Babenko’s equation for periodic gravity waves on water of finite depth: derivation and numerical solution

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Summary

The nonlinear two-dimensional problem, describing periodic steady waves on water of finite depth is considered in the absence of surface tension. It is reduced to a single pseudo-differential operator equation (Babenko’s equation), which is investigated analytically and numerically. This equation has the same form as the equation for waves on infinitely deep water; the latter had been proposed by Babenko and studied in detail by Buffoni, Dancer and Toland. Instead of the $2\pi$-periodic Hilbert transform $C$ used in the equation for deep water, the equation obtained here contains a certain operator $B_r$, which is the sum of $C$ and a compact operator whose dependence on the parameter involves on the depth of water. Numerical computations are based on an equivalent form of Babenko’s equation derived by virtue of the spectral decomposition of the operator $B_r \frac{d}{dt}$. Bifurcation curves and wave profiles of the extreme form are obtained numerically.

1. Introduction

Near the end of his remarkable career in both pure and applied mathematics (see [1] for highlights of major achievements), Konstantin Ivanovich Babenko (1919–1987) published four brief notes [2, 3, 4, 5] (the last two appeared posthumously) on a classical nonlinear problem in the mathematical theory of water waves, namely, the two-dimensional problem of steady, periodic waves on infinitely deep water. In this paper dedicated to the centenary of Babenko’s birth, we extend the approach developed in [2] to the case of water of finite depth and deduce a single pseudo-differential operator equation (Babenko’s equation) equivalent to the corresponding free-boundary problem in some sense explained below (see Section 3.3). Moreover, using the spectral decomposition of a linear operator involved in the equation, we transform it to a form convenient for discretisation and then apply a very robust numerical method that allows us to produce convincing results concerning global bifurcation branches, secondary bifurcations and free surface profiles including those of the extreme form.

It was Stokes [6], who had initiated studies in this field. On the basis of approximations developed for waves with a single crest per wavelength (now, they are referred to as Stokes waves), he made some conjectures about the behaviour of such waves on deep water. To a great extent, these conjectures had determined the development of research in the 20th century; see the paper [7] and references cited therein to get an idea how these conjectures
were proved. In particular, the so-called Nekrasov’s equation was essential for this purpose. The first version of this nonlinear integral equation for waves on deep water was derived in \( \text{(8)} \) (see also \( \text{(9)} \), Part 1). Soon after that, Nekrasov generalized his equation to the case of finite depth (see \( \text{(10)} \) and \( \text{(10)} \), Part 2). Much later, Amick and Toland \( \text{(11)} \) proposed and investigated a more sophisticated version of the latter equation.

Levi-Civita \( \text{(12)} \) and Struik \( \text{(13)} \) considered (independently of Nekrasov) the problem of periodic waves on deep water and on water of finite depth respectively. The hodograph transform allowed them to reduce the question of existence of waves to that of finding a pair of conjugate harmonic functions satisfying nonlinear Neumann boundary conditions. The existence proofs given in \( \text{(12)} \) and \( \text{(13)} \) are based on a majorant method for demonstrating the convergence of power series that provide formal solutions. In his book \( \text{(14)} \), Chapter 71, Zeidler writes about these proofs that they are ‘very complicated’ in view of ‘voluminous computations’ involved. By now, both techniques (Nekrasov’s equations and the method of Levi-Civita and Struik) are investigated in detail by means of analytic bifurcation theory. An account of this theory can be found in the books \( \text{(15)} \) and \( \text{(15)} \), whereas many authors studied its application to equations describing steady water waves; these results are summarised in \( \text{(17)} \) (deep water) and in \( \text{(14)} \), Chapter 71 (water of finite depth), where one also finds detailed historical remarks. It should be mentioned that Krasovskii \( \text{(18)} \) extended studies of water waves to the case of a periodic wavy bottom.

Another method for periodic waves on deep water (with and without surface tension) was developed in detail by Buffoni, Dancer and Toland \( \text{(19, 20)} \). In the absence of surface tension, it is based on the so-called Babenko’s equation:

\[
\mu \mathcal{C}(v') = v + v \mathcal{C}(v') + \mathcal{C}(v'^2), \quad t \in (-\pi, \pi).
\] (1.1)

Here \( \mu \) is a dimensionless bifurcation parameter (it is related to the speed of wave propagation), which must be found along with the \( 2\pi \)-periodic function \( v(t) \) that describes the wave profile parametrically in certain dimensionless coordinates; \( ' \) stands for differentiation with respect to \( t \) and \( \mathcal{C} \) is the \( 2\pi \)-periodic Hilbert transform (the conjugation operator in the theory of Fourier series; see, for example, \( \text{(22)} \)). It is defined on \( L^2(-\pi, \pi) \) by linearity from the following relations:

\[
\mathcal{C}(\cos nt) = \sin nt \quad \text{for } n \geq 0, \quad \mathcal{C}(\sin nt) = -\cos nt \quad \text{for } n \geq 1.
\] (1.2)

The original form of \( \text{(1.1)} \) was announced by Babenko \( \text{(2)} \) (see also \( \text{(21)} \), Section 3.7), where the equation is derived and expressed in terms of the self-adjoint operator \( J_0 = \mathcal{C} \frac{d}{dt} \). In his second note \( \text{(3)} \), Babenko outlines how to prove the local existence theorem for his equation in a neighbourhood of the first bifurcation point equal to unity. He demonstrates that \( \mu \) must be equal to \( 1 + \epsilon^2 \), and, after changing the unknown function by applying the invertible operator \( I + J_0 \) to the original one, a solution is sought in the form of expansion in powers of \( \epsilon \). It is shown that the expansion converges for \( \epsilon \leq 1/25 \). Some numerical computations related to the Babenko’s version of equation \( \text{(1.1)} \) are presented in \( \text{(4, 5)} \).

A generalization of Babenko’s equation was later studied in \( \text{(23)} \). Besides, Longuet-Higgins \( \text{(24)} \) derived an infinite system of algebraic equations equivalent to \( \text{(1.1)} \) (see also \( \text{(25)} \) and \( \text{(25)} \), Section 3.6). He used this system for numerical computations of Stokes-wave bifurcations (see \( \text{(26)} \) and also \( \text{(21)} \), Sections 3.2 and 3.8). It is worth mentioning that this system naturally appears from the Lagrangian formalism developed in \( \text{(27)} \). In the paper
(28), a similar system of quadratic relations between the Fourier coefficients of the wave profile was obtained in the case of water having finite depth.

Interesting results concerning the secondary or sub-harmonic bifurcations from branches describing Stokes-wave solutions of (1.1) are proved in the articles (19) and (20). In the first of these, it is shown that such bifurcations do not occur in a neighbourhood of those points, where Stokes waves bifurcate from a trivial solution. On the other hand, significant numerical evidence about the existence of steady periodic waves that distinguish from Stokes waves had appeared in the 1980s. These new waves have more than one crest per period and bifurcate from Stokes waves. Branches of sub-harmonic bifurcations for deep water were first computed by Chen and Saffman (29), whereas Vanden-Broeck (30) obtained similar result for water of finite depth by solving numerically an integro-differential system arising after the hodograph transform; this system was proposed in (31). Craig and Nicholls (32) used a different method for computing numerical results generalising those of Vanden-Broeck. Moreover, it was shown that non-symmetric periodic waves exist on water of finite depth for which purpose a weakly nonlinear Hamiltonian model was applied in (33).

References to other works containing numerical results on sub-harmonic bifurcations can be found in (21) and (34). In the latter paper, some theoretical insights concerning these bifurcations are also given. The above mentioned numerical observations were confirmed rigorously in (20), where it was ‘concluded that the sub-harmonic bifurcations [...] are an inevitable consequence of the formation of Stokes highest wave’. A characteristic property of the latter wave is the angle equal to $2\pi/3$ formed at the crest by two smooth, symmetric curves. Concerning this wave see (7) and references cited therein.

Apart of numerical approaches mentioned above, the direction of studies was quite different for water of finite depth. Of course, Nekrasov’s equation and the approach of Struik were both developed for waves of a given wavelength. On the other hand, a pseudo-differential equation in terms of variables arising after the hodograph transform was derived in (37). This equation describes all waves for which the rate of flow per unit span and the Bernoulli constant are prescribed and serves for justifying the Benjamin–Lighthill conjecture for near-critical waves. However, it is not suitable for studying the Stokes-wave and sub-harmonic bifurcations. The results obtained for equation (1.1) in (19) (20) show that Babenko’s equation serves better for this purpose. Here the analysis of equation similar to (1.1), but describing waves on water of finite depth is initiated and new numerical results are obtained.

Besides, Constantin, Strauss and Vărvărucă investigated the problem of water waves with constant vorticity in their recent paper (35). In the case of finite depth, this problem is reduced to a quasilinear pseudo-differential system provided the vorticity is non-zero. In the irrotational case (that is, for zero vorticity) and for some particular value of a parameter involved in the system, the latter turns into a single equation that has the same form as (1.1) with $\mathcal{C}$ changed to the so-called periodic Hilbert transform on a strip. In Section 5, we compare this equation with Babenko’s equation derived in this paper; see (3.15) below.

The plan of the paper is as follows. For the problem describing steady, periodic waves on water of finite depth two equivalent statements — dimensional and non-dimensional — are formulated in Section 2. Babenko’s equation is derived from the non-dimensional formulation in Section 3.1 and the existence of local bifurcation branches for this equation is proved on the basis of the Crandall–Rabinowitz theorem in Section 3.2. In Section 3.3, we outline how to obtain a solution of the non-dimensional problem from a given solution
of Babenko’s equation. Numerical procedure applied for solving Babenko’s equation is presented in Section 4 along with various bifurcation curves and wave profiles obtained with its help. Section 5 contains concluding remarks.

2. Statements of the problem

In its simplest form, the problem of steady surface waves concerns the two-dimensional, irrotational motion of an inviscid, incompressible, heavy fluid, say water, bounded above by a free surface and below by a rigid horizontal bottom. (For example, this kind of motion occurs in water occupying an infinitely long channel with rectangular cross-section and having uniform width.) In an appropriate frame of reference the velocity field of steady motion is time-independent as well as the free-surface profile, and they can be described in two equivalent ways that differ by prescribed parameters.

2.1 The Benjamin–Lighthill statement for steady waves

The classical formulation proposed by Benjamin and Lighthill is convenient for justification of their conjecture (see (36) and (37), where it had been justified for Stokes waves and all near-critical waves respectively). In this formulation, \( Q \) — the constant rate of flow per channel’s unit span — is prescribed along with the total head \( R \) also referred to as the Bernoulli constant. Let Cartesian coordinates \((X,Y)\) be chosen so that the bottom coincides with the \( X \)-axis and gravity acts in the negative \( Y \)-direction, whereas the wave profile has a crest on the \( Y \)-axis (this does not restrict generality). Thus, the profile is given by the graph of an unknown positive function \( \xi \) (that is, \( Y = \xi(X) \), \( X \in \mathbb{R} \)), attaining its maximum at \( X = 0 \). Moreover, we suppose that \( \xi \) is continuously differentiable and even.

In the longitudinal section of the water domain \( \mathcal{D} = \{ X \in \mathbb{R}, \ 0 < Y < \xi(X) \} \), the velocity field is described by the stream function \( \Psi(X,Y) \), that is, the projections of the velocity vector at \((X,Y)\) on the \( X \)- and \( Y \)-axis are \( \Psi_Y \) and \( -\Psi_X \) respectively. It is assumed that \( \Psi \) belongs to \( C^2(\mathcal{D}) \cap C^1(\overline{\mathcal{D}}) \) and is an even function of \( X \) (hence it is bounded on \( \overline{\mathcal{D}} \)).

If the surface tension is neglected, then \( \Psi \) and \( \xi \) must satisfy the following free-boundary problem:

\[
\begin{align*}
\Psi_{XX} + \Psi_{YY} &= 0, \quad (X,Y) \in \mathcal{D}; \\
\Psi(X,0) &= -Q, \quad X \in \mathbb{R}; \\
\Psi(X,\xi(X)) &= 0, \quad X \in \mathbb{R}; \\
\frac{1}{2}\left|\nabla\Psi(X,\xi(X))\right|^2 + g\xi(X) &= R, \quad X \in \mathbb{R}.
\end{align*}
\]

In the left-hand side of the last relation usually referred to as Bernoulli’s equation, \( g > 0 \) is the constant acceleration due to gravity. It is known that non-trivial solutions of problem (2.1)–(2.4) exist only when \( Q \neq 0 \) and \( R > R_c = \frac{3}{2}(Qg)^{2/3} \). For the proof of the first relation see Proposition 1.1 in (38), whereas the last inequality is proved in (39) under weaker assumptions than listed above. In what follows, these restrictions on \( Q \) and \( R \) hold; moreover, we suppose (without loss of generality) that \( Q > 0 \).

2.2 A non-dimensional statement for periodic waves

Let us assume that \( \xi \) is a \( 2\ell \)-periodic function \( (\ell > 0) \), whereas \( \Psi(X,Y) \) is \( 2\ell \)-periodic in \( X \), but the constant \( R \) is to be found along with these functions from problem (2.1)–(2.4). In
order to reduce the reformulated problem to a non-dimensional form, we average Bernoulli’s equation over \((-\ell, \ell)\). Since \(\Psi\) is constant on the free surface, we obtain that
\[
\frac{c^2}{2} = \frac{1}{2\ell} \int_{-\ell}^{\ell} \xi(X) \, dX \quad \text{and} \quad c^2 = \frac{1}{2\ell} \int_{-\ell}^{\ell} \left| \frac{\partial \Psi}{\partial n}(X, \xi(X)) \right|^2 \, dX.
\] (2.5)

Here \(n\) is the unit normal to \(Y = \xi(X)\) directed out of \(D\). One can show that the last equality (2.5) is true with the same constant \(c^2\) when this curve is changed to \(\tilde{Y} = \tilde{\xi}(X)\) — an arbitrary streamline — and \(n\) is changed to \(\tilde{n}\) — the unit normal to this streamline.

Therefore, \(c > 0\) is the unknown mean velocity of flow in the positive direction of the \(X\)-axis. It is convenient to introduce the following non-dimensional quantities: \(h = \frac{\pi H}{\ell}\) (the mean depth of flow) and \(Q_0 = \frac{Q}{\sqrt{\frac{g}{\pi}}}\left(\frac{\ell}{\pi}\right)^3\) (the rate of flow). Now we scale the dimensional variables and shift the vertical variables downwards as follows:
\[
x = \frac{\pi}{\ell} X, \quad y = \frac{\pi}{\ell} Y - h; \quad \eta(x) = \frac{\pi}{\ell} \xi(X) - h; \quad \psi(x, y) = \frac{Q_0}{Q} \Psi(X, Y).
\] (2.6)

This is advantageous because the new unknown \(\eta\) is a \(2\pi\)-periodic and even function of \(x\) satisfying the following condition:
\[
\int_{-\pi}^{\pi} \eta(x) \, dx = 0.
\] (2.7)

Furthermore, the function \(\psi\) has the same properties on \(\tilde{D}\) as \(\Psi\) has on \(\bar{D}\); namely,
\[
\psi \in C^1(\bar{D}) \cap C^2(D), \quad \text{where} \quad D = \{x \in \mathbb{R}, -h < y < \eta(x)\},
\]
and is a \(2\pi\)-periodic and even function of \(x\). Moreover, the change of variables (2.6) reduces relations (2.1)–(2.4) to the following
\[
\psi_{xx} + \psi_{yy} = 0, \quad (x, y) \in D; \quad (2.8)
\]
\[
\psi(x, -h) = -Q_0, \quad x \in \mathbb{R}; \quad (2.9)
\]
\[
\psi(x, \eta(x)) = 0, \quad x \in \mathbb{R}; \quad (2.10)
\]
\[
|\nabla \psi(x, \eta(x))|^2 + 2\eta(x) = \mu, \quad x \in \mathbb{R}. \quad (2.11)
\]

In the non-dimensional Bernoulli equation, the parameter \(\mu = \pi c^2/(g\ell)\) is the Froude number squared which must be found along with \(\eta\) and \(\psi\). Besides, \(\mu/2\) serves as the independent of \(h\) upper bound for \(\eta\) with equality achieved only for the wave of extreme form with the Lipschitz crest; see (32). Thus, the non-dimensional statement of the problem is as follows.

**Definition 2.1.** Let \(Q_0\) and \(h\) be given positive numbers, then problem \(\text{P}(Q_0, h)\) is to find a triple \((\mu, \eta, \psi)\) from relations (2.8)–(2.11) so that \(\mu\) is positive, \(\eta\) satisfies condition (2.7), whereas all other properties of \(\eta\) and \(\psi\) (smoothness, \(2\pi\)-periodicity and symmetry) are as described above.

On the other hand, having a solution of problem \(\text{P}(Q_0, h)\), formulae (2.6) yield a \(2\ell\)-periodic solution of problem (2.1)–(2.4) for any \(\ell > 0\), for which purpose one has to put
\[
Q = Q_0 \sqrt{\frac{g(\ell/\pi)^3}{h}}\] and to determine \(R\) from the equality \(c^2 = 2(R - gH)\) with \(c^2 = \mu g\ell/\pi\) and \(H = h\ell/\pi\).
3. Babenko’s equation

The aim of this section is to derive a single nonlinear pseudo-differential equation that has the same form as (1.1), but \( C \) is replaced by the sum of \( C \) and some compact operator depending on a real parameter. The equation is related to the family of problems \( P(Q_0, h) \) in the following sense. The value of operator’s parameter together with equation’s solution allow us to determine \( h \) and to obtain some solution of the water-wave problem.

3.1 Derivation of Babenko’s equation

First, we follow considerations in Section 8 of (9); see also the rather recent paper (40).

By \( w(z) = \varphi + i\psi \) \((z = x + iy)\) we denote the complex potential restricted to the one-wave domain \( D = \{ -\pi < x < \pi, -h < y < \eta(x) \} \) of some periodic wave on water of a certain depth \( h \). Here \( \varphi(x, y) \) is the odd in \( x \) harmonic conjugate to \( \psi \), for which purpose the additive constant must be chosen properly. For some \( r \in (0, 1) \) we consider a conformal mapping, say \( u(z) \), from the \( z \)-plane to the auxiliary \( w \)-plane; it maps \( D \) onto

\[
A_r = \{ r < |u| < 1; \Re u \notin (-1, -r) \text{ when } \Im u = 0 \}. \tag{3.1}
\]

This annular domain has a cut which makes it simply connected and the map is such that the images of the upper and bottom parts of \( \partial D \) are

\[
\{ |u| = 1; \Re u \neq -1 \} \text{ and } \{ |u| = r; \Re u \neq -r \}
\]

respectively, whereas the left (right) side of \( \partial D \) is mapped onto the upper (lower respectively) side of the cut \( \{ \Re u \in (-1, -r) \text{ when } \Im u = 0 \} \). Putting

\[
u = e^{-iw} \text{ and } \frac{dz}{du} = i \left[ u^{-1} + f(u) \right], \tag{3.2}
\]

where \( f(u) \) is a certain Laurent series, one obtains that

\[
-\frac{dw}{dz} = 1 + uf(u)^{-1}. \tag{3.3}
\]

In (9), Section 8, this formula serves as the basis for deriving Nekrasov’s equation in the case of finite depth. An equivalent form of this equation is given in (40); see equation (1.1) there.

According to the second equality (3.2), the general form of the inverse conformal mapping \( A_r \ni u \mapsto z \in D \) is as follows:

\[
z(u) = i \left[ \log u - a_0 + \sum_{k=1}^{\infty} a_k \left( u^k - r^{2k} u^{-k} \right) \right], \text{ where } a_k \in \mathbb{R}, \ k = 0, 1, 2, \ldots. \tag{3.4}
\]

Here, we put minus at \( a_0 \) because it is convenient in what follows. The fact that all coefficients \( a_k \) are real is a consequence of equality (3.3) because \( \psi \) is equal to a real constant on the bottom part of \( \partial D \) which corresponds to \( \{ |u| = r; \Re u \neq -r \} \) — the circumference cut on the negative real axis.

Let us derive some relations for the coefficients from (3.4). First, for \( u = r \) we obtain

\[
a_0 = h + \log r. \tag{3.5}
\]
Substituting \( u = e^{it}, \ t \in (-\pi, \pi), \) into (3.4) and separating the real and imaginary parts, we arrive at the following parametric representation of the free surface profile:

\[
x(t) = -t - \sum_{k=1}^{\infty} a_k (1 + r^{2k}) \sin kt, \quad \eta(t) = -a_0 + \sum_{k=1}^{\infty} a_k (1 - r^{2k}) \cos kt.
\] (3.6)

Now we see that another relation is equivalent to condition (2.7) written in terms of the last two expressions, namely:

\[
\int_{-\pi}^{\pi} \eta(t)x'(t) \, dt = 0 \iff a_0 = \frac{1}{2} \sum_{k=1}^{\infty} k a_k^2 (1 - r^{4k}).
\] (3.7)

It follows from the last equality that \( a_0 > 0 \) in the non-trivial case. Then equality (3.5) shows that the value of \( r \) is related not only to the depth \( h \), but also to a particular solution of problem \( P(Q_0, h) \).

It is worth mentioning that both expressions (3.6) are similar to those for the infinite depth (cf. (21), Section 3.7, where Babenko’s results are outlined), and in that case, a consequence is the relation \( x_t = -1 - \mathcal{C} \eta_t \) with

\[
(\mathcal{C}v)(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} v(\tau) \cot \frac{t - \tau}{2} \, d\tau,
\]

which is the form of the 2\(\pi\)-periodic Hilbert transform alternative to formulae (1.2).

The crucial point for obtaining a similar relation in the case of finite depth is to introduce the operator \( \mathcal{B}_r = \mathcal{C} + \mathcal{K}_r \) for \( r \in (0, 1) \), where

\[
(\mathcal{K}_r v)(t) = \frac{2}{\pi} \int_{-\pi}^{\pi} v(\tau) K_r(t - \tau) \, d\tau \quad \text{with} \quad K_r(t - \tau) = \sum_{n=1}^{\infty} \frac{r^{2n}}{1 - r^{2n}} \sin(t - \tau).
\] (3.8)

It is straightforward to calculate that \( \mathcal{B}_r \) can also be defined on \( L^2(-\pi, \pi) \) by linearity from the following relations

\[
\mathcal{B}_r(\cos nt) = \frac{1 + r^{2n}}{1 - r^{2n}} \sin nt \quad \text{for} \quad n \geq 0, \quad \mathcal{B}_r(\sin nt) = \frac{1 + r^{2n}}{1 - r^{2n}} \cos nt \quad \text{for} \quad n \geq 1
\] (3.9)

that are similar to (1.2). Combining these formulae and (3.5) yields that

\[
x_t = -1 - \mathcal{B}_r \eta_t \quad \text{for} \quad t \in (-\pi, \pi).
\] (3.10)

An important fact about the operator \( \mathcal{B}_r \) is that it is a conjugation in the following sense. If \( F(u) \) is analytic in \( A_r \) and \( 3F \) vanishes identically on \( \{|u| = r; \mathcal{R}u \neq -r\} \), then

\[
\mathcal{R}F(e^{it}) + [\mathcal{B}_r(3F)](t) = 0 \quad \text{for all} \quad t \in (-\pi, \pi).
\] (3.11)

Let us calculate the derivative \( z_\phi \) of the mapping inverse to the complex potential. In view of the first equality (3.2), we have

\[
z_\phi = z u u_\phi = -i z u e^{-i w} w_\phi = -i u z_u.
\]
Combining this and (3.4), we obtain that

\[ z_\varphi = 1 + \sum_{k=1}^{\infty} k a_k \left( u^k + r^{2k} u^{-k} \right), \]  

(3.12)

and the function on the right-hand side is analytic in \( A_r \). Since \( z_\varphi \) does not vanish in the closure of \( A_r \), we have that \( z_\varphi^{-1} = |\nabla \varphi|^2 \frac{1}{z_\varphi} \) is also analytic in \( A_r \). Moreover, the Bernoulli equation (2.11) implies that

\[ z_\varphi^{-1} = (\mu - 2\eta)(x_\varphi - iy_\varphi) = (\mu - 2\eta)(i\eta t - x_t) \quad \text{when} \quad u = e^{-i\tau}, \]  

(3.13)

(cf. formula (3.38) in (21)). Here the second equality is a consequence of the Cauchy–Riemann equations. Then equality (3.10) yields that

\[ z_\varphi^{-1} = (\mu - 2\eta)(1 + B_r \eta t + i\eta t) \quad \text{for} \quad t \in (-\pi, \pi). \]  

(3.14)

It follows from previous considerations that the constant in the Laurent expansion of \( z_\varphi^{-1} \) is equal to \( \mu \). Furthermore, \( \Im \{ z_\varphi^{-1} - \mu \} \) vanishes identically on \( \{ |u| = r; \Re u \neq -r \} \), which allows us to apply formula (3.11) to the function \( z_\varphi^{-1} - \mu \), whose trace on \( \{ u = e^{it} \} \) is equal to

\[ (\mu - 2\eta) B_r (\eta') - 2v + i(\mu - 2\eta)\eta'. \]

Here again ‘ stands for differentiation with respect to \( t \). Thus, we arrive at

\[ (\mu - 2\eta) B_r (\eta') - 2\eta + B_r [ (\mu - 2\eta)\eta' ] = 0 \quad \text{for} \quad t \in (-\pi, \pi), \]

which simplifies to Babenko’s equation for waves on water of finite depth:

\[ \mu B_r (\eta') = \eta + B_r (\eta') + B_r (\eta' \eta) \quad \text{for} \quad t \in (-\pi, \pi). \]  

(3.15)

This equation is similar to (1.1) and the derivation procedure yields that for each \( r \in (0, 1) \) it is related to some solution of problem \( P(Q_0, h) \).

3.2 Local bifurcation branches for Babenko’s equation

To show the existence of small solutions of equation (3.15), bifurcating from the zero solution, we apply the Crandall–Rabinowitz theorem (see (41), Theorem 1.7) that deals with bifurcation from simple eigenvalues of the linearised equation; its formulation is as follows.

**Theorem 1.** Let \( X, Y \) be real Banach spaces with the continuous embedding \( X \subset Y \). If a continuous map \( F(\mu, v) : \mathbb{R} \times X \to Y \) has the following properties:

(i) the equality \( F(\mu, 0) = 0 \) holds for all \( \mu \in \mathbb{R} \),

(ii) the operators \( F_\mu, F_v \) and \( F_{\mu v} \) exist and are continuous,

(iii) for some \( \mu^* \) the operator \( F_v(\mu^*, 0) \) is a Fredholm one with zero index and its null-space is one-dimensional,

(iv) if the null-space of \( F_v(\mu^*, 0) \) is generated by \( v^{(0)} \), then \( F_{\mu v}(\mu^*, 0) v^{(0)} \) does not belong to the range of \( F_v(\mu^*, 0) \),
Then a sufficiently small $\varepsilon > 0$ exists and a continuous curve
\[ \{ (\mu(s), v(s)) : |s| < \varepsilon \} \subset \mathbb{R} \times X, \]
bifurcates from $(\mu^*, 0)$; for pairs belonging to this curve
\[ \mu(s) = \mu^* + o(s) \text{ and } v(s) = s v^{(0)} + o(s) \text{ when } 0 < |s| < \varepsilon. \]

Moreover, if $\mathcal{F}_{\mu v}$ is continuous, then the curve is of class $C^1$.

As in (19), we say that a real-valued function $v$ belongs to the Sobolev space $H_0$ provided it is absolutely continuous on $[-\pi, \pi]$, $v(-\pi) = v(\pi)$, and its weak derivative $v'$ belongs to $L^2(-\pi, \pi)$. Let $\hat{H}_0$ be the subspace of $H_0$ consisting of even functions.

In terms of the map $\mathcal{F} : \mathbb{R} \times \hat{H}_0 \mapsto L^2(-\pi, \pi)$ defined by
\[ \mathcal{F}(\mu, v) = \mu \mathcal{B}_r(v') - v + v \mathcal{B}_r(v') + \mathcal{B}_r (v') v, \]
equation (3.16) takes the following form:
\[ \mathcal{F}(\mu, v) = 0, \quad (\mu, v) \in \mathbb{R} \times \hat{H}_0. \]

Let us apply the Crandall–Rabinowitz theorem to this equation to obtain local branches of Stokes-wave solutions of small amplitude, for which purpose we have to check conditions (i)–(iv) for $\mathcal{F}(\mu, v)$.

It is clear that (i) and (ii) are fulfilled and $\mathcal{F}_\mu(\mu, 0) = \mu \mathcal{B}_r (d/dt) - I$, where $I$ is the identity operator. Hence the set of bifurcation points of equation (3.17) is $\{\mu_n\}_{n=1}^{\infty}$, where
\[ \mu_n = \frac{1 - r^{2n}}{n(1 + r^{2n})}, \quad n = 1, 2, \ldots, \] (3.18)
are the characteristic values of $\mathcal{B}_r (d/dt)$. Furthermore, $\mathcal{F}_\mu(\mu_n, 0)$ is a Fredholm operator, its index is equal to zero for every $\mu_n$, and the corresponding null-space in $\hat{H}_0$ is one-dimensional being generated by $v^{(0)}_n(t) = \cos nt$, thus yielding condition (iii). Since $\mathcal{F}_{\mu v}(\mu_n, 0) = -I$ we see that $\mathcal{F}_\mu(\mu_n, 0) v^{(0)}_n(t) = -\cos nt$, and so condition (iv) for $n = 1, 2, \ldots$ is a consequence of the fact that the equation
\[ \mu_n \mathcal{B}_r (v') - v = -\cos nt \]
has no solution. Indeed, a solution of this equation exists if and only if its right-hand side is orthogonal to the null-space of the adjoint operator
\[ [\mu_n \mathcal{B}_r (d/dt) - I]^* = \mu_n (d/dt) \mathcal{B}_r - I. \]
Since its null-space is one-dimensional and generated by $\cos nt$, the orthogonality condition does not hold for $-\cos nt$. This completes verification of condition (iv).

Then the Crandall–Rabinowitz theorem yields the following.

**Theorem 2.** For every $n = 1, 2, \ldots$ there exists $\varepsilon_n > 0$ such that for $0 < |s| < \varepsilon_n$ there is the family $(\mu^{(s)}_n, v^{(s)}_n)$ of Stokes-wave solutions to equation (3.17). Together with the
Fig. 1 The branch of solutions of equation (3.15) with $r = 4/5$, bifurcating from the zero solution at $\mu_1(4/5) = 0.219512195122$. The upper bound mentioned prior to Definition 1 is also included.

The last assertion is a consequence of the fact that $F_{vv}$ is continuous which is obvious. This theorem is illustrated in Fig. 1, where we have a plot of the bifurcation branch $C_1$ in terms of $\mu$ and the norm of solution $\|v\|_\infty$ in the space $L^\infty(-\pi, \pi)$. The plotted branch bifurcating from $\mu_1(4/5)$ has no secondary bifurcation points as the analogous branch for equation (1.1); see (19, 20) for the rigorous proof and detailed discussion. Moreover, it exhibits the phenomenon of a turning point at the largest value of $\mu$ attained on $C_1$, occurring high on the branch; see further details in Section 4.3. (The fastest traveling wave of given period corresponds to this point.) By means of a different method this property was demonstrated by (32), whereas our method shows that it also takes place for equation (1.1) on the branch bifurcating from $\mu_1(0)$. This phenomenon is related to the ‘Tanaka instability’ found numerically in (42), and later investigated analytically in (43).

3.3 Solutions of Babenko’s equation define periodic waves

Let us outline a procedure demonstrating how to obtain a solution of problem (2.8)–(2.11) from that of Babenko’s equation; that is, if equation (3.15) with $r \in (0, 1)$ is satisfied by some $\mu > 0$ and an even function $v$ (the existence of such pairs — at least in the form (3.19) — follows from Theorem 2), then one can find the following:

1) a 2π-periodic, symmetric curve with zero mean and a negative number $-h$, which define the wave profile and the level of horizontal bottom, respectively, thus giving a one-period water domain, say $\Omega$, on the $(x, y)$-plane;

2) a function $\psi$ harmonic in $\Omega$ and vanishing on its top side and two positive constants serving as the right-hand side terms in the boundary conditions (2.9) and (2.11).

Let we have an even, 2π-periodic solution $v$ of equation (3.15), whose Fourier coefficients
we denote \( b_0, b_1, b_2, \ldots \) to distinguish these coefficients from those in (3.24), and let the periodic extension of \( v \) to \( \mathbb{R} \) be real-analytic. Using these coefficients, we define the following holomorphic function on \( A_r \):

\[
z(u) = i \left[ \log u - B + \sum_{k=1}^{\infty} b_k \left( u^k - r^{2k} u^{-k} \right) \right].
\] (3.20)

Here \( B \) is a real number that will be determined below in terms of the Fourier coefficients of \( v \). Let us consider the images that correspond under this mapping to the curves and segments of \( \partial A_r \). First, we see that \( z(e^{it}) = x(t) + iy(t) \) for \( t \in (-\pi, \pi) \), where

\[
x(t) = -t - \sum_{k=1}^{\infty} b_k \left( 1 + r^{2k} \right) \sin kt, \quad y(t) = -B + \sum_{k=1}^{\infty} b_k \left( 1 - r^{2k} \right) \cos kt.
\] (3.21)

Since this curve given parametrically serves as the upper part of \( \partial \Omega \), we require its mean value to vanish. This gives that

\[
B = \frac{1}{2} \sum_{k=1}^{\infty} k b_k^2 \left( 1 - r^{4k} \right),
\] (3.22)

where the series converges because the Fourier coefficients of the real-analytic \( v \) decay faster than any power of \( k \).

Now we are in a position to determine the mean depth of flow \( h \). In view of symmetry we have that \( z(r) \) is the mid-point of the bottom; that is, \( z(r) = -i h \). Then putting \( u = r \) into (3.20), we find that

\[
h = B - \log r = \frac{1}{2} \sum_{k=1}^{\infty} k b_k^2 \left( 1 - r^{4k} \right) - \log r,
\] (3.23)

and so is positive; here the last equality is a consequence of (3.22). Thus, the second expression (3.24) takes the following form:

\[
y(t) = -(h + \log r) + \sum_{k=1}^{\infty} b_k \left( 1 - r^{2k} \right) \cos kt.
\] (3.24)

Hence the curve \( z_s = \{ x = x(t) = -t - (2r) y(t), \quad y = y(t); \quad t \in [-\pi, \pi] \} \) has the zero mean value. Here, the first formula (3.24) is applied to express \( x(t) \) in terms of \( y(t) \).

Furthermore, we have

\[
z(|u|e^{\pm \pi i}) = \mp \pi + i \left[ \log |u| - h + \sum_{k=1}^{\infty} (-1)^k b_k (u^{2k} - r^{2k}) / |u|^k \right] \quad \text{for } u \in [-1, -r],
\] (3.25)

thus obtaining two vertical segments \( z_- \) and \( z_+ \) on the lines \( x = -\pi \) and \( x = \pi \) respectively.

Taking into account (3.22) and (3.24), we see that

\[
z(re^{it}) = -ih - t - 2 \sum_{k=1}^{\infty} b_k r^k \sin kt \quad \text{for } t \in [-\pi, \pi]
\] (3.26)

on the inner circumference. This defines a horizontal segment \( z_0 \) on the line \( y = -h \).

It is clear that the curve \( \Gamma = z_+ \cup z_- \cup z_- \cup z_0 \) constructed above is closed and one can check (for example, numerically) that the set \( \Omega \) enclosed within \( \Gamma \) is a domain. The next step is to show that \( z(u) \) defined with the help of the Fourier coefficients of \( v \) is a conformal mapping of \( A_r \) onto \( \Omega \). For this purpose one can use the boundary correspondence principle;
Theorem 3 (The boundary correspondence principle). Let $D$ and $D^*$ be two bounded simply connected domains with piecewise smooth boundaries and let $f$ be holomorphic in $D$ and continuous in $\overline{D}$. If $f(p)$ parametrises $\partial D^*$ counter-clockwise provided $p$ is a counter-clockwise parametrisation of $\partial D$, then $f$ is a conformal mapping of $D$ onto $D^*$.

According to this theorem $z(u)$ maps $A_r$ onto $\Omega$ conformally provided one can show (for example, numerically) that the map $\partial A_r \ni u \mapsto z \in \Gamma$ is a homeomorphism. Moreover, condition (2.7) is fulfilled for $\eta(x) = y(t(x))$; here $t(x)$ is the inverse of $x = -t - (B_r y)(t)$, existing provided the curve $z_x$ is not self-intersecting. Thus, the curve $y = \eta(x)$ defines the upper side of $\Omega$.

Figs 2–6 illustrate how the boundary correspondence principle works numerically in recovering Stokes waves from solutions of \([3.15]\). We consider the equation with $r = 4/5$ and take the solution $(\mu, v)$ with $\mu \approx 0.32671$. This solution belongs to the branch bifurcating from $\mu_1$ (equal to $0.219512195122$ for $r = 4/5$), and the value of $\mu$ in point is close to the
critical one on this branch (see Fig. 1 and Fig. 8). Substituting the Fourier coefficients of $v$ into (3.20) and (3.22), we define $z(u)$ which is holomorphic in $A_r$ and maps $A_r$ onto $\Omega$ continuously; the latter set is the closure of the prospective one-wave domain. To demonstrate that $z(u)$ is a conformal mapping we choose several points on $\partial A_r$, numbering them counter-clockwise (see Fig. 2), and calculate their images on $\partial \Omega$, assigning to each the same number as the object point has on $\partial A_r$. It occurs that the images are also numbered counter-clockwise in agreement with the boundary correspondence principle (see Fig. 3). To be sure that the counter-clockwise boundary correspondence is not violated between the chosen points we provide three figures more. In Fig. 4, the graph of

$$x_h(t) = -t - 2\sum_{k=1}^{\infty} b_k r^k \sin kt$$

(3.27)

is plotted for $r = 4/5$ and $t$ varying from 0 to $\pi$ (this parametrises the upper half of the inner circumference clockwise provided it is considered as a part of $\partial A_r$; see Fig. 2). According to (3.26), this gives the left-hand half of the bottom shown in Fig. 3 also parametrised clockwise. Since (3.27) is a monotonic function, there is no violation of the boundary correspondence on the bottom because, by symmetry, it is sufficient to check this on its right-hand half only.

Fig. 4 The graph of (3.27) with $r = 4/5$; its monotonicity confirms that the boundary correspondence is not violated on the bottom part of $\Gamma$.

Fig. 5 The graph of (3.28) with $r = 4/5$; its monotonicity confirms that the boundary correspondence is not violated on the right-hand side of $\Gamma$. 
In Fig. 5, the graph of
\[ y_+(u) = \log |u| - h + \sum_{k=1}^{\infty} (-1)^k b_k \left( u^{2k} - r^{2k} \right) / |u|^k \]  
(3.28)
is plotted for \( r = 4/5 \) and \( u \) varying from \(-r\) to \(-1\) (this parameterises the lower side of the cut counter-clockwise provided it is considered as a part of \( \partial A_r \); see Fig. 2). According to (3.25), this gives the right-hand side of \( \Gamma \) shown in Fig. 3. Since (3.28) is a monotonic function, there is no violation of the boundary correspondence on the right-hand side of \( \Gamma \).

Finally, the graph of the first function (3.21) is plotted in Fig. 6 for \( r = 4/5 \) and \( t \) varying from 0 to \( \pi \) (this parameterises the upper half of the exterior circumference of \( \partial A_r \) counter-clockwise; see Fig. 2). According to the first equation (3.21), this parameterises the left-hand part of the upper side of \( \Gamma \) shown in Fig. 3. Since (3.21) is a monotonic function, there is no violation of the boundary correspondence on this part of \( \Gamma \).

It remains to check that \( \Omega \) is a one-wave domain for some Stokes wave; that is, there exists a stream function \( \psi \) defined on \( \Omega \) so that it satisfies conditions (2.9)–(2.11) with some constant serving as the right-hand side term in (2.9), where \( \mu \) stands in (2.11). For this purpose we map \( \Omega \) conformally on an auxiliary rectangle
\[ R^* = \{ (\varphi^*, \psi^*) : -\pi < \varphi^* < \pi, -\psi_0 < \psi^* < 0 \} \]
so that the images of \( z_s \) and \( z_b \) are the top and bottom parts of \( \partial R^* \) respectively, whereas the value \( \psi_0 > 0 \) will be chosen later. Thus, there are harmonic functions \( \varphi^* \) and \( \psi^* \) defined on \( \Omega \), and for every \( \psi_0 \) the image of \( R^* \) under the mapping \( e^{-i(\varphi^* + i\psi^*)} \) is the annular domain \( A_\rho \) with some \( \rho \). It is clear that the value of \( \rho \) decreases from unity to zero as \( \psi_0 \) characterising \( R^* \) increases from zero to infinity. Requiring \( \rho \) to be equal to \( r \), we fix the value of \( \psi_0 \), thus determining \( \psi^* \) which, in its turn, gives the constant value \(-Q^*_+\) that stands on the right-hand side of condition (2.10); here the sign is chosen so that \( Q^*_+ \) is positive. It should be noted that this procedure guarantees that condition (2.10) is also fulfilled. It remains to use \( \varphi^* \) and \( \psi^* \) for determining \( \psi \) so that it satisfies condition (2.11) along with (2.9) and (2.10).

Using the Fourier coefficients \( b_1, b_2, \ldots \) of \( v \) in formula (3.12), we obtain the function \( z_{\varphi^*} \) holomorphic in \( A_r \) and non-vanishing there. According to equation (3.15), we have that
\[ \left( 1 - 2\mu^{-1} y(u) \right) z_{\varphi^*}(u) - 1 \right|_{|u|=1} \]
is the limit as $|u| \to 1$ of some holomorphic function given in $A_r$ and having its imaginary part equal to zero on $\partial A_r \cap \{|u| = r\}$. Besides, the same property holds for $z_{\varphi^*}$, and so it is also true for the function whose limit as $|u| \to 1$ is equal to

$$\left[(1 - 2\mu^{-1}y(u))|z_{\varphi^*}(u)|^2\right]_{|u|=1}.$$ 

Therefore, we have that

$$1 - 2\mu^{-1}\eta(x) = q^2|\nabla\psi^*(x, \eta(x))|^2, \quad x \in (-\pi, \pi),$$

with some $q > 0$ and $\eta$ defined above. For $\psi = q\sqrt{T}\psi^*$ the last relation coincides with (4.1).

Thus, the triple $(\mu, \eta, \psi)$ satisfies problem $P(Q_0, h)$ with $h$ defined by (3.23), whereas $Q_0 = q\sqrt{T}Q_*$ and $Q_*$ depends on $r$ implicitly as described above. This completes the description of a procedure how to obtain a solution of problem (2.8)–(2.11) from the given solution of Babenko’s equation.

4. Numerical solution of Babenko’s equation

In this section, we describe a numerical method for solving equation (3.15) in the class of even, periodic functions on $(-\pi, \pi)$. The existence of small solutions of this kind is proved in Section 3.2, whereas general solutions are discussed in Section 5. The essence of our method is to calculate the solution’s Fourier coefficients $b_0, b_1, \ldots$, which allows us to restore the conformal mapping $z(u)$ (see Section 3.3), thus demonstrating numerically the equivalence of Babenko’s equation and problem $P(Q_0, h)$.

4.1 Transformation of (3.15) to a form convenient for discretisation

Let $r \in (0, 1)$ be fixed, then $J_r = \mathcal{B}_r d/dt$ is a self-adjoint operator on $L^2_{per}(-\pi, \pi)$ of $2\pi$-periodic square integrable functions. Its domain is $H_0$ (see Section 3.2 for the definition), and it can also be defined by linearity from $J_r \cos nt = \lambda_n \cos nt$ for $n = 0, 1, \ldots$ and $J_r \sin nt = \lambda_n \sin nt$ for $n = 1, 2, \ldots$; here the eigenvalues are $\lambda_n = \mu_n^{-1}$ for $n \geq 1$ and $\lambda_0 = 0$; cf. (3.18). Since the corresponding eigenfunctions form a basis in $L^2(-\pi, \pi)$, the following spectral decomposition holds:

$$J_r = \sum_{n=1}^{\infty} \lambda_n (\hat{P}_n + \hat{\bar{P}}_n). \quad (4.1)$$

Here $\hat{P}_n$ ($\hat{\bar{P}}_n$) is the projector onto the subspace spanned by $\cos nt$ ($\sin nt$, respectively).

Seeking solutions of (3.15) in $H_0$, it is convenient to write the equation in an equivalent form to accelerate its calculations. This form arises after replacing $J_r$ in (3.15) by the right-hand side of (4.1) with omitted $\hat{P}_n$, which is possible in view of the bijection between $H_0$ and the Sobolev space $W^{1,2}(0, \pi)$; indeed, for every $w \in W^{1,2}(0, \pi)$ its even extension $v$ belongs to $H_0$ and vice versa. Therefore, it is convenient to put $\hat{\mathcal{B}}_r = \sum_{n=1}^{\infty} \lambda_n P_n$, where $P_n$ is the projector onto the subspace of $L^2(0, \pi)$ spanned by $\cos nt$. Then $\hat{\mathcal{B}}_r$ is defined on $W^{1,2}(0, \pi)$ and $\hat{\mathcal{B}}_r w = J_r v = (\mathcal{B}_r v')$ almost everywhere on $(0, \pi)$, and so (3.15) takes the following equivalent form

$$\mu \hat{\mathcal{B}}_r w = w + w\hat{\mathcal{B}}_r w + \frac{1}{2}\hat{\mathcal{B}}_r (w^2), \quad t \in (0, \pi), \quad (4.2)$$

where $w(t)$ is sought in $W^{1,2}(0, \pi)$. To solve this equation numerically, a modified version of
the software SpecTraVVave is applicable; the latter is available freely at the site indicated in \text{[45]}, whereas its detailed description can be found in \text{[46]}.

For the reason made clear below, we amend \text{1.2} further; namely, we set \( \mu_0 = 1 \) and put \( \mathcal{L}_r = \sum_{n=0}^{\infty} \mu_n P_n \). Hence \( \mathcal{L}_r \) is invertible and \( \mathcal{L}_r^{-1} = P_0 + \mathcal{J}_r \); that is, \( \mathcal{J}_r \mathcal{L}_r = I - P_0 \), where \( I \) is the identity operator. Applying \( \mathcal{L}_r \) to both sides of \text{1.2}, we obtain the following equation:

\[
\mu(I - P_0)w = \mathcal{L}_r w + \mathcal{L}_r (w \mathcal{J}_r w) + \frac{1}{2} (I - P_0) w^2, \quad t \in (0, \pi). \tag{4.3}
\]

It should be noted that the unbounded operator \( \mathcal{J}_r \) is present in the nonlinear part of the last equation only, and so one can expect that \text{1.3} would demonstrate better numerical stability. Finally, equations \text{1.3} and \text{3.15} are equivalent in the following sense. The sets \( \{b_n(w)\}_{n=0}^{\infty} \) and \( \{b_n(v)\}_{n=0}^{\infty} \) of the Fourier coefficients coincide for solutions of \text{1.3} and \text{3.15}, respectively, provided the value of \( \mu \) is the same for both solutions.

For equation \text{1.3} the existence of small solutions follows from its equivalence to \text{3.15}. It can also be established directly with the help of the Crandall–Rabinowitz theorem; see Section 3.2, which yields the asymptotic formulae \text{3.15} for the branch of solutions of \text{1.3} bifurcating from \( \mu_n \) and trivial \( w \). This can serve as a solution guess to start with in the numerical procedure.

### 4.2 Discretisation of equation \text{1.3}

We use the standard cosine collocation method, according to which solutions of \text{1.3} are sought in the form of linear combinations of \( \cos mx \), \( m = 0, 1, \ldots, \) a basis in \( L^2(0, \pi) \). For the discretisation the subspace \( S_N \) spanned by the first \( N \) cosines is used, which is defined by their values at the collocation points \( x_n = \pi \frac{2n-1}{2N} \) for \( n = 1, \ldots, N \). Thus, for any \( f \in W^{1,2}(0, \pi) \) the vector \( f^N \) given by its coordinates

\[
f_n^N = \sum_{k=0}^{N-1} (P_k f)(x_n), \quad n = 1, \ldots, N,
\]

is considered. The operator \( \mathcal{L}_r^N \), discretising \( \mathcal{L}_r \), is defined as follows:

\[
(\mathcal{L}_r^N f^N)_n = \sum_{k=0}^{N-1} (P_k \mathcal{L}_r f)(x_n), \quad n = 1, \ldots, N.
\]

Furthermore, \( \mathcal{J}_r^N \) and \( P_0^N \) are introduced as the discretisations of \( \mathcal{J}_r \) and \( P_0 \) respectively.

These definitions are correct because \( f^N \) defines the function \( f \) with values \( f(x_n) = f_n^N \) uniquely up to a projection on the subspace orthogonal to \( S_N \). It is clear that each of these discrete operators is a composition of the discrete cosine transform, some diagonal matrix and the inverse discrete cosine transform. The diagonal matrix for \( \mathcal{L}_r^N \) is \( \{1, \ldots, \mu_{N-1}\} \), whereas the diagonal for \( \mathcal{J}_r^N \) is \( \{0, \lambda_1, \ldots, \lambda_{N-1}\} \), and \( \{1, 0, \ldots, 0\} \) is the diagonal for \( P_0^N \).

The discrete analogue of \text{1.3} is as follows:

\[
\mathcal{L}_r^N w^N - \mu (I - P_0^N) w^N + \mathcal{L}_r^N (w^N \mathcal{J}_r^N w^N) + \frac{1}{2} (I - P_0^N) (w^N)^2 = 0. \tag{4.4}
\]

Since solutions \( (\mu, w^N) \) of this equation form curves in the \((\mu, a)\)-plane, where

\[
a = \|w^N\| = \max_n |w_n^N|,
\]

it is convenient to parametrise these curves for making calculations more effective. Thus,
Fig. 7 The solution branches $C_2$, $C_3$ and $C_4$ for equation (3.15) with $r = 0$, bifurcating from the zero solution at $\mu_2(0) = 1/2$, $\mu_3(0) = 1/3$ and $\mu_4(0) = 1/4$ respectively. The secondary solution branches are denoted $C_{21}$, $C_{31}$ and $C_{41}$ respectively. The upper bound mentioned prior to Definition 1 is also included.

Fig. 8 The solution branch $C_1$ for equation (3.15) with $r = 4/5$ in a vicinity of the turning point, whose characteristics are as follows: $\mu \approx 0.32671$ the solution’s $L^\infty$-norm is approximately equal to 0.15862. The bold dot marks the solution plotted in Fig. 3. The upper bound mentioned prior to Definition 1 is also included.

due to a new parameter, say $\theta$, we have $\mu = \mu(\theta)$ and $a = a(\theta)$ on each curve of solutions. Therefore, $\mu(\theta)$ must be substituted into (4.4) instead of $\mu$, and this algebraic system must be complemented by the equation:

$$
\max_{n=1,\ldots,N} |w_n^N| = a(\theta).$$

The resulting system (4.4)–(4.5) has $N + 1$ equations with the following unknowns $\theta, w_1^N, \ldots, w_N^N$. Hence the standard Newton’s iteration method is applicable for finding bifurcations from a trivial solution, and the Crandall–Rabinowitz asymptotic formula (3.19) yields an initial guess. Further details concerning the proposed parametrisation and the particular realisation of algorithm can be found in (46).

4.3 Bifurcation curves for equation (3.15)

We begin with the results of a test calculation in which the algorithm described in Section 4.2 is applied to equation (3.15) with $r = 0$, thus giving bifurcation curves for equation (1.1). The curves plotted in Fig. 7 show the bifurcations from a trivial solution and the first three secondary bifurcations for this case; the curve $C_1$ is omitted because its behaviour is
similar to that presented in Fig. 1, including the presence of a turning point. The secondary bifurcation branches $C_{21}$, $C_{31}$ and $C_{41}$ bifurcate from $C_2$, $C_3$ and $C_4$, respectively, at the points, where $\mu$ is approximately equal to 0.58768, 0.39172 and 0.29389 respectively. These values are in good agreement with those presented by Aston (47); see Table 1 in his paper.

Now we turn to numerical results obtained for equation (3.15) with $r = 4/5$. The solution branch $C_1$ is presented in Fig. 1, and some of its characteristics are described after Theorem 2. In particular, it is pointed out that it has a turning point, and so we give a zoomed plot of the curve $C_1$ in a vicinity of this point; see Fig. 8, where bold dot marks one of two solutions corresponding to $\mu \approx 0.32671$. The wave profile corresponding to this solution is plotted in Fig. 3, where some of its characteristics are given; moreover, its $L^\infty$-norm is approximately equal to 0.15862.

The last example concerns the solution branch $C_3$ for equation (3.15) with $r = 4/5$. It is presented in Fig. 9, where one observes the presence of a turning point as well as the secondary bifurcation. Indeed, the branch $C_{31}$ bifurcates from $C_3$ at the point, where $\mu$ is approximately equal to 0.25298, and shortly after that $C_3$ has its turning point. The algorithm proposed in Section 4.2 allows us to solve (4.3) up to both critical values on $C_3$ and $C_{31}$; see Fig. 10 and Fig. 11, respectively, for the plots of wave profiles corresponding to these solutions.

In Fig. 10, the wave profile corresponds to the end-point solution on the branch $C_3$; $\mu \approx 0.25175$ for this solution of equation (3.15) with $r = 4/5$. Like a small-amplitude wave characterised by the second formula (3.19), this profile has the wavelength $2\pi/3$, and so three wave periods are plotted. Moreover, this Stokes wave has the extreme form; that is, the tangents to two smooth arcs form the angle $2\pi/3$ at every crest. The tangency is demonstrated with sufficient accuracy in the figure, where the angle inscribed into the wave profile has the sides $y = y_c \pm x/\sqrt{3}$ with $y_c = y(0)$; see (3.24) for $y(t)$ and the first formula (3.21) for $x(t)$ that describe the profile parametrically. Of course, the same tangency with similar angles takes place at every crest.

In Fig. 11, the wave profile corresponds to the end-point solution on the branch $C_{31}$; $\mu \approx 0.24827$ for this solution of equation (3.15) with $r = 4/5$. The profile has the wavelength $2\pi$, and so two wave periods are plotted. Thus, the period-tripling occurs as $C_{31}$ bifurcates
5. Concluding remarks

We have considered the nonlinear problem describing steady, gravity waves on water of finite depth. This problem is reduced to a single pseudo-differential operator equation (3.15) (Babenko’s equation), which generalises the well-known equation (1.1) describing waves on infinitely deep water. The local bifurcation for (3.15) is investigated analytically with the help of the Crandall–Rabinowitz theorem, whereas a combination of analytical and numerical methods is applied for demonstrating that the initial, free-boundary problem and Babenko’s equation are equivalent in the following sense. For every solution of the initial problem one of its components, namely, the free-surface elevation, is a solution of Babenko’s equation for some value of the parameter on which the equation’s operator depends; this value is determined by the solution of the free-boundary problem. On the contrary, every
solution of Babenko’s equation defines a solution of some free-boundary problem through a certain procedure.

Besides, we outline an algorithm which allows us to solve Babenko’s equation numerically using a modification of the free software SpecTraVVave; see (45). It should be emphasised that the developed numerical procedure is not only very fast, but remarkable for its high accuracy. The latter is essential when computing solutions to which wave profiles of the extreme form correspond, thus allowing us to plot global bifurcation branches presented in Section 4.3.

This paper is just an initial step in studies of Babenko’s equation both analytically and numerically. First, it is desirable to prove rigorously that every solution of Babenko’s equation defines a solution of the free-boundary problem that describes steady waves on a flow of finite depth with certain characteristics. Second, it is natural to show that the profiles of waves below the highest, that has the extreme form being non-smooth at its highest point, are real analytic curves. Third, one has to demonstrate the absence of sub-harmonic bifurcations in a neighbourhood of every point, where the bifurcation from the zero solution occurs. Finally, a global Stokes-wave theory should be developed and used for proving that there exist sub-harmonic bifurcations on branches of smooth waves close to the highest wave. All these results had been established for waves on infinitely deep water on the basis of equation (1.1); see (19, 20).

An interesting direction for further numerical investigations is to find higher bifurcations that might exist for waves on water of finite depth as it happens in the case of deep water as had been shown in (47), where just several isolated points of higher bifurcations are listed in Table 1. Since the algorithm based on equation (3.15) and realised by using the software SpecTraVVave is a rather robust tool, one could apply it for calculating branching bifurcation curves that have more than one point of bifurcation.

In conclusion, we outline what equation (3.15) has in common with Babenko’s equation for finite depth obtained by Constantin, Strauss and Vărvărucă (35); see Remark 4 in their paper. Namely,

$$\tilde{\mu} \tilde{C}_d(\tilde{v}') = \tilde{v} + \tilde{v} \tilde{C}_d(\tilde{v}') + \tilde{C}_d(\tilde{v}' \tilde{v})$$

literally coincides with (2.50) in (35) with one exception. We use $d$ as the operator’s parameter instead of $h$. There are two reasons for this: (1) $d$ and $h$ are equal to each other in Remark 4, since $k$ is taken equal to unity there; (2) a different quantity is denoted by $h$ in Section 2.2 of our paper and $h$ will be used in that meaning below.

It is obvious that the form of the last equation is exactly the same as that of (3.15), but what about the meaning of symbols involved? First, the parameter $d > 0$ is equal to the so-called conformal mean depth and the latter is defined uniquely by the water domain; see (19), Appendix A. However, this depth, generally speaking, is not equal to the non-dimensional mean depth of the water domain $D$ introduced in Section 2.2; see, in particular, formulae (2.6). By analogy with the conformal mean depth, it would be natural to call the parameter $r \in (0, 1)$, on which the operator $B_r$ depends in (3.15), the conformal mean radius of the water domain $D$. Furthermore, the conjugation operator $C_d$ is defined for $2\pi$-periodic functions on $\mathbb{R}$ as follows. If $f$ has zero mean value over a $2\pi$ interval, that is, its Fourier series has the form

$$f(x) = \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \quad x \in \mathbb{R},$$
then
\[(\mathcal{C}_d f)(x) = \sum_{n=1}^{\infty} \coth nd (a_n \sin nx - b_n \cos nx), \quad x \in \mathbb{R}.\]

This definition is similar to that of $\mathcal{B}_r$ in (3.9), but with the multiplier $\coth nd$ instead of $(1 + r^{2n})/(1 - r^{2n})$. Moreover, $\mathcal{C}_d$ has the representation analogous to $\mathcal{B}_r = \mathcal{C} + \mathcal{K}_r$ with $\mathcal{K}_r$ given by (4.3); see formulae (A.9) and (A.12) in (35). Thus, there is a significant similarity between $\mathcal{C}_d$ and $\mathcal{B}_r$. The essential point that distinguishes $\mathcal{C}_d$ and $\mathcal{B}_r$ is that the latter operator is defined for all $2\pi$-periodic functions, whereas the domain of $\mathcal{C}_d$ is orthogonal to constants.

Finally, let us demonstrate that if $k = 1$ (this is the case in (35), Remark 4), the free surface profile satisfies the assumptions made in this paper (see Section 2.1) and $\tilde{v}$ is an even and $2\pi$-periodic solution of (5.1), then $\tilde{\mu} = \mu$. Thus, the bifurcation parameter is the same in both (5.1) and (3.15). According to Remark 4 in (35), we have
\[\tilde{\mu} = \frac{2R}{g} - 2d - 2\beta, \quad (5.2)\]
where $R$ is the Bernoulli constant in (2.1), whereas the exact value of $\beta$ is unimportant for what follows. Since $k = 1$, formulae used in Section 2.2, dealing with the derivation of the non-dimensional problem, imply that $l = \pi$ and $H = h$, and so
\[\mu = \frac{2R}{g} - 2h.\]
Combining this formula and (5.2), we see that $\tilde{\mu} = \mu$ holds, if we show that $h = d + \beta$.

In order to prove the last equality, we first notice that the definition of $\mathcal{C}_d$ implies that $\mathcal{C}_d(\tilde{v}')$ and $\mathcal{C}_d(\tilde{v}\tilde{v}')$ are orthogonal to constants. Then equation (5.1) yields that
\[\int_{-\pi}^{\pi} \tilde{v}(x)[1 + \mathcal{C}_d(\tilde{v}')(x)] \, dx = 0 \iff \int_{0}^{\pi} \tilde{v}(x)[1 + \mathcal{C}_d(\tilde{v}')(x)] \, dx = 0,\]
where the second relation is a consequence of the assumption that $\tilde{v}$ is even. To transform this relation we consider the parametric representation of the free surface profile used in (35):
\[X(x) = U(x,0) = x + \mathcal{C}_d(\tilde{v} + \beta), \quad Y(x) = V(x,0) = \tilde{v} + d + \beta, \quad x \in \mathbb{R}, \quad (5.3)\]
see (2.7), (2.8), (2.10) and Remark 4. Hence we have
\[\int_{0}^{\pi} \tilde{v}(x) \frac{dX}{dx} (x) \, dx = \int_{0}^{\pi} \tilde{v}(x(X)) \, dX = 0,\]
where it is taken into account that $X(x)$ is invertible on $(0, \pi)$ since $k = 1$. Averaging the second formula (5.3) over $(0, \pi)$, we obtain
\[h = \frac{1}{\pi} \int_{0}^{\pi} Y(x(X)) \, dX = \frac{1}{\pi} \int_{0}^{\pi} \tilde{v}(x(X)) \, dX + d + \beta = d + \beta,\]
which yields the required equality $\tilde{\mu} = \mu$.

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