ON GENUS OF DIVISION ALGEBRAS

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Abstract. The genus $\gen(D)$ of a finite-dimensional central division algebra $D$ over a field $F$ is defined as the collection of classes $[D'] \in \Br(F)$, where $D'$ is a central division $F$-algebra having the same maximal subfields as $D$. We show that the fact that quaternion division algebras $D$ and $D'$ have the same maximal subfields does not imply that the matrix algebras $M_l(D)$ and $M_l(D')$ have the same maximal subfields for $l > 1$. Moreover, for any odd $n > 1$, we construct a field $L$ such that there are two quaternion division $L$-algebras $D$ and $D'$ and a central division $L$-algebra $C$ of degree and exponent $n$ such that $\gen(D) = \gen(D')$ but $\gen(D \otimes C) \neq \gen(D' \otimes C)$.

The genus $\gen(D)$ of a finite-dimensional central division algebra $D$ over a field $F$ is defined as the collection of classes $[D'] \in \Br(F)$, where $D'$ is a central division $F$-algebra having the same maximal subfields as $D$. This means that $D$ and $D'$ have the same degree $n$, and a field extension $K/F$ of degree $n$ admits an $F$-embedding $K \hookrightarrow D$ if and only if it admits an $F$-embedding $K \hookrightarrow D'$. Different variations of the notion of the genus are mentioned in [1].

The following questions were formulated in [2, footnote 1 and Remark 2.2]:

Does the fact that division algebras $D$ and $D'$ have the same maximal subfields imply that the matrix algebras $M_l(D)$ and $M_l(D')$ have the same maximal subfields / étale subalgebras for any (or even some) $l > 1$?

Let $n_1$ and $n_2$ be relatively prime positive integers. Let also $D_i$ and $D'_i$ be central division algebras of degree $n_i$ over a field $F$ for $i = 1, 2$. Is it true that if $\gen(D_i) = \gen(D'_i)$ for $i = 1, 2$, then $\gen(D_1 \otimes D_2) = \gen(D'_1 \otimes D'_2)$?

Negative answers to these questions are given in Theorem 5 and Corollary 6 below.

We use the following notation. For a field $F$, $F^*$ denotes the multiplicative group of $F$. $F^{*2}$ denotes the subgroup of squares in $F^*$. For a field extension $K/F$ and a central simple $F$-algebra $A$, $A_K$ denotes the tensor product $A \otimes_F K$ and $\res_{K/F} : \Br(F) \to \Br(K)$ denotes the restriction homomorphism.

Let $F$ be a field, $\text{char}(F) \neq 2$. Assume that there are two non-isomorphic quaternion division $F$-algebras $A$ and $B$. We will be interested in finite separable field extensions $K/F$ that satisfy the following three conditions:

(A) There is $d \in K^* \setminus K^{*2}$ such that there is no $a \in F^*$ such that $da \in K^{*2}$;

(B) $K$ does not split $B$;

(C) $K(\sqrt{d})$ splits $B$ but does not split $A$.

Example 1. Let $M$ be a field containing a primitive $n$th root of 1, $\text{char}(M) \nmid 2n$. Let also $F := M(x, y, z, w)$ be a purely transcendental extension of $M$ of transcendence degree 4 and $A := (x, y)$, $B := (w, y)$ quaternion $F$-algebras. Then $K := F(\sqrt{z})$ is a cyclic extension of $F$ of degree $n$ and $K$ does not split $B$. Finally, let $d := w + y(\sqrt{z} + 1)^2$. 


Since \( K(\sqrt{d}) \) is a purely transcendental extension of \( M(x, y) \), then \( K(\sqrt{d}) \) does not split \( \mathcal{A} \). On the other side, \( K(\sqrt{d}) \) splits \( \mathcal{B} \) since \( d \) is represented over \( K \) by the quadratic form \( < w, y, -yw > \). Finally, note that there is no \( a \in F^* \) such that \( da \in K^{*2} \). Thus the extension \( K/F \) satisfies conditions (A) - (C). Note also that the symbol \( F \)-algebra \((x, z)_n\) of degree and exponent \( n \) is split by \( K \). Moreover, if \( \text{char}(M) = 0 \) and \( M \) contains all roots of 1, then for any \( n > 1 \), the field \( F \) has an extension of degree \( n \) satisfying conditions (A) - (C).

In the notation above, we have the following

**Proposition 2.** Let \( c \in F^*\backslash F^{*2} \). Then there exists a regular field extension \( F_c/F \) such that

1. the homomorphism \( \text{res}_{F_c/F} : \text{Br}(F) \to \text{Br}(F_c) \) is injective;
2. the field \( F_c(\sqrt{c}) \) splits the algebras \( \mathcal{A}_F \) and \( \mathcal{B}_F \).

Moreover, if a field extension \( K/F \) satisfies conditions (A) - (C) and \( c \notin K^{*2} \), then

3. there is no \( a \in F_c \) such that \( da \in F_cK^{*2} \);
4. the composite \( F_cK \) does not split \( \mathcal{B}_F \);
5. the composite \( F_cK(\sqrt{d}) \) splits \( \mathcal{B}_F \) but does not split \( \mathcal{A}_F \).

**Proof.** Let \( F(x) \) be a purely transcendental extension of \( F \) of transcendence degree 1. Let also

\( \mathcal{C} := \mathcal{A}_{F(x)} \otimes (c, x) \)

be a biquaternion \( F(x) \)-algebra and \( F_1 \) the function field of the Severi-Brauer variety of the algebra \( \mathcal{C} \).

Now let \( F_1(y) \) be a purely transcendental extension of \( F_1 \) of transcendence degree 1 and

\( \mathcal{D} := \mathcal{B}_{F_1(y)} \otimes (c, y) \)

a biquaternion \( F_1(y) \)-algebra. Let also \( F_c \) be the function field of the Severi-Brauer variety of the algebra \( \mathcal{D} \).

Since the kernel of the restriction homomorphism \( \text{res}_{F_c/F(x)} \) is generated by the class of the algebra \( \mathcal{C} \), then the homomorphism \( \text{res}_{F_c/F} \) is injective. Indeed, \( \mathcal{C} \) ramifies at the discrete valuation (trivial on \( F \)) of \( F(x) \) defined by the polynomial \( x \). Hence \( [\mathcal{C}] \neq [\mathcal{Q}_{F(x)}] \) for any central simple \( F \)-algebra \( \mathcal{Q} \).

Note that \( F_1 \) splits \( \mathcal{C} \). Then \( \mathcal{A}_{F_1} \cong (c, x)F_1 \), and \( F_1(\sqrt{c}) \) splits \( \mathcal{A}_{F_1} \).

Let \( K/F \) be a field extension satisfying conditions (A) - (C). Since \( F_1/F \) is a regular field extension, then there is no \( a \in F_1 \) such that \( da \in F_1K^{*2} \). In particular, this means that \( dc \notin F_1K^{*2} \). Moreover, \( F_1K/K \) is a regular extension of \( K \). Thus, if \( c \notin K^{*2} \), then \( c \notin F_1K^{*2} \).

The composite \( F_1K(\sqrt{d}) \) is the function field of the Severi-Brauer variety of the \( K(\sqrt{d})(x) \)-algebra \( \mathcal{C}_{K(\sqrt{d})(x)} \). Hence the kernel of the restriction homomorphism \( \text{res}_{F_1K(\sqrt{d})(x)} \) of degree \( n \) of degree and exponent \( n \) is split by \( K \). Moreover, if \( \text{char}(M) = 0 \) and \( M \) contains all roots of 1, then for any \( n > 1 \), the field \( F \) has an extension of degree \( n \) satisfying conditions (A) - (C).
Thus the extension $F_1K/F_1$ satisfies conditions (A)-(C) with respect to the algebras $A_{F_1}$ and $B_{F_1}$.

The field $F_c$ satisfies conditions (1)-(5) of the proposition by the same arguments as for the field $F_1$. We just replace the ground field $F$ by $F_1$ and the extension $K/F$ by $F_1K/F_1$. 

\[\square\]

Remark 3. Conditions (1) and (3)-(5) of Proposition 2 say that if $K/F$ is a field extension satisfying conditions (A)-(C), then the extension $F_cK/F_c$ satisfies conditions (A)-(C) with respect to the algebras $A_{F_c}$ and $B_{F_c}$.

In the notation above, we also have the following

\[\text{Proposition 4. Let } U := \{c \in F^* \setminus F_{\sqrt{c}}^2 \mid F(\sqrt{c}) \text{ splits } A \text{ or } B\}. \text{ There exists a regular field extension } E(F)/F \text{ such that} \]

(1) the homomorphism $\text{res}_{E(F)/F} : \text{Br}(F) \to \text{Br}(E(F))$ is injective;

(2) the field $E(F)(\sqrt{c})$ splits the algebras $A_{E(F)}$ and $B_{E(F)}$ for any $c \in U$.

Moreover, if a field extension $K/F$ satisfies conditions (A) - (C), then

(3) there is no $a \in E(F)$ such that $da \in E(F)K^{*2}$;

(4) the composite $E(F)K$ does not split $B_{E(F)}$;

(5) the composite $E(F)K(\sqrt{d})$ splits $B_{E(F)}$ but does not split $A_{E(F)}$.

\[\text{Proof. Note that for any field extension } K/F \text{ satisfying conditions (A) - (C), } U \cap K^{*2} = \emptyset \text{ since } K \text{ does not split } A \text{ and } B. \]

Let $<$ be a well-ordering on the set $U$ and let $c_0$ denote its least element. Set $E^{\alpha_0} := F_{c_0}$, where the field $F_{c_0}$ is constructed in Proposition 2.

For $c \in U$, $c \neq c_0$, set

$$E^{<c} := \bigcup_{c' < c} E^{c'} \text{ and } E^c := E^{<c},$$

where the field $E^c$ is obtained by applying Proposition 2 to the field $E^{<c}$ and the element $c \in E^{<c}$ and the algebras $A_{E^{<c}}$ and $B_{E^{<c}}$. Define also $E(F) := \bigcup_{c \in U} E^c$.

By Proposition 2 and transfinite induction, the field $E(F)$ satisfies conditions (1)-(5) of the proposition. \[\square\]

\[\text{Theorem 5. Let } F \text{ be a field such that there are two non-isomorphic quaternion } F\text{-algebras } A \text{ and } B. \text{ There exists a regular field extension } L/F \text{ with the following properties:} \]

(1) $A_L$ and $B_L$ are division algebras and $\text{gen}(A_L) = \text{gen}(B_L)$;

(2) If $K/F$ is a field extension of degree $n$ satisfying properties (A) - (C) with respect to the algebras $A$ and $B$, then the matrix algebras $M_n(A_L)$ and $M_n(B_L)$ do not have the same maximal subfields;

(3) If $K$ is a field from the previous item and $C$ is a central division $F$-algebra of exponent $n$ which is split by $K$, then $C_L$ is a division algebra of exponent $n$ and the algebras $A_L \otimes C_L$ and $B_L \otimes C_L$ do not have the same maximal subfields.

\[\text{Proof. Let } K_0 := F. \text{ We recursively define } K_i, i \in \mathbb{Z}_{>0}, \text{ to be the field } E(K_{i-1}) \text{ constructed by applying Proposition 4 to the field } K_{i-1} \text{ and the algebras } A_{K_{i-1}} \text{ and } B_{K_{i-1}}. \text{ Let also } L := \bigcup_{i \geq 0} K_i. \]
By induction and Proposition 4, the homomorphism \( res_{L/F} : Br(F) \rightarrow Br(L) \) is injective. Hence \( \mathcal{A}_L \) and \( \mathcal{B}_L \) are non-isomorphic division algebras.

Assume that \( M \) is a maximal subfield of \( \mathcal{A}_L \). Then there exists \( i \geq 0 \) such that \( M = L M' \), where \( M' \) is a quadratic extension of \( K_i \) that splits \( \mathcal{A}_{K_i} \). By the construction of \( K_{i+1} \), the field \( K_{i+1} M' \) splits the algebra \( \mathcal{B}_{K_{i+1}} \). Hence \( M \) splits \( \mathcal{B}_L \). Analogously, every maximal subfield of \( \mathcal{B}_L \) splits \( \mathcal{A}_L \) and the algebras \( \mathcal{A}_L \) and \( \mathcal{B}_L \) have the same family of maximal subfields, i.e., \( \text{gen}(\mathcal{A}_L) = \text{gen}(\mathcal{B}_L) \).

Assume that \( K/F \) is a field extension of degree \( n \) satisfying conditions (A) - (C) with respect to the algebras \( \mathcal{A} \) and \( \mathcal{B} \). By induction and Proposition 4, the composite \( L K(\sqrt{d}) \) splits \( \mathcal{B}_L \) but does not split \( \mathcal{A}_L \). Then \( L K(\sqrt{d}) \) embeds in \( M_n(\mathcal{B}_L) \) but does not embed in \( M_n(\mathcal{A}_L) \). Hence \( M_n(\mathcal{A}_L) \) and \( M_n(\mathcal{B}_L) \) do not have the same maximal subfields.

Finally, let \( C \) be a central division \( F \)-algebra of exponent \( n \) which is split by \( K \). Since the homomorphism \( res_{L/F} \) is injective, then the exponent of \( C_L \) is \( n \). Since the exponent of \( C_L \) divides its index, then \( C_L \) is a division algebra. The composite \( L K(\sqrt{d}) \) splits \( \mathcal{B}_L \otimes C_L \) but does not split \( \mathcal{A}_L \otimes C_L \). This means that \( \mathcal{A}_L \otimes C_L \) and \( \mathcal{B}_L \otimes C_L \) do not have the same maximal subfields.

**Corollary 6.** There exists a field \( L \) such that there are two quaternion division \( L \)-algebras \( \mathcal{D} \) and \( \mathcal{D}' \) such that \( \text{gen}(\mathcal{D}) = \text{gen}(\mathcal{D}') \), but for any \( n > 1 \) the matrix algebras \( M_l(\mathcal{D}) \) and \( M_l(\mathcal{D}') \) do not have the same maximal subfields.

**Proof.** Let \( F \) be a field such that there are two non-isomorphic quaternion \( F \)-algebras \( \mathcal{A} \) and \( \mathcal{B} \) and for any \( n > 1 \), the field \( F \) has an extension of degree \( n \) satisfying conditions (A) - (C) with respect to the algebras \( \mathcal{A} \) and \( \mathcal{B} \). Let \( L \) be the field constructed in Theorem 5. By Theorem 5, the algebras \( \mathcal{D} := \mathcal{A}_L \) and \( \mathcal{D}' := \mathcal{B}_L \) have the required properties.

**References**

[1] V.I. Chernousov, A.S. Rapinchuk, I.A. Rapinchuk, *Division algebras with the same maximal subfields*, Russian Math. Surveys 70:1 (2015), 83-112.

[2] V.I. Chernousov, A.S. Rapinchuk, I.A. Rapinchuk, *The finiteness of the genus of a finite-dimensional division algebra, and some generalizations*, http://arxiv.org/abs/1802.00299

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