Single Deep Counterfactual Regret Minimization

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Abstract

Counterfactual Regret Minimization (CFR) is the most successful algorithm for finding approximate Nash equilibria in imperfect information games. However, CFR’s reliance on full game-tree traversals limits its scalability. For this reason, the game’s state- and action-space is often abstracted (i.e. simplified) for CFR, and the resulting strategy is then translated back to the full game, which requires extensive expert-knowledge and often converges to highly exploitable policies. A recently proposed method, Deep CFR, applies deep learning directly to CFR, allowing the agent to intrinsically abstract and generalize over the state-space from samples, without requiring expert knowledge. In this paper, we introduce Single Deep CFR (SD-CFR), a simplified variant of Deep CFR that has a lower overall approximation error by avoiding the training of an average strategy network. We show that SD-CFR is more attractive from a theoretical perspective and empirically outperforms Deep CFR with respect to exploitability and one-on-one play in poker.

1. Introduction

In perfect information games, players usually seek to play an optimal deterministic strategy. In contrast, sound policy optimization algorithms for imperfect information games converge towards a Nash equilibrium, a distributional strategy characterized by minimizing the losses against a worst-case opponent. The most popular family of algorithms for finding such equilibria is Counterfactual Regret Minimization (CFR) (Zinkevich et al., 2008). Conventional CFR methods iteratively traverse the game-tree to improve the strategy played in each state. For instance, CFR+ (Tammelin, 2014), a fast variant of CFR, was used to solve two-player Limit Texas Hold’em Poker (Bowling et al., 2015; Tammelin et al., 2015), a variant of poker frequently played by humans.

However, the scalability of such tabular CFR methods is limited since they need to visit a given state to update the policy played in it. In games too large to fully traverse, practitioners hence often employ domain-specific abstraction schemes (Ganzfried & Sandholm, 2014; Brown et al., 2015) that can be mapped back to the full game after training has finished. Unfortunately, these techniques have been shown to lead to highly exploitable policies in the large benchmark game Heads-Up No-Limit Texas Hold’em Poker (HUNL) (Lisy & Bowling, 2016) and typically require extensive expert knowledge to design well. In an attempt to address these two problems, researchers started to augment CFR with neural network function approximation, resulting in DeepStack (Moravčík et al., 2017). Concurrently with Libratus (Brown & Sandholm, 2018a), DeepStack was one of the first algorithms to defeat professional poker players in HUNL, a game consisting of $10^{160}$ states and thus being far too large to fully traverse. A common weakness of tabular CFR algorithms is their poor sample-efficiency. Unfortunately, this issue carries over to DeepStack for reasons we discuss later.

While tabular CFR has to visit a state of the game to update its policy in it, a parameterized policy may be able to play an educated strategy in states it has never seen before. Purely parameterized (i.e. non-tabular) policies have led to great breakthroughs in AI for perfect information games (Mnih et al., 2015; Schulman et al., 2017; Silver et al., 2017) and were recently also applied to large imperfect information games by Deep CFR (Brown et al., 2018a) to mimic a variant of tabular CFR from samples.

Deep CFR’s strategy relies on a series of two independent neural approximations. In this paper, we introduce Single Deep CFR (SD-CFR), a simplified variant of Deep CFR that obtains its final strategy after just one neural approximation by using what Deep CFR calls value networks directly instead of training an additional network to approximate the weighted average strategy. This reduces the overall sampling- and approximation error and makes training more efficient. We show experimentally that SD-CFR improves upon the convergence of Deep CFR in poker games and outperforms Deep CFR in head-to-head matches.
2. Extensive-form games

This section introduces extensive-form games and the notation we will use throughout this work. Formally, a finite two-player extensive-form game with imperfect information is a set of histories \( \mathcal{H} \), where each history is a path from the root \( \phi \in \mathcal{H} \) to any particular state. The subset \( \mathcal{Z} \subset \mathcal{H} \) contains all terminal histories. \( A(h) \) is the set of actions available to the acting player at history \( h \), who is chosen from the set \( \{1, 2, \text{chance}\} \) by the player function \( P(h) \). In any \( h \in \mathcal{H} \) where \( P(h) = \text{chance} \), the action is chosen by the dynamics of the game itself. Let \( N = \{1, 2\} \) be the set of both players. When referring to a player \( i \in N \), we refer to his opponent by \( -i \). All nodes \( z \in \mathcal{Z} \) have an associated utility \( u_i(z) \) for each player. This work focuses on zero-sum games, defined by the property \( u_i(z) = -u_{-i}(z) \) for all \( z \in \mathcal{Z} \).

Imperfect information is represented by partitioning \( \mathcal{H} \) into information sets. An information set \( I_i \) is a subset of \( \mathcal{H} \), where histories \( h, h' \in I_i \) are in the same information set if and only if player \( i \) cannot distinguish between \( h \) and \( h' \) given his private and all available public information. Hence, histories in any information set \( I_i \) differ only by the private information of player \( -i \). For each player \( i \in N \), an information partition \( \mathcal{I}_i \) is a set of all such information sets. Let \( A(I) = A(h) \) and \( P(I) = P(h) \) for all \( h \in I \) and each \( I \in \mathcal{I}_i \).

Each player \( i \) chooses actions according to a behavioural strategy \( \sigma_i \), with \( \sigma_i(I, a) \) being the probability of choosing action \( a \) when in \( I \). We refer to a tuple \( (\sigma_1, \sigma_2) \) as a strategy profile \( \sigma \). Let \( \pi^\sigma(h) \) be the probability of reaching history \( h \) if both players follow \( \sigma \) and let \( \pi^\sigma_i(h) \) be the probability of reaching \( h \) if player \( i \) acts according to \( \sigma_i \) and player \( -i \) always acts deterministically to get to \( h \). It follows that the probability of reaching an information set \( I \) if both players follow \( \sigma \) is \( \pi^\sigma(I) = \sum_{h \in I} \pi^\sigma(h) \) and is \( \pi^\sigma_i(I) = \sum_{h \in I} \pi^\sigma_i(h) \) if \( -i \) plays to get to \( I \).

Player \( i \)'s expected utility from any history \( h \) assuming both players follow strategy profile \( \sigma \) from \( h \) onward is denoted by \( u_i^\sigma(h) \). Thus, their expected utility over the whole game given a strategy profile \( \sigma \) can be written as \( u_i^\sigma(\phi) = \sum_{z \in \mathcal{Z}} \pi^\sigma(z) u_i(z) \).

A strategy profile \( (\sigma_1, \sigma_2) \) is an \( \epsilon \)-Nash equilibrium if its optimal counter-strategy cannot beat it by more than \( \epsilon \). Formally, the following two statements have to hold for all legal strategies \( \sigma'_1 \) and \( \sigma'_2 \)

\[
\begin{align*}
 u_i^{(\sigma_1, \sigma_2)}(\phi) + \epsilon & \geq u_i^{(\sigma'_1, \sigma_2)}(\phi) \quad (1) \\
 u_i^{(\sigma_1, \sigma_2)}(\phi) + \epsilon & \geq u_i^{(\sigma_1, \sigma'_2)}(\phi) \quad (2)
\end{align*}
\]

3. Counterfactual Regret Minimization (CFR)

In this section, we provide an introduction to CFR and some accelerated variants of it.

Counterfactual Regret Minimization (CFR) (Zinkevich et al., 2008) is an iterative algorithm. On each iteration \( t \), it can either run simultaneous or alternating updates. If the former is chosen, CFR produces a new iteration-strategy \( \sigma^*_i \) for all players \( i \in N \). In contrast, alternating updates only produce a new strategy for one player per iteration, with player \( t \mod 2 \) updating his on iteration \( t \).

To understand how CFR converges to a Nash equilibrium, let us first define the instantaneous regret for player \( i \) of action \( a \in A(I) \) in any \( I \in \mathcal{I}_i \) as

\[
r^i_t(I, a) = \pi^\sigma_i(I)(v^\sigma'_i(I, a) - v^\sigma'_i(I))
\]

where

\[
v^\sigma'_i(I) = \sum_{h \in I} \sum_{a \in A(I)} \pi^\sigma_i(h) u^\sigma_i(h \rightarrow a | \sigma^\sigma_{-i}(I))
\]

Intuitively, \( r^i_t(I, a) \) quantifies how much more player \( i \) would have won (in expectation), had he always and chosen \( a \) in \( I \) and played to get to \( I \) but according to \( \sigma^t \) thereafter. The overall regret on iteration \( T \) is

\[
R^i_T(I, a) = \sum_{t=1}^T r^i_t(I, a)
\]

The iteration-strategy for player \( i \) can be derived by

\[
\sigma^t+1_i(I, a) = \begin{cases} 
\frac{R^i_t(I, a)_+}{\sum_{a' \in A(I)} R^i_t(I, a')_+} & \text{if } \sum_{a} R^i_t(I, a)_+ > 0 \\
\frac{1}{|A(I)|} & \text{otherwise}
\end{cases}
\]

where \( x_+ = \max(x, 0) \). Note that \( \sigma^0_i(I, a) = \frac{1}{|A(I)|} \).

Perhaps unintuitively at first, the iteration-strategy profile \( \sigma^t \) does not converge to an equilibrium as \( t \to \infty \) in most variants of CFR.\(^1\). The policy that has been shown to converge to an equilibrium profile is the average strategy \( \bar{\sigma}^T_i \). For all \( I \in \mathcal{I}_i \) and each \( a \in A(I) \) it is defined as

\[
\bar{\sigma}^T_i(I, a) = \frac{\sum_{t=1}^T \pi^\sigma^t_i(I) \sigma^t_i(I, a)}{\sum_{t=1}^T \pi^\sigma^t_i(I)}
\]

\(^1\)In CFR-BR (Johnson et al., 2012) \( \sigma^t \) does converge probabilistically as \( t \to \infty \) and in CFR+ (Tammelin, 2014) it often does so empirically (but without guarantees); in vanilla CFR and linear CFR (Brown & Sandholm, 2018b) \( \sigma^t \) typically does not converge.
3.1. Variations of CFR

Many improvements upon vanilla CFR have been proposed over the past years. Most research focused on improving regret-based methods to be able to solve ever bigger games (Tammelin et al., 2015; Brown & Sandholm, 2018a; Moravčík et al., 2017). This can be achieved in many ways, including improved methods for regret updates (Tammelin, 2014; Brown & Sandholm, 2018b), automated schemes for abstraction design (Ganzfried & Sandholm, 2014), or by incorporating sampling (Lanctot et al., 2009). The most successful algorithms in the recent past also often employ a process referred to as real-time solving or resolving (Brown et al., 2018b; Moravčík et al., 2017). This section will introduce the reader to variants of CFR that are directly relevant to Single Deep CFR.

Discounted Regret Minimization (DFCR) (Brown & Sandholm, 2018b) and CFR+ (Tammelin, 2014) modify equations 6 and 8, which leads to drastically faster convergence with a similar per-iteration cost. A special case of DFCR is linear CFR (LCFR). In LCFR, the contribution of the instantaneous regret of iteration $t$ as well as the contribution of $\sigma^t$ to $\bar{\sigma}^T$ is weighted by $t$. This change alone suffices to let LCFR converge up to two orders of magnitude faster than vanilla CFR does in some large games.

Monte-Carlo CFR (MC-CFR) (Lanctot et al., 2009) proposes a family of tabular methods that visit only a subset of information sets on each iteration. They do this by partitioning $Z$ into blocks (i.e. subsets) such that the union of the set of all blocks $Q = \{Q_1, Q_2, ..., Q_k\}$ spans $Z$. Different versions of MC-CFR can be constructed by choosing different sampling policies and block-sizes. Two examples presented by (Lanctot et al., 2009) are Outcome Sampling (OS) and External Sampling (ES). In OS, only one action is followed at every node, resulting in blocks of size one. ES differs from OS only by that it executes all actions not controlled by $i$ (i.e. those of $-i$ and chance actions). In practice, OS and ES significantly outperform full-width CFR in terms of sample-efficiency and run-time, with ES usually being the faster one of the two. However, in games with many player-actions, ES traverses a relatively large part of the game-tree on each iteration. For such settings, Average Strategy Sampling (Burch et al., 2012) and Robust Sampling (Hui et al., 2018) were developed. They, in different ways, sample only a sub-set of actions for $i$, thereby improving sample efficiency in many settings. For long, CFR+ was regarded as to sensitive to variance for Monte-Carlo sampling, but (Schmid et al., 2018) recently introduced variance-reduced MC-CFR+. VR-MCCFR+ employs baselines similar to those used in reinforcement learning and is two orders of magnitude faster than MC-CFR based on vanilla CFR.

4. Deep CFR

In this section, we summarize how Deep CFR works in its original form as proposed by (Brown et al., 2018a).

Deep CFR computes an approximation of linear CFR (Brown & Sandholm, 2018b) with alternating player updates. On each iteration, it fits a value network $\hat{D}_i$ for one player $i$ to approximate what we call advantage, which is defined as

$$D^T_i(I, a) = \frac{R^T_{i,linear}(I, a)}{\sum_{t=1}^{T}(t\pi^{\sigma^t}_{i}(I))} \tag{9}$$

where here, because Deep CFR approximates linear CFR,

$$R^T_{i,linear}(I, a) = \sum_{t=1}^{T}(t\pi^1_{i}(I, a)) \tag{10}$$

Note that in very large games like HUNL, reach-probabilities naturally are (on average) very small after many tree-branchings. Unfortunately, it is well-known that learning values across many orders of magnitude is difficult for neural networks, assuming that normalization cannot be used (van Hasselt et al., 2016). Dividing $R^T_{i,linear}(I, a)$ by the total linear reach $\sum_{t=1}^{T}(t\pi^{\sigma^t}_{i}(I))$ avoids this problem while still yielding correct results because $\sum_{t=1}^{T}(t\pi^{\sigma^t}_{i}(I))$ is identical for all actions $a \in A(I)$.

We can derive the iteration-strategy for $t + 1$ from $D^t$ similarly to CFR in equation 7 by

$$\sigma^{t+1}_i(I, a) = \begin{cases} \frac{D^t_i(I, a)_+}{\Sigma_{\tilde{a} \in A(I)} D^t_i(I, \tilde{a})_+} & \text{if } \sum_{\tilde{a}} D^t_i(I, \tilde{a})_+ > 0 \\ 1/|A(I)| & \text{otherwise} \end{cases} \tag{11}$$

Deep CFR obtains the training data for $\hat{D}$ via external sampling (Lanctot et al., 2009). All instantaneous regret values that external sampling produces are stored in a memory buffer $B^u_i$. After its maximum capacity is reached, $B^u_i$ is updated via reservoir sampling (Vitter, 1985). To mimic the behaviour of linear CFR, we need to weight the training losses between the predictions $\hat{D}$ makes and the sampled regret vectors in $B^u_i$ with the iteration-number on which a given datapoint was added to the buffer.

At the end of its training procedure (i.e. after the last iteration), Deep CFR fits another neural network $\hat{S}_i(I, a)$ to approximate the linear average strategy

$$\bar{\sigma}^T_i(I, a) = \frac{\sum_{t=1}^{T}(t\pi^{\sigma^t}_i(I))\sigma^t_i(I, a)}{\sum_{t=1}^{T}(t\pi^{\sigma^t}_i(I))} \tag{12}$$

Data to train $\hat{S}_i$ is collected in a separate reservoir buffer $B^s_i$ during the same traversals that data for $B^u_i$ is being collected on. Recall that external sampling always samples all
actions for the traverser, let us call him $i$, and plays according to $\sigma^t_{-i}$ for the opponent. Thus, when $i$ is the traverser, $-i$ is the one who adds his strategy vector $\sigma^t_{-i}(I)$ to $B^t_{-i}$ in every $I \in \mathcal{I}_{-i}$ visited during this traversal. This elegantly assures that the likelihood of $\sigma^t_{-i}(I)$ being added to $B^t_{-i}$ on any given traversal is proportional to $\pi^t_{-i}(I)$. Like before, we also need to weight the training loss for each datapoint by the iteration-number on which the datapoint was created.

Notice that tabular CFR achieves importance weighting between iterations through multiplying with some form of the reach probability (see equations 3 and 8). In contrast, Deep CFR does so by controlling the expected frequency of datapoints from different iterations occurring in its buffers and by weighting the neural network losses differently for data from each iteration.

5. Single Deep Counterfactual Regret Minimization (SD-CFR)

Notice that $\bar{\sigma}$ is not used during the training of linear CFR and Deep CFR. We propose that storing all iteration-strategies would allow one to compute the average strategy ‘on the fly’ during play instead of explicitly storing it for all states. In tabular methods the only gain would be that the memory requirements are halved, coming at the cost of storing $t$ very large tables (though potentially on disk). However, this is much different for Deep CFR. Not aggregating into $\hat{S}$ removes the sampling- and approximation error that $B^s$ and $\hat{S}$ introduce, respectively. In practise, storing all past value networks takes less space than a reasonably sized $B^s$ would during training. Moreover, the computational work needed to train $\hat{S}$ is no longer required. One slight downside is that we need to keep all value networks in memory during play, whereas Deep CFR discards $B^s$ after training. However, in practise, this is possible on a normal laptop. Single Deep CFR (SD-CFR) implements those changes and provides two methods for strategy inference from a buffer of past value networks.

Equation 12 states that the contribution of each iteration-strategy $\sigma^t_i$ to the average strategy $\hat{\sigma}^T_i$ should be proportional to $t\pi^t_i(I)$ in any $I \in \mathcal{I}_i$. Recall that Deep CFR with $\hat{S}$ satisfies this through the frequency of data from different iterations in $B^s$ and through weighting neural network losses of each datapoint with the iteration-number on which it was sampled. In contrast, SD-CFR stores the value networks of all past iterations in model-buffer $B^M_i$ for each player $i$ and assigns the value network created on iteration $t$ a sampling weight equal to $t$. We introduce two ways to correctly weigh by $\pi^t_i(I)$. First, we will look at querying one action-sample on each step along a freely playable trajectory. Second, we consider querying a complete action probability distribution in any $I \in \mathcal{I}_i$.

5.1. Acting on freely playable trajectories

In many cases (e.g. head-to-head evaluation and rollouts), a trajectory is played from the root of the game-tree and the agent is only required to return action-samples of the average strategy on each step forward. To sample from $\hat{\sigma}^T_i$, we first choose a value network $D_t \in B^M_i$ at the start of the game with respect to the sampling weights. The policy $\sigma_t$, which this network gives by equation 11, is now going to be used for the whole game trajectory. We call this method of sampling trajectory-sampling SD-CFR.

By applying the sampling weights when selecting a $D_t \in B^M_i$, we satisfy the linear averaging constraint of equation 12, and by using the same $D_t$ for the whole trajectory starting at the root, we ensure that the iteration-strategies are also weighted proportionally to each of their reach-probabilities in any given state along that trajectory. The latter happens naturally, since $D_t$ of any $t$ produces $\sigma_t$, which reaches each information set $I$ with a likelihood directly proportional to $\pi^T_i(I)$ when playing from the root.

5.2. Querying a complete action distribution

Let us now consider querying the complete action probability-distribution $\hat{\sigma}^T_i(I)$ in some information set $I \in \mathcal{I}_i$. This is necessary for example to evaluate an agent’s exploitability. We can calculate $\hat{\sigma}^T_i(I)$ through an analogous form of equation 12. Crucially notice that the term $\pi^T_i(I)$ is composed of two components 1) $\pi^\tau_i(I)$, the contribution of actions of player $i$ and 2) the contribution of stochastic environment transitions (i.e. chance actions), $\bar{\pi}_i,chance(I)$. Rewriting equation 12 in those terms yields

$$\hat{\sigma}^T_i(I, a) = \frac{\sum_{t=1}^{T} t \pi_{i,chance}(I) \pi^\tau_i(I) \sigma^t_i(I, a)}{\sum_{t=1}^{T} t \pi_{i,chance}(I) \pi^\tau_i(I)} \quad (13)$$

where we can factor $\pi_{i,chance}(I)$ out in both the numerator and the denominator because it does not depend on $\sigma^t_i$. The two terms then cancel each other and we arrive at

$$\hat{\sigma}^T_i(I, a) = \frac{\sum_{t=1}^{T} t \pi^\tau_i(I) \sigma^t_i(I, a)}{\sum_{t=1}^{T} t \pi^\tau_i(I)} \quad (14)$$

As opposed to $\pi^\tau_i$, we can easily compute $\pi^\tau_{i,i}$ without any domain knowledge by

$$\pi^\tau_{i,i}(I) = \prod_{I' \in I, P(I')=i,a':I' \rightarrow I} \sigma^t_{i}(I', a') \quad (15)$$

Here, $I' \in I$ means that $I'$ is on the trajectory leading to $I$ and $a' : I' \rightarrow I$ is the action selected in $I'$ that leads to $I$. 
This computation can be done with at most\(^2\) as many feedforward passes through each of the value networks in \(B^M_i\) as player \(i\) had decisions along the trajectory to \(I\). If not optimized or batched at all, this will typically take a few seconds on two CPU cores assuming \(t = 300\) and 5 decisions for \(i\). An optimization that is especially interesting in games with long trajectories can be applied if a trajectory is played forward from the root. Then, we can cache these reach-probabilities on each step \(I^k\) along the trajectory and compute \(\pi^*_i(I^{k+1}) = \sigma^*_i(I^{k+1}, a')\pi^*_i(I^k)\), where \(a'\) is the action that leads from \(I^k\) to \(I^{k+1}\). This reduces the number of value network queries necessary per step to at most the number of iterations SD-CFR trained.

\[5.3. \text{Theoretical comparison to Deep CFR}\]

We now want to compare the theoretical properties of SD-CFR with those of Deep CFR. SD-CFR always mimics \(\bar{\sigma}^T\) correctly from the iteration-strategies it is provided with. If the iteration-strategies it is given are perfect approximations of the real iteration-strategies, SD-CFR is equivalent to linear CFR according to Theorem 2. Hence, SD-CFR’s performance only depends upon the approximation quality of its value networks and on how many of them can be stored. Theorem 1 shows that this cannot be said about Deep CFR with average strategy networks when the capacity of \(B^*\) or the number of traversals per iteration is finite. Note that SD-CFR faces similar problems whenever the number of iterations trained exceeds the capacity of \(B^M\). However, when using the same neural architecture as Deep CFR (Brown et al., 2018a), this would only happen after about 60,000 iterations when assigning \(B^M\) the same maximal memory capacity that a strategy buffer \(B^*\) needs to store 40,000,000 samples as originally done by (Brown et al., 2018a). Most common applications run CFR just for a four-digit number of iterations (Tammelin et al., 2015; Moravčík et al., 2017), although some experiments run close to 30,000 (Brown & Sandholm, 2018b). However, it was found that Deep CFR’s exploitability mostly oscillates and stops improving after not more than a few hundred to a thousand iterations (Brown et al., 2018a). Conclusively, it is unlikely that keeping all value networks will represent a problem in practise. Moreover, note that both Deep CFR and SD-CFR are only truly sound if, besides a few other conditions, all buffers are of infinite size. Whenever the buffers are finite, SD-CFR starts to introduce iteration bias (through reservoir sampling on \(B^M\)) after the capacity of \(B^M\) is exceeded (e.g. after \(t = 60,000\)), while Deep CFR is likely to do so from \(t = 2\) and is guaranteed to exceed the capacity of \(B^*\). Hence, both algorithms are guaranteed to eventually introduce iteration bias as \(t \to \infty\). However, SD-CFR never introduces additional approximation error when averaging, while Deep CFR does.

Crucially observe that Deep CFR and SD-CFR depend upon the accuracy of the value networks in exactly the same way. Hence, SD-CFR is a better or equally good approximation of linear CFR than Deep CFR with average strategy networks as long as all value networks are stored. While these conclusions show that SD-CFR is largely superior to Deep CFR in theory, they do not imply that SD-CFR will always produce better or equally strong strategies than Deep CFR does. We will investigate this empirically later on.

Theorem 1. If the capacity of strategy buffer \(B^*\) is finite or if only a finite number \(K\) of traversals is executed per iteration, \(B^*\) is not guaranteed to reflect the true average strategy \(\bar{\sigma}^T(I)\) for every \(I \in I_n\) even if all value networks are perfect approximators of the true advantage after any number of training iterations \(T > 2\). Hence, even a perfect function approximator for \(\bar{S}\) is not guaranteed to model \(\bar{\sigma}^T\) without error.

Theorem 2. Assume that for all \(i \in N\), all \(I \in I_n\), all \(a \in A(I)\), and all \(t\) up to the number of iterations trained \(T\), \(\hat{D}^t_i(I, a) = D^t_i(I, a)\) (i.e. that all value networks perfectly model the true advantages). Now, SD-CFR represents \(\bar{\sigma}^T\) without error. This holds for both trajectory-sampling SD-CFR and for when SD-CFR computes \(\bar{\sigma}^T(I)\) explicitly. Furthermore, an opponent has no way of distinguishing which of the two proposed methods of sampling from \(\bar{s}\) is used solely from gameplay.

Proofs for both Theorem 1 and 2 can be found in the supplementary material.

6. Experiments

We empirically evaluate SD-CFR by comparing with Deep CFR and by analyzing the effect of sampling on \(B^M\). Recall that Deep CFR and SD-CFR are equivalent in how they train their value networks. This allows both algorithms to share the same value networks in our experiments, which makes the evaluation far less susceptible to variance over algorithm runs and conveniently guarantees that both algorithms tend to the same Nash equilibrium.

The hyperparameters used are listed separately for each experiment. Where not otherwise noted, we use those from (Brown et al., 2018a). Like them, we train all networks using the Adam optimizer (Kingma & Ba, 2014) with a learning rate of 0.001 and gradient norm clipping of 10. In addition to what (Brown et al., 2018a) describe, our environment observations also include features such as the size of the pot. We represent cards without any domain-specific optimization and concatenate one-hot vectors for each card’s rank and suit.

\(^2\)This number can often be reduced by omitting queries for any iteration’s model as soon as it assigns probability 0 to the action played on that trajectory.
6.1. Exploitability

We compare the exploitability of SD-CFR and Deep CFR in two small games. We use Leduc Hold’em as first described in \cite{Southey2005} and a bigger variant of it that uses a deck with 12 ranks and allows a maximum of 6 instead of 2 bets per round. In Leduc, slight deviations from optimal play give away a lot about one’s private information as there are just three distinguishable cards. In contrast, BigLeduc, despite having more states, might be less susceptible to approximation error as it has 12 distinguishable cards but similarly simple rules.

The networks and buffers for Leduc (Figure 1) are large enough to memorize the game, but yet we find that SD-CFR strongly outperforms Deep CFR. The results in BigLeduc (Figure 2) give evidence for our belief that this gap might get smaller in games where slight strategy deviations are a lesser problem. In both games, winnings are measured in milli-ante per game ($\text{mA/g}$), a linear metric. Although storing all value networks is most often feasible, we analyze the effect of reservoir sampling on $B^M$ in Figure 3 and find it results in plateauing and oscillation.

![Figure 1. Exploitability of Single Deep CFR and Deep CFR in Leduc Hold’em averaged over five runs. All buffers have a capacity of 1 million. On each iteration, data is collected over 1,500 external sampling traversals and a new value network is trained to convergence (750 updates of batch size 2048), initialized randomly at $t < 2$ and with the weights of the value net from iteration $t-2$ afterwards. Average-strategy networks are trained to convergence (5000 updates of batch size 2048) from a random initialization. All networks used for this evaluation have 3 fully-connected layers of 64 units each, which adds up to more parameters than Leduc Hold’em has states.](image1)

![Figure 2. Exploitability of Single Deep CFR and Deep CFR in BigLeduc averaged over five runs. All buffers have a capacity of 4 million. On each iteration, data is collected over 8,800 external sampling traversals and a new value network is trained to convergence (1200 updates of batch size 2816), initialized randomly at $t < 2$ and with the weights of the value net from iteration $t-2$ afterwards. Average-strategy networks are trained to convergence (10000 updates of batch size 5632) from a random initialization. The network architecture used is as in \cite{Brown2018}, differing only by the card-branch having 64 units per layer instead of 192.](image2)

![Figure 3. Exploitability of SD-CFR using $B^M$ with different maximum capacities in Leduc Hold’em. All hyperparameters are as in the previous experiment on Leduc. Graphs are averaged over three runs. In brackets is the memory required to store $B^M$ with the given number of networks, assuming the architecture from \cite{Brown2018}.](image3)
6.2. Head-to-Head in 5-Flop Hold’em Poker (5-FHP)

In addition to the experiments on exploitability, we evaluate the head-to-head performance between SD-CFR and Deep CFR in 5-Flop Hold’em Poker (5-FHP). 5-FHP is a large poker game similar to regular FHP, which was used to evaluate Deep CFR (Brown et al., 2018a). The only difference is that 5-FHP uses five instead of three flop cards to increase the number of states in the game and hence force the agent to abstract and generalize more.

The neural architecture is as in (Brown et al., 2018a). Both algorithms again share the same value networks. On each iteration, we run a batch of 300,000 external sampling traversals. Each value network is trained from a random initialization using a batch size of 10,240 for 4,000 updates. We train the average strategy networks with a batch size of 20,480 for 20,000 updates. Like in (Brown et al., 2018a), $B^o$ and $B^a$ have a capacity of 40 million for each player. SD-CFR’s $B^M$ is essentially infinite, since we run far fewer than the 60,000 iterations that would be needed to fill a buffer with the same memory requirements as $B^a$.

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We found that, on the earliest iterations, SD-CFR consistently defeats Deep CFR by a very large margin. This is expected since $B^o$ holds only very little data at this point. Note that $B^o_i$ for both players had reached its maximum capacity of 40 million by iteration 120 in all five runs. The three measurements in Figure 4 on which SD-CFR appears to have marginally lost to Deep CFR have values and 95% confidence intervals of $-1.1 \pm 5.8$, $-1.3 \pm 5.5$ and $-1.1 \pm 5.4$ mbb/g, and are hence statistical ties. SD-CFR won with statistical significance in 74% of all measurements, where the remaining 26% are statistical ties due to insignificant confidence intervals. SD-CFR was never defeated across the 50 measurements taken and consistently won in the mean over five independent runs.

6.3. Comparing absolute strategies in 5-FHP

In Figure 4, we show the results of head-to-head matches between SD-CFR and Deep CFR measured every 30 iterations. Each mark on the y-axis plots SD-CFR’s average winnings against Deep CFR over 3 million poker hands, measured in milli-big blinds per game (mbb/g), a linear metric conventionally used in the literature. For reference, 10 mbb/g is considered a good margin between humans in Heads-Up Limit Hold’em (HULH), a game with longer action sequences, but similar minimum and maximum winnings per game as 5-FHP. For each evaluation, a new average strategy network was trained. Measuring the performance on iteration $t$ compares how well the SD-CFR averaging procedure would do against the one of Deep CFR if the algorithm stopped training after $t$ iterations.

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### Table 1: Disagreement between SD-CFR’s and Deep CFR’s average strategies

| DEPTH | ROUND | DIF MEAN | DIF STD | N |
|-------|-------|----------|---------|---|
| 0     | PF    | 0.012±0.0001 | 0.017 | 200K |
| 1     | PF    | 0.013±0.0001 | 0.018 | 100K |
| 2     | FL    | 0.052±0.0003 | 0.048 | 80K  |
| 3     | FL    | 0.083±0.0005 | 0.075 | 83K  |
| 4     | FL    | 0.113±0.0011 | 0.109 | 37K  |
| 5     | FL    | 0.175±0.0057 | 0.206 | 5K   |

In Table 1 we analyze how far the average strategies of SD-CFR and Deep CFR are apart at different depths of the tree of 5-FHP. In particular, we measure

$$\frac{1}{2} \sum_{i \in \{1,2\}} \sum_{a \in A(I)} |\bar{\sigma}^{T,SD}_i(I,a) - \bar{\sigma}^{T,\hat{S}}_i(I,a)|$$

We ran 200,000 trajectory rollouts for each player, where player $i$ plays according to SD-CFR’s average strategy $\bar{\sigma}^{T,SD}_i$ and $-i$ plays uniformly random. Hence, we only evaluate on trajectories on which the agent should feel comfortable. The two agents again share the same value networks and thus approach the same equilibrium. We trained for 180 iterations, a little more than it takes for $B^a$ and $B^v$ to be full for both players. Table 1 shows that Deep CFR’s
approximation is good in early levels of the tree but has a larger error in information sets reached after multiple decision points.

7. Related Work

Deep CFR was not the first algorithm that used deep learning with the goal of solving large games efficiently. Regression CFR (R-CFR) (Waugh et al., 2015) applies function approximation to estimate regret values in CFR and CFR+. Unfortunately, despite promising expectations, recent work failed to apply R-CFR in combination with sampling (Srinivasan et al., 2018). Advantage Regret Minimization (ARM) (Jin et al., 2017) is similar to R-CFR but was only applied to single-player environments. Nevertheless, ARM did show that regret-based methods can be of interest in multi-player imperfect information games much bigger, less structured, and more chaotic than poker, thereby opening up interesting lines of research in multi-agent reinforcement learning.

A very successful application of deep learning to CFR in the domain of poker is DeepStack (Moravčík et al., 2017), an algorithm that was able to defeat professional poker players in head-to-head gameplay in the game of Heads-Up No-Limit Hold’em Poker (HUNL) with statistical significance. However, while DeepStack works very well in poker, it still relies on tabular CFR methods to generate data for its counterfactual value networks and for re-solving. This could be too expensive in environments, where sample complexity is a concern. Furthermore, because DeepStack does not use any card abstraction, it would have difficulties handling games with many more private information states than HUNL has.

The recent work Double Neural CFR (Hui et al., 2018) has been successfully evaluated in small games. However, its theoretical guarantees are not clear as its regret-sum-network has to learn a cumulative quantity as opposed to Deep CFR (Brown et al., 2018a), where the advantage, which tends to 0, is learned. Moreover, it seems that the target values Double Neural CFR uses to train its regret-sum-network are scaled by the reach-probability. This could make convergence difficult in large games for reasons we discussed earlier. Deep CFR avoids this by controlling importance weighting over the occurrence-frequency in its buffers. Despite these shortcomings, Double Neural CFR proposes an interesting sampling scheme that blends outcome sampling and external sampling, which is very well suited to be applied to (Single) Deep CFR in games with many actions.

To the best of our knowledge, Neural Fictitious Self-Play (NFSP) (Heinrich & Silver, 2016) was the first algorithm to soundly apply deep reinforcement learning from single trajectory samples to large extensive-form games. While not showing record-breaking results in terms of exploitability, NFSP was able to learn a competitive strategy in Limit Texas Hold’em Poker over just 14 GPU/days.

Recent literature elaborates on the convergence properties of multi-agent deep reinforcement learning (Lancot et al., 2017). A theoretical discussion around the relationship between game theoretic approaches to finding Nash equilibria and the convergence of actor-critic reinforcement learning methods in multi-agent imperfect information games (Srinivasan et al., 2018) lead to three novel policy optimization algorithms.

8. Future Work

So far, Deep CFR was only evaluated in games with three player actions. Since external sampling would likely be intractable in games with tens or more actions, one could employ robust sampling (Hui et al., 2018) or average-strategy-sampling (Burch et al., 2012) in such settings. In contrast to some recent work involving re-solving (Moravčík et al., 2017; Brown & Sandholm, 2018a), SD-CFR and Deep CFR presumably only work well with a simplified action-space in games with hundreds or thousands of actions. To aid this, we are highly interested in developing methods based on continuous approximations of large discrete action-spaces where actions are closely related (e.g., bet-size selection in No-Limit Poker games). This might be achieved by having the value networks predict parameters to a continuous function whose integral can be evaluated efficiently. The iteration-strategy could be derived by normalizing the advantage clipped below 0. The probability of action a could be calculated as the integral of the strategy on the interval corresponding to a in the discrete space.

After the recent success of real-time re-solving (Brown & Sandholm, 2017; Moravčík et al., 2017), it could be interesting to use an SD-CFR agent as a blueprint-strategy.

Perhaps given a few modifications of its neural architecture and sampling procedure, SD-CFR could potentially abstract over much less structured domains than poker. This is exciting for two reasons. First, popular abstraction schemes heavily rely on features specific to poker such as invariance in the order of cards, which many other games and real-world problems do not offer. SD-CFR, however, abstracts intrinsically and hence does not require such favourable patterns in its problem domain. Second, many successful deep reinforcement learning methods such as PPO (Schulman et al., 2017) empirically converge to good policies in 2-party imperfect information games such as Dota 2, but are not guaranteed to approach a Nash equilibrium, while SD-CFR would. A first step on this line of
research could be to evaluate whether SD-CFR is preferred over approaches such as (Srinivasan et al., 2018) in these settings.

9. Conclusions

We introduced Single Deep CFR (SD-CFR), a simplified variant of Deep CFR, that is more attractive in theory and performs better in practice. SD-CFR extracts the average strategy directly from the collection of value networks instead of training another neural network to approximate it from samples. We show that this modification improves convergence and outperforms Deep CFR in head-to-head matches. In the future, SD-CFR could be applied to more chaotic imperfect information games than poker.

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Code

https://github.com/TinkeringCode/Deep-CFR implements Deep CFR and SD-CFR, and provides scripts to reproduce the results presented in this paper.

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A. Proof of Theorem 1

Proof. Let $I$ be any information set in $\mathcal{I}_i$. Assuming that $0 < \pi^T_i(I) < 1$. Recall that external sampling only one action for player $i$ and $\text{chance}$ at any decision point, when $-i$ is the traverser. Since $(1 - \pi^T_i(I))^K > 0$ for any finite number of traversals $K$ per iteration, we cannot guarantee that $I$ will be visited. If $I$ is not visited despite $\pi^T_i(I) > 0$, the contribution of $\sigma_i^t$ to $\bar{\sigma}_i^T(I)$ is not represented in $B^t_i$.

For the second argument, we assume that $K = \infty$. Let $I$ again be any information set in $\mathcal{I}_i$ in which $|A(I)| > 1$. Assume that $\pi^T_i(I)$ is irrational and that $\pi^T_i(I)$ is rational. Clearly, because its capacity is finite, $B^t_i$ could not reflect the ratio between $\pi^T_i(I)$ and $\pi^T_i(I)$ correctly through the frequency of the appearance of samples from iterations $t$ and $j$, regardless of the number of traversals. Furthermore, in games where the number of members in the set 

$$\{I \in \mathcal{I}_i : |A(I)| > 1, \pi^T_i(I) > 0\}$$

is bigger than the capacity of $B^t_i$, not every $I \in \mathcal{I}_i$ can fit into $B^t_i$ on iteration $t$, also making $B^t_i$ an incomplete representation of $\bar{\sigma}_i^T(I)$. \qed

B. Proof of Theorem 2

Proof. Let $B^M_i$ be a buffer of all value networks up to iteration $T$ belonging to player $i$.

Since $\hat{D}^t_i(I, a) = D^t_i(I, a)$ for all $I \in \mathcal{I}_i$ and all $a \in A(I)$,

$$\sigma_i^{t+1}(I, a) = \begin{cases} \frac{D^t_i(I, a)}{|A(I)|} & \text{if } \sum_{a} D^t_i(I, a) > 0 \\ \text{otherwise} & \end{cases}$$

(16)

can be restated in terms of $\hat{D}^t_i(I, a)$.

Recall that $\pi^T_i(I)$ is composed of the two components $\pi_{i, \text{chance}}^T(I)$ and $\pi_{i, \text{chance}}^T(I)$ for the contribution of $i$ and $\text{chance}$, respectively. Hence, we can rewrite

$$\bar{\sigma}_i^T(I, a) = \frac{\sum_{t=1}^T (t \pi_{i, \text{chance}}^T(I) \pi_{i, \text{chance}}^T(I) \sigma_{i}(I, a))}{\sum_{t=1}^T (t \pi_{i, \text{chance}}^T(I) \pi_{i, \text{chance}}^T(I))}$$

(17)

$\pi_{i, \text{chance}}(I)$ does not depend on $t$ and can thus be factored out in both the numerator and the denominator

$$\bar{\sigma}_i^T(I, a) = \frac{\pi_{i, \text{chance}}(I) \sum_{t=1}^T (t \pi_{i, \text{chance}}^T(I) \sigma_{i}(I, a))}{\pi_{i, \text{chance}}(I) \sum_{t=1}^T (t \pi_{i, \text{chance}}^T(I))}$$

(18)

which can be simplified to

$$\bar{\sigma}_i^T(I, a) = \frac{\sum_{t=1}^T (t \pi_{i, \text{chance}}^T(I) \sigma_{i}(I, a))}{\sum_{t=1}^T (t \pi_{i, \text{chance}}^T(I))}$$

(19)

By definition,

$$\pi_{i,i}^T(I) = \prod_{I' \in I, P(I') = i, a': I' \to I} \sigma_{i}(I', a')$$

(20)

Since all $\sigma_i^T$ have no error by assumption, SD-CFR’s recomputation of $\pi_{i}^T(I)$ and hence also $\bar{\sigma}_i^T(I, a)$ are exact for any $I \in \mathcal{I}_i$ and all $a \in A(I)$.

To show this for trajectory-sampling SD-CFR, consider a trajectory starting at the tree’s root $\phi$ leading to an information set in $I \in \mathcal{I}_i$. Since $\sigma_i^T$ can be deduced from $\hat{D}^t_i$ as before, $B^M_i$ can be seen as a buffer of iteration-strategies. Let $f : I \rightarrow a$ be a function that first chooses a $\sigma_i^T \in B^M_i$, where each $\sigma_i^T$ is assigned a sampling weight of $t \pi_{i}^T(I)$. $f$ then returns an action sampled from the distribution $\sigma_i^T(I)$.

Since $f$ weights strategies like the numerator of the definition

$$\bar{\sigma}_i^T(I, a) = \frac{\sum_{t=1}^T (t \pi_{i}^T(I) \sigma_{i}(I, a))}{\sum_{t=1}^T (t \pi_{i}^T(I))}$$

(21)

executing $f(I)$ is equivalent to sampling directly from $\bar{\sigma}_i^T$.

Note that $\pi_{i}^T(\phi) = 1$ for all $\sigma$. Thus, $f(\phi)$ would choose a given $\sigma_i^T \in B^M_i$ with sampling weight $t$. This is what trajectory-sampling SD-CFR does at $\phi$. For each information set $I'$ from $\phi$ until the end of the trajectory, SD-CFR plays using the same iteration-strategy selected at $\phi$. Thus, SD-CFR will reach each information set $I$ with a probability proportional to $t \pi_{i}^T(I)$ conditional on knowing which iteration-strategy was selected. Combining these facts, we see that the assigned weight of $\sigma_i^T$ in any $I$ is $t \pi_{i}^T(I)$ for any $t$ up to $T$. It follows that the probability of $\sigma_i^T$ being the acting policy in any $I$ is

$$\frac{t \pi_{i}^T(I)}{\sum_{t'=1}^T (t' \pi_{i}^T(I))}$$

Since this is equivalent to the weighting scheme between iteration-strategies in the definition of $\bar{\sigma}_i^T$, trajectory-sampling SD-CFR samples correctly from $\bar{\sigma}_i^T$.

Moreover, because the opponent does not know which $\sigma_i^T$ is the acting policy, this result also shows that an opponent cannot tell whether the agent is using this sampling method or following an explicitly computed $\bar{\sigma}_i^T$. \qed