Recent advancements in conformal gravity

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Abstract. In recent years, due to the lack of direct observed evidence of cold dark matter, coupled with the shrinking parameter space to search for new dark matter particles, there has been increased interest in Alternative Gravitational theories. This paper, addresses three recent advances in conformal gravity, a fourth order renormalizable metric theory of gravitation originally formulated by Weyl, and later advanced by Mannheim and Kazanas. The first section of the paper applies conformal gravity to the rotation curves of the LITTLE THINGS survey, extending the total number of rotation curves successfully fit by conformal gravity to well over 200 individual data sets without the need for additional dark matter. Further, in this rotation curve study, we show how MOND and conformal gravity compare for each galaxy in the sample. Second, we look at the original Zwicky problem of applying the virial theorem to the Coma cluster in order to get an estimate for the cluster mass. However, instead of using the standard Newtonian potential, here we use the weak field approximation of conformal gravity. We show that in the conformal case we can get a much smaller mass estimate and thus there is no apparent need to include dark matter. We then show that this calculation is in agreement with the observational data from other well studied clusters. Last, we explore the calculation of the deflection of starlight through conformal gravity, as a first step towards applying conformal gravity to gravitational lensing.

1. Introduction

Conformal gravity is a fourth order renormalizable metric theory of gravity, where the Ricci scalar in the Einstein-Hilbert action is replaced by the square of the conformal Weyl tensor as:

\[ I_W = -\alpha_g \int d^4x(-g)^{1/2}C_{\lambda\mu\nu\kappa}C^{\lambda\mu\nu\kappa} - 2\alpha_g \int d^4x(-g)^{1/2} \left[R_{\mu\kappa}R^{\mu\kappa} - (1/3)(R^\alpha_{\alpha})^2\right], \]

where

\[ C_{\lambda\mu\nu\kappa} = R_{\lambda\mu\nu\kappa} - \frac{1}{2}(g_{\lambda\nu}R_{\mu\kappa} - g_{\lambda\kappa}R_{\mu\nu} - g_{\mu\nu}R_{\lambda\kappa} + g_{\mu\kappa}R_{\lambda\nu}) + \frac{1}{6}R^\alpha_{\alpha}(g_{\lambda\nu}g_{\mu\kappa} - g_{\lambda\kappa}g_{\mu\nu}) \]

is the conformal Weyl tensor and the gravitational coupling constant \( \alpha_g \) is dimensionless. This action, when varied with respect to the metric (as in standard gravity) [1], yields the conformal gravity field equations,

\[ 4\alpha_g W^{\mu\nu} = 4\alpha_g \left[2C^{\mu\lambda\nu\kappa}_{\lambda\mu\nu\kappa} - C^{\mu\lambda\nu\kappa}_\lambda R_{\lambda\kappa}\right] = T^{\mu\nu}. \]
These equations were later solved by Mannheim and Kazanas [2] for a spherical mass, exterior solution, which behaves as a conformal gravity Schwarzschild like solution, and allows for application of the theory to various data sets. In recent years, Mannheim and O’Brien have shown that conformal gravity can be used to accurately describe the rotation curves of a diverse set of galaxies without the need to invoke dark matter [3],[4],[5],[6]. Since all attempts to find any direct evidence of dark matter have been unsuccessful so far, alternative theories of gravity have become increasingly worth considering as an explanation for phenomena ascribed to dark matter. Its success in fitting galactic rotations makes conformal gravity an ideal candidate to explore in an effort to theoretically describe these additional effects. While the rotation curve fitting provides direct data that can be tested against the predictions of an alternative model of gravity, there are other observational motivations for the existence of dark matter, such as the motion of galaxies in clusters and gravitational lensing. The goal of this paper is to first report conformal gravity’s fits on the rotation curves of a very recent dwarf galaxy survey called the LITTLE THINGS, to show how conformal gravity can address the cluster motion problem without invoking dark matter, and develop the framework for gravitational lensing formalism in conformal gravity.

2. Rotation Curves of The Little Things Survey
In this paper, we review our recent efforts [7] in fitting the LITTLE THINGS survey [8], a survey of 25 dwarf galaxies offering ultra high resolution of modern rotation curve data. Data from the Local Irregular That Trace Luminosity Extremes in the Hi Nearby Galaxy Survey (LITTLE THINGS) was collected using the NRAO Very Large Array (VLA). The survey is a fairly local compilation, where each of the galaxies featured are within 11 Mpc of the Milky Way. The observations of the galaxies were used to create rotation curves and were combined with Spitzer archival 3.6m and ancillary optical U, B, and V images to construct mass distribution models. The paper calls for more analysis on high-resolution low mass dwarf galaxies to further test the theory of $\Lambda$CDM. For our purposes, we use the data collected and make mass model predictions for the rotation curves of the LITTLE THINGS Dwarfs using conformal gravity and MOND. Although MOND is out of the scope of this paper, its inclusion in the fits is a new mechanism for comparison of alternative theories, on a per galaxy, same input parameter basis (see Table 1).

We adopt a uniform, non-biased method of adopting the estimated distance from the mean of the NASA Extragalactic Database (NED), and all other input parameters were kept consistent with [8]. In Fig. 1, we present the General Relativity, conformal gravity and MOND rotation curve predictions for each data set. The General Relativity prediction is given by [9],

$$v_{gr}(R) = \sqrt{\frac{N^* \beta^* c^2 R^2}{2R_0^3}} \left[ I_0 \left( \frac{R}{2R_0} \right) K_0 \left( \frac{R}{2R_0} \right) - I_1 \left( \frac{R}{2R_0} \right) K_1 \left( \frac{R}{2R_0} \right) \right],$$

(4)

the conformal gravity prediction is given by [4],

$$v_{cg}(R) = \sqrt{v_{gr}^2 + \frac{N^* \beta^* c^2 R^2}{2R_0^3} I_1 \left( \frac{R}{2R_0} \right) K_1 \left( \frac{R}{2R_0} \right) + \frac{\gamma_0 c^2 R}{2} - \kappa c^2 R^2},$$

(5)

and the MOND prediction is given by [10],

$$V_{mond}(R) = \sqrt{V_{gr}^2(R) + \left( \frac{V_{gr}^2(R)}{2} \right) \frac{1 + 4\mu R}{V_{gr}^2}}.$$

(6)

In each of the above formula, $N^*$ is the estimated number of stars in the galaxy (proportional to the mass) and $R_0$ is the galactic scale length. In eq. (5), $\gamma^* = 5.42 \times 10^{-41} \text{cm}^{-1}$,
\[ \gamma_0 = 3.06 \times 10^{-30} \text{ cm}^{-1}, \quad \kappa = 9.54 \times 10^{-54} \text{ cm}^{-2}. \]

In eq. (6), \[ a_0 = 1.35 \times 10^{-8} \text{ cm s}^{-2} \]
and we refer the reader to [4] and [10] for the derivation and determination of such constants. It should be noted that in each of these models, both the luminous disk and gas were fit, such that for a given galaxy the total velocity contributions are given by

\[ v_{\text{tot}}(R) = \sqrt{v_{\text{cg}}(R)^2 + v_{\text{gas}}(R)^2}. \] (7)

The individual galaxies’ total gas mass is given in Table 1 (where all quantities with the \( \odot \) are in solar quantities). No bulges were included in any of these fits due to lack of observational evidence [8].

It can be easily seen in Fig. 1, that both MOND and conformal gravity capture the essence of the data without the need for dark matter. In particular, conformal gravity fits this sample of dwarf galaxies with high precision without any parameter modification. More specific information about the particular morphology of each galaxy can be found in [8], which helps explain some of the more exotic shapes of the rotation curves, but on the whole, conformal gravity generates mass models that return mass to light ratios consistent with physical observations for dwarf galaxies (see Table 1). Further, in the last column of Table 1, we see the same trend observed in [4],[3],and [5], where universality can be found in the data, where the number \( \left( \frac{v^2}{c^2 R^l} \right)_{\text{last}} \) seems to be of order one, for now over 200 galaxies. The fact that there seems to be this universality in the last data point of each galaxies centripetal acceleration (scaled here to \( \frac{1}{c^2} \)) provides argument for the recent work of McGaugh et al. [11], where it is stated that such a universality could be explained by new physics outside of GR, instead of by the inclusion of dark matter. Thus, the LITTLE THINGS survey adds a robust, high resolution set of dwarf galaxies to the catalogue of rotation curves successfully fit by conformal gravity without the need for dark matter.
FIG. 1: Fitting to the rotational velocities (km/s) of the LITTLE THINGS sample with quoted errors as a function of radial distance (kpc). Blue dashed curve=Newtonian, Full blue curve=conformal gravity, Full red curve=MOND. No dark matter is assumed.
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The virial theorem in order to estimate the mass. The virial theorem for a potential of the form was to use the data for the velocity dispersion of the galaxies within the Coma cluster and apply clusters it would be wise to start with the initial problem as solved by Zwicky. His approach was to use the data for the velocity dispersion of the galaxies within the Coma cluster and apply the virial theorem in order to estimate the mass. The virial theorem for a potential of the form

3. Cluster Motion in Conformal Gravity

Although recent literature on alternative gravitational models have been focused around rotation curves, the original missing mass problem was founded in the study of random motions of galaxies in galactic clusters [10]. In 1933, Zwicky applied the virial theorem to his observations on the Coma cluster and he found that the theoretical mass estimate resulted in a mass to light ratio that was several orders of magnitude larger than theoretical prediction [12]. He reasoned that this result was due to some “missing matter” that could not be seen. Over the following decades, other problems in astrophysics arose that also seemed to point to the existence of this dark matter, such as the rotation curve problem of Rubin [13]. Since the Zwicky missing mass problem was the original calculation that led astronomers to suspect that there was missing matter in the universe, it would be appropriate to revisit the Coma cluster prediction in conformal gravity. This work was originally done by Mannheim [14] but due to the recent work of Mannheim and O’Brien [3], the full fourth order solution (with the inclusion of the global quadratic potential) can be applied to the Coma cluster to give a more representative prediction.

3.1. The Coma Cluster

In order to see whether conformal gravity can solve the problem of dark matter in galactic clusters it would be wise to start with the initial problem as solved by Zwicky. His approach was to use the data for the velocity dispersion of the galaxies within the Coma cluster and apply the virial theorem in order to estimate the mass. The virial theorem for a potential of the form

Table 1. Properties of the LITTLE THINGS Galaxy Sample

| Galaxy   | $i$  | Dis. (Mpc) | Lumin. | $R_0$ | $R_\ell$ | $M_{\text{gas}}$ | $M_{\text{disk}}$ | $(M/L)_C$ | $(v^2/c^2 R_\ell)_{\text{last}}$ |
|----------|------|------------|--------|-------|---------|----------------|---------------|-----------|--------------------------------|
| CVN      | 66.5 | 3.18       | 0.11   | 1.1   | 2.5     | 2.26           | 0.03          | 0.27      | 1.16                                          |
| DDO 43   | 40.6 | 7.40       | 1.54   | 1.1   | 4.0     | 20.94          | 0.32          | 0.21      | 0.94                                          |
| DDO 46   | 27.9 | 8.78       | 2.45   | 0.9   | 5.1     | 45.77          | 18.15         | 7.42      | 0.83                                          |
| DDO 47   | 45.5 | 6.00       | 3.28   | 1.1   | 8.9     | 62.31          | 1.82          | 0.55      | 1.62                                          |
| DDO 50   | 49.7 | 2.28       | 3.06   | 0.5   | 6.6     | 6.08           | 1.95          | 0.64      | 8.92                                          |
| DDO 53   | 27.0 | 3.31       | 0.44   | 0.6   | 1.4     | 6.34           | 0.37          | 0.87      | 1.69                                          |
| DDO 70   | 50.0 | 1.59       | 1.01   | 0.7   | 2.4     | 5.67           | 0.77          | 0.76      | 2.75                                          |
| DDO 87   | 55.5 | 7.27       | 1.40   | 2.3   | 7.0     | 25.96          | 2.21          | 1.58      | 1.44                                          |
| DDO 101  | 51.0 | 11.25      | 4.85   | 1.1   | 4.1     | 10.82          | 16.98         | 3.53      | 3.60                                          |
| DDO 126  | 65.0 | 4.14       | 1.01   | 1.0   | 3.4     | 11.73          | 0.44          | 0.43      | 1.30                                          |
| DDO 133  | 43.4 | 4.95       | 2.59   | 1.6   | 4.9     | 25.72          | 2.86          | 1.10      | 1.37                                          |
| DDO 154  | 68.2 | 3.84       | 0.80   | 0.7   | 8.2     | 37.99          | 0.07          | 0.08      | 1.01                                          |
| DDO 168  | 46.5 | 4.61       | 3.41   | 1.2   | 4.3     | 29.75          | 8.56          | 2.51      | 1.70                                          |
| DDO 210  | 66.7 | 0.70       | 0.22   | 0.4   | 0.3     | 1.05           | 0.05          | 2.11      | 1.28                                          |
| DDO 216  | 63.7 | 0.76       | 0.21   | 0.9   | 0.7     | 0.20           | 0.06          | 0.23      | 1.66                                          |
| F-567 V3 | 56.5 | 8.73       | 0.68   | 0.7   | 3.7     | 4.40           | 0.79          | 0.11      | 0.55                                          |
| IC 10    | 47.0 | 0.87       | 6.85   | 0.4   | 0.6     | 2.20           | 1.77          | 0.26      | 7.45                                          |
| IC 1613  | 48.0 | 0.61       | 7.96   | 0.7   | 0.6     | 2.20           | 1.77          | 0.22      | 0.75                                          |
| NGC 1569 | 69.1 | 2.87       | 2.11   | 0.8   | 2.7     | 14.81          | 0.98          | 0.05      | 2.39                                          |
| NGC 2366 | 63.0 | 3.34       | 7.87   | 1.4   | 7.9     | 6.69           | 8.50          | 1.08      | 1.47                                          |
| NGC 3738 | 22.6 | 3.82       | 6.54   | 0.5   | 1.4     | 7.65           | 33.17         | 5.07      | 2.95                                          |
| UGC 8508 | 83.5 | 2.60       | 0.43   | 0.5   | 1.9     | 1.91           | 0.83          | 1.92      | 3.80                                          |
| WLM 74   | 40.0 | 0.93       | 0.73   | 0.8   | 3.0     | 6.45           | 0.79          | 1.08      | 0.96                                          |
| Haro 29  | 61.2 | 4.70       | 0.68   | 0.4   | 4.0     | 5.94           | 1.20          | 1.76      | 7.58                                          |
| Haro 36  | 70.0 | 8.91       | 3.27   | 0.6   | 3.0     | 10.44          | 1.00          | 0.31      | 3.45                                          |

Table 1 Entries, from left to right are: Galaxy name, inclination, NED Distance, Total Blue Luminosity, Disk Scale Length, Distance of the last data point, Mass of the gas (including helium), disk mass, mass to light ratios, scaled centripetal acceleration of the last observed data point.
\[ V = ar^n \] states that:
\[ 2\langle T \rangle = n\langle V \rangle, \]  
where \( \langle T \rangle \) is the average total kinetic energy and \( \langle V \rangle \) is the average total potential energy.

Zwicky used this fact along with the fact that:
\[ \langle v^2 \rangle \approx 3\sigma^2, \]  
where \( v^2 \) is the three dimensional systemic velocity of the cluster and \( \sigma \) is the measured velocity dispersion. The approximation sign is used because if the cluster is rotating or is not completely isotropic this calculation would only be true to an order of magnitude. It should be noted that in a cluster which is virialized this is a decent approximation used commonly in the literature [14]. Since this work is meant to only establish the correct order of magnitudes needed, we will ignore systematic effects such as rotational or tangential motions since these would only account for fine tuning of the numbers within the order of magnitude. For a complete treatment of how these effects will be implemented in the future, see [14]. When we plug eq. (9) along with the Newtonian potential into eq. (8) we arrive at the simple prediction for the mass based on the velocity observations,
\[ M \approx \frac{3\sigma^2 R}{G}, \]  
where \( R \) is the virial radius of the cluster. We can use eq. (10) along with the observations for the Coma Cluster to get an estimate of its mass,
\[ M_{\text{Coma}} \approx 9.5 \times 10^{14} M_\odot, \]  
since for the Coma Cluster, the current measured distance and velocity dispersions are \( \sigma^2_{\text{Coma}} = 3.413 \times 10^{12} \frac{m^2}{s^2} \) and \( R_{\text{Coma}} = 1.206 \text{ Mpc} \). It should be noted that this is a much larger value than that which Zwicky himself obtained [12], because he used a value for the distance that is much smaller than our current one. However, the larger mass [15] seems to only make his initial claim for missing mass stronger. Since it is estimated that there are around 1000 galaxies in the Coma Cluster and we have an estimate for its total mass, we can say that the average mass of a galaxy in the Coma Cluster is \( 9.5 \times 10^{11} M_\odot \), which could only be reasonable assuming that the galaxies within the cluster are all large spirals. Also, assuming that an average galaxy has a luminosity of about \( 5 \times 10^9 L_\odot \), we can calculate the mass to light ratio of an average galaxy in the Coma cluster:
\[ \gamma = \frac{M}{L} \approx 200 \gamma_\odot, \]  
This mass to light ratio is similar to the one Zwicky himself obtained and which caused him to consider that there was some mass inside the cluster that could not be seen. We can now repeat the same calculation using the conformal potential in order to see if this problem could be resolved without invoking a need for non luminous matter.

3.2. The Coma Cluster in Conformal Gravity

To mirror the Zwicky calculation in Conformal Gravity, we first need to establish the single point source potential. This work was originally done by Mannheim and Kazanas [2], who showed that for a spherically symmetric source, the line element for an external solution to the mass:
\[ ds^2 = -Bc^2 dt^2 + \frac{1}{B} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \]  
where
can be solved analytically. The exterior fourth order solution yields,
\[ B = 1 - \frac{2GM}{c^2r} + \frac{M\gamma^*r}{M_\odot} + \gamma_0r - 2kr^2, \]  
(14)
where \( \gamma^* = 5.42 \times 10^{-39} \text{ m}^{-1} \), \( \gamma_0 = 3.06 \times 10^{-28} \text{ m}^{-1} \) and \( k = 9.54 \times 10^{-50} \text{ m}^{-2} \) are constants of the theory determined through galactic rotation fitting [4]. From this we can immediately get the weak field approximation for the potential:
\[ V = -\frac{GM}{r} + \frac{M\gamma^*c^2r}{2M_\odot} + \frac{\gamma_0c^2r}{2} - kc^2r^2, \]  
(15)
We can now use this along with eq. (8) to get the conformal estimate for the mass of a cluster:
\[ M \approx \frac{\sigma^2 - \gamma_0c^2R}{G} + \frac{\gamma^*c^2R^2}{2M_\odot}, \]  
(16)
which for \( R \) on the scale of the virial radius of a cluster can be simplified to
\[ M \approx \frac{2k}{\gamma^*} R M_\odot. \]  
(17)
When we plug in the same distance as we did for the original Zwicky problem, we can calculate the conformal estimate for the mass which to yield \( M \approx 1.3 \times 10^{12} M_\odot \) or an average of \( 1.3 \times 10^9 M_\odot \) which is a much more reasonable average across various morphologies of galaxies (see for example [16]). Finally we can calculate the mass to light ratio which comes out to be:
\[ \gamma = \frac{M}{L} \approx 0.26 \gamma_\odot, \]  
(18)
As we can see this mass to light ratio in no way leads us to suspect that there might be missing matter.

### 3.3. Virgo Cluster

A few years after Zwicky, Smith performed a similar calculation for the inner members of the Virgo Cluster [17]. Using eq. (10), Smith found that the mass of an average galaxy from the 500 tracers in the central region of the cluster was \( M \approx 2 \times 10^{11} M_\odot \) which again was about 200 times larger than the Hubble estimate for the mass of an average nebula of \( 10^9 M_\odot \). If we use Smith’s value for the radius of that cluster of 0.2 Mpc we can plug it into eq. (17) to get the conformal approximation of the mass which comes out to be \( 2.2 \times 10^{11} M_\odot \) or \( 4.4 \times 10^8 M_\odot \) for an average galaxy. Using this and the average luminosity of the virgo cluster [15], we come to a physical mass to light ratio as,
\[ \gamma = \frac{M}{L} \approx 1.02 \gamma_\odot, \]  
(19)
which again in no way suggests that we need to incorporate dark matter.

### 3.4. Recent Cluster Observations

We can further this argument since we are now in a position to make predictions about cluster dynamics without having to make a claim about the mass at all. To this end, we note the last quadratic term in the potential is a global term which does not depend on the mass of the
Table 2. Selected SPZ-SZ Survey Clusters

| Cluster Name | \(\sigma_{\text{vel}}\) (\(\text{km} \text{s}^{-1}\)) | \(M_\Lambda \) (\(10^{14} M_\odot\)) | \(R_\Lambda\) (Mpc) | \(M_C\) (\(10^{11} M_\odot\)) | \(R_C\) (Mpc) |
|--------------|----------------|-----------------|-----------------|----------------|----------------|
| SPT-CL J0000-5748 | 935 | 4.29 | 0.707 | 7.67 | 0.566 |
| SPT-CL J0014-4952 | 1004 | 5.14 | 0.735 | 7.97 | 0.608 |
| SPT-CL J0037-5047 | 945 | 3.64 | 0.587 | 6.37 | 0.572 |
| SPT-CL J0040-4407 | 1171 | 10.18 | 1.070 | 11.60 | 0.709 |
| SPT-CL J0118-5156 | 865 | 3.39 | 0.653 | 7.08 | 0.524 |
| SPT-CL J0205-5829 | 1101 | 4.79 | 0.569 | 6.17 | 0.667 |
| SPT-CL J0205-6432 | 862 | 3.29 | 0.638 | 6.92 | 0.522 |
| SPT-CL J0233-5819 | 884 | 3.71 | 0.684 | 7.42 | 0.535 |
| SPT-CL J0234-5831 | 1076 | 7.64 | 0.951 | 10.31 | 0.652 |
| SPT-CL J0240-5946 | 948 | 5.29 | 0.848 | 9.20 | 0.574 |

This table shows selected galaxies taken from the SPZ-SZ survey. The values for the velocity dispersion \(\sigma\) are taken directly from [18]. The mass quoted as \(M_\Lambda\) in the table is the mass derived with dark matter from [18]. The \(R_C\) values are the ones derived in this paper from eq.(20). The \(R_\Lambda\) values are derived by using eq.(10) and the values of \(\sigma\) and \(M_\Lambda\). \(M_C\) is the value of the mass when if we used \(R_\Lambda\) in eq. (17).

This specific cluster but still becomes the dominant term over large distances. Thus if we take only the last term of eq. (16) and plug it into the virial theorem, we can get the following relation,

\[
3\sigma^2 \approx kc^2 R^2.
\]  

Although this is another approximation, it still allows us to make some profound claims on the order of magnitudes of the numbers in question. Moreover, we can use this as a means to make some comparisons to the standard theory. To further assess the validity of the conformal model, we can use the data for the velocity dispersions of clusters from the SPT-SZ survey [18]. In this work, the authors use the measured velocity dispersions, as well as dark matter mass estimates to predict the virial radial distance of the galaxies within the cluster. Since the numbers they obtain using the dark matter paradigm are published and hence trusted as results, we can use eq. (20) to make the same prediction without assuming any dark matter whatsoever, using only the observed measured velocities. It should be noted that the authors here have simply chosen the first 10 clusters studied in [18] and have in no way selected them based on any bias. In Table 1, we show the comparison of the two methods and show that the conformal prediction is right on par with the standard gravity prediction for the virial radius. For completion, we include the expected mass for each cluster within the ΛCDM cosmology paradigm which we reiterate is not used in obtaining the conformal prediction. We also show what the conformal mass estimate from eq. (17) would be if the virial radial distance obtained from the ΛCDM paradigm is taken to be the more precise value. As we can see the mass in the conformal case has to be several orders of magnitude smaller to provide the same prediction. We also note the luminosity estimates for these clusters in [18] are all of order \(10^{11} L_\odot\), the same order as the conformal mass estimates. Thus we can calculate the mass to light ratio of a cluster for which we have a conformal estimate and a quoted luminosity. SPT-CL J0233-5819 fits both criteria and since its luminosity is \(1.21 \times 10^{11} L_\odot\) and its conformal mass is \(7.42 \times 10^{11} M_\odot\) we get a mass to light ratio of \(6.1 \gamma_\odot\) which again in no way suggests missing matter.
4. Gravitational Bending of light in conformal gravity

One of the great observational triumphs of General Relativity was the famous 1919 experiment verifying the gravitational bending of light. Since General Relativity so accurately captured this phenomena, any competing alternative theory of gravity must be able to accommodate this feature. Since conformal gravity is still a metric theory of gravity, it should retain the same qualities of Einstein gravity in the weak field or local level. In order to establish whether conformal gravity can explain the observations from weak lensing without the need for dark matter we must first tackle the simpler problem of establishing what the bending of light by a central source would look like in conformal gravity. We start again by writing the conformal metric (14) for a spherical source. However as an initial proof of concept, we ignore the global terms [19] as we are looking at the local lensing effect. Thus, the metric of eq. (13) becomes:

\[ ds^2 = -(1 - \frac{2GM}{c^2r} + \frac{M\gamma^* r}{M_\odot})c^2 dt^2 + \frac{1}{(1 - \frac{2GM}{c^2r} + \frac{M\gamma^* r}{M_\odot})} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2. \] (21)

In order to model the situation of lensing around a spherically symmetric object (such as the sun), we set up the typical coordinate system as pictured in [20]. We look at a two dimensional deflection for simplicity and hence set \( \phi = 0 \). Following the original deflection calculation in [20], since we are using the sun as our gravitational source, we take the weak approximation of eq. (21) and expand the denominator of the radial metric term, viz:

\[ ds^2 = -(1 - \frac{2GM}{c^2r} + \frac{M\gamma^* r}{M_\odot})c^2 dt^2 + (1 + 2\frac{GM}{c^2r} - \frac{M}{M_\odot}\gamma r) dr^2 + r^2 d\theta^2 \] (22)

The simplest way to approach this problem is to use Huyghens Principle which states that:

\[ \frac{d\theta}{dx} = \frac{1}{v} \frac{dv}{dy} \] (23)

where \( \theta \) is the angle of deflection and \( v \) is the coordinate velocity of light in the direction of motion. To exploit eq. (23), we rewrite the metric of eq. (22) in terms of the \( x \) coordinate to yield:

\[ ds^2 = -(1 - \frac{2GM}{c^2r} + \frac{M}{M_\odot}\gamma^* r) c^2 dt^2 + (1 + 2\frac{GM}{c^2r} - \frac{M}{M_\odot}\gamma^* r) \left( \frac{dr}{dx} \right)^2 dx^2 + \frac{r^2}{\left( \frac{dr}{dx} \right)^2} d\theta^2 \] (24)

We can then plug in the transformation equations between Cartesian and polar coordinates:

\[ r = \sqrt{x^2 + y^2}, \quad \cos \theta = \frac{y}{r}, \] (25)

such that the final expression for the line element in this coordinate system becomes:

\[ ds^2 = -(1 - \frac{2GM}{c^2r} + \frac{M}{M_\odot}\gamma^* r) c^2 dt^2 + (1 + 2\frac{GMx^2}{c^2r^3} - \frac{M\gamma^* x^2}{M_\odot r}) dx^2. \] (26)

Since the light will follow the null geodesics, we set \( ds = 0 \) and solve for the velocity as,

\[ v = \frac{dx}{dt} = [1 + \left( \frac{M\gamma^* r}{2M_\odot} - \frac{GM}{c^2r} \right)(1 + \frac{x^2}{r^2})] c. \] (27)

The feature of eq. (27) is that treats the gravitational field as though it was a medium which allows us to essentially calculate an effective gravitational index of refraction. We then use eq.
(27) in the polar coordinate transform eq. (25) to calculate the deflection angle in terms of the x coordinate as
\[ \frac{d\theta}{dx} = \frac{GM}{c^2} \left( \frac{3x^2y}{r^5} + \frac{y}{r^3} \right) + \frac{M\gamma^* r_0}{2M_\odot} \left( \frac{y}{r} - \frac{yx^2}{r^3} \right). \] (28)

This expression can be integrated immediately to give the total deflection angle:
\[ \theta = \int_{-\infty}^{\infty} \frac{GM}{c^2} \left( \frac{3x^2y}{r^5} + \frac{y}{r^3} \right) dx + \int_{-\infty}^{\infty} \frac{M\gamma^* r_0}{2M_\odot} \left( \frac{y}{r} - \frac{yx^2}{r^3} \right) dx \] (29)

Since we know that the deflection from the straight path would be very small, we can expect that \( y \) won’t be very different from the impact parameter \( r_0 \). Thus we simplify eq. (29) to:
\[ \theta \approx \frac{GM r_0}{c^2} \int_{-\infty}^{\infty} \left( \frac{3x^2}{r^5} + \frac{1}{r^3} \right) dx + \frac{M\gamma^* r_0}{2M_\odot} \int_{-\infty}^{\infty} \left( \frac{1}{r} - \frac{x^2}{r^3} \right) dx \] (30)

These integrals are trivial to solve when we make the following substitutions:
\[ x = r_0 \tan \theta, \quad r = \frac{r_0}{\cos \theta}, \quad \frac{dx}{d\theta} = \frac{r^2}{r_0} \] (31)

which we can then plug into eq. (30) to get:
\[ \theta \approx \frac{2GM}{c^2 r_0} \int_{0}^{\pi} (3 \sin^2 \theta + 1) \cos \theta \, d\theta + \frac{M\gamma^* r_0}{2M_\odot} \int_{0}^{\pi} \cos \theta \, d\theta \]
\[ \theta \approx \frac{4GM}{c^2 r_0} + \frac{M\gamma^* r_0}{2M_\odot} \] (32)

We recognize the first term in eq. (32) as the result for gravitational bending of light in general relativity and the second term is thus the conformal correction. Reassuringly if we take \( r_0 \) to be the radius of the sun we see that the conformal correction is negligible and thus we get the same result as general relativity for a beam of light just passing by the sun. We also see that over larger distances we can expect bigger and bigger departures from general relativity which can have a profound effect on the predictions for weak lensing. Over larger distances we would also expect to start noticing the global effects of curvature and thus it would be instructive to derive the full conformal bending of light by following the same procedure described above, but using the metric from eq. (14) and including the global terms. Upon repeating the same procedure outlined above with the added terms, we can calculate the total angular deflection as:
\[ \theta \approx \frac{4GM}{c^2 r_0} + \frac{M\gamma^* r_0}{2M_\odot} + \gamma_0 r_0 - \frac{\pi k r_0^2}{M_\odot}. \] (33)

In order to bring this in alignment with the above work on clusters, we can rewrite this equation in terms of the mass:
\[ M \approx \frac{\theta - \gamma_0 r_0 + \pi k r_0^2}{\frac{4GM}{c^2 r_0} + \frac{M\gamma^* r_0}{2M_\odot}} \] (34)

which for large impact parameters reduces to:
\[ M \approx \left( \frac{\pi k r_0}{\gamma^*} \right) M_\odot \] (35)

This estimate for the mass is consistent with the one seen in eq. (16) using the virial theorem for clusters. It should be noted that authors such as Nesbet have recently been looking for
global connections of the conformal gravity constants [21], and here we have shown that we get a consistent framework for the constants which were originally described in terms of rotation curves, with clusters and gravitational bending of light. Further, the work done by Sultana et al. [22] in 2012 has shown that conformal gravity can accurately predict the precession of the perihelion of Mercury. Sultana’s result, coupled with the calculation above for the deflection of light around the sun, expand on the applications of conformal gravity to the local solar system. Sultana’s work, similar to Nesbit, has also shown that the global parameter of conformal gravity $\gamma$ can be bounded by the Mercury observations and is completely consistent with the value originally obtained from rotation curves [22]. These advances in conformal gravity are helping show the universality of the theory as a whole, and the theory is being held consistent at all gravitational scales.

5. Conclusion
In this paper we reviewed three recent important advancements in conformal gravity, which provide the framework for a lot of potential future research. Once again, conformal gravity provides an accurate description to 25 new high resolution rotation curves, increasing the net total of rotation curves fit by the theory to over 200. Further, we have also produced the framework for comparison of conformal gravity to MOND on a one to one galaxy basis. This coupled with the tools described in [23], can be extended to include other models in the future. Second, we show that conformal gravity is equipped to solve the original missing mass problem as found by Zwicky, since it can model cluster dynamics without needing any additional mass. Work on this front is currently being extended to the Abel cluster. In future work, conformal gravity will be put to a similar test in the ultra faint galaxies, studying the radial and tangential motions of individual tracer objects in young galaxies [24]. Lastly, we make the first initial steps towards applying conformal gravity to gravitational lensing, with the reproduction of the gravitational deflection of starlight calculation, which serves as a primer for a more robust lensing discussion.

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