Generalized Gompertz - Generalized Gompertz Distribution

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Abstract. In the present paper, we introduce a new generated family of continuous distributions based on generalized Gompertz distribution. Then generalized Gompertz - generalized Gompertz distribution is proposed as a special case of this new family. The probability density, cumulative distribution, reliability and hazard rate functions are introduced. Additionally, the most essential statistical properties of this new distribution such as the mean, variance, coefficient of skewness, coefficient of kurtosis, characteristic function, quantile, median, Shannon and relative entropies along with stress strength model are obtained.

1. Introduction

Many studies have been made to introduce new generalized families of distributions. The common feature of these new distributions is that they have more parameters. Eugene et al. (2002) [4] developed family of beta-generated distributions. Zografos and Balakrishnan (2009) [8] proposed generalized gamma generated G family of distributions. Cordeiro and de Castro (2011)[2] developed a new generated family of distributions based on the Kumaraswamy distribution. Bourguignon et al. (2014)[1] developed the Weibull - G family of probability distributions. Nadarajah et al. (2015)[6] introduced generated G family of distributions based on Zografos-Balakrishnan. Rezaei et al. (2017)[7] proposed a new generated family of distributions based on Topp-Leone. Hamedani et al. (2018)[5] introduced a new extended G family of continuous distributions that can be viewed as a mixture representation of the exponentiated G densities.

Now, suppose that G(x) and g(x) (from now, for simplicity, we shall use the symbols G and g instead of G(x) and g(x) respectively), are any continuous baseline cumulative distribution function (cdf) and probability density function (pdf) of X. The proposed new family of distributions is given by

\[1 - F(x) = R(x) = \int_0^{-\ln G} h(x) \, dx = H(-\ln G)\]

where \(H(.)\) and \(h(.)\) represents the cdf and pdf of any continuous distribution. Depending on Equation (1), the proposed general formula of cdf for new family will be,

\[F(x) = 1 - H(-\ln G)\] (2)

and the associated pdf, \(f(x) = \frac{d}{dx} [F(x)] = -\frac{d}{dx}[R(x)],\) will be,

\[f(x) = \frac{g}{G} h(-\ln G)\] (3)
According to the above general formulas, this paper seek to introduce a new generated family of continuous distributions based on generalized Gompertz distribution named as generalized Gompertz – G family.

2. Generalized Gompertz – G Family
El-Gohary (2013)[3] suggested the Generalized Gompertz (GGom) distribution as a new generalization of the exponential, Gompertz, and generalized exponential distributions. The GGom can be widely used for the comparison of mortality rate of different populations, especially in actuarial studies and growth models.

Let \( X \) be a random variable has GGom distribution with three parameters \( \lambda, c, \theta \). The cdf and the associated pdf are given respectively by[3]

\[
H(x; \lambda, c, \theta) = \left[1 - e^{-\frac{\lambda}{c}e^{cx-1}}\right]^{\theta} \quad \text{with } x \geq 0 ; \lambda, \theta > 0 \text{ and } c \geq 0 \tag{4}
\]

\[
h(x; \lambda, c, \theta) = \lambda \theta e^{cx} e^{-\frac{\lambda}{c}e^{cx-1}} \left[1 - e^{-\frac{\lambda}{c}e^{cx-1}}\right]^{\theta-1} \tag{5}
\]

Now, suppose that \( H(-\ln G) \) and \( h(-\ln G) \) represents the cdf and pdf of the GGom with the parameters \( a, b, c \) as

\[
H(-\ln G; a, b, c) = \left[1 - e^{-\frac{a}{b}(e^{b(-\ln G)-1})}\right]^{c} \tag{6}
\]

\[
h(-\ln G; a, b, c) = ace^{b(-\ln G)} e^{-\frac{a}{b}(e^{b(-\ln G)-1})} \left[1 - e^{-\frac{a}{b}(e^{b(-\ln G)-1})}\right]^{c-1} \tag{7}
\]

after substituting Equations (6) and (7) in Equations (2) and (3), we get a new family of continuous distributions based on interval \([0, \infty)\) generalized Gompertz distribution, named generalized Gompertz – G distributions (symbolized by GGom – G). The cdf and the associated pdf for this new family are given respectively by Equations (8) and (9)

\[
F(x) = 1 - \left[1 - e^{-\frac{a}{b}(e^{b(-\ln G)-1})}\right]^{c} \tag{8}
\]

\[
F(x) = 1 - \left[1 - e^{-\frac{a}{b}(e^{b(-\ln G)-1})}\right]^{c} \tag{9}
\]

As well as the pdf can expansion as

\[
f(x) = \frac{g}{G} ac e^{-b\ln G} e^{-\frac{a}{b}(e^{b(-\ln G)-1})} \left[1 - e^{-\frac{a}{b}(e^{b(-\ln G)-1})}\right]^{c-1} \tag{10}
\]

Now since,

\[
(1 - z)^{-k} = \sum_{l=0}^{\infty} \frac{\Gamma(k+l)}{l! \Gamma(k)} z^l \quad \text{with } |z| < 1 \text{ & } k > 0
\]

\[
(1 - z)^{b} = \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(b+1)}{m! \Gamma(b+m+1)} z^m \quad \text{with } |z| < 1 \text{ & } b > 0
\]

So that for

\[
\left[1 - e^{-\frac{a}{b}(e^{b(-\ln G)-1})}\right]^{c-1}
\]

we get two formula

If \( c - 1 > 0 \) we have

\[
\left[1 - e^{-\frac{a}{b}(e^{b(-\ln G)-1})}\right]^{c-1} = \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(c)}{m! \Gamma(c-m)} e^{-\frac{a}{b}m(G^{b-1})}
\]

\[
= \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(c)}{m! \Gamma(c-m)} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \left(\frac{am}{b}\right)^i (G^{b-1})^i
\]

\[
= \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(c)}{m! \Gamma(c-m)} \sum_{i=0}^{\infty} \frac{(am)^i}{i!} \sum_{j=0}^{\infty} C_j (G^{b-1})^j (-1)^{i-j}
\]

\[
\]
Therefor for \( c - 1 > 0 \) we get

\[
1 - e^{-\frac{a}{b}(G^{b-1})}
\]

\[
= \sum_{m=0}^{\infty} \frac{\Gamma(c-1+m)}{m! \Gamma(c-1)} \left( -\frac{a}{b} \right)^m \frac{\Gamma(c)}{\Gamma(c-m)} \left( \frac{am}{b} \right)^i G^{-bj}
\]

If \( c - 1 < 0 \) we have

\[
1 - e^{-\frac{a}{b}(G^{b-1})}
\]

\[
= \sum_{m=0}^{\infty} \frac{\Gamma(c-1+m)}{m! \Gamma(c-1)} \frac{(-1)^i}{i!} \left( \frac{am}{b} \right)^i (G^{b-1})^i
\]

\[
= \sum_{m=0}^{\infty} \frac{\Gamma(c-1+m)}{m! \Gamma(c-1)} \frac{(-1)^i}{i!} \left( \frac{am}{b} \right)^i \sum_{i=0}^{\infty} C_i G^{-bj}(-1)^i
\]

Therefor for \( c - 1 < 0 \) we get

\[
1 - e^{-\frac{a}{b}(G^{b-1})}
\]

\[
= \sum_{m=0}^{\infty} \frac{\Gamma(c-1+m)}{m! \Gamma(c-1)} \left( -\frac{a}{b} \right)^m \frac{\Gamma(c)}{\Gamma(c-m)} \left( \frac{am}{b} \right)^i G^{-bj}
\]

So the pdf, Equation (10), is given by the following two cases

**Case one:** for \( c - 1 > 0 \)

\[
f(x) = a c G^{-(b+1)} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left( \frac{a}{b} \right)^k (G^{b-1})^k \sum_{m=0}^{\infty} \frac{\Gamma(c)}{\Gamma(c-m)} \frac{am}{b}^i G^{-bj}
\]

So the pdf for \( c - 1 > 0 \) will be

\[
f(x) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{\Gamma(c)}{\Gamma(c-m)} \frac{am}{b}^i G^{-(b+1)-bt-bj}
\]

**Case two:** for \( c - 1 < 0 \)

\[
f(x) = a c G^{-(b+1)} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left( \frac{a}{b} \right)^k (G^{b-1})^k \sum_{m=0}^{\infty} \frac{\Gamma(c)}{\Gamma(c-m)} \frac{am}{b}^i G^{-bj}
\]

So the pdf for \( c - 1 < 0 \) will be

\[
f(x) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{\Gamma(c)}{\Gamma(c-m)} \frac{am}{b}^i G^{-(b+1)-bt-bj}
\]

Hence the expansion formula for the pdf of GGom – G distributions is given by

\[
f(x) = \begin{cases} 
G^{-(b+1)}(1-t)^{-1} ; & c - 1 > 0 \\
\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{i=0}^{\infty} \frac{\Gamma(c)}{\Gamma(c-m)} \frac{am}{b}^i G^{-(b+1+t)}(1-t)^{-1} ; & c - 1 < 0 
\end{cases}
\]

(11)

3. Generalized Gompertz – Generalized Gompertz Distribution

In this section, the generalized Gompertz - generalized Gompertz (GGom – GGom) distribution is proposed as a special case of GGom – G family. The cumulative distribution, probability density, reliability and hazard rate functions are introduced as follow.

Suppose that \( G \) and \( g \), in Equations (8) and (9), represents the cdf and pdf of the GGom with the parameters \( \alpha, \beta, \lambda \) as

\[
G(x; \alpha, \beta, \lambda) = \left( 1 - e^{-\frac{\alpha}{\beta} e^{\beta x-1}} \right)^\lambda \quad \text{with} \quad x \geq 0 ; \quad \alpha, \lambda > 0 \quad \text{and} \quad \beta \geq 0
\]

(12)

\[
g(x; \alpha, \beta, \lambda) = \alpha \lambda e^{\beta x} e^{-\frac{\alpha}{\beta} e^{\beta x-1}} \left[ 1 - e^{-\frac{\alpha}{\beta} e^{\beta x-1}} \right]^{\lambda-1}
\]

(13)
Then we can get the cdf and pdf of the new distribution GGom – GGom as

\[
F(x) = 1 - \left[ 1 - e^{-\frac{\alpha}{\beta}(\epsilon^{\beta x - 1})^{1-b\lambda}} \right]^c
\]

(14)

\[
f(x) = ac\alpha e^{\beta x} \left( 1 - e^{-\frac{\alpha}{\beta}(\epsilon^{\beta x - 1})^{-(b\lambda+1)}} \right) e^{-\frac{\alpha}{\beta}(\epsilon^{\beta x - 1})^{1-b\lambda}}
\]

\[
\left[ 1 - e^{-\frac{\alpha}{\beta}(\epsilon^{\beta x - 1})^{1-b\lambda}} \right]^{c-1}
\]

(15)

The reliability function of GGom – GGom distribution is given by

\[
R(x) = \left[ 1 - e^{-\frac{\alpha}{\beta}(\epsilon^{\beta x - 1})^{1-b\lambda}} \right]^c
\]

(16)

The hazard rate function, \( \lambda(x) \), of GGom – GGom distribution is given by

\[
\lambda(x) = ac\alpha e^{\beta x} \left( 1 - e^{-\frac{\alpha}{\beta}(\epsilon^{\beta x - 1})^{-(b\lambda+1)}} \right) e^{-\frac{\alpha}{\beta}(\epsilon^{\beta x - 1})^{1-b\lambda}}
\]

\[
\left[ 1 - e^{-\frac{\alpha}{\beta}(\epsilon^{\beta x - 1})^{1-b\lambda}} \right]^{-1}
\]

(17)

4. Essential Statistical Properties of GGom – GGom Distribution

In this section, the most essential statistical properties of GGom – GGom distribution such as the r-th moment, mean, variance, coefficient of skewness, coefficient of kurtosis, Shannon and relative entropies along with stress strength model are obtained.

The r-th Moment: The r-th non-central moment is given by

\[
E(X^r) = \int_0^\infty x^r f(x) dx
\]

With expansion cases of the pdf, Equation (11), the integration will only depend on \( gG^{-b(1+t+j)-1} \), so let

\[
I = \int_0^\infty x^r \left( 1 - e^{-\frac{\alpha}{\beta}(\epsilon^{\beta x - 1})^{1-b\lambda}} \right)^{\lambda-1} \left( 1 - e^{-\frac{\alpha}{\beta}(\epsilon^{\beta x - 1})^{1-b\lambda}} \right)^{\lambda-1} dx
\]

\[
I = \frac{1}{b(1+t+j)} \int_0^\infty x^r \left( \alpha\lambda[-b(1+t+j)] e^{\beta x} e^{-\frac{\alpha}{\beta}(\epsilon^{\beta x - 1})^{1-b\lambda}} \right)^{\lambda-1} \left( 1 - e^{-\frac{\alpha}{\beta}(\epsilon^{\beta x - 1})^{1-b\lambda}} \right)^{\lambda-1} dx
\]

It is clear that \( \alpha\lambda[-b(1+t+j)] e^{\beta x} e^{-\frac{\alpha}{\beta}(\epsilon^{\beta x - 1})^{1-b\lambda}} \) represents the pdf of GGom distribution as in Equation (5), with \( \lambda = \alpha, \ c = \beta, \ \theta = \lambda[-b(1+t+j)] \).
Recall that the r-th moment of the GGom distribution is given by [3]

\[ \mu_r = \int_0^\infty x^r f(x) \, dx = \theta \lambda \Gamma(r + 1) \sum_{j=0}^\infty \sum_{k=0}^\infty C_j^k \left[ \frac{\alpha}{\beta} \right]^{\alpha j+1} \left[ \frac{\beta}{\alpha} \right]^{\beta k+1} (-1)^{j+k} e^{-\frac{\alpha}{\beta} x} \left[ \frac{1}{\Gamma(r+1)} \right] \left( -1 \right)^{r+1} \]

So I will be

\[ I = -\frac{1}{b(1 + t + j)} \alpha \lambda \left[ -b(1 + t + j) \right] \Gamma(r + 1) \sum_{k=1}^\infty \sum_{j=1}^\infty C_j^k \left[ \frac{\alpha}{\beta} \right]^{\alpha j+1} \left[ \frac{\beta}{\alpha} \right]^{\beta k+1} (-1)^{j+k} \frac{\alpha}{\beta} \left[ \frac{1}{\Gamma(r+1)} \right] \left( -1 \right)^{r+1} \]

and it simplifies to

\[ I = \alpha \lambda \Gamma(r + 1) \sum_{k=1}^\infty \sum_{j=1}^\infty C_j^k \left[ \frac{\alpha}{\beta} \right]^{\alpha j+1} \left[ \frac{\beta}{\alpha} \right]^{\beta k+1} (-1)^{j+k} \frac{\alpha}{\beta} \left[ \frac{1}{\Gamma(r+1)} \right] \left( -1 \right)^{r+1} \]

Therefore the r-th moment of the proposed GGom – GGom distribution will be given by

\[ E(X^r) = \left\{ \begin{array}{ll}
\sum_{k=0}^\infty \sum_{t=0}^\infty \sum_{m=0}^\infty \sum_{l=0}^\infty \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{q=0}^\infty \sum_{p=0}^\infty C_i^k C_j^p \left[ \frac{\alpha}{\beta} \right]^{\alpha j+1} \left[ \frac{\beta}{\alpha} \right]^{\beta k+1} (-1)^{j+k} \frac{\alpha}{\beta} \left[ \frac{1}{\Gamma(r+1)} \right] \left( -1 \right)^{r+1} & ; c = 0 \\
\sum_{k=0}^\infty \sum_{t=0}^\infty \sum_{m=0}^\infty \sum_{l=0}^\infty \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{q=0}^\infty \sum_{p=0}^\infty C_i^k C_j^p \left[ \frac{\alpha}{\beta} \right]^{\alpha j+1} \left[ \frac{\beta}{\alpha} \right]^{\beta k+1} (-1)^{j+k} \frac{\alpha}{\beta} \left[ \frac{1}{\Gamma(r+1)} \right] \left( -1 \right)^{r+1} & ; c = 1
\end{array} \right\} \]

Depending on the particular E(X^r); (r = 1,2,3,4), another properties of this distribution such as the mean (\mu = E(X)), variance (V(X) = \sigma^2 = E(X^2) - [E(X)]^2), coefficient of skewness \( (CS = \frac{E(X^3) - 3\mu E(X^2) + 2\mu^3}{\sigma^3}) \) and coefficient of kurtosis \( (CK = \frac{E(X^4) - 4\mu E(X^3) + 6\mu^2 E(X^2) - 3\mu^4}{\sigma^4}) \) can be obtained.

**The Characteristic Function:** The characteristic function is given by

\[ \psi(t) = E(e^{itx}) = E \left( \sum_{r=0}^\infty \frac{(tx)^r}{r!} \right) = \sum_{r=0}^\infty \frac{(it)^r}{r!} E(X^r) \]

Therefore the characteristic function of the GGom – GGom distribution will be given by
The Quantile and Median: The 100th quantile, define as the solution of the following equation, with respect to $x_q$, where $x_q > 0$ and $0 < q < 1$

$$q = P(x \leq x_q) = F(x_q)$$

From the cdf of GGom – GGom distribution, Equation (14), we get

$$q = 1 - \left[ 1 - e^{-\frac{a}{b} \left[ e^{\frac{a}{b} (\frac{b}{a} x_q - 1)} \right]^{\frac{\lambda}{\beta}}} \right]^c$$

Then

$$1 - (1 - q)^{\frac{1}{c}} = e^{-\frac{a}{b} \left[ 1 - e^{-\frac{a}{b} \left( \frac{b}{a} e^{\frac{a}{b} (\frac{b}{a} x_q - 1)} \right)^{\frac{\lambda}{\beta}}} \right]}$$

By taking the natural logarithm of both sides we get

$$\ln \left( 1 - (1 - q)^{\frac{1}{c}} \right) = -\frac{a}{b} \left( 1 - e^{-\frac{a}{b} \left( \frac{b}{a} e^{\frac{a}{b} (\frac{b}{a} x_q - 1)} \right)^{\frac{\lambda}{\beta}}} \right)$$

$$1 - \frac{b}{a} \ln \left( 1 - (1 - q)^{\frac{1}{c}} \right) = \left[ 1 - e^{-\frac{a}{b} \left( \frac{b}{a} e^{\frac{a}{b} (\frac{b}{a} x_q - 1)} \right)^{\frac{\lambda}{\beta}}} \right]^{-\frac{1}{b\lambda}}$$

Again by taking the natural logarithm of both sides with some simplification steps we get the quantiles of the GGom – GGom distribution as

$$x_q = \frac{1}{\beta} \ln \left( 1 - \frac{\beta}{\alpha} \ln \left( 1 - \frac{b}{a} \ln \left( 1 - (1 - q)^{\frac{1}{c}} \right) \right)^{\frac{-1}{b\lambda}} \right)$$

The median of the GGom – GGom distribution can be obtained directly by putting $\frac{1}{2}$ instead of $q$ in Equation (21). Therefore the median of the GGom – GGom distribution is given by

$$\text{Median} = \frac{1}{\beta} \ln \left( 1 - \frac{\beta}{\alpha} \ln \left( 1 - \frac{b}{a} \ln \left( 1 - (\frac{1}{2})^{\frac{1}{c}} \right) \right)^{\frac{-1}{b\lambda}} \right)$$

Simulated Data: Since the GGom – GGom distribution has a closed form for cdf, therefore the simulated data can be derived from
where $U$ has the uniform $(0,1)$ distribution.

**Shannon Entropy**: The Shannon entropy can be written as

\[
SH = -\int_0^\infty \ln f(x) f(x) \, dx
\]

By taking the natural logarithm to Equation (9), we get

\[
\ln f(x) = \ln ac + \ln g - (b+1) \ln G - \frac{a}{b} (G^{-b} - 1) + (c-1) \ln \left[1 - e^{-\frac{a}{b} (G^{-b} - 1)}\right]
\]

Now

\[
SH = - \left\{ \ln ac + \int_0^\infty \ln g f(x) \, dx - (b+1) \int_0^\infty \ln G f(x) \, dx - \frac{a}{b} \int_0^\infty G^{-b} f(x) \, dx + \frac{a}{b} \right. \\
+ (c-1) \int_0^\infty \ln \left[1 - e^{-\frac{a}{b} (G^{-b} - 1)}\right] f(x) \, dx
\]

Rewrite $SH$ as

\[
SH = - \left\{ \ln ac + I_1 - (b+1) I_2 - \frac{a}{b} \lambda + \frac{a}{b} + (c-1) \lambda \right\}
\]

Now

\[
I_1 = \int_0^\infty \ln g f(x) \, dx = \int_0^\infty \ln \left[ a \rho \beta x e^{-\frac{a}{b} (e^{\beta x} - 1)} \left(1 - e^{-\frac{a}{b} (e^{\beta x} - 1)}\right)^{\lambda - 1}\right] f(x) \, dx
\]

\[
= \int_0^\infty \ln \alpha \lambda x + \beta x - \frac{a}{b} (e^{\beta x} - 1) + (\lambda - 1) \ln \left(1 - e^{-\frac{a}{b} (e^{\beta x} - 1)}\right) f(x) \, dx
\]

Again let

\[
I_{1,1} = \int_0^\infty \ln \left(1 - e^{-\frac{a}{b} (e^{\beta x} - 1)}\right) f(x) \, dx
\]

Since $\ln(1-x) = -\sum_{n=0}^\infty \frac{x^n}{n}$ with $-1 < x < 1$ and in our case $0 < e^{-\frac{a}{b} (e^{\beta x} - 1)} < 1$, then

\[
I_{1,1} = \int_0^\infty -\sum_{n=0}^\infty \frac{1}{n} e^{-\frac{a}{b} (e^{\beta x} - 1)} f(x) \, dx
\]

\[
= -\sum_{n=0}^\infty \sum_{i=0}^\infty \frac{1}{n} \frac{(-1)^i}{i!} \left(\frac{na}{\beta}\right)^i \int_0^\infty (e^{\beta x} - 1)^i f(x) \, dx
\]

\[
= \sum_{n=0}^\infty \sum_{i=0}^\infty \sum_{k=0}^\infty C_k \frac{(-1)^{2i+k+1}}{n i!} \left(\frac{n a}{\beta}\right)^i \int_0^\infty e^{\beta x} f(x) \, dx
\]

So that

\[
I_1 = \ln \alpha \lambda + \beta E(X) - \frac{\alpha}{\beta} \left(\sum_{n=0}^\infty \frac{\beta^i}{i!} \, E(X^i)\right)
\]

\[
+ (\lambda - 1) \sum_{n=0}^\infty \sum_{i=0}^\infty \sum_{k=0}^\infty C_k \frac{(-1)^{2i+k+1}}{n i!} \left(\frac{n a}{\beta}\right)^i \frac{(k \beta)^i}{i!} E(X^i)
\]

\[
I_2 = \int_0^\infty \ln G f(x) \, dx = \int_0^\infty \ln \left(1 - e^{-\frac{a}{b} (e^{\beta x} - 1)}\right) f(x) \, dx = \lambda \int_0^\infty \ln \left(1 - e^{-\frac{a}{b} (e^{\beta x} - 1)}\right) f(x) \, dx = \lambda I_{1,1}
\]

\[
I_2 = \lambda \sum_{n=0}^\infty \sum_{i=0}^\infty \sum_{k=0}^\infty C_k \frac{(-1)^{2i+k+1}}{n i!} \left(\frac{n a}{\beta}\right)^i \frac{(k \beta)^i}{i!} E(X^i)
\]
\[ I_3 = \int_0^\infty G^{-b} f(x) \, dx = \int_0^\infty \left( 1 - \frac{\alpha}{\beta} e^{(e^\beta x - 1)} \right)^{-b\lambda} f(x) \, dx \]
\[ = \sum_{j=0}^\infty \sum_{j_1=0}^\infty \sum_{j_2=0}^\infty \sum_{j_3=0}^\infty C_{j_1}^{t_1} C_{j_2}^{t_2} C_{j_3}^{t_3} \left( \frac{\alpha}{\beta} \right)^{j_1} \left( \frac{j_2}{\beta} \right)^{j_2} \left( \frac{j_3}{\beta} \right)^{j_3} \Gamma \left( b \lambda + j \right) \Gamma \left( b \lambda + j_1 \right) \Gamma \left( b \lambda + j_2 \right) \Gamma \left( b \lambda + j_3 \right) \left( \frac{\alpha}{\beta} \right)^{j_1} \left( \frac{j_2}{\beta} \right)^{j_2} \left( \frac{j_3}{\beta} \right)^{j_3} \Gamma \left( \beta \lambda + j \right) \Gamma \left( \beta \lambda + j_1 \right) \Gamma \left( \beta \lambda + j_2 \right) \Gamma \left( \beta \lambda + j_3 \right) \left( \frac{\alpha}{\beta} \right)^{j_1} \left( \frac{j_2}{\beta} \right)^{j_2} \left( \frac{j_3}{\beta} \right)^{j_3} \right) \]

Then
\[ I_4 = \int_0^\infty \ln \left[ 1 - e^{-\frac{\alpha}{\beta}(e^{\beta x} - 1)} \right] \, f(x) \, dx \]

Again by using \( \ln(1 - x) = -\frac{\sum_{n=0}^\infty x^n}{n} \), where in our case \( 0 < e^{\frac{\alpha}{\beta}(e^{\beta x} - 1)} < 1 \), we get
\[ I_4 = \int_0^\infty -\sum_{n=0}^\infty \frac{1}{n} e^{-\frac{\alpha}{\beta}(e^{\beta x} - 1)} \, f(x) \, dx \]

Again let
\[ I_{4.1} = \int_0^\infty \left[ 1 - e^{-\frac{\alpha}{\beta}(e^{\beta x} - 1)} \right]^{-\lambda b t_2} f(x) \, dx \]
\[ = \int_0^\infty \sum_{j=0}^\infty \frac{\Gamma(\lambda b t_2 + j)}{\Gamma(\lambda b t_2)} e^{-\frac{\alpha}{\beta}(e^{\beta x} - 1)} f(x) \, dx \]
\[ = \sum_{j=0}^\infty \frac{\Gamma(\lambda b t_2 + j)}{\Gamma(\lambda b t_2)} \int_0^\infty e^{-\frac{\alpha}{\beta}(e^{\beta x} - 1)} f(x) \, dx \quad (\text{see } I_3) \]
\[ I_{4.1} = \sum_{j=0}^\infty \frac{\Gamma(\lambda b t_2)}{\Gamma(\lambda b t_2)} \sum_{j_1=0}^\infty \sum_{j_2=0}^\infty \sum_{j_3=0}^\infty C_{j_1}^{t_1} C_{j_2}^{t_2} C_{j_3}^{t_3} \left( \frac{\alpha}{\beta} \right)^{j_1} \left( \frac{j_2}{\beta} \right)^{j_2} \left( \frac{j_3}{\beta} \right)^{j_3} \Gamma \left( \beta \lambda + j_1 \right) \Gamma \left( \beta \lambda + j_2 \right) \Gamma \left( \beta \lambda + j_3 \right) \left( \frac{\alpha}{\beta} \right)^{j_1} \left( \frac{j_2}{\beta} \right)^{j_2} \left( \frac{j_3}{\beta} \right)^{j_3} \Gamma \left( \beta \lambda + j_1 \right) \Gamma \left( \beta \lambda + j_2 \right) \Gamma \left( \beta \lambda + j_3 \right) \left( \frac{\alpha}{\beta} \right)^{j_1} \left( \frac{j_2}{\beta} \right)^{j_2} \left( \frac{j_3}{\beta} \right)^{j_3} \right) \]

Then
\[ I_4 = \sum_{n=0}^\infty \sum_{t_1=0}^\infty \sum_{t_2=0}^\infty \sum_{t_3=0}^\infty \sum_{t_1'=0}^\infty \sum_{t_2'=0}^\infty \sum_{t_3'=0}^\infty C_{t_1}^{t_1} C_{t_2}^{t_2} C_{t_3}^{t_3} \left( \frac{\alpha}{\beta} \right)^{t_1} \left( \frac{t_2}{\beta} \right)^{t_2} \left( \frac{t_3}{\beta} \right)^{t_3} \Gamma \left( \lambda b t_2 + j \right) \Gamma \left( \lambda b t_2 + j_1 \right) \Gamma \left( \lambda b t_2 + j_2 \right) \Gamma \left( \lambda b t_2 + j_3 \right) \left( \frac{\alpha}{\beta} \right)^{t_1} \left( \frac{t_2}{\beta} \right)^{t_2} \left( \frac{t_3}{\beta} \right)^{t_3} \Gamma \left( \beta \lambda + j_1 \right) \Gamma \left( \beta \lambda + j_2 \right) \Gamma \left( \beta \lambda + j_3 \right) \left( \frac{\alpha}{\beta} \right)^{t_1} \left( \frac{t_2}{\beta} \right)^{t_2} \left( \frac{t_3}{\beta} \right)^{t_3} \right] \]

Substituting \( I_1, I_2, I_3 \) and \( I_4 \) in Equation (24), the Shannon entropy of the GGom – GGom distribution is given by
\[ \text{SH} = - \left\{ \ln N_a + N_b + \ln N_c + \ln N_d + \ln N_e + \ln N_f + \ln N_g + \ln N_h + \ln N_i + \ln N_j + \ln N_k + \ln N_l + \ln N_m + \ln N_n + \ln N_o + \ln N_p + \ln N_q + \ln N_r + \ln N_s + \ln N_t + \ln N_u + \ln N_v + \ln N_w + \ln N_x + \ln N_y + \ln N_z \right\} \]

The Relative Entropy: The relative entropy can be written as
\[ \text{RE} = \int_0^\infty \ln \frac{f(x)}{f_1(x)} \, f(x) \, dx \]
\[
\ln \frac{f(x)}{f_1(x)} = \ln \frac{a \alpha \lambda \beta x - \alpha \beta x - \beta x - \alpha \beta^2 (e^{\beta x} - 1) + \frac{\alpha}{\beta} (e^{\beta x} - 1) - (b \lambda + 1) \ln \left[ 1 - e^{-\frac{\alpha}{\beta} (e^{\beta x} - 1)} \right] + (b \lambda + 1) \ln \left[ 1 - e^{-\frac{\alpha}{\beta} (e^{\beta x} - 1)} \right] - \frac{a}{b} \left[ 1 - e^{-\frac{\alpha}{\beta} (e^{\beta x} - 1)} \right]^{-b \lambda} \right] + (c_1 - 1) \ln \left[ 1 - e^{-\frac{\alpha}{\beta} (e^{\beta x} - 1)} \right]^{-b \lambda} \right] - (c_1 - 1) \ln \left[ 1 - e^{-\frac{\alpha}{\beta} (e^{\beta x} - 1)} \right]^{-b \lambda} \right]
\]

\[
RE = \ln \frac{a \alpha \lambda \beta x - \alpha \beta x - \beta x - \alpha \beta^2 (e^{\beta x} - 1) + \frac{\alpha}{\beta} (e^{\beta x} - 1) - (b \lambda + 1) \ln \left[ 1 - e^{-\frac{\alpha}{\beta} (e^{\beta x} - 1)} \right] + (b \lambda + 1) \ln \left[ 1 - e^{-\frac{\alpha}{\beta} (e^{\beta x} - 1)} \right] - \frac{a}{b} \left[ 1 - e^{-\frac{\alpha}{\beta} (e^{\beta x} - 1)} \right]^{-b \lambda} \right] + (c_1 - 1) \ln \left[ 1 - e^{-\frac{\alpha}{\beta} (e^{\beta x} - 1)} \right]^{-b \lambda} \right]
\]

Therefor the relative entropy of the GGom – Gogom distribution can be written as follows:

\[
RE = \ln \frac{a \alpha \lambda \beta x - \alpha \beta x - \beta x - \alpha \beta^2 (e^{\beta x} - 1) + \frac{\alpha}{\beta} (e^{\beta x} - 1) - (b \lambda + 1) \ln \left[ 1 - e^{-\frac{\alpha}{\beta} (e^{\beta x} - 1)} \right] + (b \lambda + 1) \ln \left[ 1 - e^{-\frac{\alpha}{\beta} (e^{\beta x} - 1)} \right] - \frac{a}{b} \left[ 1 - e^{-\frac{\alpha}{\beta} (e^{\beta x} - 1)} \right]^{-b \lambda} \right]
\]

\[
E \left( \ln \left[ 1 - e^{-\frac{\alpha}{\beta} (e^{\beta x} - 1)} \right] \right) = I_{1.1} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{2i-k+1}}{n!} \left( \frac{n \alpha}{\beta} \right) \Gamma(b \lambda + 1) \Gamma(b \lambda) \left( \frac{\alpha}{\beta} \right)^{i+1} \Gamma(b \lambda - i) \Gamma(b \lambda - i) E(X^i)
\]

\[
E \left( \ln \left[ 1 - e^{-\frac{\alpha}{\beta} (e^{\beta x} - 1)} \right] \right) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{2i-k+1}}{n!} \left( \frac{n \alpha}{\beta} \right) \Gamma(b \lambda + 1) \Gamma(b \lambda) \left( \frac{\alpha}{\beta} \right)^{i+1} \Gamma(b \lambda - i) \Gamma(b \lambda - i) E(X^i)
\]

\[
E \left[ 1 - e^{-\frac{\alpha}{\beta} (e^{\beta x} - 1)} \right]^{-b \lambda} = I_3
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{2i-k+1}}{n!} \left( \frac{n \alpha}{\beta} \right) \Gamma(b \lambda + 1) \Gamma(b \lambda) \left( \frac{\alpha}{\beta} \right)^{i+1} \Gamma(b \lambda - i) \Gamma(b \lambda - i) E(X^i)
\]

\[
(\text{26})
\]
**Stress Strength:** The stress strength can be written as

\[
SS = \int_0^\infty f_X(x) F_Y(x) \, dx
\]

Since the cdf can be written as \( F_Y(x) = 1 - \left[ 1 - e^{-\frac{a_{1Y}}{b_{1Y}} (c_{1Y} - 1)} \right]^{c_{1Y}} \), where \( 0 < e^{-\frac{a_{1Y}}{b_{1Y}} (c_{1Y} - 1)} < 1 \)

Let \( z = e^{-\frac{a_{1Y}}{b_{1Y}} (c_{1Y} - 1)} \), then

\[
(1 - z)^{c_{1Y}} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{\Gamma(c_{1Y} + 1)}{\Gamma(c_{1Y} - m + 1)} z^m = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{\Gamma(c_{1Y} + 1)}{\Gamma(c_{1Y} - m + 1)} e^{-\frac{a_{1Y}}{b_{1Y}} (c_{1Y} - 1)}
\]

Let \( G_{1Y}^{-b_{1Y}c_{1Y}} = \left[ 1 - e^{-\frac{a_{1Y}}{b_{1Y}} (e^{b_{1Y}x} - 1)} \right]^{-b_{1Y}c_{1Y}} \), then

\[
G_{1Y}^{-b_{1Y}c_{1Y}} = \sum_{j=0}^{\infty} \left( \frac{\alpha_{1Y}}{b_{1Y}} \right)^j \frac{(-1)^j}{j!} \frac{\Gamma(b_{1Y} + j)}{\Gamma(b_{1Y} + 1)} \left( e^{b_{1Y}x} - 1 \right)^{j + 1}
\]

Then

\[
(1 - z)^{c_{1Y}} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{\Gamma(c_{1Y} + 1)}{\Gamma(c_{1Y} - m + 1)} \sum_{i=0}^{\infty} \left( \frac{\alpha_{1Y}}{b_{1Y}} \right)^i \frac{(-1)^i}{i!} \frac{\Gamma(b_{1Y} + i)}{\Gamma(b_{1Y} + 1)} \left( e^{b_{1Y}x} - 1 \right)^{i + 1}
\]

\[
(1 - z)^{c_{1Y}} = \sum_{m=0}^{\infty} \sum_{i=0}^{\infty} \left( \frac{\alpha_{1Y}}{b_{1Y}} \right)^i \frac{(-1)^i}{i!} \frac{\Gamma(b_{1Y} + i)}{\Gamma(b_{1Y} + 1)} \left( e^{b_{1Y}x} - 1 \right)^{i + 1}
\]

Thus the stress strength of the GGom - GGom distribution can be written as

\[
SS = 1 - \sum_{m=0}^{\infty} \sum_{i=0}^{\infty} \left( \frac{\alpha_{1Y}}{b_{1Y}} \right)^i \frac{(-1)^i}{i!} \frac{\Gamma(b_{1Y} + i)}{\Gamma(b_{1Y} + 1)} \left( e^{b_{1Y}x} - 1 \right)^{i + 1}
\]

5. Concluding Remarks

A new family of continuous distributions based on \([0, \infty)\) generalized Gompertz distribution, named generalized Gompertz – G distributions has been proposed. The generalized Gompertz - generalized Gompertz (GGom - GGom) distribution is discussed as a special case of this new family. The probability density, cumulative distribution, reliability and hazard rate functions along with the essential statistical properties of GGom – GGom distribution, such as non – central rth moment, mean, variance, skewness, kurtosis, characteristic function, quantile, median, Shannon and relative entropies are derived. Furthermore, this paper deals with the determination of stress-strength \( R = P(Y < X) \) when \( X \) and \( Y \) (represents the strength and stress respectively) are two independent GGom - GGom distribution with different parameters.

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