Cramer’s estimate for the exponential functional of a Levy process

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1 Introduction

The aim of this paper is to study the asymptotic behavior of the exponential functional

\[ A_\infty = \int_0^\infty e^{\xi s} \, ds, \]

where \((\xi_t)_{t \geq 0}\) is a Lévy process such that Cramér’s condition holds, that is to say there exists \(\chi > 0\) such that \(\mathbb{E}(e^{\chi \xi_1}) = 1\). The precise result will be, under others conditions on \((\xi_t)_{t \geq 0}\), that the tail of \(A_\infty\) is decreasing like \(t^{-\chi}\) when \(t \to \infty\).

One method to understand this result is to start from the analogous problem in discrete time: the random difference equation

\[ Y_n = M_n Y_{n-1} + 1 \]

where \((M_n)_{n \in \mathbb{N}}\) is a sequence of i.i.d real variables, and whose solution \(Y_n\), under certain additional hypothesis (see [6] and [5]), converges in distribution to \(R := \sum_{k=0}^\infty M_1 \ldots M_{k-1}\). In our case, the Lévy process \((\xi_t)_{t \geq 0}\) will play the role of the random walk \((S_n := \sum_{k=1}^n \log |M_k|)_{n \geq 1}\) and \(A_\infty\) the role of the limit variable \(R\).

Let us explain a little the analogy. In the discrete case, for all stopping-time \(N\) which is finite almost-surely we have the following identity in law (see [6], lemma 1.2):

\[ R \overset{d}{=} M_1 \ldots M_N R + R_N, \]

where \((R_n)_{n \in \mathbb{N}}\) stands for the sequence of partial sums \((\sum_{k=0}^n M_1 \ldots M_{k-1})_{n \in \mathbb{N}}\). In continuous-time this identity is still valid if we replace \((R_n)_{n \in \mathbb{N}}\) by the process \((A_t := \int_0^t e^{\xi s} \, ds)_{t \geq 0}\); indeed the lemma 6.2 in [6] implies that for all stopping-time \(T\) which is finite almost-surely we have:

\[ A_\infty \overset{d}{=} e^{\xi_T} A_\infty + A_T. \]

Kesten, in the above quoted article, found the asymptotic behaviour of the distribution of \(R\):

\[ \mathbb{P}(R > t) \sim C t^{-\chi} \quad \text{for some constant } C > 0, \]
and he noticed that the proof (in the one-dimensional-case, which interests us here) relies essentially on Cramér’s estimate for the random walk $S_n$: if $\chi > 0$ satisfies the Cramér’s condition $E(M_1^\chi) = 1$, then :
\[
P(\max(S_0, S_1, \ldots) > t) \sim C e^{-\chi t},
\] (3)
for some constant $C$ when $t \to +\infty$.
But Bertoin and Doney (see [1]) have proved that Cramér’s estimate extends to Lévy processes. Their proof relies on the introduction of what they call the associated Lévy process $X^*$, whose exponent is $\Phi^*(\lambda) = \Phi(\lambda + \chi)$. It amounts to make the change of probability defined by the martingale $(e^{\chi t})_{t \geq 0}$, as we do in our proof of Proposition 4.3. They use a Wald identity for this process, namely :
\[
E(H^*_1) = E(X^*_1)E(\tau^*_1),
\]
where $\tau^*$ is the inverse process of the local time $L^*$ of the reflected process at the supremum $(S^*_t - \xi^*_t = \sup_{s \leq t} \xi^*_s - \xi^*_t, t \geq 0)$ and $H$ the ascending ladder process : $H^*_t = S^*_\tau_t = \xi^*_\tau_t$. For sake of completeness, we give a proof of this Wald identity in the Annex.

In fact, instead of the equivalence mentioned in [3], we can easily obtain an upper bound, and this suffices to show that $A_\infty$ has moments of all orders $\alpha < \chi$ where $\chi$ is again the non-negative root of $E(e^{\chi t_1}) = 1$. We will give a proof of this result (in the third section), which has an interest by itself and will be used to obtain the more precise result concerning the tail of $A_\infty$, that we give now :

**Theorem 1.1.** Let $\xi$ be a Lévy process, with Lévy exponent $\Phi$ (i.e $E(e^{\lambda \xi_t}) = e^{-t\Phi(\lambda)}$) and fulfilling the following Cramér’s condition :

\[
\exists \chi > 0 \text{ such that } \Phi(\chi) = 0
\] (4)

Notice that this can only happen if $-\infty \leq \mu := E(\xi_1) < 0$.

We make besides the stronger hypothesis that $\Phi > -\infty$ on an interval $[0, \chi + \epsilon]$, with $\epsilon > 0$.

At last we assume that the law of $\xi_1$ is not arithmetic.

Then the exponential functional $A_\infty := \int_0^\infty e^{\xi_t} ds$ is well defined and there exists some constant $C > 0$ such that when $t \to +\infty$ :

\[
P(A_\infty > t) \sim C t^{-\chi}
\] (5)

The proof we give in the third part of the article is greatly inspired by Goldie ([4]) who found a simpler proof of Kesten’s result, that extends to the continuous case as we will show here.
2 Examples

2.1 Brownian motion with drift

If \( \xi_t = \sigma B_t + \nu t \) is a brownian motion with negative drift (\( \nu < 0 \)), then \( \Phi(\lambda) = \lambda (\frac{\sigma^2}{2} \lambda + \nu) \), so it satisfies the Cramér’s condition with:

\[ \chi = \frac{-2\nu}{\sigma^2}. \]

In fact, in this case, we know explicitly the law of the exponential functional (see for instance [3]):

\[ \int_0^\infty e^{\sigma B_s + \nu s} ds \sim \frac{2}{\sigma^2 \gamma_{-2\nu/\sigma^2}}, \]

where \( \gamma_m \) denotes a gamma variable with index \( m \). This implies easily the asymptotic behaviour of \( A_\infty \) given by Theorem 1.1.

2.2 Compound Poisson process with drift

Let us take \( \xi_t = -t - \eta_t \) where \( \eta \) is a compound Poisson process with \( \mu(dx) = (a + b - 1)be^{bx} dx, x < 0 \), with \( 0 < a < 1 < a + b \). Then for \( \lambda < b \) we have \( \Phi(\lambda) = \frac{\lambda}{b-\lambda} (1 - a - \lambda) \), therefore it satisfies the Cramér’s condition with

\[ \chi = 1 - a. \]

Here we also know the law of the exponential functional:

\[ \int_0^\infty e^{\xi_s} ds \sim \frac{1}{\beta_{1-a,a+b-1}}, \]

where \( \beta_{a,b} \) is a beta variable with parameters \( a \) and \( b \), so once again we could find easily (3).

2.3 Opposite of a stable subordinator with drift

Here we consider \( \xi_t = -S_t + at \) where \((S_t)_{t\geq 0}\) denotes a standard stable subordinator with index \( 0 < \alpha < 1 \), and \( a \) is a positive real. Then \( \Phi(\lambda) = \lambda^\alpha - a\lambda \) for \( \lambda \geq 0 \) so that \( \Phi(\chi) = 0 \) for \( \chi = a^{1/(\alpha - 1)} \). Thus:

\[ \mathbb{P}(\int_0^\infty e^{-S_u+au} du > t) \sim \frac{c}{t^{\alpha/(\alpha - 1)}}, \]

This example is of greater interest since here we cannot compute the law of the exponential functional.
### 3 Moments of the exponential functional

**Proposition 3.1.** If Cramér’s condition (4) is satisfied, then

\[ \mathbb{E}(A_{\infty}^\alpha) < \infty, \quad \forall \, 0 \leq \alpha < \chi \]

**Proof.** The proof relies on the following lemma:

**Lemma 3.2.** Under the precedent hypothesis, if we note

\[ S_{\infty} = \sup_{t \geq 0} \xi_t, \]

we have

\[ \mathbb{E}(e^{\alpha S_{\infty}}) < \infty, \quad \forall \, 0 \leq \alpha < \chi \]

Let us write \( S_t = \sup_{0 \leq s \leq t} \xi_s \) for each \( t \geq 0 \). Since \( \mathbb{E}(e^{\chi \xi_t}) = 1 \), the process \((e^{\chi \xi_t})_{t \geq 0}\) is a nonnegative martingale, to which we can apply Doob’s Submartingale Inequality for fixed \( t > 0 \) and \( x \in \mathbb{R}^+ \):

\[
x \mathbb{P}(\sup_{0 \leq s \leq t} e^{\chi \xi_s} \geq x) \leq \mathbb{E}(e^{\chi \xi_t}) = 1
\]

We obtain that for fixed \( a \in \mathbb{R}^+ \) and for all \( t > 0 \), \( \mathbb{P}(S_t \geq a) \leq e^{-\chi a} \). Since \( S_{\infty} \) is finite almost-surely thanks to the fact that \( \xi_t \to -\infty \) when \( t \to +\infty \), it follows that \( \mathbb{P}(S_{\infty} > a) \leq e^{-\chi a} \), which concludes the proof of the lemma \□

Now let us fix \( 0 \leq \alpha < \chi \) and introduce the Lévy process \((\xi'_t = \xi_t + kt, t \geq 0)\) with \( k > 0 \) small enough to ensure that \( \xi'_t \) has the same properties as \( \xi \): precisely we assume that \( k + \mathbb{E}(\xi_1) < 0 \). On the one hand we then have that \( \xi'_t \to -\infty \) when \( t \to +\infty \) and on the other hand the Lévy exponent \( \Psi \) of \( \xi'_t \) still has a unique zero \( \chi' > 0 \), such that \( 0 \leq \alpha < \chi' \) is \( k \) is taken small enough.

Now we notice that :

\[
A_{\infty} = \int_0^\infty e^{\xi'_t - ks} ds \leq e^{S'_{\infty}} \int_0^\infty e^{-ks} ds = \frac{1}{k} e^{S'_{\infty}}
\]

writing as above \( S'_{\infty} = \sup_{t \geq 0} \xi'_t \). We deduce that :

\[
\mathbb{E}(A_{\infty}^\alpha) \leq \frac{1}{k^\alpha} \mathbb{E}(e^{\alpha S'_{\infty}})
\]

If suffices to apply the lemma to the Lévy process \((\xi'_t)_{t \geq 0}\) to end the proof. \□

### 4 Proof of the theorem

#### 4.1 First step

To prove (5), it suffices to prove that :

\[
\int_{-\infty}^t e^{-(t-v)r(v)} dv \xrightarrow{t \to +\infty} C \quad \text{where} \quad r(v) = e^{\chi v} \mathbb{P}(A_{\infty} > e^v)
\]

It’s a consequence of lemma 9.3 of [3], that we quote here for sake of completeness :
Lemma 4.1. Let $k > 0$ and $X$ be a real random variable. If $\int_0^t u^k \mathbb{P}(X > u) \, du \sim C t$ when $t \to \infty$, then $\mathbb{P}(X > t) \sim C t^{-k}$ when $t \to +\infty$.

If we introduce the function $K(t) = e^{-t}$ for $t > 0$ and equal to 0 for $t \leq 0$, we can write the left member of (6) as the convolution between $r$ and $K$, and we will denote it by $\tilde{r}$. More generally for all function $f$, we will note

$$\tilde{f}(t) = f \ast K(t) = \int_{-\infty}^t e^{-(t-u)} f(u) \, du.$$  

The key ingredient of the proof will be then a renewal theorem.

4.2 Second step

We are now going to write $\tilde{r}$ in the following form :

$$\tilde{r}(t) = \tilde{g} \ast \nu_{n-1}(t) + \tilde{\delta}_n(t) \quad \forall n \geq 1$$  \hspace{1cm} (7)

with appropriate functions $g$ and $\tilde{\delta}_n$ and measure $\nu_{n-1}$.

For this we need a few notations : Let $(T_i)_{i \geq 1}$ be a sequence of i.i.d variables with exponential law of parameter 1, and independent of the Lévy process $(\xi_t)_{t \geq 0}$. Let $\Theta_n = \sum_{i=1}^n T_i$ for $n \geq 1$. The process $(S_n := \xi_{\Theta_n})_{n \geq 0}$, with $S_0 = 0$, is a random walk. Let us note that :

$$\mathbb{E}(S_1) = \int_{-\infty}^{\infty} \mathbb{E}(\xi_t) e^{-t} \, dt = \mathbb{E}(\xi_1) \int_{0}^{\infty} t e^{-t} \, dt = \mathbb{E}(\xi_1) \in [-\infty, 0)$$  \hspace{1cm} (8)

so that $S_n \xrightarrow{a.s. \ t \to +\infty} -\infty$.

Lemma 4.2. For all $n \geq 1$ we have $r(t) = g \ast \nu_{n-1}(t) + \delta_n(t)$, with :

- $g(t) = e^{xt} (\mathbb{P}(A > t') - \mathbb{P}(MA > t'))$, with $M$ (respectively $A$) a random variable distributed as $e^{Si}$ (respectively $A_\infty$), independent of $(\xi_t)_{t \geq 0}$ and $(T_i)_{i \geq 1}$, and $M$ independent of $A$.
- $\nu_n(dt) = e^{xt} \sum_{k=0}^n \mathbb{P}(S_k \in dt)$
- $\delta_n(t) = e^{xt} \mathbb{P}(e^{S_n} A > te^t)$ with $A$ as above.

**Proof.** First, by the identity in law between $A$ and $A_\infty$, $\mathbb{P}(A_\infty > t') = \mathbb{P}(A > t')$; then by a different way of writing, we obtain that for all fixed $n \geq 1$ :

$$\mathbb{P}(A > t') = \sum_{k=1}^n \mathbb{P}(e^{S_{k-1}}A > e^t) - \mathbb{P}(e^{S_k}A > e^t) + \mathbb{P}(e^{S_n}A > e^t)$$

$$= \sum_{k=1}^n \mathbb{P}(e^{S_{k-1}}A > e^t) - \mathbb{P}(e^{S_k}MA > e^t) + \mathbb{P}(e^{S_n}A > e^t),$$

the last equality resulting from the independence between $(M,A)$ and $(S_n)_{n \geq 0}$, and from the fact that this process is a random walk.

Thus we have :

$$r(t) = \sum_{k=0}^{n-1} \int_{\mathbb{R}} e^{x(t-u)} (\mathbb{P}(A > e^{t-u}) - \mathbb{P}(MA > e^{t-u})) e^{xu} \mathbb{P}(S_k \in du) + e^{xt} \mathbb{P}(e^{S_n}A > e^t)$$

$$= \int_{\mathbb{R}} g(t-u) \nu_{n-1}(du) + \delta_n(t)$$

and this ends the proof \(\square\)

Since $\tilde{r} = r \ast K$, $\tilde{g} = g \ast K$ and $\tilde{\delta}_n = \delta_n \ast K$, this lemma obviously implies (7). \(\square\)
4.3 Third step

We are now going to show that:

\[ \forall \delta_n(t) \rightarrow 0 \quad \text{when} \quad n \rightarrow +\infty \]  

(9)

and then, that there is a renewal measure \( \nu \) such that:

\[ \forall \tilde{g} \ast \nu_n(t) \rightarrow \tilde{g} \ast \nu(t) \quad \text{when} \quad n \rightarrow +\infty \]  

(10)

- The first point is the easier one:

\[ e^{S_n} \xrightarrow{a.s} 0, \quad \text{so for fixed} \quad t, \quad \delta_n(t) \xrightarrow{n \rightarrow +\infty} 0. \]

We conclude by dominated convergence:

\[ \tilde{\delta}_n(t) = \int_{-\infty}^{t} e^{-(t-u)} \delta_n(u) \; du \]

but \( 0 \leq \delta_n(u) \leq e^{\chi u} \) and \( \int_{-\infty}^{t} e^{(\chi+1)u} \; du < \infty \) since \( \chi > 0 \).

- To establish (10), we shall need the following proposition:

**Proposition 4.3.** Let be \( \nu(dt) = e^{\chi t} \sum_{k=0}^{\infty} \mathbb{P}(S_k \in dt) \). Then \( \nu \) is the renewal measure associated to some random walk \( (Y_i)_{i \geq 0} \) such that \( 0 < m := \mathbb{E}(Y_1) < +\infty \) and that the law of \( Y_1 \) is not arithmetic.

**Proof.** The proof relies on the following change of probability: if for all \( t \geq 0 \) we denote by \( \mathcal{F}_t \) the natural filtration of the process \( (\xi_t)_{t \geq 0} \), since \( (e^{\chi \xi_t})_{t \geq 0} \) is a strictly positive \( (\mathcal{F}_t)_{t \geq 0} \)-martingale, we can define a probability \( Q \) on \( \bigvee_{t \geq 0} \mathcal{F}_t \) by the change of probability:

\[ E_Q(X) = E_P(e^{\chi \xi} X) \quad \text{for all bounded and} \quad \mathcal{F}_t \text{-measurable} \quad X \]  

(11)

We easily check that under \( Q \), \( (\xi_t)_{t \geq 0} \) is still a Lévy process, with Lévy exponent \( \Phi_Q(\lambda) = \Phi(\lambda + \chi) \). Since \( \Phi \) is concave, and \( \Phi(0) = \Phi(\chi) = 0 \) and last that \( \Phi'(0) = -\mu > 0 \), we have:

\[ E_Q(\xi_1) = -\Phi'_Q(0) = -\Phi'(\chi) > 0. \]

Let then \( (Y_i)_{i \geq 0} \) be a sequence of i.i.d variables, whose common distribution is the law of \( S_1 \) under \( Q \).

Let us consider the renewal measure \( U \) associated to this process \( (Y_i)_{i \geq 0} \), precisely:

\[ U(dt) = \sum_{k=0}^{\infty} Q(S_k \in dt) \]

Since \( S_k = \xi_{\Theta_k} \) is \( \mathcal{F}_{\Theta_k} \)-measurable, and since the formula (11) is still true with any almost-surely finite stopping-time \( T \) instead of \( t \), in particular for \( T = \Theta_k \), we obtain:

\[ Q(S_k \in dt) = E_P(1_{\{\xi_{\Theta_k} \in dt\}} e^{\chi \xi_{\Theta_k}}) = e^{\chi t} \mathbb{P}(\xi_{\Theta_k} \in dt) \]

and thus,

\[ U(dt) = e^{\chi t} \sum_{k=0}^{\infty} \mathbb{P}(S_k \in dt). \]
As a consequence $U$ is exactly the measure $\nu$ introduced in the proposition.

What is left to be proved is that $(Y_i)_{i \geq 0}$ fulfills the hypothesis of the proposition:

- The sign of the mean is a result of the same calculus that in (8) : $m := \mathbb{E}(Y_1) = \mathbb{E}_Q(\xi_1) > 0$. Moreover if we had $m = \infty$ that would be in contradiction with the fact that $\Phi > -\infty$ on a neighborhood of $\chi$.

- For the second point we first notice that if the law of $S_1$ is arithmetic under $\mathbb{Q}$ then it is also the case under $\mathbb{P}$ : indeed if there is some $\lambda$ such that $\mathbb{Q}(S_1 \in \lambda \mathbb{Z}) = 1$, that means that $\mathbb{E}_\mathbb{P}(1_{\{S_1 \in \lambda \mathbb{Z}\}} e^{\chi S_1}) = 1$, but the variable $e^{\chi S_1}$ being nonnegative and having a mean equal to 1 under $\mathbb{P}$ (since $\mathbb{E}(e^{\chi \xi_1}) = \int_0^\infty \mathbb{E}(e^{\chi t}) e^{-t} dt = \int_0^\infty e^{-t} dt = 1$), this implies that $\mathbb{P}(S_1 \in \lambda \mathbb{Z}) = 1$. But the law of $S_1 = \xi_{\theta_1}$ cannot be arithmetic unless that of $\xi_t$ is arithmetic for all $t$, which is excluded by hypothesis. \hfill \Box

To prove that $\tilde{g} \ast \nu_n(t) \to \tilde{g} \ast \nu(t)$, and then apply a renewal theorem to $\nu$, we have now to show the direct Riemann-integrability of $\tilde{g}$. This will be the fourth step, but for the moment let us assume this result. We then know that $|\tilde{g}| \ast \nu(t) < \infty$ for all $t$. This means that:

$$
\mathbb{E}(\sum_{k=0}^\infty e^{\chi S_k} |\tilde{g}(t - S_k)|) < \infty
$$

and we deduce that:

$$
\tilde{g} \ast \nu_n(t) = \sum_{k=0}^n \mathbb{E}(e^{\chi S_k} \tilde{g}(t - S_k)) \xrightarrow{n \to +\infty} \tilde{g} \ast \nu(t)
$$

We have thus proved the two points (9) and (10).

4.4 Fourth step

**Proposition 4.4.** $\tilde{g}$ is directly Riemann-integrable.

**Proof.** The key is the following lemma, the demonstration of which is given in [4] (p.143) :

**Lemma 4.5.** If $f \in L^1(\mathbb{R})$, then $\tilde{f}$ is directly Riemann-integrable.

We are thus going to prove that $g(t) = e^{\chi t} (\mathbb{P}(A > e^t) - \mathbb{P}(MA > e^t))$ is in $L^1(\mathbb{R})$.

Firstly, by the change of variables $u = e^t$,

$$
\int_{\mathbb{R}} |g(t)| dt = \int_0^\infty u^{-1}(|\mathbb{P}(A > u) - \mathbb{P}(MA > u)|) du .
$$

Now for any almost-surely finite stopping time $T$, by using the strong Markov property at time $T$ for the Lévy process $\xi$, we have the following decomposition of the exponential functional $A_\infty$ :

$$
A_\infty = A_T + e^{\xi_T} \int_0^\infty e^{\xi_s + \tau - \xi_T} ds = A_T + e^{\xi_T} \tilde{A}_\infty^T ,
$$

the variable $\tilde{A}_\infty^T := \int_0^\infty e^{\xi_s + \tau - \xi_T} ds$ having the same law as $A_\infty$ and being independent of $(\xi_s)_{0 \leq s \leq T}$. The identity (2) mentioned in the introduction was a direct consequence of (12).
By using \((12)\) with \(T = \Theta_1\), we first obtain \((M, A) \overset{d}{=} (e^{\xi_{\Theta_1}}, \tilde{A}^{\Theta_1}_\infty)\), and so \(\mathbb{P}(MA > u) = \mathbb{P}(e^{\xi_{\Theta_1}} \tilde{A}^{\Theta_1}_\infty > u)\); then we write \(\mathbb{P}(A > u) = \mathbb{P}(A_\infty > u)\), which eventually leads us to:

\[
|\mathbb{P}(A > u) - \mathbb{P}(MA > u)| = \mathbb{P}(e^{\xi_{\Theta_1}} \tilde{A}^{\Theta_1}_\infty \leq u < A_\infty)
\]

Hence:

\[
\int_{\mathbb{R}} |g(t)| \, dt = \int_0^\infty u^{\chi-1} |\mathbb{P}(A > u) - \mathbb{P}(MA > u)| \, du = \frac{1}{\chi} \mathbb{E} \left( A_\infty^\chi - (e^{\xi_{\Theta_1}} \tilde{A}^{\Theta_1}_\infty)^\chi \right)
\]

Two cases have to be distinguished:

- **First case**: \(0 < \chi \leq 1\)

  Then the following inequality holds:

  \[
  |x^\chi - y^\chi| \leq |x - y|^\chi \text{ for all nonnegative } x, y \text{. Thus}
  \]

  \[
  \frac{1}{\chi} \mathbb{E} \left( A_\infty^\chi - (e^{\xi_{\Theta_1}} \tilde{A}^{\Theta_1}_\infty)^\chi \right) \leq \frac{1}{\chi} \mathbb{E}(A_\infty - e^{\xi_{\Theta_1}} \tilde{A}^{\Theta_1}_\infty)^\chi = \frac{1}{\chi} \mathbb{E}(A_\Theta_1^\chi).
  \]

  To see that the last right-member is finite, we can use the same method that in the proof of Proposition \([5.1]\): indeed \(A_{\Theta_1}\) can be seen as the terminal value of the process \((\int_0^t e^{\xi(s)} \, ds)_{t \geq 0}\) where \(\xi(1)\) is the initial Lévy process killed at the independent exponential time \(\Theta_1\). His Lévy exponent is \(\Phi(1) = \Phi + 1\). Since \(\Phi(\chi) = 0\) and since there exist \(\varepsilon > 0\) for which \(\Phi > -\infty\) on \([0, \chi + \varepsilon]\), by continuity we can find \(\lambda_0 > \chi\) such that \(\Phi(\lambda_0) = 0\). Then if we now use the martingale:

  \[
  (e^{\lambda_0 \xi(1)} + \Phi(\lambda_0) s)_{t \geq 0},
  \]

  we deduce, as in Proposition \([5.1]\) that the exponential functional associated to the killed Lévy process \(\xi(1)\) has moments of all order \(\alpha < \lambda_0\); in particular with \(\alpha = \chi\) we obtain exactly that \(\mathbb{E}(A_{\Theta_1}^\chi) < \infty\).

- **Second case**: \(1 < \chi\)

  This time we use the inequality \(|x^\chi - y^\chi| \leq \chi |x - y|(|x| + |y|)^{\chi-1}\), which leads to the upper bound:

  \[
  \int_{\mathbb{R}} |g(t)| \, dt \leq \mathbb{E} \left( (A_\infty - e^{\xi_{\Theta_1}} \tilde{A}^{\Theta_1}_\infty)^\chi \right).
  \]

  The right-member can also be written

  \[
  \mathbb{E} \left( A_{\Theta_1}(A_{\Theta_1} + e^{\xi_{\Theta_1}} \tilde{A}^{\Theta_1}_\infty)^{\chi-1} \right).
  \]

  Since \(|x + y|^\chi \leq c_r(|x|^\chi + |y|^\chi)\) for some constant \(c_r\), we obtain:

  \[
  \int_{\mathbb{R}} |g(t)| \, dt \leq c_{\chi-1} \left[ \mathbb{E}(A_{\Theta_1}^\chi) + \mathbb{E} \left( A_{\Theta_1}(\tilde{A}^{\Theta_1}_\infty e^{\xi_{\Theta_1}})^{\chi-1} \right) \right].
  \]

  We have already seen that \(\mathbb{E}(A_{\Theta_1}^\chi) < \infty\). Concerning the second term, we first use the independence between \(\tilde{A}^{\Theta_1}_\infty\) and \((\xi_s)_{0 \leq s \leq \Theta_1}\):
\[
\mathbb{E}\left(A_{\Theta_1}(\tilde{A}_\infty^{\Theta_1}e^{\xi_{\Theta_1}})^{-1}\right) = \mathbb{E}\left(\left(\tilde{A}_\infty^{\Theta_1}\right)^{-1}\right)\mathbb{E}\left(A_{\Theta_1}e^{(\chi-1)\xi_{\Theta_1}}\right)
\]

But \(1 < \chi\) so by Hölder’s inequality :

\[
\mathbb{E}\left(A_{\Theta_1}e^{(\chi-1)\xi_{\Theta_1}}\right) \leq \mathbb{E}(A_{\Theta_1}^{\chi})^{1/\chi}\mathbb{E}(e^{\chi\xi_{\Theta_1}})^{1/\chi}
\]

We have already seen that \(\mathbb{E}(e^{\chi\xi_{\Theta_1}}) = 1\) so :

\[
\mathbb{E}\left(A_{\Theta_1}e^{(\chi-1)\xi_{\Theta_1}}\right) < \infty
\]

Lastly :

\[
\mathbb{E}\left((\tilde{A}_\infty^{\Theta_1})^{-1}\right) = \mathbb{E}\left(A_\infty^{\chi-1}\right) < \infty
\]

because \(A_\infty\) has finite moments of all order \(0 < \alpha < \chi\) (see Proposition 3.1). In conclusion, in this second case we still have \(\int_\mathbb{R} |g(t)| dt < +\infty\).

\[\square\]

4.5 Conclusion

The second and third step imply that :

\[\tilde{r}(t) = \tilde{g} * \nu(t) .\]

Now the Proposition 4.3 and the last step enable us to apply the renewal theorem to \(\nu\) :

\[\tilde{g} * \nu(t) = \int_\mathbb{R} \tilde{g}(t-u)\nu(du) \overset{t \to +\infty}{\longrightarrow} \frac{1}{m} \int_\mathbb{R} \tilde{g} .\]

So the proof is finished by taking \(C = \frac{1}{m} \int_\mathbb{R} \tilde{g} = \frac{1}{m} \int_\mathbb{R} g .\)

4.6 Another method

There exists a shorter proof of Theorem 1.1 that we are going to detail here. Nevertheless, the previous proof has an interest by itself since it shows that techniques used in discrete time can be adapted to continuous time.

Let us now explain this other method.

The starting point is to notice that \(A_\infty\) satisfies the random difference equation :

\[A_\infty = MA'_\infty + Q \]

where on the right hand side

\[A'_\infty = \int_0^\infty \exp(\xi_{1+s} - \xi_1) ds \quad , \quad M = e^{\xi_1} \quad \text{and} \quad Q = \int_0^1 e^{\xi_s} ds .\]

\(A'_\infty\) is distributed as \(A_\infty\) and is independent of the pair \((M, Q)\), which enables to recover the relation (2) of the introduction when \(T = 1\).

The idea is then to show that \(M\) and \(Q\) satisfy the conditions of Kesten’s Theorem (cf. e.g. Theorem 4.1 in [1]), which gives the conclusion.
– The first condition $\mathbb{E}(|M|) = 1$ is of course satisfied.
– The second one is $\mathbb{E}(|M|^\chi \log^+ |M|) < \infty$.
  This is true here thanks to the hypothesis that $\Phi > -\infty$ on an interval $[0, \chi + \epsilon]$ for some $\epsilon > 0$. Indeed one has:

$$
\mathbb{E}(|M|^\chi \log^+ |M|) = \mathbb{E}(e^{\chi \xi_1} \xi_1^+) \\
\leq \frac{1}{\epsilon} \mathbb{E}(e^{\chi \xi_1} e^{\epsilon \xi_1}) \\
= \frac{1}{\epsilon} e^{-\Phi(\chi + \epsilon)} < \infty.
$$

– The last thing to check is that $\mathbb{E}|Q|^\chi < \infty$.
  In fact one can prove that $Q^\chi \in L^{1+\epsilon_0}$ with $\epsilon_0 = \frac{\epsilon}{\chi}$ for all $\epsilon$ such that $\Phi(\chi + \epsilon) > -\infty$. Indeed one first observes that

$$
0 \leq Q^\chi \leq \sup_{0 \leq s \leq 1} e^{\chi \xi_s}.
$$

But $(e^{\chi \xi_s})_{0 \leq s \leq 1}$ is a martingale bounded in $L^{1+\epsilon_0}$, since for all $0 \leq s \leq 1$ :

$$
\mathbb{E}(e^{\chi(1+\epsilon_0)\xi_s}) = e^{-\epsilon \Phi(\chi + \epsilon)} \leq e^{-\Phi(\chi + \epsilon)} < \infty.
$$

Thus by Doob’s Inequality in $L^{1+\epsilon_0}$, one concludes that :

$$
\mathbb{E}(Q^{\chi(1+\epsilon_0)}) < \infty.
$$

5 Annex : proof of Wald identity for Lévy processes

We first need some notations before stating that we call Wald identity, since it generalizes the so-called result for random walks.

Let $X$ be a Lévy process started at 0, with Lévy exponent $\Psi$, such that $0 < \mu = \mathbb{E}(X_1) < \infty$. Let us write $S_t = \sup_{s \leq t} X_s$; let $L$ be a local time at 0 for the reflected process at the supremum $S - X$ and $\tau$ its inverse process : $\tau_t = \inf\{s > 0, L_s > t\}$. Since $\mu \mathbb{E}(X_1) > 0$, $L_\infty = +\infty$ a.s and $\tau_t < \infty$ for all $t \geq 0$. Then one can define the ascending ladder height process $H$ by $H_t = S_{\tau_t} = X_{\tau_t}$ for all $t \geq 0$.

**Proposition 5.1.** With the previous notations, if there exists $K < 0$ such that $0 < \Psi(K) < \infty$, then :

$$
\mathbb{E}(H_1) = \mu \mathbb{E}(\tau_1).
$$

**Remark** : This result applies to the Lévy process $X^*$ defined in the introduction. Indeed $\mathbb{E}(X^*_1) = -\Phi(\chi) \in \mathbb{R}_+$ (cf proof of Proposition 4.3) and $\Phi^*(\lambda) = \Phi(\lambda + \chi) \in (0, +\infty)$ for all $-\chi < \lambda < 0$.

**Proof.**

We first prove that $\tau_1$, which is a stopping-time with respect to the filtration $\mathcal{F}_t = \sigma\{X_s, 0 \leq s \leq t\}$, is integrable :

**Lemma 5.2.** Under the only assumption that $0 < \mathbb{E}(X_1) \leq \infty$, one has $\mathbb{E}(\tau_1) < \infty$. 


Proof. Let $K$ be the Lévy exponent of the subordinator $\tau$. One knows (see for example \cite{2}, p.166) that:

$$K(\lambda) = \exp\left(\int_0^\infty (e^{-t} - e^{-\lambda t})t^{-1}\mathbb{P}(X_t \geq 0) \, dt\right).$$

By writing for all $\lambda > 0$ that $\ln \lambda = \int_0^\infty (e^{-t} - e^{-\lambda t})t^{-1} \, dt$, one obtains the following expression:

$$\frac{K(\lambda)}{\lambda} = \exp\left(\int_0^\infty (e^{-\lambda t} - e^{-t})t^{-1}\mathbb{P}(X_t < 0) \, dt\right).$$

But, since $\lim_{t \to +\infty} X_t = +\infty$, $\int_1^\infty t^{-1}\mathbb{P}(X_t < 0) \, dt < \infty$, and by dominated convergence one concludes that:

$$K'(0) = \lim_{\lambda \to 0} \frac{K(\lambda)}{\lambda} = \exp\left(\int_0^\infty (1 - e^{-t})t^{-1}\mathbb{P}(X_t < 0) \, dt\right) < \infty \quad \square$$

Then we prove the integrability of the terminal value of the infimum process $(I_t = \inf_{s \leq t} X_s, t \geq 0)$:

**Lemma 5.3.** Let us write $I_\infty = \inf_{t \geq 0} X_t$. Then $\mathbb{E}(I_\infty) > -\infty$.

**Proof.** First we notice that $I_\infty$ is finite a.s. since $\mu > 0$.

We introduce the dual process $\tilde{X} = -X$, whose Lévy exponent is $\tilde{\Psi}(\lambda) = \Psi(-\lambda)$.

Writing $M_t = \sup_{s \leq t} \tilde{X}_s$ for $t \geq 0$, we are going to prove that $M_\infty$ has exponential moments of order $\alpha$ for all $0 \leq \alpha < -K$, which a fortiori implies that $M_\infty$ is integrable, and the conclusion will follow since $I_\infty = -M_\infty$.

Since $\tilde{\Psi}(-K) = \Psi(K) < \infty$, the process $(e^{-K\tilde{X}_t + \tilde{\Psi}(-K)t})_{t \geq 0}$ is well defined and is a non-negative $(\mathcal{F}_t)_{t \geq 0}$-martingale, so if we fix $t > 0$ and $x \in \mathbb{R}^+$, we have by Doob’s Submartingale Inequality:

$$x \mathbb{P}(\sup_{0 \leq s \leq t} e^{-K\tilde{X}_s + \tilde{\Psi}(-K)s} \geq x) \leq \mathbb{E}(e^{-K\tilde{X}_t + \tilde{\Psi}(-K)t}) = 1$$

Since $\tilde{\Psi}(-K) > 0$, this implies that $x \mathbb{P}(\sup_{0 \leq s \leq t} e^{-K\tilde{X}_s} \geq x) \leq 1$, and thus that for all fixed $a \in \mathbb{R}^+$ and all $t > 0$, $\mathbb{P}(M_t \geq a) \leq e^{Ka}$. It follows that $\mathbb{P}(M_\infty > a) \leq e^{Ka}$ which ends the proof of the lemma. \quad \square

We are now able to prove (15). Using that $(X_t - \mu t, t \geq 0)$ is a $(\mathcal{F}_t)_{t \geq 0}$-martingale, we deduce that for all $n \geq 0$,

$$\mathbb{E}(X_{\tau_1 \wedge n}) = \mu \mathbb{E}(\tau_1 \wedge n) \quad (16)$$

Writing $X_t = X_t^+ - X_t^-$ with $X_t^+ = \max(X_t, 0)$ and $X_t^- = \max(-X_t, 0)$, we have $X_{\tau_1 \wedge n}^+ \leq X_{\tau_1 \wedge n} - I_\infty$, hence using (16) and Fatou’s lemma, one obtains:

$$\mathbb{E}(X_{\tau_1}^+) \leq \liminf_{n \to +\infty} \mathbb{E}(\mu(\tau_1 \wedge n) - I_\infty).$$

Thanks to the previous lemmas, $\liminf_{n \to +\infty} \mathbb{E}(\mu(\tau_1 \wedge n) - I_\infty) = \mu \mathbb{E}(\tau_1) - \mathbb{E}(I_\infty) < \infty$, thus

$$0 \leq \mathbb{E}(H_1) = \mathbb{E}(X_{\tau_1}) \leq \mathbb{E}(X_{\tau_1}^+) < \infty.$$
Since $I_\infty \leq X_{\tau_1 \wedge n} \leq S_{\tau_1} = H_1$, one concludes from (10) by dominated convergence that

$$
\mathbb{E}(X_{\tau_1}) = \mu \mathbb{E}(\tau_1).
$$

\[\square\]

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