Sample-and-Forward: Communication-Efficient Control of the False Discovery Rate in Networks

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Abstract—This work concerns controlling the false discovery rate (FDR) in networks under communication constraints. We present sample-and-forward, a flexible and communication-efficient version of the Benjamini-Hochberg (BH) procedure for multihop networks with general topologies. Our method evidences that the nodes in a network do not need to communicate p-values to each other to achieve a decent statistical power under the global FDR control constraint. Consider a network with a total of \( m \) p-values, our method consists of first sampling the (empirical) CDF of the p-values at each node and then forwarding \( O(\log m) \) bits to its neighbors. Under the same assumptions as for the original BH procedure, our method has both the provable finite-sample FDR control as well as competitive empirical detection power, even with a few samples at each node. We provide an asymptotic analysis of power under a mixture model assumption on the p-values.

I. INTRODUCTION

For large-scale networks where each node performs multiple hypothesis tests, it is of theoretical and practical interest to understand the amount of communication needed among the nodes to achieve a satisfactory testing performance. We follow the false discovery rate control criterion and aim to adapt the celebrated Benjamini-Hochberg (BH) procedure [1] (also see [2]–[9] for extensions) to the network settings with a limited communication budget in mind (e.g., battery-powered sensors), where multiple hypothesis tests come in to play naturally when each node in the network is equipped with multiple sensors for various measurements (e.g., temperature, PM2.5, humidity, etc.). It should be pointed out that our focus on decentralized inference with FDR control is fundamentally different from the existing literature on distributed detection and hypothesis testing formulations [10]–[13] which focus on testing a single hypothesis.

Our study is motivated by a recent distributed FDR control method called the Query-Test-Exchange (QuTE) algorithm [14], which requires each node to transmit its p-values to all of its neighbors. This method can handle multihop networks with general topologies, however, the amount of communication needed to implement this algorithm can be a bottleneck for practical problems as the size of the network or the number of p-values per node grows. On the other hand, the authors in [15]–[18] have studied a distributed sensor network setting by assuming a broadcast model, where each sensor is allowed to broadcast its decision to the entire network. Under this model, they have proposed several (iterative) distributed BH procedures where each sensor broadcasts at most 1-bit information (which is possible as each sensor has only one p-value in [15], [16], [18]; or a transformation is needed in [17] to combine multiple dependent p-values to a single scalar).

Despite the low communication cost, this broadcast model is mainly suitable for small-scale sensor networks (e.g., see [17, Section VIII] for discussions), and the proposed iterative methods are not applicable for multihop networks with general topologies, which is the main focus of this work. In [19], [20], the authors took an asymptotic perspective and proposed a communication-efficient aggregation method for star networks to match the global performance by achieving the global BH threshold asymptotically. However, the analysis is made possible under a mixture model assumption and a fixed alternative (conditional) distribution throughout the nodes, which is different from the original setting for the BH procedure with no restrictions on the alternative distributions. It has been shown that it is possible to reduce the communication cost of the QuTE algorithm via quantization [21], [22], however, these methods still require each node to transmit the quantized version of the p-values.

In this work, we propose sample-and-forward, a sampling-based version of the BH procedure for networks, with finite-sample provable FDR control. In our method, each node samples the (empirical) CDF of p-values (see (1)) and only needs to send \( O(\log m) \) bits to its neighboring nodes. This is in contrast with sending \( O(m) \) real-valued p-values required in the QuTE method [14]; \( O(m \log m) \) bits are needed for the quantized QuTE [22] to retain the full power of BH, while \( O(m) \) bits are sufficient for the FDR control. Our method is flexible in the sense that (a) it can be coupled with the QuTE algorithm to deal with arbitrary network topologies, and (b) the FDR control holds independent of the number of samples taken at each node. Our method shows robust and competitive statistical powers in a variety of simulation settings.

II. MULTIPLE TESTING AND FDR CONTROL

Consider testing the null hypotheses \( H_{0,k}, 1 \leq k \leq m \) according to the test statistics \( X_{k}, 1 \leq k \leq m \). Let \( P_{k} = \Psi_{H_{0,k}}(X_{k}), 1 \leq k \leq m \), denote the p-values computed for the test statistics, where \( \Psi_{H_{0,k}} = 1 - F_{H_{0,k}} \) and \( F_{H_{0,k}} \) is the (hypothetical) CDF of \( X_{k} \) under \( H_{0,k} \). Let \( m_{0} \) denote the
number of statistics generated according to their corresponding null hypotheses, i.e., \( m_0 \) of the null hypotheses are true. The p-values computed for these \( m_0 \) statistics are called null p-values and we refer to the rest of them as non-nulls. If the statistics are generated according to a continuous distribution functions, by the probability integral transform, the null p-values will have a uniform distribution over \([0,1]\).

The multiple testing problem concerns testing \( m \) hypotheses while controlling a simultaneous measure of the type I error at some (prefixed) level \( \alpha \). The two most popular approaches are the family-wise error rate (FWER) control and the false discovery rate (FDR) control. The FWER is more conservative (reject fewer hypotheses) as it controls the probability of making at least one false rejection, while FDR is more liberal (reject fewer hypotheses) as it controls the probability of rejections and the number of false rejections, respectively.

Then, FWER is defined as

\[
\text{FWER} = \Pr(\exists \text{ at least } 1 \text{ false rejection})
\]

while controlling a simultaneous measure of the type I error for all non-decreasing \( D \subseteq [0,1]^m \) and \( i \in H_0 := \{ : k : H_0 \text{ is true} \} \) [23].

**Assumption I:** \( \mathbf{P} \) satisfies the PRDS condition. The PRDS condition holds trivially if the p-values are independent [2].

Let \( \mathbf{P} = (P_1, P_2, \ldots, P_m) \) denote the ascending-ordered p-values. Using this notation, BH procedure rejects \( k \) smallest p-values, i.e.,

\[
\mathcal{R}_{BH} = \{ : P_k \leq P(\hat{k})_k \},
\]

where \( \hat{k} = \max \{ 1 \leq k \leq m : P(k) \leq \tau_k \} \) with \( \tau_k = \alpha k/m \) and \( k = 0 \) if \( P(P_k) > \tau \) for all \( k \). We adopt the notation \( R_{BH} := \hat{k} \) and \( \tau_{BH} := \tau = \alpha \tau_{BH}/m \) for the rejecting index and rejecting threshold, respectively. The power of detection for a multiple testing procedure is defined as \( \mathcal{P} = E_{TDP} \) where \( TDP = R_{V \rightarrow \frac{V}{m \sqrt{2}}} \) is the true discovery proportion, where \( a \lor b := \max\{a, b\} \). Define the function

\[
F(t) = \frac{1}{m} \sum_{i=1}^{m} 1\{ P_i \leq t \}.
\]

In the rest of this work, we refer to \( F(t) \) as the pseudo-CDF since the p-values are not assumed to be i.i.d. Observe that \( F(P_k) = k/m \). Therefore \( R_{BH} = \max \{ 0 \leq k \leq m : F(P_k) \geq F(k)/\alpha \} \), with the convention \( F(0) = 0 \). Similarly, an alternative representation for the BH threshold \( \tau_{BH} \) is known [24], which can be adapted to our setting straightforwardly as \( \tau_{BH} = \sup \{ t \geq 0 : F(t) = t/\alpha \} \).

**III. SAMPLE-AND-FORWARD FOR STAR NETWORKS**

To set the stage for the general multi-hop network settings in the next section, we start with describing our method for a star network, where one center node can send (or receive) information to (or from) \( N \) other nodes. At each node \( i \in \{1, \ldots, N\} \), there are \( m_i(t) = m_0(t) + m_1(t) \) p-values \( \mathbf{P}(i) = (P^{(i)}_1, \ldots, P^{(i)}_{m_i}) \) where \( m_0(t) \) (or \( m_1(t) \)) of them correspond to the true null (or alternative) hypotheses.

There are two obvious ways to attain the global FDR control over the network. Baseline I: performing the global BH procedure, i.e., pooling all the p-values in the network at the center node and performing the BH procedure; Baseline II: performing local BH procedures with (node-wise) Bonferroni corrected test size \( \alpha_i = \frac{\alpha}{m} \). For Baseline I, it is straightforward to observe that the same performance can be achieved by only communicating the p-values less than the test size \( \alpha \) along with the number of p-values each node possesses. However, this is not communication-efficient as it still requires each node to communicate \( \mathcal{O}(m) \) p-values. Regarding the second baseline, even though no communication is needed, the method suffers from a significant power loss in comparison to the pooled inference, especially as the number of nodes in the network grows. Our proposed algorithm achieves a detection power close to Baseline I but with \( \mathcal{O}(\log m) \) bits communication cost for each node as we shall see in the section on numerical results.

**A. Sample-and-Forward: Star Networks**

For an overall targeted FDR level \( \alpha \), our sample-and-forward BH method consists of four main steps. Note that \( \tau_{BH} \leq \alpha \), thus it suffices to sample in \([0, \alpha]\).

1. **Sample the pseudo-CDF:** Each node \( i \) computes

\[
F_i(t) = \frac{1}{m} \sum_{j=1}^{m} 1\{ P_j^{(i)} \leq t \}
\]

and then take \( M \geq 2 \) samples from it on the \([0, \alpha]\) interval, at locations \( t_j := \frac{j-1}{M-1} \alpha, 1 \leq j \leq M \). Then each node \( i \) sends (I) the samples (i.e., \( F_i(t_j) \), \( 1 \leq j \leq M \)) and (II) the number of p-values \( m_i(t) \) to the center node.

2. **Approximate the pooled pseudo-CDF:** The center node computes \( S_j = \frac{1}{m} \sum_{i=1}^{N} m_i(t) F_i(t_j) \), \( 1 \leq j \leq M \) where \( m = \sum_{i=1}^{N} m_i(t) \).

3. **Compute the global threshold:** The center node computes \( \hat{\tau} = t + \alpha(S_j \Rightarrow t_i/\alpha) \) where

\[
I = \max \{ j : S_j \geq t_j/\alpha \}
\]

and broadcasts it to the other nodes.

4. **Reject according to \( \hat{\tau} \):** Upon receiving \( \hat{\tau} \), each node \( i \) rejects \( \{ 1 \leq j \leq m_i(t) : P_j^{(i)} \leq \hat{\tau} \} \).

In the statement of the algorithm, we take a fixed number of samples \( M \) at fixed locations \( t_j \), \( 1 \leq j \leq M \) for each node to simplify the presentation. Nevertheless, the proof does not rely on this structure. In fact, each node (i) can take an arbitrary number of samples \( M(i) \) at any set of locations \( \{ Z_{i}^{(1)}, Z_{i}^{(2)}, \ldots, Z_{i}^{(M(i))} \} \). However, these locations need to be communicated to the center node in step (1). To avoid this communication cost, one can restrict the sampling positions to be equispaced. In this case, the center node can compute the positions based on the number of samples it receives from the local nodes. For the general framework, in step (2), the pooled pseudo-CDF is approximated as follows,

\[
\hat{F}(t) = \frac{1}{m} \sum_{i=1}^{N} m_i(t) \hat{F}_i(t),
\]
Define $G$ the common CDF of the p-values under true alternatives. Note quality in terms of $M$. Let $F$ be a non-increasing function, with probability 1, one can take the sample size $M$ large enough such that the number of null p-values in the network.

$F^{\text{FDR}} \leq \tau \leq \alpha$.

Let $\hat{\tau}$ be the true FDR. To implement this formula, one can use the following procedure. Let $J$ be the set of jump locations of $\hat{F}$. Define $\tau^j = \sup \{ \tau : \hat{F}(\tau) \geq j/\alpha \}$. The threshold $\hat{\tau}$ can be computed via $\hat{\tau} = \tau^0 + \alpha(\hat{F}(\hat{\tau}) - \tau^0/\alpha)$ which is compatible with the formula given in step (3).

**B. FDR Control**

Let $F(t) = \sum_{i=1}^{N} m_i F_i(t)$ denote the (pooled) pseudo-CDF of the p-values in the network.

**Proposition 1:** Our sample-and-forward algorithm controls the FDR globally.

**Proof:** Let $\hat{R}$ denote the total number of rejections in the network made according to $\hat{\tau}$. We note that $\hat{R} = m \cdot \hat{F}(\hat{\tau}) \geq m \cdot \hat{F}(\hat{\tau})/\alpha$ by $\hat{F} \leq \hat{F}$ and (4). Therefore,

$$\text{FDR} = \sum_{j \in \mathcal{H}_a} \mathbb{E} \left[ \frac{1 \{ P_j \leq \hat{\tau} \}}{\hat{R} \cdot \alpha} \right] \leq \sum_{j \in \mathcal{H}_a} \mathbb{E} \left[ \frac{1 \{ P_j \leq \hat{\tau} \}}{(m/\alpha) \cdot \alpha} \right],$$

where $\mathcal{H}_a = \{1 \leq k \leq m : H_{a,k} \text{ is true} \}$. We note that $\hat{\tau}$ is a non-increasing function of any p-value in the network since decreasing a p-value will result in a stochastically smaller sampled pseudo-CDF $\hat{F}$ and thus larger $\hat{\tau}$. Therefore, according to [23], [25], we get $\mathbb{E} \left[ \frac{1 \{ P_j \leq \hat{\tau} \}}{(m/\alpha) \cdot \alpha} \right] \leq 1$. Hence,

$$\text{FDR} \leq \sum_{j \in \mathcal{H}_a} \frac{m_j}{m} \cdot \frac{\alpha}{\alpha} \leq \frac{m_0}{m} \cdot \alpha,$$

where $m_0$ denotes the total number of null p-values in the network.

**C. Power Analysis**

It is of interest to understand the approximation quality of $\hat{F}(t)$ in (2) (under the fixed sampling location scheme with $t_j = \frac{1}{m+1} \cdot \alpha$). Observe that if $\tau_{BH} = 0$, then $\hat{\tau} = 0$ since $\hat{\tau} \leq \tau_{BH}$ on the other hand, if $\tau_{BH} > 0$ and the p-values are generated according to a continuous distribution function, with probability 1, one can take the sample size $M$ large enough such that $P_i(t) < t_j < \tau_{BH} < P_i(t+1)$ for some $1 \leq j \leq M$. In this case, we get $\hat{F}(\tau_{BH}) = \hat{F}(\tau_{BH})$, which implies $\hat{\tau} = \tau_{BH}$ immediately. To shed light on the approximation quality in terms of $\tau$, suppose the p-values are generated i.i.d. according to the CDF $F = \pi_0 U + \pi_1 G$, $\pi_1 > 0$, where $U$ is the distribution function of true nulls and $G(t)$ denotes the common CDF of the p-values under true alternatives. Note that $G$ can be a mixture of various alternative distributions. Define $\tau^* := \sup \{ t \in \mathbb{R} : F(t) = \tau/\alpha \}$.

**Assumption 2:** $F(t) > \tau/\alpha$, $t \in \left( \tau^* - \delta, \tau^* \right)$ for some $\delta > 0$, $F(t)$ is continuously differentiable at $\tau^*$, and $F'(\tau^*) \neq 1/\alpha$ (part of Assumption 2).

The following lemma is a stronger version of Lemma 1, which only relies on continuous differentiability of $F(t)$ at $\tau^*$, and $F'(\tau^*) \neq 1/\alpha$ (part of Assumption 2).

**Lemma 1:** ([20, Lemma 4]) If Assumption 2 holds, then $\tau_{BH} \overset{a.s.}{\rightarrow} \tau^*$.

**Assumption 3:** $t_j < \tau^* < t_{j+1}$ for some $1 \leq j^* \leq M$, meaning, $\tau^*$ is not located exactly at any sampling point $t_j$.

The following theorem (see Appendix B for the proof) concerns the limiting value of the sample-and-forward threshold.

**Theorem 1:** If Assumptions 2 (with $\delta = \delta^*$) and 3 hold, then $\hat{\tau} \overset{a.s.}{\rightarrow} \alpha F(t_{j^*}) = \hat{\tau}_{BH}$ for all $M > (\alpha/\delta^*) \{ \tau^* > 20\delta^* \} + 1$.

**Corollary 1:** If Assumptions 2 (with $\delta = \delta^*$) and 3 hold, then $\lim \hat{\tau} \overset{a.s.}{\rightarrow} \alpha F(t_{j^*}) = \hat{\tau}_{BH} = 0$ for all $M > (\alpha/\delta^*) \{ \tau^* > 20\delta^* \} + 1$.

**Proof:** According to the proof of Theorem 1 (8) and (9), we have $t_j = \alpha F(t_{j^*}) = \tau < t_{j+1} = \tau_{BH}$ for all $M > (\alpha/\delta^*) \{ \tau^* > 20\delta^* \} + 1$. We note that $\hat{\tau} \leq \tau_{BH}$ (because $\hat{F} \leq F$). Therefore, $t_{j^*} < \tau < \tau^* < t_{j+1}$.

**Assumption 4:** $G$ is $C$-Lipschitz on $(t_{j^*}, t_{j+1})$.

Let $\mathcal{P}_B(m)$ and $\mathcal{P}(m)$ denote the detection power of the BH and sample-and-forward procedures. See Appendix C for the proof of the following corollary.

**Corollary 2:** If Assumptions 2 (with $\delta = \delta^*$), 3 and 4 (with $C = C^*$) hold, then $\lim_{m \to \infty} \left( \mathcal{P}_B(m) - \mathcal{P}(m) \right) \leq C^* \frac{\alpha}{\alpha - 1} \{ \tau^* > 20\delta^* \} + 1$.

**Remark 1:** Note that for Gaussian statistics, Assumption 2 holds, and Corollary 2 holds for all $m > 2$ and Corollary 2 holds for all $M > 3$.

**IV. MULTIHOP NETWORK WITH GENERAL TOPOLOGIES**

In this section, we present a communication-efficient algorithm for the general multihop networks, adapting our method to an existing algorithm for decentralized inference known as the QuTE [14], which controls the FDR over networks with arbitrary topology by allowing communication between adjacent nodes. The QuTE algorithm consists of three steps: query, test, and exchange. At the query step, each node needs to communicate all of its p-values to the neighboring nodes. We now present our communication-efficient version of this algorithm where nodes communicate sampled pseudo-CDFs (instead of p-values) while maintaining the FDR control.

1. **Query:** Each node queries its neighbors for the sampled pseudo-CDFs and the number of p-values they possess.

2. **Test:** Let $n(a)$ denote the number of p-values at node $a$ and all of its neighbors. In this step, each node $x$ pools the sampled pseudo-CDFs using (2) and computes the threshold $\tau_x(x)$ (via (4)) with the test size $\alpha(x) = n(x)/m \cdot \alpha$.

3. **Exchange:** Each node $x$ communicates $\tau_x(x)$ to its neighbors. Let $\{\tilde{\tau}_1(x), \ldots, \tilde{\tau}_n(x)\}$ denote the set of thresholds node $a$ receives from its neighbors where $n_a$ denote the number of the nodes neighboring node $a$. Each node $x$ decides about the p-values that originally belong to it based on $\tau_x(x) = \max_{0 \leq i \leq n_a(x)} \tilde{\tau}_i(x)$, i.e., it rejects $R(x) = \{ 1 \leq
Fig. 1. From left to right, the four sets of plots correspond to Experiments 1 to 4. M denotes the number of samples taken at each node, λ the mean of the number of p-values at each node, N the number of nodes, and ρ the correlation coefficient.

\[ j \leq m(x) : P_j(x) \leq \hat{\tau}(x) \]. Note that if \( i \neq 0 \), then \( \hat{\tau}_i(x) = \frac{m(y)}{m(x)} \) for some \( y \) that neighbors \( x \).

**Proposition 2:** Our sample-and-forward version of the QuTE algorithm controls the FDR globally.

**Proof:** We follow the approach in [14] closely. Let \( \mathcal{V} \) and \( \mathcal{H}_0^{(a)} \) denote the set of nodes and the set of null p-values at node \( a \), respectively. For node \( a \), define \( \hat{R}_0^{(a)} = n(a) \frac{\pi_0^{(a)}}{\sigma_0^{(a)}} \) and let \( \{\hat{R}_1^{(a)}, \ldots, \hat{R}_{N_a}^{(a)}\} \) denote the same quantity for its neighbors. Now define \( \hat{R}^{(a)} = \max_{0 \leq i \leq N_a} \hat{R}_i^{(a)} \) and observe \( \hat{R}^{(a)} = \sum_{x \in \mathcal{V} \cup \{x\}} A_{x,j} \), where \( A_{x,j} \) can be upper bounded as follows:

\[ A_{x,j} \leq \mathbb{E} \left[ \frac{1 \{ P_j(x) \leq \hat{\tau}(x) \}}{\hat{R}^{(a)} \vee 1} \right] = \mathbb{E} \left[ \frac{1 \{ P_j(x) \leq \hat{\tau}(x) \}}{m(x) \hat{\tau}(x) \vee 1} \right], \]

because \( \hat{R} \geq \hat{R}^{(a)} \geq \hat{R}^{(y)} \) for some \( y \in x \cup \text{neighbors}(x) \), where \( \hat{R}^{(y)} \) denotes the number of p-values at node \( y \) and its neighbors that are smaller than \( \frac{m(y)}{m(x)} \). By the same argument as in the proof of Proposition 1, \( \hat{\tau}(x) \) is a non-increasing function of any p-value at node \( x \) or at its neighbors. Thus, according to [23], [25], we get

\[ \mathbb{E} \left[ \frac{1 \{ P_j(x) \leq \hat{\tau}(x) \}}{\hat{R}^{(a)} \vee 1} \right] \leq 1. \]

Hence, FDR \( \leq \sum_{x \in \mathcal{V}} \sum_{j \in \mathcal{H}_0^{(x)}} \frac{m(x)}{m(0)} \alpha = \frac{m_0}{m} \alpha \), where \( m_0 \) denotes the total number of null p-values in the network.

The QuTE algorithm has been extended to the setting where each node can query from the neighbors that are \( c \) edges away, which requires \( c \) rounds of communication between immediate neighbors [14]. Our arguments regarding the communication efficiency and FDR control via sampling the pseudo-CDFs can be carried over to \( c \geq 2 \) as well. Note that the “Query” step plays the role of synchronization in the multi-hop setting.

V. NUMERICAL RESULTS

In this section, we evaluate the empirical performance of our algorithm for the star networks with \( \alpha = 0.2 \). The estimated FDR and power are computed by averaging over 10000 trials. The number of p-values at each local node is drawn from Pois(\( \lambda \)) and the probability of generating a non-null p-value at node \( i \) is \( \pi_1^{(i)} = 0.5 - 0.4(i/N) \), where \( N \) denotes the number of local nodes in the network. The statistics are distributed according to \( \mathcal{N}(0,1) \) under \( H_0 \) and we compute two-sided p-values. Under \( H_1 \), we consider composite alternatives, i.e., we generate samples according to \( \mathcal{N}(\mu^{(i)},1) \) at node \( i \) where \( \mu^{(i)} \sim \text{Unif}([\mu^{(i)} - 0.5, \mu^{(i)} + 0.5]) \) and \( \mu^{(i)} = 2 + 4(i/N) \). The pseudo-CDFs at local nodes are sampled at \( \frac{M}{M+1} \), \( 1 \leq j \leq M \). The sample locations in section III-A are chosen slightly differently than this just to simplify the presentation of our power analysis. We compare our method sample-and-forward, with the two baselines discussed in Section III: the Bonferroni method and the global (referred to as pooled-BH) multiple testing by carrying out the BH procedure over all the p-values from all nodes, i.e., \( \{P^{(1)}, \ldots, P^{(N)}\} \). Under this setting, we consider the following four experiments.

**Experiment 1 (vary M).** We examine the effect of varying \( M \) on the power and FDR. We fix \( N = 100 \) and \( \lambda = 3 \). Our sample-and-forward method approaches the pooled BH even when the number of samples \( M \) is as low as 3.

**Experiment 2 (vary \( \lambda \)).** We fix \( M = 3 \), \( N = 100 \) and vary \( \lambda \) from 1 to 10. We observe that for a fixed number of samples taken from the pseudo-CDFs, increasing the number of p-values improves the power of the sample-and-forward while it impacts the Bonferroni method in a negative way.

**Experiment 3 (vary N).** We fix \( M = 3 \) and \( \lambda = 3 \), and vary the number of nodes in the network from \( N = 5 \) to 205. It can be observed that increasing the number of nodes improves the power of the sample-and-forward while it impacts the Bonferroni method in a negative way.

**Experiment 4 (dependent p-values).** We fix \( M = 3 \), \( \lambda = 3 \), and \( N = 100 \), and evaluate the performance of our method when the p-values are dependent. In particular, we adopt the tapering covariance structure \( \Sigma_{i,j} = \rho^{|i-j|} \) where \( \rho \) varies from 0 to 0.9. Although these p-values are not known to satisfy
Assumption 1, it can be observed that the sample-and-forward has stable power and controls the FDR in this setting.

VI. CONCLUSION

We have proposed the sample-and-forward method, which communicates samples of pseudo-CDFs as a communication-efficient alternative for the transmission of (quantized) p-values to perform the BH (or more generally QuTE algorithm) in networks. Our method comes with the finite-sample FDR control, while the power loss has been characterized asymptotically. The numerical results confirm that a node receiving a constant number of pseudo-CDF samples from its neighbors with $O(\log m)$ bits performs similarly to the centralized procedure with $O(m \log m)$ bits.

APPENDIX A

Proofs

A. Proof of Theorem 1

Define the sequences $a_m = \frac{m}{m^2} (1 - m^{-1/4})$ and $b_m = \frac{m^2}{2} (1 + m^{-1/2})$. Recall that $R_{BH} = m/\alpha \cdot \tau_{BH}$ denotes the BH deciding index and note that $a_m \leq R_{BH} \leq b_m$, a.s. for large $m$ according to the proof of [20, Lemma 4]. Pick some $\tau^* - \delta^* < \zeta < \tau^*$ and define,

$$D_{\text{max}}(m) := \frac{m}{\alpha} \sup \{ t : F(t) \geq \tau / \alpha \}$$

$$= \max \left\{ \left. 0 \right\} \cup \left\{ 1 \leq k \leq m : P(k) \leq \frac{\alpha k}{m} \right\} \right. ,$$

$$D_{\text{min}}(m; \zeta) := \frac{m}{\alpha} \inf \{ t : \zeta < \zeta : F(t) \leq \tau / \alpha \}$$

$$= \min \left\{ \left. m \right\} \cup \left\{ \zeta_m < k < m : P(k+1) > \frac{\alpha k}{m} \right\} \right. ,$$

where $\zeta_m = m \zeta / \alpha$. We note that

$$D_{\text{max}} = R_{BH} = \frac{m}{\alpha} \sup \{ t : F(t) = \tau / \alpha \} = \max \left\{ \left. \zeta_m < k < a_m \right\} \right. .$$

Therefore, by Lemma 1 we have $a_m \leq D_{\text{max}} \leq b_m$ for large $m$ almost surely. This implies $\{ t : \zeta : F(t) = \tau / \alpha \} \neq \emptyset$ a.s. for large $m$. Thus, for large enough $m$ we have $D_{\text{min}} \leq D_{\text{max}}$ almost surely. Now we wish to prove $a_m \leq D_{\text{min}} \leq b_m$ for large $m$ almost surely. First note that, $D_{\text{max}} \leq b_m$ a.s. implies $D_{\text{min}} \leq D_{\text{max}}$ a.s. since $D_{\text{min}} \leq D_{\text{max}}$ a.s. for large $m$. So we only need to show $D_{\text{min}} \geq a_m$ a.s. We note,

$$P(D_{\text{min}} < a_m) \leq P \left( \bigcup_{\zeta_m < k < a_m} \{ P(k+1) > \alpha k / m \} \right)$$

$$\leq \sum_{\zeta_m < k < a_m} P \left( \mu(k) - \sum_{i=1}^{m} \{ P_i \leq \alpha k / m \} \geq \mu(k) - k \right) ,$$

where $\mu(k) = \mathbb{E}(\sum_{i=1}^{m} \{ P_i \leq \alpha k / m \}) = m F(\alpha k / m)$. We observe that

$$\mu(k) - k = \alpha (F(\alpha k / m) - \frac{1}{

According to Taylor’s theorem we have,

$$h(a_m) = m \left( \tau^* m^{-1/4} \left( 1 / \alpha - F^*(\tau^*) \right) + o(m^{-1/4}) \right).$$

We note,

$$\inf_{\zeta_m < k < a_m} h(k) = \inf_{\frac{a_m}{\alpha} < \frac{a_m}{m} < \frac{a_m}{2 \alpha m}} \left( \frac{F(\alpha k / m) - \frac{1}{\alpha} (ak / m)}{\zeta_m < t < \tau^* (1 - m^{-1/4})} \left( \frac{F(t - \frac{t}{\alpha})}{\alpha} \right) \right) ,$$

According to Assumption 2, we have $F(t) - t/\alpha > 0$ for all $\zeta < t < \tau^*$. But we note that,

$$F^{*} (1 - m^{-1/4}) - \frac{\tau^* (1 - m^{-1/4})}{\alpha} = \frac{h(a_m)}{m} = o(1),$$

and by Assumption 2, $F(t) - t/\alpha$ is strictly decreasing in a neighborhood of $\tau^*$. Hence, for large enough $m$ we get $\inf_{\zeta_m < k < a_m} h(k) \geq h(a_m)$ which implies $h(k) \geq h(a_m)$ for all $\zeta_m < k < a_m$. Now by Hoeffding’s inequality, we get

$$P(D_{\text{min}} < a_m) \leq \sum_{\zeta_m < k < a_m} e^{-2h(a_m)^2 / m} \leq m e^{-2h(a_m)^2 / m} \to 0 .$$

Since the upper bound is summable in $m$, we get $D_{\text{min}} \geq a_m$ a.s. for large $m$. Therefore, for any $\tau^* - \delta^* < \zeta < \tau^*$ we have

$$\inf \{ t > \zeta : F(t) \leq \tau / \alpha \} \overset{\text{a.s.}}{\to} \tau^* .$$

Also from Lemma 1 we have

$$\sup \{ t : F(t) \geq \tau / \alpha \} \overset{\text{a.s.}}{\to} \tau^* .$$

We note that if $M = (\alpha / \delta^*) \{ \tau^* > 2 \delta^* \}$ and $t_j^* < \tau^* < t_{j+1}^*$, then $\tau^* - \delta^* < t_j^* < \tau^* < t_{j+1}^*$. In this case, according to (6) and (7) we get

$$t_j^* / \alpha < F(t_{j+1}) < t_{j+1}^* / \alpha , \text{ a.s. all } m > m^l(\omega) ,$$

which implies

$$t_j^* < \alpha F(t_{j+1}) < t_{j+1}^* , \text{ a.s. all } m > m^l(\omega) .$$

According to (3), we get

$$\tilde{F} (\alpha F(t_j^*)) = \tilde{F} (t_{j+1}) = \frac{\alpha F(t_{j+1})}{\alpha} .$$

We note that $\tilde{F} \leq \tau_{BH}$ (since $\tilde{F} \leq F$) and therefore, $\tilde{F} = \alpha F(t_j^*)$. By the strong law of large numbers, we have

$$\tilde{F} (t_{j+1}) \overset{\text{a.s.}}{\to} \alpha F(t_{j+1}) ,$$

which implies $\tilde{F} \overset{\text{a.s.}}{\to} \alpha F(t_{j+1})$, completing the proof.

B. Proof of Corollary 2

Proof: According to the Glivenko-Cantelli Theorem [26] and $\tau_{BH} \overset{\text{a.s.}}{\to} \tau^*$, we get $\text{TDPA}_m \overset{\text{a.s.}}{\to} G(\tau^*)$. Dominated convergence theorem implies,

$$\lim_{m \to \infty} (\text{P}_{\text{BH}} - \tilde{\text{P}}_{\text{BH}}) = G(\tau^*) - G(\tau) \leq C^* (\tau^* - \tau) \leq \frac{C^* \alpha}{M - 1} ,$$

where the inequalities hold according to Corollary 1 and Assumption 4.

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