THE MODULI OF FLAT $U(p, 1)$ STRUCTURES ON RIEMANN SURFACES

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Abstract. For a compact Riemann surface $X$ of genus $g > 1$, $\text{Hom}(\pi_1(X), U(p, 1))/U(p, 1)$ is the moduli space of flat $U(p, 1)$-connections on $X$. There is an integer invariant, $\tau$, the Toledo invariant associated with each element in $\text{Hom}(\pi_1(X), U(p, 1))/U(p, 1)$. If $q = 1$, then $-2(g-1) \leq \tau \leq 2(g-1)$. This paper shows that $\text{Hom}(\pi_1(X), U(p, 1))/U(p, 1)$ has one connected component corresponding to each $\tau \in 2\mathbb{Z}$ with $-2(g-1) \leq \tau \leq 2(g-1)$. Therefore the total number of connected components is $2(g-1) + 1$.

1. Introduction and Results

Let $X$ be a smooth projective curve over $\mathbb{C}$ with genus $g > 1$. The deformation space

$$\mathcal{CM}_B = \text{Hom}^+(\pi_1(X),\text{GL}(n,\mathbb{C}))/\text{GL}(n,\mathbb{C})$$

is the space of equivalence classes of semi-simple $\text{GL}(n,\mathbb{C})$-representations of the fundamental group $\pi_1(X)$. This is the $\text{GL}(n,\mathbb{C})$-Betti moduli space on $X$.

A theorem of Corlette, Donaldson, Hitchin and Simpson relates $\mathcal{CM}_B$ to two other moduli spaces—the $\text{GL}(n,\mathbb{C})$-de Rham and the $\text{GL}(n,\mathbb{C})$-Dolbeault moduli spaces, respectively [2, 3, 8, 15]. The Dolbeault moduli space consists of holomorphic objects (Higgs bundles) over $X$; therefore, the classical results of analytic and algebraic geometry can be applied to the study of the Dolbeault moduli space.

Since $U(p, 1) \subset \text{GL}(n,\mathbb{C})$, $\mathcal{CM}_B$ contains the space

$$\mathcal{M}_B = \text{Hom}^+(\pi_1(X), U(p, 1))/U(p, 1).$$

The space $\mathcal{M}_B$ will be referred to as the $U(p, 1)$-Betti moduli space.

The Betti moduli spaces are of great interest in geometric topology and uniformization. When $p = q = 1$, Goldman analyzed $\mathcal{M}_B$ and determined the number of its connected components [4]. Hitchin

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subsequently considered this moduli space from the Higgs bundle perspective and determined its topology \[8\]. The case of \(p = 2, q = 1\) was treated in \[23\]. Other related results have been obtained in \[5, 21, 22\]. In this paper, we treat the general case of \(q = 1\) and determine its number of connected components.

Each element in \(\mathcal{M}_B\) is associated with a Toledo invariant \(\tau \in 2\mathbb{Z}\) \[3, 19, 20\]. The main result presented here is the following:

**Theorem 1.1.** \(\text{Hom}^+((\pi_1(X), U(p, 1))/U(p, 1)\) has one connected component for each \(\tau \in 2\mathbb{Z}\) with \(-2(g - 1) \leq \tau \leq 2(g - 1)\). Therefore the total number of connected components is \(2(g - 1) + 1\).

1.1. The \(\text{PU}(p, 1)\)-representations. One hopes to prove a similar result for the case of \(\text{PU}(p, 1)\)-representations i.e. there is one connected component for each possible \(\tau\). Here the Toledo invariant is also bounded as \(-2(g - 1) \leq \tau \leq 2(g - 1)\), but takes on values in \(\frac{2}{p+1}\mathbb{Z}\) \[23\]. Again the cases of \(p = 1, 2\) have been treated in \[4, 8, 23\]. At present, the case of general \(p\) seems rather difficult. The reason we are able to treat the \(U(p, 1)\)-representations is due to the presence of certain reducible \(U(p, 1)\)-representations. These representations correspond to semi-stable Higgs bundles that can be constructed rather explicitly.

2. **Backgrounds and Preliminaries**

The coarse moduli space \(M_{r,d}\) of semi-stable vector bundles on \(X\) of rank \(r\) and degree \(d\) exists and has dimension \(r^2(g - 1) + 1\) \[13\].

2.1. The \(U(p, 1)\)-Higgs bundles. Each element \(\rho \in \text{Hom}(\pi_1(X), \text{GL}(n, \mathbb{C}))\) acts on \(\mathbb{C}^n\) via the standard representation of \(\text{GL}(n, \mathbb{C})\). The representation \(\rho\) is called reducible (irreducible) if its action on \(\mathbb{C}^n\) is reducible (irreducible). A representation \(\rho\) is called semi-simple if it is a direct sum of irreducible representations.

**Definition 2.1.**

\[\mathbb{C}\mathcal{M}_B = \{\sigma \in \text{Hom}(\pi_1(X), \text{GL}(n, \mathbb{C})) : \sigma \text{ is semi-simple}\}\)/\(\text{GL}(n, \mathbb{C})\).

\[\mathcal{M}_B = \{\sigma \in \text{Hom}(\pi_1(X), U(p, 1)) : \sigma \text{ is semi-simple}\}\)/\(U(p, 1)\).

Let \(E\) be a rank \(n\) complex vector bundle over \(X\) with \(\text{deg}(E) = 0\). Denote by \(\Omega\) the canonical bundle on \(X\). A holomorphic structure \(\overline{\partial}\) on \(E\) induces holomorphic structures on the bundles \(\text{End}(E)\) and \(\text{End}(E) \otimes \Omega\). A Higgs bundle is a pair \((E, \Phi)\), where \(\overline{\partial}\) is a holomorphic structure on \(E\) and \(\Phi \in H^0(X, \text{End}(E) \otimes \Omega)\). Such a \(\Phi\) is called a Higgs field. We denote the holomorphic bundle \(E\) by \(V\).
Define the slope of a vector bundle $V$ to be

$$s(V) = \frac{\deg(V)}{\text{rank}(V)}.$$  

For a fixed $\Phi$, a holomorphic sub-bundle $W \subset V$ is said to be $\Phi$-invariant if $\Phi(W) \subset W \otimes \Omega$. A Higgs bundle $(V, \Phi)$ is stable (semi-stable) if $W \subset V$ being $\Phi$-invariant implies

$$s(W) < (\leq) s(V).$$

A Higgs bundle is called poly-stable if it is a direct sum of stable Higgs bundles of the same slope [8, 16].

The Dolbeault moduli space $C_M$ is the moduli space of poly-stable (or S-equivalence classes of semi-stable [12]) Higgs bundles on $X$ [8, 4, 12, 16]. A Higgs bundle is called reducible if it is poly-stable (semi-stable) but not stable.

Now we summarize the relation between the moduli spaces $C_M$ and $C_M$.

**Definition 2.2.** Let $M$ be the subset of $C_M$ consisting of Higgs bundles $(V, \Phi)$ satisfying the following two conditions:

1. $V$ is a direct sum:

   $$V = V_P \oplus V_Q,$$

   where $V_P, V_Q$ are of ranks $p, 1$, respectively.

2. The Higgs field decomposes into two maps:

   $$\Phi_1 : V_P \rightarrow V_Q \otimes \Omega,$$

   $$\Phi_2 : V_Q \rightarrow V_P \otimes \Omega.$$

Hence each $(V, \Phi) = (V_P \oplus V_Q, \Phi) \in M$ is associated with an invariant

$$d = \deg(V_P) = -\deg(V_Q).$$

The Toledo invariant is defined to be $\tau = 2d$. The subset of $M$ consisting of classes with a fixed Toledo invariant $\tau$ is denoted by $M_\tau$.

**Theorem 2.3.**

1. The moduli spaces $C_M_B$ and $C_M$ are homeomorphic.
2. The reducible representations in $C_M_B$ correspond to the poly(semi)-stable, but not stable, points.
3. The subspace $M_B$ is homeomorphic to $M$.
4. $\tau \in 2\mathbb{Z}$ and $-2(g-1) \leq \tau \leq 2(g-1)$. In particular, equality holds only if the corresponding Higgs bundle is poly(semi)-stable, but not stable.
5. The space $M_\tau$ is homeomorphic to $M_{-\tau}$.

**Proof.** See [2, 8, 13, 16, 19, 20, 23].
By Theorem 2.3 (4) (5), we assume, for the rest of the paper that
\[ 0 \leq \tau \leq 2g - 2. \]

3. THE $\mathbb{C}^*$-ACTION AND THE HODGE BUNDLES

If $(V, \Phi) \in \mathcal{CM}$, then $(V, t\Phi) \in \mathcal{CM}$ for all $t \in \mathbb{C}^*$. This defines an action
\[ \mathbb{C}^* \times \mathcal{CM} \rightarrow \mathcal{CM}. \]

By Definition 2.2, we have

**Proposition 3.1.** The $\mathbb{C}^*$-action preserves $\mathcal{M}$.

A Hodge bundle on $X$ is a direct sum of bundles $\mathbb{C}^*$
\[ V = \bigoplus_{s,t} V^{s,t} \]
together with maps (Higgs field)
\[ \Phi : V^{s,t} \rightarrow V^{s-1,t+1} \otimes \Omega. \]

Definition 2.2 implies that

**Proposition 3.2.** Suppose $(V_P \oplus V_Q, (\Phi_1, \Phi_2)) \in \mathcal{M}$. Then $(V_P \oplus V_Q, (\Phi_1, \Phi_2))$ is a Hodge bundle if and only if $(V_P \oplus V_Q, (\Phi_1, \Phi_2))$ is either binary or ternary in the following sense:

1. **Binary:** $\Phi_2 \equiv 0$.
2. **Ternary:** $V_P = V_1 \oplus V_2$ and the Higgs field consists of two maps:
   \[ \Phi_1 : V_2 \rightarrow V_Q \otimes \Omega, \]
   \[ \Phi_2 : V_Q \rightarrow V_1 \otimes \Omega. \]

**Proposition 3.3.** A Higgs bundle $(V, \Phi) \in \mathcal{CM}$ is a Hodge bundle if and only if $(V, \Phi) \cong (V, t\Phi)$ for all $t \in \mathbb{C}^*$.

**Proof.** See [16, 17, 18].

**Lemma 3.4.** The $\lim_{t \rightarrow 0} (V, t\Phi)$ exists in $\mathcal{M}$ for any $(V, \Phi)$ in $\mathcal{M}$. In other words, the $\mathbb{C}^*$-action always extends to a $\mathbb{C}$-action.

**Proof.** The $\lim_{t \rightarrow 0} (V, t\Phi)$ exists in $\mathcal{CM}$ for $(V, \Phi)$ in $\mathcal{CM}$. Since $U(p,1)$ is a closed subgroup of $GL(n, \mathbb{C})$, $\mathcal{M}_B$ is a closed subset of $\mathcal{CM}_B$ and the embedding is proper. The lemma then follows from Theorem 2.3.

From Proposition 3.3 and Lemma 3.4 and the facts that the $\mathbb{C}^*$-action preserves $\mathcal{M}$ and $\mathcal{M}$ is closed in $\mathcal{CM}$, we have
Corollary 3.5. Every Higgs bundle in $\mathcal{M}$ can be deformed to a Hodge bundle in $\mathcal{M}$.

Lemma 3.6. Let $E = (V_1 \oplus V_2 \oplus V_Q, (\Phi_1, \Phi_2))$ be a stable ternary bundle in $\mathcal{M}$ with $\Phi_2 \not\equiv 0$. Then there is a stable Higgs bundle $F = (V, \Phi) \in \mathcal{M}$ not isomorphic to $E$ such that

$$\lim_{t \to \infty} (V, t\Phi) = E.$$  

Proof. This is essentially Lemma 11.9 in [18]. There is one additional and crucial observation to be made. That is the Higgs bundle $F$ in the proof is actually in $\mathcal{M}$. This is immediate from the construction of $F$. \hfill \square

Proposition 3.7. Every Higgs bundle in $\mathcal{M}$ can be deformed to a binary Hodge bundle.

Proof. The proof parallels the proof of Corollary 11.10 in [18]. The only difference is that for the subset $\mathcal{M}_\tau$, the lowest stratum is the space of binary Hodge bundles with Toledo invariant $\tau$, instead of $\mathcal{M}_{p+1,0}$. Suppose $(V, \Phi) \in \mathcal{M}_\tau$. Then it is of the form $(W', 0) \oplus (W', \Phi')$ where $(W', \Phi')$ is stable. By Corollary 3.3, we may assume $(W', \Phi')$ is ternary. Then Lemma 3.6 implies that $(W', \Phi')$ is not in the lowest possible stratum, hence, can be deformed to a fixed point set (with respect to the $\mathbb{C}^*$-action) of the lowest stratum which consists of binary Higgs bundles (See the section titled “Actions of $\mathbb{C}^*$” in [18] for further discussions of these strata). \hfill \square

Definition 3.8. Let $B_\tau$ be the space of all poly-stable (or $S$-equivalence classes of semi-stable [12]) binary Hodge bundles $(V_P \oplus V_Q, (\Phi_1, 0))$ with $\deg(V_P) = d = -\deg(V_Q)$ and $\tau = 2d$.

The rest of the paper is devoted to showing that $B_\tau$ is connected.

4. The Deformation of Binary Hodge Bundles

A family (or flat family) of Higgs bundles $(V_Y, \Phi_Y)$ is a variety $Y$ such that there is a vector bundle $V_Y$ on $X \times Y$ together with a section $\Phi_Y \in \Gamma(Y, (\pi_Y)_*(\pi_X^*\Omega \otimes \text{End}(V_Y)))$ [12]. $\mathbb{C}\mathcal{M}$ being a moduli space implies that if $Y$ is a family of stable (poly-stable or $S$-equivalence classes of semi-stable) Higgs bundles, then there is a natural morphism $t: Y \to \mathbb{C}\mathcal{M}$.

Moreover $t$ takes every point $y \in Y$ to the point of $\mathbb{C}\mathcal{M}$ that corresponds to the Higgs bundle in the family over $y$ [10, 11, 12].
The space $\mathcal{M}$ is a subvariety of $\mathbb{CM}$; hence, to show that two stable (poly-stable or $S$-equivalence classes of semi-stable) Higgs bundles $(V_1, \Phi_1)$ and $(V_2, \Phi_2)$ belong to the same component of $\mathcal{M}$, it suffices to exhibit a connected family $Y$ (whose image under the morphism $t: Y \rightarrow \mathbb{CM}$ is contained in $\mathcal{M}$) of stable (poly-stable or $S$-equivalence classes of semi-stable) Higgs bundles containing both $(V_1, \Phi_1)$ and $(V_2, \Phi_2)$.

Let $(V_P \oplus V_Q, \Phi) \in B_\tau$ be a stable binary Hodge bundle with $d = \deg(V_P) = -\deg(V_Q)$.

4.1. The Grothendieck Quot scheme. Denote by $H_{r,d_1}$ the set of all vector bundles of rank $r$ and degree $d_1 \leq 0$ with the property that if $W \in H_{r,d_1}$ and $U \subset W$, then $\deg(U) \leq 0$.

**Proposition 4.1.** Suppose $W \in H_{r,d_1}$. Then for any line bundle $L$ with $\deg(L) > 2g - 2 - d_1$,

1. $H^1(W \otimes L) = 0$,
2. $W \otimes L$ is generated by global sections.

**Proof.** The proof resembles the proof of Lemma 20 in Chapter I of [13]. By Serre duality, $H^1(W \otimes L) \cong H^0(\Omega \otimes W^* \otimes L^*)$. Hence if $H^1(W \otimes L) \neq 0$, then $\Omega \otimes W^* \otimes L^*$ contains a line bundle $N$ of degree greater than or equal to 0:

$$0 \rightarrow N \rightarrow \Omega \otimes W^* \otimes L^* \rightarrow (\Omega \otimes W^* \otimes L^*)/N \rightarrow 0.$$

Dualizing and tensoring with $\Omega \otimes L^*$, we obtain an exact sequence

$$0 \rightarrow U \rightarrow W \rightarrow L' \rightarrow 0,$$

with $\deg(U) \geq d_1 - (2g - 2) + \deg(L)$. Since $W \in H_{r,d_1}$ and $U \subset W$, $\deg(U) \leq 0$. This implies $\deg(L) \leq 2g - 2 - d_1$ which is a contradiction. This shows that (1) is true for any $L > 2g - 2 - d_1$.

The proof of (2) essentially reduces to showing that

$$H^1(W \otimes L \otimes L_x^{-1}) = 0$$

(see the proof of Lemma 20 in Chapter I of [13]), where $L_x$ is the ideal sheaf at a point $x \in X$. Since $\deg(L_x) = 1$, $\deg(L \otimes L_x^{-1}) > 2g - 2 - d_1$. Since (1) is true for any $L > 2g - 2 - d_1$, (2) follows. \[\square\]

Let $D = 2gr + (1-r)d_1$ and $a = D + r(1-g) = r(g+1)+(1-r)d_1$. For the pair $(a, r)$, we construct the Grothendieck scheme $Q$ parameterizing the quotient sheaves of $\mathcal{O}^a$ with Hilbert polynomial $H(m) = a + rm$. The scheme $Q$ contains the sub-scheme $R$ defined by:

$$R = \{W \in Q : W \text{ is locally free and } H^1(W) = 0\}.$$
The sub-scheme \( R \) is smooth and connected (The proof is the same as that of Proposition 23 in Chapter I of [13]).

Suppose \( L \) is a line bundle of degree \(-2g + d_1\). If \( W \in R \), then \( \text{deg}(W \otimes L) = d_1 \). Hence \( R \) also parameterizes a family of vector bundles of degree \( d_1 \) and rank \( r \). By Proposition [1,4], \( R \) contains all the bundles in \( H_{r,d_1} \). We shall denote the scheme \( R \) so constructed as \( R_{r,d_1} \).

\[ \text{deg}(W \otimes L) = d_1. \]

Thus the diagram simplifies to

\[ 0 \rightarrow V_1 \overset{f_1}{\rightarrow} V_P \overset{f_2}{\rightarrow} V_2 \rightarrow 0 \]

\[ \Phi \downarrow \quad \varphi \downarrow \]

\[ 0 \leftarrow W_2 \overset{g_2}{\leftarrow} V_Q \otimes \Omega \leftarrow W_1 \leftarrow 0 \]

commutes, and the rows are exact, \( \text{rank}(V_2) = \text{rank}(W_1) \) and \( \varphi \) has full rank at a generic point of \( X \). As \( V_Q \otimes \Omega = W_1 \).

4.2. The canonical factorization. Let \((V_P \oplus V_Q, \Phi) \in B_{\tau}\) be a binary Higgs bundle with \( \Phi \not\equiv 0 \). There exist bundles \( V_1, V_2 \) and \( W_1, W_2 \) such that the following diagram (the canonical factorization [14])

\[ 0 \rightarrow V_1 \overset{f_1}{\rightarrow} V_P \overset{f_2}{\rightarrow} V_2 \rightarrow 0 \]

\[ \Phi \downarrow \quad \varphi \downarrow \]

\[ 0 \leftarrow W_2 \overset{g_2}{\leftarrow} V_Q \otimes \Omega \leftarrow W_1 \leftarrow 0 \]

Thus the diagram simplifies to

\[ 0 \rightarrow V_1 \overset{f_1}{\rightarrow} V_P \overset{f_2}{\rightarrow} V_2 \rightarrow 0 \]

\[ \Phi \downarrow \quad \varphi \downarrow \]

\[ V_Q \otimes \Omega = W_1 \]

Let \( d_1 = \text{deg}(V_1) \) and \( d_2 = \text{deg}(V_2) \). Since \((V_P \oplus V_Q, \Phi)\) is semi-stable, \( d_1/(p-1) \leq 0 \). Since \( \varphi \not\equiv 0 \), \( d_2 \leq (2g-2)-d \) (Recall that \( d = \text{deg}(V_P) \)).

Hence \( d_1 \geq 2d - (2g - 2) \). To summarize

\[ \begin{cases} 2d - (2g - 2) \leq d_1 \leq 0 \\ d \leq d_2 \leq -d + (2g - 2) \end{cases} \]

Denote by \( B_{\tau}(d_2) \) the subspace of \( B_{\tau} \) such that \((V, \Phi) \in B_{\tau}(d_2)\) implies \( \text{deg}(V_2) = d_2 \) in the above canonical factorization.

**Proposition 4.2.** The space \( B_{\tau}(d_2) \) is connected.

**Proof.** Denote by \( J^l \) the Jacobi variety identified with the set of holomorphic line bundles of degree \( l \) on \( X \). For each \( V_2 \in J^{d_2} \), the variety \( \mathbb{C}^* \times X^{-d+2(g-1)-d_2} \) parameterizes a family of pairs that contains all pairs \((V_Q, \varphi)\) such that \( V_Q \in J^{-d} \) and

\[ 0 \not\equiv \varphi \in H^0(X, V_2^{-1} \otimes V_Q \otimes \Omega). \]

Note the moduli of all such pairs is simply \( \mathbb{C}^* \times Sym^{-d+2(g-1)-d_2}X \), where \( Sym^tX \) is the \( t \)-th symmetric product of \( X \), i.e. the moduli is
$\mathbb{C}^* \times X^{-d+2(g-1)-d_2}$ quotiented by the symmetry group on $-d+2(g-1)-d_2$ letters. Hence the variety

$$S = J^d \times (\mathbb{C}^* \times X^{-d+2(g-1)-d_2})$$

parameterizes a family of triples that contains all triples $(V_2, V_Q, \varphi)$ such that

$$V_2 \xrightarrow{\varphi} V_Q \otimes \Omega,$$

with $V_2 \in J^d, V_Q \in J^{-d}$ and $\varphi \neq 0$. The variety $S$ is smooth.

Suppose $d_2 = 0$. Then $d_1 = 0$ and every Higgs bundle in $B_\tau(0)$ is reducible. There are two cases.

Case 1: $\tau > 0$. Then $\Phi \neq 0$ and every Higgs bundle in $B_\tau(0)$ is of the form $(V_1, 0) \oplus (V_2 \oplus V_Q, \Phi)$, where $V_1 \in M_{p-1,0}, V_2 \in J^d, V_Q \in J^{-d}$. Hence $M_{p-1,0} \times \mathbb{C}^* \times Sym^{-2d+(2g-2)}X \times J^{-d}$ parameterizes a family that contains every Higgs bundle in $B_\tau(0)$. Since $M_{p-1,0} \times \mathbb{C}^* \times Sym^{-2d+(2g-2)}X \times J^{-d}$ is connected, $B_\tau(0)$ is connected.

Case 2: $\tau = 0$. Then every Higgs bundle in $B_0(0)$ must be one of the following two forms:

1. $(V_P \oplus V_Q, 0)$, where $V_P \in M_{p,0}, V_Q \in J^0$,
2. $(V_1, 0) \oplus (V_2 \oplus V_Q, \Phi)$, where $V_1 \in M_{p-1,0}, V_2 \in J^0, V_Q \in J^0$.

Type 2 can be deformed to type 1 by simply deforming the Higgs field $\Phi$ to zero. Since $M_{p,0} \times J^0$ is connected, $B_0(0)$ is connected.

**Lemma 4.3.** Suppose $d_2 > 0$. Then the dimension of the space $\text{Ext}^1(V_2, V_1)$ is $(p-1)(g-1) + (p-1)d_2 - d_1$.

**Proof.** The subspace $V_1$ is $\Phi$-invariant. By semi-stability, $V' \subset V_1$ implies

$$s(V') \leq 0 < d_2 = \deg(V_2).$$

Hence $H^0(\text{Hom}(V_2, V_1)) = 0$. The lemma then follows from the fact that $\text{Ext}^1(V_2, V_1) \cong H^1(\text{Hom}(V_2, V_1))$ and from Riemann-Roch. \qed

For $d_2 > 0$, construct the universal bundle \[1, 13\]

$$U \longrightarrow X \times R_{p-1,d_1} \times S$$

such that

$$U|(X,V_1,V_2,V_Q,\varphi) = V_2^{-1} \otimes V_1.$$

Let $\pi$ be the projection

$$\pi : X \times R_{p-1,d_1} \times S \longrightarrow R_{p-1,d_1} \times S.$$

Applying the right derived functor $R^1$ to $\pi$ gives the sheaf $\mathcal{F} = R^1\pi_*(U)$ \[\#\] such that

$$\mathcal{F}|(V_1,V_2,V_Q,\varphi) = H^1(X, V_2^{-1} \otimes V_1).$$
By Lemma 4.3 and Grauert’s theorem \[7\], \( \mathcal{F} \) is locally free, hence, is associated with a vector bundle 

\[ F \mapsto R_{p-1, d_1} \times S \]

of rank \( (p - 1)(g - 1) + (p - 1)d_2 - d_1 \). Since \( R_{p-1, d_1} \) is smooth and connected, the total space \( F \) is smooth, connected and parameterizes a family of Higgs bundles that fit into the canonical decomposition with fixed \( d_2 \). By construction, the scheme \( F \) parameterizes a family of Higgs bundles that contains every member in the parameter space \( B_r(d_2) \). Moreover if a Higgs bundle in \( F \) is semi-stable, it must belong to \( B_r(d_2) \). Since the semi-stability condition is open \[17\], the subset of \( F \) parameterizing the semi-stable Higgs bundles, if not empty, is open and dense in \( F \), hence, connected. This implies \( B_r(d_2) \) is connected.

4.3. **Deformation between the** \( B_r(d_2)'s. \) Fix a set of distinct points 

\[ A = \{x_1, \ldots, x_{d_2}, y_1, \ldots, y_{d_2-1}, z_1, \ldots, z_{d_2-d-1}\} \subset X \]

and let \( Y = X \setminus A \). Fix \( y \in Y \). For \( t \in Y \), consider the following divisors on \( X \):

\[ D_2 = \{x_1, \ldots, x_{d_2}\}, \]

\[ C(t) = \{x_1, \ldots, x_{d_2}, -y_1, \ldots, -y_{d_2-1}, -t\}, \]

\[ C = \{x_1, \ldots, x_{d_2}, -z_1, \ldots, -z_{d_2-d-1}, -y\}. \]

For any divisor \( D \), denote by \([D]\) the line bundle associated with \( D \). The set \( Y \) parameterizes a family of Higgs bundles as follows. Let

\[ V_P(t) = [C] \bigoplus_{i=1}^{p-1} [C_i(t)], \]

where \( C_i(t) = C(t) \) for all \( i \) and denote the projection maps to the \([C]\) and \([C_i(t)]\) factors by \( p \) and \( p_i(t) \), respectively. The divisors \( D_2 - C(t) \) and \( D_2 - C \) define maps \( h_i(t) : [C_i(t)] \to [D_2] \) and \( h : [C] \to [D_2] \), respectively. These maps induce a map

\[ G_t : V_P(t) \to [D_2], \quad G_t = h + \sum_{i=1}^{p-1} h_i(t). \]

Let \( V_2 = [D_2] \). Since \( d_2 \leq (2g - 2) - d \), there exists \( V_Q \in J^{-d} \) and \( 0 \neq \varphi \in H^0(V_2^{-1} \otimes V_Q \otimes \Omega) \). Let \( \Phi(t) = \varphi \circ G_t \). Then \( (V_P(t) \oplus V_Q, \Phi(t)) \) is a family of Higgs bundles parameterized by \( Y \). Let \( p_P, p_Q \) be the projections onto the \( V_P(t), V_Q \) factors.

**Lemma 4.4.** If \( U \subset V_P(t) \), then \( \deg(U) \leq d \).
Proof. This is an inductive argument.

Case 1: \( p(U) \neq 0 \). Consider the sequence
\[
0 \longrightarrow U' \longrightarrow U \longrightarrow p(U) \longrightarrow 0.
\]
Then \( \deg(p(U)) \leq d \) and \( \deg(U) \leq \deg(U') + d \). Now we begin with the smallest \( i \) with \( p_i(t)(U') \neq 0 \) and construct
\[
0 \longrightarrow U_i \longrightarrow U' \longrightarrow p_i(U') \longrightarrow 0.
\]
Again \( \deg(p_i(U')) \leq \deg([C_i(t)]) = 0 \) and \( \deg(U) \leq \deg(U_i) + d \). Now we let \( j > i \) be the smallest integer with \( p_j(U_i) \neq 0 \) and construct the new sequence and obtaining \( U_j \) with \( \deg(U) \leq \deg(U_j) + d \). Note \( \text{rank}(U) > \text{rank}(U') > \text{rank}(U_i) \), so eventually the process ends and since \( \deg([C_i(t)]) = 0 \) for all \( i \), we have \( \deg(U) \leq d \).

Case 2: \( p(U) = 0 \). Here we simply begin with the smallest \( i \) with \( p_i(t)(U') \neq 0 \) as in Case 1. The rest is the same and we conclude that \( \deg(U) \leq 0 \leq d \). \( \square \)

Lemma 4.5. Suppose \( L \subset V_P(t) \) is a line bundle with \( \deg(L) > 0 \), then \( L = [C] \).

Proof. Suppose \( p_i(t)(L) \neq 0 \) for some \( i \), then \( \deg(L) \leq \deg(C_i(t)) = 0 \), a contradiction. \( \square \)

Proposition 4.6. The Higgs bundle \( (V_P(t) \oplus V_Q, \Phi(t)) \) is in \( B_\tau(d_2 - 1) \) if \( t = y \) and in \( B_\tau(d_2) \) if \( t \neq y \).

Proof. From the definition, one only needs to check that \( (V_P(t) \oplus V_Q, \Phi(t)) \) is semi-stable for all \( t \in Y \). Suppose \( W \in V_P(t) \oplus V_Q \) is \( \Phi(t) \)-invariant. There are two cases.

Case 1: \( p_Q(W) \neq 0 \). Then there is an exact sequence
\[
0 \longrightarrow U \longrightarrow W \longrightarrow p_Q(W) \longrightarrow 0.
\]
Since \( U \subset V_P \), by Lemma 4.4, \( \deg(U) \leq d \). Since \( \deg(p_Q(W)) \leq \deg(V_Q) = -d \), \( \deg(W) \leq d - d = 0 \).

Case 2: \( p_Q(W) = 0 \). Since \( W \) is \( \Phi(t) \)-invariant, \( W \subset \ker(G_t) \). It is immediate from the definition that, \([C] \not\subset \ker(G_t)\). Now we begin the construction similar to that in the proof of Lemma 4.4. Begin with the smallest \( i \) with \( p_i(t)(W) \neq 0 \) and construct
\[
0 \longrightarrow U_i \longrightarrow W \longrightarrow p_i(W) \longrightarrow 0.
\]
As \( \deg(p_i(W)) \leq \deg([C_i(t)]) = 0 \), \( \deg(W) \leq \deg(U_i) \). Continue this process as before. Eventually we reach the exact sequence
\[
0 \longrightarrow U_k \longrightarrow U_j \longrightarrow p_k(U_j) \longrightarrow 0,
\]
with
\[ \deg(W) \leq \deg(U_j) = \deg(U_k) + \deg(p_k(U_j)) \leq \deg(U_k) \]
and \( U_k \) a line bundle. Since \( [C] \not\in \ker(G_t) \) and \( U_k \subset \ker(G_t) \), by Lemma 4.5, \( \deg(U_k) \leq 0 \). Hence \( \deg(W) \leq \deg(U_k) \leq 0 \). Hence we conclude that the Higgs bundle \( (V_P(t) \oplus V_Q, \Phi(t)) \) is semi-stable. \( \square \)

Proposition 3.7 states that every Higgs bundle in \( \mathcal{M} \) may be deformed to a Higgs bundle in \( B \), which is the union of the \( B_{\tau}(d_2) \)'s. Proposition 4.2 asserts that each \( B_{\tau}(d_2) \) is connected. Theorem 1.1 then follows from Proposition 4.6.

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