Weighted norm inequalities for pseudo-differential operators with smooth symbols and their commutators

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Abstract We obtain weighted $L^p$ inequalities for pseudo-differential operators with smooth symbols and their commutators by using a class of new weight functions which include Muckenhoupt weight functions. Our results improve essentially some well-known results.

1. Introduction

Let $m$ be real number. Following [13], a symbol in $S^m_{1,\delta}$ is a smooth function $\sigma(x, \xi)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ such that for all multi-indices $\alpha$ and $\beta$ the following estimate holds:

$$|D_x^\alpha D_\xi^\beta \sigma(x, \xi)| \leq C_{\alpha,\beta} (1 + |\xi|)^{m-|\beta|+\delta|\alpha|},$$

where $C_{\alpha,\beta} > 0$ is independent of $x$ and $\xi$. A symbol in $S^{-\infty}_{1,\delta}$ is one which satisfies the above estimates for each real number $m$.

The operator $T$ given by

$$Tf(x) = \int_{\mathbb{R}^n} \sigma(x, \xi) e^{2\pi i x \cdot \xi} \hat{f}(\xi) \, d\xi$$

is called a pseudo-differential operator with symbol $\sigma(x, \xi) \in S^m_{1,\delta}$, where $f$ is a Schwartz function and $\hat{f}$ denotes the Fourier transform of $f$. As usual, $L^m_{1,\delta}$ will denote the class of pseudo-differential operators with symbols in $S^m_{1,\delta}$.

Miller [8] showed the boundedness of $L^0_{1,0}$ pseudo-differential operators on weighted $L^p(1 < p < \infty)$ spaces whenever the weight function belongs to Muckenhoupt’s class $A_p$. In particular, A. Laptev [7] proved that any $L^0_{1,0}$ pseudo-differential operator is a standard Calderón-Zygmund operator.

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Our purpose is to improve the above results of Miller. To state the main results, let us first introduce some notations.

In this paper, \( Q(x, t) \) denotes the cube centered at \( x \) and of the sidelength \( t \). Similarly, given \( Q = Q(x, t) \) and \( \lambda > 0 \), we will write \( \lambda Q \) for the \( \lambda \)-dilate cube, which is the cube with the same center \( x \) and with sidelength \( \lambda t \). Given a Lebesgue measurable set \( E \) and a weight \( \omega \), \( |E| \) will denote the Lebesgue measure of \( E \) and \( \omega(E) = \int_E \omega \, dx \). \( \|f\|_{L^p(\omega)} \) will denote \( (\int_{\mathbb{R}^n} |f(y)|^p \omega(y) \, dy)^{1/p} \) for \( 0 < p < \infty \). \( C \) denotes the constants that are independent of the main parameters involved but whose value may differ from line to line. For a measurable set \( E \), denote by \( \chi_E \) the characteristic function of \( E \). By \( A \sim B \), we mean that there exists a constant \( C > 1 \) such that \( 1/C \leq A/B \leq C \).

Throughout this paper, we let \( \varphi(t) = (1 + t)^{\alpha_0} \) for \( \alpha_0 > 0 \) and \( t \geq 0 \).

A weight will always mean a positive function which is locally integrable. We say that a weight \( \omega \) belongs to the class \( A_p(\varphi) \) for \( 1 < p < \infty \), if there is a constant \( C \) such that for all cubes \( Q = Q(x, r) \) with center \( x \) and sidelength \( r \)

\[
\left( \frac{1}{\varphi(|Q|)|Q|} \int_Q \omega(y) \, dy \right) \left( \frac{1}{\varphi(|Q|)|Q|} \int_Q \omega^{-\frac{1}{p-1}}(y) \, dy \right)^{p-1} \leq C.
\]

We also say that a nonnegative function \( \omega \) satisfies the \( A_1(\varphi) \) condition if there exists a constant \( C \) for all cubes \( Q \)

\[
M_\varphi(\omega)(x) \leq C \omega(x), \text{ a.e. } x \in \mathbb{R}^n.
\]

where

\[
M_\varphi f(x) = \sup_{x \in Q} \frac{1}{\varphi(|Q|)|Q|} \int_Q |f(y)| \, dy.
\]

We also defined the Hardy-Littlewood maximal operator \( M \) by

\[
Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| \, dy.
\]

Obviously, \( f(x) \leq M_\varphi f(x) \leq Mf(x) \) a.e. \( x \in \mathbb{R}^n \) and the function \( M_\varphi f(x) \) is lower semi-continuous.

Since \( \varphi(|Q|) \geq 1 \), so \( A_p(\mathbb{R}^n) \subset A_p(\varphi) \) for \( 1 \leq p \leq \infty \), where \( A_p(\mathbb{R}^n) \) denote the classical Muckenhoupt weights; see [5]. It is well known that if \( \omega \in A_\infty(\mathbb{R}^n) = \bigcup_{p \geq 1} A_p(\mathbb{R}^n) \), then \( \omega(x) \, dx \) be a doubling measure, that is, there exist a constant \( C > 0 \) for any cube \( Q \) such that

\[
\omega(2Q) \leq C \omega(Q).
\]

From the definition of \( A_p(\varphi) \) and (iv) of Lemma 2.1 in Section 2, it is easy to see that if \( \omega \in A_p(\varphi) \), then \( \omega(x) \, dx \) may be not a doubling measure. In fact, let \( 0 \leq \gamma \leq n\alpha_0 \), it is easy to check that \( \omega(x) = (1 + |x|)^{-(n+\gamma)} \notin A_\infty(\mathbb{R}^n) \) and \( \omega(x) \, dx \) is not a doubling measure, but \( \omega(x) = (1 + |x|)^{-(n+\gamma)} \in A_1(\varphi) \).

Now let us state our main results as follows.
Theorem 1.1. Suppose $T \in L_{1,0}^0$. Let $1 < p < \infty$ and $\omega \in A_p(\varphi)$, then there exists a constant $C > 0$ such that

$$\|Tf\|_{L^p(\omega)} \leq C\|f\|_{L^p(\omega)}.$$  

As a consequence of Theorem 1.1, we have the following result.

Corollary 1.1. Suppose $T \in L_{1,0}^0$. Let $1 < p < \infty$ and $\omega(x) = (1 + |x|)^{\gamma_1}|x|^{\gamma_2}$ with $\gamma_1 \in \mathbb{R}$ and $-n < \gamma_2 < n(p-1)$. Then there exists a constant $C > 0$ such that

$$\|Tf\|_{L^p(\omega)} \leq C\|f\|_{L^p(\omega)}.$$  

Let $b \in BMO$(bounded mean oscillation function) defined in [6]. We also consider the commutator of Coifman-Rochberg-Weiss $[b,T]$ defined by

$$[b,T]f(x) = b(x)Tf(x) - T(bf)(x).$$  

Similar to Theorem 1.1, we have

Theorem 1.2. Suppose $T \in L_{1,0}^0$. Let $b \in BMO$, $1 < p < \infty$ and $\omega \in A_p(\varphi)$, then there exists a constant $C > 0$ such that

$$\|[b,T]f\|_{L^p(\omega)} \leq C\|b\|_{BMO}\|f\|_{L^p(\omega)}.$$  

As a consequence of Theorem 1.2, we have the following result.

Corollary 1.2. Suppose $T \in L_{1,0}^0$. Let $b \in BMO$, $1 < p < \infty$ and $\omega(x) = (1 + |x|)^{\gamma_1}|x|^{\gamma_2}$ with $\gamma_1 \in \mathbb{R}$ and $-n < \gamma_2 < n(p-1)$. Then there exists a constant $C > 0$ such that

$$\|[b,T]f\|_{L^p(\omega)} \leq C\|b\|_{BMO}\|f\|_{L^p(\omega)}.$$  

We remark that the above results also hold if $T \in L_{1,\delta}^0$ with $0 < \delta < 1$.

The organization of the paper is as follows: We study some elementary properties of the new weight functions in Section 2. We proved Theorem 1.1 and Corollary 1.1 in Section 3. Theorem 1.2 and Corollary 1.2 are proved, and the weighted weak (1,1) type inequality is obtained in Section 4.

Remark: We will consider the similar results for bilinear pseudo-differential operators in the forthcoming paper.

2. Some properties of $A_p(\varphi)$

Similar to the classical Muckenhoupt weights, we give some properties for weights $\omega \in A_\infty(\varphi) = \bigcup_{p \geq 1} A_p(\varphi)$.

Lemma 2.1. For any cube $Q \subset \mathbb{R}^n$, then
(i) If $1 \leq p_1 < p_2 < \infty$, then $A_{p_1}(\varphi) \subset A_{p_2}(\varphi)$.

(ii) $\omega \in A_p(\varphi)$ if and only if $\omega^{-\frac{1}{p'}} \in A_{p'}(\varphi)$, where $\frac{1}{p} + \frac{1}{p'} = 1$.

(iii) If $\omega_1, \omega_2 \in A_p(\varphi)$, $p \geq 1$, then $\omega_1^\alpha \omega_2^{1-\alpha} \in A_p(\varphi)$ for any $0 < \alpha < 1$.

(iv) If $\omega \in A_p$ for $1 \leq p < \infty$, then
\[
\frac{1}{\varphi(|Q|)5|Q|} \int_Q |f(y)| \omega(y) dy \leq C \left( \frac{1}{\omega(5Q)} \int_Q |f|^p \omega(y) dy \right)^{1/p}.
\]

In particular, let $f = \chi_E$ for any measurable set $E \subset Q$,
\[
\frac{|E|}{\varphi(|Q||Q|)} \leq C \left( \frac{\omega(E)}{\omega(5Q)} \right)^{1/p}.
\]

Proof: (i), (ii) and (iii) are obvious. We only prove (iv). In fact,
\[
\frac{1}{\varphi(|Q|)|Q|} \int_Q |f(y)| dy = \frac{1}{\varphi(|Q|)|Q|} \int_Q |f(y)| \omega^\frac{1}{p}(y) \omega^{-\frac{1}{p'}}(y) dy
\leq \left( \frac{1}{\varphi(|Q|)|Q|} \int_Q |f(y)|^p \omega(y) dy \right)^{\frac{1}{p}} \left( \frac{1}{\varphi(|Q|)|Q|} \int_Q \omega^{-\frac{1}{p'}}(y) dy \right)^{\frac{p-1}{p}}
\leq C \left( \frac{1}{\varphi(|Q|)|Q|} \int_Q |f(y)|^p \omega(y) dy \right)^{\frac{1}{p}} \left( \frac{1}{\varphi(|5Q|)|5Q|} \int_{5Q} \omega(y) dy \right)^{-\frac{p-1}{p}}
\leq C \left( \frac{1}{\omega(5Q)} \int_Q |f(y)|^p \omega(y) dy \right)^{1/p}.
\]
Thus, (iv) is proved.

Obviously, it is easy to see that $\mathcal{S}$ (the set of all Schwartz functions) is dense in $L^p(\omega)$ for $\omega \in A_\infty(\varphi)$ and $1 \leq p < \infty$. Hence, we always assume $f \in \mathcal{S}$ if $f \in L^p(\omega)$ for $1 \leq p < \infty$.

Next, we give a result about the operator $M_\omega$ defined by
\[
M_\omega(f)(x) = \sup_{x \in Q} \frac{1}{\omega(5Q)} \int_Q |f(x)| \omega(x) dx.
\]
Lemma 2.2. Let $\omega \in A_\infty(\varphi)$, then
\[
\omega(\{x \in \mathbb{R}^n : M_\omega f(x) > \lambda\}) \leq \frac{C}{\lambda} \|f\|_{L^1(\omega)}, \quad \forall \lambda > 0, \forall f \in L^1(\omega), \quad (2.1)
\]
and for $1 < p < \infty$
\[
\|M_\omega f\|_{L^p(\omega)} \leq C_p \|f\|_{L^p(\omega)}. \quad (2.2)
\]
Proof: We set $x \in E_\lambda = \{y \in \mathbb{R}^n : M_\omega f(y) > \lambda\}$ with any $\lambda > 0$, then, there exists a cube $Q_x \ni x$ such that
\[
\frac{1}{\omega(5Q_x)} \int_{Q_x} |f(x)|\omega(x)dx > \lambda.
\]
Thus, $\{Q_x\}_{x \in E_\lambda}$ covers $E_\lambda$. By Vitali lemma, there exists a class disjoint cubes $\{Q_{xj}\}$ such that $\bigcup Q_{xj} \subset E_\lambda \subset \bigcup 5Q_{xj}$ and
\[
\omega(E_\lambda) \leq \sum_j \omega(5Q_{xj}) \leq \frac{C}{\lambda} \sum_j \int_{Q_{xj}} |f(y)|\omega(y)dy \leq \frac{C}{\lambda} \|f\|_{L^1(\omega)}.
\]
Thus, (2.1) is proved. By interpolation from (2.1) and the boundedness of $M_\omega$ in $L^\infty(\omega)$, we obtain (2.2). The proof is finished.

From (iii) of Lemma 2.1, we know that
\[
M_\varphi f(x) \leq C(M_\omega(|f|^p)(x))^{1/p}, \quad x \in \mathbb{R}^n.
\]
From this and using Lemma 2.2, we can get the following result.

Lemma 2.3. Let $1 \leq p_1 < \infty$ and suppose that $\omega \in A_{p_1}(\varphi)$. If $p_1 < p < \infty$, then the equality
\[
\int_{\mathbb{R}^n} |M_\varphi f(x)|^p\omega(x)dx \leq C_p \int_{\mathbb{R}^n} |f(x)|^p\omega(x)dx.
\]
Furthermore, let $1 \leq p < \infty$, $\omega \in A_p(\varphi)$ if and only if
\[
\omega(\{x \in \mathbb{R}^n : M_\varphi f(x) > \lambda\}) \leq \frac{C_p}{\lambda^p} \int_{\mathbb{R}^n} |f(x)|^p\omega(x)dx.
\]
From Lemma 2.3, we know that $M_\varphi$ may be not bounded on $L^p(\omega)$ for all $\omega \in A_p(\varphi)$ and $1 < p < \infty$. We now need to define a variant maximal operator $M_{\varphi,\eta}$ for $0 < \eta < \infty$ by
\[
M_{\varphi,\eta} f(x) = \sup_{x \in Q} \frac{1}{\varphi(|Q|)^{\eta/|Q|}} \int_Q |f(y)|dy.
\]
Proposition 2.1. Let $1 < p < \infty$, $p' = p/(p-1)$ and suppose that $\omega \in A_p(\varphi)$. There exists a constant $C_p > 0$ such that
\[
\|M_{\varphi,p'} f\|_{L^p(\omega)} \leq C_p \|f\|_{L^p(\omega)}.
\]
The proof can be found in [14] and [11]. We omit the details here.

As a consequence of Proposition 2.1, we have

**Corollary 2.1.** Let $1 \leq p < \infty$ and $\omega \in A_p(\varphi)$. Let $\Psi$ be a radial, positive function with compact support and total integral 1. Set $\Psi_t(x) = t^{-n} \Psi(x/t)$. Then

(i) $\sup_{0 < t < 1} |f \ast \Psi_t(x)| \leq C \eta M_{\varphi, \eta} f(x)$ for $f \in L^p(\omega)$ and $0 < \eta < \infty$;

(ii) $f \ast \Psi_t(x) \to f(x)$, as $t \to 0$, almost every for $f \in L^p(\omega)$;

(iii) $\|f \ast \Psi_t - f\|_{L^p(\omega)} \to 0$, as $t \to 0$, almost every for $f \in L^p(\omega)$.

**Lemma 2.4.** If $\omega \in A_\infty(\varphi)$, then there exists positive constants $C > 0, \delta > 0$ and $\delta_1 > 0$, such that for any $Q = Q(x_0, r) \subset \mathbb{R}^n$ with $r < 1$, we have

$$
\left(\frac{1}{|Q|} \int_Q \omega^{1+\delta} dx\right)^{1/(1+\delta)} \leq C \frac{1}{|Q|} \int_Q \omega dx
$$

and for any measurable set $E \subset Q = Q(x_0, r)$ with $r < 1$, we have

$$
\frac{\omega(E)}{\omega(Q)} \leq C \left(\frac{|E|}{|Q|}\right)^{\delta_1}.
$$

Proof: We first claim that for any $\alpha > 0$, there exists a positive constant $c_0 < 1$ such that if $|A|/|Q| < \alpha$, then $\frac{\omega(A)}{\omega(Q)} < c_0$ holds for any measurable set $A \subset Q$.

Since $|A| < |Q|$, we have for any $x \in Q$,

$$
M(\chi_{Q \setminus A})(x) > 1 - \alpha,
$$

where $M$ denotes the standard Hardy-Littlewood maximal operator.

Note that if $r < 1$, then there exists a constant $C > 0$ such that $M(\chi_{Q \setminus A})(x) \leq CM_{\varphi}(\chi_{Q \setminus A})(x)$ for any $x \in Q$. From this and by (iv) of Lemma 2.1, we have

$$
M(\chi_{Q \setminus A})(x) \leq CM_{\omega}(\chi_{Q \setminus A})^{\frac{1}{p}}(x).
$$

By Lemma 2.2, we get that

$$
\omega(Q) \leq \omega(\{x \in Q : M(\chi_{Q \setminus A})(x) > 1 - \alpha\})
$$

$$
\leq \omega(\{x \in Q : M_{\omega}(\chi_{Q \setminus A})(x) > C(1 - \alpha)^p\})
$$

$$
\leq \frac{C}{(1 - \alpha)^p} \int_{\mathbb{R}^n} \chi_{Q \setminus A}(x) \omega(x) dx
$$

$$
= \frac{C}{(1 - \alpha)^p}(\omega(Q) - \omega(A)).
$$
Obviously, \( \frac{c}{1-\alpha p} > 1 \). Hence, \( \omega(A) \leq c_0 \omega(Q) \), where \( c_0 < 1 \). Thus, the claim is proved. Using above claim and adapting the standard proof in [5] and [12], we can obtain the desired results. Thus, the proof of Lemma 2.4 is finished.

Let \( 0 < \eta < \infty \). We define the dyadic maximal operator \( \mathcal{M}_{\varphi,\eta}^\Delta f(x) \) by

\[
\mathcal{M}_{\varphi,\eta}^\Delta f(x) := \sup_{x \in Q_{\text{dyadic cube}}} \frac{1}{\varphi(|Q|)^{\eta} |Q|} \int_Q |f(x)| \, dx.
\]

**Lemma 2.5.** Let \( f \) be a locally integrable function on \( \mathbb{R}^n \), \( \eta, \lambda > 0 \), and \( \Omega_\lambda = \{x \in \mathbb{R}^n : \mathcal{M}_{\varphi,\eta}^\Delta f(x) > \lambda\} \). Then \( \Omega_\lambda \) may be written as a disjoint union of dyadic cubes \( \{Q_j\} \) with

\[
\begin{align*}
(i) & \quad \lambda < (\varphi(|Q_j|)^{\eta} |Q_j|)^{-1} \int_{Q_j} |f(x)| \, dx, \\
(ii) & \quad (\varphi(|Q_j|)^{\eta} |Q_j|)^{-1} \int_{Q_j} |f(x)| \, dx \leq 2^n \lambda, \text{ for each cube } Q_j. \text{ This has the immediate consequences:} \\
(iii) & \quad |f(x)| \leq \lambda \text{ for } a.e \ x \in \mathbb{R}^n \setminus \bigcup_j Q_j \\
(iv) & \quad |\Omega_\lambda| \leq \lambda^{-1} \int_{\mathbb{R}^n} |f(x)| \, dx.
\end{align*}
\]

The proof follows from the argument of Lemma 1 in page 150 of [11].

Let \( 0 < \eta < \infty \). The dyadic sharp maximal operator \( \mathcal{M}_{\varphi,\eta} f(x) \) is defined by

\[
\mathcal{M}_{\varphi,\eta} f(x) := \sup_{x \in Q_{\text{dyadic cube}}} \frac{1}{|Q|} \int_Q |f(x) - f_Q| \, dx + \sup_{x \in Q_{\text{dyadic cube}}} \frac{1}{\varphi(|Q|)^{\eta} |Q|} \int_Q |f(x)| \, dx
\]

\[
\simeq \sup_{x \in Q_{\text{dyadic cube}}} \frac{1}{|Q|} \int_Q |f(y) - C| \, dy + \sup_{x \in Q_{\text{dyadic cube}}} \frac{1}{\varphi(|Q|)^{\eta} |Q|} \int_Q |f(x)| \, dx
\]

where \( Q \)'s denote dyadic cubes and \( f_Q = \frac{1}{|Q|} \int_Q f(x) \, dx \). We define sharp maximal operator \( \mathcal{M}_{\varphi,\eta} f(x) \) as above if dyadic cubes replaced by any cubes.

By Lemmas 2.4 and 2.5, we establish the following “good \( \lambda \)” inequality.

**Lemma 2.6.** Let \( \omega \in A_\infty(\varphi) \) and \( 0 < \eta < \infty \). For a locally integrable function \( f \), and for \( b \) and \( \gamma \) positive \( \gamma < b < b_0 = 1/\varphi(2^n)^{\eta} \), we have the following inequality

\[
\omega(\{x \in \mathbb{R}^n : \mathcal{M}_{\varphi,\eta}^\Delta f(x) > \lambda, \mathcal{M}_{\varphi,\eta}^{\sharp\Delta} f(x) \leq \gamma \lambda\}) \leq a^\delta_1 \omega(\{x \in \mathbb{R}^n : \mathcal{M}_{\varphi,\eta}^\Delta f(x) > b \lambda\})
\]

for all \( \lambda > 0 \), where \( a = 2^n \gamma/(1 - \frac{b}{b_0}) \) and \( \delta_1 > 0 \) depends only on \( \omega \)(see Lemma 2.4).
Proof: We may assume that the set \( \{ x : M_{\varphi,\eta}^\Delta f(x) > b\lambda \} \) has finite measure, otherwise the inequality (2.3) is obvious. By Lemma 2.5, that this set is the union of disjoint maximal cubes \( \{ Q_j \} \). We let \( Q = Q(x_0, r) = Q_j \) denote one of these cubes. Thus, we need only to show that

\[
\omega(\{ x \in Q : M_{\varphi,\eta}^\Delta f(x) > \lambda, M_{\varphi,\eta}^{\sharp\Delta} f(x) \leq \gamma \lambda \}) \leq a^h \omega(Q). \tag{2.4}
\]

We consider two cases about sidelength \( r \), that is, \( r < 1 \) and \( r \geq 1 \).

Case 1. When \( r < 1 \), let \( \hat{Q} \supset Q \) be the parent of \( Q \), by the maximality of \( Q \) we have \( |f|_{\hat{Q}} \leq b\lambda \varphi(|\hat{Q}|) \leq b\lambda / b_0 \). So far all \( x \in Q \) for which \( M_{\varphi,\eta}^\Delta f(x) > \lambda \), it follows that \( M_{\varphi,\eta}^\Delta (f \chi_Q)(x) > \lambda \), and also that \( M_{\varphi,\eta}^\Delta [(f - f_{\hat{Q}}) \chi_Q](x) > (1 - b/b_0)\lambda \). By the weak type (1,1) of \( M_{\varphi,\eta}^\Delta \) (see (iv) of Lemma 2.5), we have

\[
|\{ x \in Q : M_{\varphi,\eta}^\Delta f(x) > \lambda, M_{\varphi,\eta}^{\sharp\Delta} f(x) \leq \gamma \lambda \}| \leq \frac{1}{(1 - b/b_0)\lambda} \int_Q |f - f_{\hat{Q}}| dx
\]

\[
\leq \frac{1}{(1 - b/b_0)\lambda} \int_{\hat{Q}} |f - f_{\hat{Q}}| dx
\]

\[
\leq \frac{|\hat{Q}|}{(1 - b/b_0)\lambda} \inf_{x \in Q} M_{\varphi,\eta}^{\sharp\Delta} f(x)
\]

\[
\leq \frac{2^n |\hat{Q}|}{1 - b/b_0}.
\]

if the set in question is not empty. Thus (2.4) is proved in the case \( r < 1 \) by Lemma 2.4.

Case 2. When \( r \geq 1 \), note that

\[
b\lambda < \frac{1}{\varphi(|Q|)\eta |Q|} \int_Q |f(y)| dy \leq \inf_{x \in Q} M_{\varphi,\eta}^{\sharp\Delta} f(x) \leq \gamma \lambda,
\]

but \( \gamma < b \), hence, the set in question is empty. Thus (2.4) is proved, and hence (2.3). the proof of Lemma 2.6 is complete.

As a consequence of Lemma 2.6, we have the following result.

**Corollary 2.2.** Let \( 1 < p < \infty, \omega \in A_\infty(\varphi) \) and \( 0 < \eta < \infty \). Then there exists a constant \( C > 0 \) such that

\[
\| M_{\varphi,\eta}^\Delta f \|_{L^p(\omega)} \leq C \| M_{\varphi,\eta}^{\sharp\Delta} f \|_{L^p(\omega)}.
\]

Note that \( |f(x)| \leq M_{\varphi,\eta}^\Delta f(x) \) a.e. \( x \in \mathbb{R}^n \) and \( M_{\varphi,\eta}^{\sharp\Delta} f(x) \leq M_{\varphi,\eta}^\Delta f(x) \) for all \( x \in \mathbb{R}^n \) for any \( \eta > 0 \). By Corollary 2.2, we have

**Proposition 2.2.** Let \( 1 < p < \infty, \omega \in A_\infty(\varphi), 0 < \eta < \infty \) and \( f \in L^p(\omega) \), then

\[
\| f \|_{L^p(\omega)} \leq \| M_{\varphi,\eta}^\Delta f \|_{L^p(\omega)} \leq C \| M_{\varphi,\eta}^{\sharp\Delta} f \|_{L^p(\omega)}.
\]
A variant of dyadic maximal operator and dyadic sharp maximal operator

\[ M_{\delta,\varphi,\eta}^\Delta f(x) = M_{\varphi,\eta}^\Delta (|f|^{\delta})^{1/\delta}(x) \]

and

\[ M_{\delta,\varphi,\eta}^\sharp f(x) = M_{\varphi,\eta}^\sharp (|f|^{\delta})^{1/\delta}(x), \]

which will become the main tool in our scheme. Proposition 2.2 and Lemma 2.6 imply immediately that

**Proposition 2.3.** Let \( 1 < p < \infty \), \( \omega \in A_\infty(\varphi) \), \( 0 < \eta < \infty \) and \( \delta > 0 \).

(a) Let \( \varphi : (0, \infty) \to (0, \infty) \) be doubling, that is, \( \varphi(2a) \leq C\varphi(a) \) for \( a > 0 \). Then, there exists a constant \( C \) depending upon the \( A_\infty \) condition of \( \omega \) and doubling condition of \( \varphi \) such that

\[
\sup_{\lambda > 0} \varphi(\lambda) \omega\{y \in \mathbb{R}^n : M_{\delta,\varphi,\eta}^\Delta f(y) > \lambda\} \\
\leq C \sup_{\lambda > 0} \varphi(\lambda) \omega\{y \in \mathbb{R}^n : M_{\delta,\varphi,\eta}^\sharp f(y) > \lambda\}
\]

for every function such that the left hand side is finite.

(b) If \( f \in L^p(\omega) \), then

\[
\|f\|_{L^p(\omega)} \leq \|M_{\delta,\varphi,\eta}^\Delta f\|_{L^p(\omega)} \leq C\|M_{\delta,\varphi,\eta}^\sharp f\|_{L^p(\omega)}.
\]

3. The proof of Theorem 1.1

In this section, our proof follows from [8] and [9]. It is worth pointing out that our proof here is rather complex, which is of independent interest. As in [8], we first give a result for any \( L_{1,0}^{-\infty} \) pseudo-differential operator.

**Lemma 3.1.** Suppose \( A \in L_{1,0}^{-\infty} \) and \( 0 < \eta < \infty \). Then there exists a constant \( C > 0 \) such that for all \( x_0 \in \mathbb{R}^n \) and all \( f \in S \),

\[
M_{\varphi,\eta}^\sharp (Af)(x_0) \leq CM_{\varphi,\eta}f(x_0).
\]

Proof: If \( a(x,\xi) \) is the symbol of \( A \), then for any real number \( m \), and any multi-indices \( \alpha \) and \( \beta \),

\[
|D_x^\alpha D_\xi^\beta a(x,\xi)| \leq C_{\alpha,\beta,m}(1+|\xi|)^{m-|\beta|}.
\]

we can write the operator as follows:

\[
Af(x) = \int_{\mathbb{R}^n} \hat{f}(\xi)a(x,\xi)e^{2\pi i x \cdot \xi} d\xi = \int_{\mathbb{R}^n} K(x, x - y)f(y) dy,
\]
where
\[ K(x, x - y) = \int_{\mathbb{R}^n} a(x, \xi) e^{2\pi i (x - y) \cdot \xi} \, d\xi. \]

Obviously, for any \( k > 0 \), there is a constant \( C_k > 0 \) such that
\[ |K(x, x - y)| \leq C_k(1 + |x - y|)^{-k}, \quad \text{for all } x \in \mathbb{R}^n. \]

Then
\[ |Af(x)| \leq \int_{\mathbb{R}^n} |K(x, x - y)||f(y)| \, dy \]
\[ \leq C_k \int_{\mathbb{R}^n} \frac{|f(y)|}{(1 + |x - y|)^k} \, dy. \]

From this, we obtain that
\[ \|Af\|_{L^1(\mathbb{R}^n)} \leq C\|f\|_{L^1(\mathbb{R}^n)}. \]

For any cube \( Q = Q(x_1, r) \ni x_0 \), we write \( f := f \chi_{2Q} + f \chi_{\mathbb{R}^n \setminus 2Q} := f_1 + f_2 \). Then we have
\[
\frac{1}{\varphi(|Q|)\varphi(|Q|)} \int_Q |Af(x)| \, dx \leq \frac{1}{\varphi(|Q|)\varphi(|Q|)} \int_Q |Af_1(x)| \, dx + \frac{1}{\varphi(|Q|)\varphi(|Q|)} \int_Q |Af_2(x)| \, dx
\]
\[ \leq \frac{C}{\varphi(|Q|)\varphi(|Q|)} \int_{\mathbb{R}^n} |f_1(y)| \, dy
\]
\[ + \frac{C_k}{\varphi(|Q|)\varphi(|Q|)} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f_2(y)|}{(1 + |x - y|)^k} \, dy \, dx
\]
\[ \leq \frac{C}{\varphi(|Q|)\varphi(|Q|)} \int_{2Q} |f(y)| \, dy
\]
\[ + \frac{C_k}{\varphi(|Q|)\varphi(|Q|)} \int_{Q} \int_{|y - x_0| > r} \frac{|f(y)|}{(1 + |x_0 - y|)^k} \, dy \, dx
\]
\[ \leq C M_{\varphi, \eta} f(x_0) + C_k \int_{r<|y-x_0|<1} \frac{|f(y)|}{(1 + |x_0 - y|)^k} \, dy
\]
\[ + C_k \int_{1<|y-x_0|<1} \frac{|f(y)|}{(1 + |x_0 - y|)^k} \, dy \]
\[ \leq C M_{\varphi, \eta} f(x_0). \]

Taking the supremum of the left side over all cubes \( Q \) containing \( x_0 \), we obtain the desired result.

Let us now turn to prove Theorem 1.1.

**Proof of Theorem 1.1.** To prove Theorem 1.1, from propositions 2.1 and 2.3, we need only to show that for any \( p' < \eta < \infty \) and \( 0 < \delta < 1 \) such that
\[
M_{\delta, p', \eta}^x (Tf)(x_0) \leq C_{\eta} M_{\varphi, \eta} f(x_0), \quad \text{a.e. } x_0 \in \mathbb{R}^n, \tag{3.1}
\]
where $M_\delta^{\varphi,\eta}(Tf)(x) = M_\delta^{\varphi,\eta}(|Tf|^\delta)(x)^{1/\delta}$.

Fix $x_0 \in \mathbb{R}^n$ and let $x_0 \in Q = Q(x_1, r)$. Decompose $f = f_1 + f_2$, where $f_1 = f\chi_{\bar{Q}}$, where $\bar{Q} = Q(x_1, 8r)$.

Case 1. When $r < 1$. Let $C_Q = |(Tf_2)_Q|$. Since $0 < \delta < 1$, then

\[
\left(\frac{1}{|Q|} \int_Q ||Tf(x)||^\delta - C_Q^\delta \right)^{1/\delta} \leq \left(\frac{1}{|Q|} \int_Q ||Tf(x)|| - |(Tf_2)_Q|^\delta \right)^{1/\delta}
\]
\[
\leq \left(\frac{1}{|Q|} \int_Q |Tf(x) - (Tf_2)_Q|^\delta \right)^{1/\delta}
\]
\[
\leq C \left(\frac{1}{|Q|} \int_Q |Tf_1(x)|^\delta \right)^{1/\delta} + C \left(\frac{1}{|Q|} \int_Q |Tf_2(x) - (Tf_2)_Q|^\delta \right)^{1/\delta}
\]
\[
\leq C \left(\frac{1}{|Q|} \int_Q |Tf_1(x)|^\delta \right)^{1/\delta} + C \left(\frac{1}{|Q|} \int_Q |Tf_2(x) - (Tf_2)_Q| \right)\]
\[
= I + II.
\]

For $I$, we recall that $T$ is weak type $(1, 1)$; see [7]. Note that $\varphi(|\hat{Q}|) \sim 1$, by Kolmogorov’s inequality(see[9]), we then have

\[
I \leq \frac{C}{|Q|} \|Tf_1\|_{L^{1,\infty}}
\]
\[
\leq \frac{C}{|Q|} \int_Q |f(y)| \, dy
\]
\[
\leq CM_{\varphi,\eta}f(x_0).
\]

To deal with the second term, we shall also assume that $a(x, \xi)$, the symbol of $T$, has compact $\xi$-support. The various constants that occur in the following argument will not depend on the support of $a$; at the end we show how to dispense with the assumption on the support of $a$.

We decompose the operator $T$ into a sum of simpler operators. We begin by fixing $\eta$, a nonnegative, radial, $C^\infty$ function of compact support, defined in the $\xi-$space $\mathbb{R}^n$, with the properties that $\eta(\xi) = 1$ for $|\xi| \leq 1$ and $\eta(\xi) = 0$ for $|\xi| \geq 2$. Together with $\eta$, we define another function $\phi$, by $\phi(\xi) = \eta(\xi) - \eta(2\xi)$. Then we have the following “partitions of unity” of the $\xi-$space:

\[
1 = \eta(\xi) + \sum_{j=1}^\infty \phi(2^{-j}\xi), \text{ all } \xi.
\]
Now we can write
\[ T f_2(x) = \int_{\mathbb{R}^n} \hat{f}_2(\xi)a(x, \xi)e^{2\pi i x \cdot \xi} d\xi \]
\[ = \int_{\mathbb{R}^n} \hat{f}_2(\xi)a(x, \xi)\eta(\xi)e^{2\pi i x \cdot \xi} d\xi \]
\[ + \sum_{j=1}^{\infty} \int_{\mathbb{R}^n} \hat{f}_2(\xi)a(x, \xi)\phi(2^{-j}\xi)e^{2\pi i x \cdot \xi} d\xi \]
\[ := Af_2(x) + \sum_{j=1}^{\infty} A_j f_2(x). \]

Obviously, \( A \in L^{-\infty}_{1,0} \). Using Lemma 3.1, we obtain that
\[ II \leq C |Q| \int_Q |Af_2(y) - (Af_2)_Q| dy + C \frac{1}{|Q|} \int_Q \sum_{j=1}^{\infty} |A_j f_2(x) - (A_j f_2)_Q| dx \]
\[ \leq CM_{\varphi, \eta} f(x_0) + C \sum_{j=1}^{\infty} \frac{1}{|Q|} \int_Q |A_j f_2(x) - (A_j f_2)_Q| dx. \]

It remains to examine the operators \( A_j \),

\[ A_j f_2(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a(x, \xi)\phi(2^{-j}\xi)e^{2\pi i (x-y) \cdot \xi} d\xi dy. \]

The following lemma proved in [8] allows to control the inner integral.

**Lemma 3.2.** Let \( q(x, \xi) \) be a symbol of order \( m \), and suppose \( \phi \in C_0^\infty(\mathbb{R}^n) \) has support in \( \{ \xi : \frac{1}{2} \leq |\xi| \leq 2 \} \). If \( N \geq 0 \), then there is a constant \( C_N > 0 \) such that the inequality
\[ |y|^N \left| \int_{\mathbb{R}^n} q(x, \xi)\phi(2^{-j}\xi)e^{2\pi i x \cdot \xi} d\xi \right| \leq C_N 2^{j(n+m-N)} \]
holds for all \( x \) and \( y \) in \( \mathbb{R}^n \) and every integer \( j \geq 1 \).

Let us continue to prove the theorem, noticing that
\[ \frac{1}{|Q|} \int_Q |A_j f_2(x) - (A_j f_2)_Q| dx = \frac{1}{|Q|} \int_Q \left| \frac{1}{|Q|} \int_Q A_j f_2(x) - A_j f_2(z) dz \right| dx \]
\[ = \frac{1}{|Q|} \int_Q \left| \frac{1}{|Q|} \int_{\mathbb{R}^n} f_2(y) \int_{\mathbb{R}^n} \phi(2^{-j}\xi) \right| \]
\[ \times |a(x, \xi) e^{2\pi i (x-y) \cdot \xi} - a(z, \xi) e^{2\pi i (z-y) \cdot \xi} d\xi dy| dz \] dx.

To estimate this last quantity, we consider two subcases:
Subcase 1. \(2^jr \geq 1\). Taking \(k_0\) such that \(1/2 < 2^{k_0}r \leq 1\). Then (3.2) is dominated by

\[
2^\sum_{k=1}^{\infty} \frac{1}{|Q|} \int_Q \int_{2^k r \leq |y-x_1| < 2^{k+1} r} |f(y)| \left| \int_{\mathbb{R}^n} \phi(2^{-j} \xi) a(x, \xi) e^{2\pi i (x-y) \cdot \xi} \, d\xi \right| \, dy \, dx
\]

\[
= 2^\sum_{k=1}^{\infty} \frac{1}{|Q|} \int_Q \int_{2^k r \leq |y-x_1| < 2^{k+1} r} |f(y)| \left| \int_{\mathbb{R}^n} \phi(2^{-j} \xi) a(x, \xi) e^{2\pi i (x-y) \cdot \xi} \, d\xi \right| \, dy \, dx
\]

\[
+ 2 \sum_{k=k_0+1}^{\infty} \frac{1}{|Q|} \int_Q \int_{2^k r \leq |y-x_1| < 2^{k+1} r} |f(y)| \left| \int_{\mathbb{R}^n} \phi(2^{-j} \xi) a(x, \xi) e^{2\pi i (x-y) \cdot \xi} \, d\xi \right| \, dy \, dx
\]

\[= I_1 + I_2.\]

For \(I_1\), by Lemma 3.2 with \(N = n+1\) and \(m = 0\), we obtain that

\[
I_1 \leq C \sum_{k=1}^{k_0} \int_Q \frac{2^{nk}}{|Q_k|} \int_{2^k r \leq |y-x_1| < 2^{k+1} r} \frac{|f(y)|}{|x-y|^{n+1}} \, dy \, dx
\]

\[
\leq C \sum_{k=1}^{k_0} r^{-1-2^{-k} 2^{-j}} \frac{1}{|Q_k|} \int_{Q_k} |f(y)| \, dy
\]

\[
\leq Cr^{-1-2^{-j}M_{\varphi,\eta}f(x_0)}.\]

For \(I_2\), by Lemma 3.2 with \(N = n+n_0+1\) and \(m = 0\), we have

\[
I_2 \leq C \sum_{k=k_0+1}^{\infty} \int_Q \frac{2^{nk}}{|Q_k|} \int_{2^k r \leq |y-x_1| < 2^{k+1} r} \frac{|f(y)|}{|x-y|^{n+n_0+1}} \, dy \, dx
\]

\[
\leq C \sum_{k=k_0+1}^{\infty} r^{-1-2^{-k} 2^{-j-n_0}} \frac{1}{\varphi(|Q_k|)|Q_k|} \int_{Q_k} |f(y)| \, dy
\]

\[
\leq Cr^{-1-2^{-j}M_{\varphi,\eta}f(x_0)}.\]

Subcases 2. \(2^jr < 1\). We write

\[
a(x, \xi) e^{2\pi i (x-y) \cdot \xi} - a(z, \xi) e^{2\pi i (z-y) \cdot \xi}
\]

\[
= \sum_{l=1}^{n} (x_l - z_l) \int_0^1 \frac{\partial a}{\partial x_l} (x(t), \xi) e^{2\pi i (x(t)-y) \cdot \xi} + 2\pi i \xi a(x(t), \xi) e^{2\pi i (x(t)-y) \cdot \xi} dt,
\]

where \(x(t) = z + t(x-z)\).

Using this last expression and the facts:
(a) \( \partial a / \partial x_t \) is a symbol of order 0;
(b) \( \xi_t a(x, \xi) \) is a symbol of order 1;
(c) \( |x_t - z_t| \leq r \) since both \( x \) and \( z \) in \( Q \); and
(d) if \( 2^k r \leq |y - x_t| \leq 2^{k+1} r \), then \( 2^{k-1} r \leq |x(t) - y| \leq 2^{k+2} r \) since \( x(t) \in Q \),
we let \( k_0 \) as above, by Lemma 3.2 again, then (3.2) is dominated by

\[
C \sum_{k=1}^{k_0} \frac{1}{|Q_k|} \int_Q \int_{2^k r \leq |y - x_t| < 2^{k+1} r} \frac{|f(y)|}{|x_t - y|^{n+1/2}} \sum_{l=1}^{n} |x_t - z_t| \int_0^1 |x(t) - y|^{n+1/2} \\
\times \left| \int_{\mathbb{R}^n} \phi(2^{-j} \xi) \left[ \frac{\partial a}{\partial x_t}(x(t), \xi) e^{2\pi i (x(t) - y) \cdot \xi} + 2\pi i \xi_t a(x(t), \xi) e^{2\pi i (x(t) - y) \cdot \xi} \right] d\xi \right| dt dy dz \\
+ \sum_{k=k_0+1}^{\infty} \frac{C}{|Q_k|} \int_Q \int_{2^k r \leq |y - x_t| < 2^{k+1} r} \frac{|f(y)| \sum_{l=1}^{n} |x_t - z_t|}{|x_t - y|^{n+\frac{1}{2} + \eta_0}} \int_0^1 |x(t) - y|^{n+\frac{1}{2} + \eta_0} \\
\times \left| \int_{\mathbb{R}^n} \phi(2^{-j} \xi) \left[ \frac{\partial a}{\partial x_t}(x(t), \xi) e^{2\pi i (x(t) - y) \cdot \xi} + 2\pi i \xi_t a(x(t), \xi) e^{2\pi i (x(t) - y) \cdot \xi} \right] d\xi \right| dt dy dz \\
\leq C \sum_{k=1}^{k_0} \frac{2^{nk}}{|Q_k|^{\eta_k}} \int_{Q_k} |f(y)| dy r^n (2^k r)^{-n-1/2} r^2 (2^{-j/2} + 2^{j/2}) \\
+ C \sum_{k=k_0+1}^{\infty} \frac{2^{nk}}{\varphi(|Q_k|)^{\eta_k}|Q_k|} \int_{Q_k} |f(y)| dy r^n (2^k r)^{-n-1/2} r^2 (2^{-j/2} + 2^{j/2} + 1/2) \\
\leq CM_{\varphi, \eta} f(x_0) r^{1/2} 2^{j/2} \sum_{k=1}^{\infty} 2^{-k/2} \\
\leq C r^{1/2} 2^{j/2} M_{\varphi, \eta} f(x_0).
\]

Putting the two subcases together, we get that

\[
\sum_{j=1}^{\infty} \frac{1}{|Q|} \int_Q |A_j f_2(x) - (A_j f_2)_Q| dx \leq C \left( \sum_{2^r \geq 1} r^{-1/2 - j} + \sum_{2^r < 1} r^{1/2} 2^{-j/2} \right) M_{\varphi, \eta} f(x_0) \\
\leq CM_{\varphi, \eta} f(x_0).
\]

Case 2. When \( r \geq 1 \), write \( \rho_1 := \eta/\delta > \eta + 1 \), we have

\[
\left( \frac{1}{\varphi(|Q|)^{\eta_k}|Q|} \int_Q |T f(y)|^\delta dy \right)^{1/\delta} \leq \frac{C}{\varphi(|Q|)^{\rho_1}} \left( \frac{1}{|Q|} \int_Q |T f_1(y)|^\delta dy \right)^{1/\delta} \\
+ \frac{C}{\varphi(|Q|)^{\rho_1}} \left( \frac{1}{|Q|} \int_Q |T f_2(y)|^\delta dy \right)^{1/\delta} \\
:= II_1 + II_2.
\]
For $I_1$, similar to $I_1$, we have

$$I_1 \leq \frac{C}{\varphi(|Q|)^{n_1} |Q|} \| Tf_1 \|_{L^{1,\infty}}$$

$$\leq \frac{C}{\varphi(|Q|)^{\eta} |Q|} \int_Q |f(y)| \, d\mu(y)$$

$$\leq CM_{\varphi,\eta} f(x).$$

Finally, for $I_2$, adapting the argument of $I_2$ in subcase 1, note that $r \geq 1$, we obtain

$$II_2 \leq C \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \int_Q \frac{2^{nk}}{|Q_k|} \int_{2^{k+1}r \leq |y-x| < 2^{k+1}r} \frac{|f(y)|}{|x-y|^{n+\eta \alpha_0+1}} |x-y|^{n+\eta \alpha_0+1}$$

$$\times \left| \int_{\mathbb{R}^n} \phi(2^{-j} \xi) a(x, \xi) e^{2\pi i (x-y) \cdot \xi} \, d\xi \right| dy$$

$$\leq C \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} r^{-1} 2^{-j-n_0} \frac{1}{\varphi(|Q_k|)^{\eta} |Q_k|} \int_{Q_k} |f(y)| \, dy$$

$$\leq C \sum_{j=1}^{\infty} r^{-1} 2^{-j} M_{\varphi,\eta} f(x_0)$$

$$\leq CM_{\varphi,\eta} f(x_0).$$

Since $r \geq 1$.

Finally, we show how to dispense with assumption on the support of $a$. Suppose now that the assumption that $a(x, \xi)$, the symbol of $T$, has not compact $\xi$-support. Let $b_j(x, \xi)$ be $a(x, \xi)$ multiplied by a smooth cutoff function which is 1 when $|\xi| \leq 2^j$ and 0 when $|\xi| \geq 2^{j+1}$. Let $B_j$ be the pseudo-differential operator whose symbol is $b_j(x, \xi)$. Since $b_j(x, \xi) \rightarrow a(x, \xi)$ as $j \rightarrow \infty$, the dominated convergence theorem implies that $B_j f(x) \rightarrow Tf(x)$ for all $x$. Another application of the dominated convergence theorem shows that for each cube $Q$ and $0 < \delta < 1$,

$$\left( \frac{1}{|Q|} \int_Q |B_j f(x)|^\delta - |(B_j f_2) Q|^\delta \right)^{1/\delta} \rightarrow \left( \frac{1}{|Q|} \int_Q |T f(x)|^\delta - |(T f_2) Q|^\delta \right)^{1/\delta}$$

and

$$\left( \frac{1}{\varphi(|Q|)^{\eta} |Q|} \int_Q |B_j f(y)|^\delta \, dy \right)^{1/\delta} \rightarrow \left( \frac{1}{\varphi(|Q|)^{\eta} |Q|} \int_Q |T f(y)|^\delta \, dy \right)^{1/\delta}.$$
if $|Q| < 1$, and
\[
\left( \frac{1}{\varphi(|Q|^{\eta}|Q|)} \int_{Q} |T f(y)^{\delta} dy \right)^{1/\delta} \leq CM_{\varphi, \eta} f(x_0),
\]
if $|Q| \geq 1$.

Thus (3.1) holds. Theorem 1.1 is proved.

By Proposition 2.3 and (3.1), the weighted weak-type $(1,1)$ estimate for the operator $T$ is obtain as follows:

**Theorem 3.1.** Let $T \in L_{1,0}^0$ and $\omega \in A_1(\varphi)$. There exists a constant $C > 0$ such that for any $\lambda > 0$,
\[
\omega\{x \in \mathbb{R}^n : |T f(x)| > \lambda\} \leq C \lambda \int_{\mathbb{R}^n} |f(x)| \omega(x) d\mu(x).
\]

**Proof of Corollary 1.1.** It is easy to see that $\omega_1(x) = (1 + |x|)^{\alpha_1}$ if $-n \alpha_0 < \gamma_1 < n \alpha_0 \in A_1(\varphi)$ and $\omega_2(x) = |x|^\alpha_2 \in A_1(\varphi)$ for $-n < \alpha_2 < n(p - 1)$. Then $\omega = w_2^\alpha w_1^{1-\alpha} \in A_p(\varphi)$ for $1 < p < \infty$ and $0 < \alpha < 1$ by (iii) of Lemma 2.1, by Theorem 1.1, we have
\[
\|T f\|_{L_p(\omega)} \leq C \|f\|_{L_p(\omega)}.
\]
Form this and by the arbitrary of $\alpha_0$, if $\omega(x) = (1 + |x|)^{\gamma_1} |x|^{-\gamma_2}$ with $\gamma_1 \in \mathbb{R}$ and $-n < \gamma_2 < n(1 - 1/p)$, we then have
\[
\|T f\|_{L_p(\omega)} \leq C \|f\|_{L_p(\omega)}.
\]
Thus, Corollary 1.1 is proved.

**4. The proof of Theorem 1.2**

In this section, our proof follows from [12] and [9], which is different from Section 3.

We first recall some basic definitions and facts about Orlicz spaces, referring to [10] for a complete account.

A function $B(t) : [0, \infty) \rightarrow [0, \infty)$ is called a Young function if it is continuous, convex, increasing and satisfies $\Phi(0) = 0$ and $B \rightarrow \infty$ as $t \rightarrow \infty$. If $B$ is a Young function, we define the $B$-average of a function $f$ over a cube $Q$ by means of the following Luxemburg norm:
\[
\|f\|_{B,Q} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_{Q} B \left( \frac{|f(y)|}{\lambda} \right) dy \leq 1 \right\}.
\]

The generalized Hölder’s inequality
\[
\frac{1}{|Q|} \int_{Q} |f g| dy \leq \|f\|_{B,Q} \|g\|_{B,Q}
\]
Weighted norm inequalities for pseudo-differential operators

holds, where $B$ is the complementary Young function associated to $B$. And we define the corresponding maximal function

$$M_B f(x) = \sup_{Q:x \in Q} \|f\|_{B,Q}$$

and for $0 < \eta < \infty$

$$M_{B,\varphi,\eta} f(x) = \sup_{Q:x \in Q} \varphi(|Q|)^{-\eta} \|f\|_{B,Q}.$$

The example that we are going to use is $B(t) = t(1 + \log^+ t)$ with the maximal function denoted by $M_{\log L}$. The complementary Young function is given by $\bar{B}(t) \approx e^t$ with the corresponding maximal function denoted by $M_{\exp L}$.

We need the following several key lemmas.

**Lemma 4.1.** Let $0 < \eta < \infty$ and $M_{\varphi,\eta/2} f$ be locally integral. Then there exists positive constants $C_1$ and $C_2$ independent of $f$ and $x$ such that

$$C_2 M_{\varphi,\eta} M_{\varphi,\eta/2} f(x) \leq M_{L_{\log L},\varphi,\eta} f(x) \leq C_1 M_{\varphi,\eta/2} M_{\varphi,\eta/2} f(x).$$

Proof: It suffices to show that for any cube $Q \ni x$, there is a constant $C > 0$ such that

$$\varphi(|Q|)^{-\eta} \|f\|_{L_{\log L},Q} \leq C \varphi(|Q|)^{-\eta/2} |Q|^{-1} \int_Q M_{\varphi,\eta/2} f(y) dy.$$

That is,

$$\|f\|_{L_{\log L},Q} \leq C \varphi(|Q|)^{\eta/2} |Q|^{-1} \int_Q M_{\varphi,\eta/2} f(y) dy. \quad (4.1)$$

In fact, by homogeneity we can take $f$ with $\|f\|_{L_{\log L},Q} = 1$ which implies

$$1 \leq \frac{C}{|Q|} \int_Q |f(y)|(1 + \log^+(|f(y)|)) dy$$

$$\leq \frac{C}{|Q|} \int_Q |f(y)| \int_1^{\log^+ (|f(y)|)+1} \frac{dt}{t} dy$$

$$\leq \frac{C}{|Q|} \int_1^\infty \int_{\{x \in Q: |f(y)| > t-1\}} |f\chi_Q(y)| dy \frac{dt}{t}$$

$$\leq \frac{C}{|Q|} \int_0^\infty \int_{\{x \in Q: |f(y)| > t\}} |f\chi_Q(y)| dy \frac{dt}{t}$$

$$\leq \frac{C}{|Q|} \int_0^\infty |\{x \in Q: M(f\chi_Q)(x) > t\}| dt$$

$$= \frac{C}{|Q|} \int_Q M(f\chi_Q)(x) dx$$

$$\leq C \frac{\varphi(|Q|)^{\eta/2}}{|Q|} \int_Q M_{\varphi,\eta/2}(f)(x) dx,$$
since $M(f \chi_Q)(x) \leq C \varphi(|Q|)^{\eta/2} M_{\varphi,\eta/2}(f)(x)$ for all $x \in Q$. Thus, (4.1) is proved.

Now let us turn to prove
\[ M_{\varphi,\eta} f(x) \leq C M_{L \log L, \varphi,\eta} f(x). \]  
(4.2)

For any fixed $x \in \mathbb{R}^n$ and any fixed cube $Q \ni x$, write $f = f_1 + f_2$, where $f_1 = f \chi_{3Q}$.

Thus,
\[
\varphi(|Q|)^{-\eta} |Q|^{-1} \int_Q |M_{\varphi,\eta} f(y)| dy \leq \varphi(|Q|)^{-\eta} |Q|^{-1} \int_Q |M_{\varphi,\eta} f_1(y)| dy \\
+ \varphi(|Q|)^{-\eta} |Q|^{-1} \int_Q |M_{\varphi,\eta} f_2(y)| dy \\
:= I + II.
\]

For $I$, we know that for all $g$ with $\text{supp} \ g \subset Q$ (see [9])
\[
\frac{1}{|Q|} \int_Q M g(y) dy \leq C \| g \|_{L \log L, Q}.
\]

From this and note that $M_{\varphi,\eta} f(x) \leq M f(x)$, we get
\[
I \leq C \varphi(|Q|)^{-\eta} |3Q|^{-1} \int_{3Q} |M f_1(y)| dy \\
\leq C \varphi(|Q|)^{-\eta} \| f \|_{L \log L, 3Q} \\
\leq CM_{L \log L, \varphi,\eta} f(x).
\]

Next let us estimate $II$. It is easy to see that for all $y, z \in Q$, we have
\[ M_{\varphi,\eta} f_2(y) \leq C M_{\varphi,\eta} f(z). \]  
(4.3)

In fact, for any cube $Q' \ni y$ and $Q' \cap (\mathbb{R}^n \setminus 3Q) \neq \emptyset$, noticing that $z \in Q \subset 3Q'$, we have
\[
\frac{1}{\varphi(|3Q'|)^{\eta} |Q'|} \int_{Q'} |f_2(y)| dy \leq C \frac{1}{\varphi(|3Q'|)^{\eta} |3Q'|} \int_{3Q'} |f_2(y)| dy \\
\leq C M_{\varphi,\eta} f(z).
\]

Hence, (4.3) holds.

Using (4.3), we obtain
\[
II \leq C \varphi(|Q|)^{-\eta} |Q|^{-1} \int_Q |M_{\varphi,\eta} f_2(y)| dy \\
\leq C |Q|^{-1} \int_Q \inf_{z \in Q} M_{\varphi,\eta} f(z) dz \\
\leq CM_{\varphi,\eta} f(x) \leq CM_{L \log L, \varphi,\eta} f(x).
\]

Thus, (4.2) is proved.
Lemma 4.2. Let $b \in BMO$ and $1 \leq \eta < \infty$. Let $0 < \delta < \epsilon < 1$, then
\[
M_{\delta, \varphi, \eta}^2([b, T]f)(x_0) \leq C\|b\|_{BMO}(M_{\epsilon, \varphi, \eta}(Tf)(x_0) + M_{L, \varphi, \eta}(f)(x_0)), \quad \text{a.e. } x_0 \in \mathbb{R}^n
\] holds for all $f \in C_0^\infty(\mathbb{R}^n)$.

Proof: Observe that for any constant $\lambda$
\[
[b, T]f(x) = (b(x) - \lambda)Tf(x) - T((b - \lambda)f)(x).
\]
As above we fix $x \in \mathbb{R}^n$ and let $x \in Q = Q(x_0, r)$. Decompose $f = f_1 + f_2$, where $f_1 = f_{\chi_{\bar{Q}}}$, where $\bar{Q} = Q(x, 8r)$. Let $\lambda$ be a constant and $C_Q$ a constant to be fixed along the proof.

To prove (4.2), we consider two cases about $r$.

Case 1. When $r < 1$. Since $0 < \delta < 1$, we then have
\[
\left( \frac{1}{|Q|} \int_Q |[b, T]f(y)|^\delta - |C_Q|^\delta dy \right)^{1/\delta} \\
\leq \left( \frac{1}{|Q|} \int_Q |[b, T]f(y) - C_Q|^\delta dy \right)^{1/\delta} \\
\leq \left( \frac{1}{|Q|} \int_Q |(b(y) - \lambda)Tf(y) - T((b - \lambda)f)(y) - C_Q|^\delta dy \right)^{1/\delta} \\
\leq C \left( \frac{1}{|Q|} \int_Q |(b(y) - \lambda)Tf(y)|^\delta dy \right)^{1/\delta} + C \left( \frac{1}{|Q|} \int_Q |T((b - \lambda)f_1)(y)|^\delta dy \right)^{1/\delta} + C \left( \frac{1}{|Q|} \int_Q |T((b - \lambda)f_2)(y) - C_Q|^\delta dy \right)^{1/\delta} \\
= I + II + III.
\]

To deal with $I$, we first fix $\lambda = b_{\bar{Q}}$, the average of $b$ on $\bar{Q}$. Then for any $1 < q < \epsilon/\delta$, by the John-Nirenberg inequality of $BMO$, we obtain that
\[
I \leq C \left( \frac{1}{|Q|} \int_Q |b(y) - b_{\bar{Q}}|^{\delta q'} dy \right)^{q'/\delta} \left( \frac{1}{|Q|} \int_Q |Tf(y)|^{\delta q} dy \right)^{\delta q} \\
\leq C\|b\|_{BMO} M_{\epsilon, \varphi, \eta}(Tf)(x_0),
\]
where $1/q' + 1/q = 1$.

For $II$, we recall that $T$ is weak type $(1, 1)$. By Kolmogorov’s inequality, we then have
\[
II \leq \frac{C}{|Q|} \|Tf_1\|_{L^{1, \infty}} \leq \frac{C}{|Q|} \int_Q |f(y)| dy \\
\leq CM_{\varphi, \eta}f(x_0).
\]
Finally, to estimate $III$, from page 241-250 in [12], we can express $T$ as

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y) f(y) dy,$$

and the kernel $K(x, y)$ satisfies

$$|K(x, y)| \leq C_N |x - y|^{-n-N} \text{ for } N \geq 0,$$

and

$$|\partial_x K(x, y)| \leq C_N |x - y|^{-n-1-N} \text{ for } N \geq 0.$$  \hfill (4.5)

Fixed the value of $C_Q$ by taking $C_Q = (T((b - b_Q) f_2))_Q$, the average of $T((b - b_Q) f_2)$ on $Q$. Let $b_{Q_k} = b_{Q(x_0, 2^{k+1}r)}$. Taking $k_0$ such that $1/2 < 2^{k_0}r \leq 1$. Then,

$$III \leq \frac{C}{|Q|} \int_Q |T((b - b_Q) f_2)(y) - (T((b - b_Q) f_2))_Q| dy$$

$$\leq \frac{C}{|Q|^2} \int_Q \int_Q \int_{\mathbb{R}^n} |K(y, \omega) - K(z, \omega)|| (b(\omega) - b_Q) f(\omega)| d\omega dz dy$$

$$\leq \frac{C}{|Q|^2} \int_Q \int_Q \int_{|x_0 - \omega| > 2r} |K(y, \omega) - K(z, \omega)|| (b(\omega) - b_Q) f(\omega)| d\omega dz dy$$

$$\leq \frac{C}{|Q|^2} \int_Q \int_Q \sum_{k=2}^{\infty} \int_{2^{k+1}r \leq |x_0 - \omega| < 2^{k+1}r} |K(y, \omega) - K(z, \omega)|| (b(\omega) - b_Q) f(\omega)| d\omega dz dy$$

$$\leq \frac{C}{|Q|^2} \int_Q \int_Q \sum_{k=2}^{k_0} \sum_{k=2}^{\infty} \int_{2^{k}r \leq |x_0 - \omega| < 2^{k+1}r}$$

$$\times |K(y, \omega) - K(z, \omega)|| (b(\omega) - b_Q) f(\omega)| d\omega dz dy$$

$$:= III_1 + III_2.$$  

Taking $N = 0$ in (4.6), we obtain that

$$III_1 \leq C \sum_{k=1}^{k_0} \frac{2^{-k}}{|Q_k|} \int_{Q_k} |b(\omega) - b_Q| f(\omega)| d\omega$$

$$\leq C \sum_{k=1}^{k_0} \frac{2^{-k}}{|Q_k|} \left[ \int_{Q_k} |b(\omega) - b_{Q_k}| f(\omega)| d\omega + |b_{Q_k} - b_Q| \int_{Q_k} |f(\omega)| d\omega \right]$$

$$\leq C \sum_{k=1}^{\infty} 2^{-k} \|b\|_{BMO_{L, \log L, \varphi, \eta}}(f(x_0)) + C \|b\|_{BMO_{L, \varphi, \eta}}(f(x_0)) \sum_{k=1}^{\infty} k2^{-k}$$

$$\leq C \|b\|_{BMO_{L, \log L, \varphi, \eta}}(f(x_0)),$$
in last inequality we have used that \( |b_Q - b_{Q_k}| \leq C k \|b\|_{BMO} \) and \( M_{\varphi, \eta}(f)(x) \leq M_{L, \log L, \varphi, \eta}(f)(x) \).

Taking \( N = n\eta_0 \) in (4.6), we get that

\[
III_2 \leq C \sum_{k=k_0+1}^{\infty} \frac{2^{-k}}{\varphi(|Q_k|)|Q_k|} \int_{Q_k} |b(\omega) - b_Q| f(\omega) |d\omega|
\]

\[
\leq C \sum_{k=k_0+1}^{\infty} \frac{2^{-k}}{\varphi(|Q_k|)|Q_k|} \left[ \int_{Q_k} |b(\omega) - b_Q| f(\omega) |d\omega| + |b_{Q_k} - b_Q| \int_{Q_k} |f(\omega)| |d\omega| \right]
\]

\[
\leq C \sum_{k=1}^{\infty} 2^{-k} \|b\|_{BMO} M_{L, \log L, \varphi, \eta}(f)(x) + C \|b\|_{BMO} M_{\varphi, \eta}(f)(x) \sum_{k=1}^{\infty} 2^{-k}
\]

\[
= C \|b\|_{BMO} M_{L, \log L, \varphi, \eta}(f)(x_0).
\]

Case 2. When \( r \geq 1 \). Since \( 0 < \delta < \epsilon < 1 \) and \( \eta \geq 1 \), let \( \rho_2 := \eta/\delta \geq \eta \), we then have

\[
\left( \frac{1}{\varphi(|Q|)|Q|} \int_Q |(b, T)f(y)|^\delta |dy| \right)^{1/\delta}
\]

\[
= \frac{1}{\varphi(|Q|)\rho_2} \left( \frac{1}{|Q|} \int_Q |(b(y) - \lambda)Tf(y) - T((b - \lambda)f)(y)|^\delta |dy| \right)^{1/\delta}
\]

\[
\leq C \frac{1}{\varphi(|Q|)\rho_2} \left( \frac{1}{|Q|} \int_Q |(b(y) - \lambda)Tf(y)|^\delta |dy| \right)^{1/\delta}
\]

\[
+ C \frac{1}{\varphi(|Q|)\rho_2} \left( \frac{1}{|Q|} \int_Q |T((b - \lambda)f_1)(y)|^\delta |dy| \right)^{1/\delta}
\]

\[
+ C \frac{1}{\varphi(|Q|)\rho_2} \left( \frac{1}{|Q|} \int_Q |T((b - \lambda)f_2)(y)|^\delta |dy| \right)^{1/\delta}
\]

\[
= I + II + III.
\]

To deal with \( I \), we first fix \( \lambda = b_Q \), the average of \( b \) on \( \tilde{Q} \). Then for any \( 1 < q < \epsilon/\delta \), by the John-Nirenberg inequality of \( BMO \), we then have

\[
I \leq C \left( \frac{1}{|Q|} \int_{\tilde{Q}} |b(y) - b_Q|^{5q'} |dy| \right)^{1/(5q')} \frac{1}{\varphi(|Q|)\rho_2} \left( \frac{1}{|Q|} \int_Q |Tf(y)|^{q} |dy| \right)^{1/(q)}
\]

\[
\leq C \|b\|_{BMO} M_{\varphi, \eta}(Tf)(x_0),
\]

where \( 1/q' + 1/q = 1 \).

For \( II \), since \( T \) is weak type \((1,1)\). By Kolmogorov’s inequality, we then have

\[
II \leq C \frac{1}{\varphi(|Q|)\rho_2} \frac{1}{|Q|} \left\| Tf_1 \right\|_{1,\infty}
\]

\[
\leq C \frac{1}{\varphi(|Q|)\rho_2} \frac{1}{|Q|} \int_Q |f(y)| |dy| \leq CM_{\varphi, \eta} f(x_0).
\]
Finally, for III we first fix the value of $C_Q$ by taking $C_Q = (T((b - b_Q)f_2))_Q$, the average of $T((b - b_Q)f_2)$ on $B$. Let $b_{Q_k} = b_{Q(x_0,2^{k+1}r)}$. Taking $N = n\eta\alpha_0 + 1$ in (4.5), note that $r \geq 1$, we obtain that

$$
II \leq \frac{C}{|Q|} \int_Q |T((b - b_Q)f_2)(y)| \, dy
$$

$$
\leq \frac{C}{|Q|^2} \int_Q \int_{\mathbb{R}^n \setminus \bar{Q}} |K(y, \omega)|(b(\omega) - b_Q)f(\omega)|d\omega dz dy
$$

$$
\leq C \sum_{k=1}^{\infty} 2^{-k}r^{-1} \|b\|_{BMO \log L, \varphi, \eta}(f)(x_0) + C \|b\|_{BMO \varphi, \eta}(f)(x) \sum_{k=1}^{\infty} k2^{-k}r^{-1}
$$

$$
\leq C \|b\|_{BMO \log L, \varphi, \eta}(f)(x_0).
$$

From these, we get (4.4). Hence the proof is finished.

**Lemma 4.3.** Let $\omega \in A_1(\varphi)$ and $\eta \geq 2$. There exists a positive $C$ such that for any function $f$ and $\lambda > 0$

$$
\omega(\{x \in \mathbb{R}^n : M_{L, \log L, \varphi, \eta}(f)(x) > \lambda\}) \leq C \int_{\mathbb{R}^n} \Phi(|f(y)|/\lambda)\omega(y)dy. \quad (4.5)
$$

**Proof:** Let $K$ be any compact subset in $\{x \in \mathbb{R}^n : M_{L, \log L, \varphi, \eta}(f)(x) > \lambda\}$. For any $x \in K$, by a standard covering lemma, it is possible to choose cubes $Q_1, \ldots, Q_m$ with pairwise disjoint interiors such that $K \subset \bigcup_{j=1}^{m} 3Q_j$ and with $\|f\|_{L, \log L, \varphi, Q_j} > \lambda$, $j = 1, \ldots, m$. This implies

$$
\varphi(|Q_j|^2|Q_j|) \leq \int_{Q_j} \frac{|f(y)|}{\lambda} \left(1 + \log^+ \left(\frac{|f(y)|}{\lambda}\right)\right) dy.
$$

From this, by (vi) in Lemma 2.1 with $p = 1$ and $E = Q$, we obtain that

$$
\omega(3Q_j) \leq C \varphi(|Q_j|)\omega(Q_j)
$$

$$
= C \varphi(|Q_j|^2|Q_j|) \frac{\omega(Q_j)}{\varphi(|Q_j|)}
$$

$$
\leq C \frac{\omega(Q_j)}{\varphi(|Q_j|)} \int_{Q_j} \frac{|f(y)|}{\lambda} \left(1 + \log^+ \left(\frac{|f(y)|}{\lambda}\right)\right) dy
$$

$$
\leq C \inf_{Q_j} \omega(x) \int_{Q_j} \frac{|f(y)|}{\lambda} \left(1 + \log^+ \left(\frac{|f(y)|}{\lambda}\right)\right) dy
$$

$$
\leq C \int_{Q_j} \frac{|f(y)|}{\lambda} \left(1 + \log^+ \left(\frac{|f(y)|}{\lambda}\right)\right) \omega(y) dy.
$$

Thus, (4.5) holds, hence, the proof is complete.

Using Proposition 2.3 and Lemmas 4.1, 4.2 and 4.3, adapting the same argument of [9], we can prove the following result.
Lemma 4.4. Let $b \in BMO, \eta > 2$ and $\omega \in A_1(\varphi)$. Then there exists a positive constant $C$ such that for any smooth function $f$ with compact support
\[
\sup_{t>0} \frac{1}{\Phi(1/t)} \omega(\{x \in \mathbb{R}^n : |[T, b]f(x)| > t\}) \leq C \Phi(\|b\|_{BMO}) \sup_{t>0} \frac{1}{\Phi(1/t)} \omega(\{x \in \mathbb{R}^n : M_{L\log L, \varphi, \eta} f(x) > t\}).
\]

Let us turn to prove the main theorems in this section.

Proof of Theorem 1.2: By Lemmas 4.1, 4.2 and Theorem 1.1, we have
\[
\|[T, b]\|_{L^p(\omega)} \leq C \|M_{\varphi, \eta}(Tf)\|_{L^p(\omega)} + C \|M_{L\log L, \varphi, \eta}(f)\|_{L^p(\omega)} \leq C \|Tf\|_{L^p(\omega)} + C \|M_{\varphi, \eta/2}M_{\varphi, \eta/2}f\|_{L^p(\omega)} \leq C \|f\|_{L^p(\omega)},
\]

taking $\eta = 2p'$.

The weighted weak-type $(1,1)$ estimate for the commutator is the following.

Theorem 4.1. Let $b \in BMO$ and $\omega \in A_1(\varphi)$. There exists a constant $C > 0$ such that for any $\lambda > 0$,
\[
\omega(\{x \in \mathbb{R}^n : |[T, b]f(x)| > \lambda\}) \leq C \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \left(1 + \log^+ \left(\frac{|f(x)|}{\lambda}\right)\right) \omega(x)dx.
\]

Proof: Let $\Phi(t) = t \log(e + t)$. By homogeneity, we need only to show that
\[
\omega(\{x \in \mathbb{R}^n : |[T, b]f(x)| > 1\}) \leq C \left(\int_{\mathbb{R}^n} \Phi(|f(x)|)\omega(x)dx\right).
\]

Indeed, using Lemmas 4.3 and 4.4 and adapting the same argument of [9], we can obtain
\[
\omega(\{x \in \mathbb{R}^n : |[T, b]f(x)| > 1\}) \leq C \sup_{t>0} \frac{1}{\Phi(1/t)} \omega(\{x \in \mathbb{R}^n : |[T, b]f(x)| > t\}) \leq C \Phi(\|b\|_{BMO}) \sup_{t>0} \frac{1}{\Phi(1/t)} \omega(\{x \in \mathbb{R}^n : M_{L\log L, \varphi, \eta} f(x) > t\}) \leq C \int_{\mathbb{R}^n} \Phi(|f(x)|)\omega(x)dx,
\]

taking $\eta \geq 2$.

Thus, the proof of Theorem 4.1 is complete.

Similar to the proof of Corollary 1.1, we can obtain Corollary 1.2. We omit the details here.
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