From ratios of right triangle to unit circle: an introduction to trigonometric functions

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Abstract. Knowing the two contexts of trigonometry both trigonometry as ratios of right-angle triangle and trigonometry as a function is important to students. Educational research has shown that students could not distinguish between these contexts and understand why trigonometry is a function. Study showed that a unit circle context can be a ‘glue’ between them. However, the transition from the triangle concept to unit circle concept is not well delivered. In this study we propose a lesson design for trigonometric to filling the gap between right triangle context and circular function context. This study is underpinned by the previous research proposed by Altman & Kidron (2017) that describe how is students cognitive process through constructing knowledge. We also report on the results of a classroom case study in which the design has been implemented and researched through a phenomenological investigation based on interviews, classroom observations. We discuss the task-related difficulties that students faced in their concept development. The results of the study suggest that the design can link between trigonometry as ratios of right-angle triangle context and unit circle context. A consideration of technical support in group work division is also discussed.

1. Introduction

Trigonometry is a huge material subject that needs to be learned by high school students. As it links algebra, geometry, and graphical representation, trigonometry can serve as an important precursor to calculus as well as college level courses relating to architecture, surveying, and engineering. Trigonometry presents many first time challenges for high school students, it requires students to relate diagrams of triangles to numerical relationships and manipulate the symbols involved in such relationships [1]. Trigonometry can be viewed as ratios of right-angle triangle as well as a function. Both of these contexts have different approaches and origins. Trigonometry as a ratios of right-angle triangle is defined as Sine is Opposite over Hypotenuse, Cosine is Adjacent over Hypotenuse, and Tangent is Opposite over Adjacent, as well known for teacher and students as SOH-CAH-TOA. While, trigonometry as a function means there is a correspondence between two sets (a domain and a range, in real numbers) such that every element in the domain has exactly one element in the range that corresponds to it.

However, the students can not distinguish between these contexts. Numerous studies were conducted regarding the students’s error, misconception and understanding about trigonometric function (read: [2]–[6]). Students did well in trigonometric ratios but lack on understanding and see
trigonometry as two contexts, especially trigonometry as a function. Teacher can easily deliver the trigonometry definition in right-angle triangle and so do the students [7]. The problem resides in the fact that many approaches to teaching trigonometry, such as ‘the right-angle triangle’ approach, primarily stress procedural skills and such approaches do not allow students to understand sine and cosine as a function [8]. Meanwhile, learning trigonometry as a function need in depth analysis. Our pilot study showed that the student unable to distinguish the contexts of trigonometry, for example, when they asked to find the solution for \( x, \sin x = \frac{1}{2} \) for \( 0^\circ \leq x \leq 360^\circ \), the students answered that \( x \) must be \( 30^\circ \). What was answered by the students was not totally wrong, it is true that \( 30^\circ \) is the answer but, it is not the only answer. When they were asked to explain their answer, the students said that they knew the answer from the table i.e. the table of sine, cosine and tangent values of special angle. Thus, it can be known that the students’ knowledge was limited to the ratio of right-angle triangle trigonometry. Furthermore, when the students were asked to draw the graphical representation of sine, \( y = \sin x \), they put the ‘degree’ number in x-axis instead of real number. Thus, when write the coordinate of arbitrary point in \( y = \sin x \), they wrote it as ‘(degree, real number), (e.g. \( 30^\circ, \frac{1}{2} \))’.

Hence, the students do not understand that the trigonometry is a function from \( \mathbb{R} \) to \( \mathbb{R} \).

As we analysed both of a learning trajectory of mathematics teacher and a textbook for highschool mathematics, we found that the transition from the trigonometry as a right-angle triangle ratios to trigonometry as a function is not stressed and not explained. Although the textbook provide the definition of trigonometry as a ratios of right-angle triangle and provide a unit circle approach. It is not clearly explained why trigonometry from right-angle triangle contexts can be defined in a circle especially in unit circle. Although unit circle can be a ‘glue’ between the triangle contexts in trigonometry and trigonometric function [9], however, the students need to get a clear understanding and make sense explanation from what they have learned to what they will learn. Hence, what follows in this study is a description of a didactical design of trigonometry that can be implemented which focus on from right-angle triangle trigonometry to unit circle as a bridge to a deep concept of trigonometric function. The research was conducted to find answers to two descriptive research questions: (1) What lesson design instruction proposed can teacher implement in linking the trigonometry as ratios of right-angle triangle and unit circle (2) What difficulties do students face in the learning process based on the lesson design instruction implemented

2. Lesson design instruction

Altman and Kidron [8] developed ideas in how to constructing knowledge about trigonometric functions and their geometric meaning on the unit circle through analysis of cognitive processes. Our motivation was somewhat similar to Altman and Kidron motivation: to obtain a picture of how students make sense of trigonometry from triangle to circle. In our research, we chose high school students, unlike Altman and Kidron who used an adult learner who many years past studying trigonometry in high school. In this research we also developed idea from an arbitrary radius of circle to unit circle, which is different from what Altman and Kidron did which they were directly focus on unit circle from the very beginning.

The basic idea of this didactical design is building a logical lesson instruction while introducing the unit circle in trigonometry. Since the students have known the concept of trigonometry as a ratio of the right-angle triangle, we approached the unit circle concept through the right-angle triangle. We presume that, when the transition from the right-angle triangle to the unit circle was introduced then it is easy for students to understand trigonometry as a function. It is important to be noted that in this didactical design the students should have completed their understanding in the right-angle triangle trigonometry contexts. After they have mastered this contexts, then the important part is give them some challenging questions (e.g. In right-angle triangle, what is the value of \( \cos 90^\circ \) what is the value \( \sin 200^\circ \)). The purpose of this question is to build awareness to the students that the value of either sine and cosine for an arbitrary angle especially for angles greater than \( 90^\circ \) can not be covered by using the definition of trigonometry in right-angle triangle. Thus, from this stage, the students need a
further exploration to find out the value of sine and cosine for any angles. This didactical design consists of two parts, hands-on activity task and analysis task. To get hands-on experience with the process of connecting a right-angle triangle of trigonometry contexts onto unit circle, students first draw some of the right-angle triangles in the same hypotenuse on a paper. Unlike what Altman and Kidron which used 1 cm length of hypotenuse, we used 4 cm and 5 cm length of hypotenuse. We considered that these lengths were ideal for group task so that it can reduce the technical obstacle might found because of the small size shape. Then the students put the angle 'symbol' (e.g. $\alpha$, $\beta$, or $\theta$) onto one of the angles in the triangle and cut the triangles. Next, they arrange the right-angle triangle onto cartesian plane in a particular direction (see figure 1).

In the analysis task, we expected the learner to recognize a relation between the lengths of the sides of the right-angle triangle and the coordinate of the intersect point between terminal side of an angle and the circle. In this stage students were expected to get a new definition of trigonometry based on coordinate system. Thus, in the next stage, students can measure the sine and cosine value for any of angles either positive or negative, either greater than 90° or smaller than 90°. By measuring the ratios of the distance of a point that correspond to the angle from the $x$-axis and radius of the circle, the students will get sine value, meanwhile, measuring the ratios of the distance of a point that correspond to the angle from the $y$-axis and radius of the circle, the students will get cosine value.

Until this point, the link between trigonometry as ratios of right-angle triangle and a new definition of trigonometry has been built. The next stage is to analyse what if the students do a measurement in different length of radius. Since the students get the different task of radius in each group work, then the next stage is to compare the value of either sine or cosine for the same angle measurement but different length of radius. From this activity, the students are expected to figure out that there is no difference in the value of either sine and cosine in different length of radius. Hence, the teacher can formalized the used of unit circle as an easiest representation to measure the value of sine and cosine.

In the figure 1, we provide a general overview of this didactical design. From the definition of trigonometry as ratios of right-angle triangle, the students construct numerous of right-angle triangles become a circle which lead them to the new definition of trigonometry and introduce them to the unit circle.

**Figure 1. A General Overview of the Design.**

3. **Methods**

The classroom study was conducted at a secondary school in Indonesia with a class of 33 of 10th grade students. In this grade, the students for the first time learn about trigonometry. A qualitative of phenomenographic approach was used in this study. The class was timetabled as double periods i.e. 90 minutes long, two times a week. In the learning process, students were asked to work in groups of four or five to complete in-class activities. The lessons were recorded on a small digital device and then analysed either later that day or the following day. This was necessary, by analysing as soon after the lesson as possible, we were able to supplement from memory and classroom notes any partially
inaudible comments or enquiries or comments and which student was making them. After the initial exposition, the students were usually set problems to work on from the modules and during this time we observed and recorded them as they attempted the problems.

4. Result and discussion

Concerning the first research question, before the implementation of the lesson design, we were intended to ask the students how to obtain sine and cosine value for angle $0^\circ$ or angle greater than $90^\circ$. However, before we asked them this matter, one of the students had asked first this matter. This was a good start for them since asking questions is a critical step to advance one’s learning so that they learn to think like mathematicians who often advance knowledge by asking new questions and trying to solve them. It has the potential to promote active learning of mathematics among school students through strengthening their metacognitive awareness and control [10]. From the learning processes, the students understood that there are limitations of trigonometry in the triangle contexts. Hence, they are ready to find out other definition which can cover any angles.

Every group of students was asked to draw several different right-angle triangles in different lengths of hypotenuse (e.g. 4 cm and 5 cm length of hypotenuse). Then they were asked to cut the triangles then arranged them into cartesian plane such that the vertices of the $\alpha$ angles will touch each other and the sides adjacent to the right angle will look as vertical or horizontal segments only. Most of students had difficulty in arranging the triangles. We did not find difficulties for triangles that fit in the first quadrant. The situation is different for the other quadrants with triangles oriented in different directions. So we gave an example in front of the class how to execute a procedure to accomplish the task. We took into account this situation since the quality of instructional interactions is related to student mathematics achievement [11]. As what we expected, after the students finished in arranging the triangles, they could easily find out that the shape formed a circle. In this stage, the aim to connect the triangle to circle has accomplished.

![Figure 2. Students’ activity while arranging the right-angle triangles and their results.](image)

The next stage is finding the relation between the lengths of the sides adjacent to the right-angle and the angle $\alpha$. In this step, we linked what they have known in trigonometry as ratios and the fact they have just known i.e. an angle measurement in a circle. As the circle was placed in cartesian plane, and there is an intersection between terminal side of angle $\alpha$ and circle, it can be known the coordinate of the intersection point. From this point the students figured out the new definition of trigonometry i.e.

$$\sin \alpha = \frac{y}{r} \quad \text{and} \quad \cos \alpha = \frac{x}{r}$$

We investigated students’ understanding about the new definition of trigonometry by giving them several tasks to find out the value of sine and cosine of some particular angles. From this task, we tried to make them aware that the value of sine and cosine can vary and is not limited to the special angle only, also there is negative value in a particular angle measurement. Hence, the students also analysed in which angles the sine and cosine is negative or positive. Furthermore they can conclude in which quadrant the sine or cosine has positive or negative values. Through this approach, the students also knew the sine and cosine value for angle $90^\circ$ or greater than it.
After the students had mastered to measure the sine and cosine for any angles, the next stage is to introduce them the notion of unit circle. Since every group had their own radius of circle i.e. 4 cm and 5 cm length of radius, they asked to measure same angle and compare the value with other group. The students noticed that there was no difference in value even though using the different length of radius. This is the key point to get them reflect the easiest radius might be used in measuring the sine and cosine value. One of the students said that it is better to use 1 cm of radius since we no need to divide either \( x \) or \( y \) by \( r \). It was clear that the students, from the sequence of instruction they could analyse and reflect what they have just learned. This finalizes the linking of the trigonometry as ratios and new definition of trigonometry in unit circle.

Concerning the second research question, there was no difficulty found in understanding the new definition, however we found that they did not master in determine the coordinate of a point. We were not anticipated this difficulty since the coordinate system material was thought in elementary and junior high school. Thus, it was quite shocking if they were still trying to figure this out. The difficulty that we captured in finding the coordinate of a point were such as ignoring the negative sign when a point was in quadrant II, III, or IV, reversing between \( x \) and \( y \) (e.g. the right coordinate is (0,2) but they write it as (2,0)), and using the radius of a circle in determining the coordinates.

Furthermore, the students seem unable to work together in groups. We presumed that this happens because of two things: 1) students are not accustomed to working in groups or 2) the division of group members that are not in accordance with their wishes. According to Hansen [11], how determine the group formation is one of the factors in the success of group work. There are basically two alternative ways of group division in the learning process that can be done by a teacher, i.e. teacher’s setting or student’s setting. We divided the group by spreading the high-ability students to every group, so that the students can learn from each other, as described by Hernandez [12]. However, after observed the learning process, high-ability students had a tendency to do the tasks by themselves rather than working in groups. Under these conditions Hernandez [12], Page and Donelan [13] suggested dividing tasks for each group member, so that each group member has his/her own responsibilities. Thus, the technique of group and tasks for each group member division became our consideration in the next research.

In general, the students developed a good level of understanding of aspects from the trigonometry as ratios and new definition in the unit circle context from the activities given. They were able to evaluate sine and cosine values for any angles by measuring the coordinate point which correspond to the angle. Although many students understood the concept to find out the sine and cosine value in, many students were not able not able to find the coordinate of a point.

5. Conclusion
We proposed the didactical design to link between trigonometry as ratios and unit circle in trigonometry which is important for students to get the link to trigonometric function. We used hands-on activity in learning process to get them involved in building the link between the two contexts, then in the rest of learning process was reflect and analyse on what the students had done. From the analysis step, the students were led to understand the new definition of trigonometry in other contexts i.e. unit circle. We examined our new approach in a classroom case study. It provided evidence of the effectiveness in promoting: (1) integrated understanding of new trigonometric definition by finding the coordinate point in a circle in such way that students do not have as many difficulties and misconceptions as reported before in the research literature on trigonometry; and (2) connected understanding of trigonometric as ratios as well as unit circle contexts. We found errors in students’s answer relate to coordinate system. It was not unclear whether the students made careless error or it is a misconception they might had. A depth research need to be conducted to find out this situation.

Reflecting on the didactical design either before revision and after revision, the most crucial thing in design is the matter of time consumption regarding the technical thing. Therefore, in the revision of design, several steps need to be revised to prevent a lot of time consumption, they are: 1) the researchers make sure that either every student or every group have adequate number of tools needed.
along the learning process. 2) make clear instruction with detail example, 3) make sure that the students have proper prior knowledge (e.g. coordinate system). Regarding the students’ group work evaluation, the researchers suggest to any educators or teacher that delivering a particular task to every group member can improve the student’s performance in group work. These consideration were made in order to the students can easily cooperate with each other.

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