The structure of spatial slices of three-dimensional causal triangulations

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Abstract. We consider causal 3-dimensional triangulations with the topology of $S^2 \times [0,1]$ or $D^2 \times [0,1]$ where $S^2$ and $D^2$ are the two-dimensional sphere and disc, respectively. These triangulations consist of slices and we show that these slices can be mapped bijectively onto a set of certain coloured two-dimensional cell complexes satisfying simple conditions. The cell complexes arise as the cross section of the individual slices.
1 Introduction

We investigate in this paper a class of problems that arise in the dynamical triangulation approach to 3-dimensional gravity restricted to the case of so-called causal triangulations. For an introduction to the dynamical triangulation approach to discrete quantum gravity we refer to [1] and an account of causal dynamical triangulations can be found in [2]. For the case of 3-dimensional gravity in particular, one may consult [3] and [4].

In [5] we gave a proof that the number $N(V)$ of causal 3-dimensional triangulations homeomorphic to a three-sphere and consisting of $V$ tetrahedra is exponentially bounded,

$$N(V) \leq C^V,$$

where $C$ is some positive constant. Validity of this bound is crucial in order for the relevant correlation functions to exist, and thereby defining the discretised model (see [1, 2]). The first step of the argument leading to this bound was to decompose the triangulations into slices (which is possible because of the causal structure defined below) and to show that it is sufficient to establish the bound for such causal slices. The number of causal slices was then shown to be bounded by the number of certain coloured two-dimensional cell complexes homeomorphic to the 2-sphere. These cell complexes arise as mid-sections of the slices. Finally, bounding the number of those two-dimensional cell complexes can be done using well known techniques.

The method of associating a coloured 2-dimensional cell complex with a causal triangulation has been applied earlier by other workers in the field [4] and also used in numerical simulations and in combination with matrix model techniques to extract properties of the model. It has, however, not been established exactly what class of 2-dimensional complexes can occur as mid-sections of a causal slice. As noted in [5], at least some non-trivial constraints have to be imposed on top of its homeomorphism class. It follows that the connection between the model defined in
terms of triangulations and the one realised in terms of a specific class of coloured two-dimensional cell complexes is obscure. It might be the case that the precise class of cell complexes is unimportant in a possible scaling limit of the model, but this remains to be investigated.

In this paper we provide a complete characterisation of the coloured 2-dimensional cell complexes that correspond to 3-dimensional causal slices. In fact, we shall consider not only the standard notion of causal triangulations homeomorphic to $S^2 \times [0, 1]$, where $S^2$ denotes the 2-sphere, which in this paper will be referred to as causal sphere-triangulations, but find it useful to generalise the notion of causal triangulation to manifolds homeomorphic to $D^2 \times [0, 1]$, where $D^2$ is the 2-dimensional disc, and $\partial D^2 \times [0, 1]$ may be viewed as the time-like part of the boundary while the two discs $D^2 \times \{0\}$ and $D^2 \times \{1\}$ form the spatial parts of the boundary. Such triangulations will be called causal disc-triangulations and the corresponding slices will be called causal disc-slices. The coloured cell complexes are in this case homeomorphic to $D^2$. For both cases we will give a full characterisation of the mid-sections and show that they are in bijective correspondence with the appropriate causal slices.

This result might serve as a starting point for obtaining an exact enumeration of causal slices by applying well known techniques for planar surfaces or maps to the coloured cell complexes homeomorphic to the disc. This interesting combinatorial problem is more complicated than those previously considered because of the colouring and the constraints identified in this paper. The result might also be instrumental in identifying an appropriate matrix model generating exactly the desired causal slices or as aid in designing effective numerical algorithms for simulations. These issues are, however, beyond the scope of the present paper.

2 Preliminaries and notation

We will use notation consistent with that of [5]. For the reader’s convenience we briefly recall the main conventions, restricting the discussion to the 3-dimensional case. The basic building blocks of our triangulations are tetrahedra or 3-simplices.
whose vertices have one of two colours: red or blue. Generally we will denote an unoriented simplex with vertices $x_1, \ldots, x_n$ by $(x_1 \ldots x_n)$. If all the vertices in a simplex have the same colour we say that the simplex is monocoloured and has that colour. This means that if $x$ and $y$ are red vertices and $e = (xy)$ is a 1-simplex then we say that $e$ is red, and a triangle $\Delta = (xyz)$ is red if its vertices (or edges) are red, etc. If a simplex is not monocoloured we say it is two-coloured. It is assumed that all tetrahedra are two-coloured. Thus the tetrahedra come in three types: type $(3,1)$ with three red vertices, type $(2,2)$ with two red vertices and type $(1,3)$ with one red vertex.

A triangulation is by definition an abstract simplicial complex whose formal definition can be found in [5] or [9]. For the purposes of this paper one may think of a triangulation as a collection of tetrahedra together with identifications of some pairs of triangles in the boundaries of the tetrahedra, respecting the colouring, such that any triangle is identified with at most one other triangle. When we identify triangles we also identify all their subsimplicies. This point of view will be used and explained in more detail in Section 4 below. Any pair of identified triangles is called an interior triangle of the triangulation while the other triangles are referred to as boundary triangles. It should be noted that it is implicit in the notion of a simplicial complex that

(i) two triangles contained in the same tetrahedron cannot be identified,

(ii) two different triangles in a given tetrahedron cannot be identified with two triangles contained in another tetrahedron.

It is common to speak about these two conditions as regularity conditions and about simplicial complexes as regular triangulations as opposed to singular triangulations, when one or both of these conditions are left out. In this paper we focus on regular triangulations.

One may think of a triangulation either as a purely combinatoric object or as a topological space embedded in a Euclidean space. In the former case two trian-
gulations are identified if there is a bijective correspondence between their vertices respecting the colouring and the pairwise identifications of triangles. In the latter case two triangulations are identified if there exists a homeomorphism between them mapping simplices to simplices and thus inducing a combinatorial identification. It is a fact, explained in e.g. [5], that the two points of view are equivalent.

We now introduce the basic objects of study in this paper, the two types of causal slices that form the building blocks of the general causal triangulations.

**Definition 1** A causal sphere-slice $K$ is a triangulation fulfilling the following conditions:

(i) $K$ is homeomorphic to the cylinder $S^2 \times [0,1]$

(ii) all mono-coloured simplices of $K$ belong to the boundary $\partial K$, such that the red ones belong to one boundary component $\partial K_{\text{red}}$ and the blue ones belong to the other component $\partial K_{\text{blue}}$.

The set of all causal sphere-slices is denoted by $\mathcal{CS}$.

**Definition 2** A causal disc-slice is a triangulation $K$ fulfilling the following conditions:

(i) $K$ is homeomorphic to the 3-dimensional ball $B^3$

(ii) all mono-coloured simplices of $K$ belong to the boundary $\partial K$, such that the red ones form a disc $D_{\text{red}}$ and the blue ones form a disc $D_{\text{blue}}$, which will be called the boundary discs of $K$.

The set of all causal disc-slices is denoted by $\mathcal{CD}$.

We note that the above definitions imply that all the vertices of a causal slice of either type lie on the boundary. Two-coloured edges are sometimes referred to as *timelike edges*. Two-coloured triangles can be of two types, with two red vertices and one blue or vice versa, and are called *forward directed* and *backwards directed* triangles, respectively.
Relaxing the condition in Definition 2 that $D_{\text{red}}$ and $D_{\text{blue}}$ are (homeomorphic to) discs to, say, the requirement that they are deformation retracts of discs yields a larger class of causal triangulations, that will not be discussed in detail in this paper (see, however, Section 6 for some further remarks). In the course of the proof of Theorem 2 we shall encounter particular examples of such triangulations and we use the same notation as for causal slices without further comment.

Although the main focus of this paper is on causal slices we introduce for the sake of completeness general causal triangulations in the following definition as a layered union of causal slices. In general these triangulations have interior vertices.

**Definition 3** A causal sphere-triangulation is a triangulation of the form

$$M = \bigcup_{i=1}^{N} K^i,$$

where each $K^i$ is a causal sphere-slice such that $K^i$ is disjoint from $K^j$ if $i \neq j$ except that $\partial K^i_{\text{blue}} = \partial K^{i+1}_{\text{red}}$ for $i = 1, \ldots, N-1$ as uncoloured 2-dimensional triangulations. The boundary components of $M$ are then $\partial K^1_{\text{red}}$ and $\partial K^N_{\text{blue}}$.

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The following lemma states that, apart from the two boundary discs, the boundary of a causal disc-triangulation consists of a "timelike cylinder", that is a 2-dimensional causal sphere-triangulation, which is defined as in Definitions 1 and 3 with $S^2$ replaced by $S^1$. 

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Lemma 1 Let \( K \) be a causal disc-slice and denote by \( C \) the subcomplex of \( \partial K \) consisting of two-coloured triangles. Then \( C \) is a 2-dimensional causal sphere-slice with boundary \( \partial D_{\text{red}} \cup \partial D_{\text{blue}} \), which will be called the side of \( K \).

More generally, if \( M \) is a causal disc-triangulation then the part of the boundary made up of triangles not in the red or blue boundary discs is a 2-dimensional causal sphere triangulation.

Proof. It is clearly enough to prove the first statement. Choose an orientation of the circle \( \partial D_{\text{red}} \) and consider a vertex \( v_1 \in \partial D_{\text{red}} \) and its two nearest neighbours \( v_0 \) and \( v_2 \) in \( \partial D_{\text{red}} \) with \( v_0 \) preceding \( v_1 \) and \( v_2 \) succeeding \( v_1 \). The edges \((v_0v_1)\) and \((v_1v_2)\) are each contained in exactly two triangles, one of which is red while the other one contains a blue vertex \( v_1', \) resp. \( v_2' \), both of which are in \( \partial D_{\text{blue}} \). Note that, by the Jordan Curve Theorem, \( \partial D_{\text{red}} \) divides \( \partial K \) into two discs. It follows that if we denote by \( u_1, \ldots, u_k \) the neighbours of \( v_1 \) in \( \partial D_{\text{red}} \) ordered cyclically around \( v_1 \) such that \( u_1 = v_0 \) and \( u_k = v_2 \), then all other neighbours of \( v_1 \) in \( \partial K \) are blue and constitute a segment \((w_1^1, w_2^1, \ldots, w_{\ell_1}^1)\) of \( \partial D_{\text{blue}} \) with \( w_1^1 = v_1' \) and \( w_{\ell_1}^1 = v_2' \), see Fig. [7]

Hence, the triangles

\[
(v_0w_1^1v_1), (w_1^1v_1w_2^1), \ldots, (w_{\ell_1-1}^1v_1w_{\ell_1}^1), (w_{\ell_1}^1v_1v_2)
\]

make up a segment of a two-dimensional causal slice.

Repeating this construction with \( v_0, v_1, v_2 \) replaced by \( v_1, v_2, v_3 \), where \( v_3 \) is the successor of \( v_2 \) in \( \partial D_{\text{red}} \), we obtain a unique continuation of the segment constructed above by

\[
(w_1^2v_2w_2^2), (w_2^2v_2w_3^2), \ldots, (w_{\ell_2-1}^2v_2w_{\ell_2}^2), (w_{\ell_2}^2v_2v_3)
\]

where \( w_1^2 = w_{\ell_1}^1 \) and \((w_1^2, \ldots, w_{\ell_2}^2)\) is a segment of \( \partial D_{\text{blue}} \).

If the vertices of \( \partial D_{\text{red}} \) are \( v_0, \ldots, v_n \) then after \( n \) steps we obtain a segment of a causal slice with horizontal edges in \( \partial D_{\text{red}} \cup \partial D_{\text{blue}} \) and whose first and last triangle share the vertex \( v_0 \). Finally this segment can be completed to a two-dimensional causal slice \( C \) by adjoining the triangles which contain \( v_0 \) and have one blue edge (in
Figure 1: The sequence of triangles on the side of the causal slice $K$ connecting the red and the blue boundary components.

$\partial D_{\text{blue}}$) and two non-coloured edges. By construction $C$ has boundary $\partial D_{\text{red}} \cup \partial D_{\text{blue}}$ and evidently constitutes all of $\partial K \setminus \text{int}(D_{\text{red}} \cup D_{\text{blue}})$.

The next two propositions are elementary and ensure the existence of causal disc-slices and sphere-slices with prescribed boundary discs, respectively boundary components.

**Proposition 1** Given two triangulated discs $D_1$ and $D_2$ there exists a causal disc-slice $K$ such that $D_{\text{red}} = D_1$ and $D_{\text{blue}} = D_2$.

*Proof.* We give an inductive argument. Suppose first that $D_1$ and $D_2$ are triangles $\Delta_{\text{red}}$ and $\Delta_{\text{blue}}$, respectively. In this case, we can choose $K$ to be the prism, depicted in Fig. 2, made up of one $(1,3)$ tetrahedron, one $(3,1)$ tetrahedron and one $(2,2)$ tetrahedron.

Now assume $K$ exists for some given $D_1$ and $D_2$. If $e$ is an arbitrary edge in $\partial D_1 = \partial D_{\text{red}} \subset \partial K$ one can glue a tetrahedron of type $(3,1)$ to $K$ along the unique triangle in the side of $K$ which contains $e$. Thus we obtain a causal disc-slice $K'$.
Figure 2: A prism made up of 3 tetrahedra $(xyzu)$, $(uvwy)$, and $(xyuw)$.

whose red boundary disc has an extra triangle compared to that of $K$ but the blue boundary disc is the same.

Similarly, given two neighbouring edges $e = (x_1x_2)$ and $e' = (x_2x_3)$ in $\partial D_1$, see Fig. 3 one can first identify $e$ and $e'$ and then glue a sequence of $(2, 2)$ tetrahedra sharing the identified $e$ and $e'$ and whose blue edges are

$$(y_1y_2), (y_2y_3), \ldots, (y_{s-1}y_s)$$

with the notation of Fig. 3. The resulting triangulation $K''$ is a causal disc-slice with red boundary disc $D'_{\text{red}}$ obtained from $D_{\text{red}}$ by identifying $e$ and $e'$ while $D_{\text{blue}}$ is unchanged.

Similar constructions can of course be made with $D_{\text{blue}}$ replacing $D_{\text{red}}$. Starting with an arbitrary triangle, it is well known (see e.g. [8] or [7]) that $D_1$ (and similarly $D_2$) can be constructed by repeated application of the process of either gluing on a triangle or by identifying two neighbouring boundary edges as described. We say that $D_1$ and $D_2$ are \emph{locally constructible}. Hence, the existence of $K$ follows from the preceding discussion by induction. Note that if $D_1$ and $D_2$ are regular triangulations then so is the causal disc-slice $K$. \hfill $\Box$
Proposition 2  Given two triangulated 2-spheres, $S_1$ and $S_2$, there exists a causal sphere-slice $K$ such that $S_1$ is the red boundary of $K$ and the blue boundary is $S_2$.

Proof. We remove one triangle from $S_1$ and another one from $S_2$. Then we obtain two triangulated discs $D_1$ and $D_2$. By Proposition 1 there exists a causal disc-slice $K'$ with boundary discs $D_{\text{red}} = D_1$ and $D_{\text{blue}} = D_2$. The side of $K'$, $C$, is a causal two-dimensional disc-slice whose boundary components are triangles. There are 3 different possibilities for $C$, up to combinatorial equivalence, see Fig. 4.

Note that we may assume that no pair of neighbouring triangles in $C$ belongs to the same tetrahedron. Indeed, if the two triangles are both forward (or both backward) directed, this follows from the fact that otherwise the removed triangle in either $S_1$ or $S_2$ would be glued to another triangle along two edges contradicting the regularity of $S_1$ and $S_2$. If one backward triangle and one forward triangle in $C$ belong to the same tetrahedron in $K'$ it must be a (2,2)-tetrahedron which can be removed from $K'$ without changing the red and blue boundary discs (only an edge in $C$ gets flipped). Since there are only finitely many (2,2)-tetrahedra in $K'$ the claim follows.

If now $C$ is of type (a) in Fig. 4 then we can glue to $K'$ a prism of the form indicated in Fig. 2 to obtain $K$ as desired. If $C$ is of type (b) we can glue onto the prism of Fig. 2 a (2,2)-tetrahedron to obtain a regular prism whose side $C'$ is of the same type as $C$ and hence can be glued onto $K'$ to obtain $K$ as desired. The argument when $C$ is of type (c) is similar. \hfill $\Box$
Figure 4: The 3 possible triangulations of the side of a disc-slice whose red and blue boundary components are both a single triangle.

3 Disc-slices and midsections

Given a causal disc-slice $K$ we define its midsection $S_K$ as in [5] for a causal sphere-slice. More explicitly, we view $K$ as being embedded in a Euclidean space and consider a tetrahedron $t = (v_1v_2v_3v_4)$ in $K$ with vertices $v_1, v_2, v_3, v_4$, i.e. $t$ consists of the points of the form

$$x = s_1v_1 + s_2v_2 + s_3v_3 + s_4v_4,$$

where $s_1, s_2, s_3, s_4 \geq 0$ and $s_1 + s_2 + s_3 + s_4 = 1$. Letting the height $h(x)$ be defined as the sum of the coefficients of the red vertices in $t$ the 2-cell $F$ corresponding to $t$ is defined as the set of points $x$ in $t$ of height $h(x) = \frac{1}{2}$. If $t$ is of type (3,1) then $F$ is a triangle whose edges by convention are coloured red; if $t$ is of type (1,3) then $F$ is likewise a triangle whose edges are coloured blue; if $t$ is of type (2,2) then $F$ is a quadrangle each of whose boundary edges are contained in exactly one boundary triangle of $t$ containing a monocoloured edge of $t$; by convention each edge of $F$
inherits the colour of the corresponding monocoloured edge in \( t \). It is easy to show (see [5] for details) that the 2-cells so obtained define a 2-dimensional cell complex homeomorphic to \( D^2 \) with edges coloured red or blue in such a way that triangles are monocoloured while quadrangles are two-coloured with opposite edges of the same colour. Moreover, this cell complex is uniquely defined up to combinatorial isomorphism and is called the *midsection* of \( K \), denoted by \( S_K \).

By construction each vertex \( a \) of \( S_K \) is contained in a unique edge \( e_a \) of \( K \) whose endpoints have different colours and vice versa. Similarly, any red (blue) edge of \( S_K \) is contained in a unique two-coloured triangle of \( K \) which contains a red (blue) edge in \( \partial K \), and vice versa. Finally, each 2-cell of \( S_K \) is contained in a unique tetrahedron of \( K \), the tetrahedron being of type (3,1), (2,2) or (1,3) depending on whether the cell is a red triangle, a quadrangle, or a blue triangle, respectively.

In a two-dimensional coloured cell complex as described we shall use the notation \( \langle ab \rangle \) and \( \langle abc \rangle \) for edges and triangles with vertices \( a, b \) and \( a, b, c \), respectively, whereas a 2-cell with cyclically ordered vertices \( a, b, c, d \), such that \( \langle ab \rangle \) and \( \langle cd \rangle \) are red edges, will be denoted by \( \langle abcd \rangle \). Note that with this convention we have, e.g., \( \langle abcd \rangle = \langle dcba \rangle = \langle cdab \rangle \).

By a red, resp. blue, path in \( S_K \) we mean a sequence \( e_1, \ldots, e_k \) of red, resp. blue, edges such that \( e_i = \langle a_i a_{i+1} \rangle \) for each \( i = 1, \ldots, k-1 \), and some vertices \( a_1, \ldots, a_k \). In this case we say that the path connects \( a_1 \) and \( a_k \). The path is called simple if either the vertices \( a_1, \ldots, a_k \) are all different or if \( a_1, \ldots, a_{k-1} \) are different while \( a_1 = a_k \), in which case the path is said to be closed.

**Remark 1** With the notation just introduced it follows from the definition of \( S_K \) that if \( a \) and \( b \) are two vertices of \( S_K \) then the red (blue) endpoints of \( e_a \) and \( e_b \) are identical if \( a \) and \( b \) are connected by a blue (red) path. Indeed, if \( e = \langle ab \rangle \) is a red edge of \( S_K \), then \( e \) is contained in a triangle in \( K \) two of whose edges are \( e_a \) and \( e_b \) sharing a blue vertex. Evidently, the claim follows from this, and similarly if \( e = \langle ab \rangle \) is blue.
The converse statement that \( a \) and \( b \) are connected by a red (blue) path if \( e_a \) and \( e_b \) share a blue (red) endpoint also holds as a consequence of the proof of Theorem 1 below.

Let us now note that since the monocoloured boundary discs of \( K \) are non-empty, it follows that \( S_K \) contains at least one triangle of each colour. For the same reason the boundary of \( S_K \) must contain edges of both colours. As a consequence, the boundary of \( S_K \) consists of a (cyclically ordered) alternating sequence of monocoloured paths which we shall call *boundary arcs*. The following few lemmas describe properties satisfied by any midsection \( S_K \).

**Lemma 2** Two different vertices in a midsection \( S_K \) cannot be connected by both a red and a blue path.

*Proof*. If two vertices \( a \) and \( b \) are connected by both a red and a blue path, then the two edges \( e_a \) and \( e_b \) in \( K \) have identical endpoints and hence are identical, which implies \( a = b \).

Since \( \partial D_{\text{red}} \) is homeomorphic to \( S^1 \), there does not exist a sequence of quadrangles \( q_1, q_2, \ldots, q_k \) in \( S_K \) such that \( q_i \) and \( q_{i+1} \) share a red edge for \( i = 1, 2, \ldots, k - 1 \) and the red edges in \( q_1 \) and \( q_k \), not shared with \( q_2 \) or \( q_{k-1} \), belong to \( \partial D_{\text{red}} \). A sequence of quadrangles as we have described will be called a *red path of quadrangles* connecting edges in \( \partial D_{\text{red}} \). The corresponding statement with red replaced by blue is of course also true. We define a closed path of quadrangles (red or blue) analogously. Obviously there cannot exist a closed path of quadrangles in \( S_K \) because then \( K \) would contain a blue or a red edge in its interior. The absence of paths of quadrangles just described follows from the following more restrictive conditions.

**Lemma 3** Let \( S_K \) be a midsection, let \( \rho \) denote a closed simple red (blue) path in \( S_K \) and let \( \mu \) be a simple red (blue) path connecting two vertices belonging to different blue (red) arcs in the boundary \( \partial S_K \). Then the following hold:
i) The interior of \( \rho \), i.e. the component of \( S_K \setminus \rho \) not containing any boundary edges, contains solely red (blue) edges.

ii) The two endpoints of \( \mu \) are the two endpoints of a red (blue) boundary arc.

Proof. i) It is sufficient to consider the case when \( \rho \) is red. If \( \rho \) encloses a blue edge it clearly also encloses a blue triangle. Since the boundary of \( S_K \) contains blue edges the exterior of \( \rho \) likewise contains a blue triangle. Evidently, all red paths connecting a vertex in an interior blue triangle with a vertex in an exterior blue triangle must intersect \( \rho \). For \( K \) this means that \( D_{\text{blue}} \) consists of two nontrivial subcomplexes sharing a single vertex (namely the common blue endpoint of the edges \( e_a \) for \( a \in \rho \)) which contradicts the fact that \( D_{\text{blue}} \) is a disc.

ii) Assume \( \mu \) is red and connects two vertices \( a \) and \( b \) belonging to two different blue boundary arcs \( \alpha_1 \) and \( \alpha_2 \). If the conclusion of ii) does not hold one sees that the common blue endpoint of the edges \( e_a \) and \( e_b \) in \( K \) would separate \( \partial D_{\text{blue}} \) into two non-trivial parts sharing only this vertex. This contradicts the fact that \( \partial D_{\text{blue}} \) is a simple closed curve. Obviously, a similar argument applies when \( \mu \) is blue.

Lemma 4 Let \( e = \langle ab \rangle \) and \( f = \langle a'b' \rangle \) be two disjoint blue (red) edges in the midsection \( S_K \). Suppose \( a \) and \( a' \) as well as \( b \) and \( b' \) are connected by a red (blue) path. Then there exists a blue (red) path of quadrangles connecting \( e \) and \( f \).

Proof. It suffices to prove the Lemma for blue edges. Let \( \Delta_e \) and \( \Delta_f \) be the two-coloured triangles of \( K \) that contain the blue edges \( e \) and \( f \), respectively. Then \( \Delta_e \) and \( \Delta_f \) share a blue edge \( (xy) \) in the blue boundary of \( K \) and also have red vertices \( v_e \) and \( v_f \) in \( \partial K \). Since \( e \) and \( f \) are disjoint \( v_e \neq v_f \). If \( (xy) \) belongs to \( \partial D_{\text{blue}} \), the star of \( (xy) \) consists of a sequence \( (xyv_1v_2), (xyv_2v_3), \ldots, (xyv_{k-1}v_k) \) of \((2,2)\)-tetrahedra sharing \( (xy) \) and an additional \((1,3)\)-tetrahedron \( (xyzv_k) \). We have \( v_e, v_f \in \{v_1, \ldots, v_k\} \) and the claim follows since each \((2,2)\)-tetrahedron corresponds to a quadrangle in the midsection. If \( (xy) \) is an interior edge in \( D_{\text{blue}} \) then the first \((2,2)\)-tetrahedron above has to be preceded by a \((1,3)\)-tetrahedron \( (xyzv_1') \) and then the rest of the argument is unchanged.
Proposition 3 The midsection $S_K$ of a causal disc-slice $K$ determines $K$ uniquely up to combinatorial equivalence.

Proof. This follows by arguments identical to those in [5] for causal sphere-slices. □

Definition 4 We let $SD$ denote the set of all coloured cell complexes $S$ homeomorphic to a disc (with cells as described previously) which have at least one triangle of each colour and satisfy the following conditions:

$(\alpha)$ No pair of different vertices in $S$ are connected by both a red and a blue path.

$(\beta_1)$ Each closed simple red (blue) path encloses solely red (blue) triangles in its interior.

$(\beta_2)$ Considering the division of $\partial S$ into red and blue arcs, there is no red (blue) path connecting two vertices belonging to different blue (red) arcs unless they are the two endpoints of a red (blue) arc.

$(\gamma)$ If $e = \langle ab \rangle$ and $f = \langle cd \rangle$ are two disjoint blue (red) edges in $S$ such that $a$ and $c$ as well as $b$ and $d$ are connected by a red (blue) path, then there exists a blue (red) path of quadrangles connecting $e$ and $f$.

Remark 2 As mentioned previously, condition $(\beta_1)$ implies the absence of closed red (blue) paths of quadrangles. Indeed, the outer blue (red) boundary component of such a path would violate $(\beta_1)$. Similarly, condition $(\beta_2)$ implies that no two different edges in $\partial S$ can be connected by a red or blue path of quadrangles. Likewise, it follows that two different edges in the same triangle cannot be connected by a path of quadrangles, since it would contradict $(\alpha)$.

One can demonstrate by explicit examples that property $(\gamma)$ does not follow from the first three properties.
**Definition 5** We say that a two-dimensional coloured cell complex satisfies condition \((\delta)\) if for any pair of distinct red, resp. blue, triangles it is not possible to join their vertices pairwise by blue, resp. red, paths.

**Lemma 5** Condition \((\delta)\) holds for all \(S \in SD\).

*Proof.* Given the two triangles, let us assume they are red and that three blue paths exist connecting \(a\) to \(a'\), \(b\) to \(b'\), and \(c\) to \(c'\). It follows easily from property \((\alpha)\) that the two triangles must be disjoint. Using \((\gamma)\) there exists a (non-trivial) path \(\rho_{ab}\) of quadrangles connecting \(\langle ab \rangle\) to \(\langle a'b' \rangle\). Similarly, a path \(\rho_{ac}\) of quadrangles exists connecting \(\langle ac \rangle\) to \(\langle a'c' \rangle\). Each of these two paths contains a blue path connecting \(a\) to \(a'\) which do not intersect each other (although they may touch at some vertices or edges). Hence they define a closed curve, whose interior consists of blue triangles by \((\beta_1)\). In particular, all edges of those blue triangles as well as all red edges of \(\rho_{ab}\) and \(\rho_{ac}\) are interior edges of \(S\), and the exterior of the curve contains the two original triangles and also the path \(\rho_{bc}\) of quadrangles connecting \(\langle bc \rangle\) to \(\langle b'c' \rangle\), whose existence again follows from \((\gamma)\). Similarly, considering the closed blue curves determined by \(\rho_{ab}\) and \(\rho_{bc}\), respectively \(\rho_{bc}\) and \(\rho_{ac}\), we conclude that all edges are interior in \(S\) which contradicts the fact that \(S\) is a disc. \(\Box\)

Lemmas 2 - 5 together with Proposition 3 show that the mapping \(\psi : K \mapsto S_K\) is a well-defined injective map from the set \(CD\) of causal disc-slices into \(SD\). We now aim to prove the following main result of the present paper.

**Theorem 1** The mapping \(\psi : K \mapsto S_K\) is bijective from the set \(CD\) of causal disc-slices onto the set \(SD\) of coloured two-dimensional cell complexes.

### 4 Proof of the main result

In this section we prove Theorem 1. The strategy is to show first that from any \(S \in SD\) we can construct a unique simplicial complex. We then proceed to show
that this simplicial complex has the topology of a ball and is actually a causal disc-slice. The midsection of this causal slice is the coloured cell complex we started with.

Let $S \in SD$ be given. In order to construct the corresponding $K \equiv K_S \in CD$ we start by associating to each vertex $a \in S$ a pair of new vertices $r_a, b_a$ which will form the vertex set $K^0_S$ of $K_S$ with the following identifications:

\begin{equation}
(\star) \quad r_a = r_b, \text{ resp. } b_a = b_b, \text{ if } a \text{ and } b \text{ are connected by a blue, resp. red, path},
\end{equation}

where $a, b$ are arbitrary vertices in $S$. Thus $K^0_S$ consists of all the vertices $r_a, b_a$ with $a \in S$ subject to the identifications $(\star)$. By definition we attach the colour red to the vertex $r_a$ while $b_a$ is coloured blue.

The set $K^3_S$ of coloured tetrahedra is obtained from the collection of 2-cells of $S$ as follows: for each red triangle $\Delta = \langle abc \rangle$ let $t_\Delta$ be the $(3,1)$-tetrahedron $(r_ar_br_c, b_a)$ where we notice that all 4 vertices are different by $(\alpha)$ and $b_a = b_b = b_c$; similarly, if $\Delta$ is a blue triangle, let $t_\Delta = (b_ab_bb_c, r_a)$; finally, for the quadrangle $q = \langle abcd \rangle$, we let $t_q$ be the $(2,2)$-tetrahedron $(r_ar_bb_c)$; again it follows from $(\alpha)$ that the four vertices are different and $t_q$ depends only on the quadrangle $q$.

Thus to each 2-cell $F$ of $S$ there corresponds a tetrahedron $t_F$ with vertices in $K^0_S$. This defines an abstract 3-dimensional coloured simplicial complex $K_S$ whose edges and triangles are obtained as sub-simplices of the tetrahedra.

Let us first verify that $F \mapsto t_F$ is bijective between 2-cells in $S$ and tetrahedra in $K_S$. By definition the mapping is surjective. Consider two 2-cells $F$ and $F'$ such that $t_F = t_{F'}$. Clearly $F$ and $F'$ are both triangles with the same colour or they are both quadrangles.

Suppose $F = \langle abc \rangle$ and $F' = \langle a'b'c' \rangle$ are, say, red triangles. Then $t_F = t_{F'}$ means that $b_a = b_b = b_c = b_{a'} = b_{b'} = b_{c'}$ and $\{r_a, r_b, r_c\} = \{r_{a'}, r_{b'}, r_{c'}\}$. By $(\alpha)$ this implies that $\{a, b, c\} = \{a', b', c'\}$ and hence $\langle abc \rangle = \langle a'b'c' \rangle$.

If $F = \langle abcd \rangle$ and $F' = \langle a'b'c'd' \rangle$ are quadrangles then $t_F = t_{F'}$ implies that $\{r_a, r_b\} = \{r_{a'}, r_{b'}\}$ and $\{b_a, b_c\} = \{b_{a'}, b_{c'}\}$. Using $(\alpha)$ it is then straightforward to
check that $\langle abcd \rangle$ is equal to $\langle a'b'c'd' \rangle$. We have thus established that the tetrahedra of $K_S$ are labelled by the 2-cells of $S$.

Next let us consider the triangles in $K_S$. These fall into four disjoint classes:

(i) Red triangles $(r_ar_br_c)$ where $\langle abc \rangle$ is a red triangle in $S$.

(ii) Blue triangles $(b_ab_bb_b)$ where $\langle abc \rangle$ is a blue triangle in $S$.

(iii) Triangles $(r_ar_br_a)$ where $\langle ab \rangle$ is a red edge in $S$.

(iv) Triangles $(b_ab_br_a)$ where $\langle ab \rangle$ is a blue edge in $S$.

Using property $(\delta)$, which holds by Lemma 3, we see that red triangles in $K_S$ corresponding to different red triangles in $S$ are different. Such triangles are not shared by different tetrahedra and therefore lie in the boundary of $K_S$. The corresponding statement about blue triangles is also clearly true.

Next consider a triangle $(r_ar_br_a)$ where $\langle ab \rangle$ is a red edge in $S$. By definition of $t_F$ it holds that, if $\langle ab \rangle$ belongs to a 2-cell $F$ in $S$, then $(r_ar_br_a)$ belongs to the boundary of $t_F$.

Conversely, suppose the triangle $(r_ar_br_a)$ belongs to the boundary of a tetrahedron $t_F$. If $F$ is a red triangle $\langle a'b'c' \rangle$, then $(r_ar_br_a)$ equals one of the triangles $(r_ar_br_a)$, $(r_ar_cr_br_a)$, $(r_br_cr_br_a)$. By $(\alpha)$ this implies that $\{a,b\}$ equals one of $\{a',b'\}$, $\{a',c'\}$, $\{b',c'\}$ and hence the edge $\langle ab \rangle$ equals one of the edges $\langle a'b' \rangle$, $\langle a'c' \rangle$, $\langle b'c' \rangle$. Thus, $\langle ab \rangle$ belongs to $F$. Similarly, if $F$ is a quadrangle $\langle a'b'c'd' \rangle$, it follows that $(r_ar_br_a)$ equals $(r_ar_br_a)$ or $(r_ar_br_c)$. By $(\alpha)$ this implies that $\langle ab \rangle = \langle a'b' \rangle$ or $\langle ab \rangle = \langle c'd' \rangle$ and hence $\langle ab \rangle$ is an edge in $F$.

From these observations follows that two tetrahedra $t_F$ and $t_{F'}$ share a triangle $\Delta = (r_ar_br_a)$ (or $\Delta = (b_ar_br_a)$) if and only if $F$ and $F'$ share the edge $\langle ab \rangle$. In particular, the interior triangles in $K_S$ are labelled by the interior edges in $S$. The boundary triangles in $K_S$ are labelled by the boundary edges of $S$ together with the monocoloured triangles in $S$ which label the monocoloured boundary triangles in $K_S$.  

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We next consider the edges in $K_S$. In particular, we want to show that monocoloured edges lie in $\partial K_S$. Let $(r_ar_b)$ be a red edge in $K_S$. This means that $a, b$ can be assumed to belong to some 2-cell $F$ in $S$ such that $\langle ab \rangle$ is a red edge in $F$. If $F = \langle abc \rangle$ is a red triangle it follows from the preceding paragraph that $(r_ar_b)$ belongs to $\partial K_S$ since it belongs to the red triangle $(r_ar_br_c)$. Alternatively, if $\langle ab \rangle$ belongs to a quadrangle $q$, then by ($\beta_1$) and ($\beta_2$) we have that $q$ belongs to a red path of quadrangles connecting either two red triangles or a red triangle and a red boundary edge in $S$. This shows that $(r_ar_b)$ is an edge in a red triangle in $\partial K_S$. Applying similar arguments to blue edges in $K_S$ shows that all monocoloured edges lie in the boundary.

The correspondence $a \rightarrow (r_ab)$ between vertices in $S$ and the two-coloured edges of $K_S$ is bijective by property ($\alpha$). Moreover, $(r_ab)$ belongs to $\partial K_S$ if and only if $a$ belongs to $\partial S$, since it belongs to the two boundary triangles corresponding to the edges in $\partial S$ incident on $a$.

Finally, consider a red vertex $x$ in $K_S$. It belongs to some tetrahedron $t_F$ where $F$ is a 2-cell in $S$. If $F$ is a quadrangle or a red triangle, then $x$ belongs to a red edge and hence, by the preceding paragraph, it belongs to $\partial K_S$. If $F$ is a blue triangle $\langle abc \rangle$ then $x = r_a = r_b = r_c$. Since we assume that there is a least one triangle of each colour in $S$ we can pick a path in $S$ starting at $a$ and ending at vertex $a'$ in a red triangle $\langle a'b'c' \rangle$. There is a first vertex $d$ in the path which is contained in a red edge and we have $x = r_d$. Hence, $x$ belongs to $\partial K_S$. An identical argument shows that a blue vertex in $K_S$ is necessarily contained in the boundary. This completes the argument that all monocoloured simplices in $K_S$ belong to the boundary.

The next thing to consider is the structure of $\partial K_S$. We have seen above that there is a one-to-one correspondence between the boundary edges in $S$ and the two-coloured triangles in $\partial K_S$. It follows that these triangles form a sequence $\Delta_1, \ldots, \Delta_k$ corresponding to the boundary edges $e_1, \ldots, e_k$ in $\partial S$ ordered cyclically. Setting $e_i = \langle a_ia_{i+1} \rangle$, where $a_1, \ldots, a_k$ are the cyclically ordered vertices of $\partial S$ (with $a_{k+1} = a_1$), we see that $\Delta_i$ and $\Delta_{i+1}$ share the edge $(r_{a_{i+1}}b_{a_{i+1}})$. By ($\beta_2$) the triangles $\Delta_1, \ldots, \Delta_k$
make up a two-dimensional causal slice $C$, homeomorphic to the cylinder $S^1 \times [0, 1]$, whose red and blue boundary circles are denoted $\partial C_{\text{red}}$ and $\partial C_{\text{blue}}$. At this stage we need the following lemma whose proof we will postpone a little.

**Lemma 6** The simplicial complex $K_S$ is homeomorphic to the 3-ball so its boundary is homeomorphic to the 2-sphere.

By the above lemma $\partial K_S \setminus C$ consists of two discs $D_{\text{red}}$ and $D_{\text{blue}}$ whose boundaries are $\partial C_{\text{red}}$ and $\partial C_{\text{blue}}$. We claim that $D_{\text{red}}$ is made up exactly of the red triangles in $\partial K_S$ and similarly for $D_{\text{blue}}$. Indeed, since $D_{\text{red}}$ is a two-dimensional pseudomanifold any triangle in $D_{\text{red}}$ can be edge-connected to $\partial D_{\text{red}} = \partial C_{\text{red}}$ by a sequence of triangles in $D_{\text{red}}$. The triangles in $D_{\text{red}}$ are mono-coloured and are therefore either red or blue. But a blue triangle cannot be glued to a red triangle so $D_{\text{red}}$ consists of red triangles. These must be all the red triangles in $\partial K_S$ since $D_{\text{blue}}$ consists solely of blue triangles. We have therefore shown that $K_S$ is a causal disc slice.

Now it is not hard to verify that the midsection of $K_S$ is combinatorially identical to the original midsection $S$ used to construct $K_S$. We already noted above that the vertices of $S$ are in bijective correspondence with the two-coloured edges in $K_S$. By the definition of the midsection $S'$ of $K_S$ its vertices are the midpoints of the two-coloured edges of $K_S$, and hence can be be labeled by the vertices of $S$. In this way the midsection of $t_F$ clearly gets identified with $F$ and the edge in $S'$ corresponding to an interior triangle $(r_ar_bb_a)$, respectively $(b_ab_br_a)$, of $K_S$ is the red, respectively blue, edge $\langle ab \rangle$ in $S$. Hence, the correspondence between vertices induces bijective correspondences between 2-cells and 1-cells as well, and so $S$ and $S'$ are combinatorially identical. This completes the proof of Theorem 1.

It remains to prove Lemma 6. For this purpose an alternative construction of $K_S$ by a gluing procedure is a useful tool. We begin by explaining this construction.

For each 2-cell $F$ in $S$ and each vertex $a$ of $F$ we define a red vertex $r_a^F$ and a blue vertex $b_a^F$ such that $r_a^F = r_b^F$, resp. $b_a^F = b_b^F$, if $\langle ab \rangle$ is a blue, resp. red, edge.
in F. In this way four different vertices are defined for each 2-cell F and they in turn define a coloured (abstract) tetrahedron \( \tau_F \) considered as a simplicial complex including its sub-simplices. As before \( \tau_F \) is of type \((3, 1), (2, 2),\) or \((1, 3)\) depending on whether \( F \) is a red triangle, a quadrangle, or a blue triangle, respectively. Without further identifications the tetrahedra so defined are pairwise disjoint and we note that the definition of \( K_S \) can be reformulated by stating that \( K_S \) is obtained from the collection of tetrahedra \( \tau_F \) labeled by the 2-cells of \( S \) by imposing the identifications of simplices implied by the relations

\[
\text{(**)} \quad r_a^{F_1} = r_b^{F_2}, \quad \text{resp.} \quad b_a^{F_1} = b_b^{F_2}, \quad \text{if either} \quad a = b \quad \text{or} \quad a \text{ is connected to } b \text{ by a blue, resp. red, path},
\]

where \( a, b \) are arbitrary vertices in \( S \) and \( F_1, F_2 \) are arbitrary 2-cells in \( S \) containing \( a \) and \( b \), respectively.

We next show that \( K_S \) can equivalently be obtained by applying a suitable gluing procedure to the collection of tetrahedra defined above. Given two 2-cells \( F_1 \) and \( F_2 \) sharing a red edge \( \langle ab \rangle \) we have that \( \tau_{F_1} \) contains the triangle \( \langle r_a^{F_1} r_b^{F_1} b_a^{F_1} \rangle \) and \( \tau_{F_2} \) contains the triangle \( \langle r_a^{F_2} r_b^{F_2} b_a^{F_2} \rangle \). We say that \( \tau_{F_1} \) is glued to \( \tau_{F_2} \) along \( \langle ab \rangle \) if \( r_a^{F_1} \) is identified with \( r_a^{F_2} \), \( r_b^{F_1} \) is identified with \( r_b^{F_2} \), and \( b_a^{F_1} \) is identified with \( b_a^{F_2} \) in \( \tau_{F_1} \) and \( \tau_{F_2} \) and their sub-simplices. Similarly, gluing along a blue interior edge in \( S \) is defined. In this way, given an interior edge \( \langle ab \rangle \) of \( S \), the tetrahedra \( \tau_{F_1} \) and \( \tau_{F_2} \) corresponding to the 2-cells \( F_1 \) and \( F_2 \) sharing \( \langle ab \rangle \) can be glued along \( \langle ab \rangle \).

We now define the simplicial complex \( K'_S \) by imposing the identifications of the simplices in the collection \( \{ \tau_F \} \) implied by gluing pairs of tetrahedra as described along all interior edges of \( S \). We claim that \( K_S = K'_S \). In order to prove this we need to verify that the identifications of vertices implied by the gluing conditions are identical to those given by \( \text{(**)} \).

First, consider \( r_a^{F'} \) and \( r_a^{F''} \) where \( a \) is a vertex in two different 2-cells \( F' \) and \( F'' \). Since \( S \) is a manifold there exist 2-cells \( F_1, F_2, \ldots, F_n \) such that \( F_1 = F' \) and \( F_n = F'' \) and \( F_i \) and \( F_{i+1} \) share an edge \( e_i \) containing \( a \) for each \( i = 1, \ldots, n \). From
the gluing of $\tau_{F_i}$ and $\tau_{F_{i+1}}$ along $e_i$ it follows that $r_a^{F_i} = r_a^{F_{i+1}}$ and $b_a^{F_i} = b_a^{F_{i+1}}$ for $i = 1, \ldots, n - 1$, and hence we conclude that $r_a^{F'} = r_a^{F''}$ and $b_a^{F'} = b_a^{F''}$.

Next consider two different vertices $a$ and $b$ in $S$ and assume they are connected by a blue path with edges $\langle a_1a_2 \rangle, \langle a_2a_3 \rangle, \ldots, \langle a_ma_{m+1} \rangle$, where $a_1 = a$ and $a_{m+1} = b$. Choosing arbitrary 2-cells $F_1, \ldots, F_m$ such that $\langle a_ia_{i+1} \rangle$ belongs to $F_i$, $i = 1, \ldots, m$, we have that $r_a^{F_i} = r_a^{F_{i+1}}$ and by the preceding paragraph $r_a^{F_i} = r_a^{F_{i+1}}$ for all $i$. It follows that $r_a^{F_1} = r_a^{F_m}$. Of course, the corresponding result for blue vertices holds if $a$ and $b$ are connected by a red path.

Conversely, it is clear that if $a$ and $b$ are different vertices belonging to 2-cells $F'$ and $F''$, respectively, and $r_a^{F'} = r_b^{F''}$, then $a$ and $b$ are connected by a blue path. Indeed, there exists a sequence of vertices $b_1, \ldots, b_m$ and corresponding 2-cells $F_1, \ldots, F_m$ such that $a = b_1, F' = F_1$ and $b = b_m, F'' = F_m$ and $r_a^{F_i}$ is identified with $r_{b_{i+1}}^{F_{i+1}}$ either through a gluing of $\tau_{F_i}$ to $\tau_{F_{i+1}}$ along an edge containing $b_i$, in which case $b_i = b_{i+1}$, or else $F_i = F_{i+1}$ and $b_i$ is connected to $b_{i+1}$ by a blue edge in $F_i$. Similarly, if $b_a^{F'} = b_b^{F''}$ then $a$ and $b$ are connected by a red path.

This shows that the identifications pertaining to $K_S$ and $K'_S$ are the same and hence $K_S = K'_S$ as claimed. We are now ready to prove Lemma 6.

**Proof of Lemma 6.** Let $S_1, S_2, \ldots, S_N = S$ be a local construction (see [6, 7]) of the midsection $S$. This means that $S_1$ is a coloured 2-cell and $S_{n+1}$ is obtained from $S_n$ by either (i) gluing a coloured 2-cell to $S_n$ along an edge $e$ in $\partial S_n$ or (ii) by identifying two edges $e_1$ and $e_2$ in $\partial S_n$ which have the same colour and share a vertex. The existence of such a construction is well known and easy to establish in the 2-dimensional case. For further details we refer to [7, 8]. Evidently, all the $S_n$’s have the topology of a disc and the 2-cells as well as interior edges of $S_n$ can be identified with corresponding 2-cells and interior edges in $S$.

The correspondence between tetrahedra $\tau_F$ in $K_S$ and 2-cells $F$ of $S$ and the local construction of $S$ gives rise to a sequence of coloured simplicial complexes $K_n$, $n = 1, \ldots, N$, where $K_n$ is defined by gluing the tetrahedra $\tau_F$ assigned to the 2-cells in $S_n$ along the interior edges of $S_n$. In particular, $K_1$ is a single tetrahedron and
Figure 5: The local construction of the midsection and the corresponding 3-dimensional simplicial complex. After identifying the edges $e_1$ and $e_2$ we identify the triangles $\Delta_1$ and $\Delta_2$.

$K_N = K_S$, since $S_N = S$ and $K_S = K'_S$ as shown above. Moreover, $K_{n+1}$ is obtained from $K_n$ by gluing a tetrahedron $\tau_F$ to a tetrahedron $\tau_{F'}$ in $K_n$ along an edge $\langle ab \rangle$. In case $\tau_F$ is not in $K_n$ already it is clear that the topological class of $K_n$ equals that of $K_{n+1}$. We shall now argue that the same holds if $\tau_F$ and $\tau_{F'}$ both belong to $K_n$. Since $K_1$ is homeomorphic to the 3-ball the same will consequently hold for $K_S$, and hence the proof of the lemma will be completed.

If $\tau_F$ and $\tau_{F'}$ belong to $K_n$ and are glued along the edge $e$ in $K_n$ then $F$ and $F'$ belong to $S_n$, and $S_{n+1}$ is obtained from $S_n$ by identifying two edges $e_1$ and $e_2$ in $\partial S_n$ sharing a vertex $v$, such that the identified edges equal $e$.

Let us set $e_1 = \langle vv_1 \rangle$ and $e_2 = \langle vv_2 \rangle$ and assume $e$ is red. Then $K_{n+1}$ is obtained from $K_n$ by identifying the triangles $\Delta_1 = (r_{v_1}^F r_{v_1}^b b_{v_1}^F)$ and $\Delta_2 = (r_{v_2}^F r_{v_2}^b b_{v_2}^F)$ which share the edge $(r_{v_1}^F b_{v_1}^F) = (r_{v_2}^F b_{v_2}^F)$ in $K_n$. This is illustrated on Fig. 5. We need to show that no further identifications of simplices are implied.

Additional identifications can only arise if there is a vertex $x \in \partial K_n$ which is a neighbour of both $r_{v_1}^F$ and $r_{v_2}^F$ in which case the edges $(x r_{v_1}^F)$ and $(x r_{v_2}^F)$ are identified in the step from $K_n$ to $K_{n+1}$. We claim that no such $x$ exists except $b_{v_1}^F$ and $r_{v_1}^F$. 23
First, suppose $x$ is a blue vertex, $x \neq b^F_v$. Then there are two vertices $w_1$ and $w_2$ in the midsection such that $b^{F_1}_{w_1} = x = b^{F_2}_{w_2}$ and $r^{F_1}_{w_i} = r^{F_1}_{v_i}$, $i = 1, 2$, in $K_n$ for some 2-cells $F'_1, F'_2$ in $S_n$. This implies that $w_1$ and $w_2$ are connected by a red path in $S$ while $w_i$ is connected to $v_i$ by a blue path in $S$ for $i = 1, 2$. When we now take the step from $S_n$ to $S_{n+1}$ and identify $e_1$ and $e_2$ the vertices $v_1$ and $v_2$ get identified and hence $w_1$ and $w_2$ are connected both by a blue path in $S$ and by a red path, see Fig. 6. Hence, $w_1 = w_2$, and we get a closed blue path enclosing the red edge $e$ which is impossible by $(\beta_1)$.

![Figure 6: Paths arising in the local construction of the midsection when $x$ is blue.](image)

Now suppose $x$ is a red vertex. Then there are four vertices $w_1, w_2, u_1, u_2$ in the midsection such that $\langle w_1 w_2 \rangle$ and $\langle u_1 u_2 \rangle$ are red edges and $x = r^{F'}_{w_2} = r^{F''}_{w_2}$, $r^{F'}_{w_1} = r^{F_1}_{v_1}$, $r^{F''}_{u_1} = r^{F_2}_{v_2}$ for some 2-cells $F', F''$ in $S_n$. It follows that $w_2$ and $u_2$ are connected by a blue path $\rho_1$ and similarly that $w_1$ is connected to $v_1$ and $u_1$ is connected to $v_2$ by a blue path in $S$, see Fig. 7.

Upon merging $e_1$ and $e_2$ we get as above a blue path $\rho_2$ from $w_1$ to $u_1$. It follows by condition $(\gamma)$ that the edges $\langle w_1 w_2 \rangle$ and $\langle u_1 u_2 \rangle$ are connected by a blue path of quadrangles. This path of quadrangles either lies inside the closed loop made up of $\rho_1, \rho_2$ and the edges $\langle w_1 w_2 \rangle$ and $\langle u_1 u_2 \rangle$ or outside it, see Fig. 8. In both cases we obtain a closed blue loop containing a red edge in its interior which violates $(\beta_1)$. 

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This completes the proof of Lemma 6 and consequently that $K_S$ is a 3-ball.  

Figure 8: Paths of quadrangles arising in the local construction of the midsection when $x$ is red.

5 Sphere-slices

In this section we generalise Theorem 1 to the case of sphere-slices. It is clear from the proofs of Lemmas 2 and 4 that the midsection of any sphere-slice has properties $(\alpha)$ and $(\gamma)$, while $(\beta_2)$ is irrelevant as it refers to the boundary. Furthermore, the
proof of Lemma 3 shows that an analogue of (β₁) still holds in the following form:

(β) Each simple closed red (or blue) path divides the midsection into two components, one of which is red (blue).

Since a simplicial decomposition of the 2-sphere requires at least 4 triangles it follows that \( S \) must contain at least 4 triangles of each colour. Let \( φ \) denote the mapping that takes any sphere slice \( K \) to its midsection \( S \).

**Definition 6** We let \( S \) denote the set of all coloured cell complexes \( S \) homeomorphic to the 2-sphere (with cells as described previously) which have at least four triangles of each colour and satisfy the conditions \((α)\), \((β)\), and \((γ)\).

**Lemma 7** Property \((δ)\) holds for all \( S \in S \).

**Proof.** Applying the same argument as in the proof of Lemma 5, we conclude that if \((α)\), \((β)\), and \((γ)\) hold for a coloured spherical cell complex \( S \) and there are two different red triangles in \( S \) whose vertices are pairwise connected by blue paths, then there exist three paths of quadrangles connecting the edges of the triangles pairwise and the complement of those paths consists entirely of blue triangles together with the two given red triangles in \( S \). In particular, there can be only two red triangles, which contradicts \( S \in S \). Clearly, a similar argument holds for blue triangles. \( \square \)

We now state and prove the main result in this section.

**Theorem 2** The map \( φ : K \to S \) is a bijection from the set \( CS \) of causal sphere-slices to the set \( S \) of coloured 2-dimensional cell complexes.

**Proof.** Given \( S₀ \in S \) it is enough to construct a sphere-slice \( K \) such that \( φ(K) = S₀ \). The idea of the proof is to cut a piece out of \( S₀ \) in such a way that we obtain a cell complex \( S \) in \( CD \). Then we use Theorem 1 to obtain a disc-slice with the given
midsection $S$. Finally, we fill in the "hole" of the disc-slice to obtain a sphere-slice with midsection $S_0$. We divide the argument into 5 parts.

(1) Let $R$ be a maximal edge-connected cluster of red triangles in $S_0$ and let $S = S_0 \setminus \text{int}(R)$. Condition ($\beta$) and the Jordan Curve Theorem imply that $R$ and $S$ are (closed) discs. Moreover, since $R$ is maximal, all edges in $\partial S$ are red and contained in quadrangles in $S$. Pick one of these quadrangles $q$ and let $\Gamma = (q_1, q_2, \ldots, q_n)$ be the maximal blue path of quadrangles containing $q$. Let $\rho = (v_1, v_2, \ldots, v_n)$ be the red path in $\Gamma$ that shares at least one red edge with $\partial S$, see Fig. 9. The other

![Figure 9: A maximal path of quadrangles in $S$ touching the boundary.](image)

red path in $\Gamma$, $\rho' = (w_1, w_2, \ldots, w_n)$, shares no vertex with $\partial S$ as a consequence of condition ($\alpha$). Likewise, $\rho$ and $\rho'$ do not intersect and neither $\rho$ nor $\rho'$ can have multiple vertices, since it would contradict ($\alpha$). Moreover, if $1 \leq i < j \leq n$ are such that $v_i, v_j \in \partial S$, then $v_k \in \partial S$ for all $k$ between $i$ and $j$ since otherwise $R$ is not maximal due to ($\beta$). Hence, $\Gamma$ intersects $\partial S$ along a curve segment $\mu$, see Fig. 9. Note that $\mu$ does not contain all of $\partial S$ because this would again contradict condition ($\alpha$), see Fig. 10.

The above observations show that the boundary of the strip covered by the path of quadrangles $\Gamma$, which consists of $\rho$ and $\rho'$ and the two blue edges $\langle v_1 w_1 \rangle$ and
Figure 10: If $\partial S = \mu$ then $w'$ and $w''$ are connected both by a red and by a blue path. $(v_n w_n)$, is a simple curve. This curve shares exactly a segment $\mu = (v_i, v_{i+1}, \ldots, v_j)$ with $\partial S$ where $1 \leq i < j \leq n$. Hence, removing the strip covered by $\Gamma$ from $S$ we obtain a new disc $S'$ whose boundary consists of the blue edges $\langle v_1 w_1 \rangle$ and $\langle v_n w_n \rangle$ together with the red paths $\rho', \partial S \setminus \mu$, $(v_1, \ldots, v_i)$ and $(v_j, \ldots, v_n)$, see Fig. 11. Now consider the blue triangles $\Delta_1$ and $\Delta_2$ in $S'$ containing $\langle v_1 w_1 \rangle$ and $\langle v_n w_n \rangle$, respectively. Denote the third vertex in these triangles by $u_1$ and $u_2$, respectively, see Fig. 11. It follows from condition $(\delta)$ (see Lemma 7) that $u_1 \neq u_2$ and they cannot be connected by a red path. Moreover, $u_1$ and $u_2$ are not in $\partial S'$ since this would violate $(\alpha)$. Now, removing $\Delta_1$ and $\Delta_2$ (including edges $\langle v_1 w_1 \rangle$ and $\langle v_n w_n \rangle$) from $S'$ we obtain another 2-dimensional cell complex $S''$ with the topology of a disc and a boundary consisting of the blue paths $(v_1, u_1, w_1)$ and $(v_n, u_2, w_n)$ and the same red paths as $\partial S'$. 

(2) We now claim that $S''$ belongs to $S^D$. Since paths in $S''$ are also paths in $S_0$, conditions $(\alpha)$ and $(\beta_1)$ are obviously fulfilled. Furthermore, $S''$ contains at least two blue triangles, and since there exist red edges in $\partial S''$ the existence of at least one red triangle in $S''$ will follow once $(\beta_2)$ has been established for $S''$. 

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In order to verify \((\beta_2)\) we recall that \(u_1\) and \(u_2\) cannot be connected by a red path. Connecting \(u_1\) to \(w_n\) or \(v_n\) or connecting \(u_2\) to \(w_1\) or \(v_1\) by a red path evidently contradicts \((\alpha)\). Similarly, a red path from \(v_1\) to \(w_n\) or from \(w_1\) to \(v_n\) contradicts \((\alpha)\). This proves \((\beta_2)\) for \(S''\) as far as red paths are concerned.

Now let \(\eta\) be a blue path in \(S''\) connecting the vertex \(w_k\) to a vertex \(v\) in the other red arc of \(\partial S''\). Viewing \(\eta\) as a path in \(S_0\) and extending it by the edge \(\langle w_kv_k \rangle\) we obtain a blue path \(\eta'\) in \(S_0\) connecting \(v_k\) to \(v\). If \(v \neq v_k\) then condition \((\alpha)\) is violated. If \(v = v_k\) then \(\eta'\) is a closed blue path in \(S_0\) with edges of both colours inside and outside unless \(k = 1\) or \(k = n\). This proves condition \((\beta_2)\).

In order to verify condition \((\gamma)\) for \(S''\) let \(e_1 = \langle x_1y_1 \rangle\) and \(e_2 = \langle x_2y_2 \rangle\) be two blue edges in \(S''\) such that \(x_1\) and \(x_2\) as well as \(y_1\) and \(y_2\) are connected by red paths. Then, since \((\gamma)\) holds for \(S_0\), there is a blue path of quadrangles \(\Lambda\) in \(S_0\) connecting \(e_1\) and \(e_2\). Since the blue boundary edges of \(S''\) are not contained in quadrangles outside \(S''\) the path \(\Lambda\) cannot leave \(S''\). This establishes condition \((\gamma)\) for the blue edges.

Figure 11: The cell complex \(S'\).
Let now $e_1 = \langle x_1y_1 \rangle$ and $e_2 = \langle x_2y_2 \rangle$ be two red edges in $S''$ such that $x_1$ and $x_2$ as well as $y_1$ and $y_2$ are connected by blue paths $\lambda'$ and $\lambda''$. By condition (\(\alpha\)) the paths $\lambda'$ and $\lambda''$ do not intersect and we can assume without loss of generality that they have no multiple vertices. Hence, the closed curve consisting of $\lambda'$, $\lambda''$ and the two edges $e_1$ and $e_2$ is simple and encloses a disc $D$ in $S''$. Since condition (\(\gamma\)) holds for $S_0$ there exists a red path $\Gamma'$ of quadrangles in $S_0$ connecting $e_1$ and $e_2$ and evidently $\Gamma'$ is either contained in $D$ or in its exterior. However, the latter case can be excluded as follows. Observe that neither $D$ nor $\Gamma'$ can intersect $R$. It follows that one of the closed curves made up of either $\lambda'$ or $\lambda''$ and one of the blue paths in $\Gamma'$ must separate $R$ (in $S_0$) from the union of $D$ and $\Gamma'$ in contradiction with condition (\(\beta\)), see Fig. 12. Note, however, that the curve in question is not necessarily simple, since $\lambda'$ may touch the blue path in $\Gamma'$ with the same endpoints, and similarly for $\lambda''$. Using property (\(\beta\)) one can in this case remove closed parts of the curve so as to obtain a simple closed blue curve with the same separating property. Thus $\Gamma'$ is contained in $D$ (and hence in $S''$) and this proves that condition (\(\gamma\)) holds for $S''$.

(3) We can now apply Theorem 1 to conclude that there is a unique disc-slice $K'' \in \mathcal{CD}$ whose midsection is $S''$. In view of the structure of $\partial S''$, the side of $K''$ consists of two pairs of adjacent backwards directed triangles and two arrays of forward directed triangles corresponding to the two red arcs in $\partial S''$.

We now first construct from $K''$ a new simplicial complex $C$ by gluing to $K''$ the tetrahedra corresponding to the quadrangles $q_1, \ldots, q_n$ in $\Gamma$ and those corresponding to the blue triangles at the ends of $\Gamma$. One way to accomplish this is to consider the simplicial complex $K_0$ whose midsection is $\Gamma$ with the two blue triangles attached to its ends. Clearly $K_0$ is a ball and we may think of it as a causal slice whose side consists of two pairs of adjacent backwards directed triangles and two arrays of forward directed triangles of size $n$ whose red edges are pairwise identified and constitute $D_{\text{red}}$. On the other hand, $D_{\text{blue}}$ consists of two triangles with one common edge, see Fig. 13. Now consider one array of forward directed triangles in $\partial K_0$ together with the two adjacent backwards directed triangles. These triangles form a disc in $\partial K_0$. 

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Figure 12: A path of quadrangles outside $D$ connecting the edges $e_1$ and $e_2$ and the blue paths $\lambda'$ and $\lambda''$ connecting their endpoints. A priori, parts of the path may be outside $S''$. 

and we can glue $K_0$ to $K''$ along this disc and the corresponding disc in $\partial K''$ or, more precisely, the gluing is performed along the edges $\langle u_1 w_1 \rangle, \langle w_2 w_2 \rangle, \ldots, \langle w_n u_2 \rangle$, see Fig. 11. Since both $K_0$ and $K''$ are simplicial balls, so is the resulting complex $K'' \sharp K_0$. Its Euler characteristic is therefore $-1$.

Note also that $K'' \sharp K_0$ has a side consisting of two pairs of adjacent backwards directed triangles and two arrays of of forward directed triangles, one of which has length $n$. The blue edges in the backwards directed triangles form a closed curve. We now identify the two backwards directed triangles in each pair, i.e. we glue tetrahedra in $K'' \sharp K_0$ along the edges $\langle u_1 v_1 \rangle$ and $\langle u_2 v_n \rangle$. Doing this, the number of triangles in the simplicial complex decreases by 2, the number of edges decreases by 4 and the number of vertices decreases by 1 and we obtain a new simplicial complex $C'$ with Euler characteristic $-2$.

The side of $C'$ is a closed circuit of forward directed triangles sharing a single blue
Figure 13: The cell complex $K_0$ in the case $n = 3$.

vertex. We now obtain the simplicial complex $C$ by gluing tetrahedra in $C'$ along the edges $\langle v_kv_{k+1} \rangle$ in the midsection, for $k = 1, \ldots, i - 1$ and $k = j, \ldots, n - 1$. It is easy to check that the Euler characteristic does not change with these identifications so

$$\chi(C) = -2.$$  

By construction, the midsection of $C$ is $S_0 \setminus R$.

(4) Now return to the disc $R$ that was removed from $S_0$ in the first step. Let $\overline{R}$ be the cone over $R$, i.e. $\overline{R}$ consists of the tetrahedra obtained by adding a common blue vertex to all the triangles in $R$ or alternatively, by gluing the tetrahedra corresponding to triangles in $R$ along interior edges in $R$. We can now glue $\overline{R}$ to $C$ along their sides, i.e. along the edges in $\partial S = \partial R$. The resulting simplicial complex $K$ has Euler characteristic

$$\chi(K) = \chi(C) + \chi(\overline{R}) + \chi_2(\partial R) = -2,$$

since $\chi(\overline{R}) = -1$, and $\chi_2(\partial R) = 1$ is the Euler characteristic of the cone $\partial R$ over $\partial R$, which is in fact a disc.

(5) Since $K''$ is a disc-slice it is evident from the preceding construction that $\partial K$ consists of two triangulated 2-spheres, one red and one blue. The midsection of $K$
is by construction $S_0$. By gluing on cones over the two boundary components of $K$ we obtain a simplicial complex $\tilde{K}$ with Euler characteristic 0. Since $\tilde{K}$ is clearly a pseudomanifold we conclude (see [10]) that $\tilde{K}$ is also a manifold. In fact, $\tilde{K}$ is simply connected since any closed curve can be deformed to a closed curve on the midsection which is simply connected. Hence, $\tilde{K}$ is a 3-sphere by [11] and $K$ is a cylinder as desired. This completes the proof of Theorem 2. □

6 Discussion

A few remarks pertaining to extensions or variations of the present work are in order. As already indicated, the requirement in Definition 2 that $D_{\text{red}}$ and $D_{\text{blue}}$ be discs is unnecessarily restrictive. Thus, replacing (ii) of Definition 2 by

\begin{itemize}
  \item [(ii')] all mono-coloured simplices of $K$ belong to the boundary $\partial K$, such that the red (resp. blue) ones form a connected and simply connected subsimplex $D_{\text{red}}$ (resp. $D_{\text{blue}}$) of $\partial K$,
\end{itemize}

leads to a convenient class of causal triangulations. Let us call this class $\mathcal{CD}'$. In particular, any coloured tetrahedron belongs to $\mathcal{CD}'$ (but not to $\mathcal{CD}$) and so does the complex depicted in Fig. 13.

By inspection of the proofs given above it is seen that condition ($\beta_2$) is used in the construction of a causal disc slice from a cell complex in $\mathcal{CD}$ solely to ensure that $\partial D_{\text{red}}$ and $\partial D_{\text{blue}}$ are simple curves. It follows that dropping ($\beta_2$) one obtains a one-to-one correspondence between $\mathcal{CD}'$ and the set $\mathcal{SD}'$ of coloured cell complexes homeomorphic to a disc and fulfilling conditions ($\alpha$), ($\beta_1$), and ($\gamma$).

On a different note, one may observe that condition ($\gamma$) is used partly to ensure the validity of condition ($\delta$) (see Lemma 5) and otherwise only in the final part of the proof of Lemma 6 ensuring that the complex $K$ is a simplicial ball. It is natural to consider replacing property ($\gamma$) by property ($\delta$) or dropping both of them. This would require stepping outside the category of simplicial complexes and adopt a different setting encompassing singular triangulations.
Finally, one might also envisage characterising causal triangulations in higher dimensions, which evidently would lead to more involved higher dimensional coloured cell complexes of which little is known at present. In [5] it is explained how the midsections of 4-dimensional causal triangulations are made up of tetrahedra and prisms which replace the triangles and quadrangles which make up the midsections considered in this paper.

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References

[1] J. Ambjørn, B. Durhuus and T. Jonsson, Quantum geometry: a statistical field theory approach, Cambridge University Press, Cambridge (1997)

[2] J. Ambjørn, A. Görlich, J. Jurkiewicz and R. Loll, Nonperturbative quantum gravity, Physics Reports 519 (2012) 127210

[3] J. Ambjørn, B. Durhuus and T. Jonsson, Three-dimensional simplicial quantum gravity and generalized matrix models, Mod. Phys. Lett. A 6 (1991) 1133-1146

[4] J. Ambjørn, J. Jurkiewicz and R. Loll, Nonperturbative 3D Lorentzian quantum gravity, Phys. Rev. D 64 (2001) 044011

[5] B. Durhuus and T. Jonsson, Exponential bounds on the number of causal triangulations, Commun. Math. Phys. 340 (2015) 105-124

[6] B. Durhuus and T. Jonsson, Remarks on the entropy of 3-manifolds, Nucl. Phys. B 445 (1995) 182-192

[7] B. Benedetti and G. M. Ziegler, On locally constructible spheres and balls, Acta Math. 206 (2011) 205-243

[8] B. Durhuus, Critical properties of some discrete random surface models. In proceedings of The 7th Ettore Majorana International School of Mathematical
Physics 1988, "Constructive quantum field theory II", eds. G. Velo and A.S. Wightman, Plenum Press, New York, 1990.

[9] C. P. Rourke and B. J. Sanderson, *Introduction to piecewise-linear topology*, Springer, Berlin (1972)

[10] H. Seifert and W. Threlfall, *A textbook of topology*, Academic Press, New York (1980)

[11] G. Perelman, *Finite extinction time for the solutions to the Ricci flow on certain three-manifolds*, arXiv:math.DG/0307245