The homogeneous balance of undetermined coefficients method and its application

Abstract: The homogeneous balance of undetermined coefficients method is firstly proposed to solve such nonlinear partial differential equations (PDEs), the balance numbers of which are not positive integers. The proposed method can also be used to derive more general bilinear equation of nonlinear PDEs. The Eckhaus equation, the KdV equation and the generalized Boussinesq equation are chosen to illustrate the validity of our method. The proposed method is also a standard and computable method, which can be generalized to deal with some types of nonlinear PDEs.

Keywords: Homogeneous balance of undetermined coefficients method, Bilinear equation, Eckhaus equation, KdV equation, Generalized Boussinesq equation

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1 Introduction

Nonlinear PDEs are known to describe a wide variety of phenomena not only in physics, but also in biology, chemistry and several other fields. In recent years, many powerful methods to construct the exact solutions of nonlinear PDEs have been established and developed, which are one of the most excited advances of nonlinear science and theoretical physics. The exact solutions of nonlinear PDEs play an important role since they can provide much physical information and more insight into the physical aspects of the problems and thus lead to further applications. Seeking the exact solutions of nonlinear PDEs has long been an interesting topic in the nonlinear mathematical physics. With the development of soliton theory, various methods for obtaining the exact solutions of nonlinear PDEs have been presented, such as the inverse scattering method [1], the Bäcklund and Darboux transformation method [2], the homotopy perturbation method [3], the first integral method [4], the variational iteration method [5], the Riccati-Bernoulli sub-ODE method [6], the Jacobi elliptic function method [7], the tanh-sech method [8], the \((G'/G)\)-expansion method [9, 10], the Hirota’s method [11], the homogeneous balance method (HBM) [12, 13], the differential transform method (DTM) [14–17] and so on.

In these traditional methods, the HBM and the DTM are straightforward and popular tools for handling many types of functional equations. Recently, the Adomian’s decomposition method (ADM) for solving differential and integral equations, linear or nonlinear, has been the subject of extensive analytical and numerical studies because the ADM provides the solution in a rapid convergent series with elegantly computable components [18–23]. The Jacobi pseudo spectral approximation method [24], the fully spectral collocation approximation method [25, 26], the Jacobi...
tau approximation method [27], and the homotopy perturbation Sumudu transform method [28, 29] are powerful and effective tools for solving nonlinear PDEs.

As a direct method, the HBM provides a convenient analytical technique to construct the exact solutions of nonlinear PDEs and has been generalized to obtain multiple soliton (or multiple solitary-wave) solutions [12, 13]. Fan improved this method to investigate the Bäcklund transformation, Lax pairs, symmetries and exact solutions for some nonlinear PDEs [30]. He also showed that there are many links among the HBM, Weiss-Tabor-Carnevale method and Clarkson-Kruskal method [31].

However, the HBM usually encounters complicated and tedious algebraic calculation. The exact solutions are fixed and single types when we deal with nonlinear PDEs by using the HBM. Moreover, the balance numbers of all nonlinear PDEs dealt with by this method usually are limited to positive integers.

Based on these problems, the homogeneous balance of undetermined coefficients method is proposed to improve the HBM, to derive more general bilinear equation of the KdV equation and the generalized Boussinesq equation, and to the exact solution of the Eckhaus equation, the balance numbers of which is not a positive integer.

The remainder of this paper is organized as follows: the homogeneous balance of undetermined coefficients method is described in Section 2. In Section 3, the exact solutions of the Eckhaus equation are obtained by using the homogeneous balance of undetermined coefficients method. In Section 4, the bilinear equation of the KdV equation and the generalized Boussinesq equation are derived respectively. A brief conclusion is given in Section 5.

2 Description of the homogeneous balance of undetermined coefficients method

Let us consider a general nonlinear PDE, say, in two variables,

$$P(u, u_t, u_x, u_{xx}, u_{xt}, \cdots) = 0,$$

where $P$ is a polynomial function of its arguments, the subscripts denote the partial derivatives. The homogeneous balance of undetermined coefficients method consists of three steps.

Step 1 Suppose that the solution of Eq. (1) is of the form

$$u = a_{mn} \left( \ln w \right)_{m,n} + \sum_{\substack{k=m, j=n \\text{for } k+j \neq 0, m+n}} a_{k,j} (\ln w)_{k,j} + a_{00},$$

where $u = u(x,t)$, $w = w(x,t)$, $(\ln w)_{k,j} = \frac{\partial^{k+j}(\ln w(x,t))}{\partial x^k \partial t^j}$, and $m, n, b$ (balance numbers) and $a_{k,j}$ ($k = 0, 1, \cdots, m, j = 0, 1, \cdots, n$) (balance coefficients) are constants to be determined later.

By balancing the highest nonlinear terms and the highest order partial derivative terms, $m$, $n$ and $b$ can be determined.

Step 2 Substituting Eq. (2) into Eq. (1) and arranging it at each order of $w$ yield an equation as follows:

$$\sum_{l=0} f_l w^l = 0,$$

where $f_l$ ($l = 0, 1, \cdots$) are differential-algebraic expressions of $w$ and $a_{k,j}$. Setting $f_l = 0$ and using compatible condition ($w_{xt} = w_{tx}$) yield a set of differential-algebraic equations (DAEs).

Step 3 Solving the set of DAEs, $w$ and $a_{k,j}$ ($k = 0, 1, \cdots, m, j = 0, 1, \cdots, n$) can be determined. By substituting $w$, $m$, $n$, $b$ and $a_{k,j}$ into Eq. (2), the exact solutions of Eq. (1) can be obtained.

In the following section, the exact solutions of Eckhaus equation can be obtained by using the homogeneous balance of undetermined coefficients method.
3 Application to the Eckhaus equation

Let us consider the Eckhaus equation in the form

\[ i\psi_t + \psi_{xx} + 2(|\psi|^2)_x \psi + |\psi|^4 \psi = 0, \]  

(4)

where \( i = \sqrt{-1} \) and \( \psi = \psi(x, t) \) is a complex-valued function of two real variables \( x, t \).

The Eckhaus equation was found as an asymptotic multiscale reduction of certain classes of nonlinear Schrödinger type equations [32]. In [33], a lot of the properties of the Eckhaus equation were obtained. In [34], the Eckhaus equation is linearized by making some transformations of dependent and independent variables. Exact traveling wave solutions of the Eckhaus equation can be obtained by the \((G'/G)\)-expansion method [10] and the first integral method [4].

In this section, by applying the homogeneous balance of undetermined coefficients method to the Eckhaus equation, new exact solutions of the Eckhaus equation can be obtained.

Suppose that the solution of Eq. (4) is of the form

\[ \psi = a_{mn} \left( \ln w \right)_{m,n} + \sum_{k=m,j=n}^{k=m,j=n} a_{kj} (\ln w)_{k,j} + a_{00} \right) \right)^b e^{i(cx+dt)}, \]  

(5)

where \( m, n, b, c, d \) and \( a_{kj} \) \( k = 0, 1, \ldots, m, j = 0, 1, \ldots, n \) are constants to be determined later.

Balancing \(|\psi|^4\) and \(\psi_{xx}\) in Eq. (4), it is required that \( 3nb + 1 = 5mb, 3nb = 5nb \). Solving this set of equations, we get \( m = 1, n = 0, b = \frac{1}{2} \). Then Eq. (5) can be written as

\[ \psi = a((\ln w)_x + b)^\frac{1}{2} e^{i(cx+dt)}, \]  

(6)

where \( a, b, c \) \((a_{10} = a, a_{00} = b)\) and \( d \) are real constants to be determined later.

Substituting Eq. (6) into Eq. (4), we get

\[ f_0 w^0 + f_1 w^1 + f_2 w^2 + f_3 w^3 + f_4 w^4 + i \left( g_1 w^1 + g_2 w^2 + g_3 w^3 \right) = 0, \]  

(7)

where

\[ f_0 = (2a^2 - 1) \left( 2a^2 - 3 \right) w_x^4, \]  

(7a)

\[ f_1 = 4 \left( 2a^2 - 1 \right) \left( b \left( 2a^2 - 1 \right) w_x^3 + w_{xx} \right), \]  

(7b)

\[ f_2 = \left( 24a^2 b^2 - 4c^2 - 8a^2 b^2 - 4d \right) w_x^2 + \left( 16a^2 b - 6b \right) w_{xx} + 2w_{xxx} \right) w_x - w_{xx}^2, \]  

(7c)

\[ f_3 = \left( 16a^2 b^3 - 8bd - 8bc^2 \right) w_x + 8a^2 b^2 w_{xx} + 2b w_{xxx}, \]  

(7d)

\[ f_4 = 4b^2 \left( a^4 b^2 - c^2 - d \right), \]  

(7e)

\[ g_1 = -2(w_x + 2cw_x) w_x^2, \]  

(7f)

\[ g_2 = -4bc w_x^2 + (2w_{xt} + 4cw_{xx} - 2bw_{x}) w_x, \]  

(7g)

\[ g_3 = 2b \left( w_{xt} + 2cw_{xx} \right). \]  

(7h)

Obviously, \( \psi \) is a solution of Eq. (4) provided that \( f_k = 0 \ (k = 0, 1, 2, 3, 4) \) and \( g_j = 0 \ (j = 1, 2, 3) \).

Firstly, suppose that \( w_x = 0 \), from Eqs. (6) and (7), we get an exact solution of Eq. (4) as follows:

\[ \psi_1 = C_0 e^{i(cx + (c_0^4 - c^2)t)}, \]  

(8)

where \( C_0 \) and \( c \) are arbitrary real constants.
Secondly, suppose that \( w_x \neq 0 \), setting \( f_0 = 0 \) and \( g_1 = 0 \), we get \( a = \pm \sqrt{2} \) or \( a = \pm \sqrt{\frac{b}{2}} \),
\[
  w_t = -2c w_x. \tag{9}
\]
Substituting Eq. (9) into Eqs. (7g) and (7h), we get \( g_2 = g_3 = 0 \).

**Case A** When \( a = \pm \sqrt{\frac{b}{2}} \), substituting Eq. (9) and \( a = \pm \sqrt{\frac{b}{2}} \) into Eqs. (7b), (7c), (7d), and (7e), we get
\[
  f_1 = 8 \left( w_{xx} + 2bw_x \right) w_x^2, \tag{10a}
\]
\[
  f_2 = \left( 42b^2 - 4c^2 - 4d \right) w_x^2 + \left( 18bw_{xx} + 2w_{xxx} \right) w_x - w_{xx}^2, \tag{10b}
\]
\[
  f_3 = 2b \left( \left( 18b^2 - 4d - 4c^2 \right) w_x + 6bw_{xx} + w_{xxx} \right), \tag{10c}
\]
\[
  f_4 = b^2 \left( 9b^2 - 4d - 4c^2 \right). \tag{10d}
\]

**Case A-1** When \( b = 0 \), substituting \( b = 0 \) into Eqs. (10), we get
\[
  f_3 = f_4 = 0, \quad f_1 = 8w_{xx}w_x^2, \quad f_2 = -4 \left( c^2 + d \right) w_x^2 + 2w_{xxx}w_x - w_{xx}^2. \tag{11}
\]
Setting \( f_1 = f_2 = 0 \) and noting that \( w_x \neq 0 \), we get
\[
  w_{xx} = 0, \quad d = -c^2. \tag{12}
\]
From Eqs. (9) and (12), we get
\[
  w = C_1 (x - 2ct) + C_0, \tag{13}
\]
where \( C_0, C_1 \) and \( c \) are arbitrary real constants.

Substituting \( a = \pm \sqrt{\frac{b}{2}} \), \( b = 0 \), Eqs. (12) and (13) into Eq. (6), we get an exact solution of Eq. (4) as follows:
\[
  \psi_2 = \pm \sqrt[3]{\frac{2C_1}{3(C_1 (x - 2ct) + C_0)}} e^{(cx-c^2t)}, \tag{14}
\]
where \( C_0, C_1 \) and \( c \) are arbitrary real constants.

**Case A-2** When \( b \neq 0 \), setting \( f_1 = f_4 = 0 \) and noting that \( w_x \neq 0 \), we get
\[
  w_x = \frac{-w_{xx}}{2b}, \quad d = \frac{9b^2}{4} - c^2. \tag{15}
\]
Substituting Eqs. (15) into Eqs. (10b) and (10c), we get
\[
  f_2 = b^2w_x^2, \quad f_3 = 2b^3w_x. \tag{16}
\]
Setting \( f_2 = f_3 = 0 \), we get \( bw_x = 0 \) from Eqs. (16), which contradicts with \( b \neq 0 \) and \( w_x \neq 0 \). Therefore, Eq. (4) has no solution in this case.

**Case B** When \( a = \pm \sqrt{\frac{b}{2}} \), substituting \( a = \pm \sqrt{\frac{b}{2}} \) and Eq. (9) into Eqs. (7b), (7c), (7d) and (7e), we get
\[
  f_1 = 0, \tag{17a}
\]
\[
  f_2 = \left( 2b^2 - 4c^2 - 4d \right) w_x^2 + \left( 2bw_{xx} + 2w_{xxx} \right) w_x - w_{xx}^2, \tag{17b}
\]
\[
  f_3 = 2b \left( \left( 2b^2 - 4c^2 - 4d \right) w_x + 2bw_{xx} + w_{xxx} \right), \tag{17c}
\]
\[
  f_4 = b^2 \left( b^2 - 4c^2 - 4d \right). \tag{17d}
\]

**Case B-1** When \( b = 0 \), substituting \( b = 0 \) into Eqs. (17), we get
\[
  f_1 = f_3 = f_4 = 0, \quad f_2 = -4 \left( c^2 + d \right) w_x^2 + 2w_x w_{xxx} - w_{xx}^2. \tag{18}
\]
Setting $f_2 = 0$ yields an equation as follows:

$$-\alpha w_x^2 + 2w_x w_{xx} - w_{xx}^2 = 0,$$

where $\alpha = 4(c^2 + d)$.

Using transformations $Z = w_x$ and $Y = \frac{w_{xx}}{w_x}$, Eq. (19) is reduced to

$$\frac{2YdY}{Y^2 - \alpha} = -\frac{dZ}{Z}.$$  (20)

Solving Eq. (20), we get

$$\frac{dZ}{dx} = \pm \sqrt{\alpha Z^2 + \beta Z},$$

namely

$$w_{xx} = \pm \sqrt{\alpha w_x^2 + \beta(t) w_x},$$  (21)

where $\beta(t)$ is an arbitrary function of $t$.

(1) When $\alpha = \beta(t) = 0$, from Eq. (21) we get

$$w = C_1(t)x + C_2(t),$$

where $C_1(t)$ and $C_2(t)$ are arbitrary functions of $t$.

Substituting the above equation into Eq. (9), we get

$$x \frac{dC_1(t)}{dt} + \frac{dC_2(t)}{dt} = -2cC_1(t).$$

Setting the coefficients of $x^j$ ($j = 0, 1$) to zero in the above equation, we get

$$\frac{dC_1(t)}{dt} = 0, \quad \frac{dC_2(t)}{dt} = -2cC_1(t).$$

Solving the above equations, we get

$$C_1(t) = C_1, \quad C_2(t) = C_0 - 2cC_1t,$$

where $C_k$ ($k = 0, 1$) are arbitrary real constants.

Then we get

$$w = C_1(x - 2ct) + C_0.$$  (22)

Substituting $a = \pm \frac{\sqrt{\alpha}}{2c}$, $b = 0$, $d = -c^2$ and Eq. (22) into Eq. (6), we get an exact solution of Eq. (4) as follows:

$$\psi_3 = \pm \sqrt{\frac{C_1}{2(C_1(x - 2ct) + C_0)}} e^{i(cx - c^2t)},$$  (23)

where $C_k$ ($k = 0, 1$) and $c$ are arbitrary real constants.

(2) When $\beta(t) = 0$, $\alpha > 0$, similar to (1), we get

$$w = C_0 + C_1 e^{2c \sqrt{c^2 + d}(x - 2ct)},$$

and an exact solution of Eq. (4) as follows:

$$\psi_4 = \pm \sqrt{\frac{\varepsilon C_1(c^2 + d)}{C_0 + C_1 e^{2c \sqrt{c^2 + d}(x - 2ct)}}} e^{i(cx + dt)},$$  (24)

where $\varepsilon = \pm 1$, $C_k$ ($k = 0, 1$), $c$ and $d$ are arbitrary real constants.
Especially, if $C_0C_1 > 0$, then Eq. (24) can be reduced to
\[
\psi_5 = \frac{\pm(c^2 + d)^{\frac{3}{4}}}{\sqrt{2}} \left(1 + \tanh\left(\sqrt{c^2 + d} \ (x - 2ct) + \xi_0\right)\right)^{\frac{1}{2}} e^{i(cx+dt)},
\]
where $C_k \ (k = 0, 1)$, $c$, $d$ and $\xi_0$ are arbitrary real constants.

If $C_0C_1 < 0$, then Eq. (24) can be reduced to
\[
\psi_6 = \frac{\pm(c^2 + d)^{\frac{3}{4}}}{\sqrt{2}} \left(1 + \coth\left(\sqrt{c^2 + d} \ (x - 2ct) + \xi_0\right)\right)^{\frac{1}{2}} e^{i(cx+dt)},
\]
where $C_k \ (k = 0, 1)$, $c$, $d$ and $\xi_0$ are arbitrary real constants.

(3) When $\beta(t) \neq 0$, $\alpha = 0$, similar to (1), we get
\[
w = \frac{1}{3} (x - 2ct + C_1)^3 + C_0,
\]
and an exact solution of Eq. (4) as follows:
\[
\psi_9 = \frac{\pm \sqrt{6} (x - 2ct + C_1)}{2 \sqrt{(x - 2ct + C_1)^3 + C_0}} e^{i(cx - c^2t)},
\]
where $C_k \ (k = 0, 1)$ and $c$ are arbitrary constants.

(4) When $\beta(t) \neq 0$, $\alpha > 0$, similar to (1), we get
\[
w = C_1 e^{C_4 (x - 2ct)} + C_2 e^{-C_4 (x - 2ct)} + C_3 (x - 2ct) + C_0,
\]
and an exact solution of Eq. (4) as follows:
\[
\psi_{10} = \pm \left(\frac{C_4 (C_1 e^{C_4 (x - 2ct)} - C_2 e^{-C_4 (x - 2ct)}) + C_3}{2 (C_1 e^{C_4 (x - 2ct)} + C_2 e^{-C_4 (x - 2ct)} + C_3 (x - 2ct) + C_0)}\right)^{\frac{1}{2}} e^{i \left(x + \left(\frac{C_4^2}{2} - c^2\right)\right)},
\]
where $C_4^2 - 4 (c^2 + d) = 0$, $4C_1C_2^2 C_2 + C_3^2 = 0$, $C_k \ (k = 0, 1, 2, 3, 4)$, $c$ and $d$ are arbitrary real constants.

(5) When $\beta(t) \neq 0$, $\alpha < 0$, similar to (1), we get
\[
w = \varepsilon C_1 (x - 2ct) + \cos(C_1 (x - 2ct) + C_2) + C_0,
\]
and an exact solution of Eq. (4) as follows:
\[
\psi_{11} = \pm \left(\frac{C_1 (\varepsilon - \sin(C_1 (x - 2ct) + C_2))}{2 (\varepsilon C_1 (x - 2ct) + \cos(C_1 (x - 2ct) + C_2) + C_0)}\right)^{\frac{1}{2}} e^{i \left(x - \left(\frac{C_1^2}{2} + c^2\right)\right)},
\]
where $C_1^2 + 4 (c^2 + d) = 0$, $\varepsilon = \pm 1$, $C_k \ (k = 0, 1, 2)$, $c$ and $d$ are arbitrary real constants.

Case B-2 When $b \neq 0$, setting $f_2 = f_3 = f_4 = 0$ in Eqs. (17b), (17c) and (17d), we get
\[
b = \pm 2 \sqrt{c^2 + d}, \quad w_x = -\frac{w_{xx}}{b}.
\]
Solving Eqs. (32) and noting that Eq. (9), we get
\[
w = C_0 + C_1 e^{\pm 2 \sqrt{c^2 + d} (x - 2ct)}.
\]
Substituting $a = \pm \sqrt{\frac{b}{2}}$, $b = \pm 2 \sqrt{c^2 + d}$ and Eq. (33) into Eq. (6), we find that this case is identical to Case B-1-(2).

In this section, by applying the homogeneous balance of undetermined coefficients method to the Eckhaus equation, new exact solutions of the Eckhaus equation are obtained. Among the solutions of the Eckhaus equation, $\psi_j \ (j = 3, 4, 5, 6, 7, 8)$ are the same as the results of [4, 10]. To our knowledge, other solutions of Eckhaus equation, $\psi_j \ (j = 2, 9, 10, 11)$ have not been reported in any literature.
4 Application to derive the bilinear equation of nonlinear PDEs

In this section, firstly, we modify the homogeneous balance of undetermined coefficients method to derive the bilinear equation of nonlinear PDEs. Then, more general bilinear equation of the KdV equation and the generalized Boussinesq equation are obtained by using the proposed method.

Let us still consider Eq. (1). Suppose that the solution of Eq. (1) is Eq. (2). By balancing the highest nonlinear terms and the highest order partial derivative terms, \( m, n \) and \( b \) can be determined. Substituting Eq. (2) into Eq. (1) and balancing the terms with \((\frac{u}{w})^i(\frac{w}{w})^j\)' yield a set of algebraic equations for \( a_{ij} \) \((i = 1, \cdots, m, j = 1, \cdots, n)\). Solving the set of algebraic equations and simplifying Eq. (1), we can get the bilinear equation of Eq. (1) directly or after integrating some times (generally, integrating times equal to the orders of lowest partial derivative of Eq. (1.) with respect to \( x, t \).

Next, the KdV equation and the generalized Boussinesq equation are chosen as examples to illustrate our method.

**Example 4.1.** Let us consider the celebrated KdV equation [9] in the form

\[
  u_t + u u_x + \delta u_{xxx} = 0, \tag{34}
\]

where \( \delta \) is a constant.

Suppose that the solution of Eq. (34) is Eq. (2). Balancing \( u_{xxx} \) and \( uu_x \) in Eq. (34), it is required that \( mb + 3 = 2mb + 1, \ nb = 2nb \). Choosing \( m = 2, n = 0 \) and \( b = 1 \), Eq. (2) can be written as

\[
  u = a_{20} (\ln w)_x + a_{10} (\ln w)_x + a_{00}, \tag{35}
\]

where \( a_{j0} \) \((j = 0, 1, 2)\) are constants to be determined later.

From Eq. (35), one can calculate the following derivatives:

\[
  u = a_{20} \left( \frac{w_{xx}}{w} - \frac{w_x^2}{w^2} \right) + a_{10} \frac{w_x}{w} + a_{00}. \tag{36a}
\]

\[
  u_x = a_{20} \left( \frac{w_{xxx}}{w} - \frac{3w_x w_{xx} w_x}{w^2} + \frac{2w_x^3}{w^3} \right) + a_{10} \left( \frac{w_{xx}}{w} - \frac{w_x^2}{w^2} \right). \tag{36b}
\]

\[
  u_t = a_{20} \left( \frac{w_{xxt}}{w} - \frac{w_{xx} w_t}{w^2} + \frac{2w_x w_{xxt}}{w^2} + \frac{2w_x^2 w_t}{w^3} \right) + a_{10} \left( \frac{w_{xt}}{w} - \frac{w_x w_t}{w^2} \right). \tag{36c}
\]

\[
  u_{xxx} = a_{20} \left( \frac{w_{xxxxx}}{w} - \frac{5w_{xxxx} w_x}{w^2} + \frac{10w_{xxx} w_{xx}}{w^2} + \frac{20w_{xxx} w_x^2}{w^3} + \frac{30w_x^2 w_x}{w^3} - \frac{60w_x w_x^2}{w^4} + \frac{24w_x^5}{w^5} \right) + a_{10} \left( \frac{w_{xxx}}{w} - \frac{4w_{xxx} w_x}{w^2} + \frac{3w_x^2 w_x}{w^2} + \frac{12w_x w_x^2}{w^3} - \frac{6w_x^4}{w^4} \right). \tag{36d}
\]

Equating the coefficients of \((\frac{w}{w})^5\) and \((\frac{w}{w})^4\) on the left-hand side of Eq. (34) to zero yields a set of algebraic equations for \( a_{20} \) and \( a_{10} \) as follows:

\[
-2a_{20}^2 + 24\delta a_{20} = 0, \quad 3a_{20}a_{10} - 6\delta a_{10} = 0.
\]

Solving the above algebraic equations, we get \( a_{20} = 12\delta, \ a_{10} = 0 \). Substituting \( a_{20} \) and \( a_{10} \) back into Eq. (35), we get

\[
  u = 12\delta (\ln w)_{xx} + a_{00}. \tag{37}
\]

where \( a_{00} \) is an arbitrary constant.

Substituting Eq. (37) into Eq. (34) and simplifying it, we get

\[
12\delta (K_1 + K_2 + K_3) = 0, \tag{38}
\]
where

\[ K_1 = \frac{w_{x,t}}{w} - \frac{2w_x w_{t}}{w} + \frac{w_{xx} w_t}{w} + \frac{2w_x^2}{w}, \quad K_2 = a_{00} \left( \frac{w_{xxx}}{w} - \frac{3w_{x} w_x}{w^2} + \frac{2w_3}{w^3} \right), \]

\[ K_3 = \delta \left( \frac{w_{xxxxx}}{w} + \frac{2w_{xxx} w_{xx}}{w^2} - \frac{5w_{xx} w_{xxx} w_x}{w^3} + \frac{16w_{xxx} w_x^2}{w^4} - \frac{6w_x^2}{w^5} \right). \]

Simplifying Eq. (38) and integrating once with respect to \( x \), we get

\[ \frac{\partial}{\partial x} \left( w_{x,t} w - w_x w_t \right) + \delta \left( w_{x,x,x,x} w - 4w_x w_{xx,x} + 3w_x^2 \right) + a_{00} \left( w_{xx} w - w_x^2 \right) = 0. \]  

(39)

Eq. (39) is identical to

\[ (w_{x,t} w - w_x w_t) + \delta \left( w_{x,x,x,x} w - 4w_x w_{xx,x} + 3w_x^2 \right) + a_{00} \left( w_{xx} w - w_x^2 \right) - C(t) w^2 = 0, \]  

where \( C(t) \) is an arbitrary function of \( t \) and \( a_{00} \) is an arbitrary constant.

Especially, taking \( C(t) \) as zero in Eq. (40), we get the bilinear equation of Eq. (34) as follows:

\[ (w_{x,t} w - w_x w_t) + \delta \left( w_{x,x,x,x} w - 4w_x w_{xx,x} + 3w_x^2 \right) + a_{00} \left( w_{xx} w - w_x^2 \right) = 0. \]  

(41)

Eq. (41) can be written concisely in terms of \( D \)-operator as

\[ \left( D_x D_t + \delta D_x^4 + a_{00} D_x^2 \right) w \cdot w = 0, \]  

(42)

where

\[ D_x^m D_t^n a \cdot b = (\partial_x - \partial_{x'})^m (\partial_t - \partial_{t'})^n a(x,t) b(x',t') \bigg|_{x'=x,t'=t}. \]

**Remark 4.2.** Applying the Hirota’s method \([11]\), the bilinear equation of Eq. (34) can be written as

\[ \left( D_x D_t + \delta D_x^4 \right) w \cdot w = 0. \]  

(43)

Eq. (43) is obtained by setting \( a_{00} = 0 \) in Eq. (42). Obviously, Eq. (43) is a special case of Eq. (42). Therefore, more general bilinear equation of the KdV equation is obtained by using our method.

**Remark 4.3.** Applying the perturbation method \([11]\) to Eq. (42), we can get 1-soliton solution and 2-soliton solution of Eq. (42) as follows:

\[ w_1 = 1 + e^{\eta_1}, \quad w_2 = 1 + e^{\eta_1} + e^{\eta_2} + \frac{(P_1 - P_2)^2}{(P_1 + P_2)^2} e^{\eta_1 + \eta_2}. \]  

(44)

where \( \eta_j = P_j x + (a_{00} P_j + \delta P_j^3) t + \xi_j^0 \), and \( P_j, \xi_j^0 \) (\( j = 1, 2 \)) and \( a_{00} \) are arbitrary constants.

Substituting Eqs. (44) into Eq. (37), 1-soliton solution and 2-soliton solution of Eq. (34) can be obtained. Similarly, \( N \)-soliton solution of Eq. (34) can be obtained.

**Example 4.4.** The generalized Boussinesq equation reads

\[ u_{tt} + 2\alpha u_{xt} + \left( \alpha^2 + \beta \right) u_{xx} + \gamma uu_{xx} + \delta u_{xxxx} = 0, \]  

(45)

where \( \alpha, \beta, \gamma \) and \( \delta \) are known constants.

Suppose that the solution of Eq. (45) is Eq. (2). In order to balance \( u_{xxxx} \) and \( uu_x \) in Eq. (45), it is required that \( mb + 4 = 2mb + 2, nb = 2nb \). Choosing \( m = 2, n = 0 \) and \( b = 1 \), Eq. (2) can be written as

\[ u = a_{20}(\ln w)_{xx} + a_{10}(\ln w)_x + a_{00}, \]  

(46)

where \( a_{j0} \) (\( j = 0, 1, 2 \)) are constants to be determined later.
Substituting Eq. (46) into Eq. (45) and equating the coefficients of \( \left( \frac{w_x}{w} \right)^6 \) and \( \left( \frac{w_x}{w} \right)^5 \) on the left-hand side of Eq. (45) to zero yield a set of algebraic equations for \( a_{20} \) and \( a_{10} \). Solving the algebraic equations, we get \( a_{20} = \frac{6\delta}{\gamma} \), \( a_{10} = 0 \). Substituting \( a_{20} \) and \( a_{10} \) back into Eq. (46), we get

\[
u = \frac{6\delta}{\gamma} (\ln w)_{xx} + a_{00},
\]

(47)

where \( a_{00} \) is an arbitrary constant.

Substituting Eq. (47) into Eq. (45), we get

\[
\frac{6\delta}{\gamma} (K_1 + K_2 + K_3) = 0,
\]

(48)

where

\[
K_1 = \frac{w_{xxxxx}}{w} - \frac{2w_x w_{xxt} + w_{xxx} w_{xt} + 2w_x w_{xxt} + 2w_x^2 w_{xt} + 2w_x^2 w_{xt} + 2w_x^2 w_{xt} + 8w_x w_t w_{xt}}{w^2} + \frac{6w_x^2 w_{xt}}{w^4} + \beta \left( \frac{w_{xxx} w_{xx} + 4w_x w_{xxx}}{w^2} - \frac{6w_x^4}{w^4} \right) + \alpha \left( \frac{w_{xxx} w_{xx} + 4w_x w_{xxx}}{w^2} - \frac{6w_x^4}{w^4} \right) - \alpha \left( \frac{2w_{xxx} w_{xx} + 6w_x w_{xxx} + 2w_x w_{xxx}}{w^2} + 12w_x w_t w_{xxx} + 12w_x^2 w_{xt} - 12w_x^2 w_{xxx} \right),
\]

\[
K_2 = a_{00} \left( \frac{2\gamma w_{xxx} w_x}{w} - \frac{8\gamma w_x w_{xxx} + 6\gamma w_x^2}{w^3} + \frac{24\gamma w_{xxx} w_{xt}}{w^4} - \frac{12\gamma w_x^4}{w^4} \right),
\]

\[
K_3 = \delta \left( \frac{w_{xxx} w_{xx} + 2w_x^2 w_{xxx} - 6w_x w_{xxx} - 3w_x w_{xxx}}{w^2} - \frac{18w_x^2 w_{xxx} - 6w_x^3}{w^3} + \frac{18w_x^3 w_{xxx} - 24w_x^3 w_{xxx}}{w^3} \right).
\]

Simplifying Eq. (48) and integrating twice with respect to \( x \), we get

\[
\frac{\partial^2}{\partial x^2} \left( \alpha^2 + \beta + 2\gamma a_{00} \left( \frac{w_{xxx} w_x}{w^2} \right) + 2\alpha \left( w_{xxt} - w_x w_t \right) + \left( \frac{w_x w_{xt} - w_x^2}{w^2} \right) + \delta \left( w_{xxx} w_{xx} - 4w_x w_{xxx} + 3w_x^2 \right) \right) = 0.
\]

(49)

Eq. (49) is identical to

\[
\left( \alpha^2 + \beta + 2\gamma a_{00} \left( w_{xxx} w_x - w_x^2 \right) + 2\alpha (w_{xxx} w_x - w_x w_t) + (w_{xxt} - w_x^2) \right) + \delta (w_{xxx} w_{xx} - 4w_x w_{xxx} + 3w_x^2) = 0,
\]

(50)

where \( C_1 (t) \) and \( C_2 (t) \) are arbitrary functions of \( t \), and \( a_{00} \) is an arbitrary constant.

Especially, letting \( C_1 (t) = C_2 (t) = 0 \) in Eq. (50), we get the bilinear equation of Eq. (45) as follows:

\[
\left( \alpha^2 + \beta + 2\gamma a_{00} \left( w_{xxx} w_x - w_x^2 \right) + 2\alpha (w_{xxx} w_x - w_x w_t) + (w_{xxt} - w_x^2) \right) + \delta (w_{xxx} w_{xx} - 4w_x w_{xxx} + 3w_x^2) = 0.
\]

(51)

Eq. (51) can be written concisely in terms of \( D \)-operator as

\[
(2\alpha D_x D_t + \delta D_x^4 + \left( \alpha^2 + \beta + 2\gamma a_{00} \right) D_x^2 + D_t^2) w \cdot w = 0,
\]

(52)

where \( a_{00} \) is an arbitrary constant.

So far, the proposed method is successfully used to establish the bilinear equation of the KdV equation and the generalized Boussinesq equation. Our method can be used to derive the bilinear equations of some nonlinear PDEs.
5 Conclusions

The homogeneous balance of undetermined coefficients method is successfully used to solve such nonlinear PDEs, the balance numbers of which are not positive integers. By applying our method to the Eckhaus equation, more exact solutions are obtained. The proposed method can also be used to derive the bilinear equation of nonlinear PDEs. More general bilinear equation of the KdV equation and the generalized Boussinesq equation can be obtained by applying our method.

Once the bilinear equation of nonlinear PDE is obtained, the perturbation method can be employed to obtain the \(N\)-soliton solution for the nonlinear PDE. Many exact solutions of the nonlinear PDE are obtained by using the three-wave method and the homoclinic test approach. Generally, some nonlinear PDE can be linearized or homogenized by using the homogeneous balance of undetermined coefficients method.

Many well-known nonlinear PDEs can be handled by our method. The performance of our method is found to be simple and efficient. The availability of computer systems like Maple facilitates the tedious algebraic calculations. Our method is also a standard and computable method, which allows us to solve complicated and tedious algebraic calculations.

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