A Mixed Finite-Element Method on Polytopal Mesh

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Abstract

In this paper, we introduce new stable mixed finite elements of any order on polytopal mesh for solving second-order elliptic problem. We establish optimal order error estimates for velocity and super convergence for pressure. Numerical experiments are conducted for our mixed elements of different orders on 2D and 3D spaces that confirm the theory.

Keywords Mixed finite-element methods · Second-order elliptic problem · Polytopal mesh

Mathematics Subject Classification 65N15 · 65N30 · 35B45 · 35J50

1 Introduction

The considered model problem seeks a flux function \( \mathbf{q} = \mathbf{q}(x) \) and a scalar function \( u = u(x) \) defined in an open bounded polygonal or polyhedral domain \( \Omega \subset \mathbb{R}^d \) (\( d = 2, 3 \)) satisfying

\[
a \mathbf{q} + \nabla u = 0 \quad \text{in} \quad \Omega, \tag{1}
\]

\[
\nabla \cdot \mathbf{q} = f \quad \text{in} \quad \Omega, \tag{2}
\]

\[
u = -g \quad \text{on} \quad \partial \Omega, \tag{3}
\]
where \( a \) is a symmetric, uniformly positive definite matrix on the domain \( \Omega \). A weak formulation for (1)–(3) seeks \( q \in H(\text{div}, \Omega) \) and \( u \in L^2(\Omega) \), such that
\[
(aq, v) - (\nabla \cdot v, u) = (gv \cdot n)_{\partial \Omega}, \quad \forall v \in H(\text{div}, \Omega),
\]
\[
(\nabla \cdot q, w) = (f, w), \quad \forall w \in L^2(\Omega).
\]

Here \( L^2(\Omega) \) is the standard space of square integrable functions on \( \Omega \), \( \nabla \cdot v \) is the divergence of vector-valued functions \( v \) on \( \Omega \), \( H(\text{div}, \Omega) \) is the Sobolev space consisting of vector-valued functions \( v \), such that \( v \in [L^2(\Omega)]^d \) and \( \nabla \cdot v \in L^2(\Omega) \), \( (\cdot, \cdot) \) stands for the \( L^2 \)-inner product in \( L^2(\Omega) \), and \( (\cdot, \cdot)_{\partial \Omega} \) is the inner product in \( L^2(\partial \Omega) \).

Finite-element methods based on the weak formulation (4)–(5) and finite dimensional subspaces of \( H(\text{div}, \Omega) \times L^2(\Omega) \) with piecewise polynomials are known as mixed finite-element methods (MFEMs). The MFEMs have been intensively studied [1–7, 14, 17] and many stable mixed finite elements have been developed, such as Raviart-Thomas (RT) and Brezzi-Douglas-Marini (BDM) elements. However, most of the existing mixed elements are defined on triangle/rectangle in two-dimensional space and tetrahedron/cuboid in three dimensional space.

Recently, the polytopal finite-element methods are investigated intensively for dealing with meshing complexity [9, 10, 13, 16, 18, 20]. In addition to their mathematical interest, the polytopal finite-element methods have many applications in engineering, such as material science. For example, polygonal finite elements can outperform their triangular and quadrilateral counterparts under bending and shear loadings [15].

However, construction of stable mixed finite elements on general polytopal mesh can be very challenging. A lowest order \( H(\text{div}) \) conforming elements on polytopal mesh was introduced in [8] using Wachspress coordinates. Such an element consists rational functions and requires polygon/polyhedron to be convex. In this paper, we use a different methodology to construct an \( H(\text{div}) \) conforming element on a polygon or a polyhedron. We employ piecewise polynomials defined on sub-triangles of a polygon/polyhedron, instead of using rational functions which cannot be integrated accurately. The idea is to construct a piecewise polynomial on a polygon/polyhedron so that its divergence is a one-piece polynomial on the whole polygon/polyhedron. It is a challenging task with mathematical interest.

To construct a stable mixed finite element on a polygon/polyhedron, one can use either a one-piece rational function for flux approximation [8] or a piecewise polynomial as we proposed here. Compared with the approach in [8], our method has several advantages: (i) it does not require the polygon/polyhedron to be convex; (ii) it is a high order method with any degree of polynomial \( k \); (iii) it can be exactly integrated. The new mixed element has the same number of degrees of freedom on each element as that of [8] when the degree of the polynomial \( k \) is 1. We would expect the new mixed finite element has closely the least number of degrees of freedom too when \( k \geq 2 \). The optimal convergence rate for velocity and superconvergence for pressure are obtained. Extensive numerical examples are tested for the new mixed finite elements of different degrees in two and three dimensions.

We like to point out that our new polytopal MFEM is fundamentally different from the traditional mixed method on a triangular mesh formed by the subtriangles/subtetrahedra. Mathematically, we find a subspace of the subtriangular \( P_k H(\text{div}) \) element whose divergence is a one-piece \( P_{k-1} \) polynomial on the polygon. Computationally, we save huge numbers of unknowns. For example, when using hexahedral mixed finite elements, the space dimension of the discrete pressure space is only 1/6 of that on the corresponding subtetrahedral mesh, and the velocity dimension would be reduced by more than 1/2.
when \( k > 1 \). Such mixed macro-finite elements are studied in [11, 12], but only for the degree of the polynomial \( k = 1 \). This is a relatively trivial case as there is no internal degree of freedoms on each macro-element. Also numerical computation and super-convergence are not provided in [11, 12].

### 2 Construction of an \( H(\text{div}, \Omega) \) Element

Let \( T_h \) be a partition of the domain \( \Omega \) consisting of polygons in two dimension or polyhedra in three dimension satisfying a set of conditions specified in [19]. Denote by \( \mathcal{E}_h \) the set of all edges/faces in \( T_h \), and let \( \mathcal{E}_h^I = \mathcal{E}_h \setminus \partial \Omega \) be the set of all interior edges/faces. For simplicity, we will use term edge for edge/face without confusion. Let \( P_k(T) \) consist all the polynomials degree less or equal to \( k \) defined on \( T \).

The space \( H(\text{div}; \Omega) \) is defined as the set of vector-valued functions on \( \Omega \) which, together with their divergence, are square integrable, that is

\[
H(\text{div}; \Omega) = \{ \mathbf{v} \in [L^2(\Omega)]^d : \nabla \cdot \mathbf{v} \in L^2(\Omega) \}. 
\]

For any \( T \in T_h \), we divide it into a set of disjoint triangles/tetrahedra \( T_i \) with \( T = \bigcup T_i \). We define \( \Lambda_k(T) \) as

\[
\Lambda_k(T) = \{ \mathbf{v} \in H(\text{div}; T) : \mathbf{v}|_{T_i} \in RT_k(T_i), \ \nabla \cdot \mathbf{v} \in P_k(T) \},
\]

where \( RT_k(T_i) = \{ [P_k(T_i)]^d \oplus \sum_{|\alpha|=k} a_\alpha x^\alpha \} \) is the usual Raviart-Thomas element of order \( k \).

Associated with the given mesh, we introduce two finite-element spaces

\[
V_h = \{ \mathbf{v} \in H(\text{div}; \Omega) : \mathbf{v}|_{T} \in \Lambda_k(T), \ T \in T_h \},
\]

and

\[
W_h = \{ w \in L^2(\Omega) : \mathbf{w}|_{T} \in P_k(T), \ T \in T_h \}.
\]

We next define an interpolation operator on polygon/polyhedron \( T \), \( \Pi_h : [H^1(T)]^d \rightarrow \Lambda_k(T) \). We assume no additional inner vertices/edges are required in subdividing this polygon/polyhedron \( T \) into \( n \) triangles/tetrahedrons. That is, we have precisely \( n - 1 \) internal edges/triangles which separate \( T \) into \( n \) parts, \( \{ T_i \} \). The 2D case is illustrated in Fig. 1.

The definition of \( \Pi_h \) is for 3D. We need only omit the fourth equation in (9) to get a 2D definition.

For any \( \mathbf{v} \in [H^1(T)]^d \), \( \Pi_h \mathbf{v} \in \Lambda_k(T) \) is defined by
\[
\begin{align*}
\int_{F_i \in \partial T} (\Pi_h v - v) \cdot n_{ij} p_k \, dS &= 0, \quad \forall p_k \in P_k(F_{ij}), \\
\int_{T_i} (\Pi_h v - v) \cdot n_1 p_{k-1} \, dx &= 0, \quad \forall p_{k-1} \in P_{k-1}(T), \\
\int_{T_i} (\Pi_h v - v) \cdot n_2 p_{k-1} \, dx &= 0, \quad \forall p_{k-1} \in P_{k-1}(T_i), \; i = 1, \ldots, n, \\
\int_{F_i \subset T^0} [\Pi_h v] \cdot n_{ij} p_k \, dS &= 0, \quad \forall p_k \in P_k(F_{ij}), \\
\int_{T_i} \nabla \cdot (\Pi_h v|_{T_i} - \Pi_h v|_{T_i}) p_k \, dx &= 0, \quad \forall p_k \in P_k(T_i), \; i = 2, \ldots, n,
\end{align*}
\]  
(9)

where \(F_{ij}\) is the \(j\)th face triangle of \(T_i\) with a fixed normal vector \(n_{ij}\), cf. Fig. 1, \((n_1, n_2, n_3)\) is a randomly chosen right-hand orthonormal system with \(\Pi_{ij} n = n_{ij}\) for all \(i\) and \(j\), and \([-]\) denotes the jump on a face triangle. Here “\(\Pi_h v|_{T_i}\)” in (9) stands for the restriction of a global polynomial \(w\) on \(T_i\), where \(w|_{T_i} = \Pi_h v|_{T_i}\).

**Lemma 1** The interpolation operator \(\Pi_h\) in (9) is well defined.

**Proof** On each of \(n\) sub-tetrahedrons \(\{T_i\}\) of \(T\) in (9), a function of \(\Lambda_k(T)\) can be expressed as

\[
v_h|_{T_i} = \sum_{i+j+k} \left( \begin{array}{c}
\frac{a_{1,ij}}{2} \\
\frac{a_{2,ij}}{2} \\
\frac{a_{3,ij}}{2}
\end{array} \right) x^i y^j z^k + \sum_{i+j+k} \left( \begin{array}{c}
\frac{a_{4,ij}}{a_{ij}} x^i y^j z^k, i_0 = 1, \ldots, n,
\end{array} \right)
\]  
(10)

and \(v_h|_T\) is determined by

\[
\frac{n(k + 1)(k + 2)(k + 3)}{2} + \frac{n(k + 1)(k + 2)}{2} = \frac{n(k + 1)(k + 2)(k + 4)}{2}
\]  
(11)

coefficients.

On the other side, the linear system (9) of equations has the following number of equations:

\[
(2n + 2) \frac{(k + 1)(k + 2)}{2} + (2n + 1) \frac{k(k + 1)(k + 2)}{6} + (n - 1) \frac{(k + 1)(k + 2)}{2} + (n - 1) \frac{(k + 1)(k + 2)(k + 3)}{6} = \frac{n(k + 1)(k + 2)(k + 4)}{2},
\]

which is exactly the number of coefficients for a \(v_h\) function in (11). Thus we have a square linear system in (9). The system has a unique solution if and only if the kernel is \(\{0\}\).

Let \(v = 0\) in (9). Though \(\Pi_h v\) is a \(P_{k+1}\) polynomial, \(\Pi_h v \cdot n_{ij}\) is a \(P_k\) polynomial when restricted on \(F_{ij}\). This can be seen by the normal format of plane equation for triangle \(F_{ij}\). By the first equation of (9), \(\Pi_h v \cdot n_{ij} = 0\) on \(F_{ij}\). By the sixth equation of (9), \(V \cdot \Pi_h v\) is a
one-piece polynomial on the whole $T$. Because $\nabla \cdot \Pi_h \mathbf{v}$ is continuous on inner interface triangles and is a $P_k(F_{ij})$ polynomial on the outer face triangles, by the first five equations in (9), we have

$$\int_T (\nabla \cdot \Pi_h \mathbf{v})^2 \, dx = \sum_{i=1}^{n} \left( \int_{T_i} -\Pi_h \mathbf{v} \cdot \nabla (\nabla \cdot \Pi_h \mathbf{v}) \, dx + \int_{\partial T_i} \Pi_h \mathbf{v} \cdot \mathbf{n} (\nabla \cdot \Pi_h \mathbf{v}) \, dS \right)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{3} \int_{T_i} (\Pi_h \mathbf{v} \cdot \mathbf{n}) (\nabla \cdot \Pi_h \mathbf{v}) \, dx$$

$$= 0.$$

That is,

$$\nabla \cdot \Pi_h \mathbf{v} = 0 \quad \text{on} \quad T. \quad (12)$$

Starting from a corner tetrahedron $T_1$, we have its three face triangles, $F_{11}$, $F_{12}$, and $F_{13}$, on the boundary of $T$. The fourth face triangle $F_{14}$ of $T_1$ is shared by $T_2$. By the selection of $\mathbf{n}_1$, the normal vector $\mathbf{n}_{14} = c_1 \mathbf{n}_1 + c_2 \mathbf{n}_2 + c_3 \mathbf{n}_3$ of $F_{14}$ has a nonzero $c_1$. A 2D polynomial $p_k \in P_k(F_{14})$ can be expressed as $p_k(x_2, x_3)$, where we use $(x_1, x_2, x_3)$ as the coordinate variables under the system $(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3)$. Viewing this polynomial as a 3D polynomial, i.e., extending it constantly in the $x_1$-direction, we have

$$p_k(x_1, x_2, x_3) = p_k(x_2, x_3), \quad (x_1, x_2, x_3) \in T_1.$$

By (12) and the third and fourth equations of (9), it follows that

$$0 = \int_{T_1} (\nabla \cdot \Pi_h \mathbf{v}) p_k \, dx$$

$$= -\int_{T_1} \left( (\Pi_h \mathbf{v} \cdot \mathbf{n}_1) \partial_{x_1} p_k + (\Pi_h \mathbf{v} \cdot \mathbf{n}_2) \partial_{x_2} p_k + (\Pi_h \mathbf{v} \cdot \mathbf{n}_3) \partial_{x_3} p_k \right) \, dx$$

$$+ \int_{F_{14}} (\Pi_h \mathbf{v}) \cdot \mathbf{n}_{14} p_k \, dS$$

$$= -\int_{T_1} (\Pi_h \mathbf{v} \cdot \mathbf{n}_1) \cdot 0 \, dx + 0 + 0 + \int_{F_{14}} (\Pi_h \mathbf{v}) \cdot \mathbf{n}_{14} p_k \, dS$$

$$= \int_{F_{14}} (\Pi_h \mathbf{v}) \cdot \mathbf{n}_{14} p_k \, dS, \quad \forall p_k \in P_k(F_{14}). \quad (13)$$

Next, for any $p_{k-1} \in P_{k-1}(T_1)$, we let $p_k \in P_k(T_1)$ be one of its anti-$x_1$-derivative, i.e., $\partial_{x_1} p_k = p_{k-1}$. Thus, by (12), the third and fourth equations of (9) and (13), we get

$$0 = \int_{T_1} \nabla \cdot \Pi_h \mathbf{v} p_k \, dx$$

$$= -\int_{T_1} \left( (\Pi_h \mathbf{v} \cdot \mathbf{n}_1) \partial_{x_1} p_k + 0 + 0 \right) \, dx + \int_{F_{14}} (\Pi_h \mathbf{v}) \cdot \mathbf{n}_{14} p_k \, dS$$

$$= -\int_{T_1} (\Pi_h \mathbf{v} \cdot \mathbf{n}_1) p_{k-1} \, dx, \quad \forall p_{k-1} \in P_{k-1}(T_1). \quad (14)$$

Continuing work on $T_1$, by $\nabla \cdot \Pi_h \mathbf{v} = 0$, all $a_{4,ij} = 0$ in (10), since the divergence of each such term is nonzero and independent of the divergence of other terms. Thus $\Pi_h \mathbf{v}|_{T_1}$ is in
\[ [P_k(T_i)]^d \text{, instead of } RT_{k}(T_i). \text{ It can be linearly expanded by the three projections on three linearly independent directions. In particular, on a corner tetrahedron } T_1 \text{ we have three outer triangles } F_{ij} \text{ on } \partial T. \text{ On } T_1, \]

\[
\Pi_h \mathbf{v} = A \begin{pmatrix} \Pi_h \mathbf{v} \cdot \mathbf{n}_{i1} \\ \Pi_h \mathbf{v} \cdot \mathbf{n}_{i2} \\ \Pi_h \mathbf{v} \cdot \mathbf{n}_{i3} \end{pmatrix} = A \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix},
\]

where \( p_1, p_2, \) and \( p_3 \) are scalar \( P_k \) polynomials, and \( A \) is a \( 3 \times 3 \) scalar matrix.

By the first equation in (9), \( p_1 \) vanishes on \( F_{i1} \) and

\[
p_1 = \lambda_1 q_{k-1} \text{ on } T_1, \tag{15}
\]

where \( \lambda_1 \) is a barycentric coordinate of \( T_1 \) (which is a linear function assuming 0 on \( F_{i1} \)), and \( q_{k-1} \) is a \( P_{k-1}(T) \) polynomial. Let \( p_i \in P_i(T) \) be an anti-\( x \)-derivative of \( (n_{11})_1 q_{k-1} \), i.e., \( (\nabla p_i)_1 = (n_{11})_1 q_{k-1} \). Note that \( (\nabla p_i)_2 \) and \( (\nabla p_i)_3 \) can be anything (of \( y \) and \( z \) functions) which result in zero integrals below. By (14) and the third and the fourth equations of (9), since \( \nabla \cdot \Pi_h \mathbf{v} = 0 \), we get

\[
\int_{T_1} \lambda_1 q_{k-1}^2 \, d\mathbf{x} = \int_{T_1} \Pi_h \mathbf{v} \cdot (n_{11} q_{k-1}) \, d\mathbf{x} = 0.
\]

Since \( \lambda_1 > 0 \) in \( T_1 \), we conclude with \( q_{k-1} = 0 \) and \( p_1 = 0 \). Repeating the analysis we get \( p_2 = p_3 = 0 \) and \( \Pi_h \mathbf{v} = 0 \) on \( T_1 \).

Adding the equations (13) and (14) to (9), \( T_2 \) would be a new corner tetrahedron with three no-flux boundary triangles. Repeating the estimates on \( T_1 \), it would lead \( \Pi_h \mathbf{v} = 0 \) on \( T_2 \). Sequentially, we obtain \( \Pi_h \mathbf{v} = 0 \) on all \( T_i \), i.e., on the whole \( T \). The lemma is proved.

**Lemma 2** For the projection \( \Pi_h \) defined in (9) and for \( \tau \in [H^1(\Omega)]^d \) and \( \mathbf{v} \in P_k(T) \), we have

\[
(\nabla \cdot \tau, \mathbf{v})_T = (\nabla \cdot \Pi_h \tau, \mathbf{v})_T, \tag{16}
\]

\[
\|\Pi_h \tau - \tau\| \leq C h^{k+1} |\tau|_{k+1}. \tag{17}
\]

**Proof** For \( \tau \in [H^1(\Omega)]^d \) and \( \mathbf{v} \in P_k(T) \), we have, by (9), (13), and (14),

\[
(\nabla \cdot (\tau - \Pi_h \tau), \mathbf{v})_T = \sum_{i=1}^{n} \left( \int_{T_i} (\tau - \Pi_h \tau) \cdot \nabla \mathbf{v} \, d\mathbf{x} + \int_{\partial T_i} (\tau - \Pi_h \tau) \cdot \mathbf{n} \, dS \right)
= \sum_{i=1}^{n} 0 + \int_{\partial T} (\tau - \Pi_h \tau) \cdot \mathbf{n} \, dS = 0.
\]

That is, (16) holds.

Since \([P_k(T)]^3 \subset \Lambda_k(T) \) and \( \Pi_h \) is uni-solvent, \( \Pi_h \mathbf{v} = \mathbf{v} \) for all \( \mathbf{v} \in [P_k(T)]^3 \). On one size 1 \( T \), by the finite dimensional norm-equivalence and the shape-regularity assumption on sub-triangles, the interpolation is stable in \( L^2(T) \), that is

\[
\|\Pi_h \tau\|_T \leq C \|\tau\|_T. \tag{18}
\]

After a scaling, the constant \( C \) in (18) remains same. It follows that
\[
\|\Pi_h \tau - \tau\|^2 \leq C \sum_{T \in T_h} (\|\Pi_h(\tau - p_{k,T})\|^2_T + \|p_{k,T} - \tau\|^2_T)
\]
\[
\leq C \sum_{T \in T_h} (C\|\tau - p_{k,T}\|^2_T + \|p_{k,T} - \tau\|^2_T)
\]
\[
\leq C \sum_{T \in T_h} h^{2k+2} |\tau|_{k+1,T}^2
\]
\[
= C h^{2k+2} |\tau|_{k+1}^2,
\]
where \(p_{k,T}\) is a \(k\)th Taylor polynomial of \(\tau\) on \(T\).

3 Mixed Finite-Element Method (MFEM)

In this section, we develop an MFEM on polytopal mesh by employing our new mixed elements and obtain optimal order error estimates for the method. First let \(V = H(\text{div}; \Omega)\) and \(W = L^2(\Omega)\).

**Algorithm 1** An MFEM for the problem (4)–(5) seeks \((\mathbf{q}_h, u_h) \in V_h \times W_h\) satisfying

(19)
\[
(aq_h, v) - (\nabla \cdot v, u_h) = \langle g, v \cdot \mathbf{n} \rangle_{\partial \Omega}, \quad \forall v \in V_h,
\]

(20)
\[
(\nabla \cdot q_h, w) = (f, w), \quad \forall w \in W_h.
\]

We introduce a norm \(\|v\|_V\) for any \(v \in V\) as follows:

(21)
\[
\|v\|_V^2 = \|v\|^2 + \|\nabla \cdot v\|^2.
\]

**Lemma 3** There exists a positive constant \(\beta\) independent of \(h\) such that for all \(\rho \in W_h\),

(22)
\[
\sup_{v \in V_h} \frac{(\nabla \cdot v, \rho)}{\|v\|_V} \geq \beta \|\rho\|.
\]

**Proof** For any given \(\rho \in W_h \subset L^2(\Omega)\), it is known in [7] that there exists a function \(\tilde{v} \in [H^1(\Omega)]^d\), such that

(23)
\[
\frac{(\nabla \cdot \tilde{v}, \rho)}{\|\tilde{v}\|_V} \geq C_0 \|\rho\|.
\]

where \(C_0 > 0\) is a constant independent of \(h\). By setting \(v = \Pi_h \tilde{v} \in V_h\) and using (17), we have

(24)
\[
\|v\|_V = \|\Pi_h \tilde{v}\|_V \leq C \|\tilde{v}\|_V.
\]

Using (16), (24), and (23), we have

\[
\frac{|(\nabla \cdot v, \rho)|}{\|v\|_V} = \frac{|(\nabla \cdot \Pi_h \tilde{v}, \rho)|}{\|v\|_V} \geq \frac{|(\nabla \cdot \tilde{v}, \rho)|}{C \|\tilde{v}\|_V} \geq \beta \|\rho\|
\]

for a positive constant \(\beta\). This completes the proof of the lemma.
Theorem 1 Let \((q_h, u_h) \in V_h \times W_h\) be the mixed finite-element solution of (19)–(20). Then, there exists a constant \(C\), such that

\[
\|q - q_h\|_V + \|u - u_h\| \leq C h^{k+1} (|q|_{k+1} + |u|_{k+1}).
\]  

(25)

Proof Let \(e_h = 1_l \cdot q - q_h\) and \(e_h = Q_h u - u_h\), where \(Q_h\) is the elementwise defined \(L^2\) projection onto \(P_k(T)\) on each element \(T\). The differences of (4)–(5) and (19)–(20) imply

\[
(a(q - q_h), v) - (\nabla \cdot v, u - u_h) = 0, \quad \forall v \in V_h,
\]

(26)

\[
(\nabla \cdot (q - q_h), w) = 0, \quad \forall w \in W_h.
\]

(27)

By adding \((a\Pi_h q, v)\) to the both sides of (26) and using the definition of \(Q_h\), (26) becomes

\[
(ae_h, v) - (\nabla \cdot v, e_h) = (a(1_l \cdot q - q), v).
\]

(28)

It follows from (16) and (27) that for \(w \in W_h\),

\[
(\nabla \cdot e_h, w) = (\nabla \cdot (\Pi_h q - q_h), w) = (\nabla \cdot (q - q_h), w) = 0.
\]

(29)

Combining (28)–(29), we have for all \((v, w) \in V_h \times W_h\),

\[
(ae_h, v) - (\nabla \cdot v, e_h) = (a(\Pi_h q - q), v),
\]

(30)

\[
(\nabla \cdot e_h, w) = 0.
\]

(31)

Letting \(v = e_h\) in (30) and using (29), we have

\[
(ae_h, e_h) = (a(\Pi_h q - q), e_h),
\]

which gives

\[
\|\Pi_h q - q_h\|_V \leq C h^{k+1} |q|_{k+1}.
\]

(32)

It follows from (30) and (32) that for all \(v \in V_h\),

\[
(\nabla \cdot v, e_h) \leq |(ae_h, v)| + |(a(\Pi_h q - q), v)| \leq C h^{k+1} \|q\|_{k+1} \|v\|_V.
\]

(33)

The inf-sup condition (22) and the estimate (33) yield

\[
\|Q_h u - u_h\| \leq C h^{k+1} \|q\|_{k+1}.
\]

(34)

It follows from (32) and (34)

\[
\|\Pi_h q - q_h\|_V + \|Q_h u - u_h\| \leq C h^{k+1} |q|_{k+1}.
\]

(35)

The error bound (25) follows from the triangle inequality and (35) and we have proved the theorem.

To obtain the superconvergence for \(u_h\), we consider the dual system: seek \((\psi, \theta) \in H_0(\text{div}; \Omega) \times L^2(\Omega)\) such that

\[
(a\psi, v) - (\nabla \cdot v, \theta) = 0, \quad \forall v \in H_0(\text{div}; \Omega),
\]

(36)
Assume that the following regularity holds:

$$\|\psi\|_1 + \|\theta\|_1 \leq C \|Q_h u - u_h\|.$$  \hspace{1cm} (38)

**Theorem 2** Let \((q_h, u_h) \in V_h \times W_h\) be the mixed finite-element solution of (19)–(20). Assume that (38) holds true. Then, there exists a constant \(C\), such that

$$\|Q_h u - u_h\| \leq C h^{k+2} (|q|_{k+1} + |u|_{k+1}).$$  \hspace{1cm} (39)

**Proof** Letting \(w = Q_h u - u_h\) in (37) and using (16), (26), (36), (27), (25), and (38), we have

$$\|Q_h u - u_h\|^2 = (\nabla \cdot \psi, Q_h u - u_h)$$
$$= (\nabla \cdot \Pi_h \psi, Q_h u - u_h)$$
$$= (\Pi_h \psi, a(q - q_h))$$
$$= (\Pi_h \psi - \psi, a(q - q_h)) + (\psi, a(q - q_h))$$
$$= (\Pi_h \psi - \psi, a(q - q_h)) + (\nabla \cdot (q - q_h), \theta)$$
$$= (\Pi_h \psi - \psi, a(q - q_h)) + (\nabla \cdot (q - q_h), \theta - Q_h \theta)$$
$$\leq C h^{k+2} \|q\|_{k+1} \|Q_h u - u_h\|,$$

which implies (39) and we have proved the theorem.

### 4 Numerical Example

We solve problem (1)–(3) on the unit square domain with the exact solution

$$q = \begin{pmatrix} \pi \sin(\pi y) \cos(\pi x) \\ \pi \sin(\pi x) \cos(\pi y) \end{pmatrix}, \quad u = \sin(\pi x) \sin(\pi y).$$  \hspace{1cm} (40)

We first use quadrilateral grids. To avoid asymptotic parallelograms under nested refinements, we use fixed types of quadrilaterals in our multi-level grids, as shown in Fig. 2. We list the computational results in Table 1. As proved, we have one order of super-convergence for both \(u_h\) and \(q_h\).

![Fig. 2 The first three levels of grids, for Table 1](image-url)
Next we solve the same problem (40) on a type of grids with quadrilaterals and hexagons, as shown in Fig. 3. We list the result of computation in Table 2, where we obtain one order of superconvergence in all cases.

We solve 3D problem (1)–(3) on the unit cube domain $\Omega = (0, 1)^3$ with the exact solution

$$
\begin{align*}
q &= \begin{pmatrix}
2^8(1 - 2x)^2(y - y^2)(z - z^2) \\
2^8(x - x^2)^2(1 - 2y)(z - z^2) \\
2^8(x - x^2)^2(y - y^2)(1 - 2z)
\end{pmatrix}, \\
\Omega &= 2^8(x - x^2)^2(y - y^2)(z - z^2).
\end{align*}
$$

Here we use a uniform wedge-type (polyhedron with two triangle faces and three rectangle faces) grids, as shown in Fig. 4. Here each wedge is subdivided into three tetrahedrons with three rectangular faces being cut, when defining piecewise $RT_\ell$ element $\Lambda_{\ell}$. The results are listed in Table 3, confirming the theory.

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**Table 1** Error profiles and convergence rates on grids shown in Fig. 2 for (40)

| Level | $||Q_h u - u_h||_0$ | Rate | $||\Pi_h q - q_h||_V$ | Rate |
|-------|---------------------|------|----------------------|------|
| By the $\Lambda_0P_0$ mixed element | | | | |
| 6     | 0.146 4E–03    | 2.00 | 0.518 5E–01  | 1.00 |
| 7     | 0.366 0E–04    | 2.00 | 0.259 3E–01  | 1.00 |
| 8     | 0.915 1E–05    | 2.00 | 0.129 6E–01  | 1.00 |
| By the $\Lambda_1P_1$ mixed element | | | | |
| 6     | 0.107 2E–05    | 3.00 | 0.410 3E–03  | 2.00 |
| 7     | 0.134 0E–06    | 3.00 | 0.102 5E–03  | 2.00 |
| 8     | 0.167 4E–07    | 3.00 | 0.256 3E–04  | 2.00 |
| By the $\Lambda_2P_2$ mixed element | | | | |
| 5     | 0.570 4E–06    | 4.00 | 0.187 8E–03  | 3.00 |
| 6     | 0.356 7E–07    | 4.00 | 0.234 9E–04  | 3.00 |
| 7     | 0.223 1E–08    | 4.00 | 0.293 7E–05  | 3.00 |
| By the $\Lambda_3P_3$ mixed element | | | | |
| 3     | 0.140 3E–05    | 5.84 | 0.155 9E–03  | 4.88 |
| 4     | 0.276 5E–07    | 5.66 | 0.596 9E–05  | 4.71 |
| 5     | 0.680 8E–09    | 5.34 | 0.283 7E–06  | 4.39 |

---

**Fig. 3** First three levels of quadrilateral-hexagon grids, for Table 2
Table 2  Error profiles and convergence rates on grids shown in Fig. 3 for (40)

| Level | $\|Q_h u - u_h\|_0$ | Rate | $\|\Pi_h q - q_h\|_V$ | Rate |
|-------|-----------------|------|------------------|------|
|       |                 |      |                  |      |
| By the $\Lambda_0 - P_0$ mixed element |
| 6     | 0.152 3E–03     | 2.00 | 0.528 2E–01      | 1.00 |
| 7     | 0.380 8E–04     | 2.00 | 0.264 1E–01      | 1.00 |
| 8     | 0.952 0E–05     | 2.00 | 0.132 1E–01      | 1.00 |
| By the $\Lambda_1 - P_1$ mixed element |
| 6     | 0.101 5E–05     | 3.00 | 0.395 8E–03      | 2.00 |
| 7     | 0.126 9E–06     | 3.00 | 0.989 3E–04      | 2.00 |
| 8     | 0.158 6E–07     | 3.00 | 0.247 3E–04      | 2.00 |
| By the $\Lambda_2 - P_2$ mixed element |
| 5     | 0.806 9E–07     | 4.01 | 0.228 3E–04      | 3.05 |
| 6     | 0.503 8E–08     | 4.00 | 0.283 0E–05      | 3.01 |
| 7     | 0.314 9E–09     | 4.00 | 0.353 0E–06      | 3.00 |
| By the $\Lambda_3 - P_3$ mixed element |
| 3     | 0.910 6E–06     | 5.72 | 0.117 6E–03      | 4.83 |
| 4     | 0.208 0E–07     | 5.45 | 0.484 4E–05      | 4.60 |
| 5     | 0.573 5E–09     | 5.18 | 0.248 1E–06      | 4.29 |

Fig. 4  First three levels of wedge grids used in Table 3

Table 3  Error profiles and convergence rates on grids shown in Fig. 4 for (41)

| Level | $\|Q_h u - u_h\|_0$ | Rate | $\|\Pi_h q - q_h\|_V$ | Rate |
|-------|-----------------|------|------------------|------|
|       |                 |      |                  |      |
| By the 3D $\Lambda_0 - P_0$ mixed element |
| 5     | 0.004 419 7     | 2.0  | 0.580 287 7      | 1.0  |
| 6     | 0.001 114 5     | 2.0  | 0.290 999 4      | 1.0  |
| 7     | 0.000 279 3     | 2.0  | 0.145 607 2      | 1.0  |
| By the 3D $\Lambda_1 - P_1$ mixed element |
| 4     | 0.004 910 6     | 2.9  | 0.523 468 8      | 2.0  |
| 5     | 0.000 622 8     | 3.0  | 0.131 704 7      | 2.0  |
| 6     | 0.000 078 2     | 3.0  | 0.032 983 0      | 2.0  |
| By the 3D $\Lambda_2 - P_2$ mixed element |
| 4     | 0.000 494 3     | 4.0  | 0.100 552 3      | 3.0  |
| 5     | 0.000 031 0     | 4.0  | 0.012 603 1      | 3.0  |
| 6     | 0.000 001 9     | 4.0  | 0.001 576 5      | 3.0  |
| By the 3D $\Lambda_3 - P_3$ mixed element |
| 3     | 0.000 666 8     | 5.0  | 0.198 680 5      | 3.9  |
| 4     | 0.000 020 7     | 5.0  | 0.012 564 1      | 4.0  |
| 5     | 0.000 000 6     | 5.0  | 0.000 787 5      | 4.0  |
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Compliance with Ethical Standards

Conflict of Interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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