Introduction

The 2-primary Hopf invariant 1 elements in the stable homotopy groups of spheres form the most accessible family of elements. In this paper we explore some properties of the $E_\infty$ ring spectra obtained from certain iterated mapping cones by applying the free algebra functor. In fact, these are equivalent to Thom spectra over infinite loop spaces related to the classifying spaces $BSO$, $BSpin$, $BString$.

We show that the homology of these Thom spectra are all extended comodule algebras of the form $A, \square_{A(r)}$, $P_r$ over the dual Steenrod algebra $A_*$ with $A, \square_{A(r)}, \mathbb{F}_2$ as an algebra retract. This suggests that these spectra might be wedges of module spectra over the ring spectra $HZ$, $kO$ or $tmf$, however apart from the first case, we have no concrete results on this.

**Contents**

Introduction
1. Iterated mapping cones built with elements of Hopf invariant 1
2. Some $E_\infty$ Thom spectra
3. Some coalgebra
4. The homology of $M_j r$ for $r = 1, 2, 3$
   4.1. The homology of $M_j 1$
   4.2. The homology of $M_j 2$
   4.3. The homology of $M_j 3$
5. Some other examples
   5.1. An example related to $kU$
   5.2. An example related to the Brown-Peterson spectrum
6. Some speculation
Appendix A. The homology of connective covers of $BO$
Appendix B. Dyer-Lashof operations and Steenrod coactions
References
fact, these are equivalent to Thom spectra over infinite loop spaces related to the classifying spaces $BSO$, $BSpin$, $BString$.

We show that the homology of these Thom spectra are all extended comodule algebras of the form $A_\ast \Box_{A(r)_\ast} P_\ast$ over the dual Steenrod algebra $A_\ast$ with $A_\ast \Box_{A(r)_\ast} \mathbb{F}_2$ as an algebra retract. This suggests that these spectra might be wedges of module spectra over the ring spectra $H\mathbb{Z}$, $kO$ or tmf, however apart from the first case, we have no concrete results on this.

Our results and methods of proof owe much to work of Arunas Liulevicius \cite{13, 14} and David Pengelley \cite{18–20}, and are also related to work of Tony Bahri and Mark Mahowald \cite{2}. However we use some additional ingredients: in particular we make use of formulae for the interaction between the $A_\ast$-coaction and the Dyer-Lashof operations in the homology of an $E_\infty$ ring spectrum described in \cite{6}. We also take a slightly different approach to identifying when the homology of a ring spectrum is a cotensor product of the dual Steenrod algebra $A_\ast$ over a finite quotient Hopf algebra $A(n)_\ast$, making use the fact that the dual Steenrod algebra is an extended $A(n)_\ast$-comodule; in turn this is a consequence of the $P$-algebra property of the Steenrod algebra $A_\ast$.

**Conventions:** We will work 2-locally throughout this paper, thus all simply connected spaces and spectra will be assumed localised at the prime 2, and $\mathcal{M}_S$ will denote the the category of $S$-modules where $S$ is the 2-local sphere spectrum as considered in \cite{10}. We will write $S^0$ for a chosen cofibrant replacement for the $S$-module $S$ and $S^n = \Sigma^n S^0$. When discussing CW skeleta of a space $X$ we will always assume that we have chosen minimal CW models in the sense of \cite{8} so that cells correspond to a basis of $H_\ast(X) = H_\ast(X; \mathbb{F}_2)$.

**Notation:** When working with cell complexes (of spaces or spectra) we will often indicate the mapping cone of a coextension $\tilde{g}$ of a map $g: S^n \to S^k$ by writing $X \cup_f e^k \cup_g e^{n+1}$.

\[
\begin{array}{ccc}
X & \longrightarrow & X \cup_f e^k \longrightarrow S^k \\
\downarrow & & \downarrow \\
S^n & \longrightarrow & g \\
\tilde{g} & \longleftarrow & S^0 \\
X & \longrightarrow & X \\
\end{array}
\]

Of course this notation is ambiguous, but nevertheless suggestive. When working stably with spectra we will often write $h: S^{n+r} \to S^{k+r}$ for the suspension $\Sigma^r h$ of a map $h: S^n \to S^k$. We will also often identify stable homotopy classes with representing elements.

1. **Iterated Mapping Cones Built with Elements of Hopf Invariant 1**

The results of this section can be proved by homotopy theory calculations using basic facts about the elements of Hopf invariant 1 in the homotopy groups of the sphere spectrum $S^0$,

\[
2 \in \pi_0(S^0), \quad \eta \in \pi_1(S^0), \quad \nu \in \pi_3(S^0), \quad \sigma \in \pi_7(S^0).
\]

In particular the following identities are well known, see for example \cite{21} figure A3.1a):

\[
(1.1) \quad 2\eta = \eta \nu = \nu \sigma = 0.
\]
Proposition 1.1. The following CW spectra exist:

\[ S^0_0 \cup \eta e^2 \cup 2 e^3, \ S^0_0 \cup \nu e^4 \cup \eta e^6 \cup 2 e^7, \ S^0_0 \cup \nu e^8 \cup \eta e^{12} \cup \nu e^{14} \cup 2 e^{15}. \]

Sketch of proof. In each of the iterated mapping cones below, we will denote the homology generator corresponding to the unique cell in dimension \( n \) by \( x_n \).

The case of \( S^0_0 \cup \eta \cup 2 e^3 \) is obvious.

Consider the mapping cone of \( \nu \), \( C_\nu = S^0_0 \cup \nu e^4 \). As \( \nu \eta = 0 \), there is a factorisation of \( \eta \) on the 4-sphere through \( C_\nu \).

\[
\begin{array}{cccc}
S^5 & \downarrow 2 & & S^5 \\
& \eta & \nearrow \nu \eta = 0 & \\
S^3 & \nu & \to & S^0 & \to & C_\nu & \to & S^4 & \to & \nu & \to & S^1 \\
\end{array}
\]

Also, \( 2 \eta = 0 \) and \( \pi_5(S^0_0) = 0 \), hence \( 2\eta x_4 = 0 \). A cobar representative for \( \eta x_4 \) in the classical Adams E_2-term is

\[ [\zeta_1^2 \otimes x_4 + \zeta_2^2 \otimes x_0] \in \text{Ext}^{1,6}_{\mathcal{A}_*}(\mathbb{F}_2, H_*(C_\nu)). \]

We can form the mapping cone \( C_{\eta x_4} = C_\nu \cup \eta x_4 e^6 \) and since \( 2\eta x_4 = 0 \), there is a factorisation of \( 2 \) on the 6-sphere through \( C_{\eta x_4} \).

\[
\begin{array}{cccc}
S^6 & \downarrow 2 & & S^6 \\
\nearrow & & \nearrow \nu \eta = 0 & \\
S^5 & \nu_1 & \to & C_\nu & \to & C_{\eta x_4} & \to & S^6 & \to & \Sigma C_\nu \\
\end{array}
\]

A cobar representative of \( \eta x_6 \) is

\[ [\zeta_1 \otimes x_6 + \zeta_2 \otimes x_4 + \zeta_3 \otimes x_0] \in \text{Ext}^{1,7}_{\mathcal{A}_*}(\mathbb{F}_2, H_*(C_{\eta x_4})). \]

Consider the mapping cone of \( \sigma \), \( C_\sigma = S^0_0 \cup \sigma e^8 \). As \( \sigma \nu = 0 \), there is a factorisation of \( \nu \) on the 8-cell through \( C_\sigma \).

\[
\begin{array}{cccc}
S^{12} & \downarrow \eta & & \\
\nearrow & & \nearrow \sigma \nu = 0 & \\
S^7 & \sigma & \to & S^0 & \to & C_\sigma & \to & S^8 & \to & \sigma & \to & S^1 \\
\end{array}
\]

Also, \( \nu \eta = 0 \) and \( \pi_{12}(S^0_0) = 0 = \pi_{13}(S^0_0) \), hence \( \eta(\nu x_8) = 0 \).

As \( \text{Ext}^{1,12}_{\mathcal{A}_*}(\mathbb{F}_2, H_*(S^0_0)) = 0 \), the element

\[ [\zeta_1^4 \otimes x_8 + \zeta_2^4 \otimes x_0] \in \text{Ext}^{1,12}_{\mathcal{A}_*}(\mathbb{F}_2, H_*(C_\sigma)) \]

is a cobar representative for \( \nu x_8 \).
We can form the mapping cone \( C_{\nu x^8} = C_\sigma \cup_{\nu x^8} e^{12} \) and since \( \eta \nu x^8 = 0 \), there is a factorisation of \( \eta \) on the 12-sphere through \( C_{\nu x^8} \).

\[
\begin{array}{ccc}
S^{13} & \xrightarrow{2} & S^{13} \\
\downarrow & & \downarrow \\
S^{11} & \xrightarrow{\eta^2 x^8} & C_\sigma \xrightarrow{\eta} C_{\nu x^8} \xrightarrow{\eta} S^{12} \xrightarrow{\nu x^8} \Sigma C_\sigma \\
\end{array}
\]

As part of the long exact sequence for the homotopy of mapping cone we have the exact sequence

\[ \pi_{13}(S^7) \xrightarrow{\sigma} \pi_{13}(S^0) \rightarrow \pi_{13}(C_\sigma) \rightarrow \pi_{13}(S^8) \]

we have \( \pi_{13}(S^0) = 0 = \pi_{13}(S^8) \), so \( \pi_{13}(C_\sigma) = 0 \). Therefore \( 2(\eta^2 x^8) = 0 \) and we can factorise 2 on the 14-sphere through the mapping cone of \( \eta^2 x^8 \).

\[
\begin{array}{ccc}
S^{13} & \xrightarrow{\eta^2 x^8} & C_{\nu x^8} \xrightarrow{\nu x^8} C_{\eta^2 x^8} \xrightarrow{\eta^2 x^8} S^{14} \xrightarrow{\eta^2 x^8} \Sigma C_{\nu x^8} \\
\end{array}
\]

A cobar representative of \( 2x_{14} \) is

\[ [\zeta_1 \otimes x_{14} + \zeta_2 \otimes x_{12} + \zeta_3 \otimes x_8 + \zeta_4 \otimes x_0] \in \text{Ext}^{1,15}_{A_*}(F_2, H_*(C_{\eta^2 x^8})). \]

The homology of the mapping cone \( C_{\nu x^8} \) has a basis \( x_0, x_8, x_{12}, x_{14}, x_{15} \), with coaction given by

\[
\begin{align*}
\psi x_8 &= \zeta_1^8 \otimes 1 + 1 \otimes x_8, \\
\psi x_{12} &= \zeta_2^4 \otimes 1 + \zeta_4^4 \otimes x_8 + 1 \otimes x_{12}, \\
\psi x_{14} &= \zeta_3^2 \otimes 1 + \zeta_2^2 \otimes x_8 + \zeta_1^2 \otimes x_{12} + 1 \otimes x_{14}, \\
\psi x_{15} &= \zeta_4 \otimes 1 + \zeta_3 \otimes x_8 + \zeta_2 \otimes x_{12} + 1 \otimes x_{15}.
\end{align*}
\]

These calculations show that CW spectra of the stated forms do indeed exist. \( \square \)

2. Some \( E_\infty \) Thom spectra

Consider the three infinite loop spaces \( BSO = BO(2), BSpin = BO(4) \) and \( BString = BO(8) \). The 3-skeleton of \( BSO \) is

\[ BSO^{[3]} = BO^{[2]} \cup_2 \mathbb{E}^3 \]

since \( \text{Sq}^1 w_2 = w_3 \). Similarly, the 7-skeleton of \( BSpin \) is

\[ BSpin^{[7]} = BO^{[4]} \cup_2 \mathbb{E}^7 \]

since \( \text{Sq}^2 w_4 = w_6 \) and \( \text{Sq}^1 w_6 = w_7 \). Finally, the 15-skeleton of \( BString \) is

\[ BString^{[15]} = BO^{[8]} \cup_2 \mathbb{E}^{15} \]

since \( \text{Sq}^4 w_8 = w_{12}, \text{Sq}^2 w_{12} = w_{14} \) and \( \text{Sq}^1 w_{14} = w_{15} \).
The skeletal inclusion maps induce (virtual) bundles whose Thom spectra are themselves skeleta of the universal Thom spectra $M\Sigma\Omega\infty$, $M\Sigma\Omega\infty$ skeleta of the universal Thom spectra $\Sigma\Omega\infty$ Thom spectrum $M\Sigma\Omega\infty$. Morphisms in homology whose images contain the lowest degree generators:

\[ 1, a^{(1)}_{1,0}, a^{(3)}_{3,0}, a^{(2)}_{7,0} \in H_*(M\Sigma\Omega\infty), \]
\[ 1, a^{(2)}_{1,0}, a^{(1)}_{3,0}, a^{(3)}_{7,0} \in H_*(M\Sigma\Omega\infty), \]
\[ 1, a^{(3)}_{1,0}, a^{(2)}_{3,0}, a^{(1)}_{7,0}, a^{(1)}_{15,0} \in H_*(M\Sigma\Omega\infty). \]

The natural orientations $M\Omega(n) \to \text{HFr}_2$ induce homomorphisms $H_*(M\Omega(n)) \to A_*$ under which $a^{(r)}_{1,0} \mapsto \zeta_1^{2r}$, $a^{(r)}_{3,0} \mapsto \zeta_2^{2r}$, $a^{(r)}_{7,0} \mapsto \zeta_3^{2r}$, and $a^{(r)}_{15,0} \mapsto \zeta_4^{2r}$.

We also note that the skeleta can be identified with skeleta of $HZ, kO$ and tmf, namely there are orientations inducing weak equivalences

\[ (2.1) \quad M\Omega(n)^[3] \cong HZ^n[3], \quad M\Omega(n)^[7] \cong kO^n[7], \quad M\Omega(n)^[15] \cong \text{tmf}^{[15]}. \]

The first two are induced from well known orientations, while the third relies on unpublished work of Ando, Hopkins & Rezk [1]. Actually such morphisms can be produced using the reduced free commutative $S$-algebra functor $\tilde{P}$ of [3], which has a universal property analogous to that of the usual free functor $P$ of [10].

**Proposition 2.1.** For $r = 1, 2, 3$, the natural map $M\Omega(n)^[2r+1-1] \to Mj_r$ has a unique extension to a weak equivalence of $\mathcal{E}_\infty$ ring spectra

\[ \tilde{P}M\Omega(n)^[2r+1-1] \cong Mj_r. \]

The orientations of (2.1) induce morphisms of $\mathcal{E}_\infty$ ring spectra

\[ \tilde{P}M\Omega(n)^[3] \to HZ, \quad \tilde{P}M\Omega(n)^[7] \to kO, \quad \tilde{P}M\Omega(n)^[15] \to \text{tmf}. \]

**Proof.** The existence of such morphisms depends on the universal property of $\tilde{P}$. The proof that those of the first kind are equivalences depends on a comparison of the homology rings using Theorem 2.3 below. \[ \square \]
Remark 2.2. In fact the weak equivalences of (2.1) extend to weak equivalences

\[(2.2) \quad M_{j_1} \sim H\mathbb{Z}[4], \quad M_{j_2} \sim kO[8], \quad M_{j_3} \sim \text{tmf}[16].\]

The homology of $M_{j_r}$ can be determined from that of the underlying infinite loop space using the Thom isomorphism, while that for the others depends on a general description of the homology of $H_*(\hat{\mathbb{F}}X)$ which can be found in [5].

Theorem 2.3. The homology rings of the Thom spectra $M_{j_r}$ are given by

\[
H_*(M_{j_1}) = \mathbb{F}_2[Y_3, \zeta_2 : I, J \text{ admissible, } \text{exc}(I) > 2, \text{exc}(J) > 3],
\]
\[
H_*(M_{j_2}) = \mathbb{F}_2[Y_4, \zeta_2 : I, J, K \text{ admissible, } \text{exc}(I) > 4, \text{exc}(J) > 6, \text{exc}(K) > 7],
\]
\[
H_*(M_{j_3}) = \mathbb{F}_2[Y_5, \zeta_2 : I, J, K, L \text{ admissible, } \text{exc}(I) > 8, \text{exc}(J) > 12, \text{exc}(K) > 14, \text{exc}(L) > 15].
\]

The homomorphisms $H_*(M_{j_r}) \to A_*$ induced by the $E_\infty$ orientations $M_{j_r} \to H\mathbb{F}_2$ have as images

\[
\mathbb{F}_2[\zeta_1^2, \zeta_2, \zeta_3, \ldots] \cong H_*(H\mathbb{Z}), \quad \mathbb{F}_2[\zeta_1^2, \zeta_2, \zeta_3, \zeta_4, \ldots] \cong H_*(kO), \quad \mathbb{F}_2[\zeta_1^4, \zeta_2^2, \zeta_3^2, \zeta_4, \zeta_5, \ldots] \cong H_*(\text{tmf}).
\]

3. Some coalgebra

First we recall a standard algebraic result, for example see [19, lemma 3.1]. We work over a field $k$ and with $k$-vector spaces. We will set $\otimes = \otimes_k$. There are slight modifications required for the graded case which we leave the reader to formulate, however as we work exclusively in characteristic 2, these have no significant effect in this paper. We refer to the classic paper of Milnor and Moore [16] for background material on coalgebra.

Let $A$ be a commutative Hopf algebra over a field $k$, and let $B$ be a quotient Hopf algebra of $A$. We denote the product and antipode on $A$ by $\varphi_A$ and $\chi$, and the coaction on a left comodule $D$ by $\psi_D$. We will identify the cotensor product $A \otimes_B k \subseteq A \otimes k$ with a subalgebra of $A$ under the canonical isomorphism $A \otimes k \xrightarrow{\cong} A$.

Lemma 3.1. Let $D$ be a commutative $A$-comodule algebra. Then there is an isomorphism of $A$-comodule algebras

\[(3.1) \quad (\varphi_A \otimes \Id_D) \circ (\Id_A \otimes \psi_D) : (A \otimes_B k) \otimes D \xrightarrow{\cong} A \otimes_B D; \quad a \otimes x \mapsto \sum_i a_i \otimes x_i,
\]

where $\psi_D x = \sum_i a_i \otimes x_i$ denotes the coaction on $x \in D$.

Here the codomain has the diagonal $A$-comodule structure, while the domain has the left $A$-comodule structure.

Here is an easily proved generalisation of this result.

Lemma 3.2. Let $C$ be a commutative $B$-comodule algebra and let $D$ be a commutative $A$-comodule algebra, then there is an isomorphism of $A$-comodule algebras

\[(3.2) \quad (A \otimes_B C) \otimes D \xrightarrow{\cong} A \otimes_B (C \otimes D),
\]

where the domain has the diagonal left $A$-coaction and $C \otimes D$ has the diagonal left $B$-coaction.
Explicitly, on an element
\[ \sum_r u_r \otimes v_r \otimes x \in (\mathbb{A} \Box_B \mathbb{C}) \otimes D \subseteq A \otimes C \otimes D, \]
the isomorphism has the effect
\[ \sum_r u_r \otimes v_r \otimes w \mapsto \sum_r \sum_i u_r a_i \otimes v_r \otimes w_i, \]
where \( \psi_D w = \sum_i a_i \otimes w_i \) as above. Similarly the inverse is given by
\[ \sum_r b_r \otimes y_r \otimes w_r \mapsto \sum_r \sum_i b_r \chi(a_{r,i}) \otimes v_r \otimes w_{r,i}. \]

Now suppose that \( H \) is a finite dimensional Hopf algebra. If \( K \) is a sub-Hopf algebra of \( H \), it is well known that \( H \) is a free left or right \( K \)-module, i.e., \( H \cong K \otimes_U H \) or \( H \cong U \otimes K \) for a vector space \( U \) (see [17, Theorems 31.1.5 & 3.3.1]). This dualises as follows: If \( L \) is a quotient Hopf algebra of \( H \), then \( H \) is an extended left or right \( L \)-comodule, i.e., \( H \cong L \otimes V \) or \( H \cong V \otimes L \) for a vector space \( V \); in fact, \( V = H \Box_L k \). More generally, according to Margolis [15, pps 193 & 240], if \( H \) is a \( P \)-algebra then a result of the first kind holds for any finite dimensional sub-Hopf algebra \( K \).

We need to make use of the finite dual of a Hopf algebra \( H \), namely
\[ H^\circ = \{ f \in \text{Hom}_k(H,k) : \exists I \vartriangleleft H \text{ such that } \text{codim} \ I < \infty \text{ and } I \subseteq \ker f \}. \]
Then \( H^\circ \) becomes a Hopf algebra with product and coproduct obtained from the adjoints of the coproduct and product of \( H \). We will say that \( H \) is a \( P \)-coalgebra if \( H^\circ \) is a \( P \)-algebra.

**Lemma 3.3.** Suppose that \( A \) is a commutative Hopf algebra which is a \( P \)-coalgebra. If \( B \) is a finite dimensional quotient Hopf algebra of \( A \), then \( A \) is an extended right (or left) \( B \)-comodule, i.e., \( A \cong W \otimes B \) (or \( A \cong B \otimes W \)) for some vector space \( W \), and in fact \( W \cong A \Box_B k \) (or \( W \cong k \Box_B A \)).

**Corollary 3.4.** For any right \( B \)-comodule \( L \) or left \( B \)-comodule \( M \), as vector spaces,
\[ A \Box_B M \cong (A \Box_B k) \otimes M, \quad L \Box_B A \cong L \otimes (k \Box_B A). \]

These are isomorphisms of left or right \( A \)-comodules for suitable comodule structures on the right hand sides.

To understand the relevant \( A \)-comodule structure on \((A \Box_B k) \otimes M\), note that there is an isomorphism of left \( A \)-comodules
\[ (A \Box_B k) \otimes M \xrightarrow{\text{Id} \otimes \psi_M} (A \Box_B k) \otimes B \otimes M \cong A \otimes M \]
where the right hand factor is the isomorphism of Lemma 3.3.

Crucially for our purposes, for a prime \( p \), the Steenrod algebra \( \mathcal{A}^* \) is a \( P \)-algebra in the sense of Margolis [15], i.e., it is a union of finite sub-Hopf algebras. When \( p = 2 \),
\[ \mathcal{A}^* = \bigcup_{n \geq 0} \mathcal{A}(n)^*, \]
and it follows from the preceding results that if \( n \geq 0 \), \( A^* \) is free as a right or left \( A(n)^* \)-module, see [13] pps 193 & 240. Dually, \( (A_n)^0 = A^* \) and \( A_* \) is an extended \( A(n)_* \)-comodule:

(3.3) \[ A_* \cong (A_* \square A(n), F_2) \otimes A(n)_*, \]

(3.4) \[ A_* \cong A(n)_* \otimes (F_2 \square A(n), A_*). \]

Given this, we see that for any left \( A(n)_* \)-comodule \( M_* \), as vector spaces

(3.5) \[ A_* \square A(n)_* M_* \cong (A_* \square A(n)_* F_2) \otimes M_* . \]

In fact this is also an isomorphism left \( A_* \)-comodules.

Here is an explicit description of isomorphisms of the type given by Lemma 3.3. For \( n \geq 0 \), we will use the function

\[ e_n : N \to N; \quad e_n(i) = \begin{cases} 2^{n+2-i} & \text{if } 1 \leq i \leq n + 2, \\ 1 & \text{if } i \geq n + 3. \end{cases} \]

For an natural number \( r \), write

\[ r = r'(n,i) e_n(i) + r''(n,i) \]

where \( 0 \leq r''(n,i) < e_n(i) \). We note that

\[ A_* \square A(n), F_2 = F_2[\zeta_1^{e_n(1)}, \zeta_2^{e_n(2)}, \zeta_3^{e_n(3)}, \ldots] \subseteq A_* , \]

and

\[ A(n)_* = A_* / (A_* \square A(n)_* F_2) = A_* / (\zeta_1^{e_n(1)}, \zeta_2^{e_n(2)}, \zeta_3^{e_n(3)}, \ldots) . \]

We will indicate elements of \( A(n)_* \) by writing \( |z| \) for the coset of \( z \) which is always chosen to be a sum of monomials \( \zeta_1^s \zeta_2^r \cdots \zeta_\ell^\ell \) with exponents satisfying \( 0 \leq s_i < e_n(i) \).

**Proposition 3.5.** For \( n \geq 0 \) there is an isomorphism of right \( A(n)_* \)-comodules

\[ A_* \cong (A_* \square A(n)_* F_2) \otimes A(n)_* \]

given on basic tensors by

\[ \zeta_1^r \zeta_2^r \cdots \zeta_\ell^r \longleftrightarrow \zeta_1^{r'(n,1)e_n(1)} \zeta_2^{r'(n,2)e_n(2)} \cdots \zeta_\ell^{r'(n,\ell)e_n(\ell)} \otimes \left\| \zeta_1^{r''(n,1)} \zeta_2^{r''(n,2)} \cdots \zeta_\ell^{r''(n,\ell)} \right\| . \]

We will also use the following result to construct algebraic maps in lieu of geometric ones. The proof is a straightforward generalisation of a standard one for the case where \( B = k \).

**Lemma 3.6.** Suppose that \( M \) is a left \( A \)-comodule and \( N \) is a left \( B \)-comodule. Then there is a natural isomorphism

\[ \text{Comod}_B(M, N) \cong \text{Comod}_A(M, A \square_B N); \quad f \mapsto \tilde{f}, \]

where \( \tilde{f} \) is the unique factorisation of \( (\text{Id} \otimes f)\psi_M \) through \( A \square_B N \).

Furthermore, if \( M \) is an \( A \)-comodule algebra and \( N \) is a \( B \)-comodule algebra, then if \( f \) is an algebra homomorphism, so is \( \tilde{f} \).
As an example of the multiplicative version of this result, suppose that $M$ is an $A$-comodule algebra which is augmented. Then there is a composite homomorphism of $B$-comodule algebras $\alpha: M \to k \to N$ giving rise homomorphism of $A$-comodule algebras

$$\tilde{\alpha}: M \to A \Box_B N; \quad \tilde{\alpha}(x) = a \otimes 1,$$

where $\psi_M(x) = a \otimes 1 + \cdots + 1 \otimes x$.

4. The homology of $M_{j_r}$ for $r = 1, 2, 3$

Now we analyse the specific cases for $H_*(M_{j_r})$ for $r = 1, 2, 3$. Since some of the details differ in each case we treat these separately. In each case there is a commutative diagram of commutative $A_\ast$-comodule algebras

(4.1)

\[
\begin{array}{c}
H_*(M_{j_r}) \\
\xrightarrow{\psi} \quad (A_\ast \Box_{A(r-1)_\ast} F_2) \otimes H_*(M_{j_r}) \xrightarrow{\pi} \quad (A_\ast \Box_{A(r-1)_\ast} F_2) \otimes H_*(M_{j_r}) / I_r \\
\xrightarrow{\cong} \quad A_\ast \Box_{A(r-1)_\ast} H_*(M_{j_r}) \xrightarrow{\pi} \quad A_\ast \Box_{A(r-1)_\ast} H_*(M_{j_r}) / I_r \\
\xrightarrow{\cong} \quad A_\ast \otimes H_*(M_{j_r}) \xrightarrow{\pi} \quad A_\ast \otimes H_*(M_{j_r}) / I_r
\end{array}
\]

in which $I_r \triangleleft H_*(M_{j_r})$ is a certain $A(r-1)_\ast$-comodule ideal. The proofs all involve showing that

4.1. The homology of $M_{j_1}$. By Theorem 2.3

(4.2) \quad $H_*(M_{j_1}) = F_2[Q^I x_2, Q^J x_3 : I, J \text{ admissible, exc}(I) > 2, \text{exc}(J) > 3]$, where the left $A_\ast$-coaction is determined by

\[
\psi x_2 = 1 \otimes x_2 + \zeta_1^2 \otimes 1, \quad \psi x_3 = 1 \otimes x_3 + \zeta_1 \otimes x_2 + \zeta_2 \otimes 1.
\]

To calculate the coaction on the other generators $Q^I x_2$ and $Q^J x_3$ we follow [6] and use the right coaction

$$\tilde{\psi}: H_*(M_{j_1}) \to H_*(M_{j_1}) \otimes A_\ast; \quad \tilde{\psi}(z) = \sum z_i \otimes \chi(\alpha_i),$$

where $\psi(z) = \sum \alpha_i \otimes z_i$ and $\chi$ is the antipode of $A_\ast$. So

$$\tilde{\psi} x_2 = x_2 \otimes 1 + 1 \otimes \zeta_1^2, \quad \tilde{\psi} x_3 = x_3 \otimes 1 + x_2 \otimes \zeta_1 + 1 \otimes \zeta_2.$$

In general, if $z$ has degree $m$, then

(4.3) \quad $\tilde{\psi} Q^r z = \sum_{m \leq k \leq r} \sum_{m \leq k \leq r} Q^k(\tilde{\psi}z) \left[\frac{\zeta(t)}{t}\right]^k = \sum_{m \leq k \leq r} Q^k(\tilde{\psi}z) \left[\left(\frac{\zeta(t)}{t}\right)^k\right]_{t^k}$.
We will consider the sequence of elements
\( (4.5) \)
\( (4.4) \)
Combining these we obtain
\[
\tilde{\psi} Q^4 x_3 = Q^4 (x_3 \otimes 1 + x_2 \otimes \zeta_1 + 1 \otimes \xi_2) \left[ \left( \frac{\zeta(t)}{t} \right)^3 \right]_t \\
+ Q^4 (x_3 \otimes 1 + x_2 \otimes \zeta_1 + 1 \otimes \xi_2) \\
= x_3^2 \otimes \zeta_1 + x_2^2 \otimes \zeta_1^2 + 1 \otimes \zeta_1 \xi_2^2 \\
+ Q^4 x_3 \otimes 1 + (Q^3 x_2 \otimes \zeta_1^2 + x_2^2 \otimes Q^2 \zeta_1) + 1 \otimes Q^4 \xi_2 \\
= x_3^2 \otimes \zeta_1 + x_2^2 \otimes \zeta_1^2 + 1 \otimes \zeta_1 \xi_2 + Q^4 x_3 \otimes 1 \\
+ Q^3 x_2 \otimes \zeta_1^2 + x_2^2 \otimes \zeta_2 + 1 \otimes \zeta_3 + Q^3 x_3 \otimes \zeta_1^2. \\
\]
We also have
\[
\tilde{\psi} Q^3 x_2 = Q^3 x_2 \otimes 1, \quad \tilde{\psi} Q^5 x_2 = Q^5 x_2 \otimes +Q^3 x_2 \otimes \zeta_1^2.
\]
Combining these we obtain
\[(4.4) \quad \tilde{\psi} (Q^4 x_3 + Q^5 x_2) = (Q^4 x_3 + Q^5 x_2) \otimes 1 + x_3^2 \otimes \zeta_1 + x_2^2 \otimes \xi_2 + 1 \otimes \xi_3, \]
or equivalently,
\[(4.5) \quad \psi (Q^4 x_3 + Q^5 x_2) = 1 \otimes (Q^4 x_3 + Q^5 x_2) + \zeta_1 \otimes x_3^2 + \zeta_2 \otimes x_2^2 + \zeta_3 \otimes 1. \]
We will consider the sequence of elements \( X_{1,1} \) and \( X_{1,s} \in H_{2s-1}(M_{s1}) \) \((s \geq 2)\) defined by
\[
X_{1,s} = \begin{cases} 
  x_2 & \text{if } s = 1, \\
  x_3 & \text{if } s = 2, \\
  Q^3 x_3 + Q^5 x_2 & \text{if } s = 3, \\
  Q^{(2s-1,\ldots,2s,2s)} (Q^4 x_3 + Q^5 x_2) & \text{if } s \geq 4,
\end{cases}
\]
where \( Q^{(i_1,i_2,\ldots,i_t)} = Q^{i_1} Q^{i_2} \cdots Q^{i_t} \). We claim the \( X_{1,s} \) have the following right and left coactions:
\[(4.6) \quad \tilde{\psi} X_{1,s} = X_{1,s} \otimes 1 + X_{1,s-1}^2 \otimes \zeta_1 + \cdots + X_{1,s-3}^{2s-3} \otimes \xi_{s-3} + X_{1,s-2}^{2s-2} \otimes \xi_{s-2} + X_{1,s}^{2s-1} \otimes \xi_{s-1} + 1 \otimes \xi_s, \]
\[(4.7) \quad \psi X_{1,s} = 1 \otimes X_{1,s} + \zeta_1 \otimes X_{1,s-1} + \cdots + \zeta_{s-1} \otimes X_{1,s-3}^{2s-3} + \xi_{s-2} \otimes X_{1,s-2}^{2s-2} + \xi_s \otimes X_{1,s}^{2s-1} + \xi_s \otimes 1. \]
To prove these, we use induction on $s$, where the early cases $s = 1, 2, 3$ are known already. For the inductive step, assume that (4.6) holds for some $s \geq 3$. Then

$$\psi X_{1,s+1} = \psi Q^2 X_{1,s} = (\psi X_{1,s})^2 \xi_1 + Q^2 (\psi X_{1,s})$$

$$= X_{1,s}^2 \otimes \xi_1 + X_{1,s-1}^2 \otimes \xi_1^3 + \cdots + X_{1,3}^{2s-1} \otimes \xi_s \xi_1$$

$$+ X_{1,2}^{2s-1} \otimes \xi_s^2 - 2 \xi_1 + X_{1,1}^{2s-1} \otimes \xi_s^2 - 1 \xi_1 + 1 \otimes \xi_s \xi_1$$

$$+ Q^2 \left( X_{1,s} \otimes 1 + X_{1,s-1} \otimes \xi_1 + \cdots + X_{1,3}^{2s-3} \otimes \xi_s \right)$$

$$= X_{1,s}^2 \otimes \xi_1 + X_{1,s-1}^2 \otimes \xi_1^3 + \cdots + X_{1,3}^{2s-3} \otimes \xi_s = X_{1,s}^2 \otimes \xi_1 + X_{1,s}^2 \otimes \xi_1^3 + \cdots + X_{1,3}^{2s-3} \otimes \xi_s$$

$$+ X_{1,2}^{2s-1} \otimes \xi_s - 2 \xi_1 + X_{1,1}^{2s-1} \otimes \xi_s - 1 \xi_1 + 1 \otimes \xi_s \xi_1$$

$$+ X_{1,s} \otimes 1 + X_{1,s-1} \otimes \xi_1 + \cdots + X_{1,3}^{2s-1} \otimes \xi_s = X_{1,s}^2 \otimes \xi_1 + X_{1,s}^2 \otimes \xi_1^3 + \cdots + X_{1,3}^{2s-1} \otimes \xi_s$$

$$+ X_{1,2}^{2s-1} \otimes \xi_s - 1 \xi_1 + X_{1,1}^{2s-1} \otimes \xi_s - 1 \xi_1 + 1 \otimes \xi_s \xi_1$$

$$+ X_{1,s} \otimes 1 + X_{1,s-1} \otimes (\xi_2 + \xi_3) + \cdots + X_{1,3}^{2s-1} \otimes (\xi_s - 2 \xi_1 + \xi_s - 3 \xi_1)$$

$$+ X_{1,2}^{2s-1} \otimes (\xi_s - 1 + \xi_s - 2 \xi_1) + X_{1,1}^{2s-1} \otimes (\xi_s + \xi_s - 1 \xi_1) + 1 \otimes (\xi_s + \xi_s)$$

$$= X_{1,s+1} \otimes 1 + X_{1,s}^2 \otimes \xi_1 + X_{1,s-1}^2 \otimes \xi_2 + \cdots + X_{1,3}^{2s-2} \otimes \xi_s$$

$$+ X_{1,2}^{2s-1} \otimes \xi_s - 1 + X_{1,1}^{2s-1} \otimes \xi_s + 1 \otimes \xi_s$$

giving the result for $s + 1$.

Under the homomorphism $\rho: H_*(Mj_1) \to A_*$ induced by the orientation $Mj_1 \to HF_2$, we have

$$\rho(x_2) = \xi_1^2, \quad \rho(x_3) = \xi_2, \quad \rho(X_{1,s}) = \xi_s \quad (s \geq 3).$$

Also

$$\rho(Q^2 x_2) = Q^2 (\rho x_2) = Q^2 (\xi_1^2) = 0,$$

and for each admissible monomial $I$, $\rho(Q^2 x_2) \in A_*$ is a square.

This shows that the restriction of $\rho$ to the subalgebra generated by the $X_{1,s}$ is an isomorphism of $A_*$-comodule algebras

$$\mathbb{F}_2 [X_{1,s} : s \geq 1] \xrightarrow{\sim} A_* \circ A(0), \mathbb{F}_2 \subseteq A_* ,$$

where

$$A(0) = A_*/\mathbb{F}_2 [\xi_1^2, \xi_2, \xi_3, \ldots]. \quad A_*/A(0), \mathbb{F}_2 = \mathbb{F}_2 [\xi_1^2, \xi_2, \xi_3, \ldots] \subseteq A_* .$$

In the algebra $H_*(Mj_1)$, the regular sequence $X_{1,s} (s \geq 1)$ generates an ideal

$$I_1 = (X_{1,s} : s \geq 1) \lhd H_*(Mj_1) .$$

This is not an $A_*$-subcomodule since for example,

$$\psi X_{1,3} = \psi (Q^4 x_3 + Q^5 x_2) = (1 \otimes X_{1,3} + \xi_1 \otimes X_{1,2}^2 + \xi_2 \otimes X_{1,1}^2) + \xi_3 \otimes 1 .$$
However under the induced $\mathcal{A}(0)_*$-coaction
\[
\psi': H_*(Mj_1) \to \mathcal{A}(0)_* \otimes H_*(Mj_1),
\]
the last term becomes trivial, in fact
\[
\psi'X_{1,3} = 1 \otimes X_{1,3} + \zeta_1 \otimes X_{1,2}^2,
\]
where we identify elements of $\mathcal{A}(0)_*$ with representatives in $\mathcal{A}_*$. More generally, by (4.7), for $s \geq 2$,
\[
\psi'X_{1,s} = 1 \otimes X_{1,s} + \zeta_1 \otimes X_{1,s-1}^2.
\]
It follows that $I_1$ is an $\mathcal{A}(0)_*$-invariant ideal.

**Proposition 4.1.** There is an isomorphism of commutative $\mathcal{A}_*$-comodule algebras
\[
H_*(Mj_1) \cong (\mathcal{A}_* \square \mathcal{A}(0)_* F_2) \otimes H_*(Mj_1) / I_1.
\]

**Proof.** Taking $r = 1$, from (4.1) we obtain a commutative diagram of commutative $\mathcal{A}_*$-comodule algebras
\[
\begin{array}{c}
\text{H}_*(Mj_1) \\
\downarrow \psi \\
\text{A}_* \square \text{H}_*(Mj_1) \\
\downarrow \pi \\
\text{A}_* \otimes \text{H}_*(Mj_1) \\
\downarrow \pi \\
\text{A}_* \otimes \text{H}_*(Mj_1) / I_1
\end{array}
\]
and furthermore
\[
\psi X_{1,1} = \zeta_1^2 \otimes 1 + 1 \otimes X_{1,1},
\]
\[
\psi X_{1,2} = \zeta_2 \otimes 1 + \zeta_1 \otimes X_{1,1} + 1 \otimes X_{1,1},
\]
\[
\psi X_{1,s} = \zeta_{s+1} \otimes 1 + \cdots + 1 \otimes X_{1,s} \quad (s \geq 3),
\]
giving
\[
\pi \psi X_{1,1} = \zeta_1^2 \otimes 1, \quad \pi \psi X_{1,2} = \zeta_2 \otimes 1, \quad \pi \psi X_{1,s} = \zeta_{s+1} \otimes 1 + \cdots.
\]
The latter form part of a set of polynomial generators for the polynomial ring
\[
\text{A}_* \otimes \text{H}_*(Mj_1) / I_1 \cong (\text{A}_* \square \text{A}(0)_* F_2) \otimes \text{H}_*(Mj_1) / I_1.
\]
Now a straightforward argument shows that the dashed arrow is surjective; but as the Poincaré series of $\text{H}_*(Mj_1)$ and $(\text{A}_* \square \text{A}(0)_* F_2) \otimes \text{H}_*(Mj_1) / I_1$ are equal, it is actually an isomorphism. Therefore
\[
\text{H}_*(Mj_1) \cong (\text{A}_* \square \text{A}(0)_*) \text{H}_*(Mj_1) / I_1.
\]

**Remark 4.2.** For the purposes of proving such a result, we might as well have set $X_{1,3} = Q^4 x_3$ and
\[
X_{1,s} = Q^{s-1} x_{s-1} \quad (s \geq 3),
\]
since
\[
\psi'X_{1,3} = 1 \otimes X_{1,3} + \zeta_1 \otimes x_3^2
\]
and so on. However, the cases of $Mj_2$ and $Mj_3$ will require modifications similar to the ones we have used above which give an indication of the methods required.

We have the following splitting result.

**Proposition 4.3.** There is a splitting of $\mathcal{A}_*\text{-comodule algebras}

\[
\begin{array}{ccc}
\mathcal{A}_*\square A(0)_*, F_2 & \cong & \mathcal{A}_*\square A(0)_*, F_2 \\
& \searrow & H_*\langle Mj_1 \rangle
\end{array}
\]

where $H_*(Mj_1) \rightarrow H_*(\mathbb{H}^\mathbb{Z}) = \mathcal{A}_*\square A(0)_*, F_2$ is induced by the $\mathcal{E}_\infty$ orientation $Mj_1 \rightarrow \mathbb{H}^\mathbb{Z}$.

**Proof.** This is proved using Lemma 3.6 together with the trivial $\mathcal{A}(0)_*$-comodule algebra homomorphism $\mathcal{A}_*\square A(0)_*, F_2 \rightarrow H_*(Mj_1)/I_1$. \qed

4.2. The homology of $Mj_2$. We have

\[H_*\langle Mj_2 \rangle = \mathbb{F}_2[Q^I x_4, Q^J x_6, Q^K x_7 : I, J, K \text{ admissible}, \text{exc}(I) > 4, \text{exc}(J) > 6, \text{exc}(K) > 7],\]

with right coaction satisfying

\[
\begin{align*}
\tilde{\psi}x_4 &= x_4 \otimes 1 + 1 \otimes \zeta_4^1, \\
\tilde{\psi}x_6 &= x_6 \otimes 1 + x_4 \otimes \zeta_1^1 + 1 \otimes \zeta_2^2, \\
\tilde{\psi}x_7 &= x_7 \otimes 1 + x_6 \otimes \zeta_1 + x_4 \otimes \zeta_2 + 1 \otimes \zeta_3.
\end{align*}
\]

Furthermore,

\[
\begin{align*}
\tilde{\psi}Q^8x_7 &= x_7^2 \otimes \zeta_1 + x_6^2 \otimes \zeta_1^1 + x_4^2 \otimes \zeta_1^1 \zeta_2^2 + 1 \otimes \xi_3^2 \zeta_1 + Q^8(x_7 \otimes 1 + x_6 \otimes \zeta_1 + x_4 \otimes \zeta_2 + 1 \otimes \zeta_3) \\
&= x_7^2 \otimes \zeta_1 + x_6^2 \otimes \zeta_1^1 + x_4^2 \otimes \zeta_1^2 \zeta_2^2 + 1 \otimes \xi_3^2 \zeta_1 \\
&\quad + Q^8x_7 + Q^5x_6 \otimes \zeta_2^2 + Q^5x_4 \otimes \zeta_2^2 + 1 \otimes (\zeta_4 \otimes \zeta_1^2) + x_6^2 \otimes \zeta_2 + x_4^2 \otimes (\zeta_3 + \zeta_1^2) \\
&= (Q^8x_7 + x_7^2 \otimes \zeta_1^1 + x_6^2 \otimes \zeta_2 + x_4^2 \otimes \zeta_3 + 1 \otimes \zeta_4) + Q^7x_6 \otimes \zeta_2 + Q^5x_4 \otimes \zeta_2^2,
\end{align*}
\]

so the left $\mathcal{A}(1)_*$-coproduct

\[
\psi': H_*(Mj_2) \rightarrow \mathcal{A}(1)_* \otimes H_*(Mj_2)
\]

has

\[
\begin{align*}
\psi'Q^8x_7 &= (Q^8x_7 + \zeta_1 \otimes x_7^2 + \zeta_2 \otimes x_6^2 + \zeta_3 \otimes x_4^2 + \zeta_4 \otimes 1) + \zeta_2^2 \otimes Q^7x_6 + \zeta_2^2 \otimes Q^5x_4 \\
&= (Q^8x_7 + \zeta_1 \otimes x_7^2 + \zeta_2 \otimes x_6^2) + \zeta_2^2 \otimes Q^7x_6.
\end{align*}
\]

We also have

\[
\begin{align*}
\psi'Q^8x_6 &= 1 \otimes Q^8x_6 + \zeta_1^2 \otimes Q^7x_6 + \zeta_2^2 \otimes Q^7x_4 + \zeta_2^2 \otimes Q^5x_4 \\
&= 1 \otimes Q^8x_6 + \zeta_1^2 \otimes Q^7x_6,
\end{align*}
\]

so

\[
\psi'(Q^8x_7 + Q^8x_6) = Q^8x_7 + \zeta_1 \otimes x_7^2 + \zeta_2 \otimes x_6^2 \in \mathcal{A}(1)_* \otimes H_*(Mj_2).
\]
Now we define a sequence of elements $X_{2,s}$ ($s \geq 1$) by

$$X_{2,s} = \begin{cases} 
 x_4 & \text{if } s = 1, \\
 x_6 & \text{if } s = 2, \\
 x_7 & \text{if } s = 3, \\
 Q^8x_7 + Q^9x_6 & \text{if } s = 4, \\
 Q^{(2s-1, \ldots, 2^5)}(Q^8x_7 + Q^9x_6) & \text{if } s \geq 5.
\end{cases}$$

A inductive calculation shows that for $s \geq 4$,

$$\psi X_{2,s} = 1 \otimes X_{2,s} + \zeta_1 \otimes X_{2,s-1} + \zeta_2 \otimes X_{2,s-2}^4 \in A(1)_* \otimes I_2.$$ 

So this sequence is regular and generates an $A(1)_*$-invariant ideal

$$I_2 = (X_{2,s} : s \geq 1) \triangleleft H_*(Mj_2).$$

The next result follows using similar arguments to those in the proof of Proposition 4.1 using the diagram (4.1).

**Proposition 4.4.** There is an isomorphism of $A_*$-comodule algebras

$$H_*(Mj_2) \xrightarrow{\cong} \frac{A_* \square A(1)_* \otimes I_2}{A_* \square A(1)_* \otimes I_2}.$$ 

4.3. The homology of $Mj_3$. In $H_*(Mj_3)$, consider the regular sequence

$$X_{3,s} = \begin{cases} 
 x_8 & \text{if } s = 1, \\
 x_{12} & \text{if } s = 2, \\
 x_{14} & \text{if } s = 3, \\
 x_{15} & \text{if } s = 4, \\
 Q^{16}x_{15} + Q^{17}x_{14} + Q^{19}x_{12} & \text{if } s = 5, \\
 Q^{(2s-1, \ldots, 2^5)}(Q^{16}x_{15} + Q^{17}x_{14} + Q^{19}x_{12}) & \text{if } s \geq 6.
\end{cases}$$

We leave the reader to verify that the ideal

$$I_3 = (X_{3,s} : s \geq 1) \triangleleft H_*(Mj_3)$$ 

is $A(2)_*$-invariant. The proof of the following result is similar to those of Propositions 4.1 and 4.4 using the diagram (4.1).
Proposition 4.6. There is an isomorphism of $A_\ast$-comodule algebras

$$H_\ast(M_j3) \xrightarrow{=} A_\ast \square_{A(2)} H_\ast(M_j3)/I_3.$$ 

The $E_\infty$ morphism $M_j3 \rightarrow \text{tmf}$ induces a homomorphism $H_\ast(M_j3) \rightarrow H_\ast(\text{tmf}) \subseteq (A)_\ast$ under which

$$X_{3,1} \mapsto \zeta_1^8, \quad X_{3,2} \mapsto \zeta_2^4, \quad X_{3,3} \mapsto \zeta_3^2, \quad X_{3,s} \mapsto \zeta_s (s \geq 3).$$

We have the following splitting result analogous to Propositions 4.3 and 4.5.

Proposition 4.7. There is a splitting of $A_\ast$-comodule algebras

$$A_\ast \square_{A(2)} F_2 \xrightarrow{\cong} A_\ast \square_{A(2)} F_2 \xrightarrow{\cong} H_\ast(M_j3)$$

where $H_\ast(M_j3) \rightarrow H_\ast(\text{tmf}) = A_\ast \square_{A(2)} F_2$ is induced by the $E_\infty$ orientation $M_j3 \rightarrow \text{tmf}$.

5. Some other examples

The approach we have used to proving algebraic splittings of the homology of $E_\infty$ Thom spectra can be be used to rederive many known results for classical examples such as $MO$, $MSO$, $MSpin$, $MString = MO(8)$ and $MU$. We can also derive some other new examples using algebraic similar methods.

5.1. An example related to $kU$. Our first example is based on similar ideas to those used to construct the spectra $M_{jr}$, but using $Spin^c$. The low dimensional homology of $BSpin^c$ can be read off from Theorem A.2 and Remark A.3. Passing to the Thom spectrum over the 7-skeleton ($BSpin^c)^{[7]}$ we have for its homology

$$H_\ast((BSpin^c)^{[7]}) = F_2\{1, a_{1,0}^{(1)}, a_{1,1}^{(1)}, (a_{1,0})^2, a_{3,0}^{(1)}, a_{7,0}\}.$$ 

For our purposes, the fact that there are two 4-cells is problematic, so we instead restrict to a smaller complex. The map $K(Z,2)^{[2]} \times BSpin^{[7]} \rightarrow BSpin^c$ induces an epimorphism in cohomology, and the resulting map

$$S^2 \vee BSpin^{[7]} \rightarrow BSpin^c$$

gives a monomorphism in homology with image

$$F_2\{1, a_{1,0}^{(1)}, a_{1,1}^{(1)}, a_{3,0}^{(1)}, a_{7,0}\}.$$
The Thom spectrum over this space has the cell structure of the form \((S^0 \cup_{\eta} e^2) \cup_{\nu} e^4 \cup_{\eta} e^6 \cup_2 e^7\).

We can factor the skeletal inclusion through an infinite loop map

\[
\begin{array}{ccc}
S^2 \lor B\text{Spin}^7 & \rightarrow & B\text{Spin}^c \\
\downarrow & \downarrow & \downarrow \\
Q(S^2 \lor B\text{Spin}^7) & \rightarrow & kU
\end{array}
\]

and obtain an \(E_\infty\) Thom spectrum \(Mj^c\) over \(Q(S^2 \lor B\text{Spin}^7)\). The homology of this is

\[H_*(Mj^c) = \mathbb{F}_2[Q^2, Q^4, Q^6, Q^7; \text{admissible, exc} (I_r) > r].\]

It is easy to see that there is a morphism of \(E_\infty\) ring spectra

\[\tilde{\mathbb{P}}(S^0 \cup_{\eta} e^4 \cup_{\eta} e^6 \cup_2 e^7) \rightarrow kU\]

inducing an epimorphism on \(H_*(-)\) under which

\[x_2 \mapsto \zeta_1^2, \quad x_4 \mapsto \zeta_1^4, \quad x_6 \mapsto \zeta_2^2, \quad x_7 \mapsto \zeta_3.\]

The 7-skeleton of \(Mj^c\) has the form

\[
\begin{array}{c}
x_7 \\
2 \\
x_6 \\
x_4 \\
x_2 \\
x_0
\end{array}
\]

since \(\pi_3(C_\eta) \cong \pi_3(S^0)/\eta \pi_1(S^0) = \pi_3(S^0)/4\pi_3(S^0)\) and the generators are detected by \(\text{Sq}^4\). It follows that there is an element \(\pi_4(Mj^c)\) whose Hurewicz image is \(x_4 + x_2^2\), and if \(w: S^4 \rightarrow Mj^c\)
is a representative, we can form the $\mathcal{E}_\infty$ cone $Mj^c//w$ as the pushout in the diagram

$$
\begin{array}{ccc}
\mathbb{P}S^4 & \longrightarrow & \mathbb{P}D^5 \\
\downarrow & & \downarrow \\
Mj^c & \longrightarrow & Mj^c//w
\end{array}
$$

taken in the category $\mathcal{C}_S$ of commutative $S$-algebras. There is a Künneth spectral sequence of the form

$$E^2_{s,t} = \text{Tor}^{H_*}_s(\mathbb{P}S^4, H_*(Mj^c)) \implies H_{s+t}(Mj^c//w)$$

where the $H_*(Mj^c)$ is the $H_*(\mathbb{P}S^4)$-module algebra with

$$H_*(\mathbb{P}S^4) = \mathbb{F}_2[Q^I z_4 : I \text{ admissible}, \text{exc}(I) > 4] \to H_*(Mj^c) ; \; Q^I z_4 \mapsto Q^I(x_2^2) + Q^I x_4.$$ 

Notice that the term $Q^I(x_2^2)$ is either trivial (if at least one term in $I$ is odd) or a square (if all terms in $I$ are even), hence can be used as a polynomial generator of $H_*(Mj^c)$ in place of $Q^I x_4$.

It follows that $H_*(Mj^c)$ is a free $H_*(\mathbb{P}S^4)$-module, so the spectral sequence is trivial with

$$E^2_{s,*} = \text{Tor}^{H_*}_{s,*}(\mathbb{P}S^4, H_*(Mj^c))$$

$$= H_*(Mj^c)/(Q^I(x_2^2) + Q^I x_4 : I \text{ admissible}, \text{exc}(I) > 4),$$

therefore we have

$$(5.1) \quad H_*(Mj^c//w) = \mathbb{F}_2[Q^I x_2, Q^I x_6, Q^I x_7 : I_r \text{ admissible}, \text{exc}(I_r) > r].$$

Now we define a sequence of elements $X_s$ in $H_*(Mj^c//w)$ as follows:

$$X_s = \begin{cases} 
x_2 & \text{if } s = 1, \\
x_6 & \text{if } s = 2, \\
x_7 & \text{if } s = 3, \\
Q^{(2^{s-1}, \ldots, 2^3)} x_7 & \text{if } s \geq 4.
\end{cases}$$

This forms a regular sequence and the induced coaction over the quotient Hopf algebra

$$\mathcal{E}(1,2)_* = A_*/(\zeta_1^2, \zeta_2^2, \zeta_3, \ldots) = A_*/\mathbb{F}_2[\zeta_1^2, \zeta_2^2, \zeta_3, \ldots] = \Lambda(\zeta_1, \zeta_2)$$

satisfies

$$\psi' X_s = \begin{cases} 
1 \otimes X_1 & \text{if } s = 1, 2, \\
3 + \zeta_1 \otimes X_2 + \zeta_2 \otimes X_1^2 & \text{if } s = 3, \\
1 \otimes X_s + \zeta_1 \otimes X_{s-1} + \zeta_2 \otimes X_{s-2} & \text{if } s \geq 4.
\end{cases}$$

therefore the ideal $I^c(X_s : s \geq 1) \triangleleft H_*(Mj^c//w)$ is an $\mathcal{E}(1,2)_*$-invariant regular ideal.

Recall that

$$A_*/\mathcal{E}(1,2)_*, F_2 = \mathbb{F}_2[\zeta_1^2, \zeta_2^2, \zeta_3, \ldots] \cong H_*(kU).$$

We have proved the following analogues of earlier results.

**Proposition 5.1.** There is an isomorphism of $A_*$-comodule algebras

$$H_*(Mj^c) \xrightarrow{\cong} A_*/\mathcal{E}(1,2)_*, H_*(Mj^c)/I^c.$$
Proposition 5.2. There is a splitting of \( A^* \)-comodule algebras

\[
\begin{array}{ccc}
A^* \bowtie_{\mathcal{E}(1,2)} F_2 & \cong & A^* \bowtie_{\mathcal{E}(1,2)} F_2 \\
\downarrow H_*(Mj^c) & & \downarrow H_*(Mj^c)
\end{array}
\]

where \( H_*(Mj^c) \to H_*(kU) = A^* \bowtie_{\mathcal{E}(1,2)} F_2 \) is induced by the \( E_\infty \) orientation \( Mj^c \to kU \).

5.2. An example related to the Brown-Peterson spectrum. We recall from [4, section 4] the 2-local \( E_\infty \) ring spectrum \( R_\infty \) which has a map of commutative ring spectra \( R_\infty \to BP \) inducing a rational equivalence, an epimorphism \( \pi_*^*(R_\infty) \to \pi_*^*(BP) \), and \( H_*(R_\infty) \) contains a regular sequence \( z_s \in H_{2^2+1-2}(R_\infty) \) mapping to the generators \( t_s \in H_{2^2+1-2}(BP) \) which in turn map to \( \zeta^2 \in H_{2^2+1-2}(H) = A_{2^2+1-2} \) under the induced ring homomorphisms

\[
H_*(R_\infty) \to H_*(BP) \to H_*(H) = A_* .
\]

We note that both of these homorphisms are compatible with the Dyer-Lashof operations, even though \( BP \) is not necessarily an \( E_\infty \) ring spectrum. These elements \( z_s \) have the following coactions:

\[
\psi(z_r) = 1 \otimes z_r + \zeta^2 \otimes z_r^{2^2-1} + \zeta^2 \otimes z_r^{2^3-2} - 1 + \zeta^2 \otimes z_r^{2^{r-1}} - 1 + \zeta^2 \otimes 1 ,
\]

generate an ideal \( I_\infty \triangleleft H_*(R_\infty) \).

Let

\[
\mathcal{E}_* = A_*/(\zeta^2_i : i \geq 1) ,
\]

the exterior quotient Hopf algebra. Although it \( \mathcal{E}_* \) is not finite dimensional, it is still true that \( A_* \) is an extended right \( \mathcal{E}_* \)-comodule,

\[
A_* \cong (A^* \bowtie_{\mathcal{E}_*} F_2) \otimes \mathcal{E}_* .
\]

Under the induced \( \mathcal{E}_* \)-coaction on \( H_*(R_\infty) \), \( I_\infty \) is an \( \mathcal{E}_* \)-comodule ideal and \( H_*(R_\infty)/I_\infty \) becomes an \( \mathcal{E}_* \)-comodule algebra.

Proposition 5.3. There is an isomorphism of commutative \( A_* \)-comodule algebras

\[
H_*(R_\infty) \cong A_\square_{\mathcal{E}_*} H_*(R_\infty)/I_\infty ,
\]

and a splitting \( A_* \)-comodule algebras

\[
\begin{array}{ccc}
A^* \bowtie_{\mathcal{E}_*} F_2 & \cong & A^* \bowtie_{\mathcal{E}_*} F_2 \\
\downarrow H_*(R_\infty) & & \downarrow H_*(R_\infty)
\end{array}
\]

where \( A^* \bowtie_{\mathcal{E}_*} F_2 \cong H_*(BP) \) and the right hand homomorphism is induced from the morphism of commutative ring spectra \( R_\infty \to BP \).

This result supports the view that \( R_\infty \) admits a map \( BP \to R_\infty \) extending the unit \( S^0 \to R_\infty \) and then the composition

\[
BP \to R_\infty \to BP
\]

would necessarily be a weak equivalence since \( BP \) is minimal atomic in the sense of [8].
6. Some speculation

Our algebraic splittings of $H_*(Mjr)$ are consistent with spectrum-level splittings. Indeed, in the case of $r = 1$, a result of Mark Steinberger [9] already shows that $Mj_1$ splits as a wedge of suspensions of $HZ$ and $HZ/2^s$ for $s \geq 1$, all of which are $HZ$-module spectra.

Using Lemma 3.2, it is easy to see that if a spectrum $X$ is a module spectrum over one of $HZ$, kO or tmf then its homology is a retract of the extended cocomodule $A_s \square_A A(r)_* H_*(X)$ for the relevant value of $r$; a similar observation holds for a module spectrum over $kU$ and $A_s \square_{E(1,2)} A_*(r)_* H_*(X)$. Thus our algebraic results provide evidence for the following conjectural splittings.

**Conjecture 6.1.** As a spectrum, $Mj_2$ is a wedge of kO-module spectra, $Mj_3$ is a wedge of tmf-module spectra and $Mj^c$ is a wedge of $kU$-module spectra.

Here the phrase ‘module spectra’ can be interpreted either purely homotopically, or strictly in the sense of [10].

Related to this conjecture, and indeed implied by it, is the following where we know that analogues hold for the cases $Mj_1, Mj_2, Mj^c$, i.e., the natural homomorphisms

$$
\pi_*(Mj_1) \rightarrow \pi_*(HZ), \quad \pi_*(Mj_2) \rightarrow \pi_*(kO), \quad \pi_*(Mj^c) \rightarrow \pi_*(kU)
$$

are epimorphisms.

**Conjecture 6.2.** The natural map $Mj_3 \rightarrow \text{tmf}$ induces an epimorphism $\pi_*(Mj_3) \rightarrow \pi_*(\text{tmf})$.

We already know this is true up to degree 16, and also holds rationally.

**Appendix A. The homology of connective covers of BO**

We review the structure of the homology Hopf algebras $H_*(BO(n)) = H_*(BO(n); F_2)$ for $n = 1, 2, 4, 8$. The dual cohomology rings were originally determined by Stong but later a body of literature due to Bahri, Kochman, Pengelley as well as the present author evolved describing these homology rings. We will use the Husemoller-Witt decompositions of $[3]$ to give explicit algebra generators; the actions of Steenrod and Dyer-Lashof operations on these can be determined using work of Kochman and Lance [11, 12].

We recall that there are polynomial generators $a_{k,s} \in H_{2^s k}(BO)$ ($k$ odd, $s \geq 0$) such that

$$B[k]_* = F_2[a_{k,s} : s \geq 0] \subseteq H_*(BO)$$

is a polynomial sub-Hopf algebra and

$$H_*(BO) = \bigotimes_{k \text{ odd}} B[k]_*.$$  

For each $h \geq 1$, there is a Hopf algebra monomorphism

$$B[k]_* \rightarrow B[k]_*; \quad x \mapsto x^{(h)} = x^{2^h},$$

whose image we will denote by $B^{(h)}[k]_*$. Notice that the primitives in $B^{(h)}[k]_*$ are the powers

$$(a_{k,0}^{(h)})^{2^s} = (a_{k,0})^{2^{s+h}} \quad (s \geq 0).$$

Let $\alpha = \alpha_2$ denote the dyadic number function which counts the number of non-zero coefficients in the binary expansion of a natural number.
Theorem A.1. The natural infinite loop maps $BO(n) \to BO(1) = BO$ ($n = 2, 4, 8$) induce monomorphisms of Hopf algebras $H_*(BO(n)) \to H_*(BO)$ whose images are the following sub-Hopf algebras of $H_*(BO)$:

\[
\begin{align*}
B^{(1)}[1]_s & \otimes \bigotimes_{k \geq 1 \text{ odd}} B[k]_s & \text{if } n = 2, \\
B^{(2)}[1]_s & \otimes \bigotimes_{\alpha(k)=2} B^{(1)}[k]_s \otimes \bigotimes_{\alpha(k)>2} B[k]_s & \text{if } n = 4, \\
B^{(3)}[1]_s & \otimes \bigotimes_{\alpha(k)=3} B^{(2)}[k]_s \otimes \bigotimes_{\alpha(k)>3} B^{(1)}[k]_s \otimes \bigotimes_{\alpha(k)>3} B[k]_s & \text{if } n = 8.
\end{align*}
\]

We may identify $H_*(MO(n))$ with $H_*(BO(n))$ using the Thom isomorphism, which is an isomorphism of algebras over the Dyer-Lashof algebra but not over the Steenrod algebra. To avoid excessive notation, we will often treat the Thom isomorphism as an equality and write $a^{(r)}_{k,s}$ for each of the corresponding elements.

The generators $a_{2^r-1,0}$ are particularly interesting. In $H_*(BO)$, $a_{2^r-1,0}$ is primitive, and in $H_*(MO)$ there is a simple formula for the $A_*$-coaction:

(A.1) $\psi(a_{2^r-1,0}) = 1 \otimes a_{2^r-1,0} + \zeta_1 \otimes a_{2^r-1,0}^2 + \zeta_2 \otimes a_{2^r-1,0}^4 + \cdots + \zeta_{s-1} \otimes a_{1,0}^{2s-1} + \zeta_s \otimes 1$.

The natural orientation $MO \to HF_2$ induces an algebra homomorphism over both of the Dyer-Lashof and Steenrod algebras under which

(A.2) $a_{2^r-1,0} \mapsto \zeta_s$.

For completeness, we also describe the homology of $BSpin^c$ in this algebraic form since we are not aware of this being documented anywhere.

Theorem A.2. The natural infinite loop map $BSpin^c \to BO$ induces a monomorphism of Hopf algebras $H_*(BSpin^c) \to H_*(BO)$ with image

\[
\bigotimes_{\alpha(k)\leq 2} B^{(1)}[k]_s \otimes \bigotimes_{\alpha(k)>2} B[k]_s.
\]

Sketch of proof. The cohomology ring $H^*(BSpin^c)$ can be calculated using the Serre spectral sequence

\[
E_2^{r,s} = H^r(BO; H^s(K(\mathbb{Z}, 2))) \Rightarrow H^{r+s}(BSpin^c)
\]

for the fibration sequence

\[
K(\mathbb{Z}, 2) \to BSpin^c \to BSO.
\]

Then

\[
E_2^{r,s} = F_2[w_k : k \geq 2] \otimes F_2[x],
\]

where $w_k \in H^k(BO)$ is the image of the $k$-th Stiefel-Whitney class, $x \in H^2(K(\mathbb{Z}, 2))$ and $x^{2t} \in H^{2t+1}(K(\mathbb{Z}, 2))$ transgresses to

\[
d_{2t+1}(x^{2t}) = w_{2t+1} \pmod{\text{decomposables}}.
\]

So the natural map $BSpin^c \to BO$ induces an epimorphism $H^*(BO) \to H^*(BSpin^c)$, and dually $H_*(BSpin^c) \to H_*(BO)$ is a monomorphism. Also $H^*(BSpin^c)$ is polynomial with one
generator in each degree of form \( n = 2m \) where \( \alpha(m) \leq 2 \), and \( n \) where \( \alpha(m) > 2 \). In fact there is an isomorphism of Hopf algebras

\[
H^*(B\text{Spin}^c) \cong \bigotimes_{k \text{ odd } \alpha(k) \leq 2} B[k]/(a_{k,0}) \bigotimes_{k \text{ odd } \alpha(k) > 2} B[k].
\]

For each odd \( k \) there is an isomorphism of Hopf algebras

\[
B[k]/(a_{k,0}) \cong \text{Hom}(B^{(1)}[k], \mathbb{F}_2),
\]

and using this the claimed description of the homology \( H_\ast(B\text{Spin}^c) \) follows.

**Remark A.3.** The natural maps \( K(Z,2) \to B\text{Spin}^c \) induce maps in homology where the image of \( H_\ast(K(Z,2)) \) contains \( a^{(1)}_{1,0} \) and \( a^{(1)}_{1,1} \), while the image of \( H_\ast(B\text{Spin}) \) contains \( (a^{(1)}_{1,0})^2 \), \( a^{(1)}_{3,0} \) and \( a_{7,0} \).

**Appendix B. Dyer-Lashof operations and Steenrod coactions**

For the convenience of the reader, we summarise some results from [9] which are based on work of Kochman and Steinberger [9, 11].

The mod 2 Steenrod algebra \( A_\ast = A(2)_\ast \) is the homology of the Eilenberg-Mac Lane spectrum \( H = H\mathbb{F}_2 \) which is an \( \mathcal{E}_\infty \) ring spectrum and so \( A_\ast \) supports an action of the Dyer-Lashof operations. However, when dealing with the left \( A_\ast \)-coaction on the homology of an \( \mathcal{E}_\infty \) ring spectrum it is often convenient to consider a twisted version formed using the antipode \( \chi \), given by

\[
\tilde{Q}^s = \chi Q^s \chi.
\]

Based on Steinberger’s determination of the usual action [9, by [6, lemma 4.4)] we have the following equivalent formulae for all \( s \geq 1 \):

\[
\begin{align*}
Q^{2^s} \xi_s &= \xi_{s+1} + \xi_1 \xi_s^2, \\
\tilde{Q}^{2^s} \xi_s &= \xi_{s+1} + \xi_1 \xi_s^2.
\end{align*}
\]

The spectra \( H\mathbb{Z}, k\mathbb{O} \) and \( \text{tmf} \) are all \( \mathcal{E}_\infty \) ring spectra and there are \( \mathcal{E}_\infty \) morphisms \( H\mathbb{Z} \to H\mathbb{F}_2 \), \( k\mathbb{O} \to H\mathbb{F}_2 \) and \( \text{tmf} \to H\mathbb{F}_2 \) inducing monomorphisms on \( H_\ast(-) \) identifying their homology with the subalgebras

\[
\mathbb{F}_2[\zeta^4_1, \zeta^4_2, \zeta^2_3, \zeta_4, \zeta_5, \ldots] \subseteq \mathbb{F}_2[\zeta^4_1, \zeta^2_2, \zeta_3, \zeta_4, \ldots] \subseteq \mathbb{F}_2[\zeta_1^2, \zeta_2, \zeta_3, \ldots] \subseteq A_\ast.
\]

It follows that each of these subalgebras is closed under the Dyer-Lashof operations. More generally by work of Stong [22], each of the \( \mathcal{E}_\infty \) morphisms \( MO(2^d) \to H\mathbb{F}_2 \) induces a ring homomorphism whose image is \( \mathbb{F}_2[\zeta^d_1, \zeta^{2d-1}_2, \ldots, \zeta^2_d, \zeta_{d+1}, \zeta_{d+2}, \ldots] \) and this must be closed under the Dyer-Lashof operations.

We will give a purely algebraic generalisation of these observations.

For \( n \geq 0 \), let

\[
\mathcal{I}(n) = \langle \zeta^1_1, \zeta^2_2, \zeta^3_3, \ldots, \zeta^{2n-1}_n, \zeta^{2n+1}_n, \zeta_{n+1}, \zeta_{n+2}, \zeta_{n+3}, \ldots \rangle \triangleleft A_\ast.
\]

This is a Hopf ideal and \( A_\ast/\mathcal{I}(n) \) is a well-known finite quotient Hopf algebra. We also set

\[
\mathcal{I}(n)[d] = \{ \alpha^{2^d} : \alpha \in \mathcal{I}(n) \} \triangleleft A_\ast,
\]
and observe that
\[ (B.2) \quad \mathcal{I}(n)^{[d+1]} \subseteq \mathcal{I}(n+1)^{[d]} \subseteq \mathcal{I}(n+d). \]

**Lemma B.1.** Let \( s \geq 1 \). If \( k \in \mathbb{N} \), then \( Q^k \zeta_s \in \mathcal{I}(s-1) \); more generally, for \( r \geq 0 \), \( Q^k (\zeta_s^{2^r}) \in \mathcal{I}(s+r-1) \).

**Proof.** We make use of the results of [6] section 5.

The proof is by induction on \( s \). When \( s = 1 \), for \( k \geq 1 \), write \( k = 2m \) or \( k = 2m + 1 \). Then
\[ Q^{2m} \zeta_1 = N_{2m+1}(\xi) = \xi_1 N_m(\xi)^2 + \xi_2 N_{m-1}(\xi)^2 + \xi_3 N_{m-3}(\xi)^2 + \cdots \in \mathcal{I}(0), \]
and
\[ Q^{2m+1} \zeta_1 = N_{2m+2}(\xi) = N_m(\xi)^2 + \xi_2 N_{m-2}(\xi)^4 + \xi_3 N_{m-6}(\xi)^2 + \cdots \in \mathcal{I}(0). \]

Now suppose that the result holds for all \( s < n \). Recall that for \( k \geq 2^n - 1 \), \( Q^k \zeta_n = 0 \) unless \( k \equiv 0 \mod 2^n \) or \( k \equiv 2^n - 1 \mod 2^n \) when
\[ Q^{2^m m} \zeta_n = N_{2^m m + 2^n - 1}(\xi) = \xi_1 N_{2^m m - 1}(\xi)^2 + \xi_2 N_{2^m m + 2^n - 2 - 1}(\xi)^4 + \xi_3 N_{2^m m + 2^n - 3 - 1}(\xi)^8 + \cdots \]
\[ = \xi_1 (Q^{2^m m} \zeta_{n-1})^2 + \xi_2 (Q^{2^m m} \zeta_{n-2})^4 + \xi_3 (Q^{2^m m} \zeta_{n-3})^8 + \cdots \in \mathcal{I}(n-2) \subseteq \mathcal{I}(n-2) \subseteq \mathcal{I}(n-1), \]
and similarly \( Q^{2^m m + 2^n - 1} \zeta_n \in \mathcal{I}(n-1) \).

For \( r \geq 0 \), \( Q^k (\zeta_s^{2^r}) = 0 \) unless \( 2^r \mid k \), and then by (B.2),
\[ Q^{2^r \ell} (\zeta_s^{2^r}) = (Q^r \zeta_s)^{2^r} \in \mathcal{I}(n-1)^{[r]} \subseteq \mathcal{I}(n+r-1). \]
\[ \square \]

**Corollary B.2.** For \( n \geq 0 \), the cotensor product \( \mathcal{A}_s \square_{\mathcal{A}(n)_s} \mathbb{F}_2 \subseteq \mathcal{A}_s \) is closed under the Dyer-Lashof operations, and the Dyer-Lashof operations commute with the Hopf algebra quotient homomorphism \( \mathcal{A}_s \to \mathcal{A}(n)_s \).

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22
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