Abstract

This paper considers the perturbed stochastic gradient descent algorithm and shows that it finds $\varepsilon$-second order stationary points ($\|\nabla f(x)\| \leq \varepsilon$ and $\nabla^2 f(x) \succeq -\sqrt{\varepsilon} I$) in $\tilde{O}(d/\varepsilon^4)$ iterations, giving the first result that has linear dependence on dimension for this setting. For the special case, where stochastic gradients are Lipschitz, the dependence on dimension reduces to polylogarithmic. In addition to giving new results, this paper also presents a simplified proof strategy that gives a shorter and more elegant proof of previously known results [Jin et al., 2017a] on perturbed gradient descent algorithm.

1 Introduction

Nonconvex optimization problems are ubiquitous in several fields of engineering such as control theory, signal processing, machine learning and so on. While these problems are NP-hard in the worst case, simple heuristics are often very effective in practice. For example, one such heuristic, stochastic gradient descent (SGD), has been quite successful for various problems in modern machine learning such as sparse recovery [Blumensath and Davies, 2009], recommender systems [Koren et al., 2009], supervised or unsupervised learning via deep neural networks [Goodfellow et al., 2016], etc. Why is it that problems arising in practice can be solved efficiently by simple heuristics while these problems are NP-hard in the worst case?

A series of works, some theoretical and some empirical, have uncovered a nice structure in several problems of practical interest that seems to answer the above question. These works show that even though these nonconvex problems have an enormous number of bad saddle points, all local minima are good. More precisely, they show that, for a large class of interesting nonconvex problems, second-order stationarity (i.e., $\nabla f(x) = 0$ and $\nabla^2 f(x) \succeq 0$), which is a weaker notion of local optimality, already guarantees (approximate) global optimality. Choromanska et al. [2014], Kawaguchi [2016] present such a result for learning multi-layer neural networks, Bandeira et al. [2016], Mei et al. [2017] for synchronization and Max-Cut, Boumal et al. [2016] for smooth semidefinite programs, Bhojanapalli et al. [2016] for matrix sensing, Ge et al. [2016] for matrix completion, and Ge et al. [2017] for robust PCA. Since gradient descent
(GD) is known to converge to second-order stationary points with probability one [Lee et al., 2016], it seems reasonable that simple gradient based heuristics such as SGD perform quite well in practice.

In order to make the above reasoning rigorous, one key aspect needs to be addressed: rate of convergence. More precisely, we need to quantify how many iterations of GD or SGD are required for finding an \( \epsilon \)-second-order stationary point (\( \| \nabla f(x) \| \leq \epsilon \) and \( \nabla^2 f(x) \succeq -\sqrt{\epsilon} I \)). This question has been investigated heavily starting from the work [Ge et al., 2015]. While initial results such as Ge et al. [2015], Levy [2016] gave good convergence rates, the dependence on the underlying dimension is at least cubic, which is impractical for high dimensional problems. For the case of GD without stochasticity, Jin et al. [2017a] addresses this issue and shows that if we add perturbation once in a while, convergence to \( \epsilon \)-second-order stationary points requires \( \tilde{O}(\epsilon^{-2}) \) iterations, with only polylogarithmic dependence on the dimension. While SGD is much more widely used compared to GD, obtaining such a result (convergence to \( \epsilon \)-second-order stationary points with minimal dependence on dimension) for SGD has so far remained open. This paper considers perturbed stochastic gradient descent (PSGD) and provides its convergence analysis with a sharp dependence on dimension.

Our contributions.

- This paper shows PSGD finds \( \epsilon \)-second-order stationary points in \( \tilde{O}(d\epsilon^{-4}) \) iterations, giving the first convergence rate that depends linearly on the dimension.
- Under the assumption of Lipschitz stochastic gradients, we show PSGD finds \( \epsilon \)-second-order stationary point in \( \tilde{O}(\epsilon^{-4}) \) iterations, further reducing the dimension dependence to polylogarithmic.
- This paper also devises a transparent proof strategy which in addition to giving the above results, significantly simplifies the proof of previously known results [Jin et al., 2017a] for perturbed GD.

1.1 Related Work

In this section we review the related work which provides convergence guarantees to find second-order stationary points. Classical algorithms for finding second-order stationary points require access to Hessian of the function. Some of the most well known second-order methods here are cubic regularization method [Nesterov and Polyak, 2006] and trust region methods [Curtis et al., 2014]. Owing to the size of Hessian matrices which scales quadratically with respect to dimension, these methods are extremely computational intensive especially for high dimensional problems. In the followings, we focus on the complexity of first-order methods to find second-order stationary points.

Full gradient setting. The basic setting is when the algorithm has access to exact gradient without error. In this case, Jin et al. [2017a] shows that perturbed gradient descent escapes saddle points and finds second-order stationary points in \( \tilde{O}(\epsilon^{-2}) \) iterations. Carmon et al. [2016], Agarwal et al. [2017] and Jin et al. [2017b] use acceleration techniques, and obtain faster convergence rates \( \tilde{O}(\epsilon^{-1.75}) \).

Stochastic setting. In this setting, the algorithm only has access to stochastic gradients. Most existing works assume that the stochastic gradients themselves are Lipschitz (or equivalently that the stochastic functions are gradient-Lipschitz). Under this assumption, and an additional Hessian vector product oracle, Allen-Zhu [2018], Tripuraneni et al. [2018] designed algorithms that have an iteration complexity of \( \tilde{O}(\epsilon^{-3.5}) \). Xu et al. [2018], Allen-Zhu and Li [2017] obtain similar results without the requirement for
Hessian-vector product oracle. The sharpest rate in this category is Fang et al. [2018], which shows that the iteration complexity can be further reduced to $\tilde{O}(\epsilon^{-3})$.

In the general case without assuming Lipschitz stochastic gradients, Ge et al. [2015] provides the first polynomial result for first-order algorithm showing noisy gradient descent finds second-order stationary points in $d^4\text{poly}(\epsilon^{-1})$ iterations. Daneshmand et al. [2018] shows that assuming the variance of stochastic gradient along the escaping direction of saddle points is at least $\gamma$ for all saddle points, then CNC-SGD finds SOSPs in $\tilde{O}(\gamma^{-4}\epsilon^{-5})$ iterations. We note that in general, $\gamma$ scales as $1/d$, which gives complexity $\tilde{O}(d^4\epsilon^{-5})$. Our work is the first result in this setting achieving linear dimension dependence.

While this work was under preparation, a manuscript Fang et al. [2019] was uploaded to arxiv which analyzes stochastic gradient descent with averaging, and obtains convergence rate $\tilde{O}(\epsilon^{-3.5})$ for the special case where stochastic gradient is Lipschitz. We also note that Fang et al. [2019] makes additional structural assumptions on stochastic gradients which enables them to analyze SGD directly, without adding perturbation.

### 1.2 Paper Organization

In Section 2, we present the preliminaries and assumptions. In Section 3, we present our main results and in Section 4 we present the simplified proof for the performance of perturbed GD, which illustrates some of our key ideas. The proof of our result for perturbed SGD is presented in the appendix. We conclude in Section 5.

### 2 Preliminaries

In this paper, we are interested in solving

$$\min_{x \in \mathbb{R}^d} f(x),$$
where \( f(\cdot) \) is a smooth function which can be nonconvex. More concretely, we assume that \( f(\cdot) \) has Lipschitz gradients and Lipschitz Hessians.

**Definition 1.** A differentiable function \( f(\cdot) \) is \( \ell \)-smooth (or \( \ell \)-gradient Lipschitz) if:
\[
\| \nabla f(x_1) - \nabla f(x_2) \| \leq \ell \| x_1 - x_2 \| \quad \forall \ x_1, x_2.
\]

**Definition 2.** A twice-differentiable function \( f(\cdot) \) is \( \rho \)-Hessian Lipschitz if:
\[
\| \nabla^2 f(x_1) - \nabla^2 f(x_2) \| \leq \rho \| x_1 - x_2 \| \quad \forall \ x_1, x_2.
\]

**Assumption 1.** Function \( f(\cdot) \) is \( \ell \)-gradient Lipschitz and \( \rho \)-Hessian Lipschitz.

Since finding a global (or even local) optimum is NP-hard, our goal will be to find points that satisfy second-order optimality conditions. These points are also called second-order stationary points.

**Definition 3.** For a \( \rho \)-Hessian Lipschitz function \( f(\cdot) \), \( x \) is an \( \epsilon \)-second-order stationary point if:
\[
\| \nabla f(x) \| \leq \epsilon \quad \text{and} \quad \lambda_{\min}(\nabla^2 f(x)) \geq -\sqrt{\rho \epsilon}.
\]

We consider the stochastic approximation setting, where we may not access \( f(\cdot) \) directly. Instead for any point \( x \), a gradient query will return \( g(x, \theta) \), where \( g(\cdot, \theta) \) are different functions and \( \theta \) is a random variable drawn from a distribution \( D \). The key property satisfied by these stochastic gradients \( g(\cdot, \cdot) \) is that
\[
\nabla f(x) = \mathbb{E}_{\theta \sim D} [g(x, \theta)],
\]
i.e. the expectation of stochastic gradient equals true gradient.

A standard assumption on the stochastic gradients is that of bounded variance i.e.,
\[
\mathbb{E}_{\theta \sim D} [\|g(x, \theta) - \nabla f(x)\|^2] \leq \sigma^2
\]
for some number \( \sigma^2 \). When we are interested in high probability bounds, one often makes the stronger assumption of sub-Gaussian tails.

**Assumption 2.** For any \( x \in \mathbb{R}^d \), stochastic gradient \( g(x; \theta) \) with \( \theta \sim D \) satisfies:
\[
\mathbb{E} g(x; \theta) = \nabla f(x), \quad \mathbb{P} (\|g(x; \theta) - \nabla f(x)\| \geq t) \leq 2 \exp(-t^2/(2\sigma^2)), \quad \forall t \in \mathbb{R}
\]

We note this notion is more general than the standard notion of sub-Gaussian random vector which assumes \( \mathbb{E} \exp(\langle v, X - EX \rangle) \leq \exp(\sigma^2 \|v\|^2/d) \). The latter one requires distribution to be “isotropic” while our assumption does not. By Lemma 24 we know that both bounded random vector, and standard sub-Gaussian random vector are special cases of our assumption.

In many applications in machine learning, the stochastic gradient \( g \) is often realized as gradient of a stochastic function \( g(\cdot; \theta) = \nabla f(\cdot; \theta) \) where the stochastic function itself can have better smoothness property, i.e. the stochastic gradient can be Lipschitz, which can help improve the convergence rate.

**Assumption 3.** (Optional) For any \( \theta \in \text{supp}(D) \), \( g(\cdot; \theta) \) is \( \tilde{\ell} \)-Lipschitz.

For the sake of clean presentation, this paper treats the general case where Assumption 3 does not hold by taking \( \tilde{\ell} = +\infty \). For the rest of this paper, we assume that \( f(\cdot) \) is \( \ell \)-smooth and \( \rho \)-Hessian Lipschitz (Assumption 1) and the stochastic gradients \( g(\cdot, \cdot) \) satisfy Assumptions 2 and 3 (possibly with \( \tilde{\ell} = \infty \)).
Algorithm 1 Perturbed Gradient Descent

Input: $x_0$, learning rate $\eta$, perturbation radius $r$.

for $t = 0, 1, \ldots$ do
    $x_{t+1} \leftarrow x_t - \eta(\nabla f(x_t) + \xi_t)$, $\xi_t \sim \mathcal{N}(0, (r^2/d)I)$

return $x_T$

Algorithm 2 Perturbed Stochastic Gradient Descent

Input: $x_0$, learning rate $\eta$, perturbation radius $r$.

for $t = 0, 1, \ldots$ do
    sample $\theta_t \sim D$
    $x_{t+1} \leftarrow x_t - \eta(g(x_t; \theta_t) + \xi_t)$, $\xi_t \sim \mathcal{N}(0, (r^2/d)I)$

return $x_T$

3 Main Result

In this section, we present our main results on the efficiency of escaping saddle points. Section 3.1 presents the result for PGD when the algorithm has access to full gradient, and Section 3.2 presents the main result for PSGD and its mini-batch version in the stochastic case.

3.1 Full Gradient Setting

In this setting, we are given an exact gradient oracle that we can query any point $x$, and the oracle returns its gradient $\nabla f(x)$ without any error. In this setting, we run perturbed gradient descent (Algorithm 1).

At each iteration, Algorithm 1 is almost the same as gradient descent, except it adds a small isotropic random Gaussian perturbation to the gradient. The perturbation $\xi_t$ is sampled from a zero-mean Gaussian with covariance $(r^2/d)I$ so that $E\|\xi_t\|^2 = r^2$. We note that Algorithm 1 simplifies the original version in Jin et al. [2017a] which adds perturbation only when certain conditions hold.

We are now ready to present our main result, which says that if we pick $r = \tilde{O}(\epsilon)$ in Algorithm 1 PGD will find $\epsilon$-second-order stationary point in iterations polylogarithmic in dimensions.

Theorem 4. If function $f(\cdot)$ satisfies Assumption I and we run PGD (Algorithm 1) with hyper-parameter $\eta = \tilde{O}(1/\ell), r = \tilde{O}(\epsilon)$. Then, with probability at least $1 - \delta$, PGD will visit $\epsilon$–second-order stationary point at least once in the following number of iterations:

$$\tilde{O} \left( \frac{\ell(f(x_0) - f^*)}{\epsilon^2} \right)$$

where $\tilde{O}, \tilde{O}$ hides poly-logarithmic factors in $d, \ell, \rho, 1/\epsilon, 1/\delta$ and $\Delta_f := f(x_0) - f^*$.

3.2 Stochastic Setting

We are now ready to present our main result which guarantees the efficiency of PSGD (Algorithm 2) in finding a second-order stationary point.
Algorithm 3 Mini-batch Perturbed Stochastic Gradient Descent

Input: $x_0$, learning rate $\eta$, perturbation radius $r$.

for $t = 0, 1, \ldots, \infty$ do

sample $\{\theta_t^{(1)}, \ldots, \theta_t^{(m)}\} \sim D$

$g_t(x_t) \leftarrow \sum_{i=1}^{m} g(x_t; \theta_t^{(i)}) / m$

$x_{t+1} \leftarrow x_t - \eta (g_t(x_t) + \xi_t), \quad \xi_t \sim \mathcal{N}(0, (r^2/d)I)$

return $x_T$

Theorem 5. For any $\epsilon, \delta > 0$, if function $f$ satisfies Assumption 1 and stochastic gradient $g$ satisfies Assumption 2 (and 3 optionally), and we run PSGD (Algorithm 2) with parameter $(\eta, r)$ chosen as:

$$\eta = \tilde{\Theta}\left(\frac{1}{\ell \cdot \mathfrak{N}}\right), \quad r = \tilde{\Theta}(\epsilon \sqrt{\mathfrak{N}}), \quad \text{where} \quad \mathfrak{N} = 1 + \min\left\{\frac{\sigma^2}{\epsilon^2} + \frac{\ell^2}{\ell \sqrt{\rho \epsilon}}, \frac{\sigma^2 d}{\epsilon^2}\right\}$$ (1)

Then, with probability at least $1 - \delta$, PSGD will visit $\epsilon-$second-order stationary point at least once in the following number of iterations:

$$\tilde{O}\left(\frac{\ell (f(x_0) - f^*)}{\epsilon^2} \cdot \mathfrak{N}\right)$$

Remark 6. In the general case ($\tilde{\ell} = \infty$), Theorem 5 guarantees that PSGD finds $\epsilon-$second-order stationary point in $\tilde{O}(d/\epsilon^4)$ iterations when treating other problem dependent parameters as constant. When assumption 3 holds, this iteration complexity can be further reduced to $\tilde{O}(1/\epsilon^4)$.

Remark 7 (Output a second-order stationary point). Theorem 5 provides the number of iterations required for PSGD to visit at least one $\epsilon-$second-order stationary point. It can be easily shown with the same proof that if we double the number of iterations, one half of the iterates will be $\epsilon-$second-order stationary points. Therefore, if we output a iterate uniformly at random, then with at least constant probability, it will be an $\epsilon-$second-order stationary point.

We also observe that when $\sigma = 0$ (i.e., full gradient case), Theorem 5 recovers Theorem 4. Finally, Theorem 5 can be easily extended to the minibatch setting.

Theorem 8 (Mini-batch Version). For any $\epsilon, \delta, m > 0$, if function $f$ satisfies Assumption 1 and stochastic gradient $g$ satisfies Assumption 2 (and 3 optionally), and we run mini-batch PSGD (Algorithm 3) with parameter $(\eta, r)$ chosen as:

$$\eta = \tilde{\Theta}\left(\frac{1}{\ell \cdot \mathfrak{M}}\right), \quad r = \tilde{\Theta}(\epsilon \sqrt{\mathfrak{M}}), \quad \text{where} \quad \mathfrak{M} = 1 + \frac{1}{m} \min\left\{\frac{\sigma^2}{\epsilon^2} + \frac{\ell^2}{\ell \sqrt{\rho \epsilon}}, \frac{\sigma^2 d}{\epsilon^2}\right\}$$ (2)

Then, with probability at least $1 - \delta$, mini-batch PSGD will visit an $\epsilon-$second-order stationary point at least once in the following number of iterations:

$$\tilde{O}\left(\frac{\ell (f(x_0) - f^*)}{\epsilon^2} \cdot \mathfrak{M}\right)$$

Theorem 8 says that if the minibatch size $m$ is not too large i.e., $m \leq \mathfrak{M}$, where $\mathfrak{M}$ is defined in Eq.(1), then mini-batch PSGD will reduce the number of iterations linearly, while not hurting the total number of stochastic gradient queries.
Algorithm 4 Perturbed Gradient Descent (Variant)

Input: $x_0$, learning rate $\eta$, perturbation radius $r$, time interval $T$, tolerance $\epsilon$.

$t_{\text{perturb}} = 0$

for $t = 0, 1, \ldots, T$ do

if $\|\nabla f(x_t)\| \leq \epsilon$ and $t - t_{\text{perturb}} > T$ then

$x_t \leftarrow x_t - \eta \xi_t$, $(\xi_t \sim \text{Uniform}(B_0(r)))$; $t_{\text{perturb}} \leftarrow t$

$x_{t+1} \leftarrow x_t - \eta \nabla f(x_t)$

return $x_T$

4 Simplified Proof for Perturbed Gradient Descent

In this section, we present a simple proof of the iteration complexity of PGD. While it is possible to prove Theorem 4 in this context, the addition of perturbation in each step makes the analysis slightly more complicated than the version of PGD considered in Jin et al. [2017a], where perturbation is added only once in a while. In order to illustrate the proof ideas and make the proof transparent, we present a proof for the iteration complexity of Algorithm 4, which is the one considered in Jin et al. [2017a]. Theorem 4 can be deduced as a special case of Theorem 5 (whose proof will be presented in Appendix A) by setting $\sigma = 0$.

Algorithm 4 adds perturbation only when the norm of gradient at current iterate is small, and the algorithm has not added perturbation in previous $T$ iterations. Similar guarantees as Theorem 4 can be shown for this version of PGD as follows:

Theorem 9. There is an absolute constant $c$ such that the following holds. If $f(\cdot)$ satisfies Assumption 4 and we run PGD (Variant) (Algorithm 4) with parameters $\eta, r, T$ chosen as Eq. (3) with $\iota = c \cdot \log\left(\frac{\Delta f}{\rho \epsilon^2}\right)$, then with probability at least $1 - \delta$, in the following number of iterations, at least one half of iterations of PGD (Variant) will be $\epsilon$-second order stationary points.

$$O\left(\frac{\ell \Delta f}{\epsilon^2}\right),$$

where $\Delta f := f(x_0) - f^*$. 

In order to prove this theorem, we first specify our choice of hyperparameter $\eta, r, T$, and two quantities $\mathcal{F}, \mathcal{I}$ which are frequently used:

$$\eta = \frac{1}{\ell}, \quad r = \frac{\epsilon}{400}, \quad \mathcal{F} = \frac{\ell}{\sqrt{\rho \epsilon}} \cdot \iota, \quad \mathcal{I} = \frac{1}{50 \ell^3} \sqrt{\frac{\epsilon \iota^3}{\rho}}, \quad T = \frac{1}{4 \ell} \sqrt{\frac{\mathcal{F}}{\rho}}$$

(3)

Our high-level proof strategy is to prove by contradiction: when the current iterate is not $\epsilon$-second order stationary point, it must either have large gradient or strictly negative Hessian, and we prove that in either case, PGD must decrease large amount of function value in a reasonable number of iterations. Finally since the function value can not decrease more than $f(x_0) - f^*$, we know that all iterates being non-$\epsilon$-second order stationary points can only last for a small number of iterations.

First, we show the decreasing speed when gradient is large.

Lemma 10 (Descent Lemma). If $f(\cdot)$ satisfies Assumption 4 and $\eta \leq 1/\ell$, then the gradient descent sequence $\{x_t\}$ satisfies:

$$f(x_{t+1}) - f(x_t) \leq -\eta \|\nabla f(x_t)\|^2 / 2$$
Lemma 13 (Coupling Sequence). Suppose \( f(\cdot) \) satisfies Assumption 7 and \( \bar{x} \) satisfies \( \lambda_{\min}(\nabla^2 f(\bar{x})) \leq -\sqrt{p\epsilon} \). Let \( \{x_t\}, \{x'_t\} \) be two gradient descent sequences which satisfy: (1) \( \max\{\|x_0 - \bar{x}\|, \|x'_0 - \bar{x}\|\} \leq \eta \tau; \) (2) \( x_0 - x'_0 = \eta \tau \epsilon_1 \), where \( \epsilon_1 \) is the minimum eigenvector direction of \( \nabla^2 f(\bar{x}) \) and \( \tau_0 > \omega := 2^{2-t} \ell / \sqrt{p\epsilon} \). Then:

\[
\min\{f(x_T) - f(x_0), f(x'_T) - f(x'_0)\} \leq -F.
\]

Next is our key lemma, which shows if the starting point has strictly negative Hessian, then adding perturbation and following by gradient descent will decrease a large amount of function value in \( T \) iterations.

Lemma 11 (Escaping Saddle). If \( f(\cdot) \) satisfies Assumption 7 and \( x \) satisfies \( \|\nabla f(\bar{x})\| \leq \epsilon \) and \( \lambda_{\min}(\nabla^2 f(\bar{x})) \leq -\sqrt{p\epsilon} \). Then let \( x_0 = \bar{x} + \eta \xi (\xi \sim \text{Uniform}(B_0(r))) \) and run gradient descent starting from \( x_0 \):

\[
\mathbb{P}(f(x_T) - f(\bar{x}) \leq -F/2) \geq 1 - \frac{\ell \sqrt{d}}{\sqrt{p\epsilon}} \cdot 2^{8-t},
\]

where \( x_T \) is the \( T \)th gradient descent iterate starting from \( x_0 \).

In order to prove this, we need to prove two lemmas, and the major simplification over Jin et al. [2017a] comes from the following lemma which says that if function value does not decrease too much over \( T \) iterations, then all the iterates \( \{x_T\}_{t=0}^T \) will remain in a small neighborhood of \( x_0 \).

Lemma 12 (Improve or Localize). Under the setting of Lemma 7 for any \( t \geq \tau > 0 \):

\[
\|x_T - x_0\| \leq \sqrt{2\eta t(f(x_0) - f(x_\tau))}
\]

Proof. Recall gradient update \( x_{t+1} = x_t - \eta \nabla f(x_t) \), then for any \( \tau \leq t \):

\[
\|x_T - x_0\| \leq \sum_{\tau=1}^t \|x_T - x_{T-1}\| \leq \left[ \sum_{\tau=1}^t \|x_T - x_{T-1}\| \right]^{\frac{1}{2}}
\]

\[
= \left[ \eta^2 \sum_{\tau=1}^t \|\nabla f(x_{\tau-1})\|^2 \right]^{\frac{1}{2}} \leq \sqrt{2\eta t(f(x_0) - f(x_\tau))}
\]

where step (1) uses Cauchy-Swartz inequality, and step (2) is due to Lemma 10.

Second, we show that the stuck region (where GD will get stuck for at least \( T \) iterations if initialized there) is thin. We show this by tracking any pair of points that differ only in escaping direction, and are at least \( w_0 \) far apart. We show that at least one sequence is guaranteed to escape the saddle point with high probability, so the stuck region along escaping direction has width at most \( w_0 \).

Lemma 13 (Coupling Sequence). Suppose \( f(\cdot) \) satisfies Assumption 7 and \( \bar{x} \) satisfies \( \lambda_{\min}(\nabla^2 f(\bar{x})) \leq -\sqrt{p\epsilon} \). Let \( \{x_t\}, \{x'_t\} \) be two gradient descent sequences which satisfy: (1) \( \max\{\|x_0 - \bar{x}\|, \|x'_0 - \bar{x}\|\} \leq \eta \tau; \) (2) \( x_0 - x'_0 = \eta \tau \epsilon_1 \), where \( \epsilon_1 \) is the minimum eigenvector direction of \( \nabla^2 f(\bar{x}) \) and \( r_0 > \omega := 2^{2-t} \ell / \sqrt{p\epsilon} \). Then:

\[
\min\{f(x_T) - f(x_0), f(x'_T) - f(x'_0)\} \leq -F.
\]
Proof. Assume the contrary, that is min\{f(x,\theta) - f(x_0), f(x',\theta) - f(x_0')\} > -\mathcal{F}. Lemma[12] implies localization of both sequences around \tilde{x}, that is for any \(t \leq \mathcal{T}\)

\[
\max\{\|x_t - \tilde{x}\|, \|x'_t - \tilde{x}\|\} \leq \max\{\|x_t - x_0\|, \|x'_t - x'_0\|\} + \max\{\|x_0 - \tilde{x}\|, \|x'_0 - \tilde{x}\|\} \\
\leq \sqrt{2\eta \mathcal{F} + \eta r} \leq \mathcal{F}
\]

(4)

where the last step is due to our choice of \(\eta, r, \mathcal{T}, \mathcal{F}, \mathcal{F}\) as in Eq.(3), and \(\ell/\sqrt{\mathcal{F}} \geq 1\). On the other hand, we can write the update equations for the difference \(\hat{x}_t := x_t - x'_t\) as:

\[
\hat{x}_{t+1} = \hat{x}_t - \eta [\nabla f(x_t) - \nabla f(x'_t)] = (I - \eta H)\hat{x}_t - \eta \Delta_t \hat{x}_t \\
= (I - \eta H)^{t+1} \hat{x}_0 - \eta \sum_{\tau=0}^{t} (I - \eta H)^{t-\tau} \Delta_\tau \hat{x}_\tau
\]

where \(H = \nabla^2 f(\tilde{x})\) and \(\Delta_\tau = \int_0^1 [\nabla^2 f(x'_\tau + \theta(x_t - x'_t)) - H]d\theta\). We note \(p(t)\) is the leading term which is due to initial difference \(\hat{x}_0\), and \(q(t)\) is the error term which is the result of that function \(f\) is not quadratic. Now we use induction to show that the error term is always small compared to the leading term. That is:

\[
\|q(t)\| \leq \|p(t)\|/2, \quad t \in [\mathcal{T}]
\]

The claim is true for the base case \(t = 0\) as \(\|q(0)\| = 0 \leq \|\hat{x}_0\|/2 = \|p(0)\|/2\). Now suppose the induction claim is true till \(t\), we prove it is true for \(t + 1\). Denote \(\lambda_{\min}(\nabla^2 f(x_0)) = -\gamma\). First, note \(\tilde{x}_0\) is in the minimum eigenvector direction of \(\nabla^2 f(x_0)\). Thus for any \(\tau \leq t\), we have:

\[
\|\hat{x}_\tau\| \leq \|p(\tau)\| + \|q(\tau)\| \leq 2\|p(\tau)\| = 2\|\hat{x}_0\| = 2(1 + \eta \gamma)^T \eta r_0
\]

By Hessian Lipschitz, we have \(\|\Delta_\tau\| \leq \rho \max\{\|x_t - \tilde{x}\|, \|x'_t - \tilde{x}\|\} \leq \rho \mathcal{F}\), therefore:

\[
\|q(t + 1)\| = \|\eta \sum_{\tau=0}^{t} (I - \eta H)^{t-\tau} \Delta_\tau \hat{x}_\tau\| \leq \eta \rho \mathcal{F} \sum_{\tau=0}^{t} \|\hat{x}_\tau\| \\
\leq 2\eta \rho \mathcal{F} \sum_{\tau=0}^{t} (1 + \eta \gamma)^T \eta r_0 \leq 2\eta \rho \mathcal{F} (1 + \eta \gamma)^T \eta r_0 \leq 2\eta \rho \mathcal{F} \|p(t + 1)\|
\]

where the second last inequality used \(t + 1 \leq \mathcal{T}\). By our choice of hyperparameter as in Eq.(3), we have \(2\eta \rho \mathcal{F} \leq 1/2\), which finishes the proof for induction.

Finally, the induction claim implies:

\[
\max\{\|x_\mathcal{F} - x_0\|, \|x'_\mathcal{F} - x_0\|\} \geq \frac{1}{2} \|\hat{x}(\mathcal{F})\| \geq \frac{1}{2} \|\hat{x}(\mathcal{F})\| - \|q(\mathcal{F})\| \geq \frac{1}{4} \|\hat{x}(\mathcal{F})\| \\
= \frac{(1 + \eta \gamma)^T \eta r_0}{4} \geq 2^{t+2} \eta r_0 > \mathcal{F}
\]

where step (1) uses the fact \((1 + x)^{1/x} \geq 2\) for any \(x \in (0, 1]\). This contradicts the localization fact Eq.(4), which finishes the proof.

Equipped with Lemma[12] and Lemma[13] now we are ready to prove the Lemma[11]
Proof of Lemma 11. Recall $x_0 \sim \text{Uniform}(B_{\tilde{x}}(\eta r))$. Define stuck region with $B_{\tilde{x}}(\eta r)$ starting from where GD requires more than $T$ steps to escape:

$$X_{\text{stuck}} := \{ x \in B_{\tilde{x}}(r) \mid \{x_t\} \text{ is GD sequence with } x_0 = x, \text{ and } f(x_T) - f(x_0) > -\mathcal{F} \}.$$ 

According to Lemma 13, we know the width of $X_{\text{stuck}}$ along $e_1$ direction is at most $\eta \omega$. That is, $\text{Vol}(X_{\text{stuck}}) \leq \text{Vol}(B_{d-1}(\eta r)) \eta \omega$. Therefore:

$$P(x_0 \in X_{\text{stuck}}) = \frac{\text{Vol}(X_{\text{stuck}})}{\text{Vol}(B^{d}(\eta r))} \leq \frac{\eta \omega \text{Vol}(B_{d-1}(\eta r))}{\text{Vol}(B^{d}(\eta r))} \leq \frac{\omega}{r} \sqrt{\frac{d}{\pi}} \leq \frac{\ell \sqrt{d}}{\sqrt{\rho} \epsilon} \cdot 2^{8-\iota}.$$ 

On the event that $x_0 \not\in X_{\text{stuck}}$, according to our parameter choice Eq.(3), we have:

$$f(x_T) - f(\tilde{x}) = (f(x_T) - f(x_0)) + (f(x_0) - f(\tilde{x})) \leq -\mathcal{F} + \epsilon \eta r + \frac{\ell \eta^2 r^2}{2} \leq -\mathcal{F}/2.$$ 

This finishes the proof.

With Lemma 10 and Lemma 11 it is not hard to finally prove Theorem 9.

Proof of Theorem 9. First, we set total iterations $T$ to be:

$$T = 8 \max \left\{ \frac{(f(x_0) - f^*)}{\mathcal{F}}, \frac{(f(x_0) - f^*)}{\eta \epsilon^2} \right\} = O \left( \frac{\ell (f(x_0) - f^*)}{\epsilon^2} \cdot \iota^4 \right).$$ 

Next, we choose $\iota = c \cdot \log \left( \frac{d \Delta_t}{\rho \epsilon \delta} \right)$ with large enough absolute constant $c$ so that:

$$(T \ell \sqrt{d}/\sqrt{\rho} \epsilon) \cdot 2^{8-\iota} \leq \delta.$$ 

Then, we argue with probability $1 - \delta$, algorithm will add perturbation at most $T/(4 \mathcal{F})$ times. This is because otherwise, we can use Lemma 11 every time we add perturbation, and:

$$f(x_T) \leq f(x_0) - T \mathcal{F} / (4 \mathcal{F}) < f^*$$

which can not happen. Finally, excluding those iterations that are within $\mathcal{F}$ steps after adding perturbations, we still have $3T/4$ steps left. They are either large gradient steps $\|\nabla f(x_t)\| \geq \epsilon$ or $\epsilon$-second order stationary points. Within them, we know large gradient steps can not be more than $T/4$. Because again otherwise, by Lemma 10:

$$f(x_T) \leq f(x_0) - T \eta \epsilon^2/4 < f^*$$

which again can not happen. Therefore, we conclude at least $T/2$ iterations must be $\epsilon-$second order stationary points.

5 Conclusion

In this paper, we considered the problem of finding second order stationary points with a stochastic gradient oracle, and presented the first result with linear dependence on dimension. In the special case where the stochastic gradients are Lipschitz, the linear dependence of dimension is improved to polylogarithmic. Further improvement over these bounds, especially in the dependence on accuracy is an interesting open problem.
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### A Proof for Stochastic Case

In this section, we give proofs for Theorem 5.
A.1 Notation

Recall the update equation of Algorithm (2) is \( x_{t+1} \leftarrow x_t - \eta (g(x_t; \theta_t) + \xi_t) \) where \( \xi_t \sim \mathcal{N}(0, (r^2/d)I) \). Across this section, we denote \( \zeta_t := g(x_t; \theta_t) - \nabla f(x_t) \). For simplicity, we also denote \( \tilde{\zeta}_t := \zeta_t + \xi_t \), and \( \tilde{\sigma}^2 := \sigma^2 + r^2 \). Then the update equation can be rewrite as \( x_{t+1} \leftarrow x_t - \eta (\nabla f(x_t) + \tilde{\zeta}_t) \). We also denote \( \mathcal{F}_t = \sigma(\xi_0, \xi_1, \ldots, \xi_t) \) be the corresponding filtration up to time step \( t \). Recall our choice of parameters:

\[
\eta = \frac{1}{\ell^9 \cdot \ell \mathcal{N}}, \quad r = \ell \cdot e \sqrt{\mathcal{N}}, \quad \mathcal{F} := \frac{\ell}{\eta \sqrt{pe}}, \quad \mathcal{I} := \frac{1}{\ell} \sqrt{\frac{e^3}{\rho}}, \quad \mathcal{J} := \frac{2}{\ell^2} \sqrt{\frac{e}{\rho}}
\]

where \( \mathcal{N} \) and log factor \( \ell \) are defined as follows:

\[
\mathcal{N} = 1 + \min \left\{ \frac{\sigma^2}{e^2} + \frac{\ell^2}{e \sqrt{pe}}, \frac{\sigma^2 d}{e^2} \right\}, \quad \nu = \mu \cdot \log \left( \frac{d \ell A_\mathcal{N}}{pe \delta} \right)
\]

\( \mu \) is a sufficiently large absolute constant to be determined later. Also we note \( c \) in this sections are absolute constant that does not depends on our choice of \( \mu \). The value of \( c \) may change from line to line.

A.2 Descent Lemma

Lemma 14 (Descent Lemma). There exists absolute constant \( c \), under Assumption \( 1, 2 \) for any fixed \( t, t_0, \ell > 0 \), if \( \eta \leq 1/\ell \), then with at least \( 1 - 4e^{-t} \) probability, the sequence PSGD(\( \eta, r \)) (Algorithm (2)) satisfies: (denote \( \tilde{\sigma}^2 = \sigma^2 + r^2 \))

\[
f(x_{t_0+t}) - f(x_{t_0}) \leq -\frac{\eta}{8} \sum_{i=0}^{t-1} \| \nabla f(x_{t_0+i}) \|^2 + c \cdot \eta \tilde{\sigma}^2 (\eta \ell t + \ell)
\]

Proof. Since Algorithm (2) is Markovian, the operations in each iterations does not depend on time step \( t \). Thus, it suffices to prove Lemma 14 for special case \( t_0 = 0 \). Recall the update equation:

\[
x_{t+1} \leftarrow x_t - \eta (\nabla f(x_t) + \tilde{\zeta}_t)
\]

where \( \tilde{\zeta}_t = \zeta_t + \xi_t \). By assumption, we know \( \zeta_t | \mathcal{F}_{t-1} \) is zero-mean nSG(\( \sigma \)). Also \( \xi_t | \mathcal{F}_{t-1} \) comes from \( \mathcal{N}(0, (r^2/d)I) \), and thus by Lemma 24 is zero-mean nSG(\( c \cdot r \)) for some absolute constant \( c \). By Taylor expansion, \( \ell \)-gradient Lipschitz and \( \eta \leq 1/\ell \), we know:

\[
f(x_{t+1}) \leq f(x_t) + \langle \nabla f(x_t), x_{t+1} - x_t \rangle + \frac{\ell}{2} \| x_{t+1} - x_t \|^2
\]

\[
\leq f(x_t) - \eta \langle \nabla f(x_t), \nabla f(x_t) + \tilde{\zeta}_t \rangle + \frac{\eta^2 \ell}{2} \left[ \frac{3}{2} \| \nabla f(x_t) \|^2 + 3 \| \tilde{\zeta}_t \|^2 \right]
\]

\[
\leq f(x_t) - \frac{\eta}{4} \| \nabla f(x_t) \|^2 - \eta \langle \nabla f(x_t), \tilde{\zeta}_t \rangle + \frac{3}{2} \eta^2 \ell \| \tilde{\zeta}_t \|^2
\]

Summing over the inequality above, we have following:

\[
f(x_t) - f(x_0) \leq -\frac{\eta}{4} \sum_{i=0}^{t-1} \| \nabla f(x_i) \|^2 - \eta \sum_{i=0}^{t-1} \langle \nabla f(x_i), \tilde{\zeta}_i \rangle + \frac{3}{2} \eta^2 \ell \sum_{i=0}^{t-1} \| \tilde{\zeta}_i \|^2
\]
For the second term in RHS, applying Lemma \[30\], there exists an absolute constant \(c\), with probability \(1 − 2e^{−t}\):
\[
−\eta \sum_{i=0}^{t−1} \langle \nabla f(x_i), \tilde{z}_i \rangle \leq \frac{\eta}{8} \sum_{i=0}^{t−1} \|\nabla f(x_i)\|^2 + c\eta \tilde{\sigma}^2 t
\]

For the third term in RHS of Eq.(6), applying Lemma 29 with probability \(1 − 2e^{−t}\):
\[
\frac{3}{2} \eta^2 \mathcal{E} \sum_{i=0}^{t−1} \|\tilde{z}_i\|^2 \leq 3\eta^2 \mathcal{E} \sum_{i=0}^{t−1} (\|\xi_i\|^2 + \|\xi_i\|^2) \leq c\eta^2 \mathcal{E} \tilde{\sigma}^2 (t + \mathcal{E})
\]

Substituting both above inequality into Eq.(6), and note the fact \(\eta \leq 1/\mathcal{E}\), we have with probability \(1 − 4e^{−t}\):
\[
f(x_t) − f(x_0) \leq −\frac{\eta}{8} \sum_{i=0}^{t−1} \|\nabla f(x_i)\|^2 + c\eta \tilde{\sigma}^2 (\eta \mathcal{E} t + \mathcal{E})
\]
This finishes the proof. □

**Lemma 15 (Improve or Localize).** Under the same setting of Lemma 14 with at least \(1 − 8d\mathcal{E} \cdot e^{−t}\) probability, the sequence \(\text{PSGD}(\eta, \mathcal{E})\) (Algorithm 2) satisfies:
\[
\forall \tau \leq t : \|x_{t_0 + \tau} − x_{t_0}\|^2 \leq c\eta \cdot [f(x_{t_0}) − f(x_{t_0 + \tau}) + \eta \tilde{\sigma}^2 (\eta \mathcal{E} t + \mathcal{E})]
\]

**Proof.** By similar argument as in proof of Lemma 14, it suffices to prove Lemma 15 in special case \(t_0 = 0\). According to Lemma 14 with probability \(1 − 4e^{−t}\), for some absolute constant \(c\):
\[
\sum_{i=0}^{t−1} \|\nabla f(x_i)\|^2 \leq \frac{8}{\eta}[f(x_0) − f(x_t)] + c\tilde{\sigma}^2 (\eta \mathcal{E} t + \mathcal{E})
\]
Therefore, for any fixed \(\tau \leq t\), with probability \(1 − 8d\cdot e^{−t}\):
\[
\|x_{\tau} − x_0\|^2 = \eta^2 \sum_{i=0}^{t−1} (\nabla f(x_i) + \tilde{z}_i)\|^2 \leq 2\eta^2 \left[ \sum_{i=0}^{t−1} \|\nabla f(x_i)\|^2 + \sum_{i=0}^{t−1} \|\tilde{z}_i\|^2 \right]
\leq 2\eta^2 t \sum_{i=0}^{t−1} \|\nabla f(x_i)\|^2 + c\eta^2 \tilde{\sigma}^2 t\mathcal{E}
\leq c\eta t [f(x_0) − f(x_t) + \eta \tilde{\sigma}^2 (\eta \mathcal{E} t + \mathcal{E})]
\]
Where in step (1) we use Cauchy-Swartz inequality and Lemma 27. Finally, applying union bound for all \(\tau \leq t\), we finishes the proof. □

### A.3 Escaping Saddle Points

This entire subsection will be devoted to prove following lemma:

**Lemma 16 (Escaping Saddle Point).** There exists absolute constant \(c_{\max}\), under Assumption 1 and 2 for any fixed \(t_0 > 0, \mathcal{E} \geq c_{\max} \log(\ell \sqrt{d(\rho \mathcal{E}))})\), if \(\eta, \mathcal{E}, \mathcal{F}\) are chosen as in Eq. 5, and \(x_{t_0}\) satisfies \(\|\nabla f(x_{t_0})\| \leq \epsilon\) and \(\lambda_{min}(\nabla^2 f(x_{t_0})) \leq -\sqrt{\frac{\epsilon}{\mathcal{K}^2}}\), then the sequence \(\text{PSGD}(\eta, \mathcal{E})\) (Algorithm 2) satisfies:
\[
\mathbb{P}(f(x_{t_0 + \mathcal{E}}) − f(x_{t_0}) \leq 0.1 \mathcal{E}) \geq 1 − 4e^{−t}\quad \text{and}
\mathbb{P}(f(x_{t_0 + \mathcal{F}}) − f(x_{t_0}) \leq -\mathcal{F}) \geq 1/3 − 5d\mathcal{F} \cdot \log(d(\mathcal{E}^2(\eta \mathcal{F}))e^{−t})
\]
Since Algorithm\textsuperscript{2} is Markovian, the operations in each iterations does not depend on time step $t$. Thus, it suffices to prove Lemma\textsuperscript{16} for special case $t_0 = 0$. To prove this lemma, we first need to introduce the concept of coupling sequence.

**Notation:** Across this subsection, we let $\mathcal{H} := \nabla^2 f(x_0)$, and $e_1$ be the minimum eigendirection of $\mathcal{H}$, and $\gamma := \lambda_{\text{min}}(\mathcal{H})$. We also let $\mathcal{P}_{-1}$ be the projection to subspace complement to $e_1$.

**Definition 17** (Coupling Sequence). Consider two sequences $\{x_t\}$ and $\{x'_t\}$ as two separate runs of PSGD (algorithm\textsuperscript{2}) both starting from $x_0$. They are coupled if both sequences share the same randomness $\mathcal{P}_{-1}\xi_t$ and $\theta_t$, while in $e_1$ direction $e_1^\top \xi_t = -e_1^\top \xi'_t$.

The first thing we can show is that if function of either sequence do not have sufficient decrease, then both sequence are localized in a small ball around $x_0$ within $\mathcal{F}$ iterations:

**Lemma 18** (Localization of coupling sequence). Under the notation of Lemma\textsuperscript{19} then:

$$\mathbb{P}(\min \{f(x) - f(x_0), f(x') - f(x_0)\} \leq -\mathcal{F}, \text{ or } \forall t \leq \mathcal{F} : \max \{\|x_t - x_0\|^2, \|x'_t - x_0\|^2\} \leq \mathcal{F}^2 \geq 1 - 16d\mathcal{F} \cdot e^{-t}$$

**Proof.** This lemma follows from applying Lemma\textsuperscript{15} on both sequences and union bound. \hfill \square

The overall proof strategy for Lemma\textsuperscript{16} is to show localization happens with a very small chance, thus at least one of the sequence must have sufficient descent. In order to prove so, we first study the dynamics of the difference of the coupling sequence.

**Lemma 19** (Dynamics of the difference of coupling sequence). Consider coupling sequence $\{x_t\}$ and $\{x'_t\}$ as in Definition\textsuperscript{17} and let $\tilde{x}_t := x_t - x'_t$. Then $\tilde{x}_t = -q_0(t) - q_{sg}(t) - q_p(t)$, where:

$$q_0(t) := \eta \sum_{\tau=0}^{t-1} (I - \eta\mathcal{H})^{t-1-\tau} \Delta_{\tau_t} \tilde{x}_t, \quad q_{sg}(t) := \eta \sum_{\tau=0}^{t-1} (I - \eta\mathcal{H})^{t-1-\tau} \zeta_{\tau_t}, \quad q_p(t) := \eta \sum_{\tau=0}^{t-1} (I - \eta\mathcal{H})^{t-1-\tau} \hat{\xi}_{\tau_t}$$

Here $\Delta_{\tau_t} := \int_0^1 \nabla^2 f(\psi x_t + (1 - \psi)x'_t) d\psi - \mathcal{H}$, and $\zeta_{\tau_t} := \zeta_{\tau_t} - \zeta'_{\tau_t}, \hat{\xi}_{\tau_t} := \xi_{\tau_t} - \xi'_{\tau_t}$.

**Proof.** Recall $\zeta_t = g(x_t; \theta_t) - \nabla f(x_t)$, thus, we have update formula:

$$x_{t+1} = x_t - \eta(g(x_t; \theta_t) + \xi_t) = x_t - \eta(\nabla f(x_t) + \xi_t)$$

Taking the difference between $\{x_t\}$ and $\{x'_t\}$:

$$\tilde{x}_{t+1} = x_{t+1} - x'_{t+1} = \tilde{x}_t - \eta(\nabla f(x_t) - \nabla f(x'_t) + \xi_t - \xi'_t)$$

$$= \tilde{x}_t - \eta((\mathcal{H} + \Delta_{\tau_t}) \tilde{x}_t + \hat{\xi}_{\tau_t} + \xi_t) = \eta(I - \eta\mathcal{H}) \tilde{x}_t - \eta(\Delta_{\tau_t} \tilde{x}_t + \hat{\xi}_{\tau_t} + e_1 e_1^\top \hat{\xi}_{\tau_t})$$

where $\Delta_{\tau_t} := \int_0^1 \nabla^2 f(\psi x_t + (1 - \psi)x'_t) d\psi - \mathcal{H}$. This finishes the proof. \hfill \square

**Lemma 20.** Denote $\alpha(t) := \left[\sum_{\tau=0}^{t-1}(1 + \eta\gamma)^{2(t-1-\tau)}\right]^{1/2}; \beta(t) := (1 + \eta\gamma)^{t}/\sqrt{2\eta\gamma}$. If $\eta\gamma \in [0, 1]$, then (1) $\alpha(t) \leq \beta(t)$ for any $t \in \mathbb{N}$; (2) $\alpha(t) \geq \beta(t)/\sqrt{3}$ for $t \geq \ln(2)/(\eta\gamma)$.\hfill 15
Proof. By summation formula of geometric sequence:

\[
\alpha^2(t) := \sum_{\tau=0}^{t-1} (1 + \eta\gamma)^{2(t-1-\tau)} = \frac{(1 + \eta\gamma)^{2t} - 1}{2\eta\gamma + (\eta\gamma)^2}
\]

Thus, the claim \(\alpha(t) \leq \beta(t)\) for any \(t \in \mathbb{N}\) immediately follows. On the other hand, note for \(t \geq \ln(2)/(\eta\gamma)\), we have \((1 + \eta\gamma)^{2t} \geq 2^2 \ln 2 \geq 2\), where the second claim follows by calculations.

The proof strategy is to show with constant probability, \(q_p(t)\) is the dominating term to escape, and \(q_h(t), q_{sg}(t)\) will not be too large compared to \(q_p(t)\).

**Lemma 21.** Under the notation of Lemma 19 and Lemma 20 let \(-\gamma := \lambda_{\text{min}}(H)\), then \(\forall t > 0:\)

\[
\mathbb{P}(\|q_p(t)\| \leq \frac{c\beta(t)\eta r}{\sqrt{d}} \cdot \sqrt{t}) \geq 1 - 2e^{-t},
\]

\[
\mathbb{P}(\|q_p(T)\| \geq \frac{\beta(T)\eta r}{10\sqrt{d}}) \geq \frac{2}{3}.
\]

**Proof.** Note \(\xi_t\) is one dimensional Gaussian with standard deviation \(2r/\sqrt{d}\) along \(e_1\) direction. As an immediate result, \(\eta \sum_{\tau=0}^{t} (1 - \eta H)^{-\tau} \xi_t\) also satisfies one dimensional Gaussian distribution since summation of Gaussian is again Gaussian. Finally note \(e_1\) is an eigendirection of \(H\) with corresponding eigenvalue \(-\gamma\), and by Lemma 20 that \(\alpha(t) \leq \beta(t)\). Then, the first inequality immediately follows from the standard concentration inequality for Gaussian; the second inequality follows from the fact if \(Z \sim \mathcal{N}(0, \sigma^2)\) then \(\mathbb{P}(\|Z\| \leq \lambda\sigma) \leq 2\lambda/\sqrt{2\pi} \leq \lambda\).

**Lemma 22.** There exists absolute constant \(c_{\text{max}}\) for any \(t \geq c_{\text{max}}, \) under the notation of Lemma 19 and Lemma 20 let \(-\gamma := \lambda_{\text{min}}(H)\), we have:

\[
\mathbb{P}(\min\{f(x_T) - f(x_0), f(x_T') - f(x_0)\} \leq -T, \text{ or } \forall t \leq T : \|q_h(t) + q_{sg}(t)\| \leq \frac{\beta(T)\eta r}{20\sqrt{d}}) \geq 1 - 10dT^2 \cdot \log(\frac{T\sqrt{d}}{\eta r})e^{-t}.
\]

**Proof.** For simplicity we denote \(E\) to be the event \(\forall \tau \leq t : \max\{\|x_\tau - x_0\|^2, \|x_\tau' - x_0\|^2\} \leq \mathcal{T}^2\). We use induction to prove following claims for any \(t \in [0, T]\):

\[
\mathbb{P}(E \Rightarrow \forall \tau \leq t : \|q_h(\tau) + q_{sg}(\tau)\| \leq \frac{\beta(\tau)\eta r}{20\sqrt{d}}) \geq 1 - 10dT \cdot \log(\frac{T\sqrt{d}}{\eta r})e^{-t}.
\]

Then Lemma 22 directly follow from combining Lemma 18 and this induction claim.

Clearly for the base case \(t = 0\), the claim trivially holds as \(q_{sg}(0) = q_h(0) = 0\). Suppose the claim holds for \(t\), then by Lemma 21 with probability at least \(1 - 2T^2e^{-t}\), we have for any \(\tau \leq t:\)

\[
\|\dot{x}_\tau\| \leq \eta \|q_h(\tau) + q_{sg}(\tau)\| + \eta \|q_p(\tau)\| \leq \frac{c\beta(\tau)\eta r}{\sqrt{d}} \cdot \sqrt{t}.
\]

Then, under the condition \(\max\{\|x_\tau - x_0\|^2, \|x_\tau' - x_0\|^2\} \leq \mathcal{T}^2\), by Hessian Lipschitz, we have \(\|\Delta_\tau\| = \|\int_0^1 \nabla^2 f(\psi x_\tau + (1 - \psi)x_\tau')d\psi - H\| \leq \rho \max\{\|x_\tau - x_0\|, \|x_\tau' - x_0\|\} \leq \rho \mathcal{T}\). This gives bounds on \(q_h(t + 1)\) terms as:

\[
\|q_h(t + 1)\| \leq \eta \sum_{\tau=0}^{t} (1 + \eta\gamma)^{t-\tau} \rho \mathcal{T} \|\dot{x}_\tau\| \leq \eta \rho \mathcal{T} \frac{c\beta(t)\eta r}{\sqrt{d}} \leq \frac{\beta(t)\eta r}{40\sqrt{d}}.
\]
where the last step is due to $\eta \mathcal{F} = 1/\ell$ by Eq.(5). By picking $\ell$ larger than absolute constant 40c, then we have $c \eta \mathcal{F} \leq 1/40$.

Also, recall $\hat{\mathcal{F}}_{T-1}$ is the summation of one nSG($\sigma$) random vector and one nSG($c \cdot r$) random vector, by Lemma 27 we know with probability at least $1 - 4d e^{-\ell}$:

$$\|q_{sg}(t+1)\| \leq c \beta(t + 1) \eta \sigma \sqrt{t}$$

On the other hand, when assumption 5 is available, we also have $\hat{\mathcal{F}}_{T-1} \sim$ nSG($\|\hat{x}_r\|$), by applying Lemma 28 with $B = \alpha^2(t) \cdot \eta^2 \ell^2 \mathcal{A}^2; b = \alpha^2(t) \cdot \eta^2 \ell^2 \cdot \eta^2 r^2 / d$, we know with probability at least $1 - 4d \cdot \log(\mathcal{F} \sqrt{d} / (\eta r)) \cdot e^{-\ell}$:

$$\|q_{sg}(t+1)\| \leq c \eta \ell \sqrt{\frac{t}{d}} \left( \sum_{\tau=0}^{t} (1 + \eta \gamma)^{2(t-\tau)} \cdot \max\{\|\hat{x}_r\|^2, \frac{\eta^2 r^2}{d}\} \right) \leq \eta \ell \sqrt{\mathcal{F}} \cdot \frac{c \beta(t) \eta r}{\sqrt{d}} \cdot \sqrt{t}$$

Finally, combine both cases, and by our choice of learning rate $\eta, r$ as in Eq.(5) with $\ell$ large enough:

$$\|q_{sg}(t+1)\| \leq c \eta \ell \cdot \beta(t) \frac{r}{\sqrt{d}} \cdot \min\{\eta \ell \sqrt{\mathcal{F}}, \frac{\sigma \sqrt{d} \gamma}{r}\} \leq \frac{\beta(t) r}{40 \sqrt{d}}$$

and the induction follows by triangular inequality and union bound.

**Proof of Lemma 16** We first prove the first claim $P(f(x_{\mathcal{F}}) - f(x_0) \leq 0.1 \mathcal{F}) \geq 1 - 4e^{-\ell}$. This is essentially because our choice of learning rate and Lemma 14 we have with probability $1 - 4e^{-\ell}$:

$$f(x_{\mathcal{F}}) - f(x_0) \leq c \eta \sigma^2 (\eta \ell \mathcal{F} + \ell) \leq 0.1 \mathcal{F}$$

where the last step is because of our choice of parameters as Eq.(5), we have $c \eta \sigma^2 (\eta \ell \mathcal{F} + \ell) \leq 2c \mathcal{F} / \ell$ and by picking $\ell$ larger than absolute constant 20c.

For the second claim $P(f(x_{\mathcal{F}}) - f(x_0) \leq -\mathcal{F}) \geq 1/3 - 5d. \mathcal{F}^2 \cdot \log(\mathcal{F} \sqrt{d} / (\eta r)) e^{-\ell}$. We consider coupling sequences $\{x_i\}$ and $\{x_i'\}$ as defined in Definition 17. We note Lemma 21 and Lemma 22 we know with probability at least $2/3 - 10d. \mathcal{F}^2 \cdot \log(\mathcal{F} \sqrt{d} / (\eta r)) e^{-\ell}$, if $\min\{f(x_{\mathcal{F}}) - f(x_0), f(x_{\mathcal{F}}') - f(x_0)\} > -\mathcal{F}$, i.e. both sequences stuck around the saddle point, then we have:

$$\|q_{g}(\mathcal{F})\| \geq \frac{\beta(\mathcal{F}) \eta r}{10 \sqrt{d}}, \quad \|q_{h}(\mathcal{F}) + q_{sg}(\mathcal{F})\| \leq \frac{\beta(\mathcal{F}) \eta r}{20 \sqrt{d}}$$

By Lemma 19 when $\ell \geq c \cdot \log(\ell \sqrt{d} / (\rho e))$ with large absolute constant $c$, we have:

$$\max\{\|x_{\mathcal{F}} - x_0\|, \|x_{\mathcal{F}}' - x_0\|\} \geq \frac{1}{2} \|\hat{x}(\mathcal{F})\| \geq \frac{1}{2} \|q_{g}(\mathcal{F})\| - \|q_{h}(\mathcal{F}) + q_{sg}(\mathcal{F})\| \geq \frac{\beta(\mathcal{F}) \eta r}{40 \sqrt{d}} = \frac{(1 + \eta \gamma) \mathcal{F}}{40 \sqrt{2} \eta \gamma d} \leq \frac{2 \eta r}{80 \sqrt{\eta \gamma d}} > \mathcal{F}$$

which contradicts with Lemma 18. Therefore, we can conclude that $P(\min\{f(x_{\mathcal{F}}) - f(x_0), f(x_{\mathcal{F}}') - f(x_0)\} \leq -\mathcal{F}) \geq 2/3 - 10d. \mathcal{F}^2 \cdot \log(\mathcal{F} \sqrt{d} / (\eta r)) e^{-\ell}$. We also know the marginal distribution of $x_{\mathcal{F}}$ and $x_{\mathcal{F}}'$ is the same, thus they have same probability to escape saddle point. That is:

$$P(f(x_{\mathcal{F}}) - f(x_0) \leq -\mathcal{F}) \geq \frac{1}{2} P(\min\{f(x_{\mathcal{F}}) - f(x_0), f(x_{\mathcal{F}}') - f(x_0)\} \leq -\mathcal{F}) \geq \frac{1}{2} P(\min\{f(x_{\mathcal{F}}) - f(x_0), f(x_{\mathcal{F}}') - f(x_0)\} \leq -\mathcal{F}) \geq \frac{1}{3} - 5d. \mathcal{F}^2 \cdot \log(\mathcal{F} \sqrt{d} / (\eta r)) e^{-\ell}$$

This finishes the proof.
A.4 Proof of Theorem\textsuperscript{5}

Proof of Theorem\textsuperscript{5} First, we set total iterations $T$ to be:

$$T = 100 \max \left\{ \frac{(f(x_0) - f^*)}{\mathcal{F}}, \frac{(f(x_0) - f^*)}{\eta \epsilon^2} \right\} = O \left( \frac{\ell(f(x_0) - f^*)}{\epsilon^2} \cdot \mathcal{R} \cdot \eta^9 \right)$$

We will show that the following two claims hold simultaneously with $1 - \delta$ probability:

1. at most $T/4$ iterates has large gradient, i.e. $\|\nabla f(x_t)\| \geq \epsilon$;
2. at most $T/4$ iterates are close to saddle points, i.e. $\|\nabla f(x_t)\| \leq \epsilon$ and $\lambda_{\min}(\nabla^2 f(x_t)) \leq -\sqrt{\rho \epsilon}$.

Therefore, at least $T/2$ iterates are $\epsilon$-second order stationary point. We prove two claims separately.

Claim 1. Suppose within $T$ steps, we have more than $T/4$ iterates that gradient is large (i.e. $\|\nabla f(x_t)\| \geq \epsilon$). Recall by Lemma\textsuperscript{14} we have with probability $1 - 4e^{-t}$:

$$f(x_T) - f(x_0) \leq -\frac{\eta}{8} \sum_{i=0}^{T-1} \|\nabla f(x_i)\|^2 + c\eta \delta^2 (\eta \ell T + \iota) \leq -\eta \left[ \frac{T \epsilon^2}{32} - \delta^2 (\eta \ell T + \iota) \right]$$

we note by our choice of $\eta, r, T$ and picking $\iota$ larger than some absolute constant, we have $T \epsilon^2 / 32 - \delta^2 (\eta \ell T + \iota) \geq \epsilon^2 / 64$, and thus $f(x_T) \leq f(x_0) - T \epsilon^2 / 64 < f^*$ which cannot be achieved.

Claim 2. We first define the stopping time which are the starting time we can apply Lemma\textsuperscript{16}:

$$z_i = \inf \{ \tau \mid \|\nabla f(x_T)\| \leq \epsilon \text{ and } \lambda_{\min}(f(x_T)) \leq -\sqrt{\rho \epsilon} \}$$

$$z_i = \inf \{ \tau \mid \tau > z_{i-1} + \mathcal{F} \text{ and } \|\nabla f(x_T)\| \leq \epsilon \text{ and } \lambda_{\min}(f(x_T)) \leq -\sqrt{\rho \epsilon} \}, \quad \forall i > 1$$

Clearly, $z_i$ is a stopping time, and is the $i$-th time in the sequence that we can apply Lemma\textsuperscript{16}. We also let $M$ be a stochastic variable where $M = \max \{ i | z_i + \mathcal{F} \leq T \}$. Therefore, we can decompose the decrease $f(x_T) - f(x_0)$ as follows:

$$f(x_T) - f(x_0) = \sum_{i=1}^{M} [f(x_{z_i+\mathcal{F}}) - f(x_{z_i})]$$

For the first term $T_1$, by Lemma\textsuperscript{16} and supermartingale concentration inequality, for each fixed $m \leq T$:

$$\mathbb{P} \left( \sum_{i=1}^{m} [f(x_{z_i+\mathcal{F}}) - f(x_{z_i})] \leq -(0.9m - c \sqrt{m \cdot \iota}) \mathcal{F} \right) \geq 1 - 5d \mathcal{F}^2 T \cdot \log(\mathcal{F} \sqrt{d} / (\eta r)) e^{-\iota}$$

Since random variable $M \leq T / \mathcal{F} \leq T$, by union bound, we know with probability $1 - 5d \mathcal{F}^2 T \cdot \log(\mathcal{F} \sqrt{d} / (\eta r)) e^{-\iota}$:

$$T_1 \leq -(0.9M - c \sqrt{M \cdot \iota}) \mathcal{F}$$

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For the second term, by union bound on Lemma\[14\] for all $0 \leq t_1, t_2 \leq T$, with probability $1 - 4T^2e^{-\epsilon}$:

$$T_2 \leq c \cdot \eta^2(\eta T + 2Mt)$$

Therefore, in sum if within $T$ steps, we have more than $T/4$ saddle points, then $M \geq T/4\mathcal{F}$, and with probability $1 - 10d\mathcal{F}^2T^2 \cdot \log(\mathcal{F}/(\eta T))e^{-\epsilon}$:

$$f(x_T) - f(x_0) \leq -(0.9M - c\sqrt{M \cdot t})\mathcal{F} + c \cdot \eta^2(\eta T + 2Mt) \leq -0.4M\mathcal{F} \leq -0.4T\mathcal{F}/\mathcal{F}$$

This will gives $f(x_T) \leq f(x_0) - 0.4T\mathcal{F}/\mathcal{F} < f^*$ which can not be achieved.

Finally, it is not hard to verify, by choose $\epsilon = c \cdot \log \left( \frac{dt\Delta_1\mathcal{F}}{\rho\delta} \right)$ with absolute constant $c$ large enough, we can make both claims hold with probability $1 - \delta$.

\[\square\]

### A.5 Proof of Theorem 8

**Proof of Theorem 8** The proof is essentially the same as the proof of Theorem 5. The only difference is that, up to a log factor, mini-batch PSGD reduces variance $\sigma^2$ and $\ell^2\|\hat{x}_r\|^2$ in Eq. (7) by a factor of $m$, where $m$ is the size of mini-batch.

\[\square\]

### B Concentration Inequality

In this section, we present the concentration inequalities required for this paper. Please refer to the technical note \[Jin et al., 2019\] for the proofs of Lemmas\[24, 25, 27\] and \[28\].

Recall the definition of norm-subGaussian random vector.

**Definition 23.** A random vector $X \in \mathbb{R}^d$ is norm-subGaussian (or nSG($\sigma$)), if there exists $\sigma$ so that:

$$\mathbb{P}(\|X - E[X]\| \geq t) \leq 2e^{-\frac{t^2}{2\sigma^2}}, \quad \forall t \in \mathbb{R}$$

We first note bounded random vector and subGaussian random vector are two special case of norm-subGaussian random vector.

**Lemma 24.** There exists absolute constant $c$ so that following random vectors are all nSG($c \cdot \sigma$).

1. A bounded random vector $X \in \mathbb{R}^d$ so that $\|X\| \leq \sigma$.

2. A random vector $X \in \mathbb{R}^d$, where $X = \xi e_1$ and random variable $\xi \in \mathbb{R}$ is $\sigma$-subGaussian.

3. A random vector $X \in \mathbb{R}^d$ that is $(\sigma/\sqrt{d})$-subGaussian.

Second, we have if $X$ is norm-subGaussian, then its norm square is subExponential, and its component along a single direction is subGaussian.

**Lemma 25.** There is an absolute constant $c$ so that if random vector $X \in \mathbb{R}^d$ is zero-mean nSG($\sigma$), then $\|X\|^2$ is $c \cdot \sigma^2$-subExponential, and for any fixed unit vector $v \in S^{d-1}$, $\langle v, X \rangle$ is $c \cdot \sigma$-subGaussian.

**Condition 26.** Let random vectors $X_1, \ldots, X_n \in \mathbb{R}^d$, and corresponding filtrations $\mathcal{F}_i = \sigma(X_1, \ldots, X_i)$ for $i \in [n]$ satisfy that $X_i|\mathcal{F}_{i-1}$ is zero-mean nSG($\sigma_i$) with $\sigma_i \in \mathcal{F}_{i-1}$, i.e.,

$$\mathbb{E}[X_i|\mathcal{F}_{i-1}] = 0, \quad \mathbb{P}(\|X_i\| \geq t|\mathcal{F}_{i-1}) \leq 2e^{-\frac{t^2}{2\sigma_i^2}}, \quad \forall t \in \mathbb{R}, \forall i \in [n].$$
Similar to subGaussian random variables, we can also prove Hoeffding type inequality for norm-subGaussian random vector which is tight up to a $\log(d)$ factor.

**Lemma 27** (Hoeffding type inequality for norm-subGaussian). There exists an absolute constant $c$, assume $X_1, \ldots, X_n \in \mathbb{R}^d$ satisfy condition [26] with fixed $\{\sigma_i\}$, then for any $\epsilon > 0$, with probability at least $1 - 2d \cdot e^{-\epsilon}$:

\[ \|\sum_{i=1}^n X_i\| \leq c \cdot \sqrt{\sum_{i=1}^n \sigma_i^2} \cdot \epsilon \]

**Lemma 28.** There exists an absolute constant $c$, assume $X_1, \ldots, X_n \in \mathbb{R}^d$ satisfy condition [26] then for any $\epsilon > 0$, and $B > b > 0$, with probability at least $1 - 2d \log(B/b) \cdot e^{-\epsilon}$:

\[ \sum_{i=1}^n \sigma_i^2 \geq B \quad \text{or} \quad \|\sum_{i=1}^n X_i\| \leq c \cdot \sqrt{\max\{\sum_{i=1}^n \sigma_i^2, b\}} \cdot \epsilon \]

Finally, we can also provide concentration inequalities for the sum of norm square of norm-subGaussian random vectors, and for the sum of inner product of norm-subGaussian random vectors with another set of random vectors.

**Lemma 29.** Assume $X_1, \ldots, X_n \in \mathbb{R}^d$ satisfy Condition [26] with fixed $\sigma_1 = \ldots = \sigma_n = \sigma$, then there exists absolute constant $c$, for any $\epsilon > 0$, with probability at least $1 - e^{-\epsilon}$:

\[ \sum_{i=1}^n \|X_i\|^2 \leq c \cdot \sigma^2 (n + \epsilon) \]

**Proof.** Note there exists an absolute constant $c$ such that $\mathbb{E}[\|X_i\|^2 | F_{i-1}] \leq c \cdot \sigma^2$, and $\|X_i\|^2 | F_{i-1}$ is $c \cdot \sigma^2$-subExponential. This lemma directly follows from standard Bernstein type concentration inequalities for subExponential random variables. \[\square\]

**Lemma 30.** There exists absolute constant $c$, assume $X_1, \ldots, X_n \in \mathbb{R}^d$ satisfy Condition [26] and random vectors $\{u_i\}$ satisfy $u_i \in F_{i-1}$ for all $i \in [n]$, then for any $\epsilon > 0$, $\lambda > 0$, with probability at least $1 - e^{-\epsilon}$:

\[ \sum_{i} \langle u_i, X_i \rangle \leq c \cdot \lambda \sum_{i} \|u_i\|^2 \sigma_i^2 + \frac{1}{\lambda} \cdot \epsilon \]

**Proof.** For any $i \in [n]$ and fixed $\lambda > 0$, since $u_i \in F_{i-1}$, according to Lemma [25] there exists constant $c$ so that $\langle u_i, X_i \rangle | F_{i-1}$ is $c \cdot \|u_i\| \sigma_i$-subGaussian. Thus:

\[ \mathbb{E}[e^{\lambda \langle u_i, X_i \rangle} | F_{i-1}] \leq e^{c \lambda^2 \|u_i\|^2 \sigma_i^2} \]

Therefore, consider following quantity:

\[ \mathbb{E} e^{\sum_{i=1}^i (\lambda \langle u_i, X_i \rangle - c \lambda^2 \|u_i\|^2 \sigma_i^2)} = \mathbb{E} \left[ e^{\sum_{i=1}^i -c \lambda^2 \sum_{i=1}^i \|u_i\|^2 \sigma_i^2} \cdot \mathbb{E} \left( e^{\lambda \langle u_i, X_i \rangle} | F_{i-1} \right) \right] \]

\[ \leq \mathbb{E} \left[ e^{\sum_{i=1}^i -c \lambda^2 \sum_{i=1}^i \|u_i\|^2 \sigma_i^2} \cdot e^{c \lambda^2 \|u_i\|^2 \sigma_i^2} \right] \]

\[ = \mathbb{E} e^{\sum_{i=1}^i (\lambda \langle u_i, X_i \rangle - c \lambda^2 \|u_i\|^2 \sigma_i^2)} \leq 1 \]
Finally, by Markov’s inequality, for any $t > 0$:

$$\mathbb{P}\left(\sum_{i=1}^{t}(\lambda \langle u_i, X_i \rangle - c \cdot \lambda^2 \|u_i\|^2 \sigma_i^2) \geq t\right) \leq \mathbb{P}\left(\sum_{i=1}^{t}(\lambda \langle u_i, X_i \rangle - c \cdot \lambda^2 \|u_i\|^2 \sigma_i^2) \geq e^t\right)$$

$$\leq e^{-t} \mathbb{E}\sum_{i=1}^{t}(\lambda \langle u_i, X_i \rangle - c \cdot \lambda^2 \|u_i\|^2 \sigma_i^2) \leq e^{-t}$$

This finishes the proof. $\square$