On separable states for composite systems of distinguishable fermions

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Abstract

We consider separable (i.e. classically correlated) states for composite systems of spinless fermions that are distinguishable. For a proper formulation of entanglement formation for such systems, the state decompositions to be taken should respect the univalence superselection rule. Fermion hopping always induces non-separability, while states with bosonic hopping correlation may or may not be separable. If we transform a given bipartite fermion system into a tensor product one by Jordan-Klein-Wigner transformation, any separable state for the former is also separable for the latter. There are U(1) gauge invariant states that are non-separable for the former but separable for the latter.

Key Words: CAR systems. Classically correlated (separable) states. Univalence superselection rule.

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1 Introduction

We consider characterization of separable, i.e. classically correlated states for lattice fermion systems where fermion particles on different sites are distinguishable. (For the case of indistinguishable fermions that are represented as anti-symmetric wave functions, see e.g. [1] and its references.)

Let \( N \) be a lattice of integers ordered by inclusion. The canonical anticommutation relations (CARs) are

\[
\begin{align*}
\{a_i^\dagger, a_j\} &= \delta_{i,j} 1, \\
\{a_i^\dagger, a_j^\dagger\} &= \{a_i, a_j\} = 0, \quad i, j \in \mathbb{N},
\end{align*}
\]

where \( a_i^\dagger \) and \( a_i \) are creation and annihilation spinless fermion operators on the site \( i \), and \( \{A, B\} = AB + BA \). For each subset \( I \) of \( \mathbb{N} \), the subsystem \( \mathcal{A}(I) \) is generated by all \( a_i^\dagger \) and \( a_i \) in \( I \).

Let \( I \) and \( J \) be disjoint subsets of \( \mathbb{N} \). We are interested in characterization of state correlations between the pair of subsystems \( \mathcal{A}(I) \) and \( \mathcal{A}(J) \). We shall comment on our motivation. It is sometimes useful to convert the argument for quantum spin models to that for the corresponding fermion lattice models by Jordan-Klein-Wigner transformations vice versa in quantum statistical mechanics. We hope that the comparison of tensor product systems and CAR systems in terms of state correlations would be useful for some purpose, though do not have any practical suggestion. Also this work is a sort of continuation of [7] that studied the independence of states for CAR systems.

We give notation. The even-odd grading transformation is given by

\[
\Theta(a_i^\dagger) = -a_i^\dagger, \quad \Theta(a_i) = -a_i.
\]

The even and odd parts of \( \mathcal{A}(I) \) are

\[
\mathcal{A}(I)_{±} = \left\{ A \in \mathcal{A}(I) \mid \Theta(A) = ±A \right\}.
\]

We introduce U(1) gauge transformation:

\[
\gamma_\theta(a_i^\dagger) = e^{i\theta}a_i^\dagger, \quad \gamma_\theta(a_i) = e^{-i\theta}a_i
\]

for \( \theta \in \mathbb{C} \). A state that is invariant under \( \Theta \) is called even, and a state that is invariant under \( \gamma_\theta \) for any \( \theta \in \mathbb{C} \) is called U(1)-gauge invariant.

If the cardinality \( |I| \) is finite, then \( \mathcal{A}(I) \) is isomorphic to the \( 2^{|I|} \times 2^{|I|} \) full matrix algebra. Let

\[
v_I := \prod_{i \in I} v_i, \quad v_i := a_i^\dagger a_i - a_i a_i^\dagger.
\]

This \( v_I \) gives an even self-adjoint unitary operator implementing \( \Theta \),

\[
\text{Ad}(v_I)(A) = \Theta(A), \quad A \in \mathcal{A}(I).
\]
The notion of separable states is unchanged for CAR systems: If a state is written as a convex sum of product states, then it is called a separable state [2]. It is, however, important to note that due to the CAR structure (algebraic non-independence) there are limitations on marginal states that can be prepared on disjoint regions [7] and hence on product states.

According to the univalence superselection rule [3], any realizable state is $\Theta$-invariant. Thus noneven states are out of our physical interest. However, any even state has noneven-state decompositions (i.e. state decompositions in which there are noneven component states) unless it is pure. For a natural formulation of entanglement formation for even states of CAR systems, the state decompositions should be taken from the even-state space only, not from the whole state space. Such quantity now called the entanglement formation under the univalence superselection rule is zero if and only if the given even state is separable (Proposition 4). (Later we provide another definition of entanglement formation for CAR systems that works for non-even states as well but seems not so natural.)

We compare fermion systems to tensor product systems in terms of state correlations. For fermion systems any particle hopping term between disjoint subsystems always induces non-separability (Proposition 1), while for tensor product systems, states with particle hopping correlation may or may be separable. We show that any separable state for the CAR pair $(\mathcal{A}(I), \mathcal{A}(J))$ is also separable for the tensor product pair $(\mathcal{A}(I), \mathcal{A}(I)' )$, where $\mathcal{A}(I)'$ denotes the commutant of $\mathcal{A}(I)$ in $\mathcal{A}(I \cup J)$ (Proposition 3). It was already noted in [5] that the set of all separable states for the CAR pair is strictly smaller than that for the tensor product pair. We reproduce this result by our model independent argument, which seems to have some merits. First the statement is valid for the infinite-dimensional case as well. Second it is clarified that fermionic correlation due to particle hopping is responsible for this strict inclusion which is realized in $\text{U}(1)$-gauge invariant state space as shown in § 4 by examples.

In § 5 we consider the general case including noneven states and provide a criterion of separability (Proposition 6).

## 2 Separability condition for bipartite fermion systems

We give a definition of separability for fermion systems. Let $I$ and $J$ be mutually disjoint subsets of $\mathbb{N}$, and $\omega$ be a (not necessarily even) state on $\mathcal{A}(I \cup J)$. We denote the restriction of $\omega$ to $\mathcal{A}(I)$ ($\mathcal{A}(J)$) by $\omega_1$ ($\omega_2$). Conversely, we are given a pair of states $\omega_1$ on $\mathcal{A}(I)$ and $\omega_2$ on $\mathcal{A}(J)$. If there exists a state $\omega$ on the total system $\mathcal{A}(I \cup J)$ such that its restriction to $\mathcal{A}(I)$ is equal to $\omega_1$ and that to $\mathcal{A}(J)$ is $\omega_2$, then $\omega$ is called a state extension of $\omega_1$ and $\omega_2$. If

$$\omega(A_1A_2) = \omega_1(A_1)\omega_2(A_2)$$

(6)

for all $A_1 \in \mathcal{A}(I)$ and $A_2 \in \mathcal{A}(J)$, then such $\omega$ is unique and called the product state extension of $\omega_1$ and $\omega_2$ denoted $\omega_1 \circ \omega_2$. The product property in the
Thus \(\omega\) are noneven. Hence there are odd elements \(A\)
that \(A\) is non-zero real. On the other hand, \(A\) exist a set of states \(\{\omega_{1,i}\}\) on \(A(I)\), also that \(\{\omega_{2,i}\}\) on \(A(J)\), and some positive numbers \(\{\lambda_i\}\) such that \(\sum_i \lambda_i = 1\), satisfying that

\[
\omega(A_1A_2) = \sum_i \lambda_i \omega_{1,i} \circ \omega_{2,i}(A_1A_2)
\]

for any \(A_1 \in A(I)\) and \(A_2 \in A(J)\). This formula requires the existence of the product state \(\omega_{1,i} \circ \omega_{2,i}\) for each pair of \(\omega_{1,i}\) and \(\omega_{2,i}\). For tensor product systems, the existence of product state extension for any given states on disjoint subsystems is automatic, while for fermion systems it is not always the case [7].

**Proposition 1.** Let \(I\) and \(J\) be a pair of disjoint subsets and \(\omega\) be a state on \(A(I \cup J)\). If \(\omega\) is a separable state for \(A(I)\) and \(A(J)\), then for any \(A_{1-} \in A(I)_-\) and \(A_{2-} \in A(J)_-\),

\[
\omega(A_{1-}A_{2-}) = 0.
\]

If \(\omega\) is a product state, then at least one of its restrictions to \(A(I)\) and \(A(J)\) is even.

**Proof.** First we show the second statement. Let \(\omega\) be a product state with its marginal states \(\omega_1\) on \(A(I)\) and \(\omega_2\) on \(A(J)\). Now suppose that both \(\omega_1\) and \(\omega_2\) are noneven. Hence there are odd elements \(A_{1-} \in A(I)_-\) and \(A_{2-} \in A(J)_-\) such that \(\omega_1(A_{1-}) \neq 0\) and \(\omega_2(A_{2-}) \neq 0\). We are going to derive the contradiction. By the assumed product property,

\[
\omega_1 \circ \omega_2(A_{1-}A_{2-}) = \omega_1(A_{1-})\omega_2(A_{2-}) \neq 0.
\]

Both \(A_{1-} + A_{1-}^\dagger\) and \(i(A_{1-} - A_{1-}^\dagger)\) are self-adjoint elements in \(A(I)_-\). Since \(A_{1-}\) can be written as their linear combination, the expectation value of at least one of them for \(\omega_1\) must be non-zero. Thus we can take \(A_{1-} = A_{1-}^\dagger \in A(I)_-\) such that \(\omega_1(A_{1-}) \neq 0\) and similarly \(A_{2-} = A_{2-}^\dagger \in A(J)_-\) such that \(\omega_2(A_{2-}) \neq 0\).

Now both \(\omega_1(A_{1-})\) and \(\omega_2(A_{2-})\) are non-zero real, hence \(\omega_1(A_{1-})\omega_2(A_{2-})\) is non-zero real. On the other hand, \(A_{1-}A_{2-}\) is skew self-adjoint as

\[
(A_{1-}A_{2-})^\dagger = A_{2-}^\dagger A_{1-} = A_{2-}A_{1-} = -A_{1-}A_{2-}.
\]

Thus \(\omega_1 \circ \omega_2(A_{1-}A_{2-})\) must be purely imaginary, which is a contradiction.

We assume that \(\omega\) is a separable state. By definition, \(\omega\) has a decomposition into the affine sum of product states:

\[
\omega = \sum_i \lambda_i \omega_{1,i} \circ \omega_{2,i}.
\]
Suppose that there exist $A_1^- \in \mathcal{A}(I)_-$ and $A_2^- \in \mathcal{A}(J)_-$ such that

$$\omega(A_1^- A_2^-) \neq 0.$$  

Then there exists some product state $\omega_{1,i} \circ \omega_{2,i}$ in the decomposition such that

$$\omega_{1,i} \circ \omega_{2,i}(A_1^- A_2^-) \neq 0.$$  

But this is impossible. Our assertion is now proved.  

For a given symmetry $G$, there may exist $G$-invariant separable states which have no separable decomposition that consists of all $G$-invariant product states [9], for example, $U(1)$-symmetry. The next proposition shows the nonexistence of such separable states for $\Theta$-symmetry.

**Proposition 2.** Let $I$ and $J$ be a pair of disjoint subsets and $\omega$ be an even state on $\mathcal{A}(I \cup J)$. If $\omega$ is a separable state for $\mathcal{A}(I)$ and $\mathcal{A}(J)$, then it has a separable decomposition

$$\omega = \sum_i \lambda_i \omega_{1,i} \circ \omega_{2,i},$$  

such that $\lambda_i > 0$, $\sum_i \lambda_i = 1$, and all the marginal states $\omega_{1,i}$ on $\mathcal{A}(I)$ and $\omega_{2,i}$ on $\mathcal{A}(J)$ are even.

If $I$ and $J$ are finite subsets, all $\omega_{1,i}$ and $\omega_{2,i}$ above can be taken from the set of pure even states.

**Proof.** Let $\omega = \sum_i \lambda_i \omega_i$ where $\omega_i := \omega_{1,i} \circ \omega_{2,i}$, $\omega_{1,i}$ and $\omega_{2,i}$ are some states on $\mathcal{A}(I)$ and $\mathcal{A}(J)$. We shall show that all $\omega_{1,i}$ and $\omega_{2,i}$ can be taken from pure even states.

By Proposition 1 at least one of $\omega_{1,i}$ and $\omega_{2,i}$ should be even for the existence of the product state $\omega_{1,i} \circ \omega_{2,i}$. For a given state $\psi$ let $\hat{\psi}$ denote its $\Theta$-averaged state $\frac{1}{2^{\#I}} \hat{\psi}$. By the evenness of $\omega$, we have the following identity:

$$\omega = \hat{\omega} = \sum_i \lambda_i \hat{\omega}_i.$$  

For each $i$, $\hat{\omega}_i$ is an even product state for $\mathcal{A}(I)$ and $\mathcal{A}(J)$ because $\hat{\omega}_i = \hat{\omega}_{1,i} \circ \hat{\omega}_{2,i}$. Replacing $\omega_{1,i}$ and $\omega_{2,i}$ by $\hat{\omega}_{1,i}$ and $\hat{\omega}_{2,i}$, we obtain a separable decomposition for $\omega$ consisting of all even states.

For a finite dimensional CAR system, every even state can be decomposed into an affine sum of pure even states. Hence if $I$ is finite, we have $\omega_{1,i} = \sum_{(j)} l_{(j)} \omega_{1,i(j)}$, where $l_{(j)} > 0$, $\sum_{(j)} l_{(j)} = 1$ and each $\omega_{1,i(j)}$ is a pure even state of $\mathcal{A}(I)$. Similarly, $\omega_{2,i} = \sum_k l_{(k)} \omega_{2,i(k)}$, where $l_{(k)} > 0$, $\sum_{(k)} l_{(k)} = 1$ and each $\omega_{2,i(k)}$ is a pure even state of $\mathcal{A}(J)$. Hence we have an even-pure-state decomposition $\omega_{1,i} \circ \omega_{2,i} = \sum_{(j,k)} \omega_{1,i(j)} \omega_{2,i(k)}$ for each $i$. Those for
all indexes induce a desired decomposition of $\omega$.

For the second statement of this proposition, the assumption that $I$ and $J$ are finite subsets is necessary since there is an even state that is pure on $\mathcal{A}(I)_+$ but non-pure on $\mathcal{A}(I)$ when $|I|$ is infinite [8].

Remark 1: Examples of bosonic $U(1)$-gauge invariant separable states that cannot be prepared locally under the $U(1)$-gauge symmetry are given in the above mentioned reference [9]. We now consider the lattice-fermionic counterpart of Example 1 (eq.4) given there. Let $|0\rangle$ and $|1\rangle$ be the unit vector denoting the absence and the presence of one-fermion particle. Let two disjoint subsystems under consideration be indicated by $A$ and $B$. Let

$$\rho_1 := \frac{1}{4} \left( |0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B + |1\rangle_A |0\rangle_B + |0\rangle_A |1\rangle_B \right) + 1/2 |\Psi^+\rangle_{AB} |\Psi^+\rangle, \quad (12)$$

where $|\Psi^+\rangle_{AB} := \frac{1}{\sqrt{2}} (|0\rangle_A |1\rangle_B + |1\rangle_A |0\rangle_B)$. Let $|a_{1,2}\rangle := \frac{1}{\sqrt{2}} (|0\rangle \pm |1\rangle)$ and $|a_{3,4}\rangle := |b_{3,4}\rangle := \frac{1}{\sqrt{2}} (|0\rangle \pm i|1\rangle)$, where $a$ and $b$ indicate that the states are of $A$ and $B$, respectively, and the subscripts 1 and 2 correspond to + and −, respectively.

For the bosonic case, $\rho_1$ is separable since it has its separable decomposition:

$$\rho_1 = \sum_{k=1}^{4} |a_k\rangle \langle a_k| \otimes |b_k\rangle \langle b_k|.$$

For the fermionic case, $\rho_1$ is nonseparable. Note that the notation $|a_k\rangle \langle a_k| \otimes |b_k\rangle \langle b_k|$ ($k = 1, 2, 3, 4$) makes no sense (even mathematically), because there is no product state extension for $|a_k\rangle |a_k\rangle$ on $A$ and $|b_k\rangle |b_k\rangle$ on $B$ that are both noneven states. (In fact there is no state extension at all for them by Theorem 1 (2) of [7].) Furthermore, Proposition 1 claims the nonexistence of separable decomposition of $\rho_1$ due to the particle hopping correlation by $|\Psi^+\rangle_{AB} |\Psi^+\rangle$.

Let $\mathcal{A}(I)' (\mathcal{A}(J)')$ denote the commutant algebra of $\mathcal{A}(I) (\mathcal{A}(J))$ in $\mathcal{A}(I \cup J)$. If the cardinality $|I|$ of $I$ is infinite, $\mathcal{A}(I)' = \mathcal{A}(J)_{+}$. If $|I|$ is finite, $\mathcal{A}(I)' = \mathcal{A}(J)_+ + v_{I} \mathcal{A}(J)_-$ and $\mathcal{A}(I \cup J) = \mathcal{A}(I) \otimes \mathcal{A}(I)'$ hold. As is well known, the CAR pair $(\mathcal{A}(I), \mathcal{A}(J))$ is transformed to the tensor product pair $(\mathcal{A}(I), \mathcal{A}(I)')$ and to $(\mathcal{A}(J), \mathcal{A}(J)')$ by Jordan-Klein-Wigner transformations. We consider how the properties of state correlation (separability, entanglement degrees, etc) will remain or change by the replacement of the CAR pair by the tensor-product ones, and vice versa. The following proposition shows that the separability condition for the CAR pair always implies that for the tensor product pair for even states. We have noted in Remark 1 that the converse of this proposition does not hold. Later in Proposition 8 we will see that the evenness assumption is unnecessary. We now provide the simple proof that makes use of the evenness assumption.

Proposition 3. Let $I$ and $J$ be a pair of disjoint subsets and $\omega$ be an even state on $\mathcal{A}(I \cup J)$. If it is separable for the CAR pair $\mathcal{A}(I)$ and $\mathcal{A}(J)$, then so it is for the tensor product pair $\mathcal{A}(I)$ and $\mathcal{A}(I)'$.
Proof. Since \( \omega \) is an even separable state, it has a separable decomposition in the form of (11) where each \( \omega_{1,i} \) and \( \omega_{2,i} \) is even. By CARs and the evenness of \( \omega_{1,i} \) and \( \omega_{2,i} \), we verify that \( \omega_{1,i} \circ \omega_{2,i} \) is a product state with respect to the tensor product pair \( \mathcal{A}(I) \) and \( \mathcal{A}(I)' \). Hence the separability of \( \omega \) for the pair \( (\mathcal{A}(I), \mathcal{A}(I)') \) follows.

3 The entanglement formation under the univalence superselection rule

We introduce a quantity that measures non-separability of even states between \( \mathcal{A}(I) \) and \( \mathcal{A}(J) \) for disjoint finite subsets \( I \) and \( J \). The von Neumann entropy of the density matrix \( D \) is given by

\[
-\text{Tr}(D \log D),
\]

(13)

where \( \text{Tr} \) denotes the trace which takes the value 1 on each minimal projection. The von Neumann entropy of a state \( \omega \) is given by (13) for its density matrix with respect to \( \text{Tr} \) and is denoted \( S(\omega) \).

For even state \( \omega \) of \( \mathcal{A}(I \cup J) \), we define

\[
E_{\Theta}^{(\omega)}(\omega, \mathcal{A}(I), \mathcal{A}(J)) := \inf_{\omega = \sum \lambda_i \omega_i} \sum \lambda_i S(\omega_i |_{\mathcal{A}(I)}),
\]

(14)

where the infimum is taken over all even-state decompositions of \( \omega \). Namely, each \( \omega_i \) is an even state on \( \mathcal{A}(I \cup J) \). We shall call this quantity entanglement formation under the univalence superselection rule. From [10] it follows that

\[
S(\omega_i |_{\mathcal{A}(I)}) = S(\omega_i |_{\mathcal{A}(J)}) = S(\omega_i |_{\mathcal{A}(I)_+}) = S(\omega_i |_{\mathcal{A}(J)_+}).
\]

(15)

Thus the subsystem in the r.h.s of (14) can be any of \( \mathcal{A}(I) \), \( \mathcal{A}(J) \), \( \mathcal{A}(I)_+ \) and \( \mathcal{A}(J)_+ \). We give a criterion of the separability between the CAR pair \( \mathcal{A}(I) \) and \( \mathcal{A}(J) \) in terms of this degree.

Proposition 4. Let \( I \) and \( J \) be finite disjoint subsets and \( \omega \) be an even state of \( \mathcal{A}(I \cup J) \). It is a separable state for \( \mathcal{A}(I) \) and \( \mathcal{A}(J) \) if and only if its entanglement formation under the univalence superselection rule \( E_{\Theta}^{(\omega)}(\omega, \mathcal{A}(I), \mathcal{A}(J)) \) is equal to zero.

Proof. If an even state \( \omega \) satisfies the separability condition, then by Proposition 2 there exists a product-state decomposition

\[
\omega(A_1A_2) = \sum_i \lambda_i \omega_{1,i} \circ \omega_{2,i}(A_1A_2)
\]

(16)

such that each of \( \omega_{1,i} \) and \( \omega_{2,i} \) is even and pure. Thus \( E_{\Theta}^{(\omega)}(\omega, \mathcal{A}(I), \mathcal{A}(J)) = 0 \) by definition. The converse direction is easily verified. 

4 Non-separable for the CAR pair but separable for the tensor product pair

We construct some U(1)-gauge invariant states that are separable for the tensor product pair \((A(I), A(I'))\) but non-separable for the CAR pair \((A(I), A(J))\). As will be specified below, their non-separability is purely due to fermion hopping terms.

Let \(\tau\) be the tracial state on \(A(I \cup J)\). We note the following product properties of the tracial state:

\[
\tau(A_1 A_2) = \tau(A_1) \tau(A_2),
\]

(17)

for every \(A_1 \in A(I)\) and \(A_2 \in A(J)\), and

\[
\begin{align*}
\tau(A_1 B_2) &= \tau(A_1) \tau(B_2), \\
\tau(B_1 A_2) &= \tau(B_1) \tau(A_2),
\end{align*}
\]

(18)

for every \(A_1 \in A(I)\), \(B_2 \in A(I')\), and every \(B_1 \in A(J)'\), \(A_2 \in A(J)\).

Let \(K_1\) and \(K_2\) be odd elements in \(A(I)\) and in \(A(J)\). Typically those are field operators on specified regions. Let \(K := 1/2(K_1^\dagger K_2 - K_1 K_2^\dagger)\), which is self-adjoint and may represent fermion hopping. Suppose that \(\|K_1\| \leq 1\), \(\|K_2\| \leq 1\), then \(\|K\| \leq 1\). For \(\lambda \in \mathbb{R}\), define

\[
P(\lambda) := \text{id} + \lambda K.
\]

(19)

By definition \(P(\lambda)\) is self-adjoint, and by

\[
\|\lambda K\| \leq |\lambda|,
\]

it is a positive operator if \(|\lambda| \leq 1\). From (17) and the evenness of the tracial state it follows that

\[
\tau(P(\lambda)) = \tau(\text{id} + \lambda K) = \tau\left(\text{id} + \frac{\lambda}{2}(K_1^\dagger K_2 - K_1 K_2^\dagger)\right)
\]

\[
= \tau(\text{id}) + \frac{\lambda}{2} (\tau(K_1^\dagger K_2) - \tau(K_1 K_2^\dagger))
\]

\[
= \tau(\text{id}) + \frac{\lambda}{2} (\tau(K_1^\dagger) \tau(K_2) - \tau(K_1) \tau(K_2^\dagger))
\]

\[
= \tau(\text{id}) + \frac{\lambda}{2} \cdot 0 = 1.
\]

(20)

Hence for \(\lambda \in \mathbb{R}, |\lambda| \leq 1\), \(P(\lambda)\) is a density matrix with respect to the tracial state \(\tau\). Let us define the state \(\varphi_\lambda\) on \(A(I \cup J)\) by

\[
\varphi_\lambda(A) := \tau(P(\lambda)A), \quad A \in A(I \cup J).
\]

(21)
By definition,

$$\Theta(P(\lambda)) = P(\lambda),$$

hence $\varphi_\lambda$ is an even state of $\mathcal{A}(I \cup J)$.

We now compute the expectation value of $\varphi_\lambda$ for the product element $A_1 A_2$ of $A_1 \in \mathcal{A}(I)$ and $A_2 \in \mathcal{A}(J)$. We have

$$\tau \left( (K_1^\dagger K_2) A_1 A_2 \right) = \tau \left( K_1^\dagger (K_2 A_1) A_2 \right) = \tau \left( K_1^\dagger (\Theta(A_1)) (K_2 A_2) \right) = \tau \left( (K_1^\dagger \Theta(A_1)) \tau(K_2 A_2) \right) = \tau \circ \Theta \left( \Theta(K_1^\dagger) A_1 \right) \tau(K_2 A_2) = \tau \left( (-K_1^\dagger) A_1 \right) \tau(K_2 A_2) = -\tau(K_1^\dagger A_1) \tau(K_2 A_2),$$

and similarly

$$\tau \left( (K_1 K_2^\dagger) A_1 A_2 \right) = -\tau(K_1 A_1) \tau(K_2^\dagger A_2),$$

where we have used CARs, (5), (17), and $\tau = \tau \circ \Theta$ which follows from the uniqueness of the tracial state. Thus we obtain

$$\varphi_\lambda(A_1 A_2) = \tau(A_1 A_2) - \frac{\lambda}{2} \left( \tau(K_1^\dagger A_1) \tau(K_2 A_2) - \tau(K_1 A_1) \tau(K_2^\dagger A_2) \right).$$

Since the tracial state is an even product state and $K_1 \in \mathcal{A}(I^-), K_2 \in \mathcal{A}(J^-)$, writing $A_1 = A_{1+} + A_{1-}, A_{1\pm} \in \mathcal{A}(I), A_2 = A_{2+} + A_{2-}, A_{2\pm} \in \mathcal{A}(J), \pm$, we obtain

$$\varphi_\lambda(A_1 A_2) = \tau(A_{1+})(A_{2+}) - \frac{\lambda}{2} \left( \tau(K_1^\dagger A_{1-}) \tau(K_2 A_{2-}) - \tau(K_1 A_{1-}) \tau(K_2^\dagger A_{2-}) \right).$$

Similarly we have

$$\varphi_\lambda(A_2 A_1) = \tau(A_1 A_2) + \frac{\lambda}{2} \left( \tau(K_1^\dagger A_1) \tau(K_2 A_2) - \tau(K_1 A_1) \tau(K_2^\dagger A_2) \right) = \tau(A_{1+})(A_{2+}) + \frac{\lambda}{2} \left( \tau(K_1^\dagger A_{1-}) \tau(K_2 A_{2-}) - \tau(K_1 A_{1-}) \tau(K_2^\dagger A_{2-}) \right).$$

We summarize the above computations as follows.
Proposition 5. The state \( \varphi_\lambda \) given by the density \( P(\lambda) := id + i\lambda K \) with \( \lambda \in \mathbb{R}, |\lambda| \leq 1 \), \( K := 1/2(K_1^1K_2 - K_1K_2^1) \), \( K_1 \in \mathcal{A}(I) \) and \( K_2 \in \mathcal{A}(J) \) such that \( \|K_1\| \leq 1 \) and \( \|K_2\| \leq 1 \), has the following correlation functions:

\[
\varphi_\lambda(A_1A_2)=\tau(A_1+)\langle A_2+\rangle - \frac{\lambda}{2}(\tau(K_1^1A_1-\rangle\tau(K_2A_2-\rangle - \tau(K_1A_1-\rangle\tau(K_2^1A_2-\rangle),
\]

\[
\varphi_\lambda(A_2A_1)=\tau(A_1+)\langle A_2+\rangle + \frac{\lambda}{2}(\tau(K_1^1A_1-\rangle\tau(K_2A_2-\rangle - \tau(K_1A_1-\rangle\tau(K_2^1A_2-\rangle). \tag{22}
\]

Let us recall a well known criterion of separability for tensor product systems in [11]: A state is separable for a bipartite tensor product system \( \mathcal{A}_1 \otimes \mathcal{A}_2 \) if and only if it is mapped to a positive element under \( \Lambda \otimes id \) for any positive map \( \Lambda \) from \( \mathcal{A}_1 \) to \( \mathcal{A}_2 \). By applying this criterion to the density (19) of \( \varphi_\lambda \), we verify that it is separable for \((\mathcal{A}(I), \mathcal{A}(I)')\) and also for \((\mathcal{A}(J), \mathcal{A}(J)')\) for any \( \lambda \in \mathbb{R}, |\lambda| \leq 1 \). But this is not the case for the CAR pair \((\mathcal{A}(I), \mathcal{A}(J))\). Take one-site subsets \( I = \{1\} \) and \( J = \{2\} \), and let \( K_1 = a_1 \), \( K_2 = a_2 \) for computational simplicity. Then we have

\[
\varphi_\lambda(a_1^*a_2) = \frac{\lambda}{8},
\]

\[
\varphi_\lambda(a_1a_2^*) = -\frac{\lambda}{8}. \tag{23}
\]

By Proposition 1, \( \varphi_\lambda \) is non-separable between \( \mathcal{A}(I) \) and \( \mathcal{A}(J) \) for any non-zero \( \lambda \).

5 The general case including noneven states

In this section, our state \( \omega \) on \( \mathcal{A}(I \cup J) \) can be noneven. We define the following quantity for positive number \( k, 0 \leq k \leq 1 \):

\[
E^k_{\mathcal{A}(I \cup J)}(\omega, \mathcal{A}(I), \mathcal{A}(J)) := \inf_{\omega = \sum \lambda_i \omega_i} \sum_i \lambda_i \left( kS(\omega_i|_{\mathcal{A}(I)}) + (1 - k)(\omega_i|_{\mathcal{A}(J)}) \right), \tag{24}
\]

where the infimum is taken over all the state decompositions of \( \omega \) in the state space of \( \mathcal{A}(I \cup J) \). For any pure state \( \omega \) of \( \mathcal{A}(I \cup J) \), it reduces to

\[
E^k_{\mathcal{A}(I \cup J)}(\omega, \mathcal{A}(I), \mathcal{A}(J)) = kS(\omega|_{\mathcal{A}(I)}) + (1 - k)S(\omega|_{\mathcal{A}(J)}). \tag{25}
\]

For \( k = 1, 0 \), (24) reduces to the usual definition of entanglement formation [4] denoted \( E_{\mathcal{A}(I \cup J)}(\omega, \mathcal{A}(I)) \) and \( E_{\mathcal{A}(I \cup J)}(\omega, \mathcal{A}(J)) \), respectively. We note that \( E_{\mathcal{A}(I \cup J)}(\omega, \mathcal{A}(I)) \) quantifies the non-separability of states for the tensor product pair \((\mathcal{A}(I), \mathcal{A}(I)')\), not for our target \((\mathcal{A}(I), \mathcal{A}(J))\).

Asymmetry of entanglement may arise for noneven states as shown in [6]. For example, there is a noneven pure state \( \rho \) on \( \mathcal{A}(I \cup J) \) such that \( \rho|_{\mathcal{A}(I)} \) is a pure state while \( \rho|_{\mathcal{A}(J)} \) is a tracial state, giving

\[
0 = S(\rho|_{\mathcal{A}(I)}) < S(\rho|_{\mathcal{A}(J)}) = \log 2 \tag{26}
\]
when \( I = \{1\} \) and \( J = \{2\} \). Hence for quantification of state correlation between \((A(I), A(J))\) for noneven states, we have to take both subsystems on \( I \) and on \( J \) into account. Here we take the equal probability \( k = 1/2 \) for simplicity, and denote \( E_{A(I\cup J)}^{1/2}(\omega, A(I), A(J)) \) by \( E_{A(I\cup J)}^{\text{avr}}(\omega, A(I), A(J)) \) which is called the averaged entanglement formation.

**Proposition 6.** Let \( I \) and \( J \) be finite disjoint subsets and \( \omega \) be a state on \( A(I\cup J) \). Then it is a separable state for \( A(I) \) and \( A(J) \) if and only if the averaged entanglement formation \( E_{A(I\cup J)}^{\text{avr}}(\omega, A(I), A(J)) = 0 \).

**Proof.** If \( \omega \) satisfies the separability condition (8), then there exists the product-state decomposition:

\[
\omega(A_1 A_2) = \sum_i \lambda_i \omega_{1,i} \circ \omega_{2,i}(A_1 A_2). \tag{27}
\]

For each index \( i \), at least one of \( \omega_{1,i} \) and \( \omega_{2,i} \) should be even for the existence of the product state \( \omega_{1,i} \circ \omega_{2,i} \) by Proposition 1. So let \( \omega_{1,i} \) be even. Then it can be decomposed as \( \omega_{1,i} = \sum_j t_{i(j)} \omega_{1,i(j)} \), where \( t_{i(j)} > 0, \sum_j t_{i(j)} = 1 \), and all \( \omega_{1,i(j)} \) can be taken from pure even states of \( A(I) \). (This is always possible when \( I \) is finite.) We have a decomposition of \( \omega_{2,i} \) as \( \omega_{2,i} = \sum_k l_{i(k)} \omega_{2,i(k)} \), where \( l_{i(k)} > 0, \sum_k l_{i(k)} = 1 \), and all \( \omega_{2,i(k)} \) are pure states of \( A(J) \). Since each \( \omega_{1,i(j)} \) is an even state of \( A(I) \), we are given the (unique) product state extension \( \omega_{1,i(j)} \circ \omega_{2,i(k)} \) for any \( i(j) \) and \( i(k) \). Repeating the same machinery for all \( i \), we have a state decomposition of \( \omega \) into \( \{ \omega_{1,i(j)} \circ \omega_{2,i(k)} \} \) where each \( \omega_{1,i(j)} \) and \( \omega_{2,i(k)} \) is a pure state. Hence

\[
S(\omega_{1,i(j)}|A(I)) = S(\omega_{2,i(k)}|A(J)) = 0
\]

for every \( i(j), i(k) \). Thus this decomposition gives

\[
E_{A(I\cup J)}^{\text{avr}}(\omega, A(I), A(J)) = 0. \tag{28}
\]

Conversely, assume (28). By definition, there exists a state decomposition \( \omega = \sum_i \lambda_i \omega_i \) such that

\[
S(\omega_i|A(I)) = S(\omega_i|A(J)) = 0, \tag{29}
\]

for all \( i \). This implies that \( \omega_i \) has pure state restrictions on both \( A(I) \) and \( A(J) \). By Theorem 1 (2) in [7], at least one of \( \omega_i|A(I) \) and \( \omega_i|A(J) \) should be even for the existence of their state extension \( \omega_i \) on \( A(I\cup J) \) and \( \omega_i \) is uniquely given as \( \omega_i|A(I) \circ \omega_i|A(J) \). Hence \( \omega \) can be written as the affine sum of the product states \( \{ \omega_i \} \) and hence is a separable state.

The following relationships among the introduced entanglement formations are obvious.
Lemma 7. For any state $\omega$ on $\mathcal{A}(I \cup J)$,
\begin{equation}
E_{\mathcal{A}(I \cup J)}^{\text{NT}}(\omega, \mathcal{A}(I), \mathcal{A}(J)) \geq 1/2E_{\mathcal{A}(I \cup J)}(\omega, \mathcal{A}(I)) + 1/2E_{\mathcal{A}(I \cup J)}(\omega, \mathcal{A}(J)). \tag{30}
\end{equation}
For any even state $\omega$ on $\mathcal{A}(I \cup J)$,
\begin{equation}
E_{\mathcal{A}(I \cup J)}^{\Theta}(\omega, \mathcal{A}(I), \mathcal{A}(J)) \geq E_{\mathcal{A}(I \cup J)}^{\text{NT}}(\omega, \mathcal{A}(I), \mathcal{A}(J)), \tag{31}
\end{equation}
and
\begin{equation}
E_{\mathcal{A}(I \cup J)}^{\Theta}(\omega, \mathcal{A}(I), \mathcal{A}(J)) \geq E_{\mathcal{A}(I \cup J)}(\omega, \mathcal{A}(I)), \quad E_{\mathcal{A}(I \cup J)}^{\Theta}(\omega, \mathcal{A}(I), \mathcal{A}(J)) \geq E_{\mathcal{A}(I \cup J)}(\omega, \mathcal{A}(J)). \tag{32}
\end{equation}

Proof. The inequality (30) follows directly from the definitions. The optimal decomposition of $E_{\mathcal{A}(I \cup J)}^{\Theta}(\omega, \mathcal{A}(I), \mathcal{A}(J))$ is given by some $\omega = \sum \lambda_i \omega_i$ such that all $\omega_i$ are pure and even. Since each $\omega_i$ satisfies (15), (31) and (32) follow.

The inequalities (30) and (32) are exact for $\varphi_\lambda$ of $\lambda \neq 1$ in § 4, since it is always separable for $(\mathcal{A}(I), \mathcal{A}(I)')$ and for $(\mathcal{A}(J), \mathcal{A}(J)')$ hence $E_{\mathcal{A}(I \cup J)}(\varphi_\lambda, \mathcal{A}(I)) = E_{\mathcal{A}(I \cup J)}(\varphi_\lambda, \mathcal{A}(J)) = 0$, while for the case of (23) it is non-separable for $(\mathcal{A}(I), \mathcal{A}(J))$ and hence both $E_{\mathcal{A}(I \cup J)}(\varphi_\lambda, \mathcal{A}(I), \mathcal{A}(J))$ and $E_{\mathcal{A}(I \cup J)}(\varphi_\lambda, \mathcal{A}(I), \mathcal{A}(J))$ should be nonzero.

The noneven pure state $\varrho$ with its asymmetric marginal states (26) gives $E_{\mathcal{A}(I \cup J)}(\varrho, \mathcal{A}(J)) = 0$, $E_{\mathcal{A}(I \cup J)}(\varrho, \mathcal{A}(I)) = \log 2$, and $E_{\mathcal{A}(I \cup J)}^{\text{NT}}(\varrho, \mathcal{A}(I), \mathcal{A}(J)) = 1/2(\log 2)$. Hence
\begin{equation}
E_{\mathcal{A}(I \cup J)}^{\text{NT}}(\omega, \mathcal{A}(I), \mathcal{A}(J)) \geq E_{\mathcal{A}(I \cup J)}(\omega, \mathcal{A}(I)) \quad \text{and} \quad E_{\mathcal{A}(I \cup J)}^{\text{NT}}(\omega, \mathcal{A}(I), \mathcal{A}(J)) \geq E_{\mathcal{A}(I \cup J)}(\omega, \mathcal{A}(J))
\end{equation}
is not satisfied in general.

We can now generalize Proposition 3 to the case including noneven states, assuming additionally that the systems are finite dimensional.

Proposition 8. Let $I$ and $J$ be finite subsets and $\omega$ be a state on $\mathcal{A}(I \cup J)$. If it is separable for the CAR pair $\mathcal{A}(I)$ and $\mathcal{A}(J)$, then so it is for the tensor product pair $\mathcal{A}(I)$ and $\mathcal{A}(I')$.

Proof. If it is separable, then $E_{\mathcal{A}(I \cup J)}^{\text{NT}}(\omega, \mathcal{A}(I), \mathcal{A}(J)) = 0$. Hence by (30), $E_{\mathcal{A}(I \cup J)}(\omega, \mathcal{A}(I)) = 0$. This is equivalent to the separability of $\omega$ for $(\mathcal{A}(I), \mathcal{A}(I'))$.

By Propositions 4 and 6, both $E_{\mathcal{A}(I \cup J)}^{\Theta}(\omega, \mathcal{A}(I), \mathcal{A}(J))$ and $E_{\mathcal{A}(I \cup J)}^{\text{NT}}(\omega, \mathcal{A}(I), \mathcal{A}(J))$ serve characterization of separable states for $(\mathcal{A}(I), \mathcal{A}(J))$. We do not know whether the inequality (31) can be strict.
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