COMPACT GROUPS WITH PROBABILISTICALLY CENTRAL MONOTHEtic SUBGROUPS

BY

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ABSTRACT

If \( K \) is a closed subgroup of a compact group \( G \), the probability that a randomly chosen pair of elements from \( K \) and \( G \) commute is denoted by \( \Pr(K, G) \). Say that a subgroup \( K \leq G \) is \( \epsilon \)-central in \( G \) if \( \Pr(\langle g \rangle, G) \geq \epsilon \) for any \( g \) in \( K \). Here \( \langle g \rangle \) denotes the monothetic subgroup generated by \( g \in G \). Our main result is that if \( K \) is \( \epsilon \)-central in \( G \), then there is an \( \epsilon \)-bounded number \( e \) and a normal subgroup \( T \leq G \) such that both the index \( [G : T] \) and the order of the commutator subgroup \( [K^e, T] \) are finite and \( \epsilon \)-bounded. In particular, if \( G \) is a compact group for which there is \( \epsilon > 0 \) such that \( \Pr(\langle g \rangle, G) \geq \epsilon \) for any \( g \in G \), then there is an \( \epsilon \)-bounded number \( e \) and a normal subgroup \( T \) such that both the index \( [G : T] \) and the order of \( [G^e, T] \) are finite and \( \epsilon \)-bounded.

1. Introduction

In this paper, all compact groups are Hausdorff topological spaces. By a subgroup of a topological group we mean a closed subgroup unless explicitly stated otherwise. If \( S \) is a subset of a topological group \( G \), then we denote by \( \langle S \rangle \) the subgroup (topologically) generated by \( S \). A subgroup of \( G \) is monothetic if it is generated by a single element.

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The Borel $\sigma$-algebra $\mathcal{M}$ of a compact group $G$ is the one generated by all closed subsets of $G$. We say that a measure on $(G, \mathcal{M})$ is a (left) Haar measure provided $\mu$ is both inner and outer regular, $\mu(K) < \infty$ and $\mu(xE) = \mu(E)$ for all compact subsets $K$ and measurable subsets $E$ of $G$ (see [14, Chapter 4] or [22, Chapter II]). Recall that there is a unique Haar measure $\mu$ on $(G, \mathcal{M})$ such that $\mu(G) = 1$.

Let $G$ be a compact group and let $K$ be a subgroup of $G$. Consider the set $C = \{(x, y) \in K \times G \mid xy = yx\}$. This is closed in $K \times G$ since it is the preimage of 1 under the continuous map $f : K \times G \to G$ given by $f(x, y) = [x, y]$. Denoting the normalized Haar measures of $K$ and $G$ by $\nu$ and $\mu$, respectively, the probability that a random element from $K$ commutes with a random element from $G$ is defined as $\Pr(K, G) = (\nu \times \mu)(C)$. This is a well-studied concept (see in particular [5, 6, 10, 12, 15, 18, 19, 23, 28]).

Recently, Detomi and the second author proved in [3] that if $G$ is finite and $\Pr(K, G) \geq \epsilon$ for some $\epsilon > 0$, then there is a normal subgroup $T$ of $G$ and a subgroup $B$ of $K$ such that the indices $[G : T]$ and $[K : B]$ and the order of the subgroup $[T, B]$ are $\epsilon$-bounded. Throughout the article we use the expression “$(a, b, \ldots)$-bounded” to mean that a quantity is bounded from above by a number depending only on the parameters $a, b, \ldots$. If $B$ and $T$ are subgroups of a group $G$, we denote by $[T, B]$ the subgroup generated by all commutators $[t, b]$ with $t \in T$ and $b \in B$. In the case where $K = G$, this is a well-known theorem due to P. M. Neumann [25]. Conversely, if $K$ is a subgroup of a finite group $G$, and if $T \leq G$ and $B \leq K$, then $\Pr(K, G)$ is bounded away from zero in terms of the indices $[G : T]$ and $[K : B]$ and the order of $[T, B]$. The present work grew out of a desire to understand the impact of similar probabilistic considerations on the structure of a compact group. Our first goal is to extend the main result of [3] to compact groups.

**Proposition 1.1:** Let $\epsilon > 0$ and let $G$ be a compact group having a subgroup $K$ such that $\Pr(K, G) \geq \epsilon$. Then there is a normal subgroup $T \leq G$ and a subgroup $B \leq K$ such that the indices $[G : T]$ and $[K : B]$ and the order of $[T, B]$ are $\epsilon$-bounded.

Let $G$ be a compact group and let $K$ be a subgroup of $G$. If $\Pr(\langle g \rangle, G) \geq \epsilon$ for every $g \in K$ we say that $K$ is $\epsilon$-central in $G$, and in the case where $K = G$ we say that $G$ is $\epsilon$-central. If $e$ is a positive integer, we denote by $G^e$ the subgroup generated by all $e$th powers of elements of $G$. Recall that a group $G$
has finite exponent $e$ if $G^e = 1$ and $e$ is the least positive number with this property. It is easy to see that if $G$ has exponent $e$, then $\text{Pr}(\langle g \rangle, G) \geq \frac{1}{e}$ for all $g \in G$. More generally, if $G^e \leq Z(G)$, where $Z(G)$ denotes the center of $G$, then $\text{Pr}(\langle g \rangle, G) \geq \frac{1}{e}$ for all $g \in G$. We shall prove the following theorem.

**Theorem 1.2:** Let $\epsilon > 0$ and assume that the subgroup $K$ is $\epsilon$-central in $G$. Then there is an $\epsilon$-bounded number $e$ and a finite-index normal subgroup $T \leq G$ such that the index $[G : T]$ and the order of $[K^e, T]$ are $\epsilon$-bounded.

The theorem implies that if $G$ is an $\epsilon$-central compact group, then there is an $\epsilon$-bounded number $e$ and a normal subgroup $T$ such that the index $[G : T]$ and the order of $[K^e, T]$ are $\epsilon$-bounded. Moreover, the exponent of the commutator subgroup $[T, T]$ is $\epsilon$-bounded. Indeed, passing to the quotient over $[G^e, T]$ we can assume that all $e$th powers of elements of $G$ centralize $T$. In particular, $T/Z(T)$ has exponent $e$, and a theorem of Zelmanov [34] ensures that $T/Z(T)$ is locally finite. A result of Mann [20] can then be used to deduce that $[T, T]$ has finite $\epsilon$-bounded exponent.

As usual, we denote the conjugacy class of an element $x \in G$ by $x^G$. It is easy to see that if $K$ is a subgroup of $G$ such that $|x^G| \leq n$ for every $x \in K$, then $K$ is $\frac{1}{n}$-central in $G$. More generally, let $l, n$ be positive integers and suppose that $K$ is a subgroup of a compact group $G$ such that any conjugacy class containing an $l$th power $x^l$ of an element $x \in K$ is of size at most $n$. It is not difficult to see that $K$ is $\frac{1}{ln}$-central in $G$. It turns out that this admits a converse: if $K$ is $\epsilon$-central in $G$, then there exist $\epsilon$-bounded integers $l$ and $n$ such that every conjugacy class containing an $l$th power of an element of $K$ has cardinality at most $n$. Indeed, let $x \in K$. Since $\text{Pr}(\langle x \rangle, G) \geq \epsilon$, in view of Proposition 1.1 there is a normal subgroup $T$ of $G$ and a subgroup $B$ of $\langle x \rangle$ such that the indices $[G : T]$ and $[\langle x \rangle : B]$ and the order of $[T, B]$ are $\epsilon$-bounded. Hence, as required, there are $\epsilon$-bounded numbers $l$ and $n$ such that $[G : C_G(x^l)] \leq n$ for all $x \in K$. Therefore we have proved that:

**Theorem:** For every $0 < \epsilon \leq 1$ there are positive integers $l$ and $n$ depending only on $\epsilon$ with the property that if $K$ is an $\epsilon$-central subgroup of the compact group $G$, then $[G : C_G(g^l)] \leq n$ for all $g \in K$.

Taking this into consideration, Theorem 1.2 will follow from the next proposition.
PROPOSITION 1.3: Let $G$ be a compact group and let $l, n$ be positive integers. Suppose that there is a subgroup $K$ of $G$ such that $[G : C_G(g^l)] \leq n$ for every $g \in K$. Then there exist a positive integer $e$, depending only on $l$ and $n$, and a normal subgroup $T$ of $G$, such that the index $[G : T]$ and the order of $[K^e, T]$ are $(l, n)$-bounded.

Recall that a group is said to be a BFC-group if its conjugacy classes are finite and have bounded size. A famous theorem of B. H. Neumann says that in a BFC-group the commutator subgroup $G'$ is finite [24]. It follows that if $|x^G| \leq n$ for every $x \in G$, then the order of $G'$ is bounded by a number depending only on $n$. A first explicit bound for the order of $G'$ was found by J. Wiegold [31], and the best known was obtained in [9] (see also [26] and [30]). Proposition 1.3 is an extension of the Neumann theorem (in the particular case where $K = G$ and $l = 1$ the proof shows that we can take $T = G$ and $e = 1$). Recently, some other generalizations of Neumann’s theorem have been obtained (see in particular [1, 2, 4]).

We end this introduction with the remark that Theorem 1.1 admits a converse:

THEOREM: For any positive integers $s, e, m$ there is $0 < \epsilon \leq 1$ depending only on $s, e, m$ with the property that if $K$ is a subgroup of a compact group $G$ and if $G$ has a normal subgroup $T$ of index at most $s$ such that $[K^e, T]$ has order at most $m$, then $K$ is $\epsilon$-central in $G$.

To see this simply note that if $K$ is as above, then $[G : C_G(g^e)] \leq ms$ for every $g \in K$.

2. Preliminaries

In this section we record some results needed in the proofs of the main theorems. The following lemma is [28, Lemma 3.1].

LEMMA 2.1: Let $G$ be a compact group and let $K$ be a subgroup of $G$. Then either $\mu(K) = 0$ or $\mu(K) > 0$ and $K$ is open on $G$. Furthermore, in the latter case,

$$\mu(K) = [G : K]^{-1}.$$
Lemma 2.2: Let $G$ be a compact group, and let $K$ and $H$ be subgroups of $G$ with $K \leq H$. Assume further that $\mu(K) \geq \epsilon \mu(H) > 0$ for some positive $\epsilon$. Then $[H : K] \leq \epsilon^{-1}$.

Proof. Since $\mu(K), \mu(H) > 0$, the previous lemma implies that both subgroups are of finite index and $\mu(K) = [G : K]^{-1}$ and $\mu(H) = [G : H]^{-1}$. Hence the result. \qed

For every $x \in G$, the centralizer $C_G(x)$ equals $f^{-1}_x(1)$, where $f_x$ is the continuous function $f_x(y) = [x, y]$, so this subgroup is closed and measurable.

Lemma 2.3: Let $H$ and $K$ be subgroups of a compact group $G$, with $H \leq K$. Then

$$\Pr(K, G) \leq \Pr(H, G) \leq \Pr(H, K).$$

In particular, $\Pr(G, G) \leq \Pr(K, G) \leq \Pr(K, K)$.

Proof. Let $\mu$, $\nu$ and $\lambda$ be the normalized Haar measures of $G$, $K$ and $H$, respectively. Given $x \in G$, the map $\alpha : \{hC_H(x) \mid h \in H\} \rightarrow \{kC_K(x) \mid k \in K\}$ taking $hC_H(x)$ to $hC_K(x)$ is injective. We deduce that $[H : C_H(x)] \leq [K : C_K(x)]$ and $\nu(C_K(x)) \leq \lambda(C_H(x))$. We have

$$\Pr(H, G) = \int_G \lambda(C_H(x))d\mu(x) \geq \int_G \nu(C_K(x))d\mu(x) = \Pr(K, G).$$

The other inequality is proved in an analogous way. \qed

Suppose that $G$ is a compact group and let $N$ be a normal subgroup of $G$. The normalized Haar measure on $G/N$ coincides with the one induced by the normalized Haar measure on $G$. If $A$ is a measurable subset of $G$, we denote by $\chi_A$ the characteristic function of $A$. We say that $x \in G$ is an FC-element if the conjugacy class of $x$ in $G$ is finite.

In the case of finite groups the next lemma was established in [3].

Lemma 2.4: Let $G$ be a compact group and let $N$ be a normal subgroup of $G$. For any subgroup $K$ of $G$, we have

$$\Pr(K, G) \leq \Pr(KN/N, G/N) \Pr(K \cap N, N).$$
Proof. If $X$ is a compact group, we denote by $\mu_X$ the normalized Haar measure of $X$. We have

$$
\Pr(K, G) = \int_K \mu_G(C_G(x))d\mu_K(x).
$$

If $x \in G$ is an FC-element, then

$$
\mu_G(C_G(x)N) = [C_G(x)N : C_G(x)]\mu_G(C_G(x)) = [N : C_N(x)]\mu_G(C_G(x)),
$$

so $\mu_G(C_G(x)N)\mu_N(C_N(x)) = \mu_G(C_G(x))$. Let $\text{FC}(K)$ be the abstract subgroup of $K$ consisting of elements having finite conjugacy class in $G$. Then

$$
\int_K \mu_G(C_G(x))d\mu_K(x) = \int_{\text{FC}(K)} \mu_G(C_G(x))d\mu_K(x)
$$

$$
= \int_{\text{FC}(K)} \mu_G(C_G(x)N)\mu_N(C_N(x))d\mu_K(x)
$$

$$
\leq \int_K \mu_G(C_G(x)N)\mu_N(C_N(x))d\mu_K(x).
$$

We now apply the extended Weil formula [27, p. 88] to the last integral and obtain

$$
\Pr(K, G)
$$

$$
\leq \int \left( \int_{K \cap N} \mu_G(C_G(xk)N)\mu_N(C_N(xk))d\mu_{K \cap N}(k) \right) d\mu_{K \cap N}(x(K \cap N))
$$

(1)

$$
\leq \int \left( \int_{K \cap N} \mu_G(C_G(xN))\mu_N(C_N(xk))d\mu_{K \cap N}(k) \right) d\mu_{K \cap N}(x(K \cap N))
$$

$$
= \int \mu_G(C_G(xN)) \left( \int_{K \cap N} \mu_N(C_N(xk))d\mu_{K \cap N}(k) \right) d\mu_{K \cap N}(x(K \cap N)).
$$

If $x$ is any element of $K$, define the set

$$
A_x = \{(k, n) \in (K \cap N) \times N \mid [xk, n] = 1\}
$$

$$
= \{(k, n) \in (K \cap N) \times N \mid xk \in C_G(n) \cap x(K \cap N)\}.$$
If $C_G(n) \cap x(K \cap N)$ is nonempty, then it equals $tC_{K \cap N}(k)$ for some $t \in x(K \cap N)$. Thus,

$$A_x = \{(k, n) \in (K \cap N) \times N \mid xk \in tC_{K \cap N}(k)\} = \{(k, n) \in (K \cap N) \times N \mid k \in x^{-1}tC_{K \cap N}(k)\}.$$ 

We use the Lebesgue–Fubini Theorem to give an estimate for the expression in the last line of (1):

$$\int_{K \cap N} \mu_N(C_N(xk))d\mu_{K \cap N}(k) = \int_{(K \cap N) \times N} \chi_{A_x}(k,n)(d\mu_{K \cap N} \times \mu_N)(k,n) 
\leq \int_N \mu_{K \cap N}(x^{-1}tC_{K \cap N}(n))d\mu_N(n) 
= \int_N \mu_{K \cap N}(C_{K \cap N}(n))d\mu_N(n) 
= \Pr(K \cap N, N).$$ 

Replacing this back in (1) we have

$$\Pr(K, G) \leq \int_{K \cap N} \mu_N(C_N(xK))\left(\int_{K \cap N} \mu_N(C_N(xn))d\mu_{K \cap N}(n)\right)d\mu_{\frac{K}{K \cap N}}(x(K \cap N)) 
\leq \Pr(K \cap N, N) \int_{\frac{K}{K \cap N}} \mu_N(C_N(xN))d\mu_{\frac{K}{K \cap N}}(x(K \cap N)).$$ 

Finally, since $K/K \cap N$ and $KN/N$ are isomorphic, we can apply Corollary 2.5 in [28] with respect to the last integral above to conclude that

$$\int_{\frac{K}{K \cap N}} \mu_N(C_N(xN))d\mu_{\frac{K}{K \cap N}}(x(K \cap N)) = \int_{\frac{KN}{N}} \mu_N(C_N(xN))d\mu_{\frac{KN}{N}}(xN) 
= \Pr(KN/N, G/N).$$ 

The lemma follows. ■

If $A$ and $B$ are normal subgroups of a group $G$ such that $[A : C_A(B)] \leq m$ and $[B : C_B(A)] \leq m$, then $[A, B]$ has $m$-bounded order. This well-known result is due to Baer, cf. [29, 14.5.2]. We need a variation of it, which is [3, Lemma 2.1].
Lemma 2.5: Let $m \geq 1$ and let $G$ be a group containing a normal subgroup $A$ and a subgroup $B$ such that $[A : C_A(y)] \leq m$ and $[B : C_B(x)] \leq m$ for all $x \in A, y \in B$. Assume further that $\langle B^G \rangle$ is abelian. Then $[A, B]$ has finite $m$-bounded order.

The next theorem holds in any group and plays a key role in the proof of Theorem 1.1. It is taken from [1].

Theorem 2.6: Let $m$ be a positive integer, $G$ a group having a subgroup $K$ such that $|x^G| \leq m$ for each $x \in K$, and let $H = \langle K^G \rangle$. Then the order of the commutator subgroup $[H, H]$ is finite and $m$-bounded.

The next lemma is essentially Lemma 2.1 in [5].

Lemma 2.7: Let $G$ be a compact group with normalized Haar measure $\mu$ and let $r \geq 1$. Suppose that $X$ is a closed symmetric subset of $G$ containing the identity. If $\mu(X) > \frac{1}{r+1}$, then $\langle X \rangle = X^{3^r}$.

Proof. Suppose $x_i \in X^{3^{i+1}} \setminus X^{3^i}$ for $i = 0, \ldots, r$. Then for each $i$, as long as $X^{3^{i+1}} \setminus X^{3^i}$ is nonempty, we have

$$x_i X \subseteq X^{3^{i+2}} \setminus X^{3^{i-1}}.$$ 

So, assume that the sets $X^{3^{i+1}} \setminus X^{3^i}$ are nonempty for $i = 0, \ldots, r$. Then $x_0 X, \ldots, x_r X$ are disjoint subsets of $G$, each of measure $\mu(X)$, and

$$\mu \left( \bigcup_{i=0}^{r} x_i X \right) = (r+1)\mu(X) > 1.$$ 

Therefore $X^{3^{i+1}} = X^{3^i}$ for some $i \leq r$. In particular, $X^{3^r} = \langle X \rangle$.

In a compact group $G$ the set $\{ x \in G \mid |x^G| \leq n \}$ is closed and thus measurable.

Lemma 2.8: Let $G$ be a compact group and $n$ a positive integer. The set $X = \{ x \in G \mid |x^G| \leq n \}$ is a closed subset of $G$.

Proof. It is sufficient to show that if $a \in G \setminus X$, then $a$ is contained in an open subset $U$ which has empty intersection with $X$. Since $a \notin X$, we can choose $n+1$ elements $x_1, \ldots, x_{n+1}$ in such a way that the conjugates $a^{x_i}$ are distinct for $i = 1, \ldots, n+1$. Set

$$U = \{ u \in G ; \ [u, x_i x_j^{-1}] \neq 1 \text{ for } 1 \leq i, j \leq n+1 \}.$$
Observe that $a \in U$ and every element in $U$ has at least $n + 1$ conjugates, whence $U \cap X = \emptyset$. Further, since the commutator map is continuous, $U$ is open. The proof is complete.

**Remark 2.9:** If $S$ is any subset of $G$, we denote by $x^S$ the set of $S$-conjugates of $x$ in $G$. The previous argument also proves that the set $\{ x \in G \mid |x^S| \leq n \}$ is closed in $G$ for any (not necessarily closed) subset $S$ of $G$.

### 3. Proof of Proposition 1.1

Now we are able to prove Proposition 1.1, which we restate here for the reader’s convenience.

**Proposition 1.1:** Let $\epsilon > 0$ and let $G$ be a compact group having a subgroup $K$ such that $\Pr(K, G) \geq \epsilon$. Then there is a normal subgroup $T \leq G$ and a subgroup $B \leq K$ such that the indices $[G : T]$ and $[K : B]$ and the order of $[T, B]$ are $\epsilon$-bounded.

**Proof.** Let $\mu$ and $\nu$ be the normalized Haar measures of $G$ and $K$, respectively. Set

$$X = \{ x \in K \mid |x^G| \leq 2/\epsilon \}.$$

Note that $X$ is measurable, by Lemma 2.8: it is the intersection of $K$ and the closed set $\{ x \in G \mid |x^G| \leq 2/\epsilon \}$. We have

$$K \setminus X = \{ x \in K \mid |x^G| > 2/\epsilon \}.$$

Since $\mu(C_G(x)) < \epsilon/2$ for all $x \in K \setminus X$, it follows that

$$\epsilon \leq \Pr(K, G) = \int_K \mu(C_G(x))d\nu(x)$$

$$= \int_X \mu(C_G(x))d\nu(x) + \int_{K \setminus X} \mu(C_G(x))d\nu(x)$$

$$\leq \int_X d\nu(x) + \int_{K \setminus X} \frac{\epsilon}{2}d\nu(x)$$

$$= \nu(X) + \frac{\epsilon}{2}(1 - \nu(X)) \leq \nu(X) + \frac{\epsilon}{2}.$$
This implies that $\epsilon/2 \leq \nu(X)$. Let $B$ be the subgroup generated by $X$. Then, by Lemma 2.7, every element of $B$ is a product of at most $6/\epsilon$ elements of $X$. Clearly, $\nu(B) \geq \nu(X) \geq \epsilon/2$, so the index of $B$ in $K$ is at most $2/\epsilon$, by Lemma 2.2. Furthermore, $|b^G| \leq (2/\epsilon)^{6/\epsilon}$ for every $b \in B$.

Let $L = \langle B^G \rangle$. Theorem 2.6 tells us that the commutator subgroup $[L, L]$ has finite $\epsilon$-bounded order. Let us use the bar notation for images of subgroups of $G$ in $G/[L, L]$. By Lemma 2.4, $\Pr(K, G) \geq \nu(C_K(y)) < \epsilon/2$ for all $y \in G \setminus Y$.

Observe that $Y$ is closed, by Remark 2.9. Arguing as before, since $\nu(C_K(y)) < \epsilon/2$ for all $y \in G \setminus Y$, we have

$$\epsilon \leq \Pr(K, G) = \int_{G} \nu(C_K(y)) d\mu(y)$$

$$= \int_{Y} \nu(C_K(y)) d\mu(y) + \int_{G \setminus Y} \nu(C_K(y)) d\mu(y)$$

$$\leq \int_{Y} d\mu(y) + \int_{G \setminus Y} \epsilon/2 d\mu(y)$$

$$= \mu(Y) + \epsilon/2 - (1 - \mu(Y)) \leq \mu(Y) + \epsilon/2.$$

Therefore, $\mu(Y) \geq \epsilon/2$. Let $E$ be the subgroup generated by $Y$. Lemma 2.7 ensures that every element of $E$ is a product of at most $6/\epsilon$ elements of $Y$. Also, we have $\mu(E) \geq \mu(Y) \geq \epsilon/2$, so the index of $E$ in $G$ is at most $2/\epsilon$, by Lemma 2.2. Since $|y^K| \leq \epsilon/2$ for every $y \in Y$, it follows that $|g^K| \leq (2/\epsilon)^{6/\epsilon}$ for every $g \in E$. Let $T$ be the maximal normal subgroup of $G$ contained in $E$. Then the index $[G : T]$ is $\epsilon$-bounded. Moreover, $|b^G| \leq (2/\epsilon)^{6/\epsilon}$ for every $b \in B$ and $|g^K| \leq (2/\epsilon)^{6/\epsilon}$ for every $g \in T$. As $L$ is abelian, we can apply Lemma 2.5 and deduce that $[T, B]$ has finite $\epsilon$-bounded order. The proposition follows. ■
4. About $\epsilon$-central subgroups

Let $G$ be a topological group generated by a symmetric set $X$. If it is possible to write $g \in G$ as a product of finitely many elements from $X$, we denote by $w(g)$ the shortest length of such an expression. If $g$ cannot be written as a product of finitely many elements of $X$, we simply say that $w(g)$ is infinite. The next result is Lemma 2.1 in [4].

**Lemma 4.1:** Let $G$ be a group generated by a symmetric set $X$ and let $D$ be a subgroup of index $m$ in $G$. Then every coset $Db$ contains an element such that $w(g) \leq m - 1$.

We remark that Lemma 4.1 holds for topological groups and their closed (open) subgroups of finite index. Indeed, for an integer $r \geq 0$ let $D_r$ be the union of the cosets of $D$ containing some element $g$ with $w(g) \leq r$. Then $D_r \subseteq D_{r+1}$ and $D_r X \subseteq D_{r+1}$ for all $r$. Let $R$ be the minimal integer such that $D_{R+1} = D_R$. Then $D_R$ is a closed set containing the group generated by $X$, so $D_R = G$. Since $D$ has $m$ cosets and $D_0 = D$, $R < m$.

We now proceed to the proof of Proposition 1.3, which we restate here for the reader’s convenience.

**Proposition 1.3:** Let $G$ be a compact group and let $l, n$ be positive integers. Suppose that there is a subgroup $K$ of $G$ such that $[G : C_G(g^l)] \leq n$ for every $g \in K$. Then there exists a positive integer $e$, depending only on $l$ and $n$, and a normal subgroup $T$ of $G$, such that the index $[G : T]$ and the order of $[K^e, T]$ both are finite and $(l, n)$-bounded.

Let $G$ be a compact group satisfying the hypothesis of Proposition 1.3. Let $X$ be the union of the conjugacy classes of $G$ containing an $l$th power of an element of $K$ and let $H$ be the subgroup generated by $X$. Define $m$ as the maximum of the indices of $C_H(x)$ in $H$, where $x \in X$. Obviously, $m \leq n$.

**Lemma 4.2:** For any $x \in X$ the order of the subgroup $[H, x]$ is $m$-bounded.

**Proof.** Since the index of $C_H(x)$ in $H$ is at most $m$, Lemma 4.1 guarantees that there are elements $y_1, \ldots, y_m$ in $H$ such that each $y_i$ is a product of at most $m - 1$ elements of $X$ and the subgroup $[H, x]$ is generated by the commutators $[y_i, x]$, for $i = 1, \ldots, m$. For any such $i$ write $y_i = y_{i1} \cdots y_{i(m-1)}$, where $y_{ij}$ belongs
to $X$. Using the standard commutator identities, we can rewrite $[y_i, x]$ as a product of conjugates in $H$ of the commutators $[y_{ij}, x]$. Let $\{h_1, \ldots, h_s\}$ be the set of conjugates in $H$ of all elements from the set $\{x, y_{ij} \mid 1 \leq i, j \leq m - 1\}$. Note that the number $s$ here is $m$-bounded. This follows from the fact that $C_H(x)$ has index at most $m$ in $H$ for every $x \in X$. Let $D$ be the subgroup of $H$ generated by $h_1, \ldots, h_s$. Since $[H, x]$ is contained in the commutator subgroup $D'$, it suffices to show that $D'$ has finite $m$-bounded order. Observe that the center $Z(D)$ has index at most $m^s$ in $D$, since the index of $C_H(h_i)$ is at most $m$ for every $h_i$. Thus, by Schur’s theorem [29, 10.1.4], we conclude that $D'$ has finite $m$-bounded order.  

We now argue by induction on $m$. If $[H : C_H(g^{l^2})] \leq m - 1$ for all $g \in K$, then by induction the result holds. We therefore assume that there is $d \in K$ such that $[H : C_H(d^2)] = m$. Of course, necessarily, $[H : C_H(d^l)] = m$. Set $a = d^l$ and choose $b_1, \ldots, b_m$ in $H$ such that $a^H = \{a^{b_i} \mid i = 1, \ldots, m\}$ and $w(b_i) \leq m - 1$ (the existence of the elements $b_i$ is guaranteed by Lemma 4.1).

Since $C_H(a) = C_H(a^l)$, it follows that $$ (a^l)^H = \{(a^l)^{b_i} \mid i = 1, \ldots, m\}. $$

Set $U = C_G(\langle b_1, \ldots, b_m \rangle)$. Note that the index of $U$ in $G$ is $n$-bounded. Indeed, since $w(b_i) \leq m - 1$ we can write $b_i = b_{i1} \ldots b_{i(m-1)}$, where $b_{ij} \in X$ and $i = 1, \ldots, m$. By the hypothesis, the index of $C_G(b_{ij})$ in $G$ is at most $n$ for any such element $b_{ij}$. Thus, $[G : U] \leq n^{(m-1)m}$.

**Lemma 4.3:** Suppose that $u \in U$ and $ua \in X$. Then $[H, u] \leq [H, a]$.

**Proof.** For each $i = 1, \ldots, m$ we have $(ua)^{b_i} = ua^{b_i}$, since $u$ belongs to $U$. By hypothesis, $ua \in X$. Hence, taking into account the assumption on the cardinality of the conjugacy class of $ua$ in $H$, we deduce that $(ua)^H$ consists exactly on the elements $(ua)^{b_i}$, for $i = 1, \ldots, m$. Therefore, given an arbitrary element $h \in H$, there exists $b \in \{b_1, \ldots, b_m\}$ such that $(ua)^h = (ua)^b$ and so $u^h a^h = ua^b$. It follows that $[u, h] = a^b a^{-h} \in [H, a]$, and the lemma holds.  

Let $R$ be the normal closure in $G$ of the subgroup $[H, a]$, that is, $R = [H, a^{b_1}] \ldots [H, a^{b_n}]$, where $a^{b_i}$ are all the conjugates of $a$ in $G$ (if $|a^G| \leq n - 1$, then not all the $a^{b_1}, \ldots, a^{b_n}$ are pairwise distinct). By Lemma 4.2, each of the subgroups $[H, a^{b_i}]$ has $n$-bounded order. Thus, the order of $R$ is $n$-bounded as well.
Let $Y_1 = Xa^{-l} \cap U$ and $Y_2 = Xa^{-1} \cap U$. Note that for any $y \in Y_1$, the product $ya^l$ belongs to $X$. So, by Lemma 4.3 applied with $a^l$ in place of $a$ and $y$ in place of $u$, the subgroup $[H, y]$ is contained in $[H, a^l]$, which is contained in $R$. Similarly, for any $y \in Y_2$, we have $[H, y] \leq R$. Set $Y = Y_1 \cup Y_2$. Thus, \[ [H, Y] \leq R. \]

Let $U_0$ be the maximal normal subgroup of $G$ contained in $U$. Observe that the index of $U_0$ in $G$ is $n$-bounded. Observe further that for any $u \in U_0$ the commutators $[u, a^{-l}]$ and $[u, a^{-l}]$ lie in $Y$. Since $[U_0, a^{-l}] = [U_0, a]$, we deduce that \[ [H, [U_0, a]] \leq R. \]

Let $K_0 = K \cap U_0$. We remark that $(ua)^l(a^l)^{-1} \in Y$ whenever $u \in K_0$. We pass to the quotient $\overline{G} = G/R$ and use the bar notation to denote images in $\overline{G}$. We know that $Y$ is central in $\overline{H}$. We also deduce that $[\overline{U_0}, \overline{a}] \leq Z(\overline{H})$.

Since $(ua)^l(a^l)^{-1} \in Y$ whenever $u \in K_0$ and since $Y \leq Z(\overline{H})$, it follows that in the quotient $\overline{G}/Z(\overline{H})$ the element $\overline{a}$ commutes with $\overline{U_0}$ and $(\overline{ua})^l(\overline{a})^{-l} = 1$ for every $\overline{a} \in \overline{K_0}$. This implies that $\overline{K_0}$ has exponent dividing $l$ modulo $Z(\overline{H})$. It follows that $\overline{K_0}$ is abelian and every element of $\overline{K_0}^{l^2}$ is again an $l$th power of an element in $\overline{K_0}$. We therefore deduce that \[ \Pr(\overline{K_0}^{l^2}, \overline{G}) \geq \frac{1}{n}. \]

By Proposition 1.1 there is a normal subgroup $\overline{T}$ in $\overline{G}$ and a subgroup $\overline{V}$ in $\overline{K_0}^{l^2}$ such that the indices $[\overline{G} : \overline{T}]$ and $[\overline{K_0}^{l^2} : \overline{V}]$ and the order of $[\overline{T}, \overline{V}]$ are $(l, n)$-bounded. Let $T$ be the inverse image of $\overline{T}$ in $G$ and $V$ the inverse image of $\overline{V}$ in $K_0^{l^2}$. Bearing in mind that the order of $R$ is $n$-bounded, we conclude that the indices $[G : T]$ and $[K_0^{l^2} : V]$ are $(n, l)$-bounded, as also is the order of $[T, V]$. As the index of $V$ in $K_0^{l^2}$ is bounded, there is a positive $(n, l)$-bounded integer $e$ such that $K^e \leq V$. This completes the proof of the proposition.

**Corollary 4.4:** Let $G$ be a compact group and let $G_0$ be the connected component of identity in $G$. Then, the following are equivalent:

(i) The probability $\Pr(G_0, G)$ is positive.

(ii) The centralizer $C_G(G_0)$ is open in $G$.

(iii) The connected component $G_0$ is $\epsilon$-central in $G$ for some $\epsilon > 0$. 


Proof. Suppose first that $\Pr(G_0, G) > 0$. Then there are subgroups of finite index $T$ of $G$ and $B$ of $G_0$ such that $[T, B]$ is finite. Since $G_0$ is divisible [21], the only finite index subgroup of $G_0$ is $G_0$ itself. Therefore, the set of commutators $\{[x, y] \mid x \in G_0, y \in T\}$ is connected and finite, hence is trivial and $T \leq C_G(G_0)$.

Now, assume that $C_G(G_0)$ is open in $G$ and let $\mu$ and $\mu_0$ be the normalized Haar measures of $G$ and $G_0$, respectively. For any $x \in G_0$, the inclusion $C_G(G_0) \leq C_G(x)$ holds, thus $\mu(C_G(G_0)) \leq \mu(C_G(x))$. We have

$$\Pr(\langle x \rangle, G) = \int_{G_0} \mu(C_G(x)) d\mu_0(x) \geq \mu(C_G(G_0)) > 0.$$ 

We conclude that $G_0$ is $\epsilon$-central, where $\epsilon = [G : C_G(G_0)]^{-1}$, and so (ii) implies (iii).

Now we assume the validity of (iii) and prove that (i) holds. By Theorem 1.2, there are a finite index subgroup $T$ of $G$ and a natural number $e$ such that $[G_0^e, T]$ is finite. Since $G_0$ is divisible, $G_0 = G_0^e$ and so $T$ centralizes $G_0$ and $T \leq C_G(x)$ for any $x \in G_0$. Writing $\mu$ and $\mu_0$ for the normalized Haar measures of $G$ and $G_0$, respectively, we have

$$\Pr(G_0, G) = \int_{G_0} \mu(C_G(x)) d\mu_0(x) \geq \mu(T) > 0. \quad \Box$$

5. Corollaries for Finite Groups

In this section we collect some easy corollaries of Theorem 1.2 for finite groups. Roughly, we show that many well-known results on the exponent of a finite group admit a probabilistic interpretation in the spirit of Theorem 1.2.

The restricted Burnside problem was whether the order of an $r$-generator finite group of exponent $e$ is bounded in terms of $r$ and $e$ alone. This was famously solved in the affirmative by Zelmanov [32, 33]. Theorem 1.2 enables us to obtain the following extension of Zelmanov’s theorem.

Theorem 5.1: Let $G$ be a finite $\epsilon$-central $r$-generator group. Then $G$ has a normal subgroup $N$ such that both the index $[G : N]$ and the order of the commutator subgroup $[N, N]$ are $(r, \epsilon)$-bounded.
**Proof.** By Theorem 1.2 there is an $\epsilon$-bounded number $e$ and a normal subgroup $T \leq G$ such that the index $[G : T]$ and the order of $[G^e, T]$ are $\epsilon$-bounded. The solution of the Restricted Burnside problem implies that the index $[G : G^e]$ is $(e, r)$-bounded. Therefore the subgroup $N = G^e \cap T$ has the required properties.

An important part of the eventual solution of the restricted Burnside problem was developed by Hall and Higman in their paper [13]. They proved that if $p$ is a prime and $G$ is a finite group with Sylow $p$-subgroups of exponent $p^s$, then $G$ has a normal series of $s$-bounded length all of whose factors are $p$-groups, or $p'$-groups, or direct products of nonabelian simple groups of order divisible by $p$.

**Theorem 5.2:** Let $p$ be a prime and $G$ a finite group with $\epsilon$-central Sylow $p$-subgroups. Then $G$ has a normal series of $\epsilon$-bounded length all of whose factors are $p$-groups, or $p'$-groups, or direct products of nonabelian simple groups of order divisible by $p$.

**Proof.** Let $P$ be a Sylow $p$-subgroup of $G$. By Theorem 1.2 there is an $\epsilon$-bounded number $e$ and a normal subgroup $T \leq G$ such that the index $[G : T]$ and the order of $[P^e, T]$ are $\epsilon$-bounded. Set $P_0 = P^e$. It is sufficient to show that $T$ has a normal series with the required properties. Note that $[P_0, T]$ is normal in $T$. Let $Z$ be the inverse image in $T$ of the center of $T/[P_0, T]$. Consider the normal series

$$1 \leq [P_0, T] \leq Z \leq T.$$ 

Here $[P_0, T]$ has $\epsilon$-bounded order, $Z/[P_0, T]$ is abelian, and $T/Z$ has Sylow $p$-subgroups of exponent dividing $e$. According to the Hall–Higman theory $T/Z$ has a normal series of $\epsilon$-bounded length with the required properties. Thus, the result follows.

Given a group-word $w$, we write $w(G)$ for the corresponding verbal subgroup of a group $G$, that is, the subgroup generated by the values of $w$ in $G$. The word is said to be a law in $G$ if $w(G) = 1$. In view of Theorem 1.2 it is easy to see that if $w(G)$ is $\epsilon$-central in $G$, then a law of $(\epsilon, w)$-bounded length holds in the group $G$. This simple observation provides a tool for obtaining extensions of results about finite groups satisfying certain laws. We will illustrate this with a theorem bounding the nonsoluble length of a finite group. The concept of nonsoluble length $\lambda(G)$ of a finite group $G$ was introduced in [17]. This is the
minimum number of nonsoluble factors in a normal series of \( G \) in which every factor either is soluble or is a direct product of non-abelian simple groups. (In particular, the group is soluble if and only if its nonsoluble length is 0.)

It was shown in [7] that if a word \( w \) is a law in the Sylow 2-subgroup of a finite group \( G \), then \( \lambda(G) \) is bounded in terms of the length of the word \( w \) only.

This can be extended as follows.

**Theorem 5.3:** Let \( w \) be a group-word and \( P \) a Sylow 2-subgroup of a finite group \( G \) such that \( w(P) \) is \( \epsilon \)-central in \( P \). Then \( \lambda(G) \) is \( (\epsilon, w) \)-bounded.

**Proof.** As explained above, this is straightforward combining the result in [7] and Theorem 1.2.

For a group of automorphisms \( A \) of a group \( G \) we write \( C_G(A) \) for the centralizer of \( A \) in \( G \). It is well-known that if a finite group \( G \) admits a coprime group of automorphisms \( A \), then \( C_G/N(A) = NC_G(A)/N \) for any \( A \)-invariant normal subgroup \( N \) of \( G \) (see for example [8, Theorem 6.2.2 (iv)]). Here the group \( A \) is a coprime group of automorphisms if \((|G|, |A|) = 1 \). The symbol \( A^\# \) stands for the set of nontrivial elements of the group \( A \). The main result of [16] states that if a finite group \( G \) admits an elementary abelian coprime group of automorphisms \( A \) of order \( p^2 \) such that \( C_G(\phi) \) has exponent dividing \( \epsilon \) for each \( \phi \in A^\# \), then the exponent of \( G \) is \( (\epsilon, p) \)-bounded. We can now extend this in the following way.

**Theorem 5.4:** Let \( \epsilon > 0 \), and let \( G \) be a finite group admitting an elementary abelian coprime group of automorphisms \( A \) of order \( p^2 \) such that \( C_G(\phi) \) is \( \epsilon \)-central in \( G \) for each \( \phi \in A^\# \). Then there is a \( (p, \epsilon) \)-bounded number \( e \) and an \( A \)-invariant normal subgroup \( T \) such that the index \( [G : T] \) and the order of \( [G^e, T] \) are \( (p, \epsilon) \)-bounded.

**Proof.** Let \( A_1, \ldots, A_{p+1} \) be the subgroups of order \( p \) in \( A \) and set \( G_i = C_G(A_i) \) for \( i = 1, \ldots, p+1 \). According to Theorem 1.2 there is an \( \epsilon \)-bounded number \( d \) and, for \( i = 1, \ldots, p+1 \), \( A \)-invariant normal subgroups \( T_i \leq G \) such that the index \([G : T_i]\) and the order of \([G_i^d, T_i]\) are \( \epsilon \)-bounded. Set \( T = \bigcap T_i \) and observe that \( T \) is \( A \)-invariant and the index of \( T \) in \( G \) is \( (p, \epsilon) \)-bounded.

Let \( N_i = [G_i^d, T] \) and \( N_0 = \prod N_i \). Note that the subgroup \( N_0 \) is normal in \( T \) and has \( (p, \epsilon) \)-bounded order. Let \( N = \langle N_0^G \rangle \) be the normal closure of \( N_0 \) in \( G \). Since the index of \( T \) in \( G \) is \( (p, \epsilon) \)-bounded, it follows that the order of
$N$ is $(p, \epsilon)$-bounded as well. We also observe that $N$ is $A$-invariant since the subgroups $N_i$ are. Let $C$ be the centralizer of $T$ modulo $N$, that is,

$$C = \{x \in G \mid [T, x] \leq N\}.$$ 

Clearly, the subgroup $C$ is $A$-invariant. Moreover $G^d_i \leq C$ for $i = 1, \ldots, p + 1$. Hence, $C_{G/C}(A_i)$ has exponent dividing $d$ for each $i = 1, \ldots, p + 1$. Now the main result of [16] says that the exponent of $G/C$ is $(d, p)$-bounded. Therefore there exists a $(p, \epsilon)$-bounded number $e$ such that $G^e \leq C$, that is, $[G^e, T] \leq N$. This completes the proof. □

**Corollary 5.5:** Under the hypotheses of Theorem 5.4 there exists a number $\epsilon_0 > 0$ depending only on $\epsilon$ and $p$ such that $G$ is $\epsilon_0$-central.

**Proof.** Assume the hypotheses of Theorem 5.4. The theorem tells us that there is a $(p, \epsilon)$-bounded number $e$ and a normal subgroup $T$ such that the index $[G : T]$ and the order of $[G^e, T]$ are $(p, \epsilon)$-bounded. As explained in the introduction, Theorem 1.2 admits a converse. Hence the result. □

In the spirit of the work [11] we record the following theorem.

**Theorem 5.6:** Let $\epsilon > 0$, and let $G$ be a finite group admitting an elementary abelian coprime group of automorphisms $A$ of order $p^3$ such that the commutator subgroup of $C_G(\phi)$ is $\epsilon$-central in $G$ for each $\phi \in A^\#$. Then there is a $(p, \epsilon)$-bounded number $e$ and an $A$-invariant normal subgroup $T$ such that the index $[G : T]$ and the order of $[[G, G]^e, T]$ are $(p, \epsilon)$-bounded.

**Proof.** Let $A_1, \ldots, A_s$ be the subgroups of order $p$ of $A$ and let $D_i$ denote the commutator subgroup of $C_G(A_i)$ for $i = 1, \ldots, s$. According to Theorem 1.2 there is an $\epsilon$-bounded number $d$ and, for $i = 1, \ldots, s$, $A$-invariant normal subgroups $T_i \leq G$ such that the index $[G : T_i]$ and the order of $[D_i^d, T_i]$ are $\epsilon$-bounded. Set $T = \bigcap T_i$ and observe that $T$ is $A$-invariant and the index of $T$ in $G$ is $(p, \epsilon)$-bounded. Let $N_i = [D_i^d, T]$ and $N_0 = \prod N_i$. Note that $N_0$ is normal in $T$ and has $(p, \epsilon)$-bounded order. Let $N = \langle N_0^G \rangle$ be the normal closure of $N_0$ in $G$. Since the index of $T$ in $G$ is $(p, \epsilon)$-bounded, it follows that the order of $N$ is $(p, \epsilon)$-bounded. We also observe that $N$ is $A$-invariant since the subgroups $N_i$ are. Let $C$ be the centralizer of $T$ modulo $N$, that is, $C = \{x \in G \mid [T, x] \leq N\}$. Clearly, the subgroup $C$ is $A$-invariant. Moreover $D_i^d \leq C$ for $i = 1, \ldots, s$. Hence, $C_{G/C}(A_i)$ has commutator subgroup of
exponent dividing $d$ for each $i = 1, \ldots, s$. Now the main result of [11] says that the exponent of the commutator subgroup of $G/C$ is $(d, p)$-bounded. Therefore there exists a $(p, \varepsilon)$-bounded number $e$ such that $[G, G]^e \leq C$, that is,

$$[[G, G]^e, T] \leq N.$$ 

This completes the proof. □

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