Diversity Order Gain with Noisy Feedback in Multiple Access Channels

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Abstract—In this paper, we study the effect of feedback channel noise on the diversity-multiplexing tradeoff in multiuser MIMO systems using quantized feedback, where each user has \( m \) transmit antennas and the base-station receiver has \( n \) antennas. We derive an achievable tradeoff and use it to show that in SNR-symmetric channels, a single bit of imperfect feedback is sufficient to double the maximum diversity order to \( 2mn \) compared to when there is no feedback (maximum is \( mn \) at multiplexing gain of zero). Further, additional feedback bits do not increase this maximum diversity order beyond \( 2mn \). Finally, the above diversity order gain of \( mn \) over non-feedback systems can also be achieved for higher multiplexing gains, albeit requiring more than one bit of feedback.

I. INTRODUCTION

Channel state information to the transmitters has been extensively studied in MIMO systems [1–12] to improve over the diversity-multiplexing tradeoff without feedback [13, 14]. While the earlier work often assumed noiseless feedback (possibly quantized), recent emphasis has been on studying the performance with noisy feedback [4–7] in single-user MIMO channels. Two distinct models of feedback have appeared. First is that of two-way training, suitable for symmetric time-division duplex systems and is the focus of study in [5, 6]. The other is that of quantized channel state feedback in the context of single user systems [1, 2, 4, 5, 7].

We highlight the fact that all our results are derived for a multiuser system with \( L \) users, each with \( m \) transmit antennas and a receiver with \( n \) receive antennas. This in contrast to most of the earlier work which has considered noisy channel state feedback in the context of single user systems [1, 2, 4, 5, 7].

The rest of the paper is outlined as follows. In Section II we give background on the channel model, introduce feedback model and diversity-multiplexing tradeoff. In Section III, we find the diversity-multiplexing tradeoff for Multiple Access Channel. In Section IV, we discuss these results. Section V concludes the paper.

II. SYSTEM MODEL

A. Channel Model

Consider a multiple access channel with \( L \) users where the transmitters have an array of \( m \) transmit antenna and the receiver has an array of \( n \) receive antenna. The channel is constant during a fading block of \( T \) channel uses, but changes independently from one block to the next. During a fading block \( k \), the channel is represented by \( n \times m \) random matrices \( H_{s,l} \) \((1 \leq s \leq L)\), and the received signal can be written in the matrix form as \( Y_{l} = \sum_{1 \leq i \leq L} H_{s,l} X_{s,l} + W_{l} \). Here, \( W_{l} \) of size \( n \times T \) represents additive white Gaussian noise at the
receiver with all entries i.i.d. $CN(0,1)$. We consider a richly scattered Rayleigh fading environment, i.e. elements of $H_{s,l}$ are assumed to be i.i.d $CN(0,1)$. The transmitters are subject to an average power constraint such that the long-term power is upper bounded, i.e. $\mathbb{E} \left[ X_{s,l}^2 \right] \leq SNR_i$ for $1 \leq s \leq L$.

### B. Feedback Model

We will assume that the receiver has perfect knowledge of the channel coefficients $H_{s,l}$ ($1 \leq s \leq L$). We denote $H_l = (H_{1,l}, H_{2,l}, ..., H_{L,l})$. The receiver then uses the knowledge of channel coefficients to compute a feedback signal $I(H_l)$ which is sent to the transmitters. Furthermore, we will assume that this feedback signal takes on only finite number of values from the set $\{1, 2, ..., K\}$, where $K > 1$. Note that when $K = 1$, there is no feedback, and hence the case reduces to that in [13]. Finally, the mapping $I(H_l) : H_l \mapsto \{1, 2, ..., K\}$ is a deterministic function which can potentially depend on the SNR and the rate of transmission. Due to the error in the feedback, the users do not receive the same signal as is sent by the receiver. The feedback channel is modeled as follows. Let $I(H_l) = i$ be transmitted from the receiver. User $s$ receives an index $I_s$ which takes on only finite number of values from the set $\{1, 2, ..., K\}$ and is given by

$$I_s = \{ i \text{ with probability } 1 - \epsilon \} \cup \{ i' \neq i \text{ with probability } \frac{\epsilon}{K-1} \} \quad \text{for } 1 \leq s \leq L,$$

where $\epsilon$ depends on SNR.

### C. Diversity-Multiplexing Tradeoff Definitions

A codeword $X_{s,l}$ is assumed to span a single fading block. Since we do not consider coding over multiple fading blocks, the block index $l$ will be omitted whenever this does not cause any confusion. Conditioned on indices $I_s = i_s$, the transmitter $s$ chooses a codeword $X_s$ from the codeword $C_{s,i_s} = \{X_{s,i_s}(1), X_{s,i_s}(2), ..., X_{s,i_s}(E_s)\}$ of rate $R_s$ for $1 \leq s \leq L$. All the $X_{s,i}(k)$’s are matrices of size $m \times T$.

In this paper, we will only consider single rate transmission where the rate of the codebooks does not depend on the feedback index and is known to the receiver. Therefore, regardless of which feedback index the transmitters receive, the receiver attempts to decode the received codeword from the same codebook. Outage occurs when the transmission power is less than the power needed for successful (outage-free) transmission.

The average power constraint at each transmitter can be given along the lines of [8] as follows. First define average power per codeword

$$P_i^s = \frac{1}{TE} \sum_{k=1}^{E_s} \left\| X_{s,i}(k) \right\|_F^2, \quad 1 \leq s \leq S$$

which leads to average power constraints

$$E[H_i^s P_i^s (H) \mathbb{P}(I_s(H) = i)] \leq SNR_s, \quad 1 \leq s \leq L \quad \text{(1)}$$

where $\mathbb{P}(\alpha)$ denotes probability of event $\alpha$.

Since our focus is asymptotic performance behavior in the form of diversity-multiplexing tradeoff, we will assume that

$$SNR_s \approx SNR \quad \text{for all } 1 \leq s \leq L.$$ \footnote{We adopt the notation of [14] to denote $\approx$ to represent exponential equality. We similarly use $\geq, >, \leq, <$ to denote exponential inequalities.}

Note that all the index mappings, codebooks, rates, powers are dependent on SNRs. The dependence of rates on the SNRs is explicitly given by $R_s = r_s \log SNR_s$. We refer to $r_s = (r_s)_{1 \leq s \leq L}$ as the multiplexing gains.

In point-to-point channels, outage is defined as the event that the mutual information of the channel, $I(X;Y|H)$ is less than the desired rate $R_s$ where $I(X;Y|H) = \log \det (I + \frac{P}{m}HQH^T)$ is the mutual information of a point-to-point link with $m$ transmit and $n$ receive antennas, transmit power $P$ and input distribution Gaussian with covariance matrix $Q$ [14]. Since $I(X;Y|H)$ depends on transmit power, we write this dependence explicitly as $I(X;Y|H,P)$. In a multiple access channel, corresponding outage event is defined as the event that the channel cannot support target data rate for all the users [13].

Hence, for a multiple access channel with $L$ users, each equipped with $m$ transmit antennas, and a receiver with $n$ receive antennas, the outage event is $O(R, P) \triangleq \bigcup_S O_S(R, P)$ where $P = (P_1, P_2, ..., P_L)$ and $R = (R_1, R_2, ..., R_L)$. The union is taken over all subsets $S \subseteq \{1, 2, ..., L\}$, and $O_S(R, P) \triangleq \{ H \in \mathbb{C}^{n \times m} : I(X_S;Y|X_S,H) < \sum_{i \in S} R_i \}$ where $X_S$ contains the input signals from the users in $S$ with powers $P$. As before, $I(X_S;Y|X_S,H)$ represents $I(X_S;Y|X_S,H)$ when the transmit powers are $P$. Let $\Pi(O)$ denote the probability of outage. The system is said to have diversity order of $d$ if $\Pi(O) \approx SNR^{-d}$. The diversity multiplexing for the multiple users can be described as: given the multiplexing gains $r_s$ for all the users, the diversity order that can be achieved describes the diversity-multiplexing tradeoff region.

The probability of outage with rate $R = (R_1, R_2, ..., R_L)$ and transmit power $P = (P_1, P_2, ..., P_L)$ is denoted by $\Pi(O(R, P)) = \Pi(\bigcup_S O_S(R, P))$. Also let $U(R, P)$ be defined as the indicator function of $\bigcup S O_S(R, P)$. Then, $\Pi(O(R, P))$ is the probability of event $\{U(R, P) = 1\}$ over the randomness of channel matrices. Let $P_S \triangleq SNR_{P_S}$ for all $1 \leq s \leq L$. Further let $R = (R_1, R_2, ..., R_L) \triangleq (r_1 \log SNR, r_2 \log SNR, ..., r_L \log SNR)$. Let $D(r, p)$ be defined as $\Pi(O(R, P)) = SNR^{-D(r, p)}$ where $r = (r_1, r_2, ..., r_L)$ and $p = (p_1, p_2, ..., p_L)$.

**Lemma 1.** Let $p_s = p$ for all $1 \leq s \leq L$. Also, let $\sum_{i \in S} r_i \leq \min(|S|m, n)$ for all non-empty subsets $S$ of $\{1, 2, ..., L\}$. Then,

$$D(r, p) = \min_S G_{m,n}(\sum_{i \in S} r_i, p).$$ \footnote{We adopt the notation of [14] to denote $\approx$ to represent exponential equality. We similarly use $\geq, >, \leq, <$ to denote exponential inequalities.}

where $G_{m,e}(r, p) \triangleq \inf_{\alpha \in A} \sum_{i=1}^{\min(m,n)} (2i - 1 + \max(m, n) - \min(m, n)) \alpha_i$. 

In a multiple access channel, corresponding outage event is defined as the event that the channel cannot support target data rate for all the users [13].
with \( A \triangleq \{ \alpha_1^{\min(m,n)} | \alpha_1 \geq \ldots \alpha_{\min(m,n)} \geq 0, \sum_{i=1}^{\min(m,n)} (p - \alpha_i) < r \} \).

**Proof:** Note that \( \Pi(R, P) = \Pi_H \left( \bigcup_{S \subseteq \mathcal{O}_S} O_S(R, P) \right) \). Hence, \( \Pi_H(O_S(R, P)) \leq \Pi(R, P) \leq \sum_S \Pi(H(O_S(R, P))) \). As, \( \Pi_H(O_S(R, P)) \) is probability of outage for single user with \(|S| \) transmit antennas, \( n \) receive antennas, rate \( \sum_{i \in S} r_i \log(\text{SNR}) \), power \( \geq \text{SNR}^p \), by [8], \( \Pi_H(O_S(R, P)) \geq \text{SNR}^{-\min(S|m,n) \sum_{i \in S} r_i} \). Hence, \( \Pi(R, P) \geq \text{SNR}^{-D(r,p)} \). ■

**Remark 1.** \( G_{m,n}(r,p) \) is a piecewise linear curve connecting the points \( (r, G_{m,n}(r,p)) = (kp, p(m-k)(n-k)) \), with \( k = 0,1,\ldots,\min(m,n) \) for fixed \( m, n \) and \( p > 0 \). This follows directly from Lemma 2 of [8].

### D. Feedback-based Power Control

In this section, we describe the power control policy for the optimum receiver for which successful decoding occurs if the transmission power is greater than or equal to the power needed for outage-free transmission. Recall that the sent feedback signal \( I_1 \) and the received feedback signal \( I_a \) takes values over a finite set as described in Section II-B. For each received index \( I_a = i \) at User \( s \), the transmitted power is denoted by \( P_{s,i} \). We assume that \( P_1^s \leq P_2^s \leq \ldots \leq P_K^s \). We denote the power tuple as \( P_s = (P_1^s, P_2^s, \ldots, P_L^s) \). Following [8, 11], \( I = i \) is calculated as

\[
i = \begin{cases} 1 & \text{if } U(R, P_K) = 1 \\
\min_{k \in \{1,\ldots,K\}} U(R, P_k) = 0 & \text{otherwise}
\end{cases}
\]

According to the scheme, we transmit at minimum power level needed for outage-free transmission in case outage can be avoided, and send at minimum power level in case it cannot be avoided. Using the scheme, we can compute the probability of occurrence of event \( (I = i) \) as

\[
\Pi(I = i) = \begin{cases} 1 + \Pi(R(P_K) - \Pi(R, P_1), & i = 1 \\
\Pi(R(P_{i-1}) - \Pi(R, P_i), & 2 \leq i \leq K
\end{cases}
\]

The power levels are chosen to minimize the outage probability \( \Pi(O) \) subject to power constraints (1).

### III. DIVERSITY-Multiplexing Tradeoff

In this section, we will give an achievable diversity multiplexing tradeoff with errors in feedback with certain cases when this is the best achievable. We assume that the feedback errors decay with SNR as \( \epsilon \triangleq \text{SNR}^{-y} \).

When \( K = 1 \), there is no feedback and hence no imperfection. The diversity for any multiplexing vector \( r = (r_1, r_2, \ldots, r_L) \) is given by \( D(r, 1) \) for \( \sum_{i \in S} r_i < \min(|S|m,n) \) for all non-empty subsets \( S \subseteq \{1,2,\ldots,L\} \) where \( 1 \) is a vector of length \( L \) containing all ones [13]. So, we only consider the case \( K > 1 \) in this section.

Let \( C_{m,n,K}(r) \) be given by a recursive equation

\[
C_{m,n,j}(r) = \begin{cases} 0 & \text{when } j = 0 \\
D(r, 1 + C_{m,n,j-1}(r)) & \text{when } j \geq 1
\end{cases}
\]

**Theorem 1.** Suppose that \( K > 1 \) and \( \epsilon \triangleq \text{SNR}^{-y} \) for some \( y > 0 \). Further suppose that \( \sum_{i \in S} r_i < \min(|S|m,n) \) for all non-empty subsets \( S \subseteq \{1,2,\ldots,L\} \). Then, the lower bound for diversity-multiplexing tradeoff is given by \( \epsilon_{\text{opt}} = \min(C_{m,n,K}(r), y + C_{m,n,1}(r)) \) where \( C_{m,n,K}(r) \) is given by a recursive equation

\[
C_{m,n,j}(r) = \begin{cases} 0 & \text{when } j = 0 \\
D(r, 1 + \min(y, C_{m,n,j-1}(r))) & \text{when } j \geq 1
\end{cases}
\]

**Proof:** The probability of outage for this scheme can be bounded as

\[
\Pi(O) \leq \text{SNR}^{-D(r,p)} \sum_{s=1}^{K} \Pi(I_s < i) \Pi(I = i) = \left(1 - \epsilon\right) \Pi(R, P_K) + \frac{\epsilon K}{K - 1} \sum_{s=1}^{K-1} \Pi(R, P_s)
\]

We can determine the probability that \( I_s = i \) as

\[
\Pi(I_s = i) = \frac{\epsilon}{K - 1} + \left(1 - \frac{\epsilon K}{K - 1}\right) \Pi(I = i).
\]

The power levels are selected to minimize outage probability subject to power constraints (1). Consider the power levels as:

\[
\tilde{P}_s^i = \begin{cases} \text{SNR}^{-\frac{1}{K}} & \text{when } i = 1 \\
\min(\text{SNR}^{-\frac{1}{K}}, \frac{\text{SNR}^{-y} - C_{m,n,s-1}(r)}{K - 1}) & \text{when } i > 1 \forall 1 \leq s \leq L.
\end{cases}
\]

These power levels satisfy the SNR constraints, and hence the optimal outage probability is \( \epsilon \leq \Pi(O) \). From these power levels, we find that

\[
\tilde{C}_s^i = \text{SNR}^{1+\min(r, C_{m,n,i-1}(r))}
\]

Hence,

\[
\Pi(O) \leq \left(1 - \epsilon\right) \Pi(R, \tilde{P}_K) + \frac{\epsilon K}{K - 1} \sum_{s=1}^{K-1} \Pi(R, \tilde{P}_s)
\]

\[
\leq \text{SNR}^{-C_{m,n,K}(r)} + \text{SNR}^{-y - C_{m,n,1}(r)}
\]

\[
\leq \text{SNR}^{-\min(C_{m,n,K}(r), y + C_{m,n,1}(r))}
\]

Hence, we find that \( \Pi(O) \leq \min(C_{m,n,K}(r), y + C_{m,n,1}(r)) \). Noting that \( C_{m,n,1}(r) = C_{m,n,1}(r) \) proves the theorem. ■

Theorem 1 gives a lower bound to the diversity-multiplexing tradeoff performance. We will now consider some special cases when this bound is tight.

**Lemma 2.** Suppose that \( K > 1 \), \( \epsilon = 0 \) and \( \sum_{i \in S} r_i < \min(|S|m,n) \) for all non-empty subsets \( S \subseteq \{1,2,\ldots,L\} \). Then, the diversity-multiplexing tradeoff with \( K \) indices of global feedback is given by \( C_{m,n,K}(r) \) for given multiplexing gain \( r = (r_1, r_2, \ldots, r_L) \).

**Proof:** The lower bound of the diversity multiplexing tradeoff follows from Theorem 1 by taking limit as \( y \to \infty \). We will now prove the upper bound for the diversity multiplexing
tradeoff by finding the lower bound for outage probability.

Note that $\Pi(O) = \Pi(R, P_K)$ when there is no error in the feedback. To calculate the lower bound for outage probability, we first weaken the above optimization problem as $\min \Pi(O)$ subject to the following power constraint

$$\Pi(I = i)P_i^1 \leq \text{SNR}_s \forall 1 \leq s \leq L$$

(7)

The solution of (7) is denoted by $\overline{P}_s^1$. As the constraint set is bigger compared to the original problem, it follows that $\Pi(O) \geq \overline{\Pi}(O)$ where $\overline{\Pi}(O)$ is the outage probability taking powers $\overline{P}_s^1$. Note from (7) that $\overline{P}_s^1 \leq K\text{SNR}_s$ which gives $\overline{P}_s^1 \leq \text{SNR}$. Using (7) and (3) recursively, we find that

$$\overline{P}_s^1 \leq \text{SNR}^{1+\epsilon_{m,n-1}(\epsilon)}$$

Hence, the outage probability

$$\Pi(O) \geq \Pi(R, \overline{P}_K) \geq \text{SNR}^{D(r,1(1+\epsilon_{m,n,K-1}(\epsilon)))} = \text{SNR}^{-\epsilon_{m,n,K}(\epsilon)}$$

(9)

**Remark 2.** Lemma 2 has earlier been proved in [8] for $L = 1$ and [11] for $L = 2$.

**Lemma 3.** Suppose that $K > 1$, $L = 1$ and $\epsilon = \text{SNR}^{-y}$ for some $y > 0$. Further suppose that $r_1 < \min(m, n)$. Then, the diversity-multiplexing tradeoff for the optimal receiver with $K$ indices of feedback is given by $d_{\text{opt}}^{K} = \min(C_{m,n,K}(r), y + C_{m,n,1}(r))$.

**Proof:** The lower bound follows by Theorem 1. We will now prove the upper bound for the diversity-multiplexing tradeoff by finding the lower bound for outage probability.

For this, we first weaken the above optimization problem as $\min \Pi(O)$ subject to the following power constraint

$$\frac{\epsilon}{K-1} \overline{P}_1^1 + \left(1 - \frac{\epsilon K}{K-1}\right) \Pi(I = i) \overline{P}_i^1 \leq \text{SNR}_1$$

(10)

The solution of (10) is denoted by $\overline{P}_i^1$. As the constraint set is bigger compared to the original problem, it follows that $\Pi(O) \geq \overline{\Pi}(O)$ where $\overline{\Pi}(O)$ is the outage probability taking powers $\overline{P}_i^1$.

Note that $\overline{P}_i^1 \leq K\text{SNR}$. From (10), it follows that $\frac{\text{SNR}}{\overline{P}_i^1} \geq \frac{\epsilon}{K-1} + \left(1 - \frac{\epsilon K}{K-1}\right) \Pi(I = j)$. Hence, $\overline{P}_i^1 \leq \text{SNR}^{1+y}$. Assuming that $\overline{P}_i^1 \leq \text{SNR}^{1+y}$, we find using (10) that $\overline{P}_i^1 \leq \text{SNR}^{1+\epsilon_{m,n,1}(r) - \overline{P}_{i-1}^1}$, for $\overline{P}_1^1 \leq \text{SNR}^{1+y}$. Using this recursively, we find that $\overline{P}_i^1 \leq \text{SNR}^{1+\epsilon_{m,n,j-1}(r)}$ and hence $\Pi(R, \overline{P}_K^1) \geq C_{m,n,i}(r)$.

Also, since $L = 1$,

$$\Pi(O) = \Pi(R, P_K) + \sum_{i=2}^{K} \overline{P}(1 < i) = \Pi(I = i)$$

$$= (1 - \epsilon)\Pi(R, P_K) + \frac{\epsilon}{K-1} \sum_{i=1}^{K-1} \Pi(R, P_i)$$

(11)

Hence, the outage probability $\Pi(O)$

$$\geq (1 - \epsilon)\text{SNR}^{-C_{m,n,K}(r)} + \frac{\epsilon}{K-1} \sum_{i=1}^{K-1} \text{SNR}^{-C_{m,n,i}(r)}$$

$$= \text{SNR}^{-\min(C_{m,n,K}(r), y + C_{m,n,1}(r))}$$

(12)

**Remark 3.** It was recently observed in [7] that for $K > 1$ and $L = 1$, we do not gain in diversity order with feedback if the feedback errors do not decay with SNR. This also follows as a special case of Lemma 3 with $y \rightarrow 0$ in which case, $d_{\text{opt}}^{K} = C_{m,n,1}(r)$ is the same as the diversity order without feedback.

**IV. DOUBLING OF DIVERSITY ORDER**

Theorem 1 gives achievable diversity-multiplexing tradeoff for MAC with imperfect feedback. Note that the theorem considered $\epsilon = \text{SNR}^{-y}$ for any $y > 0$. We saw in Lemma 2 the performance of MAC with perfect feedback. When the feedback error does not decay with SNR, we get $d_{\text{opt}}^{K} = C_{m,n,1}(r)$. This is same as the diversity-multiplexing tradeoff without feedback [13] and hence if the feedback errors do not decay with SNR, the feedback do not help in getting any increase in diversity order.

When the forward and the backward channel are SNR-symmetric, the feedback error from the transmitter to the receiver scales as $\epsilon = \text{SNR}^{-y}$. Thus, $y = mn$. Next, we analyze the performance loss with imperfection in feedback.

**Lemma 4 (Doubling of Diversity Order).** Let $y = mn$. When $r \rightarrow 0$, the diversity order is given by

$$d_{\text{opt}}^{K} = \begin{cases} 
    mn & \text{when } K = 1 \\
    2mn & \text{when } K > 1 
\end{cases}$$

Furthermore, as $r \rightarrow 0$, $C_{m,n,K}$ and $C_{m,n,K}$ behave as

$$C_{m,n,K}(0) = \min\left(\frac{mnK - 1}{mn - 1}\right)$$

for $K \geq 1$.

$$C_{m,n,K}(0) = \begin{cases} 
    mn & \text{when } K = 1 \\
    mn(1 + mn) & \text{when } K > 1 
\end{cases}$$

When the feedback is perfect, diversity order increases exponentially with the number of feedback indices [11] while Lemma 4 shows that if the feedback is imperfect, the diversity order do not increase with feedback (for $K > 1$). Lemma 4 also shows that diversity order of $2mn$ is achievable as multiplexing gains go to zero for any number of indices of feedback ($K > 1$), and hence also for single-bit of feedback. This further means that achievable diversity order doubles with just a single-bit of feedback compared to the case of no feedback for zero multiplexing gains.

In [5, 6], it was shown for MIMO channels that diversity of $2mn$ can be achieved by training. In this paper, by Lemma 4, we have shown that diversity order of $2mn$ can be achieved with just a single bit of feedback. Hence, the training can be replaced with just a single bit of feedback if the objective is just to achieve diversity of $2mn$ for zero multiplexing gains.

2Note that the receiver which is sending back the feedback is assumed to operate without any channel state information, especially when operating in an FDD system.
Till now, we focussed on zero multiplexing gains. Now, we consider general multiplexing gains satisfying $\sum_{i \in S} r_i < \min(|S|m,n)$.

**Lemma 5.** Let $r = (r_1, ..., r_L)$ with $\sum_{i \in S} r_i < \min(|S|m,n)$. Then, the following holds:

$$D(r, 1(1+mn)) \geq mn + D(r, 1)$$ (13)

**Proof:** We will be done if we prove that $G_{|S|m,n}(\sum_{i \in S} r_i, 1 + mn) \geq mn + G_{|S|m,n}(\sum_{i \in S} r_i, 1)$. Hence, it is enough need to prove that $G_{m,n}(t, 1 + mn) \geq mn + G_{m,n}(t, 1)$ for $t < \min(m,n)$. Note that $G_{m,n}([t], 1 + mn) = mn(1 + mn) - [t](m + n - 1) \geq mn + (m - [t])(n - [t]) = mn + G_{m,n}([t], 1)$. Similarly, $G_{m,n}([t], 1 + mn) \geq mn + G_{m,n}(x, 1)$ are linear in $x$ for $[t] \leq x \leq [t]$, the result follows.

Note that $C_{m,n,j+1}(r) > C_{m,n,j}(r)$ for $\sum_{i \in S} r_i < \min(|S|m,n)$. Also, $C_{m,n,1}(r) \leq mn$. Hence, there exist a $k \geq 1$ such that:

$$C_{m,n,j}(r) \begin{cases} \leq mn & \text{for } j < k \\ > mn & \text{for } j > k \end{cases}$$

**Lemma 6.** Let $y = mn$. Further assume that $k \geq 1$ be the maximum $j$ such that $C_{m,n,j}(r) \leq mn$. Then, the achievable diversity-multiplexing tradeoff in Theorem 1 reduces to:

$$d^j_{opt} = \begin{cases} C_{m,n,j}(r) & \text{for } j \leq k \\ \min(C_{m,n,j}(r), mn + C_{m,n,1}(r)) & \text{for } j = k + 1 \\ mn + C_{m,n,1}(r) & \text{for } j > k + 1 \end{cases}$$

**Proof:** This follows directly from Theorem 1 & Lemma 5.

Thus, we find from Lemma 6 that the diversity increases in the same fashion with imperfect feedback as it does with perfect feedback till number of feedback indices $\leq k$. For number of indices $> k + 1$, the diversity is limited by $mn+\text{the diversity without feedback}$. Hence, the gain in diversity order with imperfect feedback over no feedback is limited by $mn$ for any multiplexing gain. The maximum diversity order that can be achieved with feedback is more than double as compared to the diversity order without feedback for non-zero multiplexing gains since the diversity order without feedback is less than $mn$. To get this maximum gain in diversity order, $k + 2$ feedback indices are sufficient. Hence, although single bit was enough for multiplexing gains going to 0, for general multiplexing gains we need more feedback to attain maximum diversity. This can also be seen in Figure 1 for MIMO channel that diversity order of $2mn$ can be achieved with single bit of feedback which is the maximum possible. Higher amount of feedback indices help at higher multiplexing to get a gain in diversity order of $mn$ above no-feedback diversity order. We also see in Figure 2 for MAC with two transmitters $(L = 2)$ that the diversity order is $mn$ more than the diversity order without feedback after a certain number of feedback levels.

From these figures, we see that as the multiplexing increases, more indices are needed to get the maximum gain in diversity order of $mn$.

**V. CONCLUSION**

Channel state information at the transmitters is imperfect due to noise. Inspired by this fact, we constructed a feedback error model and characterized the diversity multiplexing tradeoff performance for MAC systems.

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