Minimal generating set of Sylow 2-subgroups commutator of alternating group. Commutator width in Sylow $p$-subgroups of $A_n$, $S_n$ and in the wreath product of groups

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Abstract

The size of a minimal generating set for commutator of Sylow 2-subgroup of alternating group was found. A couple of examples of such minimal generating sets in permutational presentation are given. It was shown that $(Syl_2 A_{2^k})^2 = Syl'_2 (A_{2^k})$, $k > 2$.

Given a permutational wreath product of finite cyclic groups sequence we prove that the commutator width of such groups is 1 and we research some properties of its commutator subgroup.

A new approach to presentation of Sylow 2-subgroups of the alternating group $A_{2^k}$ was applied. As a result the short proof that the commutator width of Sylow 2-subgroups of alternating group $A_{2^k}$, permutation group $S_{2^k}$ and Sylow $p$-subgroups of $Syl_2 A_{p^k}$ ($Syl_2 S_{p^k}$) are equal to 1 was obtained. Commutator width of permutational wreath product $B \wr C_n$ were investigated.

It was proven that the commutator length of an arbitrary element of commutator of the wreath product of cyclic groups $C_{p_i}$, $p_i \in \mathbb{N}$ equals to 1. The commutator width of direct limit of wreath product of cyclic groups is found. As a corollary, it was shown that the commutator width of Sylow $p$-subgroups $Syl_2(S_{p^k})$ of symmetric $S_{p^k}$ and alternating groups $A_{p^k}$ $p \geq 2$ also equal to 1. A recursive presentation of Sylows 2-subgroups $Syl_2(A_{2^{ak}})$ of $A_{2^k}$ was introduced.

The structure of Sylows 2-subgroups commutator of symmetric and alternating groups were investigated. For an arbitrary group $B$ an upper bound of commutator width of $C_p \wr B$ was found.

Key words: wreath product of groups, minimal generating set of the commutator subgroup of Sylow 2-subgroups, commutator width of wreath product, commutator width of Sylow $p$-subgroups, commutator subgroup of alternating group.

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1 Introduction

As well known the first example of a group $G$ with $cw(G) > 1$ was given by Fite [4]. We deduce an estimation for commutator width of wreath product of groups $C_p \wr B$ taking in consideration a $cw(B)$ of passive group $B$.

Commutator width of groups, and of elements has proven to be an important property in particular via its connections with "stable commutator length" and bounded cohomology.

A form of commutators of wreath product $A \wr B$ that was shortly considered in [2]. For more deep description of this form we take into account commutator width ($cw(G)$) which was presented in work of Muranov [1].

The form of commutator presentation [2] was presented by us in form of wreath recursion [9] and commutator width of it was studied. We impose more weak condition on the presentation of wreath product commutator then it was imposed by J. Meldrum.

In this paper we continue a researches which was stared in [17]. A research of commutator-group serve to decision of inclusion problem [5] for elements of $Syl_2 A_{2k}$ in its derived subgroup $(Syl_2 A_{2k})'$. It was known that, the commutator width of iterated wreath products of nonabelian finite simple groups is bounded by an absolute constant [3, 4]. But it was not proven that commutator subgroup of $\prod_{i=1}^{k} C_p_i$ consists of commutators. We generalize the passive group of this wreath product to any group $B$ instead of only wreath product of cyclic groups and obtain an exact commutator width. Our goal is to improve these estimations and generalize it on a bigger class of passive groups of wreath product. Also we are going to prove that the commutator width of Sylows $p$-subgroups of symmetric and alternating groups $p \geq 2$ is 1.

2 Preliminaries

Let $G$ be a group acting (from the left) by permutations on a set $X$ and let $H$ be an arbitrary group. Then the (permutational) wreath product $H \wr G$ is the semidirect product $H^X \rtimes G$, where $G$ acts on the direct power $H^X$ by the respective permutations of the direct factors. The group $C_p$ or $(C_p, X)$ is equipped with a natural action by the left shift on $X = \{1, \ldots, p\}$, $p \in \mathbb{N}$. As well known that a wreath product of permutation groups is associative construction.

The multiplication rule of automorphisms $g, h$ which presented in form of the wreath recursion [6] $g = (g(1), g(2), \ldots, g(d)) \sigma_g$, $h = (h(1), h(2), \ldots, h(d)) \sigma_h$, is given by the for-
mula:
\[ g \cdot h = (g(1)^{h(\sigma_1(1))}) \cdot g(2)^{h(\sigma_2(2))} \cdot \cdots \cdot g(d)^{h(\sigma_d(d))}) \sigma \cdot g \cdot h. \]

We define \( \sigma \) as \((1, 2, \ldots, p)\) where \( p \) is defined by context.

The set \( X^* \) is naturally a vertex set of a regular rooted tree, i.e. a connected graph without cycles and a designated vertex \( v_0 \) called the root, in which two words are connected by an edge if and only if they are of form \( v \) and \( vx \), where \( v \in X^* \), \( x \in X \). The set \( X^n \subset X^* \) is the \( n \)-th level of the tree \( X^* \) and \( X^0 = \{v_0\} \). We denote by \( v_{j,i} \) the vertex of \( X^j \), which has the number \( i \). Note that the unique vertex \( v_{k,1} \) corresponds to the unique word \( v \) in alphabet \( X \). For every automorphism \( g \in AutX^* \) and every word \( v \in X^* \) define the section (state) \( g(v) \in AutX^* \) of \( g \) at \( v \) by the rule: \( g(v)(x) = y \) for \( x, y \in X^* \) if and only if \( g(vx) = g(v)y \). The subtree of \( X^* \) induced by the set of vertices \( \cup_{i=0}^k X_i \) is denoted by \( X^{[k]} \). The restriction of the action of an automorphism \( g \in AutX^* \) to the subtree \( X^{[k]} \) is denoted by \( g(v)|_{X^{[k]}} \). A restriction \( g(v)|_{X^{[k]}} \) is called the vertex permutation (v.p.) of \( g \) in a vertex \( v \). We call the endomorphism \( \alpha|_v \), restriction of \( g \) in a vertex \( v \) \([6]\). For example, if \( |X| = 2 \) then we just have to distinguish active vertices, i.e., the vertices for which \( \alpha|_v \) is non-trivial.

Let us label every vertex of \( X^l \), \( 0 \leq l < k \) by sign 0 or 1 in relation to state of v.p. in it. Let us denote state value of \( \alpha \) in \( v_{k,i} \) as \( s_{ki}(\alpha) \) we put that \( s_{ki}(\alpha) = 1 \) if \( \alpha|_{v_{ki}} \) is non-trivial, and \( s_{ki}(\alpha) = 0 \) if \( \alpha|_{v_{ki}} \) is trivial. Obtained by such way a vertex-labeled regular tree is an element of \( AutX^{[k]} \). All undeclared terms are from \([7, 8]\).

Let us make some notations. The commutator of two group elements \( a \) and \( b \), denoted by \([a, b] = aba^{-1}b^{-1}\), conjugation by an element \( b \) as
\[ a^b = bab^{-1}, \]
We define \( G_k \) and \( B_k \) recursively i.e.
\begin{align*}
B_1 &= C_2, \quad B_k = B_{k-1} \cdot C_2 \text{ for } k > 1, \\
G_1 &= \langle e \rangle, \quad G_k = \{(g_1, g_2) \in B_k \mid g_1 g_2 \in G_{k-1}\} \text{ for } k > 1.
\end{align*}

Note that \( B_k = \prod_{i=1}^k C_2 \).

The commutator length of an element \( g \) of the derived subgroup of a group \( G \), denoted \( clG(g) \), is the minimal \( n \) such that there exist elements \( x_1, \ldots, x_n, y_1, \ldots, y_n \) in \( G \) such that \( g = [x_1, y_1] \cdots [x_n, y_n] \). The commutator length of the identity element is 0. The commutator width of a group \( G \), denoted \( cw(G) \), is the maximum of the commutator
lengths of the elements of its derived subgroup \([G,G]\).

3 Main result

We are going to prove that \(cw(Syl_2A_{2k}) = 1\) and \(cw(Syl_2S_{2k}) = 1\).

The following Lemma imposes the corollary 4.9 of the Meldrum’s book [2] and it will be deduced from the corollary 4.9.

**Lemma 1.** An element of form \((r_1, \ldots, r_{p-1}, r_p) \in W' = (B \wr C_p)'\) iff product of all \(r_i\) (in any order) belongs to \(B'\).

**Proof.** Analogously to the Corollary 4.9 of the Meldrum’s book [2] we can deduce new presentation of commutators in form of wreath recursion

\[ w = (r_1, r_2, \ldots, r_{p-1}, r_p), \]

where \(r_i \in B\). If we multiply elements from a tuple \((r_1, \ldots, r_{p-1}, r_p)\), where \(r_i = h_i g_a(i) h^{-1} g^{-1} a b^{-1}(i)\), \(h, g \in B\) and \(a, b \in C_p\), then we get a product

\[ x = \prod_{i=1}^{p} r_i = \prod_{i=1}^{p} h_i g_a(i) h^{-1} g^{-1} a b^{-1}(i) \in B', \]  

(1)

where \(x\) is a product of corespondent commutators. Therefore, we can write \(r_p = r_{p-1}^{-1} \ldots r_1^{-1} x\). We can rewrite element \(x \in B'\) as the product \(x = \prod_{j=1}^{m} [f_j, g_j], m \leq cw(B)\).

Note that we impose more weak condition on the product of all \(r_i\) to belongs to \(B'\) then in Definition 4.5. of form \(P(L)\) in [2], where the product of all \(r_i\) belongs to a subgroup \(L\) of \(B\) such that \(L > B'\).

In more detail deducing of our representation constructing can be reported in following way. If we multiply elements having form of a tuple \((r_1, \ldots, r_{p-1}, r_p)\), where \(r_i = h_i g_a(i) h^{-1} g^{-1} a b^{-1}(i)\), \(h, g \in B\) and \(a, b \in C_p\), then in case \(cw(B) = 0\) we obtain a product

\[ \prod_{i=1}^{p} r_i = \prod_{i=1}^{p} h_i g_a(i) h^{-1} g^{-1} a b^{-1}(i) \in B'. \]  

(2)

Note that if we rearrange elements in (1) as \(h_1 h_1^{-1} g_1 g_1^{-1} h_2 h_2^{-1} g_2 g_2^{-1} \ldots h_p h_p^{-1} g_p g_p^{-1}\) then by the reason of such permutations we obtain a product of corespondent commutators. Therefore, following equality holds true...
\[ \prod_{i=1}^{p} h_i g_{a(i)} h_{ab(i)}^{-1} g^{-1}_{aba^{-1}(i)} = \prod_{i=1}^{p} h_i h_i^{-1} g_i g_i^{-1} x \in B', \]  

(3)

where \( x \) is a product of correspondent commutators. Therefore,

\[ (r_1, \ldots, r_{p-1}, r_p) \in W' \text{ iff } r_{p-1} \cdot \ldots \cdot r_1 \cdot r_p = x \in B'. \]  

(4)

Thus, one element from states of wreath recursion \((r_1, \ldots, r_{p-1}, r_p)\) depends on rest of \(r_i\). This dependence contribute that the product \( \prod_{j=1}^{p} r_j \) for an arbitrary sequence \( \{r_j\}_{j=1}^{p} \) belongs to \( B' \). Thus, \( r_p \) can be expressed as:

\[ r_p = r_1^{-1} \cdot \ldots \cdot r_{p-1}^{-1} x. \]

Denote a \(j\)-th tuple, which consists of a wreath recursion elements, by \((r_{j1}, r_{j2}, \ldots, r_{jp})\). Closedness by multiplication of the set of forms \((r_1, \ldots, r_{p-1}, r_p) \in W = (B \wr C_p)'\) follows from

\[ \prod_{j=1}^{k} (r_{j1} \ldots r_{jp-1} r_{jp}) = \prod_{j=1}^{k} \prod_{i=1}^{p} r_{ji} = R_1 R_2 \ldots R_k \in B', \]  

(5)

where \( r_{ji} \) is \(i\)-th element from the tuple number \(j\), \( R_j = \prod_{i=1}^{p-1} r_{ji}, \ 1 \leq j \leq k \). As it was shown above \( R_j = \prod_{i=1}^{p-1} r_{ji} \in B' \). Therefore, the product \([5]\) of \( R_j, j \in \{1, \ldots, k\} \) which is similar to the product mentioned in \([2]\), has the property \( R_1 R_2 \ldots R_k \in B' \) too, because of \( B' \) is subgroup. Thus, we get a product of form \([1]\) and the similar reasoning as above are applicable.

Let us prove the sufficiency condition. If the set \( K \) of elements satisfying the condition of this theorem, that all products of all \( r_i \), where every \( i \) occurs in this forms once, belong to \( B' \), then using the elements of form

\[ (r_1, e, \ldots, e, r_1^{-1}), \ldots , (e, e, \ldots, e, r_1, e, r_1^{-1}), \ldots , (e, e, \ldots, e, r_{p-1}, r_{p-1}^{-1}), (e, e, \ldots, e, r_1 r_2 \cdot \ldots \cdot r_{p-1}) \]

we can express any element of form \((r_1, \ldots, r_{p-1}, r_p) \in W = (B \wr C_p)'\). We need to prove that in such way we can express all element from \( W \) and only elements of \( W \). The fact that all elements can be generated by elements of \( K \) follows from randomness of choice every \( r_i, i < p \) and the fact that equality (1) holds so construction of \( r_p \) is
determined. □

Lemma 2. For any group $B$ and integer $p \geq 2$ if $w \in (B \wr C_p)'$ then $w$ can be represented as the following wreath recursion

$$w = (r_1, r_2, \ldots, r_{p-1}, r_1^{-1} \ldots r_{p-1}^{-1} \prod_{j=1}^{k} [f_j, g_j]),$$

where $r_1, \ldots, r_{p-1}, f_j, g_j \in B$ and $k \leq cw(B)$.

Proof. According to Lemma 1 we have the following wreath recursion

$$w = (r_1, r_2, \ldots, r_{p-1}, r_p),$$

where $r_i \in B$ and $r_{p-1}r_{p-2} \ldots r_2r_1r_p = x \in B'$. Therefore we can write $r_p = r_1^{-1} \ldots r_{p-1}^{-1}x$. We also can rewrite element $x \in B'$ as product of commutators $x = \prod_{j=1}^{k} [f_j, g_j]$ where $k \leq cw(B)$. □

Lemma 3. For any group $B$ and integer $p \geq 2$ if $w \in B \wr C_p$ is defined by the following wreath recursion

$$w = (r_1, r_2, \ldots, r_{p-1}, r_1^{-1} \ldots r_{p-1}^{-1}[f, g]),$$

where $r_1, \ldots, r_{p-1}, f, g \in B$ then we can represent $w$ as the following commutator

$$w = [(a_{1,1}, \ldots, a_{1,p})\sigma, (a_{2,1}, \ldots, a_{2,p})],$$

where

$$a_{1,i} = e, \text{ for } 1 \leq i \leq p - 1,$$

$$a_{2,1} = (f^{-1})r_1^{-1} \ldots r_{p-1}^{-1},$$

$$a_{2,i} = r_{i-1}a_{2,i-1}, \text{ for } 2 \leq i \leq p,$$

$$a_{1,p} = g^{a_{2,p}^{-1}}.$$

Proof. Let us to consider the following commutator

$$\kappa = (a_{1,1}, \ldots, a_{1,p})\sigma \cdot (a_{2,1}, \ldots, a_{2,p}) \cdot (a_{1,p}^{-1}, a_{1,1}^{-1}, \ldots, a_{1,p-1}^{-1})\sigma^{-1} \cdot (a_{2,1}^{-1}, \ldots, a_{2,p}^{-1})$$

$$= (a_{3,1}, \ldots, a_{3,p}).$$
where

$$a_{3,i} = a_{1,i}a_{2,1+i \mod p}^{-1}a_{1,i}^{-1}a_{2,i}^{-1}.$$  

At first we compute the following

$$a_{3,i} = a_{1,i}a_{2,i+1}^{-1}a_{2,i}^{-1} = a_{2,i+1}a_{2,i}^{-1} = r_i a_{2,i}a_{2,i}^{-1} = r_i, \text{ for } 1 \leq i \leq p - 1.$$ 

Then we make some transformation of $a_{3,p}$:

$$a_{3,p} = a_{1,p}a_{2,1}^{-1}a_{1,p}^{-1}a_{2,p}^{-1}$$

$$= (a_{2,1}a_{2,1}^{-1})a_{1,p}a_{2,1}^{-1}a_{1,p}^{-1}a_{2,p}^{-1}$$

$$= a_{2,1}a_{1,p}^{-1}a_{2,1}$$

$$= a_{2,1}^{-1}a_{2,p}[a_{2,1}^{-1}, a_{1,p}]a_{2,p}$$

$$= (a_{2,p}a_{2,1}^{-1})^{-1}[(a_{2,1}^{-1})a_{2,p}, a_{1,p}^{-1}]$$

$$= (a_{2,p}a_{2,1}^{-1})^{-1}[(a_{2,1}^{-1}), a_{2,p}^{-1}, a_{1,p}^{-1}]$$

Now we can see that the form of the commutator $\kappa$ is similar to the form of $w$.

Let us make the following notation

$$r' = r_{p-1} \ldots r_1.$$ 

We note that from the definition of $a_{2,i}$ for $2 \leq i \leq p$ it follows that

$$r_i = a_{2,i+1}a_{2,i}^{-1}, \text{ for } 1 \leq i \leq p - 1.$$ 

Therefore

$$r' = (a_{2,p}a_{2,p-1}^{-1})(a_{2,p-1}a_{2,p-2}^{-1}) \ldots (a_{2,3}a_{2,2}^{-1})(a_{2,2}a_{2,1}^{-1})$$

$$= a_{2,p}a_{2,1}^{-1}.$$ 

And then

$$(a_{2,p}a_{2,1}^{-1})^{-1} = (r')^{-1} = r_1^{-1} \ldots r_{p-1}^{-1}.$$
And now we compute the following

\[(a_{2,1}^{-1})a_{2,p}a_{2,1}^{-1} = (((f^{-1})r_1^{-1} \cdots r_{p-1}^{-1})^{-1}r') = (f(r')^{-1})r' = f,
\]

\[a_{1,p}^{a_{2,p}} = (g_{a_{2,p}}^{-1})a_{2,p} = g.\]

Finally we conclude that

\[a_{3,p} = r_1^{-1} \ldots r_{p-1}^{-1}[f, g].\]

Thus, the commutator \(\kappa\) is presented exactly in the similar form as \(w\) has.

For future using we formulate previous Lemma for the case \(p = 2\).

**Corollary 4.** For any group \(B\) if \(w \in B \wr C_2\) is defined by the following wreath recursion

\[w = (r_1, r_1^{-1}[f, g]),\]

where \(r_1, f, g \in B\) then we can represent \(w\) as commutator

\[w = [(e, a_{1,2})\sigma, (a_{2,1}, a_{2,2})],\]

where

\[a_{2,1} = (f^{-1})r_1^{-1},\]
\[a_{2,2} = r_1a_{2,1},\]
\[a_{1,2} = g_{a_{2,2}}^{-1}.\]

**Lemma 5.** For any group \(B\) and integer \(p \geq 2\) inequality

\[cw(B \wr C_p) \leq \max(1, cw(B))\]

holds.

**Proof.** We can represent any \(w \in (B \wr C_p)'\) by Lemma \(\Box\) with the following wreath
recursion

\[ w = (r_1, r_2, \ldots, r_{p-1}, r_1^{-1} \ldots, r_{p-1}^{-1}) \prod_{j=1}^{k} [f_j, g_j] \]

\[ = (r_1, r_2, \ldots, r_{p-1}, r_1^{-1} \ldots, r_{p-1}^{-1}) \prod_{j=2}^{k} [(e, \ldots, e, f_j), (e, \ldots, e, g_j)], \]

where \( r_1, \ldots, r_{p-1}, f_j, g_j \in B \) and \( k \leq cw(B) \). Now by the Lemma 3 we can see that \( w \) can be represented as a product of \( \max(1, cw(B)) \) commutators. \( \square \)

**Corollary 6.** If \( W = C_{p_k} \cdots \wr C_{p_1} \) then \( cw(W) = 1 \) for \( k \geq 2 \).

**Proof.** If \( B = C_{p_k} \wr C_{p_{k-1}} \) then taking into consideration that \( cw(B) > 0 \) (because \( C_{p_k} \wr C_{p_{k-1}} \) is not commutative group). Since Lemma \( 5 \) implies that \( cw(C_{p_k} \wr C_{p_{k-1}}) = 1 \) then according to the inequality \( cw(C_{p_k} \wr C_{p_{k-1}} \wr C_{p_{k-2}}) \leq \max(1, cw(B)) \) from Lemma \( 5 \) we obtain \( cw(C_{p_k} \wr C_{p_{k-1}} \wr C_{p_{k-2}}) = 1 \). Analogously if \( W = C_{p_k} \cdots \wr C_{p_1} \) and supposition of induction for \( C_{p_k} \cdots \wr C_{p_2} \) holds, then using an associativity of a permutational wreath product we obtain from the inequality of Lemma \( 5 \) and the equality \( cw(C_{p_k} \cdots \wr C_{p_2}) = 1 \) that \( cw(W) = 1 \). \( \square \)

We define our partial ordered set \( M \) as the set of all finite wreath products of cyclic groups. We make of use directed set \( \mathbb{N} \).

\[ H_k = \bigwedge_{i=1}^{k} C_{p_i} \quad (6) \]

Moreover, it has already been proved in Corollary 4 that each group of the form \( \bigwedge_{i=1}^{k} C_{p_i} \) has a commutator width equal to \( 1 \), i.e \( cw(\bigwedge_{i=1}^{k} C_{p_i}) = 1 \). A partial order relation \( \leq \) will be a subgroup relationship. Define the injective homomorphism \( f_{k,k+1} \) from the \( \bigwedge_{i=1}^{k} C_{p_i} \) into \( \bigwedge_{i=1}^{k+1} C_{p_i} \) by mapping a generator of active group \( C_{p_i} \) of \( H_k \) in a generator of active group \( C_{p_i} \) of \( H_{k+1} \). In more details the injective homomorphism \( f_{k,k+1} \) is defined as \( g \mapsto g(e, \ldots, e) \), where a generator \( g \in \bigwedge_{i=1}^{k} C_{p_i} \), \( g(e, \ldots, e) \in \bigwedge_{i=1}^{k+1} C_{p_i} \).

Therefore this is an injective homomorphism of \( H_k \) onto subgroup \( \bigwedge_{i=1}^{k} C_{p_i} \) of \( H_{k+1} \).

**Corollary 7.** The direct limit \( \varinjlim_{i=1}^{k} C_{p_i} \) of direct system \( \left\langle f_{k,j}, \bigwedge_{i=1}^{k} C_{p_i} \right\rangle \) has commutator width 1.
Proof. We make the transition to the direct limit in the directed system $\langle f_{k,j}, \{ c_{p_i} \} \rangle$

of injective mappings from chain $e \to \cdots \to \{ c_{p_i} \} \to \{ c_{p_i} \} \to \{ c_{p_i} \} \to \cdots$.

Since all mappings in chains were injective homomorphisms, it have a trivial kernel, so the transition to a direct limit boundary preserves the property $\rho$: that any element of commutator subgroup is commutator, because in each group $H_k$ from the chain endowed by $\rho$.

The direct limit of the direct system is denoted by $\lim_{\to} \{ c_{p_i} \}$ and is defined as disjoint union of the $H_k$’s modulo a certain equivalence relation:

$$\lim_{\to} \{ c_{p_i} \} = \biguplus_{k} H_k / _\sim.$$

Since every element $g$ of $\lim_{\to} \{ c_{p_i} \}$ coincides with some element from one of the groups $G_m$ of directed system, then by the injectivity of the mappings for $g$ the property $cw(\{ c_{p_i} \}) = 1$ also holds. Thus, it holds for the whole $\lim_{\to} \{ c_{p_i} \}$. \qed

Corollary 8. For prime $p$ and $k \geq 2$ commutator width $cw(Syl_p(S_{p^k})) = 1$ and for prime $p > 2$ and $k \geq 2$ commutator width $cw(Syl_p(A_{p^k})) = 1$.

Proof. Since $\operatorname{Syl}_p(S_{p^k}) \simeq c_{p_i}$ see [10, 11], then $cw(Syl_p(S_{p^k})) = 1$. As well known in case $p > 2$ we have $\operatorname{Syl}_pS_{p^k} \simeq Syl_pA_{p^k}$ see [16, 19], then $cw(Syl_p(A_{p^k})) = 1$. \qed

Proposition 9. The following inclusion $B'_k < G_k$ holds.

Proof. Induction on $k$. For $k = 1$ we have $B'_1 = G_1 = \{e\}$. Let us fix some $g = (g_1, g_2) \in B'_k$. Then $g_1 g_2 \in B'_{k-1}$ by Lemma 1. As $B'_{k-1} < G_{k-1}$ by induction hypothesis therefore $g_1 g_2 \in G_{k-1}$ and by definition of $G_k$ it follows that $g \in G_k$. \qed

Corollary 10. The set $G_k$ is a subgroup in the group $B_k$.

Proof. According to recursively definition of $G_k$ and $B_k$, where $G_k = \{(g_1, g_2) \pi \in B_k \mid g_1 g_2 \in G_{k-1}\}$ $k > 1$, $G_k$ is subset of $B_k$ with condition $g_1 g_2 \in G_{k-1}$. It is easy to check the closedness by multiplication elements of $G_k$ with condition $g_1 g_2, h_1 h_2 \in G_{k-1}$ because $G_{k-1}$ is subgroup so $g_1 g_2 h_1 h_2 \in G_{k-1}$ too. A condition of existing inverse be verified trivial. \qed

Lemma 11. For any $k \geq 1$ we have $|G_k| = |B_k|/2$. 

10
Proof. Induction on $k$. For $k = 1$ we have $|G_1| = 1 = |B_1/2|$. Every element $g \in G_k$ can be uniquely write as the following wreath recursion

$$g = (g_1, g_2) \pi = (g_1, g_1^{-1} x) \pi$$

where $g_1 \in B_{k-1}$, $x \in G_{k-1}$ and $\pi \in C_2$. Elements $g_1, x$ and $\pi$ are independent therefore $|G_k| = 2 |B_{k-1}| \cdot |G_{k-1}| = 2 |B_{k-1}| \cdot |B_{k-1}|/2 = |B_k|/2$. \hfill \Box

Corollary 12. The group $G_k$ is a normal subgroup in the group $B_k$ i.e. $G_k \triangleleft B_k$.

Proof. There exists normal embedding (normal injective monomorphism) $\varphi : G_k \to B_k$ \[20\] such that $G_k \triangleleft B_k$. Indeed, according to Lemma index $|B_k : G_k| = 2$ so it is normal subgroup that is quotient subgroup $B_k/C_2 \simeq G_k$. \hfill \Box

Theorem 13. For any $k \geq 1$ we have $G_k \simeq Syl_2 A_{2k}$.

Proof. Group $C_2$ acts on the set $X = \{1, 2\}$. Therefore we can recursively define sets $X^k$ on which group $B_k$ acts

$$X^1 = X,$$

$$X^k = X^{k-1} \times X \text{ for } k > 1.$$ 

At first we define $S_{2k} = Sym(X^k)$ and $A_{2k} = Alt(X^k)$ for all integer $k \geq 1$. Then $G_k < B_k < S_{2k}$ and $A_{2k} < S_{2k}$.

We already know \[16\] that $B_k \simeq Syl_2 (S_{2k})$. Since $|A_{2k}| = |S_{2k}|/2$ therefore $|Syl_2 A_{2k}| = |Syl_2 S_{2k}|/2 = |B_k|/2$. By Lemma \[3\] it follows that $|Syl_2 A_{2k}| = |G_k|$. Therefore it is left to show that $G_k < Alt(X^k)$.

Let us fix some $g = (g_1, g_2) \sigma^i$ where $g_1, g_2 \in B_{k-1}$, $i \in \{0, 1\}$ and $g_1 g_2 \in G_{k-1}$. Then we can represent $g$ as follows

$$g = (g_1 g_2, e) \cdot (g_2^{-1}, g_2) \cdot (e, e, \cdots) \sigma^i.$$ 

In order to prove this theorem it is enough to show that $(g_1 g_2, e), (g_2^{-1}, g_2), (e, e, \cdots) \sigma \in Alt(X^k)$.

Element $(e, e, \cdots) \sigma$ just switch letters $x_1$ and $x_2$ for all $x \in X^k$. Therefore $(e, e, \cdots) \sigma$ is product of $|X^{k-1}| = 2^{k-1}$ transpositions and therefore $(e, e, \cdots) \sigma \in Alt(X^k)$.

Elements $g_2^{-1}$ and $g_2$ have the same cycle type. Therefore elements $(g_2^{-1}, e)$ and $(e, g_2)$
also have the same cycle type. Let us fix the following cycle decompositions

\[(g_2^{-1}, e) = \sigma_1 \cdots \sigma_n,\]
\[(e, g_2) = \pi_1 \cdots \pi_n.\]

Note that element \((g_2^{-1}, e)\) acts only on letters like \(x_1\) and element \((e, g_2)\) acts only on letters like \(x_2\). Therefore we have the following cycle decomposition

\[(g_2^{-1}, g_2) = \sigma_1 \cdots \sigma_n \pi_1 \cdots \pi_n.\]

So, element \((g_2^{-1}, g_2)\) has even number of even permutations and then \((g_2^{-1}, g_2) \in \text{Alt}(X^k)\).

Note that \(g_1 g_2 \in G_{k-1}\) and \(G_{k-1} \simeq \text{Alt}(X^{k-1})\) by induction hypothesis. Therefore \(g_1 g_2 \in \text{Alt}(X^{k-1})\). As elements \(g_1 g_2\) and \((g_1 g_2, e)\) have the same cycle type then \((g_1 g_2, e) \in \text{Alt}(X^k)\).

As it was proven by the author in \[\text{[16]}\] Sylow 2-subgroup has structure \(B_{k-1} \rtimes W_{k-1}\), where definition of \(B_{k-1}\) is the same that was given in \[\text{[16]}\].

Recall that it was denoted by \(W_{k-1}\) the subgroup of \(\text{Aut}X^{[k]}\) such that has active states only on \(X^{k-1}\) and number of such states is even, i.e. \(W_{k-1} \triangleleft \text{St}_{\text{Aut}X^{[k]}}(k-1)\) \[\text{[6]}\]. It was proven that the size of \(W_{k-1}\) is equal to \(2^{2^{k-1}-1}, k > 1\) and its structure is \((C_2)^{2^{k-1}-1}\).

The following structural theorem characterizing the group \(G_k\) was proven \[\text{[16]}\].

**Theorem 14.** A maximal 2-subgroup of \(\text{Aut}X^{[k]}\) that acts by even permutations on \(X^k\) has the structure of the semidirect product \(G_k \simeq B_{k-1} \rtimes W_{k-1}\) and isomorphic to \(\text{Syl}_2 A_{2^k}\).

Note that \(W_{k-1}\) is subgroup of stabilizer of \(X^{k-1}\) i.e. \(W_{k-1} < \text{St}_{\text{Aut}X^{[k]}}(k-1) \triangleleft \text{Aut}X^{[k]}\) and is normal too \(W_{k-1} \triangleleft \text{Aut}X^{[k]}\), because conjugation keeps a cyclic structure of permutation so even permutation maps in even. Therefore such conjugation induce an automorphism of \(W_{k-1}\) and \(G_k \simeq B_{k-1} \rtimes W_{k-1}\).

Thus the structure was founded by us in \[\text{[16]}\] fully consistent with the recursive group representation (which used in this paper) based on the concept of wreath recursion \[\text{[9]}\].

**Theorem 15.** Elements of \(B'_k\) have the following form \(B'_k = \{[f, l] \mid f \in B_k, l \in G_k\} = \{[l, f] \mid f \in B_k, l \in G_k\}\).

**Proof.** It is enough to show either \(B'_k = \{[f, l] \mid f \in B_k, l \in G_k\}\) or \(B'_k = \{[l, f] \mid f \in B_k, l \in G_k\}\) because if \(f = [g, h]\) then \(f^{-1} = [h, g]\).

We prove the proposition by induction on \(k\). For the case \(k = 1\) we have \(B'_1 = (e)\).
Consider case $k > 1$. According to Lemma 2 and Corollary 4 every element $w \in B'_k$ can be represented as

$$w = (r_1, r_1^{-1}[f, g])$$

for some $r_1, f \in B_{k-1}$ and $g \in G_{k-1}$ (by induction hypothesis). By the Corollary 4 we can represent $w$ as commutator of

$$(e, a_{1,2})\sigma \in B_k$$

and

$$(a_{2,1}, a_{2,2}) \in B_k,$$

where

$$a_{2,1} = (f^{-1})r_1^{-1},$$
$$a_{2,2} = r_1a_{2,1},$$
$$a_{1,2} = g^{a_{2,1}}.$$

If $g \in G_{k-1}$ then by the definition of $G_k$ and Corollary 12 we obtain $(e, a_{1,2})\sigma \in G_k$. □

**Remark 16.** Let us to note that Theorem 15 improve Corollary 8 for the case $\text{Syl}_2 S_k$.

**Proposition 17.** If $g$ is an element of the group $B_k$ then $g^2 \in B'_k$.

**Proof.** Induction on $k$. We note that $B_k = B_{k-1} \wr C_2$. Therefore we fix some element

$$g = (g_1, g_2)\sigma^i \in B_{k-1} \wr C_2,$$

where $g_1, g_2 \in B_{k-1}$ and $i \in \{0, 1\}$. Let us to consider $g^2$ then two cases are possible:

$$g^2 = (g_1^2, g_2^2) \text{ or } g^2 = (g_1g_2, g_2g_1).$$

In second case we consider a product of coordinates $g_1g_2 \cdot g_2g_1 = g_1^2g_2^2x$. Since according to the induction hypothesis $g_i^2 \in B'_k$, $i \leq 2$ then $g_1g_2 \cdot g_2g_1 \in B'_k$ also according to Lemma 3 $x \in B'_k$. Therefore a following inclusion holds $(g_1g_2, g_2g_1) = g^2 \in B'_k$. In first case the proof is even simpler because $g_1^2, g_2^2 \in B'$ by the induction hypothesis. □

Since as well known a 2-group $G'$ contains the subgroup $G$ then a product $G^2_k G'_k$ contains all elements from the $G^2_k$. In the paper 10 we obtain that $G^2_k \simeq G'_k$.

**Lemma 18.** If an element $g = (g_1, g_2) \in G'_k$ then $g_1, g_2 \in G_{k-1}$ and $g_1g_2 \in B'_{k-1}$.
Proof. As $B'_k < G_k$ therefore it is enough to show that $g_1 \in G_{k-1}$ and $g_1g_2 \in B'_{k-1}$. Let us fix some $g = (g_1, g_2) \in G'_k < B'_k$. Then Lemma implies that $g_1g_2 \in B'_{k-1}$.

In order to show that $g_1 \in G_{k-1}$ we firstly consider just one commutator of arbitrary elements from $G_k$

$$ f = (f_1, f_2)\sigma, \ h = (h_1, h_2)\pi \in G_k, $$

where $f_1, f_2, h_1, h_2 \in B_{k-1}$, $\sigma, \pi \in C_2$. The definition of $G_k$ implies that $f_1f_2, h_1h_2 \in G_{k-1}$.

If $g = (g_1, g_2) = [f, h]$ then

$$ g_1 = f_1h_1f_2^{-1}h_2^{-1} $$

for some $i, j, k \in \{1, 2\}$. Then

$$ g_1 = f_1h_1f_2(f_j^{-1})^2h_3(h_k^{-1})^2 = (f_1f_j)(h_1h_k)x(f_j^{-1}h_k^{-1})^2, $$

where $x$ is product of commutators of $f_i, h_j$ and $f_i, h_k$, hence $x \in B_{k-1}'$.

It is enough to consider first product $f_1f_j$. If $j = 1$ then $f_1^2 \in B'_{k-1}$ by Proposition 17 if $j = 2$ then $f_1^2 \in B'_{k-1}$ by definition of $G_k$, the same is true for $h_1h_2$. Thus, for any $i, j, k$ it holds $f_1f_j, h_1h_2 \in G_{k-1}$. Besides that a square $(f_j^{-1}h_k^{-1})^2 \in B'_{k}$ according to Proposition 17. Therefore $g_1 \in G_{k-1}$ because of Proposition 17 and Proposition 9 the same is true for $g_2$.

Now it lefts to consider the product of some $f = (f_1, f_2), h = (h_1, h_2)$, where $f_1, h_1 \in G_{k-1}$, $f_1h_1 \in G_{k-1}$ and $f_1f_2, h_1h_2 \in B'_{k-1}$

$$ fh = (f_1h_1, f_2h_2). $$

Since $f_1f_2, h_1h_2 \in B'_{k-1}$ by imposed condition in this item and taking into account that $f_1h_1f_2h_2 = f_1f_2h_1h_2 x$ for some $x \in B'_{k-1}$ then $f_1h_1f_2h_2 \in B'_{k-1}$ by Lemma 1. Other words closedness by multiplication holds and so according Lemma 4 we have element of commutator $G_k'$. 

Proposition 19. For arbitrary $g \in G_k$ the inclusion $g^2 \in G_k'$ holds.

Proof. Induction on $k$: for $G_k^2$ elements has form $(\sigma)^2 = e$ where $\sigma = (1, 2)$ so statement
holds. In general case when \( k > 1 \) elements of \( G_k \) has form
\[
g = (g_1, g_2)\sigma^i, \quad g_1, g_2 \in B_{k-1}, \quad i \in \{0, 1\}
\]
then we have two possibilities
\[
g^2 = (g_1^2, g_2^2) \quad \text{or} \quad g^2 = (g_1 g_2, g_2 g_1).
\]

Firstly we show that
\[
g_1^2 \in G_{k-1}, \quad g_2^2 \in G_{k-1}
\]
Since according to Proposition \ref{prop:Gk} we have \( g_1^2, g_2^2 \in B'_{k-1} \) and according to Proposition \ref{prop:Bk} we have \( B'_{k-1} < G_{k-1} \) then using Theorem \ref{thm:Gk} \( g^2 = (g_1^2, g_2^2) \in G_k \).

Consider second case \( g^2 = (g_1 g_2, g_2 g_1) \). Since \( g \in G_k \) then according to the definition of \( G_k \) we have that \( g_1 g_2 \in G_{k-1} \).

By Proposition \ref{prop:Bk} and definition of \( G_k \) we obtain
\[
g_1 g_2 = g_1 g_2 g_1^{-1} g_2^{-1} = g_1 g_2 [g_2^{-1}, g_1^{-1}] \in G_{k-1},
g_1 g_2 \cdot g_2 g_1 = g_1 g_2 g_1^{-1} g_2^{-2} \in B'_{k-1}.
\]

Note that \( g_1^2, g_2^2 \in B'_{k-1} \) according to Proposition \ref{prop:Bk} then \( g_1^2 g_2^2 [g_2^{-2}, g_1^{-1}] \in B'_{k-1} \). This is a reason why \( g_1^2 g_2^2 [g_2^{-2}, g_1^{-1}] \in B'_{k-1} \). Since \( g_1 g_2 \cdot g_2 g_1 \in B'_{k-1} \) then according to Lemma \ref{lem:Bk} we obtain \( g^2 = (g_1 g_2, g_2 g_1) \in G'_k \).

Thus, the group \( Syl_2 A_{2k} \) has the same property as special \( p \)-groups have that is \( P' = \Phi(P) \) because \( G_k^2 = G_k' \) and so \( \Phi(Syl_2 A_{2k}) = Syl'_2(A_{2k}) \).

In the following theorem we prove 2 facts at once.

**Theorem 20.** The following statements are true.

1. An element \( g = (g_1, g_2) \in G'_k \) iff \( g_1, g_2 \in G_{k-1} \) and \( g_1 g_2 \in B'_{k-1} \).

2. Commutator subgroup \( G'_k \) coincides with set of all commutators for \( k \geq 1 \)
\[
G'_k = \{ [f_1, f_2] \mid f_1 \in G_k, f_2 \in G_k \}.
\]

**Proof.** For the case \( k = 1 \) we have \( G'_1 = \langle e \rangle \). So, further we consider the case \( k \geq 2 \).
Sufficiency of the first statement of this theorem follows from the Lemma 24. So, in order to prove necessity of the first statement and the second statement it is enough to show that element

\[ w = (r_1, r_1^{-1}x), \]

where \( r_1 \in G_{k-1} \) and \( x \in B'_{k-1} \) can be represented as a commutator of elements from \( G_k \). By Proposition 14 we have \( x = [f, g] \) for some \( f \in B_{k-1} \) and \( g \in G_{k-1} \). Therefore

\[ w = (r_1, r_1^{-1}[f, g]). \]

By the Corollary 4 we can represent \( w \) as a commutator of \( (e, a_{1,2}) \in B_k \) and \( (a_{2,1}, a_{2,2}) \in B_k \), where

\[
\begin{align*}
  a_{2,1} &= (f^{-1})r_1^{-1}, \\
  a_{2,2} &= r_1a_{2,1}, \\
  a_{1,2} &= g^{a_{2,2}}.
\end{align*}
\]

It is only left to show that \( (e, a_{1,2}) \in G_k \). Let us to note the following

\[
\begin{align*}
  a_{1,2} &= g^{a_{2,2}} \in G_{k-1} \text{ by Corollary 12} \\
  a_{2,1}a_{2,2} &= a_{2,1}r_1a_{2,1} = r_1[r_1, a_{2,1}]a_{2,1}^2 \in G_{k-1} \text{ by Proposition 9 and Proposition 17}.
\end{align*}
\]

So we have \( (e, a_{1,2}) \in G_k \) and \( (a_{2,1}, a_{2,2}) \in G_k \) by the definition of \( G_k \).

**Corollary 21.** Commutator width of the group \( Syl_2 A_{2k} \) equal to 1 for \( k \geq 2 \).

We denote by \( A_l \) a set of all functions \( a_l \) such, that \([\varepsilon, \ldots, \varepsilon, a_l, \varepsilon, \ldots] \in [A]_l \). Recall that according to \( 21 \) \( l \)-coordinate subgroup \( U < G \) is the following subgroup.

**Definition 1.** We call a \( k \)-coordinate a subgroup \( U < G \) a subgroup, which is determined by \( k \)-coordinate sets \([U]_l, l \in \mathbb{N}, \) if this subgroup consists of all Kaloujnine’s tableaux \( a \in I \) for which \( k \in \mathbb{N} [a]_l \in [U]_l \).

We denote as \( G_k(l) \) such subgroup of \( AutX^{[k]} \) that contains all v.p. from \( X^l, l < k-1 \). In other words it contains all v.p. from \( Stab_{AutX^{[k]}}(l) \) and does not contains v.p. from
$\text{Stab}_{\text{Aut}X^k}(l+1)$, $l < k-1$. We denote as $G_k(k-1)$ such subgroup of $\text{Aut}X^k$ that consists of v.p. which are located on $X^{k-1}$ and isomorphic to $W_{k-1}$. Note that a $G_k(l)$ is in bijective correspondence with $l$-coordinate subgroup $[U]_l$.

Note that if we consider $B_k$ instead of $G_k$ then $B_k(l) \equiv [U]_l$ for $l < k-1$.

Recall the homomorphism from $G(l)$ onto $C_2$ which was constructed in previous our work in the following way: $\varphi_l(\alpha) = \frac{2^l}{\sum_{i=1} s_{i_l}(\alpha) \mod 2}$. Note that $\varphi_l(\alpha \cdot \beta) = \varphi_l(\alpha) \cdot \varphi_l(\beta) = (\sum_{i=1} s_{i_l}(\alpha) + \sum_{i=1} s_{i_l}(\beta)) \mod 2$.

**Remark 22.** A sections group of $\text{Rist}_{G_k}(v_{11})$ at vertex $v_{11}$ isomorphic to $G'_k X_{11}$. Analogous property for $\text{Rist}_{G_k}(v_{12})$ is true. These two subgroups are related by subdirect product with condition of parity.

**Proof.** Let $g \in G'_k$ then $g = (g_{11}, g_{12})^{\pi^i}, i = 0, \pi = (1, 2)$. According to Lemma 21 elements $g_{12}, g_{12} \in G_{k-1}$ it means that a sections group of $\text{Rist}G'_k(v_{11})$ isomorphic to $G'_k X_{11}$, because every element from first coordinate belongs to $G_{k-1}$. Then according to Proposition 21 about structure of $G'_k$ elements of $G'_k$ has a structure $g_1 = (g_{11}, g_{12}) \in G_{k-1}$, where $g_{11}, g_{12} \in G_{k-2}$ and $g_{11}g_{12} \in B'_k$. As it was proved in 10 that elements $g \in B'_k$ have even index $I_{1l}(g)$. Conversely the operation $\boxtimes$ admits all possible tuples of transpositions from multipliers of form $G_{k-1}$ with condition of parity of transpositions number in the product $G_{k-1} \boxtimes G_{k-1}$ on any level $X^l, l > k-1$. Since parity of transpositions in both factors is the same. Conversely $G'_k$ has structure of subdirect product with condition of parity.

Thus, $G'_k \simeq G_{k-1} \boxtimes G_{k-1}$, where $\boxtimes$ is even subdirect product which was presented in 10. Note that in $G'_k$ the operation $\boxtimes$ degenerates in $\times$ because of parity of level index $I_{1l}(g)$ on every subtree $v_{11} AutX^{k-1}$ and $v_{12} AutX^{k-1}$. Therefore commutator of $G'_k \simeq (G_{k-2} \boxtimes G_{k-2}) \boxtimes (G_{k-2} \boxtimes G_{k-2})$.

Analogous property for $\text{Rist}_{G_k}(v_{12})$ is true.

Let $X' = X_1 \cup X_2$ and $X_1 = \{v_{k-1,1}, v_{k-1,2}, ..., v_{k-1,2^{l-1}}\}$, $X_2 = \{v_{k-1,2^{k-2}+1}, ..., v_{k-1,2^k}\}, 1 < l < k$.

**Lemma 23.** An element $g$ belongs to $G'_k \simeq S_{yl2A_{2k}}$ iff $g$ is an arbitrary element from $G_k$ which has all even indexes on $X^l$, $l < k-1$ of $X^k$ and on $X^{k-2}$ of subtrees $v_{11} X^{k-1}$ and $v_{12} X^{k-1}$.

**Proof.** Let us prove the ampleness by induction on a number of level $l$. Conjugation by automorphism $\alpha$ from $Autv_{11}X^{k-1}$ of automorphism $\theta$, that has index $x : 1 \leq x \leq 2^{l-1}$
on $X_1$ does not change $x$. Also automorphism $\theta^{-1}$ has the same number $x$ of v. p. on $X_1$ as $\theta$ has. If $\alpha$ from $Autv_{11}X^{[k-1]}$ and $\alpha \notin v_{12}AutX^{[k]}$ then conjugation $(\alpha \theta \alpha^{-1})$ permutes on $X^l$ vertices only inside $X_1$. So this index forms as a result of multiplying of elements of commutator presented as wreath recursion $(\alpha \theta \alpha^{-1}) \cdot \theta^{-1} = (h_1, ..., h_{2^l})\pi_1 \cdot (f_1, ..., f_{2^l})\pi_2 = (h_1, ..., h_{2^l})(f_{\pi_1(1)}, ..., f_{\pi_1(2^l)})\pi_1\pi_2$, where $\pi_1, \pi_2 \in B_{l-1}$.

Thus, $\alpha \theta \alpha^{-1}$ and $\theta$ have the same parities of number of active v. p. on $X_1$ ($X_2$). Hence, a product $\alpha \theta \alpha^{-1} \theta^{-1}$ has an even number of active v. p. on $X_1$ ($X_2$) in this case.

More over a coordinate-wise sum of active v. p. from $(\alpha \theta \alpha^{-1})$ and $\theta^{-1}$ on $X_1$ ($X_2$) is congruent to 0 by mod2 and equal to $y : 0 \leq y \leq 2x$.

If conjugation by $\alpha$ permutes sets $X_1$ and $X_2$ then there are coordinate-wise sums of no trivial v. p. from $\alpha \theta \alpha^{-1} \theta^{-1}$ on $X_1$ (analogously on $X_2$) have form:

$$(s_{k-1,1}(\alpha \theta \alpha^{-1}), ..., s_{k-1,2^k-2}(\alpha \theta \alpha^{-1})) \oplus (s_{k-1,\pi_1(1)}(\theta^{-1}), ..., s_{k-1,\pi_1(2^k-2)}(\theta^{-1})).$$

This sum has even number of v. p. on $X_1$ and $X_2$ because $(\alpha \theta \alpha^{-1})$ and $\theta^{-1}$ have a same parity of no trivial v. p. on $X_1$ ($X_2$). Hence, $(\alpha \theta \alpha^{-1}) \theta^{-1}$ has even number of v. p. on $X_1$ as well as on $X_2$.

An automorphism $\theta$ from $G_k$ was arbitrary so number of active v. p. $x$ on $X_1$ is an arbitrary $0 \leq x \leq 2^l$. And $\alpha$ is and arbitrary from $AutX^{[k-1]}$ so vertices can be permuted in such way that the commutator $[\alpha, \theta]$ has arbitrary number $y$ of active v. p. on $X_1$, $0 \leq y \leq 2x$.

Hence $In_{l}(\alpha \theta \alpha^{-1}) = 2k + 2$ and coordinates of new vertices $v_{2k+1}, v_{2k+2}$ are arbitrary from 1 to $2^l$.

So multiplication $(\alpha \theta \alpha^{-1})\theta$ generates a commutator having index $y$ equal to coordinate-wise sum by mod2 of no trivial v. p. from vectors $(s_{l1}(\alpha \theta \alpha^{-1}), s_{l2}(\alpha \theta \alpha^{-1}), ..., s_{l2^l}(\alpha \theta \alpha^{-1})) \oplus (s_{l\pi_1(1)}(\theta), s_{l\pi_1(2)}(\theta), ..., s_{l\pi_1(2^l)}(\theta))$ on $X^l$. A indexes parities of $\alpha \theta \alpha^{-1}$ and $\theta^{-1}$ are same so their sum by mod2 are even. Choosing $\theta$ we can choose an arbitrary index $x$ of $\theta$ also we can choose arbitrary $\alpha$ to make a permutation of active v. p. on $X_1$. Thus, we obtain an element with arbitrary even index on $X^l$ and arbitrary location of active v. p. on $X^l$.

Check that property of number parity of v. p. on $X_1$ as well as on $X_2$ have the same parities. So action by conjugation only can permutes it, hence, we again get the same structure of element. Conjugation by automorphism $\alpha$ from $Autv_{11}X^{[k-1]}$ automorphism $\theta$, that has odd number of active v. p. on $X_1$ does not change its parity. Choosing the $\theta$ we can choose arbitrary index $x$ of $\theta$ on $X^{k-1}$ and number of active v. p. on $X_1$ and $X_2$ also we can choose arbitrary $\alpha$ to make a permutation active v. p. on $X_1$ and $X_2$. Thus, we can generate all possible elements from a commutator.

Let $\kappa_1, \kappa_2 \in K$ and each of which has even index on $X^l$ and $2^l$-tuples of v. p.
\((s_{1,1}(\kappa_1), \ldots, s_{k-1,2}(\kappa_1)), (s_{l,\kappa_{(1)}}(\kappa_2), \ldots, s_{l,\kappa_{(2)}}(\kappa_2))\) corresponds to portrait of \(\kappa_1, \kappa_2\) on \(X^l\). Then a number of non-trivial coordinates in a coordinate-wise sum
\((s_{1,1}(\kappa_1), \ldots, s_{k-1,2}(\kappa_1)) \oplus (s_{l,\kappa_{(1)}}(\kappa_2), \ldots, s_{l,\kappa_{(2)}}(\kappa_2))\) is even.

Let us check that the set of all commutators \(K\) from \(Syl_2A_{2k}\) is closed with respect to multiplication of commutators. Note that conjugation of \(\kappa\) can permute sets \(X_1\) and \(X_2\) so parities of \(x_1\) and \(X_2\) coincide. It is obviously that the parity of index of \(\alpha\kappa\alpha^{-1}\) is the same as index of \(\kappa\).

Check that a set \(K\) is a set closed with respect to conjugation.

Let \(\kappa \in K\), then \(\alpha\kappa\alpha^{-1}\) also belongs to \(K\); it is so because conjugation does not change index of an automorphism on a level. Conjugation only permutes vertices on a level because elements of \(AutX^{l-1}\) acts on vertices of \(X^l\). But as it was proved above elements of \(K\) have all possible indexes on \(X^l\), so as a result of conjugation \(\alpha\kappa\alpha^{-1}\) we obtain an element from \(K\).

Check that the set of commutators is closed with respect to multiplication of commutators. Let \(\kappa_1, \kappa_2\) be an arbitrary commutators of \(G_k\). The parity of the number of vertex permutations on \(X^l\) in the product \(\kappa_1\kappa_2\) is determined exceptionally by the parity of the numbers of active v.p. on \(X^l\) in \(\kappa_1\) and \(\kappa_2\) (independently from the action of v.p. from the higher levels). Thus \(\kappa_1\kappa_2\) has an even index on \(X^l\).

Hence, a normal closure of the set \(K\) coincides with \(K\). It means that commutator subgroup of \(Syl_2A_{2k}\) consists of commutators.

\(\square\)

**Proposition 24.** An element \((g_1, g_2)\sigma^i \in G'_k\) iff \(g_1, g_2 \in G_{k-1}\) and \(g_1g_2 \in B_{k-1}'\).

**Proof.** Since, if \((g_1, g_2) \in G'_k\) then indexes of \(g_1\) and \(g_2\) on \(X^{k-1}\) are even according to Lemma 23, thus, \(g_1, g_2 \in G_{k-1}\). A sum of indexes of \(g_1\) and \(g_2\) on \(X^l\), \(l < k - 1\) are even according to Lemma 23, too, so index of product \(g_1g_2\) on \(X^l\) is even. Thus, \(g_1g_2 \in B_{k-1}'\). Hence, necessity is proved.

Let us prove the sufficiency via Lemma 24. Wise versa, if \(g_1, g_2 \in G_{k-1}\) then indexes of these automorphisms on \(X^{k-2}\) of subtrees \(v_{11}X^{[k-1]}\) and \(v_{12}X^{[k-1]}\) are even as elements from \(G'_k\) have. The product \(g_1g_2\) belongs to \(B_{k-1}'\) by condition of this Lemma and so sum of indexes of \(g_1, g_2\) on any level \(X^l\), \(0 \leq l < k - 1\) is even. Thus, the characteristic property of \(G'_k\) described in Lemma 24 holds. \(\square\)

**Lemma 25.** A number of active vertices of any automorphism \(g \in G'_k\) on \(X_1\) and on \(X_2\) of \(X^l\) a equals by mod2.

According to Lemma 24 and Lemma 24 an index of element of \(G'_k\) on any level of \(X^{[k]}\) is even. Also it was proved in statement about Frattini subgroup of \(G_k\) in [16] acts
by all even permutations on $X^l$, where $1<l<k$. Therefore, equality

$$
\prod_{j=1}^{2^{i-1}} g_{ij} = \prod_{j=2^{i-1}+1}^{2^i} g_{ij}
$$

is true. Where $g_{ij}$ is trivial or equal $(1,2)$.

In case $i = k-1$ there are the even number of transpositions on each of last levels of $v_{11}X^{[k-2]}$ and of $v_{12}X^{[k-2]}$. Therefore the following condition

$$
\prod_{j=1}^{2^{k-2}} g_{k-1j} = \prod_{j=2^{k-2}+1}^{2^{k-1}} g_{k-1j} = e
$$

holds.

**Remark 26.** The second commutator of $G_k$ has three equations of form (7). These equations are follows:

$$
\prod_{j=1}^{2^{i-1}} g_{ij} = \prod_{j=2^{i-1}+1}^{2^i} g_{ij},
$$

equation of level.

$$
\prod_{j=1}^{2^{i-2}} g_{ij} = \prod_{j=2^{i-2}+1}^{2^{i-1}} g_{ij},
$$

equation of a left sublevel $X_1$.

$$
\prod_{j=2^{i-1}+1}^{2^{i-1}+\frac{i}{4}} g_{ij} = \prod_{j=2^{i-2}+1}^{2^{i-1}} g_{ij},
$$

equation of right sublevel $X_2$.

Thus, level subgroup $G_k'(l)$ of $G_k''$ contains in 4 times lesser number of combinations of vertexes which are active then $G_k'(l)$ of $G_k'$.

**Proof.** In fact, due to structure of second commutator described in Remark 22 level subgroup of $G_k'$ has one relation 8 but level subgroup of $G_k''$ has 4 relations of form 10 and . Therefore, a number of valid cortege of $G_k''(l)$ in 4 times lesser then a number of $G_k'(l)$ cortege.

**Proposition 27.** Any minimal generating set of $\text{Syl}_2^l A_{2k}$ consist of $2k - 3$ elements.
Proof. The proof is based on two facts $G_k^pG_k^\prime \simeq G_k^{p^2}$ and $G_k^p \trianglelefteq G_k^pG_k^\prime$ (more exact it is based on Proposition 19) and $|G' : G_k^2G_k^\prime| = 2^{k-3}$. We could construct a homomorphism from $G_k'(/l)$ onto $C_2$ in the following way as we make it in 16 but we choose more short way to show that $|G' : G_k^2G_k^\prime| = 2^{k-3}$. Since a group $G_k^2$ contains the subgroup $G'$ then a product $G^2G'$ contains all elements from the commutant. In the author’s paper 16 in Proposition 3 it was that shown $G_k^2 \simeq G_k'$ for this group $G_k$. Further for finding the Frattini factor it is enough to calculate $|G' : G_k^\prime|$ because as it was shown that $\Phi(Syl_2A_2k) = Syl_2'(A_2k)$ according to Proposition 19 and $G_{k-1}'' \simeq G_{k-2}' \boxtimes G_{k-2}'$.

Indeed, the sections subgroup of second commutator has structure

$$G_{k-1}' \simeq G_{k-2} \boxtimes G_{k-2}.$$ 

The second commutator could be presented as $G_{k-1}'' \simeq G_{k-2}' \boxtimes G_{k-2}'$. By recursive principle we express $G_{k-1}'' \simeq (G_{k-3} \boxtimes G_{k-3}) \boxtimes (G_{k-3} \boxtimes G_{k-3})$. Since in such presentation of $G_{k-1}''$ too new restrictions (constrainted) arise then order of $G_{k-1}''$ in 4 times lesser than the order of $G_{k-1}'$, where $0 < l < k$. Also this ratio is proven in Remark 26. On the level $X^l$ the order of $G_k'$ is $2^l$ and order of $G_{k-1}'$ is $2^l$. Therefore a quotient group $G_{k-1}'(1)/G_{k-1}'(1) \simeq C_2$. On the rest of levels $X^l$, $1 < l < k$, the order of the subgroup $G_{k-1}'(l)$ is $2^{l-1}$ and the order of $G_{k-1}'(l)$ is $2^{l-3}$. Only on first level $X^1$ these relations of orders are changed $|G_{k-1}'(1)| = 2^1$. Thus, the index of $G_{k-1}''$ in $G_{k-1}'$ is $2k - 3$.

Since $G_k \simeq W_k \rtimes B_k$ and we have the same homomorphism from $G_k(l)$ to $C_2$, for $l \leq k$, then $G_k'/G_{k-1}'(l) \simeq G_k^p/G_{k-1}'(l) \simeq C_2$. Thus, according to Frattini theorem, a minimal generating set of $Syl_2' A_{2k}$ consists of $2k - 3$ elements. As well known, the orders of the irreducible generating set for $p$-group are equal to each other.

This result was confirmed by the algebraic system GAP. Indeed, it was founded by GAP that the minimal generating set for $Syl_2' A_{8}$.

Example 1. Minimal generating set of commutator of $Syl_2' (A_8)$ consist of elements: 

$\{ e, (13)(24)(57)(68), (12)(34), (14)(23)(57)(68), (56)(78), (13)(24)(58)(67), (12)(34)(56)(78), (14)(23)(58)(67) \}$. The commutator $Syl_2'(A_8) \simeq C_2^3$ that is an elementary abelian 2-group of order 8.

Example 2. A minimal generating set of $Syl_2' (A_8)$ consist of 3 generators: $(1, 3)(2, 4)(5, 7)(6, 8), (1, 2)(3, 4), (1, 3)(2, 4)(5, 8)(6, 7)$.

Example 3. Minimal generating set of $Syl_2' (A_{16})$ consist of 5 (or $2 \cdot 4 - 3$) generators: 

$(1, 4, 2, 3)(5, 6)(9, 12)(10, 11), (1, 4)(2, 3)(5, 8)(6, 7), (1, 2)(5, 6)$.
(1, 7, 3, 5)(2, 8, 4, 6)(9, 14, 12, 16)(10, 13, 11, 15) (1, 7)(2, 8)(3, 6)(4, 5)(9, 16, 10, 15) ×
× (11, 14, 12, 13).

4 Conclusion

The size of minimal generating set for commutator of Sylow 2-subgroup of alternating group $A_{2k}$ was proven is equal to $2k - 3$.

A new approach to presentation of Sylow 2-subgroups of alternating group $A_{2k}$ was applied. As a result the short proof of a fact that commutator width of Sylow 2-subgroups of alternating group $A_{2k}$, permutation group $S_{2k}$ and Sylow $p$-subgroups of $Syl_2 A_{2k}$ ($Syl_2 S_{2k}$) are equal to 1 was obtained. Commutator width of permutational wreath product $B \wr C_n$ were investigated.

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