MEAN VALUE PROPERTIES AND UNIQUE CONTINUATION

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Abstract. In the first part of the paper we review some mean value properties and their connections to the Laplacian and other significant nonlinear operators like the $p$-Laplacian and the infinity-Laplacian. The second part is devoted to the unique continuation property, including a brief description of the methods, some of the main problems in the area and connections to the so called infinity mean value property.

1. Introduction and basic notation. This expository paper consists of two parts. The first part (section 2) is devoted to mean value properties and their relation to the classical laplacian and other important nonlinear operators such as the $\infty$-Laplacian and the $p$-Laplacian ($1 < p < \infty$). In section 2.1 we will review some facts about the classical mean value property, its converse and its connections to the laplacian.

Our exposition here owes much to the excellent survey [40]. Subsections 2.2 and 2.3 discuss other types of nonlinear mean value properties and their connections to the $\infty$-Laplacian and the $p$-Laplacian, respectively.

The second part (section 3) is devoted to a fundamental property of harmonic functions, the unique continuation property, and its extension to other differential operators. Subsection 3.1 deals mainly with the Laplacian and subsection 3.2 discusses the same problem for the $\infty$-Laplacian and the $p$-Laplacian, where a new variant of the problem and new terminology is required due to nonlinearity. While in the linear case there is an abundant literature on unique continuation, the problem is essentially open for both the $p$-Laplacian and the $\infty$-Laplacian and few positive results are known. The link between the two parts of the paper is provided by the analysis of unique continuation for solutions of the so called $\infty$-mean value property (see [34]).

We introduce now some basic notation that will be used in the paper. Given $a \in \mathbb{R}^n$ and $r > 0$ we will denote by $B(a,r)$, $\overline{B}(a,r)$ and $S(a,r)$ respectively the open ball, the closed ball and the sphere of center $a$ and radius $r$ in $\mathbb{R}^n$. $<,>$ denotes euclidean inner product in $\mathbb{R}^n$. Furthermore, $m$ denotes $n$-dimensional measure and $\sigma$ stands for surface measure on spheres. For $E \subset \mathbb{R}^n$, the symbol $\underline{\int}_E f$ denotes the average of $f$ in $E$ (with respect to Lebesgue measure).

Definition 1.1. For a domain $\Omega \subset \mathbb{R}^n$ and $x \in \Omega$, a radius $r > 0$ is admissible at $x$ if $0 < r < \text{dist}(x, \partial \Omega)$ and a subset of positive radii $E(x)$ is admissible if $r$ is
admissible at $x$ for each $r \in E(x)$. We will say that $\mathcal{E} = \{E(x)\}_{x \in \Omega}$ is an admissible family of radii in $\Omega$ if $E(x)$ is admissible for each $x \in \Omega$.

Note in particular that $B(x,r) \subset \Omega$ for each $x \in \Omega$ and each $r \in E(x)$ provided $\mathcal{E} = \{E(x)\}_{x \in \Omega}$ is admissible in $\Omega$.

As usual, $\Delta$ stands for the Laplace operator in $\mathbb{R}^n$:

$$\Delta \equiv \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$$

and $u : \Omega \to \mathbb{R}$ is harmonic in $\Omega$ if $u \in C^2(\Omega)$ and $\Delta u = 0$ in $\Omega$. Observe that if $n = 1$ harmonic functions are just linear functions.

2. Mean value properties: old and new.

2.1. The classical mean value property and the Laplacian. Let $\Omega \subset \mathbb{R}^n$ be a domain, $\mathcal{E} = \{E(x)\}_{x \in \Omega}$ an admissible family of radii in $\Omega$ and $u \in C(\Omega)$. We say that $u$ satisfies the (volume) mean value property with respect to $\mathcal{E}$ if

$$u(x) = \frac{1}{m(B(x,r))} \int_{B(x,r)} u(y) dm(y)$$

for each $x \in \Omega$ and each $r \in E(x)$.

Analogously, $u$ satisfies the (spherical) mean value property with respect to $\mathcal{E}$ if

$$u(x) = \frac{1}{\sigma(S(x,r))} \int_{S(x,r)} u(y) d\sigma(y)$$

In [40] the more general case of integrable $u$ is considered but continuity will be enough for our purposes in what follows. The basic direct result connecting the mean value property and harmonicity is the classical theorem of Gauss ([15]) which, with the above terminology, reads as follows:

**Theorem 2.1.1** (Gauss). If $u$ is harmonic in the domain $\Omega \subset \mathbb{R}^n$ then $u$ satisfies both the volume and the spherical mean value property with respect to the family $\mathcal{E} = \{E(x)\}$ where $E(x) = (0, \text{dist}(x,\partial\Omega))$ for all $x \in \Omega$. That is, $u$ satisfies the mean value property (either volume or spherical) at each $x \in \Omega$ for any admissible radius at $x$.

More interesting than the direct mean value property is the so called converse mean value property that asks under which conditions (on the family of radii $\mathcal{E}$, on the continuous function $u$ and on the domain $\Omega$) the mean value property with respect to $\mathcal{E}$ implies harmonicity. A theorem of Koebe ([29]) says that the converse of Gauss theorem is also true, that is, the mean value property for any admissible radius together with continuity implies harmonicity. Actually the same arguments show that the conclusion holds under weaker assumptions, as the following result, which we still refer to as Koebe’s theorem, says:

**Theorem 2.1.2** (Koebe). Let $u$ be continuous in the domain $\Omega \subset \mathbb{R}^n$. If $u$ satisfies the volume mean value property with respect to some admissible family of radii $\mathcal{E} = \{E(x)\}$ in $\Omega$ such that $\inf E(x) = 0$ for all $x \in \Omega$ then $u$ is harmonic in $\Omega$. The same conclusion holds if we replace “volume” by “spherical”.
That is, a continuous function is harmonic if it satisfies the mean value property at any point for arbitrarily small radii. The last result suggests that assuming the mean value property for less radii might still imply harmonicity. For instance, what about one single radius? If $u \in C(\Omega)$ and $u$ satisfies the mean value property at each point for one single radius (depending on the point), is $u$ harmonic? These sort of questions go back to Volterra ([45]) who proved the next theorem for regular domains; the regularity assumption was later removed by Kellogg ([25]):

**Theorem 2.1.3** (Volterra, Kellogg). Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $u \in C(\overline{\Omega})$. Suppose that for each $x \in \Omega$ there is a radius $r(x)$, with $0 < r(x) \leq \text{dist}(x,\partial \Omega)$ such that $u$ satisfies the volume mean value property at $x$ with radius $r(x)$. Then $u$ is harmonic in $\Omega$. The same conclusion holds if we replace “volume” by “spherical”.

This is a distinguished example of a one radius theorem. That the assumptions in the statement of the Volterra-Kellogg theorem are necessary, even if $n = 1$, is shown by the following examples.

**Example 2.1.1.** Let $\Omega = \mathbb{R}$ and $r > 0$. Then any $r$-periodic function $u : \mathbb{R} \to \mathbb{R}$ satisfies the spherical mean value property with respect to the family $E = \{E(x)\}$ where $E(x) = r\mathbb{N}$ for all $x \in \mathbb{R}$ which, from periodicity, is equivalent to the functional equation

$$u(x) = \frac{1}{2}(u(x + r) + u(x - r))$$

(3)

**Example 2.1.2.** As above, take $\Omega = \mathbb{R}$ and $r > 0$. Choose a complex number $z_0 \neq 0$ such that

$$\frac{\sin(rz_0)}{rz_0} = 1$$

(4)

Then a direct computation shows that the complex valued function $f : \mathbb{R} \to \mathbb{C}$ given by $f(x) = e^{ixz_0}$ satisfies

$$f(x) = \frac{1}{2r} \int_{x-r}^{x+r} f(t) dt$$

(5)

In particular $u = Re f$ and $v = Im f$ satisfy the volume mean value property with radius $r$, that is, $E(x) = \{r\}$ for all $x \in \mathbb{R}$. Both examples 2.1.1 and 2.1.2 show that boundedness of the domain in the Volterra-Kellogg is a necessary assumption, even for functions satisfying the mean value property for a constant radius.

**Example 2.1.3.** That continuity up to the boundary is also essential in the Volterra-Kellogg theorem is shown by zig-zag type functions. Pick sequences $(a_n)_{n \in \mathbb{Z} \setminus \{0\}}$, $(b_n)_{n \in \mathbb{Z}}$ with $b_0 = 1/2$ and

$$0 < \cdots < -a_{-n+1} < b_{-n} < a_{-n} < \cdots < b_0 < \cdots < a_n < b_n < a_{n+1} < \cdots < 1$$

for all $n \geq 1$ and define $u : (0,1) \to [0,1]$ such that $u(a_n) = 0$ for all $n \in \mathbb{Z} \setminus \{0\}$, $u(b_n) = 1$ for all $n \in \mathbb{Z}$ and $u$ is extended by linear interpolation between any two consecutive points of the sequences $(a_n)$, $(b_n)$. Then the sequences can be chosen in such a way (see [11] for a particular choice) that $u$ is continuous in $(0,1)$ and for each $x \in (0,1)$ there is an admissible radius $r(x) > 0$ such that $u$ satisfies (2.3) with $r(x)$ instead of $r$. Then $u$ satisfies the spherical mean value property with respect to the family $E = \{E(x)\}$ with $E(x) = \{r(x)\}$ for each $x \in (0,1)$. Of course the only points where the mean value property needs to be checked are the “peak” points.
A construction of the same style can be used to obtain a continuous function in 
\((0, 1)\) (this time unbounded) that satisfies the volume mean value property at each 
x \in (0, 1) for some admissible radius \(r(x) > 0\), that is, \(u\) satisfies the functional 
equation (5) with \(r(x)\) instead of \(r\) for all \(x \in (0, 1)\).

In higher dimensions, similar counterexamples can be constructed in the disc or 
the ball by pasting appropriate linear combinations of the fundamental solution of 
the Laplacian to show that continuity in the open ball together with the mean value 
property (either volume or spherical) for one single radius at every point does not 
necessarily imply harmonicity (see [5], exercice 18, Ch. 1 for a sketch in the case 
of the disc). Counterexamples obtained in this way are unbounded in both the 
volume and the spherical cases.

In his book of function theory ([30]), Littlewood asked whether Theorem 2.1.2. 
should be true in the unit disc if the continuity up to the boundary is replaced by 
continuity in the open disc together with boundedness of the function. The conjecture, 
which was open for a long time, was finally solved by Hansen and Nadirashvili 
(see [20], [21]) and their solution showed that, surprisingly, the volume and spherical 
cases behave in a very different way. See [40] for a detailed account of the evolution 
of Littlewood’s conjecture including a number of interesting related results.

**Theorem 2.1.4** (Hansen-Nadirashvili).

1. Let \(\Omega \subset \mathbb{R}^n\) be a bounded domain and suppose that \(u \in C(\Omega)\) is bounded and 
satisfies the volume mean value property at each \(x \in \Omega\) for a single admissible 
radius \(r(x)\). Then \(u\) is harmonic.

2. There is a continuous, bounded, non-harmonic function \(u\) defined in the unit 
disc \(D \subset \mathbb{R}^2\) such that for every \(z \in D\) there is an admissible radius \(r(z) > 0\) 
so that \(u\) satisfies the spherical mean value property at \(z\) with radius \(r(z)\).

It follows from Theorem 2.1.4. that the volume version of Littlewood’s conjecture 
is true in any dimension but the spherical version is false in the plane. The problem 
is still open for \(n > 2\).

We close this subsection with a brief discussion of the connection between the 
Laplacian and the asymptotic mean value property. Suppose that \(u \in C^2(\Omega)\), \(x \in \Omega\) 
and \(B(x, r) \subset \Omega\). Taylor’s development at \(x\) gives

\[
u(x + h) - u(x) = <\nabla u(x), h> + \frac{1}{2} <Hu(x)h, h> + o(r^2)
\]

where \(|h| \leq r\) and \(Hu(x)\) denotes the hessian matrix of \(u\) at \(x\). Now, averaging \(6\) 
over \(h\), the first order term vanishes and we get:

\[
\int_{B(x, r)} (u - u(x)) = \frac{1}{2} \int_{B(0, r)} <Hu(x)h, h> + o(r^2)
\]

(7)

An elementary computation shows that the average in the right hand of \(7\) is 
\[
\frac{\Delta u(x)}{n+2}
\]

so taking limits as \(r \to 0\) we deduce

\[
\lim_{r \to 0} \frac{1}{r^2} \left( \int_{B(x, r)} u - u(x) \right) = \frac{\Delta u(x)}{2(n + 2)}
\]

(8)

In particular we deduce that if the limit in the left hand side of \(8\) is 0 for all 
\(x\) then \(u\) is harmonic. One may suspect that this asymptotic mean value property 
opens the way to define harmonicity without using derivatives and this is exactly 
the case. The following theorem is a version of results going back to Blaschke ([7]), 
Privaloff ([41]) and Zaremba ([47]):
Theorem 2.1.5 (Blaschke, Privaloff, Zaremba). Let $u \in C(\Omega)$ and suppose that for each $x \in \Omega$ there is a sequence of radii $r_k = r_k(x) > 0$ such that $r_k \to 0$ as $k \to \infty$ and
\[
\lim_{k \to \infty} \frac{1}{r_k} \left( \int_{B(x,r_k)} u - u(x) \right) = 0 \quad (9)
\]
Then $u$ is harmonic in $\Omega$.

Observe that this result implies Theorem 2.1.2.

2.2. The infinity mean value property and the infinity Laplacian. We introduce now a nonlinear mean value property with close connections with a relevant second order nonlinear differential operator, the $\infty$-Laplacian. As in the introduction, let $\Omega \subset \mathbb{R}^n$ be a domain and $E = \{ E(x) \} \in \Omega$ an admissible family of radii in $\Omega$.

Definition 2.2.1. We say that $u \in C(\Omega)$ satisfies the $\infty$-Mean Value Property in $\Omega$ ($\infty$-MVP for short) with respect to an admissible family of radii $E = \{ E(x) \}$ in $\Omega$ if,
\[
u(x) = \frac{1}{2} \left( \sup_{B(x,r)} u + \inf_{B(x,r)} u \right) \quad (10)
\]
for each $x \in \Omega$ and each $r \in E(x)$.

Let us mention how the functional equation (10) arises in other contexts. First, the case when the sets $E(x)$ consist of a single radius $r(x)$ and the function $r(x)$ satisfies certain regularity conditions has been studied in [35] and [33] in the more general setting of metric spaces, in connection with the classical extension problem of a continuous function on a closed subset of the space respecting its modulus of continuity.

Secondly, equation (10) has recently appeared as a Dynamic Programming Principle related to some stochastic games. To get a glimpse of why (10) is connected to games and probabilistic ideas let us consider a simple, very specific model case: a finite graph $G$ with boundary $\partial G$. Suppose that $E \subset \partial G$. A token is placed at a point of the graph and at each subsequent stage it is moved to a contiguous point. Player I wins if the token reaches $\partial G$ for the first time at $E$ and player II wins if it exits at $\partial G \setminus E$.

Given $z \in G$, let $u(z)$ be the probability that player I wins the game, starting from $z$. Such probability will of course depend on the rules according to which the token moves from one point to any of the contiguous ones (the transition rules). Suppose that $z_1, \cdots, z_n$ are the contiguous points of $z$ in $G$ and let us stipulate that the token moves randomly and with equal probability to any of its contiguous points (transition rule). Then, by conditional probability we get
\[
u(z) = \frac{1}{n} \sum_{i=1}^{n} u(z_i) \quad (11)
\]
so $u$ satisfies the usual mean value property in the graph. Functions $u$ satisfying (11) at any point of $G$ are just harmonic functions in the graph and this is the starting point of classical potential theory in graphs.

The previous game is “passive” in the sense that everything relies on randomness but we could imagine games for which the role of the players is more “active” by changing the rules. Suppose, as before, that $E \subset \partial G$ and the token is placed at some initial point of the graph. Then a coin is successively tossed and the winner player
can freely choose the contiguous point towards which to move the token at every stage. The game ends whenever the token reaches \( \partial G \) for the first time. Player I wins if the token exits at \( E \), while player II wins if the token exits at \( \partial G \setminus E \). Then it is clear that, according to the rules, player I should try to maximize -at every stage-the probability of exit at \( E \) while player II should try to minimize such probability. Then, starting from a point \( z \in G \), player I, if winning, should choose a contiguous point \( z_i \) so that the probability of exit at \( E \) from \( z_i \) is maximal while player II, if winning, should choose a contiguous point \( z_j \) so that the corresponding probability from \( z_j \) is minimal. If \( u(z) \) is the probability that player I wins the game starting from \( z \) then by conditioning over the result of the coin toss we have

\[
    u(z) = \frac{1}{2} \sup u(z_i) + \frac{1}{2} \inf u(z_i)
\]

which is a discrete version of (10) in this context. Equation (12) is an example of the Dynamic Programming Principle. Two player games of this style (so called tug-of-war games) were used and applied for the first time in [43] as stochastic substitutes of brownian motion in connection to extension problems in metric spaces and the Dirichlet problem for the \( \infty \)-Laplacian in domains of \( \mathbb{R}^n \). See next subsection for further developments of the theory. The graph example above is of course an oversimplification but it is enough to get an idea of why (10) is connected to stochastic games.

As a third example, functional equations as (12) have been recently used in problems of image processing (see [13]). Here is a rough explanation: suppose that \( u \) is a function defined in a graph \( G \), for example the (light)intensity at any point of the graph. The question is what kind of reasonable mean value property should satisfy \( u \), that is, how the intensity of any point should be related to the intensities of the neighboring points. One can of course consider the usual mean value property (11) but there are models that use other nonlinear mean value properties like (12).

We list now some examples of functions satisfying the \( \infty \)-mean value property. See [34] for details.

**Example 2.2.1.** It is easy to check that if \( I \subset \mathbb{R} \) is an open interval, \( u : I \to \mathbb{R} \) satisfies the \( \infty \)-MVP in \( I \) with respect to the admissible family \( \mathcal{E} = \{ E(x) \} \) in \( I \) and \( \inf E(x) = 0 \) for each \( x \in I \) then \( u \) is linear in \( I \). This is a sort of one-dimensional Koebe-type theorem in this setting.

**Example 2.2.2.** The following construction provides a family of examples in \( \mathbb{R} \). Let \( u_0 : [0, 1] \to [0, 1] \) be continuous, nondecreasing with \( u_0(0) = 0 \), \( u_0(1) = 1 \) and extend \( u_0 \) to \( \mathbb{R} \) by defining \( u(x + k) \equiv u_0(x) + k \) for each \( x \in [0, 1] \) and \( k \in \mathbb{Z} \). Then \( u : \mathbb{R} \to \mathbb{R} \) is continuous, nondecreasing and satisfies the \( \infty \)-MVP in \( \mathbb{R} \) with respect to the family \( \mathcal{E} = \{ E(x) \} \) with \( E(x) = \mathbb{N} \) for all \( x \in \mathbb{R} \).

**Example 2.2.3.** The argument function in \( \mathbb{R}^2 \) satisfies the \( \infty \)-MVP in any domain \( \Omega \) in which a continuous determination of the argument is defined, with respect to the maximal admissible family \( \mathcal{E} = \{ E(x) \}, E(x) = (0, \text{dist}(x, \partial \Omega)) \).  

**Example 2.2.4.** Let \( F \subset \mathbb{R}^n \) be a closed, convex set. Then the distance function \( u(x) = \text{dist}(x, F) \) satisfies the \( \infty \)-MVP in \( \mathbb{R}^n \setminus F \) with respect to the maximal admissible family \( \mathcal{E} = \{ E(x) \}, E(x) = (0, \text{dist}(x, F)) \). (See [34], Prop. 4.3). Taking \( F = \{ a \} \) with \( a \in \mathbb{R}^n \) we get the cone function \( C(x) = |x - a|, x \in \mathbb{R}^n \setminus \{ a \} \). Another important example (for reasons that will appear later) arises taking \( F = \)
\{(x,0): x \leq 0\} \subset \mathbb{R}^2$, which originates the function
\[
    u(x,y) = \text{dist}((x,y),F) = \begin{cases} 
    |y|, & \text{if } x \leq 0 \\
    \sqrt{x^2 + y^2}, & \text{if } x > 0 
\end{cases}
\] (13)

We justify now the relation between the \(\infty\)-mean value property and the \(\infty\)-Laplacian. Let \(u \in C^2(\Omega), x \in \Omega\) and assume that \(\nabla u(x) \neq 0\). Let \(r > 0\) small enough so that \(\overline{B}(x,r) \subset \Omega\) and \(\nabla u \neq 0\) in \(\overline{B}(x,r)\). We follow the analysis in [38]. Pick \(x_r, x'_r \in \overline{B}(x,r)\) such that
\[
    u(x_r) = \max_{\overline{B}(x,r)} u, \quad u(x'_r) = \min_{\overline{B}(x,r)} u
\] (14)

Note that, from the hypothesis on \(\nabla u\), both \(x_r\) and \(x'_r\) can be chosen to be on the sphere \(S(x,r)\). Put \(h_r = x_r - x, h'_r = x'_r - x\) and \(\tilde{x}_r = x - h_r\). Using (6) with \(h = \pm h_r\), we get
\[
    u(x_r) - u(x) = - \nabla u(x), h_r, > + \frac{1}{2} < Hu(x)h_r, h_r > + o(r^2)
\]
\[
    u(\tilde{x}_r) - u(x) = - \nabla u(x), h_r, > + \frac{1}{2} < Hu(x)h_r, h_r > + o(r^2)
\]
In particular
\[
    \frac{1}{2} \left( u(x_r) + u(\tilde{x}_r) \right) - u(x) = \frac{1}{2} < Hu(x)h_r, h_r > + o(r^2) \tag{15}
\]

From (14) and the definition of \(x_r\) we deduce
\[
    \frac{1}{2} \left( \sup_{B(x,r)} u + \inf_{\overline{B}(x,r)} u \right) - u(x) \leq \frac{1}{2} < Hu(x)h_r, h_r > + o(r^2) \tag{16}
\]
and, repeating the same argument for \(x'_r\), we obtain
\[
    \frac{1}{2} < Hu(x)h'_r, h'_r > + o(r^2) \leq \frac{1}{2} \left( \sup_{B(x,r)} u + \inf_{\overline{B}(x,r)} u \right) - u(x) \tag{17}
\]

Now an standard argument with Lagrange multipliers shows that
\[
    h_r = r \frac{\nabla u(x_r)}{|\nabla u(x_r)|}, \quad h'_r = r \frac{\nabla u(x'_r)}{|\nabla u(x'_r)|}
\]

So
\[
    \lim_{r \to 0} \frac{h_r}{r} = - \lim_{r \to 0} \frac{h'_r}{r} = \frac{\nabla u(x)}{|\nabla u(x)|} \tag{18}
\]

and from (15),(16) and (17) we finally get the following

**Proposition 2.2.1.** Let \(u \in C^2(\Omega), x \in \Omega\) and suppose that \(\nabla u(x) \neq 0\). Then
\[
    \lim_{r \to 0} \frac{1}{2} \left( \sup_{B(x,r)} u + \inf_{\overline{B}(x,r)} u \right) - u(x) = \frac{< Hu(x)\nabla u(x), \nabla u(x) >}{2|\nabla u(x)|^2} \tag{19}
\]

**Proposition 2.2.1.** motivates the following definition.

**Definition 2.2.2.** Let \(\Omega \subset \mathbb{R}^n\) be a domain and \(u \in C^2(\Omega)\). The \(\infty\)-Laplacian of \(u\), denoted \(\Delta_{\infty} u\), is defined by
\[
    \Delta_{\infty} u = \sum_{i,j=1}^n u_{x_i}u_{x_j}u_{x_i,x_j} = < (Hu)\nabla u, \nabla u > \tag{20}
\]
where \(Hu\) is the hessian matrix of \(u\).
The ∞-Laplacian was introduced by Aronsson in the sixties (see [1], [2]) in connection to lipschitz extension problems and has been object of increasing attention since then. See [6] and [10] for general accounts of the theory. The relation with tug-of-war games appeared in [43].

The origin of the terminology can be justified as follows. Suppose that Ω ⊂ ℝⁿ is a bounded domain and 1 < p < ∞. It is well known that if u minimizes ||∇u||Lᵖ(Ω) with prescribed boundary values then it satisfies the equation
\[ \triangle_p u \equiv \text{div}(|\nabla u|^{p-2} \nabla u) = 0 \quad (21) \]
Here \( \triangle_p \) is the p-Laplacian and solutions of (21) are called p-harmonic functions.

Note that we recover the classical case if p = 2. In a similar way (see [22], §0) it turns out that minimizers (in an appropriate way) of ||∇u||L∞(Ω) satisfy \( \triangle_\infty u = 0 \) in a certain weak sense. Also, \( \triangle_\infty \) can be formally obtained from \( \triangle_p \) as p → ∞.

Indeed, assuming enough regularity, some computation shows that the p-Laplacian can be expressed in the following way
\[ \triangle_p u = \text{div}(\nabla u|\nabla u|^{p-2}) = |\nabla u|^{p-2} \left( \triangle u + \frac{p-2}{|\nabla u|^2} \triangle_\infty u \right) \]
In particular
\[ \triangle_p u = 0 \iff \frac{\triangle u}{p-2} + \frac{\triangle_\infty u}{|\nabla u|^2} = 0 \quad (22) \]
so from (22) we get, formally, that \( \triangle_p u = 0 \rightarrow \triangle_\infty u = 0 \) as p → ∞. The first example in the literature where the ∞-Laplacian is obtained as limit of the p-Laplacian as p → ∞ is [8]. See also [22] and [32].

Aronsson already observed that classical solutions of \( \triangle_\infty u = 0 \) are too rigid (see [2]) so a weaker concept of solution was needed. It turns out that the right concept is that of viscosity solution.

**Definition 2.2.3.** We say that u satisfies \( \triangle_\infty u \geq 0 \) in Ω in viscosity sense if u is upper semicontinuous in Ω and for every local maximum \( x \in \Omega \) of u − φ, where φ is \( C^2 \) in a neighbourhood of x then \( \triangle_\infty \phi(x) \geq 0 \). Analogously, \( \triangle_\infty u \leq 0 \) in Ω in viscosity sense if u is lower semicontinuous in Ω and for every local minimum \( x \in \Omega \) of u − φ where φ is \( C^2 \) in a neighbourhood of x then \( \triangle_\infty \phi(x) \leq 0 \). Finally, \( \triangle_\infty u = 0 \) in Ω in viscosity sense if \( \triangle_\infty u \geq 0 \) and \( \triangle_\infty u \leq 0 \) in viscosity sense.

**Remark 2.2.1.** Note that viscosity solutions are in particular continous.

**Remark 2.2.2.** If \( u \in C^2(\Omega) \) then \( \triangle_\infty u = 0 \) in classical sense if and only if \( \triangle_\infty u = 0 \) in viscosity sense.

Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain and \( f \in C(\partial\Omega) \). While Aronsson showed in [2] that the Dirichlet problem
\[
\begin{align*}
\triangle_\infty u &= 0 \quad \text{in } \Omega \\
u|_{\partial\Omega} &= f
\end{align*}
\]
does not necessarily have a \( C^2 \) solution, a fundamental theorem of Jensen ([22]) shows that it always has a (unique) solution in the viscosity sense.

We close the subsection by reviewing some properties of the ∞-mean value property and its relation to the ∞-Laplacian. The next proposition is a sort of asymptotic ∞-mean value property that is analogous to Theorem 2.1.5. in subsection 2.1.
Proposition 2.2.2. Let \( u \in C(\Omega) \). Suppose that for each \( x \in \Omega \) there is a sequence of positive numbers \( r_k = r_k(x) \to 0 \) such that
\[
\lim_{k \to \infty} \frac{1}{r_k^2} \left[ \frac{1}{2} \left( \sup_{B(x,r_k)} u + \inf_{B(x,r_k)} u \right) - u(x) \right] = 0
\]
Then \( \nabla u = 0 \) in \( \Omega \) in viscosity sense.

Proof. We prove first that \( \nabla u \geq 0 \) in viscosity sense. We can assume that \( \phi \) touches \( u \) from above at \( x \in \Omega \), where \( \phi \in C^2 \) in a neighbourhood of \( x \) and \( \nabla \phi(x) \neq 0 \). If \( r > 0 \) is small enough,
\[
\frac{1}{2} \left( \sup_{B(x,r)} u + \inf_{B(x,r)} u \right) - u(x) \leq \frac{1}{2} \left( \sup_{B(x,r_k)} \phi + \inf_{B(x,r_k)} \phi \right) - \phi(x)
\]
and from (19), (20) and (23) we deduce that \( \nabla \phi(x) \geq 0 \). This proves that \( \nabla u \geq 0 \) in viscosity sense. The reverse inequality is proved in the same way. \( \square \)

Corollary 2.2.1. Suppose that \( u \) satisfies the \( \infty \)-mean value property in \( \Omega \) with respect to some admissible family of radii \( E = \{E(x)\} \) such that \( \inf E(x) = 0 \) for each \( x \in \Omega \). Then \( \nabla u = 0 \) in \( \Omega \) in viscosity sense.

The next proposition (see [34]) is a version of the Maximum/Minimum Principle for solutions of the \( \infty \)-MVP. It is a consequence of Corollary 2.2.1. together with the corresponding result for viscosity solutions of \( \nabla u = 0 \) (see [6], section 2.3); however the proof is completely elementary in the \( \infty \)-MVP setting without appealing to the equation.

Proposition 2.2.3. Suppose that \( u \) satisfies the \( \infty \)-mean value property in a domain \( \Omega \subset \mathbb{R}^n \) with respect to an admissible family of radii \( E = \{E(x)\} \) such that \( \inf E(x) = 0 \) for each \( x \in \Omega \).

1. If \( u \) attains a global extremum in \( \Omega \) then \( u \) is constant.
2. For any bounded, relatively compact subdomain \( \Omega' \subset \Omega \), we have
\[
\sup_{\Omega'} u = \sup_{\partial \Omega'} u, \quad \inf_{\Omega'} u = \inf_{\partial \Omega'} u
\]

Remark 2.2.3. The converse of Proposition 2.2.2 is not true. Aronsson’s function \( u(x,y) = x^{4/3} - y^{4/3} \) provides a counterexample. It can be checked that \( \nabla u = 0 \) in \( \mathbb{R}^2 \) in viscosity sense but if \( a = (1,0) \), it was shown in [38] that
\[
\lim_{r \to 0} \frac{1}{r^2} \left[ \frac{1}{2} \left( \sup_{B(a,r)} u + \inf_{B(a,r)} u \right) - u(a) \right] = \frac{1}{18}
\]
However there is a positive result in the opposite direction if the asymptotic \( \infty \)-mean value property is interpreted in certain viscosity sense ([38]).

Remark 2.2.4. It would make a significative difference to have “local” instead of “global” in part 1 of Proposition 2.2.3. It is known that \( \infty \)-harmonic functions are locally constant around a local extremum (see [6], section 2.3) but this does not necessarily imply that they are globally constant, unless restricted unique continuation holds (see Definition 3.1 in section 3).
2.3. A mixed mean value property and the \( p \)-Laplacian. Let \( 1 < p < \infty \). It follows from (22) that, formally,

\[
\triangle_p u = 0 \Leftrightarrow (p - 2)\triangle_\infty u + \triangle u = 0 \tag{24}
\]

Equation (24) can actually be interpreted in viscosity sense (see [38], [39]; note that the equivalence of weak solutions and viscosity solutions for the \( p \)-Laplacian was obtained in [24]). Since (24) is saying that the \( p \)-Laplacian can be interpreted as a sort of average of the \( \infty \)-Laplacian and the usual Laplacian, it makes sense to consider averages of the mean value properties (1) and (10). More precisely, for \( \alpha, \beta \in \mathbb{R} \) with \( \alpha + \beta = 1 \) we may consider the following mean value property

\[
u(x) = \frac{\alpha}{2} \left( \sup_{B(x,r)} u + \inf_{B(x,r)} u \right) + \beta \int_{B(x,r)} u \tag{25}
\]

Using similar arguments to those of subsections 2.1 and 2.2 it can be seen that the right choices of \( \alpha \) and \( \beta \) associated to equation \( \triangle_p u = 0 \) are (see [38]):

\[
\alpha = \frac{p - 2}{n + p}, \quad \beta = \frac{n + 2}{n + p} \tag{26}
\]

A substantial part of the results in [38] and [39] consists of making precise the connection between solutions of \( \triangle_p u = 0 \) and (25), with the values of \( \alpha \) and \( \beta \) given by (26). The following result is a sort of analogue of Theorem 2.1.5 and Proposition 2.2.2. in this setting.

**Theorem 2.3.1 ([38]).** Let \( 1 < p \leq \infty \) and \( u \in C(\Omega) \). Then \( \triangle_p u = 0 \) in \( \Omega \) if and only if the asymptotic expansion

\[
u(x) = \frac{\alpha}{2} \left( \sup_{B(x,r)} u + \inf_{B(x,r)} u \right) + \beta \int_{B(x,r)} u + o(r^{2}) \text{ as } r \to 0 \tag{27}
\]

holds for all \( x \in \Omega \) in viscosity sense, where \( \alpha, \beta \) are determined by (26).

See [38] for the viscosity interpretation of (27).

Functions satisfying (25) for a fixed radius \( r \), with \( \alpha \) and \( \beta \) as in (26) have been called \( p \)-harmonious functions in [39] (observe that continuity is not assumed as part of the hypothesis here). Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain, \( F \in C(\partial \Omega) \) and \( r > 0 \). Define

\[
\Gamma_r = \{ x \in \mathbb{R}^n \setminus \Omega : \text{dist}(x, \partial \Omega) \leq r \}
\]

In order to deal with boundary values for \( p \)-harmonious functions the authors in [39] extend \( F \) to \( \Gamma_r \) in a continuous way and show that there is a unique \( p \)-harmonious function in \( \Omega \) with boundary values \( F \). Then they prove the following approximation result

**Theorem 2.3.2 ([39]).** Let \( 2 < p < \infty \), \( \Omega \subset \mathbb{R}^n \) a bounded domain satisfying certain boundary regularity condition (for instance an exterior cone condition) and \( F \in C(\partial \Omega) \). Let \( u \) be the (unique) solution of the Dirichlet problem

\[
\begin{cases}
\triangle_p u = 0 & \text{in } \Omega \\
u|_{\partial \Omega} = F
\end{cases}
\]

and let \( u_r \) be the unique \( p \)-harmonious function in \( \Omega \) associated to the radius \( r \) with boundary values \( F \). Then

\[
u_r \to u \quad \text{uniformly in } \overline{\Omega}
\]
as \( r \to 0 \) and the limit above does not depend on the extensions of \( F \) to the strips \( \Gamma_r \).

In [39] the authors interpret (25) as a Dynamic Programing Principle for a tug-of-war game which is an averaged version of the two games described at the beginning of subsection 2.2 and deduce Theorem 2.3.2. from this circle of ideas. A purely analytic proof of the existence and uniqueness of \( p \)-harmonious functions with continuous boundary data (in the sense specified above) has been obtained in [36]. See also [26].

3. Unique continuation. One of the most characteristic - yet striking - properties of harmonic functions is the \textit{unique continuation} property. It says that if two harmonic functions coincide in a ball contained in a domain then they must coincide in the whole domain. Because of linearity this is equivalent to saying that if a harmonic function vanishes in a ball contained in a domain then it is identically zero in the domain. Nevertheless this is not necessarily the case in the absence of linearity. This justifies the following definition.

\textbf{Definition 3.0.1.} Let \( \Omega \subset \mathbb{R}^n \) be a domain and \( A \) a class of continuous functions defined in \( \Omega \).

- We say that \( A \) satisfies the \textit{unique continuation} property in \( \Omega \) if whenever \( u, v \in A \) and \( u, v \) coincide in a ball contained in \( \Omega \) then \( u \equiv v \) in \( \Omega \).
- We say that \( A \) satisfies the \textit{restricted unique continuation} property in \( \Omega \) if whenever \( u \in A \) and \( u \) vanishes in a ball contained in \( \Omega \) then \( u \equiv 0 \) in \( \Omega \).

3.1. The classical case. Most textbooks in function theory deduce the unique continuation property for harmonic functions - in an essentially trivial manner - as a consequence of the fact that they are real analytic. Analyticity, however, is not available in many contexts and it already became apparent long ago that other methods applicable to more general situations should be investigated. There are two other methods towards unique continuation that do not involve analyticity or the explicit use of a Poisson kernel: Carleman-type estimates and variants of Almgrem’s frequency function. It should be remarked that, even for the Laplacian, the proof of the unique continuation property from any of such methods is far from being trivial.

One of the first and most celebrated unique continuation results is due to Carleman ([9]) who proved that solutions of the equation \( \triangle u + Vu = 0 \) in \( \mathbb{R}^2 \), where \( V \in L^\infty_{\text{loc}}(\mathbb{R}^2) \) satisfy the unique continuation property using what we call now Carleman estimates. Since then, Carleman estimates have been a fundamental tool in the abundant literature on unique continuation (see [27], [28] and the references therein). Roughly speaking, a Carleman estimate in this context is an inequality of the form

\[
\|w \frac{f}{d}\|_{L^p(\Omega)} \leq C \|w \triangle f\|_{L^p(\Omega)}
\]

where \( f \in C_0^\infty(\Omega) \), \( w \) is a certain weight and \( d \) denotes the distance to \( \partial \Omega \). The following Carleman estimate

\[
\int |x|^{-t-nq} |f(x)|^q dx \leq C \int |x|^{-t+2-np} |\triangle f(x)|^p dx
\]

for all \( f \in C_0^\infty(\mathbb{R}^n \setminus \{0\}) \) where \( n \geq 3 \), \( t \in \mathbb{R} \setminus \mathbb{Z} \) and \( p = \frac{2n}{n+2}, q = \frac{2n}{n-2} \) was obtained in [23]. Lin ([31]) applied (28) to unique continuation for solutions of the equation

\[
\triangle u + Vu = 0
\]
More precisely, suppose that \( u \) is a solution of (29) in the unit ball \( B_1 \subset \mathbb{R}^n \) and \( V \in L^{n/2}(B_1) \). Lin proved that the \( L^q \)-norm of \( u \) satisfies the following doubling condition
\[
||u||_{L^q_{2r}} \leq C||u||_{L^q_r}
\]
if \( 0 < r < 1/2 \), where \( C \) depends possibly on \( u \) and \( n \) but not on \( r \) (here \( B_r \) stands for the ball \( B(0,r) \)). The doubling condition (30) implies that solutions of (29) satisfy the unique continuation property. (It follows in fact that \( |u| \) is an \( A_p \) weight).

Another approach to unique continuation is based in the use of the frequency function. Suppose that \( u \) is harmonic in a neighbourhood of the unit ball \( B_1 \subset \mathbb{R}^n \) and define
\[
D(r) = \int_{B_r} |\nabla u|^2 dm
\]
\[
L(r) = \int_{\partial B_r} u^2 d\sigma
\]
\[
N(r) = \frac{rD(r)}{L(r)}
\]
where, as before, \( B_r = B(0,r) \). Since
\[
L'(r) = \frac{n-1}{r} + 2D(r)
\]
it follows that
\[
\frac{d}{dr} \log \left( \frac{L(r)}{r^{n-1}} \right) = 2 \frac{D(r)}{L(r)}
\]
and therefore
\[
\log \left( \frac{L(2r)}{2^{n-1}L(r)} \right) = 2 \int_r^{2r} \frac{D(t)}{L(t)} dt = 2 \int_r^{2r} \frac{N(t)}{t} dt
\]
(31)

Almgrem ([4]) observed that the function \( N(r) \) is nondecreasing and called it the frequency function of \( u \). (In [16] the authors mention that the reason of the name is that \( N(r) \equiv k \) if \( u(z) = Re z^k \) in \( \mathbb{R}^2 \)). From (32) and the fact that \( N \) is nondecreasing it follows that
\[
L(2r) \leq 2^{n-1}4N(1)L(r)
\]
so the spherical means \( L(r) \) (and also the volume means) satisfy a doubling condition, which implies the unique continuation property. By appropriate modifications of the frequency function, a number of unique continuation results for elliptic equations have been obtained in this way (see [16], [17] and references therein).

3.2. Unique continuation, the \( \infty \)-Laplacian and the \( \infty \)-mean value property. If \( 1 < p < \infty \) both the unique continuation and restricted unique continuation properties for \( p \)-harmonic functions are known to be true in the plane (see [37]) due to the fact that the (complex) gradient of a \( p \)-harmonic function is a quasiregular mapping if \( n = 2 \). There have been some recent attempts to define a \( p \)-harmonic frequency function (see [18]) but the problem of unique continuation for \( p \)-harmonic functions is essentially open if \( n > 2 \).

The situation is slightly different for the \( \infty \)-Laplacian. It is known that unique continuation is false, as the function \( u(x,y) = \text{dist}((x,y),F) \) given by (13) shows. (See Example 2.2.4; here \( F \) is the negative real axis in \( \mathbb{R}^2 \)). Indeed, \( u \) satisfies the \( \infty \)-MVP in \( \mathbb{R}^2 \setminus F \) with respect to the maximal family of admissible radii so it is a viscosity solution of \( \Delta_\infty u = 0 \) in \( \mathbb{R}^2 \setminus F \) according to Corollary 2.2.1. We see from
(13) that taking $v = y$ then both $u$ and $v$ are $\infty$-harmonic in the upper half plane and $u = v$ in the second quadrant. This shows that unique continuation fails for the $\infty$-Laplacian. However, restricted unique continuation might still be true. In fact, a considerably stronger conclusion holds if one assumes more regularity, as the following theorem says. It was proved by Aronsson ([2]) for $n = 2$ and by Yu ([46]) for general $n$.

**Theorem 3.2.1** (Aronsson, Yu). Let $\Omega \subset \mathbb{R}^n$ be a domain. Suppose that $u \in C^2(\Omega)$ is a nonconstant solution of $\Delta_\infty u = 0$. Then $u$ does not have critical points in $\Omega$.

According to Theorem 3.2.1 the question of restricted unique continuation is still meaningful for general viscosity solutions of $\Delta_\infty u = 0$. In [34] the simpler question of whether restricted unique continuation holds for solutions of the $\infty$-MVP is considered. Even in dimension one, some caution is necessary if the sets of radii are too sparse. With the notation of Example 2.2.2, choose $\Omega_0 : [0, 1] \to [0, 1]$ such that $u_0 \equiv 0$ in $[0, 1/2]$. Then the extended function $u$ satisfies the $\infty$-MVP in $\mathbb{R}$ with respect to the family $\mathcal{E} = \{E(x)\}$ with $E(x) = N$ for all $x \in \mathbb{R}$. In particular $u$ vanishes in an interval but $u \neq 0$. In [34] it is shown that this cannot happen if the sets $E(x)$ contain enough radii and verify certain mild regularity assumption.

**Theorem 3.2.2** ([34]). Let $\Omega \subset \mathbb{R}^n$ be a domain and $\rho : \Omega \to (0, +\infty)$ a lower semicontinuous function such that $0 < \rho(x) \leq \text{dist}(x, \partial \Omega)$ for all $x \in \Omega$. Define $\mathcal{E} = \{E(x)\}_{x \in \Omega}$ where $E(x) = (0, \rho(x))$. Then the class of functions satisfying the $\infty$-mean value property with respect to $\mathcal{E}$ has the restricted unique continuation property.

Thus, Theorem 3.2.2 supports the conjecture that restricted unique continuation should be true for viscosity solutions of $\Delta_\infty u = 0$.

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