Distribution of Transmitted Charge through an Ultrasmall Double-Tunnel Junction

Heinz–Olaf Müller

Institut für Festkörperphysik, Friedrich–Schiller–Universität, Lessingstraße 8, DE–07743 Jena, Germany

(March 24, 2022)

The transmission of charge through an ultrasmall double junction is considered with Coulomb effects but at zero temperature. We construct an equation which describes the time development of the transmission probability of charges and solve this equation in terms of a recursion relation. The results are compared with the double junction without charging effects.

73.23.Hk, 73.40Gk, 73.40Rw

I. INTRODUCTION

During more than one decade, single-electron tunneling (SET) has experienced considerable progress in both experiment and theory. Basically, SET is a stochastic tunneling process of charges occurring at ultrasmall low-capacitance high-resistance tunnel junctions. Those tunnel junctions ensure that, on the one hand, the charging energy is large (compared to the thermal energy) whereas, on the other hand, charge fluctuations are suppressed. Thus, the number of tunnelled charges forms a classical whereas the tunneling probabilities (rates) are calculated quantum mechanically, for instance in terms of Fermi’s Golden Rule.

This calculation restricts the application of the semi-classical approach to slow time dependencies on the scale of the RC–time of the tunnel junction. It is in the range of $10^{-10}$s for nowadays devices. There are approaches beyond the semi-classical theory that allow the exploration of short time scales (for instance Ref. 1), however for the purpose of this work simpler Golden Rule approach is sufficient.

This paper is devoted to the study of the ultrasmall double tunnel junction, which is one of the cornerstones of single electronics in experiment and theory. It consists of two junction connected in series, thus forming a small island inbetween. This system is capable of displaying pronounced effects of single-electron tunneling like Coulomb blockade, Coulomb staircase, and shot-noise suppression for suitable parameters of the applied bias voltage $V$.

Throughout this paper zero temperature is assumed. On the one hand, this assumption simplifies our consideration considerably and enhances its clarity. On the other hand, many SET experiments are done at sub-Kelvin temperatures, where a zero-temperature approximation is reasonable.

In the next Section we develop the basics of our approach and display the connection to the well-known “orthodox” theory and to the double junction without charging effects. It follows a discussion of these results in the third Section.

II. TIME DEVELOPMENT

For the double junction the semi-classical approach results in a master equation,

$$\frac{d\sigma(m,t)}{dt} = r_1(m-1)\sigma(m-1,t) + r_2(m+1)\sigma(m+1,t) - [r_1(m) + r_2(m)]\sigma(m,t),$$

which describes the time dependence of the probability $\sigma(m)$ of $m$ extra charges on the island. $r_{1,2}$ describe the rates of a charge tunneling event at the first and second junction. Tunneling against the bias voltage does not occur for zero temperature. The number of the charge states $m$ is finite, their energy lies between the Fermi levels of the banks. We denote the lowest and highest of these states by min and max, respectively.

With regard to the stochastic process of single-electron tunneling, the first moment describes the average number of tunnelled charges, $\langle n(t) \rangle$, which is connected with the stationary electrical current through the double junction, $\langle I \rangle = e\langle n(t) \rangle/t$, where $e$ is the unit charge. Using the stationary solution of (1)

$$\sigma(m) = \frac{1}{Z} \prod_{i=\text{min}}^{m-1} r_1(i) \prod_{i=m+1}^{\text{max}} r_2(i),$$

with the normalization constant $Z$, the stationary current reads

$$\langle I \rangle = e \sum_{m=\text{min}}^{\text{max}} r_1(m)\sigma(m) = e \sum_{m=\text{min}}^{\text{max}} r_2(m)\sigma(m).$$

We note that continuity levels the current across the first and the second junction.

The second moment of the stochastic process under consideration is connected with the current noise in a double junction, and has been investigated theoretically in recent years and its low-frequency limit was measured in experiment.

Further moments were - to our knowledge - not subject of theoretical efforts, however, an approach to the stochastic process of tunneling through a double junction (without charging effects and at zero temperature)
was performed recently. It is based on the calculation of the probability $P_n(t)$ of $n$ transmitted charges in terms of the equations

\[ \frac{dP_0(t)}{dt} = -\gamma_2 P_0(t) \]  
\[ \frac{dP_{n}(t)}{dt} = \gamma_2 P_{n-1}(t) - \gamma_2 P_{n}(t). \]  

$\gamma_{1,2}$ are the transmission probabilities of the two junctions. According to the Pauli exclusion principle there are only two different island states that have to be considered, unoccupied ($0$) and occupied ($1$) by one charge. The initial occupation probability of them is given by

\[ P_{0}^{(0,1)}(t = 0) = \frac{\gamma_{2,1}}{\gamma_1 + \gamma_2} \]  

and the probability of $n$ transmitted charges at arbitrary time $t$ is

\[ P_{n}(t) = \frac{\gamma_2}{\gamma_1 + \gamma_2} P_{n}^{(0)}(t) + \frac{\gamma_1}{\gamma_1 + \gamma_2} P_{n}^{(1)}(t). \]  

This problem is solved analytically in Ref. 12, i.e. the transmission probabilities $P_{n}(t)$ can be calculated. The description of the stochastic process, however, is much clearer in terms of its characteristic function $\chi(\lambda, t)$

\[ \chi(\lambda, t) = \sum_{n=0}^{\infty} P_{n}(t) e^{i n \lambda}, \]  

from which in turn all moments $\mu_k(t) = \langle n(t)^k \rangle$ of the stochastic process are obtained

\[ \mu_k(t) = \lim_{\lambda \to 0} \left( \frac{\partial}{\partial \lambda} \right)^k \chi(\lambda, k). \]  

Equivalently, the cumulants follow from the logarithmic expansion of the characteristic function $\chi(\lambda, t)$.

In our approach we include charging effects, however, still assume zero temperature. Therefore, a finite number of island charge states $m$ is considered and the description is done in terms of the probability $\rho(m, n, t)$ of $n$ transmitted charges through both junctions and simultaneously $m$ excess charges on the island between them. Thus, the following sum rules hold

\[ 1 = \sum_{m=\min}^{\max} \sum_{n=0}^{\infty} \rho(m, n, t), \]  
\[ \sigma(m, t) = \sum_{n=0}^{\infty} \rho(m, n, t), \]  
\[ P_{n}(t) = \sum_{m=\min}^{\max} \rho(m, n, t). \]  

For the time development of $\rho(m, n, t)$ the following first-order differential equation is found

\[ \frac{d\rho(m, n, t)}{dt} = r_1(m-1)\rho(m-1, n, t) + r_2(m+1)\rho(m+1, n-1, t) - \left[ r_1(m) + r_2(m) \right] \rho(m, n, t). \]  

The discussion of this equation and its solution is the main point of this paper.

The initial condition for the solution of (9) follows from (8) using $P_0(t = 0) = 1$,

\[ \rho(m, 0, t = 0) = \sigma(m) \]  
\[ \rho(m, n, t = 0) = 0 \quad n > 0. \]  

Summing (8) over $n$ and $m$, respectively, results by the use of (3) and (4) in the former Eqs. (8) and (4). Even (8) includes Coulomb effects in our case indicated by $m$ entering into (8).

Let us consider (8) in the long-time limit ($t \to \infty$) shortly. Single-electron tunneling is usually considered in this limit and several simplifications apply. The probability $\rho(m, n, t)$ factorizes $\rho(m, n, t) = P_m(t)\sigma(m)$, where $\sigma(m)$ is the stationary solution (2) of the master equation (8). Furthermore, the transmission probabilities become periodic, $P_{n \pm 1}(t) = P_n(t \mp e/\langle I \rangle)$. Under these conditions the stationary-current formula (8) results from (8), even for non-zero temperatures.

In general, the assumptions of the last paragraph do not hold and the solution takes a more complicated form, however, it can be expressed semi-analytically in terms of a recursion formula. The general solution of (8) can be written as

\[ \rho(m, n, t) = \sum_{k=\min}^{n} A_{nm}^{nk} \frac{t^n}{n!} \exp(-\gamma_k t) \]

\[ + \sum_{l=0}^{n-1} \sum_{k=\min}^{\max} A_{nm}^{lk} \frac{t^l}{l!} \exp(-\gamma_k t), \]

where $\gamma_k = r_1(k) + r_2(k)$ is used for abbreviation. Thus, the solution turns out to be a superposition of different poisson-like probabilities. The stochastic process as a whole, however, is not poisson-like. For $n = 0$ the second term in (11) vanishes.

The coefficients $A_{nm}^{lk}$ have to fulfil a recursion relation which can be derived from (8) by introducing (11). For a short-hand notation of this recursion relation we make use of the following decomposition

\[ \frac{1}{(s + \gamma_m)(s + \gamma_k)^{l+1}} = \frac{\alpha_{lk}^{lm}}{s + \gamma_m} + \sum_{i=0}^{l} \frac{\alpha_{lk}^{lm}}{(s + \gamma_k)^{i+1}}, \]

which defines the numbers $\alpha_{lk}^{lm}$ in a unique way as solution of a linear equation system. In terms of this notation we find
\[ A_{nm}^{nk} = r_1(m-1)A_{nm-1}^{nk} \]
\[ A_{nm}^{nm} = r_1(m-1)A_{nm-1}^{n-1m} + r_2(m+1)A_{n-1m+1}^{n-1m} \]
\[ A_{nm}^{lm} = r_1(m-1)A_{nm-1}^{l-1m} + r_2(m+1)A_{n-1m+1}^{l-1m} \]
\[ A_{nm}^{ik} = r_1(m-1)A_{nm-1}^{ik} + r_2(m+1)A_{n-1m+1}^{ik} \]

\[ k < m \]

For simplicity we made use of the If\[ \ldots \]. notation: the corresponding terms contribute only if the given condition is fulfilled. Eq. 12 is, in connection with (11), our main result.

In addition to (12), the initialization of \( A_{nm}^{ik} \) follows from (9) for \( n = 0 \) using the given initial condition (10),

\[ A_{0m}^{ik} = -\frac{r_1(m-1)A_{0m-1}^{ik}}{\gamma_k - \gamma_m} \quad k < m \]
\[ A_{0m}^{0m} = \sigma(m) + r_1(m-1) \sum_{k=\min}^{m-1} \frac{A_{0m-1}^{0k}}{\gamma_k - \gamma_m}. \]

Eq. 13 does not hold in the very symmetric case. Then, the solution might derived from the given set of equation in terms of a limit procedure \( \gamma_k \to \gamma_m \).

Eqs. 12 and 13 allow the construction of all \( A_{nm}^{ik} \). These coefficients tell the important contributions within (11) from the unimportant ones. If a sole term dominates the stochastic process will approach a Poisson process, similar to the case without Coulomb effects. In general, however, more terms contribute and the total process is a superposition of Poisson processes. The result (12), however, does not seem to provide clear guidance to a simplification by the selection of relevant terms. Even if the given recursion relation (12) looks clumsy its numeric implementation is neat.

III. DISCUSSION

Owing to the complicated expressions (12) we discuss our results by means of numerical computations. The most important question concerns the time domain where (11) is valid. Due to the derivation of the rates \( r_1,2(m) \) by Fermi’s Golden Rule, the time scale is limited to \( t \gg R_1,2C_{1,2} \). On the other hand, for very long times the system will approach the mentioned stationary state of the island and the investigation loses its thrill.

For simplicity we made use of the If\[ \ldots \]. notation: the corresponding terms contribute only if the given condition is fulfilled. Eq. 12 is, in connection with (11), our main result.

In addition to (12), the initialization of \( A_{nm}^{ik} \) follows from (9) for \( n = 0 \) using the given initial condition (10),

\[ A_{0m}^{ik} = -\frac{r_1(m-1)A_{0m-1}^{ik}}{\gamma_k - \gamma_m} \quad k < m \]
\[ A_{0m}^{0m} = \sigma(m) + r_1(m-1) \sum_{k=\min}^{m-1} \frac{A_{0m-1}^{0k}}{\gamma_k - \gamma_m}. \]

Eq. 13 does not hold in the very symmetric case. Then, the solution might derived from the given set of equation in terms of a limit procedure \( \gamma_k \to \gamma_m \).

Eqs. 12 and 13 allow the construction of all \( A_{nm}^{ik} \). These coefficients tell the important contributions within (11) from the unimportant ones. If a sole term dominates the stochastic process will approach a Poisson process, similar to the case without Coulomb effects. In general, however, more terms contribute and the total process is a superposition of Poisson processes. The result (12), however, does not seem to provide clear guidance to a simplification by the selection of relevant terms. Even if the given recursion relation (12) looks clumsy its numeric implementation is neat.

III. DISCUSSION

Owing to the complicated expressions (12) we discuss our results by means of numerical computations. The most important question concerns the time domain where (11) is valid. Due to the derivation of the rates \( r_1,2(m) \) by Fermi’s Golden Rule, the time scale is limited to \( t \gg R_1,2C_{1,2} \). On the other hand, for very long times the system will approach the mentioned stationary state of the island and the investigation loses its thrill.
is shown. For short times it approximates the total probability \( 1 \), whereas later neglected states get more important. We note that the first ten \( P_n(t) \) are sufficient to describe times up to \( \approx 20 R_2 C_2 \) in the case with Coulomb interaction, but only up to \( \approx 2 R_2 C_2 \) in the case without charging. For short times the applied description of the system fails, however.

\[
\sum_{n=0}^{\infty} P_n(t)
\]

For the studied case with Coulomb interaction noticeable changes in \( r(t) \) are observed even in the time range of \( t \approx 10 R_2 C_2 \). This is well above the critical short-time range, where the approach in terms of the Golden Rule is insufficient. This fact supports justification of the presented approach.

In order to apply the solution (11), we plot the Fano factor

\[
r(t) = \frac{\mu_2(t)}{\mu_1(t)} - \mu_1(t)
\]

for the case with and without Coulomb interaction in Fig. 2. The Fano factor describes the correlation of single tunneling events. Small \( r(t) \) indicate a large degree of correlation. Therefore, a high Fano factor is expected at \( t = 0 \), whereas lateron it should approach a lower stationary limit.

Again, our computation makes use of the first ten \( P_n(t) \) only. This corresponds to Fig. 1. Owing to the limited number \( n \) the plotted approximation fails at longer times, i.e., when the total probability (as shown in the inset of Fig. 1) differs from unity. Besides this flaw the expected behavior of the Fano factor is recovered. For the case without Coulomb interaction we plot the long-time limit of Ref. 12 as well.

Coulomb interaction favors correlated tunneling. This is seen by the lower value of \( r(t) \) in comparison to the stationary case without Coulomb interaction. The degree of correlation depends on the symmetry of the double junction [. . .]. It gets maximal in case of a symmetric double junction \( (R_1 = R_2, C_1 = C_2) \), where the Fano factor reaches 1/2. The double junction under consideration, however, is rather asymmetric. Thus, the decrease of the Fano factor is small.

Finally, in Fig. 3 we display the time-dependence of the characteristic function \( |\chi(\lambda, t)| \) for three different values of \( \lambda (0.5, 1.0, 2.0) \), from top to bottom. The double junction parameters correspond to the other figures. Again, solid and dashed lines indicate the computation with and without Coulomb energy, respectively.

IV. CONCLUSION

In this paper we have studied the stochastic process of single-electron tunneling in an ultrasmall normal-metal double-tunnel junction at zero temperature. We have introduced an equation, (11), which enables the investigation of transmission probabilities of charge corresponding to the “orthodox theory” of single-electron tunneling. The time-dependent solution of this equation is derived in terms of a recursion relation, (12), with the appropriate initialization (13). The results are discussed in comparison to the known investigation of the double junction without charging effects. We observe a suppression of tunneling due to Coulomb repulsion and an enhanced correlation. The recursion relation provides a suitable
tool for computation of stochastic properties of the double junction. The absolute value of the characteristic function $\chi(\lambda, t)$ serves as an example.

ACKNOWLEDGMENT

Hospitality at the Friedrich-Schiller-Universität Jena, Germany, is gratefully acknowledged.

1 phm@rz.uni-jena.de.
2 D. V. Averin and K. K. Likharev, J. Low Temp. Phys. 62, 345 (1986).
3 T. A. Fulton and G. J. Dolan, Phys. Rev. Lett. 59, 109 (1987).
4 D. V. Averin and K. K. Likharev, in Mesoscopic Phenomena in Solids, Modern Problems in Condensed Matter Sciences (30), edited by B. L. Altshuler, P. A. Lee, and R. A. Webb (Elsevier, Amsterdam, 1991), pp. 173–271.
5 M. Amman et al., Phys. Rev. B 43, 1146 (1991).
6 H. Schoeller and G. Schön, Phys. Rev. B 50, 18436 (1994), cond-mat/9409053.
7 F. Seume and W. Krech, Ann. Phys. (Leipzig) 1, 198 (1992).
8 A. N. Korotkov, Phys. Rev. B 49, 10381 (1994).
9 U. Hanke, Y. M. Galperin, K. A. Chao, and N. Zou, Phys. Rev. B 48, 17209 (1993).
10 H.-O. Müller, A. Hädicke, U. Hanke, and K.-A. Chao, Z. Phys. e–B 1, 1 (1997), http://science.springer.de/zfp-e-b/tocs/t7001001.htm, cond-mat/9603098.
11 H. Birk, M. J. M. de Jong, and C. Schönberger, Phys. Rev. Lett. 75, 1610 (1995).
12 M. J. M. de Jong, Phys. Rev. B 54, 8144 (1996), cond-mat/9605094.
13 N. G. van Kampen, Stochastic Processes in Physics and Chemistry (Elsevier, Amsterdam, 1981).