Existence of convolution maximizers in $L_p(\mathbb{R}^n)$ for kernels from Lorentz spaces

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Abstract

The paper extends an earlier result of G.V. Kalachev and the author (Sb. Math. 2019) on the existence of a maximizer of convolution operator acting between two Lebesgue spaces on $\mathbb{R}^n$ with kernel from some $L_q$, $1 < q < \infty$. In view of Lieb’s result of 1983 about the existence of an extremizer for the Hardy-Littlewood-Sobolev inequality it is natural to ask whether a convolution maximizer exists for any kernel from weak $L_q$. The answer in the negative was given by Lieb in the above citation. In this paper we prove the existence of maximizers for kernels from a slightly more narrow class than weak $L_q$, which contains all Lorentz spaces $L_{q,s}$ with $q \leq s < \infty$.

Keywords: convolution, existence of extremizer, weak $L_p$ space, tight sequence, Hardy-Littlewood-Sobolev inequality, best constants.

MSC: 44A35, 46E30, 41A44.

1 Introduction

In this paper we extend the main result of [2]. We will mostly follow [2] in notation and terminology.

Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwarz space and let $k \in \mathcal{S}'(\mathbb{R}^n)$, i.e. $k$ is a tempered distribution in $\mathbb{R}^n$. We write, following [9], $k \in \text{Cnv}(p, r)$, if the convolution operator $K_k : f \mapsto k * f$ defined initially for $f \in \mathcal{S}(\mathbb{R}^n)$ extends continuously to the map from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$, that is,

$$\|k * f\|_{L_q(\mathbb{R}^n)} \leq C\|f\|_{L_p(\mathbb{R}^n)}$$

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1 corresponds to $L_p^*$ in [6]
for any \( f \in \mathcal{S}(\mathbb{R}^n) \) with \( C \) independent of \( f \).

In the sequel we write \( \| f \|_p \) instead of \( \| f \|_{L_p(\mathbb{R}^n)} \).

We address the question of the existence of a maximizer for the operator \( K_k \). A maximizer is any function \( f \) with \( \| f \|_p = 1 \) such that

\[
\| k * f \|_q = \| K_k \| \quad (= \| K_k \| \| f \|_p).
\]

Convolution operators considered in this paper will have kernels from \( \text{Cnv}(p,r) \) where \( 1 < p < r < \infty \). (In [2], the exponent of the target space was denoted \( r' \).)

If \( k \in L_q \) and the triple \((p,q,r)\) satisfies the Young condition

\[
\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}, \tag{1}
\]

then \( k \in \text{Cnv}(p,r) \) by Young’s convolution inequality.

A weaker sufficient condition is provided by the Hardy-Littlewood-Sobolev theorem: \( k \in \text{Cnv}(p,r) \) if (1) holds and \( k \in L_{q,\infty} \), the weak \( L_q \) space, see e.g. [1] Theorem 1.4.24. (We recall the relevant definitions in Sec. 2). The norm of the operator \( K_k \) is estimated by the weak \( q \)-norm of the kernel \( k \); this important fact follows from the Hardy-Littlewood-Sobolev inequality (see e.g. [7] Theorem 4.5.3) or [5] Theorem 4.3) and the Riesz rearrangement inequality for convolutions [5] Theorem 3.7).

The main result of [2], sharpening the earlier result of Pearson [8] (by removing extraneous assumptions), is

**Theorem A.** If \( p, q, r \) are related by (1), \( 1 < p < r < \infty \), and \( k \in L_q \), then the operator \( K_k: L_p \to L_r \) has a maximizer.

The “boundary” cases \( p = 1, r = \infty \), and \( p = r \) are discussed in [2], too. The full analysis of the case \( p = q = r = 1 \) is given in [3].

The present paper is motivated by a desire to connect Theorem A with another maximizer existence result due to Lieb [4], see also [5] Sec. 4.8:

**Theorem B.** If \( p, q, r \) are related by (1), \( 1 < p < r < \infty \), and

\[
h(x) = |x|^{-n/q},
\]

then the operator \( K_h: L_p(\mathbb{R}^n) \to L_r(\mathbb{R}^n) \) has a maximizer.
Here \( h \not\in L_q \), but \( h \in L_{q,\infty} \). One is tempted to conjecture that the condition \( k \in L_{q,\infty} \) is always sufficient for the existence of a maximizer. However this is wrong as was pointed out already by Lieb [4, p. 352].

We prove the existence result for kernels from a slightly more narrow class \( L_{q,\infty,0} \subset L_{q,\infty} \) defined in Sec. 2. It is Theorem 2. We obtain it as a consequence of Theorem 1, where the assumptions on the kernel have more abstract form and are not very convenient for immediate applications. In its turn, Theorem 2 yields Theorem 3, where the sufficient condition is simply \( k \in L_{q,s}, q \leq s < \infty \), i.e. the kernel is assumed to belong to a Lorentz space between the strong \( L_q \) and the weak \( L_q \). The paper’s title refers precisely to the latter result for in that case the class of suitable kernels is most easily understood.

In comparison with [2], the novelty in this paper is primarily in formulations. For the proof of the base result, Theorem 1, we have to make only modest adjustments of some lemmas from [2].

To be clear, our result does not imply Lieb’s. The existence of an extremizer in the Hardy-Littlewood-Sobolev inequality is due to the dilatational symmetry of the kernel; it does sustain truncations of the kernel.

\section{Statement of results}

\textbf{Theorem 1.} Let \( 1 < p < r < \infty \) and let \( p, q, r \) be related by (1). Suppose that the distribution \( k \in \text{Cnv}(p,r) \) has the following approximation property. For any \( \varepsilon > 0 \), there exists a measurable function \( k_\varepsilon \) such that

(i) \( k_\varepsilon \) has finite support;

(ii) \( k_\varepsilon \in L_\infty \);

(iii) \( \| K_k - k_\varepsilon \|_{L_p \to L_r} \leq \varepsilon \).

Then the operator \( K_k : L_p \to L_r \) has a maximizer.

Note that (i) and (ii) imply \( k_\varepsilon \in L_q \), hence \( k_\varepsilon \in L_p^r \), so \( k - k_\varepsilon \in L_p^r \) and the condition (iii) is meaningful.

Next we exhibit some regular class of measurable functions \( k \in L_p^r \) for which the operator \( K_k \) has a maximizer. The word \textit{regular} means that the function \( k \) has the described property if and only if the function \( |k| \) has has the property.

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\[ ^2 \text{by analogy with definition of a regular integral operator} \]
Recall the definitions of the weak $L_q$ space $L_{q,\infty}$ and the Lorentz spaces $L_{q,s}$.

Given a measurable function $f$ defined on $\mathbb{R}^n$, its distribution function is

$$d_f(\lambda) = |\{x \mid |f(x)| > \lambda\}|,$$

where $|\Omega|$ denotes the Lebesgue measure of the set $\Omega$.

The decreasing rearrangement of $f$ is the function

$$f^*(t) = \inf\{\lambda > 0 \mid d_f(\lambda) \leq t\},$$

defined on $[0, +\infty)$.

Put

$$\|f\|_{q,\infty} = \sup_{t>0} t^{1/q} f^*(t)$$
and

$$\|f\|_{q,s} = \left( \int_0^\infty \left( t^{1/q} f^*(t) \right)^s \frac{dt}{t} \right)^{1/s}$$
if $0 < s < \infty$.

The Lorentz space $L_{q,s}$, $0 < s \leq \infty$, consists of measurable functions for which $\|f\|_{q,s} < \infty$.

It is known that $s > s' \Rightarrow L_{q,s} \supset L_{q,s'}$, see e.g. [1, Sec. 1.4.2].

Also, $L_{q,q} = L_q$. The largest space (for the fixed $q$) $L_{q,\infty}$ is called the weak $L_q$ space. We will now define its special subspace.

**Definition.** The space $L_{q,\infty,0}$ is the subspace of $L_{q,\infty}$ that consists of functions $f$ such that

$$\lim_{t \to 0^+} t^{1/q} f^*(t) = \lim_{t \to \infty} t^{1/q} f^*(t) = 0.$$

It is easy to see that the equivalent condition is

$$\lim_{\lambda \to 0^+} \lambda^q d_f(\lambda) = \lim_{\lambda \to \infty} \lambda^q d_f(\lambda) = 0.$$

**Theorem 2.** If $p, q, r$ are related by (1), $1 < p < r < \infty$, and $k \in L_{q,\infty,0}$, then the operator $K_k : L_p \to L_r$ has a maximizer.

We will show that the classical Lorentz spaces $L_{q,s}$, $s < \infty$, are contained in $L_{q,\infty,0}$ and thus obtain

**Theorem 3.** If $p, q, r$ are related by (1), $1 < p < r < \infty$, and $k \in L_{q,s}$ where $q \leq s < \infty$, then the operator $K_k : L_p \to L_r$ has a maximizer.
3 Proofs

Proof of Theorem 1. The examination of the proof of Theorem 1 in [2] shows that the assumption \( k \in L^q \) is confined to following lemmas: 3.7, 4.1, and 4.2–4.3. We will state and prove appropriate modifications of those lemmas in sections (a) to (c) below. As a matter of fact, the proofs will go essentially along the same lines as in [2]. The difference, in essence, can be described as follows. Where “tails” are cut from the kernel obtain a core which is bounded and has finite support, Young’s inequality was invoked in the preceding proofs to estimate the operator norms of the corresponding “small perturbations” on the spot. Presently for the same purpose we can just refer to the condition (iii) of Theorem 1.

(a) Tightness of a maximizing sequence

Recall the definition of \( \delta \)-diameter of a function \( f \in L_p(\mathbb{R}^n) \) in the direction \( v \in \mathbb{R}^n, \|v\| = 1 \) ([2], Definition 2.2):

\[
D_{\delta,v}(f) = \inf_{b>a} \left\{ b - a \left| \int_{a<(x,v)<b} |f(x)|^p \, dx \geq \|f\|_p^p - \delta \right. \right\}.
\]

Lemma 1. (Substitute for Lemma 3.7 in [2])
Put \( N = \| K_k \|_{L_p \to L_r} \). Under the assumptions of Theorem 1, suppose \( \epsilon \in (0, N/3) \) is given and \( k_{\epsilon}(x) = 0 \) for \( |x| > R \) (such an \( R \) exists by condition (i)). Let \( \epsilon_1 = \epsilon/N \) and \( f \in L_p \) be any \( \epsilon_1 \)-maximizer of the operator \( K_k \), i.e. \( \|f\|_p = 1 \) and \( \|K_k f\|_r \geq N(1-\epsilon_1) \). Then for

\[
\delta = \frac{6\epsilon_1}{1 - 2^{1-r/p}}
\]

and any unit vector \( v \in \mathbb{R}^n \) we have

\[
D_{\delta,v}^p(f) \leq 8R \epsilon_1^{-p/r}.
\]

Remark. The condition (ii) in the formulation of Theorem 1 is not needed for this Lemma.

Proof. We have the decomposition \( K_k = A + B \), where \( A = K_{k_{\epsilon}} \) is the operator of convolution with bounded function supported in the ball \( |x| \leq R \), while \( \|B\| \leq \epsilon \) (condition (iii)).
Since $f$ is an $\varepsilon$-maximizer for $K_k$, we have

$$\|Af\|_r \geq \|K_k f\|_r - \|B f\|_r \geq \|K_k\|(1 - \varepsilon_1) - \varepsilon.$$ 

On the other hand, $\|A\| \leq \|K_k\| + \varepsilon$. Hence

$$\|Af\|_r \geq \|A\|(1 - \varepsilon_2),$$

where

$$1 - \varepsilon_2 = \left(1 - \varepsilon_1 - \frac{\varepsilon}{N}\right) \left(1 + \frac{\varepsilon}{N}\right)^{-1}.$$ 

Thus $f$ is an $\varepsilon_2$-maximizer for $A$.

By the choice of $\varepsilon_1$ we obtain

$$\varepsilon_2 = \frac{3\varepsilon/N}{1 + \varepsilon/N} < 3\varepsilon_1.$$ 

By Lemma 3.6 of [2], there exist $\delta > 0$ and $L > 0$ such that for any unit vector $v \in \mathbb{R}^n$

$$D_{\delta,v}^p(f) \leq L.$$ 

The expressions for $\delta$ and $L$ are available in explicit form. The cited Lemma 3.6 is applied with parameters $\tau = \varepsilon_2$, $\gamma = r/p > 1$, $a = R$. We take

$$\delta = 2\frac{\tau}{1 - 2^{1-\gamma}} = \frac{2\varepsilon_2}{1 - 2^{1-r/p}};$$

then the parameter $\kappa$ in Lemma 3.6 is $\kappa = 2\tau$ and a suitable bound $L$ for $D_{\delta,v}^p(f)$ can be taken in the form

$$L = 8a(\kappa - \tau)^{-1/\gamma} = 8R\varepsilon_2^{-p/r}.$$ 

The same upper bound is valid for $D_{\delta',v}^p$ for any $\delta' \geq \delta$. In particular, we may take $\delta' = 6\varepsilon_1(1 - 2^{1-r/p})^{-1}$.

Renaming $\delta'$ into $\delta$ and replacing $\varepsilon_2$ in the above expression for $L$ by $\varepsilon_1$, which only makes the upper bound coarser (larger) since $3/(1 + \varepsilon_1) > 1$, we come to the result as stated.

A sequence of functions $f_j$ with $\|f_j\|_p = 1$ is relatively tight (Definition 2.5 in [2]) if for any $\delta > 0$ there holds

$$\sup_j \sup_{\|v\| = 1} D_{\delta,v}^p(f_j) < \infty.$$ 

As a simple consequence of Lemma 1, we deduce the analog of Corollary 3.2 of [2]:

If an operator $K_k \in \text{Cuv}(p,r)$ satisfies the conditions of Theorem 1 then any maximizing sequence for $K_k$ is relatively tight.
(b) Compactness lemma

This lemma is used to assert that the operator $K_k$ maps a weakly convergent in $L_p$ sequence to a sequence convergent in $L_r$-norm on any bounded set in $\mathbb{R}^n$.

**Lemma 2.** (Substitute for Lemma 4.1 and Corollary 4.1 in [2])

Under the assumptions of Theorem 1, suppose that the sequence $(f_n)$ with $\|f_n\| = 1$ weakly converges in $L_p$ to $f$. Then for any function $\chi \in L_r \cap L_\infty(\mathbb{R}^n)$ we have $\|\chi(x) \cdot (K_k f_j - K_k f)\|_r \to 0$.

**Proof.** Given $\varepsilon > 0$, we want to find $n_0$ such that $\|\chi(x) \cdot (K_k f_n(x) - K_k f(x))\|_p < \varepsilon$ for all $j \geq j_0$.

Consider the decomposition $k = k_{\varepsilon/3} + (k - k_{\varepsilon/3})$ provided by the assumptions of Theorem 1 (with $\varepsilon/3$ in place of $\varepsilon$). Let $K_k = A + B$ be the corresponding decomposition of the operator $K_k$.

Without loss of generality, we may assume that $\|\chi\|_\infty \leq 1$.

The first part of the proof, dealing with operator $A$ (convolution with $k_{\varepsilon/3}$), is the same as in the proof of Lemma 4.1 of [2]. Since $k_{\varepsilon/3} \in L_1 \cap L_\infty \subseteq L_{p'}$, the sequence $Af_n$ converges pointwise. Moreover, $\|\chi \cdot Af_j\|_r \leq \|\chi\|_r \cdot \|k_{\varepsilon/3}\|_{p'} \cdot \|f_j\|_p$ is uniformly bounded (w.r.t. $j$), so by the dominated convergence theorem there exists $j_0$ such that

$$\|\chi \cdot A(f_j - f)\|_r < \frac{\varepsilon}{3}, \quad j \geq j_0.$$  

The final step of the proof differs from that in [2] in that now it refers to the condition (iii) of Theorem 1 that is, $\|B\| \leq \varepsilon/3$. We get

$$\|\chi \cdot B(f_j - f)\|_r < \|B\|\|f_j\|_p + \|f\|_p \leq \frac{2\varepsilon}{3}$$

for any $n$, due to the fact that $\|f\|_p \leq \lim \|f_j\|_p = 1$.

We conclude that for $j \geq j_0$

$$\|\chi \cdot K_k (f_j - f)\|_r < \varepsilon,$$

as required. \qed
(c) Preservation of tightness property under the action of $K_k$

In [2], Lemma 4.3 is the result that eventually applies in the global scheme of the proof. It is derived as a consequence of Lemma 4.2. Here we will not need an analog of Lemma 4.2 and will derive the required analog of Lemma 4.3 using that very lemma in the form established in [2].

**Lemma 3.** (Substitute for Lemma 4.3 in [2]) Suppose that $(f_j), \|f_j\|_p = 1$, is a tight sequence in $L_p(\mathbb{R}^n)$. That is, for any $\delta > 0$, there is a cube $Q \in \mathbb{R}^n$ such $\int_{\mathbb{R}^n \setminus Q} |f_j|^p \leq \delta$ for all $j$. If $k \in \text{Cnv}(p,r)$ satisfies the assumptions of Theorem 1, then the sequence $g_j = K_k f_j$ is tight in $L_r(\mathbb{R}^n)$, that is, for any $\delta > 0$ there exists a cube $Q$ such that $\int_{\mathbb{R}^n \setminus Q} |g_j|^r \leq \delta$.

**Proof.** Consider the decomposition $K_k = A + B$ provided by the conditions of Theorem 1, where $A = K_{k\varepsilon}$ and $\|B\|_{L_p \to L_r} \leq \varepsilon$. The value of $\varepsilon$ will be specified later.

We have $g_j = Af_j + Bf_j$, where $\|Bf_j\|_r \leq \varepsilon$ and $A$ is a convolution operator with kernel that certainly lies in $L_q$, so that Lemma 4.3 of [2] is applicable to it.

Now, given $\delta > 0$, let us choose $\varepsilon = \frac{1}{2}\delta^{1/r}$. Put $\delta_1 = 2^{-r}\delta$. By Lemma 4.3 of [2], there exists a cube $Q$ such that for all $j$

$$\int_{\mathbb{R}^n \setminus Q} |Af_j|^r \leq \delta_1.$$

By Minkowski’s inequality,

$$\int_{\mathbb{R}^n \setminus Q} |g_j|^r \leq (\|Af_j\|_{L_r(\mathbb{R}^n \setminus Q)} + \|Bf_j\|_{L_r(\mathbb{R}^n \setminus Q)})^r \leq \delta_1^r + \varepsilon^r = \delta,$$

as required. \qed

We have revised all the lemmas that needed revision. The structure of the proof of Theorem 1 in [2] and all other details stay as is. Thus Theorem 1 of this paper is proved. \qed

**Proof of Theorem 2.** Given $\varepsilon > 0$, we are going to exhibit the function $k_\varepsilon$ satisfying the conditions (i)–(iii) of Theorem 1.
By definition of the class $L_{q,\infty,0}$, there exists $M > 0$ such that

$$\lambda d_k(\lambda) < \varepsilon$$

for all $\lambda > M$.

Therefore the function

$$u(x) = \begin{cases} k(x) & \text{if } |k(x)| > M, \\ 0 & \text{if } |k(x)| \leq M \end{cases}$$

satisfies the inequality

$$\lambda d_u(\lambda) < \varepsilon$$

for all $\lambda > 0$.

Put $v = k - u$. Then $|v|_{\infty} \leq M$.

Also, $d_v(\lambda) \leq d_k(\lambda)$. Hence, by definition of the class $L_{q,\infty,0}$, there exists $\delta > 0$ such that

$$\lambda d_w(\lambda) < \varepsilon$$

for all $0 < \lambda < \delta$.

Therefore the function

$$w(x) = \begin{cases} u(x) & \text{if } |u(x)| < \delta, \\ 0 & \text{if } |u(x)| \geq \delta \end{cases}$$

satisfies the inequality

$$\lambda d_w(\lambda) < \varepsilon$$

for all $\lambda > 0$.

By the Young-like form of the Hardy-Littlewood-Sobolev inequality we have

$$\|K_u\|_{L_p \to L_r} \leq C\varepsilon$$

and

$$\|K_w\|_{L_p \to L_r} \leq C\varepsilon,$$

where $C$ depends only on $n$ (the dimension of the space), $p$ and $r$.

The function $y(x) = v(x) - w(x)$ is bounded: $\|y\|_{\infty} \leq M$ and has support of finite measure: $d_y(0) \leq d_v(\delta) \leq \|k\|_{q,\infty}\delta^{-q}$. Therefore there exists $R > 0$ such that

$$\int_{|x| > R} |y|^q dx < \varepsilon.$$
Put

\[ z(x) = \begin{cases} y(x) & \text{if } |x| > R, \\ 0 & \text{if } |x| \leq R. \end{cases} \]

By Young’s inequality, \( \| K_z \|_{L^p \to L^r} \leq \varepsilon. \)

The function

\[ \tilde{k} = y - z = k - (v + w + z) \]

is bounded, has finite support, and

\[ K_{k-\tilde{k}} \leq (2C + 1)\varepsilon. \]

Re-denoting \( (2C + 1)\varepsilon \) into \( \varepsilon \), we obtain \( \tilde{k} = k_{\varepsilon} \) with all the required properties.

**Proof of Theorem 3.** We will show that if \( s < \infty \), then \( L_{q,s} \subset L_{q,\infty,0} \), hence Theorem 3 will follow from Theorem 2.

We assume that \( f \in L_{q,s} \), that is, \( \int_0^\infty (f^*(t))^s t^{s/q - 1} dt < \infty \). Given \( \varepsilon > 0 \), there exists \( T_\varepsilon > 0 \) such that the integral from \( T_\varepsilon \) to \( \infty \) is less than \( \varepsilon \). Suppose that \( T \geq 2T_\varepsilon \). Then

\[
\varepsilon > \int_{T_\varepsilon}^{T} (f^*(t))^s t^{s/q - 1} dt \geq (f^*(T))^s \int_{T_\varepsilon}^{T} t^{s/q - 1} dt \geq C \left( f^*(T) T^{1/q} \right)^s,
\]

where

\[ C = \frac{q}{s} \left( 1 - 2^{-s/q} \right). \]

Hence \( \lim \sup_{t \to \infty} f^*(t) t^{1/q} \leq \left( \varepsilon / C \right)^{-1/s} \). Since \( \varepsilon \) is arbitrary, we obtain \( \lim_{t \to \infty} f^*(t) t^{1/q} = 0 \).

We use a similar (actually simpler) argument to prove that \( \lim_{t \to 0^+} f^*(t) t^{1/q} = 0 \). Given \( \varepsilon > 0 \), there exists \( T_\varepsilon > 0 \) such that \( \int_0^{T_\varepsilon} (f^*(t))^s t^{s/q - 1} dt < \varepsilon \). For any \( T \in (0, T_\varepsilon) \) we have

\[
\varepsilon > \int_0^{T} (f^*(t))^s t^{s/q - 1} dt \geq (f^*(T))^s \int_0^{T} t^{s/q - 1} dt \geq \frac{q}{s} \left( f^*(T) T^{1/q} \right)^s,
\]

whence the desired conclusion follows. \( \square \)
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