Duality Theorem and Hom Functor in Braided Tensor Categories

Yange Xu \textsuperscript{a,b}, Shouchuan Zhang\textsuperscript{a}, Jing Cheng \textsuperscript{a}

\textsuperscript{a}: Department of Mathematics, Hunan University, Changsha 410082, P.R. China.
\textsuperscript{b}: Department of Mathematics, Pingdingshan University, Pingdingshan 467000, P.R. China.

Abstract

The Blatter-Montgomery duality theorem is generalized into braided tensor categories. It is shown that \(\text{Hom}(V,W)\) is a braided Yetter-Drinfeld module for any two braided Yetter-Drinfeld modules \(V\) and \(W\).

Keywords: braided Hopf algebra, \(\text{Hom}\) functor, duality theorem.

0 Introduction

The duality theorems play an important role in actions of Hopf algebras (see \cite{13}). In \cite{4} and \cite{13}, Blattner and Montgomery proved the following duality theorem for an ordinary Hopf algebra \(H\) and some Hopf subalgebra \(U\) of \(H^*\):

\[(R\#H)^\#U \cong R \otimes (H\#U)\] as algebras,

where \(R\) is a \(U\)-comodule algebra. The dual theorems for co-Frobenius Hopf algebra \(H\),

\[(R\#H)^\#H^{*\text{rat}} \cong M^f_H(R) \quad \text{and} \quad (R\#H^{*\text{rat}})^\#H \cong M^f_H(R)\] as \(k\)-algebras

were proved in \cite{6} and \cite{7} (see \cite{7} Corollary 6.5.6 and Theorem 6.5.11). Braided tensor categories become more and more important. They have been applied in conformal field, vertex operator algebras, isotopy invariants of links (see \cite{8,10,6} and \cite{9,11,15}). One of the authors in \cite{18} generalized the duality theorem to the braided case, i.e., for a finite Hopf algebra \(H\) with \(C_{H,H} = C_{H,H}^{-1}\),

\[(R\#H)^\#H^* \cong R \otimes (H \otimes H^*)\] as algebras in \(C\).
The Blattner-Montgomery duality theorem was also generalized into Hopf algebras over commutative rings [3]. Hom functor also has extensive use in homological algebra and representation theory.

We know that $H$ is an infinite braided Hopf algebra if it has no left duals (See [16]). In this paper we generalize the above results to infinite braided Hopf algebras. In section 1, we introduce quasi-dual $H^d$ of $H$ and prove the duality theorem in a braided tensor category $\mathcal{D}$; In section 2, we prove that if $V, W$ are in $B^\mathcal{D}(\mathcal{C})$, then $\text{Hom}(V, W)$ is also in $B^\mathcal{D}(\mathcal{C})$; In section 3, we concentrate on the Yetter-Drinfeld module category $B^\mathcal{B}$. Some notations. Let $(\mathcal{D}, \otimes, I, C)$ be a braided tensor category, where $I$ is the identity object and $C$ is the braiding, its inverse is $C^{-1}$. If $f : U \to V$, $g : V \to W$, $h : I \to V$, $k : U \to I$, $\alpha : U \otimes V \to P$, $\alpha_I : U \otimes V \to I$ are morphisms in $\mathcal{D}$, we denote them by:

$$
\begin{align*}
  f &= \begin{array}{c}
    U \\
    V
  \end{array}
  \begin{array}{c}
    1 \\
    g
  \end{array}
  \begin{array}{c}
    V \\
    W
  \end{array},
  g f &= \begin{array}{c}
    U \\
    V
  \end{array}
  \begin{array}{c}
    g \\
    f
  \end{array}
  \begin{array}{c}
    W \\
    T
  \end{array},
  h &= \begin{array}{c}
    U \\
    T
  \end{array}
  \begin{array}{c}
    h \\
    V
  \end{array}
  \begin{array}{c}
    V \\
    U
  \end{array},
  k &= \begin{array}{c}
    U \\
    T
  \end{array}
  \begin{array}{c}
    k \\
    V
  \end{array}
  \begin{array}{c}
    T \\
    I
  \end{array},
  \alpha &= \begin{array}{c}
    U \\
    T
  \end{array}
  \begin{array}{c}
    \alpha \\
    P
  \end{array}
  \begin{array}{c}
    V \\
    U
  \end{array},
  \alpha_I &= \begin{array}{c}
    U \\
    T
  \end{array}
  \begin{array}{c}
    \alpha_I \\
    I
  \end{array}
  \begin{array}{c}
    V \\
    U
  \end{array},
  C_{U,V} &= \begin{array}{c}
    U \\
    V
  \end{array}
  \begin{array}{c}
    V \\
    U
  \end{array},
  C_{U,V}^{-1} &= \begin{array}{c}
    V \\
    U
  \end{array}
  \begin{array}{c}
    U \\
    V
  \end{array},
  C_{U,V} = C_{U,V}^{-1} = \begin{array}{c}
    U \\
    V
  \end{array}
  \begin{array}{c}
    V \\
    U
  \end{array},
\end{align*}
$$

where $U, V, W$ are in $\mathcal{D}$.

Since every braided tensor category is always equivalent to a strict braided tensor category, we can view every braided tensor as a strict braided tensor and use braiding diagrams freely.

1 Duality theorem for braided Hopf algebras

In this section, we obtain the duality theorem for braided Hopf algebras living in the braided tensor category $(\mathcal{D}, C)$. Although the most results in this section appeared in [19], we write them by means of braided diagrams.

If $U, V, W \in \text{ob} \mathcal{D}$ and $f, g$ act are morphisms in $\mathcal{D}$, we call act : $\text{Hom}(V, W) \otimes V \to W$ satisfy elimination, if:

$$
\begin{align*}
  f &= \begin{array}{c}
    U \\
    V
  \end{array}
  \begin{array}{c}
    f \\
    g
  \end{array}
  \begin{array}{c}
    V \\
    W
  \end{array} \quad \Rightarrow f = g.
\end{align*}
$$

Definition 1.1. Let $(H, m, \eta, \Delta, \epsilon)$ is a braided Hopf algebra in braided tensor category $\mathcal{D}$. If there is a braided Hopf algebra $H^d$ in $\mathcal{D}$ and a morphism $\langle, \rangle : H^d \otimes H \to I$ in $\mathcal{D}$ satisfy:
then $H^d$ is called a quasi-dual of $H$ under $<,>$, and $<,>$ is called a quasi-evaluation of $H^d$ on $H$.

No other statement, all the objects and morphisms of this paper are in $\mathcal{D}$. The act always exists and satisfy elimination, the braiding is symmetric on $H$ and $H^d$.

**Lemma 1.2.** (i) $(H, \rightarrow)$ is a left $H$-module algebra;
(ii) $(H, \rightarrow)$ is a left $H^d$-module algebra;
(iii) $(H^d, \leftarrow)$ is a right $H$-module algebra;
(iv) $(H, \leftarrow)$ is a right $H^d$-module algebra;

where

Consequently, we construct two smash products $H\#H^d$ and $H^d\#H$ [17, Chapter 4].
Definition 1.3. We say CRL-condition holds on $H$ and $H^d$ under $<,>$ if the following conditions are satisfied:

CRL1 $E =: End_C H \in ob\mathcal{D}$ is an algebra under multiplication of composition in $\mathcal{D}$ and satisfy:

\[
\begin{align*}
E \cdot EH &= EH, \\
H \cdot H &= H.
\end{align*}
\]

CRL2 There are two morphisms $\lambda: H \# H^d \to E$ and $\rho: H^d \# H \to E$ such that

\[
\begin{align*}
H^d \cdot H &= H^d, \\
H \cdot H^d &= H^d.
\end{align*}
\]

CRL3 there exists an algebra morphism $\bar{\lambda}$ from $E$ to $H \# H^d$ such that $\bar{\lambda}\lambda = id_{H \# H^d}$;

Lemma 1.4. If CRL-condition holds on $H$ and $H^d$ under $<,>$, then $\lambda$ is an algebra morphism from $H \# H^d$ to $E$ and $\rho$ is an anti-algebra morphism from $H^d \# H$ to $E$.

Proof. We only consider $\lambda$. It is straightforward to prove that $\lambda \eta_{H \# H^d} = \eta_E$ and

\[
act((\lambda \otimes id_H)(m \otimes id_H)) = act((m \otimes id_H)(\lambda \otimes \lambda \otimes id_H)). \quad \Box
\]

Lemma 1.5. If CRL-condition holds on $H$ and $H^d$ under $<,>$, then $\lambda$ and $\rho$ satisfy
Proof. We show (1) by following five steps. It is easy to check the following (i) and (ii).

(i) \[ H\#_H^d \quad \eta_H^d \#_H \]
\[ \eta_H^d \#_H \quad H\#_H^d \]

(ii) \[ \eta_H \#_H^d \quad H^d \#_\eta_H \]
\[ H^d \#_\eta_H \quad \eta_H \#_H^d \]
Thus (iii) holds.
(iv)\[ H \# \eta_H \cdash H \# \eta_H \] = \ \text{End} H

\[ H \# \eta_H \cdash H \# \eta_H \] = \ \text{End} H

\[ \eta_H \cdash H \eta_H \] = \ \text{End} H

\[ \eta_H \cdash H \eta_H \] = \ \text{End} H

\[ \text{by (iii)} \] = \ \text{End} H

\[ \text{In fact,} \]
Thus (iv) holds.

(v)

In fact,
Thus (v) holds. Next, we show (1) holds.

the left side of (1) by Lemma [4].
If \((R, \psi)\) is a right \(H^d\)-comodule algebra, then \((R, \alpha)\) becomes a left \(H\)-module algebra (similar to [4, Example 1.6.4]) under the module operation:

\[
H \overset{R}{\leftarrow} H^d \overset{\psi}{\leftarrow} R
\]

Lemma 1.6. \(R\#H\) becomes an \(H^d\)-module algebra under the module operation:

\[
H^d \overset{R\#H}{\leftarrow} H \overset{\psi}{\leftarrow} H
\]

Proof. It is straightforward. \(\Box\)

Consequently, we obtain another smash product \((R\#H)\#H^d\).

Theorem 1.7. Let \(H\) and \(H^d\) be Hopf algebra with invertible antipodes, and the CRL-condition holds on \(H\) and \(H^d\) under \(<,>\). Let \(R\) be an \(H^d\)-comodule algebra such that \(R\)
is an $H$-module algebra defined as above, $H^d$ act on $R \# H$ by acting trivially on $R$ and via $\rightarrow$ on $H$, then

$$(R \# H) \# H^d \cong R \otimes (H \# H^d)$$

as algebras in $\mathcal{D}$.

**Proof.** By (CRL)-condition, there exists a algebra morphism $\bar{\lambda}$ from $E$ to $H \# H^d$ such that $\bar{\lambda} \lambda = id_{H \# H^d}$. We first define a morphism $w = \bar{\lambda} \rho (S^{-1} \otimes \eta_H)$ from $H^d$ to $H \# H^d$. Since $\rho$ and $S^{-1}$ are anti-algebra morphisms by Lemma 1.4, $w$ is an algebra morphism.

We now define two morphisms:

It is straightforward to check that $\Psi \Phi = id$, $\Phi \Psi = id$. Now we show that $\Phi$ is algebraic. Define

$$(R \# H) \# H^d \cong R \otimes (H \# H^d)$$

where $\xi = \eta_H$. It is clear that $\Phi = (id \otimes \bar{\lambda}) \Phi'$. Consequently, we only need to show that $\Phi'$ is algebraic.

We claim that

......(*)
Thus relation (*) holds.
Now, we check that $\xi$ is algebraic. We see that

$$R \otimes \text{End} H = R \otimes \text{End} H$$

since $\rho(S \otimes \eta_H)$ is algebraic

Thus $\xi$ is algebraic.

Now we show that $\Phi'$ is algebraic.

$$(R \# H) \# H^d (R \# H) \# H^d$$

by Lemma 1.4

since $\xi$ is algebraic
Thus $\Phi'$ is algebraic.  

We obtain the following by Theorem 1.7:

**Corollary 1.8.** Let $H$ be a finite braided Hopf algebra with a left dual $H^*$. If the braiding is symmetric on $H$, then

$$(R\#H)\#H^* \cong R \otimes (H\#H^*)$$ as algebras in $D$.

This corollary reproduces the main result in [18].

### 2 Hom functor in braided Yetter-Drinfeld module category

In this section, we prove that if $V, W$ are in $^B\mathcal{YD}(C)$, then $Hom(V, W)$ is also in $^B\mathcal{YD}(C)$. Let $B$ be a Hopf algebra in braided tensor category $(\mathcal{C}, C)$, and $(M, \alpha, \phi)$ be a left $B$-module and left $B$-comodule in $(\mathcal{C}, C)$. If

(YD):

then $(M, \alpha, \phi)$ is called a braided Yetter-Drinfeld $B$-module in $\mathcal{C}$, written as braided YD $B$-module in short. Let $^B\mathcal{YD}(C)$ denote the category of all braided Yetter-Drinfeld $B$-modules in $\mathcal{C}$. 

16
Throughout this section, the braiding is in $\mathcal{C}$ and is symmetric on set $\{B, V, W, \text{Hom}(V, W)\}$, where $B, V, W \in \mathcal{C}$. Assume that $\otimes =: \otimes_\mathcal{C}$. $H$ and $H^d$ be braided Hopf algebra in $B \mathcal{YD}(\mathcal{C})$ and $H^d$ is a quasi-dual of $H$ under the operations:

![Diagram of braiding operations](image)

**Lemma 2.1.** If $B$ has left duals and invertible antipode, then

(i) If $(V, \alpha_V, \phi_V)$ and $(W, \alpha_W, \phi_W)$ are in $B \mathcal{YD}(\mathcal{C})$, then $\text{Hom}_\mathcal{C}(V, W)$ is in $B \mathcal{YD}(\mathcal{C})$ under the following module operation and comodule operation:

![Diagram of module and comodule operations](image)

(ii) $\text{End}M$ is an algebra in $B \mathcal{YD}(\mathcal{C})$, where $M$ is in $B \mathcal{YD}(\mathcal{C})$.

**Proof.** Let $B^*$ denote the left deal of $B$, the definitions of $\alpha$ and $\phi$ are reasonable, since $\text{act}$ satisfy elimination. In fact, let

![Diagram of additional operations](image)
Now we show that $\text{Hom}_C(V,W)$ is a module and a comodule.
Now we show that

\[ \phi \alpha = B \text{ Hom} \]

In fact,

\[ B \text{ HomV} = B \text{ Hom} \]

by [17 Proposition 1.0.13].
So $Hom_c(V, W)$ is in $B \mathcal{YD}(C)$.

(ii) It is straightforward. □

**Lemma 2.2.** Let $H$ be a braided Hopf algebra in $B \mathcal{YD}(C)$, $E =: \text{End}H$, then

(i) $\text{act} : E \otimes H \rightarrow H$ is a morphism in $B \mathcal{YD}(C)$.

(ii) The evaluation $<,>$ is in $B \mathcal{YD}(C)$.

**Proof**
so act is a $B$-module homomorphism, similarly, we can show that act is a $B$-comodule homomorphism.

(ii) It is similar to (i). □

If act satisfy elimination, let $D = \mathcal{YD}(C)$, $CRL1$ could be instead of:

$CRL1'$ $E =: End_C H$ satisfy:

$$
\begin{array}{c}
\begin{array}{c}
E \\
\downarrow \text{Ev}
\end{array} \\
\begin{array}{c}
EH \\
\downarrow \text{Ev}
\end{array}
\end{array}
\begin{array}{c}
= \\
\downarrow
\end{array}
\begin{array}{c}
EE \\
\downarrow \\
\downarrow
\end{array} \\
\begin{array}{c}
H \\
\downarrow \\
\downarrow
\end{array},

\begin{array}{c}
\begin{array}{c}
H \\
\downarrow
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
H \\
\downarrow
\end{array}
\end{array}.
\end{array}$$

3 Duality theorems in Yetter-Drinfeld module categories

In this section, we present the duality theorem for braided Hopf algebras in the Yetter-Drinfeld module category $B\mathcal{YD}$ (i.e. if $C$ is the category of vector spaces, we write $B\mathcal{YD}(C) = B\mathcal{YD}$). Throughout this section, $H$ is a braided Hopf algebra in $B\mathcal{YD}$ with Hopf algebra $B$ and $H^d$ is a quasi-dual of $H$ under a left faithful $<,>$ (i.e. $<x, H > = 0$ implies $x = 0$) such that $(b \cdot f)(x) = \sum b_2 \cdot f(S(b_1) \cdot x)$ and $\sum < f(0), x > f(-1) = \sum < f, x(0) > S^{-1}(x(-1))$ for any $x \in H, b \in B, f \in H^d$. Let $<,>_ev$ the ordinary evaluation of any spaces.

Let $A$ be any braided algebra in $D = B\mathcal{YD}$. Define

$$A^*_D = \{ f \in A^* \mid Ker(f) \text{ contains an ideal of finite codimension in } B\mathcal{YD} \}$$

Consequently, let $H$ be a Hopf algebra in $B\mathcal{YD}$, $U$ be a subHopfalgebra of $H^*_D$. Then we call that $U$ satisfy the $RL$-condition with respect to $H$ if $\rho(U \# 1) \subseteq \lambda(H \# U)$ [13, Definition 9.4.5].

Lemma 3.1. If braided algebra $A \in ob\mathcal{D}$, then $A^*_D \in ob\mathcal{D}$.

Proof. By Lemma 3.1, $A^* \in ob(D)$. For any $f \in A^*_D$, there exists an ideal $I$ of $A$ and $I$ is a $B$-submodule and a $B$-subcomodule of $A$ with finite codimension and $f(I) = 0$. Since $(b \cdot f)(x) = \sum b_2 \cdot f(S(b_1) \cdot x) = 0$ for any $b \in B, x \in I$, we have $b \cdot f \in A^*_D$. Thus $A^*_D$ is a $B$-submodule of $A^*$. By Lemma 3.1, we can assume $\phi_{A^*}(f) = \sum_i u_i \otimes u_i$ with linear independent $u_i$'s. Since $\sum_i u_i v_i(x) = \sum f(x(0)) S^{-1}(x(-1)) = 0$ for any $x \in I$, we have that $v_i(x) = 0$ and $v_i(I) = 0$, which implies $v_i \in A^*_D$, thus $A^*_D$ is a $B$-subcomodule of $A^*$. Consequently, it is clear that $A^*_D \in ob\mathcal{D}$.
Lemma 3.2. If $f$ is a morphism from $U$ to $V$ in $\mathcal{D}$, then $f^*$ is a morphism from $V^*$ to $U^*$ in $\mathcal{D}$.

Proof. For any $v^* \in V^*, u \in U, b \in B$, see that

\[
(b \cdot f^*(v^*))(u) = \sum b_2 \cdot f^*(v^*)(S(b_1) \cdot u) = \sum b_2 \cdot v^*(f(S(b_1) \cdot u)) \quad \text{since } f \text{ is a } B\text{-module homomorphism} \\
= \sum b_2 \cdot v^*(S(b_1) \cdot f(u)) = f^*(b \cdot v^*)(u).
\]

Thus $b \cdot f^*(v^*) = f^*(b \cdot v^*)$ and $f^*$ is a $B$-module homomorphism. Similarly, we can show that $f^*$ is a $B$-comodule homomorphism. □

By Lemma 3.1, Lemma 3.2 and [13, Theorem 9.1.3], we get the following:

Theorem 3.3. If $H$ is a Hopf algebra in $\mathcal{D}$, then $H^*_D$ is a Hopf algebra in $\mathcal{D}$.

Next, we give an example which showed that there exists a Hopf algebra $H$ in Yetter-Drinfeld module category such that $H^*_D$ is nontrivial.

Let $T = T(G, g_i, \chi_i; J)$ be the free algebra generated by set $X = \{x_i \mid i \in J\}$ where $G$ is a group, $J = \{1, 2, \cdots, \theta\}$, $g_i \in Z(G)$ and $\chi_i \in \hat{G}$ with $i \in J$. We present the construction of $T$ as follows: Denote by $T_0 = \emptyset$, $T_1 = X$, and for $n \geq 2$ by $T_n = X \otimes X \otimes \cdots \otimes X$, the tensor product of $n$ copies of the set $X$, then $T = \oplus_{n \geq 0} T_n$. Define coalgebra operations and $kG$-(co-)module operations in $T$ as follows:

\[
\Delta x_i = x_i \otimes 1 + 1 \otimes x_i, \quad \epsilon(x_i) = 0,
\]

\[
\delta^-(x_i) = g_i \otimes x_i, \quad h \cdot x_i = \chi_i(h)x_i.
\]

Then $T$ is called a quantum tensor algebra in $\mathcal{D} = kG^* \mathcal{YD}$.

If $\chi_i(g_j)\chi_j(g_i) = 1$ for $i, j \in \{1, 2, \cdots, \theta\}$, it is easily to check that $T$ is quantum cocommutative.

For quantum tensor algebra $T$, we can find an ideal $D$ of $T^*$ such that $D \subseteq T^*_D$. We know that $T^* \cong \prod_{n \geq 0} T_n^*$, denote $D = \oplus_{n \geq 0} T_n^* \subseteq T^*$. For any $f = \sum_{i=1}^s f_i \in D$, denote $L_n = \oplus_{i \leq n} T_i$ with $n \geq s$, it is easily to prove that $L_n$ is a finite codimension ideal of $T$ and $f(L_n) = 0$. Consequently, we only need to show that $L_n$ is a sub-(co-)module, it is sufficient to prove that $T_m$ is a sub-(co-)module for $m \geq 0$. In fact, for any $g \in G$, $y = y_1y_2 \cdots y_m \in T_m$ (i.e.the multiplication of $T$ is $m(x \otimes y) = xy.$), where $y_i \in X$ for any $i \in J$, then

\[
g \cdot y = g \cdot (y_1y_2 \cdots y_m) \quad \text{since } T \text{ is a } kG\text{-module algebra}
\]

\[
= (g \cdot y_1)(g \cdot y_2) \cdots (g \cdot y_m)
\]

\[
= \chi_1(g)\chi_2(g) \cdots \chi_m(g)y_1y_2 \cdots y_m \quad \text{Let } \alpha = \chi_1(g)\chi_2(g) \cdots \chi_m(g)
\]

\[
= \alpha y \in T_m
\]
and

\[
\delta^-(y) = \delta(y_1y_2\cdots y_m) \quad \text{since } T \text{ is a kG-comodule algebra}
\]

\[
= (g_1 \cdot y_1)(g_2 \cdot y_2)\cdots (g_m \cdot y_m)
\]

\[
= \chi_1(g_1)\chi_2(g_2)\cdots \chi_m(g_m)y_1y_2\cdots y_m
\]

Let \( \beta = \chi_1(g_1)\chi_2(g_2)\cdots \chi_m(g_m) \)

\[
\beta y \in T_m \subseteq KG \otimes T_m,
\]

so \( T_m \) is a sub-(co-)module for \( m \geq 0 \).

**Definition 3.4.** Let \( H \) be a braided Hopf algebra in \( \mathcal{D} = \mathcal{B} \mathcal{YD} \), \( U \) is a Hopf subalgebra of \( H \cdot H \), we define morphisms \( \lambda : H \# U \rightarrow \text{End}_H \) via \( \lambda(h \# f)(k) = \Sigma hk_1 < f, k_2 > \), and \( \rho : U \# H \rightarrow \text{End}_H \) via \( \rho(f \# h)(k) = \Sigma k_2 h < f, k_1 > \) for all \( h, k \in H, f \in U \).

It is clear that \( \lambda, \rho \in \mathcal{D} \), then, since \( \check{\lambda} \lambda = \text{id}_{H \# H \cdot H} \in \mathcal{D} \) (by Lemma 3.5), \( \check{\lambda} \in \mathcal{D} \).

**Lemma 3.5.** (i) Define \( \text{act} : \text{End}_H \otimes H \rightarrow H \) via \( \text{act}(f \otimes h) = f(h) \), then \( \text{act} \) satisfy elimination.

(ii) If the antipode of \( H \) is invertible, then there exists morphism \( \check{\lambda}_1 \) from \( \text{Im}\lambda_1 \) to \( H \cdot H \cdot H \) such that \( \check{\lambda}_1 \lambda = \text{id}_{H \cdot H \cdot H} \). Furthermore, let \( E = E_1 \oplus \text{Im}\lambda_1 \) where \( E_1 \) is a subspace of \( E \), define \( \check{\lambda} = 0 + \check{\lambda}_2 : E \rightarrow H \cdot H \cdot H \), then \( \check{\lambda} \lambda = \text{id}_{H \cdot H \cdot H} \).

(iii) If \( H \) is quantum cocommutative, then RL-condition holds on \( H \) and \( U \) under \( <,> \).

**Proof.** (i) It is straightforward.

(ii) We define a morphism \( \lambda' \) and \( \check{\lambda}_2 \) as follows:
obviously, $\lambda'$ is a injective. Now we show that $\bar{\lambda}^2 \lambda = \lambda'$.

(iii) It follows from the simple fact $\rho(f \# 1) = \lambda(1 \# f)$ for any $f \in U$ (see [13, Example 9.4.7]). □

**Theorem 3.6.** (Duality Theorem) Let $H$ be a braided Hopf algebra in $D = B_B V D$ with invertible antipode, $U$ is a braided Hopf subalgebra of $H_B^*$, the braiding is symmetric on set $\{B, H, H_B^*\}$. And let $R$ in $B_B V D$ be an $U$-comodule algebra such that $R$ is an $H$-module algebra defined as in section [4], $U$ act on $R \# H$ by acting trivially on $R$ and via $\rightarrow$ on $H$. Then

(i) If $U$ satisfy the $RL$-condition with respect to $H$, then

$$(R \# H) \# U \cong R \otimes (H \# U) \quad \text{as algebras in } D.$$

(ii) If $H$ is quantum cocommutative, then

$$(R \# H) \# U \cong R \otimes (H \# U) \quad \text{as algebras in } D.$$

**Proof.** It is clear that $U$ is a quasi-dual of $H$ under evaluation $\langle, \rangle_{ev}$. $U$ has an invertible antipode since $H$ has an invertible antipode and $U$ satisfy the $RL$-condition with respect to $H$ implies that $w$ is a algebra morphism, so $CRL'$-condition is hold. By Theorem [1.7] we complete the proof. □

This theorem reproduces the main result in [13].

**Example 3.7.** Let $H$ be quantum tensor algebra in $D = k_0^G V D$ with $\chi_i(g_j)\chi_j(g_i) = 1$ for $i, j \in \{1, 2, \cdots \theta\}$ and has invertible antipode, or let $H$ be a quantum cocommutative braided Hopf algebra in $D = B_B V D$ with finite-dimensional commutative and cocommutative $B$ (for example, $H$ is the universal enveloping algebra of a Lie superalgebra). Set $U = H_B^* = R$. It is clear that $(R, \phi)$ is a right $U$-comodule algebra with $\phi = \Delta$. By theorem [3.6] we have

$$(R \# U) \# H \cong R \otimes (U \# H) \quad \text{as algebras in } D.$$
References

[1] N. Andruskewisch and H.J. Schneider, Pointed Hopf algebras, new directions in Hopf algebras, edited by S. Montgomery and H.J. Schneider, Cambridge University Press, 2002.

[2] N. Andruskiewitsch and H. J. Schneider, Lifting of quantum linear spaces and pointed Hopf algebras of order $p^3$, J. Algebra, 209 (1998), 658-691.

[3] J.Y. Abuhlail, J. Gomez-Torrecillas and F.J. Lobillo, Duality and rational modules in Hopf algebras over commutative rings, Journal of Algebra, 240 (2001), 165–184.

[4] R. J. Blattner and S. Montgomery, A duality theorem for Hopf module algebras, J. Algebra, 95 (1985), 153–172.

[5] B. Bakalov, A. Kirillov, Lectures on tensor categories and modular functors, University Lecture Series Vol. 21, American Mathematical Society, Providence, 2001.

[6] A. Van Daele and Y. Zhang, Galois theory for multiplier Hopf algebras with integrals, Algebra Representation Theory, 2 (1999), 83-106.

[7] S. Dascalescu, C. Nastasescu and S. Raianu, Hopf algebras: an introduction, Marcel Dekker Inc., 2001.

[8] Yi-Zhi Huang, Vertex operator algebras, the Verlinde conjecture and modular tensor categories, Proc. Nat. Acad. Sci. 102 (2005) 5352-5356.

[9] M. A. Hennings, Hopf algebras and regular isotopy invariants for link diagrams, Proc. Cambridge Phil. Soc., 109 (1991), 59-77.

[10] Yi-Zhi Huang, Liang Kong, Open-string vertex algebras, tensor categories and operads, Commun.Math.Phys. 250 (2004) 433-471.

[11] L. Kauffman, Invariants of links and 3-manifolds via Hopf algebras, in Geometry and Physics, Marcel Dekker Lecture Notes in Pure and Appl. Math. 184 (1997), 471-479.

[12] T. Kerler, Bridged links and tangle presentations of cobordism categories, Adv. Math. 141 (1999), 207–281.

[13] S. Montgomery, Hopf algebras and their actions on rings, CBMS Number 82, Published by AMS, 1993.

[14] S. Majid, Foundations of quantum group theory, Cambridge University Press, 1995.
[15] D.E. Radford, The trace function and Hopf algebras, J. Algebra, 163 (1994), 583-622.

[16] M. Takeuchi, Finite Hopf algebras in braided tensor categories, J. Pure and Applied Algebras, 138 (1999), 59-82.

[17] Shouchuan Zhang, Braided Hopf Algebras, Hunan Normal University Press, Changsha, Second edition. Also in math. RA/0511251.

[18] Shouchuan Zhang, Duality theorem and Drinfeld double in braided tensor categories, Algebra Colloq. 10 (2003)2, 127-134. math.RA/0307255.

[19] Shouchuan Zhang, Yao-Zhong Zhang, Yanying Han, Duality Theorems for Infinite Braided Hopf Algebras, e-print, math/0309007