Macroscopic Entangled States in Superconducting Flux Qubits

Mun Dae Kim\(^1\) and Sam Young Cho\(^2\)

\(^1\) Korea Institute for Advanced Study, Seoul 130-722, Korea
\(^2\) Center for Modern Physics and Department of Physics, Chongqing University, Chongqing 400044, China

E-mail: mdkim@kias.re.kr

Abstract. We theoretically study the macroscopic quantum entanglement in superconducting flux qubits. A phase-coupling scheme is proposed to offer enough strength of interactions between qubits. It is shown that due to the two-qubit tunneling processes both the ground state and excited states of coupled two flux qubits can be Bell type states, maximally entangled, in experimentally accessible regimes. The parameter regimes for the Bell states are discussed in terms of external magnetic flux and Josephson coupling energies. We also investigate two types of genuine three-qubit entanglement, known as the Greenberger-Horne-Zeilinger (GHZ) and W states. While an excited state can be the W state, the GHZ state is formed at the ground state of the coupled three flux qubits. The GHZ and W states are shown to be robust against external flux fluctuations for feasible experimental realizations.

1. Introduction

Entanglement is the key concept in the quantum information science. Recently the realization of entanglement in experiment has been widely studied. For photonic, atomic and ionic systems the bipartite and tripartite entanglements have been demonstrated experimentally. However, in solid state systems the experimental realization has hardly been achieved. Only partial entanglements have been reported [1, 2, 3, 4, 5]. Recently two coupled phase qubits showed a Bell state at an excited level (eigenstate) with high fidelity [6]. In that experiment the interaction between two phase qubits is tunnel-type coupling which corresponds to simultaneous two-qubit flipping.

In this paper we study the multi-qubit tunnelling processes to achieve the maximally entangled states for tripartite as well as bipartite systems consisting of superconducting flux qubits. We show that the multi-qubit flipping processes are essential for obtaining the maximally entangled states in appropriate parameter regimes. When the strength of (Ising-type) coupling is sufficiently strong, we found that the multi-qubit tunnellings become dominant over the usual single-qubit tunnelling. We introduce a phase-coupling scheme [7, 8, 9, 10, 11] for a strong coupling. For the coupled two-qubit system it is found that the Bell state can be formed at the ground state of the Hamiltonian. On the other hand, the three-qubit system show the GHZ state at the ground state whereas it produces the W state at an excited level.

We use the Q-measure [12] for the entanglement of tripartite system, while calculating the concurrence [13] for the bipartite system. By calculating the peak width of these entanglement measures we show that the maximally entangled states in our study are robust against noises.
2. Bell states

The Hamiltonian of two coupled flux qubits in Fig. 1(a) is given by

$$\hat{H} = \frac{1}{2} \hat{P}_i^T M_{ij}^{-1} \hat{P}_j + U_{\text{eff}}(\hat{\phi}),$$

(1)

where $M_{ij} = (\Phi_0/2\pi)^2 C_i \delta_{ij}$ with the capacitance $C_i$ of the Josephson junction is the effective mass, $\hat{\phi} = (\phi_{L1}, \phi_{L2}, \phi_{L3}, \phi_{R1}, \phi_{R2}, \phi_{R3}, \phi')$, and the effective potential $U_{\text{eff}}(\hat{\phi}) = \sum_{i=1}^{3} E_{Ji}(1 - \cos \phi_{Li}) + \sum_{i=1}^{3} E_{Ji}(1 - \cos \phi_{Ri}) + E'_{J}(1 - \cos \phi')$. The boundary conditions in the left and right qubit loops and the connecting loop are given by $2\pi(n_{L(R)} + f_{L(R)}) - (\phi_{L(R)1} + \phi_{L(R)2} + \phi_{L(R)3}) = 0$ and $2\pi r + (\phi_{L1} - \phi_{R1}) - \phi' = 0$ with integers $n_L, n_R$ and $r$. Introducing the rotated coordinates $\phi_p \equiv (\phi_{L3} + \phi_{R3})/2$ and $\phi_m \equiv (\phi_{L3} - \phi_{R3})/2$ with the usual assumption $E_{J2} = E_{J3} = E_J$ and $\phi_{L(R)2} = \phi_{(L(R)3)}$, the Hamiltonian can be rewritten in the form, $\hat{H} = \hat{P}_p^2/2M_p + \hat{P}_m^2/2M_m + U_{\text{eff}}(\phi_p, \phi_m)$, where the effective potential becomes in the rotated coordinates

$$U_{\text{eff}}(\phi_p, \phi_m) = 2E_{J1}(1 + \cos 2\phi_p \cos 2\phi_m) + 4E_J(1 - \cos \phi_p \cos \phi_m) + E'_J(1 - \cos 4\phi_m).$$

(2)

Here, $M_p \equiv \hbar^2 (2C_1 + C)/e^2$, $M_m \equiv \hbar^2 (2C_1 + C + 4C')/e^2$ and the conjugate momentum is $\hat{P}_{p(m)} = -i\hbar \partial / \partial \phi_{p(m)}$ with commutation relation $[\phi_{p(m)}, \hat{P}_{p(m)}] = i\hbar$.

In Fig. 2 we show the effective potential $U_{\text{eff}}(\phi_p, \phi_m)$ for a weak and a strong coupling. For a weak coupling, $E'_J \approx 0$, the single-qubit tunnelling amplitudes between two nearest neighbor states are dominant over the two-qubit tunnelling through a potential barrier denoted as a solid line. However, for a strong coupling in Fig. 2(b) the same pseudo-spin states have much lower energies than the opposite spin states. Then as shown in Fig. 3(a) the potential barrier for the single-qubit tunnelling $t_1$ is higher than that for $t_2^2$ and the energy difference between two states $|\downarrow\downarrow\rangle$ and $|\uparrow\downarrow\rangle$ becomes large. As a result, the two-qubit tunnelling $t_2^2$ between $|\downarrow\downarrow\rangle$ and $|\uparrow\downarrow\rangle$ states become dominant over $t_1$. In the meanwhile, since the wavefunction overlap between $|\downarrow\downarrow\rangle$ and $|\uparrow\downarrow\rangle$ states in Fig. 3(a) is too weak, the two-qubit tunnelling amplitude $t_2^2$ between these states is negligible.

In Fig. 3(b) we show the concurrence calculated from the tight binding Hamiltonian using the tunnelling amplitudes obtained above for a strong coupling, $E'_J = 0.6E_J$. The peak width along the $f_p$ axis is not so large, but we found that it is larger than the flux fluctuation $10^{-6}\Phi_0$ in usual experiments [14]. Thus the maximally entangled states will be robust against noises.

Figure 1. (a) Phase-coupled two flux qubits where two superconducting qubit loops are connected by a connecting loop. The gray squares are the Josephson junctions with phase difference $\phi_i$'s across the junction and the Josephson coupling energy $E_{J_i}$'s. The arrows show the flow of Cooper pairs. We introduce a pseudo-spin notation $|\downarrow\rangle$ for the diamagnetic and $|\uparrow\rangle$ for paramagnetic current states. Here the external magnetic fluxes $f_{L(R)}$ threading the left (right) qubit loops are set to be in the opposite directions, which makes the coupling between the pseudo-spins ferromagnetic. (b) Three coupled flux qubits which are connected at a three-way junction. The Josephson coupling energy of each Josephson junction in the connecting loop is $E_{J_3}$. 

Figure 2. The effective potential $U_{\text{eff}}(\phi_p, \phi_m)$ in Eq. (2) for two coupled flux qubits for (a) a negligible coupling ($E'_{j} \approx 0$) and (b) a strong coupling ($E'_{j} = 0.6E_{j}$). Here the solid line denote the two-qubit tunnelling $t_{2}^{\text{a}}$ between $|\downarrow\downarrow\rangle$ and $|\uparrow\uparrow\rangle$ states and the dashed line the single-qubit tunnelling $t_{1}$.

3. GHZ and W states
For a tripartite system we introduce the coupled three flux qubits in Fig. 1(b). By using the boundary conditions, $\phi_{q1} + \phi_{q2} + \phi_{q3} = 2\pi(n_q + f_q)$, $(\phi_{a1} - \phi_{c1}) - \phi'_b + \phi'_c = 2\pi r$, and $(\phi_{b1} - \phi_{c1}) - \phi'_b + \phi'_c = 2\pi s$, with $q \in \{a, b, c\}$ and integers $n_q$, $r$, $s$, we obtain the effective potential $U_{\text{eff}}(\phi_{q3}, \phi_{b3}, \phi_{c3})$ as shown in Fig. 4(a). Here we introduce rotated coordinates, $\phi_{p} \equiv (\phi_{a3} + \phi_{b3})/\sqrt{2}$ and $\phi_{m} \equiv (\phi_{a3} - \phi_{b3})/\sqrt{2}$. In Fig. 4(b) we plot the potential wells corresponding to the arrows in Fig. 4(a), which shows that for a strong coupling of $E'_{j}/E_{j} = 0.9$ the three-qubit tunnelling $t_{3}^{\text{a}}$ is dominant over the other tunnellings.

In this study we use the Q factor for the global entanglement measure; for tripartite system $Q = 1$ for the GHZ state and $Q = 8/9$ for the W state. In Fig. 4(c) we show the value of Q measure in the $(f_{\alpha}, f_{\beta})$ plane. For a strong coupling we can see a asterisk shape of maximally entangled state of $Q = 1$ (GHZ state). However, along the axis of $f_q \equiv (f_{a} + f_{b} + f_{c})/\sqrt{3}$ the Q-measure shows a rather sharp peak. We analyzed the peak width of the Q-measure envelop to find that the peak width depends on the three-qubit tunnelling amplitude $t_{3}^{\text{a}}$. Since for a strong coupling $t_{3}^{\text{a}}$ is large as shown in Fig. 4(b), the GHZ state is robust against external fluctuations.

On the other hand, the W state $|\Psi_{W}\rangle = (|\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle)/\sqrt{3}$ can be formed by virtue of the two-qubit tunnelling processes. For a strong coupling, the three-qubit rather than two-qubit tunnelling amplitude is enhanced. Thus we need somewhat weak coupling. However, if

Figure 3. (a) Effective potential $U_{\text{eff}}$ as a function of $\phi_{L3}$ along the corresponding lines in Fig. 2(b). (b) Concurrence in $(f_{m}, f_{p})$ plane. Around the coresonance line we observe the maximal entanglement.
Figure 4. (a) The effective potential of three coupled qubits in Fig. 1(b) in the \((\phi_p, \phi_c)\) plane for a strong coupling of \(E'_c/E_J = 0.9\). Here \(t_i^q\) is \(i\)-qubit tunnelling amplitude. (b) The potential wells corresponding to the lines in Fig. 4(a) as a function of \(\phi_q = (\phi_p + \phi_c)/\sqrt{3}\), which shows that the potential barrier for three-qubit tunnelling is much lower than those for other tunnelling processes. (c) \(Q\)-measure for the entangled three-qubit system as a function of rotated fluxes, \(f_0 \equiv (f_a - f_b)/\sqrt{2}\) and \(f_\beta \equiv (f_a + f_b - 2f_c)/\sqrt{6}\). Around the central region we see the maximally entangled state \((Q = 1)\) corresponding to the GHZ state, \(|\Psi_{GHZ}\rangle = (|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle) / \sqrt{2}\).

the coupling is too weak, the states, \(|\uparrow\downarrow\rangle\), \(|\uparrow\uparrow\rangle\), \(|\downarrow\downarrow\rangle\), and \(|\downarrow\uparrow\rangle\), in a W state become mixed with the states \(|\uparrow\uparrow\rangle\) and \(|\downarrow\downarrow\rangle\) in the GHZ state, reducing the entanglement. Therefore, for some intermediate coupling we can obtain the W states in excited levels [15].

4. Summary
We study the entanglement of coupled superconducting flux qubits. It is found that the multi-qubit tunnelling plays a crucial role for achievement of the maximally entangled states. By obtaining the potential wells for single-, two-, three-qubit tunnelling processes we found that for strong coupling cases the multi-qubit tunnelling becomes dominant over the other tunnelling processes. As a result, the maximally entangled states can be obtained as the eigenstates of the system and, moreover, are robust against the external flux fluctuations.

5. References
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