Phase diagram of hot and dense QCD constrained by the Statistical Model

Kenji Fukushima

Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan

We propose a prescription to constrain the chiral effective model approach to the QCD phase diagram using the thermal Statistical Model, which is a description consistent with the experimental data at the freeze-out. In the transition region where thermal quantities of hadrons blow up, deconfined quarks and gluons should smoothly take over the degrees of freedom from hadrons in the Statistical Model. We use the Polyakov-loop coupled Nambu–Jona-Lasinio (PNJL) model as an effective description in the quark side. We require that the validity regions of these descriptions should have an overlap on the phase diagram, which constrains model uncertainty. Our results favor a phase diagram with the chiral phase transition located at slightly higher temperature than deconfinement.

PACS numbers: 12.38.Aw, 11.10.Wx, 11.30.Rd, 12.38.Gc

Introduction

Exploration of the QCD (Quantum Chromodynamics) phase diagram, particularly toward higher baryon-density regime, is of increasing importance in both theoretical and experimental sides [1]. From the theoretical point of view, so far, only the lattice-QCD simulation [1, 2] is the first-principle approach at work to the QCD phase transitions — chiral restoration and quark deconfinement. The functional renormalization group method is also developing as a promising non-perturbative tool [3]. The chiral condensate \( \langle \bar{\psi} \psi \rangle \) and the Polyakov loop \( \Phi \) are the (approximate) order parameters for chiral restoration and quark deconfinement, respectively, which are gauge invariant and measurable on the lattice. The lattice-QCD simulation is, however, of practical use only when the baryon chemical potential \( \mu_B \) is sufficiently smaller than the temperature \( T \). For \( \mu_B/T \gtrsim 1 \) the notorious sign problem prevents us from extracting any reliable information from the lattice-QCD data [1, 4].

The effective model study is an alternative and pragmatic approach toward the phase diagram of dense QCD. The idea is the following: one starts with some models that yield a reasonable description of hadron properties in the vacuum and then puts them in a finite-\( T \) and/or finite-\( \mu_B \) environment. What is recognized nowadays as the “QCD phase diagram” is actually a theoretical conjecture based on various effective model studies.

Along this line the Polyakov-loop coupled chiral models such as the PNJL (Polyakov–Nambu–Jona-Lasinio) [5, 6] and the PQM (Polyakov-Quark-Meson) [7, 8] models are successful to handle \( \langle \bar{\psi} \psi \rangle \) and \( \Phi \) on the equal footing. Besides, the Polyakov loop potential \( U(\Phi) \) is determined by \( \Phi \) and the pressure \( p \) measured in the lattice simulation of the pure gluonic theory. This means that the model includes the pressure contribution from gluons as well as quarks, so that the model is able to deal with the full thermodynamics comparable with the full lattice-QCD simulation. The point is that the dynamics of transverse gluons \( A_T^G \) is under the control of the deconfinement order parameter \( \Phi \) and thus is to be encompassed in the Polyakov loop potential \( U(\Phi) \), while the Polyakov loop itself is expressed in terms of the longitudinal gluon \( A_4 \).

Since theory instruments to examine both \( \langle \bar{\psi} \psi \rangle \) and \( \Phi \) are now in our hands, it is intriguing to address the following question: whether the chiral and deconfinement phase transitions would go on simultaneously or separate after all when the baryon density increases. There are then two key issues. One is the so-called QCD (chiral) critical point (which is often called the critical end-point) at which the chiral and the baryon number susceptibilities diverge [9, 11] and the higher moments are even more singular [12]. The other one is a triple-point-like region associated with the appearance of quarkyonic matter [13, 16] where the baryon abundance surpasses mesons.

One reasonable way to characterize quarkyonic matter for finite-\( N_c \) QCD is to use two order parameters \( \Phi = 0 \) and the quark (baryon) number density \( \langle \bar{\psi} \psi \rangle \neq 0 \), which would definitely work for \( N_c = \infty \) [13]. In principle this statement is not directly related to chiral symmetry, but a substantially large value of \( \langle \bar{\psi} \psi \rangle \) is favored by light quarks existing in the chiral symmetric phase. In this sense, practically, one can identify the quarkyonic phase as an exotic state where chiral symmetry is restored first (\( \langle \bar{\psi} \psi \rangle \neq 0 \)) and still the confining property remains (\( \Phi \approx 0 \)). In other words the bulk pressure is mostly dominated by light quarks and, nevertheless, excited quarks on top of the Fermi sphere feel a confining force. [There is an argument that the confining force might cause inhomogeneous chiral condensation [14]. Such a possibility is beyond our current scope.]

There is no strong evidence for such an exotic window. Phenomenological considerations could lead to a different scenario [15], though some suggested arguments have been reported [16, 17] and some model studies are supportive [5, 18]. In general the PNJL and PQM models rather favor the quarkyonic picture; the model predicts the deconfinement temperature weakly dependent on \( \mu_B \). The Polyakov loop tends to be small for any \( \mu_B \).
as long as $T$ is vanishingly small, whereas the chiral condensate melts at high $\mu_B$. However, the serious problem in any model studies is that the model-parameter choice is largely uncertain. The PNJL and PQM models are not exceptions. The situation is worse at higher $\mu_B$ because the lattice-QCD data is unavailable then. It should fatally depends on model assumptions whether the phase diagram has the critical point(s) and/or quarkyonic matter or even nothing at all. To make any solid statement, it is indispensable to impose some constraints on the effective model. In this work we attempt to deduce the phase structure from the phenomenological point of view.

**Thermodynamics from the Statistical Model** Regarding the QCD phase diagram at finite $T$ and $\mu_B$ useful information is quite limited. Only the chemical freeze-out points in the heavy-ion collisions are experimental hints about the phase diagram. Although the freeze-out points shape an intriguing curve on the $\mu_B$-$T$ plane, as plotted by error-bar dots in Fig. 1, one should carefully interpret it.

The freeze-out points are not the raw experimental data but an interpretation through the Statistical Model [19, 20]. In view of the fact that the Statistical Model is such successful to fit various particle ratios with $\mu_B$ and $T$ only ($\mu_Q, \mu_s, \mu_c$ are determined by the initial condition), it should be legitimate to take the freeze-out points for experimental data, which in turn validates the Statistical Model (though why it works lacks for an explanation from QCD).

Let us proceed further accepting that the Statistical Model is a valid description of the state of matter until the freeze-out curve or slightly above. It is then a straightforward application of the Statistical Model to estimate thermodynamic quantities such as the pressure $p$, the entropy density $s$, the baryon number density $n$, etc. We shall utilize the open code THERMUS ver.2.1 to calculate $s$ and $n$ at various $T$ and $\mu_B$ [21].

Figure 1 shows the chemical freeze-out points taken from Refs. [19, 20], on which $s$ and $n$ are overlaid. For convenience we normalized these quantities by

$$s_{\text{free}} = \left\{ \left( N_c^2 - 1 \right) + \frac{7}{4} N_c N_f \right\} \frac{4\pi^2}{45} T^3 + \frac{N_c N_f}{3} \mu^2 T^2,$$

$$n_{\text{free}} = N_f \left( \frac{\mu^3}{3\pi^2} + \frac{\mu c T^2}{3} \right).$$

These are the entropy density and the baryon number density of free massless $N_c^2 - 1$ gluons and $N_c N_f$ quarks.

Here we note that, in drawing Fig. 1 we have intentionally relaxed the neutrality conditions for electric charge and heavy flavors and simply set $\mu_Q = \mu_s = \mu_c = 0$. We have done so to make it possible to compare the results from the Statistical Model to the chiral effective model approach in later discussions. [We note that one can force the chiral model to satisfy neutrality but it would be technically involved [22].] Nevertheless, we would emphasize that the neutrality conditions have only minor effects on the bulk thermodynamics and make only small differences in any case.

We should mention that we used Eq. (1) with $N_c = N_f = 3$. The choice of $s_{\text{free}}$ and $n_{\text{free}}$ is arbitrary and the following discussions do not rely on this choice, for we will use $s_{\text{free}}$ and $n_{\text{free}}$ just as common denominators to display the Statistical Model and the PNJL model results.

The Statistical Model cannot tell us about the QCD phase transitions. Still, Fig. 1 is suggestive enough. We can clearly see the thermodynamic quantities from the Statistical Model blowing up in a relatively narrow region. The red and blue (upper and lower) bands indicate the regions where $s/s_{\text{free}}$ and $n/n_{\text{free}}$, respectively, grow quickly from 0.3 to 0.8. In the Hagedorn’s picture [22] this rapid and simultaneous rise in $s$ and $n$ has a natural interpretation as the Hagedorn limiting temperature, above which color degrees of freedom is liberated, i.e. color deconfinement.

**Thermodynamics from the PNJL Model** Figure 1 is useful to have a guess-estimate about the deconfinement boundary but we can deduce no information about the chiral property. So, to address the QCD phase transitions, we must find a way to connect the thermodynamics in Fig. 1 to the order parameters $\langle \bar{\psi} \psi \rangle$ and $\Phi$. Here let us go into details of the chiral effective model for that purpose.

It is essential to adopt the Polyakov-loop augmented model here because the entropy density should contain the contribution from gluons which is taken care of by the Polyakov loop potential. The PNJL model that we use...
As shown in Fig. 1) and that in the PNJL model with a choice $T_0 = 200$ MeV (top band with green color). The blue band between two represents the results with the ansatz (3).

In what follows is defined with the following potential:

$$U[\Phi, \bar{\Phi}] = T^4 \left\{ -\frac{a(T)}{2} \bar{\Phi} \Phi + b(T) \ln \left[ 1 - 6\bar{\Phi} \Phi + 4(\bar{\Phi}^3 + \Phi^3) - 3(\bar{\Phi} \Phi)^2 \right] \right\} \tag{2}$$

with $a(T) = a_0 + a_1(T_0/T) + a_2(T_0/T)^2$ and $b(T) = b_3(T_0/T)^3$. There are thus five parameters one out of which is constrained by the Stefan-Boltzmann limit. These parameters are fixed by the pure-gluonic lattice data as $a_0 = 3.51$, $a_1 = -2.47$, $a_2 = 15.2$, $b_3 = -1.75$, and $T_0 = 270$ MeV from the deconfinement temperature of first order in the pure-gluonic theory. It is important to note that only $T_0$ is a parameter with the mass dimension, so that the energy scale is set by this $T_0$.

In the presence of dynamical quarks, if we keep using $T_0 = 270$ MeV, the simultaneous crossover temperature of deconfinement and chiral restoration is over 200 MeV, which is too high as compared to the lattice-QCD value. It is argued in Ref. [2] that the back-reaction from quark loops affects the mass scale $T_0$ which changes from $T_0 = 270$ MeV for $N_f = 0$ reduced down to $T_0 = 208$ MeV for $N_f = 2$ and $T_0 = 187$ MeV for $N_f = 2 + 1$ [6]. In this work we choose to use $T_0 = 200$ MeV throughout.

In Fig. 2 we show the entropy density calculated in the mean-field PNJL model with $T_0 = 200$ MeV in the same way as presented in Fig. 1. The bottom (top) band in red (green) color is the result from the Statistical Model (PNJL model). From the figure it is obvious that the PNJL model is not consistent with the Statistical Model even qualitatively. With the properly scaled $T_0$ from 270 MeV down to 200 MeV, the blow-up behavior in $s$ from the Statistical Model could be smoothly connected to the PNJL model description only for $\mu_B \lesssim 400$ MeV. The curvature of the band as a function of $\mu_B$ is so different; the PNJL model result is too flat horizontally and it eventually take apart from the region where the Statistical Model breaks down.

**Problem and Solution.** Such a manifest discrepancy from the Statistical Model is a crucial drawback of the PNJL model. To make the entropy density at $\mu_B \gtrsim 500$ MeV get saturated, quark degrees of freedom must be released at smaller temperature than predicted by the PNJL model.

One can imagine why this happens in the following way: the energy scale in the Polyakov loop potential is specified by the parameter $T_0$ that may differ depending on the effects of $T$ and $\mu_B$ in the quark sector. We have shifted $T_0$ from 270 MeV down to 200 MeV, through which we have incorporated the scale change induced by $N_f$ quarks at finite $T$. In this way we may well consider that $T_0$ should decrease with increasing $\mu_B$ [7].

Our idea proposed here is to make use of Fig. 2 to fix $T_0(\mu_B)$ for consistency with phenomenology. [One can pick up other thermodynamic quantities than the entropy density like the internal energy density, which would anyway make little change in the final result.] In Ref. [19] the freeze-out curve is parametrized as $T_f(\mu_B) = a - b \mu_B - c \mu_B^3$ with the fitting result $a = 166(2)$ MeV, $b = 1.39(16) \times 10^{-4}$ MeV$^{-1}$, and $c = 5.3(21) \times 10^{-11}$ MeV$^{-3}$. Because the behavior of the entropy density must be dominantly controlled by deconfinement, we postulate that the change in $T_0$ is to be correlated with $T_f(\mu_B)$. [We see that the freeze-out points and the entropy band in Fig. 1 have roughly same curvature.] Let us simply use same $b$ and make an ansatz as

$$\frac{T_0(\mu_B)}{T_0} = 1 - (bT_0)(\frac{\mu_B}{T_0})^2, \tag{3}$$

which yields the blue band in the middle of Fig. 2. [To prevent unphysical negative $T_0$ for large $\mu_B$ we set a threshold at 10 MeV so that $T_0 \geq 10$ MeV. Hence, the results at $T < 10$ MeV are not meaningful.] We see at a glance that the results from this modified PNJL model have a reasonable overlap with the Statistical Model results in the whole density region as plotted.

At this point one might have thought that the energy scale in the quark (NJL) sector should be modified as well. We will come back to this question after discussing the phase diagram next.
Phase Diagram  Now we get ready to proceed to the possible QCD phase diagram that is at least consistent with the Statistical Model outputs in Fig. 1. Using the standard computational procedure of the mean-field PNJL model we can solve $\langle \bar{\psi}\psi \rangle$ and $\Phi$ as functions of $T$ and $\mu_B$, from which the phase boundaries of chiral restoration and deconfinement are located.

Figure 3 shows the phase diagram from the modified PNJL model. The blue (red) band is a region where the Polyakov loop (normalized light-quark chiral condensate) increases from 0.4 to 0.6. In contrast to the old PNJL model ones, the new results show that the chiral phase transition is almost parallel to and entirely above the deconfinement, which agrees with the situation considered recently in Ref. [12]. We have found the critical point at $(\mu_B, T) \simeq (45 \text{ MeV}, 330 \text{ MeV})$, but would not take it seriously since its location is easily affected in $\Phi$. Still, it is a good news for the critical point search that two QCD phase transitions stay close to each other, for the experimental signature would be detectable only if the critical point sits sufficiently near the freeze-out point.

Discussions  It is an intriguing observation that the chiral phase transition occurs later than deconfinement. This is quite consistent with the Statistical Model assumption. In the Statistical Model the hadron masses are just the vacuum values and any hadron mass/width modifications are neglected, which would be a reasonable treatment only if the chiral phase transition is above the Hagedorn temperature. Under such a phase structure, besides, our assumption of neglecting $\mu_B$-dependence in the NJL-model parameters turns out to be as acceptable as the Statistical Model treatment. This can be understood from the fact that the NJL part yields the hadron masses in the vacuum which are intact in the Statistical Model.

The failure of the standard PNJL model is attributed to baryonic degrees of freedom missing in a non-confining quark description. Hence, one may say that a modification made in $\Phi$ stems from such crossover between baryons and quarks, which is presumably parametrized by the Polyakov loop alone, similarly to the transverse gluon pressure. It is an important question how our phenomenological input is validated/invalidated from the first-principle QCD calculation, which will be answered by future developments in the functional renormalization group method.

Finally, our conclusion is that, if quarkyonic matter is defined by restored chiral symmetry with confinement, it is not consistent with the Statistical Model and is unlikely to occur. However, to complete our analysis it should be necessary to think of the quarkyonic spiral, which is an important future problem.

Acknowledgments  The author thanks Y. Hidaka for numerical assistance. He also thanks K. Redlich and J.M. Pawlowski for discussions and A. Andronic for the numerical data of his freeze-out points. This work is supported by Japanese MEXT grant No. 20740134 and in part by Yukawa International Program for Quark Hadron Sciences.
[14] T. Kojo, Y. Hidaka, L. McLerran and R. D. Pisarski, arXiv:0912.3800 [hep-ph].
[15] P. Castorina, R. V. Gavai and H. Satz, arXiv:1003.6078 [hep-ph].
[16] A. Andronic et al., Nucl. Phys. A 837, 65 (2010).
[17] S. Hands, S. Kim and J. I. Skullerud, arXiv:1001.1682 [hep-lat].
[18] C. Sasaki, B. Friman and K. Redlich, Phys. Rev. D 75, 074013 (2007); L. McLerran, K. Redlich and C. Sasaki, Nucl. Phys. A 824, 86 (2009).
[19] J. Cleymans, H. Oeschler, K. Redlich and S. Wheaton, Phys. Rev. C 73, 034905 (2006).
[20] F. Becattini, J. Manninen and M. Gazdzicki, Phys. Rev. C 73, 044905 (2006); A. Andronic, P. Braun-Munzinger and J. Stachel, Phys. Lett. B 673, 142 (2009).
[21] S. Wheaton and J. Cleymans, Comput. Phys. Commun. 180, 84 (2009) [arXiv:hep-ph/0407174].
[22] K. Fukushima, Phys. Rev. D 79, 074015 (2009).
[23] N. Cabibbo and G. Parisi, Phys. Lett. B 59, 67 (1975).
[24] T. Hatsuda and T. Kunihiro, Phys. Rept. 247, 221 (1994).