NEW CLASSES OF STRICTLY OPTIMAL LOW HIT ZONE FREQUENCY HOPPING SEQUENCE SETS

HONGYU HAN∗
School of Computer Science
Sichuan Normal University
Chengdu, Sichuan 610066, China

SHENG ZHANG
School of Information Science and Technology
Southwest Jiaotong University
Chengdu, Sichuan 611756, China

(Communicated by Zhengchun Zhou)

Abstract. Low hit zone frequency hopping sequences (LHZ FHSs) with favorable partial Hamming correlation properties are desirable in quasi-synchronous frequency hopping multiple-access systems. An LHZ FHS set is considered to be strictly optimal when it has optimal partial Hamming correlation for all correlation windows. In this study, an interleaved construction of new sets of strictly optimal LHZ FHSs is proposed. Strictly optimal LHZ FHS sets with new and flexible parameters are obtained by selecting suitable known optimal FHSs and appropriate shift sequences.

1. Introduction

Throughout this paper, let \( l \) denote a positive integer and \( \mathcal{F} = \{f_1, f_2, \cdots, f_l\} \) be a set of \( l \) available frequencies, also called the alphabet. A sequence \( X = \{x_k\}_{k=0}^{N-1} \) is called a frequency hopping sequence (FHS) of length \( N \) over \( \mathcal{F} \) if \( x_k \in \mathcal{F} \) for all \( 0 \leq k \leq N-1 \). For any two FHSs \( X = \{x_k\}_{k=0}^{N-1}, Y = \{y_k\}_{k=0}^{N-1} \) of length \( N \) over \( \mathcal{F} \), their Hamming correlation \( H_{X,Y}(\tau) \) at a time delay \( \tau \) is defined by

\[
H_{X,Y}(\tau) = \sum_{k=0}^{N-1} h(x_k, y_{k+\tau}), \quad 0 \leq \tau < N
\]

where \( h(a, b) = 1 \) if \( a = b \), and 0 otherwise, and all operations among the position indices are performed modulo \( N \). If \( x_k = y_k \) for all \( 0 \leq k \leq N-1 \), then we say \( X = Y \) and call \( H_{X,X}(\tau) \) Hamming autocorrelation of \( X \), denoted by \( H_X(\tau) \) for short. Define \( H(X) \) as

\[
H(X) = \max_{1 \leq \tau < N} \{H_X(\tau)\}
\]

2010 Mathematics Subject Classification: Primary: 94A05; Secondary: 94B60.
Key words and phrases: Frequency hopping sequences, low hit zone, partial Hamming correlation, interleaved technique, quasi-synchronous frequency hopping communication.

This work is supported in part by the National Science Foundation of China under Grants 61701331, 61801401, and in part by the Project of Sichuan Education Department under Grant 18ZB0496.

∗ Corresponding author: Hongyu Han.
for an FHS $X$, and $H(X,Y)$ as

$$H(X,Y) = \max_{0 \leq \tau < N} \{ H_{X,Y}(\tau) \}$$

for two FHSs $X$ and $Y$ such that $X \neq Y$.

Throughout this paper, let $(N,l,\alpha)$ denote an FHS $X$ of length $N$ over an alphabet of size $l$, with $\alpha = H(X)$.

Let $\mathcal{V}$ be a set of $M$ FHSs of length $N$ over the alphabet $\mathcal{F}$. The maximum Hamming correlation $H(\mathcal{V})$ of the sequence set $\mathcal{V}$ is defined by

$$H(\mathcal{V}) = \max \left\{ \max_{X \in \mathcal{V}} \{ H(X) \}, \max_{X,Y \in \mathcal{V}, X \neq Y} \{ H(X,Y) \} \right\}.$$ 

The partial Hamming correlation of two FHSs $X, Y \in \mathcal{V}$, for a correlation window length $L$ starting at $k$ is defined by

$$H_{X,Y}(k|L; \tau) = \sum_{j=k}^{k+L-1} h(x_j, y_{j+\tau}), \quad 0 \leq \tau < N, 1 \leq L \leq N$$

where all operations among the position indices are performed modulo $N$. When $X = Y$, $H_{X,Y}(k|L; \tau)$ is called the partial Hamming autocorrelation of $X$, and denoted by $H_{X}(k|L; \tau)$ for short. In particular, if $L = N$ and $k = 0$, the partial Hamming correlation defined in (1) is exactly the conventional Hamming correlation $H_{X,Y}(\tau)$. Define $H(X;L)$ as

$$H(X;L) = \max_{1 \leq \tau < N, 0 \leq k < N} \{ H_{X}(k|L; \tau) \}$$

for an FHS $X$, and $H(X,Y;L)$ as

$$H(X,Y;L) = \max_{0 \leq \tau, k < N} \{ H_{X,Y}(k|L; \tau) \}$$

for two FHSs $X$ and $Y$ such that $X \neq Y$.

For the given FHS set $\mathcal{V}$, the maximum partial Hamming correlation $H(\mathcal{V};L)$ of $\mathcal{V}$ for a correlation window length $L$ is defined by

$$H(\mathcal{V};L) = \max \left\{ \max_{X \in \mathcal{V}} \{ H(X;L) \}, \max_{X,Y \in \mathcal{V}, X \neq Y} \{ H(X,Y;L) \} \right\}.$$ 

For the FHS set $\mathcal{V}$, let $\gamma$ be a given nonnegative integer. Then the low hit zone (LHZ) $L_H$ is defined by

$$L_H = \min \{ L_{AH}, L_{CH} \}$$

where

$$L_{AH} = \max \{ R \mid H_X(\tau) \leq \gamma, \forall X \in \mathcal{V} \},$$

$$L_{CH} = \max \{ R \mid H_{X,Y}(\tau) \leq \gamma, \forall X \in \mathcal{V}, \forall Y \in \mathcal{V}, X \neq Y \}.$$ 

An FHS set $\mathcal{V}$ with $L_H \geq 0$ is called an LHZ FHS set.

Throughout this paper, we use $(N,M,l,L_H,\gamma)$ to denote a set of $M$ FHSs of length $N$ over an alphabet of size $l$, with the LHZ $L_H$ and maximum Hamming correlation $\gamma$ within the LHZ.

For the FHS set $\mathcal{V}$ and a given correlation window length $L$ ($L \leq N$), let $\varepsilon$ be a given nonnegative integer. Then the partial Hamming correlation low hit zone $L_{PH}$ is defined by

$$L_{PH} = \min \{ L_{PAH}, L_{PCH} \}$$
where
\[ L_{PAH} = \max\{ R \mid H_X(k; \tau) \leq \varepsilon, \text{for } 0 \leq k < N, 1 \leq \tau \leq R, \forall X \in \mathcal{V} \}, \]
\[ L_{PCH} = \max\{ R \mid H_{X,Y}(k; \tau) \leq \varepsilon, \text{for } 0 \leq k < N, 0 \leq \tau \leq R, \forall X, Y \in \mathcal{V}, X \neq Y \}. \]

Frequency hopping spread spectrum techniques have been widely used in ultra-wideband (UWB) communication, Bluetooth, military applications, and other areas [1, 3, 24]. FHSs with low Hamming correlation are employed in such systems to reduce the multiple-access interference caused by frequency hits [8]. Therefore, designing FHSs with favorable Hamming correlation properties is crucial. The periodic Hamming correlation of FHSs has received extensive attention in literature. The study of the periodic Hamming autocorrelation of FHSs can be traced back to the seminal work of Lempel and Greenberger [14], in which the well-known Lempel-Greenberger bound was established. Peng and Fan [19] made a subsequent contribution by developing the bounds on the periodic Hamming correlation of an FHS set. Over the past decades, numerous constructions of FHSs achieving those bounds have been proposed [6, 10].

While a great deal of knowledge exists for the periodic Hamming correlation, relatively little is known about the partial Hamming correlation of FHSs. FHSs with favorable partial Hamming correlation properties are important for applications wherein an appropriate window length is shorter than the period of the selected FHSs to reduce hardware complexity or minimize the synchronization time of receivers [7, 21]. The partial Hamming correlation of FHSs is also an important theoretical object of interest because it includes the periodic Hamming correlation as a special case [2]. Therefore, it is necessary to consider the partial Hamming correlation, rather than the period Hamming correlation. In 2004, Eun et al. [7] derived the bound on the partial Hamming autocorrelation of individual FHSs and obtained a class of individual FHSs with optimal partial Hamming autocorrelation. In 2010, Niu et al. [16] generalized the bound in [7] to the case of an FHS set and obtained the bound on the partial Hamming correlation of an FHS set. Subsequently, individual FHSs with optimal partial autocorrelation and FHS sets with optimal partial correlation were constructed [2, 17, 27].

The design of an FHS set with LHZ aims to minimize Hamming correlation values within a correlation zone [20, 23]. The importance of an LHZ FHS set is that the number of collisions between difference FHSs is small as long as the relative delay does not exceed the fixed limit (zone), thereby reducing the mutual interference. LHZ FHS sets with low Hamming correlation within the fixed zone are useful in quasi-synchronous frequency hopping multiple-access (FHMA) systems wherein relative delays between distinct users are restricted within a zone around the origin [9]. Several optimal LHZ FHS sets with respect to the Peng-Fan-Lee bound [20] have been constructed in literature [4, 13, 15, 18].

LHZ FHS sets with optimal partial Hamming correlation are desirable in many quasi-synchronous FHMA systems [12]. In 2010, Niu et al. [16] obtained the bound on the partial Hamming correlation of an LHZ FHS set. However, only a few constructions of LHZ FHS sets with optimal partial Hamming correlation with respect to the bound in [16] have been reported. In 2015, Wang et al. [22] constructed two classes of LHZ FHS sets with strictly optimal partial Hamming correlation based on Cartesian product. The purpose of the current work is to construct new sets of LHZ FHSs with strictly optimal partial Hamming correlation with respect to the bound in [16] by using the interleaving technique.
The remainder of this paper is organized as follows. In Section 2, we recall some preliminaries. In Section 3, interleaving technique is reviewed. In Section 4, an interleaving construction of strictly optimal LHZ FHS sets is presented. Finally, we give some concluding remarks in Section 5.

2. Preliminaries

For a real number $x$, let $\lceil x \rceil$ denote the least integer no less than $x$. In 2010, Niu et al. [16] obtained the following lower bound on the maximum partial Hamming correlation of an FHS set with LHZ.

Lemma 2.1. Let $\mathcal{V}$ be a set of $M$ FHSs of length $N$ over an alphabet of size $l$, and $L_{PH}$ the partial Hamming correlation low hit zone of $\mathcal{V}$ with respect to the maximum partial Hamming correlation $\varepsilon$. Then for each correlation window length $1 \leq L \leq N$, we have

$$\varepsilon \geq \left\lceil \frac{(ML_{PH} + M - l)L}{(ML_{PH} + M - 1)l} \right\rceil.$$  

Definition 2.2. An FHS set $\mathcal{V}$ is optimal with respect to the bound in (2) and the given correlation window length $L$ when the equality in (2) is achieved for the given correlation window length $L$. In this case, the FHS set $\mathcal{V}$ is called an LHZ FHS set with optimal partial Hamming correlation. When the equality in (2) is achieved for all correlation window lengths $1 \leq L \leq N$, the FHS set $\mathcal{V}$ is strictly optimal with respect to the bound in (2). In this case, $\mathcal{V}$ is called an LHZ FHS set with strictly optimal partial Hamming correlation.

In (2), let $L_{PH} = N - 1$ and $M = 1$, Niu et al. [16] derived the following lower bound on the maximum partial Hamming autocorrelation of an FHS.

Lemma 2.3. Let $X$ be an FHS of length $N$ over an alphabet of size $l$. Then for each correlation window length $1 \leq L \leq N$, we have

$$H(X; L) \geq \left\lceil \frac{(N - l)L}{(N - 1)l} \right\rceil.$$  

Definition 2.4. An FHS $X$ is optimal with respect to the bound in (3) and the given correlation window length $L$ when the equality in (3) is achieved for the given correlation window length $L$. In this case, the FHS $X$ is called an FHS with optimal partial Hamming autocorrelation. When the equality in (3) is achieved for all correlation window lengths $1 \leq L \leq N$, the FHS $X$ is strictly optimal with respect to the bound in (3). In this case, $X$ is called an FHS with strictly optimal partial Hamming autocorrelation.

3. Interleaving technique

Interleaving technique was introduced by Gong in [11], and it has been applied to the construction of sequences with good Hamming correlation [5, 25] and other sequences with good correlation (but not Hamming correlation) [26]. We provide a brief introduction of interleaving technique.
Let $C = (c_0, c_1, \cdots, c_{N-1})$ be an $(N, l, \alpha)$ FHS, and $D = (d_0, d_1, \cdots, d_{I-1})$ a shift sequence of length $I$ over $Z_N$, i.e., $d_k \in Z_N$, $0 \leq k < I$. An $N \times I$ matrix can be formed by placing the shifts of sequence $C$ referred to $D$ as follows

$$
\mathcal{E} = \begin{pmatrix}
    c_0 + d_0 & c_0 + d_1 & \cdots & c_0 + d_{I-1} \\
    c_1 + d_0 & c_1 + d_1 & \cdots & c_1 + d_{I-1} \\
    \vdots & \vdots & \ddots & \vdots \\
    c_{N-1} + d_0 & c_{N-1} + d_1 & \cdots & c_{N-1} + d_{I-1} \\
\end{pmatrix}
$$

where $\mathcal{E}$ is referred to the shift sequence, and $E$ elements in matrix $\mathcal{E}$ is used for an array form of $C$. We write the interleaved sequence $E$ as

$$
E = \mathcal{I}(\mathcal{L}^{d_0}(C), \mathcal{L}^{d_1}(C), \cdots, \mathcal{L}^{d_{I-1}}(C)),
$$

where $\mathcal{I}$ is the interleaving operator, and $\mathcal{L}$ is the (left cyclical) shift operator, i.e., $\mathcal{L}^2(C) = (c_2, \ldots, c_{N-1}, c_0, c_1)$.

4. **An Interleaving Construction of Strictly Optimal LHZ FHS Sets**

In this section, we present a construction of strictly optimal LHZ FHS sets via interleaving technique. New classes of strictly optimal LHZ FHS sets are obtained with the new design.

**Construction 1.**

**Step 1:** Select an $(N, l, \alpha)$ FHS

$$
C = (c_0, c_1, \cdots, c_{N-1})
$$

with strictly optimal partial Hamming autocorrelation, where $H_C(k_c|L_c; \tau) = 1$ for all $k_c, \tau \ (0 \leq k_c < N, 1 \leq \tau < N)$ and the given correlation window length $L_c$.

**Step 2:** Let $R, I, v$ and $s$ be four positive integers such that $R \geq 2$, $Iv = N$, $\gcd(s, N) = 1$, $\gcd((v + 1)s^{-1} \pmod{N}, L_c) = 1$, and $(v + 1)Rs^{-1} \equiv 1 \pmod{N}$. Generate a shift sequence set $\mathcal{G} = \{G^t \ | \ 0 \leq t < I\}$, where for each $0 \leq t < I$,

$$
G^t = (g^t_0, g^t_1, \cdots, g^t_{R-1})
$$

$$
= (tv^s^{-1} \pmod{N}), (tv^s + (v + 1)s^{-1} \pmod{N}), \cdots, (tv^s + (v + 1)(R-1)s^{-1} \pmod{N}).
$$

**Step 3:** Construct the desired FHS set $\mathcal{P} = \{P^t \ | \ 0 \leq t < I\}$, where for each $0 \leq t < I$,

$$
P^t = (P^t_0, P^t_1, \cdots, P^t_{R-1})
$$

$$
= \mathcal{I}(\mathcal{L}^{tv^s \pmod{N}}(C), \mathcal{L}^{(tv^s + (v + 1))s^{-1} \pmod{N}}(C), \cdots, \mathcal{L}^{(tv^s + (v + 1)(R-1))s^{-1} \pmod{N}}(C))
$$

**Theorem 4.1.** The FHS set $\mathcal{P}$ constructed by Construction 1 is an $(RN, I, l, v - 1, R\alpha)$ LHZ FHS set with strictly optimal partial Hamming correlation when $R \equiv s \pmod{v}$. 
Proof. From Theorem 2 in [18], we know that \( \mathcal{P} \) is an \((RN, I, l, v - 1, R\alpha)\) LHZ FHS set when \( R = s \text{(mod } v) \).

By the matrix representation, \( P^t \) can be written as

\[
\begin{pmatrix}
C_{tvu-1} & C_{(tv+(v+1))z} & \cdots & C_{(tv+(v+1)(R-1))z}
\end{pmatrix}
\]

where all operations in subscripts are performed modulo \( N \). Let \( w = yR + z \), \( 0 \leq y < N \), \( 0 \leq z < R \). From \( (v+1)Rs^{-1} \equiv 1 \text{(mod } N) \), we have

\[
P^t_w = C_{(tv+(v+1))z} + y = C_{(tv+(v+1)(w-yR))z}
\]

\[
= C_{(tv+(v+1)w)s-1} + [1 - (v+1)Rs^{-1}]y = C_{(tv+(v+1)w)s-1}.
\]

The partial Hamming correlation between two sequences \( P_{t_1}, P_{t_2} \in \mathcal{P}, 0 \leq t_1, t_2 < I \), for the correlation window length \( L_p \) starting at \( k_p \) is

\[
H_{P_{t_1}, P_{t_2}}(k_p|L_p; \tau) = \sum_{w=k_p}^{k_p+L_p-1} h(p^t_{t_1}, p^t_{t_2})
\]

where \( 1 \leq \tau \leq v - 1 \) if \( t_1 = t_2 \), and \( 0 \leq \tau \leq v - 1 \) otherwise.

Given that \( H_C(k_c|L_c; \tau) = 1 \) for all \( k_c \), \( 0 \leq k_c < N \), \( 1 \leq \tau < N \) and the given correlation window length \( L_c \). From the definition of partial Hamming correlation, we obtain

\[
H_C(k_c|L_c; \tau) = \sum_{j=k_c}^{k_c+L_c-1} h(c_j, c_{j+\tau}) = 1.
\]

Evidently, \( j^* (0 \leq j^* < N) \) exists such that \( c_{j^*} - c_{j^*+\tau} = 0 \) and \( c_{j^*+n_1} - c_{j^*+\tau+n_1} \neq 0 \) for all \( n_1, 1 \leq n_1 < L_c \), where \( L_c|N \). This produces

\[
c_{j^*+n} - c_{j^*+\tau+n} = \begin{cases} 0, & \text{if } n \equiv 0 \text{(mod } L_c) \\ \text{nonzero}, & \text{otherwise} \end{cases}
\]

Therefore, the maximum partial Hamming autocorrelation \( H(C; L) \) of FHS \( C \) for the correlation window length \( L \) can be given by

\[
H(C; L) = \max \{H_C(k_c|L; \tau) \mid 0 \leq k_c < N, 1 \leq \tau < N\}
\]

\[
= \max \{|n \mid c_{k_c+n} - c_{k_c+n+\tau} = 0, 0 \leq k_c < N, 1 \leq \tau < N, 0 \leq n < L\}
\]

\[
= \max \{|n \mid n \equiv 0 \text{(mod } L_c), 0 \leq n < L\}
\]

\[
= \left\lfloor \frac{L}{L_c} \right\rfloor.
\]

From (4), we determine that \( c_{(tv+(v+1)w)s-1} \) runs over all frequencies in \((c_0, c_1, \ldots, c_{N-1})\) as \( w \) takes all elements in \((z, R + z, \ldots, yR + z, \ldots, (N-1)R + z)\) for any fixed \( 0 \leq z < R \). Then, \( w^* (0 \leq w^* < RN) \) exists such that

\[
c_{(t_1v+(v+1)w^*)s-1} - c_{(t_2v+(v+1)(w^*+\tau))s-1} = 0
\]
and
\[ c_{(t_1v+(v+1)(w^*+n_2))s^{-1}} - c_{(t_2v+(v+1)(w^*+\tau+n_2))s^{-1}} = c_{(t_1v+(v+1)w^*)s^{-1}+(v+1)n_2s^{-1}} - c_{(t_2v+(v+1)(w^*+\tau))s^{-1}+(v+1)n_2s^{-1}} \neq 0 \]
for all \( n_2, 1 \leq (v+1)n_2s^{-1}(mod\ N) < L_c \) where \( L_c|N \). This produces
\[ (7) \]
\[ c_{(t_1v+(v+1)(w^*+n))s^{-1}} - c_{(t_2v+(v+1)(w^*+\tau+n))s^{-1}} = \begin{cases} 
0, & \text{if } (v+1)ns^{-1} \equiv 0(\text{mod } L_c) \\
nonzero, & \text{otherwise}.
\end{cases} \]
Given that \( \text{gcd}((v+1)s^{-1}(mod\ N), L_c) = 1 \), from (5) and (7) we have
\[ (8) \]
\[ p_{w^*+n}^{t_1} - p_{w^*+\tau+n}^{t_2} = \begin{cases} 
0, & \text{if } n \equiv 0(\text{mod } L_c) \\
nonzero, & \text{otherwise}.
\end{cases} \]

Therefore, the maximum partial Hamming correlation \( H(\mathcal{D}; L) \) of LHZ FHS set \( \mathcal{D} \) for the correlation window length \( L \) can be given by
\[ H(\mathcal{P}; L) = \max \{ H_{p_{k_1}, p_{k_2}}(k_p|L; \tau) \mid 0 \leq t_1, t_2 < I, 0 \leq k_p < R_N, \]
\[ 1 \leq \tau \leq v-1 \] if \( t_1 = t_2, 0 \leq \tau \leq v-1 \) if \( t_1 \neq t_2 \}
\[ = \max \{|n| \mid p_{k_1+n}^{t_1} - p_{k_2+n}^{t_2} = 0, 0 \leq t_1, t_2 < I, 0 \leq k_p < R_N, 0 \leq n < L, \]
\[ 1 \leq \tau \leq v-1 \] if \( t_1 = t_2, 0 \leq \tau \leq v-1 \) if \( t_1 \neq t_2 \}
\[ = (8) \max \{|n| \mid n \equiv 0(\text{mod } L_c), 0 \leq n < L\} \]
\[ = \left\lfloor \frac{L}{L_c} \right\rfloor. \]

According to the bound in (2), for the \( (RN, I, l, v-1, Ro) \) LHZ FHS set \( \mathcal{P} \), given that \( Iv = N \), the maximum partial Hamming correlation for the correlation window length \( L \) can be given by
\[ H(\mathcal{P}; L) \geq \left\lfloor \frac{(I(v-1)+I-l)L}{(I(v-1)+I-1)l} \right\rfloor = \left\lfloor \frac{(N-l)L}{(N-1)l} \right\rfloor. \]

Given that sequence \( C \) is an \( (N, l, \alpha) \) FHS with strictly optimal partial Hamming autocorrelation, in accordance with the bound in (3) and Definition 2.4, we derive
\[ H(C; L) = \left\lfloor \frac{(N-l)L}{(N-1)l} \right\rfloor. \]
Because of \( H(\mathcal{P}; L) = H(C; L) = \left\lfloor \frac{L}{L_c} \right\rfloor \), we get
\[ H(\mathcal{P}; L) = \left\lfloor \frac{(N-l)L}{(N-1)l} \right\rfloor. \]

Therefore, the \( (RN, I, l, v-1, Ro) \) LHZ FHS set \( \mathcal{P} \) meets the bound in (2) with equality. That is, set \( \mathcal{P} \) is an LHZ FHS set with strictly optimal partial Hamming autocorrelation and appropriate shift sequences. 

Based on the Construction 1, new LHZ FHS sets with strictly optimal partial Hamming correlation are obtained in Table 1 by utilizing known individual FHSs with strictly optimal partial Hamming autocorrelation and appropriate shift sequences.
Table 1. New sets of strictly optimal LHZ FHSs

| Based on individual FHSs with strictly optimal partial Hamming autocorrelation | $L_c$ | $(RN, I, l, v - 1, R\alpha)$ | Maximum partial Hamming correlation for the correlation window length $L$ | Constraints |
|---|---|---|---|---|
| [2] | $g$ | $(Reg, I, g, v - 1, Re)$ | $\left\lfloor \frac{L}{g} \right\rfloor$ | $R \geq 2, Iv = N, \gcd(s, N) = 1,$ |
| [7] | $q+1$ | $(R(q^2 - 1), I, q, v - 1, R(q - 1))$ | $\left\lfloor \frac{L}{q+1} \right\rfloor$ | $\gcd((v + 1)s^{-1}(\text{mod } N), L) = 1,$ |
| [27] | $T$ | $(R(q^n - 1), I, q^{n-1}, v - 1, R(q - 1))$ | $\left\lfloor \frac{T}{T} \right\rfloor$ | $(v+1)Rs^{-1} \equiv 1(\text{mod } N), R = s(\text{mod } v)$ |

$g$ is any odd integer with the prime factor decomposition $g = p_1^{m_1}p_2^{m_2} \cdots p_k^{m_k}$; $e > 1, e|\gcd(p_1 - 1, p_2 - 1, \cdots, p_k - 1)$; $q$ is a prime power and $T = \frac{q^{n-1}}{q-1}$.

In the following we give an example of Construction 1.

**Example 1.** Select an $(48, 7, 6)$ FHS $C = \{126221605323352041311430651556102454425036466340\}$ meeting the bound in (3) for all correlation window length $1 \leq L \leq 48$. We can check that $H_C(k_c|L_c; \tau) = 1$ for all $k_c, \tau (0 \leq k_c < 48, 1 \leq \tau < 48)$ and the correlation window length $L_c = 8$.

Generate a shift sequence set $G = \{G^t = (g^t_0, g^t_1, g^t_2, g^t_3, g^t_4) | 0 \leq t < 4\}$ by the Construction 1, such that

$G^0 = (0, 29, 10, 39, 20), G^1 = (12, 41, 22, 3, 32), G^2 = (24, 5, 34, 15, 44), G^3 = (36, 17, 46, 27, 8)$

where $R = 5, I = 4, v = 12$, and $s = 17$.

We can construct the FHS set $\mathcal{P} = \{P^0, P^1, P^2, P^3\}$ by the Construction 1, where

$P^0 = \{16201213346036322540242661506564431041 \cdots\}$,  
$P^1 = \{36322540242661506564431041455230535116 \cdots\}$,  
$P^2 = \{61506564431041455230535116201213346036 \cdots\}$,  
$P^3 = \{41455230535116201213346036322540242661 \cdots\}$.

It can be verified that for all $\tau$ ($\tau \leq 11$), the maximum partial Hamming correlations of FHS set $\mathcal{P}$ for all correlation window lengths $L$, $0 < L \leq 240$, are given by
Optimal low hit zone frequency hopping sequences

\[ H(\mathcal{P}; L) = \begin{cases} 
1, & 0 < L \leq 8, \\
2, & 8 < L \leq 16, \\
\vdots & \vdots \\
i, & 8(i - 1) < L \leq 8i, \\
\vdots & \vdots \\
30, & 232 < L \leq 240. 
\end{cases} \]

The maximum partial Hamming correlations of \( \mathcal{P} \) for the correlation window length \( L = 8 \) are shown in Fig. 1. It can be seen that \( H(\mathcal{P}; 8) = 1 \) for all \( \tau (\tau \leq 11) \).

We verify that the \((240, 4, 7, 11, 30)\) LHZ FHS set \( \mathcal{P} \) meets the bound in (2) for all correlation window lengths \( 1 \leq L \leq 240 \). That is, set \( \mathcal{P} \) is an LHZ FHS set with strictly optimal partial Hamming correlation.

**Figure 1.** Maximum partial Hamming correlations of \( \mathcal{P} \) for the correlation window length \( L = 8 \) in Example 1

5. Conclusions

In this paper, a new design is proposed for strictly optimal LHZ FHS sets by using interleaving technique. New LHZ FHS sets with strictly optimal partial Hamming correlation are obtained with the new design. The proposed FHS sets may be useful in eliminating MA interference in quasi-synchronous FHMA systems.

References

[1] “Specification of the Bluetooth SystemsCore,” Bluetooth Special Interest Group (SIG), 2003. Available from: [http://www.bluetooth.org](http://www.bluetooth.org).
[2] H. Cai, Z. C. Zhou, Y. Yang and X. H. Tang, A new construction of frequency-hopping sequences with optimal partial Hamming correlation, IEEE Trans. Inform. Theory, 60 (2014), 5782–5790.

[3] H. H. Chen, “The Next Generation CDMA Technologies”, John Wiley & Sons, London, 2007.

[4] J. H. Chung and K. Yang, New classes of optimal low-hit-zone frequency-hopping sequence sets by Cartesian product, IEEE Trans. Inform. Theory, 59 (2013), 726–732.

[5] J. H. Chung, Y. K. Han and K. Yang, New classes of optimal frequency-hopping sequences by interleaving techniques, IEEE Trans. Inform. Theory, 55 (2009), 5783–5791.

[6] C. Ding, R. Fuji-Hara, Y. Fujiwara, M. Jimbo and M. Mishima, Sets of frequency hopping sequences: Bounds and optimal constructions, IEEE Trans. Inform. Theory, 55 (2009), 3297–3304.

[7] Y. C. Eun, S. Y. Jin, Y. P. Hong and H. Y. Song, Frequency hopping sequences with optimal partial autocorrelation properties, IEEE Trans. Inform. Theory, 50 (2004), 2438–2442.

[8] P. Z. Fan and M. Darnell, "Sequence Design for Communications Applications," John Wiley & Sons, London, 1996.

[9] R. D. Gaudenzi, C. Elia and R. Viola, Bandlimited quasi-synchronous CDMA: A novel satellite access technique for mobile and personal communication systems, IEEE J. Sel. Areas Commun., 10 (1992), 328–343.

[10] G. Ge, Y. Miao and Z. Yao, Optimal frequency hopping sequences: Auto- and cross-correlation properties, IEEE Trans. Inform. Theory, 55 (2009), 867–879.

[11] G. Gong, Theory and applications of q-ary interleaved sequences, IEEE Trans. Inform. Theory, 41 (1995), 400–411.

[12] H. Y. Han, D. Y. Peng and X. Liu, On low-hit-zone frequency-hopping sequence sets with optimal partial Hamming correlation, in Sequences and Their Applications - SETA 2014, Lecture Notes in Comput. Sci., 8665, Springer, Cham, 2014, 293–304.

[13] H. Y. Han, D. Y. Peng and U. Parampalli, New sets of optimal low-hit-zone frequency-hopping sequences based on m-sequences, Cryptogr. Commun., 9 (2017), 511–522.

[14] A. Lempel and H. Greenberger, Families of sequences with optimal Hamming correlation properties, IEEE Trans. Information Theory, 20 (1974), 90–94.

[15] W. P. Ma and S. H. Sun, New designs of frequency hopping sequences with low hit zone, Des. Codes Cryptogr., 60 (2011), 145–153.

[16] X. H. Niu, D. Y. Peng, F. Liu and X. Liu, Lower bounds on the maximum partial correlations of frequency hopping sequence set with low hit zone, IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences, 93-A (2010), 2227–2231.

[17] X. H. Niu, D. Y. Peng and Z. C. Zhou, Frequency/time hopping sequence sets with optimal partial Hamming correlation properties, Sci. China Inf. Sci., 55 (2012), 2207–2215.

[18] X. H. Niu, D. Y. Peng and Z. C. Zhou, New classes of optimal frequency hopping sequences with low hit zone, Adv. Math. Commun., 7 (2013), 293–310.

[19] D. Y. Peng and P. Z. Fan, Lower bounds on the Hamming auto- and cross correlations of frequency-hopping sequences, IEEE Trans. Inform. Theory, 50 (2004), 2149–2154.

[20] D. Y. Peng, P. Z. Fan and M. H. Lee, Lower bounds on the periodic Hamming correlations of frequency hopping sequences with low hit zone, Sci. China Ser. F, 49 (2006), 208–218.

[21] M. K. Simon, J. K. Omura, R. A. Scholtz and B. K. Levitt, "Spread Spectrum Communications Handbook,” McGraw-Hill, New York, NY, 2001.

[22] C. Y. Wang, D. Y. Peng, H. Y. Han and L. M. N. Zhou, New sets of low-hit-zone frequency-hopping sequence with optimal maximum periodic partial Hamming correlation, Sci. China Inf. Sci., 58 (2015), 1–15.

[23] X. N. Wang and P. Z. Fan, A class of frequency hopping sequences with no hit zone, in Proc. of the 4th International Conference on Parallel and Distributed Computing, Applications and Technologies, 2003, 896–898.

[24] S. Zhang, J. Zhang, W. Zheng and H. So, Widely-linear complex-valued estimated-input LMS algorithm for bias-compensated adaptive filtering with noisy measurements, IEEE Trans. Signal Process., 67 (2019), 3592–3605.
[25] X. Y. Zeng, H. Cai, X. H. Tang and Y. Yang, A class of optimal frequency hopping sequences with new parameters, *IEEE Trans. Inform. Theory*, 58 (2012), 4899–4907.

[26] Z. C. Zhou, X. H. Tang and G. Gong, A new class of sequences with zero or low correlation zone based on interleaving technique, *IEEE Trans. Inform. Theory*, 54 (2008), 4267–4273.

[27] Z. C. Zhou, X. H. Tang, X. H. Niu and P. Udaya, New classes of frequency-hopping sequences with optimal partial correlation, *IEEE Trans. Inform. Theory*, 58 (2012), 453–458.

Received February 2018; 1st revision August 2018; 2nd revision March 2019.

*E-mail address:* hyhan@my.swjtu.edu.cn

*E-mail address:* dr.s.zhang@ieee.org