Hypotheses Founded Semantics of Logic Programs for Information Integration in Multi-Valued Logics

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Abstract

We address the problem of integrating information coming from different sources. The information consists of facts that a central server collects and tries to combine using (a) a set of logical rules, i.e. a logic program, and (b) a hypothesis representing the server's own estimates. In such a setting incomplete information from a source or contradictory information from different sources necessitate the use of many-valued logics in which programs can be evaluated and hypotheses can be tested. To carry out such activities we propose a formal framework based on bilattices such as Belnap’s four-valued logics. In
this framework we work with the class of programs defined by Fitting and we develop a theory for information integration. We also establish an intuitively appealing connection between our hypothesis testing mechanism on the one hand, and the well-founded semantics and Kripke-Kleene semantics of Datalog programs with negation, on the other hand.

**Keywords**: deductive databases and knowledge bases, information integration, logics of knowledge, inconsistency, bilattices.

1 Introduction

In several information oriented activities there is a need for combining (or “integrating”) information coming from different sources.

A typical example of such information-oriented activity is building a data warehouse, i.e. a special kind of very large database for decision-making support in big enterprises [1]. The information stored in a data warehouse is obtained from queries to operational databases internal to the enterprise, and from remote information sources external to the enterprise accessed through the Internet. The answers to all such queries are then combined (by the so-called “integrator”) to derive the information to be stored in the data warehouse.

The basic pattern of the data warehouse paradigm, i.e. collection of information then integration, is encountered in many different situations. What changes usually from one situation to another is the type (and volume) of the collected information and the means used for the integration.

In this paper we address a specific problem of information integration, namely, the information consists of facts that a central server collects from a number of autonomous sources and then tries to combine them using:

- a set of logical rules, i.e. a logic program, and
- a hypothesis, representing the server’s own estimates.

In such a setting incomplete information from a source or contradictory information coming from different sources necessitate the use of many-valued logics, in which programs can be evaluated and hypotheses can be tested. Let us see a simple example.
Example 1 Consider a legal case where a judge (the “central server”) has to decide whether to charge a person named John accused of murder. To do so, the judge first collects facts from two different sources: the public prosecutor and the person’s lawyer. The judge then combines the collected facts using a set of rules in order to reach a decision. For the sake of our example let us suppose that the judge has collected a set of facts $F$ that he combines using a set of rules $R$ as follows:

$$F = \begin{cases} \text{witness(John)} & \text{false} \\ \text{friends(John, Ted)} & \text{true} \end{cases}$$

$$R = \begin{cases} \text{suspect}(X) & \leftarrow \text{motive}(X) \lor \text{witness}(X) \\ \text{innocent}(X) & \leftarrow \exists Y (\text{alibi}(X, Y) \land \neg \text{friends}(X, Y)) \\ \text{friends}(X, Y) & \leftarrow \text{friends}(Y, X) \lor (\text{friends}(X, Z) \land \text{friends}(Z, Y)) \\ \text{charge}(X) & \leftarrow \text{suspect}(X) \oplus \neg \text{innocent}(X) \end{cases}$$

The first fact of $F$ says that there is no witness, i.e. the fact witness(John) is false. The second fact of $F$ says that Ted is a friend of John, i.e. the fact friends(John, Ted) is true.

Turning now to the set of rules, the first rule of $R$ describes how the prosecutor works: in order to support the claim that a person $X$ is a suspect, the prosecutor tries to provide a motive and/or a witness.

The second rule of $R$ describes how the lawyer works: in order to support the claim that $X$ is innocent, the lawyer tries to provide an alibi for $X$ by a person who is not a friend of $X$. This rule depends on the third rule which defines the relation friends.

Finally, the fourth rule of $R$ is the “decision making rule” and describes how the judge works: in order to reach a decision as to whether to charge $X$, the judge examines the premises suspect($X$) and $\neg$innocent($X$). As explained earlier, the values of these premises come from two different sources: the prosecutor and the lawyer. Each of these premises can have the value true or false. However, it is also possible that the value of a premiss is undefined. For example, if a motive is not known and a witness has not been found, then the value of suspect($X$) will be undefined.

\footnote{That notation means that the only facts which are defined (or assigned to logical values different from unknown) are witness(John) which is false and friends(John, Ted) which is true.}
In view of these observations, the question is what value is appropriate to associate with charge(X).

What we propose is to collect together the values of the premises suspect(X) and ¬innocent(X), and to consider the resulting set of values as the value of charge(X). This is precisely what the notation

\[ \text{charge}(X) \leftarrow \text{suspect}(X) \oplus \neg \text{innocent}(X) \]

means, where \( \oplus \) denotes the “collecting together” operation.

It follows that there are four possible values for charge(X): \( \emptyset \), \{true\}, \{false\} and \{true, false\}. We shall call these values: Underdefined, True, False and Overdefined, and we shall denote them by \( U \), \( T \), \( F \) and \( O \), respectively.

The value Underdefined for a premiss means that the premiss is true or false but its actual value is currently unknown. For the purpose of this paper we shall assume that any premiss whose value is not known is associated with the value Underdefined.

We note that the value Underdefined is related to the so-called “null values” of attributes in database theory. In database theory, however, a distinction is made between two types of null values [22]:

- the attribute value exists but is currently unknown;
- the attribute value does not exist.

An example of the first type is the Department-value for an employee that has just been hired but has not yet been assigned to a specific department, and an example of the second type is the maiden name of a male employee. The value Underdefined corresponds to the first type of null value.

Returning now to our example, the decision whether to charge John depends on the value that charge(John) will receive when collecting the values of the premises together. Looking at the facts of \( F \) and the rules of \( R \) (and using intuition) we can see that suspect(John) and innocent(John) both receive the value \( U \) and so then does charge(John).

This is clearly a case where the judge cannot decide whether to actually charge John!

In the context of decision making, however, one has to reach a decision (based on the available facts and rules) even if some values are not defined. This can be accomplished by assuming values for some or all underdefined
premises. Such an assignment of values to underdefined premises is what we call a hypothesis.

Thus in our example, if the judge assumes the innocence of John, then charge(John) receives the value false and John is not charged. We note that this is precisely what happens in real life under similar circumstances, i.e. the defendant is assumed innocent until proved guilty.

Clearly, when hypothesizing on underdefined premises we would like our hypothesis to be “reasonable” in some sense, with respect to the available information, i.e., with respect to the given facts and rules. Roughly speaking, we define a hypothesis $H$ to be “reasonable” or sound using the following test:

1. calling a fact $f$ defined under $H$ if $H(f) \neq U$;
2. add $H$ to $F$ to produce a new set of facts $F' = F \cup H$;
3. apply the rules of $R$ to $F'$ to produce a new assignment of values $H'$;
4. if the facts defined under $H$ are assigned to the same values in $H'$ then $H$ is sound, otherwise $H$ is unsound.

That is, if there is no fact of $H$ that has changed value as a result of rule application then $H$ is a sound hypothesis; otherwise $H$ is unsound.

In our example, for instance, consider the following hypothesis:

$$H_1 = \left[\begin{array}{c}
innocent(John) \\
\text{charge}(John) \\
\end{array}\right]$$

Applying the above test we find the following values for the facts of $H_1$:

$$H'_1 = \left[\begin{array}{c}
innocent(John) \\
charge(John) \\
\end{array}\right]$$

As we can see, the fact charge(John) has changed value, i.e. this fact had the value $T$ in $H_1$ and now has the value $F$ in $H'_1$. Therefore, $H_1$ is not a sound hypothesis.

Next, consider the following hypothesis:

$$H_2 = \left[\begin{array}{c}
innocent(John) \\
\text{charge}(John) \\
\end{array}\right]$$

Applying again our test we find:
That is, the values of the facts of $H_2$ remain unchanged in $H'_2$, thus $H_2$ is a sound hypothesis.

Intuitively, if our hypothesis is sound this means that what we have assumed is compatible with the given facts and rules.

From now on let us denote $\mathcal{P}$ the facts of $F$ together with the rules of $R$, i.e. $\mathcal{P} = \langle F, R \rangle$, and let us call $\mathcal{P}$ a program.

In principle, we may assume or hypothesize values for every possible ground atom. However, given a program $\mathcal{P}$ and a hypothesis $H$, we cannot expect $H$ to be sound with respect to $\mathcal{P}$, in general. What we can expect is that some “part” of $H$ is sound with respect to $\mathcal{P}$.

More precisely, given two hypotheses $H$ and $H'$, call $H$ a part of $H'$, denoted $H \leq H'$, if $H(f) \neq \mathcal{U}$ implies $H(f) = H'(f)$, i.e., if $H$ agrees with $H'$ on every defined fact. It is then natural to ask, given program $\mathcal{P}$ and hypothesis $H$, what is the maximal part of $H$ that is sound with respect to $\mathcal{P}$. We call this maximal part the support of $H$ by $\mathcal{P}$, and we denote it by $s_H^P$. Intuitively, the support of $H$ indicates how much of $H$ can be assumed safely, i.e., remaining compatible with the facts and rules of $\mathcal{P}$.

We show that the support $s_H^P$ can be used to define a hypothesis-based semantics of $\mathcal{P} = \langle F, R \rangle$, denoted by $sem_H^P$. This is done by a fixpoint computation that uses an immediate consequence operator $T$ as follows:

- $F_0 = F$;
- $F_{i+1} = T(F_i) \oplus s^H_{\langle F_i, R \rangle}$.

We also show that there is an interesting connection between hypothesis based semantics and the semantics of Datalog programs with negation. More precisely, we show that if $\mathcal{P}$ is a Datalog program with negation then:

- if $H$ is the everywhere false hypothesis then $sem_H^P$ coincides with the well-founded semantics of $\mathcal{P}$ [20], and
- if $H$ is the everywhere underdefined hypothesis then $sem_H^P$ coincides with the Kripke-Kleene semantics of $\mathcal{P}$ [5].
As we shall see, these results allow us to extend the well-founded semantics and the Kripke-Kleene semantics of Datalog program with negation to the broader class of Fitting programs [8].

Motivation for this work comes from the area of knowledge acquisition, where contradictions may occur during the process of collecting knowledge from different experts. Indeed, in multi-agent systems, different agents may give different answers to the same query. It is then important to be able to process the answers so as to extract the maximum of information on which the various agents agree, or to detect the items on which the agents give conflicting answers.

Motivation also comes from the area of deductive databases. Updates leading to a certain degree of inconsistency should be allowed because inconsistency can lead to useful information, especially within the framework of distributed databases. In particular, Fuhr and Rölleke showed in [9] that hypermedia information retrieval requires the handling of inconsistent information and non-uniform hypotheses.

The remaining of the paper is organized as follows. In Section 2 we recall very briefly some definitions and notations from well-founded semantics, Belnap’s logic \textit{FOUR}, bilattices and Fitting programs. We then proceed, in Section 3, to define sound hypotheses and their support by a Fitting program \( P \); we also discuss computational issues and we present algorithms for computing the support of a hypothesis by a program \( P \) and the hypothesis-founded semantics of \( P \). In Section 4 we show that the notion of support actually unifies the notions of well-founded semantics and Kripke-Kleene semantics and extends them from Datalog program with negation to the broader class of Fitting programs. Section 5 contains concluding remarks and suggestions for further research.

\section{Preliminaries}

\subsection{Three-valued logics}

\subsubsection{Well founded semantics}

Well-founded semantics of logic programs were first proposed in [20]. The well founded semantics of a Program \( P \) is based on the closed world assumption, i.e. every atom is supposed to be \textit{false} by default. In the approach of [20] an interpretation \( I \) is a set of ground literals that does not contain literals of the
form $A$ and $\neg A$. Now, if we consider an instantiated program $P$ defined as in \[2\], its well-founded semantics is defined using the following two operators on partial interpretations $I$:

- the immediate consequence operator $T_P$, defined by
  \[
  T_P(I) = \{\text{head}(r) \mid r \in P \land \forall B \in \text{body}(r), B \in I\},
  \]
  and

- the unfounded operator $U_P$, where $U_P(I)$ is defined to be the greatest unfounded set with respect to the partial interpretation $I$.

We recall that a set of instantiated atoms $U$ is said to be unfounded with respect to $I$ if for all instantiated atoms $A \in U$ and for all rules $r \in P$ the following holds:

\[
\text{head}(r) = A \Rightarrow \exists B \in \text{body}(r) (\neg B \in I \lor B \in U)
\]

In \[4\] it is proven that $U_P(I) = \mathcal{HB} \setminus SPF_P(I)$, where $\mathcal{HB}$ is the Herbrand Base and $SPF_P(I)$ is the limit of the increasing sequence $[SPF^i(I)]_{i \geq 1}$ defined by:

- $SPF^1_P(I) = \{\text{head}(r) \mid r \in P \land \neg B \in I \lor B \in U\}$

- $SPF^{i+1}_P(I) = \{\text{head}(r) \mid r \in P \land \neg B \in I \lor B \in U\}, i > 0$.

The atoms of $SPF_P(I)$ are called potentially founded atoms.

The operator $W_P$, called the well-founded operator, is then defined by $W_P(I) = T_P(I) \cup \neg U_P(I)$ and is shown to be monotone with respect to set inclusion. The well-founded semantics of $P$ is defined to be the least fixpoint of $W_P$ \[20\].

2.1.2 Kripke-Kleene semantics

The Kripke-Kleene semantics was introduced in \[5\]. In the approach of \[5\], a valuation is a function from the Herbrand base to the set of logical values \{true, false, unknown\}. Now, given an instantiated program $P$ defined as in \[5\], its Kripke-Kleene semantics is defined using an operator $\Phi_P$ on valuations, defined as follows: given a ground atom $A$,
• if there is a rule in $\mathcal{P}$ with head $A$, and the truth value of the body under $v$ is \textit{true}, then $\Phi_{\mathcal{P}}(v)(A) = \text{true}$;

• if there is a rule in $\mathcal{P}$ with head $A$, and for every rule in $\mathcal{P}$ with head $A$ the truth value of the body under $v$ is false, then $\Phi_{\mathcal{P}}(v)(A) = \text{false}$;

• else $\Phi_{\mathcal{P}}(v)(A) = \text{unknown}$.

The Kripke-Kleene semantics of a Program $\mathcal{P}$ is based on the open world assumption, i.e. every atom is supposed to be $\text{unknown}$ by default, and is defined to be is the iterated fixpoint of $\Phi_{\mathcal{P}}$ obtained by beginning the iteration with the everywhere unknown valuation.

2.2 Multi-valued logics

2.2.1 Belnap’s four-valued logic

In \cite{3}, Belnap defines a logic called $\textit{FOUR}$ intended to deal with incomplete and inconsistent information. Belnap’s logic uses four logical values that we shall denote by $\mathcal{F}$, $\mathcal{T}$, $\mathcal{U}$ and $\mathcal{O}$, i.e. $\textit{FOUR} = \{\mathcal{F}, \mathcal{T}, \mathcal{U}, \mathcal{O}\}$. These values can be compared using two orderings, the knowledge ordering and the truth ordering.

In the knowledge ordering, denoted by $\leq_k$, the four values are ordered as follows: $\mathcal{U} \leq_k \mathcal{F}$, $\mathcal{U} \leq_k \mathcal{T}$, $\mathcal{F} \leq_k \mathcal{O}$, $\mathcal{T} \leq_k \mathcal{O}$. Intuitively, according to this ordering, each value of $\textit{FOUR}$ is seen as a possible knowledge that one can have about the truth of a given statement. More precisely, this knowledge is expressed as a set of classical truth values that hold for that statement. Thus, $\mathcal{F}$ is seen as $\{\text{false}\}$, $\mathcal{T}$ is seen as $\{\text{true}\}$, $\mathcal{U}$ is seen as $\emptyset$ and $\mathcal{O}$ is seen as $\{\text{false}, \text{true}\}$. Following this viewpoint, the knowledge ordering is just the set inclusion ordering.

In the truth ordering, denoted by $\leq_t$, the four logical values are ordered as follows: $\mathcal{F} \leq_t \mathcal{U}$, $\mathcal{F} \leq_t \mathcal{O}$, $\mathcal{U} \leq_t \mathcal{T}$, $\mathcal{O} \leq_t \mathcal{T}$. Intuitively, according to this ordering, each value of $\textit{FOUR}$ is seen as the degree of truth of a given statement. $\mathcal{U}$ and $\mathcal{O}$ are both less false than $\mathcal{F}$, and less true than $\mathcal{T}$, but $\mathcal{U}$ and $\mathcal{O}$ are not comparable.

The two orderings are represented in the double Hasse diagram of Figure 1.

Both $\leq_t$ and $\leq_k$ give $\textit{FOUR}$ a lattice structure. Meet and join under the truth ordering are denoted by $\land$ and $\lor$, and they are natural generalizations
of the usual notions of conjunction and disjunction. In particular, $\mathcal{U} \land \mathcal{O} = \mathcal{F}$ and $\mathcal{U} \lor \mathcal{O} = \mathcal{T}$. Under the knowledge ordering, meet and join are denoted by $\otimes$ and $\oplus$, and are called the consensus and gullibility, respectively: $x \otimes y$ represents the maximal information on which $x$ and $y$ agree, whereas $x \oplus y$ adds the knowledge represented by $x$ to that represented by $y$. In particular, $\mathcal{F} \otimes \mathcal{T} = \mathcal{U}$ and $\mathcal{F} \oplus \mathcal{T} = \mathcal{O}$.

There is a natural notion of negation in the truth ordering denoted by $\neg$, and we have: $\neg \mathcal{T} = \mathcal{F}$, $\neg \mathcal{F} = \mathcal{T}$, $\neg \mathcal{U} = \mathcal{U}$, $\neg \mathcal{O} = \mathcal{O}$. There is a similar notion for the knowledge ordering, called conflation, denoted by $\sim$, and we have: $\sim \mathcal{U} = \mathcal{O}$, $\sim \mathcal{O} = \mathcal{U}$, $\sim \mathcal{F} = \mathcal{F}$, $\sim \mathcal{T} = \mathcal{T}$.

The operations $\lor$, $\land$, $\neg$ restricted to the values $\mathcal{T}$ and $\mathcal{F}$ are those of classical logic, and if we add to these operations and values the value $\mathcal{U}$ then they are those of Kleene’s strong three-valued logic.

### 2.2.2 Bilattices

In [6, 14], bilattices are used as truth-value spaces for integration of information coming from different sources. The bilattice approach is a basic contribution to many-valued logics. Bilattices and their derived sublogics are useful in expressing uncertainty and inconsistency in logic programming and databases [2, 6, 8, 10, 15, 18]. The simplest non-trivial bilattice is called FOUR, and it is basically Belnap’s four-valued logic [3].

**Definition 1** A bilattice is a triple $\langle \mathcal{B}, \leq_t, \leq_k \rangle$, where $\mathcal{B}$ is a nonempty set and $\leq_t, \leq_k$ are each a partial ordering giving $\mathcal{B}$ the structure of a lattice with a top and a bottom.

In a bilattice $\langle \mathcal{B}, \leq_t, \leq_k \rangle$, meet and join under $\leq_t$ are denoted $\lor$ and $\land$, and meet and join under $\leq_k$ are denoted $\oplus$ and $\otimes$. Top and bottom under
\( \leq_t \) are denoted \( T \) and \( F \), and top and bottom under \( \leq_k \) are denoted \( I \) and \( U \). If the bilattice is complete with respect to both orderings, infinitary meet and join under \( \leq_t \) are denoted \( \bigvee \) and \( \bigwedge \), and infinitary meet and join under \( \leq_k \) are denoted \( \bigoplus \) and \( \bigotimes \).

**Definition 2** A bilattice \( \langle B, \leq_t, \leq_k \rangle \) is called distributive if all 12 distributive laws connecting \( \lor \), \( \land \), \( \oplus \) and \( \otimes \) hold. It is called infinitely distributive if it is a complete bilattice in which all infinitary, as well as finitary, distributive laws hold.

An example of a distributive law is \( x \otimes (y \lor z) = (x \otimes y) \lor (x \otimes z) \). An example of an infinitary distributive law is \( x \otimes \bigvee \{y_i | i \in S\} = \bigvee \{x \otimes y_i | i \in S\} \).

**Definition 3** A bilattice \( \langle B, \leq_t, \leq_k \rangle \) satisfies the interlacing conditions if each of the operations \( \lor \), \( \land \), \( \oplus \) and \( \otimes \) is monotone with respect to both orderings. If the bilattice is complete, it satisfies the infinitary interlacing conditions if each of the infinitary meet and join is monotone with respect to both orderings.

An example of an interlacing condition is: \( x_1 \leq_t y_1 \) and \( x_2 \leq_t y_2 \) implies \( x_1 \otimes x_2 \leq_t y_1 \otimes y_2 \). An example of an infinitary interlacing condition is: \( x_i \leq_t y_i \) for all \( i \in S \) implies \( \bigoplus \{x_i | i \in S\} \leq_t \bigoplus \{y_i | i \in S\} \). A distributive bilattice satisfies the interlacing conditions.

\( \text{FOUR} \) is an infinitary distributive bilattice which satisfies the infinitary interlacing laws. A bilattice is said to be **nontrivial** if the bilattice \( \text{FOUR} \) can be isomorphically embedded in it.

There are two principal ways for constructing bilattices that were introduced in \[12\], and then developed in details in \[6\]. The first one consists in considering two lattices \( \langle L_1, \leq_1 \rangle \) and \( \langle L_2, \leq_2 \rangle \). We can see \( L_1 \) as the set of values used for representing the degree of belief (evidence, confidence, etc.) of an information and \( L_2 \) as the set of values used for representing the degree of doubt (counter-evidence, lack of confidence, etc.) of the information.

We define the structure \( L_1 \odot L_2 \) to be the structure \( \langle L_1 \times L_2, \leq_t, \leq_k \rangle \) where:

- \( \langle x, y \rangle \leq_t \langle z, w \rangle \) iff \( x \leq z \) and \( w \leq y \),
- \( \langle x, y \rangle \land \langle z, w \rangle = \langle \min(x, z), \max(y, w) \rangle \), and
• \((x, y) \leq_k (z, w)\) iff \(x \leq z\) and \(y \leq w\).

\((x, y) \otimes (z, w) = (\min(x, z), \min(y, w))\).

\(L_1 \otimes L_2\) is a bilattice satisfying the interlacing conditions; it is a complete bilattice satisfying the infinitary interlacing conditions if \(L_1\) and \(L_2\) are complete; it is infinitely distributive if \(L_1\) and \(L_2\) are complete and infinitely distributive. Moreover, if \(L = L_1 = L_2\), then a negation can be defined by \(\neg (x, y) = (y, x)\).

The following example illustrates possible uses of such a bilattice.

**Example 2** Suppose that we have two information sources: two veterinaries \(v_1\) and \(v_2\), and that we want to know the answer to the query: Is Marguerite a crazy cow?

If \(v_1\) asserts that the probability she is mad is 70\%, and \(v_2\) asserts that the probability she is not mad is 40\%, then this knowledge can be represented by assigning to the atom \(\text{Mad(Marguerite)}\) the logical value \((0.7, 0.4) \in [0; 1] \times [0; 1]\).

Such values could also be useful when each source can only answer by true or false, but is associated to a specific degree of reliability. The value \((0.7, 0.4)\) could then represent the fact that \(v_1\) asserts that she is mad whereas \(v_2\) asserts that she is not mad, but that we are more confident in diagnostics of \(v_1\) than in those of \(v_2\). That difference of reliability or confidence being represented by the assignation of different degrees of reliability to information sources, in our example, \(v_1\) would be supposed reliable for 70\% and \(v_2\) for 40\%.

The second way of constructing a bilattice consists in interpreting values as approximations of exact values. Suppose we have a lattice \(\langle L, \leq_L \rangle\) of truth values. An approximation of a truth value can be seen as an interval \([a, b] = \{ x \mid a \leq_L x \leq_L b \}\) containing that value. We can provide to the set of intervals a structure of bilattice \(\langle O(L), \leq_t, \leq_k \rangle\) such that, for \([a, b], [c, d] \in O(L)\):

• \([a, b] \leq_k [c, d]\) if \(a \leq_L c\) and \(d \leq_L b\), and

• \([a, b] \leq_t [c, d]\) if \(a \leq_L c\) and \(b \leq_L d\).

The intuition is that knowledge increases if the interval become shorter and truth increases if the interval contains greater values.
By abuse of notation we will sometimes talk about the bilattice $\mathcal{B}$ when the orders are irrelevant or understood from the context. From now on, we assume that $\mathcal{B}$ is an infinitely distributive bilattice that satisfy the infinitary interlacing conditions and has a negation unless explicitly stated otherwise.

### 2.2.3 Fitting programs

Conventional logic programming has the set $\{F, T\}$ as its intended space of truth values but since not every query may produce an answer partial models are often allowed (i.e. $U$ is added). If we want to deal with inconsistency as well then $O$ must be added. Thus Fitting asserts that $\text{FOUR}$ can be thought as the “home” of ordinary logic programming and extends the notion of logic program, as follows:

**Definition 4 (Fitting program)**

- A formula is an expression built up from literals and elements of $\mathcal{B}$, using $\land, \lor, \otimes, \oplus, \exists, \forall$.  

- A clause is of the form $P(x_1, ..., x_n) \leftarrow \phi(x_1, ..., x_n)$, where the atomic formula $P(x_1, ..., x_n)$ is the head, and the formula $\phi(x_1, ..., x_n)$ is the body. It is assumed that the free variables of the body are among $x_1, ..., x_n$.

- A program is a finite set of clauses with no predicate letter appearing in the head of more than one clause (this apparent restriction causes no loss of generality [6]).

We shall represent a Fitting program as a pair $(F, R)$ where $F$ is a function from the Herbrand base into $\mathcal{B}$ and $R$ a set of clauses. This is possible because every fact can be seen as a rule of the form $A \leftarrow v$, where $A$ is an atom and $v$ is a value in $\mathcal{B}$.

A Datalog program with negation can be seen as a Fitting program whose underlying truth-value space is the subset $\{F, T, U\}$ of $\mathcal{B}$ and which does not involve $\otimes, \oplus, \forall, U, O, F$.

### 3 Hypothesis Testing

In the remaining of this paper, in order to simplify the presentation, we assume that all Fitting programs are instantiated programs. Moreover, we
use the term “program” to mean “Fitting program”, unless explicitly stated otherwise.

3.1 Interpretations

First, we introduce some terminology and notation that we shall use throughout the paper. Given a program $P$, call interpretation of $P$ any function $I$ over the Herbrand base $\mathcal{HB}_P$ such that, for every atom $A$ of $\mathcal{HB}_P$, $I(A)$ is a value from $B$.

Two interpretations $I$ and $J$ are compatible if, for every ground atom $A$, $(I(A) \neq \mathcal{U} \text{ and } J(A) \neq \mathcal{U}) \Rightarrow I(A) = J(A)$.

An interpretation $I$ is a part of an interpretation $J$, denoted $I \leq J$, if $I(A) \neq \mathcal{U}$ implies $I(A) = J(A)$, for every ground atom $A$. Clearly, the part-of relation just defined is a partial ordering on the set $\mathcal{V}(\mathcal{B})$ of all interpretations over $\mathcal{B}$. Given an interpretation $I$, we denote by $\text{def}(I)$ the set of all ground atoms $A$ such that $I(A) \neq \mathcal{U}$. Moreover, if $S$ is any set of ground atoms, we define the restriction of $I$ to $S$, denoted by $I / S$ as follows: for all $A \in \mathcal{HB}_P$,

$$I / S(A) = \begin{cases} I(A) & \text{if } A \in S, \\ \mathcal{U} & \text{otherwise.} \end{cases}$$

We can extend the two orderings of $\mathcal{B}$ (i.e. the truth ordering and the knowledge ordering) to the set $\mathcal{V}(\mathcal{B})$ as follows: Let $I_1$ and $I_2$ be in $\mathcal{V}(\mathcal{B})$, then

- $I_1 \leq_k I_2$ if and only if $I_1(A) \leq_k I_2(A)$ for all ground atoms $A$;

- $I_1 \leq_t I_2$ if and only if $I_1(A) \leq_t I_2(A)$ for all ground atoms $A$.

Under these two orderings $\mathcal{V}(\mathcal{B})$ becomes a bilattice, and we have $(I \land J)(A) = I(A) \land J(A)$, and similarly for the other operators. $\mathcal{V}(\mathcal{B})$ is distributive, satisfies the interlacing conditions and has a negation and a conflation.

The operations of $\mathcal{B}$ can be extended naturaly to $\mathcal{V}(\mathcal{B})$ in the following way: $I \oplus J(A) = I(A) \oplus J(A)$ and similarly for the other operations.

The actions of interpretations can be extended from atoms to formulas as follows:

- $I(X \land Y) = I(X) \land I(Y)$, and similarly for the other operators,
• $I(\exists x \phi(x)) = \forall_{t=\text{closedterm}} I(\phi(t))$, and

• $I(\forall x \phi(x)) = \land_{t=\text{closedterm}} I(\phi(t))$.

If $B$ is a closed formula then we say that $B$ evaluates to the logical value $\alpha$, with respect to an interpretation $I$, denoted by $B \equiv \alpha$ w.r.t. $I$ or by $B \equiv_I \alpha$, if $J(B) = \alpha$ for any interpretation $J$ such that $I \leq J$ (i.e. if the value of $B$ is equal to $\alpha$ with respect to the defined atoms of $I$ whatever the values of underdefined atoms could be). There are formulas $B$ in which underdefined atoms do not matter for the logical value that can be associated with $B$. For example let us take $B = A \lor C$ and let the interpretation $I$ be defined by $I(A) = \mathcal{U}$, $I(C) = \mathcal{T}$; then no matter how $A$ is interpreted $B$ is evaluated to $\mathcal{T}$, that is, $B \equiv_I \mathcal{T}$.

Given an interpretation $I$, let $I_\mathcal{O}$ be the interpretation defined by: if $I(A) \neq \mathcal{U}$ then $I_\mathcal{O}(A) = I(A)$ else $I_\mathcal{O}(A) = \mathcal{O}$, for every atom $A$. Using the interlacing conditions, we have the following lemma that provides a method of testing whether $B \equiv_I \alpha$, based on the interpretation $I_\mathcal{O}$.

**Lemma 1** Given a closed formula $B$, $B \equiv_I \alpha$ iff $I(B) = \alpha$ and $I_\mathcal{O}(B) = \alpha$.

### 3.2 The Support of a Hypothesis

Given a program $P = \langle F, R \rangle$, we consider two ways of inferring information from $P$. First by activating the rules of $R$ in order to derive new facts from those of $F$, through an immediate consequence operator $T$. Second, by a kind of default reasoning based on a given hypothesis.

**Definition 5 (immediate consequence operator $T$)** The immediate consequence operator $T$ takes as input the facts of $F$ and returns an interpretation $T(F)$, defined as follows: for all ground atoms $A$,

- if there is a rule $A \leftarrow B \in R$, then $T_R(F)(A) = \alpha$ if $B \equiv_F \alpha$,

- $T_R(F)(A) = \mathcal{U}$, otherwise.

What we call a hypothesis is actually just an interpretation $H$. However, we use the term “hypothesis” to stress the fact that the values assigned by $H$ to the atoms of the Herbrand base are assumed values - and not values that have been computed using the facts and rules of the program. As such, a hypothesis $H$ must be tested against the “sure” knowledge provided by $P$. 

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The test consists of “adding” $H$ to $F$, then activating the rules of $\mathcal{P}$ (using $T$) to derive an interpretation $H'$. If $H \leq H'$, then the hypothesis $H$ is a sound one, i.e. the values defined by $H$ are not in contradiction with those defined by $\mathcal{P}$. Hence the following definition:

**Definition 6 (Sound Hypothesis)** Let $\mathcal{P} = \langle F, R \rangle$ be a program and $H$ a hypothesis. $H$ is sound w.r.t. $\mathcal{P}$ if

- $F$ and $H$ are compatible, and
- $H_{/\mathrm{Head}(\mathcal{P})} \leq T(F \oplus H)$, where $\mathrm{Head}(\mathcal{P}) = \{A \mid \exists A \leftarrow B \in \mathcal{P}\}$.

We use the restriction of $H$ to $\mathrm{Head}(\mathcal{P})$ before making the comparison with $T(F \oplus H)$ because all atoms which are not head of any rule of $\mathcal{P}$ will be assigned to the value *Underdefined* by $T(F \oplus H)$. Then $H$ and $T(F \oplus H)$ are compatible on these atoms.

The following example illustrates the definition of sound hypothesis with respect to a logic program.

**Example 3** We consider the program $\mathcal{P} = \langle F, R \rangle$ defined by:

$$
F = [\begin{array}{l}
\text{witness(Jean)} \quad T \\
\end{array}]
$$

$$
R = \begin{cases}
\text{suspect}(X) & \leftarrow \text{motive}(X) \lor \text{witness}(X) \\
\text{innocent}(X) & \leftarrow \exists Y (\text{alibi}(X, Y) \land \neg \text{friends}(X, Y)) \\
\text{friends}(X, Y) & \leftarrow \text{friends}(Y, X) \lor (\text{friends}(X, Z) \land \text{friends}(Z, Y)) \\
\text{charge}(X) & \leftarrow \text{suspect}(X) \oplus \neg \text{innocent}(X)
\end{cases}
$$

Let $H$ be the following hypothesis:

$$
H = [\begin{array}{l}
\text{witness(Jean)} \quad F \\
\text{motive(Jean)} \quad F \\
\text{suspect(Jean)} \quad F \\
\text{innocent(Jean)} \quad T
\end{array}]
$$

\[2\text{That notation means that the only atom which is assigned to a logical value different from } U \text{ in } F \text{ is witness(Jean) and that its value is } T.\]
We can easily note that \( H \) is not sound with respect to \( P \). The atom \( \text{witness(Jean)} \) is defined in \( H \) and in \( F \), but with different values, so \( H \) and \( F \) are not compatible.

The maximal part of \( H \) that is compatible with \( F \) is
\[
H' = \begin{bmatrix}
\text{motive(Jean)} & \mathcal{F} \\
\text{suspect(Jean)} & \mathcal{F} \\
\text{innocent(Jean)} & \mathcal{T}
\end{bmatrix}
\]

\( F \) and \( H' \) are compatible, so it is possible to collect the knowledge defined by these two interpretation in a new one without creating conflicts or inconsistencies.

\[
F \oplus H' = \begin{bmatrix}
\text{witness(Jean)} & \mathcal{T} \\
\text{motive(Jean)} & \mathcal{F} \\
\text{suspect(Jean)} & \mathcal{F} \\
\text{innocent(Jean)} & \mathcal{T}
\end{bmatrix}
\]

Then we activate the rules of \( R \) on the interpretation \( F \oplus H' \):
\[
T_R(F \oplus H') = \begin{bmatrix}
\text{witness(Jean)} & \mathcal{T} \\
\text{motive(Jean)} & \mathcal{F} \\
\text{suspect(Jean)} & \mathcal{T} \\
\text{charge(Jean)} & \mathcal{F}
\end{bmatrix}
\]

We observe that \( H' \) is not sound with respect to \( P \) because \( H' \) is not a part of \( T_R(F \oplus H) \) and is in contradiction with the derived knowledge.

Even if a hypothesis \( H \) is not sound w.r.t. \( P \), it may be that some part of \( H \) is sound w.r.t. \( P \). Of course, we are interested to know what is the maximal part of \( H \) that is sound w.r.t. \( P \). We shall call this maximal part the “support” of \( H \). To see that the maximal part of \( H \) is unique (and thus that the support is a well-defined concept), we give the following lemma:

**Lemma 2** If \( H_1 \) and \( H_2 \) are two sound parts of \( H \) w.r.t. \( P \), then \( H_1 \oplus H_2 \) is sound w.r.t. \( P \).

**Proof.** \( H_1 \) and \( H_2 \) are both restrictions of \( H \) so they are compatible. Moreover, \( H_1 \) and \( H_2 \) are two sound parts of \( H \) with respect to \( P \), so \( H_1 \) and
$H_2$ are both compatible with $F$. It follows that $H_1 \oplus H_2$ is compatible with $F$.

We also have $H_1/\text{Heads}(\mathcal{R}) \leq T(F \oplus H_1)$, i.e., for all atom $A$ head of a rule $A \leftarrow B \in \mathcal{P}$, if $H_1(A) \neq \mathcal{U}$ then $H_1(A) = T(F \oplus H_1)(A)$. The same property is verified by $H_2$. If $H_1 \oplus H_2(A) \neq \mathcal{U}$ then we have:

- either $H_1(A) \neq \mathcal{U}$ and $H_2(A) = \mathcal{U}$,
- either $H_2(A) \neq \mathcal{U}$ and $H_1(A) = \mathcal{U}$,
- either $H_1(A) = H_2(A) \neq \mathcal{U}$.

In the first case, we have $T(F \oplus H_1)(A) = H_1(A) = H_1 \oplus H_2(A)$, i.e. $B \equiv H_1 \oplus H_2(A)$ p.r. $F \oplus H_1$, so $B \equiv H_1 \oplus H_2(A)$ w.r.t. $F \oplus H_1 \oplus H_2$. We have the same result in the two other cases, so

$$(H_1 \oplus H_2)/\text{Heads}(\mathcal{R}) \leq T(F \oplus H_1 \oplus H_2).$$

Thus the maximal sound part of $H$ is defined by $\bigoplus\{H' \mid H' \leq H \text{ and } H' \text{ is sound w.r.t. } \mathcal{P}\}$.

**Definition 7 (Support)** Let $\mathcal{P}$ be a program and $H$ a hypothesis. The support of $H$ w.r.t. $\mathcal{P}$, denoted $s^H_{\mathcal{P}}$, is the maximal sound part of $H$ w.r.t. $\mathcal{P}$ (where maximality is understood w.r.t. the part-of ordering $\leq$).

**Example 4** Let $\mathcal{P}$ be the program and $H$ the hypothesis defined in the example 3, then the support of $H$ with respect to $\mathcal{P}$ is:

$$s^H_{\mathcal{P}} = \left[ \text{motive(Jean)} \ F \right]$$

We can remark that the support of a hypothesis with respect to a program $\mathcal{P} = \langle R, F \rangle$ is compatible with the interpretation obtained by activating the rules of $R$ on the facts of $F$.

**Lemma 3** Let $\mathcal{P} = \langle R, F \rangle$ be a logic program and $H$ a hypothesis. $T_R(F)$ and $s^H_{\mathcal{P}}$ are compatible.
Proof. For all atom \( A \), if \( A \) is not the head of any rule of \( R \), then \( T_R(F)(A) \leq s_H^p(A) \). If there is in \( R \) a rule forme \( A \leftarrow B \), then:

- if \( s_H^p(A) = H(A) \neq \mathcal{U} \), then \( B \equiv \alpha \text{ w.r.t. } F \oplus s_H^p \), and if \( T_R(F)(A) \neq \mathcal{U} \), then \( T_R(F)(A) = H(A) \);
- if \( T_R(F)(A) = \alpha \neq \mathcal{U} \), then \( B \equiv \alpha \text{ w.r.t. } F \) and \( B \equiv \alpha \text{ w.r.t. } F \oplus s_H^p \), and if \( s_H^p(A) \neq \mathcal{U} \), then \( s_H^p(A) = \alpha \).

\( \Box \)

We now give an algorithm for computing the support \( s_H^p \) of a hypothesis \( H \) w.r.t. a program \( \mathcal{P} \).

Consider the following sequence \( \langle PF_i \rangle \), \( i \geq 0 \):

- \( PF_0 = \emptyset \);
- \( PF_i = \{ A \mid A \leftarrow B \in \mathcal{P} \text{ and } B \neq H(A) \text{ w.r.t. } F \oplus H_{/(H_{B\cap IF(F,H)}\backslash PF_{i-1})} \} \) for all \( i \geq 0 \),
  where \( IF(F,H) \) is the set of facts that are incompatible with \( H \), defined by \( IF(F,H) = \{ A \mid (F(A) \neq \mathcal{U}) \land (H(A) \neq \mathcal{U}) \land (F(A) \neq H(A)) \} \).

The intuition here is that we want to evaluate step by step the atoms that could potentially have a logical value different than their values in \( H \).

We have the following results:

**Proposition 1** The sequence \( \langle PF_i \rangle \), \( i \geq 0 \) is increasing with respect to set inclusion and it has a limit reached in a finite number of steps. This limit is denoted \( PF \).

Proof. We show by recurrence that for all \( n \), \( PF_{n-1} \subseteq PF_n \).

\( PF_0 = \emptyset \) so the property is satisfied for \( n = 1 \).

Suppose that \( PF_{i-1} \subseteq PF_i \). Thus we have

\[
H_{/(H_{B\cap IF(F,H)}\backslash PF_{i-1})} \leq H_{/(H_{B\cap IF(F,H)}\backslash PF_{i-1})}. 
\]

For all atom \( A \), if there is in \( \mathcal{P} \) a rule \( A \leftarrow B \), then if \( B \neq H(A) \) with respect to \( F \oplus H_{/(H_{B\cap IF(F,H)}\backslash PF_{i-1})} \), then \( B \neq H(A) \) with respect to \( F \oplus H_{/(H_{B\cap IF(F,H)}\backslash PF_{i-1})} \), and consequently, \( PF_i = \{ A \mid A \leftarrow B \in \mathcal{P} \) and
\[ B \not\equiv H(A) \text{ with respect to } F \oplus H_{HBF} \backslash IP_{F,H} \backslash PF_{i-1} \] \subseteq \{ A \mid A \leftarrow B \in P \}
and
\[ B \not\equiv H(A) \text{ with respect to } F \oplus H_{HBF} \backslash IP_{F,H} \backslash PF_{i} \} = PF_{i+1}. \]

If an atom of the Herbrand base is not in \( PF \), then it means that, with respect to \( P \), there is no way of inferring for that atom a logical value different than its value in \( H_B \).

**Theorem 1** Let \( P \) a logic program and \( H \) a hypothesis, we have

\[ s^H_P = H_{HBF} \backslash IP_{F,H} \backslash PF \]

**Proof.** We note \( X = H_{HBF} \backslash IP_{F,H} \backslash PF \). Firstly, we show that \( X \) is a sound part of \( H \) with respect to \( P \). By definition, \( X \) and \( F \) are compatible. Let \( A \) be an atom such that there exists in \( R \) a rule \( A \leftarrow B \) and such that \( X(A) = H(A) \not\equiv U \). Then \( A \not\in PF \) and \( B \equiv H(A) \) w.r.t. \( F \oplus X \), so \( X \) is sound.

Secondly, we prove that \( X \) is the maximal sound part of \( H \) with respect to \( P \). Let \( Y \) be a sound part of \( H \). We show by recurrence that for all atom \( A \), if \( Y(A) = H(A) \not\equiv U \), then \( A \not\in PF_n \).

If \( A \) is not the head of any rule in \( R \), then \( A \not\in PF_i \), for all \( i \). \( PF_0 = \emptyset \) so \( A \not\in PF_0 \). Suppose the property satisfied for \( n = i - 1 \). We have, for all atom \( A \), if \( Y(A) = H(A) \not\equiv U \), then \( A \not\in PF_{i-1} \) and \( Y \leq H_{HBF} \backslash IP_{F,H} \backslash PF_{i-1} \).

If there is a rule \( A \leftarrow B \) in \( R \) and \( Y(A) = H(A) \not\equiv U \), then
\[ B \equiv H(A) \text{ w.r.t. } F \oplus Y \text{ because } Y \text{ is sound, and it follows that} \]
\[ B \equiv H(A) \text{ w.r.t. } F \oplus H_{HBF} \backslash IP_{F,H} \backslash PF_{i-1} \]. We can conclude that \( A \not\in PF_i \).

We have shown that for all atom \( A \), if \( Y(A) = H(A) \not\equiv U \), then \( A \not\in PF \)
and consequently, \( X(A) = H(A) \).

For all sound part \( Y \) of \( H \), we have \( Y \leq X \).

\[ \square \]

### 4 Hypothesis Founded Semantics

As we explained earlier, given a program \( P = \langle F, R \rangle \), we derive information in two ways: by activating the rules (i.e. by applying the immediate consequence operator \( T \)) and by making a hypothesis \( H \) and computing its
support $s^H_{\mathcal{P}}$ w.r.t. $\mathcal{P}$. In the whole, the information that we derive comes from $T(F) \oplus s^H_{\mathcal{P}}$.

Now, roughly speaking, the semantics that we would like to associate with a program $\mathcal{P}$ is the maximum of information that we can derive from $\mathcal{P}$ under a sound hypothesis $s^H_{\mathcal{P}}$ but without any other information. To implement this idea we proceed as follows:

1. As we don’t want any extra information (other than $\mathcal{P}$ and $s^H_{\mathcal{P}}$), we use the everywhere undefined interpretation, call it $I_U$.

2. In order to actually derive the maximum of information from $\mathcal{P}$ and $I_U$, we collect together the knowledge inferred by activating the rules of $R$, i.e. by applying the operator $T_R$, and as much of assumed knowledge as possible, i.e. the support of $H$ w.r.t. $\mathcal{P}$.

**Proposition 2** The sequence $\langle F_n \rangle$, $n \geq 0$ defined by:

- $F_0 = F$, and
- $F_{n+1} = T_R(F_n) \oplus s^H_{(F_n,R)}$,

is increasing with respect to $\leq$ and has a limit denoted by $\text{sem}^H_{\mathcal{P}}$.

**Proof.** It is straightforward that $T_R$ is monotonic with respect to $\leq$, so for all $n$, $T_R(F_n) \leq T_R(F_{n+1})$.

We prove by recurrence that for all $n$, $s^H_{(F_n,R)} \leq s^H_{(F_{n+1},R)}$.

For $n = 0$, if $s^H_{(F,R)}(A) = \alpha \neq U$ then :

- if $A$ is not the head of any rule in $R$, then $s^H_{(F_1,R)}(A) = \alpha$ ;
- if there is a rule $A \leftarrow B$ in $R$, then $B \equiv \alpha$ w.r.t. $s^H_{(F,R)}$, so $B \equiv \alpha$ w.r.t. $s^H_{(F,R)} \oplus T_R(F)$ and $B \equiv \alpha$ w.r.t. $F_1$.

It follows that $B \equiv \alpha$ w.r.t. $F_1 \oplus s^H_{(F_1,R)}$. Consequently, $s^H_{(F_1,R)} = \alpha$.

The property is true for $n = 0$.

Now, suppose that $s^H_{(F_{n-1},R)} \leq s^H_{(F_n,R)}$. If $s^H_{(F_n,R)}(A) = \alpha \neq U$, then:

- if $A$ is not the head of any rule in $R$, then $s^H_{(F_n,R)}(A) = \alpha$ ;
• if there is a rule $A \leftarrow B$ in $R$, then $B \equiv \alpha$ w.r.t. $F_{n-1} \oplus s^H_{(F_{n-1},R)}$.

For all $n$, $F_n \oplus s^H_{(F_n,R)} \leq T_R(F_n) \oplus s^H_{(F_n,R)}$. Indeed, if $F_n \oplus s^H_{(F_n,R)}(A) = \alpha \neq \emptyset$, then:

- $T_R(F_{n-1})(A) = \alpha$, and it follows that $T_R(F_n)(A) = \alpha$, because $T_R(F_{n-1}) \leq T_R(F_n)$; or
- $s^H_{(F_{n-1},R)}(A) = \alpha$, and it follows that $s^H_{(F_n,R)}(A) = \alpha$, because $s^H_{(F_{n-1},R)} \leq s^H_{(F_n,R)}$; or
- $s^H_{(F_n,R)}(A) = \alpha$.

Consequently, $B \equiv \alpha$ w.r.t. $T_R(F_{n-1}) \oplus s^H_{(F_{n-1},R)}$, i.e. $B \equiv \alpha$ w.r.t. $F_n$, thus $B \equiv \alpha$ w.r.t. $F_n \oplus s^H_{(F_n,R)}$. Finally $s^H_{(F_n,R)}(A) = \alpha$.

So we have for all $n$, $T_R(F_n) \leq T_R(F_{n+1})$ and $s^H_{(F_n,R)} \leq s^H_{(F_{n+1},R)}$, following the lemma \[3\] we can conclude that for all $n$, $F_n \leq F_{n+1}$.

\[ \square \]

**Proposition 3** The interpretation $\text{sem}^H_P$ is a model of $\mathcal{P}$.

**PROOF.** $\text{sem}^H_P = T_R(\text{sem}^H_P) \oplus s^H_{(\text{sem}^H_P,R)}$. But $\text{sem}^H_P$ is a sound hypothesis, so, by definition, we have

$$s^H_{(\text{sem}^H_P,R)} / \text{Heads}(R) \leq T_R(\text{sem}^H_P \oplus s^H_{(\text{sem}^H_P,R)}).$$

It follows that

$$s^H_{(\text{sem}^H_P,R)} / \text{Têtes}(R) \leq T_R(T_R(\text{sem}^H_P) \oplus s^H_{(\text{sem}^H_P,R)}),$$

i.e.

$$s^H_{(\text{sem}^H_P,R)} / \text{Têtes}(R) \leq T_R(\text{sem}^H_P).$$

Thus, for all rule $A \leftarrow B \in R$, $\text{sem}^H_P(A) = T_R(\text{sem}^H_P)(A) = \alpha$ if $B \equiv \alpha$ with respect to $\text{sem}^H_P$, so $\text{sem}^H_P$ is a model of $\mathcal{P}$.

\[ \square \]

This justifies the following definition of semantics for $\mathcal{P}$.  

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Definition 8 (Hypothesis founded semantics of \( \mathcal{P} \)) The interpretation \( \text{sem}^H_{\mathcal{P}} \) is defined to be the semantics of \( \mathcal{P} \) w.r.t. \( H \) or the \( H \)-founded semantics of \( \mathcal{P} \).

Following this definition, any given program \( \mathcal{P} \) can be associated with different semantics, one for each possible hypothesis \( H \). Theorem 2 below asserts that this approach extends the usual semantics of Datalog programs with negation to a broader class of programs, namely the Fitting programs.

Two remarks are in order here before stating Theorem 2. First, if we restrict our attention to three values only, i.e. \( \mathcal{F}, \mathcal{T} \) and \( \mathcal{U} \), then our definition of interpretation is equivalent to the one used by Van Gelder et al. [20], in the following sense: given an interpretation \( I \) following our definition, the set \( \{ A \mid I(A) = \mathcal{T} \} \cup \{ \neg A \mid I(A) = \mathcal{F} \} \) is a partial interpretation following [20]; conversely, given a partial interpretation \( J \) following [20], the function \( I \) defined by: \( I(A) = \mathcal{T} \) if \( A \in J \), \( I(A) = \mathcal{F} \) if \( \neg A \in J \), and \( I(A) = \mathcal{U} \) otherwise, is an interpretation in our sense.

Second, if we restrict our attention to Datalog programs with negation (recall that the class of Fitting programs strictly contains the Datalog programs with negation) then the concept of sound interpretation for the everywhere false hypothesis reduces to that of unfounded set of Van Gelder et al. [20]. The difference is that the definition in [20] has rather a syntactic flavor, while ours has a semantic flavor. Moreover, our definition not only extends the concept of unfounded set to multi-valued logic, but also generalizes its definition to any given hypothesis \( H \) (not just the everywhere false hypothesis).

Theorem 2 Let \( \mathcal{P} \) be a Datalog programs with negation.

1. If \( H_\mathcal{F} \) is the everywhere false hypothesis, then \( \text{sem}^{H_\mathcal{F}}_{\mathcal{P}} \) coincides with the well-founded semantics of \( \mathcal{P} \);

2. If \( H_\mathcal{U} \) is the everywhere underdefined hypothesis, then \( \text{sem}^{H_\mathcal{U}}_{\mathcal{P}} \) coincides with the Kripke-Kleene semantics of \( \mathcal{P} \).

Proof. A Datalog programs with negation can be seen, in our approach, as a set of rules of the form

\[
A \leftarrow (L_{1,1} \land ... \land L_{1,n}) \lor ... \lor (L_{i,1} \land ... \land L_{i,m}),
\]

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where the $L_{p,q}$ are litterals.

First part. Let $wfs(\mathcal{P})$ be the well-founded semantics of $\mathcal{P}$. We use in this part the definition of $wfs(\mathcal{P})$ given in section 2.1.1, considering all the interpretations as functions from $\mathcal{HB}_\mathcal{P}$ in $\{ \mathcal{F}, \mathcal{U}, \mathcal{T} \}$.

Firstly, we show that $wfs(\mathcal{P}) = T_R(wfs(\mathcal{P})) \oplus s^{H_F}_{(wfs(\mathcal{P}), R)}$.

We know that $wfs(\mathcal{P})(A) = T$ if and only if there is in $\mathcal{P}$ a rule

$$A \leftarrow (L_{1,1} \land ... \land L_{1,n}) \lor ... \lor (L_{i,1} \land ... \land L_{i,m}),$$

such that there is $(L_{p,1} \land ... \land L_{p,q}) \in \{(L_{1,1} \land ... \land L_{1,n}); ...; (L_{i,1} \land ... \land L_{i,m})\}$ such that $wfs(\mathcal{P})(L_{p,1}) = ... = wfs(\mathcal{P})(L_{p,q}) = T$, i.e. if and only if $T_R(wfs(\mathcal{P}))(A) = T$.

We know that $wfs(\mathcal{P})(A) = \mathcal{F}$ if and only if:

- either $A$ is not the head of any rule of $R$, and then $s^{H_F}_{(wfs(\mathcal{P}), R)}(A) = \mathcal{F}$;
- either there is in $\mathcal{P}$ a rule $A \leftarrow (L_{1,1} \land ... \land L_{1,n}) \lor ... \lor (L_{i,1} \land ... \land L_{i,m})$, such that for all $(L_{p,1} \land ... \land L_{p,q}) \in \{(L_{1,1} \land ... \land L_{1,n}); ...; (L_{i,1} \land ... \land L_{i,m})\}$ there is $L_{p,k} \in \{L_{p,1}, ..., L_{p,q}\}$ such that $wfs(\mathcal{P})(L_{p,k}) = \mathcal{F}$, i.e. if and only if $T_R(wfs(\mathcal{P}))(A) = \mathcal{F}$.

Consequently, $wfs(\mathcal{P}) = T_R(wfs(\mathcal{P})) \oplus s^{H_F}_{(wfs(\mathcal{P}), R)}$. It follows that

$$sem_{\mathcal{P}}^{H_F} \leq wfs(\mathcal{P}).$$

We prove now than $sem_{\mathcal{P}}^{H_F} \leq T_\mathcal{P}^e(sem_{\mathcal{P}}^{H_F}) \cup \mathcal{U}_\mathcal{P}(sem_{\mathcal{P}}^{H_F})$.

We know that $sem_{\mathcal{P}}^{H_F}(A) = T$ if and only if there is in $\mathcal{P}$ a rule

$$A \leftarrow (L_{1,1} \land ... \land L_{1,n}) \lor ... \lor (L_{i,1} \land ... \land L_{i,m}),$$

such that $((L_{1,1} \land ... \land L_{1,n}) \lor ... \lor (L_{i,1} \land ... \land L_{i,m})) \equiv T \ w.r.t. \ sem_{\mathcal{P}}^{H_F}$, i.e. such that there is $(L_{p,1} \land ... \land L_{p,q}) \in \{(L_{1,1} \land ... \land L_{1,n}); ...; (L_{i,1} \land ... \land L_{i,m})\}$ such that $sem_{\mathcal{P}}^{H_F}(L_{p,1}) = ... = sem_{\mathcal{P}}^{H_F}(L_{p,q}) = T$, i.e. if and only if $T_\mathcal{P}^e(sem_{\mathcal{P}}^{H_F})(A) =$
We know that \( \text{sem}^H_{\mathcal{P}}(A) = \mathcal{F} \) if and only if:

- either \( A \) is not the head of any rule of \( R \), and then \( \mathcal{U}_P(\text{sem}^H_{\mathcal{P}})(A) = \mathcal{F} \);

- either there is in \( \mathcal{P} \) a rule \( A \leftarrow (L_{1,1} \land ... \land L_{1,n}) \lor ... \lor (L_{i,1} \land ... \land L_{i,m}) \), such that \((L_{1,1} \land ... \land L_{1,n}) \lor ... \lor (L_{i,1} \land ... \land L_{i,m})) \equiv \mathcal{F} \) w.r.t. \( \text{sem}^H_{\mathcal{P}} \), i.e. for all \((L_{p,1} \land ... \land L_{p,q}) \in \{(L_{1,1} \land ... \land L_{1,n});...;(L_{i,1} \land ... \land L_{i,m})\} \) there is \( L_{p,k} \in \{L_{p,1}, ..., L_{p,q}\} \) such that \( \text{sem}^H_{\mathcal{P}}(L_{p,k}) \equiv \mathcal{F} \), i.e. if and only if \( \mathcal{U}_P(\text{sem}^H_{\mathcal{P}})(A) = \mathcal{F} \).

Consequently, \( \text{sem}^H_{\mathcal{P}} \leq T^\mathcal{F}_P (\text{sem}^H_{\mathcal{P}}) \cup \mathcal{U}_P(\text{sem}^H_{\mathcal{P}}) \), thus \( \text{wfs}(\mathcal{P}) \leq \text{sem}^H_{\mathcal{P}} \).

**Second part.** Let \( I \) be an interpretation, we show that \( T_R(I) = \Phi_\mathcal{P}(I) \) where \( \Phi_\mathcal{P} \) is the Kripke-Kleene operator.

Then \( \Phi_\mathcal{P}(I)(A) = \alpha \neq \mathcal{U} \) if and only if there is in \( \mathcal{P} \) a rule \( A \leftarrow B \), where \( B \) is defined by \((L_{1,1} \land ... \land L_{1,n}) \lor ... \lor (L_{i,1} \land ... \land L_{i,m}) \), and if:

- either there is \((L_{j,1} \land ... \land L_{j,k}) \subseteq B \) such that \( L_{j,1} = ... = L_{j,k} = \mathcal{T} \);

- either for all \((L_{j,1} \land ... \land L_{j,k}) \subseteq B \), there is \( L_{j,l} \in \{L_{j,1}, ..., L_{j,k}\} \) such that \( L_{j,l} = \mathcal{F} \).

Thus \( \Phi_\mathcal{P}(I)(A) = \alpha \neq \mathcal{U} \) if and only if there is in \( \mathcal{P} \) a rule \( A \leftarrow B \) such that \( B \equiv_I \alpha \), i.e. if and only if \( T_R(I)(A) = \alpha \neq \mathcal{U} \).

As we consider the hypothesis \( H_U \), we will infer information only with \( T_R \). \( \text{sem}^H_{\mathcal{P}} \) is the least fixpoint of \( T_R \) with respect to the knowledge ordering, thus \( \text{sem}^H_{\mathcal{P}} \) coincides with the Kripke-Kleene semantics of \( \mathcal{P} \) which is the least fixpoint of \( \Phi_\mathcal{P} \) with respect to the knowledge ordering.

\[ \square \]

**5 Concluding remarks**

We have defined a formal framework for information integration based on hypothesis testing. A basic concept of this framework is the support provided by a program \( \mathcal{P} = \langle F, R \rangle \) to a hypothesis \( H \). The support of \( H \) is the maximal
part of $H$ that does not contradict the facts of $F$ or the facts derived from $F$ using the rules of $R$.

We have then used the concept of support to define hypothesis-based semantics for the class of Fitting programs, and we have given an algorithm for computing these semantics.

Finally, we have shown that our semantics extends the well-founded semantics and the Kripke-Kleene semantics to multi-valued logics with lattice structure, and also generalizes them in the following sense: if we restrict our attention to three-valued logics then for $H_F$ the everywhere false interpretation our semantics reduces to the well-founded semantics, and for $H_U$ the everywhere underdefined interpretation our semantics reduces to the Kripke-Kleene semantics.

We believe that hypothesis-based semantics can be useful not only in the context of information integration but also in the context of explanation-based systems. Indeed, assume that a given hypothesis $H$ turns out to be a part of the $H$-semantics of a program $P$. Then $P$ can be seen as an “explanation” of the hypothesis $H$. We are currently investigating several aspects of this explanation oriented viewpoint.

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