We prove some new relations between weak approximation and some rational equivalence relations (Brauer and R-equivalence) in algebraic groups over arithmetical fields. By using weak approximation and local - global approach, we compute completely the group of Brauer equivalence classes of connected linear algebraic groups over number fields, and also completely compute the group of R-equivalence classes of connected linear algebraic groups $G$, which either are defined over a totally imaginary number field, or contains no anisotropic almost simple factors of exceptional type $^{3,4}D_4$, nor $E_6$. We discuss some consequences derived from these, e.g., by giving some new criteria for weak approximation in algebraic groups over number fields, by indicating a new way to give examples of non stably rational algebraic groups over local fields and application to norm principle.

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**Introduction.** Let $G$ be a linear algebraic group defined over a field $k$. There are two closely related questions in the arithmetic theory of algebraic groups over fields: the question of weak approximation and that of rationality of a given $G$. Of course it is not real to study such questions for arbitrary groups over arbitrary fields. One should restrict to some class of groups and fields which are convenient in application.

Let $X$ be an algebraic variety defined over a field $k$, $X = X \times \bar{k}$, where $\bar{k}$ is an algebraic closure of $k$. Denote by $\text{Br}(X)$ the cohomological Brauer group $H^2_{\text{et}}(X, \mathbb{G}_m)$ of $X$, $\text{Br}_1(X) = \text{Ker} (\text{Br}(X) \to \text{Br}(\bar{X}))$. Following Manin, Colliot-Thélène and Sansuc (see [M1,2], [CTS1,2]), one defines the Brauer equivalence and R-equivalence as follows. First one defines a pairing

$$X(k) \times \text{Br}_1(X) \to \text{Br}(k), (x, b) \mapsto b(x),$$

where $b \in H^2_{\text{et}}(X, \mathbb{G}_m)$ is considered (see [S], Section 6) as an equivalence class of Azumaya algebras over $X$ and $b(x)$ is the equivalence class of central simple algebras over $k$, which is considered as an element of $\text{Br}(k)$.

Two points $x, y \in X(k)$ are said to be *Brauer equivalent* (*Br-equivalent*) if for any $b \in \text{Br}_1(X)$, we have $b(x) = b(y)$ (*resp.* if there is a sequence of points $z_i \in X(k), x = z_1, y = z_n$, such that for each pair $z_i, z_{i+1}$ there is a $k$-rational map $f: \mathbb{P}^1 \to X$, regular at 0 and 1, with $f(0) = z_i, f(1) = z_{i+1}, 1 \leq i \leq n - 1$). It is known [CTS1, Prop. 16] that the Brauer equivalence is weaker than R-equivalence, i.e., two points of $X(k)$, being R-equivalent, are necessarily Br-equivalent. In [loc.cit], basic theory of Brauer equivalence on tori defined over a field $k$ of characteristic 0 has been developed. In particular, in the arithmetic case, i.e., when $k$ is a local or global field, formulae for computations of the group $T(k)/\text{Br}$ are given and it turns out to be in general a birational invariant of $T$. Though $T(k)/\text{Br}$ is "computable", the group itself and its computation is in general non-trivial.

In a subsequent paper [S], Sansuc developed Brauer theory of linear algebraic groups $G$ over number fields, and applied it to obtain certain fundamental sequences connecting various arithmetic (obstruction to weak approximation), cohomological (Tate - Shafarevich group) and geometric invariants (the first Galois cohomology of the Picard group of a smooth compactification of $G$ over an algebraic closure of $k$) for connected linear algebraic groups $G$ over number fields $k$.

In this paper we continue the approach taken by Colliot-Thélène and Sansuc, to obtain certain connections between the above arithmetic, cohomological and birational (Brauer and R-) invariants of connected linear algebraic groups $G$ over local and global fields of characteristic 0. As it was pointed out above, in general, the group of Brauer equivalence classes of $G$ is non-trivial, even in the case of tori. Therefore it is natural to ask what kind of analogs in the case of arbitrary connected linear algebraic groups one can have. The main result of this paper, among others, is Theorem 4.12, which allows us to compute completely the group $G(k)/R$ for those connected linear algebraic groups $G$, which either are defined over a totally imaginary number field $k$, or contain no anisotropic almost simple factors of exceptional type $^3\!D_4$, nor $E_6$. The group $G(k)/\text{Br}$ can be computed completely, which in fact gives apriori (or preliminary information) on the group $G(k)/R$. The content of the paper after this introduction is as follows.

1. Recall of some basic facts from Brauer theory of algebraic groups.
2. A Brauer relative of exact sequence (R) for algebraic tori.
3. Some reductive analogs.
4. Some variations and applications.
5. An application to norms principle.
6. Remarks, problems and conjectures.

Bibliography.

Notation. Let $S$ be a finite set of valuations of a global field $k$, and $G$ a connected linear algebraic group defined over $k$. Denote by $\text{Cl}_S(G(k))$ the closure of $G(k)$ in the product $\prod_{v \in S} G(k_v)$ in the product topology, where $G(k)$ is embedded diagonally into the direct product, and $G(k_v)$ has the induced $v$–adic topology. We say that $G$ has weak approximation with respect to $S$ (or in $S$) if $\text{Cl}_S(G(k)) = \prod_{v \in S} G(k_v)$, and has weak approximation over $k$ if it is so for any finite $S$. Let

$$A(S, G) = \prod_{v \in S} G(k_v)/\text{Cl}_S(G(k)), \quad A(G) = \prod_v G(k_v)/\text{Cl}(G(k)),$$

the obstruction of weak approximation in $S$ and over $k$, respectively. Let $G(k)/R$ (resp. $G(k)/Br$) denote the group of R-equivalence (resp. Br-equivalence) classes of $G$ over $k$. Let $A$ be the group $\text{Hom}(A, \mathbb{Q}/\mathbb{Z})$, and $\hat{A}$ be the group $\text{Hom}(A, \mathbb{G}_m)$. $\text{III}(\cdot)$ denotes the Tate - Shafarevich group of $\cdot$. We denote also by $\text{Br}_a X := \text{Br} X/\text{Br}_1 X$, the arithmetic Brauer group of $X$. By $\text{H}^2(k, G)$ we denote the Galois cohomology of $G$.

1 Recall of some basic facts from Brauer theory of algebraic groups [CTS1], [S]

Let $T$ be a torus defined over field $k$ of characteristic 0 with splitting field $K$. Consider a flasque resolution

$$1 \to S \to N \to T \to 1$$

of $T$, where $S$ is a flasque $k$-torus, and $N$ is an induced $k$-torus. Denote by $\text{Br}(k, K)$ the kernel of $\text{H}^2(k, \mathbb{G}_m) \to \text{H}^2(K, \mathbb{G}_m)$. The exact sequence (1) induces a homomorphism

$$\text{H}^1(k, \hat{S}) \overset{\Delta}{\to} \text{H}^2(k, \hat{T}),$$

which is injective, since $N$ has trivial 1-cohomology.

One has a cup-product

$$T(k) \times \text{H}^2(K/k, \hat{T}) \overset{\cup}{\to} \text{Br}(k, K),$$

which defines, via (2), a pairing

$$\beta : T(k) \times \text{H}^1(K/k, \hat{S}) \overset{\cup}{\to} \text{Br}(k, K).$$

We have

1.1. Theorem. ([CTS1], Prop. 17 and Corol.)
1) The map $\beta$ defines the Brauer equivalence relation over $T(k)$, hence also a map

$$\gamma : T(k)/\text{Br} \to \text{Hom}(\text{H}^1(K/k, \hat{S}), \text{Br}(k, K)).$$
2) We have the following anti-commutative diagram

\[
\begin{array}{ccc}
T(k)/R & \xrightarrow{\delta} & T(k)/Br \\
\downarrow & & \downarrow \\
H^1(k, S) & \xrightarrow{\omega} & \text{Hom}(H^1(K/k, \hat{S}), Br(k, K))
\end{array}
\]

Here \(\delta\) is an isomorphism ([CTS1, Theorem 2]), and \(\omega\) comes from the cup-product

\[
H^1(k, S) \times H^1(k, \hat{S}) \rightarrow H^2(k, G_m).
\]

3) \(T(k)/Br \simeq \text{Im} \ (\omega)\) and \(T(k)/Br\) is a birational invariant in the class of \(k\)-tori, stably equivalent to \(T\).

4) If \(k\) is a \(p\)-adic local field, the Brauer equivalence on \(T(k)\) coincides with \(R\)-equivalence on \(T(k)\) and

\[
T(k)/Br \simeq H^1(k, S)^-.
\]

5) If \(k\) is a number field, and \(\mu\) is the composition map

\[
H^1(k, S) \xrightarrow{\Delta} \prod_v H^1(k_v, \hat{S}) \xrightarrow{\Delta} \prod_v H^1(k_v, \hat{S})
\]

then \(T(k)/Br \simeq [\text{Im} \ (\lambda)/\text{Im} \ (\mu)]^-\).

1.2. Theorem. ([CTS1], Prop. 19) With above notation, let \(k\) be a number field. We have the following exact sequences:

\[(R)\] 0 \(\rightarrow\) \(\Pi(S) \rightarrow T(k)/R \xrightarrow{\text{pr}} \prod_v T(k_v)/R \rightarrow A(T) \rightarrow 0,\]

\[(V)\] 0 \(\rightarrow\) \(A(T) \rightarrow H^1(k, \hat{S})^- \rightarrow \Pi(S) \rightarrow 0.\]

The following result gives us the group structure on \(G(k)/Br\), induced from that of \(G(k)\).

1.3. Proposition. ([CTS1], p. 216, [S], Lem. 6.9(1)) Let \(K\) be a field and \(G\) a connected linear algebraic group over \(K\), assumed to be reductive if \(K\) is not perfect. Then the pairing

\[
G(K) \times Br_e G \rightarrow Br K
\]

is biadditive. In particular, \(G(K)/Br\) has a natural group structure induced from \(G(K)\).

The following well-known fact (which is a direct consequence of the Hasse principle for Brauer group of global fields) was mentioned in [MT]:

1.4. Proposition. [MT] Let \(X\) be a smooth variety defined over a number field \(k\). Then the restriction map

\[
X(k)/Br \rightarrow \prod_v X(k_v)/Br
\]
is injective.

1.5. Remarks. 1) Notice that in Theorem 1.2, we have identified $\mathbb{H}(S)$ with a subgroup of $T(k)/R$ via the isomorphism $\delta$ of Theorem 1.1, 2). The exact sequence (V), which is due to Voskresenskii (see e.g. [V1], [S]), has been extended to the case of arbitrary connected linear algebraic groups over number fields by Sansuc [S].

2) We are interested in Brauer equivalence relation for connected linear algebraic groups, which are rational over algebraic closure $\overline{k}$ of $k$, hence the $Br$-equivalence and $Br_1$-equivalence are the same.

Our objective is to study the analogs of the exact sequence (R) in the case of Brauer and $R$-equivalence over local and global fields, for tori in particular, and for connected linear algebraic groups in general.

2 A Brauer relative of exact sequence (R) for algebraic tori

Let $S$ be a finite set of valuations of a number field $k$, $T$ a $k$-torus, $T_S := \prod_{v \in S} T(k_v)$. Denote by $RT(L)$ (resp. $BT(L)$) the set of elements of $T(L)$ which are $R$- (resp. $Br$-) equivalent to 1 in $T(L)$, where $L$ is a field extension of $k$. Let $RT_S = \prod_{v \in S} RT(k_v)$, $BT_S = \prod_{v \in S} BT(k_v)$. The following result was mentioned in [V2] (which is valid also for any field $k$).

2.1. Proposition. $RT_S \subset Cl(T(k))$ and is an open subgroup in $T_S$.

From above one derives the following

2.2. Corollary. $A(S,T) \simeq \text{Coker}(T(k)/R \to \prod_{v \in S} T(k_v)/R)$,

$A(T) \simeq \text{Coker}(T(k)/R \to \prod_{v} T(k_v)/R)$.

2.3. Corollary. $BT_S \subset Cl(T(k))$.

Proof. If $v$ is a non-archimedean, then Theorem 1.1, (4) tells us that $BT(k_v) = RT(k_v)$. If $v$ is archimedean, then it is well-known that $T$ is rational over $k_v$, hence has trivial groups $T(k_v)/Br$ and $T(k_v)/R$, i.e., $BT(k_v) = RT(k_v)$.

In what follows we identify $T(k)$ with a subgroup of $T_S$ via diagonal embedding.

2.4. Proposition. We have

1) $A(S,T) \simeq \text{Coker}(T(k)/Br \to \prod_{v \in S} T(k_v)/Br)$.

2) $A(T) \simeq \text{Coker}(T(k)/Br \to \prod_{v} T(k_v)/Br)$.
Proof. Notice that
\[
\text{Coker}(T(k)/Br \to T_S/\text{BT}_S) = T_S/T(k)\text{BT}_S \\
= T_S/\text{Cl}(T(k)),
\]

since \(\text{BT}_S\) contains \(R\text{BT}_S\) so is also an open subgroup of \(T_S\). So 1) and 2) follow by noticing that for almost all \(v\)

\[T(k_v)/Br = 1,\]

by [CTS1], p. 205. 

We have the following close analog of an exact sequence in [CTS1] (see Theorem 1.2 above) in the case of \(R\)-equivalence of algebraic tori over number fields.

2.5. Proposition. With above notation we have the following exact sequences

1) \[1 \to T(k)/Br \to \prod_v T(k_v)/Br \to A(T) \to 1.\]

2) \[0 \to \ker[\text{H}^1(k, S) \to \text{Hom}(\text{H}^1(k, S), \text{Br}(k, K)) \to \text{III}(S) \to \alpha \to 1].\]

Here the map \(\beta: \text{III}(S) \to T(k)/R \to T(k)/Br\) is the composite map, which is a trivial homomorphism, and \(\alpha\) is an isomorphism.

Proof. 1) follows directly from Propositions 1.4 and 2.4. To prove 2), it suffices to prove the exactness of the sequence at the first three terms.

a) \(\ker \alpha = 0\). Consider the following commutative diagram

\[
\begin{array}{ccc}
\text{H}^1(k, S) & \to & \text{Hom}(\text{H}^1(k, \hat{S}), \text{Br}(k)) \\
\downarrow q & & \downarrow \zeta \\
\prod_v \text{H}^1(k_v, S) & \cong & \prod_v \text{Hom}(\text{H}^1(k_v, \hat{S}), \text{Br}(k_v))
\end{array}
\]

where \(\zeta\) is the natural morphism of restriction. Here we have \(p\) an isomorphism since by Tate - Nakayama duality we have

\[
\text{H}^1(k_v, S) \cong \text{H}^1(k_v, \hat{S})^- \\
\cong \text{Hom}(\text{H}^1(k_v, \hat{S}), \mathbb{Q}/\mathbb{Z}) \\
= \text{Hom}(\text{H}^1(k_v, \hat{S}), \text{Br}(k_v))
\]
Therefore

\[ \text{Ker} (\omega) \subset \text{Ker} (q) = \text{III}(S). \]

b) \( \text{Im} (\alpha) = \text{Ker} (\beta) \). Let \( x \in \text{III}(S) \). Then

\[ x \in \text{Im} (\alpha) \iff x \in \text{Ker} (\omega) \]

\[ \iff (x) \cup (y) = 0 \in \text{Br}(k, K), \forall y \in H^1(k, \hat{S}). \]

But

\[ \text{Ker} (\beta) = \{ x \in H^1(k, S) = T(k)/R | \beta(x) \in BT \} \]

\[ = BT/RT \]

\[ = \{ x \in H^1(k, S) : (x) \cup (y) = 0, \forall y \in H^1(k, \hat{S}) \} \]

hence \( \text{Ker} (\beta) = \text{Im} (\alpha) \).

c) \( \text{Im} (\beta) = \text{Ker} (\gamma_T) \). Consider the following commutative diagram

\[
\begin{array}{ccc}
\text{III}(S) & \xrightarrow{i} & T(k)/R \xrightarrow{\rho_T} \prod_v T(k_v)/R \\
\beta \downarrow & & \downarrow p \quad \simeq \quad \downarrow q \\
T(k)/Br & \xrightarrow{\gamma_T} & \prod_v T(k_v)/Br \\
\end{array}
\]

If \( x \in \text{Im} (\beta) \), then \( x = p(i(s)), s \in \text{III}(S) \). Since \( \rho_T \circ i = 0 \), we have

\[ q(\rho_T((i(s)))) = \gamma_T(p(i(s))) \]

\[ = \gamma_T(x) \]

\[ = 0, \]

i.e. \( x \in \text{Ker} (\gamma_T) \).

Conversely, if \( x \in \text{Ker} (\gamma_T) \), \( x = p(t) \), since \( p \) is surjective. Then \( 0 = \gamma_T(p(t)) = q(\rho_T(t)) \), so \( \rho_T(t) = 0 \), since \( q \) is an isomorphism (see Theorem 1.1 (4)). Hence \( t \in \text{Ker} (\rho_T) = \text{Im} (i) \) since the upper row is exact by Theorem 1.2.

Since \( \gamma_T \) is injective (see Proposition 1.4), \( \beta \) is trivial and \( \alpha \) is an isomorphism. Hence 2) is proved. 

As a consequence of the above proposition, we have the following

2.6. Proposition. We have the following exact sequence connecting the two groups of rational equivalence classes

\[ 0 \to \text{III}(S) \to T(k)/R \to T(k)/Br \to 0. \]
In particular, the order of $T(k)/Br$ is equal to $n_T$.

Proof. We have (with above notation) the following commutative diagram.

$$
\begin{array}{ccc}
1 & \to & \Pi(S) \\
\downarrow & & \downarrow \\
\Pi \to T(k)/R & \xrightarrow{\rho_T} & \Pi \to \Lambda(T) \\
\downarrow & & \downarrow \\
1 & \to & T(k)/Br \xrightarrow{\gamma_T} \Pi \to \Lambda(T) \\
\end{array}
$$

In this diagram, $\lambda_T$ is induced from $\lambda_T'$ and is just the quotient map. Indeed, we have the vertical isomorphism $\approx$ due to Theorem 1.1, 4), and it is clear that

$$
\lambda_T'(<\rho_T>) \subset Ker(\gamma_T).
$$

Therefore it follows that

$$
T(k)/R/Ker(\rho_T) \approx T(k)/Br
$$

and we are done. •

3 Some reductive analogs

In this section we prove some analogs of results in Section 2 for the case of connected reductive groups $G$ over number fields $k$. First we recall the following

3.1. Proposition. [T3] Let $G$ be a connected linear algebraic groups defined over a number field $k$, $S$ a finite set of valuations of $k$. For each $v \in S$ denote by $RG_v$ the subgroup of $G(k_v)$ consisting of elements $R$-equivalent to $1$, and by $RG_S$ the direct product of $RG_v$ for $v \in S$. Then $RG_S \subset Cl(G(k))$.

3.2. Proposition. [T3] Let $G$, $k$, $S$ be as above. Then we have the following canonical isomorphisms

1) $A(S,G) \simeq \text{Coker}(G(k)/R \to \Pi \in S G(k_v)/R)$.

2) $A(G) \simeq \text{Coker}(G(k)/R \to \Pi \in \in G(k_v)/R)$.

We have the following analog in the case of Brauer equivalence relation.

3.3. Theorem. Let $G$, $k$, $S$ be as above. Let $BG_v$ be the subgroup of $G(k_v)$ consisting of elements which are $Br$-equivalent to $1$, and $BG_S$ be the direct product of $BG_v$. Then

$$
BG_S \subset Cl(G(k)).
$$

Proof. We follow the proof given in [T3]. We know by [CTS1] that for a torus $T$ over $k$, $T(k_v)/R = T(k_v)/Br$, hence it follows that the Theorem holds for tori. Now we may assume that $G$ is not a torus. Further we just follow the proof given in [T3], where $R$ is replaced by $Br$ everywhere. •
3.4. **Theorem.** With notation as above we have the following canonical isomorphisms

1) \( A(S,G) \cong \text{Coker}(G(k)/Br) \rightarrow \prod_{e \in S} G(k_e)/Br \).

2) \( A(G) \cong \text{Coker}(G(k)/Br) \rightarrow \prod_{e} G(k_e)/Br \).

**Proof.** The same as in 2.4, by making use of Proposition 3.3. \( \blacksquare \)

We need the following technical result.

3.5. **Proposition.** [O] Let \( G \) be a connected reductive group defined over a field \( K \). There exists a connected reductive \( K \)-group \( H \) with simply connected semisimple part and an induced \( K \)-torus \( Z \) such that the following sequence is exact.

\[
1 \rightarrow Z \rightarrow H \rightarrow G \rightarrow 1.
\]

(Such \( H \) is called also a \( z \)-extension of \( G \) over \( K \).)

The relation between the groups of Brauer equivalence classes of \( G \) and \( H \) is shown in the following statement, where we restrict ourselves only to the case of a field \( k \) of characteristic 0.

3.6. **Proposition.** Let \( k \) be a field of characteristic 0.

1) If \( H \) is a \( z \)-extension of a connected reductive \( k \)-group \( G \) then there is a canonical isomorphism \( H(k)/Br \cong G(k)/Br \).

2) If for any torus over \( k \), \( T(k)/Br \) is finite, then for any connected linear algebraic group \( G \), the group of Brauer equivalence classes \( G(k)/Br \) is finite. In particular, it is so if \( k \) is of finite type over \( \mathbb{Q} \) or a local field of characteristic 0.

First we need the following (perhaps well-known for experts, but I do not know of any reference).

3.6.1. **Lemma.** Let \( X \rightarrow Y \) be a morphism of smooth varieties all defined over a field \( k \) of characteristic 0, \( \pi^* : Br_1 Y \rightarrow Br_1 X \) is an induced homomorphism. The following diagram is commutative

\[
\begin{array}{ccc}
X(k) \times Br_1 X & \xrightarrow{\pi} & Br_k \\
\downarrow \pi & & \nearrow \pi^* \\
Y(k) \times Br_1 Y & \xrightarrow{} & Br_k
\end{array}
\]

i.e. the pairing \( X(k) \times Br_1 X \rightarrow Br_k \)

is functorial in \( X \), i.e., given \( x \in X(k), b \in Br'X \), where \( Br' \) denotes the usual Brauer group of \( X \), which we identify with the cohomological one (see [S, Sec. 6]), then

\( \pi^*(b)(x) = b(\pi(x)) \).
Proof. Let \( x \in X(k), y = \pi(x) \in Y(k) \). Denote by \( \mathcal{O}_{X,x}, \mathcal{O}_{Y,y} \) the local ring of \( X \) (resp. \( Y \)) at \( x \) (resp. \( y \)). One has the following commutative diagram

\[
\begin{array}{ccc}
\mathcal{O}_{Y,y} & \xrightarrow{\pi^*} & \mathcal{O}_{X,x} \\
\downarrow{\pi''} & & \downarrow{\pi'} \\
& k & \\
\end{array}
\]

We denote

\[
\mathfrak{g} := \text{Gal}(k/k), (f_s, t) \in Z^2(\mathfrak{g}, \mathcal{O}_{X,x}^*)
\]

the absolute Galois group of \( k \) and a 2-cocycle representative of \( b \in \text{Br}_1 Y \), respectively. Then by [S], Lemme 6.2, \( b(y) \) is the class \([f_s, t(y)] \) in \( \text{Br} k \). This 2-cocycle gives rise to a 2-cocycle \([f_s, t] \) in \( Z^2(\mathfrak{g}, \mathcal{O}_{X,x}^*) \) which is nothing else than the \( a = \pi^*(b) \). Then \( a(x) \) is just the class of

\[
[(f_s, t)(\pi(x))] = [(f_s, t(\pi(x))] = [f_s, t(y)] \in \text{Br} k.
\]

Therefore

\[
\pi^*(b)(x) = b(\pi(x))
\]

as required. The proof of the lemma is complete. •

Proof of Proposition 3.6.

1) Let \( \pi : H \to G \) be the projection. It induces an epimorphism

\[
H(k) \to G(k)
\]

hence also an epimorphism

\[
\pi' : H(k)/\text{Br} \to G(k)/\text{Br}.
\]

We show that \( \pi' \) is injective. Since \( Z \) is an induced \( k \)-torus, it is well-known that there is a \( k \)-section

\[
i : G \to H, \pi \circ i = id_G.
\]

By Lemma 3.6.1, we have the following commutative diagram

\[
\begin{array}{ccc}
H(k) & \times & \text{Br}_1 H \\
\downarrow{i^*} & & \downarrow{\pi^*} \\
G(k) & \times & \text{Br}_1 G \\
\end{array}
\]

Since \( \pi \circ i = id_G \), it follows that we have

\[
(\pi \circ i)^* = i^* \circ \pi^* = id_{\text{Br}_1 G}.
\]
Therefore $i^*$ is surjective. Now assume that $h \in H(k)$ such that

$$\pi(h) \cup g = 0, \forall g \in \text{Br}_1 G.$$ 

Let $g = \pi(h)$. Then by Lemma 3.6.1 we have

$$i(g) \cup \hat{h} = \pi(i(g)) \cup i^*(\hat{h}) = 0, \forall \hat{h} \in \text{Br}_1 H$$

since $i^*$ is surjective. This implies that $i(g) \in BH(k)$. Since

$$\pi(h) = g = \pi(i(g)),$$

one deduces that $h \equiv i(g) (\text{mod.} Z(k))$. Since $Z$ is $k$-rational, it follows that $Z(k) \subset BH(k)$, hence we are done. 

(In the case 2) we can prove our statement by using 2). Indeed, we have a birational equivalence

$$H \simeq G \times Z,$$

and this induces a bijection (see [CTS1], Section 7)

$$H(k)/\text{Br} \simeq (G(k)/\text{Br} \times Z(k)/\text{Br}) \simeq G(k)/\text{Br}.$$

since $H(k)/\text{Br}$ is finite by 2), $\pi'$ is injective, hence also an isomorphism.)

2) One reduces easily to the case where $G$ is a connected reductive group. Take a $\mathbb{Z}$-extension $H$ of $G$ as above. We will show that $H(k)/\text{Br}$ is finite. Let $H = SG$, where $\hat{G}$ is a simply connected semisimple group and $S$ a central torus of $H$. We have the following exact sequence of $k$-groups

$$1 \rightarrow \hat{G} \rightarrow H \rightarrow T \rightarrow 1$$

where $T$ is a torus. One derives from [S], Corollaire 6.11, the following exact sequence

$$(3) \quad \text{Pic}T \rightarrow \text{Pic}H \rightarrow \text{Pic}\hat{G} \rightarrow \text{Br}_a T \rightarrow \text{Br}_a H \rightarrow \text{Br}_a \hat{G}.$$ 

By [S], Lemme 6.9 (iv), we know that $\text{Pic}(\hat{G})$ and $\text{Br}_a \hat{G}$ are trivial, so we obtain from (3) the following isomorphism

$$\text{Br}_a T \simeq \text{Br}_a H.$$ 

Since $\text{Br}_1 T \hookrightarrow \text{Br}_1 H$ ([S], Corollaire 6.11), it follows that

$$(4) \quad \text{Br}_1 T = \text{Br}_1 H.$$ 

(One can use also Prop. 6.10 (loc.cit) to get this equality.)

We continue the proof of the proposition. By the lemma we have the following commutative diagram

$$
\begin{array}{ccc}
H(k) \times \text{Br}_1 H & \xrightarrow{\pi} & \text{Br}_1 k \\
\| & \| & \| \\
\pi & \xrightarrow{i^*} & \text{Br}_1 k \\
\end{array}
$$

11
Assume that $h \in H(k)$ with $\pi(h) \in BT(k)$. Then we have
\[ \pi^*(a)(h) = a(\pi(h)) = 0 \]
for all $a \in Br_1 T$, and from (4) we derive that $h \in BH(k)$. Thus we have proved that $\pi$ induces an injective homomorphism
\[ H(k)/Br \rightarrow T(k)/Br. \]
Therefore, the first statement of 2) follows. Now if $k$ is a field of finite type over $\mathbb{Q}$ or a local field, then by [CTS1], Corollaire 1, p. 217, $T(k)/Br$ is finite. Therefore $H(k)/Br$, and a fortiori $G(k)/Br$, is also finite. 

3.6.2. Remark. It was proved in [G2] that if $k$ is a number field then $G(k)/R$ is finite, hence $G(k)/Br$ is also. Here we did not use the result of Gille in the proof.

3.6.3. Corollary. (of the proof.) Let
\[ 1 \rightarrow A \rightarrow B \xrightarrow{\pi} C \rightarrow 1 \]
be an an exact sequence of connected linear algebraic groups defined over a field $k$ of characteristic 0, where $A$ has trivial Picard and arithmetic Brauer groups. Then the canonical homomorphism $B(k)/Br \rightarrow C(k)/Br$ is injective. In particular, if $k$ is a non-archimedean local field and $A$ is simply connected then we have
\[ B(k)/Br \simeq C(k)/Br. \]

Proof. Only the second part requires a proof, which follows from Kneser’s Theorem on the triviality of $H^1$ of simply connected groups. Indeed, from the exact sequence of cohomology we see that $\pi$ is surjective on $k$-points, thus gives a surjective map
\[ B(k)/Br \rightarrow C(k)/Br, \]
which is also bijective. 

Before we give the formulation of one of main results, we recall some definition and notation of Section 2.4.

For a reductive group $H$, we call torus quotient of $H$ the factor group of $H$ by its semisimple part. Given a torus $T$ defined over a number field $k$ we denote by $V(T)$ a smooth compactification of $T$ over $k$ and by $S$ the Neron - Severi $k$-torus of $T$, which is by definition the dual to the Picard group of $V(T)$, $\hat{S} \simeq \text{Pic}(V(T))$. The first Galois cohomology of $S$, which depends on $T$, does not depend on the chosen smooth compactification, and so is the Shafarevich - Tate group $\text{III}(S)$. Thus by Proposition 2.5 we have the following exact sequence
\[ 1 \rightarrow T(k)/Br \xrightarrow{\gamma_T} \prod_v T(k_v)/Br \rightarrow \Lambda(T) \rightarrow 1. \]
A generalization of this sequence is given by the following theorem.

3.7. Theorem. Let $G$ be a connected linear algebraic group defined over a number field $k$. Then we have the following commutative diagram, all rows of which are exact sequences

$$1 \to G(k)/Br \overset{\gamma_G}{\to} \prod_v G(k_v)/Br \to \Lambda(G) \to 1$$

(5) $$1 \to T(k)/Br \overset{\gamma_T}{\to} \prod_v T(k_v)/Br \to \Lambda(T) \to 1$$

where $T$ is the torus quotient of any z-extension $H$ of the reductive part of $G$, and all vertical maps are (functorial) isomorphisms (including local components ones). In particular, the image of $G(k)/Br$ via $\gamma_G$ is a finite group of order $n_T$.

Proof. The exactness of the above sequences follows from Propositions 2.5 and 3.4. By Proposition 3.6 (1), there is a canonical (functorial) isomorphism $G(K)/Br \simeq H(K)/Br$ for any extension field $K/k$. Therefore it suffices to prove theorem 3.7 for $H$.

We have the following commutative diagram (see the notation as above).

$$1 \to H(k)/Br \overset{\gamma_H}{\to} \prod_v H(k_v)/Br \to \Lambda(H) \to 1$$

$$1 \to T(k)/Br \overset{\gamma_T}{\to} \prod_v T(k_v)/Br \to \Lambda(T) \to 1$$

where $p,q,r$ are natural maps, induced from the projection $pr : H \to T$. By Corollary 3.6.3, $p$ is injective and $q$ and all its local components of are isomorphisms.

Next we need the following

3.8. Lemma. With above notation, we have a canonical isomorphism of finite groups

$$\Lambda(H) \simeq \Lambda(T).$$

Proof. Consider the following commutative diagram
\[
\begin{align*}
H(k) & \rightarrow T(k) \xrightarrow{\delta} H^1(k, \hat{G}) \\
H_S & \rightarrow T_S \xrightarrow{\delta} \prod_{v \in S} H^1(k_v, \hat{G})
\end{align*}
\]

where we take \( S \) a sufficiently large finite set of valuations of \( k \) containing all the archimedean ones, such that

\[
\text{A}(H) = H_S/\text{Cl}(H(k)),
\]
\[
\text{A}(T) = T_S/\text{Cl}(T(k)).
\]

It is a general and well-known fact (see, e.g. [T3] for a discussion with references) that for any linear algebraic group \( P \) over \( k \), \( \text{Cl}(P(k)) \) is an open subgroup of \( P_S \). Also any connected linear algebraic group satisfies weak approximation with respect to the set \( \infty \) of archimedean valuations.

Let \( t_S \in T_S \), \( t_S = (t_\infty, t_f) \), where \( t_\infty \) (resp. \( t_f \)) is the \( \infty \)-(resp. finite) component of \( t_S \). Let \( t_n \in T(k) \) such that \( \lim_n t_n = t_\infty \). Then for \( n \) large enough, the element \( (t_n, t_n) \) is very close to \( (t_\infty, t_n) \) in the \( S \)-adic (product) topology, i.e., \( (t^{-1}_n t_\infty, 1) \) is approaching \( 1 \) in \( T_S \). Since \( \text{Cl}(T(k)) \) is open, there is \( N \) such that if \( n > N \), then

\[
(t^{-1}_n t_\infty, 1) \in \text{Cl}(T(k)),
\]

i.e.,

\[
(t_\infty, t_n) \in \text{Cl}(T(k)).
\]

Since \( t_S = (t_\infty, t_n)(1, t^{-1}_n t_f) \), it follows that

(7) each coset of \( T_S/\text{Cl}(T(k)) \) has a representative from \( 1 \times T_{S-\infty} \).

Since \( H^1(k_v, \hat{G}) \) is trivial for \( v \) non-archimedean by Kneser’s Theorem [Kn2], it follows that

\[
T_{S-\infty} = \pi(H_{S-\infty})
\]

and from (7) we derive that the natural homomorphism

\[
\text{A}(S, H) = \text{A}(H) \xrightarrow{\pi} \text{A}(T) = \text{A}(S, T)
\]

is surjective.

Next we show that \( \pi \) is injective. Let \( h_S \in H_S \) such that \( \pi(h_S) \in \text{Cl}(T(k)) \). Then

\[
\pi(h_S) = \lim_n t_n, t_n \in T(k),
\]

hence from the commutative diagram (6) we derive

\[
1 = \delta \pi(h_S) = \lim_n \delta(t_n).
\]

(Notice that here one endows \( H^1(k, \hat{G}), H^1(k_v, \hat{G}) \) with discrete topologies, and one checks readily that all maps in the diagram (6) are continuous.) Since \( \prod_{v \in S} H^1(k_v, \hat{G}) \) is finite, there is \( N_1 \) such that if \( n > N_1 \) then \( \delta(t_n) = 1 \). Let \( h_n \in H(k) \), such that \( \pi(h_n) = t_n \). Then \( \lim_n \pi(h^{-1}_n h_S) = 1 \) since \( \text{Cl}(H(k)) \) is open in \( H_S \), and we deduce that

\[
h^{-1}_n h_S \in \pi(\hat{G}_S)\text{Cl}(H(k)).
\]
for \( n \) large.

Since \( \bar{G} \) has weak approximation property (Kneser - Harder Theorem, see e.g. [S]), we have \( \pi(\bar{G}_S) \subset Cl(H(k)) \), hence \( h_S \in Cl(H(k)) \) as required. Hence \( A(H) \cong A(T) \) and the lemma is proved. \( \blacksquare \)

Continuation of the proof of Theorem 3.7. In the diagram (5) we know that \( r \) is an isomorphism by Lemma 3.8, \( q \) is an isomorphism and \( p \) is injective by Corollary 3.6.3. It follows that \( \gamma_H \) and \( \gamma_T \) have isomorphic images. Therefore

\[
H(k)/Br \cong T(k)/Br.
\]

In particular, the order of \( G(k)/Br \) is equal to \( n_T \). \( \blacksquare \)

4 Some variations and applications

In this section we consider some applications of results obtained in previous sections, and also of those obtained in [T3], where we have not given any. We keep our notation as above. First we derive from Proposition 3.3 the following.

4.1. Proposition. Let \( S \) be a finite set of valuations of \( k \) and \( G \) a connected linear algebraic group over \( k \). If \( G \) has trivial group \( G(k_v)/Br \) of Brauer equivalence classes for all \( v \in S \), then \( G \) has weak approximation property in \( S \).

4.2. Conversely, assume that \( G \) does not have weak approximation property with respect to some \( v \) (necessarily non-archimedean). Then \( G \) has non-trivial group \( G(k_v)/Br \) by Proposition 3.3, hence also non-trivial group \( G(k_v)/R \). Therefore \( G \) is not stably rational over \( k_v \), and a fortiori, over \( k \).

4.2.1. Remarks. 1) Usually the counter-examples to weak approximation over number fields \( k \) serve as examples of linear algebraic \( k \)-groups, which are non-stably rational over \( k \) only. The statement 4.2 above shows that these examples are, in fact, stronger in the sense that they serve also as examples of non-stably rational groups over bigger fields (say \( k_v \)). The reader may consult a variety of examples in [S].

2) We mention the following one of the main results due to Sansuc [S], Corollaire 9.7, in the case \( G \) has no simple component of type \( E_8 \) (and also goes back to Voskresenski in torus case). Since the Hasse principle is also holds for \( E_8 \) by Chernousov, the following holds.

4.2.2. Theorem. ([S], Cf.Thm 9.5.) If \( G \) is a connected linear algebraic group, defined over a number field \( k \), then we have the following exact sequence

\[
0 \to A(G) \to H^1(k, PicV(G)^\sim) \to II(G) \to 0.
\]

In particular, if \( H^1(k, PicV(G)) \) is trivial, then \( G \) has weak approximation over \( k \) and satisfies Hasse principle for \( H^1 \).
One may ask, by comparing with 4.2, if we have a similar situation if we assume that 
\[ A(G) = 0, \quad \text{III}(G) = 0. \]
However it is not true as the following classical example shows.

**Example.** Let \( a, b \in \mathbb{Z} \) (the integers), and let \( K = \mathbb{Q}(\sqrt{a}, \sqrt{b}) \), be a bi-quadratic extension of \( \mathbb{Q} \), where \( \mathbb{Q} \) denotes the rational numbers. Denote by \( T(a, b) = R_{K/\mathbb{Q}}^{(1)}(G_m) \). Then (see [CTS1], Prop. 7, or [V1], p.157) we have

\[ H^1(\mathbb{Q}, \text{Pic}^0(T)) = \mathbb{Z}/2\mathbb{Z}. \]

If we choose \( a, b \) such that all the decomposition groups for \( K \) are cyclic then it is known (by Serre) that \( A(T) = 0 \). For example, \( a = 5, b = 29 \) satisfy this condition. However, one checks that \( T(5, 29) \) is rational over all completions of \( \mathbb{Q} \), but \( \text{III}(T(a, b)) = \mathbb{Z}/2\mathbb{Z} \).

In the next result we consider some applications to weak approximation in semisimple groups defined over number fields \( k \).

**4.3. Theorem.** Let \( G \) be a semisimple \( k \)-group such that \( G \) is of inner type over \( k_v \) for all \( v \in S \). Then \( G \) has weak approximation over \( k \) with respect to \( S \).

The theorem follows from the following

**4.4. Proposition.** If as a group over \( k_v \), \( G \) is an inner type then \( G \) has trivial group \( G(k_v)/R \).

First we need the following result due to Gille [G1], Prop. 2.3.

**4.4.1. Proposition.** [G1] Let

\[ 1 \to F \to G_1 \xrightarrow{\lambda} G_2 \to 1 \]

be an isogeny of connected reductive groups, all defined over a field \( k \) of characteristic 0 and \( C_\lambda(k) = G_2(k)/\lambda(G_1(k)) \). Then the following sequence of groups is exact

\[ G_1(k)/R \to G_2(k)/R \to H^1(k, F)/R, \]

where the image of \( \delta_R \) is the factor group \( C_\lambda(k)/R \) and the \( R \)-equivalence relation on \( C_\lambda(k) \) (\( \to H^1(k, F) \)) is induced from that on \( H^1(k, F) \).

**Note.** In the case that \( H^1(k, G_1) \) is trivial, we identify \( C_\lambda(k) \) with \( H^1(k, F) \) and also write the above exact sequence in the form

\[ G_1(k)/R \to G_2(k)/R \to H^1(k, F)/R \to 0. \]

**Proof of Proposition 4.4.** We distinguish two cases.

1) \( v \in \infty \). It is well-known that any connected linear algebraic group \( G \) over \( \mathbb{R} \) has trivial group of \( \mathbb{R} \)-equivalences. Here is a short indication of proof. One reduces easily
to proving that any semisimple element \( s \in G(\mathbb{R}) \) is \( \mathbb{R} \)-equivalent to 1. But this follows from the fact that \( s \) is belong to some torus defined over \( \mathbb{R} \), and any such torus is rational over \( \mathbb{R} \).

2) \( v \) is non-archimedean. Let \( G_s \) be a \( k_v \)-split form of \( G \), which is obtained from \( G \) by an inner twist. Denote by \( F \) the fundamental group of \( G \), which is the same for \( G_s \). One can check that simply connected groups have trivial groups of \( \mathbb{R} \)-equivalence classes, we have by Proposition 4.4.1 ([G1], Prop.2.3) the following exact sequences

\[
0 \to G_s(k_v)/R \to H^1(k_v, F)/R \to 0,
\]

\[
0 \to G(k_v)/R \to H^1(k_v, F)/R \to 0.
\]

Here we identify \( H^1(k_v, F) \) with the factor group \( G_s(k_v)/\pi(\tilde{G}_s(k_v)) \) (resp. \( G(k_v)/\pi(\tilde{G}(k_v)) \)), where \( \pi : \tilde{G}_s \to G_s \) (resp. \( \pi : \tilde{G} \to G \)) is the simply connected covering of \( G_s \) (resp. \( G \)) and take the factor group modulo the rational relation induced on \( H^1(k_v, F) \). We have also used the Kneser Theorem on the vanishing of \( H^1(k_v, G_s) \) and \( H^1(k_v, \tilde{G}) \). Since \( G_s \) is rational over \( k_v \), the second group in (8) is trivial, therefore by (9) the group \( G(k_v)/R \) is also trivial.

Thus by this proposition, and by Proposition 3.3, \( G \) has weak approximation with respect to \( S \) and Theorem 4.3 is proved.

4.5. Remarks. 1) One cannot simply drop the inner type assumption, since there are examples of semisimple quasi-split groups over number fields which do not have weak approximation property. First examples of such groups were given by Serre (see [Kn1] and [S] for more information).

2) In the case of number field, this result also extends the previously known (but more general) result by Harder, namely we have

4.5.1. Corollary. (Harder [H1]) If a semisimple group \( G \) defined over a number field \( k \) is split over \( k_v \) for all \( v \in S \), then \( G \) has weak approximation in \( S \).

Now we apply our results to give new proofs (and also discuss some extension) of some results due to Harder and Sansuc. In the following theorem, the first result is due to Harder ([H2], Satz 2.2.3) and the second is due to Sansuc ([S], Cor. 5.4). The third, in the case that the given group is semisimple and split over a metacyclic extension of the given number field, is also due to Sansuc.

Let \( \pi : \tilde{G} \to G \) be a central isogeny of semisimple groups, all defined over a field \( k \), where \( \tilde{G} \) is simply connected covering of \( G \). \( \pi \) is called a normal isogeny (after Harder [H2]) if \( \mu := \text{Ker } \pi \) can be embedded into an induced \( k \)-torus \( M \), such that \( M/\mu \) is also an induced \( k \)-torus. One can show, for example, that adjoint groups \( G \) have normal isogenies.

4.6. Theorem. The following groups have weak approximation property over number fields.

1) ([H2]) Semisimple groups which are images of normal isogenies;
2) ([S]) Absolutely almost simple groups;
3) Inner forms of connected reductive groups which are split over a metacyclic extension of \( k_v \) for all non-archimedean \( v \). Moreover, two connected reductive groups, which are inner form of each other have the same group of \( R \)-equivalence classes over local non-archimedean fields.

**Proof.** 1) Let \( \pi : \tilde{G} \to G \) be a normal isogeny defined over a number field \( k \), and \( S \) any finite set of valuations of \( k \). We show that for all \( v \in (S - \infty) \), \( G(k_v)/R \) is trivial. Indeed, let \( \mu = \text{Ker } \pi \), \( M \) be an induced \( k \)-torus, such that \( M/\mu \) is also an induced \( k \)-torus. As above (see Proposition 4.4.1), we have the following exact sequences

\[
\tilde{G}(k_v) \to G(k_v) \to H^1(k_v, \mu) \to 0,
\]

\[
\tilde{G}(k_v)/R \to G(k_v)/R \to H^1(k_v, \mu)/R \to 0.
\]

One can show easily that \( \tilde{G}(k_v)/R \) is trivial. (Here is a short argument. One reduces to almost simple case. If \( G \) is isotropic, then it is well-known that \( G(k_v) \) has no nontrivial normal subgroup, i.e. \( G(k_v) = R\tilde{G}(k_v) \), since \( R\tilde{G}(K_v) \) is a normal Zariski dense subgroup of \( G(k_v) \). Otherwise, \( G \) is of inner type \( A_n \) by a result of Kneser, and in this case the result is well-known.)

Hence we have an isomorphism

\[
G(k_v)/R \simeq H^1(k_v, \mu)/R.
\]

By considering similar exact sequences

\[
1 \to \mu \to M \to M' \to 1,
\]

\[
M(k_v)/R \to M'(k_v)/R \to H^1(k_v, \mu)/R \to 0,
\]

where \( M, M' \) are induced tori and using that \( M, M' \) are rational, we get that \( H^1(k_v, \mu)/R \) is trivial, and so is \( G(k_v)/R \). Therefore \( G \) has weak approximation over \( k \) by Proposition 3.3.

In the case \( G \) is an adjoint group, we can use a direct argument as follows. We may use the following (easy to show) fact : adjoint groups over local \( p \)-adic fields are rational. This was mentioned in the preprints [T1-2]. One may also argue as follows. Let \( G_q \) be a quasi-split inner form of \( G \) defined over \( k \), and \( F \) be its fundamental group. By using the same argument we have in the proof of Proposition 4.4, one concludes that \( G(k_v)/R \) is trivial for all \( v \). Therefore by Proposition 3.3, \( G \) has weak approximation over \( k \).

2) Let \( G \) be an (absolutely) almost simple \( k \)-group. We want to show that for any finite set \( S \) of valuations of \( k \), \( G(k_v)/R \) is trivial for all \( v \in S \). [In fact, one can show a much stronger result in this case : see the paper by Chernousov and Platonov : The rationality problem of simple algebraic groups, C. R. Acad. Sci. Paris 322 (1996), 245 - 250, which have many results overlapped with results of [T2], where also other results were mentioned]

If \( v \) is a non-archimedean valuation, then \( G \) is rational over \( k_v \) if \( G \) is not of type \( A_n \).]

Here we can use the following simple argument as follows. Let \( G_q \) be an almost simple quasi-split inner form of \( G \). As in the case 1) we are reduced to proving the statement
for quasi-split groups. Assuming that \( G \) is not split, then \( G \) is of type \( A_n, D_n, E_6, \) or \( 3.6D_4 \). Let \( T \) be a maximal \( k \)-torus of \( G \) containing a maximal \( k_v \)-split torus of \( G \). If \( G \) is not of trialitarian type, then we know by Tits [Ti] that \( T \) is split by a quadratic extension of \( k_v \). The structure of tori split over a quadratic extensions are well-known: they are direct product of groups of type \( G_m, R_{K/k_v}(G_m), \) or \( R_{K/k_v}^{(1)}(G_m) \) where \( K/k_v \) is a quadratic extension of \( k_v \). In particular they are rational over \( k_v \), and so is \( G \) by Bruhat decomposition. In the trialitarian case one proves in the same way that maximal tori containing a maximal split torus are rational. Thus by Proposition 3.3, \( G \) has weak approximation in \( S \) for any \( S \), thus also over \( k \).

3) a) First we show that if \( G \) is a connected reductive group which is split over metacyclic extension \( l_v \) of \( k_v \) for each non-archimedean \( v \in S \) then \( G \) has weak approximation with respect to any such finite set \( S \) of valuations. In fact we prove the following stronger result.

4.7. Proposition. If \( G \) is a connected reductive group defined over non-archimedean \( k_v \) and split over a metacyclic extension \( l_v \) of \( k_v \) then \( G(k_v)/Br \) is trivial.

Proof. It can be shown that there exists a maximal \( k_v \)-torus \( T \) of \( G \) which is split over \( l_v \). (And in the case of number field \( k \), one can show that there exists a maximal \( k \)-torus \( T \) such that \( T \) is \( l_v \)-split for \( v \in S \).) Let \( H \) be a \( z \)-extension of \( G \),\n
\[
1 \to Z \to H \xrightarrow{\pi} G \to 1.
\]

Let \( T_H \) be the maximal \( k_v \)-torus of \( G \) such that \( T_H \) is mapped onto \( T \) via \( \pi \). Let \( \hat{G} \) be the simply connected covering of the semisimple part \( G' \) of \( G \), and let \( \hat{T} \) be the maximal \( k_v \)-torus of \( \hat{G} \) which is mapped into \( T \) via the composite map

\[
\hat{G} \to G' \to G.
\]

We have the following exact sequences of tori.

(10) \[
1 \to Z \to T_H \to T \to 1,
\]

(11) \[
1 \to \hat{T} \to T_H \to T_0 \to 1.
\]

It is clear from (10), (11) that \( \hat{T} \) is also split over \( l_v \) for all \( v \in S \). By [CTS1], Corollaire 3, p. 200, we have

\[
\hat{T}(k_v)/R = T(k_v)/R = \{1\}, \forall v \in S,
\]

therefore \( T_H(k_v)/R = \{1\}, \forall v \in S \).

Now we consider a maximal \( k_v \)-split torus \( T_1 \) of \( G \). Then

\[
Z_G(T_1) = T'H',
\]

where \( T' \) is the connected center and \( H' \) is a semisimple \( k_v \)-group, anisotropic over \( k_v \). If \( H' \) is trivial, i.e., \( G \) is quasi-split over \( k_v \), then the torus \( T' \) is split over metacyclic extension \( l_v \), so has trivial group of \( R \)-equivalence classes by [CTS1], Corollaire 3, p. 200, and so is \( G \), since \( G \) and \( Z_G(T_1) \) are birationally equivalent over \( k_v \) (using Bruhat decomposition). Therefore \( G(k_v)/Br \) is trivial also. One may therefore assume that \( H' \)
is non-trivial, and by replacing $G$ by $Z_G(T_1)$, one may assume that $G$ has semisimple part $G'$ anisotropic over $k_v$.

By using a consequence of the Kneser’s Vanishing Theorem [Kn2], we see that $G'$ is necessarily a product of almost simple $k_v$-factors of type $1A$, which may be taken to be absolutely almost simple. So we have

$$\tilde{G} = H_1 \times \cdots \times H_t,$$

where $H_i$ is simply connected of type $1A_{m_i-1}$ for all $i$.

We now recall the construction of the $z$-extension $H$ of $G$. Let $G = G'P$, where $P$ is a $k_v$-torus. Let $F = G' \cap P$, $F_1 = \{(f, f) : f \in F\}$ and we have the following exact sequence

$$1 \to F_1 \to G' \times P \to G \to 1.$$  

By taking the composite of two isogenies

$$\tilde{G} \times P \to G' \times P \to G,$$

we have an isogeny

$$1 \to F' \to \tilde{G} \times P \to G \to 1.$$  

Thus one sees that since $\tilde{G}$ is of inner type (in fact the product of groups SL), the group $F'$ can be embedded into a split torus $Z$ defined over $k_v$. Then we take

$$F'_1 := \{(f, f) : f \in F'\}$$

and take

$$H = (Z \times (\tilde{G} \times P))/F'_1.$$  

From the very construction it follows from (10), (11) that $T_H$ is split over $l_v$. Since $\tilde{T}$ and $T_H$ are split over $l_v$, the same holds for $T_0$. Therefore $T_0(k_v)/R$ is trivial by [CTS1], p. 200. Since $T_0(k_v)/R = T_0(k_v)/Br$ ([CTS1], p. 217) and $H(k_v)/Br = T_0(k_v)/Br$ by Corollary 3.6.3 and the proof given there, we see that $H(k_v)/Br = G(k_v)/Br = \{1\}$. The proof of 4.7 is complete.  

Now we see that $G(k_v)/Br = 1$ for all $v \in S$. Thus $G$ has weak approximation for any given finite set $S$, which means that $G$ has weak approximation over $k$.

3) b) Now we assume that $G_1$ is an inner form of a group $G$. First we prove the last statement in 3) of the theorem.

We need the following very useful

4.8. Lemma. (Ono - Sansuc [S]) Let $G$ be a connected reductive group defined over a field $k$. There exists a number $n$, induced $k$-tori $T$ and $T'$ such that we have the following central $k$-isogeny

$$1 \to F \to \tilde{G}^n \times T' \to G^n \times T \to 1,$$

where $\tilde{G}$ is the simply connected covering of the semisimple part $G'$ of $G$.

The finite covering of an algebraic group by a direct product of simply connected group with an induced torus (such as $\tilde{G}^n \times T' \to G^n \times T$ above) is called after Sansuc a
special covering. It is obvious that to prove our statement we may assume that the group $G_1$ itself has a special covering

$$1 \to F \to \tilde{G} \times T' \xrightarrow{\pi} G \to 1$$

defined over $k$. Since the inner twist does not effect the center it is obvious that we have also a special covering

$$(13) \quad 1 \to F \to \tilde{G}_1 \times T' \to G_1 \to 1,$$

where $\tilde{G}_1$ is the simply connected covering of the semisimple part of $G_1$. The exact sequences (12) and (13) induce the following exact sequences of groups of $R$-equivalences

$$(14) \quad (\tilde{G}(k_v) \times T'(k_v))/R \to G(k_v)/R \to H^1(k_v, F)/R \to 0,$$

$$(15) \quad (\tilde{G}_1(k_v) \times T'(k_v))/R \to G_1(k_v)/R \to H^1(k_v, F)/R \to 0,$$

(compare with (8) and (9)). Since the first groups in the exact sequences (14), (15) are trivial, we obtain

$$G(k_v)/R \simeq G_1(k_v)/R \simeq H^1(k_v, F)/R.$$ 

Now assume that $G_1$ is an inner form of a group $G$ satisfying 3) a) above. We will show that $G_1$ has weak approximation with respect to any finite set $S$ of valuations $v$ where $G$ has metacyclic splitting field extension $l_v/k_v$. By Proposition 3.4, it suffices to show that $G_1(k_v)/Br$ is trivial for all $v \in S$.

Let $G = G'S$, where $G'$ is the semisimple part of $G$, and $S$ a central torus. Let $F = S \cap G'$, $\tilde{G}$ be the simply connected covering of $G'$, $\pi_G : \tilde{G} \to G'$ the canonical isogeny. As in part a) we denote

$$F_1 = \{(f, f) : f \in F\} \hookrightarrow S \times G',$$

so we have a central isogenies

$$1 \to F_1 \to S \times G' \xrightarrow{\alpha} G = SG' \to 1,$$

$$1 \to F_2 \to S \times \tilde{G} \xrightarrow{\beta} S \times G' \to 1,$$

where $F_2 = \{(x, 1) : x \in \text{Ker} (\pi_G)\} \simeq \text{Ker} \pi_G$. Denote by

$$\pi : S \times \tilde{G} \to SG'$$

the composite of isogenies $\alpha$ and $\beta$. Then one checks that

$$\mu := \text{Ker} \pi = \{(g, \pi_G(g)) : g \in \pi_G^{-1}(F)\}$$

$$\simeq \pi_G^{-1}(F)$$

$$\hookrightarrow \tilde{F} := \text{Cent}(\tilde{G})$$
Since $G_1$ is an inner twist of $G$, $G_1 = SG'_1$, where $G'_1$ is the semisimple part of $G_1$, and $S \cap G'_1 = S \cap G = F$. We define
\[
\mu_1 := \{(x, x) : x \in \mu\},
\]
\[
H = (Z \times (S \times \tilde{G}))/\mu_1,
\]
\[
H_1 = (Z \times (S \times \tilde{G}_1))/\mu_1.
\]
Then from the construction it follows that
\[
H = \tilde{G}P, H_1 = \tilde{G}_1P,
\]
where $P$ is the connected center of $H$ and $H_1$, and also
\[
(16) \quad \tilde{G} \cap P = \tilde{G}_1 \cap P.
\]
Therefore we have
\[
(17) \quad T_0 := H/\tilde{G} \simeq T_1 := H_1/\tilde{G}_1.
\]
From the proof of the part a) above (Proposition 4.7) we see that $T_1(k_v)/Br$ is trivial since $T_0(k_v)/Br (= H(k_v)/Br = G(k_v)/Br)$ is trivial (see the proof of 4.7 of part a)). Therefore by Corollary 3.6.3 and relations (16), (17) $H_1(k_v)/Br$ (which is isomorphic to $T_1(k_v)/Br$) is also trivial, thus so is $G_1(k_v)/Br$. The proof of Theorem 4.6 is complete.

Now we consider some other analogs related with $R$-equivalence relations. We have the following extension of similar property of tori (see Theorem 1.1, (4)).

**4.9. Theorem.** Let $G$ be a connected linear algebraic group defined over a $p$-adic field $k_v$. Then the group of $R$-equivalence classes and the group of Brauer equivalence classes coincide:
\[
G(k_v)/R = G(k_v)/Br.
\]

**Proof.** As above, we may assume that $v$ is non-archimedean and $G$ is reductive.

**Step 1.** Let $G$ be a connected reductive $k_v$-group, $G_q$ be its quasi-split inner form defined over $k_v$. Then we have
\[
(18) \quad G(k_v)/R \simeq G_q(k_v)/R.
\]
This has been proved in Theorem 4.6, 3).

**Step 2.** Let $G_q$ be a connected reductive quasi-split $k_v$-group. Then
\[
(19) \quad G_q(k_v)/R = G_q(k_v)/Br.
\]

**Proof.** Take a maximal $k_v$-torus $T$ of $G_q$ containing a maximal $k_v$-split torus $S$ of $G_q$. Then we have
\[
T = Z_{G_q}(S),
\]
and Bruhat decomposition for $G_q$ shows that (see [CTS1], Section 7) we have
\[ T(k_v)/R = G_q(k_v)/Br, \]
\[ T(k_v)/Br = G_q(k_v)/Br. \]
Since for tori $T$ we have
\[ T(k_v)/R = T(k_v)/Br, \]
by Theorem 1.1, 4), hence $G_q(k_v)/R = G_q(k_v)/Br$.

**Step 3.** If $G_q$ is a quasi-split inner form of a connected reductive $k_v$-group $G$, then
\[ (20) \quad G(k_v)/Br \simeq G_q(k_v)/Br. \]
Indeed, by Proposition 3.6 we may assume that the semisimple part $\tilde{G}$ (resp. $\tilde{G}_q$) of $G$ (resp. $G_q$) is simply connected. From the proof of Theorem 4.6, 3) b (see (16), (17)), it follows that we have the following canonical isomorphism
\[ G_q/\tilde{G}_q \simeq G/\tilde{G} \simeq T, \]
where $T$ is quotient torus of $G$ (and $T$ is defined on the same field as $G$). We know by Corollary 3.6.3 that
\[ G_q(k_v)/Br \simeq T(k_v)/Br \simeq G(k_v)/Br, \]
hence (20) holds.

Now the theorem follows from the combination of (18), (19), (20).

\subsection*{4.10. Remarks.}
1) In [CTS2], Remarque 2.8.17, it was shown that for a given smooth variety $X$ over a local $p$-adic field and under some condition $(H'_x)$ on the universal torsor under some torus, the Brauer and $R$-equivalence are the same. Also, it is a very general method to obtain such kind of results (e.g. one may obtain similar results for tori over $p$-adic fields (see [CTS2], Section 2, for details).

2) All results above tell us that if a connected reductive group $G$ over a number field $k$ fails to have weak approximation over $k$, then for some valuation $v$ (which is necessarily non-archimedean), and the quasi-split inner form $G_q$ of $G$ (which is necessarily non-split), we have $G_q(k_v)/Br \neq 1$.

3) Our assumption in Theorem 4.6 on the existence of metacyclic extension of $k_v$ splitting $G$ has local character, so it is weaker than that of Sansuc [Sa], Corollaire 5.4, p. 34.

The following local-global statement (or principle) would show that our result is equivalent to that of Sansuc:

A connected reductive group $G$ defined over a number field $k$ has a metacyclic splitting field if and only if it is so over all completions $k_v$ of $k$.

4) Equally it is natural (and important) to ask for which class $\mathcal{G}$ of finite groups the following holds. We say that a finite Galois extension $k'/k$ is a $\mathcal{G}$-extension if $Gal(k'/k) \in \mathcal{G}$. 23
We require that \( G \) be a kind of formation of groups, i.e., it is closed with respect to the operations of taking subgroups, factor groups and finite direct product. Then we ask when the following holds:

A connected reductive group over a number field \( k \) has a \( G \)-splitting field if and only if it is so over all completions \( k_v \).

Now we are able to formulate and prove a close analog of the exact sequence (\( Br \)) for groups of \( R \)-equivalence classes.

4.11. Theorem. Let \( G \) be a connected linear algebraic group defined over a number field \( k \). Let \( H \) be a \( z \)-extension of the reductive part of \( G \), \( T \) be its torus quotient and \( S \) be the Neron - Severi torus of \( T \). Then in the following exact sequence

\[
(R') \quad 1 \rightarrow \text{Ker} \rho_G \rightarrow G(k)/R \overset{\rho_G}{\rightarrow} \prod_v G(k_v)/R \rightarrow A(G) \rightarrow 1
\]

the subgroup \( \text{Ker} \rho_G \) has finite index \( n_T = [T(k)/R : \Pi(S)] \), (or the same, \( \text{Card}(T(k)/Br) \)) in \( G(k)/R \). Moreover the following sequence is exact

\[
1 \rightarrow \text{Ker} \rho_G \rightarrow G(k)/R \rightarrow G(k)/Br \rightarrow 1,
\]

and the image of \( G(k)/R \) in \( \prod_v G(k_v)/R \), being isomorphic to \( G(k)/Br \), is also isomorphic to \( T(k)/Br \).

Proof. The fact that the first sequence is exact follows from Proposition 3.2. From Theorem 3.7 we have the following commutative diagram with exact rows

\[
(21) \quad 1 \rightarrow \text{Ker} \rho_G \rightarrow G(k)/R \overset{\rho_G}{\rightarrow} \prod_v G(k_v)/R \rightarrow A(G) \rightarrow 1
\]

\[
\downarrow \lambda_G \quad \downarrow \lambda'_G \quad \downarrow \simeq \quad \downarrow =
\]

\[
1 \rightarrow G(k)/Br \overset{\gamma_G}{\rightarrow} \prod_v G(k_v)/Br \rightarrow A(G) \rightarrow 1.
\]

In the above diagram, the homomorphism \( \lambda_G \) is induced from \( \lambda'_G \) since we have the vertical isomorphism "\( \simeq \)" due to Theorem 4.9, and it is clear that

\[
\lambda'_G(\text{Ker} (\rho_G)) \subset \text{Ker} (\gamma_G).
\]

Therefore it follows that

\[
G(k)/R/\text{Ker} (\rho_G) \simeq G(k)/Br
\]

and the image of \( G(k)/R \) in the product \( \prod_v G(k_v)/R \) is isomorphic to the group \( G(k)/Br \simeq T(k)/Br \) and has order equal to \( n_T = [T(k)/R : \Pi(S)] \) by Proposition 2.6 and Theorem 3.7.

From Theorem 4.11 it follows that to determine the structure of \( G(k)/R \) one needs to understand the structure of \( \text{Ker} \rho_G \), which is given in the following theorem. We also derive the following analog of the exact sequence (\( R \)) in Section 1 in the case the number field \( k \) is totally imaginary and also in many other cases, namely if the semisimple part of
$G$ does not contain anisotropic factors neither of (exceptional, trialitarian) type $D_4$ nor $E_6$.

4.12. **Theorem.** 1) Let $G$ be a connected linear algebraic group defined over number field $k$. Then we have the following commutative diagram, where all rows and columns are exact sequences

\[
\begin{array}{ccc}
\tilde{G}(k)/R & \rightarrow & \tilde{G}(k)/R \\
\downarrow & & \downarrow \pi_R \\
1 & \rightarrow & \text{Ker } \rho_G \rightarrow G(k)/R \xrightarrow{\chi^*_G} G(k)/Br \rightarrow 1 \\
\downarrow p & & \downarrow q \quad \downarrow r \\
1 & \rightarrow & \text{III}(S) \rightarrow T(k)/R \xrightarrow{\chi^*_T} T(k)/Br \rightarrow 1 \\
\end{array}
\]

and $\tilde{G} \xrightarrow{\pi} G_s$ is the simply connected covering of the semisimple part $G_s$ of $G$, $T$ is the torus quotient of the reductive part of a $z$-extension $H$ of $G$, $S$ the Neron-Severi torus of $T$ and $r$ is an isomorphism.

2) If the semisimple part of $G$ does not contain anisotropic almost simple factors neither of types $D_4$ (trialitarian) nor $E_6$ then the following exact sequence $(R')$ holds for $G$.

\[(R') \quad 1 \rightarrow \text{III}(S) \rightarrow G(k)/R \xrightarrow{\rho_G} \prod_v G(k_v)/R \rightarrow \Lambda(G) \rightarrow 1.\]

In general, $(R')$ holds for all connected linear algebraic groups $G$ if and only if all simply connected almost simple groups have trivial group of $R$-equivalence classes.

3) If $k$ is totally imaginary number field, then $p$, $q$ are also isomorphisms and the exact sequence $(R')$ holds for $G$.

First we need the following results.

4.13. **Theorem.** ([CM], Thm. 4.3.) Let $G$ be an almost simple algebraic group of outer type $2A_n$ defined over a field $k$, $G(k) = \text{SU}(\Phi, D)$, where $\Phi$ is the associated hermitian form with respect to an involution $J$ of second kind over a division algebra $D$ of center $K$. Let $\Sigma_J$ (resp. $\Sigma'_J$) be the group of elements which are $J$-symmetric (resp. with $J$-symmetric reduced norm) of $D$. Then

\[G(k)/R \cong \Sigma'_J/\Sigma_J.\]
4.14. **Theorem.** [H3] Assume that $k$ is totally imaginary number field. Then any simply connected semisimple $k$-group has trivial (Galois) $1$-cohomology, and anisotropic almost simple $k$-groups are of type $A_n$.

4.15. **Proposition.** [T3] Let $H$ be a $z$-extension of a connected reductive group $G$, all defined over a field $k$. Then the natural projection $H \rightarrow G$ induces canonical isomorphism of abstract groups

$$H(k)/R \simeq G(k)/R.$$ 

**Proof of Theorem 4.12.**

1) The assertion regarding the exactness of two rows and that $r$ is an isomorphism in the above diagram follows from the commutative diagram (21), and the bijectivity of $r$ follows from Theorem 3.7.

4.16. **Lemma.** With above notation, $\text{Ker } q \subset \text{Ker } \rho_G$.

**Proof.** Indeed, if $x \in \text{Ker } q$ then $r(\chi'_G(x)) = \chi'_f(q(x)) = 0$, hence $\chi'_G(x) = 0$ since $r$ is an isomorphism. As we mentioned above, the rows in the diagram are exact, so $x \in \text{Ker } \rho_G$. ■

The following is a well-known (and trivial) result from homological algebra.

4.17. **Lemma.** In the above diagram, $p$ is surjective if and only if $q$ is surjective.

4.18. **Lemma.** Let $T$ be a torus defined over a field $k$, $S$ a finite set of discrete valuations of $k$. Then

$$\text{Cl}_S(RT(k)) = RT_S.$$ 

In particular, if $S$ consists of real valuations then $RT(k)$ is dense (in the $S$-adic topology) in $T(k)$.

**Proof.** Let

$$1 \rightarrow N \rightarrow P \rightarrow T \rightarrow 1$$

be a flasque resolution of $T$ over $k$. Here $P$ is an induced $k$-torus and $S$ is flasque. By [CTS1], Théorème 2, in the above exact sequence we have

$$q(P(k)) = RT(k),$$

hence

$$\text{Cl}_S(RT(k)) = \text{Cl}_S(q(P(k))).$$

If $x \in q(\text{Cl}_S(P(k)))$, $x = q(y)$, where $y = \lim_{n} p_n$, $p_n \in P(k)$ then

$$x = q(\lim_{n} p_n) = \lim_{n} q(p_n)$$

$$\in \text{Cl}_S(q(P(k))) = \text{Cl}_S(RT(k)),$$

hence $q(\text{Cl}_S(P(k))) \subset \text{Cl}_S(RT(k))$. 26
Since $P$ has weak approximation property, $\text{Cl}_S(P(k)) = P_S$, and as mentioned above, $q(P_S) = RT_S$, hence

$$RT_S \subset \text{Cl}_S(\text{RT}(k)).$$

On the other hand, by Proposition 2.1, $RT_S$ is an open subgroup of $T_S$ containing $\text{RT}(k)$, hence the first assertion follows. The rest of the lemma follows from the previous one and also from the fact that any torus over the real numbers are rational. ■

4.19. Lemma. With notation as in the theorem, $q$ (hence also $p$) is surjective.

Proof. We need only show that

$$T(k) = q(G(k)) \text{RT}(k),$$

where we may assume that $G$ has simply connected semisimple part $\hat{G}$, and $q : G \to T = G/\hat{G}$ is the projection.

By Lemma 4.18 we know that

$$T_\infty = \text{Cl}_\infty(\text{RT}(k)).$$

For $x \in T(k)$ we have

$$x = \lim_n r_n, r_n \in \text{RT}(k)$$

(the limit is taken with respect to the archimedean $\infty$-topology). We have the following commutative diagram similar to (6):

$$G(k) \to T(k) \xrightarrow{\delta} H^1(k, \hat{G})$$

(22)

where all rows are exact and all arrows are continuous with respect to the topologies induced from $G_\infty$ and $T_\infty$. We have

$$\delta_\infty(\beta(x)) = \lim_n \delta_\infty(\beta(r_n))$$

$$= \delta_\infty(\beta(r_n)), \forall n > N,$$

for some fixed $N$, since $\prod_{v \in \infty} H^1(k_v, \hat{G})$ is finite. Hence

$$\delta_\infty(\beta(x)) = \gamma(\delta(x))$$

$$= \gamma(\delta(r_n))$$

for $n > N$. Since $\gamma$ is an isomorphism (i.e. bijection) by Hasse principle, one concludes that

$$\delta(x) = \delta(r_n), \forall n > N.$$
for some $g \in G(k)$. Thus

$$T(k) = RT(k)q(G(k)),$$

i.e., $q$ (and $p$, by Lemma 4.17) is surjective. ■

With this lemma, the proof of the exactness of the first column in Theorem 4.12 is complete. Next we consider the exactness of the second column. We have the following general result.

4.20. Lemma. Let $k$ be a field of characteristic 0, $G$ a connected reductive group with simply connected semisimple part $\tilde{G}$, $T = G/\tilde{G}$. Then we have the following exact sequence of groups

$$\tilde{G}(k)/R \to G(k)/R \to T(k)/R.$$  

Proof. It is obvious that if the lemma is true for some power $G^n = G \times \cdots \times G$, then it is also true for $G$, so by virtue of Lemma 4.8 we may assume that $G$ has a special covering $\tilde{G} \times T'$, where $T'$ is an induced $k$-torus, and we have the following exact sequence of algebraic groups, all defined over $k$:

$$1 \to F \to \tilde{G} \times T' \to G \to 1,$$

where $F$ is a finite central subgroup of $\tilde{G} \times T'$. From this we derive the following $3 \times 3$-commutative diagram

\[
\begin{array}{ccc}
1 & 1 & 1 \\
\downarrow & \downarrow & \downarrow \\
1 & F \cap \tilde{G} & F \\
\downarrow & \downarrow & \downarrow \\
\tilde{G} & \tilde{G} \times T' & T' \\
\downarrow l & \downarrow \pi & \downarrow \\
1 & \tilde{G} & G \\
\downarrow & \downarrow & \downarrow \\
1 & 1 & 1
\end{array}
\]

Since $\tilde{G}$ is simply connected, $l$ is an isomorphism, hence $F \cap \tilde{G} = 1$, and $u$ is also an isomorphism. From the diagram above we derive the following commutative diagram.
\[
\begin{gathered}
(G(k) \times T'(k))/R \xrightarrow{\pi} G(k)/R \to H^1(k, F)/R \\
\downarrow p' \quad \downarrow q' \quad \simeq \downarrow r' \\
T'(k)/R \to T(k)/R \to H^1(k, F)/R
\end{gathered}
\]

where all rows are exact (see Proposition 4.4.1), and \( r' \) is an isomorphism. Since \( T' \) is an induced \( k \)-torus, \( T'(k)/R = 1 \), and

\[
(\hat{G}(k) \times T'(k))/R \simeq \hat{G}(k)/R.
\]

If \( x \in G(k)/R \) such that \( q'(x) = 1 \), then by chasing on this diagram we see that \( x \in \text{Im } \pi \), thus

\[
\text{Ker } q' = \text{Im } (\hat{G}(k)/R \to G(k)/R)
\]

as required.

From this the exactness of the second column of the diagram in the theorem is proved, hence we have finished the proof of 1).

2) The "general" part of 2) follows directly from 1). Next we show that if the semisimple part of \( G \) does not contain anisotropic almost simple factors of exceptional types \( D_4 \) and \( E_6 \) then (R') holds for \( G \).

To see this, we reduce the proof to the following situation. Namely, one may assume that the group \( G \) above is connected reductive with anisotropic semisimple part. Indeed, it is clear that we may assume \( G \) to be reductive.

**Step 1.** If \( H \) is an almost simple simply connected group over \( k \) then \( H(k) \) has no proper noncentral normal subgroups except possibly for the following types:

- anisotropic \( A_n, D_4 \) (exceptional), \( E_6 \);
- isotropic exceptional types \( 2E_6^{35}, 2E_6^{29} \).

This is the result of many authors where the readers are referred to Chapter 9, Section 9.1 of [PR] for further information. (Quite recently Y. Segev and G. Seitz announced that the Platonov - Margulis conjecture is true for the case \( 1A_n \).)

**Step 2.** If \( H \) is as in Step 1, but \( H \) can be of anisotropic type \( A_n \), then \( H(k)/R = 1 \).

This follows from Step 1, the well-known fact that \( RH(k) \) is an infinite normal subgroup of \( H(k) \), Theorem 4.13 in combination with results of Wang (that the group \( \text{SK}_1(A) = 1 \) for any central simple algebra \( A \) over number field \( k \), and the result of Platonov - Yanchevski i (that \( \Sigma f/\Sigma f = 1 \) for number field \( k \)).

4.21. **Lemma.** If \( H \) is simply connected either of isotropic type \( 2E_6^{35} \) or \( 2E_6^{29} \) then \( H(k)/R = 1 \).
Proof. If $S'$ is a maximal $k$-split torus of $H$, then it is well-known by [CTS1] that we have the following functorial isomorphism of abstract groups

$$H(K)/R \simeq Z_H(S')(K)/R,$$

for any field extension $K$ of $k$. Hence by Theorem 3.4, we have

$$A(H) \simeq A(Z_H(S')).$$

(One can show in general that this last isomorphism holds for any field $k$, see [T4].)

Case $E_6$. Let $S'$ be a maximal $k$-split torus of $H$. The Tits index of $H$ is as follows

$$\begin{array}{ccc}
\circ & \longrightarrow & \bullet \\
\downarrow & & \\
\bullet & \longrightarrow & \circ \\
\end{array}$$

One can check that the centralizer $Z := Z_H(S')$ of $S'$ in $H$ is

$$Z_H(S') = S'L,$$

where $L$ is an almost simple simply connected $k$-group of type $^2A_5$. Since $L(k)/R = 1$ by Step 2, from the result of 1) (namely from the commutative diagram in the theorem), we have the following exact sequence (R') for $Z$:

$$(23) \quad 1 \rightarrow \Xi(S) \rightarrow Z(k)/R \rightarrow \prod_v Z(k_v)/R \rightarrow A(Z) \rightarrow 1,$$

where $S$ is the Neron - Severi torus of the torus $Z/L$ since $Z$ is the $z$-extension of itself. Since $Z/L$ is a $k$-split torus, $S$ has trivial cohomology, so $\Xi(S)$ is trivial. As we notice earlier that

$$Z(k_v)/R \simeq H(k_v)/R$$

is trivial for all $v$ since $H$ is simply connected, hence from (23) it follows that

$$1 = Z(k)/R = H(k)/R$$

as required.

Case $E_6$. The Tits index of $H$ is as follows

$$\begin{array}{ccc}
\bullet & \longrightarrow & \circ \\
\downarrow & & \\
\bullet & \longrightarrow & \bullet \\
\end{array}$$

We have

$$Z := Z_H(S') = S'T_0L,$$

where $L$ is a simply connected $k$-group of (classical) type $D_4$ (hence satisfies $L(k)/R = 1$), and $T_0$ is a one-dimensional $k$-torus. As above we have the exact sequence (R') for the group $Z$. In this case, the torus quotient

$$T = Z/L = S'T_0/(S'T_0 \cap L)$$
is a two-dimensional $k$-torus, which is rational over $k$ by a classical result of Voskresenskiĭ [V1]. Therefore

$$T(k)/R = 1.$$ 

By [CTS1], Proposition 19(ii), we have the following exact sequence

$$0 \to \text{III}(T) \to \text{Br}_a X \to \prod_v \text{Br}_a X_v \to T(k)/R \to \text{III}(S) \to 0.$$ 

(Here $X$ denotes a smooth compactification of $T$ over $k$ and $\hat{S} = \text{Pic}(\bar{X})$.) In particular, $\text{III}(S) = 0$. Further we argue as above to obtain that $Z(k)/R$ is trivial, hence so is $H(k)/R$. □

So from 1), Steps 1, 2 and from Lemma 4.21 it follows that the exact sequence (R') holds for $G$ except possibly the case the semisimple part of $G$ contains anisotropic almost simple factors of exceptional types $D_4$ and/or $E_6$. Hence 2) is proved.

3) We claim that $H(k)/R = 1$ for any almost simple simply connected group $H$ over $k$. If $H$ is anisotropic then by Theorem 4.14, $H$ is of type $\Lambda_n$ and the claim follows from Steps 1, 2 in 2). Also from there, by combing with Lemma 4.21, we know that the claim holds for any isotropic group $H$. So over totally imaginary fields $k$, $H(k)/R$ is trivial for all simply connected semisimple $k$-groups $H$. To obtain (R') we may use the result of 1) and Theorem 4.11. We supply also a proof of this fact independent of 1) as follows.

By using the same argument as in the proof of Theorem 4.4 (or 4.6) and by using the Harder’s result on the triviality of the Galois cohomology of simply connected groups (Theorem 4.14), we can show that

$$G(k)/R \simeq G_q(k)/R,$$

where $G_q$ is a quasi-split inner form of $G$ over $k$ and we may assume also that $G$ has simply connected reductive part and that $G$ is reductive. Let $T_q$ be a maximal $k$-torus of $G_q$ containing a maximal $k$-split torus $S$ of $G_q$. By the same argument as in the proof of 4.6, 4.7, it follows that $G$ and $G_q$ have isomorphic torus quotients $T$. It is also well-known that for simply connected quasi-split semisimple groups $G_q'$ any maximal torus containing a maximal $k$-split torus is also quasi-split (i.e. induced) torus. Denote such a torus by $T_q'$. Then we have

$$1 \to T_q' \to T_q \to T \to 1$$

is an exact sequence of $k$-tori, and since $T_q'$ is cohomologically trivial. Since this is a $z$-extension, from Proposition 4.15 it follows that

$$T_q(k)/R \simeq T(k)/R,$$

hence

$$G(k)/R \simeq T(k)/R,$$

and this is true for any field extension of $k$. In particular,

$$G(k_v)/R \simeq T(k_v)/R$$
for all \( v \), hence from (R) we deduce the exact sequence

\[
(R') \quad 1 \to \mathbb{H}(S) \to G(k)/R \to \prod_v G(k_v)/R \to \Lambda(G) \to 1,
\]

where \( T \) is the torus quotient of any \( z \)-extension of \( G \) and \( S \) is its Neron - Severi torus.

The proof of Theorem 4.12 is therefore complete. \( \blacksquare \)

We derive the following consequence from Theorem 4.12 which gives a relation between \( RG(k) \) and \( RG_s \), extending the corresponding result for tori (Lemma 4.18) over number fields.

4.22. Theorem. Let \( k \) be a number field, \( S \) a finite set of valuations of \( k \), \( G \) a connected linear algebraic \( k \)-group. Then

\[
RG_S = Cl_S(RG(k)).
\]

Proof. We claim that

\[
RG_S = \pi(\tilde{G}_S)Cl_S(RG(k)),
\]

where \( \pi : \tilde{G} \to G_s \) is the universal covering of the semisimple part \( G_s \) of \( G \).

We may assume that \( G \) is reductive. We first show that \( RG_S \) contains the set on right-hand side. It is known and easy to see that \( RG_S \) is an open subgroup in \( \prod_{v \in S} G(k_v) \), hence also closed. We know that for simply connected groups

\[
\tilde{G}_S = R\tilde{G}_S = Cl_S(\tilde{G}(k))
\]

hence

\[
\pi(\tilde{G}_S) = \pi(R\tilde{G}_S) \subset RG_S,
\]

and

\[
\pi(\tilde{G}_S)Cl_S(RG(k)) \subset RG_S.
\]

To prove the other inclusion, first we assume that \( \tilde{G} \) is the semisimple part of \( G \). Let \( T = G/\tilde{G} \). We have the following commutative diagram

\[
\begin{array}{ccc}
G(k) & \overset{p}{\longrightarrow} & T(k) \overset{\delta}{\longrightarrow} H^1(k, \tilde{G}) \\
\downarrow \alpha & & \downarrow \beta & \downarrow \gamma \\
G_S & \overset{p_S}{\longrightarrow} & T_S \overset{\delta}{\longrightarrow} \prod_{v \in S} H^1(k_v, \tilde{G})
\end{array}
\]

Let \( x \in RG_S \). Then \( p(x) \in RT_S = Cl_S(RT(k)) \) by Lemma 4.18, so

\[
p(x) = \lim_n r_n, \quad r_n \in RT(k).
\]

Then

\[
\delta(r_n) \to \delta(p(x)) = 1, \quad n \to \infty
\]

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hence
\[ \delta(r_n) = 1, \forall n > N, \]
for some fixed \( N \). Therefore \( r_n \in p(G(k)), r_n = p(g_n), g_n \in G(k) \) for \( n > N \). Thus
\[ \lim_n p(xg_n^{-1}) = 1. \]

Moreover, from Lemma 4.20 and the fact that \( p(g_n) \in RT(k) \) we deduce
\[ (24) \quad g_n \in \tilde{G}(k)RG(k), \forall n > N. \]

Let
\[ G = \tilde{G}T', \]
where \( T' \) is a \( k \)-torus. The natural isogeny
\[ p' : \tilde{G} \times T' \to G \]
induces an open map
\[ p' : G_S \times T'_S \to G_S. \]
In particular, \( \tilde{G}_S T'_S \) is an open subgroup of \( G_S \). Since \( CL_S(RT'(k)) = RT'_S \) by Lemma 4.18, and \( RT'_S \) is open in \( T'_S \) by Proposition 2.1, it follows that \( CL_S(RT'(k)) \) is open in \( T'_S \). Hence \( \tilde{G}_S CL_S(RT'(k)) \) is an open subgroup of \( G_S \), and so is \( \tilde{G}_S CL_S(RG(k)) \). Let \( V_n, n = 1, 2, \ldots \) be a nested system of open neighbourhoods of 1 in \( T_S \) such that \( V_{n+1} \subset V_n \) for all \( n \),
\[ \cap_n V_n = \{1\}, \]
and \( p(xg_n^{-1}) \in V_n \), for all \( n \). Then
\[ xg_n^{-1} \in p^{-1}(V_n), \forall n, \]
or for all \( n \) we have
\[ x \in p^{-1}(V_n)g_n \subset p^{-1}(V_n)\tilde{G}(k)RG(k), \]
\[ \subset p^{-1}(V_n)\tilde{G}_S CL_S(RG(k)) \]
by (24). Since
\[ \cap_n p^{-1}(V_n) = p^{-1}(\cap_n V_n) \]
\[ = p^{-1}(1) \]
\[ = \tilde{G}_S, \]
so \( (V_n) \) form a nested system of open neighbourhoods of \( \tilde{G}_S \). Therefore for some \( N \),
\[ p^{-1}(V_n) \subset \tilde{G}_S CL_S(RG(k)), \forall n > N, \]
since \( \tilde{G}_S CL_S(RG(k)) \) is an open subgroup of \( G_S \). Thus
\[ x \in \tilde{G}_S CL_S(RG(k)) = CL_S(\tilde{G}(k))CL_S(RG(k)). \]
since, by Kneser’s and Harder’s results, \( \tilde{G} \) has weak approximation property over \( k \).

In general case, let \( H \) be a \( z \)-extension of \( G \),

\[
1 \to Z \to H \xrightarrow{\pi} G \to 1,
\]

where \( \pi \) induces the covering isogeny \( \tilde{G} \to G_s \subset G \). By Proposition 4.15 the projection \( \pi \) induces surjections

\[
RH_S \to RG_S, \quad RH(k) \to RG(k),
\]

hence

\[
RG_S = \pi(RH_S)
\]

\[
= \pi(\tilde{G}_S)\text{Cl}_S(RH(k)))
\]

\[
= \pi(\tilde{G}_S)\pi(\text{Cl}_S(RH(k)))
\]

\[
\subset \pi(\tilde{G}_S)\text{Cl}_S(\pi(RH(k)))
\]

\[
= \pi(\tilde{G}_S)\text{Cl}_S(RG(k)),
\]

and the claim is proved. Since \( R\tilde{G}(k) \) is an infinite normal subgroup of \( \tilde{G}(k) \), \( \text{Cl}_S(R\tilde{G}(k)) \) is an infinite normal subgroup of \( \text{Cl}_S(\tilde{G}(k)) = G_s \). It is known that \( G_s \) has no proper infinite normal subgroup (consequence of Kneser - Tits conjecture over local fields). Thus

\[
\text{Cl}_S(\tilde{G}(k)) = \tilde{G}_S,
\]

hence

\[
\tilde{G}_S \subset \text{Cl}_S(RH(k))
\]

and \( RG_S = \text{Cl}_S(RG(k)) \). Theorem 4.22 is proved. ■

We derive also the following finiteness result of the group of \( R \)-equivalence classes.

**4.23. Corollary.** [G2] If \( k \) is a number field then \( G(k)/R \) is finite.

**Proof.** A well-known theorem of Margulis - Prasad says that if \( \tilde{G} \) is a simply connected semisimple \( k \)-group then \( \tilde{G}(k)/R \) is finite. From the commutative diagram in Theorem 4.12 and from the finiteness of \( T(k)/R \) (see [CTS1, Corol. 2, p. 200] it follows that \( G(k)/R \) is finite. ■

**4.24. Corollary.** Let \( G \) be an adjoint semisimple group defined over a number field \( k \) and \( \tilde{G} \) be its simply connected covering. If \( \tilde{G}(k)/R = 1 \) then \( G(k)/R = 1 \). In particular, it is so if \( G \) contains no anisotropic factors of exceptional types \( D_4, E_6 \).

**Proof.** Let \( F = \text{Ker}(\tilde{G} \to G), \tilde{G}_q, G_q \) be quasi-split inner forms of \( \tilde{G}, G \) respectively. Denote by \( \tilde{T}_q, T_q \) their corresponding maximal \( k \)-torus containing maximally \( k \)-split torus, which are known to be induced tori. From the exact sequence

\[
\tilde{G}(k)/R \to G(k)/R \to H^1(k, F)/R,
\]
and from the assumption it follows that \( G(k)/R \to H^1(k, F)/R \) is injective. By considering the corresponding sequence for tori

\[
\tilde{T}_q(k)/R \to T_q(k)/R \to H^1(k, F)/R \to 0,
\]

(we use the fact that \( H^1(k, \tilde{T}_q) = 0 \)) and from the fact that these tori are induced, so we have trivially \( H^1(k, F)/R = 0 \), thus \( G(k)/R \) is trivial. 

5 An application to norm principle

Let \( A \) be a central simple associative algebra of finite dimension over its center \( k \), we denote by \( K_1(A) = A^*/[A^*, A^*] \) the Whitehead group of \( A \) and by \( SK_1(A) = SL_1(A)/[A^*, A^*] \) the reduced Whitehead group of \( A \). Then by a theorem of Draxl [D], for any finite extension \( K \) of \( k \) we have a functorial norm maps

\[
K_1(A \otimes K) \to K_1(A), \ SK_1(A \otimes K) \to SK_1(A).
\]

If \( G \) denotes the algebraic \( k \)-group defined by \( G(k) = SL_1(A) \), then by a theorem of Voskresenski (see e.g. [V1])

\[
SK_1(A) \cong G(k)/R
\]

so we have also a norm map \( G(K)/R \to G(k)/R \).

If \( A \) is endowed with an involution of the second kind \( J \), which is not trivial on \( k \), then one can introduce the unitary Whitehead group

\[
UK_1(A) = A^*/\Sigma_J,
\]

and the reduced unitary Whitehead group

\[
USK_1(A) = \Sigma_J'/\Sigma_J,
\]

where \( \Sigma_J \) is the subgroup of \( A^* \) generated by elements with \( J \)-symmetric reduced norm and \( \Sigma_J \) denotes the subgroup of \( A^* \) generated by \( J \)-symmetric elements. It is easy to show that in this case we also have norm maps

\[
UK_1(A \otimes K) \to UK_1(A), \ USK_1(A \otimes K) \to USK_1(A).
\]

If \( A \) is equipped with an isotropic nondegenerate \( J \)-hermitian form \( \Phi \) and \( k_0 \) is the subfield of \( J \)-symmetric elements of \( K \) then a result of Yanchevskii (see e.g. [MY]) showed that for the group \( G \) defined by \( G(k_0) = SU(A, \Phi) \), we have

\[
USK_1(A) \cong G(k_0)/R,
\]

so we have also a norm map for \( G(k_0)/R \) in this case.

In general, as suggested by these examples, one may ask

Is there a functorial norm map

\[
N_{k'/k} : G(k')/R \to G(k)/R,
\]

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for any reductive group $G$ defined over a field $k$ and for any finite extension $k'/k$.

For an arbitrary field $k$ it seems to be a difficult question. Here we apply our results in previous sections to answer this question in the case $k$ is a local or global field of characteristic 0.

It turns out that the investigation (and the existence) of the Norm Principle for the group of $R$-equivalence classes touches upon deep and interesting properties of semisimple algebraic groups over fields and it has also some interesting applications (see e.g. [ChM]).

One of our main ingredients is the following extension of a result of P. Deligne [De] about the existence of a norm maps between certain factor groups of connected reductive groups over local and global fields. (In fact this result of Deligne motivates our study in the norm principle. There is a more general result in [T1] but we restrict ourselves only to the case which we need.)

5.1. Theorem. [T1] Let $k$ be a local or global field of characteristic 0.

1) Let $\pi : G \to H$ be a homomorphism of connected reductive groups, all defined over $k$, such that the restriction of $\pi$ to the semisimple part $G'$ of $G$ is an isogeny onto that of $H$. Then for any finite extension $k'$ of $k$ there exists a functorial corestriction (norm) map

$$H(k')/\pi(G(k')) \to H(k)/\pi(G(k)).$$

2) Let $1 \to F \to G \to H \to 1$ be an exact sequence of linear algebraic $k$-groups, with $G$ and $H$ connected reductive groups and $F$ a central subgroup of $G$. Let $k'$ be any finite extension of $k$, and $\delta_{k'}$, $\delta_k$ be boundary homomorphism $H(k') \to H^1(k', F)$, $H(k) \to H^1(k, F)$, respectively. Then the corestriction homomorphism

$$\text{Cor}_{k'/k} : H^1(k', F) \to H^1(k, F)$$

induces a functorial corestriction (norm) map

$$N_{k'/k} : \delta_{k'}(H(k')) \to \delta_k(H(k)).$$

5.2. Remark. The proof of this result in [T1] is based on cohomological consideration, and the map we constructed comes indeed from a corestriction map in certain cohomology (in terminology of [De], from Picard category). It was also the starting point for abelianized Galois cohomology of Borovoi (to appear).

The following result provides one application of Theorem 5.1.

5.3. Theorem. Let $G$ be a connected reductive group over a local or global field $k$, $k'$ a finite extension of $k$. Assume that the $K$-rational points of the simply connected covering $\tilde{G}$ of the semisimple part $G_s$ of $G$ have images lying in $RG(K)$ when $K = k'$ or $k$. Then there is a functorial (corestriction) norm map

$$G(k')/R \to G(k)/R.$$
Proof. By Lemma 4.8 there exists induced $k$-tori $P, Q$ and a natural number $n$ such that we have a central isogeny of connected reductive groups

$$1 \to F \to \hat{G}^n \times P \xrightarrow{\pi} G^n \times Q \to 1,$$

where $F$ is a finite central subgroup. We denote by $G_1 = \hat{G} \times P$, $G_2 = G^n \times Q$. It is clear that it suffices to prove the theorem for $G_2$. By assumption, we have

$$\pi(G_1(K)) \subset RG_2(K)$$

for $K = k, k'$. We have the following induced exact sequence of cohomology

$$G_1(K) \to G_2(K) \to H^1(K, F),$$

and the following exact sequence (see Proposition 4.4.1)

$$G_1(K)/R \to G_2(K)/R \to \text{Im } \delta/R \to 1,$$

where $K$ is any field extension of $k$ and $\text{Im } \delta/R$ denotes the set of $R$-equivalence classes of $\text{Im } (\delta)$ with the natural $R$-equivalence induced from that of $G(K)$. Now we take $K = k'$ and $K = k$. By taking the induced cohomology we have the following commutative diagram with exact sequences rows, where $R := RG_2(k)$, $R' := RG_2(k')$ are subgroups in $G_2(k)$, $G_2(k')$, respectively, which are generated by elements $R$-equivalent to 1.

$$
\begin{array}{c}
1 \to R'/\text{Im } \pi \to G_2(k')/\text{Im } \pi \to G_2(k')/R' \to 1 \\
\left|\begin{array}{c}
\alpha \\
\beta \\
(?)
\end{array}\right|
\end{array}
$$

$$1 \to R/\text{Im } \pi \to G_2(k)/\text{Im } \pi \to G_2(k)/R \to 1.$$

Here we have used the assumption that the image of $G_1(K)$ under $\pi$ lies in the group $RG_2(K)$ for $K = k$ and $k'$ and $\beta$ is the functorial norm map which exists by Theorem 2.3 and the functorial norm map $\alpha$ exists by a result of Gille [G1], Prop. 3.3.2. Therefore one obtains also a functorial norm map $(?) : G_2(k')/R \to G_2(k)/R$ as required.

From results of previous sections we derive the following.

5.4. Proposition. Let $k$ be a local $p$-adic field and $G$ a connected linear $k$-group. Then for any finite extension $k'$ of $k$ there is a functorial norm map

$$G(k')/R \to G(k)/R.$$

Proof. Let $\hat{G}$ be the simply connected covering of the semisimple part of $G$. It follows from above that as any simply connected semisimple $k$-group, $\hat{G}$ has trivial group of $R$-equivalence classes. By applying Theorem 5.3 there is a functorial norm map for $G$ as indicated.

Now we assume that $k$ is a number field.
5.5. Proposition. Let $G$ be a connected linear algebraic group over a number field $k$. If any almost simple factor of $G$ is neither of anisotropic exceptional type $D_4$, nor $E_6$, then for any finite extension $k'$ of $k$ there is a functorial norm map

$$G(k')/R \to G(k)/R.$$  

Proof. It follows from Section 4 that the simply connected covering of the semisimple part of $G$ has trivial group of $R$-equivalence classes. Hence there exists a functorial norm map as indicated by Theorem 5.3.  

We can prove the analog of Theorem 5.1, 1) for the group of $R$-equivalence classes, i.e., the Corestriction (Norm) Principle in its relative form.

5.6. Proposition. Let $\rho : G \to H$ be a homomorphism of connected reductive groups, all defined over a local or global field $k$ of characteristic 0, such that the restriction of $\rho$ to the semisimple part of $G$ is a central isogeny onto that of $H$. Denote $M_G(k') = G(k')/R$ for any field extension $k'$ of $k$, $\rho_R(\rho'_H)$ the induced homomorphism $M_G(k) \to M_H(k)(M_G(k') \to M_H(k'))$. Then for any finite extension $k'$ of $k$ there is a functorial norm map

$$M_H(k')/\text{Im} \ (\rho'_R) \to M_H(k)/\text{Im} \ (\rho_R).$$

Proof. There are few ways to prove the theorem and we just indicate one of them. Let $\tilde{G}$ be the simply connected covering of the semisimple part $G'$ of $G$, hence also of $H'$. Let $\alpha : \tilde{G} \to G$ and $\beta : \tilde{G} \to H$ be the corresponding homomorphisms.

Step 1. We show that there exists a norm map

$$M_G(k')/\text{Im} \ (\alpha'_R) \to M_G(k)/\text{Im} \ (\alpha_R),$$

(and also similar statement for the pair $(H, \beta)$). We know that there exist a natural number $n$, induced $k$-tori $P$ and $Q$ such that there is a central isogeny

$$1 \to F \to \tilde{G}^n \times P \xrightarrow{\beta} G^n \times Q \to 1,$$

all defined over $k$. Here $F$ is a finite central subgroup of $\tilde{G}^n \times P$ of multiplicative type.

By Theorem 5.1, the Corestriction Principle for the image of

$$\delta_k : (G^n \times Q)(k) \to H^1(k, F)$$

holds, i.e., for any finite extension $k'$ we have

$$N_{k'/k}(\text{Im} \ (\delta_{k'})) \subset \text{Im} \ (\delta_k).$$

Since the similar also holds for the image of the set of elements $R$-equivalent to 1 by [G1], Prop. 3.3.2, it follows that for the induced norm map

$$N : H^1(k', F)/R \to H^1(k, F)/R,$$

$$A := \text{Im} \ ((G^n \times Q)(k')/R \to H^1(k', F)/R),$$

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and

\[ B := \text{Im} \ ((G^n \times Q)(k)/R \rightarrow H^1(k, F)/R), \]

we have

\[ N(A) \subset B. \]

Since \( P, Q \) are induced tori, they are all rational over \( k \), so \( \pi_R \) is just \( \alpha_R^P \), and we can finish the proof by using the exact sequence

\[(G^n \times P)(K)/R \xrightarrow{\pi_R} (G^n \times Q)(K)/R \rightarrow \text{Im} \ \delta_K/R \rightarrow 0 \]

for any field extension \( K \) of \( k \).

Step 2. From Step 1 we have the following commutative diagram with exact rows

\[
\begin{array}{ccc}
M_G(k')/\text{Im} \alpha'_R & \xrightarrow{\rho'_R} & M_H(k')/\text{Im} \beta'_R \\
\downarrow N_1 & & \downarrow N_2 \\
M_G(k)/\text{Im} \alpha_R & \xrightarrow{\rho_R} & M_H(k)/\text{Im} \beta_R
\end{array}
\]

where the functorial norm maps \( N_1, N_2 \) exist by Step 1, hence the same is true for (?) and the proposition follows. \( \blacksquare \)

6 Remarks, problems and conjectures

6.1. From Theorem 4.12 it follows that over an arbitrary field \( k \), the finiteness of groups \( G(k)/R \) for connected reductive groups \( G \) depends only on the finiteness of tori and simply connected groups over \( k \). It seems natural to state the following

**Problem 1.** Study the finiteness of \( G(k)/R \) for simply connected groups \( G \) over finitely generated (over the prime field) \( k \).

**Problem 2.** Same problem as above, but only for purely transcendental extensions of \( Q, Q_p, R, C \).

6.2. It is natural to make the following

**Conjecture 1.** The exact sequence (R') holds for any connected linear algebraic group \( G \) over any number field \( k \).

Notice from above that this conjecture is equivalent to the following conjecture

**Conjecture 2.** \( G(k)/R \) is trivial for anisotropic almost simple simply connected group \( G \) of exceptional type \( D_4 \) or \( E_6 \) over a number field \( k \).

As we have seen from above, the last conjecture is a consequence of another stronger conjecture due to Platonov and Margulis. (See [PR, Chapter 9] for more information.)
6.3. In our earlier preprint [T3], we propose another way to express an exact sequence connecting groups of R-equivalence classes, weak approximation obstruction $A(G)$ and the Tate-Shafarevich group of some finite Galois module. It is clear that the above exact sequence (R') is, in a sense, more true (or natural) analog of the initial exact sequence (R) for tori established by Colliot-Thélène and Sansuc.

6.4. All main results of this paper remain true in the case of global function field, except perhaps few exceptions regarding purely inseparable isogenies. This case will be considered, together with other related problems, in another paper under preparation.

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