OPTIMAL CONTROL PROBLEM FOR A VISCOELASTIC BEAM AND ITS GALERKIN APPROXIMATION

ANDRZEJ JUST
Centre of Mathematics and Physics, Lodz University of Technology
Al.Politechniki 11
90-924 Lodz, Poland

ZDZISLAW STEMPIEŃ *
Institute of Mathematics, Lodz University of Technology
ul. Wolczanska 215
90-924 Lodz, Poland

ABSTRACT. This paper is concerned with the optimal control problem of the vibrations of a viscoelastic beam, which is governed by a nonlinear partial differential equation. We discuss the initial-boundary problem for the cases when the ends of the beam are clamped or hinged. We define the weak solution of this initial-boundary problem. Our control problem is formulated by minimization of a functional where the state of a system is the solution of viscoelastic beam equation. We use the Galerkin method to approximate the solution of our control problem with respect to a spatial variable. Based on the finite dimensional approximation we prove that as the discretization parameters tend to zero then the weak accumulation points of the optimal solutions of the discrete family control problems exist and each of these points is the solution of the original optimal control problem.

1. Introduction. In this paper, we discuss the optimal control problem with nonlinear state equation for a viscoelastic beam in 1D. The equation represents a nonlinear beam model developed by J.Ball in [5]. The equation of motion in the vertical displacement of points of this beam has a form

\[
\frac{\partial^2 y}{\partial t^2} + \alpha \frac{\partial^4 y}{\partial x^4} - \left[ \beta + \gamma \int_0^l \left( \frac{\partial y(\xi,t)}{\partial \xi} \right)^2 d\xi \right] \frac{\partial^2 y}{\partial x^2} + \delta \frac{\partial^5 y}{\partial x^5 \partial t} - \sigma \int_0^l \frac{\partial y(\xi,t)}{\partial \xi} \cdot \frac{\partial^2 y(\xi,t)}{\partial \xi \partial t} d\xi \cdot \frac{\partial^2 y}{\partial x^2} + \eta \frac{\partial y}{\partial t} = f(x,t).
\]

The parameters \(\alpha, \gamma, \delta, \sigma\) are positive and \(\beta, \eta \in \mathbb{R}\). These physical constants depend on the Young modulus, the cross-sectional area, the cross-sectional second moment of inertia, and the density of material. The position \(x \in (0,l)\) and the time \(t \in (0,T)\) for \(l, T < \infty\). Later on we put assumptions on \(f\). We shall further make clear in which sense this equation is understood.

2010 Mathematics Subject Classification. Primary: 49J20, 49M25; Secondary: 49K20, 93C20.
Key words and phrases. Nonlinear beam equation, viscoelastic beam, optimal control, Galerkin approximation, convergence.

* Corresponding author: zdzislaw.stempien@p.lodz.pl.
We consider, from the mechanical point of view, the boundary conditions corresponding to clamped ends, when
\[ y(0, t) = y(l, t) = \frac{\partial y(0, t)}{\partial x} = \frac{\partial y(l, t)}{\partial x} = 0 \quad (2) \]
or the boundary hinged ends, when
\[ y(0, t) = y(l, t) = \frac{\partial^2 y(0, t)}{\partial x^2} = \frac{\partial^2 y(l, t)}{\partial x^2} = 0. \quad (3) \]

We consider the initial-boundary value problem consisting of (1), the initial conditions
\[ y(x, 0) = y_0(x) \quad \text{and} \quad \frac{\partial y(x, 0)}{\partial t} = y_1(x) \quad (4) \]
and the boundary conditions (2) or (3).

Mathematical models for nonlinear beam have a long history. In the bibliography, a number of papers have discussed the problem of the existence and uniqueness of solution to a nonlinear beam equation. For example J. Ball’s papers [5, 6] and the papers [1, 3, 11, 18, 19]. In [2] Authors present existence results for one-dimensional viscoelastic beam with the Signorini contact condition.

Concerning the control problems governed by PDEs we refer for the basic theory to Lions [17]. Some questions of optimal control problems for the beam equation were studied by M. Barboteu et al. [7], M. Galewski [12], I. Hlaváček and J. Lovišek [13], J. Hwang [14], I. Sadek et al. [20], and by many others. The Galerkin method can be applied to the boundary problems as well as to control systems. Galerkin method in solving the boundary beam systems was investigated in articles [1, 5, 6, 18, 19]. Semidiscrete Galerkin approximation of nonlinear control problems was studied for example in [4, 16, 20, 21] and in our papers [8, 9, 15].

In the present paper, we extend our discussions in [15] from elastic beam to viscoelastic beam. Section 2 establishes some notations and studies the properties of an operator from the control space into the state space. The optimal control problem is studied in Section 3. In Section 4 we consider the Galerkin approximation of our optimal control problem and in Section 5 we prove the theorem of convergence for semidiscretization.

2. Preliminaries. Let \( \Omega = (0, l) \), where \( l > 0 \) is the natural length of the beam, \( S = (0, T) \) and \( Q = \Omega \times S \). We shall need the following spaces:

- Lebesgue spaces \( L^2(\Omega) \), \( L^2(Q) \),
- Lebesgue-Bochner spaces
  \[ L^2(S; W) = \left\{ \omega : S \to W \mid \int_S \|\omega(t)\|_W^2 dt < \infty \right\} \]
and
  \[ L^\infty(S; W) = \left\{ \omega : S \to W \mid \text{ess sup}_{t \in S} \|\omega(t)\|_W < \infty \right\} \]
with the standard norms, where \( W \) is any Banach space (see [17] p. 108),
- Sobolev spaces \( H^2(\Omega) \), \( H^2_0(\Omega) \), \( H^1_0(\Omega) \) with the standard norms.

Let \( V = H^2_0(\Omega) \) for clamped ends or \( V = H^2(\Omega) \cap H^1_0(\Omega) \) (the closed subspace of \( H^2(\Omega) \)) for hinged ends and \( H = L^2(\Omega) \). These spaces are equipped with standard norms. The embedding \( V \subset H \) is dense and compact. Identifying \( H \) with its dual we have the evolution triple \( V \subset H \subset V^* \) (see [10] p. 391). The duality pairing
(see [22] p.251) \( \langle \varphi, \psi \rangle \) on \( V^* \times V \) is identical with the inner product \( \langle \varphi, \psi \rangle \) on \( H \) if \( \varphi \in H \).

We define a weak solution of the equation (1) with the initial condition (4) and the boundary conditions (2) or (3) (see [6], [14]) as a solution of following variational equation

\[
\langle \dot{y}(t), \psi \rangle + \alpha(y_{xx}(t), \psi_{xx}) - (\beta + \gamma \| y_x(t) \|^2_H)(y_{xx}(t), \psi) + \delta(\dot{y}_{xx}(t), \psi_{xx}) \\
- \sigma(y_x(t), \dot{y}_x(t))(y_{xx}(t), \psi) + \eta(\dot{y}(t), \psi) = (f(t), \psi) \quad \forall \psi \in V \quad \text{for a.e. } t \in S, \quad (5)
\]

\[
y(0) = y_0 \quad \text{and} \quad \dot{y}(0) = y_1,
\]

where \( \langle \varphi, \psi \rangle = \int_0^T \varphi(x)\psi(x)dx \) (the inner product on \( H \)). For simplicity we use the notation \( \dot{y} = \frac{dy}{dt}, \ddot{y} = \frac{d^2y}{dx^2} \) and the subscript \( x \) denotes the derivative with respect to \( x \).

**Remark 1.** In equation (5) the derivatives of the weak solution of the equation (1) are satisfied in the sense of distribution and the solution \( y \) of (5) does not have to satisfy the boundary conditions (3) in any classical sense (see [6]).

In our first theorem we state the existence and uniqueness of weak solution (1)-(4) (see [5, 11]).

**Theorem 2.1.** Suppose \( f \in L^2(Q), y_0 \in V \) and \( y_1 \in H \). Then, there exists a unique solution \( y \) of equations (5), such that \( y \in L^\infty(S; V) \) and \( \dot{y} \in L^\infty(S; H) \cap L^2(S; V) \).

Let us put in (5) \( f = g + Bu \), where \( g \in L^2(Q), u \in U \) (the control space which we assume to be a separable Hilbert space) and \( B \in L(U; L^2(Q)) \). The equation (5) has a form

\[
\langle \dot{y}(t), \psi \rangle + \alpha(y_{xx}(t), \psi_{xx}) - (\beta + \gamma \| y_x(t) \|^2_H)(y_{xx}(t), \psi) + \delta(\dot{y}_{xx}(t), \psi_{xx}) \\
- \sigma(y_x(t), \dot{y}_x(t))(y_{xx}(t), \psi) + \eta(\dot{y}(t), \psi) = (g(t) + (Bu)(t), \psi), \quad (6)
\]

\[
\forall \psi \in V \quad \text{for a.e. } t \in S, y(0) = y_0 \quad \text{and} \quad \dot{y}(0) = y_1.
\]

Let us consider a space \( \mathcal{W} = \{ \omega \in L^2(S; H) \mid \omega, \omega_{xx}, \ddot{\omega} \in L^2(S; H) \} \). We equip \( \mathcal{W} \) with a classical norm

\[
\| \omega \|_\mathcal{W} = \sqrt{\| \omega \|^2 + \| \omega_x \|^2 + \| \omega_{xx} \|^2 + \| \ddot{\omega} \|^2}
\]

where \( \| \cdot \| \) is a norm \( L^2(S; H) \). Now we define a nonlinear operator \( F \) from \( U \) into the space \( \mathcal{W} \) by

\[
F(u) = y,
\]

where \( y = y(u) \) is the unique solution of (6) for \( u \in U \) and

\[
\| F(u) \|_\mathcal{W} = \| y \|_\mathcal{W} = \int_0^T \left[ \| y(t) \|_H^2 + \| y_x(t) \|_H^2 + \| y_{xx}(t) \|_H^2 + \| \ddot{y}(t) \|_H^2 \right] dt
\]

\[
= \| y \|_{L^2(S; V)} + \| \ddot{y} \|_{L^2(S; H)}.
\]

In order to prove that the optimal control process exists, we first indicate the continuous dependence on control parameter.

**Lemma 2.2.** Suppose \( g \in L^2(Q), y_0 \in V, y_1 \in H \) and that the operator \( B : U \to L^2(Q) \) is linear and bounded. Then the operator \( F \) is locally Lipschitz continuous and a weakly continuous map.
Proof. The proof is divided into three steps.

Firstly, we prove the energy estimates. We note that for equations (6) the system energy is given by
\[
\frac{1}{2} \frac{d}{dt} \left[ ||\dot{y}(t)||^2_H + \alpha ||y_{xx}(t)||^2_H + \beta ||y_x(t)||^2_H + \frac{1}{2} \gamma ||y_x(t)||^4_H \right] + \delta ||y_{xx}(t)||^2_H + \sigma (y_x(t), \dot{y}_x(t))^2 + \eta ||\dot{y}(t)||^2_H = (g(t) + (Bu)(t), \dot{y}(t))
\]
This energy expression (7) can be obtained, formally, by choosing \( \psi = \dot{y}(t) \) in (6) and using integration by parts. Integrating (7) over \([0, t]\) for \( t < T \), we arrive at
\[
||\dot{y}(t)||^2_H + \alpha ||y_{xx}(t)||^2_H + \beta ||y_x(t)||^2_H + \frac{1}{2} \gamma ||y_x(t)||^4_H + 2 \delta \int_0^t ||y_{xx}(s)||^2_H ds + 2 \sigma \int_0^t (y_x(s), y_x(s))^2 ds + 2 \eta \int_0^t ||\dot{y}(s)||^2_H ds
\]
\[
= 2 \int_0^t (g(s) + (Bu)(s), \dot{y}(s))ds + ||y_1||^2_H + \alpha ||y_{0xx}||_H^2 + \beta ||y_{0x}||_H^2 + \frac{1}{2} \gamma ||y_{0x}||_H^4.
\]
This implies by Schwartz’s (see [22] p. 8) and Young’s (see [22] p. 36) inequalities that
\[
||\dot{y}(t)||^2_H + \alpha ||y_{xx}(t)||^2_H \leq C_1 (1 + ||u||_V^2) + C_2 \int_0^t \left[ ||\dot{y}(s)||^2_H + \alpha ||y_{xx}(s)||^2_H \right] ds
\]
a.e. \( t \in S \)
with constants \( C_1, C_2 > 0 \) depending on \( ||y_0||_V, ||y_1||_H \) and \( ||g||_{L^2(Q)} \).

From (8) by Gronwall’s lemma (see [10] p. 127) and Poincare’s inequality (see [22] p. 59) we obtain
\[
||y(t)||^2_H + ||y_x(t)||^2_H + ||y_{xx}(t)||^2_H + ||\dot{y}(t)||^2_H \leq C_3 (1 + ||u||_V^2)
\]
a.e. \( t \in S \) with a constant \( C_3 > 0 \).

Integrating this inequality over the interval \([0, T]\), we obtain the following estimate
\[
||F(u)||_V \leq C_4 (1 + ||u||_V) \quad \text{and for some constant } C_4 > 0.
\]

Secondly, we shall prove the operator \( F \) is a locally Lipschitz’s map. Let \( u_1, u_2 \in U \)
and \( y_1 = y_1(u_1), y_2 = y_2(u_2) \) be the unique solution of (6) that is
\[
\langle \dot{y}_i(t), \psi \rangle + \alpha \langle y_{1xx}(t), \psi_{xx} \rangle - (\beta + \gamma ||y_{1x}(t)||^2_H) \langle y_{1xx}(t), \psi \rangle + \delta \langle y_{1xx}(t), \psi_{xx} \rangle - \sigma \langle y_{1x}(t), \dot{y}_{1x}(t) \rangle \langle y_{1xx}(t), \psi \rangle + \eta \langle y_i(t), \psi \rangle = (g(t) + (Bu_i)(t), \psi),
\]
\( \forall \psi \in V \quad \text{for a.e. } t \in S, \quad y_i(0) = y_0 \quad \text{and } \dot{y}_i(0) = y_1 \quad \text{for } i = 1, 2. \)

From Theorem 2.1 we know, that the equation (10) for \( i = 1, 2 \) has exactly one solution \( y_i \in L^\infty(S; V) \) and \( \dot{y}_i \in L^\infty(S; H) \cap L^2(S; V) \). Subtracting the two equations (10) we have
\[
\langle \dot{y}_1(t) - \dot{y}_2(t), \psi \rangle + \alpha (y_{1xx}(t) - y_{2xx}(t)), \psi_{xx} \rangle - \beta (y_{1xx}(t) - y_{2xx}(t), \psi)
\]
\(- \gamma (\|y_{1x}(t)\|^2_{H} y_{1xx}(t) - \|y_{2x}(t)\|^2_{H} y_{2xx}(t)) + \delta (\dot{y}_{1xx}(t) - \dot{y}_{2xx}(t), \psi_{xx})
- \sigma ((y_{1x}(t), \dot{y}_{1x}(t)) y_{1xx}(t) - (y_{2x}(t), \dot{y}_{2x}(t)) y_{2xx}(t)) + \eta (\dot{y}_{1x}(t) - \dot{y}_{2x}(t), \psi)
= \langle (B(u_1 - u_2))(t), \psi \rangle ,
\forall \psi \in V \text{ for a.e. } t \in S.
\)

We take the inner product in (11) with \( \dot{y}_1(t) - \dot{y}_2(t) \) as a test function \( \psi \) (which is possible since \( y_1 \in L^2(S; V) \)). Thus
\[
\langle \dot{y}_1(t) - \dot{y}_2(t), \dot{y}_1(t) - \dot{y}_2(t) \rangle + \alpha (y_{1xx}(t) - y_{2xx}(t), y_{1xx}(t) - y_{2xx}(t))
- \beta (y_{xx}(t) - y_{x2x}(t), y_{1x}(t) - y_{2x}(t)) - \gamma (\|y_{1x}(t)\|^2_{H} y_{1xx}(t))
- \|y_{xx}(t)\|^2_{H} y_{2xx}(t), \dot{y}_1(t) - \dot{y}_2(t)) + \delta (\dot{y}_{1xx}(t) - \dot{y}_{2xx}(t), y_{1xx}(t) - y_{2xx}(t))
- \sigma ((y_{1x}(t), \dot{y}_{1x}(t)) y_{1xx}(t) - (y_{2x}(t), \dot{y}_{2x}(t)) y_{2xx}(t), \dot{y}_1(t) - \dot{y}_2(t))
+ \eta (\dot{y}_1(t) - \dot{y}_2(t), \dot{y}_1(t) - \dot{y}_2(t)) + \eta \|\dot{y}_1(t) - \dot{y}_2(t)\|_{H}^2
= \langle (B(u_1 - u_2))(t), \dot{y}_1(t) - \dot{y}_2(t) \rangle \text{ for a.e. } t \in S.
\]

Then as in [5] and [6] the nonlinear parts of (12) may be estimated
\[
\gamma (\|y_{1x}(t)\|^2_{H} y_{1xx}(t) - \|y_{2x}(t)\|^2_{H} y_{2xx}(t), \dot{y}_1(t) - \dot{y}_2(t))
\leq C_5 \left( \|y_{1xx}(t) - y_{2xx}(t)\|^2_{H} + \|\dot{y}_1(t) - \dot{y}_2(t)\|^2_{H} \right)
\]
and
\[
\sigma ((y_{1x}(t), \dot{y}_{1x}(t)) y_{1xx}(t) - (y_{2x}(t), \dot{y}_{2x}(t)) y_{2xx}(t), \dot{y}_1(t) - \dot{y}_2(t))
= -\sigma (y_{2xx}(t), \dot{y}_1(t) - \dot{y}_2(t))^2
- \sigma (y_{xx}(t), \dot{y}_1(t) - \dot{y}_2(t))(y_{1x}(t), y_{1xx}(t) - y_{2xx}(t))
- \sigma (y_{1x}(t), y_{1xx}(t) - y_{2xx}(t), y_{1x}(t) - y_{2x}(t))
\leq -\sigma (y_{2xx}(t), \dot{y}_1(t) - \dot{y}_2(t))^2 + C_6(|\dot{y}_1(t) - \dot{y}_2(t)|^2_{H} + |y_{1xx} - y_{2xx}|^2_{H})
\]
where \( C_5, C_6 > 0 \) are the constants depending only on the data \( (u_1, u_2, y_1, y_0) \).

Finally, combining (13) and (14) with (12) we arrive at the following inequality
\[
\frac{1}{2} \frac{d}{dt} \|\dot{y}_1(t) - \dot{y}_2(t)\|^2_{H} + \alpha (|y_{1xx}(t) - y_{2xx}(t)|^2_{H}) + \delta (\dot{y}_{1xx}(t) - \dot{y}_{2xx}(t), \psi_{xx})
+ \sigma (y_{xx}(t), \dot{y}_1(t) - \dot{y}_2(t))^2 + \eta \|\dot{y}_1(t) - \dot{y}_2(t)\|^2_{H}
\leq C_7 |\dot{y}_1(t) - \dot{y}_2(t)|^2_{H} + |y_{1xx}(t) - y_{2xx}(t)|^2_{H} + \|(B(u_1 - u_2))(t)\|^2_{H}
\]
for a.e. \( t \in S \) and a constant \( C_7 > 0 \).

Next from (15) by the integration over a subinterval \( [0, t] \) of \( [0, T] \) and by applying Gronwall’s lemma (see [10] p. 127) and Poincare’s inequality (see [22] p.59) we have that
\[
\|\dot{y}_1(t) - \dot{y}_2(t)\|^2_{H} + |y_1(t) - y_2(t)|^2_{V} \leq C_8 \int_0^t \| (B(u_1 - u_2))(s) \|^2_{H} ds.
\]

This implies that the operator \( F \) is locally Lipschitz’s map because a constant \( C_8 > 0 \) depends only on the data and the operator \( B \) is linear and bounded.

Thirdly, we shall prove that the operator \( F \) is a weakly continuous mapping. Let \( (u_n) \) denote a sequence such that
\[
u_n \rightarrow \pi \text{ weakly in } U.
\]
Let \( y_n = y(u_n) \) satisfy the equation (6) with \( u = u_n \), i.e.

\[
\langle \dot{y}_n(t), \psi \rangle + \alpha (y_{nxx}(t), \psi_{xx})
- (\beta + \gamma \| y_{nx}(t) \|_H^2) (y_{nxx}(t), \psi) + \delta (\dot{y}_{nxx}(t), \psi_{xx})
- \sigma (y_{nx}(t), y_{nx}(t)) (y_{nxx}(t), \psi) + \eta (\dot{y}_n(t), \psi)
= (g(t) + (Bu_n)(t), \psi),
\]

\( \forall \psi \in \mathcal{V} \) for a.e. \( t \in S \), \( y_n(0) = y_0 \) and \( \dot{y}_n(0) = y_1 \) \( \forall n \in \mathbb{N} \).

From Theorem 2.1 we know that the problem (17) has exactly one weak solution \( y_n \) for \( n \in \mathbb{N} \). From the assumptions of Lemma and from the first part of this proof we have that the sequences \( (y_n), (y_{nx}), (y_{nxx}), (\dot{y}_n) \) and \( (\ddot{y}_{nxx}) \) are bounded in \( L^2(S; H) \).

Using the diagonal procedure we may thus extract subsequence (also denoted by \( (y_n) \)) such that

\[
y_n \rightharpoonup \bar{y} \text{ weakly in } L^2(S; V)
\]

\[
y_n \rightharpoonup \tilde{y} \text{ weakly in } L^2(S; H)
\]

\[
y_n \rightharpoonup \bar{y} \text{ strongly in } L^2(S; H),
\]

as proved in [5] and [6] we have also the following convergences

\[
\| y_{nx} \|^2 y_{nxx} \rightharpoonup \| \bar{y}_x \|^2 \bar{y}_{xx} \text{ weakly in } L^2(S; H)
\]

\[
(y_{nx}, \bar{y}_x)y_{nxx} \rightharpoonup (\bar{y}_x, \tilde{y}_x)\bar{y}_{xx} \text{ weakly in } L^2(S; H).
\]

Starting from the first equation (17) we deduce that for all test functions \( \varphi \in C_0^\infty(0, T) \) (see [17] p. 26) we have

\[
\int_0^T \langle \dot{y}_n(t), \psi \rangle \varphi(t) dt + \int_0^T [\alpha (y_{nxx}(t), \psi_{xx})
- (\beta + \gamma \| y_{nx}(t) \|_H^2) (y_{nxx}(t), \psi) + \delta (\dot{y}_{nxx}(t), \psi_{xx})
- \sigma (y_{nx}(t), y_{nx}(t)) (y_{nxx}(t), \psi) + \eta (\dot{y}_n(t), \psi)] \varphi(t) dt
= \int_0^T (g(t) + (Bu_n)(t), \psi) \varphi(t) dt
\]

\( \forall \psi \in \mathcal{V} \) and \( \forall \varphi \in C_0^\infty(0, T) \).

For the first term in (18), by integration by parts, we have

\[
\int_0^T \langle \dot{y}_n(t), \psi \rangle \varphi(t) dt = - \int_0^T \langle \dot{y}_n(t), \psi \rangle \dot{\varphi}(t) dt
\]

for any function \( \varphi \in C_0^\infty(0, T) \) or

\[
\int_0^T \langle \dot{y}_n(t), \psi \rangle \varphi(t) dt = (y_1, \psi) \varphi(0) - \int_0^T \langle \dot{y}_n(t), \psi \rangle \dot{\varphi}(t) dt
\]

for any function \( \varphi \in C^1(0, T) \) such that \( \varphi(T) = 0 \).

Using (19) in (18) we obtain

\[
- \int_0^T \langle \dot{y}_n(t), \psi \rangle \dot{\varphi}(t) dt + \int_0^T [\alpha (y_{nxx}(t), \psi_{xx})
- (\beta + \gamma \| y_{nx}(t) \|_H^2) (y_{nxx}(t), \psi) + \delta (\dot{y}_{nxx}(t), \psi_{xx})
- \sigma (y_{nx}(t), y_{nx}(t)) (y_{nxx}(t), \psi) + \eta (\dot{y}_n(t), \psi)] \varphi(t) dt
= \int_0^T (g(t) + (Bu_n)(t), \psi) \varphi(t) dt
\]
- σ(y_{nxx}(t), y_{xx}(t), \psi) + \eta(y_{n}(t), \psi) \right) \varphi(t) dt \\
= \int_0^T \left( g(t) + (Bu_n)(t), \psi \right) \varphi(t) dt \quad \forall \varphi \in V \quad \text{and} \quad \forall \psi \in C_0^\infty(0,T).

Passing to the limit in (21) we obtain (as the operator B is linear)

\[ - \int_0^T \left( \bar{y}(t), \psi \right) \varphi(t) dt + \int_0^T \left[ \alpha(\bar{y}_{xx}(t), \psi_{xx}) \\
- (\beta + \gamma \|y_{x}(t)\|_H^2)(\bar{y}_{xx}(t), \psi) + \delta(\bar{y}_{xx}(t), \psi_{xx}) \\
- \sigma(\bar{y}_{x}(t), \bar{y}_{x}(t))(\bar{y}_{xx}(t), \psi) + \eta(\bar{y}(t), \psi) \right] \varphi(t) dt \quad \forall \varphi \in V \quad \text{and} \quad \forall \psi \in C_0^\infty(0,T). \]

Therefore, by using the distributional derivative in the first term of (22), we deduce that the function \( \bar{y} \) verifies the first equation (6) with \( u = \bar{u} \), because the functions \( \psi \in V \) and \( \varphi \in C_0^\infty(0,T) \) are arbitrary. Finally, combining (20) and (18) and passing to the limit in (18), we obtain that the function \( \bar{y} \) verifies the initial conditions in (6) too.

This completes the proof of this lemma. \( \square \)

**Remark 2.** Lemma 2.2 is one of the most important result of our paper. It permits to prove the main result of our paper - convergence of solutions of the approximated family control problems to the solution of original control problem.

3. Optimal control problem. The state of system of our control problem is described by an equation

\[ \langle \dot{y}(t), \psi \rangle + \alpha(y_{xx}(t), \psi_{xx}) - (\beta + \gamma \|y_{x}(t)\|_H^2)(y_{xx}(t), \psi) + \delta(y_{xx}(t), \psi_{xx}) \]
\[ - \sigma(y_{x}(t), \bar{y}_{x}(t))(y_{xx}(t), \psi) + \eta(\bar{y}(t), \psi) = \left( g(t) + (Bu)(t), \psi \right), \]
\[ \forall \psi \in V \quad \text{for a.e.} \ t \in S, \quad y(0) = y_0 \quad \text{and} \quad \dot{y}(0) = y_1. \]

The optimal control problem \( (P) \) can be formulated as follows: find an optimal pair \((u^0, y^0) \in U \times \mathcal{W} \) which minimizes a functional \( J(u, y) \) where \( J : U \times \mathcal{W} \to \mathbb{R} \) and \( y = y(u) \) is a unique solution of (23) for \( u \in U \) and the space \( U \times \mathcal{W} \) is equipped with a norm

\[ \|(u, \omega)\|_{U \times \mathcal{W}} = \left( \|u\|_U^2 + \|\omega\|_{L^2(Q)}^2 + \|\omega_x\|_{L^2(Q)}^2 + \|\omega_{xx}\|_{L^2(Q)}^2 + \|\omega_{xx}\|_{L^2(Q)}^2 \right)^{1/2}. \]

For any control \( u \in U \), from Theorem 2.1 and (6), there is a unique state \( y = y(u) = F(u) \). Then, we can define for the functional \( J : U \times \mathcal{W} \to \mathbb{R} \) the reduced functional \( I : U \to \mathbb{R} \) as \( I(u) = J(u, y(u)) \) for any \( u \in U \).

**Definition 3.1.** The functional \( J : U \times \mathcal{W} \to \mathbb{R} \) is coercive as the reduced functional \( I : U \to \mathbb{R} \) is coercive on the control space \( U \) i.e. \( \lim_{\|u\|_U \to \infty} I(u) = \infty \).

**Theorem 3.2.** Let the assumptions of Lemma 2.2 be satisfied, i.e. \( g \in L^2(Q) \), \( y_0 \in V \), \( y_1 \in H \) and assume that the operator \( B \) is linear and bounded from the separable Hilbert space \( U \) into \( L^2(Q) \). Assume the functional \( J \) is continuous and convex on \( U \times \mathcal{W} \) and the reduced functional \( I \) is coercive on \( U \). Then, there exists
at least one optimal pair \((u^0, y^0)\) \(\in U \times \mathcal{W}\) such that \(\inf_{u \in U} J(u, y(u)) = J(u^0, y^0)\) where \(y^0 = y(u^0)\) is the solution of \((23)\) for \(u = u^0\).

\[\text{Proof.}\] Let \((u_n)\) be a minimizing sequence for the functional \(J\) i.e. \(u_n \in U\) for \(n \in \mathbb{N}\) and

\[\lim_{n \to \infty} J(u_n, y_n) = \inf_{u \in U} J(u, y(u))\]

where \(y_n = y(u_n)\) is the solution of \((23)\) for \(u = u_n\). Since \(J\) is coercive, then sequence \((u_n)\) is bounded in \(U\). It follows that there exists a subsequence, which we also denote by \((u_n)\), such that \(u_n \xrightarrow{n \to \infty} \pi\) weakly in \(U\). From Lemma 2.2 the sequence \(y_n \to \overline{y}\) weakly convergences in \(\mathcal{W}\) where \(\overline{y} = y(\pi)\) is unique solution of \((23)\) for \(u = \pi\). The functional \(J\) is convex and continuous then it is weakly lower semicontinuous on \(U \times \mathcal{W}\), therefore

\[\inf_{u \in U} J(u, y(u)) = \lim_{n \to \infty} J(u_n, y(u_n)) = \lim_{n \to \infty} \inf_{n \to \infty} J(u_n, y(u_n)) \geq J(\pi, \overline{y}).\]

Hence \(J(\pi, \overline{y}) = J(u^0, y^0).\) \(\Box\)

Remark 3. In many engineering applications \(J\) may be quadratic functional in the form

\[J(u, y) = ||u||^2_H + \lambda_1 \int_0^T \int_0^l |y(t, x) - y_d|^2 dx dt + \lambda_2 \int_0^T \int_0^l |y_x(t, x) - y_d|^2 dx dt\]

\[+ \lambda_3 \int_0^T \int_0^l |y_{xx}(t, x) - y_d|^2 dx dt + \lambda_4 \int_0^T \int_0^l |\ddot{y}(t, x) - y_d|^2 dx dt\]

where \(\lambda_i \geq 0\) for \(i = 1, \ldots, 4\) and \(\sum_{i=1}^4 \lambda_i = 1\) and \(y_d, y_d^1, y_d^2, y_d^3 \in L^2(S; H)\) are desired functions. This functional represents the total energy of the beam.

4. Galerkin approximation of the optimal control problem. Here we recall some known results concerning the finite dimensional Galerkin approximation (see [22] p. 271). They are basic for the convergence analysis of our optimal problem.

We consider a family \(\{V_h\}_{h \in G}\) of finite dimensional subspaces of \(V\) which satisfies the following conditions:

\[\forall h_1, h_2 \in G \quad (h_1 > h_2 \implies V_{h_1} \subset V_{h_2}) \quad \text{and} \quad \bigcup_{h \in G} V_h = V; \quad (24)\]

where the set \(G \subset (0, 1]\) of parameters \(h\) has an accumulation point at 0. The approximation of space \(H\) is the same family \(\{V_h\}_{h \in G}\) with an induced norm with \(H\). The approximation of the spaces \(L^2(S; V)\) and \(L^2(S; H)\) is understood here as a family of space \(\{L^2(S; V_h)\}_{h \in G}\) with respective norms.

As approximate solutions of equation \((23)\) we mean the family of functions \(y_h \in L^2(S; V_h)\) which are the solutions of the following system

\[\langle \ddot{y}_h(t), \psi_h \rangle + \alpha \langle y_{h,xx}(t), \psi_{h,xx} \rangle + (\beta + \gamma ||y_{h,x}(t)||^2_H) \langle y_{h,x}(t), \psi_h \rangle + \delta \langle y_{h,xxx}(t), \psi_{h,xxx} \rangle\]
If the following conditions hold: 2.2 and the properties of Galerkin approximation (24) and (26) are satisfied, then corresponding sequence of solution of equation (27). Let proof of Lemma 2.2 and our assumptions of Galerkin approximation, we may prove.

Theorem 3.2 because the equation (27) has for every \( \omega \) is a solution of the system

\[
\begin{align*}
\begin{bmatrix}
\sigma(\dot{y}(t), \dot{y}(t)) & \gamma(\dot{y}(t), \psi) \\
\beta & \gamma
\end{bmatrix}
\begin{bmatrix}
\dot{y}(t) \\
\psi(t)
\end{bmatrix}
+ 
\begin{bmatrix}
\alpha & \beta \\
\gamma & \delta
\end{bmatrix}
\begin{bmatrix}
y(t) \\
\psi(t)
\end{bmatrix}
+ 
\begin{bmatrix}
\eta & \delta
\end{bmatrix}
\begin{bmatrix}
y(t) \\
\dot{y}(t)
\end{bmatrix}
+ 
\begin{bmatrix}
\mu & \nu
\end{bmatrix}
\begin{bmatrix}
y(t) \\
\dot{y}(t)
\end{bmatrix}
+ 
\begin{bmatrix}
J(t) \\
\int \omega(t) dt
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\end{align*}
\]

and \( \mathcal{W} = \{ \omega \in L^2(S; V_h) \mid \omega_{xx}, \omega_{xxx}, \omega_t \in L^2(S; V_h) \} \) with an induced norm from \( \mathcal{W} \).

The control problem \( P_{hk} \) is the lumped parameter system.

**Theorem 4.1.** Under the assumptions of Theorem 3.2 and the properties of Galerkin approximation (24) and (26), the approximated control problem \( P_{hk} \) has at least one solution \( u_{hk}^0 \in U_k \).

The proof of this theorem can be constructed in the same way as the proof of Theorem 3.2 because the equation (27) has for every \( u \in U_k \) the unique solution \( y_{hk} = y_h(u_k) \).

5. **Convergence of approximation.** In this section, we prove the main result of our paper, convergence of solutions of approximated optimal control problems \( P_{hk} \) to a solution of original problem \( P \). Using the same techniques as in the proof of Lemma 2.2 and our assumptions of Galerkin approximation, we may prove.

**Lemma 5.1.** Let \( (u_k) \) be a sequence of elements in \( U_k \subset U \) and \( y_{hk} \) be the corresponding sequence of solution of equation (27). If the assumptions of Lemma 2.2 and the properties of Galerkin approximation (24) and (26) are satisfied, then the following conditions hold:

(i) If \( u_k \xrightarrow[k \to 0]{} \pi \) weakly in \( U \), then

\[
y_{hk} \xrightarrow[h,k \to 0]{} y \text{ weakly in } L^2(S; H)
\]
where the function $\mathcal{y}$ is unique solution of system (6) for $u = \pi$.

Let us now consider the convergence of approximation for problem (P).

**Theorem 5.2.** If the assumptions of Theorem 4.1 are satisfied, then there exist weakly condensation points of a set of solutions of the optimization problem $(P_{hk})$ in $U \times \mathcal{W}$ and each of these points is a solution of the optimization problem (P).

**Proof.** We want to prove that the sequence $(u_{hk}^0)$ is a minimizing sequence for the functional $I(u) = J(u, y(u))$. According to the approximation of the space of controls $U$ for $u^0 \in U$ (solution of problem (P)) there exists a sequence $(u_k^0)$ such that $u_k^0 \in U_k$ for $k \in K$, and $u_k \xrightarrow[k \to 0]{} u^0$ strongly in $U$. Lemma 5.1 (ii) implies that the solution $y_{hk}$ of equation (27) for $u = u_{hk}^0$ converges strongly in $\mathcal{W}$ to $y^0 = y(u^0)$ i.e. $y_h(u_{hk}^0) \xrightarrow[h,k \to 0]{} y^0$ strongly in $\mathcal{W}$, and pair $(u^0, y^0) \in U \times \mathcal{W}$ is optimal pair for problem (P). Then, as

$$
\inf_{u \in U} J(u, y(u)) = J(u^0, y^0) \leq J(u_{hk}^0, y(u_{hk}^0)) \leq J(u_k^0, y_h(u_k^0))
$$

and that functional $J$ is continuous, we have

$$
\lim_{h,k \to 0} J(u_{hk}^0, y_{hk}^0) = J(u^0, y^0).
$$

Since the functional $I(u) = J(u, y(u))$ is coercive, then the sequence $(u_{hk}^0)$ is bounded in $U$. There is a subsequence, denoted again by $(u_{hk}^0)$, such that $u_{hk}^0 \xrightarrow[h,k \to 0]{} \pi$ converges weakly in $U$. Let $y_{hk}^0 = y_h(u_{hk}^0)$ be a solution of equation (27) for $u_k = u_{hk}^0$. From Lemma 5.1 (i) we have that

$\quad y_{hk}^0 \xrightarrow[h,k \to 0]{} y = y(\pi)$ weakly in $L^2(S; H),
$

$\quad y_{hkx}^0 \xrightarrow[h,k \to 0]{} y_{xx}(\pi)$ weakly in $L^2(S; H),
$

$\quad y_{hkkx}^0 \xrightarrow[h,k \to 0]{} y_{xxx}(\pi)$ weakly in $L^2(S; H),
$

$\quad \dot{y}_{hk} \xrightarrow[h,k \to 0]{} \dot{y} = \dot{y}(\pi)$ weakly in $L^2(S; H)$

and $\mathcal{y}$ is the unique solution of (6) for $u = \pi$.

Because functional $J$ is convex and continuous then it is weakly lower semicontinuous on $U \times \mathcal{W}$, therefore

$$
\inf_{u \in U} J(u, y(u)) = \lim_{h,k \to 0} J(u_{hk}^0, y_{hk}^0) = \lim_{h,k \to 0} J(u_{hk}^0, y_{hk}^0) \geq J(\pi, \mathcal{y}).
$$
This implies that a pair \((u, y)\) is one of the solutions of the optimal control problem \((P)\).

**Corollary 1.** Theorem 5.2 also proves the existence of a solution of the control problem \((P)\) because the solutions sequence of approximated problems \((P_{hk})\), obtained with Galerkin method, is one of the minimizing sequences.

**Acknowledgments.** We would like to thank very much the Referees for their important remarks and comments which allow us to correct and improve this paper.

**Conflict of interest.** The authors declare that they have no conflict of interest.

**REFERENCES**

[1] A. S. Ackleh, H. T. Banks and G. A. Pinter, A Nonlinear Beam Equation, *Appl. Math. Letters*, 15 (2002), 381–387.

[2] J. Ahn and D. E. Stewart, A viscoelastic Timoshenko beam with dynamic frictionless impact, *Discr. Cont. Dynam. Sys. B*, 12 (2009), 1–22.

[3] K. T. Andrews, Y. Dumont, M. F. M’Bengue, J. Purcell and M. Shillor, Analysis and simulations of nonlinear elastic dynamic beam, *ZAMP*, 63 (2012), 1005–1019.

[4] N. Arada, E. Casas and F. Tröltzsch, Error estimates for the numerical approximation of a semilinear elliptic control problem, *Comp. Optim. Applic.*, 23 (2002), 201–229.

[5] J. M. Ball, Stability theory for an extensible beam, *J. Differ. Equat.*, 14 (1973), 399–418.

[6] J. M. Ball, Initial-boundary value problems for an extensible beam, *J. Math. Anal. Appl.*, 42 (1973), 61–90.

[7] M. Barboteu, M. Sofonea and D. Tiba, The control variational method for beams in contact with deformable obstacles, *ZAMM*, 92 (2012), 25–40.

[8] A. Dębinka-Nagórska, A. Just and Z. Stempień, Approximation of an optimal control problem governed by a differential parabolic inclusion, *Optim.*, 59 (2010), 707–715.

[9] A. Dębinka-Nagórska, A. Just and Z. Stempień, Galerkin method for optimal control of second-order evolution equations, *Math. Meth. Appl. Sci.*, 27 (2004), 221–230.

[10] Z. Denkowski, S. Migórski and N. Papageorgiou, Nonlinear Analysis. Applications, Kluwer Academic Publishers, Boston/Dordrecht/London, 2002.

[11] E. Feireisl and L. Herrman, Oscillations of a nonlinearly dumped extensible beam, *Appl. Math.*, 37 (1992), 469–478.

[12] M. Galewski, On the optimal control problem governed by the nonlinear elastic beam equation, *Appl. Math. Comput.*, 203 (2008), 916–920.

[13] I. Hlaváček and J. Lovišek, Optimal control of semi-coercive variational inequalities with application to optimal design of beams and plates, *ZAMM*, 78 (1998), 405–417.

[14] J. Hwang, Optimal control problems for an extensible beam equation, *J. Math. Anal. Appl.*, 353 (2009), 436–448.

[15] A. Just and Z. Stempień, Pareto optimal control problem and its Galerkin approximation for a nonlinear one-dimensional extensible beam equation, *Opuscula Math.*, 36 (2016), 239–252.

[16] I. Lasiecka, Galerkin approximation of infinite-dimensional compensators for flexible structure with unbounded control action, *Acta Appl. Math.*, 28 (1992), 101–133.

[17] J. L. Lions, *Optimal Control of Systems Governed by Partial Differential Equations*, Springer-Verlag, Berlin/Heidelberg/New York, 1971, (Russian Edition, Mir, Moscow 1972).

[18] M. L. Oliveira and O. A. Lima, Exponential decay of the solutions of the beam system, *Nonlinear Anal.*, 42 (2000), 1271–1291.

[19] D. C. Pereira, Existence, uniqueness and asymptotic behaviour for solutions of the nonlinear beam equation, *Nonlinear Anal.*, 14 (1990), 613–623.

[20] I. Sadek, M. Abukhaled and T. Abdurah, Coupled Galerkin and parametrization methods for optimal control of discretely connected parallel beams, *Appl. Math. Modelling*, 34 (2010), 3949–3957.

[21] F. Tröltzsch, Semidiscrete Ritz-Galerkin approximation of nonlinear parabolic boundary control problems - strong convergence of optimal controls, *Appl. Math. Optimz.*, 29 (1994), 309–329.
[22] E. Zeidler, *Nonlinear Functional Analysis and its Applications*, II Springer-Verlag, Berlin, 1990.

Received November 2016; revised May 2017.

E-mail address: andrzej.just@p.lodz.pl
E-mail address: zdzislaw.stempien@p.lodz.pl