An approach to Quantum Conformal Algebra

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Abstract

We aim to explore if inside a quantum vertex algebras, we can find the right notion of a quantum conformal algebra.

1 Introduction

Since the pioneering papers [BPZ, Bo1], there has been a great deal of work towards understanding of the algebraic structure underlying the notion of the operator product expansion (OPE) of chiral fields of a conformal field theory. The singular part of the OPE encodes the commutation relations of fields, which leads to the notion of a conformal algebra [K1].

In [BK], they develop foundation of the theory of field algebras, which are a “non-commutative version” of a vertex algebra. Among other results they show that inside certain field algebras, more precisely strong field algebras (where the \( n \)-product axiom holds) we have a conformal algebra and a differential algebra toghether with certain compatibility equations, and conversely, having this two structures plus those equations we can recover a strong field algebra. One of these equations is the conformal analog of the Jacobi Identiy. They call a conformal algebra satisfying this equation Leibnitz conformal algebra.

A definition of a quantum vertex algebra, which is a deformation of a vertex algebra, was introduced by Etingof and Kazhdan in 1998, [EK]. Roughly speaking, a quantum vertex algebra is a braided state-field correspondence which satisfies associativity and braided locality axioms. Such braiding is a one-parameter braiding with coefficients in Laurent series.

Recently in [DGK], they developed a structure theory of quantum vertex algebras, parallel to that of vertex algebras. In particular, they introduce braided \( n \)-products for a braided state-field correspondence and prove for quantum vertex algebras a version of the Borcherds identity.

Following [BK], in this article, we try to determine the quantum analog of the notion of conformal algebra inside a quantum vertex algebra \( V \). For this purpose, we introduced new products parametrized by Laurent polinomials \( f \),

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and we showed that all this products are determined by those corresponding $f = 1$ and $f = z^{-1}$. The case $f = 1$ coincides with the $\lambda$-product defining a conformal algebra [BK]. This allows us to deal with the coefficients of the braiding in $V$. An important remark is that $V$ together with the $\lambda$-product is no longer a Leibnitz conformal algebra, since due to the braiding, the analog of the Jacobi identity involves not only the products corresponding to $f = 1$ (as in [BK]), but those of $f = z^{-1}$. We translate to this language the hexagon axioms, quasi-associativity and associativity relations, and the braided skew-symmetry in a quantum vertex algebra, and all this allows us to give an equivalent definition of quantum vertex algebra and present a candidate of a quantum conformal algebra.

The article is organized as follows. In Section 2 we review all the definitions and basic notion of field algebras and braided field algebras. In Section 3 we introduce the ($\lambda, f$)-product and prove some of its properties and we finish the section proving in Theorem 3 that shows that having a strong braided field algebra is the same of having a conformal algebra, a differential algebra with unit with some compatibility equations. In Section 4, we translate the hexagon axiom, quasi-associativity, and associativity relations, and the braided skew-symmetry in a quantum vertex algebra, we give an equivalent definition of quantum vertex algebra and present a candidate of a quantum conformal algebra with an example of a Lie quantum conformal algebra.

2 Preliminaries

In this section review some basic definitions followig [BK, DCK]. Throughout the paper all vector spaces, tensor products, etc are over a field $K$ of characteristic zero, unless otherwise specified.

2.1 Calculus of formal distribution

Given a vector space $V$, we let $V[[z, z^{-1}]]$ be the space of formal power series with coefficients in $V$; they are called formal distributions. A quantum field over $V$ is a formal distribution $a(z) \in (\text{End}V)[[z, z^{-1}]]$ with coefficients in $\text{End}V$, such that $a(z)v \in V((z))$ for every $v \in V$. Hereafter $V((z)) = V[[z]][z^{-1}]$ stands for the space of Laurent series with coefficients in $V$.

Throughout the article $\iota_{z,w}$ (resp $\iota_{w,z}$) denotes the geometric series expansion in the domain $|z| > |w|$ (resp $|w| > |z|$), namely we set for $n \in \mathbb{Z}$,

$$\iota_{z,w}(z + w)^n = \sum_{l \in \mathbb{Z}_+} \binom{n}{l} z^{n-l} w^l$$

where

$$\binom{n}{l} = \frac{n(n-1) \cdots (n-l+1)}{l!}.$$
For an arbitrary formal distribution \( a(z) \), we have
\[
\text{Res}_z(a(z)) = a_{-1},
\]
which is the coefficient of \( z^{-1} \). Denote by \( \text{gl}(V) \) the space of all \( \text{End} V \)-valued fields. We also need the Taylor’s Formula (cf. Proposition 2.4,[K1]), namely,
\[
\iota_{z,w}a(z + w) = \sum_{j \in \mathbb{Z}^+} \frac{\partial_j}{j!} a(z) w^j = e^{w \partial_z} a(z).
\]
(2)

For each \( n \in \mathbb{Z} \) one defines the \( n \)-th product of fields \( a(z) \) and \( b(z) \) by the following formula:
\[
a(z)_{(n)} b(z) = \text{Res}_x(a(x)b(z))\iota_{x,z}(x - z)^n - b(z)a(x)\iota_{z,x}(x - z)^n.
\]
(3)

2.2 Conformal algebras and Field Algebras

In this subsection we recall the definition of a field algebra, conformal algebras and its properties following [BK].

A state-field correspondence on a pointed vector space \((V, |0\rangle)\) is a linear map \( Y : V \otimes V \to V((z)) \) satisfying

(i) (vacuum axioms) \( Y(z)(|0 \rangle \otimes a) = a, Y(z)(a \otimes |0\rangle) \in a + V[[z]]z; \)

(ii) (translation covariance) \( T Y(z)(a \otimes b) - Y(z)(a \otimes T b) = \partial_z Y(z)(a \otimes b), \)

(iii) \( Y(z)(Ta \otimes b) = \partial_z Y(z)(a \otimes b), \)

where \( T(a) := \partial_z(Y(z)(a \otimes |0\rangle)) |_{z=0} = a_{(-2)} |0\rangle \), is called the translation operator.

Note that we will also denote by \( Y \) the map \( Y : V \to \text{End}V[[z, z^{-1}]], a \mapsto Y(a, z) = \sum_{k \in \mathbb{Z}} a(k) z^{-k-1} \), such that \( Y(a, z)b = Y(z)(a \otimes b). \)

Note that \( Y(a, z) \) is a quantum field, i.e \( Y(a, z)b \in V((z)) \) for any \( b \in V. \)

The following results, proved in [BK], will be useful in the sequel.

**Proposition 1.** (cf. [BK], Prop.2.7). Given \( Y : V \otimes V \to V((z)) \) satisfying conditions (i) and (ii) above, we have:

(a) \( Y(z)(a \otimes |0\rangle) = e^{zT} a; \)

(b) \( e^{wT} Y(z)(1 \otimes e^{-wT}) = \iota_{z,w} Y(z + w). \)

If, moreover, \( Y \) is a state-field correspondence, then

(c) \( Y(z)(e^{wT} \otimes 1) = \iota_{z,w} Y(z + w). \)
Given a state field correspondence $Y$, define

$$Y^{\text{op}}(z)(u \otimes v) = e^{zT}Y(-z)(v \otimes u).$$

(4)

Then $Y^{\text{op}}$ is also a state-field correspondence, called the opposite to $Y$. (cf. [BK], Prop 2.8).

Let $(V, |0\rangle)$ be a pointed vector space and let $Y$ be a state-field correspondence. Recall that $Y$ satisfies the $n$-th product axiom if for all $a, b \in V$ and $n \in \mathbb{Z}$

$$Y(a_{(m)}b, z) = Y(a, z)Y_{(m)}(b, z).$$

(5)

We say that $Y$ satisfies the associativity axiom if for all $a, b, c \in V$, there exists $N \gg 0$ such that

$$(z - w)^N Y(-w)(Y(z \otimes 1))(a \otimes b \otimes c) = (z - w)^N z,w Y(z - w)(1 \otimes Y(-w))(a \otimes b \otimes c).$$

(6)

Let $(V, |0\rangle)$ be a pointed vector space. As in [BK], a field algebra $(V, |0\rangle, Y)$ is a state-field correspondence $Y$ for $(V, |0\rangle)$ satisfying the associativity axiom. A strong field algebra $(V, |0\rangle, Y)$ is a state-field correspondence $Y$ satisfying the $n$-th product axiom. Note that, every strong field algebra is a field algebra.

Let $(V, |0\rangle)$ be a pointed vector space and let $Y$ be a state-field correspondence. For $a, b \in V$, [BK] defined the $\lambda$-product given by

$$a_\lambda b = \text{Res}_z e^{\lambda z}Y(z)(a \otimes b) = \sum_{n \geq 0} \frac{\lambda^n}{n!} a_{(n)}b.$$ 

(7)

and the $\cdot$-product on $V$, which is denote as

$$a \cdot b = \text{Res}_z z^{-1}Y(z)(a \otimes b) = a_{(-1)}b.$$ 

(8)

The vacuum axioms for $Y$ implies

$$|0\rangle \cdot a = a = a \cdot |0\rangle,$$

(9)

while the translation invariance axioms imply

$$T(a \cdot b) = T(a) \cdot b + a \cdot T(b),$$

(10)

and

$$T(a_\lambda b) = (T a)_\lambda b + a_\lambda (T b), \quad (T a)_\lambda b = -\lambda a_\lambda b$$

(11)

for all $a, b \in V$. Notice that from these equations we can derive that $T(|0\rangle) = 0$ and $|0\rangle \lambda a = 0 = a_\lambda |0\rangle$ for $a \in V$.

Conversely, if we are given a linear operator $T$, a $\lambda$-product and a $\cdot$-product on $(V, |0\rangle)$, satisfying the above properties ([9] - [11]), we can reconstruct the state-field correspondence $Y$ by the formulas

$$Y(a, z)_+ b = (e^{zT}a)_+ b, \quad Y(a, z)_- b = (a_{-\partial_z}b)(z^{-1}),$$

(12)
where $Y(a, z) = Y(a, z)_+ + Y(a, z)_-$. 

A $\mathbb{K}[T]$-module $V$, equipped with a linear map $V \otimes V \to \mathbb{K} \otimes V$, $a \otimes b \to \alpha \lambda b$, satisfying (11) is called a ($\mathbb{K}[T]$)-conformal algebra. On the other hand with respect to the $\cdot$-product, $V$ is a ($\mathbb{K}[T]$)-differential algebra (i.e an algebra with derivation $T$) with a unit $|0\rangle$.

Summarizing, (Cf. [BK], Lemma 4.1), we have that, giving a state-field correspondence on a pointed vector space $(V, |0\rangle)$ is equivalent to provide $V$ with a structure of a $\mathbb{K}[T]$-conformal algebra and a structure of a $\mathbb{K}[T]$-differential algebra with a unit $|0\rangle$.

Now, recall the following results. Later on, we will prove some analogous result for the braided environment.

**Lemma 1.** ([BK], Lemma 4.2) Let $(V, |0\rangle)$ be a pointed vector space and let $Y$ be a state-field correspondence. Fix $a, b, c \in V$. Then the collection of $n$-th product identities $Y(a_{(n)}b, z)c = (Y(a, z)(n)Y(b, z))c$ (for $n \geq 0$) implies

$$ (a_{\lambda} b)_{\lambda+\mu} = a_{\lambda} (b_{\mu} c) - b_{\mu} (a_{\lambda} c), \quad (13) $$

and

$$ a_{\lambda} (b, c) = (a_{\lambda} b).c + b.(a_{\lambda} c) + \int_0^\lambda (a_{\lambda} b)_{\mu} c d\mu. \quad (14) $$

The $(-1)$-st product identity $Y(z)(a_{(-1)}b \otimes c) = (Y(z)(-1)Y(z))(a \otimes b \otimes c)$ implies

$$ (a - b)_{\lambda} c = (e^{T \partial_x} a).(b_{\lambda} c) + (e^{T \partial_x} b).(a_{\lambda} c) + \int_0^\lambda b_{\mu} (a_{\lambda - \mu} c) d\mu, \quad (15) $$

and

$$ (a \cdot b, c - a \cdot (b, c) = \left( \int_0^T d\lambda a \right) \cdot (b_{\lambda} c) + \left( \int_0^T d\lambda b \right) \cdot (a_{\lambda} c). \quad (16) $$

Identity (13) is called the (left) Jacobi identity. A conformal algebra satisfying this identity for all $a, b, c \in V$ is called a (left) Leibnitz conformal algebra. Equation (14) is known as the “non-commutative” Wick formula, while (15) is called the quasi-associativity formula.

Finally, we also recall the following result.

**Theorem 1.** ([BK], Theorem 4.4) Giving a strong field algebra structure on a pointed vector space $(V, |0\rangle)$ is the same as providing $V$ with a structure of Leibnitz $\mathbb{K}[T]$-conformal algebra and a structure of a $\mathbb{K}[T]$-differential algebra with a unit $|0\rangle$, satisfying (14)-(16).

Recall also the following result.

**Theorem 2.** ([BK], Theorem 6.3) A vertex algebra is the same as a field algebra $(V, |0\rangle, Y)$ for which $Y = Y^{op}$.

Therefore we may assume this as a definition of vertex algebra.
2.3 Braided Field Algebras

We will follow the notation and presentation introduced in [DGK].

Throughout the rest of the paper we shall work over the algebra \( \mathbb{K}[[h]] \) of formal series in the variable \( h \), and all the algebraic structures that we will consider are modules over \( \mathbb{K}[[h]] \).

A topologically free \( \mathbb{K}[[h]] \)-module is isomorphic to \( W[[h]] \) for some \( \mathbb{K} \)-vector space \( W \).

Note that \( W[[h]] \not\cong W \otimes \mathbb{K}[[h]] \), unless \( W \) is finite-dimensional over \( \mathbb{K} \), and that the tensor product \( U[[h]] \otimes_{\mathbb{K}[[h]]} W[[h]] \) of topologically free \( \mathbb{K}[[h]] \)-modules is not topologically free, unless one of \( U \) and \( W \) are finite dimensional. For any vector space \( U \) and \( W \), the completed tensor product by \( U \)

\[
U[[h]] \hat{\otimes}_{\mathbb{K}[[h]]} W[[h]] := (U \otimes W)[[h]]
\]

This is a completion in \( h \)-adic topology of \( U[[h]] \otimes_{\mathbb{K}[[h]]} W[[h]] \).

Given a topologically free \( \mathbb{K}[[h]] \)-module \( V \), we let \( V_h((z)) = \{ a(z) \in V[[z, z^{-1}]] \mid a(z) \in V((z)) \bmod h^M \text{ for every } M \in \mathbb{Z}_{\geq 0} \} \).

Namely, expanding \( a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1} \), we ask that \( \lim_{n \to +\infty} a_n = 0 \) in \( h \)-adic topology.

Let \( V \) be a topologically free \( \mathbb{K}[[h]] \)-module. Following [DGK], we call an \( \text{End}_{\mathbb{K}[[h]]} V \)-valued formal distribution \( a(z) \) such that \([T, a(z)] = \partial_z a(z)\) (translation covariance), we have

\[
ea(z)[0] = e^{z^\tau a} = \sum_{k \geq 0} \frac{T^k a}{k!} z^k,
\]

where \( a = \text{Res}_z z^{-1} a(z)[0] \).

**Lemma 2.** Let \( |0\rangle \in V \) and \( T : V \to V \) be a \( \mathbb{K}[[h]] \)-linear map such that \( T(|0\rangle) = 0 \). Then for any \( \text{End}_{\mathbb{K}[[h]]} V \)-valued quantum field \( a(z) \) such that \([T, a(z)] = \partial_z a(z)\) (translation covariance), we have

\[
ea(z)[0] = e^{z^\tau a} = \sum_{k \geq 0} \frac{T^k a}{k!} z^k,
\]

where \( a = \text{Res}_z z^{-1} a(z)[0] \).

**Lemma 3.** Let \( T : V \to V \) be a \( \mathbb{K}[[h]] \)-linear map and let \( a(z) \) be an \( \text{End}_{\mathbb{K}[[h]]} V \)-valued quantum field such that \([T, a(z)] = \partial_z a(z)\). We have

\[
e^{wT} a(z) e^{-wT} = t_{z,w} a(z+w).
\]
Let \( V \) be a topologically free \( \mathbb{K}[[h]] \)-module, with a given non-zero vector \( |0\rangle \in V \) (vacuum vector) and a \( \mathbb{K}[[h]] \)-linear map \( T : V \rightarrow V \) such that \( T(|0\rangle) = 0 \) (translation operator). Again, following [DGK],

(a) A topological state-field correspondence on \( V \) is a linear map
\[
Y : V \hat{\otimes} V \rightarrow V_h((z)),
\]
satisfying

(i) (vacuum axioms) \( Y(z)(|0\rangle \otimes v) = v \) and
\[
Y(z)(v \otimes |0\rangle) \in v + V[[z]]z, \text{ for all } z \in V;
\]

(ii) (translation covariance)
\[
\partial_z Y(z) = TY(z) - Y(z)(1 \otimes T) = Y(z)(T \otimes 1),
\]

(b) A braiding on \( V \) is a \( \mathbb{K}[[h]] \)-linear map
\[
S(z) : V \hat{\otimes} V \rightarrow V \hat{\otimes} V \hat{\otimes} (\mathbb{K}((z))[[h]])
\]
such that \( S = 1 + O(h) \).

A braided state-field correspondence is a quintuple \((V, |0\rangle, T, Y, S)\) where \( Y \) is a topological state-field correspondence and \( S \) is a braiding as above.

We will use the following standard notation: given \( n \geq 2 \) and \( i, j \in \{1, \cdots, n\} \), we let
\[
S^{i,j}(z) : V \hat{\otimes}^n \rightarrow V \hat{\otimes}^{n} \hat{\otimes} (\mathbb{K}((z))[[h]]),
\]
act in the \( i \)-th and \( j \)-th factors (in this order) of \( V \hat{\otimes}^n \), leaving the other factors unchanged.

A braided vertex algebra is a quintuple \((V, |0\rangle, T, Y, S)\) where \( Y \) is a topological state-field correspondence and \( S \) is a braiding as above, satisfying the following \( S \)-locality: for every \( a, b \in V \) and \( M \in \mathbb{Z}_{\geq 0} \), there exists \( N = N(a, b, M) \geq 0 \) such that
\[
(z - w)^N Y(z)(1 \otimes Y(w))S^{12}(z - w)(a \otimes b \otimes c) = (z - w)^N Y(w)(1 \otimes Y(z))(b \otimes a \otimes c),
\]
where this equality holds mod \( h^M \), for all \( c \in V \).
Again, given a topological state-field correspondence \(Y\), set
\[ Y^{\text{op}}(z)(u \otimes v) = e^{zT}Y(-z)(v \otimes u). \tag{26} \]

It was shown in [DGK], Lemma 3.6, that in a braided vertex algebra \(V\) we have
\[ Y(z)S(z)(a \otimes b) = Y^{\text{op}}(z)(a \otimes b) \tag{27} \]
for all \(a, b \in V\).

After the proof of this result, (cf. Remark 3.7, [DGK]) they point out that it is enough to have the \(S\)-locality (25) holding just for \(c = |0\rangle\), to prove that \(YS = Y^{\text{op}}\) in a braided vertex algebra. We will use this remark later.

We recall at this point two important Propositions for our sequel.

**Proposition 2.** ([EK], Prop. 1.1) Let \(V\) be a braided vertex algebra. for every \(a, b, c \in V\) and \(M \in \mathbb{Z}_{\geq 0}\), there exists \(N \geq 0\) such that
\[ \imath_{z,w}(z + w)^N Y(z + w)(1 \otimes Y(w))S^{23}(w)S^{13}(z + w)(a \otimes b \otimes c)) \]
\[ = (z + w)^N Y(w)S(w)(Y(z) \otimes 1)(a \otimes b \otimes c) \mod M. \tag{28} \]

**Proposition 3.** ([DGK], Proposition 3.9) Let \((V, |0\rangle, T, Y, S)\) be a braided vertex algebra. Extend \(Y(z)\) to a map \(V \hat{\otimes} V \hat{\otimes} (\mathbb{K}([z])[[h]])\) in the obvious way. Then, modulo \(\text{Ker}Y(z)\), we have

(a) \(S(|0\rangle \otimes a) \equiv |0\rangle\), and \(S(z)(|0\rangle \otimes a) \equiv |0\rangle \otimes a\);

(b) \([T \otimes 1, S(z)] \equiv -\partial_z S(z)\) (left shift condition);

(c) \([1 \otimes T, S(z)] \equiv \partial_z S(z)\) (right shift condition);

(d) \([T \otimes 1 + 1 \otimes T, S(z)] \equiv 0\);

(e) \(S(z)S^{21}(-z) = 1\) (unitary).

Moreover, we have the quantum Yang-Baxter equation:

(f) \(S^{12}(z_1 - z_2)S^{13}(z_1 - z_3)S^{23}(z_2 - z_3) \equiv S^{23}(z_2 - z_3)S^{13}(z_2 - z_3)S^{12}(z_1 - z_2),\)

modulo \(\text{Ker}(Y(z_1)(1 \otimes Y(z_2))(1^{\otimes 2} \otimes Y(z_3)(- \otimes - \otimes - |0\rangle))).\)

8
3 On the structure of braided state-field correspondence

As in [BK], we aim to show that there are, inside certain braided vertex algebras, a “braided conformal algebra” and a “differential algebra” satisfying some family of equation. Conversely, we will show that given such structures under some nice conditions, we can give some reconstruction theorem.

Let \((V, |0\rangle, T, Y, S)\) be a braided state-field correspondence. For \(n \in \mathbb{Z}\), the quantum \(n\)-product \(Y(z)^S_{(n)} Y(z)\) is defined as

\[
(Y(z)^S_{(n)} Y(z))(a \otimes b \otimes c) = \text{Res}_x (\iota_{x,z}(x-z)^n Y(x)(1 \otimes Y(z)))(a \otimes b \otimes c) - \iota_{z,x}(x-z)^n Y(z)(1 \otimes Y(x))S^{12}(z-x)(b \otimes a \otimes c)).
\]  

(29)

Note that this definition differs but is equivalent to the definition introduced in [DGK] when you ask \(S\) to satisfies unitary relation \(S_{21}(-x)S(x) = 1\) (cf. proposition 3e) which holds in a braided vertex algebra, where we are going to work. Now, we have the following result.

Lemma 4. Given \((V, |0\rangle, T, Y, S)\) a braided state-field correspondence satisfying the equations

\[
[T \otimes 1, S(z)] = -\partial_z S(z),
\]

(30)

\[
[1 \otimes T, S(z)] = \partial_z S(z).
\]

(31)

The quantum \(n\)-product (29) satisfies the following equation

\[
\partial_z (Y(a, z)^S_{(n)} Y(b, z)) = (\partial_z Y(a, z))^S_{(n)} Y(b, z) + Y(a, z)^S_{(n)} (\partial_z Y(b, z)).
\]

(32)

Proof. Applying the definition of quantum \(n\)-product (29), using integration by parts and translation covariance (22), the LHS becomes
On the other hand using translation covariance, RHS becomes

\[
\begin{align*}
\Res_{t,x,z} \partial_z ((x-z)^n Y(x)(1 \otimes Y(z))) (a \otimes b \otimes c) \\
-\Res_{t,x,z} \partial_z ((x-z)^n Y(z)(1 \otimes Y(x)) S^{12}(z-x)) (b \otimes a \otimes c) \\
= \Res_{t,x,z} \partial_z ((x-z)^n Y(x)(1 \otimes Y(z))) (a \otimes b \otimes c) \\
+ \Res_{t,x,z} \partial_z ((x-z)^n Y(z)(1 \otimes Y(x)) S^{12}(z-x)) (b \otimes a \otimes c) \\
-\Res_{t,x,z} \partial_z (x-z)^n Y(z)(1 \otimes Y(x)) S^{12}(z-x) (b \otimes a \otimes c) \\
-\Res_{t,x,z} (x-z)^n Y(z)(1 \otimes Y(x)) \partial_z S^{12}(z-x) (b \otimes a \otimes c) \\
= -\Res_{t,x,z} \partial_z ((x-z)^n Y(x)(1 \otimes Y(z))) (a \otimes b \otimes c) \\
+ \Res_{t,x,z} ((x-z)^n Y(x)(1 \otimes Y(z))) (1 \otimes T \otimes 1)(a \otimes b \otimes c) \\
+ \Res_{t,x,z} \partial_z ((x-z)^n Y(z)(1 \otimes Y(x)) S^{12}(z-x)) (b \otimes a \otimes c) \\
-\Res_{t,x,z} (x-z)^n Y(z)(1 \otimes Y(x)) (T \otimes 1)(1 \otimes Y(z)) S^{12}(z-x) (b \otimes a \otimes c) \\
+ \Res_{t,x,z} (x-z)^n Y(z)(1 \otimes Y(x)) \partial_z S^{12}(z-x) (b \otimes a \otimes c) \\
= \Res_{t,x,z} ((x-z)^n Y(x)(T \otimes 1)(1 \otimes Y(z))) (a \otimes b \otimes c) \\
+ \Res_{t,x,z} ((x-z)^n Y(x)(1 \otimes Y(z))(1 \otimes T \otimes 1)(a \otimes b \otimes c) \\
-\Res_{t,x,z} (x-z)^n Y(z)(1 \otimes Y(x)) (1 \otimes T \otimes 1)(b \otimes a \otimes c) \\
-\Res_{t,x,z} (x-z)^n Y(z)(1 \otimes Y(x)) T \otimes 1 \otimes 1)(b \otimes a \otimes c) \\
-\Res_{t,x,z} (x-z)^n Y(z)(1 \otimes Y(x))(T \otimes 1 \otimes 1)(b \otimes a \otimes c).
\end{align*}
\]

(33)

Due to equations (30) and (31) we get

\[
(T \otimes 1 \otimes 1) S^{12}(z-x)(b \otimes a \otimes c) = S^{12}(z-x)(T \otimes 1 \otimes 1)(b \otimes a \otimes c) + \partial_x S^{12}(z-x)(b \otimes a \otimes c),
\]

(35)

and

\[
(1 \otimes T \otimes 1) S^{12}(z-x)(b \otimes a \otimes c) = S^{12}(z-x)(1 \otimes T \otimes 1)(b \otimes a \otimes c) - \partial_x S^{12}(z-x)(b \otimes a \otimes c).
\]

(36)
Applying equations (35) and (36) to RHS, we get

\[
\text{Res}_{x,z}(x-z)^n Y(x)(T \otimes 1)(1 \otimes Y(z))(a \otimes b \otimes c) \\
- \text{Res}_{x,z}(x-z)^n Y(z)(1 \otimes Y(x))(1 \otimes T \otimes 1)S^{12}(z-x)(b \otimes a \otimes c) \\
+ \text{Res}_{x,z}(x-z)^n Y(z)(1 \otimes Y(x))\partial_x S^{12}(z-x)(b \otimes a \otimes c) \\
+ \text{Res}_{x,z}(x-z)^n Y(x)(1 \otimes Y(z))(1 \otimes T \otimes 1)(a \otimes b \otimes c) \\
- \text{Res}_{x,z}(x-z)^n Y(z)(1 \otimes Y(x))(T \otimes 1 \otimes 1)S^{12}(z-x)(b \otimes a \otimes c) \\
+ \text{Res}_{x,z}(x-z)^n Y(x)(1 \otimes Y(z))(1 \otimes T \otimes 1)(a \otimes b \otimes c) \\
- \text{Res}_{x,z}(x-z)^n Y(z)(1 \otimes Y(x))(T \otimes 1 \otimes 1)S^{12}(z-x)(b \otimes a \otimes c).
\]

(37)

Then equations (33) and (37) are equal, therefore the claim follows. \(\blacksquare\)

**Remark 1.** Recall that, as we quote in Proposition 8 it was shown by DGK that in a braided vertex algebra, equations (30) and (31) hold mod Ker\(Y\). In [EK], condition (30) is asked as part of the definition of a braided vertex operator algebra. In this context, asking (31) is equivalent to ask \(T\) to be a derivation of a braided vertex operator algebra. It is shown in [Li], that if in addition we ask the underlying field algebra to be non-degenerate (cf. definition 5.12, [Li]), we have that (31) holds in a braided vertex algebra where the associativity relation (6) holds (cf. [EK]).

Let \((V, |0\rangle, T, Y, S)\) be a braided-state field correspondence. \(Y\) satisfies the quantum \(n\)-th product identities if for all \(a, b, c \in V\) and \(n \in \mathbb{Z}\)

\[
Y(z)^S_{(n)} Y(z)(a \otimes b \otimes c) = Y(z)(a^S_{(n)} b \otimes c),
\]

where

\[
a^S_{(n)} b = \text{Res}(z^n Y(z)S(z)(a \otimes b)).
\]

(38)

(39)

\(Y\) satisfies the associativity relation if for any \(a, b, c \in V\) and \(M \in \mathbb{Z}_{\geq 0}\) there exists \(N \in \mathbb{Z}_{\geq 0}\) such that

\[
\epsilon_{z,w}(z + w)^N Y(z + w)((1 \otimes Y(w))(a \otimes b \otimes c)
\]

\[
= (z + w)^N Y(w)(Y(z) \otimes 1)(a \otimes b \otimes c) \mod \hbar^M,
\]

(40)

Let \((V, |0\rangle)\) be a pointed vector space. We define a braided field algebra \((V, |0\rangle, Y, T, S)\) is a braided state-field correspondence \(Y\) satisfying the associativity relation (6). We also introduce a strong braided field algebra \((V, |0\rangle, Y, T, S)\)
as a state-field correspondence $Y$ satisfying the quantum $n$-th product identities $[\mathbb{K}]$. This are the braided versions of field algebra and strong field algebra introduced by $[\mathbb{BK}]$.

Let $(V, |0\rangle, T, S)$ be a braided-state field correspondence. For $a, b \in V, f \in \mathbb{K}((z))[h]$, we define the $(\lambda, f)$-product by the

$$a(\lambda, f)b = \text{Res}_z e^{\lambda z} f(z) Y(z)(a \otimes b) = \sum_{n \in \mathbb{Z}_{\geq 0}, \text{finite}} \sum_{i \in \mathbb{Z}} f_i(h) a_{(n+i)} b \frac{\lambda^n}{n!} \in V \otimes \mathbb{K}[\lambda][[h]]$$

where $f(z) = \sum_{i \in \mathbb{Z}} f_i(h) z^i$, $f_i(h) \in \mathbb{K}[[h]]$. Note that $f_i(h) = 0$ for $i << 0$.

**Remark 2.** (i) If in addition we ask $V$ to have a structure of $\mathbb{K}((z))$-module structure, more precisely $z^k(a_{(n)} b) = a_{(n+k)} b$, this $(\lambda, f)$-product resembles the operations introduced in $[\mathbb{GKK}]$. Instead, we are asking $V$ to have a braiding that involves some elements of $\mathbb{K}((z))[[h]]$.

(ii) The product $a(\lambda, f)$ coincides with $X_{\lambda, -\lambda, z_0}^z(a, b; f(z_1 - z_0))$ from Sec. 6.4 in the paper $[\mathbb{BDHK}]$.

We have the following useful Lemma.

**Lemma 5.** Given $(V, |0\rangle, T, S)$ be a braided-state field correspondence, we have

(a) $a_{(\lambda, z^m)} b = \partial^m a_{(\lambda, f)} b$ for $m \geq 0$, and $f \in \mathbb{K}((z))[h]$. In particular, $a_{(\lambda, z^m)} b = \partial^m a_{(\lambda, 1)} b$ for $m \geq 0$,

(b) $a_{(\lambda, z^k)} b = ((\lambda + T)_{(k-1)} a_{(\lambda, z)} b$, for $k \geq 1$.

**Proof.** Let $f(z) = \sum_i f_i(h) z^i$, item (a) follows from the definition of $(\lambda, f)$-product:

$$a_{(\lambda, z^m)} b = \text{Res}_z e^{\lambda z} z^m f(z) Y(z)(a \otimes b)$$

$$= \sum_i f_i(h) \text{Res}_z \sum_{k \geq 0} \lambda^k / k! \sum_{j \in \mathbb{Z}} a_{(j)} b z^{-j+k+m+i}$$

$$= \sum_i f_i(h) \sum_{k \geq 0} \lambda^{k-m} / (k - m)! a_{(k+i)} b$$

$$= \partial^m a_{(\lambda, f)} b.$$
lation covariance we get item \((b)\), namely:

\[
\begin{align*}
a_{(\lambda, z^{-k})} b &= a_{(\lambda, (-\partial_z)^{z^{-1}})} b \\
&= \text{Res}_z e^{\lambda z} (-\partial_z)^{z^{-1}} Y(z) (a \otimes b) \\
&= \text{Res}_z e^{\lambda z} (-\partial_z)^{k-1} e^{\lambda z} Y(z) (a \otimes b) \\
&= \text{Res}_z e^{\lambda z} \sum_{r=0}^{k-1} \lambda^r Y(z) (T^{(k-1-r)} a \otimes b) \\
&= \text{Res}_z e^{\lambda z} Y(z) ((\lambda + T)^{(k-1)} a \otimes b) \\
&= ((\lambda + T)^{(k-1)} a)_{(\lambda, z^{-1})} b.
\end{align*}
\]

Note that if \(f = 1\) in \((41)\), we recover the \(\lambda\)-product introduced in \((7)\) for a state-field correspondence. We will denote \(a_{(\lambda, 1)} = a\lambda b\). Observe also that, due to the Lemma above, any \((\lambda, f)\)-product can be written in terms of the \(\lambda\)-product and the \((\lambda, z^{-1})\)-product.

The vacuum axioms for \(Y\) imply that,

\[
|0\rangle_{(\lambda, z^{-1})} a = a = a_{(\lambda, z^{-1})} |0\rangle,
\]

while the translation invariance axioms show that,

\[
T(a_{(\lambda, f)} b) = T(a)_{(\lambda, f)} b + a_{(\lambda, f)} T(b)
\]

and

\[
T(a)_{(\lambda, f)} b = -\lambda a_{(\lambda, f)} b - a_{(\lambda, f')} b
\]

for all \(a, b \in V\) and \(f \in \mathbb{K}((z))\). Note that, when \(f = 1\) in \((43)\) and \((44)\), we recover equation \((11)\).

Conversely, if we are given a pointed topologically free \(\mathbb{K}[[h]]\)-module \((V, |0\rangle))\), together with a \(\mathbb{K}[[h]]\)-linear map \(T\), a braiding \(S\), a \((\lambda, 1)\)-product and a \((\lambda, z^{-1})\)-product on \(V\) satisfying the properties \((12)\)-\((14)\), we can reconstruct the braided state-field correspondence \(Y\) by the formulas:

\[
Y(a, z)_{+} b = (e^{zT} a)_{(\lambda, z^{-1})} b|_{\lambda = 0}, \quad Y(a, z)_{-} b = (a_{(-\partial_z, 1)} b)(z^{-1}),
\]

where \(Y(a, z) = Y(a, z)_{+} + Y(a, z)_{-}\).

We will need the following Lemma.

**Lemma 6.** We have that

\[
a_{(\lambda, f'(z))} b = ((-\lambda - T)^l a)_{(\lambda, f)} b,
\]

for all \(a\) and \(b \in V\) and \(l \geq 0\). Here and further \(f^{(l)}(z) = \partial_z^l f(z)\).
Proof. Straightforward using (44).

For the following Proposition it will be useful to introduce the following notation:

\[ a_{(\cdot),f}b := a_{(\lambda,z-1)f}|_{\lambda=0} = \operatorname{Res}_z z^{-1} f(z) Y(z)(a \otimes b) = \sum_{i \in \mathbb{Z}} f_i(h) a_{(i-1)} b, \]  

(46)

for \( a, b \in V, f \in \mathbb{K}((z))[[h]], f(z) = \sum_{i \in \mathbb{Z}} f_i(h) z^i, f_i(h) \in \mathbb{K}[[h]] \). Note that in the case \( f = 1 \) we obtain the \( \cdot \) product in [BK], (cf. (8)), namely

\[ a_{(\cdot,1)}b = a_{(\lambda,z-1)}|_{\lambda=0} = a \cdot b, \]

since it is easy to show that

\[ a_{(\lambda,z-1)} b = a_{\lambda} b + \int_0^\lambda a_{\mu} b d\mu. \]  

(47)

Remark 3. The \( a_{\lambda,z-1} \) from (44) is the so called integral of \( \lambda \)-bracket, which appeared earlier in [DK].

Whith all this, we can state the following result.

Proposition 4. Let \((V, |0\rangle, T, Y, S)\) be a braided state field correspondence such \( S \)-locality holds for \( c = |0\rangle \). Then the collection of the \( n \)-th quantum product identities (29) for \( n \geq -1 \) implies:

\[
(a_{-\alpha-T} b)_{\alpha+\beta} c = -b_{\alpha} (a_{\beta}) c + \sum_{i=1}^{r} \sum_{l \geq 0} (-1)^{l} a^i_{(\beta, (f_{i}(z))(l))} (b^i_{(\alpha,z^l)} c),
\]  

(48)

\[
(a_{-\lambda} b) c = -b_{(\lambda-T)}(a, c) + \sum_{i=1}^{r} \sum_{l \geq 0} (-1)^{l} a^i_{(\cdot, (f_{i}(z))(l))} (b^i_{(\lambda-T,z^l)} c)
\]

\[ + \int_0^{T-\lambda} (a_{-\lambda} b)_{\mu} c d\mu, \]  

(49)

\[
(a b)_{\lambda c} = (e^{T \partial_{\lambda}} b)(a c) - \int_0^{-T} (a_{-\mu-T} b)_{\lambda} c d\mu + \sum_{i=1}^{r} [(e^{T \partial_{\lambda}} a^i)_{(\cdot, (f_{i}(z))}(b^i_{\lambda c})
\]

\[ - \sum_{l \geq 0} \int_0^{\lambda} a^i_{(\mu, (f_{i}(z))(l))} (b^i_{(\lambda-\mu,z^l)} c) d\mu], \]  

(50)
(a \cdot b) \cdot c = b \cdot (a \cdot c) + \text{Res}_z \left( \int_0^T d\lambda b \right) (b \lambda c) - \int_0^T (a_{\mu-Tb}) \cdot c d\mu

+ \sum_{i=1}^r \sum_{l \geq 0} \left( \int_0^T d\lambda a^i \right) (b^i_{\lambda, D, f_i(z)} c)

+ \sum_{i=1}^r \sum_{m,l \geq 0} (-1)^i a^i(z, z^{m+1}) (b^i_{l, D, f_i(z)} z^{m} c), \quad (51)

where \( D_l = z^l v^{(l)} \) and \( S(z)(a \otimes b) = \sum_{i=0}^r f_i(z) a^i \otimes b^i \).

**Proof.** Recall that the fact that the \( S \)-locality holds for \( c = |0\rangle \), implies that \( Y(z) S(z) = Y^\text{op}(z) \). Applying definitions of \( \lambda \)-product and the definition of \( Y^\text{op} \), due to Lemma 27 we get

\[ a_{\lambda} b = \text{Res}_z e^{\lambda z} Y(z)(a \otimes b) = \text{Res}_z e^{(\lambda + T)z} Y^\text{op}(-z)(b \otimes a) = -\text{Res}_z e^{-(\lambda + T)z} Y(z) S(z)(b \otimes a) := -b^{S}_{-(\lambda + T)} a. \quad (52) \]

The collection of \( n \)-th product identities [38] together [52] are equivalent to:

\[
Y(a_{-\lambda} b, z)c = -Y(b^{S}_{(\lambda-T)} a, z)c
\]

\[
= -\sum_{n \geq 0} Y(z)(b^{S}_{(n)} a \otimes c) \frac{(\lambda - T)^n}{n!}
\]

\[
= -\sum_{n \geq 0} Y(z)^{S}_{(n)} Y(z)(b \otimes a \otimes c) \frac{(\lambda - T)^n}{n!}
\]

\[
= -\sum_{n \geq 0} \text{Res}_x \iota_{x,z}(x - z)^n Y(x)(1 \otimes Y(z))(b \otimes a \otimes c)
\]

\[
+ \iota_{z,x}(z - x)^n Y(z)(1 \otimes Y(x)) S^{12}(z - x)(a \otimes b \otimes c) \frac{(\lambda - T)^n}{n!}
\]

\[
= -\text{Res}_x e^{(\lambda - T)(x-z)} Y(x)(1 \otimes Y(z))(b \otimes a \otimes c)
\]

\[
+ \text{Res}_x e^{(\lambda - T)(z-x)} \sum_{i=1}^r e^{-x \partial_i(f_i(z))} Y(z)(1 \otimes Y(x))(a^i \otimes b^i \otimes c)
\]

\[
= -e^{(\lambda + T)z} [b^{S}_{(\lambda-T)} (Y(a, z) c)]
\]

\[
- \sum_{i=1}^r \sum_{l \geq 0} (-\partial_x)^{(l)}(f_i(z)) Y(a^i, z)(b^i_{l, \lambda - T, \lambda}) c]. \quad (53)
\]

Taking \( \text{Res}_z e^{(\lambda + T)z} \), and changing \( \lambda - T \) by \( \alpha \) and \( \mu + T \) by \( \beta \), we obtain [48].
Taking Res$_z^{-1}$ in (53) and using $e^{(-\lambda+T)z^{-1}} = z^{-1} + \int_0^{-\lambda+T} e^{\mu z} d\mu$, we get
\[
(a-\lambda b).c = -b(\lambda-T)(a.c) - \int_0^{T-\lambda} b(\lambda-T)(a_\mu c) d\mu \\
+ \sum_{i=1}^{r} \sum_{l \geq 0} (-1)^l [a^i_{\mu, (f_i(z))^{(l)}} (b^i_{\lambda-T, x^i} c)] \\
+ \int_0^{T-\lambda} a^i_{\mu, (f_i(z))^{(l)}} (b^i_{\lambda-T, x^i} c) d\mu].
\] (54)
This, together with (48), implies (49) (after the substitution $\mu' = \lambda + \mu - T$).

Applying definitions of $(-1)$-st product and $Y^{op}$, due to (27) we get
\[
a \cdot b = \text{Res}_z z^{-1} Y(z)(a \otimes b) \\
= \text{Res}_z z^{-1} e^{zT} Y^{op}(-z)(b \otimes a) \\
= \text{Res}_z z^{-1} e^{-zT} Y(z) S(z)(b \otimes a) \\
= \text{Res}_z z^{-1} Y(z) S(z)(b \otimes a) + \int_0^{-T} e^{\mu z} Y(z) S(z)(b \otimes a) d\mu \\
= b^S a + \int_0^{-T} b^S a d\mu.
\] (55)

Then this equation together the quantum $(-1)$- product we get
\[
Y(a \cdot b, z)c = Y(z)(b^S_{\mu} a \otimes c) + \int_0^{-T} Y(z)(b^S_{\mu} a \otimes c) d\mu \\
= (Y(z)^{S}_{\mu-1} Y(z))(b \otimes a \otimes c) + \int_0^{-T} Y(z)(b^S_{\mu} a \otimes c) d\mu \\
= Y(b, z) a Y(a, z)c + \sum_{i=1}^{r} Y(a^i, z)(b^i_{\mu} a \otimes c)(z^{-1} f_i(z)) \\
+ \int_0^{-T} Y(b^S_{\mu} a, z)c d\mu.
\] (56)
Taking $\text{Res}_z e^{\lambda z}$ and using integration by parts, we get:

\[ (a,b)_\lambda = \text{Res}_z (e^{T\partial_\lambda} e^{\lambda z} b) (a_\lambda c) + \text{Res}_z \sum_{i=1}^{r} Y(a^i, z) (b^i_{\lambda - \partial_\lambda} c (e^{\lambda z} z^{-1} f_i(z))) \\
+ \int_{0}^{-T} (b^i_S a)_\lambda c \, d\mu \]

\[ = (e^{T\partial_\lambda} b) (a_\lambda c) + \text{Res}_z \sum_{i=1}^{r} Y(a^i, z) (b^i_{\lambda - \partial_\lambda} c (z^{-1} f_i(z) + \int_{0}^{\lambda} f_i(z) e^{\mu z} d\mu)) \\
- \int_{0}^{-T} (a_{-\mu - T} b)_\lambda c \, d\mu \]

\[ = (e^{T\partial_\lambda} b) (a_\lambda c) - \int_{0}^{-T} (a_{-\mu - T} b)_\lambda c \, d\mu + \sum_{i=1}^{r} ((e^{T\partial_\lambda} a^i) (f_i(z)) (b^i_{\lambda} c) \\
+ \sum_{l \geq 0} \int_{0}^{\lambda} a^i_{(\mu, f_i(z)) (l)} (b^i_{\lambda - \mu - x} c) \, d\mu), \quad (57) \]

Due to (56) and taking $\text{Res}_z z^{-1}$, we get

\[ (a,b)_c = \text{Res}_z z^{-1} Y(b, z)_+ Y(a, z)_+ c + \text{Res}_z z^{-1} Y(b, z)_- Y(a, z)_- c \\
+ \text{Res}_z z^{-1} \int_{0}^{-T} Y(b^i_S a, z) c \, d\mu \]

\[ + \sum_{i=1}^{r} \sum_{l \geq 0} (-1)^l \text{Res}_z z^{-1} Y(a^i, z)(f_i(z))^l (\partial_z)^l Y(b^i, z)_- c \\
= b.(a,c) + \text{Res}_z z^{-1} ((e^{zT} - 1)b) (Y(a, z)_- c) + \int_{0}^{-T} (b^i_S a)_c \, d\mu \]

\[ + \sum_{i=1}^{r} \sum_{l \geq 0} \text{Res}_z z^{-1} ((e^{zT} - 1)a^i) (f_i(z))^l (\partial_z)^l Y(b^i, z)_- c \\
+ \sum_{i=1}^{r} \sum_{l \geq 0} \text{Res}_z z^{-1} Y(a^i, z)_-(f_i(z))^l (\partial_z)^l Y(b^i, z)_- c \\
= b.(a,c) + \text{Res}_z \left( \int_{0}^{T} e^{\lambda z} d\lambda b \right) (Y(b, z)_- c) - \int_{0}^{-T} (a_{-\mu - T} b)_c \, d\mu \]

\[ + \sum_{i=1}^{r} \sum_{l \geq 0} \text{Res}_z \left( \int_{0}^{T} e^{\lambda z} d\lambda a^i \right) (f_i(z))^l (\partial_z)^l Y(b^i, z)_- c \\
+ \sum_{i=1}^{r} \sum_{m,l \geq 0} (-1)^l a^i_{(\mu, z_{m+1})} (b^i_{\lambda - D_i(f_i(z)) z - m} c), \quad (58) \]

which proves (51).  

\[ \square \]
Note that $V$ together with the $\lambda$-product is what [BK] called conformal algebra, and $V$ with $(\lambda, z^{-1})$-product is also a $\mathbb{K}[T]$-differential algebra with unit due to (42) and (43).

An important remark is that $V$ together with the $\lambda$-product is no longer a Leibnitz conformal algebra, since due to the braiding, the analog of the Jacoby identity involves $(\lambda, z^{-1})$-products.

With this in mind, we can prove the following

**Theorem 3.** Giving a braided state field correspondence $(V, |0\rangle, Y, \mathcal{S})$, satisfying the $\mathcal{S}$-locality for $|0\rangle$ and the axiom of quantum $(n)$-product (2) implies to provide $V$ with a structure of a conformal algebra and a structure of a $\mathbb{K}[T]$-differential algebra with a unit $|0\rangle$, satisfying (48)-(51).

Conversely, given $V$ a topologically free $\mathbb{K}[[\hbar]]$-module, a $\mathbb{K}[[\hbar]]$-linear map $T$ and a braiding $\mathcal{S}$. Assume that $V$ has a structure of conformal algebra and a structure of a $\mathbb{K}[T]$-differential algebra with a unit $|0\rangle$, satisfying (48)-(51) and $\mathcal{S}$ satisfies (30)-(31), then $(V, |0\rangle, Y, \mathcal{S})$, is a braided state field correspondence satisfying the axiom of quantum $(n)$-product, namely a strong braided field algebra.

**Proof.** If $(V, |0\rangle, Y, \mathcal{S})$ is a braided state field algebra satisfying the axiom of quantum $n$-product, then by the above discussion we can define a $(\lambda, f)$-product on $V$ satisfying all the requirement. Conversely, given a $(\lambda, f)$-product we define a braided state field correspondence $Y$ by (45). In the proof of Lemma 4, we have seen that the equations (48)-(49) are equivalent to the identities

$$\text{Res}_z \left[ \sum_{k \geq 0} \left[ (-\partial_z)^k Y(b^{\mathcal{S}}(k-1)_a, z) - Y(b, z) \partial_z^k \mathcal{S}(k-1) Y(a, z) \right] F \right] = 0,$$

while the equations (50)-(51) are equivalent to the identities

$$\text{Res}_z \sum_{k \geq 0} \left[ (-\partial_z)^k Y(b^{\mathcal{S}}(k-1)_a, z) - Y(b, z) \partial_z^k \mathcal{S}(k-1) Y(a, z) \right] F = 0,$$

for $a, b \in V$ and $F = e^{\lambda z}$ or $z^{-1}$.

Due to Lemma 4 and using translation invariance of $Y$, this identity is equivalent to

$$\text{Res}_z \sum_{k \geq 0} \left[ Y(b^{\mathcal{S}}(k-1)_a, z) - Y(b, z) \partial_z^k \mathcal{S}(k-1) Y(a, z) \right] (\partial_z)^k F = 0,$$

$a, b \in V$, $F = e^{\lambda z}$ or $z^{-1}$.

Using the translation invariance of $Y$ and integration by parts, we see that identity (59) holds also with $F$ replaced with $\partial_z F$. Hence equations (59) and (60) hold for all $F = z^l, l < 0$. For $F = e^{\lambda z}$, taking coefficients at power of $\lambda$ shows that they are satisfied also for $F = z^l, l \geq 0$. This implies the $n$-th quantum product axioms for $n \geq -1$. The theorem proof remains the same as that proof of Theorem 4.4 [BK].

\[\square\]
4 Quantum conformal algebra

In this section, based on what we have seen in Section 3, we aim to give a definition of braided conformal algebra. Until now, we didn’t ask any further structure for the braiding $S$ besides \[[60]\] and \[[61]\]. We have the following results.

**Proposition 5.** If the hexagon relation

$$S(x)(Y(z) \otimes 1) = (Y(z) \otimes 1)S^{23}(x)\iota_{x,z}S^{13}(x+z)$$  \hfill \((62)\)

holds in a braided state field correspondence, then we have that:

$$S(x)(a_{(\lambda,f)}b \otimes c) = \sum_{l \geq 0} \partial^l_\lambda((\iota_{(\lambda,f)} \otimes 1)S^{23}(x)\partial^l_\lambda S^{13}(x)(a \otimes b \otimes c))$$  \hfill \((63)\)

$$= e^{\partial^l_\lambda \partial^m_\lambda}((\iota_{(\lambda,f)} \otimes 1)S^{23}(x)S^{13}(x_1)(a \otimes b \otimes c)|_{x_1 = d})$$  \hfill \((64)\)

for all $a, b, c \in V$.

**Proof.** Applying the definition of $(\lambda, f)$-product and using the hexagon relation \[[62]\], definition of $S$, Taylor expansion and change of variables we get,

$$S(x)(a_{(\lambda,f)}b \otimes c) = S(x)\text{Res}_z e^{\lambda z} f(z)(Y(z) \otimes 1)(a \otimes b \otimes c)$$

$$= \text{Res}_z e^{\lambda z} f(z)S(x)(Y(z) \otimes 1)(a \otimes b \otimes c)$$

$$= \sum_{i, j \in Z} h_i(x)\text{Res}_z e^{\lambda z} f(z)(Y(z) \otimes 1)S^{23}(x)\iota_{x,z}S^{13}(x+z)(a \otimes b \otimes c)$$

$$= \sum_{i, j \in Z} h_i(x)\text{Res}_z e^{\lambda z} f(z)(e^{z\partial_\lambda} g_j(x))(Y(a, z)b^{(i)} \otimes (c^{(j)})^{(i)})$$

$$= \sum_{i, j, k, l \in Z} h_i(x)\text{Res}_z e^{\lambda z} f(z)(e^{z\partial_\lambda} g_j(x))(Y(a, z)b^{(i)} \otimes (c^{(j)})^{(i)})$$

$$= \sum_{i, j, k, l \in Z} h_i(x)\lambda^{(k)} f_r \text{Res}_z g_r(\lambda^{(l)}(a^{(j)}_{(m)}, b^{(i)} \otimes (c^{(j)})^{(i)}))$$

$$= \sum_{i, j, k, l \in Z} h_i(x)\lambda^{(k)} f_r \text{Res}_z g_r(\lambda^{(l)}(a^{(j)}_{(m)}, b^{(i)} \otimes (c^{(j)})^{(i)}))$$

$$= \sum_{i, j, k, l \in Z} \partial^l_\lambda((\iota_{(\lambda,f)} \otimes 1)S^{23}(x)\partial^l_\lambda S^{13}(x)(a \otimes b \otimes c),$$  \hfill \((65)\)

where $S^{23}(x)(a \otimes b \otimes c) = \sum_j h_i(x)a \otimes b^{j} \otimes c^{j}$ and $S^{13}(x)(a \otimes b \otimes c) = \sum_j g_r(x) a^{j} \otimes b \otimes c^{j}$.

Similarly, we have the following results.

**Proposition 6.** If the associativity relation holds, namely, there exists $N \in \mathbb{Z}_{\geq 0}$ such that

$$\iota_{z,w}(z+w)^N Y(z+w)((1 \otimes Y(w)))(a \otimes b \otimes c) = (z+w)^N Y(w)(Y(z) \otimes 1)(a \otimes b \otimes c)$$

$$= \sum_{i, j, k, l \in Z} \partial^l_\lambda((\iota_{(\lambda,f)} \otimes 1)S^{23}(x)\partial^l_\lambda S^{13}(x)(a \otimes b \otimes c),$$  \hfill \((65)\)

where $S^{23}(x)(a \otimes b \otimes c) = \sum_j h_i(x)a \otimes b^{j} \otimes c^{j}$ and $S^{13}(x)(a \otimes b \otimes c) = \sum_j g_r(x) a^{j} \otimes b \otimes c^{j}$.  \hfill \[\square\]
mod \, h^M, \text{ for any } a, b, c \in V \text{ and } M \in \mathbb{Z}_{\geq 0} \text{ in a (braided) state field correspondence, then}

\begin{equation}
\partial^N_{\lambda} a_{\lambda}(b_{\mu}c) = \partial^N_{\lambda}(a_{\lambda}b_{\lambda+\mu}c),
\end{equation}

\text{mod } h^M, \text{ for all } a, b, c \in V.

Proof. Changing \( z + w \) by \( x \) in the associativity relation we have
\begin{equation}
x^NY(x)(1 \otimes Y(w))(a \otimes b \otimes c) = x^NY(w)(Y(x) \otimes 1)(a \otimes b \otimes c).
\end{equation}

Taking \( \text{Res}_x \text{Res}_w e^{\lambda x} e^{\mu w} \) to the LHS of \( (67) \) and using Lemma 5 (a), we have
\begin{equation}
\text{Res}_x \text{Res}_w e^{\lambda x} e^{\mu w} x^NY(x)(1 \otimes Y(w))(a \otimes b \otimes c) = (a_{(\lambda x)}(b_{\mu}c)) = \partial^N_{\lambda}(a_{\lambda}b_{\mu}c).
\end{equation}

Now, taking \( \text{Res}_x \text{Res}_w e^{\lambda x} e^{\mu w} \) to the RHS of \( (67) \), using Taylor’s formula \( (2) \), translation covariance and Lemma 5 (a),
\begin{align}
\text{Res}_x \text{Res}_w e^{\lambda x} e^{\mu w} x^NY(w)h_{x}(x-w) \otimes 1(a \otimes b \otimes c) & = \text{Res}_x \text{Res}_w e^{\lambda x} e^{\mu w} x^NY(w)e^{-\partial x}Y(x) \otimes 1(a \otimes b \otimes c) \\
& = \text{Res}_x \text{Res}_w e^{\lambda x} e^{\mu w} x^NY(w)Y(x) \otimes 1(1 \otimes Y(w)) \otimes b \otimes c \\
& = \text{Res}_x \text{Res}_w e^{\mu w} Y((e^{-wT}a) \otimes c)(w)c \\
& = \text{Res}_x \text{Res}_w e^{\mu w} \partial^N_{X}Y((e^{-wT}a) \otimes \lambda, b, w)c \\
& = \partial^N_{\lambda}(a_{\lambda}b_{\mu}c).
\end{align}

Equating \( (68) \) and \( (69) \), we finish the proof. \( \square \)

Now, we will show a similar result but for the quasi-associativity \( (70) \).

Proposition 7. Let \( V \) be a braided state field correspondence. Suppose that for every \( a, b, c \in V \) and \( M \in \mathbb{Z}_{\geq 0} \) there exists \( N \geq 0 \) such that
\begin{equation}
\lambda_{x, w}(z + w)N Y(z) Y(w)(1 \otimes Y(w)) S^{23}(w) S^{13}(z + w) (a \otimes b \otimes c) \\
= (z + w)N Y(w) S(w) Y(z) (a \otimes b \otimes c) \mod h^M.
\end{equation}

holds mod \( h^M \). Then
\begin{equation}
\sum_{i,j} \partial^N_{\lambda} a^j_{(x, g_j)((-\lambda + \mu, h_j)(c^i))} |_{\sigma = \lambda} = (\partial_{\lambda} + \partial_{\mu})^N \sum_{r} (a_{\lambda} b_{(\mu, f_r)} c^r) \mod h^M,
\end{equation}

where
\begin{align}
S(x)(a_{\lambda} b) \otimes c &= \sum_{r} f_r(x)(a_{\lambda} b)^r \otimes c^r), \\
S^{13}(x)(a \otimes b \otimes c) &= \sum_{j} g_j(x)(a^j \otimes b \otimes c^j), \\
S^{23}(x)(a^j \otimes b \otimes c^j) &= \sum_{i} h_i(x)(a^j \otimes b^i \otimes (c^j)^i).
\end{align}
Proposition 8. Suppose we have a state-field correspondence $V$ where $Y(z)\mathcal{S}(z) = Y^{op}(z)$ holds. Then, for $a$ and $b$ in $V$,

$$-b_{-\lambda -\tau} = \sum_i a^i_{(\lambda, f_i)} b^i.$$  

(75)
Proof. We have that
\begin{equation}
\text{Res}_z e^{\lambda z} Y(z) S(z)(a \otimes b) = \sum_i \text{Res}_z e^{\lambda z} f_i(z) Y(a^i, z)b^i
= \sum_i a^i_{(\lambda, f_i)} b^i.
\end{equation}
(76)

On the other hand, using $Y(z) S(z) = Y^{op}(z)$,
\begin{equation}
\text{Res}_z e^{\lambda z} Y(z) S(z)(a \otimes b) = \text{Res}_z e^{\lambda z} Y^{op}(z)(a \otimes b)
= \text{Res}_z e^{\lambda z} e^{T z} Y(-z)(b \otimes a)
= -\text{Res}_z e^{(-\lambda - T)z} Y(z)(b \otimes a)
= -b_{-\lambda - T} a,
\end{equation}
(77)
finishing the proof. \qed

A braided vertex algebra where the associativity relation holds, is called quantum vertex algebra. (Cf. Definition 3.12, [DGK]). In the Characterization Theorem (cf. Theorem 5.13, [DGK]) they proved, among other equivalences, that a quantum vertex algebra is a braided state field correspondence such that the associativity relation and $Y S = Y^{op}$ holds. We have shown in the discussion before Lemma 6, combined with the fact that all $(\lambda, f)$ products can be rewritten in terms of $\lambda$-products and $(\lambda, z^{-1})$-products, that having a braided state field correspondence is the same of having topologically free $\mathbb{K}[[h]]$-module $V$, together with a $\mathbb{K}[[h]]$-linear map $T : V \to V$, a distinguished vector $|0\rangle$, a braiding $S$ on $V$ and linear maps $(\lambda, f) : V \otimes V \to \mathbb{K}[[\lambda]][[h]], a \otimes b \to a_{(\lambda, f)} b$ for $f \in \mathbb{K}((Z))[[h]]$, such that
\begin{equation}
|0\rangle_{(\lambda, z^{-1})} a = a = a_{(\lambda, z^{-1})}|0\rangle,
\end{equation}
\begin{equation}
T(a_{(\lambda, f)} b) = T(a)_{(\lambda, f)} b + a_{(\lambda, f)} T(b)
\end{equation}
and
\begin{equation}
T(a)_{(\lambda, f)} b = -\lambda a_{(\lambda, f)} b - a_{(\lambda, f')} b
\end{equation}
for all $a, b \in V$. Combining this with Proposition 6 and Proposition 8 we have the following.

Theorem 4. Let $V$ be topologically free $\mathbb{K}[[h]]$-module, together with a $\mathbb{K}[[h]]$-linear map $T : V \to V$, a distinguished vector $|0\rangle$, a braiding $S$ on $V$. Define in $V$ linear maps $(\lambda, f) : V \otimes V \to \mathbb{K}[[\lambda]][[h]], a \otimes b \to a_{(\lambda, f)} b$ for $f \in \mathbb{K}((Z))[[h]]$, such that the equation above hold. Let $Y$ be a topological state-field correspondence. The following statements are equivalent:
(i) $(V, T, |0\rangle, Y, S)$ is a quantum vertex algebra.
(ii) $(V, T, |0\rangle, (\cdot_{(\lambda, f)\cdot}), S)$ satisfies the equations:
\begin{equation}
|0\rangle_{(\lambda, z^{-1})} a = a = a_{(\lambda, z^{-1})}|0\rangle,
\end{equation}
(81)
\[ T(a_{\lambda,f} b) = T(a)_{\lambda,f} b + a_{\lambda,f} T(b) \]  

(82)

and

\[ T(a_{\lambda,f} b) = -\lambda a_{\lambda,f} b - a_{\lambda,f'} b \]  

(83)

for all \( a, b \in V \), and

\[ -b_{-\lambda-T} a = \sum_i a_{\lambda,f_i}^i b^i, \]  

(84)

where \( S(z)(a \otimes b) = \sum_i f_i(z) a^i \otimes b^i \), and there exists \( N >> 0 \) such that

\[ \partial_N^N a_{\lambda}(b)_{\mu} c = \partial_N^N (a_{\lambda} b)_{\lambda+\mu} c, \]  

(85)

\[ \mod h^N, \text{ for all } a, b, c \in V. \]

If was proved in Proposition 3.13 in [DGK] that if a braided vertex algebra satisfies the hexagon relation then the associativity relation holds.

Assume that we have a braided vertex algebra \( V \) and the hexagon relation holds, thus we have a quantum vertex algebra. I we also ask in \( V \) the condition

\[ [T \otimes 1, S(x)] = -\partial_x S(x) \quad \text{and} \quad [1 \otimes T, S(x)] = \partial_x S(x), \]

(which hold, for instance, in what [DK] called non-degenerate quantum vertex algebra), and consider here the \( \lambda \)-product above, we showed that \( (V, T, S) \) together with the \( \lambda \)-product is a conformal algebra (in the sense of [BK]), sitting inside our quantum vertex algebra such that (64) holds. All these, leads us to the following definition.

**Definition 1.** A quantum conformal algebra is a topologically free \( \mathbb{K}[[h]] \)-module \( V \), together with a \( \mathbb{K}[[h]] \)-linear map \( T : V \to V \), a braiding \( S \) on \( V \), namely a map

\[ S : V \otimes V \to \mathbb{C}((x)) \otimes V \otimes V \]

\[ a \otimes b \mapsto \sum_{i=1}^r f_i(x) a^i \otimes b^i, \]

and a linear map \( \lambda : V \otimes V \to \mathbb{K}[\lambda], a \otimes b \to a_{\lambda} b \) called \( \lambda \)-product, such that:

(a, b, c \in V)

(i) \([T \otimes 1, S(x)] = -\partial_x S(x) \) (left shift condition);

(ii) \([1 \otimes T, S(x)] = \partial_x S(x) \) (right shift condition);

(iii) \( T(a_{\lambda} b) = (Ta)_{\lambda} b + a_{\lambda} (Tb) \);

(iv) \( \lambda \)-product

\[ S(x)(a_{\lambda} b \otimes c) = e^{\partial_{\lambda} \partial_x} ((\lambda \cdot) \otimes 1) S^{23}(x) S^{13}(x) (a \otimes b \otimes c)|_{x_1 = x}, \]

(hexagon relation).
Moreover if we ask $a_\lambda(b_\mu c) = (a_\lambda b)_{\lambda+\mu} c$, we call $V$ associative quantum conformal algebra.

In particular, when $S : V \otimes V \to \mathbb{C}[x] \otimes V \otimes V$ and the Jacobi identity

\[(a_{-\lambda - T} b)_{\alpha + \beta} c = -b_a(a_{(\beta)} c) + \sum_{i=1}^{r} \sum_{l \geq 0} f_i^{(l)}(\partial_\beta)(-\partial_\alpha)^i a_\beta^i(b_\alpha^i c), \quad (86)\]

holds it is called Leibnitz quantum conformal algebra. Moreover, if the Jacobi identity together with the skew-symmetry $-b_{-\lambda - T} a = \sum_{i=1}^{r} f_i(\partial_\lambda)(a_\lambda b^i)$ hold we say that $V$ is a Lie quantum conformal algebra.

Note that, due to (48) and (75), combined with Proposition 5(a), the Jacobi identity and the skew-symmetry hold when we are in the case when the quantum vertex algebra has a braiding $S : V \otimes V \to \mathbb{C}[x] \otimes V \otimes V$, as in the following example.

### 4.1 Lie quantum conformal algebra associated with associative algebras of Zamolodchikov-Faddeev type

In [Li1] the quantum vertex algebras associated with associative algebras of Zamolodchikov-Faddeev type is constructed as follows. Let $H$ be a vector space equipped with a bilinear form $\langle \cdot, \cdot \rangle$ and let $S(x)$ be a linear map from $H \otimes H$ to $H \otimes H \otimes \mathbb{C}[x]$. Let $T(H \otimes \mathbb{C}[t, t^{-1}])^+$ be the subspace spanned by the vectors

\[(a^{(1)} \otimes t^{n_1}) \cdots (a^{(r)} \otimes t^{n_r})\]

for $r \geq 1$, $a^{(i)} \in H$, $n_i \in \mathbb{Z}$ with $n_1 + \cdots + n_r \geq 0$. Set

\[J = T(H \otimes \mathbb{C}[t, t^{-1}])T(H \otimes \mathbb{C}[t, t^{-1}])^+,
\]

a left ideal of $T(H \otimes \mathbb{C}[t, t^{-1}])$. We then set

\[\bar{V}(H, S) = T(H \otimes \mathbb{C}[t, t^{-1}])/J, \quad (88)\]

a left $T(H \otimes \mathbb{C}[t, t^{-1}])$-module. Recall that we denote $a(n) = a \otimes t^n$ for $n \in \mathbb{Z}$. Clearly, $\bar{V}(H, S)$ is cyclic on the vector $\bar{1} = 1 + J$ where $a(n)\bar{1} = 0$ for $a \in H$, $n \geq 0$. Furthermore, for $a \in H$, $w \in \bar{V}(H, S)$, we have

\[a(m)w = 0 \quad \text{for } m \text{ sufficiently large}. \quad (89)\]

Now we define $V(H, S)$ to be the quotient $T(H \otimes \mathbb{C}[t, t^{-1}])$-module of $\bar{V}(H, S)$ module the following relations:

\[a(x_1)b(x_2)w - \sum_{i=1}^{r} t_{x_2,x_1}(f_i(x_2-x_1)b^{(i)}(x_2)a^{(i)}(x_1)w = x_2^{-1} \delta \left(\frac{x_1}{x_2}\right) (a, b) w \quad (90)\]

for $a, b \in H$, $w \in \bar{V}(H, S)$, where $S(b \otimes a) = \sum_{i=1}^{r} b^{(i)} \otimes a^{(i)} \otimes f_i(x)$. Denote by $\bar{1}$ the image of $1$ in $V(H, S)$.

The following Theorem in [Li1], shows the existence of a non-degenerate quantum vertex algebra structure on $V(H, S)$:
Theorem 5. Let $H$ be a finite-dimensional vector space over $\mathbb{C}$ equipped with a bilinear form $\langle \cdot, \cdot \rangle$ and let $S : H \otimes H \to H \otimes H \otimes \mathbb{C}[x]$ be a bilinear map with $S(0) = 1$. Assume that $V(H, S)$ is of PBW type in the sense that $S(H \otimes t^{-1}\mathbb{C}[t^{-1}])$-module $grV(H, S) = \bigcup_{k \geq 0} V(H, S)[k]/V(H, S)[k-1]$ is a free module, where $V(H, S)[k]$ is the span of the vectors $a^{(1)}(-m_1) \cdots a^{(r)}(-m_r)\mathbf{1}$, for $r \geq 0$, $a^{(1)}, \cdots, a^{(r)} \in H$, $m_1 \geq 1$ with $m_1 + \cdots + m_r \geq k$. Then there exist a unique weak quantum structure on $V(H, S)$ with $\mathbf{1}$ as the vacuum vector such that

$$Y(a(-1)\mathbf{1}, x) = a(x) \quad \text{for } a \in H \quad (91)$$

Furthermore $V(H, S)$ is a nondegenerate quantum vertex algebra.

Consider $V(H, S)$ as in the theorem above as the base vector space together with the $\lambda$-product defined as $a \lambda b = \text{Res}_z e^{\lambda z}Y(a, z)b$. Note in particular, that given $a \in H$ and $w \in V(H, S)$, due to (90) we have that

$$(a(-1)\mathbf{1})_\lambda w = \sum_m (a(m)w) \lambda^m,$$

where the sum is finite since (88) holds. As we saw in Section 3, this will be a quantum conformal algebra sitting inside the quantum vertex algebras associated with associative algebras of Zamolodchikov-Faddeev type since: the non-degeneracy of the quantum vertex algebra gives us conditions (i) and (ii), namely the left and right shift conditions in Definition 1. We showed in Section 3, that with this definition of $\lambda$-product in a general quantum vertex algebra (iii) always holds. As the hexagon relation holds in a quantum vertex algebra, we have (iv) by Proposition 5, and since the braiding $S : H \otimes H \to H \otimes H \otimes \mathbb{C}[x]$ we also have Jacobi and skew-symmetry. Thus we have what we called a Lie quantum conformal algebra.

In general, each time you consider a non-degenerate quantum vertex algebra with a braiding $S : H \otimes H \to H \otimes H \otimes \mathbb{C}[x]$, you will have sitting inside a Lie quantum conformal algebra.

5 Author’s contributions

All authors contributed equally to this work.

6 Data availability

Data sharing is not applicable to this article as no new data were created or analyzed in this study.
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