Dispersion Compensation with a Prism-pair

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Abstract

We present a detailed calculation of the total dispersion (spectral phase) from a pair of Brewster-cut prisms. This paper aims to aid advanced students in tracking the operation of this major configuration for dispersion-compensation and phase-control of ultra-short pulses.

1 Introduction

Prism pairs are commonly applied in the field of ultrafast optics as a mechanism to control the spectral phase of ultrashort pulses. For example, many mode-locked lasers that produce femtosecond-scale pulses [1] employ an intracavity prism-pair to compensate for the dispersion from intra-cavity elements, such as gain medium, lenses, etc. The core concept of a prism-pair is to generate a tunable, frequency dependent phase that will compensate for the phase accumulated by other dispersive elements in the setup. In the following, the total spectral-phase accumulated through a prism-pair is calculated.
2 The Brewster Prism

When a ray passes through a prism at minimum deviation, the angles of refraction through the prisms are symmetric (see Fig.1), such that

$\theta_1 = \theta_4$
$\theta_2 = \theta_3 = \frac{\alpha}{2}$.

If the entrance angle is also the Brewster angle $\theta_B$ (for a certain wavelength $\lambda_0$), we obtain

$\theta_B = \theta_1 = \theta_4$
$\theta_B + \theta_2 = \frac{\pi}{2}$.

and the relation between the prism’s apex angle $\alpha$ and Brewster angle becomes

$\alpha = 2\theta_2 = 2\left(\frac{\pi}{2} - \theta_B\right)$,
$\alpha = \pi - 2\theta_B$.
3 The Optical Path Through a Prism-Pair

We now wish to calculate the total optical path (and phase) through a prism-pair. The coordinate system used for the prism-pair has two degrees of freedom, as illustrated in Fig. 2: R the separation between the prisms (segment AB), and H the penetration of the prism into the beam (segment BC). The red line represents a Brewster ray (at the design wavelength $\lambda_0$), i.e. a ray that enters and exits the prisms at Brewster angle $\theta_B$. The blue line is a deviated ray at $\lambda \neq \lambda_0$, which deviates from the Brewster ray by an angle $\delta \theta$ due to the prism dispersion.

To calculate the phase accumulated by an arbitrary wavelength $\lambda$ (blue line) through the prism-pair, one usually calculates the optical path of the deviated beam along a continuous ray, starting from point A, through the refractions in the second prism until a final reference plane perpendicular to the beam, which is located after the second prism, where all the colors are parallel [2, 3]. This method results in somewhat complicated expressions for the optical path, leaving little room for intuition. An elegant alternative that avoids altogether calculations of refraction is to use the concept of wave-fronts (outlined in Fig. 3), which was originally presented in [4].

Each ray within a plane-wave beam that propagates through the prism-pair has the same optical path (and phase) between any two reference planes perpendicular to the beam that coincide with the phase-front (see Fig. 3; the reference planes are marked R1 through R5, and all rays have the same wavelength $\lambda$). Changing the position of plane R1 only adds a constant delay to all frequencies, hence, R1 can be chosen freely to pass through point A.
Figure 3: Planes of equal phase in a plane-wave beam propagating through the prism-pair.

Figure 4: Complete optical path through a prism-pair.

The same argument applies to R5, which can be chosen to pass through point C. Consequently, the optical path through the entire prism-pair can be reduced to the segment

$$l_{opt} = AA' + CC'. \quad (4)$$

Note that this path already takes into account refraction of the beam in the material, although it is composed of two rays that are free of refraction. As illustrated in Fig.4 (where only the rays are shown without the prisms for clarity), the optical path in Eq.4 is identical to the segment $EC$, where the point C is the vertex of the second prism. Thus, the optical path of the deviated ray through the entire prism pair is simply given by

$$l_{opt} = EC = ED + DC, \quad (5)$$

4
where $ED = R \cos \delta \theta$, and $DC = H \cos \theta_H$ ($\theta_H = \angle DCB$).

Since $EC$ is parallel to $AA'$, and R2 is parallel to R4, $\theta_H$ is also the angle between R2 and the deviated ray (blue line) in the first prism (see Fig.5)

$$\theta_H = \frac{\alpha}{2} + \frac{\pi}{2} - \theta_B - \delta \theta. \quad (6)$$

By substituting Eq.3 into Eq.6 we obtain

$$\theta_H = \pi - 2\theta_B - \delta \theta = \alpha - \delta \theta, \quad (7)$$

indicating that $DC = H \cos(\alpha - \delta \theta) = H \cos \alpha \cos \delta \theta + H \sin 2\theta_B \sin \delta \theta$.

By substituting the above relations into Eq.5, the total optical path is

$$l_{opt} = R \cos \delta \theta + H \cos \alpha \cos \delta \theta + H \sin 2\theta_B \sin \delta \theta. \quad (8)$$

and the optical phase is $\phi(\omega) = \frac{\omega}{c} l_{opt}(\omega)$.

Since $\delta \theta$ is typically small, we can approximate $\cos \delta \theta \approx 1 - (\delta \theta^2 / 2)$ and $\sin \delta \theta \approx \delta \theta$, yielding

$$l_{opt} \approx (R + H \cos \alpha) \left(1 - \frac{\delta \theta^2}{2}\right) + (H \sin 2\theta_B) \delta \theta. \quad (9)$$

Equations 8 and 9 are a generalization of the method presented in [4], where only the case of $H = 0$ was presented.
4 Prism angular dispersion

Let us now express the angle $\delta \theta$ using Snell’s law, assuming small angles $\delta \theta$. We assume that the beam enters the first prism at minimum deviation angle that coincides with the Brewster angle $\theta_B$ for a certain wavelength $\lambda_0$, and that the refractive index of the prisms is $n_0$ for $\lambda_0$. The refractive index for the deviated beam ($\lambda \neq \lambda_0$) is: $n = n_0 + \delta n$. We define the angles of refraction inside the prism for the Brewster beam as $\beta_0$ (equal at both faces of the prism). As illustrated in Fig.6, the angle of refraction for the deviated beam inside the prism is $\beta_1 = \beta_0 + \delta \beta$. It is easy to show that the deviated beam will hit the exit face of the prism at an angle of $\beta_2 = \beta_0 - \delta \beta$. The exit angle for the deviated beam will be $\theta_{out} = \theta_B + \delta \theta$. In addition, since at Brewster angle we have $\tan \theta_B = n_0$ and $\theta_B = \pi/2 - \beta_0$, we obtain the following relations: $\sin \theta_B = \cos \beta_0$ and $\cos \theta_B = \sin \beta_0$.

We now follow Snell’s law from the entrance face to the exit. At the entrance

$$\sin \theta_B = n \sin \beta_1. \quad (10)$$

We expand the righthand side as

$$n \sin \beta_1 = (n_0 + \delta n) \sin (\beta_0 + \delta \beta)$$

$$= (n_0 + \delta n)(\sin \beta_0 \cos \delta \beta + \cos \beta_0 \sin \delta \beta)$$

$$\approx (n_0 + \delta n)(\sin \beta_0 + \delta \beta \cos \beta_0) \quad (11)$$

$$= n_0 \sin \beta_0 + \delta n \sin \beta_0 + \delta \beta n_0 \cos \beta_0$$

$$= n_0 \cos \theta_B + \delta n \cos \theta_B + \delta \beta n_0 \sin \theta_B.$$
Substituting Eq. 11 into Eq.10 and dividing by \( \sin \theta_B \), (noting that \( \tan \theta_B = n_0 \)) provides

\[
1 + \frac{\delta n}{n_0} + n_0 \delta \beta = 1 \\
\boxed{n_0^2 \delta \beta = -\delta n}.
\]

Following similar logic at the exit face of the prism, we obtain

\[
n \sin \beta_2 = \sin \theta_{out},
\]

and expanding both sides of Eq.13 yields

\[
n \sin \beta_2 = (n_0 + \delta n) \sin(\beta_0 - \delta \beta) \\
\approx n_0 \sin \beta_0 + \delta n \sin \beta_0 - \delta \beta n_0 \cos \beta_0 \\
= n_0 \cos \theta_B + \delta n \cos \theta_B - \delta \beta n_0 \sin \theta_B \\
\sin \theta_{out} = \sin(\theta_B + \delta \theta) = \sin \theta_B \cos \delta \theta + \cos \theta_B \sin \delta \theta \\
\approx \sin \theta_B + \delta \theta \cos \theta_B.
\]

Equating both sides provides

\[
n_0 \cos \theta_B + \delta n \cos \theta_B - \delta \beta n_0 \sin \theta_B = \sin \theta_B + \cos \theta_B \delta \theta,
\]

and dividing Eq.15 by \( \sin \theta_B \) yields

\[
1 + \frac{\delta n}{n_0} - n_0 \delta \beta = 1 + \frac{\delta \theta}{n_0} \\
\boxed{\delta n - n_0^2 \delta \beta = \delta \theta}.
\]

Finally, substituting Eq.12 into Eq.16 yields

\[
\boxed{\delta \theta = 2 \delta n}.
\]

The wavelength dependent phase through the prism-pair can be now obtained by substituting \( \delta \theta = 2 \delta n = 2(n(\lambda) - n_0) \) into Eq.8. The wavelength dependent refractive index of the prisms \( n(\lambda) \) can be obtained from Sellmeier data\(^1\). Now, the total phase and any of its derivatives (dispersion orders) can be easily calculated.

\[^1\text{http://refractiveindex.info}\]
References

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