Asymptotic analysis of powers of matrices

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Abstract

We analyze the representation of $A^n$ as a linear combination of $A^j$, $0 \leq j \leq k - 1$, where $A$ is a $k \times k$ matrix. We obtain a first order asymptotic approximation of $A^n$ as $n \to \infty$, without imposing any special conditions on $A$.

We give some examples showing the application of our results.

Keywords: Matrix powers, asymptotic approximations of integrals, generating functions.

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1 Introduction

In a recent article [1], Abu-Saris and Ahmad showed how to compute the powers of a matrix without having to compute its eigenvalues. Their main result was:

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Theorem 1  If $A$ is a $k \times k$ matrix with characteristic polynomial

$$P(x) = x^k + \sum_{j=0}^{k-1} a_j x^j,$$  \hspace{1cm} (1)

then,

$$A^n = \sum_{j=0}^{k-1} b_j(n) A^j, \hspace{0.5cm} n \geq k$$  \hspace{1cm} (2)

where

$$b_j(k) = -a_j, \hspace{0.5cm} 0 \leq j \leq k - 1, \hspace{0.5cm} b_{-1}(n) = 0, \hspace{0.5cm} n \geq k,$$  \hspace{1cm} (3)

$$b_j(n+1) = b_{j-1}(n) - a_j b_{k-1}(n), \hspace{0.5cm} n \geq k, \hspace{0.5cm} 0 \leq j \leq k - 1.$$

The purpose of this paper is to find an asymptotic representation for the numbers $b_j(n)$ as $n \to \infty$, which using (2) will give an asymptotic representation of $A^n$ for large $n$. Since the coefficients $b_j(n)$ depend only on $P(x)$, our estimates will be valid for similar matrices.

The asymptotic behavior of powers of matrices has been considered before by other authors. In [5] and [6], Gautschi computed upper bounds for $A^n$ and $\|A^n\|$, where $\|A\|$ is a norm of $A$. Estimates of $\|A^n\|$ were also studied in [2], [3], [11], [13] and [14].

In [4], Friedland and Schneider considered the matrix

$$B^{(m)} = A^m (I + \cdots + A^{q-1}), \hspace{0.5cm} m \geq 1$$

where $A$ is a nonnegative matrix and $q$ is a certain positive integer. They proved a theorem on the growth of $B^{(m)}$ under the assumption that the spectral radius of $A$ is equal to one. Powers of nonnegative matrices were also analyzed by Lindqvist in [8]. Rothblum [10], obtained Cesaro asymptotic expansions of $\sum_{i=0}^{N} A^i$, where $A$ is a complex matrix with spectral radius less than or equal to one.

This paper is organized as follows: In Section 2 we find an integral representation for the exponential generating function $G_j(z)$ of the coefficients $b_j(n)$. We obtain exact formulas for $G_j(z)$ and $b_j(n)$ in the special case of the matrix $A$ having $k$ distinct eigenvalues. We conclude the section with some examples.
In Section 3 we give an exact representation and a first order asymptotic approximation for $b_j(n)$, as $n \to \infty$. We consider the cases of simple and multiple eigenvalues. Our formulas are relatively easy to implement and offer very accurate estimates of $b_j(n)$, and therefore of $A^n$, for large $n$. We present some examples for different cases of $P(x)$.

2 Generating function

In this section we shall find an exponential generating function for the coefficients $b_j(n)$. First, let us define the spectral radius $\rho(A)$ of the matrix $A$ by
\[ \rho(A) = \max \{|\lambda| \mid P(\lambda) = 0\}. \quad (4) \]

**Theorem 2** Let $G_j(z)$ be defined by
\[ G_j(z) = \sum_{n \geq 0} b_j(n+k) \frac{z^n}{n!}, \quad (5) \]

Then, we have
\[ G_j(z) = -\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{s^{k-j-1} \rho_j(s)}{P(s)} e^{sz} ds, \quad (6) \]

where $c > \rho(A)$,
\[ p_j(s) = \sum_{l=0}^{j} a_l x^l, \quad 0 \leq j \leq k - 1, \quad (7) \]

and $P(s)$ is the characteristic polynomial of $A$ defined in (7).

**Proof.** If we use (5) in (3), we obtain
\[ G_j' = G_{j-1} - a_j G_{k-1}, \quad G_j(0) = -a_j, \quad 0 \leq j \leq k - 1 \quad (8) \]

with $G_{-1}(z) = 0$. Taking the Laplace transform of $G_j(z)$,
\[ L_j(s) = \int_0^\infty G_j(z) e^{-zs} dz \]
in (8) we get
\[ sL_j + a_j = L_{j-1} - a_j L_{k-1}, \quad 0 \leq j \leq k - 1 \] (9)
and \( L_{-1}(z) = 0 \). ■

The solution of (9) is given by
\[ L_j(s) = -s^{k-j-1} \frac{P_j(s)}{P(s)}, \quad 0 \leq j \leq k - 1 \] (10)
where
\[ p_j(s) = \sum_{l=0}^{j} a_l x^l, \quad 0 \leq j \leq k - 1, \]
and \( p_{-1}(s) = 0 \). Inverting the Laplace transform in (10) the theorem follows.

**Remark 3** Since
\[ \lim_{|s| \to \infty} L_j(s) = 0, \quad 0 \leq j \leq k - 1 \]
we can replace the Bromwich contour in (7) with a circle \( C \) of radius \( R \) centered at the origin \( \mathbb{C} \).

\[ G_j(z) = -\frac{1}{2\pi i} \int_C s^{k-j-1} \frac{P_j(s)}{P(s)} e^{zs} ds \] (11)
with \( R > \rho(A) \).

**Corollary 4** If
\[ P(s) = (s - \lambda_1)(s - \lambda_2) \cdots (s - \lambda_k), \]
where the eigenvalues \( \lambda_i \) are all distinct, then
\[ G_j(z) = -\sum_{l=1}^{k} (\lambda_i)^{k-j-1} \frac{P_j(\lambda_i)}{P'(\lambda_i)} \exp(\lambda_i z) \] (12)
and
\[ b_j(n) = -\sum_{l=1}^{k} (\lambda_i)^{n-j-1} \frac{P_j(\lambda_i)}{P'(\lambda_i)}. \] (13)
Proof. Applying the residue theorem to (11) we obtain
\[ G_j(z) = - \sum_{P(\lambda) = 0} \text{Res} \left[ s^{k-j-1} \frac{P_j(s)}{P(s)} e^{zs}; \lambda \right] \] (14)
which in turn gives (12) after computing
\[ \text{Res} \left[ s^{k-j-1} \frac{P_j(s)}{P(s)} e^{zs}; \lambda_l \right] = \lim_{s \to \lambda_l} s^{k-j-1} \frac{P_j(s)}{P(s)} \frac{e^{zs}}{P'(s)} = (\lambda_l)^{k-j-1} \frac{P_j(\lambda_l)}{P'(\lambda_l)} \exp(\lambda_l z). \]

Writing (12) as
\[ G_j(z) = - \sum_{l=1}^{k} (\lambda_l)^{k-j-1} \frac{P_j(\lambda_l)}{P'(\lambda_l)} \sum_{n \geq 0} (\lambda_l)^n \frac{z^n}{n!} \]
and changing the order of summation, we have
\[ G_j(z) = \sum_{n \geq 0} \left[ - \sum_{l=1}^{k} (\lambda_l)^{n+k-j-1} \frac{P_j(\lambda_l)}{P'(\lambda_l)} \right] \frac{z^n}{n!} \]
which implies
\[ b_j (n + k) = - \sum_{l=1}^{k} (\lambda_l)^{n+k-j-1} \frac{P_j(\lambda_l)}{P'(\lambda_l)}. \]

Example 5 In [1] the authors considered the following examples:
1. \[ P(x) = x^3 - 7x^2 + 16x - 12 = (x - 2)^2 (x - 3). \]
Using (11) we have
\[ L_0(s) = \frac{-s^2 (12)}{(s - 2)^2 (s - 3)} \]
\[ L_1(s) = \frac{-s (16s - 12)}{(s - 2)^2 (s - 3)} \]
\[ L_2(s) = \frac{-(7x + 16x - 12)}{(s - 2)^2 (s - 3)} \]
and inverting we obtain

\[ G_0(z) = -12 (8 + 4z) e^{2z} + 108e^{3z} \]
\[ G_1(z) = 4 (23 + 10z) e^{2z} - 108e^{3z} \]
\[ G_2(z) = -4 (5 + 2z) e^{2z} + 27e^{3z}. \]

Expanding in series we get

\[ G_0(z) = -96 \sum_{n=0}^{\infty} \frac{2^n z^n}{n!} - 48 \sum_{n=0}^{\infty} \frac{2^{n-1} n z^n}{n!} + 108 \sum_{n=0}^{\infty} \frac{3^n z^n}{n!} \]

and from (5) we conclude that

\[ b_0(n+3) = -96 \times 2^n - 48 \times 2^{n-1} n + 108 \times 3^n \]

or

\[ \begin{align*}
    b_0(n) &= -96 \times 2^{n-3} - 48 \times 2^{n-4} (n - 3) + 108 \times 3^{n-3} \\
    &= -3 (1 + n) \times 2^n + 4 \times 3^n.
\end{align*} \]

Similar calculations give

\[ b_1(n) = \left( 4 + \frac{5}{2} n \right) \times 2^n - 4 \times 3^n \]
\[ b_2(n) = -\left( 1 + \frac{1}{2} n \right) \times 2^n + 3^n, \]

in agreement with the results shown in [11].

2.

\[ P(x) = x^3 - 5x^2 + 6x = x (x - 2) (x - 3). \]

We can apply (13) directly and obtain

\[ b_0(n) = 0 \]
\[ b_1(n) = \frac{3}{2} \times 2^n - \frac{2}{3} \times 3^n \]
\[ b_2(n) = -\frac{1}{2} \times 2^n + \frac{1}{3} \times 3^n. \]
3.  

\[ P(x) = x^5 - 5x^4 + 10x^3 - 20x^2 - 15x - 4 = (x - 4)(x^4 - x^3 + 6x^2 + 4x + 1) \].

Although (as the authors noted) MAPLE is unable to compute the zeros of \( P(x) \) exactly, it can provide us with very accurate numerical approximations

\[
\begin{align*}
\lambda_1 &= 4 \\
\lambda_2 &= 0.8090169944 + 2.489898285i \\
\lambda_3 &= 0.8090169944 - 2.489898285i \\
\lambda_4 &= -0.3090169944 + 0.2245139883i \\
\lambda_5 &= -0.3090169944 - 0.2245139883i
\end{align*}
\]

which we can use in (13) to get

\[
b_j(n) = \frac{C_j}{305} 4^n - \sum_{l=2}^{5} (\lambda_l)^{n-j} \frac{p_j(\lambda_l)}{P'(\lambda_l)}
\]

with

\[
C_0 = 1, C_1 = 4, C_2 = 6, C_3 = -1, C_4 = 1.
\]

Note that, for \( 0 \leq j \leq 4 \), we have

\[
\sum_{l=2}^{5} (\lambda_l)^{n-j} \frac{p_j(\lambda_l)}{P'(\lambda_l)} = O (|\lambda_2|^n) = O (2.618^n)
\]

as \( n \to \infty \).

**Example 6**  Let \( A \) be the matrix

\[
A = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}
\]

with characteristic polynomial

\[ P(x) = x^2 + 1 = (x - i)(x + i). \]

Using (13) we have

\[
\begin{align*}
b_0(n) &= \cos \left( \frac{\pi}{2} n \right) \\
b_1(n) &= \sin \left( \frac{\pi}{2} n \right)
\end{align*}
\]
and from (2) we get

\[ A^n = \begin{pmatrix} \cos \left( \frac{\pi}{2} n \right) + \sin \left( \frac{\pi}{2} n \right) & 2 \sin \left( \frac{\pi}{2} n \right) \\ -\sin \left( \frac{\pi}{2} n \right) & \cos \left( \frac{\pi}{2} n \right) - \sin \left( \frac{\pi}{2} n \right) \end{pmatrix}. \]

In particular, we have

\[ A^n = \begin{cases} I, & n \equiv 0(4) \\ A, & n \equiv 1(4) \\ -I, & n \equiv 2(4) \\ -A, & n \equiv 3(4) \end{cases} \]

where \( I \) denotes the identity matrix.

**Example 7** This example appeared in [9]. Let \( A \) be the matrix

\[ A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \]

with characteristic polynomial

\[ P(x) = x^2 - x - 1 = (x - \alpha)(x - \beta), \]

where

\[ \alpha = \frac{1}{2} \left( 1 + \sqrt{5} \right), \quad \beta = \frac{1}{2} \left( 1 - \sqrt{5} \right). \]

Then, from [13], we have

\[ b_0(n) = \frac{1}{\sqrt{5}} (\alpha^{n-1} - \beta^{n-1}) = f_{n-1} \]

\[ b_1(n) = \frac{1}{\sqrt{5}} (\alpha^n - \beta^n) = f_n \]

where \( f_n \) is the \( n \)th Fibonacci number. Thus,

\[ A^n = \begin{pmatrix} f_n + f_{n-1} & f_n \\ f_n & f_{n-1} \end{pmatrix} = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix}. \]
3 Asymptotic analysis

We begin by finding an integral representation of the coefficients $b_j(n)$.

**Lemma 8** The numbers $b_j(n)$ can be represented as

$$b_j(n) = -\frac{1}{2\pi i} \int_C s^{n-j-1} \frac{p_j(s)}{P(s)} ds$$

(15)

where $C$ is a circle of radius $R > \rho(A)$ centered at the origin and the polynomials $p_j(s)$ were defined in (7).

**Proof.** Since the power series

$$e^{zs} = \sum_{n=0}^{\infty} s^n z^n / n!$$

converges uniformly on $|s| \leq R$, we can interchange integration and summation in (11) and obtain

$$G_j(z) = \sum_{n=0}^{\infty} \left[ -\frac{1}{2\pi i} \int_C s^{n+k-j-1} \frac{p_j(s)}{P(s)} ds \right] \frac{z^n}{n!}.$$ 

Then, (5) implies

$$b_j(k + n) = -\frac{1}{2\pi i} \int_C s^{n+k-j-1} \frac{p_j(s)}{P(s)} ds, \quad 0 \leq j \leq k - 1$$

and the result follows. 

**Remark 9** An alternative method for approximating the coefficients $b_j(n)$ is to write (15) as

$$b_j(n) = -\frac{R^{n-j}}{2\pi} \int_0^{2\pi} \exp [it(n - j)] \frac{p_j(Re^{it})}{P(Re^{it})} dt$$

(16)

with $R > \rho(A)$ and to compute the integral (16) numerically. This approach offers the advantage of avoiding the computation of the eigenvalues of $A$. 

We now have all the necessary elements to establish our main theorem.

**Theorem 10** Let

\[ \rho(A) = |\lambda| > |\lambda_2| > \cdots > |\lambda_r| \]

be the eigenvalues of the matrix \( A \), i.e.,

\[ P(x) = (x - \lambda)^m (x - \lambda_2)^{m_2} \cdots (x - \lambda_r)^{m_r} \tag{17} \]

with \( r \leq k \). Then,

\[ b_j(n) \sim -\lambda^{n-m-j} \frac{p_j(\lambda)}{P'(\lambda)} m! \binom{n-k}{m-1}, \quad n \to \infty, \tag{18} \]

where

\[ P'(\lambda) = \frac{d^n P}{ds^n} \bigg|_{s=\lambda}. \]

**Proof.** To find an asymptotic approximation of (15), we shall use a modified version of Darboux’s Method [12]. We write

\[ s^{n-j-1}p_j(s) \frac{P(s)}{P(s)} = s^{n-k} \times \frac{s^{k-j-1}p_j(s)}{P(s)}, \]

so that \( \deg (s^{k-j-1}p_j) = k - 1 \) and \( \deg (P) = k \).

From (17) we have

\[ \frac{s^{k-j-1}p_j(s)}{P(s)} \sim \frac{\lambda^{k-j-1}p_j(\lambda)}{(s - \lambda)^m g(\lambda)}, \quad s \to \lambda \tag{19} \]

where

\[ g(x) = (x - \lambda_2)^{m_2} \cdots (x - \lambda_r)^{m_r}. \]

Using the Binomial Theorem, we obtain

\[ s^{n-k} = \sum_{l=0}^{n-k} (s - \lambda)^l \binom{n-k}{l} \lambda^{n-k-l}. \tag{20} \]

Combining (19) and (20), we get

\[ s^{n-j-1}p_j(s) \frac{P(s)}{P(s)} \sim \lambda^{n-n-j} \frac{p_j(\lambda)}{g(\lambda)} \binom{n-k}{m-1} \frac{1}{(s - \lambda)}, \quad s \to \lambda \]
and therefore
\[
b_j(n) = -\frac{1}{2\pi i} \int_C s^{n-j-1} \frac{p_j(s)}{P(s)} \, ds \sim -\lambda^{n-m-j} \frac{p_j(\lambda)}{g(\lambda)} \frac{n-k}{m-1}, \quad n \to \infty. \tag{21}
\]

To find the value of \(g(\lambda)\), we use L’Hopital’s Theorem
\[
g(\lambda) = \lim_{s \to \lambda} \frac{P(s)}{(s-\lambda)^m} = \lim_{s \to \lambda} \frac{P^{(m)}(s)}{m!} = \frac{P^{(m)}(\lambda)}{m!}. \tag{22}
\]

Replacing (22) in (21), we obtain (18).

\[\text{Remark 11} \quad \text{Note that when } m = 1 \text{ we recover the leading term in (13).}\]

If more than one eigenvalue has absolute value equal to the spectral radius of \(A\), the asymptotic behavior of \(b_j(n)\) can be obtained by adding the contributions from each eigenvalue. We state this formally in the following corollary.

\[\text{Corollary 12 If } \rho(A) = |\lambda_1| = |\lambda_2| = \cdots = |\lambda_r|, \]

with respective multiplicities \(m_1, m_2, \ldots, m_r\), then
\[
b_j(n) \sim -\sum_{l=1}^{r} (\lambda_l)^{n-m_l-j} \frac{p_j(\lambda_l)}{P^{(m_l)}(\lambda_l)} \left( \frac{n-k}{m_l-1} \right), \quad n \to \infty. \tag{23}
\]

\[\text{Remark 13 Since} \]

\[
\left( \frac{n-k}{m_l-1} \right) \sim \frac{1}{(m_l-1)!} n^{m_l-1}, \quad n \to \infty
\]

we have
\[
b_j(n) \sim -\sum_{l=1}^{r} (\lambda_l)^{n-m_l-j} \frac{p_j(\lambda_l)}{P^{(m_l)}(\lambda_l)} m_l n^{m_l-1}, \quad n \to \infty.
\]

Therefore, in the case of several eigenvalues located on the circle \(|s| = \rho(A)|\), the dominant term in (23) will correspond to the eigenvalue with the greatest multiplicity.
Example 14 In Example 1 (1) we consider

\[ P(x) = x^3 - 7x^2 + 16x - 12 = (x - 2)^2 (x - 3). \]

In this case, \( \lambda = 2 \), \( m = 2 \) and \( k = 3 \). From (13), we get

\[ b_0(n) \sim 4 \times 3^n \]
\[ b_1(n) \sim -4 \times 3^n \]
\[ b_2(n) \sim 3^n \]

which are the leading terms in the solution previously obtained.

Example 15 We now consider the case of more than one eigenvalue having absolute value equal to \( \rho(A) \). Let

\[ P(x) = x^4 + x^3 - 15x^2 - 9x + 54 = (x - 2)(x - 3)(x + 3)^2. \]

In this case, \( \lambda_1 = -3 \), \( m_1 = 2 \), \( \lambda_2 = 3 \), \( m_2 = 1 \) and \( k = 4 \). From (23), we have

\[ b_0(n) \sim -\frac{1}{5} n (-3)^n + \frac{4}{5} (-3)^n - \frac{1}{2} 3^n \]
\[ b_1(n) \sim \frac{1}{10} n (-3)^n - \frac{2}{5} (-3)^n - \frac{1}{12} 3^n \]
\[ b_2(n) \sim \frac{1}{45} n (-3)^n - \frac{1}{45} (-3)^n + \frac{1}{9} 3^n \]
\[ b_3(n) \sim -\frac{1}{90} n (-3)^n + \frac{2}{45} (-3)^n + \frac{1}{36} 3^n. \]

The exact values are

\[ b_0(n) = -\frac{1}{5} n (-3)^n + \frac{21}{50} (-3)^n - \frac{1}{2} 3^n + \frac{27}{25} 2^n \]
\[ b_1(n) = \frac{1}{10} n (-3)^n - \frac{83}{300} (-3)^n - \frac{1}{12} 3^n + \frac{9}{25} 2^n \]
\[ b_2(n) = \frac{1}{45} n (-3)^n + \frac{2}{225} (-3)^n + \frac{1}{9} 3^n - \frac{3}{25} 2^n \]
\[ b_3(n) = -\frac{1}{90} n (-3)^n + \frac{11}{900} (-3)^n + \frac{1}{36} 3^n - \frac{1}{25} 2^n. \]

As we observed before, the main contribution comes from the eigenvalue of maximum multiplicity, in this case \( \lambda_1 = -3 \).
Example 16  Finally, let’s consider the case of complex eigenvalues of multiplicity greater than one located on the circle \(|s| = \rho(A)\). Let

\[
P(x) = x^5 - 9x^4 + 34x^3 - 66x^2 + 65x - 25
\]

\[
= (x - 1) [x - (2 + i)]^2 [x - (2 - i)]^2.
\]

In this case, \(\lambda_1 = 2 + i, \ m_1 = 2, \lambda_2 = 2 - i, \ m_2 = 2\) and \(k = 5\). From (23), we obtain

\[
b_0(n) \sim \frac{1}{4} \left(\sqrt{5}\right)^n (n - 5) \left[\cos(\theta_n) - 7 \sin(\theta_n)\right]
\]

\[
b_1(n) \sim -\frac{1}{10} \left(\sqrt{5}\right)^n (n - 5) \left[2 \cos(\theta_n) - 39 \sin(\theta_n)\right]
\]

\[
b_2(n) \sim -\frac{1}{10} \left(\sqrt{5}\right)^n (n - 5) \left[2 \cos(\theta_n) + 31 \sin(\theta_n)\right]
\]

\[
b_3(n) \sim \frac{1}{10} \left(\sqrt{5}\right)^n (n - 5) \left[2 \cos(\theta_n) + 11 \sin(\theta_n)\right]
\]

\[
b_4(n) \sim -\frac{1}{20} \left(\sqrt{5}\right)^n (n - 5) \left[\cos(\theta_n) + 3 \sin(\theta_n)\right]
\]

with

\[
\theta = \arctan \left(\frac{1}{2}\right).
\]

The exact values are

\[
b_0(n) = \frac{1}{4} \left(\sqrt{5}\right)^n [(n - 21) \cos(\theta_n) + (-7n + 22) \sin(\theta_n)] + \frac{25}{4}
\]

\[
b_1(n) = -\frac{1}{10} \left(\sqrt{5}\right)^n [2(n - 50) \cos(\theta_n) + (-39n + 125) \sin(\theta_n)] - 10
\]

\[
b_2(n) = -\frac{1}{10} \left(\sqrt{5}\right)^n [(2n + 65) \cos(\theta_n) + (31n - 100) \sin(\theta_n)] + \frac{13}{2}
\]

\[
b_3(n) = \frac{1}{10} \left(\sqrt{5}\right)^n [2(n + 10) \cos(\theta_n) + (11n - 35) \sin(\theta_n)] - 2
\]

\[
b_4(n) = -\frac{1}{20} \left(\sqrt{5}\right)^n [(n + 5) \cos(\theta_n) + (3n - 10) \sin(\theta_n)] + \frac{1}{4}.
\]

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