A DISCRETE-TIME DYNAMICAL SYSTEM OF STAGE-STRUCTURED WILD AND STERILE MOSQUITO POPULATION

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Abstract. We study the discrete-time dynamical systems associated to a stage-structured wild and sterile mosquito population. We describe all fixed points of the evolution operator (which depends on five parameters) of mosquito population and show that depending on the parameters this operator may have unique, two and infinitely many fixed points. Under some general conditions on parameters we determine type of each fixed point and give the limit points of the dynamical system. Moreover, for a special case of parameters we give full analysis of corresponding dynamical system. We give some biological interpretations of our results.

1. Introduction

In [9] the authors gave a mathematical model of mosquito dispersal, which is a continuous-time dynamical systems of mosquito populations. Recently, in [11] a discrete-time dynamical system, generated by an evolution operator of this mosquito population is studied.

In this paper following [7] we consider another model of the mosquito population. Mosquitoes undergo complete metamorphosis going through four distinct stages of development during a lifetime: egg, larva, pupa and adult [3]. After drinking blood, adult females lay eggs in water. Within a week, the eggs hatch into larvae that breathe air through tubes which they poke above the surface of the water. Larvae eat bits of floating organic matter and each other. Larvae molt four times as they grow; after the fourth molt, they are called pupae. Pupae also live near the surface of the water, breathing through two horn-like tubes on their back. Pupae do not eat. When the skin splits after a few days from a pupa, an adult emerges. The adult lives for only a few weeks and the full life-cycle of a mosquito takes about a month [6], [1, 2].

Consider a wild mosquito population without the presence of sterile mosquitoes. For the simplified stage-structured mosquito population, we group the three aquatic stages into the larvae class by $x$, and divide the mosquito population into the larvae class and the adults, denoted by $y$. We assume that the density dependence exists only in the larvae stage [7].

We let the birth rate, that is, the oviposition rate of adults be $\beta(\cdot)$; the rate of emergence from larvae to adults be a function of the larvae with the form of $\alpha(1 - k(x))$, where $\alpha > 0$ is the maximum emergence rete, $0 \leq k(x) \leq 1$, with $k(0) = 0, k'(x) > 0$, and $\lim_{x \to \infty} k(x) = 1$, is the functional response due to the intraspecific competition [8]. We let the death rate of larvae be a linear function, denoted by $d_0 + d_1x$, and the death rate of adults be constant, denoted by $\mu$. Then we arrive at, in the absence of sterile mosquitoes, the following system
of equations:
\[
\begin{align*}
\frac{dx}{dt} &= \beta(\cdot)y - \alpha(1 - k(x))x - (d_0 + d_1)x, \\
\frac{dy}{dt} &= \alpha(1 - k(x))x - \mu y
\end{align*}
\] (1.1)

We further assume a functional response for \(k(x)\), as in [8], in the form
\[k(x) = \frac{x}{1 + x}\]

Suppose mosquito adults have no difficulty to find their mates such that no Allee effects are concerned, and hence the adults birth is constant, simply denoted as \(\beta(\cdot) = \beta\) [7]. The interactive dynamics for the wild mosquitoes are governed by the following system:
\[
\begin{align*}
\frac{dx}{dt} &= \beta y - \frac{\alpha x}{1 + x} - (d_0 + d_1)x, \\
\frac{dy}{dt} &= \frac{\alpha x}{1 + x} - \mu y
\end{align*}
\] (1.2)

Denote
\[r_0 = \frac{\alpha \beta}{(\alpha + d_0)\mu}\] (1.3)

The dynamistic generated by the system (1.2) can be summarized as follows.

**Theorem 1. (Theorem 3.1 in [7]):** If \(r_0 \leq 1\), where \(r_0\) is defined in equation (1.3), the trivial equilibrium \((0; 0)\) of system (1.2) is a globally asymptotically stable, and there is no positive equilibrium. If \(r_0 > 1\), the trivial equilibrium \((0; 0)\) is unstable, and there exists a unique positive equilibrium \((x(0), y_0)\) with
\[x(0) = \frac{\sqrt{(d_0 + d_1)^2 - 4d_1(\alpha + d_0)(1 - r_0)} - d_0 - d_1}{2d_1}, \quad y_0 = \frac{\alpha x(0)}{\mu(1 + x(0))},\]
which is a globally asymptotically stable.

In this paper (as in [11], [12]) we study the discrete time dynamical systems associated to the system (1.2).

Define the operator \(W : \mathbb{R}^2 \to \mathbb{R}^2\) by
\[
\begin{align*}
x' &= \beta y - \frac{\alpha x}{1 + x} - (d_0 + d_1)x + x, \\
y' &= \frac{\alpha x}{1 + x} - \mu y + y
\end{align*}
\] (1.4)

where \(\alpha > 0, \beta > 0, \mu > 0, d_0 \geq 0, d_1 \geq 0\).

We would like to study dynamical systems corresponding to the operator (1.4).

The paper is organized as follows. In Section 2 we describe all fixed points of the operator (1.4) of mosquito population and show that depending on the parameters this operator may have unique, two and infinitely many fixed points (laying on the graph of a continuous function). In Section 3 we determine type of each fixed point and give the limit points of the dynamical system under some general conditions on parameters. In Section 4 we consider a special case of parameters and give full analysis of corresponding dynamical system. In the last section we give some biological interpretations of the results.
2. Fixed points

Let $\mathbb{R}_+^2 = \{(x, y) : x, y \in \mathbb{R}, x \geq 0, y \geq 0\}$. A point $z \in \mathbb{R}_+^2$ is called a fixed point of $W$ if $W(z) = z$. The set of fixed points is denoted by $\text{Fix}(W)$.

Let us find fixed points of the operator $W$. For this we solve the following system

\[
\begin{align*}
  x &= \beta y - \frac{\alpha x}{1+x} - (d_0 + d_1)x + x, \\
  y &= \frac{\alpha y}{1+x} - \mu y + y
\end{align*}
\]

i.e.,

\[
\begin{align*}
  \beta y &= \frac{\alpha x}{1+x} + (d_0 + d_1)x, \\
  \mu y &= \frac{\alpha y}{1+x}
\end{align*}
\]

Independent from parameters this system has a solution $(0, 0)$. To find other solutions, from the second equation of (2.2) we obtain $y = \frac{\alpha x}{\mu(1+x)}$.

Denote $\gamma(x) = \frac{\alpha x}{\mu(1+x)}$.

Then from the first equation we get the following form

\[d_1x^2 + (d_0 + d_1)x + d_0 + \alpha(1 - \frac{\beta}{\mu}) = 0\] (2.3)

There are the following cases:

a) Let $d_1 = 0$. Then (2.3) has the form

\[d_0x + d_0 + \alpha(1 - \frac{\beta}{\mu}) = 0\] (4.4)

a.1) if $d_0 = 0$ and $\beta = \mu$ then (2.4) has infinitely many roots.

a.2) if $d_0 = 0$ and $\beta \neq \mu$ then (2.4) has no roots.

a.3) if $d_0 \neq 0$ and $\beta = \mu(1 + \frac{d_0}{\alpha})$ then $x = 0$.

a.4) if $d_0 \neq 0$ and $\beta > \mu(1 + \frac{d_0}{\alpha})$ then $x = \frac{\alpha(\beta - \mu)}{\mu d_0} - 1 \in \mathbb{R}_+ \setminus \{0\}$.

b) Let $d_1 \neq 0$. The discriminant of (2.3) is

\[\Delta = (d_0 - d_1)^2 + \frac{4\alpha d_1(\beta - \mu)}{\mu} \geq 0.\]

b.1) if $\Delta < 0$ then (2.3) has no roots.

b.2) if $\Delta = 0$ then $x = -\frac{d_0 + d_1}{2d_1} \notin \mathbb{R}_+$.

b.3) if $\Delta > 0, \beta > \mu(1 + \frac{d_0}{\alpha})$ then $x = \frac{\sqrt{\Delta} - d_0 - d_1}{2d_1} \in \mathbb{R}_+ \setminus \{0\}$.

Denote

\[
\begin{align*}
  \Omega &= \left\{(\alpha, \beta, \mu, d_0, d_1) \in \mathbb{R}^5 : \alpha > 0, \beta > 0, \mu > 0, d_0 \geq 0, d_1 \geq 0\right\}, \\
  \Phi_1 &= \left\{(\alpha, \beta, \mu, d_0, d_1) \in \Omega : d_0 \neq 0, d_1 = 0, \beta > \mu(1 + \frac{d_0}{\alpha})\right\}, \\
  \Phi_2 &= \left\{(\alpha, \beta, \mu, d_0, d_1) \in \Omega : d_1 \neq 0, \beta > \mu(1 + \frac{d_0}{\alpha})\right\}, \\
  \Psi &= \left\{(\alpha, \beta, \mu, d_0, d_1) \in \Omega : d_0 = d_1 = 0, \beta = \mu\right\}, \\
  \Omega^* &= \Omega \setminus (\Phi_1 \cup \Phi_2 \cup \Psi).
\end{align*}
\]

Summarizing we formulate the following
Theorem 2.  

a. **Uniqueness of fixed point:** If \((\alpha, \beta, \mu, d_0, d_1) \in \Omega^*\) then the operator \((1.4)\) has a unique fixed point \((0, 0)\).

b. **Two fixed points,** \((x_i, \gamma(x_i)), \text{ with } \gamma(x) = \frac{\alpha x}{\mu(1+x)}, i = 1, 2:\)

b.1) If \((\alpha, \beta, \mu, d_0, d_1) \in \Phi_1\) then mapping \((1.4)\) has two fixed points with

\[ x_1 = 0, \ x_2 = \frac{\alpha(\beta - \mu)}{\mu d_0} - 1. \]

b.2) If \((\alpha, \beta, \mu, d_0, d_1) \in \Phi_2\) then the fixed points are with

\[ x_1 = 0, \ x_2 = \frac{\sqrt{\Delta - d_0 - d_1}}{2d_1}. \]

c. If \((\alpha, \beta, \mu, d_0, d_1) \in \Psi\) then any point \((x, \gamma(x)), x \in \mathbb{R}_+\) is a fixed points of \((1.4)\).

3. **Types of the fixed points**

To interpret values of \(x\) and \(y\) as probabilities we assume \(x \geq 0\) and \(y \geq 0\). Moreover, to define a dynamical system we need that \(W\) maps \(\mathbb{R}^2_+\) to itself. It is easy to see that if

\[
\alpha \leq 1 - d_0, \ \beta > 0, \ 0 < \mu \leq 1, \ 0 \leq d_0 < 1, \ d_1 = 0
\]

then operator \((1.4)\) maps \(\mathbb{R}^2_+\) to itself. In this case the system \((1.4)\) becomes

\[
\begin{align*}
x' &= \beta y - \left(\frac{\alpha}{1+x} + d_0 - 1\right)x, \\
y' &= \frac{\alpha x}{1+x} + (1 - \mu)y
\end{align*}
\]

Now we shall examine the type of the fixed points.

**Definition 1.** (see [4]) A fixed point \(s\) of the operator \(W\) is called hyperbolic if its Jacobian \(J\) at \(s\) has no eigenvalues on the unit circle.

**Definition 2.** (see [4]) A hyperbolic fixed point \(s\) called:

1) attracting if all the eigenvalues of the Jacobi matrix \(J(s)\) are less than 1 in absolute value;

2) repelling if all the eigenvalues of the Jacobi matrix \(J(s)\) are greater than 1 in absolute value;

3) a saddle otherwise.

To find the type of a fixed point of the operator \((3.2)\) we write the Jacobi matrix:

\[
J(z) = J_W = \begin{pmatrix}
1 - d_0 - \frac{\alpha}{(1+x)^2} & \beta \\
\frac{\alpha}{(1+x)^2} & 1 - \mu
\end{pmatrix}.
\]

The eigenvalues of the Jacobi matrix are

\[
\lambda_{1,2} = \frac{1}{2} \left(2 - g(x) \pm \sqrt{f(x)}\right),
\]

where \(g(x) = \mu + d_0 + \frac{\alpha}{(1+x)^2}, \ f(x) = (\mu - d_0 - \frac{\alpha}{(1+x)^2})^2 + \frac{4\alpha\beta}{(1+x)^2} \).

If

\[
|\lambda_{1,2}| = \left|\frac{1}{2} \left(2 - g(x) \pm \sqrt{f(x)}\right)\right| < 1
\]

(3.3)
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then fixed points are attractive.

The inequality (3.3) is equivalent to the following

\[
\begin{cases}
0 < g(x) \leq 2 \\
\sqrt{f(x)} < g(x)
\end{cases} \quad \text{or} \quad \begin{cases}
2 < g(x) < 4 \\
\sqrt{f(x)} < 4 - g(x)
\end{cases}
\]

The fixed points are repelling if

\[
|\lambda_{1,2}| = \frac{1}{2} \left( 2 - g(x) \pm \sqrt{f(x)} \right) > 1.
\]

The inequality (3.5) is equivalent to the following

\[
\begin{cases}
g(x) < 0 \\
\sqrt{f(x)} < -g(x)
\end{cases} \quad \text{or} \quad \begin{cases}
g(x) > 4 \\
\sqrt{f(x)} < g(x) - 4
\end{cases}
\]

Denote

\[
\Theta = \{ (\alpha, \beta, \mu, d_0, d_1) \in \Omega : d_1 = 0, \alpha \leq 1 - d_0, 0 < \mu \leq 1, 0 \leq d_0 < 1 \},
\]

\[
\Theta_1 = \{ (\alpha, \beta, \mu, d_0, d_1) \in \Omega : d_1 = 0, \mu + d_0 + \alpha \leq 2, \beta < \mu(1 + \frac{d_0}{\alpha}) \},
\]

\[
\Theta_2 = \{ (\alpha, \beta, \mu, d_0, d_1) \in \Omega : d_1 = 0, \mu + d_0 + \alpha \leq 2, \beta > \mu(1 + \frac{d_0}{\alpha}) \},
\]

Note that the set \( \Theta \) is the condition (3.1), to work under this condition we need to introduce the following sets

\[
\Theta^* = \Omega^* \cap \Theta, \quad \Phi^* = \Theta \cap \Phi_1, \quad \Psi^* = \Theta \cap \Psi.
\]

By solving (3.4) and (3.6) at each fixed point we obtain the following

**Theorem 3.**

a. The type of the unique fixed point, \((0,0)\), for (3.2) is as follows:

\((0,0)\) is

\[
\begin{cases}
\text{attractive} \quad \text{if} \quad (\alpha, \beta, \mu, d_0, d_1) \in \Theta^* \cap \Theta_1, \\
\text{saddle} \quad \text{if} \quad (\alpha, \beta, \mu, d_0, d_1) \in \Theta^* \setminus \Theta_1.
\end{cases}
\]

b. The point \((x_1,y_1) = (0,0)\) is saddle if \((\alpha, \beta, \mu, d_0, d_1) \in \Phi^*\) and

\[
(x_2, \gamma(x_2)) \quad \text{with} \quad x_2 = \begin{cases}
\frac{\alpha(\beta - \mu)}{\mu d_0} - 1 \text{ is attractive} \quad \text{if} \quad (\alpha, \beta, \mu, d_0, d_1, d_2) \in \Phi^* \cap \Theta_2,
\end{cases}
\]

\[
\text{or} \quad \begin{cases}
\frac{\alpha(\beta - \mu)}{\mu d_0} - 1 \text{ is saddle} \quad \text{if} \quad (\alpha, \beta, \mu, d_0, d_1, d_2) \in \Phi^* \setminus \Theta_2.
\end{cases}
\]

c. For any \(x \in \mathbb{R}_+\) the point \((x, \gamma(x))\) is saddle if \((\alpha, \beta, \mu, d_0, d_1) \in \Psi^*\).

**Remark 1.** We have the following

- if \((\alpha, \beta, \mu, d_0, d_1) \in \Theta\) then the fixed point with \(x_2 = \sqrt[2d_1]{\frac{-d_0-\lambda}{d_1}}\) is outside of \(\mathbb{R}_+^2\)
- if \((\alpha, \beta, \mu, d_0, d_1) \in \Theta^*\) then \(\text{Fix}(W) \cap \{(x, \gamma(x)) : x \in \mathbb{R}_+\} = \{(0,0)\}\).

From the known theorems (see \([4]\) and \([5]\)) we get the following result
Proposition 1. For the operator \( W \) given by (1.4), under condition (3.1) the following holds
\[
\lim_{n \to \infty} W^n(z_0) = \begin{cases} 
(0, 0), & \text{if } \beta \leq \mu \left(1 + \frac{d_0}{\alpha}\right), \text{ and } z_0 \in U_0 \\
(x^*, y^*), & \text{if } \beta > \mu \left(1 + \frac{d_0}{\alpha}\right), \text{ and } z_0 \in U^* 
\end{cases}
\]
where \( W^n \) is \( n \)-th iteration of \( W \), \( x^* = \frac{\alpha (\beta - \mu)}{\mu d_0} - 1 \), \( y^* = \frac{\alpha x^*}{\mu (1 + x^*)} \) and \( U_0 \) is a neighborhood of \((0,0)\), \( U^* \) is a neighborhood of \((x^*, y^*)\).

In the following examples we show that if the condition (3.1) is not satisfied then the dynamical system may have several kind of limit points.

Example 1. With parameters \((\alpha, \beta, \mu, d_0, d_1) = (1.5, 0.4, 0.5, 0, 0)\) belong to the set \(\Omega^* \setminus \Theta\). If the initial point is \((x_0, y_0) = (5, 4)\) then the trajectory of system (1.4) is shown in the Fig. 1, i.e., it converges to \((0,0)\).

Example 2. With parameters \((\alpha, \beta, \mu, d_0, d_1) = (1.5, 0.5, 0.4, 0, 0)\) belong to set \(\Omega^* \setminus \Theta\). If the initial point is \((x_0, y_0) = (10, 9)\) then the trajectory of system (1.4) is shown in the Fig. 2. In this case the first coordinate of the trajectory goes to infinite and the second coordinate has limit point approximately 3.75.

Example 3. With parameters \((\alpha, \beta, \mu, d_0, d_1) = (6, 0.5, 0.4, 0.6, 0)\) belong to set \(\Phi_1 \setminus \Theta\). If the initial point is \((x_0, y_0) = (50, 80)\) then the trajectory of system (1.4) is shown in the Fig. 3, i.e., it converges to the fixed point \((1.5, 9)\).

4. Dynamics for a special case

In this section we assume
\[
\beta = \mu, \quad d_0 = d_1 = 0
\]
then (1.4) has the following form

\[ W_0 : \begin{cases} x' = \beta y - \frac{\alpha x}{1 + x} + x, \\ y' = \frac{\alpha x}{1 + x} - \beta y + y. \end{cases} \] (4.1)

We denote

\[ S = \{(x, y) \mid x, y \in \mathbb{R}_+, x + y = 1\}, \]
\[ A = \{(\alpha; \beta) : \beta \in \left(0; \frac{1}{2}\right), \alpha \in \left(0; 1 + 2\sqrt{\beta(1 - \beta)}\right)\}, \]
\[ B = \{(\alpha; \beta) : \beta \in \left[\frac{1}{2}; 1\right], \alpha \in (0; 2]\}. \]

The following lemma is useful

**Lemma 1.** The operator \( W_0 \) maps the set \( S \) to itself if and only if \((\alpha, \beta) \in A \cup B\).

**Proof.** Necessity. Let \( z = (x, y) \in S \). \( z' = W_0(z) = (x', y') \). If we add equations of (4.1) then \( x' + y' = x + y = 1 \). So, we have \( y = 1 - x \) and \( x' = \beta(1 - x) - \frac{\alpha x}{1 + x} + x \). If \( x = 0 \) then \( x' = \beta \). Since \( x' \in [0; 1] \) we get \( \beta \in (0; 1] \). If \( x = 1 \) then \( x' = 1 - \frac{\alpha}{2} \) and \( \alpha \in (0; 2] \) because \( x' \in [0; 1] \).
Moreover, for $x' \in [0; 1]$ it should be true the inequalities $0 \leq \beta(1-x) - \frac{\alpha x}{1+x} + x \leq 1$. Let us write these inequalities as
\[
\begin{aligned}
(1-\beta)x^2 + (1-\alpha)x + \beta &\geq 0, \\
(1-\beta)x^2 - \alpha x + \beta - 1 &\leq 0.
\end{aligned}
\tag{4.2}
\]
The second inequality in (4.2) is always true for all $\alpha > 0$ and $\beta \in (0; 1]$. Hence we solve the first inequality under conditions $\alpha > 0$ and $\beta \in (0; 1]$. 

\textbf{i)} if $\beta \in (0; 1]$, $(1-\alpha)^2 - 4\beta(1-\beta) \leq 0$ for (4.2), then $x \in [0; 1]$. Also we obtain $\alpha \in [1 - 2\sqrt{\beta(1-\beta)}; 1 + 2\sqrt{\beta(1-\beta)}]$. 

\textbf{ii)} if $\beta \in (0; 1]$, $(1-\alpha)^2 - 4\beta(1-\beta) \geq 0$ for (4.2), then $x \in [0; 1] \subset (-\infty; x^{(1)})$ or $x \in [0; 1] \subset [x^{(2)}; \infty)$. 

So, it is possible that $x^{(1)} = \frac{\alpha - 1 - \sqrt{(1-\alpha)^2 - 4\beta(1-\beta)}}{2(1-\beta)} \geq 1$ or $x^{(2)} = \frac{\alpha - 1 + \sqrt{(1-\alpha)^2 - 4\beta(1-\beta)}}{2(1-\beta)} \leq 0$. 

From the inequality $x^{(1)} \geq 1$ we get
\[
\begin{aligned}
\alpha - 1 - 2(1-\beta) &\geq (1-\alpha)^2 - 4\beta(1-\beta), \\
(1-\alpha)^2 - 4\beta(1-\beta) &\geq 0.
\end{aligned}
\]

Consequently, $\beta \in \left[\frac{1}{2}; 1\right]$, $\alpha \in [1 + 2\sqrt{\beta(1-\beta)}; 2]$. 

From the inequality $x^{(2)} \leq 0$
\[
\begin{aligned}
\sqrt{(1-\alpha)^2 - 4\beta(1-\beta)} &\leq 1 - \alpha, \\
(1-\alpha)^2 - 4\beta(1-\beta) &\geq 0
\end{aligned}
\]
and it follows that $\beta \in (0; 1]$, $\alpha \in (0; 1 - 2\sqrt{\beta(1-\beta)}]$. 

So, $x \in [0; 1]$ holds for (4.2) when $\alpha > 0$ and $\beta \in (0; 1]$ and in cases \textbf{i)}, \textbf{ii)} it holds $\beta \in (0; 1]$, $\alpha \in [1 - 2\sqrt{\beta(1-\beta)}; 1 + 2\sqrt{\beta(1-\beta)}]$ or $\beta \in \left[\frac{1}{2}; 1\right]$, $\alpha \in [1 + 2\sqrt{\beta(1-\beta)}; 2]$ or $\beta \in (0; 1]$, $\alpha \in (0; 1 - 2\sqrt{\beta(1-\beta)}]$ for parameters $\alpha$ and $\beta$. 

Sufficiency. It is easy to check if $(\alpha, \beta) \in A \cup B$ then $W_0 : S \to S$. 

The restriction on $S$ of the operator $W_0$, denoted by $U$, has the form
\[
U : x' = \beta(1-x) - \frac{\alpha x}{1+x} + x. \tag{4.3}
\]

$U : S_1 \to S_1$, $S_1 = \{x : x \in [0; 1]\}$.

For fixed point of $U$ the following lemma holds.

\textbf{Lemma 2.} (4.3) has unique fixed point $x^* = \frac{\sqrt{\alpha^2 + 4\beta^2} - \alpha}{2\beta}$. 

\textbf{Proof.} We need to solve $x = \beta(1-x) - \frac{\alpha x}{1+x} + x$. It is easy to see that $x_{1,2} = \frac{-\alpha \pm \sqrt{\alpha^2 + 4\beta^2}}{2\beta}$ are roots. Since these roots should be in $S_1$, one checks that $x_1 = \frac{-\alpha + \sqrt{\alpha^2 + 4\beta^2}}{2\beta} \in S_1$, $x_2 = \frac{-\alpha - \sqrt{\alpha^2 + 4\beta^2}}{2\beta} \notin S_1$. \hfill \Box
Note that the fixed points of $U$ are solutions of the equation $\alpha$. The fixed points of $U$ can be reduced to description of 2-periodic points of the function $z$. Solutions to (4.5) are $x$. Consider

$$1 + x^* = 1 + \frac{-\alpha + \sqrt{\alpha^2 + 4\beta^2}}{2\beta} = \frac{2\alpha}{\alpha - 2\beta + \sqrt{\alpha^2 + 4\beta^2}}.$$

Thus we obtain $U'(x^*) = 1 - \frac{\alpha}{(1+x^*)^2} = 1 - \frac{\alpha^2 + 4\beta^2 + (\alpha - 2\beta)\sqrt{\alpha^2 + 4\beta^2}}{2\alpha}$.

1) Let $|U'(x^*)| < 1$. Then we have $\beta(2\beta - \alpha) + (1 - \beta)(2\beta - \alpha + \sqrt{\alpha^2 + 4\beta^2}) > 0$. This inequality is always true in $(\alpha; \beta) \in (A \cup B) \setminus \{(2; 1)\}$.

2) Let $|U'(x^*)| = 1$. If $U'(x^*) = 1$, then $\alpha^2 + 4\beta^2 + (\alpha - 2\beta)\sqrt{\alpha^2 + 4\beta^2} = 0$. So we have $\alpha = 0$, $\beta = 0$. If $U'(x^*) = -1$, then $\alpha^2 - 4\alpha + 4\beta^2 + (\alpha - 2\beta)\sqrt{\alpha^2 + 4\beta^2} = 0$. This equality holds when $\alpha = 2$, $\beta = 1$.

3) Let $|U'(x^*)| > 1$. This inequality does not hold in $(\alpha; \beta) \in (A \cup B) \setminus \{(2; 1)\}$.

For type of $x^*$ the following lemma holds.

**Lemma 3.** The type of the fixed point $x^*$ for (4.3) are as follows:

i) if $(\alpha; \beta) \in (A \cup B) \setminus \{(2; 1)\}$ then $x^*$ is attracting;

ii) if $\alpha = 2$ and $\beta = 1$ then $x^*$ is saddle;

**Periodic points**

A point $z$ in $W_0$ is called periodic point of $W_0$ if there exists $p$ so that $W_0^p(z) = z$. The smallest positive integer $p$ satisfy $W_0^p(z) = z$ is called the prime period or least period of the point $z$. Denote by $Per_p(W_0)$ the set of periodic points with prime period $p$.

Let us first describe periodic points with $p = 2$ on $S$, in this case the equation $W_0(W_0(z)) = z$ can be reduced to description of 2-periodic points of the function $U$ defined in (4.3), i.e., to solution of the equation

$$U(U(x)) = x. \quad (4.4)$$

Note that the fixed points of $U$ are solutions to (4.4), to find other solution we consider the equation

$$\frac{U(U(x)) - x}{U(x) - x} = 0,$$

simple calculations show that the last equation is equivalent to the following

$$(1 - \beta)x^2 + (2 - \alpha)x + 1 + \beta + \frac{\alpha}{\beta - 2} = 0. \quad (4.5)$$

Solutions to (4.5) are $x_{1,2} = \frac{\alpha - 2 + \sqrt{\alpha^2 + 4\beta^2}}{2(1 - \beta)}$. Since $\alpha \in (0; 2]$ we have $x_1 = \frac{\alpha - 2 + \sqrt{\alpha^2 + 4\beta^2}}{2(1 - \beta)} < 0$ and $x_2 \notin S_1$. It holds that $x_2 = \frac{\alpha - 2 + \sqrt{\alpha^2 + 4\beta^2}}{2(1 - \beta)} \in S_1$ when parameters $\alpha$ and $\beta$ satisfy the attitude $(1 + \beta)(2 - \beta) \leq \alpha \leq \frac{4(2 - \beta)}{3 - \beta}$. This attitude holds when parameters $(\alpha; \beta) \in A \cup B$ are only $\alpha = 2$, $\beta = 1$.

Thus we have

**Lemma 4.** The set of two periodic points of (4.3):
\[\begin{align*}
\text{if } \alpha = 2, \beta = 1 \text{ then } Per_2(U) = S_1 \\
\text{if } (\alpha; \beta) \in (A \cup B) \setminus \{(2; 1)\} \text{ then } Per_2(U) = \emptyset
\end{align*}\]

The following describes the trajectory of any point \(x_0\) in \(S_1\).

**Lemma 5.** Let \(x_0 \in S_1\) be an initial point

1) If \((\alpha; \beta) \in (A \cup B) \setminus \{(2; 1)\}\) then \(\lim_{m \rightarrow \infty} U^m(x_0) = x^*\).

2) If \(\alpha = 2, \beta = 1\) then

\[\lim_{n \rightarrow \infty} U^n(x_0) = \begin{cases} x_0, & \text{for } n = 2k, k = 0, 1, 2, \\
\frac{1-x_0}{1+x_0}, & \text{for } n = 2k - 1 \end{cases}\]

**Proof.** Denote

\[C = \{(\alpha; \beta): \beta \in (0; 1], \alpha \in (0; 1-\beta)\},\]

\[D = \{(\alpha; \beta): \beta \in [\frac{1}{2}; 1], \alpha \in [4(1-\beta); 2]\},\]

\[E = \{(\alpha; \beta): \beta \in (0; 1], \alpha \in (1-\beta; 2(1-\beta)]\},\]

\[F = \{(\alpha; \beta): \beta \in (0; 1], \alpha \in [2(1-\beta); 4(1-\beta)] \cap [0; 2]\},\]

\[\alpha^* = (A \cup B) \cap E, \quad F^* = (A \cup B) \cap F,\]

where \(A \cup B = C \cup D \cup E^* \cup F^*, C \cap E^* = \emptyset, C \cap F^* = \emptyset, C \cap D = \emptyset, E^* \cap F^* = \emptyset, E^* \cap D = \emptyset, D \cap F^* = \emptyset\)

If \((\alpha; \beta) \in A \cup B\), then \(U(x)\) maps \(S_1\) to itself.

Let us find minimum points of \(U(x)\). By solving \(U'(x) = 0\) we have \(x_{\text{min}} = \sqrt{\frac{\alpha}{1-\beta}} - 1\). If \((\alpha; \beta) \in C \cup D\) then \(x_{\text{min}} \notin \text{int}S_1 = \{(x, y) \in (0; 1)^2: x + y = 1\}\) if \((\alpha; \beta) \in E^* \cup F^*\) then \(x_{\text{min}} \in \text{int}S_1\).

1) **Case:** \(x_{\text{min}} \notin \text{int}S_1\). (cf. with proof of Lemma 3.4 of [10]) In this case for set \(C\) we have \(U'(x) > 0\), i.e., \(U(x)\) is an increasing function (see Fig. 5). Here we consider the case when the function \(U\) has unique fixed point \(x^*\). We have that the point \(x^*\) is attractive, i.e., \(|U'(x^*)| < 1\). Now we shall take arbitrary \(x_0 \in S_1\) and prove that \(x_n = U(x_{n-1}), n \geq 1\) converges as \(n \rightarrow \infty\). Consider the following partition \([0; 1] = [0; x^*) \cup \{x^*\} \cup (x^*; 1]\). For any \(x \in [0; x^*)\) we have \(x < U(x) < x^*\), since \(U\) is an increasing function, from the last inequalities we get \(x < U(x) < U^2(x) < \ldots < U^{n-1}(x) < U^n(x) < x^*\), iterating this argument we obtain \(U^{n-1}(1) < U^n(x) < x^*\), which for any \(x_0 \in [0; x^*)\) gives \(x_{n-1} < x_n < x^*\), i.e., \(x_n\) converges and its limit is a fixed point of \(U\), since \(U\) has unique fixed point \(x^*\) in \([0; x^*)\) we conclude that the limit is \(x^*\). For \(x \in (x^*; 1]\) we have \(1 > x > U(x) > x^*\), consequently \(x_n > x_{n+1}\), i.e., \(x_n\) converges and its limit is again \(x^*\).

Note that \(D \subset B\). For set \(D\) we have \(U'(x) < 0\), i.e., \(U(x)\) is a decreasing function (see Fig. 3). Let \(g(x) = U(U(x))\). \(g\) is increasing since \(g'(x) = U'(U(x))U'(x) > 0\). By Lemma 2 and Lemma 4 we have that \(g\) has at most unique fixed point (including \(x^*\)). Hence one can repeat the same argument of the proof of part 1) for the increasing function \(g\) and complete the proof.

**Case:** \(x_{\text{min}} \in \text{int}S_1\).

a) Let \(x_{\text{min}} < x^*\). Consider the following partition \([0; 1] = [0; x_{\text{min}}) \cup [x_{\text{min}}; 1]\). The function \(U(x)\) is decreasing in \([0; x_{\text{min}})\) and is increasing in \([x_{\text{min}}; 1]\) (see Fig. 7). For...
all \( x \in [0; x_{\text{min}}) \), \( x < U(x), \) \( U(x) > x_{\text{min}} \). For \( U(x) \in [x_{\text{min}}; 1] \) it can be proved that \( x_n \) converges to the attractive fixed point \( x^* \) (see Lemma 3) like previous case.

b) Let \( x_{\text{min}} > x^* \). Consider the following partition \([0; 1] = [0; x_{\text{min}}) \cup [x_{\text{min}}; 1]\). For all \( x \in [x_{\text{min}}; 1) \), \( x > U(x) > U^2(x) > \ldots > U^k(x) \), \( U^k(x) < x_{\text{min}} \) (see Fig. 3). If \( U(x) \in [0; x_{\text{min}}) \) then the sequence \( x_n \) converges to \( x^* \).

2) When \( \alpha = 2 \), \( \beta = 1 \) the function becomes \( U(x) = \frac{1-x}{1+x} \). Besides, \( U^{2n}(x) = x \), \( U^{2n-1}(x) = \frac{1-x}{1+x} \). This completes the proof.

\[ \square \]

As a corollary of proved lemmas we obtain

**Theorem 4.** Let \( z_0 = (x_0, 1-x_0) \in S \) be an initial point

1) If \( (\alpha; \beta) \in (A \cup B) \setminus \{(2; 1)\} \) then

\[
\lim_{m \to \infty} W^m_0(z_0) = (x^*, 1-x^*).
\]

2) If \( \alpha = 2, \beta = 1 \) then

\[
\lim_{n \to \infty} W^n_0(z_0) = \begin{cases} (x_0, 1-x_0) & \text{for } n = 2k, k = 0, 1, 2, \ldots \\ \left( \frac{1-x_0}{1+x_0}, \frac{2x_0}{1+x_0} \right) & \text{for } n = 2k - 1 \end{cases}
\]

5. **Biological interpretations**

In biology an population biologist is interested in the long-term behavior of the population of a certain species or collection of species. Namely, what happens to an initial population of
members. Does the population become arbitrarily large as time goes on? Does the population tend to zero, leading to extinction of the species? In this section we briefly give some answers to these questions related to our model of the mosquito population.

Each point (vector) $z = (x; y) \in \mathbb{R}^2_+$ can be considered as a state (a measure) of the mosquito population. In case $x + y \neq 0$ one can consider $z$ as a probability measure (after a normalization if needed). If, for example, the value of $x$ is close to zero, biologically this means that the contribution of the larvae class is small in future of the population.

Let us give some interpretations of our main results:

(a) (Case Theorem 2, part a.) The population has a unique equilibrium state;
(b) (Case Theorem 2, part b.) Under conditions on parameters the population has exactly two equilibrium states;
(c) (Case Theorem 2, part c.) The population has a continuum set of equilibrium states.
(d) (Case Proposition 1) If parameters of the model satisfy the conditions of the proposition then the trajectory of the population has limit $(0, 0)$ (i.e. vanishing of the population) or $(x^*, y^*)$ (i.e. both species will survive).
(e) (Case Theorem 4) Under conditions of the part 1) of the theorem both stages of the population will survive with probability $x^*$ and $1 - x^*$ respectively. Under conditions of part 2) of the theorem, for each initial state the population will have 2-periodic state. Thus any state is 2-periodic.

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