GEOMETRICAL EXPRESSION FOR
SHORT-DISTANCE SINGULARITIES IN FIELD THEORY*

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ABSTRACT

We consider the linear space of composite fields as an infinite dimensional vector
bundle over the theory space whose coordinates are simply the parameters of a
renormalized field theory. We discuss a geometrical expression for the short dis-
tance singularities of the composite fields in terms of beta functions, anomalous
dimensions, and a connection.

1. Introduction

The idea of theory space was introduced to understand continuum limits in field
theory. Though we have gained much qualitative understanding of field theory
(e.g., renormalization group (RG) flows, fixed points, universality), very few quan-
titative results have been obtained so far. A notable exception is the c-theorem
on the monotonic decrease of the c-function along the RG flows on the space of
two-dimensional field theories.

It is important to understand the theory space quantitatively, since the theory
space is the natural framework in which we discuss field theory. For example, the
continuum limits of field theories can be defined non-perturbatively as the image of
an infinite number of RG transformations on the theory space.

Another reason we should study the theory space is related to a yet unknown
non-perturbative formulation of string theory. It has been advocated for long that
the string action should be a function on the space of all possible two dimensional
field theories. We hardly know anything about the space.

The theory space we will discuss below is more restricted than the full theory
space that includes all possible field theories with a UV cutoff. We will only consider
the space of renormalizable field theories. Then, given a renormalizable theory with
$N$ parameters, we obtain an $N$-dimensional theory space. We will first clarify the
geometrical objects on the theory space and next show that certain short distance
singularities adopt a geometrical interpretation.

* Based upon a talk given at the International Colloquium on Modern Quantum Field Theory, 5–11
January 1994, Bombay, India. This work was supported in part by the U.S. Department of Energy, under
Contract DE-AT03-88ER 40384 Mod A006 Task C.
2. Theory Space

We consider a renormalized field theory in $D$-dimensional euclidean space with parameters $g^i(i = 1, ..., N)$. Let the origin $g^i = 0$ be an UV fixed point. We regard $g^i$'s as local coordinates of a finite dimensional theory space. The parameters satisfy the renormalization group equations

$$\frac{d}{dt} g^i = \beta^i(g).$$

The beta functions $\beta^i$ form a vector field on the theory space.

Let $\mathcal{O}_i$ be a scalar composite field conjugate to $g^i$: the field $\mathcal{O}_i$ generates a deformation of the theory in the $g^i$ direction. The conjugate fields make a basis of the tangent vector bundle over the theory space, since the fields transform as

$$\mathcal{O}_i \rightarrow \mathcal{O}'_i = \frac{\partial g^j}{\partial g'^i} \mathcal{O}_j$$

under an arbitrary coordinate change $g \rightarrow g'$.

We introduce a basis of composite fields $\{\Phi_a\}$ at each point $g$ on the theory space. Since the composite fields make a linear space at every $g$, they form an infinite dimensional vector bundle over the theory space. We let $\Gamma^b_a(g)$ be the full scale dimension of the composite fields:

$$\frac{d}{dt} \Phi_a = \Gamma^b_a(g) \Phi_b.$$

The choice of a basis is not unique, and we can transform the basis as

$$\Phi_a \rightarrow \Phi'_a = N^b_a(g) \Phi_b,$$

where $N(g)$ is an invertible matrix. For an arbitrary choice of $n$ different points $r_k(k = 1, ..., n)$ in space, the correlation function $\langle \Phi_{a_1}(r_1) ... \Phi_{a_n}(r_n) \rangle_g$ make a basis of rank-$n$ tensor fields over the theory space.

We now consider the operator product expansion (OPE)

$$\mathcal{O}_i(r) \Phi_a(0) = \frac{1}{\text{vol}(S^{D-1})} (C_i)_a^b(r; g) \Phi_b(0) + o\left(\frac{1}{r^D}\right),$$

where $C_i(r; g)$ denotes the part of the OPE coefficient which cannot be integrated over volume. We define a matrix by

$$H_i(g) \equiv C_i(r = 1; g).$$

This is a tensor over the theory space, since it transforms as

$$H_i(g) \rightarrow N(g) H_i(g) N(g)^{-1}$$

under Eq. (4).
3. Geometrical Expression for Short Distance Singularities

The following geometrical expression for the tensor $H_i$, defined by Eq. (6), has been derived:\[^3,4\]  
\[ H_i(g) = \frac{\partial \Psi(g)}{\partial g^i} + [c_i(g), \Psi(g)] + \beta^j(g) \Omega_{ji}(g). \]  
(8)  
where  
\[ \Psi \equiv \Gamma + \beta^i c_i, \quad \Omega_{ji} \equiv \partial_j c_i - \partial_i c_j + [c_j, c_i]. \]  
(9)  
It is easy to check that $\Psi(g)$ has the same transformation property as $H_i$ under Eq. (4). The tensor $\Omega_{ji}$ is the curvature of the connection $c_i$, and it has a field theoretic expression:  
\[ \Omega_{ji}(\Phi_g) = \int_{r \leq 1} d^D r \ F.P. \int_{r' \leq 1} d^D r' \ \left\{ \left( O_i(r) \left( O_j(r') - \frac{1}{\text{vol}(S^{D-1})} C_j(r') \right) - (r \leftrightarrow r') \right) \Phi(0) \right\}_{\psi}, \]  
(10)  
where F.P. stands for taking the integrable part in $r$. We will not repeat the derivation of Eqs. (8) and (10) here. They have been derived on the basis of an assumption that $O_i$ generates deformation of the theory under a change of $g^i$. Then the consistency with the RG gives the above expressions.

If we restrict our attention to the conjugate fields alone, then the matrix $\Gamma$ is given by  
\[ \Gamma_{ij} = D\delta^j_i - \frac{\partial \beta_j}{\partial g^i}. \]  
(11)  
Hence,  
\[ \Psi_{ij} = D\delta^j_i - \frac{\partial \beta_j}{\partial g^i} + \beta^k [c_k, \beta^j] = D\delta^j_i - \nabla_i \beta^j, \]  
(12)  
and  
\[ (H_i)_j^k = -\nabla_i \nabla_j \beta^k + \beta^j (\Omega_{ij})_j^k, \]  
(13)  
where $\nabla_i$ is the covariant derivative, and we have used the symmetry of the connection  
\[ (c_i)_j^k = (c_j)_i^k. \]  
(14)  
Eq. (13) is a recent result of Dolan.\[^5\] Note the symmetry  
\[ (H_i)_j^k = (H_j)_i^k. \]  
(15)  
due to the cyclic symmetry of the curvature:  
\[ (\Omega_{ij})_j^k + (\Omega_{ji})_j^k + (\Omega_{ij})_i^k = 0. \]  
(16)  
It is natural to ask if the connection $(c_i)_j^k$ is the unique Riemann connection of some metric $G_{ij}$ on the theory space. We do not know the answer in general, though on the space of two-dimensional conformal field theories the metric is given by the two-point function of the conjugate fields.\[^2,6\]
The connection arises naturally in field theory by considering infinitesimal changes of the correlation function \(\langle \Phi_a(r_1) \ldots \Phi_a(r_n) \rangle_g\). The naked derivative \(\partial/\partial g^i\) simply spoils the covariance of the correlation function under Eq. (4). To restore covariance we must introduce a connection. Then the covariant derivative
\[
\partial_i \langle \Phi_a(r_1) \ldots \Phi_a(r_n) \rangle_g + \sum_{k=1}^{n} (c_i)_{ab} \langle \Phi_a(r_1) \ldots \Phi_b(r_k) \ldots \Phi_a(r_n) \rangle_g
\]
is given by an integral of the conjugate field \(O_i\) with the UV singularities properly subtracted. Thus, the connection arises as finite counterterms.\[^3\]

4. Examples

We consider two examples here using dimensional regularization.

4.1. \((\phi^4)_4\) Theory

The lagrangian is given by
\[
\mathcal{L} = \frac{1}{2} \partial_\mu \phi_0 \partial_\mu \phi_0 + \frac{Z_m m^2}{2} \phi_0^2 + \frac{Z_\lambda \lambda}{4!} \phi_0^4,
\] (17)
where \(\phi_0\) is a bare field. Hence, the conjugate fields are
\[
O_\lambda = \frac{\partial \mathcal{L}}{\partial \lambda} \bigg|_{m^2, \phi_0} = \frac{\partial (Z_\lambda \lambda) \phi_0^4}{4!} + \frac{\partial Z_m}{\partial \lambda} m^2 \phi_0^2,
\]
\[
O_m = \frac{\partial \mathcal{L}}{\partial m^2} \bigg|_{\lambda, \phi_0} = Z_m \frac{\phi_0^2}{2}.
\] (18)
These fields are renormalized.\[^7\]

We can choose the parameters \(\lambda, m^2\) such that the following gauge conditions are satisfied:
\[
(c_\lambda)_{\lambda} \phi_0 = (c_\lambda)_{m} \phi_0 = 0.
\] (19)
This may amount to choosing a non-minimal subtraction scheme for \(Z_\lambda, Z_m\). In the above gauge, we find
\[
(c_\lambda)_{\lambda} m = m^2 (\Omega_m)_{\lambda}^m,
\] (20)
which turns out to be nonvanishing.\[^3\] Eq. (8) (or (13)) implies
\[
(H_m)_{\lambda}^m = (H_\lambda)_{m}^m = -\beta_m',
\]
\[
(H_\lambda)_{\lambda} = -\beta_m'(\lambda),
\] (21)
where \(\beta_\lambda\) is the beta function, and \(\beta_m\) is the anomalous dimension of the mass parameter. More details can be found elsewhere.\[^3\]

4.2. \(O(N)\) Nonlinear Sigma Model in \(D = 2\)

The lagrangian is given by
\[
\mathcal{L} = \frac{\partial_\mu \Phi_0^i \partial_\mu \Phi_0^i}{2Z_0 g},
\] (22)

\[^3\] In fact \(O_\lambda\) may still need counterterms proportional to \(\partial^2\phi^2\) and \(m^4\). Likewise \(O_m\) may need a term proportional to \(m^2\).
where $\Phi_0^I(I = 1, ..., N)$ is a bare field normalized by $\Phi_0^I \Phi_0^I = 1$, and $g$ is a renormalized temperature. The field conjugate to $g$ is

$$O_E = \frac{\partial \mathcal{L}}{\partial \Phi_0^I} = \frac{\partial}{\partial g} \left( \frac{1}{Z_g g} \right) \frac{1}{2} \partial_\mu \Phi_0^I \partial_\mu \Phi_0^I. \quad (23)$$

This is a renormalized field with anomalous dimension $-\beta_E(g)$, where $\beta_E(g)$ is the beta function.

Let us consider a dimensionless scalar field $\Phi^I \ldots^I_n$ which is a traceless rank-n symmetric tensor under $O(N)$. Let $\gamma_n(g)$ be its anomalous dimension. Then,

$$O_E(r) \Phi^I \ldots^I_n(0) = \frac{1}{2\pi} (C_E)^n_n(r; g) \Phi^I \ldots^I_n(0) + o \left( \frac{1}{r^2} \right), \quad (24)$$

where the OPE coefficient is given by

$$(C_E)^n_n(r; g) = \frac{\beta_E(g(\ln r))}{\beta_E(g)} (H_E)^n_n(g(\ln r)), \quad (25)$$

and $g(t)$ is the running parameter satisfying

$$\frac{d}{dt} g(t) = \beta_E(g(t)) \quad (g(0) = g). \quad (26)$$

Eq. (8) implies that

$$(H_E)^n_n(g) = \frac{d}{dg} (\gamma_n + \beta_E(c_E)^n_n). \quad (27)$$

We can change the normalization of the field $\Phi^I \ldots^I_n$ so that

$$(c_E)^n_n(g) = 0. \quad (28)$$

Then, we simply get

$$(H_E)^n_n(g) = \gamma'_n(g). \quad (29)$$

More details will be provided elsewhere.  

5. Conclusion

The results we have presented in the above are modest: the short distance singularities in the products of the conjugate fields and composite fields can be expressed geometrically in terms of beta functions, anomalous dimensions, and a connection. We would like to think that this is an indication that field theory awaits geometrical interpretation.

A more ambitious program is to consider the full theory space, i.e., the space of all possible field theories with a UV cutoff. In this case, however, we may not obtain any interesting local geometry. As we have remarked at the end of sect. 2, the connection arises as finite counterterms necessary for UV subtracted integrals. The cutoff theories do not need any counterterm, so we expect that a scheme exists
in which the connection vanishes locally on the theory space. But it is possible to find interesting **global** geometry with the flat connection.

We do not have any a priori reason that field theory should be interpreted geometrically, but we think that any geometrical structure of theory space is worth studying for its own sake.

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