Explicit formulas using partitions of integers for numbers defined by recursion

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Abstract

In this article we obtain an explicit formula in terms of the partitions of the positive integer \( n \) to express the \( n \)-th term of a wide class of sequences of numbers defined by recursion. Our proof is based only on arithmetics. We compare our result with similar formulas obtained with different approaches already in the XIX century. Examples are given for Bernoulli, Euler and Fibonacci numbers.

1 Introduction and main result

Consider two sequences of numbers \( a = \{a_0, a_1, \ldots \} \) and \( b = \{b_0, b_1, \ldots \} \) such that for all integers \( n \geq 0 \) the following condition holds:

\[
\sum_{h=0}^{n} a_{n-h} b_h = \delta_{0,n},
\]

in which \( \delta_{0,n} \) is 1 if \( n = 0 \) and 0 otherwise. We stress that for \( n = 0 \), Condition (1) implies \( a_0 b_0 = 1 \). It is convenient to consider \( a_0 = b_0 = 1 \). If this is not the case, one may consider the sequences obtained from \( a \) and \( b \) by factorizing out the factors \( a_0 \) and \( b_0 \), that is with coefficients \( a_i/a_0 \) and \( b_i/b_0 \), \( \forall i \geq 0 \). Therefore, from now on we will suppose \( a_0 = b_0 = 1 \).

If, with the two given sequences, one builds the formal power series

\[
a(x) = \sum_{i=0}^{\infty} a_i x^i, \quad b(x) = \sum_{i=0}^{\infty} b_i x^i,
\]

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and makes the “Cauchy product” of the two series (with formal power series one does not bother about convergence problems), then Condition (1) is equivalent to the equation

\[ a(x) b(x) = 1. \]

Condition (1) is also equivalent to a recursive definition of the elements of one of the two sequences. One has, in fact:

\[ b_n = - \sum_{h=0}^{n-1} a_{n-h} b_h, \quad \forall n > 0, \quad b_0 = 1, \quad (3) \]

that allows to determine the \( n \)-th element of the sequence \( b \) from all the preceding ones.

Many sequences of numbers are defined through Eqs. (1) or (3) (some examples are given in Table 1), and in those cases to calculate the \( n \)-th element of the sequence one has to calculate all the elements of the sequence with index smaller than \( n \).

Formulas that allow to calculate the \( n \)-th element of the sequence only in terms of \( n \), and not in terms of other elements of the sequence, are then useful and welcome, often more for theoretical than for practical interest. These formulas are often called \textit{closed} formulas, but we prefer to call them \textit{explicit} formulas.

In this article we present, in Theorem 1, an explicit formula that allows to calculate the \( n \)-th element of a sequence of numbers satisfying Condition (1) in terms of the partitions of the integer \( n \). This is not a new result, as we will see, but we will prove it in an elementary way that seems absent in the literature. Later on we will list and comment alternative ways to get the same results.

Before to state our theorem we recall some basic definitions concerning partitions and compositions.

Given a positive integer \( n \), a \textit{composition} \( c \) of \( n \) is an unordered set of \( l(c) \) positive integers \( n_i \) that have sum \( n \):

\[ c = \{n_1, n_2, \ldots, n_{l(c)}\}, \quad n = \sum_{i=1}^{l(c)} n_i. \]

The number \( l(c) \) is called the \textit{length} of the composition \( c \).

A \textit{partition} \( p \) is a composition in which all the \( n_i \) are ordered, usually in decreasing order. The length \( l(p) \) of a partition \( p \) is the number of elements
of $p$. If $p$ is a partition obtained from the composition $c$, one obviously has $l(p) = l(c)$. If $p = \{n_1, n_2, \ldots, n_{l(p)}\}$, one then has:

$$n_i \geq n_{i+1}, \quad \forall i = 1, \ldots, l(p) - 1.$$  

For any partition $p$ there are many compositions formed with the same $l(p)$ elements of $p$. Let $\mu(p)$ be the number of different compositions formed by the same numbers in the partition $p$. It is an easy exercise to verify that the number $\mu(p)$ is given by the formula:

$$\mu(p) = l(p)! \prod_{n_i \in \cup(p)} m_p(n_i)!,$$  \hspace{1cm} (4)

where $m_p(n_i)$ is the multiplicity of $n_i$ in $p$, that is the number of times the number $n_i$ appears in $p$, and the product at the denominator has to be done for all different numbers $n_i$ that appear in $p$, that is, for all numbers $n_i$ that appear in the union $\cup(p)$ of the elements of $p$. The multiplicities $m_p(n_i)$ that appear at the denominator of (4) always form a composition of the length $l(p)$ that appears at the numerator. The second member of Eq. (4) can also be written using the multinomial coefficient:

$$\mu(p) = \binom{l(p)}{m_p(n_1), m_p(n_2), \ldots},$$

where the numbers $m_p(n_1), m_p(n_2), \ldots$ are those appearing in the denominator of the second member of Eq. (4).

Let $C(n)$ and $P(n)$ be the set of all compositions of $n$ and the set of all partitions of $n$, respectively.

We are ready to state our main result.

**Theorem 1.** Let $\{a_0, a_1, \ldots\}$ and $\{b_0, b_1, \ldots\}$ be sequences such that $a_0 = b_0 = 1$, and such that Condition (1) holds, then one may calculate the elements $b_n$, $\forall n > 0$, from the following explicit formulas involving the sequence $\{a_0, a_1, \ldots\}$ and the compositions or the partitions of $n$:

$$b_n = \sum_{c \in C(n)} \prod_{n_i \in c} (-a_{n_i}),$$  \hspace{1cm} (5)

$$b_n = \sum_{p \in P(n)} \mu(p) \prod_{n_i \in p} (-a_{n_i})$$  \hspace{1cm} (6)
Formulas (5) and (6) in Theorem 1 differ only for the use of compositions in place of partitions. In general, the use of partitions is preferable in calculations because it involves fewer terms. In Table 2 we list explicit formulas for the general terms of some sequences of numbers in terms of partitions only; but there exist similar formulas using compositions, too.

It must be said that the explicit formulas that use partitions of integers to calculate the general $n$-th term of a sequence of numbers are often not convenient for practical computations, because the cardinality of $\mathcal{P}(n)$ grows very rapidly with $n$.

**Proof of Theorem 1 (elementary version).** We want to solve the linear system expressed by Condition (1) in the indeterminates $b_1, b_2, \ldots$. We recall that we suppose $a_0 = b_0 = 1$. Following Scherk [7], we do this by substitution, using recursive formula (3) to solve the equation for $b_1$, then that one for $b_2$, and so on. At each step, the equation number $n$ contains only the indeterminates $b_1, \ldots, b_n$, so the substitution method is easy to apply.

The first few equations are not difficult to solve:

$$
\begin{align*}
b_1 &= -a_1 \\
b_2 &= -a_2 - a_1 b_1 = -a_2 + a_1^2 \\
b_3 &= -a_3 - a_2 b_1 - a_1 b_2 = -a_3 + 2a_2a_1 - a_1^3 \\
b_4 &= -a_4 - a_3 b_1 - a_2 b_2 - a_1 b_3 = -a_4 + 2a_3a_1 + a_2^2 - 3a_2a_1^2 + a_1^4 \\
b_5 &= -a_5 - a_4 b_1 - a_3 b_2 - a_2 b_3 - a_1 b_4 = \\
&= -a_5 + 2a_4a_1 + 2a_3a_2 - 3a_3a_1^2 - 3a_2^2a_1 + 4a_2a_1^3 - a_1^5 \\
\end{align*}
$$

(7)

At this point one notes that, at least for $h = 1, 2, 3, 4, 5, \ldots$, $b_h$ is obtained as the sum of all products like

$$
a_{n_1}a_{n_2}\ldots a_{n_k},
$$

whose indices satisfy the conditions

$$1 \leq k \leq h, \quad n_1 + n_2 + \ldots + n_k = h,$$

and each term is taken with the + sign if $k$ is even, with the − sign if $k$ is odd. The $k$ positive numbers $n_1, \ldots, n_k$ form then a composition of length $k$ of the positive integer $h$. We are then led to write the formula:

$$
b_n = \sum_{c \in \mathcal{C}(n)} \prod_{n_i \in c} (-a_{n_i}), \quad \forall n \geq 1, \tag{8}
$$

where $\mathcal{C}(n)$ is the set of all compositions of the integer $n$. Let’s verify by induction that Formula (8) is true for all values of $n \geq 1$. From (7) we see...
that Eq. (8) is true for \( n \leq 5 \). Supposing (8) true up to a given \( n \geq 1 \), let’s prove that it is true also for \( n \) substituted by \( n + 1 \). Starting from (8), and using (8), one finds:

\[
b_{n+1} = -a_{n+1} - \sum_{h=1}^{n} a_{n+1-h} b_{h} = \
= -a_{n+1} - \sum_{h=1}^{n} \left( \sum_{c \in \mathcal{C}(h)} \prod_{n_i \in c} (-a_{n_i}) \right) a_{n+1-h}.
\]

For any composition \( c = \{n_1, n_2, \ldots, n_{l(c)}\} \in \mathcal{C}(h) \), consider the set obtained by appending to the \( l(c) \) numbers \( n_i \) in \( c \) the number \( n_0 = (n + 1) - h \), that appears in the index of the last factor \( a_{n+1-h} \) (so that it can be rewritten in the form \( a_{n_0} \)). The set \( c' = \{n_0, n_1, n_2, \ldots, n_{l(c)}\} \) is now a composition of \( (n + 1) \) of length \( l(c') = l(c) + 1 \). Considering then all possible values of \( h \) resulting from the sum of the last expression, we obtain in this way all possible compositions of \( (n+1) \) of length greater than 1. Considering the term outside the summation symbol as corresponding to the unique composition \( \{n + 1\} \) of length 1 of \( n + 1 \), we can rewrite the last expression in the following form:

\[
b_{n+1} = \sum_{c' \in \mathcal{C}(n+1)} \prod_{n_i \in c'} (-a_{n_i}).
\]

This is Formula (8) written for the integer \( (n + 1) \) instead of \( n \), so our proof by induction is now complete and (8) is true for all values of \( n \geq 1 \).

Usually it is more convenient to use partitions in place of compositions. Using the partitions, Formula (8) becomes

\[
b_{n} = \sum_{p \in \mathcal{P}(n)} \mu(p) \prod_{n_i \in p} (-a_{n_i}), \quad \forall n \geq 1,
\]

where the factor \( \mu(p) \) is given in Equation (4).

\[\square\]

2 Different approaches

As already remarked, one may find results similar to those expressed by Theorem 1 also in other papers. We will give here a short historical review and with it we will stress how a simple proof of Theorem 1 is not available. In fact, the result expressed by Theorem 1 is usually seen in the context of the theory of formal power series or in relation with the Taylor series expansion
of functions.

In 1825, Scherk (pages 59–66 of [7]) showed how to solve with iterated substitutions a triangular linear system like the one corresponding to Eq. (3) (that we present in Eq. (7)), and found correctly that the \( n \)-th indeterminate \( b_n \) is obtained by summing up \( 2^{n-1} \) terms involving products of the sequence elements \( a_i, \forall i = 1, \ldots, n \). The number \( 2^{n-1} \) is equal to the number of compositions of \( n \), but Scherk did not recognize the sum over the compositions of \( n \) in his result.

The triangular system considered by Scherk was written in matrix form by Brioschi in 1858 [2], and solved using Cramer’s rule, arriving at the following expression for the \( n \)-th term of the sequence:

\[
b_n = (-1)^n \det \begin{pmatrix}
a_1 & 1 & 0 & \ldots & 0 \\
a_2 & a_1 & 1 & \ldots & 0 \\
a_3 & a_2 & a_1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_n & a_{n-1} & a_{n-2} & \ldots & a_1 \\
\end{pmatrix}.
\]

Brioschi then obtained the following explicit formula:

\[
b_n = \sum_{q_1+2q_2+3q_3+\ldots+nq_n=n} (-1)^q \frac{q!}{q_1!q_2!\cdots q_n!} a_1^{q_1}a_2^{q_2}\cdots a_n^{q_n}, \quad (9)
\]

where \( q = q_1 + q_2 + \ldots + q_n \), and the sum is over all sets of non-negative integers \( \{q_1, q_2, \ldots, q_n\} \) that are solutions of the diophantine equation

\[
q_1 + 2q_2 + 3q_3 + \ldots + nq_n = n. \quad (10)
\]

In his work Brioschi also reported that the same formula (9) was obtained one year before by Fergola [5].

Indeed, Fergola found Eq. (9) in 1857 [5] using an explicit formula he earlier found in [4] for the expansion of the \( n \)-th derivative of the inverse function \( f^{-1}(x) \) in terms of the derivatives of the function \( f(x) \) up to the \( n \)-th order, where \( f(x) \) is any differentiable function. At the beginning of his paper [5], Fergola said that Sylvester remarked the importance of partitions in connection with his work and the end of the same paper he stated without proof that the number of solutions \( \{q_1, q_2, \ldots, q_n\} \) of the diophantine equation (10), with \( q_i \geq 0 \), is equal to the number of partitions of \( n \).

Sylvester in 1871 [8] stated without proof that the set of solutions of the diophantine equation (10) are in a one to one correspondence with the set of partitions of \( n \). This fact is crucial to show that Eq. (9) and the result
expressed by our Theorem 1 are equivalent, so we will give a simple proof of this fact in the following Proposition.

**Proposition 1.** The sets of solutions \( \{q_1, q_2, \ldots, q_n\} \) of the diophantine equation (10), with \( q_i \geq 0 \), are in a one to one correspondence with the partitions \( p = \{n_1, n_2, \ldots, n_k\} \) of \( n \). The correspondence is obtained by taking \( q_i = m_p(n_i) \), if \( n_i \in p \), and 0 otherwise. It then follows that \( l(p) = \sum_{i=1}^{n} q_i \).

**Proof.** Let \( \{q_1, q_2, \ldots, q_n\} \) be a set of non negative integers that are solutions of the diophantine equation (10), then \( p = \{q_1, 2q_2, \ldots, nq_n\}\{0\} \) is a partition of \( n \). Viceversa, given a partition \( p = \{n_1, n_2, \ldots\} \) of \( n \), with \( n_i \geq 1 \) and multiplicity \( m_p(n_i) \), the set \( \{q_1, q_2, \ldots, q_n\} \), where \( q_i = m_p(n_i) \), if \( n_i \in p \), and 0 otherwise, is a set of non negative integers that are solutions of the diophantine equation (10). By the discussion following Eq. (4) one then has that \( l(p) = \sum_{i=1}^{n} q_i \). \( \square \)

Proposition 1 can now be applied to prove the following Proposition.

**Proposition 2.** Eq. (9) is equivalent to Eq. (6).

**Proof.** Proposition 1 implies that the sum in Eq. (9) is over all partitions \( p \) of \( n \), where \( p = \{q_1, 2q_2, \ldots, nq_n\}\{0\} \). The factor \( \frac{q^1}{q_1!q_2! \cdots q_n!} \) in Eq. (9) is then equal to the factor \( \mu(p) \) defined in Eq. (4), because \( q = q_1 + q_2 + \ldots + q_n = l(p) \). Moreover, the product \( (−1)^q a_1^{q_1} a_2^{q_2} \cdots a_n^{q_n} \) in Eq. (9) can be rewritten as \( (−a_1)^{q_1} (−a_2)^{q_2} \cdots (−a_n)^{q_n} \), and this is equal to the product \( \prod_{n_i \in p} (−a_{n_i}) \) in Eq. (6). \( \square \)

Nowadays, links between the theory of formal power series and partitions are encountered in many textbooks and papers (see, for example, [3]). We present, for example, a proof of Theorem 1 in the context of the theory of formal power series.

**Proof of Theorem 1 (formal power series version).** Consider the formal power series (2) satisfying \( b(x) = 1/a(x) \). Being \( a_0 = 1 \), one has \( a(x) = 1 + \sum_{i=1}^{\infty} a_i x^i \), so one can write:

\[
b(x) = \frac{1}{a(x)} = \left(1 + \sum_{i=1}^{\infty} a_i x^i\right)^{-1} = 1 + \sum_{k=1}^{\infty} \left(-\sum_{i=1}^{\infty} a_i x^i\right)^k
\]

where in the last member the geometric series expansion has been used. Using the multinomial theorem

\[
(x_1 + x_2 + \ldots)^k = \sum_{r_1+r_2+\ldots=k} \binom{k}{r_1, r_2, \ldots} x_1^{r_1} x_2^{r_2} \cdots,
\]
where \( r_1, r_2, \ldots \) are non-negative integers, to expand the terms \((-\sum_{i=1}^{\infty} a_i x^i)^k\), and factoring out the same powers of \( x \), one may write:

\[
b(x) = 1 + \sum_{k=1}^{\infty} \sum_{r_1+2r_2+\ldots=k} \left( \frac{k}{r_1, r_2, \ldots} \right) (-a_1)^{r_1} (-a_2)^{r_2} \ldots = 1 + \sum_{k=1}^{\infty} \sum_{r_1+2r_2+\ldots=k} x^{r_1+2r_2+\ldots} \left( \frac{k}{r_1, r_2, \ldots} \right) (-a_1)^{r_1} (-a_2)^{r_2} \ldots = 1 + \sum_{n=1}^{\infty} x^n \sum_{r_1+2r_2+\ldots+nr_n=n} \left( \frac{r_1 + r_2 + \ldots + r_n}{r_1, r_2, \ldots, r_n} \right) (-a_1)^{r_1} (-a_2)^{r_2} \ldots (-a_n)^r_n.
\]

Using Proposition 1 and Eq. (4) one has:

\[
b(x) = 1 + \sum_{n=1}^{\infty} x^n \sum_{p \in \mathcal{P}(n)} \mu(p) \prod_{n_i \in p} (-a_{n_i}),
\]

so that, for Eq. (2), one has \( \forall n > 0: \)

\[
b_n = \sum_{p \in \mathcal{P}(n)} \mu(p) \prod_{n_i \in p} (-a_{n_i}),
\]

This expression for \( b_n \) is equal to Eq. (6) in Theorem 1.

We listed such a different methods to obtain explicit formulas using partitions of integers for the elements of sequences satisfying Condition (1), to point out that no complete elementary method, not involving calculus or formal power series, is available in the literature. We think moreover that our proof is simple because it only uses some basic arithmetics and combinatorics.

3 Examples

We already observed that the equation \( a(x)b(x) = 1 \) for two formal power series is equivalent to Condition (1). This applies in the particular case of two functions \( f(x) \) and \( 1/f(x) \) for which a Taylor series expansion exists in \( x = x_0 \), and leads to following expressions for the sequence elements in terms of partitions of integers:

\[
a_n = \frac{1}{n!} \left. \frac{d^n}{dx^n} f(x) \right|_{x=x_0},
\]

\[
b_n = \frac{1}{n!} \left. \frac{d^n}{dx^n} \frac{1}{f(x)} \right|_{x=x_0} = \sum_{p \in \mathcal{P}(n)} \mu(p) \prod_{n_i \in p} \left( \frac{1}{n_i!} \left. \frac{d^{n_i}}{dx^{n_i}} f(x) \right|_{x=x_0} \right).
\]
Many sequences of numbers can be defined as the coefficients of the Taylor series expansions of convenient generating functions \( g(x) \). In all these cases, the sequence elements can be written in terms of partitions of integers using Eq. (11) for the reciprocal \( f(x) = 1/g(x) \) of the generating function.

We report in Table 1 a few sequences of numbers satisfying Condition (1). In the Table one should consider \( n > 0 \), because for \( n = 0 \) not all the listed entries are equal to 1, as we have supposed. The corresponding expansions in terms of partitions of integers are reported in Table 2. These expansions are usually found using Eq. (11), but in the following we will prove some of them using a purely arithmetic method.

The aim of the cited papers \([5, 2]\) was to give an explicit formula for the Bernoulli and Euler numbers. For this reason we list Entries 1–4 in Table 1. The proof of Condition (1) for Entries 1, 3 and 4 is classical and can easily be found in the literature (for example in \([6]\), starting from Eq. (3), p. 238, and Eq. (1) p. 256). We only report below a proof of Condition (1) for Entry 2.

Explicit formulas for the Fibonacci numbers in terms of partitions of integers are less known. Therefore, we report below a proof of Condition (1) for Entries 5 and 6.

| Entry | Name          | \( a_n \)            | \( b_n \)   |
|-------|---------------|-----------------------|-------------|
| 1     | Bernoulli numbers | \( \frac{1}{(n+1)!} \) | \( \frac{1}{n} B_n \) |
| 2     | Even Bernoulli numbers | \( \frac{2}{(2n+2)!} \) | \( -\frac{2n-1}{(2n)!} B_{2n} \) |
| 3     | Euler numbers  | \( \frac{1+(-1)^n}{2n!} \) | \( \frac{1}{n!} E_n \) |
| 4     | Even Euler numbers | \( \frac{1}{(2n)!} \) | \( \frac{1}{(2n)!} E_{2n} \) |
| 5     | Fibonacci numbers | \( \frac{(-1)^n-1}{2} \) | \( F_n \) |
| 6     | Even Fibonacci numbers | \( -n \) | \( F_{2n} \) |

Table 1: Some sequences satisfying Condition (1).
Proof of Condition (1) for Entry 2. We need Condition (1) for Entry 1 of Table 1, that is:
\[
\sum_{h=0}^{n} \frac{1}{(n+1-h)!} \frac{B_h}{h!} = 0.
\]
(12)

It is well-known that \( B_{2n+1} = 0, \forall n > 0 \). A purely arithmetic proof of this fact is outlined in [6] p. 238. It is based on a formula published by J. Faulhaber in 1631 and proved by B. Pascal in 1654, as reported in [1], Theorem 3.1 (i). \[ \]

Using the fact that \( B_{2n+1} = 0, \forall n > 0 \), it is convenient in Eq. (12) to separate the sums over the odd and over the even indices. We can write Eq. (12) for \( n = 2m \) and for \( n = 2m + 1 \). When \( n = 2m \), Eq. (12) gives:

\[
0 = \sum_{h=0}^{2m} \frac{1}{(2m+1-h)!} \frac{B_h}{h!} = \sum_{k=0}^{m} \frac{1}{(2m+1-2k)!} \frac{B_{2k}}{(2k)!} + \frac{B_1}{(2m)!},
\]
that is:

\[
- \frac{B_1}{(2m)!} = \sum_{k=0}^{m} \frac{2m+2-2k}{(2m+2-2k)!} \frac{B_{2k}}{(2k)!}
\]
(13)

When \( n = 2m + 1 \), Eq. (12) gives:

\[
0 = \sum_{h=0}^{2m+1} \frac{1}{(2m+2-h)!} \frac{B_h}{h!} = \sum_{k=0}^{m} \frac{1}{(2m+2-2k)!} \frac{B_{2k}}{(2k)!} + \frac{B_1}{(2m+1)!},
\]
that is:

\[
- \frac{B_1}{(2m+1)!} = \sum_{k=0}^{m} \frac{2m+1}{(2m+2-2k)!} \frac{B_{2k}}{(2k)!}
\]
(14)

The difference between Eqs. (13) and (13) gives:

\[
0 = \sum_{k=0}^{m} \left( \frac{2m+1}{(2m+2-2k)!} - \frac{2m+2-2k}{(2m+2-2k)!} \right) \frac{B_{2k}}{(2k)!} = \sum_{k=0}^{m} \frac{2k-1}{(2m+2-2k)!} \frac{B_{2k}}{(2k)!}
\]

\[ ^{1}\text{A standard proof uses the generating function for the Bernoulli numbers}
\]
\[
g_B(x) = \frac{x}{e^x - 1} = \sum_{h=0}^{\infty} B_h \frac{x^h}{h!}
\]
and the observation that \( g_B(x) + x/2 \) is an even function.
Proof of Condition (11) for Entries 5 and 6. We first recall the standard definition of the Fibonacci numbers: $F_1 = F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$, $\forall n > 2$. We then prove by induction that the following formulas are true $\forall n \geq 1$:

$$F_{2n} = \sum_{h=1}^{n} F_{2h-1},$$  \hspace{1cm} (15) \\

$$F_{2n+1} = 1 + \sum_{h=1}^{n} F_{2h},$$  \hspace{1cm} (16) \\

$$F_{2n} = n + \sum_{h=1}^{n-1} F_{2h}(n-h).$$  \hspace{1cm} (17)

Eqs. (15), (16) and (17) are true for $n = 1$. Let’s suppose they are true for a certain $n \geq 2$ and let’s prove they are true also for $n$ substituted by $n + 1$. One has:

$$F_{2(n+1)} = F_{2n+2} = F_{2n+1} + F_{2n} = F_{2n+1} + \sum_{h=1}^{n} F_{2h-1} = \sum_{h=1}^{n+1} F_{2h-1},$$

so Eq. (15) is true for all $n$. One has:

$$F_{2(n+1)+1} = F_{2n+3} = F_{2n+1} + F_{2n+2} = 1 + \sum_{h=1}^{n} F_{2h} + F_{2n+2} = 1 + \sum_{h=1}^{n+1} F_{2h},$$

so Eq. (16) is true for all $n$. Using this result one has:

$$F_{2(n+1)} = F_{2n+2} = F_{2n+1} + F_{2n} = 1 + \sum_{h=1}^{n} F_{2h} + n + \sum_{h=1}^{n-1} F_{2h}(n-h) =$$

$$= n + 1 + \sum_{h=1}^{n} F_{2h} + \sum_{h=1}^{n} F_{2h}(n-h) = n + 1 + \sum_{h=1}^{n} F_{2h}(n+1-h),$$

and this proves that Eq. (17) is true for all $n$. Let’s now form the sequence $b$, with elements $b_n = F_{2n}$, $\forall n > 0$, and $b_0 = 1$, and the sequence $a$, with $a_n = -n$, $\forall n > 0$, and $a_0 = 1$. Then Eq. (17) has the form of Eq. (3) and is clearly equivalent to Condition (11) reported in
Table 2: Explicit formulas using partitions for the sequences in Table 1.

Entry 6 of Table 1

Eqs. (15) and (16) can be summarized by the following:

\[
F_n = \frac{1}{n!} B_n = \sum_{p \in \mathcal{P}(n)} \mu(p) \prod_{n_i \in p} \left(-\frac{1}{(n_i+1)!}\right)
\]

Let’s now form the sequence \(b,\) with elements \(b_n = F_n, \forall n > 0,\) and \(b_0 = 1,\)
and the sequence \(a,\) with \(a_n = \frac{(-1)^n}{2}\), \(\forall n > 0,\) and \(a_0 = 1.\) Then Eq. (18) has the form of Eq. (3)
and is clearly equivalent to Condition (1) reported in Entry 5 of Table 1. □

Theorem 1 can now be applied to all the sequences listed in Table 1 above and allows to write the explicit formulas using partitions reported in Table 2.

Finally, we present two explicit calculations using \(\mathcal{P}(4).\) One has:

\[
\mathcal{P}(4) = \{\{4\}, \{3, 1\}, \{2, 2\}, \{2, 1, 1\}, \{1, 1, 1, 1\}\}
\]

so from the first and sixth rows of Table 2 one finds:

\[
B_4 = 4! \left[ \frac{1!}{1!} \cdot \frac{1}{5!} - \frac{2!}{1! 1!} \cdot \frac{(-1)^2}{4! 2!} + \frac{2!}{3! 3!} \cdot \frac{(-1)^3}{1! 2!} + \frac{4!}{2! 2! 2!} \cdot \frac{(-1)^4}{3! 2! 2!} \right] = -\frac{1}{30},
\]
\[ F_n = \left[ \frac{1!}{1!} \cdot 4 + \frac{2!}{1!1!} \cdot 3 \cdot 1 + \frac{2!}{2!} \cdot 2 \cdot 2 + \frac{3!}{1!2!} \cdot 2 \cdot 1 \cdot 1 + \frac{4!}{4!} \cdot 1 \cdot 1 \cdot 1 \cdot 1 \right] = 21. \]

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