\mu_i, B^m_{\alpha_j}(c_j))\) satisfies (a2). For every \( i \in [k] \), assume \( r_i < D_i \), let \( \alpha_i \in (r_i, D_i) \), let \( \tau_i > 0 \), and assume that \( c_i \) is the unique point that achieves \( \max\{G^\alpha(z) \mid z \in B^m_{\tau_i}(c_i)\} \). Then there exists \( \xi > 0 \) such that with high probability, for every \( \alpha' \in \mathbb{R}^k \) with \( \|\alpha' - \alpha\|_{\infty} \leq \xi \) and for every \( i \in [k] \), \( \arg\max\{C^\alpha(z) \mid z \in \{x_{\ell}^{(i)}\}_{\ell \in [n_i]}\} \subseteq \text{int} B^m_{\tau_i}(c_i) \).

**Proof** Since \( G^\alpha(z) \) is continuous in \( z \) according to Observation 3, and \( B^m_{\tau_i}(c_i) \) is compact, we know that for every \( i \in [k] \), \( \max\{G^\alpha(z) \mid z \in B^m_{\tau_i}(c_i) \} \text{ int} B^m_{\tau_i}(c_i) \) is achieved. Since, by assumption, for every \( i \in [k] \), \( c_i \) is the unique point that achieves \( \max\{G^\alpha(z) \mid z \in B^m_{\tau_i}(c_i) \} \), we obtain that \( G^\alpha(c_i) - \max\{G^\alpha(z) \mid z \in B^m_{\tau_i}(c_i) \} > 0, \forall i \in [k] \).

Let

\[
L := \min_{i \in [k]} \left\{ G^\alpha(c_i) - \max\{G^\alpha(z) \mid z \in B^m_{\tau_i}(c_i) \} \right\} > 0.
\]

Since for every \( i \in [k] \), \( G^\alpha(z) \) is continuous in \( z = c_i \), we know that for every \( i \in [k] \), there exists \( 0 < \tau_i' < \tau_i \) such that for every \( z \in B^m_{\tau_i}(c_i) \), we have \( G^\alpha(z) > G^\alpha(c_i) - L/2 \). Hence, for every \( z \in B^m_{\tau_i}(c_i) \), we have

\[
G^\alpha(z) - \max\{G^\alpha(z) \mid z \in B^m_{\tau_i}(c_i) \} \text{ int} B^m_{\tau_i}(c_i) > G^\alpha(c_i) - \frac{L}{2} - \max\{G^\alpha(z) \mid z \in B^m_{\tau_i}(c_i) \} \text{ int} B^m_{\tau_i}(c_i) \geq L - \frac{L}{2} = \frac{L}{2} > 0.
\]

(21)
Let \( \beta := \max_{i \in [k]} \beta_i \) and let \( \xi := L/(8k\beta) \). Notice that for every \( \alpha' \in \mathbb{R}^k \) with \( \|\alpha' - \alpha\|_{\infty} \leq \xi \) and for every \( z \in \mathbb{R}^m \), we have

\[
\left| \frac{1}{n} C^\alpha(z) - \frac{1}{n} C^\alpha(z') \right| = \frac{1}{n} \left| \sum_{i \in [k]} \sum_{\ell \in [n_i]} \left( (\alpha'_i - d(\ell, x^{(i)}_{\ell}))_+ - (\alpha_i - d(\ell, x^{(i)}_{\ell}))_+ \right) \right|
\leq \frac{1}{n} \sum_{i \in [k]} \sum_{\ell \in [n_i]} \left| (\alpha'_i - d(\ell, x^{(i)}_{\ell}))_+ - (\alpha_i - d(\ell, x^{(i)}_{\ell}))_+ \right|
\leq \frac{1}{n} \sum_{i \in [k]} \sum_{\ell \in [n_i]} |\alpha'_i - \alpha_i| \leq \frac{1}{n} \sum_{i \in [k]} \sum_{\ell \in [n_i]} \frac{L}{8k\beta} \leq \frac{L}{8}.
\]

(22)

For every \( i \in [k] \), denote by \( A_i \) the event that for every \( \alpha' \in \mathbb{R}^k \) with \( \|\alpha' - \alpha\|_{\infty} \leq \xi \), we have \( \text{argmax} \{ C^\alpha(z) \mid z \in \{ x^{(i)}_{\ell} \}_{\ell \in [n_i]} \} \subseteq \text{int} B^{m}_{\tau_i}(c_i) \). For every \( i \in [k] \), denote by \( T_i \) the event that there is some \( w \in [n_i] \) such that \( x^{(i)}_w \in B^{m}_{\tau_i}(c_i) \). For every \( i \in [k] \) and \( w \in [n_i] \), denote by \( M_{i,w} \) the event that at least one of the following event happens:

- \( x^{(i)}_w \in B^{m}_{\tau_i}(c_i) \) and \( \frac{1}{n} C^\alpha \left( x^{(i)}_w \right) - G^\alpha \left( x^{(i)}_w \right) \geq -L/8 \);
- \( x^{(i)}_w \in \text{int} B^{m}_{\tau_i}(c_i) \backslash B^{m}_{\tau_i}(c_i) \);
- \( x^{(i)}_w \in B^{m}_{\tau_i}(c_i) \backslash \text{int} B^{m}_{\tau_i}(c_i) \) and \( \frac{1}{n} C^\alpha \left( x^{(i)}_w \right) - G^\alpha \left( x^{(i)}_w \right) \leq L/8 \).

Note that for every \( i \in [k] \), if \( T_i \) is true and \( M_{i,w} \) is true for every \( w \in [n_i] \), then \( A_i \) is true. This is because \( B^{m}_{\tau_i}(c_i) \cap \{ x^{(i)}_w \}_{w \in [n_i]} \) is nonempty and, for every \( z \in B^{m}_{\tau_i}(c_i) \cap \{ x^{(i)}_w \}_{w \in [n_i]} \) and for every \( z' \in (B^{m}_{\tau_i}(c_i) \backslash \text{int} B^{m}_{\tau_i}(c_i)) \cap \{ x^{(i)}_w \}_{w \in [n_i]} \), we have \( C^\alpha(z) > C^\alpha(z') \) for every \( \alpha' \) with \( \|\alpha' - \alpha\|_{\infty} \leq \xi \). To see the last inequality, we use (22), the definition of the events \( M_{i,w} \), and (21) to obtain

\[
\frac{1}{n} C^\alpha(z) - \frac{1}{n} C^\alpha(z') = \left( \frac{1}{n} C^\alpha(z) - \frac{1}{n} C^\alpha(z) \right) + \left( \frac{1}{n} C^\alpha(z) - G^\alpha(z) \right)
- \left( \frac{1}{n} C^\alpha(z') - G^\alpha(z') \right)
+ \left( \frac{1}{n} C^\alpha(z') - \frac{1}{n} C^\alpha(z') \right) + (G^\alpha(z) - G^\alpha(z'))
> -\frac{L}{8} - \frac{L}{8} - \frac{L}{8} + \frac{L}{2} = 0.
\]

To prove the lemma we just need to show that the event \( \bigcap_{i \in [k]} A_i \) happens with high probability. In the remainder of the proof, we denote by \( \bar{E} \) the complement of an event \( E \). From the above discussion, we have that for every \( i \in [k] \),

\[
P(A_i) \geq P(T_i \cap \bigcap_{w \in [n_i]} M_{i,w}) = 1 - P(\bar{T}_i \cup \bigcup_{w \in [n_i]} \bar{M}_{i,w}) \geq 1 - P(\bar{T}_i).
\]
Hence, in the remainder of the proof we will provide a lower bound for \( \mathbb{P}(A_i) \) by providing upper bounds for \( \mathbb{P}(\bar{T}_i) \) and \( \mathbb{P}(\bar{M}_{iw}) \).

Let \( p := \max_{i \in [k]} \mathbb{P}(x^{(i)}_i \notin B^m_{\tau_i}(c_i)) \). Since for every \( i \in [k] \), the probability space \((\mu_i, B^m_{\tau_i}(c_i))\) satisfies (a2), we know that \( p \in (0, 1) \). So we get

\[
\mathbb{P}(\bar{T}_i) = \prod_{\ell \in [n_i]} \mathbb{P}(x^{(i)}_\ell \notin B^m_{\tau_i}(c_i)) \leq p^{n_i} \leq p^n. \tag{24}
\]

Next, we derive an upper bound for \( \mathbb{P}(\bar{M}_{iw}) \). We start by observing that for every \( i \in [k] \) and \( w \in [n_i] \), the event \( \bar{M}_{iw} \) is true if and only if at least one of the events \( P_{iw} \) and \( Q_{iw} \) happens, where the events \( P_{iw} \) and \( Q_{iw} \) are defined below.

\[
P_{iw}: x^{(i)}_w \in B^m_{\tau_i}(c_i) \text{ and } \frac{1}{n} C^a(x^{(i)}_w) - G^a(x^{(i)}_w) < -L/8;
\]

\[
Q_{iw}: x^{(i)}_w \in B^m_{\tau_i}(c_i) \backslash \text{int} B^m_{\tau_i}(c_i) \text{ and } \frac{1}{n} C^a(x^{(i)}_w) - G^a(x^{(i)}_w) > L/8.
\]

Next, we upper bound the probability of the event \( P_{iw} \). We notice that

\[
\mathbb{P}(P_{iw}) = \int_{B^m_{\tau_i}(c_i)} \mathbb{P}\left(\frac{1}{n} C^a(z) - G^a(z) < -\frac{L}{8} \mid x^{(i)}_w = z\right) d\mu_i(z).
\]

\[
\leq \sup \left\{ \mathbb{P}\left(\frac{1}{n} C^a(z) - G^a(z) < -\frac{L}{8} \mid x^{(i)}_w = z\right) \mid z \in B^m_{\tau_i}(c_i) \right\}. \tag{25}
\]

For every \( z \in \mathbb{R}^m \) and for every \( i \in [k] \), \( w \in [n_i] \), we define the random variable \( X_{iw}(z) := (\alpha_i - d(z, x^{(i)}_w))_+ \). We know that for every \( z \), \( X_{iw}(z) \) are independent random variables since \( x^{(i)}_w \) are independent. We then obtain

\[
C^a(z) = \sum_{i \in [k]} \sum_{w \in [n_i]} (\alpha_i - d(z, x^{(i)}_w))_+ = \sum_{i \in [k]} \sum_{w \in [n_i]} X_{iw}(z).
\]

Note that, if we fix \( z = x^{(i)}_w \), we can then rewrite \( C^a(z) \) in the form

\[
C^a(z) = \sum_{j \in [k] \backslash \{i\}} \sum_{\ell \in [n_w]} X_{j\ell}(z) + \sum_{\ell \in [n_i] \backslash \{w\}} X_{i\ell}(z) + \alpha_i. \tag{26}
\]

Also, we have

\[
\mathbb{E}\left(\sum_{j \in [k] \backslash \{i\}} \sum_{\ell \in [n_w]} X_{j\ell}(z) + \sum_{\ell \in [n_i] \backslash \{w\}} X_{i\ell}(z)\right) = \mathbb{E}(C^a(z) - X_{iw}(z))
\]

\[
= nG^a(z) - I(z), \tag{27}
\]

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where \( I(z) := \int_{B_m^m(z) \cap B_{\tau_i}^m(c_i)} (\alpha_i - d(z, x)) \, d\mu_i(x) \) and the last equality follows from the definition of \( G^\alpha(z) \) and using the same argument in the proof of Observation 2. We then define \( M := \max_{i \in [k]} \alpha_i \) and observe that \( X_{j\ell}(z) \in [0, M] \) for every \( j \in [k], \ell \in [n_j] \).

Let \( \beta := \max_{i \in [k]} \beta_i \). We obtain

\[
\mathbb{P}\left( \frac{1}{n} C_\alpha - G^\alpha(z) < -L \left| x_w^{(i)} = z \right. \right) = \mathbb{P}\left( \frac{1}{n} \left( \sum_{j \in [k] \setminus \{i\}} \sum_{\ell \in [n_j]} X_{j\ell}(z) + \sum_{\ell \in [n_i] \setminus \{w\}} X_{i\ell}(z) \right) - \left( G^\alpha(z) - \frac{1}{n} I(z) \right) < -L \left| x_w^{(i)} = z \right. \right)
\]

\[
= \mathbb{P}\left( \left( \sum_{j \in [k] \setminus \{i\}} \sum_{\ell \in [n_j]} X_{j\ell}(z) + \sum_{\ell \in [n_i] \setminus \{w\}} X_{i\ell}(z) \right) - \left( nG^\alpha(z) - I(z) \right) < \left( nL - 8I(z) + 8\alpha_i \right) \right)
\]

\[
\leq \exp\left( \frac{(nL - 8I(z) + 8\alpha_i)^2}{32M^2(\sum_{i \in [k]} n_i - 1)} \right) \leq \exp\left( \frac{(nL)^2}{32M^2(\sum_{i \in [k]} n_i - 1)} \right)
\]

\[
\leq \exp\left( \frac{nL^2}{32k\beta M^2} \right).
\]

The first equality follows from (26) and by adding on both sides \( \frac{1}{n} I(z) \). In the second equality, we use the fact that for every \( z \), \( X_{j\ell}(z) \) are independent. In the first inequality, we use Hoeffding’s inequality and (27). The second inequality holds because \( \alpha_i - I(z) \geq 0 \) and the last inequality follows by the definition of \( \beta \). Using (25), we obtain the following upper bound on the probability of the event \( P_{iw} \).

\[
\mathbb{P}(P_{iw}) \leq \sup \left\{ \mathbb{P}\left( \frac{1}{n} C_\alpha - G^\alpha(z) < -L \left| x_w^{(i)} = z \right. \right) \mid z \in B_{\tau_i}^m(c_i) \right\}
\]

\[
\leq \exp\left( \frac{nL^2}{32k\beta M^2} \right).
\]
In a similar fashion, we now obtain an upper bound on the probability of the event $Q_{iw}$. We have

$$
\mathbb{P}\left(\frac{1}{n} C^\alpha(z) - G^\alpha(z) > \frac{L}{8} \mid x_w^{(i)} = z\right)
\leq \mathbb{P}\left(\frac{1}{n} \left( \sum_{j \in [k] \setminus [i]} \sum_{\ell \in [n_\ell] \setminus [w]} X_{j\ell}(z) + \sum_{\ell \in [n_\ell] \setminus [w]} X_{i\ell}(z) \right) - (G^\alpha(z) - \frac{1}{n} I(z)) > \frac{L}{8} + \frac{1}{n} I(z) - \frac{\alpha_i}{n} \mid x_w^{(i)} = z\right)
\leq \mathbb{P}\left(\left( \sum_{j \in [k] \setminus [i]} \sum_{\ell \in [n\ell]} X_{j\ell}(z) + \sum_{\ell \in [n_\ell] \setminus [w]} X_{i\ell}(z) \right) - (nG^\alpha(z) - I(z)) > \frac{nL}{8} + I(z) - \alpha_i\right)
\leq \exp\left(-\frac{(nL + 8I(z) - 8\alpha_i)^2}{32M^2(\sum_{i \in [k]} n_i - 1)}\right) \leq \exp\left(-\frac{(nL - 8\alpha_i)^2}{32M^2(\sum_{i \in [k]} n_i - 1)}\right)
\leq \exp\left(-\frac{(nL - 8\alpha_i)^2}{32k\beta n M^2}\right),
$$

where the second inequality holds because $I(z) \geq 0$. We obtain the following upper bound on the probability of the event $Q_{iw}$.

$$
\mathbb{P}(Q_{iw}) \leq \sup \left\{ \mathbb{P}\left(\frac{1}{n} C^\alpha(z) - G^\alpha(z) > \frac{L}{8} \mid x_w^{(i)} = z\right) \mid z \in B^m_{r_1}(c_i) \setminus \text{int } B^m_{n_1}(c_i) \right\}
\leq \exp\left(-\frac{(nL - 8\alpha_i)^2}{32k\beta n M^2}\right),
$$

where the last inequality holds when $n > 16\alpha_{\max} / ((2 - \sqrt{2})L)$, and $\alpha_{\max} := \max_{i \in [k]} \alpha_i$.

Using the union bound, when $n > 16\alpha_{\max} / ((2 - \sqrt{2})L)$, we have

$$
\mathbb{P}(\tilde{M}_{iw}) \leq \mathbb{P}(P_{iw}) + \mathbb{P}(Q_{iw}) \leq \exp\left(-\frac{nL^2}{32k\beta M^2}\right) + \exp\left(-\frac{nL^2}{64k\beta M^2}\right)
\leq 2 \exp\left(-\frac{nL^2}{64k\beta M^2}\right).
$$
Using (23) and (24), when \( n > 16\alpha_{\text{max}} / ((2 - \sqrt{2})L) \), we have

\[
\mathbb{P}(A_i) \geq 1 - p^n - 2n_i \exp \left( - \frac{nL^2}{64k\beta M^2} \right) \geq 1 - p^n - 2\beta n \exp \left( - \frac{nL^2}{64k\beta M^2} \right).
\]

The latter quantity goes to 1 as \( n \) goes to infinity because \( p < 1 \) and \( p, k, \beta, L, M \) are all parameters that do not depend on \( n \). Hence each event \( A_i \), for \( i \in [k] \), happens with high probability. Therefore, also \( \bigcap_{i \in [k]} A_i \) happens with high probability. So with high probability, for every \( \alpha' \) with \( \| \alpha' - \alpha \|_\infty \leq \xi \) and for every \( i \in [k] \), we have that \( \arg\max \{ C^{\alpha'}(z) \mid z \in \{ x^{(i)}_\ell \}_{\ell \in [n_i]} \} \subseteq \text{int} \ B^{m}_{r_i} (c_i) \).

We are now ready to prove Theorem 2. In the proof we use Theorem 1 and Lemmas 1 and 4 to 8.

### 4.3 Proof of Theorem 2

In this proof, for every \( i \in [k] \), we denote by \( x^{(i)}_* \) a median of \( \{ x^{(i)}_\ell \}_{\ell \in [n_i]} \). Let \((\bar{y}, \bar{z})\) be the feasible solution to \((\text{IP})\) that assigns each point \( x^{(i)}_\ell \) to the ball \( B^{m}_{r_i} (c_i) \) from which it is drawn. In particular, in this solution we have \( y_p = 1 \) if and only if \( p \in \{ x^{(i)}_* \mid i \in [k] \} \). Furthermore, we have \( z_{pq} = 1 \) if and only if \( y_p = 1 \) and \( p, q \) are drawn from the same ball.

To prove the theorem, we show that \((\bar{y}, \bar{z})\) is the unique optimal solution to \((\text{LP})\) with high probability. Clearly, \((\bar{y}, \bar{z})\) is a feasible solution to \((\text{IP})\). We know from Theorem 1 that \((\bar{y}, \bar{z})\) is the unique optimal solution to \((\text{LP})\) if there exists \( \bar{a} \in \mathbb{R}^p \) such that

\[
C^{\bar{a}}(a_1) = \cdots = C^{\bar{a}}(a_k)
\]

(28)

\[
C^{\bar{a}}(q) < C^{\bar{a}}(a_1)
\]

(29)

\[
\bar{a}_q \geq d(a_i, q)
\]

(30)

\[
\bar{a}_q < d(a_i, q)
\]

(31)

Let \( \gamma, \alpha \) as in the statement of Theorem 2. For all \( i \in [k] \), we obtain \( r_i < D_i \), and using the definition of \( a_i \), we obtain \( a_i \in (r_i, D_i) \). Hence there exists \( \xi_1 > 0 \) such that \([a_i - \xi_1, a_i + \xi_1] \subseteq (r_i, D_i) \) for all \( i \in [k] \). From Lemma 6 with \( a_i = a_i - \xi_1 \) and \( b_i = a_i + \xi_1 \), we obtain that there exist \( \tau_i > 0, \forall i \in [k] \), such that \( \forall i, j \in [k] \), \( \forall z \in \text{int} B^{m}_i (c_i) \), and \( \forall \alpha' \in [\alpha_j - \xi_1, \alpha_j + \xi_1] \), we have

\[
B^{m}_j (z) \cap B^{m}_r (c_j) = \begin{cases} B^{m}_r (c_i) & \text{if } j = i \\ \emptyset & \text{otherwise.} \end{cases}
\]

(32)

Since the assumptions of Lemma 8 are satisfied, there exists \( \xi_2 > 0 \) such that with high probability, for every \( \alpha' \in \mathbb{R}^k \) with \( \| \alpha' - \alpha \|_\infty \leq \xi_2 \) and for every \( i \in [k] \), \( \arg\max \{ C^{\alpha'}(z) \mid z \in \{ x^{(i)}_\ell \}_{\ell \in [n_i]} \} \subseteq \text{int} B^{m}_{\epsilon_i} (c_i) \). Let \( \xi := \min \{ \xi_1, \xi_2 \} \). For every...
\( i \in [k] \), let \( \OPT_i := \sum_{\ell \in [n_i]} d(x_*^{(i)}, x_*^{(i)}) \). We then know from Lemma 5 that with high probability, for every \( i \in [k] \), we have \( |\OPT_i / n_i - E| < \xi \). From Lemma 4, we know that with high probability, for every \( i \in [k] \), we have \( x_*^{(i)} \in \text{int} B_{r_i}^m(c_i) \).

For every \( i \in [k] \), fix \( \alpha_i := \alpha_i + \epsilon_i \), where \( \epsilon_i := \OPT_i / n_i - E_i \). For every \( q \in P \), we set \( \tilde{\alpha}_q := \alpha_i \), where \( i \) is the unique index in \( [k] \) with \( q \in B_{r_i}^m(c_i) \). We next show that, with this choice of \( \tilde{\alpha} \), (28)–(31) are satisfied with high probability. Using the definition of \( \tilde{\alpha} \), it suffices to show that

\[
\begin{align*}
C^{\alpha'}(x_*^{(i)}) &= \cdots = C^{\alpha'}(x_*^{(k)}) \quad (33) \\
C^{\alpha'}(x_*^{(i)}) &< C^{\alpha'}(x_*^{(i)}) \quad \forall i \in [k], \forall \ell \in [n_i] \text{ with } x_*^{(i)} \neq x_*^{(i)} \quad (34) \\
\alpha_i' &\geq d(x_*^{(i)}, x_*^{(i)}) \quad \forall i \in [k], \forall \ell \in [n_i] \quad (35) \\
\alpha_i' &< d(x_*^{(i)}, x_*^{(i)}) \quad \forall i, j \in [k], i \neq j, \forall \ell \in [n_i]. \quad (36)
\end{align*}
\]

Since for every \( i \in [k] \), we have \( |\epsilon_i| < \xi \), we know that \( \alpha_i' \in (r_i, D_i) \). Since for every \( i \in [k] \), we have \( x_*^{(i)} \in \text{int} B_{r_i}^m(c_i) \), we have that (32) holds with \( z = x_*^{(i)} \). Thus from Lemma 7 (with \( s_i = x_*^{(i)} \)) we obtain (35), (36), and

\[
\begin{align*}
C^{\alpha'}(x_*^{(i)}) &= n_i \alpha_i' - \sum_{\ell \in [n_i]} d(x_*^{(i)}, x_*^{(i)}) = n_i (\alpha_i + \epsilon_i) - \OPT_i \\
&= n_i \left( \frac{\gamma}{\beta_i} + \OPT_i \right) - \OPT_i = \gamma n + \OPT_i - \OPT_i = \gamma n \quad \forall i \in [k].
\end{align*}
\]

which implies (33). For every \( i \in [k] \), let \( s_i \in \text{int} B_{r_i}^m(c_i) \cap \{x_*^{(i)}\}_{\ell \in [n_i]} \). Since \( s_i \in \text{int} B_{r_i}^m(c_i) \), for every \( i, j \in [k] \), we have that \( z = s_i \) satisfies (32). From Lemma 7 we obtain

\[
C^{\alpha'}(s_i) = n_i \alpha_i' - \sum_{\ell \in [n_i]} d(s_i, x_*^{(i)}) \leq n_i \alpha_i' - \sum_{\ell \in [n_i]} d(x_*^{(i)}, x_*^{(i)}) = C^{\alpha'}(x_*^{(i)}) \forall i \in [k].
\]

Since from Lemma 1, the vector \( x_*^{(i)} \) is the unique median of \( \{x_*^{(i)}\}_{\ell \in [n_i]} \) with probability one when \( n_i \geq 3 \), the above inequality achieves equality if and only if \( s_i = x_*^{(i)} \). Since \( \|\alpha' - \alpha\|_{\infty} = \|\epsilon\|_{\infty} < \xi \leq \xi_2 \), for every \( i \in [k] \) we have \( \arg\max\{C^{\alpha'}(z) \mid z \in \{x_*^{(i)}\}_{\ell \in [n_i]}\} \subseteq \text{int} B_{r_i}^m(c_i) \). Thus, we know that \( x_*^{(i)} \) is the unique point that achieves \( \max\{C^{\alpha'}(z) \mid z \in \{x_*^{(i)}\}_{\ell \in [n_i]}\} \). This implies (34).

Hence \( (\tilde{y}, \tilde{z}) \) is the unique optimal solution to (LP) with high probability.

\[\square\]

### 4.4 Proof of Corollary 1

Let \( \gamma := \alpha' - E_1 \). Since for every \( i \in [k] \), \( \beta_i = 1, r_i = 1 \), and \( E_1 = \cdots = E_k \), we have

\[\max_{i \in [k]} \beta_i (r_i - E_i) = 1 - E_1 \text{ and } \min_{i \in [k]} \beta_i (D_i - E_i) = \min_{i \neq j} d(c_i, c_j) - 1 - E_1.\]
From $1 < \alpha' < \min_{i \neq j} d(c_i, c_j) - 1$ we then obtain $\max_{i \in [k]} \beta_i (r_i - E_i) < \gamma < \min_{i \in [k]} \beta_i (D_i - E_i)$. Notice that for every $i \in [k]$ we have

$$\alpha_i = \alpha' = E_i + \gamma = E_i + \frac{\gamma}{\beta_i}.$$  

Since for $i \in [k]$, $c_i$ is the unique point that achieves $\max\{G^\alpha(z) \mid z \in B_{r_i}(c_i)\}$, Theorem 2 implies that (LP) achieves exact recovery with high probability. \hfill \Box

## 5 Exact recovery in the ESBM

In this section, we present our recovery results for the ESBM. We also show that for the ESBM with some special structure, including the SBM, (LP) can perform even better.

For completeness, we start with the simple case $m = 1$. The next two theorems can be seen as corollaries of Theorem 2.

**Theorem 3** Consider the ESBM with $m = 1$. For every $i \in [k]$, assume that the probability space $(\mu_i, B_{r_i}(c_i))$ satisfies (a1), (a2), (a3). Let $R := \max_{i \in [k]} r_i$ and $\beta := \max_{i \in [k]} \beta_i$. If for every $i \neq j$ we have $d(c_i, c_j) > r_i + r_j + (1 + 2\beta)R$, then (LP) achieves exact recovery with high probability.

**Proof** It suffices to check that all assumptions of Theorem 2 are satisfied. For every $i \in [k]$, denote by $E_i := \mathbb{E}d(x, c_i)$, where $x$ is a random vector drawn according to $\mu_i$. Let $\gamma := \max_{i \in [k]} \beta_i (2r_i - E_i)$. Using the fact that $r_i \in (0, R]$, $\beta_i \in [1, \beta]$, and $E_i \in (0, R]$, we can bound $\gamma$ and obtain

$$\max_{i \in [k]} \beta_i (r_i - E_i) < \gamma < 2\beta R < \min_{j \in [k]} r_j + (1 + 2\beta)R - R < \min_{i \neq j} d(c_i, c_j) - r_i - R \leq \min_{i \in [k]} \beta_i (D_i - E_i).$$

Let $\alpha_i := E_i + \frac{\gamma}{\beta_i}$ for every $i \in [k]$. It remains to show that for every $i \in [k]$, $c_i$ is the unique point that achieves $\max\{G^\alpha(z) \mid z \in B_{r_i}(c_i)\}$. For every $i \in [k]$, from the definition of $\gamma$ we have $\alpha_i \geq 2r_i$, thus for every $z \in B_{r_i}(c_i)$, we have $B_{\alpha_i}(z) \cap B_{r_i}(c_i) = B_{r_i}(c_i)$. For every $i \in [k]$ we have $\alpha_i \leq R + \gamma < (1 + 2\beta)R$ which implies $B_{\alpha_i}(z) \cap B_{r_i}(c_i) = \emptyset$ for every $z \in B_{r_i}(c_i)$ and every $j \in [k]\{i\}$. From Observation 2, we know that for every $i \in [k]$ and for every $z \in B_{r_i}(c_i)$ with $z \neq c_i$, we have

$$G^\alpha(z) = \int_{-r_i}^{r_i} (\alpha_i - d(z, x))d\mu_i(x) = \alpha_i - \mathbb{E}d(z, x) < \alpha_i - \mathbb{E}d(c_i, x) = G^\alpha(c_i),$$

where the inequality follows from Lemma 2.

The assumptions of Theorem 2 are satisfied, and so (LP) achieves exact recovery with high probability. \hfill \Box
Theorem 4 Consider the ESBM with \( m = 1 \). For every \( i \in [k] \), assume that the probability space \((\mu_i, B^m_{r_i}(c_i))\) satisfies (a1), (a2), (a3). For every \( i \in [k] \), assume \( n_i = n, r_i = 1 \), and denote by \( E_i := \mathbb{E}d(x, c_i) \), where \( x \) is a random vector drawn according to \( \mu_i \). We further assume \( E_1 = \cdots = E_k \). If for every \( i \neq j \) we have \( d(c_i, c_j) > 2 + 2 \), then (LP) achieves exact recovery with high probability.

Proof It suffices to check that all assumptions of Corollary 1 are satisfied. Let \( \Theta := \min_{i \neq j} d(c_i, c_j) - 2 > 2 \), let \( \epsilon \in (0, \Theta - 2) \), and let \( \alpha' := 2 + \epsilon \). Note that we have \( 2 < \alpha' < \Theta \), which in particular implies

\[
1 < \alpha' < \min_{i \neq j} d(c_i, c_j) - 1.
\]

Let \( \alpha_i := \alpha' \) for every \( i \in [k] \). It remains to show that for every \( i \in [k] \), \( c_i \) is the unique point that achieves \( \max\{G^\alpha(z) \mid z \in B^m_{r_i}(c_i)\} \). For every \( i \in [k] \) and for every \( z \in B^m_{r_i}(c_i) \), \( \alpha' > 2 \) implies \( B^m_{r_i}(z) \cap B^m_{r_i}(c_i) = B^m_{r_i}(c_i) \) and \( \alpha' \sim \Theta \) implies \( B^m_{r_i}(z) \cap B^m_{r_i}(c_j) = \emptyset \) for every \( j \in [k] \setminus \{i\} \). From Observation 2, we know that for every \( i \in [k] \) and for every \( z \in B^m_{r_i}(c_i) \) with \( z \neq c_i \), we have

\[
G^\alpha(z) = \int_{-1}^1 (\alpha_i - d(z, x))d\mu_i(x) = \alpha_i - \mathbb{E}d(z, x) < \alpha_i - \mathbb{E}d(c_i, x) = G^\alpha(c_i),
\]

where the inequality follows from Lemma 2.

The assumptions of Corollary 1 are satisfied, and so (LP) achieves exact recovery with high probability.

Theorem 4 implies that in the SBM with \( m = 1 \), a sufficient condition for (LP) to achieve exact recovery is that the distance between any pair of points from the same ball is always smaller than the distance between any pair of points from different balls. We remark that under this assumption a simple threshold algorithm can also achieve exact recovery. It is currently unknown if in the SBM with \( m = 1 \) a pairwise distance smaller than 4 may be sufficient to guarantee exact recovery.

Next, we present our most interesting results, which consider exact recovery for the ESBM and the SBM with \( m \geq 2 \).

Theorem 5 Consider the ESBM with \( m \geq 2 \). For every \( i \in [k] \), assume that the probability space \((\mu_i, B^m_{r_i}(c_i))\) satisfies (a1), (a2), (a3). Let \( \beta, r, R \in \mathbb{R} \) such that for every \( i \in [k] \) we have \( r_i \in [r, R] \) and \( \beta_i \leq \beta \). Then there is a function \( \epsilon(k, m) = C\sqrt{k \log m/m} \), where \( C \) is a positive constant, such that, if for every \( i \neq j \) we have \( d(c_i, c_j) > (1 + \beta)R + \max\{r_i, r_j\} + \epsilon(k, m) \), then (LP) achieves exact recovery with high probability.

Theorem 6 Consider the ESBM with \( m \geq 2 \). For every \( i \in [k] \), assume that the probability space \((\mu_i, B^m_{r_i}(c_i))\) satisfies (a1), (a2), (a3). For every \( i \in [k] \), assume \( n_i = n, r_i = 1 \), and denote by \( E_i := \mathbb{E}d(x, c_i) \), where \( x \) is a random vector drawn according to \( \mu_i \). We further assume \( E_1 = \cdots = E_k \). If for every \( i \neq j \) we have \( d(c_i, c_j) > 2 + 1.29 \), then (LP) achieves exact recovery with high probability.
Theorem 7 Consider the ESBM with $m \geq 2$. For every $i \in [k]$, assume that the probability space $(\mu_i, B_m(c_i))$ satisfies (a1), (a2), (a3). For every $i \in [k]$, assume $n_i = n$, $r_i = 1$, and denote by $E_i := \mathbb{E}d(x, c_i)$, where $x$ is a random vector drawn according to $\mu_i$. We further assume $E_1 = \cdots = E_k$. Then there is a function $\epsilon(k, m) = C \sqrt{k \log m/m}$, where $C$ is a positive constant, such that, if for every $i \neq j$ we have $d(c_i, c_j) > 2 + \epsilon(k, m)$, then (LP) achieves exact recovery with high probability.

Theorem 8 Consider the SBM with $m \geq 2$. Assume that the probability space $(\mu, B_1^m(0))$ satisfies (a1), (a2), (a3). Assume that $\mu$ has a density function $p(x)$ and assume that, for $x_1, x_2 \in B_1^m(0)$ with $\|x_1\| < \|x_2\|$ we have $p(x_1) > p(x_2)$. If for every $i \neq j$ we have $d(c_i, c_j) > 2$, then (LP) achieves exact recovery with high probability.

For the SBM, Theorem 6 gives the best known sufficient condition for exact recovery, which does not depend on $k$ or $m$. Furthermore, if $k$ does not grow fast, Theorem 7 gives a near optimal condition for exact recovery in high dimension. Furthermore, Theorem 8 corrects the result in [9] by adding assumptions on the probability measure. Beyond the SBM, Theorem 5 shows that if we consider the much more general ESBM, exact recovery still happens, as long as the numbers of points drawn from each ball have the same order. We already discussed in Sect. 3 that this assumption is necessary and cannot be dropped (see Example 1 in “Appendix A”).

The remainder of the section is devoted to proving Theorems 5 to 8.

5.1 Analysis of $G^\alpha(z)$

According to Sect. 4, we know that exact recovery is closely related to the function $G^\alpha(z)$. In this section, we present an in-depth study of the function $G^\alpha(z)$. These results on $G^\alpha(z)$ will be used to prove our exact recovery results in dimension $m \geq 2$, i.e., Theorems 5 to 8. This is why in several results in this section we assume $m \geq 2$.

5.1.1 The random variable $\theta$

In the following, for $r \geq 0$, we denote by $\mu^r$ the uniform probability measure with support $S_{m-1}^r(0)$. Let $v$ be a fixed unit vector in $\mathbb{R}^m$ and let $x$ be a random vector in $\mathbb{R}^m$ drawn according to $\mu^1$. We define the random variable $\theta(x)$ to be the angle between $v$ and $x$. Since both $v$ and $x$ are unit vectors we can write

$$\theta(x) := \arccos \langle v, x \rangle \in [0, \pi].$$

We can then use $\tilde{\mu}(\theta)$ to denote the probability measure of $\theta$. In the next observation we show that the probability measure $\tilde{\mu}(\theta)$ also arises from probability measures more general than $\mu^1$. We recall that two random variables $A, B$ have the same probability measure if for every $\psi \in \mathbb{R}$, we have $\mathbb{P}(A \leq \psi) = \mathbb{P}(B \leq \psi)$.

Observation 4 Let $(\mu, B_m^m(0))$ be a probability space that satisfies (a1), (a3). Let $v$ be a fixed unit vector in $\mathbb{R}^m$, and let $x$ be a random vector in $\mathbb{R}^m$ drawn according to $\mu$. 

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We define the random variable $\theta'(x)$ to be the angle between $v$ and $x$ if $x \neq 0$, and $\theta'(x) := \pi/2$ if $x = 0$. Then $\theta'$ has the same probability measure as $\theta$.

**Proof** Since $v$ is a unit vector we have for every $x \neq 0$,

$$\theta'(x) = \arccos \left( \frac{v \cdot x}{\|x\|} \right) \in [0, \pi].$$

If $x = 0$, we have $\theta'(0) = \pi/2$.

Since $(\mu, B^m_r(0))$ satisfies (a3), we have $P(x = 0) = 0$. Since $(\mu, B^m_r(0))$ satisfies (a1), we have that, for $x \neq 0$, $x / \|x\|$ is a random vector drawn according to $\mu_1$.

So for every $\psi \in [0, \pi]$, we have

$$P(\theta'(x) \leq \psi) = P(\theta'(x) \leq \psi, x \neq 0) + P(\theta'(x) \leq \psi, x = 0) = P(\theta'(x) \leq \psi | x \neq 0) P(x \neq 0) = P(\theta(x) \leq \psi).$$

We then obtain that $\theta'$ and $\theta$ have the same probability measure. \qed

When $m \geq 2$ we note that the random variable $\theta$ has a density function and we denote it by $p^{(m)}(\theta) := d\tilde{\mu} / d\theta$. In the remainder of this section we study the density function $p^{(m)}(\theta)$ thus we always assume $m \geq 2$. In the following, we denote by $\Gamma(x)$ the gamma function.

**Observation 5** Let $m \geq 2$. We have

$$p^{(m)}(\theta) = \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m-1}{2}\right)} \sin^{m-2}\theta.$$ 

**Proof** Let $\psi$ be a fixed angle in $[0, \pi]$. We know that those $x \in S^{m-1}_1(0)$ such that $\theta(x) = \psi$ form a $m - 2$ dimensional sphere in $\mathbb{R}^m$ centered at $v \cos \psi$ with radius $\sin \psi$, which we denote by $S^{m-2}_{\sin \psi}(v \cos \psi)$. Formally, we define

$$S^{m-2}_{\sin \psi}(v \cos \psi) := \left\{ x \in S^{m-1}_1(0) \mid \theta(x) = \psi \right\}.$$

In the following we denote by $\lambda^{m-1}(\cdot)$ the $(m-1)$-dimensional volume and by $\lambda^{m-2}(\cdot)$ the $(m-2)$-dimensional volume. Then

$$\lambda^{m-1}\left(\{x \in S^{m-1}_1(0) \mid \theta(x) \leq \psi\}\right) = \int_0^\psi \lambda^{m-2}(S^{m-2}_{\sin \theta}(v \cos \theta)) d\theta = \lambda^{m-2}(S^{m-2}_1(0)) \int_0^\psi \sin^{m-2}\theta d\theta.$$

In particular,

$$\lambda^{m-1}(S^{m-1}_1(0)) = \lambda^{m-1}\left(\{x \in S^{m-1}_1(0) \mid \theta(x) \leq \pi\}\right) = \lambda^{m-2}(S^{m-2}_1(0)) \int_0^\pi \sin^{m-2}\theta d\theta.$$
\[
\frac{\lambda^{m-2} \left( S_{1}^{m-2}(0) \right) \sqrt{\pi} \Gamma \left( \frac{m-1}{2} \right)}{\Gamma \left( \frac{m+1}{2} \right)}.
\]

Since \( x \) is drawn uniformly from \( S_{1}^{m-1}(0) \), we know that
\[
P(\theta \leq \psi) = \frac{\lambda^{m-1} \left( \{ x \in S_{1}^{m-1}(0) \mid \theta(x) \leq \psi \} \right)}{\lambda^{m-1} \left( S_{1}^{m-1}(0) \right)} = \frac{1}{\sqrt{\pi} \Gamma \left( \frac{m-1}{2} \right)} \int_{0}^{\psi} \sin^{m-2} \theta \, d\theta.
\]

Thus, we obtain
\[
p^{(m)}(\theta) = \frac{1}{\sqrt{\pi} \Gamma \left( \frac{m-1}{2} \right)} \sin^{m-2} \theta.
\]

\[\square\]

**Observation 6** Let \( m \geq 2 \). Then there exists a threshold \( s_{m} \in (0, 1) \) such that
\[
p^{(m)}(\theta) - p^{(m+1)}(\theta) \begin{cases} 
\geq 0 & \text{if } 0 \leq \sin \theta \leq s_{m}, \\
< 0 & \text{if } s_{m} < \sin \theta \leq 1.
\end{cases}
\]

**Proof** Using Observation 5 we can write
\[
p^{(m)}(\theta) - p^{(m+1)}(\theta) = \frac{1}{\sqrt{\pi} \Gamma \left( \frac{m-1}{2} \right)} \sin^{m-2} \theta - \frac{1}{\sqrt{\pi} \Gamma \left( \frac{m+1}{2} \right)} \sin^{m-1} \theta
\]
\[
= \left( \frac{\Gamma \left( \frac{m}{2} \right)}{\Gamma \left( \frac{m-1}{2} \right)} - \frac{\Gamma \left( \frac{m+1}{2} \right)}{\Gamma \left( \frac{m}{2} \right)} \sin \theta \right) \frac{1}{\sqrt{\pi} \Gamma \left( \frac{m+1}{2} \right)} \sin^{m-2} \theta.
\]

We set
\[
s_{m} := \frac{\Gamma \left( \frac{m}{2} \right)^{2}}{\Gamma \left( \frac{m-1}{2} \right) \Gamma \left( \frac{m+1}{2} \right)}.
\]

Since \( \Gamma(x) \) is a positive strictly logarithmically convex function for \( x \in (0, \infty) \), we have \( s_{m} \in (0, 1) \). We note that if \( \sin \theta = s_{m} \), then \( p^{(m)}(\theta) - p^{(m+1)}(\theta) = 0 \). Since the gamma function is positive, when \( 0 \leq \sin \theta < s_{m} \), from (37) we obtain
\[
p^{(m)}(\theta) - p^{(m+1)}(\theta) \geq 0.
\]

On the other hand, when \( s_{m} < \sin \theta \leq 1 \), from (37) we obtain
\[
p^{(m)}(\theta) - p^{(m+1)}(\theta) < 0.
\]

\[\square\]
Observation 7 Let $m \geq 2$ and let $\bar{\theta} \leq \frac{\pi}{2}$. Let $g(\theta)$ be a nonnegative decreasing function on $(0, \bar{\theta})$. Then we have $\int_0^{\bar{\theta}} g(\theta)(p^{(m)}(\theta) - p^{(m+1)}(\theta))d\theta \geq 0$.

Proof Let $s_m \in (0, 1)$ be the threshold for $p^{(m)} - p^{(m+1)}$ from Observation 6. Let $\psi \in (0, \pi/2)$ such that $\sin \psi = s_m$. We then have

$$p^{(m)}(\theta) - p^{(m+1)}(\theta) \begin{cases} \geq 0 & \text{if } 0 \leq \theta \leq \psi, \\ < 0 & \text{if } \psi < \theta \leq \pi/2. \end{cases}$$

We consider separately two cases. In the first case we assume $\bar{\theta} \leq \psi$. Since $p^{(m)}(\theta) - p^{(m+1)}(\theta) \geq 0$ when $\theta \in (0, \bar{\theta})$ and since $g(\theta)$ is a nonnegative decreasing function on $(0, \bar{\theta})$, we obtain

$$\int_0^{\bar{\theta}} g(\theta)(p^{(m)}(\theta) - p^{(m+1)}(\theta))d\theta \geq g(\bar{\theta}) \int_0^{\bar{\theta}} (p^{(m)}(\theta) - p^{(m+1)}(\theta))d\theta \geq 0.$$  

In the second case we assume $\psi < \bar{\theta} \leq \frac{\pi}{2}$. We have

$$\int_0^{\bar{\theta}} g(\theta)(p^{(m)}(\theta) - p^{(m+1)}(\theta))d\theta = \int_0^{\psi} g(\theta)(p^{(m)}(\theta) - p^{(m+1)}(\theta))d\theta + \int_{\psi}^{\bar{\theta}} g(\theta)(p^{(m)}(\theta) - p^{(m+1)}(\theta))d\theta \geq g(\psi) \int_0^{\psi} (p^{(m)}(\theta) - p^{(m+1)}(\theta))d\theta + g(\psi) \int_{\psi}^{\bar{\theta}} (p^{(m)}(\theta) - p^{(m+1)}(\theta))d\theta = g(\psi) \int_0^{\bar{\theta}} (p^{(m)}(\theta) - p^{(m+1)}(\theta))d\theta = -g(\psi) \int_{\bar{\theta}}^{\frac{\pi}{2}} (p^{(m)}(\theta) - p^{(m+1)}(\theta))d\theta \geq 0.$$  

The last equality follows from the fact that $\int_0^{\frac{\pi}{2}} p^{(m)}(\theta)d\theta = \int_0^{\frac{\pi}{2}} p^{(m+1)}(\theta)d\theta = \frac{1}{2}$.

The last inequality uses the fact that $g(\psi) \geq 0$ and $\int_{\bar{\theta}}^{\frac{\pi}{2}} (p^{(m)}(\theta) - p^{(m+1)}(\theta))d\theta \leq 0$. 

Lemma 9 Let $m \geq 2$ and let $[\phi_1, \phi_2] \subseteq [0, \pi]$ such that $\frac{\pi}{2} \notin [\phi_1, \phi_2]$. Denote by $\phi$ an angle $\theta \in [\phi_1, \phi_2]$ for which $\sin \theta$ is the largest. Then $\mathbb{P}(\theta \in [\phi_1, \phi_2]) < \frac{\sqrt{\pi}}{2} \sqrt{\frac{m}{\pi}} \sin^{m-2} \phi$.

Proof Gautschi’s inequality implies that for every $x > 0$ and every $s \in (0, 1)$, the following inequality holds

$$\frac{\Gamma(x+1)}{\Gamma(x+s)} < (x+1)^{1-s}.$$
We apply Gautschi’s inequality with $x = m/2 - 1$, for $m \geq 3$, and $s = 1/2$. Hence, when $m \geq 3$ we have

$$\frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m-1}{2}\right)} < \sqrt{\frac{m}{2}}.$$  

Notice that the above inequality also holds for $m = 2$ since

$$\frac{\Gamma(1)}{\Gamma\left(\frac{1}{2}\right)} = \frac{1}{\sqrt{\pi}} < 1 = \sqrt{\frac{2}{2}}.$$

Using Observation 5 we obtain

$$\mathbb{P}(\theta \in [\phi_1, \phi_2]) = \int_{\phi_1}^{\phi_2} p^{(m)}(\theta) d\theta = \int_{\phi_1}^{\phi_2} \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m-1}{2}\right)} \sin^{m-2} \theta d\theta$$

$$\leq \frac{1}{\sqrt{\pi}} \sqrt{\frac{m}{2}} (\phi_2 - \phi_1) \sin^{m-2} \phi < \frac{\sqrt{\pi}}{2} \sqrt{\frac{m}{2}} \sin^{m-2} \phi,$$

where the last inequality holds because $\phi_2 - \phi_1 < \frac{\pi}{2}$.  

5.1.2 Three functions related to $G^\alpha(z)$

According to Observation 2, we have

$$G^\alpha(z) = \sum_{i \in [k]} \beta_i \int_{B_{m_i}^r(z) \cap B_{m_i}^r(c_i)} (\alpha_i - d(z, x)) d\mu_i(x).$$

Then, for every vector $z \in \mathbb{R}^m$, the function $G^\alpha(z)$ can be seen as the sum of the contributions that $z$ gets from each singular ball $B_{m_i}^r(c_i)$. Motivated by this observation, in this section we will analyze $G^\alpha(z)$ by defining three new functions. The first function can be seen as $G^\alpha(c_1) - G^\alpha(z)$ for $z \in B_{m_1}^r(c_1)$, in the case $k = 1$, $c_1 = 0$, $\alpha_1 > r_1$, and $\beta_1 = 1$.

**Definition 8** Let $(\mu, B_r^m(0))$ be a probability space that satisfies (a1) and let $\alpha > r$. We define the function $H^{(\alpha, \mu, m)}(z) : B_r^m(0) \rightarrow \mathbb{R}$ as

$$H^{(\alpha, \mu, m)}(z) := \int_{B_r^m(0)} (\alpha - \|x\|) d\mu(x) - \int_{B_m^r(z) \cap B_r^m(0)} (\alpha - d(z, x)) d\mu(x).$$

The second function is the special case of $H^{(\alpha, \mu, m)}$ where $\mu = \mu^r$.

**Definition 9** Let $r, \alpha \in \mathbb{R}_+$ with $\alpha > r$. We define the function $T^{(\alpha, m)}(z) : B_r^m(0) \rightarrow \mathbb{R}$ as

$$T^{(\alpha, m)}(z) := \int_{B_r^m(0)} (\alpha - \|x\|) d\mu^r(x) - \int_{B_m^r(z) \cap B_r^m(0)} (\alpha - d(z, x)) d\mu^r(x).$$
The third function can be seen as the part of \(G^\alpha(z)\) for \(z \notin B_r^m(c_1)\), coming from the ball 1, in the case \(c_1 = 0, \alpha_1 \geq r_1, \beta_1 = 1\).

**Definition 10** Let \((\mu, B_r^m(0))\) be a probability space that satisfies (a1) and let \(\alpha > r\). We define the function \(R^{(\alpha, \mu, m)}(z) : \mathbb{R}^m \setminus B_r^m(0) \rightarrow \mathbb{R}\) as

\[
R^{(\alpha, \mu, m)}(z) := \int_{B_r^m(z) \cap B_r^m(0)} (\alpha - d(z, x)) d\mu(x).
\]

Since the probability measures considered in Definitions 8–10 are invariant under rotations centered in the origin, we obtain that \(H^{(\alpha, \mu, m)}(z), T^{(\alpha, m)}(z)\), and \(R^{(\alpha, \mu, m)}(z)\) are also invariant under rotations centered in the origin. Therefore, in some parts of this section, we fix a unit vector \(v\), and we study the three above functions evaluated in points of the form \(z = tv\), where \(t = \|z\| \geq 0\).

The rest of the section is devoted to deriving bounds for \(H^{(\alpha, \mu, m)}, T^{(\alpha, m)},\) and \(R^{(\alpha, \mu, m)}\). We start with an observation that will be used several times in the analysis of \(H^{(\alpha, \mu, m)}\) and \(T^{(\alpha, m)}\).

**Observation 8** Let \(r, \alpha \in \mathbb{R}_+\) with \(\alpha > r\) and let \(s \in [0, r]\). Let \(v\) be a unit vector in \(\mathbb{R}^m\) and let \(t \in [0, r]\). Then \(H^{(\alpha, \mu, s, m)}(tv)\) can be written in the form

\[
H^{(\alpha, \mu, s, m)}(tv) = \begin{cases} 
\alpha - s - \int_0^\pi \left(\alpha - \sqrt{s^2 + t^2 - 2st \cos \theta}\right) d\hat{\mu}(\theta) = Ed(tv, x) - s & \text{if } s = 0 \\
\alpha - s - \int_0^\theta \left(\alpha - \sqrt{s^2 + t^2 - 2st \cos \theta}\right) d\hat{\mu}(\theta) & \text{if } 0 < s \leq \alpha - t \\
\alpha - s - \int_0^t \left(\alpha - \sqrt{s^2 + t^2 - 2st \cos \theta}\right) d\hat{\mu}(\theta) & \text{if } s \geq \alpha - t.
\end{cases}
\]

In the second case \(x\) is a random vector drawn according to \(\mu^s\). In the third case

\[
\bar{\theta} := \arccos \frac{s^2 + t^2 - \alpha^2}{2st} \leq \pi.
\]

**Proof** Note that we can write \(H^{(\alpha, \mu, s, m)}(tv)\) in the form

\[
H^{(\alpha, \mu, s, m)}(tv) = \alpha - s - \int_{B_r^m(tv) \cap S^{m-1}_s(0)} (\alpha - d(tv, x)) d\mu^s(x).
\]  \hspace{1cm} (38)

If \(s = 0\) we have \(S^{m-1}_s(0) \subseteq B_r^m(tv)\) and we obtain

\[
H^{(\alpha, \mu, 0, m)}(tv) = \alpha - \int_{S^{m-1}_s(0)} (\alpha - d(tv, x)) d\mu^0(x) = \alpha - \alpha + t = t.
\]

In the rest of the proof we assume \(s > 0\). For \(x \in S^{m-1}_s(0),\) we have \(d(tv, x) = \sqrt{s^2 + t^2 - 2st \cos \theta'},\) where \(\theta'\) is the angle between \(v\) and \(x\). This implies that the function under the integral sign in (38) can be written as a function of \(\theta'\). Let \(x\) be a random vector in \(\mathbb{R}^m\) drawn according to \(\mu^s\) and denote by \(\hat{\mu}\) the probability measure of \(\theta'\). According to Observation 4, the random variable \(\theta'\) has the same probability measure as the random variable \(\theta\) studied in Sect. 5.1.1.
Lemma 10 Let \( r \in \mathbb{R}_+ \) with \( \alpha > r \). Let \( z \in B_r^{m}(0) \setminus \{0\} \) with \( \|z\| \leq \alpha - r \). Then we have \( T^{(\alpha,m)}(z) > 0 \).

Proof From Observation 8 with \( s = r \) and \( tv = z \) we have \( T^{(\alpha,m)}(z) = \mathbb{E}d(z, x) - r = \mathbb{E}d(z, x) - \mathbb{E}\|x\| \), where \( x \) is a random vector drawn according to \( \mu^t \). Since \( z \neq 0 \), from Lemmas 2 and 3, we obtain \( \mathbb{E}d(z, x) - \mathbb{E}\|x\| > 0 \).

Lemma 11 Let \( m \geq 2 \) and let \( r \in \mathbb{R}_+. \) Let \( z \in B_r^{m}(0) \). Then \( T^{(\alpha,m)}(z) \) is strictly increasing in \( \alpha \) when \( \alpha \in (r, r + \|z\|) \) and is constant in \( \alpha \) when \( \alpha \geq r + \|z\| \).

Proof Let \( v \) be a unit vector in \( \mathbb{R}^m \). Since \( T^{(\alpha,m)}(z) \) is invariant under rotations centered in the origin, it suffices to consider vectors \( z \in B_r^{m}(0) \) of the form \( z = tv \), for \( t \in [0, r] \).

Consider first the case \( \alpha \geq r + t \). From Observation 8 with \( s = r \) we have \( T^{(\alpha,m)}(tv) = \mathbb{E}d(tv, x) - r \), where \( x \) is a random vector drawn according to \( \mu^t \). Hence in this case \( T^{(\alpha,m)}(tv) \) is constant in \( \alpha \).
Next, consider the case \(\alpha \in (r, r + t)\). From Observation 8 with \(s = r\) we have

\[
T^{(\alpha, m)}(tv) = \alpha - r - \int_0^{\tilde{\theta}} \left( \alpha - \sqrt{r^2 + t^2 - 2rt \cos \theta} \right) d\tilde{\mu}(\theta)
\]

\[
= \alpha - r - \int_0^{\tilde{\theta}} \left( \alpha - \sqrt{r^2 + t^2 - 2rt \cos \theta} \right) p^{(m)}(\theta) d\theta,
\]

where

\[
\tilde{\theta} := \arccos \frac{r^2 + t^2 - \alpha^2}{2rt} < \pi.
\]

We derive with respect to the variable \(\alpha\) and obtain

\[
\frac{\partial T^{(\alpha, m)}}{\partial \alpha}(tv) = 1 - \int_0^{\tilde{\theta}} d\tilde{\mu}(\theta) - \left( \alpha - \sqrt{r^2 + t^2 - 2rt \cos \tilde{\theta}} \right) p^{(m)}(\tilde{\theta}) \frac{\partial \tilde{\theta}}{\partial \alpha}
\]

\[
= 1 - \int_0^{\tilde{\theta}} d\tilde{\mu}(\theta) = 1 - P(\theta \leq \tilde{\theta}) > 0.
\]

Here, the second equality holds because \(\alpha - \sqrt{r^2 + t^2 - 2rt \cos \tilde{\theta}} = 0\) and the second equality holds because \(\tilde{\theta} < \pi\). Hence in this case \(T^{(\alpha, m)}(tv)\) is strictly increasing in \(\alpha\). \(\square\)

**Lemma 12** Let \(m \geq 2\) and let \(r, \alpha \in \mathbb{R}_+\) with \(\alpha > r\). Let \(z \in B^m_r(0)\) and \(z' \in B^{m+1}_r(0)\) with \(\|z\| = \|z'\|\). Then we have \(T^{(\alpha, m+1)}(z') \geq T^{(\alpha, m)}(z)\).

**Proof** Let \(z = tv\) and \(z' = tv'\), where \(v\) is a unit vector in \(\mathbb{R}^m\) and \(v'\) is a unit vector in \(\mathbb{R}^{m+1}\). Then according to Observation 8 with \(s = r\), we have

\[
T^{(\alpha, m+1)}(tv') - T^{(\alpha, m)}(tv) = \int_0^{\tilde{\theta}} \left( \alpha - \sqrt{r^2 + t^2 - 2rt \cos \theta} \right) \left( p^{(m)}(\theta) - p^{(m+1)}(\theta) \right) d\theta,
\]

where

\[
\tilde{\theta} := \begin{cases} 
\pi & \text{if } t \leq \alpha - r, \\
\arccos \frac{r^2 + t^2 - \alpha^2}{2rt} < \pi & \text{if } t > \alpha - r.
\end{cases}
\]

We let \(f(t, \theta) := \alpha - \sqrt{r^2 + t^2 - 2rt \cos \theta}\) and write

\[
T^{(\alpha, m+1)}(tv') - T^{(\alpha, m)}(tv) = \int_0^{\tilde{\theta}} f(t, \theta)(p^{(m)}(\theta) - p^{(m+1)}(\theta)) d\theta. \tag{40}
\]

We define \(\hat{f}(t, \theta) := f(t, \pi - \theta) = \alpha - \sqrt{r^2 + t^2 + 2rt \cos \theta}\). It can be checked that \(f(t, \theta)\) is a decreasing function in \(\theta\), when \(\theta \in (0, \pi)\) and \(t\) is fixed in \([0, r]\).
Furthermore, we have $f(t, \tilde{\theta}) \geq 0$. Thus $f(t, \theta) \geq 0$ when $\theta \leq \tilde{\theta}$. Next, we will discuss several cases for $\tilde{\theta}$.

In the first case we assume $\tilde{\theta} \leq \frac{\pi}{2}$. From (40) and Observation 7, we obtain $T^{(\alpha, m+1)}(tv') - T^{(\alpha, m)}(tv) \geq 0$.

In the second case we assume $\tilde{\theta} > \frac{\pi}{2}$. Let $s_m \in (0, 1)$ be the threshold for $p^m(\theta) - p^{(m+1)}(\theta)$ from Observation 6. Let $\psi \in (\pi/2, \pi)$ such that $\sin \psi = s_m$.

We first show that

$$
\int_{\tilde{\theta}}^{\pi} f(t, \theta)(p^m(\theta) - p^{(m+1)}(\theta))d\theta \leq 0. \tag{41}
$$

If $\tilde{\theta} = \pi$, (41) obviously hold, so we assume $\tilde{\theta} < \pi$. Now assume $\tilde{\theta} \in [\psi, \pi)$. We have $f(t, \tilde{\theta}) = 0$, thus $f(t, \theta) \leq 0$ for every $\theta \in [\tilde{\theta}, \pi]$. On the other hand we have $p^m(\theta) - p^{(m+1)}(\theta) \geq 0$ for every $\theta \in [\tilde{\theta}, \pi]$. Hence (41) holds also in this case. So we now assume $\frac{\pi}{2} < \tilde{\theta} < \psi$. We notice that

$$
\int_{\tilde{\theta}}^{\pi} f(t, \theta)(p^m(\theta) - p^{(m+1)}(\theta))d\theta = -\int_{\tilde{\theta}}^{\pi} f(t, \theta)(p^m(\theta) - p^{(m+1)}(\theta))d\theta
$$

$$
= -\int_{0}^{\pi-\tilde{\theta}} -f(t, \theta)(p^m(\theta) - p^{(m+1)}(\theta))d\theta
$$

$$
= -\int_{0}^{\pi-\tilde{\theta}} -\tilde{f}(t, \theta)(p^m(\theta) - p^{(m+1)}(\theta))d\theta.
$$

Here, in the second equality we perform the change of variable $\theta = \pi - \xi$ and in the third equality we use the fact that $p^m(\xi) = p^m(\pi - \xi)$ for every $m$ and every $\xi$.

We observe that, for $\xi \in [0, \pi - \tilde{\theta}]$, $-\tilde{f}(t, \xi)$ is a decreasing function and

$$
-\tilde{f}(t, \xi) = \sqrt{r^2 + t^2 + 2rt \cos \xi} - \alpha \geq \sqrt{r^2 + t^2 + 2rt \cos(\pi - \tilde{\theta})}
$$

$$
-\alpha = \sqrt{r^2 + t^2 - 2rt \cos \tilde{\theta}} - \alpha = 0.
$$

By Observation 7, we conclude that $\int_{0}^{\pi-\tilde{\theta}} -\tilde{f}(t, \xi)(p^m(\xi) - p^{(m+1)}(\xi))d\xi \geq 0$, thus (41) holds. This concludes the proof of (41).

Next, we show

$$
T^{(\alpha, m+1)}(tv') - T^{(\alpha, m)}(tv) \geq \int_{0}^{\frac{\pi}{2}} (f(t, \theta) + \tilde{f}(t, \theta))(p^m(\theta) - p^{(m+1)}(\theta))d\theta. \tag{42}
$$

From (40) we have

$$
T^{(\alpha, m+1)}(tv') - T^{(\alpha, m)}(tv) = \int_{0}^{\hat{\theta}} f(t, \theta)(p^m(\theta) - p^{(m+1)}(\theta))d\theta
$$

$$
\hat{\theta} = \begin{cases} 
\frac{\pi}{2}, & \text{if } \tilde{\theta} \leq \frac{\pi}{2}; \\
\psi, & \text{if } \tilde{\theta} > \frac{\pi}{2}.
\end{cases}
$$
\[
\int_0^{\pi} f(t, \theta)(p^{(m)}(\theta) - p^{(m+1)}(\theta))d\theta \\
+ \int_{\frac{\pi}{2}}^\theta f(t, \theta)(p^{(m)}(\theta) - p^{(m+1)}(\theta))d\theta.
\]

Now note that
\[
\int_{\frac{\pi}{2}}^\theta f(t, \theta)(p^{(m)}(\theta) - p^{(m+1)}(\theta))d\theta \geq \int_0^{\pi} f(t, \theta)(p^{(m)}(\theta) - p^{(m+1)}(\theta))d\theta \\
= \int_0^{\pi} f(t, \pi - \xi)(p^{(m)}(\pi - \xi) - p^{(m+1)}(\pi - \xi))d(\pi - \xi) \\
= \int_0^{\pi} f(t, \pi - \xi)(p^{(m)}(\xi) - p^{(m+1)}(\xi))d\xi.
\]

Here, in the inequality we use (41), in the first equality we perform the change of variable \( \theta = \pi - \xi \), and in the last equality, we use the fact that \( p^{(m)}(\theta) = p^{(m)}(\pi - \theta) \) for every \( m \) and every \( \theta \). Thus we continue
\[
T^{(\alpha, m+1)}(tv') - T^{(\alpha, m)}(tv) \geq \int_0^{\pi} f(t, \theta)(p^{(m)}(\theta) - p^{(m+1)}(\theta))d\theta \\
+ \int_0^{\frac{\pi}{2}} f(t, \pi - \xi)(p^{(m)}(\xi) - p^{(m+1)}(\xi))d\xi \\
= \int_0^{\pi} (f(t, \theta) + \hat{f}(t, \theta))(p^{(m)}(\theta) - p^{(m+1)}(\theta))d\theta,
\]

where in the equality we use the fact that \( \hat{f}(t, \theta) = f(t, \pi - \theta) \). This concludes the proof of (42).

To finish the proof it suffices to show that \( f(t, \theta) + \hat{f}(t, \theta) \) is a nonnegative decreasing function for \( \theta \in (0, \frac{\pi}{2}) \). In fact, using (42) and Observation 7, we can then conclude that \( T^{(\alpha, m+1)}(tv') - T^{(\alpha, m)}(tv) \geq 0 \).

We derive \( f(t, \theta) + \hat{f}(t, \theta) \) with respect to the variable \( \theta \) and obtain
\[
\frac{\partial (f + \hat{f})}{\partial \theta}(t, \theta) = \frac{\partial}{\partial \theta} \left( 2\alpha - \sqrt{r^2 + t^2 - 2rt \cos \theta} - \sqrt{r^2 + t^2 + 2rt \cos \theta} \right) \\
= rt \sin \theta \left( \frac{1}{\sqrt{r^2 + t^2 - 2rt \cos \theta}} - \frac{1}{\sqrt{r^2 + t^2 + 2rt \cos \theta}} \right).
\]

Hence the derivative is nonpositive for \( \theta \in (0, \frac{\pi}{2}) \) and so \( f(t, \theta) + \hat{f}(t, \theta) \) is decreasing for \( \theta \in (0, \frac{\pi}{2}) \). This implies that, for \( \theta \in (0, \frac{\pi}{2}) \), we have \( f(t, \theta) + \hat{f}(t, \theta) \geq f(t, \frac{\pi}{2}) + \hat{f}(t, \frac{\pi}{2}) = 2\alpha - 2\sqrt{r^2 + t^2} \). The latter quantity is nonnegative. In the case
\(t > \alpha - r\), this is because \(\cos \bar{\theta} = \frac{r^2 + t^2 - \alpha^2}{2rt} < 0\) when \(\bar{\theta} > \frac{\pi}{2}\). In the case \(t \leq \alpha - r\), this is because we have \(\alpha^2 \geq (r + t)^2 \geq r^2 + t^2\).

In the next lemma, we use Lemma 9 to bound the function \(T^{(\alpha,m)}(z)\).

**Lemma 13** Let \(m \geq 2\), let \(r \in \mathbb{R}_+\), let \(\epsilon \in (0, 1)\), and let \(\alpha = r(1 + \epsilon)\). Let \(z \in B^m_r(0)\) with \(\|z\| \geq \epsilon r\). Then we have

\[
T^{(\alpha,m)}(z) \geq \frac{r \epsilon^2}{8} - r \sqrt{\frac{\pi m}{2} \left(1 - \frac{\epsilon^2}{16}\right)^{m-2}}.
\]

**Proof** Let \(v\) be a unit vector in \(\mathbb{R}^m\). Since \(T^{(\alpha,m)}(z)\) is invariant under rotations centered in the origin, it suffices to consider vectors \(z \in B^m_r(0)\) of the form \(z = tv\), for \(t \in [\epsilon r, r]\). We define

\[
\theta_\epsilon := \arccos \frac{\epsilon}{4} < \frac{\pi}{2}, \quad \bar{\theta} := \arccos \frac{r^2 + t^2 - \alpha^2}{2rt} < \pi.
\]

We consider two cases. In the first case we assume \(\bar{\theta} \leq \theta_\epsilon\). Then, according to Observation 8, we have

\[
T^{(\alpha,m)}(tv) = \alpha - r - \int_0^{\bar{\theta}} \left(\alpha - \sqrt{r^2 + t^2 - 2rt \cos \theta}\right) p^{(m)}(\theta)d\theta \\
\geq \alpha(1 - \mathbb{P}(\theta \in (0, \bar{\theta}))) - r \geq \alpha(1 - \mathbb{P}(\theta \in (0, \theta_\epsilon))) - r \\
\geq r \sqrt{1 + \frac{\epsilon^2}{2} (1 - \mathbb{P}(\theta \in (0, \theta_\epsilon)))} \\
- r \geq r \sqrt{1 + \frac{\epsilon^2}{2} \left(1 - \frac{\sqrt{\pi}}{2} \sqrt{\frac{m}{2} \sin^{m-2} \theta_\epsilon}\right)} - r \\
= r \sqrt{1 + \frac{\epsilon^2}{2} \left(1 - \frac{\sqrt{\pi}}{2} \sqrt{\frac{m}{2} \left(1 - \frac{\epsilon^2}{16}\right)^{m-2}}\right)} - r.
\]

The third inequality holds because \(\alpha = r(1 + \epsilon) > r\sqrt{1 + \epsilon^2/2}\), and the fourth inequality follows from Lemma 9.

In the second case we assume \(\bar{\theta} > \theta_\epsilon\). Then, according to Observation 8, we have

\[
T^{(\alpha,m)}(tv) = \alpha - r - \int_0^{\bar{\theta}} \left(\alpha - \sqrt{r^2 + t^2 - 2rt \cos \theta}\right) p^{(m)}(\theta)d\theta \\
= \alpha - r - \int_0^{\theta_\epsilon} \left(\alpha - \sqrt{r^2 + t^2 - 2rt \cos \theta}\right) p^{(m)}(\theta)d\theta \\
- \int_{\theta_\epsilon}^{\bar{\theta}} \left(\alpha - \sqrt{r^2 + t^2 - 2rt \cos \theta}\right) p^{(m)}(\theta)d\theta.
\]
\[ \geq \alpha - r - \alpha P(\theta \in (0, \theta_\epsilon)) - \int_{\theta_\epsilon}^{\hat{\theta}} \left( \alpha - \sqrt{r^2 + t^2 } - 2rt \cos \theta \right) p^{(m)}(\theta) d\theta \]

\[ \geq \alpha - r - \alpha P(\theta \in (0, \theta_\epsilon)) - \left( \alpha - r \sqrt{1 + \frac{\epsilon^2}{2}} \right) P(\theta \in (\theta_\epsilon, \hat{\theta})) \]

\[ \geq \alpha - r - \alpha P(\theta \in (0, \theta_\epsilon)) - \left( \alpha - r \sqrt{1 + \frac{\epsilon^2}{2}} \right) (1 - P(\theta \in (0, \theta_\epsilon))) \]

\[ = r \left( 1 + \frac{\epsilon^2}{2} \right) (1 - P(\theta \in (0, \theta_\epsilon))) \]

\[ - r \geq r \left( 1 + \frac{\epsilon^2}{2} \right) \left( 1 - \frac{\sqrt{\pi}}{2} \sqrt{\frac{m}{2}} \sin^{m-2} \theta_\epsilon \right) - r \]

\[ = r \left( 1 + \frac{\epsilon^2}{2} \right) \left( 1 - \frac{\sqrt{\pi}}{2} \sqrt{\frac{m}{2}} \left( 1 - \frac{\epsilon^2}{16} \right)^{\frac{m-2}{2}} \right) - r. \]

Here, the second inequality holds because

\[ \sqrt{r^2 + t^2 - 2rt \cos \theta} \geq \sqrt{r^2 + t^2 - 2rt \cos \theta_\epsilon} \geq \sqrt{r^2 + \epsilon^2 r^2 - 2\epsilon r^2 \cos \theta_\epsilon} \]

\[ = r \sqrt{1 + \epsilon^2/2} \]

when \( \theta \geq \theta_\epsilon \) and \( t \geq \epsilon r \) and the last inequality follows from Lemma 9.

Since \( (1 + \epsilon^2/8) \leq \sqrt{1 + \epsilon^2/2} \leq 2 \), we obtain

\[ T^{(\alpha, m)}(z) \geq r \left( 1 + \frac{\epsilon^2}{2} \right) \left( 1 - \frac{\sqrt{\pi}}{2} \sqrt{\frac{m}{2}} \left( 1 - \frac{\epsilon^2}{16} \right)^{\frac{m-2}{2}} \right) - r \geq \frac{r \epsilon^2}{8} \]

\[ - r \sqrt{\frac{\pi m}{2}} \left( 1 - \frac{\epsilon^2}{16} \right)^{\frac{m-2}{2}}. \]

\[ \square \]

**Analysis of the function** \( H^{(\alpha, \mu, m)} \). Our next goal is to derive a lower bound on \( H^{(\alpha, \mu, m)} \) using the lower bound for \( T^{(\alpha, m)} \) given in Lemma 13.

**Lemma 14** Let \((\mu, B^m_r(0))\) be a probability space with \( m \geq 2 \) that satisfies (a1) and let \( \alpha > r \). Let \( z \in B^m_r(0) \). Then we have \( H^{(\alpha, \mu, m)}(z) \geq T^{(\alpha, m)}(z) \).

**Proof** Let \( x \) be a random vector drawn according to \( \mu \). Since \((\mu, B^m_r(0))\) satisfies (a1), we know that, conditioned on the event that \( \|x\| = s \), \( x \) is drawn according to \( \mu^s \).
In the following, we let $D := B^m_{\alpha}(z) \cap B^m_r(0)$, we denote by $I_D(x)$ the indicator function of $D$, and by $\nu$ be the probability measure of $\|x\|$. Then we have

$$H^{(\alpha, \mu, m)}(z) = \int_{B^m_{\alpha}\cap B^m_r(0)} (\alpha - \|x\| - (\alpha - d(z, x))I_D(x)) \, d\mu(x) = \int_0^r d\nu(s) \int_{S^m_{\nu}(0)} (\alpha - \|x\| - (\alpha - d(z, x))I_D(x)) \, d\mu^x(x) = \int_0^r H^{(\alpha, \mu^x, m)}(z) \, d\nu(s).$$

To complete the proof of the lemma it suffices to show that the scalar $r$ achieves

$$\min \left\{ H^{(\alpha, \mu^x, m)}(z) \mid s \in [0, r] \right\}. \quad (43)$$

In fact, this implies

$$H^{(\alpha, \mu, m)}(z) = \int_0^r H^{(\alpha, \mu^x, m)}(z) \, d\nu(s) \geq H^{(\alpha, \mu^x, m)}(z) = T^{(\alpha, m)}(z).$$

Let $v$ be a unit vector in $\mathbb{R}^m$. Since $H^{(\alpha, \mu^x, m)}(z)$ is invariant under rotations centered in the origin, it suffices to consider vectors $z \in B^m_r(0)$ of the form $z = tv$, for $t \in [0, r]$.

If $s = 0$, then from Observation 8 we have $H^{(\alpha, \mu^x, m)}(tv) = t$. If $0 < s \leq \alpha - t$, Observation 8 implies $H^{(\alpha, \mu^x, m)}(tv) = \mathbb{E}d(tv, x) - s = \mathbb{E}d(tv, x) - \mathbb{E}\|x\| = \mathbb{E}(d(tv, x) - \|x\|) \leq \mathbb{E}t = t$, where $x$ is a random vector drawn according to $\mu^x$. So we only need to show (43) for $s \in (0, r]$ rather than $s \in [0, r]$.

We now consider separately two cases. In the first case we assume $s \in (0, \alpha - t]$. From Observation 8 we can write

$$H^{(\alpha, \mu^x, m)}(tv) = \alpha - s - \int_0^\pi \left( \alpha - \sqrt{s^2 + t^2 - 2st \cos \theta} \right) \, d\tilde{\mu}(\theta).$$

We derive with respect to the variable $s$ and obtain

$$\frac{\partial H^{(\alpha, \mu^x, m)}}{\partial s}(tv) = \int_0^\pi \frac{s - t \cos \theta}{\sqrt{s^2 + t^2 - 2st \cos \theta}} \, d\tilde{\mu}(\theta) - 1 \leq 0,$$

because $(s - t \cos \theta)/\sqrt{s^2 + t^2 - 2st \cos \theta} \leq 1$. This implies that the function $H^{(\alpha, \mu^x, m)}(tv)$ is decreasing in $s$, when $s \in (0, \alpha - t]$.

In the second case we assume $s \in (\alpha - t, r]$. From Observation 8 we can write

$$H^{(\alpha, \mu^x, m)}(tv) = \alpha - s - \int_0^{\tilde{\theta}} \left( \alpha - \sqrt{s^2 + t^2 - 2st \cos \theta} \right) \, d\tilde{\mu}(\theta)$$

$$= \alpha - s - \int_0^{\tilde{\theta}} \left( \alpha - \sqrt{s^2 + t^2 - 2st \cos \theta} \right) p^{(m)}(\theta) \, d\theta,$$
where

\[ \bar{\theta} := \arccos \frac{s^2 + t^2 - \alpha^2}{2st} < \pi. \]

We derive with respect to the variable \( s \) and obtain

\[
\frac{\partial H^{(\alpha, \mu, m)}}{\partial s}(t v) = -1 + \int_0^{\bar{\theta}} \frac{s - t \cos \theta}{\sqrt{s^2 + t^2 - 2st \cos \theta}} d\tilde{\mu}(\theta) - \left( \alpha - \sqrt{s^2 + t^2 - 2st \cos \bar{\theta}} \right) p^{(m)}(\bar{\theta}) \frac{\partial \bar{\theta}}{\partial s}
\]

\[
= -1 + \int_0^{\bar{\theta}} \frac{s - t \cos \theta}{\sqrt{s^2 + t^2 - 2st \cos \theta}} d\tilde{\mu}(\theta) \leq -1 + \mathbb{P}(\theta \leq \bar{\theta}) < 0.
\]

Here, the second equality holds because \( \alpha - \sqrt{s^2 + t^2 - 2st \cos \bar{\theta}} = 0 \) and the first inequality holds because \( (s - t \cos \theta)/\sqrt{s^2 + t^2 - 2st \cos \bar{\theta}} \leq 1 \) and \( \bar{\theta} < \pi \). So we conclude that \( H^{(\alpha, \mu, m)}(t v) \) is also decreasing in \( s \), when \( s \in (\alpha - t, r] \).

The above two cases imply that \( H^{(\alpha, \mu, m)}(t v) \) is decreasing in \( s \), when \( s \in (0, r] \).

Thus, for every \( z \in B_{r}^m(0) \), the scalar \( r \) achieves (43).

According to Lemma 14, we know that every lower bound for \( T^{(\alpha, \mu)}(z) \) is also a lower bound for \( H^{(\alpha, \mu, m)}(z) \).

**Lemma 15** Let \((\mu, B_r^m(0))\) be a probability space with \( m \geq 2 \) that satisfies (a1), let \( \epsilon \in (0, 1) \), and let \( \alpha = r(1 + \epsilon) \). Let \( z \in B_r^m(0) \) with \( \|z\| \geq \epsilon r \). Then we have

\[
H^{(\alpha, \mu, m)}(z) \geq \frac{r \epsilon^2}{8} - r \sqrt{\frac{\pi m}{2}} \left( 1 - \frac{\epsilon^2}{16} \right)^{m-2}. 
\]

**Proof** Directly from Lemmas 13 and 14. \(\square\)

**Analysis of the function \( R^{(\alpha, \mu, m)} \).** In the next lemma, we will provide an upper bound for \( R^{(\alpha, \mu, m)}(z) \)

**Lemma 16** Let \((\mu, B_r^m(0))\) be a probability space with \( m \geq 2 \) that satisfies (a1), (a3) and let \( \alpha > r \). Let \( z \in \mathbb{R}^m \) with \( \|z\| \in (\alpha, \alpha + r) \). Then we have

\[
R^{(\alpha, \mu, m)}(z) \leq (\alpha + r - \|z\|) \frac{\sqrt{\pi}}{2} \sqrt{\frac{m}{2}} \left( \frac{\alpha}{\|z\|} \right)^{m-2}.
\]

**Proof** For every \( x \in B_r^m(0) \), let \( \theta'(x) \) be the angle between \( x \) and \( z \). Let \( D := B_{\alpha}^m(z) \cap B_r^m(0) \). For every \( x \in D \), we denote by \( \Pi_{x}(z) \) the orthogonal projection of \( z \) on the line containing 0 and \( x \). Then we know that for every \( x \in D \) we have
Let \( \theta^* := \arcsin(\alpha/\|z\|) \in (0, \pi/2) \). Thus we obtain \( D \subseteq \{ x \in B^m_\alpha(0) \mid \theta'(x) \leq \theta^* \} \).

According to Observation 4, the random variable \( \theta' \) has the same probability measure as the random variable \( \theta \) studied in Sect. 5.1.1. So we obtain

\[
R^{(\alpha, \mu, m)}(z) = \int_{B^m_\alpha(z) \cap B^m_\beta(0)} (\alpha - d(z, x)) d\mu(x) \leq (\alpha + r - \|z\|) P(x \in D)
\]

\[
\leq (\alpha + r - \|z\|) P(\theta \leq \theta^*) \leq (\alpha + r - \|z\|) \frac{\sqrt{\pi}}{2} \sqrt{\frac{m}{\alpha}} \left( \frac{\|z\|}{\alpha} \right)^{-m/2},
\]

where the first inequality holds because \( d(z, x) \geq \|z\| - r \) and the last inequality holds by Lemma 9.

\[\square\]

### 5.2 Proof of Theorem 5

It suffices to check that all assumptions of Theorem 2 are satisfied. For every \( i \neq j \), we define \( \Theta_{ij} > 0 \) so that \( d(c_i, c_j) = (1 + \beta)R + \max\{r_i, r_j\} + 2\Theta_{ij} \). We also define \( \Theta := \min_{i \neq j} \Theta_{ij} \) and \( \gamma := \beta R + \Theta \). For every \( i \in [k] \), denote by \( E_i := B^m(x, c_i) \), where \( x \) is a random vector drawn according to \( \mu_i \). We can then bound \( \gamma \) as follows.

\[
\max_{i \in [k]} \beta_i (r_i - E_i) < \beta R < \gamma < (1 + \beta)R + 2\Theta - R
\]

\[
\leq \min_{i \neq j} d(c_i, c_j) - \max_{i \in [k]} d(c_i, c_j) - R \leq \min_{i \in [k]} \beta_i (D_i - E_i).
\]

For every \( i \in [k] \), we define \( \alpha_i := E_i + \frac{\gamma}{\beta_i} \). It remains to show that for every \( i \in [k], c_i \) is the unique point that achieves \( \max\{G^\alpha(z) \mid z \in B^m_\alpha(c_i) \} \). Using the fact that \( r_i \in [R, R] \), \( \beta_i \in [1, \beta] \), and \( E_i \in (0, R) \) for \( i \in [k] \), we obtain

\[
r_i < R + \frac{\Theta}{\beta} = \frac{\gamma}{\beta} < \alpha_i \leq R + \gamma = (1 + \beta)R + \Theta < \min_{j \in [k] \setminus \{i\}} d(c_i, c_j) - r_i = D_i.
\]

(44)

Lemma 6 implies that \( B^m_\alpha(c_i) \cap B^m_\beta(c_j) = \emptyset \) for every \( j \in [k] \setminus \{i\} \). From Observation 2, we know that for every \( i \in [k] \),

\[
G^\alpha(c_i) = \beta_i \int_{B^m_\alpha(c_i)} (\alpha_i - d(c_i, x)) d\mu_i(x).
\]

From Observation 2 we obtain that for every \( z \in B^m_\alpha(c_i) \),

\[\square\ Springer\]
\[ G^\alpha(c_i) - G^\alpha(z) \]
\[ = \beta_i \left( \int_{B^m_{r_i}(c_i)} (\alpha_i - d(c_i, x))d\mu_i(x) - \int_{B^m_{r_i}(z) \cap B^m_{r_j}(c_i)} (\alpha_i - d(z, x))d\mu_i(x) \right) \]
\[ - \sum_{j \in [k] \setminus \{i\}} \beta_j \int_{B^m_{\alpha_j}(z) \cap B^m_{r_j}(c_j)} (\alpha_j - d(z, x))d\mu_j(x). \] (45)

It then suffices to show that, when \( \Theta \) is large, the right hand side of (45) is positive for every \( z \in B^m_{r_i}(c_i) \setminus \{c_i\} \). So we now fix a vector \( z \in B^m_{r_i}(c_i) \setminus \{c_i\} \).

From (44) we obtain
\[ \alpha_i > R + \frac{\Theta}{\beta} = \left( 1 + \frac{\Theta}{\beta R} \right) R \geq \left( 1 + \frac{\Theta}{\beta R} \right) r_i. \] (46)

We now consider separately two cases.

In the first case we assume \( d(c_i, z) \leq \Theta r_i / (\beta R) \). Notice that under this assumption, for every \( j \in [k] \setminus \{i\} \) and for every \( x \in B^m_{\alpha_j}(z), y \in B^m_{r_j}(c_j) \), we have
\[
\begin{align*}
d(x, y) &\geq d(z, c_j) - r_j - \alpha_j \geq d(c_i, c_j) - d(z, c_i) - r_j - \alpha_j \\
&\geq d(c_i, c_j) - d(z, c_i) - r_j - (1 + \beta) R - \Theta \\
&\geq 2\Theta r_{ij} - \Theta - d(z, c_i) \geq \Theta - d(z, c_i) > 0,
\end{align*}
\]

where the third inequality follows from (44). So we must have \( B^m_{\alpha_j}(z) \cap B^m_{r_j}(c_j) = \emptyset \) for \( j \in [k] \setminus \{i\} \). Therefore, from (45) we have
\[ G^\alpha(c_i) - G^\alpha(z) \]
\[ = \beta_i \left( \int_{B^m_{r_i}(c_i)} (\alpha_i - d(c_i, x))d\mu_i(x) - \int_{B^m_{r_i}(z) \cap B^m_{r_i}(c_i)} (\alpha_i - d(z, x))d\mu_i(x) \right) \]
\[ = \beta_i H^{(\alpha_i, \mu_i', m)}(z - c_i) \geq \beta_i T^{(\alpha_i, m)}(z - c_i) > 0, \]

where \( \mu_i' \) is the image of \( \mu_i \) under the translation \( x' = x - c_i \). The first inequality above follows from Lemma 14 and the last inequality follows from Lemma 10 because from (46) we have \( d(c_i, z) \leq \Theta r_i / (\beta R) < \alpha_i - r_i \). Thus, in the first case Theorem 2 implies that (LP) achieves exact recovery with high probability.

In the remainder of the proof we only need to consider the second case, where we assume \( d(c_i, z) > \Theta r_i / (\beta R) \). We notice that in this case \( d(c_i, z) \leq r_i \) implies \( \Theta / (\beta R) < 1 \). We first show that for every \( j \in [k] \setminus \{i\} \), we have
\[
\int_{B^m_{\alpha_j}(z) \cap B^m_{r_j}(c_j)} (\alpha_j - d(z, x))d\mu_j(x) \leq R \sqrt{\frac{\pi}{2}} \sqrt{\frac{m}{2}} \left( 1 - \frac{\Theta}{(1 + \beta)R + 2\Theta} \right)^{m-2}.
\] (47)
If \( d(z, c_j) \geq \alpha_j + r_j \), then \( B^m_{r_j}(c_j) \cap B^m_{d_j}(z) \) contains at most one point and (47) clearly holds because (a3) implies

\[
\int_{B^m_{d_j}(z) \cap B^m_{r_j}(c_j)} (\alpha_j - d(z, x)) d\mu_j(x) = 0.
\]

If \( d(z, c_j) < \alpha_j + r_j \), we can apply Lemma 16 to \( z \), since we also have

\[
d(z, c_j) \geq d(c_i, c_j) - r_i \geq (1 + \beta)R + 2\Theta \geq \alpha_j + \Theta > \alpha_j,
\]

where the last inequality follows by (48). If we denote by \( \mu_j' \) the image of \( \mu_j \) under the translation \( x' = x - c_j \), we then obtain

\[
\int_{B^m_{d_j}(z) \cap B^m_{r_j}(c_j)} (\alpha_j - d(z, x)) d\mu_j(x) = R^{(\alpha_j, \mu_j', m)}(z - c_j)
\]

\[
\leq (\alpha_j + r_j - d(z, c_j)) \sqrt{\frac{\pi}{2}} \sqrt{\frac{m}{2}} \left( \frac{\alpha_j}{d(z, c_j)} \right)^{m-2}
\]

\[
\leq (r_j - \Theta) \sqrt{\frac{\pi}{2}} \sqrt{\frac{m}{2}} \left( \frac{\alpha_j}{\alpha_j + \Theta} \right)^{m-2}
\]

\[
= (r_j - \Theta) \sqrt{\frac{\pi}{2}} \sqrt{\frac{m}{2}} \left( 1 - \frac{\Theta}{\alpha_j + \Theta} \right)^{m-2}
\]

\[
\leq R \sqrt{\frac{\pi}{2}} \sqrt{\frac{m}{2}} \left( 1 - \frac{\Theta}{(1 + \beta)R + 2\Theta} \right)^{m-2},
\]

where in the second inequality we use \( d(z, c_j) \geq \alpha_j + \Theta \) from (48) and the last inequality follows because \( \alpha_j \leq (1 + \beta)R + \Theta \) from (44). This concludes the proof of (47).

From (47) we obtain

\[
\sum_{j \in [k] \setminus \{i\}} \beta_j \int_{B^m_{d_j}(z) \cap B^m_{r_j}(c_j)} (\alpha_j - d(z, x)) d\mu_j(x)
\]

\[
\leq k\beta R \sqrt{\frac{\pi}{2}} \sqrt{\frac{m}{2}} \left( 1 - \frac{\Theta}{(1 + \beta)R + 2\Theta} \right)^{m-2}
\]

\[
\leq k\beta R \sqrt{\frac{\pi}{2}} \sqrt{\frac{m}{2}} \exp \left( -\frac{(m-2)\Theta}{(1 + \beta)R + 2\Theta} \right),
\]

(49)

where the last inequality we use the fact that \( 1 - x \leq e^{-x} \) for every \( x \).
Now let $\alpha'_i := r_i (1 + \Theta/\beta R)$. We know from (46) that $\alpha_i > \alpha'_i$. If we denote by $\mu'_i$ the image of $\mu_i$ under the translation $x' = x - c_i$, we obtain

$$
\int_{B^m_r(c_i)} (\alpha_i - d(c_i, x)) d\mu_i(x) - \int_{B^m_r(\alpha'_i) \cap B^m_r(c_i)} (\alpha_i - d(z, x)) d\mu_i(x) \\
= H^{(\alpha_i, \mu'_i, m)}(z - c_i) \\
\geq T^{(\alpha_i, m)}(z - c_i) \geq T^{(\alpha'_i, m)}(z - c_i) \geq \frac{r_i \Theta^2}{8 \beta^2 R^2} - r_i \sqrt{\frac{\pi m}{2}} \left(1 - \frac{\Theta^2}{16 \beta^2 R^2}\right)^{m-2}.
$$

(50)

The first inequality holds by Lemma 14, the second inequality holds by Lemma 11, and the third inequality holds by Lemma 13 with $\epsilon := \Theta/(\beta R)$ which satisfies $\epsilon \in (0, 1)$. In the last inequality we use $1 - x \leq e^{-x}$ for every $x$.

To show $G^\alpha(c_i) - G^\alpha(z) > 0$, it is sufficient to show $(G^\alpha(c_i) - G^\alpha(z))/(\beta_i r_i) > 0$. From (45), (49), and (50) we then obtain

$$
\frac{G^\alpha(c_i) - G^\alpha(z)}{\beta_i r_i} \geq \frac{\Theta^2}{8 \beta^2 R^2} - \sqrt{\frac{\pi m}{2}} \exp\left(-\frac{(m-2)\Theta^2}{32 \beta^2 R^2}\right) \\
- \frac{k \beta R \sqrt{\pi}}{\beta_i r_i} \sqrt{\frac{m}{2}} \exp\left(-\frac{(m-2)\Theta}{(1+\beta)R+2\Theta}\right) \\
\geq \frac{\Theta^2}{8 \beta^2 R^2} - \frac{k \sqrt{\pi}}{2} \sqrt{\frac{m}{2}} \exp\left(-\frac{(m-2)\Theta}{(1+\beta)R+2\Theta}\right) \\
- \frac{k \beta R \sqrt{\pi}}{r} \sqrt{\frac{m}{2}} \exp\left(-\frac{(m-2)\Theta}{(1+\beta)R+2\Theta}\right)
$$

(51)

Note that the lower bound on $(G^\alpha(c_i) - G^\alpha(z))/(\beta_i r_i)$ obtained in (51) does not depend on the index $i$ and is an increasing function in $\Theta$. Next we show that $(G^\alpha(c_i) - G^\alpha(z))/(\beta_i r_i)$ is positive when $\Theta > C \sqrt{\log m}/m$, where $C$ is a large constant. To do so we use the lower bound in (51) and the fact that $\beta, r, R$ are fixed constants. We have

$$
\frac{G^\alpha(c_i) - G^\alpha(z)}{\beta_i r_i} \geq \frac{\Theta^2}{8 \beta^2 R^2} - k \sqrt{\frac{\pi m}{2}} \exp\left(-\frac{(m-2)\Theta^2}{32 \beta^2 R^2}\right)
$$

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\[ -k\beta R \sqrt{\pi \frac{m}{2}} \exp \left( -\frac{(m-2)\Theta}{(1+\beta)R+2\Theta} \right) \]

\[ > k \left( \frac{C^2 \log m}{8m\beta^2 R^2} - \sqrt{\pi \frac{m}{2}} \exp \left( -\frac{C^2(m-2)k \log m}{32m\beta^2 R^2} \right) \right) \]

\[ -\beta R \sqrt{\pi} \sqrt{\frac{m}{2}} \exp \left( -\frac{C(m-2)\sqrt{k \log m/m}}{(1+\beta)R+2C\sqrt{k \log m/m}} \right) \]

\[ \geq k \left( \frac{C^2 \log m}{8m\beta^2 R^2} - \sqrt{\pi \frac{m}{2}} \exp \left( -\frac{C^2(m-2) \log m}{32m\beta^2 R^2} \right) \right) \]

\[ -\beta R \sqrt{\pi} \sqrt{\frac{m}{2}} \exp \left( -\frac{C(m-2)\sqrt{\log m/m}}{(1+\beta)R+2C\sqrt{\log m/m}} \right) \]

\[ = \frac{k \log m}{m} \left( \frac{C^2}{8\beta^2 R^2} - F(C, m) \right), \]

where, to simplify the notation, we let

\[ F(C, m) := \frac{m \log m}{\log m} \sqrt{\pi \frac{m}{2}} \exp \left( -\frac{C^2(m-2) \log m}{32m\beta^2 R^2} \right) \]

\[ + \frac{m}{\log m} \beta R \sqrt{\pi} \sqrt{\frac{m}{2}} \exp \left( -\frac{C(m-2)\sqrt{\log m/m}}{(1+\beta)R+2C\sqrt{\log m/m}} \right). \]

It then suffices to show that, for every \( m \geq 2 \), we have \( C^2/(8\beta^2 R^2) > F(C, m) \) for some constant \( C \) large enough. It can be checked that for every \( m \geq 2 \), \( F(C, m) \) is a decreasing function in \( C \). Also it can be checked that there is some threshold \( C' > 0 \) such that if \( C \geq C' \) then \( \lim_{m \to \infty} F(C, m) = 0 \). This implies that \( \sup\{F(C, m) \mid C \geq C', m \geq 2\} = \sup\{F(C', m) \mid m \geq 2\} < \infty \). Therefore it suffices to choose \( C > C' \) large enough so that \( C^2/(8\beta^2 R^2) > \sup\{F(C, m) \mid C \geq C', m \geq 2\}. \]

### 5.3 Proof of Theorem 6

It suffices to check that all assumptions of Corollary 1 are satisfied. Let \( \Theta := \min_{i \neq j} d(c_i, c_j) - 2 > 1.29 \), \( \alpha' := 1.29 \), and let \( \alpha_i := \alpha' \) for every \( i \in [k] \). It remains to show that for every \( i \in [k] \), \( c_i \) is the unique point that achieves \( \max\{G^x(z) \mid z \in B^m_i(c_i)\} \).

Since \( \min_{i \neq j} d(c_i, c_j) = 2 + \Theta > 3.29 \), we know that for every \( i \in [k] \) and for every \( z \in B^m_i(c_i) \), we have \( B^m_{1.29}(z) \cap B^m_j(c_j) = \emptyset \) for every \( j \in [k] \) with \( j \neq i \). Thus according to Observation 2, for every \( i \in [k] \) and for every \( z \in B^m_i(c_i) \) we have

\[ G^x_i(z) = \int_{B^m_{1.29}(z) \cap B^m_i(c_i)} (1.29 - d(z, x))d\mu_i(x). \]

Now we fix \( i \in [k] \) and \( z \in B^m_i(c_i) \). Then we have

\[ G^x_i(c_i) - G^x_i(z) = H^{(1.29, \mu_i, m)}(z - c_i) \geq T^{(1.29, m)}(z - c_i) \geq T^{(1.29, 2)}(z - c_i), \]

\( \square \) Springer
Fig. 2 The graph of the function $T^{(1.29,2)}(z)$ in the proof of Theorem 6

where $\mu'_i$ is the image of $\mu_i$ under the translation $x' = x - c_i$. The first inequality follows by Lemma 14 and the second inequality follows by Lemma 12. Since $T^{(1.29,2)}$ is invariant under rotations centered in $c_i$, we define a unit vector $v \in \mathbb{R}^m$ and a scalar $t \in (0, 1]$ such that $z = tv$. If $t \leq 0.29$, then Lemma 10 implies $T^{(1.29,2)} > 0$. Hence, in the remainder of the proof we assume $t > 0.29$. According to Observation 5, we know that $p^{(2)}(\theta) = 1/\pi$. So applying Observation 8 with $r = s = 1$ and $\alpha = 1.29$, we get

$$T^{(1.29,2)}(tv) = H^{(1.29,\mu',2)}(tv) = 0.29 - \frac{1}{\pi} \int_{0}^{\bar{\theta}} \left( 1.29 - \sqrt{1 + t^2 - 2t \cos \theta} \right) d\theta,$$

where

$$\bar{\theta} = \arccos \frac{1 + t^2 - 1.29^2}{2t}.$$

Using the above formula it can be checked that $T^{(1.29,2)}(z) > 0$ for every $t \in (0.29, 1]$. The graph of the function $T^{(1.29,2)}(z)$ can be seen in Fig. 2. Thus, for every $i \in [k]$, $c_i$ is the unique point that achieves $\max\{G^\alpha(z) \mid z \in B^m_1(c_i)\}$. \hfill \Box

5.4 Proof of Theorem 7

It suffices to check that all assumptions of Corollary 1 are satisfied. Let $\Theta := \min_{i \neq j} d(c_i, c_j) - 2$, $\alpha' := 1 + \Theta/2 \in (1, 1 + \Theta)$, and let $\alpha_i := \alpha'$ for every $i \in [k]$. It remains to show that for every $i \in [k]$, $c_i$ is the unique point that achieves $\max\{G^\alpha(z) \mid z \in B^m_1(c_i)\}$.

For every $i \in [k]$, from Observation 2 and Lemma 6 (with $a_i = b_i = \alpha_i$) we obtain

$$G^\alpha(c_i) = \sum_{j \in [k]} \int_{B^m_1(c_j) \cap B^m_1(c_i)} (\alpha_j - d(c_i, x)) d\mu_j(x).$$
\[ = \int_{B_1^m(c_i)} (\alpha_i - d(c_i, x)) d\mu_i(x). \]

So for every \( i \in [k] \) and for every \( z \in B_1^m(c_i) \) we have

\[
G^\alpha(c_i) - G^\alpha(z) \]
\[
= \left( \int_{B_1^m(c_i)} (\alpha_i - d(c_i, x)) d\mu_i(x) \right) - \int_{B_1^m(c) \cap B_1^m(c_j)} (\alpha_i - d(z, x)) d\mu_i(x) \right) \]
\[
- \sum_{j \in [k] \setminus \{i\}} \int_{B_1^m(z) \cap B_1^m(c_j)} (\alpha_j - d(z, x)) d\mu_j(x). \]

We will show that, under the assumptions of the theorem, the right hand side of (52) is positive for every \( z \in B_1^m(c_i) \setminus \{c_i\} \).

We now fix \( i \in [k] \) and \( z \in B_1^m(c_i) \setminus \{c_i\} \). If \( d(c_i, z) \leq \Theta/2 \), then for every \( j \in [k] \setminus \{i\} \), the set \( B_1^m(z) \cap B_1^m(c_j) \) contains at most one point. In this case, (a3) implies

\[
G^\alpha(c_i) - G^\alpha(z) = H(\alpha_i, \mu', m)(z - c_i) \geq T(\alpha_i, m)(z - c_i) > 0, \]

where \( \mu' \) is the image of \( \mu_i \) under the translation \( x' = x - c_i \). The first inequality follows by Lemma 14 and the last inequality follows by Lemma 10. Hence, in the remainder of the proof we assume \( d(c_i, z) > \Theta/2 \). Then, we must have \( \Theta/2 < 1 \), since \( 1 \geq d(c_i, z) > \Theta/2 \).

Next, we show that for every \( j \in [k] \setminus \{i\} \) we have

\[
\int_{B_1^m(z) \cap B_1^m(c_j)} (\alpha_j - d(z, x)) d\mu_j(x) \leq \frac{\sqrt{\pi}}{2} \frac{m}{2} \left( 1 - \frac{\Theta}{2(1 + \Theta)} \right)^{m-2}. \]

First consider the case \( d(c_j, z) \geq \alpha_j + 1 \). Then \( B_1^m(z) \cap B_1^m(c_j) \) contains at most one point and (a3) implies

\[
\int_{B_1^m(z) \cap B_1^m(c_j)} (\alpha_j - d(z, x)) d\mu_j(x) = 0 \leq \frac{\sqrt{\pi}}{2} \frac{m}{2} \left( 1 - \frac{\Theta}{2(1 + \Theta)} \right)^{m-2}. \]

Now consider the case \( d(c_j, z) < \alpha_j + 1 \). We obtain

\[
\int_{B_1^m(z) \cap B_1^m(c_j)} (\alpha_j - d(z, x)) d\mu_j(x) = R(\alpha_j, \mu', m)(z - c_j) \]
\[
\leq (\alpha_j + 1 - d(c_j, z)) \frac{\sqrt{\pi}}{2} \frac{m}{2} \left( \frac{\alpha_j}{d(c_j, z)} \right)^{m-2} \]
\[
\leq \left( 1 - \frac{\Theta}{2} \right) \frac{\sqrt{\pi}}{2} \frac{m}{2} \left( \frac{1 + \Theta}{1 + \Theta} \right)^{m-2}. \]
\[ \leq \frac{\sqrt{\pi}}{2} \sqrt{\frac{m^2}{2}} \left( 1 - \frac{\theta}{2(1 + \theta)} \right)^{m-2}, \]

where \( \mu'_j \) is the image of \( \mu_j \) under the translation \( x' = x - c_j \). The first inequality follows from Lemma 16 since \( d(c_j, z) \geq 1 + \theta > \alpha_j \) and in the second inequality we use \( d(c_j, z) \geq 1 + \theta \). This concludes the proof of (53).

On the other hand, we know from Lemma 15, with \( \epsilon := \theta/2 \in (0, 1) \), that

\[
\int_{B_i^n(c_i)} (\alpha_i - d(c_i, x)) d\mu_i(x) - \int_{B_i^n(z) \cap B_i^n(c_i)} (\alpha_i - d(z, x)) d\mu_i(x) = H^{(\alpha_i, \mu'_i, m)}(z - c_i) \geq \frac{\theta^2}{32} - \frac{\sqrt{\pi m^2}}{2} \left( 1 - \frac{\theta^2}{64} \right)^{m-2}.
\]

From (52), (53), and (54), we obtain

\[
G^\alpha(z) - G^\alpha(c_i) \geq \frac{\theta^2}{32} - \frac{\sqrt{\pi m^2}}{2} \left( 1 - \frac{\theta^2}{64} \right)^{m-2} - \frac{\sqrt{\pi}}{2} \sqrt{\frac{m^2}{2}} \left( 1 - \frac{\theta}{2(1 + \theta)} \right)^{m-2} - k \frac{\sqrt{\pi}}{2} \sqrt{\frac{m^2}{2}} \left( 1 - \frac{\theta}{2(1 + \theta)} \right)^{m-2}.
\]

Notice that the lower bound on \( G^\alpha(z) - G^\alpha(c_i) \) obtained in (55) is an increasing function in \( \theta \). We next show that \( G^\alpha(z) - G^\alpha(c_i) \) is positive when \( \theta > C \sqrt{k \log m/m} \), where \( C \) is a large constant. We have

\[
G^\alpha(z) - G^\alpha(c_i) \geq \frac{\theta^2}{32} - \frac{\sqrt{\pi m^2}}{2} \exp \left( -\frac{(m - 2)\theta^2}{128} \right) - k \frac{\sqrt{\pi}}{2} \sqrt{\frac{m}{2}} \exp \left( -\frac{(m - 2)\theta}{2(1 + \theta)} \right) - k \frac{\sqrt{\pi}}{2} \sqrt{\frac{m}{2}} \exp \left( -\frac{(m - 2)\theta}{2(1 + \theta)} \right).
\]
where, to simplify the notation, we let

\[
F(C, m) := \frac{m}{\log m} \sqrt{\frac{\pi m}{2}} \exp \left( - \frac{C^2(m - 2) \log m}{128m} \right) + \frac{m}{\log m} \frac{\sqrt{\pi}}{2} \sqrt{\frac{m}{2}} \exp \left( - \frac{(m - 2)C \sqrt{\log m/m}}{2(1 + C \sqrt{\log m/m})} \right).
\]

It then suffices to show that, for every \( m \geq 2 \), we have \( C^2/32 > F(C, m) \) for some constant \( C \) large enough. It can be checked that for every \( m \geq 2 \), \( F(C, m) \) is a decreasing function in \( C \). Also it can be checked that there is some threshold \( C' > 0 \) such that if \( C \geq C' \) then \( \lim_{m \to \infty} F(C, m) = 0 \). This implies that \( \sup\{ F(C, m) \mid C \geq C', m \geq 2 \} = \sup\{ F(C', m) \mid m \geq 2 \} < \infty \). Therefore it suffices to choose \( C > C' \) large enough so that \( C^2/32 > \sup\{ F(C, m) \mid C \geq C', m \geq 2 \} \). □

### 5.5 Proof of Theorem 8

For every \( i \in [k] \), let \( \mu_i := \mu + c_i \). We first show that for every \( i \in [k] \) and for every \( z \in \bigcup_{j \in [k]} B_1^m(c_j) \setminus \{c_j\} \) such that \( B_1^m(z) \cap B_1^m(c_i) \) has positive measure, we have

\[
\int_{B_1^m(z) \cap B_1^m(c_i)} (d(z, x) - d(c_i, x)) d\mu_i(x) > 0. \tag{56}
\]

Let \( H \) be the unique hyperplane that contains \( S_{i-1}^m(z) \cap S_{i-1}^m(c_i) \) (see Fig. 3). We obtain that the balls \( B_1^m(z) \) and \( B_1^m(c_i) \) are the reflection of each other with respect to \( H \). Let \( f(x) : B_1^m(z) \cap B_1^m(c_i) \to B_1^m(z) \cap B_1^m(c_i) \) be the function that reflects \( x \) with respect to \( H \). Let \( S_+ := \{ x \in B_1^m(z) \cap B_1^m(c_i) \mid d(z, x) - d(c_i, x) > 0 \} \) and \( S_- := \{ x \in B_1^m(z) \cap B_1^m(c_i) \mid d(z, x) - d(c_i, x) < 0 \} \). Let \( x_+ \in S_+ \) and let \( x_- := f(x) \). We then have \( x_- \in S_- \) since \( d(z, x_-) - d(c_i, x_-) = d(x_+, c_i) - d(x_+, z) < 0 \). Let \( p_i(x) \) be the density function of \( \mu_i(x) \). Since \( d(x_+, c_i) < d(c_i, x_-) \), the assumption of the theorem on \( p(x) \) implies that we have \( p_i(x_+) > p_i(x_-) \). We obtain

\[
\int_{B_1^m(z) \cap B_1^m(c_i)} (d(z, x) - d(c_i, x)) d\mu_i(x) = \int_{S_+} (d(z, x) - d(c_i, x)) p_i(x) dx + \int_{S_-} (d(z, x) - d(c_i, x)) p_i(x) dx = \int_{S_+} (d(z, x) - d(c_i, x))(p_i(x) - p_i(f(x))) dx > 0.
\]

This concludes the proof of (56).

Next we show that for every \( i \in [k] \) and for every \( z \in \bigcup_{j \in [k]} B_1^m(c_j) \setminus \{c_j\} \) such that \( B_1^m(z) \cap B_1^m(c_i) \) has positive measure, we have

\[
\int_{B_1^m(z) \cap B_1^m(c_i)} (d(z, x) - d(c_i, x)) d\mu_i(x) > 0 \quad \forall \tilde{a} > 1. \tag{57}
\]
Let $\tilde{S}_+ := \{ x \in B^m_\alpha(z) \cap B^m_1(c_i) \mid d(z, x) - d(c_i, x) > 0 \}$ and $\tilde{S}_- := \{ x \in B^m_\alpha(z) \cap B^m_1(c_i) \mid d(z, x) - d(c_i, x) < 0 \}$. Furthermore, let $S_+$ and $S_-$ be defined as in the proof of (56). Clearly we have $S_+ \subseteq \tilde{S}_+$. We observe that $S_- = \tilde{S}_-$. This is because $S_- \subseteq \tilde{S}_-$ and for every $x \in B^m_1(c_i)$ with $d(z, x) > 1$ we must have $d(z, x) - d(c_i, x) > 0$, which implies $x \notin \tilde{S}_-$. Then we have

$$\int_{B^m_\alpha(z) \cap B^m_1(c_i)} (d(z, x) - d(c_i, x))\,d\mu_i(x)$$

$$= \int_{S_+} (d(z, x) - d(c_i, x))\,p_i(x)\,dx + \int_{S_-} (d(z, x) - d(c_i, x))\,p_i(x)\,dx$$

$$\geq \int_{S_+} (d(z, x) - d(c_i, x))\,p_i(x)\,dx + \int_{S_-} (d(z, x) - d(c_i, x))\,p_i(x)\,dx$$

$$= \int_{B^m_1(z) \cap B^m_1(c_i)} (d(z, x) - d(c_i, x))\,d\mu_i(x) > 0,$$

where the last inequality holds by (56). This concludes the proof of (57).

Next, we claim that there exists $\varepsilon \in (0, \min_{i \neq j} d(c_i, c_j) - 2)$ such that, for every $i \in [k]$ and for every $z \in B^m_1(c_i)$, there exists a set $D_j$, for each $j \in [k]$, obtained from $B^m_{1+\varepsilon}(z) \cap B^m_1(c_j)$ via a rotation centered in $c_j$ followed by the translation $c_j - c_i$, such that the sets $D_j$, for $j \in [k]$, do not intersect. We now prove our claim. Let $i \in [k]$ and $z \in B^m_1(c_i)$. Note that, since the balls $B^m_1(c_j)$, for $j \in [k]$, do not intersect, we have that the sets $B^m_1(z) \cap B^m_1(c_j)$, for $j \in [k]$, do not intersect. Let $H$ be the unique hyperplane that contains $S^{m-1}_1(z) \cap S^{m-1}_1(c_i)$. It follows that also the reflections with respect to $H$ of the sets $B^m_1(z) \cap B^m_1(c_j)$, for $j \in [k]$, do not intersect. Note that the reflection with respect to $H$ of each set $B^m_1(z) \cap B^m_1(c_j)$ can be seen as the set
obtained from \( B^m_1(z) \cap B^m_1(c_j) \) by first applying a rotation centered in \( c_j \) and then the translation \( c_j - c_i \). Hence, we have shown that there exists a set \( D_j \), for each \( j \in [k] \), obtained from \( B^m_1(z) \cap B^m_1(c_j) \) via a rotation centered in \( c_j \) followed by the translation \( c_j - c_i \), such that the sets \( D_j \), for \( j \in [k] \), do not intersect. By continuity, for every \( i \in [k] \) and for every \( z \in B^m_1(c_i) \), there exists \( \epsilon_{i,z} > 0 \) small enough such that there exists a set \( D_j \), for each \( j \in [k] \), obtained from \( B^m_1(z+\epsilon_{i,z}) \cap B^m_1(c_j) \) via a rotation centered in \( c_j \) followed by the translation \( c_j - c_i \), such that the sets \( D_j \), for \( j \in [k] \), do not intersect. Since \( \cup_{i \in [k]} B^m_1(c_i) \) is a compact set, we can define \( \epsilon := \min(\epsilon_{i,z} \mid i \in [k], z \in B^m_1(c_i)) > 0 \). By eventually decreasing \( \epsilon \), we can also assume \( \epsilon < \min_{i \neq j} d(c_i, c_j) - 2 \), and this concludes the proof of our claim.

Let \( \alpha' := 1+\epsilon < \min_{i \neq j} d(c_i, c_j) - 1 \) and define \( \alpha_i := \alpha' \) for every \( i \in [k] \). In order to apply Corollary 1, it remains to show that for every \( i \in [k] \), \( c_i \) is the unique point that achieves \( \max\{ G^\alpha(z) \mid z \in B^m_1(c_i) \} \). We now fix \( i \in [k] \) and \( z \in B^m_1(c_i) \) for each \( j \in [k] \), let \( D_j \) be the set obtained from \( B^m_1(z) \cap B^m_1(c_j) \) as stated in the previous claim. Note that \( D_j \subseteq B^m_1(c_i) \). Since \( \mu_i \) is a translation of \( \mu_j \), we know that

\[
\int_{B^m_1(z) \cap B^m_1(c_j)} d(c_j, x) d\mu_j(x) = \int_{D_j} d(c_i, x) d\mu_i(x).
\]  

(58)

We obtain

\[
G^\alpha(c_i) = \sum_{j\in[k]} \int_{B^m_1(c_i) \cap B^m_1(c_j)} (\alpha' - d(c_i, x)) d\mu_j(x)
\]

\[
= \int_{B^m_1(c_i)} (\alpha' - d(c_i, x)) d\mu_i(x)
\]

\[
> \sum_{j\in[k]} \int_{D_j} (\alpha' - d(c_i, x)) d\mu_i(x)
\]

\[
= \sum_{j\in[k]} \int_{B^m_1(z) \cap B^m_1(c_j)} (\alpha' - d(c_j, x)) d\mu_j(x).
\]  

(59)

In the first equality we use Observation 2, in the second equality Lemma 6 (with \( a_i = b_i = \alpha' \) and \( z = c_i \)), in the inequality we use the fact that the sets and \( D_j \), for \( j \in [k] \) are disjoint subsets of \( B^m_1(c_i) \), and in the last equality we use (58). Therefore from (59) and Observation 2 we obtain

\[
G^\alpha(c_i) - G^\alpha(z) \geq \sum_{j\in[k]} \int_{B^m_1(z) \cap B^m_1(c_j)} (d(z, x) - d(c_j, x)) d\mu_j(x) > 0,
\]

where the inequality follows from (57). \( \square \)
In this section, we perform two sets of numerical experiments to illustrate the empirical performance of (LP) under the SBM and the ESBM.

In the first set of experiments, we consider the ESBM. Our goal is to show under the ESBM, even if balls have different radii and different probability measures, exact recovery can still happen. We draw $N = 20$ data points $\{x_i^{(1)}\}_{i=1}^N$ uniformly from $B_2^1(0)$ and $N$ data points $\{x_i^{(2)}\}_{i=1}^N$ uniformly from $B_R^2(\Delta, 0)$. We take $P = \{x_i^{(j)} \mid i \in \{1, 2\}, j \in [N]\}$ as the set of data points and perform an experiment by solving the corresponding (LP). We say that an experiment succeeds, if the (LP) achieves exact recovery. In our experiments, $\Delta \in [2, 4]$ and $R \in [1, 3]$. For each fixed pair of parameters $(\Delta, R)$, we perform 10 independent experiments and compute the empirical probability of success. Figure 4 shows the change of empirical probability of success according to the parameter pairs $(\Delta, R)$. In Fig. 4, each point below the diagonal of the figure corresponds to a pair of parameters $(\Delta, R)$ such that the two balls $B_1^2(0)$ and $B_R^2(\Delta, 0)$ are separated. In such case, it is clear from Fig. 4 that (LP) achieves exact recovery with high probability. Also, when we fix the parameter $\Delta$, the probability of success becomes lower as $R$ increases, which implies that a larger separation is needed when the radii of the two balls are significantly different.

In the second set of experiments, we consider the SBM. Our goal is to show that, when the balls are significantly separated, (LP) achieves exact recovery with high probability; furthermore, such a significant separation is also necessary. We construct a probability measure $\mu$ over $B_1^2(0)$, that is invariant under rotations centered in 0. We let $x$ be a point drawn according to $\mu$. With probability 0.9, $\|x\|$ is uniformly distributed over $[0, 0.99]$, and with probability 0.1, $\|x\|$ is uniformly distributed over $[0.99, 1]$. We first draw $N$ points $\{x_i^{(0)}\}_{i=1}^N$ according to $\mu$. For $i \in [N]$, we take $x_i^{(1)} := x_i^{(0)} + (\Delta, 0)$ as an input data point in $B_1^2(\Delta, 0)$. Next, we draw $N$ more input
Fig. 5  Numerical experiments for Theorem 6. We plot the empirical probability of success of (LP) under the SBM for two unit balls, with parameters $\Delta \in [2, 4]$ and $N \in [5, 35]$. Deeper color represents higher probability (color figure online).

points $\{x_i^{(2)}\}_{i=1}^N$ according to $\mu$. We take $P = \{x_j^{(i)} \mid i \in \{1, 2\}, j \in [N]\}$ as the set of input data points and we perform an experiment by solving the corresponding (LP) as we discussed above. Let $\Delta \in [2, 4]$ and $N \in [5, 35]$. For each fixed pair of parameters $(\Delta, N)$, we perform 10 independent experiments and we compute the empirical probability of success. We can see from Fig. 5 that, as stated in Theorem 6, when $\Delta > 3.29$, (LP) achieves exact recovery with high probability and almost all experiments succeed. However, exact recovery does not happen very often if two balls are not separated significantly. In fact, in Fig. 5, a phase transition happens around $\Delta \approx 2.25$. When $\Delta > 2.25$, as $N$ increases, the probability of success becomes higher. On the contrary, when $\Delta < 2.25$, as $N$ increases, the probability of success becomes even lower. In particular, we can see that no experiment succeeds when $\Delta < 2.25$ and $N > 30$. This implies that for (LP) to succeed with high probability, a significant separation between ball centers is not only sufficient but also necessary.

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Appendix A: On the assumption $n_i = \beta_i n$ in the ESBM

In this section, we present an example which justifies the assumption that in the ESBM the number $n_i$ of points drawn from each ball $i \in [k]$ satisfies $n_i = \beta_i n$. In fact, Example 1 shows that if in the SBM we allow to draw different numbers $n_i$ of data points from different balls, and the $n_i$ are of different orders, then with high probability (LP) does not achieve exact recovery, no matter how distant the balls are. In the following example we denote by $e_1, \ldots, e_m$ the vectors of the standard basis of $\mathbb{R}^m$.
Example 1 Consider the SBM with $k = 2$. Let $c_1 := 0$ and let $c_2 := d e_1$ where $d > 2$. Let $\mu$ be the uniform probability measure on $B_{<0}^1(0)$.

For each $i \in [2]$, we draw $n_i(n)$ random vectors instead of $n$ as in the definition of the SBM, and we assume that $\lim_{n \to \infty} n_1/n_2 = \infty$. Then with high probability every feasible solution to (IP) that assigns each point to the ball from which it is drawn is not optimal to (IP).

Proof For every $i \in [k]$, we denote by $x^{(i)}_*$ the median of $\{x^{(i)}_{\ell}\}_{\ell \in [n_i]}$. Among all the feasible solutions $(y, z)$ to (IP) that assign each point to the ball from which it is drawn, the ones with the smaller objective function have the property that, for every $i \in [k]$, the component of the vector $y^*$ corresponding to $x^{(i)}_*$ is equal to one. Let $(y^*, z^*)$ be such a solution. It then suffices to show that with high probability $(y^*, z^*)$ is not optimal to (IP).

We first evaluate the objective value $\text{obj}^*$ of $(y^*, z^*)$. We have
\[
\frac{\text{obj}^*}{n_1 + n_2} = \frac{\sum_{\ell \in [n_1]} d(x^{(1)}_{\ell}, x^{(1)}_*)}{n_1} \frac{n_1}{n_1 + n_2} + \frac{\sum_{\ell \in [n_2]} d(x^{(2)}_{\ell}, x^{(2)}_*)}{n_2} \frac{n_2}{n_1 + n_2} \geq \frac{\sum_{\ell \in [n_1]} d(x^{(1)}_{\ell}, x^{(1)}_*)}{n_1} \frac{n_1}{n_1 + n_2}.
\]

Let $x$ be a random vector drawn according to $\mu$. Since $\mu$ is the uniform probability measure, we know that $\mathbb{E} \|x\| \in (0, 1)$. Let $\epsilon \in (0, 1)$ be a small number. From Lemma 5, we know that with high probability we have $\left| \sum_{\ell \in [n_1]} d(x^{(1)}_{\ell}, x^{(1)}_*) / n_1 - \mathbb{E} \|x\| \right| < \epsilon$. Since $\lim_{n \to \infty} n_1/n_2 = \infty$, we know that when $n$ is large enough, with high probability, we have
\[
\frac{\text{obj}^*}{n_1 + n_2} \geq \frac{\sum_{\ell \in [n_1]} d(x^{(1)}_{\ell}, x^{(1)}_*)}{n_1} \frac{n_1}{n_1 + n_2} > \mathbb{E} \|x\| - 2\epsilon.
\]

Consider now the point $s := -e_1/2 \in B_{>0}^1(0)$, and define the sets $S_1 := \{x \in B_{>0}^1(0) \mid x_1 \leq -\frac{1}{2}\}$ and $S_2 := \{x \in B_{>0}^1(0) \mid x_1 > -\frac{1}{2}\}$. Let $x$ be a random vector drawn according to $\mu$. We know that when $x \in S_1$, we have $d(s, x) < \|x\|$. To simplify the notation, let $\xi := \min\{\|x\| - d(s, x) \mid x \in S_1\} > 0$. Then we have
\[
\mathbb{E} \min\{d(s, x), \|x\|\} = \int_{B_{>0}^1(0)} \min\{d(s, x), \|x\|\} d\mu(x) \leq \int_{S_1} d(s, x) d\mu(x) + \int_{S_2} \|x\| d\mu(x) \leq \int_{B_{>0}^1(0)} \|x\| d\mu(x) - \xi \mathbb{P}(x \in S_1) = \mathbb{E} \|x\| - \xi \mathbb{P}(x \in S_1).
\]
(60)

Note that $\min\{|d(x^{(1)}_{\ell}, s), \|x^{(1)}_*\|\}$, for $\ell \in [n_1]$, are independent random variables bounded by the interval $[0, 1]$. From Hoeffding’s inequality, with high probability we
have
\[
\left| \sum_{\ell \in [n_1]} \min \{d(x^{(1)}_{\ell}, s), \|x^{(1)}_{\ell}\| \} \right| < \epsilon.
\]

So with high probability, we obtain
\[
\sum_{\ell \in [n_1]} \min \{d(x^{(1)}_{\ell}, s), \|x^{(1)}_{\ell}\| \} \leq \mathbb{E} \min \{d(s, x), \|x\| \} < \epsilon.
\]

where the last inequality follows from (60). Since \( \mu \) is the uniform probability measure on \( B^m_\epsilon(0) \), we know that with high probability there is some point \( \tilde{x} \in B^m_\epsilon(s) \cap \{x^{(1)}_{\ell}\}_{\ell \in [n_1]} \) and there is some point \( \bar{x} \in B^m_\epsilon(0) \cap \{x^{(1)}_{\ell}\}_{\ell \in [n_1]} \). Now we construct a feasible solution \((y', z')\) to (IP) with objective value \( \text{obj}' \) such that \( \text{obj}' < \text{obj}^* \). We choose \( \tilde{x} \) and \( \bar{x} \) as the centers of the two clusters. We then assign each point \( x^{(i)}_{\ell} \), for \( i \in [2] \) and \( \ell \in [n_1] \), to the cluster with the closest center. Let \((y', z')\) be the feasible solution to (IP) corresponding to this choice. Then we have
\[
\frac{\text{obj}'}{n_1 + n_2} = \frac{\sum_{\ell \in [n_1]} \min \{d(x^{(1)}_{\ell}, \tilde{x}), d(x^{(1)}_{\ell}, \bar{x})\} n_1}{n_1 + n_2} + \frac{\sum_{\ell \in [n_2]} \min \{d(x^{(2)}_{\ell}, \tilde{x}), d(x^{(2)}_{\ell}, \bar{x})\} n_2}{n_1 + n_2} \leq \frac{\sum_{\ell \in [n_1]} \min \{d(x^{(1)}_{\ell}, \tilde{x}), d(x^{(1)}_{\ell}, \bar{x})\} n_1}{n_1 + n_2} + (d + 2) \frac{n_2}{n_1 + n_2}.
\]

Here, the inequality follows because \( \min \{d(x^{(2)}_{\ell}, \tilde{x}), d(x^{(2)}_{\ell}, \bar{x})\} \leq d + 2 \) for every \( \ell \in [n_2] \). Next, we use the fact that when \( n \) is large enough \( (d + 2)n_2/(n_1 + n_2) \) can be arbitrarily small and thus can be bounded by \( \epsilon \). So with high probability we obtain
\[
\frac{\text{obj}'}{n_1 + n_2} \leq \frac{\sum_{\ell \in [n_1]} \min \{d(x^{(1)}_{\ell}, \tilde{x}), d(x^{(1)}_{\ell}, \bar{x})\} n_1}{n_1} + \epsilon
\leq \frac{\sum_{\ell \in [n_1]} \min \{d(x^{(1)}_{\ell}, s), \|x^{(1)}_{\ell}\|\} n_1}{n_1} + \epsilon
\leq \mathbb{E} \|x\| - \xi \mathbb{P}(x \in S_1) + 3\epsilon,
\]

where the second inequality follows by the triangle inequality, and in the last inequality we use (61).

Notice that when \( \epsilon < \xi \mathbb{P}(x \in S_1)/5 \), we have \( \text{obj}' < \text{obj}^* \), which implies that with high probability, \((y^*, z^*)\) is not an optimal solution to (IP).  
\( \square \)
Appendix B: Counterexample to Theorem 7 in [9]

In this section we present an example which shows that Theorem 7 in [9] is false. In Example 2, we construct a probability measure that satisfies all assumptions in the statement of Theorem 7 in [9] and $\min_{i \neq j} d(c_i, c_j) = 2.2$. We then show that with high probability (LP) does not achieve exact recovery. The key problem in the proof of Theorem 7 in [9] is discussed in Appendix C.

Example 2 There is an instance of the SBM with $m = 2$, $k = 7$, $\min_{i \neq j} d(c_i, c_j) = 2 + 0.2$, where $\mu$ has a continuous density function and the probability space $(\mu, B_1^m(0))$ satisfies (a1), (a2), such that with high probability (LP) does not achieve exact recovery.

Proof Let $\mu$ be a probability measure that has a continuous density function and the probability space $(\mu, B_1^m(0))$ satisfies (a1), (a2). Let $\epsilon \in (0, 1)$ be a small number. We further assume that $\mu$ and $\epsilon$ satisfy

$$(0.292 - 8\epsilon) \Pr(\|x\| \geq 1 - \epsilon) > 0.279 + 6\epsilon + (3 + 2\epsilon) \Pr(\|x\| < 1 - \epsilon).$$

(62)

Note that assumption (62) can be fulfilled as long as $\Pr(\|x\| < 1 - \epsilon)$ and $\epsilon$ are small enough. We define $c_1 := 0$ and, using polar coordinates, $c_i := (2.2, -(i - 2)\pi/3)$ for every $i \in [7] \setminus \{1\}$ (see Figure 6).

In particular, for every $i, j \in [7]$ with $i \neq j$, we have $d(c_i, c_j) = 2.2$. This concludes the description of the instance of the SBM that we consider. In the remainder of the example we show that with high probability (LP) does not achieve exact recovery.
For every $z \in B^m_1(0)$, let $c(z)$ be a point among $c_2, \ldots, c_7$ that is closest to $z$ and we define $f(z) := d(z, c(z))$. For every $z \in B^m_1(0)$, let $\theta(z)$ be the angle between the vectors $z$ and $c(z)$. Clearly, for every $z \in B^m_1(0)$, we have $\theta(z) \in [0, \pi/6]$. Let $z$ be a random vector drawn according to $\mu$. Since $\mu$ satisfies (a1), we know that the random variable $\theta(z)$ is uniform on $[0, \pi/6]$, thus its density function is constant on $[0, \pi/6]$ and equal to $6/\pi$.

Next, we show the upper bound

$$
\int_{B^m_1(0)} (f(x) + 2\epsilon - \|x\|) d\mu(x) < 0.279 + 4\epsilon + (3 + 2\epsilon)\mathbb{P}(\|x\| < 1 - \epsilon). \tag{63}
$$

Let $z \in S^{m-1}_1(0)$, then $f(z) = \sqrt{(2.2)^2 + 1 - 4.4 \cos \theta}$. Let $L := B^m_1(0) \setminus B^m_{1-\epsilon}(0)$. The triangle inequality implies that for every $z \in L$ we have

$$
f(z) < \sqrt{(2.2)^2 + 1 - 4.4 \cos \theta(z)} + \epsilon, \quad d(z, 0) \geq 1 - \epsilon. \tag{64}
$$

So we obtain

$$
\int_{B^m_1(0)} (f(x) + 2\epsilon - \|x\|) d\mu(x) = \int_{B^m_{1-\epsilon}(0)} (f(x) + 2\epsilon - \|x\|) d\mu(x)
+ \int_{L} (f(x) + 2\epsilon - \|x\|) d\mu(x)
\leq (3 + 2\epsilon)\mathbb{P}(\|x\| < 1 - \epsilon) + \int_{L} (f(x) + 2\epsilon - \|x\|) d\mu(x)
\leq (3 + 2\epsilon)\mathbb{P}(\|x\| < 1 - \epsilon)
+ \int_{L} \left(\sqrt{(2.2)^2 + 1 - 4.4 \cos \theta(x)} + \epsilon + 2\epsilon - (1 - \epsilon)\right) d\mu(x)
\leq (3 + 2\epsilon)\mathbb{P}(\|x\| < 1 - \epsilon)
+ \frac{6}{\pi} \int_{0}^{\pi/6} \left(\sqrt{(2.2)^2 + 1 - 4.4 \cos \theta} + \epsilon + 2\epsilon - (1 - \epsilon)\right) d\theta
= (3 + 2\epsilon)\mathbb{P}(\|x\| < 1 - \epsilon)
+ \frac{6}{\pi} \int_{0}^{\pi/6} \left(\sqrt{(2.2)^2 + 1 - 4.4 \cos \theta} - 1\right) d\theta + 4\epsilon
< 0.279 + 4\epsilon + (3 + 2\epsilon)\mathbb{P}(\|x\| < 1 - \epsilon).
$$

Here, the first inequality uses the fact that $f(x) \leq 3$ for every $x \in B^m_{1-\epsilon}(0)$ and the second inequality holds because of (64). The third inequality follows by the fact that $\theta(x)$ does not depend on $\|x\|$ and has a density function $\pi/6$. In the last inequality, we use the fact that

$$
\frac{6}{\pi} \int_{0}^{\pi/6} \left(\sqrt{(2.2)^2 + 1 - 4.4 \cos \theta} - 1\right) d\theta < 0.279.
$$
This concludes the proof of (63).

Let $s := e_1$, where $e_1$ is the first vector of the standard basis of $\mathbb{R}^m$, and let $\mu_i := \mu + c_i$ for every $i \in [k]$. Next, we prove the lower bound

$$\sum_{i=1}^{7} \int_{B^m_1(c_i)} (d(x, c_i) - 2\epsilon - d(x, s))_+ d\mu_i(x) > (0.292 - 8\epsilon)\mathbb{P}(\|x\| \geq 1 - \epsilon).$$

(65)

For ease of notations we give the following definitions. For every $x \in B^m_1(0)$, let $\psi(x)$ be the angle between the vectors $x$ and $s$. For every $x \in B^m_1(c_2)$, let $\phi(x)$ be the angle between $x - c_2$ and $s - c_2$. We also define $L_1 := \{x \in B^m_1(0) \mid \psi(x) \leq \pi/3, \|x\| \geq 1 - \epsilon\}$ and $L_2 := \{x \in B^m_1(c_2) \mid \phi(x) \leq \theta', d(x, c_2) \geq 1 - \epsilon\}$, where $\theta' := \arccos 0.6$.

Notice that for every $x \in S^m_{i-1}(0)$, we have $d(x, s) = \sqrt{2 - 2\cos\psi(x)}$ and for every $x \in S^m_{i-1}(c_2)$, we have $d(x, s) = \sqrt{(1.2)^2 + 1 - 2.4\cos\phi(x)}$. Using the triangle inequality, we obtain

$$d(x, s) \leq \sqrt{2 - 2\cos\psi(x) + \epsilon} \quad \forall x \in L_1,$$

$$d(x, s) \leq \sqrt{(1.2)^2 + 1 - 2.4\cos\phi(x) + \epsilon} \quad \forall x \in L_2.$$ 

(66) (67)

We obtain

$$\sum_{i=1}^{7} \int_{B^m_1(c_i)} (d(x, c_i) - 2\epsilon - d(x, s))_+ d\mu_i(x)$$

$$\geq \int_{L_1} (\|x\| - 2\epsilon - d(x, s))_+ d\mu(x) + \int_{L_2} (d(x, c_2) - 2\epsilon - d(x, s))_+ d\mu_2(x)$$

$$\geq \int_{L_1} (1 - 3\epsilon - d(x, s)) d\mu(x) + \int_{L_2} (1 - 3\epsilon - d(x, s)) d\mu_2(x)$$

$$\geq \int_{L_1} \left(1 - 4\epsilon - \sqrt{2 - 2\cos\psi(x)}\right) d\mu(x)$$

$$+ \int_{L_2} \left(1 - 4\epsilon - \sqrt{(1.2)^2 + 1 - 2.4\cos\phi(x)}\right) d\mu_2(x)$$

$$= \mathbb{P}(\|x\| \geq 1 - \epsilon) \frac{1}{\pi} \int_0^{\pi/3} \left(1 - 4\epsilon - \sqrt{2 - 2\cos\psi}\right) d\psi$$

$$+ \mathbb{P}(\|x\| \geq 1 - \epsilon) \frac{1}{\pi} \int_0^{\theta'} \left(1 - 4\epsilon - \sqrt{(1.2)^2 + 1 - 2.4\cos\phi}\right) d\phi$$

$$> (0.292 - 8\epsilon)\mathbb{P}(\|x\| \geq 1 - \epsilon).$$

Here, the second inequality follows from the definition of $L_1$ and $L_2$. The third inequality follows by (66) and (67). The equality holds because $\psi(x)$ does not depend on $\|x\|$.
and $\phi(x)$ does not depend on $d(c_2, x)$. The last inequality holds because

$$
\frac{1}{\pi} \int_{0}^{\pi} \left(1 - \sqrt{2 - 2\cos \psi}\right) d\psi + \frac{1}{\pi} \int_{0}^{\theta'} \left(1 - \sqrt{(1.2)^2 + 1 - 2.4\cos \phi}\right) d\phi > 0.292.
$$

This completes the proof of (65).

Using Hoeffding’s inequality, with high probability we have

$$
\frac{1}{n} \sum_{\ell \in [n]} (f(x^{(1)}_{\ell}) + 2\epsilon - ||x^{(1)}_{\ell}||) - \int_{B^m(0)} (f(x) + 2\epsilon - ||x||) d\mu(x) < \epsilon,
$$

and using (63) with high probability we have

$$
\frac{1}{n} \sum_{\ell \in [n]} (f(x^{(1)}_{\ell}) + 2\epsilon - ||x^{(1)}_{\ell}||) < \int_{B^m(0)} (f(x) + 2\epsilon - ||x||) d\mu(x) + \epsilon < 0.279 + 5\epsilon + (3 + 2\epsilon)\mathbb{P}(||x|| < 1 - \epsilon).
$$

Using Hoeffding’s inequality, with high probability we have

$$
\sum_{i=1}^{7} \int_{B^m(c_i)} (d(x, c_i) - 2\epsilon - d(x, s))_+ d\mu_i(x) - \frac{1}{n} \sum_{i \in [7]} \sum_{\ell \in [n]} (d(x^{(i)}_{\ell}, c_i) - 2\epsilon - d(x^{(i)}_{\ell}, s))_+ < \epsilon,
$$

and using (65) with high probability we have

$$
\frac{1}{n} \sum_{i \in [7]} \sum_{\ell \in [n]} (d(x^{(i)}_{\ell}, c_i) - 2\epsilon - d(x^{(i)}_{\ell}, s))_+ \geq \sum_{i=1}^{7} \int_{B^m(c_i)} (d(x, c_i) - 2\epsilon - d(x, s))_+ d\mu_i(x) - \epsilon \geq (0.292 - 8\epsilon)\mathbb{P}(||x|| \geq 1 - \epsilon) - \epsilon.
$$

For every $i \in [k]$, we denote by $x^{(i)}_*$ the median of $\{x^{(i)}_{\ell}\} \in [n]$. Among all the feasible solutions $(y, z)$ to (IP) that assign each point to the ball from which it is drawn, the ones with the smaller objective function have the property that, for every $i \in [k]$, the component of the vector $y^*$ corresponding to $x^{(i)}_*$ is equal to one. Let $(y^*, z^*)$ be such a solution. It then suffices to show that with high probability $(y^*, z^*)$ is not optimal to (LP).

We know from Lemma 4 that with high probability, for every $i \in [7]$, we have $d(x^{(i)}_*, c_i) < \epsilon$. Next, we show that we can use Theorem 1 to prove that $(y^*, z^*)$ is not
optimal to (LP) with high probability. To do so, we just need to show that there is no \( \alpha \) that satisfies conditions (7)–(10).

For ease of notation we denote by \( \alpha^{(i)}_\ell \) the component of \( \alpha \) corresponding to the point \( x^{(i)}_\ell \). Suppose that \( \alpha \) satisfies (9) and (10), thus \( d(x^{(i)}_\star, x^{(i)}_\ell) \leq \alpha^{(i)}_\ell \leq d(x^{(j)}_\star, x^{(i)}_\ell) \) for every \( i, j \in [k] \) with \( i \neq j \) and for every \( \ell \in [n] \). Then we have

\[
\frac{1}{n} C^\alpha(x^{(1)}_\star) = \frac{1}{n} \sum_{\ell \in [n]} (\alpha^{(i)}_\ell - d(x^{(i)}_\star, x^{(i)}_\ell)) \leq \frac{1}{n} \sum_{\ell \in [n]} (f(x^{(i)}_\ell) + 2\epsilon - \|x^{(i)}_\ell\|) < 0.279 + 5\epsilon + (3 + 2\epsilon)P(\|x\| < 1 - \epsilon),
\]

where in the first inequality we use the fact that \( \alpha^{(i)}_\ell \leq d(x^{(j)}_\star, x^{(i)}_\ell) \leq d(c_j, x^{(i)}_\ell) + \epsilon \) for every \( j \in [7] \setminus \{1\} \) and \( d(x^{(i)}_\star, x^{(i)}_\ell) \geq \|x^{(i)}_\ell\| - \epsilon \), and the second inequality follows from (68).

Let \( N := B^m_\epsilon(s) \cap L \) and note that the assumption (62) on \( \mu \) imply that with high probability there exists a point \( x' \in N \cap \{x^{(i)}_\ell\}_{\ell \in [n_\ell]} \). We have

\[
\frac{1}{n} C^\alpha(x') = \frac{1}{n} \sum_{i \in [7]} \sum_{\ell \in [n]} (\alpha^{(i)}_\ell - d(x', x^{(i)}_\ell))_+ \geq \frac{1}{n} \sum_{i \in [7]} \sum_{\ell \in [n]} (d(c_i, x^{(i)}_\ell) - 2\epsilon - d(s, x^{(i)}_\ell))_+ > (0.292 - 8\epsilon)P(\|x\| \geq 1 - \epsilon) - \epsilon,
\]

where in the first inequality we use that for every \( i \in [7] \) we have \( \alpha^{(i)}_\ell \geq d(x^{(i)}_\ell, x^{(i)}_\ell) \geq d(c_i, x^{(i)}_\ell) - \epsilon \) and \( d(x', x^{(i)}_\ell) \geq d(s, x^{(i)}_\ell) - \epsilon \), and the second inequality follows from (69). The inequalities (70) and (71) imply \( C^\alpha(x') > C^\alpha(x^{(1)}_\star) \) due to assumption (62). This implies that conditions (7),(8) cannot hold. Thus, according to Theorem 1, with high probability (LP) does not achieve exact recovery. □

**Appendix C: Problem in the proof of Theorem 7 in [9]**

In this section we point out the key problem in the proof of Theorem 7 in [9]. To prove this theorem, the authors introduce two conditions: the separation condition and the central dominance condition. When the two conditions happen together, then (LP) achieves exact recovery. We refer the reader to [9] for more details about these two conditions. In the proof of Theorem 7 the authors show that the separation condition happens with high probability according to the law of large number, while the central dominance condition happens in expectation and thus happens with high probability. Formally, the authors prove the following lemma about the central dominance condition.
Lemma 17 (Lemma 13 in [9]) In the hypothesis of Theorem 7, there exists $\alpha > 1$ such that for all $j \in [k]$, $\mathbb{E} P^{(\alpha, \ldots, \alpha)}(z)$ restricted to $z \in B_1^m(c_j)$ attains its maximum in $z = c_j$.

In the proof of Lemma 17, the goal of the authors is to obtain some $\alpha > 1$ such that $c_i$ achieves $\max \{\mathbb{E} P^{(\alpha, \ldots, \alpha)}(z) \mid z \in B_1^m(c_i)\}$ for every $i \in [k]$, where

$$\mathbb{E} P^{(\alpha, \ldots, \alpha)}(z) = \sum_{i \in [k]} \int_{x \in B_1^m(c_i)} (\alpha - d(z, x)) + d\mu_i(x).$$

In order to do so, they select some $\alpha > 1$ such that for every $i \in [k]$ and for every $z \in B_1^m(c_i)$, the sets $B_1^m(z) \cap \bigcup_{j \neq i} B_1^m(c_j)$, for $j \in [k] \setminus \{i\}$, can be copied isometrically inside $B_1^m(c_i)$ along the boundary without intersecting each other. Their goal is to use the fact that $B_1^m(c_i)$ contains all these copies to show that $\mathbb{E} P^{(\alpha, \ldots, \alpha)}(c_i) > \mathbb{E} P^{(\alpha, \ldots, \alpha)}(z)$. The problem is that, although the copies have the same area of the original sets, the density function may differ from a point $x \in B_1^m(z) \cap \bigcup_{j \neq i} B_1^m(c_j)$ to the corresponding point $x' \in B_1^m(c_i)$ with $d(z, x) = d(c_i, x')$. If the probability measure is anti-concentrated, which means that the area near the boundary of each ball has a very large probability, then the choice of $\alpha$ given by the authors may cause $\mathbb{E} P^{(\alpha, \ldots, \alpha)}(c_i) < \mathbb{E} P^{(\alpha, \ldots, \alpha)}(z)$ for some $z \in B_1^m(c_i) \setminus \{c_i\}$.

We also remark that there is also a requirement omitted in the statement of Lemma 17. In fact, in the statement the authors require $\alpha > 1$. However, in order to satisfy the central dominance condition, $\alpha$ cannot be chosen too large. In particular, the requirement $\alpha < 1 + \Theta$, where $\Theta = \min_{j \neq i} d(c_i, c_j) - 2$, should be added to the lemma.

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