Area Preserving Transformations in Non-commutative Space and NCCS Theory

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We propose an heuristic rule for the area transformation on the non-commutative plane. The non-commutative area preserving transformations are quantum deformation of the classical symplectic diffeomorphisms. Area preservation condition is formulated as a field equation in the non-commutative Chern-Simons gauge theory. The higher dimensional generalization is suggested and the corresponding algebraic structure - the infinite dimensional sin-Lie algebra is extracted. As an illustrative example the second-quantized formulation for electrons in the lowest Landau level is considered.

I. INTRODUCTION

In the recent papers [1, 2] it is raised an intriguing question on the connection between hydrodynamics of the incompressible fluid and the gauge field theory on the non-commutative (NC) space. Practical realization of this idea for the planar ($D = 2$) electron system was considered earlier in the context of Chern-Simons (CS) description of the quantum Hall effect [3].

Introduction of the vector potential as an hydrodynamical variable together with the requirement of invariance under the classical area preserving transformations leads to the CS gauge theory based on the group of the symplectic transformations $Sdiff$ in $\mathbb{R}^2$. Non-commutative Chern-Simons (NCCS) theory is obtained subjecting the classical symplectic structure to the quantum deformation.

In the present paper we propose to attribute the above deformation of the classical algebra to the non-commutativity of the two-dimensional surface under consideration. In other terms we consider a counterpart of area preserving diffeomorphisms (APD’s) in the NC space and extract the corresponding symplectic structure, which, as one may expect turns out to be the Moyal-type deformation of the classical Poisson bracket.

The non-commutative plane is represented by the pair of Hermitian operators $\hat{x}_i$ obeying

$$[\hat{x}_i, \hat{x}_k] = i\theta_{ik} = i\theta\epsilon_{ik} \quad (i, k = 1, 2)$$

with the constant anti-symmetric non-commutativity matrix $\theta$ (for the review of NC geometry and adopted notations see e.g. [4]).

In order to establish the non-commutative analogue of APD’s let us remind some basic definitions concerning classical symplectic structures and APD’s [5, 6].

Let $\Delta \subset \mathbb{R}^2$ be some compact domain, described by the Cartesian coordinates $x_i$. The Poisson bracket is defined by

$$\{f(x), g(x)\}_P = \theta_{ik} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_k}. \quad (2)$$

Consider a diffeomorphism

$$x_i \rightarrow x'_i = F_i(x), \quad \Delta \rightarrow \Delta'. \quad (3)$$

Under this map the area

$$\Omega_\Delta = \int_\Delta d^2x = \int_\Delta d^2x \text{Pf} \omega \text{Pf}\{x_i, x_k\}_P \quad \omega = \theta^{-1}$$

changes according to the rule

$$\Omega_\Delta \rightarrow \Omega'_\Delta = \int_{\Delta'} d^2x' = \int_\Delta d^2x \text{Pf} \omega \text{Pf}\{F_i, F_k\}_P. \quad (4)$$
Here we use that
\[ J(x) = \text{Pf} \omega \text{Pf} \{ F_i, F_k \}_P \]
(6)
is the \textit{Jacobian} determinant corresponding to the transformation \( J \). The \textit{Pfaffian} is defined by \( \text{Pf} \omega_{ik} = (\det M_{ik})^{1/2} \).

Infinitesimal transformations
\[ F_i(x) = x_i + \xi_i(x) \]
(7)
are generated by the divergenceless vector fields \( \xi_i \)
\[ \xi_i = \theta_{ik} \partial_k \xi, \quad \partial_i \xi_i = 0. \]
(8)
The sought after algebraic structure can be revealed considering the variation (a \textit{Lie} derivative) of the scalar function
\[ \delta _\xi f(x) = -L_\xi f(x) = \{ \xi, f \}_P. \]
(9)
Generators
\[ t[\xi] = -i L_\xi \]
(10)
satisfy commutation relations
\[ [t[\xi], t[\eta]] = t[i \{ \xi, \eta \}_P] \]
(11)
which define the \textit{Lie} algebra of the group \( \text{Diff} \).

In the case of \( D = 2N \)-dimensional \textit{Euclidean} space one may assume that \( x_i \) are canonical coordinates, \textit{i.e.} only non-vanishing \textit{Poisson} brackets are
\[ \{ x_{2\alpha-1}, x_{2\alpha} \} = \theta_{\alpha} \equiv \theta_{2\alpha-1,2\alpha} > 0 \quad (\alpha = 1, 2, \ldots, N). \]
(12)
In general the canonical coordinate system is not an ortho-normal one, and the constant metric tensor \( h_{ik} \) is not diagonal. In that case under the diffeomorphism \( x_i \rightarrow F_i(x) \) the D-volume changes according to the formula
\[ \Omega_\Delta = \int_\Delta d^D x \sqrt{\det h_{ik} \text{Pf} \omega \text{Pf} \{ x_i, x_k \}_P} \quad \rightarrow \quad \Omega'_\Delta = \int_\Delta d^D x \sqrt{\det h_{ik} \text{Pf} \omega \text{Pf} \{ F_i, F_k \}_P}. \]
(13)
For \( D > 2 \) divergenceless vector fields \( \xi \) constitute a symplectic (\textit{i.e.} \( \theta \) conserving) subgroup of the volume preserving transformations.

\section*{II. AREA PRESERVING TRANSFORMATIONS IN NC \( \mathbb{R}^2 \)}

The formula \( \xi \) may be used with the aim to state the area transformation rule on the NC plane. But first one has to give a mathematical substance to the notion of the area on the NC space.

With this purpose consider the realization of the commutation relation \( [\mathbb{I}] \) in the \textit{Hilbert} space \( \mathcal{H} \). Operators \( \hat{z} = \hat{x}_1 + i \hat{x}_2 \) and \( \hat{\bar{z}} = \hat{x}_1 - i \hat{x}_2 \) satisfy the oscillator algebra
\[ [\hat{z}, \hat{\bar{z}}] = 2\theta. \]
(14)
Introduce the normalized coherent states
\[ |\zeta\rangle = e^{-|\zeta|^2} e^{\frac{i}{\sqrt{2}} \zeta \hat{\bar{z}}}|0\rangle, \quad \hat{\bar{z}}|0\rangle = 0 \quad \langle \zeta|\zeta\rangle = 1 \]
(15)
such that
\[ \hat{z}|\zeta\rangle = \zeta |\zeta\rangle \quad \zeta = \zeta_1 + i \zeta_2. \]
(16)
Here by \( \zeta_i \) we denote the averages
\[ \zeta_i = \langle \zeta|\hat{\bar{x}}_i|\zeta\rangle. \]
(17)
Note that (17) establishes the 1-1 correspondence between the coherent states (15) and the points in $\mathbb{R}^2$ ($\zeta_1, \zeta_2$) $\in \mathbb{R}^2$ $\leftrightarrow$ $|\zeta\rangle$ $\in \mathcal{H}$.

Using the isomorphism between the domain $\Delta \subset \mathbb{R}^2$ and the subspace $\mathcal{H}_\Delta \subset \mathcal{H}$, the area $\Omega_\Delta$ can be presented as an integral in the $\zeta$-plane

$$\Omega_\Delta = \int_\Delta d^2\zeta \text{Pf} \langle \zeta | \hat{\Omega} | \zeta \rangle \equiv \int d^2\zeta = \Omega_\Delta$$

where

$$\hat{\Omega} = -\frac{i}{2} \epsilon_{ij} [\hat{x}_i, \hat{x}_j].$$

The matrix element in (19) mimics the Pfaffian Pf{$x_i, x_k$} in (4). Acting by analogy with (5) we set, that the operator homomorphism

$$\hat{W}[x_i] = \hat{x}_i$$

induces the “area transformation”

$$\Omega_\Delta \rightarrow \Omega'_\Delta = \int_\Delta d^2\zeta \text{Pf} \omega\langle \zeta | \hat{\Omega}' | \zeta \rangle$$

where

$$\hat{\Omega}' = -\frac{i}{2} \epsilon_{ik} [\hat{W}[F_i], \hat{W}[F_k]] = -\frac{i}{2} \epsilon_{ik} \hat{W}[F_i \star F_k].$$

Here by

$$\hat{W}[F_i] = \frac{1}{(2\pi)^2} \int d^2p \int d^2xe^{-ip(\hat{x}_i - x_i)} F_i(x)$$

we denote the symbol of the Weyl ordering and

$$f(x) \star g(x) = e^{\frac{i}{\hbar} \int \partial_i \partial' k f(x') \cdot g(x') |_{x' = x}}$$

is the Groenewold-Moyal star product. It must be noted, that the area transformation Ansatz (22) is not unique as well as one can use other types of operator ordering.

Remark now, that in the formula (22) there figures an integral

$$I_\Delta[f] \equiv \int_\Delta d^2\zeta \langle \zeta | \hat{W}[f] | \zeta \rangle$$

and from the definitions of coherent states and Weyl symbols one easily finds that

$$I_\Delta[f] = \int_\Delta d^2\zeta \int d^2x D(\zeta - x)f(x),$$

where

$$D(\zeta - x) = \frac{1}{\theta} e^{-\frac{1}{\theta}[(x_i - \zeta_i)^2].}$$

Thus the area transformation rule (22) takes the following form

$$\Omega_\Delta \rightarrow \Omega'_\Delta = \int_\Delta d^2\zeta \int d^2x D(\zeta - x) \text{Pf} \omega(-i) \text{Pf}\{F_i, F_k\}_M(x)$$

where

$$\{F_i, F_k\}_M(x) = F_i(x) \star F_k(x) - F_k(x) \star F_i(x)$$
is the *Moyal* bracket.

Now we see, that transition to the NC case is realized by the substitution

\[
\{ F_1, F_k \}_\rho(\zeta) \to \{ F_1, F_2 \}_{NC}(\zeta) \equiv -i \int d^2 x D(\zeta - x) \{ F_1, F_k \}_\rho(x).
\]  

In the commutative limit

\[
\lim_{\theta \to 0} \text{Pf} \omega \text{Pf} \{ F_i, F_k \}_{NC} = \lim_{\theta \to 0} \text{Pf} \omega \{ F_i, F_k \}_\rho = J
\]

is a *Jacobian* and the expression (29) gives the classical result 45. Note that the earlier proposed heuristic rule for the area transformation \( \delta^2(\zeta - x) \) may be presented in the form (29) if one sets \( D(\zeta - x) = \delta^2(\zeta - x). \)

The requirement that (21) is an area preserving operator transformation results

\[
\left[ \hat{W}[F_1], \hat{W}[F_2] \right] = \left[ \hat{x}_1, \hat{x}_2 \right] = i\theta.
\]  

The same condition may be rewritten in terms of the *Moyal* bracket

\[-i\text{Pf} \omega \text{Pf} \{ F_i, F_k \}_M = 1\]  

Equations (33)-(34) are non-commutative counterparts of the classical area preservation condition \( \{ F_1, F_2 \}_\rho = \theta \) and corresponding operator homomorphisms can be referred to as the NC or quantum APD’s.

The last equations can be generalized for the \( 2N \)-dimensional non-commutative space defined by the commutators

\[
\left[ \hat{x}_{2\alpha-1}, \hat{x}_{2\alpha} \right] = i\theta_{2\alpha-1,2\alpha} \equiv i\theta_\alpha \quad (\alpha = 1, 2, \ldots, N).
\]  

The *Fock* space operators are identified by

\[
\hat{x}_\alpha = \hat{x}_{2\alpha-1} + i\hat{x}_{2\alpha}, \quad \hat{z}_\alpha = \hat{x}_{2\alpha-1} - i\hat{x}_{2\alpha} \quad \left[ \hat{x}_\alpha, \hat{z}_\alpha \right] = 2\theta_\alpha.
\]

Define coherent states

\[
|\zeta_\alpha\rangle = e^{-\frac{1}{4\theta_\alpha}|\zeta_\alpha|^2} \sum_{n=0}^\infty \frac{1}{\sqrt{n!}} \zeta_\alpha^n |0\rangle, \quad \zeta_\alpha |0\rangle = 0 \quad \langle \zeta_\alpha | \zeta_\alpha \rangle = 1
\]

such that

\[
\hat{z}_\alpha |\zeta_\alpha\rangle = \zeta_\alpha |\zeta_\alpha\rangle.
\]

Construct the state vector

\[
|\{\zeta\}\rangle = \prod_\alpha \otimes |\zeta_\alpha\rangle
\]

and introduce averages

\[
\zeta_i = \langle \{\zeta\} | \hat{x}_i | \{\zeta\} \rangle, \quad 1 \leq i \leq D.
\]

The last relation states the isomorphism between points in \( \mathbb{R}^D \) and vectors in the *Hilbert* space.

The volume transformation may be presented as

\[
\hat{W} \Delta \to \hat{W} \Delta' = \int_{\Delta} d^D \zeta \sqrt{\det h_{ij}} \text{Pf} \omega \langle \{\zeta\} | \hat{W} \Delta' | \{\zeta\} \rangle
\]

where

\[
\hat{W} \Delta' = \hat{W} [\text{Pf} \{ F_i, F_k \}_M].
\]

The star-product modified *Pfaffian* is defined by

\[
\text{Pf} \{ F_i, F_j \}_M = \frac{1}{2N!} \epsilon_{i_1 j_1 \cdots i_N j_N} A_{i_1 j_1} \ast A_{i_N j_N} \ast \cdots \ast A_{i_1 j_1},
\]

where \( \ast \) and \( \{ F_i, F_j \}_M \) correspond to the matrix \( \theta \) in (35).

The multi-dimensional analogue of the volume preservation condition (35) may be presented as follows

\[
(-i)^N \text{Pf} \omega \text{Pf} \{ F_i, F_k \}_M = 1
\]
III. NCCS

Consider the map

\[ x_i \rightarrow F_i(x) = x_i + \theta_{ik}a_k(x) \]  \hspace{1cm} (i, k = 1, 2, ..., 2N). \hspace{1cm} (45)

The basic Moyal bracket is given by

\[ \{F_i, F_k\}_M = i\theta_{ik} + i\Phi_{ik} \]  \hspace{1cm} (46)

where

\[ \Phi_{ik} = \theta_{im}\theta_{kn}[\partial_m a_n - \partial_n a_m - i(a_m * a_n - a_n * a_m)] \]  \hspace{1cm} (47)

In terms of these hydrodynamical variables the volume preservation condition (44) looks as follows

\[ \epsilon_{i_1 j_1 ... i_N j_N} \sum_{l=0}^{N-1} \frac{N!}{l!(N-l)!} \theta_{i_1 j_1} \cdots \theta_{i_{l+1} j_{l+1}} \Phi_{i_{l+1} j_{l+1} i_N j_N} = 0. \]  \hspace{1cm} (49)

In \( D = 2 \)

\[ Pf(\theta_{ik} + \Phi_{ik}) = \theta + \frac{1}{2} \theta^2 \epsilon_{mn} F_{mn} \]  \hspace{1cm} (50)

and equation (49) takes the form

\[ \theta_{ik} D_i a_k \equiv Da = \theta_{ik}(\partial_i a_k - ia_i * a_k) = 0. \]  \hspace{1cm} (51)

One can notice the evident resemblance between (51) and the Gauss law in the NCCS gauge theory. This theory is described by the Lagrangian

\[ \mathcal{L}_{NCCS} = \frac{\kappa}{2} \varepsilon^{\mu\nu\lambda} a_\mu \left( \partial_\nu a_\lambda - \frac{i}{3} \{a_\nu, a_\lambda\}_M \right) \]  \hspace{1cm} (\mu, \nu, \lambda = 0, 1, 2) \hspace{1cm} (52)

and (51) turns out to be the Euler-Lagrange equation

\[ \frac{\delta \mathcal{L}_{NCCS}}{\delta a_0} = \kappa Da = 0. \]  \hspace{1cm} (53)

The infinitesimal operator transformation

\[ \Delta_\lambda \hat{W}[F_i] = -i \left[ \hat{W}[\lambda], \hat{W}[F_i] \right] = -\theta_{ik} \hat{W}[\partial_k \lambda + i\{\lambda, a_k\}_M] = -\theta_{ik} \hat{W}[\delta_{\text{gauge}} a_k] \]

induces the gauge transformation of the vector field \( a_k \).

In the commutative limit one recovers the CS theory with the classical gauge group \( S_{\text{diff}} \). The corresponding non-linear symplectic CS (SCS) Lagrangian is given by

\[ \mathcal{L}_{SCS} = \frac{\nu}{2} \varepsilon^{\mu\nu\lambda} A_\mu \left( \partial_\nu A_\lambda + \frac{1}{3} \{A_\nu, A_\lambda\}_P \right) \]  \hspace{1cm} (54)

and gauge transformations look as follows

\[ \delta_{\text{gauge}} A_i = \partial_i \lambda - \{\lambda, A_i\}_P. \]  \hspace{1cm} (55)

Demanding the gauge invariance under the group \( S_{\text{diff}} \) one arrives at the Lagrangian (54) which could be interpreted as an approximation for the total NCCS Lagrangian (52). Transition from the SCS to the NCCS theory is accomplished by the replacement \( i\{f, g\}_P \rightarrow \{f, g\}_M \). This circumstance is exploited with the goal to promote NCCS theory as an adequate scheme for the description of non-compressible quantum Hall fluids [3].
IV. ALGEBRAIC STRUCTURE AND ELECTRONS IN LLL

In this item we pass to the algebraic structure associated with the group of NC APD’s and its explicit quantum-mechanical realization.

Consider the infinitesimal operator transformation

\[ \hat{W}[x_k] \rightarrow \hat{W}[x_k + \theta_{kl}\partial_l \xi] = \hat{x}_k + i \left[ \hat{W}[\xi], \hat{x}_k \right] \] (56)

which is a non-commutative version of (5).

The corresponding variation of the Weyl symbol of the scalar function \( f(x) \) will be given by

\[ \Delta_\xi \hat{W}[f] = -i \left[ \hat{W}[\xi], \hat{W}[f] \right] = \hat{W}[-i\{\xi, f\}_M]. \] (57)

Generators

\[ T[\xi] = \hat{W}[\xi] \] (58)

obey commutation relation

\[ \left[ T[\xi], T[\eta] \right] = T[\{\xi, \eta\}_M] \] (59)

in accord with (11).

The commutation relation (59) describes the algebraic structure of the group of symplectic diffeomorphisms in the non-commutative space. The corresponding structure constants could be fixed considering a special bases in the function space. In the case of operators

\[ T_p = T[e^{ipx}] \] (60)

the commutation relation

\[ \left[ T_p, T_q \right] = -2i \sin \left( \frac{1}{2} \theta_{ik} p_i q_k \right) T_{p+q} \] (61)

reproduces the well known algebra with a trigonometric structure constants [11]. Remind, that originally the Lie brackets for the trigonometric sin-algebra were postulated by analogy with the Virasoro-type commutators and corresponding structure constants were calculated imposing the Jacobi identities.

Application to the theory of the quantum Hall effect is based on the assumption, that the configuration space of the system of electrons is a NC space. In the quantum Hall states the planar system of electrons is exposed to the intense orthogonal magnetic field \( B = (0, 0, -B) \) and electrons are constrained to lie in the lowest Landau level (LLL). Non-commutative coordinates satisfy

\[ [\hat{r}_i, \hat{r}_k] = -i \frac{B}{\theta_{ik}} \] (62)

(we use natural units \( c = \hbar = 1 \) taking electron charge \( e = -1 \)). Remind, that the commutator (62) arises from the Dirac bracket for the system with a second class constraints [12], [13].

For the one-particle quantum-mechanical density operator we take the Weyl symbol

\[ \hat{\rho}_{QM}(x) = \hat{W}[\delta(x - r)] = \frac{1}{(2\pi)^2} \int dke^{-ik(\hat{r}-x)}. \] (63)

The subscript \( r \) in (63) means that the Weyl ordering is taken with respect to the operators \( \hat{r}_i \) satisfying (62). In the same time coordinates \( x_i \) are considered as classical variables parameterizing the plane.

Remark that

\[ \int d^2 x \hat{\rho}_{QM}(x) = \hat{W}[1] \] (64)
and
\[
\int d^2 x \hat{\rho}_{QM}(x_i + \theta_{ik}a_k(x)) = \hat{W}_r \left[ 1 - \frac{1}{2} \theta_{ij} f_{ij} + \frac{1}{2} \theta_{ij} \theta_{mn} (a_n \partial_m f_{ij} - f_{mj} f_{ni}) \right] + O(\theta^2)
\] (65)
in accord with the Seiberg-Witten map [14],[2].

Operator (63) obey commutation relation
\[
\left[ \hat{\rho}_{QM}(x'), \hat{\rho}_{QM}(x'') \right] = \int d^2 K(x', x''|x) \hat{\rho}_{QM}(x).
\] (66)

In the kernel
\[
K(x', x''|x) = \delta(x - x') \ast \delta(x - x'') - \delta(x - x'') \ast \delta(x - x')
\] (67)
the star product is implied with respect to the variable x and the non commutativity parameter \(\theta = -1/B\).

Charge operators
\[
\hat{Q}_{QM}\{\xi\} = \int d^2 x \hat{\rho}_{QM}(x) \xi(x) = \hat{W}_r[\xi]
\] (68)
generate algebra (59)
\[
\left[ \hat{Q}_{QM}\{\xi\}, \hat{Q}_{QM}\{\eta\} \right] = \hat{Q}_{QM}\{\{\xi,\eta\}_M\}.
\] (69)

Up to now our consideration was restricted to the one-particle quantum mechanics and it would be instructive to develop corresponding field theory setup.

Introduce operators
\[
\hat{b} = \sqrt{B/2} (\hat{r}_1 - i \hat{r}_2), \quad \hat{b}^+ = \sqrt{B/2} (\hat{r}_1 + i \hat{r}_2) \quad [\hat{b}, \hat{b}^+] = 1
\] (70)
oscillator
\[
|n\rangle = \frac{1}{\sqrt{n!}} \hat{b}^+ n |0\rangle, \quad \hat{b}|0\rangle = 0
\] (71)
and coherent states
\[
\langle z | = \langle 0 | e^{\sqrt{B/2} \hat{b} e^{-i B/4 |z|^2}}.
\] (72)

The LLL second quantized field is given by
\[
\hat{\psi}(x) = \sum_{n=0}^{\infty} \hat{f}_n u_n(x).
\] (73)

Here by \(\hat{f}_n\) we denote the Fermi operators satisfying
\[
[\hat{f}_n, \hat{f}_m^+] = \delta_{mn}
\] (74)
and \(u_n\) are one-particle LLL wave functions
\[
u_n(x) = \langle z | n\rangle
\] (75)
which obey the LLL condition
\[
\left( \partial_{\bar{z}} + \frac{B}{4} z \right) u_n(x) = 0
\] (76)
(we adopt the standard complex notations \(z = x_1 + ix_2, \partial_{\bar{z}} = \frac{1}{2} (\partial_1 + i \partial_2)\)).
Reminding that as a one-particle density operator we use the Weyl symbol define corresponding second quantized objects: density
\[
\hat{\rho}(x) = \sum_{m,n} \langle m|\hat{\rho}_{QM}(x)|n\rangle \hat{f}_m \hat{f}_n = \int dx' \int dx'' \psi^+(x')\langle z'|\hat{\rho}_{QM}(x)|z''\rangle \psi(x'')
\] (77)
and charge
\[
\hat{Q}\{\xi\} = \int dx' \xi(x') \hat{\rho}(x').
\] (78)
One easily verifies that
\[
[\hat{Q}\{\xi\}, \hat{Q}\{\eta\}] = \hat{Q}\{\{\xi,\eta\}_M\}.
\] (79)
Operator transformation
\[
\Delta \xi \hat{r}_k = -i \left[ \hat{W}_r[\xi], \hat{r}_k \right]
\] (80)
induces the transformation of the oscillator states
\[
|n\rangle \rightarrow (1 - i\hat{W}_r[\xi])|n\rangle
\] (81)
and corresponding variation of the matter field
\[
\delta \xi \hat{\psi}(x) = -i \sum_{n=0}^{\infty} \langle z|\hat{W}_r[\xi]|n\rangle \hat{f}_n.
\] (82)
One easily finds, that
\[
\left( \partial \bar{z} + \frac{B}{2} \bar{z} \right) \delta \xi \hat{\psi}(x) = 0
\] (83)
i.e. the transformation does not violate the LLL condition. Note, that essentially the same transformation was used in with the aim to establish algebra satisfied by the LLL projected density operators \(\hat{\rho}^L(x)\). The Fourier components of these densities obey the commutation relation
\[
\left[ \hat{\rho}^L_p, \hat{\rho}^L_q \right] = 2i \sin \left( \frac{p \wedge q}{2B} \right) e^{\frac{\pi}{2} p \cdot q} \hat{\rho}^L_{p+q}.
\] (84)
Note, that rescaled operators
\[
\tilde{T}_p = e^{-\frac{\pi}{2} p^2} T_p
\] (85)
obey the same algebra as the operators \(\hat{\rho}^L_p\). Different forms of commutation relations are related to the various possible ways of the operator ordering and definitions of corresponding symbols. In the present paper we use Weyl symbols with the symmetric ordering of the Fock operators. Equally well one can apply other types of orderings and symbols (e.g. Wick normal and anti-normal orderings) accompanied by the appropriate modifications of the star product.

V. SUMMARY

In the present paper we introduce the notion of the finite area on the NC plane and suggest an heuristic rule for its transformations under the operator homomorphisms. Algebraic structure corresponding to APD’s on the NC plane coincides with the quantum deformation of the algebra of the group of the classical symplectic diffeomorphisms.

Invariance under NC APD’s is equivalent of the Gauss law in the NCCS theory. Otherwise speaking area preserving transformations in the NC space are induced by gauge potentials satisfying field equations of the NCCS gauge theory. The corresponding gauge group corresponds to geometric transformations in the NC space.

APD’s constitute an invariance group for the incompressible fluids like strongly interacting electrons in Laughlin states. This symmetry is embodied in the infinite-dimensional algebra generated by the LLL projected density operators [15, 17]. On the other hand the standard CS gauge theory seems to be an adequate model for the description of the quantum Hall effect (see e.g. [18]). In the present paper we argue, that the CS and the infinite symmetry approaches can be unified in the framework of the NCCS theory, where the gauge symmetry has the geometric origin.
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[1] R.Jackiw, *Noncommuting fields and non-Abelian fluids*, arXiv: hep-th/0305027
[2] R.Jackiw, S.-Y. Pi and A.Polychronakos, *Ann. Phys. (NY)* **301**, (2002), 157.
[3] L.Susskind, *The Quantum Hall Fluid and Non-Commutative Chern-Simons Field Theory*, arXiv: hep-th/0101029
[4] R.J.Szabo, *Phys. Rep.* **378**, (2003), 207.
[5] R.Jost, *Rev. Mod. Phys* **36**, (1964), 572.
[6] V.I.Arnold, *Mathematical Methods of Classical Mechanics*, (Springer-Verlag, Berlin, 1978).
[7] A.M.Perelomov, *Generalized Coherent States and Their Applications*, (Springer-Verlag, Berlin, 1986).
[8] J.Madore, *An Introduction to Noncommutative Geometry and its Physical Applications*, (Cambridge University Press, 1999).
[9] M.Eliashvili and G.Tsitsishvili, *Int. J. Mod. Phys.* **B16**, (2002), 3725.
[10] R.Manvelyan and R.Mkrtchyan, *Phys.Lett.* **B327**, 47 (1994).
[11] D.B.Fairlie, P.Fletcher and C.K.Zachos, *Phys. Lett.* **B218**, (1989), 203.
[12] P. Dirac, *Lectures on Quantum Mechanics* (Belfer Graduate School of Science, Yeshiva University, New York, 1964).
[13] R.Jackiw, *Nucl. Phys. Proc. Suppl.* **108**, (2002), 30.
[14] N.Seiberg and E.Witten, *JHEP* **9909**, (1999), 032.
[15] S.Iso, D.Karabali and B.Sakita, *Phys. Lett.* **B196**, (1992), 142.
[16] J.Martinez and M.Stone, *Int. J. Mod. Phys.* **B7**, (1993), 4389.
[17] A.Cappelli, C.Trugenberger and G.Zemba, *Nucl. Phys.* **B396**, (1993), 465.
[18] Z.F.Ezawa, *Quantum Hall Effects*, (World Scientific, Singapore, 2000).