Gravitational Collapse and Cosmic Censorship

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ABSTRACT

This article gives an elementary overview of the end-state of gravitational collapse according to classical general relativity. The focus of discussion is the formation of black holes and naked singularities in various physically reasonable models of gravitational collapse. Possible implications for the cosmic censorship hypothesis are outlined.

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I. Introduction

It is expected that a very massive star will not end up either as a white dwarf or as a neutron star, and that it will undergo an intense gravitational collapse towards the end of its life history. The very late stages in the evolution of such a star will necessarily be determined by quantum gravitational effects. Since we do not yet have a quantum theory of gravity, we cannot give a definite description of these very late stages. However, we can at least ask what classical general relativity predicts for the final stages of the evolution. It is possible that the answer given by the classical theory will have some connection with the answer coming from quantum gravity - perhaps the latter will provide a sort of quantum correction to the former.

It is remarkable that seventy years after Einstein proposed the general theory of relativity we do not have a complete understanding of the theory’s prediction for the end-state of gravitational collapse. This situation is intimately related with our lack of understanding of the general global properties of the solutions of Einstein’s equations. The most significant developments to date in the study of gravitational collapse have been the singularity theorems of Hawking and Penrose [1]. In the context of gravitational collapse, the theorems show that if a trapped surface forms during the collapse of a compact object made out of physically reasonable matter, the spacetime geometry will develop a gravitational singularity (assuming the non-existence of closed time-like curves). By a gravitational singularity one means that the evolution of geodesics in the spacetime will be incomplete. It is plausible that the formation of a gravitational singularity in a collapsing star will be accompanied by a curvature singularity - one or more curvature scalars will diverge.

The general conditions which will ensure the formation of a trapped surface are not well understood. This is one of the aspects in which our understanding of gravitational collapse is incomplete. In this article, however, we will not be concerned with this particular issue. We will assume that a gravitational singularity does form, either because the conditions of the singularity theorems have been met, or otherwise. All the same it should be mentioned that the astrophysical parameters for very massive collapsing stars are usually such that a trapped surface can be expected to form during gravitational collapse.

It maybe the case that the singularity is not visible to a far-away observer because light is not able to escape the collapsing star. This is essentially what we mean when we say that a black-hole has formed. The singularity is hidden from view by the event horizon, which is the boundary of that spacetime region surrounding the singularity which cannot communicate with the far-away observer (Figure 1).

The singularity theorems of Hawking and Penrose do not imply that the collapsing star which develops a singularity will necessarily become a black-hole. That is, even if a trapped surface and hence a singularity does form during collapse, the trapped surface need not hide the singularity from a far-away observer. This alternate possibility is called a naked singularity (Figure 2). In this case, the event-horizon fails to cover the singularity and light from the singularity could escape to infinity. One could form a rough picture of this situation
by imagining that a singularity has formed at the center of a collapsing spherical star before its boundary has entered the Schwarzschild radius. Such a singularity could be visible to an observer watching the collapse. Also, if a singularity forms without the conditions of the singularity theorems having been met, it could be naked, as the event horizon may not form at all.

![Diagram of black hole formation](image)

**Figure 1.** The formation of a black-hole in gravitational collapse. The event horizon starts forming at O and covers the singularity. The boundary of the trapped region inside the star is the apparent horizon.

Does gravitational collapse end in a black-hole or a naked singularity? Einstein’s equations must give a definite answer to this question, but at present we do not know what that answer is. One could well ask, what difference does it make? The difference is very significant. If a singularity is naked, one could prescribe arbitrary data on the singular surface - this would result in total loss of predictability in the future of the singularity. If a singularity is hidden behind an event horizon, predictability would be preserved at least in the spacetime region outside the horizon.

Furthermore, the issue has great importance for black-hole astrophysics and for the theory of black-holes. In terms of their astrophysical properties, naked singularities could be very different from black-holes. While (classical) black-holes are one-way membranes and inflowing matter simply gets sucked into the singularity, quite the opposite may hold for a naked singularity; matter might be thrown out with great intensity from near a naked singularity. The validity of many theorems on black-hole dynamics depends on the assumption of absence of naked singularities. The extensive and successful applications of black-holes in astrophysics, and the detailed studies of their profound and elegant properties does lend strong support to the belief in their existence. Nonetheless, it is an open question as to
whether gravitational collapse necessarily ends in a black-hole or could in some cases lead to a naked singularity. If the latter is the case, then one needs to know what kind of stars end as naked singularities.

![Figure 2. The formation of a naked singularity. The singularity begins to form at S.](image)
The event horizon starts forming at O and fails to cover the singularity entirely. Else, the singularity may form without the event horizon forming at all. Light rays are able to escape from the singularity.

In the absence of an answer to the above question, the basis for black-hole physics and its applications is the Cosmic Censorship Hypothesis. In somewhat non-rigorous terms, the hypothesis could be stated as

- Gravitational collapse of physically reasonable matter starting from generic initial data leads to the formation of a black-hole, not a naked singularity.

We will elaborate in a moment on the italics. Such a hypothesis was first considered by Penrose in [2], where he asked as to whether there exists a Cosmic Censor who would always cloth a singularity with an event horizon. (It is interesting, though probably not well-known, that on occasion Penrose has also written in support of naked singularities [3]). The censorship hypothesis has remained unproved despite many serious efforts (for a review of some of the attempts see [4]). Part of the difficulty lies in not having a unique rigorous statement one could try to prove. Studying gravitational collapse using Einstein equations is a formidable task, and evidence for formation of a black-hole is as limited as for a naked singularity. This is simply because very few examples of dynamical collapse have been worked out, and not many exact solutions are known. Until recently, the widely held belief has been that the hypothesis must be right; however studies over the last few years have caused at least
some relativists to reconsider their earlier view. It is obvious, though not often emphasized, that falsification of the hypothesis does not necessarily rule out black-holes altogether but might allow only a subset of the initial data to result in naked singularities. The former alternative (i.e. no black-holes) is extremely unlikely, given the great richness of black-hole physics and astrophysics.

By ‘physically reasonable matter’ one usually means that the (classical) matter obeys one or more of the energy conditions (for a discussion see [5]). The weak energy condition for instance requires that the pressures in the collapsing matter not be too negative, and the energy density be positive. By ‘generic initial data’ one means that the solution of Einstein equations being used to study collapse has as many free functions as are required for the initial data to be arbitrary.

Many people consider naked singularities to be a disaster for general relativity and for physics as such. We have mentioned that naked singularities could result in a total loss of predictability to their future. However, if we make the natural assumption that the ultimate theory of gravity will preserve predictability (in some suitable sense of the concept), the occurrence of naked singularities in general relativity will signal a real need for modification of the theory. Such a definite (and rare) signal could only help improve our understanding of gravitation, instead of being a disaster! To many other people, naked singularities are a positive asset for astrophysics - the possibility of emission of light from high curvature regions close to the singularity might make such singularities extraordinary energy sources. Sometimes a view is expressed that in any case a quantum theory of gravity will avoid singularities altogether - in that case how does it matter whether the classical theory (general relativity) predicts the singularities to be naked or covered? Our discussion above suggests that even if quantum gravity were to avoid singularities, the spacetime regions that are naked according to the classical theory will behave very differently from black-holes in the quantized theory as well. Besides, the relevance of the hypothesis to black-hole astrophysics cannot be overlooked.

Perhaps it is important to mention that the validity of the hypothesis could be discussed at two distinct levels. Firstly, we need to find out if it holds in general relativity. However, even if relativity theory allows naked singularities, it could be that actual stars may not end up as naked singularities. This could happen if the initial conditions necessary for a naked singularity to form are simply not observed in the real world. Thus violation of the hypothesis at a theoretical level might compel us to consider replacing general relativity by a better theory, but may not have observational consequences.

The failure to prove the hypothesis has led, over the last ten years or so, to a change in the approach towards the problem. Attention has shifted to studies of specific examples of gravitational collapse. For instance, people have been studying spherical gravitational collapse, with a particular choice of the matter stress tensor - like dust, perfect fluids, and massless scalar fields. These and other examples often show that collapse of physically reasonable matter can end either in a black-hole or a naked singularity, depending on the choice of initial data. To a degree, such examples go against expectations that the censorship
hypothesis is correct. It, however, does remain to be seen as to whether or not generic initial data will lead to naked singularities. These examples can be regarded as good learning exercises - at the very least we learn about the properties of naked singularities which form. Perhaps these very properties might suggest that the singularity, though naked, does not violate the spirit of the hypothesis.

In this article we will attempt an overview of the end state of gravitational collapse according to general relativity, the focus of discussion being the censorship hypothesis. Rather than reviewing the attempts to prove the hypothesis, we will concern ourselves with recent studies of models of collapse. It is relevant to note that these models typically examine the properties of curvature singularities that form during collapse. If a naked curvature singularity does form, the issue of its geodesic incompleteness is to be handled separately - an aspect we will consider briefly towards the end of the review. Also, we will not discuss examples of static naked singularities in general relativity, although many such are known.

In Section II, spherical gravitational collapse for various forms of matter is reviewed. We also discuss some properties of the naked singularities found in these models. Section III is a brief discussion of the limited results on non-spherical collapse. In the last section, a critical comparison of the various existing results and their interpretation is attempted.

2. Spherical Gravitational Collapse

It is only natural that most of the examples studied are for the idealized case of spherical collapse, in asymptotically flat backgrounds. Even here, one does not yet know the conditions for formation of black-holes and naked singularities. Some results are known for specific forms of energy-momentum tensors. Often there are striking similarities amongst results for different kinds of matter, suggesting an underlying pattern. We review results for collapse of dust, perfect fluids with pressure, a radiating star described by the Vaidya metric, massless scalar fields, and spherical collapse for matter with no restriction on the energy-momentum tensor, except the weak energy condition.

2.1 Dust collapse

By dust one means a perfect fluid for which the pressure is negligible and is set to zero. This highly idealized description has the advantage that an exact solution of Einstein equations is known, which describes the collapse of a spherical dust cloud in an asymptotically flat spacetime. This is the Tolman-Bondi solution, given independently by the two authors [6].

The evolution of the cloud is determined once the initial density and velocity distribution of the fluid has been given. Because of spherical symmetry these distributions will be functions only of the radial coordinate $r$, so that the initial data consists of two arbitrary functions of $r$. The cloud is assumed to extend up to a finite radius and the interior dust solution is matched to a Schwarzschild exterior. Since we are interested in collapse, the velocity of each fluid element is taken to be towards the center of the cloud. It can be shown that starting from regular initial data, the collapse leads to the formation of a curvature singularity.

A very special case of the Tolman-Bondi solution is a dust cloud whose initial density
distribution is homogeneous, and the velocity increases linearly with the physical distance from the center. This of course is the Friedmann solution matched to a Schwarzschild exterior and was used by Oppenheimer and Snyder [7] to provide the first example of dynamical collapse in general relativity. Starting from regular initial data, the cloud develops a curvature singularity at its center which is not visible. This was the first theoretical example of black-hole formation (Figure 3(i)).

For many years, the work of Oppenheimer and Snyder has remained a model for how a black-hole might form in gravitational collapse. The collapsing star will enter its Schwarzschild radius, become trapped and proceed to become singular, and the singularity will be hidden behind the event horizon. The censorship hypothesis was also originally inspired essentially by this work, because not much more was known anyway about properties of gravitational collapse. In hindsight, one might be surprised with the degree of generality attributed to results arising from the study of a model star that is spherically symmetric, homogeneous and made of dust - none of the three properties are obviously true for a real star!

\[\text{Figure 3.}
\text{Comparing the formation of the black-hole in homogeneous dust collapse (Fig. 1) with the formation of a globally naked singularity in inhomogeneous dust collapse (Fig. 2). The event horizon begins to form at O. For inhomogeneous collapse leading to a naked singularity, the center is the first point to get trapped, unlike in the homogeneous case. In (a), a family of rays escapes from the singularity at } O.\]

Since the exact solution of Tolman and Bondi was known, it would have been quite natural to extend the work of Oppenheimer and Snyder to inhomogeneous dust collapse, described by this exact solution. However, for nearly three decades after the paper of Oppenheimer and Snyder was published, very little work appears to have been done on gravitational collapse. Presumably, there were not enough reasons for interest in the subject until the discovery of quasars in the sixties, and the development of singularity theorems. It is also true though that analysis of light propagation in the collapsing inhomogeneous dust star is a difficult
task, and tractable methods have been developed only in recent years. The first work to deal with the inhomogeneous Tolman-Bondi model appeared in the seventies [8]. Since then, many investigations of inhomogeneous dust collapse have taken place [9], from the point of view of the censorship hypothesis, and the last word on this particular model has not yet been said.

Yodzis et al. [8] were investigating what are called shell-crossing singularities (or caustics) which form due to the intersection of two collapsing dust-shells at a point other than the center. These are curvature singularities and they are also naked - but are not regarded as violation of censorship since there is evidence [10] that such singularities are gravitationally weak (as discussed later in the article). They are similar to the shell-crossing singularities that occur in Newtonian gravity also and it is believed that spacetime can be extended through such singularities.

Of a more serious nature are the shell-focussing singularities which form at the center of the cloud - they result from the shrinking of collapsing shells to zero radius. It was found by various people that for some of the initial density and velocity distributions, the collapse ends in a naked singularity, whereas for other distributions it ends in a black-hole. Also, both the black-hole and naked singularity solutions result from a non-zero measure set of initial data. In particular, there was found a one-parameter family of solutions (described say by the parameter $\xi$), such that for $\xi < \xi_c$ the collapse leads to a black-hole, whereas for $\xi > \xi_c$ it leads to a visible singularity.

The space-time diagram for inhomogeneous dust collapse leading to a naked shell-focussing singularity is shown in Figure 3(ii), and should be contrasted with Figure 3(i) for Oppenheimer-Snyder collapse. Of particular importance is the difference in the evolution of trapped surfaces in the two cases, and the fact that for inhomogeneous collapse different shells become singular at different times, unlike in the homogeneous case. For a discussion of trapped surfaces in dust collapse see Jhingan et al. in [9].

At this stage we need to distinguish between a *locally* naked singularity and a *globally* naked singularity. We say the singularity is locally naked if light-rays do emerge from the singularity but fall back to the center without escaping the event-horizon. Such a singularity will be visible to an infalling observer who has entered the Schwarzschild radius, but cannot be seen by an asymptotic observer (Figure 4). A singularity is called globally naked if light-rays emerging from the singularity escape the event-horizon and reach an asymptotic observer. We say the singularity is visible if there are light-rays emerging from the singularity - a visible singularity may be locally or globally naked. We say that a piece of the singularity is covered if it is not even locally naked. In our terminology, a locally naked singularity is also a black-hole. In this article when we call a singularity naked, we mean it is globally naked. The *weak* censorship hypothesis allows for the occurrence of locally naked singularities but not globally naked ones, whereas *strong* censorship does not allow either.
As an illustration we describe in some detail the gravitational collapse of an inhomogeneous dust cloud, starting from rest. It can be shown that the singularity resulting from the collapse of shells with $r > 0$ is covered. At most, the singularity forming at $r = 0$ (the central singularity) can be naked. The conditions for the central singularity to be naked are given as follows. Let the initial density $\rho(R)$ as a function of the physical radius $R$ be given as a power-series, near the center:

$$\rho(R) = \rho_0 + \frac{1}{2} \rho_2 R^2 + \frac{1}{6} \rho_3 R^3 + ...$$

where $\rho_0$ is the initial central density, and $\rho_2$ and $\rho_3$ are respectively the second and third derivatives at the center. We assume that the density decreases with increasing $R$, hence the first non-vanishing derivative is negative. It turns out that if $\rho_2 < 0$ the singularity is visible. If $\rho_2 = 0$ and $\rho_3 < 0$ then one defines a parameter $\xi = |\rho_3|/\rho_0^{5/2}$. The singularity is visible for $\xi > 25.47$ and covered for $\xi < 25.47$. If $\rho_2 = \rho_3 = 0$ the singularity is covered and we have the formation of a black-hole. The Oppenheimer-Snyder example is a subset of this case. In the case of a visible singularity, entire families of light-rays emerge from the singularity. Note that $\rho_2 < 0$ is generic and $\rho_2 = 0$ non-generic, hence dust collapse starting from rest leads to a visible singularity for generic initial density profiles.

We find that there is a transition from naked singularity type behaviour to black-hole type behaviour as the density profile is made less inhomogeneous by setting more and more density derivatives to zero. In the cases where the singularity is visible, the initial density distribution through the star then determines whether the singularity is locally or globally
naked - examples of both kind occur. When the collapsing dust cloud has an initial velocity, the overall picture regarding the nature of the singularity is essentially similar to the one described here for the case of a cloud collapsing from rest. Both black-holes and visible singularities now arise from generic initial data. A Penrose diagram for the naked singularity is shown in Figure 5. In summary, when one allows for inhomogeneity in the density distribution, the nature of gravitational collapse is quite different from what Oppenheimer and Snyder found for the homogeneous case. Some open issues relating to dust collapse are pointed out later, in the discussion.

Figure 5. A globally naked central singularity. The dashed line is the Cauchy horizon and the dotted line is the event horizon.

2.2 Including Pressure

The description of collapsing matter as dust is an idealization and a realistic study of collapse must take pressure gradients into account. However, useful exact solutions of Einstein equations describing spherical collapse of matter with pressure are hard to come by, and as a result a clear picture of the kind presented above for dust does not yet exist. It is in fact remarkable that we do not even know how the Tolman-Bondi solution would change under the introduction of “small” pressures. For instance if we are considering an equation of state $p = k \rho$ for a perfect fluid, with $k$ very close to zero, it is not clear whether the solution is a perturbation to the dust solution. In this section we summarize the few known results on collapse of relativistic fluids, vis a vis the censorship hypothesis.

When pressure is included, the energy momentum tensor $T_{ik}$ for matter undergoing spherical collapse is conveniently described in a comoving coordinate system. In these coordinates,
$T_{ik}$ is diagonal and its components are the energy density, the radial pressure and the tangential pressure. (The only exception to this general description of the matter is the case of null dust, described by the Vaidya spacetime, and reviewed below). For a perfect fluid, the two pressures are identical. Since perfect fluids are easier to study compared to imperfect ones, they have inevitably received greater attention.

An important early paper on the collapse of perfect fluids is that of Misner and Sharp [11]. They set up the Einstein equations for this system in a useful physical form, bringing out the departures from the Oppenheimer-Volkoff equations of hydrostatic equilibrium. However, they did not consider solutions of these equations. Lifshitz and Khalatnikov, and Podurets [12] worked out the form of the solution near the singularity for the equation of state of radiation, $p = \rho/3$. Their approach has not received much attention, but appears to offer a promising starting point for investigating censorship.

A significant development was the work of Ori and Piran [13], who investigated the self-similar gravitational collapse of a perfect fluid with an equation of state $p = k\rho$. It is readily shown that the collapse leads to the formation of a curvature singularity. The assumption of self-similarity reduces Einstein equations to ordinary differential equations which are solved numerically, along with the equations for radial and non-radial null geodesics. It is then shown that for every value of $k$ (in the range investigated: $0 \leq k \leq 0.4$) there are solutions with a naked singularity, as well as black-hole solutions. Each kind of solution has a non-zero measure in the space of spherical self-similar solutions for this equation of state. The issue as to which initial data (density profile, velocity profile, and value of $k$) leads to naked singularities has not been fully worked out though, and is an interesting open problem. (We have in mind a comparison with the results for dust, quoted above).

An analytical treatment for this problem was developed by Joshi and Dwivedi [14]. After deriving the Einstein equations for the collapsing self-similar perfect fluid they reduce the geodesic equation, in the neighborhood of the singularity, to an algebraic equation. The free parameters in this algebraic equation are in principle determined by the initial data. The singularity will be naked for those values of the parameters for which this equation admits positive real roots. Since this is an algebraic equation, it will necessarily have positive roots for some of the values of the parameters, and for the initial data corresponding to such values of the parameters the singularity is naked. It is shown that families of non-spacelike geodesics will emerge from the naked singularity. As in the case of Ori and Piran’s work, the explicit relation between the initial data and the naked singularity has not yet been worked out. Also it is not clear as to what is the measure of the subset of solutions leading to a naked singularity or a black-hole. This analysis was extended to a self-similar spacetime with a general form for $T_{ik}$ [15].

On physical grounds, imperfect fluids are more realistic than perfect ones; very little is known about their collapse properties though. An interesting paper is the one by Szekeres and Iyer [16], who do not start by assuming an equation of state. Instead they assume the metric components to have a certain power-law form, and also assume that collapse of physically reasonable fluids can be described by such metrics. The singularities resulting in
the evolution are analysed, with attention being concentrated on shell-focussing singularities at $r > 0$. They find that although naked singularities can occur, this requires that the radial or tangential pressure must either be negative or equal in magnitude to the density.

Lastly we mention the collapse of null dust (directed radiation), described by the Vaidya spacetime [17]. Since the exact solution is known, the collapse has been thoroughly investigated [18] for the occurrence of naked singularities. One considers an infalling spherical shell of radiation and imagines it as being made of layers of thin shells. A thin shell becomes singular when its radius shrinks to zero. Let the shells be labelled by the advanced time coordinate $v$, with $v = 0$ for the innermost shell, and let $m(v)$ be the Vaidya mass function. It can be shown that the singularity at $v > 0$ is covered. For $v = 0$ the singularity is naked if $\lambda \equiv 2 \frac{dm(v)}{dv}|_{v=0}$ is less than or equal to $1/8$, and covered otherwise. These results bear an interesting similarity with those for dust, described in the previous section.

It will be evident from the previous few paragraphs that our understanding of spherical collapse, when pressure gradients are included, is rather incomplete. Ultimately, one would like to develop the kind of clear picture that is available for dust collapse. It is interesting however that the collapse of a self-similar perfect fluid, and of the fluids considered by Szekeres and Iyer, admits both black-hole and naked singularity solutions. This also brings forth an astrophysical issue - what is the relevant equation of state in the final stages of collapse of a star? Could it be that the initial data leading to a naked singularity is not being realised astrophysically?

2.3 Collapse of a scalar field

One could take the viewpoint that the description of matter as a relativistic fluid is phenomenological, and that the censorship hypothesis should be addressed by considering matter as fundamental fields. As a first step, one could study the spherical collapse of a self-gravitating massless scalar field. A good deal of work has been done on this problem over the last few years, and exciting results have been found. Here we can provide only a very brief overview.

In a series of papers, Christodoulou has pioneered analytical studies of scalar collapse [19]. He established the global existence and uniqueness of solutions for the collapsing field, and also gave sufficient conditions for the formation of a trapped surface. For a self-similar scalar collapse he showed that there are initial conditions which result in the formation of naked singularities. He has claimed that such initial conditions are a subset of measure zero and hence that naked singularity formation is an unstable phenomenon.

Christodoulou was also interested in the question of the mass of the black-hole which might form during the collapse of the scalar wave-packet. Given a one parameter family $S[p]$ of solutions labelled by the parameter $p$ which controls the strength of interaction, it was expected that as $p$ is varied, there would be solutions with $p \to p_{\text{weak}}$ in which the collapsing wave-packet disperses again, and solutions with $p \to p_{\text{strong}}$ which have black-hole formation. For a given family there was expected to be a critical value $p = p_*$ for which
the first black-hole appears as \( p \) varies from the weak to the strong range. Do the smallest mass black holes have finite or infinitesimal mass [20]? This issue would be of interest for censorship, since an infinitesimal mass would mean one could probe arbitrarily close to the singularity. Of course when one is considering real stars, a finite lower limit on the mass of the collapsing object arises because non-gravitational forces are also involved.

This problem was studied by Choptuik [21] numerically and some remarkable results were found. He confirmed that the family \( S[p] \) has dispersive solutions as well as those forming black-holes, and a transition takes place from one class to the other at a critical \( p = p_* \). The smallest black-holes have infinitesimal mass. Near the critical region, the mass \( M_{bh} \) of the black-hole scales as \( M_{bh} \approx (p - p_*)^\gamma \), where \( \gamma \) is a universal constant (i.e. same for all families) having a value of about 0.37. The near critical evolution can be described by a universal solution of the field equations which also has a periodicity property called echoing, or discrete self-similarity. That is, it remains unchanged under a rescaling \((r, t) \to (e^{-n\Delta}r, e^{-n\Delta}t)\) of spacetime coordinates. \( n \) is an integer, and \( \Delta \) is about 3.4. Subsequently, these results have been confirmed by others [22]. The occurrence of black-holes with infinitesimal mass goes against the spirit of censorship. The critical solution \((p = p_*)\) is a naked singularity. However, since the naked singularity is being realised for one specific solution in the one parameter family, it is a measure zero subset.

Attempts have been made to construct analytical models which will reproduce Choptuik’s numerical results [23]. Since it is difficult to make a model with discrete self-similarity, continuous self-similarity is assumed instead. Brady showed that there are solutions which have dispersal, as well as solutions which contain a black-hole or a naked singularity. It would be of interest to relate his results to the naked singularity solutions found by Christodoulou for self-similar collapse. Recently, Gundlach has constructed a solution with discrete self-similarity, which agrees with the critical universal solution found numerically by Choptuik [24].

Similar critical behaviour has also been found in numerical studies of collapse with other forms of matter. Axisymmetric collapse of gravitational waves was shown to have a \( \gamma \) of about 0.36, and \( \Delta \simeq 0.6 \) [25]. For spherical collapse of radiation (perfect fluid with equation of state \( p = \rho/3 \)) the critical solution has continuous self-similarity, and \( \gamma \) of about 0.36 [26]. However it has become clear now that the critical exponent \( \gamma \) is not independent of the choice of matter. A study of collapse for a perfect fluid with an equation of state \( p = k\rho \) shows that \( \gamma \) depends on \( k \) [27]. For a given form of matter, there appears to be a unique \( \gamma \), but the value changes as the form of \( T_{ik} \) is changed. Further studies of critical behaviour are reported in [28].

The models described in this section exhibit a naked-singularity like behaviour for near-critical solutions - such solutions are presumably of measure zero on the space of all solutions. In the supercritical region the collapse is said to lead to the formation of a black-hole. This raises a question as to how these results, say the supercritical solutions for radiation fluid collapse, are consistent with those of Ori and Piran, who do find naked singularities. (The Ori-Piran naked singularity lies in the supercritical region). It maybe that when one
numerically finds a singularity at the center $r = 0$ one is not easily able to tell whether this is a black-hole or a naked singularity, and this may have to be investigated further.

It is perhaps also relevant to note that collapse of real stars which proceed to become singular is expected to be described by supercritical solutions. Thus the naked singularity observed near the critical region, while of major theoretical interest, may not have astrophysical implications. This further emphasizes the need for investigating whether the singularity being observed numerically in the supercritical region is necessarily covered by the horizon, or could be naked.

It is undoubtedly true that these studies of critical behaviour have opened up an entirely new aspect of gravitational collapse and the related issue of censorship. Obtaining a theoretical understanding of these numerically observed phenomena is an important open problem.

2.4 Spherical collapse with general form of matter

One finds a certain degree of similarity in the collapse behaviour of dust, perfect fluids and scalar fields - in all cases some of the initial distributions lead to black holes, and other distributions lead to naked singularities. This would suggest an underlying pattern which is probably characterized, not by the form of matter, but by some invariants of the gravitational field. The role of matter may simply be that of determining which part of the parameter space these invariants lie in. Hence studies of collapse which put no restriction on $T_{ik}$ apart from an energy condition should prove useful (still maintaining spherical symmetry).

An interesting attempt in this direction has been made by Dwivedi and Joshi in [29], where they generalized their earlier work on dust collapse and self-similar fluids. They assumed a general $T_{ik}$ obeying the weak energy condition, and also that the collapsing matter forms a curvature singularity at $r = 0$ (the central singularity). As we noted earlier, in the comoving coordinate system, matter is described by its energy density and the radial and tangential pressures. Along with these three functions, three functions describing the metric enter a set of five Einstein equations, which are augmented with an equation of state in order to close the system. The geodesic equation for radial null geodesics is written in the limit of approach to the singularity, and it is shown that the occurrence of a visible singularity is equivalent to the occurrence of a positive real root for the geodesic equation, suitably written. Since this equation depends on free initial data, it follows that for a subset of the initial data there will be positive real roots and the singularity will be visible.

This approach needs to be pursued further in order to find out whether or not the naked singularities are generic. Also, it is of interest to work out as to exactly which kind of initial data lead to naked singularities. These are difficult problems, in the absence of known exact solutions. Another interesting attempt at treating spherical collapse without restricting $T_{ik}$ is due to Lake [30]. He concluded that a visible central singularity could form if the mass function in the neighborhood of the singularity satisfies certain conditions. The relation of these conditions with the initial data is not yet apparent.
2.5 Properties of naked singularities

It is evident that energy conditions by themselves do not restrict the occurrence of naked singularities. One would then like to examine in some detail properties of these naked singularities, so as to see if these properties might contain clues for a censorship hypothesis. We review below some of the important features of the naked singularities found in various models.

Curvature strength: When a collapsing star develops a curvature singularity, the energy density also becomes singular. However, finite physical volumes may or may not be crushed to zero volume as the singularity is approached. This could be used as a criterion for judging the physical seriousness of the singularity, and also for the possible extendibility of spacetime through the singularity [31]. We call a singularity a strong curvature singularity if collapsing volume elements do get crushed to zero at the singularity, and a weak curvature singularity if they do not. (The terms weak and strong singularity are sometimes used in the literature with a different meaning. We use them here in Tipler’s sense). It is believed that spacetime cannot be extended through a strong singularity, but is possibly extendible through a weak one. A rigorous proof for this is not yet available but is being attempted by some relativists (for detailed studies see [32]). Clarke and Krolak [33] gave necessary and sufficient quantitative criteria for the singularity to be strong, in terms of the rate of growth of curvature along outgoing geodesics, as the singularity is approached.

A strong naked singularity is regarded as a more serious violation of censorship as compared to a weak one. For instance, the shell-crossing type singularities are gravitationally weak [10]. Newman [34] studied the naked central singularity in Tolman-Bondi dust collapse for a wide class of initial data and showed it to be weak. On this basis it was conjectured in [34] that strong naked singularities do not occur in collapse. It was however shown by various people [35] that inclusion of initial data not considered by Newman gives rise also to strong central naked singularities in dust. Interestingly, it has recently become clear that the initial data leading to these strong naked singularities is non-generic, whereas the data leading to a weak naked singularity is generic [36]. In this sense Newman’s conjecture does hold for dust collapse! But strong naked singularities have been found in other models - for instance in the naked singularities in the Vaidya spacetime, where they arise from generic initial data. They were also found by Ori and Piran in their study of the self-similar perfect fluid. No results on strength seem to be known for scalar collapse. The general $T_{ik}$ studied by Dwivedi and Joshi would lead to a strong naked singularity for some initial data - however the genericity of such initial data is an open issue. Thus the generality of strong naked singularities remains unclear and it still might be possible to formulate a censorship hypothesis along the lines of Newman’s conjecture.

Are naked singularities massless?: In all known examples of naked singularities, the mass of the collapsing object (well-defined in spherical symmetry, with a vacuum exterior) is found to be zero at the point where the singularity forms. There is evidence that this is a general property of naked singularities in spherical collapse [30]. On the other hand the black hole singularity is always found to be massive. Since a massless singularity might be thought of as
having no associated gravitational field, this has led to the suggestion that such singularities do not violate censorship. Note however that even from this massless naked singularity entire families of geodesics emerge, and it is not clear whether it is the mass or the outgoing geodesics which are a more important property of the naked singularity!

*Redshift:* In known examples of naked singularities for dust and perfect fluids, the redshift along outgoing geodesics emerging from the singularity is found to be infinite (when calculated for observers in the vacuum region). This could be interpreted to mean that no “information” is being transmitted from the naked singularity and could be yet another approach to preserving censorship.

*Stability and Genericity:* This of course is the most important issue relating to the naked singularity examples, and a notion of stability is hard to define. The most direct definition of stability (equivalently, genericity) of naked singularities would be simply as follows. If a solution of Einstein equations describing collapse leading to a naked singularity has as many free functions as required for arbitrary initial data, the solution is stable. (One is reminded here of the methods adopted by Belinskii, Lifshitz and Khalatnikov to show that general solutions of Einstein equations contain singularities). Of course progress in such a broad sense is hopelessly difficult, and one talks of stability of a given solution under specific kinds of perturbations. For instance, one would consider stability of the solution against change of initial data, against change of equation of state, and against non-spherical metric perturbations. From these viewpoints, very little is known about the stability of the naked singularity models mentioned in this article. (It is important to note that equally little is known about the stability of the black-holes which form in these models during collapse! Various studies show that the event horizon is stable to small perturbations [37], and hence the singularity is stable, but it could be either naked or covered.)

One very useful way to address stability of naked singularities is to study the blue-shift instability of the Cauchy horizon. One considers ingoing waves starting from null infinity, as they approach the Cauchy horizon. If they develop an infinite blue-shift along the Cauchy horizon, this in some sense is like saying this horizon would be ‘destroyed’ and the spacetime region beyond, which is exposed to the naked singularity, would no longer be accessible. Hence predictability will be preserved in the observable spacetime region. Interestingly enough, it has been found that the Cauchy horizon in the dust and perfect fluid examples does not have a blue-shift instability.

*Quantum effects:* The censorship hypothesis as such is concerned with the nature of singularities in classical general relativity. However, even if naked singularities do occur in the classical theory, one could ask if their formation would be avoided when quantum effects in their vicinity are taken into account. This would be a quantum cosmic censorship. Some investigations have taken place in this direction [38, 39] and this very interesting question deserves to be pursued further. Essentially the idea is to repeat for a naked singularity the kind of calculation Hawking carried out to show that quantum effects cause black-holes to radiate. Since regions of very high curvature are exposed near the naked singularity, intense particle production can be expected. It is typically found in these calculations that as a re-
sult of the produced particles, the energy-momentum flux at infinity diverges - in spite of the fact that the naked singularity is massless and the classical outgoing geodesics have infinite redshift! Although back-reaction calculations are hard to carry out, the infinite flux would suggest that the naked singularity formation will be avoided. From the quantum viewpoint, naked singularities appear to be explosive events, and the outgoing flux might be their only possible observational signature. It is worth studying the properties of this flux in detail to understand what observations, if any, can detect naked singularities if they do occur in nature.

Thus we find that properties like curvature strength, masslessness, redshift, blue-shift instability, and quantum effects give a mixed sort of picture regarding the significance of these naked singularity examples. An optimistic assessment of this situation is that there is a good deal of richness in the problem, and much to think about, before we can decide one way or the other.

3. Non-spherical gravitational collapse

As we have seen, there are examples of naked singularities in spherical collapse. By assuming that the evolution can be continued beyond the Cauchy horizon, one can conclude that the collapsing star will eventually shrink below its Schwarzschild radius and the event horizon will form (according to the infalling observer). There is also evidence that the horizon is stable to small perturbations. However, if there are large departures from spherical symmetry, the picture could be different, and the horizon may not form at all. The naked singularity so forming would qualitatively be of a different kind, compared to the ones seen in spherically symmetric spacetimes.

Our knowledge of exact solutions in the non-spherical case is inevitably even more limited than for spherical systems, and one must again resort to introducing some symmetry. An important early study was due to Thorne [40], and was motivated by the work of Lin, Mestel and Shu [41] on the collapse of Newtonian spheroids. Thorne examined the collapse of an infinite cylinder and showed that it develops a curvature singularity without an event horizon forming. Considerations such as these led him to propose what came to be known as the hoop conjecture, which he stated as follows [40]:

• Horizons form when and only when a mass $M$ gets compacted into a region whose circumference in EVERY direction is $C \lesssim 4\pi M$.

According to the conjecture, collapsing objects which become so asymmetric as to attain a circumference which is greater than the bound will not develop horizons; hence if a singularity forms it will be naked. We note that even if the conjecture holds, a naked singularity can form, (as it does sometimes in spherical collapse), but an event horizon will also form. One could say that the naked singularities which form when the hoop conjecture holds are of a less serious nature than those which form when the conjecture does not hold. In the former case an infalling observer cannot communicate with asymptotic observers after crossing the horizon, while in the latter case no such restriction arises.
Important numerical simulations were carried out by Shapiro and Teukolsky [42] to test the hoop conjecture. They studied the gravitational collapse of homogeneous non-rotating oblate and prolate spheroids of collisionless gas, starting from rest. Maximal slicing is used, and the evolution of the matter particles is followed with the help of the Vlasov equation in the self-consistent gravitational field. The development of a singularity is detected by measuring the Riemann invariant at every point on the spatial grid. Since an event horizon can be observed only by tracking null rays indefinitely, they instead search for the formation of an apparent horizon (the boundary of trapped surfaces) - the existence of the apparent horizon can be determined locally. If a spacetime region has an apparent horizon, it will also have an event horizon, to its outside.

They found that oblate spheroids first collapse to thin pancakes, but then the particles overshoot and ultimately the distribution becomes prolate and collapses to a thin spindle. An apparent horizon develops to enclose the spindle, which eventually becomes singular, and a black hole is formed. The minimum exterior circumference in the polar and equatorial directions is consistent with the requirements of the hoop conjecture. The collapse of prolate spheroids leads however to the formation of a spindle singularity, with no evidence for an apparent horizon covering the singularity. The initial dimensions of the spheroid are such that the minimum circumference, at the time of formation of the singularity, exceeds $4\pi M$. The collapse of smaller prolate spheroids leads to spindle singularities that are covered by a horizon, again favouring the conjecture. Shapiro and Teukolsky suggested, noting the absence of an apparent horizon for large prolate configurations, that the resulting spindle singularity is naked, and hence that the hoop conjecture holds.

They were also careful to point out that the absence of an apparent horizon up until the time of singularity formation does not necessarily imply the absence of an event horizon. Hence, strictly one could not conclude that the singularity is necessarily naked. Wald and Iyer [43] showed this mathematically with an example - they demonstrated that Schwarzschild spacetime can be sliced with nonspherical slices, which approach arbitrarily close to the singularity, but do not have any trapped surfaces. Another analytical example showing that absence of apparent horizon does not imply the singularity is naked can be found in [36], where inhomogeneous spherical dust collapse was studied. However, in support of their conclusion Shapiro and Teukolsky pointed out that null rays continue to propagate away from the region of the singularity until when the simulations are terminated, and that the formation of an event horizon is unlikely. It is perhaps fair to conclude that while their numerical simulations are of major importance and their results suggestive, further investigations are necessary to settle the issue. An analytical demonstration analogous to these simulations was worked out in [44].

Another interesting analytical example is the quasi-spherical dust solution due to Szekeres, which also admits naked singularities [45], including those having strong curvature [46].
4. Discussion

We now attempt a critical comparison of the results reviewed here, and discuss their implications for the censorship hypothesis. (For other recent reviews of cosmic censorship see Clarke [4] and Joshi [47]). Let us begin with a quick summary, even though it amounts to repetition.

Very massive stars are expected to end their gravitational collapse in a singularity. There has been around an unproven conjecture that the singularity will be hidden behind an event horizon, and hence such stars will become black-holes. If the conjecture is false, some stars can end up as naked singularities - this will have major implications for black-hole physics and astrophysics, and for classical general relativity. Since a proof for the conjecture has not been forthcoming, relatively modest attempts have been made to study specific examples of gravitational collapse. These studies, which so far have been mostly for spherical collapse, have thrown up some surprises. The collapse does not always end in a black-hole; for some initial data it ends in a naked singularity, and this is true for various forms of matter.

The spherical gravitational collapse of inhomogeneous dust leads to weak naked singularities for generic initial data; the strong naked singularities which do form for some data are non-generic. The naked singularity, irrespective of whether it is weak or strong, is massless. The outgoing geodesics have an infinite redshift, and the Cauchy horizon does not have a blue-sheet instability. The collapse of a self-similar perfect fluid also exhibits strong curvature naked singularities for some initial data - these are again massless, have infinitely redshifted outgoing geodesics, and the Cauchy horizon is stable. Numerical studies of scalar field collapse suggest that the critical solution is a naked singularity; elsewhere in the data space the collapse ends either in dispersal or a singularity. It seems unclear as to whether this singularity is definitely covered, or could be naked. For collapse of a general form of matter, there is an existence proof that for some of the data the singularity will be naked and strong - its genericity is an open issue. There appear to be no conclusive studies on non-spherical perturbations of these solutions, or on non-spherical collapse as such - the simulations of Shapiro and Teukolsky are however an important progress in this direction.

A broad conclusion is that at this early stage in collapse studies, one does not have enough evidence to take a grand decision about the validity of the censorship hypothesis, one way or the other. Further, it does not appear very useful right now to try and prove a specific version of the hypothesis. Consider, just as an example, the following proposal:

- Gravitational collapse of physically reasonable matter starting from generic initial data leads to the formation of a black-hole or a naked singularity. The naked singularity is massless and gravitationally weak, the Cauchy horizon does not have a blue-shift instability and the redshift along outgoing geodesics is infinite.

This does not appear to be a very interesting proposal to prove, given the number of properties attached to the naked singularity, nor is it certain that it will survive further studies.
of collapse models. It is not even clear whether this proposal proves or disproves cosmic censorship!

What is noteworthy however is that naked singularities do occur in dynamical collapse, side by side with black-hole solutions, so to say. In order to address the censorship hypothesis, one has to assess the significance of their properties, and also generalise the models studied so far. There are perhaps three important questions: (i) what is the role of the form of matter? (ii) are the naked singularities genuine features of the spacetime geometry? (iii) do they have observational effects? We respond to these questions briefly.

The form of matter in these models is dust, a perfect fluid, a scalar field, or where an existence proof has been given, a general $T_{ik}$ obeying the weak energy condition. There are quite a few views on using dust as a form of matter in these studies, which we try to enumerate. Firstly, since dust collapse can give rise to singularities even in Newtonian gravity or in special relativity, it is said that the (naked) singularities being observed in general relativity have nothing to do with gravitational collapse. This view appears acceptable for shell-crossing and the weak shell-focussing naked singularities. But it is difficult to accept it for the strong naked singularities which crush physical volumes to zero and hence ought to be a genuine general relativistic feature. Secondly, the evolution of collisionless matter is described by the Einstein-Vlasov equations; at any given point in space the particles have a distribution of momenta. Dust is a very special case of these equations, defined by the assumption that all particles have exactly the same momentum. It has been shown [48] that the spherically symmetric Einstein-Vlasov system has global solutions which do not contain singularities, naked or otherwise, and hence censorship is preserved. However, we recall that dust collapse itself has a rich structure, admitting both black-holes and naked singularities, a variety in trapped surface dynamics, and of course includes the classic Oppenheimer-Snyder model of black-hole formation. It would be a little surprising if these dust features turn out to have no connection at all with more realistic collapse models, which do have singularities. Thirdly, it has been suggested, though certainly not universally accepted, that during late stages of collapse matter will effectively behave like dust. In my view, a useful attitude towards the dust collapse results is to treat them as a learning exercise and see if they will survive when more general forms of matter are considered.

Quite naturally, perfect fluids with pressure get more serious consideration than dust. Yet, the naked singularities found in their collapse can be objected to by saying that a fluid description is phenomenological and not fundamental. This objection has been weakened by the results showing naked singularities in scalar field collapse, and also by the existence proofs for naked singularities with a general form of $T_{ik}$. Also, the assumption of self-similarity made in the explicit examples given by Ori and Piran [13] could be considered as restrictive. The existence proofs by Dwivedi and Joshi relax this assumption. It appears safe to conclude at this stage that the form of matter is not crucial in the examples of naked singularities that have been found so far.

The second question deserves a more serious consideration, and holds the key to the validity of the censorship hypothesis. Are these naked singularities genuine features of the
spacetime geometry? What is the relative importance of the properties like masslessness of the naked singularity, curvature strength, redshift along outgoing geodesics, and instability of the Cauchy horizon? Do one or more of these features establish that the naked singularity is not genuine? There is a need to develop what one might call the theory of naked singularities, in order to answer a difficult and important question of this sort. Stability against non-spherical perturbations will also help decide the significance of these examples. The singularity theorems showed that gravitational singularities are not restricted to spherical symmetry; if this is any guide then one would expect both the black-hole as well as naked singularity solutions to persist when sphericity is relaxed. It is indeed difficult to visualize how naked singularities will convert themselves to black-holes when asymmetry is introduced.

The third question, as to whether naked singularities might have observational effects, also deserves attention in view of the examples now available. In spite of limited theoretical evidence for their formation, black-holes have been successfully used to model many observed astrophysical processes. One did not have to wait for the censorship hypothesis to be proved before applications of black-holes could begin. In a similar vein, one ought to give naked singularities a chance, so as to examine if they will or will not have a significant flux emission, due to quantum effects or otherwise.

While there is no clearly defined line of attack for further investigations of the censorship hypothesis, two specific approaches appear to hold promise. Firstly, the methods used by Belinskii, Lifshitz and Khalatnikov to prove the generality of singularities [49] involve construction of solutions of Einstein equations near spacetime singularities. Perhaps one could study propagation of light rays using these solutions, to investigate if the singularities could be naked. Secondly, major advances in numerical relativity have made the subject ripe for studies on the censorship hypothesis. For instance, it should be possible to check numerically if the naked singularities of spherical inhomogeneous dust collapse persist if the collapse is non-spherical. Or, to check for naked singularities in spherical perfect fluid collapse, when the self-similarity assumption is relaxed.

How does a very massive star evolve during the final stages of its collapse? Does it choose to hide behind the event horizon to die a silent death, or does it explode dramatically, exposing the singularity, as if the Big Bang was being reenacted for our benefit? Recent developments in our understanding of classical general relativity leave room for both possibilities. Further studies on gravitational collapse should prove to be exciting, and it remains to be seen whether naked singularities will come to play a role in physics and astrophysics.

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