ITERATED MONODROMY GROUPS OF INTERMEDIATE GROWTH

ASHLEY S. DOUGHERTY, LYDIA R. KINDELIN, AARON M. REAVES, ANDREW J. WALKER, AND NATHANIEL F. ZAKAHI

ABSTRACT. We give new examples of groups of intermediate growth, by a method that was first developed by Grigorchuk and later adapted by Bux and Pérez. Our examples are the groups generated by the automata with the kneading sequences 11(0)ω and 0(011)ω. By results of Nekrashevych, both of these groups are iterated monodromy groups of complex post-critically finite quadratic polynomials.

We include a complete, systematic description of Bux and Pérez’s adaptation of Grigorchuk’s method. We also prove, as a sample application of this method, that the groups determined by the automata with kneading sequence 1(0k)ω (k ≥ 2) have intermediate growth, although this result is implicit in a survey article by Bartholdi, Grigorchuk, and Sunik.

The paper concludes with an example of a group with no admissible length function; i.e., the group in question admits no length function with the properties required by the arguments of Bux and Pérez. Whether the group has intermediate growth appears to be an open question.

1. INTRODUCTION

Let p : C → C be a complex polynomial. We say that p is post-critically finite if, for each critical point c, the set of all forward iterates \{p(p(...(c)))\} of c is a finite set. Nekrashevych [7] has shown how to associate a group, called an iterated monodromy group, to any post-critically finite complex polynomial. The iterated monodromy group of p, denoted IMG(p), acts on an infinite rooted n-ary tree if the degree of p is n.

We first began this project because of our interest in the following conjecture from [2], where it is attributed to Nekrashevych:

Conjecture 1.1. If p : C → C is a post-critically finite quadratic polynomial with pre-periodic kneading sequence, then IMG(p) has intermediate growth.

The first positive evidence was obtained by Bux and Pérez [2], who showed that IMG(z^2+i) has subexponential growth. (The proof that IMG(z^2+i) also has superpolynomial growth is comparatively straightforward, so their work proves that IMG(z^2+i) has intermediate growth.) There are known counterexamples, however. Grigorchuk and Zuk [6] showed that IMG(z^2−1) has exponential growth. The tuning of z^2−1 by z^2+i results in a post-critically finite quadratic polynomial g(z) = z^2 + c with pre-periodic kneading sequence such that IMG(z^2−1) embeds in IMG(g). It follows easily that IMG(g) also has exponential growth, making it a counterexample to Conjecture 1.1. (Note that z^2−1 has a periodic kneading sequence.)

2000 Mathematics Subject Classification. 20F65, 37F20.
Key words and phrases. Iterated monodromy group, intermediate growth, automaton.
sequence, so it is not a counterexample in itself.) The following conjecture appears to be open:

**Conjecture 1.2.** If \( p : \mathbb{C} \to \mathbb{C} \) is a non-renormalizable post-critically finite quadratic polynomial with pre-periodic kneading sequence, then \( \text{IMG}(p) \) has intermediate growth.

The hypothesis of non-renormalizability rules out the counterexamples to Conjecture 1.1 that arise from tuning.

Our goal here is to give two more examples in support of the latter conjecture, namely the iterated monodromy groups of polynomials with the kneading sequences \( 11(0)^\omega \) and \( 0(011)^\omega \). (The kneading sequence of a kneading automaton is described in Definition 2.15. Theorem 2.21 says that the latter definition agrees with the classical definition of the kneading sequence of a polynomial up to relabeling.) We also give a short proof that the groups generated by the automata with the kneading sequences of the form \( 1(0^k)^\omega \) have intermediate growth, although a proof that these groups have intermediate growth can be obtained from Theorem 10.5 of [1]. A secondary goal is to provide an exposition of the methods of Bux and Pérez. While formulated differently, the key idea in their arguments, Proposition 10 of [2], is based upon the work of Grigorchuk in his proof that the First Grigorchuk group has subexponential growth [5]. We attempt to isolate the precise hypotheses that are necessary to make their arguments work, and state general theorems. The main result in this direction is Theorem 3.23 which gives a simple sufficient condition for the group of a kneading automaton over an alphabet with two letters to have subexponential growth.

The paper is structured as follows. In Section 2, we review the definition of automata, and explain how an automaton can be used to define a group that acts by automorphisms on a rooted tree. Section 3 contains an exposition of Bux and Pérez’s formulation of Grigorchuk’s method. Section 4 contains proofs that the groups determined by the automata with the kneading sequences \( 1(0^k)^\omega \) \((k \geq 1)\), \( 11(0)^\omega \), and \( 0(011)^\omega \) have intermediate growth. In Section 5 we give an example of a group defined by an automaton with pre-periodic kneading sequence to which this method does not apply. (Specifically, the group in question has no admissible length function – see Definition 3.9.) All of the results in Sections 4 and 5 were proved by the authors at SUM-SRI, an REU program based at Miami University, during the summer of 2011. The authors also produced an article as part of the REU, which can be found at www.units.muohio.edu/sumsri/sumj/2011/fp_alg.pdf. The material in Section 3 was contributed by Daniel Farley.

The authors would like to thank Rodrigo Pérez for clarifying the status of Conjecture 1.1 to us.

2. Background

Essentially all of the material in this section has appeared in [7]. We gather it here (sometimes in slightly altered form) for the reader’s convenience.

2.1. Automata, Moore Diagrams, Trees.

**Definition 2.1.** Let \( X \) be an alphabet. An automaton \( A \) over \( X \) is given by:

(1) a set of states, usually also denoted \( A \);
(2) a map $\tau: A \times X \to X \times A$.

For each state $a \in A$, we define a function $\tau_a: X \to X$ by the rule $\tau_a(x) = \pi_1(\tau(a, x))$, where $\pi_1: X \times A \to X$ is projection on the first coordinate. If $\tau_a \neq \text{id}_X$, we say that $a$ is an active state. We say that the automaton $A$ is invertible if each $\tau_a: X \to X$ is a bijection.

Automata can be conveniently described using Moore diagrams.

**Definition 2.2.** Let $A$ be an automaton over the alphabet $X$. The Moore diagram for $A$ is a directed labelled graph $\Gamma$, defined as follows. The vertices of $\Gamma$ are the states of $A$. If $a, b \in A$ and $\tau(a, x) = (y, b)$, then there is a directed edge from $a$ to $b$ with the label $(x, y)$.

**Example 2.3.** We define an automaton $A$ as follows. The states are $a$, $b$, $t$, and $\text{id}$. The alphabet is $X = \{0, 1\}$. We define the function $\tau: A \times X \to X \times A$ by the rule:

- $\tau(a, 0) = (0, \text{id})$
- $\tau(b, 0) = (0, b)$
- $\tau(a, 1) = (1, t)$
- $\tau(b, 1) = (1, a)$
- $\tau(t, 0) = (1, \text{id})$
- $\tau(id, 0) = (0, id)$
- $\tau(t, 1) = (0, id)$
- $\tau(id, 1) = (1, id)$

The Moore diagram for this automaton is pictured in Figure 1.

![Figure 1. The Moore diagram for the automaton from Example 2.3.](image)

**Definition 2.4.** Let $X$ be an alphabet. We let $X^*$ denote the free monoid generated by $X$. Thus, $X^*$ is the collection of all positive strings in the letters of $X$, including the empty string. We can associate to $X^*$ a tree, which we also denote $X^*$, as follows: the vertices of the tree are the members of $X^*$, and there is an edge connecting two vertices $w_1, w_2 \in X^*$ if and only if $w_2 = w_1 x$, for some $x \in X$ (or $w_1 = w_2 x$, for some $x \in X$).

It is easy to see that if $|X| = n$, then $X^*$ is a complete rooted $n$-ary tree, i.e., there is a root vertex, $\emptyset$, of degree $n$, and all other vertices have degree $n + 1$.

**2.2. The group determined by an automaton.** We will now explain how an invertible automaton over $X$ determines a group of automorphisms of the tree $X^*$.

**Definition 2.5.** Let $A$ be an automaton over $X$. Let $a \in A$ and $x \in X$. We define $a|x = \pi_2 \tau(a, x)$, where $\pi_2: X \times A \to A$ is projection on the second factor.
For each \( a \in A \), we can define a function \( a : X^* \to X^* \) by induction on the length of a word \( w \in X^* \). If \( |w| = 0 \) (so \( w \) is the null string), then we set \( a(w) = w \). If \( |w| > 0 \), then we can write \( w = xy_1 \), for some \( x \in X \) and \( w_1 \in X^* \). We define \( a(w) = \tau_a(x)a|_x(w_1) \).

It is easy to see that the function \( a : X^* \to X^* \) preserves parents and children: if \( w_2 = w_1x \) for some \( x \in X \) (so \( w_2 \) is a child of \( w_1 \)), then \( a(w_2) \) is also a child of \( a(w_1) \). If the automaton \( A \) is invertible then \( a : X^* \to X^* \) is an automorphism for each \( a \in A \) ([7]: pg. 7).

**Definition 2.6.** Let \( A \) be an invertible automaton. The group defined by \( A \), \( G(A) \), is the subgroup of \( \text{Aut}(X^*) \) generated by:

\[
\{a : X^* \to X^* \mid a \in A\}.
\]

While our definition of \( a|_x \) (for \( a \in A \) and \( x \in X \)) is good enough to define the action of \( G(A) \) on its associated tree, we will often need a definition of \( w|_x \), where \( w \in A^* \) and \( x \in X \).

**Definition 2.7.** Let \( a_1a_2 \ldots a_n \in A^* \). For \( x \in X \), we define \((a_1a_2 \ldots a_n)|_x \) in \( A^* \) by the rule:

\[
(a_1a_2 \ldots a_n)|_x = a_1|_{\tau_{a_1} \tau_{a_2} \ldots \tau_{a_n}(x)} \cdots a_{n-1}|_{\tau_{a_{n-1}}(x)} a_n|_x.
\]

**Note 2.8.** An easy way to compute \((a_1a_2 \ldots a_n)|_x \) is as follows. Form the concatenation \( a_1a_2 \ldots a_nx \) (in \((A \cup X)^*\)) and then regard the function \( \tau : A \times X \to X \times A \) as a rule telling us to replace a substring \( \hat{a}\hat{x} \) (\( \hat{a} \in A \), \( \hat{x} \in X \)) by the string \( \hat{x}\hat{a} \), where \( \tau(\hat{a}, \hat{x}) = (\hat{x}, \hat{a}) \). Once we have rewritten the original word \( a_1a_2 \ldots a_nx \) in the form \( y\hat{a}_1 \ldots \hat{a}_n \), where \( y \in X \), it follows that \((a_1 \ldots a_n)|_x = \tilde{a}_1 \ldots \tilde{a}_n \).

For instance, if \( A \) is the automaton from Example 2.3, then

\[
abta \cdot 1 = ab1t = ab0(id)t = a0b(id)t = 0(id)b(id)t.
\]

We omit occurrences of the identity state (Definition 2.12) from \((a_1a_2 \ldots a_n)|_x \). (Note also that \((a_1a_2 \ldots a_n)|_x \) is a string in \((A \setminus \{id\})^* \), not an element of a group – this distinction is important in Definition 2.14.) It follows that \( abta_1 = bt \).

**Definition 2.9.** Let \( G \) be a group defined by an automaton \( A \). We let \( G_n \) denote the \( n \)th level stabilizer, defined as follows:

\[
G_n = \{g \in G \mid g \cdot w = w, \text{ for all } w \in X^* \text{ such that } |w| \leq n\}.
\]

**Definition 2.10.** Let \( g_0, g_1, \ldots, g_{n-1} \) be automorphisms of the tree \( X^* \), where \( |X| = n \). Assume, without loss of generality, that \( X = \{0, 1, \ldots, n-1\} \). We let \( g = (g_0, g_1, \ldots, g_{n-1}) \) be the automorphism defined by the rule \( g(jw) = jg_j(w) \), for \( j \in X \) and \( w \in X^* \). (Thus, \( g \) fixes the top level of the tree and acts like the automorphism \( g_j \) on the \( j \)th branch from the root.)

Note that if \( G \) is a group defined by an automaton \( A \) over \( X = \{0, 1, \ldots, n-1\} \), and \( g \in G_1 \) can be represented by a word in \((A \setminus \{id\})^* \), then \( g = (g_0, \ldots, g_{n-1}) \). We will make extensive use of this fact in subsequent sections.

We will always work with reduced automata.

**Definition 2.11.** An automaton \( A \) is reduced if different states \( a \) of \( A \) induce different functions \( a : X^* \to X^* \).
Any automaton can be reduced, i.e., there is an algorithm which finds a reduced automaton whose states define the same set of functions \( a : X^* \to X^* \) as the given automaton ([7]; pg. 8).

**Definition 2.12.** A state \( a \) is called an identity state if \( a : X^* \to X^* \) is the identity automorphism.

It is clear from the definition that a reduced automaton can have at most one identity state.

2.3. Kneading automata of quadratic polynomials. Throughout this section, we assume that \( X \) is a two-letter alphabet. The definitions in this section are all drawn from [7]. In most cases, our definitions look simpler than the ones in [7] because we are restricting our attention to a two-letter alphabet, while the discussion in [7] is more general. Note in particular that Definition 2.13 would be an incorrect definition of kneading automata if we replaced \( X \) with a larger set.

**Definition 2.13.** ([7]; pg. 167) Let \( A \) be an invertible reduced automaton over \( X = \{0, 1\} \). We say that \( A \) is a kneading automaton if

1. there is only one active state;
2. in the Moore diagram of \( A \), each non-identity state has exactly one incoming arrow;
3. at most one outgoing arrow from the active state leads to a non-identity state.

**Definition 2.14.** ([7]; pg. 177) A kneading automaton \( A \) over \( X = \{0, 1\} \) is planar if there is some linear ordering \( a_1 \ldots a_m \) of the non-trivial states of \( A \) such that \((a_1 \ldots a_m)^2\) is a cyclic shift of \( a_1 \ldots a_m \) for each \( x \in X \).

If \( A \) is a kneading automaton over the two-letter alphabet \( X \), then there are two general forms that \( A \) might take. Consider the result of deleting the identity state and all arrows in the Moore diagram for \( A \) that lead to the identity state. It is not too difficult to see that the resulting directed graph \( \Gamma_A \) is topologically either a circle, or a circle with a sticker \( ([0, 1]) \) attached at one of its ends. (In the latter case, the active state of \( A \) is the unique vertex of degree 1.) It is also clearly possible to reconstruct the Moore diagram of \( A \) from \( \Gamma_A \) (since all of the arrows that are missing from \( \Gamma_A \) must lead to the identity state).

**Definition 2.15.** ([7]; pg. 183) Let \( A \) be a kneading automaton. We define the kneading sequence of \( A \) as follows.

If \( \Gamma_A \) is topologically a circle with \( m \) edges, then the kneading sequence for \( A \) has the form \((\ell_1 \ell_2 \ldots \ell_m)^{\omega}\), where \( \ell_1 \) is the label of the (unique) arrow leading from \( a_1 \) (say) into the active state, \( \ell_2 \) is the label of the arrow leading from \( a_2 \) into \( a_1 \), and so forth, so that \( \ell_m \) leads from the active state into \( a_{m-1} \). (In other words, we trace the arrows backwards from the active state, while recording the labels in the order that they are encountered. When we reach the active state again, having recorded the string \( \ell_1 \ell_2 \ldots \ell_m \), we define the kneading sequence to be \((\ell_1 \ldots \ell_m)^{\omega}\).)

If \( \Gamma_A \) is topologically a circle with a sticker, then we similarly trace the arrows backward from the active state (which is necessarily the unique vertex of degree 1 in \( \Gamma_A \)) and record the labels. The kneading sequence takes the form \( u(v)^{\omega} \), where \( u \) is the (non-empty) label of the sticker, and \( v \) is the label of the circle. The latter label \( v \) is read from \( \Gamma_A \) in essentially the same way as before.
We abbreviate the four possible labels \((0,0), (1,1), (0,1),\) and \((1,0)\) by \(0, 1, *_0,\) and \(*_1\) (respectively).

**Example 2.16.** The automaton \(A\) in Example 2.3 is a kneading automaton. Its associated graph \(\Gamma_A\) is topologically a circle with a sticker. If we trace the arrows backwards from \(t\) to \(a\), then to \(b\), and then to \(b\) again, we read the labels \((1,1), (0,0)\) (respectively). Since following the arrow backwards from \(b\) leads to \(b\), the kneading sequence repeats after this. It follows that the kneading sequence is \(11(0)\omega\).

**Note 2.17.** It is straightforward to check that a kneading automaton can be recovered from its kneading sequence. Note also that the automaton with the kneading sequence \(1(0)^\omega\) is different from the automaton with the kneading sequence \(1(00)^\omega\) (for example).

**Definition 2.18.** ([7]; pg. 184) We say that a kneading sequence is pre-periodic if it has the form \(u(v)^\omega\), where \(u, v \in X^*\) are non-trivial strings. (That is, if the graph \(\Gamma_A\) is topologically a circle with a sticker attached.)

We use a characterization of “bad isotropy groups” on page 184 of [7] as a definition:

**Definition 2.19.** A kneading automaton \(A\) over \(X = \{0, 1\}\) has bad isotropy groups if and only if its kneading sequence is pre-periodic and the word \(v\) from Definition 2.18 is a proper power.

**Definition 2.20.** A complex polynomial \(p: \mathbb{C} \to \mathbb{C}\) is called post-critically finite if for each critical point \(c\) (i.e., \(p'(c) = 0\)), the set
\[
\{p(p(\ldots p(c)))\}
\]

is finite.

**Theorem 2.21.** Let \(A\) be an invertible reduced kneading automaton over \(X\), where \(|X| = 2\). If \(A\) is planar and does not have bad isotropy groups, then \(G(A)\) is the iterated monodromy group of a post-critically finite quadratic polynomial \(p: \mathbb{C} \to \mathbb{C}\), and the kneading sequence of \(A\) is also the kneading sequence of \(p\) (up to relabeling of \(X\)).

**Proof.** This follows from Nekrashevych’s Theorem 6.9.6 [7] and the discussion on page 187 of [7].

3. A SUFFICIENT CONDITION FOR SUBEXPONENTIAL GROWTH

Let \(A\) be a kneading automaton with pre-periodic kneading sequence. It frequently happens that an element \(g \in G(A)_1 = G_1\) becomes shorter in total length when it is written as an ordered pair, i.e., as \(g = (g_0, g_1)\). The arguments of Bux and Pérez [2] show that if some positive proportion of group elements \(g\) possess this property, then \(G\) will have subexponential growth.

The goal of this section is to give a careful statement and proof of this fact. Our approach is based on [2].
3.1. Length Functions, rewriting rules and weak reduced forms. We assume, from now on, that $A$ is a kneading automaton with a pre-periodic kneading sequence, and $X = \{0, 1\}$. We let $t$ denote the (unique) active state of $A$. Our assumptions imply that $t : X^* \to X^*$ is defined by the rule $t(0w) = 1w; \quad t(1w) = 0w$.

We let $S$ be the set of non-trivial, non-active states of $A$. We set $H = \langle S \rangle \leq G(A)$. We will frequently write $G$ in place of $G(A)$ when the automaton $A$ is understood.

It is straightforward to check that every generator $a \in A - \{id\}$ has order 2 under the current assumptions. We will assume this fact in what follows. We can therefore represent each group element in $G$ (or $H$) by a word in $(A - \{id\})^*$.

**Definition 3.1.** Let $\hat{\ell} : A - \{id\} \to \mathbb{R}^+$ (where $\mathbb{R}^+$ is the set of positive real numbers). The assignment $\hat{\ell}$ determines two functions from $(A - \{id\})^*$ to $\mathbb{R}^+ \cup \{0\}$.

We write $C$ in place of $\ell(t)$.

**Definition 3.2.** Let $s \in S$. We say that $(s_{[0]}, s_{[1]})$ is the first-line string production of $s$. More generally, if $w \in (A - \{id\})^*$, we replace each letter $\bar{s} \in S$ of $w$ with the pair $(\bar{s}_{[0]}, \bar{s}_{[1]})$, and then collect all powers of the active state $t$ at the end of the word using the relation $t(a, b) = (b, a)t$. Finally, we multiply all of the pairs coordinate-wise, without any cancellation or application of relations from $G(A)$ (i.e., the multiplication takes place in $(A - \{id\})^*$). The resulting expression has the form $(\bar{w}_1, \bar{w}_2)(t)$ for some words $\bar{w}_1, \bar{w}_2 \in (A - \{id\})^*$; it is called the first-line string production of $w$. (Note that, here and in what follows, the $t$ in $(\bar{w}_1, \bar{w}_2)(t)$ appears between parentheses because it may be present or not.) If $w \in (A - \{id\})^*$, then we let $(\bar{w}_1, \bar{w}_2)(t)$ denote the first-line string production.

**Example 3.3.** We consider the automaton $A$ from Example 2.3 and find the first-line string production of $w = tabbtabt$. We find

$$tabbtabt = t(1, t)(b, a)t(1, t)(b, a)t = (tabl, bata)t,$$

which is the first-line string production of $w$.

**Definition 3.4.** For each element $h$ in the group $H = \langle S \rangle$ we fix a word $w_h \in S^*$ such that

1. $w_h = h$ in $G$;
2. If $\bar{w} \in S^*$ is equal to $h$ in $G$, then $|w_h| \leq |\bar{w}|$.

(In other words, $w_h$ is a representative for $h \in G$ that uses only letters from $S$, and such that $\cdot : S^* \to \mathbb{R}^+ \cup \{0\}$ is at a minimum over all such representatives.)

We let $T$ denote the set of choices for $w_h$, where $h$ ranges over all of $H$.

Let $w = (t^{a_1}w_1t^{a_2}w_2t^{a_3} \ldots t^{a_{n-1}}w_{n-1}t^{a_n})$ be an arbitrary word in $(A - \{id\})^*$, where each $w_i \in S^*$ and each $a_i \geq 1$. (The parenthesized terms may be present or not.) We let $r(w)$ be the result of replacing each subword $w_i$ with its representative in $T$. The latter assignment determines a function $r : (A - \{id\})^* \to (A - \{id\})^*$.

A word $w \in (A - \{id\})^*$ is in weak reduced form if it is in the range of $r$. 
Note 3.5. In other words, we put an arbitrary string \( w \in (A - \{id\})^* \) into weak reduced form by replacing each maximal string from \( S^* \) by its representative from \( T \), without cancelling any \( ts \). Thus, it is perfectly acceptable for a substring of the form \( t^n \ (n \geq 2) \) to appear in \( r(w) \).

Note 3.6. It is easy to check that \( r : (A - \{id\})^* \to (A - \{id\})^* \) has the property \( r(r(u)r(v)) = r(uv) \), where the multiplication takes place in the free semigroup \( (A - \{id\})^* \).

Definition 3.7. Let \( w \in (A - \{id\})^* \). Let \( (\tilde{w}_0, \tilde{w}_1)(t) \) be the first-line string production of \( w \). The first-line production of \( w \) is \( (r(\tilde{w}_0), r(\tilde{w}_1))(t) \).

We write \( (w_0, w_1)(t) \) for the first-line production of \( w \).

Suppose that \( w = (t^{\alpha_0})w_1t^{\alpha_1}w_2t^{\alpha_2} \ldots w_mt^{\alpha_m} \) is in weak reduced form. We set

\[
|w|_* = (C) + (m - 1)C + (C) + \sum_{i=1}^{m} |w_i|,
\]

where the parenthesized terms will be absent if the corresponding factors \( t^{\alpha_0}, t^{\alpha_m} \) are absent in \( w \).

Lemma 3.8. For all \( w \in (A - \{id\})^* \) in weak reduced form, we have \( \ell(w) \leq |w|_* \leq |w| \).

Proof. Let \( w = (w_0)t^{\alpha_0}w_1t^{\alpha_1} \ldots t^{\alpha_m-1}w_mt^{\alpha_m}(w_{m+1}) \) be a word in weak reduced form. (Thus, \( w_i \in T \), for all \( i \).) We assume, for the sake of simplicity, that \( w_0 = w_{m+1} = 1 \).

\[
|w| = \sum_{i=0}^{m} \alpha_i C + \sum_{i=1}^{m} |w_i| \geq \sum_{i=0}^{m} C + \sum_{i=1}^{m} |w_i| = |w|_*.
\]

For each \( i \in \{0, \ldots, m\} \), let \( \beta_i \in \{0, 1\} \) be the result of reducing \( \alpha_i \) modulo 2. It follows that \( t^{\alpha_0}w_1t^{\alpha_1} \ldots t^{\alpha_m-1}w_mt^{\alpha_m} = t^{\beta_0}w_1t^{\beta_1} \ldots t^{\beta_{m-1}}w_mt^{\beta_m} \) in the group \( G(A) \). We have

\[
|w|_* \geq \sum_{i=0}^{m} \beta_i C + \sum_{i=1}^{m} |w_i| = |t^{\beta_0}w_1t^{\beta_1} \ldots t^{\beta_{m-1}}w_mt^{\beta_m}| \geq \ell(w),
\]

where the final inequality follows from the definition of \( \ell \). \( \square \)

Definition 3.9. The length function \( \ell : (A - \{id\})^* \to \mathbb{R}^+ \cup \{0\} \) is called admissible if \( |t| + |w| \geq |w_0| + |w_1| \), for all \( w \in T \), where \( (w_0, w_1)(t) \) is the first-line production of \( w \).

Lemma 3.10. Let \( w \in (A - \{id\})^* \) be in weak reduced form, and let \( (w_0, w_1)(t) \) be the first-line production of \( w \).

(1) \( |w|_* + C \geq |w_0| + |w_1| \) if \( w \) contains fewer blocks of \( ts \) than blocks of letters from \( S \).
Proof. We prove (2) first. Let us assume that $t$ is in weak reduced form. We will confine our attention to this subcase, since the subcase $w = \hat{w}_1 t^{\alpha_1} \ldots t^{\alpha_{m-1}} \hat{w}_m$ is essentially similar, and the subcase $w = t^{\alpha_0} \hat{w}_1 t^{\alpha_1} \ldots t^{\alpha_{m-1}} \hat{w}_m$ is easier.

We have

$$|w|_* = \sum_{i=1}^m (|t| + |\hat{w}_i|) \geq \sum_{i=1}^m (|\hat{w}_i|_0 + |\hat{w}_i|_1).$$

The desired conclusion $|w|_* \geq |w_0| + |w_1|$ now follows from the observation that there is a partition $\{P_0, P_1\}$ of $\{(\hat{w}_i)_j | i = 1, \ldots, m; j = 0, 1\}$ such that each of the words $w_k (k = 0, 1)$ in the first-line production $(w_0, w_1)(t)$ of $w$ is obtained by multiplying the elements of $P_k$ in some order (with each word in $P_k$ appearing exactly once), and then applying $r$.

To prove (1), we note that $w$ must have the form $w = \hat{w}_1 t^{\alpha_1} \ldots t^{\alpha_{m-1}} \hat{w}_m t^{\alpha_m} \hat{w}_{m+1}$. We consider the word $wt$. By case (2),

$$|wt|_* \geq |(wt)_0| + |(wt)_1|,$$

but $|wt|_* = |w|_* + C$, and $(wt)_i = w_i$ for $i = 0, 1$. \qed

3.2. Good and $\epsilon$-good.

Definition 3.11. Let $w \in (A - \{id\})^*$ be in weak reduced form. A subword of $w$ is called protected if it begins and ends with $t$, and contains at least one letter from $S$.

Definition 3.12. Write $\hat{w} \preceq w$ if there is some protected subword $\hat{w}$ of $w$ such that $\hat{w}$ is a protected subword of some word in the first-line production of $\hat{w}$.

Lemma 3.13. Let $w$ be a word representing an element of $G_1$, and let $u_1, \ldots, u_n$ be protected subwords meeting (at most) in an initial or terminal block of $ts$. Let $\hat{u}_i \preceq u_i$ for $i = 1, \ldots, n$. The first-line production of $w$ contains occurrences of each of the $\hat{u}_i$. These occurrences meet (at most) in an initial or terminal block of $ts$ and there are at least $n$ distinct occurrences in all (one of each).

Proof. Consider the words $\hat{u}_1, \ldots, \hat{u}_n$ obtained from $u_1, \ldots, u_n$ (respectively) by omitting the initial and terminal blocks of $ts$. We note that $\hat{u}_i$ and $u_i$ have the same first-line string productions (although they might produce the words in opposite coordinates). We write $w = v_1 \hat{u}_1 v_2 \hat{u}_2 \ldots \hat{u}_n v_{n+1}$, where: (i) the $v_i$ (for $1 \leq i \leq n$) end with blocks of $ts$ (and may consist entirely of $ts$), and (ii) the $v_i$ (for $2 \leq i \leq n + 1$) begin with blocks of $ts$ (and may consist entirely of $ts$), but are not trivial strings in either case. We form the first-line string production of $w$. Note that this is done word-by-word in the product $w = v_1 \hat{u}_1 v_2 \hat{u}_2 \ldots \hat{u}_n v_{n+1}$. Each word $v_i$, $\hat{u}_i$ contributes a string to both the left- and right-hand coordinates (although each may contribute the right-half of its first-line string production to the left word in the first-line string production of $w$, or vice versa).

Assume that

$$(\hat{u}_1)_{i_1} (\hat{u}_2)_{j_2} \ldots (\hat{u}_n)_{j_n} (\hat{u}_{n+1})_{i_{n+1}}$$
is the left-half of the first-line string production of $v_1 \hat{u}_1 \ldots \hat{u}_m v_{m+1}$, where each string $(\hat{u}_k)_{j_k}$. $(\hat{u}_k)_{j_k}$ above is either the left or right half of the first-line string production of $v_k$, $\hat{u}_k$ (respectively). Choose a word $(\hat{u}_k)_{j_k}$ ($k \in \{1, \ldots, m\}$). We assume without loss of generality that $\hat{u}_k$ occurs as a reduced subword in $(\hat{u}_k)_{j_k}$, where $(\hat{u}_k)_{j_k}$ denotes either the left- or right-half of the first-line production of $\hat{u}_k$. Since $\hat{u}_k \not\preceq \hat{u}_k$, there is some protected subword $u$ such that $(\hat{u}_k)_{j_k} = xuy$ and $r(u) = \hat{u}_k$. Thus $r((\hat{u}_k)_{j_k}) = r(x)r(u)r(y)$. Since no $ts$ are cancelled in an application of $r$, we get that $r(u) = \hat{u}_k$ occurs as a subword in $w_0$. That is, $\hat{u}_k$ occurs as a protected subword in $w_0$. The lemma now follows.

**Definition 3.14.** Let $w$ be a word in weak reduced form. A subword $v$ of $w$ is called **reducing** if $v \equiv tw_1tw_2tw_3 \ldots tw_m$, where each $w_i \in T$, the word $w_m$ is followed immediately by a $t$ in $w$, $m \geq 1$, and $\ell(v_0) + \ell(v_1) < |v|_s$.

**Definition 3.15.** A word $u$ is **good at depth** $m$ if there are $u_1, \ldots, u_m$ such that

$$u \succ u_1 \succ \ldots \succ u_m,$$

where $u_m$ contains a reducing subword $v$.

We set

$$C_u = \frac{\ell(v_0) + \ell(v_1)}{|v|_s},$$

and

$$\sigma_u = |v|_s.$$

**Definition 3.16.** A word $w \in (A - \{id\})^*$ is **reduced** if $|w| = \ell(w)$.

**Definition 3.17.** Let $0 < \epsilon < 1$. A reduced word $w$ of length $N$ (i.e., $\ell(w) = N$) is **$\epsilon$-good with respect to the good word $u$** if at least $\epsilon N$ of its length is taken up by occurrences of $u$ which meet in an initial or terminal $t$.

**Definition 3.18.** If $w \in (A - \{id\})^*$ represents an element of $G_k$ and $\alpha \in \{0, 1\}^k$, then we let $w_\alpha$ denote the production of $w$ in position $\alpha$. For instance, $w_0$ denotes the left word in the first-line production of $w$. (Here we consider the “0” branch the left, and the “1” branch the right, half of the binary tree $X^*$.) If $k \geq 2$, then $w_{01}$ would denote the right word in the first-line production of $w_0$. We also say that $w_{01}$ is in the second-line production of $w$, $w_{011}$ is in the third-line production, and so forth.

**Proposition 3.19.** Let $u$ be a good-at-depth-$m$ word ($m \geq 0$). There are $0 < \theta < 1$ and $K > 0$ such that, for any $w \in (A - \{id\})^*$ satisfying

1. $|w| = \ell(w),$
2. $\pi(w) \in G_{m+1},$ and
3. $w$ is $\epsilon$-good with respect to $u$,

$$\sum_\delta \ell(w_\delta) \leq \theta \ell(w) + K.$$

The sum on the left is over all strings $\delta \in \{0, 1\}^{m+1}$.

**Proof.** Since occurrences of $u$ contribute at least $\epsilon |w|$ to the total length of $w$, there must be at least $\epsilon |w|/|u|$ of them. We let $\beta$ be the number of such occurrences; thus, $\beta \geq \epsilon |w|/|u|$. We let $v, u_1, \ldots, u_m$ be the words from Definition 3.15.
We choose $\theta$ so that $0 < \theta < 1$ and
\[ \theta + \frac{\epsilon(1 - C_u)\sigma_u}{|u|} > 1. \]

It is clearly possible to find such $\theta$, since the second term on the left is positive.
We first produce the $m$th line production of $w$. We have either:

1. $\sum_{\gamma} |w_{\gamma}|_* \leq \theta |w|$, for all $w$ sufficiently large, or
2. $\sum_{\gamma} |w_{\gamma}|_* > \theta |w|$, for some sequence of words $w$ such that $|w| \to \infty$.

The sums on the left are taken over all strings $\gamma \in \{0, 1\}^m$.

In the first case, we have
\[ \theta|w| + 2^m C \geq \sum_{\gamma} (|w_{\gamma}|_* + C) \geq \sum_{\gamma} |w_{\gamma}0| + |w_{\gamma}1| \geq \sum_{\delta} \ell(w_{\delta}), \]

where the final sum is over all strings $\delta$ of length $m + 1$ in $\{0, 1\}$, and $(w_{\gamma0}, w_{\gamma1})$ is the first-line string production of $w$. Thus there is nothing left to prove in this case.

Now assume that we are in the second case. For each $\gamma \in \{0, 1\}^m$, we let $\beta_\gamma$ denote the number of occurrences of $u_m$ in $w_\gamma$. By Lemma 3.13 we have $\sum_{\gamma} \beta_\gamma \geq \beta$.

The word $w_\gamma$ has the form
\[ w_\gamma = x_0vx_1vx_2v\ldots x_{\beta_\gamma-1}vx_{\beta_\gamma}, \]

where each $x_i$ is a word in weak reduced form, any one of which may be trivial, with the exception of $x_{\beta_\gamma}$, which must begin with a $t$ if $\beta_\gamma > 0$. We note that each occurrence of $v$ must be followed immediately by a $t$ according to Definition 3.14. It follows that each $x_i$ (for $i > 0$) begins with a $t$ if it is not the null string. If some $x_i$ ends with a block of $t$s, then we can combine this block with the word $v$ following it (if any). The altered occurrence of $v$, $v'$, still contributes the same strings to the next line string production (though possibly in different places) and even satisfies $|v'|_* = |v|_*$, so we may ignore the additional powers of $t$ for the sake of the following argument. One can now easily verify that each $x_i$ ($i > 0$) contains at least as blocks of $t$s as blocks of letters from $S$.

We first claim that
\[ |w_\gamma|_* + C + (C_u - 1)\beta_\gamma\sigma_u \geq \ell(w_{\gamma0}) + \ell(w_{\gamma1}), \]

for each $\gamma \in \{0, 1\}^m$. From the definition of $| \cdot |_*$ we see that
\[ * \Rightarrow |w_\gamma|_* = \sum_{i=0}^{\beta_\gamma} |x_i|_* + \sum_{i=1}^{\beta_\gamma} |v|_*. \]
It follows that
\[
|w_\gamma| + C + \sum_{i=1}^{\beta_\gamma} |x_i| + (|x_0| + C) + \beta_\gamma \sigma_u
\geq \sum_{i=0}^{\beta_\gamma} [\ell((x_i)_0) + \ell((x_i)_1)] + \beta_\gamma C_u \sigma_u
\geq \ell(w_\gamma 0) + \ell(w_\gamma 1).
\]
Here the first equality follows directly from \(\ast\). The expression on the second line is the \(|\cdot|\)-length of the strings that result from taking the first-line productions of the words \(x_i\) and \(v\), and reducing the results (i.e., substituting shortest-length strings) individually. The first inequality follows from Lemma 3.10. The final inequality is now immediate. We also have
\[
|w_\gamma| + C + \beta_\gamma C_u \sigma_u - \beta_\gamma \sigma_u = \sum_{i=0}^{\beta_\gamma} |x_i| + C + \beta_\gamma C_u \sigma_u
\geq \sum_{i=0}^{\beta_\gamma} [\ell((x_i)_0) + \ell((x_i)_1)] + \beta_\gamma C_u \sigma_u.
\]
The claim follows readily.

Applying the claim (and Lemma 3.10 repeatedly) we get
\[
|w| + (1 + 2 + 2^2 + \ldots + 2^m)C \geq \sum_{\gamma} (|w| + C)
\geq \sum_{\gamma} (|w| + C + \beta_\sigma_u (C_u - 1))
\geq \sum_{\delta} \ell(w_{\delta}).
\]
At worst,
\[
\sum_{\gamma} (|w| + C + \beta_\sigma_u (C_u - 1)) > \theta |w|
\]
for infinitely many \(w\) such that \(|w| \to \infty\) (otherwise there is nothing left to prove).
We conclude that
\[
|w| + (1 + 2 + 2^2 + \ldots + 2^m)C \geq \theta |w| + (1 - C_u) \beta_\sigma_u
\]
\[
\geq \theta |w| + (1 - C_u) \frac{|w|}{|u|} \sigma_u
\]
\[
= |w| \left( \theta + \frac{\sigma_u (1 - C_u) \epsilon}{|u|} \right).
\]
for all such \(w\). It follows that
\[
|w| + (1 + 2 + \ldots + 2^m)C \geq D |w|,
\]
where \(C, D,\) and \(m\) are constants, \(D > 1\), and \(|w| \to \infty\). This is a contradiction. \(\square\)

**Definition 3.20.** Let \(0 < \epsilon < 1\). Let \(U = \{u_1, \ldots, u_n\}\) be such that \(u_i\) is a good-at-depth-\(m_i\) word, possibly for varying \(m_i\). A reduced word \(w\) of length \(N\) is \(\epsilon\)-good with respect to \(U\) if at least \(\epsilon N\) of its length is taken up by occurrences of words from \(U\) which meet (at most) in initial or terminal blocks of \(s\).
Theorem 3.21. Let $\mathcal{U} = \{u_1, \ldots, u_n\}$ be as in Definition 3.20. Let $M = \max\{m_i\}$. There are $0 < \theta < 1$ and $K > 0$ such that, for all reduced words $w \in (A - \{id\})^*$ satisfying

1. $|w| = \ell(w)$,
2. $\pi(w) \in G_{M+1}$, and
3. $w$ is $\epsilon$-good with respect to $\mathcal{U}$,

$$\sum_\delta \ell(w_\delta) \leq \theta \ell(w) + K.$$

The sum on the left is over all strings $\delta \in \{0,1\}^{M+1}$.

Proof. Let $w$ be an arbitrary word satisfying the given conditions. It follows that $w$ is $\epsilon/n$-good with respect to some word $u_i$, which is good at depth $m_i$. It follows from Proposition 3.19 that

$$\sum_\delta \ell(w_\delta) \leq \theta(u_i, \epsilon/n)\ell(w) + K(u_i, \epsilon/n),$$

where the sum on the left is over all $\delta_i \in \{0,1\}^{m_i+1}$.

Lemma 3.10 implies that

$$\sum_\delta \ell(w_\delta) \leq \sum_\delta \ell(w_\delta) + (2^{m_i+1} + 2^{m_i+2} + \ldots + 2^M)C.$$

We set $K_2(u_i, \epsilon/n) = (2^{m_i+1} + \ldots + 2^M)C$. (In fact, $K_2$ depends only on $u_i$.)

We have

$$\sum_\delta \ell(w_\delta) \leq \theta(u_i, \epsilon/n)\ell(w) + K(u_i, \epsilon/n) + K_2(u_i, \epsilon/n).$$

We set $\theta(\mathcal{U}, \epsilon) = \max\{\theta(u_i, \epsilon/n)\}$, $K(\mathcal{U}, \epsilon) = \max\{K(u_i, \epsilon/n)\}$, and $K_2(\mathcal{U}, \epsilon) = \max\{K_2(u_i, \epsilon/n)\}$.

It now follows easily that

$$\sum_\delta \ell(w_\delta) \leq \theta(\mathcal{U}, \epsilon)\ell(w) + K(\mathcal{U}, \epsilon) + K_2(\mathcal{U}, \epsilon)$$

for all words $w$ satisfying the hypotheses. \hfill $\square$

3.3. Subexponential Growth.

Definition 3.22. Let $\mathcal{U}$ be a collection of good words. A reduced word $w$ is $\mathcal{U}$-bad if no word of $\mathcal{U}$ occurs as a subword of $w$.

Theorem 3.23. For $i = 1, \ldots, n$, let $u_i$ be a good-at-depth $m_i$ word. Let $\mathcal{U} = \{u_1, \ldots, u_n\}$. If there is some $M > 0$ such that, for all $L > 0$, there are at most $M$ $\mathcal{U}$-bad words $w$ of length $L$ (i.e., $\ell(w) = L$), then $G(A)$ has subexponential growth.

Proof. For each $r > 0$ and all small $\epsilon > 0$, we estimate the number $b_\epsilon(r)$ of $\epsilon$-bad words $w \in G_{N+1}$ of length $\ell$ precisely $r$. We will make the (harmless) assumption that each word has integral length. We have the following estimate:

$$b_\epsilon(r) \leq \binom{r+1}{\lfloor \epsilon r \rfloor + 1} M^{1+\lfloor \epsilon r \rfloor}\mathcal{U}^{\lfloor \epsilon r \rfloor}.$$
bad words. The binomial coefficient counts the number of solutions in non-negative integers to the inequality

\[ \ell_1 + \ldots + \ell_{\lfloor \epsilon r \rfloor + 1} \leq r - \lfloor \epsilon r \rfloor, \]

which over-counts the number of possible choices for the lengths of these bad words. Since the number of bad words of a given length is uniformly bounded by \( M \), we find that there are at most \( M^{1 + \lfloor \epsilon r \rfloor} \) choices for these words.

Next, we consider the number \( B_\epsilon(n) \) of reduced words \( w \in G_{N+1} \) of length less than or equal to \( n \).

\[
B_\epsilon(n) = \sum_{r=0}^{n} b_\epsilon(r) \\
\leq \sum_{r=0}^{n} \binom{r + 1}{\lfloor \epsilon r \rfloor + 1} M^{1 + \lfloor \epsilon r \rfloor} |\mathcal{U}|^{\lfloor \epsilon r \rfloor} \\
\leq M^{1 + \epsilon n} |\mathcal{U}|^{\epsilon n} \sum_{r=0}^{n} \binom{r + 1}{\lfloor \epsilon r \rfloor + 1} \\
\leq M^{1 + \epsilon n} |\mathcal{U}|^{\epsilon n}(n + 1)\left( \frac{n + 1}{\lfloor \epsilon n \rfloor + 1} \right) \\
\leq M^{1 + \epsilon n} |\mathcal{U}|^{\epsilon n}(n + 1)^{\lfloor \epsilon n \rfloor + 1}/(\lfloor \epsilon n \rfloor + 1)!.
\]

For large \( n \), the latter quantity is approximately

\[
M^{1 + \epsilon n} |\mathcal{U}|^{\epsilon n}(n + 1)^{\lfloor \epsilon n \rfloor + 1}/\sqrt{2\pi(\lfloor \epsilon n \rfloor + 1)} \left( \frac{n + 1}{\lfloor \epsilon n \rfloor + 1} \right)^{\lfloor \epsilon n \rfloor + 1},
\]

by an application of Stirling's Formula to the quantity \((\lfloor \epsilon n \rfloor + 1)!\).

Now we suppose (for a contradiction) that \( G \) (thus, \( G_{N+1} \)) has exponential growth. Thus, there is some \( \lambda > 1 \) such that the ball of radius \( n \) in \( G_{N+1} \) has roughly \( \lambda^n \) elements. The estimate of \( B_\epsilon(n) \) implies that we can choose \( \epsilon \) sufficiently small that \( B_\epsilon(n) \leq (\lambda_1)^n \), for some \( 1 < \lambda_1 < \lambda \). Thus, the proportion of \( \epsilon \)-good elements in the ball of radius \( n \) to the total is at the least (roughly) \( 1 - \lambda_1/\lambda \). It follows that there is some positive proportion of reduced words \( w \in G_{N+1} \) that satisfy the inequality in Theorem 3.21. In view of Proposition 10 from [2], we are done. \( \square \)

4. Examples of groups with intermediate growth

In practice, we will want to have one more type of next-line production, which is intermediate between the first-line string production and the first-line production.

**Definition 4.1.** Let \( v = tw_1tw_2t \ldots tw_{n} \), where each \( w_i \in T \). Consider the following operation: Replace each \( w_i \) with its first-line production \( (w_i)_0, (w_i)_1 \) and collect all powers of \( t \) at the end of the word, using the relation \( t(a,b) = (b,a)t \). Multiply the pairs coordinate-by-coordinate with no cancellation (i.e., the multiplication occurs in \( (A - \{id\})^+ \)). The result is called the special production of \( v \).

**Lemma 4.2.** Let \( v \) be as in Definition 4.1 and suppose that \( \langle \tilde{v}_0, \tilde{v}_1 \rangle(t) \) is its special production. If \( |\tilde{v}_i| > \ell(\tilde{v}_i) \) for \( i = 0 \) or \( 1 \), then \( |v| > \ell(v_0) + \ell(v_1) \).
Proof. We note that the inequality $|v|_s \geq |\tilde{v}_0| + |\tilde{v}_1|$ follows immediately from Lemma 3.10 from which the desired conclusion follows by Note 3.6. □

The groups associated to the automata with the kneading sequences $1(0^k)\omega$, $11(0)\omega$, and $0(011)\omega$ will be proved to have subexponential growth in this section. In fact, all also have superpolynomial growth, since each group $G$ is commensurable with $G \times G$, and this is condition is known to imply superpolynomial growth (see [4], for instance).

4.1. The Case of $1(0^k)\omega$. We consider the automata $A_k$ ($k \geq 2$) with kneading sequence $1(0^k)\omega$.

The graph $\Gamma_{A_k}$ is depicted in Figure [2] which also indicates our convention for naming the states of $A_k$ ($k \geq 2$). Namely, $x_0$ is the state adjacent to $t$ (the unique active state), $x_1$ is the first state we encounter while tracing directed edges backwards from $x_0$, $x_2$ is the second such state, and so forth.

![Graph Γ_A for the automaton with the kneading sequence 1(0^k)ω](image)

**Figure 2.** The graph $\Gamma_A$ for the automaton with the kneading sequence $1(0^k)\omega$.

**Lemma 4.3.** Let $x_0, x_1, \ldots, x_{k-1}$ be the inactive states in the automaton $A_k$.

$$\langle x_0, \ldots, x_{k-1} \rangle \cong (\mathbb{Z}/2\mathbb{Z})^k.$$ 

Proof. The action of $x_i$ on a string $n_1 n_2 \ldots n_m \in X^*$ can be described as follows. If $n_1 n_2 \ldots n_m$ begins with a string of exactly $j$ 0’s ($j \geq 0$) followed by a 1, then $x_i \cdot n_1 \ldots n_m = n_1 \ldots n_m$ if $j \neq i$ modulo $k$. If $j \equiv i$ modulo $k$, then $x_i \cdot n_1 n_2 \ldots n_j n_{j+1} \tilde{n}_{j+2} \ldots$ where $\tilde{n}_{j+2}$ is 0 if $n_{j+2}$ is 1, and 1 if $n_{j+2}$ is 0. (If $n_1 n_2 \ldots n_m$ contains no 1, or if $m = j + 1$, then $x_i \cdot n_1 \ldots n_m = n_1 \ldots n_m$.) It follows easily from this description that each $x_i$ has order 2, and that any two elements of $\{x_0, \ldots, x_{k-1}\}$ commute (since they have disjoint supports).

We claim that the words $x_{i_1} \ldots x_{i_\alpha}$ ($\alpha \geq 0$) are all distinct, where $i_1, \ldots, i_\alpha$ is an increasing sequence and $\{i_1, \ldots, i_\alpha\} \subseteq \{0, \ldots, k-1\}$. In fact $w := x_{i_1} x_{i_2} \ldots x_{i_\alpha}$ has a description analogous to that of $x_i$: If $n_1 n_2 \ldots n_m$ begins with a string of exactly $j$ 0’s ($j \geq 0$) followed by a 1, then $w \cdot n_1 \ldots n_m = n_1 \ldots n_m$ if $j \neq i$ modulo $k$, for any $i \in \{i_1, \ldots, i_\alpha\}$. If $j \equiv i$ modulo $k$ for some such $i$, then $w \cdot n_1 n_2 \ldots n_j n_{j+1} \tilde{n}_{j+2} \ldots$ where $\tilde{n}_{j+2}$ is 0 if $n_{j+2}$ is 1, and 1 if $n_{j+2}$ is 0. (If $n_1 n_2 \ldots n_m$ contains no 1, or if $m = j + 1$, then $w \cdot n_1 \ldots n_m = n_1 \ldots n_m$.) It follows immediately that all such $x_{i_1} \ldots x_{i_\alpha}$ are distinct, for distinct sequences $i_1, \ldots, i_\alpha$.

Thus, $\langle x_0, \ldots, x_{k-1} \rangle$ contains at least $2^k$ elements, which implies that $\langle x_0, \ldots, x_{k-1} \rangle \cong (\mathbb{Z}/2\mathbb{Z})^k$, since our earlier argument shows that the former group is a quotient of the latter. □
We assign weights to the states \( t, x_0, \ldots, x_{k-1} \) as follows:

\[
\hat{\ell}(t) = (k + 2)^2 \\
\hat{\ell}(x_i) = (k + 1 - i) \quad (0 \leq i \leq k - 1)
\]

We let \( T = \{ x_{i_1} \ldots x_{i_\alpha} \mid \alpha \geq 0; 0 \leq i_1 < i_2 < \ldots < i_\alpha \leq k - 1 \} \). It is not difficult to see that this choice \( T \) has the property required by Definition 3.4.

**Lemma 4.4.** The length function \( \ell : (A - \{id\})^* \to \mathbb{R}^+ \cup \{0\} \) induced by \( \hat{\ell} \) is admissible. In fact, \( |t| + |w| > |w_0| + |w_1| \) for all \( w \in T - \{ x_0x_1x_2 \ldots x_{k-1} \} \) (i.e., for all words in \( T \) except for the unique one using all \( k \) states \( x_0, \ldots, x_{k-1} \)).

**Proof.** Note that

\[
x_0 = (x_{k-1}, t); \\
x_i = (x_{i-1}, 1) \quad (1 \leq i \leq k - 1).
\]

We first consider the case in which \( w \) contains no occurrence of \( x_0 \) (i.e., \( i_1 > 0 \)). In this case \( |w_1| = 0 \) and \( w_0 \in T \). It follows that

\[
|w_0| \leq \sum_{i=0}^{k-1} |x_i| = \frac{k^2 + 3k}{2} < |t|,
\]

from which the strict inequality \( |t| + |w| > |w_0| + |w_1| \) follows immediately.

Next, we suppose that \( w \) contains an occurrence of \( x_0 \) (i.e., \( i_1 = 0 \)), but \( w \neq x_0x_1 \ldots x_{k-1} \). In this case \( |w_1| = |t| \). Thus, to establish \( |t| + |w| > |w_0| + |w_1| \), we want to show that \( |w| > |w_0| \). We write \( w = x_0x_{i_2} \ldots x_{i_\alpha} \). Note that \( \alpha < k \).

\[
|w_0| = |x_{k-1}x_{i_2-1}x_{i_3-1} \ldots x_{i_\alpha-1}|
\]

\[
= 2 + \sum_{\beta=2}^{\alpha} |x_{i_\beta-1}|
\]

\[
= 2 + \sum_{\beta=2}^{\alpha} (k + 1 - (i_\beta - 1))
\]

\[
= (\alpha + 1) + \sum_{\beta=2}^{\alpha} (k + 1 - i_\beta)
\]

\[
< (k + 1) + \sum_{\beta=2}^{\alpha} (k + 1 - i_\beta)
\]

\[
= |w|,
\]

as required.

Finally, one easily sees that \( |t| + |w| = |w_0| + |w_1| \) if \( w = x_0x_1 \ldots x_{k-1} \). \( \square \)

**Theorem 4.5.** Each \( G(A_k) \) \((k \geq 2)\) has subexponential growth.

**Proof.** We first nominate a set \( \mathcal{U} \) of good words. Let \( \mathcal{U} = \{ twt \mid w \in T - \{ x_0x_1 \ldots x_{k-1} \} \} \). Each word in this collection is good at depth 0 (the subword \( tw \) is a reducing word in each case, by Lemma 4.4).

By Theorem 3.23, to prove that the growth is subexponential it is enough to show that there is an \( M > 0 \) such that, for all \( L > 0 \), there are at most \( M \mathcal{U} \)-bad words of length \( L \). Let us consider a \( \mathcal{U} \)-bad word of the form \( tw_1tw_2t \ldots tw_mt \) \((m > 0)\). It is
clear that each $w_i$ must be $x_0x_1\ldots x_{k-1}$, so there is exactly one $U$-bad word of this form for any $m$. A general bad word has the form $(w_0)tw_1t\ldots tw_m t(w_{m+1})$, which shows that the number of such words is bounded above by a uniform constant that is independent of $m$. This easily implies the existence of the required $M$. □

**Note 4.6.** All of the groups in this class have bad isotropy groups, so Theorem 2.21 does not guarantee that these groups are the iterated monodromy groups of complex polynomials.

### 4.2. The Case of $11(0)\omega$

We now consider the group $G(A)$ of the kneading automaton $A$ with kneading sequence $11(0)\omega$. This automaton has already appeared as Example 2.3. The group $G(A)$ is generated by the automorphisms $t$, $a = (1, t)$, and $b = (b, a)$.

**Lemma 4.7.** The group $\langle a, b \rangle$ is isomorphic to $D_4$, the dihedral group of order 8.

*Proof.* We first note that $a^2 = (1, t^2) = (1, 1) = 1$. We also have $b^2 = (b^2, a^2) = (b^2, 1)$. It follows easily by induction on the length $m$ of a word $n_1\ldots n_m \in \{0, 1\}^*$ that $b^2$ also acts as the identity. Thus, $b^2 = 1$.

$$ (ab)^4 = (b, at)^4 = (1, (at)^4). $$

Also,

$$ (at)^4 = (atat)^2 = [(1, t)t(1, t)t]^2 = (t, t)^2 = 1. $$

It follows that $(ab)^4 = 1$.

We’ve now shown that $\langle a, b \rangle$ is a quotient of $D_4$. We define a homomorphism $\phi : \langle a, b \rangle \to \langle a, t \rangle$ by $\phi(a) = t$; $\phi(b) = a$. (This homomorphism is restriction to the second coordinate, or restriction to the right branch of the tree $\{0, 1\}^*$.) Now one notes that $\langle a, t \rangle$ has order 8 as follows. We consider $00$, $01$, $10$, $11 \in \{0, 1\}^*$.

Relabel these vertices 1, 2, 3, and 4, respectively. It is straightforward to check that $a$ acts as the permutation $(34)$ and $t$ acts as $(13)(24)$. It follows that $|\langle a, t \rangle| \geq 8$, so $|\langle a, b \rangle| \geq 8$. Thus, $|\langle a, b \rangle| = 8$, so $\langle a, b \rangle \cong D_4$. □

We define $\hat{\ell} : \langle a, b, t \rangle \to \mathbb{R}^+$ by the rule $\hat{\ell}(a) = \hat{\ell}(b) = \hat{\ell}(t) = 1$. We now fix representatives $w_h \in S^*$ for each $h \in \langle a, b \rangle$ as in Definition 3.4 Set

$$ T = \{1, a, ab, aba, abab, b, ba, bab\}. $$

The first-line productions of the elements of $T$ are (respectively) as follows:

$$ (1, 1), (1, t), (b, ta), (b, tat), (1, tata), (b, a), (b, at), (1, ata). $$

One can easily check that the length function $\ell : (A - \{id\})^* \to \mathbb{R}^+ \cup \{0\}$ associated to $\hat{\ell}$ is admissible. We summarize the relevant calculations in a table.

We let $\alpha_n$ be the element of $T$ of word-length $n$ beginning with $a$. Let $\beta_n$ denote the element of word-length $n$ beginning with $b$. Thus, $T = \{1, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3\}$.

**Proposition 4.8.** The following families $P_i$ of words in the generators $\langle a, b, t \rangle$ are good, where each box $\square$ represents an occurrence of a string from $\{\beta_1, \beta_2, \alpha_2\}$:

- $P_0$: $t\alpha_1 t, \alpha_2 t, \alpha_3 t$;
- $P_1$: $t\alpha_1 t \square t\beta_1 \diamond t t \square t$;
- $P_2$: $t\beta_1 t \square t \beta_2 \diamond t$;
- $P_3$: $t \alpha_2 t \square t \beta_2 \diamond t$;
- $P_4$: $t \beta_2 t \square t \beta_2 \diamond t \alpha_2 t$;
Theorem 4.9. The group $G(A)$ of the automaton $A$ with kneading sequence $11(0)\omega$ has subexponential growth. The group $G(A)$ is also the iterated monodromy group of a complex post-critically finite quadratic polynomial.
discussed forms. or

\( \beta \)
cannot contain a subsequence of the form

\( \alpha \) contains 2 or fewer occurrences of

with a long string of

\( \beta \)

only possibilities are that

\( w \) begins

\( \alpha \) follows

\( \beta \)

Proof. We prove the second statement first. In fact, by Theorem 2.21, it is enough to show that \( A \) is planar. This follows easily from the observation that \( abtabt_0 = bta \) and \( abtabt_1 = tab \).

We now turn to the first statement. Consider the collection of all reduced words that are \( U \)-bad (where \( U \) is as in Proposition 4.8). By Theorem 3.23 it is enough to show that there is \( M > 0 \) such that, for a given \( L \), the number of \( U \)-bad words of length \( L \) is less than \( M \). It is clear that we may restrict our attention to large \( L \).

We first consider bad words \( w \) that begin and end with \( t \). Thus \( w = tw_1tw_2 \ldots tw_m t \) (for some large integer \( m \)). It is clear (from the description of \( P0 \)) that \( w_i \in \{ \beta_1, \beta_2, \alpha_2 \} \) for each \( i \). We consider the possibilities for the sequence \( w_1, w_3, w_5, \ldots, w_{2k-1} \), where \( 2k-1 \) is the largest odd number less than or equal to \( m \).

First, suppose that \( w_1 = \alpha_2 \). It follows that \( w_1, w_3, w_5, \ldots, w_{2k-1} \) has one of the forms:

1. \( \alpha_2, \alpha_2, \ldots, \alpha_2, \alpha_2 \);
2. \( \alpha_2, \alpha_2, \ldots, \alpha_2, \beta_1 \);
3. \( \alpha_2, \alpha_2, \ldots, \alpha_2, \beta_1, \beta_1 \) (if \( 2k-1 < m \)).

Indeed, \( \beta_2 \) cannot follow \( \alpha_2 \), since that would create a subword from \( P7 \). If \( \beta_1 \) follows \( \alpha_2 \) except in the last place (or in the second-to-last place, if \( 2k-1 < m \)), then the next word in the sequence begins with three or fewer occurrences of \( \beta_2 \) followed by a sequence of one of the forms (1)-(3), or that \( w_1, w_3, w_5, \ldots, w_{2k-1} \) begins with a long string of \( \beta_2 \)'s, followed by a sequence of one of the forms (1)-(3) that contains 2 or fewer occurrences of \( \alpha_2 \).

Now we consider the sequences \( w_1, \ldots, w_{2k-1} \) such that \( w_1 = \beta_2 \) and some subsequent \( w_i \) is \( \alpha_2 \) (for an odd subscript \( i \)). We claim that no occurrence of \( \beta_1 \) can appear between \( w_1(= \beta_2) \) and the earliest occurrence of \( \alpha_2 \). If \( w_1 = \beta_2 \) is the earliest such occurrence, then \( w_{j-2} = \beta_2 \) and any choice of \( w_{j+2} \in \{ \beta_1, \beta_2, \alpha_2 \} \) yields a subword from \( P1, P2, \) or \( P4 \) (respectively). This proves the claim. Now note that, once an \( \alpha_2 \) occurs in \( w_1, w_3, \ldots, w_{2k-1} \), the remainder of the sequence takes one of the forms enumerated above, from the case in which \( w_1 = \alpha_2 \). In view of \( P6 \), the only possibilities are that \( w_1, w_3, \ldots, w_{2k-1} \) begins with 2 or fewer occurrences of \( \beta_2 \) followed by a sequence of one of the forms (1)-(3), or that \( w_1, w_3, \ldots, w_{2k-1} \) begins with a long string of \( \beta_2 \)'s, followed by a sequence of one of the forms (1)-(3) that contains 2 or fewer occurrences of \( \alpha_2 \).

Now we consider the sequences \( w_1, \ldots, w_{2k-1} \) such that \( w_1 = \beta_2 \) and \( w_1, \ldots, w_{2k-1} \notin \{ \beta_1, \beta_2 \} \). In view of \( P1 \), \( w_1, \ldots, w_{2k-1} \) contains neither a subsequence of the form \( \beta_1, \beta_1, \beta_2 \) nor one of the form \( \beta_1, \beta_1, \beta_1 \). In view of \( P2 \), it cannot contain a subsequence of the form \( \beta_2, \beta_1, \beta_2 \). It follows that \( w_1, \ldots, w_{2k-1} \) is a sequence of \( \beta_2 \)'s ending with two or fewer occurrences of \( \beta_1 \).

Now suppose that \( w_1 = \beta_1 \). If \( w_1, \ldots, w_{2k-1} \) begins with three or more occurrences of \( \beta_1 \), then we create a subword from \( P1 \), an impossibility. Thus, either \( w_3 \) or \( w_5 \) is \( \{ \beta_2, \alpha_2 \} \), and the remainder of the sequence takes one of the previously-discussed forms.

We have now completely described the possibilities for \( w_1, \ldots, w_{2k-1} \). Our discussion shows that \( w_1, \ldots, w_{2k-1} \) is essentially a constant sequence of \( \alpha_2 \)'s or of \( \beta_2 \)'s, with a small amount of variation possible at the beginning and end. It follows that the number of such sequences is bounded by a constant that is independent of \( k \). A similar analysis establishes a similar form for the sequence \( w_2, w_4, \ldots \). It follows that the number of \( U \)-bad words of the form \( tw_1 t \ldots tw_m t \) is bounded, by a bound that is independent of \( m \).
A general $U$-bad word has the form $(w_0)tw_1t\ldots tw_m(t(w_{m+1}))$, for some $w_0, w_{m+1} \in T$, and it follows immediately that the number of such words is uniformly bounded, regardless of $m$. Theorem 3.23 now implies that $G(A)$ has subexponential growth. □

4.3. The Case of 0(011)$\omega$. We now consider the automaton $A$ with kneading sequence 0(011)$\omega$. The graph $\Gamma_A$ appears in Figure 3, which also indicates our convention for labeling the states.

![Figure 3. The graph $\Gamma_A$ for the automaton $A$ with kneading sequence 0(011)$\omega$.](image)

Lemma 4.10. \langle a, b, c \rangle \cong \langle a, b \rangle \times \langle c \rangle \cong D_8 \times \mathbb{Z}/2\mathbb{Z}$, where $D_8$ is the dihedral group of order 16.

Proof. First, we note that $a^2 = (t^2, c^2) = (1, c^2); c^2 = (1, b^2); b^2 = (a^2, 1)$. It follows from this that the automorphism $a^2$ has the following inductive definition: $a^2 \cdot n_1n_2\ldots n_k = n_1n_2\ldots n_k$ if $n_1n_2n_3 \neq 110$ (or if $k \leq 3$), and $a^2 \cdot 110n_4\ldots n_k = 110 \cdot (a^2 \cdot n_4\ldots n_k)$. It follows directly that $|a| = 2$. Therefore, $|b| = 2$ and $|c| = 2$ by the above computations.

It follows immediately that $\langle a, b \rangle$ is a dihedral group. One can easily check that $(ta)^2 = (ct, tc), (ta)^4 = ((b, b), (b, b))$, and (therefore) $(ta)^8 = 1$. Now $(ab)^n = ((ta)^n, c^n)$, so $(ab)^8 = 1$. We’ve shown that $(ta)^4 \neq 1$, so $(ab)^4 \neq 1$, implying that $|ab| = 8$. Thus, $\langle a, b \rangle \cong D_8$.

Next we show that $c$ commutes with $a$ and $b$. The relation $bc = cb$ follows because $c$ and $b$ have disjoint support, and

$ac = (t, cb) = (t, bc) = ca$.

Finally, we show that $\langle c \rangle \cap \langle a, b \rangle = 1$. It is enough to show that $c \notin \langle a, b \rangle$. The elements of $\langle a, b \rangle$ all have the form $(\_ , c)$ or $(\_ , 1)$. Since $c = (1, b)$ and $b \neq c$, $c \notin \langle a, b \rangle$. It follows that $\langle c \rangle \cap \langle a, b \rangle = 1$, so $\langle a, b, c \rangle \cong \langle a, b \rangle \times \langle c \rangle$, as claimed. □

We define $\widehat{\ell} : A - \{id\} \rightarrow \mathbb{R}^+$ by the rule

$\widehat{\ell}(a) = 7; \; \widehat{\ell}(b) = 7; \; \widehat{\ell}(c) = 6 \; \widehat{\ell}(t) = 3$.

We set

$T = \{1, \alpha_1, \alpha_2, \ldots, \alpha_8, \beta_1, \ldots, \beta_7\} \cup \{c, c\alpha_1, \ldots, c\alpha_8, c\beta_1, \ldots, c\beta_7\}$.

It is straightforward to check that $T$ satisfies the conditions of Definition 3.4.
We list the words $\alpha_1, \alpha_2, \ldots, \alpha_8, \beta_1, \ldots, \beta_7$, their first-line productions, and the corresponding weights $|t| + |w|$ and $|w_0| + |w_1|$ in the table below. The first-line productions of the remaining words are obtained from the entries in the table simply by post-multiplying the second coordinates by $b$. Similarly, the weights $|t| + |w|$ and $|w_0| + |w_1|$ can be obtained by adding (respectively) 6 and 7 to the totals below. It follows easily that the length function $\ell$ is admissible.

| $w$ | $(w_0, w_1)$ | $|t| + |w|$ | $|w_0| + |w_1|$ | $w$ | $(w_0, w_1)$ | $|t| + |w|$ | $|w_0| + |w_1|$ |
|-----|-------------|-------------|----------------|-----|-------------|-------------|----------------|
| $\alpha_1$ | $(t, c)$ | 10 | 9 | $\beta_1$ | $(a, 1)$ | 10 | 7 |
| $\alpha_2$ | $(ta, c)$ | 17 | 16 | $\beta_2$ | $(at, c)$ | 17 | 16 |
| $\alpha_3$ | $(tat, 1)$ | 38 | 36 | $\beta_3$ | $(ata, c)$ | 38 | 27 |
| $\alpha_4$ | $(tatata, c)$ | 45 | 43 | $\beta_4$ | $(atat, c)$ | 45 | 36 |
| $\alpha_5$ | $(tatata, c)$ | 59 | 57 | $\beta_5$ | $(atat, c)$ | 59 | 43 |

Proposition 4.11. The following families of words in the generators $\{a, b, c, t\}$ are good, where each box $\Box$ represents an occurrence of a string from $\{c\alpha_1, c\alpha_2, c\beta_2, c\beta_3\}$:

- $P_0$. $twt$, $w \in T - \{c\alpha_1, c\alpha_2, c\beta_2, c\beta_3\}$;
- $P_1$. $tc\alpha_1\Box t c\alpha_1 t$;
- $P_2$. $tc\alpha_1\Box t c\alpha_2 t$;
- $P_3$. $t\Box t c\alpha_1\Box t c\beta_2\Box t\Box t t$;
- $P_4$. $t\Box t c\alpha_2\Box t c\alpha_1\Box t\Box t t$;
- $P_5$. $tc\alpha_2\Box t c\beta_2 t$;
- $P_6$. $tc\alpha_2\Box t c\beta_3 t$;
- $P_7$. $t c\beta_3\Box t c\alpha_1 t$;
- $P_8$. $t c\beta_3\Box t c\alpha_2 t$;
- $P_9$. $t\Box t c\beta_2\Box t c\beta_3 t\Box t t t$;
- $P_{10}$. $t\Box t c\beta_3 t\Box t c\alpha_2 t\Box t t t$;
- $P_{11}$. $t c\beta_3\Box t c\beta_2 t$;
- $P_{12}$. $t c\beta_3\Box t c\beta_3 t$.

The union $\mathcal{U}$ of the above families is finite.

Proof. We check that each family is made up of good words:

- $P_0$. In this case, one can check that $tw$ is a reducing word in each $twt$, so the members of this family are good at depth 0.
- $P_1$. The first-line production of each word $w$ in this family has the form $(\Box, tc\beta_1 t)$. Thus $tc\beta_1 t \ll w$, so each $w$ is good at depth 1, since $tc\beta_1 t \in P_0$ is good at depth 0.
- $P_2$. The first-line production of each word $w$ in this family has the form $(\Box, tc\beta_3 t a)$. Thus $tc\beta_3 t \ll w$, so each $w$ is good at depth 1, since $tc\beta_3 t \in P_0$ is good at depth 0.
- $P_3$. Let $w \in P_3$. We consider the first line production $(w_0, w_1)$ of $w$. If the first $\Box$ in $w$ is filled by either $c\alpha_1$ or $c\beta_2$, then $w_1$ contains a copy of $tc\beta_1 t$, and so $w$ is good at depth 1. If the final $\Box$ in $w$ is filled with either $c\alpha_1$ or
$\alpha_2$, then $w_1$ again contains a copy of $t\beta_1t$, which makes $w$ good at depth 1. In all other cases, $w_1$ contains a copy of $t\alpha_2 t\beta_2 t\beta_2 t$, which is good at depth 1 (see P5). Therefore, in this last case, $w$ is good at depth 2.

P4. Let $w \in P4$. We again consider $w_1$ in the first line production of $w$. If the first $\square$ in $w$ is filled by either $\alpha_1$ or $\beta_2$, then $w_1$ contains a copy of $t\beta_1 t$, and so $w$ is good at depth 1 (see P0). If the final $\square$ in $w$ is filled by either $\alpha_1$ or $\alpha_2$, then $w_1$ contains a copy of $t\beta_1 t$, so again $w$ is good at depth 1. In every other case, $w_1$ contains a copy of $t \alpha_2 t \alpha_2 t \beta_2 t \in P5$. Since the latter word is good at depth 1, $w$ is good at depth 2.

P5. All words $w \in P5$ produce the word $w_1 = t\alpha_3 t$ on the first line. Since the latter word is good at depth 0 (see P0), we conclude that each $w \in P5$ is good at depth 1.

P6. All words $w \in P6$ produce the word $w_1 = t \alpha_3 t a$. Since $w_R$ contains the protected subword $t \alpha_3 t$, we conclude that each $w \in P6$ is good at depth 1.

P7. All words $w \in P7$ produce the word $w_1 = a t \alpha_3 t$ on the first line. Since the protected subword $t \beta_2 t$ of $w_1$ is in $P0$, we conclude that $w$ is good at depth 1.

P8. All words $w \in P8$ produce $w_1 = a t \beta_1 t a$ on the first line. It follows that $w$ is good at depth 1.

P9. Let $w \in P9$. If the first $\square$ is filled by an occurrence of $\alpha_2$ or $\beta_3$, then $w_1$ contains an occurrence of $t \alpha_3 t$, which makes $w$ good at depth 1. If the last $\square$ is filled by an occurrence of $\beta_2$ or $\beta_3$, then $w_1$ again contains an occurrence of $t \alpha_3 t$, making $w$ good at depth 1. In all other cases, $w_1$ contains an occurrence of $t \beta_2 t \beta_2 t \alpha_2 t \in P8$, which makes $w$ good at depth 2.

P10. Let $w \in P10$. If the first $\square$ is filled by an occurrence of $\alpha_2$ or $\beta_3$, then $w_1$ contains an occurrence of $t \alpha_3 t$, which makes $w$ good at depth 1. If the last $\square$ is filled by an occurrence of $\beta_2$ or $\beta_3$, then $w_1$ again contains an occurrence of $t \alpha_3 t$, making $w$ good at depth 1. In all other cases, $w_1$ contains an occurrence of $t \beta_2 t \alpha_2 t \alpha_2 t \in P8$, which makes $w$ good at depth 2.

P11. All words $w$ in this family produce $w_1 = a t \alpha_3 t$ on the first line. It follows that all words $w \in P11$ are good at depth 1, since $t \alpha_3 t \in P0$ is good at depth 0.

P12. All words $w$ in this family produce $w_1 = a t \alpha_3 t a$ on the first line. It follows that all words $w \in P12$ are good at depth 1, since $t \alpha_3 t \in P0$ is good at depth 0.

Finally, it is clear that each of these families is finite, so their union $U$ is finite. □

**Theorem 4.12.** The group $G(A)$ determined by the automaton $A$ with kneading sequence $0(011)^*$ has subexponential growth. The group $G(A)$ is the iterated monodromy group of a complex post-critically finite quadratic polynomial.

**Proof.** We prove the second statement first. By Theorem 2.21 it is enough to show that the automaton is planar (the other conditions being obvious). The planarity of $A$ follows from the equalities $(t ach b)_{|0} = cbta$ and $(tachb)_{|1} = ta cb$, both valid in $(A - \{id\})^*$. 

We turn to a proof of the first statement. It is sufficient to show that there is $M > 0$ such that, for any $L > 0$, there are at most $M$ $U$-bad reduced words of length exactly $L$.

We first consider reduced words of the form $tw_1tw_2\ldots tw_m t$. Indeed, it is sufficient to consider words of this form, since a general reduced word has the form $(w_0)tw_1\ldots tw_m t(w_{m+1})$, and such a word is $U$-bad if and only if $tw_1\ldots tw_m t$ is. Thus, the total number of bad words of the form $(w_0)tw_1\ldots tw_m t(w_{m+1})$ is a constant multiple of the number of bad words of the form $tw_1t\ldots tw_m t$.

As in the proof of Theorem 4.9, we will consider the sequences $w_1, w_3, w_5, \ldots$ and $w_2, w_4, w_6, \ldots$. The description of subfamily $P0$ shows that each $w_i$ ($i = 1, \ldots, m$) must be taken from $\{c\alpha_1, c\alpha_2, c\beta_2, c\beta_3\}$. The descriptions of the remaining subfamilies $P1 - P12$ show (essentially; see below) that certain words from $\{c\alpha_1, c\alpha_2, c\beta_2, c\beta_3\}$ must not be followed by certain other words from $\{c\alpha_1, c\alpha_2, c\beta_2, c\beta_3\}$ in the sequences $w_1, w_3, \ldots$ and $w_2, w_4, \ldots$. Thus, for instance, the description of $P1$ implies that $c\alpha_1$ cannot follow $c\alpha_1$ in $w_1, w_3, \ldots$ or $w_2, w_4, \ldots$.

The exceptions are $P3$, $P4$, $P9$, and $P10$. These subfamilies also forbid one word from following another, except possibly at the immediate end or beginning of $w$. We can ignore this distinction for the sake of this argument, however, since these exceptions allow only a finite amount of variation at the end and beginning of $w$, and this variation will simply increase the uniform bound $M$.

With this understanding, we can make the following observations. In $w_1, w_3, \ldots$ and $w_2, w_4, \ldots$

1. $c\alpha_1$ can be followed only by $c\beta_3$;
2. $c\alpha_2$ can be followed only by $c\alpha_2$;
3. $c\beta_2$ can be followed only by $c\beta_2$;
4. $c\beta_3$ can be followed only by $c\alpha_1$.

Thus, modulo the above considerations, the only possibilities for the sequences $w_1, w_3, \ldots$ and $w_2, w_4, \ldots$ are those that alternate between $c\alpha_1$ and $c\beta_3$ and constant sequences of either $c\alpha_2$’s or $c\beta_2$’s. Thus the number of possible sequences $tw_1t\ldots tw_m t$ is bounded above, by a constant independent of $m$. The existence of the uniform bound $M$ now follows easily, and Theorem 3.23 establishes that $G(A)$ has subexponential growth. \hfill \qed

5. A GROUP $G(A)$ WITH NO ADMISSIBLE LENGTH FUNCTION

We consider the kneading automaton with kneading sequence $01(10)^\omega$. We label the active state $t$, and label the remaining states $a$, $b$, and $c$, in the order that they are encountered while tracing directed edges backward from the active state in the Moore diagram. Thus $a = (t, 1)$, $b = (c, a)$, and $c = (1, b)$. Our goal in this section is to sketch a proof that $G(A)$ has no admissible length function.

**Proposition 5.1.** The group $G(A)$ admits no admissible length function.

**Proof.** Choose an arbitrary $\tilde{\ell} : A - \{id\} \to \mathbb{R}^+$ and an arbitrary $T$ satisfying the conditions from Definition 3.3.

It turns out that: (1) the word $c$ must be in $T$; (2) one of the words $cbcb$, $bcab$ must be in $T$, and (3) one of the words $baba$, $abab$ must be in $T$. To prove this, it helps to use the homomorphism $\phi : \langle a, b, c \rangle \to (\mathbb{Z}/2\mathbb{Z})^3$, where $\phi(a) = (1, 0, 0)$, $\phi(b) = (0, 1, 0)$, and $\phi(c) = (0, 0, 1)$. One establishes that $\phi$ is well-defined as follows. The subgroup $N = \langle abab, bcba \rangle$ is central in $\langle a, b, c \rangle$, any two of the
generators $a$, $b$, $c$ commute modulo $N$, and the set $\{1, a, b, c, ab, ac, bc, abc\}$ is a transversal for $N$ in $\langle a, b, c \rangle$. The existence of $\phi$ now follows directly from the First Isomorphism Theorem. One proves (1), (2), and (3) by first arguing that every other representative $w'$ of the word $w$ in question must have at least as many occurrences of each letter as $w$, and then arguing that no other permutations of the letters of $w$ can represent the same group element. We omit the details.

Suppose, without loss of generality, that $\{c, baba, cbacb\} \subseteq T$, and assume that the length function $\ell : (A - \{id\})^* \rightarrow \mathbb{R}^+ \cup \{0\}$ is admissible. The first-line production of $cbacb$ is $(c, t, c, b, a, b, c)$, so

$$|t| + 2|c| + 2|b| + |a| \geq |t| + 2|a| + 2|b| + 2|c|,$$

from which we conclude that $|a| \leq 0$. This is a contradiction. $\Box$

References

[1] Laurent Bartholdi, Rostislav I. Grigorchuk, and Zoran Šunić. Branch groups. In Handbook of algebra, Vol. 3, pages 989–1112. North-Holland, Amsterdam, 2003.

[2] Kai-Uwe Bux and Rodrigo Pérez. On the growth of iterated monodromy groups. In Topological and asymptotic aspects of group theory, volume 394 of Contemp. Math., pages 61–76. Amer. Math. Soc., Providence, RI, 2006.

[3] A. Douady. Chirurgie sur les applications holomorphes. In Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986), pages 724–738, Providence, RI, 1987. Amer. Math. Soc.

[4] Rostislav Grigorchuk and Igor Pak. Groups of intermediate growth: an introduction. Enseign. Math. (2), 54(3-4):251–272, 2008.

[5] Rostislav I. Grigorchuk. Degrees of growth of finitely generated groups and the theory of invariant means. (Russian). Izv. Akad. Nauk SSSR Ser. Mat., 48(5):939–985, 1984.

[6] Rostislav I. Grigorchuk and Andrzej Žuk. On a torsion-free weakly branch group defined by a three state automaton. Internat. J. Algebra Comput., 12(1-2):223–246, 2002. International Conference on Geometric and Combinatorial Methods in Group Theory and Semigroup Theory (Lincoln, NE, 2000).

[7] Volodymyr Nekrashevych. Self-similar groups, volume 117 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2005.