Differential spaces in integrable Hamiltonian systems
Richard Cushman and Jędrzej Śniatycki

In this paper we use differential space to obtain some new results for completely integrable Hamiltonian systems. Differential spaces were introduced by Sikorski [4] in 1967 with a comprehensive treatment given in his book [5] (in Polish). The theory of differential spaces allows the differential geometric study of arbitrary subsets of \( \mathbb{R}^n \) [6]. In particular it allows one to study the singularities of classical integrable systems [1]. In a recent paper [3] on completely integrable systems the authors avoid the theory of differential spaces using ad hoc definitions. They obtain weaker results than the ones given here.

A completely integrable Hamiltonian system is a triple \((M, \omega, F = (f_1, \ldots, f_n))\), where 1) \((M, \omega)\) is a 2n-dimensional smooth symplectic manifold; 2) the Hamiltonian vector fields \(X_{f_i}\) for \(1 \leq i \leq n\) are complete; 3) the components of the integral mapping

\[ F : M \to \mathbb{R} : x \mapsto (f_1(x), \ldots, f_n(x)) \]

are in pairwise involution, that is, \(L_{X_{f_j}}f_i = \{f_i, f_j\} = 0\); and 4) the components are functionally independent almost everywhere, that is, the rank of \(DF(x)\) is \(n\) for all points \(x\) in a dense open subset \(U\) of \(M\).

Let \(M/X_F\) be the space of orbits of the family of Hamiltonian vector fields \(X_F = \{X_{f_i}\}_{i=1}^n\) on \((M, \omega)\), see [6, Chpt 3, p.44].

**Claim 1** Each orbit of a family of Hamiltonian vector fields \(X_F\) associated to the completely integrable system \((M, \omega, F)\) is an immersed submanifold of \(M\).

**Proof.** This follows from the first Sussmann theorem [7, theorem 4.1 p.179]. See also Śniatycki [6, theorem 3.4.5 p.46]. \(\square\)

The commutant of the integrable system \((M, \omega, F)\) is the set \(C^\infty(M)^F\) of smooth functions on \(M\), which Poisson commute with \(f_i\) for \(1 \leq i \leq n\),

---

1email: rcushman@gmail.com and sniatycki@gmail.com
Department of Mathematics and Statistics, University of Calgary
that is, $g \in C^\infty(M)^F$ if and only if $\{g, f_i\} = 0$ for every $1 \leq i \leq n$. The notion of commutant was introduced in [3]. Equivalently,

**Claim 2** The commutant $C^\infty(M)^F$ is a differential structure on the space of orbits $M/X_F$.

**Proof.** Let $\pi : M \to M/X_F$ be the canonical projection map. By definition $h \in C^\infty(M/X_F)$ if and only if $\pi^*h \in C^\infty(M)^F$. We show that $C^\infty(M/X_F)$ satisfies the conditions for a differential structure [6, Chpt 2, p.15].

Consider the topology on $M/X_F$ generated by the subbasis

$$\{h^{-1}(I) \mid I \text{ open interval in } \mathbb{R} \text{ and } h \in C^\infty(M/X_F)\}.$$  

This topology satisfies condition 1. Condition 2 is automatic. To verify condition 3 let $g : M/X_F \to \mathbb{R}$ be a function such that for every $x \in M/X_F$ there are functions $f, f_1, \ldots, f_n \in C^\infty(M/X_F)$ and open intervals $I_1, \ldots, I_\ell \subseteq \mathbb{R}$ such that

$$x \in U_x = f^{-1}_1(I_1) \cap \cdots \cap f^{-1}_\ell(I_\ell) \quad (1)$$

and

$$g|_{U_x} = f|_{U_x}. \quad (2)$$

Since $f_1, \ldots, f_\ell \in C^\infty(M/X_F)$, it follows that

$$\pi^{-1}(U_x) = \pi^{-1}(f^{-1}_1(I_1) \cap \cdots \cap f^{-1}_\ell(I_\ell))$$

$$= \pi^{-1}(f^{-1}_1(I_1)) \cap \cdots \cap \pi^{-1}(f^{-1}_\ell(I_\ell))$$

$$= (f_1 \circ \pi)^{-1}(I_1) \cap \cdots \cap (f_\ell \circ \pi)^{-1}(I_\ell)$$

is open in $M$. Hence the topology of the orbit space $M/X_F$ is coarser than that of $M$. Equation (2) implies that $\{\pi^{-1}(U_x) \mid x \in M/X_F\}$ is an open covering of $M$ such that $\pi^*(g|_{U_x})$ is smooth and is preserved by the vector fields $X_{f_1}, \ldots, X_{f_n}$ restricted to $\pi^{-1}(U_x)$. Hence $\pi^*g \in C^\infty(M)^F$. □

**Corollary 2a** The orbit space $M/X_F$ with its differential structure $C^\infty(M)^F$ is the differential space $(M/X_F, C^\infty(M/X_F))$.

Because the hypothesis that the vector fields in the family $X_F$ are complete is not used in the proofs above, we introduce a weaker notion of integrable Hamiltonian system, which satisfies hypotheses 1), 3) and 4) of the definition of completely integrable Hamiltonian system. We have proved

**Corollary 2b** Let $(M, \omega, F = (f_1, \ldots, f_n))$ be an integrable Hamiltonian system. Then each orbit of $X_F$ is an immersed submanifold of $M$. The orbit
space $M/X_\mathbf{F}$ with its differential structure $C^\infty(M/X_\mathbf{F})$ is a differential space
and the projection mapping $\pi : M \to M/X_\mathbf{F}$ is smooth.

Let $(M, \omega, \mathbf{F})$ and $(M, \omega, \mathbf{G})$ be two integrable Hamiltonian systems on $(M, \omega)$. Following Seppe and Vu Ngoc [3], we say that these integrable systems are equivalent, that is, $\mathbf{F} \sim \mathbf{G}$, if and only if $C^\infty(M)^{\mathbf{F}} = C^\infty(M)^{\mathbf{G}}$. Let $X_{C^\infty(M)^{\mathbf{F}}}$ be the set of all Hamiltonian vector fields $X_f$ on $(M, \omega)$ where $f$ lies in the commutant $C^\infty(M)^{\mathbf{F}}$. Observe that the orbit of the families $X_\mathbf{F}$ and $X_{C^\infty(M)^{\mathbf{F}}}$ through each point $x$ in $M$ coincide. Hence, $\mathbf{F} \sim \mathbf{G}$ is equivalent to saying that on $M$ the orbits of the families $X_\mathbf{F}$ and $X_\mathbf{G}$ coincide.

As in [3], we say that two integrable systems $(\mathbf{F}, M, \omega)$ and $(\mathbf{F}', M', \omega')$ are symplectically equivalent if and only if there is a symplectic diffeomorphism $\varphi : M \to M'$ such that $\mathbf{F} = \varphi^* \mathbf{F}'$. In other words, $C^\infty(M)^{\mathbf{F}} = C^\infty(M)^{\varphi^* \mathbf{F}'}$. Because the diffeomorphism $\varphi$ induces the map $\varphi^* : C^\infty(M') \to C^\infty(M)$ being symplectically equivalent means that we have $C^\infty(M)^{\mathbf{F}} = \varphi^* (C^\infty(M)^{\mathbf{F}'})$. If $(\mathbf{F}, M, \omega)$ and $(\mathbf{F}', M', \omega')$ are symplectically equivalent by the diffeomorphism $\varphi : M \to M'$, then $\varphi$ induces the diffeomorphism of differential spaces

$$
\varphi^\vee : (M/X_\mathbf{F}, C^\infty(M/X_\mathbf{F})) \to (M'/\mathbf{F}', C^\infty(M'/\mathbf{F}'))
$$

defined by sending the orbit of $X_\mathbf{F}$ through $x \in M$ to the orbit of $X_{\mathbf{F}'}$ in $M'$ through $\varphi(x)$. Moreover the following diagram commutes.

$$
\begin{array}{ccc}
M & \xrightarrow{\varphi} & M' \\
\downarrow{\pi} & & \downarrow{\pi'} \\
M/X_\mathbf{F} & \xrightarrow{\varphi^\vee} & M/\mathbf{F}'
\end{array}
$$

Claim 3 If $(\mathbf{F}, M, \omega)$ and $(\mathbf{F}', M', \omega')$ are symplectically equivalent by the symplectic diffeomorphism $\varphi$ and $(M', \omega', \mathbf{F}')$ is completely integrable, then $(M, \omega, \mathbf{F})$ is completely integrable.

Proof. Suppose that $\varphi : M \to M'$ is a symplectic diffeomorphism such that $\mathbf{F} = \varphi^* \mathbf{F}'$. Then

$$
X_{\mathbf{F}} = (X_{f_1}, \ldots, X_{f_n}) = (X_{\varphi^* f_1'}, \ldots, X_{\varphi^* f_n'}) = X_{\varphi^* \mathbf{F}'}.
$$

So if the Hamiltonian vector fields in $X_{\varphi^* \mathbf{F}'}$ are complete then those in $X_{\mathbf{F}}$ are also. □
We now look at the relation between the space of orbits $M/X_F$ of the family $X_F$ of Hamiltonian vector fields associated to the integrable Hamiltonian system $(M, \omega, F)$ and the image $F(M)$ of its integral map. Since $F(M) \subseteq \mathbb{R}^n$, we have a differential structure $C^\infty_i(F(M))$ on $F(M)$ defined by $g \in C^\infty_i(F(M))$ if and only if for every $y \in F(M)$ there is an open neighborhood $V_y$ of $y$ in $\mathbb{R}^n$ and a function $G_y \in C^\infty(\mathbb{R}^n)$ such that $g|_{V_y \cap F(M)} = G_y|_{V_y \cap F(M)}$. The differential space $(F(M), C^\infty_i(F(M)))$ is a differential subspace of $(\mathbb{R}^n, C^\infty(\mathbb{R}^n))$.

**Claim 4** The integral map

$$F : (M, C^\infty(M)) \to (F(M), C^\infty_i(F(M)))$$

is a smooth mapping of differential spaces.

**Proof.** Suppose that $g \in C^\infty_i(F(M))$. We need to show that $F^*g \in C^\infty(M)$. By definition for every $y \in F(M)$ there is an open neighborhood $V_y$ of $y$ in $\mathbb{R}^n$ and a function $G_y \in C^\infty(\mathbb{R}^n)$ such that $g|_{V_y \cap F(M)} = G_y|_{V_y \cap F(M)}$. Hence

$$(F^*g)|_{F^{-1}(V_y \cap F(M))} = (F^*G_y)|_{F^{-1}(V_y \cap F(M))}. \quad (3)$$

Since $G_y \in C^\infty(\mathbb{R}^n)$ and $F : M \to F(M) \subseteq \mathbb{R}^n$ is smooth, it follows that $F^*G_y = G_y \circ F$ is smooth. Thus for every $x \in M$ there is a $y = F(x) \in F(M)$, an open neighborhood $F^{-1}(V_y)$ of $x$ in $M$, and a function $F^*G_y \in C^\infty(M)$ such that equation (3) holds. Hence $F^*g \in C^\infty(M)$. \hfill \Box

For each $x \in M$ let $L_x$ be the connected component of the fiber $F^{-1}(F(x))$ containing the point $x$. Let $N = \{L_x \mid x \in M\}$ and let $\rho : M \to N : x \mapsto L_x$. The integral mapping $F$ induces the map

$$\mu : N \to F(M) \subseteq \mathbb{R}^n : L_x \mapsto F(x). \quad (4)$$

**Claim 5** For every $x \in M$ the connected component $L_x$ of the fiber $F^{-1}(F(x))$ containing $x$ is the orbit of the family $X_F$ through $x$.

**Proof.** For each $1 \leq i \leq n$ let $\varphi^i_t$ be the local flow of the vector field $X_{f_i}$ on $(M, \omega)$. Fix $x_0 \in M$. Then $f_j(\varphi^i_t(x_0)) = f_j(x_0)$ for every $1 \leq j \leq n$. Hence the orbit of $X_F$ through $x_0$ is contained in $F^{-1}(F(x_0))$. Since orbits of $X_F$ are connected, they are the connected components of $F^{-1}(F(x_0))$.

To finish the argument we must show that the orbits of $X_F$ are open in the fibers of the integral mapping $F$. Let $O_x$ be the orbit of $X_F$ through $x$. Suppose that $\text{rank } dF(x) = k$. By the implicit function theorem, there is a neighbourhood $U$ of $x$ in $M$ such that $U \cap F^{-1}(F(x))$ is a $k$-dimensional
submanifold of $M$. On the one hand, since $O_x$ is a manifold contained in $F^{-1}(F(x))$, it follows that its dimension is at most $k$. On the other hand, rank $dF(x) = k$ implies that there exist $k$ linear combinations of vectors $X_{f_1}(x), ..., X_{f_n}(x)$ that are linearly independent at $x$. Therefore, the dimension of the orbit $O_x$ is at least $k$. Hence, $\dim O_x = k$ and there exists a neighbourhood $U'$ of $x$ in $M$ such that $U' \subseteq U$ and $U' \cap O_x = U' \cap F^{-1}(F(x))$ is an open subset of $F^{-1}(F(x^0))$. This holds for every $x \in F^{-1}(F(x^0))$, which implies that orbits of $X_F$ that are contained in $F^{-1}(F(x^0))$ are open subsets of $F^{-1}(F(x^0))$. □

Claim 5 enables us to identify the space $M/X_F$ of orbits of the family $X_F$ of vector fields on $M$ with the space $N$ of connected components of the fibers of the integral mapping $F : M \rightarrow F(M) \subseteq \mathbb{R}^n$. The identification $M/X_F = N$ leads to the identification of the projection map $\pi : M \rightarrow M/X_F$ with the map $\rho : M \rightarrow N$. In papers on reduction of symmetries in Hamiltonian systems, $M/X_F$ is call the orbit space and $\pi : M \rightarrow M/X_F$ the orbit map, see [1]. In papers on completely integrable Hamiltonian systems, the space $N$ of connected components of the fibers of the integral map $F$ is called the base space, see [2].

Since $F$ is constant on the orbits of $X_F$, which are connected components of the fibers of $F$, it follows that the integral map $F : M \rightarrow F(M) \subseteq \mathbb{R}^n$ factors into the composition of $\pi : M \rightarrow M/X_F$ and the map

$$\mu : M/X_F = N \rightarrow F(M) : L_x \mapsto F(x).$$ (5)

In other words, $F = \mu \circ \pi$.

**Corollary 5a** The mapping

$$\mu : \big( M/X_F = N, C^\infty(M/X_F) \big) \rightarrow \big( F(M), C^\infty_i(F(M)) \big)$$

is smooth.

**Proof.** Suppose that $g \in C^\infty_i(F(M))$. From claim 4, we get $F^*g \in C^\infty(M)$. Clearly $F^*g = g_0 F$ is constant on the fibers of $F$. Since the orbits of $X_F$ are connected subsets in the fibers of $F$, it follows that $F^*g \in C^\infty(M)^F$. Hence there is a function $h \in C^\infty(M/X_F)$ such that $F^*g = \pi^*h$. The equality $F = \mu \circ \pi$ yields

$$\pi^*h = (\mu \circ \pi)^*g = \pi^*(\mu^*g).$$

Since $\pi^* : C^\infty(M/X_F) \rightarrow C^\infty(M)^F$ is bijective, it follows that $\mu^*g = h \in C^\infty(M/X_F)$. Hence the mapping $\mu$ is smooth. □
Suppose that every fiber of the integral map $\mathbf{F}$ of an integrable Hamiltonian system $(\mathcal{M}, \omega, \mathbf{F})$ is connected. Then the smooth mapping $\mu : M/X_{\mathbf{F}} \to \mathbf{F}(M)$ is bijective with inverse $\mu^{-1} : \mathbf{F}(M) \to M/X_{\mathbf{F}}$. Define a differential structure $C^\infty(\mathbf{F}(M))$ on $\mathbf{F}(M)$ by $C^\infty(\mathbf{F}(M)) = (\mu^{-1})^*C^\infty(M/X_{\mathbf{F}})$. By construction

**Claim 6** The map

$$\mu : (M/X_{\mathbf{F}}, C^\infty(M/X_{\mathbf{F}})) \to (\mathbf{F}(M), C^\infty(\mathbf{F}(M)))$$

is a diffeomorphism of differential spaces.

The topology $\mathcal{T}$ on $\mathbf{F}(M)$ coming from the differential structure $C^\infty(\mathbf{F}(M))$ is the same as the topology $\mathcal{S}$ on $M/X_{\mathbf{F}}$ coming from the differential structure $C^\infty(M/X_{\mathbf{F}})$ because the mapping $\mu$ is a diffeomorphism and hence homeomorphism. The topology $\mathcal{S}$ is the same as the topology $\mathcal{T}_1$ on $\mathbf{F}(M)$ coming from the differential structure $C^\infty_i(\mathbf{F}(M))$, since the mapping $\mu : (M/X_{\mathbf{F}}, \mathcal{S}) \to (\mathbf{F}(M), \mathcal{T}_1)$ is a continuous bijective map onto a locally compact Hausdorff space and thus is a homeomorphism. Consequently, the topologies $\mathcal{T}$ and $\mathcal{T}_1$ on $\mathbf{F}(M)$ are the same. However, the differential spaces $(\mathbf{F}(M), C^\infty(\mathbf{F}(M)))$ and $(\mathbf{F}(M), C^\infty_i(\mathbf{F}(M)))$ are not diffeomorphic, since the identity map $\text{id}_{\mathbf{F}(M)}$, which is the composition of the smooth maps $\mu^{-1}$ from $(\mathbf{F}(M), C^\infty(\mathbf{F}(M)))$ to $(M/X_{\mathbf{F}}, C^\infty(M/X_{\mathbf{F}}))$ and the map $\mu$ from $(M/X_{\mathbf{F}}, C^\infty(M/X_{\mathbf{F}}))$ to $(\mathbf{F}(M), C^\infty(\mathbf{F}(M)))$, is a smooth map, whose inverse is not necessarily smooth as $\mathbf{F}(M)$.

**Corollary 6a** If $\mathbf{F}(M)$ is a closed subset of $\mathbb{R}^n$, then the identity mapping

$$\text{id}_{\mathbf{F}(M)} : (\mathbf{F}(M), C^\infty(\mathbf{F}(M))) \to (\mathbf{F}(M), C^\infty_i(\mathbf{F}(M)))$$

is a diffeomorphism of differential spaces.

**Proof.** Let $g \in C^\infty(\mathbf{F}(M))$. Because $\mathbf{F}(M)$ is a closed subset of $\mathbb{R}^n$, by the Whitney extension theorem there is a smooth function $G$ on $\mathbb{R}^n$ such that $g = G|_{\mathbf{F}(M)}$. Hence $g \in C^\infty_i(\mathbf{F}(M))$. So $C^\infty_i(\mathbf{F}(M)) \subseteq C^\infty_i(\mathbf{F}(M))$. By the above discussion, we have $\text{id}_{\mathbf{F}(M)}^*C^\infty_i(\mathbf{F}(M)) \subseteq C^\infty_i(\mathbf{F}(M)) \subseteq C^\infty(\mathbf{F}(M))$. Let $g \in C^\infty_i(\mathbf{F}(M))$. Then for every $y \in \mathbf{F}(M)$ we have $g(y) = g(\text{id}_{\mathbf{F}(M)}(y)) = (\text{id}_{\mathbf{F}(M)}^*g)(y)$. So $C^\infty_i(\mathbf{F}(M)) \subseteq \text{id}_{\mathbf{F}(M)}^*C^\infty_i(\mathbf{F}(M))$. Thus $\text{id}_{\mathbf{F}(M)}^*C^\infty_i(\mathbf{F}(M)) = C^\infty_i(\mathbf{F}(M))$, which implies $C^\infty_i(\mathbf{F}(M)) = C^\infty(\mathbf{F}(M))$. In other words, the mapping $\text{id}_{\mathbf{F}(M)}$ is a diffeomorphism of differential spaces.

**Corollary 6b** If $\mathbf{F}(M)$ is a closed subset of $\mathbb{R}^n$, then the map

$$\mu : (M/X_{\mathbf{F}}, C^\infty(M/X_{\mathbf{F}})) \to (\mathbf{F}(M), C^\infty_i(\mathbf{F}(M)))$$

is a diffeomorphism.
References

[1] R.H. Cushman and L.M. Bates, *Global aspects of classical integrable systems*, second edition, Birkhauser, Basel, 2015.

[2] T. Ratiu, C. Wacheux, and N.T. Zung, Convexity of singular affine structures and toric-focus integrable Hamiltonian systems, arXiv:math.SG.1706.01093v1.

[3] D. Seppe and S. Vu Ngoc, Integrable systems, symmetries, and quantization, arXiv:math.SG.1704.0668v1.

[4] R. Sikorski, Abstract covariant derivative, *Colloq. Math.* 18 (1967) 252–272.

[5] R. Sikorski, *Wstęp do Geometrii Różniczkowej*, PWN, Warsaw, 1972.

[6] J. Śniatycki, *Differential geometry of singular spaces and reduction of symmetry*, New mathematical monographs 23, Cambridge University Press, Cambridge, UK, 2013.

[7] H. Sussmann, Orbits of families of vector fields and foliations with singularities, *Trans. Amer. Math. Soc* 180 (1973) 171–188.

[8] H. Whitney, Analytic extensions of functions defined in closed sets, *Trans Amer Math Soc* 36 (1934) 63–89.