THE LAN PROPERTY FOR MCKEAN-VLASOV MODELS IN A MEAN-FIELD REGIME

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ABSTRACT. We establish the local asymptotic normality (LAN) property for estimating a multidimensional parameter in the drift of a system of $N$ interacting particles observed over a fixed time horizon in a mean-field regime $N \to \infty$. By implementing the classical theory of Ibragimov and Hasminski, we obtain in particular sharp results for the maximum likelihood estimator that go beyond its simple asymptotic normality thanks to Hájek’s convolution theorem and strong controls of the likelihood process that yield asymptotic minimax optimality (up to constants). Our structural results shed some light to the accompanying nonlinear McKean-Vlasov experiment, and enable us to derive simple and explicit criteria to obtain identifiability and non-degeneracy of the Fisher information matrix. These conditions are also of interest for other recent studies on the topic of parametric inference for interacting diffusions.

Mathematics Subject Classification (2010): 62C20, 62F12, 62F99, 62M99.

Keywords: Parametric estimation; LAN property; maximum likelihood estimation; statistics and PDE; interacting particle systems; McKean-Vlasov models.

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Date: May 13, 2022.
1. INTRODUCTION

1.1. Motivation. Collective dynamics models are becoming increasingly popular in modelling complex stochastic systems, with a versatility of applications, ranging from mathematical biology (neurosciences, Baladron et al. [2], structured models in population dynamics, Mogilner et al. [40], Burger et al. [8]) to social sciences (opinion dynamics, Chazelle et al. [13], cooperative behaviours, Canuto et al. [9]) and finance (systemic risk, Fouque and Sun [17]), or more recently, mean-field games (Cardaliaguet et al. [10], Cardaliaguet and Lehalle [11]). Whereas stochastic systems of interacting particles and associated nonlinear Markov processes in the sense of McKean [38] date back to the 1960’s and have been studied extensively over more than half a century, see e.g. [7, 44, 45, 39, 47] among a myriad of references, the development of statistical inference in this setting is only emerging, (with some notable exceptions like Löcherbach [35] in large time or e.g. Kasonga [27] or Bishwal [4]) in a mean-field limit. Recently, Giesecke et al. [22] and Sharrock, Kantas, Parpas and Grigorios [43] revisit the work of Kasonga and consider a parametric framework where convergent and asymptotically normal contrast estimators are constructed. Several other parametric frameworks (that consider various observation schemes and asymptotic frameworks) have also been recently considered, like [14, 34, 49] or Genon-Catalot and Laredo [20, 21]. There also exist recent results in nonparametric inference: we mention our work [16] and Belometsny et al. [3], together with studies in identification like [31, 32, 33] or learning [30, 36, 37].

The present paper, close in spirit to [4, 22, 27] and [13] (in their so-called offline case) considers a parametric framework in a mean-field regime over a fixed time horizon. We take a deeper look at the asymptotic structure of the associated statistical experiment, in the sense of local asymptotic normality or LAN, in order to derive strong results for the maximum likelihood, both in asymptotic distribution and in an asymptotic minimax sense (up to constants) for various loss functions. For simplicity, we keep-up with continuous observations, but we briefly explain how to move to a discrete data setting. Also, we look for simple and explicit criteria that enable us to verify identifiability and non-degeneracy of the model. This is a non-trivial issue in the context of nonlinear McKean-Vlasov models that is usually a bit overlooked in the literature.

1.2. Setting. We have a parameter of interest \( \vartheta \) lying in a compact set \( \Theta \subset \mathbb{R}^p \) (with non empty interior), for some fixed \( p \geq 1 \). For some fixed time horizon \( T > 0 \), we continuously observe a stochastic system of \( N \) interacting particles

\[
X^{(N)} = (X^1_t, \ldots, X^N_t)_{t \in [0,T]},
\]
evolving in an Euclidean ambient space $\mathbb{R}^d$, that solves

\begin{equation}
\begin{cases}
  dX^i_t = b(\vartheta; t, X^i_t, N_i(t))dt + \sigma(t, X^i_t)dB^i_t, & 1 \leq i \leq N, \ t \in [0, T], \\
  \mathcal{L}(X^1_0, \ldots, X^N_0) = \mu_0^\otimes N,
\end{cases}
\end{equation}

where $\mu_t^N = N^{-1} \sum_{i=1}^N \delta_{X^i_t}$ is the empirical measure of the system. The $(B^i_t)_{t \in [0, T]}$ are independent $\mathbb{R}^d$-valued Brownian motions. The initial condition $\mu_0$, the drift $b$ and the diffusion coefficient $\sigma$ are at least sufficiently regular so that

$$
\mu^N(t) = (\mu^N_t)^{t \in [0, T]} \to \mu = (\mu_t)^{t \in [0, T]}
$$

weakly as $N \to \infty$, where $\mu$ is a family of probability measures that solves (in a weak sense) the parabolic nonlinear equation

\begin{equation}
\begin{cases}
  \partial_t \mu + \text{div}(b(\vartheta; \cdot, \mu)) = \frac{1}{2} \sum_{k,k'} \partial^2 \sigma_{kk'}(c_{kk'} \mu), & t \in [0, T], \\
  \mu_{t=0} = \mu_0,
\end{cases}
\end{equation}

with $c = \sigma \sigma^T$. We will write $\mu^\vartheta = (\mu^\vartheta_t)^{t \in [0, T]}$ to emphasise the dependence in $\vartheta$. In this context, we are interested in estimating from data (1) the parameter $\vartheta \in \Theta$ of the function $(\vartheta; t, x, \nu) \mapsto b(\vartheta; t, x, \nu) \in \mathbb{R}^d$. Asymptotics are taken as $N \to \infty$.

A particular case of interest that covers many examples is when the dependence in the measure variable for $b$ is linear: we then have

\begin{equation}
  b(\vartheta; t, X^i_t, N_i(t)) = \int_{\mathbb{R}^d} \tilde{b}(\vartheta; X^i_t, y) \mu^N_t(dy) = N^{-1} \sum_{j=1}^N \tilde{b}(\vartheta; X^i_t, X^j_t),
\end{equation}

for some function $\tilde{b} : \Theta \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$. A typical form is $\tilde{b}(\vartheta; t, x, y) = G_\vartheta(x) + F_\vartheta(x - y)$ where $G_\vartheta, F_\vartheta : \mathbb{R}^d \to \mathbb{R}^d$ play the role of a common external force to the system and an interaction force respectively.

1.3. Results and organisation of the paper. In Section 2 we rigorously construct the (sequence of) statistical experiment(s) generated by the observation (1) under the dynamics (2) that we denote $(E^N)^{N \geq 1}$. It is well defined and regular in the classical sense of Ibragimov and Hasminskii [24] under strong integrability of the initial condition $\mu_0$ and standard smoothness assumptions on the drift $\vartheta \mapsto b(\vartheta; \cdot)$ and the diffusion matrix $c = \sigma \sigma^T$, see Assumptions 1, 2, 3 and 4 and Proposition 6. The deep study of the identifiability of $E^N$ and the non-degeneracy of its information matrix $I_{E^N, N}(\vartheta)$ is simplified via the accompanying experiment $\tilde{G}^N$, where $\tilde{G}$ is generated by the continuous observation of a solution to the McKean-Vlasov equation

\begin{equation}
\begin{cases}
  dX^i_t = b(\vartheta; t, X^i_t, \mu^\vartheta_t)dt + \sigma(t, X^i_t)dB_t, & t \in [0, T], \\
  \mathcal{L}(X_0) = \mu_0,
\end{cases}
\end{equation}

for a standard Brownian motion $(B_t)_{t \in [0, T]}$ on $\mathbb{R}^d$ and where $\mu^\vartheta_t$ is the marginal distribution of the solution at time $t$. In particular, in the case of representation 4 we have that $E^N$ and $\tilde{G}^N$ do not separate asymptotically by a simple entropy argument, see Proposition 10 and we always have the convergence of the corresponding Fisher information matrices:

$$
N^{-1}I_{E^N}(\vartheta) \to I_{\tilde{G}}(\vartheta)
$$

in a mean-field limit $N \to \infty$, as established in Proposition 11. This approximation is the gateway to obtain explicit identifiability and non-degeneracy criteria, as detailed in Section 2.4. In particular, under additional regularity assumptions, we obtain a quite simple criterion for $I_{\tilde{G}}(\vartheta)$ to be
non-degenerate in Proposition [15] namely the property that one of the functions
\[ x \mapsto \nabla \varphi(c^{-1/2} b)(\vartheta; 0, x, \mu_0)^\top z, \quad j = 1, \ldots, d \]
is not identically vanishing, for every \( z \in \mathbb{R}^p \) with \(|z| = 1\), with \( c^{-1/2} \) a square root of \( c = \sigma \sigma^T \). We use the notation \( f = (f^j)_{1 \leq j \leq d} \) componentwise, the \( f^j \) being real-valued functions. In particular, [5] has the advantage to only relate to the initial condition \( \mu_0 \) in the measure argument and not the whole \((\mu^o)_t \in [0, T] \) which is (almost) never explicit. Having a simple criterion to achieve the non-degeneracy of the Fisher information seems to have been a bit overlooked in the literature (where it is usually simply assumed to hold true) and our result is thus of interest for other studies.

In Section 3 we state the main results of the paper, Theorem 17 where we establish the LAN property: if we reparametrise the experiments via \( \vartheta = \vartheta_0 + N^{-1/2} u \) locally around a fixed point \( \vartheta_0 \), with \( u \in \mathbb{R}^p \) being now the unknown parameter, then both \( \mathcal{E}^N \) and \( G^{\otimes N} \) look like a Gaussian shift: we observe
\[ Y^N = u + I_g(\vartheta_0)^{-1/2} \xi, \]
where \( \xi \) is a standard Gaussian random vector in \( \mathbb{R}^p \). This has important consequences in terms of existence and properties of optimal procedures: we have Hájek’s convolution theorem (Corollary 18), namely for any estimator \( \tilde{\vartheta}_N \),
\[ \liminf_{N \to \infty} \sup_{|\vartheta' - \vartheta| \leq \delta} \mathbb{E}_{\vartheta_0}[w(N^{1/2} I_g(\vartheta)^{1/2}(\tilde{\vartheta}_N - \vartheta'))] \geq \mathbb{E}[w(\xi)], \]
for small enough \( \delta > 0 \), where \( \mathbb{P}^N_{\vartheta'} \) is the distribution of the data when the parameter is \( \vartheta' \) and \( w \) is an arbitrary loss function satisfying some regularity properties. The bound (6) is achieved by the maximum likelihood estimator \( \hat{\vartheta}^{\text{MLE}}_N \) obtained by maximising the contrast
\[ \vartheta \mapsto \ell^N(\vartheta; X^{(N)}) = \sum_{i=1}^N \int_0^T ((c^{-1/2} b)(\vartheta; t, X^i_t, \mu^N_t)^\top dX^i_t - \frac{1}{2}(c^{-1/2} b)(\vartheta; t, X^i_t, \mu^N_t)^2 dt), \]
This implies in particular the convergence
\[ \sqrt{N}(\hat{\vartheta}^{\text{MLE}}_N - \vartheta) \to N(0, I_g(\vartheta)^{-1}) \]
in distribution. Moreover, we have in Theorem 19 the minimax asymptotic optimality of \( \hat{\vartheta}^{\text{MLE}}_N \), in the sense that
\[ \mathcal{R}^N_{\text{w}}(\hat{\vartheta}^{\text{MLE}}_N; \Theta) = \inf_{\tilde{\vartheta}_N} \mathcal{R}^N_{\text{w}}(\tilde{\vartheta}_N; \Theta)(1 + o(1)) \]
where \( \mathcal{R}^N_{\text{w}}(\tilde{\vartheta}_N; \Theta) = \sup_{\vartheta \in \Theta} \mathbb{E}_{\vartheta^N}[w(N^{1/2} I_g(\vartheta)^{1/2}(\tilde{\vartheta}_N - \vartheta))] \) is the classical minimax risk. Thus the LAN property enables us to obtain considerably stronger results than simply (8). In Section 4 we investigate several non-trivial examples that generalise the results of [27], and where our identifiability and non-degeneracy criteria easily apply. We treat in particular the case of a kinetic mean-field double layer potential that may serve as a representative model for swarming models, see in particular [6] and the references therein. The proofs are delayed until Sections 5 and 6 with an appendix (Section 7) that contains useful technical results.

In practice, maximising the function (7) is not feasible, since only discrete data are available. It is then reasonable to replace the ideal observation (1) by the more realistic
\[ X^{(N,m)} = (X^1_t, \ldots, X^N_t)_{t \in \{t^1_m, \ldots, t^m_m\}}, \]
where \((0 = t_0^m < t_1^m < \ldots < t_m^m = T)\) is a subdivision of \([0, T]\) with mesh
\[
\max_{1 \leq j \leq m} (t_j^m - t_{j-1}^m) \leq Cm^{-1}.
\]
We thus have \((m + 1) \times N\) data with values in \(\mathbb{R}^d\). We may then replace \(\vartheta\) by
\[
\vartheta \mapsto N^{-1} \sum_{i=1}^N \sum_{j=0}^m \left((c^{-1/2}b)(\vartheta; t_j^m, X_{t_j-1}^{i, m}, \mu_{t_j-1}^{(N)})^\top (X_{t_j}^{i, m} - X_{t_j-1}^{i, m})
- \frac{1}{2}(c^{-1/2}b)(\vartheta; t_j^m, X_{t_j-1}^{i, m}, \mu_{t_j-1}^{(N)})^2 (t_j^m - t_{j-1}^m)\right).
\]
Assuming the function \((t, x) \mapsto (c^{-1/2}b)(\vartheta; t, x, \mu_t^{(N)})\) to be smooth, we may safely expect the discrete approximation to be close to its continuous counterpart up to an additional error of order \(m^{-1/2}\), by standard high-frequency discretisation techniques, see the textbooks of Jacod and co-authors [11, 25, 26]. In particular, if \(m \gg N\), the same results as for continuous observations are likely to hold true.

2. Construction and properties of the statistical model

2.1. Notation. The dimension \(d \geq 1\) of the state space \(\mathbb{R}^d\) and the dimension \(p \geq 1\) of the parameter space \(\Theta\) as well as the time horizon \(T > 0\) are fixed once for all. We write \(\| \cdot \|\) for the Euclidean distance on \(\mathbb{R}^q\) (\(q = p, d\) or any other integer, depending on the context) or for a matrix norm on \(\mathbb{R}^p \otimes \mathbb{R}^p\) fixed throughout.

We consider functions that are mappings defined on products of metric spaces (typically \(\Theta \times [0, T] \times \mathbb{R}^d \times \mathcal{P}_1\) or subsets of these) with values in \(\mathbb{R}\) or \(\mathbb{R}^d\). Here, \(\mathcal{P}_1\) denotes the set of probability measures on \(\mathbb{R}^d\) with a first moment, endowed with the Wasserstein 1-metric
\[
W_1(\mu, \nu) = \inf_{\phi \in \Gamma(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y| \mu(dx, dy) = \sup_{|\phi|_{Lip} \leq 1} \int_{\mathbb{R}^d} \phi d(\mu - \nu),
\]
where \(\Gamma(\mu, \nu)\) denotes the set of probability measures on the product space \(\mathbb{R}^d \times \mathbb{R}^d\) with marginals \(\mu\) and \(\nu\). For a probability measure \(\mu\) on \(\mathbb{R}^d\), we also set
\[
m_r(\mu) = \int_{\mathbb{R}^d} |y|^r \mu(dy)
\]
for its moment of order \(r \geq 1\) and we say that \(\mu \in \mathcal{P}_r\) if \(m_r(\mu)\) is finite. All the functions in the paper are implicitly measurable with respect to the Borel-sigma field induced by the product topology. A \(\mathbb{R}^d\)-valued function \(f\) is written componentwise as \(f = (f^k)_{1 \leq k \leq d}\) where the \(f^k\) are real-valued. We denote by \(\partial_{\vartheta_t}, \nabla_\vartheta, \partial^2_{\vartheta_t \vartheta_t}\) respectively the partial derivative of a function with respect to the \(k\)-th component \(\vartheta_k\), the gradient of a real-valued function with respect to \(\vartheta\), the second order partial derivative of a function with respect to \(k\)-th and \(l\)-th components \(\vartheta_k, \vartheta_l\).

Finally, we repeatedly use the notation \(C\) for a positive number that does not depend on \(N\), nor \(\vartheta\), that may vary from line to line and that we call a constant, although it usually depends on some other (fixed) quantities of the model. In most cases, it is explicitly computable.

2.2. Model assumptions.
Well-posedness of the model and its associated statistical experiment. We work under the following strong integrability property for the initial condition $\mu_0$.

**Assumption 1.** For every $r \geq 1$, we have $\mu_0 \in \mathcal{P}_r$.

As for the diffusion matrix $\sigma : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$, we make the following strong ellipticity and Lipschitz smoothness assumption.

**Assumption 2.** The diffusion matrix $\sigma$ is measurable and for some $C \geq 0$, we have

$$|\sigma(t, x') - \sigma(t, x)| \leq C|x' - x|.$$ 

Moreover, $c = \sigma \sigma^\top$ is such that $\sigma_+^2 |y|^2 \leq (c(t, x)y)^\top y \leq \sigma_+^2 |y|^2$ for some $\sigma_+ > 0$.

As for the drift part $b : \Theta \times [0, T] \times \mathbb{R}^d \times \mathcal{P}_1 \to \mathbb{R}^d$, we work under usual Lipschitz smoothness assumptions.

**Assumption 3.** The drift $b$ is measurable and for some $C \geq 0$, we have

$$\sup_{t \in [0, T], \vartheta \in \Theta} |b(\vartheta; t, x', \nu') - b(\vartheta; t, x, \nu)| \leq C(|x' - x| + W_1(\nu', \nu)),$$

and there exists some $\vartheta_0 \in \Theta$ such that

$$b_0 = \sup_{t \in [0, T]} |b(\vartheta_0; t, 0, \delta_0)| < \infty .$$

We let $|b|_{\text{Lip}}$ denote the smallest $C \geq 0$ for which Assumption 3 holds.

Assumptions 1, 2, 3 together are sufficient to guarantee the well-posedness of the statistical model: there exists a unique weak solution to (2) for every $(\vartheta \in \Theta, \mu_0 \in \mathcal{P}_r)$ under the dynamics (2) and that we realise as $\mu_t \in \mathcal{P}_r$ for every $t \in [0, T]$. Moreover, for every $\vartheta \in \Theta$, the parabolic nonlinear equation (3) has a unique probability solution $\mu = (\mu_t)_{t \in [0, T]}$ and we have the weak convergence $\mu_t \overset{(\text{N})}{\to} \mu_\vartheta$ under $\mathbb{P}_\vartheta$ for every $\vartheta \in \Theta$.

We thus study under Assumptions 1, 2, 3 the (sequence of) statistical experiment(s) generated by the observation (1) under the dynamics (2) and that we realise as

$$(\mathcal{E}_N)_{N \geq 1} = \left(\mathcal{E}_N ; \mathcal{F}_N, (\mathbb{P}_\vartheta^N, \vartheta \in \Theta)\right)_{N \geq 1} .$$

Note that at that stage, we do not impose any identifiability assumption i.e. we do not assume that the mapping $\vartheta \mapsto \mathbb{P}_\vartheta^N$ is one-to-one. We will discuss that matter together with the non-degeneracy of the model later in Section 2.4.
Regularity of the experiment $\mathcal{E}^N$. In order to study the regularity of the model, we need specific smoothness properties for the function $\vartheta \mapsto b(\vartheta, \cdot)$.

**Assumption 4.** There exist $r_1, r_2 \geq 1$ and $C > 0$ such that for every point $\vartheta$ in the interior of $\Theta$, the function $\vartheta \mapsto b(\vartheta; t, x, \nu)$ is twice differentiable and for every $1 \leq \ell, \ell' \leq p$,

$$
\sup_{t \in [0,T]} (|\partial_{\vartheta_\ell} b(\vartheta; t, x, \nu)| + |\partial_{\vartheta_\ell \vartheta_\ell'} b(\vartheta; t, x, \nu)|) \leq C(1 + |x|^{r_1} + m_{r_2}(\nu)),
$$

$$
\sup_{t \in [0,T]} (|\partial_{\vartheta_\ell} b(\vartheta; t, x', \nu') - \partial_{\vartheta_\ell} b(\vartheta; t, x, \nu)|) \leq C(|x' - x| + \mathcal{W}_1(\nu', \nu)).
$$

The smoothness properties of the map $\vartheta \mapsto b(\vartheta; \cdot)$ granted by Assumption 4 enables us to explore further the regularity of the experiment $\mathcal{E}^N$. First, note that we have a log-likelihood by setting

$$
\ell^N(\vartheta; X^{(N)}) = \sum_{i=1}^N \int_0^T (c^{-1}b)(\vartheta; t, X_i^t, \mu_i^{(N)})^\top dX_i^t - \frac{1}{2} \sum_{i=1}^N \int_0^T |(c^{-1/2}b)(\vartheta; t, X_i^t, \mu_i^{(N)})|^2 dt,
$$

where $c^{-1/2}$ is fixed once for all. Indeed, by Girsanov’s theorem again, the laws $\mathbb{P}_0^N$ are all absolutely continuous w.r.t. $\mathcal{W}^N$, defined as the unique probability on $(\mathcal{E}^N, \mathcal{F}^N)$ under which the processes

$$
\left( \int_0^t c^{-1/2}(s, X_s^i) dX_s^i \right)_{t \in [0,T]} \quad 1 \leq i \leq N
$$

are independent standard Brownian motions on $\mathbb{R}^d$, together with $\mathcal{L}(X_0^0, \ldots, X_0^N) = \mu_0^{\otimes N}$. In turn, for every $\vartheta \in \Theta$,

$$
\frac{d\mathbb{P}_0^N}{d\mathbb{W}^N}(X^{(N)}) = \exp \left( \ell^N(\vartheta; X^{(N)}) \right)
$$

holds $\mathcal{W}^N$-almost-surely. We further write $L^N(\vartheta; X^{(N)}) = \exp \left( \ell^N(\vartheta; X^{(N)}) \right)$ for the likelihood process, indexed by the parameter $\vartheta \in \Theta$. We recall one possible classical definition of a regular statistical experiment, following [24].

**Definition 5.** The dominated (sequence of) experiment(s) $(\mathcal{E}^N)_{N \geq 1}$ is regular if

(i) $\vartheta \mapsto L^N(\vartheta; X^{(N)})$ is differentiable for every $\vartheta$ in (the interior of) $\Theta$, $\mathcal{W}^N$-almost surely,

(ii) $\vartheta \mapsto \nabla_\vartheta L^N(\vartheta; X^{(N)})$ is continuous in quadratic $\mathcal{W}^N$-mean, for every $\vartheta$ in (the interior of) $\Theta$,

(iii) we have finite Fisher information

$$
\mathbb{E}_{\mathbb{P}_0^N} \left[ |\nabla_\vartheta \ell^N(\vartheta; X^{(N)})|^2 \right] < \infty
$$

for every $\vartheta$ in (the interior of) $\Theta$.

**Proposition 6.** Under Assumptions 2, 3 and 4, the (sequence of) experiment(s) $(\mathcal{E}^N)_{N \geq 1}$ is regular.

**(Sketch of) Proof.** By exchanging the order of the differentiation with respect to $\vartheta$ and the stochastic integral we have

$$
\partial_{\vartheta_k} \ell^N(\vartheta; X^{(N)}) = \sum_{i=1}^N \int_0^T \partial_{\vartheta_k}(c^{-1}b)(\vartheta; t, X_i^t, \mu_i^{(N)})^\top dX_i^t
$$

$$
- \sum_{i=1}^N \int_0^T \partial_{\vartheta_k}(c^{-1/2}b)(\vartheta; t, X_i^t, \mu_i^{(N)})^\top (c^{-1/2}b)(\vartheta; t, X_i^t, \mu_i^{(N)})dt.
$$
We obtain the representation

\[ \partial_{\theta_k} \ell^N(\theta; X^{(N)}) = \sum_{i=1}^{N} \int_0^T \partial_{\theta_k} (c^{-1/2} b)(\theta; t, X^i_t, \mu^{(N)}_t) \partial_{\theta'} (c^{-1/2} b)(\theta; t, X^i_t, \mu^{(N)}_t) \, dB^i_{t, \theta}, \]

where

\[ (B^i_{t, \theta})_{t \in [0, T]} = \left( \int_0^t c^{-1/2}(s, X^i_s)(dX^i_s - b(\theta; s, X^i_s, \mu^{(N)}_s)ds) \right)_{t \in [0, T]}, \quad 1 \leq i \leq N \]

are independent Brownian motions on \( \mathbb{R}^d \) under \( \mathbb{P}^N_{\theta} \). The properties (i), (ii) and (iii) are then a simple consequence of Assumption 4 together with the following moment bound.

\textbf{Lemma 7.} Under Assumptions \( \mathcal{A} \), \( \mathcal{B} \), \( \mathcal{C} \) for every \( r \geq 1 \), we have

\[ \sup_{\theta \in \Theta, t \in [0, T], N \geq 1} \mathbb{E}^N_{\theta}[|X^i_t|^r] < \infty. \]

Note that \( \mathbb{E}^r_{\theta}[|X^i_t|^r] \) does not depend on \( i \). The proof of Lemma 7 is given in Appendix 7.1.

Finally, we have a notion of Fisher information matrix by setting

\[ I_{\mathcal{F}}(\theta) = \mathbb{E}^N_{\theta}[\nabla_{\theta} \ell^N(\theta, X^{(N)}) \nabla_{\theta} \ell^N(\theta, X^{(N)})^\top]. \]

Thanks to (10), we also have

\[ I_{\mathcal{F}}(\theta) = \left( \sum_{i=1}^{N} \mathbb{E}^N_{\theta} \left[ \int_0^T \partial_{\theta i} (c^{-1/2} b)(\theta; t, X^i_t, \mu^{(N)}_t) \partial_{\theta j} (c^{-1/2} b)(\theta; t, X^j_t, \mu^{(N)}_t) \, dt \right] \right)_{1 \leq i, j \leq p}. \]

\subsection*{2.3. The companion McKean-Vlasov product experiment.}

We let \( \mathcal{C} = \mathcal{C}([0, T], \mathbb{R}^d) \) denote the space of continuous functions on \( \mathbb{R}^d \), equipped with the natural filtration \( \{\mathcal{F}_t\}_{0 \leq t \leq T} \) induced by the canonical mapping \( X_t(\omega) = \omega_t \). For every \( \theta \in \Theta \), we let \( \mathbb{F}_\theta \) denote the unique law under which the process

\[ (B^\theta_t)_{t \in [0, T]} = \left( \int_0^t c^{-1/2}(s, X_s)(dX_s - b(\theta; s, X_s, \mu^\theta_s)ds) \right)_{t \in [0, T]} \]

is a standard Brownian motion on \( \mathbb{R}^d \), appended with the condition \( \mathcal{L}(X_0) = \mu_0 \), and \( \mu^\theta = (\mu^\theta_t)_{t \in [0, T]} \) is a probability solution of (3). The family \( \{\mathbb{F}_\theta\}_{\theta \in \Theta} \) is well-defined under Assumptions \( \mathcal{A} \), \( \mathcal{B} \), \( \mathcal{C} \). In particular, the canonical process \( X \) on \( (\mathcal{C}, \mathcal{F}_t) \) is a solution to the McKean-Vlasov equation

\[ \left\{ \begin{array}{l}
\text{d}X_t = b(\theta; t, X_t, \mu^\theta_t) \text{d}t + \sigma(t, X_t) \text{d}B^\theta_t, \quad t \in [0, T],
\mathcal{L}(X_0) = \mu_0.
\end{array} \right. \]

The following result is the counterpart of Lemma 7. Note in particular that the marginals of \( \mathbb{F}_\theta \) coincide with the solution \( \mu^\theta = (\mu^\theta_t)_{t \in [0, T]} \) of the Fokker-Planck equation (3).

\textbf{Lemma 8.} Under Assumptions \( \mathcal{A} \), \( \mathcal{B} \), \( \mathcal{C} \) for every \( r \geq 1 \), we have

\[ \sup_{\theta \in \Theta, t \in [0, T]} \mathbb{E}^\theta_{\mathcal{F}}[|X^i_t|^r] = \sup_{\theta \in \Theta, t \in [0, T]} \int_{\mathbb{R}^d} |x|^r \mu^\theta_t(dx) < \infty. \]

The proof is given in Section 7.2. We also have the following smoothness property in the parameter \( \theta \), proof of which is delayed until Section 6.1.

\textbf{Proposition 9.} Under Assumption \( \mathcal{A} \), \( \mathcal{B} \), \( \mathcal{C} \) and \( \mathcal{D} \) the mapping \( \theta \mapsto \mu^\theta_t \) is Lipschitz continuous in the Wasserstein-1 metric \( \mathcal{W}_1 \), uniformly in \( t \in [0, T] \).
We next consider the limit experiment
\[ \mathcal{G} = (\mathcal{E}, \mathcal{F}, (\mathbb{P}_\theta)_{\theta \in \Theta}) \]
and its \(N\)-fold counterpart
\[ \mathcal{G}^{\otimes N} = \left( \mathcal{E}^N, \mathcal{F}_T^N, (\mathbb{P}_\theta^{\otimes N})_{\theta \in \Theta} \right) \]
that serves as an approximation for the experiment \( \mathcal{E}^N \). Inspired by classical propagation of chaos techniques (see in particular [29]), we can easily show that the measures \( \mathbb{P}_\theta^N \) and \( \mathbb{P}_\theta^{\otimes N} \) are indistinguishable when the drift is of the form
\[ b(\hat{\theta}; t, x, \nu) = \int_{\mathbb{R}^d} \tilde{b}(\hat{\theta}; t, x, y)\nu(dy), \]
for some kernel \( \tilde{b}(\hat{\theta}; \cdot) : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d \) such that
\[ \sup_{t \in [0, T], \theta \in \Theta} |\tilde{b}(\hat{\theta}; t, x; y)| \leq C(1 + |x|^{r_1} + |y|^{r_2}) \]
for some \( r_1, r_2 \geq 1 \), a situation that covers most of our examples, see Section 4 below. More precisely, we have the following

Proposition 10. Under Assumptions [123] if \( b \) has moreover the form (13)-(14), we have
\[ \limsup_{N \to \infty} \sup_{\theta \in \Theta} \mathbb{E}_{\mathbb{P}_\theta^{\otimes N}} \left[ \log \frac{d\mathbb{P}_\theta^{\otimes N}}{d\mathbb{P}_\theta^N} \right] < \infty. \]
In particular, if
\[ \sup_{\theta \in \Theta} \int_0^T \int_{\mathbb{R}^d \times \mathbb{R}^d} |\tilde{b}(\hat{\theta}; t, x, y)|^2 (\mu_t^g \otimes \mu_t^g)(dx, dy)dt < 4, \]
then
\[ \limsup_{N \to \infty} \sup_{\theta \in \Theta} \|\mathbb{P}_\theta^N - \mathbb{P}_\theta^{\otimes N}\|_{TV} < 1, \]
where \( \| \cdot \|_{TV} \) denotes total variation distance.

The proof is given in Section 6.2 Some remarks are in order: 1) The estimate (15) tells us that it is impossible to statistically discriminate between \( \mathbb{P}_\theta^N \) and \( \mathbb{P}_\theta^{\otimes N} \) asymptotically. More precisely, inequality (16) shows in particular that provided \( \tilde{b} \) is not too big or \( T \) not too large, then there exists no test of the null \( H_0 : \mathbb{P}_\theta^N = \mathbb{P}_\theta^{\otimes N} \) against the alternative \( H_1 : \mathbb{P}_\theta^N \neq \mathbb{P}_\theta^{\otimes N} \) with asymptotically arbitrarily small first and second kind error in the limit \( N \to \infty \). 2) We will actually prove a stronger result in Section 3 below, showing that both \( (\mathcal{E}^N)_{N \geq 1} \) and \( (\mathcal{G}^{\otimes N})_{N \geq 1} \) share the LAN property, with same asymptotic Fisher information. 3) Finally, (15) may hold in wider generality when the dependence in the measure variable in the drift is nonlinear, as soon as we have some differentiability in the following sense: there exists \( \partial_{\nu} b(\hat{\theta}; t, x, \cdot) : \mathbb{R}^d \times \mathcal{P}_1 \to \mathbb{R}^d \) such that
\[ b(\hat{\theta}; t, x, \nu) - b(\hat{\theta}; t, x, \nu') = \int_0^1 \partial_{\nu} b(\hat{\theta}; t, x, \nu + (1 - \lambda)\nu')(\nu - \nu')(dy) \]
for every \( \nu, \nu' \in \mathcal{P}_1 \) and \( \partial_{\nu} b(\hat{\theta}; t, x, \cdot) \) satisfies additional smoothness properties. Iterating the operator \( \partial_{\nu} \), if \( \partial_{\nu}^k b(\hat{\theta}; t, x, \cdot) : (\mathbb{R}^d)^k \times \mathcal{P}_1 \to \mathbb{R}^d \) exists and satisfies some smoothness and integrability properties, we may expect (15) to hold as soon as \( k \geq d/2 \). We refer to Assumption 4 and Proposition 19 of [16] where this approach is developed.
We also have a log-likelihood in the experiment \( \mathcal{G}^\otimes N \) by setting

\[
\ell^N(\theta; X^{(N)}) = \sum_{i=1}^N \int_0^T (c^{-1/2}b)(\theta; t, X_t^i, \mu_t^\theta) dX_t^i - \frac{1}{2} \sum_{i=1}^N \int_0^T |(c^{-1/2}b)(\theta; t, X_t^i, \mu_t^\theta)|^2 dt.
\]

This is the same argument as before: the laws \( \mathbb{P}^\otimes N_\theta \) are all absolutely continuous w.r.t. \( \mathbb{W}^N \), and for every \( \theta \in \Theta \),

\[
\frac{d\mathbb{P}^\otimes N_\theta}{d\mathbb{W}^N}(X^{(N)}) = \exp(\ell^N(\theta; X^{(N)}))
\]

holds \( \mathbb{W}^N \)-almost-surely.

Finally under Assumptions \([1, 2, 3, 4]\) the (sequence of) experiment(s) \( \mathcal{G}^\otimes N \) is also a regular model and its (normalised) Fisher information \( \mathbb{I}_\mathcal{G}(\theta) = N^{-1}\mathbb{I}_{\mathcal{G}^\otimes N}(\theta) \) is given by

\[
N^{-1}\mathbb{E}_{\mathbb{P}_\theta} \left[ \nabla_\theta \ell^N(\theta; X^{(N)}) \nabla_\theta \ell^N(\theta; X^{(N)})^\top \right] = \sum_{j=1}^d \int_0^T \int_{\mathbb{R}^d} \nabla_\theta (c^{-1/2}b)j(\theta; t, x, \mu_t^\theta) \nabla_\theta (c^{-1/2}b)j(\theta; t, x, \mu_t^\theta)^\top \mu_t^\theta(dx)dt
\]

\[
= \left( \int_0^T \int_{\mathbb{R}^d} \partial_{\theta j}(c^{-1/2}b)(\theta; t, x, \mu_t^\theta) \partial_{\theta j}(c^{-1/2}b)(\theta; t, x, \mu_t^\theta)^\top \mu_t^\theta(dx)dt \right)_{1 \leq \ell, \ell' \leq p}.
\]

Moreover, the mapping \( \theta \mapsto \mathbb{I}_\mathcal{G}(\theta) \) is smooth and appears as the (normalised) asymptotic information of \( \mathcal{E}^N \):

**Proposition 11.** Under Assumptions \([1, 2, 3, 4]\) the mapping \( \theta \mapsto \mathbb{I}_\mathcal{G}(\theta) \) is Lipschitz continuous. Moreover, for every \( \theta \) in (the interior of) \( \Theta \), we have

\[
N^{-1}\mathbb{I}_{\mathcal{G}^\otimes N}(\theta) \to \mathbb{I}_\mathcal{G}(\theta)
\]

as \( N \to \infty \), where \( \mathbb{I}_{\mathcal{G}^\otimes N}(\theta) \) is the Fisher information matrix of the experiment \( \mathcal{E}^N \) defined in \( (11) \) above.

The proof is given in Section \([6, 3]\).

### 2.4. Identifiability and non-degeneracy of the Fisher information.

**Motivation.** In the preceding section, we have built \( \mathcal{E}^N \) and \( \mathcal{G}^\otimes N \) (equivalently \( \mathcal{G} \)) as possibly redundant, in the sense that the mappings \( \theta \mapsto \mathbb{P}^N_\theta \) and \( \theta \mapsto \mathbb{P}_\theta \) are not necessarily one-to-one on \( \Theta \). Having a well-posed parametrisation is required since we wish to have at least consistent estimators. Arguing asymptotically, we only need to work in the limit model \( \mathcal{G} \).

Also, asymptotic identifiability is somehow linked to the non-degeneracy of the (normalised) Fisher information matrix \( \mathbb{I}_\mathcal{G} \). Following \([42]\), see also \([48]\), we say that a point \( \theta \) in (the interior of) \( \Theta \) is regular if \( \theta' \mapsto \mathbb{I}_\mathcal{G}(\theta') \) has constant rank in a neighbourhood of \( \theta \) and the experiment \( \mathcal{G} \) is called locally identifiable at \( \theta \) if the mapping \( \theta' \mapsto \mathbb{P}_{\mathcal{G}, \theta'} \) is injective in a neighbourhood of \( \theta \). We have the following classical result (that goes back at least to Cramer \([15]\)):

**Proposition 12 (Theorem 1 in [42]).** If \( \theta \) is regular, then \( \mathcal{G} \) is locally identifiable at \( \theta \) if and only if \( \mathbb{I}_\mathcal{G}(\theta) \) has full rank.

Unfortunately, there is no hope to obtain a global result that links the two notions unless in very specific cases, see Proposition \([16]\) below. We next give ad-hoc assumptions that give sufficient (and independent) condition for both identifiability and non-degeneracy of the Fisher information.
An identifiability assumption. We first have a relatively weak assumption that guarantees global identifiability in $\mathcal{G}$.

**Assumption 13.** For all $\vartheta \in \Theta$, for $\mathbb{P}_\vartheta$-almost all $\omega$, for all $\vartheta' \neq \vartheta$, the functions $t \mapsto b(\vartheta; t, X_t(\omega), \mu^\vartheta_0)$ and $t \mapsto b(\vartheta'; t, X_t(\omega), \mu^{\vartheta'}_0)$ are not $dt$-a.e. equal.

Assumption [13] is relatively standard in the literature of statistics of random processes and minimal (see e.g. [19] in a somewhat analogous context). Indeed, by Girsanov’s theorem, for two different parameters $\vartheta, \vartheta' \in \Theta$, the laws $\mathbb{P}_\vartheta$ and $\mathbb{P}_{\vartheta'}$ are absolutely continuous and

$$\log \frac{d\mathbb{P}_\vartheta}{d\mathbb{P}_{\vartheta'}}(X) = \int_0^T (c^{-1}b)(\vartheta; s, X_s, \mu^\vartheta_0) - (c^{-1}b)(\vartheta'; s, X_s, \mu^{\vartheta'}_0) \, dX_s^i \leftarrow \frac{1}{2} \int_0^T (|(c^{-1/2}b)(\vartheta; s, X_s^i, \mu^\vartheta_0)|^2 - (c^{-1/2}b)(\vartheta'; s, X_s^i, \mu^{\vartheta'}_0)|^2)ds.$$

Having Assumption [13] fail for some $\vartheta'$ implies $\mathbb{P}_\vartheta(\frac{d\mathbb{P}_\vartheta}{d\mathbb{P}_{\vartheta'}}(X) = 1)$, i.e. $\mathbb{P}_\vartheta = \mathbb{P}_{\vartheta'}$. Assumption [13] may be difficult to check in practice. Yet, it is satisfied as soon as the mapping $\vartheta \mapsto ((t, x) \mapsto b(\vartheta; t, x, \mu^\vartheta_0))$ is one-to-one. Also, for certain form of the likelihood, we have other criteria, see Proposition [16] below.

**Non-degeneracy of the information.** We need some notation. For any $\vartheta, \vartheta' \in \Theta$ such that the segment $[\vartheta, \vartheta'] = \{\vartheta + \lambda(\vartheta' - \vartheta), \lambda \in [0, 1]\} \subset \Theta$ and a function $\phi$ defined on $\Theta$, we set

$$\phi([\vartheta, \vartheta']) = \int_0^1 \phi(\vartheta + \lambda(\vartheta' - \vartheta))d\lambda.$$

**Definition 14.** The statistical experiment $\mathcal{G}$ is non-degenerate if

$$\inf_{[\vartheta, \vartheta'] \subset \Theta} \det \mathbb{E}_{\mathbb{P}_\vartheta} \left[ \nabla_{\vartheta} \ell^1([\vartheta, \vartheta']) \nabla_{\vartheta} \ell^1([\vartheta, \vartheta'])^\top \right] > 0,$$

where $\det$ denotes the determinant.

Equivalently, we can rewrite (18) as

$$\inf \det \left( \sum_{j=1}^d \int_0^T \int_{\mathbb{R}^d} \nabla_{\vartheta} (c^{-1/2}b)^j([\vartheta, \vartheta']; t, x, \mu^\vartheta_t) \nabla_{\vartheta} (c^{-1/2}b)^j([\vartheta, \vartheta']; t, x, \mu^{\vartheta'}_t) \, \mu^\vartheta_t(dx)dt \right) > 0,$$

where the infimum is taken over all segments $[\vartheta, \vartheta'] \subset \Theta$. Obviously, if $\mathcal{G}$ is non-degenerate, taking $\vartheta = \vartheta'$, Definition [14] boils down to

$$\inf_{\vartheta \in \Theta} \det \mathbb{I}_\mathcal{G}(\vartheta) > 0$$

i.e. $\vartheta \mapsto \mathbb{I}_\mathcal{G}(\vartheta)$ has full rank uniformly in $\vartheta$ and we find back the usual non-degeneracy of the Fisher information. The somewhat stronger non-degeneracy criterion that we pick in Definition [14] enables us to check the assumptions of the theory of Ibragimov and Hasminski for obtaining sharp properties for the maximum likelihood estimator (see in particular Step 2 of the proof of Theorem [19] in Section 5.3 below). In explicit examples, proving (18) is no more difficult than proving (19), see Section [14] below.
Checking (18) or (19) in practice. A special difficulty for the statistical analysis of $\mathcal{E}^N$ or rather $\mathcal{G}$ lies in the asymptotic form (12) with the presence of $(\mu^0_t)_{0 \leq t \leq T}$ in the drift, which is never explicit, except in very special cases with a specific moment structure in the measure dependence, see Section 4 below.

It is noteworthy that (18) can usually be tested in a simple way given an explicit parametrisation. Indeed, Definition 14 is equivalent to show that for every segment $[\vartheta, \vartheta'] \subset \Theta$,

$$\inf_{[\vartheta, \vartheta'] \subset \Theta} \min_{|z| = 1} \sum_{j=1}^d \int_0^T \int_{\mathbb{R}^d} \left( \nabla_\vartheta (c^{-1/2} b)^j ([\vartheta, \vartheta']; t, x, \mu^0_t)^T z \right)^2 \mu^0_t (dx) dt > 0.$$ 

Under Assumptions 1, 2, 3, we have that $\mu^0_t(dx) = \mu^0_d(x) dx$ is absolutely continuous on $\mathbb{R}^d$ for $t > 0$, and we may pick a version $\mu^0$ of the density that is continuous and positive on $\mathbb{R}^d$. This follows from classical Gaussian tail estimates for the solution of parabolic equations. We refer for example to Corollary 8.2.2 of [15]. By a simple continuity argument, it is then sufficient to show that there cannot exist a segment $[\vartheta, \vartheta'] \subset \Theta$ and some $|z| = 1$, such that the function

$$x \mapsto \int_0^T \sum_{j=1}^d \left( \nabla_\vartheta (c^{-1/2} b)^j ([\vartheta, \vartheta']; t, x, \mu^0_t)^T z \right)^2 dt$$

vanishes asymptotically, or, as soon as we have continuity in $t$ as $t \to 0$, if one of the functions

$$x \mapsto \nabla_\vartheta (c^{-1/2} b)^j ([\vartheta, \vartheta']; 0, x, \mu^0_0)^T z, \quad j = 1, \ldots, d$$

does not identically vanishes. This last criterion has the advantage to avoid the term $\mu^0_t$ for $t > 0$.

We gather these observations in the following:

**Proposition 15.** Work under Assumptions 1, 2, 3, and 4. Assume moreover that the functions

$$t \mapsto \nabla_\vartheta (c^{-1/2} b)^j ([\vartheta, \vartheta']; t, x, \mu^0_t), \quad j = 1, \ldots, d$$

are all continuous at $t = 0$ for every $[\vartheta, \vartheta'] \subset \Theta$ and a.e.-almost $x \in \mathbb{R}^d$.

If, for every $[\vartheta, \vartheta'] \subset \Theta$ and any $z \in \mathbb{R}^p$ with $|z| = 1$, one of the functions

(20) $$x \mapsto \nabla_\vartheta (c^{-1/2} b)^j ([\vartheta, \vartheta']; 0, x, \mu^0_0)^T z, \quad j = 1, \ldots, d$$

does not identically vanishes, then $\mathcal{G}$ is non-degenerate in the sense of Definition 14.

We specifically apply this criterion in the examples Section 4 and check that the criterion (20) is particularly simple to establish when the dependence in the measure argument of the function $b$ is of the form (13).

A case of equivalence between global identifiability and non-degeneracy of the information. We revisit Theorem 3 in [42] to obtain the following criterion:

**Proposition 16.** Work under Assumptions 1, 2, 3, and 4. Assume that the log-likelihood $\ell^N(\vartheta; X^{(N)})$ in $\mathcal{E}^N$ defined by (9) has the form

(21) $$\ell^N(\vartheta; X^{(N)}) = \vartheta^T G^N (X^{(N)}) + \vartheta^T H^N (X^{(N)}) \vartheta,$$

where $G^N$ and $H^N$ are functions of the trajectory $X^{(N)}$ with values in $\mathbb{R}^p$ and $\mathbb{R}^p \otimes \mathbb{R}^p$ respectively, and $(H^N)^T = H^N$ is symmetric. If $\Theta_0 \subset \Theta$ is a convex set such that $I_{\Theta}(\vartheta)$ is non-singular for every $\vartheta \in \Theta_0$, then, both $(\mathcal{E}^N)_{N \geq 1}$ and $\mathcal{G}$ are identifiable on $\Theta_0$. 


By identifiability of the sequence of experiment \((\mathcal{E}_N)_{N \geq 1}\), we mean injectivity of the mapping \(\vartheta \mapsto (\mathbb{P}_\vartheta^{N})_{N \geq 1}\) (i.e. simultaneously for every \(N \geq 1\)). The proof is given in Section 6.4. In the specific case of McKean type models that date back to \([38, 44, 46]\) and widely used in practice (see e.g. \([12, 17]\) or \([27]\) in statistics), we have in some instances a representation like \((21)\) and explicit formulas for \(\mathbb{I}_G(\vartheta)\), which gives global identifiability for free as soon as \(\mathbb{I}_G(\vartheta)\) is non-degenerate. See the examples in Section 4.

3. Main results

3.1. The LAN property. The local asymptotic normality property of a statistical model characterises its regularity: it expresses the fact that the experiment locally resembles a Gaussian shift in an optimal scale driven by the Fisher information. It has powerful consequences in terms of properties of optimal procedures via the celebrated Hájek convolution theorem \([23]\). More precisely the sequence of experiments \((\mathcal{E}_N)_{N \geq 1}\) satisfies the LAN property at \(\vartheta \in \Theta\) with information rate \(N\mathbb{I}_G(\vartheta)\) if

\[
\log \frac{d\mathbb{P}_\vartheta^{N}}{d\mathbb{P}_\vartheta} \xrightarrow{\vartheta \to \vartheta_0} \mathbb{E}_\vartheta^{N}[\mathbb{I}_G(\vartheta_0)^{-1/2}u] = u^\top \xi_\vartheta^{N} - \frac{1}{2}|u|^2 + r_N(\vartheta, u),
\]

where \(\xi_\vartheta^{N}\) converges in distribution under \(\mathbb{P}_\vartheta^{N}\) to standard Gaussian variable in \(\mathbb{R}^p\) and \(r_N(\vartheta, u) \to 0\) in \(\mathbb{P}_\vartheta^{N}\)-probability. Of course, the convergence \((22)\) is meaningful only if \(\vartheta + (N\mathbb{I}_G(\vartheta))^{-1/2}u \in \Theta\) and is well-defined, i.e. if \(\det \mathbb{I}_G(\vartheta) > 0\). This is granted for instance for \(\vartheta\) in the interior of \(\Theta\) for large enough \(N\) and under \((19)\).

**Theorem 17.** Work under Assumptions \([1, 2, 3, 4]\) and \([13]\) Assume moreover that \(G\) is non-degenerate according to Definition \([14]\). For every \(\vartheta\) in (the interior of) \(\Theta\), the sequence of experiments \((\mathcal{E}_N)_{N \geq 1}\) is locally asymptotically normal at \(\vartheta\) with information rate \(N\mathbb{I}_G(\vartheta)\).

The same result holds for \((G^{\otimes N})_{N \geq 1}\).

Several remarks are in order: 1) Theorem \([17]\) is the most powerful result one can obtain about the structure of \((\mathcal{E}_N)_{N \geq 1}\) and \((G^{\otimes N})_{N \geq 1}\): it tells us that around a given point \(\vartheta_0\), if we parametrise locally the experiment via \(\bar{\vartheta} = \vartheta_0 + N^{-1/2}u\) with \(u \in \mathbb{R}^p\) being the unknown parameter, then the experiments look like the simplest possible experiment, namely a Gaussian shift

\[
Y_N = u + \mathbb{I}_G(\vartheta_0)^{-1/2}\xi + o(1)
\]

where \(\xi\) is a standard normal \(\mathcal{N}(0, \mathbb{I}_G(\vartheta_0))\) and \(o(1)\) is a small term that vanishes in \(\mathbb{P}_\vartheta^{N}\) or \(\mathbb{P}_\vartheta^{\otimes N}\) probability, locally uniformly in \(u\). 2) The fact that both \((\mathcal{E}_N)_{N \geq 1}\) and \((G^{\otimes N})_{N \geq 1}\) share the LAN property with same asymptotic Fisher variance quantifies their asymptotic similarity, see in particular Proposition \([10]\). 3) The LAN property has several consequences in terms of strong properties of the maximum likelihood estimator, see Theorem \([19]\) below. In particular, the first simple consequence is given in terms of exact asymptotic minimax lower bounds: call a centrally symmetric function \(w: \mathbb{R}^p \to [0, \infty)\) such that the sets \(\{w < c\}, c > 0\) are all convex a polynomial loss function if it admits a polynomial majorant.

**Corollary 18.** In the setting of Theorem \([17]\) let \(w\) be a polynomial loss function. Then, for any estimator \(\hat{\vartheta}_N\) in \(\mathcal{E}_N\) and any sufficiently small \(\delta > 0\), for every \(\bar{\vartheta} \in (\text{the interior of})\) \(\Theta\) for which \(\det \mathbb{I}_G(\bar{\vartheta}) > 0\), we have

\[
\liminf_{N \to \infty} \sup_{|\vartheta - \vartheta_0| \leq \delta} \mathbb{E}_{\vartheta_0}^{N}[w(N^{1/2}\mathbb{I}_G(\vartheta)^{1/2}(\vartheta_N - \vartheta))] \geq (2\pi)^{-p/2} \int_{\mathbb{R}^p} w(x) \exp(-\frac{1}{2}|x|^2)dx.
\]

The same result holds true for \(\hat{\vartheta}_N\) in \(G^{\otimes N}\) replacing \(\mathbb{P}_\vartheta^{N}\) by \(\mathbb{P}_\vartheta^{\otimes N}\).
Corollary [18] is a simple application of Hájek convolution theorem, given the LAN property of Theorem [17] see e.g. Theorem II.12.1 (in particular Remark III.12.1) in [24]. It provides with a sharp local asymptotically minimax bound, up to constants. We shall see below that the maximum likelihood estimator achieves this bound.

3.2. Maximum likelihood estimation and properties. We elaborate on the properties of the maximum likelihood estimator by relying on (a uniform version of) the LAN property of Theorem 17. It implies several fine results that go beyond the usual asymptotic weak expansions given by an ad-hoc study of the form of the estimator, as is usually the case in the literature.

**Theorem 19.** Work under Assumptions [1], [2], [3], [4] and [13] Then, for large enough $N$, the solution $\hat{\vartheta}_N$ to
\begin{equation}
L_N(\hat{\vartheta}_N; X^{(N)}) = \sup_{\vartheta \in \Theta} L_N(\vartheta; X^{(N)})
\end{equation}
is well-defined. Moreover, the following asymptotic upper bounds are valid:

(i) if $S$ is non-degenerate in the sense of Definition 14,
\begin{equation}
\sqrt{N}(\hat{\vartheta}_N - \vartheta) \to N(0, I_S(\vartheta)^{-1})
\end{equation}
in $P_{\vartheta}$-distribution as $N \to \infty$.

(ii) For every polynomial loss function $w$ and any $\vartheta$ in the interior of $\Theta$, we have exact local asymptotic minimax optimality:
\begin{equation}
\limsup_{N \to \infty} \sup_{|\vartheta' - \vartheta| \leq \delta} E_{P_{\vartheta}}[w(N^{1/2} I_S(\vartheta)^{1/2}(\hat{\vartheta}_N^{\text{MLE}} - \vartheta'))] \to (2\pi)^{-p/2} \int_{\mathbb{R}^p} w(x) \exp(-1/2 |x|^2) dx
\end{equation}
as $\delta \to 0$.

(iii) For every polynomial loss function $w$ and any (non empty) open set $\Theta_0 \subset \Theta$, we have global asymptotic minimax optimality:
\begin{equation}
R_w(\hat{\vartheta}_N; \Theta_0) = \inf_{\vartheta \in \Theta_0} R_w(\vartheta; \Theta_0)(1 + o(1))
\end{equation}
as $N \to \infty$, where
\begin{equation}
R_w(\vartheta; \Theta_0) = \sup_{\vartheta \in \Theta_0} E_{P_{\vartheta}}[w(N^{1/2} I_S(\vartheta)^{1/2}(\hat{\vartheta}_N - \vartheta))].
\end{equation}

Some further remarks: 1) We find back the classical asymptotic properties (i) of the maximum likelihood estimator that are given in the literature, but the result is appended by a much stronger convergence in (ii), that matches in particular the lower bound of Corollary 18. 2) We finally obtain global asymptotic minimax optimality by (iii), which is the parametric analog (in a much more precise way) of our minimax results of Section 4 in [16] in the nonparametric case.

4. Examples

In this section, we elaborate on specific examples that appear in the literature and in applications. We first revisit the linear McKean model studied at length in [27]. We slightly extend in Section 4.1 this example (1.3) from $p = 2$ to $p = 3$. In Section 4.2, we develop an example of a generalised linear form and show in particular how our identifiability and non-degeneracy criteria of Section 2.4 are easily implementable and avoid to use the machinery of [27]. In Section 4.3, we develop a non-trivial example of kinetic mean-field model with a double layer potential that may serve in many applications, like swarming models or more general individual based-models, see [6] and the references therein. We finally develop a genuinely non-linear example, i.e. when the...
measure argument is not linear like in (4), as for instance in the examples of [41]. Assumption [1] is in force throughout.

4.1. McKean-like models. In many applications, (2) takes the explicit form

\[ dX_t^i = (\vartheta_1 X_t^i + \vartheta_2)dt - \vartheta_3 N^{-1} \sum_{j=1}^{N} (X_t^i - X_t^j)dt + dB_t^i, \quad i = 1, \ldots, N \]

with \( X_t^i \in \mathbb{R} \). The parameter is \( \vartheta = (\vartheta_1, \vartheta_2, \vartheta_3)^T \). In [27] the case \( \vartheta_2 = 0 \) is studied at length in particular. In our setting, we can encompass a more general situation with \( X_t^i \in \mathbb{R}^d \) for some arbitrary \( d \geq 1 \) and replace \( \vartheta_3 \) by a parameter in \( \mathbb{R}^d \otimes \mathbb{R}^d \) as well as \( \vartheta_2 \) by a parameter in \( \mathbb{R}^d \). In this case, Assumptions 2, 3 and 4 are readily checked. Likewise, the identifiability and non-degeneracy assumptions can be obtained with some extra care on the initial condition. We elaborate on a specific case below.

Likelihood equations. To keep-up with notational simplicity, we detail the case \( p = 3 \) with \( \vartheta = (\vartheta_1, \vartheta_2, \vartheta_3)^T \in \Theta \) as a compact subset of \( \mathbb{R}^3 \) for an ambient dimension \( d = 1 \), with \( \vartheta_1 \neq \vartheta_3 \) and \( \vartheta_1 \neq 0 \). Introduce

\[
A_T^N(x) = \int_0^T \langle A_t^N(x), \mu_t^{(N)} \rangle dt,
\]

and

\[
B_T^N = N^{-1} \sum_{i=1}^{N} \int_0^T B_t^N(X_t^i) dX_t^i.
\]

Thanks to the linearity in \( \vartheta \) of the drift \( b(\vartheta; t, x, \nu) = \vartheta_1 x + \vartheta_2 - \vartheta_3 \int_{\mathbb{R}} (x - y) \nu(dy) \), the likelihood equations are explicit and the maximum likelihood estimator \( \hat{\vartheta}_{\text{MLE}}^N \) solves

\[
A_T^N \hat{\vartheta}_{\text{MLE}}^N = B_T^N.
\]

Moreover, the Fisher information matrix is given by

\[
I_{\vartheta}(\vartheta) = \int_0^T \langle A_t(\vartheta; x), \mu_t^{\vartheta} \rangle dt,
\]

with

\[
A_t(\vartheta; x) = \begin{pmatrix}
\frac{x^2}{x} & x & -\langle \cdot - x, \mu_t^{\vartheta} \rangle^2 \\
x & 1 & 0 \\
-\langle \cdot - x, \mu_t^{\vartheta} \rangle^2 & 0 & \langle \cdot - x, \mu_t^{\vartheta} \rangle^2
\end{pmatrix}.
\]
Non-degeneracy and identifiability. The property $\det \mathbb{I}_2(\vartheta) > 0$ can be verified on the explicit form of its matrix:

$$
\det \mathbb{I}_2(\vartheta) = \int_0^T \text{Var}(\mu^0_t) dt \left( T \int_0^T \overline{m}_1(\mu^0_t)^2 dt - \left( \int_0^T \overline{m}_1(\mu^0_t) dt \right)^2 \right)
$$

with $\text{Var}(\nu) = \int_{\mathbb{R}} (x - \overline{m}_1(\nu))^2 \nu(dx)$ and $\overline{m}_1(\nu) = \int_{\mathbb{R}} x \nu(dx)$. Therefore $\det \mathbb{I}_2(\vartheta) > 0$ unless $\mu^0_t$ is degenerate for all $t$ or stationary. In the case of a linear equation of the type (24), the accompanying limiting measure $\mu^0$ associated to the McKean-Vlasov equation

$$
dX_t = (\vartheta_1X_t + \vartheta_2) dt - \vartheta_3(X_t - E_{\vartheta_1}[X_t]) dt + dB_t
$$

is a Gaussian process that can be thought of as an inhomogeneous Ornstein-Uhlenbeck model for which we have closed-form moment formulas:

$$
(28) \quad \overline{m}_1(\mu^0_t) = -\vartheta_1^{-1} \vartheta_2 + (\overline{m}_1(\mu_0) + \vartheta_1^{-1} \vartheta_2) \exp(\vartheta_1 t)
$$

and

$$
(29) \quad m_2(\mu^0_t) = \exp \left( -2(\vartheta_1 - \vartheta_3)t \right) \text{Var}(\mu_0) + \frac{1 - \exp(-2(\vartheta_1 - \vartheta_3)t)}{2(\vartheta_1 - \vartheta_3)} + \overline{m}_1(\mu^0_t)^2.
$$

In particular, having

$$
(30) \quad \overline{m}_1(\mu_0) + \vartheta_1^{-1} \vartheta_2 \neq 0, \quad \vartheta_1 \neq 0, \quad \vartheta_1 \neq \vartheta_3
$$

yields the non-degeneracy of $\mathbb{I}_2(\vartheta)$ as well as the non-degeneracy in the sense of Definition 14 since $\nabla \phi b(\vartheta; t, x, \nu)$ does not depend on $\vartheta$. Also, the convergence $A^N_T \to \mathbb{I}_2(\vartheta)$ in $\mathbb{P}_{\vartheta}^N$-probability as $N \to \infty$ tells us that (33) has a well defined and unique solution with $\mathbb{P}_{\vartheta}^N$ probability tending to one as $N \to \infty$. This last statement can be quantified via the convergence Lemma 21 below. Writing $\phi(x, \nu) = (x - \int_{\mathbb{R}} (x - y) \nu(dy))$, the log-likelihood

$$
(31) \quad \ell^N(\vartheta; X^{(N)}) = \sum_{i=1}^N \int_0^T \vartheta^T \phi(X_i^t, \mu_i^0(N)) dX_i^t - \frac{1}{2} \sum_{i=1}^N \int_0^T \left( \vartheta^T \phi(X_i^t, \mu_i^0(N)) \right)^2 dt
$$

has representation (21) with

$$
G^N(X^{(N)}) = \sum_{i=1}^N \int_0^T \phi(X_i^t, \mu_i^0(N)) dX_i^t \quad \text{and} \quad H^N(X^{(N)}) = -\frac{1}{2} \sum_{i=1}^N \int_0^T \phi(X_i^t, \mu_i^0(N)) \phi(X_i^t, \mu_i^0(N))^T dt
$$

and we obtain the identifiability of $E^N$ and $\mathcal{G}$ over compact parameter sets $\Theta \subset \mathbb{R}^2$ that are moreover convex and satisfy the constraint (30) by Proposition 16. Finally, explicit formulas for $\mathbb{I}_2(\vartheta)$ and its inverse can be derived thanks to (28) and (29).

4.2. Generalised linear like models. We push further the preceding linear structure by considering the following model

$$
dX_i^t = \vartheta_1 f(X_i^t) dt + \vartheta_2 N^{-1} \sum_{j=1}^N g(X_i^t - X_j^t) dt + dB_i^t, \quad 1 \leq i \leq N
$$

where $f$ and $g$ are known and Lipschitz continuous real-valued functions and $X_i^t \in \mathbb{R}$ for simplicity. The parameter is $\vartheta = (\vartheta_1, \vartheta_2) \in \mathbb{R}^2$. Thus $p = 2$ which again yields simple and explicit formulas. Assumptions 2, 3 and 4 are readily checked. Writing $\phi(x, \nu) = (f(x) - g(x \nu(x))^T$, with $g * \nu(x) = \int_{\mathbb{R}} g(x - y) \nu(dy)$ the log-likelihood function $\ell^N(\vartheta; X^{(N)})$ has again representation (31) hence (21) holds as well and we obtain identifiability of $E^N$ and $\mathcal{G}$ over compact parameter sets
of the form two preceding linear-like models and study the model \( P \) by the non-degeneracy property established above thanks to Proposition 15. Hence \( P \) by taking the limit in \( \mathbb{P} \rho \). Again, we have \( \nabla \phi (\partial) = \frac{1}{\mathbb{P} \rho} \mathbb{E} \mathbb{P} \rho [g(X_i)] \), the fact that \( t \mapsto X_t \) is continuous in probability at \( t = 0 \) under \( \mathbb{P} \rho \) and Lebesgue dominated convergence. Now, let \( z = (z_1, z_2) \) with \( z_1^2 + z_2^2 = 1 \). In order to obtain non-degeneracy, it is sufficient by Proposition 15 to show that the function

\[ x \mapsto f(x)z_1 + g* \mu_0(x)z_2 \]

is non identically zero. If \( f \) does not identically vanishes, we may assume that \( z_2 \neq 0 \). Then it is sufficient to have that

\[ x \mapsto \lambda f(x) + g* \mu_0(x) \]

does not vanish identically for every \( \lambda \neq 0 \). It is then very easy to build families of functions and initial condition \( (f, g, \mu_0) \) such that (32) is satisfied. For instance, if \( \mu_0 = \delta_{x_0} \) for some arbitrary \( x_0 \), then having \( f \) non-identically equal to a constant is sufficient.

Likelihood equations. Finally, we explicitly solve the likelihood equations. Again, they are of a simple form, and the maximum likelihood estimator \( \hat{\vartheta} \) solves

\[ (A_N^T \phi (\vartheta))^N = B_T^N, \]

where \( A_N^T \) and \( B_T^N \) are defined via (25) and (26), with

\[ A_N^T (x) = \begin{pmatrix} f(x)^2 & f(x) (g(x - \cdot), \mu_{i}^{(N)}) & (g(x - \cdot), \mu_{i}^{(N)})^2 \end{pmatrix}, \]
\[ B_T^N (x) = \begin{pmatrix} f(x) \end{pmatrix}. \]

Again, we have

\[ \mathbb{E} [g(\vartheta)] = \begin{pmatrix} \int_0^T (f^2, \mu_{i}^{(N)}) dt \int_0^T (f (g * \mu_{i}^{(N)}), \mu_{i}^{(N)}) dt \int_0^T ((g * \mu_{i}^{(N)})^2, \mu_{i}^{(N)}) dt \end{pmatrix}, \]

by taking the limit in \( \mathbb{P} \rho \)-probability of \( A_N^T = \int_0^T (A_N^T (x), \mu_{i}^{(N)}) dt \). We also know that \( \text{det} \mathbb{I}_G \rho (\vartheta) > 0 \) by the non-degeneracy property established above thanks to Proposition 15. Hence \( A_N^T \) is invertible with \( \mathbb{P} \rho \)-probability that goes to one as \( N \to \infty \) and \( \phi \) is asymptotically well defined.

4.3. A double layer potential model. We depart from the structure (21) of the likelihood as in the two preceding linear-like models and study the model

\[ dX_t = N^{-1} \sum_{j=1}^N \nabla U_0(X_t - X_j) dt + dB_t, \quad 1 \leq i \leq N, \]

with ambient space \( \mathbb{R}^d \) for \( d \geq 1 \) and where \( U_0 : \mathbb{R}^d \to \mathbb{R} \) is a family of pairwise potentials of the form

\[ U_0 (x) = -\vartheta_1 \exp (-\vartheta_2 |x|^2) + \vartheta_3 \exp (-\vartheta_4 |x|^2) \]
modelling short range repulsion and long range attraction, where \( \vartheta_1, \vartheta_3 \) and \( \vartheta_2, \vartheta_4 \) are respectively the strengths and the lengths of attraction and repulsion. The parameter is \( \vartheta = (\vartheta_1, \ldots, \vartheta_4) \in (0, \infty)^4 \). As minimal identifiability condition, we impose \( \vartheta_2 \neq \vartheta_4 \). Such models are commonly used (in their kinetic version) in swarming modelling, see e.g. \([6]\).

We have \( b(\vartheta; t, x, \nu) = \nabla U_\vartheta * \nu(x) \) and it is readily verified that Assumptions 2, 3 and 4 are met. Here, there is no hope to explicitly solve the likelihood equations, and a numerical scheme has to be implemented. We further investigate the identifiability and non-degeneracy of the model. Note that the assumptions on the drift and the initial condition ensure that for every \( \vartheta \in \Theta \), \( \mu_0^\vartheta \) is absolutely continuous with a nowhere vanishing density for \( t > 0 \) (we refer to \([5]\) and in particular Corollary 8.2.2).

**Identifiability.** We study the injectivity of the mapping \( \vartheta \mapsto \mathfrak{F}_\vartheta \) via the injectivity of \( \vartheta \mapsto \left( (t, x) \mapsto \nabla U_\vartheta * \mu_t^\vartheta \right) \), see Assumption 13. If \( \vartheta, \vartheta' \) are such that \( \mathfrak{F}_\vartheta = \mathfrak{F}_{\vartheta'} \), this also implies \( \mu_t^\vartheta = \mu_t^{\vartheta'} \) for every \( t \in [0, T] \). Henceforth, we plan to apply Proposition 15. From (35), we have

\[
\mathcal{F}(\nu)(\xi) = \int_{\mathbb{R}^d} \exp(\iota x^\top \xi) \nu(dx)
\]

for almost every \( \xi \in \mathbb{R}^d \). Let \( \mathcal{F}(\nu)(\xi) \) denote a Fourier transform of \( \nu \) (well defined if \( \nu \) is a probability measure or an integrable function). Since \( \xi \mapsto \mathcal{F}(\mu_t)(\xi) \) is continuous and \( \mathcal{F}(\mu_t)(\xi) = 1 \) (the \( \mu_t \) are all probability measures on \( \mathbb{R}^d \)), there are infinitely many points \( \xi \in \mathbb{R}^d \) such that \( \mathcal{F}(\mu_t)(\xi) \neq 0 \). Applying \( \mathcal{F} \) on both side of (34), we obtain

\[
\mathcal{F}(\nabla U_\vartheta)(\xi) = \mathcal{F}(\nabla U_{\vartheta'})(\xi)
\]

for such points \( \xi \). Moreover

\[
\nabla U_\vartheta(x) = 2\vartheta_1 \vartheta_2 x \exp(-\vartheta_2|x|^2) - 2\vartheta_3 \vartheta_4 x \exp(-\vartheta_4|x|^2),
\]

and

\[
\mathcal{F}(\nabla U_\vartheta)(\xi) = i\xi^\top \vartheta^\top \vartheta^d/2 \left( \vartheta_1 \vartheta_2^{-d/2} \exp(-\frac{1}{4\vartheta_2}|\xi|^2) - \vartheta_3 \vartheta_4^{-d/2} \exp(-\frac{1}{4\vartheta_4}|\xi|^2) \right).
\]

It follows that the condition \( \vartheta_2 \neq \vartheta_4 \) is sufficient to achieve identifiability, i.e. \( \vartheta = \vartheta' \). Henceforth, we may parametrise our model via any compact \( \Theta \subset (0, \infty)^4 \) such that \( \vartheta_2 \neq \vartheta_4 \).

**Non-degeneracy.** We plan to apply Proposition 15. From (35), we have

\[
\nabla_\vartheta b(\vartheta; t, x, \nu)^j = \nabla_\vartheta (\nabla U_\vartheta * \nu(x))^j = G(\vartheta; x)^j * \nu(x),
\]

with

\[
G(\vartheta; x)^j = \begin{pmatrix}
2\vartheta_2 x_j \exp(-\vartheta_2|x|^2) \\
2\vartheta_1 x_j (1 - \vartheta_2|x|^2) \exp(-\vartheta_2|x|^2) \\
-2\vartheta_4 x_j \exp(-\vartheta_4|x|^2) \\
-2\vartheta_3 x_j (1 - \vartheta_4|x|^2) \exp(-\vartheta_4|x|^2)
\end{pmatrix}, \quad j = 1, \ldots, d
\]

The mapping

\[
t \mapsto \nabla_\vartheta b([\vartheta, \vartheta']; t, x, \mu_t^\vartheta)^j = \mathbb{E}_{\vartheta'} \left[ G([\vartheta, \vartheta']; X_t)^j \right]
\]

is continuous at \( t = 0 \), as a simple consequence of the fact that \( t \mapsto X_t \) is continuous in \( \mathbb{P}_\vartheta \)-probability at \( t = 0 \). It is then sufficient to prove that for any \( z \in \mathbb{R}^d \) with \( z_1^2 + z_2^2 + z_3^2 + z_4^2 = 1 \), one of the functions

\[
x \mapsto (G([\vartheta, \vartheta']; \cdot)^j * \mu_0(x))^\top z, \quad j = 1, \ldots, d.
\]
does not vanish identically. Assume on the contrary that \((G([\vartheta, \vartheta']_j; \cdot^{\dagger}) \ast \mu_0)^\top z\) is identically 0 for every \(1 \leq j \leq d\). Then this is also the case for \(\mathcal{F}(G([\vartheta, \vartheta']_j; \cdot^{\dagger}) \ast \mu_0)^\top z\). Assume now that \(\mathcal{F}(\mu_0)(\xi) \neq 0\) a.e. Then from
\[
\mathcal{F}(G([\vartheta, \vartheta']_j; \cdot^{\dagger}) \ast \mu_0)^\top z(\xi) = \mathcal{F}(G([\vartheta, \vartheta']_j; \cdot^{\dagger})(\xi))^\top z \cdot \mathcal{F}(\mu_0)(\xi)
\]
we must have
\[
\xi \mapsto \mathcal{F}(G([\vartheta, \vartheta']_j; \cdot^{\dagger}))(\xi)^\top z = 0 \quad d\xi - a.e.
\]
for every \(1 \leq j \leq d\), or equivalently
\[
x \mapsto (G([\vartheta, \vartheta']_j; x^{\dagger})^\top z = 0 \quad dx - a.e.
\]
This is not possible, as proved by an inspection of the equation
\[
\int_0^1 \left( [\vartheta_2, \vartheta'_2]_t x j \exp(-[\vartheta_2, \vartheta'_2]_t x |2^2 z_1 + [\vartheta_1, \vartheta'_1]_t x j (1 - [\vartheta_2, \vartheta'_2]_t x |2^2 \exp(-[\vartheta_2, \vartheta'_2]_t x |2^2 z_2) d\lambda
\]
\[
= \int_0^1 \left( [\vartheta_4, \vartheta'_4]_t x j (1 - [\vartheta_4, \vartheta'_4]_t x |2^2 \exp(-[\vartheta_4, \vartheta'_4]_t x |2^2 z_4) d\lambda,
\]
for almost every \(x \in \mathbb{R}^d\), by the mean-value theorem for some \(\lambda_x, \lambda'_x \in [0, 1]\), that also respectively depend on \((\vartheta_1, \vartheta_2, \vartheta'_1, \vartheta'_2, z_1, z_2)\) and \((\vartheta_3, \vartheta_4, \vartheta'_3, \vartheta'_4, z_3, z_4)\). A simple sufficient condition is \(\vartheta_2 \neq \vartheta_4\): indeed, if \(\vartheta_2\) and \(\vartheta_4\) take values in disjoint intervals for instance, then for every \(\vartheta, \vartheta' \in \Theta\), with \([\vartheta, \vartheta'] \subset \Theta\), we have \([\vartheta_2, \vartheta'_2] \cap [\vartheta_4, \vartheta'_4] = \emptyset\). Then one easily checks that (36) cannot hold for sufficiently large \(|x|\).

If we only need to verify that \(\mathcal{F}(\vartheta)\) is non-degenerate, it is sufficient to take \(\vartheta = \vartheta'\) in (36) that simply becomes
\[
(\vartheta_2 x_j z_1 + \vartheta_1 x_j (1 - \vartheta_2 |x|2^2 z_2) \exp(-\vartheta_2 |x|^2) = (\vartheta_4 x_j z_3 + \vartheta_3 x_j (1 - \vartheta_4 |x|2^2 z_4) \exp(-\vartheta_4 |x|^2)
\]
for almost every \(x \in \mathbb{R}^d\), in which case having \(\vartheta_2 \neq \vartheta_4\) is sufficient (and somewhat easier to obtain then the non-degeneracy of \(\mathcal{F}\) in the sense of Definition [4]). In conclusion, as soon as \(\mathcal{F}(\mu_0)\) is non-vanishing almost everywhere and \(\Theta \subset (0, \infty)^4\) is a compact such that \(\vartheta_2 \neq \vartheta_4\), we obtain non-degeneracy of \(\mathcal{F}\).

4.4. A genuinely non-linear example. We end-up this section by inspecting an example where the parametrisation \(\nu \mapsto b(\vartheta; t, x, \nu)\) is genuinely non-linear in the measure argument. Consider the model
\[
dX^i_t = F\left(\vartheta N^{-1} \sum_{j=1}^N g(X^i_t - X^j_t)\right) dt + dB^i_t, \quad 1 \leq i \leq N,
\]
with \(X^i_t \in \mathbb{R}\) and \(\vartheta > 0\) for simplicity. The functions \(F, g : \mathbb{R} \rightarrow \mathbb{R}\) are known and smooth, \(g\) is nonnegative, integrable, with positive mass and \(F\) is one-to-one on the positive axis. We have \(b(\vartheta; t, x, \nu) = F(\vartheta g \ast \nu(x))\).
The smoothness of $F$ and $g$ yields Assumptions \[2, 3\] and \[4\]. As for the identifiability of the model, assume that $\mathbb{P}_\theta = \mathbb{P}_{\theta'}$, and so $\mu_\theta = \mu_{\theta'}$ as well. If, for almost every $x \in \mathbb{R}$, we have

$$F(\partial g * \mu_\theta(x)) = F(\partial g * \mu_{\theta'}(x)),$$

then, since $g(x) \geq 0$ and $\mu_\theta(x) > 0$ for every $x \in \mathbb{R}$ and $t > 0$, the function $g * \mu_\theta$ is positive on the whole real line $\mathbb{R}$. Since $F$ is one-to-one on the positive axis, it follows that

$$\partial g * \mu_\theta(x) = \partial g * \mu_{\theta'}(x)$$

and that can only be true if $\theta = \theta'$ since $g * \mu_\theta(x)$ is nowhere vanishing. As for the non-degeneracy, we have

$$\partial_0 b(\theta; t, x, \mu_\theta) = g * \mu_\theta(x)F'(\partial g * \mu_\theta(x)) = \mathbb{E}_{\mathbb{P}_\theta}[g(x - X_t)]F'(\partial \mathbb{E}_{\mathbb{P}_\theta}[g(x - X_t)])$$

which is continuous at $t = 0$ by the continuity in $\mathbb{P}_{\theta'}$-probability of $t \mapsto X_t$ at $t = 0$. Finally, assume that $\mu_0$ has a positive density. Then $x \mapsto g * \mu_0(x)F'(\partial g * \mu_0(x))$ is non-identically vanishing since $g * \mu_0$ is positive and $F'$ is non-vanishing on $(0, \infty)$. We conclude by Proposition \[15\].

5. Proof of the main results

5.1. Preliminaries: couplings. For technical reasons, we will need certain couplings on the canonical space $(\mathbb{C}^N, \mathcal{F}^N)$. We now fix $\theta \in \Theta$, and introduce, for every $\theta' \in \Theta$, the following two processes

$$X^{(N), \theta'} = (X_i^{1, \theta'}, \ldots, X_i^{N, \theta'})_{t \in [0, T]}$$

and

$$\overline{X}^{(N), \theta'} = (\overline{X}_i^{1, \theta'}, \ldots, \overline{X}_i^{N, \theta'})_{t \in [0, T]}$$

defined on $(\mathbb{C}^N, \mathcal{F}^N)$ by

$$X_i^{i, \theta'} = X_i^0 + \int_0^t b(\theta'; s, X_s^{i, \theta'}, \mu_i^{(N), \theta'}) ds + \int_0^t \sigma(s, X_s^{i, \theta'}) dB_s^{i, N, \theta},$$

with $\mu_i^{(N), \theta'} = N^{-1} \sum_{i=1}^N \delta_{X_i^{i, \theta'}}$ and

$$\overline{X}_i^{i, \theta'} = X_i^0 + \int_0^t b(\theta'; s, \overline{X}_s^{i, \theta'}, \mu_i^{\theta'}) ds + \int_0^t \sigma(s, \overline{X}_s^{i, \theta'}) dB_s^{i, N, \theta},$$

and

$$B^{i, N, \theta'}_t = \int_0^t c^{-1/2}(s, X_s^i)(dX_s^i - b(\theta, s, X_s^i, \mu_s^{(N)}) ds).$$

Note that $X^{(N), \theta'}$ and $\overline{X}^{(N), \theta'}$ actually depend on $\theta$ pathwise via $(B^{i, N, \theta'}_t)_{t \in [0, 1]}$ under $\mathbb{P}^N_\theta$ but not their laws! Indeed, the $(B^{i, N, \theta'}_t)_{t \in [0, 1]}$ are standard Brownian motions under $\mathbb{P}^N_{\theta'}$. For notational simplicity, we omit the dependence upon $\theta$ here. Otherwise, we write $X_{t, \theta'}^i$ or $\overline{X}_{t, \theta'}^i$. Thus $X^{(N), \theta'}$ and $\overline{X}^{(N), \theta'}$ are defined as functions of $X^{(N)}$ (as strong solutions of \[37\] and \[38\]) and have law $\mathbb{P}^N_\theta$ and $\mathbb{P}^N_{\theta'}$ under $\mathbb{P}^N_{\theta'}$. This is a convenient way to couple $\mathbb{P}^N_\theta$ and $\mathbb{P}^N_{\theta'}$ while still working with the canonical process under $\mathbb{P}^N_{\theta'}$. We write $\mu^{(N)}_t = N^{-1} \sum_{i=1}^N \delta_{X_{t, \theta'}^i}$ for the empirical measure of the canonical process. We also introduce

$$\overline{\mu}^{(N), \theta'}_t = N^{-1} \sum_{i=1}^N \delta_{\overline{X}_{t, \theta'}^i}.$$
and write $\mu_i^{(N),\vartheta,\vartheta}$ and $\overline{\mu}_i^{(N),\vartheta,\vartheta}$ whenever we want to emphasise that the coupling is constructed with the processes $(B_i^{t,N,\vartheta})_{t \in [0,T]}$ for $1 \leq i \leq N$. We have the following approximation results:

**Lemma 20.** For every $\vartheta, \vartheta' \in \Theta$ and every $r \geq 1$, we have

$$
\sup_{t \in [0,T]} E_{\vartheta,\vartheta'} [W_1(\mu_t^{(N),\vartheta,\vartheta}, \mu_t^{(N),\vartheta,\vartheta})] \leq \sup_{t \in [0,T]} E_{\vartheta,\vartheta'} [N^{-1} \sum_{i=1}^N |X_t^i,\vartheta - X_t^i,\vartheta'|] \leq C|\vartheta' - \vartheta|^r.
$$

(40) \sup_{t \in [0,T]} E_{\vartheta,\vartheta'} [W_1(\overline{\mu}_t^{(N),\vartheta,\vartheta}, \overline{\mu}_t^{(N),\vartheta,\vartheta})] \leq \sup_{t \in [0,T]} E_{\vartheta,\vartheta'} [N^{-1} \sum_{i=1}^N |\overline{X}_t^i,\vartheta - \overline{X}_t^i,\vartheta'|] \leq C|\vartheta' - \vartheta|^r.

There exists $\delta > 0$ such that for every $r \geq 1$:

$$
\sup_{t \in [0,T], \vartheta \in \Theta} E_{\vartheta,t} [W_1(\mu_t^{(N),\vartheta,\vartheta})] \leq CN^{-\delta r},
$$

(42) \sup_{t \in [0,T], \vartheta \in \Theta} N^{-1} \sum_{i=1}^N E_{\vartheta,t} [|X_t^i - \overline{X}_t^i,\vartheta|^r] \leq CN^{-\delta r}

as $N \to \infty$.

The proof is given in Appendix 7.3.

5.2. **Proof of Theorem 17** We prove a slightly stronger result, namely a uniform type LAN condition, following Chapter III of [24]. Let $(u_N)_{N \geq 1}$ be a sequence of $\mathbb{R}^p$ such that $u_N \to u$ and $(\vartheta_N)_{N \geq 1}$ a sequence of $\Theta$ such that $\vartheta_N + (N \mathbb{I}_G(\vartheta_N))^{-1/2}u_N \to \vartheta$ for large enough $N$ and such that $\vartheta_N \to \vartheta$, for some $\vartheta$ such that $\mathbb{I}_G(\vartheta) > 0$. We claim that

$$
\zeta_N(\vartheta_N; u_N) = \log \frac{dE_{\vartheta_N + (N \mathbb{I}_G(\vartheta_N))^{-1/2}u_N}}{dP^N_\vartheta} = u^\top \Gamma_N - \frac{1}{2} |u|^2 + r_N(\vartheta_N, u_N),
$$

where $\Gamma_N \to \mathcal{N}(0, \text{Id}_{\mathbb{R}^p})$ in distribution under $P^N_\vartheta$ and $r_N(\vartheta_N, u_N) \to 0$ in $P^N_\vartheta$-probability. Clearly, (45) implies (22) and thus Theorem 17. Note that since $\mathbb{I}_G(\vartheta) > 0$ we have that $\mathbb{I}_G(\vartheta_N)$ is invertible for large enough $N$, thanks to the continuity of the mapping $\vartheta \to \mathbb{I}_G(\vartheta)$, recall Proposition 11. The asymptotic expansion (45) is therefore meaningful for large enough $N$.

We further write $\mathbb{I}(\vartheta)$ for $\mathbb{I}_G(\vartheta)$.

**Step 1. (Preliminary expansion.)** We have

$$
\zeta_N(\vartheta_N; u_N) = \sum_{i=1}^N \int_0^T \left((c^{-1/2}b)(\vartheta_N + (N \mathbb{I}(\vartheta_N))^{-1/2}u_N; t, X_t^i; \mu_t^N) - (c^{-1/2}b)(\vartheta; t, X_t^i; \mu_t^N)\right)^\top dB_t^i,\vartheta_N
$$

$$
- \frac{1}{2} \sum_{i=1}^N \int_0^T \left|(c^{-1/2}b)(\vartheta_N + (N \mathbb{I}(\vartheta_N))^{-1/2}u_N; t, X_t^i; \mu_t^N) - (c^{-1/2}b)(\vartheta; t, X_t^i; \mu_t^N)\right|^2 dt,
$$

where the $B_t^i,\vartheta_N = \int_0^t c^{-1/2}(s, X_s^i)(dX_s^i - b(\vartheta_N; s, X_s^i; \mu_s^N))ds$, $1 \leq i \leq N$ are independent Brownian motions on $\mathbb{R}^d$ under $P^N_\vartheta$. A first-order Taylor’s expansion therefore yields the representation,

$$
\zeta_N(\vartheta_N; u_N) = u_N^\top (\mathbb{I}(\vartheta_N))^{-1/2} \Delta_N,\vartheta_N(u_N) - \frac{1}{2} u_N^\top (\mathbb{I}(\vartheta_N))^{-1/2} \mathbb{I}_{N,\vartheta_N}(u_N)(\vartheta_N)^{-1/2} u_N
$$
where
\[
\Delta_{N,\vartheta_N}(u) = N^{-1/2} \sum_{i=1}^{N} \sum_{j=1}^{d} \int_{0}^{T} \nabla_{\vartheta} (c^{-1/2}b)^j ([\vartheta_N, \vartheta_N + (N\mathbb{I}(\vartheta_N))^{-1/2}u]; t, X_t^i, \mu_t^N) d(B_t^{i,N,\vartheta_N})^j
\]
and
\[
\tilde{\mathbb{I}}_{N,\vartheta_N}(u) = N^{-1} \sum_{i=1}^{N} \sum_{j=1}^{d} \int_{0}^{T} \nabla_{\vartheta} (c^{-1/2}b)^j ([\vartheta_N, \vartheta_N + (N\mathbb{I}(\vartheta_N))^{-1/2}u]; t, X_t^i, \mu_t^N) \\
\times (\nabla_{\vartheta} (c^{-1/2}b)^j ([\vartheta_N, \vartheta_N + (N\mathbb{I}(\vartheta_N))^{-1/2}u]; t, X_t^i, \mu_t^N)) \; dt,
\]
with the notation \( \phi((\vartheta, \vartheta')) = \int_{0}^{1} \phi(\vartheta + \lambda(\vartheta' - \vartheta)) d\lambda \) that we introduced before. We rewrite the above expansion as
\[
\xi_N(\vartheta_N; u_N) = u_N^\top (\mathbb{I}(\vartheta_N)^{-1/2})^\top \Delta_{N,\vartheta_N}(0) - \frac{1}{2} |u|^2
\]
\[
+ u_N^\top (\mathbb{I}(\vartheta_N)^{-1/2})^\top \Delta_{N,\vartheta_N}(u_N) - u_N^\top (\mathbb{I}(\vartheta_N)^{-1/2})^\top \Delta_{N,\vartheta_N}(0)
\]
\[
- \frac{1}{2} (u_N^\top (\mathbb{I}(\vartheta_N)^{-1/2})^\top \tilde{\mathbb{I}}_{N,\vartheta_N}(u_N))\mathbb{I}(\vartheta_N)^{-1/2}u_N - |u|^2
\]
and thus (44) follows from
\[
(\mathbb{I}(\vartheta_N)^{-1/2})^\top \Delta_{N,\vartheta_N}(0) \to N(0, \text{Id}_{\mathbb{R}^p})
\]
under \( \mathbb{P}_{\vartheta_N}^N \) in distribution together with the convergence to 0 of the last two components.

**Step 2. (Convergence of the Gaussian part.)** We prove (44) or equivalently, the convergence
\[
\xi^\top (\mathbb{I}(\vartheta_N)^{-1/2})^\top \Delta_{N,\vartheta_N}(0) = \sum_{q,q' = 1}^{d} \xi_q (\mathbb{I}(\vartheta_N)^{-1/2}) q^\top q' \Delta_{N,\vartheta_N}(0) \to N(0, |\xi|^2)
\]
in distribution under \( \mathbb{P}_{\vartheta_N}^N \) for every \( \xi \in \mathbb{R}^p \). We apply a classical semimartingale convergence result, following for instance Jacod and Shiryaev [26] (Corollary 3.24). For \( t \in [0, T] \), the process
\[
\Delta_{N,\vartheta_N}(0)_t = N^{-1/2} \sum_{i=1}^{N} \sum_{j=1}^{d} \int_{0}^{t} \nabla_{\vartheta} (c^{-1/2}b)^j ([\vartheta_N, \vartheta_N + (N\mathbb{I}(\vartheta_N))^{-1/2}u]; s, X_s^i, \mu_s^N) d\lambda d(B_s^{i,N,\vartheta_N})^j
\]
is a continuous local martingale under \( \mathbb{P}_{\vartheta_N}^N \) and so is \( \xi^\top (\mathbb{I}(\vartheta_N)^{-1/2})^\top \Delta_{N,\vartheta_N}(0)_t \). It coincides with \( \xi^\top (\mathbb{I}(\vartheta_N)^{-1/2})^\top \Delta_{N,\vartheta_N}(0) \) at \( t = T \) and has predictable compensator
\[
\begin{align*}
\left\langle \xi^\top (\mathbb{I}(\vartheta_N)^{-1/2})^\top \Delta_{N,\vartheta_N}(0)_t \right\rangle_t = \sum_{q,q' = 1}^{d} \xi_q q^\top q' (\mathbb{I}(\vartheta_N)^{-1/2}) q^\top q' \Delta_{N,\vartheta_N}(0)_t \\
N^{-1} \sum_{i=1}^{N} \sum_{j=1}^{d} \int_{0}^{t} \partial_{\vartheta q^i} (c^{-1/2}b)^j ([\vartheta_N; s, X_s^i, \mu_s^N]) \partial_{\vartheta q'^i} (c^{-1/2}b)^j ([\vartheta_N; s, X_s^i, \mu_s^N]) ds,
\end{align*}
\]
that converges to
\[
\sum_{q,q' = 1}^{d} \xi_q q^\top q' \mathbb{I}(\vartheta)^{-1/2} q^\top q' \Delta_{N,\vartheta_N}(0)_t.
\]
in $\mathbb{P}_{\theta_N}^T$-probability, but that last quantity is exactly $\|\xi\|^2$ at $t = T$, which proves (44). As for the last convergence in probability, it is a simple consequence of the continuity of $\vartheta \mapsto \mathbb{I}(\vartheta)$, see Proposition 11 and the following lemma

**Lemma 21.** Let $\beta > 0$ and $\phi : \Theta \times [0,T] \times \mathbb{R}^d \times \mathcal{P}_\beta \to \mathbb{R}$ be such that for some $C, \alpha > 0$, we have

$$
\sup_{t \in [0,T]} |\phi(\vartheta', t, x', \nu') - \phi(\vartheta; t, x, \nu)| \leq C(|\vartheta' - \vartheta| + |x' - x| + \mathcal{W}_1(\nu', \nu))(1 + |x|^{\alpha} + |x'|^{\alpha} + \mathcal{M}_\beta(\nu) + \mathcal{M}_\beta(\nu')).
$$

Then, there exists $0 < \delta \leq 1/2$ such that for every $t \in [0,T]$ and every $m > 0$, we have

$$
\mathbb{E}_{\mathbb{P}_{\vartheta_N}^T} \left[ \left| N^{-1} \sum_{i=1}^N \int_0^t \phi(\vartheta_N; s, X_{i,s}, \mu_s^{(N)}) ds - \int_0^t \int_{\mathbb{R}^d} \phi(\vartheta; s, x, \mu_s^0)(dx) ds \right|^m \right] \leq C(|\vartheta_N - \vartheta|^m + N^{-\delta m}).
$$

We apply Lemma 21 to $\phi(\vartheta; s, x, \nu) = \partial_{\vartheta s} (c^{-1/2}b^j)(\vartheta; s, x, \nu) \partial_{\vartheta s} (c^{-1/2}b^j)(\vartheta; s, x, \nu)$, thanks to Assumption 4. The proof is given in Appendix 7.4.

**Step 3. (Convergence of the remainder terms.)** We first prove

(45) \(u_N^\beta(\mathbb{P}(\vartheta_N)^{-1/2})^\top \Delta_{N, \vartheta_N}(u_N) - u^\top (\mathbb{E}(\vartheta_N)^{-1/2})^\top \Delta_{N, \vartheta_N}(0) \to 0\)

in $\mathbb{P}_{\theta_N}^T$-probability. Since $\mathbb{P}(\vartheta_N)^{-1/2}$ is well defined for large enough $N$ and converges to $\mathbb{P}(\vartheta)^{-1/2}$ and $u_N \to u$, it is sufficient to prove $\Delta_{N, \vartheta_N}(u_N) - \Delta_{N, \vartheta_N}(0) \to 0$ in $\mathbb{P}_{\theta_N}^T$-probability. Introduce the process

$$
G_{N,t}^r(\vartheta, u)_t = \int_0^1 (\partial_{\vartheta r} (c^{-1/2}b^j)(\vartheta + \lambda(\mathbb{P}(\vartheta))^{-1/2}u; t, X_{i,t}^\beta, \mu_t^{(N)}) - \partial_{\vartheta r} (c^{-1/2}b^j)(\vartheta; t, X_{i,t}^\beta, \mu_t^{(N)})) d\lambda
$$

for $1 \leq r \leq p$ and $t \in [0,T]$. By Itô’s isometry

$$
\mathbb{E}_{\mathbb{P}_{\theta_N}^T} \left[ |\Delta_{N, \vartheta_N}(u_N) - \Delta_{N, \vartheta_N}(0)|^2 \right] = \sum_{r=1}^p \mathbb{E}_{\mathbb{P}_{\theta_N}^T} \left[ |N^{-1/2} \sum_{i=1}^N \int_0^T G_{N,t}^r(\vartheta_N, u_N)_t dB_{i,t}^r, \vartheta_N(0)|^2 \right]
$$

$$
= \sum_{r=1}^p N^{-1} \sum_{i=1}^N \int_0^T \mathbb{E}_{\mathbb{P}_{\theta_N}^T} \left[ |G_{N,t}^r(\vartheta_N, u_N)_t|^2 \right] dt.
$$

Moreover

$$
|G_{N,t}^r(\vartheta_N, u_N)_t|^2 \leq \sum_{j=1}^d \sup_{\vartheta \in \Theta} |(\nabla_{\vartheta j} \partial_{\vartheta r} (c^{-1/2}b^j)(\vartheta; t, X_{i,t}^\beta, \mu_t^{(N)})\lambda(\mathbb{P}(\vartheta))^{-1/2}u_N)|^2 \leq CN^{-1}(1 + |X_{i,t}^\beta|^{2r_1} + \mathcal{M}_\beta(\mu_t^{(N)})^2),
$$

for large enough $N$, thanks to Assumption 4. We conclude

$$
\sup_{t \in [0,T]} \mathbb{E}_{\mathbb{P}_{\theta_N}^T} \left[ |G_{N,t}^r(\vartheta_N, u_N)_t|^2 \right] \leq CN^{-1}
$$

for large enough $N$ by Lemma 7 and (45) follows.

The convergence of the second remainder term is a simple consequence of $\mathbb{I}_{N, \vartheta_N}(u_N) \to \mathbb{I}(\vartheta)$ thanks to Lemma 21 together with the continuity of $\vartheta \mapsto \mathbb{I}(\vartheta)$ and Proposition 11. The Proof of
Theorem [17] is complete for the experiment $E^N$.

**Step 4. (The case of the experiment $G^{\otimes N}$.)** We now easily extend the previous results to the experiment $G^{\otimes N}$. Since it is a product of the same experiment $G$, it is tempting to use classical criterions for IID data. However, from a simple glance at the structure of the previous computations, it suffices to retrace Steps 1 to 3 replacing $\mu_i^{(N)}$ by $\mu_i^{\theta_N}$ and $P_{\theta_N}^N$ by $P_{\theta_N}^{G^{\otimes N}}$ that actually turn out to be simpler. We omit the details.

### 5.3. **Proof of Theorem [19]**

We plan to apply the classical theory of Ibragimov–Hasminski, and more specifically Theorem III.1.1 of [24]. We introduce the notation

$$Z_N(\theta; u) = \frac{dP_{\theta}^N \left( \omega_{\theta}((N\|_2(\theta))^{-1/2}u) \right)}{dP_{\theta}^N}.$$  

We first establish two key regularity properties of the likelihood process.

**Step 1. (A regularity property for the likelihood process.)** Here, we prove that for any $r \geq 2$

$$\mathbb{E}_{\theta_N} \left[ \left| Z_N(\theta; u)^{1/r} - Z_N(\theta; v)^{1/r} \right| \right] \leq C(1 + \kappa^r)|u - v|^r,$$

for some positive $\gamma, \gamma'$, uniformly in $u, v$ such that $Z_N(\theta; u)$ and $Z_N(\theta; v)$ are well-defined and $|u|, |v|$ are bounded by $\kappa > 0$. Pick any $r \geq 2$. By a first-order expansion

$$\mathbb{E}_{\theta_N} \left[ \left| Z_N(\theta; u)^{1/r} - Z_N(\theta; v)^{1/r} \right| \right] \leq |u - v|^r \mathbb{E}_{\theta_N} \left[ \int_0^1 \nabla u(Z_N)^{1/r}(\theta; u + \lambda(v - u))d\lambda \right]^r$$

(47)

$$\leq C|u - v|^r \int_0^1 \mathbb{E}_{\theta_N} \left[ |\partial u_q((Z_N)^{1/r})(\theta; u + \lambda(v - u))| \right]d\lambda.$$

Define, for $t \in [0, T]$, the random process

$$\phi_t(\theta; X^i, \mu^{(N)}) = \exp \left( \int_0^t (c^{-1}b)(\theta; s, X_s^i, \mu_s^{(N)})dX_s^i - \frac{1}{2} \int_0^t (c^{-1/2}b)(\theta; s, X_s^i, \mu_s^{(N)})^2 ds \right).$$

Since

$$Z_N(\theta; u) = \prod_{i=1}^N \phi_{1/r}^{(\mu^{(N))}}(\theta + (N\|_2(\theta))^{-1/2}u, X^i, \mu^{(N))}),$$

we have

$$\partial u_q((Z_N)^{1/r})(\theta; u) = \sum_{i=1}^N \partial u_q(\phi_{1/r}^{(\mu^{(N))}}(\theta + (N\|_2(\theta))^{-1/2}u, X^i, \mu^{(N))})) \times \prod_{i' \neq i} \phi_{1/r}^{(\mu^{(N))}}(\theta + (N\|_2(\theta))^{-1/2}u, X^{i'}, \mu^{(N))})$$

$$= \sum_{i=1}^N \nabla q(\phi_{1/r}^{(\mu^{(N))}}) \times \prod_{i' \neq i} \phi_{1/r}^{(\mu^{(N))}}(\theta + (N\|_2(\theta))^{-1/2}u, X^i, \mu^{(N))})$$

$$= (Z_N)^{1/r}(\theta; u) \left( (N\|_2(\theta))^{-1/2} \right)_q \times$$
\[
\sum_{i=1}^{N} \nabla_{\vartheta} \left( \log(\phi_{T}^{1/r}) \right) \left( \vartheta + (N\bar{I}_{0}(\vartheta))^{-1/2}u, X^{i}, \mu^{(N)} \right).
\]

Interpreting \(Z_{\vartheta}(\vartheta; u)\) as a Radon-Nikodym derivative entails
\[
\mathbb{E}_{\vartheta}^{P_{\vartheta}^{N}} \left[ \left( \partial_{u_{q}}((Z_{\vartheta})^{1/r})(\vartheta; u) \right) \right]^{*}
\]
\[
(48) \quad = \left( (\bar{I}_{0}(\vartheta))^{-1/2} \right)_{u}^{*} \mathbb{E}_{\vartheta}^{P_{\vartheta}^{N}}_{\vartheta + (N\bar{I}_{0}(\vartheta))^{-1/2}u} \left[ N^{-1/2} \sum_{i=1}^{N} \nabla_{\vartheta} \left( \log(\phi_{t}^{1/r}) \right) \left( \vartheta + (N\bar{I}_{0}(\vartheta))^{-1/2}u \right) \right]^{*}
\]
by a change of probability between \(P_{\vartheta}^{N}\) and \(P_{\vartheta + (N\bar{I}_{0}(\vartheta))^{-1/2}u}^{N}\). Next, by definition of \(\phi_{t}\), we have, for every \(\vartheta' \in \Theta\)
\[
\partial_{\vartheta'}^{(1/r)}(\log(\phi_{t}^{1/r}))(\vartheta'; X^{i}, \mu^{(N)}) = \frac{1}{r} \int_{0}^{T} \partial_{\vartheta'}^{(1/r)}(c^{-1}b)(\vartheta'; t; X^{i}_{t}, \mu^{(N)}_{t}) dX^{i}_{t}
\]
\[
- \frac{1}{2r} \int_{0}^{T} 2\partial_{\vartheta'}^{(1/r)}(c^{-1}b)(\vartheta'; t; X^{i}_{t}, \mu^{(N)}_{t})^{T} (c^{-1/2}b)(\vartheta'; t; X^{i}_{t}, \mu^{(N)}_{t}) dt
\]
\[
= \frac{1}{r} \int_{0}^{T} \partial_{\vartheta'}^{(1/r)}(c^{-1}b)(\vartheta'; t; X^{i}_{t}, \mu^{(N)}_{t}) dB^{i,N,\vartheta'}_{t},
\]
where the \((B^{i,N,\vartheta'}_{t})_{t \in [0,T]} = (\int_{0}^{t} c^{-1/2}(s, X^{i}_{s}) (dX^{i}_{s} - b(\vartheta'; s, X^{i}_{s}, \mu^{(N)}_{s}) ds)_{t \in [0,T]}\) \(1 \leq i \leq N\) are independent Brownian motions on \(\mathbb{R}^{d}\) under \(P_{\vartheta'}^{N}\). Plugging-in this representation in \((48)\) at \(\vartheta' = \vartheta + (N\bar{I}_{0}(\vartheta))^{-1/2}u\), we infer
\[
\mathbb{E}_{\vartheta}^{P_{\vartheta}^{N}} \left[ \left( \partial_{u_{q}}((Z_{\vartheta})^{1/r})(\vartheta; u) \right) \right]^{*}
\]
\[
\leq C \mathbb{E}_{\vartheta}^{P_{\vartheta}^{N}} \left[ N^{-1/2} \sum_{i=1}^{N} \int_{0}^{T} \partial_{\vartheta'}^{(1/r)}(c^{-1}b)(\vartheta'; t; X^{i}_{t}, \mu^{(N)}_{t}) dB^{i,N,\vartheta'}_{t} \right]^{*}
\]
\[
\leq C \mathbb{E}_{\vartheta}^{P_{\vartheta}^{N}} \left[ N^{-1} \sum_{i=1}^{N} \int_{0}^{T} \left| \partial_{\vartheta'}^{(1/r)}(c^{-1}b)(\vartheta'; t; X^{i}_{t}, \mu^{(N)}_{t}) \right|^{2} dt \right]^{r/2}
\]
\[
\leq CN^{-1} \sum_{i=1}^{N} \int_{0}^{T} \left( 1 + \mathbb{E}_{\vartheta}^{P_{\vartheta}^{N}} \left[ |X^{i}_{t}|^{r+1} \right] \right) + m_{r_{2}}(\mu^{(N)}) dt
\]
\[
\leq C(1 + \sum_{i=1}^{N} \int_{0}^{T} \mathbb{E}_{\vartheta}^{P_{\vartheta}^{N}} \left[ |X^{i}_{t}|^{r \max(r_{1}, r_{2})} \right] dt),
\]
where we successively used the Burkholder-Davis-Gundy, Assumption 3, and the fact that \(r \geq 2\).

Now, we claim that with \(\vartheta' = \vartheta + (N\bar{I}_{0}(\vartheta))^{-1/2}(u + \lambda(v - u))\), we have
\[
\int_{0}^{T} \mathbb{E}_{\vartheta}^{P_{\vartheta}^{N}} \left[ |X^{i}_{t}|^{r \max(r_{1}, r_{2})} \right] dt \leq C(1 + \kappa^{r})
\]
for some \(\gamma > 0\), uniformly in \(|u|, |v|\) bounded by \(\kappa\) and where \(C\) depends on \(\Theta\) only. Indeed, keeping up with the abbreviation \(\vartheta'\), we have
\[
\mathbb{E}_{\vartheta'} \left[ |X^{i}_{t}|^{r \max(r_{1}, r_{2})} \right] \leq C \mathbb{E}_{\vartheta'} \left[ |X^{i}_{t} - X^{i,\vartheta'}_{t}|^{r \max(r_{1}, r_{2})} \right] + \mathbb{E}_{\vartheta'} \left[ |X^{i,\vartheta'}_{t}|^{r \max(r_{1}, r_{2})} \right].
\]

By (39) of Lemma 20
\[
\mathbb{E}_{\vartheta'} \left[ |X^{i}_{t} - X^{i,\vartheta'}_{t}|^{r \max(r_{1}, r_{2})} \right] \leq C |\vartheta - \vartheta'|^{r \max(r_{1}, r_{2})}
\]
and the second term that only depends on \( \vartheta \) by coupling is uniformly bounded by Lemma 7. The estimate (49) follows. Going back to (47), we conclude

\[
\mathbb{E}^N\left[ |Z_N(\vartheta; u) - u|^{1/r} \right] \leq C|u - v|^r (1 + \kappa^r \max(r_1, r_2))
\]

and (46) is established with \( \gamma = r \max(r_1, r_2) \) and \( \gamma' = r \).

Step 2. Here we prove a moment bound for the likelihood ratio process, namely, for every \( r > 0 \)

\[
(50) \quad \mathbb{E}^N\left[ |Z_N(\vartheta; u)\right] \leq C|u|^{-r},
\]

uniformly in \( \vartheta \in \Theta \) and \( u = (N_{1}\vartheta)\)\((\vartheta' - \vartheta) \) with \( \vartheta' \in \Theta \). Introducing for \( t \in [0, T] \) the \( \mathbb{P}_\vartheta \)-martingale

\[
M_t^N(\vartheta; u) = \sum_{i=1}^{N_t} \int_0^t \left( (c^{-1/2}b)(\vartheta + (N_{1}\vartheta)(\vartheta - \vartheta); s, X^i_s, \mu^i_s(N)) - (c^{-1/2}b)(\vartheta; s, X^i_s, \mu^i_s(N)) \right) dB^i_s,\]

we have

\[
Z_N(\vartheta; u) = \exp \left( M_T^N(\vartheta; u) - \frac{1}{2} \langle M^N(\vartheta; u) \rangle_T \right).
\]

It follows that

\[
\mathbb{E}^N\left[ Z_N(\vartheta; u)\right] = \mathbb{E}^N\left[ \exp \left( -\frac{1}{2} M_T^N(\vartheta; u) - \frac{3}{16} \langle M^N(\vartheta; u) \rangle_T \right) \right]
\]

\[
\leq \mathbb{E}^N\left[ \exp \left( -\frac{1}{2} M_T^N(\vartheta; u) - \frac{3}{16} \langle M^N(\vartheta; u) \rangle_T \right) \right]^{\frac{1}{2}} \mathbb{E}^N\left[ \exp \left( -\frac{3}{16} \langle M^N(\vartheta; u) \rangle_T \right) \right]^{\frac{1}{2}}
\]

\[
\leq \mathbb{E}^N\left[ \exp \left( -\frac{3}{16} \langle M^N(\vartheta; u) \rangle_T \right) \right]^{1/3},
\]

thanks to Hölder’s inequality and the martingale property of \( \{ M_t^N(\vartheta; u) \}_{t \in [0, T]} \). With the help of the parametrisation \( u = (N_{1}\vartheta)\)\((\vartheta' - \vartheta) \), we rewrite \( \langle M^N(\vartheta; u) \rangle_T \) as

\[
\sum_{i=1}^{N} \int_0^T \left| (c^{-1/2}b)(\vartheta'; t, X^i_t, \mu^i_t(N)) - (c^{-1/2}b)(\vartheta; t, X^i_t, \mu^i_t(N)) \right|^2 dt
\]

\[
= \sum_{i=1}^{N} \sum_{j=1}^{d} \int_0^T \left( (N_{1}\vartheta)(\vartheta - \vartheta'; t, X^i_t, \mu^i_t(N)) \right)^2 dt
\]

\[
= u^\top \mathbb{I}_G(\vartheta')^{-1/2} \Sigma(N(\vartheta, \vartheta'; u) \mathbb{I}_G(\vartheta)^{-1/2} u,
\]

with

\[
\Sigma(N(\vartheta, \vartheta'; u) = N^{-1} \sum_{i=1}^{d} \sum_{j=1}^{d} \int_0^T \nabla \varphi(c^{-1/2}b)(\vartheta; t, X^i_t, \mu^i_t(N))(\nabla \varphi(c^{-1/2}b)(\vartheta; t, X^i_t, \mu^i_t(N)))^\top dt
\]

\[
\Sigma(\vartheta, \vartheta') = \sum_{j=1}^{d} \int_0^T \int_{\mathbb{R}^d} \nabla \varphi(c^{-1/2}b)(\vartheta; t, x, \mu^i_t(N))(\nabla \varphi(c^{-1/2}b)(\vartheta; t, x, \mu^i_t(N)))^\top \mu_t(dx) dt
\]

which converges by Lemma 21 to

\[
\Sigma(\vartheta, \vartheta') = \sum_{j=1}^{d} \int_0^T \int_{\mathbb{R}^d} \nabla \varphi(c^{-1/2}b)(\vartheta; t, x, \mu^i_t(N))(\nabla \varphi(c^{-1/2}b)(\vartheta; t, x, \mu^i_t(N)))^\top \mu_t(dx) dt
\]
in $\mathbb{P}_0^N$-probability. Abbreviating further $\hat{\Sigma}^N(\vartheta; u) = I_3(\vartheta)^{-1/2} \Sigma^N([\vartheta, \vartheta']; u) I_3(\vartheta)^{-1/2}$ and $\bar{\Sigma}(\vartheta; u) = I_3(\vartheta)^{-1/2} \Sigma([\vartheta, \vartheta']; u) I_3(\vartheta)^{-1/2}$, we have

$$
\mathbb{E}_{\mathbb{P}_0^N}[Z_N(\vartheta; u)]^{1/2} \leq \mathbb{E}_{\mathbb{P}_0^N}\left[\exp\left(-\frac{3}{16} u^\top \hat{\Sigma}^N(\vartheta; u) u\right)\right]^{1/3}
\leq \mathbb{P}_0^N\left(u^\top \hat{\Sigma}^N(\vartheta; u)u \leq \frac{1}{2} u^\top \bar{\Sigma}(\vartheta; u)u + \exp\left(-\frac{1}{32} u^\top \bar{\Sigma}(\vartheta; u)u\right)\right).
$$

(51)

The non-degeneracy assumption (recall Definition [14]) ensures $u^\top \bar{\Sigma}(\vartheta; u)u \geq \varepsilon |u|^2$ for some $\varepsilon > 0$ that does not depend on $\vartheta$ nor $u$ (but that depends on $\Theta$), hence the remainder term decays faster than any power of $|u|$. The first term in the right-hand side of (51) is bounded above by

$$
\mathbb{P}_0^N\left(u^\top \hat{\Sigma}^N(\vartheta; u) - \bar{\Sigma}(\vartheta; u)\right)\leq \mathbb{P}_0^N\left(|u^\top \Sigma^N(\vartheta; u) - \bar{\Sigma}(\vartheta; u)| \leq \frac{1}{2} \varepsilon |u|^2\right)^{1/3}
\leq C|u|^{-\frac{2m}{3}} \mathbb{E}_{\mathbb{P}_0^N}\left[|u^\top \Sigma^N(\vartheta; u) - \bar{\Sigma}(\vartheta; u)|^m\right]^{1/3}
$$

for every $m > 0$ by Markov’s inequality and where we used the non-degeneracy assumption again. For $1 \leq l, l' \leq p$, introduce

$$
\phi_{l,l'}([\vartheta, \vartheta']; t, x, \nu) = (\partial_{l'}(e^{1/2l})([\vartheta, \vartheta']; t, x, \nu)) \partial_{l'}(e^{1/2l})([\vartheta, \vartheta']; t, x, \nu),
$$

so that $(\Sigma^N(\vartheta; u) - \Sigma(\vartheta; u))_{l,l'}$ is simply

$$
N^{-1} \sum_{i=1}^N \int_0^T \phi_{l,l'}([\vartheta, \vartheta']; t, X_{i,t}^{(N)}) dt - \int_0^T \int_{\mathbb{R}^d} \phi_{l,l'}([\vartheta, \vartheta']; t, x, \mu_t^\vartheta) \mu_t^\vartheta(dx) dt.
$$

By Lemma [21] we derive

$$
\mathbb{E}_{\mathbb{P}_0^N}\left[|\Sigma^N(\vartheta; u) - \Sigma(\vartheta; u)|^{m}\right] \leq C N^{-\delta m},
$$

therefore

$$
\mathbb{E}_{\mathbb{P}_0^N}\left[|u^\top \hat{\Sigma}^N(\vartheta; u) - \bar{\Sigma}(\vartheta; u)|^m\right]^{1/3} \leq C|u|^{2m/3} N^{-\delta m/3},
$$

and finally

$$
\mathbb{P}_0^N\left(|u^\top \hat{\Sigma}^N(\vartheta; u) - \bar{\Sigma}(\vartheta; u)| \geq \frac{1}{2} u^\top \bar{\Sigma}(\vartheta; u)u \right)^{1/3} \leq C N^{-\delta m/3}
$$

(52)

Pick $r \geq 1$. Combining (51) and (52), we infer

$$
|u|^r \mathbb{E}_{\mathbb{P}_0^N}\left[Z_N(\vartheta; u)^{1/2}\right] \leq C|u|^r N^{-\delta m/3} + |u|^r \exp\left(-\frac{\varepsilon}{32} |u|^2\right).
$$

For $u = (NI_3(\vartheta))^{1/2}(\vartheta' - \vartheta)$ with $\vartheta' \in \Theta$, we have $|u| \leq C N^{1/2}$. The first term in the right-hand side of the previous estimate is thus bounded as soon a $m \geq 3r/(2\delta)$. The second term is bounded uniformly in $|u|$. We thus have established (50).

**Step 3.** We are now ready to apply Theorem III.1.1 of [24] and gather several properties of the maximum likelihood estimator. Note that the continuity of the likelihood function $\vartheta \mapsto \mathcal{L}^N(\vartheta; X^{(N)})$ and the compactness of $\Theta$ ensures that a solution $\hat{\vartheta}^{(N)}$ to (23) exists.

The uniform LAN condition given in the proof of Theorem [17] is the Condition N1 of Chapter III [24]. The non-degeneracy assumption (according to Definition [14]) is related to the uniform use of Condition N2 of [24]. Step 1 and Step 2 are respectively Condition N3 and N4 of [24].
We may thus apply Theorem III.1.1 of [24] and we readily obtain Statement (i) of Theorem [19]. Statement (ii) is a consequence of Corollary III.1.1 of [24] while Statement (iii) is a consequence of Theorem III.1.3 of [24].

The proof of Theorem [19] is complete.

6. Remaining proofs

6.1. Proof of Proposition [9] Anticipating the proof of Lemma [20] we prove a slightly stronger result, namely

\[(53) \quad \left( \mathbb{E}_{\mathcal{P}^N} \left[ \left| \bar{X}^{t, \theta}_{t} - \bar{X}^{t, \theta'}_{t} \right|^r \right] \right)^{1/r} \leq C |\theta - \theta'| \]

for \( r \geq 1. \) Indeed, Proposition [9] is then a consequence of

\[ \mathcal{W}_1(\mu^t_0, \mu^t_0') \leq \mathbb{E}_{\mathcal{P}^N} \left[ \left| \bar{X}^{t, \theta}_{t} - \bar{X}^{t, \theta'}_{t} \right| \right] \leq \left( \mathbb{E}_{\mathcal{P}^N} \left[ \left| \bar{X}^{t, \theta}_{t} - \bar{X}^{t, \theta'}_{t} \right|^{2r} \right] \right)^{1/2r} \]

for any \( r \geq 1. \) From \( \bar{X}^{t, \theta}_{t} = \bar{X}^{t, \theta'}_{t}, \) we have

\[ \bar{X}^{t, \theta}_{t} - \bar{X}^{t, \theta'}_{t} = \int_0^t \left( b(\theta; s, \bar{X}^{s, \theta}_{s}, \mu^s_0) - b(\theta'; s, \bar{X}^{s, \theta'}_{s}, \mu^s_0') \right) ds + \int_0^t \left( \sigma(s, \bar{X}^{s, \theta}_{s}) - \sigma(s, \bar{X}^{s, \theta'}_{s}) \right) d\mathbb{B}^{i,N, \theta}_s. \]

Using Assumption [3] and the Burkholder-Davis-Gundy inequality, we infer

\[ \mathbb{E}_{\mathcal{P}^N} \left[ \int_0^t \left( \sigma(s, \bar{X}^{s, \theta}_{s}) - \sigma(s, \bar{X}^{s, \theta'}_{s}) \right) d\mathbb{B}^{i,N, \theta}_s \right]^{2r} \leq C \mathbb{E}_{\mathcal{P}^N} \left[ \int_0^t \left( \left| \bar{X}^{s, \theta}_{s} - \bar{X}^{s, \theta'}_{s} \right|^2 ds \right)^r \right] \]

since \( r \geq 1. \) Thanks to the smoothness properties of \( b \) granted by Assumptions [3] and [4] and incorporating the previous estimate, we obtain

\[ \mathbb{E}_{\mathcal{P}^N} \left[ \left| \bar{X}^{t, \theta}_{t} - \bar{X}^{t, \theta'}_{t} \right|^{2r} \right] \leq C \int_0^t \mathbb{E}_{\mathcal{P}^N} \left[ |\theta - \theta'|^{2r} (1 + \left| \bar{X}^{s, \theta}_{s} - \bar{X}^{s, \theta'}_{s} \right|^{2r} + m_{r_2}(\mu^s_0)^{2r} + \left| \bar{X}^{s, \theta}_{s} - \bar{X}^{s, \theta'}_{s} \right|^{2r} + \mathcal{W}_1(\mu^s_0, \mu^s_0'))^{2r}] ds \]

\[ \leq C (|\theta - \theta'|^{2r} + \int_0^t \mathbb{E}_{\mathcal{P}^N} \left[ \left| \bar{X}^{s, \theta}_{s} - \bar{X}^{s, \theta'}_{s} \right|^{2r} + \mathcal{W}_1(\mu^s_0, \mu^s_0')^{2r}] ds \right) \]

where we used that \( \mathbb{E}_{\mathcal{P}^N} \left[ \left| \bar{X}^{t, \theta}_{t} \right|^{2r} \right] = m_r(\mu^t_0) \) is bounded uniformly in \( t \in [0, T] \) and \( \theta \in \Theta \) for all values of \( r' \geq 1 \) by Lemma [8]. We obtain \( 53 \) for \( 2r \) by Grönwall’s lemma, hence for every \( r \geq 1 \) by Cauchy-Schwarz’s inequality. The proposition follows.

6.2. Proof of Proposition [10] By Girsanov’s theorem,

\[ \mathbb{E}_{\mathcal{P}^N} \left[ \log \frac{d\mathbb{P}_{\mathcal{P}^N}}{d\mathbb{P}_{\mathcal{P}_\theta}} \right] = \frac{1}{2} \sum_{i=1}^N \int_0^T \left[ b(\theta; t, X^i_t, \mu^i_1(N)) - b(\theta; t, X^i_t, \mu^i_0) \right]^2 dt \]

\[ = \frac{1}{2} \sum_{i=1}^N \int_0^T \mathbb{E}_{\mathcal{P}^N} \left[ \left| N^{-1} \sum_{j=1}^N \left( \tilde{b}(\theta; t, X^i_t, X^j_t) - \mathbb{E}_{\mathcal{P}_\theta} [\tilde{b}(\theta; t, \zeta, X^j_t)]_{\zeta=X^i_t} \right) \right|^2 \right] dt. \]
We plan to use the following decomposition
\[
N^{-1} \sum_{j=1}^{N} \tilde{b}(\theta; t, X_i^j, X_i^j) - \mathbb{E}_{\mathcal{F}_t} \tilde{b}(\theta; t, \zeta, X_i^j)\big|_{\zeta=X_i^j} \\
= N^{-1} \tilde{b}(\theta; t, X_i^j, X_i^j) - \mathbb{E}_{\mathcal{F}_t} \tilde{b}(\theta; t, \zeta, X_i^j)\big|_{\zeta=X_i^j} \\
+ \frac{N-1}{N} \sum_{j=1, j \neq i}^{N} \left( \tilde{b}(\theta; t, X_i^j, X_i^j) - \mathbb{E}_{\mathcal{F}_t} \tilde{b}(\theta; t, X_i^j, X_i^j) \big|_{X_i^j} \right).
\]

Using the elementary inequality \((a+b)^2 \leq (1+\rho)a^2 + (1+\rho^{-1})b^2\) valid for every \(\rho > 0\), we obtain
\[
\mathbb{E}_{\mathcal{F}_t} \left[ \left| N^{-1} \sum_{j=1}^{N} \tilde{b}(\theta; t, X_i^j, X_i^j) - \mathbb{E}_{\mathcal{F}_t} \tilde{b}(\theta; t, \zeta, X_i^j)\big|_{\zeta=X_i^j} \right|^2 \right] \\
\leq (1+\rho^{-1})N^{-2} \mathbb{E}_{\mathcal{F}_t} \left[ \tilde{b}(\theta; t, X_i^j, X_i^j)^2 \right] + (1+\rho)N^{-1} \mathbb{E}_{\mathcal{F}_t} \left[ \tilde{b}(\theta; t, X_i^j, X_i^j)^2 \right],
\]
and therefore
\[
\limsup_{N \to \infty} \sup_{\theta \in \Theta} \frac{1}{2} \mathbb{E}_{\mathcal{F}_t} \left[ \log \frac{d\mathbb{P}^{\otimes N}_{\theta}}{d\mathbb{P}^{\otimes N}_\phi} \right] \leq \frac{1}{2} \mathbb{E}_{\mathcal{F}_t} \left[ \tilde{b}(\theta; t, x, y)^2 (\mu^{\phi}_t \otimes \mu^{\phi}_t) (dx, dy) dt \right].
\]

By assumption,
\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} \tilde{b}(\theta; t, x, y)^2 (\mu^{\phi}_t \otimes \mu^{\phi}_t) (dx, dy) \leq C (1 + \sup_{t \in [0, T]} (m_{r_1}(\mu^{\phi}_t) + m_{r_2}(\mu^{\phi}_t)))
\]
which is finite by Lemma 8. We thus obtain (15). In order to obtain (16), we simply apply Pinsker’s inequality:
\[
\limsup_{N \to \infty} \sup_{\theta \in \Theta} \left\| \mathbb{P}^{\otimes N}_{\theta} - \mathbb{P}^{\otimes N}_\phi \right\|_{TV} \leq \frac{1}{2} \limsup_{N \to \infty} \mathbb{E}_{\mathcal{F}_t} \left[ \log \frac{d\mathbb{P}^{\otimes N}_{\theta}}{d\mathbb{P}^{\otimes N}_\phi} \right] \\
\leq \frac{1}{2} \sup_{\theta \in \Theta} \int_{0}^{T} \int_{\mathbb{R}^d \times \mathbb{R}^d} \tilde{b}(\theta; t, x, y)^2 (\mu^{\phi}_t \otimes \mu^{\phi}_t) (dx, dy).
\]

The conclusion follows by taking \(\rho\) sufficiently small.

6.3. Proof of Proposition 11

Let
\[
(\phi_{\ell, t}(\vartheta; t, x, \nu))_{1 \leq \ell, t, \nu \leq p} = (\partial_{\theta_{\ell}}(c^{-1/2} b(\theta; t, x, \mu^\theta_t)) \partial_{\theta_{t'}}(c^{-1/2} b(\theta; t, x, \mu^\theta_t)))_{1 \leq \ell, t, \nu \leq p}.
\]

We have
\[
(\mathbb{I}_g(\vartheta))_{\ell, t'} - (\mathbb{I}_g(\vartheta'))_{\ell, t'} = \int_{0}^{T} \left( \mathbb{E}_{\mathcal{F}_t} \tilde{b}(\theta; t, X_i^j, \mu^\theta_t) - \mathbb{E}_{\mathcal{F}_t} \tilde{b}(\theta'; t, X_i^j, \mu^{\theta'}_t) \right) dt \\
= \int_{0}^{T} \left( \mathbb{E}_{\mathcal{F}_t} \tilde{b}(\theta; t, \overline{X}_1^\theta, \mu^\theta_t) - \mathbb{E}_{\mathcal{F}_t} \tilde{b}(\theta'; t, \overline{X}_1^{\theta'}, \mu^{\theta'}_t) \right) dt.
\]

Thanks to the smoothness properties of \(b\) and \(\sigma\) granted by Assumptions 2, 3, and 4, we have
\[
|\phi_{\ell, t}(\vartheta; t, \overline{X}_1^\theta, \mu^\theta_t) - \phi_{\ell, t}(\vartheta'; t, \overline{X}_1^{\theta'}, \mu^{\theta'}_t)| \\
\leq C|\vartheta - \vartheta'| + |\overline{X}_1^\theta - \overline{X}_1^{\theta'}| + \mathcal{W}_1(\mu^\theta_t, \mu^{\theta'}_t)(1 + |\overline{X}_1^\theta|^\alpha + |\overline{X}_1^{\theta'}|^\alpha + \mathcal{W}_1(\mu^\theta_t, \mu^{\theta'}_t)).
\]
We have $\mathbb{E}_{P_N} \left[ |\mathbf{X}_t^1|^{\theta} \right] = m_r(\mu^\theta)$, which is uniformly bounded in $t \in [0, T]$, $\theta \in \Theta$ for every $r \geq 1$ by Lemma 8. Likewise $\mathbb{E}_{P_N} \left[ |\mathbf{X}_t^1|^{\theta} \right] \leq C(m_r(\mu^\theta)) + \mathbb{E}_{P_N} \left[ |\mathbf{X}_t^1|^{\theta} \right]$, therefore, by Cauchy-Schwarz’s inequality

$$
\left| \mathbb{E}_{P_N} \left[ \phi_{\ell, t} (\theta, t, \mathbf{X}_t^1, \mu^\theta) - \phi_{\ell, t} (\theta', t, \mathbf{X}_t^1, \mu^\theta') \right] \right|
$$

$$
\leq C \left( (\mathbb{E}_{P_N} \left[ |\mathbf{X}_t^1|^{2\theta} \right])^{1/2} + (\mathbb{E}_{P_N} \left[ |\mathbf{X}_t^1|^{2\theta'} \right])^{1/2} \right)
$$

and the Lipschitz smoothness follows by applying (40) of Lemma 20 and Proposition 9. The convergence $N^{-1} \| \mathbb{E}_N (\theta) \rightarrow \mathbb{E}_G (\theta)$ is a simple consequence of Lemma 21.

6.4. Proof of Proposition 16

Note that for every $0 \leq \ell \leq p$:

$$
\partial_{\theta_{\ell}} \mathcal{E}_N (\theta; X^N) = G_N (X^N)_{\ell} + 2 \sum_{\ell' = 1}^p \partial_{\theta_{\ell'}} H_N (X^N)_{\ell, \ell'}.
$$

Let $\theta, \theta' \in \Theta_0$ be such that $\mathbb{P}^N_\theta = \mathbb{P}^N_{\theta'}$ for every $N \geq 1$. (This also implies $\mathbb{P}_\theta = \mathbb{P}_{\theta'}$ by Lemma 21 for instance.) Using the symmetry of $H_N (X^N)$, we note that

$$
\partial^T H_N (X^N) \theta - (\theta')^T H_N (X^N) \theta' = \sum_{\ell = 1}^p (\theta_{\ell} - \theta'_{\ell}) \sum_{\ell' = 1}^p H_N (X^N)_{\ell, \ell'} (\theta_{\ell'} + \theta'_{\ell}').
$$

It follows that

$$
0 = \mathcal{E}_N (\theta; X^N) - \mathcal{E}_N (\theta'; X^N) = \sum_{\ell = 1}^p (\theta_{\ell} - \theta'_{\ell}) \left( G_N (X^N)_{\ell} + 2 \sum_{\ell' = 1}^p H_N (X^N)_{\ell, \ell'} \frac{\theta_{\ell'} + \theta'_{\ell'}}{2} \right)
$$

$$
= \sum_{\ell = 1}^p \left( G_N (X^N)_{\ell} + 2 \sum_{\ell' = 1}^p \frac{\theta_{\ell'} + \theta'_{\ell'}}{2} H_N (X^N)_{\ell, \ell'} \right) (\theta_{\ell} - \theta'_{\ell})
$$

$$
= \sum_{\ell = 1}^p \partial_{\theta_{\ell}} \mathcal{E}_N (\theta^*; X^N) \xi_{\ell} = \nabla_{\theta} \mathcal{E}_N (\theta^*; X^N)^T \xi,
$$

with $\xi = \theta - \theta'$ and $\theta^* = \frac{1}{2} (\theta + \theta') \in \Theta_0$ by the convexity of $\Theta_0$ and that does not depend on $X^N$. Assume now that $\theta \neq \theta'$. This implies that for some $\xi \neq 0$, we have

$$
0 = \left( \nabla_{\theta} \mathcal{E}_N (\theta^*; X^N)^T \xi \right)^T \mathcal{E}_N (\theta^*; X^N)^T \xi = \xi^T \mathbb{I}_N (\theta^*) \xi.
$$

Thus $\mathbb{I}_N (\theta^*)$ is degenerate for every $N \geq 1$. Letting $N \rightarrow \infty$ and applying Proposition 11, we infer that $\mathbb{I}_G (\theta^*)$ is degenerate as well, a contradiction. The conclusion follows for $\mathcal{I}$ likewise.

7. Appendix

7.1. Proof of Lemma 7

By Assumption 3, we have $b_0 = \sup_{t \in [0, T]} |b(\theta_0; t, 0, \delta_0)| < \infty$ for some $\theta_0 \in \Theta$. Combined with Assumption 4, since $\Theta$ is compact, we infer

$$
|b(\theta; s, X^i_s, \mu^\theta_s) - b(\theta; s, X^i_s, \mu^\theta_s)| \leq C(1 + |X^i_s|) + N^{-1} \sum_{i = 1}^N |X^i_s|
$$

uniformly in $s \in [0, T]$ and $\theta \in \Theta$. For $M > 0$, define $\tau_M = \inf \{ s \geq 0, \max_{1 \leq i \leq N} |X^i_s| \geq M \} \wedge T$ and note that $\tau_M$ is a $(\mathcal{F}_t)$-stopping time. We have

$$
|X^i_{\tau_M} - X^i_0| \leq |X^i_0| + \int_0^{\tau_M} |b(\theta; s, X^i_s, \mu^\theta_s)| ds + \int_0^{\tau_M} \sigma(s, X^i_s) dB^i_N(s, \theta)
$$
\[ \leq |X_0^i| + C \int_0^{t \wedge \tau_M} (1 + |X_s^i| + N^{-1} \sum_{i=1}^N |X_s^i|)ds + \int_0^{t \wedge \tau_M} \sigma(s, X_s^i)dB^{i,N,\vartheta}_s \]

Taking \( \mathbb{P}_0^N \)-expectation of order \( r \geq 1 \), we obtain

\[ \mathbb{E}_{\mathbb{P}_0^N} [ |X_{t \wedge \tau_M}^i|^r ] \leq C(1 + \mathbb{E}_{\mathbb{P}_0^N} [ |X_0^i|^r ] + \int_0^t (1 + \mathbb{E}_{\mathbb{P}_0^N} [ |X_{s \wedge \tau_M}^i|^r ])ds), \]

using Jensen’s inequality, the exchangeability of \( \mathbb{P}_0^N \) and the Burckholder-Davis-Gundy inequality together with Assumption 2 to obtain

\[ \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| \int_0^t \sigma(s, X_s^i)dB^{i,N,\vartheta}_s \right|^r \right] \leq C \mathbb{E} \left[ \left( \int_0^t |\sigma(s, X_s^i)|^2 s \right)^{r/2} \right] \leq C'. \]

By Grönwall’s lemma, we infer

\[ \mathbb{E}_{\mathbb{P}_0^N} [ |X_{t \wedge \tau_M}^i|^r ] \leq (\mathbb{E}_{\mathbb{P}_0^N} [ |X_0^i|^r ] + C') \exp(Ct). \]

Letting \( M \to \infty \), we conclude by Fatou’s lemma.

### 7.2. Proof of Lemma 8

It is a slight variation of the proof of Lemma 7. By Theorem 4.21 in [12], we have \( \mathbb{E}_{\mathbb{P}_0^N} [ \sup_{0 \leq t \leq T} |X_t^i|^r ] < \infty \) for every \( r \geq 1 \). (Their argument is developed for \( r = 2 \) but the extension to any \( r \geq 1 \) is straightforward.) Therefore, only the uniformity in \( \vartheta \) requires a proof. From

\[ \mathbb{E}_{\mathbb{P}_0^N} [ |X_t^i|^r ] \leq C \mathbb{E}_{\mathbb{P}_0^N} [ |X_t^i - X_t^i| + |X_t^i|^r ], \]

and the fact that \( \mathbb{E}_{\mathbb{P}_0^N} [ |X_t^i|^r ] = \mathbb{E}_{\mathbb{P}_0} [ |X_t^i|^r ] \), Lemma 8 is now a simple consequence of (42) in Lemma 20 together with Lemma 7.

### 7.3. Proof of Lemma 20

Proof of (39). The first inequality is obvious. Then, since \( X_0^i = X_0^{i,\vartheta} \), we have

\[ X_t^i - X_t^{i,\vartheta} = \int_0^t \left( b(\vartheta; s, X_s^i, \mu_s^{(N)}) - b(\vartheta; s, X_s^{i,\vartheta}, \mu_s^{(N)}) \right)ds + \int_0^t \left( \sigma(s, X_s^i) - \sigma(s, X_s^{i,\vartheta}) \right)dB_{s}^{i,N,\vartheta}. \]

Thanks to the smoothness properties of \( b \) and \( \sigma \) granted by Assumptions 2, 3, 4, taking expectation to the power \( r \) on both side and applying the Burckholder-Davis-Gundy inequality, we infer

\[ \mathbb{E}_{\mathbb{P}_0^N} [ |X_t^i - X_t^{i,\vartheta}|^r ] \]

\[ \leq C \int_0^t \mathbb{E}_{\mathbb{P}_0^N} [ |\vartheta - \vartheta'| |X_s^i| + m_{\vartheta}(\mu_s^{(N)})]ds + \int_0^t \left( \mathbb{E}_{\mathbb{P}_0^N} [ |X_s^i - X_s^{i,\vartheta}|^r + W_1(\mu_s^{(N)}, \mu_{s}^{(N)}) \right)ds \]

\[ \leq C |\vartheta - \vartheta'|^r + \int_0^t \left( \mathbb{E}_{\mathbb{P}_0^N} [ |X_s^i - X_s^{i,\vartheta}|^r + W_1(\mu_s^{(N)}, \mu_{s}^{(N)}) \right)ds \]

where we used that \( \mathbb{E}_{\mathbb{P}_0^N} [ m_{\vartheta}(\mu_t^{(N)}) \right] \leq \mathbb{E}_{\mathbb{P}_0^N} [ |X_t|^r ] \) which is bounded uniformly in \( t \in [0, T] \) and \( \vartheta \in \Theta \) by Lemma 7. Also, using the first part of (39), namely

\[ \mathbb{E}_{\mathbb{P}_0^N} [ W_1(\mu_s^{(N)}, \mu_{s}^{(N)}) \right] \leq C N^{-1} \sum_{i=1}^N \mathbb{E}_{\mathbb{P}_0^N} [ |X_s^i - X_s^{i,\vartheta}|^r ] \]
and taking averages over \( i = 1, \ldots, N \) on both sides, we infer
\[
N^{-1} \sum_{i=1}^{N} \mathbb{E}_{\mathbb{P}_N} [ |X_i^t - X_i^{i'; \theta'}|^r ] \leq C (|\theta - \theta'|^r + \int_0^t N^{-1} \sum_{i=1}^{N} \mathbb{E}_{\mathbb{P}_N} [ |X_i^s - X_i^{i'; \theta'}|^r ] ds).
\]

We obtain the second part of (39) by Grönwall’s lemma.

**Proof of (40).** The first inequality is obvious. The second part is simply (53) from the proof of Proposition 9.

**Proof of (41) and (42).** By triangle inequality,
\[
(54) \quad W_1(\mu_t^{(N)}, \mu_t^0) \leq W_1(\mu_t^{(N)}, \overline{\mu}_t^{(N)}, \mu_t^0) + W_1(\overline{\mu}_t^{(N)}, \mu_t^0) \leq N^{-1} \sum_{i=1}^{N} |X_i^t - \overline{X}_i^t| + W_1(\overline{\mu}_t^{(N)}, \mu_t^0).
\]

By Theorem 2 of [18], we have \( \sup_{t \in [0, \tau], \theta \in \Theta} \mathbb{E}_{\mathbb{P}_N} [ W_1(\overline{\mu}_t^{(N)}, \mu_t^0)^r ] \leq C N^{-\delta r} \) for every \( r \geq 1 \) and some \( \delta > 0 \). The value of \( \delta \) depends on the dimension \( d \) of the state space. The uniformity in \((t, \theta)\) follows in particular from the uniform moment bounds of Lemma 8 (see the conditions of Theorem 2 of [18]). Therefore (41) is a consequence of (42).

In order to establish (42), since \( X_0 = \overline{X}_0^0 \), we write
\[
X_i^t - \overline{X}_i^t = \int_0^t \left( b(\theta; X_i^s, \mu_s^{(N)}) - b(\theta; s, \overline{X}_i^{i'; \theta'}, \mu_s^0) \right) ds + \int_0^t \left( \sigma(s, X_i^s) - \sigma(s, \overline{X}_i^{i'; \theta'}) \right) dB_i^{s, N}.
\]

Taking expectation to the power \( r \) on both side and applying the Burckholder-Davis-Gundy inequality, we infer
\[
\mathbb{E}_{\mathbb{P}_N} [ |X_i^t - \overline{X}_i^t|^r ] \leq C \int_0^t \mathbb{E}_{\mathbb{P}_N} [ |X_i^s - \overline{X}_i^s|^r + W_1(\mu_s^{(N)}, \mu_s^0)^r ] ds
\]
\[
\leq C \int_0^t \mathbb{E}_{\mathbb{P}_N} [ |X_i^s - \overline{X}_i^s|^r + W_1(\overline{\mu}_s^{(N)}, \mu_s^0)^r ] ds
\]
\[
\leq \varepsilon_N + C \int_0^t \mathbb{E}_{\mathbb{P}_N} [ |X_i^s - \overline{X}_i^s|^r ] ds,
\]
arguing as in (54), and where \( \varepsilon_N = C T \sup_{t \in [0, \tau], \theta \in \Theta} \mathbb{E}_{\mathbb{P}_N} [ W_1(\overline{\mu}_t^{(N)}, \mu_t^0)^r ] \leq C N^{-\delta r} \). Note that the constants in each line are uniformly bounded in \( \theta \in \Theta \). We obtain (42) by Grönwall’s lemma.

### 7.4. Proof of Lemma 21

We plan to use the following decomposition
\[
N^{-1} \sum_{i=1}^{N} \int_0^t \phi(\theta_N; s, X_i^s, \mu_s^{(N)}) ds - \int_0^t \int_{\mathbb{R}^d} \phi(\theta; s, x, \mu_s^0) \mu_s^0(dx) ds = I + II + III,
\]
with
\[
I = N^{-1} \sum_{i=1}^{N} \int_0^t \left( \phi(\theta_N; s, X_i^s, \mu_s^{(N)}) - \phi(\theta_N; s, \overline{X}_i^{i', \theta'}, \mu_s^{(N)}) \right) ds
\]
\[
II = N^{-1} \sum_{i=1}^{N} \int_0^t \left( \phi(\theta_N; s, \overline{X}_i^{i', \theta'}, \mu_s^{(N)}) - \phi(\theta; s, \overline{X}_i^{i', \theta'}, \mu_s^0) \right) ds
\]
\[
III = N^{-1} \sum_{i=1}^{N} \int_0^t \phi(\theta; s, \overline{X}_s^{i', \theta'}, \mu_s^0) ds - \mathbb{E}_{\mathbb{P}_\theta} \left[ \int_0^t \phi(\theta; s, X_s^{i', \theta'}, \mu_s^0) ds \right].
\]
Thanks to the properties of $\phi$ the term $I$ is bounded by a constant times
\[
N^{-1} \sum_{i=1}^{N} \int_{0}^{t} (|X_s^i - \bar{X}_s^{i,\partial N}| + \mathcal{W}_1(\mu_s^{(N)}, \mu_s^{\partial N})) (1 + |\bar{X}_s^{i,\partial N}|^\alpha + |X_s^i|^\alpha) m(\mu_s^{\partial N}) + m_\beta(\mu_s^{(N)})) ds
\]
\[
\leq \left( N^{-1} \sum_{i=1}^{N} \int_{0}^{t} (|X_s^i - \bar{X}_s^{i,\partial N}| + \mathcal{W}_1(\mu_s^{(N)}, \mu_s^{\partial N}))^2 ds \right)^{1/2}
\]
\[
x \times C \left( N^{-1} \sum_{i=1}^{N} \int_{0}^{t} (1 + |\bar{X}_s^{i,\partial N}|^2 + |X_s^i|^2 + m_2(\mu_s^{\partial N}) + m_2(\mu_s^{(N)})) ds \right)^{1/2}.
\]
by Cauchy-Schwarz’s inequality. Applying Cauchy-Schwarz’s inequality again together with Jensen’s inequality, the $\mathbb{P}^{N}_{\partial N}$-expectation to the power $m$ of the first term is then bounded by a constant times
\[
\mathbb{E}_{\mathbb{P}^{N}_{\partial N}} \left[ \left| N^{-1} \sum_{i=1}^{N} \int_{0}^{t} (|X_s^i - \bar{X}_s^{i,\partial N}|^2 + \mathcal{W}_1(\mu_s^{(N)}, \mu_s^{\partial N}))^2 ds \right]^{1/2} \right]^{m/2}
\]
\[
\times \left( \int_{0}^{t} (1 + \mathbb{E}_{\mathbb{P}^{N}_{\partial N}} [\bar{X}_s^{i,\partial N}]^{2m\alpha} + |X_s^i|^{2m\alpha}) + m_{2\alpha}(\mu_s^{\partial N}) + m_{2\alpha}(\mu_s^{(N)})) ds \right)^{1/2}.
\]
The first term is bounded by a constant times $N^{-m\delta}$ by (41) and (42) of Lemma 20. Also the $\mathbb{P}^{N}_{\partial N}$-expectation of $[\bar{X}_s^{i,\partial N}]^{2m\alpha}$, $|X_s^i|^{2m\alpha}$ and $m_{2\alpha}(\mu_s^{\partial N})$ is uniformly bounded in $s \in [0, T]$ by Lemma 7 and so is $m_{2\alpha}(\mu_s^{\partial N})$ by Lemma 8. We conclude
\[
\mathbb{E}_{\mathbb{P}^{N}_{\partial N}} \left[ |I|^m \right] \leq CN^{-\delta m}.
\]
The second term $II$ is bounded by a constant times
\[
N^{-1} \sum_{i=1}^{N} \int_{0}^{t} (|\bar{X}_s^{i,\partial N} - \bar{X}_s^{i,\partial N}| + \mathcal{W}_1(\mu_s^{\partial N}, \mu_s^{\partial N})) ds.
\]
Taking $\mathbb{P}^{N}_{\partial N}$-expectation to the power $m$ and applying successively the first and second part of (40) in Lemma 20 we obtain
\[
\mathbb{E}_{\mathbb{P}^{N}_{\partial N}} \left[ |II|^m \right] \leq C|\bar{\vartheta} - \vartheta_N|^m.
\]
Finally, the third and last term converges to 0 by the law of large numbers, applying for instance Rosenthal’s inequality for a precise bound in $N$. We obtain
\[
\mathbb{E}_{\mathbb{P}^{N}_{\partial N}} \left[ |III|^m \right] \leq CN^{-m/2}.
\]
The proof of Lemma 21 is complete.

References

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