THE MILNOR–CHOW HOMOMORPHISM REVISITED

MORITZ KERZ, STEFAN MÜLLER-STACH

Abstract. The aim of this note is to give a simplified proof of the surjectivity of the natural Milnor–Chow homomorphism \( \rho : K^M_n(A) \to CH^n(A, n) \) between Milnor \( K \)–theory and higher Chow groups for essentially smooth (semi–)local \( k \)–algebras \( A \) with \( k \) infinite. It implies the exactness of the Gersten resolution for Milnor \( K \)–theory at the generic point. Our method uses the Bloch-Levine moving technique and some properties of the Milnor \( K \)–theory norm for fields.

Introduction

In \([4, 3]\) the surjectivity of the natural Milnor–Chow homomorphism

\[ \rho : K^M_n(A) \to CH^n(A, n) \]

between Milnor \( K \)–theory and higher Chow groups for any essentially smooth (semi–)local \( k \)–algebra \( A \) with \( k \) infinite was shown. This morphism associates to a symbol \( \{f_1, \ldots, f_n\} \) the graph cycle of the map \( f = (f_1, \ldots, f_n) \).

In this note we want to give a very simple argument which uses two basic ingredients. The first is a new argument derived from fairly elementary properties of the norm–map for the Milnor \( K \)–theory of rings which were sketched in \([7]\) and build up on the theory of Bass and Tate \([1]\) (see section 2). The idea in \([7]\) was to use a Milnor \( K \)–group which is not induced directly from a ring (or algebra) but only from certain generic elements of a ring. The same technique can also be used to show the Gersten conjecture for Milnor \( K \)–theory of regular (semi–)local rings \([7]\). The second input is a standard application of the easy moving lemma of Bloch–Levine \([2, 8]\) which implies that we can restrict to the case of cycles with smooth components. This was also used in the proof in \([3]\).

Our main theorem is:

Theorem 0.1. Let \( A \) be an essentially smooth (semi–)local \( k \)–algebra with infinite residue fields. Then the homomorphism \( \rho : K^M_n(A) \to CH^n(A, n) \) is surjective for \( n \geq 1 \).

Here for a field \( k \) we say that a \( k \)–algebra \( A \) is essentially smooth if \( A \) is the localization of a smooth affine \( k \)–algebra. In fact under the conditions of the theorem one can show \( \rho \) is bijective \([7]\). This theorem has a few beautiful applications:

Corollary 0.2. Let \( A \) be as above and \( X = \text{Spec}(A) \) integral (i.e., \( A \) a domain with quotient field \( F \)). Then the Gersten resolution for Milnor \( K \)–theory is exact.

Date: 9.1.06.

Supported by Studienstiftung des deutschen Volkes and Deutsche Forschungsgemeinschaft.
at the generic point:

\[ K^M_n(A) \xrightarrow{i_*} K^M_n(F) \xrightarrow{T} \bigoplus_{x \in X^{(1)}} K^M_{n-1}(k(x)) \to \ldots, \]

i.e., \( \ker(T) = \text{im}(i_*) \), where \( T \) is the tame symbol.

The exactness of this complex is well known in codimensions \( p \geq 1 \) by the work of Gabber and Rost \([10]\) and follows in degree zero with the same proof as in \([3]\) by comparing with the corresponding sequence for higher Chow groups \([2]\). Note that the work of Kerz \([7]\) implies also the Gersten conjecture, i.e. the injectivity of \( i_* \) for such \( k \)-algebras \( A \).

There is another nice application to étale cohomology:

**Corollary 0.3.** Assume the Bloch–Kato conjecture \([13]\). Let \( A \) be a (semi–)local ring containing an infinite field and \( l > 0 \) prime to \( \text{char}(A) \). Then the graded ring \( H^*_\text{et}(A, \mu_l^{\otimes n}) \) is generated by elements of degree one.

**Proof.** First assume that \( A \) is essentially smooth over an infinite field. The Bloch–Kato conjecture implies that we have an isomorphism

\[ CH^n(A, n)/l \xrightarrow{\sim} H^n_{\text{et}}(A, \mu_l^{\otimes n}) \]

for any \( l \) prime to \( \text{char}(A) \). Composing with \( \rho \) we get a surjective ring homomorphism \( K^N_n(A)/l \to H^*_\text{et}(A, \mu_l^{\otimes n}) \) which shows the corollary in this case, because Milnor \( K \)-theory is generated in degree one.

Let \( A \) be arbitrary. By a direct limit argument we can assume \( A \) to be a localization of an affine algebra. Now Hoobler’s trick \([5]\) can be applied: There is a Henselian pair \((A', I)\) with \( A' \) essentially smooth and \( A = A'/I \). In this situation \( H^*_\text{et}(A, \mu_l^{\otimes n}) \) and \( H^*_\text{et}(A', \mu_l^{\otimes n}) \) are isomorphic \([5]\). The commutative diagram

\[ \begin{array}{ccc}
K^M_n(A')/l & \xrightarrow{\text{nat}} & K^M_n(A)/l \\
\downarrow \rho & & \downarrow \rho \\
H^*_\text{et}(A', \mu_l^{\otimes n}) & \xrightarrow{\text{nat}} & H^*_\text{et}(A, \mu_l^{\otimes n})
\end{array} \]

implies immediately that \( \rho : K^N_n(A)/l \to H^*_\text{et}(A, \mu_l^{\otimes n}) \) is surjective. \( \square \)

**Corollary 0.4** (Bloch). Again assuming the Bloch–Kato conjecture, let \( X/\mathbb{C} \) be a variety and \( \xi \in H^i(X, \mathbb{Z}) \) an element of prime exponent \( l \). Fix some points \( x_1, \ldots, x_n \in X \). Then there exists an effective divisor \( D \subset X \) such that \( \xi \) restricted to \( X - D \) vanishes and \( x_j \notin D \) for all \( j = 1, \ldots, n \).

**Proof.** This is essentially the same argument as in the proof of Corollary 7.7 of \([12]\). \( \square \)

1. **The Milnor–Chow map \( \rho \)**

1.1. **Higher Chow groups.** S. Bloch \([2]\) defined higher Chow groups as a candidate for motivic cohomology, i.e. an algebraic singular (co)homology. They form
a Borel–Moore homology theory for schemes over a field \( k \), which we fix from now on. In order to define them we use the algebraic \( n \)--cube

\[ \Box^n = (\mathbb{P}^1_k \setminus \{1\})^n. \]

The \( n \)--cube has \( 2^n \) codimension one faces, defined by \( x_i = 0 \) and \( x_i = \infty \) for \( 1 \leq i \leq n \). An integral subvariety \( W \subseteq \Box^n \) of codimension \( p \) is called admissible if its intersection with all faces is again of codimension \( p \) or empty. For each face \( F = \{ x_i = 0 \} \) or \( F = \{ x_i = \infty \} \) we have a pull–back map \( \partial_0^i \) resp. \( \partial_\infty^i \) which sends a subvariety \( W \subseteq \Box^n \) to the intersection product of cycles \( W \cdot F \) with appropriate multiplicities in the sense of Serre’s Tor–formula. A total differential is given by

\[ \partial = \sum_{i=1}^{n} (-1)^{i-1} (\partial_0^i - \partial_\infty^i). \]

Let \( X \) be a quasi–projective variety over \( k \) (standard techniques allow to extend this definition to equidimensional schemes over \( k \) and even, but much harder, to schemes over Dedekind rings). The notion of faces, restriction maps, and differentials extends to \( \Box^n_X = X \times_k \Box^n \). \( Z^p_c(X, n) \) is defined to be the quotient of the group of admissible cycles of codimension \( p \) in \( X \times \Box^n \) by the group of degenerate cycles as defined in [11], p.180 (where they are denoted by \( d^p(X, n) \)). Let \( CH^p(X, n) \) be the \( n \)--th homology of the complex \( Z^p_c(X, \cdot) \) with differential \( \partial \).

1.2. Milnor \( K \)--theory. Milnor \( K \)--theory of a ring \( A \) is defined as the quotient

\[ T(A)/S(A) \]

of the free graded tensor algebra \( T(A) = \mathbb{Z} \oplus A^\times \otimes A^\times \otimes \cdots \) over the units \( A^\times \) of \( A \) by the ideal \( S(A) \) generated by the degree two relations of the form \( (f, 1-f) \) for all \( f, 1-f \in A^\times \) and \( (f, -f) \) for all \( f \in A^\times \). Note that in the case of fields or (semi–)local rings with large residue fields the relation \( (f, -f) \) follows from the usual Steinberg relation \( (f, 1-f) \).

1.3. The map \( K^M_n(A) \to CH^n(A, n) \). Now we consider the special case where \( A \) is a localization of an affine \( k \)--algebra with \( k \) an arbitrary ground field. Denote by \( CH^p(A, n) \) the higher Chow groups of \( \text{Spec}(A) \). In particular we have the series of abelian groups \( CH^n(A, n) \). To any tuple \( f = (f_1, \ldots, f_n) \) of elements \( f_i \in A^\times \) we can associate a map

\[ f = (f_1, \ldots, f_n) : \text{Spec}(A) \to (\mathbb{P}^1)^n \]

and hence by restricting to the cube a graph cycle

\[ \Gamma_f = \text{graph}(f_1, \ldots, f_n) \cap \Box^n_A. \]

Since such graph cycles have no boundary, we immediately get a map

\[ \rho : (A^\times)^n \to CH^n(A, n). \]

One can show that \( \rho \) preserves bilinearity, is skew–commutative, and obeys the Steinberg relations \( \rho(f, 1-f, f_3, \ldots, f_n) = 0 \) and \( \rho(f, -f, f_3, \ldots, f_n) = 0 \). Therefore it descends to a well–defined homomorphism

\[ \rho : K^M_n(A) \to CH^n(A, n) \]
for all \( n \geq 0 \). If \( A \) is essentially smooth \( CH^*(A,*) \) has a ring structure and \( \rho \) becomes a ring homomorphism. In the special case where \( A \) is a field \( F \) the following result is classical.

**Theorem 1.1** (Nesterenko/Suslin, Totaro). \( \rho \) is an isomorphism for every field \( F \).

**Proof.** Totaro’s proof \([11]\) uses cubical higher Chow groups as defined above. He shows that any cycle \( Z \in CH^n(F,n) \) is equivalent (cobordant) to a norm–cycle which has all coordinate entries in \( F \). This already gives the surjectivity of \( \rho \). The inverse map \( \rho^{-1} \) is defined using the norm as follows: By linearity it is sufficient to define \( \rho^{-1} \) for \( Z \) irreducible. In this case we choose a minimal finite field extension \( L/F \) such that \( Z \) corresponds to an \( L \)–valued point \( (z_1, \ldots, z_n) \). Then \( \rho^{-1}(Z) = \mathcal{N}_{L/F}(\{z_1, \ldots, z_n\}) \) as an element of \( K^M_n(F) \), where \( \mathcal{N}_{L/F} \) is the norm map of Bass and Tate \([1]\). \( \Box \)

2. **Symbols in general position**

The main result of this section is Proposition 2.8 which in some sense represents the idea that for good extensions of (semi–)local rings there should be norms of Milnor \( K \)–groups as in the field case. In fact such norms can be constructed by an extension of the methods described below \([7]\).

2.1. **The group** \( K^t_n(A) \). Let \( A \) be a (semi–)local UFD and \( F = Q(A) \) its quotient field. The group \( K^t_n(A) \), we are going to define, should be thought of as the proper Milnor \( K \)–group of the ring \( A[t]_S \), where \( S \) denotes the multiplicative system of all monic polynomials.

**Definition 2.1.** An \( n \)–tuple of rational functions

\[
\left( \frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots, \frac{p_n}{q_n} \right) \in F(t)^n
\]

with \( p_i, q_i \in A[t] \) for \( i = 1, \ldots, n \) is called feasible if

1. The highest nonvanishing coefficients of \( p_i, q_i \) are invertible in \( A \) for \( i = 1, \ldots, n \).
2. For every irreducible factor \( u \) of \( p_i \) or \( q_i \) and \( v \) of \( p_j \) or \( q_j \) \( (i, j = 1, \ldots, n, i \neq j) \) \( u \) is either equivalent or coprime to \( v \).

Before stating the definition of \( K^t_n(A) \) we have to replace ordinary tensor product.

**Definition 2.2.** Define

\[
\mathcal{T}^t_n(A) = \mathbb{Z} < \{(p_1, \ldots, p_n) \mid (p_1, \ldots, p_n) \text{ feasible, } p_i \in A[t] \text{ irreducible or unit} \} > /L
\]

Here \( L \) denotes the subgroup generated by elements

\[
(p_1, \ldots, ap_i, \ldots, p_n) - (p_1, \ldots, a, \ldots, p_n) - (p_1, \ldots, p_i, \ldots, p_n)
\]

with \( a \in A^x \).
By bilinear factorization the element
\[(p_1, \ldots, p_n) \in T^t_n(A)\]
is defined for every feasible \(n\)-tuple with \(p_i \in F(t)\).
Now define the subgroup \(St \subset T^t_n(A)\) to be generated by feasible \(n\)-tuples
(1) \((p_1, \ldots, p, 1 - p, \ldots, p_n)\)
and
(2) \((p_1, \ldots, p, -p, \ldots, p_n)\)
with \(p, p_i \in F(t)\).

**Definition 2.3.** Define
\[K^t_n(A) = T^t_n(A) / St\]
We denote the image of \((p_1, \ldots, p_n)\) in \(K^t_n(A)\) by \(\{p_1, \ldots, p_n\}\).

2.2. The tame symbol. Recall that Milnor constructed so called tame symbols
\[\partial_\pi : K^M_n(F(t)) \longrightarrow K^M_{n-1}(F[t]/(\pi))\]
for every irreducible \(\pi \in F[t]\) \(\not| t\) – in fact this construction works for all discrete valuation rings in contrast to our generalization below.

**Proposition 2.4** (Tame symbol). For every irreducible, monic polynomial \(\pi \in A[t]\) and \(n > 0\) one has a unique well defined tame symbol
\[\partial_\pi : K^t_n(A) \longrightarrow K^M_{n-1}(A[t]/(\pi))\]
which satisfies
(3) \[\partial_\pi : \{\pi, x_2, \ldots, x_n\} \mapsto \{\bar{x}_2, \ldots, \bar{x}_n\}\]
for \(x_i \in A[t]\) and \(x_i\) coprime to \(\pi\).
For \(\pi = 1/t\) there is a similar tame symbol
\[\partial_\pi : K^t_n(A) \longrightarrow K^M_{n-1}(A)\]
which satisfies (3) for \(x_i \in A[1/t]\).

**Proof.** Assume \(\pi \in A[t]\). Uniqueness is easy to check. In order to show existence, introduce according to an idea of Serre a formal skew–commutative element \(\xi\) with \(\xi^2 = \xi \{ -1 \}\) and \(\text{deg}(\xi) = 1\). Define a formal map (which is clearly not well defined)
\[\theta_\pi : T^t_*(A) \longrightarrow K^M_*(A[t]/(\pi))[\xi]\]
by
\[\theta_\pi(u_1 \pi^{i_1}, \ldots, u_n \pi^{i_n}) = (i_1 \xi + \{\bar{u}_1\}) \cdots (i_n \xi + \{\bar{u}_n\}) .\]
We define \(\partial_\pi\) by taking the (right–)coefficient of \(\xi\). This is a well defined homomorphism. So what remains to be shown is that \(\partial_\pi\) factors over the Steinberg relations.
Let \(x = (\pi^iu, -\pi^iu)\) be feasible, then
\[\theta_\pi(x) = (i\xi + \{\bar{u}\})(i\xi + \{-\bar{u}\}) = i\xi\{-1\} - i\xi\{\bar{u}\} + i\xi\{-\bar{u}\} + \{\bar{u}, -\bar{u}\} = 0 .\]
For \( i > 0 \) and \( x = (\pi^i u, 1 - \pi^i u) \) feasible one has
\[
\theta_n(x) = (i\xi + \{\bar{u}\})\{1\} = 0.
\]
For \( i < 0 \) and \( x = (\pi^i u, 1 - \pi^i u) \) feasible one has
\[
\theta_n(x) = (i\xi + \{\bar{u}\})(i\xi + \{-\bar{u}\}) = i\xi\{-1\} + i\xi\{-\bar{u}\} - i\xi\{\bar{u}\} + \{\bar{u}, -\bar{u}\} = 0.
\]
\[\square\]

The tame symbols from Proposition 2.4 are compatible with the corresponding symbols of the quotient field of \( A \). This is the content of the next lemma.

**Lemma 2.5.** Fix either an irreducible, monic \( \pi \in A[t] \) as above and let \( B = A[t]/(\pi) \) and \( L = Q(B) \) or set \( \pi = 1/t \), \( B = A \) and \( L = F \). The square

\[
\begin{array}{c}
K'^t_n(A) \xrightarrow{\partial_n} K'^{M}_{n-1}(B) \\
\downarrow \quad \downarrow
\end{array}
\]

\[
K'^M_n(F(t)) \xrightarrow{\partial_n} K'^{M}_{n-1}(L)
\]

is commutative.

Moreover for monic \( \pi \in F[t] \) but \( \pi \notin A[t] \) the composition

\[
K'^t_n(A) \longrightarrow K'^M_n(F(t)) \xrightarrow{\partial_n} K'^{M}_{n-1}(F[t]/(\pi))
\]

vanishes.

**Proposition 2.6.** If the residue fields of \( A \) are infinite the map

\[
\bigoplus_{\pi} \partial_n : K'^t_n(A) \longrightarrow \bigoplus_{\pi} K'^{M}_{n-1}(A[t]/(\pi))
\]

is surjective, where the sum is over all monic, irreducible \( \pi \in A[t] \).

In fact the kernel of \( \bigoplus_{\pi} \partial_n \) is precisely \( K'^{M}_{n}(A) \), but this is more difficult to show.

**Proof.** Consider the filtration \( L_d \subset K'^t_n(A) \), where \( L_d \) is generated by the feasible \( (x_1, \ldots, x_n) \) with \( x_i \in A[t] \) of degree at most \( d \). One has to show

\[
\bigoplus_{\deg(\pi) = d} \partial_n : L_d \longrightarrow \bigoplus_{\deg(\pi) = d} K'^{M}_{n-1}(A[t]/(\pi))
\]

is surjective. Fix \( \pi \) of degree \( d \). For a symbol \( \xi = \{\bar{x}_2, \ldots, \bar{x}_n\} \in K'^{M}_{n-1}(A[t]/(\pi)) \) we can according to the following sublemma suppose without restriction that \( \xi = \{\pi, x_2, \ldots, x_n\} \in K'^t_n(A) \) is well defined assuming \( x_i \) to be choosen of degree \( d - 1 \). As we have \( \partial_n(\xi) = 0 \) for \( \xi \neq \pi \), \( \deg(\pi') = d \), and \( \partial_n(\xi) = \xi \), this proves the proposition.

**Sublemma 2.7** (Gabber). Given monic \( y_1, \ldots, y_k \in A[t] \) and an arbitrary \( x \in A[t] \) coprime to \( \pi \) there exists a factorization

\[
x \equiv x'x'' \mod (\pi)
\]

such that \( x', x'' \in A[t] \) have invertible highest coefficients, \( \deg(x') = \deg(x'') = d-1 \) and \( x', x'' \) are coprime to \( y_j \) for \( j = 1, \ldots, k \).
Proof. Using the Chinese remainder theorem and reduction modulo all maximal ideals we can assume that \( A \) is an infinite field. The moduli space of factorizations \( x \equiv x'x'' \mod (\pi) \) is a nonempty Zariski open subset of \( \mathbb{A}^d_A \). As finite intersections of such subsets contain a rational point, the sublemma is proven.

2.3. Norms. With the notation as above \((B = A[t]/(\pi), F = Q(A), L = Q(B))\) let \( i : A \to F \) and \( j : B \to L \) be the natural embeddings. For the convenience of the reader we recall the construction of norms

\[
N_{L/F} : K_n^M(L) \to K_n^M(F)
\]

from [1].

Given \( \xi \in K_n^M(L) \) choose \( \zeta \in K_{n+1}^M(F(t)) \) such that \( \partial_{\pi'}(\zeta) = 0 \) for \( \pi' \neq \pi \) and \( \partial_\pi(\zeta) = \xi \). Set \( N_{L/F}(\xi) = -\partial_1/t(\zeta) \). Kato showed this norm depends only on the isomorphism class of \((L, \xi)\) over \( F \) and is functorial [6].

Proposition 2.8. We have

\[
N_{L/F}(\text{im}(j_*)) \subset \text{im}(i_*)
\]

with \( i_*, j_* \) the homomorphisms induced on Milnor \( K \)-groups.

Proof. Given \( \xi \in K_n^M(B) \) choose by Lemma 2.6 \( \zeta \in K_{n+1}^t(A) \) such that \( \partial_{\pi'}(\zeta) = 0 \) for \( \pi' \neq \pi \) and \( \partial_\pi(\zeta) = \xi \). Set \( \xi' = -\partial_{1/t}(\zeta) \in K_n^M(A) \). It follows from Lemma 2.5 that

\[
N_{L/F}(j_*(\xi)) = i_*(\xi') .
\]

\( \square \)

3. Proof of Theorem 0.1

Assume that \([Z] \in CH^n(A, n)\) is a higher Chow cycle. We want to construct an element \( \xi \in K_n^M(A) \) such that \( \rho(\xi) = [Z] \).

Lemma 3.1. \( Z \) is cobordant to a sum of irreducible cycles \( Z' \) such that

1. \( Z' \subset \square^n_A \) does not intersect any face.
2. With the coordinate functions \( t_1, \ldots, t_n \in O_{Z'} \) one has \( A[t_1, \ldots, t_i] \) essentially smooth over \( k \) and finite over \( A \) for every \( 1 \leq i \leq n \).
3. \( O_{Z'} = A[t_1, \ldots, t_n] \).

Proof. This follows immediately from the “easy moving lemma” of Bloch and Levine [8, chap. II, 3.5] and is also applied and explained in [3]. \( \square \).

Without loss of generality we may therefore assume that \( Z \) is irreducible and already in good position as in the lemma. Look at the following diagram:

\[
\begin{array}{ccc}
K_n^M(A) & \xrightarrow{i_*} & K_n^M(F) \\
\rho \Big| & & \rho \Big| \\
CH^n(A, n) & \xrightarrow{i_*} & CH^n(F, n)
\end{array}
\]
where – by abuse of notation – we use the same symbols $\rho$ and $i_*$ for the corresponding maps of rings or fields and $F$ is the quotient field of $A$. Since $\rho$ is an isomorphism on the level of fields, we know that there is an element $\tau \in K_n^M(F)$ such that $\rho(\tau) = i_*[Z]$. By the description of $\rho^{-1}$ in Totaro’s proof of Theorem 1.1, we know that one has $\tau = N_{L/F}(\{t_1, \ldots, t_n\})$ where $L$ is the quotient field of $\mathcal{O}_Z$ and $N_{L/F}$ is the norm on Milnor $K$-theory of fields. Now look at the consecutive extensions

$$A \subset A[t_1] \subset A[t_1, t_2] \subset \ldots \subset A[t_1, \ldots, t_i] \subset \ldots$$

These rings are all essentially smooth and hence factorial. Each extension is of the type

$$A[t_1, \ldots, t_{i+1}] = A[t_1, \ldots, t_i][t]/(t_{i+1}).$$

Therefore we may apply Proposition 2.8 and conclude that there is an element $\xi$ with $i_*(\xi) = \tau$. But the map $i_* : CH^n(A, n) \to CH^n(F, n)$ is injective by [2] and therefore we have $\rho(\xi) = [Z]$, since $i_*(\rho(\xi)) = i_*[Z]$. \hfill \Box

References

[1] Bass, H.; Tate, J.: The Milnor ring of a global field. Algebraic $K$-theory II (Proc. Conf., Seattle, Wash., Battelle Memorial Inst., 1972), pp. 349–446. Lecture Notes in Math., Vol. 342, Springer, Berlin, 1973.

[2] Bloch, Spencer: Algebraic cycles and higher $K$-theory, Adv. in Math. 61 (1986), no. 3, 267–304.

[3] Elbaz-Vincent, Philippe; Müller-Stach, Stefan: Milnor $K$-theory of rings, higher Chow groups and applications, Invent. Math. 148 (2002), no. 1, 177–206.

[4] Gabber, Ofer: Letter to Bruno Kahn (1998).

[5] Hoobler, Raymond: The Merkuriev-Suslin theorem for any semi-local ring, preprint $K$-theory Preprint Archives No. 731 (2005).

[6] Kato, Kazuya: A generalization of local class field theory by using $K$-groups II, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 27 (1980), no. 3, 603–683.

[7] Kerz, Moritz: Diploma thesis, Universität Mainz 2005

[8] Levine, Marc: Mixed motives. Mathematical Surveys and Monographs 57, American Mathematical Society, Providence, RI, 1998.

[9] Milnor, John: Algebraic $K$-theory and quadratic forms, Invent. Math. 9 1969/1970 318–344.

[10] Rost, Markus: Chow groups with coefficients, Doc. Math. 1 (1996), 319–393.

[11] Totaro, Burt: Milnor $K$-theory is the simplest part of algebraic $K$-theory $K$-Theory 6 (1992), no. 2, 177–189.

[12] Voevodsky, Vladimir: Motivic cohomology with $\mathbb{Z}/2$-coefficients Publ. Math. Inst. Hautes Etudes Sci. No. 98 (2003), 59–104.

[13] Voevodsky, Vladimir: On motivic cohomology with $\mathbb{Z}/l$-coefficients, $K$-theory Preprint Archives No. 639 (2003).

E-mail address: m.kerz@dpmms.cam.ac.uk, mueller-stach@uni-mainz.de

University of Cambridge, Department of Pure Mathematics and Mathematical Statistics, Wilberforce Road, Cambridge CB3 0WB, United Kingdom

Johannes Gutenberg-Universität Mainz, Institut für Mathematik, Staudingerweg 9, 55099 Mainz, Germany