1/f-noise in the Bak-Sneppen model

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Abstract

We calculate time correlation functions in the Bak-Sneppen model (Phys. Rev. Lett. 71 4083 (1993)), a model showing self-organised criticality. For a random neighbour version of the model, analytical results are presented, while on a one dimensional lattice we give numerical results. The power spectrum of these correlation functions shows 1/f- behaviour in both cases.

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A large diversity of physical systems show $1/f$-noise [1]. The power spectra of time correlation functions of such systems show powerlaw behaviour $f^{-\beta}$ over many orders of magnitude with an exponent $\beta$ in the range $0.6 \sim 1.6$. A possible explanation for the wide occurrence of this phenomenon was put forward in a paper entitled "Self-organised criticality: An explanation of $1/f$ noise" [2]. In that paper, Bak, Tang and Wiesenfeld [2] argue that many open non-linear dynamical systems with large number of degrees of freedom evolve to a state where they show critical behaviour characterised by powerlaw correlations both in space and time. Bak, Tang and Wiesenfeld (BTW) illustrate their ideas using a simple model, the so called sandpile model [3]. While this model shows many interesting properties, detailed investigations [4, 5] showed that its power spectrum has $f^{-2}$ behaviour in any finite dimension. A mean-field calculation of the model did however show the expected $1/f$-behaviour exactly [6].

Following the work of BTW a great variety of models (deterministic and stochastic, conservative and dissipative, . . .) have been introduced which show the phenomenon of self-organised criticality (SOC). A common feature of these models is the presence of a separation of time scales; the system is driven at a very slow rate until one of his elements reaches a threshold. This triggers a burst of activity (avalanche) which occurs on a very short timescale. When the avalanche is over, the system evolves again according to the slow drive until a next avalanche is triggered, . . . . The activity of the system in this way consists of a series of independent avalanches. A generic signature of SOC is the presence of a powerlaw in the size (or duration) distribution of
the avalanches. If one increases the external driving rate of the system this powerlaw disappears. It was however shown by Hwa and Kardar [4], that if one increases the rate at which sand is dropped in the sandpile model, and one thus obtains the possibility of interacting avalanches, there appears a region in the power spectrum where the behaviour is $1/f$.

The BTW-sandpile model is a stochastic and conservative model. Olami, Feder and Christensen (OFC) [8] introduced a deterministic and dissipative model, related to spring-block models of earthquakes, which shows signatures of SOC, such as the occurrence of powerlaw distributions for the sizes of the avalanches, with an exponent which depends on the degree of non-conservation in the model. In a subsequent study [9] it was shown that this model shows $1/f$-noise with an exponent $\beta$ which also depends on the degree of non-conservation in the model. In a sense then, the OFC-model fulfills more than the sandpile model the original requirements of the concept of SOC.

In the present paper we study the question of $1/f$-noise in the Bak-Sneppen model (BS) [10]. This model was introduced to describe the coevolution of species in the Earth’s ecology. Indeed the model shows many qualitative similarities with data from the real world, but fails on a quantitative level (see e.g. [11]). In this paper we are only interested in the BS-model as an interesting physical model and do not discuss its possible biological relevance.

In the BS-model one has a system of $N$ interacting species, each of which is represented by a real variable $x_i \in [0,1]$ ($i : 1, \ldots, N$) which is a mea-
sure of the fitness of the species. Initially, all $x_i$ are given a random value, taken from a uniform distribution on $[0, 1]$. The dynamics of the model is defined as follows. First one looks for the site $j$ where the fitness takes its lowest value. One then assigns a new random variable (taken again from the uniform distribution) $x_j$ to species $j$. At the same time, the fitness of $K$ other species is changed randomly. Several versions of the BS-model can be defined, depending on the way in which these other species are chosen. In the lattice version of the model, the species are arranged on a lattice and the $K$ species are taken as nearest neighbours. A random neighbour version, in which the $K$ neighbours are chosen at random at each timestep, was introduced in [12]. This version of the model has the advantage that several of its properties can be calculated exactly [13]. In this paper we will study both this random neighbour version (with $K = 1$) and a one dimensional version of the model in which we only modify the fitness of the neighbour to the right of the species with lowest fitness.

Analytical calculations and extensive simulations have shown that the BS-model evolves to a state in which the probability distribution $p(x)$ that a species has a fitness $x$ becomes a step function, which is zero for $x$ less than some threshold value $x_c < 1$, and which is $1/(1 - x_c)$ for $x > x_c$. In the random neighbour model it is known that $x_c = 1/(K + 1)$ exactly. The exact value of $x_c$ is not known for any lattice version of the model, but precise numerical estimates exist, especially in $d = 1$, for the case in which both neighbours are updated [14, 15]. For the case of the one-dimensional model in which one neighbour is updated, we know of no estimate for $x_c$ in the
literature. From our numerical results, we estimate $x_c = 0.710 \pm 0.005$ for this case (details of our numerical work are described below).

Once the system has reached the equilibrium state, its dynamics is characterised by periods (identified with avalanches) in which at least one of the species has a fitness less than $x_c$, separated by periods in which all species have a fitness above threshold. The avalanches can be characterised either by their duration or by their total activity. Let us denote by $n(t)$ the number of species which are below threshold as a function of (discrete) time $t$. The total activity $s$ of an avalanche lasting from $t = t_-$ to $t = t_+$ (so its total duration is $T = t_+ - t_- + 1$) is then given by

$$s = \sum_{t=t_-}^{t_+} n(t)$$  \hspace{1cm} (1)

The distributions $P(T)$ of avalanche durations and $P(s)$ of avalanche sizes follow a powerlaw

$$P(T) \sim T^{-\tau}, \quad P(s) \sim s^{-y}$$  \hspace{1cm} (2)

For the random neighbour model, it is known exactly that $\tau = 3/2$ [13] while for the one dimensional model (2 neighbour updating) the most accurate numerical estimate is $\tau = 1.073 \pm .003$ [14]. Our simulations of the one-dimensional one neighbour model lead to the estimate $\tau = 1.08 \pm .01$ giving strong evidence that, as could be expected, both one-dimensional models are in the same universality class. We don’t know of any existing estimates of the exponent $y$ for the BS-model. We will return to this exponent at the end of the paper.
It is of importance to remark that in the BS-model as described so far there is no explicit time separation between a fast time scale for avalanches and a slow time scale for inter-avalanche periods. Such a separation is however implicitly present in the definition of the model since one assumes that one time step in the model is related to a step in 'geological' time $t_g = \exp x_{\text{min}}/T$ (where $x_{\text{min}}$ is the lowest value of $x$ at a given time and $T$ is a measure of mutation rate, see e.g. [11]). When $1/T \gg 1$, avalanches occur on time scales which are short compared to the timescale of the external drive which is set by the mutation rate.

In order to study spectral properties of the BS-model it is necessary to introduce a dynamical correlation function $G_N(t)$. In this model a natural definition of a correlation function is

$$ G_N(t) = <n(t_0)n(t_0 + t)>_{t_0} - <n(t_0)>_{t_0}^2 $$

(3)

where the average is taken over time $t_0$ in the equilibrium state. According to the dynamical scaling hypothesis [16] one expects the Fourier transform $\hat{G}_N(\omega)$ of a correlation function such as (3) to scale as

$$ \hat{G}_N(\omega) = \omega^{-\sigma}H(\omega N^z) $$

(4)

where $H$ is a scaling function and $z$ the dynamical exponent. Or equivalently, in real space

$$ G_N(t) = N^{z(\sigma-1)}\tilde{H}(t/N^z) $$

(5)

We have calculated $G_N(t)$ analytically for the random neighbour version
(K = 1) of the BS-model and numerically for the one-dimensional one neighbour version of the model. In both cases we find the presence of 1/f-noise. We now turn to the details of these calculations, and we start with the analytical results.

In [13] a master equation approach to the random neighbour model was introduced. Let \( P_n(t) \) be the probability that at time \( t \), \( n \) species have a fitness which is below a certain value \( \lambda \). In the end we will be most interested in the case when \( \lambda = x_c \) but for the moment we look at the more general case. It is then rather easy to write down a master equation for \( P_n(t) \)

\[
P_n(t + 1) = \sum_{m=0}^{N} M_{nm} P_m(t)
\]

where the matrix elements \( M_{nm} \) can be written down in terms of \( \lambda \) and \( N \) [13]. For \( t \to \infty \), \( P_n(t) \) evolves to an equilibrium distribution \( P_n^* \). The correlation function \( G_N(t) \) can also be written down in terms of the matrix \( M \). One has;

\[
G_N(t) = \lim_{t_0 \to \infty} \sum_{m=0}^{N} \sum_{k=0}^{N} mkP_m(t_0)[M^tP(t_0)]_k - \left[ \lim_{t_0 \to \infty} \sum_{m=0}^{N} mP_m(t_0) \right]^2
\]

This expression in fact allows a (numerically) exact calculation of \( G_N(t) \) in finite systems by simple iteration of the master equation (6). We have performed such calculations for \( \lambda = x_c \) for systems with \( N \) up to 4000 and times \( t \) up to \( 2N \) (results are discussed below).

More interesting is the scaling limit in which \( N \to \infty \) and \( \lambda \to x_c \). In that limit it is possible to get a closed expression for the dynamic correlation
function. It is therefore convenient to rewrite (7) as

\[ G_N(t) = \sum_{m=0}^{N} \sum_{k=0}^{N} mkP^*_m Q_{mk}(t) - \left[ \sum_{m=0}^{N} mP^*_m \right]^2 \] (8)

where \( Q_{mk}(t) \) is the probability that in \( t \)-timesteps the number of species with fitness below \( \lambda \) changes from \( m \) to \( k \). The authors of [13] assume that in the scaling limit \( P^*_n \) becomes a scaling function \( f \) of the variable \( n/\sqrt{N} \)

\[ P^*_n = \frac{1}{\sqrt{N}} f \left( \frac{n}{\sqrt{N}} \right) \] (9)

Inserting (9) into (8) and taking \( t \rightarrow \infty \), \( N \rightarrow \infty \) and \( \lambda \rightarrow x_c \) then gives a differential equation from which \( f \) can be calculated (see eqn. (21) of [13]). Using this result we immediately get the second term on the rhs of (8)

\[ \left[ \sum_{m=0}^{N} mP^*_m \right]^2 = N \frac{N}{2\pi} \] (10)

What remains is a calculation of \( Q_{mk}(t) \) in the scaling limit. We therefore assume that this probability scales as

\[ Q_{mk}(t) = \frac{1}{\sqrt{N}} g \left( \frac{m}{\sqrt{N}}, \frac{k}{\sqrt{N}}, \frac{t}{N} \right) \] (11)

If we insert this assumption in (8) and take the scaling limit, we obtain a differential equation for \( g \) (with \( x = k/\sqrt{N}, y = m/\sqrt{N} \) and \( \tau = t/N \));

\[ \frac{\partial g}{\partial \tau} = g + x \frac{\partial g}{\partial x} + \frac{1}{4} \frac{\partial^2 g}{\partial x^2} \] (12)

which has to solved with the initial condition

\[ g(x, y, \tau = 0) = \delta(x - y) \] (13)

and reflecting boundary conditions in \( x = 0 \).
The solution is
\[ g(x, y, \tau) = h(x, y, \tau) + h(x, -y, \tau) \]  \hspace{1cm} (14)

where
\[ h(x, y, \tau) = \sqrt{\frac{2}{\pi}} \left( \frac{1}{1 - \exp(-2\tau)} \right)^{1/2} \exp 2y^2 \cdot \exp \left\{ -\frac{2}{1 - \exp(-2\tau)}(y^2 + x^2 - 2xy \exp(-\tau)) \right\} \]  \hspace{1cm} (15)

This result has to be used, together with (11), in the first term on the rhs of (8). Taking the scaling limit and using the expression of \( P_m^* \) from [13] we can rewrite this term as
\[ N \frac{2\sqrt{2}}{\sqrt{\pi}} \left[ \int_0^\infty dx \int_0^\infty dy \ x \cdot y \ \exp(-2y^2) g(x, y, \tau) \right] \]

Inserting our result for \( g(x, y, \tau) \) and performing the integration then finally gives;
\[ G_N(t) = N \left\{ \frac{1}{8\pi} (1 - \exp -2\tau)^{3/2} [F(1, 2, 3/2, r_-(\tau)) + F(1, 2, 3/2, r_+(\tau)) - F(1, 2, 5/2, r_-(\tau))/3 - F(1, 2, 5/2, r_+(\tau))/3] - \frac{1}{2\pi} \right\} \]  \hspace{1cm} (16)

where \( F(a, b, c, z) \) is the hypergeometric function and where
\[ r_\pm(\tau) = \frac{1}{2} (1 \pm \exp -\tau) \]

We thus see that the correlation function has indeed the scaling form \( (5) \) with \( z = 1 \) and \( \sigma = 2 \). In figure 1 we show our result \( (16) \) for \( G_N(t)/N \) versus
\( \tau \), together with the numerical results obtained from direct computation of (8) in finite systems. The agreement is perfect thus lending support to the scaling assumptions we made.

To obtain the power spectrum we only have to Fouriertransform (16). Unfortunately, we were not able to obtain an analytical expression for this transform. The result of a numerical transform using \textcopyright\textsc{MATHEMATICA} is shown in figure 2. We show \( \hat{G}_N(\omega)/N^2 \) versus \( \omega N \), which are the natural scaling variables according to (4). The straight line shown has a slope \(-1\). These results then show that over many order of magnitude

\[
\hat{G}_N(\omega) \sim \frac{N}{\omega}
\]

(17)

so that indeed there is \( 1/f \)-noise in the model.

It is interesting to remark here that the random neighbour versions of both the BTW-sandpile model \cite{17} and the BS-model \cite{12} can be related to the critical branching process \cite{18}. Within this approximation both models are thus in the same universality class. Since it is known that in a mean-field theory the sandpile model shows \( 1/f \)-noise \cite{6} it is not so surprising to find the same results for the BS-model.

We now turn to a discussion of the one-dimensional one neighbour version of the BS-model. Due to long range correlations which are present between subsequent species that have lowest fitness \cite{10} a master equation approach is no longer possible. So far, the only approach known for these lattice versions of the BS-model is numerical. We have therefore performed extensive
numerical calculations of the model on one dimensional lattices with $N$ up to 8192 and for time $t$ up to $2^{32}$. Using these data the values of $x_c$ and $\tau$ for the one-dimensional one neighbour model mentionned above were obtained. Figure 3 shows numerical results for the correlation function $G_N(t)$ for various system sizes. Surprisingly, for large system sizes the correlation function seems to become independent of $N$ implying that $z$ becomes 0. We don’t fully understand this result, but it may be connected with similar behaviour found for an other exponent ($\eta$) in [15].

Figure 4 shows the power spectrum of the correlation function for the system with $N = 8192$. As can be seen the behaviour is of the form $\omega^{-\beta}$ over many orders of magnitude. We estimate $\beta = .97 \pm .05$. Thus contrary to the sandpile model, the BS-model has $1/f$ behaviour also in a lattice version of the model. The exponent $\beta$ is furthermore remarkably close to its mean-field value.

We conclude by interpreting our results in the light of a general theory for $1/f$-noise for systems with self-organised criticality, put forward in [4, 5]. We therefore have to introduce first one more exponent, denoted $\mu$ which relates the (average) duration $< T >_s$ of an avalanche to its size $s$ as:

$$< T >_s \sim s^\mu$$

(18)

In [5] it is shown on quite general grounds that when $2\mu + \tau > 3$ a model shows $1/f$-noise with

$$\beta = \frac{3 - \tau}{\mu}$$

(19)
The value of $\mu$ for the one-dimensional BS model can be obtained from [14, 15] as follows. Let $< n(t_a) >$ denote the average number of species below threshold a time $t_a$ after the start of an avalanche. In [14, 15] it is shown (numerically) that $< n(t_a) >$ grows slower than any power (and even becomes a constant according to [15]). Therefore, for long living avalanches, it follows from (1) that $< T > \sim s$, or $\mu = 1$ for the one-dimensional model. Turning to the random neighbour BS-model, we don’t know how to calculate $\mu$ exactly from the master equation approach to this model. From the precise form of the transition probability matrix $M$ as derived in [13], it is however clear that for $\lambda = x_c$, $n(t)$ performs a random walk, from which it can be concluded that for long times after the start of an avalanche $< n(t_a) > \sim t^{1/2}$, so that using (1) we obtain $\mu = 2/3$. We have indeed verified this $\mu$-value in simulations of the random neighbour BS model. (This is an appropriate place to remark that these estimates of $\mu$ allow us to determine the exponent $y$ (see (2) from the obvious relation $y = [(\tau - 1)/\mu] + 1$ so that for the one dimensional BS-model we obtain $y = \tau$ while for the random neighbour model $y = 7/4$.)

Using these values we find for the random neighbour version of the BS-model $2\mu + \tau = 17/6 < 3$ so that the results of [5] predict $\beta = 2$, or absence of $1/f$-noise. For the one dimensional model the results of this reference do predict $1/f$-noise but with an exponent $\beta \approx 1.92$. Both predictions are in contradiction with the numerical results which we presented here. A possible, heuristic, explanation for this is the following. In the arguments of [5] time correlations within one avalanche are considered. This is the natural thing to do, since in a generic SOC-model the avalanches occur instantly on the long
time scale, and therefore correlations including the inter-avalanche period would be trivially zero. As argued above this explicit time separation is not present in the BS-model. Our calculation of the time correlation function therefore include periods in which there is no activity. These lead to a more rapid decorrelation as compared to the case where one only studies intra-avalanche correlations. This more rapid decorrelation in turn leads to a decrease in the power of the low frequency components of the power spectrum and therefore to a $\beta$-exponent which is lower than that following from simple application of the results in [4, 5]. In a sense then, the BS-model is somewhat akin to the sandpile model with a finite driving rate as studied in [7], but without the possibility of interacting avalanches. This mechanism seems to lead to a model which shows both powerlaws in space and time correlations without destroying the powerlaws in the size (and/or time) distribution of avalanches. It is an interesting project to investigate the generality of this scheme.

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Figure Captions

Fig.1 The exact correlation function of the random neighbour Bak-Sneppen model. The figure shows the exact result (16) together with appropriately scaled finite system results obtained using (8). The results are for $N = 250(\Diamond), N = 500(\times), N = 1000(\square), N = 2000(+) \text{ and } N = 4000(\triangle)$.

Fig.2 Numerical Fourier transform (open circles) of the exact correlation function (12). The straight line represents a best fit through the linear part of the data and has a slope of -1.

Fig.3 Numerical results for the correlation function of the one-dimensional one neighbour Bak-Sneppen model. The different curves represent results for (bottom to top) $N = 128, 256, 1024 \text{ and } 4096$ respectively. The upper two curves almost completely coincide.

Fig.4 Fourier transform (crosses) of the numerically calculated correlation function of the one-dimensional one neighbour Bak-Sneppen model. The results are for a system of $N = 8192$ species. The straight line represents a best fit through the linear part of the data and has a slope of -.972.
\[ \hat{\ln G_N(\omega)} \]

\[ \ln N = 8192 \]