Non-abelian Reciprocity Laws on a Riemann Surface

I. Horozov

Mathematisches Institut,
Universität Tübingen, Auf der Morgenstelle 10,
72076 Tübingen, Germany

Abstract

On a Riemann surface there are relations among the periods of holomorphic differential forms, called Riemann’s relations. If one looks carefully in Riemann’s proof, one notices that he uses iterated integrals. What I have done in this paper is to generalize these relations to relations among generating series of iterated integrals. Since the main result is formulated in terms of generating series, it gives infinitely many relations - one for each coefficient of the generating series. The lower order terms give the well known classical relations. The new result is reciprocity for the higher degree terms, which give non-trivial relations among iterated integrals on a Riemann surface. As an application we refine the definition of Manin’s non-commutative modular symbol in order to include Eisenstein series. Finally, we have to point out that this paper contains some constructions needed for multidimensional reciprocity laws like a refinement of one of the Kato-Parshin reciprocity laws.

Contents

0 Introduction 2

1 Background on iterated integrals 4

1.1 Definition of iterated integrals 4

1.2 Homotopy invariance of iterated integrals 4

1.3 Differential equation 4

1.4 Multiplication formulas 5

1.5 Shuffle relations 5

1.6 Reversing the path 6

2 Non-abelian reciprocity law on a Riemann surface 6

2.1 Iterated integrals over a loop around a pole 6

2.2 Generating series of iterated integrals over a loop around a pole 9

2.3 Generating series over α and β cycles of a Riemann surface 10

2.4 Non-abelian reciprocity law 11

*E-mail: horozov@uni-tuebingen.de
0 Introduction

This paper is the first one from a series of papers on reciprocity laws on a complex variety based on properties of a fundamental group. Here we limit ourselves to the case of Riemann surfaces. We obtain infinitely many reciprocity laws on a Riemann surface, which we write as a reciprocity law for a generating series. The first few reciprocity laws are known, but the rest are new. The second paper from this series gives a reciprocity law for a new symbol, which is a refinement of Parshin symbol (see [P2], [P3]) for a complex surface.

In both papers we use properties of a certain fundamental group by examining iterated integrals in the sense of K.-T. Chen (see [Ch]).

We define a generating series of iterated integrals $F$ with good analytic properties. And we prove a reciprocity law for such a generating series $F$. Since we have a generating series, each coefficient gives us a reciprocity law. The first few terms give us known reciprocity laws. For a term of degree 3, we obtain a new reciprocity law [H], which we use in the next paper on refinement of one of the Kato-Parshin ([Ka]) reciprocity laws for the Parshin symbol ([P2], [P3]). The lower degree terms give well known reciprocity laws. In degree 1 we recover the following reciprocity: the sum of the residues of a differential form on a Riemann surface is zero. In degree 2, we recover Riemann relations among differential forms of the 3-rd kind [GH]. Also from degree 2, we recover Weil reciprocity. If one looks carefully at a proof of Riemann’s relations (see [GH]), one notices that they use iterated integrals. What I have done in this paper is to generalize these relations to relations among generating series of iterated integrals $F$. Since it is formulated in terms of generating series, it gives infinitely many relations - one for each coefficient of the generating series. This paper establishes the main tools, which will be used in papers that establish a new reciprocity law for portion of Kato-Parshin symbols. For a complex surface see [H]. For refinement of Parshin symbol for higher dimensional complex varieties, we have the a prove for the new reciprocity law. This in the 3-rd paper from the series, which is in preparation.

We construct the generating series of iterated integrals $F$ as a solution of an ordinary differential equation, following an idea of Manin [M]. For each closed loop $\sigma$ on a Riemann surface, we consider the generating series of the iterated integrals $F_\sigma$. On a Riemann surface there is ”good” choice of generators of the fundamental group, with only one relation. For each of the generators of the fundamental group we consider the generating series of iterated integrals $F_\sigma$. Composition of paths corresponds to a composition of generating series of iterated integrals. For two loops $\sigma_1$ and $\sigma_2$ with common base point, we have $F_{\sigma_1}F_{\sigma_2} = F_{\sigma_1\sigma_2}$ for the generating series. The relation among the generators in the fundamental group gives us a relation among the generating series of iterated
integrals $F_\sigma$ for various loops $\sigma$. We call this relation a reciprocity law. This is the key idea in the construction of the reciprocity law.

Before we formulate the final statement of the reciprocity law, we consider several simplifications. The first one is for a simple loop around a pole of any of the differential forms. We present constructively the process of taking residues of iterating integrals, which we generalize to taking residues of generating series of iterated integrals. Another simplification that we make is for $F_{[\alpha_i, \beta_i]}$, where $[\alpha_i, \beta_i]$ is commutator of $\alpha_i$ and $\beta_i$ loops on a Riemann surface. When all this is done, we formulate the reciprocity law.

In the last section we apply the developed ideas about non-commutative reciprocity law to Manin’s non-commutative modular symbol. We are able to extend his construction so that we can include Eisenstein series.

Where is the non-abelian group in the reciprocity law? An iterated integral over a loop depends on the loop. Since iterated integrals are homotopy invariant, we have that an iterated integral over a loop depends only on the element of the fundamental group that this loop represents. How much do iterated integrals distinguish elements of the fundamental group? I thank one of the referees, who pointed out that conjecture by Parshin [P1] that iterated integrals of differential forms of the third kind on a Riemann surface capture precisely the pro-unipotent part of the fundamental group. For some progress and intuition in this direction one might look at [Q]. In terms of nilpotent variations of mixed Hodge structures related to the fundamental group, one can look at the following papers [G] and [DG].

There are non-commutative reciprocity laws by Brylinski and McLaughlin (see [BrMc]) on a complex variety. There reciprocity laws are non-abelian, since the authors consider not only a classifying space $BGL$ of the structure sheaf, but also a certain central extension. In this paper the reciprocity laws are non-abelian for a different reason. The reciprocity laws are based on properties of the fundamental group on a punctured Riemann surface $X$, which is a non-abelian group in most of the interesting situations. Also, with the use of iterated integrals, we capture non-abelian quotient of the fundamental group $\pi_1(X, P)$, which is strictly larger than the maximal abelian quotient $H_1(X)$.

I would like to make one remark about the reciprocity laws in this paper. Instead of working on a Riemann surface, one can consider a smooth algebraic curve over the algebraic closure of the rational numbers. Using the construction in [G] and [DG] we can conclude that the iterated integrals of algebraic differential forms with logarithmic poles over a path give periods in the sense of algebraic geometry. So the reciprocity laws that we describe in this paper are relations among periods.

We are going to use explicit integrals. However, in a more theoretical approach, one can consider framed mixed Hodge structures associated to the integrals that we consider. The process of taking residues of iterated integrals corresponds to taking co-product of the corresponding framed mixed Hodge structure (see [G]).

The non-abelian reciprocity law in this paper has a generalization to higher dimensions. Even in dimension 2, new phenomenons occur: instead of reciprocity law for the Parshin symbol, we obtain a new reciprocity law for a refinement of the Parshin symbol (see [H]).
1 Background on iterated integrals

This section establishes both the notation and the main properties of iterated integrals, which we are going to use throughout the paper. We recall well known properties of iterated integrals, which we are going to use heavily in the rest of the paper. One can look at K.-T. Chen [Ch] and Goncharov [G] for proofs of the statements and for more properties of iterated integrals. For the differential equation (1.1), we follow the idea of Manin [M].

1.1 Definition of iterated integrals

Definition 1.1 Let $\omega_1, \ldots, \omega_n$ be holomorphic 1-forms on a simply connected open subset $U$ of the complex plane $\mathbb{C}$. Let

$$\gamma : [0, 1] \to U$$

be a path. We define an iterated integral of the forms $\omega_1, \ldots, \omega_n$ over the path $\gamma$ to be

$$\int_{\gamma} \omega_1 \circ \cdots \circ \omega_n = \int_{0\leq t_1 \leq \cdots \leq t_n \leq 1} \gamma^* \omega_1(t_1) \wedge \cdots \wedge \gamma^* \omega_n(t_n).$$

It is called iterated because it can be defined inductively by

$$\int_{\gamma} \omega_1 \circ \cdots \circ \omega_n = \int_{0}^{1} \left( \int_{0}^{1} \omega_1 \circ \cdots \circ \omega_{n-1} \right) \gamma^* \omega_n(t).$$

1.2 Homotopy invariance of iterated integrals

Theorem 1.2 Let $\omega_1, \ldots, \omega_n$ be holomorphic 1-forms on a simply connected open subset $U$ of the complex plane $\mathbb{C}$. Let

$$H : [0, 1] \times [0, 1] \to U$$

be a homotopy, fixing the end points, of paths

$$\gamma_s : [0, 1] \to U$$

such that $\gamma_s(t) = H(s, t)$, and for fixed $s$, we have a path $\gamma_s : [0, 1] \to U$. Then

$$\int_{\gamma_s} \omega_1 \circ \cdots \circ \omega_n$$

is independent of $s$.

1.3 Differential equation

When we consider an iterated integral, we can let the end point vary in a small neighborhood. Then the iterated integral becomes an analytic function.

Let $\omega_1, \ldots, \omega_n$ be differentials of the 3rd kind on a Riemann surface $X$. Following an idea of Manin [M], we consider the differential equation

$$dF = \sum_{i=1}^{n} A_i \omega_i,$$

(1.1)
where $A_1, \ldots, A_n$ are non-commuting formal variables. Let $P$ be a point on $X$ such that none of the differential forms has a pole at $P$. It is easy to check that the solution of $F(z)$ with initial condition $F(P) = 1$ is

$$F(z) = 1 + \sum_i (A_i \int_P^z \omega_i) + \sum_{i,j} A_i A_j \int_P^z \omega_i \circ \omega_j + \sum_{i,j,k} A_i A_j A_k \int_P^z \omega_i \circ \omega_j \circ \omega_k + \ldots.$$ 

The summation is over all iterated integrals of the given differential forms. Note that

$$d \int_P^z \omega_i \circ \ldots \circ \omega_j \circ \omega_k = (\int_P^z \omega_i \circ \ldots \circ \omega_j) \omega_k.$$

### 1.4 Multiplication formulas

We can take a path $\gamma$ from $P$ to $z$. We denote the solution of the differential equation by $F_\gamma$. If $\gamma_1$ is a path that ends at $Q$ and $\gamma_2$ is a path that starts at $Q$ we can compose them. Denote the composition by $\gamma_1 \gamma_2$.

**Theorem 1.3** (Composition of paths) With the above notation, we have

$$F_{\gamma_1} F_{\gamma_2} = F_{\gamma_1 \gamma_2}.$$

**Corollary 1.4** (Composition of paths) Let $\omega_1, \ldots, \omega_n$ be differential forms, some of them could repeat. Let also $\gamma_1$ be a path that ends at $Q$ and $\gamma_2$ be a path that starts at $Q$. We can compose them. Denote the composition by $\gamma_1 \gamma_2$. Then

$$\int_{\gamma_1 \gamma_2} \omega_1 \circ \ldots \circ \omega_n = \sum_{i=0}^n \int_{\gamma_1} \omega_1 \circ \ldots \circ \omega_i \int_{\gamma_2} \omega_{i+1} \circ \ldots \circ \omega_n,$$

where for $i = 0$ we define $\int_{\gamma_1} \omega_1 \circ \ldots \circ \omega_i = 1$, and similarly, for $i = n$ we define $\int_{\gamma_2} \omega_{i+1} \circ \ldots \circ \omega_n = 1$.

### 1.5 Shuffle relations

**Definition 1.5** Denote by $\text{Sh}(m,n)$ the shuffles, which are permutations $\tau$ of the set \{1, \ldots, m, m+1, \ldots, m+n\} such that

$$(\forall i < j \leq m) \quad \tau(i) < \tau(j)$$

and

$$(\forall i < j \leq m+n) \quad \tau(m+i) < \tau(m+j).$$

**Theorem 1.6** (Shuffle relation) Let $\omega_1, \ldots, \omega_m, \omega_{m+1}, \ldots, \omega_{m+n}$ be differential $1$-forms, some of them could repeat. Let also $\gamma$ be a path that does not pass through any of the poles of the given differential forms. Then

$$\int_{\gamma} \omega_1 \circ \ldots \circ \omega_m \int_{\gamma} \omega_{m+1} \circ \ldots \circ \omega_{m+n} = \sum_{\tau \in \text{Sh}(m,n)} \int_{\gamma} \omega_{\tau(1)} \circ \omega_{\tau(2)} \circ \ldots \circ \omega_{\tau(m+n)}.$$
1.6 Reversing the path

**Lemma 1.7 (Reversing the path)** Let $\gamma$ be a path. Let $\gamma^{-1}$ be the same path but going to the opposite direction. Then

$$\int_{\gamma^{-1}} \omega_1 \circ \omega_2 \circ \ldots \circ \omega_n = (-1)^n \int_{\gamma} \omega_n \circ \omega_{n-1} \circ \ldots \circ \omega_1. $$

2 Non-abelian reciprocity law on a Riemann surface

This section is the heart of the article. It continues with the topic of iterated integrals. However, now this is done in the direction of building the reciprocity law. It is more technical than the previous section. And it ends with the statement of the non-abelian reciprocity law on Riemann surfaces.

2.1 Iterated integrals over a loop around a pole

Let $U$ be an open simply connected subset of the complex plane. We can assume that 0 belongs to $U$. Let $f_1dz, \ldots, f_n dz$ be holomorphic differentials on $U$. We are going to iterate these differential forms together with the form $dz/z$. Let $P$ be a point in $U$ different from 0. And let $\sigma$ be a simple loop in $U$ that starts and ends at $P$ and goes around 0 once in a counterclockwise direction. We can assume that $\sigma$ does not intersect itself. Let $\gamma_\epsilon$ be a path that starts at $P$ and ends at $\epsilon$ for a point $\epsilon \neq 0$ in $U$ very close to 0. For convenience we take $\epsilon$ to be a positive real number. Let $\gamma$ be a path starting from $P$ and ending at 0, which is the limit of $\gamma_\epsilon$ when $\epsilon$ tends to zero. We define also $\sigma_\epsilon$ to be a loop that starts and ends at $\epsilon$ and goes around 0 once in a counterclockwise direction along a circle of radius $\epsilon$. We can deform $\sigma$ homotopic in $U - \{0\}$ to $\gamma_\epsilon \sigma_\epsilon \gamma_\epsilon^{-1}$. Note that iterated integrals are invariant with respect to homotopic deformation of the path of integration (see theorem 1.2). We are going to use corollary 1.4 for the composition of the paths $\gamma_\epsilon \sigma_\epsilon \gamma_\epsilon^{-1}$.

**Lemma 2.1** With the above notation

$$\int_{\sigma_\epsilon} \frac{dz}{z} \circ \ldots \circ \frac{dz}{z} = \left(\frac{2\pi i}{r!}\right)^r,$$

where we iterate the form $dz/dz$ with itself $r$-times

**Proof.** We can take the following parametrization of $z$ along $\sigma_\epsilon$. Let $z = \epsilon e^{2\pi it}$ for $0 \leq t \leq 1$. Then $dz/z = 2\pi idt$. Therefore,

$$\int_{\sigma_\epsilon} \frac{dz}{z} \circ \ldots \circ \frac{dz}{z} = (2\pi i)^r \int_0^1 dt \circ \ldots \circ dt = \left(\frac{2\pi i}{r!}\right)^r.$$

**Corollary 2.2** If $\omega_1, \ldots, \omega_r$ are holomorphic forms on $U - \{0\}$ with simple poles at 0 then

$$\int_{\sigma_\epsilon} \omega_1 \circ \ldots \circ \omega_r = \frac{1}{r!} \prod_{i=1}^r \text{Res}_0 \omega_i.$$
Lemma 2.3 Consider an iterated integral over $\sigma_{\epsilon}$ of the differential forms

$$\frac{dz}{z}, f_1 dz, \ldots, f_n dz$$

in various order with possible repetition, so that not all of them need to appear in the integral. If at least one of the holomorphic differentials $f_1 dz, \ldots, f_n dz$ on $U$ is present in the integral then the limit of the iterated integral, as $\epsilon$ goes to 0, is 0.

Proof. Expand the holomorphic differential forms around $z = 0$. Consider the following parametrization of the variable $z$ along the path $\sigma_{\epsilon}$:

$$z = \epsilon e^{2\pi it}$$

for $0 \leq t \leq 1$. It is enough to prove the statement of the lemma for $f_i(z) = z^{n_i}$. (The general statement will follow since we have an uniform convergence of the power series of $f_i$ on compact subsets of $U$.) Note that $dz/z = 2\pi i dt$ and $z^n dz = 2\pi i \epsilon^{n+1} e^{2\pi i(n+1)t} dt$ for some $n \geq 0$. Consider the parametrization in terms of $t$ for $0 \leq t \leq s$, where $s$ is close to 1. Then the iterated integral over $0 \leq t \leq s$ becomes

$$Ce^N e^{2\pi i N s} g(s),$$

where $C$ is a constant $N > 0$ and $g(t)$ is a polynomial in $t$. So the limit as $s$ approaches 1 will be $Ce^N g(1)$. So the iterated integral has value $Ce^N g(1)$. Finally, this value approaches zero as $\epsilon$ tends to zero, which proves the lemma.

Lemma 2.4 Let $f_1 dz, \ldots, f_n dz$ be holomorphic forms on $U - \{0\}$. Let $f_i$ and $f_{i+1}$ be also holomorphic at 0. Denote by $\omega^\sigma r$-fold iteration of $\omega$ Then

$$\lim_{\epsilon \to 0} \sum_{j=0}^{r} \int_{\gamma_{\epsilon}} f_1 dz \circ \ldots \circ f_i dz \circ \left( \frac{dz}{z} \right)^{\circ j} \int_{\gamma_{\epsilon}} \left( \frac{dz}{z} \right)^{\circ (r-j)} \circ f_{i+1} dz \circ \ldots \circ f_n dz = 0.$$

The limit is still zero in the cases when the set $\{ f_1 dz, \ldots, f_i dz \}$ is empty and/or the set $\{ f_{i+1} dz, \ldots, f_n dz \}$ is empty.

Proof. Consider the integral

$$\int_{\gamma_{\epsilon}} f_1 dz \circ \ldots \circ f_i dz$$

as a function of $\epsilon$. It is a sum of terms of the type a constant times $\epsilon^k \log^l(\epsilon)$. Since $f_i$ is holomorphic, we have that if $l \geq 0$ then $k \geq 0$. Also if $l > 0$ then $k > 0$, because the integral is convergent for $\epsilon = 0$. When $\epsilon$ tends to zero $\epsilon^k \log^l(\epsilon)$ tend to zero. Now, consider the integral

$$\int_{\gamma_{\epsilon}} f_1 dz \circ \ldots \circ f_i dz \circ \left( \frac{dz}{z} \right)^{\circ j}.$$

As a function of $\epsilon$ it is a sum of terms of the type constant times $\epsilon^k \log^{l+j}(\epsilon)$ for $k \geq 0$ and $l \geq 0$. From the previous considerations if $l > 0$ then $k > 0$ and the term tends to zero as $\epsilon$ approaches zero. Thus, the only term that does not tend to zero as $\epsilon$ approaches zero is the constant term of

$$\int_{\gamma_{\epsilon}} f_1 dz \circ \ldots \circ f_i dz$$
\[
\lim_{\varepsilon \to 0} \int_{\gamma} f_1dz \circ \ldots \circ f_idz \circ \left(\frac{dz}{z}\right)^{oj} - \frac{1}{j!} \log^j(\varepsilon) \int_{\gamma} f_1dz \circ \ldots \circ f_idz = 0.
\]

For the other integral in this lemma we use
\[
\int_{\gamma^{-1}} \left(\frac{dz}{z}\right)^{(r-j)} \circ f_{i+1}dz \circ \ldots \circ f_ndz = (-1)^{n-i+r-j} \int_{\gamma} f_ndz \circ \ldots \circ f_{i+1}dz \circ \left(\frac{dz}{z}\right)^{(r-j)}.
\]

Using the same arguments as in the beginning of the proof we find that
\[
\lim_{\varepsilon \to 0} \int_{\gamma} f_ndz \circ \ldots \circ f_{i+1}dz \circ \left(\frac{dz}{z}\right)^{(r-j)} - \frac{1}{(r-j)!} \log^{r-j}(\varepsilon) \int_{\gamma} f_ndz \circ \ldots \circ f_{i+1}dz = 0.
\]

Note also that
\[
(-1)^{n-i+r-j} \int_{\gamma} f_ndz \circ \ldots \circ f_{i+1}dz = (-1)^{r-j} \int_{\gamma^{-1}} f_{i+1}dz \circ \ldots \circ f_ndz.
\]

Therefore,
\[
\lim_{\varepsilon \to 0} \sum_{j=0}^{r} \int_{\gamma} f_1dz \circ \ldots \circ f_jdz \circ \left(\frac{dz}{z}\right)^{oj} \int_{\gamma^{-1}} \left(\frac{dz}{z}\right)^{(r-j)} \circ f_{i+1}dz \circ \ldots \circ f_ndz = \int_{\gamma} f_1dz \circ \ldots \circ f_jdz \int_{\gamma^{-1}} f_{i+1}dz \circ \ldots \circ f_ndz \sum_{j=0}^{r} \lim_{\varepsilon \to 0} \frac{1}{j!} (-1)^{r-j} \frac{1}{(r-j)!} \log^{r-j}(\varepsilon).
\]

Finally, we have
\[
\sum_{j=0}^{r} \frac{1}{j!} \log^j(\varepsilon) (-1)^{r-j} \frac{1}{(r-j)!} \log^{r-j}(\varepsilon) = 0,
\]

using binomial coefficients, after multiplying by \(r!\). Thus, the limit in the lemma is zero.

Now we are ready to give the general algorithm for expressing an iterated integral over a loop \(\sigma\) around 0 in terms of integrals over a path \(\gamma\) starting at \(P\) and ending at 0 and residues at \(z = 0\).

**Theorem 2.5** Let \(\omega_1, \ldots, \omega_n\) be holomorphic forms on \(U\). Let \(i_1, \ldots, i_m\) be integers such that
\[
0 \leq i_1 < i_2 < \ldots < i_m \leq n.
\]

Let \(j_1, \ldots, j_m\) be positive integers. Then
\[
\int_{\sigma} \omega_1 \circ \ldots \circ \omega_{i_1} \circ \left(\frac{dz}{z}\right)^{oj_1} \circ \omega_{i_1+1} \circ \ldots \circ \omega_{i_2} \circ \left(\frac{dz}{z}\right)^{oj_2} \circ \omega_{i_2+1} \circ \ldots \circ \omega_{i_m} \circ \left(\frac{dz}{z}\right)^{oj_m} \circ \omega_{i_m+1} \circ \ldots \circ \omega_n = \sum_{k=1}^{m} \frac{(2\pi i)j_k}{j_k!} \int_{\gamma} \omega_1 \circ \ldots \circ \omega_{i_1} \circ \left(\frac{dz}{z}\right)^{oj_k} \circ \omega_{i_k+1} \circ \ldots \circ \omega_{i_k+1} \circ \ldots \circ \omega_{i_m} \circ \left(\frac{dz}{z}\right)^{oj_m} \circ \omega_{i_m+1} \circ \ldots \circ \omega_n.
\]
Proof. First we use that $\sigma$ is homotopic to $\gamma_\epsilon \sigma_\epsilon \gamma_\epsilon^{-1}$. Then we use the formula for composition of paths in corollary 1.4 expressing the integral over $\sigma$ in terms of a sum of products of an integral over $\gamma_\epsilon$, an integral over $\sigma_\epsilon$ and an integral over $\gamma_\epsilon^{-1}$. Consider the portion of this sum, where there are no differential forms integrated over $\sigma_\epsilon$. The sum of all such terms is zero because by corollary 1.4 this is the same as the integral over $\gamma_\epsilon \gamma_\epsilon^{-1}$, which is homotopic to the constant loop at $P$.

Next, we examine what possible iterated integrals over $\sigma_\epsilon$ we can have. If we have a holomorphic differential form at zero in the iterated integral over $\sigma_\epsilon$ then, by lemma 2.3 the value of the integral tends to zero as $\epsilon$ approaches zero. Therefore, it is enough to consider only iterations of $dz/z$ over $\sigma_\epsilon$.

Consider iterated integrals of $dz/z$ over $\sigma_\epsilon$. If the corresponding iterated integral over $\gamma$ ends with $dz/z$ and/or the corresponding iterated integral over $\gamma^{-1}$ starts with $dz/z$ then by lemma 2.3 the sum of all such integrals tends to zero as $\epsilon$ approaches zero.

Therefore, the only terms we have to consider are the ones in the theorem.

2.2 Generating series of iterated integrals over a loop around a pole

Consider $n$ differential 1-forms of with simple poles $\omega_1, \ldots, \omega_n$, defined on an open and simply connected set $U \subset \mathbb{C}$. Consider the differential equation (1.1)

$$dF = F \sum_{i=1}^{n} A_i \omega_i$$

where $A_1, \ldots, A_n$ are non-commuting formal variables. Let $Q \in U$ be a point where at least one of the differential forms has a pole. Let $P \in U$ be a point, which is not a pole for any of the differential forms. Consider a simple loop $\sigma$ with the following properties: it starts at $P$; it does not self-intersect; it bounds a region $V$ homeomorphic to a disk; and the only pole of the differential forms that lies inside $V$ is $Q$.

Let us simplify the solution $F_\sigma$, using theorem 2.5. Define $\gamma$ to be the path, starting at $P$ and ending at $Q$, that sits in the region $V$, bounded by $\sigma$. Using theorem 2.5 we can decompose $F_\sigma$. Let $F_{\gamma}^{\text{reg}}$ be regularization of $F_\gamma$, which contains only the iterated integrals over the path $\gamma$, whose iteration does not end with a differential form that has a pole at $P$. The series $F_\gamma$ contains the summand 1 and also all the iterated integrals mentioned above times the corresponding non-commuting variables. Similarly, $F_{\gamma^{-1}}^{\text{reg}}$ is regularization of $F_{\gamma^{-1}}$, which contains only the iterated integrals over the path $\gamma$, whose iteration does not start with a differential form that has a pole at $Q$, where $\gamma^{-1}$ is the reversed path of $\gamma$.

Lemma 2.6 With the notation in this subsection, we have

$$(F_{\gamma}^{\text{reg}})^{-1} = F_{\gamma^{-1}}^{\text{reg}}.$$ 

Proof. It follows from lemma 1.7.

We call $F_\sigma^{\text{Res}}$ the residual part of $F_\sigma$, defined by the portion of $F_\sigma$ that contains only the iterated integrals of differential forms that have a pole at $Q$. In particular, $F_\sigma^{\text{Res}}$ does contain the constant 1. Let

$$F_\sigma^{\text{Res}+} = -1 + F_\sigma^{\text{Res}}$$
be the residual part without the constant term 1. Recall, \( Q \) is a point, where at least one of the differential forms has a pole. Then we have the following version of theorem 2.5 in terms of generating series.

**Theorem 2.7** With the notation from this subsection, we have

\[
F_\sigma = 1 + F^\text{reg}_\gamma F^\text{Res}_{\gamma} F_{\gamma-1}^\text{reg}.
\]

**Proof.** From theorem 2.5 we have

\[
F_\sigma = F^\text{reg}_\gamma F^\text{Res}_{\gamma} F_{\gamma-1}^\text{reg}.
\]

We simplify the right hand side, using lemma 2.6

\[
F^\text{reg}_\gamma F^\text{reg}_{\gamma-1} = 1.
\]

\[
F^\text{reg}_\gamma F^\text{Res}_{\gamma} F^\text{reg}_{\gamma-1} = F_{\sigma} = F^\text{reg}_\gamma (1 + F^\text{Res}_{\gamma+}) F^\text{reg}_{\gamma-1} = 1 + F^\text{reg}_\gamma F^\text{Res}_{\gamma} F^\text{reg}_{\gamma-1}.
\]

### 2.3 Generating series over \( \alpha \) and \( \beta \) cycles of a Riemann surface

Let \( X \) be a Riemann surface of genus \( g \). Let \( \omega_1, \ldots, \omega_n \) be differential forms of the third kind on \( X \). Let also \( \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g \) be loops on \( X \) starting at \( P \), which do not pass through a pole of any of the differential forms, such that they generate the fundamental group \( \pi_1(X, P) \) with only one relation

\[
[\alpha_1, \beta_1] \ldots [\alpha_g, \beta_g] = 1.
\]

We used the notation \([\alpha, \beta] = \alpha \beta \alpha^{-1} \beta^{-1}\).

We are going to simplify

\[
F_{[\alpha, \beta]} = F_{\alpha} F_{\beta} F_{\alpha^{-1}} F_{\beta^{-1}}.
\]

Note that if we take the constant term 1 from \( F_{\beta} \) then we will have

\[
F_{\alpha} F_{\alpha^{-1}} F_{\beta^{-1}} = F_{\beta^{-1}},
\]

which follows from the homotopy invariance of \( F_{\alpha \alpha^{-1}} \) (theorem 1.2) In order to capture such a cancellation, we define for a path \( \gamma \)

\[
F^+_{\gamma} = F_{\gamma} - 1.
\]

**Lemma 2.8** With the above notation

\[
F_{[\alpha, \beta]} = 1 + F_{\beta}^+ F_{\alpha^{-1}}^+ - F_{\alpha}^+ F_{\beta^{-1}}^+ + F_{\alpha}^+ F_{\beta}^+ F_{\alpha^{-1}}^+ + F_{\beta}^+ F_{\alpha}^+ F_{\beta^{-1}}^+ + F_{\alpha}^+ F_{\beta}^+ F_{\alpha^{-1}}^+ F_{\beta^{-1}}^+.
\]
Proof. We are going to use many times $F_\gamma^+ = F_\gamma - 1$. We have

$$F_{[\alpha,\beta]} = F_\alpha F_\beta F_{\alpha^{-1}} F_{\beta^{-1}} = F_\alpha F_\beta^+ F_{\alpha^{-1}} F_{\beta^{-1}} + F_\alpha F_{\alpha^{-1}} F_{\beta^{-1}} =$$

$$= F_\alpha F_\beta^+ F_{\alpha^{-1}} F_{\beta^{-1}} + F_{\alpha^{-1}} F_{\beta^{-1}} = F_\alpha F_\beta^+ F_{\alpha^{-1}} F_{\beta^{-1}} + F_\alpha F_{\beta^{-1}} F_{\beta^{-1}} + F_{\alpha^{-1}} F_{\beta^{-1}} =$$

$$= F_\alpha F_\beta^+ F_{\alpha^{-1}} F_{\beta^{-1}} + F_\alpha F_{\beta^{-1}} F_{\beta^{-1}} - F_\alpha F_{\beta^{-1}} F_{\beta^{-1}} + F_{\beta^{-1}} =$$

$$= F_\alpha F_\beta^+ F_{\alpha^{-1}} F_{\beta^{-1}} + F_{\beta^{-1}} = (F_\alpha - F_\alpha + F_{\beta^{-1}} - 1) + 1 =$$

$$= F_\alpha F_\beta^+ F_{\alpha^{-1}} F_{\beta^{-1}} - F_\alpha F_{\beta^{-1}} + 1.$$

Also

$$F_\alpha F_\beta^+ F_{\alpha^{-1}} F_{\beta^{-1}}^+ = F_\alpha F_\beta^+ F_{\alpha^{-1}} F_{\beta^{-1}} + F_\alpha F_{\alpha^{-1}} F_{\beta^{-1}} =$$

$$= F_\alpha F_\beta^+ F_{\alpha^{-1}} F_{\beta^{-1}} + F_\alpha F_{\alpha^{-1}} F_{\beta^{-1}} + F_\beta^+ F_{\alpha^{-1}} F_{\beta^{-1}} + F_\beta^+ F_{\alpha^{-1}} F_{\beta^{-1}}.$$

From these two sequences of equalities the lemma follows.

2.4 Non-abelian reciprocity law

Here we keep the notation from the previous section. Let $\omega_1 \ldots \omega_n$ be differential forms of the third kind on a Riemann surface $X$. Let $Y$ be the open subset of $X$ obtained by removing the poles of $\omega_1 \ldots \omega_n$. Consider the differential equation \([\square]\) on $Y$

$$dF = F(\sum_{i=1}^{n} A_i \omega_i),$$

where $A_i$ for $i = 1, \ldots, n$ are non-commuting formal variables. Fix a point $P$ in $Y$. Let

$$\gamma: [0, 1] \to Y$$

be a piecewise smooth path that starts at $P$ and ends at $z$. Then the solution of the differential equation with initial condition $F(P) = 1$ is

$$F_\gamma = 1 + \sum_{i=1}^{n} A_i \int_0^1 \gamma^* \omega_i + \sum_{i,j=1}^{n} A_i A_j \int_0^1 \gamma^* \omega_i \circ \gamma^* \omega_j + \ldots.$$

If $\gamma_1$ and $\gamma_2$ are two paths such that the end point of $\gamma_1$ is the beginning point of $\gamma_2$. Let, $\gamma_1 \gamma_2$ be the composition of the two paths. Then, $F_{\gamma_1 \gamma_2} = F_{\gamma_1} F_{\gamma_2}$ (theorem \([\square]\)).

Consider a simple loop $\sigma_i$ in $Y$ with the following properties: it starts at $P$; it does not self-intersect; it bounds a open region $V_i$ homeomorphic to a punctured disk; and the only pole of the differential forms that lies inside the closure of $V_i$ is $P_i$. Next we define a path $\gamma_i$ with the following properties: $\gamma_i$ lies in the closure of $V_i$; $\gamma_i$ starts at $P$ and ends at $P_i$. We can choose the loops $\sigma_i$ so that in the counterclockwise we start with loops around points of the poles of $\omega_1$. Then it continues with loops around the points of the poles of $\omega_2$, which are not poles of $\omega_1$, then it continues with loops around the poles of $\omega_3$, which are not poles of $\omega_1$ or $\omega_2$ and so on. Call these loops $\sigma_1, \ldots, \sigma_N$. We assume that $\sigma_i$ bounds a disk - the closure of $V_i$, containing only one point $P_i$ from the poles of the differential forms. Let also $\alpha_i$ and $\beta_i$ be loops on $Y$ for $i = 1, \ldots, g$, where $g$ is the genus of $Y$. We can choose such that $\alpha_i$ and $\beta_i$ so that $\alpha_i$ and $\beta_i$ for $i = 1, \ldots, g$ and $\sigma_1, \ldots, \sigma_N$ generate $\pi_1(Y)$ and the only relation between them is

$$\sigma_1 \ldots \sigma_N [\alpha_1, \beta_1] \ldots [a_g, b_g] = 1.$$
We define a pro-unipotent tame symbol to be $F_\sigma$ where $\sigma$ is one of the loops that we have defined above. Note that we have made some choices of loops. If we multiply all the loops going in counterclockwise direction, then we will obtain a loop homotopic to the zero loop at $P$. This gives the global reciprocity law for pro-unipotent tame symbols after simplification that uses section 1. We can write this reciprocity law in the following way. Let $\sigma_1, \ldots, \sigma_N$ in this order be the above loops counted in counterclockwise direction. Consider $F_\sigma$ as an element of the formal power series $\mathbb{C} << A_1, \ldots, A_n >>$.

In the formulation of the pro-unipotent reciprocity law, we are going to use the notation from this section and from subsections 2.2 and 2.3.

**Theorem 2.9** The non-abelian reciprocity law on a Riemann surface is

$$(\prod_{i=1}^{N} 1 + F_{\gamma_i}^{\text{reg}} F_{\gamma_i}^{\text{Res}} + F_{\gamma_i}^{\text{reg}}) \times \left( \prod_{j=1}^{g} 1 + F_\beta^+ F_\alpha^+ - F_\beta^- F_\alpha^- + F_\alpha^+ F_\beta^+ F_\alpha^- F_\beta^- + F_\beta^+ F_\alpha^- F_\beta^- F_\alpha^+ + F_\alpha^+ F_\beta^+ F_\alpha^- F_\beta^- \right) = 1.$$  

**Proof.** For the solutions of this differential equation we have that for two loops $\sigma$ and $\tau$ starting at the same point $P$ we have that $F_\sigma F_\tau = F_{\sigma\tau}$ (see theorem 1.3). The composition $$\sigma_1 \ldots \sigma_N [\alpha_1, \beta_1] \ldots [\alpha_g, \beta_g]$$ is homotopic to the trivial loop at $P$. Therefore,

$$F_{\sigma_1} \ldots F_{\sigma_N} F_{[\alpha_1, \beta_1]} \ldots F_{[\alpha_g, \beta_g]} = 1.$$  

Using lemma 2.6 in subsection 2.2, we have

$$F_{\sigma_i} = 1 + F_{\gamma_i}^{\text{reg}} F_{\gamma_i}^{\text{Res}} F_{\gamma_i}^{\text{reg}}.$$  

From theorem 2.7 from subsection 2.3, we have

$$F_{[\alpha_j, \beta_j]} = 1 + F_\beta^+ F_\alpha^+ - F_\beta^- F_\alpha^- + F_\alpha^+ F_\beta^+ F_\alpha^- F_\beta^- + F_\beta^+ F_\alpha^- F_\beta^- F_\alpha^+ + F_\alpha^+ F_\beta^+ F_\alpha^- F_\beta^-.$$  

The following two lemmas are useful for explicit computations, which we are going to consider in section 4.

**Lemma 2.10** The coefficient next to the linear terms in the formal variables in

$$\prod_{i=1}^{N} F_{\sigma_i}$$

is zero.

**Proof.** It is enough to prove the lemma for the coefficient $A_1$. The coefficient next to $A_1$ is

$$\sum_{i=1}^{N} \int_{\sigma_i} \omega_1.$$  

The sum is zero because it is equal to the sum of the residues of $\omega_1$.

**Lemma 2.11** The coefficient next to $A$, $B$ or $C$ in

$$F_{[\alpha_l, \beta_l]}$$

is zero for every $l = 1, \ldots, g$.

**Proof.** From lemma 2.8 we see that there is no term in degree 1 in the formal variables.
3 Explicit formulas for reciprocity laws

3.1 Classical reciprocity laws

Let $\omega_1$ and $\omega_2$ be differential forms of the third kind on a Riemann surface $X$ of genus at least $g \geq 1$. Consider the differential equation

$$dF = F(A\omega_1 + B\omega_2),$$

(see the differential equation (1.1)) where $A$ and $B$ are non-commuting formal variables. Let $P$ be point on $X$, which is not a pole for $\omega_1$ and $\omega_2$. Assume that there are no common poles between $\omega_1$ and $\omega_2$.

Let $\gamma$ is a path starting at $P$ and ending at $Q$. Let $F_\gamma = F(Q)$ be the solution of the differential equation with initial conditions $F(P) = 1$ solved along the path $\gamma$ and $F(Q)$ is the evaluation of that solution at the point $Q$.

Let $P_1, \ldots, P_p$ be the poles of $\omega_1$. Let $Q_1, \ldots, Q_q$ be the poles of $\omega_2$. Let $Y = X - \{P_1, \ldots, P_p, Q_1, \ldots, Q_q\}$. Consider a simple loop $\sigma_{P_i}$ in $Y$ with the following properties: it starts at $P$; it does not self-intersect; it bounds a open region $V_{P_i}$ homeomorphic to a punctured disk; and the only pole of the differential forms that lies inside the closure of $V_{P_i}$ is $P_i$. We can choose these loops so that in the counterclockwise we start with loops $\sigma_{P_1}, \ldots, \sigma_{P_p}$ around the poles of $\omega_1$ in this order, followed by the loops $\sigma_{Q_1}, \ldots, \sigma_{Q_q}$ around the poles of $\omega_2$. Let also $\alpha_l$ and $\beta_l$ be loops on $Y$ for $l = 1, \ldots, g$, where $g$ is the genus of $Y$. We can choose $\alpha_l$ and $\beta_l$ so that $\alpha_l$ and $\beta_l$ for $l = 1, \ldots, g$ and $\sigma_{P_1}, \ldots, \sigma_{P_p}, \sigma_{Q_1}, \ldots, \sigma_{Q_q}$ generate $\pi_1(Y, P)$ and the only relation between them is

$$\sigma_{P_1} \ldots \sigma_{P_p} \sigma_{Q_1} \ldots \sigma_{Q_q} [\alpha_1, \beta_1] \ldots [\alpha_g, \beta_g] = 1.$$

Let $\gamma_{P_i}$ be a path from $P$ to $P_i$ that lies inside the disk bound by $\sigma_{P_i}$. Similarly, let $\gamma_{Q_j}$ be a path from $P$ to $Q_j$ that lies inside the disk bound by $\sigma_{Q_j}$.

Let $\sigma$ be any of the above loops $\sigma_{P_i}$, $\sigma_{Q_j}$, $\alpha_k$ or $b_k$. Consider the tame symbol $F_\sigma$.

Consider the coefficients contributing to $AB$ in the reciprocity law associated to the differential equation

$$dF = F(A\omega_1 + B\omega_2).$$

**Theorem 3.1** Let $\omega_1$ and $\omega_2$ be two differential forms of the third kind without common poles. Then with the above notation we have a reciprocity law for differential forms of the third kind

$$\sum_{i=1}^p \text{Res}_{P_i} \omega_1 \int_{\gamma_{P_i}} \omega_2 + \sum_{j=1}^q \text{Res}_{Q_j} \omega_2 \int_{\gamma_{Q_j}} \omega_1 + \sum_{k=1}^g (\int_{\alpha_k} \omega_1 \int_{\beta_k} \omega_2 - \int_{\beta_k} \omega_1 \int_{\alpha_k} \omega_2) = 0.$$

**Proof.** We have that

$$F_{\sigma_{P_1} \ldots \sigma_{P_p} \sigma_{Q_1} \ldots \sigma_{Q_q} [\alpha_1, \beta_1] \ldots [\alpha_g, \beta_g]} = 1.$$

From lemma [2.10] and lemma [2.11] we have that there is no linear terms in $A$ or $B$ in $F_{\sigma_{P_1} \ldots \sigma_{P_p} \sigma_{Q_1} \ldots \sigma_{Q_q}}$ and in $F_{[\alpha_1, \beta_1] \ldots [\alpha_g, \beta_g]}$. Using theorem [2.9], lemma [2.8] and lemma [2.6]
we obtain that the coefficient next to $AB$ in $F_{\sigma_{P_i}}$ is

$$Res_{P_i}\omega_1 \int_{\gamma_{P_i}^{-1}} \omega_2.$$ 

Similarly, the coefficient next to $AB$ in $F_{\sigma_{Q_j}}$ is

$$Res_{Q_j}\omega_2 \int_{\gamma_{Q_j}} \omega_1.$$ 

And finally, the coefficient next to $AB$ in $F_{[\alpha_k,\beta_k]}$ is

$$\int_{\alpha_k} \omega_1 \int_{\beta_k} \omega_2 - \int_{\beta_k} \omega_1 \int_{\alpha_k} \omega_2.$$ 

When the two forms do not have a common pole we obtain a reciprocity law for differential forms of the third kind (see [GH]).

This also implies Weil reciprocity law. For similar approaches see [KL].

**Corollary 3.2** *(Weil reciprocity)* With the above notation, consider $\omega_1 = df/f$ and $\omega_2 = dg/g$, where $f$ and $g$ are meromorphic functions on the Riemann surface with disjoint divisors. Let the corresponding divisors $(f)$ and $(g)$ be

$$(f) = \sum_{i=1}^{p} a_i P_i$$ 

and

$$(g) = \sum_{j=1}^{q} b_j Q_j.$$ 

Then

$$\prod_{i=1}^{p} g(P_i)^{-a_i} \prod_{j=1}^{q} f(Q_j)^{b_j} = 1.$$ 

**Proof.** Consider the above theorem with $\omega_1 = df/f$ and $\omega_2 = dg/g$ and exponentiate.

### 3.2 Non-abelian reciprocity laws involving three differential forms

Consider the differential equation (1.1) in the following case

$$dF = F(A\omega_1 + B\omega_2 + C\omega_3),$$

where $A$, $B$ and $C$ are non-commuting formal variables. Assume $\omega_1$, $\omega_2$ and $\omega_3$ have distinct poles. Let $P$ be point on $X$, which is not a pole for $\omega_1$, $\omega_2$ or $\omega_3$.

Let $\gamma$ is a path starting at $P$ and ending at $Q$. Let $F_\gamma = F(Q)$ be the solution of the differential equation with initial conditions $F(P) = 1$ solved along the path $\gamma$ and $F(Q)$ is the evaluation of that solution at the point $Q$.

Let $P_1,\ldots,P_p$ be the poles of $\omega_1$, $Q_1,\ldots,Q_q$ be the poles of $\omega_2$ and $R_1,\ldots,R_r$ be the poles of $\omega_3$. Let $Y = X - \{P_1,\ldots,P_p,Q_1,\ldots,Q_q,R_1,\ldots,R_r\}$. Consider a simple
Similarly, let \( \sigma \) be a path from \( P \) to \( P_i \) that lies inside the punctured disk \( V_{P_i} \) bound by \( \sigma_{P_i} \). Similarly, let \( \gamma_{Q_j} \) be a path from \( P \) to \( Q_j \) that lies inside the punctured disk \( V_{Q_j} \) bound by \( \sigma_{Q_j} \) and let \( \gamma_{R_k} \) be a path from \( P \) to \( R_k \) that lies inside the punctured disk \( V_{R_k} \) bound by \( \sigma_{R_k} \).

Let \( \sigma \) be any of the above loops \( \sigma_{P_i}, \sigma_{Q_j}, \sigma_{R_k} \) \( \alpha_l \) or \( \beta_l \). Consider the tame symbol \( F_\sigma \). For them we have the following reciprocity.

**Theorem 3.3** Using the above notation, for three differential forms of the third kind, with disjoint poles, we have

\[
\sum_{i=1}^{p} \text{Res}_{P_i} \omega_1 \int_{\gamma_{P_i}^{-1}} 2 \circ \omega_3 + \sum_{j=1}^{q} \text{Res}_{Q_j} \omega_2 \int_{\gamma_{Q_j}^{-1}} \omega_1 \int_{\gamma_{Q_j}} \omega_3 + \sum_{k=1}^{r} \text{Res}_{R_k} \omega_3 \int_{\gamma_{R_k}} \omega_1 \circ \omega_2 + \sum_{l=1}^{g} (\int_{\alpha_l} \omega_1 \circ \omega_2 \int_{\beta_l} \omega_3 - \int_{\beta_l} \omega_1 \circ \omega_2 \int_{\alpha_l} \omega_3 + \int_{\alpha_l} \omega_3 \circ \omega_2 \int_{\beta_l} \omega_1 - \int_{\alpha_l} \omega_1 \int_{\beta_l} \omega_2 \int_{\alpha_l} \omega_3 + \int_{\beta_l} \omega_1 \int_{\alpha_l} \omega_2 \int_{\beta_l} \omega_3) = 0.
\]

**Proof.**

Consider only the coefficients of the tame symbol \( F_\sigma \) next to the \( A, B, C, AB, BC \) and \( ABC \) terms in the non-commuting power series in \( A, B \), and \( C \).

The coefficient next to \( ABC \) in \( F_{\sigma_{P_i}} \) is

\[
\int_{\sigma_{P_i}} \omega_1 \circ \omega_2 \circ \omega_3 = \text{Res}_{P_i} \omega_1 \int_{\gamma_{P_i}^{-1}} \omega_2 \circ \omega_3.
\]

Let \( \sigma_1 = \sigma_{P_1} \sigma_{P_2} \ldots \sigma_{P_p} \) be a product of loops that go around the poles of \( \omega_1 \). Then the coefficient next to \( ABC \) in \( F_{\sigma_1} = F_{\sigma_{P_1}} \ldots F_{\sigma_{P_p}} \) is

\[
\int_{\sigma_1} \omega_1 \circ \omega_2 \circ \omega_3 = \sum_{i=1}^{p} \text{Res}_{P_i} \omega_1 \int_{\gamma_{P_i}^{-1}} \omega_2 \circ \omega_3.
\]

Also the coefficient next to \( A \) in \( F_{\sigma_1} \) is equal to the sum of the residues of \( \omega_1 \), which is zero. Also the coefficient next to \( B \) and \( C \) in \( F_{\sigma_1} \) is zero because the disks that \( \sigma_1 \) bounds do not contain any of the poles of \( \omega_2 \) or \( \omega_3 \).
Let $\sigma_3 = \sigma_{R_1}\sigma_{R_2} \ldots \sigma_{R_r}$ be the product of the loops around the poles of $\omega_3$. Consider the coefficients next to $A$, $B$ and $C$ in $F_{\sigma_3}$. The coefficient next to $A$ and $B$ is zero because the loop does not bound any poles of $\omega_1$ or $\omega_2$. The coefficient next to $B$ is zero because the sum of the residues of $\omega_3$ is zero. Similarly to the commutation in the previous paragraph the coefficient next to $ABC$ in $F_{\sigma_3}$ is

$$\int_{\sigma_3} \omega_1 \circ \omega_2 \circ \omega_3 = \sum_{k=1}^{r} \text{Res}_{\gamma_k} \omega_3 \int_{\gamma_k} \omega_1 \circ \omega_2.$$

Let $\sigma_2 = \sigma_{Q_1}\sigma_{Q_2} \ldots \sigma_{Q_q}$ be the product of the loops around the poles of $\omega_2$. Consider the coefficients next to $A$, $B$ and $C$ in $F_{\sigma_2}$. The coefficient next to $A$ and $C$ is zero because the loop does not bound any poles of $\omega_1$ or $\omega_3$. The coefficient next to $B$ is zero because the sum of the residues of $\omega_2$ is zero.

The coefficient next to $ABC$ in $F_{\sigma_2}$ is

$$\int_{\sigma_2} \omega_1 \circ \omega_2 \circ \omega_3 = \sum_{j=1}^{q} \text{Res}_{\gamma_j} \omega_2 \int_{\gamma_j} \omega_1 \int_{\gamma_j} \omega_3.$$

Therefore, the coefficient next to $ABC$ in $F_{\sigma_1\sigma_2\sigma_3}$ is

$$\sum_{i=1}^{p} \text{Res}_{P_i} \omega_1 \int_{\gamma_{P_i}} \omega_2 \circ \omega_3 + \sum_{j=1}^{q} \text{Res}_{Q_j} \omega_2 \int_{\gamma_{Q_j}} \omega_1 \int_{\gamma_{Q_j}} \omega_3 + \sum_{k=1}^{r} \text{Res}_{R_k} \omega_3 \int_{\gamma_{R_k}} \omega_1 \circ \omega_2.$$

Consider the coefficients next $A$, $B$, $C$ and $ABC$ of $F_{[\alpha_i, \beta_j]}$. Using lemma 2.8, we notice that the coefficient next to $A$, $B$ or $C$ in $F_{[\alpha_i, \beta_j]}$ is zero. Also, the linear coefficients next to $F_{\sigma_1}$, $F_{\sigma_2}$ and $F_{\sigma_3}$ are zero. Therefore, there will be any contribution from the quadratic terms in $A$, $B$ and $C$ in the series $F_{[\alpha_i, \beta_j]}$, since it has to be multiplied by a linear term, which is zero. From the reciprocity law, theorem 2.9, we have consider only the coefficient next to $ABC$ of. We only need to consider the sum of the coefficients next to $ABC$ in $F_{[\alpha_i, \beta_j]}$, $F_{\sigma_1}$, $F_{\sigma_2}$ and $F_{\sigma_3}$, because the linear terms are zero. By lemma 2.11 (see also theorem 2.9), we have that the coefficient next to $ABC$ in $F_{[\alpha_i, \beta_j]}$ is

$$\int_{\beta_i} \omega_1 \int_{\alpha_i^{-1}} \omega_2 \circ \omega_3 + \int_{\beta_i} \omega_1 \circ \omega_2 \int_{\alpha_i^{-1}} \omega_3 -$$

$$- \int_{\alpha_i} \omega_1 \int_{\beta_i^{-1}} \omega_2 \circ \omega_3 - \int_{\alpha_i} \omega_1 \circ \omega_2 \int_{\beta_i^{-1}} \omega_3 +$$

$$+ \int_{\alpha_i} \omega_1 \int_{\beta_i^{-1}} \omega_2 \int_{\alpha_i^{-1}} \omega_3 + \int_{\beta_i} \omega_1 \int_{\alpha_i^{-1}} \omega_2 \int_{\beta_i^{-1}} \omega_3.$$

Using lemma 1.7 for reversing a path of the above iterated integrals, we finish the proof of the theorem.

**Remark 3.4** We need the last formula, for the coefficient next to $ABC$ in $F_{[\alpha_i, \beta_j]}$, in the paper on a refinement of the Parshin symbol for surfaces [H].

4 Manin’s non-commutative modular symbol, involving Eisenstein series

In this section we give an application of our reciprocity law to the non-commutative modular symbol that Manin has defined (see [M]). Before we can apply the reciprocity law, we make a generalization of Manin’s non-commutative modular symbol. We generalize it so that we can consider modular symbols involving Eisenstein series.
4.1 Construction and properties of the symbols

In this section we enrich Manin’s non-commutative modular symbol so that it includes Eisenstein series, not only cusp forms. This was a question raised by Glen Stevens.

Let \( \Gamma \) be an arithmetically defined, torsion free, discrete group, acting on the upper half plane \( \mathcal{H} \), whose quotient is a Riemann surface \( X \) without a divisor \( D \). (We allow compact quotients. That is \( D = 0 \).) We will consider loops on the modular curve \( X - D \).

Let \( f_1dz, \ldots, f_n dz \) be cusp or Eisenstein modular forms on \( \mathcal{H} \) with respect to \( \Gamma \). Consider the differential equation

\[
dJ(z) = J(z) \sum_{i=1}^{n} A_i f_i dz,
\]

where \( A_i \)'s are non-commuting formal variables. This is an equation of the type of equation (1.1), which Manin uses to define the non-commutative modular symbol \([M]\). Originally Manin considered only the case where all \( f_i \) are cup forms. Here we allow them to be either cusp forms or Eisenstein series.

Let \( P \) be the image of \( i = \sqrt{-1} \) in the projection \( pr : \mathcal{H} \rightarrow X - D \). That is,

\[
P = pr(\sqrt{-1}).
\]

Usually, by a modular form \( f(z) \) people denote a holomorphic function on the upper half plane that has 'good' transformation properties under the action of the arithmetic group \( \Gamma \). We are going to use a different trivialization of the modular form. A modular form will be a holomorphic 1-form on the upper half plane that has a 'good' transformation properties under the action of the group \( \Gamma \).

We are going to identify the upper half plane \( \mathcal{H} \) with \( SL_2(\mathbb{R})/SO_2(\mathbb{R}) \) by identifying \( i \) in the upper half plane \( \mathcal{H} \) with the projection of the identity element \( I \) in \( SL_2(\mathbb{R}) \). Let \( P \in X - D \) be the projection of \( i \) from the upper half plane. Then

\[
\pi_1(X - D, P) = \Gamma.
\]

Let \( Q_1, \ldots, Q_N \) be the points of the divisor \( D \). They are called cusp points. Consider the punctured Riemann surface \( X \). We can slice it to \((N - 1 + 4g)\)-gon, so that the points of the divisor \( D \) are vertices of this polygon.

Denote by \( \gamma_j \) a path, which is a piece of one of the sides of this polygon, that starts at an orbit of \( P \) and ends at a vertex with label \( Q_j \). Denote by \( \tilde{\gamma}_j^{-1} \) a path, which is a piece of one of the sides of this polygon, that starts at \( Q_j \) and ends at the other orbit of \( P \) compared to \( \gamma_j \).

Let \( \omega_1, \ldots, \omega_n \) be modular 1-forms. Consider a neighborhood \( U \) of a cusp point \( Q = Q_j \) on the modular curve \( X \) such that the modular 1-forms \( \omega_1, \ldots, \omega_n \) have trivialization on \( U \). Let \( Q' \) be in \( U \) and \( Q' \neq Q \). As before, let \( \sigma = \sigma_j \) be a simple loop on the modular curve \( X - D \) that starts at \( P \) and bounds a disc, which contains \( Q \). Let \( Q_\epsilon \) be a point inside \( U \) in an \( \epsilon \)-neighborhood of \( Q \). The loop \( \sigma \) is homotopic to

\[
\gamma' \gamma \epsilon \sigma \epsilon^{-1} \gamma'^{-1},
\]

17
where $\gamma'$ is a path from $P$ to $Q'$, $\gamma_e$ a path from $Q'$ to $Q_e$ and $\sigma_e$ a loop, starting at $Q_e$ around $Q$ in an $e$-neighborhood of $Q$. Let $\tilde{\sigma} = \tilde{si_j}$ be a lift of $\sigma$ on the upper half plane. Define lifts $\tilde{\gamma}'$, $\tilde{\gamma}_e$, $\tilde{\sigma}_e$, $\tilde{\gamma}^{(-1)}$, $\tilde{\gamma}^{(-1)}$ from the modular curve $X - D$ to the upper half plane of $\gamma'$, $\gamma_e$, $\sigma_e$, $\gamma_e^{-1}$ and $\gamma'^{-1}$, respectively. We also require that the ending point of a lift path is the staring point of the next path.

**Definition 4.1** Given modular 1-forms $(\omega_1, \ldots, \omega_n) = \Omega$ and a path $\gamma$ in the upper half plane $\mathcal{H}$, define

$$J_{\gamma}(\Omega)$$

to be the generating series of iterated integrals of $\omega_1, \ldots, \omega_n$ over the path $\gamma$, if the modular forms do not have poles at the starting point and at the ending point. If the starting point of $\gamma$ is a pole of any of the modular 1-forms, then we define

$$J_{\gamma}^{\text{Reg}}(\Omega)$$

to be the generating series of iterated integrals whose first 1-form in the iteration has no pole at the starting point of $\gamma$. Similarly, if the ending point of $\gamma$ is a pole of any of the modular 1-forms, then we define

$$J_{\gamma}^{\text{Reg}}(\Omega)$$

to be the generating series of iterated integrals whose last 1-form in the iteration has no pole at the ending point of $\gamma$. We define

$$J_{\gamma}^{\text{Res}}(\Omega)$$

where $Q$ is a cusp point, to be exponential of the sum of the residues of the modular 1-forms at the point $Q$ on the modular curve, computed in a small neighborhood of $Q$. And finally,

$$J_{\gamma}^{\text{Res}+}(\Omega) = -1 + J_{\gamma}^{\text{Res}}(\Omega).$$

We have

**Theorem 4.2** With the notation from this subsection, we have

$$J_{\tilde{\gamma}}(\Omega) = 1 + J_{\tilde{\gamma}}^{\text{Reg}}(\Omega) J_{\tilde{\gamma}}^{\text{Res}+}(\Omega) J_{\gamma^{(-1)}}^{\text{Reg}}(\Omega),$$

where $\gamma$ is a path starting at $P$ and ending at $Q$, $\tilde{\gamma}$ and $\tilde{\gamma}^{(-1)}$ are lifts of $\gamma$ and $\gamma^{-1}$, so that $\tilde{\gamma}$ starts at the initial point of $\tilde{\sigma}$ and ends at a cusp corresponding to $Q$ and $\tilde{\gamma}^{(-1)}$ at the ending point of $\gamma$ and ends at the ending point of $\tilde{\sigma}$.

**Proof.** We apply theorem 2.7 to the loop $\gamma_e\sigma_e\gamma_e^{-1}$. Then we use that $J_{\gamma'}(\Omega)J_{\gamma_e}(\Omega) = J_{\gamma'\gamma_e}(\Omega)$. The same equality hold for regularized generating series.

We have a non-abelian reciprocity law on the upper half plane involving holomorphic modular 1-forms, which are either cusp forms or Eisenstein series (as 1-forms). It is similar to the non-abelian reciprocity law on Riemann surface involving differential forms of he third kind. See theorem 2.9. In the following theorem we are going to omit $\Omega$ from the notation. That is, we are going to write $J_{\tilde{\gamma}}$ instead of $J_{\tilde{\gamma}}(\Omega)$.
Theorem 4.3 With the above notation, we have the following reciprocity law for Eisenstein and cusp forms

\[
\left( \prod_{j=1}^{M} 1 + J_{\tilde{\gamma}_j}^{\text{reg}} J_{Q_j}^{\text{Res}} J_{\tilde{\gamma}_j(-1)}^{\text{reg}} \right) = 1,
\]

where the composition of the paths \( \tilde{\gamma}_j \) and \( \tilde{\gamma}_j(-1) \) for all \( j \), namely,

\[
\prod_{j=1}^{M} \tilde{\gamma}_j \tilde{\gamma}_j(-1)
\]

is contractible and \( M = N + 4g \).

Definition 4.4 Let \( P \) and \( P' \) be in the orbit of \( i \) in the upper half plane under the action of the arithmetic group \( \Gamma \). Let \( Q \) and \( Q' \) be a cusp points. Let also \( \tilde{\gamma} \), \( \tilde{\gamma}'(-1) \) and \( \tilde{\alpha} \) be paths in the upper half plane that connect \( P \) with \( Q \), \( Q' \) with \( P \) and \( P \) with \( P' \), respectively. We define non-commutative modular symbols to be

\[
J_{\tilde{\gamma}}^{\text{reg}}(\Omega), \quad J_{\tilde{\gamma}'(-1)}^{\text{reg}}(\Omega), \quad J_{\tilde{\alpha}}(\Omega)
\]

and

\[
J_{Q}^{\text{Res}}(\Omega).
\]

The non-commutative symbol that Manin has defined can be recovered by considering the product

\[
J_{\tilde{\gamma}'(-1)}^{\text{reg}}(\Omega) J_{\tilde{\gamma}}^{\text{reg}}(\Omega),
\]

where the regularization is not needed for cusp forms.

4.2 Main Example

A very important example occurs, when we consider generating series of iterated integrals \( J \) of 1-forms \( f dz \) and \( dz \) for a torsion free arithmetic groups \( \Gamma \) commensurable to \( SL_2(\mathbb{Z}) \), where \( f dz \) is a cusp form.

Remark 4.5 Note that \( f dz \) is a cusp 1-form but \( f \) is not a cusp form. The poles of \( f \) cancel with the poles of \( dz \). In terms of algebraic geometry a cusp form is a section of a certain line bundle on the modular curve \( X \). Instead of considering local trivialization in terms of the rational functions \( O_X \), we consider local trivialization in terms of 1-forms on \( X, \Omega_X \). The cusp form \( F \) trivialized as a function on the upper half plane is related to the cusp form trivialized as a holomorphic 1-form \( f dz \) by

\[
F(x) = \int_{0}^{x} f dz
\]

for a point \( x \) in the chosen fundamental domain.

Definition 4.6 We denote by

\[
dz^{\circ n}
\]

the \( n \)-fold iteration of the form \( dz \).
With this notation we have the following theorem.

**Theorem 4.7** Let

\[
L(F, n_1, F, n_2, \ldots, F, n_k) := \int_{i\infty}^0 F(z)z^{n_1-1}dz \circ \ldots \circ F(z)z^{n_k-1}dz.
\]

Then

(a) \( L(F, n) = (n + 1)! \int_{i\infty}^0 f dz \circ dz^{on}; \)

(b) \( L(F, n_1, F, n_2, \ldots, F, n_k) = \prod_{j=1}^k (n_j + 1)! \int_{i\infty}^0 f dz \circ dz^{on_1} \circ \ldots \circ f dz \circ dz^{on_k}; \)

(c) \( J_{reg}^\gamma(fdz, dz) \) is the generating series of the \( L \)-functions in part (b).

**Proof.** The form \( dz \) on the upper half plane corresponds to a differential form of the third kind on the modular curve. Also, at the point 0 of the upper half plane, both \( f dz \) and \( dz \) are holomorphic. Thus, the regularization is well defined.

**Acknowledgments.** I would like to thank Yuri Manin for his inspiring talk on non-commutative modular symbol. Also, I would like to thank Alexander Goncharov, from whom I learned iterated integrals. And finally, I would like to thank Glen Stevens for raising the question about non-commutative modular symbols involving Eisenstein series.

I would like to thank Max Planck Insitut für Mathematik in Bonn, the University of Durham, Brandeis University and Universität Tübinbgen for the kind hospitality and the creative atmosphere. Also, I would like to thank the Arithmetic Algebraic Geometry Marie Curie Network for the financial support.

**References**

[BrMc] Brylinski, J.-L., McLaughlin, D.A., *Characteristic classes and multidimensional reciprocity laws*, Math. Res. Let., 3 (1996), 19-30.

[Ch] Chen, K.-T. Iterated path integrals, *Bull. AMS*, 83(1977), 831-879.

[DG] Deligne, P., Goncharov, A. B.: *Groupes fondamentaux motiviques de Tate mixte*, Ann. Sci. École Norm. Sup. (4) 38 (2005) no. 1, 1-56.

[G] Goncharov, A. B.: *Multiple polylogarithms and mixed Tate motives*, math.AG/0103059, 82 pages.

[GH] Griffiths P., Harris, J.: *Principles of algebraic geometry*, Pure and Applied Mathematics. Wiley-Interscience [John Wiley & Sons], New York, 1978. xii+813 pp.

[H] Horozov, I.: *A refinement of the Parshin symbol for surfaces*, preprint.

[Ka] Kato, K.: *Milnor K-theory and the Chow group of zero cycles*, Contemp. Math. 55, A.M.S. (1986), 241-253.
[Kh] Khovanskii, A. Logarithmic functional and the Weil reciprocity laws, Proceedings of the Waterloo Workshop on Computer Algebra, 2006, Ed. Kot-zscheas I, Zima E., World Scientific Pres, 85-108, 2007.

[M] Manin, Yu. I.: Iterated integrals of modular forms and noncommutative modular symbols, Algebraic geometry and number theory, 565-597, Progr. Math. 253, Brikhäuser Boston, Boston MA, 2006.

[P1] Parshin, A. N.: On a certain generalization of Jacobian manifold, (Russian) Izv. Acad. Nauk SSSR Ser. Mat. 30, 1966 175-182.

[P2] Parshin A.N.: Local class field theory, Trudy Mat. Inst. Steklov, vol. 165, 1984.

[P3] Parshin A.N.: Galois cohomology and Brauer group of local fields, Trudy Mat. Inst. Steklov, vol. 183, 1984.

[Q] Quillen, D.: On associated graded ring of a group ring, J. Algebra 10, 1968, p.411-418.