1 Introduction

We review efforts to unify both the Bardeen, Cooper & Schrieffer (BCS) and Bose-Einstein condensation (BEC) pictures of superconductivity. We have finally achieved this in terms of a “complete boson-fermion (BF) model” (CBFM) that reduces in special cases to all the main continuum (as opposed to “spin”) statistical theories of superconductivity. Our BF model is “complete” in the sense that not only two-electron (2e) but also two-hole (2h) Cooper pairs (CPs) are allowed in arbitrary proportions. In contrast, BCS-Bogoliubov theory—which can also be considered as the theory of a mixture of kinematically independent electrons, 2e- and 2h-CPs—allows only equal, 50%-50%, mixtures of the two kinds of CPs. This is obvious from the perfect symmetry about $\mu$, the electron chemical potential, of the well-known Bogoliubov [1] $v^2(\epsilon)$ and $u^2(\epsilon)$ coefficients, where $\epsilon$ is the electron energy. The CBFM is then applied to see: a) whether the BCS model interaction for the electron-phonon dynamical mechanism is sufficient to predict the unusually high values [2] of $T_c$ (in units of the Fermi temperature) of $\simeq 0.01-0.1$ exhibited by the so-called “exotic” superconductors [3] in both 2D and 3D—relative to the low values of $\lesssim 10^{-3}$ more or less correctly predicted by BCS theory for conventional, elemental superconductors; and b) whether it can at least suggest, if not explain, why “hole superconductors” have higher $T_c$’s.
Boson-fermion (BF) models of superconductivity as a BEC [4, 5] go back to the mid-1950’s [6]-[9], pre-dating even the BCS-Bogoliubov theory [10, 11]. Although BCS theory only contemplates the presence of “Cooper correlations” of single-particle states, BF models [6]-[9], [12]-[20] posit the existence of actual bosonic CPs. Such pair charge carriers have been observed in magnetic flux quantization experiments with elemental [21, 22] as well as with cuprate [23] superconductors. But apparently no experiment has yet been done that distinguishes between electron and hole CPs. The fundamental drawback of early [6]-[9] BF models, which considered 2e bosons in analogy with diatomic molecules in a classical gas mixture, is the notorious absence of an electron energy gap $\Delta$. The gap first began to appear in later BF models [12]-[16]. With two [15, 16] exceptions, however, all BF models neglect the effect of hole CPs formulated on an equal footing with electron CPs to give us a complete BF model (CBFM) consisting of both bosonic CP species coexisting with unpaired electrons.

Without going into a detailed justification we merely list several common “myths” in the theory of superconductivity that we tacitly disbelieve:

1. With the electron-phonon dynamical mechanism transition temperatures $T_c \lesssim 45$ K at most. For higher $T_c$’s one needs magnons or excitons or plasmons or other electronic mechanisms.

2. Cooper pairs (CPs):
   a) consist of negative-energy stable (i.e., stationary) bound states [24];
   b) propagate in the Fermi sea with energy $\hbar^2 K^2/2(2m)$ (Ref. [9], p. 94) where $\hbar K$ is the total or center-of-mass momentum (CMM) of the composite pair;
   c) have a linear dispersion $E \propto v_F \hbar K$, with $v_F$ the Fermi velocity, which is merely the acoustic mode in the ideal Fermi gas (+ interactions) with sound speed $v_F/\sqrt{d}$ in any dimensionality $d$;
   d) “...with $K \neq 0$ represent states with net current flow [25];”
   e) are not bosons (Ref. [26], p. 38). And most notoriously, that:

3. Superconductivity is unrelated to BEC [27].

We question all of these assertions which will be discussed in greater detail elsewhere.

2  The CBFM Hamiltonian

The CBFM [15, 16] is described by $H = H_0 + H_{int}$ where the unperturbed Hamiltonian $H_0$ corresponds to an ideal (i.e., noninteracting) gas mixture of fermions and both types of CPs,
two-electron (2e) and two-hole (2h), namely

\[ H_0 = \sum_{k_{1,s_1}} \varepsilon_{k_1} a_{k_{1,s_1}}^+ a_{k_{1,s_1}} + \sum_K E_\pm(K) b_K^+ b_K - \sum_K E_-(K) c_K^+ c_K, \]  

where \( K \equiv k_1 + k_2 \) is the CMM wavevector, \( k \equiv \frac{1}{2}(k_1 - k_2) \) being the relative one, while \( \varepsilon_k \equiv \hbar^2 k^2/2m \) are the electron and \( E_\pm(K) \) the 2e-/2h-CP energies. Here \( a_{k_{1,s_1}}^+ \) (\( a_{k_{1,s_1}} \)) are creation (annihilation) operators for fermions and similarly \( b_K^+ \) (\( b_K \)) and \( c_K^+ \) (\( c_K \)) for 2e- and 2h-CP bosons, respectively.

Two-hole CPs are considered distinct and kinematically independent from 2e-CPs as their Bose commutation relations involve a relative sign change, in sharp contrast with electron or hole fermions whose Fermi anticommutation relations do not. In fact, holes have a dramatic effect even in the simple, elementary CP problem where they were originally neglected thereby giving [24] a negative-real-energy, stationary (i.e., infinite-lifetime) two-fermion bound-state. But if electrons and holes are treated equally and simultaneously through a Bethe-Salpeter (BS) equation (see, e.g., Ref. [28] p. 131) in the ideal Fermi gas (IFG) ground-state about which the CPs are defined, the resulting energy is pure imaginary [11],[29]—implying an obvious instability. The IFG-based CP problem is thus meaningless if particles are taken on an equal footing with holes, as consistency would demand. However, a similar BS treatment not about the IFG but about the BCS ground-state yields [30] real (but positive) 2e- and 2h-CP energies, along with an imaginary part that is nonzero only for \( K \neq 0 \) signifying a finite lifetime, but zero for \( K = 0 \) implying permanent pairs. Thus, the CP problem is vindicated in a very natural, physical way via the BS equation.

The interaction Hamiltonian \( H_{\text{int}} \) consists of four distinct interaction vertices, each with two-fermion/one-boson creation or annihilation operators, depicting how unpaired electrons (subindex +) [or holes (subindex −)] combine to form the 2e- (and 2h-CPs) assumed in the system of size \( L \), namely

\[
H_{\text{int}} = L^{-3/2} \sum_{k,K} f_+(k) \{ a_{k+\frac{1}{2}K,\uparrow}^+ a_{-k+\frac{1}{2}K,\downarrow}^+ b_K + a_{-k+\frac{1}{2}K,\downarrow} a_{k+\frac{1}{2}K,\uparrow} b_K^+ \} \\
+ L^{-3/2} \sum_{k,K} f_-(k) \{ a_{k+\frac{1}{2}K,\uparrow}^+ a_{-k-\frac{1}{2}K,\downarrow}^+ c_K + a_{-k-\frac{1}{2}K,\downarrow} a_{k+\frac{1}{2}K,\uparrow} c_K^+ \}. 
\]

(2)

Note that the fermion-pair interaction \( H_{\text{int}} \) is reminiscent of the Fröhlich (or Dirac QED) interaction Hamiltonian involving two fermion and one boson operators but with two types of CPs instead of phonons (or photons). But in contrast with Fröhlich and Dirac there is no a conservation law for the number of unpaired electrons, i.e., \([H_{\text{int}}, \sum_{k_{1,s_1}} \varepsilon_{k_1} a_{k_{1,s_1}}^+ a_{k_{1,s_1}}] \neq 0\). (Note too that \([H_{\text{int}}, \sum_{k_{1,s_1}} K_1 a_{k_{1,s_1}}^+ a_{k_{1,s_1}}] = 0\) and \([H_{\text{int}}, \sum_{k_{1,s_1}} s_1 a_{k_{1,s_1}}^+ a_{k_{1,s_1}}] = 0\). Just as the Fröhlich (or Dirac) interaction Hamiltonians are the most natural ones to use in a many-electron/phonon
(or photon) system, one can conjecture the same of (2) for the BF system under study. Indeed, this \( H_{\text{int}} \) has \textit{formally} already been employed under various guises by several authors [12],[31]-[34]. The energy form factors \( f_{\pm}(k) \) are essentially the Fourier transforms of the 2e- and 2h-CP intrinsic wavefunctions, respectively, in the relative coordinate between the paired fermions of the CP. Here they are taken simply as

\[
f_{+}(\varepsilon) = \begin{cases} f & \text{for } E_f < \varepsilon < E_f + \delta \varepsilon, \\ 0 & \text{otherwise,} \end{cases} \tag{3}
\]

\[
f_{-}(\varepsilon) = \begin{cases} f & \text{for } E_f - \delta \varepsilon < \varepsilon < E_f, \\ 0 & \text{otherwise,} \end{cases} \tag{4}
\]

with \( E_f \) and \( \delta \varepsilon \) phenomenological dynamical energy parameters (in addition to the positive coupling parameter \( f \)) related to the bosonic CPs through

\[
E_f \equiv \frac{1}{4}[E_+(0) + E_-(0)] \quad \text{and} \quad \delta \varepsilon \equiv \frac{1}{2}[E_+(0) - E_-(0)],
\]

where \( E_{\pm}(0) \) are the (empirically unknown) zero-CMM energies of the 2e- and 2h-CPs, respectively. Clearly \( E_{\pm}(0) = 2E_f \pm \delta \varepsilon \). The quantity \( E_f \) will serve as a convenient energy scale and is not to be confused with the Fermi energy \( E_F = \frac{1}{2}mv_F^2 \equiv k_B T_F \) where \( T_F \) is the Fermi temperature. The Fermi energy \( E_F \) equals \( \pi \hbar^2 n/m \) in 2D and \( (\hbar^2/2m)(3\pi^2 n)^{2/3} \) in 3D, with \( n \) the total number-density of charge-carrier electrons. The quantities \( E_f \) and \( E_F \) coincide \textit{only} when perfect 2e/2h-CP symmetry holds.

### 3 Main statistical theories as special cases of CBFM

The interaction Hamiltonian (2) can be further simplified by keeping only the \( K = 0 \) terms. One then applies the Bogoliubov “recipe” (see, e.g., [28] p. 199) of replacing in the full hamiltonian \( H = H_0 + H_{\text{int}} \) all zero CMM creation and annihilation operators by c-numbers: \( \sqrt{N_0} \) and \( \sqrt{M_0} \) for 2e- and 2h-CP operators, where \( N_0(T) \) and \( M_0(T) \) are the number of zero-CMM 2e- and 2h-CPs, respectively. Minimizing with respect to the independent variables \( N_0 \) and \( M_0 \) the so-called thermodynamic (or grand) potential associated with the full Hamiltonian \( H \), as well as keeping the total number of electrons fixed and thereby introducing the electron chemical potential \( \mu \), yields a set of three coupled, transcendental, integral equations (Ref. [15], Eqs. 7-8). These three equations embody the CBFM. Two of these are coupled gap-like equations involving the 2e-CP and 2h-CP BE-condensed boson number densities \( n_0(T) \equiv N_0(T)/L^d \) and \( m_0(T) \equiv M_0(T)/L^d \), linked together through an electron energy gap \( \Delta \). The third equation can be cast as a number equation of the form 

\[
2n_B(T) - 2m_B(T) + n_f(T) = n
\]

involving both 2e and 2h boson number-densities but now for all energy states, where \( n_f(T) \) is the number-density of unpaired electrons. Most significantly, these three equations contain \textit{five} different theories as special cases, see flow chart in Fig. 1. For perfect 2e/2h CP symmetry \( n_B(T) = m_B(T) \) implies [15] that \( n_0(T) = m_0(T) \) and that \( E_f \) coincides with \( \mu \). The CBFM then reduces to: i) the gap
and number equations of the BCS-Bose crossover picture [35] for the BCS model interaction—if the BCS parameters \( V \) and Debye energy \( h \omega_D \) are properly identified with the CBFM dynamical parameters through \( f^2/2\delta\varepsilon \) and \( \delta\varepsilon \), respectively. The crossover picture for unknowns \( \Delta(T) \) and \( \mu(T) \) is now supplemented by the key relation \( \Delta(T_c) = f \sqrt{n_0(T_c)} = f \sqrt{m_0(T_c)} \). If in addition one imposes that \( \mu \simeq E_F \), as occurs for weak coupling from the number equation, the crossover picture is well-known to reduce to: ii) ordinary BCS theory. Thus, the BCS condensate is precisely a BE condensate whenever \( n_B(T_c) = m_B(T_c) \) and the coupling is weak.

On the other hand, for no 2h-CPs present the CBFM reduces [15] also to: iii) the BEC BF model in 3D of Friedberg and Lee [13, 14] characterized by the relation \( \Delta(T) = \sqrt{n_0(T)} \). With just one adjustable parameter (the ratio of perpendicular to planar boson masses) this theory fitted [14] cuprate \( T_c/T_F \) data quite well. When \( f = 0 \) the CBFM reduces to both: iv) the ideal BF model of Ref. [19, 20] that predicts nonzero BEC \( T_c \)'s even in 2D, as well as to: v) the familiar \( T_c \)-formula of ordinary BEC in 3D, albeit as an implicit equation with the boson number-density being \( T \)-dependent. Figure 2 displays the \( T_c \) prediction [20] in 2D for cuprate superconductors of special case (iv), with no adjustable parameters.

4 BEC limit of all electrons paired

The general BEC \( T_c \)-formula for noninteracting bosons in \( d \)-dimensions of energy \( \varepsilon_K = C_s K^s \), \( s > 0 \), is [18]

\[
T_c = \frac{C_s}{k_B} \left[ \frac{s \Gamma(d/2)(2\pi)^d}{2\pi^{d/2} \Gamma(d/s) g_{d/s}(1) n_B} \right]^{s/d}
\]

(5)

where \( n_B \) is the boson number-density and the Bose integral \([36] g_{\sigma}(z) \equiv \sum_{l=1}^{\infty} z^l/l^\sigma \), with \( z \equiv e^{\mu_B/k_bT} \) is the “fugacity” and \( \mu_B \) the boson chemical potential. For \( z = 1 \), \( g_0(1) \equiv \zeta(\sigma) \), the Riemann Zeta-function, if \( \sigma > 1 \), while for \( 0 < \sigma \leq 1 \) the infinite series \( g_{\sigma}(1) \) diverges. Eq. (5) is formally valid for all \( d > 0 \) and \( s > 0 \). Hence, for \( 0 < d \leq s \), \( T_c = 0 \) since \( g_{d/s}(1) = \infty \) for \( d/s \leq 1 \) but \( T_c \) is otherwise finite. We stress that as a consequence of the former all 2D \( T_c \) predictions in Fig. 2 (except the BCS one that survives for all \( d > 0 \)) would collapse to zero had \( s = 2 \) been used in 2D instead of the correct \( s = 1 \). For \( s = 2 \), \( C_2 = \hbar^2/2m_B \) (5) leads to the familiar 3D result \( T_c \simeq 3.31 \hbar^2 n_B^{2/3} / m_B k_F \) since \( \zeta(3/2) \simeq 2.612 \). Recalling that \( k_B T_F = \hbar^2 k_F^2/2m \) with \( k_F = [2^{d-2}\pi^{d/2} \Gamma(d/2)n]^{1/d} \), then if \( m_B = 2m \) and \( n_B = n/2 \) (all electrons paired) for \( s = 2 \) (5) gives \( T_c/T_F = \frac{1}{2}[2/d \Gamma(d) g_{d/2}(1)]^{2/d} = 0 \) for \( d \leq 2 \) since \( g_{d/2}(1) = \infty \) for \( d/2 \leq 1 \). For \( d = 3 \) we arrive at another familiar result \( T_c/T_F = \frac{1}{5}[2/3 \Gamma(3/2) \zeta(3/2)] \simeq 0.218 \) (see dashed line in “Uemura plot” of Ref. [2], Fig. 2). This value appears marked in Fig. 4 as a black triangle.

We now focus on \( s = 1 \). For the boson energy \( \eta \) to be used below the leading term in the
many-body Bethe-Salpeter (BS) CP dispersion relation is linear, i.e., \( \eta \simeq (\lambda/2\pi)\hbar v_F K \) [see Ref. [30] for the derivation in 3D which gives \( \eta \simeq (\lambda/4)\hbar v_F K \)]. Here \( \lambda \equiv VN(E_F) \) where \( N(E_F) \) is the electron density of states (DOS) (for one spin) at the Fermi surface. Note that the boson energy \( \eta \) is linear in CMM \( K \)—and not the quadratic \( \hbar^2 K^2/4m \) appropriate for a composite boson of mass \( 2m \) moving not in the Fermi sea but in vacuum [6]-[9], [13]-[16]. The quadratic holds only when \( E_F \) is strictly zero [17], i.e., when no Fermi sea is present. These linearly-dispersive CPs are commonly confused with the also linearly-dispersive sound phonons of the collective excitation sometimes referred to as the Anderson-Bogoliubov-Higgs (ABH) (Ref. [11] Sec. 3; [37, 38]) mode, which for zero coupling reduces [39] to the IFG result \( \hbar v_F K/\sqrt{d} \). The IFG sound speed \( c = \sqrt{E/d} \) also follows directly from the zero-temperature IFG pressure \( P = n^2[d(E/N)/dn] = 2nE_F/(d + 2) \) via the familiar thermodynamic relation \( dP/dn = mc^2 \), where \( E \) is the ground-state energy and as before \( n \equiv N/L^d = k_F^d/(2d^{d-2} \pi^{d/2} \Gamma(d/2)) \) is the fermion-number density. But the above results with \( \eta \propto \lambda \hbar v_F K \) in fact refer to actual “moving” (or “excited”) CPs in the Fermi sea. Both kinds of distinct soundwave-like solutions—moving CPs and ABH phonons—appear in the many-body BS ladder-summation scheme of Ref. [30]. Note also that the BS CP linear dispersion coefficient \( \lambda/2\pi \) in 2D (or \( \lambda/4 \) in 3D) contrasts markedly with the coupling-independent \( 2/\pi \) coefficient in 2D (or \( 1/2 \) in 3D, as first quoted in Ref. [26], p. 33) obtained [40] in the simple CP problem [24] which ignores holes. Thus (5) again with \( n_B = n/2 \), for \( s = 1 \) and \( C_1 = \lambda b(d)\hbar v_F \) with \( b(2) = 1/2\pi \) and \( b(3) = 1/4 \), yields \( T_c/T_F = 2\lambda b(d)/[d\Gamma(d)\zeta(d)]^{1/d} \) which (for \( \lambda = 1/2 \)) is \( \simeq 0.088 \) if \( d = 2 \) since \( \zeta(2) = \pi^2/6 \), and \( \simeq 0.129 \) if \( d = 3 \) since \( \zeta(3) \simeq 1.202 \). These two values for \( T_c/T_F \) will appear as the uppermost black squares in Fig. 4 marking the BEC limiting values if all electrons in our 2D or 3D many-electron system were imagined paired into noninteracting bosons formed with the BCS model interelectron interaction. For \( \lambda = 1/4 \) the lowermost black squares in Fig. 4 apply.

## 5 Enhanced \( T_c \)’s from the CBFM

We now apply this very general CBFM to exhibit the sizeable enhancements in \( T_c \)s over BCS theory that emerge for moderate departures from perfect 2e/2h-pair symmetry for the same interaction model. The pair-fermion interaction (2) with (3) and (4) bears a one-to-one correspondence with the more familiar “direct” interfermion electron-phonon interaction, mimicked, e.g., in the BCS model interaction (whose double Fourier transform is a negative constant \(-V\) nonzero only within an energy shell \( 2\hbar \omega_D \) about the Fermi surface, with \( \omega_D \) the Debye frequency) if [15, 16] we set \( f^2/2\delta \varepsilon \equiv V \) and \( \delta \varepsilon \equiv \hbar \omega_D \). The familiar dimensionless BCS model interaction parameters \( \lambda \equiv N(E_F)V \) and \( \hbar \omega_D/E_F \) are then recovered.

The three coupled equations of the CBFM determining the \( d \)-dimensional BE-condensate number-densities \( n_0(T) \) and \( m_0(T) \) of 2e- and 2h-CPs, respectively, as well as the fermion
chemical potential \( \mu(T) \), were solved numerically in 3D for \( \lambda = 1/5 \) and \( \hbar \omega_D/E_F = 0.001 \) in Ref. [16] assuming a quadratic boson dispersion relation \( \eta = \hbar^2 K^2/4m \). For this case Figure 3 maps the phase diagram in the vicinity of the BCS \( T_c \) value (marked BCS-B in the figure) at \( \Delta n \equiv n/n_f - 1 = 0 \) (corresponding to perfect \( 2e/2h \)-CP symmetry) where \( n_f \) is the number of unpaired electrons for zero gap and zero temperature, provided that \( n_f \leq n \) [15, 16]. Besides the normal phase \( (n) \) consisting of the ideal BF gas described by \( H_0 \), three different types of stable (plus several metastable, i.e., of higher Helmholtz free energy) BEC-like phases emerged. These are two pure phases of either \( 2e \)- (s+) or \( 2h \)-CP (s–) BE-condensates, and a lower temperature mixed phase \( (ss) \) with arbitrary proportions of \( 2e \)- and \( 2h \)-CPs. Of greater physical interest are the two higher-\( T_c \) pure phases so that we focus below only on them. For each pure phase at a critical temperature we have either \( \Delta(T_{cs+}) = f \sqrt{n_0(T_{cs+})} \equiv 0 \) or \( \Delta(T_{cs-}) = f \sqrt{m_0(T_{cs-})} \equiv 0 \), where \( \Delta(T) \) is the electronic (BCS-like) energy gap. Their intersection gives the BCS \( T_c \) value of \( 7.64 \times 10^{-6}T_f \) where \( k_B T_f \equiv E_f = (\hbar^2/2m)(3\pi^2 n_f)^{2/3} \).

5.1 Two dimensions (2D)

In 2D the one-spin electronic density of states (DOS) is constant, namely \( N(\varepsilon) = m/2\pi \hbar^2 \). Using the aforementioned BS CP linear dispersion relation \( \eta \simeq (\lambda/2\pi)\hbar v_F K \) we get for the bosonic DOS \( M(\eta) \equiv (1/2\pi)K(dK/d\eta) \simeq (2\pi/\lambda^2 \hbar^2 v_F^2)\eta \) instead of the constant that follows in 2D from quadratic dispersion. Employing \( E_f \equiv \pi \hbar^2 n_f/m = k_BT_f \) as energy/density/temperature scaling factors, and the relation \( n/n_f = (E_F/E_f)^{d/2} \) to convert quantities such as \( T_c/T_f \) to \( T_c/T_F \), where \( E_F \equiv k_B T_F \), the two working equations for the pure \( 2e \)-CP phase [i.e., \( m_B(T_c) \equiv 0 \)] become (with all quantities dimensionless, energies in units of \( E_f \) and electron particle-densities in units of \( n_f \))

\[
1 + \hbar \omega_D/2 - \mu = \lambda(\hbar \omega_D/2) \int_1^{1 + \hbar \omega_D} dx \frac{1}{|x - \mu|} \tanh \frac{|x - \mu|}{2T_c}, \tag{6}
\]

\[
\frac{1}{2} \int_0^\infty dx [1 - \tanh \frac{x - \mu}{2T_c}] + \frac{\pi^2}{\lambda^2} \frac{1}{n} \int_0^\infty dx [\coth \frac{x + 2(1 + \hbar \omega_D/2 - \mu)}{2T_c} - 1] = n. \tag{7}
\]

These are just the gap-like equation associated with \( 2e \)-CPs and its corresponding number equation.

For the pure \( 2h \)-CP phase (i.e., \( n_B \equiv 0 \)) the two working equations are

\[
\mu - 1 + \hbar \omega_D/2 = \lambda(\hbar \omega_D/2) \int_{1 - \hbar \omega_D}^1 dx \frac{1}{|x - \mu|} \tanh \frac{|x - \mu|}{2T_c}, \tag{8}
\]
\[
\frac{1}{2} \int_{0}^{\infty} dx \left[ 1 - \tanh \frac{x - \mu}{2T_c} \right] - \frac{\pi^2}{12} \int_{0}^{\infty} dx x \left[ \coth \frac{x - 2(1 - \hbar \omega_D/2 - \mu)}{2T_c} - 1 \right] = n.
\]

(9)

which are the gap-like equation associated with 2h-CPs and its corresponding number equation. Note that (7) and (9) are quadratic in \(n\), and that all integrals there are exact, namely

\[
\int_{0}^{\infty} dx \left[ 1 - \tanh \frac{x - \mu}{2T_c} \right] = \mu + 2T_c \ln[2 \cosh(\mu/2T_c)],
\]

(10)

and

\[
\int_{0}^{\infty} dx x \left[ \coth \frac{x + \delta}{2T_c} - 1 \right] = 2T_c^2 g_2(e^{-\delta/T_c})
\]

(11)

where the Bose function \(g_\sigma(z)\) was defined after (5); it is designated in Ref. [41] as PolyLog\([\sigma, z]\).

The integrals in (6) and (8) were performed numerically. In 2D we use the two extreme values of \(\lambda = 1/4\) (lower set of curves in Fig. 4) and \(= 1/2\) (upper set of curves), and \(\hbar \omega_D/E_F = 0.05\) (a typical value for cuprates), to compute from equations (6) to (9) the \(T_c/T_F\) vs. \(n/n_f\) phase diagram which is graphed in the figure (left panel for 2D) for both 2e-CP (dashed curve) and 2h-CP (full curve) pure, stable BEC-like phases. The value \(n/n_f = 1\) corresponds to perfect 2e/2h-CP symmetry. The \(T_c\) value where both curves \(n_0(T_c) = m_0(T_c) = 0\) intersect is marked by the large dots in the figure; these values are consistent with those gotten from the familiar BCS expression \(T_c/T_F \simeq 1.134(h\omega_D/E_F) \exp(-1/\lambda) \simeq 0.001\) for \(\lambda = 1/4\), and 0.008 for \(\lambda = 1/2\), for \(\hbar \omega_D/E_F = 0.05\). These values differ little from those from the exact BCS (implicit) \(T_c\)-formula (Ref. [28], p. 447) \(1 = \lambda \int_{0}^{\hbar \omega_D/2k_B T_c} dx x^{-1} \tanh x.\) Cuprate data empirically [42] fall within the range \(T_c/T_F \simeq 0.03 - 0.09\). Thus, moderate departures from perfect 2e/2h-CP symmetry enable the CBFM to reach quasi-2D cuprate empirical \(T_c\) values, and quite likely also room temperature superconductivity, without abandoning electron-phonon dynamics—contrary to popular belief. Compelling evidence for a strong, if not sole, phonon dynamical component in cuprates has recently been reported [43] from angle-resolved-photoemission data.

### 5.2 Three dimensions (3D)

In 3D \(N(\varepsilon) \equiv (1/2\pi^2)k^2(dk/d\varepsilon) = (m^2/2\pi^2\hbar^3)\varepsilon\) and, analogously as before, \(E_f = (\hbar^2/2m)\times (3\pi^2n_f)^{2/3} \equiv k_B T_f\) which again differs from \(E_F = (\hbar^2/2m)(3\pi^2n)^{2/3} \equiv k_B T_F\) except when perfect 2e/2h-CP symmetry holds when they coincide, whereas the leading term in the BS CP boson dispersion energy is now the linear expression \(\eta \simeq (\lambda/4)\hbar v_F K\) [30] so that \(M(\eta) \equiv (1/2\pi^2)K^2(dK/d\eta) \simeq (32/\pi^2\lambda^3\hbar^3 v_F^3)\eta^2\). The above working equations for the pure 2e-CP
phase now in 3D (all quantities again dimensionless) are

\[ 1 + \hbar \omega_D/2 - \mu = \lambda (\hbar \omega_D/2) \frac{1}{n^{1/3}} \int_{1}^{\infty} dx \sqrt{x} \frac{1}{|x - \mu|} \tanh \frac{|x - \mu|}{2T_c}, \quad (12) \]

\[ \frac{3}{4} \int_{0}^{\infty} dx \sqrt{x} \left[ 1 - \tanh \frac{x - \mu}{2T_c} \right] + \frac{12}{\lambda^3 n} \int_{0}^{\infty} dx x^2 \left[ \coth x + 2 \left( 1 \frac{\hbar \omega_D/2 - \mu}{2T_c} \right) - 1 \right] = n, \quad (13) \]

while for the pure 2h-CP phase they are

\[ \mu - 1 + \hbar \omega_D/2 = \lambda (\hbar \omega_D/2) \frac{1}{n^{1/3}} \int_{1-h\omega_D}^{1} dx \sqrt{x} \frac{1}{|x - \mu|} \tanh \frac{|x - \mu|}{2T_c}, \quad (14) \]

\[ \frac{3}{4} \int_{0}^{\infty} dx \sqrt{x} \left[ 1 - \tanh \frac{x - \mu}{2T_c} \right] - \frac{12}{\lambda^3 n} \int_{0}^{\infty} dx x^2 \left[ \coth x - 2 \left( 1 \frac{\hbar \omega_D/2 - \mu}{2T_c} \right) - 1 \right] = n. \quad (15) \]

Results in 3D are reported only for the 2e-CP BEC case for the same extreme values of \( \lambda = 1/4 \) and 1/2 as in 2D but now for \( \hbar \omega_D/E_F = 0.005 \). In Fig. 4 for 3D (right panel) the dashed curves are the 2e-CP BEC phase boundaries. The large dot again marks the BCS \( T_c/T_F \) values of 0.0001 for \( \lambda = 1/4 \) and 0.0008 for \( \lambda = 1/2 \). Empirical data for both exotic and conventional, elemental superconductors in 3D are taken from Ref. [2]. We see that whereas BCS theory can roughly reproduce \( T_c/T_F \) values well for the latter, it takes moderate departures from perfect 2e/2h-CP symmetry to access 3D exotic superconductor \( T_c/T_F \) values, which empirically [2] fall within the range \( \simeq 0.01 - 0.1 \). This is much larger than the range \( \lesssim 0.001 \) for conventional (elemental) superconductors, also shaded in the right panel of the figure.

6 Hole superconductivity

Finally, we address the unique but mysterious role played by holes in superconductors in general. For example: a) of the cuprates those that are hole-doped have transition temperatures \( T_c \) about six times higher than electron-doped ones; and b) in fullerite (an fcc crystal of \( C_{60} \) fullerenes) \( T_c \) is now claimed to be more than three times higher with hole rather than electron doping, as recently observed [44] with the so-called “field-effect transistor” technique of injecting holes. And even in conventional superconductors [45] c) over 80% of all superconducting elements have positive Hall coefficients (meaning hole charge carriers); and d) over 90% of non-superconducting metallic, non-magnetic elements have electron charge carriers. This greater “efficiency” of holes in producing higher \( T_c \)'s is clearly predicted in Fig. 4 at least for 2D superconductors, at least insofar as 2h-CP BE condensates exhibit higher \( T_c \)'s than 2e-CP ones.
7 Conclusions

This brief review has sketched how five statistical continuum theories of superconductivity—including both the BCS and BEC theories—are contained as special cases in a single theory, the “complete boson-fermion model” (CBFM). This model includes, for the first time, both two-electron and two-hole pairs in freely variable proportions, along with unpaired electrons, all in chemical/thermal equilibrium. The BCS condensate (characterized by a single equation, namely the $T$-dependent gap equation) follows directly as a BE condensate through the condition for phase equilibria when both 2e and 2h pair numbers are equal at a given temperature and coupling—provided the coupling is weak enough such that the electron chemical potential is roughly the Fermi energy. Ordinary BEC theory, on the other hand, is recovered from the CBFM when hole pairs are neglected, fermion-pair coupling is made to vanish, and the limit of all electrons being paired into bosons is taken.

The practical outcome of this BCS-BEC unification via the CBFM is then threefold: a) enhancements in $T_c$, by more than an order-of-magnitude in 2D, and more than two orders-of-magnitude in 3D, are obtained for the same electron-phonon dynamics mimicked by the BCS model interaction—provided only that one departs moderately from the perfect 2e/2h-pair symmetry to which BCS theory is intrinsically restricted; b) these enhancements in $T_c$ fall within empirical ranges for 2D and 3D “exotic” superconductors, whereas BCS $T_c$ values continue to lie low and within the empirical ranges for conventional, elemental superconductors; and c) hole-doped superconductors are predicted to have higher $T_c$’s than electron-doped ones, in agreement with observation.

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Figure 1: Flow chart of how the CBFM reduces in special cases to the statistical, continuum models of superconductivity discussed in text, thereby displaying how both BCS and BEC theories can be unified.
Figure 2: Critical 2D BEC-like temperature $T_c$ in units of $T_F$ for the BCS model interaction with $\lambda = 1/2$ for varying $\hbar \omega_D/E_F \equiv \Theta_D/T_F$ for: the pure unbreakable-boson gas with some and with all fermions paired; for the breakable-boson gas; and for the boson-fermion mixture (thick full curve labeled “binary gas”) in thermal/chemical equilibrium, all as described in Ref. [20] for the original (simple) CPs where $C_1 = (2/\pi)h v_F$. Dashed curve is the BCS theory $T_c$, and cuprate experimental data are taken from Ref. [42].
Figure 3: Phase diagram [16] with 3D superconducting critical temperatures $T_{cs+}$, $T_{cs-}$, $T_{css+}$, and $T_{css-}$ as functions of $\Delta n \equiv n/n_f - 1$ as defined in text, in the vicinity of the BCS $T_c$ value, assuming a quadratic boson dispersion, for $\lambda = 1/5$ and $\hbar \omega_D/E_F = 0.001$. 
Figure 4: Phase diagrams in 2D and 3D for temperature (in units of $T_F$) and electron density (in units of $n_f$ as defined in text) showing the phase boundaries of $T_c$'s for the pure 2e-CP BEC phases (dashed curves) determined by $\Delta(T_c) = f \sqrt{n_0(T_c)} \equiv 0$, and the pure 2h-CP BEC phase (in 2D only) given by $\Delta(T_c) = f \sqrt{m_0(T_c)} \equiv 0$ for $\lambda = 1/4$ and $1/2$ with $\hbar \omega_D/E_F = 0.05$ in 2D and 0.005 in 3D. Intersections corresponding to $n_0(T_c) = m_0(T_c)$ giving the BCS $T_c$ approximately are marked by black dots, while black squares mark the BEC limit where all electrons are imagined paired into 2e-CP bosons, and the black triangle marks the familiar 3D limit as determined in Sec. 4.