ON REDEI’S BIQUADRATIC ARTIN SYMBOL

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Abstract. We explain the origin, definition and symmetry properties of a trilinear symbol introduced by Rédei in order to describe 2-class groups of quadratic number fields. It has regained interest in view of the recent work of A. Smith [11,12] on the ‘average behavior’ of such groups, and the general interest in trilinear maps [1].

We correct Corsman’s 2007 modification of Rédei’s original definition, and prove that the Rédei symbol is perfectly symmetric in its arguments.

1. Introduction

In a 1939 Crelle paper [10], the Hungarian mathematician László Rédei introduced a trilinear quadratic symbol \([a,b,c] \in \{\pm 1\}\) for quadratic discriminants \(a,b \in \mathbb{Z}\) and positive squarefree integers \(c\) satisfying a number of conditions. He used his symbol to describe the 8-rank of quadratic class groups, much in the way he had described the 4-rank of these class groups in terms of Legendre symbols in his earlier work [9]. His definition of the symbol, as a Jacobi symbol in \(\mathbb{Q}(\sqrt{a})\), is somewhat involved, and seems to depend on many choices. Moreover, it only allows for limited ‘symmetry’ of the symbol in its arguments, as infinite primes are disregarded in the definition.

An improved definition in class field theoretic terms was provided in 2007 by Jens Corsman [3]. He imposed fewer conditions on the arguments of the symbol, but failed to notice the dyadic ramification complications that this gives rise to, both in the definition of the symbol and in the proof of the striking feature that we call Rédei’s reciprocity law in Theorem 7.7: the trilinear symbol \([a,b,c]\) is perfectly symmetric in its arguments. We give the first complete proof of this result.

In the case of prime arguments \(a,b,c \equiv 1 \mod 4\), no dyadic ramification subtleties arise, and the symbol can be interpreted following Morishita [7, Section 8.2] as an arithmetic Milnor invariant, leading to a description as a triple Massey product that is useful in the study of pro-2-extensions of \(\mathbb{Q}\) with given ramification locus [4].

1.1. The linearity properties of the symbol make it one of the rare trilinear maps that ‘naturally occur’ in mathematics, and it is an interesting question whether it has variants having properties of cryptographic interest in the sense of [1].

We approach Rédei’s symbol along historical lines, showing how it naturally arises in the study of the 2-part of the narrow class group \(C\) of the quadratic field \(K\) of discriminant \(D\). Starting from classical results in Section 2 on the 2-rank of \(C\) that go back to Gauss, we first describe the 4-rank of \(C\) in terms of the Rédei matrix \(R_4 = R_4(D)\), which is defined in terms of the relative Legendre symbols.
of the primes dividing $D$ (Theorem 3.1). A similar linear algebra argument works for the 8-rank of $C$ (Theorem 4.1), and in order to describe the associated matrix $R_8 = R_8(D)$, one is led to the definition of the Rédei symbol as an Artin symbol in some unramified biquadratic extension $K \subset F$ (Definition 4.4). In Section 5, we explain how to explicitly compute the Rédei symbol $[a, b, c]$, by constructing $F$ from a projective point on the plane conic $x^2 - ay^2 - bz^2 = 0$. Section 6 shows how Rédei’s reciprocity law is suggested by the behavior of small examples, and indicates that Rédei’s definition should be extended to make reciprocity possible. Making this precise involves a careful treatment of the dyadic primes in the key Definition 7.4, which specifies the auxiliary field $F$ used in defining $[a, b, c]$. Our proof of Rédei’s reciprocity law in Section 8 then shows it to follow from quadratic reciprocity over $\mathbb{Q}(\sqrt{a})$, when phrased as a product formula for Hilbert symbols.

Although Galois cohomology does play a role in Corsman’s approach to the Rédei symbol, its relation to Massey products and the applications of Rédei reciprocity to the average behavior of the 2-part of imaginary quadratic $C$ in the work of Smith [11, 12], neither the definition of the symbol nor the proof of its symmetry properties requires it, and we will not use this point of view in the current paper. As an immediate application in the line of Rédei’s own work, our final Section 9 shows, following Corsman, that Rédei reciprocity easily implies the existence result [13] for governing fields for the 8-rank of class groups in 1-parameter families $C(dp)$, with $d$ a fixed integer and $p$ a variable prime.

2. Classical results

Let $d \neq 1$ be a squarefree integer, $K = \mathbb{Q}(\sqrt{d})$ the corresponding quadratic field, $D \in \{d, 4d\}$ the discriminant of $K$, and $C = \text{Cl}_K^+ = \text{Cl}^+(\mathcal{O}_K)$ the narrow class group of $K$, i.e., the quotient $C = I/P^+$ of the group $I$ of fractional $\mathcal{O}_K$-ideals by the subgroup of principal ideals $(x) = x\mathcal{O}_K$ with generator of positive norm $N(x)$. If $K$ is real quadratic with fundamental unit $\varepsilon_d$ of norm $N(\varepsilon_d) = 1$, the ideal class $F_{\infty} = [(\sqrt{d})] \in C$ is of order 2, and we have an exact sequence

$$0 \to \langle F_{\infty} \rangle \to C \to \text{Cl}_K \to 0,$$

showing that $C$ has twice the size of the ordinary class group $\text{Cl}_K$ of $K$. In the case $N(\varepsilon_d) = -1$, or when $K$ is imaginary quadratic, every principal ideal has a generator of positive norm, and $C = \text{Cl}_K$ is the ordinary class group of $K$.

Describing the 2-part of the finite abelian group $C$ can be done by specifying, for $k \geq 1$, its $2^k$-rank

$$r_{2^k} = r_{2^k}(D) = \dim_{\mathbb{F}_2} C[2^k]/C[2^{k-1}] = \dim_{\mathbb{F}_2} 2^{k-1}C/2^kC.$$ 

In the case $k = 1$, the 2-rank $r_2$ of $C$ was already determined by Gauss, who defined $C$ in terms of binary quadratic forms. To state the result, we factor $D$ as a product

$$D = \prod_{i=1}^{t} p_i^* = t_D \prod_{p \mid D \text{ odd}} p^*$$

of signed prime discriminants $p^* = (-1)^{(p-1)/2} p \equiv 1 \mod 4$ and a discriminantal 2-part $t_D \in \{1, -4, \pm 8\}$ that we sloppily denote by $2^*$ in case $D$ is even. Let $p_i | p$ be the prime of $K$ lying over $p_i$.

**Theorem 2.1.** We have $r_2 = t - 1$, with $t$ the number of prime divisors of $D$. 

Proof: There are two fundamentally different proofs of this result, describing $C[2]$ and $C/2C$, respectively. The first uses the ambiguous ideal classes $[p_i] \in C$ of order dividing 2 coming from the $t$ ramifying primes $p_i | p_i$ of $K$, the second the $t$ genus characters corresponding to the discriminantal divisors $p_i^* \in (2)$.

In the first proof, one exploits the Galois action on $C$ of $\text{Gal}(K/Q) = \langle \sigma \rangle$, noting that $\sigma$ acts by inversion as the norm map $N = 1 + \sigma$ annihilates $C$. Some Galois cohomology shows that the 2-torsion subgroup $C[2] = C[\sigma - 1]$ is generated by the $t$ classes $[p_i]$, subject to a single relation. This yields $r_2 = t - 1$.

For the second proof, one views $C = \text{Gal}(H/K)$ under the Artin isomorphism as the Galois group over $K$ of the narrow Hilbert class field $H$ of $K$. Then $H$ is Galois over $Q$ with dihedral Galois group

$$\text{Gal}(H/Q) \cong \text{Gal}(H/K) \times \text{Gal}(K/Q) = C \rtimes \langle \sigma \rangle,$$

as the surjection $\text{Gal}(H/Q) \rightarrow \text{Gal}(K/Q) = \langle \sigma \rangle$ is split and $\sigma$ acts by inversion. The genus field $H_2 \subset H$ of $K$, which is defined as the maximal subfield of $H$ that is abelian over $Q$, has as its Galois group over $Q$ the elementary abelian 2-group

$$\text{Gal}(H/Q)^{ab} = C/2C \rtimes \langle \sigma \rangle.$$

One may generate $H_2$ explicitly over $Q$ by $t$ independent square roots as

$$H_2 = Q(\{\sqrt{p_i^*} : i = 1, 2, \ldots, t\}),$$

so $\text{Gal}(H_2/Q)$ is an $F_2$-vector space of dimension $t$, and the subgroup $C/2C = \text{Gal}(H_2/K) \subset \text{Gal}(H_2/Q)$ has dimension $r_2 = t - 1$. \qed

The second proof of Theorem 2.1 shows that the prime power discriminants $p_i^* | D$ in (2) yield a basis of the quadratic characters on $\text{Gal}(H_2/Q)$, with

$$\chi_{p_i^*} : \text{Gal}(H_2/Q) \rightarrow \text{Gal}((Q(\sqrt{p_i^*})/Q) \cong F_2$$

giving the Galois action on $\sqrt{p_i^*}$. Similarly, the character $\chi_{d_1} = \prod_{i \in S} \chi_{p_i}$ for a subset $S \subset \{1, 2, \ldots, t\}$ corresponding to the discriminantal divisor $d_1 = \prod_{i \in S} p_i^*$ of $D$ gives the action on $\sqrt{d_1}$. When restricted to $C/2C = \text{Gal}(H_2/K) \subset \text{Gal}(H_2/Q)$, it yields a quadratic character in the character group

$$\widehat{C} = \text{Hom}(C, Q/Z)$$

of $C$ that coincides with the character $\chi_{d_2}$ corresponding to the complementary divisor $d_2 = D/d_1 = \prod_{i \in S} p_i^*$. Rédei calls an unordered pair $(d_1, d_2)$ of quadratic discriminants satisfying

$$D = d_1 d_2$$

a discriminantal decomposition of $D$. It is the quadratic character $\chi_{d_1} = \chi_{d_2}$ on $C$, and the corresponding finitely unramified quadratic extension $K \subset E$ inside $H$ is

$$E = K(\sqrt{d_1}) = Q(\sqrt{d_1}, \sqrt{d_2}) = K(\sqrt{d_2}).$$

3. The 4-rank

The first proof of Theorem 2.1 describes the subgroup $C[2]$ of ambiguous ideal classes in $C$ as a quotient of $F_2^t$ by a surjection

$$\alpha : F_2^t \rightarrow C[2]$$

that sends the $j$-th basis vector to the class $[p_j]$. The one-dimensional kernel of $\alpha$ encodes the non-trivial relation between the classes of the ramifying primes of $K$. 


The principal ideal \((\sqrt{d})\), which factors as a non-trivial product of ramifying primes for \(d \neq -1\), yields the desired relation for \(d < -1\), and also for \(d > 0\) in case we have \(N(\varepsilon_d) = -1\). For \(d > 0\) and \(N(\varepsilon_d) = 1\), it doesn’t in view of (1), and there is a different relation coming from the factorization of the \(\sigma\)-invariant ideal \((1 + \varepsilon_d)\) in the unit class of \(C\) as the product of an integer and certain \(p_i|D\).

The second proof of Theorem 2.1 describes the quotient \(C/2C = \Gal(H_2/K)\) of \(C\) as a subspace of \(\Gal(H/Q) = F_2^4\) under the inclusion map
\[
\gamma : C/2C = \Gal(H_2/K) \rightarrow \Gal(H_2/Q) = F_2^4,
\]
with the \(i\)-th component of \(\gamma(a) \in F_2^4\) for \([a] \in C\) describing the action of the Artin symbol \(\Art(a,H/K) \in \Gal(H/K)\) on \(\sqrt{p_i^2}\). As all Artin symbols fix the product \(\prod_{i=1}^r \sqrt{p_i^2} = \sqrt{D}\), the map \(\gamma\) embeds \(C/2C\) as the ‘sum-zero-hyperplane’ in \(F_2^4\).

The 4-rank of \(C\) equals the \(F_2\)-dimension of the kernel \(C[2]\cap 2C\) of the natural map
\[
\varphi_4 : C[2] \rightarrow C/2C,
\]
and we can find it by combining \(\varphi_4\) with the surjection \(\alpha\) and the injection \(\gamma\) from (8) and (9) into a single \(F_2\)-linear map
\[
(10) \quad R_4 : \ F_2^4 \xrightarrow{\alpha} C[2] \xrightarrow{\varphi_4} C/2C \xrightarrow{\gamma} \Gal(H_2/Q) = F_2^4.
\]
We have \(1 + \dim_{F_2} \ker \varphi_4 = \dim_{F_2} \ker R_4 = t - \rank_{F_2} R_4\), and writing \(r_2 = t - 1\) as in Theorem 3.1, we obtain the following result.

**Theorem 3.1.** The 4-rank of \(C\) equals \(r_4 = r_2 - \rank_{F_2} R_4\). \(\square\)

Explicit matrix entries for \(R_4 = (\varepsilon_{ij})_{i,j} \in \Mat_{t \times t}(F_2)\) are given for \(i \neq j\) by the Legendre symbols
\[
(11) \quad (-1)^{\varepsilon_{ij}} = \left( \frac{p_i^2}{p_j} \right),
\]
which are defined as Kronecker symbols for \(p_j = 2\) and describe the action of the Artin symbol \(\Art(p_j,H/K)\) on \(\sqrt{p_i^2} \in H_2 \subset H\). By the sum-zero-property of \(\gamma(p_j)\) we have \(\varepsilon_{ij} = \sum_{i \neq j} \varepsilon_{ij}\). This simple description of \(r_4\) in terms of the relative quadratic behavior of the primes \(p_i\) dividing \(D\) goes back to Rédei [9]. It is different from his earlier proof with Reichardt [8], which explicitly constructed the narrow 4-Hilbert class field of \(K\). A combination of both proofs leads to the description of the 8-rank of \(C\) that follows.

4. **The 8-rank**

The 8-rank \(r_8\) of \(C\) equals the \(F_2\)-dimension of the kernel of the natural map
\[
\varphi_8 : C[2]\cap 2C \rightarrow 2C/4C
\]
between \(r_4\)-dimensional vector spaces over \(F_2\). Under the Artin isomorphism, the group \(2C/4C\) is the Galois group \(\Gal(H_4/H_2)\), with \(H_4 \subset H\) the narrow 4-Hilbert class field of \(K\). We can restrict \(\alpha\) in (8) to the kernel of the 4-rank map \(R_4\) from (10) and compose with \(\varphi_8\) to obtain an \(F_2\)-linear map
\[
(12) \quad R_8 : \ker R_4 \xrightarrow{\alpha} C[2] \cap 2C \xrightarrow{\varphi_8} 2C/4C = \Gal(H_4/H_2) \cong F_2^{r_4}
\]
defined on the \((r_4 + 1)\)-dimensional space \(\ker R_4\). As \(r_8 = \dim_{F_2} \ker \varphi_8\) is the codimension of the image of \(\varphi_8\) in \(2C/4C\), we obtain the following analogue of Theorem 3.1.
From the rank of an explicit matrix (13) \( \eta \) entries.

Remark 4.3. As an element of \( \hat{H} \) that, together with the all-one vector in \( r \)

In order to compute the 8-rank in Theorem 4.1, the map \( \hat{H} \) characterizes these quadratic characters.

\[ \psi \in \hat{E} \text{ and only if } \chi \] vanishing on the 2-torsion subgroup \( C[2] \). This leads to the following characterization of these quadratic characters.

Lemma 4.2. For a quadratic character \( \chi \in \hat{C} \) defining \( E = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2}) \) as in (6) and (7), the following are equivalent:

1. there exists a cyclic quartic extension \( K \subset F \) inside \( H \) containing \( E \);
2. \( \chi|_d \circ R_4 = \chi|_d \circ R_4 \) is the zero map;
3. all ramified primes of \( K \) split completely in \( K \subset E \);
4. for \( i = 1, 2 \) and \( p|d_i \) prime we have \( (\frac{D_{d_i}}{p}) = 1 \).

Proof. Having \( \chi = 2\psi \) for a quartic character \( \psi \in \hat{C} \) defining \( F \) as in (1) is equivalent to \( \chi \) vanishing on the subgroup \( C[2] \) of ambiguous ideal classes generated by the classes of the ramifying primes \( p|D = d_1d_2 \) of \( K \) as in (8). One can phrase this using the map \( R_4 \) from (10) as in (2), or in terms of the splitting of the ramifying primes in \( K \subset E \) as in (3). A ramifying prime of \( K \) divides exactly one of \( d_1, d_2 \).

Remark 4.3. The identity \( \chi = 2\psi \) determines \( \psi \in \hat{C} \) up to a quadratic character, as an element of \( \hat{C}[4]/\hat{C}[2] \), and this means that the quadratic extension \( E \subset F \) it gives rise to in (1) is not uniquely determined by \( \chi \). However, the quadratic extension \( H_2 \subset H_2 \) it generates over the genus field \( H_2 \) corresponding to the group \( \hat{C}[2] \) of quadratic characters is unique. \( \diamond \)

In order to compute the 8-rank in Theorem 4.1 from the rank of an explicit matrix describing the map \( R_8 \) in (12), we choose an \( F_2 \)-basis for the \((r_4+1)\)-dimensional subspace \( \ker R_4 \subset F_2 \), and write

\[ [m_j] \in C[2] \cap 2C \quad \text{for} \quad j = 1, 2, \ldots, r_4 + 1 \]

for the images of the basis vectors under the map \( \alpha \) from (8). The classes \([m_j]\) span \( C[2] \cap 2C \), subject to a single relation. Note that \( m_j \) is a product of ramified primes of \( K \), and that the Artin symbol of \([m_j]\) is \( 2C \) acts trivially on the genus field \( H_2 \).

Similarly, we pick quartic characters \( \psi_i \) for \( i = 1, 2, \ldots, r_4 \) spanning \( \hat{C}[4]/\hat{C}[2] \), and denote by \( F_{4,i} \) the corresponding unramified quartic extensions of \( K \). By condition (2) of Lemma 4.2, the quadratic characters \( \chi_i = 2\psi_i \) correspond to vectors that, together with the all-one vector in \( F_2 \), span the kernel ker \( R_4^T \) of the transpose of the Rédei matrix. The \( r_4 \) quadratic extensions \( H_2F_{4,i}/H_2 \) span \( H_2 \subset H_4 \), and the map \( R_8 \) is represented by a matrix \( R_8 = (\eta_{ij})_{i,j} \in \text{Mat}_{r_4 \times (r_4+1)}(F_2) \) with entries

\[ \eta_{ij} = \psi_i[m_j] = \text{Art}(m_j, H_2F_{4,i}/K) \in \text{Gal}(H_2F_{4,i}/H_2) = F_2. \]
In cases where we know the kernel of the map $\alpha$ in (8), i.e., the non-trivial relation between the ramified primes of $K$ in $C$, we can use it to leave out the column of $R_8$ corresponding to a ‘superfluous’ generator $[m_j]$ of $C[2] \cap 2C$, and work with an $(r_4 \times r_4)$-matrix to describe $\varphi_8$ in (12).

A product $m$ of distinct ramified primes of $K$ is characterized by the squarefree divisor $m|D$ arising as its norm, and the residue class of a quartic character $\psi$ in $\widetilde{C}/\widetilde{C}[2]$ by the invariant field $E = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ of the quadratic character $2\psi$ corresponding to a decomposition $D = d_1d_2$ ‘of the second kind’. This leads to a classical notation for the entries $\psi([m])$ in (13) as Rédei symbols.

**Definition 4.4.** Let $D = d_1d_2$ be a decomposition of the second kind, $K \subset F$ a corresponding extension as in condition (1) of Lemma 4.2, and $m|D$ the squarefree norm of an integral ideal $m$ in $K$ with $[m] \in C[2] \cap 2C$. Then the Rédei symbol associated to $d_1$, $d_2$, and $m$ is the Artin symbol

$$[d_1, d_2, m] = \text{Art}(m, H_2F/K) \in \text{Gal}(H_2F/H_2) \cong \mathbb{F}_2.$$ 

For the purposes of linear algebra, it is convenient to take the values of the Rédei symbol in $\mathbb{F}_2$, as they arise as the entries $\eta_{ij}$ of the matrix $R_8$ in (11). However, as Rédei symbols describe the Galois action on certain square roots, just like the entries $e_{ij}$ of $R_4$ in (11), their values are traditionally taken in $\{\pm 1\}$.

**5. Computing Rédei-Symbols**

Definition 4.4 of the Rédei symbols $[d_1, d_2, m]$ arising as the entries of the matrix $R_8$ does not immediately show that these symbols can actually be computed fairly easily from the prime factorisations of $d_1$, $d_2$, and $m$. Their computation requires the actual construction of the dihedral number fields $F$ occurring in Definition 4.4.

**Lemma 5.1.** Let $Q$ be a field of characteristic different from 2, and $Q(\sqrt{a})$ a quadratic extension. For $\beta \in Q(\sqrt{a})^*$ a non-square element of norm $N\beta = b \in Q^*$, let $F$ be the normal closure of the quartic extension $Q(\sqrt{a}, \sqrt{b})$ of $Q$. Then

1. for $\overline{\beta} \notin \{\overline{a}, \overline{b}\} \subset Q^*/Q^*^2$, the field $F$ is quadratic over $Q(\sqrt{a}, \sqrt{b})$, cyclic of degree 4 over $Q(\sqrt{ab})$, and dihedral of degree 8 over $Q$;
2. for $\overline{b} = \overline{a} \in Q^*/Q^*^2$, the field $F$ is quadratic over $Q(\sqrt{a})$ and cyclic of degree 4 over $Q$;
3. for $\overline{b} = \overline{a} \in Q^*/Q^*^2$, the field $F$ is quadratic over $Q(\sqrt{a})$ and non-cyclic abelian of degree 4 over $Q$.

Conversely, every field $F$ having the properties in (1), (2), or (3) is obtained in this way for some $\beta \in Q(\sqrt{a})$ of norm $b$.

**Proof.** Basic Galois theory. \(\square\)

**Corollary 5.2.** For $a, b \in Q^*$ as in (1) of Lemma 5.1, a quadratic extension $F$ of $E = Q(\sqrt{a}, \sqrt{b})$ is cyclic over $Q(\sqrt{ab})$ and dihedral of degree 8 over $Q$ if and only if there exists a non-zero solution $(x, y, z) \in Q^3$ to the equation

$$x^2 - ay^2 - bz^2 = 0$$

such that for the elements $\beta = x + y\sqrt{a} \in Q(\sqrt{a})$ and $\alpha = 2(x + z\sqrt{b}) \in Q(\sqrt{b})$ of norm $\beta\beta' \in b \cdot Q^*^2$ and $\alpha\alpha' \in a \cdot Q^*^2$, we have

$$F = E(\sqrt{\beta}) = E(\sqrt{\alpha}).$$
Given one \( F = E(\sqrt{\beta}) \), any other such extension is of the form \( F_t = E(\sqrt{t\beta}) \) for some unique \( t \in Q^*/\langle a, b, Q^{*2} \rangle \).

**Proof.** The first statement follows if we write \( \beta = x+y\sqrt{a} \in Q(\sqrt{a}) \) in the dihedral case (1) of Lemma 5.1, and observe that \( E(\sqrt{\beta}) \) is not only the normal closure over \( Q \) of the quartic extension \( Q(\sqrt{a}, \sqrt{\beta}) \), but also of the quartic extension \( Q(\sqrt{b}, \sqrt{\alpha}) \): it contains a square root of the non-square element

\[
(\sqrt{x+y\sqrt{a}} + \sqrt{x-y\sqrt{a}})^2 = 2(x+z\sqrt{b}) = \alpha \in Q(\sqrt{b})^*.
\]

For the second statement, let \( F \) and \( F_0 \) be distinct dihedral fields coming from \( \beta \) and \( \beta_0 \). Then \( \beta/\beta_0 \in Q(\sqrt{\alpha})^* \) is not a square, and by case (3) of Lemma 5.1, \( Q \subset Q(\sqrt{\alpha}, \sqrt{\beta/\beta_0}) \) is a non-cyclic abelian extension of degree 4, of the form \( Q(\sqrt{\alpha}, \sqrt{t}) \) for \( t \in Q^* \). The three quadratic subextensions of \( E \subset E(\sqrt{\beta}, \sqrt{\beta_0}) \) are \( F = E(\sqrt{\beta_3}), E(\sqrt{\beta}) \) and \( F_t = E(\sqrt{t\beta_3}) = E(\sqrt{t\beta}) \). Moreover, \( t \in Q^* \) is unique up to multiplication by elements of \( E^{*2} \cap Q^* = \langle a, b, Q^{*2} \rangle \). \( \Box \)

**Remark 5.3.** The dihedral group \( D_4 \) of order 8 can be viewed as the Heisenberg group \( U_3(F_2) \) of upper triangular \( 3 \times 3 \) matrices with coefficients in \( F_2 \), and extending an extension \( Q \subset Q(\sqrt{a}, \sqrt{b}) \) to a \( D_4 \)-extension amounts to an embedding problem that can be treated in terms of Massey symbols \( [\cdot, \cdot, \cdot] \). For our purposes, the basic Galois theory of Lemma 5.1 and Corollary 5.2 is already sufficient. \( \diamondsuit \)

In order to construct an unramified extension \( K \subset F \) containing \( E = Q(\sqrt{d_1}, \sqrt{d_2}) \) for \( D = d_1d_2 \) satisfying the conditions of Lemma 4.2, we apply Corollary 5.2 for \( Q = Q \) and \( (a, b) = (d_1, d_2) \). It shows that \( F \) can be explicitly generated as

\[
F = F(x, y, z) = E(\sqrt{\delta_2}) = E(\sqrt{d_1}),
\]

for elements \( \delta_2 = x+y\sqrt{d_1} \in Q(\sqrt{d_1})^* \) and \( \delta_1 = 2(x+y\sqrt{d_1}) \in Q(\sqrt{d_1})^* \) coming from a solution \( (x, y, z) \) to the equation

\[
x^2 - d_1y^2 - d_2z^2 = 0.
\]

By Corollary 5.2, scaling any non-zero solution \( (x, y, z) \) with an appropriate element \( t \in Q^* \), which amounts to replacing \( F(x, y, z) \) by the *quadratic twist* \( F(tx, ty, tz) \), will make \( K \subset F \) unramified. As we will show in a slightly more general setting in Corollary 7.2, every primitive integral solution \( (x, y, z) \) to (16) yields an extension \( K \subset F(x, y, z) \) that is unramified at all odd primes. Ramification over 2 can be avoided by twisting the extension with a suitable choice of \( t \in \{1, \pm 2\} \).

In the case of *even* \( D \), non-trivial solvability of (16) over \( Q \) may not guarantee the existence of unramified extensions \( K \subset F(x, y, z) \), but the slightly stronger conditions of Lemma 4.2 do.

**Example 5.4.** Take \( K = Q(\sqrt{-5 \cdot 41}) \) of discriminant \( D = -4 \cdot 5 \cdot 41 = -820 \), which has \( t = 3 \) and \( r_2 = 2 \). The columns of the Rédei matrix

\[
R_4 = \begin{pmatrix}
1 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

describe the action of the Artin symbols of the ramified primes \( p_2, p_5, \) and \( p_{41} \) on the square roots of \(-4, 5\) and \( 41 \) generating \( H_2 = Q(i, \sqrt{5}, \sqrt{41}) \) as in (11). From the matrix \( R_4 \) we read off that \( r_4 \) equals \( r_2 - \text{rank}(R_4) = 1 \), that \( [p_5] \) and \([p_{41}] \) span
\[ C[2] \cap 2C, \text{ and that } D = -20 \cdot 41 \text{ is the unique decomposition of the second kind. The equation} \]
\[ x^2 + 20y^2 - 41z^2 = 0 \]
\[ \text{has a primitive solution } (12, 1, 2) \text{ for which the element } \delta = 12 + 2\sqrt{-5} \text{ of norm } 2^2 \cdot 41 \text{ is ‘primitive outside 2’ and satisfies } \delta \equiv (1 + \sqrt{-5})^2 \text{ mod 4. This shows that } \delta = 2(6 + \sqrt{-5}) \text{ has an unramified square root over } E = \mathbb{Q}(\sqrt{-5}, \sqrt{41}), \text{ whereas the primitive elements } \pm 6 + \sqrt{-5} \text{ yield extensions } E \subset E(\sqrt{\pm 2\delta}) \text{ ramified over 2.} \]

The solution \((17, 2, 3)\) defining the primitive element \(\delta_0 = 17 + 4\sqrt{-5}\) of norm \(3^2 \cdot 41\) satisfying \(\delta_0 \equiv 1 \text{ mod 4}\) also has an unramified square root over \(E\). We have
\[ \delta_0 = 164 + 82\sqrt{-5} = -|\sqrt{41}(1 - \sqrt{-5})|^2 \in -1 \cdot E^{\times^2}, \]
and \(E(\sqrt{\delta_0}) = E(\sqrt{i\delta})\) for \(t = -1\). Over \(H_2\), both \(\sqrt{\delta_0}\) and \(\sqrt{i\delta}\) generate \(H_4 = H_2(\sqrt{\delta_0}) = H_2(\sqrt{i\delta})\).

As we know that \((\sqrt{-5} \cdot 41) = p_5 p_{41}\) is trivial in \(C\), the class of either \(p_5\) or \(p_{41}\) generates \(C[2] \cap 2C\). The Rédei matrix \(R_8\) consists of a single Rédei symbol
\[ [-20, 41, 5] = [-20, 41, 41] \]
describing whether the prime \(p_5\) (or, equivalently, \(p_{41}\)) of \(K\) splits completely in \(H_4\). It does not, as \(\delta = 12 + 2\sqrt{-5}\) (like \(\delta_0 = 17 + 4\sqrt{-5}\)) is congruent to the quadratic non-residue 2 modulo every prime over 5 in \(H_2\). We conclude that we have \(r_8 = 0\), and that the 2-part of \(C\) is isomorphic to \(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}\).

In this case, the decomposition \(D = d_1 d_2 = -4 \cdot 205\) is not of the second kind, but the conic
\[ x^2 + 4y^2 - 205y^2 = 0 \]
derived by \((16)\) does have infinitely many rational points \((x, y, z)\), such as \((3, 7, 1)\). None of them defines an unramified quartic extension \(F(x, y, z)\) over \(K\).

6. DISCOVERING RÉDEI RECIPROCITY

Explicit computations of Rédei symbols exhibit a ‘reciprocity law’ that we can discover by looking at one of the most classical examples.

**Example 6.1.** Take \(K = \mathbb{Q}(\sqrt{-p})\) for \(p\) an odd prime number. Then we have \(r_2 = 1\) if and only if \(p \equiv 1\) mod 4 by Theorem 2.1, and looking at the matrix \(R_4\) in Theorem 3.1, we see that we have \(r_4 = 1\) if and only if we have \((\frac{2}{p}) = 1\), i.e., if we have \(p \equiv 1\) mod 8. We further assume \(p \equiv 1\) mod 8.

The class of \(p_2\) generates \(C[2] = C[2] \cap 2C\), and we have \(r_8 = 1\) if and only if the Rédei symbol \([-4, p, 2]\) vanishes, i.e., if and only if the prime \(p_2\) of \(K\) splits completely in the 4-Hilbert class field \(H_4(-p)\). Solving \(x^2 + 4y^2 - p\overline{z}^2 = 0\) with \(z = 1\), we can generate \(H_4(-p)\) over \(E = \mathbb{Q}(i, \sqrt{p})\) by adjoining a square root of \(\pi = x + 2iy\). Now \(p_2\) splits into 4 primes in the extension \(K \subset H_4(-p)\) if and only if the prime \((1 + i)\) over 2 in \(\mathbb{Q}(i)\) splits into 4 primes in the extension
\[ H_4(-p) = \mathbb{Q}(i, \sqrt{\pi}, \sqrt{\overline{\pi}}). \]
This shows that \([-4, p, 2]\) is a ‘Kronecker symbol’ \((\frac{-p}{4})\). By class field theory (or quadratic reciprocity) over \(\mathbb{Q}(i)\), this is the same as the Legendre symbol \((\frac{-4}{p})\), which is well defined for \(p \equiv 1\) mod 8, and we have \(r_8 = 1\) for those \(p\) that split completely in \(\mathbb{Q}(\zeta_8, \sqrt{\overline{\Gamma} + i})\). We deduce that the prime over 2 splits completely in the unramified extension \(K \subset H_4(-p)\) if and only if \(p\) splits completely in
ON RÉDEI’S BIQUADRATIC ARTIN SYMBOL

\[ \mathbb{Q}(\zeta_8, \sqrt{1+i}), \text{ a field totally ramified at 2. By the case } (a,b) = (-1,2) \text{ of Lemma 5.1, } \mathbb{Q}(\zeta_8, \sqrt{1+i}) \text{ is a dihedral field like } H_4(-p). \text{ In fact, these fields are abelian of exponent 2 over } \mathbb{Q}(i), \text{ quadratic over respectively } \mathbb{Q}(i, \sqrt{2}) \text{ and } \mathbb{Q}(i, \sqrt{-p}), \text{ and cyclic over respectively } \mathbb{Q}(\sqrt{-2}) \text{ and } \mathbb{Q}(\sqrt{-p}). \text{ This is a special case of Rédéi reciprocity, and in terms of Rédéi symbols it can suggestively be formulated as}

\begin{equation}
(17) \quad [-1, p, 2] = [-1, 2, p].
\end{equation}

The symbol on the left is defined by Definition 4.4, at least upon identifying the discriminant \(-4\) with the radicand \(-1\), but the symbol on the right is not, as it refers to a ramified quadratic extension of \(\mathbb{Q}(i, \sqrt{2})\) in which the primes over \(p \equiv 1 \mod 8\) are either split or inert, depending on the value of the symbol.

As we can swap the arguments \(-1\) and \(p\) in the left hand side by the symmetry in the definition of the symbol, one naturally wonders whether both symbols are also equal to a symbol \([p, 2, -1]\) that describes the splitting of \(\{-1\}\) in the narrow 4-Hilbert class field \(H_4(2p)\) of \(\mathbb{Q}(\sqrt{2p})\). By Theorem 3.1, the field \(H_4(2p)\) is quadratic over the totally real field \(\mathbb{Q}(\sqrt{2}, \sqrt{p})\) for \(p \equiv 1 \mod 8\). Now Frobenius symbols at \(\{-1\}\), which over \(\mathbb{Q}\) raise roots of unity to the power \(-1\), arise in class field theory as complex conjugations, acting trivially on totally real fields. The dihedral field \(H_4(2p)\) is abelian of exponent 2 over \(\mathbb{Q}(\sqrt{2})\), and it is totally real if and only if its conductor is \(p\), not \(p \cdot \infty\). Looking at the ray class group

\[ (\mathbb{Z}/\sqrt{2})/p\mathbb{Z}/(\sqrt{2})^*/(-1, 1 + \sqrt{2}) \]

modulo \(p\) of \(\mathbb{Q}(\sqrt{2})\), we see that \(H_4(2p)\) is real exactly when the fundamental unit \(1 + \sqrt{2} \in \mathbb{Q}(\sqrt{2})\) is a square modulo \(p\), and this happens for the primes that split completely in the dihedral field \(\mathbb{Q}(\zeta_8, \sqrt{1+i})\) which, by equation (14) for \((a,b) = (-1,2)\) and \((x,y,z) = (1,1,1)\), is the same field as \(\mathbb{Q}(\zeta_8, \sqrt{1+i})\). This shows that the Rédéi symbol

\begin{equation}
(18) \quad [-1, p, 2] = [-1, 2, p] = [p, 2, -1],
\end{equation}

when properly defined, is invariant under all permutations of the arguments. In Rédéi’s own definition, \([−1, 2, p]\) does not exist, and \([p, 2, -1]\) equals 1 for all \(p\). Our adapted definition in the next section introduces a notion of minimal ramification for extensions \(K \subset F\) as in (15), correcting the definition found in [3].

7. RÉDEI SYMBOLS

In order to formulate Rédéi reciprocity, we will generalize the symbol \([d_1, d_2, m]\) in Definition 4.4 beyond the setting of dihedral fields \(F\) containing \(\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})\) that are cyclic and unramified over \(K = \mathbb{Q}(\sqrt{d_1d_2})\) and norms \(m\) of ambiguous ideals \(m\) of \(K\) with trivial Artin symbol in the genus field of \(K\). As \(d_1\) and \(d_2\) encode quadratic fields, and \(m\) is the norm of an ideal with ideal class in a group of exponent 2, it will not come as a surprise that the general Rédéi symbol takes its arguments in \(\mathbb{Q}^*/\mathbb{Q}^{*2}\), and that it is linear in each of the arguments.

Every element \(\mathfrak{a} \in \mathbb{Q}^*/\mathbb{Q}^{*2}\) is uniquely represented by a squarefree integer \(a\), and corresponds to a number field \(\mathbb{Q}(\sqrt{a})\) that is quadratic for \(a \neq 1\). Given non-trivial elements in \(\mathbb{Q}^*/\mathbb{Q}^{*2}\) represented by squarefree integers \(a, b\), the number field

\[ E = \mathbb{Q}(\sqrt{a}, \sqrt{b}) \]
is quadratic over $K = \mathbb{Q}(\sqrt{ab})$, and biquadratic over $\mathbb{Q}$ for $a \neq b$. It is unramified over $\mathbb{Q}$ at primes that do not divide the discriminants $\Delta(a)$, $\Delta(b)$ of the quadratic fields corresponding to $a$ and $b$, and we have

$$\text{(19)} \quad \mathbb{Q}(\sqrt{ab}) = K \subset E \quad \text{is unramified over } p \iff p \nmid \gcd(\Delta(a), \Delta(b)).$$

By Lemma 5.1 and Corollary 5.2, every non-zero rational solution $(x, y, z)$ to

$$\text{(20)} \quad x^2 - ay^2 - bz^2 = 0$$

generates a cyclic extension

$$\text{(21)} \quad \mathbb{Q}(\sqrt{ab}) = K \subset F = F(x, y, z) = E(\sqrt{\beta}) = E(\sqrt{\alpha})$$

of degree 4 defined by $\beta = x + y\sqrt{a}$ and $\alpha = 2(x + z\sqrt{b})$. It is dihedral over $\mathbb{Q}$ for $a \neq b$, and uniquely determined by $a$ and $b$ up to twisting by rational quadratic characters: if $(x_1, y_1, z_1)$ is any other non-zero solution of (20), there exists $t \in \mathbb{Q}^*$ such that $F(x_1, y_1, z_1)$ is equal to the quadratic twist

$$\text{(22)} \quad F_t = F(tx, ty, tz) = E(\sqrt{t\beta}) = E(\sqrt{t\alpha})$$

of $F$. Note that $F$ and $F_t$ are ramified over $\mathbb{Q}$ at the same rational primes $p \nmid \Delta(t)$.

It follows from (19) that a prime of $K$ that divides both $\Delta(a)$ and $\Delta(b)$ is totally ramified in every cyclic tower $K \subset E \subset F_t$. If a rational prime $p$ does not divide $\gcd(\Delta(a), \Delta(b))$, it is often possible to avoid ramification over $p$ in $K \subset F$ by twisting, if necessary, by $p \in \mathbb{Q}^*/\mathbb{Q}^{*2}$ and, in case $p = 2$, also by $-1$.

**Proposition 7.1.** Let $a, b \in \mathbb{Z}_{\neq 1}$ be squarefree, $p \nmid \Delta(b)$ a prime number, and $K = \mathbb{Q}(\sqrt{ab}) \subset F = E(\sqrt{\beta})$ an extension as in (21). Define the quadratic twist $F_t$ of $F$ for $t \in \mathbb{Q}^*$ as in (22).

1. If $p$ does not divide $\Delta(a)$, then $Q \subset F_t$ is unramified at $p$ for a unique value $t \in \{1, p\}$ if $p$ is odd, and for a unique value $t \in \{\pm 1, \pm 2\}$ if $p = 2$.
2. If $p$ is odd and divides $\Delta(a)$, then $K \subset F$ is unramified over $p$.
3. If $\Delta(a)$ is even and $\Delta(b)$ is $1 \mod 8$, then $K \subset F_l$ is unramified over $2$ for exactly two values of $t \in \{\pm 1, \pm 2\}$.
4. If $\Delta(a)$ is even and $\Delta(b)$ is $5 \mod 8$, then $K \subset F_t$ is ramified over $2$ for all $t \in \mathbb{Q}^*$, and $\Delta(a)$ is $4 \mod 8$.

**Proof.** We consider $F$ as a quartic extension of $K_a = \mathbb{Q}(\sqrt{\alpha})$. The intermediate field $E = K(\sqrt{\beta}) = K_a(\sqrt{\beta})$ is a quadratic extension of both $K$ and $K_a$ that is unramified at primes dividing $p$, as we have $p \nmid \Delta(b)$. It follows that $K \subset F$ is unramified over $p$ if and only if $K_a \subset F$ is.

Write $F = K_a(\sqrt{\beta}, \sqrt{\beta'})$, with $\beta' \in K_a$ the $\mathbb{Q}$-conjugate of $\beta$. For $K_a \subset F$ to be unramified at $p$, a necessary condition, which for odd $p$ is also sufficient, is that both $\beta$ and $\beta'$ are, up to squares in $K_a^*$, units at the primes over $p$ in $K_a$. In view of the transitive Galois action of $\text{Gal}(K_a/\mathbb{Q})$ on such primes, it suffices to check this for a single prime $p|p$ of $K_a$, and since $\beta\beta' \in b \cdot \mathbb{Q}^{*2}$ has even valuation at $p$, the condition holds if and only if $\text{ord}_p(\beta)$ is even.

In the unramified case $p \nmid \Delta(a)$, we have $\text{ord}_p(p\beta) = \text{ord}_p(\beta) + 1$, so exactly one of $\beta$ and $p\beta$ has even valuation at $p$. For odd $p$, this shows that exactly one of the fields $F'$ and $F_p$ is unramified over $K_a$ (and $\mathbb{Q}$) at the primes over $p$, as claimed in (1). In the ramified case $p|\Delta(a)$, we have $(p) = p^2$ in $K_a$, and $\text{ord}_p(\beta) = \text{ord}_p(\beta\beta') \equiv \text{ord}_p(b) \mod 2$ is automatically even, showing that for $p$ odd, $K \subset F$ is unramified over $p$ — which proves (2).
For $p = 2$, we also find that up to squares, $\beta$ is a 2-unit in $K_a$ if $\Delta(a)$ is even, and exactly one of $\beta$ and $2\beta$ is a 2-unit in the ring of integers $O$ of $K_a$ if $\Delta(a)$ is odd. However, for a 2-unit to have a square root that is unramified at 2, we need the stronger condition it is a square modulo 4.

Suppose $\beta$ is a 2-unit in the ring of integers $O$ of $K_a$. For $2 \nmid \Delta(a)$, the group $(O/4O)^*$ has order 4 or 12, depending on whether 2 is split or inert in $O$, and the squares in it have index 4. Together with $-1$, they generate the kernel of the surjective norm map $N: (O/4O)^* \rightarrow (\mathbb{Z}/4\mathbb{Z})^*$. We have $\beta \beta' \equiv b \equiv 1 \mod 4O$, so the residue classes $\beta, \beta' \in \ker N$ are squares in $(O/4O)^*$ for a unique ‘sign choice’ of $\beta$, and $Q \subset F_t$ is unramified at 2 for a unique value $t \in \{\pm1, \pm2\}$, proving (1).

For $2|\Delta(a)$, the group $(O/4O)^* = (O/p^4O)^*$ is of order 8, and its subgroup of squares, of index 4, of order 2. The norm map $O = \mathbb{Z}[\sqrt{a}] \rightarrow \mathbb{Z}$ induces a homomorphism

$$N: (O/4O)^* \rightarrow (\mathbb{Z}/8\mathbb{Z})^*$$

for which the image, of order 2, is generated by $1 - a \mod 8$ when $a \equiv \pm2 \mod 8$ is even, and by 5 mod 8 when $a \equiv -1 \mod 4$ is odd.

In the case where $a$ is even, ker $N$ is non-cyclic of order 4, generated by $-1$ and the squares in $(O/4O)^*$, and it contains $\beta \mod 4O$ as $\beta \beta' \equiv b \mod 8$ is not 5 mod 8 $\notin$ im $N$. In this case, we have $\Delta(b) = b \equiv 1 \mod 8$, and we conclude just as before that exactly one of $F$ and $F_{-1}$ is unramified over $K$ at 2. By the same argument applied to $F_2$, one of $F_2$ and $F_{-2}$ is unramified over $K$ at 2, so $K \subset F_t$ is unramified over 2 for exactly two values $t \in \{\pm1, \pm2\}$, as stated in (3).

In the remaining case $a \equiv -1 \mod 4$, or $\Delta(a) \equiv 4 \mod 8$, the residue class of

$$t = (1 + \sqrt{a})^2/2 = (1 + a)/2 + \sqrt{a}$$

in $(O/4O)^*$, which equals $\sqrt{a} \mod 4O$ for $a \equiv -1 \mod 8$ and $2 + \sqrt{a} \mod 4O$ for $a \equiv 3 \mod 8$, has square $-1 \mod 4O$, so it is of order 4 and generates ker $N$.

We now have 2 cases. For $\Delta(b) = b \equiv 1 \mod 8$ we have $\beta \mod 4O \in \ker N$, and twisting by $t = 2$, which replaces $\beta$ by $\beta/\tau$, may be used to move $\beta$ into the subgroup $\pm1 \mod 4O$ of squares in $(O/4O)^*$. In this case either $F$ and $F_{-1}$ or $F_2$ and $F_{-2}$ are unramified over $K$ at 2, finishing the proof of (3).

The final case $a \equiv -1 \mod 4$ and $\Delta(b) = b \equiv 5 \mod 8$ is the case occurring in (4). Here twisting by $-1$ or 2 cannot move $\beta$ or $\beta'$ into ker $N$, and the extension $K_a \subset F = K_a(\sqrt{3}, \sqrt{7})$ is ramified at the prime $p|2$ of $K_a$. This implies that $K \subset F_t$ is ramified over 2 for all $t \in Q^*$, proving (4). Alternatively, one can argue that if the ramified prime over 2 in $K$, which is inert in $K \subset E$, were unramified in the cyclic quartic extension $K \subset F_t$, the primes over 2 in $F_t$ would have ramification index 2 and inertia degree 4 over $Q^*$; but the dihedral group of order 8 has no cyclic quotient of order 4.

The ramified case (4) of Proposition 7.1 does not occur when $D = \Delta(a)\Delta(b)$ is a decomposition satisfying the conditions of Lemma 4.2, as for even $D$, the prime 2 splits in either $Q(\sqrt{a})$ or $Q(\sqrt{b})$, by condition (4) of Lemma 4.2.

**Corollary 7.2.** Let $(x, y, z)$ be a primitive integral solution to (16) for $D = d_1d_2$ satisfying the conditions of Lemma 4.2. Then there exists $t \in \{\pm1, \pm2\}$ such that $F_t = Q(\sqrt{d_1}, \sqrt{d_2}, \sqrt{xt + y\sqrt{d_1}})$ is unramified and cyclic of degree 4 over $Q(\sqrt{D})$.

**Proof.** For $(x, y, z)$ primitive and $p$ odd, $\beta = x + y\sqrt{d_1}$ and $\alpha = 2(x + z\sqrt{d_2})$ are not divisible by $p$, hence units at a prime over $p$ in $Q(\beta)$ and $Q(\alpha)$, making
\( \mathbb{Q}(\sqrt{7}) \subset F_1 \) unramified outside 2. Twisting by \( t \in \{ \pm 1, \pm 2 \} \) as in (1) and (3) of Proposition 7.1 makes it unramified at 2 as well. \( \square \)

In the ramified case \( a \equiv -1(4) \) and \( b \equiv 5 \mod 8 \) of Proposition 7.1, which is essential for Rédei reciprocity, the extension \( K \subset F \) in (21) gives rise to a local field \( F \otimes \mathbb{Q}_2 \) that is dihedral of degree 8 over \( \mathbb{Q}_2 \), and quadratic over

\[
E_2 = \mathbb{Q}_2(\sqrt{a}, \sqrt{b}) = \mathbb{Q}_2(i, \sqrt{5}).
\]

It is cyclic over \( \mathbb{Q}_2(\sqrt{-5}) \) for \( a \equiv -1(8) \), and cyclic over \( \mathbb{Q}_2(i) \) for \( a \equiv -5(8) \).

Although ramification in \( E_2 \subset E_2(\sqrt{5}) \) cannot be avoided, one can obtain minimal ramification using twisting by the generator \( t = 2 \) of \( \mathbb{Q}_2^*/(-1, 5, \mathbb{Q}_2^2) \cong \mathbb{Z}/2\mathbb{Z} \). In view of (23), this amounts to replacing \( \beta \) by \( \tau \beta \). In this way we can make \( \beta \) trivial in the group \( (\mathcal{O}/2\mathcal{O})^* \cong \langle \sqrt{a} \rangle = (\sqrt{a} \mod 2\mathcal{O}) \) of order 2, and we can even change the sign of \( \beta \) – this does not change the extension \( E_2(\sqrt{5}) \) – to achieve \( \beta \equiv 1 \mod \mathfrak{p}^3 \), with \( \mathfrak{p}\mid 2 \) in \( K_a \). This is not quite the congruence \( \beta \equiv 1 \mod \mathfrak{p}^4 \) that would make \( K_a = \mathbb{Q}(\sqrt{a}) \subset F \) unramified over 2, but it does ensure that the local extension \( \mathbb{Q}_2(\sqrt{a}) \subset F \otimes \mathbb{Q}_2 \) is of conductor 2, the minimum for a ramified quadratic extension of \( \mathbb{Q}_2(/\sqrt{a}) \). One has \( F \otimes \mathbb{Q}_2 = E_2(\sqrt{5}) \) with \( x = -1 + 2t \) for \( a \equiv -1 \mod 8 \) and \( x = 3 + 2\sqrt{-5} \) for \( a \equiv -5 \mod 8 \).

**Definition 7.3.** The extension \( K \subset F \) in (21) obtained for \( \Delta(a) \equiv 4 \mod 8 \) and \( \Delta(b) \equiv 5 \mod 8 \) is said to be 2-minimally ramified if the local extension \( \mathbb{Q}_2(\sqrt{a}) \subset F \otimes \mathbb{Q}_2 \) is of conductor 2.

The requirement in Definition 7.3 means that we have \( F = E(\sqrt{5}) \) for an element \( \beta \in 1 + 2\mathcal{O} \subset K_a^* \). Any \( F \) in case (4) of Proposition 7.1 has a twist \( F_t \) with \( t \in \{ \pm 1, \pm 2 \} \), unique up to sign, that is 2-minimally ramified. For arbitrary non-trivial elements \( a, b \in \mathbb{Q}^*/\mathbb{Q}^2 \) for which (20) admits non-zero solutions, we are led to the following global notion of minimal ramification.

**Definition 7.4.** For \( a, b \in \mathbb{Q}^*/\mathbb{Q}^2 \setminus \{ 1 \} \), the extension \( K \subset F \) in (21) defined by a non-zero rational solution to (20) is said to be minimally ramified if it is

1. unramified over all odd primes \( p \mid \gcd(\Delta(a), \Delta(b)) \);
2. unramified over 2 when \( \Delta(a) \Delta(b) \) is odd, or one of \( \Delta(a), \Delta(b) \) is 1 mod 8;
3. 2-minimally ramified if \( (\Delta(a), \Delta(b)) \equiv (5, 4) \) or \( (4, 5) \) modulo 8.

Every extension \( K \subset F \) in (21) can be twisted by some \( t \in \mathbb{Q}^*/\mathbb{Q}^2 \) to obtain a minimally ramified extension \( K \subset F_t \), but \( F_t \) is not uniquely determined by \( a, b \in \mathbb{Q}^*/\mathbb{Q}^2 \). More precisely, we define the twisting subgroup

\[
T_{a,b} \subset \mathbb{Q}^*/\mathbb{Q}^2
\]

as follows. Write \( \Delta(a) = t_a \prod_{p \mid \Delta(a) \text{ odd}} p^* \) as in (2) as a product of signed primes \( p^* \equiv 1 \mod 4 \) and a discriminantal 2-part \( t_a \in \{ 1, -4, \pm 8 \} \), and similarly for \( \Delta(b) \). Then \( T_{a,b} \) is the subgroup of \( \mathbb{Q}^*/\mathbb{Q}^2 \) generated by the residue classes of the signed primes \( p^* \) with \( p \mid ab \) odd, the 2-parts \( t_a \) and \( t_b \), and, in case \( \Delta(a) \) and \( \Delta(b) \) are both even, the elements \(-1\) and \( 2 \). It is tailored to get the following.

**Lemma 7.5.** Given \( a, b \in \mathbb{Q}^*/\mathbb{Q}^2 \setminus \{ 1 \} \), there exists \( K \subset F \) as in (21) that is minimally ramified. For such an \( F \) and \( t \in \mathbb{Q}^*/\mathbb{Q}^2 \), we have

\[
K \subset F_t \text{ is minimally ramified} \iff t \in T_{a,b}.
\]
Proof. We have already seen that a minimally ramified extension \( K \subset F \) exists. If \( \Delta(a) \) and \( \Delta(b) \) are not both even, it follows from (4) that the elements \( t \in T_{a,b} \) yield exactly the Dirichlet characters corresponding to the quadratic extensions \( \mathbb{Q} \subset \mathbb{Q}(\sqrt{\alpha}, \sqrt{\beta}) \) that become unramified over \( E = \mathbb{Q}(\sqrt{\alpha}, \sqrt{\beta}) \), and preserve the minimal ramification of \( F \) under twisting. If both \( \Delta(a) \) and \( \Delta(b) \) are even, inclusion of both generators \(-1 \) and \( 2 \) ’at 2’ ensures that for \( t_a = t_b \neq 1 \), when Definition 7.4 imposes no restriction on ramification at 2 on \( K \subset F \), we do allow all possible twists of 2-power conductor. 

The condition of non-trivial solvability of (20), i.e., the existence of a rational point on the projective conic it defines, is by the classical local-global principle for conics equivalent to non-trivial solvability over every \( p \)-adic completion \( \mathbb{Q}_p \) of \( \mathbb{Q} \), including the archimedean completion ‘at infinity’ \( \mathbb{Q}_\infty = \mathbb{R} \). Solvability over \( \mathbb{Q}_p \) is usually phrased in terms of the Hilbert symbol \( (a, b)_p \in \{ \pm 1 \} \), which is defined for \( a, b \in \mathbb{Q}_p^* \) and \( p \leq \infty \) prime, and equals 1 if and only if (20) admits a non-zero solution in \( \mathbb{Q}_p \). It yields a perfect symmetric pairing

\[
(\cdot, \cdot)_p : \mathbb{Q}_p^*/\mathbb{Q}_p^{*2} \times \mathbb{Q}_p^*/\mathbb{Q}_p^{*2} \longrightarrow \{ \pm 1 \}
\]

in which \( (a, b)_p \) is the local Artin symbol of the element \( b \in \mathbb{Q}_p^* \) for the extension \( \mathbb{Q}_p \subset \mathbb{Q}_p(\sqrt{a}) \). It equals 1 if and only if \( b \) is a norm from \( \mathbb{Q}_p(\sqrt{a}) \), as in (20). The local symbol \( (a, b)_p \) equals 1 at all odd \( p \) for which \( a \) and \( b \) are \( p \)-adic units, and quadratic reciprocity follows from the global product formula

\[
\prod_{p \leq \infty} (a, b)_p = 1 \quad \text{for all} \quad a, b \in \mathbb{Q}^*.
\]

Non-trivial solvability of (20) amounts to having \( (a, b)_p = 1 \) for all \( p \).

Definition 7.6. The Rédei symbol \([a, b, c] \in \{ \pm 1 \}\) is defined for \( a, b, c \in \mathbb{Q}^*/\mathbb{Q}^{*2} \) satisfying

\[
(a, b)_p = (a, c)_p = (b, c)_p = 1 \quad \text{for all} \quad p \leq \infty;
\]

\[
gcd(\Delta(a), \Delta(b), \Delta(c)) = 1.
\]

If one of \( a, b, \) or \( c \) is trivial in \( \mathbb{Q}^*/\mathbb{Q}^{*2} \), we take \([a, b, c] = 1\). Otherwise, we let \( K = \mathbb{Q}(\sqrt{ab}) \subset F \) be any extension in (21) that is minimally ramified, and define

\[
[a, b, c] \in \text{Gal}(F/E) = \{ \pm 1 \}
\]

in terms of \( F \) as

\[
[a, b, c] = \text{Art}_c(F/K) = \begin{cases} 
\text{Art}(c, F/K) & \text{if} \ c > 0; \\
\text{Art}(c\infty, F/K) & \text{if} \ c < 0.
\end{cases}
\]

Here \( c \) is an integral \( \mathcal{O}_K \)-ideal of norm \( |c_0| \), with \( c_0 \) the squarefree integer in the class of \( c \), and \( \infty \) denotes an infinite prime of \( K \).

With this definition, which we show in Corollary 8.2 to be independent of the choice of minimally ramified extension \( K \subset F \), Rédei’s reciprocity law is the following.

Theorem 7.7. For \( a, b, c \in \mathbb{Q}^*/\mathbb{Q}^{*2} \) satisfying (26) and (27), the symbol (28) is well-defined, multiplicative in each of its arguments, and satisfies

\[
[a, b, c] = [b, a, c] = [a, c, b] \in \{ \pm 1 \}.
\]
8. Proving Rédei reciprocity

We already mentioned that Rédei’s original definition is different from (28). Not only does it omit a contribution of the infinite prime, putting \([a, b, -c] = [a, b, c]\), it also requires at least one of \(\Delta(a)\) and \(\Delta(b)\) to be odd, making a symbol like \([-1, 2, p]\) in (18) undefined. The resulting reciprocity law [10, Satz 4] has superfluous 2-adic restrictions on the entries, and for \(bc < 0\) the symbols \([a, b, c]\) and \([a, c, b]\), which are only both defined for \(\Delta(a)\) without prime factors congruent to 3 mod 4, differ by a product of four quadratic and biquadratic symbols.

In his 2007 thesis, Corsman found that including an Artin symbol at infinity for \(c < 0\) leads to a perfectly symmetric version of the reciprocity law. However, both the definition of the symbol and his proof of the law rely heavily on an incorrect lemma [3, Lemma 5.1.2] claiming that the assumptions (26) and (27) guarantee the existence of an extension \(K \subset F\) in (28) that is unramified at all primes \(p \nmid \gcd(\Delta(a), \Delta(b))\). Smith’s recent paper on the average 8-rank behavior of imaginary quadratic class groups has an incorrect version of the reciprocity law [11, Proposition 2.1] that disregards the subtleties both at the infinite and at the dyadic primes.

We let \(a, b, c\) be squarefree integers different from 1 satisfying (26) and (27). To see that \([a, b, c]\) is well-defined, and independent of the many choices that go into the definition of the symbol, we first note that a minimally ramified extension \(K \subset F\) as is used in (28) exists, as (20) is non-trivially solvable for \(a, b \in \mathbb{Q}^\ast\) satisfying \((a, b)_p = 1\) for all \(p\), and Proposition 7.1 shows how to obtain a minimally ramified twist starting from any non-zero solution to (20).

Let \(K = \mathbb{Q}(\sqrt{ab}) \subset F\) be minimally ramified, and \(p\) a prime dividing \(c\). Then \(p\) is split or ramified in \(\mathbb{Q}(\sqrt{a})\) and in \(\mathbb{Q}(\sqrt{b})\) by (26), and unramified in at least one of these fields by (27). For a prime \(p_K|p\) in \(K\), this implies that \(p_K\) is of degree 1, and split in the extension \(K \subset E = \mathbb{Q}(\sqrt{a}, \sqrt{b})\). Moreover, \(p_K\) is unramified in \(K \subset F\) for primes \(p|c\). Indeed, (27) implies that we are in case (1) of Definition 7.4 for \(p\) odd. For \(2|c\) it implies that at least one of \(\Delta(a), \Delta(b)\) is odd, say \(\Delta(b)\), and then the condition \((b, c)_2 = (\Delta(b), 2) = 1\) from (26) shows that we have \(\Delta(b) \equiv 1 \mod 8\), putting us in case (2) of the Definition 7.4. Thus \(\Art(p_K, F/K) \in \Gal(F/Q)\) is well-defined. As \(\Gal(F/E)\) is contained in the center of \(\Gal(F/Q)\), and equal to it if \(Q \subset F\) is dihedral,

\[
[a, b, c]_{F,p} = \Art(p_K, F/K) \in \Gal(F/E)
\]

only depends on \(F\) and \(p\), not on \(p_K|p\) in \(K\). For \(p \nmid c\) we put \([a, b, c]_{F,p} = 1\).

For \(c < 0\), we have \(a, b > 0\) by condition (26) for \(p = \infty\), so \(E = \mathbb{Q}(\sqrt{a}, \sqrt{b})\) is totally real, and the decomposition group at every infinite prime of \(F\) is generated by the Frobenius at infinity

\[
[a, b, c]_{F, \infty} = \Art(\infty, F/K) \in \Gal(F/E).
\]

For \(c > 0\) we put \([a, b, c]_{F, \infty} = 1\).

With this notation, the Rédei symbol in (28) becomes a product

\[
[a, b, c] = \prod_{p \leq \infty} [a, b, c]_{F,p} \in \Gal(F/E) = \{\pm 1\}
\]

of its \(p\)-parts. The infinite product (30) is well-defined as \([a, b, c]_{F,p} = -1\) only occurs for primes \(p|c\), with \(\infty|c\) having the meaning \(c < 0\).
As the prime \( p_K \) in the Artin symbol \( \text{Art}(p_K, F/K) = [a, b, c]_{F,p} \) for \( p \in (29) \) splits in \( K \subset E \), we can view it as the Artin symbol of a prime \( p_E \) of \( E \) in the quadratic extension \( E \subset F = E(\sqrt{\beta}) = E(\sqrt{\beta'}) \). As \( p_E \) is unramified in \( E \subset F \), its norm to \( K_a \) is a prime \( p \) of degree 1 over \( p \) in \( K_a \) that is unramified in at least one of the quadratic extensions \( K_a(\sqrt{\beta}) \) and \( K_a(\sqrt{\beta'}) \) of \( K_a \). Replacing \( p \) by a conjugate prime in \( K_a \) if necessary, we can take it to be unramified in \( K_a \subset K_a(\sqrt{\beta}) \). We can then compute the \( p \)-part of \([a, b, c] \) as

\[
[a, b, c]_{F,p} = \text{Art}(p, K_a(\sqrt{\beta})/K_a).
\]

This shows that \([a, b, c]_{F,p} \) is essentially a Legendre symbol \((\frac{\beta}{p})\) in the field \( K_a \).

More precisely, for \( p \) \( p \)-unramified in \( K_a \subset K_a(\sqrt{\beta}) \), it is the quadratic Hilbert symbol

\[
[a, b, c]_{F,p} = (\beta, \pi)_p
\]

of \( \beta \) and a uniformizer \( \pi \) in the completion of \( K_a \) at \( p \). For \( c < 0 \) and \( p = \infty \), we have \([a, b, c]_{F,\infty} = (\beta, -1)_p \) as the archimedean nature of \( F = E(\sqrt{\beta}) \) is determined by the sign of \( \beta \) at a real prime \( p \) of \( K_a \).

It is clear from the symmetry in \( a \) and \( b \) of the definition of the Rédéi symbol \([a, b, c] \) that we have \([a, b, c] = [b, a, c] \) whenever the symbol is defined. In order to prove the non-trivial reciprocity law \([a, b, c] = [a, c, b] \) in Theorem 7.7, we choose a minimally ramified extension \( F = E(\sqrt{\beta}) \) of \( K = \mathbb{Q}(\sqrt{ab}) \) as in (21) in order to express \([a, b, c] \) as a product of \( p \)-parts \([a, b, c]_{F,p} \) as in (30), and similarly a minimally ramified extension \( F' = E'(\sqrt{\gamma}) \) of \( K' = \mathbb{Q}(\sqrt{ac}) \) in order to express \([a, c, b] \) as a product of \([a, b, c]_{F',p} \). Here \( \beta, \gamma \in \mathbb{Q}(\sqrt{a})' \) are elements of norm \( b, c \in \mathbb{Q}(\sqrt{ac})' \), respectively, and the fields \( F \) and \( F' \) are the normal closures of \( \mathbb{Q}(\sqrt{a}, \sqrt{\beta}) \) and \( \mathbb{Q}(\sqrt{a}, \sqrt{\gamma}) \). In the spirit of (32), we then have the following key lemma.

**Lemma 8.1.** Let \( a, b, c \in \mathbb{Q}(\sqrt{\gamma})' \) be non-trivial elements satisfying (26) and (27), and \( F = E(\sqrt{\beta}) \) and \( F' = E'(\sqrt{\gamma}) \) minimally ramified extensions of \( K = \mathbb{Q}(\sqrt{ab}) \) and \( K' = \mathbb{Q}(\sqrt{ac}) \) defined as above. For all rational primes \( p \leq \infty \), we then have

\[
[a, b, c]_{F,p} \cdot [a, c, b]_{F',p} = \prod_{p|p \text{ in } \mathbb{Q}(\sqrt{a})} (\beta, \gamma)_p.
\]

**Proof.** We denote the left and right hand side of (33) by \( L_p \) and \( R_p \), respectively, and note that \( L_p \) and \( R_p \) are symmetric in \( b \) and \( c \). Moreover, we can replace \( \beta \) (or \( \gamma \)) in \( R_p \) by its conjugate without changing the value of \( R_p \), as the expression \( R'_p \) obtained by replacing \( \beta \) by \( \beta' \) satisfies \( R'_p R_p = \prod_{p|p \text{ in } \mathbb{Q}(\sqrt{a})} (b, \gamma)_p = (b, c)_p = 1 \).

For \( p = \infty \), condition (26) implies that at most one of \( a, b, c \) is negative. If they are all positive, we have \( L_\infty = 1, \) and both \( \beta \) and \( \gamma \) are totally positive or negative in the real quadratic field \( K_a = \mathbb{Q}(\sqrt{a}) \). The symbols \((\beta, \gamma)_p \) at the two infinite primes of \( K_a \) then have the same value, so we also have \( R_\infty = 1 \). If only \( a \) is negative, we have \( L_\infty = 1 = R_\infty \), as the unique infinite prime of \( K_a \) is complex.

If \( a \) is positive and exactly one of \( b \) and \( c \), say \( c \), is negative, \( L_\infty \) is the Frobenius at infinity in \( E \subset F = E(\sqrt{\beta}) \), which equals 1 if \( \beta \in \mathbb{R} \) is totally positive, and \(-1 \) if \( \beta \) is totally negative. As \( \gamma \) has a positive and a negative embedding in \( \mathbb{R} \), the same is true for the product \( R_\infty = (\beta, \gamma)_\infty \cdot (\beta, \gamma)_\infty \) of the Hilbert symbols at the infinite primes of \( K_a \). This settles the case \( p = \infty \).

For \( p \) a finite prime, take \( a, b, c \) to be squarefree integers. Condition (27) implies that \( p \) divides at most two of \( a, b, c \). If \( p \) divides \( b \), it is split or ramified in \( K_a \),
and \( \beta \) is, up to squares in \( K_a^* \), a uniformizer at a prime \( p_1|p \) and, in the split case \((p) = p_1p_2 \), a unit at the other prime \( p_2|p \) in \( K_a \). If \( p \) does not divide \( b \), then the minimal ramification of \( K \subset F \) implies that \( \beta \) is a \( p \)-unit, up to squares in \( K_a^* \).

For odd \( p \) this means that \( \sqrt{\beta} \in F \) generates an extension of \( K_a \) that is unramified over \( p \). Analogous statements apply to \( c \) and \( \gamma \).

Suppose first that \( p \) is odd. If \( p \) does not divide \( bc \), we have \( L_p = 1 = R_p \), as the Hilbert symbols \((\beta, \gamma)_p\) at \( p|p \) are equal to 1 for \( p \)-units \( \beta \) and \( \gamma \). If \( p \) divides exactly one of \( b, c \), say \( c \), we can take \( \beta \) to be a \( p \)-unit, with square root in \( F \) that is unramified over \( p \), and \( \gamma \) a uniformizer at a prime \( p_1|p \). By (32), we then have

\[
L_p = [a, b, c]_{F, p} = (\beta, \gamma)_{p_1}.
\]

In the split case \((p) = p_1p_2 \), we further have \((\beta, \gamma)_{p_2} = 1 \), as both \( \beta \) and \( \gamma \) are units at \( p_2 \). This yields \( L_p = R_p \) both in the ramified and in the split case.

If \( p \) divides both \( b \) and \( c \), it does not divide \( a \), so we are in the split case \((p) = p_1p_2 \) in \( K_a \). After replacing \( \beta \) by its conjugate, if necessary, \( \beta \) is a unit at \( p_1 \) and a uniformizer at \( p_2 \), whereas \( \gamma \) is a uniformizer at \( p_1 \) and a unit at \( p_2 \). Again by (32),

\[
L_p = [a, b, c]_{F, p}[a, c, b]_{F, p} = (\beta, \gamma)_{p_1}(\beta, \gamma)_{p_2} = R_p,
\]

so we have proved our lemma for odd \( p \).

For \( p = 2 \), we need a finer distinction as \( 2 \nmid b \), and even \( 2 \nmid \Delta(b) \), does not imply that the minimally ramified extension \( K \subset F \) is unramified over 2, and that \( \sqrt{\beta} \) generates a subextension of \( K_a \subset F \) that is unramified over 2. For \( 2 \mid \Delta(b) \), or \( b \equiv 1 \) mod 4, Definition 7.4 shows that it does in all cases except in the case \( a \equiv -1(4) \) and \( b \equiv 5 \) mod 8. For \( b \equiv -1 \) mod 4, when \( 2 \) divides \( \Delta(b) \) but not \( b \), we do know that \( \beta \) is, up to squares in \( K_a^* \), a 2-adic unit. Moreover, for \( \Delta(b) \) even and 2 split in \( K_a \), the extension \( K_a \subset K_a(\sqrt{\beta}) \) is unramified at one prime over 2, and ramified at the other. Same for \( c \) and \( \gamma \).

Suppose first that \( bc \) is odd. Then we have \( L_2 = 1 \), and we take \( \beta \) and \( \gamma \) to be 2-units. By the condition \((b, c)_2 = 1 \) at least one of \( b, c \), say \( b \), is 1 mod 4. For \( c \equiv -1 \) mod 4, the condition \((a, c)_2 = 1 \) implies \( a \not\equiv -1 \) mod 4, so the minimally ramified extension \( K \subset F \) is unramified over 2, and all Hilbert symbols \((\beta, \gamma)_p \) at primes \( p|2 \) in \( K_a \) occurring in \( R_2 \) equal 1, as \( \gamma \) is a unit at \( p \) and \( K_a \subset K_a(\sqrt{\beta}) \) is unramified at \( p \). For \( c \equiv 1 \) mod 4, Definition 7.4 tells us that we are in the same situation, with \( R_2 = 1 \) because one of \( \beta, \gamma \) is a \( p \)-unit and the other has a \( p \)-unramified square root, provided that either we have \( a \not\equiv -1 \) mod 4 or one of \( b, c \) is 1 mod 8. The remaining special case \( a \equiv -1 \) mod 4 and \( b \equiv c \equiv 5 \) mod 8 is when both \( K_a \subset F \) and \( K_a \subset F' \) are ramified at the prime \( p|2 \) of \( K_a \). This is where the minimal ramification at 2 of the extensions \( K \subset F \) and \( K \subset F' \) from Definition 7.3 is essential: once more we have \( R_2 = (\beta, \gamma)_p = 1 \), as \( \sqrt{\beta} \) generates an extension of conductor 2 of the completion \( \mathbb{Q}_2(\sqrt{\beta}) \) of \( K_a \) at \( p \), and \( \gamma \) is 1 modulo \( p^2 \equiv (2) \) in \( K_a \). This proves \( L_2 = 1 = R_2 \) for \( bc \) odd.

If exactly one of \( b, c \) is even, say \( c \), the condition \((a, c)_2 = (b, c)_2 = 1 \) implies \( a, b \not\equiv 5 \) mod 8. For \( b \equiv 1 \) mod 8, the minimally ramified extension \( K \subset F \), and therefore \( K_a \subset K_a(\sqrt{\beta}) \), is unramified over 2. In this case, we have

\[
L_2 = [a, b, c]_{F, 2} = (\beta, \gamma)_{p_1} = R_2
\]

just as in the case of odd \( p \), as we can take \( \gamma \) to be a uniformizer at \( p_1|2 \) and, in the split case, a unit at the other prime \( p_2 \). In the other case \( b \equiv -1 \) mod 4 both
\( \Delta(b) \) and \( \Delta(c) \) are even, so we have \( a \equiv 1 \mod 8 \) and \( (2) = p_1 p_2 \) in \( K_a \). In this case \( \sqrt{b} \) and \( \sqrt{c} \) generate extensions of \( K_a \) that are ramified at one prime over 2, and unramified at the other. Replacing \( \beta \) or \( \gamma \) by their conjugate if necessary, we can assume that \( K_a \subset K_a(\sqrt{b}) \) is unramified at \( p_1 \) and \( K_a \subset K_a(\sqrt{c}) \) unramified at \( p_2 \). Up to squares, \( \gamma \) is a then a uniformizer at \( p_1 \) and \( \beta \) a unit at \( p_2 \), so we have

\[
L_2 = [a, b, c]_{F, 2} = (\beta, \gamma)_{p_1} = (\beta, \gamma)_{p_2} = R_2.
\]

Finally, for \( b \) and \( c \) both even, we are also in the split case, as \( \Delta(a) \) is odd and \( (a, b)_2 = (a, 2)_2 = 1 \) implies \( a \equiv 1 \mod 8 \). As above, we can choose \( K_a \subset K_a(\sqrt{b}) \) unramified at \( p_1 \) and \( K_a \subset K_a(\sqrt{c}) \) unramified at \( p_2 \). Up to squares, this makes \( \beta \) a uniformizer at \( p_2 \) and \( \gamma \) a uniformizer at \( p_1 \). We obtain

\[
L_2 = [a, b, c]_{F, 2} = (\beta, \gamma)_{p_1} (\beta, \gamma)_{p_2} = R_2,
\]

and we have finished the proof of Lemma 8.1.

**Proof of Theorem 7.7.** By Lemma 8.1, the product of the Rédéi symbols \([a, b, c]\) and \([a, c, b]\), when defined as in (28) with the help of \( F = E(\sqrt{b}) \) and \( F' = E'(\sqrt{c}) \), respectively, equals \( \prod_{p \leq \infty} (\beta, \gamma)_p \), where the product ranges over all primes \( p \leq \infty \) of \( \mathbb{Q}(\sqrt{a}) \). By the product formula for Hilbert symbols, this value is equal to 1, so we have \([a, b, c] = [c, b, a] \), as desired.

**Corollary 8.2.** The value of the symbol \([a, b, c]\) in (7.6) is the same for all minimally ramified extensions \( K \subset F \).

**Proof.** By Theorem 7.7, the symbol is equal to \([a, c, b]\), which is defined independently of a choice of \( K \subset F \).

Even though the symbol \([a, b, c]\) itself is independent of the choice of \( F \) in (7.6), its \( p \)-parts \([a, b, c]_{F, p}\) in (30) do depend on the minimally ramified extension \( K \subset F \).

It is also possible to define \([a, b, c]\) as an Artin symbol in an abelian extension \( K \subset F_{a,b} \) that is uniquely defined in terms of \( a \) and \( b \). For any minimally ramified extension \( K \subset F \) as in (21), we can take the compositum

\[
F_{a,b} = FG_{a,b},
\]

of \( F \) with the multiquadratic extension \( G_{a,b} \) obtained by adjoining the square roots \( \sqrt{t} \) of the elements \( t \in T_{a,b} \) from (24). By Lemma 7.5, \( F_{a,b} \) is the compositum of all minimally ramified extensions \( K \subset F \), so it is uniquely defined in terms of \( a \) and \( b \).

We now replace \( F \) by \( F_{a,b} \) in (28) and define \([a, b, c] \in \text{Gal}(F_{a,b}/G_{a,b}) = \{\pm 1\}\) as

\[
[a, b, c] = \text{Art}_c(F_{a,b}/K) = \begin{cases} 
\text{Art}(c, F_{a,b}/K) & \text{if} \ c > 0; \\
\text{Art}(c \infty, F_{a,b}/K) & \text{if} \ c < 0.
\end{cases}
\]

By the following Lemma, this is an equivalent definition.

**Lemma 8.3.** Let \( a, b, c \in \mathbb{Z} \setminus \{1\} \) be squarefree integers satisfying (26) and (27). Then \( \text{Art}_c(F_{a,b}/K) \) in (36) restricts to the identity on \( G_{a,b} \).

**Proof.** It follows from Corollary 8.2 that this has to be the case. Alternatively, one can prove directly that the conditions \((a, c)_p = (b, c)_p = 1\) imply that \( \text{Art}_c \) in (36) acts trivially on the square roots \( \sqrt{t} \) for \( t \in T_{a,b} \).
Although Definition (36) is in many ways the ‘correct’ definition of \([a, b, c]\), it has the algorithmic disadvantage of being defined in a field that is potentially very large. For the proof of the reciprocity of the symbol, and for actual computations of Rédei symbols, the \(p\)-parts of \([a, b, c]\), which are Legendre symbols in quadratic fields such as \(K_a = \mathbb{Q}(\sqrt{a})\) by (31), are handled more easily.

9. Governing fields

An immediate application of Rédei’s reciprocity law in the form we have stated it is the existence of governing fields for the 8-rank of the narrow class group \(C(dp)\) of the quadratic field \(\mathbb{Q}(\sqrt{dp})\), with \(d\) a fixed squarefree integer and \(p\) a variable prime. By this, we mean that there exists a normal number field \(\Omega_{8,d}\) with the property that for primes \(p, p' \nmid d\) that are coprime to its discriminant and have the same Frobenius conjugacy class in \(\text{Gal}(\Omega_{8,d}/\mathbb{Q})\), the groups \(C(dp)/C(dp')\) are isomorphic.

Theorem 2.1 trivially implies that we \(\Omega_{2,d} = \mathbb{Q}(i)\) is a governing field for the 2-rank of \(C(dp)\). By the explicit form (11) of Theorem 3.1, we see that that we can take

\[
\Omega_{4,d} = \mathbb{Q}(i, \sqrt{p : p\mid d \text{ prime}})
\]

for the field governing the 4-rank of \(C(dp)\). Suppose \(p\) and \(p'\) are primes that are unramified in \(\Omega_{4,d}\) with the same Frobenius conjugacy class in \(\text{Gal}(\Omega_{4,d}/\mathbb{Q})\). Then the Rédei matrices \(R_4\) and \(R_4'\) for \(C(dp)\) and \(C(dp')\) as given in (11) coincide, and this implies that the 8-rank maps in (12) can be described by matrices \(R_8\) and \(R_8'\) for \(C(dp)\) and \(C(dp')\) with entries given by (13) that may be compared ‘entry-wise’.

In other words, for every entry in \(R_8\) corresponding to the Rédei symbol \([d_1, d_2, m]\) in Definition 4.4, with \(d_1d_2 \in \{dp, 4dp\}\) a decomposition of the second kind and \(m|dp\) the norm of an integral ideal \(m\) with \(m \in C(dp)[2] \cap C(dp)^2\), we have a corresponding Rédei symbol \([d_1', d_2', m']\) in which the arguments are obtained by replacing prime factors \(p\) dividing the entries by \(p'\).

Possibly switching the role of \(d_1\) and \(d_2\), we may suppose that we have \(p \nmid d_1\), and therefore \(d_1 = d_1'\). We may also suppose that we have \(p \nmid m\) and \(m = m'\), since in the case \(p|m\) we can multiply \([d_1, d_2, m]\) with the trivial Rédei symbol

\[\quad [d_1, d_2, -d_1d_2] = 1\]

to rewrite it as

\[
[d_1, d_2, m] = [d_1, d_2, -d_1d_2/m] = [d_1, d_2, dp/m].
\]  

(37)

The triviality of \([d_1, d_2, -d_1d_2]\) follows by taking \(\epsilon = (\sqrt{d_1d_2}) = (\sqrt{dp})\) in the definition (28). For \(d_1d_2 = dp < 0\) this is the trivial ideal class in \(C(dp)\), which acts trivially on every subfield of the Hilbert class field. For \(d_1d_2 = dp > 0\) it is an ideal in the class \(F_\infty\) in (1), which acts as the Frobenius at infinity and becomes trivial when multiplied by itself as we do in (28) for \(-d_1d_2 < 0\).

In order to show that the value of \([d_1, d_2, m]\) for \(p \nmid d_1\) is governed by the splitting behavior of \(p\) in some finite extension of \(\Omega_{4,d}\), it suffices to rewrite it using Theorem 7.7 as

\[\quad [d_1, d_2, m] = [d_1, m, d_2],\]

and observe that we now have \([d_1, m, d_2] = [d_1, m, d_2']\) for \(d_2' = p'd_2/p\) whenever \(p\) and \(p'\) have the same splitting behavior in \(\Omega_{4,d}(\sqrt{m})\), with \(\mu \in \mathbb{Q}(\sqrt{d_1})\) an element with norm in \(m \cdot \mathbb{Q}^{+2}\) that generates a minimally ramified extension of
ON REDEI’S Biquadratic Artin Symbol

Let $K = \mathbb{Q}(\sqrt{md})$ as in (21) for $(a, b) = (d_1, m)$. Taking the compositum of all the fields arising in this way, we arrive at the following theorem, which was proved in a more involved way in 1988 in [13]. The short proof we gave above already occurs in [3].

**Theorem 9.1.** A governing field $\Omega_{8,d}$ for the 8-rank of $C(dp)$ exists, and one can take for it the maximal exponent 2 extension of $\Omega_{4,d}$ unramified outside 2d.

Cohn and Lagarias [2] conjectured in 1983 that such governing fields should also exist for all higher 2-power ranks of $C(dp)$. The recent work of Milovic [5], which proves the first density results for 16-ranks of class groups $C(dp)$ with cyclic 2-part, such as $C(-2p)$, with error terms that are “too good” to come from a governing field, makes it unlikely that the conjecture holds for $2^k$-ranks with $k > 3$.

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