Pointwise summability of Fourier–Laguerre series of integrable functions

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Abstract

We present an approximation version of the results of D. P. Gupta [J. of Approx. Theory, 7 (1973), 226-238] A. N. S. Singoura [Proc. Japan Acad., 39 (4) (1963), 208-210] and G. Szegő [Math. Z., 25 (1926), 87-115]. Some corollaries and examples will also be given.

Key words: Rate of approximation, summability of Fourier–Laguerre series

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1 Introduction

Let $L_w$ be the class of all real–valued functions, integrable in the Lebesgue sense over $\mathbb{R}^+$ with the weight $w(t) = e^{-t^\alpha} \ (\alpha > -1)$, i.e.

$$
\int_{\mathbb{R}^+} e^{-t^\alpha} |f(t)| \ dt < \infty.
$$

We will consider the Fourier–Laguerre series

$$
S^{(\alpha)} f(x) := \sum_{\nu=0}^{\infty} a^{(\alpha)}_{\nu}(f) L^{(\alpha)}_{\nu}(x), \text{ with } \alpha > -1,
$$

where

$$
L^{(\alpha)}_{n}(x) = \frac{x^{-\alpha} e^x}{n!} \frac{d^n}{dx^n} \left( x^{n+\alpha} e^{-x} \right) = \sum_{\nu=0}^{n} \frac{(-1)^\nu}{\nu!} \binom{n+\alpha}{n-\nu} x^{\nu}
$$
and
\[ a^{(\alpha)}_\nu(f) = \frac{1}{\Gamma(\alpha + 1) A^{(\alpha)}_n} \int_0^{\infty} e^{-y^\alpha} L^{(\alpha)}_\nu(y) f(y) dy, \text{ with } A^{(\alpha)}_n = \left( \frac{n + \alpha}{n} \right). \]

Let define the \((C, \gamma)\) - means of partial sums
\[ S^{(\alpha)}_k(f(x)) = \sum_{\nu=0}^{n} a^{(\alpha)}_\nu(f) L^{(\alpha)}_\nu(x) \]
of \( S^{(\alpha)} f \) as follows
\[ S^{(\gamma, \alpha)}_n(f(x)) = \frac{1}{A^{(\gamma)}_n} \sum_{k=0}^{n} A^{(\gamma-1)}_{n-k} S^{(\alpha)}_k(f(x)), \quad (n = 0, 1, 2, ...) \]
and let
\[ \Delta_x f(t) = f(x + t) - f(x). \]

The deviation \( S^{(\gamma, \alpha)}_n(0) - f(0) \) was examined in the papers \([4, 6, 1]\) as follows:

**Theorem A.** \([1, \text{Theorem 1}]\) Let \( f \in L_w, \alpha > -1 \) and \( \gamma > \alpha + \frac{1}{2} \). If a function \( f \) satisfies the conditions
\[ \int_0^u e^{-u^\alpha} |\Delta_0 f(t)| dt = o(u^{\alpha+1}) \quad \text{as } u \to 0^+ \]
and
\[ \int_1^{\infty} e^{-t^\alpha - \gamma - \frac{1}{2}} |\Delta_0 f(t)| dt < \infty, \]
then
\[ |S^{(\gamma, \alpha)}_n(0) - f(0)| = o(1) \quad \text{as } n \to \infty. \]

Similar results in a case of norm approximation due of C. Markett and E. L. Poiani in papers \([2, 3]\) were obtained.

We will say that a nonnegative function \( \omega \) is a function of the modulus of continuity type if it is nondecreasing continuous function on \([0, \infty)\) having the following conditions: \( \omega(0) = 0 \) and \( \omega(\delta_1 + \delta_2) \leq \omega(\delta_1) + \omega(\delta_2) \) for any \( \delta_1, \delta_2 \in [0, \infty). \)

In this paper, we will study the upper bound of the quantity \( |S^{(\gamma, \alpha)}_n(0) - f(0)| \) by some means of a function of the modulus of continuity type \( \omega \). From our result we will derive some corollaries, remark and construct some examples.

## 2 Statement of the results

First we present the estimate of the quantity \( |S^{(\gamma, \alpha)}_n(0) - f(0)| \).
Theorem 1. Let $f \in L_w$, $\alpha > -1$, $\gamma > \alpha + \frac{1}{2}$ and let a function $\omega$ of the modulus of continuity type satisfy the conditions:

$$
\frac{u^{-(\alpha+1)}}{\Gamma(\alpha+1)} \int_0^u e^{-\frac{t}{2}t^\alpha} |\Delta_0 f(t)|\,dt = O(\omega(u)) \quad (u > 0)
$$

and

$$
\frac{1}{\Gamma(\alpha+1)} \int_u^\infty e^{-\frac{t}{2}t^\alpha} |\Delta_0 f(t)|\,dt = O(\omega(1/u)) \quad (u \geq 1).
$$

Then

$$
|S_n^{(\gamma,\alpha)} f(0) - f(0)| = O\left(n^{\eta + \frac{2(\alpha-\gamma)+1}{4}} \sum_{k=1}^\infty \frac{\omega(1/k)}{k^\frac{2(\alpha-\gamma)+1}{4}} \right) + O(\omega(1/n^\eta))
$$

for $0 < \eta < -\frac{2(\alpha-\gamma)+1}{4}$.

Now, we formulate some corollaries and remark.

Corollary 1. Under the assumptions of the above theorem

$$
|S_n^{(\gamma,\alpha)} f(0) - f(0)| = o(1) \quad \text{as } n \to \infty.
$$

Remark 1. Using Theorem 1 and Corollary 1 we obtain the result of D. P. Gupta from Theorem A.

Corollary 2. Analyzing the proof of Theorem 1 we can obtain, under the assumptions of this theorem, the following more precise estimate

$$
|S_n^{(\gamma,\alpha)} f(0) - f(0)| = O\left(n^{\eta + \frac{2(\alpha-\gamma)+1}{4}} \sum_{k=1}^\infty \frac{\omega(1/k)}{k^\frac{2(\alpha-\gamma)+1}{4}} \right) + O(\omega(1/n^\eta)) + O(\omega(1/n))
$$

when $\frac{2(\alpha-\gamma)+1}{4} + 1 < 0$.

In the special case, taking $\eta = -\frac{2(\alpha-\gamma)+1}{8}$, we have

$$
|S_n^{(\gamma,\alpha)} f(0) - f(0)| = O(\omega(1/n^\eta)),
$$

when $\eta \leq 1$.

3 Examples

Let $f_1(t) = e^{-\frac{t}{2}}$ and $\omega_1(t) = t$ for $t \geq 0$.

It is clear that $f_1 \in L_w$. Moreover, applying the Lagrange mean value theorem we get that

$$
|\Delta_0 f_1(t)| = |e^{-\frac{t}{2}} - 1| \leq \frac{t}{2}
$$

for $t \geq 0$. Therefore, by elementary calculations we get

$$
\frac{u^{-(\alpha+1)}}{\Gamma(\alpha+1)} \int_0^u e^{-\frac{t}{2}t^\alpha} |\Delta_0 f_1(t)|\,dt
$$

3
\[
\leq \frac{u^{-(\alpha+1)}}{2\Gamma(\alpha+1)} \int_0^u t^{\alpha+1} dt = \frac{1}{2\Gamma(\alpha+1)(\alpha+2)} \omega_1(u)
\]

for \( u > 0 \) and

\[
\frac{1}{\omega_1 \left( \frac{1}{\gamma} \right) \Gamma(\alpha+1)} \int_u^\infty e^{-\frac{t}{\gamma}} t^{\alpha-\gamma-\frac{s}{2}} |\Delta_0 f_1(t)| dt
\]

\[
\leq \frac{u}{2\Gamma(\alpha+1)} \int_u^\infty e^{-\frac{t}{\gamma}} t^{\alpha-\gamma-\frac{s}{2}} dt \leq \frac{1}{2\Gamma(\alpha+1)} \int_u^\infty e^{-\frac{t}{\gamma}} t^{\alpha+\gamma+\frac{s}{2}} dt
\]

\[
\leq \frac{1}{2\Gamma(\alpha+1)} \int_0^u e^{-\frac{t}{\gamma}} t^{\alpha+\gamma+\frac{s}{2}} dt = \frac{1}{2\Gamma(\alpha+1)} \int_0^\infty e^{-\frac{t}{\gamma}} t^{-1+\alpha-\gamma+\frac{s}{2}} dt
\]

\[
= \frac{1}{\Gamma(\alpha+1)} 2^{\alpha-\gamma+\frac{s}{2}} \Gamma \left( \alpha - \gamma + \frac{8}{3} \right) < \infty
\]

for \( u \geq 1 \) and \( \alpha - \gamma + \frac{8}{3} > 0 \).

Hence the function \( f_1 \) satisfies the conditions (1) and (2). Using Theorem 1 we get the following estimate for \( |S_n(t, 0) - f_1(0)| \):

**Example 1.** Let \( \alpha > -1, \alpha + \frac{1}{2} < \gamma < \alpha + \frac{8}{3} \) and \( 0 < \eta < -\frac{2(\alpha-\gamma+1)}{\delta} \). Then

\[
|S_n(t, 0) - f_1(0)| = O \left( n^{\eta+\frac{2(\alpha-\gamma+1)}{\delta}} \sum_{k=1}^{\infty} \frac{1}{k^{\eta+\frac{2(\alpha-\gamma+1)}{\delta}+2}} + O \left( \frac{1}{n^{\eta}} \right) \right).
\]

Suppose \( f_2(t) = t^{\delta} \) and \( \omega_2(t) = t^{\delta} \) for \( \delta \in (0, 1] \) and \( t \geq 0 \).

Obviously \( f_2 \in L_\omega \). In addition, it is easy to show that

\[
\frac{u^{-(\alpha+1)}}{\Gamma(\alpha+1)} \int_0^u t^{\alpha+\delta} dt = \frac{1}{\Gamma(\alpha+1)(\alpha+\delta+1)} \omega_2(u)
\]

for \( u > 0 \) and

\[
\frac{1}{\omega_2 \left( \frac{1}{\gamma} \right) \Gamma(\alpha+1)} \int_u^\infty e^{-\frac{t}{\gamma}} t^{\alpha-\gamma-\frac{s}{2}} |\Delta_0 f_2(t)| dt
\]

\[
\leq \frac{u^\delta}{\Gamma(\alpha+1)} \int_u^\infty e^{-\frac{t}{\gamma}} t^{\alpha-\gamma-\frac{s}{2}} dt \leq \frac{1}{\Gamma(\alpha+1)} \int_u^\infty e^{-\frac{t}{\gamma}} t^{-1+\alpha-\gamma+2\delta+\frac{s}{2}} dt
\]

\[
\leq \frac{1}{\Gamma(\alpha+1)} \int_0^\infty e^{-\frac{t}{\gamma}} t^{-1+\alpha-\gamma+2\delta+\frac{s}{2}} dt
\]

\[
= \frac{1}{\Gamma(\alpha+1)} 2^{\alpha-\gamma+2\delta+\frac{s}{2}} \Gamma \left( \alpha - \gamma + 2\delta + \frac{2}{3} \right) < \infty
\]

for \( u \geq 1 \) and \( \alpha - \gamma + 2\delta + \frac{2}{3} > 0 \).

Therefore the function \( f_2 \) satisfies the conditions (1) and (2). Using Theorem 1 we get the following estimate for \( |S_n(t, 0) - f_2(0)| \):
Example 2. Let $\alpha > -1$, $\delta \in (0, 1]$, $\alpha + \frac{1}{2} < \gamma < \alpha + 2\delta + \frac{2}{3}$ and $0 < \eta < -\frac{2(\alpha - \gamma + 1)}{4}$. Then

$$|S_n^{(\gamma, \alpha)}f_2(0) - f_2(0)| = O\left(n^{\eta \frac{2(\alpha - \gamma + 1)}{4}}\sum_{k=1}^{n} k^{\eta \frac{2(\alpha - \gamma + 1)}{4} + 1 + \delta} + O\left(\frac{1}{n^{\delta}}\right)\right).$$

4 Auxiliary results

We begin this section by some notations from [5]. We have

$$L_k^{(\alpha+1)}(y) = \sum_{\nu=0}^{k} L_{\nu}^{(\alpha)}(y), \quad L_{\nu}^{(\alpha)}(0) = \left(\begin{array}{c} \nu + \alpha \\ \nu \end{array}\right)$$

and therefore

$$S_k^{(\alpha)}f(0) = \frac{1}{\Gamma(\alpha + 1)} \int_{0}^{\infty} e^{-y} y^{\alpha} L_k^{(\alpha+1)}(y) f(y) dy,$$

$$S_n^{(\gamma, \alpha)}f(0) = \frac{1}{\Gamma(\alpha + 1) A_n^{(\gamma)}} \int_{0}^{\infty} e^{-y} y^{\alpha} L_n^{(\alpha+\gamma+1)}(y) f(y) dy.$$

Hence, by evidence equality

$$\frac{1}{\Gamma(\alpha + 1)} \int_{0}^{\infty} e^{-y} y^{\alpha} L_{\nu}^{(\alpha+1)}(y) dy = \begin{cases} 1 & \text{if } \nu = 0, \\ 0 & \text{if } \nu \neq 0, \end{cases}$$

we have

$$S_k^{(\alpha)}f(0) - f(0) = \frac{1}{\Gamma(\alpha + 1)} \int_{0}^{\infty} e^{-y} y^{\alpha} L_k^{(\alpha+1)}(y) \Delta_0 f(y) dy,$$

$$S_n^{(\gamma, \alpha)}f(0) - f(0) = \frac{1}{\Gamma(\alpha + 1) A_n^{(\gamma)}} \int_{0}^{\infty} e^{-y} y^{\alpha} L_n^{(\alpha+\gamma+1)}(y) \Delta_0 f(y) dy.$$

Next, we present the useful estimates:

Lemma 1. [5, p. 172] Let $\beta$ be an arbitrary real number, $c$ and $\delta$ be fixed positive constants. Then

$$\left|L_n^{(\beta)}(x)\right| = \begin{cases} O\left(n^\beta\right) & \text{if } 0 \leq x \leq \frac{c}{n}, \\ O\left(x^{-(2\beta+1)/4}n^{(2\beta-1)/4}\right) & \text{if } \frac{c}{n} \leq x \leq \delta. \end{cases}$$

Lemma 2. [5, p. 235] Let $\beta$ and $\lambda$ be arbitrary real numbers, $\delta > 0$ and $0 < \theta < 4$. Then

$$\max_{x} e^{-x^2/2\lambda} \left|L_n^{(\beta)}(x)\right| = \begin{cases} O\left(n^{\max\left(\lambda-\frac{1}{2}, \frac{4}{3}-\frac{\lambda}{2}\right)}\right) & \text{if } \delta \leq x \leq (4 - \theta) n, \\ O\left(n^{\max\left(\lambda-\frac{1}{2}, \frac{4}{3}-\frac{\lambda}{2}\right)}\right) & \text{if } x \geq \delta. \end{cases}$$
Lemma 3. [7, Vol. I, (1.15) and Theorem 1.17] If \( \gamma > -1 \), then
\[
A_n^{(\gamma)} = \left( \frac{n + \gamma}{n} \right) \approx O((n + 1)^{\gamma})
\]
and \( A_n^{(\gamma)} \) is positive for \( \gamma > -1 \) increasing (as a function of \( n \)) for \( \gamma > 0 \) and decreasing for \(-1 < \gamma < 0\).

5 Proofs of Theorems

5.1 Proofs of Theorem 1

It is clear that if
\[
S_n^{(\gamma, \alpha)} f(0) - f(0) = \int_0^1 e^{-y} y^\alpha \left| L_n^{(\alpha+\gamma+1)} (y) \right| |\Delta_0 f(y)| dy
\]
then
\[
|S_n^{(\gamma, \alpha)} f(0) - f(0)| \leq |J_1| + |J_2| + |J_3| + |J_4|.
\]

By Lemma 1, Lemma 3 and (1)
\[
|J_1| \leq \frac{1}{\Gamma(\alpha + 1) A_n^{(\gamma)}} \int_0^{1/n} e^{-y} y^\alpha \left| L_n^{(\alpha+\gamma+1)} (y) \right| |\Delta_0 f(y)| dy
\]
\[
= \frac{O\left(n^{\alpha+\gamma+1}\right)}{\Gamma(\alpha + 1) A_n^{(\gamma)}} \int_0^{1/n} e^{-y} y^\alpha |\Delta_0 f(y)| dy
\]
\[
= \frac{O\left(n^{\alpha+1}\right)}{\Gamma(\alpha + 1)} \int_0^{1/n} e^{-y} y^\alpha |\Delta_0 f(y)| dy = O\left(\omega\left(1/n\right)\right)
\]
\[
\leq O\left(n^{2(\alpha-2\gamma+1)/4}\right) \sum_{k=1}^{n} \frac{\omega(1/k)}{k^{(2\alpha-2\gamma+1)/4+1}}
\]
\[
\leq O\left(n^{\eta+2(\alpha-\gamma+1)/4}\right) \sum_{k=1}^{n} \frac{\omega(1/k)}{k^{n+2(\alpha-\gamma+1)/4+1}}
\]
with \(0 < \eta < \frac{-2(\alpha-\gamma)+1}{4}\).

Using Lemma 1, we get
\[
|J_2| \leq \frac{1}{\Gamma(\alpha + 1) A_n^{(\gamma)}} \int_{1/n}^{1} e^{-y} y^\alpha \left| L_n^{(\alpha+\gamma+1)} (y) \right| |\Delta_0 f(y)| dy
\]
\[
= \frac{O\left(n^{2(\alpha+\gamma+1)/4}\right)}{\Gamma(\alpha + 1) A_n^{(\gamma)}} \int_{1/n}^{1} e^{-y} y^\alpha |\Delta_0 f(y)| y^{-2(\alpha+\gamma+1)/4} dy.
\]
Let $F_\alpha(y) = \frac{\omega^{-(\alpha+1)}}{(\alpha+1)} \int_0^1 e^{-\frac{2}{\eta}y^{(2\alpha-2\gamma-3)/4}} |\Delta_0 f(u)| du$. Applying Lemma 3 and integrating by parts with $\gamma > \alpha + \frac{1}{2}$ and $\alpha > -1$ we have

$$|J_2| = O \left( n^{(2\alpha-2\gamma+1)/4} \right) \int_{1/n}^1 e^{-\eta/2y^{(2\alpha-2\gamma-3)/4}} |\Delta_0 f(y)| dy$$

$$= O \left( n^{(2\alpha-2\gamma+1)/4} \right) \left\{ \int_{1/n}^1 F_\alpha(y) y^{2(\alpha-\gamma+1)/4} dy \right\}$$

$$+ \frac{2(\alpha+\gamma)+3}{4} \int_{1/n}^1 F_\alpha(y) y^{2(\alpha-\gamma+1)/4 - 1} dy \right\}$$

$$\leq O \left( n^{(2\alpha-2\gamma+1)/4} \right) \left\{ F_\alpha(1) + \int_{1/n}^1 F_\alpha(y) y^{2(\alpha-\gamma+1)/4 - 1} dy \right\}$$

$$= O \left( n^{(2\alpha-2\gamma+1)/4} \right) \left\{ F_\alpha(1) + \int_{1/n}^1 F_\alpha(y) y^{2(\alpha-\gamma+1)/4 - 1} dy \right\}$$

$$= O \left( n^{(2\alpha-2\gamma+1)/4} \right) \left\{ F_\alpha(1) + \int_{1/n}^1 F_\alpha(y) y^{2(\alpha-\gamma+1)/4 - 1} dy \right\}$$

$$\leq O \left( n^{(2\alpha-2\gamma+1)/4} \right) \left\{ F_\alpha(1) + \sum_{k=1}^{n-1} F_\alpha(1/k) k^{2(\alpha-\gamma+1)/4 - 1} \right\}$$

$$\leq O \left( n^{(2\alpha-2\gamma+1)/4} \right) \sum_{k=1}^{n-1} 2F_\alpha(1/k) k^{2(\alpha-\gamma+1)/4 - 1}$$

$$\leq O \left( n^{(2\alpha-2\gamma+1)/4} \right) \sum_{k=1}^{n-1} F_\alpha(1/k) \frac{k^{2(\alpha-\gamma+1)/4 - 1}}{k(2\alpha-2\gamma+1)/4 + 1}.$$

By (1) we obtain

$$|J_2| = O \left( n^{(2\alpha-2\gamma+1)/4} \right) \sum_{k=1}^n \frac{\omega(1/k)}{k^{(2\alpha-2\gamma+1)/4 + 1}}$$

$$\leq O \left( n^{(2\alpha-2\gamma+1)/4} \right) \sum_{k=1}^n \frac{\omega(1/k)}{k^{(2\alpha-2\gamma+1)/4 + 1}}.$$

with $0 < \eta < \frac{2(\alpha-\gamma+1)}{4}.$

Applying Lemma 2 with $\alpha + \gamma + 1$ instead of $\beta$ and $\lambda = \frac{2\alpha+2\gamma+3}{4}$ (since $\max \left( \lambda - \frac{1}{2}, \frac{\alpha+\gamma+1}{2} - \frac{1}{4} \right) = \frac{2\alpha+2\gamma+1}{4}$) we have

$$|J_3| \leq \frac{1}{\Gamma(\alpha+1)A_n^{(\gamma)}} \int_1^n e^{-\eta/2y^{(2\alpha-2\gamma-3)/4}} |\Delta_0 f(y)| e^{-\eta/2y^{(2\alpha+2\gamma+3)/4}} |L_n^{(\alpha+\gamma+1)}(y)| dy$$

$$= \frac{O \left( n^{(2\alpha+2\gamma+1)/4} \right)}{\Gamma(\alpha+1)A_n^{(\gamma)}} \int_1^n e^{-\eta/2y^{(2\alpha-2\gamma-3)/4}} |\Delta_0 f(y)| dy.$$
Using Lemma 3 and integrating by parts with $\gamma > \alpha + \frac{1}{2}$ and $\alpha > -1$, we get

\[
|J_3| = \frac{O\left(n^{(2\alpha-2\gamma+1)/4}\right)}{\Gamma(\alpha+1)} \int_1^n \left[ e^{-y/2} y^{\alpha} |\Delta_0 f(y)| \right] y^{(-2\alpha-2\gamma-3)/4} dy
\]

\[
= O\left(n^{(2\alpha-2\gamma+1)/4}\right) \int_1^n \frac{d}{dy} \left[ \int_0^y \frac{e^{-u/2} u^{\alpha} |\Delta_0 f(u)|}{\Gamma(\alpha+1)} du \right] y^{(-2\alpha-2\gamma-3)/4} dy
\]

\[
= O\left(n^{(2\alpha-2\gamma+1)/4}\right) \left\{ \left[ y^{(-2\alpha-2\gamma-3)/4} \int_0^y \frac{e^{-u/2} u^{\alpha} |\Delta_0 f(u)|}{\Gamma(\alpha+1)} du \right]_1^n + \frac{2\alpha + 2\gamma + 3}{4} \int_1^n \frac{d}{dy} \left[ \int_0^y \frac{e^{-u/2} u^{\alpha} |\Delta_0 f(u)|}{\Gamma(\alpha+1)} du \right] y^{(-2\alpha-2\gamma-3)/4} dy \right\}
\]

\[
\leq O\left(n^{(2\alpha-2\gamma+1)/4}\right) \left\{ F_\alpha(n^\eta) n^{2(\alpha-\gamma+1)/4} + \frac{2\alpha + 2\gamma + 3}{4} \int_1^n F_\alpha(y) y^{2(\alpha-\gamma+1)/4} dy \right\}.
\]

By (1) we obtain

\[
|J_3| \leq O\left(n^{(2\alpha-2\gamma+1)/4}\right) \left\{ \omega(n^\eta) n^{2(\alpha-\gamma+1)/4} + \frac{2\alpha + 2\gamma + 3}{4} \int_1^n \omega(y) y^{2(\alpha-\gamma+1)/4} dy \right\}
\]

\[
\leq O(1) \left\{ n^{(2\alpha-2\gamma+1)/4} n^\eta \omega(1) n^{2(\alpha-\gamma+1)/4} + n^{(2\alpha-2\gamma+1)/4} \omega(n^\eta) \int_1^n y^{2(\alpha-\gamma+1)/4} dy \right\}
\]

\[
\leq O\left(n^{(2\alpha-2\gamma+1)/4} n^\eta \right) \left\{ n^{2(\alpha-\gamma+1)/4} + n^{2(\alpha-\gamma+1)/4} + n^{2(\alpha-\gamma+1)/4} \omega(n) \right\} \leq O\left(n^{2(\alpha-\gamma+1)/4} \right) \omega(1)
\]

\[
\leq O\left(n^{2(\alpha-\gamma+1)/4} \right) \sum_{k=1}^n \frac{\omega(1/k)}{k^{2(\alpha-\gamma+1)+1}}.
\]

with $0 < \eta < \frac{-2(\alpha-\gamma)+1}{4}$.

If $\lambda = \gamma + \frac{1}{3}$ then $\lambda - \frac{1}{3} = \gamma > \frac{\alpha+\gamma+1}{3} - \frac{1}{4}$ since $\gamma > \alpha + \frac{1}{2}$. So, applying Lemma 2 with $\alpha + \gamma + 1$ instead of $\beta$ and $\lambda = \gamma + \frac{1}{3}$ we obtain

\[
|J_4| \leq \frac{1}{\Gamma(\alpha+1) A_n^{(7)}} \int_1^n e^{-y/2} y^{(3\alpha-3\gamma-1)/3} |\Delta_0 f(y)| e^{-y/2} y^{(3\gamma+1)/3} \left| L_n^{(\alpha+\gamma+1)}(y) \right| dy
\]

\[
= \frac{O(n^\gamma)}{\Gamma(\alpha+1) A_n^{(7)}} \int_1^n e^{-y/2} y^{(3\alpha-3\gamma-1)/3} |\Delta_0 f(y)| dy.
\]

Next, using Lemma 3 and (2) we get

\[
|J_4| = \frac{O(1)}{\Gamma(\alpha+1)} \int_1^n e^{-y/2} y^{(3\alpha-3\gamma-1)/3} |\Delta_0 f(y)| dy = O\left(\omega(1/n^\eta)\right).
\]
Finally, collecting the above estimates we have

\[ \left| S_n^{(\gamma, \alpha)} f(0) - f(0) \right| = O \left( n^{\eta + \frac{2(\alpha - \gamma) + 1}{2}} \right) \sum_{k=1}^{n} \frac{\omega(1/k)}{k^{\eta + \frac{2(\alpha - \gamma) + 1}{2} + 1}} + O \left( \omega(1/n^\eta) \right). \]

and our proof is completed. ■

6 Conclusions

We investigated pointwise approximation of real–valued functions, integrable in the Lebesgue sense over \( \mathbb{R}^+ \) with the weight \( w(t) = e^{-t^\alpha} \) \( (\alpha > -1) \) by the \((C, \gamma)\) - means of partial sums of their Fourier–Laguerre series. In particular, we estimated the deviation \( \left| S_n^{(\gamma, \alpha)} f(0) - f(0) \right| \) by means of a function of the modulus of continuity type \( \omega \). From our result some corollaries were derived and some examples were constructed.

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