Long-term evolution of discs around magnetic stars

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ABSTRACT

We investigate the evolution of a thin viscous disc surrounding a magnetic star, including the spin-down of the star by the magnetic torques it exerts on the disc. The transition from an accreting to a non-accreting state, and the change of the magnetic torque across the corotation radius $r_c$ are included in a generic way, the widths of the transition taken in the range suggested by numerical simulations. In addition to the standard accreting state, in which the star gradually moves into spin equilibrium with the disc, two more states are found. An accreting state can develop into a ‘dead’ disc state, with inner edge $r_{in}$ well outside corotation. More often, a ‘trapped’ state develops, in which $r_{in}$ stays close to corotation even at very low accretion rates. The long-term evolution of these two states is different. In the dead state the star spins down incompletely, retaining much of its initial spin. In the trapped state the star can asymptotically spin-down to arbitrarily low rates, with its angular momentum transferred to the disc. We identify these outcomes with respectively the rapidly rotating and the very slowly rotating classes of Ap stars and magnetic white dwarfs.

Key words: accretion, accretion discs – instabilities – MHD – stars: formation – stars: magnetic field – stars: oscillations – stars: rotation.

1 INTRODUCTION

Accreting stars with strong magnetic fields are generally observed to rotate more slowly than their less-magnetic or discless counterparts. In protostars, T Tauri systems (which often have strong surface magnetic fields of $\sim 10^2$–$10^3$ G) with discs rotate more rapidly than systems without discs (Getman et al. 2008). Most (but not all) main-sequence Ap stars (with surface fields of up to $10^4$ G) are observed to rotate much slower than normal A stars (Stephan & Landstreet 2002), and a recent work suggests that this relationship extends down to their pre-main-sequence progenitors, the Herbig Ae stars (Alecian et al. 2008). In high-energy systems the result is similar: accreting neutron stars with weak ($\sim 10^3$ G) fields rotate up to $10^4$ times faster than neutron stars with strong ($\sim 10^{12}$ G) fields. These observations suggest that the interaction between the accretion disc and stellar magnetic field plays a critical role in regulating the spin-rate of the star.

Early theoretical studies of accretion predicted that a strong stellar field would truncate the accretion disc some distance from the surface of the star, with the truncation radius located roughly where the magnetic pressure $(B^2/4\pi)$ equals the ram pressure of the infalling gas $(m_v/2\pi r^2)$, so that infalling matter is channelled on to the surface via magnetic field lines, causing the star to spin-up (Pringle & Rees 1972). This assumes that the disc is truncated inside the corotation radius $[r_c \equiv (GM_*/\Omega_0^2)^{1/3}]$, where the star’s spin frequency is equal to the disc’s Keplerian frequency. If instead the magnetic field spins faster than the inner edge of the disc, a centrifugal barrier prevents accretion. Interaction between the magnetic field and the disc will then spin down the star (Illarionov & Sunyaev 1975; Mineshige, Rees & Fabian 1991; Lovelace, Romanova & Bisnovatyi-Kogan 1999; Romanova et al. 2004; Ustyugova et al. 2006).

The presence of the centrifugal barrier is often equated in the literature with the idea that the accreting gas will be flung out, or ‘propellered’ out of the system so as to maintain a steady state. This assumption turns out to be both arbitrary and unnecessary. For example, in order for the accreting material to be flung out of the system, the disc must be truncated a sufficient distance away from $r_c$. Otherwise the rotational velocity difference between the disc and the magnetosphere is too small (Spruit & Taam 1993) to expel matter.

Steady disc solutions with a centrifugal barrier at the inner edge were first described by Sunyaev & Shakura (1977), who called them ‘dead discs’, because even though the disc is actively transporting angular momentum outwards, no accretion on to the star takes place and the disc itself is very dim.

As pointed out already in Sunyaev & Shakura (1977) and Spruit & Taam (1993), in a system with an externally imposed mass flux, the likely effect of a centrifugal barrier is to cause the accretion on to the star to be cyclic. Accretion phases alternate with quiescent periods during which mass piles up outside the barrier, without mass having to leave the system. In the quiescent phase, the angular momentum extracted from the star by the disc–field interaction is
carried outwards through the disc by viscous stress. This alters the surface density profile of the disc from the usual accreting solution.

In our previous paper (D’Angelo & Spruit 2010; hereafter DS10) we studied this form of cyclic accretion with numerical solutions of the viscous diffusion equation for a thin disc subject to a magnetic torque. As in the (somewhat more ad hoc) model of Spruit & Taam (1993), limit cycles of the relaxation oscillator type were found. The cycle period of these oscillations depends on the accretion rate, from fast oscillations at higher mass flux to arbitrarily long periods at low accretion rates.

Instead of the two states, accreting and dead as suggested above, the results in DS10 are actually described better by including a third, intermediate state that we call here ‘trapped’ state:

(i) \( r_{\text{in}} < r_c \): accreting state, star spins up;
(ii) \( r_c - \Delta < r_{\text{in}} < r_c + \Delta \): trapped state, spin-up or spin-down;
(iii) \( r_{\text{in}} - r_c > \Delta \): star spins down, no accretion (dead disc);

where \( \Delta \ll r_c \) is a narrow range around corotation, to be specified later. In state (ii), the inner edge of the disc remains close to corotation over a range of accretion rates on to the star, and the net torque on the star can be of either sign, depending on the precise location of the inner edge of the disc.

For a given accretion rate, a disc that starts in state (i) will gradually move into state (ii) or (iii) as the star spins up and \( r_c \) moves inwards. In state (ii) accretion can proceed steadily or happen in bursts, depending on the disc–field interaction at \( r_{\text{in}} \). For steady externally imposed accretion a disc in this state will eventually move into spin equilibrium with the star, so that the net torque on the star is zero. In the dead state (iii) a steady state can exist if the torque exerted by the star is taken up at the outer edge of the disc by a companion star. If we neglect the transition to the propeller regime, then in theory the dead disc solution can exist for a disc truncated at any distance outside \( r_c \). Such a disc will remain static as the star spins down and \( r_c \) moves outwards. Our model is thus qualitatively different from the conventional ‘propeller’ picture since at very low accretion rates a considerable amount of mass remains confined in the disc, and the star can be efficiently spun down.

In the following we study the long-term evolution of the star-disc system by using the description of magnetospheric accretion in DS10, allowing the star’s spin rate to evolve. Of special interest will be the trapped state (ii), since in many cases the evolution of the system ends in it. The accretion cycles found in DS10 also take place essentially within a trapped state. The inner edge of the disc is near corotation in the trapped state, as is the case also for a disc in spin equilibrium with the accreting star. Spin equilibrium is only a special case of a trapped state, however. In general, however, a trapped state is not one of spin equilibrium; spin-up as well as spin-down is possible. We focus particular attention on cases in which the mean accretion rate through the disc is very low and the star cannot easily move into spin equilibrium with the disc. This is most applicable for isolated stars, or accreting binaries in quiescence.

This scenario poses a number of questions which we address in the course of this paper. These include: under what conditions does the disc get into a trapped state, and when does it instead evolve into a dead state? It will turn out that this is determined by the details of the disc–field interaction and the ratio of the spin-down time-scale of the star \( (T_{\text{spin}}) \) to the viscous time-scale of the disc \( (T_{\text{visc}}) \). The initial conditions of the disc also significantly influence the outcome. In Section 5.2 we ask how an initially trapped disc could become untrapped as a dead disc state. In particular, does this depend on the initial location of the inner edge of the disc, the initial accretion rate, the presence or absence of a companion or the size of the disc?

Finally, in Section 5 we discuss the physics that determines whether a disc will become trapped, and in Section 5.3.1 we ask whether a trapped disc could plausibly regulate the slow spins observed in Ap stars and magnetic white dwarfs, some of which have spin periods of decades.

In a companion paper we investigate the observable consequences of a trapped disc, focusing in particular on how the burst instability studied in DS10 will change the spin evolution and observable properties of the star. In that paper we also discuss our model’s predictions in terms of observations of magnetospherically regulated accretion in both protostars and X-ray binaries.

We use the code developed in DS10, adding the star’s moment of inertia as a parameter of the problem in order to follow the spin evolution of the star in response to the disc interaction. We can then simultaneously follow the viscous evolution of the disc and spin evolution of the star as the star’s spin changes, and explore how these two interact with each other. We describe our model in more detail in the following section.

2 MAGNETOSPHERIC INTERACTIONS WITH A THIN DISC

2.1 Magnetic torque

The interaction between a strong stellar magnetic field and surrounding accretion disc truncates the disc close to the star, and forces the incoming matter to accrete along closed field lines on to the surface of the star in a region called the magnetosphere. At the outer edge of the magnetosphere (termed here the magnetospheric radius), the field lines become strongly embedded in the disc over some small radial extent that we term the interaction region, \( \Delta r \). The differential rotation between the star and the Keplerian disc will cause the field lines to be twisted, which will generate a toroidal component to an initially poloidal field (e.g. Ghosh, Pethick & Lamb 1977). This will allow the transfer of angular momentum between the disc and star, with the torque per unit area exerted by the field on the disc given by \( \tau = r S_\phi \dot{\phi} \), where

\[
S_\phi = \frac{B_\phi B_z}{4\pi}
\]  \hspace{1cm} (1)

is the magnetic stress generated by the twisted field lines. Both theoretical arguments (e.g. Aly 1985; Lovelace, Romanova & Bisnovatyi-Kogan 1995) and numerical simulations (such as Wang & Robertson 1985; Hayashi, Shibata & Matsumoto 1996; Goodson, Winglee & Boehm 1997; Miller & Stone 1997) suggest that the strong coupling between magnetic field lines and the disc will cause the field lines to inflate and open. The inflation and opening of field lines limit the growth of the \( B_\phi \) component for the field to \( B_\phi = \eta B_z \), with \( \eta \) of order unity, and reduces the radial extent of the interaction region, since beyond a given radius the field lines are always open and the disc–field connection will be severed. We take the interaction region to be narrow, \( \Delta r / r < 1 \) (as found in numerical simulations, see Section 2 of DS10 for a more detailed discussion). Assuming the star’s dipole field strength \( B_d(r) \) as an estimate of \( B_z \), and taking into account the fact that \( \dot{\phi} \) acts on both sides of the disc, (1) yields the magnetic torque \( T_0 \) exerted on the disc:

\[
T_0 = 4\pi \eta^2 \Delta r^2 = \eta^2 \Delta r^2 B_z^2.
\]  \hspace{1cm} (2)

This torque exists only if the inner edge \( r_{\text{in}} \) of the disc is outside the corotation radius \( r_c \). For \( r_{\text{in}} < r_c \), we have instead a disc accreting...
on an object rotating slower than the Kepler rate at \( r_m \). By the standard theory of thin viscous discs, the torque exerted on the disc by the accreting object then vanishes, independent of the nature of the object. The torque \( T_B(r_c) \) thus changes over a narrow range around \( r_c \). To model this transition we introduce a ‘connecting function’ \( y_\Sigma \):

\[
T_B(r_m) = y_\Sigma(r_m) \pi (r_m)^2 \cdot \Omega_1(r_m)
\]

with the properties \( y_\Sigma \rightarrow 0 \) \( (r_c - r_m \gg \Delta r) \), \( y_\Sigma \rightarrow 1 \) \( (r_m - r_c \gg \Delta r) \). As in DS10, we take for this function

\[
y_\Sigma = \frac{1}{2} \left[ 1 + \tanh \left( \frac{r_m - r_c}{\Delta r} \right) \right].
\]

The width of the transition is thus described by \( \Delta r \). We take the same value for it as used in 2.

2.2 Model for disc–magnetosphere interaction

In DS10 we derived a description of the interaction between a disc and magnetic field for a disc truncated either inside or outside \( r_c \), and introduced two numerical parameters to connect the two regimes. To keep the problem axisymmetric, we assumed a dipolar magnetic field, with the dipole axis aligned with the stellar and disc rotation axis. Since the region of interaction between the disc and the field is small, we use our description of the interaction as a boundary condition for a standard thin accretion disc (Shakura & Sunyaev 1973).

To evolve a thin disc in time, we must choose a description for the effective viscosity (\( \nu \)) that allows transport of angular momentum. We adopt an \( \alpha \) prescription for the viscosity and assume a constant scaleheight (\( H \)) for the disc, so that

\[
\nu = \alpha (GM)^{1/2} (H/r)^2 r^{1/2}.
\]  

At the inner edge of the disc the behaviour is regulated by the disc–field interaction. However, since the interaction region is small, we incorporate the interaction as a boundary condition on the inner disc, and assume that the majority of the disc is shielded from the magnetic field and then evolves as a standard viscous disc, albeit with a very different inner boundary condition from the standard one. Below we summarize our analysis of the disc–field interaction and how these translate into boundary conditions on the disc. (For the detailed derivation of our boundary conditions, see sections 2.3, 2.4 and 3.2 of DS10.)

2.2.1 Surface density at \( r_m \)

In a dead disc, the disc–field interaction prevents matter from accreting or being expelled from the system, instead retaining matter that interacts with the magnetic field. This implies that the angular momentum injected via magnetic torques in the interaction region \( \Delta r \) must be transported outwards by viscous torques in the disc. A dead disc will therefore have a maximum in surface density at \( r_m \), and \( \Sigma(r_m) \) will depend on the amount of angular momentum being added by the disc–field interaction.

By equating the amount of angular momentum added by the field to the amount carried outwards by viscous processes, we can calculate the surface density at the inner boundary of the disc needed for the injected angular momentum to be carried away. This yields (see DS10) a value for the surface density \( \Sigma \) at the inner edge of a dead disc, proportional to the magnetic torque (3):

\[
3\pi \nu \Sigma(r_m) = \frac{T_B}{r_m^2 \Omega_k}.
\]

where \( \Omega_k \) is the Keplerian rotation frequency. If the stellar field is a dipole and we use (5) to describe the viscosity, then for \( r_m > r_c \), \( \Sigma(r_m) \propto r_m^{-4} \). \( \Sigma(r_m) \) thus decreases rapidly with an increasing \( r_m \).

2.2.2 Accretion rate across \( r_c \)

If the inner edge is well inside the corotation radius \( r_c \), we use a standard result to estimate the location of \( r_m \) as a function of the accretion rate \( \dot{m} \). It is obtained from the azimuthal equation of motion for gas at the point at which it is forced to corotate with the star (cf. Spruit & Taam 1993). This gives

\[
\dot{m} = \pi \Sigma(r_m) \dot{r}_m = \pi \dot{r}_m \frac{\mu I^2}{4 \Omega_k r_m^3}.
\]

(7)

(Note that we take the sign of \( \dot{m} \) positive for inward mass flow.) It is not necessary that stationarity holds: (7) can also be applied when the inner edge of the disc moves. However, since it describes the accretion through the magnetosphere–disc boundary \( r_m \), it has to be applied in a frame comoving with \( r_m \). If \( \dot{m} \) is the mass flow rate in a fixed frame, it is related to the accretion rate in this comoving frame \( \dot{m}_c \) by

\[
\dot{m}_c = \dot{m} + 2\pi \Sigma(r_m) \dot{r}_m.
\]

(8)

where \( \dot{r}_m \) is the rate of change of the inner disc edge.

To connect the accreting case with the dead disc case, for which \( \dot{m} = 0 \), we need one more prescription, this time for the accretion rate as a function of the inner edge radius. We introduce a connecting function \( y_m \) for this (DS10):

\[
y_m(r_m) = \frac{y_m(r_m) \dot{m}(r_m)}{\dot{m}_c}
\]

(9)

with the properties \( y_m \rightarrow 1 \) \( (r_c - r_m \gg \Delta r_2) \), \( y_m \rightarrow 0 \) \( (r_m - r_c \gg \Delta r_2) \), with

\[
y_m = \frac{1}{2} \left[ 1 - \tanh \left( \frac{r_m - r_c}{\Delta r_2} \right) \right].
\]

(10)

where \( \Delta r_2 \) describes the width of the transition (different in general from \( \Delta r \)).

With the star’s assumed field of dipole moment \( \mu, B_d = \mu I r^3 \) and Keplerian orbits in the disc, (8) becomes, with the viscous thin-disc expression for \( \dot{m} \):

\[
6\pi \nu \Sigma r_m \frac{\partial (v \Sigma r^{1/2})}{\partial r} \bigg|_{r_m} = y_m \frac{\mu I^2}{4 \Omega_k r_m^3} - 2\pi r_m \Sigma(r_m) \dot{r}_m.
\]

(11)

where \( \eta \) is a numerical factor of order unity and \( \mu \) the dipole moment of the star (see DS10 for details).

Along with our description for the viscosity, (6) and (11) define a boundary condition at \( r_m \) and an equation for \( r_m(t) \), for a disc over a continuous range of accretion rates, from strongly accreting systems \( (r_m \ll r_c) \) to dead-disc systems \( (\dot{m} \approx 0) \).

2.2.3 Evolution of corotation radius

In order to study the response of a disc to changes in spin of the star, we must incorporate the angular momentum exchange between the star and the disc:

\[
I_s \frac{d\Omega_k}{dt} = \frac{dJ}{dt}
\]

(12)

which introduces the moment of inertia of the star, \( I_s = k^2 M_\star R_\star^2 \), as an additional parameter of the problem.

The disc–star angular momentum exchange \( dJ/dt \) has two components: matter accreting on to the star adds angular momentum at the rate of \( m \omega^2 r_m^2 \Omega_k(r_m) \), while the disc–field coupling outside

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corotation extracts angular momentum, spinning the star down. The rate of angular momentum exchange between the disc and the star will thus be (with 2, 3, 4)
\[
\frac{dJ}{dt} = m_{\text{cir}} r_{\text{in}}^2 \Omega_B(r_{\text{in}}) - T_B
\]
\[
= \frac{\eta \mu^2}{r_{\text{in}}^3} \left[ \frac{1}{4} \left( \frac{r}{r_{\text{in}}} \right)^{3/2} y_{\text{in}} - \frac{\Delta r}{r_{\text{in}}} y_{\text{in}} \right].
\]
(13)

The corotation radius (a function of \(\Omega_c\)) evolves as
\[
\frac{dr_c}{dt} = -\frac{2}{3} \frac{dJ}{r_{\text{in}}^3} \left( \frac{G M_*}{r_c^2} \right)^{1/2}.
\]
(14)

Equation 13 shows that there is a value of \(\dot{m}\) for which there is no net angular momentum exchange with the star. This is the ‘spin equilibrium’ state discussed in previous work. In addition to the mean accretion rate in the disc, this zero-point will depend on our adopted connecting functions as well as the size of the transition widths, \(\Delta r\) and \(\Delta r_2\). If \(\dot{m} = 0\), there is no spin-equilibrium solution: the star will spin down by the magnetic torque.

2.2.4 Steady-state solutions

In the presence of a magnetic torque at the disc inner edge, the steady solutions (\(\partial/\partial r = 0\)) of the thin viscous disc diffusion equation with the above boundary conditions have the form (cf. DS10)
\[
3\pi \nu \Sigma = \frac{T_B}{\Omega(r_{\text{in}}) r_{\text{in}}^2} \left( \frac{r_{\text{in}}}{r} \right)^{3/2} + \dot{m} \left[ \frac{r_{\text{in}}}{r} \right]^{1/2}.
\]
(15)

where \(\dot{m}\) is the accretion rate on to the star, given by (9). If the inner edge is inside corotation (\(T_B = 0\)), \(\Sigma\) has the standard form for steady accretion on an object rotating below the Keplerian rate (second term on the right-hand side).

For \(r_{\text{in}}\) well outside corotation (\(r_{\text{in}} - r_c \gg \Delta r\)), \(\dot{m} \downarrow 0\) and we have a dead disc. The surface density is then determined by the first term on the right-hand side. The steady outward flux of angular momentum in this case has to be taken up by a sink at some larger distance, otherwise the disc cannot be stationary as assumed. This sink can be the orbital angular momentum of a companion star, or the disc can be approximated as infinite. The latter is a good approximation for changes in the inner regions of the disc, if timescales short compared with the viscous evolution of the outer disc are considered.

2.3 Numerical method

We use the one-dimensional numerical code described in DS10 to evolve the standard diffusive thin-disc equation with our viscosity prescription (5) and our description of the disc–field interaction (which gives the inner boundary conditions the boundary conditions 6 and 11). At the outer boundary a mass flux and a flux of angular momentum in this case has to be taken up by a sink at some larger distance, otherwise the disc cannot be stationary as assumed. This sink can be the orbital angular momentum of a companion star, or the disc can be approximated as infinite. The latter is a good approximation for changes in the inner regions of the disc, if timescales short compared with the viscous evolution of the outer disc are considered.

2.3.1 Outer boundary condition

The lifetime and evolution of a star surrounded by a dead disc is an inherently time-dependent problem, so the initial conditions in the disc can be critical for its evolution. Since the spin-down time-scale for the star can be much larger than viscous timescales throughout the disc, the conditions in the outer disc will also strongly influence the evolution of the system.

We thus consider the effect of varying the outer boundary conditions for the disc. The first condition we study is the simplest: a fixed mass flux \(\dot{m} = \dot{m}_0 (>0\), corresponding to accretion). As discussed in Section 1, a key aspect of disc–magnetosphere interaction is that accretion is possible even as the star is spun down. At fixed \(\dot{m} > 0\), the angular momentum flux can be either inward or outward.

If the mass flux specified vanishes at \(r_{\text{out}}\), the boundary condition is
\[
\frac{\partial}{\partial r} \left( r^{1/2} \Sigma \right) \bigg|_{r_{\text{out}}} = 0.
\]
(16)

On long evolution timescales, the finite extent of the disc itself could be relevant in a star without a companion where the disc can spread outwards. To model this as our final boundary condition, we take \(\Sigma(r_{\text{out}}) = 0\), so that the angular momentum added at \(r_{\text{in}}\) is taken in by the outer parts of the disc, causing the disc to spread outwards. In Section 5.2 we discuss the consequences of these assumptions in limiting the lifetime of a trapped disc.

2.3.2 The evolution of \(r_c\)

The final modification to our code used in DS10 is to allow \(r_c\) to evolve as the spin rate of the star changes (14). The characteristic evolution time-scale for \(r_c\), the spin-down time-scale for the star, is much longer than the nominal viscous time-scale in the disc (see the next section). The code updates \(r_c\) by an explicit time-step, rather than implicitly, as we do for the other variables. Rather than discretizing (14) and adding it to our system of linearized equations that are solved numerically at each time-step, we approximate the evolution in \(r_c\) to first order in time, that is
\[
r_c(t_0 + \Delta t) = r_c(t_0) + \frac{dr_c}{dt} \bigg|_{t_0} \Delta t + O(\Delta t^2).
\]
(17)

This scheme is simpler than adding additional equations to the code, and is sufficient to describe the co-evolution of the disc and stellar spin-rate. However, as we discuss in Section 4.3, it is not accurate enough when \(r_{\text{in}}\) is close to \(r_c\) and the spin-down time-scale is comparable to the viscous time-scale in the inner regions of the disc.
Table 1. Spin-down and viscous time-scales for different types of magnetic star.

| Source                        | Mass (M_⊙) | Radius (R_⊙) | B_⊙ (G) | P_* (s) | I_* (M_⊙ R_⊙^2) | T_SD (yr) | T_vis(r_c)/T_SD |
|-------------------------------|-------------|---------------|---------|---------|------------------|-----------|-----------------|
| Slow pulsar                   | 1.4         | 1.4 × 10^-5   | 10^{12} | 5.0     | 2.9 × 10^{-11}  | 4400      | 3 × 10^{-7}     |
| ms pulsar                     | 1.4         | 1.4 × 10^-5   | 10^{4}  | 0.1     | 2.9 × 10^{-11}  | 2.6 × 10^{11} | 2 × 10^{-17}    |
| White dwarf                   | 0.95        | 0.02          | 1.6 × 10^{7} | 20 d   | 3.8 × 10^{-5}  | 5.5 × 10^{4} | 1.6 × 10^{-6} |
| Magnetic Ae star              | 3.0         | 5.5           | 10^{4}  | 10 yr   | 4.0              | 3 × 10^{5}  | 0.06            |
| T Tauri star                  | 0.6         | 3.0           | 1500    | 7 d     | 0.54             | 2.3 × 10^{4} | 0.001           |

*a* Parameters from Wickramasinghe & Ferrario (2000)

*b* B_⊙ and I_* from Stepien (2000)

*c* Sipos et al. (2009)

3 CHARACTERISTIC TIME-SCALES OF DISC–STAR EVOLUTION

Three kinds of time-scale play a role in the evolution of a star coupled magnetically to an accretion disc. These are the spin period of the star, the time-scale for changes in the spin period of the star and viscous evolution time-scales of the disc. The viscous evolution does not have a single characteristic time-scale; it can vary over many orders of magnitude depending on which regions in the disc participate in the evolution.

The spin period of the star (P_*) determines the location of the corotation radius, which in turn sets the accretion rate at which the magnetic torque changes sign. P_* also determines the time-scale for magnetospospheric variability (from processes like reconnection of field lines), which can lead to variability in m_*, Δr/Δr_2 and the B_⊙ component of the magnetic field (which sets the magnitude of the torque). P_* is much shorter than the other characteristic time-scales studied in this paper, and the complex variability processes are best studied with detailed magnetohydrodynamics (MHD) simulations, so in this work we assume time-averaged values for m_*, Δr/Δr_2 and B_⊙ and neglect time-scales much shorter than the characteristic viscous time-scale at R_*.

A convenient unit of time for measuring changes in a viscous disc at a distance r from the centre is t_c = r^2/ν(r), sometimes called the accretion- or viscous time-scale at distance r. If α is the viscosity parameter and H the disc thickness, it is longer than the orbital time-scale Ω_∗^{-1} by a factor α^{-1}(r/ν)^2, a large number for most observed discs. Natural choices for r in this expression would be the disc’s inner edge r_in (t_in) or the corotation radius r_c(t_c). Both of these are functions of time. The actual time-scales of variation in our discs can be much shorter than t_c and t_in, however, since the extent of the disc that participates in the variation can be much smaller than r_in.

If the cyclic accretion mode described in DS10, for instance, cycle times as short as 0.01t_c are found. The time-scale for viscous adjustment in the outer disc regions, on the other hand, can be very large compared to t_c.

The longest time-scale is the rate at which the star’s spin will change, which is determined both by the rate of angular momentum exchange with the disc and by the star’s moment of inertia. The spin-down torque of a dead disc (with m = 0 and r_in = r_c) is, from equation (13),

$$I_* \frac{dΩ_*}{dt} = -\frac{ημ^2δ}{r_in^3},$$

where δ = Δr/r_in. The characteristic spin-down time is

$$T_SD \equiv \frac{P_*}{Ω_∗} \sim \frac{I_∗Ω_∗r_in^3}{ημ^2δ}.$$  

If the inner edge stays near corotation, Ω_κ = Ω_∗, replacing r_in by r_c, this yields

$$T_SD = \frac{GM_∗I_*}{2πημ^2δ} P_∗, \quad [Ω_κ(r_c) = Ω_∗].$$  

The spin-down time-scale varies considerably between different sources. Adopting η = 1 and Δr/r_in = 0.3, this spin-down time-scale is short enough to account for spin-regulation in slowly rotating magnetic stars. In Table 1 we summarize the predicted spin-down time-scales for a slowly rotating X-ray pulsar, a millisecond pulsar, a slowly rotating Ap star, and a typical T Tauri star. For all these examples except for the millisecond pulsar, the spin-down time-scale is much shorter than the lifetime of the star. Provided the conditions are such that the inner edge of the disc can stay near corotation (i.e. what we have called the ‘trapped disc’ state), it will be able to spin-down a star to very slow rotation periods. In the following sections we will explore how this could work in detail by evolving a viscous disc in time numerically.

The last column of Table 1 lists the ratio of the viscous time-scale r_in^2/ν and (19). Note that for our description of viscosity, the viscous and spin-down time-scales both scale as r^{-3/2}. The quantity T_vis(r_c)/T_SD thus defines the ratio of the time that gas at that radius takes to travel inwards on to the star and the time it would take to spin-down so that r_c(r) independent of radius. In all cases, the spin-down time-scale is much longer than the viscous time-scales in the disc, so that at least part of the disc is able to adjust to the new spin rate of the star. However, the exact ratio between the two time-scales varies over 10 orders of magnitude, from 0.06 for a strongly magnetic Herbig Ae star to 10^{-17} for an accreting millisecond pulsar. This ratio implies that the extent of spin-down will be influenced by the viscous evolution of the disc itself in response to the disc–field interaction, and that this evolution will vary substantially between different systems, breaking the scale invariance usually assumed in disc–magnetosphere interactions. In Section 4.3 we demonstrate how the ratio of these two time-scales is critical in determining the ratio of r_in to r_c in a trapped disc.

3.1 Representative model

In Sections 4 and 5.2 we study how a trapped disc can form and evolve, as well as how it can become untrapped. In order to simplify comparison between different simulations, we adopt a set of parameters for a representative model, which we then vary between solutions as necessary.

For the dimensionless parameters we adopt Δr/r_in = 0.1, Δr_c/r_in = 0.04 and B_⊙/B_⊙ = η = 0.1. The values of Δr/r_in and Δr_c/r_in are small enough to provide an abrupt transition between an accreting and a non-accreting disc, but do not show the cyclic instability discussed in DS10. Neglecting the star’s spin change, the problem

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has a scale invariance (DS10), whereby the parameters $\mu$, $M_*$, $\Omega_0$, and $\dot{m}$ can be rewritten as the ratio $\dot{m}/\dot{m}_c$, and $\dot{m}_c$ is the accretion rate in (7) that puts the magnetospheric radius at $r_c$, a natural unit of $\dot{m}$ for magnetospheric accretion. The results in DS10 were presented in this unit.

As discussed in the previous section, the variation of $r_c$ with time during spin-down of the star makes this unit impractical. Instead, we present the representative model in units suitable for a protostellar system with $T_{\text{visc}}/T_{\text{SD}} = 2.6 \times 10^{-3}$ (which is as large a ratio as the present version of the code allows), and explore the effect of varying this ratio. As a unit of length we use $r_*$, the star’s radius, and for time-scale $t_*$, the nominal viscous time-scale of the disc at the star’s radius.

4 TRAPPED DISCS

4.1 Trapped disc evolving from an accreting disc

For our description of the disc–field interaction (which ignores outflows), once the accretion rate falls to zero, the inner edge of the disc could be located anywhere outside $r_c$, depending on the amount of mass in the disc. What then determines the location of the inner radius of a dead disc? To answer this question, we simulate an initially steadily accreting disc in which the accretion rate at $r_{\text{out}}$ suddenly decreases to zero. As the disc runs out of its reservoir of gas, the accretion rate on to the star declines, and the inner radius of the disc moves outwards.

In the simulation we use our representative disc parameters described above, and set the initial inner radius of the disc to be just inside $r_c$, $r_{\text{in}}(t = 0) = 0.88r_*$, and the outer radius $r_{\text{out}} = 100r_*$.

We can calculate the corresponding accretion rate from (7), and use the static solution for $\Sigma$ given by (15) as our initial surface density profile. At $t = 0$, we set $\delta_0((1/\Sigma)^{1/3})|_{r_{\text{out}}} = 0$, so that no mass is added to the disc or allowed to escape. This sets a constant angular momentum flux at $r_{\text{out}}$.

The results are shown in Fig. 1. The bottom panel of Fig. 1 shows the change in accretion rate on to the star, scaled to $\dot{m}_c$. The top panel of Fig. 1 shows the evolution of the inner radius (solid black curve) and $r_c$ (dashed red curve) in response to the changing accretion rate. After an initial steady accretion over about $30t_*$ (less than $1/100t_*$, the viscous time-scale at $r_{\text{out}}$), $\dot{m}$ through the inner edge of the disc begins to decrease as the reservoir of mass in the disc is accreted on to the star, and $r_{\text{in}}$ begins to move outwards. From $30t_*$ to $1500t_*$, $\dot{m}$ decreases exponentially with a decay time-scale of about $240t_*$, and the disc moves outwards. However, $r_{\text{in}}$ increases by only about 20 per cent as the accretion rate decreases by 3 orders of magnitude. The structure of $r_{\text{in}}$ around $r_c$ is an artefact of the tanh connecting functions we adopted to describe $r_{\text{in}}$ and $\dot{m}$ across the transition region.

After $\sim 1500t_*$ the star begins to spin-down (so that $r_c$ moves outwards), and the inner radius of the disc begins to track $r_c$, so that the ratio $r_{\text{in}}/r_c$ remains nearly constant thereafter. The behaviour of the accretion rate at this point also changes. Although it continues to decrease exponentially, the delay time-scale lengthens considerably and the accretion rate ($\sim 10^{-4}$ of the initial $\dot{m}$) is regulated by the spin-down rate of the star. Instead of continuing to move away from $r_c$ into the ‘dead disc’ regime (in which $\dot{m} = 0$), the inner radius instead remains trapped at a nearly constant fraction of $r_c$ while the star continues to spin-down. We thus call this disc solution a ‘trapped disc’, since rather than continue to move outwards, $r_{\text{in}}$ becomes trapped at a nearly constant fraction of $r_c$.

At the outer edge of the disc $\dot{m} = 0$ and there is an outward angular momentum flux. The accretion on to the star comes from the disc being slowly eroded (although at a very low rate) as $r_{\text{in}}$ moves outwards. The evolution of $r_{\text{in}}$ and of the inner parts of the disc is determined by the spin-down rate of the star itself, which is itself influenced by how close $r_c$ can stay to $r_c$. In Section 4.3 we demonstrate how $r_{\text{in}}/r_c$ is mostly determined by the parameters $\Delta r_2$ and $\Delta T_2$, and the ratio $T_{\text{visc}}/T_{\text{SD}}$. However, the main conclusion of this section is clear: if a trapped disc forms and can efficiently carry away the angular momentum of the star, over spin-down time-scales the disc will accrete at such a rate that the inner edge of the disc can move outwards together with $r_c$ and the star could in theory spin down completely.

4.2 Trapped disc evolving from a dead disc

Consider next a case where the initial condition is a dead disc ($r_{\text{in}} > r_c$) given by the steady profile (15 with $\dot{m} = 0$ and $\Sigma_2 = 1$). As the star spins down, $r_c$ moves out until it catches up with the inner edge $r_{\text{in}}$. From then on, the evolution is the same as the previous case: $r_{\text{in}}$ and $r_c$ move outwards together indefinitely. A small amount of mass accretes while the star’s angular momentum is transferred to the disc.

The asymptotic evolution of this dead disc can be compared with the case when a fixed mass flux is imposed at the outer edge. The asymptotic state is then a steady state with spin equilibrium: the spin-up torque due to the accreted mass is balanced by the magnetic torque at $r_c$ transferring angular momentum outwards.

We illustrate the distinction between these two cases in Fig. 2. It shows the evolution of an initially dead disc (black, thick curves) and accreting disc (red, thin curves). Both the discs have the same representative parameters (Section 3.1) and $r_{\text{out}} = 100r_*$, but with different initial inner radii (a few times the stellar radius), accretion rates and appropriate initial surface density profiles given by (15). For the dead disc, we take $r_{\text{in}} = 1.3r_{\text{in,0}}$ (where $r_{\text{in,0}}$ is the initial corotation radius), which corresponds to $\dot{m} \approx 0$ for our chosen value of $\Delta r_2/r_{\text{in,0}}$. For the accreting disc, we choose $r_{\text{in}} = 1.1r_{\text{in,0}}$, which corresponds to a low but non-zero accretion rate ($\dot{m} = 8 \times 10^{-3}\dot{m}_c$).

Since $r_{\text{in}}$ for the accreting disc is initially low compared to $r_c$, at early times $r_c$ evolves at the same rate in both simulations (accreting: red, triple-dashed curve; non-accreting: black dashed curve), and
...the spin-down will effectively cease. Since $r_{\text{in}} \sim r_\star$ can begin. Since $r_{\text{in}} - r_\star \simeq 1$ will move quickly away is of the order 0.5 and moves close enough to $v_\star$ and $r_\star(y_\Sigma)$, which causes $r_{\text{in}} \partial r/\partial t = 0$ will move outwards. (This is because the magnetic torque between a dead disc and $r_{\text{in}}$ no longer satisfies (15). Since $r_{\text{in}}$ moves closer to $r_\star$ (within $\Delta r$ or $\Delta r_2$), this static state is no longer possible: either matter at $r_{\text{in}}$ starts accreting on to the star ($y_\Sigma \neq 0$), or the surface density at $r_{\text{in}}$ declines ($y_\Sigma \neq 1$), which causes $r_{\text{in}}$ to move closer to $r_\star$ until accretion through $r_{\text{in}}$ can begin. Since the viscous time-scale in the inner part of the disc is much shorter than the spin-down time-scale, after accretion through $r_{\text{in}}$ begins, a pseudo-steady-state develops, and $r_{\text{in}}$ moves slowly outwards with $r_\star$. If the disc can maintain a steady state for the given $m$, then $r_{\text{in}}$ will track $r_\star$, and the disc will remain a nearly dead disc as the star spins down to a small fraction of its initial spin period.

We can study this quantitatively by considering the equations for $r_{\text{in}}$ and $r_c$. The evolution of the inner edge of the disc (11 from Section 2.2.2), defining $u \equiv \Sigma r$ for convenience is

$$2\pi u(r_\star)\dot{r}_{\text{in}} = y_\Sigma \frac{\eta_\Sigma^2}{4\Omega_\star} - 6v_\text{visc}u^{1/2} \frac{\partial u}{\partial r} \bigg|_{r_{\text{in}}} .$$

(21)

As long as the two terms on the right-hand side of (21) balance, $r_{\text{in}} = 0$ even after accretion through $r_{\text{in}}$ begins. This will continue until the surface density profile near $r_{\text{in}}$ no longer satisfies (15). Since the change in $r_c$ is the only source of variability in the problem, $r_{\text{in}}$ will approximately track $r_\star$.

Equation (21) cannot be solved as is, since $\partial u/\partial r$ depends on solving the fully time-dependent diffusion problem. As an estimate we assume that $u$ changes as a result of the changing boundary condition (which will increase the surface density gradient), divided by the rate at which the rest of the disc can respond to that change (which will smooth it out). This can be approximated by

$$\frac{\partial u}{\partial r} \bigg|_{r_{\text{in}}} \sim - \frac{\partial u(r_{\text{in}})}{\partial r} v_{\text{visc}}^{-1},$$

(22)

where $v_{\text{visc}}$ is the viscous speed at $r_{\text{in}}$, of the order of $v_r \sim v/\ell$.

The time derivative for $u(r_{\text{in}})$ follows from the boundary condition for $u$:

$$\frac{\partial u}{\partial r} \bigg|_{r_{\text{in}}} = \frac{\partial u}{\partial r_{\text{in}}} + \frac{\partial u}{\partial r_c} \dot{r}_c.$$  

(23)

The equation for $r_{\text{in}}$ then becomes

$$r_{\text{in}} = \frac{v_\text{visc} y_{\Sigma} f^{3/2}}{8\Omega_\star \partial r_{\text{in}} y_{\Sigma}} \left( \frac{\Delta r}{r_{\text{in}}} \right)^{-1} \left( \frac{10}{3} \frac{y_{\Sigma}}{\partial_r y_{\Sigma} r_{\text{in}}} - 1 \right),$$

(24)

where $f \equiv (r_c/r_{\text{in}})$, and we have used the definition of $y_{\Sigma}$ from Section 2.2.1 so that $\partial_r y_{\Sigma} = -\partial_y y_{\Sigma}$.

The evolution of $r_c$ depends on the rate of angular momentum exchange with the star. Matter falling on to the star spins it up, while the interaction with the disc outside $r_c$ transfers angular momentum outwards and spins the disc down. The equation for this evolution
is given by (14), which can be rewritten as
\[ \dot{r}_c = \frac{2}{3} \frac{\eta \mu^2}{(GM_y)^{1/2} \dot{I}} f^{7/2} \left( \frac{\Delta r}{r_m \gamma} - \frac{\gamma_m}{4} f^{1/2} \right) r_m^{-1/2}. \] (25)

We can study the evolution of \( r_c \) and \( r_m \) in two limiting cases. In the limit where \( r_m = r_c \gg \Delta r, \Delta r_2 \):
\[ \dot{r}_c \rightarrow \frac{2}{3} \frac{\eta \mu^2 r_m^{-4}}{(GM_y)^{1/2} \dot{I} r_m} \Delta r^{7/2}, \]
\[ \dot{r}_m \rightarrow 0, \] (26)
so that
\[ r_c = \left( \frac{3}{2} \frac{\eta \mu^2 r_m^{-4}}{(GM_y)^{1/2} \dot{I} r_m} \Delta r + r_c^{-5/2} \right)^{-2/5}, \]
\[ r_m = r_m,0. \] (27)

This is the limiting ‘dead disc’ case, where the amount of angular momentum being injected at \( r_m \) can be extracted at \( r_m \) and the disc remains steady while the star is spun down. It predicts a slightly smaller spin-down torque than was estimated in Section 3 because the torque scales with \( r_m \).

The inner radius of the disc will remain approximately constant until either \( \frac{\eta y_m}{\gamma} \) or \( \frac{\Delta r}{\Delta r_2} \) becomes non-negligible (24). Based on our assumption that the disc will remain in a quasi-steady-state while \( \dot{r}_m \) moves, \( r_m \) will evolve only in response to changes in \( r_c \), which means that \( f \) is a constant. This simplifies (24) and (25) considerably:
\[ \dot{r}_c = \left( A_0 f^{7/2} - A_1 f^{9/2} \right) r_m^{-1/2}, \]
\[ \dot{r}_m = \frac{B_0 f^{3/2} r_m^{-1/2} - r_c}{B_1 - 1}, \] (28)
where
\[ A_0 = \frac{2}{3} \frac{\eta \mu^2}{(GM_y)^{1/2} \dot{I} r_m \gamma}, \]
\[ A_1 = \frac{1}{6} \frac{\eta \mu^2}{(GM_y)^{1/2} \dot{I} \gamma}, \]
\[ B_0 = \frac{10 y_m}{8 r_m \gamma_m \gamma(y_\gamma)} \left( \frac{\Delta r}{r_m} \right)^{-1}, \]
\[ B_1 = \frac{10 y_\gamma}{3 r_m \gamma_m \gamma(y_\gamma)}. \] (29)

The solution is then
\[ r_c = \frac{3}{2} \left( A_0 f^{3/2} - A_1 f^{9/2} \right) t + r_c,0^{2/3}, \]
\[ r_m = \frac{3}{2} \left( B_0 f^{3/2} - A_0 f^{7/2} + A_1 f^{9/2} \right) t + r_m,0^{2/3}. \] (30)

We can use (28) to calculate \( f \), that is, how close \( r_c \) can move towards \( r_m \) before the disc will start moving outwards in response. Setting \( \dot{r}_m = f \dot{r}_m \), we can re-express the constants in (30) as
\[ f = \left( \frac{2 \Delta r}{r_m \gamma(y_\gamma)} - \frac{y_m f^{1/2}}{2} \right) \left( \frac{10 y_\gamma}{3 r_m \gamma_m \gamma(y_\gamma)} + f - 1 \right) \]
\[ \times \left( \frac{r_m \gamma_m \gamma(y_\gamma)}{y_m} \right) = \frac{3}{8} \frac{T_{SD}}{T_{visc}}. \] (31)

and solve \( f \) numerically to give the approximate evolution for \( r_c \) and \( r_m \) in time.

In Fig. 3 we compare our estimates for the evolution of \( r_c \) and \( r_m \) to the solution from our numerical simulation. We consider a rapidly spinning star (\( r_c = 1.7 \)), with \( \Delta r r_m = 0.1 \) and \( \Delta r_2 r_m = 0.05 \). In

![Figure 3. Comparison between numerical solution and analytic estimate for an idealized dead disc around a star spinning close to break-up, \( r_c = 1.7 \), with \( \Delta r r_m = 0.1 \) and \( \Delta r_2 r_m = 0.05 \). Top: numerical solution of the evolution of \( r_c \) (black, dashed curve) and \( r_m \) (black, solid curve) in time. Overplotted is our analytic estimate for \( r_m \) (red dash-dotted curve) and \( r_c \) (red dash-triple dotted curve) for the same physical conditions. Bottom: the ratio \( r_c/r_m \) for our numerical solution (solid curve) and analytic estimate (dashed–dotted curve). The disagreement between the two solutions arises from our simplified treatment of the viscous disc.](https://academic.oup.com/mnras/article-abstract/416/2/893/1055394)
5 TRAPPED AND UNTRAPPED

5.1 Accreting discs evolving to trapped or dead disc states

The results found so far and in DS10 indicate the strong tendency for the inner edge of the disc to track the corotation radius, what we call here a ‘trapped disc’. This does not happen in all cases, however. We would like to find out under what conditions a disc gets stuck in this way, and when instead the inner edge proceeds to move well outside corotation into the ‘dead disc’ state.

Armed with a qualitative understanding from the previous section, we can now address this question with a few numerical experiments. We take the case of a neutron star with field strength $B_5 = 10^{12} \, G$, initial spin period $P_s = 5 \, s$, and initial inner edge radius $r_m = 0.95 r_o$. The disc is thus initially in an accreting state. The (initial) outer boundary is located at $r_out = 100r_m$, and we set $m = 0$ there so that no matter can escape from the system. The other parameters are the same as those for our representative model.

We first investigate the effect of varying the viscosity on the way the transition from an accreting to a dead disc takes place. This is shown in Fig. 4. The spin-down time-scale (20) for the initial $r_c$ is $T_{SD} = 10^7 \, yr$. From top to bottom, we plot this transition for decreasing values of viscosity (and hence increasing viscous time-scales). To compare these two quantities, we define $T_{visc}$ as the viscous time-scale at the initial $r_c$. In the different plots, the ratio $T_{visc}/T_{SD}$ increases from $2.5 \times 10^{-9}$ to $10^{-4}$.

The disc, the initial evolution is the same: as $\dot{m}$ decreases, $r_m$ moves outwards, crossing $r_c$ over about $10^3 \, yr$. Once that happens, however, the subsequent evolution of the disc and star varies substantially between simulations. In the most viscous discs (top), $r_m$ continues moving steadily outwards over $10^{10}$ years – roughly $10^6$ times longer than the nominal spin-down time-scale of the disc. Since $r_m$ keeps moving outwards, the torque on the star decreases too, so that the disc moves far away from $r_c$ before it is able to spin down the star.

As the viscosity is reduced, the ratio between the two time-scales becomes smaller, and $r_m$ does not move so far away from $r_c$ before $\dot{r}_m \sim \dot{r}_c$. Thus for the disc with the lowest viscosity (bottom), $r_m$ and $r_c$ begin to move outwards after about $10^3 \, yr$, and the trapped disc (where $\dot{m}$ is regulated by $\dot{r}_c$) has a much larger accretion rate on to the star (7 orders of magnitude larger after $10^3 \, yr$) than for higher viscosities.

With our chosen initial conditions the magnetic torque pushes the inner edge out across corotation. This causes mass to pile up outside $r_m$. The higher the viscosity, the faster this pile-up is reduced by spreading outwards, and the faster the inner edge can continue to move outwards in response. The experimental result is then that disc-trapping behaviour is avoided when the transition from accreting to non-accreting disc takes place fast enough. We will return to this topic in Section 7.

The pile-up is also influenced by the way in which accretion on the star changes as $r_m$ crosses $r_c$, hence we expect that the parameter controlling this, $\Delta r_2$, will have a strong effect as well.

In Fig. 5 we show three discs in which the initial ratio $T_{SD}/T_{visc}$ is kept fixed, but the value of $\Delta r_2$ changes, from top to bottom, $\Delta r_2/r_m = [0.4, 0.04, 0.004]$. The larger $\Delta r_2$, the farther away $r_m$ must move from $r_c$ in order for the accretion rate to decrease sufficiently to form the trapped disc. In the top panel (when $\Delta r_2$ is largest), the disc must move out a considerable distance before becoming trapped, sufficiently far to significantly decrease the efficiency of the spin-down torque (and hence increase the spin-down time-scale of the star). As well as decreasing the spin-down efficiency, if $\Delta r_2$ is large enough, the inner radius could move far enough away from $r_c$ that material could begin to be launched from the disc in an outflow, and the disc would become untrapped.

As the results reported above show, the evolution of a disc with $\dot{m} = 0$ can end either in a dead disc state in which the star has lost only a fraction of its angular momentum, or in a trapped state in which corotation is maintained and the star can spin-down much farther. Which outcome depends on the details of the interaction between the star and the disc, parametrized in our model by the transition widths $\Delta r$ and $\Delta r_2$. It also depends on the rate at which the disc can respond viscously compared to the spin change rate of

![Figure 4](https://academic.oup.com/mnras/article-abstract/416/2/893/1055394/1055394)

The evolution from an accreting to a non-accreting disc, for increasing (top to bottom) ratios of $T_{visc}/T_{SD}$ (with $\Delta r/r_m = 0.1$, $\Delta r_2/r_m = 0.04$). From top to bottom, the ratio $T_{visc}/T_{SD}$ is $2.5 \times [10^{-9}, 10^{-7}, 10^{-5}, 10^{-4}]$. As the ratio between the two time-scales decreases, the disc is not able to move outwards as quickly before the star begins to spin-down, so that $r_m$ will always remain close to $r_c$. Black solid curve: evolution of $r_m$. Red dashed curve: evolution of $r_c$. 

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the star. A fast response of the disc makes the transition through corotation faster than \(r_c\) changes, and the disc is more likely to enter the dead state. This makes it far more likely to occur in young stellar systems than in neutron star binaries.

As was shown in Section 3, the ratio \(T_{\text{visc}}/T_{\text{SD}}\) itself can vary over many orders of magnitude in different systems, from \(10^{-17}\) in neutron stars with weak magnetic fields to \(10^{-2}\) in discs around massive young stars. The size of this ratio will also determine whether a trapped disc can form over the lifetime of the star. This analysis would suggest that, assuming \(\Delta r_f/r_m\) does not vary much from system to system, trapped discs are much more likely to form in protostellar discs than in strongly ionized discs around neutron stars.

The situation changes somewhat when \(\dot{m}_{\text{visc}} \neq 0\). A large \(T_{\text{visc}}/T_{\text{SD}}\) means that the accretion rate through \(r_m\) will be higher to trap the disc near \(r_c\). If this accretion rate is initially smaller than \(\dot{m}_{\text{visc}}\), then the evolution will proceed according to the accretion rate prescribed by \(T_{\text{visc}}/T_{\text{SD}}\) until the disc moves far enough out that \(\dot{m}_{\text{visc}} = \dot{m}_{\text{visc}}\). After this the evolution towards spin equilibrium will be determined by \(\dot{m}_{\text{visc}}\). In general, the star will spin-down faster than it would for a smaller \(T_{\text{visc}}/T_{\text{SD}}\). However, the final state will still be spin equilibrium where the net torque on the star is zero.

In the results presented above, the initial conditions were taken from steady solutions of the viscous thin disc equation. These included dead discs in which a steady state was made possible by a sink of angular momentum at the outer boundary of the numerical grid, which takes up the angular momentum added by the magnetic torques at the inner edge.

### 5.2 Dead discs evolving into trapped discs

In this example we investigate the opposite case of Section 5.1: discs with \(r_m\) initially outside corotation, and conditions chosen such that the disc begins by spreading inwards. The mass flux at the outer boundary is set to zero. Varying the initial outer radius, \(r_{\text{out},0}\), varies the amount of mass in it, and the time-scale of its long-term evolution can change.

![Figure 5](link)

**Figure 5.** The evolution from an accreting to a non-accreting disc, for stable discs (\(\Delta r/r_m = 1\)) with different \(\Delta r_2\). From top to bottom, \(\Delta r_2/r_m = [0.4, 0.04, 0.004]\). Curves show \(r_\text{in}\) and \(r_c\) as in Fig. 4.

![Figure 6](link)

**Figure 6.** Discs starting outside corotation and spreading inwards. Evolution of \(r_f/r_c\) for different initial \(r_{\text{out}}\). From top to bottom: \(r_{\text{out},0} = 10r_{\text{in},0}\) (dotted curve), \(10^2\) (dashed curve), \(10^3\) (dash–dotted curve), and \(10^4\) (triple dash–dotted curve) \(r_m\) at \(t = 0\). Larger discs have a larger reservoir of matter, so that they can sustain a larger \(\dot{m}\) (so smaller \(r_m\)) as \(r_c\) increases due to the spin-down of the star.

We adopt the representative model parameters used before, with an initial inner radius set to \(1.3r_c\) (corresponding to a negligible accretion rate for our chosen \(\Delta r_2/r_m\)), and set \(r_{\text{out},0} = [10, 100, 10^3, 10^4]r_{\text{in},0}\). The evolution of \(r_{\text{in}}\) and \(r_c\) is plotted in Figs 6 and 7. Fig. 6 compares the evolution of \(r_{\text{in}}/r_c\) for different sizes of disc, while Fig. 7 compares the evolution of \(r_{\text{in}}\) and \(r_c\) for different disc sizes to the simulation where \(r_{\text{out},0} = 10r_{\text{in},0}\). In both simulations, the different curves correspond to different initial \(r_{\text{out}}\): \(10r_{\text{in}}\) (dotted curve), \(100r_{\text{in}}\) (dashed curve), \(1000r_{\text{in}}\) (dash–dotted curve) and \(10^4r_{\text{in},0}\) (dash–triple–dotted curve).

Fig. 6 shows the evolution of \(r_{\text{in}}/r_c\) in time for different initial \(r_{\text{out}}\). For the simulations with \(r_{\text{out},0}/r_{\text{in},0} = [10, 100, 1000]\), the ratio \(r_{\text{in}}/r_c\) declines to a minimum value that decreases as \(r_{\text{out}}\) is taken larger, before again increasing approximately logarithmically. In the largest disc, \(r_{\text{out},0}/r_{\text{in},0} = 10^4\), the evolution is the same as in the smaller discs at early times, but \(r_{\text{in}}/r_c\) continues to decline for much
longer until it reaches a minimum at around $10^7t_\star$ when it finally turns over.

The minimum value of $r_{\text{in}}/r_c$ is thus determined by the amount of mass in the disc available for accretion. The larger discs have more mass, which sustains the accretion rate on to the star for a longer time before a drop in surface density causes $r_{\text{in}}$ to move outwards again.

The bottom panel of Fig. 7 shows the evolution of $r_{\text{in}}$ and $r_c$ for $r_{\text{out},0} = 10r_{\text{in},0}$. The evolution is qualitatively the same as we derived in the analytic approximation in Section 4.3. Initially $r_{\text{in}}$ remains fixed as $r_c$ starts to evolve outwards, until $r_c$ moves close enough to $r_{\text{in}}$ that accretion can begin. The inner radius then evolves outwards at approximately the same rate as $r_c$ as the star spins down. The variation between different simulations is emphasized in the top panel of Fig. 7. Here we plot the evolution of $r_{\text{in}}$ (thick curves, red) and $r_{\text{in}}$ (thin curves, black) for the discs with $r_{\text{out},0}/r_{\text{in},0} = [10^2, 10^3, 10^4]$, divided by the solution for $r_{\text{out},0}/r_{\text{in},0} = 10$. For larger discs the accretion rate is higher, so that $r_{\text{in}}$ can move closer to $r_c$. In the three smaller discs, the accreted mass adds a negligible amount of angular momentum to the star, so that the value of $r_{\text{in}}$ moves closer to that of $r_c$. As a result, the spin-down torque simply becomes more efficient, and $r_c$ spins down faster. After $10^5t_\star$, $r_c$ in the disc with $r_{\text{out},0}/r_{\text{in},0} = 10^2$ is more than 10 per cent larger than that in the smallest disc. However, for the largest disc, the accretion rate puts $r_{\text{in}}$ close enough to $r_c$ that the efficiency of the spin-down torque starts to drop and the spin-up from accretion becomes non-negligible. Although $r_c$ still increases, after $10^7t_\star$, the $r_c$ is 30 per cent smaller than that for a small disc.

These results show that the size of the disc can considerably influence the efficiency of spin-down, emphasizing the fact that the spin-down of a star is an initial-value problem. The initial size of the disc can be as important as the ratio $T_{\text{SP}}/T_{\text{visc}}$ and the parameter $\Delta r_2$ in determining whether the disc can become trapped, and the efficiency of the spin-down torque. The results of this section would suggest that larger discs (with their larger reservoirs of mass) are more likely to become trapped than smaller discs.

5.3 Long-term behaviour of discs of finite size

As we have already seen in this section, the long-term evolution of the disc+star system tends to bifurcate. The end state is either a star that has lost little angular momentum, surrounded by a dead disc, or a star continuously spun down by a disc trapped at corotation with the star.

We investigate this further by a set of simulations in which the initial state is a disc of finite size $r_0$, evolving in a grid that is 10 times larger than $r_{\text{out}}$. Apart from this change we use the standard parameters (Section 3.1), setting the initial value of $r_0/r_c$ at 1.3 (for comparison with the results of the previous section). Whether the disc will be able to substantially spin-down the star depends on the ratio of the time-scales $T_{\text{visc}}$ at $r_{\text{out}}$ to $T_{\text{SP}}$. If the disc is very small ($T_{\text{visc}} \ll T_{\text{SP}}$), $r_{\text{in}}$ will move outwards too quickly to spin-down the star, while if $T_{\text{visc}} \gg T_{\text{SP}}$ the moment of inertia in the disc is sufficiently high to be able to absorb much more angular momentum from the star, so that the disc can operate as an efficient sink for the star’s angular momentum.

Fig. 8 shows the results for three different sizes of disc. In each curve, the dashed red line shows the evolution of $r_c$, while the solid black curve shows the evolution of $r_{\text{in}}$. The top panel of Fig. 8 shows the disc evolution when the viscosity is chosen such that $T_{\text{visc}} \ll T_{\text{SP}}$, with $r_0 = 100 r_{\text{in}}$. Since the angular momentum injected at $r_{\text{in}}$ is moved away by viscous spreading of the disc, the disc quickly evolves away from its initial configuration. Since the moment of inertia in the disc is much smaller than in the star, the disc becomes too spread out to absorb the angular momentum injected at $r_{\text{in}}$, and $r_{\text{in}}$ moves outwards before the star is able to slow down substantially.

In the middle panel of Fig. 8 we show the evolution of $r_{\text{in}}$ and $r_c$ when the spin-down time-scale is comparable to the viscous time-scale at $r_0$. The disc also initially diffuses outwards (as seen by the increase in $r_{\text{in}}$ around $t = 5 \times 10^3$), so that the rate of angular momentum exchange from the star to $r_{\text{in}}$ decreases. This allows $r_c$ to catch up, and the two radii start to evolve together. Eventually, however, viscous spreading of the disc wins, and $r_{\text{in}}$ begins to move outwards.

The bottom panel of Fig. 8 shows the evolution of $r_{\text{in}}$ and $r_c$ when $T_{\text{visc}}(r_0) \gg T_{\text{SP}}$. The result is essentially the same as in Section 5.2, since the disc is now so large that additional angular momentum from the star does not reach $r_{\text{out}}$ on the spin-down time-scale. In other words, the moment of inertia of the disc itself is large enough...
that the star is able to spin down without causing $r_m$ to move rapidly outwards.

5.3.1 The rotation of Ap stars and magnetic white dwarfs

An intriguing clue to the spin-down of magnetic stars comes from slowly rotating Ap stars. As a class these stars are observed to have very strong dipolar magnetic fields (up to 10 kG). A few of them have extremely long rotation periods (up to 10–100 yr), while others have rotation periods as short as 0.5 d. A similar phenomenon is observed in isolated magnetic white dwarfs. Most of these have periods of a few days or weeks, but some rotate as slowly as the slowest Ap stars (e.g. Wickramasinghe & Ferrario 2000). We suggest that the bifurcation of outcomes we found in the above is the underlying reason for the remarkable range of spin periods of magnetic stars.

6 CONCLUSIONS

As found before in Sunyaev & Shakura (1977) and DS10, a disc in contact with the magnetosphere of a star can be in a ‘dead’ state, with its inner edge well outside the corotation radius so that the accretion rate on to the star vanishes, and the torque exerted by the magnetic field is transmitted outwards by the viscous stress. In the calculations reported here, an additional state was found, intermediate between the accreting and dead state. In this state, the inner edge of the disc stayed close to the corotation radius $r_c$, even as the accretion rate on to the star varied by large factors. We call this phenomenon trapping of the disc. Accretion can be stationary or in the form of a limit cycle in this state.

One of the goals of this investigation was to find out under what conditions this trapping takes place, using a series of numerical experiments with varying initial and boundary conditions. If initial conditions are such that the inner edge starts inside corotation and slowly moves outwards, we find that the disc gets stuck in a trapped state for a long time if the disc viscosity $\nu$ is low (up to about $10^3$ times shorter than the spin-down time-scale of the star). The accretion rate then slowly vanishes but the inner edge always stays close to corotation. At a higher viscosity ($>10^4 \times$ the spin-down time-scale) on the other hand, the disc evolves through corotation into a ‘dead’ disc state, with the inner edge well outside the corotation radius. In terms of the standard viscosity parametrization $\nu = \alpha(H\rho)^{\kappa}$, compared with the spin-down time-scale of the disc, a trapped state is more likely to happen in strongly magnetic ($B_8 \sim 10^{12}$ G) X-ray pulsars and protostellar discs than the weaker magnetic fields ($B_8 \sim 10^8$ G) of millisecond X-ray pulsars.

Our second goal was to find out how a star spins down in the long term, under the influence of the angular momentum it loses by the magnetic torque exerted on the disc. The results show an interesting ‘bifurcation’ of long-term outcomes: if the disc evolves into a dead state, the star loses only a fraction of its initial angular momentum, and can remain spinning rapidly throughout its life. If on the other hand it enters a trapped state at some point, it remains in this state. The star can then slow down to very low rotation rates, the inner edge of the disc tracking the corotation radius outwards.

We suggest that these two outcomes can be identified respectively with the rapidly rotating and slowly rotating classes of magnetic Ap stars and magnetic white dwarfs. The evolution of the trapped state could also be reproduced with a simplified model that does not require solving the full viscous diffusion equation.

This picture of magnetosphere–disc interaction differs from the standard view that mass will be ‘propelled’ out of the system instead of accreting, once the star rotates more rapidly than the inner edge of the disc. As shown already by Sunyaev & Shakura (1977) and again argued in Spruit & Taam (1993) and DS10, this assumption is not necessary, in many cases unlikely, and ignores some of the theoretically and observationally most interesting aspects of the disc–magnetosphere interaction.

Though the dead state is a regime where a significant fraction of the disc mass could in principle be expelled ($r_m \gg r_c$), the results presented here show that magnetospherically accreting systems often avoid this regime. Instead, they end up in the trapped state, in which the disc–field interaction keeps the inner radius truncated very close to the corotation radius, even at very low accretion rates. Both the trapped and ($r_m \approx r_c$) and the dead state ($r_m \gg r_c$) allow the disc to efficiently spin down the star. The disc retains a large amount of mass, but in the absence of accretion on to the central star appears quiescent.

7 DISCUSSION

By assuming a given dependence of viscosity on distance, we have bypassed the physics that determines it. In terms of the standard $\alpha$-parametrization of viscosity, we have left out the physics that determines the disc temperature and hence its thickness $Hr$. Additional time dependence or instabilities may arise from feedback between accretion and disc temperature. The radiation produced by matter accreting on the star could be large enough to change temperature, the ionization state and hence the thickness at the inner edge of the disc. In a trapped disc, this is just the region that controls the accretion rate on to the star. The size of the transition region, which we have parametrized with the widths $\Delta r$, $\Delta r_c$, may well depend on disc thickness. Positive feedback may be thus possible. We leave this possibility for future work.

In systems such as the accreting X-ray pulsars the accretion is episodic on long time-scales. This is attributed to the instability of viscous discs that is also responsible for the outbursts of cataclysmic variables. In these cases, in which there is a large change in $m$ and $r_m$ is far from $r_c$ in the quiescent phase, there is the possibility of hysteresis: the same accretion rate will lead to a different value of $r_m$ (and therefore disc torque) depending on whether the source is moving into or out of outburst. As the source goes into outburst, the disc will not have as much mass in its inner regions, so that $r_m$ will move inwards gradually from large radii until it crosses $r_c$ and the outburst begins. In the decline phase, however, the disc will become trapped around the inner radius of the disc when the accretion rate drops, allowing for a larger spin-down in the disc and accretion bursts via the instability of DS10. The net effect of such an episodic accretion on the spin history of the star, as compared with the case of steady accretion, is not obvious. We discuss this in more depth in the companion paper.

Some work on disc–magnetosphere interaction assumes magnetic torques to act over a significant part of the disc (Königl 1991; Armitage & Clarke 1996). More recent work (and in particular numerical MHD simulations of the disc–field interaction) finds the interaction region to be much narrower (as we have also assumed here), and spin-down torques on the star are correspondingly smaller. The difference for the long-term spin evolution is not dramatic, however, as our trapped disc results demonstrate.

Interestingly, Armitage & Clarke (1996) also observed that their discs would become trapped around $r_c$ as the accretion rate in the disc decreased by several orders of magnitude. In their model, the magnetic field–disc interaction also acts like a boundary condition.
at low accretion rates and the disc evolved viscously in response. This is presumably because of the reason that although the disc in their model is threaded by a magnetic field everywhere, the disc–field interaction is by far the strongest in the inner regions, so that they see a similar behaviour to the one described in this paper.

MHD simulations of interaction between the disc and the magnetic field are becoming increasingly realistic. These simulations can only run for very short time-scales (the longest of order $T_{\text{visc}}$ at $r_m$), so they tend to emphasize initial transients. Still, they offer an insight into how the disc and the magnetic field will interact. To date, most simulations have concentrated on strongly accreting (Hayashi et al. 1996; Goodson et al. 1997; Miller & Stone 1997) or propelling (Romanova et al. 2004) cases. Simulations that come closest to the conditions of a trapped disc are the study of a so-called ‘weak propeller’ regime by Ustyugova et al. (2006). The authors found that discs in which $r_m$ was initially truncated close to $r_c$ launch much weaker outflows than discs truncated further away. They also found some evidence of the field changing the disc structure (as shown from mass piling up in the inner regions), although in their simulation the majority of the angular momentum in the star was dispersed via a wind, rather than through the disc. These simulations also did not run for very long, however, so it is hard to separate transient behaviour due to the initial conditions from the longer-term systematic effects we are interested in.

Perna, Bozzo & Stella (2006) did calculations for a case where the star’s field is inclined with respect to the disc axis. The authors’ results suggest that the transition width which we have parametrized by $\Delta r_2$ would increase with the inclination of the magnetic field, since there will be some values for $r_m$ at which both accretion and disc mass trapping can occur. This then would suggest that dead discs and unstable discs are more likely to form in systems with small inclinations between the spin axis and the magnetic axis. This prediction is supported by the observation that Ap stars with the longest periods tend to have the lowest inclination angles for the magnetic field (Landstreet & Mathys 2000).

The transition widths of disc–magnetosphere interaction that can be inferred from these simulations are significant, and are in the range we have assumed here. They are much larger than the very narrow interaction regions assumed by Matt et al. (Matt & Pudritz 2004, 2005; Matt et al. 2010), mainly because in the interaction region matter can deviate significantly from Keplerian orbits and the thin-disc solution is no longer relevant.

We have found that the distance from $r_c$ at which the inner edge of the disc gets trapped is determined by the ratio of two time-scales for the disc’s evolution: the viscous evolution time-scale in the disc (which determines the rate at which disc density profile can change) and the star’s spin-down time-scale (which sets the rate at which $r_c$ moves outwards). The viscous time-scale is in general much shorter than the spin-down time-scale, and the ratio of the two time-scales varies from $10^{-3}$ (in protostellar discs) to $10^{-17}$ (in millisecond X-ray pulsars). The larger the ratio, the longer the disc will take to respond to changes in $r_c$. As a result $r_m$ remains closer to $r_c$ than it would if the ratio were smaller, and the trapped disc has a higher accretion rate on to the star. If the ratio is too low, then $r_m$ moves much further away from $r_c$, and the system likely enters the dead disc regime.

For the parameters characterizing the interaction region between the disc and magnetosphere in our model, we have used here values such that the cyclic accretion behaviour found in DS10 does not develop. This was done for convenience, since the short time-steps needed to follow these cycles makes it harder to calculate the long-term evolution. The time-averaged effect of these cycles is not expected to make a big difference for the long-term evolution.

These cyclic accretion bursts can persist in the trapped state, when the star is spinning down efficiently. They are observed to occur both over several orders of magnitude of accretion rates and transiently (over a small range of accretion rate), and depend on the ratio of time-scales discussed above. Whether or not the instability occurs is determined by the detailed disc–field interaction (the parameters $\Delta r$ and $\Delta r_2$ in our model). The peak of the accretion bursts is typically much larger ($>10 \times$) than the mean accretion rate for the system, and the period is typically between $0.01T_{\text{visc}}(r_m)$ and $10^2T_{\text{visc}}(r_m)$. The properties and conditions for the occurrence of these cycles are studied further in a companion paper.

7.1 ‘Propelling’

In our calculations we have left out the possibility that interaction of the magnetosphere with the disc will cause mass ejection from the system. The point is that, contrary to common belief, such an interaction can function without mass ejection by ‘propelling’, as pointed out already by Sunyaev & Shakura (1977). An understanding of this restricted case, as we have developed here, is a prerequisite for understanding the much less well defined case of mass-shedding discs.

On energetic grounds, mass-loss from the system is necessarily limited, unless the inner edge is well outside corotation (Spruit & Taam 1993). This point has also been made by Perna et al. (2006), who propose that mass lifted at $r_m$ may fall back on to the disc at some finite distance. This would create a feedback loop in the mass flux through the disc, opening the possibility of additional forms of time-dependent behaviour that do not exist in accretion on to non-magnetic stars. In the trapped disc state we have studied here whether the difference in rotation between disc and star is small, so any significant amount of mass kicked up from the interaction region cannot move very far before returning to the disc. Its effects are then secondary, at least for the long-term evolution of the disc.

On the other hand, a significant amount of mass-loss from a disc wind would enhance the trapped disc effect, since if most of the mass flux through the disc is dispersed in a wind rather than accreted on to the star, a trapped disc could continue spinning down the star for much longer without moving towards spin equilibrium.

The possibility of significant effects of mass-loss is more realistic for the dead disc states, where the distance of the inner edge from corotation can become much larger. Real propelling is expected to happen when mass transfer from a companion star sets in for the first time on to a rapidly spinning magnetic star. The cataclysmic binary AE Aqr is evidently such a case (Pearson, Horne & Skidmore 2003). A disc is absent in this CV, and all mass transferred appears to be ejected in a complex outflow associated with a strong radio emission.

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