Abstract. It is well known that in the case of three or more particles the Lipmann-Schwinger (LS) integral equation is not of Fredholm type. This is the consequence of the existence of bounded subsystems. By using identical transformations of the kernel of LS equation for certain classes of potentials Faddeev obtained Fredholm type integral equations for the three-body problem [1]. The motion of for bodies is described by equations of Yakubovsky and Alt-Grassberger-Sandhas-Khelashvili [2,3], which are obtained as a result of two subsequent transformations of the kernel of LS equation. It has been shown that for $N > 4$, on the $z$ complex plane wherever $\text{Im}z \neq 0$, after $N-2$ iterations the kernel of $N$-particle integral equations of the collision theory became compact. For $\text{Im}z = 0$ the compactness has not been proven yet [4]. In turned out that, in the case of $N \geq 3$, the mentioned equations have correct asymptotic form the derivation and detailed analysis of which is presented this paper.

Keywords and phrases: Collision theory, asymptotic integral equations, Unitary asymptotic solution.

Few-body equations, general formalism

Let us consider a system of $N$ particles in non-relativistic approximation. Interaction operator $V$ in the approximation of two-particle forces is given as:

$$V = \sum_{\alpha} v_{\alpha}, \quad \alpha \equiv (mn); \quad m, n = 1, 2, \ldots, N; \quad m < n, N \geq 3.$$  \hfill (1)

Here $v_{\alpha}$ are potentials of pair-interactions.

$N$-particle scattering $T$-matrix is determined by the LS equation:

$$T(z) = V + V G_0(z) T(z).$$  \hfill (2)

Here $G_0$ is the Green’s function of $N$ free particles:

$$G_0(z) = (z - H_0)^{-1}, \quad z = E_0 + i \varepsilon,$$  \hfill (3)

where $H_0$ is the operator of kinetic energy and $E_0$ is the energy of the system’s free motion. $G_0$ can be presented as a sum of anti-Hermitian and Hermitian parts:

$$G_0(z) = G_1(z) + G_2(z), \quad G_1(z) = \frac{1}{2} (G_0(z) - G_0(\bar{z})), \quad G_2(z) = \frac{1}{2} (G_0(z) + G_0(\bar{z})).$$  \hfill (4)

Here $\bar{z}$ stands for the complex conjugate of the $z$ parameter.

To solve equation (2) formally one applies the standard Faddeev approach and obtains:

$$T(z) = \sum_{\alpha} T_{\alpha}(z),$$  \hfill (6)

where auxillary operators $T_{\alpha}$ are define by the following system of equations:

$$T_{\alpha}(z) = T_{\alpha}(z) + T_{\alpha}(z) G_0(z) \sum_{\beta \neq \alpha} T_{\beta}(z),$$  \hfill (7)

and for two-particle operators $T_{\alpha}$ one can write:

$$T_{\alpha}(z) = v_{\alpha} + v_{\alpha} G_0(z) T_{\alpha}(z).$$  \hfill (8)
Equation (2) can be also solved formally by applying the Heitler formalism of separation [6] resulting in:

\[ T(z) = K(z) + K(z)G_1(z)T(z), \]  
\[ K(z) = V + VG_2(z)K(z). \]  

(9)  
(10)

From the Heitler equation (9) \( T \) can be expressed in terms of a Hermitian operator \( K \) which in turn is determined by Eq. (10). If Eqs. (9) and (10) are correct equations (the two-particle case) then the \( T \) operator constructed this way satisfies the corresponding condition of unitarity [6] for any approximation of \( K \).

Similarly to Eq. (2) let us look for the solution of Eq. (10) in the form of a sum:

\[ K(z) = \sum_{\alpha} K^\alpha(z), \]  

(11)

where auxiliary operators \( K^\alpha \) are given by equation:

\[ K^\alpha(z) = K_\alpha(z) + K_\alpha(z)G_2(z)\sum_{\beta \neq \alpha} K^\beta(z), \]  

(12)

and for two particle operators \( K_\alpha \) we have:

\[ K_\alpha(z) = v_\alpha + u_\alpha G_2(z)K_\alpha(z). \]  

(13)

It is well known that Eqs. (7) and (12) are not compact in general.

Below we show that by using the Heitler formalism it is possible to find the correct asymptotic (high energy) form of the system of multi-particle integral equations (7).

**Multi-particle equations in asymptotics**

Using the behaviour of Eq. (12) at sufficiently high energies: \( \text{Re}(z) \gg E_{\text{min}}^B \) (\( E_{\text{min}}^B \) denotes the absolute magnitude of the minimum of all allowed binding energies in the system to be considered), let us restrict Eq. (9) to its high-energy approximation and using the obtained result let us rewrite Eq. (7) in asymptotics.

It is well-known that the square of the of Eq. (12) can be expressed as a sum of terms of the following form:

\[ K_\alpha(z)G_2(z)K_\beta(z)G_2(z), \quad \alpha \neq \beta, \]  

(14)

which in the approximation of Eq. (36) can be neglected. This is why the kernel of Eq. (12) at sufficiently high energies is square integrable so that the corresponding Neuman series converges. To obtain the desired system of equations let us keep the first order asymptotic terms in Eq. (12) (\( K \)-matrix impulse approximation, see [7]):

\[ K^\alpha(z) = K_\alpha(z), \quad K(z) = \sum_{\alpha} K_\alpha(z). \]  

(15)

We substitute the \( K \) operator of Eqs. (15) into Eq. (9) and obtain:

\[ T(z) = \sum_{\alpha} K_\alpha(z) + \sum_{\alpha} K_\alpha(z)G_1(z)T(z). \]  

(16)

Equation (16) represents the Heitler equation in the \( K \)-matrix impulse approximation. It is defined in the kinematic area: \( \text{Re}(z) \gg E_{\text{min}}^B \). Let us demonstrate that by using the standard Faddeev approach Eq. (16) reduces to the desired system of integral equations. For this purpose we decompose \( T \) in the form of Eq. (6) where, according to Eq. (16)

\[ T^\alpha(z) = K_\alpha(z) + K_\alpha(z)G_1(z)T(z) \]  

(17)
and insert Eq. (6) into Eq. (17). Inverting the diagonal elements in the resulting expression we obtain:

\[(1 - K_\alpha (z) G_1 (z))^{-1} K_\alpha (z) + (1 - K_\alpha (z) G_1 (z))^{-1} K_\alpha (z) G_1 (z) \sum_{\beta \neq \alpha} T^\beta (z). \quad (18)\]

Using two-particle equations:

\[T_\alpha (z) = K_\alpha (z) + K_\alpha (z) G_1 (z) T_\alpha (z), \quad (19)\]

one can write:

\[K_\alpha (z) = (1 + T_\alpha (z) G_1 (z))^{-1} T_\alpha (z). \quad (20)\]

Inserting Eq. (20) into Eq. (18) we obtain:

\[T^\alpha (z) = T_\alpha (z) + T_\alpha (z) G_1 (z) \sum_{\beta \neq \alpha} T^\beta (z). \quad (21)\]

Eq. (21) represents asymptotic \((\text{Re} z \gg E^B_{\min})\) form of the system (7) of \(N\)-particle integral equations, which indicates that at high energies in the Green’s function \(G_0\) entering in Eq. (7) its Hermitian part can be neglected (for the two-particle case see Ref. [8]). Equation (21) is a system of equations for auxiliary operators \(T^\alpha\) in the \(K\)-matrix impulse approximation. According to the estimation (see appendix, Eq. (38))

\[||T_\alpha (z) G_1 (z) T_\beta (z) G_1 (z)|| = O (z^2) \approx 0, \quad \alpha \neq \beta, \quad (22)\]

the kernel of Eq. (21) is square integrable and therefore it is the Fredholm type system of integral equations, iterative series of which with the accuracy of Eq. (22) converges to the following finite sum:

\[T^\alpha (z) = T_\alpha (z) + T_\alpha (z) G_1 (z) \sum_{\beta \neq \alpha} T_\beta (z), \quad \text{Re} (z) \gg E^B_{\min}. \quad (23)\]

It can be shown that the specified accuracy of convergence guarantees the unitarity of \(T\)-matrix [9].

Equation (21) describes scattering on weakly bound system \((\text{Re} (z) \gg E^B_{\min})\). According to the approximation of Eq. (15), equation (21) corresponds to the processes in which single collision dominate. The expression of the kernel of the obtained asymptotic equations with the Green’s function \(G_1\) indicates that these equations are adequate for elastic and quasi-elastic scattering reactions (the internal energy of the system does not change).

Let us insert Eq. (23) in Eq. (6):

\[T (z) = \sum_{\alpha} T_\alpha (z) \left(1 + G_1 (z) \sum_{\beta \neq \alpha} T_\beta (z)\right), \quad \text{Re} (z) \gg E^B_{\min}, \quad (24)\]

\[\alpha, \beta \equiv (mn), \quad m, n = 1, 2, \ldots, N; \quad m < n, N \geq 3.\]

Equation (24) is a particular unitary solution of Eq. (16), which at the same time represents an asymptotic solution of Eqs. (2) and (9).

Thus, by using the Heitler formalism in multi-particle problem we got multi-particle asymptotic equations. In particular, keeping in the right-hand side of Eq. (9) the high-energy terms we obtained asymptotic equation (16). Based on this equation we constructed the Faddeev type compact the system of \(N\)-particle integral equations in asymptotic, Eq. (21). Using the convergence of the iterative series corresponding to Eq. (21) we obtained Eq. (24), the unitary asymptotic solution of multi-particle Eqs. (2) and (9), which is identical to the result proposed by us earlier [9].

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Appendix

Below we establish the high-energy behavior of operators determined by Eqs. (8) and (13). The obtained result is used for the estimation of the kernel of multi-particle equations.

Let us show that

\[ T_\alpha (z) \approx v_\alpha, \quad \text{Re} (z) > \text{Re} (\bar{z}). \]  \hspace{1cm} (25)

Here \( z = E_0 + i\varepsilon \). \( E_0 \) is a kinetic energy of the considered system. \( \bar{z} = \bar{E} + i\varepsilon \). \( \bar{E} \) is the fixed energy, which satisfies the condition: \( \bar{E} \gg E_{\text{min}}^B \). \( E_{\text{min}}^B \) denotes the absolute value of the minimum of all allowed binding energies in the considered system.

Using the symbolic notation \( \alpha \equiv (mn) \), we write the connection among \( t_\alpha \) and \( T_\alpha \) operators:

\[ \langle q'_1 q'_2 \cdots q'_N | T_\alpha (z) | q_1 q_2 \cdots q_N \rangle = \prod_{l \neq m \neq n \neq l} \delta (q'_l - q_l) \langle q'_m q'_n | t_\alpha (z') | q_m q_n \rangle, \]  \hspace{1cm} (26)

where the pure two particle interaction operator \( t_\alpha \) satisfies the LS integral equation:

\[ t_\alpha (z') = v_\alpha + v_\alpha g_0 (z') t_\alpha (z'). \]  \hspace{1cm} (27)

Here \( g_0 \) is the Green’s function of two free particles.

\[ z' = E' + i\varepsilon, \quad E' = E_0 - \sum_{n \neq m} \frac{q_l}{2m_l}. \]  \hspace{1cm} (28)

\( E_0 \) and \( q_l \) are fixed parameters. According to the Klein-Zemach theorem [10] we have:

\[ t_\alpha (z') \approx v_\alpha, \quad \text{Re} (z') > \text{Re} (\bar{z}). \]  \hspace{1cm} (29)

Inserting the relation (29) into Eq. (26) we get Eq. (25).

Let us express \( T_\alpha \) from Eq. (8) and substitute the obtained result in Eq. (25):

\[ T_\alpha (z) = (1 - v_\alpha G_0 (z))^{-1} v_\alpha \approx v_\alpha, \quad \text{Re} (z) > \text{Re} (\bar{z}). \]  \hspace{1cm} (30)

Due to Eqs. (25) and (30) we obtained:

\[ ||T_\alpha (z) G_0 (z)|| \approx ||v_\alpha G_0 (z)|| \approx \varepsilon, \quad \text{Re} (z) > \text{Re} (\bar{z}). \]  \hspace{1cm} (31)

Similarly to Eq. (26) we write:

\[ \langle q'_1 q'_2 \cdots q'_N | K_\alpha (z) | q_1 q_2 \cdots q_N \rangle = \prod_{l \neq m \neq n \neq l} \delta (q'_l - q_l) \langle q'_m q'_n | k_\alpha (z') | q_m q_n \rangle. \]  \hspace{1cm} (32)

Here \( k_\alpha \) is a two particle Hermitian operator:

\[ k_\alpha (z') = v_\alpha + v_\alpha g_2 (z') k_\alpha (z'), \]  \hspace{1cm} (33)

where \( g_2 \) can be expressed as follows:

\[ g_2 (z') = \frac{1}{2} (g_0 (z') + g_0 (\bar{z}')). \]  \hspace{1cm} (34)

The parameter \( z' \) in Eq. (34) is defined from equalities (28).

Due to the mentioned Klein-Zemach theorem, one can show similarly to Eq. (25):

\[ K_\alpha (z) \approx v_\alpha, \quad \text{Re} (z) > \text{Re} (\bar{z}). \]  \hspace{1cm} (35)

According to Eqs. (13) and (35) we have:

\[ ||K_\alpha (z) G_2 (z)|| \approx ||v_\alpha G_2 (z)|| \approx \varepsilon, \quad \text{Re} (z) > \text{Re} (\bar{z}). \]  \hspace{1cm} (36)

From Eq. (19) in the approximation of Eq. (35) we obtain [8]:

\[ T_\alpha (z) = v_\alpha + v_\alpha G_1 (z) T_\alpha (z). \]  \hspace{1cm} (37)

Using Eqs. (37) and (25) one can write:

\[ ||T_\alpha (z) G_1 (z)|| \approx ||v_\alpha G_1 (z)|| \approx \varepsilon, \quad \text{Re} (z) > \text{Re} (\bar{z}). \]  \hspace{1cm} (38)

Thus, it is shown that in the asymptotic \( \text{Re} (z) \gg E_{\text{min}}^B \), the kernels of multi-particle equations are sufficiently small, so that the corresponding iterative series converge.
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