MHD equilibria with incompressible flows: symmetry approach

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We identify and discuss a family of azimuthally symmetric, incompressible, magneto-hydrodynamic plasma equilibria with poloidal and toroidal flows in terms of solutions of the Generalized Grad Shafranov (GGS) equation. These solutions are derived by exploiting the incompressibility assumption, in order to rewrite the GGS equation in terms of a different dependent variable, and the continuous Lie symmetry properties of the resulting equation and in particular a special type of “weak” symmetries.

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I. INTRODUCTION

There is a rich literature concerning the transformation properties and the symmetry analysis of the general form of the equations that describe the magnetohydrodynamic (MHD) equilibrium of a plasma with flows. See e.g., Ref.1 where transformation properties of a generalized Grad–Shafranov (GS) equation are analyzed for cases with and without spatial symmetry in a framework that makes it possible to recover both the magnetostatic and the “stationary hydromechanic” limits. In Ref.2 as an application of the method, solutions of the stationary three-dimensional MHD equations with field-aligned incompressible flows are calculated. Other applications of the transformation properties as a useful method for finding explicit solutions for the general MHD equilibrium equations starting from known solutions can be found in Refs.3–5, whereas in Ref.6 the equivalence is shown of this method with the classical approach to infinitesimal Lie point symmetries.

The symmetries of interest in this analysis are described by continuous Lie groups of transformations and we refer e.g., to the books 7–9 for all details about this method and for some of its typical applications in different physical problems. It can be remarked that the presence of a symmetry has twofold applications: i) it makes it possible to obtain new solutions starting from a known one, ii) it makes it possible to look for solutions which are left invariant by the symmetry transformation (this may be obtained considering suitable reduced equations).

The analysis of the symmetry properties of the specific GS equation that describes the case of stationary, azimuthally symmetric MHD plasma equilibria without flows (static equilibria) was developed in previous papers, see e.g. Refs.10–13 and Ref.14 for the Euler equations with swirl. In Ref.11 an analysis of the symmetry properties of a class of partial differential equations of relevance for plasma physics, including the GS equation as a special case, has been performed. In particular, in Ref.10 the notion of “weaker” symmetries (conditional symmetries and similar) was introduced for the GS equation; the presence of such symmetries made it possible to identify for the first time some additional classes of solutions to the GS equation that describe D-shaped toroidal plasma equilibria. The present paper is devoted to the investigation of a generalized form of the GS equation that describes two-dimensional azimuthally symmetric plasma equilibria in the presence of poloidal and toroidal plasma flows, see Refs.15–17, but we will restrict our study to incompressible flows. The rationale
behind the incompressibility assumption stems from the fact that it has been known in the
literature, Refs.18–22, that in this limit the equation that takes the role of the GS in the
presence of plasma flows, Eq. (1) in Sec.II, can be transformed into a much simpler equation
of the GS-type. This is made possible by a redefinition of the dependent variable (the
magnetic flux function \( \psi \)) and can be applied as long as the poloidal flow is sub-alfvènic.23
This method allows us to generalize to a plasma with flows a class of solutions identified in
Ref.10 using this symmetry approach.

The present paper is organized as follows. In Sec.II we introduce the Generalized Grad
Shafranov (GGS) equation which includes the effect of poloidal and toroidal flows, take
the incompressible plasma limit, define the new dependent variable \( \chi = \chi(\psi) \) and obtain a
modified GS equation with an additional term that has an \( r^4 \) dependence of the cylindrical
radius \( r \).
In Sec.III we apply the concept of weak conditional symmetry to the GS equation. First we
recover the D-shaped solution of the GS equation described in Ref.10 as a special case of
a wider class of solutions. The common geometrical structure in the poloidal \( r \)-\( z \) plane of
this class of solutions is discussed, and shown to arise directly from its symmetry properties.
On the other hand, determining the specific spatial shape of these solutions involves the
(num)erical) integration of a nonlinear second order ODE whose coefficients depend of the
physical parameters of the specific equilibrium configuration. A qualitative classification of
the different types of solutions of this ODE is given.
In Sec.IV A we extend this analysis to the GGS equation and describe a novel class of equi-
libria solutions with flows. These solutions share the same common geometrical structure of
the solutions without flows because the symmetry properties are the same. Specific exam-
pies of equilibria with flows are then shown, including solutions that exhibit three magnetic
axes along the toroidal direction. These latter solutions correspond to configurations made
of three separate sets of tori, each set nested around one of the magnetic axes. By imposing
appropriate boundary conditions corresponding to \( \psi = \bar{\psi} = \text{const} \) surfaces, each of these sets
can be considered as an independent equilibrium configuration.
In the Conclusions we note that in the case where poloidal flows are present the reconstruc-
tion from the solutions of the GGS equation of the explicit spatial dependence of the plasma
pressure, of the poloidal and toroidal velocity fields and of the toroidal field from that of the
variable $\chi$ involves the inversion of the integral relationship between $\psi$ and $\chi$ together with the solution of a set of nonlinear algebraic equations. This inversion is not required in the case of toroidal flows and an explicit example of the effect of the toroidal flows and their gradients on the equilibrium configuration is shown.

Finally, we indicate further extensions of the solution procedure of the GS and of the GGS equations employed in the present article.

II. GENERALIZED GRAD SHAFRANOV EQUATION

We consider an axisymmetric Magnetohydrodynamic plasma equilibrium with poloidal ($v_p$) and azimuthal ($v_\varphi$) flows and write the corresponding generalized Grad Shafranov equation (see Refs.15–17) in the form

$$\nabla \cdot \left[ \left( 1 - \frac{F^2}{4\pi \rho} \right) \frac{1}{r^2} \nabla \psi \right] + \frac{F F'}{4\pi \rho} \frac{|\nabla \psi|^2}{r^2} = -4\pi \rho \left( J' - U_S I' + r v_\varphi G' \right) - \frac{1}{r^2} \left( H + r v_\varphi F \right) \left( H' + r v_\varphi F' \right).$$

(1)

Here the notation of Ref.[17] has been adopted with coordinates $(r, \varphi, z)$ where the azimuthal angle $\varphi$ is an ignorable coordinate. In Eq.(1) a prime denotes differentiation with respect to the magnetic flux function $\psi$, $\rho(r, z)$ is the plasma density and the magnetic field $B$ is expressed in terms of its azimuthal and poloidal components, $B_\varphi$ and $B_p$ through the flux function $\psi(r, z)$ as

$$B = B_\varphi \hat{\varphi} + \nabla \psi \times \nabla \varphi,$$

(2)

where $\hat{\varphi} = r \nabla \varphi$ is the unit vector in the azimuthal direction. The flux functions in Eq.(1) are defined as

$$\mathcal{F} (\psi) = 4\pi \rho v_p / B_p,$$

(3)

$$\mathcal{G} (\psi) = v_\varphi / r - \mathcal{F} B_\varphi / (4\pi \rho r),$$

(4)

$$\mathcal{H} (\psi) = r B_\varphi - r \mathcal{F} v_\varphi,$$

(5)

$$I (\psi) = S,$$

(6)

$$J (\psi) = U + \rho U_\rho + v^2 / 2 - r v_\varphi G.$$  

(7)

where $S$ is the plasma entropy, $U$ is the plasma internal energy and $U_\rho$ its derivative at constant entropy.
In the following we consider an incompressible equilibrium with $\rho = \text{const.}$ in which case $\rho(J' - U_S I') = (p + \rho v^2/2 - \rho v_\varphi G)'$ and define the Alfvénic Mach number $M(\psi) = F/(4\pi \rho)^{1/2}$. Inverting Eqs.(4-5) we obtain

\[ r B_\varphi = \left[ r^2 F G + \mathcal{H} \right] / (1 - M^2), \]
\[ r v_\varphi = \left[ r^2 G + \mathcal{H} F / (4\pi \rho) \right] / (1 - M^2). \]

The r.h.s. of Eq.(1) contains functions of $\psi$ and powers of $r$. By substituting for $r v_\varphi$ its expression from Eq.(9) and collecting powers of $r^2$, we can rewrite the r.h.s. of Eq.(1) as

\[ -\frac{1}{2r^2} \left[ \frac{\mathcal{H}^2}{1 - M^2} \right]' - \left[ \frac{G F H}{1 - M^2} \right]' - 4\pi \rho (J' - U_S I') \right] r^2 \left[ \frac{(G^2 F^2/2)/2}' \right] \left[ \frac{1}{1 - M^2} + 4\pi \rho (G^2/2)' \right]. \]

In the limit of zero plasma flow the $1/r^2$ term reduces to the toroidal magnetic field contribution to the standard GS equation and the $r$-independent term to the plasma pressure contribution. The plasma poloidal flow modifies the operator on the l.h.s. of Eq.(1) as discussed in Refs.15 and 16. On the r.h.s. the poloidal flow introduces the multiplication factors depending on $(1 - M^2)^{-1}$ and both flows modify the flux functions in the $1/r^2$ in the $r$-independent term. In addition they introduce a new term proportional to $r^2$. In the remaining part of this article we will consider only sub-alfvénic poloidal flows so that $M^2 < 1$.

A. A new dependent variable: $\chi = \chi(\psi)$

The fact that the alfvénic Mach number $M$ in a constant density equilibrium is a flux function allows us to bring back the operator on the l.h.s. of Eq.(1) to its standard GS form by redefining the dependent variable $\psi$. Following Refs.18-22 we define the new dependent variable $\chi = \chi[\psi(r, z)]$ as

\[ \chi(\psi) = \int_{\psi}^{\psi'} [1 - M^2(\eta)]^{1/2} d\eta, \]

where $M^2(\eta) < 1$. Then, we obtain

\[ \nabla \cdot \left[ \left( 1 - \frac{F^2}{4\pi \rho} \right) \frac{1}{r^2} \nabla \psi \right] + \frac{F F'}{4\pi \rho} \frac{\nabla \psi}{r^2} = (1 - M^2)^{1/2} \nabla \cdot \left[ \frac{1}{r^2} \nabla \chi \right]. \]

Finally, re-expressing the flux functions of $\psi$ on the r.h.s. of Eq.(10) as functions of $\chi$ through their dependence on $\psi(\chi)$ and using cylindrical coordinates, we obtain (see also
Ref.24 and 25)

\[ \frac{\partial^2 \chi}{\partial r^2} - \frac{1}{r} \frac{\partial \chi}{\partial r} + \frac{\partial^2 \chi}{\partial z^2} = \mathcal{A}_0(\chi) + r^2 \mathcal{A}_2(\chi) + r^4 \mathcal{A}_4(\chi), \]  

(13)

where

\[ \mathcal{A}_0(\chi) \equiv \frac{1}{2[1 - M^2]^{1/2}} \left[ \frac{\mathcal{H}^2}{1 - M^2} \right]’, \]

(14)

\[ \mathcal{A}_2(\chi) \equiv \frac{1}{[1 - M^2]^{1/2}} \left[ \left( \frac{G \mathcal{F} \mathcal{H}}{1 - M^2} \right)’ + 4\pi \rho (\mathcal{J}’ - \mathcal{U}\mathcal{S})’ \right], \]

(15)

\[ \mathcal{A}_4(\chi) \equiv \frac{1}{[1 - M^2]^{1/2}} \left[ \left( \frac{G^2 \mathcal{F}^2}{2} / 1 - M^2 \right)’ + 4\pi \rho (G^2/2)’ \right]. \]

(16)

III. SYMMETRIES, CONDITIONAL AND “WEAK” CONDITIONAL SYMMETRIES.

A. GS equation with no flow: \( \mathcal{A}_4 = 0 \)

Let us start by considering the “standard” GS equation (i.e. with no flow so that in particular \( \chi(\psi) = \psi \))

\[ \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} = r^2 F(\psi) + G(\psi). \]  

(17)

The classification presented in Ref.10 exhausts all the possible Lie point-symmetries admitted by this equation. The presence of these symmetries is strictly related to precise choices of the functions \( F(\psi) \) and \( G(\psi) \), as fully discussed in Ref.10. Other papers devoted to the symmetry analysis of this equation (or of particular cases of it) are e.g. Refs.11–14. In Ref.26, also an example of generalized symmetry is proposed.

As shown in Ref.10, one can also look for weaker notions of symmetries, and in particular for the existence of conditional symmetries Refs.27–29 (see also Refs.13 and 30 for other references). Let us recall that a conditional symmetry \( Y \) generates a transformation which does not map solutions into solutions, but defines a \( Y \)-invariant variable with the property that the solutions of the reduced equation obtained writing the initial one in terms of this variable, are also solutions of the initial equation.

In Ref.10 we have shown to be useful to introduce even weaker type of conditional symmetry (see Ref.13, 29, and 30 for this notion of symmetry); in particular, the choice

\[ Y = z \frac{\partial}{\partial r} + r \frac{\partial}{\partial z} \]  

(18)
has revealed to be especially convenient (see Ref.10 for a discussion and other details).

Writing the GS equation in terms of the \( Y \)-invariant variable

\[
s = r^2 - z^2
\]

one obtains the following equation, where the variable \( r \) still appears (playing here the role of a parameter, as happens in the case of weak conditional symmetries)

\[
(8r^2 - 4s)\psi_{ss} - 2\psi_s = r^2F(\psi) + G(\psi).
\] (19)

This equation can be naturally split into two ODE’s involving only the new variable \( s \):

\[
8\psi_{ss} = F(\psi) \quad \text{and} \quad 4s\psi_{ss} + 2\psi_s = -G(\psi). \tag{20}
\]

Clearly, these equations admit solutions only if there is a precise relationship between \( F(\psi) \) and \( G(\psi) \). A simultaneous solution of these equations can be found of the form

\[
\psi(s) = (r^2 - z^2)^{-q} \tag{21}
\]

for any real \( q \neq 0 \), with

\[
F(\psi) = 8q(q+1)\psi^{1+(2/q)} \quad , \quad G(\psi) = -2q(2q+1)\psi^{1+(1/q)}. \tag{22}
\]

Let us notice that with the choice (22) for \( F, G \), the GS equation admits the Lie point scaling symmetry

\[
X_1 = r \frac{\partial}{\partial r} + z \frac{\partial}{\partial z} - 2q \psi \frac{\partial}{\partial \psi} \tag{23}
\]

and that the above solution (21) is invariant under this symmetry.

B. A D-shaped equilibrium solution.

The choice \( q = -1/4 \) in (21) is of special interest as it makes it possible to construct a new class of solutions using an exceptional symmetry of the GS equation. We consider the solution

\[
\psi = (r^2 - z^2)^{1/4} = s^{1/4} \tag{24}
\]

which clearly holds only in the region \(|z| \leq r\), and solves the GS equation with

\[
F(\psi) = -(3/2)\psi^{-7} \quad , \quad G(\psi) = (1/4)\psi^{-3} \tag{25}
\]
As shown in Ref. [10] with the above choice for $F$, $G$ and with $q = -1/4$, the GS equation admits also the “exceptional” symmetry (see also Ref. [14])
\[
X_2 = 2rz \frac{\partial}{\partial r} + (z^2 - r^2) \frac{\partial}{\partial z} + z\psi \frac{\partial}{\partial \psi}.
\]
(26)
The presence of this symmetry implies that if $\psi(r, z)$ is a solution of the GS equation, then also
\[
\tilde{\psi}(r, z) = [C(r, z, \lambda)]^{1/2} \psi(\tilde{r}(r, z, \lambda), \tilde{z}(r, z, \lambda))
\]
(27)
where
\[
C(r, z, \lambda) = 1 + \lambda^2(r^2 + z^2) + 2\lambda z
\]
\[
\tilde{r} = r[C(r, z, \lambda)]^{-1} \quad \text{and} \quad \tilde{z} = [z + \lambda(r^2 + z^2)] [C(r, z, \lambda)]^{-1}
\]
is a solution of the equation for any value of the real parameter $\lambda$. Taking advantage from this symmetry, we can construct from the solution (24) a continuous family of solutions (see Ref. [10]): we obtain, for any $\lambda$,
\[
\psi(r, z) = [r^2 - (z + \lambda(r^2 + z^2))^2]^{1/4}
\]
(28)
which holds in the interior of the two circles centered resp. in $r_0 = 1/(2\lambda)$, $z_0 = -1/(2\lambda)$ and $r_0 = -1/(2\lambda)$, $z_0 = -1/(2\lambda)$, both of radius $1/(\sqrt{2}\lambda)$, excluding their intersection (it is not restrictive to assume $\lambda > 0$). Thanks to the obvious invariance of the GS equation under translations $z \rightarrow z + \text{const.}$, we can shift this solution and choose $z_0 = 0$. The resulting “D-shaped” solutions and configurations are shown and fully discussed in Ref. [10].

C. Additional “exceptional symmetry” solutions.

It is possible to generalize the D-shaped solution described above by taking
\[
F(\psi) = a_2\psi^{-7}, \quad G(\psi) = a_0\psi^{-3}
\]
(29)
where $a_0$ and $a_2$ are free real coefficients and by setting
\[
\psi(r, z) = s^{1/4} \phi(y), \quad y = r^2/s \geq 1.
\]
(30)
The variable $y$ is an invariant variable under the symmetry (23) and is a trigonometric function of the “latitude” angle $\theta$, $(1 - 1/y = z^2/r^2 = \tan^2 \theta)$ with $y = 1$ corresponding to
the equatorial \((z = 0)\) plane. For the sake of definiteness we restrict ourselves to positive values of \(a_0\) and conveniently rescale the variable \(\psi\) such that for all cases \(a_0 = 1/4\). Inserting Eq. (30) into Eq. (17) one obtains, thanks to the invariance of Eq. (17) under the scaling symmetry \(X_1\) , an O.D.E., which is given by

\[
(8y^3 - 12y^2 + 4y) \phi'' + (12y^2 - 10y) \phi' + (-3y/2 + 1/4) \phi = \\
= 1/(4 \phi^3) + a_2 y/\phi^7.
\]

For \(a_2 = -3/2\), Eq. (31) is solved by \(\phi \equiv 1\) and returns the result of Sec. III B. For different values of \(a_2\), keeping \(\phi(1) = 1\) as a normalization condition on \(\psi\), Eq. (31) can be easily solved numerically. Note that the exceptional symmetry (26) depends only on the scaling of \(F(\psi)\) and of \(G(\psi)\) on \(\psi\) and not on the specific values of the coefficients \(a_0\) and \(a_2\) (see also Sec. IV A). Then the transformation (27) applies also to the solutions of Eqs. (30, 31) and sends solutions into new solutions of the form

\[
\psi(r, z) = [r^2 - (z + \lambda(r^2 + z^2))^2]^{1/4} \phi(r^2/[r^2 - (z + \lambda(r^2 + z^2))^2]),
\]

which can be rewritten in a geometrically more transparent way as

\[
\psi(r, z)/r^{1/2} = \phi(y_{\lambda}, a_2)/y_{\lambda}^{1/4},
\]

where we have made explicit the dependence of the solution on the parameter \(a_2\), and

\[
y_{\lambda} \equiv r^2/[r^2 - (z + \lambda(r^2 + z^2))^2], \quad 1/y_{\lambda} = 1 - (\tan \theta + \lambda R/ \cos \theta)^2,
\]

with \(R^2 = r^2 + z^2\). Notice that \(y = \text{const.}\) corresponds in the \(r \geq 0\) half-plane to the lines \(r = k |z|\), with \(k > 1\), whereas \(y_{\lambda} = \text{const.}\) corresponds to the borders of the lunes (crescents) defined by the circles

\[
(r \mp (1 - 1/y_{\lambda})^{1/2}/(2\lambda))^2 + z^2 = (2 - 1/y_{\lambda})/(2\lambda)^2 \quad \text{and} \quad r \geq 0,
\]

as shown in Fig. 1 (here the variable \(z\) has been translated, see below Eq. (28)). Note that the shape of these lunes does not depend on the value of \(a_2\) and that the function \(\phi(y_{\lambda}, a_2)/y_{\lambda}^{1/4}\) is constant on their borders. Thus the shape in the \(r-z\) plane of the class of solutions discussed below can be visualized as given by the level curves of \(r^{1/2}\) times a function that depends on \(a_2\) and that is constant on the borders of the lunes in Fig. 1 whose shape depends only on the transformation parameter \(\lambda\).
Figure 1: Lunes with borders given by the circles in Eq.(35) for $\lambda = 1$ and $1 - 1/y_\lambda = 1$, red, $1 - 1/y_\lambda = 1/2$, green and $1 - 1/y_\lambda = 0$, black. The red border is the transformed of the lines $|z| = r$, the black one of $z = 0$. The two (common) vertices of the lunes are the transformed of the origin and of the point at $\infty$.

Noting that the coefficient of $\phi''$ in Eq.(31) vanishes at $y = 1$ and restricting to solutions where $\phi''$ does not diverge, keeping as mentioned above $\phi(1) = 1$, we find that the value of $\phi'$ at $y = 1$ is given by

$$2 \phi'(1) = \frac{3}{2} + a_2.$$  

(36)

Two classes of solutions of Eq.(31) can be identified depending on the value of $a_2$. For $a_2 > -3/2$, and then $\phi'(1) > 0$, the function $\phi$ remains positive over the whole interval $1 \leq y \leq \infty$ and behaves asymptotically as $\phi(y) \propto y^{1/4}$ for $y \to \infty$ i.e., for $z^2 \to r^2$. Note that for $a_2 = -1$, one has the exact solution $\phi = y^{1/4}$ of Eq.(31), which yields the special solution $\psi(r, z) = r^{1/2}$.

For $a_2 < -3/2$, the function $\phi(y)$ reaches zero at a finite value $\bar{y} = \bar{y}(a_2)$. For $y \to \bar{y}$, i.e., for $z^2 \to r^2(1 - 1/\bar{y})$, it behaves locally as $\phi(y) \propto (\bar{y} - y)^{1/4}$ which corresponds to $\psi(r, z) \propto [z^2 - r^2(1 - 1/\bar{y})]^{1/4}$. In this case a complementary solution can be found for $\bar{y} < y < \infty$ which starts as $\phi(y) \propto (y - \bar{y})^{1/4}$ and behaves as $\propto y^{1/4}$ for $y \to \infty$.

The structure in the $r$-$z$ half-plane of the solutions with $a_2 > -3/2$, after being transformed according to Eq.(27), is similar to that of the D-shaped solution obtained in Ref.10 with the magnetic axis shifting towards larger values of $r$ with increasing values of $a_2$. The structure
of the solutions with \( a_2 < -3/2 \) will be discussed in the next section in the frame of the solutions of the GGS equation.

IV. EQUILIBRIA WITH FLOWS

A. Symmetries of the GGS equation

We now consider the case of the generalized GS equation \([13]\).

It is easy to show that the two symmetries \( X_1 \) and \( X_2 \) given above \([23]\) and \([26]\) are still valid: more precisely, the symmetry \( X_1 \) is admitted by any equation of the form

\[
\frac{\partial^2 \chi}{\partial r^2} - \frac{1}{r} \frac{\partial \chi}{\partial r} + \frac{\partial^2 \chi}{\partial z^2} = \sum_{\ell} a_\ell r^\ell \chi^{1+\frac{2+\ell}{2q}}
\]  

(37)

and, similarly, the exceptional symmetry \( X_2 \) is admitted by any equation of the form

\[
\frac{\partial^2 \chi}{\partial r^2} - \frac{1}{r} \frac{\partial \chi}{\partial r} + \frac{\partial^2 \chi}{\partial z^2} = \sum_{\ell} a_\ell r^\ell \chi^{-3-2\ell}
\]  

(38)

where at the r.h.s. any combination of terms \( r^\ell \chi^{1+\frac{2+\ell}{2q}} \) (resp. \( r^\ell \chi^{-3-2\ell} \)) with arbitrary constants \( a_\ell \) and – in principle – any real value of \( \ell \) is admitted

Restricting to the cases relevant to Eq.(13), we can write instead of (37)

\[
\frac{\partial^2 \chi}{\partial r^2} - \frac{1}{r} \frac{\partial \chi}{\partial r} + \frac{\partial^2 \chi}{\partial z^2} = a_0 \chi^{\frac{3}{2}} + a_2 r^2 \chi^{\frac{3}{2}} + a_4 r^4 \chi^{\frac{3}{2}}
\]  

(39)

and finally, with \( q = -1/4, \)

\[
\frac{\partial^2 \chi}{\partial r^2} - \frac{1}{r} \frac{\partial \chi}{\partial r} + \frac{\partial^2 \chi}{\partial z^2} = \frac{a_0}{\chi^{\frac{5}{2}}} + \frac{a_2 r^2}{\chi^{\frac{5}{2}}} + \frac{a_4 r^4}{\chi^{11}}.
\]  

(40)

B. Solutions of the GGS equation

We proceed as in Sec. [III C] by defining

\[
\chi(r, z) = s^{1/4} \phi(y), \quad y = r^2/s \geq 1, \quad \phi(1) = 1,
\]  

(41)

and obtain an O.D.E. (thanks again to the invariance of Eq.(40) under the symmetry \( X_1 \)) which is of the form of Eq.(31) with an additional nonlinear term (as before we put \( a_0 = 1/4 \)):

\[
(8y^3 - 12y^2 + 4y) \phi'' + (12y^2 - 10y) \phi' + (-3y/2 + 1/4) \phi = \]

(42)
\[ = 1/(4\phi^3) + a_2 y/\phi^7 + a_4 y^2/\phi^{11}. \]

As in Sec. (III C) the exceptional symmetry allows us to construct a family of D-shaped equilibria which now include plasma flows

\[ \chi(r, z)/r^{1/2} = \phi(y_\lambda, a_2, a_4)/y_\lambda^{1/4}. \]  

These solutions can again be visualized as the level curves of \( r^{1/2} \) times a function that now depends on both \( a_2 \) and \( a_4 \) and that is constant on the borders of the lunes defined in Sec. (III C)

Note that for \( a_4 \neq 0 \) there is no constant solution \( \phi = 1 \), while Eq.(36) is changed into

\[ 2 \phi'(1) = 3/2 + a_2 + a_4. \]  

Three classes of solutions of Eq.(42) can be identified depending on the values of \( a_2 \) and \( a_4 \) as indicated in Fig.2.

Figure 2: Regions defining the three different types of solutions of Eq.(42) in the \( a_2 \) and \( a_4 \) parameter plane. For \( a_4 = 0 \) see also Sec. (III C)

For \( \phi'(1) = (a_4 + a_2 + 3/2)/2 \geq 0 \) with the additional condition indicated by numerical calculations \( a_4 + \rho(a_2 + 3/2) \geq 0 \) with \( \rho \approx 0.85 \), region \( A \) of Fig.2 the function \( \phi \) remains positive monotonically growing over the whole interval \( 1 \leq y \leq \infty \) and behaves asymptotically as \( \phi(y) \propto y^{1/4} \) for \( y \to \infty \) as in the case without flow.

For \( a_4 + \rho(a_2 + 3/2) \leq 0 \) and \( a_4 < 0 \), region \( B \) of Fig.2, the solution reaches zero at a finite value \( y = \bar{y} \), as shown in Fig.3 in terms of the variable \( 1 - 1/y = z^2/r^2 \). In this case, \( \phi(y) \) behaves locally as \( \phi(y) \propto (\bar{y} - y)^{1/6} \) and, as in the case without flow, a complementary solution can be found for \( \bar{y} < y < \infty \) which starts as \( \phi(y) \propto (y - \bar{y})^{1/6} \) and behaves for
Figure 3: Plot of $\phi(y, 0.5, -1.85)/y^{1/4}$, see Eq.(43), expressed as a function of the variable $1 - 1/y$ whose value determines the lune borders in Fig.1. Part of the solution for $y > \hat{y}$ is not shown.

Figure 4: Plot of $\phi(y, -3.5, 0.5)/y^{1/4}$, versus the variable $1 - 1/y$.

$y \to \infty$ as $\propto y^{1/4}$.

For $a_4 + a_2 + 3/2 < 0$ and $a_4 > 0$, region C of Fig.2, we have a solution which remains strictly positive but does not behave monotonically, and asymptotically grows as $\propto y^{1/4}$, as shown in Fig.4 in terms of the variable $1 - 1/y$. For $a_4 + \rho(a_2 + 3/2) \simeq 0$, there are “separating solutions” which behave asymptotically as $y^{1/12}$. Notice the particular case $a_4 = 0$, $a_2 = -3/2$ where one recovers the known solution $\phi \equiv 1$.

In order to illustrate the different shape in the $r-z$ plane of the solutions belonging to region A and to regions B-C, when transformed according to Eq.(43) and shifted in $z$, we may use simple analytical fitting formulae of the numerical solutions of $\phi(y, a_2, a_4)$. 

\[ \phi(y) = \frac{1}{y^{1/4}} \]

$1 - 1/y$
Figure 5: Shaded contour plot of the solution $\chi(r, z) = r^{1/2} \phi(y_\lambda, -1.8, 0.7)/y_\lambda^{1/4}$ for $\lambda = 1$.

The shaded contour plots of two such solutions are shown in Figs. 5 and 6 over the whole domain within the red lune in Fig. 1. Obviously, these solutions can be restricted to within a reduced domain inside this lune by imposing (conducting) boundary conditions on the $\chi = \text{const.}$ ($\rightarrow \psi = \text{const.}$) border of the chosen domain.

The solution from region $A$ with $a_2 = -1.8$, $a_4 = 0.7$ and $\lambda = 1$ is shown in Fig. 5: it is D-shaped with its magnetic axis shifted toward large $r$ and steep gradients of $\chi$ near the border of its outermost magnetic surface.

The solution from region $C$ with $a_2 = -3.5$, $a_4 = 0.5$ (the same of Fig. 4) and $\lambda = 1$ is shown in Fig. 6. It exhibits three magnetic axes along the toroidal direction. Thus this solution corresponds to a configuration made of three sets of surfaces, each set nested around one of the magnetic axes. Each set can be considered as an independent equilibrium configuration.

The configuration centered at the middle axis is D shaped while the outer configuration has a crescent shape. A similar structure, but with steeper gradients, is obtained for solutions from region $B$. In this latter case the three sets of surfaces are separated by the $\chi = 0$ surface.
V. CONCLUSIONS

In this article we have developed a method based on the Lie symmetries of the GS and of the GGS equations that we have identified by imposing specific relationships between the flux functions of the magnetic flux variable $\psi$ in these equations.

In the case of the GS equation (plasma equilibria without flows) we have extended the class of the D-shaped solutions found in Ref. [10]. Given the numerical solution of an ODE whose general properties have been discussed in Sec. [III C] these solutions can be worked out explicitly in order to obtain the dependence of the plasma pressure and of the toroidal field on the cylindrical coordinate $r$ and $z$ as done in Ref. [10].

In the case of the GGS equation, when treating equilibria with poloidal flows, we have made use of an implicit integral transformation from $\psi$ to a new dependent variable $\chi(\psi)$ that can be applied when the alfvenic Mach number of the poloidal flow is a flux function and is smaller than unity. Although the shape of $\chi$ (and thus of $\psi$) can be found with a relatively minor generalization of the procedure adopted for the GS equation, finding the explicit dependence of the plasma pressure, of the poloidal and toroidal velocity fields and of the toroidal field on $r$ and $z$ in the GGS case requires the solution of a set of nonlinear equations and, in the case of poloidal flows, the inversion of the integral relationship between

Figure 6: Shaded contour plot of the solution $\chi(r, z) = r^{1/2} \phi(y_\lambda, -3.5, 0.5)/y^{1/4}_\lambda$ for $\lambda = 1$. 
\( \phi \) and \( \chi \) once the flux function \( \mathcal{F}(\psi) \) is chosen. The system of Eqs. (14, 15, 16) must in fact be solved for the remaining flux functions to make the functions \( A_0, A_2, A_4 \) compatible with Eq. (39).

![Shaded contour plots](image)

Figure 7: Shaded contourplots of \( \psi(r, z) \) for \( \lambda = 1, a_2 = -3/2 \) and \( a_4 = 0 \) top frame, \( a_4 = 0.1 \) center frame and \( a_4 = -0.1 \) bottom frame.

This aspect of the problem, together with possible restrictions it might bring on the physical range over which the coefficients \( a_2 \) and \( a_4 \) can vary, has not been addressed explicitly in the present article. However if physical boundary conditions are imposed, for example by external conductors, that restrict the solution to the “flat” central domain of e.g., the configuration shown in Fig. 5, that is to say to a domain inside \( \chi = \text{const} \) (and thus \( \psi = \text{const} \)) surfaces within which \( \chi \) is sufficiently larger than zero and the Mach number sufficiently
smaller than one, Eq.(11) can be inverted by a simple perturbation procedure. On the contrary, the inverse power dependence on \( \chi \) of the terms of the r.h.s. of GGS equation (40) implies that the equilibrium quantities, including the flow profiles, will develop large gradients, and/or the Mach number will approach unity, at the borders of the lunes in Fig.1 where \( \chi \to 0 \) (and, for solutions in region B, near the internal \( \chi = 0 \) curves). This can be seen by combining \( d/d\psi = [1 - M^2(\psi)]^{1/2} d/d\chi \) with the \( \chi \) derivatives of the flux functions \( A_0(\chi), A_2(\chi) \) and \( A_4(\chi) \) in Eqs.(14-16).

Furthermore, it is instructive to compare the D shaped equilibrium configuration without flows discussed in Ref.10 with two corresponding equilibria (at fixed \( J(\psi) \), see Eq.(7)) with toroidal flows but without poloidal flows so that the functions \( \chi \) and \( \psi \) coincide. This comparison is shown in Fig.7 where the shaded contour plot of the D-shaped equilibrium of Ref.10 is shown in the top frame (\( a_2 = -3/2, a_4 = 0 \)), that of a corresponding equilibrium with angular velocity increasing with \( \psi (a_2 = -3/2, a_4 > 0) \) is shown in the center frame and one with angular velocity decreasing with \( \psi (a_2 = -3/2, a_4 < 0) \) in the bottom frame. We see that a positive gradient of the centrifugal term due to the toroidal flow (\( a_4 > 0 \)) leads to a shift of the maximum of the flux function \( \psi(r, z) \) towards larger values of \( r \) and to the formation of large gradients at the outer border, as characteristic of the equilibrium solutions in region A (see e.g., Fig.5). On the contrary, a negative gradient of the centrifugal term (\( a_4 < 0 \)) causes \( \psi(r, z) \) to vanish not only at the boundary of the solution domain but also along two curves inside the domain as is the case for the solutions in region C. This splits the solution into “separate” solutions with different magnetic axes, as characteristic of the equilibrium solutions in regions B and C (see e.g., Fig.6). From the bottom frame of Fig.7 we see that the inner solution is, for the chosen parameters, vanishingly thin, the central one has a rather flat radial profile while the outer solution has large gradients.

As a concluding remark we observe that the procedure developed in the present paper can be further extended to a wider class of solutions by adapting it to the different choices of the variable \( s = s(r, z) \) made in Sec.IV of Ref.10.

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31 The “complementary” solution $\psi = (z^2 - r^2)^{1/4}$, see Ref.10 could have also been considered here and in the developments of the following sections. For the sake of brevity this case is not discussed explicitly in the present article.

32 Since $\psi = \psi(\chi)$, as follows from inverting Eq.(11), the shape of the $\psi$ surfaces is the same as that of the $\chi$ surfaces.

33 In Fig.7 the exact numerical solutions of Eq.(31) are used while in Figs.5 and 6 simple analytical fitting formulae are used.