Recent T2K results indicate a sizeable reactor angle $\theta_{13}$ which would rule out exact tri-bimaximal lepton mixing. We study the vacuum alignment of the Altarelli-Feruglio $A_4$ family symmetry model including additional flavons in the $1'$ and $1''$ representations and show that it leads to trimaximal mixing in which the second column of the lepton mixing matrix consists of the column vector $(1, 1, 1)^T/\sqrt{3}$, with a potentially large reactor angle. In order to limit the reactor angle and control the higher order corrections, we propose a renormalisable $S_4$ model in which the $1'$ and $1''$ flavons of $A_4$ are unified into a doublet of $S_4$ which is spontaneously broken to $A_4$ by a flavon which enters the neutrino sector at higher order. We study the vacuum alignment in the $S_4$ model and show that it predicts accurate trimaximal mixing with approximate tri-bimaximal mixing, leading to a new mixing sum rule testable in future neutrino experiments. Both $A_4$ and $S_4$ models preserve form dominance and hence predict zero leptogenesis, up to renormalisation group corrections.
1 Introduction

Recently T2K have published evidence for a large non-zero reactor angle \[\theta_{13} \neq 0\] which, when combined with data from MINOS and other experiments in a global fit yields \[\theta_{13} = 8^\circ \pm 1.5^\circ, \quad (1.1)\]

where the errors indicate the one \(\sigma\) range, although the statistical significance of a non-zero reactor angle is about 3\(\sigma\).

If confirmed this would rule out the hypothesis of exact tri-bimaximal (TB) mixing \[3\]. However other schemes such as trimaximal (TM) mixing remain viable \[4–8\]:

\[
U_{TM} = P' \begin{pmatrix}
\frac{2}{\sqrt{6}} \cos \vartheta & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \sin \vartheta e^{i\rho} \\
-\frac{1}{\sqrt{6}} \cos \vartheta - \frac{1}{\sqrt{2}} \sin \vartheta e^{-i\rho} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \cos \vartheta + \frac{1}{\sqrt{6}} \sin \vartheta e^{i\rho} \\
-\frac{1}{\sqrt{6}} \cos \vartheta + \frac{1}{\sqrt{2}} \sin \vartheta e^{-i\rho} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \cos \vartheta - \frac{1}{\sqrt{6}} \sin \vartheta e^{i\rho}
\end{pmatrix} P, \quad (1.2)
\]

where \(\frac{2}{\sqrt{6}} \sin \vartheta = \sin \theta_{13}\), \(P'\) is a diagonal phase matrix required to put \(U_{TM}\) into the PDG convention \[9\], and \(P = \text{diag}(1, e^{i\frac{\pi}{2}}, e^{i\frac{3\pi}{2}})\) contains the usual Majorana phases. In particular TM mixing approximately preserves the successful TB mixing for the solar mixing angle \(\theta_{12} \approx 35^\circ\) as the correction due to a non-zero but relatively small reactor angle is of second order. Although TM mixing reduces to TB mixing in the limit that \(\vartheta \to 0\), it is worth emphasising that in general TM mixing involves a reactor angle \(\theta_{13}\) which could in principle be large or even maximal (e.g. 45\(^\circ\)). The observed smallness of the reactor angle \(\theta_{13}\) compared to the atmospheric angle \(\theta_{23} \approx 45^\circ\) and the solar angle \(\theta_{12} \approx 34^\circ\) \[2\] is therefore not explained by the TM hypothesis alone. Clearly the relative smallness of the reactor angle can only be explained with additional model dependent input. We shall show that, although an \(A_4\) family symmetry can provide an explanation of trimaximal mixing, the observed smallness of the reactor angle as compared to the atmospheric and solar angles may be naturally explained by an \(S_4\) family symmetry.

In the original \(A_4\) models of TB mixing Higgs fields \[10\] or flavons \[11,12\] transforming under \(A_4\) as \(3\) and \(1\) but not \(1'\) or \(1''\) were used to break the family symmetry and to lead to TB mixing. However there is no good reason not to include flavons transforming as \(1'\) or \(1''\) and once included they will lead to deviations from TB mixing \[13,14\] in particular it has been noted that they lead to TM mixing \[15\]. However, as remarked above, TM mixing by itself does not account for the smallness of the reactor angle, and in particular the model in \[15\] does not provide any explanation for this. Another aspect is that higher order operators in \(A_4\) models lead to deviations from TB mixing in a rather uncontrolled way \[12\], and in such models any simple TM structure may be washed out unless the theory is promoted to a renormalisable one in which the higher order operators are under control. For example, this was done for two \(A_4\) models of TB mixing in \[16\], and one may follow an analogous strategy including also \(1'\) or \(1''\) flavons.

In this paper we make a first study of the vacuum alignment of an \(A_4\) family symmetry model of leptons with additional flavons in the \(1'\) and/or \(1''\) representations and show that it leads to accurate trimaximal mixing at leading order (LO). In order to constrain the reactor angle and control the higher order corrections, we then propose a renormalisable
$S_4$ model of leptons in which the $1'$ and $1''$ flavons of $A_4$ are unified into a doublet of $S_4$ which is spontaneously broken to $A_4$ by a flavon which enters the neutrino sector at higher order. We study the vacuum alignment of the $S_4$ model and show that it predicts accurate trimaximal mixing with approximate tri-bimaximal mixing, leading to a new mixing sum rule testable in future neutrino experiments. We also remark that both $A_4$ and $S_4$ models preserve form dominance [17] and hence predict zero leptogenesis [18], up to renormalisation group corrections.

The layout of the rest of the paper is as follows. In Section 2 we first revisit the Altarelli-Feruglio $A_4$ model [12] with regard to the possibility of generating deviations from TB mixing which respect TM mixing. As the obvious ideas do not yield TM mixing, we then consider the model augmented by extra flavons in the $1'$ and/or $1''$ representation. We study the vacuum alignment and show that this model leads to TM mixing, with an unconstrained reactor angle. In Section 3 we propose a renormalisable $S_4$ model of leptons and study its vacuum alignment leading to accurate trimaximal mixing with approximate tri-bimaximal mixing. In Section 4 we give an analytic discussion of the perturbative deviations to TB mixing arising from any TM model with a physical reactor angle leading to a new mixing sum rule testable in future neutrino experiments. Section 5 concludes the paper. Appendix A contains Clebsch-Gordon coefficients for $S_4$ and $A_4$, and Appendix B describes a perturbative diagonalisation of the right-handed neutrino mass matrix.

2 $A_4$ models of trimaximal mixing

2.1 The Altarelli-Feruglio $A_4$ model of tri-bimaximal mixing

The original and well studied Altarelli-Feruglio (AF) model of lepton masses and mixings [12] is formulated as an effective theory, defined purely by the particle content and the symmetries. There exist two versions, one with right-handed neutrinos and one without. For definiteness we will only consider the former which makes use of the elegant seesaw mechanism to obtain effective light neutrino masses around the eV scale. The particle content and the symmetries of the AF model we consider is presented in Table 1, including the $\xi$ flavon, but excluding the $\xi'$ and $\xi''$ flavons which we shall consider later. In addition the vacuum alignment in the AF model requires a further auxiliary flavon singlet $\tilde{\xi}$ which does not acquire a VEV and is not shown in the table. Particles with a $U(1)_R$ charge of 2 are called driving fields. Setting their $F$-terms to zero leads to the $F$-term conditions which control the alignment of the flavon fields.

The relevant effective superpotential terms of the Yukawa sector of the AF model are

$$W_{A_4}^{\text{eff}} \sim LH_u N^c + (\varphi_S + \xi) N^c N^c + \frac{1}{M} H_d \left[ (L\varphi_T)_1 e^c + (L\varphi_T)_1 \mu^c + (L\varphi_T)_1 \tau^c \right]. \quad (2.1)$$

Here $(\cdots)_r$ denotes the contraction of the $A_4$ indices to the representation $r$. When the three flavon fields $\varphi_S, \xi, \varphi_T$ acquire their VEVs [12],

$$\langle \varphi_S \rangle = v_S \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \langle \xi \rangle = u, \quad \langle \varphi_T \rangle = v_T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad (2.2)$$

$$\langle \varphi_T \rangle = v_T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad (2.2)$$
at some high scale, the flavour structure of the Yukawa couplings is generated which, after electroweak symmetry breaking, gives rise to a trivial Dirac neutrino mass matrix $m_D$, a TB heavy right-handed neutrino mass matrix $M_R$ as well as a diagonal charged lepton mass matrix $m_l$. Applying the seesaw formula yields a TB light neutrino mass matrix which is diagonalised by $U_{\text{TB}} = U_{\text{TM}}(\theta = 0)$. With the charged leptons being already diagonal, the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) mixing matrix becomes $U_{\text{PMNS}} = U_{\text{TB}}$. The reason why the AF $A_4$ model preserves TB mixing is that the flavon $\varphi_S$ preserves the $S$ generator of $A_4$ and the absence of $\xi'$ and $\xi''$ flavons leads to an accidental $U$ symmetry, where $S,U$ symmetry in the neutrino sector and $T$ symmetry in the charged lepton sector implies TB mixing (see e.g. [19]).

This result is correct at LO. It is argued in [12] that higher order operators which are allowed by the imposed symmetries should be considered as well, both in $W_{\text{A}_4}$ as well as in the superpotential that generates the flavon alignments in Eq. (2.2). In general such additional terms would lead to deviations from TB mixing which are important in the light of the latest experimental hints for a non-zero reactor angle [1, 2]. In order to account for a (relatively) large value of $\theta_{13}$ the higher order corrections need to be large enough which can be arranged by assuming different suppression scales for different non-renormalisable operators. This assumption, however, raises the question about the origin of the suppression scales.

In this context, it was pointed out recently that particular ultraviolet (UV) completions of flavour models do not necessarily give rise to all effectively allowed terms in the superpotential [16]. In fact, it was found that the minimal UV completion of the AF model does not generate any deviations from TB mixing. The simplest way to obtain the desired deviations is to add new messengers to the minimal model such that certain higher order Yukawa operators are switched on. In the neutrino sector, the effective non-trivial operators allowed by the symmetries read

$$\frac{1}{M} \left[ (N^c N^c)_{1'} (\varphi_S \varphi_T)_{1''} + (N^c N^c)_{1''} (\varphi_S \varphi_T)_{1'} + (N^c N^c)_{3} (\varphi_S \varphi_T)_{3_{1,0}} + (N^c N^c)_{3} (\varphi_S \varphi_T)_{3} \right] , \quad (2.3)$$

all of which break TB neutrino mixing. Inspection of the flavour structure of these terms shows that only the first two terms lead to mass matrices that have an eigenvector $\frac{1}{\sqrt{3}}(1,1,1)^T$, thus retaining TM neutrino mixing. However, requiring the messengers that

\[1\] Note that the third term, corresponding to two independent contractions, was not listed in Eq. (32) of [12] despite giving non-trivial contributions to the mass matrix, i.e. contributions that differ from the tri-bimaximal structure.

\[\begin{array}{cccccccccccccc}
\text{Particle} & N^c & L & e^c & \mu^c & \tau^c & H_u & H_d & \varphi_T & \varphi_S & \xi & \xi' & \xi'' & \varphi_0^T & \varphi_S^0 & \xi^0 \\
A_4 & 3 & 3 & 1 & 1'' & 1' & 1 & 1 & 3 & 3 & 3 & 3 & 1 & 1' & 1'' & 3 & 3 & 1 \\
Z_3 & \omega^2 & \omega^2 & \omega^2 & \omega^2 & 1 & 1 & 1 & \omega^2 & \omega^2 & \omega^2 & \omega^2 & 1 & \omega^2 & \omega^2 \\
U(1)_R & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 \\
\end{array}\]

Table 1: The particle content and symmetries of the $A_4$ model with extra $\xi'$, $\xi''$ flavons.
mediate these effective operators to be matter-like, i.e. they should have a $U(1)_R$ charge of 1, we find that we will always get a contribution from the third term of Eq. (2.3) as well. Therefore, without introducing new flavon fields, we cannot find a simple UV completion of the AF model where the neutrino mixing features a trimaximal pattern while breaking the tri-bimaximal one.

2.2 Trimaximal mixing from $A_4$ with extra $\xi'$ and $\xi''$ flavons

The discussion in the previous subsection leads us to consider the case where extra flavons $\xi'$ in the $1'$ representation and $\xi''$ in the $1''$ representations of $A_4$ are added to the model as already shown in Table 1. This has previously been suggested (without the see-saw mechanism) in [15] where the phenomenological consequences of the LO terms were studied numerically. However the flavon alignment was not derived in [15] but simply postulated. Remarkably, although the difference between the $\xi'$ and $\xi''$ flavon VEVs breaks the accidental $U$ symmetry and thereby violates TB mixing, the presence of these flavons respects the $S$ symmetry and leads to TM mixing.

In this subsection we consider the effect on the neutrino mass matrices of adding flavons $\xi'$ in the $1'$ representation and/or $\xi''$ in the $1''$ representations. In the subsequent subsection we consider the vacuum alignment problem including these flavons. Assuming the flavon alignments in Eq. (2.2), it is straightforward to find the structure of the charged lepton and the light neutrino mass matrices. As the charged lepton Yukawa couplings are non-renormalisable, a particular set of messengers is necessary to generate the required couplings. Following [16], one can show that a minimal messenger completion does not generate any off-diagonal entries in $m_\ell$. Therefore, the leptonic mixing matrix $U_{\text{PMNS}}$ is solely determined by the neutrino sector. Given the symmetries of Table 1 the corresponding renormalisable neutrino part of the superpotential is extended to,

$$W_{A_4}'' = y_L H_u N^c + (y_1 \varphi_S + y_2 \xi + y'_3 \xi' + y''_3 \xi'') N^c N^c .$$

Inserting the flavon vacuum alignments in Eq. (2.2), and assuming both $\xi'$ and $\xi''$ as well as SM Higgs VEVs, we obtain the Dirac neutrino mass matrix $m_D$ as well as the right-handed neutrino mass matrix $M_R$,

$$m_D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} y_{v_u} ,$$

$$M_R = \begin{bmatrix} \alpha \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} + \beta \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \gamma' \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \gamma'' \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{bmatrix} ,$$

where $\alpha = y_1 v_S$, $\beta = y_2 \langle \xi \rangle$, $\gamma' = y'_3 \langle \xi' \rangle$, $\gamma'' = y''_3 \langle \xi'' \rangle$.

The complex symmetric matrix $M_R$ is diagonalised by a unitary transformation $U_R$,

$$U_R^T M_R U_R = M_R^{\text{diag}} .$$
One can easily verify that $\frac{1}{\sqrt{3}}(1, 1, 1)^T$ is an eigenvector of $M_R^T M_R$, the eigenvalue being $|\beta + \gamma' + \gamma''|^2$. Therefore the matrix $U_R$ will have a trimaximally mixed column which, for phenomenological reasons, will be identified as the second column. The other two columns are more difficult to determine and we will discuss the analytic approximate calculation in Section 4. However, they can be parameterised as a linear combination of the first and third columns of the TB mixing matrix $U_{TB}$. Thus $U_R$ is of the form given in Eq. (1.2), with the parameters $\vartheta$ and $\rho$ being functions of the Yukawa couplings and VEVs. Note that a Majorana phase matrix would still have to be multiplied on the right in order to render the right-handed neutrino masses real. Disregarding this and other issues about phase conventions, we can now rewrite the effective light neutrino mass matrix,

$$m_{\nu}^{\text{eff}} = -m_D M_R^{-1} m_T D = -m_D U_R (M_R^{\text{diag}})^{-1} U_R^T m_D = -U_\nu (y_{v_u})^2 \frac{m_T}{M_R^{\text{diag}}} U_\nu^T,$$

where the matrix $U_\nu$ is defined as

$$U_\nu = \frac{m_D}{y_{v_u}} U_R.$$  

Due to the trivial structure of $m_D$, see Eq. (2.5), it is unitary and again of trimaximal form. In fact, $U_\nu$ differs from $U_R$ only in the interchange of the second and the third row. Assuming diagonal charged leptons, the matrix $U_\nu$ is just the lepton mixing matrix $U_\nu = U_{PMNS}$. From Eq. (2.8) it is then apparent that the Dirac neutrino mass matrix in the basis of diagonal right-handed neutrinos and charged leptons, is given by $m_D' = y_{v_u} U_{PMNS}$. In this basis, the columns of $m_D'$ are proportional to the columns of $U_{PMNS}$, and thus we see that the $A_4$ model with additional $\xi'$ and $\xi''$ flavons satisfies form dominance [17] and hence leptogenesis vanishes [18]. However, form dominance does not imply that the effective light neutrino mass matrix is form diagonalisable, and indeed this is not the case since the mixing matrix $U_\nu$ depends on the Yukawa couplings and VEVs (which in turn determine the neutrino masses). Only in the limit where $\gamma' = \gamma''$ do we recover a form diagonalisable mass matrix corresponding to the TB case.

The size of the deviations from TB mixing are controlled by the parameters $y_3'$ and $y_3''$ as well as the VEVs $\langle \xi' \rangle$ and $\langle \xi'' \rangle$. From the vacuum alignment discussion in the next subsections, however, there is no reason to believe that these VEVs should be small, i.e. one would expect a large reactor angle as well as large deviations from maximal atmospheric mixing. In order to meet the experimental bounds, one would therefore have to assume the input parameters of the flavon and Yukawa superpotential to conspire so as to yield only small deviations from the TB mixing pattern. Eventually this will motivate us to go beyond $A_4$ to $S_4$ where the two extra flavons $\xi'$ and $\xi''$ are unified into a doublet representation.

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2 This conclusion can be avoided by taking into account renormalisation group corrections which effect the entries of the Dirac neutrino Yukawa matrix differently so that the orthogonality of the columns is broken [20].
2.3 Vacuum alignment in $A_4$ with extra $\xi'$ and $\xi''$ flavons

In this subsection we scrutinise the modifications to the alignment mechanism of Altarelli and Feruglio due to the addition of flavons $\xi'$ in the $1'$ and/or $\xi''$ in the $1''$ representation of $A_4$. Our main result is that contrary to [12], one need not introduce the auxiliary flavon singlet $\xi$ which does not acquire a VEV and so plays no direct role in the flavour structure of the neutrinos. In that sense the introduction of a $\xi'$ and/or $\xi''$ flavon does not only serve the purpose of breaking the TB structure to a TM one, but also does not complicate the model at all.

The renormalisable terms of the driving superpotential, replacing the $\tilde{\xi}$ flavon of the AF model by the $\xi'$ and $\xi''$ flavons, are,

\[ W^\text{flavon}_{A_4} = \varphi^0_T (M\varphi_T + g\varphi_T\varphi_T) + \varphi^0_S (g_1\varphi_S\varphi_S + g_2\varphi_S\xi' + g_3'\varphi_S\xi'' + g_3''\varphi_S\xi'') + \xi^0 (g_4\varphi_S\varphi_S + g_5\xi\xi + g_6\xi'\xi'') . \] (2.10)

As the $F$-term equations of $\varphi^0_T$ remain unchanged from the AF model, we obtain - up to some $A_4$ transformed solutions - the well known alignment

\[ \langle \varphi_T \rangle = v_T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_T = -\frac{M}{2g} . \] (2.11)

On the other hand, the $F$-term conditions of $\varphi^0_S$ and $\xi^0$ are slightly modified in this new setup. Writing $\langle \varphi_{S1} \rangle = s_1$ and $\langle \xi \rangle = u$, $\langle \xi' \rangle = u'$ and $\langle \xi'' \rangle = u''$, we get

\[ 2g_1 \begin{pmatrix} s^2_1 - s_2s_3 \\ s^2_2 - s_1s_3 \\ s^2_3 - s_1s_2 \end{pmatrix} + g_2u \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} + g_3'u' \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} + g_3''u'' \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} , \] (2.12)

\[ g_4(s^2_1 + 2s_2s_3) + g_5u^2 + g_6u'u'' = 0 . \] (2.13)

For simplicity, let us first consider the case where only one of the two non-trivial one-dimensional flavons is present. For definiteness we assume this to be the $\xi'$. It is then straightforward to work out the most general solutions to Eqs. (2.12)-(2.13). Again, disregarding the ambiguity caused by $A_4$ symmetry transformations we find two possible non-trivial solutions,

\[ \langle \varphi_S \rangle = v_S \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad v^2_S = -\frac{g_5}{3g_4}u^2 , \quad u' = -\frac{g_2}{g_3}u , \] (2.14)

as well as

\[ \langle \varphi_S \rangle = v_S \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_S = -\frac{g_4'}{2g_1}u' , \quad u = 0 . \] (2.15)

Choosing the soft mass parameter $m^2_\xi < 0$, the second solution is eliminated and the VEV $u$ slides to a large scale [12]. In this way we are able to get the original alignment of $\varphi_S$ and non-vanishing VEVs for $\xi$ and $\xi'$, see Eq. (2.14).
We now consider the effect of having both flavons $\xi'$ and $\xi''$ in the $1'$ and $1''$ representations of $A_4$. Then the terms proportional to $g''_6$ and $g_6$ would be switched on in the flavon superpotential of Eq. (2.10). The corresponding extra terms in the $F$-term conditions of Eqs. (2.12,2.13) would thus modify the physical solution of Eq. (2.14) to

$$
\langle \varphi_S \rangle = v_S \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad v_S^2 = -\frac{g_5 u^2 + g_6 u' u''}{3g_4}, \quad u = -\frac{g'_3 u' + g''_3 u''}{g_2}.
$$

(2.16)

This solution has the unpleasant feature of leading to arbitrary physics. For instance, if $y'_3 u' = y''_3 u''$, where $y''_3$ denotes the Yukawa coupling of $\xi''$ to $N^c N^c$, see Eq. (2.1), then this implies that the mass matrix $M_R$ in Eq. (2.6) has a tri-bimaximal structure, since $\gamma' = \gamma''$, and thus the reactor angle vanishes identically. It is for this reason that models with either $\xi'$ or $\xi''$ essentially yield the same physics. However, adding both types of flavons in $A_4$ generates a bothersome ambiguity in physical predictions. In the next section the above ambiguity is removed by unifying the flavons $\xi'$ and $\xi''$ into an $S_4$ doublet $\eta_\nu$, whose VEV components are aligned along a $U$ preserving direction, thereby restoring TB mixing, at least approximately.

### 3 S₄ model of trimaximal mixing

#### 3.1 The effective $S_4$ model of leptons

As pointed out in the previous section, the $A_4$ model with $\xi'$ and/or $\xi''$ flavons cannot explain the smallness of the deviations from TB mixing. Furthermore, adding both non-trivial one-dimensional flavons leads to an ambiguity in physical predictions. In order to cure these shortcomings we consider an $S_4$ model in which the $1'$ and $1''$ representations of $A_4$ are unified into the $\eta_\nu$ doublet of $S_4$ while the triplet representations remain. The complete list of lepton, Higgs and flavon fields is given in Table 2. Similar to the $A_4$ model we have a $U(1)_R$ symmetry as well as a $Z_3$ symmetry which separates the neutrino and the charged lepton sector.

In the neutrino sector of the $S_4$ model there are three flavon fields: $\varphi_\nu$ and $\xi_\nu$ (analogous to $\varphi_S$ and $\xi$ of the AF model) and $\eta_\nu$ (which unifies the two $A_4$ flavon fields $\xi'$ and $\xi''$). The neutrino part of the effective superpotential is then,

$$
W^{\nu,\text{eff}}_{S_4} \sim L H_u N^c + (\varphi_\nu + \xi_\nu + \eta_\nu) N^c N^c + \frac{\xi_\nu}{M_\chi} \eta_\nu N^c N^c,
$$

(3.1)

analogous to Eq. (2.4), where, as in the $A_4$ model, the Dirac neutrino mass matrix takes the trivial form $m_D$ of Eq. (2.5). However an additional flavon $\xi_\nu$ in the $1'$ representation

---

3 In principle the triplet fields can either be identified with the $3$ or the $3'$ of $S_4$. They differ from each other only in the sign of the $U$ generator (see Appendix A) such that all representation matrices of the $3$ have determinant $+1$, while this is not the case for the $3'$. In the case of the right-handed neutrinos $N^c$ and the lepton doublet $L$, we are free to choose the type of $S_4$ triplet as long as it is the same for both fields, and we choose the $3$. Note that the triplet flavon $\varphi_\nu$ must furnish a $3'$ of $S_4$ because it is coupled to the symmetric product $N^c N^c$.
of $S_4$ serves the purpose of breaking $S_4$ to $A_4$. The $U$ violating flavon $\zeta_\nu$ is forbidden by $S_4$ to couple to $N^c N^c$ at renormalisable level but appears at higher order, thereby breaking the TB structure by a small amount, leading to a non-zero but suppressed reactor angle, while preserving the TM structure due to the unbroken $S$ symmetry. Inserting the flavon VEVs, whose alignment is discussed later,

$$
\langle \varphi_\nu \rangle = v_\nu \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \langle \eta_\mu \rangle = w_\mu \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \langle \xi_\nu \rangle = u_\nu, \quad \langle \zeta_\nu \rangle = z_\nu ,
$$

(3.2)

$W_{\nu,eff}^{\nu,eff}$ leads to TB neutrino mixing at LO, broken to TM mixing by the non-vanishing VEV for the $\zeta_\nu$ flavon at next-to-leading order (NLO). Note that the neutrino sector above may readily be incorporated without change into an $SU(5)$ GUT as was done for $A_4$ and $S_4$ models of TB mixing along the lines of [21–24].

In order to account for the charged lepton mass hierarchy we identify the right-handed charged leptons with one-dimensional representations of $S_4$, and distinguish them by an extra $Z'_3$ family symmetry, broken by a triplet flavon $\varphi_\ell$ as well as the doublet flavons $\eta_\mu, \eta_e$, resulting in the effective charged lepton superpotential,

$$
W_{\ell,eff}^{\ell,eff} \sim \frac{\varphi_\ell}{M_{\Omega_1}} H_d \left[ L \tau^c + \frac{\eta_\mu}{M_{\Omega_2}} L \mu^c + \frac{\eta_e}{M_{\Omega_3}} L e^c \right],
$$

(3.3)

where $M_{\Omega_i}$ are a priori independent mass scales, although two of the mass scales will turn out to be equal, $M_{\Omega_2} = M_{\Omega_3}$. Inserting the flavon VEVs,

$$
\langle \varphi_\ell \rangle = v_\ell \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \langle \eta_\mu \rangle = w_\mu \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \langle \eta_e \rangle = w_e \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},
$$

(3.4)

whose alignment is discussed later, and using the Clebsch-Gordan coefficients listed in Appendix A leads to

$$
W_{\ell,eff}^{\ell,eff} \sim \frac{v_\ell H_d}{M_{\Omega_1}} \left[ L_3 \tau^c + \frac{w_\mu}{M_{\Omega_2}} L_2 \mu^c + \frac{w_e}{M_{\Omega_3}} L_1 e^c \right],
$$

(3.5)

resulting in a diagonal charged lepton mass matrix $m_\ell$. The larger $\tau$ lepton mass is explained by the fact that the $e$ and $\mu$ masses are only provided at higher order by two
additional doublet flavons $\eta_\mu, \eta_e$. The electron to muon mass hierarchy is accounted for by a hierarchy of flavon VEVs $w_e \ll w_\mu$, which can be obtained consistently assuming certain hierarchies in the mass parameters of the flavon superpotential. Such large hierarchies of VEVs are of course stable in SUSY models, and are already familiar due to the hierarchy between the weak scale and the flavour or GUT scales. Finally note that the charged lepton sector above will eventually require modification as was done for $A_4$ and $S_4$ models of TB mixing in order to be incorporated into an $SU(5)$ GUT since quark mixing requires some small off-diagonal mixing also in the charged lepton sector \[21–24\].

### 3.2 The renormalisable $S_4$ model of leptons

As emphasised earlier, any non-renormalisable term of an effective superpotential should be understood in terms of a more fundamental underlying renormalisable theory. Without such a UV completion of a model, higher order terms which are allowed by the symmetries may or may not be present. Thus a purely effective formulation would leave room for different physical predictions. In order to remove any such ambiguity within our $S_4$ model we have constructed a fully renormalisable theory of the lepton sector. The required messengers are listed in Table 3 together with the driving fields which control the alignment of the flavons.

With the particle content and the symmetries specified in Tables 2 and 3, we can replace the effective neutrino superpotential in Eq. (3.1) by the sum of two renormalisable pieces, a LO piece and a messenger piece,

$$W^{\nu}_{S_4} = W^{\nu,\text{LO}}_{S_4} + W^{\nu,\text{mess}}_{S_4},$$

where,

$$W^{\nu,\text{LO}}_{S_4} = y_L H_u N^c + (y_1 \varphi_\nu + y_2 \xi_\nu + y_3 \eta_\nu) N^c N^c ,$$

$$W^{\nu,\text{mess}}_{S_4} = x_1 N^c \zeta_\nu \chi + \chi^c (x_2 \eta_\nu + x_3 \varphi_\nu) N^c + M_{\chi \chi^c} \chi \chi^c .$$

The leading order contribution to the right-handed neutrino mass matrix $M_R$ is given by $W^{\nu,\text{LO}}_{S_4}$ and leads to a TB structure. Given a non-vanishing VEV for the $\zeta_\nu$ flavon, $W^{\nu,\text{mess}}_{S_4}$ is responsible for breaking the TB structure to a TM one at higher order. The corresponding LO and NLO diagrams are depicted in Figure 1.

4We remark that we have suppressed the additional messenger terms $(\varphi_\nu + \eta_\nu + \xi_\nu) \chi \chi^c$ which are
Figure 1: Leading and next-to-leading order right-handed neutrino mass contributions.

The relative minus sign arises as the $\zeta_{\nu}$ VEV breaks the $U$ symmetry of $S_4$. Its presence leads to a deviation from the TB structure which would exist if the two matrices were added instead.

The $S_4$ model thus gives rise to TB neutrino mixing at LO which is broken to a TM mixing pattern by NLO corrections induced by the VEV of the $S_4$ breaking flavon field $\zeta_{\nu}$. This naturally explains why the reactor angle as well as the deviations from maximal atmospheric mixing are relatively small. Provided the charged leptons are diagonal, the lepton mixing matrix is purely determined by the structure of the right-handed neutrino mass matrix $M_R$, given in Eq. (2.6), with

$$\gamma' = \gamma - \Delta, \quad \gamma'' = \gamma + \Delta,$$

where we have defined

$$\gamma = y_3 w_{\nu}, \quad \Delta = x_1 x_2 \frac{w_{\nu} z_{\nu}}{M_X}.$$  \hspace{1cm} (3.10)

Notice that $\gamma'$ and $\gamma''$ are equal at LO. The deviations from a TB mass matrix arise only at NLO which is parameterised by $\frac{\Delta}{\gamma} \sim \frac{\nu}{M_X}.$

The charged lepton sector is formulated at the renormalisable level, using two new pairs of messengers, $\Omega_i$ and $\Omega_i^c$ ($i = 1, 2$). With the particles and symmetries listed in allowed in $W_{S_4}^\nu$ since they become relevant only at next-to-next-to-leading order with the $\zeta_{\nu}$ flavon entering quadratically; as the Klein symmetry $Z_S \times Z_U$ of the neutrino sector [19] is restored in this diagram by the quadratic appearance of $\zeta_{\nu},$ such a higher order term yields a TB contribution to $M_R.$ Therefore, the only significant term contributing to the breaking of TB to TM is the one shown in Figure II.
Table 2 and 3, we get the renormalisable superpotential for the charged leptons,

\[
W^\ell_{S_4} \sim LH_d \Omega_1 + \Omega_1^c \varphi^c = \varphi_\tau + \Omega_2^c \eta_\mu \mu^c + \Omega_2^c \eta_e e^c \\
+ M_{\Omega_1} \Omega_1^c \Omega_1^c + (M_{\Omega_2} + \zeta_{\nu}) \Omega_2^c \Omega_2^c ,
\]

(3.12)

where we have suppressed all order one coupling constants. The appearance of \( \zeta_{\nu} \) in the second line leads to an irrelevant correction of the \( \Omega_2^c \) messenger mass which we ignore in the following. Integrating out the messengers, we obtain the diagrams of Figure 2 leading uniquely to \( W^\ell_{S_4} \) in Eq. (3.3), with \( M_{\Omega_2} = M_{\Omega_1} \).

3.3 Vacuum alignment in the \( S_4 \) model

So far we have only postulated the particular alignments of the neutrino-type flavons, given in Eq. (3.2), and the flavons of the charged lepton sector, Eq. (3.4). In this subsection we first explore the driving potential relevant for the neutrino sector and prove that the assumed alignment can in fact be obtained in an elegant way which is similar to the \( A_4 \) case. Then we perform a similar study of the flavon alignments for the charged leptons.

The renormalisable superpotential involving the driving fields necessary for aligning the neutrino-type flavons is given as

\[
W^{\text{flavon},\nu}_{S_4} = \varphi^0_\nu (g_1 \varphi_\nu \varphi_\nu + g_2 \varphi_\nu \xi_\nu + g_3 \varphi_\nu \eta_\nu) + \xi^0_\nu (g_4 \varphi_\nu \varphi_\nu + g_5 \xi_\nu \xi_\nu + \frac{g_6}{2} \eta_\nu \eta_\nu) \\
+ g_7 \varphi^0_\nu \varphi_\nu \eta_\nu + D^0 (f_1 \xi_\nu \xi_\nu + f_2 \eta_\mu \eta_\mu + f_3 H_u H_d + f_4 M_0^2) .
\]

(3.13)

Identifying the two components of the doublet flavon \( \eta_\nu \) with \( \xi''_\nu \) and \( \xi'_\nu \), the first line of Eq. (3.13) is identical to the corresponding terms in the \( A_4 \) driving superpotential of Eq. (2.10). Due to the \( S_4 \) structure, we additionally get \( g_3' = g_3'' = g_3 \). All other common coupling constants \( g_i \) \( (i = 1, ..., 6) \) appear in exactly the same way once the \( S_4 \) and \( A_4 \) indices are expanded out. As identical equations yield identical results, the first line of Eq. (2.10) gives rise to an alignment which suffers from the same ambiguity as the alignment of Eq. (2.16): the ratio of the VEVs of the two doublet flavon components, i.e. \( \frac{\langle \eta_{\nu,1} \rangle}{\langle \eta_{\nu,2} \rangle} \), remains a free parameter. This ambiguity can be removed by introducing the driving field \( \varphi^0_\nu \) in the \( 3 \) representation of \( S_4 \). As \( \varphi_\nu \) furnishes a \( 3' \) representation, we only get one coupling, proportional to \( g_7 \), at the renormalisable level. With \( \varphi_\nu \) already being aligned in the \( (1, 1, 1)^T \) direction, the resulting \( F \)-term condition enforces equal VEVs for
As in the AF model, the VEV \( u \) of the superpotential takes the form

\[
\langle \phi \rangle = \begin{pmatrix} t \\ t \\ t \\ 0 \end{pmatrix} \quad \text{and} \quad \langle \eta \rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\]

As a result one finds two solutions, unique solution for the VEV of the \( \eta \), the remaining terms of the flavon superpotential of Eq. (3.13) are responsible for driving the VEV of the \( \zeta \) flavon in the \( 1' \) representation of \( S_4 \). The idea is to introduce a second driving field with identical quantum numbers as \( D^0 \). Then one obtains two F-term conditions which are identical in their structure but involve independent coupling constants, \( f_i \) for one driving field and \( f'_i \) for the other. As both conditions must be satisfied, one can linearly combine the equations and thus find a unique solution for the VEV of the \( \zeta \) flavon,

\[
v^2 = -\frac{1}{3g_4} \left[ g_5 + g_6 \left( \frac{g_2}{2g_3} \right)^2 \right] u^2, \quad w = -\frac{g_2}{2g_3} u. \tag{3.14}
\]

Turning to the flavon alignment of the charged lepton sector, the renormalisable driving superpotential takes the form

\[
W_{S_4}^{\text{flavon,}\ell} = \varphi^0 (h_1 \varphi \varphi + h_2 \varphi \eta) + \xi^0 (h_3 \varphi \varphi + h_4 \eta \eta) + \eta^0 (M_2 \eta + h_5 \zeta \eta + h_6 \eta \eta + h_7 \varphi \varphi) . \tag{3.16}
\]

The alignments of the flavon fields \( \varphi \) and \( \eta \) are constrained by the first line of Eq. (3.16). Writing \( \langle \varphi \rangle = t_i \) and \( \langle \eta_i \rangle = r_i \), the F-term conditions read

\[
2h_1 \begin{pmatrix} t_1^2 - t_2 t_3 \\ t_2^2 - t_1 t_2 \\ t_3^2 - t_1 t_2 \end{pmatrix} + h_2 r_1 \begin{pmatrix} t_2 \\ t_3 \\ t_1 \end{pmatrix} + h_2 r_2 \begin{pmatrix} t_3 \\ t_1 \\ t_2 \end{pmatrix} = 0 \quad \text{and} \quad h_3 (t_1^2 + 2t_2 t_3) + 2h_4 r_1 r_2 = 0 . \tag{3.17}
\]

The most general solution to these equations can be obtained by a straightforward calculation. As a result one finds two solutions,

\[
\langle \varphi \rangle = v_\ell \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \langle \eta \rangle = w_\mu \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad v_\ell = -\frac{h_2}{2h_1} w_\mu . \tag{3.19}
\]
as well as
\[
\langle \varphi_\ell \rangle = v_\ell \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \langle \eta_\mu \rangle = w_\mu \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad v_\ell^2 = \frac{2h_4}{3h_3} w_\mu^2 .
\] (3.20)

Clearly, alignments which are derived from these by application of an arbitrary $S_4$ transformation also satisfy the $F$-term conditions of Eqs. (3.17,3.18). In order to get rid of the second solution we can use the idea of adding a second $\xi^0_\ell$ driving field [24], thus effectively splitting the $F$-term condition of Eq. (3.18) into two equations,
\[
(t_1^2 + 2t_2t_3) = 0 , \quad r_1 r_2 = 0 .
\] (3.21)

Then the remaining solution, i.e. Eq. (3.19), is just what we have used in the discussion of the charged lepton flavour structure, see Eq. (3.4).

Having fixed the alignment of $\varphi_\ell$ and $\eta_\mu$ we can now proceed with the alignment of the flavon doublet $\eta_e$. This is determined from the $F$-term condition of the driving field $\eta^0_\ell$ in Eq. (3.16). Inserting the VEVs of $\varphi_\ell$, $\eta_\mu$ and $\zeta_\nu$, and writing $\langle \eta_{e,i} \rangle = q_i$ we obtain
\[
M_2 \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} + h_5 z_\nu \begin{pmatrix} q_1 \\ -q_2 \end{pmatrix} + h_6 w_\mu^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + h_7 v_\ell^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} .
\] (3.22)

As $z_\nu$ is already fixed, $q_2 = 0$ and we find the $\eta_e$ alignment of Eq. (3.4) with
\[
w_e = -\frac{h_6 w_\mu^2 + h_7 v_\ell^2}{M_2 + h_5 z_\nu} .
\] (3.23)

Finally, we have to consider the terms of $W_{S_4}^{\text{flavon},\nu}$ proportional to the driving field $D^0$, see Eq. (3.13). Repeating the discussion above Eq. (3.15), the $\eta_\mu$ and $\eta_e$ VEVs are simultaneously driven to non-vanishing values with
\[
w_\mu w_e = -\frac{f_4 f_1’ - f_1 f_4’}{f_2 f_1’ - f_1 f_2’} M_0^2 .
\] (3.24)

We conclude this section by sketching how the hierarchy of charged lepton masses might be obtained in our setup. We will adopt a very rough approximation, dropping all order one coefficients and keeping only orders of magnitude. Everything must be explained in terms of the input mass parameters $M_0$ and $M_2$ of Eqs. (3.13,3.16) as well as the involved messenger masses. Assuming
\[
M_2 \sim 1000 M_0 ,
\] (3.25)
we get
\[
v_\ell \sim w_\mu \sim 10 M_0 , \quad w_e \sim \frac{1}{10} M_0 .
\] (3.26)

This yields a hierarchy of about 100 between the muon and the electron mass. Setting the messenger masses $M_{\Omega_i}$ at a scale of around $100 M_0$ fixes the ratio between the tau and the muon mass at a value of order 10. The tau mass itself is then set to be approximately $\frac{10}{10}$.

\footnote{For a discussion of why such $S_4$ symmetry transformed solutions lead to the same physics see [23].}
4 Trimaximal mixing in terms of perturbative deviations to tri-bimaximal mixing

In this section we make a perturbative and analytic study of the deviations to TB mixing which are predicted by the $A_4$ and $S_4$ models of TM mixing, subject to the phenomenological constraint that the reactor angle is perturbatively small, which enables TM mixing to be viewed as a perturbative expansion around the TB limit. This is natural from the point of view of $S_4$ models where TB mixing arises at LO, broken by higher order corrections which preserve TM mixing. It also enables alternative phenomenological proposals to be viewed and compared on the same footing.

Our starting point is the right-handed neutrino mass matrix of the $A_4$ model in Eq. (2.6). It can be written as the sum of a matrix that preserves TB mixing and a matrix that violates it,

$$M_R = M_{R}^{TB} + \Delta M_R ,$$

(4.1)

where

$$M_{R}^{TB} = \alpha \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} + \beta \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} ,$$

(4.2)

and

$$\Delta M_R = \Delta \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix} ,$$

(4.3)

with

$$\Delta = \frac{1}{2} (\gamma'' - \gamma') , \quad \gamma = \frac{1}{2} (\gamma' + \gamma'') .$$

(4.4)

In the $S_4$ model the explicit form of $\Delta M_R$ is given in Eq. (3.9). In both the $A_4$ and $S_4$ models the TB violating matrix is required to be small,

$$|\Delta| \ll |\alpha|, |\beta| .$$

(4.5)

This assumption is necessary in order to meet the experimental constraints that any deviations from TB mixing should be small. The parameter $\gamma$ on the other hand need not be small as it does not break the TB pattern.

Since $M_{R}^{TB}$ is diagonalised by the TB mixing matrix $U_{TB}$, this enables $M_R$ to be diagonalised perturbatively by expanding about the TB mixing case. Writing the matrix $U_R$ which diagonalises the right-handed neutrino mass matrix in terms of its column vectors $\Phi_i$,

$$U_R = (\Phi_1, \Phi_2, \Phi_3) ,$$

(4.6)

we can expand $U_R$, for small deviations from TB mixing, in linear approximation around its TB form using

$$\Phi_i = \Phi_i^{TB} + \Delta \Phi_i , \quad \Delta \Phi_i = \sum_j \alpha_{ij} \Phi_j^{TB} ,$$

(4.7)
where $\Phi^{\text{TB}}_i$ are just the columns of the TB mixing matrix. As shown in Appendix B, due to the unitarity of $U_R$ and the special form of the mass matrix $M_R$ in Eq. (4.1), the only non-zero parameter is $\alpha_{13} = -\alpha^*_3$ whose dependence on the input parameters $\alpha, \beta, \gamma, \Delta$ is given in Eqs. (B.15, B.16). The fact that only $\alpha_{13} = -\alpha^*_3$ is non-zero implies that $U_R$ is of TM form as expected. Furthermore, since,

$$U^T_R M_R U_R = M^\text{diag}_R,$$  

it is then straightforward to derive the lepton mixing matrix $U_{\text{PMNS}}$, as in Eq. (2.9),

$$U_{\text{PMNS}} = \frac{m_D}{y v_u} U_R.$$  

Due to the trivial structure of $m_D$ as well as a diagonal charged lepton sector, the PMNS mixing matrix can thus be directly obtained from $U_R$ by permuting the second and the third row as well as multiplying the Majorana phase matrix $P$ on the right and another phase matrix $P'$ on the left, leading to $U_{\text{PMNS}} = U_{\text{TM}}$ where,

$$U_{\text{TM}} \approx P' \left( \begin{array}{ccc} \frac{\sqrt{2}}{\sqrt{6}} & \frac{\sqrt{1}}{\sqrt{3}} & -\frac{\sqrt{2}}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} + \frac{\sqrt{1}}{\sqrt{2}} \alpha_{13} & \frac{\sqrt{1}}{\sqrt{3}} & \frac{1}{\sqrt{2}} + \frac{\sqrt{1}}{\sqrt{6}} \alpha^*_3 \\ -\frac{1}{\sqrt{6}} - \frac{\sqrt{1}}{\sqrt{2}} \alpha_{13} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} + \frac{\sqrt{1}}{\sqrt{6}} \alpha^*_3 \end{array} \right) P.$$  

The matrix $P'$ has to be chosen such that the PMNS matrix without Majorana phases is brought to the standard PDG form where the 2-3 and 3-3 elements are real and the mixing angles are all between $0^\circ$ and $90^\circ$. In linear approximation, the required form of $P'$ becomes

$$P' \approx \text{diag}(1, a_+, -a_-), \quad a_\pm = 1 \pm i \cdot \frac{\text{Im}(\alpha_{13})}{\sqrt{3}}.$$  

Multiplying this explicit form of the phase matrix $P'$ we obtain a mixing matrix that is consistent with the standard PDG phase conventions.

It is useful to compare the TM mixing matrix in Eq. (4.10) to a general parametrisation of the PMNS mixing matrix in terms of deviations from TB mixing [25],

$$U_{\text{PMNS}} \approx \left( \begin{array}{ccc} \frac{2}{\sqrt{6}} (1 - \frac{1}{2} s) & \frac{1}{\sqrt{3}} (1 + s) & \frac{1}{\sqrt{2}} r e^{-i \delta} \\ -\frac{1}{\sqrt{6}} (1 + s - a + re^{i \delta}) & \frac{1}{\sqrt{3}} (1 - \frac{1}{2} s - a - \frac{1}{2} re^{i \delta}) & \frac{1}{\sqrt{2}} (1 + a) \\ \frac{1}{\sqrt{6}} (1 + s + a - re^{i \delta}) & -\frac{1}{\sqrt{3}} (1 - \frac{1}{2} s + a + \frac{1}{2} re^{i \delta}) & \frac{1}{\sqrt{2}} (1 - a) \end{array} \right) P,$$  

where the deviation parameters $s, a, r$ are defined as [25],

$$\sin \theta_{12} = \frac{1}{\sqrt{3}} (1 + s), \quad \sin \theta_{23} = \frac{1}{\sqrt{2}} (1 + a), \quad \sin \theta_{13} = \frac{r}{\sqrt{2}}.$$  

This comparison yields

$$s \approx 0, \quad a \approx \frac{\text{Re} (\alpha_{13})}{\sqrt{3}}, \quad r \cos \delta \approx -\frac{2}{\sqrt{3}} \text{Re} (\alpha_{13}), \quad \delta \approx \text{arg} (\alpha_{13}) + \pi.$$  

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where $\delta$ is the CP violating oscillation phase of the lepton sector, and $\alpha_{13}$ is proportional to $\Delta$ as shown in Eqs. (B.15, B.16). Independently of the value of $\alpha_{13}$ we confirm the TM sum rules given in [26],

$$s \approx 0, \quad a \approx -\frac{1}{2} r \cos \delta.$$ (4.15)

We emphasise that the sum rules in Eq. (4.15) hold for any model of trimaximal mixing, since it is always the case that phenomenology requires that it has approximate tri-bimaximal form. The above perturbative form of TM mixing in terms of TB deviation parameters is useful when comparing TM mixing to other proposed forms of the mixing matrix. For example, if the reactor angle is measured to be sizeable, but the solar and atmospheric angles remain close to their tri-bimaximal values, i.e. the deviation parameters in Eq. (4.13) take the form $s \approx a \approx 0$ but $r \neq 0$, then the mixing matrix takes the “tri-bimaximal-reactor” (TBR) form [26]. Such a mixing has recently been obtained in an $S_4$ setup [27]. Alternative proposals [28–40] that have been put forward to accommodate the T2K result could similarly be compared using the deviation parameters $s, a, r$.

With future neutrino oscillation experiments being able to not only accurately measure the reactor angle, parametrised here as $r$, but also the atmospheric and solar deviation parameters $a, s$ and eventually the CP violating oscillation phase $\delta$, it is clear that relating these deviation parameters via sum rules comprise the next step in discriminating different models of lepton masses and mixings.

## 5 Conclusions

In the well known direct models of tri-bimaximal (TB) mixing, based on $A_4$ and $S_4$, the TB mixing is enforced by a Klein symmetry $Z_2^S \times Z_2^U$ in the neutrino sector, together with a $Z_3^T$ symmetry in the charged lepton sector, where a common basis corresponds to a diagonal charged lepton mass matrix. It is also well known that TB mixing can emerge from either $S_4$, which contains the generators $S, T, U$, or $A_4$, which contains $S, T$. In the case of $A_4$ the $U$ symmetry emerges accidentally as a result of the absence of flavons in the $1'$ or $1''$ representations of $A_4$. Such models are called “direct models” since (some of) the group generators remain unbroken in different sectors of the low energy effective theory. Although this simple and appealing picture is apparently shattered by the T2K results, which indicate a sizeable reactor angle $\theta_{13}$, simple alternative possibilities such as trimaximal (TM) mixing remain.

We have argued that, in the framework of direct $A_4$ models, the T2K results motivate adding the “missing” flavons $1'$ or $1''$, whose VEVs break the accidental $U$ symmetry but preserve the $S$ symmetry in the neutrino sector corresponding to a $Z_2^S$ subgroup of $A_4$, leading to a TM pattern of mixing as in Eq. (1.2). We have studied the vacuum alignment of such an extended Altarelli-Feruglio $A_4$ family symmetry model including additional flavons in the $1'$ and $1''$ representations and show that it leads to TM mixing in which the second column of the lepton mixing matrix consists of the column vector $(1, 1, 1)^T/\sqrt{3}$. However there are drawbacks to this approach. To begin with, the higher order corrections need to be kept under control otherwise they can wash out the desirable
TM structure, however this can in principle be done by formulating the theory at the renormalisable level. Furthermore, this approach within $A_4$ provides no explanation for why the reactor angle should be smaller than the atmospheric and solar angles.

In order to overcome these drawbacks, we have proposed a renormalisable $S_4$ model of leptons in which the $1'$ and $1''$ flavons of $A_4$ are unified into a doublet of $S_4$ which is spontaneously broken to $A_4$ by a $1'$ flavon of $S_4$. We have studied the vacuum alignment in the $S_4$ model and shown that it predicts accurate TM neutrino mixing due to a preserved $Z_2^S$ symmetry in the neutrino sector, where $Z_2^S \subset Z_2^S \times Z_2^U \subset S_4$. In the $S_4$ model, the $U$ generator corresponding to $Z_2^U$ is broken by the $1'$ flavon of $S_4$, which however only enters the neutrino sector at higher order, resulting in approximate TB mixing.

Although the $A_4$ and $S_4$ models of leptons presented here involve diagonal charged lepton mass matrices, when the models are extended to include quarks, for example in the framework of $SU(5)$ unification, we would expect the charged lepton sectors (but not the neutrino sectors) of these models to be modified. This could introduce additional contributions to lepton mixing from the charged lepton sector. Interestingly both the $A_4$ and $S_4$ models here preserve form dominance and hence predict zero leptogenesis, independently of charged lepton mixing, up to renormalisation group corrections.

It should be clear that a sizeable reactor angle as indicated by T2K does not rule out the family symmetry approach, on the contrary it provides additional input which helps to refine the symmetry approach. In particular we have explored an $S_4$ model in which the leading order Klein symmetry $Z_2^S \times Z_2^U$ associated with TB mixing is broken at higher order down to a $Z_2^S$ subgroup capable of enforcing a simple TM mixing pattern. We emphasise that the smoking gun signature of TM mixing is accurate trimaximal solar mixing $s \approx 0$, together with the sum rule $2a + r \cos \delta \approx 0$, which can be tested in forthcoming high precision neutrino oscillation experiments.

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Appendix

A $S_4$ and $A_4$ Clebsch-Gordan coefficients

Finite groups can be defined in terms of their generating elements which satisfy certain product rules, the so called presentation. Starting from these rules it is possible to work out the matrix representations of these elements up to an ambiguity related to the choice of basis. In the literature there exist different 'standard' choices which all have advantages but also disadvantages. As the Clebsch-Gordan coefficients depend on the chosen basis, it is necessary to carefully define the basis in which a model is constructed.
The generators and Clebsch-Gordan coefficients of $S_4 = \Delta(24)$ and $A_4 = \Delta(12)$ in a basis where the triplets are explicitly real were derived in a general way in \cite{10, 11}. As was argued in \cite{12}, there exists a more suitable triplet basis for $A_4$ models in which the order-three generator $T$ is brought to a diagonal and complex form. By now this choice has become the standard or physical basis for direct models \cite{19}. The corresponding basis in the case of $S_4$ can be found for instance in \cite{20}. As $A_4$ is a subgroup of $S_4$ it is natural to express this relation also in terms of the generators where the elements $S, T, U$ generate $S_4$, while $A_4$ is obtained by simply dropping the $U$ generator \cite{21}.

Table 4 lists the generators of $S_4$ and $A_4$ in the physical basis. The primed representations of $S_4$ differ only in the sign of the $U$ generator. Dropping the $U$ generator we obtain $A_4$. It is clear from the table that the doublet of $S_4$ becomes a reducible representation under $A_4$, denoted by $1''$ and $1'$.

The $S_4$ product rules in the chosen basis are listed below, where we use the number of primes within the expression

$$\alpha^{(i)} \otimes \beta^{(i)} \rightarrow \gamma^{(i)} ,$$

(A.1)

to classify the results. We denote this number by $n$, e.g. in $3 \otimes 3' \rightarrow 3'$ we get $n = 2$.

| $S_4$ | $A_4$ | $S$ | $T$ | $U$ |
|-------|-------|-----|-----|-----|
| 1, 1' | 1     | 1   | 1   | ±1  |
| 2     | \(\begin{pmatrix} 1'' \\ 1' \end{pmatrix}\) | \(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\) | \(\begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}\) | \(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\) |
| 3, 3' | \(\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}\) | \(\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}\) | \(\pm \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\) |

Table 4: The generators $S, T, U$ of $S_4$ and $S, T$ of $A_4$ as used in this article.
\begin{align*}
2 \otimes 2 & \rightarrow 2 \quad \{ \begin{array}{ll}
n = & \text{even} \\
n = & \text{odd}
\end{array} \} \quad (\alpha_2 \beta_2)_{\alpha_1 \beta_1}, \\
2 \otimes 3^{(i)} & \rightarrow 3^{(i)} \quad \{ \begin{array}{ll}
n = & \text{even} \\
n = & \text{odd}
\end{array} \} \quad \alpha_1 \left( \frac{\beta_2}{\beta_1} \right) + (-1)^n \alpha_2 \left( \frac{\beta_3}{\beta_2} \right), \\
3^{(i)} \otimes 3^{(i)} & \rightarrow 1^{(i)} \quad \{ \begin{array}{ll}
n = & \text{even} \\
n = & \text{odd}
\end{array} \} \quad \alpha_1 \beta_1 + \alpha_2 \beta_3 + \alpha_3 \beta_2, \\
3^{(i)} \otimes 3^{(i)} & \rightarrow 2 \quad \{ \begin{array}{ll}
n = & \text{even} \\
n = & \text{odd}
\end{array} \} \quad (\alpha_2 \beta_2 + \alpha_3 \beta_1 + \alpha_1 \beta_3) - (\alpha_2 \beta_3 + \alpha_1 \beta_2 + \alpha_2 \beta_1), \\
3^{(i)} \otimes 3^{(i)} & \rightarrow 3^{(i)} \quad \{ \begin{array}{ll}
n = & \text{odd} 
\end{array} \} \quad (2 \alpha_3 \beta_1 - \alpha_2 \beta_3 - \alpha_3 \beta_2), \\
3^{(i)} \otimes 3^{(i)} & \rightarrow 3^{(i)} \quad \{ \begin{array}{ll}
n = & \text{even} 
\end{array} \} \quad (\alpha_2 \beta_3 - \alpha_3 \beta_2), \\
\end{align*}

The $A_4$ Clebsch-Gordan coefficients can be obtained from the above expressions by simply dropping all primes and identifying the components of the $S_4$ doublet $2$ as the $1''$ and $1'$ representations of $A_4$, see Table \ref{table}. We thus find for the non-trivial products

\begin{align*}
1' \otimes 1'' & \rightarrow 1 \quad \alpha \beta, \\
1' \otimes 3 & \rightarrow 3 \quad \alpha \left( \frac{\beta_2}{\beta_1} \right), \\
1'' \otimes 3 & \rightarrow 3 \quad \alpha \left( \frac{\beta_3}{\beta_1} \right), \\
3 \otimes 3 & \rightarrow 1 \quad \alpha_1 \beta_1 + \alpha_2 \beta_3 + \alpha_3 \beta_2, \\
3 \otimes 3 & \rightarrow 1' \quad \alpha_3 \beta_3 + \alpha_1 \beta_2 + \alpha_2 \beta_1, \\
3 \otimes 3 & \rightarrow 1'' \quad \alpha_2 \beta_2 + \alpha_3 \beta_1 + \alpha_1 \beta_3, \\
3 \otimes 3 & \rightarrow 3 + 3 \quad \left( \frac{2 \alpha_1 \beta_1 - \alpha_2 \beta_3 - \alpha_3 \beta_2}{2 \alpha_3 \beta_3 - \alpha_1 \beta_2 - \alpha_2 \beta_1} \right) + \left( \frac{\alpha_2 \beta_3 - \alpha_3 \beta_2}{\alpha_1 \beta_2 - \alpha_2 \beta_1} \right) + \left( \frac{\alpha_3 \beta_1 - \alpha_1 \beta_3}{\alpha_3 \beta_1 - \alpha_1 \beta_3} \right).
\end{align*}
B Perturbative diagonalisation of $M_R$

In this Appendix we present a general method for a perturbative diagonalisation of the right-handed neutrino mass matrix and the determination of the PMNS matrix in the diagonal charged lepton mass basis. The method is analogous to the perturbative expansion of the effective neutrino mass matrix developed in [45], which is mainly applicable to the case of a diagonal right-handed neutrino mass matrix as encountered in indirect models. We show that, in the case of the direct models in this paper, it is sufficient to consider the matrix $U_R$ which perturbatively diagonalises the right-handed neutrino mass matrix, since the PMNS matrix is then simply obtained from $U_R$ due to the trivial structure of the Dirac mass matrix.

Writing the matrix $U_R$ as in Eq. (4.6),

$$U_R = \Phi_1, \Phi_2, \Phi_3,$$

we can express the mass matrix $M_R$, see Eq. (4.8), as

$$M_R = U_R^* M_R^{\text{diag}} U_R^\dagger = \sum_i M_i \Phi_i^* \Phi_i^\dagger,$$

(B.1)

where $M_i$ are the eigenvalues of $M_R$. For small deviations from TB mixing, we can expand this equation in linear approximation around its TB form using

$$M_R = M_R^{\text{TB}} + \Delta M_R, \quad M_i = M_i^{\text{TB}} + \Delta M_i, \quad \Phi_i = \Phi_i^{\text{TB}} + \Delta \Phi_i.$$  

(B.2)

Here $M_R^{\text{TB}}$ is the part of $M_R$ that does not depend on the small parameter $\Delta$. Its eigenvalues are

$$M_1^{\text{TB}} = 3\alpha + \beta - \gamma, \quad M_2^{\text{TB}} = \beta + 2\gamma, \quad M_3^{\text{TB}} = 3\alpha - \beta + \gamma,$$

(B.3)

with eigenvectors

$$\Phi_1^{\text{TB}} = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \quad \Phi_2^{\text{TB}} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \Phi_3^{\text{TB}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$  

(B.4)

Following [45], we parameterise $\Delta \Phi_i$ in the basis of these TB column vectors $\Phi_i^{\text{TB}}$,

$$\Delta \Phi_i = \sum_j \alpha_{ij} \Phi_j^{\text{TB}}.$$  

(B.5)

The parameters $\alpha_{ij}$, which are a measure of how much the mixing deviates from the TB pattern, are small. In order to determine their dependence on the input parameters $\alpha, \beta, \gamma, \Delta$, we first observe that $U_R$ must be unitary, i.e.

$$\delta_{ij} = \left( \Phi_i^{\text{TB}} + \sum_k \alpha_{ik} \Phi_k^{\text{TB}} \right)^\dagger \left( \Phi_j^{\text{TB}} + \sum_l \alpha_{jl} \Phi_l^{\text{TB}} \right) \approx \delta_{ij} + \alpha_{ij}^* + \alpha_{ji},$$

(B.6)

where we have dropped second order terms. As a consequence of unitarity we thus get

$$\alpha_{ji} \approx -\alpha_{ij}^*. $$

(B.7)
Expanding Eq. (B.1) around its TB structure and keeping only terms linear in small deviations, we get

\[
\Delta M_R = M_R - M^\text{TB}_R \\
\approx \sum_i \left( \Delta M_i \Phi^\text{TB}_i \Phi^\text{TB}_i^T + M^\text{TB}_i \Delta \Phi^* \Phi^\text{TB}_i^T + M^\text{TB}_i \Phi^\text{TB}_i \Delta \Phi^\dagger \Phi^\text{TB}_i \right) \tag{B.8}
\]

\[
\approx \sum_i \left( \Delta M_i \Phi^\text{TB}_i \Phi^\text{TB}_i^T + M^\text{TB}_i \sum_j \alpha^*_ij \Phi^\text{TB}_j \Phi^\text{TB}_i^T + M^\text{TB}_i \sum_j \Phi^\text{TB}_i \alpha^*_ij \Phi^\text{TB}_j^T \right).
\]

With \(\Delta M_R\) corresponding to the TB breaking contribution to \(M_R\), as in Eq. (3.9),

\[
\Delta M_R = \Delta \begin{pmatrix}
0 & 1 & -1 \\
1 & -1 & 0 \\
-1 & 0 & 1
\end{pmatrix}, \tag{B.9}
\]

it is now possible to calculate the parameters \(\alpha_{ij}\) and \(\Delta M_i\) from Eq. (B.8). Counting the number of unknowns we get 6 real parameters for \(\Delta M_i\) as well as 3 + 6 real parameters for \(\alpha_{ij}\) where we have accounted for the unitarity constraint of Eq. (B.7). On the other hand Eq. (B.8) yields 6 complex conditions, 3 from the diagonal entries and 3 from the off-diagonals (\(\Delta M_R\) is symmetric). As the number of unknowns is bigger than the number of conditions we do not find a unique solution. Note however that the effect of \(\alpha_{ii}\) can always be absorbed in \(\Delta M_i\). Hence we can remove 3 unknowns by setting \(\alpha_{ii} = 0\) without loss of generality.

The remaining 12 unknowns can be determined by sandwiching Eq. (B.8) between \(\Phi^\text{TB}_k^T\) and \(\Phi^\text{TB}_l\),

\[
\Phi^\text{TB}_k^T \Delta M_R \Phi^\text{TB}_l \approx \left( \Delta M_k \delta_{kl} + M^\text{TB}_l \alpha^*_{lk} + M^\text{TB}_k \alpha_{kl} \right). \tag{B.10}
\]

Explicit calculation of the left-hand side gives

\[
0 \approx M^\text{TB}_1 \alpha^*_{12} - M^\text{TB}_2 \alpha_{12}, \tag{B.11}
\]

\[
0 \approx M^\text{TB}_2 \alpha^*_{23} - M^\text{TB}_3 \alpha_{23}, \tag{B.12}
\]

\[
-\sqrt{3} \Delta \approx M^\text{TB}_1 \alpha^*_{13} - M^\text{TB}_3 \alpha_{13}, \tag{B.13}
\]

\[
0 \approx \Delta M_1 \approx \Delta M_2 \approx \Delta M_3. \tag{B.14}
\]

Notice that the TB breaking parameter \(\Delta\) does not give rise to corrections to the mass eigenvalues in linear approximation. Eqs. (B.11) and (B.12) tell us that \(\alpha_{12} = \alpha_{23} = 0\) unless \(|M^\text{TB}_1| = |M^\text{TB}_2|\) and \(|M^\text{TB}_2| = |M^\text{TB}_3|\), respectively. Eq. (B.13) can be used to determine the complex valued \(\alpha_{13}\) in terms of the parameters \(\alpha, \beta, \gamma\) and \(\Delta\) of the right-

\footnote{Note that \(\alpha_{ii}\) must be purely imaginary.}
handed neutrino mass matrix $M_R$. A straightforward but tedious calculation yields

$$\text{Re} (\alpha_{13}) \approx - \frac{\sqrt{3}}{2} \cdot \left[ \text{Re} \left( \frac{\Delta}{\beta - \gamma} \right) + \text{Im} \left( \frac{\Delta}{\beta - \gamma} \right) \cdot \text{Im} \left( \frac{3\alpha}{\beta - \gamma} \right) \right], \quad (B.15)$$

$$\text{Im} (\alpha_{13}) \approx \frac{\sqrt{3}}{2} \cdot \frac{\text{Im} \left( \frac{\Delta}{\beta - \gamma} \right)}{\text{Re} \left( \frac{3\alpha}{\beta - \gamma} \right)}, \quad (B.16)$$

Notice that $\alpha_{13}$ is proportional to the TB violating parameter $\Delta$, which, in the $S_4$ model, is proportional to the $\zeta_\nu$ flavon VEV as shown in Eq. (3.11). Once $\alpha_{13}$ is determined, the matrix $U_R$, and hence the PMNS matrix, are fully determined, as discussed in Section 4.

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