Spectrum generating algebra of the $C_\lambda$-extended oscillator and multiphoton coherent states

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Abstract

The $C_\lambda$-extended oscillator spectrum generating algebra is shown to be a $C_\lambda$-extended $(\lambda - 1)$th-degree polynomial deformation of $\text{su}(1,1)$. Its coherent states are constructed. Their statistical and squeezing properties are studied in detail. Such states include both some Barut-Girardello and the standard $\lambda$-photon coherent states as special cases.

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1 Introduction

Coherent states (CS) of the harmonic oscillator [1], as well as generalized CS associated with various algebras [2], have found considerable applications in quantum optics. The former, defined as the eigenstates of the annihilation operator $a$, have properties similar to those of the classical radiation field. The latter, on the contrary, may exhibit some nonclassical properties, such as photon antibunching [3] or sub-Poissonian photon statistics [4], and squeezing [5, 6], which have given rise to an ever-increasing interest during the last few years.

As examples of CS with nonclassical properties, we may quote the eigenstates of $a^2$ [7], which were introduced as even and odd CS or cat states [8], and are a special case of generalized CS associated with the Lie algebra $su(1,1)$ [2, 9]. We may also mention the eigenstates of $a^\lambda (\lambda > 2)$ or kitten states [10], which may be generated in $\lambda$-photon processes. Many alternative multiphoton CS have been constructed and studied (see e.g. [11]).

Recently there has also been much interest in the study of nonlinear CS, defined as the eigenstates of the annihilation operator of a deformed oscillator (or $f$-oscillator) [12, 13, 14, 15, 16]. It has indeed been shown [15] that for a particular class of nonlinearities they are useful in the description of a trapped ion and that they have strong nonclassical properties. Subsequently, even and odd nonlinear CS [17], as well as the eigenstates of an arbitrary power of the $f$-oscillator annihilation operator [18], have also been considered in connection with nonclassical effects.

The purpose of the present Letter is to construct and study the nonclassical properties of some multiphoton CS, which may be associated with the recently introduced $C_\lambda$-extended oscillator [13]. The latter, which has proved very useful in the context of supersymmetric quantum mechanics and some of its variants [19, 20], may be considered as a deformed oscillator with a $Z_\lambda$-graded Fock space. Hence, its CS will be a special case of the nonlinear CS of Ref. [18]. However, their connection with the $C_\lambda$-extended oscillator spectrum generating algebra to be determined in the first part of this Letter will endow them with some extra properties. As a consequence, they will satisfy Klauder’s minimal set of conditions for generalized CS [21], which is not the case in general for all the states of Ref. [18].
2 The $C_\lambda$-extended oscillator and its spectrum generating algebra

The $C_\lambda$-extended oscillator Hamiltonian is defined (in units wherein $\hbar \omega = 1$) by \[ H_0 = \frac{1}{2} \{ a, a^\dagger \}, \] (1)

where the creation and annihilation operators $a^\dagger, a$ satisfy the relations

$$[N, a^\dagger] = a^\dagger, \quad [a, a^\dagger] = I + \sum_{\mu=0}^{\lambda-1} \alpha_\mu P_\mu, \quad a^\dagger P_\mu = P_{\mu+1} a^\dagger, \tag{2}$$

together with their Hermitian conjugates. Here $N = N^\dagger$ is the number operator, $\alpha_\mu$ are some real parameters subject to the conditions $\sum_{\mu=0}^{\lambda-1} \alpha_\mu = 0$ and

$$\sum_{\nu=0}^{\mu-1} \alpha_\nu > -\mu, \quad \mu = 1, 2, \ldots, \lambda - 1, \tag{3}$$

and the operators $P_\mu = \lambda^{-1} \sum_{\nu=0}^{\lambda-1} \exp[2\pi i \nu (N - \mu)/\lambda]$, which are linear combinations of the operators of a cyclic group $C_\lambda$, project on the subspaces $F_\mu \equiv \{ |k\lambda + \mu\rangle | k = 0, 1, \ldots \}$ of the $\mathbb{Z}_\lambda$-graded Fock space $F = \bigoplus_{\mu=0}^{\lambda-1} F_\mu$. Throughout this paper, we use the convention $P_\mu' = P_\mu$ if $\mu' - \mu = 0 \mod \lambda$ (and similarly for other operators or parameters labelled by $\mu, \mu'$).

The operators $N, a^\dagger, a$ are related to each other through the structure function $F(N)$, which is a fundamental concept of deformed oscillators $[12, 22]$: $a^\dagger a = F(N), \ a a^\dagger = F(N + 1)$. In the present case, $F(N)$ is given by $F(N) = N + \sum_{\mu=0}^{\lambda-1} \beta_\mu P_\mu$, where $\beta_\mu \equiv \sum_{\nu=0}^{\mu-1} \alpha_\nu$ (with $\beta_0 \equiv 0$) \[19\].

The Fock space basis states $|n\rangle = |k\lambda + \mu\rangle, \ k = 0, 1, \ldots, \mu = 0, 1, \ldots, \lambda - 1,$ are given by $|n\rangle = \mathcal{N}_n^{-1/2} (a^\dagger)^n |0\rangle$, where $|0\rangle$ is a vacuum state (i.e., $a|0\rangle = 0$), and $\mathcal{N}_n$ is some normalization coefficient \[19\]. The number, creation, and annihilation operators act on $|n\rangle$ as

$$N|n\rangle = n|n\rangle, \quad a^\dagger |n\rangle = \sqrt{F(n+1)} |n+1\rangle, \quad a|n\rangle = \sqrt{F(n)} |n-1\rangle. \tag{4}$$

From the restriction (3) on the parameters, it follows that $F(\mu) = \beta_\mu + \mu > 0$, so that all the states $|n\rangle$ are well defined.
The eigenstates of $H_0$ are the states $|n\rangle = |k\lambda + \mu\rangle$ and their eigenvalues are given by $E_{k\lambda+\mu} = k\lambda + \mu + \gamma_\mu + \frac{1}{2}$, where $\gamma_\mu \equiv \frac{1}{2} (\beta_\mu + \beta_{\mu+1})$. In each $F_\mu$ subspace of $F$, the spectrum of $H_0$ is harmonic, but the $\lambda$ infinite sets of equally spaced energy levels, corresponding to $\mu = 0, 1, \ldots, \lambda - 1$, are shifted with respect to each other by some amounts depending upon the parameters $\alpha_0, \alpha_1, \ldots, \alpha_{\lambda-1}$.

For vanishing parameters $\alpha_\mu$, the $C_\lambda$-extended oscillator reduces to the standard harmonic oscillator. In such a case, $[a, a^\dagger] = I$, $\beta_\mu = \gamma_\mu = 0$, $F(N) = N$, and $E_n = n + \frac{1}{2}$. The operators

$$ J_+ = \frac{1}{2} (a^\dagger)^2, \quad J_- = \frac{1}{2} a^2, \quad J_0 = \frac{1}{2} H_0 = \frac{1}{4} \{a, a^\dagger\} \tag{5} $$

are known to generate the whole spectrum from the zero- and one-quantum states $|23\rangle$. They satisfy the commutation relations

$$ [J_0, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = -2J_0, \tag{6} $$

and the Hermiticity properties $J_0^\dagger = J_0$, $J_\pm^\dagger = J_\mp$, characteristic of the Lie algebra $su(1,1)$. The Casimir operator

$$ C = J_+ J_- - J_0 (J_0 - 1) = J_- J_+ - J_0 (J_0 + 1), \tag{7} $$

which commutes with $J_0, J_+, J_-$, has the same eigenvalue $c = 3/16$ in the two $su(1,1)$ unitary irreducible representations (unirreps) corresponding to even and odd states. The latter are distinguished by the lowest $J_0$ eigenvalue, equal to $1/4$ and $3/4$, respectively.

For nonvanishing parameters and $\lambda = 2$, the $C_2$-extended oscillator is equivalent to the Calogero-Vasiliev oscillator, which provides an algebraic formulation of the two-particle Calogero problem (see $[24]$ and references quoted therein), and an alternative description of parabosons $[25]$. In such a case, $[a, a^\dagger] = I + \alpha_0 (-1)^N$, $\alpha_0 = -\alpha_1 = \beta_1 = 2\gamma_0 = 2\gamma_1$, $F(N) = N + \alpha_0 [1 - (-1)^N]/2$, and $E_n = n + (\alpha_0 + 1)/2$. The spectrum is that of a shifted oscillator and the operators $[4]$ still form an $su(1,1)$ spectrum generating algebra $[24, 26]$. The Casimir operator $[4]$ has now distinct eigenvalues $c_\mu = (1 + \alpha_\mu) (3 - \alpha_\mu)/16$ for even ($\mu = 0$) and odd ($\mu = 1$) states, and the lowest $J_0$ eigenvalue is $(1 + \alpha_0)/4$ and $(3 + \alpha_0)/4$, respectively.
When going to \(\lambda\) values greater than two, the operators (5) are replaced by the operators

\[
J_+ = \frac{1}{\lambda} (a^\dagger)^\lambda, \quad J_- = \frac{1}{\lambda} a^\lambda, \quad J_0 = \frac{1}{\lambda} H_0 = \frac{1}{2\lambda} \{a, a^\dagger\},
\]

which connect among themselves all the equally spaced levels characterized by a given \(\mu\) value. By using (2) or (4), it is easy to show that they satisfy the commutation relations

\[
[J_0, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = f(J_0, P_\mu),
\]

where

\[
f(J_0, P_\mu) = -\frac{1}{\lambda} \left\{ \prod_{l=0}^{\lambda-2} \left( \lambda J_0 + \frac{1}{2} \sum_\mu \left( 2l + 1 + \alpha_\mu + 2 \sum_{m=1}^l \alpha_{\mu+m} \right) P_\mu \right) 
\]

\[
+ \sum_{i=1}^{\lambda-1} \left( \lambda J_0 - \frac{1}{2} \sum_\mu (1 + \alpha_\mu) P_\mu \right) \left[ \prod_{j=1}^{i-1} \left( \lambda J_0 + \frac{1}{2} \sum_\mu \left( -2j - 1 + \alpha_\mu + 2 \sum_{k=1}^{\lambda-j-1} \alpha_{\mu+k} \right) P_\mu \right) \right] 
\]

\[
\times \left[ \prod_{l=0}^{\lambda-2} \left( \lambda J_0 + \frac{1}{2} \sum_\mu \left( 2l + 1 + \alpha_\mu + 2 \sum_{m=1}^l \alpha_{\mu+m} \right) P_\mu \right) \right] \right\}
\]

(10)
is a \((\lambda - 1)\)th-degree polynomial in \(J_0\) with \(P_\mu\)-dependent coefficients, \(f(J_0, P_\mu) = \sum_{i=0}^{\lambda-1} s_i(P_\mu) J_0^i\). The definition of the spectrum generating algebra is completed by the commutation relations

\[
[J_0, P_\mu] = [J_+, P_\mu] = [J_-, P_\mu] = 0.
\]

(11)

Since the \(P_\mu\)'s are linear combinations of \(C_\lambda\) operators, we conclude that the algebra is a \(C_\lambda\)-extended polynomial deformation of \(\text{su}(1,1)\): in each \(F_\mu\) subspace, it reduces to a standard polynomial deformation of \(\text{su}(1,1)\) [27].

Its Casimir operator can be written as

\[
C = J_- J_+ + h(J_0, P_\mu) = J_+ J_- + h(J_0, P_\mu) = f(J_0, P_\mu),
\]

(12)

where

\[
h(J_0, P_\mu) = \frac{1}{\lambda^2} \left\{ - \left[ \lambda J_0 + \frac{1}{2} \sum_\mu (2\lambda - 1 - \alpha_\mu) P_\mu \right] \right.
\]

\[
\times \left[ \prod_{k=1}^{\lambda-1} \left( \lambda J_0 + \frac{1}{2} \sum_\mu \left( 2k - 1 + \alpha_\mu + 2 \sum_{l=1}^{k-1} \alpha_{\mu+l} \right) P_\mu \right) \right] 
\]

\[
+ \frac{1}{2\lambda} \sum_\mu \left[ (2\lambda - 1 - \alpha_\mu) \prod_{k=1}^{\lambda-1} \left( 2k - 1 + \alpha_\mu + 2 \sum_{l=1}^{k-1} \alpha_{\mu+l} \right) P_\mu \right] \right\}
\]

(13)
is a λ-th-degree polynomial in $J_0$ with $P_\mu$-dependent coefficients, $h(J_0, P_\mu) = \sum_{i=0}^\lambda t_i(P_\mu)J_0^i$.

Each $F_\mu$ subspace is the carrier space of a unirrep characterized by the eigenvalue

$$c_\mu = \frac{1}{\lambda^2} (2\lambda - 1 - \alpha_\mu) \prod_{k=1}^{\lambda-1} \left( 2k - 1 + \alpha_\mu + 2 \sum_{i=1}^{k-1} \alpha_{\mu+i} \right)$$

of $C$, and by the lowest eigenvalue $(\mu + \gamma_\mu + \frac{1}{2}) / \lambda$ of $J_0$.

For $\lambda = 2$, equations (9), (10), (12), and (13) reduce to equations (3) and (7), as it should be. Nonlinearities make their appearance for $\lambda = 3$, for which

$$f(J_0, P_\mu) = -9J_0^2 - J_0 \sum_\mu (\alpha_\mu + 2\alpha_{\mu+1})P_\mu - \frac{1}{12} \sum_\mu (1 + \alpha_\mu)(5 - \alpha_\mu)P_\mu,$$

$$h(J_0, P_\mu) = -J_0 \left[ 3J_0^2 + \frac{1}{2} J_0 \sum_\mu (9 + \alpha_\mu + 2\alpha_{\mu+1})P_\mu + \frac{1}{12} \sum_\mu (23 + 10\alpha_\mu + 12\alpha_{\mu+1} - \alpha_\mu^2)P_\mu \right],$$

$$c_\mu = \frac{1}{12} (1 + \alpha_\mu)(5 - \alpha_\mu)(3 + \alpha_\mu + 2\alpha_{\mu+1}),$$

and the lowest $J_0$ eigenvalues are $(1 + \alpha_0)/6$, $(3 - \alpha_1 - 2\alpha_2)/6$, $(5 - \alpha_2)/6$ for $\mu = 0, 1, 2$, respectively.

As a by-product of our analysis, it is also worth mentioning that for $\alpha_\mu = 0$, the operators (8) close a polynomial deformation of su(1,1), characterized by equations (8) and (12), where $f(J_0, P_\mu)$ and $h(J_0, P_\mu)$ are replaced by

$$f(J_0) = \frac{1}{\lambda^2} \sum_{j=0}^\lambda \left( 1 - (-1)^{\lambda-j} \right) (\lambda J_0)^j \sum_{i=j}^\lambda \binom{i}{j} \left( -\frac{1}{2} \right)^{i-j} S_\lambda^{(i)},$$

$$h(J_0) = -\frac{1}{\lambda^2} \sum_{j=1}^\lambda (\lambda J_0)^j \sum_{i=j}^\lambda (-1)^{\lambda-i} \binom{i}{j} \left( \frac{1}{2} \right)^{i-j} S_\lambda^{(i)},$$

and $\binom{i}{j}$, $S_\lambda^{(i)}$ denote a binomial coefficient and a Stirling number of the first kind, respectively. Note that the function $f(J_0)$ has a given parity, opposite to that of $\lambda$, and that the eigenvalues of $C$ do not depend on $\mu$ and are given by $c = (2\lambda - 1)!! / (\lambda^2 2^\lambda)$.

## 3 Coherent states associated with the $C_\lambda$-extended oscillator spectrum generating algebra

As mentioned in the previous section, the $C_2$-extended oscillator spectrum generating algebra is the Lie algebra su(1,1), with which one can associate various types of generalized
CS (see e.g. [2, 9]). Of special relevance in quantum optics are the Barut-Girardello CS [9],
which are the eigenstates of the generator $J_-$, defined in [4].

For the $C_\lambda$-extended oscillator with $\lambda > 2$, it therefore seems appropriate to consider as
CS the eigenstates $|z; \mu\rangle$ of the operator $J_-$, defined in (8),
\begin{equation}
J_-|z; \mu\rangle = z|z; \mu\rangle, \quad z \in \mathbb{C}, \quad \mu = 0, 1, \ldots, \lambda - 1.
\end{equation}
Here $\mu$ distinguishes between the $\lambda$ independent (and orthogonal) solutions of equation (17),
belonging to the various subspaces $F_\mu$.

The CS $|z; \mu\rangle$ may be considered as special cases of the nonlinear CS of Ref. [18], since
$J_-$ may be written in terms of the creation and annihilation operators $b^\dagger, b$ of a standard
harmonic oscillator as
\begin{equation}
J_- = (bf(N_b))^\lambda, \quad N_b \equiv b^\dagger b = N, \quad f(N_b) = \lambda^{-1/\lambda} \left[ \frac{F(N_b)}{N_b} \right]^{1/2}.
\end{equation}
More interesting for our purposes, however, is the similarity existing between $|z; \mu\rangle$ and
some CS of nonlinear algebras [18], when disregarding the discrete label $\mu$ occurring in the former.

By using (4) and (8), it is easy to construct the CS $|z; \mu\rangle$ in terms of the basis states
$|k\lambda + \mu\rangle$ of $F_\mu$. The result reads
\begin{equation}
|z; \mu\rangle = [N_\mu(|z|)]^{-1/2} \sum_{k=0}^{\infty} \frac{\left( z/\lambda^{(\lambda-2)/2} \right)^k}{k! \left( \Pi_{\nu'=1}^{\mu} (\bar{\beta}_\nu + 1) \right) \left( \Pi_{\nu'=\mu+1}^{\lambda-1} (\bar{\beta}_{\nu'} k) \right)^{1/2}} |k\lambda + \mu\rangle,
\end{equation}
where $\bar{\beta}_\mu \equiv (\beta_\mu + \mu)/\lambda$, $(a)_k$ denotes Pochhammer’s symbol, and the normalization factor
$N_\mu(|z|)$ can be expressed in terms of a generalized hypergeometric function,
\begin{equation}
N_\mu(|z|) = {}_0F_{\lambda-1} \left( \bar{\beta}_1 + 1, \ldots, \bar{\beta}_\mu + 1, \bar{\beta}_{\mu+1}, \ldots, \bar{\beta}_{\lambda-1}; y \right), \quad y \equiv |z|^2/\lambda^{\lambda-2}.
\end{equation}
There exists an alternative form of (19) in terms of the generator $J_+$ of the spectrum
generating algebra,
\begin{equation}
|z; \mu\rangle = [N_\mu(|z|)]^{-1/2} {}_0F_{\lambda-1} \left( \bar{\beta}_1 + 1, \ldots, \bar{\beta}_\mu + 1, \bar{\beta}_{\mu+1}, \ldots, \bar{\beta}_{\lambda-1}; zJ_+/\lambda^{\lambda-2} \right) |\mu\rangle.
\end{equation}
The set of CS $\{|z, \mu\rangle \mid \mu = 0, 1, \ldots, \lambda - 1\}$ satisfies a unity resolution relation, which can be written as

$$
\sum_{\mu} \int d\rho_{\mu}(z, z^*) \langle z; \mu | \langle z; \mu | = I,
$$

where $d\rho_{\mu}(z, z^*)$ is a positive measure. Making the polar decomposition $z = |z| \exp(i\phi)$ and the ansatz $\rho_{\mu}(z, z^*) = 0 F_{\lambda-1}(\bar{\beta}_1 + 1, \ldots, \bar{\beta}_\mu + 1, \bar{\beta}_{\mu+1}, \ldots, \bar{\beta}_{\lambda-1}; y) h_{\mu}(y)|d|z|d\phi$, where $h_{\mu}(y)$ is a yet unknown density on the positive half-line, we find that equation (22) reduces to the relations

$$
\int_{0}^{\infty} dy y^k h_{\mu}(y) = \frac{k!}{\pi \lambda^{\lambda-2}} \left( \prod_{\nu=1}^{\mu} (\bar{\beta}_\nu + 1) \right) \left( \prod_{\nu'=\mu+1}^{\lambda-1} (\bar{\beta}_{\nu'}) \right) (23)
$$

where $k = 0, 1, \ldots, \mu = 0, 1, \ldots, \lambda - 1$. Hence $h_{\mu}(y)$ is the inverse Mellin transform of the right-hand side of (23) and is proportional to a Meijer G-function [29]:

$$
h_{\mu}(y) = \frac{O_{\lambda-1}^0 \left( y \mid 0, \bar{\beta}_1, \ldots, \bar{\beta}_\mu, \bar{\beta}_{\mu+1} - 1, \ldots, \bar{\beta}_{\lambda-1} - 1 \right)}{\pi \lambda^{\lambda-2} \left( \prod_{\nu=1}^{\mu} \Gamma(\bar{\beta}_\nu + 1) \right) \left( \prod_{\nu'=\mu+1}^{\lambda-1} \Gamma(\bar{\beta}_{\nu'}) \right)}. (24)
$$

There are two main consequences arising from the latter result. First, we can express any CS in terms of the others corresponding to the same $\mu$ value:

$$
|z; \mu \rangle = \int d\rho_{\mu}(z', z'^*) |z'; \mu \rangle \langle z'; \mu | z; \mu \rangle. (25)
$$

The reproducing kernel $\langle z'; \mu' | z; \mu \rangle$ can be evaluated from (25) and is given by

$$
\langle z', \mu' | z; \mu \rangle = \delta_{\mu', \mu} [N_{\mu}(|z|)N_{\mu}(|z'|)]^{-1/2}
\times 0 F_{\lambda-1}(\bar{\beta}_1 + 1, \ldots, \bar{\beta}_\mu + 1, \bar{\beta}_{\mu+1}, \ldots, \bar{\beta}_{\lambda-1}; z'^* z/\lambda^{\lambda-2}). (26)
$$

Second, an arbitrary element $|\psi\rangle$ of the Fock space $\mathcal{F}$ can be written in terms of the CS:

$$
|\psi\rangle = \sum_{\mu} \int d\rho_{\mu}(z, z^*) \tilde{\psi}_{\mu}(z, z^*) |z; \mu \rangle, (27)
$$

where

$$
\tilde{\psi}_{\mu}(z, z^*) = [N_{\mu}(|z|)]^{-1/2} \sum_{k=0}^{\infty} \frac{\left(z^*/\lambda^{(\lambda-2)/2}\right)^k}{k! \left( \prod_{\nu=1}^{\mu} (\bar{\beta}_\nu + 1) \right) \left( \prod_{\nu'=\mu+1}^{\lambda-1} (\bar{\beta}_{\nu'}) \right)} 1/2 \langle k\lambda + \mu | \psi \rangle. (28)
$$

All these properties show that the CS form an overcomplete basis of $\mathcal{F}$.
From the previous results, it follows that the CS, defined in equation (17), satisfy Klauder’s minimal set of conditions for generalized CS [21]: they are normalizable, continuous in the label $z$, and they allow a resolution of unity. It is also worth mentioning that the other discrete label $\mu$ is analogous to the vector components of vector (or partially) CS [30].

We conclude the present section by presenting two important special cases of our CS. The first one corresponds to $\lambda = 2$ and $\alpha_\mu \neq 0$. In such a case, the generalized hypergeometric function $\, _0F_1$ and the Meijer G-function $G^{(0)}_{02}$ reduce to modified Bessel functions $I_\nu$ and $K_\nu$ for some appropriate $\nu$ value [31], respectively, so that

$$|z; \mu\rangle = \left(\frac{|z|^{(\alpha_0-1+2\mu)/2}}{I_{(\alpha_0-1+2\mu)/2}(2|z|)}\right)^{1/2} \sum_{k=0}^{\infty} \frac{z^k}{k! \Gamma((\alpha_0 + 1 + 2\mu + 2k)/2)} |2k + \mu\rangle,$$

$$d\rho_\mu (z, z^*) = 2\pi^{-1} I_{(\alpha_0-1+2\mu)/2}(2|z|)K_{(\alpha_0-1+2\mu)/2}(2|z|) |z| d|z| d\phi,$$

where $\mu = 0, 1$. This gives back Barut-Girardello results for the $su(1,1)$ unirrep characterized by the lowest $J_0$ eigenvalue $(\alpha_0 + 1 + 2\mu)/4$.

The second case corresponds to an arbitrary value of $\lambda$ and $\alpha_\mu = 0$. By taking into account that now $\bar{\beta}_\mu = \mu/\lambda$, the CS given in equation (19) reduce to the standard $\lambda$-photon CS [10],

$$|z; \mu\rangle = N_\mu(|z|)^{-1/2} \sum_{k=0}^{\infty} \left(\frac{\mu!}{(k\lambda + \mu)!}\right)^{1/2} (\lambda z)^k |k\lambda + \mu\rangle.$$

From (22), it follows that such CS satisfy a unity resolution relation, which, as far as the author knows, is a new result. To find the special form taken by $h_\mu(y)$ in (24), the easiest thing is to go back to equation (23) and to rewrite it as

$$\int_0^{\infty} dy \, y^k h_\mu(y) = \lambda^{2-\lambda(k+1)} (\pi\mu!)^{-1} \Gamma(k\lambda + \mu + 1),$$

by using Gauss’ multiplication formula. Then an inverse Mellin transform [29] directly leads to

$$h_\mu(y) = \lambda^{\mu-\lambda+2} (\pi\mu!)^{-1} y^{(\mu-\lambda+1)/\lambda} \exp \left(-\lambda y^{1/\lambda}\right).$$

It should be noted that the states (30) provide us with a simple example of Mittag-Leffler CS [32], since $N_\mu(|z|)$ and $|z; \mu\rangle$ can be written as

$$N_\mu(|z|) = \mu! \, E_{\lambda, \mu+1} \left(\lambda^2 |z|^2\right),$$

where $E_{a,b} (x)$ is the Mittag-Leffler function.
\[ |z; \mu \rangle = \left( \frac{\mu!}{E_{\lambda,\mu+1} (\lambda^2 |z|^2)} \right)^{1/2} E_{\lambda,\mu+1} \left( \lambda^2 z J_+ \right) |\mu \rangle, \quad (34) \]

where \( E_{\alpha,\beta}(x) \equiv \sum_{k=0}^{\infty} x^k / \Gamma(\alpha k + \beta) \) is a generalized Mittag-Leffler function [31]. The weight function (32) agrees with the principal solution of Ref. [32], obtained for arbitrary positive values of \( \alpha, \beta \). In addition, our results show that for \( \alpha = \lambda, \beta = \mu + 1 \), the deformed boson operators \( \hat{b}_{\alpha,\beta}, \hat{b}_{\alpha,\beta}, \) and \( \hat{N} \) of Ref. [32] can be realized as \( \left( \hat{b}^\dagger \right)^\lambda, \hat{b}^\lambda, \) and \( (\hat{N} - \mu) / \lambda \), respectively, where \( \hat{b}^\dagger, \hat{b}, \) and \( \hat{N} \) are the standard boson operators considered in (18). Here the roles of \( z \) and of the vacuum state \( |0 \rangle \) are played by \( \lambda z \) and \( |\mu \rangle \), respectively.

4 Nonclassical properties of coherent states

In quantum optics, the properties of the CS \( |z; \mu \rangle \) may be analyzed in two different ways. In both approaches, they are considered as exotic states defined in terms of the deformed operators \( a^\dagger, a \), but in the first one considers “real” photons, described by the operators \( \hat{b}^\dagger, \hat{b} \) of (18) satisfying the canonical commutation relation, while in the second one considers “dressed” photons, described by the operators \( a^\dagger, a \) satisfying a more general commutation relation. Such generalized photons may be invoked in phenomenological models explaining some non-intuitive observable phenomena.

We use here the latter approach, leaving the former for future work. It should be stressed that this choice only affects the squeezing properties (to be studied in Subsec. 4.2) through the definition of the quadratures. On the contrary, since \( N \) and \( \hat{N} \) coincide (see (18)), the deformed photon statistics (to be studied in Subsec. 4.1) is actually the same as the photon statistics, which would result from the other approach.

4.1 Photon statistics

A convenient measure of the deviation of the photon number statistics from the Poisson distribution is the Mandel parameter

\[ Q = \frac{(\Delta N)^2 - \langle N \rangle}{\langle N \rangle}, \quad (35) \]
which vanishes for the Poisson distribution. It is positive or negative according to whether the distribution is super-Poissonian (bunching effect) or sub-Poissonian (antibunching effect).

From (4) and (19), we obtain

\[
Q = \lambda \left[ 1 - \bar{\beta}_{\lambda - 1} - \left( \prod_{\nu=1}^{\lambda-1} \bar{\beta}_\nu \right)^{-1} y \Phi_0^{\lambda-1}(y) + \bar{\beta}_{\lambda - 1} \Phi_{\lambda - 1}^{\lambda-2}(y) \right] - 1 \quad \text{if } \mu = 0,
\]

\[
= \left[ \lambda^{-1} - \bar{\beta}_1 + \bar{\beta}_1 \Phi_0^0(y) \right]^{-1} \left\{ \left( \bar{\beta}_1 - \lambda^{-1} \right) \left[ 1 + \lambda \bar{\beta}_1 \Phi_0^0(y) \right] - \lambda \bar{\beta}_1^2 \left[ \Phi_0^0(y) \right]^2 \right. \\
+ \lambda \left( \prod_{\nu=2}^{\lambda-1} \bar{\beta}_\nu \right)^{-1} y \Phi_1^{\lambda-1}(y) \left\} \quad \text{if } \mu = 1,
\]

\[
= \left[ \mu \lambda^{-1} - \bar{\beta}_\mu + \bar{\beta}_\mu \Phi_{\mu-1}^\mu(y) \right]^{-1} \left\{ \bar{\beta}_\mu - \mu \lambda^{-1} + \lambda \bar{\beta}_\mu \left( \bar{\beta}_\mu - \bar{\beta}_{\mu-1} - \lambda^{-1} \right) \Phi_{\mu-1}^\mu(y) \right.
\]

\[
- \lambda \bar{\beta}_\mu^2 \left[ \Phi_{\mu-1}^\mu(y) \right]^2 + \lambda \bar{\beta}_{\mu-1} \bar{\beta}_\mu \Phi_{\mu-2}^{\mu-2}(y) \right\} \quad \text{if } \mu = 2, 3, \ldots, \lambda - 1,
\]

(36)

where \( y = |z|^2/\lambda^{\lambda-2} \) and \( \Phi_{\mu}^\mu(y) = N_\mu^\mu(|z|)/N_\mu(|z|) \).

Standard even (resp. odd) CS, corresponding to \( \lambda = 2, \alpha_0 = -\alpha_1 = 0, \mu = 0 \) (resp. \( \mu = 1 \)) are known to exhibit a bunching (resp. antibunching) effect. For the even (resp. odd) CS associated with the Calogero-Vasiliev oscillator, i.e., for \( \lambda = 2, \alpha_0 = -\alpha_1 \neq 0, \mu = 0 \) (resp. \( \mu = 1 \)), this trend is enhanced for positive (resp. negative) values of \( \alpha_0 \). However, as shown in Fig. 1, for negative (resp. positive) values of \( \alpha_0 \) and sufficiently high values of \( |z| \), the opposite trend can be seen. In particular, for well-chosen values of \( \alpha_0 \), it is possible to get antibunching for even CS over almost the whole \( |z| \) range.

For higher values of \( \lambda \), more or less similar results are obtained for \( \mu = 0 \), on one hand, and \( \mu \neq 0 \), on the other hand. From (36), it is straightforward to show that for any values of \( \alpha_\mu \) and \( |z| = 0 \), \( Q = \lambda - 1 \) if \( \mu = 0 \), and \( Q = -1 \) if \( \mu = 1, 2, \ldots, \lambda - 1 \). Hence there is bunching (resp. antibunching) for \( \mu = 0 \) (resp. \( \mu \neq 0 \)) for sufficiently small values of \( |z| \).

Fig. 2, corresponding to \( \lambda = 3 \), shows that considering negative (resp. positive) values of \( \alpha_0 = \beta_1 \) or/and \( \alpha_0 + \alpha_1 = \beta_2 \) allows one to reverse the trend and to get antibunching (resp. bunching) for \( \mu = 0 \) (resp. \( \mu \neq 0 \)) for sufficiently high values of \( |z| \). Note however that the behaviour of \( Q \) is more complicated for \( \mu = 1 \) than for \( \mu = 0 \) or 2.
4.2 Squeezing effect

Let us define the deformed electromagnetic field components $x$ and $p$ as

$$x = \frac{1}{\sqrt{2}} (a^t + a), \quad p = \frac{i}{\sqrt{2}} (a^t - a).$$  \hfill (37)

In any state belonging to $\mathcal{F}_\mu$, their dispersions $\langle (\Delta x)^2 \rangle$ and $\langle (\Delta p)^2 \rangle$, where $\Delta x \equiv x - \langle x \rangle$ and $\Delta p \equiv p - \langle p \rangle$, satisfy the uncertainty relation

$$\langle (\Delta x)^2 \rangle \langle (\Delta p)^2 \rangle \geq \frac{1}{4} |\langle [x, p] \rangle|^2 = \frac{\lambda^2}{4} (\bar{\beta}_{\mu + 1} - \bar{\beta}_\mu)^2.$$ \hfill (38)

We note that the right-hand side of this inequality becomes smaller than the conventional value $1/4$ if $\alpha_0 < 0$ for $\mu = 0$ or $-2 < \alpha_\mu < 0$ for $\mu = 1, 2, \ldots, \lambda - 1$. In the latter case, it even vanishes for $\alpha_\mu = -1$.

Here we are interested in the dispersions in the CS $|z; \mu\rangle$, which are obtained as

$$\langle (\Delta x)^2 \rangle = \langle H_0 \rangle + \delta_{\lambda, 2} (z + z^*), \quad \langle (\Delta p)^2 \rangle = \langle H_0 \rangle - \delta_{\lambda, 2} (z + z^*),$$ \hfill (39)

where

$$\langle H_0 \rangle = \lambda \left[ \frac{1}{2} \bar{\beta}_1 + \left( \prod_{\nu=1}^{\lambda-1} \bar{\beta}_\nu \right)^{-1} y \Phi_0^{\lambda - 1}(y) \right] \quad \text{if } \mu = 0,$$

$$= \lambda \left[ \frac{1}{2} (\bar{\beta}_{\mu + 1} - \bar{\beta}_\mu) + \bar{\beta}_\mu \Phi_\mu^{\mu - 1}(y) \right] \quad \text{if } \mu = 1, 2, \ldots, \lambda - 1. \hfill (40)$$

In $\mathcal{F}_\mu$, the role of the vacuum state is played by the number state $|\mu\rangle = |0; \mu\rangle$, which is annihilated by $J_-$. The corresponding dispersions are given by

$$\langle (\Delta x)^2 \rangle_0 = \langle (\Delta p)^2 \rangle_0 = \frac{\lambda}{2} (\bar{\beta}_{\mu + 1} + \bar{\beta}_\mu) = \gamma_\mu + \mu + \frac{1}{2}.$$ \hfill (41)

Comparing with the uncertainty relation (38), we conclude that the state $|\mu\rangle$ satisfies the minimum uncertainty property in $\mathcal{F}_\mu$, i.e., gives rise to the equality in (38), only for $\mu = 0$ because $\bar{\beta}_0 = 0$ and $\bar{\beta}_\mu > 0$ for $\mu = 1, 2, \ldots, \lambda - 1$. On the other hand, the dispersions in the vacuum are smaller than the conventional value $1/2$ if $\gamma_\mu < -\mu$. However, from condition (3) and the definitions of $\beta_\mu$ and $\gamma_\mu$, it follows that $\gamma_\mu$ is also restricted by the condition $\gamma_\mu > -\mu - \frac{1}{2}$ if $\mu = 0, 1, \ldots, \lambda - 2$, or $\gamma_\mu > -\mu/2$ if $\mu = \lambda - 1$. Since both
types of conditions on $\gamma_{\mu}$ are compatible only for $\mu = 0, 1, \ldots, \lambda - 2$, we conclude that $\langle (\Delta x)^2 \rangle_0 = \langle (\Delta p)^2 \rangle_0$ can be less than $1/2$ for such $\mu$ values only.

In the following, we shall restrict ourselves to $\mu = 0$, for which the vacuum state $|0\rangle$ satisfies the minimum uncertainty property. According to the usual definition [12], we say that the quadrature $x$ (resp. $p$) is squeezed in $|z; 0\rangle$ if $\langle (\Delta x)^2 \rangle < \langle (\Delta x)^2 \rangle_0$ (resp. $\langle (\Delta p)^2 \rangle < \langle (\Delta p)^2 \rangle_0$) or, in other words, if the ratio $X \equiv \langle (\Delta x)^2 \rangle / \langle (\Delta x)^2 \rangle_0$ (resp. $P \equiv \langle (\Delta p)^2 \rangle / \langle (\Delta p)^2 \rangle_0$) is less than one. From (39) and (41), it is obvious that the results will be different according to whether $\lambda = 2$ or $\lambda > 2$.

For $\lambda = 2$, we first note that $X$ and $P$ are related with each other by the transformation $Re z \rightarrow -Re z$. Then it is clear that the maximum squeezing in $x$ will be achieved for real, negative values of $z$. So hereafter we only consider $X$ for such values. We find

$$X \simeq 1 - \frac{2}{\beta_1}(-z) + \cdots \quad \text{if} \quad -z < 1, \quad X \simeq \frac{1}{2\beta_1} + \cdots \quad \text{if} \quad -z > 1,$$

(42)

showing that for sufficiently small values of $-z$, $X$ is always smaller than one and closer to zero for small values of $\beta_1$ than for large ones, while for large values of $-z$, $X < 1$, $X \simeq 1$, or $X > 1$ according to whether $\beta_1 > \frac{1}{2}$, $\beta_1 = \frac{1}{2}$, or $\beta_1 < \frac{1}{2}$. Hence, as displayed in Fig. 3, we obtain a large squeezing effect over the whole range of real, negative values of $z$ for positive values of $\alpha_0$, whereas for $\alpha_0 = 0$ or $\alpha_0 < 0$, the squeezing effect becomes negligibly small or disappears for large values of $-z$.

For $\lambda > 2$, the ratios $X$ and $P$ are equal and only depend on $|z|$. From (39), (41), and (41), it is then obvious that $X = P > 1$ if $|z| \neq 0$, so that there is no squeezing effect in this case.

It is also interesting to study higher-order squeezing [3] in the CS $|z; 0\rangle$. The quadrature $x$ (resp. $p$) is said to be squeezed to the $2N$th order if $\langle (\Delta x)^{2N} \rangle < \langle (\Delta x)^{2N} \rangle_0$ (resp. $\langle (\Delta p)^{2N} \rangle < \langle (\Delta p)^{2N} \rangle_0$). Considering fourth-order squeezing, we have to determine whether the ratio $Y \equiv \langle (\Delta x)^4 \rangle / \langle (\Delta x)^4 \rangle_0$ (resp. $Q \equiv \langle (\Delta p)^4 \rangle / \langle (\Delta p)^4 \rangle_0$) is less than one. For $\mu = 0$, we obtain

$$\frac{\langle (\Delta x)^4 \rangle}{\langle (\Delta p)^4 \rangle} = \frac{3}{2}H_0^2 - \frac{\lambda}{4} \left( 1 + \beta_1 - \beta_2 - \beta_{\lambda-1} \right) \langle H_0 \rangle + \frac{\lambda^2}{8} \beta_1 \left( 1 + \beta_2 - \beta_{\lambda-1} \right)$$

$$+ \delta_{\lambda, 2} \left[ z^2 + (z^*)^2 \pm 2 (z + z^*) \langle (H_0) + 1 \rangle \right] + \delta_{\lambda, 4} (z + z^*),$$

(43)

$$\langle (\Delta x)^4 \rangle_0 = \langle (\Delta p)^4 \rangle_0 = \frac{\lambda^2}{4} \beta_1 (\beta_1 + \beta_2),$$

(44)
where on the right-hand side of (43), the upper (resp. lower) sign applies to $\langle (\Delta x)^4 \rangle$ (resp. $\langle (\Delta p)^4 \rangle$), and

$$\langle H_0^2 \rangle = \lambda^2 \left\{ \frac{1}{4} \bar{\beta}_1^2 + \left( \prod_{\nu=1}^{\lambda-1} \bar{\beta}_\nu \right)^{-1} y \left[ (1 + \bar{\beta}_1 - \bar{\beta}_{\lambda-1}) \Phi_0^{\lambda-1}(y) + \bar{\beta}_{\lambda-1} \Phi_0^{\lambda-2}(y) \right] \right\}. \quad (45)$$

For $\lambda = 2$, $Y$ and $Q$ are related with each other by the transformation $\text{Re} z \rightarrow -\text{Re} z$ and the maximum fourth-order squeezing in $x$ is achieved for real, negative values of $z$. So we only consider $Y$ for such values and find

$$Y \simeq 1 - \frac{4}{\bar{\beta}_1} (-z) + \cdots \quad \text{if } -z \ll 1, \quad Y \simeq \frac{3}{4\bar{\beta}_1(1 + \bar{\beta}_1)} + \cdots \quad \text{if } -z \gg 1, \quad (46)$$

showing that the behaviour of $Y$ in terms of $-z$ and $\bar{\beta}_1$ should be roughly similar to that of $X$. This is confirmed numerically, as displayed in Fig. 3. There are however two main differences between the results for $X$ and $Y$. First, for $\alpha_0 = 0$, there is fourth-order squeezing only for $-z < 3/4$, as compared with second-order squeezing over the whole range of $-z$ values. Second, for $\alpha_0 > 0$ and any given $-z$ value, the fourth-order squeezing is larger than the second-order one.

For $\lambda > 2$, we have only investigated the case $\lambda = 4$, for which it is known that there is fourth-order squeezing when $\alpha_\mu = 0$ \[10\]. The present calculations show that the latter is small ($Y_{\text{min}} \simeq 0.933$) and is obtained for very small values of $-z$ ($-z < 0.1$). Considering nonvanishing values of $\alpha_\mu$ can enhance the fourth-order squeezing, but it always remains confined to rather small values of $-z$ and its dependence on the parameters is rather weak. For instance, we obtained fourth-order squeezing for $-z \leq 0.558$ and $Y_{\text{min}} \simeq 0.632$ for $\alpha_0 = \alpha_1 = 0$ and an $\alpha_2$ value as high as 30.

While retrieving the second- and fourth-order squeezing properties of standard $\lambda$-photon CS \[11\], we have therefore shown that they can be improved by considering nonvanishing values of $\alpha_\mu$. The most striking effect is obtained for $\lambda = 2$ and $\alpha_0 > 0$. Note that these values of $\alpha_0$ are precisely those for which the conventional uncertainty constraint is respected.
5 Conclusion

In the present Letter, we established that the $C_\lambda$-extended oscillator spectrum generating algebra is a $C_\lambda$-extended $(\lambda - 1)$th-degree polynomial deformation of $su(1,1)$, and we characterized its $\lambda$ unirreps corresponding to the various subspaces $F_\mu$, $\mu = 0, 1, \ldots, \lambda - 1$, of $F$ by the eigenvalues of its Casimir operator and the lowest $J_0$ eigenvalue.

We then constructed the CS $|z; \mu\rangle$ of the spectrum generating algebra, defined as the eigenstates of the lowering generator $J_-$. We proved that they are normalizable, continuous in the parameter $z$, and that they allow a resolution of unity. We showed that they contain as special cases both the CS of the Calogero-Vasiliev oscillator (equivalent to some Barut-Girardello CS [9]) and the standard $\lambda$-photon CS of Ref. [10] (equivalent to some Mittag-Leffler CS [32]).

Finally, we established that the $C_\lambda$-extended oscillator parameters have a striking influence on the CS nonclassical properties, which may be rather different from those of standard $\lambda$-photon CS. Especially for $\mu = 0$, the CS exhibit strong nonclassical properties, such as antibunching and quadrature squeezing, on a considerable parameter range.

The CS presented here are not the only ones that can be associated with the $C_\lambda$-extended oscillator. Other possibilities are under current investigation, and we hope to report on them in the near future.

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Figure captions

Fig. 1. Mandel’s parameter $Q$ as a function of $|z| \equiv r$ for $\lambda = 2$ and various parameters: (a) $\mu = 0$ and $\alpha_0 = 0$ (solid line), $\alpha_0 = -4/5$ (dashed line), $\alpha_0 = -24/25$ (dotted line), or $\alpha_0 = 1$ (dot-dashed line); (b) $\mu = 1$ and $\alpha_0 = 0$ (solid line), $\alpha_0 = 1$ (dot-dashed line), $\alpha_0 = 9$ (dotted line), or $\alpha_0 = 19$ (dashed line).

Fig. 2. Mandel’s parameter $Q$ as a function of $|z| \equiv r$ for $\lambda = 3$ and various parameters: (a) $\mu = 0$ and $\alpha_0 = \alpha_1 = 0$ (solid line), $\alpha_0 = -\alpha_1 = -7/10$ (dashed line), $\alpha_0 = -47/50$, $\alpha_1 = -1$ (dotted line), or $\alpha_0 = -\alpha_1 = 2$ (dot-dashed line); (b) $\mu = 1$ and $\alpha_0 = \alpha_1 = 0$ (solid line), $\alpha_0 = -\alpha_1 = 2$ (dashed line), $\alpha_0 = 0$, $\alpha_1 = 13$ (dotted line), or $\alpha_0 = -47/50$, $\alpha_1 = -1$ (dot-dashed line); (c) $\mu = 2$ and $\alpha_0 = \alpha_1 = 0$ (solid line), $\alpha_0 = -\alpha_1 = 2$ (dashed line), $\alpha_0 = 0$, $\alpha_1 = 28$ (dotted line), or $\alpha_0 = -47/50$, $\alpha_1 = -1$ (dot-dashed line).

Fig. 3. The ratios $X \equiv \langle (\Delta x)^2 \rangle / \langle (\Delta x)^2 \rangle_0$ and $Y \equiv \langle (\Delta x)^4 \rangle / \langle (\Delta x)^4 \rangle_0$ as functions of $-z$ for real $z$, $\lambda = 2$, and $\mu = 0$. The parameter value is $\alpha_0 = 0$ (solid lines), $\alpha_0 = -2/5$ (dashed lines), $\alpha_0 = 1$ (dotted lines), or $\alpha_0 = 3$ (dot-dashed lines).