A Formulation of the Yang-Mills Theory
as a Deformation of a Topological Field Theory
Based on the Background Field Method
and Quark Confinement Problem

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Abstract

By making use of the background field method, we derive a novel reformulation of the Yang-Mills theory which was proposed recently by the author to derive quark confinement in QCD. This reformulation identifies the Yang-Mills theory with a deformation of a topological quantum field theory. The relevant background is given by the topologically non-trivial field configuration, especially, the topological soliton which can be identified with the magnetic monopole current in four dimensions. We argue that the gauge fixing term becomes dynamical and that the gluon mass generation takes place by a spontaneous breakdown of the hidden supersymmetry caused by the dimensional reduction. We also propose a numerical simulation to confirm the validity of the scheme we have proposed. Finally we point out that the gauge fixing part may have a geometric meaning from the viewpoint of global topology where the magnetic monopole solution represents the critical point of a Morse function in the space of field configurations.

Key words: quark confinement, topological field theory, magnetic monopole, background field method, topological soliton

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1 Introduction

More than twenty years ago many authors \cite{1} have proposed various strategies of deriving quark confinement in quantum chromodynamics (QCD). One of them is to show that the QCD vacuum is the dual superconductor which squeezes the color electric flux between quarks and anti-quarks. The evidences have been accumulated by recent investigations. Especially, recent numerical simulations have confirmed this picture, see \cite{2, 3}. In this scenario, the magnetic monopole \cite{4, 5, 6} obtained by the Abelian projection \cite{7} in QCD plays the essential role \cite{8, 9}. These results suggest that the low-energy effective theory of QCD is given by the dual Ginzburg-Landau theory \cite{10}. In fact, it has been shown that the dual Ginzburg-Landau theory can be derived starting from the QCD Lagrangian at least in the strong coupling region, see e.g. \cite{11}.

In the previous paper \cite{12}, we have proposed a novel formulation of the Yang-Mills theory as a (perturbative) deformation of a topological quantum field theory (TQFT). We have shown \cite{12, 13} that the quark confinement in QCD in the sense of area law of the Wilson loop (or equivalently, the linear static potential between quark and anti-quark) can be derived from the formulation at least in the maximal Abelian (MA) gauge. The MA gauge realizes the Abelian magnetic monopole in Yang-Mills theory without introducing the scalar field as an elementary field (Hence it is realized as a composite field constructed from the gauge degrees of freedom). In the similar way, it has been shown \cite{14} that the four-dimensional Abelian gauge theory can have the confining phase in the strong coupling region. This can be used to give another derivation of quark confinement in QCD based on the low-energy effective Abelian gauge theory, see \cite{15}.

The above results are consistent with those of lattice gauge theory \cite{16}, though our formulation is given directly on the continuum space-time. This similarity is due to a fact that the ingredients of confinement in our formulation lies in the compactness of the gauge group (or the periodicity in the gauge potential) and the existence of topological soliton. Thus, the existence of magnetic monopole is a sufficient condition for explaining quark confinement, as confirmed by analytical and numerical results \cite{2}.

In this paper, we re-derive the formulation proposed in \cite{12, 14} based on the background field method (BGFM) \cite{17, 18, 19, 20, 21}. A purpose of this paper is to fill the gap in the previous presentation \cite{12} without any ad hoc argument. This derivation enables us to discuss various topological soliton or topological defect other than the magnetic monopole, which might equally play the important role in explaining the origin of quark confinement. Such a viewpoint is necessary to answer the question: what are the most relevant degrees of freedom for quark confinement, since the necessary and sufficient condition for quark confinement is not yet known. Therefore, our formulation can also be applied to other scenarios of quark confinement based on this reformulation, the gauge-fixed Yang-Mills theory is decomposed into the TQFT part and the remaining part. Then we assume that the remaining part can be treated in perturbation theory in the gauge coupling constant. This assumption is nothing but the meaning of the perturbative deformation.
various confiners, e.g. instanton, center vortex or non-Abelian magnetic monopole, although the details will be given in a subsequent paper. Another advantage of BGFM is that it simplifies the proof \[22, 11\] that the renormalization group beta function of the Abelian-projected effective gauge theory is the same as the original Yang-Mills non-Abelian gauge theory.

Just as the topological Yang-Mills theory \[23\] describes the gauge field configurations satisfying the self-dual equation, i.e., instantons,

$$F_{\mu\nu} = \pm F_{\mu\nu}, \quad \tilde{F}_{\mu\nu} := \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma},$$

the TQFT that we have proposed deals with the gauge field configurations which obeys the MA gauge equation,

$$D_\pm^\mu [a] A_\mu^\pm = 0,$$

which is nothing but the background field equation. Both equations are the 1st order partial differential equations. They may have some properties in common. In fact, a class of classical solutions of the MA gauge equation (1.2) simultaneously satisfies the self-dual equation (1.1) and vice versa \[24, 25\]. It is obtained from the same ansatz as that of 't Hooft for the multi-instanton. The instanton is the point defect in four dimensions, while the magnetic monopole is the point defect in three dimensions. In four dimensions, therefore, the magnetic monopole is a one-dimensional object, i.e., a current $k_\mu$ (a closed loop due to the topological conservation law $\partial_\mu k_\mu = 0$). This is a Lorentz covariant generalization of the observation that the static monopole in three dimensions draws the straight line in the time direction in four dimensions where the monopole charge is given by the integral $Q_m := \int d^3 x k_0(x)$ from the monopole density $k_0$. Since we frequently use the ‘magnetic’ monopole as implying the solution of the MA gauge equation, the solution can describe the object which looks like the magnetic monopole and the instanton at the same time. Therefore, the magnetic monopole and the instanton are not the disjoint concept in four dimensions. Actually, strong correlations between monopoles and instantons are shown in the analytical studies \[24, 25\] and observed in the lattice simulations \[26, 27, 28, 29, 30\]. It is easy to see that the instanton is also a solution of the field equation (2nd order partial differential equation)

$$D_\nu [A] F_{\mu\nu} = 0.$$  

However, it is not yet clarified which solution of the MA gauge equation (1.2) becomes that of the field equation besides the solution mentioned above. The solution of (1.2) may contain the solution which is not the solution of the field equation (1.3).

When we see the intersection of the magnetic monopole current with the two-dimensional plane, the classical configuration satisfying (1.2) looks like the instanton in two-dimensional nonlinear sigma model (NLSM$_2$), as shown in \[12\]. Therefore the condensation of the magnetic monopole current in four dimensions can be examined on the two-dimensional subspace which can be chosen arbitrarily. The condensation of the two-dimensional instanton in NLSM$_2$ leads to that of the four-dimensional
magnetic monopole current in Yang-Mills theory. So the instanton condensation in \( NLSM_2 \) is a sufficient condition of quark confinement based on the dual superconductor scenario.

In the scenario \([12]\) of deriving quark confinement, the gauge fixing part for the gauge fixing condition \((1.2)\) has played the essential role. Since the quark confinement must be a gauge invariant concept, it is better to derive it based on the gauge invariant formulation. In contrast to the lattice gauge theory, however, the continuum formulation of the ordinary gauge theory free from the gauge fixing is not available except for special cases. Then we are forced to deal with the formulation based on the specific choice of gauge fixing. The readers might think the claim strange that the essence of quark confinement lies in the gauge-fixing part. Recall that, in the level of the classical theory, the action part of the gauge theory is well understood from a viewpoint of the geometry of connection. In quantum theory, however, we need to include the gauge fixing term in order to correctly quantize the gauge theory. Usually, the gauge fixing term introduced in this way is not considered to have any geometric meaning. However, this observation is not necessarily correct. In fact, the gauge fixing term plus the associated Faddeev-Popov ghost term can have the very geometric meaning from the viewpoint of global topology, as will be discussed in this paper. In the quantum gauge theory, therefore, the action part and the gauge fixing part should be treated on equal footing. Unfortunately, we must discuss the topology of the infinite dimensional manifold for gauge field configurations. Then the mathematically rigorous analysis will be rather hard, so that we can at best analyze the finite dimensional analog.

Usually, we consider that, even if the gauge fixing term has a geometric interpretation, it can not have any local dynamics (propagating mode) and describe only the topological objects, since it is written as the Becchi-Rouet-Stora-Tyupin (BRST) exact form, i.e., \( S_{GF} = \{ Q_B, \kappa \Psi \} \) using the BRST charge \( Q_B \). In the manifestly covariant formalism of gauge theory, the physical state \( |phys \rangle \) is specified by the condition, \( Q_B |phys \rangle = 0 \). If we consider the theory with the action \( S_{GF} = \{ Q_B, \kappa \Psi \} \) alone by neglecting the Yang-Mills action (this theory is identified with the TQFT), the expectation value of the gauge invariant quantity \( \langle O \rangle \) does not depend on the coupling \( \kappa \), since \( \frac{\partial}{\partial \kappa} \langle 0 | O | 0 \rangle = - \langle 0 | O \{ Q_B, \Psi \} | 0 \rangle + \langle 0 | O | 0 \rangle \{ Q_B, \Psi \} | 0 \rangle = - \langle 0 | \{ Q_B, O \Psi \} | 0 \rangle = 0 \) where we have used the BRST invariance of \( O \) and \( |0 \rangle \in |phys \rangle \). However, taking into account the action \( S_{YM} \) in addition to \( S_{GF} \), we can not draw the same conclusion. This is the usual situation of quantized gauge theory. A subtle point is that the above consideration is based on the assumption that the BRST symmetry is not broken. If the BRST symmetry happen to be spontaneously broken \([31]\), the physical state including the vacuum is not annihilated by the BRST charge, i.e., \( Q_B |phys \rangle \neq 0 \). In this case, the TQFT with the action \( S_{TQFT} = \{ Q_B, \kappa \Psi \} \) can have local dynamics and the expectation value can depend on the coupling constant \( \kappa \).

In our scenarios, the spontaneous breaking of the hidden supersymmetry \( OSp(4|2) \) rather than the BRST symmetry can take place by the dimensional reduction (in the sense of Parisi-Sourlas \([32, 33]\)), at least for a special choice of the MA gauge \([12]\). Here it is worth remarking that the equivalence of the correlation functions hold only for a class of them and hence the Hilbert space of the reduced theory is different.
from the original theory [12]. The symmetry breaking occurs spontaneously in the following sense. We can choose arbitrary $(D - 2)$-dimensional subspace from the $D$-dimensional spacetime. Once the specific subspace is chosen, however, the hidden supersymmetry $OSp(4|2)$, i.e., the rotational symmetry in the superspace is broken by this procedure.

Another purpose of this paper is to propose a numerical simulation in order to confirm the dimensional reduction and examine its implications to quark confinement problem. The result will prove or disprove the validity of our scenario for deriving quark confinement based on the above reformulation.

This paper is organized as follows. In section 2, we briefly review the BGFM for the Yang-Mills theory and its BRST version based on the functional integral formalism. In section 3, we explain how the quantum theory of topological soliton can be obtained in the framework of BGFM. We discuss a relationship between the instanton and the magnetic monopole in this construction. In section 4, by making the change of gauge field variable, we show that the formulation proposed in [12] is recovered from the BFGM. This is the main result of this paper. In section 5, we give a strategy of deriving quark confinement based on the above formulation. We take up some issues which have not been mentioned in the previous publications. We give a proposal of numerical calculation for checking the validity of the strategy. In section 6, we examine the mass generation for the gluon field in the MA gauge. We discuss a possibility of mass generation caused by the dimensional reduction as a result of breakdown of the hidden supersymmetry. In section 7, we discuss that the gauge fixing part in the quantum theory of gauge fields can have a geometric meaning from the viewpoint of global topology. In the final section, we summarize the results and discuss the role of various topological solitons other than the magnetic monopole for explaining color confinement in QCD.

2 Background field method

2.1 Path integral for Yang-Mills field

We consider the functional integral approach to the Yang-Mills gauge field theory with the action

$$S_{YM}[A] := \int d^D x L_{YM}[A] = -\int d^D x \frac{1}{4} (F^A_{\mu\nu}[A])^2;$$

(2.1)

where $F^A_{\mu\nu}[A]$ is the field strength for the gauge field $A^A_{\mu}$ defined by

$$F^A_{\mu\nu}[A] := \partial_\mu A^A_{\nu} - \partial_\nu A^A_{\mu} + gf^{ABC} A^B_{\mu} A^C_{\nu}.$$  

(2.2)

In the quantum theory of the Yang-Mills gauge field, the generating functional is defined by

$$Z[J] := \int [dA] \delta(\vec{F}^A[A]) \det \left[ \frac{\delta \vec{F}^A}{\delta A^B} \right] \exp \left\{ i S_{YM}[A] + (J_{\mu} \cdot A^A_{\mu}) \right\},$$

(2.3)

2The tilde is used only for later convenience (in section 4) and does not have particular physical meaning.
where \((J \cdot A)\) is the source term

\[
(J_{\mu} \cdot A_{\mu}) := \int d^{D}x J_{\mu}(x) A_{\mu}(x).
\]  

(2.4)

In (2.3), the gauge-fixing condition is imposed by

\[
\tilde{F}^{A}[A] = 0,
\]  

(2.5)

and \(\det \left[ \frac{\delta \tilde{F}^{A}}{\delta \omega^{B}} \right]\) is the so-called Faddeev-Popov (FP) determinant which is the determinant of the derivative of the gauge-fixing function \(\tilde{F}^{A}\) under an infinitesimal gauge transformation,

\[
\delta A_{\mu} = D^{AB}_{\mu}[A] \tilde{\omega}^{B} := \partial_{\mu} \tilde{\omega}^{A} + g f^{ABC} A^{B}_{\mu} \tilde{\omega}^{C},
\]  

(2.6)

\[
D^{AB}_{\mu} := \partial_{\mu} \delta^{AB} - g f^{ABC} A^{C}_{\mu}.
\]  

(2.7)

The delta function is made less singular by introducing the gauge-fixing parameter \(\tilde{\alpha}\) as

\[
\delta (\tilde{F}^{A}[A]) := \prod_{x,A} \delta (\tilde{F}^{A}[A(x)]) \rightarrow \exp \left\{-i \frac{1}{2\tilde{\alpha}} (\tilde{F}^{A}[A] \cdot \tilde{F}^{A}[A]) \right\}.
\]  

(2.8)

For example, a common choice is the Lorentz gauge,

\[
\tilde{F}^{A}[A] = \partial_{\mu} A^{\mu}_{A}.
\]  

(2.9)

Then the FP determinant is given by

\[
\det \left[ \frac{\delta \tilde{F}^{A}}{\delta \omega^{B}} \right] = \det (\partial_{\mu} D^{AB}_{\mu}[A] \delta^{D}(x - y)).
\]  

(2.10)

The connected Green’s functions are generated by

\[
W[J] := -i \ln Z[J].
\]  

(2.11)

The effective action is defined by making the Legendre transformation

\[
\Gamma[\bar{Q}] := W[J] - (J_{\mu} \cdot \bar{Q}_{\mu}),
\]  

(2.12)

where

\[
\bar{Q}^{A} := \frac{\delta W}{\delta J_{\mu}^{A}}.
\]  

(2.13)

It is well known that the derivative of the effective action with respect to \(\bar{Q}\) are the one-particle irreducible (1PI) Green’s function.
2.2 BGFM

Next, we consider the quantization on a given background gauge field \( \Omega_\mu \),

\[
A_\mu = \Omega_\mu + Q_\mu, \tag{2.14}
\]

where \( Q_\mu \) denotes the field to be quantized. The generating functional is given by

\[
\tilde{Z}[J, \Omega] := \int [dQ] \det \left[ \frac{\delta \tilde{F}^A}{\delta \tilde{\omega}^B} \right] \exp \left\{ i \left[ S_{YM}[\Omega + Q] + (J_\mu \cdot Q_\mu) - \frac{1}{2\alpha}(\tilde{F}[Q] \cdot \tilde{F}[Q]) \right] \right\}. \tag{2.15}
\]

where the gauge invariance for \( Q_\mu \) is broken by the the gauge fixing condition \( \tilde{F}^A[Q] = 0 \) which is supposed to fix completely the gauge degrees of freedom and the \( \frac{\delta \tilde{F}^A}{\delta \tilde{\omega}^A} \) is the derivative of the gauge-fixing term under the infinitesimal gauge transformation given by

\[
\delta Q^A_\mu = (D_\mu[\Omega + Q]\tilde{\omega})^A. \tag{2.16}
\]

In (2.15), we do not couple the background field to the source following 't Hooft \[18\].

In the background field method (BGFM) \[17, 18, 19, 20, 21\], the the following gauge fixing condition is chosen,

\[
\tilde{F}^A[Q] := D^{AB}_\mu[\Omega]Q^B_\mu = 0, \tag{2.17}
\]

which is called the background field (BGF) gauge. An advantage of the BGF gauge is that the BGF gauge condition retains explicit gauge invariance for the background gauge field \( \Omega_\mu \) even after the gauge fixing for the field \( Q_\mu \).

**Proposition**[19]: Under the BGF gauge condition (2.17), the BGF generating functional \( \tilde{Z}[J, \Omega] \) and \( \tilde{W}[J, \Omega] := -i \ln \tilde{Z}[J, \Omega] \) are invariant under the (infinitesimal) transformation,

\[
\delta \Omega^A_\mu = (D_\mu[\Omega]\omega)^A := (\partial_\mu \omega + ig[\omega, \Omega_\mu])^A, \tag{2.18}
\]

\[
\delta J^A_\mu = ig[\omega, J_\mu]^A := -gf^{ABC} \omega^B J^C_\mu. \tag{2.19}
\]

This is shown as follows. By making the change of integration variables, \( Q_\mu \rightarrow Q_\mu + i[\omega, Q_\mu] \), i.e.,

\[
\delta Q^A_\mu = ig[\omega, Q_\mu]^A = \omega \times Q. \tag{2.20}
\]

Eq. (2.19) and (2.20) represent an adjoint group rotation for \( J_\mu \) and \( Q_\mu \) respectively, so the term \( (J_\mu \cdot Q_\mu) \) is clearly invariant. Adding (2.18) and (2.20), we find

\[
\delta (\Omega_\mu + Q_\mu)^A = (D_\mu[\Omega + Q]\omega)^A. \tag{2.21}
\]

This is just a gauge transformation on the field variable \( A_\mu = \Omega_\mu + Q_\mu \), so the action \( S_{YM}[\Omega + Q] \) is also invariant. Note that the BGF gauge condition \( \tilde{F}^A[Q] \) is just the covariant derivative of \( Q_\mu \) with respect to the BGF \( \Omega_\mu \). Eq. (2.18) is a gauge transformation on \( \Omega_\mu \) and (2.20) is an adjoint rotation of \( Q_\mu \). Then the gauge fixing
term \((\tilde{F} \cdot \tilde{F})\) is invariant under such transformations. The FP determinant is also invariant, since the determinant is invariant under the adjoint rotation. Thus the BGF generating functional \(\tilde{Z}[J, Q]\) is invariant under (2.18) and (2.19).

By using
\[
\tilde{W}[J, \Omega] := -i \ln \tilde{Z}[J, \Omega],
\] (2.22)
we define the background effective action
\[
\tilde{\Gamma}[\tilde{Q}, \Omega] := \tilde{W}[J, \Omega] - (J_\mu, \tilde{Q}_\mu),
\] (2.23)
where
\[
\tilde{Q}_\mu^A = \frac{\delta \tilde{W}}{\delta J_\mu^A}.
\] (2.24)

From the invariance of \(\tilde{Z}[J, Q]\), it follows that \(\tilde{\Gamma}[\tilde{Q}, \Omega]\) is invariant under
\[
\delta \Omega_\mu = (D_\mu[\Omega] \tilde{\omega}),
\] (2.25)
\[
\delta \tilde{Q}_\mu = ig[\tilde{\omega}, \tilde{Q}_\mu],
\] (2.26)
Since (2.26) is a homogeneous transformation, \(\tilde{\Gamma}[0, \Omega]\) is invariant under the transformation (2.25) alone. Hence the effective action \(\tilde{\Gamma}[0, \Omega]\) in the BGFM is an explicitly gauge invariant functional of \(\Omega\), since (2.23) is just an ordinary gauge transformation. As a result, 1PI Green’s functions generated by differentiating \(\tilde{\Gamma}[0, \Omega]\) with respect to \(\Omega\) will obey the naive Ward-Takahashi identities of gauge invariance. Hence, \(\tilde{\Gamma}[0, \Omega]\) calculated in the BGFG is equal to the conventional effective action \(\Gamma[\bar{Q}]\) with \(\bar{Q} = \Omega\) calculated in an unconventional gauge which depends on \(\Omega\).

Then we obtain
\[
\tilde{\Gamma}[0, \Omega] = \Gamma[\bar{Q}]|_{\bar{Q}=\Omega},
\] (2.28)
as a special case of
\[
\tilde{\Gamma}[\tilde{Q}, \Omega] = \Gamma[\bar{Q}]|_{\bar{Q}=\tilde{Q}+\Omega}.
\] (2.29)
The 1PI Green functions calculated from the gauge invariant effective action \(\tilde{\Gamma}[0, \Omega]\) will be very different from those calculated by conventional method in normal gauges. Nevertheless, the relation assures us that all gauge-invariant physical quantities will come out the same in either approach [21]. Thus \(\tilde{\Gamma}[0, \Omega]\) can be used to generate the S-matrix of a gauge theory in exactly the same way as the usual effective action is employed.
2.3 BRST version of the BGFM

Now we give the Becchi-Rouet-Stora-Tyupin (BRST) version of the BGFM. The BGF generating functional is rewritten into

\[ \hat{Z}[J, \Omega] := \int [dQ][d\tilde{C}][d\tilde{\tilde{C}}][d\tilde{B}] \exp \left\{ iS_{YM}[\Omega + Q] + i\tilde{S}_{GF}[Q, \tilde{C}, \tilde{\tilde{C}}, \tilde{B}] + i(J_{\mu} \cdot Q_{\mu}) \right\} \]  

where \( \tilde{B} \) is the auxiliary scalar field and \( \tilde{C}, \tilde{\tilde{C}} \) are Hermitian anticommuting scalar field called the FP ghost and anti-ghost field, \( \tilde{C}^\dagger = \tilde{C}, \tilde{\tilde{C}}^\dagger = \tilde{\tilde{C}} \). Using the BRST transformation,

\[
\begin{align*}
\delta_B \Omega_{\mu}(x) &= 0, \\
\delta_B Q_{\mu}(x) &= D_{\mu}[\Omega + Q] \tilde{C}(x) := \partial_{\mu} \tilde{C}(x) - i g[\Omega_{\mu}(x) + Q_{\mu}(x), \tilde{C}(x)], \\
\delta_B \tilde{C}(x) &= i \frac{1}{2} g[\tilde{C}(x), \tilde{C}(x)], \\
\delta_B \tilde{\tilde{C}}(x) &= i \tilde{B}(x), \\
\delta_B \tilde{B}(x) &= 0,
\end{align*}
\]

the gauge fixing and the FP ghost terms for the BGG are combined into a compact form,

\[ \tilde{S}_{GF}[Q, \tilde{C}, \tilde{\tilde{C}}, \tilde{B}] := - \int d^D x \ i \delta_B \ \text{tr}_G \left[ \tilde{C} \left( \tilde{F}[Q] + \tilde{\tilde{\alpha}} \tilde{B} \right) \right], \]  

or

\[ \tilde{S}_{GF}[Q, \tilde{C}, \tilde{\tilde{C}}, \tilde{B}] = \int d^D x \ \text{tr}_G \left[ B D_{\mu}[\Omega] Q_{\mu} + \frac{\tilde{\tilde{\alpha}}}{2} \tilde{B} \tilde{B} + i \tilde{C} D_{\mu}[\Omega] D_{\mu}[\Omega + Q] \tilde{C} \right], \]  

where \( \tilde{\alpha} \) is the gauge-fixing parameter and \( \tilde{\alpha} = 0 \) corresponds to the Landau gauge (delta function gauge). This is clearly BRST invariant \( \delta_B S_{GF} = 0 \) due to nilpotency of the BRST transformation, \( \delta_B^2 \equiv 0 \). If the auxiliary field \( \tilde{B} \) is integrated out, the gauge-fixing part reads

\[ \tilde{S}_{GF}[Q, \tilde{C}, \tilde{\tilde{C}}] = \int d^D x \ \text{tr}_G \left[ \frac{1}{2 \tilde{\alpha}} (D_{\mu}[\Omega] Q_{\mu})^2 + i \tilde{C} D_{\mu}[\Omega] D_{\mu}[\Omega + Q] \tilde{C} \right]. \]

In fact, this recovers the original form (2.13), since

\[ \det \left[ \frac{\delta \tilde{F}^A}{\delta \tilde{G}^B} \right] = \int [d\tilde{C}][d\tilde{\tilde{C}}] \exp \left[ i \int d^D x \ \text{tr}_G \left( i \tilde{C} D_{\mu}[\Omega] D_{\mu}[\Omega + Q] \tilde{C} \right) \right]. \]

The explicit form of the FP ghost term is

\[ \text{tr}_G \left[ i \tilde{C} D_{\mu}[\Omega] D_{\mu}[\Omega + Q] \tilde{C} \right] = i \tilde{\tilde{C}} \left[ \partial_{\mu} \partial_{\mu} \delta^{AB} - gf^{ACB} \partial_{\mu} (\Omega_{\mu} + Q_{\mu}) C + g f^{ACB} \Omega_{\mu} B \partial_{\mu} + g^2 f^{ACE} f^{FBD} \Omega_{\mu} (\Omega_{\mu} + Q_{\mu}) D \tilde{C} B, \right) \]  

8
and the gauge fixing term is

$$\text{tr}_G \left[ -\frac{1}{2\alpha} \left( D_\mu \Omega Q_\mu \right)^2 \right]$$

$$= -\frac{1}{2\alpha} \left[ \left( \partial_\mu Q_\mu^A \right)^2 + 2gf^{ABC} \Omega^B \partial_\mu Q_\mu^C \partial_\mu Q_\mu^A + g^2 f^{ABC} f^{ADE} \Omega^B \Omega^C \partial_\mu Q_\mu^D \partial_\mu Q_\mu^E \right]. \quad (2.37)$$

Feynmann rule for the BGFM is derived from the shifted action $S_{YM}[\Omega + Q]$ and (2.34), see Abbott [19]. In the limit $\Omega_\mu \to 0$, the BRST version of BGFM reduces to the usual BRST formulation of the Yang-Mills theory in the Lorentz gauge, $F^A[Q] = \partial_\mu Q_\mu$.

The advantage of the BGFM becomes apparent when the two-loop $\beta$ function is calculated. The BGFM makes the calculation much easier than previous calculations using the conventional approach, see [20, 21].

3 Quantum theory of topological soliton and BGFM

3.1 Summation over topological soliton background

In the conventional approach, the background field $\Omega_\mu$ is chosen to be a solution of the classical field equation. In Yang-Mills theory, the equation of motion is given by

$$\frac{\delta S_{YM}[A]}{\delta A_\mu^A} \equiv D_\nu^A [A] F_{\mu\nu}^A [A] = 0. \quad (3.1)$$

Then, under the identification (2.14),

$$A_\mu = \Omega_\mu + Q_\mu, \quad \text{(3.2)}$$

the quantization is performed around arbitrary but fixed background $\Omega_\mu$ which satisfies (3.1). In this paper, we consider the topologically nontrivial field configuration as a background field $\Omega_\mu$, around which the quantization of the Yang-Mills theory is performed. Once a specific type of field configurations is chosen as the background $\Omega_\mu$, we will include all possible configurations of the same type, in other words, we sum up all contributions coming from such a type of configurations. Therefore, in our formulation, a candidate for the generating functional of the total Yang-Mills theory is given by

$$Z[J] = \int [d\Omega_\mu] Z[J, \Omega] =: \int [d\Omega_\mu] \exp(i\tilde{S}_{eff}[J, \Omega]), \quad (3.3)$$

where we have defined

$$\tilde{S}_{eff}[J, \Omega] := -i \ln \tilde{Z}[J, \Omega], \quad (3.4)$$

and $[d\Omega_\mu]$ is the integration measure specified later.

3 Such a procedure was performed so far in various forms, e.g., by summing up the monopole-currents trajectories [34, 35, 36].
Note that the action $\tilde{S}_{\text{eff}}[J, \Omega]$ can have the local gauge invariance (2.18) for $\Omega$, by virtue of the BGFM. Hence, the total Yang-Mills theory defined in this way is identified with the (quantized) gauge theory with the action $\tilde{S}_{\text{eff}}[J, \Omega]$, provided that the integration measure $[d\Omega_{\mu}]$ is gauge invariant. However, in order to quantize the total Yang-Mills theory correctly, we need to fix the local gauge invariance for the non-Abelian gauge field $\Omega_{\mu}$. Thus, instead of (3.3), we define the generating functional of the total Yang-Mills theory by

$$Z[J] = \int [d\Omega_{\mu}] \delta(F^A[\Omega]) \det \left[ \frac{\delta F^A}{\delta \omega^B} \right] \tilde{Z}[J, \Omega],$$  \hspace{1cm} (3.5)

or

$$Z[J] = \int [d\Omega_{\mu}] \det \left[ \frac{\delta F^A}{\delta \omega^B} \right] \exp \left( i\tilde{S}_{\text{eff}}[J, \Omega] - \frac{i}{2\alpha} (F[\Omega] \cdot F[\Omega]) \right),$$  \hspace{1cm} (3.6)

where the gauge fixing function $F^A$ is not necessarily equal to the BGF gauge $\tilde{F}^A$. The choice of $F^A$ is quite important in our formulation for realizing topological soliton background, as explained below. In order to be able to incorporate the topological soliton, the gauge fixing function $F[\Omega]$ should be nonlinear in $\Omega$. The measure $[d\Omega_{\mu}]$ must be chosen appropriately for the topological soliton in question. In the final stage the measure is replaced by the integration over the collective coordinates of the soliton.

### 3.2 Yang-Mills instanton

In four-dimensional Euclidean space, the most popular topologically nontrivial field configuration of pure Yang-Mills theory is the instanton (anti-instanton) \cite{37, 38, 39, 40, 41, 42, 43} which is a solution of the self-dual (self-antidual) equation,

$$\mathcal{F}_{\mu\nu}[A] = \pm \mathcal{F}^*_{\mu\nu}[A], \quad \mathcal{F}^*_{\mu\nu}[A] := \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \mathcal{F}^*_{\rho\sigma}[A],$$  \hspace{1cm} (3.7)

with a finite action

$$S_{YM}[A] < \infty.$$  \hspace{1cm} (3.8)

The self-dual equation (3.7) is a first order nonlinear partial differential equation (NL PDE), whereas the field equation (3.1) is a second order nonlinear partial differential equation. The instanton is a kind of topological soliton which is possible due to the nonlinearity of the self-dual equation. Due to the Bianchi identity,

$$\mathcal{D}^\nu[A] \mathcal{F}^*_{\mu\nu}[A] \equiv 0,$$  \hspace{1cm} (3.9)

any instanton (anti-instanton) solution is also a solution of the Yang-Mills field equation,

$$\mathcal{D}^\nu[A] \mathcal{F}_{\mu\nu}[A] = 0,$$  \hspace{1cm} (3.10)
Figure 1: Moduli space, i.e., space of solutions for the Yang-Mills equation of motion (3.1), self-dual instanton equation (3.7) and the magnetic monopole equation (3.17) in the MA gauge in four dimensions. The instanton solution of the self-dual equation (3.7) is also a solution of the Yang-Mills equation of motion (3.1). The converse is not necessarily true. In fact, the Sibner-Sibner-Uhlenbeck (SSU) solution \[44\] on \(S^4\) is a solution of the Yang-Mills field equation which is not a solution of the self-dual equation (3.1). The general instanton solution on \(S^4\) can be constructed according to the Atiyah, Drinfeld, Hitchin and Mannin (ADHM) \[41\]. The explicit form for the multi-instanton is known in the specific cases, e.g., 't Hooft type \[43\] or Witten type \[39\]. Both types include one-instanton solution of Belavin, Polyakov, Schwartz and Tyupkin (BPST) \[37\]. The multi-instanton solution of 't Hooft type is also the solution of the magnetic monopole equation (A.1). Some solutions are known for (A.1), Chernodub and Gubarev (CG) \[25\] and Brower, Orginos and Tan (BOT) \[24\]. See Appendix A. The general solution of (A.1) is not yet known. In principle, there may exist a solution of the monopole equation which is not a solution of Yang-Mills field equation, indicated by ? in the figure.
but the converse does not hold. In fact, the instanton and the anti-instanton do not exhaust the solution of the Yang-Mills field equation, since there exists at least one solution of the Yang-Mills field equation (Sibner-Sibner-Uhlenbeck (SSU) solution \[44\]) which is not a solution of the self-dual equation, see Fig. 1. The existence of the instanton solution is suggested from the non-triviality of Homotopy group \[45, 46, 47\]

\[\pi_3(G) = \pi_3(SU(N)) = \mathbb{Z} \quad (N = 2, 3, \cdots). \quad (3.11)\]

It is possible to construct the instanton background by choosing the gauge fixing condition,

\[F^A[\Omega] = F_{\mu \nu}^\pm A[\Omega], \quad F_{\mu \nu}^\pm A[\Omega] := F_{\mu \nu}^A[\Omega] \mp F_{\mu \nu}^A[\Omega]. \quad (3.12)\]

It is shown \[48\] that this choice leads to the topological Yang-Mills theory \[23, 49\] which is an example of the TQFT of Witten type. Since the topological Yang-Mills theory is derived from the N=2 Supersymmetric Yang-Mills theory by the procedure called the twisting, this might shed more light on the quark confinement based on the dual Meissner effect or the magnetic monopole \[50\]. However, quark confinement will be realized only when the \[N=2\] supersymmetry is broken down to \[N=1\] by adding the mass perturbation. Since we do not have any convincing argument to justify such a scenario, we do not consider this possibility anymore in this paper.

### 3.3 Magnetic monopole current

In our formulation, however, the background field \(\Omega_\mu\) is not a priori required to be the classical solution of the field equation (3.1), when we consider the quantum theory of the background field \(\Omega_\mu\). In quantum theory, it is not necessarily true that the most dominant contribution is given by the solution of the field equation. This is obvious in the functional integral approach because we must take into account the entropy associated with the relevant field configurations, which comes from the integration measure \([d\Omega_\mu]\) of the functional integral. In fact, whether the phase transition occurs or not is determined according to the balance between the action (energy) and the entropy, which is called the action (energy)-entropy argument. What kind of field configuration is important may vary from problem to problem.

In our approach, we take the magnetic monopole current as the topologically nontrivial background \(\Omega_\mu\). This choice is suggested from the recent result \[9\] of Monte Carlo simulations in lattice gauge theories; the (Abelian) magnetic monopole after Abelian projection \[7\] plays the dominant role in quark confinement. This fact is called the (Abelian) magnetic monopole dominance \[8\].

The magnetic monopole in pure Yang-Mills theory (without the elementary Higgs scalar field) is obtained as follows. First, we restrict the Non-Abelian gauge group \(G\) to the subgroup \(H\) \((G \rightarrow H)\) and retain only the gauge invariance for \(H\), in other words, the gauge group element \(U(x) \in G\) is restricted to the coset \(G/H\). To obtain Abelian magnetic monopole, \(H\) is chosen to be the maximal torus subgroup of \(G\) (We will discuss other choices in the final section). We realize this restriction by the

\[12\]
partial gauge fixing. The MAG is a partial gauge fixing so that $G/H$ is fixed and $H$ is retained by choosing $F^A[\Omega]$ appropriately. The MA gauge condition is obtained by minimizing the $\mathcal{R}[A]$ with respect to the gauge rotation $U$ where

$$
\mathcal{R}[A] := \int d^D x \ tr_{G/H} \left( \frac{1}{2} A_\mu(x) A_\mu(x) \right) \equiv \int d^D x \frac{1}{2} A^a_\mu(x) A^a_\mu(x), \quad (3.13)
$$

where we have used the Cartan decomposition which decomposes the non-Abelian gauge field into the diagonal and the off-diagonal pieces,

$$
A_\mu = A^{A} T^A = a^a_\mu T^a + A^a_\mu T^a. \quad (3.14)
$$

Note that the trace is taken only on the coset part, see Appendix B. A geometric meaning of this function is given in section 7. According to the Cartan decomposition, the Abelian gauge potential is defined by

$$
A^a_\mu(x) := \text{tr}[H^a A_\mu(x)], \quad (3.15)
$$

where $H^a = T^a (i = 1, \ldots, \text{rank}G)$ is the Cartan subalgebra. For $G = SU(2)$, the differential MA gauge is obtained as

$$
F^a[A] := (\partial_\mu \delta^{ab} - \epsilon^{abc} A^c_\mu) A^b_\mu := D_\mu a^b[A^3] A^b_\mu \quad (a, b = 1, 2), \quad (3.16)
$$

where $a_\mu = A^3_\mu$. Note that the equation

$$
D_\mu a^b[A^3] A^b_\mu = 0 \quad (a, b = 1, 2). \quad (3.17)
$$

is a 1st order nonlinear partial differential equation. We call this equation the monopole equation in what follows.

Next, using the solution of $F^a[A] = 0$, the magnetic monopole current is defined by

$$
k^a_\mu := \partial_\nu \tilde{f}^a_{\mu \nu} = \frac{1}{2} \epsilon_{\mu \nu \rho \sigma} \partial_\nu f^a_{\rho \sigma}, \quad (3.18)
$$

where the Abelian field strength is given by

$$
f^a_{\mu \nu} := \partial_\mu a^a_\nu - \partial_\nu a^a_\mu. \quad (3.19)
$$

Due to the topological conservation law, $\partial_\mu k^a_\mu = 0$ the magnetic monopole current denotes a closed loop in four dimensions. The respective magnetic monopole is characterized by an integer-valued topological (magnetic) charge $Q^a_m := \int d^3 x k^a_\mu(x) \in \mathbb{Z}$ ($\alpha = 1, \ldots, N - 1 = \text{rank}SU(N)$). For the static monopole, the monopole current is given by $k^a_\mu(x) = Q^a_m \delta^3(\mathbf{x}) \delta_\mu \rho$ with $Q^a_m$ being the magnetic charge.

Finally, we must check the finiteness of $\mathcal{R}[A]$, which is necessary to define the Morse function, see section 7. The instanton solution give a finite Yang-Mills action, i.e., $S_{YM}[A] < \infty$ irrespective of the gauge choice, as a consequence of self-duality of the equation. On the other hand, the magnetic monopole solution of $F^a[A] = 0$
must give a finite \( R[A] \), i.e., \( R[A] < \infty \). This condition leads to the finiteness of the gauge-fixing action,

\[
S_{GF}[\Omega] < \infty,
\]

in the MA gauge. In the gauge-fixed formulation of the quantum gauge field theory, the gauge fixing part \( S_{GF} \) is important as well as the Yang-Mills action, \( S_{YM} \). In what follows, it is very convenient to separate the pure gauge piece in the gauge potential,

\[
\tilde{\Omega}_\mu(x) := \frac{i}{g} \tilde{U}(x) \partial_\mu \tilde{U}^\dagger(x) = \tilde{\Omega}_\mu^A(x) T^A, \quad \tilde{U} \in G/H.
\]

(3.21)

For \( G = SU(2) \) and \( H = U(1) \), it is shown that the Abelian gauge potential calculated as

\[
a_i(x) = \text{tr}[\frac{1}{2} \sigma_3 \tilde{\Omega}_i(x)](i = 1, 2, 3)
\]

(3.22)

agrees exactly with the well-known static potential for the Dirac magnetic monopole [4, 5, 6], see e.g. [11].

From the mathematical point of view, the existence of magnetic monopole is consistent with the following relation for the Homotopy group,

\[
\pi_2(G/H) = \pi_1(H) \quad \text{when} \quad \pi_2(G) = 0.
\]

(3.23)

Usually, the pure Yang-Mills theory does not have magnetic monopole as a stable topological soliton. This is consistent with

\[
\pi_2(G) = 0.
\]

(3.24)

Therefore, for the existence of the magnetic monopole in pure Yang-Mills theory, the coset structure \( G/H \) is an indispensable ingredient. For \( G = SU(N) \), magnetic monopoles of \( N-1 \) species are expected for the maximal torus group \( H = U(1)^{N-1} \), since

\[
\pi_2(SU(N)/U(1)^{N-1}) = \pi_1(U(1)^{N-1}) = \mathbb{Z}^{N-1},
\]

(3.25)

whereas

\[
\pi_2(SU(N)) = 0 (N = 2, 3, \cdots).
\]

(3.26)

4 \quad Deformation of a topological field theory

4.1 \quad Change of field variables

For the decomposition of field variable,

\[
A_\mu(x) = \Omega_\mu(x) + Q_\mu(x),
\]

(4.1)
it is possible to identify the gauge transformation
\[
\delta \mathcal{A}_\mu(x) = D_\mu[A]\omega(x) := \partial_\mu \omega(x) - ig[A_\mu(x), \omega(x)],
\] (4.2)
with a set of transformations
\[
\begin{align*}
\delta \Omega_\mu(x) &= D_\mu[\Omega]\omega(x), \\
\delta Q_\mu(x) &= ig[\omega(x), Q_\mu(x)].
\end{align*}
\] (4.3)
(4.4)

Here (4.3) and (4.4) correspond to (2.18) and (2.20) respectively. Note that \(\Omega_\mu\) transforms as a gauge field, while \(Q_\mu\) as a adjoint matter field.

When \(A_\mu\) is given by a finite gauge rotation (large gauge transformation) \(U(x)\) of \(V_\mu\), we take the following identification,
\[
\begin{align*}
\Omega_\mu(x) &= i g U(x) \partial_\mu U^\dagger(x), \\
Q_\mu(x) &= U(x) Q_\mu(x) U^\dagger(x).
\end{align*}
\] (4.5)

Therefore, the BGF gauge for \(Q_\mu\),
\[
D_\mu[\Omega] Q_\mu(x) = 0,
\] (4.8)
is equivalent to the Lorentz gauge for \(V_\mu\),
\[
\partial_\mu V_\mu(x) = 0,
\] (4.9)
under the identification of the variables (4.5). Under (4.5), we can rewrite (4.3) as
\[
\delta U \partial_\mu U^\dagger + U \partial_\mu \delta U^\dagger = ig \omega U \partial_\mu U^\dagger - ig U \partial_\mu (U^\dagger \omega),
\] (4.10)
and (4.4) as
\[
\delta U V_\mu U^\dagger + U V_\mu \delta U^\dagger + U \delta V_\mu U^\dagger = ig \omega U V_\mu U^\dagger - ig U V_\mu U^\dagger \omega.
\] (4.11)

Therefore, in the BGF gauge, (4.3) and (4.4) reduce to a set of transformations,
\[
\begin{align*}
\delta U(x) &= i g \omega(x) U(x), \\
\delta V_\mu(x) &= 0,
\end{align*}
\] (4.12)
(4.13)
since the gauge degrees of freedom for \(V_\mu\) (small gauge transformation) is fixed by the Lorentz gauge (4.9). In what follows, we assume that the non-compact gauge
field variable $V_{\mu}(x)$ does not have topologically nontrivial configuration and all topologically nontrivial contributions come from the compact gauge group variable $U(x)$ alone. We treat $V_{\mu}(x)$ and $U(x)$ as if they are independent variables. The topological soliton (magnetic monopole) is derived as a solution of the nonlinear equation for $\Omega_{\mu}$ which follows from the nonlinear gauge fixing condition (MA gauge). The local gauge invariance of $\tilde{Z}[J, \Omega]$ written in terms of $\Omega_{\mu}$ and $Q_{\mu}$ reduces to the invariance under the transformation (4.12), i.e.

$$U(x) \to e^{ig\omega(x)}U(x).$$

The measure $[d\Omega]$ invariant under (2.18) is replaced by the invariant Haar measure $[dU]$ which is invariant under the local gauge rotation (4.14).

### 4.2 BRST formalism

First, we rewrite the BRST formulation of BGFM in terms of new variables. By making the change of variable (4.5) which is a gauge transformation of $V_{\mu}(x)$ by $U(x)$, it turns out that the BRST transformation (2.31) for the variables $\Omega_{\mu}, Q_{\mu}, \tilde{C}, \bar{C}, \tilde{B}$ is rewritten into

$$\tilde{\delta}_B U(x) = 0,$$

$$\tilde{\delta}_B V_{\mu}(x) = \mathcal{D}_\mu[V] \gamma(x),$$

$$\tilde{\delta}_B \gamma(x) = \frac{i}{2} g[\gamma(x), \gamma(x)],$$

$$\tilde{\delta}_B \bar{\gamma}(x) = i \beta(x),$$

$$\tilde{\delta}_B \beta(x) = 0,$$

(4.15)

where $V_{\mu}, \gamma, \bar{\gamma}, \beta$ are the adjoint rotation of $Q_{\mu}, \tilde{C}, \bar{C}, \tilde{B}$ respectively,

$$V_{\mu} := U^\dagger Q_{\mu} U, \quad \gamma := U^\dagger \tilde{C} U, \quad \bar{\gamma} := U^\dagger \bar{C} U, \quad \beta := U^\dagger \tilde{B} U. \quad (4.16)$$

Under the adjoint rotation (4.16), the measure is invariant,

$$[dV][d\gamma][d\bar{\gamma}][d\beta] = [dQ][d\tilde{C}][d\bar{C}][d\tilde{B}].$$

(4.17)

The Yang-Mills action is invariant

$$ S_{YM}[A] = S_{YM}[\Omega + Q] = S_{YM}[V].$$

(4.20)

For the definition of the field strength

$$F_{\mu\nu}[A] := \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} - ig[A_{\mu}, A_{\nu}],$$

(4.18)

the change of variable $A_{\mu} = U V_{\mu} U^\dagger + \frac{i}{g} U \partial_{\mu} U^\dagger$ leads to

$$F_{\mu\nu}[A] = U F_{\mu\nu}[V] U^\dagger + \frac{i}{g} U [\partial_{\mu}, \partial_{\nu}] U^\dagger.$$

(4.19)

Note that the second term $F^\ast_{\mu\nu} := \frac{i}{g} U [\partial_{\mu}, \partial_{\nu}] U^\dagger$ can have a nonzero value and may modify the action. If $U(x) \in G$ is restricted to the coset $G/H$, it may yield a line-like singularity. For example,
The gauge fixing part (2.32) for the BGF gauge is transformed into
\[
\tilde{S}_{GF}[\mathcal{V}, \gamma, \bar{\gamma}, \beta] := -\int d^Dx \, i\tilde{\delta}_B \operatorname{tr}_G \left[ \bar{\gamma} \left( \partial_\mu \mathcal{V}_\mu + \frac{\bar{\alpha}}{2} \beta \right) \right]
\]
(4.21)
\[
= \int d^Dx \, \operatorname{tr}_G \left[ \beta \partial_\mu \mathcal{V}_\mu + \frac{\bar{\alpha}}{2} \beta + i\bar{\gamma} \partial_\mu \mathcal{D}_\mu[\mathcal{V}] \gamma \right],
\]
(4.22)
where we have used (4.7) and (4.16). It turns out that the gauge fixing condition for \(\mathcal{V}_\mu\) field is given by the Lorentz gauge (4.9). Note that (4.22) agrees with the form given in [12]. This BRST transformation corresponds to the small gauge transformation which does not change the topology of the gauge field. Thus the generating functional (2.30) is transformed as
\[
\tilde{Z}[J, \Omega] = \int [d\mathcal{V}] [d\gamma] [d\bar{\gamma}] [d\beta] \exp \left\{ i[S_{YM}[\mathcal{V}] + \tilde{S}_{GF}[\mathcal{V}, \gamma, \bar{\gamma}, \beta] + (J_\mu \cdot \mathcal{U} \mathcal{V}_\mu U^\dagger)] \right\}
\]
(4.23)
Next, we consider the total generating functional
\[
Z[J] = \int [d\Omega_\mu][d\bar{C}][dB] \tilde{Z}[J, \Omega] \exp(iS_{GF}[\Omega, C, \bar{C}, B]) \exp[i(J_\mu \cdot \Omega_\mu)]
\]
(4.24)
\[
= \int [d\Omega_\mu][dC][d\bar{C}][dB] \exp\{i\tilde{S}_{efl}[J, \Omega] + iS_{GF}[\Omega, C, \bar{C}, B] + i(J_\mu \cdot \Omega_\mu)\}
\]
(4.25)
where we have introduced the source term \((J_\mu \cdot \Omega_\mu)\) for the background field. We introduce the BRST transformation,
\[
\delta_B \Omega_\mu(x) = \mathcal{D}_\mu[\Omega] C(x) := \partial_\mu C(x) - ig[\Omega_\mu(x), C(x)],
\]
\[
\delta_B C(x) = i\frac{1}{2} g[C(x), C(x)],
\]
\[
\delta_B \bar{C}(x) = iB(x),
\]
\[
\delta_B B(x) = 0,
\]
(4.26)
and the anti-BRST transformation,
\[
\bar{\delta}_B \Omega_\mu(x) = \mathcal{D}_\mu[\Omega] \bar{C}(x) := \partial_\mu \bar{C}(x) - ig[\Omega_\mu(x), \bar{C}(x)],
\]
\[
\bar{\delta}_B \bar{C}(x) = i\frac{1}{2} g[C(x), \bar{C}(x)],
\]
\[
\bar{\delta}_B C(x) = i\bar{B}(x),
\]
\[
\bar{\delta}_B \bar{B}(x) = 0,
\]
(4.27)
where \(\bar{B}\) is defined by
\[
B(x) + \bar{B}(x) = g[C(x), \bar{C}(x)].
\]
(4.28)

It is possible to have \(F_{x y} := \frac{1}{E^2} \delta(x) \delta(y) \delta(z) \sigma_3\) which corresponds to the presence of Dirac string extending into the direction of the negative \(z\) axis from the origin, see e.g. Appendix C of [11]. Here the factor \(\frac{1}{E^2}\) corresponds to the magnetic charge. The same contribution as the Dirac string can be incorporated by taking into account the magnetic monopole instead of the Dirac string, as shown in [11]. Moreover, the existence of such terms introduces rather singular terms in the action. Thus, we do not consider the effect of this term in what follows.
The BRST and anti-BRST transformations have the following properties,
\[
(\delta_B)^2 = 0, \quad (\bar{\delta}_B)^2 = 0, \quad \{\delta_B, \bar{\delta}_B\} := \delta_B \bar{\delta}_B + \bar{\delta}_B \delta_B = 0. \tag{4.29}
\]

In what follows, we consider the variable \(U(x)\) as the fundamental variable instead of \(\Omega(\mu)(x)\). Then the BRST and anti-BRST transformations for \(U\) and \(\mathcal{V}_{\mu}\) are given by
\[
\delta_B U(x) = i g C(x) U(x), \quad \bar{\delta}_B U(x) = i g \bar{C}(x) U(x), \tag{4.30}
\]
and
\[
\delta_B \mathcal{V}_{\mu}(x) = 0 = \bar{\delta}_B \mathcal{V}_{\mu}(x), \tag{4.31}
\]
which are the BRST version of (4.12) and (4.13) respectively. In fact, (4.30) reproduces the usual BRST (4.26) and anti-BRST (4.27) transformations of the gauge field,
\[
\Omega_{\mu}(x) := \frac{i}{g} U(x) \partial_{\mu} U^\dagger (x). \tag{4.32}
\]
Note that (4.30) and (4.31) lead to
\[
\delta_B \Omega_{\mu}(x) = \mathcal{D}_{\mu}[\Omega] C(x), \tag{4.33}
\]
\[
\delta_B \mathcal{Q}_{\mu}(x) = i g [C(x), \mathcal{Q}_{\mu}(x)]. \tag{4.34}
\]
These are the BRST version of (4.3) and (4.4) respectively, since within the BGFM, \(\Omega_{\mu}, C, \bar{C}, B\) or \(U, C, \bar{C}, B\) are external fields in the sense that they are not integrated out in the measure \([d\mathcal{V}][d\bar{C}][d\bar{\mathcal{C}}][dB]\). Thus the generating functional of the total Yang-Mills theory reads
\[
Z[J] = \int [dU][dC][d\bar{C}][dB] \exp\{i\tilde{S}_{\text{eff}}[J, U] + iS_{GF}[\Omega, C, \bar{C}, B] + i(J_{\mu} \cdot \Omega_{\mu})\}, \tag{4.35}
\]
where \([dU]\) is the invariant Haar measure and we have redefined
\[
\tilde{S}_{\text{eff}}[J, U] := -i \ln \tilde{Z}[J, \Omega]. \tag{4.36}
\]

In order to realize the magnetic monopole background, we adopt the MA gauge for which the gauge fixing and the FP ghost terms are written in the form [11]
\[
S_{GF}[\Omega, C, \bar{C}, B] := - \int d^D x \ i \delta_B \mathrm{tr}_{G/H} \left[ \bar{C} \left( F[\Omega] + \frac{\alpha}{2} B \right) \right], \tag{4.37}
\]
where the trace is taken on the coset \(G/H\), not the entire \(G\). For \(G = SU(2)\),
\[
F^a[\Omega] := (\partial^\mu \delta^{ab} - \epsilon^{a3b} \Omega^3_{\mu}) \Omega^b_{\mu} := D^{\mu ab} [\Omega^3_{\mu}] \Omega^b_{\mu} \quad (a, b = 1, 2). \tag{4.38}
\]

By adding an BRST-exact ghost self-interaction term, (4.37) is cast into the more convenient form [12],
\[
S_{GF}'[\Omega, C, \bar{C}, B] := \int d^D x \ i \delta_B \bar{\delta}_B \mathrm{tr}_{G/H} \left[ \frac{1}{2} \Omega_{\mu}(x) \Omega_{\mu}(x) - \frac{\alpha}{2} i C(x) \bar{C}(x) \right]. \tag{4.39}
\]
From (4.29), \( S'_{GF} \) is invariant under the BRST and anti-BRST transformations,

\[
\delta_B S'_{GF} = 0 = \bar{\delta}_B S'_{GF}.
\]

(4.40)

This action \( S'_{GF} \) describes the topological soliton derived from the nonlinear equation \( F[\Omega] = 0 \), the monopole equation. This action is BRST exact and hence there is no local degrees of freedom propagating in spacetime. It describes the quantity related to the global topology, as though the Chern-Simons theory describes the linking of knots [51]. We call the theory with the BRST exact action \( S'_{GF} = S_{TQFT} \) alone the topological quantum field theory (TQFT). The generating functional is given by

\[
Z_{TQFT}[J] = \int [dU][dC][d\bar{C}][dB] \exp\{iS_{TQFT}[\Omega, C, \bar{C}, B] + i(J_\mu \cdot \Omega_\mu)\}.
\]

(4.41)

In view of this, the above reformulation of the Yang-Mills theory was called the deformation of the TQFT.

In the above rederivation, the fact that the field \( \Omega_\mu \) behaves as if it is a gauge field \( A_\mu \) is essential. This is guaranteed by the BGFM.

### 4.3 Expectation value

In our formulation, an arbitrary function \( f(A) \) of \( A \) is written as \( f(A) = g(V_\mu, U) h(U) \) by making the change of variable (1.1) and (1.3). Then the expectation value is evaluated as

\[
\langle f(A) \rangle_{YM} = \langle g(V_\mu, U) h(U) \rangle_{pYM}^{\text{TQFT}} = \langle g(V_\mu, U) h(U) \rangle_{pYM}^{\text{TQFT}}.
\]

(4.42)

Here, taking the expectation value \( \langle \cdot \rangle_{\text{TQFT}}^{U} \) corresponds to summing over topological soliton contributions by making use of the TQFT described by the variable \( U \), whereas \( \langle \cdot \rangle_{pYM}^{U} \) denotes the expectation value for the deformation piece which is described by the usual Yang-Mills theory with the variable \( V_\mu \) (Here \( p \) denotes the perturbative). Of course, we can change the ordering of taking the expectation value,

\[
\langle f(A) \rangle_{YM} = \langle g(V_\mu, U) h(U) \rangle_{pYM}^{\text{TQFT}}.
\]

(4.43)

Both expressions should give the same result, if they are calculated exactly.

Under the assumption of perturbative deformation, the expectation \( \langle g(V_\mu, U) \rangle_{pYM} \) is calculated by expanding the integrand \( g(V_\mu, U) \) into power series in \( V_\mu \). This is a minimal assumption in the practical calculation. After that, \( \langle g(V_\mu, U) \rangle_{pYM} \) is still a function of \( U \), say, \( p(U) \). Finally, the expectation \( \langle p(U) h(U) \rangle_{\text{TQFT}} \) must be evaluated in the non-perturbative way, since this piece estimates the soliton contribution. Perturbative deformation is an assumption that the deformation part is evaluated in the perturbation theory in the coupling constant \( g \). In other words, all the essential non-perturbative contributions are provided with the topological soliton described by the TQFT. Actually, this strategy was performed in the evaluation of the Wilson loop [12, 13].

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5 Different reformulations based on the similar idea have been presented by many authors, e.g., by Hata and Taniguchi [52], and Fucito, Martellini and Zeni [53].
4.4 Abelian-projected effective gauge theory

The above result should be compared with the previous formulation [22, 11] which begins with the generating functional,

\[ Z[J] = \int [dA][dC][dB] \exp \{ iS_{YM}[A] + iS_{GF}[A, C, \bar{C}, B] + i(J_\mu \cdot A_\mu) \}. \] (4.44)

First, following the Cartan decomposition (3.14), the non-Abelian gauge field was decomposed into the diagonal and the off-diagonal pieces,

\[ A_\mu = A_\mu^A T^A + a_\mu^a T^a. \] (4.45)

Then the MA gauge was imposed as a gauge fixing condition. Finally, all the off-diagonal fields taking values in the Lie algebra of the coset \( G/H \) were integrated out in the functional integral,

\[ Z[J] = \int [da][dC][dB] \exp \{ iS_{\text{diag}}[a^i, C^i, \bar{C}^i, B^i] + i(J_\mu \cdot a_\mu) \}, \] (4.46)

where

\[ Z[a^i, C^i, \bar{C}^i, B^i] := \exp \{ iS_{\text{diag}}[a^i, C^i, \bar{C}^i, B^i] \} \] (4.47)

\[ := \int [dA][dC][dB] \exp \{ iS_{YM}[A] + iS_{GF}[A, C, \bar{C}, B] + i(J_\mu \cdot A_\mu) \}. \] (4.48)

The theory with the action \( S_{\text{diag}}[a^i, C^i, \bar{C}^i, B^i] \) was called the Abelian-projected effective gauge theory (APEGT). It has been shown [22, 11] that the APEGT has the same beta function as the original Yang-Mills theory, exhibiting the asymptotic freedom, although the APEGT is an Abelian gauge theory.

It turns out that the previous strategy presented in [22, 11] is equivalent to the above formulation presented in this paper and that the results obtained in the previous works are the immediate consequence of the present formulation, if we identify the diagonal and off-diagonal fields with the background field and the quantum fluctuation respectively, i.e.,

\[ \Omega_\mu = a_\mu^i T^i, \quad Q_\mu = A_\mu^a T^a. \] (4.49)

The theory with an action \( S_{\text{diag}}[a^i, C^i, \bar{C}^i, B^i] \) is written in terms of only the diagonal fields. As long as the BGF gauge is imposed on the off-diagonal field \( A_\mu \), this theory becomes the Abelian gauge theory, since the BGFM guarantees that the background field \( a_\mu \) transforms as a gauge field (Of course, the diagonal field is reduced to the Abelian gauge field in this case). Indeed, the BGF gauge \( D^{ab}[a] A^b = 0 \) is nothing but the MA gauge. Hence the coincidence of the beta function is understood from the BGFM.

5 Strategy of a derivation of quark confinement

We consider the D-dim. QCD (QCD\(_D\)) with a gauge group G for \( D > 2 \). The (full) non-Abelian Wilson loop is defined as the path-ordered exponent along a loop \( C \),

\[ W^C[A] := \text{tr} \left[ P \exp \left( ig \oint_C A_\mu^A(x) T^A dx^\mu \right) \right] / \text{tr}(1). \] (5.1)
We define the (full) string tension $\sigma$ by

$$\sigma := - \lim_{A(C) \to \infty} \frac{1}{A(C)} \ln \langle W^C[A] \rangle,$$

(5.2)

where $A(C)$ is the minimal area spanned by the Wilson loop $C$. The non-zero string tension $\sigma \neq 0$ implies that the Wilson loop expectation value behaves for large loop as

$$\langle W^C[A] \rangle \sim \exp(-\sigma A(C)).$$

(5.3)

This is called the area (decay) law. The static potential $V(R)$ for a pair of quark and anti-quark is evaluated from the rectangular Wilson loop $C$ with sides $T$ and $R$ ($A(C) = TR$) according to

$$V(R) = - \lim_{T \to \infty} \frac{1}{T} \ln \langle W^C[A] \rangle.$$

(5.4)

The area law of the Wilson loop or non-zero string tension $\sigma \neq 0$ implies the existence of the linear part $\sigma R$ in the static potential $V(R)$, leading to quark confinement.

In a series of papers \cite{11, 12, 14, 13, 15}, a derivation of the area law of the Wilson loop in \textit{QCD} has been given in the following steps.

Step 1: Reformulating the Yang-Mills theory as a deformation of a TQFT in MA gauge \cite{12}

Step 2: Parisi-Sourlas Dimensional reduction \cite{12}

Step 3: Abelian magnetic monopole dominance \cite{13}

Step 4: Instanton calculus \cite{12} or large $N$ expansion \cite{54}

The first two steps are shown schematically as follows.

The following analyses within the above reformulation (the perturbative deformation of a TQFT) is based on an assumption that the contribution from $V_\mu(x)$ can be treated in perturbation theory in the gauge coupling constant $g$. This assumption leads to the topological sector dominance in the sense that the area law contribution comes solely from the contribution of $U(x)$ described by the TQFT and the remaining part $V_\mu(x)$ does not contribute to the area law of the Wilson loop.
5.1 Step 1: Reformulating the Yang-Mills theory as a deformation of a TQFT in MA gauge

QCD$_D$ is reformulated as a deformation of a TQFT$_D$ in MA gauge. The MA gauge is a partial gauge fixing such that the coset part $G/H$ of the gauge group $G$ is fixed and the maximal torus group $H$ is left as a residual gauge group.

For $G = SU(2)$, it has been shown that the expectation value of the non-Abelian Wilson loop is rewritten using the non-Abelian Stokes theorem into

$$\langle W^C[A] \rangle_{YM} = \langle \exp \left[ igJ \int_C dx^\mu n^A(x) A^\mu(x) \right] \rangle_{pYM} \exp \left[ iJ \int_S d^2 z \epsilon_{\mu\nu} \cdot \left( \partial_\mu n \times \partial_\nu n \right) \right]_{TQFT}$$

where $S$ is a surface with a boundary $C$ and $n(x) = (n^1(x), n^2(x), n^3(x))$ is the three-dimensional unit vector ($n(x) \cdot n(x) = 1$) defined by

$$n^A(x)T^A = U^\dagger(x)T^3U(x), \quad T^A = \frac{1}{2} \sigma^A \quad (A = 1, 2, 3).$$

Here $J$ specifies the representation of the fermion in the definition of the Wilson loop and $J = 1/2$ corresponds to the fundamental representation.

5.2 Step 2: Parisi-Sourlas dimensional reduction

It has been shown that TQFT$_D$ is equivalent to the coset $G/H$ nonlinear sigma model (NLSM) in $(D-2)$ dimensions, NLSM$_{D-2}$. This is a consequence of Parisi-Sourlas dimensional reduction due to the supersymmetry hidden in TQFT (4.41). This is an advantage that we have chosen the MA gauge.

As extensively discussed more than 20 years ago, QCD$_4$ and NLSM$_2$ have various common properties: renormalizability, asymptotic freedom (i.e., negative beta function $\beta(g) < 0$), dynamical mass generation, existence of instanton solution, no phase transition for any value of coupling constant (i.e., one phase), etc. This similarity between two theories can be understood from this correspondence,

$$QCD_4 \supset TQFT_4 \leftrightarrow G/H \text{ NLSM}_2.$$  (5.7)

For the $SU(N)$ gauge group, the existence of 2D instanton is guaranteed for any $N$, because $\pi_2(SU(N)/U(1)^{N-1}) = \pi_1(U(1)^{N-1}) = \mathbb{Z}^{N-1}$. See the second paper in [54] for details.

For $G = SU(2)$, G/H NLSM is nothing but the O(3) NLSM. For the planar Wilson loop, the evaluation of the expectation value $\langle \cdot \rangle_{TQFT}$ in TQFT$_4$

$$\langle W^C[A] \rangle_{YM} = Z_{TQFT}^{-1} \int [dU(x)]_{x \in \mathbb{R}^4} \exp(-S_{TQFT}(U))p(U)h(U)$$

is reduced to that in the coset $G/H$ NLSM$_2$

$$\langle W^C[A] \rangle_{YM} = Z_{NLSM}^{-1} \int [dn(x)]_{x \in \mathbb{R}^2} \exp(-S_{NLSM}(n))p(U)h(U),$$  (5.9)
where we have used the notation (4.42) with

\[
  h(U) := \exp \left[ iJ \int_S d^2z \epsilon_{\mu\nu} \mathbf{n} \cdot (\partial_\mu \mathbf{n} \times \partial_\nu \mathbf{n}) \right], 
\]

(5.10)

\[
p(U) := \left\langle \exp \left[ igJ \oint_C dx^\mu n^A(\mathbf{x}) \mathcal{V}^A_\mu(\mathbf{x}) \right] \right\rangle_{pYM}.
\]

(5.11)

In the original Lagrangian of QCD, the scalar field is not included as an elementary field, but it appears as a composite field according to (5.6). The unit vector \( \mathbf{n}(x) \) plays the same role as the monopole scalar field \( \phi(x) \) which describes the 't Hooft-Polyakov monopole \[55, 56\], for \( G = SU(2) \), i.e.,

\[
n^A(\mathbf{x}) \leftrightarrow \hat{\phi}^A(\mathbf{x}) := \frac{\phi^A(\mathbf{x})}{|\phi(\mathbf{x})|}, \quad |\phi(\mathbf{x})| := \sqrt{\phi^A(\mathbf{x})\phi_A(\mathbf{x})}.
\]

(5.12)

### 5.3 Step 3: Abelian magnetic monopole dominance

The diagonal (or Abelian) string tension \( \sigma_{Abel} \) is defined by

\[
\sigma_{Abel} := - \lim_{A(C) \to \infty} \frac{1}{A(C)} \ln \left\langle W^C[\Omega] \right\rangle_{TQFT4},
\]

(5.13)

by making use of the diagonal Wilson loop,

\[
W^C[\Omega] = \exp \left( igJ \oint_C dx^\mu a^\Omega_\mu(\mathbf{x}) \right),
\]

(5.14)

\[
a^\Omega_\mu(\mathbf{x}) := \Omega^3_\mu(\mathbf{x}) := \text{tr}(T^3\Omega_\mu(\mathbf{x})), \quad \Omega_\mu(\mathbf{x}) := \frac{i}{g} U(\mathbf{x}) \partial_\mu U(\mathbf{x})^\dagger.
\]

(5.15)

Owing to the dimensional reduction, we find

\[
\left\langle W^C[\Omega] \right\rangle_{TQFT4} = \left\langle W^C[\Omega] \right\rangle_{NLSM2},
\]

(5.16)

where

\[
\left\langle W^C[\Omega] \right\rangle_{NLSM2} = Z_{NLSM2}^{-1} \int [d\mathbf{n}(\mathbf{x})]_{\mathbf{x} \in \mathbb{R}^2} \exp(-S_{NLSM2}[\mathbf{n}]) W^C[\Omega].
\]

(5.17)

Then it is shown that, in the limit of large Wilson loop, two string tensions agree with each other, \( \sigma = \sigma_{Abel} \), since

\[
\frac{1}{A(C)} \left[ \ln \left\langle W^C[\mathbf{A}] \right\rangle_{YM4} - \ln \left\langle W^C[\Omega] \right\rangle_{NLSM2} \right] \downarrow 0 \quad (A(C) \uparrow \infty),
\]

(5.18)

if we identify the deformation with the perturbative one. Thus, for the large planar (non-intersecting) Wilson loop, the full string tension \( \sigma \) is saturated by the diagonal string tension \( \sigma_{Abel} \). This explains the Abelian dominance and magnetic monopole dominance.

This result is derived as follows. In calculating (5.9), if we put \( p(U) \equiv 1 \), then \( \left\langle W^C[\mathbf{A}] \right\rangle_{YM4} \) coincides with \( \left\langle W^C[\Omega] \right\rangle_{NLSM2} \), since it is shown \[12\] that \( h(U) \equiv
\[ W^C[a^\Omega] \]. In our framework called the perturbative deformation of TQFT, \( p(U) \) is estimated by making use of the power-series expansion in the coupling constant \( g \) (or in the ’t Hooft coupling \( \lambda := g^2 N \) in the framework of large \( N \) expansion, see [54]) as

\[
p(U) = 1 + \sum_{n=1}^\infty \frac{(ig)^n}{n!} \left\langle \left( \int_C dx^\mu n^\mu A(x) V^A(x) \right)^n \right\rangle_{pYM} = 1 + \sum_{n=1}^\infty \frac{(igJ)^n}{n!} \int_C dx_1^\mu \cdots \int_C dx_n^\mu n^\mu A_1(x_1) \cdots n^\mu A_n(x_n)
\times \left\langle V^A_{\mu_1}(x_1) \cdots V^A_{\mu_n}(x_n) \right\rangle_{pYM}.
\] (5.19)

By calculating the expectation value \( \left\langle V^A_{\mu_1}(x_1) \cdots V^A_{\mu_n}(x_n) \right\rangle_{pYM} \), we can express \( p(U) \) in terms of the \( n(x) \) fields which are defined on the loop \( C \) embedded in the two-dimensional space. However, it is shown that the additional contribution from \( p(U) - 1 \) to the expectation value of the Wilson loop does not have the the area decay part. Incidentally, although the perturbative deformation part is insufficient to derive the area law, it leads to the running coupling constant which is governed by the renormalization group \( \beta \) function of the original Yang-Mills theory in consistent with the asymptotic freedom.

### 5.4 Step 4: Instanton calculus

The whole problem is reduced to calculating the diagonal Wilson loop in NLSM2,

\[
\left\langle W^C[a^\Omega] \right\rangle_{\text{NLSM}_2} = \left\langle e^{2\pi JQ_S} \right\rangle_{\text{NLSM}_2}, \quad Q_S = \frac{1}{8\pi} \int_S d^2z \epsilon_{\mu\nu} n \cdot (\partial_\mu n \times \partial_\nu n). \quad (5.20)
\]

Note that the integrand of \( Q_S \) is the instanton density in NLSM2. Therefore, \( Q_S \) counts the number of instantons minus that of anti-instantons inside the area \( S(\subset \mathbb{R}^2) \) bounded by the Wilson loop \( C \). This suggests that the quark confinement follows from the condensation of topological soliton, the magnetic monopole.

In this step we have employed the naive instanton calculus to calculate the diagonal Wilson loop. In the dilute gas approximation the two-dimensional instanton contributions are summing up according to

\[
\sum_{n_+ = 0}^{\infty} \sum_{n_- = 0}^{\infty} \frac{1}{n_+! n_-!} \int \prod_{i=1}^n d^2z_i \int \prod_{i=1}^n d\mu(\rho_i) \exp[-(n_+ + n_-)S_1(g)] e^{2\pi JQ_S}, \quad (5.21)
\]

where the action of NLSM2 is replaced by \( (n_+ + n_-)S_1(g) \) using the numbers of instanton and anti-instanton \( n_+ , n_- \) and the action for one instanton \( S_1(g) = 4\pi^2/g^2 \) in NLSM2. Thus the (infinite dimensional) functional integral measure \( \left| d\mathbf{n}(x) \right|_{x \in \mathbb{R}^2} \) has been replaced with the (finite dimensional) integration with respect to the collective coordinates, \( z_i \) (position of the instanton) and \( \rho_i \) (size of the instanton). Such reduction of degrees of freedom in the functional integration is a common feature in TQFT as shown in section 7.

This leads to the area law of the diagonal Wilson loop and the non-zero diagonal string tension \( \sigma_{\text{Abel}} \) for half odd integer \( J \) or the fractional charge \( q \).
5.5 Area law and quark confinement

In the framework of the deformation of a TQFT for the Yang-Mills theory, the non-zero string tension $\sigma$ in QCD$_4$ follows from the non-zero diagonal string tension $\sigma_{Abel}$ in NLSM$_2$. The problem of proving area law in QCD$_4$ is reduced to the corresponding problem in NLSM$_2$.

All the above steps are exact except for the instanton calculus of the Wilson loop in NLSM$_2$. For sufficiently large and planar Wilson loop, it was shown [12, 13] that the string tension is given by

$$\sigma = 2Be^{-S_1[1 - \cos(2\pi J)]}, \quad S_1 = \frac{2\pi^2\alpha}{g^2},$$

where $B$ is a constant with the mass-squared dimension coming from the integration over the instanton size $\int d\mu(\rho)$ and $S_1$ is the action for one instanton in NLSM$_2$.

The result (5.22) shows that for half odd integers $J = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots$, the Wilson loop exhibits area law for sufficiently large Wilson loop $C$, whereas the area law and the linear potential disappears for integers $J = 1, 2, 3, \cdots$. Therefore, the fundamental fermion $J = \frac{1}{2}$ is confined, while the adjoint fermion $J = 1$ can not be confined.

In the above formulation using the MA gauge, it is the compact residual Abelian group that plays the essential role in evaluating the gauge invariant quantity. This feature is very similar to the situation in the lattice gauge theory. In fact, the result (5.22) is a consequence of the periodicity (or compactness) of the residual Abelian gauge group, i.e., maximal torus group $U(1)$ of $SU(2)$, in the variable $U$ after the MA gauge is chosen. On the other hand, the gauge degrees of freedom for the non-compact field $V_\mu$ have been completely fixed by the gauge fixing condition of Lorentz type. The explicit expression (5.22) depends on the approximation taken in the instanton calculus, the periodicity of the string tension (hence the absence of string tension for $J = 1, 2, \cdots$) does not depend on the approximation.

It is the perturbative part that gives the running of the coupling constant $g$. The running is governed by the renormalization group beta function $\beta(g)$. The usual Yang-Mills$_4$ theory exhibits asymptotic freedom, e.g., for $G = SU(N_c)$ at one-loop level,

$$\beta(g) := \frac{dg(\mu)}{d\mu} = -\frac{b_0}{16\pi^2}g(\mu)^3 + \cdots, \quad b_0 = \frac{11N_c}{3} > 0.$$  

(5.23)

In our framework, the correct beta function is derived based on the BGFM, see [22, 11]. For the static potential $V(R)$, the perturbative part gives a Coulomb potential contribution $\alpha(\mu)/R$ where $\alpha(\mu) := g^2(\mu)/4\pi$ runs according to the $\beta(g)$.

Similar strategy can also be applied to QED$_4$ ($G = U(1)$) to prove the existence of strong coupling confinement phase [4]. This follows from the existence of Berezinski-Kosterlitz-Thouless transition of the O(2) NLSM$_2$. The corresponding steps are shown as follows.
This result enables us to give another derivation of quark confinement in QCD based on the low-energy effective Abelian gauge theory [11], see [13]. This viewpoint is more interesting in the sense that the confinement-deconfinement transition can be discussed within the same framework.

5.6 Remarks and unresolved issues

The dilute gas approximation can be improved. More systematic instanton calculations enable us to identify an instanton solution with the Coulomb gas of vortices [57, 58, 59, 60, 61]. Consequently, the low-energy effective Abelian gauge theory belongs to the strong coupling phase where the quark confinement is realized, see [15].

The absence of intermediate Casimir scaling region (i.e., $\sigma = 0$ for integer $J$) may be due to our simplified treatment of the instanton size. In order to obtain the result (5.22) we have treated the instanton as if it is exactly a point-like object in the dilute gas approximation. The Casimir scaling will be explained by taking into account the size effect of the instanton, as performed for the center vortex by Greensite et al. [62].

Recent investigations show that the QCD vacuum is a dual superconductor caused by the condensation of magnetic monopole and that the low-energy effective gauge theory is given the dual Ginzburg-Landau theory. However, numerical simulations claim that the dual superconductor is near type I, rather than type II, see [3]. This result seems to contradict with the analytical studies.

It is desirable to extend the above analyses into more general gauge groups. The case of $G = SU(3)$ will be discussed in forthcoming paper in detail.

5.7 A proposal of numerical calculations

Some of the implications from the above strategy will be checked by direct numerical simulations on the lattice. Due to difficulties of defining supersymmetry on the lattice, it might be impossible to check directly the equivalence between the 4D TQFT and the 2D NLSM. Nevertheless, it is desirable to check the following statements:
1. Validity of perturbative deformation of TQFT: The expectation value of the diagonal Wilson loop in NLSM$_2$,
\[
\langle W^C[a^\Omega] \rangle_{NLSM_2} = \left\langle \exp \left[ i 2\pi J \frac{1}{8\pi} \int_S d^2 z \, \epsilon_{\mu \nu} \mathbf{n} \cdot (\partial_\mu \mathbf{n} \times \partial_\nu \mathbf{n}) \right] \right\rangle_{NLSM_2},
\]
behaves as that of the non-Abelian Wilson loop in Yang-Mills$_4$,
\[
\langle W^C[A] \rangle_{YM_4} = \left\langle \text{tr} \left[ \mathcal{P} \exp \left( ig \oint_C A^A(x) T^A dx^\mu \right) \right] \right\rangle_{YM_4}.
\]
Two string tensions $\sigma_{Abel}$ and $\sigma$ agree with each other.

2. Validity of instanton calculus: Only the instanton contribution in NLSM$_2$ is sufficient to recover the Abelian string tension, $\sigma_{Abel}$.

3. Existence of the scale: The asymptotic scaling holds for the Abelian string tension $\sigma_{Abel}$ calculated from the NLSM$_2$.

The results will prove or disprove validity of our strategy of deriving quark confinement.

6 Gauge fixing and gluon mass

6.1 A naive MA gauge

A simple but ad hoc way to give the mass for the off-diagonal gluon is to introduce the following mass term to the Yang-Mills action,
\[
S_m = \int d^D x tr_{G/H} \left( \frac{1}{2} m^2 A_\mu A_\mu \right).
\]
This introduce the mass of the off-diagonal gluons in the tree level and this explicitly break the gauge invariance corresponding to $G/H$. Indeed, the mass term (6.1) is derived as a gauge fixing term as follows. The simplest MA gauge where the off-diagonal part is made as small as possible will be the following gauge,
\[
F^a = A^a_\mu = 0.
\]
In order to write the gauge-fixing action, we must introduce the vector auxiliary field $B_\mu$ and the vector FP ghost $C_\mu$ and anti-ghost $\bar{C}_\mu$, so that
\[
S_{GF} = -\int d^D x \, i \delta_B tr_{G/H} \left[ C_\mu \left( A_\mu + \frac{\alpha}{2} B_\mu \right) \right],
\]
where the nilpotent BRST transformation is constructed as
\[
\delta_B A_\mu \quad = \quad C_\mu,
\]
\[
\delta_B C_\mu \quad = \quad 0,
\]
\[
\delta_B \bar{C}_\mu \quad = \quad i B_\mu,
\]
\[
\delta_B B_\mu \quad = \quad 0.
\]

$^6$ This BRST transformation does not leave the Yang-Mills action invariant, unless the equation of motion is used. So, it is unusual when we include the Yang-Mills action.
Eliminating the auxiliary field $B_\mu$, we reproduce the mass term,

$$ S_{GF} = \int d^4x \, \text{tr}_{G/H} \left[ \frac{m_\alpha^2}{2} A_\mu(x) A_\mu(x) + i \bar{C}_\mu(x) C_\mu(x) \right], \quad (6.5) $$

where we have put $m_\alpha^2 = 1/\alpha$. In four-dimensions, the parameter $m_\alpha$ looks like a mass which is arbitrary and cannot be determined. The BRST transformation is highly unusual, since it corresponds to the gauge transformation much larger than the SU(N) gauge transformation. Note that $C_\mu^A$ has the same number of indices as $A_\mu^A$. Hence we can use $C_\mu^A$ to eliminate the fields $A_\mu^A$ to obtain the vacuous theory. The ghost field $C_\mu^A$ has its own remaining ghost symmetry, parameterized by the ghost field, $\phi^A$, the ghost for ghost, so the ghosts themselves require more gauge fixing. Note that the gauge fixing condition (6.2) does not allow the topological soliton, since it is linear in the field.

Making the change of variables with the adjoint orbit parameterization

$$ n^A(x) = \text{tr} \left[ U^\dagger(x) T^3 U(x) T^A \right], \quad T^A = \frac{1}{2} \sigma^A \quad (A = 1, 2, 3) \quad (6.6) $$

lead to the four-dimensional coset $G/H$ NLSM.

$$ S_{GF} = \int d^4x \left[ \frac{m_\alpha^2}{2} \partial_\mu n(x) \partial_\mu n(x) + \cdots \right], \quad (6.7) $$

since $A_\mu = U \partial_\mu U^\dagger + \cdots$. This is similar to a piece of the effective theory for the low-energy QCD proposed by Faddeev and Niemi [63] based on Cho’s works [64].

### 6.2 The MA gauge

The naive MA gauge above should be compared with the MA gauge. The MA gauge

$$ F^a[\Omega] := (\partial^\mu \delta_8^{ab} - \epsilon^{a3b} \Omega^3) \Omega^b_\mu := D^{ab}[\Omega^3] \Omega^b_\mu \quad (a, b = 1, 2) \quad (6.8) $$

is obtained by minimizing the $\mathcal{R}[\mathcal{A}^U]$ with respect to the gauge rotation $U$ where

$$ \mathcal{R}[\mathcal{A}] := \int d^Dx \, \text{tr}_{G/H} \left( \frac{k}{2} A_\mu(x) A_\mu(x) \right), \quad (6.9) $$

where $k$ is a constant. The MA gauge fixing leads to the gauge-fixing action (1.37) where the gauge fixing parameter is arbitrary at this stage. In our formulation, we demand the supersymmetry [12] of the gauge fixing action. Then the dimensional reduction [12] occurs as a spontaneous breaking of the supersymmetry (as explained below). This symmetry requirement has determined the form of the gauge-fixing term (4.39) and the result is independent from the coefficient $k$ in $\mathcal{R}[\mathcal{A}]$. The explicit action (1.39) after taking the BRST transformation is rather complicated and does

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7 For $D = 4$, the mass dimension is given as follows, $\dim[A_\mu] = \dim[C_\mu] = 1$, $\dim[B_\mu] = \dim[\bar{C}_\mu] = 3$ and $\dim[\alpha] = -2$. 

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not have any apparent mass term, see [11, 12]. However, the dimensional reduction \[12\] leads to

\[
S'_{GF} := \int d^{D-2}z \ 2\pi \tr_{G/H} \left[ \frac{1}{2} A_a(z, 0) A_a(z, 0) + i C(z, 0) \bar{C}(z, 0) \right]
\]

\[
= \int d^D x \ \tr_{G/H} \left[ \frac{2\pi}{2} A_a(x) A_a(x) + i 2\pi C(x) \bar{C}(x) \right] \delta^2(\hat{x}),
\]

where \( x = (z, \hat{x}) \in \mathbb{R}^D \) and \( z \in \mathbb{R}^{D-2}, \ \hat{x} \in \mathbb{R}^2, \ a = 1, \ldots, D - 2 \). Hence, the MA gauge leads to the unusual mass term, \( m(x) = m(z, \hat{x}) = 2\pi \delta^2(\hat{x}) \). The mass is anisotropic and the gauge field is massive only in \( D - 2 \) dimensions. However, the choice of the \( (D-2) \)-dimensional subspace is arbitrary. For \( D = 4 \), the equivalent action is given in the form of two-dimensional NLSM,

\[
S_{GF} = \int d^2z \ \left[ \frac{2\pi/g^2}{2} \partial_a n(z) \cdot \partial_a n(z) + \tr_{G/H} \left( i 2\pi C_\mu(z) C_\mu(z) \right) \right].
\]

It is known that the two-dimensional NLSM exhibits dynamical mass generation, that is to say, the spectrum has a mass gap, although the initial lagrangian does not have the usual mass term. In this sense, in the subspace \( R^2 \) the gauge field can have the mass. \[8\]

### 6.3 Spontaneous breakdown of hidden supersymmetry

A step of dimensional reduction is a little bit subtle. In the TQFT, the action is BRST exact by definition, so the partition function and the expectation value of the gauge

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\[8\] The mass generation due to dimensional reduction to the NLSM was first demonstrated by Hata and Kugo [33] in the context of color confinement.

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\[33\]
invariant operator do not depend on the coupling constant. However, the NLSM$_{D-2}$ obtained after the dimensional reduction from the TQFT$_D$ is not topological and may depend on the coupling constant. This seems at first glance inconsistent.

This problem will be resolved as follows. The dimensional reduction is a consequence of the hidden supersymmetry in TQFT obtained in the MA gauge, see [12]. Here the supersymmetry implies the invariance under the super rotation in the superspace ($x^\mu, \theta, \bar{\theta}$), i.e., the orthosymplectic group $OSp(D|2)$. The advantage of introducing the superspace is to give a geometric meaning to the BRST transformation. In fact, the BRST symmetry becomes the translational invariance in the superspace. The BRST charges $Q_B$ and $\bar{Q}_B$ are generators of the translations in the direction of the Grassmann variables $\theta$ and $\bar{\theta}$, i.e., they are identified as $\frac{\partial}{\partial \theta}$ and $\frac{\partial}{\partial \bar{\theta}}$, respectively. In the process of the dimensional reduction we can choose arbitrary two dimensions $\mathbb{R}^2$ from $D$ dimensions $(x^1, \cdots, x^D) \in \mathbb{R}^D$, since there is no privileged direction. However, once we have chosen specific two dimensions, the rotation symmetry is partially broken by this procedure. In this sense, the dimensional reduction causes the spontaneous breakdown of the supersymmetry hidden in TQFT.

This becomes more clear in evaluating the expectation value of an operator based on the dimensional reduction. For this strategy to work, the support of all the operators must be contained in the $(D-2)$ dimensional subspace to which the dimensional reduction occurs. Such an expectation value is obtained from the generating functional by restricting the external source $\mathcal{J}(x, \theta, \bar{\theta})$ to a $(D-2)$-dimensional subspace $\mathcal{J}((z, \hat{x} = 0), \theta = 0, \bar{\theta} = 0)$ i.e., by putting $\hat{x} = \theta = \bar{\theta} = 0$. This is obtained as a consequence of restricting the orthosymplectic $OSp(D|2)$ rotation to the orthogonal $O(D-2)$ rotation, see section IV of the paper [12]. Of course, when a pair of quark and anti-quark exists, it is convenient to choose the $(D-2)$-dimensional subspace so that their trajectories are contained in the subspace when $D \geq 4$. The supersymmetry (i.e., rotational invariance $OSp(D|2)$ in the BRST superspace) is broken into the orthogonal $O(D-2)$ rotation, whereas the BRST symmetries (i.e., translational invariances in the direction of $\theta, \bar{\theta}$ in the superspace) is broken by putting $\hat{x} = \theta = \bar{\theta} = 0$. Note that the Hilbert space of the $(D-2)$-dimensional bosonic theory is different from the original D-dimensional supersymmetric theory. Consequently, $(D-2)$-dimensional bosonic theory is no longer topological. Thus, the NLSM$_2$ can be obtained without contradiction from TQFT$_4$ by dimensional reduction.

Finally we consider the above result from a different point of view. We consider the $4 + \epsilon$ dimensional Yang-Mills theory and $2 + \epsilon$ dimensional NLSM. For $\epsilon > 0$, both theories have two phases, the disordered (high-temperature) phase in the strong coupling region $g > g_c$ and the ordered (low-temperature) phase in the weak coupling region $g < g_c$ where $g_c \sim O(\epsilon)$. The beta function is expected to be positive for $0 < g < g_c$ and negative for $g > g_c$. See Fig. 2(a). As $\epsilon$ decreases, the ordered phases shrinks and finally disappears ($g_c(\epsilon) \downarrow 0$ as $\epsilon \downarrow 0$). In this limit, the beta function becomes negative for any value of $g$, leading to the asymptotic freedom for YM$_4$ and NLSM$_2$. See Fig. 2(b). In this limit, the massless Nambu-Goldstone particle associated to the spontaneous breaking of $G$ to $H$ also disappear, as examined by Bardeen, Lee and Shrock [63]. For $D = 4$, thus, the massless Nambu-Goldstone (NG) particle associated with the spontaneous breaking of supersymmetry, if any, can not
exist in two dimensions.

7 Geometric meaning of gauge fixing term

In quantizing the gauge theory, the procedure of gauge fixing is indispensable to avoid infinities due to overcounting of gauge equivalent configurations. So in the quantized gauge theory we must treat the gauge fixing term seriously as well as the gauge field action. Already at the level of classical theory, it is well known that the gauge theory has a geometric meaning, i.e., gauge theory is nothing but the geometry of connection. In this section we want to emphasize that the gauge fixing term may have a geometric meaning from a viewpoint of global topology.

7.1 FP determinant

The usual procedure of gauge fixing is to insert the identity

$$1 = \Delta_{FP}[A] \int [dU] \prod_x \delta(F^a[A^U])$$

(7.1)

into the functional integral

$$Z = \int [dA^U] \exp(-S_{YM}[A^U]).$$

(7.2)

Then we obtain

$$Z = \int [dU] \int [dA^U] \Delta_{FP}[A^U] \prod_x \delta(F^a[A^U]) \exp(-S_{YM}[A^U]),$$

(7.3)

since $\Delta_{FP}$ is gauge invariant, $\Delta_{FP}[A] = \Delta_{FP}[A^U]$. The $\Delta_{FP}$ is calculated as follows.

$$\Delta_{FP}[A]^{-1} = \int [d\omega] \prod_x \delta(F^a[A^\omega])$$

$$= \int [d\omega] \sum_k \left| \det \left( \frac{\delta F^a[A^\omega]}{\delta \omega} \right) \right|_{\omega=\omega_k} \delta (\omega - \omega_k)$$

$$= \sum_k \left| \det \left( \frac{\delta F^a[A^\omega]}{\delta \omega} \right) \right|_{\omega=\omega_k}. \quad (7.4)$$

When this result in the presence of Gribov copies is substituted into (7.3), the BRST formulation does not work. Even when there is no Gribov copies, we have the absolute value of the determinant,

$$\Delta_{FP}[A] = \left| \det \left( \frac{\delta F^a[A^\omega]}{\delta \omega} \right) \right|_{\omega=\omega_k}. \quad (7.5)$$

9 The above argument is clearly insufficient in the following sense. The BRST symmetry is a global continuous symmetry. If it is spontaneously broken, the NG particle should appear, otherwise the Higgs mechanism should occur. It has been shown that the massless NG boson in question does not appear in the physical state and that the dynamical Higgs mechanism does occur. As a result, the off-diagonal gluons and the off-diagonal ghosts (anti-ghosts) become massive. See the recent paper [66] and references cited there.
Figure 3: Various two-dimensional surfaces with different topology. The Euler number \( \chi \) of the two-dimensional surface is determined by the topology of the surface, i.e., \( \chi = 2 - 2g \) for the surface with the genus \( g \). The Morse index \( \nu_p = (-1)^{\lambda_p} \) is a local quantity which is determined at the critical points (black dots in the figures) of the Morse function. The Euler number is equal to the sum of the Morse indices over all the critical points, i.e., \( \chi = n_0 - n_1 + n_2 \). For example, it is easy to see that (a) \( \chi = 2 \) for the sphere \( S^2 \), (b) \( \chi = 0 \) for the torus \( T^2 \).

This expression is difficult to be used. Therefore, we do not adopt this approach. Rather we start from the expression,

\[
Z = \int [dU] \int [d\mathcal{A}^U] \prod_x \delta(F^a[\mathcal{A}^U]) \det \left( \frac{\delta F^a[\mathcal{A}^\omega]}{\delta \omega} \right) \exp(-S_{YM}[\mathcal{A}^U]).
\]

Such a formulation was proposed by Fujikawa [67]. We will show that such a proposal is very natural from the viewpoint of global topology.

### 7.2 Gauge fixing and global topology

In the following, we use the notation \( \phi^a = \{ A_\mu^a(x) \} \) where \( a \) denotes collectively \( x, \mu, A \). If we take the gauge fixing condition

\[
F^a[\phi] := \frac{\delta f(\phi)}{\delta \phi^a} = 0,
\]

the partition function in the gauge theory is given by

\[
Z := \int [d\phi] \prod_a \delta(F^a[\phi]) \det \left( \frac{\delta F^a[\phi]}{\delta \phi^b} \right) e^{-S[\phi]}
= \int [d\phi] \prod_a \delta \left( \frac{\delta f(\phi)}{\delta \phi^a} \right) \det \left( \frac{\delta^2 f(\phi)}{\delta \phi^a \delta \phi^b} \right) e^{-S[\phi]}.
\]
Let $M$ be the manifold $M$ of field configurations $\{\phi^a\}$ and $f$ a continuous function from $M$ to $\mathbb{R}$, $f : M \to \mathbb{R}$. In order to see the geometric meaning of the gauge fixing term, we consider the limit $S[\phi] \to 0$, i.e., only the gauge fixing and the corresponding FP ghost term,

$$\chi_G := \int [d\phi] \prod_a \delta \left( \frac{\delta f(\phi)}{\delta \phi^a} \right) \det \left( \frac{\delta^2 f(\phi)}{\delta \phi^a \delta \phi^b} \right).$$  \hspace{1cm} (7.9)$$

Hence, from the property of the Dirac delta function, we have

$$\chi_G = \int [d\phi] \sum_k \frac{\delta (\phi - \phi_k)}{|\det (\delta^2 f(\phi)/\delta \phi^a \delta \phi^b)|_{\phi = \phi_k}} \det \left( \frac{\delta^2 f(\phi)}{\delta \phi^a \delta \phi^b} \right) \hspace{1cm} (7.10)$$

$$= \sum_{k : \nabla f(\phi_k) = 0} \operatorname{sign} \left[ \det \left( \frac{\delta^2 f(\phi)}{\delta \phi^a \delta \phi^b} \right)_{\phi = \phi_k} \right], \hspace{1cm} (7.11)$$

where $\phi_k$ is a solution of $\nabla f(\phi) = 0$ or $F^a[\phi] = 0$. Thus we obtain

$$\chi_G = \sum_{p : \nabla f(\phi_p) = 0} \nu_p, \quad \nu_p := \operatorname{sign}(\det H_f) = (-1)^{\lambda_p}, \hspace{1cm} (7.12)$$

where $H_f$ is the Hessian defined by

$$H_f := \frac{\delta^2 f(\phi)}{\delta \phi^a \delta \phi^b}. \hspace{1cm} (7.13)$$

For a smooth function $f$, the Hessian is a symmetric matrix and hence its eigenvalues are all real. The index $\lambda_p$ is equal to the number of negative eigenvalues of the Hessian. Note that $\chi$ is an integer. In order to obtain this simple expression, the existence of the FP determinant is indispensable. The function $f$ is called the Morse function \cite{[68]}, if all the critical points $P$ (i.e., $\nabla f(P) = 0$) of $f$ are non-degenerate, i.e., $\det(H_f) \neq 0$. For a finite dimensional case, it is known that all non-degenerate critical points of $f$ are isolated critical points, i.e., there is no other critical point in the neighborhood of a critical point. Whether the critical point is degenerate or non-degenerate does not depend on how to choose the coordinate systems in the manifold $M$ of field configurations $\{\phi^a\}$. A convenient way to see the above situation is to use the standard form. The quantity $\chi$ is obtained as the sum of the index $\nu_p$ over all the critical points, once the function $f$ is given. See Fig. 3 for a simple case of a two-dimensional surface $M_2$. In that case, we have

$$\chi(M_2) = n_0 - n_1 + n_2, \hspace{1cm} (7.14)$$

where $n_0, n_1, n_2$ are the total numbers of the minimal, saddle and maximum points respectively.

For a given field configuration $\phi^a$, we can consider the global topology. The measure $[d\phi(x)]$ includes various field configurations with various global topology,
each of which is characterized by an appropriate topological invariant. By repeating the similar calculation, we obtain for the partition function,

\[ Z = \sum_{\nu_p e^{-S[\phi_p]}} = \sum_{\lambda_p e^{-S[\phi_p]}} (-1)^\nu_p e^{-S[\phi_p]}, \tag{7.15} \]

The two-dimensional case is rather simple, see Fig. 3. The Poincare'-Hopf theorem for the two-dimensional surface states that \( \chi \) defined in (7.12) is equal to the Euler number \( \chi \) of the two-dimensional surface \( M_2 \),

\[ \chi(M_2) = 2 - 2g, \tag{7.16} \]

where \( g \) is the genus, i.e., number of handles. The Morse index \( \nu_p \) or \( \lambda_p \) is a local quantity, but, the sum \( \chi \) given by (7.12) is determined only from the global topology of the surface (7.16) without any local information. The Poincare-Hopf theorem gives a bridge between the local geometry and global topology:

Local \( \to \sum_p (\text{local index})_p = \text{topological invariant} \leftarrow \text{Global}. \tag{7.17} \]

This is a very important result for our purpose. Because, by deforming the surface in a continuous way, the location of the critical point change and the Morse index at the new critical point may also change, but the total sum of the Morse index is a topological invariant which is determined only by the global topology of the surface irrespective of the way of continuous deformation.

It is well known that the two-dimensional manifold is completely classified by the genus or the Euler number. This is not the case in higher dimensions. In fact, we must treat the infinite dimensional case. Even in the infinite dimensional case, for a specific field configuration \( M^\alpha_\infty \), a topological invariant \( Q(M^\alpha_\infty) \) will be determined. Then it is expected that the \( \chi_G \) is expressed as a sum of topological invariants,

\[ \chi_G = \sum_Q w_Q Q(M^\alpha_\infty). \tag{7.18} \]

In view of this, the functional we have chosen to derive the MA gauge,

\[ \mathcal{R}[\Omega] := \int d^D x \ tr_{G/H} \left( \frac{1}{2} \Omega_\mu(x) \Omega_\mu(x) \right) \tag{7.19} \]

is considered to be a Morse function. Indeed, the MA gauge condition is obtained as the gradient of the Morse function \( \mathcal{R} \) with respect to the large gauge transformation,

\[ \frac{\delta \mathcal{R}[\Omega^\omega]}{\delta \omega} = F[\Omega]. \tag{7.20} \]

The topology change in the field configurations may be caused by the large (or finite) gauge transformation allowed in the measure \( dU \), since the measure is invariant under the global gauge rotation \( U \to e^{i\omega} U \). Thus, the gauge fixing and the associated FP term when integrated out by the functional measure \( d\phi \) can have a geometric
meaning which is related to the global topology of the field configurations. The Morse function is a tool of probing the global topology allowed in the functional space of the field configurations by gathering the local information at all the critical points.

The partition function of Yang-Mills theory is given by
\[
Z_{YM}[0] = \sum_{p:F(\Omega_p)=0} \nu_p e^{-S_{\text{eff}}[\Omega_p]} = \sum_{p:F(\Omega_p)=0} (-1)^{\lambda_p} e^{-S_{\text{eff}}[\Omega_p]}. \tag{7.21}
\]

Finally, we consider how \( \chi \) changes when we change the Morse function \( f \). In two-dimensional case, \( \chi(M_2) \) is determined by the topology of manifold \( M_2 \) irrespective of the choice of Morse function \( f \). If this feature survives in the infinite dimensional case, we can conclude that the global topology of the Yang-Mills theory does not depend on the way of gauge fixing.

### 7.3 Morse function and BRST transformation

By introducing the auxiliary field \( B \) and the FP ghost and anti-ghost fields, \( \psi, \bar{\rho} \), we can write
\[
\chi := \int [d\phi] \prod_a \delta(F^a[\phi]) \det \left( \frac{\delta F^a[\phi]}{\delta \phi^b} \right) = \int [d\phi][dB][d\psi][d\bar{\rho}]e^{-S_{\text{GF}}[\phi]}, \tag{7.22}
\]

where the gauge-fixing action reads
\[
S_{\text{GF}} = \int d^Dx \left[ \frac{\alpha}{2} B^2 + BF - \bar{\rho}F'[\phi]\psi \right] = \int d^Dx \delta_B \left[ \bar{\rho} \left( F[\phi] + \frac{\alpha}{2} B \right) \right] \tag{7.23}
\]

with the nilpotent BRST transformation,
\[
\delta_B \phi = \psi, \tag{7.24}
\]
\[
\delta_B \psi = 0, \tag{7.25}
\]
\[
\delta_B \bar{\rho} = B, \tag{7.26}
\]
\[
\delta_B B = 0. \tag{7.26}
\]

By eliminating the auxiliary field \( B \), we have
\[
\chi = \int [d\phi][d\psi][d\bar{\rho}]e^{-S'_{\text{GF}}}, \tag{7.27}
\]
\[
S'_{\text{GF}} = \int d^Dx \left[ -\frac{1}{2\alpha}(F[\phi])^2 - \bar{\rho}F'[\phi]\psi \right], \tag{7.28}
\]

where
\[
\delta_B \phi = \psi, \tag{7.29}
\]
\[
\delta_B \psi = 0, \tag{7.30}
\]
\[
\delta_B \bar{\rho} = \frac{F[\phi]}{\alpha}. \tag{7.30}
\]

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The critical point $F[\phi] := \frac{\partial f(\phi)}{\partial \phi} = 0$ corresponds to the fixed point of BRST transformation. Therefore, the integration $\int [d\phi] \delta(F[\phi]) \cdots$ is localized on the fixed point of BRST transformation. This is a characteristic feature of topological quantum field theory. The condition $F^a[\phi] = 0$ is regarded as a non-linear partial differential equation. The space of parameters characterizing the solution of this equation is called the moduli space. In the TQFT, the above argument shows that the infinite dimensional functional integral reduces to finite dimensional integral on moduli space. For example, for the Yang-Mills instanton with $Q = k$, $\dim M = 8k < \infty$.

8 Conclusion and discussion

In this paper we have derived a reformulation of the Yang-Mills theory based on the background field method. The reformulation identifies the Yang-Mills theory as a deformation of a topological quantum field theory as proposed in [12]. The background field is given by a topological soliton.

In order to show quark confinement, the condensation of a topological soliton is necessary to occur. This has been actually derived by summing up the topological soliton contributions, provided that the topological soliton is described by the topological field theory. The topological field theory has been derived from the gauge fixing term corresponding to the nonlinear gauge fixing condition, the maximal Abelian gauge. The maximal Abelian gauge implies that the topological soliton in question is nothing but the magnetic monopole current, the four-dimensional version of the magnetic monopole. The result ensures that the quark confinement is realized in the QCD vacuum as a dual superconductor. Furthermore, we have proposed a numerical simulation which is able to confirm the validity of the above reformulation.

We have discussed a novel mechanism for the mass generation for the gauge field, i.e., dynamical mass generation as the dimensional reduction which causes the spontaneous breakdown of the hidden supersymmetry in the topological field theory. Moreover, we have suggested that the gauge fixing action may have the geometric meaning from the view point of global topology by making use of the Morse function.

In this paper we have restricted our consideration to the maximal Abelian gauge where the residual gauge group $H$ is the maximal torus group of the non-Abelian gauge group $G$ ($H = \text{U}(1)^{N-1}$ for $G = \text{SU}(N)$), although our formulation can be applied to any choice of $H$. Therefore the topological soliton is given by the Abelian magnetic monopole. However, it is possible to consider other choices for the residual gauge group $H$ (especially for $G = \text{SU}(N)(N \geq 3)$) which leads to the topological soliton other than the Abelian magnetic monopole, e.g., non-Abelian magnetic monopole, center vortex. Either choice will lead to the quark confinement. From the viewpoint of color confinement, however, the maximal torus group for $H$ is not necessarily the best choice. The details will be given in the subsequent paper [36].
A Overlapping between monopole current and instanton solutions

The MA gauge (3.16) is written as

\[(\partial_\mu \mp iA_\mu^A)A_\mu^\pm = 0, \quad (A.1)\]

where

\[A_\mu^\pm := \frac{1}{\sqrt{2}}(A_\mu^1 \pm iA_\mu^2). \quad (A.2)\]

We show that the gauge potential of the form,

\[A_\mu^A(x) = \eta_{\mu\nu}\partial_\nu f(x), \quad (A.3)\]

satisfies the monopole equation (A.1) for arbitrary function \(f\) as long as \([\partial_\mu, \partial_\nu]f = 0\).

Here the \(\eta\)-symbol ('t Hooft symbol) is defined by

\[\eta_{\mu\nu} := \epsilon_{\mu\nu\alpha\beta}\eta_{\alpha\beta}, \quad (A.4)\]

for \(\mu, \nu = 1, 2, 3, 4, A = 1, 2, 3\) (we have assumed \(\epsilon_{A4\nu} = \epsilon_{A4\mu} = 0\)).

Substituting the ansatz (A.3) into the definition of \(A_\mu^\pm\), we have

\[A_\mu^A A_\mu^\pm = \frac{1}{\sqrt{2}}\eta_{\mu\rho}\eta_{\nu\sigma}(\eta_{\mu\rho}^1 \pm i\eta_{\mu\rho}^2)\partial_\rho f\partial_\sigma f = \frac{1}{\sqrt{2}}(\eta_{\rho\sigma}^1 \pm i\eta_{\rho\sigma}^2)\partial_\rho f\partial_\sigma f = 0, \quad (A.5)\]

where we have used the relations \[38\]

\[\eta_{\mu\nu}\eta_{\mu\sigma} = \delta_{\nu\sigma}\delta_{\rho\rho} + \epsilon_{ABC}\eta_{\rho\sigma}^C, \quad (A.6)\]

\[\eta_{\mu\nu} = -\eta_{\nu\mu}. \quad (A.7)\]

On the other hand, we find

\[\partial_\mu A_\mu^A(x) = \eta_{\mu\nu}\partial_\mu \partial_\nu f(x), \quad (A.8)\]

and hence

\[\partial_\mu A_\mu^\pm := \frac{1}{\sqrt{2}}(\partial_\mu A_\mu^1 \pm i\partial_\mu A_\mu^2) = \frac{1}{\sqrt{2}}(\eta_{\mu\nu}^1 \partial_\mu \partial_\nu f \pm i\eta_{\mu\nu}^2 \partial_\mu \partial_\nu f). \quad (A.9)\]

Note that \(\eta_{\mu\nu}^A \partial_\mu \partial_\nu f(x) = \frac{1}{2}\eta_{\mu\nu}^A [\partial_\mu, \partial_\nu]f(x)\). Therefore, the MA gauge (A.1) is satisfied for any function \(f\) as long as \([\partial_\mu, \partial_\nu]f = 0\).

The ansatz (A.3) is the same as the multi-instanton solution of 't Hooft type. The \(\eta\)-symbols are self-dual in the vector indices,

\[\eta_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\eta_{\alpha\beta}. \quad (A.10)\]
The instanton solution is obtained assuming \([\partial_\mu, \partial_\nu]f = 0\), since the instanton is the point defect in four dimensions. For the magnetic monopole current (the line defect) in four dimensions, \([\partial_\mu, \partial_\nu]f \neq 0\) can happen, see e.g. Appendix C of [11]. Such a possibility has not been studied so far.

The \(n\)-instanton solution in the singular gauge is given by

\[
f(x) = \ln \left[ 1 + \sum_{k=1}^{n} \frac{\rho_k^2}{(x-z_k)^2} \right],
\]

where \(z_i\) is the position and \(\rho_i\) is the size of the \(i\)-th instanton \((k = 1, \cdots, n)\), and all the instantons have the same color orientations. For the \(n\)-instanton, \(f = \ln \phi\) and \(\phi\) are singular at \(n\) points, \(x = z_k(k = 1, \cdots, n)\). The gauge potential reads

\[
A^A_\mu(x) = \frac{-2 \sum_{k=1}^{n} \frac{\rho_k^2}{(x-z_k)^2} \eta^A_{\mu\nu}(x-z_k)_\nu}{1 + \sum_{k=1}^{n} \frac{\rho_k^2}{(x-z_k)^2}}.
\]

The one-instanton solution [37] is obtained as a special case, \(n = 1\),

\[
A^A_\mu(x) = \frac{-2\rho^2 \eta^A_{\mu\nu}(x-z)_\nu}{(x-z)^2 \left[(x-z)^2 + \rho^2\right]},
\]

The singular solution behaves as a pure gauge near the singular point \(x = z\),

\[
A_\mu(x) := A^A_\mu(x) \frac{\sigma^A}{2} \to U(y) \partial_\mu U(y),
\]

since \(A_\mu\) is written as

\[
A_\mu(x) = U(y) \partial_\mu U(y) \frac{\rho^2}{y^2 + \rho^2},
\]

with \(y = x - z\),

\[
U(x) = \frac{x_4 + ix_A \sigma^A}{|x|}, \quad |x| := \sqrt{x_4 x_4 + x^A x^A}.
\]

Since the self-duality and field equations are gauge covariant, the gauge transformed potential also satisfies them. This holds also for the monopole equation \((\text{A.1})\). By the appropriate inverse gauge transformation \(U\), we can get rid of the singularity and the resulting solution vanishes at \(x = z\). In fact, the singularity at \(x = z\) can be removed by a singular gauge transformation,

\[
A_\mu'(x) = U(y)[A_\mu(x) + \partial_\mu]U(y),
\]

and the non-singular solution is obtained [13]

\[
A^A_\mu'(x) = \frac{-2\eta^A_{\mu\nu}(x-z)_\nu}{(x-z)^2 + \rho^2},
\]
where $\bar{\eta}$-symbol is defined by
\[ \bar{\eta}_A^{\mu\nu} := \epsilon_A^{\mu\nu} - \delta_A^\mu \delta_4^\nu + \delta_A^\nu \delta_4^\mu. \]  
(A.19)

Indeed, the solution (A.18) has no singularity at any $x$. The non-singular solution approaches the pure gauge as $x \to \infty$,
\[ A'_\mu(x) \to U(x)\partial_\mu U^\dagger(x). \]  
(A.20)

Note that the multi-antiinstanton is obtained by interchanging $\eta$-symbol and $\bar{\eta}$-symbol which is self-antidual,
\[ \bar{\eta}_A^{\mu\nu} = -\frac{1}{2} \epsilon_{\mu\alpha\beta} \bar{\eta}_A^{\alpha\beta}. \]  
(A.21)

If we restrict the diagonal component
\[ A_3^\mu(x) = \eta_3^{\mu\nu} \partial_\nu f(x) = \epsilon_3^{\mu\nu} \partial_\nu f(x), \]  
(A.22)

and $f$ is independent from 4,
\[ A_i^3(x) := a_i(x) = \epsilon_{ij} \partial_j f(x) (i, j = 1, 2), \]  
(A.23)

This solution is similar to the Witten solution [39] for the multi-instanton with cylindrical symmetry, see section V of [12].

It is known [70] that various well-known equations in lower dimensions are obtained by (dimensional) reduction of the self-dual equation in four dimensions. For example, if the field is static, i.e., $\partial_0 A_\mu = 0$, the identification $A_0(x_0, x) = \phi(x)$ with $x = (x_1, x_2, x_3)$ leads to the Bogomol’nyi equation describing the magnetic monopole in three dimensions,
\[ F_{ij}(x) = \pm \epsilon_{ijk} D_k \phi(x) \quad (i, j, k = 1, 2, 3). \]  
(A.24)

In this sense, the self-dual equation contains a kind of magnetic monopole.

Under the ansatz (A.3), the self-dual equation (3.12) for the instanton is satisfied only for the function $f$ given by (A.11) which is also a solution of the Yang-Mills field equation (3.1). The same ansatz gives a solution of the monopole equation for arbitrary $f$ which is not necessarily the solution of the Yang-Mills field equation. The general solution of the self-dual equation is given according to the method of Atiyah, Drinfeld, Hitchin and Mannin (ADHM). To author’s knowledge, the general solution is not known for the monopole equation (A.1). In order for the solution of (A.1) to give a magnetic monopole, we must check whether the solution gives non-trivial $k_\mu$.

The explicit monopole solution was constructed in [25] and [24]. Only the solution of Brower, Orginos and Tan (BOT) [24] satisfies $R[A] < \infty$. For more details, see [24].
B Comparison with the Cho-Faddeev-Niemi variables

If we choose

$$\mathcal{R}[A] := \int d^D x \ tr_G \left( \frac{1}{2} A_\mu(x) A_\mu(x) \right),$$

(B.25)

the variation is given by

$$\delta_\omega \mathcal{R}[A] := \int d^D x \ tr_G (A_\mu(x) \delta_\omega A_\mu(x)) = \int d^D x \ tr_G (A_\mu(x) D_\mu[A] \omega(x))$$

(B.26)

$$= - \int d^D x \ tr_G (D_\mu[A] A_\mu(x) \cdot \omega(x)),$$

(B.27)

where we have used the partial integration by parts. The requirement $\delta_\omega \mathcal{R}[A] = 0$ for arbitrary $\omega$ yields $0 = D_\mu[A] A_\mu(x) = \partial_\mu A_\mu(x)$. This is the familiar Lorentz gauge which is a linear gauge. The linear equation $\partial_\mu A_\mu(x) = 0$ can not have a soliton solution.

Another way to obtain $\delta_\omega \mathcal{R}[A] = 0$ is to restrict the gauge transformation $\omega$ such that $D_\mu[A] \omega(x) = 0$. This equation is solved for $A_\mu$, see [64]. For example, in the case of $G = SU(2)$,

$$D_\mu[A] \omega(x) := \partial_\mu \omega^A(x) + g \epsilon^{ABC} A^C_\mu(x) \omega^B(x) = 0.$$  

(B.28)

A solution is given by

$$A^A_\mu(x) = a_\mu \omega^A(x) - \frac{1}{2g} \epsilon^{ABC} \omega^B(x) \partial_\mu \omega^C(x),$$

(B.29)

where $a_\mu$ is arbitrary Abelian vector field and $\omega^A$ is chosen to be a unit vector in three dimensions, $\omega^A(x) \omega^A(x) = 1$. In this way, it is possible to obtain a subset of a non-Abelian gauge theory. This formalism gives essentially the same result as the partial gauge fixing, the MA gauge. The details will be given in a forthcoming publication.

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References

[1] Y. Nambu, Strings, monopoles, and gauge fields, Phys. Rev. D 10, 4262-4268 (1974).

G. ’t Hooft, in: High Energy Physics, edited by A. Zichichi (Editorice Compositori, Bologna, 1975).

S. Mandelstam, Vortices and quark confinement in non-abelian gauge theories, Phys. Report 23, 245-249 (1976).
[2] T. Suzuki, Monopole condensation in lattice SU(2) QCD, [hep-lat/9506010].
R.W. Haymaker, Dual Abrikosov vortices in U(1) and SU(2) lattice gauge theories, [hep-lat/9510035].
A. DiGiacomo, Monopole condensation and color confinement, [hep-lat/9802008].
M.I. Polikarpov, Recent results on the abelian projection of lattice gluodynamics, [hep-lat/9609020].
M.N. Chernodub and M.I. Polikarpov, Abelian projections and monopoles, [hep-th/9710203].

[3] G.S. Bali, The mechanism of quark confinement, [hep-ph/9809351].

[4] P.A.M. Dirac, Quantized singularities in the electromagnetic field, Proc. Roy. Soc., London, A 133, 60-72 (1931).
P.A.M. Dirac, The theory of magnetic poles, Phys. Rev. 74, 817-830 (1948).

[5] T.T. Wu and C.N. Yang, Concept of nonintegrable phase factors and global formulation of gauge fields, Phys. Rev. D 12, 3845-3857 (1975).
T.T. Wu and C.N. Yang, Dirac monopole without strings: monopole harmonics, Nucl. Phys. B 107, 365 (1976).

[6] P. Goddard and D.I. Olive, Magnetic monopoles in gauge field theories, Rep. Prog. Phys. 41, 1357-1437 (1978).

[7] G. 't Hooft, Topology of the gauge condition and new confinement phases in non-Abelian gauge theories, Nucl. Phys. B 190 [FS3], 455-478 (1981).

[8] Z.F. Ezawa and A. Iwazaki, Abelian dominance and quark confinement in Yang-Mills theories, Phys. Rev. D 25, 2681-2689 (1982).
Abelian dominance and quark confinement in Yang-Mills theories, II. Oblique confinement and \(\eta'\) mass, Phys. Rev. D 26, 631-647 (1982).

[9] T. Suzuki and I. Yotsuyanagi, Possible evidence of abelian dominance in quark confinement, Phys. Rev. D 42, 4257-4260 (1990).

[10] T. Suzuki, Prog. Theor. Phys. 80, 929-934 (1988).
S. Maedan and T. Suzuki, Prog. Theor. Phys. 81, 229-240 (1989).
T. Suzuki, Prog. Theor. Phys. 81, 752-757 (1989).

[11] K.-I. Kondo, Abelian-projected effective gauge theory of QCD with asymptotic freedom and quark confinement, [hep-th/9709103].
K.-I. Kondo, [hep-th/9803063], Prog. Theor. Phys. Supplement, No. 131, 243-255 (1998).

[12] K.-I. Kondo, Yang-Mills Theory as a Deformation of Topological Field Theory, Dimensional Reduction and Quark Confinement, [hep-th/9801024].

[13] K.-I. Kondo, Abelian Magnetic Monopole Dominance in Quark Confinement, [hep-th/9805153].

[14] K.-I. Kondo, Existence of confinement phase in quantum electrodynamics, [hep-th/9803133].
[15] K.-I. Kondo, Quark Confinement and Deconfinement in QCD from the Viewpoint of Abelian-Projected Effective Gauge Theory, [hep-th/9810167], Phys. Lett. B 455, 251-258 (1999).

[16] M. Creutz, Quarks, gluons and lattices (Cambridge Univ. Press, 1983).
H.J. Rothe, Lattice gauge theories, an introduction (World Scientific, Singapore, 1992).
I. Montvay and G. Münster, Quantum fields on a lattice (Cambridge Univ. Press, 1994).

[17] B.S. DeWitt, Phys. Rev. 162, 1195, 1239 (1967).
B.S. DeWitt, in Dynamical theory of groups and fields (Gordon and Breach, 1965).

[18] G. ’t Hooft, in Acta Universitatis Wratislaviensis, no.38, 12th Winter School of Theoretical Physics in Karpacz; Functional and probabilistic methods in quantum field theory, vol.1 (1975).

[19] L.F. Abbott, Introduction to the background field method, Acta Physica Polonica B13, 33-50 (1982).

[20] L.F. Abbott, The background field method beyond one loop, Nucl. Phys. B 185, 189-203 (1981).

[21] L.F. Abbott, M.T. Grisaru and R.K. Schaefer, The background field method and the S-matrix, Nucl. Phys. B 229, 372-380 (1983).

[22] M. Quandt and H. Reinhardt, Field strength formulation of SU(2) Yang-Mills theory in the maximal abelian gauge: perturbation theory, [hep-th/9707185], Int. J. Mod. Phys. A13, 4049-4076 (1998).

[23] E. Witten, Topological quantum field theory, Commun. Math. Phys. 117, 353-386 (1988).

[24] R.C. Brower, K.N. Orginos and C.-I. Tan, Magnetic monopole loop for the Yang-Mills instanton, [hep-th/9610101], Phys. Rev. D 55, 6313-6326 (1997).

[25] M.N. Chernodub and F.V. Gubarev, Instantons and monopoles in maximal abelian projection of SU(2) gluodynamics, [hep-th/9506026].

[26] A. Hart and M. Teper, Instantons and monopoles in the maximally abelian gauge, [hep-lat/9511016], Phys. Lett. B371, 261-269 (1996).

[27] V. Bornyakov and G. Schierholz, Instantons or monopoles? Dyons, [hep-lat/9605013].

[28] H. Suganuma, A. Tanaka, S. Sasaki and O. Miyamura, Evidence of strong correlation between instanton and QCD-monopole on SU(2) lattice, [hep-lat/9512024].

[29] M. Fukushima, S. Sasaki, H. Suganuma, A. Tanaka, H. Toki and D. Diakonov, [hep-lat/9608084], Phys. Lett. B 399, 141-147 (1997).
[30] M. Feurstein, H. Markum and S. Thurner, Coexistence of monopoles and instantons for different topological charge definitions and lattice actions, hep-lat/9702004.
M. Feurstein, H. Markum and S. Thurner, Instantons and monopoles in lattice QCD. hep-lat/9702006.

[31] K. Fujikawa, Dynamical stability of the BRS supersymmetry and the Gribov problem, Nucl. Phys. B 223, 218-234 (1983).

[32] G. Parisi and N. Sourlas, Random magnetic fields, supersymmetry, and negative dimensions, Phys. Rev. Lett. 43, 744-745 (1979).

[33] H. Hata and T. Kugo, Color confinement, Becchi-Rouet-Stora symmetry, and negative dimensions, Phys. Rev. D 32, 938-944 (1985).

[34] M. Stone and P.R. Thomas, Condensed monopoles and abelian confinement, Phys. Rev. Lett. 41, 351-353 (1978).

[35] K. Bardakci and S. Samuel, Local field theory for solitons, Phys. Rev. D 18, 2849-2860 (1978). K. Bardakci, Local field theory for solitons. II, Phys. Rev. D 19, 2357-2366 (1979).

[36] D. Antonov and D. Ebert, Dual formulation and confining properties of the SU(2)-gluodynamics, hep-th/9902177.

[37] A.A. Belavin, A.M. Polyakov, A.S. Schwartz and Yu.S. Tyupkin, Pseudoparticle solutions of the Yang-Mills equations, Phys. Lett. B 59, 85-87 (1975).

[38] G. ’t Hooft, Computation of the quantum effects due to a four-dimensional pseudoparticle, Phys. Rev. D 14, 3432-3450 (1976).

[39] E. Witten, Some exact multipsedoparticle solutions of classical Yang-Mills theory, Phys. Rev. Lett. 38, 121-124 (1977).

[40] R. Jackiw and C. Rebbi, Conformal properties of a Yang-Mills psedoparticle, Phys. Rev. D 14, 517-523 (1976).
R. Jackiw, C. Nohl and C. Rebbi, Conformal properties of pseudoparticle configurations, Phys. Rev. D 15, 1642-1646 (1977).
R. Jackiw and C. Rebbi, Degrees of freedom of pseudoparticle systems, Phys. Lett. B 67, 189-192 (1977).
E.F. Corrigan, D.B. Fairlie, P. Goddard and R.G. Yates, The construction of self-dual solutions to SU(2) gauge theory, Commun. Math. Phys. 58, 223 (1978).
C.W. Bernard, N.H. Christ, A.H. Guth and E.J. Weinberg, Pseudoparticle parameters for arbitrary gauge groups, Phys. Rev. D 16, 2967-2977 (1977).
N. Christ, E.J. Weinberg and N.K. Stanton, General Yang-Mills solutions, Phys. Rev. D 18, 2013-2025 (1978).
C. Bernard, Gauge zero modes, instanton determinants and quantum-chromodynamic calculations, Phys. Rev. D 19, 3013-3019 (1979).

[41] M. Atiyah, V. Drinfeld, N. Hitchin and Y. Manin, Construction of instantons, Phys. Lett. A 65, 185-187 (1978).

[42] S. Coleman, *Aspect of Symmetry* (Cambridge Univ. Press, New York, 1985).
[43] R. Rajaraman, *Solitons and Instantons* (North-Holland, Amsterdam, 1989).

[44] L. Sibner, R. Sibner and K. Uhlenbeck, Solution of Yang-Mills equation that are not self-dual, Proc. Nath. Acad. Sci. USA, 86, 8610-8613 (1989).

[45] C. Nash and S. Sen, *Topology and Geometry for Physicists* (Academic Press, New York, 1983).

[46] M. Nakahara, *Geometry, Topology and Physics* (IOP, Bristol, 1990).

[47] N.D. Mermin, The topological theory of defects in ordered media, Rev. Mod. Phys. 51, 591-648 (1979).

[48] J. Kalkkinen and A.J. Niemi, Supersymmetric structures in 4-D Yang-Mills theory, Eur. Phys. J. C4, 723-739 (1998).

[49] D. Birmingham, M. Blau, M. Rakowski and G. Thompson, Topological field theory, Phys. Report 209, 129-340 (1991).

[50] N. Seiberg and E. Witten, Electric-magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory, hep-th/9407087, Nucl. Phys. B 426, 19-52 (1994).

[51] E. Witten, Quantum field theory and the Jones polynomial, Commun. Math. Phys. 121, 351-399 (1989).

[52] H. Hata and Y. Taniguchi, Color confinement in perturbation theory from a topological model, hep-th/9502083, Prog. Theor. Phys. 94, 435-444 (1995).

[53] F. Fucito, M. Martellini and M. Zeni, The BF formalism for QCD and quark confinement, hep-th/9605018, Nucl. Phys. B 496, 259-284 (1997).

[54] K.-I. Kondo and Y. Taira, Non-Abelian Stokes Theorem and Quark Confinement in SU(3) Yang-Mills gauge theory, hep-th/9906129, Mod. Phys. Lett. A, to appear; Non-Abelian Stokes Theorem and Quark Confinement in SU(N) Yang-Mills gauge theory, hep-th/9911242.

[55] G. ’t Hooft, Magnetic monopoles in unified gauge theories, Nucl. Phys. B 79, 276-284 (1974).

[56] A.M. Polyakov, Particle spectrum in the quantum field theory, Pisma Zh. Eksp. Teor. Fiz. 20, 430-433 (1974) [JETP Lett. 20, 194-195 (1974)].

[57] B. Berg and M. Lüscher, Computation of quantum fluctuations around multi-instanton field from exact Green’s functions: the $\mathbb{CP}^{n-1}$ case, Commun. Math. Phys. 69, 57-80 (1979).

[58] V.A. Fateev, I.V. Frolov and A.S. Schwarz, Quantum fluctuations of instantons in the non-linear $\sigma$ model, Nucl. Phys. B 154, 1-20 (1979).

[59] A.P. Bukhvostov and L.N. Lipatov, Instanton–Anti-instanton interaction in the O(3) non-linear $\sigma$ model and an exactly soluble fermion theory, Nucl. Phys. B 180 [FS2], 116-140 (1981).
[60] P.G. Silvestrov, A new way of instanton–anti-instanton interaction description. The nonlinear O(3) sigma model example, Sov. J. Nucl. Phys. 51, 1121-1127 (1990).

[61] A.M. Polyakov, *Gauge fields and strings* (Harwood Academic Publishers, London, 1987).

[62] M. Faber, J. Greensite and S. Olejnik, Evidence for a center vortex origin of the adjoint string tension, hep-lat/9807008. M. Faber, J. Greensite and S. Olejnik, Casimir scaling from center vortices: towards an understanding of the adjoint string tension, hep-lat/9710039, Phys. Rev. D 57, 2603-2609 (1998).

[63] L. Faddeev and A.J. Niemi, Partially dual variables in SU(2) Yang-Mills theory, hep-th/9807009. Partial duality in SU(N) Yang-Mills theory, hep-th/9812090.

[64] Y.M. Cho, Restricted gauge theory, Phys. Rev. D 21, 1080-1088 (1980). Y.M. Cho, Extended gauge theory and its mass spectrum, Phys. Rev. D 23, 2415-2426 (1981).

[65] W.A. Bardeen, B.W. Lee and R.E. Schrock, Phase transition in the nonlinear $\sigma$ model in a $(2+\epsilon)$-dimensional continuum, Phys. Rev. D 14, 985-1005 (1976).

[66] K.-I. Kondo and T. Shinohara, Abelian dominance in low-energy gluodynamics due to dynamical mass generation, CHIBA-EP-120, hep-th/0004158.

[67] K. Fujikawa, Comment on the covariant path integral formalism in the presence of Gribov ambiguities, Prog. Theor. Phys. 61, 627-632 (1979). K. Fujikawa, BRST symmetric formulation of a theory with Gribov-type copies, hep-th/9510111, Nucl. Phys. B 468, 355-379 (1996). K. Fujikawa, Gribov problem and BRST symmetry, hep-th/9511124. K. Fujikawa, Dynamical stability of the BRS supersymmetry and the Gribov problem, Nucl. Phys. B 223, 218-234 (1983).

[68] J. Milnor, *Morse Theory* (Princeton Univ. Press, 1963).

[69] K.-I. Kondo, in preparation.

[70] L.J. Mason and N.M.J. Woodhouse, *Integrability, self-duality and twistor theory* (Oxford Univ. Press, Oxford, 1996).