MINIMAL NON-ODD-TRANSVERSAL HYPERGRAPHS AND MINIMAL NON-ODD-BIPARTITE HYPERGRAPHS

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Abstract. Among all uniform hypergraphs with even uniformity, the odd-transversal or odd-bipartite hypergraphs are more close to bipartite simple graphs from the viewpoint of both structure and spectrum. A hypergraph is called minimal non-odd-transversal if it is non-odd-transversal but deleting any edge results in an odd-transversal hypergraph. In this paper we give an equivalent characterization of the minimal non-odd-transversal hypergraphs by the degrees and the rank of its incidence matrix over $\mathbb{Z}_2$. If a minimal non-odd-transversal hypergraph is uniform, then it has even uniformity, and hence is minimal non-odd-bipartite. We characterize 2-regular uniform minimal non-odd-bipartite hypergraphs, and give some examples of $d$-regular uniform hypergraphs which are minimal non-odd-bipartite. Finally we give upper bounds for the least H-eigenvalue of the adjacency tensor of minimal non-odd-bipartite hypergraphs.

1. Introduction

Let $G = (V, E)$ be a hypergraph, where $V =: V(G)$ is the vertex set, and $E =: E(G)$ is the edge set whose elements $e \subseteq V$. If for each edge $e$ of $G$, $|e| = k$, then $G$ is called a $k$-uniform hypergraph. The degree $d(v)$ of a vertex $v$ of $G$ is defined to be the number of edges of $G$ containing $v$. If $d(v) = d$ for all vertices $v$ of $G$, then $G$ is called $d$-regular.

A hypergraph $G$ is called 2-colorable if there exists a 2-coloring of the vertices of $V(G)$ such that $G$ contains no monochromatic edges; and it is called minimal non-2-colorable if it is non-2-colorable but deleting any edge from $E(G)$ results in a 2-colorable hypergraph. Seymour [25] proved that if $G$ is minimal non-2-colorable and $V(G) = \bigcup \{e \in E(G)\}$, then $|E(G)| \geq |V(G)|$. Aharoni and Linial [1] presented an infinite version of Seymour’s result. Alon and Bregman [3] proved that if $k \geq 8$ then every $k$-regular $k$-uniform hypergraph is 2-colorable. Henninga and Yeo [14] showed that Alon-Bergman result is true for $k \geq 4$.

A subset $U$ of $V(G)$ is called a transversal (also called vertex cover or hitting set) of $G$ if each edge of $G$ has a nonempty intersection with $U$. The transversal number of $G$ is the minimum size of transversals in $G$, which was well studied by Alon [2], Chvátal and McDiarmid [6], Henninga and Yeo [15]. $G$ is called bipartite if for some nonempty proper subset $U \subseteq V(G)$, $U$ and its complement $U^c$ are both transversal; or equivalently the vertex set $V(G)$ has a bipartition into two parts such that every edge of $E(G)$ intersects both parts. Surely, $G$ is bipartite if and only if $G$ is 2-colorable.

A subset $U$ of $V(G)$ is called an odd transversal of $G$ if each edge of $G$ intersects $U$ in an odd number of vertices [8, 24]. A hypergraph $G$ is called odd-transversal if it...
has an odd transversal. Nikiforov [21] firstly uses odd transversal to investigate the spectral symmetry of tensors and hypergraphs. Hu and Qi [16] introduce the notion of odd-bipartite hypergraph to study the zero eigenvalue of the signless Laplacian tensor.

**Definition 1.1** ([16]). Let $G$ be a $k$-uniform hypergraph, where $k$ is even. If there exists a bipartition $\{U, U^c\}$ of $V(G)$ has such that each edge of $G$ intersects $U$ (and also $U^c$) in an odd number of vertices, then $G$ is called *odd-bipartite*, and $\{U, U^c\}$ is an *odd-bipartition* of $G$.

So, odd-bipartite hypergraphs are surely odd-transversal hypergraphs and bipartite hypergraphs. For the uniform hypergraphs with even uniformity, the notion of odd-bipartite hypergraphs is equivalent to that of odd-transversal hypergraphs.

From the viewpoint of spectrum, a simple graph is bipartite if and only if its adjacency matrix has a symmetric spectrum. However, the adjacency tensor of a bipartite uniform hypergraph does not possess such property. We note that the hypergraphs under consideration are uniform when discussing their spectra. Shao et al. [26] proved that the adjacency tensor of a $k$-uniform hypergraph $G$ has a symmetric H-spectrum if and only if $k$ is even and $G$ is odd-bipartite. So, the odd-bipartite hypergraphs are more close to bipartite simple graphs than the bipartite hypergraphs based on the following two reasons. First they both have a structural property, namely, there exists a bipartition of the vertex set such that every edge intersects the each part of the bipartition in an odd number of vertices. Second they both have a symmetric H-spectrum.

There are some examples of odd-bipartite hypergraphs, e.g. power of simple graphs and cored hypergraphs [17], hm-hypergraphs [16], $m$-partite $m$-uniform hypergraphs [7]. Nikiforov [21] gives two classes of non-odd-transversal hypergraphs. Fan et al. [18] construct non-odd-bipartite generalized power hypergraphs from non-bipartite simple graphs. It is known that a connected bipartite simple graph has a unique bipartition up to isomorphism. However, an odd-bipartite hypergraph can have more than one odd-bipartition. Fan et al. [11] given a explicit formula for the number of odd-bipartition of a hypergraph by the rank of its incidence matrix over $\mathbb{Z}_2$. So, it seems hard to give examples of non-odd-bipartite hypergraphs.

To our knowledge, there is no characterization of non-odd-transversal or non-odd-bipartite hypergraphs. We observe that non-odd-transversal hypergraphs have a hereditary property, that is, if $G$ contains a non-odd-transversal sub-hypergraph, then $G$ is non-odd-transversal. $G$ is called *minimal non-odd-transversal*, if $G$ is non-odd-transversal but deleting any edge from $G$ results in an odd-transversal hypergraph, or equivalently, any nonempty proper edge-induced sub-hypergraph of $G$ is odd-transversal. In this paper we give an equivalent characterizaton of the minimal non-odd-transversal hypergraphs by the degrees and the rank of its incidence matrix over $\mathbb{Z}_2$. If a minimal non-odd-transversal hypergraph is uniform, then it has even uniformity, and hence is minimal non-odd-bipartite. We characterized 2-regular uniform minimal non-odd-bipartite hypergraphs, and give some examples of $d$-regular uniform hypergraphs which are minimal non-odd-bipartite. Finally we give upper bounds for the least H-eigenvalue of the adjacency tensor of minimal non-odd-bipartite hypergraphs.

### 2. Basic notions

Unless specified somewhere, all hypergraphs in this paper contain no multiple edges or isolated vertices, where vertex is called *isolated* if it is not contained in any edge of the hypergraph. Let $G = (V, E)$ be a hypergraph. $G$ is called *square* if $|V| = |E|$. A *walk* of length $t$ in $G$ is a sequence of alternate vertices and
edges: \( e_{i_1}e_{i_2} \ldots e_{i_t}, \) where \( \{v_i, v_{i+1}\} \subseteq e_i \) for \( i = 0, 1, \ldots, t - 1. \) \( G \) is said to be connected if every two vertices are connected by a walk.

The vertex-induced sub-hypergraph of \( G \) by the a subset \( U \subseteq V(G), \) denoted by \( G[U], \) is a hypergraph with vertex set \( U \) and edge set \( \{e \cap U : e \in E(G), e \cap U \neq \emptyset\}. \) For a connected hypergraph \( G, \) a vertex \( v \) is called a cut vertex of \( G \) if \( G[V(G) \setminus \{v\}] \) is disconnected. The edge-induced sub-hypergraph of \( G \) by a subset \( F \subseteq E(G), \) denoted by \( G[F], \) is a hypergraph with vertex set \( \cup_{e \in F} e \) and edge set \( F. \)

Let \( G \) be a hypergraph and let \( e \) be an edge of \( G. \) Denote by \( G - e \) the hypergraph obtained from \( G \) by deleting the edge \( e \) from \( E(G). \) For a connected hypergraph \( G, \) an edge \( e \) is called a cut edge of \( G \) if \( G - e \) is disconnected.

A matching \( M \) of \( G \) is a set of pairwise disjoint edges of \( G. \) In particular, if \( G \) is a bipartite graph with a bipartition \( \{V_1, V_2\}, \) a vertex subset \( U_1 \subseteq V_1 \) is matched to \( U_2 \subseteq V_2 \) in \( M, \) if there exists a bijection \( f : U_1 \rightarrow U_2 \) such that \( \{\{v, f(v)\} : v \in U_1\} \subseteq M. \) A subset \( U_1 \subseteq V_1 \) (or \( V_2 \)) is matched by \( M \) if every vertex of \( U \) is incident with an edge of \( M. \) \( M \) is called a perfect matching if \( V_1 \) and \( V_2 \) are both matched by \( M. \)

The incidence bipartite graph \( \Gamma_G \) of \( G \) is a bipartite simple graph with two parts \( V(G) \) and \( E(G) \) such that \( \{v, e\} \in E(\Gamma_G) \) if and only if \( v \in e. \)

The edge-vertex incidence matrix of \( G, \) denoted by \( B_G = (b_{e,v}), \) is a matrix of size \( |E(G)| \times |V(G)|, \) whose entries \( b_{e,v} = 1 \) if \( v \in e, \) and \( b_{e,v} = 0 \) otherwise.

The dual of \( G, \) denoted by \( G^*, \) is the hypergraph whose vertex set is \( E(G) \) and edge set is \( \{\{e \in E(G) : v \in e\} : v \in V(G)\}. \) If no two vertices of \( G \) are contained in precisely the same edges of \( G, \) then \( (G^*)^* \) is isomorphic to \( G. \) In this situation, the incidence bipartite graph \( \Gamma_{G^*} \cong \Gamma_G^*, \) and the incidence matrix \( B_{G^*} = B_{G^*}^T, \) where the latter denotes the transpose of \( B_{G^*}. \)

Let \( G \) be a simple graph, and let \( k \) be an even integer greater than \( 2. \) Denote by \( G^{k, 2} \) the hypergraph obtained from \( G \) whose vertex set is \( \cup_{e \in V(G)} v \) and edge set \( \{u \cup v : \{u, v\} \in E(G)\}, \) where \( v \) denotes an \( \frac{k}{2} \)-set corresponding to \( v, \) and all those sets are pairwise disjoint; intuitively \( G^{k, 2} \) is obtained from \( G \) by blowing up each vertex into a \( \frac{k}{2} \)-set and preserving the adjacency relation \( [15]. \) It is proved that \( G^{k, 2} \) is non-odd-bipartite if and only if \( G \) is non-bipartite \([13].\)

Next we will introduce some knowledge of eigenvalues of a tensor. For integers \( k \geq 2 \) and \( n \geq 2, \) a tensor (also called hypermatrix) \( T = (t_{i_1i_2 \ldots i_k}) \) of order \( k \) and dimension \( n \) refers to a multidimensional array \( t_{i_1i_2 \ldots i_k} \in \mathbb{C} \) for all \( i_1, i_2, \ldots, i_k \in [n] := \{1, 2, \ldots, n\} \) and \( j \in [k]. \) \( T \) is called symmetric if its entries are invariant under any permutation of their indices.

Given a vector \( x \in \mathbb{C}^n, \) \( Tx^k \in \mathbb{C}^n, \) and \( Tx^{k-1} \in \mathbb{C}^n, \) which are defined as follows:

\[
T x^k = \sum_{i_1, i_2, \ldots, i_k \in [n]} t_{i_1i_2 \ldots i_k} x_{i_1} x_{i_2} \cdots x_{i_k},
\]

\[
(T x^{k-1})_i = \sum_{i_2, \ldots, i_k \in [n]} t_{i_2i_3 \ldots i_k} x_{i_2} x_{i_3} \cdots x_{i_k}, \quad \text{for } i \in [n].
\]

Let \( I \) be the identity tensor of order \( k \) and dimension \( n, \) that is, \( t_{i_1i_2 \ldots i_k} = 1 \) if and only if \( i_1 = i_2 = \cdots = i_k \in [n] \) and zero otherwise.

**Definition 2.1.** \([19, 23]\) Let \( T \) be a \( k \)-th order \( n \)-dimensional real tensor. For some \( \lambda \in \mathbb{C}, \) if the polynomial system \( (\lambda I - T)x^{k-1} = 0, \) or equivalently \( T x^{k-1} = \lambda x^{k-1}, \) has a solution \( x \in \mathbb{C}^n \setminus \{0\}, \) then \( \lambda \) is called an eigenvalue of \( T \) and \( x \) is an eigenvector of \( T \) associated with \( \lambda, \) where \( x^{k-1} := (x_1^{k-1}, x_2^{k-1}, \ldots, x_n^{k-1}) \in \mathbb{C}^n. \)

The characteristic polynomial \( \varphi_T(\lambda) \) of \( T \) is defined as the resultant of the polynomials \( (\lambda I - T)x^{k-1}; \) see \([23, 15, 13]. \) It is known that \( \lambda \) is an eigenvalue of \( T \)
if and only if it is a root of $\varphi_T(\lambda)$. The spectrum of $T$ is the multi-set of the roots of $\varphi_T(\lambda)$.

Suppose that $T$ is real. If $x$ is a real eigenvector of $T$, surely the corresponding eigenvalue $\lambda$ is real. In this case, $x$ is called an $H$-eigenvalue and $\lambda$ is called an $H$-eigenvalue. The $H$-spectrum of $T$ is the set of all $H$-eigenvalues of $T$, denoted by $\text{HSpec}(T)$. The spectral radius of $T$ is defined as the maximum modulus of the eigenvalues of $T$, denoted by $\rho(T)$. Denote by $\lambda_{\max}(T), \lambda_{\min}(T)$ the largest $H$-eigenvalue and the least $H$-eigenvalue of $T$, respectively.

For a symmetric tensor, we have the following result.

Lemma 2.2. Let $T$ be a real symmetric tensor of order $k$ and dimension $n$. Then

1. [30, Theorem 3.6] If $T$ is also nonnegative, then

$$\lambda_{\max}(T) = \min\{T x^k : x \in \mathbb{R}^n, x \geq 0, \|x\|_k = 1\},$$

where $\|x\|_k = \left(\sum_{i=1}^n |x_i^k| \right)^{\frac{1}{k}}$. Furthermore, $x$ is an optimal solution of the above optimization if and only if it is an eigenvector of $T$ associated with $\lambda_{\max}(T)$.

2. [23, Theorem 5] If $k$ is also even, then

$$\lambda_{\min}(T) = \min\{T x^k : x \in \mathbb{R}^n, \|x\|_k = 1\},$$

and $x$ is an optimal solution of the above optimization if and only if it is an eigenvector of $T$ associated with $\lambda_{\min}(T)$.

Let $G$ be a $k$-uniform hypergraph on $n$ vertices $v_1, v_2, \ldots, v_n$. The adjacency tensor of $G$ [7] is defined as $A(G) = (a_{i_1i_2\ldots i_k})$, an order $k$ dimensional $n$ tensor, where

$$a_{i_1i_2\ldots i_k} = \begin{cases} \frac{1}{\binom{n}{k}} & \text{if } \{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\} \in E(G); \\ 0, & \text{otherwise.} \end{cases}$$

The spectral radius, the least $H$-eigenvalue of $G$ are referring to its adjacency tensor $A(G)$, denoted by $\rho(G), \lambda_{\min}(G)$ respectively. The $H$-spectrum of $A(G)$ is denoted by $\text{HSpec}(G)$.

The spectral hypergraph theory has been an active topic in algebraic graph theory recently; see e.g. [6, 9, 10, 21, 22]. By the Perron-Frobenius theorem for non-negative tensors [4, 12, 27, 28, 29], $\rho(G)$ is exactly the largest $H$-eigenvalue of $A(G)$. If $G$ is connected, there exists a unique positive eigenvector up to scales associated with $\rho(G)$, called the Perron vector of $G$. Noting that the adjacency tensor $A(G)$ is nonnegative and symmetric, so $\rho(G)$ holds (1) of Lemma 2.2 and $\lambda_{\min}(G)$ holds (2) of Lemma 2.2 if $k$ is even. By Perron-Frobenius theorem, $\lambda_{\min}(G) \geq -\rho(G)$. By the following lemma, if $G$ is connected and non-odd-bipartite, then $\lambda_{\min}(G) > -\rho(G)$.

Lemma 2.3. [21, 20, 29, 11] Let $G$ be a $k$-uniform connected hypergraph. Then the following results are equivalent.

1. $k$ is even and $G$ is odd-bipartite.
2. $\lambda_{\min}(G) = -\rho(G)$.
3. $\text{HSpec}(G) = -\text{HSpec}(G)$.

Finally, we introduce some notations used throughout out the paper. Denote by $C_n$ a cycle of length $n$ as a simple graph. Denote by $\mathbb{1}$ an all-one vector whose size can be implicated by the context, $\text{rank}A$ the rank of a matrix $A$ over $\mathbb{Z}_2$, and $\mathbb{F}_q$ a field of order $q$.

3. Characterization of minimal non-odd-transversal hypergraphs

In this section we will give some equivalent conditions in terms of degrees and rank of the incidence matrix over $\mathbb{Z}_2$ for a hypergraph to be minimal non-odd-transversal.
Lemma 3.1. If $G$ is a minimal non-odd-transversal hypergraph, then $G$ is connected and contains no cut vertices.

Proof. If $G$ contains more than one connected component, then at least one of them is non-odd-transversal, a contradiction to the definition. So $G$ itself is connected. Suppose $G$ contains a cut vertex. Then $G$ is obtained from two connected nontrivial sub-hypergraphs $G_1, G_2$ sharing exactly one vertex (the cut vertex). So, at least one of $G_1, G_2$ is non-odd-transversal, also a contradiction. \hfill \Box

Lemma 3.2. Let $G$ be a connected hypergraph, and $B_G$ be the edge-vertex incidence matrix of $G$. Then $G$ is odd-transversal if and only if the equation

\begin{equation}
B_Gx = \mathbb{1} \text{ over } \mathbb{Z}_2
\end{equation}

has a solution, or equivalently

\begin{equation}
\text{rank } B_G = \text{rank}(B_G, \mathbb{1}) \text{ over } \mathbb{Z}_2.
\end{equation}

Proof. If $G$ is odd-transversal, then there is an odd-transversal $U$ of $G$. Define a vector $x \in \mathbb{Z}_2^{V(G)}$ such that $x_v = 1$ if $v \in U$, and $x_v = 0$ otherwise. By the definition, it is easy to verify that $x$ is a solution of the equation (3.1). On the other hand, if $x$ is a solution of the equation (3.1), define $U = \{v : x_v = 1\}$. Then $U \neq \emptyset$, and for each edge $e$ of $G$, $|e \cap U|$ is odd, implying that $G$ is odd-transversal. \hfill \Box

For each edge $e \in E(G)$, define an indicator vector $\chi_e \in \mathbb{Z}_2^{V(G)}$ such that $\chi_e(v) = 1$ if $v \in e$ and $\chi_e(v) = 0$ otherwise. Then $B_G$ consists of those $\chi_e$ as row vectors for all $e \in E(G)$.

Lemma 3.3. Let $G$ be a connected hypergraph with $m$ edges. If $m$ is odd, and each vertex has an even degree, or equivalently $\sum_{e \in E(G)} \chi_e = 0$ over $\mathbb{Z}_2$, then $G$ is non-odd-transversal.

Proof. Let $e_1, \ldots, e_m$ be edges of $G$. Write $(B_G, \mathbb{1})$ as the following form:

\begin{equation}
(B_G, \mathbb{1}) = \begin{pmatrix}
\chi_{e_1} & 1 \\
\chi_{e_2} & 1 \\
\vdots & \vdots \\
\chi_{e_m} & 1
\end{pmatrix}
\end{equation}

Adding the first row to all other rows over $\mathbb{Z}_2$, we will have

\begin{equation}
\begin{pmatrix}
\chi_{e_1} \\
\chi_{e_2} + \chi_{e_1} \\
\vdots \\
\chi_{e_m} + \chi_{e_1}
\end{pmatrix} = \begin{pmatrix}
\chi_{e_1} & 1 \\
C & O
\end{pmatrix}.
\end{equation}

So, $\text{rank}(B_G, \mathbb{1}) = 1 + \text{rank} C$. As $m$ is odd and $\sum_{e \in E(G)} \chi_e = 0$,

\begin{equation}
\chi_{e_1} = \sum_{i=2}^{m} (\chi_{e_i} + \chi_{e_1}),
\end{equation}

implying that $\text{rank} B_G = \text{rank} C$. By Lemma 3.2, $G$ is non-odd-transversal. \hfill \Box

Theorem 3.4. Let $G$ be a connected hypergraph with $m$ edges. The following are equivalent.

1. $G$ is minimal non-odd-transversal.
2. $m$ is odd, $\sum_{e \in E(G)} \chi_e = 0$ over $\mathbb{Z}_2$, and $\sum_{e \in F} \chi_e \neq 0$ over $\mathbb{Z}_2$ for any nonempty proper subset $F$ of $E(G)$.
3. $m$ is odd, $\sum_{e \in E(G)} \chi_e = 0$ over $\mathbb{Z}_2$, and $\text{rank } B_G = m - 1$ over $\mathbb{Z}_2$. 

\begin{align*}
& \text{(1) } G \text{ is minimal non-odd-transversal.} \\
& \text{(2) } m \text{ is odd, } \sum_{e \in E(G)} \chi_e = 0 \text{ over } \mathbb{Z}_2, \text{ and } \sum_{e \in F} \chi_e \neq 0 \text{ over } \mathbb{Z}_2 \text{ for any nonempty proper subset } F \text{ of } E(G). \\
& \text{(3) } m \text{ is odd, } \sum_{e \in E(G)} \chi_e = 0 \text{ over } \mathbb{Z}_2, \text{ and } \text{rank } B_G = m - 1 \text{ over } \mathbb{Z}_2.
\end{align*}
(4) $m$ is odd, each vertex of $G$ has an even degree, and any nonempty proper edge-induced sub-hypergraph of $G$ contains vertices of odd degrees.

Proof. (1) $\Rightarrow$ (2). Suppose that $G$ is minimal non-odd-transversal. By Eq. 3.3 and Eq. 3.4, as $\text{rank}_{\mathbb{Z}} G \neq \text{rank}_{\mathbb{Z}} (B_G, \mathbb{I})$ over $\mathbb{Z}_2$ by Lemma 3.2, $\chi_e$ is a linear combination of $\chi_{e_i} + \chi_{e_j}$ for $i = 2, \ldots, m$. So there exist $a_i \in \mathbb{Z}_2$ for $i = 2, \ldots, m$ such that

$$
\chi_{e_1} = \sum_{i=2}^{m} a_i (\chi_{e_1} + \chi_{e_2}) = \left( \sum_{i=2}^{m} a_i \right) e_1 + \sum_{i=2}^{m} a_i \chi_{e_i}.
$$

We assert that $a_1 = 1$ for $i = 2, \ldots, m$. Otherwise, there exists a $j$, $2 \leq j \leq m$, such that $a_j = 0$. Then $\chi_{e_j}$ is a linear combination of $\chi_{e_i} + \chi_{e_j}$ for $i = 2, \ldots, m$ and $i \neq j$. So, $\text{rank}_{\mathbb{Z}} (B_{G-e_j}, \mathbb{I}) = \text{rank}_{\mathbb{Z}} (B_{G-e_j})$, implying that $G - e_j$ is odd-transversal by Lemma 3.2, a contradiction to the definition.

If $m$ is even, then $\sum_{i=2}^{m} a_i = 0$ by Eq. 3.5, implying the vertices of $V(G)$ all have even degrees in $G - e_1$. So the vertices of $e_1$ all have odd degrees in $G$. By the arbitrariness of $e_1$, each vertex has an odd degree in $G$. However, there exists a vertex $v \notin e_1$ so that $d_G(v) = d_{G-e_1}(v)$, which is an even number, a contradiction.

So, $m$ is odd, and $\sum_{e \in E(G)} \chi_e = 0$ by Eq. 3.5. Assume to the contrary there exists a nonempty proper subset $F$ of $E(G)$. $\sum_{e \in F} \chi_e = 0$ over $\mathbb{Z}_2$. If $|F|$ is odd, then by Lemma 3.3, the sub-hypergraph $G|_{E(G) \setminus F}$ induced by the edges of $F$ is non-odd-transversal, a contradiction to the definition. Otherwise, $|F|$ is even, then $\text{rank}_{\mathbb{Z}} (B_{G \setminus e_1}, \mathbb{I})$ is odd as $m$ is odd, and the sub-hypergraph $G|_{E(G) \setminus F}$ is non-odd-transversal, also a contradiction.

(2) $\Rightarrow$ (3). As $\sum_{e \in E(G)} \chi_e = 0$, $\text{rank}_{\mathbb{Z}} G \leq m - 1$ over $\mathbb{Z}_2$. If $\text{rank}_{\mathbb{Z}} G \leq m - 2$ over $\mathbb{Z}_2$, then $\chi_{e_1}, \ldots, \chi_{e_{m-1}}$ are linear dependent. So there exists $a_1, \ldots, a_{m-1} \in \mathbb{Z}_2$, not all being zero, such that $\sum_{i=1}^{m-1} a_i \chi_{e_i} = 0$. Taking $F = \{ e_i : a_i = 1, 1 \leq i \leq m - 1 \}$, then $\sum_{e \in F} \chi_e = \sum_{i=1}^{m-1} a_i \chi_{e_i} = 0$, a contradiction to (2). So $\text{rank}_{\mathbb{Z}} G = m - 1$ over $\mathbb{Z}_2$.

(3) $\Rightarrow$ (1). By Lemma 3.3, $G$ is non-odd-transversal. Let $e$ be an arbitrary edge of $G$. Adding all rows $\chi_f$ for $f \neq e$ to the row $\chi_e$ will yield a zero row as $\sum_{e \in E(G)} \chi_e = 0$. So $\text{rank}_{\mathbb{Z}_2} G - e = \text{rank}_{\mathbb{Z}_2} G = m - 1$ over $\mathbb{Z}_2$, implying that $G - e$ has full rank over $\mathbb{Z}_2$ with respect to rows. Hence, $\text{rank}_{\mathbb{Z}_2} G - e \geq \text{rank}_{\mathbb{Z}_2} (B_{G-e}, \mathbb{I})$ over $\mathbb{Z}_2$, and $G - e$ is odd-transversal by Lemma 3.2. So $G$ is minimal non-odd-transversal.

Of course (2) is equivalent to (4). \qed

Remark 3.5. From the proof of (3) $\Rightarrow$ (1) in Theorem 3.4 if $G$ is minimal non-odd-transversal hypergraphs with $m$ edges, then any $m - 1$ rows of $B_G$ are linear independent over $\mathbb{Z}_2$.

Example 3.6. The following are minimal non-odd-transversal hypergraphs by verifying the degrees and the rank of incidence matrix over $\mathbb{Z}_2$ according to Theorem 3.4, where the last two hypergraphs are square.

$$(1) \{1, 2, 3, 4\}, \{2, 3, 4, 5\}, \{1, 5\}.$$  

$$(2) \{1, 2, 3\}, \{2, 3, 4\}, \{3, 4, 5\}, \{1, 4, 5\}, \{3, 4\}.$$  

$$(3) \{1, 2, 3\}, \{1, 3, 4, 5\}, \{1, 2, 4, 6\}, \{1, 5, 6, 7\}, \{2, 4, 7\}, \{2, 5, 6, 7\}, \{4, 5, 6, 7\}.$$  

From Example 3.6 we know a minimal non-odd-transversal hypergraph can contain both even-sized edges and odd-sized edges. In the following we will discuss minimal non-odd-transversal hypergraphs only with even-sized edges.

Corollary 3.7. Let $G$ be an minimal non-odd-transversal hypergraph only with even-sized edges, which has $n$ vertices and $m$ edges. Then the following results hold.
(1) \( n \geq m \).
(2) For \( 1 \leq t \leq m - 1 \), any \( t \) edges intersect at least \( t + 1 \) vertices.
(3) The incidence bipartite graph \( \Gamma_G \) has a matching \( M \) such that \( E(G) \) is matched by \( M \), namely, there exists an injection \( f : E(G) \to V(G) \) such that \( f(e) \in e \) for each \( e \in E(G) \).

Proof. Consider the incidence matrix \( B_G \) of \( G \). As \( G \) contains only even-sized edges, each row sum of \( B_G \) is zero over \( \mathbb{Z}_2 \), which implies \( \text{rank} B_G \leq n - 1 \). By Theorem 3.4(3), \( \text{rank} B_G = m - 1 \), yielding the result (1).

Let \( e_1, \ldots, e_t \) be \( t \) edges of \( G \), where \( 1 \leq t \leq m - 1 \). Let \( U = \bigcup_{i=1}^t e_i \). Let \( B_G[e_1, \ldots, e_t|U] \) be the sub-matrix of \( B_G \) with rows indexed \( e_1, \ldots, e_t \) and columns indexed by the vertices of \( U \). By Remark 3.5, \( \text{rank} B_G[e_1, \ldots, e_t|U] = t \leq |U| - 1 \), as each row sum of the sub-matrix is zero. So we have \( |U| \geq t + 1 \), yielding the result (2).

The result (3) follows from Hall’s Theorem. □

Corollary 3.8. Let \( G \) be a square minimal non-odd-transversal hypergraph only with even-sized edges. Then

(1) The incidence bipartite graph \( \Gamma_G \) has a perfect matching, namely, there exists a bijection \( f : E(G) \to V(G) \) such that \( f(e) \in e \) for each \( e \in E(G) \).
(2) For each nonempty proper subset \( U \) of \( V(G) \), \( G|U \) contains at least \( |U| + 1 \) edges, and also contains odd-sized edges.

Proof. Surely (1) comes from (3) of Corollary 3.7 as \( G \) is square. Now let \( U \) be a nonempty proper subset of \( V(G) \). Let \( F \) be the set of edges that intersect \( U \) so that \( G|U \) has edges \( e \cap U \) for all \( e \in F \). If \( F = E(G) \), then \( |F| = |V(G)| \geq |U| + 1 \) as \( G \) is square. Otherwise, we consider the submatrix \( B_G[F^c|U^c] \), which has rank \( |F^c| \) from the its rows by Remark 3.5. So, \( |F^c| = n - |F| \leq n - |U| - 1 \) as each row sum of \( B_G[F^c|U^c] \) is zero over \( \mathbb{Z}_2 \), implying that \( |F| \geq |U| + 1 \).

Assume to the contrary that each edge of \( G|U \) has even size. Then \( B_G[E(G)|U] \), and \( B_G[E(G)|U^c] \) as well, has zero row sums. So

\[
\text{rank} B_G \leq \text{rank} B_G[E(G)|U] + \text{rank} B_G[E(G)|U^c] \\
\leq |U| - 1 + |U^c| - 1 = |V(G)| - 2 = |E(G)| - 2,
\]

a contradiction to Theorem 3.4(3). □

Corollary 3.9. Let \( G \) be a square hypergraph only with even-sized edges and even-degree vertices. Then \( G \) is minimal non-odd-transversal if and only if its dual \( G^* \) is minimal non-odd-transversal.

Proof. Suppose \( G \) is minimal non-odd-transversal with \( n \) vertices (edges). By Corollary 3.2, no two vertices of \( G \) lie in precisely the same edges of \( G \). So \( G^* \) is also square, and \( B_{G^*} = B_G^T \). As each edge of \( G \) is even sized, each vertex of \( G^* \) has even degree. So \( G^* \) is minimal non-odd-transversal by Theorem 3.4. As \( G \) is isomorphic to \( (G^*)^* \), \( G \) is minimal non-odd-transversal if \( G^* \) is. □

4. MINIMAL NON-ODD-BIPARTITE REGULAR HYPERGRAPHS

In this section we mainly discuss minimal non-odd-transversal \( k \)-uniform hypergraphs \( G \). By the following lemma, \( k \) is necessarily even. So the minimal non-odd-transversal uniform hypergraphs are exactly the minimal non-odd-bipartite hypergraphs.

Lemma 4.1. Let \( G \) be a minimal non-odd-transversal \( k \)-uniform hypergraphs \( G \), which has \( n \) vertices and \( m \) edges. Then \( k \) is even. If \( G \) is further \( d \)-regular, then \( d \) is even and \( d \leq k \).
As a simple generalization, the generalized power hypergraph \( G^n \) is a minimal non-bipartite simple graph if an odd cycle is constructed as in Construction 4.3, where

\[ k \geq 2. \]

Let \( H \) be a nonempty proper edge-induced sub-hypergraph of \( G \). As \( G \) is connected, \( H \) contains a vertex \( v \), which is also contained in some edge not in \( H \). So \( v \) has degree 1 in \( H \). The result follows by Theorem 3.4(4).

Next we give a construction of 2-regular \( k \)-uniform hypergraphs, where \( k \) is an even integer greater than 2.

**Construction 4.3.** Let \( k \) be an even integer greater than 2, and let \( n, m \) be positive integers such that \( n = \frac{km}{2} \). Let \( K_{n,m} \) be a complete bipartite simple graph with two parts \( U_1 \) and \( U_2 \), where \( U_1 = [n] \) and \( U_2 = \cup_{t=1}^m \{ e_t^1, \ldots, e_t^k \} \). Let \( K_{n,m} \) be obtained from \( K_{n,m} \) by deleting the edges between the vertices of \( V_t := \{ \frac{k}{2}(t-1) + 1, \ldots, \frac{k}{2}t \} \) and the vertices of \( E_t := \{ e_t^1, \ldots, e_t^k \} \) for \( t \in [m] \).

Let \( M \) be a perfect matching of \( K_{n,m} \) such that \( W_t := \{ i_{1t}, \ldots, i_{kt} \} \) are matched to \( E_t \) respectively for \( t \in [m] \), and if \( W_t = V_s \) for some \( s \neq t \), then \( W_s = V_t \).

Define a hypergraph \( G \) with vertex set \([n]\), whose edges are

\[ e_t = V_t \cup W_t, \quad \text{for } t \in [m]. \]

**Lemma 4.4.** The hypergraph \( G \) defined in Construction 4.3 is a 2-regular \( k \)-uniform hypergraph on \( n \) vertices.

**Proof.** As there is no edge between \( V_t \) and \( E_t \) in \( K_{n,m} \), \( W_t \cap V_t = \emptyset \) for each \( t \in [m] \). So each edge \( e_t \) contains exactly \( k \) vertices. Note that \( \{ V_1, \ldots, V_t \} \) and \( \{ W_1, \ldots, W_t \} \) both form a \( t \)-partition of \([n]\). For each vertex \( v \) of \( G \), \( v \in V_s \) for a unique \( s \in [m] \) and \( v \in W_t \) for a unique \( t \in [m] \), where \( t \neq s \) as \( V_s \cap W_t = \emptyset \). So \( v \) contained in exactly two edges \( e_s \) and \( e_t \), implying \( v \) has degree 2. Finally we note that \( G \) contains no multiple edges; otherwise, if \( e_s = e_t \) for \( s \neq t \), then \( V_s \cup W_t = V_t \cup W_s \), which implies that \( V_s = W_t \) and \( V_t = W_s \), as \( V_s \cup V_t = \emptyset \) and \( W_s \cup W_t = \emptyset \), contradicting the assumption. The result follows.

**Corollary 4.5.** Any 2-regular \( k \)-uniform hypergraph on \( n \) vertices can be constructed as in Construction 4.3, where \( k \) is even integer greater than 2.

**Proof.** Let \( G \) be a 2-regular \( k \)-uniform hypergraph with \( V(G) = [n] \) and \( E(G) = \{ e_1, \ldots, e_m \} \). Surely, \( n = \frac{km}{2} \). Let \( G := \frac{k}{2} \) be a \( k \)-uniform hypergraph with vertex set \( V(G) \) and edge set \( \frac{k}{2} \) \( E(G) := \{ \frac{k}{2} e : e \in E(G) \} \), where \( \frac{k}{2} e \) means the \( \frac{k}{2} \) copies
of \(e\), written as \(e^1, \ldots, e^\ell\). Then \(G\) is a \(k\)-uniform \(k\)-regular multi-hypergraph on \(n\) vertices. The incidence bipartite graph \(\Gamma_G\) of \(G\) is \(k\)-regular.

Let \(K_G\) be a complete bipartite graph with two parts \(V(G)\) and \(E(G)\). Let \(\hat{K}_G\) be obtained from \(K_G\) by deleting the edges between the vertices of \(V_t := \{\frac{k}{2}(t-1) + 1, \ldots, \frac{k}{2}t\}\) and the vertices of \(E_t := \{e^1_t, \ldots, e^\ell_t\}\) for \(t \in [m]\). Then \(\hat{K}_G\) is an \((n, m, k, \frac{k}{2})\)-regular bipartite graph.

Considering the \(k\)-regular bipartite graph \(\Gamma_G\), it contains a perfect matching \(M\). By a possible relabeling of the vertices, we may assume that for \(t \in [m]\), \(V_t := \{\frac{k}{2}(t-1) + 1, \ldots, \frac{k}{2}t\}\) is matched to \(E_t := \{e^1_t, \ldots, e^\ell_t\}\) in \(M\). By the construction of \(\hat{G}\), returning to \(G\), \(V_t \subseteq e_t\) for \(t \in [m]\).

Now deleting the edges between the vertices of \(V_t\) and the vertices of \(E_t\) from \(\Gamma_G\) for \(t \in [m]\), we arrive at a \(\frac{k}{2}\)-regular bipartite graph denoted by \(\hat{\Gamma}_G\), which is a subgraph of \(\hat{K}_G\). Now \(\hat{\Gamma}_G\) and hence \(\hat{K}_G\) has a perfect matching \(\hat{M}\), where, for \(t \in [m]\), \(W_t := \{i_1, \ldots, i_{\ell_t}\}\) is matched to \(E_t\) in \(\hat{M}\). So, returning to \(G\), \(W_t \subseteq e_t\) for \(t \in [m]\). As there is no edge between \(V_t\) and \(E_t\) in \(\hat{\Gamma}_G\), \(W_t \cap V_t = \emptyset\) for each \(t \in [m]\), which implies that \(e_t \cap V_t\) for \(t \in [m]\).

As \(G\) contains no multiple edges, if \(W_t = V_s\) for some \(s \neq t\), surely \(W_s \neq V_t\); otherwise \(e_t = e_s = V_t \cup V_s\), a contradiction.

From the above discussion, \(K_G\) and \(\hat{K}_G\) are respectively isomorphic to \(K_{n,n}\) and \(\hat{K}_{n,n}\). A perfect matching \(M\) in \(\hat{K}_{n,n}\) is isomorphic to a perfect matching in \(K_{n,n}\). So \(G\) can be constructed as in Construction \([4.3]\) \(\Box\)

**Theorem 4.6.** Let \(G\) be a 2-regular \(k\)-uniform hypergraphs with \(n\) vertices and \(m\) edges, where \(m\) is odd and \(k\) is even. Then \(G\) is a minimal non-odd-bipartite hypergraph if and only if \(G\) can be constructed as in Construction \([4.3]\) and \(G\) is connected.

**Proof.** The sufficiency follows from Lemmas \([4.3]\) and \([4.2]\) and the necessity follows from Corollary \([4.6]\) and Lemma \([5.1]\) \(\Box\)

**Remark 4.7.** The hypergraph constructed as in Construction \([4.3]\) may not be connected. However, by Lemma \([4.2]\) at least one component is minimal non-odd-bipartite as the total number of edges is odd. For example, the following 4-uniform hypergraph \(G\) on 18 vertices with edges
\[
e_t = \{2t - 1, 2t, 2t + 5, 2t + 6\}, t \in [9],
\]
where the labels of the vertices are modulo 18. \(G\) has 3 connected components \(G_1, G_2, G_3\) with edge sets listed below, each of which is isomorphic to \(C^4_{12}\) (a minimal non-odd-bipartite hypergraph).

\[
E(G_1) := \{1, 2, 7, 8\}, \quad \{7, 8, 13, 14\}, \quad \{13, 14, 1, 2\}.
\]
\[
E(G_2) := \{3, 4, 9, 10\}, \quad \{9, 10, 15, 16\}, \quad \{15, 16, 3, 4\}.
\]
\[
E(G_3) := \{5, 6, 11, 12\}, \quad \{11, 12, 17, 0\}, \quad \{17, 0, 5, 6\}.
\]

In Fig. \([4.4]\) we give an illustration of \(G\) constructed as in the way of Construction \([4.3]\) where the dotted lines indicate a perfect matching in \(K_{18,18}\), and the solid lines indicate a perfect matching in \(K_{18,18}\).

### 4.2. Examples of \(d\)-regular minimal non-odd-bipartite hypergraphs

We first give an example of \(k\)-regular \(k\)-uniform minimal non-odd-bipartite hypergraph by using Cayley hypergraph. Let \(G = (\mathbb{Z}_n; \{1, 2, \ldots, k-1\})\) be a Cayley hypergraph, where \(V(G) = \mathbb{Z}_n\), and \(E(G)\) consists of edges \(\{i, i+1, \ldots, i+k\}\) for \(i \in \mathbb{Z}_n\). Then \(G\) is connected, \(k\)-uniform and \(k\)-regular, with \(n\) vertices and \(n\) edges.
Theorem 4.8. Let $k$ be an even integer greater than 2, and $n$ be an odd integer greater than $k$. The $G = (\mathbb{Z}_n; \{1, 2, \ldots, k-1\})$ is minimal non-odd-bipartite if and only if $\gcd(k, n) = 1$.

Proof. By Theorem 3.3, it suffices to show that $\text{rank}B_G = n-1$ over $\mathbb{Z}_2$ if and only if $\gcd(k, n) = 1$. Consider the equation $B_Gx = 0$ over $\mathbb{Z}_2$. For each $i \in \mathbb{Z}_n$, as $\{i, \ldots, i+k-1\}$ and $\{i+1, \ldots, k\}$ are edges of $G$, by the above equation we have

$$x_i + \cdots + x_{i+k-1} = x_{i+1} + \cdots + x_{i+k} = 0.$$ 

So $x_i = x_{i+k}$ for each $i \in \mathbb{Z}_n$. Let $t := \gcd(k, n)$. Then there exist integers $p, q$ such that $pk + qn = t$. Note that $t$ is odd as $n$ is odd, and if writing $k = st$, then $s$ is even as $k$ is even.

For each $i \in \mathbb{Z}_n$,

$$x_i = x_{i+k} = \cdots = x_{i+pk} = x_{i+tqn} = x_{i+t},$$

As $s$ is even, for any $x_1, \ldots, x_t \in \mathbb{Z}_2$, and any edge

$$x_1 + \cdots + x_k = (x_1 + \cdots + x_t) + \cdots + (x_{(s-1)t+1} + \cdots + x_{st}) = s(x_1 + \cdots + x_t) = 0.$$ 

So, the solution space of $B_Gx = 0$ over $\mathbb{Z}_2$ has dimension $t$, which implies that $\text{rank}B_G = n-t$ over $\mathbb{Z}_2$. The result now follows.

Let $G$ be a $k$-uniform hypergraph with $n$ vertices and $m$ edges. Let $G^1, G^2, \ldots, G^t$ be $t$ disjoint copies of $G$. For each vertex $v$ (or each edge $e$) of $G$, it has $t$ copies $v^1, \ldots, v^t$ (or $e^1, \ldots, e^t$) in $G^1, \ldots, G^t$ respectively. Let $t\circ G$ be a hypergraph whose vertex set is $\bigcup_{i=1}^t V(G^i)$, and edge set is $\{e^1 \cup \cdots \cup e^t : e \in E(G)\}$. Then $t \circ G$ is $tk$-uniform hypergraph with $tn$ vertices and $m$ edges, and the degree of $v^i$ in $t \circ G$ is same as the degree of $v$ in $G$ for each $v \in V(G)$ and $i \in [t]$. If further $G$ is $d$-regular, then $t \circ G$ is also $d$-regular.

Lemma 4.9. Let $G$ be a $k$-uniform hypergraph. Then $G$ is minimal non-odd-bipartite if and only if $t \circ G$ is minimal non-odd-bipartite.

Proof. By a suitable labeling of the vertices of $t \circ G$, we have $B_{t\circ G} = (B_G, B_G, \ldots, B_G)$, where $B_G$ occurs $t$ times in the latter matrix. As $\text{rank}B_G = \text{rank}B_{t\circ G}$, the result follows by Theorem 3.3.

Next we give an example of $d$-regular $k$-uniform minimal non-odd-bipartite hypergraph $G$ with $n$ vertices and $m$ edges, where $m$ is odd and $d$ is even such that $\gcd(d, m) = 1$. Obviously $nd = km$, and $d|k$ as $\gcd(d, m) = 1$. Suppose $k = td$, where $t > 1$. By Theorem 1.8, the hypergraph $H = (\mathbb{Z}_m; \{1, 2, \ldots, d-1\})$ is minimal non-odd-bipartite, which is $d$-regular, $d$-uniform, with $m$ edges. By Lemma 4.9, $t \circ H$ is minimal non-odd-bipartite with $m$ edges, which is $d$-regular and $td(= k)$-uniform.
Corollary 4.10. Let $H = (\mathbb{Z}_m; \{1, 2, \ldots, d-1\}$, where $m$ is odd and $d$ is even such that $\gcd(d, m) = 1$. Then $t \circ H$ is minimal non-odd-bipartite with $m$ edges, which is $d$-regular and $td$-uniform.

Note that in Corollary 4.10 if $d = 2$, then $H$ is an odd cycle $C_m$, and $t \circ C_m = C_m^{2t}$ (a generalized power hypergraph), both of which are minimal non-odd-bipartite.

Thirdly we use a projective plane $(X, B)$ of order $q$ to construct a regular minimal non-odd-bipartite hypergraph. Recall a projective plane of order $q$ consists of a set $X$ of $q^2 + q + 1$ elements called points, and a set $B$ of $(q + 1)$-subsets of $X$ called lines, such that any two points lie on a unique line. It can be derived from the definition that any points lies on $q + 1$ lines, and two lines meet in a unique point, and there are $q^2 + q + 1$ lines. Now define a hypergraph based on $(X, B)$, denoted by $G = (X, B)$, whose vertices are the points of $X$ and edges are the lines of $B$. Then $G = (X, B)$ is a $(q + 1)$-regular $(q + 1)$-uniform hypergraph with $q^2 + q + 1$ vertices.

Theorem 4.11. Let $(X, B)$ be a projective plane of order $q$, and let $G = (X, B)$ be a hypergraph defined as in the above. If $q$ is odd, then $G = (X, B)$ is minimal non-odd-bipartite.

Proof. Let $e$ be an edge of $G = (X, B)$ or a line of $(X, B)$. Then $B_{G-e}B^T_{G-e} = qI + J,$

where $I$ is the identity matrix, and $J$ is an all-ones matrix, both of size $q^2 + q$. So $\det B_{G-e}B^T_{G-e} = \det(qI + J) = (q^2 + 2q)q^{q^2+q-1} \equiv 1 \mod 2,$

implying that $\text{rank} B_G = m - 1$ over $\mathbb{Z}_2$. The result follows by Theorem 3.3(3). \hfill \Box

It is known that if $q$ is an odd prime power, then there always exists a projective plane of order $q$ by using the vector space $\mathbb{F}_q^3$. By Lemma 3.9 and Theorem 4.11, we easily get the following result.

Corollary 4.12. Let $q$ be an odd prime power. There exists a $(q + 1)$-regular $(q + 1)$-uniform minimal non-odd-bipartite hypergraph $G$ with $q^2 + q + 1$ edges. For any positive integer $t > 1$, there exists a $(q + 1)$-regular $(t(q + 1))$-uniform minimal non-odd-bipartite hypergraph with $q^2 + q + 1$ edges.

Remark 4.13. From Corollaries 4.8 and 4.12, the minimal non-odd-bipartite hypergraphs $G$ have degree $d$ and edge number $m$ such that $\gcd(d, m) = 1$. (Note that $\gcd(q + 1, q^2 + q + 1) = 1$.) As $\gcd(d, m) = 1$, from the equality $nd = mk$, we have $d \mid k$, where $n, k$ are the number of vertices and the uniformity of $G$ respectively.

In fact, there exist $d$-regular minimal non-odd-bipartite hypergraphs with $m$ edges such that $\gcd(d, m) > 1$. For example, let $G$ be a 6-uniform 6-regular hypergraph with 9 edges below:

$\{1, 2, 3, 4, 5, 6, 7, 8\}, \quad \{1, 4, 5, 6, 7, 9\}, \quad \{1, 3, 5, 6, 7, 8\}, \quad \{1, 2, 4, 6, 7, 8\}, \quad \{1, 3, 5, 7, 8, 9\},

\{1, 2, 6, 7, 8, 9\}, \quad \{2, 3, 4, 5, 7, 9\}, \quad \{2, 3, 4, 5, 8, 9\}, \quad \{2, 3, 4, 6, 8, 9\}.$

By Theorem 3.3, it is also easy to verify that $G$ is minimal non-odd-bipartite.

There also exist $d$-regular $k$-uniform minimal non-odd-bipartite hypergraphs such that $d \nmid k$. For example, let $G$ be a 6-regular 8-uniform hypergraph with 9 edges below:

$\{1, 2, 3, 4, 5, 6, 7, 8\}, \quad \{1, 3, 4, 5, 6, 7, 9, 11\}, \quad \{1, 4, 5, 6, 7, 8, 9, 10\},

\{1, 5, 7, 8, 9, 10, 11, 12\}, \quad \{1, 2, 3, 6, 7, 9, 10, 12\}, \quad \{1, 2, 4, 5, 8, 10, 11, 12\},

\{2, 3, 4, 6, 8, 10, 11, 12\}, \quad \{2, 3, 4, 5, 8, 9, 11, 12\}, \quad \{2, 3, 6, 7, 9, 10, 11, 12\}.$

By Theorem 3.3, it is easy to verify that $G$ is minimal non-odd-bipartite.
Example 4.14. The minimal non-odd-bipartite uniform hypergraphs with fewest edges. By Theorem 3.3 if $G$ is a $k$-uniform minimal non-odd-bipartite hypergraph with $n$ vertices and $m$ edges, then $m$ is odd. If $m = 1$, $G$ is surely odd-bipartite. So, $m \geq 3$, and hence the maximum degree is at most 3 if $m = 3$. Assume that $m = 3$. By Theorem 3.4 each vertex has an even degree, implying that $G$ is 2-regular. So $2n = 3k$, and $3|n$. Letting $n = 3l$, we have $k = 2l$. So $G = C_{2l}$, which is the unique example of minimal non-odd-bipartite hypergraph with 3 edges. It is consistent with the fact that $C_{3}$ is the unique minimal non-bipartite simple graph with 3 edges by taking $k = 2$.

Example 4.15. The minimal non-odd-bipartite uniform hypergraphs with fewest vertices. If $G$ is a $k$-uniform minimal non-odd-bipartite hypergraph with $n$ vertices and $m$ edges. Then $n \geq k + 1$, as an edge is odd-bipartite. Assume that $n = k + 1$. Then $m \leq \binom{k+1}{k} = k + 1$, with equality if and only if $G$ is a $(k + 1)$-simplex [7], i.e. any $k$ vertices of $G$ forms an edge. Let $\Delta$ be the maximum degree of $G$, which is even by Theorem 3.4. As $m$ is odd by Theorem 3.4 we have

$$m - 1 \geq \Delta \geq \frac{km}{k+1} = m - \frac{m}{k+1} \geq m - 1,$$

which implies that $m = k + 1$ and $k$ is even. So, the $(k + 1)$-simplex is the unique example of $k$-uniform minimal non-odd-bipartite hypergraph with $k + 1$ vertices by Theorem 3.4. If taking $k = 2$, then $C_{3}$ is the the unique minimal non-bipartite simple graph with 3 vertices.

Example 4.16. Example of non-regular minimal non-odd-bipartite hypergraph. Let $G$ be a 4-uniform hypergraph on vertices $1, \ldots, 9$ with edges

$$\{1, 2, 3, 4\}, \{1, 2, 4, 5\}, \{1, 3, 6, 7\}, \{1, 5, 8, 9\}, \{6, 7, 8, 9\}.$$

It is easy to verify that $G$ is non-regular minimal non-odd-bipartite by Theorem 3.4

Remark 4.17. A minimal non-odd-bipartite hypergraph may contain cut edges. For example, the following 4-uniform hypergraph $G$ with vertex set $[18]$ and 9 edges:

$$\{1, 3, 4, 5\}, \{2, 3, 4, 6\}, \{5, 7, 8, 9\}, \{6, 7, 8, 9\}, \{1, 2, 10, 11\}, \{10, 12, 13, 14\}, \{11, 12, 13, 15\}, \{14, 16, 17, 18\}, \{15, 16, 17, 18\},$$

where $\{1, 2, 10, 11\}$ is a cut edge of $G$. By Theorem 3.4 $G$ is minimal non-odd-bipartite.

5. Least H-eigenvalue of minimal non-odd-bipartite hypergraphs

Let $G$ be a $k$-uniform minimal hypergraph. Let $x \in \mathbb{C}^{V(G)}$ whose entries are indexed by the vertices of $G$. For a subset $U$ of $V(G)$, denote $x^{U} := \Pi_{u \in U} x_{u}$. Then we have

$$\mathcal{A}(G) x^{k} = \sum_{e \in E(G)} k x^{e}, \quad (5.1)$$

Theorem 5.1. Let $G$ be $k$-uniform minimal non-odd-bipartite hypergraph with $n$ vertices and $m$ edges, where $k$ is even. Then

(1) $\lambda_{\text{min}}(G) \leq -\rho(G) + \frac{2k}{m}$

(2) $\lambda_{\text{min}}(G) \leq -(1 - \frac{2}{m}) \rho(G)$

Proof. As $G$ is connected by Lemma 3.4 by Perron-Frobenius theorem, there exists a positive eigenvector $x$ of $\mathcal{A}(G)$ associated with the spectral radius $\rho(G)$. We may assume $\|x\|_{k} = 1$. Then

$$\rho(G) = \mathcal{A}(G) x^{k} = \sum_{e \in E(G)} k x^{e}. \quad (5.2)$$
Define a vector \( y \) otherwise. Note that \( \bar{e} \) is odd-bipartite with an odd-bipartition \( \{U, U^c\} \). Now define a vector \( y \) on the vertices of \( G - \bar{e} \) such that \( y_v = x_v \) if \( v \in U \) and \( y_v = -x_v \) otherwise. Note that \( \bar{e} \) intersects \( U^c \) in an even number of vertices as \( G \) is non-odd-bipartite, which implies that \( y^e = x^e > 0 \). By a similar discussion as the above, we have

\[
\lambda_{\min}(G) \leq A(G)y^k = -A(G)x^k - 2kx^\bar{e} \leq -\rho(G) + \frac{2k}{m}. 
\]

For the second result, from Eq. (5.2), there exists one edge \( \hat{e} \) such that \( kx^\hat{e} \) is not greater than the average of the summands \( kx^e \) over all \( m \) edges \( e \) of \( G \), that is,

\[
kx^\hat{e} \leq \frac{\rho(G)}{m}. 
\]

Note that \( G - \hat{e} \) is also odd-bipartite with an odd-bipartition say \( \{W, W^c\} \). Now define a vector \( z \) on the vertices of \( G - \hat{e} \) such that \( z_v = x_v \) if \( v \in W \) and \( z_v = -x_v \) otherwise. By a similar discussion as the above, we have

\[
\lambda_{\min}(G) \leq A(G)z^k = -A(G)x^k - 2kx^\hat{e} \leq -(1 - \frac{2}{m})\rho(G).
\]

\[\square\]

**Corollary 5.2.** Let \( k \) be a positive even integer. For any \( \epsilon > 0 \), for any \( k \)-uniform minimal non-odd-bipartite hypergraph \( G \) with sufficiently larger number of vertices or edges,

1. \(-\rho(G) < \lambda_{\min}(G) < -\rho(G) + \epsilon\),
2. \(-1 < \lambda_{\min}(G)/\rho(G) < -1 + \epsilon\).

For a connected \( k \)-uniform hypergraph \( G \), where \( k \) is even, if we denote

\[
\alpha(G) := \rho(G) + \lambda_{\min}(G), \quad \beta(G) := -\lambda_{\min}(G)/\rho(G),
\]

then by Lemma 2.3 \( \alpha(G) \geq 0 \), with equality if \( G \) is odd-bipartite; and \( 0 < \beta(G) \leq 1 \), with right equality if and only if \( G \) is odd-bipartite. So we can use \( \alpha(G) \) and \( \beta(G) \) to measure the non-odd-bipartiteness of an even uniform hypergraph.

Furthermore, by Theorem 5.1 and Corollary 5.2 if \( G \) is minimal non-odd-bipartite hypergraph, then \( \alpha(G) \to 0 \) and \( \beta(G) \to 1 \) when the number of vertices or edges of \( G \) goes to infinity. So, the minimal non-odd-bipartite hypergraphs are very close to be odd-bipartite in this sense.

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