Warped AdS$_3$ black holes in new massive gravity

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Abstract

We investigate stationary, rotationally symmetric solutions of a recently proposed three-dimensional theory of massive gravity. Along with BTZ black holes, we also obtain warped AdS$_3$ black holes, and (for a critical value of the cosmological constant) AdS$_2 \times S^1$ as solutions. The entropy, mass and angular momentum of these black holes are computed.

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1. Introduction

In a recent paper [1], a new theory of massive gravity in three dimensions has been proposed. In this theory, as in the case of the older topologically massive gravity (TMG) [2], the linearized excitations about the Minkowski (or de Sitter or anti-de Sitter in the case of a non-vanishing cosmological constant) vacuum describe a propagating massive graviton. To the difference of TMG, which achieves this goal through the addition to the Einstein–Hilbert action of a parity-violating Chern–Simons term, new massive gravity (NMG) is a parity-preserving, higher-derivative extension of three-dimensional general relativity. The possibility of generalizing such an extension to higher dimensions has been explored in [3], with the finding that only the three-dimensional model is unitary in the tree level.

Cosmological TMG admits two very different kinds of black hole solutions, BTZ black holes [4], which are discrete quotients of AdS$_3$, and warped AdS$_3$ black holes [5–8], which are discrete quotients of warped AdS$_3$. Similarly to BTZ black holes, these have a 4-parameter local isometry algebra, which generically is $sl(2, R) \times R$, and may be generated from the corresponding vacua by local coordinate transformations [7, 8]. The ADM lapse function of these warped black holes goes to a constant value at spacelike infinity (with the ADM shift function going to zero) [5], which makes them closer in this respect to four-dimensional black holes than the BTZ black holes. But the warped AdS$_3$ black holes have the very special property of being intrinsically non-static, their ergosphere extending to infinity [5].
It is straightforward to show that cosmological NMG also admits the BTZ black holes as solutions [1]. Because TMG and NMG have much in common, and indeed may be unified in a ‘general massive gravity’ model [1], the existence of warped AdS3 black hole solutions to cosmological NMG may be conjectured. The purpose of the present paper is to construct these black hole solutions, and to compute their mass, angular momentum and entropy.

In the next section we investigate NMG with two Killing vectors, write in compact form the dimensionally reduced field equations, and exhibit four constants of the motion. A simple ansatz reduces in the third section these fourth-order derivative equations to a system of algebraic equations with two generic solutions, leading either to BTZ black holes or warped AdS3 black holes. In the fourth section, we compute the mass, angular momentum and entropy of these black holes, which satisfy the first law of black hole thermodynamics, as well as a Smarr-like relation. Our results are summarized in the conclusion. Solutions with a horizon and without naked closed timelike curves, but which may be transformed to the vacuum by a global coordinate transformation and thus are not genuine black holes, are discussed in the appendix.

2. New massive gravity with two Killing vectors

The action of the cosmological new massive gravity theory is1 [1]

\[ I_3 = \frac{1}{16\pi G} \int d^3x \sqrt{|g|} \left[ R - \frac{1}{m^2} K - 2\Lambda \right], \tag{2.1} \]

where \( R \) is the trace of the Ricci tensor \( R_{\mu\nu} \), the quadratic curvature invariant \( K \) may be defined in terms of the Schouten tensor

\[ K = \epsilon_{\lambda\mu\nu}\epsilon_{\rho\sigma\tau} g^{\lambda\rho} S^{\mu\sigma} S^{\nu\tau} = R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2 \tag{2.2} \]

and, for sake of generality, we will consider both signs of the real squared mass parameter \( m^2 \).

We search for stationary circularly symmetric solutions of this theory using the dimensional reduction procedure of [9]. A three-dimensional metric with two commuting Killing vectors \( \partial_t, \partial_\phi \) may be parametrized as

\[ ds^2 = \lambda_{ab}(\rho) dx^a dx^b + \zeta^{-2}(\rho) R^{-2}(\rho) d\rho^2 \tag{2.3} \]

\((x^0 = t, x^1 = \phi)\), where \( R^2 = -\det \lambda \), and the scale factor \( \zeta(\rho) \) allows for arbitrary reparametrizations of the radial coordinate \( \rho \). The local isomorphism between the \( SL(2, R) \) group of transformations in the 2-Killing vector space and the Lorentz group \( SO(2, 1) \) suggests the parametrization of the \( 2 \times 2 \) matrix \( \lambda \)

\[ \lambda = \begin{pmatrix} T + X & Y \\ Y & T - X \end{pmatrix}, \tag{2.4} \]

such that \( R^2 = X^2 \) is the Minkowski pseudo-norm of the ‘vector’ \( X(\rho) = (T, X, Y) \),

\[ X^2 = \eta_{ij} X^i X^j = -T^2 + X^2 + Y^2. \tag{2.5} \]

We shall use the notations \( X \cdot Y = \eta_{ij} X^i X^j \) for the scalar product of two vectors, and

\[ (X \wedge Y)^i = \eta^{ij} \epsilon_{jkl} X^k Y^l \tag{2.6} \]

(with \( \epsilon_{012} = +1 \)) for their wedge product.

The Ricci tensor components are [6]

\[ R^a_{\ b} = -\frac{\zeta}{2} ((\zeta RR')^a \ 1 + (\zeta \ell')^a \ 1)_{\ b}, \quad R^2_{\ 2} = -\zeta (\zeta RR')' + \frac{1}{2} \zeta^2 (X^2), \tag{2.7} \]
where \( \ell \) denotes the derivative \( df/d\rho \), and \( \ell \) is the matrix

\[
\ell = \begin{pmatrix} -L^1 & -L^2 + L^3 \\ L_1^T + L^3 & L_1^T \end{pmatrix}
\]

(2.8)

associated with the vector

\[
L \equiv X \wedge X'.
\]

(2.9)

It follows that

\[
\mathcal{R} = \zeta^2 \left[ -2(\mathcal{R} R')' + \frac{1}{2}(X^2)^2 \right] - 2 \zeta \zeta' \mathcal{R} R',
\]

\[
K = \zeta^4 \left[ \frac{1}{2}(L^2)^2 - \frac{1}{4}(\mathcal{R} R') (X^2) + \frac{2}{m^2}(X^2)^2 \right] + \zeta^3 \zeta' \left[ (L \cdot L') - \frac{1}{4} \mathcal{R} R'(X^2) \right] + \zeta^2 \zeta^2 \frac{1}{2}(L^2),
\]

(2.10)

reducing the action (2.1) to the one-dimensional form

\[
I_1 = \int d\rho \left[ A \zeta^3 + B \zeta^2 \zeta' + C \zeta \zeta^2 + D \zeta + E \zeta' + 2 \zeta^{-1} \Lambda \right]
\]

(2.11)

\[
= \int d\rho \left[ \left( A - \frac{1}{3} B' \right) \zeta^3 + C \zeta \zeta^2 + (D - E') \zeta + 2 \zeta^{-1} \Lambda \right]
\]

(2.12)

(up to a surface term), where

\[
A = \frac{1}{m^2} \left[ \frac{1}{2}(X \cdot X')(X \cdot X'') - \frac{1}{2}(X^2)(X'')^2 - \frac{1}{4}(X \cdot X')(X^2) - \frac{3}{32}(X^2)^2 \right],
\]

\[
B = \frac{1}{m^2} \left[ (X \cdot X')(X \cdot X'') - (X^2)(X'')(X'') - \frac{1}{4}(X \cdot X')(X^2) \right]
\]

(2.13)

(the expression of \( C \) shall not be needed in the following, and those of \( D \) and \( E \) are obvious from (2.10)).

Rather than reducing the three-dimensional equations of massive gravity [1], it is more convenient to derive the reduced equations directly from the reduced action. Taking advantage of the reparametrization invariance of (2.3), we shall fix the gauge \( \zeta = \text{constant} \) after variation of \( \zeta \). The first form (2.11) of the reduced action is most convenient for variation relative to \( X \), which leads to the equations

\[
X \wedge (X \wedge X'') + \frac{5}{2} X \wedge (X' \wedge X'') + \frac{3}{2} X' \wedge (X \wedge X'') + \frac{9}{4} X' \wedge (X' \wedge X'')
\]

\[
- \frac{1}{2} X'' \wedge (X \wedge X') - \left[ \int \frac{1}{8} (X^2) + \frac{m^2}{\zeta^2} \right] X'' = 0.
\]

(2.14)

The wedge product \( X \wedge (2.14) \) can be first integrated, leading to the constancy of the superangular momentum vector

\[
J = -\frac{\zeta^2}{m^2} \left[ (X^2)[X \wedge X'' - X' \wedge X''] + 2(X \cdot X')X \wedge X'' \right]
\]

\[
+ \left[ \int \frac{1}{8} (X^2) - \frac{5}{2} (X \cdot X') \right] X \wedge X' \right] + X \wedge X'
\]

(2.15)

associated with the \( SL(2, \mathbb{R}) \) invariance of the reduced action (2.11) [13]. The second form (2.12) of the reduced action is more convenient for variation relative to \( \zeta \), leading (in the gauge \( \zeta = \text{constant} \)) to the Hamiltonian constraint

\[
H \equiv (X \wedge X') \cdot (X \wedge X'') - \frac{1}{2} (X \wedge X'')^2 + \frac{3}{2} (X \wedge X') \cdot (X' \wedge X'')
\]

\[
+ \frac{1}{32} (X^2)^2 + \frac{m^2}{2 \zeta^2} (X^2)^2 + \frac{2 m^2}{\zeta^4} = 0.
\]

(2.16)
The combination $2X \cdot (2.14) - 3(2.16)$ leads to the simpler scalar equation

$$\frac{1}{2} (X \cdot X'')^2 - \frac{1}{2} (X^2)(X''^2) - \frac{1}{4} (X^2)(X \cdot X'') - \frac{3}{32} (X^2)^2$$

$$- \frac{m^2}{\xi^2} \left[ \frac{3}{2} (X^2) + 2(X \cdot X'') \right] - \frac{6m^2 \Lambda}{\xi^4} = 0,$$

which is equivalent to the trace of the three-dimensional field equations [1]

$$K + m^2 (R - 6 \Lambda) = 0.$$

### 3. Regular black holes

As in the case of topologically massive gravity [10–13], equations (2.14) and (2.16) presumably admit a variety of exact solutions. We shall only construct here the solutions generated from the quadratic ansatz [13]

$$X = \alpha \rho^2 + \beta \rho + \gamma,$$

($\alpha$, $\beta$ and $\gamma$ being linearly independent constant vectors), which in the case of TMG leads to warped AdS$_3$ black holes [8]. Inserting this ansatz into the vector equation (2.14), we find that the fourth- and third-order components vanish provided the vectors $\alpha$ and $\beta$ are constrained by

$$\alpha^2 = 0, \quad (\alpha \cdot \beta) = 0.$$  

(3.2)

It is easy to show that the two constraints (3.2) further imply

$$\alpha \wedge \beta = b \alpha, \quad \beta^2 = b^2,$$

for some real constant $b$. The lower order components of (2.14) and the scalar equation (2.17) are then satisfied provided

$$[z + \frac{1}{4} b^2 - \bar{m}^2] \alpha = 0,$$

$$z^2 + \frac{1}{4} b^2 z - \frac{3}{64} b^4 - \bar{m}^2 \left[ \frac{1}{2} b^2 - 2z \right] - 3\bar{m}^2 \bar{X} = 0,$$

where we have put

$$(\alpha \cdot \gamma) = -z,$$

(3.6)

and $\bar{m}^2 \equiv m^2 / \xi^2$, $\bar{X} \equiv \Lambda / \xi^2$.

Equation (3.4) has two solutions. The first is $\alpha = 0$ (implying $z = 0$), so that our ansatz (3.1) reduces to

$$X = \beta \rho + \gamma,$$

(3.7)

which leads to the BTZ black hole metric

$$ds^2 = (-2l^{-2} \rho + M/2) dt^2 - 2 dt \ d\phi + (2 \rho + M l^2 / 2) \ d\phi^2$$

$$+ [4l^{-2} \rho^2 - (M^2 l^2 - J^2) / 4l^{-1}] \ d\rho^2,$$

(3.8)

for $\xi = 1, b^2 = 4l^{-2}$ [5, 13]. The AdS$_3$ curvature parameter $l^{-2}$ is obtained by solving (3.5),

$$l^{-2} = 2m^2 \left[ -1 \pm \sqrt{1 - \Lambda / m^2} \right]$$

(3.9)

(compare with equation (29) of [1]). For the upper sign, this parameter is positive if $\Lambda < 0$ (with the restriction $\Lambda > m^2$ for $m^2 < 0$), and reduces as it should to $l^{-2} = -\Lambda$ for $m^2 \to \infty$. 

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The other branch of the solution (lower sign) exists only in the tachyonic case $m^2 < 0$ for
$\Lambda > m^2$.

The second solution of (3.4) is
\[ z = m^2 - \frac{17}{8} b^2. \]  
(3.10)
Inserting this in (3.5), we obtain the equation
\[ \bar{m}^4 - 3b^2 m^2 + \frac{17}{8} b^4 - m^2 k = 0, \]  
(3.11)
which (assuming $\Lambda/m^2 \neq 1$) is solved by
\[ \frac{m^2}{\zeta^2 b^2} = \frac{6 \pm \sqrt{3 (5 + 7 \Lambda/m^2)}}{4(1 - \Lambda/m^2)} \]  
(3.12)
with $b^2 \neq 0$. At this point, we can without loss of generality fix the scale $\zeta$ and the spatial
orientation\(^2\) so that
\[ b = -1, \]  
(3.13)
which enables us to make contact with the results of [8]. Squaring (3.1), we obtain
\[ R^2 = (1 - 2z)\rho^2 + 2(\beta \cdot \gamma)\rho + \gamma^2 = \beta^2 (\rho^2 - \rho_0^2), \]  
(3.14)
where we have set
\[ z = (1 - \beta^2)/2, \]  
(3.15)
translated the radial coordinate so that $(\beta \cdot \gamma) = 0$, and defined $\gamma^2 \equiv -\beta^2 \rho_0^2$. As shown in
the appendix, we can choose a rotating frame and a length-time scale such that
\[ \alpha = (1/2, -1/2, 0), \quad \beta = (\omega, -\omega, -1), \]
\[ \gamma = (z + u, z - u, -2\omega z) \]  
(3.16)
(other possible choices either can be reduced to this by a global coordinate transformation, or
lead to non-black-hole solutions). This choice leads to the metric
\[ ds^2 = -\beta^2 \frac{\rho^2 - \rho_0^2}{r^2} \, dt^2 + r^2 \left[ d\varphi - \frac{\rho + (1 - \beta^2)\omega}{\beta^2 \rho^2} \, dt \right]^2 + \frac{1}{\beta^2 \rho^2} \, d\rho^2, \]  
(3.17)
where
\[ r^2 = \rho^2 + 2\omega \rho + \omega^2 (1 - \beta^2) + \frac{\beta^2 \rho_0^2}{1 - \beta^2}, \]  
(3.18)
and the constants $\beta^2$ and $\zeta$ are given by
\[ \beta^2 = \frac{9 - 21\Lambda/m^2 \mp \sqrt{3 (5 + 7 \Lambda/m^2)}}{4(1 - \Lambda/m^2)}, \quad \zeta^2 = \frac{21 - 4\beta^2}{8m^2}. \]  
(3.19)
The solutions in the special cases $\beta^2 = 1$ and $\beta^2 = 0$ are given in [8] (where $\mu_E$ should be
replaced by $\zeta$).

The metric (3.17) represents a black hole of the warped AdS$_3$ type, with two horizons at
$\rho = \pm \rho_0$, if $\beta^2 > 0$ and $\rho_0^2 \geq 0$. As discussed in [8], naked closed timelike curves (CTC) do
not occur provided $\beta^2 < 1$ and $\omega > -\rho_0/\sqrt{1 - \beta^2}$. Including the limiting cases $\beta^2 = 1$ and
$\beta^2 = 0$ (which are also causally regular in a certain parameter range), the necessary condition
for the existence of causally regular warped AdS$_3$ black holes is therefore
\[ 0 \leq \beta^2 \leq 1 \]  
(3.20)
\(^2\) The sign of $b$ can be reversed by reversing the angular orientation $\varphi \rightarrow -\varphi$.\footnote{The sign of $b$ can be reversed by reversing the angular orientation $\varphi \rightarrow -\varphi$.}
(implying, from the second equation (3.19), $m^2 > 0$). This corresponds to the upper sign in (3.19) and

$$\frac{-35m^2}{289} \leq \Lambda \leq \frac{m^2}{21}. \quad (3.21)$$

Special values are $\Lambda = -35m^2/289$, corresponding to $\beta^2 = 1$, $\Lambda = 0$, corresponding to $\beta^2 = (9 - 2\sqrt{3})/4$, and $\Lambda = m^2/21$, corresponding to $\beta^2 = 0$.

The case $\Lambda = m^2$, for which equation (3.12) becomes undeterminate, deserves a special investigation. This is carried out in the appendix, with the result that (in the framework of our quadratic ansatz) the only solution without naked CTC is $\text{AdS}_2 \times S^1$.

4. Entropy, mass and angular momentum

In this section, we shall compute the various thermodynamical observables for the black holes of the previous section, and check that they obey the first law. We start with the computation of the entropy, which is a straightforward application of Wald’s general formula [14–16]

$$S = 2\pi \int_h \sqrt{\gamma} \frac{\delta L}{\delta R_{\mu\nu\rho\sigma}} \epsilon_{\mu
u} \epsilon_{\rho\sigma}, \quad (4.1)$$

where $h$ is the spatial section of the event horizon, $\gamma$ is the determinant of the induced metric on $h$, $L$ is the Lagrangian in (2.1) and $\epsilon_{\mu\nu}$ is the binormal to $h$. For a stationary circularly metric of the form (2.3), this gives

$$S = 4\pi A_h \left( \frac{\delta L}{\delta R_{0020}} (g^{(0)}_{22})^{-1} \right)_h = \frac{A_h}{4G} \left( 1 - \frac{1}{m^2} \left[ (g^{(0)}_{00} - 1) R^{00} + g^{02} R^{22} - \frac{3}{4} R \right]_h \right), \quad (4.2)$$

where $A_h = 2\pi r_h$ is the horizon ‘area’. Computing the Ricci tensor components from (2.7), we obtain, on the horizon $R^2 = 0$,

$$S = \frac{A_h}{4G} \left( 1 - \frac{1}{2m^2} \left[ (X \cdot X'') - \frac{1}{4} (X'^2) \right] \right). \quad (4.3)$$

For the BTZ black hole (3.7), this leads to

$$S = \frac{A_h}{4G} \left( 1 - \frac{1}{2m^2 \beta^2} \right), \quad (4.4)$$

which could also be obtained directly from the results of [17–19]. For the warped AdS$_3$ black holes (3.17), the result can be simplified with the aid of (3.10) to

$$S = \frac{A_h}{2Gm^2}. \quad (4.5)$$

It is remarkable that, although their curvature is not constant$^3$ as in the BTZ case, their entropy is simply the Bekenstein–Hawking entropy renormalized by a factor $2/m^2 = 16/(21 - 4\beta^2)$, independently of the black hole parameters $\rho_0$ and $\omega$.

The computation of the mass and angular momentum of these black holes is straightforward in the BTZ case, using e.g. the Abbott–Deser–Tekin (ADT) approach to the computation of the energy of asymptotically AdS solutions to higher curvature gravity theories [20, 21]. In the case of the warped AdS$_3$ black holes, an extension of the ADT approach to the case of massive gravity with non-constant curvature backgrounds, similar to

$^3$ But their curvature invariants are constant and depend only on the parameter $\beta^2$ [7, 8].
that carried out in [6] for topologically massive gravity, is required. In the present work, we shall make the educated guess that, as in the case of generic three-dimensional Einstein-scalar field theories [22] and TMG [6], the mass and angular momentum can be computed from

\[ M = -\frac{\zeta}{8\pi G} (\delta J^Y + \Delta), \quad J = \frac{\zeta}{8\pi G} (\delta J^T - \delta J^X), \]  

(4.6)

where \( \delta J \) is the difference between the values of the super-angular momentum (2.15) for the black hole and for the background. The term \( \Delta \), which depends on the specific theory considered, is not known in the case of new massive gravity. However we can use (4.6) to compute the angular momentum \( J \) (which does not require the knowledge of \( \Delta \)), and derive the mass \( M \) by integrating the first law of black hole thermodynamics

\[ dM = T_H dS + \Omega_b dJ, \]  

(4.7)

where the Hawking temperature and the horizon angular velocity, computed from the metric in the ADM form, are

\[ T_H = \frac{1}{4\pi} \frac{\zeta}{r_h} (N^2)'|_h, \quad \Omega_b = -N^\nu|_h. \]  

(4.8)

In the case of the BTZ black hole (3.7), we obtain

\[ J = \left( 1 - \frac{1}{2m^2l^2} \right) L, \]  

(4.9)

leading (if we assume that, as in the case of TMG, \( \Delta = 0 \) for the BTZ black hole solution of new massive gravity) to

\[ M = \left( 1 - \frac{1}{2m^2l^2} \right) \frac{M}{8G}, \quad J = \left( 1 - \frac{1}{2m^2l^2} \right) \frac{J}{8G}, \]  

(4.10)

The mass, angular momentum and entropy are renormalized by the same factor \( 1 - 1/(2m^2l^2) \) [19] so that, contrary to the case of TMG where the modification of the mass, angular momentum and entropy from the case of cosmological gravity is nontrivial [5, 23]), the first law (4.7) is trivially satisfied. The integral Smarr-like relation

\[ M = \frac{1}{2} T_H S + \Omega_b J \]  

(4.11)

satisfied by the usual BTZ black holes [24] and black hole solutions to generic three-dimensional Einstein-scalar field theories [22], as well as by BTZ black holes in TMG [25], is also satisfied by BTZ black holes in new massive gravity.

In the case of the warped AdS_3 black hole (3.17), we find for the super-angular momentum \( J \)

\[ J = -\frac{2\beta^2}{m} \left[ \beta \wedge \gamma + \rho_0^2 \alpha \right], \]  

(4.12)

leading (if the spacetime (3.17) with \( \rho_0 = \omega = 0 \) is chosen as a background) to

\[ J = \frac{\zeta \beta^2}{4Gm^2} \left[ \omega^2(1 - \beta^2) - \frac{\rho_0^2}{1 - \beta^2} \right]. \]  

(4.13)

Inserting this into the first law (4.7), together with the entropy (4.5) and the temperature and horizon angular velocity

\[ T_H = \frac{\zeta \beta^2 \rho_0}{A_h}, \quad \Omega_h = \frac{2\pi}{A_h} \left( \frac{\rho_0}{\sqrt{1 - \beta^2}} + (\rho_0 + \omega(1 - \beta^2)) \right), \]  

(4.14)

we obtain

\[ M = \frac{\zeta \beta^2 (1 - \beta^2)}{2Gm^2} \omega, \]  

(4.15)
which, as in the case of TMG [6], is twice the ‘naive’ super-angular momentum value \((4.6)\) with \(\Delta = 0\). The fact that the integrability conditions for \((4.7)\) are satisfied is a non-trivial check of our formula \((4.6)\) for the angular momentum. These values of the mass, angular momentum and entropy satisfy the modified Smarr-like formula \([8]\)

\[
M = T_H S + 2\Omega_\theta J
\]  

\((4.16)\)

appropriate for warped AdS_3 black holes. Let us also note that the values \((4.15), (4.13)\) and \((4.5)\) for \(M, J\) and \(S\) coincide with the corresponding values for the warped black holes of gravitating Chern–Simons electrodynamics (equation (5.15) of \([8]\) with \(\lambda \equiv \mu E/2\mu G = 0\) renormalized by a factor \(2/m^2\).

5. Conclusion

In this paper, we have shown that, besides the BTZ black holes, the cosmological new massive gravity theory of \([1]\) also admits warped AdS_3 black hole solutions, causally regular in the range \((3.20)\), and computed their entropy, mass and angular momentum. An interesting side result of our analysis, discussed at the end of the appendix, is that for the critical value \(\Lambda = m^2 < 0\) for which the two branches of the BTZ black hole solution coincide, NMG also admits AdS_2 \(\times S^1\) as a solution.

The fact that our values for the warped black hole mass and angular momentum satisfy non-trivially both the first law of black hole thermodynamics and the modified Smarr relation \((4.11)\) indicates that these are very likely to be correct. This should be confirmed by a computation from first principles, e.g. an extension of the ADT approach to the case of new massive gravity with non-constant curvature backgrounds, along the lines followed in \([6]\).

Another, related, problem which we feel should be addressed is that of the sign of the energy. The massive excitations of TMG \([2]\) or of NMG \([1]\) linearized about the Minkowski vacuum have negative energy, unless the gravitational coupling constant \(G\) is chosen to be negative. In cosmological TMG linearized around a constant curvature background, either the massive gravitons or the BTZ black holes have negative energy, except for a critical, ‘chiral’ value of the Chern–Simons coupling constant \([26]\). Similarly, irrespective of the sign of the gravitational coupling constant, the sign of the energy of massive excitations of NMG linearized around an AdS_3 background has been found to be opposite to the sign of the mass \((4.10)\) and entropy \((4.4)\) of the BTZ black holes, except for the critical value \(m^2\ell^2 = 1/2\) \([27]\). However for this value the BTZ black hole mass and entropy and the energy of the massive modes all vanish, and the theory seems to be trivial. On the other hand, the entropy \((4.5)\) of the warped AdS_3 black holes of NMG is positive definite for \(G > 0\), and their mass \((4.15)\) is non-negative if \(\omega \geq 0\). A difficult task which should be addressed is that of the linearization of the theory around an appropriate, non-constant curvature background, e.g. warped AdS_3 or the warped vacuum \(\rho_0 = \omega = 0\), and the determination of the sign of the energy of the corresponding massive excitations.

Appendix. Non-black-hole solutions

A generic null vector \(\alpha\) can be parametrized by

\[
\alpha = (c, c\cos\alpha, c\sin\alpha),
\]

\((A.1)\)

with \(c\) real and \(0 \leq \alpha < 2\pi\). From \((2.3)\) and \((3.1)\),

\[
g_{\psi\psi} \sim c(1 - \cos\alpha)\rho^2 \quad (\rho \to \infty).
\]

\((A.2)\)
This is non-negative (absence of CTC at infinity) provided either 1) $\alpha \neq 0$ and $c > 0$, or 2) $\alpha = 0$.

In the first case ($\alpha \neq 0$), transition to a rotating frame $d\varphi \to d\varphi' = d\varphi - \Omega dt$ preserves (A.2), but transforms $\alpha^i$ to 0 for $\Omega = \cot(\alpha/2)$. The transformed null vector $\alpha'$ is of the form (A.1), where $\alpha' = \pi$ and $c'$ can be set to the value $1/2$ by a combined length and time rescaling [8], leading to the parametrization (3.16).

In the second case ($\alpha = 0$), the sign of $c$ remains arbitrary, while its absolute value can again be set to $1/2$ by a similar combined rescaling. The generic vector $\beta$ satisfying (3.3) with $b = -1$ is of the form $\beta = (\omega, \omega, 1)$, where $\omega$ can be transformed to 0 by transition to a co-rotating frame (which preserves $\alpha$ and $\beta^T = 1$) with angular velocity $\Omega = \omega$. In this frame the vectors $\alpha$, $\beta$ and $\gamma$ are

$$\alpha = \epsilon(1/2, 1/2, 0), \quad \beta = (0, 0, 1),$$
$$\gamma = \epsilon(u + z, u - z, 0) \quad (u = \beta^2 \rho_0^2/4z),$$

with $\epsilon = \text{sign}(z)$. This leads to the metric

$$ds^2 = -\frac{\beta^2}{|1 - \beta^2|}(\rho^2 - \rho_0^2)dt^2 + |1 - \beta^2|[d\varphi + \frac{\rho}{|1 - \beta^2|}dt]^2 + \frac{1}{\beta^2 \rho^2 - \rho_0^2} \frac{d\rho^2}{\rho^2 - \rho_0^2},$$

for all positive $\beta^2 \neq 1$. This metric, similar to the rotating Bertotti–Robinson metric of [28], can be obtained from the first form of the warped AdS3 black hole metric given in equation (3.16) of [8] (with $c = \epsilon$ and $\omega = 0$) by exchanging the time and angular coordinates, $t \leftrightarrow -\varphi$. It follows that the metric (A.4) can be generated from the vacuum ((A.4) with $\rho_0 = 0$) by a global coordinate transformation (equation (6.12) of [8] with $t \leftrightarrow -\varphi$), and so is not a black hole. Indeed, for $\rho_0^2 = -\gamma^2 < 0$, equation (A.4) can be transformed to equation (3.3) of [7] by putting $\rho = \gamma \sinh \sigma$ and rescaling the $t$ and $\varphi$ coordinates, showing that this spacetime is spacelike-warped AdS3. If, disregarding this fact, we compute the entropy (from (4.5)), and the mass and angular momentum (from (4.6), where $\Delta = \delta J^T$ is assumed), we obtain

$$S = -\frac{16\pi^2 \sqrt{|1 - \beta^2|}}{\kappa^2 m^3}, \quad M = 0, \quad J = -\frac{4\pi \zeta \beta^2 |1 - \beta^2|}{\kappa^2 m^2},$$

which, being independent of the (spurious) parameter $\rho_0$, trivially satisfy the first law (4.7) and, together with the temperature and horizon angular velocity

$$T_H = \frac{\zeta \beta^2 \rho_0}{2\sqrt{|1 - \beta^2|}}, \quad \Omega_h = -\frac{\rho_0}{|1 - \beta^2|},$$

also satisfy the modified Smarr-like relation (4.16).

In the case $\beta^2 = 0$, the solution (A.4) is replaced by equation (3.22) of [8] with $c = 1$, $\omega = 0$ and $t \leftrightarrow -\varphi$.

$$ds^2 = -2\nu \rho dt^2 + [d\varphi + (\rho + \nu) dt]^2 + \frac{d\rho^2}{2\xi^2 \nu \rho}$$

($\nu > 0$). This can similarly be generated from the corresponding vacuum metric by a global coordinate transformation (equation (6.16) of [8] with $t \leftrightarrow -\varphi$).

In the limit $\Lambda/m^2 \to 1$, equation (3.12) still makes sense if either (1) $b$ is fixed and the lower sign is chosen, leading in the gauge $b^2 = 1$ to a non-causally regular black hole with $\beta^2 = 35/8$, or (2) $b \to 0$. For $\Lambda/m^2 = 1$ and $b^2 = 0$, equations (3.4) and (3.5) are both solved by

$$z = \overline{m}^2.$$
Note that $b^2 = 0$ in (3.3) implies $\beta \propto \alpha$, so that the ansatz (3.1) can be reduced by a translation of $\rho$ to

$$X = \alpha \rho^2 + \gamma.$$  

(A.9)

This leads to

$$R^2 = -2m^2(\rho^2 - \rho_0^2),$$

(A.10)

with $\rho_0^2 = (\gamma^2)/2m^2$, so that the metric (3.1) has the correct signature only for $m^2 < 0$. The constant vectors $\alpha$ and $\gamma$ can, without loss of generality, be chosen in the form

$$\alpha = (\pm 1/2, -1/2, 0), \quad \gamma = (\pm (m^2 - u), m^2 + u, v)$$

(A.11)

$$2u = \rho_0^2 - v^2/2m^2). \quad \text{In the case of the upper sign, } g_{\varphi\varphi} = T = X = \rho^2 - 2u < \rho^2 - \rho_0^2,$$

so the metric has closed timelike curves outside the horizon $\rho = \rho_0$. In the case of the lower sign, $g_{\varphi\varphi} = T = X$ and $g_{\psi\psi} = Y$ are both constant, so the parameter $v$ can be set to 0 by a frame rotation, leading to the Bertotti–Robinson-like metric (in the gauge $\zeta = 1$)

$$\begin{align*}
\text{ds}^2 &= -(\rho^2 - \rho_0^2)\, dt^2 - 2m^2\, d\varphi^2 - \frac{d\rho^2}{2m^2(\rho^2 - \rho_0^2)} \quad (m^2 < 0),
\end{align*}$$

(A.12)

which, for all real $\rho_0^2$, is a parametrization of AdS$_2 \times S^1$. In this case, the contribution to the entropy from the second, quadratic term in the right-hand side of (4.3) exactly compensates the Bekenstein–Hawking entropy, leading to a vanishing net entropy. Similarly, the mass and angular momentum computed from (4.12) and (4.6) (with the same assumption as above) also vanish,

$$S = 0, \quad M = 0, \quad J = 0,$$

(A.13)

and both the first law and the modified Smarr formula are trivially satisfied.

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