A NOTE ON TWISTOR INTEGRALS

SPENCER BLOCH

1. INTRODUCTION

This paper is a brief introduction to twistor integrals from a mathematical point of view. It was inspired by a paper of Hodges [H] which we studied in a seminar at Cal Tech directed by Matilde Marcoli. The idea is to write the amplitude for a graph with \(n\) loops and \(2n + 2\) propagators using the geometry of pfaffians for sums of rank 2 alternating matrices. (Hodges considers the case of 1 loop and 4 edges).

Why is this of interest to a mathematician? The Feynman amplitude is a period in the sense of arithmetic algebraic geometry. In parametric form, the amplitude integral associated to a graph \(\Gamma\) with \(N\) edges and \(n\) loops has the form

\[
(1.1) \quad c(N, n) \int_\delta \frac{S_1^{N-2n-2} \Omega}{S_2^{N-2n}}.
\]

Here \(S_1\) and \(S_2\) are the first and second Symanzik polynomials [BK, BEK, IZ], and \(\Omega = \sum \pm A_i dA_1 \wedge \cdots \wedge \hat{dA}_i \wedge \cdots \wedge dA_N\) is the integration form on \(\mathbb{P}^{N-1}\), the projective space with homogeneous coordinates indexed by edges of \(\Gamma\). The chain of integration \(\delta\) is the locus of points on \(\mathbb{P}^{N-1}\) where all the \(A_i \geq 0\). Note \(\Omega, S_1, S_2\) are homogeneous of degrees \(N, n, n + 1\) in the \(A_i\), so the integrand is homogeneous of degree 0 and represents a rational differential form. Finally, \(c(N, n)\) is some elementary constant depending only on \(N\) and \(n\).

Two special cases suggest themselves. In the log divergent case when \(N = 2n\), the integrand is simply \(\Omega/S_1^2\). The first Symanzik polynomial depends only on the edge variables \(A_i\), so the result in this case is a constant. (If the graph is non-primitive, i.e. has log divergent subgraphs, the integral will diverge. We do not discuss this case.) Inspired by the conjectures of Broadhurst and Kreimer [BrK], there has been a great deal of work done on the primitive log divergent amplitudes.

The polynomial \(S_1\) itself is the determinant of an \(n \times n\)-symmetric matrix with entries linear forms in the \(A_i\). The linear geometry of this determinant throws an interesting light on the motive of the hypersurface \(X(\Gamma) : S_1 = 0\). For example, one has a “Riemann-Kempf” style
theorem that the dimension of the null space of the matrix at a point is equal to the multiplicity of the point on $X(\Gamma)$, $[P]$, $[K]$. Furthermore, the projectivized fibre space $Y(\Gamma)$ of these null lines maps birationally onto $X(\Gamma)$ and in some sense “resolves” the motive. Whereas the motive of $X(\Gamma)$ can be quite subtle, the motive of $Y(\Gamma)$ is quite elementary. In particular, it is mixed Tate $[B]$. (The Riemann-Kempf theorem refers to the map $\pi : Sym^{g-1}C \rightarrow \Theta \subset J_{g-1}(C)$ where $C$ is a Riemann surface and $\Theta$ is the theta divisor. The dimension of the fibre of $\pi$ at a point of $\Theta$ equals the multiplicity of the divisor $\Theta$ at the point.)

The second case is $N = 2n + 2$, e.g. one loop and 4 edges. The amplitude is $\int_\delta \Omega / S^2_2$ and is a function of external momenta and masses. The second Symanzik has the form

$$S_2 = S^0_2(A, q) - \left( \sum_{i=1}^N m_i^2 A_i \right) S_1(A)$$

Here $q$ denotes the external momenta, and $S^0_2(A, q)$ is homogeneous of degree 2 in $q$ and of degree $n+1$ in the $A$. Moreover, $S^0_2$ is a quaternionic pfaffian associated to a quaternionic hermitian matrix, $[BK]$, so in the case of zero masses there is again the possibility of linking the motive to the geometry of a linear map. In this note we go further and show for the case $N = 2n + 2$ that $S_2$ is itself a pfaffian via the calculus of twistors.

To avoid issues with convergence for the usual propagator integral, I assume in what follows that the masses are positive and the propagators are euclidean. Note that in (1.4) the pfaffian can vanish where some of the $a_i = 0$. The issues which arise are analogous to issues of divergence already familiar to physicists. They will not be discussed here.

**Theorem 1.1.** Let $\Gamma$ be a graph with $n$ loops and $2n + 2$ edges as above. We fix masses $m_i > 0$ and external momenta $q$ and consider the amplitude

$$A(\Gamma, q, m) = \int_{\mathbb{R}^{4n}} \frac{d^{4n}x}{\prod_{i=1}^{2n+2} P_i(x, q, m_i)}$$

where the $P_i$ are euclidean. Then there exist alternating bilinear forms $Q_i$ on $\mathbb{R}^{2n+2}$ where $Q_i$ depends on $P_i$, $1 \leq i \leq 2n + 2$, and a universal constant $C(n)$ depending only on $n$ such that

$$A(\Gamma, q, m) = C(n) \int_\delta Pfaffian(\sum_{i=1}^{2n+2} a_i Q_i)^2$$
Here $\Omega_{2n+1} = \sum \pm a_i da_1 \wedge \cdots \wedge da_i \cdots da_{2n+2}$ and $\delta$ is the locus on $\mathbb{P}^{2n+1}$ with coordinate functions $a_i$ where all the $a_i \geq 0$.

By way of analogy, the first Symanzik polynomial is given by
\begin{equation}
S_1(\Gamma)(a_1, \ldots, a_N) = \det(\sum_{\text{edge}} a_\epsilon M_\epsilon)
\end{equation}
where $M_\epsilon$ is a rank 1 symmetric $n \times n$-matrix associated to $(e^\vee)^2$, where $e^\vee : H_1(\Gamma, \mathbb{R}) \to \mathbb{R}$ is the functional which associates to a loop the coefficient of $e$ in that loop. Thus, the amplitude in the case of $n$ loops and $2n$ edges is given by
\begin{equation}
A(\Gamma) = C'(n) \int_{\delta} \Omega_{2n-1} \det(\sum a_i M_i)^2
\end{equation}
where $C'(n)$ is another constant depending only on $n$.

I want to acknowledge help from S. Agarwala, M. Marcolli, and O. Ceyhan. Much of this work was done during June, 2012 when I was visiting Cal Tech.

2. Linear Algebra

Fix $n \geq 1$ and consider a vector space $V = k^{2n+2} = ke_1 \oplus \cdots \oplus ke_{2n+2}$. (Here $k$ is a field of characteristic 0.) We write $O = ke_1 \oplus ke_2$ and $I = ke_3 \oplus \cdots \oplus ke_{2n+2}$, so $V = O \oplus I$. $G(2, V)$ will be the Grassmann of 2-planes in $V$.

We have
\begin{equation}
\text{Hom}_k(O, I) \hookrightarrow G(2, V) \to \mathbb{P}(\text{\bigwedge}^2 V).
\end{equation}

Here $i(\psi) = k(e_1 + \psi(e_1)) \oplus k(e_2 + \psi(e_2))$ and $j(W) = \text{\bigwedge}^2 W \hookrightarrow \text{\bigwedge}^2 V$.

Write $V^*$ for the dual vector space with dual basis $e_i^*$. We identify $\text{\bigwedge}^2 V^*$ with the dual of $\text{\bigwedge}^2 V$ in the evident way, so $\langle e_i^* \wedge e_j^*, e_i \wedge e_j \rangle = 1$.

For $\alpha \in \text{\bigwedge}^2 V^*$, the assignment
\begin{equation}
\psi \mapsto \langle (e_1 + \psi(e_1)) \wedge (e_2 + \psi(e_2)), a \rangle
\end{equation}
defines a quadratic map $q_\alpha : \text{Hom}(O, I) \to k$.

**Lemma 2.1.** Assume $0 \neq \alpha = v \wedge w$ with $v, w \in V^*$. Then the quadratic map $q_\alpha$ has rank 4.

**Proof.** It suffices to show $\langle (\sum x_i e_i) \wedge (\sum y_j e_j), v \wedge w \rangle$, viewed as a quadric in the $x_i$ and $y_j$ variables, has rank 4. By assumption $v, w$ are
linearly independent. We can change coordinates so \( v = \varepsilon^*_i, w = \varepsilon^*_j \), and \( \sum x_i e_i = \sum x'_i e_i, \sum y_j e_j = \sum y'_j e_j \). The polynomial is then
\[
\langle \left( \sum x'_i e_i \right) \wedge \left( \sum y'_j e_j \right), \varepsilon^*_i \wedge \varepsilon^*_j \rangle = x'_i y'_j - x'_j y'_i.
\]
This is a quadratic form of rank 4.

Returning to the notation in (2.1), we can write \( I = \bigoplus_{i=1}^n I_i \) with \( I_i = ke_{2i+1} \oplus ke_{2i+2} \). We can think of \( \text{Hom}(O, I) = \bigoplus \text{Hom}(O, I_i) \) as the decomposition of momentum space into a direct sum of Minkowski spaces. We identify \( \text{Hom}(O, I_i) \) with the space of \( 2 \times 2 \)-matrices, and the propagator with the determinant. With these coordinates, an element in \( \text{Hom}(O, I) \) can be written as a direct sum \( A_1 \oplus \cdots \oplus A_n \) of \( 2 \times 2 \)-matrices. The propagators have the form \( \det(a_1 A_1 + \cdots + a_n A_n) \) with \( a_i \in k \). The map \( \psi : O \to I \) given by \( \psi(e_1) = x_3 e_3 + \cdots + x_{2n+2} e_{2n+2} \) and \( \psi(e_2) = y_3 e_3 + \cdots + y_{2n+2} e_{2n+2} \) corresponds to the matrices
\[
A_i = \begin{pmatrix} x_{2i+1} & x_{2i+2} \\ y_{2i+1} & y_{2i+2} \end{pmatrix}.
\]

Lemma 2.2. Let \( A_i \) be as in (2.4). Let
\[
\alpha = \left( \sum_{i=1}^n a_i e^*_{2i+1} \right) \wedge \left( \sum_{i=1}^n a_i e^*_{2i+2} \right) \in \bigwedge^2 V^*.
\]
Then the quadratic map \( q_\alpha \) in lemma 2.1 is given by
\[
q_\alpha(A_1 \oplus \cdots \oplus A_n) = \det(a_1 A_1 + \cdots + a_n A_n).
\]
Proof. This amounts to the identity
\[
\det \left( \sum_{i=1}^n a_i x_{2i+1} y_{2i+2} - \sum_{i=1}^n a_i y_{2i+1} x_{2i+2} \right) =
\langle \left( \sum_{i \geq 3} x_i e_i \right) \wedge \left( \sum_{i \geq 3} y_i e_i \right), \left( \sum_{i=1}^n a_i e^*_{2i+1} \right) \wedge \left( \sum_{i=1}^n a_i e^*_{2i+2} \right) \rangle.
\]
For \( i = j \) (resp. \( i \neq j \)) the coefficient of \( a_i a_j \) in this expression is
\[
x_{2i+1} y_{2i+2} - x_{2i+2} y_{2i+1}
\]
resp. \( x_{2i+1} y_{2j+2} - x_{2i+2} y_{2j+1} + x_{2j+1} y_{2j+2} - x_{2j+2} y_{2i+1} \).

The full inhomogeneous propagator, which in physics notation would be written \( (p_1, \ldots, p_n) \mapsto (\sum a_i p_i + s)^2 \) with the \( p_i \) and \( s \) 4-vectors,
becomes in the twistor setup
\begin{equation}
(c_1e_1^* + c_2e_2^* + \sum_{i \geq 3} a_i e_{2i+1}^*) \land (d_1e_1^* + d_2e_2^* + \sum_{i \geq 1} a_i e_{2i+2}^*)) =
\end{equation}
\begin{align*}
\det \begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \end{pmatrix} + c_1 \sum a_i y_{2i+2} - c_2 \sum a_i x_{2i+2} - d_1 \sum a_i y_{2i+1} + \\
& d_2 \sum a_i x_{2i+1} + \det \left( \sum a_i x_{2i+1} \sum a_i y_{2i+1} \sum a_i y_{2i+2} \sum a_i x_{2i+2} + d_1 \right) \\
& \det \left( \sum a_i x_{2i+1} + c_1 \sum a_i x_{2i+2} + d_1 \right).
\end{align*}

Remark 2.3. In (2.9), our $\alpha \in \wedge^2 V^*$ is of rank 2, i.e. it is decomposable as a tensor and corresponds to an element in $G(2, V) \subset \mathbb{P}(\wedge^2 V^*)$, (2.1). If we want to add mass to our propagator, we simply replace $\alpha$ by $\alpha + m^2 e_1^* \land e_2^*$, yielding $(\sum a_i p_i + m)^2 + m^2$. The massive $\alpha$ represents a point in $\mathbb{P}(\wedge^2 V^*)$ but not necessarily in $G(2, V^*)$.

3. The Twistor Integral

In this section we take $k = \mathbb{C}$. Consider the maps
\begin{equation}
V \times V - S \xrightarrow{\rho} G(2, V) \xrightarrow{\varphi} \mathbb{P}(\wedge^2 V).
\end{equation}
Here $S = \{(v, w) \mid v \land w = 0 \}$ and $\rho(v, w)$ is 2-plane spanned by $v, w$.

Lemma 3.1. $V \times V - S/G(2, V)$ is the principal $GL_2(\mathbb{C})$-bundle (frame bundle) associated to the rank 2 vector bundle $W$ on $G(2, V)$ which associates to $g \in G(2, V)$ the corresponding rank 2 subspace of $V$.

Proof. With notation as in (2.1), let $U = \text{Hom}_\mathbb{C}(O, I) \subset G(2, V)$. We have
\begin{equation}
\rho^{-1}(U) = \{(z_1, \ldots, z_{2n+2}, v_1, \ldots, v_{2n+2}) \mid \det \begin{pmatrix} z_1 & z_2 \\ v_1 & v_2 \end{pmatrix} \neq 0 \}.
\end{equation}
We can define a section $s_U : U \to \rho^{-1}(U)$ by associating to $a : O \to I$ its graph
\begin{equation}
s_U(a) := (1, 0, a_1^1, \ldots, a_{2n}^1; 0, 1, a_1^2, \ldots, a_{2n}^2).
\end{equation}
Using this section and the evident action of $GL_2(\mathbb{C})$ on the fibres of $\rho$, we can identify $\rho^{-1}(U) = GL_2(\mathbb{C}) \times U$. The fibre $\rho^{-1}(u)$ for $u \in U$ is precisely the set of framings $w = \mathbb{C}z \oplus \mathbb{C}v$ as claimed. $\square$
Lemma 3.2. The canonical bundle $\omega_{G(2,V)} = \mathcal{O}(-2n-2)$ where $\mathcal{O}(-1)$ is the pullback $j^*\mathcal{O}_{\mathbb{P}(\Lambda^2 V)}(-1)$.

Proof. The tautological sequence on $G(2,V)$ reads
\[ 0 \rightarrow \mathcal{W} \rightarrow V_{G(2,V)} \rightarrow V_{G(2,V)}/\mathcal{W} \rightarrow 0. \]
Here $\mathcal{W}$ is the rank 2 sheaf with fibre over a point of $G(2,V)$ being the corresponding 2-plane in $V$. One has
\[ \Omega^1_{G(2,V)} = \text{Hom}(V_{G(2,V)}/\mathcal{W}, \mathcal{W}) = (V_{G(2,V)}/\mathcal{W})^\vee \otimes \mathcal{W}. \]
By definition of the Plucker embedding $j$ above we have $\mathcal{O}_{G}(-1) = \Lambda^2 \mathcal{W}$. The formula for calculating chern classes of a tensor product yields
\[ c_1(\Omega^1_{G}) = c_1((V_{G(2,V)}/\mathcal{W})^\vee) \otimes c_1(\mathcal{W}) \otimes 2n = \mathcal{O}_{G}(-2n-2). \]
\[ \square \]

We now fix a point $a \in \mathbb{P}(\Lambda^2 V)$. Upto scale, $a$ determines a non-zero alternating bilinear form on $V$ which we denote by $Q: (x,y) \mapsto \sum_{\mu,\nu} x^\mu Q_{\mu\nu} y^\nu$. By restriction we may view $Q \in \Gamma(G(2,V),\mathcal{O}(1))$. By the lemma $\omega_G \otimes \mathcal{O}(2n+2) \cong \mathcal{O}_G$, so upto scale there is a canonical meromorphic form $\xi$ on $G(2,V)$ of top degree $4n$ with exactly a pole of order $2n+2$ along $Q = 0$. We write
\[ \xi = \frac{\Xi}{Q^{2n+2}}; \quad 0 \neq \Xi \in \Gamma(G,\omega_G(2n+2)) = \mathbb{C}. \]

Lemma 3.3. We have
\[ H^i(V \times V - S, \mathbb{Q}) = \begin{cases} \mathbb{Q} & i = 0, 4n+1, 4n+3, 8n+4 \\ (0) & \text{else} \end{cases}. \]

Proof. We compute the dual groups $H^*_c(V \times V - S, \mathbb{Q})$. Note a complex vector space has compactly supported cohomology only in degree twice the dimension. Also, $H^1_c(V - \{0\}) \cong H^0_c(\{0\}) = \mathbb{Q}$. Let $p: S \rightarrow V$ be projection onto the first factor. The fibre $p^{-1}(v) \cong \mathbb{C}$ for $v \neq 0$ and $p^{-1}(0) = V$. It follows that
\[ H^i_c(S - \{0\} \times V) \cong H^{i-2}_c(V - \{0\}) = (0); \quad i \neq 3, 4n+6. \]
Now the exact sequence
\[ H^i_c(S - \{0\} \times V) \rightarrow H^i_c(S, \mathbb{Q}) \rightarrow H^i_c(V, \mathbb{Q}) \]
yields $H^i_c(S) = \mathbb{Q}$, $i = 3, 4n+4, 4n+6$ and vanishes otherwise. Thus, $H^j_c(V \times V - S) = \mathbb{Q}$; $j = 4, 4n+5, 4n+7, 8n+8$ and vanishes otherwise. Dualizing, we get the lemma. \[ \square \]
Let \( R \subset V \times V \) be the zero locus of the alternating form \( Q \) on \( V \) defined above. Clearly \( S \subset R \).

**Lemma 3.4.** Assume the alternating form \( Q \) is non-degenerate. Then we have

\[
H^i(V \times V - R, \mathbb{Q}) = \begin{cases} 
\mathbb{Q} & i = 0, 1, 4n + 3, 4n + 4 \\
(0) & \text{else}
\end{cases}
\]

**Proof.** Again let \( p : R \to V \) be projection onto the first factor. We have \( p^{-1}(0) = V \) and \( p^{-1}(v) \cong \mathbb{C}^{2n+1} \) for \( v \neq 0 \). It follows that \( H^i_c(R - \{0\} \times V) = (0), i \neq 4n + 3, 8n + 6 \). As before, this yields \( H^j_c(V \times V - R) = \mathbb{Q}, j = 4n + 4, 4n + 5, 8n + 7, 8n + 8 \) and the lemma follows by duality. \( \square \)

Note that in the case \( n = 0 \), \( \text{dim} \, V = 2 \) we have \( S = R \) and the two lemmas give the same information, which also describes the cohomology of the fibres of the map \( \rho \). Namely, \( H^i(\rho^{-1}(pt)) = \mathbb{Q}, i = 0, 1, 3, 4 \) and \( H^1 = (0) \) otherwise.

The form \( Q \) induces a quadratic map on \( V \times V \) given by \( (v, v') \mapsto vQv' \).

**Lemma 3.5.** Choose a basis for \( V \) and write \( dv \) for the evident holomorphic form of degree \( 4n + 4 \) on \( V \times V \). Then \( \mu := dv/Q^{2n+2} \) is homogeneous of degree 0 and represents a non-trivial class in \( H^{4n+4}_{\text{DR}}(V \times V - R) \).

**Proof.** \( V \times V - R \) is affine, so we can calculate de Rham cohomology using algebraic forms. There is an evident \( \mathbb{G}_m \)-action which is trivial on cohomology. Writing a form \( \nu \) as a sum of eigenforms for this action, we can assume the \( \mathbb{G}_m \)-action is trivial on \( \nu \), which therefore is written \( \nu = Fdv/Q^{2n+2+N} \) for some \( N \geq 0 \) and \( \deg F = 2N \). Since \( Q \) is non-degenerate, we can write \( F = \sum_i F_i \partial Q/\partial v_i \). Let \( (dv)_i \) be the form obtained by contracting \( dv \) against \( \partial/\partial v_i \). Then

\[
(3.12) \quad \nu + d\left( \frac{1}{2n+1+N} \sum F_i (dv)_i/Q^{2n+1+N} \right) = Gdv/Q^{2n+1+N}.
\]

where \( G \) is homogeneous of degree \( 2(N-1) \). Continuing in this way, we conclude that \( \nu \) is cohomologous to a constant times \( dv/Q^{2n+2} \). Since by the lemma \( H^{4n+4}(V \times V - R) = \mathbb{Q} \), we conclude that \( \mu := dv/Q^{2n+2} \) is not exact. \( \square \)

If one keeps track of the Hodge structure, lemma 3.4 can be made more precise. One gets e.g. \( H^{4n+4}(V \times V - R, \mathbb{Q}) \cong \mathbb{Q}(-2n-3) \). For a
suitable choice of coordinatizations for the two copies of $V$ and a suitable rational scaling for the chain $\sigma$ representing a class in $H_{4n+4}(V \times V - R, \mathbb{Q})$ we can write the corresponding period as

\[(3.13) \quad \int_{\sigma} d^{2n+2}z \wedge d^{2n+2}v/(\sum z_{\mu}v_{\mu})^{2n+2} = (2\pi i)^{2n+3}.\]

Now we make the change of coordinates $v_{\mu} = \sum_{p} Q_{\mu p}w_{p}$ and deduce

\[(3.14) \quad \int_{\sigma} d^{2n+2}z \wedge d^{2n+2}w/(\sum z_{\mu}Q_{\mu p}w_{p})^{2n+2} = \frac{(2\pi i)^{2n+3}}{\det Q}.\]

Here $Q$ is alternating in our case, so $\det Q = \text{Pfaffian}(Q)^2$.

The “Feynman trick” in this context is the integral identity

\[(3.15) \quad \prod_{i=1}^{2n+2} \frac{1}{A_{i}} = (2n + 1)! \int_{0}^{\infty} \frac{da_{1} \cdots da_{2n+2}\delta(1 - \sum a_{i})}{(\sum a_{i}A_{i})^{2n+2}}.\]

We apply the Feynman trick with $A_{i} = \sum_{\mu, p} z_{\mu}Q_{\mu p}w_{p}$ and integrate over $\sigma$

\[(3.16) \quad \int_{\sigma} \frac{d^{2n+2}z \wedge d^{2n+2}w}{\prod_{i=1}^{2n+2}(\sum_{\mu, p} z_{\mu}Q_{\mu p}w_{p})} = (2n + 1)! \int_{0}^{\infty} \frac{da_{1} \cdots da_{2n+2}\delta(1 - \sum a_{i})}{(\sum a_{i}(\sum_{\mu, p} z_{\mu}Q_{\mu p}w_{p}))^{2n+2}} \int_{0}^{\infty} \frac{d^{2n+2}z \wedge d^{2n+2}w}{(\sum_{\mu, p} z_{\mu}(\sum a_{i}Q_{\mu p}w_{p})^{2n+2}} = (2n + 1)!(2\pi i)^{2n+3} \int_{0}^{\infty} \frac{da_{1} \cdots da_{2n+2}\delta(1 - \sum a_{i})}{\text{Pfaffian}(\sum a_{i}Q_{i})^{2}}.\]

The integral on the right in (3.16) can be rewritten as a projective integral as on the right in (1.4):

\[(3.17) \quad \int_{0}^{\infty} \frac{da_{1} \cdots da_{2n+2}\delta(1 - \sum a_{i})}{\text{Pfaffian}(\sum a_{i}Q_{i})^{2}} = \int \frac{\Omega_{2n+1}}{\text{Pfaffian}(\sum_{i=1}^{2n+2} a_{i}Q_{i})^{2}}.\]

4. PROOF OF THEOREM 1.1

To finish the proof of theorem 1.1 we need to understand the chain of integration $\sigma$ in (3.16). We also need to choose the alternating forms $Q_{i}$ on the left side of (3.16) so the resulting integral coincides up to a constant with the Feynman integral in the statement of the theorem (1.3).
Put an hermitian metric $|| \cdot ||$ on $V$. The induced metric on the bundle of 2-planes defines a submanifold $M \subset V \times V - S$ where $M$ is the set of pairs $(z, v) \in V \times V - S$ such that $||z|| = ||v|| = 1$ and $\langle z, v \rangle = 0$. $M$ is a $\mathbb{U}_2$-bundle which is a reduction of structure of the $GL_2(\mathbb{C})$ bundle $V \times V - S$. The inclusion $M \subset V \times V - S$ is a homotopy equivalence. In particular, the fibre

$$
(\mathbb{R}^4 \rho_\ast \mathbb{Z})_w \cong H^4(M_w) = H^4(\mathbb{U}_2) = \mathbb{Z} \cdot [\mathbb{U}_2].
$$

($\mathbb{U}_2$ is a compact orientable 4-manifold, so this follows by Poincaré duality.)

For the base, write $G^0 := G(2, V) - \{ Q = 0 \}$ where $Q \in \bigwedge^2 V^\vee$ is of rank $2n + 2$. $G^0$ is affine (and hence Stein) of dimension $4n$, so $H^i(G^0, \mathbb{Z}) = (0)$ for $i > 4n$. Let $\rho^0 : V \times V - R \to G^0$ be the $GL_2$ principal bundle obtained by restriction from $\rho$. We are interested in the class in $H^{4n+4}(V \times V - R, \mathbb{Q})$ (cf. lemma [3.4]) dual to $\sigma$. The grassmann is simply connected, so by (1.1), necessarily $R^4 \rho_\ast \mathbb{Z} \cong \mathbb{Z}_G$. Since the fibres of $\rho$ have cohomological dimension 4, we have also

$$
Q = H^{4n+4}(V \times V - R, \mathbb{Q}) \cong H^{4n}(G^0, R^4 \rho_\ast \mathbb{Q}) \cong H^{4n}(G^0, \mathbb{Q}).
$$

It is not hard to show in fact that $H^{4n}(G^0, \mathbb{Q}) = \mathbb{Q} \cdot c_2(\mathcal{W})^n$ where $\mathcal{W}$ is the tautological rank 2 bundle on $G(2, V)$ as in (3.4). The interesting question is what if anything this class has to do with the topological closure of real Minkowski space in $G(2, V)$ which is classically the chain of integration for the Feynman integral.

Recall we have $\Gamma$ a graph with no self-loops and no multiple edges. External edges will play no role in our discussion, so assume $\Gamma$ has none. The chain of integration for the Feynman integral is $\mathbb{R}^{4n}$ where $n$ is the loop number of $\Gamma$. This vector space is canonically identified with $H := H_1(\Gamma, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{R}^4$. In particular, an edge $e \in \text{Edge}(\Gamma)$ yields a functional $e^\vee : H_1(\Gamma, \mathbb{R}) \to \mathbb{R}$ associating to a loop $\ell$ the coefficient of $e$ in $\ell$.

To avoid divergences, the theorem is formulated for euclidean propagators. Let $q : \mathbb{R}^4 \to \mathbb{R}$ be $q(x_1, \ldots, x_4) = x_1^2 + \cdots + x_4^2$. The propagators which appear in the denominator of the integral have the form

$$
H = H_1(\Gamma, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{R}^4 \xrightarrow{e^\vee \otimes \text{id}_4} \mathbb{R}^4 \xrightarrow{q} \mathbb{R}.
$$

We take complex coordinates in $\mathbb{C}^4 = \mathbb{R}^4 \otimes \mathbb{C}$ of the form

$$
z_1 = x_1 + ix_2, \quad z_2 = ix_3 + x_4, \quad w_1 = ix_3 - x_4, \quad w_2 = x_1 - ix_2;
$$

$$
x_1 = \frac{z_1 + w_2}{2}, \quad x_2 = \frac{z_1 - w_2}{2}, \quad x_3 = \frac{z_2 + w_1}{2}, \quad x_4 = \frac{z_2 - w_1}{2}.
$$
In these coordinates \( q = z_1 w_2 - z_2 w_1 \) and the real structure is \( \mathbb{R}^4 = \{ (z_1, z_2, -\pi^2, \pi_1) \mid z_j \in \mathbb{C} \} \).

Now take real coordinates for \( H_1(\Gamma, \mathbb{R}) \) and let \( (z^k_1, z^k_2, w^k_1, w^k_2) \), \( k \geq 1 \) be the resulting coordinates on \( H_\mathbb{C} \). It is then the case that for each edge \( e \) there are real constants \( \alpha_k = \alpha_k(e) \in \mathbb{R} \) not all zero, and the propagator for \( e \) is

\[
(4.6) \quad \det \left( \sum_{k \geq 1} \alpha_k z^k_1 \sum_{k \geq 1} \alpha_k z^k_2 \sum_{k \geq 1} \alpha_k z^k_1 \sum_{k \geq 1} \alpha_k z^k_2 \right) = | \sum_{k} \alpha_k z^k_1 |^2 + | \sum_{k} \alpha_k z^k_2 |^2.
\]

Since the linear functionals associated to the various edges \( e \) span the dual space to \( H_1(\Gamma, \mathbb{R}) \), we see that a positive linear combination of the propagators is necessarily positive definite on \( H_\mathbb{R} \) (i.e. \( > 0 \) except at 0.) Using the coordinates \( z^k_1, w^k_1 \) we can identify \( H_\mathbb{C} \) with an open set in \( G = G(2, 2n + 2) \); namely the point with coordinates \( z, w \) is identified with the 2-plane of row vectors

\[
(4.7) \quad \begin{pmatrix} 1 & 0 & z^1_1 & z^1_2 & z^1_3 & \cdots \\ 0 & 1 & w^1_1 & w^1_2 & w^1_3 & \cdots \end{pmatrix}.
\]

We throw in two more coordinates \( z^0_1, z^0_2 \) (resp. \( w^0_1, w^0_2 \)) and view the \( z^k_j \) (resp. \( w^k_j \)) as coordinates of points in \( V_\mathbb{C} = \mathbb{C}^{2n+2} \). The fact that the set of non-zero matrices of the form \( \begin{pmatrix} z^1_1 & z^1_2 & \cdots \\ -z^0_2 & z^0_2 & \cdots \end{pmatrix} \) is a group under multiplication means that the set of non-zero \( 2 \times (2n + 2) \)-matrices

\[
(4.8) \quad \begin{pmatrix} z^1_0 & z^0_1 & z^1_2 & \cdots & z^0_1 & z^0_2 \\ -z^0_2 & z^0_1 & -z^1_2 & \cdots & -z^0_1 & -z^0_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ z^{2n}_0 & z^{0n}_1 & \cdots & z^{2n}_0 & z^{0n}_1 & \cdots \\
\end{pmatrix}
\]

is closed in \( G \). It is clearly the closure in \( G \) of the real Minkowski space whose complex points are given in (1.7). It will be convenient to scale the rows by a positive real scalar and assume \( \sum_{j,k} |z^k_j|^2 = 1 \), so the resulting locus is compact in \( V \times V - R \). We also scale the bottom row by a constant \( e^{i\theta} \) of norm 1. The resulting locus

\[
(4.9) \quad \sigma := \left\{ \begin{pmatrix} z^0_1 & z^0_2 & z^1_1 & z^1_2 & \cdots & z^0_1 & z^0_2 \\ -e^{i\theta} z^0_2 & e^{i\theta} z^0_1 & -e^{i\theta} z^1_2 & e^{i\theta} z^1_1 & \cdots & -e^{i\theta} z^0_1 & e^{i\theta} z^0_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ z^{2n}_0 & z^{0n}_1 & \cdots & z^{2n}_0 & z^{0n}_1 & \cdots \\
\end{pmatrix} \mid \sum_{j,k} |z^k_j|^2 = 1 \right\}
\]

is compact and depends on \( 4n + 4 \) real parameters.

Let \( Q_\epsilon \in \bigwedge^2 V^\vee \) be the form which associates to (1.7) the determinant

\[
\det \left( \sum_{k \geq 1} \alpha_k(e) z^k_1 \sum_{k \geq 1} \alpha_k(e) z^k_2 \sum_{k \geq 1} \alpha_k(e) w^k_1 \sum_{k \geq 1} \alpha_k(e) w^k_2 \right).
\]
Let $a_e > 0$ be constants, and let $\tilde{Q} = \sum_e a_e Q_e \in \bigwedge^2 V^\vee$. Finally, let $Q_0 \in \bigwedge^2 V^\vee$ associate to the matrix \(4.8\) the minor $z_1^0 z_1^0 + z_2^0 z_2^0$. It is clear that $Q := Q_0 + \tilde{Q}$ doesn’t vanish on any non-zero matrix of the form \(4.8\). We conclude:

**Proposition 4.1.** Let $G(\mathbb{R}) \subset G$ be the set of points \(4.8\). Then with $Q$ as above, we have $G(\mathbb{R}) \subset G^0 = G - \{Q = 0\}$.

The locus $\sigma$, \(4.9\), projects down to $G(\mathbb{R})$ with fibre the group $U_2$.

**Proposition 4.2.** With this choice of $\sigma$ we have

\[
\int_{\sigma} d^{2n+2}z \wedge d^{2n+2}w \neq 0.
\]

**Proof.** Let $v_j^{k,\vee}$ be the basis of $V^\vee$ which is dual to the coordinate system $z_j^k$ introduced above. Then one checks that $Q$ as described above is associated to an element

\[
Q = \sum_{k=0}^{n} b_k v_1^{k,\vee} \wedge v_2^{k,\vee} \in \bigwedge^2 V^\vee, \quad b_k > 0.
\]

Applied to the matrix on the right in \(4.9\),

\[
Q(\cdots) = e^{i\theta} \sum_{k=0}^{n} b_k (|z_1^k|^2 + |z_2^k|^2)
\]

Computing $d^{2n+2}z \wedge d^{2n+2}w$ on the right hand side of \(4.9\) yields

\[
e^{(2n+2)i\theta} d\theta \wedge \\
\wedge d z_1^0 \wedge \cdots \wedge d z_2^n \wedge \sum_k \left( \bar{z}_2^k dz_1^k - \bar{z}_1^k dz_2^k \right) \wedge \bigwedge_{j \neq k} (dz_1^j \wedge dz_2^j).
\]

The crucial point is that the $e^{i\theta}$ factor in the integrand \(4.10\) cancels. Rescaling we can reduce to the case where all the $b_k = 1$. Integrating over $\sigma$ yields a $2\pi i$ from the $id\theta$ and then an integral over the volume form of the $4n + 3$ sphere $\sum_{k=0}^{n} (|z_1^k|^2 + |z_2^k|^2) = 1$. This is non-zero. □

The proof of theorem \(1.1\) is now complete. To summarize, given $\Gamma$, one uses the change of coordinates \(4.4\) in order to rewrite the euclidean propagators $P_i$ as determinants of alternating matrices $Q_i$. One uses the discussion in section \(2\) particularly formula \(2.9\) and remark \(2.3\) to interpret these propagators with external momenta and masses as elements in $\bigwedge^2 V^\vee$, where $V \cong \mathbb{C}^{\text{Edge}(\Gamma)} \cong \mathbb{C}^{2n+2}$. Using \(4.6\), one sees that a positive linear combination of the $Q_i$ does not vanish on the locus $\sigma$ defined in \(4.9\). This means that the integrand
on the right in (3.16) has poles only on the boundary of the chain of integration where some of the $a_i = 0$. The integral on the left, given our definition of $\sigma$, is a constant (depending only on $n$) times the euclidean amplitude integral.

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