3D ANGLE-OF-ARRIVAL POSITIONING USING VON MISES–FISHER DISTRIBUTION

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ABSTRACT

We propose modeling an angle-of-arrival (AOA) positioning measurement as a von Mises–Fisher (VMF) distributed unit vector instead of the conventional normally distributed azimuth and elevation measurements. Describing the 2-dimensional AOA measurement with three numbers removes discontinuities and reduces nonlinearity at the poles of the azimuth–elevation coordinate system. Our computer simulations show that the proposed VMF measurement noise model based filters outperform the normal distribution based algorithms in accuracy in a scenario where close-to-pole measurements occur frequently.

Index Terms— angle-of-arrival; positioning; von Mises–Fisher distribution; particle filter; extended Kalman filter

1. INTRODUCTION

Many future positioning systems will use angle-of-arrival (AOA) measurements, as the coming 5G networks can be equipped with antenna arrays that enable measuring the AOA of the received electromagnetic signal [1]. An AOA measurement consists of two components: azimuth and elevation. A conventional approach is to model the measurements as noisy versions of the true azimuth and elevation [2–6], and use extended Kalman filter (EKF) or unscented Kalman filter (UKF) that assume that the measurement noises of azimuth and elevation follow normal distributions. However, this model is problematic in a number of ways: 1) In this model the solid angle of measurement uncertainty is smaller close to the “pole” directions, i.e. the two directions where azimuth is not defined. 2) The measurement model is highly nonlinear close to the poles and discontinuous in the pole, which makes gradient-based approximations for optimisation and extended Kalman filtering unstable. 3) Rotations of the spherical coordinate system in which the azimuth and elevation are expressed change the measurement error model.

To remedy these problems, we propose expressing the 2-dimensional spherical AOA measurement as a 3-dimensional Cartesian unit vector, and modeling the measurement error with the von Mises–Fisher (VMF) distribution [7, 8]. This idea and its advantages are analogous to modeling a rotation with a 4-dimensional Bingham-distributed unit quaternion instead of a 3-dimensional Euler angle set [9, Ch. 3.10], [10]. We also propose a particle filter (PF) algorithm [11] based on the VMF measurement error model, and EKF [12, Ch. 8.3] and UKF [13] algorithms that approximate the VMF update with the assumption that the unit vector measurement is the true direction’s unit vector plus a trivariate normal noise. Our simulations show that the proposed positioning algorithms outperform the conventional algorithms in accuracy. VMF filters have also been proposed in [14–17], but in these filters both the state and measurements are VMF-distributed unit vectors.

2. MODELLING OF AOA MEASUREMENT

2.1. Von Mises–Fisher distribution

The support of the VMF’s probability density function (PDF) is the unit (hyper-)sphere. A unit vector \( x \in \mathbb{R}^n \) follows the distribution \( \text{VMF}(\mu, \kappa) \) with mean direction \( \mu \in \mathbb{R}^n, \| \mu \| = 1 \), and concentration parameter \( \kappa \in \mathbb{R}_+ \) if its PDF is

\[
p(x) = C_\kappa e^{\kappa \mu^T x}.
\]

The larger the parameter \( \kappa \) is, the more the probability mass is concentrated around the direction \( \mu \). For \( \kappa > 0 \) the distribution is unimodal and for \( \kappa = 0 \) it is uniform on the sphere. For a 3-dimensional variable the normalisation constant is

\[
C_\kappa = \left\{ \begin{array}{ll}
\frac{\kappa}{4\pi \sinh \kappa}, & \kappa > 0 \\
\frac{1}{4\pi}, & \kappa = 0
\end{array} \right.
\]

The VMF distribution is suitable for modelling directional data because a direction can be bijectively mapped to a unit vector. The distribution is rotation invariant in the sense that if \( x \sim \text{VMF}(\mu, \kappa) \), then for \( y = Rx \) holds \( y \sim \text{VMF}(R\mu, \kappa) \) for a rotation matrix \( R \). The PDF of \( \text{VMF}(\mu, \kappa) \) is the restriction of the PDF of the multivariate normal distribution \( N(\mu, \frac{1}{\kappa} I_n) \) into the origin-centered unit hyper-sphere [18, Ch. 9.3.2].

2.2. Comparison of normal and VMF models

In this paper an AOA measurement consists of azimuth measurement \( y_{\text{azi}} \in (-\pi, \pi] \) and elevation measurement \( y_{\text{elf}} \in\)
where $e$ proposed VMF based measurement model is terms that are statistically mutually independent and indepen-
dencies of
$$
\cos(\theta) = \frac{x}{\|x\|}, \quad \sin(\theta) = \frac{y}{\|y\|},
$$

Given user position $\theta \in \mathbb{R}^3$ and anchor position $s \in \mathbb{R}^3$, the conventional normal distribution based measurement model is

$$
y_{\text{AZI}} = \tan(\theta, s_1, s_2) + \epsilon_{\text{AZI}} \quad \text{(4a)}
y_{\text{ELE}} = \tan(\theta, s_3) + \epsilon_{\text{ELE}} \quad \text{(4b)}
$$

where $\epsilon_{\text{AZI}} \sim N(0, \sigma_{\text{AZI}}^2)$ and $\epsilon_{\text{ELE}} \sim N(0, \sigma_{\text{ELE}}^2)$ are noise terms that are statistically mutually independent and independent from $\theta$, and $\sigma_{\text{AZI}}$ and $\sigma_{\text{ELE}}$ are model parameters. The proposed VMF based measurement model is

$$
\text{T0 UNIVECTOR}(y_{\text{AZI}}, y_{\text{ELE}}) \sim \text{VMF}(\frac{\pi}{2}, \kappa),
$$

where the concentration $\kappa$ is a model parameter.

In order to compare the normal and VMF based estimation algorithms, we seek a simple rule-of-thumb formula that converts one model to the other. When $\alpha_{x,\mu}$ is the angle between unit vectors $x$ and $\mu$, the PDF of $x \sim \text{VMF}(\mu, \kappa)$ is

$$
p(x) \propto e^{\kappa \cos(\alpha_{x,\mu})} \approx e^{\kappa(1 - \frac{1}{2} \cos^2\alpha_{x,\mu})} \approx N(\alpha_{x,\mu}; 0, \frac{1}{\kappa}),
$$

which follows from the second order truncated MacLaurin series of $\cos(\alpha_{x,\mu})$ and holds for small $\alpha_{x,\mu}$. We thus recommend to implement the normal distribution based filters for VMF-distributed errors and VMF based filters for normally distributed errors with the conversion rules

$$
[\sigma^2_{\text{AZI}}]_{\text{filter}} = \frac{1}{\kappa_{\text{true}}}, \quad [\sigma^2_{\text{ELE}}]_{\text{filter}} = \frac{1}{\max([\sigma^2_{\text{AZI}}]_{\text{true}}, [\sigma^2_{\text{ELE}}]_{\text{true}})}.
$$

3. BAYESIAN FILTERING

We assume a normal initial prior $x_0 \sim N(x_{0|0}, P_{0|0})$ and a linear–normal state transition model for the state $x \in \mathbb{R}^n$

$$
x_k = A_{k|k-1} x_{k|k-1} + w_{k|k-1}, \quad w_{k|k} \sim N(0, Q_{k|k}),
$$

where $A_{k|k-1}$ is state transition matrix, $w_{k|k}$ is process noise, and $Q_{k|k}$ is process noise covariance matrix. In this paper the state includes the 3-dimensional user position, and the three position components in the state are denoted by $[x_k]_{\text{pos}}$.

In the PF algorithm [19, Ch. 3] random samples (“particles”) are generated from the initial prior, propagated in time using the state transition model, weighted using the measurement information, and resampled when the weight concentrates too much. PF is flexible in modeling and can be applied to both normal distribution based measurement model (4) and VMF model (5) without any application-specific tweaks. PF for the VMF model is given in Algorithm 1.

Algorithm 1 Particle filter for VMF measurement noise

1: Inputs: initial prior $x_{0|0}$, $P_{0|0}$, state-transition model $A_{1:K}$, $Q_{1:K}$; concentration parameter $\kappa$; $y_{1|K}$, $y_{1|K}$; anchor positions \& rotations $s_1, s_2, R_1, R_2$
2: Outputs: estimates $x_{k|k}$ for $k = 1, \ldots, K$
3: $x_0^{(i)} \sim N(x_{0|0}, P_{0|0}), \quad w_0^{(i)} \leftarrow \frac{1}{N_p}$ for $i = 1, \ldots, N_p$
4: for $k = 1 : K$ do
5: $u_j \leftarrow \text{T0 UNIVECTOR}([y_{j|k}], y_{j|k})$ for $j = 1, \ldots, n_s$
6: for $i = 1 : N_p$ do
7: $x_k^{(i)} \sim N(A_k x_k^{(i)} - 1, Q_k)$
8: $\bar{w}_k^{(i)} \leftarrow \exp \left( \kappa \sum_{j=1}^{n_s} w_j R_j \|w_j\| \right)$, $w_k^{(i)}$
9: end for
10: $w_k^{(i)} \leftarrow \frac{\bar{w}_k^{(i)}}{\sum_{j=1}^{N_p} w_j^{(i)}}$ for $i = 1, \ldots, N_p$
11: $x_k|k \leftarrow \sum_{i=1}^{N_p} w_k^{(i)} x_k^{(i)}$
12: if $1 / \sum_{i=1}^{N_p} w_k^{(i)} < 0.1 N_p$ then
13: $[x_k^{(1:N_p)}, w_k^{(1:N_p)}] = \text{RESAMPLE}(x_k^{(1:N_p)}, w_k^{(1:N_p)})$
14: end if
15: end for

Algorithm 2 Measurement model function

1: Inputs: position $\theta$; anchor positions \& rotations $s_1, s_2, R_1, R_2$
2: Outputs: measurement model function value $c$ and Jacobian $C$
3: for $j = 1 : n_s$ do
4: $d_j \leftarrow \frac{1}{R_j} (\theta - s_j)$
5: $c_{3j-2,3j} \leftarrow R_j d_j$
6: $C_{3j-2,3j,\text{pos}} \leftarrow \frac{1}{R_j} R_j (I_3 - d_j d_j^T)$
7: $C_{3j-2,3j,\text{pos}} \leftarrow O \triangleright \text{pos}$: indices excluding pos indices
8: end for

EKF and UKF are nonlinear Kalman filter extensions for state-space models where the noises are normally distributed but the model functions can be nonlinear. Application of EKF and UKF to the normal model (4) is straightforward, except that the angle wrappings have to be taken into account when computing the angular differences. Because EKF and UKF assume normally distributed measurement noise, they are not applicable to the VMF measurement model (5), but we approximate the VMF model with

$$
\text{T0 UNIVECTOR}([y_{j|k}], y_{j|k}) \sim N([x_{k|pos}]_j - s_j, \frac{1}{\kappa} I_3).
$$

The details of the computation of the measurement model function and its Jacobian are given in Algorithm 2.

4. SIMULATIONS

We compare the proposed filters based on the VMF and unit vector model (5) with the filters based on the normal distribution and azimuth–elevation model (4). The comparisons rely on numerical simulations computed with MATLAB.

We study two different measurement models, from which the measurements are generated. In Model I the measure-
ments are generated from the normal model (4) such that each direction has a unique azimuth–elevation representation; i.e. if the generated elevation measurement \( y_{ELE}^k \) is negative (resp. larger than \( \pi \)), the elevation is flipped to its absolute value \( |y_{ELE}^k| \) (resp. to the angle \( 2\pi - |y_{ELE}^k| \)) and the azimuth measurement \( y_{AZI}^k \) is flipped to \( y_{AZI}^k - \pi \). The flipping reflects a real-world equipment’s behavior to have a unique representation of direction. In Model II the measurements are generated from the VMF model (5).

We compare four different positioning filters:
- AE-nominal: normal model (4); with Model I
- AE-adaptive: normal model (4); with Model I
- AE-fitted: normal model (4); \( \sigma_{AZI} \) and \( \sigma_{ELE} \) fitted as the maximum likelihood parameters given \( 10^5 \) simulated measurements generated for \( 10^5 \) random directions.
- AE-adaptive: normal model (4); \( \sigma_{AZI} \) and \( \sigma_{ELE} \) chosen as the standard deviations of the normal distribution with flipping at the given elevation; these standard deviations are pre-computed using a grid with 1-degree grid size for the elevation, and shown in Fig. 1.
- VMF: VMF model (5); with Model I

4.1. Model comparison

In this test we compute the expectations of the normal and VMF log-likelihoods over the distribution \( p(y_{AZI}, y_{ELE} | \theta) \). The conditional measurement distribution \( p(y_{AZI}, y_{ELE} | \theta) \) is either the normal distribution with flipping (Model I) or the azimuth–elevation distribution implied by the VMF distribution (Model II), and 3-dimensional position’s distribution is the uniform distribution over the unit sphere \( p(x) = \frac{1}{4\pi} \). The quantitative goodness of the normal fit is measured with

\[
\mathcal{L}_N = \int_0^\pi \int_{-\pi}^\pi \int_{S_2(1)} \left[ \log N(-\pi, \pi)(y_{AZI} - \text{atan}_2(\theta_2, \theta_1); 0, \sigma_{AZI}^2) + \log N_0(0, \sigma_{ELE}^2)(y_{ELE} - \text{atan}_2(\theta_3, \|\theta_1\|); 0, \sigma_{ELE}^2) \right] \times p(y_{AZI}, y_{ELE} | \theta) \frac{1}{4\pi} d\theta_1 d\theta_2 d\theta_3.
\]

where \( S_2(1) \) is the 3-dimensional unit sphere, and \( N_A \) is the normal distribution truncated to the set \( A \). The goodness of the VMF fit is measured with the number

\[
\mathcal{L}_{VMF} = \int_0^\pi \int_{-\pi}^\pi \int_{S_2(1)} \left[ \log VMF(TQ UNITVECTOR(y_{AZI}^*, y_{ELE}^*); \theta, \kappa) + \log(\sin(y_{ELE}^* + \frac{\pi}{2})) \right] p(y_{AZI}^*, y_{ELE}^* | \theta) \frac{1}{4\pi} d\theta_1 d\theta_2 d\theta_3.
\]

The term \( \log(\sin(y_{ELE}^* + \frac{\pi}{2})) \) in (11) comes from the transformation from Cartesian space’s unit sphere into spherical coordinates’ area with radius one. We computed (10) and (11) using Monte Carlo integration with \( 10^5 \) samples.

Table 1 gives the used parameters as well as the model comparison numbers \( \exp(\mathcal{L}) \) normalised to sum to unity. The results show that VMF explains the data better than AE-nominal and AE-fitted models for both Model I and Model II. AE-adaptive attempts to fix the problem of underestimating the azimuth’s variance close to the poles, and has indeed the model comparison number close to the VMF models especially for the normal model with flipping. The fitted maximum likelihood parameters of the AE model show large \( \sigma_{AZI} \) compared to the nominal value because of the influence of the pole areas.

4.2. A single measurement update example

In this subsection we illustrate the difference of the azimuth–elevation and unit vector based filter updates. We use the prior distribution for the position \( \theta \sim N \left( \begin{pmatrix} 0.3 \\ -0.3 \\ 0.75^2 I_3 \end{pmatrix} \right) \) and a direction measurement \( y_{AZI} = -\pi, y_{ELE} = -\frac{\pi}{2} \), \( \sigma_{AZI} = \sigma_{ELE} = 5^\circ \frac{\pi}{180} \), which gives \( \kappa = (\frac{180^\circ}{\pi})^2 \). The anchor is in the origin. The used UKF parameter is \( \lambda = 0.5 \), which provides equally weighted sigma points. We have intentionally placed the prior mean and the direction measurement close to and on different sides of the pole.

Fig. 2 shows the example scenario and the filter updates as line segments whose one end is in the prior mean and the other end is in the filtering posterior mean. The figure shows that the VMF filter estimates are in directions close to the measurement direction, which is desirable because the prior distribution is quite diffuse and measurement is the only additional piece of information. The AE filters, on the contrary, update the estimate to an incorrect direction because the
The results show that the VMF based algorithms greatly and systematically outperform the AE algorithms in accuracy. The differences are emphasised with EKF and UKF, which is probably due to the high nonlinearity of the measurement function close to the pole directions as explained in Section 4.2. AE-adaptive filters are closer in accuracy to VMF than AE-nominal and AE-fitted, but in EKF and UKF the adaptivity does not necessarily improve the accuracy. This is probably due to the fact that these filters choose the measurement noise variance locally, in a single point in EKF and in the sigma points in UKF. Furthermore, VMF based EKF and UKF are close to VMF based PF in accuracy.

5. CONCLUSION

In this article we propose modelling an angle-of-arrival (AOA) positioning measurement as a von Mises–Fisher (VMF)-distributed unit vector instead of the conventional normally distributed azimuth and elevation measurements. Describing the 2-dimensional AOA measurement as a von Mises–Fisher (VMF) provides a more robust model to handle the physical errors invariantly of rotations of the spherical coordinate system in which the azimuth and elevation measurements are expressed, which is sound if there is no reason to assume narrower and asymmetric error distributions in solid angle space close to the pole directions. The presented simulations show that when the user moves close to the pole directions, the proposed VMF based particle filter (PF), extended Kalman filter (EKF), and unscented Kalman filter (UKF) algorithms show substantial improvement in the positioning accuracy.
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