1/(N − 1) Expansion Approach to Full-counting Statistics for the SU(N) Anderson Model

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We apply a recently developed 1/(N − 1) expansion to the full-counting statistics for the N-fold degenerate Anderson impurity model in the Kondo regime. This approach is based on the perturbation theory in the Coulomb interaction $U$ and is different from the conventional large-$N$ theories, such as the usual 1/N expansion and non-crossing approximation. We have confirmed that the calculations carried out up to order 1/(N − 1)$^2$ agree closely with those of the numerical renormalization group at $N = 4$, where the degeneracy is still not so large. This ensures the applicability of our approach for $N \geq 4$. We present the results of the cumulants of the probability distribution function for a nonequilibrium current through a quantum dot in the particle-hole symmetric case.

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I. INTRODUCTION

The universal Kondo behavior in quantum dots has been studied intensively for a nonequilibrium current [1–8] and shot noise [9–13]. The effects of orbital degeneracy on the universality have also been a subject of current interest [14–18].

The essential features of the low-energy properties of orbital systems may be deduced from the SU(N) Anderson model, for which two or three interacting orbitals, excluding the one prohibited by the Pauli principle. With this scaling, the perturbation series in $U$ can be reorganized as an expansion in powers of 1/(N − 1). To leading order in 1/(N − 1), it describes the Hartree-Fock random phase approximation (HF-RPA). The higher-order corrections systematically describe the fluctuations beyond the HF-RPA. As the unperturbed Hamiltonian includes the tunneling matrix element between the impurity and the conduction bands, this approach naturally describes the Fermi-liquid state.

In this report, we discuss dependences of the renormalized parameters on the degeneracy $N$, carrying out the calculations up to order 1/(N − 1)$^2$. Furthermore, we apply this approach to the full-counting statistics [27] for the nonequilibrium current distribution through quantum dots in the Kondo regime.

II. MODEL AND FORMULATION

We consider the SU(N) impurity Anderson model with a finite interaction $U$ connected to two leads ($\nu = L, R$):

$$
\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_U,
$$

$$
\mathcal{H}_0 = \sum_{m=1}^{N} \xi_d d_m^\dagger d_m + \sum_{\nu=L,R} \sum_{m=1}^{N} \int_{D} d\epsilon \epsilon c_{\nu,m}^\dagger c_{\nu,m} + \sum_{\nu=L,R} \sum_{m=1}^{N} v_\nu \left( d_m^\dagger \psi_\nu + \psi_\nu^\dagger d_m \right),
$$

$$
\mathcal{H}_U = \sum_{m \neq m'} U \left( n_{dm} - \frac{1}{2} \right) \left( n_{dm'} - \frac{1}{2} \right).
$$

Here, $\xi_d = \epsilon_d + (N - 1)U/2$, $d_m^\dagger$ is the creation operator for an electron with energy $\epsilon_d$ and orbital $m$ $(= 1, 2, \cdots, N)$ in the impurity site, and $n_{dm} = d_m^\dagger d_m$. The operator $c_{\nu,m}^\dagger$ for a conduction electron in the lead $\nu$ is normalized as $\{c_{\nu,m}, c_{\nu',m'}^\dagger\} = \delta_{\nu\nu'} \delta_{mm'} \delta(\epsilon - \epsilon')$, and $\psi_\nu = \int_{D} d\epsilon \sqrt{\rho} c_{\nu}$.

The hybridization energy scale is given by $\Delta = \Gamma_L + \Gamma_R$, with $\Gamma_\nu = \pi \rho v_\nu^2$ and $\rho = 1/(2D)$.

We use the imaginary-frequency Green’s function that is given, for $|\omega| \ll D$, by

$$
G(\omega) = \frac{1}{i\omega - \xi_d + i\Delta \mathrm{sgn} \omega - \Sigma(i\omega)}.
$$

Here, $\Sigma(i\omega)$ is the self-energy due to $\mathcal{H}_U$. The ground-state average of the local charge, $\langle n_{dm} \rangle = \delta/\pi$, can be deduced from the phase shift $\delta \equiv \cot^{-1}(E_d^2/\Delta)$ with $E_d^2 = \xi_d + \Sigma(0)$. The renormalized parameters are defined by $1/z \equiv 1 - \partial \Sigma(i\omega)/\partial(i\omega)|_{\omega=0}$, $\epsilon_d \equiv z E_d^2$, and...
\[ \Delta \equiv z \Delta. \]

Furthermore, the enhancement factors for the spin and charge susceptibilities, \( \tilde{\chi}_s \equiv \tilde{\chi}_{mm} - \tilde{\chi}_{mm'} \) and \( \tilde{\chi}_c \equiv \tilde{\chi}_{mm} + (N-1) \tilde{\chi}_{mm'} \), can be expressed in terms of \( z \) and the vertex function \( \Gamma_{mm';mm'}(i\omega_1, i\omega_2; i\omega_3, i\omega_4) \) for \( m \neq m' \) \[28, 29\]:

\[
\tilde{\chi}_{mm} = \frac{1}{z}, \quad \tilde{\chi}_{mm'} = -\frac{\sin^2 \delta}{\pi \Delta} \Gamma_{mm';mm'}(0,0;0,0), \quad (4)
\]

and \( \tilde{U} \equiv z^2 \Gamma_{mm';mm'}(0,0;0,0) \) for \( m \neq m' \) represents the residual interaction between the quasi-particles.

We introduce a scaling for the bare and the renormalized interactions with a factor \( N-1 \) of \[23, 24\]:

\[
g \equiv \frac{(N-1)U}{\pi \Delta}, \quad \tilde{g} \equiv \frac{(N-1)\tilde{U}}{\pi \Delta}. \quad (5)
\]

With these parameters, the Wilson ratio \( R \equiv z \tilde{\chi}_s \) and that for the charge sector can be expressed in the form,

\[
R = 1 + \frac{\tilde{g}}{N-1} \sin^2 \delta, \quad z \tilde{\chi}_c = 1 - \tilde{g} \sin^2 \delta. \quad (6)
\]

One of the merits of this scaling is that the perturbation expansion with respect to \( H_U \) can be classified according to the power of \( 1/(N-1) \), taking \( g \) as an independent variable. In the present report, we consider the particle-hole symmetric case: \( \xi_d = -(N-1)U/2 \) and \( \delta = \pi/2 \).

### III. Expansion up to Order \( 1/(N-1)^2 \)

At zero order with respect to \( 1/(N-1) \), the limit of \( N \to \infty \) is taken at fixed \( g \), and it gives the Hartree-Fock result. Specifically, at half-filling, the unperturbed Green’s function is given by \( G_0(i\omega) = [i\omega + i\Delta \text{sgn} \omega]^{-1} \) as \( \xi_d = 0 \).

The leading-order corrections in the \( 1/(N-1) \) expansion arise from a series of bubble diagrams, \( U_{\text{bub}}(i\omega) \), of the RPA type as shown in Fig. 2.

\[
U_{\text{bub}}(i\omega) = U + U \sum_{k=1}^{\infty} A_k \left[ -U \chi_0(i\omega) \right]^k. \quad (7)
\]

Here, \( \chi_0(i\omega) = -\int \frac{d\omega'}{2\pi} G_0(i\omega + i\omega')G_0(i\omega') \), and the coefficient \( A_k = (N-1)^k \sum_{p=0}^{k} [-1/(N-1)]^p \) arises from the summation over the orbital indices for a series of fermion loops. Correspondingly, the order \( 1/(N-1) \) contributions of \( \Gamma_{mm';mm'}(0,0;0,0) \) arise from the first diagram in Fig. 2 and give the leading-order correction to the renormalized coupling as

\[
\tilde{g} = \frac{g}{1 + g} + O\left(\frac{1}{N-1}\right). \quad (8)
\]

This correction determines the Wilson ratio \( R \) to order \( 1/(N-1) \) through Eq. (5). Similarly, the order \( 1/(N-1) \) self-energy arises from the second diagram in Fig. 2.

Higher-order fluctuations beyond the RPA appear first through the next-leading-order contributions. Figures 3 and 4 show the order \( 1/(N-1)^2 \) diagrams for the vertex function and the self-energy in the particle-hole symmetric case, respectively. These contributions and the higher-order components from Eq. (5) determine the renormalization factor \( z \) and the Wilson ratio \( R \) to order \( 1/(N-1)^2 \). In the present work, we have calculated all these next-leading-order contributions. An outline of the calculations is given in a previous paper \[24\]. We have also carried out the NRG calculations for \( N = 4 \) to demonstrate the reliability of the \( 1/(N-1) \) expansion.

Figure 5 shows the next-leading-order results for \( \tilde{g} \) and \( z \), plotted as functions of \( g \). We see the very close agreement between the NRG and the next-leading-order results in the \( 1/(N-1) \) expansion for \( N = 4 \), especially for the renormalized coupling \( \tilde{g} \). The two curves for \( \tilde{g} \), for \( N = 4 \), almost overlap each other over the whole range of \( g \) although the next-leading-order results (green dashed line) are slightly smaller than those of the NRG (solid circles). As \( N \) increases, \( \tilde{g} \) converges rapidly to the RPA value, \( \tilde{g} \to g/(1 + g) \), which is asymptotically exact in the limit of \( N \to \infty \). We also see that the value that \( \tilde{g} \) can take is bounded in a very narrow region between the curve for \( N = 4 \) and that for the \( N \to \infty \) limit.
The average and the fluctuations of the steady current at low energies can be described by the local Fermi-liquid theory, which follows from this form of the Green’s function. The Fermi-liquid theory, or the related renormalized perturbation theory (RPT) [32], can also be applied to the full-counting statistics for nonequilibrium transport. Specifically, we consider the probability distribution $P(q)$ of the transferred charge $q = (q_1, q_2, \cdots, q_n)$ from the left lead to the orbital $m$ of the Anderson impurity during a time interval $T$. The generating function for this probability distribution is defined by

$$\chi(\lambda) = \sum_n e^{i\lambda n} P(q)$$

with the counting fields $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_N)$, and can be expressed in the form [22]

$$\chi(\lambda) = \left\langle T_C \exp \left\{ -i \int_C d\tau [\mathcal{H}^T_\tau(t) + \mathcal{H}_U(t)] \right\} \right\rangle. \quad (10)$$

Here, $T_C$ is the time-ordering operator along the Keldysh contour $C$, and the time evolution is defined with respect to an extended Hamiltonian $\mathcal{H}^\lambda(t) = \mathcal{H}_0 + \mathcal{H}^T_\tau(t) + \mathcal{H}_U$:

$$\mathcal{H}^\lambda(t) = \sum_{km} \left[ \nu_L e^{i\lambda_m(t)/2} d^\dagger_m c_{kLm} + \nu_R d^\dagger_m c_{kRm} \right] + \text{H.c.} \quad (11)$$

Here, the counting field depends on the path such that $\lambda_m(t_\pm) = \pm \lambda_m$ for the forward ($t_-$) and backward ($t_+$) paths, respectively, and is switched on during a time interval $[t : 0 \rightarrow T \rightarrow 0]$.

We obtained an explicit expression of $\chi(\lambda)$, which is asymptotically exact at low energies up to order $(eV)^2$ for general $N$ and $U$, using the RPT [17, 18]. The cumulant for the full current can be derived from $\chi(\lambda)$, choosing the counting fields to be $\lambda_m = \lambda$ for all $m$ and then taking a derivative $C_n = (-i)^n \frac{d^n}{d\lambda^n} \ln \chi(\lambda)$:

$$C_n = \frac{J_n}{T} \left[ \delta_{1n} + \frac{(-1)^n}{12} \left\{ 1 + \left( \frac{1 + 2^{n+1}}{(N-1)} \right) \left( \frac{2eV}{\Delta} \right)^2 \right\} \right]. \quad (12)$$

Here, $J_n = Nc^2V/(2\pi\hbar)$ is the linear-response current of the unitary limit, and $\delta_{nm'}$ is the Kronecker’s delta. Equation (12) shows that the nonequilibrium properties can also be characterized by the two renormalized parameters, $\bar{g}$ and the Kondo energy scale $\Delta$, at low energies in the particle-hole symmetric case. For $n = 1$, the cumulant $eC_1/T$ expresses the steady current. The universal properties of the cumulants for $n \geq 2$ can be extracted from the Fano-factor-inspired ratio (FFIR), which is normalized with respect to the backscattering current $J_b \equiv J_n - eC_1/T$ [12]:

$$C_n \equiv \frac{1 + \left( \frac{1 + 2^{n+1}}{N-1} \right) g^2}{\frac{5g^2}{N-1}}, \quad \frac{C_n}{\bar{g}} \equiv \left( -\frac{1}{n} \right) \frac{J_n}{e}. \quad (13)$$

Here, $C_n^P$ can also be regarded as the Poisson value of the cumulant, and the FFIR for $n = 2$ corresponds to the Fano factor [10].

Figure 6 shows the ratios $C_n^P/C_n^*$ for the noise $n = 2$ (top), the skewness $n = 3$ (middle), and sharpness $n = 4$ (bottom). The results are compared with the RPT results [17, 18]. The backscattering current $J_b \equiv 4$ at $g = 4$ in the unitary limit, and $\delta_{nm'}$ is the Kronecker’s delta. Equation (12) shows that the nonequilibrium properties can also be characterized by the two renormalized parameters, $\bar{g}$ and the Kondo energy scale $\Delta$, at low energies in the particle-hole symmetric case. For $n = 1$, the cumulant $eC_1/T$ expresses the steady current. The universal properties of the cumulants for $n \geq 2$ can be extracted from the Fano-factor-inspired ratio (FFIR), which is normalized with respect to the backscattering current $J_b \equiv J_n - eC_1/T$ [12]:

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FIG. 6: (Color online) The FFIR $C_n/C_P^n$ for $n = 2$ (top), $n = 3$ (middle), and $n=4$ (bottom) are plotted as functions of $g$ for $N = 2$ (Bethe ansatz), and $N = 4, 6, 8, 10, 20$ (next-leading-order results in the $1/(N-1)$ expansion). The NRG results (solid circles) are also shown for $N = 4$.

In summary, carrying out the $1/(N-1)$ expansion up to order $1/(N-1)^2$, we have obtained the cumulants for the probability distribution of the current through a quantum dot with orbital degeneracy $N > 4$ at low energies. The results of the renormalized coupling $\tilde{g}$ show excellent agreement at $N = 4$ with the exact NRG results in the particle-hole symmetric case. This enables us to gain almost exact numerical results for the Fano-factor-inspired ratio $C_n/C_P^n$ for $N > 4$.

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