On congruence modular varieties and Gumm categories

Dominique Bourn

Laboratoire de Mathématiques Pures et Appliquées Joseph Liouville, Univ. Littoral Côte d’Opale, UR 2597, LMPA, Calais, France

ABSTRACT
Starting from a characterization of the so-called Shifting Lemma by a property of the fibers of the fibration \( \mathcal{E} \) of split epimorphisms in the category \( \mathcal{E} \), on the model of what happens for Mal’tsev categories, we shall investigate it in two directions: (1) a new one: in following the golden thread of the abelian split epimorphisms naturally provided by this characterization; (2) a more expected one: in measuring the distance between the consequences of the Shifting Lemma in the varietal context of congruence modular varieties and in the much more general categorical one. Along the way we shall emphasize a quite amazing phenomenon we called algebraic crystallography.

ARTICLE HISTORY
Received 7 June 2021
Revised 9 November 2021
Communicated by Alberto Facchini

KEYWORDS
Jónsson–Tarski and subtractive variety; Mal’tsev and congruence modular variety; Mal’tsev and Gumm category; split epimorphism; unital; subtractive; and strongly unital categories

2020 MATHEMATICS SUBJECT CLASSIFICATION
08A30; 08B05; 18A20; 18C10; 18C40; 18E13

0. Introduction

In [21], Gumm characterized the congruence modular varieties among all the varieties of Universal Algebra by the validity of the Shifting Lemma. Doing this, he “translated” the modular formula for congruences:

\[(T \vee S) \wedge R = T \vee (S \wedge R),\]

for any triple \((T, S, R)\) such that \(T \subset R\)

in geometric terms: given any triple of equivalence relations \((T, S, R)\) such that \(R \cap S \subset T\), the following left-hand side situation implies the right-hand side one:

Thanks to the Yoneda embedding, one of the main interest of the Shifting lemma is that, being freed of any condition involving finite colimits (as the existence of suprema for instance), it allows us to
recover in any category $\mathcal{V}(\mathcal{E})$ of internal $\mathcal{V}$-algebras in a finitely complete category $\mathcal{E}$ all the properties satisfied by the finite limits of the variety $\mathcal{V}$, when this variety is congruence modular.

More generally, this shifting property keeps a meaning in any finitely complete category $\mathcal{E}$; a very synthetic categorical characterization was given in [13] where its first consequences were investigated, mainly concerning the centralization of equivalence relations, the internal groupoids and the internal categories. Later on, the categories satisfying the Shifting Lemma were called Gumm categories in [7, 8, 14].

Now, in the same way as the Mal’tsev categories [4], the Gumm categories were characterized in [7] by some properties of the pointed fibers $\text{Pt}_Y \mathcal{E}$ (=split epimorphisms above the object $Y$) of their fibrations of points $\mathbb{1}_\mathcal{E}$, see recalls in Sections 1 and 2.1. The aim of this work is to start from this characterization in order to deepen the investigations on Gumm categories.

A first point is that any regular Gumm category shares with the (conceptually stronger) context of regular Mal’tsev categories the following property: given any pullback of split epimorphisms along a regular epimorphism $h$

$$
\begin{array}{c}
X \\
\downarrow^g \\
X' \\
\downarrow^{g'} \\
Y \\
\downarrow^h \\
Y'
\end{array}
$$

the upward and rightward square is necessarily a pushout in $\mathcal{E}$.

A second point is that the characterization theorem provide us with a golden thread inside the Gumm categories: an intrinsic notion of abelian split epimorphism, see Section 3, which naturally leads to an intrinsic notion of abelian equivalence relation and to a class of morphisms $f : X \to Y$ called special, which determines a naturally Mal’tsev subcategory $\text{APt}\mathcal{E}/Y$ of the slice category $\mathcal{E}/Y$ and a naturally Mal’tsev subcategory $\text{Aff}\mathcal{E}$ of affine objects in $\mathcal{E}$.

Then, coming back to the varietal context and still following this thread, we shall be again led to emphasize two Mal’tsev remainders inside the congruence modular varieties:

1. a new one: some specific class of reflexive and symmetric relations necessarily produces equivalence relations, see Theorem 4.1;
2. a partially known one from [21] and a bit generalized here: some specific pairs of equivalence relations do permute, see Proposition 5.2.

These two Mal’tsev ingredients naturally lead to an original construction of the universal abelian split epimorphism, see Theorem 4.3. Some other kinds of relationship between Shifting Lemma and Mal’tsev categories are asserted in [20] as well.

Another natural effect of these investigations will be to make it possible to measure the distance between the consequences of the Shifting Lemma in the varietal context and in the much more general categorical one. The point 1) above necessarily holds in any Gumm category $\mathcal{E}$ (with some further inductive condition), see Theorem 4.1, but there is no reason why the point 2) would do. In Section 5 a property (Axiom $\exists$) which realizes this second point in any regular category $\mathcal{E}$ is made explicit. With these two points in $\mathcal{E}$, the previous construction of the universal abelian split epimorphism holds in full generality, see Theorem 5.1.

Now, beyond that, these investigations draw our attention to a very specific phenomenon which we call algebraic crystallography and which we describe in Section 6.

Here is the organization of the article. After a first section devoted to conventions and notation, Section 2 recalls the Characterization Theorem and investigates the three directions opened, within the pointed categories, by this characterization. The most important one seems to be the setting of congruence hyperextensible categories whose main structural property is that it produces an intrinsic notion of abelian object. Section 3 investigates the structural outcomes of the Characterization
Theorem in following the golden thread provided by the notion of abelian split epimorphisms which are nothing but the intrinsic abelian objects of the congruence hyperextensible fibers \(Pt_Y\). Section 4 introduces the notion of diagonal punctuation which leads to the construction of the universal abelian split epimorphisms in the congruence modular varieties. Section 5 is devoted to the Axiom which allows to recover in an abstract way the results of Section 4 concerning this universal construction. Finally, Section 6 introduces and discusses the notion of algebraic crystallography. Finally, Section 7 collects all the needed informations about the technically useful equivalence between regular pushouts and congruence permutations in any regular category \(E\).

1. Conventions, notations

In this article, any category \(E\) will be supposed finitely complete. Given any map \(f : X \to Y\), we use the following simplicial notations for its kernel equivalence relation \(R[f]\):

\[
\begin{array}{cccccc}
R_2[f] & \xrightarrow{d_2^f} & R[f] & \xleftarrow{t_0^f} & X & \xrightarrow{f} & Y \\
R_2(g) & \downarrow & & \downarrow & g & \downarrow & h \\
R_2[f'] & \xrightarrow{d_2^{f'}} & R[f'] & \xleftarrow{t_0^{f'}} & X' & \xrightarrow{f'} & Y'
\end{array}
\]

and the notation \(R(g)\) for the morphism induced by the right-hand side commutative square. A regular epimorphism is a map \(f\) which is the coequalizer of the two legs of it kernel equivalence relation \(R[f]\), namely its quotient map.

Given any category \(E\), the slice category \(E/Y\) is the category whose objects are the maps with codomain \(Y\) and morphisms the commutative triangles above \(Y\), while the coslice category \(Y/E\) is the category whose objects are the maps with domain \(Y\) and morphisms are the commutative triangles under \(Y\). As we shall see, specially important are the properties of \(E\) which are stable under slicing and coslicing.

A category \(E\) is regular, see [1], when the two first following properties hold and exact when the three properties hold:

1. the regular epimorphisms are stable under pullback;
2. any kernel equivalence relation \(R[f]\) admits a quotient map;
3. any equivalence relation \(R\) is effective, i.e. of the form \(R[f]\) for some map \(f\).

Any variety \(V\) is exact and a regular epimorphism is nothing but a surjective homomorphism. Consider now any commutative quadrangle of regular epimorphisms in a regular category \(E\):

\[
\begin{array}{cccccc}
X & \xrightarrow{f} & Y \\
\downarrow & (g,f) & \downarrow \\
X' \times_Y Y' & \xleftarrow{(g',f')} & Y'
\end{array}
\]

When the induced morphism \((g,f)\) toward the pullback of \(h\) along \(f'\) is a regular epimorphism as well, the square in question is necessarily a pushout; we shall need the following:
**Definition 1.1.** Let $E$ be a regular category. A commutative square of regular epimorphims as above is said to be a regular pushout when the induced morphism $(g, f)$ toward the pullback is a regular epimorphism.

The main interest of this notion is due to the fact that, on the one hand, any regular pushout insures that the induced morphism $R(f) : R[g] \to R[h]$ and all the induced morphisms $R_n(f) : R_n[g] \to R_n[h]$ are regular epimorphisms as well and that, on the other hand, this notion characterizes the permutation of the kernel equivalence relations $R[f]$ and $R[g]$, see Proposition 7.1 in Section 7.2.

### 1.1. The fibration of points

We denote by $PtE$ the category whose objects are the split epimorphisms in $E$ with a given splitting and morphisms the commutative squares between these data; by $\mathfrak{p}_E : PtE \to E$ we denote the functor associating its codomain with any split epimorphism. It is left exact and a fibration whose cartesian maps are the pullbacks of split epimorphisms; it is called the **fibration of points** [3]. It has an important classification power in Algebra, see [2]. Let us recall two of them we shall need here [4]:

**Theorem 1.1.** Given any category $E$:

1. it is a Mal’tsev category if and only if any fiber $PtYE$ is unital;
2. it is a naturally Mal’tsev category if and only if any fiber $PtYE$ is additive.

A category is a Mal’tsev one if and only if any reflexive relation in it is an equivalence relation, see [17, 18]. It is a naturally Mal’tsev category when any object $X$ is endowed with a natural Mal’tsev operation, see [23]. A unital category is a pointed category which fulfills the categorical characterization of a Jónsson–Tarsky variety, namely it is a pointed category $E$ in which the only punctual relation between $X$ and $Z$, see next section, is given by the projections of the product $X \times Z$, see [4]. The more usual example is given by the category $Mon$ of monoids. We shall recall, in Section 2.1, how this same fibration $\mathfrak{p}_E$ characterizes the congruence modular varieties and the Gumm categories as well [7]. The fiber above an object $Y \in E$, denoted by $PtYE$, is nothing but the coslice of a slice category, namely: $1_Y/(E/Y)$.

### 1.2. Ternary operation and fibration of points

In any fiber $PtXE$ there is, with the split epimorphism $(p^X_1, s^X_0) : X \times X \twoheadrightarrow X$, an object which has a remarkable property with respect to the fibration $\mathfrak{p}_E$. It is described by the following diagram where the right-hand side part is a pullback:

![Diagram](image)

and means, among other things, that any split epimorphisms with domain $X$ is “embedded” in this split epimorphism $(p^X_1, s^X_0)$, see [9].

**Lemma 1.1.** Given any category $E$, a binary operation on $(p^X_1, s^X_0)$ inside the fiber $PtXE$ is nothing but a ternary $p$ operation on $X$ in $E$ such that $p(x, x, x) = x$. This binary operation is a subtraction in $PtXE$ if and only if it satisfies the Mal’tsev identities, $p(x, y, y) = x = p(y, y, x)$. This subtraction is the one associated with the difference mapping of a group in $PtXE$ if and only if the Mal’tsev law is associative. In this case, the group is abelian if and only if the Mal’tsev law is autonomous.
Proof. Straightforward is the first assertion: such a binary operation is necessarily of the form \( d((x,z),(y,z)) = (p(x,y,z),z) \); being pointed by \( s_0^X \) in the fiber, this operation has to satisfy \( d((z,z),(z,z)) = (z,z) \), namely \( p(z,z,z) = z \).

We get \( d((x,z),(x,z)) = (z,z) \) if and only if we have \( p(x,x,z) = z \). We get \( d((x,z),(z,z)) = (x,z) \) if and only if we have \( p(x,z,z) = x \). See [5] for the two last points.

1.3. Punctual spans in pointed categories

A category \( \mathcal{E} \) is pointed when the terminal object 1 is initial as well. Any fiber \( Pt_Y \mathcal{E} \) is pointed.

In a pointed category \( \mathcal{E} \), a span \((f,g) : W \to X \times Z\):

\[
\begin{array}{ccc}
W & \xleftarrow{g} & Z \\
\downarrow{f} & & \downarrow{\tau_Z} \\
\tau_X \downarrow{s} & & \downarrow{0_Z} \\
X & \xleftarrow{0_X} & 1
\end{array}
\]

is said to be right punctual (resp. left punctual) when there is a section \( t \) of \( g \) (resp. \( s \) of \( f \)) such that \( f.t = 0 \) (resp. \( g.s = 0 \)). It is said to be punctual when it is both right and left punctual. A relation is a span such that the map \((f,g) : W \to X \times Z\) is a monomorphism.

Later on, we shall be highly interested in the punctual spans in the pointed fibers \( Pt_Y \mathcal{E} \), namely in the following commutative squares of split epimorphisms in \( \mathcal{E} \):

\[
\begin{array}{ccc}
W & \xleftarrow{g} & Z \\
\downarrow{f} & & \downarrow{\tau_Z} \\
\tau_X \downarrow{s} & & \downarrow{0_Z} \\
X & \xleftarrow{0_X} & 1
\end{array}
\]

is said to be right punctual (resp. left punctual) when there is a section \( t \) of \( g \) (resp. \( s \) of \( f \)) such that \( f.t = 0 \) (resp. \( g.s = 0 \)). It is said to be punctual when it is both right and left punctual. A relation is a span such that the map \((f,g) : W \to X \times Z\) is a monomorphism.

Later on, we shall be highly interested in the punctual spans in the pointed fibers \( Pt_Y \mathcal{E} \), namely in the following commutative squares of split epimorphisms in \( \mathcal{E} \):

\[
\begin{array}{ccc}
W & \xleftarrow{g} & Z \\
\downarrow{f} & & \downarrow{\tau_Z} \\
\tau_X \downarrow{s} & & \downarrow{0_Z} \\
X & \xleftarrow{0_X} & 1
\end{array}
\]

Proposition 1.1. Given any punctual span in the fiber \( Pt_Y \mathcal{E} \), the idempotent morphisms \( sf \) and \( tg \) on \( W \) do commute in \( \mathcal{E} \).

Proof. The proof is straightforward: \( sftg = s\sigma_X\phi_Zg = t\sigma_Z\phi_Xf = tgsf \).

So, the whole diagram originates from a pair commutative idempotents on the object \( W \).

2. Pointed variations on the Shifting Lemma

2.1. The characterization theorem

As recalled in the introduction a Gumm category is a category satisfying the Shifting Lemma. Clearly, and importantly, this property is stable under slicing and coslicing, and consequently under the passage to the fibers \( Pt_Y \mathcal{E} \). We have also the following:

Lemma 2.1 ([14]). Let \( F : \mathcal{C} \to \mathcal{D} \) be left exact and conservative functor (i.e. reflecting the isomorphisms). If \( \mathcal{D} \) is a Gumm category, so is \( \mathcal{C} \).

As already noticed, thanks to the Yoneda embedding any result in [21] which is proved from the Shifting Lemma by using only projective properties remains valid in any Gumm category. It is the case, in particular, for the Cube Lemma (Proposition 2.4 in [21]) we shall need later on:
Proposition 2.1 (Cube Lemma). Let \( \mathcal{E} \) be a Gumm category. Given any triple of equivalence relations \((T, S, R)\) on an object \(X\) such that \(R \cap S \subseteq T\), the plain arrows imply the dotted one:

\[
\begin{array}{c}
\begin{array}{ccc}
X & \xrightarrow{T} & X'
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{ccc}
S & \xrightarrow{R} & S'
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{ccc}
R & \xrightarrow{R} & R'
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{ccc}
T & \xrightarrow{R} & T'
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{ccc}
S & \xrightarrow{R} & S'
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{ccc}
R & \xrightarrow{R} & R'
\end{array}
\end{array}
\end{array}
\]

As recalled above, the fibration of points \(\mathcal{E}\) allows to characterize the Gumm categories:

Theorem 2.1 ([7]). Given a category \(\mathcal{E}\), the following conditions are equivalent:

1. \(\mathcal{E}\) is Gumm category;
2. any fiber \(Pt_Y \mathcal{E}\) is congruence hyperextensible;
3. any fiber \(Pt_Y \mathcal{E}\) is congruence hypoextensible;
4. any fiber \(Pt_Y \mathcal{E}\) is punctually congruence modular.

Any of these notions are recalled and investigated in the next sections. The point 2) implies that a regular Mal’tsev category is necessarily a Gumm one [7].

2.2. Congruence hyperextensible categories

Definition 2.1. A pointed category \(\mathcal{E}\) is said to be congruence hyperextensible when given any punctual span in \(\mathcal{E}\) and any equivalence relation \(T\) on \(W\) such that \(R[f] \cap R[g] \subseteq T\), we get \(R[f] \cap g^{-1}(t^{-1}(T)) \subseteq T\).

In other words, when the following situation holds in \(W\):

\[
\begin{array}{c}
\begin{array}{ccc}
x & \xrightarrow{R[g]} & x'
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{ccc}
x' & \xrightarrow{R[f]} & x'
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{ccc}
tg(x) & \xrightarrow{R[f]} & tg(x')
\end{array}
\end{array}
\]

Any Jónsson–Tarsky variety \(V\) and more generally any regular unital category \(\mathcal{E}\) is congruence hyperextensible [7]. Here is an example of non-unital congruence hyperextensible variety:

Example 2.1. Let us call \(Hex_3\) the pointed variety defined by three ternary terms \(p_i\) such that:

\[
\begin{align*}
p_1(a,0,0) &= a \\
p_2(a,0,a) &= a \\
p_3(0,0,a) &= a \\
p_1(a,a,b) &= p_2(a,a,b) \\
p_2(a,b,b) &= p_3(a,b,b) \\
p_1(a,a,a) &= a, \quad \forall i \quad 1 \leq i \leq 3
\end{align*}
\]

It is a non-unital congruence hyperextensible variety.

Proof. In the varietal context, it is enough to check the property for the punctual relations. So let \(T\) be an equivalence relation defined on the punctual relation \(W\) on \(X \times Z\). Suppose we have \((0Wb)T(0Wc)\). Then we get: \(aWb = p_1(a,0,0)Wp_1(b,b,b) = p_1(aWb,0Wb,0Wb)\). It is in the \(T\)-equivalence class of \(p_1(aWb,0Wb,0Wc) = p_1(a,0,0)Wp_1(b,b,c) = p_2(a,0,a)Wp_2(b,b,c) = \)
\[ p_2(aWb,0Wb,aWc) \] which, in turn, is in the \( T \)-class of \( p_2(aWb,0Wc,aWc) = p_2(a,0,a)Wp_2(b,c,c) = p_3(0,0,a)Wp_3((b,c,c)) = p_3(0Wb,0Wc,aWc). \) It is itself in the \( T \)-class of \( p_3(0Wc,0Wc, aWc) = p_3(0,0,a)Wp(c,c,c) = aWc. \)

It remains to show that this variety is not a Jónsson–Tarsky one. For that we shall exhibit a \( Hex_3 \)-algebra structure \( X \) on the set \( \{0, 1\} \) producing a strict subalgebra structure \( W \) of \( X \) which contains the two canonical injections \( X \rightarrow X \times X \). Let us define the three ternary operations \( p_1, p_2, p_3 \) from the following table:

|         | (0,0,0) | (1,0,0) | (0,1,0) | (0,0,1) |
|---------|---------|---------|---------|---------|
| (1,1,1) | (0,1,1) | (1,0,1) | (1,1,0) |

by the following ones:

\[
\begin{array}{cccccc}
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
\end{array}
\]

Then, on the subset \( \{(0,0),(1,0),(0,1)\} \) of \( X \times X \), there is a structure of \( Hex_3 \)-subalgebra since the element \((1, 1)\) is obtained as the result of the three ternary operations \( p_i \) on \( X \times X \) only if these operations do operate on it.

**Lemma 2.2.** Let \( \mathbb{E} \) be any congruence hyperextensible category. Consider any punctual relation:

\[
\begin{array}{ccc}
W & \xleftarrow{\alpha_W} & Z \\
\pi_X & \downarrow & \pi_Z \\
X & \xleftarrow{\alpha_X} & 1
\end{array}
\]

Then the map \( \pi_Z \) is the cokernel of the map \( 0_W^X \). Moreover, it is the unique map \( h : W \rightarrow Z \) in \( \mathbb{E} \) such that: (1) \( h \cdot 0_W^X = 1_Z \) and (2) \( h \cdot 0_X^W = 0 \).

**Proof.** We shall use the Yoneda embedding for the first step: namely showing that a map \( f : W \rightarrow V \) is such that \( f \cdot 0_W^X = 0 \) if and only if it annihilates the kernel equivalence relation \( R[\pi_Z] \).

With \( f \cdot 0_X^W = 0 \) and consequently \( f(xW0) = f(0W0) \), we are in the following situation:

\[
\begin{array}{ccc}
xWz & \xrightarrow{R[\pi_X]} & xW0 \\
\downarrow & & \downarrow \\
R[f] \circ R[\pi_Z] & \xrightarrow{R[\pi_Z]} & R[f] \\
\downarrow & & \downarrow \\
0Wz & \xrightarrow{R[\pi_X]} & 0W0 \\
\end{array}
\]

whence, by the congruence hyperextensibility, the bended upper arrow since we have: \( R[\pi_X] \cap R[\pi_Z] = \Delta_W \subseteq R[f] \), where \( \Delta_W \) denotes the identity relation on \( W \). So, we get \( f(xWz) = f(0Wz) = f(x'Wz) \) which means that \( f \) annihilates \( R[\pi_Z] \). Since \( \pi_Z \) is a split epimorphism, it is the quotient of its kernel equivalence relation, and we get the desired morphism: \( h = f \cdot 0_Z^W : Z \rightarrow V \) such that \( f = h \cdot \pi_Z \).

Given any map \( h : W \rightarrow Z \) satisfying (2), we get a (unique) map \( \phi : Z \rightarrow Z \) such that \( \phi \cdot \pi_Z = h \). From (1), by composition with \( 0_Z^W \), we get \( \phi = 1_Z \) and \( h = \pi_Z \).

The previous lemma holds, in particular, when the square in question is a pullback, and allows to make explicit a very important property which is shared by the (conceptually stronger) unital categories:
Theorem 2.2. Let $\mathcal{E}$ be a congruence hyperextensible category. Given any object $Y$, the base-change functor $\mathcal{E} \to \text{Pt}_Y \mathcal{E}$ along the terminal map $\tau_Y: Y \to 1$ is fully faithful.

Proof. It is clearly faithful since $p^Z_1$ is a (split) epimorphism. Now consider the following diagram:

Given any map $(p^X_0, \phi): X \times Z \to X \times Z'$ making commute the vertical left-hand side quadrangle, the previous lemma determines a unique morphism $\psi: Z \to Z'$ necessarily making the upper horizontal quadrangle a pullback, which means: $(p^X_0, \phi) = X \times \psi$. \hfill \Box

Proposition 2.2. Let $\mathcal{E}$ be a congruence hyperextensible category. Any subtraction $d$ on an object $X$ is the difference mapping associated with an internal group structure on $X$ which is necessarily abelian. On any object $X$, there is at most one subtraction $d$; any morphism $f: X \to Y$ in $\mathcal{E}$ between two objects equipped with a subtraction is a subtraction homomorphism. This subtraction is an opsubtraction as well (namely $d(0, x) = x$ and $d(x, x) = 0$) if and only if the associated abelian group $(X, \circ, 0)$ is such that $x \circ x = 0$.

Proof. Let $d$ be any subtraction on an object $X$. Consider the following punctual relation:

Setting $\beta(x, y, z) = d(d(x, z), d(y, z))$, we get the following situation associated with the previous punctual relation:

since we have the right-hand side bended arrow by $d(t, t) = 0$. When $\mathcal{E}$ is congruence hyperextensible, we get: $\beta(x, y, z) = \beta(x, y, 0)$, namely $d(d(x, z), d(y, z)) = d(d(x, 0), d(y, 0)) = d(x, y)$. So we get the first point with the group law given by $x \circ y = d(x, d(0, y))$. The inverse of $x$ is $d(0, x)$; so we get $(\ast): d(0, d(0, x)) = x$.

We know that the group in question is commutative if and only if we have $d(x, d(x, y)) = y$. Then define $\psi: X \times X \to X$ by $\psi(x, y) = d(x, d(x, y))$, so that, by $(\ast)$, we have $\psi(0, y) = y$; we also get: $\psi(x, 0) = d(x, d(0, 0)) = d(x, x) = 0$. We then get the following situation:
Whence $d(x, d(x, y)) = \psi(x, y) = \psi(0, y) = y$.

Accordingly, when $(X, d)$ is a subtraction, it is right cancellable as being the difference mapping of a group structure. So, the following punctual span:

$$
\begin{array}{c}
X \times X & \xrightarrow{d} & X \\
\downarrow \scriptstyle{p_1^X} & & \downarrow \scriptstyle{\tau_X} \\
X & \xleftarrow{\sigma_X} & 1
\end{array}
$$

is a punctual relation. Then the uniqueness of $d$ follows from Lemma 2.2.

Now, let $(Y, d)$ another subtraction and $f : X \to Y$ any map in $\mathbb{E}$. Then consider the following diagram:

$$
\begin{array}{c}
X & \xrightarrow{s_0^X} & X \times X & \xrightarrow{d} & X \\
\downarrow \scriptstyle{f} & & \downarrow \scriptstyle{\phi} \\
Y & \xleftarrow{s_0^Y} & Y \times Y & \xleftarrow{d} & Y
\end{array}
$$

Since we have horizontal cokernel diagrams, we get a unique morphism $\phi$, satisfying $\phi(d(a, b)) = d(f(a), f(b))$. Whence $\phi(a) = \phi(d(a, 0)) = d(f(a), 0) = f(a)$. Finally we have: $x \circ x = 0$ $\iff$ $d(x, d(0, x)) = 0$ $\iff$ $x = d(0, x)$.

The uniqueness property asserted by this proposition makes intrinsic the notion of internal abelian group object in a congruence hyperextensive category $\mathbb{E}$, whence the following:

**Definition 2.2.** Let $\mathbb{E}$ be a congruence hyperextensible category. An object $X$ is said to be abelian when it is endowed with a (necessarily unique) subtraction $d$. We shall denote by $\text{Ab} \mathbb{E}$ the full subcategory of abelian objects in $\mathbb{E}$ which is additive and closed under finite limits in $\mathbb{E}$.

When $\mathbb{V}$ is a congruence hyperextensible variety, the subcategory $\text{Ab} \mathbb{V}$ is actually abelian. Here is an unexpected and remarkable situation where two distinct algebras are made abelian by subtractive homomorphisms which are defined by the same formula:

**Example 2.2.** In the variety $\text{Hex}_3$ given by Example 2.1, let us consider the following two algebra structures on the same set $\mathbb{Q}$ of rational numbers, where the first one is given by $p_1(x, y, z) = x - y + z$ and $p_2(x, y, z) = z = p_3(x, y, z)$, and the second one by $\bar{p}_1(x, y, z) = x + \frac{y + z}{2}$, $\bar{p}_2(x, y, z) = \frac{x + z}{2}$ and $\bar{p}_3(x, y, z) = \frac{x - y}{2} + z$. They are both abelian by the existence of the subtractive homomorphism $d(a, b) = a - b$.

Later on, we shall need the following classical observation:

**Proposition 2.3.** Let $\mathbb{E}$ be a regular pointed category such that the category $\text{Ab} \mathbb{E}$ of abelian groups in $\mathbb{E}$ is a full subcategory of $\mathbb{E}$. Suppose the inclusion $\text{Ab} \mathbb{E} \hookrightarrow \mathbb{E}$ has a left adjoint $A$. Then the following conditions are equivalent:
(1) the universal comparison \(\eta_X : X \to A(X)\) is a regular epimorphism;
(2) \(\text{Ab}\mathcal{E}\) is closed under monomorphism;
(3) \(\text{Ab}\mathcal{E}\) is Birkhoff subcategory of \(\mathcal{E}\).

Proof. \(\mathcal{E}\) being finitely complete, \(\text{Ab}\mathcal{E}\) is closed under product. Suppose 1). Let \(A\) be an abelian object and \(u : U \to A\) a monomorphism. Then the group homomorphism \(A(u) : A(U) \to A\) such that \(u = A(u)\eta_U\). The morphism \(u\) being a monomorphism, so is \(\eta_U\), which, being a regular epimorphism as well, is an isomorphism. Accordingly, \(U\) is abelian. Suppose 2). Take the canonical decomposition of \(\eta_X\), it produces a subobject \(v : V \to A(X)\), which according to 2) is a subgroup of the abelian group \(A(X)\). The universal property of \(A(X)\) makes \(v\) an isomorphism and \(\eta_X\) a regular epimorphism.

We are now going to show that under the condition 2), any regular epimorphism \(f : A \to Y\) with \(A\) abelian makes \(Y\) abelian. Under the Condition 2), the object \(R[f]\), as a subobject of \(A \times A\), is abelian. Consider now the following diagram:

\[
\begin{array}{cccccc}
R[f] & \times & R[f] & \xrightarrow{d_1^i \times d_1^i} & A & \xrightarrow{f \times f} & Y \times Y \\
p_0 & \searrow & p_1 & \downarrow & p_0 & \swarrow & p_1 \\
R[f] & \searrow & 0_A & \downarrow & 0_Y & \swarrow & \tau_Y \\
& 1 & \tau_{R[f]} & \downarrow & 1 & \end{array}
\]

In a regular category, the regular epimorphisms being closed under product, the morphism \(f \times f\) is a regular epimorphism which, therefore, is the quotient of the upper horizontal left-hand side equivalence relation. Accordingly the composition \(\circ\) extends to the quotient and makes \(Y\) an abelian group. \(\square\)

**Proposition 2.4.** Let \(\mathcal{E}\) be a congruence hyperextensible category. Any binary operation \(\circ\) with unit on an object \(X\) is necessarily associative.

**Proof.** Let \(\circ\) be any binary operation with unit on \(X\). Consider the following punctual relation:

\[
\begin{array}{cccc}
X \times X \times X & \xleftarrow{p_1^X, p_2^X} & X & \\
p_0^X & \uparrow & 0_{X \times X \times X} & \tau_X & 0_X \\
X \times X & \xleftarrow{\tau_X} & 0_{X \times X} & \\
& & \end{array}
\]

Setting \(\gamma(x, y, z) = x + (y + z)\), we get the following situation:

\[
\begin{array}{ccc}
(x, y, z) & \xrightarrow{R[\tau_0]} & (x, y, 0) \\
R[\gamma] & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
R[\gamma] & \xrightarrow{R[\gamma]} & (x + y, z) & \xrightarrow{R[\gamma]} & (0, x + y, 0) \\
(0, x + y, z) & \xrightarrow{R[\gamma]} & (0, x + y, 0) & \end{array}
\]

When \(\mathcal{E}\) is congruence hyperextensible, we get: \(\gamma(x, y, z) = \gamma(0, x + y, z)\), namely \(x + (y + z) = (x + y) + z\). \(\square\)
2.3. Congruence hypoextensible categories

Definition 2.3. A pointed category \( E \) is said to be congruence hypoextensible when given any punctual span in \( E \) and any equivalence relation \( T \) on \( W \) such that \( R[f] \cap R[g] \subset T \subset R[f] \), we get: \( T \subset g^{-1}(r^{-1}(T)) \).

In other words, when the following situation holds:

![Diagram](https://via.placeholder.com/150)

Immediately, we get the following remarkable categorical property:

**Proposition 2.5.** Let \( E \) be a congruence hypoextensible category. Given any punctual span in \( E \), then any equivalence relation \( T \) on \( W \) such that \( R[f] \cap R[g] \subset T \subset R[f] \) has a direct image along \( g \) which is nothing but \( r^{-1}(T) \).

**Proof.** The previous diagram provides us with the dotted morphism \( g \) of equivalence relations:

![Diagram](https://via.placeholder.com/150)

defined by \( g(uTv) = (g(u), g(v)) \), which is regular epimorphism in \( E \) since it is split by \( f \).

Any pointed subtractive variety \( V \) [24] or regular subtractive category \( E \) is congruence hypoextensible [7], where the concept of subtractive category, introduced in [22], is a way to categorically characterize the pointed subtractive varieties:

**Definition 2.4.** A pointed category \( E \) is subtractive when any split left punctual relation is a punctual one.

**Proposition 2.6.** If \( E \) is congruence hypoextensible, then any internal binary operation \( \circ \) on an object \( X \) is left cancellable as soon as it has a left unit. In particular any opsubtraction is left cancellable, and, by duality, any subtraction is right cancellable.

**Proof.** We get the following situation:

![Diagram](https://via.placeholder.com/150)

which means the left cancellability: \( a \circ c = a \circ c' \Rightarrow c = c' \).

2.4. Punctually congruence modular categories

**Definition 2.5.** A punctually congruence modular category \( E \) is a pointed category which is both congruence hyperextensible and hypoextensible; i.e. such that, given any punctual span and any equivalence relation \( T \) on \( W \) such that \( R[f] \cap R[g] \subset T \subset R[f] \), we get: \( T = R[f] \cap g^{-1}(r^{-1}(T)) \).
A more categorial translation of this definition is given by the following:

**Proposition 2.7.** Let $\mathbb{E}$ be a congruence hypextensible category. Then it is punctually congruence modular if and only if, given any punctual span in $\mathbb{E}$ and any equivalence relation $T$ on $W$ such that $R[f] \cap R[g] \subset T \subset R[f]$, the following diagram where $g$ is defined by Proposition 2.5:

$$
\begin{array}{ccc}
T & \xleftarrow{g} & t^{-1}(T) \\
\downarrow & & \downarrow \\
R[f] & \xleftarrow{R(g)} & \nabla Z
\end{array}
$$

is a pullback in the category $\text{Equ}\mathbb{E}$ of equivalence relations in $\mathbb{E}$.

Any strongly unital variety (= subtractive Jónsson–Tarsky variety) $\mathbb{V}$ or any regular strongly unital (=unital + subtractive, see [3, 22]) category $\mathbb{E}$ is punctually congruence modular [7].

**Proposition 2.8.** Let $\mathbb{E}$ be a punctually congruence category. A binary operation $(X, \circ)$ with a right-hand side unit has a unique unary right canceler $!: X \to X$, namely a unique unary operation such that $x \circ !x = 0$.

An object $X$ is abelian in $\mathbb{E}$ if and only if it is endowed with a binary operation $\circ$ which has a right-hand side unit and a unary right canceler.

**Proof.** Any binary operation $(X, \circ)$ with a right-hand side unit and a right canceler $!: X \to X$ produces a subtraction $d(x, y) = x \circ !y$ on $X$. Since $d$ is unique, then $!(x) = d(0, x)$ is unique. This subtraction $d$ makes $X$ abelian. \qed

### 3. Outcomes of the characterization theorem

By the point 2) of the characterization theorem and by Proposition 2.2, a Gumm category $\mathbb{E}$ is immediately equipped with an intrinsic class of abelian split epimorphisms: namely those split epimorphisms $(f, s): X \to Y$ which are abelian objects in the congruence hyperextensible fiber $Pt_Y \mathbb{E}$.

So, a split epimorphism $(f, s): X \to Y$ is abelian in $\mathbb{E}$ if and only if there is a (unique) subtraction on it in the fiber $Pt_Y \mathbb{E}$, namely if and only if it is endowed with a morphism $d: R[f] \to X$ such that: $d.s.f = s.f$ and $d.(1_X, s.f) = 1_X$ (i.e. $d(x, x) = sf(x)$ and $d(x, sf(x)) = x$); then the morphism $\circ_s: R[f] \to X$ defined by $x \circ_s x' = d(x, d(sf(x), x'))$ necessarily produces an internal (abelian) group structure with unit $s$ on the object $f$ in the slice category $\mathbb{E}/Y$. We shall denote this class by $\text{APt}\mathbb{E}$ (or $\text{APt}$ when there is no ambiguity about the environment). According to what is now usual in the partial Mal'tsev and protomodular context, it is natural to introduce the following:

**Definition 3.1.** Let $\mathbb{E}$ be any Gumm category. An equivalence relation $R$ on $X$ is said to be abelian, when the split epimorphism $(d^R_0, s^R_0)$ is abelian in $Pt_X \mathbb{E}$, a morphism $f: X \to Y$ is said to be $\text{APt}$-special when its kernel equivalence relation $R[f]$ is abelian. An object $X$ is said to be affine when the terminal map $\tau_X$ is $\text{APt}$-special or, equivalently, when its indiscrete equivalence relation $\nabla_X$ is abelian.

According to this definition and to Lemma 1.1, an object $X$ is affine if and only if there is a (unique) internal ternary operation $p: X \times X \times X \to X$ satisfying the Mal’tsev identities which is then necessarily autonomous. In this way, we get immediately a characterization which was already obtained in [13], but after a long computational way.

More generally, an equivalence relation $R$ on $X$ is abelian if and only if there is a (unique) morphism $p: R[d^R_0] \to X$ in $\mathbb{E}$ such that:
The aim of this section is to set up the properties of the class \( \text{APt}_E \) and of its derivations introduced by the previous definition. We get immediately:

**Lemma 3.1.** The class \( \text{APt}_E \) is fibrational, i.e. stable under pullback in \( \text{Pt}_E \).

**Proof.** Straightforward since, for any map \( f : W \to Y \) the base change \( f^* : \text{Pt}_Y E \to \text{Pt}_W E \) is left exact; so, it preserves any subtraction. \( \square \)

Before going any further, we shall need the following observation whose last assertion makes explicit a very important property which is shared by the (conceptually stronger) Mal’tsev categories:

**Proposition 3.1.** Let \( E \) be a Gumm category and the following diagram be any commutative square of split epimorphims in \( E \) such that \( R[f] \cap R[g] = \Delta_W \):

\[
\begin{array}{ccc}
W & \xrightarrow{g} & Z \\
\downarrow f & & \downarrow \phi_Z \\
X & \xleftarrow{\phi_X} & Y
\end{array}
\]

Then the upward and rightward square is a pushout in \( E \). The map \( g : W \to Z \), is the unique map \( \gamma : W \to Z \) such that \( \gamma.s = \sigma_Z.\phi_X \) and \( \gamma.t = 1_Z \). This result holds, in particular, when the previous square is a pullback, and shows that the base-change functor \( \phi_X^* : \text{Pt}_Y E \to \text{Pt}_X E \) along any split epimorphism \( \phi_X \) is fully faithful.

**Proof.** It is a direct consequence of Lemma 2.2 and of the fact that the fiber \( \text{Pt}_Y E \) is congruence hyperextensible since our data is nothing but a punctual span in this fiber.

Suppose given a pair \( (h : W \to V, k : Y \to V) \) of morphisms in \( E \) such that \( h.s = k.\phi_X \). Then consider the following morphism of split epimorphisms:

\[
\begin{array}{ccc}
W & \xrightarrow{(\phi_X.f,h)} & Y \times V \\
\downarrow f & & \downarrow (1_Y,k) \\
X & \xleftarrow{\phi_X} & Y
\end{array}
\]

This commutative diagram says that the map \((\phi_X.f,h)\) anihilates the map \( s \) in the fiber \( \text{Pt}_Y E \). Since \( g \) is the cokernel of \( s \) in this fiber, there is a unique morphism \((\phi_Z,\psi) : Z \to Y \times V\) such that \((\phi_Z,\psi).g = (\phi_Z,f.h)\) and \((\phi_Z,\psi).\sigma_Z = (1_Y,k)\), namely such that \( \psi.g = h \) and \( \psi.\sigma_Z = k \). The last sentence is a consequence of Theorem 2.2. \( \square \)

Whence an immediate extension to the regular context:

**Theorem 3.1.** Let \( E \) be a regular Gumm category and the following diagram be any commutative square of split epimorphims in \( E \) where 1) \( R[f] \cap R[g] = \Delta_W \) and 2) the horizontal arrows are regular epimorphisms:

\[
\begin{array}{ccc}
W & \xrightarrow{g} & Z \\
\downarrow f & & \downarrow \phi_Z \\
X & \xleftarrow{\phi_X} & Y
\end{array}
\]
Then the upward and rightward square is a pushout in $E$. This result holds, in particular, when the previous square is a pullback along the regular epimorphism $\phi_X$. It makes fully faithful any base-change functor $\phi_X : \text{Pt}_Y \rightarrow \text{Pt}_X E$ when $\phi_X$ is a regular epimorphism.

**Proof.** Complete the diagram with the kernel equivalence relations:

```
\[
\begin{array}{c}
R[g] \xleftarrow{d_1^0} W \xrightarrow{g} Z \\
\downarrow \quad \quad \downarrow f \quad \quad \downarrow \phi_Z \\
R(f) \xleftarrow{d_0^0} X \xrightarrow{s} Y \\
\downarrow \quad \quad \downarrow \phi_X \\
R[\phi_X] \xleftarrow{s_0^X} X \xrightarrow{\phi_X} Y \\
\end{array}
\]
```

When the pair $(f, g)$ is jointly monic, so are the pairs $(R(f), d_0^0)$ and $(R(f), d_1^0)$. Now, according to the previous proposition, any of the upward and rightward left-hand side commutative square is a pushout. Consider the following two whole rectangles which are exactly the same:

```
\[
\begin{array}{c}
R[\phi_X] \xrightarrow{R(s)} R[g] \xrightarrow{d_1^0} W \\
\downarrow \quad \quad \downarrow g \\
R[\phi_X] \xrightarrow{d_0^X} X \xrightarrow{s} W \\
\downarrow \quad \quad \downarrow \phi_X \\
R[\phi_X] \xrightarrow{s_0^X} X \xrightarrow{\phi_X} Y \\
\downarrow \quad \quad \downarrow \sigma_Z \\
X \xrightarrow{s} W \xrightarrow{g} Z \\
\end{array}
\]
```

The two squares on the left-hand side are pushouts; so both rectangles are pushouts. Since the left-hand side square of the right-hand side rectangle is a pushout, so is the right-hand side square of this rectangle.

**Lemma 3.2.** Let $E$ be a Gumm category. Any morphism in $\text{Pt}_E$ between two split epimorphisms in $\text{APt}_E$ preserves the associated abelian group structures in the fibers. In particular, any morphism between two affine objects preserves the affine structures.

**Proof.** Consider any map in $\text{Pt}_E$ between abelian objects in the fibers:

```
\[
\begin{array}{c}
X \xleftarrow{\phi} X' \\
\downarrow f \quad \quad \downarrow s' \\
Y \xrightarrow{\psi} Y' \\
\end{array}
\]
```

The base change $\psi^* : \text{Pt}_Y \rightarrow \text{Pt}_Y$ being left exact produces an abelian structure on the pullback $\psi^* (f', s')$ of $(f', s')$ along $\psi$; now, by Proposition 2.2, the induced morphism $(f, s) \rightarrow \psi^* (f', s')$ is necessarily a group homomorphism in the congruence hyperextendible fiber $\text{Pt}_Y E$. As for the last assertion of the lemma, apply the first assertion to the following morphism in $\text{Pt}_E$:

```
\[
\begin{array}{c}
X \times X \xrightarrow{f \times f} Z \times Z \\
\downarrow \quad \quad \downarrow \pi_1^Z \\
X \xrightarrow{f} Z \\
\end{array}
\]
```
From now on, \(\text{APt}\) can, and will, denote the full subcategory of \(\text{Pt}\) whose objects are the abelian split epimorphisms. Similarly \(\text{AbEqu}\) will denote the full subcategory of \(\text{Equ}\) whose objects are the abelian equivalence relations, \(\text{APt}/Y\) and \(\text{Aff}\) will denote the full subcategories of \(E/Y\) and \(E\) whose objects are respectively the \(\text{APt}\)-special maps and the affine objects.

**Corollary 3.1.** Let \(E\) be a regular Gumm category. Any base change functor \(h^*: \text{Pt}Y \to \text{Pt}Y\) along a regular epimorphism \(h\) reflects the abelian split epimorphisms; in other words, given any pullback of split epimorphism:

\[
\begin{array}{ccc}
X' & \xrightarrow{\tilde{h}} & X \\
\downarrow{f'} & & \downarrow{f} \\
Y' & \xrightarrow{h} & Y
\end{array}
\]

the split epimorphism \((f, s)\) is abelian as soon as so is \((f', s')\). Accordingly, given any fibrant morphism \((f, \tilde{f}) : R \to S\) of equivalence relations above a regular epimorphism \(f : X \to Y\), the equivalence relation \(S\) is abelian if and only if so is \(R\). More generally, any base change functor \(h^*: E/Y \to E/Y\) along a regular epimorphism \(h\) reflects the \(\text{APt}\)-special morphisms.

**Proof.** Complete the diagram by the horizontal kernel equivalence relations:

\[
\begin{array}{ccc}
R[\tilde{h}] & \xleftarrow{p_1^0} & X' & \xrightarrow{\tilde{h}} & Y' \\
\downarrow{R(f')} & & \downarrow{f} & & \downarrow{f} \\
R[h] & \xleftarrow{p_0^0} & Y' & \xrightarrow{h} & Y
\end{array}
\]

The right-hand side square being a pullback, so are the left-hand side ones; accordingly, the split epimorphism \((R(f'), R(s'))\) is abelian, and the two left-hand side squares preserve the group structures in the fibers, so that the passage to the quotient gives \((f, s)\) a group structure in the fiber \(\text{Pt}Y\).

**Proposition 3.2.** Let \(E\) be a Gumm category. The class \(\text{APt}\) is closed under finite limits in \(\text{Pt}\) and contains the terminal object \(1\).

Accordingly, the subcategories \(\text{AbEqu}\) and \(\text{APt}/Y\) (so, in particular, the subcategory \(\text{Aff}\)) are respectively closed under finite limits in \(\text{Equ}\) and \(E/Y\) (resp. in \(E\)). Any category \(\text{APt}/Y\) (and in particular \(\text{Aff}\)) is a naturally Mal’tsev category (see Theorem 1.1).

**Proof.** The stability under product in \(\text{Pt}\) is straightforward. Let us check the stability under equalizers. Consider the following diagram where the split epimorphisms \((f, s)\) and \((f', s')\) are abelian and where the two horizontal levels are equalizers:
since the squares indexed by 0 and 1 preserve the subtractions in the fibers, this subtraction naturally extends to the induced split epimorphism \((\phi, \sigma)\). The end of the first sentence is straightforward consequence of the fact that, in any of the categories in question, the finite limits are componentwise.

Now, in any category \(\text{APt}\mathcal{E}/\mathcal{Y}\), each object \(f : X \to Y\) is equipped with an internal natural autonomous Mal’tsev operation; so, any category \(\text{APt}\mathcal{E}/\mathcal{Y}\) is naturally Mal’tsev.

**Corollary 3.2.** Let \(\mathcal{E}\) be a Gumm category. Suppose that the morphisms \(g\) and \(g.f\) are APt-special, then so is \(f\).

**Proof.** We get the following pullback in \(\text{Equ}\mathcal{E}\), where \(Y\) is the codomain of \(f\):

\[
\begin{array}{ccc}
R[f] & \xrightarrow{\Delta_Y} & R[f] \\
\downarrow & & \downarrow \\
R[g.f] \xrightarrow{R(f)} R[g]
\end{array}
\]

So \(R[f]\) is abelian as soon as so are \(R[g.f]\) and \(R[g]\).

**Proposition 3.3.** Let \(\mathcal{E}\) be an exact Gumm category and \(f : X \to Y\) be any abelian extension, namely any regular APt-special morphism \(f\). Then the construction given by the diagram below in the proof produces, on the vertical right-hand side, an abelian split epimorphism in \(\mathcal{E}\). It is called the direction of this abelian extension. This construction applies in particular to the case of affine objects \(X\) with global support, namely such that the terminal map \(\tau_X : X \to 1\) is a regular epimorphism \((X \neq \emptyset\) in the varietal context); and so, it associates with any such affine object an abelian group \(\tilde{A}_X\) in \(\mathcal{E}\).

**Proof.** Consider the following diagram where the map \(p : R[f] \times_1 R[f] \to X\) is defined by \(p(x, y, z) = \partial(x, \partial(y, z))\). Since, in a Gumm context, this ternary operation is autonomous, it makes pullback any commutative square on the left-hand side:

\[
\begin{array}{ccc}
R[f] \times_1 R[f] & \xrightarrow{p} & R[f] \\
\downarrow & & \downarrow \\
R[f] & \xrightarrow{qf} & \tilde{A}_f
\end{array}
\]

so that the left-hand side part of the upper horizontal diagram is an equivalence relation. Let \(q_f\) be the quotient map of this equivalence relation. Since the left-hand side commutative squares are pullbacks, so is the right-hand side one in the exact category \(\mathcal{E}\) thanks to the Barr–Kock theorem [12]. Then, as in any category \(\mathcal{E}\), the autonomous law \(p\) gives \((\theta_f, \psi_f)\) an abelian structure defined by \(\tilde{a}b + \tilde{bc} = \tilde{ac}\), see [5].

**3.1. Baer sum of abelian extensions with a given direction**

So, again according to [5] and similarly to the exact Mal’tsev context, when \(\mathcal{E}\) is an exact Gumm category, there is an abelian group structure on any set \(\text{Ext}_A\) of isomorphic classes of abelian
extensions \( f : X \rightarrow Y \) having a given direction \((\psi, \theta) : \vec{A} \Rightarrow \vec{Y}\) in \(Pt_Y \overline{E}\); the binary operation of this group is produced by the so-called classical Baer sum processing.

### 4. Diagonal extension and punctuation

In this section, we shall be now interested in the construction of the universal abelian split epimorphism associated with any split epimorphism \((f, s) : X \twoheadrightarrow Y\).

We shall be inspired by what happens in the pointed protomodular categories \([6]\) as well as in the subtractive categories \([16]\). For that we shall need the following guide-line tool:

**Definition 4.1.** Let \(R\) be any equivalence relation on an object \(X\) in a category \(E\). Given any map \(f : X \rightarrow Y\), the diagonal extension of \(R\) along \(f\) is defined, when it exists, as the following left-hand side pushout:

\[
\begin{array}{c}
\xymatrix{ R \ar[r]^{\gamma} & f! [R] \\
R^0 \ar@<1ex>[u] \ar[r] & f!(s^0_R) \\
X \ar[r]^f \ar@<1ex>[u] & Y \\
}
\end{array}
\]

Then the retraction \(d^0_R\) produces the downward pushout indexed by 0 on the right-hand side which insures that \(f! (s^0_R)\) is a monomorphism. Of course, we have a similar downward pushout produced by the retraction \(d^1_R\); so, this construction produces a vertical reflexive graph on the right-hand side and makes the pair \((f, !f_R)\) a morphism of reflexive graphs.

When \(R = R[f]\), we shall speak of the diagonal punctuation of the map \(f\):

\[
\begin{array}{c}
\xymatrix{ R[f] \ar[r]^{\omega_f} & Dpf \\
R^0 \ar@<1ex>[u] \ar[r] & \theta_f \\
X \ar[r]^f \ar@<1ex>[u] & Y \\
}
\end{array}
\]

and the two retractions on the right-hand side will coincide in a morphism denoted by \(\psi_f\).

The symmetric isomorphism \(\sigma_{\gamma}(uRu) = vRu\) induces an isomorphism \(\gamma_R\) on \(f ![R]\) such that \(\gamma_X !f_R = !f_R \sigma_R \gamma\) and \(\gamma_R !f(d^0_R) = f! (d^1_R)\). This morphism is involutive since such is \(\sigma_R\), so that the pair \((f, !f_R)\) is actually underlying a morphism of involutive reflexive graphs. We get immediately:

**Lemma 4.1.** For any couple \((f, g), (f', g')\) of parallel pairs of morphisms between \(Z\) and \(X\) factorizing through \(R\) we have: \(!f_R.(f, g) = !f_R.(f', g') \iff !f_R.(g, f) = !f_R.(g', f')\).

**Proof.** We get: \(!f_R.(g, f) = !f_R \sigma.(f, g) = \gamma_R !f_R.(f, g) = \gamma_R !f_R.(f', g') = !f_R \sigma.(f', g') = !f_R.(g', f')\). \(\square\)

Let us now complete the previous extension diagram with the horizontal kernel equivalence relations:

\[
\begin{array}{c}
\xymatrix{ R[f] \ar[r]^{\omega_f} & Dpf \\
R^0 \ar@<1ex>[u] \ar[r] & \theta_f \\
X \ar[r]^f \ar@<1ex>[u] & Y \\
}
\end{array}
\]

Then this diagram produces a vertical left-hand side reflexive relation on the object \(R[f]\) we shall denote by \(Df\). In set-theoretical terms, the two following conditions are equivalent:
Lemma 4.2. 1) The reflexive relation $Df$ is symmetric; 2) we have: $(s_0^R)^{-1}(R[f_R]) = R[f]$; 3) we have: $(s_0^R)^{-1}(D[f]) = R$.

Proof. The first point is a straightforward consequence of Lemma 4.1; and the second one of:

$$xRx \iff f(x) = f(y).$$

Finally we have $R[f_R] = f_R(x)$.

Whence the following theorem whose assumptions are satisfied by any congruence modular variety:

Theorem 4.1. Let $E$ be a Gumm category with diagonal extensions. Then:

1) we get: $R[f_R](tRt') \iff R(f_R(x)t) = R(f_R(x)t')$, \forall xRt \in R;
2) the reflexive and symmetric relation $Df$ is necessarily an equivalence relation on the object $R[f]$.

Proof. Thanks to the Yoneda embedding it is enough to check it in $Set$.

Consider the following diagram in $R$:

(1)

where the upper horizontal arrow is obtained from $xRx$ by the equality $f_R(x) = f_R(y)$.

Then the existence of the desired dotted arrow is a consequence of the Cube Lemma, see Proposition 2.1.

We can think of this theorem as a kind of Mal’tsev remainder in the Gumm context: a particular class of reflexive relations produces equivalence relations. Now let us come back to the strict varietal context. In any congruence modular variety $\mathbb{V}$, the following result holds (Corollary 4.4 in [21]):

Proposition 4.1. Let $(R, S, T)$ be any triple of equivalence relations on an algebra $X$ in $\mathbb{V}$. Suppose $R \wedge S \leq T \leq R \vee S$; then $S$ and $T$ permute as soon as $S$ and $R$ permute.

It is again a Mal’tsev remainder: a particular class of pairs of equivalence relations do permute. From that we shall deduce three major consequences:
**Proposition 4.2.** Let \( \mathcal{V} \) be a congruence modular variety. Then for any non-empty algebra \( X \), when \( f \) is a surjective homomorphism and \( R[f] \subset R \), the diagonal extension diagram makes the downward square indexed by 0:

\[
\begin{array}{c}
R \\
\downarrow d_R^0 \\
X \\
\downarrow f \\
Y
\end{array} \rightarrow \begin{array}{c}
f!R \\
\downarrow f!d_R^0 \\
f!X \\
\downarrow f \\
f!Y
\end{array}
\]

a regular pushout.

**Proof.** According to Theorem 7.1, our assertion is equivalent to saying that the pair \( (R[d_0^R], R[f_R]) \) permutes, which is a consequence of the previous proposition: any equivalence relation \( R \) is such that the pair \( (R[d_0^R], R[d_1^R]) \) has a sharp intersection, see Definition 7.1; so that the pair \( (R[d_0^R], R[d_1^R]) \) does permute; and we have \( R[d_0^R] \land R[d_1^R] = \Delta_R \subset R[f_R] \). It remains to check: \( R[f_R] \subset R[d_0^R] \lor R[d_1^R] = (d_1^R)^{-1}(R) \). Now, according to the equivalent conditions asserted above Lemma 4.2, we get \( R[f_R] \subset R[f] \square R \); so that \( f_R(tR'f) = f_R(zRz') \) implies the following situation:

\[
\begin{array}{c}
t \\
\downarrow R[f] \\
z \\
\downarrow R[f] \\
t' \lor z'
\end{array}
\]

and when \( R[f] \subset R \), we get \( t'Rz' \) and thus \( (tR'(R[d_0^R] \lor R[d_1^R])(zRz') \).

\( \square \)

**Theorem 4.2.** Let \( \mathcal{V} \) be a congruence modular variety. Then for any non-empty algebra \( X \), when \( f \) is a surjective homomorphism and \( R[f] \subset R \), the above diagonal extension diagram provides the right-hand side reflexive graph with a unique internal (affine) groupoid structure, which makes the morphism of reflexive graphs \( (f, f_R) \) an internal functor. This functor is actually the cocartesian map with domain \( R \) above \( f \) with respect to the fibration \( (\_)_0 : Grd \rightarrow \mathbb{E} \).

In particular, for any non-empty algebra \( X \) and any surjective homomorphism \( f : X \rightarrow Y \), the diagonal punctuation of \( f \) makes abelian the split epimorphism \( (\psi_f, \theta_f) : Dpf \rightarrow Y \).

**Proof.** Consider now the previous diagram completed, at the upper level, by kernel equivalence relations of the vertical maps indexed by 0. This produces the upper horizontal left-hand side equivalence relation.
The upper horizontal right-hand side map \( R(\!f_R!) \) is the one which makes commute the upper right-hand side squares indexed by 0 and 1. So, by commutation of limits, the upper horizontal diagram produces an effective equivalence relation, namely a map completed with its kernel equivalence relation. Now, according to the previous proposition this same map \( R(\!f_R!) \) is induced by the lower right-hand side regular pushout, and thus it is a regular epimorphism (a surjective homomorphism) in \( V \); namely \( R(\!f_R!) \) is the quotient of its kernel equivalence relation. So the left-hand side squares indexed by 2 induces the vertical right-hand side \( \partial_2 \) which gives a groupoid structure on the right-hand side reflexive graph and makes the pair \((f, \!f_R!)\) an internal functor. This structure is unique according to [13] and affine (or abelian) according to [8]. Let us show that this functor is cocartesian. So, let \((f, f_1) : R \rightarrow Y_1\) be an internal functor:

\[
\begin{array}{ccc}
R & \xrightarrow{\!f_R!} & Y_1 \\
\downarrow^{d_R^0} & & \downarrow^{f_1} \\
X & \xrightarrow{f} & Y
\end{array}
\]

The pushout definition of the diagonal extension produces a dotted morphism \( \phi_1 : R[\!f!(d_R^0)] \rightarrow Y_1 \) such that \( \phi_1 \cdot \!f!(d_R^0)^{s_0^Y} = s_0^X \) and \( \phi_1 \cdot \!f_R = f_1 \). We check that \( (1_Y, \phi_1) \) is underlying a morphism of reflexive graphs by composition with the regular epimorphism \( \!f_R \) and underlying a functor by composition with the regular epimorphism \( R(\!f_R!) \).

In the particular case of the diagonal punctuation, the reflexive graph turns into a mere split epimorphism \((\!s f, 1_X) : Dp f \rightrightarrows Y\) and the groupoid structure on it makes it an abelian split epimorphism in the pointed fiber \( Pt_Y V \).

\[\text{Theorem 4.3.} \quad \text{Let } V \text{ be a congruence modular variety. Given any split epimorphism } (f, s) : X \rightrightarrows Y, \text{ consider the diagonal punctuation of the map } f:\]

\[\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow^{d_0^X} & & \downarrow^{s_0^Y} \\
X & \xrightarrow{\!f!(d_0^X)} & Y
\end{array}
\]

Then the split epimorphism \((\psi_f, \theta_f) : Dp f \rightrightarrows Y\) is the universal abelian split epimorphism associated with \((f, s)\) in the fiber \( Pt_Y V \).

The subcategories \( APrV \subset PtV \) of abelian split epimorphisms in \( V \) and \( AbEquV \subset EquV \) of abelian equivalence relations in \( V \) are closed under monomorphisms.

\[\text{Proof.} \quad \text{Since } (f, s) \text{ is a split epimorphism in } V, \text{ the homomorphism } f \text{ is a regular epimorphism, and we know that the split epimorphism } (\phi_f, \theta_f) \text{ is abelian. The pair } (f, \omega_f) \text{ underlying an internal functor, we get: } (1) \quad \omega_f(a, b) + \omega_f(b, c) = \omega_f(a, c), \text{ and the following identities when } f(a) = f(b):
\]

\[
(1) \quad \omega_f(a, b) + \omega_f(b, c) = \omega_f(a, s f(b)) = \omega_f(a, s f(a)) \\
(2) \quad \omega_f(a, b) = \omega_f(a, s f(a)) - \omega_f(b, s f(b)); \quad (3) \quad \omega_f(s f(b), b) = -\omega_f(b, s f(b))
\]

Let us show now that the homomorphism \( \omega_f(s f, 1_X) : X \rightarrow Dp(f, s) \), associating \( \omega_f(s f(a), a) \) with \( a \), produces the universal comparison with the abelian split epimorphism \((\psi_f, \theta_f)\). First let
us show it surjective. For that, when \( f(x) = f(x') \), apply the permutation diagram of Proposition 4.2:

\[
\begin{array}{ccc}
(x, x')^R[\omega_f] & \rightarrow & (sf(x), z) \\
\downarrow R[d_0] & & \downarrow R[d_0] \\
(x, x) & \rightarrow & (sf(x), sf(x))
\end{array}
\]

Now, given any abelian split epimorphism \((g, t) : A \rightarrow Y\) and any \( V \) homomorphism \( h : X \rightarrow A \) between \((f, s)\) and \((g, t)\), let us consider the following diagram where \( \partial : R[g] \rightarrow A \) is the opsubtraction (namely: \( \partial(a, a') = g(a') - g(a) \)) which gives the abelian structure to the split epimorphism \((g, t)\):

The pushout definition of \( Dp[f] \) produces a unique morphism \( \tilde{h} \), making commute the two adjacent diagrams. The commutativity of the right-hand side triangle necessarily makes \( \tilde{h} \) a group homomorphism in the congruence hyperextensible fiber \( Pt_Y \), while the commutativity of the upper quadrangle means:

\[
\begin{align*}
\omega_f(sf(x), x) &= \partial(hsf(x), h(x)) = \\
&\partial(tgh(x), h(x)) = h(x).
\end{align*}
\]

From that we get \( g.\tilde{h} = \psi_f \) and \( \tilde{h}.t_f = t \) by composition with the surjective homomorphism \( \omega_f \). The uniqueness of this morphism comes from the fact that \( \omega_f(sf, 1_X) \) is a surjective homomorphism as well. This same fact, by Proposition 2.3, insures that \( APt_Y \) is closed under monomorphisms in \( Pt_Y \), and thus, that \( APt \) is closed under monomorphism in \( Pt \). This fact obviously implies that the inclusion \( AbEqu \subset Equ \) is closed under monomorphisms as well.

**Corollary 4.1.** Let \( V \) be a congruence modular variety. Given any non-empty algebra \( X \) and the diagonal punctuation of the indiscrete equivalence relation \( \nabla_X \), the following conditions are equivalent:

1. \( s_X^{\nabla} \) is the “kernel” of \( \omega_X \), namely we have: \( \omega_X(u, v) = 0 \iff u = v \);
2. the diagonal \( s_X^{\nabla} \) is a congruence class of \( R[\omega_X] \);
3. the following downward square is a pullback:

\[
\begin{array}{ccc}
X \times X & \rightarrow & Dp_X \\
\downarrow p_0^{\nabla} & & \downarrow \theta_X \\
X & \rightarrow & 1
\end{array}
\]

4. \( X \) is an affine object in \( V \).

**Proof.** We have 1) \( \Rightarrow \) 2) in any case. The converse comes from Lemma 4.2.2, namely: \( (s_X^{\nabla})^{-1}(R[\omega_X]) = \nabla_X \). Since the above downward square is a regular pushout, it is a pullback if and only if the morphism \( (p_0^{\nabla}, \omega_X) \) is a monomorphism. By Theorem 4.1.1, we then get:
\(\omega_X(t, t) = \omega_X(t, t') \iff \omega_X(x, t) = \omega_X(x, t'), \forall x \in X;\) whence: \(0 = \omega_X(t, t') \iff \omega_X(x, t) = \omega_X(x, t')\). So, we get (1) \(\iff\) (3). Now, when we have (3), the comparison \((p_0^X, \omega_X) : X \times X \to X \times DpX\) is an isomorphism; let us denote \((p_0^X, \chi)\) its inverse. Then the ternary operation \(p(u, v, w) = \chi(u, \omega_X(v, w))\) gives \(X\) a Mal’tsev operation which insures that \(X\) is affine.

Conversely, suppose that \(X\) is affine. Then consider the following diagram, where \(\tilde{A}_X\) is the direction of the affine object \(X\) (see [5] and Proposition 3.3), and consequently where the downward quadrangle with \(\tilde{A}_X\) is a pullback:

\[
\begin{array}{c}
X \times X \\
\downarrow p^X_0 \\
X
\end{array} \quad \begin{array}{c}
\downarrow \subseteq \\
\downarrow \omega_X \\
\downarrow \subseteq
\end{array} \quad \begin{array}{c}
DpX \\
\downarrow q_X \\
\tilde{A}_X
\end{array}
\]

We get a morphism \(\epsilon\) making the two triangles commute: \(\epsilon \omega_X = q_X\) and \(\epsilon \partial_X = 0\). The first equality implies that \(\epsilon\) is a regular epimorphism, since so is \(q_X\). This \(\epsilon\) determines a regular epimorphic morphism \(\bar{\epsilon} : X \times DpX \to X \times X\) such that \(q_X \bar{\epsilon} = \epsilon \circ p_1\) and \(p^X_0 \circ \bar{\epsilon} = p_0\). So, \((p^X_0, \omega_X) \circ \bar{\epsilon} = 1_{X \times X}\) (by composition with the pair \((p^X_0, q_X)\)), and \(\bar{\epsilon}\) is a monomorphism as well. Consequently it is an isomorphism, and so is \((p^X_0, \omega_X)\). Whence (3).

\[\square\]

So, we recover a result of [21], but, now, with a categorical proof:

**Corollary 4.2.** Let \(V\) be a congruence modular variety. A non-empty algebra \(X\) is affine in \(V\) if and only if its diagonal is a congruence class.

**Proof.** Let \(\Sigma\) be a congruence on \(X \times X\) such that \(s^X_0\) is the class of \((0, 0)\). This means that the upward left-hand side square indexed by 0 (and consequently the other one indexed by 1) is a pullback. Now, let \(q : X \to Q\) be the quotient of the equivalence relation \(\Sigma\) and \(0\) be the induced morphism. Then, by the Barr-Kock Theorem, we get a pullback on the right-hand side since we have pullbacks on the left-hand side:

\[
\begin{array}{c}
\Sigma \\
\downarrow d^\Sigma_1 \\
X \times X \\
\downarrow d^\Sigma_0 \\
\downarrow p^X_1 \\
X \times X
\end{array} \quad \begin{array}{c}
\downarrow q \\
1
\end{array} \quad \begin{array}{c}
\downarrow q_X \\
\downarrow \tau_X \\
X
\end{array}
\]

Accordingly there is a unique morphism \(\gamma : DpX \to Q\) such that \(q = \gamma \omega_X\), so that we get \(R[\omega_X] \subset R[q] = \Sigma\). And, since \((s^X_0)^{-1}(R[\omega_X]) = \nabla_X\), the diagonal \(s^X_0\) is a congruence class of \(R[\omega_X]\) as well, which means that \(\omega_X(x, y) = 0\) if and only if \(x = y\) or, in other words, that \(s^X_0\) is the" kernel" of \(\omega_X\).

\[\square\]

5. Axiom

In this section, the aim will be to find a categorical property such that:

(1) it is fulfilled by any congruence modular variety \(V\);
applied to a Gumm category $\mathbb{E}$, it allows to achieve in a categorical way the proof of
Theorems 4.2 and 4.3 concerning the diagonal extensions and punctuations. With this idea
in mind, let us introduce the following:

[Axiom $\xi$]: Given any pullback in $\mathbb{E}$:

$$
\begin{array}{c}
W \xrightarrow{\pi_1^q} Z \\
\pi_0^f \downarrow \quad \quad \downarrow \delta \\
X \xrightarrow{f} Y
\end{array}
$$

and any equivalence relation $T$ on $W$ such that $T \subseteq R[f, \pi_0^f]$, the canonical induced morphism $\xi_{R[\pi_0^f], T} : R[\pi_0^X] \Box T \rightarrow R[\pi_0^X] \times_1 T$ is an extremal epimorphism.

In the regular context, the meaning of Axiom $\xi$ is again a Mal’tsev remainder; namely, some
specific pairs of equivalence relations in $\mathbb{E}$ do permute:

**Proposition 5.1.** Let $\mathbb{E}$ be a regular category. Then $\mathbb{E}$ satisfies Axiom $\xi$, if and only if, following the
notations of $\xi$, the equivalence relations $R[\pi_0^X]$ and $T$ do permute.

**Proof.** When the category $\mathbb{E}$ is regular, extremal epimorphisms do coincide with regular epimor-
phisms. Then Axiom $\xi$ is equivalent to our assertion by Proposition 7.1.

The condition $T \subseteq R[f, \pi_0^f]$ means that we are working in the slice category $\mathbb{E}/Y$. Since we
have a pullback in $\mathbb{E}$, it means a product in $\mathbb{E}/Y$. So, when $\mathbb{E}$ is regular, the axiom $\xi$ is equivalent
to asking that any slice category $\mathbb{E}/Y$ is factor permutable in the sense of [19]. Our first goal is
achieved with the following example:

**Proposition 5.2.** Let $\mathbb{V}$ be any congruence modular variety. Then the category $\mathbb{V}$ satisfies the
Axiom $\xi$.

**Proof.** Consider any pullback in $\mathbb{V}$:

$$
\begin{array}{c}
W \xrightarrow{\pi_1^q} Z \\
\pi_0^f \downarrow \quad \quad \downarrow \delta \\
X \xrightarrow{f} Y
\end{array}
$$

and any equivalence relation $T$ on $W$ satisfying the required inclusion. Now suppose that
$(a, c)T(x, z)$; thus we get $g(c) = f(a) = f(x) = g(z)$; so, $(a, z)$ and $(x, c)$ are in $W$. Now, suppose
that $(a, b)$ is in $W$, so that $(x, b)$ is in $W$. Then consider the following situation depicted by the
plain arrows:

Through the ternary term $t$ on the congruence modular variety $\mathbb{V}$ given by Theorem 4.3 in [21],
we get the dotted arrows from the plain ones, which insures the permutation of the equivalence relations $R[r_0^f]$ and $T$.

Our second goal is achieved with the following:

**Theorem 5.1.** Let $E$ be an exact Gumm category satisfying the Axiom $\mathfrak{A}$. If, in addition, $E$ admits diagonal extensions along regular epimorphisms, then, given any equivalence $T$ on $X$ such that $R[f] \subset T$, the diagonal extension:

\[
\begin{array}{c}
T \\
\downarrow f \\
X \quad \rightarrow \quad Y
\end{array}
\]

produces on the right-hand side a (unique) vertical structure of (affine) internal groupoid.

When $E$ is only a regular category, the diagonal extension of $R[f]$ produces an abelian split epi-morphism $(\psi_f, 0_f) : Dp[f] \Rightarrow Y$. Then the subcategories $\text{AbPt}_Y \subset \text{Pt}_Y E$ and $\text{AbEqu}_E \subset \text{Equ}_E$ are closed under monomorphisms.

**Proof.** Consider the following diagram where $q : X \rightarrow Q$ is the quotient map of the equivalence relation $T$ in $E$:

\[
\begin{array}{c}
R[f] \\
\downarrow \quad R(d_0^f) \\
R[0_f] \\
\downarrow \quad R(0_f) \\
X \quad \rightarrow \quad Q
\end{array}
\]

and $\tilde{q}$ the morphism induced by the inclusion $R[f] \subset T$. The category $E$ being a Gumm category, the left-hand side vertical reflexive relation is an equivalence relation by Theorem 4.1. Since $E$ satisfies the Axiom $\mathfrak{A}$, and since $R[f] \subset R[\tilde{q}, f!(d_0^T)].f[f] = R[\tilde{q}, f.d_0^T] = R[\tilde{q}.d_0^T]$ we can apply Proposition 5.1; so $R[d_0^T]$ and $T$ permute. According to Proposition 7.1, the right-hand side square indexed by $0$ is a regular pushout. We can now follow step by step the end of the proof of the first part of Theorem 4.2. For the affine (or abelian) nature of the groupoid, see [8].

When $T = R[f]$, the assumption “$E$ regular” is enough to achieve the proof of the first part of the second assertion. Now, the second part is checked on the model of the proof of Theorem 4.3.

We can now extend Corollaries 4.1 and 4.2 to our context and get the following characterizations:

**Corollary 5.1.** Let $E$ be a regular Gumm category satisfying the Axiom $\mathfrak{A}$. If, in addition, $E$ admits diagonal punctuations, given any object $X$ with global support, the following conditions are equivalent for any object $X$:

1. $s_0^X$ is the “kernel” of $\omega_X$;
2. the diagonal $s_0^X$ is an equivalence class of $R[\omega_X]$;
the following downward square is a pullback:

\[
\begin{array}{ccc}
X \times X & \xrightarrow{\omega_X} & DpX \\
\downarrow p_0^X & & \downarrow \theta_X \\
X & \xrightarrow{\tau_X} & 1
\end{array}
\]

(4) the object \( X \) is affine in \( E \).

So, an object \( X \) with global support is affine in \( E \) if and only if its diagonal is an equivalence class.

**Proof.** Just follow the categorical proofs of Corollaries 4.1 and 4.2.

A last remark: all the constructions given here are obviously valid in any regular or exact Mal’tsev category \( E \) with diagonal extensions and punctuations; actually we can even get rid of the condition \( R[f] \subseteq R \), see [10].

6. **Algebraic crystallography**

Here, we shall focus our attention on the uniqueness property of an abelian group structure on an object \( X \) in a congruence hyperextensible category \( E \).

We were already confronted to this kind of uniqueness in the unital and strongly unital categories [6] and in the subtractive categories as well [15]. But, in these contexts, it does not seem so surprising because the three cases were closely related (strongly unital = unital + subtractive) and because of the kind of their varietal origins: this uniqueness property arose naturally because, and when, some term of the varietal examples in question became a homomorphism in this variety. This phenomenon being now extended to a much larger context, we cannot keep on accepting this uniqueness so easily.

Among other things, it reveals that, besides the potential (not yet known) interest of a \( Hex_3 \)-algebra in itself, this notion appears to be also interesting by what the whole variety \( Hex_3 \) of these algebras is able to reveal through the environment it creates, which could be thought as a kind of *photographic negative* of the abelian group structure or, perhaps better, as a kind of *pressurized environment* which, under certain conditions, produces them.

So, we propose to call **crystallographic for a given algebraic structure** any categorical setting in which, on any object \( X \) of this setting, there is at most one internal algebraic structure of this kind. Accordingly:

(1) any pointed Jónsson–Tarski variety and any unital category is crystallographic for the structure of commutative monoid;

(2) any strongly unital variety and any pointed subtractive variety in the sense of [24], any strongly unital or subtractive category in the sense of [22] and, now, any congruence hyperextensible category is crystallographic for the structure of abelian group;

(3) any Mal’tsev variety or Mal’tsev category is crystallographic for the structure of commutative and associative (=autonomous) Mal’tsev operations;

(4) we can even widen the range of examples by saying that the setting \( RGhE \) of internal reflexive graphs in a Mal’tsev category \( E \) is crystallographic for the notion of internal groupoid in \( E \) [18].

A first easy remark is that when there is a "duality operator" on the algebraic structure in question, the uniqueness property implies that in a crystallographic context this algebraic structure is necessarily "commutative" as it is the case for the three first examples above.

On the other hand, the category \( CoM \) of commutative monoids is crystallographic for the structure of commutative monoid itself. In such a situation we shall speak of **intensive**
crystallographic context. It is also the case for the categories $Ab$ of abelian groups and $AutMal$ of autonomous Mal’tsev operations.

The above example 2) is also particularly interesting because it shows that some very heterogeneous environments as, i) the subtractive categories, ii) the congruence hyperextensible categories which do not seem to be related to each other, can both work as a crystallographic environment for the same notion of abelian group.

So a first order of questions would be:

- if an algebraic structure has a non-intensive (let us say extensive) crystallographic environment (as it is the case, in 2), for abelian groups), is there a weaker one or an extremal one relatively to some aspect?
- and even more generally: when does a given algebraic structure admit a crystallographic environment?

### 7. Congruence permutation

This section is mainly devoted to recalls and precisions about the relationship between regular pushouts and permutation of pairs $(R, S)$ of equivalence relations in a regular category $E$.

#### 7.1. The square construction

Given a pair $(R, S)$ of equivalence relations on an object $X$ in a category $E$, we denote by $R □ S$, see [11], the inverse image of the equivalence relation $S × S$ on $X × X$ along the inclusion $(d_0^R, d_1^R) : R → X × X$. This defines a double equivalence relation as on the following left-hand side diagram:

In set theoretical terms, $R □ S$ is the set of quadruples $(x, y, t, z) ∈ X^4$ such that $xRy$, $tRz$, $xSt$, and $ySz$, often depicted as on the above right-hand side diagram. A very useful tool is given by the following forgetful morphism:

$$ζ_{R,S} : R □ S → R ×_1 S : (x, y, t, z) → xRySz.$$  \hspace{1cm} (1)

where $R ×_1 S$ is defined by the following pullback:

If $R ∩ S = Δ_X$ (the diagonal equivalence relation on $X$), this morphism $ζ_{R,S}$ is a monomorphism. Indeed, if $(x, y, t, z)$ and $(x, y', t', z)$ are in $R □ S$, then $tRzRt'$ and $tSxSt'$, showing that $t( R ∩ S) t'$ and thus $t = t'$. 

---

D. BOURN
**Definition 7.1.** We shall say that a pair \((R, S)\) of equivalence relations on \(X\) has a *sharp* intersection when \(R \cap S = \Delta_X\) and when, in addition, the monomorphism \(\zeta_{R,S}\) is an isomorphism.

Accordingly, a relation \((f,g) : W \to X \times Z\) is *difunctional* if and only if the pair \((R[f],R[g])\) has a sharp intersection. It is well known that a reflexive relation is an equivalence relation if and only if it is difunctional, see [17] for instance. When \(R\) and \(S\) have a sharp intersection, then any commutative square determined by the above diagram of the double equivalence relation associated with \(R \sqcap S\) is a pullback. Straightforward is then the following observation:

**Lemma 7.1.** Given any pair \((R, S)\) with a sharp intersection in a category \(\mathcal{E}\), the following diagram is underlying an equivalence relation:

\[
\begin{array}{ccc}
R \sqcap S & \xleftarrow{\zeta_{R,S}} & X \\
\downarrow{\zeta_{S,R}} & & \downarrow{d^R_1 \cdot d^S_1} \\
S \times_1 R & \xrightarrow{(d^R_0 \cdot d^S_0, d^R_1 \cdot d^S_1)} & X \times X
\end{array}
\]

which is the supremum \(R \vee S\) of \(R\) and \(S\) among the equivalence relations on \(X\). So, we get: \(R \circ S = R \vee S = S \circ R\) without any regular assumption on the category \(\mathcal{E}\).

Straightforward as well, from the construction in \(\text{Set}\), is the following:

**Lemma 7.2.** Given any pair \((R, S)\) of equivalence relations on \(X\) in a category \(\mathcal{E}\), the following commutative square is a pullback in \(\mathcal{E}\):

\[
\begin{array}{ccc}
R \sqcap S & \xleftarrow{\zeta_{R,S}} & R \times_1 S \\
\downarrow{\zeta_{S,R}} & & \downarrow{d^R_0 \cdot d^R_1, d^S_0 \cdot d^S_1} \\
S \times_1 R & \xrightarrow{(d^R_0 \cdot d^S_0, d^R_1 \cdot d^S_1)} & X \times X
\end{array}
\]

### 7.2. Congruence permutation

In this section, we shall suppose \(\mathcal{E}\) regular, so that, in any case, we can compose the relations in \(\mathcal{E}\). The composition \(S \circ R\) of two equivalence relations on \(X\) is given by the canonical decomposition of the above vertical right-hand side map:

\[
(d^R_0 \cdot d^R_1, d^S_0 \cdot d^S_1) : R \times_1 S \to R \circ S \to X \times X
\]

We have \(S \circ R \subset R \circ S\), when we get the following dotted morphism making commutative the lower right-hand side triangle and, consequently, the vertical square as well:

\[
\begin{array}{ccc}
R \sqcap S & \xleftarrow{\zeta_{R,S}} & R \times_1 S \\
\downarrow{\zeta_{S,R}} & & \downarrow{d^R_0 \cdot d^R_1, d^S_0 \cdot d^S_1} \\
S \circ R & \xrightarrow{\rho} & R \circ S \\
\downarrow{\rho} & & \downarrow{\gamma} \\
S \times_1 R & \xrightarrow{\gamma} & X \times X
\end{array}
\]

**Proposition 7.1.** Let \(\mathcal{E}\) be any regular category. Given any pair \((R, S)\) of equivalence relations on an object \(X\), the following conditions are equivalent:
In this case \( R \circ S \) is an equivalence relation which is the supremum of the pair \( (R, S) \) is the fiber \( \text{Equ}_X \).

**Proof.** When \( R \) and \( S \) are equivalence relations, then it is clear that \( S \circ R \subseteq R \circ S \) if and only if \( R \circ S \subseteq S \circ R \), namely (1) \( \iff \) (2), since \( (S \circ R)^{op} = R^{op} \circ S^{op} \).

Suppose (2). The above whole quadrangle being a pullback by the previous lemma, and the map \( j \) being a monomorphism the vertical square is a pullback as well; whence (3). Suppose (3), the map \( \rho \) being a regular epimorphism, so is \( \zeta_{R,S} \) since the category \( \mathcal{E} \) is regular; whence (4). Suppose (4). Then the map \( \tilde{\rho} \cdot \zeta_{R,S} \) is a regular epimorphism. So, the presence of the monomorphism \( j \) in the commutative quadrangle induces the dotted morphism, since, in any regular category, regular epimorphisms and strong epimorphisms coincide. Whence (2). We get (1) \( \iff \) (5) by exchanging the role of \( R \) and \( S \). The last point is well known. \( \square \)

**Proposition 7.2.** Let \( \mathcal{E} \) be a regular category. Consider any commutative square of regular epimorphisms in \( \mathcal{E} \) as on the right hand side:

\[
\begin{array}{ccc}
R[f] & \xrightarrow{g} & X' \\
\downarrow d_1^g & & \downarrow f' \\
R[f] & \xrightarrow{f} & Y' \\
\end{array}
\]

The following two conditions are equivalent:
1) the right hand side square is a regular pushout;
2) the left hand side square indexed by 0 (resp. by 1) is a regular pushout.

Under any of these conditions, we get:
1) \( R[f] \) and \( R[g] \) permute;
2) the induced morphisms \( R_n(g) : R_n[f] \to R_n[f'] \) are regular epimorphisms.

**Proof.** Introduce the non-labelled quadrangles respectively given by the pull-back \( \tilde{f}_1 \) of the regular epimorphism \( f \) along \( d_1^g \) and the pull-back \( \tilde{f}'_1 \) of the regular epimorphism \( f' \) along \( h \).
So, they are both regular epimorphisms. Let \( \phi_1 \) and \( \phi \) be the natural comparisons induced by these pullbacks. There is also a dotted induced morphism \( d_0^0 \) above \( d_0^1 \) which, by commutation of limits, makes the upward quadrangle a pullback as well. So, \( d_0^0 \) is a pullback of the morphism \( g \). Accordingly, when \( g \) is a regular epimorphism, so is \( d_0^0 \). Then, since we have \( d_0^0.\phi_1 = \phi.d_0^0 \) (*), the map \( \phi \) is a regular epimorphism as soon as so is \( \phi_1 \).

On the other hand, the square satisfying (*) is a pullback as well, and \( \phi_1 \) appears as the pullback of \( \phi \) along \( d_0^0 \). In this way, when \( \phi \) is a regular epimorphism, so is \( \phi_1 \). So now, when \( g \) is a regular epimorphism, the map \( \phi \) is a regular epimorphism if and only if so is \( \phi_1 \).

Suppose the right hand side square is a regular pushout, then \( f_1 \) and \( \phi_1 \) being regular epimorphisms, so is \( R(f) \) and since \( f \) and \( \phi_1 \) are regular epimorphism, the left hand side square indexed by 0 is a regular pushout. Conversely suppose the left hand side square indexed by 0 is a regular pushout; this is the case if and only if so are \( f \), \( R(f) \) and \( \phi_1 \) namely if and only if so are \( f \) and \( \phi_1 \). By the first part of the proof, we know that this implies that \( \phi \) is a regular epimorphism. Accordingly, the right hand side square is a a regular pushout.

Now complete the diagram by the vertical kernel equivalence relations:

So, when the right hand side lower square is a regular pushout, we can apply the same result, but upward, to the left hand side vertical diagram; then the following square of regular epimorphisms in \( \mathcal{E} \) is regular pushout:

which means that the equivalence relations \( R[g] \) and \( R[f] \) do permute by Proposition7.1. The last assertion is straightforward in any regular category.

\[ \text{Theorem 7.1. Let } \mathcal{E} \text{ be a regular category. Consider any commutative square of regular epimorphisms in } \mathcal{E}: \]

\[ X \xrightarrow{g} X' \]
\[ f \downarrow \quad \downarrow s' \]
\[ \underleftarrow{f} \quad \downarrow \text{Y} \xrightarrow{h} \]

The following two conditions are equivalent:
1) \( R[f] \) and \( R[g] \) permute; 2) the commutative square is a regular pushout.
Proof. According to the previous proposition, it remains to show that when $R[f]$ and $R[g]$ permute, this square is a regular pushout. Again let us complete the diagram by the horizontal and vertical kernel equivalence relations as above in the diagram involving $R[g] \Box R[f]$. Saying that $R[f]$ and $R[g]$ permute is saying that the following square is a regular pushout:

$$\begin{array}{c}
\begin{array}{ccc}
R[g] & \xrightarrow{R(d_i^0)} & R[f] \\
R(d_i^0) & \downarrow & \downarrow R(s_0^0) \\
R[g] & \xleftarrow{d_i^1} & X \\
\end{array}
\end{array}$$

The splitting $s$ and $s'$ gives a splitting $R(s)$ to $R(f)$ and makes it a regular epimorphism. We can apply the previous proposition. Accordingly the following left hand side square indexed by 1 is a regular pushout:

$$\begin{array}{c}
\begin{array}{ccc}
R[g] & \xrightarrow{d_i^0} & X \\
R(f) & \downarrow & \downarrow g \\
R[0] & \xleftarrow{d_i^1} & X' \\
\end{array}
\end{array}$$

And, again by the previous proposition the right hand side square is a regular pushout as well. □

**ORCID**

Dominique Bourn [ORCID 0000-0002-6774-1896]

**References**

[1] Barr, M. (1971). Exact categories. In: *Exact Categories and Categories of Sheaves*. Lecture Notes in Mathematics, Vol. 236. Berlin, Heidelberg: Springer, pp. 1–120.
[2] Borceux, F., Bourn, D. (2004). *Mal’cev, Protomodular, Homological and Semi-Abelian Categories*. Mathematics and Its Applications, Vol. 566. Dordrecht: Kluwer.
[3] Bourn, D. (1991). Normalization equivalence, kernel equivalence and affine categories. In: Carboni, A., Pedicchio, M. C., Rosolini, G. eds. *Category Theory*. Lecture Notes in Mathematics, Vol. 1488. Berlin, Heidelberg: Springer, pp. 43–62.
[4] Bourn, D. (1996). Mal’cev categories and fibration of pointed objects. *Appl. Categ. Structures* 4(2–3): 307–327. DOI: 10.1007/BF00122559.
[5] Bourn, D. (1999). Baer sums and fibered aspects of Mal’cev operations. *Cah. Topol. Géom. Différ. Catég.* 40: 297–316.
[6] Bourn, D. (2002). Intrinsic centrality and associated classifying properties. *J. Algebra* 256(1):126–145. DOI: 10.1016/S0021-8693(02)00149-7.
[7] Bourn, D. (2005). Fibration of points and congruence modularity. *Algebra Universalis* 52(4):403–429. DOI: 10.1007/s00012-004-1880-2.
[8] Bourn, D. (2008). Abelian groupoids and non-pointed additive categories. *Theory Appl. Categ.* 20:48–73.
[9] Bourn, D. (2013). On the monad of internal groupoids. *Theory Appl. Categ.* 28:150–165.
[10] Bourn, D. (2021). A remarkable aspect of internal groupoids in regular Mal’tsev categories. *Theory Appl. Categ.* 37:326–336.
[11] Bourn, D., Gran, M. (2002). Centrality and connectors in Mal’tsev categories. *Algebra Universalis* 48(3): 309–343. DOI: 10.1007/s000120200003.
12. Bourn, D., Gran, M. (2004). Regular, protomodular and abelian categories. In: Pedicchio, M. C., Tholen, W., eds. Categorical Foundations. Cambridge: Cambridge University Press.

13. Bourn, D., Gran, M. (2004). Categorical aspects of modularity. In: Janelidze, G., Pareigis, B., Tholen, W., eds. Galois Theory, Hopf Algebras, and Semiabelian Categories. Fields Institute Communications, Vol. 43. Providence, RI: American Mathematical Society, pp. 77–100.

14. Bourn, D., Gran, M. (2004). Normal sections and direct product decompositions. Commun. Algebra 32(10): 3825–3842. DOI: 10.1080/10610020410001727750.

15. Bourn, D., Janelidze, Z. (2009). Subtractive categories and extended subtractions. Appl. Categ. Structures 17(4):317–327. DOI: 10.1007/s10485-008-9182-z.

16. Bourn, D., Janelidze, Z. (2016). A note on the abelianization functor. Commun. Algebra 44(5):2009–2033. DOI: 10.1080/00927872.2014.982808.

17. Carboni, A., Lambek, J., Pedicchio, M. C. (1991). Diagram chasing in Mal’cev categories. J. Pure Appl. Algebra 69(3):271–284. DOI: 10.1016/0022-4049(91)90022-T.

18. Carboni, A., Pedicchio, M. C., Pirovano, N. (1992). Internal graphs and internal groupoids in Mal’cev categories. CMS Conference Proceedings 13:97–109.

19. Gran, M. (2004). Applications of Categorical Galois Theory in Universal Algebra, In: Janelidze, G., Pareigis, B., Tholen, W., eds. Galois Theory, Hopf Algebras, and Semiabelian Categories. Fields Institute Communications, Vol. 43. Providence, RI: American Mathematical Society, pp. 243–280.

20. Gran, M., Rodelo, D., Tchoffo Nguefeu, I. (2019). Variations of the Shifting Lemma and Goursat categories. Algebra Universalis 80(1):2. DOI: 10.1007/s00012-018-0457-9.

21. Gumm, H. P. (1983). Geometrical methods in congruence modular varieties. Mem. Amer. Math. Soc. 45:80.

22. Janelidze, Z. (2005). Subtractive categories. Appl. Categ. Structures 13(4):343–350. DOI: 10.1007/s10485-005-0934-8.

23. Johnstone, P. T. (1989). Affine categories and naturally Mal’cev categories. J. Pure Appl. Algebra 61(3): 251–256. DOI: 10.1016/0022-4049(89)90075-3.

24. Ursini, A. (1994). On subtractive varieties. Algebra Universalis 31(2):204–222. DOI: 10.1007/BF01236518.