I. INTRODUCTION

Unhalted, gravitational collapse in Einstein’s theory of general relativity will lead to the formation of singularities where the spacetime curvature blows up, and the theory breaks down. Remarkably, it is conjectured that generically such singularities occur only within black holes, where they are hidden from outside observers. This behavior is referred to as cosmic censorship [1]. (Here, we refer specifically to the weak cosmic censorship conjecture.) Violations of cosmic censorship have been shown to occur in the fine-tuned configurations of critical collapse [2], as well as in spacetimes with dimension higher than four [3–5]. However, for more generic initial data in 3+1 dimensions, cosmic censorship has shown to hold in the numerous cases it has been studied [6].

One possible exception to cosmic censorship is the work of Shapiro and Teukolsky [7] following the collapse of prolate configurations of collisionless particles. In that work, one configuration was found to exhibit a blowup in the spacetime curvature in two spindle regions lying outside the collapsing matter, an indication this was not due to just a shell-crossing singularity in the matter. No apparent horizon was found before the simulation became unreliable, and this, along with the lack of turn around of null geodesics in the vicinity of the blowup, was taken as evidence that this was a naked singularity, in violation of cosmic censorship. However, since the calculation could not be continued past the blow-up, the possibility of the spacetime containing an event horizon was not conclusively ruled out. Though the presence of an apparent horizon implies the existence of an event horizon, the absence of an apparent horizon in some gauge does not rule out an event horizon. In fact, one can slice a Schwarzschild spacetime in a way that approaches the singularity without containing outer trapped surfaces [8].

The study of such prolate configurations as a candidate for violating cosmic censorship was motivated by Thorne’s hoop conjecture [9]. Though lacking a precise formulation, the hoop conjecture roughly states that a black hole will form if and only if some mass $M$ can be localized in a region whose circumference in every direction satisfies $C \lesssim 4\pi M$. (Here and throughout we use geometric units with $G = c = 1$.) For example, the collapse of an infinite cylindrical distribution of matter will not form a black hole (but will form a singularity) [9]. The hoop conjecture can be violated in the presence of negative energy, for example by cylindrical black holes in Anti-de Sitter spacetimes [10–12], or due to a scalar field with negative potential [12], where arbitrarily elongated black holes can be formed, but seems to be robust otherwise. Hence the motivation to study the collapse of very prolate distributions of matter which (at least initially) lie outside the hoop conjecture bound to form a black hole [13].

Despite follow-up work by numerous authors, including relaxing the requirement of axisymmetry and using higher resolution [14–15], utilizing excision to the causal future of the curvature blow-up [16], and extending to 5 dimensional spacetimes [17], the question of whether the configurations studied in [9] violate cosmic censorship has remained unanswered. Here, we revisit the problem, using somewhat different methods that allows us to determine the ultimate fate of such spacetimes. We show that in fact black holes do form to rescue cosmic censorship.

II. METHODOLOGY

We consider the same family of initial conditions as in [5], consisting of a prolate spheroidal distribu-
tion of collisionless matter, initially at rest, that is axisymmetric and has no angular momentum. This family is parameterized by a semi-major axis length $b$ (in units of the total mass $M$), and eccentricity $e = \sqrt{1 - a^2/b^2}$ (where $a$ is in the equatorial radius). In the Newtonian limit, the spheroids have homogeneous density. At $t = 0$, the spatial metric is conformally flat $\gamma_{ij} = \Psi^{-4}$, and the extrinsic curvature is zero $K_{ij} = 0$. See [13, 15] for further details on the initial data.

In this work, we focus on very prolate cases. In [3], the cases considered were $e = 0.9$ with $b/M = 2$ (prompt collapse to a black hole) and $b/M = 10$ (candidate for cosmic censorship violation). Here we consider a number of cases with $e = 0.9$ and $2 \leq b/M \leq 20$. We also consider one case with larger eccentricity, namely $b/M = 10$ with $e = 0.95$.

We evolve the Einstein-Vlasov equations describing a distribution of collisionless matter coupled to gravity using the methods described in [18] for evolving massive particles. For gauge conditions at $t = 0$, we choose the lapse to be $\alpha = \Psi^{-4}$ and the shift to be zero $\beta^i = 0$. However, we carry out the initial part of the evolution in harmonic gauge. Around the time of collapse, we transition to a damped harmonic gauge [19, 20] (specifically the $p = 1/4$ version used in [21]) which we find helps control the strong oscillations in the coordinate shape of the black hole as it settles down. In contrast, in [2], maximal slicing and isotropic spatial coordinates were used.

We search for apparent horizons using a flow method. When found, we measure several properties of the horizon including its area—from which a mass $M_{\text{BH}}$ can be calculated—and its proper circumferences in the polar and equatorial directions, $C_p$ and $C_{\text{eq}}$. The gravitational radiation is measured by calculating the Newman-Penrose scalar $\psi_4$. We also compute the Kretschmann scalar, obtained from contracting the Riemann tensor with itself $I = R_{abcd}R^{abcd}$, as well as the matter density, computed from the stress-energy tensor $\rho = -T^{ab}_a$.

We restrict to axisymmetry, which allows us to use a computational domain with two spatial dimensions. Most results presented below are obtained using $N = 1.6 \times 10^6$ particles and an adaptive mesh refinement simulation grid where the finest resolution is $dx \approx 0.02M$ (for $b/M \leq 12$ and $e \leq 0.9$) or $dx \approx 0.01M$ (otherwise). For select cases, we also perform resolutions studies to establish convergence using $0.75 \times$ and $1.5 \times$ the grid resolution, and $0.75^4 \times$ and $1.5^4 \times$ as many particles. Details on numerical convergence can be found in the appendix.

In Fig. 1, we show the polar and equatorial proper circumferences of the apparent horizons found for various values of the semi-major axis length $b$ (in units of the total mass). The circumference is normalized by $4\pi M_{\text{BH}}$, where $M_{\text{BH}}$ is the mass of the apparent horizon, to indicate the relation to the hoop conjecture bound. All curves are for initial data with $e = 0.9$, except for the dotted (orange) curves which corresponds to $e = 0.95$ and $b = 10$.

### III. RESULTS

Our main result is that we find that black holes form from the collapsing matter in all cases considered here. Configurations with smaller values of $b$ (or larger values of eccentricity, in the case with $e = 0.95$) form black holes more promptly, while those with larger values take longer to collapse—both perpendicular to the symmetry axis, and along the symmetry axis. In Fig. 1, we show the polar and equatorial circumferences of the apparent horizon, beginning when one is first found, for a number of cases. The matter configuration has collapsed sufficiently that the values are not more than $\sim 25\%$ above $4\pi M_{\text{BH}}$, and thus not in serious violation of the approximate inequality of the hoop conjecture. (Though we note that, since our horizon finding algorithm relies on having a sufficiently good guess for the shape, we cannot exclude the existence of a more distorted horizon at earlier times.) After formation, the horizons then exhibit damped oscillations between being prolate and oblate as they ring down.

All the matter ends up in the black hole for every case studied here. However, a non-negligible amount of energy is radiated away in gravitational radiation. In Fig. 2, we show the gravitational wave power for select cases. The power peaks around the time of black
hole formation—reaching as high as $P_{GW} \sim 0.002$ in some cases—and then dies away exponentially, again showing the characteristic quasinormal mode ringing.

We also show the total energy in gravitational radiation as a function of semi-major axis length in Fig. 2. Cases with smaller values of $b/M$ are already close to being black holes at the initial time and do not emit significant radiation. For $e = 0.9$, this is maximized at $b/M \sim 12–14$, with $E_{GW} \sim 0.015M$. A similar amount of energy is radiated for $b = 10$ and $e = 0.95$.

The difference from the total mass $M - E_{GW}$ matches the measured mass of the black hole at late times to better than 0.2% for all cases. The amount of gravitational radiation is significant for an axisymmetric spacetime. For comparison, an equal mass head-on collision of two black holes falling from rest releases 0.06% in gravitational radiation [22], while an ultra-relativistic collision releases 15% of the total mass in gravitational waves, and has a peak luminosity of $P_{GW} \sim 0.01$ [21, 23].

Most of the gravitational wave energy is due to the $\ell = 2$ angular component, but in Fig. 2 we also show the subdominant contributions from the $\ell = 4$ and 6 components. (The odd $\ell$ components are suppressed by the symmetry of the initial data, though we do not explicitly enforce the equatorial symmetry in the placement of particles, nor during evolution.)

Focusing on the $b/M = 10, e = 0.9$ case considered in Fig. 2, we also find a blow up in the curvature around the same time. As shown in Fig. 3, the maximum value obtained at blow-up increases with resolution. However, in contrast to Fig. 2, we find that the maximum in $I$ always occur in a region where the matter density is non-zero (this was also found in [15]), and in fact tracks the blow-up in density, as shown in Fig. 4. This suggests this is just due to shell-crossing in the matter (see e.g. [24]). We are able to continue the calculation past this mild type of singularity, and a short time later an apparent horizon is found which envelops all the matter, and the curvature quickly approaches the value of an isolated black hole.

IV. CONCLUSION

We have followed the relativistic collapse of very prolate spheroidal configurations of matter, revisiting a scenario originally studied in [7], and put forth as evidence against cosmic censorship. With our different choice of coordinates, we do not find a blowup of curvature outside the matter region as in [7], and we are able to follow the evolution through to the asymptotic end state. We see that a black hole does form, swallowing the matter, and censoring the interior singularity.

The original motivation for investigating this scenario for possible violations of cosmic censorship was that it seemingly pitted the hoop conjecture, which dictates that sufficiently elongated matter configurations should not form black hole horizons, against the generic tendency of unhalted relativistic collapse to
form singularities. We find here that the spacetime dynamics unfolds in such a way that, even in the case of collisionless particles, the matter collapses to where it can be surrounded by a horizon that is not too elongated. Thus, neither the hoop conjecture, nor cosmic censorship appear to be violated. This rapid and violent collapse does, however, leave its imprint in the strong gravitational radiation, which for some cases is comparable to a quasicircular binary black hole merger in peak luminosity and the fraction of the total mass of the spacetime.

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FIG. 5. The L2 norm of the generalized harmonic constraint violation $C_a = H_a - \Box x_a$ (average value in the $(x,y) \in [-20M, 20M] \times [0, 20M]$ central portion of the domain) at three resolutions for $b/M = 10$ and $e = 0.9$. In comparison to the low resolution, the grid resolution is $4/3 \times$ and $2 \times$ higher in the medium and high resolutions, respectively. The number of particles is $(4/3)^4 \times$ and $2^4 \times$ higher.

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Appendix A: Numerical convergence

In Fig. 5, we demonstrate the convergence of the Einstein constraints for an example case with $b/M = 10$ and $e = 0.9$. During the initial and final stages of evolution, the rate of convergence is close to second order, as expected (with the number of particles also increased to give this scaling). Around the time the density blows up due to shell crossing (as described in the main text), the convergence is closer to first order, though this quickly improves as an apparent horizon is found (the interior of which is excluded from this calculation).

Comparing the total energy in gravitational waves for the different resolutions in this case, we find estimate that the error in the dominant $\ell = 2$ contribution is sub-percent for the medium resolution. The error in the very sub-dominant $\ell = 4$ and 6 components (see Fig. 3) is larger, approximately 2% and 30%, respectively.