THE STAR SEQUENCE AND THE GENERAL FIRST ZAGREB INDEX

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Abstract. For a simple graph we introduce a notion of the star sequence and prove that the star sequence and the frequently sequences of a graph are inverses of each other from a combinatorial point of view. As a consequence, we express the general first Zagreb index in terms of the star sequence. Also, we calculate the ordinary generating function and find a linear recurrence relation for the sequence of the general first Zagreb indexes.

1. Introduction

Let $G$ be a simple graph whose vertex and edge sets are $V(G)$ and $E(G)$, respectively. Let $\deg(v)$ be the degree of the vertex $v \in V(G)$. For any real $p$ the general Zagreb index is defined by

$$Z_p(G) = \sum_{v \in V(G)} \deg(v)^p,$$

see [1] for more details.

Put $n = |V(G)|$ and $m = |E(V)|$. We have $Z_0(G) = n$ and $Z_1(G) = 2m$ due to the Handshaking lemma.

The well-known ordinary first Zagreb index $M_1$ is a special case of the general Zagreb index $Z_p(G)$ for $p = 2$.

It is easy to see ([2, Theorem 8.1]) that

$$Z_2(G) = 2m + 2q,$$

where and $q$ is the number of subgraphs that are isomorphic to the path graph $P_3$. Taking into account that $m$ is the number of subgraphs that are isomorphic to the star graph $S_1$ and that $P_3 \cong S_2$ we may rewrite the expression for $Z_2(G)$ as follows

$$Z_2(G) = 2S_1(G) + 2S_2(G),$$

where $S_1(G), S_2(G)$ denote the number of subgraph of $G$ that are isomorphic to the star graphs $S_1$ and $S_2$, respectively.
In the paper, we generalize this expression for the general first Zagreb index \( Z_p(G) \) for any natural \( p \).

Let \( S_k \) be the star graph defined as the complete bipartite graph \( K_{1,k} \).

Denote by \( S_k(G) \) the number of subgraphs of \( G \) that are isomorphic to the star \( S_k \).

For instance \( S_1(G) \) is equal to the number of edges of \( G \) and \( S_{n-1}(G) \) is equal to the number of vertices of maximal degrees \( n-1 \), \( n = |V(G)| \).

The sequence

\[
2S_1(G), S_2(G), \ldots, S_{n-1}(G)
\]

is called a star sequence of a graph \( G \).

We can define some types of graphs in terms of its sequence star. For example, for a graph \( G \) to be the path \( P_n \) it is necessary and sufficient to have \( S_1(G) = n - 1, S_2(G) = n - 2 \), and \( S_i(G) = 0 \) for \( i > 2 \). Similarly, a graph \( G \) is a \( k \)-regular iff the following conditions hold:

\[
2S_1 = \binom{k}{1} S_k(G), S_i = \binom{k}{i} S_k(G), \text{ and } S_i(G) = 0 \text{ for } i > k.
\]

The main result of the paper is the formula for expressing of the general first Zagreb index in terms of the star sequence

\[
Z_p(G) = 2S_1(G) + \sum_{i=2}^{p} i! \{\binom{p}{i}\} S_i(G),
\]

here \( \{\binom{p}{i}\} \) are the Stirling numbers of the second kind.

The result can be considered as a wide generalization of the Handshaking Lemma.

Also, we calculate the generating function for the integer sequence \( \{Z_p(G)\} \):

\[
\sum_{p=0}^{\infty} Z_p(G) t^p = \sum_{k=0}^{n-1} \left( \sum_{i=0}^{k} \binom{n+1}{n+1-k-i} Z_i(G) \right) \frac{t^k}{(1-t)(1-2t)\cdots(1-nt)},
\]

and find the recurrence relation for the integer sequence \( \{Z_p(G)\} \):

\[
Z_p(G) + \sum_{i=1}^{n} \left[ \binom{p+1}{p+1-i} Z_{p-i}(G) \right] = 0, \ p \geq n,
\]

here \( \left[ \binom{p}{i} \right] \) are the signed Stirling numbers of the first kind.
2. Star and frequently sequences

Let \( f_i \) denotes the number of vertices of degree \( i, i = 0 \ldots n - 1 \).

The integer sequence

\[
f_1, f_2, \ldots, f_{n-1},
\]

is called the frequency sequence of a graph.

The frequency sequence were studied intensively by several authors, see [3], [4].

There exists a close connection between the star sequence and the frequency sequence of a graph.

Let us recall that two sequences \( \{a_n\}, \{b_n\} \) that satisfies the following conditions

\[
a_i = \sum_{k=i}^{n} \binom{k}{i} b_k, b_i = \sum_{k=i}^{n} (-1)^{k-i} \binom{k}{i} a_k,
\]

are an example of a pair of inverse sequences, see [5] for more details.

The following theorem holds.

**Theorem 1.** Let \( G \) be a simple graph. Then its star and frequency sequences are inverses of each other:

\[
f_i = \sum_{k=i}^{n-1} (-1)^{k-i} \binom{k}{i} S_k(G), 1 < i \leq n - 1,
\]

\[
f_1 = 2S_1(G) + \sum_{k=2}^{n-1} (-1)^{k-i} kS_k(G),
\]

and

\[
2S_1(G) = \sum_{i=1}^{n-1} if_i, \quad S_k(G) = \sum_{i=k}^{n-1} \binom{i}{k} f_i, 1 < k \leq n - 1.
\]

**Proof.** Let us count the number of vertices that have the degree \( k > 1 \). It is clear that the number of vertices of the maximal degree \( n - 1 \) is equal to the number of \( G \)-subgraphs that are isomorphic to the star graph \( S_{n-1} \). Thus \( f_{n-1} = S_{n-1}(G) \). To count the number of vertices of degree \( n - 2 \), observe that the number \( f_{n-2} \) is not equal to \( S_{n-2}(G) \). In fact, the star graph \( S_{n-2} \) is subgraph of \( S_{n-1} \) and some of the vertices of degree \( n - 2 \) will be counted twice. Since each graph \( S_{n-1} \) consists of exactly \( \binom{n-1}{n-2} \) subgraphs \( S_{n-2} \) then
\[ f_{n-2} = S_{n-2}(G) - \binom{n-1}{n-2} S_{n-1}(G). \]

It is becoming obvious that for arbitrary \( f_k, k > 1 \) we can use the inclusion-exclusion principle:

\[ f_k = S_k(G) - \binom{k+1}{k} S_{k+1}(G) + \binom{k+2}{k} S_{k+2}(G) + \cdots + (-1)^{n-k} \binom{n-1}{k} S_{n-1}(G). \]

For the case \( k = 1 \) there exist \( S_1(G) \) edges and every one of them has 2 vertices of degree 1. Thus

\[ f_1 = 2S_1(G) - 2S_2(G) + 3S_3(G) - 4S_4(G) + \cdots + (-1)^{n-2} S_{n-1}(G). \]

Now we are able to express the star sequence of graph \( G \) in terms of its frequency sequence.

\[ 2S_1(G) = \sum_{i=1}^{n-1} \binom{i}{1} f_i, \text{ the Handshaking lemma} \]

\[ S_k(G) = \sum_{i=k}^{n-1} \binom{i}{k} f_i. \]

The following theorem can be proved in the same way as Theorem 1.

**Theorem 2.** We have

\[ 2S_1(G) + \sum_{i=2}^{n-1} (-1)^{i-1} S_i(G) = \sum_{i=1}^{n-1} f_i, \]

\[ 2S_1(G) + \sum_{i=2}^{n-1} (-1)^{i-1} i^m S_i(G) = \sum_{k=1}^{m} (-1)^{k} k! \binom{m}{k} f_k. \]

For the case \( m = 0 \) we get

\[ \sum_{i=1}^{n-1} f_i = 2S_1(G) + \sum_{i=2}^{n-1} (-1)^{i-1} S_i(G), \]

This fact implies the following interesting result:
Theorem 3.
\[ \sum_{uv \in G(V)} \left( \frac{1}{d_u} + \frac{1}{d_v} \right) = S_1(G) + \sum_{i=1}^{n-1} (-1)^{i-1} S_i(G). \]

Proof. From [7] we know that
\[ \sum_{uv \in G(V)} \left( \frac{1}{d_u} + \frac{1}{d_v} \right) = n - f_0. \]

The number of isolated vertices \( f_0 \) of a graph can be determined from its star sequence. Indeed
\[ 2S_1(G) + \sum_{i=2}^{n-1} (-1)^{i-1} S_i(G) = \sum_{i=1}^{n-1} f_i = \sum_{i=0}^{n-1} f_i - f_0 = n - f_0. \]

Thus
\[ \sum_{uv \in G(V)} \left( \frac{1}{d_u} + \frac{1}{d_v} \right) = S_1(G) + \sum_{i=1}^{n-1} (-1)^{i-1} S_i(G). \]

\[ \square \]

3. The first general Zagreb index

Now we can express the first general Zagreb index in term of star sequence.

Theorem 4.
\[ Z_p(G) = 2S_1(G) + \sum_{i=2}^{p} i^p \left\{ \begin{array}{c} p \\ i \end{array} \right\} S_i(G), \]

where \( \left\{ \begin{array}{c} p \\ i \end{array} \right\} \) is the Stirling number of the second kind.

Proof. It is easy to see that
\[ Z_p(G) = \sum_{i=1}^{n-1} i^p f_i. \]

Then
\[ \sum_{i=1}^{n-1} i^p f_i = S_1(G) + \sum_{i=1}^{n-1} i^p \sum_{k=i}^{n-1} (-1)^{k-i} \binom{k}{i} S_k(G) = \]
\[ = S_1(G) + \sum_{k=1}^{n-1} \left( \sum_{i=1}^{k} i^p (-1)^{k-i} \binom{k}{i} \right) S_k(G). \]

By using the well-known identity (see [6], the identity (6.19))
\[ \sum_{i=1}^{k} i^p (-1)^{k-i} \binom{k}{i} = k! \left\{ \begin{array}{l} p \\ k \end{array} \right\}, \]

we get

\[ \sum_{i=1}^{n-1} i^p f_i = S_1(G) + \sum_{k=1}^{n-1} k! \left\{ \begin{array}{l} p \\ k \end{array} \right\} S_k(G). \]

Since \( \left\{ \begin{array}{l} p \\ k \end{array} \right\} = 0 \) for \( k > p \) we thus obtain

\[ \sum_{i=1}^{n-1} i^p f_i = S_1(G) + \sum_{k=1}^{p} k! \left\{ \begin{array}{l} p \\ k \end{array} \right\} S_k(G). \]

\[ \square \]

Let

\[ G(Z, t) = \sum_{p=0}^{\infty} Z^p(G)t^p, \]

be the ordinary generating functions of the sequence of the first general Zagreb indexes.

Let us express the generating functions \( G(Z, t) \) in terms of the frequently and star sequences. The following theorem holds.

**Theorem 5.** Let \( G(Z, t) \) be the ordinary generating functions of the sequence of the first general Zagreb indexes. Then

\[ (i) \quad G(Z, t) = \sum_{k=0}^{n-1} \left( \sum_{i=0}^{k} \left[ \begin{array}{l} n+1 \\ n+1-(k-i) \end{array} \right] Z_i(G) \right) \frac{t^k}{(1-t)(1-2t) \cdots (1-nt)}, \]

\[ (ii) \quad Z_p(G) + \sum_{i=1}^{n} \left[ \begin{array}{l} p+1 \\ p+1-i \end{array} \right] Z_{p-i}(G) = 0, \quad p > n, \]

where \( \left[ \begin{array}{l} p \\ i \end{array} \right] \) is the signed Stirling numbers of the first kind.

**Proof.** (i) Since

\[ Z_p(G) = 2S_1(G) + \sum_{i=2}^{p} i! \left\{ \begin{array}{l} p \\ i \end{array} \right\} S_i(G), \]

then
The function $G(Z, t)$ is defined as:

$$G(Z, t) = \sum_{p=0}^{\infty} Z_p(G)t^p = Z_0(G) + \sum_{p=1}^{\infty} \left( 2S_1(G) + \sum_{i=2}^{p} i! \left\{ \frac{p}{i} \right\} S_i(G) \right) t^p =$$

$$= Z_0(G) + \sum_{p=1}^{\infty} 2S_1(G)t^p + \sum_{p=1}^{\infty} \sum_{i=2}^{p} i! \left\{ \frac{p}{i} \right\} S_i(G)t^p =$$

$$= Z_0(G) + 2S_1(G) \sum_{p=1}^{\infty} t^p + \sum_{i=2}^{\infty} \left( \sum_{p=1}^{\infty} \left\{ \frac{p}{i} \right\} t^p \right) i! S_i(G) =$$

$$= n + \frac{2S_1(G)t}{1-t} + \sum_{i=2}^{n} \frac{i! S_i(G)t^i}{(1-t)(1-2t) \cdots (1-it)}.$$

Here we used the well-known formula for the generating function for the Stirling numbers of the second kind:

$$\sum_{p=1}^{\infty} \left\{ \frac{p}{i} \right\} t^p = \frac{t^i}{(1-t)(1-2t) \cdots (1-it)}.$$

After simplification we get

$$G(Z, t) = \frac{a_0 + a_1 t + a_2 t^2 + \cdots + a_{n-1} t^{n-1}}{(1-t)(1-2t) \cdots (1-nt)},$$

for some unknown numbers $a_0, a_1, \ldots, a_{n-1}$. To define the numbers let us observe that

$$(1-t)(1-2t) \cdots (1-nt) = \sum_{i=0}^{n} \left[ \begin{array}{c} n+1 \\ n+1-i \end{array} \right] t^i.$$

Now

$$a_0 + a_1 t + a_2 t^2 + \cdots + a_{n-1} t^{n-1} =$$

$$= (1-t)(1-2t) \cdots (1-nt) (Z_0(G) + Z_1(G)t + \cdots + Z_n(G)t^n + \cdots) =$$

$$= \left( \sum_{i=0}^{n} \left[ \begin{array}{c} n+1 \\ n+1-i \end{array} \right] t^i \right) (Z_0(G) + Z_1(G)t + \cdots + Z_n(G)t^n + \cdots).$$

Equating coefficients of $t^k$ yields

$$a_k = \sum_{i=0}^{k} \left[ \begin{array}{c} n+1 \\ n+1-(k-i) \end{array} \right] Z_i(G).$$

Therefore
\[ G(Z, t) = \sum_{k=0}^{n-1} \left( \sum_{i=0}^{k} \left[ \frac{n + 1}{n + 1 - (k - i)} \right] Z_i(G) \right) \frac{t^k}{(1-t)(1-2t) \cdots (1-nt)}. \]

\[(ii)\]

The generating function \( G(Z, t) \) is a rational function with the denominator

\[ (1-t)(1-2t) \cdots (1-nt) = \sum_{i=0}^{n} \left[ \frac{n + 1}{n + 1 - i} \right] t^i. \]

Then Theorem 4.1.1 [8] immediately implies that the sequence of the first general Zagreb indices \( Z_0(G), Z_1(G), \ldots, Z_p(G), \ldots \) satisfies the recurrence relation

\[ Z_p(G) + \sum_{i=1}^{n} \left[ \frac{n + 1}{n + 1 - i} \right] Z_{p-i}(G) = 0, \]

for all \( p \geq n. \)

\[ \square \]

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