Loop Variables and Gauge Invariant Exact Renormalization Group Equations for Closed String Theory

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Abstract

We formulate the Exact Renormalization Group on the string world sheet for closed string backgrounds. The same techniques that were used for open strings is used here. There are some subtleties. One is that holomorphic factorization of the closed string vertex operators does not hold in the presence of a cutoff on the Euclidean world sheet. This introduces extra terms in the Lagrangian at the cutoff scale and they turn out to be crucial for implementing gauge invariance. This naive generalization from open string to closed strings requires a massive graviton and the gauge symmetry is Abelian, just as in open string theory. Interestingly, it turns out that if one introduces a non-dynamical background metric (as in background field formalism) and combines a gauge transformation on the field with a transformation on the coordinates and background metric, the graviton can be massless. Some examples of background coordinate covariant equations are worked out explicitly. A preliminary discussion of massive modes, massive gauge transformations and the role of world sheet regulator terms is given. Some of the gauge transformations can be given a geometric meaning if space time is assumed to be complex at some level.
1 Introduction

Equations of Motion (EOM) for specific open or closed string modes can be obtained as Renormalization Group (RG) equations (i.e. $\beta$-functions) for the world sheet action of a string propagating in a non trivial background [11-14]. The Exact Renormalization Group (ERG) [15, 16, 17, 18] on the other hand, would include all the modes of the string at once and be equivalent to string field theory as first suggested in [7, 8]. Some aspects of this and the connection with the proper time formalism was worked out in [9].

String theory possesses an infinite tower of gauge symmetries and one would like the EOM to be gauge invariant. Ideally one would like an action too. In string field theory this was elegantly solved using the BRST formalism [19, 20, 21, 22] and an action was written down.

The problem of gauge invariance can also be posed in the RG formalism. The RG formalism is potentially capable of being manifestly background independent, so a solution to the problem of gauge invariance should provide insights into fundamental aspects of string theory. In the RG formalism, loop variable techniques have been used to make equations gauge invariant [23] in the free case. Interacting RG equations were also made gauge invariant [24], though not in a form that is conveniently written down in terms of space-time fields. A convenient form was derived more recently in [26] and [27], (hereafter I and II) where gauge invariant interacting equations of motion for open strings were derived. This was obtained by applying the ERG to the world sheet action for open strings propagating in an arbitrary background. Loop variable techniques were used to ensure that the equations are gauge invariant. Since the world sheet action can be written for any background, this method is manifestly background independent [3]. The equations are quadratic, as expected from open string field theory. This can be traced to the fact that the ERG is always quadratic in coupling constants. Thus if equations of motion can be obtained from an ERG they are guaranteed to be quadratic.

The unexpected feature of the equations is that the interactions between all modes, including massive ones, are in the form of gauge invariant "field strengths" just like the Dirac-Born-Infeld equations for the massless sector in open string theory. Furthermore the gauge transformations are of the same form as in the free theory. This is characteristic of an Abelian theory. In the absence of Chan-Paton factors, open string gauge invariance is, in fact Abelian. In the BRST formulation, however, this is not the case. It is possible that a match between the two can be achieved with some field redefinitions.

Another interesting feature was that the equations seem to have their origin in a massless theory in one higher dimension. This pattern was verified

1In the BRST formalism background independence has been discussed in [28, 29, 30, 31]
in some detail for the first few levels and depended on the existence
of solutions to a highly over complete set of linear (algebraic) equations and,
a priori, is quite non trivial. Further insight is required to understand why
this is the case.

A natural question is to enquire whether any of this can carry over to
closed strings. In particular if a gauge invariant ERG can be written, it
is guaranteed to be quadratic. Of course the basic interaction vertices in
closed string theory are also cubic. We also know that the OPE (operator
product expansion) of string vertex operators carries all the information
about string interactions. From this point of view it should be possible to
write a cubic action and thus, quadratic equations for closed string also.
In the RG approach, one expects that the ERG is quadratic in coupling
constants. It is only after solving for all the irrelevant couplings that the
full non polynomial \( \beta \) function for the marginal coupling emerges.

However in BRST closed string field theory the action is non polynomial
[22]. It is at first sight surprising that the main conclusion of this paper is
that indeed gauge invariant quadratic equations can be written down start-
ing from an ERG of the world sheet theory for closed strings, just as in open
string theory. However the gauge invariance here involves transforming the
background field. This is not the same as the original invariance. Never-
theless (as in usual background field formalism) because it is very similar to
the full symmetry it is useful to have manifest at intermediate stages of the
calculation.

The technique used for open strings in I and II, continues to be appli-
cable here with one new ingredient. In closed string theory the world sheet
equation \( \partial_z \bar{\partial}_{\bar{z}} X(z, \bar{z}) = 0 \) ensures that the vertex operators are all of the
form \( \partial^n z \partial^n \bar{z} X \) and do not involve any mixed derivatives: \( \partial^m z \partial^n \bar{z} X \). This can
also be seen from the fact that Green function (in the plane)

\[
G(z, \bar{z}; 0) = \langle X(z, \bar{z}) X(0) \rangle = \ln (z \bar{z}) = \ln z + \ln \bar{z} = \langle X(z) X(0) \rangle + \langle \bar{X}(\bar{z}) \bar{X}(0) \rangle
\]

breaks up into a holomorphic and anti holomorphic part. However in the
presence of a world sheet cutoff this is not true in general. For instance a
short distance cutoff Green function would be \( G(z, \bar{z}; 0, a) = \ln (|z|^2 + a^2) \),
which does not break up into a holomorphic and anti holomorphic part.

This suggests that at least away from the continuum limit, when a finite
cutoff is present, one should have vertex operators involving mixed deriva-
tives. Indeed we will see below that it is essential for gauge invariance.
These vertex operators will not contribute to the S-matrix because their
correlators vanish in the continuum limit by using the equations of motion.

The final result is very similar to that of open strings: We get quadratic
equations of motion with interactions in terms of gauge invariant “field

\[ \text{If we use a Lorentzian world sheet metric, it is possible to regulate the left and right sectors separately: } G(x_L, x_R; 0) = \ln (x_L^2 + a^2) + \ln (x_R^2 + a^2). \text{ Since } x_L, x_R \text{ are real, this is a valid regularization.} \]
strengths” and gauge transformation law unmodified by interactions. This last fact is difficult to reconcile with what we know about gauge transformations in gravity: general coordinate transformations (GCT), which are definitely non-Abelian in form.

At first sight, another problematic feature in this construction is that the field strength construction for the lowest mass state, viz the graviton, makes sense only if the graviton is massive. There is a vector field at this level coming from the mixed (non-holomorphic) vertex operator which, in the unitary gauge is "eaten up" by the graviton, (much as a Goldstone boson is eaten up by a vector field).

Both these problems are resolved by modifying the gauge transformation by including transformations of the coordinates. The first problem (i.e. that the transformation is Abelian) is solved because this induces a tensor rotation of the fields (since they multiply vertex operators). The second problem is solved because the kinetic term involving \( \eta_{\mu\nu} \partial_\mu X^\mu \partial_\nu X^\nu \) is not invariant, and making it invariant involves introducing a background metric and induces an extra term in the definition of the field strength. This non dynamical term (essentially a Christoffel connection for the non dynamical background metric) plays the role of the extra vector field, which is not needed anymore. The graviton thus remains massless.

At this point one makes contact with standard general relativity. The massive graviton phase is analogous to the unbroken phase of a scalar field theory (where the scalar field is the graviton). The massless phase is analogous to the Goldstone phase where \( \langle g_{\mu\nu} \rangle = \eta_{\mu\nu} \). The equations for the massless graviton is covariant under background GCT with a background metric. The original gauge transformation is now part of this symmetry.

This idea can be transcribed in terms of loop variables, which is required for gauge invariance of massive higher spin fields \[32, 33\]. Massive field equations can be written down. One has to suitably define general coordinate transformations such that these massive fields are tensors. If one assumes this, it is easy to write down generally covariant equations. They are also invariant under the (massive) gauge transformations independently.

Thus when all the dust settles, we have gauge invariant and background generally covariant equations for all modes. There is also possibly a space time interpretation for massive gauge transformation. This however requires further analysis.

This paper is organized as follows: In Section 2 we give a brief summary of the techniques used in I and II and the generalization of this technique to closed strings. In Section 3 we give some explicit calculations, for the

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3This somewhat similar to what was suggested in \[25\] where in the presence of Chan Paton factors, there is an extra group rotation symmetry, which give rise to the non Abelian rotation term.

4Note that the gauge transformation of the (massless) graviton is tied to the background coordinate transformations, but the massive gauge transformations are independent.
lowest and second mass levels. In Section 4 we discuss the connection of the
gauge transformations described here and general coordinate transforma-
tions. Section 5 discusses the massive modes. Section 6 contains a summary
and conclusions.

2 Background

We assume the background material in I, II and earlier papers, suitably gen-
eralized for the present discussion. One difference in notation is that we
have consistently used lower indices for $k_{\mu}(t)$ and upper indices for $X^\mu$. In-
dex contractions can be done using $\eta_{\mu\nu}$ because we are in flat space. But
in Section 4 we will introduce a background (but still flat) metric $g_{\mu\nu}^R$
and then one has to be more careful. Typically contractions will be done using
the background metric unless otherwise specified.

2.1 ERG

We first write down an ERG in position space. \footnote{For discussions on various aspects of ERG see \cite{38,39,40,41} in addition to the original references. The position space ERG for the string world sheet has also been discussed in \cite{8,42,43}.} We start with a Euclidean
field theory on the world sheet. $z$ describes the world sheet coordinates. For
open strings, $z = x$ is on the $x$-axis. For closed strings $z = x + iy$. Thus for
closed strings $\int dz$ should be understood as $\int d^2 z$ and $X(z) = X(z, \bar{z})$. The
action is:

$$S = -\frac{1}{2} \int dz \int dz' Y^\mu(z)(G^{-1})_{\mu\nu}(z, z'; \tau)Y^\nu(z') + \int dz L[Y^\mu(z), Y^\mu_n, \bar{m}(z)]$$

Here, $G^{\mu\nu}(z, z'; \tau) \equiv \langle Y^\mu(z)Y^\nu(z') \rangle$ is a cutoff propagator, where \(\tau\) parametrizes
the cutoff. Thus for instance we can take $\tau = \ln a$ where $a$ is a short
distance cutoff or lattice spacing. $\mu = 0, ..., D - 1$ are the usual space
coordinates and $\mu = D$ (we refer to the coordinate as $\theta$), is the co-
ordinate that plays the role of the bosonized ghost of the BRST formal-
ism. We will take $G^{\mu\nu}(z, z'; \tau) = \eta^{\mu\nu}G(z, z'; \tau)$ for $\mu = 0, ..., D - 1$ and
$G^{DD}(z, z'; \tau) = \langle \theta(z)\theta(z') \rangle$. We take $\theta$ to be a massive
field, with a mass of the order of the cutoff $m = O(\frac{1}{a})$. This choice is made in order to reproduce
the S-matrix of string theory \cite{23}. The details of this dimensional reduction
will be discussed later. We need however to keep in mind that $Y^D = \theta$ has
to be treated on different footing from $Y^\mu$, $\mu = 0, ..., D - 1$. Then the ERG
is (suppressing $\tau$):

$$\int du \frac{\partial L[X(u)]}{\partial \tau} =$$
\[
\int dz \int dz' \frac{1}{2} \hat{G}^{\mu\nu}(z, z') \left( \int du \frac{\delta^2 L[X(u)]}{\delta X^\nu(z') \delta X^\mu(z)} + \int dv \frac{\delta L[X(u)]}{\delta X^\mu(z)} \frac{\delta L[X(v)]}{\delta X^\nu(z')} \right)
\]

Here \( \hat{G}^{\mu\nu} \equiv \frac{\partial G^{\mu\nu}}{\partial \tau} \).

### 2.2 Functional Derivatives and Loop Variables

When we use this equation in the loop variable formalism we have to generalize the equation. This is done as follows: In the loop variable formalism we have an infinite number of "time coordinates" \( x_n, n = 1, 2, ... \)

\[
e^{i \int \alpha(t)k(t) \partial_0 X(z(t)) dt + ik_0 X}
\]

with

\[
k(t) = k_0 + \frac{k_1}{t} + \frac{k_2}{t^2} + ... + \frac{k_n}{t^n} + ...
\]

and

\[
\alpha(t) = e^{\sum x_n t^{-n}} = 1 + \frac{\alpha_1}{t} + ... + \frac{\alpha_n}{t^n} + ...
\]

\( \alpha_n \) satisfy: \( \frac{\partial \alpha_n}{\partial x_p} = \alpha_{n-p} \).

In open string theory we had introduced \( Y(z, x_n) \) (we suppress \( \mu \) index for convenience).

\[
Y \equiv X(z) + \alpha_1 \partial_z X(z) + \alpha_2 \partial_z^2 X(z) + \frac{\alpha_3 \partial_z^3 X(z)}{2!} + ... + \frac{\alpha_n \partial_z^n X(z)}{(n-1)!} + ...
\]

with \( Y_n = \frac{\partial Y}{\partial x_n} \). Thus

\[
e^{i \int \alpha(t)k(t) \partial_0 X(z(t)) dt + ik_0 X} = e^{i \sum k_n Y_n}
\]

Furthermore in I we had introduced \( Y_{n_1, n_2} = \frac{\partial^2 Y}{\partial x_{n_1} \partial x_{n_2}} \) and so on. For closed strings we have in addition, \( \tilde{\alpha}_n, \tilde{x}_n, n = 1, 2, ... \). Thus we will let \( z \) stand for the full set \( \{z, x_n, \tilde{z}, \tilde{x}_n\} \). Also \( Y(z, x_n) \) will be extended to \( Y(z, \tilde{z}, x_n, \tilde{x}_n) \) and additionally \( Y_n = \frac{\partial Y}{\partial x_n} \) and \( Y_{n_1, n_2} = \frac{\partial^2 Y}{\partial x_{n_1} \partial x_{n_2}} \), and mixed derivatives, \( Y_{n_1, \tilde{n}_1; n_2, \tilde{n}_2} = \frac{\partial^2 Y}{\partial x_{n_1} \partial \tilde{x}_{\tilde{n}_1}} \) and also higher mixed derivatives, \( Y_{n_1, n_2; \tilde{n}_1, \tilde{n}_2, \tilde{n}_3} \) etc. The closed string loop variable is described below.

Thus in the (2.1.1) we will read \( Y(z) \) for \( X(z) \), with the meaning of \( z \) generalized as above. We can define a cutoff Green function \( \langle Y(z)Y(z') \rangle = G(z, z'; \tau) \) (where also by \( z \) we mean the full set \( \{z, \tilde{z}, x_n, \tilde{x}_n\} \)). Finally we can define the delta function

\[
\delta(z - z') \equiv \delta^2(z - z') \prod_{n=1, 2, ...} \delta(x_n - x'_n) \delta(\tilde{x}_n - \tilde{x}'_n)
\]

and

\[
dz \equiv dz \, d\tilde{z} \prod_{n=1, 2, ...} dx_n \, d\tilde{x}_n
\]
Then we can define the functional derivatives in the usual way:

\[
\frac{\delta}{\delta X(z)} X(u) = \delta(z - u)
\]

but now with the generalized meaning for the fields, coordinates and delta function. Apply this to functional \( \int du \ L[Y(u), Y_{n\tilde{m}}(u)] \): \((x_n \text{ will be associated with } u, \ x'_m \text{ with } z', \text{ and } x''_{n'} \text{ with } z'')\)

\[
\frac{\delta}{\delta Y(z')} \int du \ L[Y(u), Y_{n\tilde{m}}(u)] =
\]

\[
\int du \left\{ \frac{\partial L[Y(u), Y_{n\tilde{m}}(u)]}{\partial Y(u)} \delta(u - z') +
\sum_{n=1,2,...} \frac{\partial L[Y(u), Y_{n\tilde{m}}(u)]}{\partial Y_n(u)} \partial_{x_n} \delta(u - z') +
\sum_{n_1,n_2=1,2,...} \frac{\partial L[Y(u), Y_{n\tilde{m}}(u)]}{\partial Y_{n_1,n_2}(u)} \partial_{x_{n_1}} \partial_{x_{n_2}} \delta(u - z') +
\sum_{m=1,2,...} \frac{\partial L[Y(u), Y_{n\tilde{m}}(u)]}{\partial Y_{n\tilde{m}}(u)} \partial_{x_{n}} \partial_{x_{m}} \delta(u - z') + \ldots \right\}
\]

We have refrained from writing a completely general expression so as not to clutter the equation, but the pattern should be clear to the reader.

### 2.3 Closed Strings and Mixed Derivative Vertex Operators

We now apply this to the closed string action. We need to specify the Lagrangian and then apply the ERG (2.1.1). In the loop variable formalism for closed strings normally we would start with

\[
e^{ik_0 \cdot X(z) + \oint c \, dt \ k(t) \alpha(t) \partial_z X(z + t) + \oint c \, dt \ k(t) \beta(t) \partial_t X(z + t)}
\]

(2.3.8)

In earlier papers on loop variables for closed strings [34] we used the notation \( \tilde{k}_n \). We will use \( k_{\tilde{n}} \) in this paper. This is convenient here, because we are soon going to introduce loop variables with mixed indices. Thus (2.3.8) gives:

\[
\int dz \ e^{i(k_0 Y + \sum_{n,\tilde{n}=1,2,...}(k_n Y_n + k_{\tilde{n}} Y_{\tilde{n}}))} = \int dz \left( e^{ik_0 Y (1 + ik_n Y_n + ik_{\tilde{n}} Y_{\tilde{n}}) \ldots} \right)
\]

(2.3.9)

If in the above set we restrict ourselves to the vertex operators satisfying \( L_0 = L_0 \), that would give the complete closed string Lagrangian level by level in the old covariant (Polyakov) formalism. In the BRST formalism there are states involving the ghost oscillators. Corresponding to these, in the loop variable formalism there are the generalized loop momenta with Lorentz
index in the extra dimension. These were called \( q_n, \bar{q}_n \) in the earlier papers, and will be called \( q_n, \bar{q}_n \) here. It turns out that we also need vertex operators involving mixed derivatives. In \[34\] where the lowest level (graviton) free equation was derived it was seen that there were essential contributions from terms involving \( \frac{\partial^2 \sigma}{\partial z \partial \bar{z}} \). The Liouville mode does not in general factorize into holomorphic and anti holomorphic parts. This is also reflected in the observation made in the introduction that the regulated Green function (eg \( ln (z \bar{z} + a^2) \)) does not split into holomorphic and anti holomorphic part, since dependence on the Liouville mode arises when we regulate the theory. Thus as long as we have a finite cutoff it is not correct to impose \( \partial \partial X = 0 \) and therefore in the ERG we have to introduce mixed derivative vertex operators for consistency.

Indeed when one applies the techniques used in I and II for obtaining gauge invariant ERG in open strings, it becomes clear that we need these extra operators. We recapitulate the basic idea as applied now to closed strings: The gauge variation of the Lagrangian at level \( N = \{ n, \bar{m} \} \) has to be derivatives of lower level terms in the Lagrangian. Thus if \( L_N \) denotes the Lagrangian at level \( N \) its gauge variation has to be of the form:

\[
\delta L_N = \sum_{n,\bar{n}=1,2,...} \lambda_n \frac{\partial L_{N-n}}{\partial x_n} + \lambda_{\bar{n}} \frac{\partial L_{N-\bar{n}}}{\partial \bar{x}_{\bar{n}}} \tag{2.3.10}
\]

In the case of open strings this had the consequence that we had to introduce separately \( Y_{n,m,...} = \frac{\partial^{n+m+...}}{\partial z^n \partial \bar{z}^m \partial z^...} \), although in the original loop variable formalism \( Y_{n,m} = Y_{n+m} \). We refer the reader to II for the full details of this construction at all levels, involving vertex operators of the form \( K_{n,m,...}^\mu Y_{n,m,...}^\mu \). The Lagrangian at any level involves products of these. They can be obtained by expanding the generalized loop variable vertex operator:

\[
e^{i \left( k_0 Y^\mu + \sum_{n,m,...} K_{n,m,...}^\mu Y_{n,m,...}^\mu \right)}
\]

Thus we had at level two\[3\]

\[
L_2 = (iK_{2\mu} Y_2^\mu + iK_{11\mu} Y_{11}^\mu - \frac{1}{2} k_{1\mu} Y_{1}^\mu k_{1\nu} Y_{1}^\nu) e^{ik_0 Y}
\]

with \( K_{2\mu} = y_2 k_{0\mu}, \ K_{11\mu} = k_{2\mu} - y_2 k_{0\mu} \). These were defined with simpler gauge transformation properties: \( \delta K_2 = \lambda_2 k_0, \ \delta K_{11} = \lambda_1 k_1 \). (This is summarized in the next subsection below.) The gauge transformation of this level two Lagrangian \( (L_2) \) is:

\[
\delta L_2 = \lambda_1 \frac{\partial}{\partial x_1} \left( i k_{1\mu} \frac{\partial Y_1^\mu}{\partial x_1} e^{ik_0 Y} \right) + \lambda_2 \frac{\partial}{\partial x_2} \left( e^{ik_0 Y} \right)
\]

\[^9\text{Level 0 is the tachyon which has no gauge transformation properties, so we ignore it in this paper.}\]
which is of the form in (2.3.10) specialized to open strings where we only have \( \lambda_n \).

In the case of closed strings, we need not only \( Y_{n,m..} \) but also \( Y_{n,m..;\bar{n},\bar{m}..} \). This is clear from (2.3.10), where clearly there are mixed derivatives. Thus (2.3.9) is generalized to

\[
\int dz \ e^{i\left(k_0 Y + \sum_{(n,m..;\bar{n},\bar{m}..)} K_{n,m..;\bar{n},\bar{m}..} Y^{n,m..;\bar{n},\bar{m}..}\right)}
\]

(2.3.11)

Now in II we had explicit expressions for \( K_{n,m..} \). In the notation of this paper \( K_{n,m..} = K_{n,m..;0} \). The complex conjugate \( K_{0;\bar{n},\bar{m}..} \) is obviously given by the same expression, complex conjugated. Let us proceed to obtain expressions for the remaining \( K \)’s.

### 2.3.1 Loop Variables for closed strings

We have seen the need for mixed derivatives. Thus the loop variable defined in [34] which was a simple generalization of the open string loop variable is not sufficient. Motivated by this we will generalize our loop variable.

\[
\text{Exp} \left( i \left( k_0 X(z) + \oint_C dt \ k(t) \alpha(t) \partial_z X(z + t) + \oint_C dt \bar{l}(\bar{t}) \bar{\alpha}(\bar{t}) \partial_{\bar{z}} X(\bar{z} + \bar{t}) + \right. \\
\left. + \oint_C dt \oint_C d\bar{t} \ K(t, \bar{t}) \alpha(t) \partial_z \partial_{\bar{z}} X(z + t, \bar{z} + \bar{t}) \right) \right)
\]

(2.3.12)

Expansion for \( k(t), \alpha(t) \) are as given earlier and \( \bar{k}(\bar{t}), \bar{\alpha}(\bar{t}) \) are anti-holomorphic versions of the same. The first three terms in the exponent are the terms given in (2.3.9). The fourth term involves \( K(t, \bar{t}) \) defined below:

\[
K(t, \bar{t}) \equiv K_{0;0} + \sum_{m=1}^{\infty} K_{0;m} t^{-m} + \sum_{n=1}^{\infty} K_{n;0} \bar{t}^{-n} + \sum_{n=1, m=1}^{\infty} K_{n;m} t^{-n} \bar{t}^{-m}
\]

(2.3.13)

Expanding \( X(z + t, \bar{z} + \bar{t}) \) gives

\[
\partial_z \partial_{\bar{z}} X(z + t, \bar{z} + \bar{t}) = \partial_z \partial_{\bar{z}} X + t \partial_z^2 \partial_{\bar{z}} X + \bar{t} \partial_{\bar{z}}^2 \partial_z X + t^2 \frac{\partial_z^3 \partial_{\bar{z}} X}{2!} + \bar{t}^2 \frac{\partial_{\bar{z}}^3 \partial_z X}{2!} + ...
\]

(2.3.14)

Plugging all this in (2.3.12) gives:

\[
k_0 \left( X + \alpha_1 \partial_z X + \alpha_2 \partial_{\bar{z}}^2 X + \frac{\alpha_3 \partial_z^3 X}{2!} + ... + \bar{\alpha}_1 \partial_{\bar{z}} X + \bar{\alpha}_2 \partial_z^2 X + ... + \frac{\alpha_n \partial_z \partial_{\bar{z}}^n X}{(n-1)!} \frac{\partial_{\bar{z}}^n X}{(m-1)!} + ...ight)
\]

\[
+ K_{1;0} \left( \partial_z X + \alpha_1 \partial_{\bar{z}}^2 X + \frac{\alpha_2 \partial_{\bar{z}}^3 X}{2!} + ... + \bar{\alpha}_1 \partial_z X + \bar{\alpha}_2 \partial_{\bar{z}}^2 X + ...ight)
\]
Let us first recollect the construction of $K_{2.4}$. Constructing $K_{2.4}$ will be given in the next subsection. The open string we will nevertheless introduce separately, $K_{2.4}$'s for open strings - Recapitulation

Note that this definition makes sense only if $q$ is not defined. Thus, for instance, $\bar{q}_1$ by itself is not defined.
for open strings. Fortunately this is not needed (for open strings) - only the higher levels $\bar{q}_1^2$ etc, are needed.

\[
= e^{\sum_n y^n t^{-n}} = 1 + \frac{y_1}{t} + \frac{y_2 + y_1^2}{2t^2} + \frac{y_3 + y_1y_2 + y_1^3}{3t^3} + \ldots \tag{2.4.18}
\]

If we solve for $y_n$ in terms of $q_m$ we get

\[
\bar{q}_1 = y_1; \quad \bar{q}_2 = y_2 + \frac{y_1^2}{2} \implies y_2 = \bar{q}_2 - \frac{\bar{q}_1^2}{2},
\]

Similarly

\[
y_3 = \bar{q}_3 - \bar{q}_2 \bar{q}_1 + \frac{\bar{q}_1^3}{3}
\]

In general $\sum_{n=0}^{\infty} \frac{y_n}{t^n} = \ln (\bar{q}(t))$.

The gauge transformation of $q(t)$ (see I,II and [23]) is $q(t) \rightarrow q(t)\lambda(t)$, where

\[
\lambda(t) = 1 + \frac{\lambda_1}{t} + \ldots + \frac{\lambda_n}{t^n} + \ldots = e^{\sum_n z^n t^{-n}}
\]

Thus the gauge transformation on $y_n$ is $y_n \rightarrow y_n + z_n$. To linear order in $\lambda_m$ this becomes:

\[
\delta y_n = \lambda_n
\]

Thus the $K$’s can be defined as follows:

\[
K_{n,m,p\ldots\mu} = y_n y_m y_p \ldots k_{0\mu} : \quad n, m, \ldots \geq 2, \quad n \neq m \neq p\ldots
\]

For repeated indices the rule is

\[
K_{\underbrace{i\times i\times \ldots i\times}_{n \times} m, m, \ldots, \underbrace{i\times i\times \ldots i\times}_{m \times} p, \ldots} = \frac{y_m}{i!} y_n y_p \ldots k_{0\mu}, \quad m, n, p, \ldots \geq 2, \quad m \neq n \neq p\ldots
\]

When some of the indices are equal to 1:

\[
K_{\underbrace{i\times i\times \ldots i\times}_{n \times} 1, 1, \ldots 1, \mu} = y_n y_m y_p \ldots K_{\underbrace{i\times i\times \ldots i\times}_{n \times} 1, 1, \ldots 1, \mu}, \quad n \neq m \neq p\ldots \quad n, m, p\ldots \geq 2
\]

Again with repeated indices:

\[
K_{\underbrace{j\times j\times \ldots j\times}_{m \times} 1, 1, \ldots 1, \mu} = \frac{y_m}{j!} y_p \ldots K_{\underbrace{i\times i\times \ldots i\times}_{m \times} 1, 1, \ldots 1, \mu}, \quad m \neq p\ldots \quad m, p\ldots \geq 2
\]

This recursively defines all the $K$’s provided we give a prescription for $K_{1,1,\ldots,1,\mu}$:

\[
K_{1\mu} = k_{1\mu} \quad K_{11\mu} = k_{2\mu} - K_{2\mu}, \quad K_{111\mu} = k_{3\mu} - K_{21\mu} - K_{3\mu}
\]
The general rule proved in II is
\[ K_{\mu \underbrace{1 \ldots 1}_n} = k_{\mu n} - \sum_{[n]\in [n]' } K_{[n], \mu} \] (2.4.19)
where \([n]'\) indicates all the partitions of \(n\) except \(1 \ldots 1\). 

2.4.2 \(K\)'s for closed strings

Finally for closed strings, we make the identification
\[ K_{\mu n, m, \ldots; 0} = K_{\mu n, m, \ldots} \]
where the RHS are the \(K\)'s that we have just defined. Similarly \(K_{\mu 0; \bar{n}, \bar{m}, \ldots}\) is given by the same expressions with bars i.e. \(\bar{n}\) instead of \(n\), \(k_1\mu\) instead of \(k_1\mu\) etc.

Now we come to the mixed \(K\)'s:
\[ K_{1; \bar{1}} = \bar{y}_1 k_{1\mu} + y_1 k_{1\mu} = \bar{y}_1 K_{1; 0\mu} + y_1 K_{0; \bar{1}\mu} - y_1 \bar{y}_1 k_{0\mu} \] (2.4.20)
One can check that
\[ \delta K_{1; \bar{1}} = \lambda_1 k_{1\mu} + \bar{\lambda}_1 k_{1\mu} \]
Similarly
\[ K_{1; 1; \bar{1}} = \bar{y}_1 (k_{2\mu} - y_2 k_{0\mu}) + \frac{y_1^2}{2} k_{1\mu} - \frac{y_1^2}{2} \bar{y}_1 k_{0\mu} \]
This can be rewritten as
\[ K_{1; 1; \bar{1}} = \bar{y}_1 K_{1; 0\mu} + \frac{y_1^2}{2} K_{0; \bar{1}\mu} - \frac{y_1^2}{2} \bar{y}_1 k_{0\mu} \] (2.4.21)
One can easily verify that
\[ \delta K_{1; 1; \bar{1}} = \lambda_1 K_{1; \bar{1}\mu} + \bar{\lambda}_1 K_{1; 1\mu} \]
as required. The pattern is clear:
\[ K_{\underbrace{1, 1, \ldots, 1}_{n}, \underbrace{\bar{1}, \ldots, \bar{1}}_{m}} = \frac{\bar{y}_1^n}{m!} K_{\underbrace{1, 1, \ldots, 1}_{n}, \underbrace{\bar{1}, \ldots, \bar{1}}_{m}} + \frac{y_1^n}{n!} K_{\underbrace{0, 1, \ldots, 1}_{n}, \underbrace{\bar{1}, \ldots, \bar{1}}_{m}} - \frac{y_1^n \bar{y}_1^m}{n!} k_{0\mu} \] (2.4.22)
Let us check the variation:
\[ \delta K_{\underbrace{1, 1, \ldots, 1}_{n}, \underbrace{\bar{1}, \ldots, \bar{1}}_{m}} = \bar{\lambda}_1 \left( \frac{y_1^n}{(m - 1)!} K_{\underbrace{1, 1, \ldots, 1}_{n}, \underbrace{\bar{1}, \ldots, \bar{1}}_{m}} + \frac{y_1^n}{n!} K_{\underbrace{0, 1, \ldots, 1}_{n}, \underbrace{\bar{1}, \ldots, \bar{1}}_{m}} - \frac{y_1^n \bar{y}_1^m}{n!} k_{0\mu} \right) \]
\[
\lambda_1 \left( \frac{\bar{y}_m}{m!} K_{1,\ldots,1,0} + \frac{y_1^{n-1}}{(n-1)!} K_{0,1,\ldots,1,\mu} - \frac{y_1^{n-1}}{(n-1)!} \frac{\bar{y}_m}{m!} k_{0\mu} \right)
\]

\[
= \lambda_1 K_{1,\ldots,1,\bar{1},\bar{1},\ldots,\bar{1},\mu} + \lambda_1 K_{1,\ldots,1,\bar{1},\bar{1},\ldots,\bar{1},\mu}
\]

One can then check that

\[
K_{p_1,p_2,\ldots,1,\bar{1},\bar{1},\ldots,\bar{1},\mu} = y_{p_1}y_{p_2}\cdots\bar{y}_{q_1}\bar{y}_{q_2}\cdots K_{1,1,\bar{1},\bar{1},\ldots,\bar{1},\mu}
\]

with

\[p_1, p_2, \ldots, q_1, q_2, \geq 2, \quad p_1 \neq p_2 \neq \ldots; \bar{q}_1 \neq \bar{q}_2 \neq \ldots\]

has the right gauge transformation. If any of the \(p\) are repeated \(i\) times, then \(y_p\) is replaced by \(\frac{y_{i\mu}}{i!}\). Similarly for the \(\bar{y}_q\).

This completes the construction of \(K_{\mu[n];[\bar{m}]}\). The loop variables that are involved are the same as for the physical vertex operators of closed string theory, so no new degrees of freedom have been added. However, in principle one could add to \(K_{\mu[n];[\bar{m}]\mu}\), new variables of the form \(k_{\mu[n];[\bar{m}]\mu}\) with transformation rule

\[
\delta k_{\mu[n];[\bar{m}]\mu} = \lambda_p k_{\mu[n];p[\bar{m}]\mu} + \bar{\lambda}_p k_{\mu[n];p[\bar{m}]\mu} (2.4.24)
\]

where as earlier \([n]_i/p\) stands for the particular partition \([n]_i\) with the one \(p\) removed. (If \([n]_i\) does not contain \(p\), that term does not contribute to the gauge transformation, and can be set to zero.) In fact we will do precisely this later on, although the extra variable will be determined algebraically by the existing ones, so new degrees of freedom are still not being added.

### 2.4.3 An interesting relation

The \(K\)’s obey an interesting relation of the form:

\[
\tilde{K}_{n;\bar{m}\mu} = \sum_{i,j} K_{[n]_i;[\bar{m}]_j,\mu} = \bar{q}_n k_{\bar{m}\mu} + \bar{q}_{\bar{m}} k_{n\mu} - \bar{q}_n \bar{q}_{\bar{m}} k_{0\mu} (2.4.25)
\]

Here, as earlier \([n]_i\) denotes a particular partition of \(n\) denoted by \(i\) and \(\bar{q}\) was defined in (2.4.17). Thus for instance

\[
\tilde{K}_{2,1\mu} = K_{2;1\mu} + K_{1,1;1\mu} = \bar{q}_2 k_{1\mu} + \bar{q}_1 k_{2\mu} - \bar{q}_1 \bar{q}_2 k_{0\mu}
\]

The gauge transformation of \(\tilde{K}_{n;\bar{m}\mu}\) under \(\lambda_p\) is easily seen to be:

\[
\delta \tilde{K}_{n;\bar{m}\mu} = \lambda_p \tilde{K}_{n-p;\bar{m}\mu} (2.4.26)
\]
This can be reasoned as follows: The only partitions \([n]_i\) that contribute to the gauge transformation, are the ones that have at least one \(p\). Take these partitions and remove one \(p\). The remaining numbers are all possible ways of making \(n - p\) - so we get all the partitions of \(n - p\). The gauge transformation law then forces (2.4.25) to be true. This relation will be used in the construction of the free equations.

For the free equations one has to keep only single derivatives in the loop variable. Thus we write \(\partial_\alpha \partial_\beta \cdots \partial_\gamma Y\) for \(\partial_\alpha \partial_\beta \cdots \partial_\gamma \bar{m}_n \cdots \partial_\gamma \bar{m}_m Y\). Thus the coefficient of \(Y_\mu n_1 + n_2 + \cdots \bar{m}_1 + \bar{m}_2 + \cdots\) is \(\tilde{K}_{n\bar{m}\mu}\).

Of course one can still add some new variables \(k_{[n]}[\bar{m}][\mu]\) with the correct gauge transformation law (2.4.24), as mentioned earlier and then this would contribute to \(\tilde{K}_{n\bar{m}\mu}\) also. This is in fact done in Appendix B.

### 3 Gauge Invariant Equations and the Problem of the Massive Graviton

We now proceed to evaluate the ERG acting on the closed string world sheet action. The free part involves second derivatives and is evaluated in the Appendix. As in the case of open strings, the interacting part is in the form of products of two gauge invariant field strengths. These are evaluated below and the gauge invariance is manifest. One then has to evaluate the OPE of these in the standard fashion. We do not work this out since the details are not really important at this point.

#### 3.1 Level 2 (1;1)

We start with level 2. The interaction Lagrangian \(L\) at level 2 is best obtained by starting with the generalized loop variable, which we denote by \(L\).

\[
L = e^{i \left( k_0 Y + K_{1,0}Y_{1.0} + K_{1,1}Y_{1.1} + K_{2,0}Y_{2.0} + K_{0,2}Y_{0.2} + K_{1,0}Y_{1.1} + K_{0,1}Y_{0.1} + K_{0,1}Y_{0.1} + \cdots \right)}
\]

(3.1.27)

Let us evaluate the functional derivative (2.2.7) on \(L\). Acting once it gives the gauge invariant field strength. Since we need \(L = \bar{L}\) for the Lagrangian, we can act on \(L\) and extract terms proportional to \(k_{1\mu}k_{1\nu}\) or \(K_{1,1\mu}\), and satisfying \(L = \bar{L}\). This means that the vertex operator has to be \(Y_1^\mu Y_1^{\nu}\) or \(Y_1^\nu Y_1^{\mu}\). The latter will drop out of all integrated correlations functions in the continuum limit so we can drop those terms.
3.1.1 Field Strength

We get

\[
\int du \frac{\delta}{\delta Y^\mu(z')} \mathcal{L}(u) = \int du \left\{ \frac{\partial \mathcal{L}[Y(u), Y_{\bar{n};\bar{m}}(u)]}{\partial Y^\mu(u)} \delta(u - z') + \frac{\partial \mathcal{L}[Y(u), Y_{\bar{n};\bar{m}}(u)]}{\partial \partial_{\bar{x}_1} \delta(u - z')} \right\} \\
+ \frac{\partial \mathcal{L}[Y(u), Y_{\bar{n};\bar{m}}(u)]}{\partial Y_{1,\bar{1}}^\mu(u)} \partial_{\bar{x}_1} \delta(u - z') + \frac{\partial \mathcal{L}[Y(u), Y_{\bar{n};\bar{m}}(u)]}{\partial \partial_{\bar{x}_1} \partial_{\bar{x}_1} \delta(u - z')} \right\}
\]

\[
= \int du \left\{ \frac{\partial \mathcal{L}[Y(u), Y_{\bar{n};\bar{m}}(u)]}{\partial Y^\mu(u)} \delta(u - z') - [\partial_{\bar{x}_1} \frac{\partial \mathcal{L}[Y(u), Y_{\bar{n};\bar{m}}(u)]}{\partial Y_{1,\bar{1}}^\mu(u)} \delta(u - z') + \frac{\partial^2 \mathcal{L}[Y(u), Y_{\bar{n};\bar{m}}(u)]}{\partial \partial_{\bar{x}_1} \partial_{\bar{x}_1} \delta(u - z')}
\]

\[
- [\partial_{\bar{x}_1} \frac{\partial \mathcal{L}[Y(u), Y_{\bar{n};\bar{m}}(u)]}{\partial Y_{1,\bar{1}}^\mu(u)} \delta(u - z') + \frac{\partial^2 \mathcal{L}[Y(u), Y_{\bar{n};\bar{m}}(u)]}{\partial \partial_{\bar{x}_1} \partial_{\bar{x}_1} \delta(u - z')} \right\}
\]

\[
+ [\partial^2 \frac{\partial \mathcal{L}[Y(u), Y_{\bar{n};\bar{m}}(u)]}{\partial \partial_{\bar{x}_1} \partial_{\bar{x}_1} \delta(u - z')} \right\}
\]

\[
= \left\{ iK_{0;\bar{1}} \mathcal{L}(z') - iK_{1,\bar{1}} \partial_{\bar{x}_1} \mathcal{L}(z') - iK_{0;\bar{1}} \partial_{\bar{x}_1} \mathcal{L} + iK_{1,\bar{1}} \partial_{\bar{x}_1} \mathcal{L}(z') + iK_{1,\bar{1}} \partial_{\bar{x}_1} \mathcal{L} + iK_{0;\bar{1}} \partial_{\bar{x}_1} \mathcal{L} - iK_{1,\bar{1}} \partial_{\bar{x}_1} \mathcal{L} + iK_{1,\bar{1}} \partial_{\bar{x}_1} \mathcal{L} \right\}
\]

\[
(3.1.29)
\]

We have kept only terms that contribute to level (1; \bar{1}) and (1; \bar{1}, \bar{1}, \bar{1}). From the structure of \mathcal{L} we can see that

\[
\delta \mathcal{L} = \sum_{n,\bar{n}=1,2,...} (\lambda_n \frac{\partial \mathcal{L}}{\partial x_n} + \lambda_{\bar{n}} \frac{\partial \mathcal{L}}{\partial \bar{x}_n})
\]

\[
(3.1.30)
\]

Using (3.1.30) we can easily check that (3.1.29) is invariant under \lambda_1, \lambda_{\bar{1}} variations, and at level 2, is the gauge invariant field strength for closed strings. We write it explicitly below:

\[
-iK_{0;\bar{1}} \mathcal{L}(z') - iK_{1,\bar{1}} \partial_{\bar{x}_1} \mathcal{L}(z') - iK_{0;\bar{1}} \partial_{\bar{x}_1} \mathcal{L} + iK_{1,\bar{1}} \partial_{\bar{x}_1} \mathcal{L} - iK_{1,\bar{1}} \partial_{\bar{x}_1} \mathcal{L} + iK_{1,\bar{1}} \partial_{\bar{x}_1} \mathcal{L} \]

\[
(3.1.29)
\]

At level 2 the physical fields are the graviton, antisymmetric tensor and dilaton. Since \(K_{1,\bar{1}}\) involves \(q_1\), this field strength is well defined only if
the graviton and dilaton are massive and $q_0 \neq 0$. Thus as things stand, this cannot describe the usual closed string states which are massless at this level. We will describe the resolution of this problem later.

Let us write this equation in terms of space-time fields and analyze the gauge transformations: Define

$$\langle \frac{1}{2} k_{1(\mu} k_{1\nu)} \rangle = h_{\mu\nu} ;$$

$$\langle \frac{1}{2} k_{1[\mu} k_{1\nu]} \rangle = B_{\mu\nu}$$

Let us also define

$$\langle \frac{1}{2} (\lambda_1 k_{1\mu} + \bar{\lambda}_1 k_{1\nu}) \rangle = \epsilon_\mu ;$$

$$\langle \frac{1}{2} (\lambda_1 k_{1\mu} - \bar{\lambda}_1 k_{1\nu}) \rangle = \Lambda_\mu$$

Then the gauge transformation laws are

$$\delta_G h_{\mu\nu} = \partial_{(\mu} \epsilon_{\nu)} ;$$

$$\delta_G B_{\mu\nu} = \partial_{[\mu} \Lambda_{\nu]}$$

which are the expected forms for the linearized transformation for the metric perturbation and antisymmetric tensor associated with coordinate transformations. However, the non-linear part, which should be a tensorial "rotation", we don’t see here. This is a problem if we are to identify these gauge transformations with general coordinate transformations. There is also another problem:

The coefficient of $Y_1^{\mu} Y_1^{\nu}$ can be seen to be

$$- k_{0\mu} k_{1\mu} k_{1\nu} + k_{1\mu} k_{0\mu} k_{1\nu} + k_{1\nu} k_{1\mu} k_{0\nu} - K_{1;1\mu} k_{0\mu} k_{0\nu}$$

In terms of space time fields this is

$$G_{\rho\mu\nu} \equiv \left( - \partial_\rho (h_{\mu\nu} + B_{\mu\nu}) + \partial_\mu (h_{\rho\nu} + B_{\rho\nu}) + \partial_\nu (h_{\rho\mu} + B_{\rho\mu}) \right) - \partial_\mu \partial_\nu S_\rho$$

$$= \Gamma_{\rho\mu\nu} + H_{\rho\mu\nu} - \partial_\mu \partial_\nu S_\rho$$

$H = dB$ is the gauge invariant 3-form field strength for $B$ and $\Gamma$ is the Christoffel connection for gravity. $\Gamma_{\rho\mu\nu} \to \Gamma_{\rho\mu\nu} + 2 \partial_\rho \partial_\nu \epsilon_\rho$ is the linearized gauge transformation for the Christoffel connection. Since $S_\rho \to S_\rho + (\lambda_1 k_{1\rho} + \bar{\lambda}_1 k_{1\mu}) = S_\rho + 2 \epsilon_\rho$ our "gauge invariant field strength" is indeed gauge invariant. This construction thus requires an auxiliary field $S_\mu$ that transforms by an inhomogeneous term (a shift). One could use this shift, $\epsilon$ gauge transformation, to gauge away this field ($S_\rho$). This would use up the gauge transformation and result in extra polarization components for $h_{\mu\nu}$. Thus in effect the graviton would be massive. This is consistent with
the observation made above that the construction of such an $S_{\mu}$ field is only possible if it is massive and $q_0 \neq 0$. Thus as it stands this theory cannot describe gravity. This is the second problem. It will turn out that these problems are interrelated. This analysis also identifies the form of the gauge transformation of $\partial_{\mu} \partial_{\nu} S_{\rho}$ as being that of the Christoffel connection, a fact that will be extremely pertinent in the resolution of the two problems of the massive graviton and the Abelian gauge transformation.

### 3.1.2 Free Equation

The second functional derivative $\int dz' \int dz'' \frac{1}{2} \dot{G}(z', z'') \int du \frac{\delta^2 \mathcal{L}[X(u)]}{\delta X(z')} \delta X(z'')$ gives the free equation.

We need to evaluate

$$\eta^{\mu\nu} \frac{\delta}{\delta X^{(z')}} \int du \left\{ \frac{\partial \mathcal{L}[Y(u), Y_{n,m}(u)]}{\partial Y^{(u)}} \delta(u-z') - \partial_{x_1} \frac{\partial \mathcal{L}[Y(u), Y_{n,m}(u)]}{\partial Y^{(u)}} \delta(u-z') \right\}$$

$$- \partial_{x_1} \frac{\partial \mathcal{L}[Y(u), Y_{n,m}(u)]}{\partial Y^{(u)}} \delta(u-z') + \partial_{x_1} \partial_{x_1} \frac{\partial \mathcal{L}[Y(u), Y_{n,m}(u)]}{\partial Y^{(u)}} \delta(u-z') \}$$

$$= \eta^{\mu\nu} \int du \left[ \frac{\partial}{\partial Y^{(u)}} + \frac{\partial}{\partial x_1} \delta(u-z') \frac{\partial}{\partial Y^{(u)}} + \frac{\partial}{\partial x_1} \frac{\partial}{\partial Y^{(u)}} \right]$$

$$+ \partial_{x_1} \frac{\partial}{\partial Y^{(u)}} \delta(u-z') + \partial_{x_1} \partial_{x_1} \frac{\partial}{\partial Y^{(u)}} \delta(u-z') \}$$

$$= \eta^{\mu\nu} \left\{ \frac{\partial \mathcal{L}[Y(u), Y_{n,m}(u)]}{\partial Y^{(u)}} \delta(u-z') - \partial_{x_1} \frac{\partial \mathcal{L}[Y(u), Y_{n,m}(u)]}{\partial Y^{(u)}} \delta(u-z') \right\}$$

$$+ \partial_{x_1} \frac{\partial \mathcal{L}[Y(u), Y_{n,m}(u)]}{\partial Y^{(u)}} \delta(u-z') + \partial_{x_1} \partial_{x_1} \frac{\partial \mathcal{L}[Y(u), Y_{n,m}(u)]}{\partial Y^{(u)}} \delta(u-z') \}$$

This is evaluated in the Appendix. The result is

$$[ -k_0^2 k_{1 \mu} k_{1 \nu} + k_0 k_1 k_{0 \mu} k_{1 \nu} + k_0 k_1 k_{0 \nu} k_{1 \mu} - K_{1;1} k_0 k_{0 \mu} k_{0 \nu} ] Y_{1;0;0}^{\mu \nu} Y_{1;0}^{\nu \mu} \quad (3.1.39)$$

If we use the constraint $K_{1;1} k_0 = k_1$.k_1 (see Appendices) this equation is just

$$[ -k_0^2 k_{1 \mu} k_{1 \nu} + k_0 k_1 k_{0 \mu} k_{1 \nu} + k_0 k_1 k_{0 \nu} k_{1 \mu} - k_1 k_1 k_{0 \mu} k_{0 \nu} ] Y_{1;0;0}^{\mu \nu} Y_{1;0}^{\nu \mu} \quad (3.1.39)$$

If we use the constraint $K_{1;1} k_0 = k_1$.k_1 (see Appendices) this equation is just

$$\partial^\alpha \Gamma_{\rho \mu \nu} - \partial_{\lambda} \partial_{\mu} h_{\rho}^{\lambda} = -\partial^2 h_{\mu \nu} + \partial_{\mu} \partial_{\nu} h_{\rho}^{\rho} + \partial_{\nu} \partial_{\rho} h_{\mu}^{\rho} - \partial_{\rho} \partial_{\mu} h_{\nu}^{\rho} = 0 \quad (3.1.40)$$

which is gauge invariant. This is the usual linearized equation for the metric perturbation.
3.2 Level 4 (2;2)

3.2.1 Interacting Equation

Let us evaluate the field strength proportional to $Y_{1;0}^\mu Y_{1;0}^\nu Y_{0;1}^\rho Y_{0;1}^\sigma e^{ik_0Y}$. We can extract it from (3.1.29).

The field strength is given by:

$$ik_{0\mu} \frac{(iK_{0;1,1}^1 Y_1)^2}{2!} - iK_{1;0;1}^1(Y_1)(iK_{0;0;1}^1 Y_1) \frac{(iK_{0;1,1}^1 Y_1)^2}{2!}$$

$$-iK_{0;1,1}^1(Y_1)(iK_{0;0;1}^1 Y_1) \frac{(iK_{0;1,1}^1 Y_1)^2}{2!} + iK_{1;1,1}^1(Y_1)(iK_{0;0;1}^1 Y_1)(iK_{0;1,1}^1 Y_1)$$

$$+ iK_{1;1,1}^1(Y_1)(iK_{0;0;1}^1 Y_1)(iK_{0;1,1}^1 Y_1)$$

$$-iK_{1;1,1}^1(Y_1)(iK_{0;0;1}^1 Y_1)(iK_{0;1,1}^1 Y_1) - iK_{1;1,1}^1(Y_1)(iK_{0;0;1}^1 Y_1)^2 + iK_{1;1,1}^1(Y_1)^2$$

$$+ iK_{1;1,1}^1(Y_1)^2$$

$$= (3.2.41)$$

It is easily verified that it is gauge invariant.

Explicit expressions for the $K$'s is given in Section 2. $K_{1;0} = k_1$ and $K_{0;1} = k_1$.

3.2.2 Free Equation

The free equation is evaluated in the Appendix. The result is given below. We have assumed a metric for contraction of indices, this is discussed in the description of the ERG:

$$-\frac{1}{4} k_0^2 (k_1Y_1)^2(k_1Y_1)^2 + \frac{1}{2} k_0k_1(k_0Y_1)(k_1Y_1)(k_1Y_1)^2 + \frac{1}{2} k_0k_1(k_0Y_1)(k_1Y_1)(k_1Y_1)(k_1Y_1)^2$$

$$- k_1^2(k_0Y_1)^2(k_1Y_1)^2 - k_1k_1^2(k_0Y_1)^2(k_1Y_1)^2$$

$$- k_1k_1^2(k_0Y_1)^2(k_1Y_1)^2$$

$$= (3.2.42)$$

One can extract the coefficient of $Y_{1}^\mu Y_{1}^\nu Y_{1}^\rho Y_{1}^\sigma$ to get:

$$-k_0^2 k_1 k_{1\mu} k_{1\rho} k_{1\sigma} + k_0 k_1 k_{0(\mu} k_{1\nu)\lambda k_{1\sigma}) + k_0 k_1 k_{0(\mu} k_{1\rho)k_{1\lambda}k_{1\sigma} - k_0 k_1 k_{1\mu} k_{1\nu}$$

$$- k_1 k_{1\mu} k_{1\rho} k_{0\nu} - k_1 k_{1\mu} k_{1\rho} k_{0\nu}$$

$$= (3.2.43)$$

Let us define $\langle k_{1\mu} k_{1\nu} k_{1\rho} k_{1\sigma} \rangle = S_{\mu\rho\sigma\lambda}$. This is a "spin 4" tensor symmetric in the first two and last two indices.

Defining $\partial_{(\sigma} \partial_{(\nu} S_{\mu\lambda\rho)}$ as the sum of four terms symmetrized in $\mu\nu$ and $\rho\sigma$ and $\partial^\lambda \partial_{(\sigma} S_{(\mu\lambda\rho)}$ as the sum of two terms symmetrized in $\rho\sigma$, we can write this equation as

$$-\partial^2 S_{\mu\rho\sigma\lambda} + \partial^\lambda \partial_{(\mu} S_{\nu)} + \partial^\lambda \partial_{(\sigma} S_{(\mu\lambda\rho)}$$
\[ - \partial_\mu \partial_\nu S^\lambda_{\rho \sigma \lambda} - \partial_\mu \partial_\nu S^\lambda_{\mu \nu \rho \sigma} + \partial_\sigma (\partial_\nu S^\lambda_{\rho \mu \sigma}) = 0 \tag{3.2.44} \]

The gauge transformation is

\[ k_1^{\mu} \rightarrow k_1^{\mu} + \lambda_1 k_0^{\mu}; \quad k_\bar{1}^{\mu} \rightarrow k_\bar{1}^{\mu} + \lambda_\bar{1} k_0^{\mu}. \]

Defining \( \langle \lambda_1 k_1^{\mu} k_1^{\rho} k_1^{\sigma} \rangle = \Lambda_{\mu \rho \sigma} \) and \( \langle \lambda_\bar{1} k_\bar{1}^{\mu} k_1^{\nu} k_\bar{1}^{\rho} \rangle = \bar{\Lambda}_{\mu \nu \rho} \) the gauge transformation is

\[ \delta S_{\mu \nu \rho \sigma} = \partial_{(\mu} \Lambda_{\nu)\rho\sigma} + \partial_{(\rho} \bar{\Lambda}_{\mu\nu)\sigma} \tag{3.2.45} \]

The gauge parameter obeys a tracelessness constraint: the trace on any two indices is zero.

### 3.3 Equation for Spin 4 Interacting with Two Gravitons

We can write down a term in the full interacting equation corresponding to \( Y_1^\mu Y_1^\nu Y_1^\rho Y_1^\sigma \) by combining (3.1.36) and (3.2.44):

\[ \int dz \, \dot{G}(z, z'; \tau) (-\partial^2 S_{\mu \nu \rho \sigma} + \partial^\lambda (\partial_\mu S_{\nu} \rho \lambda) + \partial^\lambda (\partial_\mu S_{\nu} \rho \lambda) + \partial_{(\rho} \bar{\Lambda}_{\mu\nu)\sigma}) + \int dz \int dz' \, \dot{G}(z, z'; \tau) G^\lambda_{\mu \rho} G_{\lambda \nu \sigma} + \ldots = 0 \tag{3.3.46} \]

The three dots indicate other interactions. \( G_{\mu \nu \rho} \) is modified in the next section to (4.6.62). The terms described above correspond to a cubic interaction between two lowest level fields (either a graviton or antisymmetric tensor or dilaton) and a massive spin 4 field. Elucidating the full structure requires doing the dimensional reduction, which discussion we postpone.

As mentioned earlier, the field strength \( G_{\mu \nu \rho} \) will be modified in the next section. As things stand the graviton described by this field strength is massive. But once \( G_{\mu \nu \rho} \) is modified the above equation is correct.

### 4 General Coordinate Transformation and Massless Graviton

Let us restate our problems: One was that a gauge invariant EOM for the graviton can be written only when it is massive - the construction of the field \( S_{\rho \mu \nu} \) in terms of loop variables required a non zero mass. The second problem is that the gauge transformations are Abelian and seem to have nothing to do with coordinate transformations. Let us focus on the second problem first.

#### 4.1 Combining Coordinate transformations and Gauge transformation

The gauge transformation, which we call \( \delta_G \), on the graviton has the form

\[ \delta_G h_{\mu \nu} = \tilde{\epsilon}_{(\mu, \nu)} \]
We should compare this with what we know from General Relativity (GR).
In GR the metric tensor obeys:

\[ g_{\mu\nu}(X) dX^\mu dX^\nu = g'_{\mu\nu}(X') dX'^\mu dX'^\sigma \]

We then find that (for infinitesimal \(\epsilon\))

\[ \delta_{\text{GCT}} g_{\mu\nu}(X) \equiv g'_{\mu\nu}(X) - g_{\mu\nu}(X) = \epsilon^\lambda g_{\mu\nu,\lambda} + \epsilon^\lambda_{\mu} g_{\lambda\nu} + \epsilon^\lambda_{\nu} g_{\mu\lambda}; \quad \delta_{\text{GCT}} X^\mu = -\epsilon^\mu(X) \]

The subscript ”GCT” stands for General Coordinate Transformation. This is the standard tensor transformation law. We can now extract from this the transformation law for \(h_{\mu\nu}\).

Write \(g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\). Then (4.1.47) becomes

\[ \delta_{\text{GCT}} g_{\mu\nu} = \delta_{\text{GCT}} h_{\mu\nu} = \epsilon^\lambda h_{\mu\nu,\lambda} + \epsilon^\lambda_{\mu} (\eta_{\lambda\nu} + h_{\lambda\nu}) + \epsilon^\lambda_{\nu} (\eta_{\mu\lambda} + h_{\mu\lambda}) \]

where we have defined \(\epsilon_\mu \equiv \eta_\mu \epsilon^\nu\).

We see that \(\delta_{\text{GCT}}\) acting on \(h_{\mu\nu}\) has two parts: an Abelian inhomogeneous term \(\epsilon_{(\mu,\nu)}\) and a non-Abelian ”rotation”: \(\epsilon^\lambda h_{\mu\nu,\lambda} + \epsilon^\lambda_{\mu} h_{\lambda\nu} + \epsilon^\lambda_{\nu} h_{\mu\lambda}\). This is analogous to Yang-Mills where there is an Abelian part to the gauge transformation \(\delta A_\mu = \partial_\mu \Lambda\) and a non-Abelian rotation \(\delta A_\mu = \Lambda \times A_\mu\). The non-Abelian part is just a tensorial transformation. We will refer to this tensorial transformation as \(\delta_T\), ”T” stands for ”tensor". Thus

\[ \delta_T h_{\mu\nu} = \epsilon^\lambda h_{\mu\nu,\lambda} + \epsilon^\lambda_{\mu} h_{\lambda\nu} + \epsilon^\lambda_{\nu} h_{\mu\lambda}; \quad \delta_T X^\mu = -\epsilon^\mu \]

Thus acting on proper tensors, \(\delta_T = \delta_{\text{GCT}}\). But on non tensorial fields such as \(h_{\mu\nu}\) they are not the same. Note however that

\[ (\delta_G + \delta_T) h_{\mu\nu} = \bar{\epsilon}_{(\mu,\nu)} + \epsilon^\lambda h_{\mu\nu,\lambda} + \epsilon^\lambda_{\mu} h_{\lambda\nu} + \epsilon^\lambda_{\nu} h_{\mu\lambda} \]

Thus if we identify the gauge transformation parameter \(\bar{\epsilon}_\mu = \epsilon_\mu\), then, the combined transformation \((\delta_G + \delta_T) h_{\mu\nu} = \delta_{\text{GCT}} h_{\mu\nu}\).

This suggests the following: It is easy to make EOM for proper tensor fields covariant under coordinate transformations. We can introduce a background metric and background covariant derivatives and write covariant EOM. But \(h_{\mu\nu}\) is not quite a tensor. However we know that \(\delta_G\) is already a symmetry of the EOM. So what is left is to implement \(\delta_T\), which is a tensorial transformation, and therefore easy to implement. Then we should have general covariance for the \(h_{\mu\nu}\) equation.

In order to implement \(\delta_T\) we need to check if the Lagrangian is invariant. The interaction part \(L_{\text{int}}\) is manifestly invariant: \(h_{\mu\nu} \partial_z X^\mu \partial_z X^\nu\) is clearly invariant under \(\delta_T\). (This is also true for the massive higher spin fields with
some modifications discussed later.) However the kinetic term $\eta_{\mu\nu}\partial_z X^\mu \partial_z X^\nu$ is not because $\eta_{\mu\nu}$ is a fixed matrix - and does not transform.

$$\delta_T(\eta_{\mu\nu}\partial_z X^\mu \partial_z X^\nu) = -\epsilon_{(\mu,\nu)} \partial_z X^\mu \partial_z X^\nu \quad (4.1.49)$$

Non invariance of the kinetic term is very inconvenient because then the Green function is not covariant. Thus we make the kinetic term invariant and transfer this non invariance to the interaction Lagrangian.

The kinetic term can be made invariant by the standard technique of introducing a background ”reference” metric (and then covariantize the ERG) which we call $g^R_{\mu\nu}(X)$.

For the purposes of this paper, we will keep the geometry flat, so that $g^R_{\mu\nu}$ is equivalent up to coordinate transformation to $\eta_{\mu\nu}$. (Thus for instance, for infinitesimal $\xi$, $g^R_{\mu\nu}$ can be parametrized as $g^R_{\mu\nu} \equiv \eta_{\mu\nu} + h^R_{\mu\nu} = \eta_{\mu\nu} + \xi_{(\mu,\nu)}$, where we take the new coordinates to be given by $X^\mu = X^\mu - \xi^\mu(X)$.)

Now let us add and subtract $h^R_{\mu\nu}(X)\partial_z X^\mu \partial_z X^\nu$ to the action:

$$S = \int dz \left( \eta_{\mu\nu} + h^R_{\mu\nu}(X) \right) \partial_z X^\mu \partial_z X^\nu + \frac{h^R_{\mu\nu}}{L_{\text{kinetic}}} \partial_z X^\mu \partial_z X^\nu + \frac{h^R_{\mu\nu}}{L_{\text{interactions}}} \partial_z X^\mu \partial_z X^\nu + \ldots$$

The three dots stand for other vertex operators. Now let us perform the coordinate transformation $\delta_T X^\mu \equiv -\epsilon^\mu$. We will choose an action on $h^R_{\mu\nu}(X)$ such that the kinetic term is invariant. Clearly we need to cancel (4.1.49) so we need

$$\delta_T(h^R_{\mu\nu}(X)\partial_z X^\mu \partial_z X^\nu) = \epsilon_{(\mu,\nu)} \partial_z X^\mu \partial_z X^\nu \quad (4.1.50)$$

This is more explicitly written as

$$\delta_T h^R_{\mu\nu}(X) = \epsilon^\lambda h^R_{\mu\lambda} + \epsilon^\lambda h^R_{\nu\lambda} + \epsilon^\lambda h^R_{\mu\nu} + \epsilon_{(\mu,\nu)}; \quad \delta_T X = -\epsilon^\mu(X) \quad (4.1.51)$$

This gives the usual transformation of the background metric $g^R_{\mu\nu}$ under GCT. Thus we have $\delta_T L_{\text{kin}} = 0$. It is important to observe that although $g^R_{\mu\nu}$ is a background metric in the usual sense, $h_{\mu\nu}$ continues to defined as a fluctuation about $\eta_{\mu\nu}$, and not about $g^R_{\mu\nu}$. Thus $L_{\text{int}}$ involves $h_{\mu\nu} - h^R_{\mu\nu}$ now. Another way of seeing this is that since we have added and subtracted $h^R_{\mu\nu}(X)\partial_z X^\mu \partial_z X^\nu$, we really have not done anything physically different, so the physical interpretation of $h_{\mu\nu}$ remains the same.

Now $L_{\text{int}}$ is not invariant:

$$\delta_T L_{\text{int}} = \delta_T[(h_{\mu\nu}(X) - h^R_{\mu\nu}(X))\partial_z X^\mu \partial_z X^\nu] = -\epsilon_{(\mu,\nu)} \partial_z X^\mu \partial_z X^\nu$$

We can recover invariance if we combine $\delta_C$ with $\delta_T$ i.e. choose the gauge parameter $\tilde{\epsilon}_\mu$ to be related to the coordinate transformation parameter $\epsilon^\mu$:
\[ \tilde{c}_\mu = \eta_{\mu\nu} c^\nu \equiv c_{(\mu\nu)} \]

Then using \( \delta_G h_{\mu\nu}(X) = \epsilon_{(\mu\nu)} \) we get

\[
(\delta_T + \delta_G) L_{\text{int}} = (\delta_T + \delta_G)[(h_{\mu\nu}(X) - h_{\mu\nu}^R(X)) \partial_\mu X^\mu \partial_\nu X^\nu] = 0 \quad (4.1.52)
\]

Thus \((\delta_G + \delta_T) L_{\text{Kin}} = 0 = (\delta_G + \delta_T) L_{\text{int}}\). Since both terms are separately invariant, we can expect that the EOM obtained from the ERG will have this invariance manifest. Note that the combination \( h_{\mu\nu} - h_{\mu\nu}^R \equiv \tilde{h}_{\mu\nu} \) transforms as a proper tensor under \( \delta_T + \delta_G \).

4.2 Summary

Let us summarize the above: Our starting point is

\[
Z[h_{\mu\nu}, S_{\mu\nu\rho\sigma}, ...] = \int \mathcal{D}X e^{\int dz \left( (\eta_{\mu\nu} + h_{\mu\nu}^R) \partial_\mu X^\mu \partial_\nu X^\nu + S_{\mu\nu\rho\sigma} \partial_\mu X^\mu \partial_\nu X^\nu \partial_\rho X^\rho \partial_\sigma X^\sigma + ... \right)}
\]

We would like to treat \( h_{\mu\nu} \) as an interaction term. This is because we would like an exact RG equation that treats all the string modes in the same way. General covariance is then not manifest at any finite order. So we turn to a background field approach and introduce a non dynamical background \( h_{\mu\nu}^R \) as follows:

\[
\begin{align*}
\int \mathcal{D}X e^{\int dz \left( (\eta_{\mu\nu} + h_{\mu\nu}^R) \partial_\mu X^\mu \partial_\nu X^\nu + (h_{\mu\nu} - h_{\mu\nu}^R) \partial_\mu X^\mu \partial_\nu X^\nu + S_{\mu\nu\rho\sigma} \partial_\mu X^\mu \partial_\nu X^\nu \partial_\rho X^\rho \partial_\sigma X^\sigma + ... \right)} \\
= \int \mathcal{D}X e^{\int dz (g_{\mu\nu}^R \partial_\mu X^\mu \partial_\nu X^\nu + \tilde{h}_{\mu\nu} \partial_\mu X^\mu \partial_\nu X^\nu + S_{\mu\nu\rho\sigma} \partial_\mu X^\mu \partial_\nu X^\nu \partial_\rho X^\rho \partial_\sigma X^\sigma + ...)}
\end{align*}
\]

Since the action does not depend on \( h_{\mu\nu}^R \), the final answer should be independent of \( h_{\mu\nu}^R(X) \). Thus formally

\[
\frac{\delta Z}{\delta h_{\mu\nu}^R(k)} = 0 \quad (4.2.54)
\]

Equivalently

\[
Z[g_{\mu\nu}^R, \tilde{h}_{\mu\nu}] = Z[g_{\mu\nu}^R + \tilde{h}_{\mu\nu}] = Z[g_{\mu\nu}]
\]

We have a non dynamical fixed background metric \( g_{\mu\nu}^R = \eta_{\mu\nu} + h_{\mu\nu}^R \). This will be treated as a background field and will be incorporated into the propagator. \( g_{\mu\nu}^R, \tilde{h} \) are proper tensors under background GCT, which, in addition to transforming \( h_{\mu\nu} \) (and the coordinate \( X^\mu \)), also includes transforming \( h_{\mu\nu}^R \). (In fact \( h \) and \( h^R \) have similar transformations, which is why the difference \( \tilde{h}_{\mu\nu} \) transforms as a proper tensor.) \( g_{\mu\nu}^R \) is incorporated into the kinetic term.

\footnote{Note that \( \eta_{\mu\nu} \) is being used in this equation and not \( g_{\mu\nu}^R \).}
and is treated non perturbatively, while $\tilde{h}_{\mu\nu}$ will be treated perturbatively like any other ‘matter’ field as an interaction term.

$Z$ is evaluated by expanding in a power series in $\tilde{h}$ and will be non polynomial in $\tilde{h}$. If we let $\tilde{h}_{\mu\nu}(X) = \int dk \tilde{h}_{\mu\nu}(k)e^{ik.X}$, we can think of $\tilde{h}_{\mu\nu}(k)$ as an infinite number of coupling constants. There is a finite UV cutoff in place and the functional integral is well defined. Thus, assuming a non zero radius of convergence for the perturbation series, the formal statement ([4.2.54]) is expected to hold (on replacing $\tilde{h} = h - h^R$ in all the terms) after summing the series. We do not have a proof of this however.

Symmetry under GCT on the physical $h$ is not manifest when we treat $h$ perturbatively. However the background GCT symmetry, which also involves $h^R$ is manifest order by order. This is because, as mentioned above, under background transformations $\tilde{h}$, and $g^R_{\mu\nu}$ are tensors and both interaction and kinetic terms are separately invariant. This is the gist of the earlier paragraphs of this section. The EOM for $\tilde{h}$ can thus be made (background) covariant. This construction is very similar to what is done in the background field method for gauge theories [51]. There the field $A_\mu$ is replaced by $\tilde{Q}_\mu + W_\mu$, where $W_\mu$ is an arbitrary background. The gauge transformation of the background field has the inhomogeneous term and $\tilde{Q}_\mu$ transforms homogeneously:

$$\delta W_\mu = \partial_\mu \Lambda + \Lambda \times W_\mu; \quad \delta \tilde{Q}_\mu = \Lambda \times \tilde{Q}_\mu$$

The analog of $W + \tilde{Q}$ is $h_{\mu\nu}$.

Since the final answer for $Z$ cannot depend on $h^R$ (after the substitution $\tilde{h} = h - h^R$) because of ([4.2.54]), or equivalently, depends only on the sum $g^R + \tilde{h}$, background covariance of the equations also implies general covariance.

In the $\beta$ function method, rather than $Z$ one calculates $\frac{\partial Z}{\partial (\ln a)}$. But the covariance arguments are the same. When we include all other massive modes, one has to solve for all the massive modes in terms of $\tilde{h}$ first. This is equivalent to starting with the ERG that includes all the irrelevant coupling constants and obtaining the low energy $\beta$ function. (This is explained in [16]). Assuming that the massive modes are all tensors, background covariance continues to hold order by order, and once again if we invoke ([4.2.54]), this implies that the final low energy $\beta$ function for $h$ must be covariant.

The role played by the background field is illustrated by a toy example in Appendix C.

4.3 Normal Coordinates

This subsection is a digression describing in some detail how normal coordinates are introduced. The main result is that one can obtain results that are manifestly background covariant. If the reader is willing to accept this, this section may be skipped.
To obtain the background covariant equations one can use standard techniques of Riemann Normal Coordinates [48, 49, 50]. However in our case since we start with a flat geometry, we can set all (background) curvature tensors to zero in their equations. This then becomes a rather trivial application of RNC.

We begin with a point O with coordinates $x^\mu_0$ in some coordinate system, and another point P, with coordinates $x^\mu$. The geodesic that connects the two points has a geometric meaning independent of the coordinate system. Similarly the tangent vector $\xi^\mu$ (of unit length) to this geodesic at O is also a geometric object. One can derive from the geodesic equation [49]:

$$x^\mu = x_0^\mu + t\xi^\mu - \frac{t^2}{2!}\xi^\rho \xi^\sigma \Gamma^\mu_{\rho\sigma} - \frac{t^3}{3!}\xi^\rho \xi^\sigma \xi^\lambda \Gamma^\mu_{\rho\sigma\lambda} + \ldots$$

Here $t$ is the length of the geodesic from O to P and is also a geometric quantity. We can let $t\xi^\mu = y^\mu$ which also has a geometric meaning independent of the coordinate system (it is a vector at O) and write

$$x^\mu(x_0, y) = x_0^\mu + y^\mu - \frac{1}{2!}y^\rho y^\sigma \Gamma^\mu_{\rho\sigma} - \frac{1}{3!}y^\rho y^\sigma y^\lambda \Gamma^\mu_{\rho\sigma\lambda} + \ldots$$

Finally we can introduce RNC, $Y^\mu = x_0^\mu + y^\mu$. The coordinate transformation from $x$ to $Y$ defines a matrix, which varies from point to point, $T^\mu_{\rho}(x) = \frac{\partial Y^\mu}{\partial x^\rho}$ and can be used to transform tensors from one coordinate system to another.

In the RNC, $Y$, one can perform an ordinary Taylor expansion for any tensor $\bar{W}(Y) = \bar{W}(x_0 + y)$ in powers of $y^\mu$ and obtain:

$$\bar{W}_{\alpha_1 \ldots \alpha_p}(Y) = \bar{W}_{\alpha_1 \ldots \alpha_p}(x_0) + \bar{W}_{\alpha_1 \ldots \alpha_p, \mu}(x_0)y^\mu +$$

$$\frac{1}{2!}\{\bar{W}_{\alpha_1 \ldots \alpha_p, \mu\nu}(x_0) - \frac{1}{3} \sum_{k=1}^{p} \bar{R}^\beta_{\mu\alpha_k\nu}(x_0)\bar{W}_{\alpha_1 \ldots \alpha_{k-1}\beta\alpha_{k+1} \ldots \alpha_p}(x_0)\}y^\mu y^\nu +$$

$$\frac{1}{3!}\{\bar{W}_{\alpha_1 \ldots \alpha_p, \mu\nu\rho}(x_0) - \sum_{k=1}^{p} \bar{R}^\beta_{\mu\alpha_k\nu\rho}(x_0)\bar{W}_{\alpha_1 \ldots \alpha_{k-1}\beta\alpha_{k+1} \ldots \alpha_p\rho}(x_0)\}y^\mu y^\nu y^\rho + \ldots$$

(4.3.55)

where the derivatives are covariant derivatives. We have put bars over all the tensors to indicate that this expansion requires a normal coordinate system. The LHS is a tensor at $Y = x_0 + y$, whereas the RHS is a sum of tensors at $x_0$. Hence this expansion makes sense only in this particular coordinate system. In the case of a scalar, the LHS is invariant: $\phi'(x') = \phi(x)$ and also every term on the RHS is individually a scalar and hence invariant. Thus if the expansion is true in one coordinate system it is true in any
coordinate system. For a tensor, under a coordinate change, the LHS will have to be multiplied by an appropriate number of $T^\mu_\rho(x)$, to transform it to a new frame, whereas the RHS will need factors of $T^\mu_\rho(x_0)$. Thus the equation holds in any coordinate system if the RHS is multiplied by factors of $T^\mu_\rho(x)(T^{-1})^\rho_\sigma(x_0)$.

We can apply this to the metric tensor $g^R_{\rho\sigma}$. Thus the equation becomes:

$$\bar{g}_{\rho\sigma}(Y) = \bar{g}_{\rho\sigma}(x_0) - \frac{1}{3} \bar{R}_{\sigma\mu\rho\nu}(x_0)y^\mu y^\nu + \ldots$$

In particular if $R = 0$ we get $\bar{g}_{\rho\sigma}(Y) = \bar{g}_{\rho\sigma}(x_0)$, which is just the statement that in RNC, a flat metric is constant. We can transform to arbitrary coordinates and get the expected result, that the (flat) metric is no longer constant:

$$g_{\mu\nu}(x) = (T(x)T^{-1}(x_0))^{\rho}_\mu (T(x)T^{-1}(x_0))^{\rho}_\sigma g_{\rho\sigma}(x_0)$$

Now consider the kinetic term $g^R_{\mu\nu}(X)\partial_\xi X^\mu \partial_\xi X^\nu$. This is a scalar, so we will first write it in the RNC coordinates as

$$g^R_{\mu\nu}(Y)\partial_\xi Y^\mu \partial_\xi Y^\nu = g^R_{\mu\nu}(x_0 + y)\partial_\xi y^\mu \partial_\xi y^\nu$$

$$= (g^R_{\mu\nu}(x_0) - \frac{1}{3} \bar{R}^R_{\sigma\alpha\beta}(x_0)y^\alpha y^\beta + \ldots)\partial_\xi y^\mu \partial_\xi y^\nu$$

Let us once again specialize to the case where the metric $g^R$ is flat. Then we have

$$= g^R_{\mu\nu}(x_0)\partial_\xi y^\mu \partial_\xi y^\nu$$

(4.3.57)

Since this expression is a scalar (remember that $y^\mu$ is a geometric object and has a meaning independent of coordinates), the kinetic term is the same in any coordinate system, so we can remove the bars:

$$= g^R_{\mu\nu}(x_0)\partial_\xi y^\mu \partial_\xi y^\nu$$

(4.3.56)

$y^\mu$ will change to a transformed vector $y'^\mu$, but it is a variable of integration, so we have retained the same notation. Now let us turn to the interaction term

$$\tilde{h}_{\mu\nu}(X)\partial_\xi X^\mu \partial_\xi X^\nu$$

This is a scalar and we can write it directly in RNC to get

$$= \tilde{h}_{\mu\nu}(Y)\partial_\xi Y^\mu \partial_\xi Y^\nu = \tilde{h}_{\mu\nu}(x_0 + y)\partial_\xi y^\mu \partial_\xi y^\nu$$

We can Taylor expand using (4.3.56) and setting background curvature to zero, to get

$$= (\tilde{h}_{\mu\nu}(x_0) + y^\rho \nabla^R_\rho \tilde{h}_{\mu\nu}(x_0) + \frac{1}{2!} y^\rho y^\sigma \nabla^R_\rho \nabla^R_\sigma \tilde{h}_{\mu\nu}(x_0) + \ldots)\partial_\xi y^\mu \partial_\xi y^\nu$$

25
Once again each term in the expansion is a scalar and has the same value in any coordinate system, so we can remove the bars:

\[ \langle h_{\mu \nu}(x_0) + y^\rho \nabla^R_{\mu} \tilde{h}_{\mu \nu}(x_0) + \frac{1}{2!} y^\rho y^\sigma \nabla^R_{\rho} \nabla^R_{\sigma} \tilde{h}_{\mu \nu}(x_0) + \ldots \rangle \partial_\mu y^\nu \partial_\nu y^\nu \]

We can also write the above as

\[ \int dk \, \langle h_{\mu \nu}(x_0, k_0) e^{ik_\mu y_\mu} \partial_\mu y^\nu \partial_\nu y^\nu \rangle \]

where each power of \( k_\mu \) is a background covariant derivative \( \nabla^R_\mu \). Since curvature tensors are all zero, the covariant derivatives commute, so this is consistent. Similar expansions have to be made for the massive modes. This involves some subtlety because loop variables are necessary. This is discussed in the section 5. Now we have manifestly background covariant kinetic and interaction terms and we can apply the ERG.

### 4.4 Free Equation for Graviton

Let us now write down the free equation for \( h_{\mu \nu} \). We set \( \langle K_{1;1} \rangle = S_\mu = 0 \) and \( \langle k_1 k_1 \rangle = \tilde{h}_{\mu \nu} = (h_{\mu \nu} - h_{R, \mu \nu}) \). We also replace \( k_0 \rightarrow \nabla^R_\mu \) in (3.1.40). This gives

\[ [-k_0^2 k_{1,\mu} k_{1,\nu} + k_0 k_1 k_{0,\mu} k_{1,\nu} - k_0 k_1 k_{0,\mu} k_{0,\nu}] Y_{1;0}^{\mu} Y_{0;1}^{\nu} = 0 \]

\[ \Rightarrow - (\nabla^R)^2 (h_{\mu \nu} - h_{R, \mu \nu})(x_0) + \nabla^R_\mu \nabla^R_\nu (h_{\mu \nu} - h_{R, \mu \nu})(x_0) = 0 \]

The argument of the fields, \( x_0 \), is explicitly indicated - this is a tensor equation at \( x_0 \). We have set \( K_{1;1} = 0 \) so the K-constraint is \( k_0, K_{1;1} = k_1, k_1 = 0 \) - see the next paragraph.

### 4.5 The dilaton and K-constraint for the graviton

At this point we can observe the following: Parametrizing \( h_{R, \mu \nu} \) by \( \xi_{(\mu, \nu)} \) (for small \( h_{R} \)) we can write, integrating by parts on either \( z \) or \( \bar{z} \):

\[ \xi_{(\mu, \nu)} \partial_z \partial_{\bar{z}} X^\mu = - \partial_{\bar{z}} (\xi_{(\mu, \nu)}) \partial_z X^\mu = - \xi_{(\mu, \nu)} \partial_z X^\mu \partial_{\bar{z}} X^\nu = - \xi_{(\mu, \nu)} \partial_z X^\mu \partial_{\bar{z}} X^\nu \]

\[ = - \frac{1}{2} \xi_{(\mu, \nu)} \partial_z X^\nu \partial_{\bar{z}} X^\mu \]

Thus adding the reference metric term is equivalent, as far as symmetry properties are concerned, to adding the mixed derivative loop variable term \( K_{1;1}^\mu \partial_z \partial_{\bar{z}} X^\mu e^{ik_0 X} \). Thus in the present construction we do not need it and we can set \( K_{1;1}^\mu = 0 \).

The K-constraint for the graviton now reads as

\[ K_{1;1} k_0 = k_1 k_1 = k_1 k_1 + q_1 q_1 = 0 \]  

(4.5.59)
where we have separated out the $D+1$th coordinate. $\langle q_1 q_1 \rangle = \Phi_D$ is the dilaton. So the constraint reads

$$h^{R\mu}_\mu - h^\mu_\mu + \Phi_D = 0 \quad (4.5.60)$$

Index contractions are done using $g^{R\mu\nu}$. This equation is gauge covariant under $\delta_T + \delta_G$. It relates the trace of the metric to the dilaton. We remind the reader that in the old covariant formulation of string theory, the role of the dilaton is played by the trace of the metric. The physical graviton is transverse and traceless. In the present formalism, we have both the trace of the metric and a dilaton. They are related by the constraint but the constraint does not fix gauge because it is gauge covariant.

4.6 Interactions and Gauge Invariant Field Strength

The expression for the gauge invariant field strength at level 2 is

$$-k_0 k_{1\mu} k_{1\nu} + k_{1\rho} k_{0\mu} k_{0\nu} + k_{1\rho} k_{1\mu} k_{0\nu} - K_{1;1;\rho} k_{0\mu} k_{0\nu} \quad (4.6.61)$$

In terms of space time fields this is

$$\left( -\nabla^R_\rho (h_{\mu\nu} - h^{R\mu\nu}) + \nabla_\mu (h_{\rho\nu} - h^{R\rho\nu}) + B_{\rho\nu} \right) + \left( -\partial^\rho B_{\mu\nu} - \partial^\mu B_{\rho\nu} - \partial^\nu B_{\rho\mu} \right)$$

$$G_{\rho\mu\nu} \equiv \left( -\nabla^R_\rho (h_{\mu\nu} - h^{R\mu\nu}) + \nabla_\mu (h_{\rho\nu} - h^{R\rho\nu}) + \nabla_\rho (h_{\mu\nu} - h^{R\mu\nu}) \right) - H_{\rho\mu\nu} \quad (4.6.62)$$

where $H = dB$ is a gauge invariant 3-form field strength for $B$. Thus $G$ is a tensor and one can easily write down interaction terms from the ERG in terms of $G$ and other modes. One such equation is given below (5.1.64).

We thus get background gauge covariant interacting equations for $h_{\mu\nu}$. One can legitimately ask whether we can identify $h$ as the graviton since we do not see Einstein’s equation here. Einstein’s equation for $h$ (without $h^R$) is expected to emerge after integrating out all the massive modes. The fact that we have a massless symmetric rank 2 tensor with the correct gauge transformation properties guarantees that we will obtain Einstein’s equations as the low energy limit of the ERG.

As explained in section 4.2, the background covariance also helps to ensure that the final equation has to be generally covariant: This is because the background metric, $h^R$, is completely arbitrary and, at least formally, cannot appear in the final answer as expressed by (4.2.54). Then the manifest background covariance guarantees the full covariance of the final result. The background covariance of the massive modes is discussed in the next section. In Appendix C we give a toy model illustrating some of these points using Yang-Mills theory.
5 Massive Field Equations and some Speculations

5.1 Massive Field Equations

The above technique for achieving covariance should go through for massive modes as well, provided they are tensors. This requires a modification of what we mean by "GCT" and what our manifold is. We outline the arguments although the details have not been worked out yet.

The action of $\delta T + \delta G$ on massive modes is also quite simple, if $\delta T$ is defined suitably. We use loop variables as usual, so instead of $\partial_n z X^\mu$ we have $Y^\mu_n$. This ensures the invariance under $\delta G$ as explained in Section 3. To understand the action of $\delta T$ we notice that the final equation of motion has products of $Y^\mu_n$ and $Y^\nu_{\bar{m}}$. We need to define what we mean by $\delta T$ on $Y$.

We will take it to be $Y^\mu_n \rightarrow Y'^\mu_n$, which is not a consequence of the GCT on $X$:

$X^\mu \rightarrow X'^\mu$. Since the $Y$ is a very particular combination of $X$ and its derivatives.

We will assume then that our differential manifold is labelled by $Y^\mu_n$ and assume that diffeomorphisms of this manifold are a symmetry. Thus $\delta_{\text{GCT}}$ acts on $Y$ rather than $X$. We can easily see that it is a symmetry of the EOM. It is easy to see that $Y^\mu_n$ transforms as a tensor (more specifically, a vector) under:

$$
Y^\mu(z) \rightarrow Y'^\mu(z) \Rightarrow \frac{\partial Y^\mu}{\partial x_n} \rightarrow \frac{\partial Y'^\mu}{\partial x^\rho} \frac{\partial Y^\rho}{\partial x_n}
$$

(This is to be contrasted with the more complicated transformation of $Y^\mu_{[n];[\bar{m}]}$. However the final equations for physical modes do not involve these vertex operators, so we do not have to worry about them at this stage. We will touch upon this in the next section.) Thus the massive modes transform as ordinary tensors under $\delta T$. So the equations of motion are manifestly invariant (or covariant, if we remove the vertex operators multiplying the equations), provided we assign the usual tensorial properties to the fields. Contraction of indices is done using $g_{\mu\nu}$. Ordinary derivatives can be replaced by covariant derivatives: $\partial_\mu \rightarrow \nabla^R_\mu = \partial_\mu + \Gamma^R_\mu$ in the usual way. Since the curvature is zero, the covariant derivatives commute: $[\nabla^R_\mu, \nabla^R_\nu] = 0$, and the gauge invariance under $\delta G$ is preserved. Thus in (5.2.43), we simply replace $k_0$ by $\nabla^R$ and raise and lower indices with $g^{R\mu}_{\nu}$.}

\[
\int dz \, \hat{G}(z, z; \tau)(-(\nabla^R)^2 S_{\mu\rho\sigma} + \nabla^{RA}(\nabla^R S_{\nu})_{\lambda\rho\sigma} + \nabla^{RA}\nabla^R S_{\lambda\mu\nu\rho})
\]

\[
-\nabla^R \nabla^R S_{\mu\nu\lambda\rho} - \nabla^R \nabla_{\rho\sigma} S_{\mu\nu\lambda} + \nabla^R (\nabla^R S_{\mu\nu\lambda})_{\lambda\rho} + \int dz \, dz' \hat{G}(z, z'; \tau)G^\lambda_{\rho\sigma}G_{\lambda\nu\sigma} + \ldots = 0
\]

Noting that if we set $x_n = 0$, then $Y = X$. 

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This procedure gives covariant and gauge invariant equations for all the modes. However a detailed understanding of the meaning of coordinate transformations on $Y^\mu$ (as against $X^\mu$) is still lacking.

5.2 Speculations on space time interpretation of massive gauge transformations

The transformation $\delta X^\mu(z) = -\epsilon^\mu(X(z))$ on massive vertex operators gives a transformation that is more complicated than that of $\partial_z X^\mu$. For instance,

$$\delta X^\mu(z) = \partial_z X^\mu(z) - \epsilon^\mu,_{\rho} \partial_z X^\rho$$

The same transformation rule is obtained if we consider $\delta Y^\mu(z) = -\epsilon^\mu(Y(z))$:

$$\delta Y^\mu,_{\rho} \partial_z Y^\rho = \partial_z Y^\mu - \epsilon^\mu,_{\rho} \partial_z Y^\rho - \epsilon^\mu,_{\rho} \partial_z^2 Y^\rho$$ (5.2.65)

The transformation has a term that mixes different tensor structures (but at the same mass level), in addition to the usual tensorial transformation. This is to be contrasted with the transformation of $Y^\mu$: thus $Y^\mu_1$ and $Y^\mu_2$ although they are equal, transform differently. In the loop variable formalism as described in Section 3 for closed strings and I and II for open strings, the intermediate equations involve only $Y^\mu_n$, which also transforms as a vector. But the intermediate stages involve $Y^\mu_{[n]}$ ($[n]$ stands for a partition of $n$), which has a more complicated transformation. Indeed for closed strings we also need $Y^\mu_{1,1}$, which also does not transform as a simple vector. These transformations mix all the tensors at a given mass level. The question arises as to whether these more complicated transformations have any connection with the massive gauge transformations, just as the GCT is related to massless graviton transformations. We give an argument that is suggestive of such a connection but is not definitive. It involves the transformation of terms involved in regularizing the world sheet theory.

5.3 Regularization and Massive Modes

In order to derive a Renormalization Group we need to regularize the theory. Thus our starting point is

$$\int dz_1 dz_2 \eta_{\mu\nu} f(z_1 - z_2, a) \partial_z X^\mu(z_1) \partial_{z_2} X^\nu(z_2) \equiv \int dz \eta_{\mu\nu} \sum_{n,m} f_n m(a) \partial_z^n X^\mu(z) \partial_{z_2}^m X^\nu(z)$$
In (world sheet) momentum space this would correspond to
\[
\int dp \sum_{n,m} f_{n,m} X^\mu(p) p^\mu p^\nu(-p) \eta_{\nu\mu}
\]
which would cutoff the high momentum region in the loop integrals.

Let us now consider the effect of one of these terms. We will write in terms of \(Y\):
\[
c_2 \eta_{\mu\nu} \partial^2_{x_1} Y^\mu \partial^2_{x_1} Y^\nu
\] (5.3.67)

Let us transform \(Y^\mu\).
\[
c_2 \eta_{\mu\nu} \left[ (\partial^2_{x_1} Y^\mu - \epsilon^\mu_{\rho\sigma} \partial_{x_1} Y^\rho \partial_{x_1} Y^\sigma) \partial^2_{x_1} Y^\nu - (\epsilon^\nu_{\rho\sigma} \partial^2_{x_1} Y^\rho + \epsilon^\nu_{\rho\sigma} \partial_{x_1} Y^\rho \partial_{x_1} Y^\sigma) \partial^2_{x_1} Y^\mu \right]
\] (5.3.68)

"c.c." stands for the transformation of \(\partial^2_{x_1} Y^\nu\). We will not write this out explicitly, in order to keep the algebra simple. Thus in analogy with what was done for the kinetic term, let us introduce the reference fields \(h^2_{\mu\nu}\) and \(h^2_{\mu\nu,\rho}\) by adding and subtracting the terms
\[
c_2 h^2_{\mu\nu,\rho} \partial^2_{x_1} Y^\mu \partial^2_{x_1} Y^\nu + c_2 h^2_{\mu\rho,\sigma} (\partial_{x_1} Y^\rho \partial_{x_1} Y^\sigma \partial^2_{x_1} Y^\mu + \partial_{x_1} Y^\rho \partial_{x_1} Y^\sigma \partial^2_{x_1} Y^\mu)
\]

Thus if we assume that \(h^2_{\mu\nu}\) and \(h^2_{\mu\nu,\rho}\) transform as tensors and additionally have the variation (defining as before \(\epsilon_\mu \equiv \eta_{\mu\nu} \epsilon^\nu\))
\[
\delta h^2_{\mu\nu} = \epsilon_{(\nu,\mu)}; \quad \delta h^2_{\mu\rho,\sigma} = \epsilon_{(\mu,\rho,\sigma)}
\]
then this term is invariant.

The interaction Lagrangian now contains the same terms with the opposite sign:
\[
\Delta L = - (c_2 h^2_{\mu\nu,\rho} \partial^2_{x_1} Y^\mu \partial^2_{x_1} Y^\nu + c_2 h^2_{\mu\rho,\sigma} (\partial_{x_1} Y^\rho \partial_{x_1} Y^\sigma \partial^2_{x_1} Y^\mu + \partial_{x_1} Y^\rho \partial_{x_1} Y^\sigma \partial^2_{x_1} Y^\mu))
\] (5.3.69)

We already have a term
\[
K_{1,1;1,1} \partial^2_{x_1} \partial^2_{x_1} Y^\mu e^{ik_0 Y}
\]
Integrating by parts on \(x_1\) we get
\[
K_{1,1;1,1} \left( \frac{1}{2} k_0^2 k_0 \partial_{x_1} Y^\rho \partial_{x_1} Y^\sigma \partial^2_{x_1} Y^\mu + \partial_{x_1} Y^\rho \partial_{x_1} Y^\sigma \partial^2_{x_1} Y^\mu \right) + i k_0 \partial^2_{x_1} Y^\rho \partial^2_{x_1} Y^\mu e^{ik_0 Y}
\] (5.3.70)

Comparing (5.3.69) with (5.3.70) we see that we can identify \(k_{0\mu} k_{0\nu} K_{1,1;1,1}\) with \(2c_2 h^2_{\mu\nu,\rho}\) and \(\lambda_{1} K_{\nu,1;1,1}\) with \(2c_2 h^2_{\mu\nu,\rho}\) and also the gauge transformation \(K_{\nu,1;1,1} \rightarrow K_{\nu,1;1,1} + \lambda_1 K_{\nu,1;1,1}\) with that of \(h^2\) if we identify \(\lambda_{1} K_{\nu,1;1,1} + \lambda_1 K_{\nu,1;1,1} = 2\epsilon_\nu\).

This suggests that just as for the graviton, the extra terms can be associated with variations of the regulator kinetic terms. We find this extremely
interesting because the same argument that allowed us to have a massless graviton can be applied here, but now to the genuinely massive modes: they can be massless as well. Thus somehow at the scale of the cutoff, these modes could be massless. Alternatively, the physical interpretation could be that this is another consistent phase of string theory.

This analysis is clearly incomplete because the parameter \( \epsilon \) is the same as for the GCT considered in Section 4. Whereas the gauge parameter \( \langle \lambda_1 K_{\nu,1;1} + \bar{\lambda}_1 K_{\nu,1;1} \rangle \) should be independent. This is because we have restricted \( \epsilon \) to be only a function of \( Y \). More generally it could be a function of all the \( Y_n \). This will generate transformations that mix different mass levels and tensor structures. We leave this as an open question.

### 5.4 Complexification of Space time coordinates?

We identified

\[
\langle \lambda_1 k_{1\mu} + \bar{\lambda}_1 k_{1\mu} \rangle = \epsilon_\mu = -\eta_{\mu\nu} \delta X^\nu
\]

Also

\[
\langle \lambda_1 k_{1\mu} - \bar{\lambda}_1 k_{1\mu} \rangle = \Lambda_1 \mu
\]

is the gauge transformation parameter of the antisymmetric tensor. Comparing the two strongly suggests that \( X^\mu \) be thought of as the real part of a complex coordinate and that (5.4.72) be identified with the variation of the imaginary part. That space time coordinates could at some level be complex was suggested in [46, 47]. This certainly requires further investigation.

### 6 Summary and Conclusions

In this paper we have extended the construction in I and II, of a gauge invariant ERG for open strings to closed strings. The salient features are the following:

1. The construction is restricted to flat geometry for simplicity. Thus the graviton is a perturbation about flat space. Nevertheless we introduce a reference metric, so that arbitrary coordinate transformations can be made and we need not restrict the flat metric to be of the form \( \eta_{\mu\nu} \). (We chose the reference metric to be of zero curvature in order to simplify the results. This can be relaxed.) The final result is an EOM for closed strings that has general coordinate invariance in the sense of background field theory: Transform coordinates and the background reference metric (of zero curvature) and transform all other fields as tensors. The physical graviton \( h_{\mu\nu} \) occurs in combination with \( h^R_{\mu\nu} \) in the form \( \tilde{h} = h - h^R \) which is a tensor. The original Abelian gauge invariance is embedded in the general coordinate transformation of \( h \).
2. In order to apply the technique of I and II to closed strings it was found necessary to include in the intermediate stages of the calculation, vertex operators involving $\partial_r \partial_s X$. This is also expected for independent reasons: Euclidean world sheet regularization breaks the holomorphic factorization. Thus on the scale of the cutoff one should expect to add such terms. They decouple in the continuum limit. These fields are not there in the BRST formalism.

3. The EOM for physical vertex operators are gauge invariant under the natural generalization of the open string gauge transformations to closed strings. These are Abelian. While this is expected for open strings, this is not expected for closed strings. The resolution of this lies in another problem with this construction: as it stands the graviton cannot be massless. Gauge invariance requires an auxiliary field which can be written down in terms of loop variables only if the graviton is massive. Both these problems are resolved by extending the symmetry transformations to act on the space time coordinates as well. This introduces a Christoffel connection term for a background metric, that obviates the need for an independent auxiliary field. This makes the graviton massless and also makes the symmetry that of general coordinate transformations, as is appropriate for a theory of gravity.

4. The gauge transformations of massive fields continue to be Abelian, if one considers only those equations connected with the physical vertex operators. However if one considers the vertex operators at intermediate stages of the calculation, there are more complicated transformations. There is some preliminary indication of a more elaborate space time interpretation that mixes different tensor structures and mass levels.

5. The EOM are quadratic as expected from an ERG. This is different from the BRST formalism. The can possibly be attributed to the background field formalism. We have seen that the extra fields involving mixed derivatives is related to the background metric.

There are many conceptual and technical questions that need to be answered. We list a few:

1. The loop variable formalism is formally written with an extra dimension. The massive equations are obtained by dimensional reduction. This is in principle straightforward, but the details need to be worked out.

2. In the case of open strings it was found at the first and second massive levels, that the gauge transformations and constraints can be mapped
on to those of the "Old Covariant" formalism (only) in D=26 and with the correct mass levels. Thus we expect that in the critical dimension the S-matrix of this theory should coincide with that of string theory. Some arguments for the equality of the S-matrix were given in [37]. This needs to be made more rigorous and a similar analysis needs to be done for the closed string.

3. In dimensions other than 26, this seems to be a consistent classical theory of massive higher spins interacting with gravity. This is because the gauge symmetries are the same in all dimensions. Whether there are inconsistencies at the quantum level is an open question. These issues need to be sorted out.

4. One may also expect a more direct connection to the critical dimension by studying the EOM of the dilaton [2]. This involves technical issues related to overall normalization of the partition function that we have not worried about in this paper.

5. A flat background metric was chosen to avoid the additional complication of modifying the map from loop variables to space time fields for the massive modes. Non flat background metrics can be chosen. In this case one has the option of setting \( h_{\mu\nu} - h^{R}_{\mu\nu} = \tilde{h}_{\mu\nu} = 0 \) at any point. This will then give equations of motion that are quadratic in the massive fields, but non polynomial in the graviton. This is also worth exploring.

6. The connection between massive gauge transformations and space time coordinate transformations (or generalizations thereof) need to be worked out. The interplay between coordinate transformations \( Y \rightarrow Y' \) and gauge transformations \( Y \rightarrow Y + \frac{\partial Y}{\partial x^a} \) needs to be fully understood.

7. We have now background gauge covariant equations of motion. The problem of constructing a gauge invariant action is unsolved (for open strings also).

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A Appendix: Free Equation

The details of the calculation of the Level 2 (graviton) and Level 4 free equation are given here.
We have to evaluate the second derivative, which is given by the action of a functional derivative on (3.1.28):

$$
\eta^{\mu \nu} \int du \left[ \frac{\partial}{\partial Y^{\nu}(u)} \delta(u - z') + \frac{\partial}{\partial x_1} \delta(u - z') \right] \frac{\partial}{\partial Y^{\nu}_{1;0}(u)} + \frac{\partial}{\partial x_2} \delta(u - z') \right] \frac{\partial}{\partial Y^{\nu}_{2;0}(u)} + \right]
$$

$$
\left\{ \frac{\partial L[Y(u), Y_{n,m}(u)]}{\partial Y^{\mu}(u)} \delta(u - z'') - \partial_{x_1} \frac{\partial L[Y(u), Y_{n,m}(u)]}{\partial Y^{\mu}_{1;0}(u)} \delta(u - z'') \right\}
$$

Let us evaluate the action of the derivatives on each of the four terms labeled I, II, III and IV. We can reduce the number of independent terms to be evaluated by realizing that the result has to be symmetric in $z' \leftrightarrow z''$ and also that for every term, there is also a corresponding complex conjugate term. (Our notation is: $x_n$ refers to $u$, $x'_n$ refers to $z'$ and $x''_n$ refers to $z''$. Thus for instance, $\frac{\partial \delta(u - z')}{\partial x_n} = - \frac{\partial \delta(u - z')}{\partial x'_n}$)

1. 

$$
\int du \eta^{\mu \nu} \frac{\partial}{\partial Y^{\mu}(u)} \frac{\partial L[Y(u), Y_{n,m}(u)]}{\partial Y^{\nu}(u)} \delta(u - z') \delta(u - z'')
$$

$$
= - k_0^2 L(z') \delta(z' - z'')
$$

(A.2)

2. 

$$
\int du \eta^{\mu \nu} \left[ \frac{\partial}{\partial x_1} \delta(u - z') \right] \frac{\partial}{\partial Y^{\nu}_{1;0}(u)} \frac{\partial L[Y(u), Y_{n,m}(u)]}{\partial Y^{\nu}(u)} \delta(u - z'') + \left( z' \leftrightarrow z'' \right)
$$

$$
= \eta^{\mu \nu} \left( - \frac{\partial}{\partial x'_1} [\delta(z'' - z') i k_0 i K_{1;0} L[z'']] \right) + \left( z' \leftrightarrow z'' \right)
$$

$$
= \eta^{\mu \nu} \left( - [\frac{\partial}{\partial x'_1} + \frac{\partial}{\partial x''_1}] [\delta(z'' - z') i k_0 i K_{1;0} L[z'']] \right)
$$
We restore the integrals over $z', z''$, and use $G(z', z'') = \langle Y(z')Y(z'') \rangle$ and integrate by parts on $x', x''$ to get

$$\frac{d}{d\ln a} \int dz' dz'' \left( \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_1'} \right) (Y(z')Y(z'')) [\delta(z'' - z')i k_{0, i} K_{1, 0} L[z'']] =$$

$$= \frac{d}{d\ln a} \int dz' \left[ \frac{\partial}{\partial x_1} (Y(z')Y(z')) \right] i k_{0, i} K_{1, 0} L[z']$$

$$= - \int dz' \dot{G}(z', z')i k_{0, i} K_{1, 0} \frac{\partial}{\partial x_1} [L[z']]$$

In the last step we have integrated by parts again.

Finally we can add the complex conjugate to obtain:

$$= \int dz' \dot{G}(z', z') \left( k_{0, K_{1, 0}} \frac{\partial}{\partial x_1} [L[z']] + k_{0, K_{0, 1}} \frac{\partial}{\partial x_1'} [L[z']] \right) \quad (A.3)$$

3.  

$$\eta^{\mu \nu} \int dz' \int dz'' \dot{G}(z', z'') \int du \left[ \frac{\partial}{\partial x_1} \delta(u-z') \right] \left[ \frac{\partial}{\partial x_1'} \delta(u-z'') \right] \frac{\partial^2 L[u]}{\partial Y_{1, 0}^\mu(u) \partial Y_{1, 0}^\nu(u)}$$

$$= \frac{d}{d\ln a} \eta^{\mu \nu} \int dz' \int dz'' \langle Y_{1, 0}(z') Y_{1, 0}(z'') \rangle \delta(z' - z'') \frac{\partial^2 L[z']}{\partial Y_{1, 0}^\mu(z') \partial Y_{1, 0}^\nu(z'')}$$

$$= \frac{d}{d\ln a} \eta^{\mu \nu} \int dz' \dot{G}(z', z') \frac{1}{2} \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial}{\partial x_2} \right) \frac{\partial^2 L[z']}{\partial Y_{1, 0}^\nu(z')}$$

$$= - \int dz' \dot{G}(z', z') K_{1, 0} \frac{1}{2} \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial}{\partial x_2} \right) L[z'] \quad (A.4)$$

4.  The complex conjugate is:

$$= - \int dz' \dot{G}(z', z') K_{0, 1} \frac{1}{2} \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial}{\partial x_2} \right) L[z'] \quad (A.5)$$

5.  

$$\int dz' dz'' \dot{G}(z', z'') \left( \eta^{\mu \nu} \int du \left[ \frac{\partial}{\partial x_2} \delta(u-z') \right] \left[ \frac{\partial}{\partial x_2'} \delta(u-z'') \right] \frac{\partial^2 L[u]}{\partial Y_{2, 0}^\mu(u) \partial Y_{0}^\nu(u)} + z' \leftrightarrow z'' \right)$$

$$= \int dz' dz'' \dot{G}(z', z'') \left( \eta^{\mu \nu} \left[ - \frac{\partial}{\partial x_2'} \delta(z'' - z') \right] \frac{\partial^2 L[z']}{\partial Y_{2, 0}^\mu(z'') \partial Y_{0}^\nu(z''')} + z' \leftrightarrow z'' \right)$$

$$= \int dz' dz'' \frac{d}{d\ln a} \left[ \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_2'} \right] \dot{G}(z', z'') \left( \eta^{\mu \nu} [\delta(z'' - z')] \frac{\partial^2 L[z''']}{\partial Y_{2, 0}^\mu(z''') \partial Y_{0}^\nu(z'')} \right)$$

$$= \int dz' \dot{G}(z', z') \left( K_{2, 0} k_0 \left( \frac{\partial}{\partial x_2} L[z'''] \right) \right) \quad (A.6)$$
6. Complex conjugate gives:

\[ = \int dz' \, \hat{G}(z', z') \left( K_{0; 2} \cdot k_0 \left( \frac{\partial}{\partial \bar{x}'_2} \mathcal{L}[z'']\right) \right) \tag{A.7} \]

7. \[
\int dz' dz'' \hat{G}(z', z'') \left( \eta^{\mu\nu} \int \frac{d}{\mu} \left[ \frac{\partial^2}{\partial x_1 \partial \bar{x}'_1} \delta(u-z') \delta(u-z'') \frac{\partial^2 \mathcal{L}[u]}{\partial Y_{1;1}^\mu(u) \partial Y_{1;1}^\nu(u)} + z' \leftrightarrow z'' \right] \right) \\
= \frac{d}{d \ln a} \int dz' dz'' \left( Y(z')_{1;1} Y(z'') + Y(z') \bar{Y}_{1;1}(z'') \right) \left( \eta^{\mu\nu} \delta(z'-z'') \frac{\partial^2 \mathcal{L}[z']}{\partial Y_{1;1}^\mu(z') \partial Y_{1;1}^\nu(z')} \right) \\
= \frac{d}{d \ln a} \int dz' dz'' \left( Y(z')_{1;1} Y(z'') + Y(z') \bar{Y}_{1;1}(z'') \right) \left( (iK_{1;1} \cdot iK_{0;1}) \delta(z'-z'') \mathcal{L}[z'] \right) \tag{A.8} \]

(A.8) and (A.9) can be added to give

\[
\frac{d}{d \ln a} \int dz' \left[ \frac{\partial^2}{\partial x_1 \partial \bar{x}'_1} \right] \left( Y(z') \bar{Y}(z') \right) \left( (iK_{1;0} \cdot iK_{0;1}) \mathcal{L}[z'] \right) \\
= \int dz' \hat{G}(z', z') \left( (iK_{1;0} \cdot iK_{0;1}) \left[ \frac{\partial^2}{\partial x_1 \partial \bar{x}'_1} \mathcal{L}[z'] \right] \right) \tag{A.10} \]

provided the following constraint is imposed:

\[ K_{1;0} \cdot K_{0;1} \mathcal{L} = K_{1;1} \cdot k_0 \mathcal{L} \tag{A.11} \]

The constraint is gauge covariant since both sides have identical gauge transformation properties. Since \( K_{1;1} \) is an auxiliary field (i.e. not physical) we are free to impose this constraint. In fact since \( K_{1;1} \cdot k_0 \) contains \( q_{1;1} q_0 \) (for \( q_0 \neq 0 \)), this can be treated as an algebraic constraint on \( q_{1;1} \).

The massless case (Graviton) is discussed in Section 3 and Section 4.

Similar constraints on \( K_{n;n} \) occur at every level. We will refer to them as K-constraints. They are described in the next Appendix.

The terms calculated above are sufficient to extract the coefficient of the graviton multiplet vertex operators at level \( (1; 1) \), \( Y_{1;0} \bar{Y}_{0;1} \) and the next
massive level, $Y^\mu_{i;0} Y^\nu_{i;1} Y^\rho_{i;1} Y^\sigma_{i;1} \epsilon^{k_0 Y}$, a vertex operator in closed string theory at level $(2;2)$. We revert to the notation $k_1 = K_{i;0}, k_1 = K_{i;1}, \ldots$ below.

We get for level $(1;1)$:

$$[-k_1^2 k_1 k_0 + k_0 k_1 k_0 k_1 + k_0 k_1 k_0 k_1 - k_1 k_1 k_0 k_0] Y^\mu_{i;0} Y^\nu_{i;1} \quad (A.12)$$

At level $(2;2)$ for $Y^\mu_{i;0} Y^\nu_{i;1} Y^\rho_{i;1} Y^\sigma_{i;1} \epsilon^{k_0 Y}$ we get:

$$\frac{-1}{4} k_0^2 (k_1 Y_1)^2 (k_1 Y_1)^2 + \frac{1}{2} k_0 (k_0 Y_1) (k_1 Y_1) (k_1 Y_1) + \frac{1}{2} k_0 (k_0 Y_1) (k_1 Y_1) (k_1 Y_1)^2$$

$$\frac{-1}{4} k_1 k_1 (k_0 Y_1)^2 (k_1 Y_1)^2 - \frac{k_1 k_1}{4} (k_0 Y_1)^2 (k_1 Y_1)^2 - k_1 k_0 (k_0 Y_1) (k_1 Y_1) (k_1 Y_1) (k_1 Y_1)$$

$$(A.13)$$

This can easily be seen to be gauge invariant under $k_{1\mu} \rightarrow k_{1\mu} + \lambda_1 k_{0\mu}$ and $k_{1\mu} \rightarrow k_{1\mu} + \lambda_1 k_{0\mu}$, after using the tracelessness condition on the gauge parameter, $\lambda_1 k_1 k_1 = 0 = \lambda_1 k_1 k_1$ and the same for its complex conjugate.

### B Appendix: K-constraints

We derive the K-constraints that occur in the free equations.

For the free part of the equation we do not need the individual $K_{[n]c;[\bar m];\mu}$. We can write $\mathcal{L}$ in terms of $Y^\mu_{n;\bar m}$. Thus the coefficient of $Y^\mu_{n;\bar m}$ is $\sum_{i,j} K_{[n]c;[\bar m];\mu} = \bar K_{n;\bar m}$ as defined in $(2.125)$.

The general case involves combining the following two terms:

$$\int dz' \int dz'' \mathcal{G}(z', z'') \frac{\partial}{\partial x_n} \frac{\partial}{\partial Y_{n;\bar m}} \delta(u - z') \frac{\partial}{\partial x_{\bar m}} \frac{\partial}{\partial Y_{n;\bar m}} \delta(u - z'') \mathcal{L} + z' \leftrightarrow z''$$

$$= \int dz' \frac{d}{dt} \left[ \langle Y_{n;\bar m}(z') Y_{n;\bar m}(z') \rangle (ik_n ik_{\bar m}) \right] \mathcal{L} \quad (B.1)$$

and

$$\int dz' \mathcal{G}(z', z'') \frac{\partial^2}{\partial x_n \partial x_{\bar m}} \frac{\partial}{\partial Y_{n;\bar m}} \delta(u - z') \frac{\partial}{\partial Y} \delta(u - z'') \mathcal{L} + z' \leftrightarrow z''$$

$$= \int dz' \frac{d}{dt} \left[ \langle Y_{n;\bar m}(z') Y_{n;\bar m}(z') \rangle (i \bar K_{n;\bar m} ik_0) \right] \mathcal{L} \quad (B.2)$$

Now if

$$(ik_n ik_{\bar m}) \mathcal{L} = (i \bar K_{n;\bar m} ik_0) \mathcal{L} \quad (B.3)$$

then we can combine the two terms, $(B.1)$ and $(B.2)$, and write

$$\int dz' \frac{d}{dt} \frac{\partial^2}{\partial x_n \partial x_{\bar m}} \langle Y(z') Y(z') \rangle \langle -k_n k_{\bar m} \rangle \mathcal{L}$$

$$= 37$$
\[
= \int dz' G(z', z') (-k_n.k_{\bar{m}}) \frac{\partial^2}{\partial x_n \partial x_{\bar{m}}} \mathcal{L}
\] (B.4)

Since the $\tilde{K}_{n;m\mu}$ are made of the usual loop variables and no new degrees are involved, the K-constraints (B.3) would seem to reduce the number of independent degrees of freedom. However we also have the option of adding one new loop variable $k_{n;m\mu}$ (with $\mu$ chosen to be $D$, so we can call it $q_{n;m}$) to $\tilde{K}_{n;m\mu}$ so that the constraint plays the role of determining this variable.

$q_{n;m}$ should be defined to have the same gauge transformation as $\tilde{K}_{n;m\mu}$, viz:

\[
q_{n;m} \rightarrow q_{n;m} + \lambda_p q_{n;m;p} + \bar{\lambda}_p q_{n;m;p}
\]

Then the constraint does not affect the degrees of freedom count.

We have

\[
\tilde{K}_{n;m\mu} = q_n k_{\mu\bar{m}} + q_{\bar{m}} k_{n\mu} - q_n \bar{q}_{\bar{m}} k_{0\mu}
\]

\[
\tilde{Q}_{n;m} = \frac{q_n}{q_0} q_{\bar{m}} + \frac{q_{\bar{m}}}{q_0} q_n - \frac{q_n}{q_0} \frac{q_{\bar{m}}}{q_0} q_0 + q_{n;m} = \frac{q_n q_{\bar{m}}}{q_0} + q_{n;m}
\]

The constraint (B.3) becomes

\[
q_{n;m} = k_n k_{\bar{m}} - \bar{q}_n k_{\bar{m}} k_0 + q_{\bar{m}} k_n k_0 - \bar{q}_n \bar{q}_{\bar{m}} k_0^2
\]

thus fixing $q_{n;m}$ in terms of the others.

C Appendix: Utility of Background Fields: Toy Model

The exact ERG involves an infinite number of irrelevant coupling constants (massive modes), in addition to the marginal one, the graviton field. It is only after solving for all the massive modes in terms of the graviton, and plugging back in, that one gets the full non polynomial equation for the graviton, which is the covariant Einstein equation along with covariant $\alpha'$ corrections. Each equation by itself does not have the general covariance. However if we introduce background fields, it is possible to make each equation by itself, generally covariant under a transformation which includes not only the usual transformations, but also a general coordinate transformation of the background field. The main advantage of this is that a symmetry, which is very similar to the original symmetry, is manifest throughout the calculation. We illustrate this with a toy example.

Consider an $SU(2)$ Yang-Mills theory, with action.

\[
\frac{1}{4} Tr [F_{\mu\nu} F^{\mu\nu}]
\]
where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] \). Our conventions and notation are as follows: \( A_\mu = A^a_\mu \tau^a/2 \) where \( \tau^a \) are the Pauli matrices. If \( \phi = \phi^a \tau^a/2 \) is an adjoint field, it’s gauge rotation is

\[
\delta \phi = -i[\Lambda, \phi]
\]

Gauge rotation of \( A_\mu \):

\[
\delta A_\mu = \partial_\mu \Lambda + i[A_\mu, \Lambda] \tag{C.1}
\]

Thus the covariant derivative is \( D_\mu \phi = \partial_\mu \phi + i[A_\mu, \phi] \).

Now we write \( A_\mu = W_\mu + \tilde{Q}_\mu \), where \( W_\mu \) is some reference background field and \( \tilde{Q}_\mu \) is the quantum field, which we will set to be \( A_\mu - W_\mu \) at the end (by analogy with the \( \hbar = \hbar - h^R \)). So in fact nothing should depend on \( W_\mu \). Then we can set

\[
\delta W_\mu = \partial_\mu \Lambda + i[W_\mu, \Lambda]; \quad \delta \tilde{Q}_\mu = -i[\Lambda, \tilde{Q}_\mu] \tag{C.2}
\]

The inhomogeneous term has been assigned to \( W_\mu \). \( \tilde{Q} \) transforms homogeneously. If we set \( \tilde{Q}_\mu = A_\mu - W_\mu \), then this is still correct because the inhomogeneous terms cancels out.

\[
F_{\mu\nu} = (\partial_\mu W_\nu - \partial_\nu W_\mu + i[W_\mu, W_\nu]) + D_\mu^R \tilde{Q}_\nu + i[\tilde{Q}_\mu, \tilde{Q}_\nu]
\]

where we have introduced the background covariant derivative \( D_\mu^R \): \( D_\mu^R \phi = \partial_\mu \phi + i[W_\mu, \phi] \). Let us set \((\partial_\mu W_\nu - \partial_\nu W_\mu + i[W_\mu, W_\nu]) = V_{\mu\nu} \). The action becomes:

\[
S[A] = S[W + \tilde{Q}] = \frac{1}{4} Tr[F_{\mu\nu}]^2 = \frac{1}{4} Tr[V_{\mu\nu}^2 + D_\mu^R \tilde{Q}_\nu + i[\tilde{Q}_\mu, \tilde{Q}_\nu]]^2
\]

Once the background is treated separately we denote it by:

\[
S[W, \tilde{Q}] = \frac{1}{4} Tr[V_{\mu\nu}]^2 + \frac{1}{2} Tr[V_{\mu\nu} D_\mu^R \tilde{Q}_\nu] + \frac{1}{2} Tr[V_{\mu\nu}^2 i[\tilde{Q}_\mu, \tilde{Q}_\nu]] + \frac{1}{4} Tr[D_\mu^R \tilde{Q}_\nu]^2 - \frac{1}{4} Tr[[\tilde{Q}_\mu, \tilde{Q}_\nu]]^2 + \frac{1}{2} Tr[D_\mu^R \tilde{Q}_\nu][\tilde{Q}_\mu, \tilde{Q}_\nu]] \tag{C.3}
\]

The action is manifestly background gauge covariant - since each term is. The sum of the terms has the property that it can be expressed as a gauge invariant function of \( A \) because \( S[W, \tilde{Q}] = S[A] \). Another way of seeing this is that if we replace \( \tilde{Q} = A - W \), the \( W \) dependence cancels out in the sum. This fact coupled with manifest background gauge invariance (under which \( \delta A_\mu = \partial_\mu \Lambda + i[A_\mu, \Lambda]; \delta W_\mu = \partial_\mu \Lambda + i[W_\mu, \Lambda] \) guarantees that the sum is gauge invariant under (C.1). Note that

\[
S[W, 0] = \frac{1}{4} Tr[V_{\mu\nu}]^2
\]
is manifestly gauge invariant, because $\tilde{Q} = 0$ is preserved by the gauge transformation, and this sets $A = W$.  

Thus the Yang-Mills equation of motion is

$$D^\mu F_{\mu\nu} = 0 = D^{R\mu}(V_{\mu\nu} + D^{R}[\tilde{Q}_{\mu\nu}] + i[\tilde{Q}_\mu, \tilde{Q}_\nu]) + i[\tilde{Q}_\mu, (V_{\mu\nu} + D^{R}[\tilde{Q}_{\mu\nu}] + i[\tilde{Q}_\mu, \tilde{Q}_\nu])]$$

These equations are cubic. But we can imagine starting with a n action:

$$\frac{1}{4} Tr[V_{\mu\nu}]^2 + \frac{1}{2} Tr[V_{\mu\nu}D^{R}[\tilde{Q}_\mu]] + \frac{1}{2} Tr[V^{\mu\nu}i[\tilde{Q}_\mu, \tilde{Q}_\nu]] +$$

$$\frac{1}{4} Tr[D^{R}[\tilde{Q}_\mu]]^2 - \frac{1}{4} Tr[\Phi_{\mu\nu}]^2 + \frac{1}{4} Tr[i[\tilde{Q}_\mu, \tilde{Q}_\nu]\Phi_{\mu\nu}] + \frac{1}{2} Tr[D^{R}[\tilde{Q}_\mu]\tilde{Q}_\nu]$$

(C.4)

where $\Phi$ is a very massive mode, an adjoint of $SU(2)$, and we neglect the derivative part of the kinetic term at low energies. Solving for the $\Phi$ equation would give back the original low energy action. Now the EOM are quadratic:

$$\Phi_{\mu\nu} = i[\tilde{Q}_\mu, \tilde{Q}_\nu]$$

$$D^{R\mu}D^{R}[\tilde{Q}_\mu] + [\tilde{Q}_\mu, \Phi_{\mu\nu}] + [\tilde{Q}_\mu, D^{R}[\tilde{Q}_\nu]] + D^{R\mu}\tilde{Q}_\nu +$$

$$i[\tilde{Q}_\mu, V_{\mu\nu}] + D^{R\mu}V_{\mu\nu} = 0$$

Notice also that the equations are background gauge covariant. If we set $\tilde{Q} = A_\mu - W_\mu$ in the action, we would get back the original Yang-Mills action without $W$. So in principle we could therefore choose $W = 0$. However then the first equation becomes

$$\Phi_{\mu\nu} = i[A_\mu, A_\nu]$$

and we do not see any manifest\(^9\) background (or other) gauge covariance. The same is true in the second equation. Thus without background fields the individual equations do not have any manifest symmetry. Nevertheless if we substitute for $\Phi$ we get the original Yang Mills gauge covariant equation.

The lesson is that the role played by the arbitrary reference field $W$ is to make each equation manifestly covariant under a background gauge transformation. Thus in the intermediate stages of the calculation, some covariance property is manifest. This guarantees that when $\Phi$ is eliminated by its equation of motion, then the result will continue to be background covariant. Then as we have seen, the property $S[W, \tilde{Q}] = S[W + \tilde{Q}]$ guarantees that the result has the original gauge invariance.

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\(^9\)This last statement is true for a suitably defined quantum effective action also. This is what makes the background field formalism useful\(^{51}\). However we will need only the classical action in this paper.

\(^{10}\)Technically, "manifest" here means being able to write equations in terms of fields transforming in linear representations of the group.
In the problem at hand, $A$ is replaced by $h_{\mu\nu}$, $W$ by $h^R$ and $\tilde{Q}$ by $\tilde{h}$. We break up our original action into a kinetic and interaction term.

$$S(h) = \int dz \left( \eta_{\mu\nu} + h_{\mu\nu}(X) \right) \partial_z X^\mu \partial_{\bar{z}} X^\nu$$

$$= \int dz \left( \eta_{\mu\nu} + h^R_{\mu\nu}(X) \right) \partial_z X^\mu \partial_{\bar{z}} X^\nu + \tilde{h}_{\mu\nu}(X) \partial_z X^\mu \partial_{\bar{z}} X^\nu$$

where $\tilde{h} \equiv h - h^R$. The transformation rules are as given earlier (for infinitesimal $\epsilon^\mu$, and $\epsilon_\mu \equiv \eta_{\mu\nu} \epsilon^\nu$)

$$\delta h_{\mu\nu} = \epsilon_{(\mu,\nu)} + \epsilon^\lambda h_{\nu,\lambda} + \epsilon^\lambda_{,\mu} h_{\lambda,\nu} + \epsilon^\lambda_{,\nu} h_{\mu,\lambda} \quad (C.5)$$

$$\delta h^R_{\mu\nu} = \epsilon_{(\mu,\nu)} + \epsilon^\lambda h^R_{\nu,\lambda} + \epsilon^\lambda_{,\mu} h^R_{\lambda,\nu} + \epsilon^\lambda_{,\nu} h^R_{\mu,\lambda} \quad \delta \tilde{h}_{\mu\nu} = \epsilon^\lambda \tilde{h}_{\nu,\lambda} + \epsilon^\lambda_{,\mu} \tilde{h}_{\lambda,\nu} + \epsilon^\lambda_{,\nu} h^R_{\mu,\lambda} \quad (C.6)$$

Thus $\tilde{h}$ (like $\tilde{Q}$ above) transforms homogeneously. While the sum of kinetic plus interaction term is coordinate invariant (C.5), each term individually is invariant only under the background gauge transformation (C.6). We expect that $\beta(h) = \beta(h^R + \tilde{h}) = \beta(h^R, \tilde{h})$. This is because both terms correspond to the same beta function. In one we treat $h$ perturbatively to all orders, in the other we treat $h^R$ as a background and $\tilde{h}$ perturbatively. The sum of the infinite series should add up (formally, i.e. in some region of convergence) to satisfy this equation.

Now we can also calculate the ERG, which is only quadratic in fields, and has all the massive modes. If we solve for the massive modes (as in the Yang Mills example) we should recover the low energy non polynomial beta function. If we use the background field formalism, each equation is guaranteed to be background gauge covariant. Then the result of solving for the massive modes gives us the low energy $\beta$ function, which also has background gauge covariance. Now using $\beta(h^R + \tilde{h}) = \beta(h)$ we see that the result in fact has full covariance. Equivalently, the initial action does not depend on $h^R$ and $h^R$ is completely arbitrary. So if we write the final answer entirely in terms of $h$ and $h^R$, $h^R$ has to drop out of the final result as expressed by (4.2.54). Then the background covariance reduces to ordinary covariance.

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