GVZ-GROUPS

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Abstract. A finite group $G$ is called a GVZ-group if every character $\chi \in \text{Irr}(G)$ vanishes on $G \setminus Z(\chi)$, and is called flat if every conjugacy class is a coset of some subgroup. We will show that these two notions coincide, thereby obtaining a character-free definition of GVZ-groups. We obtain several other characterizations of GVZ-groups, and then use a Taketa-type argument to prove that the nilpotence class of a GVZ-group (such groups are necessarily nilpotent) is bounded above by the number of distinct degrees of its irreducible characters.

1. Introduction

All groups to be considered are finite. In Problem #30 of Research problems and themes I of [1], Berkovich suggests the study of the $p$-groups $G$ such that $\chi(1)^2 = |G : Z(\chi)|$ for every character $\chi \in \text{Irr}(G)$. In [22] and [23], Nenciu studies groups that satisfy this property and a second property called nested. Nenciu defines a group $G$ to be a generalized VZ-group (or a GVZ-group) if it satisfies $\chi(1)^2 = |G : Z(\chi)|$ for every character $\chi \in \text{Irr}(G)$. Recall that if $\chi$ is a character of $G$, then the center (quasi-kernel) of $\chi$ is $Z(\chi) = \{g \in G \mid |\chi(g)| = \chi(1)\}$.

In [18], the second author proved a number of results regarding nested groups. In particular in Theorem 1.3 of [18], he gave a characterization of nested groups that was character-free. He also proved some results regarding nested GVZ-groups. In this paper, our goal is to obtain a number of results regarding GVZ-groups. In particular, we will deduce several new characterizations of GVZ-groups.

For our first characterization of GVZ-groups, we turn to the flat groups that were defined in [25]. In [25], a group $G$ is defined to be flat if every conjugacy class is a coset of a (necessarily normal) subgroup. In [25], the authors show that a flat nilpotent group is necessarily a GVZ-group (cf. [25, Theorem 4.2]). We will show that the converse of this statement is also true.

**Theorem A.** A group $G$ is a GVZ-group if and only if $G$ is flat.

In particular, since GVZ-groups are nilpotent ([22, Proposition 1.2]), this shows that flat groups are always nilpotent. Also, since the definition of flat does not mention characters, this gives a character-free description of GVZ-groups.

It has been observed that there is a parallelism between the irreducible characters of a group and the conjugacy classes of group. David Chillag has a nice exposition about these parallels in [4]. Note we have defined GVZ-groups in terms the irreducible characters of the group. It makes sense to ask if there is an equivalent condition in terms of the conjugacy classes of $G$. For our second equivalent
condition, we provide such an equivalence. To do this, we need to introduce some notation. Let $G$ be a group. Define $\gamma_G(g) = \{ [g,x] \mid x \in G \}$ for every element $g \in G$. We let $[g,G]$ denote the subgroup generated by $\gamma_G(g)$. We will prove that $[g,G]$ is normal in $G$, and it follows that $[g,G]$ is determined by the conjugacy class of $g$. With these definitions, we prove the following:

**Theorem B.** Let $G$ be a group. Then the following are equivalent:

1. $G$ is a GVZ-group.
2. $\gamma_G(g)$ is a group for every element $g \in G$.
3. $\gamma_G(g) = [g,G]$ for every element $g \in G$.

In Section 3, we will state the definition of fully-ramified characters. We will show that if $G$ is a GVZ-group, then every irreducible character is fully-ramified over the center of the character. We note that in the literature, a group is called *central type* if there is an irreducible character that is fully-ramified over the center. Central type groups have been studied extensively in the literature. We mention a few of the important papers: [6], [7], [8], and [9]. We believe that we have a new characterization of when an irreducible character is fully-ramified over the center, and so, we have a new characterization of central type groups. Due to the technicality in stating this condition, we postpone the statement of our condition until Section 3. Now, if $G$ is a GVZ-group, then $G/\ker(\chi)$ is central type for every nonlinear irreducible character $\chi \in \text{Irr}(G)$. Thus, we can use our condition to obtain another equivalent condition for GVZ-groups.

Since nilpotent groups are $M$-groups, Taketa’s theorem implies that if $G$ is a GVZ-group, then derived length of $G$ is bounded by the number distinct degrees of the irreducible characters of $G$. That is, if $dl(G)$ is the derived length of $G$ and $\text{cd}(G) = \{ \chi(1) \mid \chi \in \text{Irr}(G) \}$ is the set of irreducible character degrees, then $dl(G) \leq |\text{cd}(G)|$. In general, there is not a bound between the number of character degrees and the nilpotence class of a nilpotent group. In particular, all dihedral, semidihedral and generalized quaternion 2-groups have the character degree set $\{1,2\}$ and there exist examples of each of these groups with arbitrarily large nilpotence classes. In fact, there has been some research on which sets of character degrees bound the nilpotence class of a $p$-group. (See [12], [13], and [14].)

We will adapt the classical Taketa argument to show that the nilpotence class of a GVZ-group is bounded by the number of irreducible character degrees. If $G$ is a nilpotent group, then we write $c(G)$ for the nilpotence class of $G$.

**Theorem C.** If $G$ is a GVZ-group, then $c(G) \leq |\text{cd}(G)|$.

It is known that if $G$ is a nilpotent group, then $dl(G) \leq \log_2(c(G)) + 1$. Hence, Theorem C implies that if $G$ is a GVZ-group, then $dl(G) \leq \log_2(|\text{cd}(G)|) + 1$. It has been conjectured that if $G$ is a solvable group, then $dl(G)$ is bounded by a logarithmic function in $|\text{cd}(G)|$. (See [15], [20], and page 110 of [21].) Hence, it would seem that GVZ-groups would give positive evidence for such a conjecture. However, at this time, it is an open question as to whether or not there are GVZ-groups of arbitrarily large derived length. Despite the fact that all of the GVZ-groups we know of at this time have derived length 2, we feel it is likely the case that there are GVZ-groups of arbitrarily large derived length. In any case, for nilpotent groups, this reduces the problem of determining if the derived length is bounded by a logarithmic function the number of character degrees to the non GVZ-groups.
We close this paper with an application. On page 254 of [2], they define a group \( G \) to be a \( CM_n - \text{group} \) if every normal subgroup is the kernel of at most \( n \) irreducible characters of \( G \). We will provide a characterization of \( CM_p^0 - \text{groups} \) that are \( p \)-groups for a prime \( p \) in terms of GVZ-groups and the field of values of the characters. A group \( G \) is a \( CM \)-group if it is a \( CM_1 \)-group. We prove that \( G \) is a \( CM \)-group that is a 2-group if and only if \( G \) is a GVZ-group and all the irreducible characters of \( G \) are rational.

\[ \text{2. Flat elements} \]

We begin with a lemma that is motivated by Lemma 1 of [3], Proposition 3.1 of [5], Lemma 2.1 of [16], and Lemma 2.1 of [19]. We also refer the reader to Lemmas 2.1 and 2.2 of the first author’s expository paper [17].

**Lemma 2.1.** Let \( M \) be a normal subgroup of \( G \) and let \( g \in G \setminus M \). Then the following are equivalent:

1. \( g \) is conjugate to every element in \( gM \).
2. For every element \( z \in M \), there exists an element \( x \in G \) so that \([g, x] = z\).
3. \(|C_G(g)| = |C_{G/M}(gM)|\).
4. \( \chi(g) = 0 \) for every character \( \chi \in \text{Irr}(G \mid M)\).

**Proof.** We first show (1) and (2) are equivalent. Notice that if \( z \in M \), then \( g \) and \( gz \) are conjugate if and only if there exists \( x \in G \) so that \( g^x = gz \). However, we see that \( g^x = gz \) if and only if \([g, x] = g^{-1}g^x = g^{-1}gz = z\). Hence, \( g \) will be conjugate to every element in \( gM \) if and only if for every element \( z \in M \), there is an element \( x \in G \) so that \([g, x] = z\).

We next show that (1) and (3) are equivalent. Note that

\[
\text{cl}(g) \subseteq \bigcup_{x \in G} (gM)^x.
\]

We know that \(|G : C_g(g)| = |\text{cl}(g)|\). On the other hand, \(|\bigcup_{x \in G} (gM)^x|\) will equal the number of conjugates of \( gM \) times the size of \( M \). Thus, we have the equality \(|\bigcup_{x \in G} (gM)^x| = |G/M : C_{G/M}(gM)||M|\). It follows that \( \text{cl}(g) = \bigcup_{x \in G} (gM)^x \) if and only if \( |C_G(g)| = |C_{G/M}(gM)|\). On the other hand, \( \text{cl}(g) = \bigcup_{x \in G} (gM)^x \) if and only if \( g \) is conjugate to all elements in \( gM \). This implies that (1) and (3) are equivalent.

Now, we show (3) and (4) are equivalent. By the second Orthogonality relation, we have

\[
|C_G(g)| = \sum_{\chi \in \text{Irr}(G)} |\chi(g)|^2 = \sum_{\chi \in \text{Irr}(G/M)} |\chi(g)|^2 + \sum_{\chi \in \text{Irr}(G \setminus M)} |\chi(g)|^2,
\]

and

\[
|G_{G/M}(gM)| = \sum_{\chi \in \text{Irr}(G \setminus M)} |\chi(g)|^2.
\]

This implies that we have \( \sum_{\chi \in \text{Irr}(G \setminus M)} |\chi(g)|^2 = 0 \) if and only if the equality \( |C_G(g)| = |C_{G/M}(gM)| \) holds. Since \( |\chi(g)|^2 \) is a nonnegative real number for all \( \chi \in \text{Irr}(G \mid M) \), we conclude that \( \sum_{\chi \in \text{Irr}(G \mid M)} |\chi(g)|^2 = 0 \) if and only if \( \chi(g) = 0 \) for all \( \chi \in \text{Irr}(G \mid M) \). This shows that (3) and (4) are equivalent. \( \square \)
Let $G$ be a group. Following the introduction, define $\gamma_G(g) = \{[g, x] \mid x \in G\}$ for every element $g \in G$. We let $[g, G]$ denote the subgroup generated by $\gamma_G(g)$. (Note that this is not the convention taken in [25].) It is not difficult to see that if $\text{cl}_G(g) = gN$ for a subgroup $N$ of $G$, then $N$ is the subgroup $[g, G]$.

This next lemma is an easy proof of the well-known fact that $[g, G]$ is normal in $G$. We note that $\gamma_G(g)$ need not be a normal subset of $G$. To see this consider $\gamma_S(1, 2)$ and $\gamma_S(1, 3)$. This is unusual, since usually when we show that a subgroup generated by a subset is normal, we show that the subset is normal.

Lemma 2.2. Let $G$ be a group. If $g \in G$, then $[g, G]$ is normal in $G$.

Proof. Consider elements $x, y \in G$. It suffices to show that $[g, xy] \in [g, G]$. Since $[g, xy] = [g, y][g, x]^y$, we have $[g, x]^y = [g, y]^{-1}[g, xy] \in [g, G]$. □

We can now apply Lemma 2.1 to obtain information about $[g, G]$ and $\gamma_G(g)$ for a fixed element $g \in G$.

Lemma 2.3. Let $G$ be a group and fix an element $g \in G \setminus Z(G)$. Then the following are equivalent:

1. $\gamma_G(g)$ is a subgroup of $G$.
2. $\text{cl}_G(g) = g[g, G]$.
3. $\chi(g) = 0$ for all characters $\chi \in \text{Irr}(G \mid [g, G])$.

Proof. We first show that (1) and (2) are equivalent. Observe that the maps $g^x \mapsto g^{-1}g^x = [g, x]$ and $[g, x] \mapsto g[g, x] = g^x$ are inverse bijections between the sets $\text{cl}_G(g)$ and $\gamma_G(g)$. Notice that this shows that $\text{cl}_G(g)$ is a subset of $g[g, G]$ and these two sets are equal if and only if $\gamma_G(g) = [g, G]$. Note that $\gamma_G(g)$ is a subgroup of $G$ if and only if $\gamma_G(g) = [g, G]$. Hence, we conclude that $\gamma_G(g)$ is a subgroup if and only if $\text{cl}_G(g) = g[g, G]$, and this proves the equivalence of (1) and (2).

Notice that if (2) holds, then $g$ is conjugate to all of $g[g, G]$ and so applying Lemma 2.1, we have that $\chi(g) = 0$ for all characters $\chi \in \text{Irr}(G \mid [g, G])$. Hence, (3) holds. Conversely, if (3) holds, then $\chi(g) = 0$ for all characters $\chi \in \text{Irr}(G \mid [g, G])$, and we can use Lemma 2.1 to see that $g$ is conjugate to all of $g[g, G]$. This implies that $g[g, G] \subseteq \text{cl}_G(g)$. Since the reverse containment always holds, we must equality and this gives (2). □

If the element $g \in G$ satisfies the conditions of Lemma 2.3, then we will say that $g$ is a flat element. Note that Lemma 2.3 (1) implies that $g$ is flat if and only if $\text{cl}_G(g) = gN$ for a normal subgroup $N$ and we noted above, $N$ must be $[g, G]$. This next lemma connects commutators and centers of characters.

Lemma 2.4. Let $G$ be a group. Fix an element $g \in G$ and a character $\chi \in \text{Irr}(G)$. Then $g \in Z(\chi)$ if and only if $[g, G] \leq \ker(\chi)$.

Proof. This follows immediately from the definition that states: $Z(\chi)/\ker(\chi) = Z(G/\ker(\chi))$. □

With this mind, we can find a different characterization of Lemma 2.3 (3) in terms of centers of characters.

Corollary 2.5. Let $G$ be a group and fix an element $g \in G \setminus Z(G)$. Then $g$ is flat if and only if $\chi(g) = 0$ for all characters $\chi \in \text{Irr}(G)$ such that $g \notin Z(\chi)$.  

\[ \text{cl}_G(g) = gN \]
3. FULLY RAMIFIED CHARACTERS

It is often convenient to describe GVZ-groups in terms of fully ramified characters. We now suppose that $G$ is a group and that $N$ is a normal subgroup of $G$. A character $\chi$ of $G$ is called fully ramified over $N$ if $\chi$ vanishes on $G \setminus N$ and restricts homogeneously to $N$. (See [10, Exercise 6.3]). If $\chi$ is fully ramified over $N$ and $\vartheta$ is the unique irreducible constituent of $\chi_N$, then we say that $\vartheta$ is also fully ramified with respect to $G/N$.

Recall that $G$ is a GVZ-group if every irreducible character $\chi$ vanishes on $G \setminus Z(\chi)$. Since $\chi$ restricts homogeneously to $Z(\chi)$ for each character $\chi \in \text{Irr}(G)$, we have that $G$ is a GVZ-group if and only if every character $\chi \in \text{Irr}(G)$ of $G$ is fully ramified over $Z(\chi)$. We therefore seek a necessary and sufficient condition for an irreducible character to be fully ramified over its center.

We now set some notation so that we may describe our first result in this direction. Let $N$ be a normal subgroup of $G$. For each element $g \in G$, we define the subgroup $D_{g,N}$ by $D_{g,N} = C_{G/N}(gN)$. It is easy to see that $D_{g,N} = \{x \in G \mid [x,g] \in N\}$ as a set. In the event $N = Z(G)$, we write $D_G(g)$ instead of $D_{g,Z(G)}$; when $N = Z(\chi)$ for a character $\chi \in \text{Irr}(G)$, we write $D_{\chi}$ instead of $D_{\chi,Z(\chi)}$. We will see that the following lemma is an immediate consequence of [11, Theorem 1.19 and Lemma 1.20].

**Lemma 3.1.** Let $G$ be a group. If the character $\vartheta \in \text{Irr}(Z(G))$ is faithful, then $\vartheta$ is fully ramified with respect to $G/Z(G)$ if and only if $[g,D_G(g)] \neq 1$ for every element $g \in G \setminus Z(G)$.

**Proof.** By Theorem 1.19 and Lemma 1.20 of [11], the number of irreducible constituents of $\vartheta^G$ equals the number of conjugacy classes of cosets $gZ(G) \in G/Z(G)$ that satisfy $[g,D_G(g)] = 1$. Observe that if $g \in Z(G)$, then $[g,D_G(g)] = 1$. Hence, the only way that there can be only one conjugacy class of elements of in $G/Z(G)$ satisfying this condition is if $[g,D_G(g)] \neq 1$ for all elements $g \in G \setminus Z(G)$. Since $\vartheta$ is fully ramified with respect to $G/Z(G)$ if and only if $\vartheta^G$ has a unique irreducible constituent, it follows that $\vartheta$ is fully-ramified with respect to $G/Z(G)$ if and only if there is only one conjugacy class satisfying the condition. This gives the desired result. □

We get a slightly stronger statement without much difficulty.

**Lemma 3.2.** Let $G$ be a group. If $\lambda \in \text{Irr}(Z(G))$ is a character, then $\lambda$ is fully ramified with respect to $G/Z(G)$ if and only if $[g,D_G(g)] \notin \ker(\lambda)$ for every element $g \in G \setminus Z(G)$.

**Proof.** Let $Z = Z(G)$ and let $K = \ker(\lambda)$. Suppose first that $\lambda$ is fully ramified with respect to $G/Z$. Since $\lambda$ is fully ramified with respect to $G/Z$, it follows that $Z/K = Z(G/K)$. Applying Lemma 3.1, we have that $[gK,D_{G/K}(gK)] \neq 1$ for all cosets $gK \in G/K \setminus Z/K$. It is not difficult to see that this implies that $[g,D_G(g)] \notin K$ for all elements $g \in G \setminus Z$. Conversely, suppose that $[g,D_G(g)] \notin K$ for all $g \in G \setminus Z$. Hence, we have $[gK,D_{G/K}(gK)] \neq 1$ for all $gK \in G/K \setminus Z/K$. This implies that $[gK,G/K] = 1$ for all cosets $gK \in G/K \setminus Z/K$, and so $Z(G/K) \leq Z/K$. Since $Z/K \leq Z(G/K)$ obviously holds, we have $Z(G/K) = Z/K$. Notice that $\lambda$ is a faithful character of $Z/K$, so we may apply Lemma 3.1 to see that $\lambda$ is fully ramified with respect to $G/Z$. □
Suppose $G$ is a group, fix a character $\chi \in \text{Irr}(G)$, and write $\chi_Z(G) = \chi(1)\lambda$ for some character $\lambda \in \text{Irr}(Z(G))$. Note that $\ker(\lambda) = \ker(\chi) \cap Z(G)$. Consider an element $g \in G$. Since $[g, D_G(g)] \leq Z(G)$, we have $[g, D_G(g)] \not\leq \ker(\lambda)$ if and only if $[g, D_G(g)] \not\leq \ker(\chi)$. Hence, Lemma 3.2 can be equivalently stated as follows.

**Lemma 3.3.** Let $G$ be a group. If $\chi \in \text{Irr}(G)$ is a character, then $\chi$ is fully ramified over $Z(G)$ if and only if $[g, D_G(g)] \not\leq \ker(\chi)$ for every element $g \in G \setminus Z(G)$.

We may now give a necessary and sufficient condition for an irreducible character to be fully ramified over its center.

**Theorem 3.4.** Let $G$ be a group, and consider a character $\chi \in \text{Irr}(G)$. Then $\chi$ is fully ramified over $Z(\chi)$ if and only if $\ker(\chi) < [g, D_{g,\chi}]\ker(\chi)$ for every element $g \in G \setminus Z(\chi)$.

**Proof.** Since $\chi$ is a faithful irreducible character of $G/\ker(\chi)$ and $Z(G/\ker(\chi)) = Z(\chi)/\ker(\chi)$, we have by Lemma 3.3 that $\chi$ is fully ramified over $Z(\chi)$ if and only if $[g, D_{g,\chi}] \not\leq \ker(\chi)$ for every element $g \in G \setminus Z(\chi)$. This happens if and only if $\ker(\chi) < [g, D_{g,\chi}]\ker(\chi)$ for every element $g \in G \setminus Z(\chi)$.

\[ \Box \]

4. GVZ-GROUPS

In this section, our main goal is to prove Theorem 4.2. We next include the following lemma, which gives another characterization of finite nilpotent groups which we will make use of.

**Lemma 4.1.** Let $G$ be a group and let $M$ be a minimal normal subgroup of $G$. Then $M \leq Z(G)$ if and only if $\ker(\chi) < Z(\chi)$ for all characters $\chi \in \text{Irr}(G | M)$.

**Proof.** Suppose $M \leq Z(G)$ and consider a character $\chi \in \text{Irr}(G | M)$. Hence, there exists a character $1_M \neq \lambda \in \text{Irr}(M)$ so that $\chi_M = \chi(1)\lambda$. It follows that $M \leq Z(\chi)$ and $M \not\leq \ker(\chi)$. Thus, we have $\ker(\chi) < Z(\chi)$. Conversely, assume that $\ker(\chi) < Z(\chi)$ for all characters $\chi \in \text{Irr}(G | M)$. We work by induction on $|G|$. Suppose there is some minimal normal subgroup $N$ of $G$ that is different from $M$. Then $NM/M$ will be a minimal normal subgroup of $G/M$ and $\text{Irr}(G/N | MN/N)$ is a subset of $\text{Irr}(G | M)$. Thus, $G/M$ satisfies the induction hypothesis. By induction, $MN/N$ is central in $G/N$. This implies that $[M, G] \leq N$. Since $M$ is normal, we see that $[M, G] \leq M$. It follows that $[M, G] \leq N \cap M = 1$, and we conclude that $M \leq Z(G)$. Thus, we may assume that $M$ is the unique minimal normal subgroup of $G$. Consider a character $\chi \in \text{Irr}(G | M)$. Since $M$ is not contained in $\ker(\chi)$ and $M$ is the unique minimal normal subgroup of $G$, we see that $\chi$ is faithful. We have that $Z(G) = Z(\chi) > 1 = \ker(\chi)$. Since $Z(G) > 1$ and $M$ is the unique minimal normal subgroup $G$, we conclude that $M \leq Z(G)$.

Our main theorem contains our alternate characterizations of GVZ-groups including the characterization that the group is flat. Notice that condition (2) is really the observation that $G$ is a GVZ-group if and only if every irreducible character of $G$ is fully-ramified over its center. As we saw before Lemma 2.2, condition (3) is equivalent to the group being flat; so the equivalence of conditions (1) and (3) is Theorem A from the introduction. Theorem B is the equivalence of conditions (1) and (4) combined with the observation that $\gamma_G(g) = [g, G]$ if and only if $\gamma_G(g)$ is a subgroup of $G$. Notice that condition (5) follows from the characterization of groups of central type in Lemma 3.2. To state the final condition, we introduce the
following notation: for a normal subgroup \( N \) of \( G \), define \( Z_N \) by \( Z_N/N = Z(G/N) \) and following the usual convention in the literature, we say that a group \( G \) is a monolithic or is monolithic if it has a unique minimal normal subgroup. Finally, condition (6) gives a characterization of GVZ-groups in terms of monoliths.

**Theorem 4.2.** Let \( G \) be a nonabelian group. Then the following are equivalent:

1. \( G \) is a GVZ-group.
2. \( \text{cd}(G) = \{ |g : Z(\chi)|^{1/2} \mid \chi \in \text{Irr}(G) \} \).
3. For every element \( g \in G \setminus Z(G) \), the following equality holds: \( \text{cl}_G(g) = g[g,G] \). (I.e., \( G \) is flat.)
4. For every element \( g \in G \), the set \( \gamma_G(g) \) is a subgroup of \( G \).
5. For every character \( \chi \in \text{Irr}(G) \) and for every element \( g \in G \setminus Z(\chi) \), we have \( \ker(\chi) < [g,D_{g,\chi}] \ker(\chi) \).
6. \( G \) is nilpotent, and for every normal subgroup \( N \) of \( G \) for which \( G/N \) is monolithic and for every element \( g \in G \setminus Z_N \), we have \( N < [g,D_{g,Z_N}] \).

**Proof of Theorem 4.2.** The fact that \( \chi(1) = |G : Z(\chi)|^{1/2} \) for a character \( \chi \in \text{Irr}(G) \) if and only if \( \chi \) vanishes on \( G \setminus Z(\chi) \) is well-known (e.g., see [10, Corollary 2.30]). Thus (1) and (2) are equivalent. The equivalence of (1) and (5) is Theorem 3.4. Note that Lemma 2.3 shows that (3) and (4) are equivalent for every element \( g \in G \setminus Z(G) \). Observe using Corollary 2.5 that (1) is equivalent to \( \chi(g) = 0 \) for every element \( g \in G \setminus Z(G) \) and every character \( \chi \in \text{Irr}(G \mid [g,G]) \), and by Lemma 2.3, we see that this condition is equivalent to condition (3). Hence, (1) and (3) are equivalent.

Next we show that (5) implies (6). Since \( [g,D_{g,\chi}] \leq Z(\chi) \) for all characters \( \chi \in \text{Irr}(G) \) and for all elements \( g \in G \setminus Z(\chi) \), condition (5) implies that \( \ker(\chi) < Z(\chi) \) for every character \( \chi \in \text{Irr}(G) \). Thus, we have that \( G \) is nilpotent by Lemma 4.1. Now, let \( N \) be a normal subgroup of \( G \) for which \( G/N \) is a monolith. Then \( G/N \) has a faithful irreducible character \( \chi \); i.e. there exists a character \( \chi \in \text{Irr}(G) \) satisfying \( \ker(\chi) = N \). Hence, \( N = \ker(\chi) \leq \ker(\chi)/[g,D_{g,\chi}] = \ker(\chi)/[g,D_{g,Z_N}] \), and (6) follows.

To complete the proof, we show that (6) implies (1). Fix a prime \( p \) that divides \( |G| \), a Sylow subgroup \( P \in \text{Syl}_p(G) \), and a character \( \psi \in \text{Irr}(P) \). Consider the character \( \xi = \psi \times 1_H \in \text{Irr}(G) \), where \( H \) is a normal \( p \)-complement of \( G \). Then \( G/\ker(\xi) \cong P/\ker(\psi) \) is monolithic, by [10, Theorem 2.32]. So \( \xi \) is fully ramified over \( Z(\xi) = Z(\psi) \times H \), and this implies that \( \psi \) is fully ramified over \( Z(\psi) \). Now, consider a character \( \chi \in \text{Irr}(G) \). To show that \( G \) is a GVZ-group, it suffices to show that \( \chi \) is fully ramified over \( Z(\chi) \). Suppose that \( G = P_1 \times \cdots \times P_r \) is a factorization of \( G \) into a direct product of its Sylow subgroups. Then there exist characters \( \nu_i \in \text{Irr}(P_i) \) so that \( \chi = n_{\nu_1} \times \cdots \times \nu_r \). Observe that \( Z(\chi) = Z(\nu_1) \times \cdots \times Z(\nu_r) \). We have already shown that each \( \nu_i \) is fully ramified over \( Z(\nu_i) \) and so it follows that \( \chi \) is fully ramified over \( Z(\chi) \), as desired. This proves (1).

Theorem 4.2 yields the following corollary, which contains a converse to [25, Theorem 4.2].

**Corollary 4.3.** Let \( G \) be a finite group. Then \( G \) is flat if and only if \( G \) is a GVZ-group. In particular, flat groups are nilpotent.

We conclude this section with a proof of Theorem C. For convenience, we set some notation. We let \( G_i \) denote the \( i \)-th member of the lower central series. That
is, we set $G_1 = G$ and we inductively define $G_{i+1} = [G_i, G]$ for every integer $i \geq 1$.
For a nilpotent group $G$, recall that $e(G)$ is the nilpotence class of $G$—the smallest integer for which $G_{e(G)+1} = 1$. The reader should compare this proof with the proof of Taketa’s theorem (see [10, Theorem 5.12]).

**Proof of Theorem C.** Let $1 = d_1 < d_2 < \cdots < d_n$ be the distinct degrees in cd$(G)$.
We work by induction on $|G|$. If $G$ is abelian, then $e(G) = 1 = |\text{cd}(G)|$, and the result holds. Thus, we may assume that $G$ is nonabelian. Consider a character $\chi \in \text{Irr}(G)$. If $\ker(\chi) > 1$, then $|G/\ker(\chi)| < |G|$ and $\text{cd}(G/\ker(\chi)) \subseteq \text{cd}(G)$. By the inductive hypothesis, we have that $G_n \leq \ker(\chi)$. Thus, if $G$ does not have a faithful irreducible character, then $G_n \leq \bigcap_{\chi \in \text{Irr}(G)} \ker(\chi) = 1$. Therefore, we may assume that there exists a character $\chi \in \text{Irr}(G)$ with $\ker(\chi) = 1$. This implies that $Z(\chi) = Z(G)$. We have $d^2 \leq |G : Z(G)| = \chi(1)^2$ for every integer $i$ with $1 \leq i \leq n$, and so, $\chi(1) = d_n$. Notice that if $a \in \text{cd}(G/Z(G))$, then $a^2 < |G : Z(G)| = d_n^2$.
It follows that $|\text{cd}(G/Z(G))| \leq n - 1$. By the inductive hypothesis, this implies $G_{n-1} \leq Z(G)$. We conclude that $G_n = 1$, as desired. \[\square\]

5. **CM$_n$-groups and GVZ-groups**

As we stated in the Introduction, CM$_n$-groups were defined in [2]. So far as we can tell, CM-groups were initially studied in [26]. The reader may also want to consult [27] for further results regarding these groups. We begin with a lemma that gives a lower bound on the number of faithful characters of a group.

**Lemma 5.1.** Let $G$ be a $p$-group and suppose that $Z(G)$ is cyclic, then the number of faithful characters in $\text{Irr}(G)$ is at least $\phi(|Z(G)|)$ where $\phi$ is the Euler $\phi$-function.

**Proof.** Note that for each faithful character $\lambda \in \text{Irr}(Z(G))$, we see that $\lambda^G$ has at least one irreducible constituent $\chi$. Note that $\ker(\chi \cap Z(G)) = \ker = 1$, and so $\ker(\chi) = 1$. Hence, $\chi$ is a faithful character. On the other hand, any character in $\text{Irr}(G)$ will have a character of $Z(G)$ as its unique irreducible constituent when restricted to $Z(G)$. This implies that the number of faithful characters of $G$ is at least the number of faithful irreducible characters of $Z(G)$. Since $Z(G)$ is cyclic, the number of faithful irreducible characters equals $\phi(|Z(G)|)$.

We say $G$ is a CM$_n$-group if for every normal subgroup $N$, there are at most $n$ characters in $\text{Irr}(G)$ that have $N$ as their kernel. Note for every group $G$, there is a minimal positive integer $n$ so that $G$ is a CM$_n$-group.

**Lemma 5.2.** If $p$ is a prime, $G$ is a $p$-group, and $G$ is a CM$_n$-group, then $n \geq p - 1$.

**Proof.** If $N = \ker(\chi)$ for some $\chi \in \text{Irr}(G)$, then $Z(\chi)/N = Z(G/N)$ is cyclic. By Lemma 5.1, we know that $G/N$ has at least $\phi(|Z(\chi)/N|)$ faithful irreducible characters. Since $Z(\chi)/N$ is a $p$-group, we know $\phi(|Z(\chi)/N|) \geq p - 1$. \[\square\]

If $p$ is a prime, then we write $Q_p$ for the field obtained by adjoining a $p$th root of unity to the rationals. We now characterize the $p$-groups that are CM$_{p-1}$-groups.

**Lemma 5.3.** Let $p$ be a prime and let $G$ be a $p$-group. Then the following are equivalent:

1. $G$ is a CM$_{p-1}$-group.
2. $G$ is a GVZ-group and $|Z(\chi)/\ker(\chi)| = p$ for all $1_G \neq \chi \in \text{Irr}(G)$.
3. $G$ is a GVZ-group and every character in $\text{Irr}(G)$ has values in $Q_p$. 

Proof. Suppose that $G$ is a $CM_{p-1}$ group. Consider a character $1_G \neq \chi \in \text{Irr}(G)$. By Lemma 5.1, we know that $G/\ker \chi$ has at least $\phi(|Z(\chi)/\ker \chi|)$ faithful irreducible characters. As we saw in Lemma 5.2, $\phi(|Z(\chi)/\ker \chi|) \geq p - 1$. Since $G/\ker \chi$ has at most $p - 1$ faithful irreducible characters, we $\phi(|Z(\chi)/\ker \chi|) \leq p - 1$, and thus, $\phi(|Z(\chi)/\ker \chi|) = p - 1$ if and only if $|Z(\chi)/\ker \chi| = p$. We now have that $Z(\chi)/\ker \chi$ has $p - 1$ nonprincipal irreducible characters. Since $G/\ker \chi$ has at most $p - 1$ faithful irreducible characters, we conclude that $\gamma^G$ has a unique irreducible constituent for each character $\gamma \in \text{Irr}(Z(\chi)/\ker \chi)$. Observe that $\chi$ is a constituent of $\gamma^G$ for such a character $\gamma$, and it is not difficult to see that this implies that $\chi$ vanishes on $G \setminus Z(\chi)$. Since $\chi$ was arbitrary, this implies that $G$ is a GVZ-group and $|Z(\chi)/\ker \chi| = p$ for all $1_G \neq \chi \in \text{Irr}(G)$.

Now, suppose that $G$ is a GVZ-group and $|Z(\chi)/\ker \chi| = p$ for all $1_G \neq \chi \in \text{Irr}(G)$. Consider a character $\chi \in \text{Irr}(G)$. We see that all the nonzero values of $\chi$ are on elements of $Z(\chi)$ and since $|Z(\chi)/\ker \chi| = p$, it follows that all the values of $\chi$ lie in $Q_p$.

Finally, suppose that $G$ is a GVZ-group and every character in $\text{Irr}(G)$ has values in $Q_p$. Let $N$ be a normal subgroup of $G$. If no irreducible character of $G$ has $N$ as a kernel, then the result is true with respect to $N$. Thus, we may assume that there exists $\chi \in \text{Irr}(G)$ so that $\ker \chi = N$. Observe that $Z(\chi)/N = Z(G/N)$ is cyclic. Since $\chi$ has values in $Q_p$, it follows that $Z(\chi)/N$ must have exponent $p$, and so, $Z(\chi)/N$ has order $p$. Now, every irreducible character in $\text{Irr}(G)$ have $N$ as its kernel will have a unique character in $\text{Irr}(Z(\chi)/N)$ as an irreducible constituent. Since $G$ is a GVZ-group, we see that each irreducible character in $Z(\chi)/N$ has a unique irreducible constituent upon $\square$

We say $G$ a $CM$-group if $G$ is a $CM_1$-group. Following the usual convention in the literature, we say $G$ is a rational group if all of the irreducible characters of $G$ are rational. We note in Lemma 1.2 of [24] that it is proved that $CM$-groups that are 2-groups are rational and several of the other results in that paper suggest that $G$ will be a GVZ-group.

Corollary 5.4. Let $G$ be a 2-group. Then $G$ is a $CM$-group if and only if $G$ is a GVZ-group and rational group.

It is easy to find examples to see that if $m > 1$, then the direct product of two $CM_n$-groups need not be a $CM_n$-group. However, when $n = p - 1$ and the groups are $p$-groups, then the story is different.

Corollary 5.5. Let $p$ be a prime and suppose that $H$ and $K$ are $CM_{p-1}$-groups. Then $H \times K$ is a $CM_{p-1}$-group.

Proof. Observe since $H$ and $K$ are GVZ-groups that $H \times K$ is a GVZ-group. Also, since all characters in $\text{Irr}(H)$ and $\text{Irr}(K)$ have values in $Q_p$, it follows that all characters in $\text{Irr}(H \times K)$ have values in $Q_p$. Applying Lemma 5.3, we see that $H \times K$ is a $CM_{p-1}$-group. $\square$

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