On the weak* continuity of $LUC(\mathcal{G})^*$-module action on $LUC(X, \mathcal{G})^*$ related to $G$-space $X$

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Abstract

For a locally compact group $\mathcal{G}$ and a $G$-space $X$, we use the notation $LUC(X, \mathcal{G})$ to denote the Banach space of all bounded left uniformly continuous functions on $X$. We introduce a left action of $LUC(\mathcal{G})^*$ on $LUC(X, \mathcal{G})^*$ to make it a Banach left module and then we study the topological centre related to this module action. We propound some main properties of that topological centre, extend the main results of topological centre related to a locally compact group to a $G$-space, and then we apply our results to some special homogeneous spaces.

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1. Introduction

Let $\mathcal{G}$ be a locally compact group. Then the Banach space $LUC(\mathcal{G})^*$, the topological dual of the space of bounded left uniformly continuous functions on $\mathcal{G}$, is a Banach algebra equipped with the first Arens product "$\circ". In general, this product is not separately weak* to weak* continuous on $LUC(\mathcal{G})^*$, and in recent years there has been shown considerable interest by harmonic analysts in the characterization of the following space

$$\mathcal{Z}(\mathcal{G}) = \{ m \in LUC(\mathcal{G})^* : n \mapsto m \circ n \text{ is weak* to weak* continuous} \}.$$ 

As far as we know the subject, the starting point of the study of the space $\mathcal{Z}(\mathcal{G})$ is the paper by Zappa [3]. In this paper Zappa proved that $\mathcal{Z}(\mathbb{R})$ is precisely $M(\mathbb{R})$, where $\mathbb{R}$ is the additive group of real number and $M(\mathbb{R})$ is the Banach algebra of all complex Radon measures on $\mathbb{R}$. This result was extended to all abelian locally compact groups by Grosser and Losert in [1], and to all locally compact groups by Lau in [2].

In this paper, considering $X$ as a locally compact Hausdorff space on which $\mathcal{G}$ acts continuously from the left, we introduce the Banach space $LUC(X, \mathcal{G})$ as well $LUC(\mathcal{G})$. Then we present a left action of $LUC(\mathcal{G})^*$ on $LUC(X, \mathcal{G})^*$, as an extension

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of the natural action of $M(G)$ on $M(X)$, to make a Banach left $LUC(G)^*$-module. Then we investigate the topological centre related to this module action and extend the main results of Lau [2] from locally compact groups to $G$-spaces.

2. The definitions and some basic results

Throughout this paper, $G$ is a locally compact group. A locally compact Hausdorff space $X$ is a $G$-space if there is a continuous action map $G \times X \to X$, denoted by $(s, x) \mapsto s \cdot x$, satisfying $(st) \cdot x = s \cdot (t \cdot x)$ and $e \cdot x = x$, for all $s, t \in G$ and $x \in X$. The action of $G$ on $X$ is said to be effective, when the following closed normal subgroup of $G$ is trivial

$$\mathfrak{N}(X, G) = \{ s \in G : s \cdot x = x \text{ for all } x \in X \}.$$ 

Moreover, if $M(X)$ denotes the Banach space of all complex Radon measures on $X$ with total variation norm, then the action of $G$ on $X$ induces an action of $M(G)$ on $M(X)$ which makes $M(X)$ as a Banach left $M(G)$-module.

**Proposition 2.1.** Let $G$ be a locally compact group and $X$ be a $G$-space. Then $M(X)$ is a faithful Banach left $M(G)$-module if and only if $G$ acts effectively on $X$.

When $X$ is a $G$-space, we define the left translation of a function $F \in C_b(X)$ by $l_s F(y) = F(s \cdot y)$ for all $y \in X$ and consider

$$LUC(X, G) = \{ F \in C_b(X) : s \in G \mapsto l_s F \in (C_b(X), \| \cdot \|_\infty) \text{ is continuous} \}.$$ 

Then $LUC(X, G)$ is a closed subspace of $C_b(X)$ which is invariant under left translation.

**Lemma 2.2.** Let $G$ be a locally compact group and $X$ be a $G$-space. Then the Banach space $C_0(X)$ is contained in $LUC(X, G)$.

Moreover, if $LUC(X, G)^*$ denotes the first dual space of the Banach space $LUC(X, G)$, $M$ is an arbitrary element of $LUC(X, G)^*$ and $F \in LUC(X, G)$, we define the function $MF : G \to \mathbb{C}$ by $MF(s) = \langle M, l_s F \rangle$, for all $s \in G$, which belongs to $LUC(G)$. Also, the mapping

$$\begin{cases} 
LUC(X, G)^* \times LUC(X, G) \to LUC(G) \\
(M, F) \mapsto MF 
\end{cases}$$

is a bounded bilinear map with $\|MF\|_\infty \leq \|M\| \|F\|_\infty$. So, we can define a bounded bilinear map as follows

$$\begin{cases} 
LUC(G)^* \times LUC(X, G)^* \to LUC(X, G)^* \\
(m, M) \mapsto m \cdot M 
\end{cases}$$

with $\|m \cdot M\| \leq \|m\| \|M\|$, where $\langle m \cdot M, F \rangle = \langle m, MF \rangle$, for all $F \in LUC(X, G)$.

**Proposition 2.3.** Let $G$ be a locally compact group and $X$ be a $G$-space. Under the mapping $(m, M) \mapsto m \cdot M$, $LUC(X, G)^*$ becomes a Banach left $LUC(G)^*$-module with $\|m \cdot M\| \leq \|m\| \|M\|$ and $\delta_e \cdot M = M$. 


If now, for a given \( \sigma \in M(\mathcal{X}) \), we define a linear functional on \( \text{LUC}(\mathcal{X}, \mathcal{G}) \), denoted again by \( \sigma \), which assigns to each \( F \in \text{LUC}(\mathcal{X}, \mathcal{G}) \) the value \( \int_{\mathcal{X}} F(x) \, d\sigma(x) \). Then, \( M(\mathcal{X}) \) may be regarded as a subspace of \( \text{LUC}(\mathcal{X}, \mathcal{G})^\ast \). Moreover, it is not hard to check that the inclusion \( \text{LUC}(\mathcal{G})^\ast \cdot M(\mathcal{X}) \subseteq M(\mathcal{X}) \) can fail even if \( \mathcal{X} = \mathcal{G} \), where

\[
\text{LUC}(\mathcal{G})^\ast \cdot M(\mathcal{X}) = \left\{ m \cdot \sigma : m \in \text{LUC}(\mathcal{G})^\ast, \, \sigma \in M(\mathcal{X}) \right\}.
\]

In other words, the Banach space \( M(\mathcal{X}) \) is not in general an \( \text{LUC}(\mathcal{G})^\ast \)-submodule of \( \text{LUC}(\mathcal{X}, \mathcal{G})^\ast \). On the other hand, if \( \mathcal{X} \) is compact, then \( M(\mathcal{X}) \) is a \( \text{LUC}(\mathcal{G})^\ast \)-submodule of \( \text{LUC}(\mathcal{X}, \mathcal{G})^\ast \); this is because of, in this case \( M(\mathcal{X}) = \text{LUC}(\mathcal{X}, \mathcal{G})^\ast \). We do not know if the converse of this fact is valid in general; here, we prove the converse under an extra assumption. The notation \( \text{CLS}(\mathcal{X}, \mathcal{G}) \) in this proposition and in the sequel denotes the norm closure of the linear span of the set

\[
\text{LUC}(\mathcal{X}, \mathcal{G})^\ast \cdot \text{LUC}(\mathcal{X}, \mathcal{G}) := \left\{ MF : M \in \text{LUC}(\mathcal{X}, \mathcal{G})^\ast, \, F \in \text{LUC}(\mathcal{X}, \mathcal{G}) \right\},
\]

with respect to the norm topology of \( \text{LUC}(\mathcal{G}) \). Also, we say that the action of \( \text{LUC}(\mathcal{G})^\ast \) on \( \text{LUC}(\mathcal{X}, \mathcal{G})^\ast \) is faithful if \( m \in \text{LUC}(\mathcal{G})^\ast \) is so that \( m \cdot M = 0 \) for all \( M \in \text{LUC}(\mathcal{X}, \mathcal{G})^\ast \), then \( m = 0 \).

**Proposition 2.4.** Let \( \mathcal{G} \) be a locally compact group and \( \mathcal{X} \) be a \( \mathcal{G} \)-space. Then the following statements hold.

(a) \( \text{CLS}(\mathcal{X}, \mathcal{G}) = \text{LUC}(\mathcal{G}) \) if and only if the action of \( \text{LUC}(\mathcal{G})^\ast \) on \( \text{LUC}(\mathcal{X}, \mathcal{G})^\ast \) is faithful.

(b) If the action of \( \text{LUC}(\mathcal{G})^\ast \) on \( \text{LUC}(\mathcal{X}, \mathcal{G})^\ast \) is faithful, then \( \mathcal{G} \) acts effectively on \( \mathcal{X} \).

(c) If \( \mathcal{X} \) is compact, then \( M(\mathcal{X}) \) is an \( \text{LUC}(\mathcal{G})^\ast \)-submodule of \( \text{LUC}(\mathcal{X}, \mathcal{G})^\ast \). The converse is also true if \( \mathcal{X} \) is a transitive \( \mathcal{G} \)-space.

3. **The weak* continuity of the left \( \text{LUC}(\mathcal{G})^\ast \)-module action**

A problem which is of interest is that for which element \( m \in \text{LUC}(\mathcal{G}) \) the map \( M \mapsto m \cdot M \) on \( \text{LUC}(\mathcal{X}, \mathcal{G})^\ast \) is weak* to weak* continuous? Therefore, it seems valuable to define

\[
\mathfrak{Z}(\mathcal{X}, \mathcal{G}) = \left\{ m \in \text{LUC}(\mathcal{G})^\ast : M \mapsto m \cdot M \text{ is weak* to weak* continuous on } \text{LUC}(\mathcal{X}, \mathcal{G})^\ast \right\},
\]

the topological centre of the module action induced by \( \text{LUC}(\mathcal{G})^\ast \) on \( \text{LUC}(\mathcal{X}, \mathcal{G})^\ast \). In the special case that we let \( \mathcal{G} \) act on itself by left multiplication, the set \( \mathfrak{Z}(\mathcal{G}, \mathcal{G}) \) coincides with \( \mathfrak{Z}(\mathcal{G}) \). This section studies the subspace \( \mathfrak{Z}(\mathcal{X}, \mathcal{G}) \) of \( \text{LUC}(\mathcal{G})^\ast \) in the case where \( \mathcal{X} \) is a \( \mathcal{G} \)-space and, in particular, the question when the subspace \( \mathfrak{Z}(\mathcal{X}, \mathcal{G}) \) is \( M(\mathcal{G}) \). Before proceeding further in this section, we should note that if \( \mathcal{X} \) is a \( \mathcal{G} \)-space, then \( \text{LUC}(\mathcal{X}, \mathcal{G})^\ast \) is a left Banach \( \mathcal{G} \)-module; in fact, it is suffices to define a left action of \( \mathcal{G} \) on \( \text{LUC}(\mathcal{X}, \mathcal{G})^\ast \) by \( (s, M) \mapsto \delta_s \cdot M \).
Now, let $X$ be a $G$-space. Given $F \in LUC(X,G)$ and $x \in X$, we define $r_x F$ on $G$ by $(r_x F)(s) = F(s \cdot x), s \in G$. A routine computation shows that $r_x F$ is a function in $LUC(G)$. Hence, if $m$ is an arbitrary element of $LUC(G)^*$, then we can define a complex-valued function $Fm$ on $X$ by $Fm(x) = \langle m, r_x F \rangle, x \in X$. Obviously that $Fm$ is a bounded function on $X$ with $\|Fm\|_\infty \leq \|m\| \|F\|_\infty$. Moreover, suppose that $m$ is an element of $LUC(G)^*$ such that for each $F \in LUC(X,G)$ the function $Fm$ is in $LUC(X,G)$, then every $M$ in $LUC(X,G)^*$ gives a linear functional $M \cdot m$ on $LUC(X,G)^*$ by $\langle M \cdot m, F \rangle = \langle M, Fm \rangle$. One may easily check that, if $M \cdot m = m \cdot M$, for all $M \in LUC(X,G)^*$, then $m \in \mathcal{Z}(X,G)$. The following theorem consider the converse of this fact whose proof is inspired by [2, Lemma 2.2].

**Theorem 3.1.** Let $G$ be a locally compact group and $X$ be a $G$-space. Then $\mathcal{Z}(X,G)$ is precisely the set of all $m \in LUC(G)^*$ for which the following conditions are satisfied

(a) $Fm \in LUC(X,G)$ for all $F \in LUC(X,G)$,

(b) $M \cdot m = m \cdot M$ for all $M \in LUC(X,G)^*$.

**Corollary 3.2.** Let $G$ be a locally compact group and $X$ be a $G$-space. Then $m \in \mathcal{Z}(X,G)$ if and only if the map $M \mapsto m \cdot M$ is weak$^*$ continuous on all bounded parts of $LUC(X,G)^*$.

**Proposition 3.3.** Let $G$ be a locally compact group and $X$ be a $G$-space. If the index of $\mathfrak{M}(X,G)$ in $G$ is finite, then $\mathcal{Z}(X,G) = LUC(G)^*$.

The following results, gives some of the main properties of the set $\mathcal{Z}(X,G)$.

**Proposition 3.4.** Let $G$ be a locally compact group and $X$ be a $G$-space. Then the following assertions hold.

(a) $\mathcal{Z}(X,G)$ is a subalgebra of $LUC(G)^*$.

(b) $\mathcal{Z}(X,G)$ is closed with respect to the norm topology of $LUC(G)^*$.

(c) $M(G)$ is contained in $\mathcal{Z}(X,G)$.

**Theorem 3.5.** Let $G$ be a locally compact non-compact group and let $X$ be a $G$-space. Then the following assertions are equivalent.

(a) $\mathcal{Z}(X,G) = M(G)$.

(b) $CLS(X,G) = LUC(G)$.

(c) The action of $LUC(G)^*$ on $LUC(X,G)^*$ is faithful.

**Corollary 3.6.** Let $G$ be a locally compact non-compact group and $X$ be a $G$-space. If $G$ does not act effectively on $X$, then $M(G)$ is properly contained in $\mathcal{Z}(X,G)$.

**Example 3.7.** Let $G$ be a locally compact group and $X$ be a $G$-space.
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(a) If $X = G$, then $M(G) = \mathfrak{z}(X,G)$.

(b) If $G$ is compact, then $\mathfrak{z}(X,G) = M(G)$.

(c) If $X$ is a finite discrete space, then $\mathfrak{z}(X,G) = LUC(G)^*$. In particular, if $G$ is non-compact, then $M(G) \subsetneq \mathfrak{z}(X,G) = LUC(G)^*$.

Example 3.8. Let $G$ be a locally compact, non-compact group and let $H$ and $K$ are two non-trivial closed normal subgroup of $G$.

(a) If $X = G/H$, then $M(G) \subsetneq \mathfrak{z}(X,G)$. In particular, if the index of $H$ in $G$ is finite, then $\mathfrak{z}(X,G) = M(G)^*$.

(b) If $X = K$, then, obviously, $G$ acts on $X$ by conjugation. In this case, $\mathfrak{z}(X,G) = \mathfrak{c}_G(K)$. Hence, we can say if either the centralizer $\mathfrak{c}_G(K)$ or $\mathfrak{c}_G(K)$ is non-trivial (for example, if $K$ is abelian), then we have $M(G) \subsetneq \mathfrak{z}(X,G)$.

Example 3.9. Suppose that $Y$ is an arbitrary locally compact Hausdorff space and $G$ is a locally compact group. Then $X = G \times Y$, equipped with the product topology and the action defined by $s \cdot (t,y) = (st,y)$, is a $G$-space for which $\mathfrak{z}(X,G) = M(G)^*$.

Example 3.10. Let $G = Q_8 \times \mathbb{F}_2$, where $Q_8 = \langle a, b : a^4 = 1, a^2 = b^2, b^{-1}ab = a^{-1} \rangle$ is the quaternion group. Take $K = \langle a \rangle$, which is a normal subgroup of $Q_8$. Then $X = K \times \mathbb{F}_2$ is a closed normal subgroup of discrete topological group $G$ and so, $G$ acts on $X$ by $(s,x) \mapsto sx^{-1}$. For the $G$-space $X$, we have $M(G) \subsetneq \mathfrak{z}(X,G) \subsetneq LUC(G)^*$.

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