MARINARI-PARISI AND SUPERSYMMETRIC COLLECTIVE FIELD THEORY

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ABSTRACT

A field theoretic formulation of the Marinari-Parisi supersymmetric matrix model is established and shown to be equivalent to a recently proposed supersymmetrization of the bosonic collective string field theory. It also corresponds to a continuum description of super-Calogero models. The perturbation theory of the model is developed and, in this approach, an infinite sequence of vertices is generated. A class of potentials is identified for which the spectrum is that of a massless boson and Majorana fermion. For the harmonic oscillator case, the cubic vertices are obtained in an oscillator basis. For a rather general class of potentials it is argued that one cannot generate from Marinari-Parisi models a continuum limit similar to that of the $d = 1$ bosonic string.
1. INTRODUCTION

As is well known, the double scaling limit [1] of large-$N$ matrix models describing sums over discretized random surfaces has provided much insight into the properties of $d \leq 1$ strings. Of these, the one dimensional string [2] exhibits the richest and most interesting structure (see [3] for a recent review).

The supersymmetric extension of these models has not been as straightforward as might have been expected, particularly in so far as their space-time properties are concerned.

Preliminary investigations seem to indicate that the way to lower dimensional superstrings is not via supermatrices [4]. From a continuum point of view, one can compute $d \leq 1$ correlation functions [5, 6], but for $d = 1$ one finds [6] that the requirement of locality for the supercharges reduces the space-time spectrum of the theory to two bosons. This is substantiated by a BRST analysis of the theory [7].

In reference [8] Marinari and Parisi introduced a supersymmetric model which can be thought of as a discretization of one dimensional superspace (it is to this aspect that we will refer when denoting these models as Marinari-Parisi models, and not to the specific choice of potential in [8]). This model provides a way to stabilize $d = 0$ bosonic potentials. This results from a well known mechanism of dimensional reduction [9] (as would be the case for a corresponding one dimensional Fokker-Planck system), and indeed a complete analysis of $d \leq 0$ bosonic no-unitary multicritical points via Marinari-Parisi has been carried out recently [10]. However, since Marinari-Parisi models have an interpretation as sums over discretized surfaces in one dimensional superspace, it is an interesting question to investigate if a double scaling limit can be identified leading to a continuum theory of $d = 1$ superstrings. In this sense these models are often referred to in the literature as possible models of one dimensional superstrings.

Further related work is indicated in reference [11].
Collective field theory [12], in the form developed by Das and Jevicki [13], provides a very successful description of the one dimensional bosonic string. Jevicki and one of the authors [14] have recently introduced a supersymmetric extension of the collective field theory by exploiting the metric structure of the bosonic model. It was emphasized in reference [14] that the theory that one obtains, provides, at the classical level, a continuum description of super-Calogero models of the type introduced by Freedman and Mende [15]. The relationship between the eigenvalue dynamics of single matrix models and Calogero models [16] is not new [17] and has been a very fruitful one [17, 18].

More recently, Dabholkar [19] has been able to show that the Marinari-Parisi model, when restricted to a suitable symmetric Hilbert space, yields a model which can be expressed in terms of the $N$ eigenvalues of the bosonic matrix and their superpartners. It is not difficult to show that this model is of the super-Calogero type.

Therefore, as will be shown in chapter 2 of this paper, supersymmetric collective field theory [14], Marinari-Parisi [8] and super-Calogero models [15] are essentially the same system. In other words, Dabholkar’s reduction of the Marinari-Parisi model yields a super-Calogero system of which the supersymmetric collective field theory provides a continuum description.

It is the purpose of this paper to study the properties of the supersymmetric collective field theory in the presence of an arbitrary potential. In this way, as follows from our previous discussion, we obtain a continuum description of super-Calogero models and develop a field theory of Marinari-Parisi models.

In chapter 3, the perturbative expansion of the model is developed. As a result of the nontrivial commutation relations amongst continuum fields established in reference [14], the hamiltonian of the model acquires an infinite set of vertices, in contrast to the cubic bosonic collective string field hamiltonian [20].

Assuming that supersymmetry is preserved at the level of the leading semiclassical configuration, it follows from the super-Calogero analogue that the spectrum
should be supersymmetric. In chapter 4 we show how this can be proved in the continuum. We also describe a class of potentials for which the spectrum consists of a massless boson and Majorana fermion.

In chapter 5 we obtain a manifestly supersymmetric oscillator expansion for the cubic vertices in the case of the harmonic superpotential.

Chapter 6 amounts to a no-go theorem for the possibility of generating $d = 1$ superstrings from Marinari-Parisi models with a rather general class of potentials.

We will first argue that even if one regards the true statement of stationarity as one obtained with respect to the original variables of the theory (“master variables”) [21], one cannot introduce a lagrange multiplier at the level of the stationarity condition if one wishes to preserve supersymmetry. For the one-dimensional bosonic string, and for a suitable choice of potential, there exists a rescaling of variables such that the normalization condition on the density of eigenvalues becomes essentially a free one. However, in the supersymmetric case we will show that for potentials that are homogeneous of arbitrary degree in the cosmological constant one can identify a rescaling in terms of which the cosmological constant is scaled out while the normalization condition remains fixed. This implies that for these potentials the time of flight remains finite in the scaling limit and the mechanism of generation of an infinite Liouville-like dimension is not present.

In chapter 7 we reexamine the bosonic sector cubic oscillator vertices of chapter 5 and point out that they are different from those of the bosonic collective string field theory [20]. We show how this discrepancy can be traced back to a difference between the turning point regularization prescriptions of chapter 5 and reference [20]. Adjusting the regularization to agree with that of [20], we argue that a bosonic sector compatible with the bosonic collective field theory cannot be generated from a corresponding regularization of the supercharges considered in this article.

Chapter 8 is reserved for conclusions.

A large portion of this work is contained in AvT’s PhD thesis [23]. During the write-up, we received a preprint by Cohn and Dykstra [24] in which related issues
2. DISCUSSION OF THE MODELS

2.1. Supersymmetric matrix model

We take as our starting point the supersymmetric matrix model of Marinari and Parisi. Our analysis is based on that of Dabholkar’s in [19].

Consider a theory of $N \times N$ hermitian matrices on a $(1, 1)$-dimensional superspace with action

$$S = \int dt \, d\theta \, d\bar{\theta} \left( \frac{1}{2} \text{Tr} \left( \bar{D} \Phi \, D \Phi \right) + \bar{W}(\Phi) \right). \quad (2.1)$$

Here $D \equiv \partial_{\bar{\theta}} - i \theta \, \partial_t$, and we can expand $\Phi$ as

$$\Phi \equiv M + \Psi^\dagger \theta + \bar{\Psi} \bar{\theta} + \bar{\theta} \theta F, \quad (2.2)$$

where $M$ and $F$ are hermitian, and where we have used the conventions $(\alpha \beta)^* = \beta^* \alpha^*$ and $(\partial_{\bar{\theta}})^* = -\partial_\theta$.

The Feynman diagrams of a matrix theory can be topologically classified according to the genus. If one takes $\bar{W}$ to be of the form

$$\bar{W}(\Phi) = \text{Tr} \left( g_2 \Phi^2 + \frac{g_3}{\sqrt{N}} \Phi^3 + \cdots + \frac{g_p}{N^{p/2}-1} \Phi^p \right), \quad (2.3)$$

then by a topological argument [27] it follows that a diagram of genus $\Gamma$ carries a factor $N^{2-2\Gamma}$ and can therefore naively be interpreted as a supertriangulation of the corresponding diagram in the perturbation expansion of a string theory with coupling constant $1/N^2$. 

are discussed.
Upon quantization one finds the hamiltonian

\[ H = \frac{1}{2} \text{Tr} \left( P^2 + \frac{\partial \bar{W}(M)}{\partial M^*} \frac{\partial \bar{W}(M)}{\partial M} \right) + \sum_{ijkl} [\Psi_{ji}^*, \Psi_{kl}] \frac{\partial \bar{W}(M)}{\partial M_{ij}^*} \frac{\partial \bar{W}(M)}{\partial M_{kl}}, \] (2.4)

and supercharges

\[ Q = \sum_{ij} \Psi_{ij}^* \left( P_{ij}^* + i \frac{\partial \bar{W}(M)}{\partial M_{ij}^*} \right), \]
\[ Q^\dagger = \sum_{ij} \Psi_{ij} \left( P_{ij} + i \frac{\partial \bar{W}(M)}{\partial M_{ij}} \right), \] (2.5)

where \([P_{ij}, M_{kl}] = -i \delta_{ik} \delta_{jl}\) and \(\{\Psi_{ij}, \Psi_{kl}\} = \delta_{ik} \delta_{jl}\).

Now let \(U\) be the unitary transformation that diagonalizes the bosonic component \(M\) of the matrix \(\Phi\). In general the fermionic components \(\Psi\) and \(\Psi^\dagger\) of \(\Phi\) will not be diagonalized by \(U\). Nevertheless, writing the diagonal elements as

\[ (U \Phi U^\dagger)_{ii} \equiv \lambda_i + \bar{\theta} \psi_i + \psi_i^\dagger \theta + \bar{\theta} \theta f_i, \] (2.6)

we shall see that we can define an invariant subspace consisting of those wave functions that depend only on \((\lambda, \psi^\dagger)\). Therefore it will make sense to restrict the theory to this subspace.

Changing variables from \(M_{ij}\) to the \(N\) eigenvalues \(\lambda_i\) and the \(N(N - 1)/2\) angular variables on which \(U\) depends, one gets the decomposition (see Dabholkar [19])

\[ \frac{\partial}{\partial M_{ij}} = \sum_m U_{jm}^\dagger U_{mi} \frac{\partial}{\partial \lambda_m} + \sum_{k \neq l} \frac{U_{kl} U_{jl}^\dagger \tilde{A}_{kl}}{(\lambda_k - \lambda_l)}, \] (2.7)

where the angular derivative \(\tilde{A}_{kl}\) is defined by \(\tilde{A}_{kl} \equiv \sum_m U_{lm} \partial / \partial U_{km} \).
In terms of the new variables the supercharges (2.5) become

\[
Q = \sum_m \hat{\Psi}^*_{mm} \left( -i \frac{\partial}{\partial \lambda_m} - i \frac{\partial \bar{W}(\lambda)}{\partial \lambda_m} \right) - i \sum_{k \neq l} \hat{\Psi}^*_{kl} \bar{A}^*_{kl} (\lambda_k - \lambda_l),
\]

\[
Q^\dagger = \sum_m \hat{\Psi}_{mm} \left( -i \frac{\partial}{\partial \lambda_m} + i \frac{\partial \bar{W}(\lambda)}{\partial \lambda_m} \right) - i \sum_{k \neq l} \hat{\Psi}_{kl} \bar{A}_{kl} (\lambda_k - \lambda_l),
\]

(2.8)

where \( \hat{\Psi} \equiv U\Psi U^\dagger \). Using \( \hat{\Psi}_{mm} = \psi_m \), one can write a wave function depending only on \((\lambda, \psi^\dagger)\) as

\[
\phi(\lambda, \psi^\dagger) = f(\lambda) \prod_k \psi^\dagger_{m_k} |0 \rangle = f(\lambda) \prod_k \hat{\Psi}^*_{m_k} |0 \rangle.
\]

(2.9)

With a little patience one can now show that the subspace of such wavefunctions is indeed invariant under the action of the supercharges (2.8) and that in the subspace the supercharges reduce to

\[
Q = \sum_i \psi_i^\dagger \left( -i \frac{\partial}{\partial \lambda_i} - i \frac{\partial \bar{W}(\lambda)}{\partial \lambda_i} \right),
\]

\[
\bar{Q} = \sum_i \psi_i \left( -i \frac{\partial}{\partial \lambda_i} + i \frac{\partial \bar{W}(\lambda)}{\partial \lambda_i} - i \sum_{l \neq i} \frac{1}{\lambda_i - \lambda_l} \right).
\]

(2.10)

At first glance these expressions may seem inconsistent in that \( \bar{Q} \) appears to have lost its property of conjugacy to \( Q \). This paradox is resolved by the observation that \( Q \) and \( \bar{Q} \) are indeed hermitian conjugates with respect to the inner product on the original Hilbert space. The original inner product reduces to a nontrivial inner product on the subspace.

To see how this works, let us review the analysis of Jevicki and Sakita [12], modified here to include fermionic degrees of freedom. Take \( \phi_1 \) and \( \phi_2 \) to be of the
form (2.9). The inner product on the original Hilbert space can be written as

\[ \langle \phi_1 | \phi_2 \rangle = \int (dM)(d\Psi)(d\bar{\Psi}) e^{-\text{Tr}(\Psi^\dagger \Psi)} \phi_1^*(\lambda, \psi) \phi_2(\lambda, \psi^\dagger) \]

(2.11)

where \( dM = \prod_i dM_{ii} \prod_{j>i} dM_{ij} \) integrates over the independent degrees of freedom of a hermitian matrix only and where we have used the fact that the fermionic measure \( \exp[-\text{Tr}(\Psi^\dagger \Psi)] \) is invariant under \( \Phi \to U \Phi U^\dagger \) (this measure has to be included when one writes the fermionic part of the inner product as a Berezinian integral [30]).

One sees that the inner product in (2.11) differs from the trivial inner product on the subspace in that one has to include the measure \( J(\lambda) \). In general, this measure is given by the integral over the spurious degrees of freedom of the jacobian associated to the change of variables.

If we now rescale the wave functions as \( \phi \to J^{1/2} \phi \), we can use the trivial inner product on the subspace provided we rescale the momenta as \( p_i \to J^{1/2} p_i J^{-1/2} = p_i + \frac{i}{2} \partial (\ln J) / \partial \lambda_i \).

Applying this transformation to the supercharges (2.10) we find

\[ Q \to \sum_i \psi_i^\dagger \left( -i \frac{\partial}{\partial \lambda_i} + \frac{1}{2} i \frac{\partial \ln J}{\partial \lambda_i} - i \frac{\partial \bar{W}(\lambda)}{\partial \lambda_i} \right) \]

\[ \bar{Q} \to \sum_i \psi_i \left( -i \frac{\partial}{\partial \lambda_i} + \frac{1}{2} i \frac{\partial \ln J}{\partial \lambda_i} + i \frac{\partial W(\lambda)}{\partial \lambda_i} - i \sum_{l \neq i} \frac{1}{\lambda_i - \lambda_l} \right) \]

(2.12)

The most efficient way to solve for \( J \) is simply to use the fact that now \( \bar{Q}^\dagger = Q \) with respect to the trivial inner product in the subspace. We find

\[ \frac{\partial \ln J}{\partial \lambda_i} = \sum_{l \neq i} \frac{1}{\lambda_i - \lambda_l} \]

(2.13)
and we get an effective theory with trivial inner product and supercharges

\[ Q = \sum_i \psi_i^\dagger \left( p_i - i \frac{\partial W(\lambda)}{\partial \lambda_i} \right), \]

\[ Q^\dagger = \sum_i \psi_i \left( p_i + i \frac{\partial W(\lambda)}{\partial \lambda_i} \right), \] (2.14)

where the effective potential \( W \) is given by

\[ W = \bar{W} - \sum_{l<k} \ln(\lambda_k - \lambda_l). \] (2.15)

2.2. The super-Calogero model

The bosonic Calogero model [16] provides an example of an exactly solvable \( N \)-particle quantum mechanical system. Its hamiltonian in the centre of mass frame of reference is given by

\[ H_B = \frac{1}{2} \sum_{i=1}^N p_i^2 + V_B(x_1, \ldots, x_N), \] (2.16)

where the potential \( V_B \) is chosen to be

\[ V_B(x_1, \ldots, x_N) = \frac{\omega^2}{2} \sum_i x_i^2 + \frac{\varepsilon^2}{2} \sum_{i \neq j} \frac{1}{(x_i - x_j)^2}. \] (2.17)

We see that the potential consists of a harmonic piece as well as a singular term describing a repulsive interparticle force. The equivalence with matrix models follows from the fact (demonstrated above for the supersymmetric case) that when one quantizes the singlet sector of a \( U(N) \) invariant matrix model, there appears a jacobian that can be reinterpreted as an effective repulsion between the eigenvalues [27]. This effective interaction is identical to the singular term in (2.17). This will be seen more clearly in the supersymmetric case discussed below.
The supersymmetric generalization of the Calogero model was first investigated by Freedman and Mende in [15]. For completeness, we repeat their construction here. Using the approach of Witten [31] one introduces, in addition to the bosonic coordinates $x_i$, the fermionic coordinates $\psi_i$ and $\psi_i^\dagger$ satisfying the standard anti-commutation relations $\{\psi_i, \psi_j\} = 0$, $\{\psi_i^\dagger, \psi_j^\dagger\} = 0$ and $\{\psi_i, \psi_j^\dagger\} = \delta_{ij}$. One then defines supercharges in terms of a so-called superpotential $W(x_1, \ldots, x_N)$ as

$$Q \equiv \sum_i \psi_i^\dagger \left( p_i - i \frac{\partial W}{\partial x_i} \right),$$
$$Q^\dagger \equiv \sum_i \psi_i \left( p_i + i \frac{\partial W}{\partial x_i} \right)$$

(2.18)
satisfying $\{Q, Q\} = 0 = \{Q^\dagger, Q^\dagger\}$. The hamiltonian is constructed as

$$H_S \equiv \frac{1}{2} \{Q, Q^\dagger\}$$
$$= \frac{1}{2} \sum_i \left( p_i^2 + \left( \frac{\partial W}{\partial x_i} \right)^2 \right) + \frac{1}{2} \sum_{i,j} [\psi_i^\dagger, \psi_j] \frac{\partial^2 W}{\partial x_i \partial x_j}$$

(2.19)
and commutes with $Q$ and $Q^\dagger$.

The model studied in [15] corresponded to choosing

$$W(x_1, \ldots, x_N) = \frac{\omega}{2} \sum_i x_i^2 + \frac{\epsilon}{2} \sum_{i \neq j} \ln |x_i - x_j|,$$

(2.20)
in which case one finds, after some algebra [15],

$$H_S = \frac{1}{2} \sum_i p_i^2 + \frac{\omega^2}{2} \sum_i x_i^2 + \frac{\epsilon^2}{2} \sum_{i \neq j} \frac{1}{(x_i - x_j)^2}$$
$$+ \omega \sum_i \psi_i^\dagger \psi_i - \frac{\epsilon}{2} \sum_{i \neq j} [\psi_i^\dagger - \psi_j^\dagger, \psi_i] \frac{1}{(x_i - x_j)^2}$$

(2.21)
$$- \frac{\omega}{2} (1 - \varepsilon N)(N - 1).$$

The bosonic part of this supersymmetric hamiltonian coincides, apart from an additive constant, with the original Calogero model (2.16) and (2.17).
The superpotential (2.20) is just a special case of the effective potential (2.15) found to describe the singlet sector of the Marinari-Parisi model in the previous section. At this level one therefore sees a correspondence between the Marinari-Parisi and super-Calogero models.

### 2.3. Continuum description

We now follow references [14] and [17] to set up a continuum approximation of the discrete model (2.14), or equivalently (2.18) and (2.19). This approximation is assumed to become exact in the limit \( N \rightarrow \infty \), where the discrete distribution \( x_i \) will approximate a continuous density. With this in mind, we introduce the continuum index \( x \) and define the fields

\[
\phi(x) \equiv \partial_x \varphi \equiv \sum_i \delta(x - x_i), \quad \phi\sigma(x) \equiv -\sum_i \delta(x - x_i) p_i,
\]

\[
\psi(x) = -\sum_i \delta(x - x_i) \psi_i, \quad \psi^\dagger(x) = -\sum_i \delta(x - x_i) \psi_i^\dagger.
\]  

(2.22)

These fields satisfy commutation relations

\[
[\sigma(x), \varphi(y)] = -i \delta(x - y),
\]

\[
[\sigma(x), \psi^\dagger(y)] = i \frac{\psi^\dagger}{\phi}(x) \partial_x \delta(x - y),
\]

\[
[\sigma(x), \psi(y)] = i \frac{\psi}{\phi}(x) \partial_x \delta(x - y),
\]

\[
\{\psi(x), \psi^\dagger(y)\} = \phi(x) \delta(x - y).
\]

(2.23)

These equalities are proved using the identity

\[
\delta(x - x_i) f(x) = \delta(x - x_i) f(x_i),
\]

(2.24)

where \( f \) is an arbitrary function.
Let us now rewrite the supercharges (2.14) or (2.18) in terms of the continuum fields. At the classical level one has \( \int dx \delta(x - x_i) \delta(x - x_j) / \phi(x) = \delta_{ij} \), which can be established via (2.24). Also, by the chain rule \( \partial W / \partial x_i = \delta W / \delta \varphi(x_i) \). Therefore the supercharges can equivalently be expressed as

\[
Q = \int dx \psi^\dagger(x) (\sigma(x) - i W_i; x),
\]

\[
Q^\dagger = \int dx \psi(x) (\sigma(x) + i W_i; x),
\]

where we have used the notation \( W_i; x \equiv \delta W / \delta \varphi(x) \).

In a careful quantum mechanical treatment of bosonic matrix models, additional terms arise in the corresponding continuum hamiltonian [12, 35]. However, we wish to establish here the supersymmetrization of the cubic collective field theory of [13]. For this model, perturbative calculations of bosonic scattering amplitudes [20] reproduce exactly those obtained in the continuum approach. In the free potential case [26] an exact solution exhibits two single particle branches, one of which can always be described semiclassically by the inclusion of the extra terms. We assume that a similar mechanism would apply here and postulate (2.25) to be the full quantum mechanical supercharges. The extra terms are easily taken into account by adding to the supercharges a term proportional to \( \int dx \psi^\dagger(x) \frac{\phi_x}{\phi} \).

We now construct the continuum hamiltonian from the supercharges (2.25). We find, using \( H = \frac{1}{2}\{Q, Q^\dagger\} \), that

\[
H = \frac{1}{2} \int dx \phi \sigma^2 + \frac{1}{2} \int dx \phi (W_i; x)^2 \\
- \frac{1}{2} \int \frac{dx}{\phi} [\psi^\dagger, \psi] \partial_x W_i; x + \frac{1}{2} \int dx \int dy [\psi^\dagger(x), \psi(y)] W_i; xy.
\]

From the definition of the field \( \phi \), we see that the continuum hamiltonian must be accompanied by the constraint

\[
\int dx \phi(x) = N.
\]

We now have a language in which to compare the Marinari-Parisi/super-Calogero...
model with the supersymmetrization of the bosonic collective field theory as constructed in [14], which we do in the next section.

2.4. Supersymmetrized collective field model

In [14] a supersymmetric extension of the bosonic collective field theory was described. This was done by noting that the collective field theory can be seen as a metric theory, which can be supersymmetrized via a standard procedure.

To see how this works, observe that the kinetic term of the bosonic collective lagrangian

\[ L = \frac{1}{2} \int \frac{dx}{\phi} \dot{\phi}^2 - \frac{\pi^2}{6} \int dx \phi^3 - \int dx v\phi \]  

(2.27)

can be written in the form

\[ L_T = \frac{1}{2} \int dx \int dy \dot{\varphi}(x, t) g_{xy}(y, t), \]

where the continuous index metric is given by

\[ g_{xy}(\varphi) = \frac{1}{\phi(x)} \delta(x - y). \]  

(2.28)

A standard supersymmetrization of a theory of this type is given by

\[ L = \frac{1}{2} (\dot{q}^a g_{ab} \dot{q}^b - g^{ab} \partial_a W \partial_b W) + i \psi^\dagger a g_{ab} \psi^b + i \psi^\dagger a \Gamma_{bc,a} q^c \psi^b + \frac{1}{2} [\psi^\dagger a, \psi^b] \Gamma_{cd}^a \partial_d W - \frac{1}{2} [\psi^\dagger a, \psi^b] \partial_a \partial_b W, \]  

(2.29)

where \( q^a \) are the bosonic variables, \( g_{ab} \) is a metric, \( W \) is a superpotential and \( \Gamma_{cd}^a \) are the Christoffel symbols. This can be obtained by a classical point transformation \( x^i \equiv x^i(q^a) \), \( \psi^i = (\partial x^i/\partial q^a) \psi^a \equiv e^i_a \psi^a \) from the lagrangian

\[ L = \sum_{i=1}^N \left( \frac{1}{2} i_{\dot{\psi}_i}^2 - \frac{1}{2} (\partial_i W)^2 \right) + i \sum_{i=1}^N \dot{\psi}_i^i \psi_i - \sum_{ij} \dot{\psi}_i^i \psi_j^j \partial_i \partial_j W, \]

which is a multidimensional generalization of one-dimensional supersymmetric quantum mechanics [31].
Note the expressions of the form $\frac{1}{2} [\psi^\dagger, \psi]$ here. Classically, taking $\psi$ and $\psi^\dagger$ to be Grassman variables, we have the identity $\frac{1}{2} [\psi^\dagger, \psi] = \psi^\dagger \psi$. At the quantum level, however, this replacement would destroy supersymmetry unless an additional term were added to the bosonic potential in (2.29). Thus (2.29) is the proper lagrangian to use for quantization.

In our case the $q^a$ are replaced by $\varphi(x)$ and $g_{xy}$ is given by (2.28). One obtains the lagrangian

$$L = \frac{1}{2} \int \frac{dx}{\phi} \varphi'^2 - \frac{1}{2} \int dx \phi (W; x)^2$$

$$+ i \int \frac{dx}{\phi} \psi^\dagger \dot{\psi} + i \int \frac{dx}{\phi} \varphi \partial_x \left( \frac{\psi^\dagger}{\phi} \right) \psi$$

$$+ \frac{1}{2} \int \frac{dx}{\phi} [\psi^\dagger, \psi] \partial_x W; x - \frac{1}{2} \int dx \int dy [\psi^\dagger(x), \psi(y)] W; xy.$$

(2.30)

This is equivalent, via a partial integration, to

$$L = \frac{1}{2} \int \frac{dx}{\phi} \varphi'^2 - \frac{1}{2} \int dx \phi (W; x)^2$$

$$+ \frac{i}{2} \int \frac{dx}{\phi} (\psi^\dagger \dot{\psi} - \dot{\psi}^\dagger \psi) + \frac{i}{2} \int \frac{dx}{\phi} \varphi \left[ \partial_x \left( \frac{\psi^\dagger}{\phi} \right) \psi - \psi^\dagger \partial_x \left( \frac{\psi}{\phi} \right) \right]$$

$$+ \frac{1}{2} \int \frac{dx}{\phi} [\psi^\dagger, \psi] \partial_x W; x - \frac{1}{2} \int dx \int dy [\psi^\dagger(x), \psi(y)] W; xy.$$

(2.31)

Our motivation for this rewriting is that in the form (2.31) the momentum conjugate to the field $\varphi$ is manifestly hermitian, while in (2.30) it is nonhermitian. Even so, one would like the respective theories to be equivalent at the quantum level. Though it would take us too far afield to show it here, this indeed turns out to be the case: in both cases we can define a hermitian $\sigma$ which satisfies the commutation relations (2.23) and in terms of which the hamiltonian can be written in the form (2.26).
The conjugate momenta are given by

\[ p(x) = \frac{\partial L}{\partial \dot{\phi}} = \dot{\phi} + \frac{i}{2} \left[ \partial_x \left( \frac{\psi^\dagger}{\phi} \right) \frac{\psi}{\phi} - \frac{\psi^\dagger}{\phi} \partial_x \left( \frac{\psi}{\phi} \right) \right], \quad (2.32) \]

\[ \Pi = \frac{\partial L}{\partial \dot{\psi}} = \frac{i}{2} \frac{\psi^\dagger}{\phi}, \quad \Pi^\dagger = \frac{\partial L}{\partial \dot{\psi}^\dagger} = -\frac{i}{2} \frac{\psi}{\phi}. \quad (2.33) \]

Identifying the second class constraints

\[ \chi = \Pi - \frac{i}{2} \frac{\psi^\dagger}{\phi}, \quad \bar{\chi} = \Pi^\dagger + \frac{i}{2} \frac{\psi}{\phi}, \quad (2.34) \]

one uses Dirac brackets \[14, 29\] to obtain the following equal time brackets

\[ [\varphi(x), \varphi(y)] = 0, \quad [\varphi(x), p(y)] = i\delta(x-y), \quad \{\psi(x), \psi^\dagger(y)\} = \phi(x)\delta(x-y), \quad (2.35) \]

\[ [\varphi(x), \psi(y)] = [\varphi(x), \psi^\dagger(y)] = 0, \quad (2.36) \]

\[ [p(x), \psi(y)] = \frac{i}{2} \frac{\psi(y)}{\phi(y)} \partial_x \delta(x-y), \quad [p(x), \psi^\dagger(y)] = \frac{i}{2} \frac{\psi^\dagger(y)}{\phi(y)} \partial_x \delta(x-y). \quad (2.37) \]

In terms of

\[ \sigma(x) \equiv p(x) - \frac{i}{2} \left[ \partial_x \left( \frac{\psi^\dagger}{\phi} \right) \frac{\psi}{\phi} - \frac{\psi^\dagger}{\phi} \partial_x \left( \frac{\psi}{\phi} \right) \right] \quad (2.38) \]

the Hamiltonian then takes the form

\[ H = \frac{1}{2} \int dx \sigma \phi \sigma + \frac{1}{2} \int dx \phi (W_{xx})^2 \]

\[ - \frac{1}{2} \int \frac{dx}{\phi} [\psi^\dagger, \psi] \partial_x W_{xx} + \frac{1}{2} \int dx \int dy [\psi^\dagger(x), \psi(y)] W_{xy}, \quad (2.39) \]

while from equations (2.35), (2.37) and (2.38) it follows that

\[ [\sigma(x), \psi(y)] = i \frac{\psi}{\phi} (x) \partial_x \delta(x-y), \quad [\sigma(x), \psi^\dagger(y)] = i \frac{\psi^\dagger}{\phi} (x) \partial_x \delta(x-y). \quad (2.40) \]
The collective field $\phi$ satisfies the constraint
\[ \int dx \phi(x) = N. \]

This Hamiltonian and these commutation relations are identical with those derived in the continuum description of the Marinari-Parisi/super-Calogero model in (2.23) and (2.26), thus completing the chain of equivalences between the Calogero model, the supersymmetrized collective field theory and the supersymmetric matrix model.

3. PERTURBATION THEORY

In this chapter we write down the general perturbation theory of the supersymmetric model given an arbitrary potential. We show that in this approach an infinite sequence of higher order interactions is generated, in contrast to the bosonic collective field theory, where there are only cubic interactions.

3.1. Perturbative expansion

As follows from our discussion of the previous section, we are led to consider general effective superpotentials of the form
\[ W = \int dx \tilde{W}(x) \partial_x \varphi - \frac{1}{2} \int dx \int dy \ln|x-y| \partial_x \varphi \partial_y \varphi. \]  

(3.1)

For the special case $\tilde{W}(x) = \frac{\omega}{2} x^2$, this is just a rewriting in terms of the continuum fields (2.22) of the harmonic plus repulsive superpotential (2.20) of the super-Calogero model. In general $\tilde{W}$ will depend on $N$ as in (2.3).

What is unusual about the commutations (2.40) is the fact that the bosonic momentum does not commute with the fermionic fields. We observe that from
(2.35) and (2.37) we have

\[
\left[ p(x), \frac{\psi(y)}{\sqrt{\phi(y)}} \right] = 0, \quad \left[ p(x), \frac{\psi^{\dagger}(y)}{\sqrt{\phi(y)}} \right] = 0.
\]

We therefore rescale \( \psi(x) \rightarrow \sqrt{\phi(x)} \psi(x) \); \( \psi^{\dagger}(x) \rightarrow \sqrt{\phi(x)} \psi^{\dagger}(x) \) and, defining \( \tilde{v}(x) \equiv \tilde{W}'(x) \), we obtain for the lagrangian

\[
L = \frac{1}{2} \int dx \Phi^2 + \frac{1}{2} \int dx \Phi \left( W_{\dot{x}x} \right)^2
\]

\[
+ \frac{i}{2} \int dx \left( \psi^{\dagger} \dot{\psi} - \dot{\psi} \psi^{\dagger} \right) + \frac{i}{2} \int dy \Phi \left( \partial_x \psi^{\dagger} \psi - \psi^{\dagger} \partial_x \psi \right)
\]

\[
+ \frac{1}{2} \int dx \left[ \psi^{\dagger}, \psi \right] \partial_x W_{\dot{x}x} - \frac{1}{2} \int dx \int dy \left[ \psi^{\dagger}(x), \psi(y) \right] \sqrt{\phi(x)} W_{\dot{x}y} \sqrt{\phi(y)},
\]

for the hamiltonian

\[
H = \frac{1}{2} \int dx \Phi \left( \phi p - \frac{i}{2} \left[ (\partial_x \psi^{\dagger}) \psi - \psi^{\dagger} (\partial_x \psi) \right] \right)^2
\]

\[
+ \frac{1}{2} \int dx \Phi \left( \int dy \frac{\partial_y \varphi}{x-y} - \tilde{v}(x) \right)^2
\]

\[
- \frac{1}{2} \int dx \left[ \psi^{\dagger}, \psi \right] \frac{d}{dx} \left( \int dy \frac{\partial_y \varphi}{x-y} - \tilde{v}(x) \right)
\]

\[
+ \frac{1}{2} \int dx \left[ \psi^{\dagger}(x) \sqrt{\phi(x)}, \frac{d}{dx} \int dy \frac{\psi(y) \sqrt{\phi(y)}}{x-y} \right]
\]

and for the supercharges

\[
Q \equiv \int dx \psi^{\dagger}(x) \sqrt{\phi(x)} \left( p(x) - \frac{i}{2\varphi} \left( (\partial_x \psi^{\dagger}) \psi - \psi^{\dagger} (\partial_x \psi) \right) \right)
\]

\[
+ i \tilde{v}(x) - i \int dy \frac{\partial_y \varphi}{x-y},
\]

\[
Q^{\dagger} \equiv \int dx \psi(x) \sqrt{\phi(x)} \left( p(x) - \frac{i}{2\varphi} \left( (\partial_x \psi^{\dagger}) \psi - \psi^{\dagger} (\partial_x \psi) \right) \right)
\]

\[
- i \tilde{v}(x) + i \int dy \frac{\partial_y \varphi}{x-y},
\]

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with standard commutators

\[
\begin{align*}
[\varphi(x), \varphi(y)] &= 0, \quad [\varphi(x), p(y)] = i \delta(x - y), \quad \{\psi(x), \psi^\dagger(y)\} = \delta(x - y), \\
[\varphi(x), \psi(y)] &= \{\varphi(x), \psi^\dagger(y)\} = [p(x), \psi(y)] = [p(x), \psi^\dagger(y)] = 0.
\end{align*}
\]

(3.6)

The square root factors appearing in equations (3.3) to (3.5), when expanded around the large \(N\) background configuration, will generate an infinite set of vertices.

This Hamiltonian and these supercharges depend implicitly on \(N\) through \(v\) and the constraint \(\int \phi = N\). To make the \(N\)-dependence explicit, observe that for \(W\) of the form (2.3) the expressions \(\tilde{v}(\sqrt{N}x)/\sqrt{N}\) and \(\tilde{v}'(\sqrt{N}x)\) are independent of \(N\). We are therefore motivated to rescale \(x \rightarrow \sqrt{N}x\). Then to get rid of the \(N\)-dependence in the constraint, we must rescale \(\phi \rightarrow \sqrt{N}\phi\).

The complete rescaling is given by \(x \rightarrow \sqrt{N}x\), \(\phi \rightarrow \sqrt{N}\phi\), \(\varphi \rightarrow N\varphi\), \(p \rightarrow p/N^{3/2}\) and \(\psi^{(i)} \rightarrow \psi^{(i)}/N^{1/4}\). Defining \(v(x) \equiv \tilde{v}(\sqrt{N}x)/\sqrt{N}\), which is independent of \(N\), we get for the supercharges

\[
\begin{align*}
Q &\equiv \int dx \psi^\dagger(x) \sqrt{\phi(x)} \left( \frac{p(x)}{N} - \frac{i}{2\phi N} \left( (\partial_x \psi^\dagger) \psi - \psi^\dagger (\partial_x \psi) \right) \right) \\
&+ iN v(x) - iN \int dy \frac{\partial_y \varphi}{x - y}, \\
Q^\dagger &\equiv \int dx \psi(x) \sqrt{\phi(x)} \left( \frac{p(x)}{N} - \frac{i}{2\phi N} \left( (\partial_x \psi^\dagger) \psi - \psi^\dagger (\partial_x \psi) \right) \right) \\
&- iN v(x) + iN \int dy \frac{\partial_y \varphi}{x - y},
\end{align*}
\]

(3.7)
and for the hamiltonian

\[
H = \frac{N^2}{2} \int dx \phi(x) \left( \int dy \frac{\partial_y \varphi}{x-y} - v(x) \right)^2 \\
+ \frac{1}{2N^2} \int \frac{dx}{\phi} \left( \phi p - \frac{i}{2} \left[ (\partial_x \psi^\dagger) \psi - \psi^\dagger (\partial_x \psi) \right] \right)^2 \\
- \frac{1}{2} \int dx [\psi^\dagger, \psi] \frac{d}{dx} \left( \int dy \frac{\partial_y \varphi}{x-y} - v(x) \right) \\
+ \frac{1}{2} \int dx \left[ \psi^\dagger(x) \sqrt{\phi(x)}, \frac{d}{dx} \int dy \frac{\psi(y) \sqrt{\phi(y)}}{x-y} \right].
\] (3.8)

The rescaled collective field satisfies the constraint

\[ \int dx \phi(x) = 1. \] (3.9)

The leading term in the hamiltonian (3.8) is just a constant. If it is nonzero, then supersymmetry is broken to leading order in \( N \). Conversely, if the vacuum configuration of the density field satisfies

\[ V_{\text{eff}}(\phi_0) = \frac{N^2}{2} \int dx \phi_0(x) \left( \int dy \frac{\phi_0(y)}{x-y} - v(x) \right)^2 = 0. \] (3.10)

then supersymmetry is preserved to leading order. This will be the case if

\[ \int dy \frac{\phi_0(y)}{x-y} - v(x) = 0. \] (3.11)

Throughout this paper we will assume that supersymmetry is unbroken at the classical level and that equation (3.11) applies.

To set up the perturbation theory, we expand around the vacuum configuration \( \phi_0 \) as

\[ \phi = \partial_x \varphi = \phi_0 + \frac{1}{N} \partial_x \eta, \quad p \rightarrow Np. \] (3.12)

The factor \( 1/N \) makes all bosonic propagators of order unity in position space, and absorbs the explicit \( N \)-dependence into the vertices. This is purely a matter
of convenience, and the factor $1/N$ could be left out without changing the theory, a fact that can be seen by power counting of graphs in position space, or most simply by writing the respective hamiltonians in terms of normalized creation and annihilation operators and noting that the resulting expressions are identical.

Expanding about $\phi_0$, terms linear in $\partial \eta$ cancel and the hamiltonian (3.8) becomes

$$H = \frac{1}{2} \int dx \phi_0 p^2 + \frac{\pi^2}{2} \int dx \phi_0 \partial_x \eta^2 - \frac{1}{2} \int dx \int dy \frac{v(x) - v(y)}{x - y} \partial_x \eta \partial_y \eta$$

$$+ \frac{1}{2} \int dx \left[ \psi^\dagger(x) \sqrt{\phi_0(x)}, \frac{d}{dx} \int dy \frac{\psi(y) \sqrt{\phi_0(y)}}{x - y} \right]$$

$$+ \frac{1}{2N} \int dx (\partial_x \eta)^2 + \frac{1}{2} \int dx \partial_x \eta \left( \int dy \frac{\partial_y \eta}{x - y} \right)^2$$

$$- \frac{i}{2N} \int dx \left[ (\partial_x \psi^\dagger) \psi - \psi^\dagger (\partial_x \psi) \right] p$$

$$- \frac{1}{2N} \int dx [\psi^\dagger(x), \psi(x)] \frac{d}{dx} \int dy \frac{\partial_y \eta}{x - y}$$

$$+ \frac{1}{2} \int dx \left[ \psi^\dagger(x) \sqrt{\phi_0(x)} \left( \sqrt{\phi/\phi_0(x)} - 1 \right), \frac{d}{dx} \int dy \frac{\psi(y) \sqrt{\phi_0(y)}}{x - y} \right]$$

$$+ \frac{1}{2} \int dx \left[ \psi^\dagger(x) \sqrt{\phi_0(x)}, \frac{d}{dx} \int dy \frac{\psi(y) \sqrt{\phi_0(y)}}{x - y} \left( \sqrt{\phi/\phi_0(y)} - 1 \right) \right]$$

$$+ \frac{1}{8N^2} \int \frac{dx}{\phi_0 + \frac{1}{N} \partial_x \eta} \left[ \psi (\partial_x \psi^\dagger) (\partial_x \psi) \psi^\dagger + \psi^\dagger (\partial_x \psi) (\partial_x \psi^\dagger) \psi \right]$$

$$+ \frac{1}{2} \int dx \left[ \psi^\dagger(x) \sqrt{\phi_0(x)} \left( \sqrt{\phi/\phi_0(x)} - 1 \right), \frac{d}{dx} \int dy \frac{\psi(y) \sqrt{\phi_0(y)}}{x - y} \left( \sqrt{\phi/\phi_0(y)} - 1 \right) \right].$$

Some remarks are in order at this point: In the bosonic collective field theory, cubic terms such as appear in the bosonic part of the hamiltonian (3.8) are simplified using the identity

$$\int dx \phi(x) \left( \int dy \frac{\phi(y)}{x - y} \right)^2 = \frac{\pi^2}{3} \int dx \phi^3(x).$$
which can readily be demonstrated [26] in Fourier space using
\[
\int dy \frac{e^{iky}}{x-y} = -\pi i \epsilon(k) e^{ikx}.
\] (3.15)

In applying (3.14) one must, however, be careful. As it stands it cannot be valid if the integral \( \int \phi^3 \) on the right hand side diverges. This can indeed happen, both at the level of the vacuum density \( \phi_0 \), as happens in the potential free case (not discussed here) and at the level of the fluctuations \( \partial \eta \), as we shall see in the harmonic case. To be general, we therefore avoid using the identity (3.14) in this chapter, and treat issues of regularization later, as they arise.

On the other hand, in (3.13) we have rewritten the quadratic terms by applying the identity
\[
\int dx \phi_0(x) \left( \int dy \frac{\partial_y \eta}{x-y} \right)^2 + 2 \int dx \partial_x \eta \left( \int dy \frac{\phi_0(y)}{x-y} \right) \left( \int dz \frac{\partial_z \eta}{x-z} \right) = \pi^2 \int dx \phi_0(\partial_x \eta)^2,
\] (3.16)

which is easy to show in Fourier space. For all the cases that we will consider, the integrals in (3.16) are well behaved and no special regularizations are needed.

Using
\[
\sqrt{\phi/\phi_0} - 1 = \frac{1}{2N\phi_0} \partial_x \eta - \frac{1}{8N^2\phi_0^2} (\partial_x \eta)^2 + o \left( \frac{1}{N^2} \right),
\] (3.17)

the hamiltonian can be written up to cubic order as
\[ H = \frac{1}{2} \int dx \, \phi_0 p^2 + \frac{\pi^2}{2} \int dx \, \phi_0 (\partial_x \eta)^2 - \frac{1}{2} \int dx \int dy \, \frac{\psi(x) - \psi(y)}{x-y} \partial_x \eta \partial_y \eta \\
+ \frac{1}{2} \int dx \left[ \psi^\dagger(x) \sqrt{\phi_0(x)} \frac{d}{dx} \int dy \, \psi(y) \sqrt{\phi_0(y)} \right] \\
+ \frac{1}{2N} \int dx (\partial_x \eta) p^2 + \frac{\pi^2}{6N} \int dx (\partial_x \eta)^3 - \frac{i}{2N} \int dx \left[ (\partial_x \psi^\dagger) \psi - \psi^\dagger (\partial_x \psi) \right] p \\
- \frac{1}{2N} \int dx [\psi^\dagger(x), \psi(x)] \frac{d}{dx} \int dy \frac{\partial_y \eta}{x-y} \\
+ \frac{1}{4N} \int dx \left[ \psi^\dagger(x) \sqrt{\phi_0(x)} \partial_x \eta, \frac{d}{dx} \int dy \frac{\psi(y) \sqrt{\phi_0(y)}}{x-y} \right] \\
+ \frac{1}{4N} \int dx \left[ \psi^\dagger(x) \sqrt{\phi_0(x)} \frac{d}{dx} \int dy \frac{\psi(y) \sqrt{\phi_0(y)}}{x-y} \partial_y \eta \right] + o \left( \frac{1}{N^2} \right), \]

(3.18)

and the supercharges (3.7) can be written as

\[
Q = \int dx \sqrt{\phi_0} \psi^\dagger(x) \left\{ p(x) - i \int dy \frac{\partial_y \eta}{x-y} \right\} \\
+ \frac{1}{2N} \int \frac{dx}{\sqrt{\phi_0}} \psi^\dagger(x) \left\{ \partial_x \eta \left[ p(x) - i \int dy \frac{\partial_y \eta}{x-y} \right] \\
- i \left( (\partial_x \psi^\dagger) \psi - \psi^\dagger (\partial_x \psi) \right) \right\} + o \left( \frac{1}{N^2} \right), \]

(3.19)

\[ Q^\dagger = h.c. \]

We see that the system develops an infinite sequence of polynomial interactions in the bare string coupling constant \(1/N^2\). This is in contrast to the cubic bosonic collective field theory hamiltonian [13, 20] and does not depend on the presence of a potential \(v\). The supercharges also acquire expansions to all orders in perturbation theory, typical of supersymmetric theories expanded about background configurations. In general, the presence of a nontrivial background \(\phi_0\) may “dress” the coupling constant and possibly allow us to take a suitable double scaling limit.

One of the remarkable properties of the cubic bosonic hamiltonian is that [20] the integral representation of a given amplitude can in general be reinterpreted as a sum of standard tachyon exchange diagrams, plus contact terms, in a one to
one correspondence to first quantized Liouville computations [25]. In studies of
critical closed string field theory, an infinite sequence of polynomial interactions
seems to be required to obtain agreement with the first quantized integrations over
moduli space [34]. In the model discussed in this paper, the need to also include
an infinite set of vertices is unavoidable. It should be clear that the reason for the
supersymmetrized version of the bosonic cubic hamiltonian to contain an infinite
sequence of higher order vertices with derivative couplings is that the supercharges
themselves have an infinite expansion.

It is conceivable that other supersymmetric extensions of the bosonic collective
string field hamiltonian (possibly formulated directly in the continuum) may exist,
with properties different from those described here. Ultimately, once a genuine field
theory of $d = 1$ superstrings is formulated, the correct choice would be selected by
requiring agreement with the super-Liouville theory [6].

4. SPECTRUM

4.1. Semiclassical spectrum

In the super-Calogero description of the model [15] it is straightforward to show
that if a supersymmetric classical configuration can be found, the semiclassical
spectrum is supersymmetric. A similar argument is developed here in the contin-
umum description of the model.

Referring back to the previous chapter, we note that the condition (3.11) for
supersymmetry at the level of the classical vacuum can equivalently be restated as

$$W_{,x}|_{\phi_0} = 0,$$

(4.1)

where $W$ stands for the effective superpotential of (3.1). When this condition is
satisfied, the continuum hamiltonian (2.26) can be expanded around $\phi_0$ to give a
quadratic contribution

\[ H_0 = H_0^B + H_0^F, \]  

where

\[ H_0^B = \frac{1}{2} \int dx \phi_0(x) \left( p^2 + \int dy \int dz W_{xy} W_{xz} \eta(y) \eta(z) \right), \]
\[ H_0^F = \frac{1}{2} \int dx \int dy \left[ \psi^\dagger(x), \sqrt{\phi_0(x)} W_{xy} \sqrt{\phi_0(y)} \psi(y) \right], \]  

and where the expressions of the form \( W_{xy} \) are evaluated at \( \phi_0 \).

Defining the change of variables \( dq = dx/\phi_0 \) and rescaling \( p \rightarrow p/\phi_0, \psi \rightarrow \psi/\sqrt{\phi_0}, \psi^\dagger \rightarrow \psi^\dagger/\sqrt{\phi_0} \), the hamiltonian simplifies to give

\[ H_0^B = \frac{1}{2} \int dq \left( p^2 + \left( \int dq' W_{qq'} \eta(q') \right)^2 \right), \]
\[ H_0^F = \frac{1}{2} \int dq \left[ \psi^\dagger(q), \int dq' W_{qq'} \psi(q') \right], \]  

where we have used the identity \( \delta/\delta \varphi(q) = \phi_0(q) \delta/\delta \varphi(x) \), which follows by the chain rule, keeping in mind the fact that \( \delta(q - q') = \phi_0(q) \delta(x - x') \).

Now let \( \tilde{W} \) be the kernel defined by

\[ \tilde{W}(\phi)(q) \equiv \int dq' W_{qq'} \phi(q'). \]

Then the quadratic hamiltonian can be written as

\[ H_0^B = \frac{1}{2} \int dq \left( p^2 + \left( \tilde{W}(\eta)(q) \right)^2 \right), \]
\[ H_0^F = \frac{1}{2} \int dq \left[ \psi^\dagger(q), \tilde{W}(\psi)(q) \right], \]  

Let \( \{ \phi_n \} \) be a complete set of normalised eigenfunctions of \( \tilde{W} \), i.e.,

\[ \tilde{W} \phi_n = \lambda_n \phi_n. \]
Assuming $\tilde{W}$ to be positive definite, we can expand the fields as

\[
\eta = \sum_n \frac{1}{\sqrt{2\lambda_n}} (a_n \phi_n + a_n^\dagger \phi_n^*),
\]
\[
p = \sum_n -i \sqrt{\lambda_n/2} (a_n \phi_n - a_n^\dagger \phi_n^*),
\]
\[
\psi = \sum_n b_n \phi_n,
\]
\[
\psi^\dagger = \sum_n b_n^\dagger \phi_n^*,
\]

where $[a_m, a_n^\dagger] = \delta_{mn}$ and $\{b_m, b_n^\dagger\} = \delta_{mn}$, all other commutators vanishing.

It is then trivial to show that

\[
H_0^B = \sum_n \lambda_n (a_n^\dagger a_n + \frac{1}{2}),
\]
\[
H_0^F = \sum_n \lambda_n (b_n^\dagger b_n - \frac{1}{2}),
\]

thus demonstrating explicit supersymmetry of the quadratic spectrum.

Restricting attention to the bosonic piece, in addition to the terms associated in the bosonic string theory [13] to a massless scalar particle, one has the additional contribution

\[
-\frac{1}{2} \int dx \int dy \frac{v(x) - v(y)}{x - y} \partial_x \eta \partial_y \eta,
\]

which may in general affect the dynamics in unexpected ways. However, using the fact that $\int dx \partial_x \eta = 0$, it is easy to show that for potentials of the general form $v(x) = a + bx + cx^2$, this term falls away and the quadratic spectrum is that of a massless scalar. These potentials include the free case, the harmonic case, and those of references [10, 24]. We will also be interested in a potential of the general form $v(x) = bx + dx^3$. In this case a symmetric ansatz [14] can be consistently implemented and again the bosonic quadratic spectrum is that of a massless scalar.
Explicitly, in these cases the quadratic part of the bosonic sector of the hamiltonian (3.18) reads

\[ H_0^B = \frac{1}{2} \int dx \phi_0 p^2 + \frac{\pi^2}{2} \int dx \phi_0 (\partial_x \eta)^2, \quad (4.10) \]

or, in \( q \)-space,

\[ H_0^B = \frac{1}{2} \int dq \ (p^2 + \pi^2 (\partial_q \eta)^2). \quad (4.11) \]

In the light of the above discussion, we already know that assuming that there exists a supersymmetric classical configuration, the fermionic spectrum will also be that of a massless scalar particle. One should therefore be able to rewrite the fermionic piece

\[ H_0^F = \frac{1}{2} \int dx \left[ \psi^\dagger(x) \sqrt{\phi_0(x)} \frac{d}{dx} \int dy \frac{\psi(y) \sqrt{\phi_0(y)}}{x - y} \right] \]

\[ = \frac{1}{2} \int dq \left[ \psi^\dagger(q) \frac{d}{dq} \int dq' \frac{\phi_0(q') \psi(q')}{x(q) - x(q')} \right] \quad (4.12) \]

in a form in which this property is manifest.

Assuming \( q \) to be defined on \([0, L]\), we expand \( \eta \) and \( p \) as

\[ \eta(q) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{2\pi^2 n}} (a_n + a_n^\dagger) \sin \frac{n\pi q}{L}, \]

\[ p(q) = \sum_{n=1}^{\infty} -i \sqrt{\frac{\pi^2 n}{2L^2}} (a_n - a_n^\dagger) \sin \frac{n\pi q}{L}, \quad (4.13) \]

where \([a_m, a_n^\dagger] = \delta_{mn}\). In the above Dirichlet boundary conditions have been assumed.
The bosonic part of the quadratic Hamiltonian is then simply given by

\[ H_0^B = \sum_{n=0}^{\infty} \frac{n\pi^2}{L} (a_n^\dagger a_n + \frac{1}{2}). \]  

(4.14)

In (4.6) the Hamiltonian was expressed in terms of a kernel \( \tilde{W} \) as

\[ H_0^B = \frac{1}{2} \int dq \left( p^2 + \left( \tilde{W}(\eta)(q) \right)^2 \right), \]
\[ H_0^F = \frac{1}{2} \int dq \left[ \psi^\dagger(q), \tilde{W}(\psi)(q) \right]. \]  

(4.15)

The same kernel therefore appears in both the bosonic and the fermionic part of the Hamiltonian. Remembering that we want to rewrite the fermionic piece in a simpler form, our motivation is therefore to solve for \( \tilde{W} \) by determining its properties.

Expanding with respect to a complete set of normalised Dirichlet eigenfunctions of \( \tilde{W} \) on \([0, L]\), we had

\[ \eta = \sum_n \frac{1}{\sqrt{2\lambda_n}} (\tilde{a}_n \phi_n + \tilde{a}_n^\dagger \phi_n^*), \]
\[ p = \sum_n -i \sqrt{\frac{\lambda_n}{2}} (\tilde{a}_n \phi_n - \tilde{a}_n^\dagger \phi_n^*), \]  

(4.16)

and

\[ H_0^B = \sum_n \lambda_n (\tilde{a}_n^\dagger \tilde{a}_n + \frac{1}{2}). \]  

(4.17)

Comparing (4.17) and (4.14), one immediately sees that in the present case \( \tilde{W} \) has eigenvalues \( \lambda_n = \pi^2 n/L \). All that remains is therefore to explicitly solve for its eigenfunctions \( \phi_n \).

As both (4.13) and (4.16) are expansions of the same fields in terms of normalised oscillator coordinates, it follows that the \( a \)'s are unitarily related to the
\(\tilde{a}'s, \text{i.e.,}\)

\[a_n = U_{ni} \tilde{a}_i, \quad a_n^\dagger = U_{ni}^* \tilde{a}_i^\dagger,\]  

(4.18)

where \(U^\dagger U = 1\). Comparing (4.17) and (4.14) one finds

\[
\sum_n \lambda_n \tilde{a}_n^\dagger \tilde{a}_n = \sum_{nij} \lambda_{nij} U_{nj}^* \tilde{a}_j^\dagger U_{ni} \tilde{a}_i,
\]

(4.19)

from which it follows by comparing coefficients that \(U^\dagger \Lambda U = \Lambda\), where \(\Lambda\) is the diagonal matrix with \(\Lambda_{ii} = \lambda_i\). Using the fact that \(U\) is unitary, this is equivalent to \([U, \Lambda] = [U^\dagger, \Lambda] = 0\). Thus \(U\) leaves the eigenspaces of \(\Lambda\) invariant, and as the eigenvalues \(\lambda_n = \pi^2 n/L\) are nondegenerate, we conclude that \(U\) is diagonal with complex phase factors on the diagonal.

Thus each \(a_n\) is related to \(\tilde{a}_n\) by a phase factor. Comparing (4.13) and (4.16), it follows that the eigenfunctions of \(\tilde{W}\) are given by

\[
\phi_n(q) = \frac{1}{\sqrt{L}} \sin \frac{\pi nq}{L},
\]

(4.20)

modulo an inessential phase. In other words,

\[
\tilde{W} \sin \frac{\pi nq}{L} = \frac{\pi^2 n}{L} \sin \frac{\pi nq}{L}.
\]

(4.21)

Using the identity

\[
\int dq \ e^{ikq} \frac{e^{ikq'}}{q'-q} = -\pi i \epsilon(k) e^{ikq'},
\]

(4.22)

one verifies that

\[
\frac{d}{dq'} \int dq \ \frac{\sin kq}{q'-q} = \pi |k| \sin kq',
\]

(4.23)

so that the kernel can be written as

\[
\tilde{W}(\phi)(q') = \frac{d}{dq'} \int dq \ \frac{\phi(q)}{q'-q}
\]

(4.24)

The conclusion is therefore that the fermionic piece of the hamiltonian (4.6)
can be written as
\[
H_0^F = \frac{1}{2} \int dq \left[ \psi^\dagger(q), \tilde{W}(\psi)(q) \right],
\]
\[
= \frac{1}{2} \int dq \left[ \psi^\dagger(q), \frac{d}{dq} \int dq' \frac{\psi(q')}{q - q'} \right]. \tag{4.25}
\]

In other words, for the class of potentials \(v(x)\) described above, the quadratic Hamiltonian can be rewritten in a form manifestly that of a free Majorana fermion. We have also proved the identity
\[
\int dq \psi^\dagger(q) \frac{d}{dq} \int dq' \frac{\phi_0(q') \psi(q')}{x(q) - x(q')} = \int dq \psi^\dagger(q) \frac{d}{dq} \int dq' \frac{\psi(q')}{q - q'}, \tag{4.26}
\]
for the same class of potentials. This identity may look deceptively trivial to prove via contour integration arguments, due to the fact that \(\phi_0 = dx/dq\), so that both integrands have the same residues where both have poles. However, in general the function \(x\) may have additional poles in unfavourable positions for the naive argument to be valid.

## 5. THE HARMONIC CASE

In this chapter, we study the model characterised by the effective superpotential
\[
W = \int dx \tilde{W}(x) \partial_x \varphi - \frac{1}{2} \int dx \int dy \ln |x - y| \partial_y \varphi, \partial_x \varphi. \tag{5.1}
\]
in the case where \(\tilde{W}(x) = \frac{\omega}{2} x^2\). As remarked before (see (3.1)), this model is a continuum version of the original super-Calogero system (2.20) and (2.21). Equivalently, it can be seen as a continuum formulation of a matrix model of the form (2.1), where the matrix superpotential \(\tilde{W}(\Phi) = \frac{\omega}{2} \Phi^2\) is purely harmonic. This system has been studied in the super collective field formalism before in [14], and an expanded version of the analysis will be presented here.
5.1. Vacuumb density and spectrum

For supersymmetry to be unbroken to leading order in $N$ the vacuum configuration of the density field must satisfy

$$\int dy \frac{\phi_0(y)}{x-y} - \omega x = 0,$$

(5.2)

where $\phi_0$ must satisfy the constraint $\int \phi_0 = 1$.

We can use the identity (3.14) to express the effective bosonic potential

$$V_{\text{eff}}(\phi) = \frac{N^2}{2} \int dx \phi(x) \left( \int dy \frac{\phi(y)}{x-y} - \omega x \right)^2$$

(5.3)

of the hamiltonian (3.8) in the form

$$V_{\text{eff}}(\phi) = N^2 \left( \frac{\pi^2}{6} \int \phi^3 - \frac{\omega}{2} \left( \int \phi \right)^2 + \frac{\omega^2}{2} \int x^2 \phi \right).$$

(5.4)

The background $\phi_0$ satisfies the stationarity condition

$$\frac{\pi^2 \phi_0^2}{2} - \omega \int \phi_0 + \frac{\omega^2}{2} x^2 = 0.$$  

(5.5)

Thus one gets

$$\phi_0(x) = \frac{1}{\pi} \sqrt{\mu_F - \omega^2 x^2},$$

(5.6)

where $\mu_F \equiv 2\omega \int \phi_0$. Requiring that $\int \phi_0 = 1$, one finds $\mu_F = 2\omega$, which is consistent with the standard analytic solutions of (5.2).

We recall that the quadratic piece of the hamiltonian (3.18) is

$$H_0 = \frac{1}{2} \int dx \phi_0 p^2 + \frac{\pi^2}{2} \int dx \phi_0 (\partial_x \eta)^2$$

$$+ \frac{1}{2} \int dx \left[ \psi^\dagger(x) \sqrt{\phi_0(x)}, \frac{d}{dx} \int dy \frac{\psi(y) \sqrt{\phi_0(y)}}{x-y} \right]$$

(5.7)

or, after changing variables $dq = dx/\phi_0$ and rescaling $p \rightarrow p/\phi_0$, $\psi \rightarrow \psi/\sqrt{\phi_0}$,
\[ \psi^\dagger \rightarrow \psi^\dagger / \sqrt{\phi_0}, \]

\[
H_0 = \frac{1}{2} \int dq p^2 + \frac{\pi^2}{2} \int dq (\partial_q \eta)^2 \\
+ \frac{1}{2} \int dq \left[ \psi^\dagger (q), \frac{d}{dq} \int dq' \frac{\phi_0(q') \psi(q')}{x(q) - x(q')} \right] \\
= \frac{1}{2} \int dq p^2 + \frac{\pi^2}{2} \int dq (\partial_q \eta)^2 \\
+ \frac{1}{2} \int dq \left[ \psi^\dagger (q), \phi_0(q) \int dq' \frac{\partial_q \psi}{x(q) - x(q')} \right].
\] (5.8)

Explicitly, the above change of variables reads

\[
q = \pi \int \frac{dx}{\sqrt{\mu_F - \omega^2 x^2}} \\
= \frac{\pi}{\omega} \arccos \left( -\frac{x}{a} \right),
\] (5.9)

where \( q \) has been chosen to be zero at the turning point \( x = -a, \ a = \sqrt{\mu_F}/\omega = \sqrt{2/\omega} \). Inverting (5.9), one finds

\[
x = -\sqrt{\frac{2L}{\pi^2}} \cos \frac{\pi q}{L}; \quad q \in [0, L],
\]

where \( L = \pi^2/\omega = \pi^2 a^2/2 \) is the half period.

Expanding \( \eta, p \) and \( \psi^{(t)} \) on \( [0, L] \) as

\[
\eta(q) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{\pi^2 n}} (a_n + a_n^\dagger) \sin \frac{n\pi q}{L},
\]

\[
p(q) = \sum_{n=1}^{\infty} -i \sqrt{\frac{\pi^2 n}{L^2}} (a_n - a_n^\dagger) \sin \frac{n\pi q}{L},
\] (5.10)

\[
\psi^{(t)}(q) = \sqrt{\frac{2}{L}} \sum_{n=1}^{\infty} b_n^{(t)} \sin \frac{n\pi q}{L},
\]

where \( [a_m, a_n^\dagger] = \delta_{mn} \) and \( \{b_m, b_n^\dagger\} = \delta_{mn} \), the bosonic part of the quadratic
Hamiltonian becomes

\[ H_0^B = \sum_{n=1}^{\infty} n\omega (a_n^\dagger a_n + \frac{1}{2}). \tag{5.11} \]

For the fermionic part, one needs to evaluate

\[ \int_{-L}^{L} dq' \frac{\cos \frac{\pi nq'}{L}}{\cos \frac{\pi q'}{L} - \cos \frac{\pi q}{L}} = \frac{1}{2} \int_{-L}^{L} dq' \exp \frac{i\pi nq'}{L} \frac{\cos \frac{\pi q'}{L}}{\cos \frac{\pi q'}{L} - \cos \frac{\pi q}{L}}. \tag{5.12} \]

This can be done by closing the contour in the upper half plane, exploiting the fact that the integrand is periodic with period \(2L\). The only relevant poles are on the real line at \(+q\) and \(-q\). The principal value prescription corresponds to taking half of the residues of any poles that lie on the contour of integration. Applying this, one obtains

\[ \int_{0}^{L} dq' \frac{\cos \frac{\pi nq'}{L}}{\cos \frac{\pi q'}{L} - \cos \frac{\pi q}{L}} = L \epsilon(n) \frac{\sin \frac{\pi nq}{L}}{\sin \frac{\pi q}{L}}. \tag{5.13} \]

Using this result, \(H_0^F\) is now easily rewritten in the form

\[ H_0^F = \sum_{n=1}^{\infty} n\omega (b_n^\dagger b_n - \frac{1}{2}), \tag{5.14} \]

thus explicitly demonstrating supersymmetry of the semiclassical spectrum.

We notice that equation (5.13) is all that is required for both the spectrum and the ground state configuration. Indeed, it is straightforward to verify that equation (5.2) is self-consistently satisfied provided one defines \(\epsilon(n = 0) = 0\). Equation (5.13) is also related to an identity involving Chebychev polynomials of the second type [14].
5.2. Three point functions

We now compute the supercharges and three point functions of the harmonic theory in the oscillator basis.

Using (5.2) and changing variables to \(q\)-space as in the previous section, the supercharges (3.19) become

\[
Q = \int_0^L dq \, \psi^\dagger(q) \left\{ p(q) - i \phi_0(q) \int_0^L dq' \frac{\partial q' \eta}{x(q) - x(q')} \right\} + \frac{1}{2N} \int_0^L dq \frac{\phi_0^2(q)}{\phi_0^2} \partial_q \psi^\dagger(q) \left\{ p(q) - i \phi_0(q) \int_0^L dq' \frac{\partial q' \eta}{x(q) - x(q')} \right\}
\]

\[
-1 \frac{i}{2N} \int_0^L dq \frac{\phi_0^2(q)}{\phi_0^2} \partial_q \psi^\dagger(q) \psi + o \left( \frac{1}{N^2} \right),
\]

\(Q^\dagger = \text{h.c.}\).

For the quadratic term in \(Q\) one uses equation (5.13) to readily obtain

\[
Q_0 = -i \sqrt{2\pi} \sum_{m > 0} \sqrt{k_m} b_m^\dagger a_m.
\]  \(\text{ (5.16)}\)

For the cubic terms in \(Q\) one needs to compute the integrals

\[
\int_{-L}^L dq \frac{e^{ik_0 q}}{\phi_0^2(q)}.
\]

For \(n > 0\) this can be done by extending the contour in the upper half plane vertically upwards at \(-L\) and \(L\). Exploiting the fact that the integrand is periodic with period \(2L\), it follows that the vertical contributions cancel. For \(n < 0\) one can similarly close the contour in the lower half plane. The function \(1/\phi_0^2 = L/2 \sin^2(\pi q/L)\) has singularities on the contour at \(-L\), at 0 and at \(L\). Applying
a principal part regularization, which corresponds to taking half of the residues of
any poles that lie fully on the contour and 1/4 of the residues of poles that are
located at the corners $-L$ and $L$, one finds that for $n$ odd the integral vanishes
while for $n$ even it is given by

$$\int_{-L}^{L} dq \frac{e^{i k_n q}}{\phi_0^2(q)} = -\frac{L^3}{\pi} |k_n|.$$ (5.17)

Using this result one finds, after some algebra, that the cubic piece of $Q$ is
given by

$$Q_3 = \frac{i L \sqrt{L}}{8 \pi N} \sum_{m,n,p \neq 0} \epsilon(n) \sqrt{|k_n k_p| |k_m + k_n + k_p|} (b_{-m}^\dagger a_n a_p - b_{-m}^\dagger a_{-n} a_p)$$
$$+ \frac{i L \sqrt{L}}{8 \pi N} \sum_{m,n,p \neq 0} \epsilon(n) \sqrt{|k_n k_m + k_n - k_p|} b_{m}^\dagger b_{n}^\dagger b_p,$$ (5.18)

where the primes on the sums indicate that one only sums over indices such that
the arguments of the absolute value signs are even.

The cubic piece of the Hamiltonian can now be generated from (5.16) and (5.18)
using $H = \frac{1}{2} \{Q, Q^\dagger\}$. One finds, after some algebra, that

$$H_3 = H_3^B + H_3^F,$$

where

$$H_3^B = \frac{L}{4N} \sqrt{\frac{L}{2\pi}} \sum_{n,p,l \geq 0} \sqrt{k_l k_n k_p} (|k_l + k_n + k_p|$$
$$- |k_l - k_n - k_p|) a_n^\dagger a_p a_l + \text{h.c.,}$$

$$H_3^F = \frac{L}{8N} \sqrt{\frac{L}{2\pi}} \sum_{n,p,l \geq 0} \sqrt{k_l (k_p - k_n)} (|k_l + k_n + k_p| - |k_l + k_n - k_p|$$
$$+ |k_l - k_n + k_p| - |k_l - k_n - k_p|) b_n^\dagger b_p a_l^\dagger + \text{h.c.}$$ (5.19)
6. STATIONARY POINT AND D=1 STRINGS

We discuss here whether it is possible to identify in this model a continuum limit which, in the bosonic sector of the theory, reduces to the standard continuum limit of \( d = 1 \) bosonic strings [2].

Consider the bosonic part of the potential in (3.8):

\[
V_{\text{eff}}(\phi) = \frac{N^2}{2} \int dx \phi(x) \left( \int dy \frac{\phi(y)}{x-y} - v(x) \right)^2,
\]

(6.1)

where \( v(x) \) is an arbitrary polynomial. Since \( \phi \) is positive, a necessary and sufficient condition for supersymmetry to be preserved to leading order in \( N \) is that there exist a classical configuration \( \phi_0 \) such that

\[
\int dx \frac{\phi_0(y)}{x-y} - v(x) = 0,
\]

(6.2)

with \( \int \phi_0 = 1 \). There is a well-known way of solving this equation using analytic methods [1]. These solutions correspond to \( d = 0 \) matrix models [10, 36].

What characterizes the standard formulation of the continuum limit of bosonic \( d = 1 \) strings is that the theory can be scaled in such a way that (a) the potential depends on a chemical potential \( \mu \) (fermi energy); (b) this chemical potential is a free parameter since all dependence on the matrix coupling constant \( g \) (which we will refer to as the cosmological constant) is scaled out with the normalization condition becoming \( g = \int \phi(x) dx \).

As is well known, in the collective field theory approach a lagrange multiplier (chemical potential) is introduced to enforce the constraint \( \int \phi(x) dx = 1 \). Therefore let us consider

\[
\tilde{V}_{\text{eff}}(\phi)/N^2 = V_{\text{eff}}(\phi)/N^2 - \mu \left( \int \phi - 1 \right).
\]

(6.3)

Notice that \( \tilde{V}_{\text{eff}} \) is not positive definite any longer. If we require \( \tilde{V}_{\text{eff}} \) to be stationary
with respect to $\phi$, i.e.,

$$\delta_{\phi} \tilde{V}_{\text{eff}} = \delta_{\phi} V_{\text{eff}} - \mu = 0,$$

(6.4)

it immediately follows that if supersymmetry is to be preserved (if equation (6.2) is to be satisfied), then $\mu = 0$. This argument would seem to immediately rule out the possibility of introducing an arbitrary parameter $\mu$.

However, the argument is not that simple. The correct equation of stationarity is

$$\delta_{\phi} \tilde{V}_{\text{eff}} = \partial_x \delta_{\phi} \tilde{V}_{\text{eff}} = 0.$$  

(6.5)

Indeed, this is also the case in the standard collective field theory description of $d = 0$ strings, and it is known that the above equation is the statement of stationarity with respect to the original ("master") variables of the theory [21]. This equation is compatible with the presence of a lagrange multiplier. But, if $\partial_x \delta_{\phi} \tilde{V}_{\text{eff}} = 0$, this implies that $\delta_{\phi} \tilde{V}_{\text{eff}} = \text{constant} \equiv c$. So we recover equation (6.4) with $\mu \rightarrow \mu - c$. However, from our discussion following equation (6.4), for supersymmetry to be preserved the equation of stationarity remains

$$\delta_{\phi} V_{\text{eff}} = 0$$

(6.6)

without allowing for an extra parameter to be added.

We now use the identity (3.14) to rewrite this last equation as

$$\frac{\pi^2}{2} \phi_0^2(x) + \frac{1}{2} v^2(x) - \int dy \frac{v(x) - v(y)}{x - y} \phi_0(y) = 0.$$  

(6.7)

When the classical vacuum configuration is supersymmetric then the last term in (6.7) may generate an effective chemical potential, an example of which we saw in the harmonic case (see equation (5.5) and the subsequent remarks). This should not be too surprising, since the universal genus zero contribution to the ground state energy of the $d = 1$ bosonic string theory [2] is different from zero and therefore one should presumably not expect $g \rightarrow g_c$ with $g_c$ independent of $\Delta \mu$.  

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Let us now consider the scaling behaviour of equation (6.7). We recall that the bosonic analogue of the stationarity condition (6.7) is given by

\[ \frac{\pi^2}{2} \phi_0^2(x) + w(x) - \mu = 0, \]  

with the constraint \( \int \phi_0 = 1 \). The bosonic potential \( w \) is in general chosen to depend on a coupling constant \( g \) in such a way that \( w(x/\sqrt{g}) = \tilde{w}(x)/g \), where the rescaled potential \( \tilde{w} \) has no explicit dependence on \( g \). Rescaling \( x \to x/\sqrt{g} \), \( \phi \to \phi/\sqrt{g} \) and \( \mu \to \mu/g \), all explicit dependence on \( g \) scales out of the stationarity condition, which becomes

\[ \frac{\pi^2}{2} \phi_0^2(x) + \tilde{w}(x) - \mu = 0, \]  

while the constraint becomes \( \int \phi_0 = g \). We see that by changing the coupling constant \( g \), which now appears in the normalization condition, we can freely adjust the fermi level \( \mu \) with respect to the rescaled potential \( \tilde{w} \). In particular, as \( \mu \) approaches a \(-x^2\) maximum, the time of flight \( L \propto \int dx/\phi_0 \) becomes infinite.

Let us now try to duplicate this in the fermionic case. Taking the potential in (6.7) to have the general scaling \( v(g^a x) = g^b \tilde{v}(x) \), where \( \tilde{v} \) is independent of \( g \) (note that this is always the case for a monomial), and rescaling \( x \to g^a x \), \( \phi \to g^c \phi \) and \( \mu \to g^{2c} \mu \), the stationarity condition (6.7) becomes

\[ g^{2c} \frac{\pi^2}{2} \phi_0^2(x) + g^{2b} \frac{1}{2} \tilde{v}^2(x) - g^{2c} \mu - g^{b+c} \int dy \frac{\tilde{v}(x) - \tilde{v}(y)}{x-y} \phi_0(y) = 0, \]  

while the constraint becomes \( \int \phi_0 = g^{-a-c} \). For generality we have reintroduced a finite \( \mu \), allowing for supersymmetry breaking vacuum configurations. Expanding \( \tilde{v} \) about \( x \) and using this constraint, the stationarity condition becomes

\[ 0 = g^{2c} \frac{\pi^2}{2} \phi_0^2(x) + g^{2b} \frac{1}{2} \tilde{v}^2(x) - g^{2c} \mu \\
- g^{b+c-a-c} \tilde{v}'(x) + \frac{1}{2} g^{b+c-a-c} \tilde{v}''(x) x - \frac{1}{2} g^{b+c} \tilde{v}''(x) \int dy \phi_0(y) \]  

(6.11)

We see that if we choose \( a = -b = -c \), an overall factor \( g^{b+c} \) divides out of the
equation and we find

\[ 0 = \frac{\pi^2}{2} \phi_0^2(x) + \frac{1}{2} \tilde{v}^2(x) - \mu 
- \tilde{v}'(x) + \frac{1}{2} \tilde{v}''(x) x - \frac{1}{2} \tilde{v}''(x) \int dy \phi_0(y) 
+ \ldots, \tag{6.12} \]

while the constraint is given by \( \int \phi_0 = g^{-a-c} = 1 \). We have thus succeeded in getting rid of all explicit and implicit dependence of \( \phi_0 \) on \( g \). Rewriting this we therefore get

\[ \frac{\pi^2}{2} \phi_0^2(x) + \frac{1}{2} \tilde{v}^2(x) - \mu - \int dy \frac{\tilde{v}(x) - \tilde{v}(y)}{x - y} \phi_0(y) = 0, \tag{6.13} \]

with the constraint \( \int \phi_0 = 1 \). A point we want to emphasize is that in contrast to the bosonic case the parameter \( g \) does not appear in the constraint, so that the effective fermi level is not a free parameter relative to the rescaled effective potential in (6.13) and cannot be adjusted to approach a maximum. (The effective fermi level is given by \( \mu \) plus any \( x \)-independent terms arising from the cross term in (6.13)). The time of flight, given by \( L \propto \int dx/\phi_0 \), remains finite and no true extra dimension is generated.

The above argument assumes that \( v(x) \) is homogeneous of arbitrary order in only one parameter \( g \), and does not make a statement on the possibility or not of reproducing an analogue of the bosonic scaling limit when this is not the case. This seems very hard to do in specific examples that have been considered (not reproduced here), but more work needs to be done to gain a better understanding of this issue.
7. TURNING POINT DIVERGENCES

We now discuss the regularization of turning point contributions as it applies to the calculation of the three point functions of chapter 5.

In chapter 3 it was mentioned that there may in general be subtleties in the application of the identity (3.14)

\[
\int dx \partial_x \eta \left( \int dy \frac{\partial_y \eta}{x - y} \right)^2 = \frac{\pi^2}{3} \int dx (\partial_x \eta)^3 \tag{7.1}
\]
given in \(q\)-space by

\[
\int dq \partial_q \eta \left( \int_0^L dq' \frac{\partial_q \eta}{x(q) - x(q')} \right)^2 = \frac{\pi^2}{3} \int \frac{dq}{\phi_0} (\partial_q \eta)^3, \tag{7.2}
\]
used to simplify the cubic piece of the hamiltonian.

In the calculation of the oscillator expansions in chapter 5, we effectively started from the left hand side of (7.2). Our regularization scheme consisted of taking principal parts at the poles. On the other hand, the authors of [20] started from the right hand side of (7.2), their regularization also coming down to taking principal parts. Comparing the bosonic part of our Hamiltonian (5.19) to the one that was obtained in [20] for the bosonic collective field theory, we see that our expression differs from theirs in that we have no terms of the form \(a_m a_n a_p\).

To see this difference more clearly, note that starting from the left hand side of (7.2), assuming that \(\eta\) obeys Dirichlet boundary conditions so that we can expand \(\partial_q \eta = \sum_{n>0} \phi_n \cos k_n q\), and using the principal part prescription, one finds

\[
\int dq \partial_q \eta \left( \int_0^L dq' \frac{\partial_q \eta}{x(q) - x(q')} \right)^2 = -\frac{\pi L^3}{24} \sum_{mnp>0} \phi_m \phi_n \phi_p \left( -3|k_m + k_n + k_p| + |k_m - k_p - k_n| \right. \\
+ \left. |k_m + k_p - k_n| + |k_m - k_p + k_n| \right), \tag{7.3}
\]
where the result (5.17) has been used. Starting from the right hand side of (7.2)
as in the bosonic collective field theory, one finds
\[
\frac{\pi^2}{3} \int \frac{dq}{\phi^2_0} (\partial_q \eta)^3 
= -\frac{\pi L^3}{24} \sum' \phi_m \phi_n \phi_p \left( |k_m + k_n + k_p| + |k_m - k_p - k_n| + |k_m + k_p - k_n| + |k_m - k_p + k_n| \right),
\] (7.4)
which is manifestly different from the left hand side.

It is therefore evident that the regularization scheme of chapter 5 differs from the one used in the bosonic collective field theory. The reason for the difference between (7.3) and (7.4) is as follows: In [20] the turning point divergence in (7.4) was regulated essentially by introducing a cutoff \( \epsilon \) so that the integration range becomes \( \int_{\epsilon}^{L-\epsilon} \). Keeping epsilon small but finite throughout and only at the end discarding all \( \epsilon \)-dependent quantities (which are argued to be nonuniversal), one obtains the principal part prescription. Applying the same cutoff to (7.3) corresponds to identifying the \( \epsilon \)-independent terms in
\[
\int_{\epsilon}^{L-\epsilon} dq \ldots \left( \int_{\epsilon}^{L-\epsilon} dq' \ldots \right)^2.
\] (7.5)
As it stands, in (7.3) we calculated not the quantities (7.5) but instead
\[
\int_{\epsilon}^{L-\epsilon} dq \ldots \left( \int_0^L dq' \ldots \right)^2.
\] (7.6)

If we want correspondence with the bosonic cubic collective field theory, we must therefore subtract corrections of the typical form
\[
\int_{\epsilon}^{L-\epsilon} dq \ldots \left( \int_0^L dq' \ldots \right) \left( \int_0^\epsilon dq'' \ldots \right),
\] (7.7)
from (7.3).
Explicitly, using (5.13) and remembering that \( x \propto \cos \pi q/L \) and \( \phi_0 \propto \sin \pi q/L \), we have

\[
\int_{\epsilon}^{L-\epsilon} dq \cos k_m q \frac{\cos k_n q'}{x(q) - x(q')} \int_{0}^{\epsilon} dq'' \frac{\cos k_p q''}{x(q) - x(q'')}
\]

\[
\propto \int_{\epsilon}^{L-\epsilon} dq \frac{\cos k_m q \sin k_n q}{\phi_0(q)} \int_{0}^{\epsilon} dq'' \frac{\cos k_p q''}{x(q) - x(q'')}
\]

\[
\sim \int_{\epsilon}^{L-\epsilon} dq \frac{\cos k_m q \sin k_n q}{\sin \frac{\pi q}{L}} \epsilon \left( \frac{1}{\cos \frac{\pi q}{L} - 1} \right).
\]

To see how we obtain a finite contribution from this, note that one gets an \( o(\epsilon) \) contribution from the integral \( \int_{0}^{\epsilon} dq'' \) and an \( o(1/\epsilon) \) contribution (see below) from the lower limit of the integral \( \int_{\epsilon}^{L-\epsilon} dq \). The \( \epsilon \)-dependence will therefore cancel. More precisely, the small \( q \) behaviour of the integrand in (7.8) is given by

\[
\cos k_m q \frac{\sin \frac{n\pi q}{L}}{\sin \frac{\pi q}{L}} \frac{1}{\cos \frac{\pi q}{L} - 1} \sim \frac{n}{\frac{\pi^2 q^2}{L^2}} \epsilon \frac{k_n q^2}{q^2} \propto \epsilon k_n \frac{1}{\epsilon} \sim k_n.
\]

Inserting this into the integral in (7.8), one gets contributions of the form

\[
\epsilon \int_{\epsilon}^{L-\epsilon} dq \frac{k_n}{q^2} \sim \epsilon k_n \frac{1}{\epsilon} \sim k_n.
\]

We therefore see that these subtractions give rise to finite (\( i.e., \epsilon \)-independent) corrections to the expansion (7.3) proportional to

\[
\sum_{mnp>0} k_n \phi_m \phi_n \phi_p = \frac{1}{3} \sum_{mnp>0} (k_m + k_n + k_p) \phi_m \phi_n \phi_p.
\]

These corrections are exactly of the right form and the right sign to restore equality between (7.3) and (7.4). There are two such terms, corresponding to the lower limit of the integrations over each of \( q' \) and \( q'' \). Due to the crudeness of the approximation
in (7.8), the above argument is not expected to give the correct overall coefficient of the correction term. However, this coefficient can be inferred from the fact that for a finite cutoff $\epsilon$, all integrals are well behaved and the identity (7.2) will be valid. We therefore expect the correction term to be exactly equal to the difference between (7.3) and (7.4), which is given by

$$-\frac{\pi L^3}{24} \sum_{mnp>0}^\prime 4 (k_m + k_n + k_p) \phi_m \phi_n \phi_p.$$  

Once the coefficients have been determined, we can work backwards and write down the regulated expressions

$$\int dq \frac{\cos k_m q \sin k_n q}{\phi_0^2} \left( \phi_0 \int dq'' \frac{\cos k_p q''}{x(q) - x(q'')} \right) = -\frac{L^3}{8} (|k_m + k_n - k_p| + |k_m - k_n + k_p|$$

$$- |k_m + k_n + k_p| - |k_m - k_n - k_p| + 2k_n)$$

(7.10)

and

$$\int dq \frac{\cos k_m q}{\phi_0^2} \left( \phi_0 \int dq' \frac{\cos k_n q'}{x(q) - x(q')} \right) \left( \phi_0 \int dq'' \frac{\cos k_p q''}{x(q) - x(q'')} \right) = -\frac{\pi L^3}{8} (|k_m + k_n - k_p| + |k_m - k_n + k_p|$$

$$- |k_m + k_n + k_p| - |k_m - k_n - k_p| + 2k_n + 2k_p),$$

(7.11)

where the final terms are cutoff independent corrections to the naive expressions obtained from the principal part prescription.

Using the second of these two equalities in the calculation of the expansion of (7.3), we indeed find equivalence between (7.3) and (7.4).

Equation (7.10) can be used in the calculation of the cubic part of the supercharge $Q$ of the previous section. However this regularization is inconsistent with supersymmetry in the following sense: because the quadratic part of the supercharge receives no turning point correction, if one evaluates $H = \frac{1}{2} \{Q, Q^\dagger\}$, one
finds coefficients which are inconsistent with (7.11). In other words, the regularization does not commute with taking the bracket.

This is not unrelated to what is happening in the continuum approach to $d = 1$ superstrings [6]. If one uses the identity (7.1) in $x$-space first and then changes variables to $q$-space, in the process rendering the interactions local, one is not able, at least in our approach, to generate this term from the supercharges. This does not rule out the possibility that one may find a local supersymmetric extension of the bosonic collective string theory in $x$-space (this can certainly be done at the quadratic level [23]).

It should be emphasized if the integrals are regulated as described in chapter 5, the theory is fully supersymmetric.

8. CONCLUSIONS

In this paper an investigation was made of the Marinari-Parisi model in an arbitrary potential with particular emphasis on its continuum formulation and possible application in the context of supersymmetrizing the bosonic collective $d = 1$ string field theory.

We started by indicating the equivalence of the Marinari-Parisi model, the super-Calogero system and the supersymmetrization of the collective field theory defined in [14]. Developing the perturbation theory, we saw that, as a result of nontrivial commutation relations amongst continuum fields, the hamiltonian acquired an infinite sequence of vertices, in contrast to the cubic bosonic collective field theory where only a cubic vertex was required.

Assuming that supersymmetry is preserved at the level of the leading semiclassical configuration, we demonstrated supersymmetry of the semiclassical spectrum and, for a specific class of potentials, we showed that the spectrum consists of a massless boson and Majorana fermion.
An investigation of the scaling behaviour of the theory showed that for potentials which are homogeneous of arbitrary degree in the cosmological constant one can identify a rescaling of variables in terms of which the cosmological constant is scaled out while the normalization condition remains the same. This implies that in general the time of flight is finite and that the mechanism of generation of an infinite Liouville-like dimension [13, 22] is not present in this case. The space-time interpretation of the Marinari-Parisi model therefore remains problematic and more work is needed on this issue.

An examination of the bosonic sector cubic oscillator vertices in the harmonic case showed a discrepancy with those of the collective bosonic string field theory [20]. This discrepancy resulted from a difference between the turning point regularization prescriptions of chapter 5 and reference [20]. Adjusting the regularization to agree with that of [20], we found that a bosonic sector compatible with the bosonic collective field theory cannot be generated from a corresponding regularization of the supercharges. In other words, the regularization does not commute with the bracket $H = \{Q, Q^\dagger\}$.

We have indicated that, provided a local supersymmetrization of the bosonic collective field theory can be found, it is likely that this problem will be overcome. Probably the best context in which this can be attempted is the potential free case [23]. In the bosonic case, this corresponds to the background independent formulation of the theory, and a W-algebra structure has been identified [18, 32]. The presence of this large symmetry is related to the existence of an infinite number of conserved quantities in the underlying Calogero model. In the super-Calogero model with a harmonic oscillator potential a few low lying exact states have already been constructed [15]. An extension of this analysis may provide a link to a suitable supersymmetric generalization of the W-algebra [33].

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