General Lagrangian of Non-Covariant Self-dual Gauge Field

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ABSTRACT

We present the general formulation of non-covariant Lagrangian of self-dual gauge theory. After specifying the parameters therein the previous Lagrangian in the decomposition of spacetime into $6 = D_1 + D_2$ and $6 = D_1 + D_2 + D_3$ can be obtained. The self-dual property of the general Lagrangian is proved in detail. We furthermore show that the new non-covariant actions give field equations with 6d Lorentz invariance. The method can be straightforward extended to any dimension and we also give a short discussion about the 10D self-dual gauge theory.

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1 Introduction

The problem in Lagrangian description of chiral p-forms, i.e. antisymmetric boson fields with self-dual had been known before thirty years ago. As first noted by Marcus and Schwarz [1], the manifest duality and spacetime covariance do not like to live in harmony with each other in one action.

Historically, the non-manifestly spacetime covariant action for self-dual 0-form was proposed by Floreanini and Jackiw [2], which is then generalized to p-form by Henneaux...
and Teitelboim [3]. In general the field strength of chiral p-form $A_{1\ldots p}$ is split into electric density $\mathcal{E}_{i_{1}\ldots i_{p+1}}$ and magnetic density $\mathcal{B}_{i_{1}\ldots i_{p+1}}$:

$$\mathcal{E}_{i_{1}\ldots i_{p+1}} \equiv F_{i_{1}\ldots i_{p+1}} \equiv \partial_{[i_{1}} A_{i_{2}\ldots i_{p+1}]}$$

$$\mathcal{B}_{i_{1}\ldots i_{p+1}} \equiv \frac{1}{(p+1)!} \epsilon^{i_{1}\ldots i_{2p+2}} F_{i_{p+2}\ldots i_{2p+2}} \equiv \tilde{\mathcal{F}}_{i_{1}\ldots i_{p+1}}$$

in which $\tilde{\mathcal{F}}$ is the dual form of $\mathcal{F}$. The Lagrangian is described by

$$L = -\frac{1}{p!} \tilde{\mathcal{B}} \cdot (\mathcal{E} - \mathcal{B}) = \frac{1}{p!} \tilde{\mathcal{F}}_{i_{1}\ldots i_{p+1}} \mathcal{F}^{i_{1}\ldots i_{p+1}}$$

in which we define

$$\mathcal{F}^{i_{1}\ldots i_{p+1}} \equiv \tilde{\mathcal{F}}^{i_{1}\ldots i_{p+1}} - F^{i_{1}\ldots i_{p+1}}$$

Note that in order for self-dual fields to exist, i.e. $\tilde{\mathcal{F}} = F$, the field strength $\mathcal{F}$ and dual field strength $\tilde{\mathcal{F}}$ should have the same number of component. As the double dual on field strength shall give the original field strength the spacetime dimension have to be 2 modulo 4.

Four years ago, a new non-covariant Lagrangian formulation of a chiral 2-form gauge field in 6D, called as $6 = 3 + 3$ decomposition, was derived in [4] from the Bagger-Lambert-Gustavsson (BLG) model [5]. Later, a general non-covariant Lagrangian formulation of self-dual gauge theories in diverse dimensions was constructed [6]. In this general formulation the $6 = 2 + 4$ decomposition of Lagrangian was found.

In the last year we have constructed a new kind of non-covariant actions of self-dual 2-form gauge theory in the decomposition of $6 = D_{1} + D_{2} + D_{3}$ [7]. In this paper we will present the general formulation of non-covariant Lagrangian of self-dual gauge theory.

In section II we first present the general non-covariant Lagrangian of self-dual gauge theory. We see that after specifying the parameters therein the all known Lagrangian in the decomposition of $6 = D_{1} + D_{2}$ [6] and $6 = D_{1} + D_{2} + D_{3}$ [7] can be obtained. In section III We discuss some properties which are crucial to formulate the general Lagrangian of non-covariant forms of self-dual gauge theory. We then prove in detail the self-dual property of the general Lagrangian. In section IV we follow Perry and Schwarz [8] to show that the general non-covariant Lagrangian gives field equations with 6d Lorentz invariance. Our prescription can be straightforward extended to any dimension and we also give a brief description about the 10D self-dual gauge theory in section V. Last section is devoted to a short conclusion.
2 Lagrangian of Self-dual Gauge Fields in Simple Decomposition

In this section we first present the general non-covariant Lagrangian $L_G$ of self-dual gauge theory in (2.2) and table 1. Then we collect all know non-covariant Lagrangian of self-dual gauge theory in six dimension [6,7] and compare them with $L_G$. Table 2 and table 3 are just those in our previous paper [7], while for convenience we reproduce them in this paper. We will see that, after specifying the parameters in $L_G$ the previous Lagrangian in the decomposition of spacetime into $6 = D_1 + D_2$ and $6 = D_1 + D_2 + D_3$ can be obtained.

To begin with, let us first define a function $L_{ijk}$:

$\tilde{L}_{ijk} = \tilde{F}_{ijk} \times (F_{ijk} - \tilde{F}_{ijk})$, without summation over indices $i, j, k$ (2.1)

which is useful in the following formulations.

In terms of $L_{ijk}$ the most general non-covariant Lagrangian of self-dual gauge theory we found is

$$L_G(\alpha_i) = \sum_a L_{12a} + (\frac{1}{2} + \frac{\alpha_1}{2})L_{134} + (\frac{1}{2} - \frac{\alpha_1}{2})L_{256} + (\frac{1}{2} + \frac{\alpha_2}{2})L_{135} + (\frac{1}{2} - \frac{\alpha_2}{2})L_{246} + (\frac{1}{2} + \frac{\alpha_3}{2})L_{136} + (\frac{1}{2} - \frac{\alpha_3}{2})L_{245} + (\frac{1}{2} + \frac{\alpha_4}{2})L_{145} + (\frac{1}{2} - \frac{\alpha_4}{2})L_{236} + (\frac{1}{2} + \frac{\alpha_5}{2})L_{146} + (\frac{1}{2} - \frac{\alpha_5}{2})L_{235} + (\frac{1}{2} + \frac{\alpha_6}{2})L_{156} + (\frac{1}{2} - \frac{\alpha_6}{2})L_{234}$$

(2.2)

Let us make three comments about above Lagrangian.

First, From table 1 we see that $L_G$ does not picks up $L_{456}$, $L_{356}$, $L_{346}$ nor $L_{345}$, which is denoted as $L_{abc}$. This can ensure to the existence of gauge symmetry $\delta A_{12} = \Phi_{12}$, which is crucial to prove the self-duality of $L_G$, as shown in next section.

Next, We have chosen the coefficient before $L_{12a}$ to be one. This is because that the overall constant of $L_G$ does not affect the self-duality.

Third, we choose coefficient $(\frac{1}{2} + \frac{\alpha_1}{2})$ before $L_{134}$ while choose coefficient $(\frac{1}{2} - \frac{\alpha_1}{2})$ before $L_{256}$. This can ensure that adding the two coefficient $(\frac{1}{2} + \frac{\alpha_1}{2}) + (\frac{1}{2} - \frac{\alpha_1}{2}) = 1$, which give a proper normalization. This property is also crucial to prove the self-duality of $L_G$, as shown in next section.
Table 1: Lagrangian in the general decompositions: \( D = 6 \).

\[
\begin{array}{|c|c|c|}
\hline
D=6 & L_{12a} & 123  \\
      &       & 456  \\
      &       & 356  \\
      &       & 346  \\
      &       & 345  \\
134 & \left(\frac{1}{2} + \frac{a_1}{2}\right)L_{134} & 256  \\
135 & \left(\frac{1}{2} + \frac{a_2}{2}\right)L_{135} & 246  \\
136 & \left(\frac{1}{2} + \frac{a_3}{2}\right)L_{136} & 245  \\
145 & \left(\frac{1}{2} + \frac{a_4}{2}\right)L_{145} & 236  \\
146 & \left(\frac{1}{2} + \frac{a_5}{2}\right)L_{146} & 235  \\
156 & \left(\frac{1}{2} + \frac{a_6}{2}\right)L_{156} & 234  \\
\hline
\end{array}
\]

2.1 Lagrangian in Decomposition: \( D = 1 + 5 \)

In the \((1+5)\) decomposition the spacetime index \( A = (1, \cdots, 6) \) is decomposed as \( A = (1, \hat{a}) \), with \( \hat{a} = (2, \cdots, 6) \). Then \( L_{ABC} = (L_{1\hat{a}\hat{b}}, L_{\hat{a}\hat{b}\hat{c}}) \). In terms of \( L_{ABC} \), the Lagrangian is expressed as [6]

\[
L_{1+5} = -\frac{1}{4} \sum L_{1\hat{a}\hat{b}} = -\frac{1}{4} \tilde{F}_{1\hat{a}\hat{b}}(F^{1\hat{a}\hat{b}} - \tilde{F}^{1\hat{a}\hat{b}}), \text{ has summation over } \hat{a} \hat{b} \quad (2.3)
\]

From table 2 we see that \( L_{1+5} \) picks up only \( L_{1\hat{a}\hat{b}} \). Note that that \( L_G(\alpha_i = 1) = L_{1+5} \).

Table 2: Lagrangian in various decompositions: \( D = D_1 + D_2 \).

\[
\begin{array}{|c|c|}
\hline
D=1+5 & 456  \\
      & 356  \\
      & 346  \\
      & 345  \\
L_{1\hat{a}\hat{b}} & 256  \\
\hat{a}\hat{b} & 246  \\
\hat{a}\hat{b} & 245  \\
134 & 236  \\
135 & 235  \\
136 & 234  \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|}
\hline
D=3+3 & 456  \\
      & 356  \\
      & 346  \\
L_{ab\hat{a}} & 345  \\
\hat{a}\hat{b} & 256  \\
\hat{a}\hat{b} & 246  \\
134 & 245  \\
135 & 236  \\
136 & 235  \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|}
\hline
D=2+4 & 456  \\
      & 356  \\
      & 346  \\
L_{a\hat{a}\hat{b}} & 345  \\
\hat{a}\hat{b} & 256  \\
\hat{a}\hat{b} & 246  \\
134 & 245  \\
135 & 236  \\
136 & 235  \\
\hline
\end{array}
\]

\[5\]
2.2 Lagrangian in Decomposition: 6 = 2 + 4

In the (2+4) decomposition [6] the spacetime index $A$ is decomposed as $A = (a, \dot{a})$, with $a = (1, 2)$ and $\dot{a} = (3, \cdots, 6)$. Then $L_{ABC} = (L_{ab\dot{a}}, L_{a\dot{a}b}, L_{\dot{a}b\dot{c}})$. From table 2 it is easy to see that in terms of $L_{ABC}$ the Lagrangian can be expressed as [7]

$$L_{2+4} = -\frac{1}{4} \left( \sum L_{ab\dot{a}} + \frac{1}{2} \sum L_{a\dot{a}b} \right)$$  \hspace{1cm} (2.4)

The $\frac{1}{2}$ factor before $L_{a\dot{a}b}$ arising from the property that $L_{a\dot{a}b}$ contains both of left-hand side element and right-hand side element (for example, it includes $L_{134}$ and $L_{256}$), thus there is double counting. Note that that $L_G(\alpha_i = 0) = L_{2+4}$.

2.3 Lagrangian in Decomposition: 6 = 3 + 3

In the (3+3) decomposition [6] the spacetime index $A$ is decomposed as $A = (a, \dot{a})$, with $a = (1, 2, 3)$ and $\dot{a} = (4, 5, 6)$. Then $L_{ABC} = (L_{abc}, L_{ab\dot{a}}, L_{a\dot{a}b}, L_{\dot{a}b\dot{c}})$. Using table 2 it is easy to see that in terms of $L_{ABC}$ the Lagrangian can be expressed as

$$L_{3+3} = -\frac{1}{12} \left( \sum L_{abc} + 3 \sum L_{ab\dot{a}} \right)$$  \hspace{1cm} (2.5)

The “3” factor before $L_{ab\dot{a}}$ arising from the property that we have to include three kinds of $L_{ijk} : L_{ab\dot{a}}, L_{a\dot{a}b}$ and $L_{\dot{a}ab}$. Note that $L_G(\alpha_1 = \alpha_2 = \alpha_3 = 1; \alpha_4 = \alpha_5 = \alpha_6 = -1) = L_{3+3}$.

Self-dual property of above decomposition had been proved in [6].

2.4 Lagrangian in Decomposition: 6 = 1 + 1 + 4

In the (1+1+4) decomposition the spacetime index $A$ is decomposed as $A = (1, 2, \dot{a})$, with $\dot{a} = (3, 4, 5, 6)$, and $L_{ABC} = (L_{12a\dot{a}}, L_{1\dot{a}b\dot{c}}, L_{1\dot{a}b\dot{c}}, L_{2\dot{a}b\dot{c}})$. From table 3 it is easy to see that, in terms of $L_{ABC}$, the Lagrangian can be expressed as [7]

$$L_{1+1+4} = 6 \sum L_{12a\dot{a}} + \frac{3(1 + \alpha)}{2} \sum L_{1\dot{a}b\dot{c}} + \frac{3(1 - \alpha)}{2} \sum L_{2\dot{a}b\dot{c}}$$  \hspace{1cm} (2.6)

We neglect overall constant in Lagrangian, which is irrelevant to the self-duality.

It is easy to see that $L_{1+1+4}$ in the case of $\alpha = 0$ is just $L_{2+4}$, in the case of $\alpha = 1$ is just $L_{1+5}$, and in the case of $\alpha = -1$ is just $L_{1+5}$ while exchanging indices 1 and 2, as can be seen from table 2. Note that $L_G(\alpha_i = \alpha) = L_{1+1+4}$. 

6
Table 3: Lagrangian in various decompositions: \( D = D_1 + D_2 + D_3 \).

| D=1+1+4 | D=1+2+3 | D=2+2+2 |
|---------|---------|---------|
| 123     | 123     | 123     |
| 124     | 124     | 124     |
| 125     | 125     | 125     |
| 126     | 126     | 126     |
| 134     | 134     | 134     |
| 135     | 135     | 135     |
| 136     | 136     | 136     |
| 145     | 145     | 145     |
| 146     | 146     | 146     |
| 156     | 156     | 156     |
| 123     | 456     | 456     |
| 124     | 356     | 356     |
| 125     | 346     | 346     |
| 126     | 345     | 345     |
| 134     | 256     | 256     |
| 135     | 246     | 246     |
| 136     | 245     | 245     |
| 145     | 236     | 236     |
| 146     | 235     | 235     |
| 156     | 234     | 234     |

2.5 Lagrangian in Decomposition: \( 6 = 1 + 2 + 3 \)

In the \((1+2+3)\) decomposition the spacetime index \( A \) is decomposed as \( A = (1, a, \dot{a}) \), with \( a = (2, 3) \), \( \dot{a} = (4, 5, 6) \), and \( L_{ABC} = (L_{1ab}, L_{1a\dot{a}}, L_{1\dot{a}b}, L_{\dot{a}bc}, L_{\dot{a}\dot{a}b}, L_{\dot{a}ab}) \). From table 3 it is easy to see that, in terms of \( L_{ABC} \), the Lagrangian can be expressed as [7]

\[
L_{1+2+3} = \sum L_{\dot{a}bc} + 6 \sum L_{1a\dot{a}} + 3 \sum L_{1\dot{a}b} \tag{2.7}
\]

Choosing \( L_{1ab} + L_{1a\dot{a}} + L_{1\dot{a}b} \) is just \( L_{1+5} \), and choosing \( L_{1ab} + L_{1a\dot{a}} + L_{\dot{a}ab} \) is just \( L_{3+3} \), as can be seen from table 2. Note that, exchanging indices 2 with 5 and 3 with 6 then \( L_G(\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 1, \alpha_6 = -1) = L_{1+2+3} \).

2.6 Lagrangian in Decomposition: \( 6 = 2 + 2 + 2 \)

In the \((2+2+2)\) decomposition the spacetime index \( A \) is decomposed as \( A = (a, \dot{a}, \ddot{a}) \), with \( a = (1, 2) \), \( \dot{a} = (3, 4) \) and \( \ddot{a} = (5, 6) \). Now, from table 3 we see that \( L_{ABC} = (L_{a\ddot{a}b}, L_{a\ddot{a}\dot{a}}, L_{a\dot{a}\ddot{a}}, L_{a\dot{a}\dot{a}b}, L_{a\ddot{a}b}, L_{a\ddot{a}\dot{a}}) \). Then, in terms of \( L_{ABC} \) the Lagrangian can be expressed as [7]

\[
L_{2+2+2} = \sum L_{a\ddot{a}b} + \sum L_{a\dot{a}\ddot{a}} + \sum L_{a\dot{a}\dot{a}b} + \sum L_{a\ddot{a}b} \tag{2.8}
\]

Note that \( L_G(\alpha_1 = 1, \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 0, \alpha_6 = -1) = L_{2+2+2} \).

Self-dual property of above decompositions had been proved in [6, 7].
3 General Lagrangian of Self-dual Gauge Fields

In order to understand how to find the general formulation we need to find some constrains in formulating the non-covariant Lagrangian of self-dual gauge theory \([6,7]\).

3.1 General Lagrangian and Gauge Symmetry

From previous studies \([6,7]\) we see that the proof of the self-dual property has used a Gauge symmetry. The property tells us that terms involved \(A_{12}\), for example, only through total derivative terms and we have gauge symmetry

\[
\delta A_{12} = \Phi_{12} \tag{3.1}
\]

for arbitrary functions \(\Phi_{12}\). In order to have this property we must not choose the dual transformation of \(L_{12a}\) in table 1, where \(a=(3,4,5,6)\). Thus the general non-covariant Lagrangian of self-dual gauge theory is

\[
L = \sum_a L_{12a} + C_1(\frac{1}{2} + \alpha_1)L_{134} + C_1(\frac{1}{2} - \tilde{\alpha}_1)L_{256} + C_2(\frac{1}{2} + \alpha_2)L_{135} + C_2(\frac{1}{2} - \tilde{\alpha}_2)L_{246} \\
C_3(\frac{1}{2} + \alpha_3)L_{136} + C_3(\frac{1}{2} - \tilde{\alpha}_3)L_{245} + C_4(\frac{1}{2} + \alpha_4)L_{145} + C_4(\frac{1}{2} - \tilde{\alpha}_4)L_{236} \\
C_5(\frac{1}{2} + \alpha_5)L_{146} + C_5(\frac{1}{2} - \tilde{\alpha}_5)L_{235} + C_6(\frac{1}{2} + \alpha_6)L_{156} + C_6(\frac{1}{2} - \tilde{\alpha}_6)L_{234} \tag{3.2}
\]

in which we have let the coefficient before \(L_{12a}\) to be one as the overall constant of \(L\) does not affect the self-duality. We will now show that above Lagrangian has desired gauge symmetry, and after proper choosing the parameters \(C_i\) it becomes \(L_G\) in (2.2) and the associated field strength has self-dualtiy property.

First, the variation of the action gives

\[
\frac{\delta L}{\delta A_{12}} = -\partial_3 \tilde{F}^{312} - \partial_4 \tilde{F}^{412} - \partial_5 \tilde{F}^{512} - \partial_6 \tilde{F}^{612} = \partial_a \tilde{F}^{a12} = 0 \tag{3.3}
\]

which is identically zero. This means that terms involved \(A_{12}\) only through total derivative terms and we have gauge symmetry

\[
\delta A_{12} = \Phi_{12} \tag{3.4}
\]

for arbitrary functions \(\Phi_{12}\).
3.2 Self-duality

Next, simply using above gauge symmetry does not guarantee that the Lagrangian has self-dual property. Let us find the another constrain.

The variation of the Lagrangian $L$ gives

$$0 = \frac{\delta L}{\delta A_{34}} = -\left[ C_1 \partial_1 \tilde{F}^{134} + C_6 \partial_2 \tilde{F}^{234} + \partial_5 \tilde{F}^{534} + \partial_6 \tilde{F}^{634} \right]$$

$$+ C_1 \left(1 - \frac{\alpha_1 + \tilde{\alpha}_1}{2}\right) \partial_1 \tilde{F}^{134} - C_1 \left(1 - \tilde{\alpha}_1\right) \partial_1 F^{134}$$

$$+ C_6 \left(1 - \frac{\alpha_6 + \tilde{\alpha}_6}{2}\right) \partial_2 \tilde{F}^{234} - C_6 \left(1 - \tilde{\alpha}_6\right) \partial_2 F^{234}$$

$$+ 2(\partial_5 \mathcal{F}^{534} + \partial_6 \mathcal{F}^{634})$$

(3.5)

Now, if we require each $L_{i,j,k}$ has a same normalization then $C_i = 1$. Under this constrain we find that

$$0 = \frac{\delta L}{\delta A_{34}} = \left(1 - \frac{\alpha_1 + \tilde{\alpha}_1}{2}\right) \partial_1 \tilde{F}^{134} - \left(1 - \tilde{\alpha}_1\right) \partial_1 F^{134}$$

$$+ \left(1 - \frac{\alpha_6 + \tilde{\alpha}_6}{2}\right) \partial_2 \tilde{F}^{234} - \left(1 - \tilde{\alpha}_6\right) \partial_2 F^{234}$$

$$+ 2(\partial_5 \mathcal{F}^{534} + \partial_6 \mathcal{F}^{634})$$

(3.6)

where we have used the property $\partial_a \tilde{F}^{a34} = 0$.

To proceed, we can from table 2 and table 3 see that, for example, the difference between the Lagrangian in decomposition $D = 2 + 4$ and $D = 1 + 5$ is that we have chosen left-hand (electric) part and right-hand (magnetic) part in $D = 2 + 4$, while in $D = 1 + 5$ we choose only left-hand (electric) part. However, in the self-dual theory the electric part is equal to magnetic part. Thus the Lagrangian choosing electric part is equivalent to that choosing magnetic part. In the decomposition into different direct-product of spacetime one can choose different part of $L_{ijk}$ to mixing to each other and we have many kind of decomposition. This observation lead us to find more decomposition $6 = D_1 + D_2 + D_3$ in [7].

This property can be applied to find the more general formulation of non-covariant Lagrangian of self-dual gauge theory. Thus the another constrain is that

$$\left(\frac{1}{2} + \frac{\alpha_i}{2}\right) + \left(\frac{1}{2} - \frac{\tilde{\alpha}_i}{2}\right) = 1 \quad \Rightarrow \quad \alpha_i = \tilde{\alpha}_i$$

(3.7)

From now on we will use this property and Lagrangian $L$ becomes $L_G$ in (2.2).
Thus
\[ 0 = \frac{\delta L}{\delta A_{34}} = (1 - \alpha_1)\partial_1 F^{134} + (1 + \alpha_6)\partial_2 F^{234} + 2\partial_5 F^{534} + 2\partial_6 F^{634} \]
\[ = \partial_1 \bar{F}_{134} + \partial_2 \bar{F}_{234} + \partial_5 \bar{F}_{534} + \partial_6 \bar{F}_{634} \] (3.8)
in which we have normalized each \( F \) by the associated factor \( (1 - \alpha_i) \) or \( 2 \) for convenience.

Similarly, we have the relations
\[ 0 = \frac{\delta L}{\delta A_{35}} = \partial_1 \bar{F}_{135} + \partial_2 \bar{F}_{235} + \partial_4 \bar{F}_{435} + \partial_6 \bar{F}_{635} \] (3.9)
\[ 0 = \frac{\delta L}{\delta A_{36}} = \partial_1 \bar{F}_{136} + \partial_2 \bar{F}_{236} + \partial_4 \bar{F}_{436} + \partial_5 \bar{F}_{536} \] (3.10)
\[ 0 = \frac{\delta L}{\delta A_{45}} = \partial_1 \bar{F}_{145} + \partial_2 \bar{F}_{245} + \partial_3 \bar{F}_{345} + \partial_6 \bar{F}_{645} \] (3.11)
\[ 0 = \frac{\delta L}{\delta A_{46}} = \partial_1 \bar{F}_{146} + \partial_2 \bar{F}_{246} + \partial_3 \bar{F}_{346} + \partial_5 \bar{F}_{546} \] (3.12)
\[ 0 = \frac{\delta L}{\delta A_{56}} = \partial_1 \bar{F}_{156} + \partial_2 \bar{F}_{256} + \partial_3 \bar{F}_{356} + \partial_4 \bar{F}_{456} \] (3.13)

Above six equations can be expressed as
\[ \partial_a \bar{F}^{abc} = 0, \quad a, b, c \neq 1, 2 \] (3.14)
which has solution
\[ \bar{F}^{abc} = \epsilon^{12abcd} \partial_d \Phi_{12} \] (3.15)
for arbitrary functions \( \Phi_{12} \). As the gauge symmetry of \( \delta A_{12} = \Phi_{12} \) can totally remove \( \Phi_{12} \) in \( \mathcal{F}^{abc} \) we have found a self-dual relation
\[ \mathcal{F}_{abc} = 0, \quad a, b, c \neq 1, 2 \] (3.16)

In the same way, the variation of the action gives
\[ 0 = \frac{\delta L}{\delta A_{13}} = (1 - \alpha_1)\partial_4 \mathcal{F}^{413} + (1 - \alpha_2)\partial_5 \mathcal{F}^{513} + (1 - \alpha_3)\partial_6 \mathcal{F}^{613} \]
\[ = \partial_4 \bar{F}^{413} + \partial_5 \bar{F}^{513} + \partial_6 \bar{F}^{613} \] (3.17)
where we have normalized each \( \mathcal{F} \) by the associated factor \( (1 - \alpha) \). In the same way we have the relations
\[ 0 = \frac{\delta L}{\delta A_{14}} = \partial_3 \bar{F}^{314} + \partial_5 \bar{F}^{514} + \partial_6 \bar{F}^{614} \] (3.18)
\[ 0 = \frac{\delta L}{\delta A_{15}} = \partial_3 \bar{F}^{315} + \partial_4 \bar{F}^{415} + \partial_6 \bar{F}^{615} \] (3.19)
\[ 0 = \frac{\delta L}{\delta A_{16}} = \partial_3 \bar{F}^{316} + \partial_4 \bar{F}^{416} + \partial_5 \bar{F}^{516} \] (3.20)
Above 4 equations give the solution of $F^{1ab}$ ($a, b \neq 2$)

$$F^{1ab} = \epsilon^{12abcd} \partial_c \Phi_d$$  \hspace{1cm} (3.21)

In the same way, we can find

$$0 = \frac{\delta L}{\delta A_{23}} = \partial_4 \tilde{F}^{423} + \partial_5 \tilde{F}^{523} + \partial_6 \tilde{F}^{623}$$ \hspace{1cm} (3.22)

$$0 = \frac{\delta L}{\delta A_{12}} = \partial_3 \tilde{F}^{324} + \partial_5 \tilde{F}^{524} + \partial_6 \tilde{F}^{624}$$ \hspace{1cm} (3.23)

$$0 = \frac{\delta L}{\delta A_{25}} = \partial_3 \tilde{F}^{325} + \partial_4 \tilde{F}^{425} + \partial_6 \tilde{F}^{625}$$ \hspace{1cm} (3.24)

$$0 = \frac{\delta L}{\delta A_{26}} = \partial_3 \tilde{F}^{326} + \partial_4 \tilde{F}^{426} + \partial_5 \tilde{F}^{526}$$ \hspace{1cm} (3.25)

Above 4 equations give the solution of $F^{2ab}$ ($a, b \neq 1$)

$$F^{2ab} = \epsilon^{12abcd} \partial_c \Phi_d$$  \hspace{1cm} (3.26)

We can now follow [6,7] to find another self-dual relation. First, taking the Hodge-dual of both sides in above equation we find that

$$F^{1ab} = \partial[a \Phi^b] \hspace{1cm} (a, b \neq 2)$$ \hspace{1cm} (3.27)

Identifying this solution with previous found solution of $F^{1ab}$, then

$$\partial[a \Phi^b] = \epsilon^{12abcd} \partial_c \Phi_d$$ \hspace{1cm} (3.28)

Acting $\partial_a$ on both sides gives

$$\partial_a \partial[a \Phi^b] = 0$$ \hspace{1cm} (3.29)

under the Lorentz gauge $\partial_a \Phi^{a1} = 0$. Now, following [6,7], imposing the boundary condition that the field $\Phi^{b1}$ be vanished at infinities will lead to the unique solution $\Phi^{b1} = 0$ and we arrive at the self-duality conditions

$$F^{2ab} = 0, \hspace{1cm} a, b \neq 1$$ \hspace{1cm} (3.30)

In the same way, we can find another self-duality conditions

$$F^{1ab} = 0, \hspace{1cm} a, b \neq 2$$ \hspace{1cm} (3.31)

These complete the proof.
4 Lorentz Invariance of Self-dual Field Equation

In this we follow the method of Perry and Schwarz [8] to show that the above general non-covariant actions give field equations with 6d Lorentz invariance. Note that sec. 4.1, 4.2.1 and 4.2.2 are just those in our previous paper [7], while for completeness we reproduce them in below.

4.1 Lorentz transformation of 2-form Field strength

We first describe the Lorentz transformation of 2-form field strength. For the coordinate transformation : \( x_a \rightarrow \bar{x}_a \equiv x_a + \delta x_a \) the tensor field \( H_{MNP}(x_a) \) will becomes

\[
H_{MNP}(x_a) \rightarrow H_{\bar{M}\bar{N}\bar{P}}(x_a + \delta x_a) \equiv \frac{\partial x^Q}{\partial \bar{x}^M} \frac{\partial x^R}{\partial \bar{x}^N} \frac{\partial x^S}{\partial \bar{x}^P} H_{QRS}(x_a + \delta x_a)
\]

\[
\approx H_{MNP}(x_a + \delta x_a) + \frac{\partial x^Q}{\partial \bar{x}^M} \frac{\partial x^R}{\partial \bar{x}^N} \frac{\partial x^S}{\partial \bar{x}^P} H_{QRS}(x_a)
\]

(4.1)

For the transformation mixing between \( x_1 \) with \( x_\mu (\mu \neq 1) \) the relation \( \delta x_a = \omega_{ab} x^b \)

leads to \( \delta x_1 = -\Lambda_\mu x^\mu \) and \( \delta x_\mu = \Lambda_\mu x^1 \) in which we define \( \omega_{1\mu} = -\omega_{\mu 1} = \Lambda_\mu \).

The orbital part of transformation [8] is defined by

\[
\delta_{orb} H_{MNP} \equiv H_{MNP}(x_a + \omega_{ab} x^b) - H_{MNP}(x_a) \approx [\delta x_a] \cdot \partial^a H_{MNP}
\]

\[
= [\Lambda_\mu x^\mu \partial^1] H_{MNP} - x^1 [\Lambda_\mu \partial^\mu] H_{MNP}
\]

\[
= [(\Lambda \cdot x) \partial^1 - x^1 (\Lambda \cdot \partial)] H_{MNP} \equiv (\Lambda \cdot L) H_{MNP}
\]

(4.2)

Note that \( \delta_{orb} \) is independent of index \( MNP \) and is universal for any tensor.

The spin part of transformation [8] becomes

\[
\delta_{spin} H_{\mu\nu\lambda} \equiv \frac{\partial x^Q}{\partial x^\mu} \frac{\partial x^R}{\partial x^\nu} \frac{\partial x^S}{\partial x^\lambda} H_{QRS}(x) - H_{\mu\nu\lambda}
\]

\[
= \frac{\partial (\delta x^1)}{\partial x^\mu} \frac{\partial x^R}{\partial x^\nu} \frac{\partial x^S}{\partial x^\lambda} H_{1RS}(x) + \frac{\partial x^Q (\delta x^1)}{\partial x^\mu} \frac{\partial x^R}{\partial x^\nu} \frac{\partial x^S}{\partial x^\lambda} H_{QRS}(x) + \frac{\partial x^Q}{\partial x^\mu} \frac{\partial x^R}{\partial x^\nu} \frac{\partial (\delta x^1)}{\partial x^\lambda} H_{RQS}(x)
\]

\[
= \left[ -\Lambda_\mu H_{1\nu\lambda} \right] + \left[ -\Lambda_\nu H_{\mu1\lambda} \right] + \left[ -\Lambda_\lambda H_{\mu\nu1} \right]
\]

(4.3)

and \( \delta H_{\mu\nu\lambda} = \delta_{orb} H_{\mu\nu\lambda} + \delta_{spin} H_{\mu\nu\lambda} \)

In a same way

\[
\delta_{spin} H_{\mu\nu1} \equiv \frac{\partial x^Q}{\partial x^\mu} \frac{\partial x^R}{\partial x^\nu} \frac{\partial x^1}{\partial x^1} H_{QRS}(x) - H_{\mu\nu1}
\]

\[
= \frac{\partial (\delta x^1)}{\partial x^\mu} \frac{\partial x^R}{\partial x^\nu} \frac{\partial x^1}{\partial x^1} H_{1RS}(x) + \frac{\partial x^Q (\delta x^1)}{\partial x^\mu} \frac{\partial x^R}{\partial x^\nu} \frac{\partial x^1}{\partial x^1} H_{QRS}(x) + \frac{\partial x^Q}{\partial x^\mu} \frac{\partial x^R}{\partial x^\nu} \frac{\partial (\delta x^1)}{\partial x^1} H_{RQS}(x)
\]

\[
= 0 + 0 + \Lambda^\lambda H_{\mu\nu\lambda}
\]

(4.4)

and \( \delta H_{\mu\nu1} = \delta_{orb} H_{\mu\nu1} + \delta_{spin} H_{\mu\nu1} = (\Lambda \cdot L) H_{\mu\nu1} + \Lambda^\lambda H_{\mu\nu\lambda} \).
4.2 Lorentz Invariance of Self-dual Field Equation

We now use above Lorentz transformation need to examine transformations (I) mixing \( x_1 \) with \( x_2 \), (II) mixing \( x_1 \) with \( x_a \) and (IV) mixing \( x_a \) with \( x_b \), \( a, b = 3, 4, 5, 6 \).

4.2.1 Mixing \( x_1 \) with \( x_2 \)

(I) Consider first the mixing \( x_1 \) with \( x_2 \). The transformation is

\[
\begin{align*}
\delta x^1 &= \omega^{12} x_2 \equiv \Lambda x_2, \\
\delta x^2 &= \omega^{21} x_1 = -\Lambda x_1
\end{align*}
\]

Define

\[
\Lambda \cdot L \equiv (\Lambda x_2) \partial_1 - x_1 (\Lambda \partial_2)
\]

then

\[
\begin{align*}
\delta F_{12a} &= (\Lambda \cdot L) F_{12a} \\
\delta F_{abc} &= (\Lambda \cdot L) F_{abc} \\
\delta F_{1ab} &= (\Lambda \cdot L) F_{1ab} - \Lambda F_{2ab} \\
\delta F_{2ab} &= (\Lambda \cdot L) F_{2ab} + \Lambda F_{1ab}
\end{align*}
\]

Using above transformation we can find

\[
\begin{align*}
\delta \tilde{F}_{12a} &= (\Lambda \cdot L) \tilde{F}_{12a} + \frac{1}{6} \epsilon_{12abcd} (\delta_{\text{spin}} F^{bced}) = (\Lambda \cdot L) \tilde{F}_{12a} \\
\delta \tilde{F}_{1ab} &= (\Lambda \cdot L) \tilde{F}_{1ab} + \frac{1}{6} \epsilon_{1ab2cd} (\delta_{\text{spin}} F^{2cd} \cdot 3) \\
&= (\Lambda \cdot L) \tilde{F}_{1ab} + \frac{1}{2} \epsilon_{1ab2cd} [\Lambda F^{1cd}] \\
&= (\Lambda \cdot L) \tilde{F}_{1ab} - \Lambda \tilde{F}_{2ab}
\end{align*}
\]

Thus

\[
\begin{align*}
\delta (F_{12a} - \tilde{F}_{12a}) &= (\Lambda \cdot L)(F_{12a} - \tilde{F}_{12a}) = 0 \\
\delta (F_{1ab} - \tilde{F}_{1ab}) &= (\Lambda \cdot L)(F_{1ab} - \tilde{F}_{1ab}) - \Lambda(F_{2ab} - \tilde{F}_{2ab}) = 0
\end{align*}
\]

which are zero for self-dual theory. Taking Hodge of above relations we also get

\[
\delta (F_{abc} - \tilde{F}_{abc}) = 0, \quad \delta (F_{2ab} - \tilde{F}_{2ab}) = 0
\]

and the non-covariant action gives field equations with 6d Lorentz invariance under transformation mixing \( x_1 \) with \( x_2 \).
4.2.2 Mixing $x_1$ with $x_a$

(II) For the mixing $x_1$ with $x_a$, $a = 3, 4, 5, 6$, we shall consider the transformation

$$\delta x^a = \omega^{a1} x_1 \equiv \Lambda^a x_1, \quad (4.17)$$

$$\delta x^1 = \omega^{1a} x_a = -\Lambda^a x_a = -\Lambda \cdot x \quad (4.18)$$

Define

$$\Lambda \cdot L \equiv (\Lambda \cdot x) \partial_1 - x_1(\Lambda \cdot \partial) \quad (4.19)$$

then

$$\delta F_{12a} = (\Lambda \cdot L) F_{12a} + \Lambda^b F_{b2a} \quad (4.20)$$

$$\delta F_{abc} = (\Lambda \cdot L) F_{abc} - \Lambda_a F_{1bc} - \Lambda_b F_{a1c} - \Lambda_c F_{ab1} \quad (4.21)$$

$$\delta F_{1ab} = (\Lambda \cdot L) F_{1ab} + \Lambda^c F_{cab} \quad (4.22)$$

$$\delta F_{2ab} = (\Lambda \cdot L) F_{2ab} - \Lambda_a F_{21b} - \Lambda_b F_{2a1} \quad (4.23)$$

Use above transformation we can find

$$\delta \tilde{F}_{12a} = (\Lambda \cdot L) \tilde{F}_{12a} + \frac{1}{6} \epsilon_{12abc} (\delta_{\text{spin}} F^{bcd})$$

$$= (\Lambda \cdot L) \tilde{F}_{12a} + \frac{1}{6} \epsilon_{12abc} [-\Lambda^b F^{1cd} - \Lambda^c F^{b1d} - \Lambda^d F^{bc1}]$$

$$= (\Lambda \cdot L) \tilde{F}_{12a} + \Lambda^b \tilde{F}_{ab2} \quad (4.24)$$

$$\delta \tilde{F}_{1ab} = (\Lambda \cdot L) \tilde{F}_{1ab} + \frac{1}{6} \epsilon_{1ab2cd} (\delta_{\text{spin}} F^{2cd} \cdot 3)$$

$$= (\Lambda \cdot L) \tilde{F}_{1ab} + \frac{1}{2} \epsilon_{1ab2cd} [-\Lambda^c F^{21d} - \Lambda^d F^{2c1}]$$

$$= (\Lambda \cdot L) \tilde{F}_{1ab} + \Lambda^c \tilde{F}_{cab} \quad (4.25)$$

Thus

$$\delta (F_{12a} - \tilde{F}_{12a}) = (\Lambda \cdot L) (F_{12a} - \tilde{F}_{12a}) + \Lambda^b (F_{ab2} - \tilde{F}_{ab2}) = 0 \quad (4.26)$$

$$\delta (F_{1ab} - \tilde{F}_{1ab}) = (\Lambda \cdot L) (F_{1ab} - \tilde{F}_{1ab}) + \Lambda^c (F_{cab} - \tilde{F}_{cab}) = 0 \quad (4.27)$$

which are zero for self-dual theory. Taking Hodge of above relations we also get

$$\delta (F_{abc} - \tilde{F}_{abc}) = 0, \quad \delta (F_{2ab} - \tilde{F}_{2ab}) = 0 \quad (4.28)$$

and the non-covariant action gives field equations with 6d Lorentz invariance under transformation mixing $x_1$ with $x_a$. 

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4.2.3 Mixing $x_a$ with $x_b$

(II) For the mixing $x_a$ with $x_b$, $a = 3, 4, 5, 6$, we shall consider the transformation

$$\delta x^a = \omega^{ab} x_b \equiv \Lambda^{ab} x_b$$

(4.29)

Define

$$\Lambda \cdot L \equiv \Lambda^{ab}(x_a \partial_b - x_b \partial_a)$$

(4.30)

then

$$\delta F_{12a} = (\Lambda \cdot L) F_{12a} - \Lambda_a e F_{12e}$$

(4.31)

$$\delta F_{abc} = (\Lambda \cdot L) F_{abc} - \Lambda_a e F_{ebc} - \Lambda_b e F_{ace} - \Lambda_c e F_{abe} - \Lambda F_{ab1}$$

(4.32)

$$\delta F_{1ab} = (\Lambda \cdot L) F_{1ab} - \Lambda_a e F_{1eb} - \Lambda_b e F_{1ae}$$

(4.33)

$$\delta F_{2ab} = (\Lambda \cdot L) F_{2ab} - \Lambda_a e F_{2eb} - \Lambda_b e F_{2ae}$$

(4.34)

Using above transformations we can find

$$\delta \tilde{F}_{12a} = (\Lambda \cdot L) \tilde{F}_{12a} + \frac{1}{6} \epsilon_{12abcd}(\delta_{\text{spin}} F^{abcd})$$

$$= (\Lambda \cdot L) \tilde{F}_{12a} - \frac{1}{6} \epsilon_{12abcd}(\Lambda^b e F^{becd} + \Lambda^c e F^{bced} + \Lambda^d e F^{bcde})$$

$$= (\Lambda \cdot L) \tilde{F}_{12a} - \Lambda_a e \tilde{F}_{12e}$$

(4.35)

$$\delta \tilde{F}_{1ab} = (\Lambda \cdot L) \tilde{F}_{1ab} + \frac{1}{6} \epsilon_{12abcd}(\delta_{\text{spin}} F^{2cde} \cdot 3)$$

$$= (\Lambda \cdot L) \tilde{F}_{1ab} - \frac{1}{6} \epsilon_{12abcd}(\Lambda^c e F^{2cde} + 3 + \Lambda^d e F^{2ce} \cdot 3)$$

$$= (\Lambda \cdot L) \tilde{F}_{1ab} - \Lambda_a e \tilde{F}_{1eb} - \Lambda_b e \tilde{F}_{1ae}$$

(4.36)

Therefore

$$\delta(F_{12a} - \tilde{F}_{12a}) = (\Lambda \cdot L)(F_{12a} - \tilde{F}_{12a}) - \Lambda_a e (F_{12e} - \tilde{F}_{12e}) = 0$$

(4.37)

$$\delta(F_{1ab} - \tilde{F}_{1ab}) = (\Lambda \cdot L)(F_{1ab} - \tilde{F}_{1ab}) - \Lambda_a e (F_{1eb} - \tilde{F}_{1eb}) - \Lambda_b e (F_{1ae} - \tilde{F}_{1ae}) = 0$$

(4.38)

which are zero for self-dual theory. Taking Hodge of above relations we also get

$$\delta(F_{abc} - \tilde{F}_{abc}) = 0, \quad \delta(F_{2ab} - \tilde{F}_{2ab}) = 0$$

(4.39)

and the non-covariant action gives field equations with 6d Lorentz invariance under transformation mixing $x_a$ with $x_b$. 

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5 10D Self-dual Gauge Theory

The extension above prescription to 10D self-dual gauge theory is straightforward. As before, let us first define a function \( L_{ijktmn} : \)

\[
L_{ijktm} \equiv \tilde{F}_{ijktm} \times (F^{ijktm} - \tilde{F}^{ijktm}), \text{ without summation over indices} \quad (5.1)
\]

In terms of \( L_{ijktm} \) the most general non-covariant Lagrangian of self-dual gauge theory is

\[
L_G(\alpha_i) = \sum_a L_{1234a} + \left( \frac{1}{2} + \frac{\alpha_1}{2} \right) L_{13456} + \left( \frac{1}{2} - \frac{\alpha_1}{2} \right) L_{2789Q}
\]

\[
+ \left( \frac{1}{2} + \frac{\alpha_2}{2} \right) L_{13457} + \left( \frac{1}{2} - \frac{\alpha_2}{2} \right) L_{2689Q}
\]

\[
+ \cdots \quad (5.2)
\]

in which \( Q \) denotes as tenth dimension hereafter.

Table 4: Lagrangian in the general decompositions: \( D = 10. \)

| D=10 | \( 12345 \) | \( 12346 \) | \( 12347 \) | \( 12348 \) | \( 12349 \) | \( 1234Q \) | \( 13456 \) | \( 13457 \) | \( 1345Q \) | \( 6789Q \) | \( 5789Q \) | \( 5689Q \) | \( 5679Q \) | \( 5678Q \) | \( 56789 \) | \( 2789Q \) | \( 2689Q \) | \( 2689Q \) |
|------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
|      | \( L_{1234a} \) |          |          |          |          |          |          | \( \frac{1}{2} + \frac{\alpha_1}{2} L_{13456} \) | \( \frac{1}{2} + \frac{\alpha_2}{2} L_{13457} \) | \( \frac{1}{2} + \frac{\alpha_2}{2} L_{1345Q} \) | \( \frac{1}{2} + \frac{\alpha_1}{2} L_{2789Q} \) | \( \frac{1}{2} + \frac{\alpha_2}{2} L_{2689Q} \) | \( \frac{1}{2} + \frac{\alpha_1}{2} L_{2789Q} \) |

From table 4 we see that \( L_G \) does not picks up \( L_{6789Q}, \ldots, L_{56789} \). This is a crucial property to have a gauge symmetry. We now summarize the proof of self-duality of above Lagrangian.
5.1 Self-duality of 10D Self-dual Gauge Theory

First, the variation of the action gives

$$\frac{\delta L_G(\alpha_i)}{\delta A_{1234}} = \partial_a \tilde{F}^{a1234} = 0 \quad (5.3)$$

which is identically zero. This means that terms involved $A_{1234}$ only through total derivative terms and we have gauge symmetry

$$\delta A_{1234} = \Phi_{1234} \quad (5.4)$$

for arbitrary functions $\Phi_{1234}$.

Next, we can find that

$$\partial_a \tilde{F}^{abcde} = 0, \quad a, b, c, d, e \neq 1, 2, 3, 4 \quad (5.5)$$

which has solution

$$\tilde{F}^{abcde} = \epsilon^{1234abcde} \partial_f \Phi_{1234} \quad (5.6)$$

for arbitrary functions $\Phi_{1234}$. As the gauge symmetry of $\delta A_{1234} = \Phi_{1234}$ can totally remove $\Phi_{1234}$ in $F^{abcde}$ we have found a self-dual relation

$$F^{abcde} = 0, \quad a, b, c, d, e \neq 1, 2, 3, 4 \quad (5.7)$$

In the same way, we can find that

$$\partial_a \tilde{F}^{1abcd} = 0, \quad a, b, c, d \neq 2, 3, 4 \quad (5.8)$$

which has solution

$$\tilde{F}^{1abcd} = \epsilon^{1234abcde} \partial_f \Phi_{234e} \quad a, b, c, d \neq 2, 3, 4 \quad (5.9)$$

for arbitrary functions $\Phi_{234e}$. We can also find that

$$\partial_a \tilde{F}^{234ab} = 0, \quad a, b \neq 1 \quad (5.10)$$

which has solution

$$\tilde{F}^{234ab} = \epsilon^{1234abcde} \partial_c \Phi_{1def}, \quad a, b \neq 1 \quad (5.11)$$

for arbitrary functions $\Phi_{234e}$. 

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We can now follow [6,7] to find another self-dual relation. First, taking the Hodge-dual of both sides in above equation we find that

$$F^{abcd} = \partial[a\Phi^{1bcd}], \quad (a, b, c, d \neq 1, 2, 3, 4) \quad (5.12)$$

Identifying this solution with previous found solution of $F^{1abcd}$, then

$$\partial[a\Phi^{1bcd}] = \epsilon^{1234abcdf}\partial_f\Phi_{2345e}, \quad (a, b, c, d \neq 1, 2, 3, 4) \quad (5.13)$$

Acting $\partial_a$ on both sides gives

$$\partial_a\partial^a\Phi^{1bcd} = 0, \quad (a, b, c, d \neq 1, 2, 3, 4) \quad (5.14)$$

under the Lorentz gauge $\partial_a\Phi^{1abc} = 0$. Now, following [6,7], imposing the boundary condition that the field $\Phi^{1bcd}$ be vanished at infinities will lead to the unique solution $\Phi^{1bcd} = 0$ and we arrive at the self-duality conditions

$$F^{234ab} = 0, \quad a, b \neq 1 \quad (5.15)$$

In the same way, we can find all other self-duality conditions.

### 5.2 Lorentz invariance of 10 D Self-dual Field Equation

Finally, the method in section IV can be easily applied to prove that general non-covariant actions give field equations with 10d Lorentz invariance. Essentially, we merely add more index in field strength.

For example, in considering mixing $x_1$ with $x_2$ we can find that

$$\delta F^{12abc} = (\Lambda \cdot L)F_{12abc} \quad (5.16)$$
$$\delta F^{abde} = (\Lambda \cdot L)F_{abde} \quad (5.17)$$
$$\delta F^{abcd} = (\Lambda \cdot L)F_{abcd} - \Lambda F_{2abcd} \quad (5.18)$$
$$\delta F^{2abde} = (\Lambda \cdot L)F_{2abde} + \Lambda F_{1abcd} \quad (5.19)$$

Using above transformation we can find that

$$\delta(F^{12abc} - \tilde{F}^{12abc}) = (\Lambda \cdot L)(F_{12abc} - \tilde{F}_{12abc}) = 0 \quad (5.20)$$
$$\delta(F^{abcd} - \tilde{F}^{abcd}) = (\Lambda \cdot L)(F_{abcd} - \tilde{F}_{abcd}) - \Lambda(F_{2abcd} - \tilde{F}_{2abcd}) = 0 \quad (5.21)$$

which are zero for self-dual theory. Taking Hodge of above relations we also get

$$\delta(F^{abde} - \tilde{F}^{abde}) = 0, \quad \delta(F_{2abde} - \tilde{F}_{2abde}) = 0 \quad (5.22)$$

and the non-covariant action gives field equations with 10d Lorentz invariance under transformation mixing $x_1$ with $x_2$. 
6 Conclusion

In this paper we have reviewed the various non-covariant formulations Lagrangian of self-dual gauge theory in 6D and then use the crucial property of the existence of gauge symmetry $\delta A = \Phi$ to present a general Lagrangian of non-covariant forms of self-dual gauge theory. We have followed previous prescription [6,7] to prove the self-dual property in the general Lagrangian. We furthermore follow the method of Perry and Schwarz [8] to show that the general non-covariant Lagrangian give field equations with 6d Lorentz invariance. Our method can be straightforward extended to any dimension and we also give a short description about the 10D self-dual gauge theory.

The result and property found in this paper have shown that there are many kinds of non-covariant Lagrangian of self-dual gauge theory and we can easily construct the general Lagrangian.

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