A note on the refinement of Hermite-Hadamard inequality

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Abstract. In this paper, we give the sharper bounds for the mean value of a convex function using dyadic decomposition. Our result is related with classical Hermite-Hadamard inequality. Moreover, using the result, we can determine the maximum error before calculating the numerical (trapezoidal) integral of the convex function.

1. Introduction
Let $I$ be the closed interval $[s,t]$ where $s,t \in \mathbb{R}$ and $s < t$. We have convex function $h : I \to \mathbb{R}$ while $h$ satisfies
\[
h(\alpha u + (1 - \alpha)v) \leq \alpha h(u) + (1 - \alpha)h(v),
\]
for every $u, v \in I$ and $\alpha \in [0,1]$. The convex functions was introduced by J L W V. Jensen in 1906 (see[1]). Meanwhile, the reverse of the above inequality causes $h$ to be concave. If $h$ has twice differentiable on $(s,t)$, then
\begin{enumerate}[(i)]
  \item $h''(x) \geq 0$ for every $x \in (a,b)$ if and only if $h$ is convex function.
  \item $h''(x) \leq 0$ for every $x \in (a,b)$ if and only if $h$ is concave function.
\end{enumerate}
The above function is very important, particulary, it becames a sufficient to get Jensen’s inequality. Besides that, here we recall the classical Hermite-Hadamard inequality as follows
\[
h\left(\frac{s + t}{2}\right) \leq \frac{1}{t - s} \int_{s}^{t} h(x)dx \leq \frac{h(s) + h(t)}{2},
\]
where $h$ be a covex function on $[s,t]$. The classical Hermite-Hadamard inequality was first noticed by Ch. Hermite [2] in 1883. Ten years later, it rediscovered by J. Hadamard (see [3]). For applications, convex function and classical Hermite-Hadamard inequality are used to observe the inequalities of several mean formulas (see [4, 5, 6]).

Here, we will show the proof of (1) by C.P. Niculescu and L.E. Persson [7]. Now, pay attention to the following lemma

Lemma 1.1. If $h$ be convex function on $[s,t]$ with $s, t \in \mathbb{R}$, $s < t$, then
\[
h(x) \leq h(s) + \frac{h(t) - h(s)}{t - s}(x - s),
\]
for every $x \in [s,t]$. 

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Proof. Let \( h \) be convex function on \([s, t]\) with \( s, t \in \mathbb{R}, s < t \). There are three cases.

Case 1: For \( x = s \), we have \( h(x = s) \leq h(s) = h(s) + \frac{h(t) - h(s)}{t - s}(s - s) \).

Case 2: For \( x = t \), we have \( h(x = t) \leq h(t) = h(s) + \frac{h(t) - h(s)}{t - s}(t - s) \).

Case 3: For \( x \in (s, t) \), there is \( \gamma \in (0, 1) \) such that \( x = \gamma s + (1 - \gamma)t \). Because \( h \) be convex function, then

\[
\frac{h(x) - h(s)}{x - s} \leq \frac{\gamma h(s) + (1 - \gamma)h(t) - h(s)}{\gamma s + (1 - \gamma)t - s} = \frac{(1 - \gamma)(h(t) - h(s))}{(1 - \gamma)(t - s)} = \frac{h(t) - h(s)}{t - s}.
\]

Consequently, \( h(x) \leq h(s) + \frac{h(t) - h(s)}{t - s}(x - s) \).

Hence, \( h(x) \leq h(s) + \frac{h(t) - h(s)}{t - s}(x - s) \), for every \( x \in [s, t] \).

The proof is complete.

Next, we present the proof of (1).

Proof. The proof of (1).

Let \( h \) be convex function on \([s, t]\) with \( s, t \in \mathbb{R}, s < t \). Using Lemma 1.1, we obtain

\[
\frac{1}{t - s} \int_s^t h(x)dx \leq \frac{1}{t - s} \int_s^t \left( h(s) + \frac{h(t) - h(s)}{t - s}(x - s) \right)dx
\]

\[
= h(s) + \left( \frac{h(t) - h(s)}{t - s} \right) \frac{(t - s)^2}{2(t - s)}
\]

\[
= h(s) + \frac{h(t)}{2}.
\]

Meanwhile

\[
\frac{1}{t - s} \int_s^t h(x)dx = \frac{1}{t - s} \left( \int_s^{s + \frac{t}{2}} h(x)dx + \int_{s + \frac{t}{2}}^t h(x)dx \right)
\]

\[
= \frac{1}{2} \left( \int_0^1 h \left( \frac{s + t - y(t - s)}{2} \right) + h \left( \frac{s + t - y(t - s)}{2} \right) dy \right)
\]

\[
\geq \int_0^1 h \left( \frac{s + t}{2} \right) dy = h \left( \frac{s + t}{2} \right).
\]

Hence, \( h \left( \frac{s + t}{2} \right) \leq \frac{1}{t - s} \int_s^t h(x)dx \leq \frac{h(s) + h(t)}{2} \).

The proof is complete.

In [8], D.Ş. Marinescu and M. Monea said that over 5000 articles have been published on the topic of the Hermite-Hadamard inequalities. Some Authors proved it using the integral operator. Their results can be read in [9, 10, 11, 12]. We also have seen that in [13], A.E. Farissi gave a simple proof of this inequality. He also has obtained new bounds for the mean value of the convex function \( f \) on \([a, b]\) as follows

\[
h \left( \frac{s + t}{2} \right) \leq l \leq \frac{1}{t - s} \int_s^t h(x)dx \leq L \leq \frac{h(s) + h(t)}{2},
\]

with \( l = \frac{1}{2} \left( h \left( \frac{3s + t}{4} \right) + h \left( \frac{s + 3t}{4} \right) \right) \) and \( L = \frac{1}{2} \left( h \left( \frac{s + t}{2} \right) + h \left( \frac{s + h(t)}{2} \right) \right) \). By [13], here we have used \( \lambda = \frac{1}{2} \).
Theorem 2.1. Let \( h \) be convex function on \([s, t]\) with \( s, t \in \mathbb{R}, s < t \). On \([s, \frac{t+s}{2}]\), we use (1) to obtain
\[
\frac{(t-s)}{2} h \left( \frac{3s+t}{4} \right) \leq \int_s^{\frac{t+s}{2}} h(x)dx \leq \frac{(t-s)}{2} h\left( \frac{s+h(\frac{t+s}{2})}{2} \right).
\]
(3)

Meanwhile, on \([\frac{t+s}{2}, t]\), we also use (1) to obtain
\[
\frac{(t-s)}{2} h \left( \frac{s+3t}{4} \right) \leq \int_{\frac{t+s}{2}}^t h(x)dx \leq \frac{(t-s)}{2} h\left( \frac{t+h(\frac{t+s}{2})}{2} \right).
\]
(4)

Sum (3) and (4). Next, divide by \((t-s)\), so we have
\[
l \leq \frac{1}{t-s} \int_s^t h(x)dx \leq L
\]
with \( l = \frac{1}{2} \left( h \left( \frac{3s+t}{4} \right) + h \left( \frac{s+3t}{4} \right) \right) \) and \( L = \frac{1}{2} \left( h \left( \frac{s+t}{2} \right) + \frac{f(s)+f(t)}{2} \right) \). Because \( h \) be convex function, then \( h \left( \frac{s+t}{2} \right) \leq l \) and \( L \leq \frac{h(s)+h(t)}{2} \). The proof is complete.

In this paper, our aim is to investigate the sharper bounds for the mean value of a convex function. We shall obtain new upper bound which is less than \( L \) and new lower bound which is greater than \( l \).

2. Main Results

The refinement of the Hermite-Hadamard inequality was done by S. Khalid and J. Pečarić (see [14]). Their proof use the differentiable convex functions. Other result can be seen in [15] which is produced by M. Nowicka and A. Witkowski. They proved it using \( n \) sub intervals of bounded interval. Here we refine the bounds of the mean value of a convex function using dyadic sub interval of bounded interval. Let an arbitrary \( I = [a, b] \) and \( n \in \mathbb{N} \). Following an advice of C.P. Niculescu and L.E. Persson [7], we use dyadic decomposition to make partitions of \( I \) as follows
\[
I_k := \left[ \frac{(2^n - k + 1)s + (k-1)t}{2^n}, \frac{(2^n - k)s + (k)t}{2^n} \right] \quad \text{and} \quad I_{2^n} := [s + (2^n - 1)t, t],
\]
where \( k : 1, 2, \ldots, (2^n - 2), (2^n - 1) \).

Our method to attain the goal is explained by the following steps. First step, we give a convex function on interval \( I \) which is partitioned with dyadic decomposition. Second step, in every sub interval, we calculate lower bound and upper bound for mean values of \( h \) using classical Hermite-Hadamard inequality. Third step, all the lower bounds of the subintervals are added up, as are all the upper bounds. Next, finally step, we calculate the distance from the number of lower bounds to the number of upper bounds and we will make a conclusion. Now it is time to present the following theorem.

Theorem 2.1. Let \( h \) be convex function on \([s, t]\) with \( s, t \in \mathbb{R}, s < t \). For every \( n \in \mathbb{N} \), we have
\[
l(2^n) \leq \frac{1}{t-s} \int_s^t h(x)dx \leq L(2^n)
\]
where \( l(2^n) = \frac{1}{2^n} \sum_{k=1}^{2^n} h \left( \frac{(2^n-2k+1)s + (2k-1)t}{2^n+1} \right) \) and \( L(2^n) = \frac{1}{2^n} \left( \frac{h(s)+h(t)}{2} + \sum_{k=1}^{2^n-1} h \left( \frac{(2^n-k)s + kt}{2^n} \right) \right) \).
Proof. Let an arbitrary $I = [s, t]$ and $n \in \mathbb{N}$. Take dyadic interval by partition of $I$ as follows

$$I_k := \left[ \frac{(2^{n} - k + 1)s + (k - 1)t}{2^n}, \frac{(2^n - k)s + kt}{2^n} \right], \quad I_{2^n} := \left[ \frac{s + (2^n - 1)t}{2^n}, t \right],$$

where $k : 1, 2, \ldots, (2^n - 2), (2^n - 1)$.

By using the classical Hermite-Hadamard inequality and inspiring the proof of (2), we have

$$\left( \frac{t - s}{2^n} \right) h \left( \frac{(2^{n+1} - 2k + 1)s + (2k - 1)t}{2^{n+1}} \right) \leq \int_{I_k} h(x)dx \leq \left( \frac{t - s}{2} \right) h \left( \frac{(2^{n} - k)s + (k + 1)t}{2^n} \right) + h \left( \frac{(2^{n} - k + 1)s + (k + 1)t}{2^n} \right),$$

for every $k : 1, 2, \ldots, (2^n - 2), (2^n - 1), 2^n$. Now, sum the above inequality for all $k$ and divide by $t - s$, so

$$l(2^n) \leq \frac{1}{t - s} \int_s^t h(x)dx \leq L(2^n)$$

where $l(2^n) = \sum_{k=1}^{2^n} h \left( \frac{(2^{n+1} - 2k + 1)s + (2k - 1)t}{2^{n+1}} \right)$ and $L(2^n) = \frac{h(s) + h(t)}{2} + \sum_{k=1}^{2^n-1} h \left( \frac{(2^{n} - k)s + (k + 1)t}{2^n} \right)$.

The proof is complete.

It can be checked that the lower and upper bounds of the Theorem 2.1 are better than the result of A.E. Farissi at [13]. The two results are the same for $n = 1$, meaning $l(2^n) = l$ and $L(2^n) = L$. Furthermore, it is important to show that the larger $n$ makes $l(2^n)$ and $L(2^n)$ closer to the mean value of $f$ over the interval $[s, t]$. The following theorem presents it.

**Theorem 2.2.** Let $h$ be convex function on $[s, t]$ with $s, t \in \mathbb{R}$, $s < t$. For every $n \in \mathbb{N}$, we have

$$l(2^n) \leq l(2^{n+1})$$

and

$$L(2^{n+1}) \leq L(2^n).$$

Proof. With $I = [s, t]$ and $n \in \mathbb{N}$, we take dyadic interval by partition of $I$ as follows

$$I_k := \left[ \frac{(2^{n+1} - k + 1)s + (k - 1)t}{2^{n+1}}, \frac{(2^{n+1} - k)s + (k)t}{2^{n+1}} \right], \quad I_{2^{n+1}} := \left[ \frac{s + (2^{n+1} - 1)t}{2^{n+1}}, t \right],$$

where $k : 1, 2, \ldots, (2^{n+1} - 2), (2^{n+1} - 1)$. Now, check that

$$l(2^{n+1}) = \frac{1}{2^{n+1}} \sum_{k=1}^{2^{n+1}} f \left( \frac{(2^{n+2} - (2k - 1))a + (2k - 1)b}{2^{n+2}} \right)$$

$$= \frac{1}{2^n} \sum_{j=1}^{2^n} \left( \frac{1}{2} f \left( \frac{(2^{n+2} - (4j - 3))a + (4j - 3)b}{2^{n+2}} \right) + \frac{1}{2} f \left( \frac{(2^{n+2} - (4j - 1))a + (4j - 1)b}{2^{n+2}} \right) \right)$$

$$\geq \frac{1}{2^n} \sum_{j=1}^{2^n} f \left( \frac{(2^{n+1} - (2j - 1))a + (2j - 1)b}{2^{n+1}} \right).$$
Thus \(l(2^{n+1}) \geq l(2^n)\). Meanwhile, we also obtain

\[
L(2^{n+1}) = \frac{1}{2^{n+1}} \left( \sum_{k=1}^{2^{n+1}-1} h \left( \frac{(2^{n+1} - k)s + kt}{2^{n+1}} \right) \right) + \frac{1}{2^{n+1}} \left( h(s) + h(t) \right)
\]

\[
= \frac{1}{2^{n+1}} \left( \sum_{j=0}^{2^n-1} h \left( \frac{(2^n + 1 - 2j)s + (2j + 1)t}{2^{n+1}} \right) \right)
\]

\[
+ \frac{1}{2^{n+1}} \left( \sum_{j=0}^{2^n-1} \frac{1}{2} h \left( \frac{(2^n - j)s + 2jt}{2^{n+1}} \right) + \sum_{j=0}^{2^n-1} \frac{1}{2} h \left( \frac{(2^n - 2(j + 1)s + 2(j + 1)t}{2^{n+1}} \right) \right)
\]

\[
= \frac{1}{2^n} \left( \sum_{j=0}^{2^n-1} h \left( \frac{(2^n - j)s + jt}{2^n} \right) \right)
\]

\[
+ \frac{1}{2^n} \left( \sum_{j=0}^{2^n-1} \frac{1}{2} h \left( \frac{(2^n - j)s + (j + 1)t}{2^n} \right) \right)
\]

\[
= \frac{1}{2^n} \left( \sum_{k=1}^{2^n-1} h \left( \frac{(2^n - k)s + kt}{2^n} \right) \right) + \frac{1}{2^n} \left( h(s) + h(t) \right).
\]

We conclude that \(L(2^n) \geq L(2^{n+1})\).

The proof is complete.

Theorem 2.2 states that \(l(\cdot)\) became a sequence which is ascending monotone and finite with an upper bound \(\frac{1}{s} \int_s^t h(x)dx\). Meanwhile, \(L(\cdot)\) also became a sequence which is descending monotone and finite with \(\frac{1}{t} \int_s^t h(x)dx\) as a lower bound of. Look again in real analysis (see [16]), that a monotonous and finite sequence (real-valued) is a convergent sequence. As a result, \(l(\cdot)\) and \(L(\cdot)\) are a convergent sequence. What is the difference between \(l(\cdot)\) and \(L(\cdot)\) when \(n\) goes to infinity? The following theorem explains it.

**Theorem 2.3.** Let \(h\) be convex function on \([s, t]\) with \(s, t \in \mathbb{R}, s < t\). For every \(n \in \mathbb{N}\), we have

\[
0 \leq L(2^n) - l(2^n) \leq \frac{1}{2^n+1} \left( h(s) + h(t) - h \left( \frac{(2^n+1 - 1)s + t}{2^n+1} \right) - h \left( \frac{s + (2^n+1 - 1)t}{2^n+1} \right) \right).
\]

**Proof.** Assume that \(h\) is convex function on \([s, t]\) with \(s, t \in \mathbb{R}, s < t\). Take an arbitrary \(n \in \mathbb{N}\) and evaluate that

\[
L(2^n) - l(2^n) = \frac{1}{2^n} \left( \frac{h(s) + h(t)}{2} \right) + \frac{1}{2^n} \left( \sum_{k=1}^{2^n-1} h \left( \frac{(2^n - k)s + kt}{2^n} \right) \right)
\]

\[
- \frac{1}{2^n} \sum_{k=1}^{2^n} h \left( \frac{(2^n+1 - (2k - 1))s + (2k - 1)t}{2^n+1} \right).
\]
Now, rearrange such that
\[
L(2^n) - l(2^n) = \frac{1}{2^{n+1}} \left( h(s) - h \left( \frac{(2^{n+1} - 1)s + t}{2^{n+1}} \right) \right) \\
+ \frac{1}{2^{n+1}} \sum_{k=1}^{2^n-1} \left( h \left( \frac{(2^n - k)s + kt}{2^n} \right) - h \left( \frac{(2^{n+1} - (2k + 1))s + (2k + 1)t}{2^{n+1}} \right) \right) \\
+ \frac{1}{2^{n+1}} \sum_{k=1}^{2^n-1} \left( h \left( \frac{(2^n - k)s + kt}{2^n} \right) - h \left( \frac{(2^{n+1} - (2k - 1))s + (2k - 1)t}{2^{n+1}} \right) \right) \\
+ \frac{1}{2^{n+1}} \left( h(t) - h \left( \frac{s + (2^{n+1} - 1)t}{2^{n+1}} \right) \right) \\
= \frac{1}{2^{n+1}} \sum_{k=0}^{2^n-1} \left( h \left( \frac{(2^{n+1} - 2k)s + 2kt}{2^{n+1}} \right) - h \left( \frac{(2^{n+1} - (2k + 1))s + (2k + 1)t}{2^{n+1}} \right) \right) \\
+ \frac{1}{2^{n+1}} \sum_{k=1}^{2^n} \left( h \left( \frac{(2^{n+1} - 2k)s + 2kt}{2^{n+1}} \right) - h \left( \frac{(2^{n+1} - (2k - 1))s + (2k - 1)t}{2^{n+1}} \right) \right).
\]

By assumption that \( h \) is convex function, so we have
\[
h \left( \frac{(2^{n+1} - 2k)s + 2kt}{2^{n+1}} \right) \leq \frac{1}{2} h \left( \frac{(2^{n+1} - (2k - 1))s + (2k - 1)t}{2^{n+1}} \right) + \frac{1}{2} h \left( \frac{(2^{n+1} - (2k + 1))s + (2k + 1)t}{2^{n+1}} \right).
\]

Consequently,
\[
0 \leq L(2^n) - l(2^n) \\
\leq \frac{1}{2^{n+1}} \left( h(s) + h(t) - h \left( \frac{(2^{n+1} - 1)s + t}{2^{n+1}} \right) - h \left( \frac{s + (2^{n+1} - 1)t}{2^{n+1}} \right) \right).
\]

The proof is complete.

Because \( h \) is bounded on a bounded interval, then Theorem 2.3 also says that
\[
\lim_{n \to \infty} (L(2^n) - l(2^n)) = 0.
\]

**Corollary 2.4.** Let \( h \) be convex function on \([s, t]\) with \( s, t \in \mathbb{R}, s < t \). We have
\[
\lim_{n \to \infty} l(2^n) = \frac{1}{t-s} \int_s^t h(x)dx = \lim_{n \to \infty} L(2^n)
\]

**Proof.** Using Theorem 2.3 and *squeeze test* in [16], we can state that the sequences \( l(\cdot) \) and \( L(\cdot) \) converge to
\[
\frac{1}{t-s} \int_s^t h(x)dx.
\]

The proof is complete.

Now, exactly that if we continue the division process, then mean value of \( h \) can be approximated by lower bound or upper bound of Hermite-Hadamard inequality with all sub-intervals of \([s, t]\) by the dyadic decomposition.
3. Application
We will give an application of our result for calculating finite integral of $f$. Not all bounded functions can be calculated its integral over $[s,t]$. The available options, we use the numerical integral to solve it, even though it is an approximate solution. Let $h$ be integrable function on $[s,t]$. Choose $\varepsilon > 0$ as a maximum error. For $h$ be convex (concave) function, we take $n \in \mathbb{N}$ such that

$$\frac{(t - s)}{2^{n+1}} \left| h(s) + h(t) - h \left( \frac{(2^{n+1} - 1)s + t}{2^{n+1}} \right) - h \left( \frac{s + (2^{n+1} - 1)t}{2^{n+1}} \right) \right| \leq \varepsilon.$$ 

Using trapezoidal rule in numerical integral, we obtain that the "distance" between $\int_s^t h(x)dx$ and $(t - s)L_h(2^n)$ are less than $\varepsilon$, so do $(t - s)L_h(2^n)$. In other words, $(t - s)L_h(2^n)$ and $(t - s)L_h(2^n)$ approach (close) to $\int_s^t h(x)dx$.

4. Conclusion
The technique of partitioning intervals with dyadic decomposition accelerates the formation of small sub-intervals. The impact is very powerful to improve the lower bound and upper bound of the mean value of the convex function. In practical use, the approximation of the integral value on a finite interval can be achieved well, although only using a simple technique in the numerical method (trapezoidal method).

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