Every countable compact subset of $S^n$ is tame

Agelos Georgakopoulos*

Mathematics Institute
University of Warwick
CV4 7AL, UK

August 25, 2022

Abstract

We prove that any two countable, compact, subsets of $S^n$, $n \geq 2$ that are homeomorphic also have homeomorphic complements. Thus any wild subspace like the classical construction of Antoine must contain a Cantor set.

Keywords: Cantor set, tame, wild, ambiently homeomorphic, strongly homogeneously embedded.

MSC 2020 Classification: 57N45, 57N35, 54C20.

1 Introduction

Just over a century ago, Antoine [3, 4] found a homeomorph $A$ of the Cantor set inside the 3-sphere $S^3$ the complement $S^3 \setminus A$ of which is not homeomorphic to the complement of a standard Cantor set in $S^3$, i.e. one contained in a linear arc inside $S^3$. Antoine’s discovery triggered a wealth of constructions of wild Cantor sets in $S^n$ with complements having various properties [6, 9, 12, 10, 15, 16, 20, 24, 25]. In particular, Blankinship [8] generalised Antoine’s construction to all dimensions $n \geq 3$. Antoine’s work also led Alexander to the discovery of his horned sphere [2], and there are interesting connections between such spheres and wild Cantor sets [11].

The aim of this note is to show that such constructions are not possible, in any dimension, if we replace the Cantor set by any topological space that does not contain it as a subspace. Our main result is

Theorem 1.1. Let $A, Z$ be two countable, compact, subsets of $S^n$, $n \geq 2$, and let $h : A \to Z$ be a homeomorphism. Then there is an automorphism $h'$ of $S^n$ extending $h$.

*Supported by EPSRC grants EP/V048821/1 and EP/V009044/1.
Moreover, $h'$ can be chosen to be orientation reversing (or preserving).

It is well-known that every uncountable closed subspace of a Polish — i.e. completely metrizable and separable — space contains a perfect set, and therefore a homeomorph of the Cantor set \([19]\). Combined with Theorem 1.1 and the aforementioned constructions of \([8, 8]\), this means that the Cantor set is the smallest space $X$ such that that there are two homeomorphs of $X$ in $S^n$, $n \geq 2$ with non-homeomorphic complements.

Thus the countability condition of Theorem 1.1 is indispensable. To see that the compactness conditions is also necessary, consider $A, Z \subset S^n$ that are both homeomorphic to $Q$, so that $A$ is dense in $S^n$ while $B$ is not\(^1\).

For $n = 2$, the first statement of Theorem 1.1 can be deduced from a classical result of Richards’ \([22]\), which applies more generally to any totally disconnected, compact, subspace of $S^2$. In particular, Richards’ result implies that every Cantor set $C \subset S^2$ is tame, i.e. $S^2 \setminus C$ is homeomorphic to the complement of a standard Cantor set in $S^2$.

As an example application of Theorem 1.1, we can use it to answer the following question that became popular in on-line forums \([5, 1]\):

**Question 1.1. Does every bijection of $\mathbb{Z}^2$ extend to a homeomorphism of $\mathbb{R}^2$?**

It may look surprising at first sight that e.g. the bijection of $\mathbb{Z}^2$ that coincides with the identity in the upper half-plane and coincides with the bijection $(n, m) \mapsto (-n, m)$ in the lower half-plane extends to a homeomorphism $h$ of $\mathbb{R}^2$. This may look more surprising if one takes into account that $h$ would have to map the standard Cayley graph $G$ of $\mathbb{Z}^2$ —i.e. the square lattice— onto a quite distorted image $h(G)$. But Theorem 1.1 applied with $A = Z$ being the 1-point compactification of $\mathbb{Z}^2$ says that it is always possible, and in any dimension. Moreover, the second statement says that it is possible to extend the identity on $\mathbb{Z}^2$ to an orientation reversing homeomorphism $h$ of $\mathbb{R}^2$. Again, it is difficult to imagine the image $h(G)$ of the square lattice: its vertices are fixed, but for every vertex $v \in \mathbb{Z}^2$, the clockwise cyclic ordering of its edges is reversed. (Conversely, given such an embedding $h(G)$ on the square lattice, one can deduce the existence of the desired automorphism of $\mathbb{R}^2$ by applying the Jordan–Schoenflies theorem to each of the faces of $h(G)$; see \([23]\) for a more general statement.)

Proving Theorem 1.1 would be easy if we could find bases $B^A, B^Z$ for the topologies of $A, Z$, ideally induced by open metric balls of $S^n$, such that $h$ maps each element of $B^A$ to one of $B^Z$. In Section 2 we define a notion of nested basis, which is possible to find in every countable, compact, metric space (Lemma 2.1), that will allow us to reduce to the aforementioned ideal situation; $h$ will map each element of $B^A$ to a finite disjoint union of elements of $B^Z$, which union is always possible to engulf inside a single topological ball. The rest of the argument is an adaptation of an idea of Richards that ping-pongs between closed submanifolds of $A, Z$ bordered by topological spheres.

\(^1\)I thank Wojtek Wawrow for this example.
We prove Theorem 1.1 in the more general setup where countability is relaxed to the existence of a nested basis (Theorem 3.1). As every tame Cantor set $C \subset \mathbb{S}^n$ has a nested basis, we recover the well-known fact that $C$ is strongly homogeneously embedded, i.e. every automorphism of $C$ extends to an automorphism of $\mathbb{S}^n$.

2 Preliminaries and proof ingredients

Let $X$ be a topological space, and $U$ a set of open subsets of $X$. We say that $U$ is nested, if for every $U, V \in U$ we have either $U \cap V = \emptyset$, or $U \subseteq V$, or $V \subseteq U$.

An open metric ball of a metric space $(Y,d)$ is a set of the form $B_r(x) := \{y \in Y \mid d(x,y) < r\}$.

Lemma 2.1. Let $(Y,d)$ be a metric space, and $X \subseteq Y$ a countable subspace. Then there is a nested set of open metric balls of $Y$ that form a basis of the topology of $X$ when restricted to it.

Proof. Given some constant $r_0 \in \mathbb{R}_{>0}$, we can produce a nested cover $U = U_r^0$ of $X$ by disjoint open balls of $Y$ of radius at most $r_0$ as follows. Let $\{x_0, x_1, \ldots\}$ be an enumeration of $X$, and choose $r < r_0$ such that the boundary of the ball $U_0 = B_r(x_0)$ is disjoint from $X$. This is possible because $X$ is countable and there are uncountably many values of $r$ to choose from. Proceed inductively, letting $U_i, i = 1, 2, \ldots$ be a ball, of radius less than $r_0$, around the next still uncovered element of $\{x_1, x_2, \ldots\}$, such that $U_i$ is disjoint from all $U_j, j < i$, and $\partial U_i \cap X = \emptyset$.

This completes the definition of a cover $U_r^0$ of $X$. We proceed to define a sequence of such covers $U_r^i$, where $r_i \to 0$ inductively: in each step $i$, we repeat the above construction, with $X$ replaced by its intersection with each element $U$ of $U_r^{i-1}$, to obtain a refinement $U_r^i$ of $U_r^{i-1}$. Nestedness is easily preserved by our inductive step, and so the union $\bigcup_{i \in \mathbb{N}} U_r^i$ yields a nested basis of $X$ as desired.

Fix some $n > 1 \in \mathbb{N}$ for the rest of this paper. A (closed) $k$-punctured sphere in $\mathbb{S}^n$ is a connected sub-manifold with $k \in \mathbb{N}_{>0}$ boundary components, each of which is piecewise flat and homeomorphic to $\mathbb{S}^{n-1}$. By a piecewise flat subspace we mean a finite union of open metric balls. We will prove Theorem 1.1 by decomposing $\mathbb{S}^n \setminus A$ (and $\mathbb{S}^n \setminus Z$) as a union of punctured spheres with non-intersecting interiors, and applying the following basic fact to each of them.

Proposition 2.2. Let $M, M'$ be two $k$-punctured spheres in $\mathbb{S}^n$, and $f$ a homeomorphism from one of the boundary components of $M$ onto one of the boundary components of $M'$. Then there is a homeomorphism $f'$ from $M$ onto $M'$ extending $f$. Moreover, $f'$ preserves orientation if and only if $f$ does.

Proof (sketch). Continuously deform each of $M, M'$ to a standard $k$-punctured sphere, i.e. one with its boundary components having specific sizes and shapes,
and lying in specific locations. Then compose one of these deformations with the inverse of the other.

The condition of piecewise flatness that we imposed on the boundary components of $M, M'$ is probably stronger than needed, but some condition is necessary to avoid obstructions such as Alexander’s horned sphere, a subset of $S^3$ homeomorphic to $S^2$, the two sides of which are not homeomorphic to each other [2], [18, p. 169].

An automorphism of a topological space $Y$ is a homeomorphism from $Y$ onto itself.

3 Proofs of main results

Theorem 1.1 is an immediate consequence of the following statement, which replaces countability by the weaker —by Lemma 2.1— condition of having nested bases consisting of open balls.

Theorem 3.1. Let $A, Z$ be two compact subsets of $S^n$, $n \geq 2$ that admit nested bases $B^A, B^Z$ consisting of open metric balls of $S^n$, and let $h : A \to Z$ be a homeomorphism. Then there is an automorphism $h'$ of $S^n$ extending $h$.

Moreover, $h'$ can be chosen to be orientation reversing (or preserving).

Proof. We will construct $h'$ inductively, in infinitely many steps, with each step extending the previous one by mapping punctured spheres of $S^n \setminus A$ to punctured spheres of $S^n \setminus Z$ via Lemma 2.2. The punctures will have radii converging to 0, which will ensure that eventually all of $S^n \setminus A$ is mapped onto $S^n \setminus Z$.

To start our inductive process, we choose two closed metric balls $M_0, M'_0$ in $S^n$ that avoid $A, Z$, respectively, and a homeomorphism $h_0 : M_0 \to M'_0$. In fact, since $A \cup Z$ is closed, we can let $M_0 = M'_0$ be a small enough ball of any point outside $A \cup Z$, and we can let $h_0$ be the identity or any orientation reversing isometry of $M_0$. We make the latter choice depending on whether we want the homeomorphism $h'$ in the statement to be orientation preserving or reversing.

For the inductive step $n = 1, 2, \ldots$, assume that we have already defined a homeomorphism $h_{n-1} : M_{n-1} \to M'_{n-1}$ between punctured spheres of $S^n$ with the following properties. For a punctured sphere $M$, we let $BC(M)$ denote the set of boundary components of $M$.

(i) $M_{n-1}$ is a punctured sphere in $S^n \setminus A$; is the boundary of a ball in $B^A$;

(ii) each $S \in BC(M_{n-1})$ has radius at most $1/n$ (any sequence converging to 0 would do).

Given $S \in BC(M_{n-1})$, notice that exactly one of the two components of $S^n \setminus S$ meets $M_{n-1}$; we let $S'$ denote the other component (which is a $1$-punctured sphere). Notice that $h_{n-1}(S)$ must coincide with one of the boundary components of $M'_{n-1}$. Thus $h_{n-1}$ induces a bijection $S \mapsto S'$ from $BC(M_{n-1})$ to $BC(M'_{n-1})$. Our next property says that this bijection respects $h$:}
(iii) for every $S \in \mathcal{BC}(M_{n-1})$, we have $h(\tilde{S} \cap A) = \tilde{S'} \cap Z$.

Moreover, we assume — and ensure inductively — that properties $\text{(i)}$ and $\text{(iii)}$ hold with the roles of $A, M_{n-1}, h_{n-1}$ exchanged with $Z, M'_{n-1}, h_{n-1}^{-1}$, respectively.

Notice that these properties are satisfied for $n = 1$. In particular, there is a unique $S \in \mathcal{BC}(M_{n-1})$, and we have $\tilde{S} \cap A = A$ and $\tilde{S'} \cap Z = Z$.

Assuming that such an $h_{n-1}$ has already been defined, we proceed to define $M_n$ and $h_n$ as follows. If $n$ is odd, for every $S \in \mathcal{BC}(M_{n-1})$, let $C_S$ be a cover of $\tilde{S} \cap A$ by elements of $\mathcal{B}^A$ of radius at most $1/(n+2)$ that are contained in $\tilde{S}$. Since $A$, and hence $\tilde{S} \cap A$, is compact, we may assume that $C_S$ is finite. Moreover, its elements can be chosen to be pairwise disjoint since $\mathcal{B}^A$ is nested. Let $M_S := \tilde{S} \setminus \bigcup C_S$, and notice that $M_S$ is a $k$-punctured sphere with $k := 1 + |C_S|$; indeed, the boundary components of $M_S$ are $\partial S$ and the boundaries of the balls in $C_S$.

We are looking for an appropriate $k$-punctured sphere $M'_S$ in $\mathbb{S}^n \setminus Z$ to map $M_S$ onto. To find it, for every $U \in C_S$, let $C_U$ be a finite cover of $h(U \cap A)$ by pairwise disjoint elements of $\mathcal{B}^Z$, chosen small enough that $\text{(ii)}$ is satisfied and $\bigcup C_U \subset S'$ holds; the latter is possible by $\text{(iii)}$. Notice that $S' \setminus \bigcup_{U \in C_S} C_U$ is a $k'$-punctured sphere, but we cannot yet use it as the desired $M'_S$ because $k'$ will typically be larger than $k$. To amend this, we combine, for each $U \in C_S$, the balls in $C_U$ into one topological ball $S_U$ as follows. First, we join the elements of $C_U$ by piecewise linear arcs in $\tilde{S}'$, so as to form a connected and simply connected set (we have to use exactly $|C_U| - 1$ arcs for this). Then, we blow up each of these arcs into a sufficiently small tube consisting of open metric balls. It is not hard to do this so that the resulting union $S_U \setminus \bigcup C_U$ and all these tubes is homeomorphic to a ball in $\mathbb{S}^n$, and so that $S_U$ is disjoint from $S_V$ for $U \neq V \in C_S$ (Figure 1). Moreover, it is easy to ensure that $\partial S_U$ is piecewise flat. Here, we used the fact that the elements of $C_U$ are pairwise disjoint. We can now let $M'_S := S' \setminus \bigcup_{U \in C_S} S_U$, which is a $k$-punctured sphere as desired.

Let $M_n$ be the union of $M_{n-1}$ with all these punctured spheres of the form $M_S$ for $S \in \mathcal{BC}(M_{n-1})$. Recall that $h_{n-1}$ restricts to a homeomorphism from $S$ onto $S'$, which is one of the punctures of $M'_S$. Applying Proposition 2.2 to each $S \in \mathcal{BC}(M_{n-1})$, we thus obtain an extension $h_n$ of $h_{n-1}$ to $M_n$, with image $M'_n := M'_{n-1} \cup \bigcup_{S \in \mathcal{BC}(M_{n-1})} M'_S$. Moreover, $h_n$ will be orientation preserving if and only if $h_{n-1}$ is,

$$h_n = \begin{cases} \text{orientation preserving} & \text{if and only if } h_{n-1}, \\ \text{orientation reversing} & \text{otherwise} \end{cases}$$

because this property is witnessed by the restriction of $h_{n-1}$ to any $S \in \mathcal{BC}(M_{n-1})$.

This completes the definition of $h_n$ for odd $n$. For even $n$ we repeat the construction verbatim, except that we exchange the roles of $A$ and $Z$; this will define homeomorphisms from punctured spheres of $\mathbb{S}^n \setminus (Z \cup M'_n)$ to punctured spheres of $\mathbb{S}^n \setminus (A \cup M'_n)$, and we use the inverses of these homeomorphisms to extend $h_{n-1}$ into $h_n$. (The reason for treating odd and even $n$ differently is to make sure that the aforementioned tubes do not cause a problem by converging to points in $\mathbb{S}^n \setminus A$, which points would then fail to be in the image of $\bigcup_{n \in \mathbb{N}} h_n$.)
Figure 1: Mapping the punctured sphere $M_S$ onto a punctured sphere $M'_S$ inside $S'$. Each element $U$ of $C_S$ (there are 3 in this example picture) gives rise to a topological ball $S_U$, obtained by joining metric balls with tubes.

Note that (i) is satisfied, because $M_n$ and $M'_n$ have been obtained by glueing internally disjoint punctured spheres along a common boundary component. For $n$ odd, $M_n$ satisfies (ii) because each of its boundary components coincides, by construction, with the boundary $\partial U$ of some element of $B^A$ of radius at most $1/(n+2)$, which is less than the desired $1/(n+1)$. The boundary components of $M'_n$ can be bigger because of the tubes, but still they inherit the desired bound from the previous (even) step. For even $n$ the same argument with the roles interchanged applies. To see that (iii) is satisfied, notice that $BC(M_n)$ consists of spheres $U \in C_S$ as in the above construction, and we have $h_n(\partial U) = \partial S_U$; in other words, we have $\partial U' = \partial S_U$. Moreover, we have $S_U \cap Z = \bigcup C_U = h(U \cap A)$ as required by (iii).

Having defined $h_n$ for all $n$, we let $h' := h \cup \bigcup_{n \in \mathbb{N}} h_n$.

Notice that every point $p \in S^n \setminus A$ is eventually in $M_n$, hence in the domain of $h'$, because of (iii), the fact that $d(p, A) > 0$ as $A$ is closed, and the fact that $p \in S_U$ for some $S \in BC(M_n)$ for every $n$ by (i). By the same argument, $h'$ surjects onto $S^n$.

Easily, $\bigcup_{n \in \mathbb{N}} h_n$ is a homeomorphism, because it is one locally by the construction of the $h_n$. It is straightforward to check that $h'$ is a homeomorphism at any $p \in S^n \setminus A$ too, because arbitrarily small elements $U$ of $B^A$ containing $p$ are mapped to $h(U)$ by (iii).

The final statement about orientation follows from (i) and the fact that $h_0$ gives us the desired choice, as mentioned above.

**Remark 1.** In Theorem 1.1 we can construct $h'$ to be piecewise linear (or smooth), because all objects involved in its construction can be chosen to be piecewise linear.
Recall that a Cantor set $C \subset S^n$ is called \textit{tame}, if $S^n \setminus C$ is homeomorphic to the complement of a standard Cantor set in $S^n$. It is well-known that $C$ is tame if and only if there is an automorphism $h : S^n \to S^n$ such that $h(C)$ lies in a piecewise-linear arc \cite{7}. Easily, $h(C)$ admits a nested basis in this case, and so Theorem \ref{thm:main} provides another proof of the well-known fact that every tame Cantor set is $C$ strongly homogeneously embedded \cite[21, p. 93]{21}.

It follows from Theorem \ref{thm:main} that for every countable, compact, subset $Z$ of $S^n$, $n \geq 2$, there is an automorphism $h$ of $S^n$ such that $h(Z)$ lies in a linear arc. This is because every such space $Z$ —more generally, every zero-dimensional separable metric space— is homeomorphic with a subset of a real interval \cite[1.3.17]{13}.

4 \ A converse

In the converse direction of Theorem \ref{thm:main} we remark that if $A, Z$ are two compact, totally disconnected subsets of $S^n$ such that $S^n \setminus A$ is homeomorphic to $S^n \setminus Z$, then $A$ is homeomorphic to $Z$. One way to see this is to use the fact that homeomorphic spaces have homeomorphic Freudenthal compactifications, combined with

\begin{proposition}
Let $A$ be a compact and totally disconnected subspace of $S^n$. Then the identity map on $S^n \setminus A$ extends to a homeomorphism between the Freudenthal compactification of $S^n \setminus A$ and $S^n$.
\end{proposition}

In particular, $A$ is homeomorphic to the space of ends of $S^n \setminus A$. Proving Proposition \ref{prop:freudenthal} is easy when $A$ admits a nested basis of open metric balls of $S^n$, and slightly more difficult in general.

5 \ Final remarks

A topological space homeomorphic to the complement of a tame Cantor set in $S^3$ is called a \textit{Cantor 3-sphere}. We remark that this space is important, as it is the only simply connected, infinitely-ended, open 3-manifold that can cover a closed 3-manifold; see \cite{17} for details.

The interested reader will find many interesting problems on wild Cantor sets in \cite{14}.

In this paper we worked with $S^n$ as the host space. How far can we extend Theorem \ref{thm:main} beyond spheres?

\begin{problem}
For which $n$ is it true that for every $n$-manifold $M$, and every two countable, compact, homeomorphic subsets $A, Z$ of $M$, the complements $M \setminus A, M \setminus Z$ are homeomorphic?
\end{problem}

\footnote{Moise \cite{21} proves this in 2-dimensions, but the higher dimensional case follows easily by induction, because we may assume that $C$ is contained in the equator of $S^n$ using the aforementioned automorphism $h$.}
Theorem 1.1 tells us that we can extend each homeomorphism \( h : A \to Z \) to a homeomorphism between their host spaces. It could be interesting to try to extend several such homeomorphisms simultaneously, so that they form a specific group. We propose two problems of this kind:

**Problem 5.2.** Let \( Z \) be a countable subset of \( \mathbb{R}^n, \ n \geq 2 \) with no accumulation points in \( \mathbb{R}^n \). Which groups of bijections of \( Z \) extend to groups of automorphisms of \( \mathbb{R}^n \)?

For example, suppose \( n = 3 \) and \( \pi \) is a bijection of \( Z \) that has a finite cycle \((z_1, z_2, \ldots, z_k)\) and fixes all other elements of \( Z \) pointwise. Let \( \Gamma \) be the finite cyclic group generated by \( \pi \). Then it is possible to realise \( \Gamma \) as a group of homeomorphisms of \( \mathbb{R}^3 \) as follows. Find a 2-way infinite arc \( A \) that contains all of \( Z \) except \( \{z_1, z_2, \ldots, z_k\} \), and find a rotation \( h \) of \( \mathbb{R}^3 \) around \( A \) that maps each \( z_i \) to \( z_{i+1 \pmod k} \). Then \( h \) generates a finite cyclic group that acts on \( Z \) the same way as \( \Gamma \).

**Problem 5.3.** Is there \( n \in \mathbb{N} \) such that for every countable, closed subset \( Z \) of \( S^n \), every group of automorphisms of \( Z \) extends into a group of automorphisms of \( S^n \)?

Call a subset \( Z \) of \( S^n \) *ambiently-reversible*, if there is an orientation-reversing automorphism \( h : S^n \to S^n \) fixing \( Z \) pointwise. Is there an ambiently-reversible wild Cantor set in \( S^n, n \geq 3 \)? Are all Cantor sets in \( S^n \) ambiently-reversible? More generally, we have

**Problem 5.4.** Which subsets of \( S^n, n \geq 2 \) are ambiently-reversible?

**References**

[1] Extending homeomorphisms from closed countable sets to \( S^2 \). MathOverflow, https://mathoverflow.net/q/257108.

[2] J. W. Alexander. An Example of a Simply Connected Surface Bounding a Region which is not Simply Connected. *Proceedings of the National Academy of Sciences*, 10(1):8–10, 1924.

[3] L. Antoine. Sur la possibilité d’étendre l’homéomorphie de deux figures à leur voisinage. *C. R. Acad. Sci. Paris*, 171:661–663, 1920.

[4] L. Antoine. *Sur l'homéomorphie de deux figures et de leurs voisinages*, PhD Thesis (available on-line). Faculté des sciences de Strasbourg, 1921.

[5] J. Belk. Does every bijection of \( \mathbb{Z}^2 \) extend to a homeomorphism of \( \mathbb{R}^2 \)? Mathematics Stack Exchange, https://math.stackexchange.com/q/1320536.

[6] M. Bestvina and D. Cooper. A wild Cantor set as the limit set of a conformal group action on \( S^3 \). *Proc. Am. Math. Soc.*, 99(4):623–626, 1987.
[7] R. H. Bing. Tame Cantor sets in $E^3$. Pacific J. Math., 11:435–446, 1961.

[8] W. A. Blankinship. Generalization of a Construction of Antoine. Annals of Mathematics, 53(2):276–297, 1951.

[9] J. W. Cannon and D. G. Wright. Slippery Cantor sets in $E^n$. Fund. Math., 106(2):89–98, 1980.

[10] D. G. DeGryse and R. P. Osborne. A wild Cantor set in $E^n$ with simply connected complement. Fund. Math., 86(1):9–27, 1974.

[11] R. J. Daverman. Wild Spheres in $E^n$ that are Locally Flat Modulo Tame Cantor Sets. Trans. Am. Math. Soc., 206:347–359, 1975.

[12] R. J. Daverman. Embedding phenomena based upon decomposition theory: wild Cantor sets satisfying strong homogeneity properties. Proc. Am. Math. Soc., 75(1):177–182, 1979.

[13] Ryszard Engelking. Dimension Theory. North-Holland Publishing Company, 1978.

[14] D. Garity and D. Repovš. Cantor set problems. In E. Pearl, editor, Open Problems in Topology II, Chapter 62, pages 675–678. Elsevier, 2007.

[15] D. Garity, D. Repovš, and M. Željko. Uncountably many Inequivalent Lipschitz Homogeneous Cantor sets in $R^3$. Pacific J. Math., 222(2):287–299, 2005.

[16] D. Garity, D. Repovš, and M. Željko. Rigid Cantor sets in $R^3$ with Simply Connected Complement. Proc. Am. Math. Soc., 134(8):2447–2456, 2006.

[17] A. Georgakopoulos and G. Kontogeorgiou. Discrete group actions on 3-manifolds and embeddable Cayley complexes. arXiv:2208.09918.

[18] Allen Hatcher. Algebraic Topology. Cambridge Univ. Press, 2002.

[19] Alexander S. Kechris. Classical descriptive set theory. Graduate texts in mathematics, no. 156. Springer-Verlag, 1995.

[20] A. Kirkor. Wild 0-dimensional sets and the fundamental group. Fund. Math., 45:237–246, 1958.

[21] Edwin E. Moise. Geometric Topology in Dimensions 2 and 3. Graduate Texts in Mathematics. Springer, 1977.

[22] I. Richards. ON THE CLASSIFICATION OF NONCOMPACT SURFACES. Trans. Am. Math. Soc., 106(2):259–269, 1963.

[23] R. B. Richter and C. Thomassen. 3-connected planar spaces uniquely embed in the sphere. Trans. Am. Math. Soc., 354:4585–4595, 2002.
[24] R. B. Sher. Concerning Wild Cantor Sets in $E^3$. *Proc. Am. Math. Soc.*, 19(5):1195–1200, 1968.

[25] R. Skora. Cantor sets in $S^3$ with simply connected complements. *Topology and its Applications*, 24(1):181–188, 1986.
This figure "figSU.png" is available in "png" format from:

http://arxiv.org/ps/2208.11534v1