On the Spectral Decomposition of Dichotomous and Bisectorial Operators

Monika Winklmeier, Christian Wyss

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Abstract
For an unbounded operator \( S \) on a Banach space the existence of invariant subspaces corresponding to its spectrum in the left and right half-plane is proved. The general assumption on \( S \) is the uniform boundedness of the resolvent along the imaginary axis. The projections associated with the invariant subspaces are bounded if \( S \) is strictly dichotomous, but may be unbounded in general. Explicit formulas for these projections in terms of resolvent integrals are derived and used to obtain perturbation theorems for dichotomy. All results apply, with certain simplifications, to bisectorial operators.

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1 Introduction
Let \( S \) be a densely defined linear operator \( S \) on a Banach space \( X \) such that the imaginary axis belongs to the resolvent set \( \varrho(S) \). A fundamental question is whether there exist closed invariant subspaces \( X_+ \) and \( X_- \) which correspond to the spectrum of \( S \) in the right and left half-plane \( \mathbb{C}_+ \) and \( \mathbb{C}_- \), respectively. \( S \) is called dichotomous if these subspaces exist and yield a decomposition \( X = X_+ \oplus X_- \). If in addition the restrictions \( -S|X_+ \) and \( S|X_- \) generate exponentially decaying semigroups, then \( S \) is exponentially dichotomous. Dichotomy and exponential dichotomy have found a wide range of applications, e.g. to canonical factorisation and Wiener-Hopf integral operators [BGK86a, BGK86b], and to block operator matrices and Riccati equations

\*Departamento de Matemáticas, Universidad de los Andes, Cra. 1a No 18A-70, Bogotá, Colombia. mwinklme@uniandes.edu.co.
\†Fachgruppe Mathematik und Informatik, Bergische Universität Wuppertal, Gaußstr. 20, 42097 Wuppertal, Germany. wyss@math.uni-wuppertal.de.
An extensive account may be found in the monograph [vdM08].

The investigation of dichotomous operators was started by H. Bart, I. Gohberg and M.A. Kaashoek in [BGK86b], where they established a sufficient condition for dichotomy: If a strip around the imaginary axis belongs to $\rho(S)$, the resolvent $(S - \lambda)^{-1}$ is uniformly bounded on this strip, i.e.,

$$\sup_{|\text{Re}\lambda| \leq h} \|(S - \lambda)^{-1}\| < \infty$$

for some $h > 0$, and the integral

$$P x = \frac{1}{2\pi i} \int_{h-i\infty}^{h+i\infty} \frac{1}{\lambda^2} (S - \lambda)^{-1} S^2 x d\lambda, \quad x \in \mathcal{D}(S^2),$$

extends to a bounded operator $P$ on $X$, then $S$ is dichotomous and $P$ is the projection onto $X_+ \oplus X_-$. On the other hand, there are simple examples where (1) holds, $X_\pm$ exist, but $X_+ \oplus X_- \subset X$ is only dense and $S$ is not dichotomous.

In our first main result, Theorem 4.1, we prove that (1) is in fact sufficient for the existence of invariant subspaces, even if $S$ is not dichotomous: If (1) holds, then the subspaces

$$G_\pm = \{ x \in X : (S - \lambda)^{-1} x \text{ has a bounded analytic extension to } \overline{\mathbb{C}_\pm} \}$$

are closed, $S$-invariant and satisfy $\sigma(S|G_\pm) \subset \mathbb{C}_\pm$. Moreover, the integral (2) extends to a closed, possibly unbounded operator $P$, which is the projection onto $G_+ \oplus G_-$. Finally, $P$ is bounded if and only if $S$ is dichotomous with respect to $X = G_+ \oplus G_-$. This decomposition has the additional property that the resolvents of $S|G_\pm$ are uniformly bounded on $\mathbb{C}_\mp$, and we call $S$ strictly dichotomous in this case.

One important class of operators satisfying (1) are bisectorial and almost bisectorial operators, for which $\mathbb{i} \mathbb{R} \subset \rho(S)$ and

$$\|(S - \lambda)^{-1}\| \leq \frac{M}{|\lambda|^{\beta}}, \quad \lambda \in \mathbb{i} \mathbb{R} \setminus \{0\},$$

with some $M > 0$ and $\beta = 1$ in the bisectorial, $0 < \beta < 1$ in the almost bisectorial case. In Theorem 5.6 we show that for bisectorial and almost bisectorial $S$ equation (2) simplifies to

$$P x = \frac{1}{2\pi i} \int_{h-i\infty}^{h+i\infty} \frac{1}{\lambda} (S - \lambda)^{-1} S x d\lambda, \quad x \in \mathcal{D}(S),$$

and that the restrictions $S|G_\pm$ are (almost) sectorial, i.e., they satisfy the resolvent estimate (3) on $\mathbb{C}_\mp$. The results for bisectorial $S$ were to some extent obtained by W. Arendt and A. Zamboni in [AZ10]. In particular, they constructed closed projections by using a rearranged version of (4); our construction of $P$ in
Theorem 4.1 is in fact a generalisation of their method to the weaker setting of (1). Note that bisectoriality does not imply dichotomy and hence unbounded $P$ are still possible here as Example 5.8 shows.

In addition to the characterisation of dichotomy in Theorems 4.1 and 5.6, we also derive perturbation results. Theorem 7.1 states that if $S$ is strictly dichotomous and $T$ is another densely defined operator on $X$ such that

(i) $\{ \lambda \in \mathbb{C} : |\text{Re}\lambda| \leq h \} \subset \varrho(S) \cap \varrho(T)$,

(ii) $\sup_{|\text{Re}\lambda| \leq h} |\lambda|^{1+\varepsilon} \|(S - \lambda)^{-1} - (T - \lambda)^{-1}\| < \infty$ for some $\varepsilon > 0$, and

(iii) $\mathcal{D}(S^2) \cap \mathcal{D}(T^2) \subset X$ is dense,

then $T$ is strictly dichotomous too. A similar result was obtained in [BGK86b], but under significantly stronger conditions, namely $\varepsilon = 1$ in (ii) and $\mathcal{D}(T^2) \subset \mathcal{D}(S^2)$ instead of (iii). It is precisely the existence of the closed projections which allows us to use the more general condition (iii). If in addition $S$ is (almost) bisectorial, then $T$ is (almost) bisectorial too and condition (iii) may be relaxed to

(iii') $\mathcal{D}(S) \cap \mathcal{D}(T) \subset X$ is dense,

see Theorem 7.3. In [TW14] this result was obtained for bisectorial $S$ and $T = S + R$ where $R$ is $p$-subordinate to $S$, i.e., $\mathcal{D}(S) \subset \mathcal{D}(R)$ and $\|R x\| \leq c\|x\|^{1-p} \|S x\|^p$, $x \in \mathcal{D}(S)$, with constants $c > 0$ and $0 \leq p < 1$. In this case (ii) holds with $\varepsilon = 1 - p$ and (iii') is trivially satisfied since $\mathcal{D}(S) = \mathcal{D}(T)$.

On the other hand, (iii) and (iii') allow for a wider class of perturbations with $\mathcal{D}(S) \neq \mathcal{D}(T)$, e.g. if $T = S + R$ holds only in an extrapolation space, see Example 8.8.

Most of our results remain valid for non-densely defined $S$. In particular, this is true for the main Theorems 4.1 and 5.6, but not for the perturbation results. Unless explicitly stated otherwise, linear operators are are not assumed to be densely defined.

There exist different approaches to dichotomy: For bisectorial $S$ an equivalent condition for dichotomy in terms of complex powers of $S$ is given in [DV89]. For generators of $C_0$-semigroups, exponential dichotomy is equivalent to the hyperbolicity of the semigroup [KVL94], the latter being important e.g. in the study of nonlinear evolution equations. Finally, our approach is connected to (bounded as well as unbounded) functional calculus.

This article is organised as follows: In Section 2 we collect general facts about unbounded projections, in particular Lemma 2.3 on the existence of closed projections corresponding to invariant subspaces. Section 3 contains the definition and basic properties of dichotomous operators. Here we also show that dichotomy alone does not uniquely determine the decomposition $X = X_+ \oplus X_-$ while strict dichotomy does. In Section 4 we derive our main Theorem 4.1 in the general setting (1). The case of bisectorial and almost bisectorial operators is then considered in Section 5. For such operators we also provide some results on the location of their spectrum and derive yet another integral representation for
Section 6 is devoted to the subtle problem that the restrictions \( S|G_\pm \) are not necessarily densely defined even if \( S \) is. Following [BGK86b] we consider certain subspaces \( M_\pm \subset G_\pm \) for which the parts of \( S \) in \( M_\pm \) are densely defined. We derive conditions for \( M_\pm = G_\pm \), which is in turn equivalent to \( S|G_\pm \) being densely defined. The perturbation results are contained in Section 7, and in Section 8 we provide some additional examples to illustrate our theory. Finally, as an application, we show the dichotomy of a Hamiltonian operator matrix from control theory whose off-diagonal entries map into extrapolation spaces. In the previous results [LRvdR02, WJZ12, TW14] only settings without extrapolation spaces or with the additional assumption of a Riesz basis of eigenvectors could be handled.

We use the following notation: The open right and open left half-plane is denoted by \( \mathbb{C}_+ \) and \( \mathbb{C}_- \), respectively. If \( X, Y \) are Banach spaces, then \( T(X \rightarrow Y) \) denotes a linear operator from a (not necessarily dense) domain in \( X \) into \( Y \). If \( Z \subset X \), then we denote the restriction of \( T \) to \( Z \) by \( T|Z \). The set of all bounded linear operators from \( X \) to \( Y \) is denoted by \( L(X, Y) \) and we set \( L(X) = L(X, X) \).

2 Unbounded projections

Our main tool for investigating the dichotomy of an operator \( S \) are projections corresponding to invariant subspaces. In the general case, these projections will be unbounded and the direct sum of the corresponding subspaces is not the whole space \( X \). Unbounded projections associated with bisectorial operators have been studied in [AZ10].

**Definition 2.1.** Let \( X \) be a Banach space. A (possibly unbounded) operator \( P(X \rightarrow X) \) is called a *projection* if \( \text{Im}(P) \subset \mathcal{D}(P) \) and \( P^2 = P \).

In other words, \( P \) is a projection in the algebraic sense on the vector space \( \mathcal{D}(P) \). If \( P \) is a projection, then

\[ \mathcal{D}(P) = \text{Im}(P) \oplus \ker(P). \]  

(5)

The *complementary projection* is given by \( Q = I - P \), \( \mathcal{D}(Q) = \mathcal{D}(P) \). In this case \( \text{Im}(Q) = \ker(P) \), \( \ker(Q) = \text{Im}(P) \).

Conversely, if \( X_1, X_2 \subset X \) are linear subspaces such that \( X_1 \cap X_2 = \{0\} \), then there are corresponding complementary projections \( P_1, P_2 \) with \( \mathcal{D}(P_1) = \mathcal{D}(P_2) = X_1 \oplus X_2 \), \( \text{Im}(P_1) = X_1 \) and \( \text{Im}(P_2) = X_2 \).

**Remark 2.2.**

(i) A projection \( P \) is closed if and only if \( \text{Im}(P) \) and \( \ker(P) \) are closed subspaces.

(ii) A closed projection \( P \) is bounded if and only if \( \text{Im}(P) \oplus \ker(P) \) is closed.
The next lemma gives a sufficient condition on a linear operator $S$ that allows for the construction of a pair of closed complementary projections which commute with $S$ and yield $S$-invariant subspaces.

**Lemma 2.3.** Let $S(X \to X)$ be a closed operator with $0 \in \varrho(S)$. Suppose that there are bounded operators $A_1, A_2 \in L(X)$ satisfying

\begin{align*}
A_1 + A_2 &= S^{-2}, \\
A_1 A_2 &= A_2 A_1 = 0, \\
A_j (S - \lambda)^{-1} &= (S - \lambda)^{-1} A_j, \quad \lambda \in \varrho(S), \; j = 1, 2.
\end{align*}

Then

(i) $P_j := S^2 A_j, \; j = 1, 2,$ are closed complementary projections, $\mathcal{D}(S^2) \subset \mathcal{D}(P_1) = \mathcal{D}(P_2)$ and $P_j x = A_j S^2 x$ for $x \in \mathcal{D}(S^2)$;

(ii) the subspaces $X_j = \text{Im}(P_j)$ are closed, $S$- and $(S - \lambda)^{-1}$-invariant for $\lambda \in \varrho(S)$ and satisfy $X_1 = \ker(A_2), \; X_2 = \ker(A_1)$. Moreover

\[ \sigma(S) = \sigma(S|X_1) \cup \sigma(S|X_2). \]

**Proof.** (i) Since $S^2$ is closed and $A_j$ is bounded, $P_j$ is closed. We have $x \in \mathcal{D}(P_j)$ if and only if $A_j x \in D(S^2)$. Hence (6) implies that $\mathcal{D}(P_1) = \mathcal{D}(P_2)$ and that $P_1 + P_2 = I$ on $\mathcal{D}(P_1)$. From (8) it follows that

\[ A_j S x = S A_j x \quad \text{for} \quad x \in \mathcal{D}(S), \; j = 1, 2. \]  

For $x \in \mathcal{D}(P_1)$, this implies $A_2 P_1 x = A_2 S^2 A_1 x = S^2 A_2 A_1 x = 0$ and hence $\text{Im}(P_1) \subset \mathcal{D}(P_2), \; P_2 P_1 = 0$ and $P_1^2 = (I - P_2) P_1 = P_1$. Thus $P_1$ and $P_2$ are complementary projections. Additionally, (10) yields that if $x \in \mathcal{D}(S^2)$, then $x \in \mathcal{D}(P_1)$ and $A_j S^2 x = S^2 A_1 x = P_j x$.

(ii) Since $S^2$ is invertible, $X_1 = \ker(P_2) = \ker(A_2)$ and analogously $X_2 = \ker(A_1)$. Consequently these subspaces are closed and (8) and (10) imply that they are also $S$- and $(S - \lambda)^{-1}$-invariant. To obtain (9), we show now the equivalent identity

\[ \varrho(S) = \varrho(S|X_1) \cap \varrho(S|X_2). \]

From the invariance of $X_j$ it is easily seen that for $\lambda \in \varrho(S)$ the operator $S|X_j - \lambda$ is bijective with inverse $(S - \lambda)^{-1}|X_j$; hence $\lambda \in \varrho(S|X_j)$. For the other inclusion let $\lambda \in \varrho(S|X_1) \cap \varrho(S|X_2)$. If $(S - \lambda) x = 0$, then $x \in \mathcal{D}(S^2) \subset \mathcal{D}(P_j)$ and therefore

\[ 0 = (S - \lambda)x = (S|X_1 - \lambda) P_1 x + (S|X_2 - \lambda) P_2 x. \]

This yields $P_1 x = P_2 x = 0$, i.e. $x = 0$. Hence $S - \lambda$ is injective. To show surjectivity, set

\[ T = (S|X_1 - \lambda)^{-1} A_1 + (S|X_2 - \lambda)^{-1} A_2. \]

Then $(S - \lambda) T = A_1 + A_2 = S^{-2}$ from which we conclude that $\text{Im}(T) \subset \mathcal{D}(S^2)$ and $(S - \lambda) S^2 T = I$. \qed
The following example illustrates a typical situation in which the projections from Lemma 2.3 are unbounded. It also shows that for fixed $S$ there may be several possible choices for $A_1, A_2$.

**Example 2.4.** Let $S$ be the block diagonal operator on the sequence space $X = l^2$ given by

$$ S = \text{diag}(S_1, S_2, \ldots), \quad S_n = \begin{pmatrix} n & 2n^2 \\ 0 & -n \end{pmatrix}. $$

First observe that

$$ (S_n - \lambda)^{-1} = \begin{pmatrix} (n - \lambda)^{-1} & 2n^2(n - \lambda)^{-1}(n + \lambda)^{-1} \\ 0 & -(n + \lambda)^{-1} \end{pmatrix}, \quad \lambda \neq \pm n. $$

Then $\sup_n \|(S_n - \lambda)^{-1}\| < \infty$ for $\lambda \not\in \mathbb{Z} \setminus \{0\}$. Hence $\sigma(S) = \mathbb{Z} \setminus \{0\}$ and $(S - \lambda)^{-1} = \text{diag}((S_1 - \lambda)^{-1}, (S_2 - \lambda)^{-1}, \ldots), \lambda \in \sigma(S)$. The spectral projections $P_n^+, P_n^-$ corresponding to the eigenvalues $n$ and $-n$ of $S_n$ are

$$ P_n^+ = \begin{pmatrix} 1 & n \\ 0 & 0 \end{pmatrix}, \quad P_n^- = \begin{pmatrix} 0 & -n \\ 0 & 1 \end{pmatrix}. $$

Moreover

$$ S_n^{-1} = \begin{pmatrix} n^{-1} & 2 \\ 0 & -n^{-1} \end{pmatrix}, \quad S_n^{-2} = \begin{pmatrix} n^{-2} & 0 \\ 0 & n^{-2} \end{pmatrix}. $$

Let $A_n^\pm := S_n^{-2}P_n^\pm$. Then

$$ A_n^+ = \begin{pmatrix} n^{-2} & n^{-1} \\ 0 & 0 \end{pmatrix}, \quad A_n^- = \begin{pmatrix} 0 & -n^{-1} \\ 0 & n^{-2} \end{pmatrix}, $$

and we have $A_n^+ + A_n^- = S_n^{-2}, A_n^+A_n^- = A_n^-A_n^+ = 0$ and $A_n^\pm (S_n - \lambda)^{-1} = (S_n - \lambda)^{-1}A_n^\pm$ for $\lambda \not\in \mathbb{Z} \setminus \{0\}$.

Now we define an operator $A_1 \in L(X)$ by choosing for each $n$ either $A_n^+$ or $A_n^-$; for $A_2 \in L(X)$ we take the complementary choice. More explicitly let $\Lambda_1 \subset \mathbb{N}$ and $\Lambda_2 = \mathbb{N} \setminus \Lambda_1$. Set $\epsilon_n^+ = +$ if $n \in \Lambda_j$ and $\epsilon_n^j = -$ if $n \not\in \Lambda_j$. Then the operators

$$ A_j = \text{diag}(A_j^{1^j}, A_j^{2^j}, A_j^{3^j}, \ldots), \quad j = 1, 2, $$

satisfy all conditions in Lemma 2.3. The closed projections $P_j = S^2 A_j$ are unbounded and block diagonal with $P_j = \text{diag}(P_j^{1^j}, P_j^{2^j}, P_j^{3^j}, \ldots)$. By Lemma 2.3, the subspaces $X_j = \text{Im}(P_j)$ are $S$-invariant. Clearly $X_j$ is the closed linear hull of the eigenvectors of $S$ for the eigenvalues $n \in \Lambda_j$ and the eigenvalues $-n$ for $n \not\in \Lambda_j$. Note that $X_1 \oplus X_2 \neq X$. We will investigate this example further in Example 4.5.

**Remark 2.5.** Lemma 2.3 continues to hold if $S^2$ and $S^{-2}$ are replaced throughout by $S$ and $S^{-1}$. That is, if there exist $B_1, B_2 \in L(X)$ such that

$$ B_1 + B_2 = S^{-1}, \quad B_1B_2 = B_2B_1 = 0, $$

$$ B_j(S - \lambda)^{-1} = (S - \lambda)^{-1}B_j, \quad \lambda \in \sigma(S), \quad j = 1, 2, $$
then we obtain closed complementary projections $P_j = SB_j$ where, in particular, $\mathcal{D}(S) \subset \mathcal{D}(P_j)$, $X_j = \text{Im}(P_j)$ are $S$- and $(S - \lambda)^{-1}$-invariant and (9) holds.

3 Dichotomous operators

**Definition 3.1.** Let $X$ be a Banach space and let $X_\pm$ be closed subspaces with $X = X_+ \oplus X_-$. An operator $S(X \to X)$ is called **dichotomous** with respect to the decomposition $X = X_+ \oplus X_-$ if

(i) $i\mathbb{R} \subset \sigma(S)$,
(ii) $X_+$ and $X_-$ are $S$-invariant,
(iii) $\sigma(S|_{X_+}) \subset \mathbb{C}_+$, $\sigma(S|_{X_-}) \subset \mathbb{C}_-$. 

An operator $S$ is called **strictly dichotomous** with respect to $X = X_+ \oplus X_-$ if, in addition,

(iv) $\| (S|_{X_+} - \lambda)^{-1} \|$ is bounded on $\mathbb{C}_+$, $\| (S|_{X_-} - \lambda)^{-1} \|$ is bounded on $\mathbb{C}_-$.

A dichotomous operator $S$ is called **exponentially dichotomous** if $-S|_{X_+}$ and $S|_{X_-}$ generate exponentially decaying semigroups.

Clearly, exponential dichotomy implies strict dichotomy. Note that all dichotomous operators appearing in [LT01, LRvdR02, TW14] are in fact strictly dichotomous, see Remark 4.4.

**Remark 3.2.** For a given operator $S$, there may exist several decompositions of $X$ with respect to which it is dichotomous (Example 3.3), but there exists at most one with respect to which it is strictly dichotomous (Lemma 3.7).

**Example 3.3.** Let $X$ be a Banach space and let $S(X \to X)$ be any operator satisfying $\sigma(S) = \emptyset$. Then evidently $S$ is dichotomous with respect to the two decompositions

$$X_+ = X, \quad X_- = \{0\} \tag{12}$$

and

$$X_+ = \{0\}, \quad X_- = X. \tag{13}$$

Examples for such operators are generators of nilpotent contraction semigroups, e.g. shift semigroups on bounded intervals. In this case, the resolvent $(S - \lambda)^{-1}$ is uniformly bounded on $\mathbb{C}_+$ and thus $S$ is strictly dichotomous with respect to the decomposition (13) but not with respect to (12).

To obtain an example of a dichotomous operator with non-empty spectrum, we consider an operator given as the direct sum $S = S_0 \oplus S_1 \oplus S_2$ on $X = X_0 \oplus X_1 \oplus X_2$ where the $S_j$ are linear operators on $X_j$ such that

$$\sigma(S_0) = \emptyset, \quad \sigma(S_1) \subset \{ \lambda \in \mathbb{C} : \text{Re} \lambda \geq h \}, \quad \sigma(S_2) \subset \{ \lambda \in \mathbb{C} : \text{Re} \lambda \leq -h \}$$
for some \( h > 0 \). Then \( S \) is dichotomous with respect to the decompositions

\[
X_+ = X_1 \oplus X_0, \quad X_- = X_2
\]  
(14)

and

\[
X_+ = X_1, \quad X_- = X_2 \oplus X_0.
\]  
(15)

**Remark 3.4.** Even in the case of bisectorial operators, dichotomy is not sufficient to determine the subspaces \( X_\pm \) uniquely, see Section 8.1.

**Remark 3.5.** Let \( S \) be a dichotomous operator with respect to \( X = X_+ \oplus X_- \).

(i) If \( x \) is a (generalised) eigenvector of \( S \) with eigenvalue \( \lambda \in \mathbb{C}_\pm \), then \( x \in X_\pm \).

(ii) Suppose that \( S \) has a complete system of generalised eigenvectors. Then the spaces \( X_\pm \) are uniquely determined by \( S \) as the closures of the span of generalised eigenvectors of \( S \) whose eigenvalues belong to \( \mathbb{C}_\pm \).

**Lemma 3.6.** (i) If \( S \) is dichotomous with respect to \( X = X_+ \oplus X_- \), then \( D(S) = (D(S) \cap X_+) \oplus (D(S) \cap X_-) \), and \( S \) admits the block matrix representation

\[
S = \begin{pmatrix}
S|_{X_+} & 0 \\
0 & S|_{X_-}
\end{pmatrix}
\]

with respect to \( X = X_+ \oplus X_- \). In particular, \( \sigma(S) = \sigma(S|_{X_+}) \cup \sigma(S|_{X_-}) \) and the subspaces \( X_\pm \) are \((S - \lambda)^{-1}\)-invariant for \( \lambda \in \varrho(S) \).

(ii) If \( S \) is strictly dichotomous, then there exist \( h > 0 \) and \( M > 0 \) such that \( \{ \lambda \in \mathbb{C} : |\Re \lambda| \leq h \} \subset \varrho(S) \) and

\[
M = \sup_{|\Re \lambda| \leq h} \| (S - \lambda)^{-1} \| < \infty.
\]  
(16)

**Proof.** (i) is proved as in [TW14, Lemma 2.4]. If \( S \) is strictly dichotomous, then a Neumann series argument implies that there exist \( h > 0 \) and \( M > 0 \) such that \( \lambda \in \varrho(S|_{X_\pm}) \) and \( \| (S|_{X_\pm} - \lambda)^{-1} \| \leq M \) whenever \( |\Re \lambda| \leq h \). (ii) then follows from the block matrix decomposition in (i).

We will now establish the uniqueness of the decomposition \( X = X_+ \oplus X_- \) of a strictly dichotomous operator. To this end, let \( S(X \to X) \) with \( i\mathbb{R} \subset \varrho(S) \). We consider the two subspaces \( G_+ \) and \( G_- \) defined by

\[
G_\pm = \{ x \in X : (S - \lambda)^{-1}x \text{ has a bounded analytic extension to } \mathbb{C}_\pm \}. \]  
(17)

More explicitly, for \( x \in X \) we consider the analytic function

\[
\varphi_x : \varrho(S) \to X, \quad \varphi_x(\lambda) = (S - \lambda)^{-1}x.
\]

Then \( x \in G_+ \) if and only if \( \varphi_x \) admits a bounded analytic extension to the closed left half-plane

\[
\varphi_x^+ : \mathbb{C}^- \cup \varrho(S) \to X.
\]
Analogously, $x \in G_-$ if and only if $\varphi_x$ admits a bounded analytic extension to the closed right half-plane
\[
\varphi_-^- : \overline{C_+} \cup g(S) \to X.
\]

**Lemma 3.7.** Let $S(X \to X)$ with $i\mathbb{R} \subset g(S)$. Then:

(i) $G_+ \cap G_- = \{0\}$.

(ii) If $S$ is strictly dichotomous for the decomposition $X = X_+ \oplus X_-$, then $X_\pm = G_\pm$. In particular, $X_\pm$ are uniquely determined.

**Proof.** (i): Let $x \in G_+ \cap G_-$. Then $\varphi_x(\lambda) = (S - \lambda)^{-1}x$ can be extended to a bounded analytic function on $\mathbb{C}$ which must be constant by Liouville’s theorem. Hence $(S - \lambda_1)^{-1}x = (S - \lambda_2)^{-1}x$ for all $\lambda_1, \lambda_2 \in g(S)$ which is only the case if $g(S) = \emptyset$ or $x = 0$.

(ii): For $x \in X_+$, strict dichotomy of $S$ implies that $\varphi_x^+(\lambda) = (S|X_+ - \lambda)^{-1}x$ is a bounded analytic extension of $(S - \lambda)^{-1}x$ to $\overline{C_+}$. Hence $x \in G_+$, i.e. $X_+ \subset G_+$; similarly $X_- \subset G_-$. From $X = X_+ \oplus X_-$ and $G_+ \cap G_- = \{0\}$ we thus obtain $X_\pm = G_\pm$. \hfill $\Box$

**Remark 3.8.** (i) The subspaces $G_\pm$ have been introduced in [BGK86b], but with the roles of $G_+$ and $G_-$ exchanged and the additional conditions $\varphi_\pm^x(\lambda) \in \mathcal{D}(S)$ and $(S - \lambda)\varphi_\pm^x(\lambda) = x$. In the proof of Theorem 4.1 we will show that if in addition to $i\mathbb{R} \subset g(S)$ an estimate (16) holds, then $\varphi_\pm^x(\lambda) = (S|G_\pm - \lambda)^{-1}x$ and hence both conditions are automatically fulfilled.

(ii) The estimate (16) also implies that the condition on the boundedness of the extensions $\varphi_\pm^x$ in the definition of $G_\pm$ can be weakened: If the analytic extensions of the resolvent to the left and right half-plane satisfy
\[
\|\varphi_+^x(\lambda)\| \leq C|\lambda|^k, \quad \text{Re} \ \lambda < -h,
\]
and
\[
\|\varphi_-^x(\lambda)\| \leq C|\lambda|^k, \quad \text{Re} \ \lambda > h,
\]
respectively, with some constants $k \in \mathbb{N}$ and $C > 0$, then they are bounded by the Phragmén-Lindelöf theorem below.

**Theorem 3.9** (Phragmén-Lindelöf [Con78, Corollary VI.4.2]). Let $a \geq 1/2$ and $\Sigma = \{z \in \mathbb{C} : |\arg z| < \frac{\pi}{2a}\}$. Consider an analytic function $f : \Sigma \to \mathbb{C}$ which is continuous on $\overline{\Sigma}$. If there exist constants $M, C > 0$ and $0 < b < a$ such that $|f(z)| \leq M$ for $z \in \partial \Sigma$ and $|f(z)| \leq Ce^{\beta|z|^b}$ for $z \in \Sigma$, then also $|f(z)| \leq M$ for $z \in \Sigma$. 

9
4 Spectral splitting along the imaginary axis

In this section we prove our first spectral splitting result: If the resolvent of $S$ is uniformly bounded along the imaginary axis, then the subspaces $G_{\pm}$ defined in (17) are closed invariant subspaces of $S$ corresponding to the spectrum in $\mathbb{C}_{\pm}$. Moreover we construct a pair of closed complementary projections $P_{\pm}$ onto $G_{\pm}$.

We make the following assumption: there exists $h > 0$ such that
\begin{equation}
\{ \lambda \in \mathbb{C} : |\text{Re}\lambda| \leq h \} \subset \varrho(S)
\end{equation}
and
\begin{equation}
M := \sup_{|\text{Re}\lambda| \leq h} \| (S - \lambda)^{-1} \| < \infty.
\end{equation}

For this assumption to hold it is sufficient that $i\mathbb{R} \subset \varrho(S)$ and that the resolvent $(S - \lambda)^{-1}$ is uniformly bounded on $i\mathbb{R}$. If (18) and (19) hold for $h > 0$, then they also hold for some $h' > h$ with a possibly larger $M$. Both statements follow from a standard Neumann series argument.

Note that by Lemma 3.6 (ii) every strictly dichotomous operator satisfies (18) and (19).

For every $S$ which satisfies (18) and (19) we can define the operators
\begin{equation}
A_{\pm} = \pm\frac{1}{2\pi i} \int_{\pm h - i\infty}^{\pm h + i\infty} \frac{1}{\lambda^2} (S - \lambda)^{-1} d\lambda
\end{equation}
as in [BGK86b]. Note that by (19) the integrals converge in the uniform operator topology, hence $A_{\pm}$ are well-defined bounded linear operators. Due to Cauchy’s theorem, the integrals on the right hand side are independent of $h$ as long as (18) and (19) hold.

The next theorem extends the results from [BGK86b, Theorem 3.1]. In particular we obtain a spectral splitting also in the case of unbounded projections $P_{\pm}$.

**Theorem 4.1.** Let $S(X \rightarrow X)$ be a linear operator on the Banach space $X$ satisfying (18) and (19). Then:

(i) The subspaces $G_{\pm}$ defined in (17) are closed, $S$- and $(S - \lambda)^{-1}$-invariant and satisfy
\begin{align}
\sigma(S) &= \sigma(S|G_{\pm}) \cup \sigma(S|G_{\pm}^C), \\
\sigma(S|G_{\pm}) &= \sigma(S) \cap \mathbb{C}_{\pm}, \\
\| (S|G_{\pm} - \lambda)^{-1} \| &\leq M \quad \text{for} \quad \lambda \in \mathbb{C}_{\mp},
\end{align}
where $M$ is given by (19).

(ii) Let $A_{\pm}$ as in (20). Then the operators $P_{\pm} = S^2 A_{\pm}$ are closed complementary projections satisfying $\mathcal{D}(P_{\pm}) = \mathcal{D}(P_{\pm}^C) = G_{\pm} \oplus G_{\mp}$ and
\begin{equation}
G_{\pm} = \text{Im}(P_{\pm}) = \ker(A_{\mp}).
\end{equation}
Hence $P_\pm$ are the complementary projections corresponding to the direct sum $G_+ \oplus G_- \subset X$.

(iii) $D(S^2) \subset D(P_\pm)$ and

$$P_\pm x = \frac{\pm 1}{2\pi i} \int_{\pm h - i\infty}^{\pm h + i\infty} \frac{1}{\lambda^2} (S - \lambda)^{-1} S^2 x \, d\lambda, \quad x \in D(S^2).$$

Proof. We first show (ii). In [BGK86b] it has been shown that $A_+ A_- = A_- A_+ = 0$ and $A_+ + A_- = S^{-2}$. For the convenience of the reader we sketch the proof. Using the resolvent identity and Fubini’s theorem, we get

$$A_+ A_- = \int_{-h - i\infty}^{h + i\infty} \int_{-h - i\infty}^{h + i\infty} \frac{1}{\lambda^2 \mu^2} (S - \lambda)^{-1}(S - \mu)^{-1} \, d\mu \, d\lambda$$

$$= \int_{-h - i\infty}^{h + i\infty} \int_{h + i\infty}^{-h + i\infty} \frac{d\mu}{\mu^2(\lambda - \mu)} \, \frac{1}{\lambda^2} (S - \lambda)^{-1} \, d\lambda 
- \int_{-h - i\infty}^{-h + i\infty} \int_{h + i\infty}^{-h + i\infty} \frac{d\lambda}{\lambda^2(\lambda - \mu)} \, \frac{1}{\mu^2} (S - \mu)^{-1} \, d\mu$$

and hence $A_+ A_- = 0$ since the integrals in parentheses vanish. Clearly $(S - \lambda)^{-1}$ commutes with $A_\pm$, so $A_+$ and $A_-$ commute too and $A_- A_+ = 0$. By Cauchy’s theorem we obtain

$$A_+ + A_- = \frac{1}{2\pi i} \int_{\Gamma} \frac{1}{\lambda^2} (S - \lambda)^{-1} \, d\lambda = S^{-2}$$

where $\Gamma$ is a small positively oriented circle around the origin. Therefore, by Lemma 2.3, $P_\pm = S^2 A_\pm$ are closed complementary projections with $\text{Im}(P_\pm) = \ker(A_\mp)$.

We show next that $G_\pm = \ker(A_\mp)$. Let $x \in G_+$. Then $(S - \lambda)^{-1}x$ has a bounded analytic extension $\varphi_+^{*}$ to $C_-$. Consequently

$$A_- x = -\frac{1}{2\pi i} \int_{-h - i\infty}^{-h + i\infty} \frac{1}{\lambda^2} (S - \lambda)^{-1} x \, d\lambda = 0$$

by Cauchy’s theorem. We thus have $G_+ \subset \ker(A_-)$. To show the converse inclusion, let $\text{Re} \, z < -h$ and consider the bounded operator

$$R_-(z) = \frac{1}{2\pi i} \int_{-h - i\infty}^{h + i\infty} \frac{z^2}{\lambda^2(\lambda - z)} (S - \lambda)^{-1} \, d\lambda.$$  \hspace{1cm} (25)

From $(S - z)(S - \lambda)^{-1} = I + (\lambda - z)(S - \lambda)^{-1}$ and the closedness of $S$ we obtain $\text{Im}(R_-(z)) \subset D(S)$ and

$$(S - z)R_-(z) = \frac{1}{2\pi i} \int_{-h - i\infty}^{h + i\infty} \frac{z^2}{\lambda^2(\lambda - z)} \, d\lambda - z^2 A_- = I - z^2 A_-.$$
thus $(S-z)R_-(z) = I$ on $\ker(A_-)$. By Lemma 2.3, $\ker(A_-)$ is $S$- and $(S-\lambda)^{-1}$-invariant. Since $(S-z)R_-(z) = R_-(z)(S-z)$ on $D(S)$, we conclude that $\mathbb{C}_- \subset \rho(S)\ker(A_-)$ and for all $x \in \ker(A_-)$

$$(S|\ker(A_-) - z)^{-1}x = \begin{cases} (S-z)^{-1}x, & \text{Re } z \leq h, \\ R_-(z)x, & \text{Re } z < -h. \end{cases} \quad (26)$$

The definition of $R_-(z)$ in conjunction with (19) implies that $\|R_-(z)\| \leq C|z|^2$ for $\text{Re } z < -h$ with some constant $C$; as remarked earlier, we may replace $h$ by $h/2$ in (25). Together with (19), the Phragmèn-Lindelöf Theorem 3.9 yields

$$\|(S|\ker(A_-) - z)^{-1}\| \leq M, \quad z \in \overline{\mathbb{C}_-}. \quad (27)$$

As a consequence, if $x \in \ker(A_-)$, then $(S|\ker(A_-) - z)^{-1}x$ is a bounded analytic extension of $(S-z)^{-1}x$ to $\overline{\mathbb{C}_-}$ and hence $x \in G_+$. We have thus shown that $G_+ = \ker(A_-)$ and the proof of $G_- = \ker(A_+)$ is analogous. In particular, we proved $\sigma(S|G_\pm) \subset \mathbb{C}_\pm$, (23) and $D(P_+) = D(P_-) = \text{Im}(P_+) \oplus \text{Im}(P_-) = G_+ \oplus G_-$. All remaining statements in (i) and (iii) follow from Lemma 2.3.

**Corollary 4.2.** Suppose $S(X \to X)$ satisfies (18) and (19). Then the following are equivalent:

(i) $S$ is strictly dichotomous;
(ii) $X = G_+ \oplus G_-;
(iii) P_+ \in L(X).

In this case $X = G_+ \oplus G_-$ is the corresponding spectral decomposition.

**Proof.** In Lemma 3.7 we have already seen that strict dichotomy implies $X_\pm = G_\pm$ and hence $X = G_+ \oplus G_-$. Conversely, if $X = G_+ \oplus G_-$, then Theorem 4.1 implies that $S$ is strictly dichotomous for the choice $X_\pm = G_\pm$. Finally (ii)$\iff$(iii) follows from $D(P_+) = G_+ \oplus G_-$ and the closed graph theorem.

In Theorem 4.1 we did not assume that $S$ is densely defined. If it is, then the projections $P_\pm$ are densely defined too, and we obtain a nice criterion for strict dichotomy.

**Corollary 4.3.** Let $S(X \to X)$ be densely defined and satisfy (18) and (19). Then $G_+ \oplus G_- \subset X$ is dense and the closed projections $P_\pm$ are densely defined. Moreover, the following assertions are equivalent:

(i) $S$ is strictly dichotomous;
(ii) $P_+$ is bounded on some dense subspace of $D(P_+);
(iii) the operator $P$ defined by

$$Px = \frac{1}{2\pi i} \int_{h-i \infty}^{h+i \infty} \frac{1}{\lambda^2} (S-\lambda)^{-1}S^2x d\lambda, \quad x \in D(S^2), \quad (28)$$

is bounded on some dense subspace of $D(S^2)$. 

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In this case \( P_+ \) is the unique bounded extension of \( P \) to \( X \).

**Proof.** The first assertions are immediate from \( D(S^2) \subset D(P_+) = G_+ \oplus G_- \).

(i)\(\Rightarrow\) (ii): This follows from Corollary 4.2.

(ii)\(\Rightarrow\) (i): If \( P_+ \) is bounded on a dense subspace, then the closedness of \( P_+ \) implies that \( D(P_+) = X \). Thus \( P_+ \in L(X) \) and \( S \) is strictly dichotomous.

(ii)\(\Leftrightarrow\) (iii) and the final assertion: This is clear since (24) implies that \( P \) is a restriction of \( P_+ \).

\( \square \)

**Remark 4.4.** Theorem 4.1 and Corollaries 4.2 and 4.3 generalise and extend the results from [BGK86b, Theorem 3.1].

(i) Probably the main result of [BGK86b, Theorem 3.1] on spectral splitting for densely defined \( S \) is that if the operator \( P \) in (28) is bounded on \( D(S^2) \), then \( S \) is dichotomous. Corollary 4.3 shows that the boundedness on some dense subspace of \( D(S^2) \) is sufficient and that \( S \) is even strictly dichotomous in this case. Note that since [LT01, LRvdR02, TW14] all use [BGK86b, Theorem 3.1] to prove dichotomy, the corresponding operators in those papers are in fact strictly dichotomous.

(ii) We have shown that also in the non-dichotomous case, i.e., when \( P_\pm \) are unbounded, \( G_\pm \) are closed invariant subspaces with \( \sigma(S|G_\pm) \subset \mathbb{C}_\pm \) and \( \sigma(S|G_+) \cup \sigma(S|G_-) = \sigma(S) \). In [BGK86b] the closedness and \( S \)-invariance of \( G_\pm \) has only been obtained for exponentially dichotomous \( S \).

(iii) Using the Phragmén-Lindelöf theorem, we showed \( G_\pm = \ker(A_\mp) \) whereas in [BGK86b] only the inclusion \( G_\pm \subset \ker(A_\mp) \) was obtained.

(iv) We showed that \( P_\pm \) as defined in Theorem 4.1 (ii) are closed projections even if they are unbounded. This is used e.g. in Corollary 4.3 where it is sufficient to check boundedness on any dense subspace. In Section 7 this allows us to prove perturbation results under weaker conditions on the domains of the involved operators.

(v) In [BGK86b] it is always assumed that \( S \) is densely defined. Theorem 4.1 and Corollary 4.2 are valid also if \( S \) is not densely defined.

**Example 4.5.** We continue Example 2.4. A straightforward calculation shows that \( \sup_{x \in \mathbb{R}} \| (S - \lambda)^{-1} \| \leq 3 \) and hence (18) and (19) are satisfied. From Theorem 4.1, in particular (20), it follows that \( A_\pm \) and \( P_\pm \) are block diagonal and given by

\[
A_+ = \text{diag}(A_1^+, A_2^+, \ldots), \quad A_- = \text{diag}(A_1^-, A_2^-, \ldots),
\]

\[
P_+ = \text{diag}(P_1^+, P_2^+, \ldots), \quad P_- = \text{diag}(P_1^-, P_2^-, \ldots),
\]

where \( A_n^\pm \) and \( P_n^\pm \) are as in Example 2.4. \( P_+ \) and \( P_- \) are unbounded and \( S \) is not dichotomous, compare Remark 3.5.
Remark 4.6. Since $P_\pm$ are closed operators with $\mathcal{D}(S^2) \subset \mathcal{D}(P_\pm)$, their restrictions $P_\pm|\mathcal{D}(S^2)$ are bounded in the graph norm of $S^2$. In the almost bisectorial case, we even have $\mathcal{D}(S) \subset \mathcal{D}(P_\pm)$, and therefore $P_\pm|\mathcal{D}(S)$ are bounded in the graph norm of $S$ (see Theorem 5.6). In the general case it is not clear if $\mathcal{D}(S) \subset \mathcal{D}(P_\pm)$ and the restriction $P_\pm|\mathcal{D}(S)$ is bounded.

5 Bisectorial and almost bisectorial operators

In this section we investigate the spectral splitting of bisectorial and almost bisectorial operators. Their resolvent norm is not only bounded on the imaginary axis as assumed in Section 4, it even decays like $|\lambda|^{-\beta}$.

For $0 \leq \theta \leq \pi$ we define the sectors

$$\Sigma_\theta = \{ z \in \mathbb{C} : |\arg z| \leq \theta \}. \quad (29)$$

Let us first recall the notion of sectorial and almost sectorial operators, see e.g. [Haa06, PS02].

Definition 5.1. A linear operator $S: X \to X$ is called sectorial if there exist $0 < \theta < \pi$ and $M > 0$ such that $\sigma(S) \subset \Sigma_\theta$ and

$$\| (S - \lambda)^{-1} \| \leq \frac{M}{|\lambda|} \quad \text{for} \quad \lambda \in \mathbb{C} \setminus \Sigma_\theta. \quad (30)$$

$S$ is called almost sectorial if there exist $0 < \beta < 1$, $0 < \theta < \pi$ and $M > 0$ such that $\sigma(S) \subset \Sigma_\theta$ and

$$\| (S - \lambda)^{-1} \| \leq \frac{M}{|\lambda|^{\beta}} \quad \text{for} \quad \lambda \in \mathbb{C} \setminus \Sigma_\theta. \quad (31)$$

Remark 5.2. There are subtle differences in the behaviour of sectorial and almost sectorial operators:

(i) If $S$ is sectorial with angle $\theta$, then (30) also holds with a smaller angle $0 < \theta' < \theta$, though the constant $M$ may be bigger for $\theta'$ as can be shown by a simple Neumann series argument.

(ii) If $S$ is almost sectorial, then $0 \in \rho(S)$ (see e.g. [PS02, Remark 2.2]).

(iii) If $S$ is sectorial and $0 \in \rho(S)$, then $S$ is almost sectorial. This is true because $\| (S - \lambda)^{-1} \|$ is bounded in a neighbourhood of zero, and we easily obtain (31) from (30) (with a different constant $M$).

Note that (i) is not necessarily true for almost sectorial operators and that sectorial operators may have zero in their spectrum.

If we require the resolvent estimates only on the imaginary axis and allow spectrum on both sides of it, we obtain the definition of bisectorial and almost bisectorial operators.
**Figure 1:** Location of the spectrum of a bisectorial and an almost bisectorial operator.

**Definition 5.3.** An operator $S(X \to X)$ is called **bisectorial** if $\mathbb{iR} \setminus \{0\} \subset \sigma(S)$ and

$$
\|(S - \lambda)^{-1}\| \leq \frac{M}{|\lambda|}, \quad \lambda \in \mathbb{iR} \setminus \{0\}.
$$

$S$ is called **almost bisectorial** if $\mathbb{iR} \setminus \{0\} \subset \sigma(S)$ and there exists $0 < \beta < 1$ such that

$$
\|(S - \lambda)^{-1}\| \leq \frac{M}{|\lambda|^{\beta}}, \quad \lambda \in \mathbb{iR} \setminus \{0\}.
$$

Bisectorial operators have been studied e.g. in [AZ10, TW14].

The following results are analogous to the sectorial case. They imply that almost bisectorial operators and bisectorial operators with $0 \in \sigma(S)$ fulfil the assumptions of Theorem 4.1.

**Remark 5.4.** (i) If $S$ is bisectorial, then there exists $0 < \theta < \pi/2$ such that the bisector

$$
\Omega_\theta = \mathbb{C} \setminus \left( \Sigma_\theta \cup (-\Sigma_\theta) \right) = \{ \lambda \in \mathbb{C} : \theta < |\arg \lambda| < \pi - \theta \}
$$

belongs to $\sigma(S)$ and (32) holds on $\Omega_\theta$, see Figure 1.

(ii) If $S$ is almost bisectorial, then $0 \in \sigma(S)$.

(iii) If $S$ is bisectorial and $0 \in \sigma(S)$, then $S$ is almost bisectorial.

(iv) If $S$ is almost bisectorial or bisectorial with $0 \in \sigma(S)$, then $S$ satisfies (18) and (19) from Section 4.

Similar to Remark 5.4 (i), the resolvent estimate of an almost bisectorial operator actually holds inside a whole region around $\mathbb{iR}$:

**Lemma 5.5.** Let $S(X \to X)$ be almost bisectorial with constants $0 < \beta < 1$ and $M > 0$ as in (33). Then for every $\alpha < 1/M$ the parabola shaped region

$$
\Omega = \{ a + ib \in \mathbb{C} \setminus \{0\} : |a| \leq \alpha |b|^\beta \}
$$

...
belongs to the resolvent set, $\Omega \subset \varrho(S)$, and (33) holds for all $\lambda \in \Omega$ (typically with a larger constant $M$), see Figure 1.

**Proof.** Let $\lambda = a + ib \in \Omega$. Then the identity

$$S - \lambda = (I - a(S - ib)^{-1})(S - ib)$$

and the estimate

$$\|a(S - ib)^{-1}\| \leq \frac{M|a|}{|b|^\beta} \leq \alpha M < 1$$

imply $\lambda \in \varrho(S)$ and thus $\Omega \subset \varrho(S)$. Moreover

$$\|(S - \lambda)^{-1}\| \leq \frac{1}{1 - \|a(S - ib)^{-1}\|\|(S - ib)^{-1}\|} \leq \frac{M}{(1 - \alpha M)|b|^\beta}.$$  

Now if also $|b| \geq 1$, then $|\lambda| \leq \alpha|b|^\beta + |b| \leq (1 + \alpha)|b|$ and hence

$$\|(S - \lambda)^{-1}\| \leq \frac{M(1 + \alpha)^\beta}{(1 - \alpha M)|b|^\beta}, \quad \lambda = a + ib \in \Omega, \ |b| \geq 1.$$ 

Since $0 \in \varrho(S)$ by Remark 5.4(ii) and $(S - \lambda)^{-1}$ is uniformly bounded on compact subsets of $\varrho(S)$, the proof is complete. 

A similar result can be shown for an almost sectorial operator $S$: Let $\theta$ as in Definition 5.1. Then $\varrho(S)$ contains a parabola around every ray $\{re^{i\varphi} : r \geq 0\}$ with $\theta \leq |\varphi| \leq \pi$.

In the rest of this section we will investigate the spectral splitting properties of bisectorial and almost bisectorial operators. Compared with Theorem 4.1, we obtain simplified formulas for the projections $P_\pm = S^2 A_\pm$ and show that the restrictions $S|G_\pm$ to the spectral subspaces are sectorial and almost sectorial, respectively.

For an almost bisectorial operator $S$ let

$$B_\pm = \pm \frac{1}{2\pi i} \int_{\pm h - i\infty}^{\pm h + i\infty} \frac{1}{\lambda} (S - \lambda)^{-1} d\lambda \quad (34)$$

with $h > 0$ small enough. By (33) and Lemma 5.5 the integrals converge in the uniform operator topology. Hence $B_\pm$ are well-defined bounded linear operators and, due to Cauchy’s theorem, the integrals on the right hand side are independent of $h$ for $h$ small enough.

**Theorem 5.6.** Let $S(X \to X)$ be almost bisectorial and let $P_\pm$ as in Theorem 4.1. Then:

(i) $P_\pm = SB_\pm$, $D(S) \subset D(P_\pm)$ and

$$P_\pm x = \pm \frac{1}{2\pi i} \int_{\pm h - i\infty}^{\pm h + i\infty} \frac{1}{\lambda} (S - \lambda)^{-1} Sx d\lambda, \quad x \in D(S). \quad (35)$$
(ii) $\pm S|G_{\pm}$ are almost sectorial with angle $\theta = \pi/2$ and unchanged constants $M, \beta$.

(iii) Let $S$ be bisectorial with $0 \notin \rho(S)$ and let $\theta$ as in Remark 5.4 (i). Then $\pm S|G_{\pm}$ are sectorial with angle $\theta$ and unchanged constant $M$.

Proof. (i): Using the resolvent identity $(S - \lambda)^{-1} = S^{-1} + \lambda(S - \lambda)^{-1}S^{-1}$ we obtain from (20)

$$A_{\pm} = \frac{\pm 1}{2\pi i} \int_{\pm h - i\infty}^{\pm h + i\infty} \frac{1}{\lambda^2} S^{-1} d\lambda + \frac{\pm 1}{2\pi i} \int_{\pm h - i\infty}^{\pm h + i\infty} \frac{1}{\lambda}(S - \lambda)^{-1}S^{-1} d\lambda = B_{\pm} S^{-1}$$

because the first integral vanishes by Cauchy’s theorem. Moreover $S^{-1}B_{\pm} = B_{\pm}S^{-1}$. Therefore $P_{\pm} = S^2 A_{\pm} = S^2 S^{-1}B_{\pm} = SB_{\pm}$. For $x \in D(S)$ we get $B_{\pm}Sx = SB_{\pm}x$ and in particular $x \in D(P_{\pm})$.

(ii): Consider $\lambda^\beta = \exp(\beta \log \lambda)$ where log is a branch of the logarithm on $\mathbb{C}_-$. Then the mapping $\lambda \mapsto \lambda^\beta$ is analytic on $\mathbb{C}_-$ and continuous on $\overline{\mathbb{C}_-}$. The almost bisectoriality of $S$ yields $\|\lambda^\beta(S|G_{\pm} - \lambda)^{-1}\| \leq M$ for $\lambda \in i \mathbb{R}$ since $(S|G_{\pm} - \lambda)^{-1} = (S - \lambda)^{-1}|G_{\pm}$. Here we used that $0 \notin \rho(S)$ by Remark 5.4 (ii). Since $\|\lambda^\beta(S|G_{\pm} - \lambda)^{-1}\|$ is bounded on $\mathbb{C}_-$ by Theorem 4.1, the Phragmén-Lindelöf theorem implies $\|\lambda^\beta(S|G_{\pm} - \lambda)^{-1}\| \leq M$ for $\lambda \in \overline{\mathbb{C}_-}$. The proof for $S|G_{\pm}$ is analogous. With $\beta = 1$ we obtain (iii). \hfill \Box

Statements (ii) and (iii) of the Theorem remain true when $S|G_{\pm}$ are replaced by the operators $S_{M_{\pm}}$ from Section 6 as follows from Lemma 6.1 (iii).

Remark 5.7. The operators $B_{\pm}$ satisfy the relations

$$B_+ + B_- = S^{-1}, \quad B_+ B_- = B_- B_+ = 0.$$ 

These identities can be obtained either from the corresponding relations for $A_{\pm}$ via $A_{\pm} = S^{-1}B_{\pm} = B_{\pm}S^{-1}$ or from (34) by direct computation. The latter approach, together with Remark 2.5, yields an alternative proof of Theorem 4.1 for almost bisectorial operators. In the bisectorial case, this was used in [AZ10]. There the sectoriality of the spectral parts $S|G_{\pm}$ was obtained ([AZ10, p. 215]), but not the $S$-invariance of $G_{\pm}$ and the decomposition (21) of the spectrum.

To illustrate the situation of Theorem 5.6, we consider an almost bisectorial operator which is not dichotomous and whose projections $P_{\pm}$ are (therefore) unbounded. This is a variant of Example 2.4 and 4.5.

Example 5.8. Let $0 < p < 1$ and consider the block diagonal operator $S$ on $X = l^2$ given by

$$S = \text{diag}(S_1, S_2, \ldots), \quad S_n = \begin{pmatrix} n & 2n^{1+p} \\ 0 & -n \end{pmatrix}.$$

Direct calculations similar to Example 2.4 yield $\lim_{n \to \infty} \|(S_n - \lambda)^{-1}\| = 0$ whenever $\lambda \notin \mathbb{Z} \setminus \{0\}$. Hence $\sigma(S) = \mathbb{Z} \setminus \{0\}$ and $S$ has a compact resolvent. Moreover,

$$\|(S - \lambda)^{-1}\| \leq \frac{M}{|\lambda|^{1-p}}, \quad \lambda \in i \mathbb{R} \setminus \{0\},$$

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i.e., $S$ is almost bisectorial. The spectral projections for $S_n$ corresponding to the eigenvalues $n$ and $-n$, respectively, are

$$P^+_n = \begin{pmatrix} 1 & n \\ 0 & 0 \end{pmatrix}, \quad P^-_n = \begin{pmatrix} 0 & -n \\ 0 & 1 \end{pmatrix}.$$ 

Consequently, $P_\pm = \text{diag}(P^+_1, P^+_2, \ldots)$ are unbounded and $S$ is not dichotomous, compare Remark 3.5.

If in the above example we choose $p = 0$, then $S$ becomes bisectorial and strictly dichotomous. In general however, bisectoriality (with $0 \in \rho(S)$) does not imply dichotomy. A counterexample was given in [MY90] (see Example 8.2).

The identity (35) for the projections $P_\pm$ from Theorem 5.6 can be rearranged to yield another integral representation. In the bisectorial case, this representation has been obtained and used extensively in [LT01, TW14].

**Corollary 5.9.** Let $S(X \to X)$ be almost bisectorial. Then:

(i) For every $x \in \mathcal{D}(S)$,

$$2P_+ x - x = P_+ x - P_- x = \frac{1}{\pi i} \int_{-\infty}^{i\infty} (S - \lambda)^{-1} x d\lambda; \quad (36)$$

in particular, the integral exists for all $x \in \mathcal{D}(S)$. Here the prime denotes the Cauchy principal value at infinity.

(ii) If, in addition, $S$ is densely defined and there exists a dense subspace $D \subset \mathcal{D}(S)$ such that

$$\int_{-\infty}^{i\infty} (S - \lambda)^{-1} x d\lambda, \quad x \in D,$$

defines a bounded operator, then $P_+$ is bounded and hence $S$ is strictly dichotomous.

**Proof.** (i) From (35) we get for $x \in \mathcal{D}(S)$,

$$P_+ x = \frac{1}{2\pi i} \int_{h-i\infty}^{h+i\infty} \frac{1}{\lambda} (S - \lambda)^{-1} S x d\lambda = \frac{1}{2\pi i} \int_{h-i\infty}^{h+i\infty} \left( \frac{1}{\lambda} x + (S - \lambda)^{-1} x \right) d\lambda$$

$$= \frac{1}{2} x + \frac{1}{2\pi i} \int_{h-i\infty}^{h+i\infty} (S - \lambda)^{-1} x d\lambda = \frac{1}{2} x + \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} (S - \lambda)^{-1} x d\lambda.$$ 

Note that in the last step we used Cauchy’s integral theorem and (33). The assertion follows from $x = P_+ x + P_- x$ for $x \in \mathcal{D}(P_+)$. 

(ii) The assumption together with (i) implies that $P_+$ is bounded on the dense subspace $D$. Corollary 4.3 yields the claim.

**Remark 5.10.** In [LT01] the representation (36) was derived under the weaker condition $\lim_{t \to \pm\infty} \|(S - it)^{-1}\| = 0$ but with the additional assumption that the integral in (36) exists for every $x \in X$. By the uniform boundedness principle, the projections $P_\pm$ are then bounded and $S$ is strictly dichotomous.
Remark 5.11. An operator $S$ is called \textit{sectorially dichotomous} if it is dichotomous and $S|X_+$ and $-S|X_-$ are sectorial operators with angle $\theta \leq \pi/2$. Note that sectorial dichotomy implies bisectoriality [TW14, Lemma 2.12] as well as strict dichotomy. A question asked in [TW14] is the following: Is every bisectorial and dichotomous operator also sectorially dichotomous? If we assume strict dichotomy, the answer is yes since in this case $\pm S|G_\pm$ are sectorial by Theorem 5.6 and strict dichotomy implies $G_\pm = X_\pm$.

The stronger assumption of strict dichotomy seems reasonable, for otherwise the spaces $X_\pm$ are not unique. Moreover the main theorems in [TW14] actually yield strictly dichotomous operators, compare Remark 4.4.

6 The subspaces $M_\pm$

Recall that the subspaces $G_\pm$ are $S$- and $(S - \lambda)^{-1}$-invariant (Theorem 4.1). However, even if $S$ is densely defined, the restrictions of $S$ to $G_\pm$ do not need to be densely defined, see Example 8.3. Let

$$M_\pm := \text{Im}(A_\pm)$$

and let $S_{M_\pm}$ be the part of $S$ in $M_\pm$, i.e. $S$ is the restriction of $S$ to $D(S_{M_\pm}) = \{x \in D(S) \cap M_\pm : Sx \in M_\pm\}$. In [BGK86b] it is shown that $D(S_{M_\pm})$ is dense in $M_\pm$ if $S$ is densely defined, cf. Lemma 6.2. In the following lemma we do not assume density of $D(S)$.

Lemma 6.1. Let $S(X \to X)$ be such that (18) and (19) hold. Then:

(i) $M_\pm \subset G_\pm$.

(ii) $M_\pm$ is $(S - \lambda)^{-1}$-invariant for $\lambda \in \varrho(S)$ and $D(S_{M_\pm}) = S^{-1}(M_\pm)$.

(iii) $M_\pm$ is $(S|G_\pm - \lambda)^{-1}$-invariant for $\lambda \in \varrho(S|G_\pm)$,

$$\sigma(S_{M_\pm}) = \sigma(S|G_\pm),$$

$$(S_{M_\pm} - \lambda)^{-1}x = (S|G_\pm - \lambda)^{-1}x, \quad x \in M_\pm, \quad \lambda \in \varrho(S_{M_\pm}).$$

Proof. (i) follows from $\text{Im}(A_\pm) \subset \ker(A_\pm) = G_\pm$ and the closedness of $G_\pm$.

(ii) From $(S - \lambda)^{-1}A_\pm = A_\pm(S - \lambda)^{-1}$ it follows that $\text{Im}(A_\pm)$ and hence $M_\pm$ are $(S - \lambda)^{-1}$-invariant. In particular $S^{-1}(M_\pm) \subset M_\pm$, which implies $D(S_{M_\pm}) = S^{-1}(M_\pm)$.

(iii) Let us prove the invariance of $M_\pm$. Recall that $\varrho(S|G_\pm) = \varrho(S) \cup \mathbb{C}_-$. For $\lambda \in \varrho(S)$ we have $(S|G_\pm - \lambda)^{-1} = (S - \lambda)^{-1}|G_\pm$, so the invariance of $M_\pm$ follows from (ii) in this case. For $\lambda \in \mathbb{C}_-$, the invariance follows from (26). The proof for $M_-$ is analogous.

The invariance property of $M_\pm$ immediately yields $\varrho(S|G_\pm) \subset \varrho(S_{M_\pm})$ and $(S_{M_\pm} - \lambda)^{-1}x = (S|G_\pm - \lambda)^{-1}x$ for $x \in M_\pm$ and $\lambda \in \varrho(S|G_\pm)$. Now

$$\sigma(S) = \sigma(S|G_+) \cup \sigma(S|G_-) \supset \sigma(S_{M_+}) \cup \sigma(S_{M_-}).$$

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So to prove $\sigma(S|G_{\pm}) \subset \sigma(S_{M_{\pm}})$ it suffices to show $\sigma(S) \subset \sigma(S_{M_{+}}) \cup \sigma(S_{M_{-}})$ or, equivalently, $\varrho(S_{M_{+}}) \cap \varrho(S_{M_{-}}) \subset \varrho(S)$. The proof is similar to the one for (9) in Lemma 2.3: If $\lambda \in \varrho(S_{M_{+}}) \cap \varrho(S_{M_{-}})$ and $(S - \lambda)x = 0$, then $x \in \mathcal{D}(S^2)$ and hence

$$x = y_+ + y_- \quad \text{with} \quad y_\pm = P_\pm x = A_\pm S^2 x \in \text{Im}(A_\pm) \subset M_\pm.$$ 

In fact $y_\pm \in \mathcal{D}(S_{M_{\pm}})$ since $x \in \mathcal{D}(S)$ and $S y_\pm = A_\pm S^3 x$. Consequently

$$(S - \lambda)x = (S_{M_+} - \lambda)y_+ + (S_{M_-} - \lambda)y_- = 0$$

and thus $y_+ = y_- = x = 0$. The surjectivity of $S - \lambda$ is obtained by considering $T = (S_{M_+} - \lambda)^{-1} A_+ + (S_{M_-} - \lambda)^{-1} A_-$. Noting $(S - \lambda) S^2 T = I$. \hfill \Box

The following result has been obtained as part of [BGK86b, Theorem 3.1 and Lemma 3.3]. For the convenience of the reader we include the proof.

**Lemma 6.2.** If $S(X \to X)$ is densely defined and satisfies (18) and (19), then $M_+ \oplus M_- \subset X$ is dense and the operators $S_{M_{\pm}}$ are densely defined.

**Proof.** From $S^{-2} = A_+ + A_-$ we obtain $\mathcal{D}(S^2) \subset \text{Im}(A_+) \oplus \text{Im}(A_-) \subset M_+ \oplus M_-$. Since the first assertion holds. For the second one note that $S^{-1}(\text{Im}(A_{\pm})) = A_{\pm}(\mathcal{D}(S))$ is dense in $\text{Im}(A_{\pm})$ and hence in $M_{\pm}$, and that $S^{-1}(\text{Im}(A_{\pm})) \subset S^{-1}(M_{\pm}) = \mathcal{D}(S_{M_{\pm}})$. \hfill \Box

Despite the invariance of $M_{\pm}$ under $(S - \lambda)^{-1}$, and the invariance of $G_{\pm}$ under $S$ and $(S - \lambda)^{-1}$, the subspaces $M_{\pm}$ are in general not invariant under $S$ itself; that is, the inclusion $S(\mathcal{D}(S) \cap M_{\pm}) \subset M_{\pm}$ does not need to hold. For densely defined operators $S$ the $S$-invariance of $M_{\pm}$ can be characterised as follows.

**Theorem 6.3.** Let $S(X \to X)$ be densely defined satisfying (18) and (19). Then:

(i) The following equivalences hold:

- $S|G_+$ densely defined $\iff M_+ = G_+ \iff M_+ \text{ is } S\text{-invariant}$,
- $S|G_- \text{ densely defined} \iff M_- = G_- \iff M_- \text{ is } S\text{-invariant}$.

In particular, $S_{M_{\pm}} = S|G_{\pm}$ if $M_{\pm} = G_{\pm}$.

(ii) If $P_+$ (or equivalently $P_-$) is bounded, then $M_{\pm} = G_{\pm}$.

(iii) Suppose there exist $Q_n \in L(X)$, $n \in \mathbb{N}$, such that $\text{Im}(Q_n) \subset \mathcal{D}(S)$, 

$$Q_n(S - \lambda)^{-1} = (S - \lambda)^{-1} Q_n \quad \text{for all } \lambda \in \varrho(S)$$

and 

$$Q_n x \to x \quad \text{for all } x \in X.$$ 

Then $M_{\pm} = G_{\pm}$.  

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Proof. (i) From $G_+ = \text{Im}(P_+)$ and $P_+ = S^2A_+$ we obtain
\[ \mathcal{D}(S^2) \cap G_+ = S^{-2}(G_+) = \text{Im}(A_+) \subset M_+. \] (38)
Suppose first that $S|G_+$ is densely defined. Since $0 \in \varrho(S|G_+)$, $(S|G_+)^2$ is densely defined too, i.e. $\mathcal{D}(S^2) \cap G_+ \subset G_+$ is dense. Taking closures in (38), we thus obtain $G_+ \subset M_+$ and hence $G_+ = M_+$. Now let $M_+$ be $S$-invariant. From (38) we see $S^{-2}(G_+) \subset \mathcal{D}(S^2) \cap M_+$, hence, by the $S$-invariance of $M_+$, we obtain $G_+ \subset M_+$, i.e. $G_+ = M_+$. The other implications are trivial and the case of $M_-, G_-$ is analogous.

(ii) If $P_{\pm}$ are bounded, then $X = G_+ \oplus G_-$. Since $M_{\pm}$ are closed, it follows that $M_+ \oplus M_-$ is closed. By Lemma 6.2 it is dense in $X$, so we obtain $M_+ \oplus M_- = X$ and hence $M_\pm = G_\pm$.

(iii) The assumption that $Q_n$ commutes with the resolvent implies $Q_n A_\pm = A_\pm Q_n$ and consequently $Q_n x \in \ker(A_-) = G_+$ for any $x \in G_+ = \ker(A_-)$. Since additionally $Q_n x \in \mathcal{D}(S)$ and $Q_n x \to x$, we obtain that $G_+ \cap \mathcal{D}(S)$ is dense in $G_+$ and thus $M_+ = G_+$ by (i). The proof of $M_- = G_-$ is analogous. \qed

Remark 6.4. The conditions of Theorem 6.3(iii) hold for example if $X$ is a Hilbert space which can be decomposed into an orthogonal direct sum of finite-dimensional subspaces $X_k$ where each $X_k$ is spanned by a set of eigenvectors of $S$. Then $Q_n$ can be chosen as the orthogonal projection onto the subspace $X_1 \oplus \cdots \oplus X_n$. Block diagonal operators as in Example 2.4 admit such orthogonal decompositions.

Although for a general densely defined operator $S$ its restrictions $S|G_\pm$ may fail to be densely defined, they always are if $S$ is bisectorial with $0 \in \varrho(S)$.

Lemma 6.5. Let $S(X \to X)$ be densely defined and bisectorial with $0 \in \varrho(S)$. Then $S|G_\pm$ are densely defined and $M_\pm = G_\pm$.

Proof. Since $S$ is densely defined and $\|\text{im}(it - S)^{-1}\|$ is bounded for $t \in \mathbb{R}$, the identity
\[ x = \lim_{t \to \infty} \text{im}(it - S)^{-1} x \]
holds for all $x \in X$, compare [Haa06, Proposition 2.1.1]. Moreover, $(it - S)^{-1} x \in G_\pm \cap \mathcal{D}(S) \subset G_\pm$, therefore $S|G_\pm$ are densely defined and hence $M_\pm = G_\pm$ by Theorem 6.3. \qed

Remark 6.6. If $X$ is reflexive, then every sectorial (or bisectorial) operator is automatically densely defined, see e.g. [Haa06, Proposition 2.1.1]. From Theorem 5.6 we already know that if $S$ is bisectorial, then the restrictions $\pm S|G_\pm$ are sectorial; so they are densely defined if $X$ is reflexive. The previous lemma ensures that this is true also in non-reflexive spaces.

Finally we show that the spaces $M_\pm$ can be expressed in terms of the operators $B_\pm$ from (34) if $S$ is densely defined and almost bisectorial.
Lemma 6.7. Let \( S(X \to X) \) be densely defined and almost bisectorial. Let \( A_\pm \) and \( B_\pm \) be the operators defined in (20) and (34). Then
\[
M_\pm = \text{Im}(A_\pm) = \text{Im}(B_\pm).
\]

Proof. Since \( B_\pm \) is bounded and \( D(S) \) is dense,
\[
\text{Im}(A_\pm) = \text{Im}(B_\pm S^{-1}) = \text{Im}(B_\pm |D(S)) = \text{Im}(B_\pm).
\]

7 Perturbation results

Our first perturbation result generalises [BGK86b, Theorem 5.1] where the stronger assumptions \( \varepsilon = 1 \), \( D(T^2) \subset D(S^2) \) and exponential dichotomy of \( S \) were required.

Theorem 7.1. Let \( S(X \to X) \) be a densely defined and strictly dichotomous operator on the Banach space \( X \). Suppose that \( T(X \to X) \) is densely defined and that there exist \( h > 0 \), \( \varepsilon > 0 \) such that the following conditions hold:

(i) \( \{ \lambda \in \mathbb{C} : |\text{Re}\lambda| \leq h \} \subset \rho(S) \cap \rho(T) \);

(ii) \( \sup_{|\text{Re}\lambda| \leq h} |\lambda|^{1+\varepsilon} \|(S - \lambda)^{-1} - (T - \lambda)^{-1}\| < \infty \);

(iii) \( D(S^2) \cap D(T^2) \subset X \) dense.

Then \( T \) is strictly dichotomous too.

Proof. Since \( S \) is strictly dichotomous, \( (S - \lambda)^{-1} \) is uniformly bounded for \( |\text{Re}\lambda| \leq h' \) by (16) for some \( 0 < h' \leq h \). Condition (ii) implies that \( (T - \lambda)^{-1} \) is also uniformly bounded for \( |\text{Re}\lambda| \leq h' \). Let \( P_\pm^S \), \( P_\pm^T \) be the projections corresponding to the spectrum in \( \mathbb{C}_+ \) of \( S \) and \( T \), respectively. Using (24),
\[
\frac{1}{\lambda^2} (S - \lambda)^{-1} S^2 = \frac{1}{\lambda^2} S + \frac{1}{\lambda} (S - \lambda)^{-1} S = \frac{1}{\lambda^2} S + \frac{1}{\lambda} S - \lambda^{-1} (S - \lambda)^{-1} - \frac{1}{\lambda},
\]
and the respective identity for \( T \), we obtain for \( x \in D(S^2) \cap D(T^2) \),
\[
P_\pm^S x - P_\pm^T x = \frac{1}{2\pi i} \int_{h'-i\infty}^{h'+i\infty} \left( \frac{1}{\lambda^2} (Sx - Tx) + ((S - \lambda)^{-1} x - (T - \lambda)^{-1} x) \right) d\lambda
\]
\[
= \frac{1}{2\pi i} \int_{h'-i\infty}^{h'+i\infty} ((S - \lambda)^{-1} x - (T - \lambda)^{-1} x) d\lambda.
\]
By (ii) the last integral defines a bounded linear operator. Since \( P_\pm^S \) is bounded, \( P_\pm^T \) is bounded on the dense subspace \( D(S^2) \cap D(T^2) \), and therefore \( T \) is strictly dichotomous by Corollary 4.3. \( \square \)

Remark 7.2. In [BGK86b, Theorem 5.1] it has been shown that, if \( S \) is exponentially dichotomous then so is \( T \). This implication remains true in our more general setting, where we require (ii) for some \( \varepsilon > 0 \) instead of \( \varepsilon = 1 \), and (iii) instead of \( D(T^2) \subset D(S^2) \). The proof of the exponential dichotomy of \( T \) is largely identical to the one in [BGK86b].
If the operator $S$ is almost bisectorial, condition (iii) of Theorem 7.1 can be relaxed:

**Theorem 7.3.** Let $S(X \rightarrow X)$ be densely defined, almost bisectorial and strictly dichotomous. Let $T(X \rightarrow X)$ be a densely defined operator and $\varepsilon > 0$ such that the following conditions hold:

(i) $\text{i} \mathbb{R} \subset \sigma(T)$;

(ii) $\sup_{\lambda \in \text{i} \mathbb{R}} |\lambda|^{1+\varepsilon} \| (S - \lambda)^{-1} - (T - \lambda)^{-1} \| < \infty$;

(iii) $\mathcal{D}(S) \cap \mathcal{D}(T) \subset X$ dense.

Then $T$ is also strictly dichotomous and almost bisectorial with the same exponent $\beta$ in the resolvent estimate (33). In particular, if $S$ is bisectorial, then so is $T$.

**Proof.** Condition (ii) immediately implies that $T$ satisfies an estimate (33) with the same $\beta$ as for $S$. By Corollary 5.9 it suffices to show that

$$\int_{-\infty}^{\infty} (T - \lambda)^{-1} x \, d\lambda$$

defines a bounded linear operator on $\mathcal{D}(S) \cap \mathcal{D}(T)$. Since $S$ is strictly dichotomous, the corresponding integral for $S$ is bounded on $\mathcal{D}(S)$ by Corollary 5.9 (i). On the other hand, the difference of both integrals

$$\int_{-\infty}^{\infty} ((S - \lambda)^{-1} - (T - \lambda)^{-1}) \, d\lambda$$

converges in the uniform operator topology by (ii) and thus defines a bounded linear operator. \qed

**Remark 7.4.** It is especially the relatively weak condition (iii) which makes Theorem 7.1 and 7.3 more generally applicable than comparable theorems from [BGK86b, LT01, TW14] where $\mathcal{D}(T^2) \subset \mathcal{D}(S^2)$ or $\mathcal{D}(T) = \mathcal{D}(S)$ was assumed. A situation where this generality is needed is the Hamiltonian operator matrix defined via extrapolation spaces in Example 8.8; in particular $\mathcal{D}(T) \neq \mathcal{D}(S)$ there.

We give an example showing that the unusual condition $\mathcal{D}(T^2) \subset \mathcal{D}(S^2)$ from [BGK86b] may fail even if $\mathcal{D}(S) = \mathcal{D}(T)$.

**Example 7.5.** Let $S$ be an unbounded selfadjoint operator with strictly positive pure point spectrum, for instance the multiplication operator

$$S(l^2 \rightarrow l^2), \quad S(x_n)_{n \in \mathbb{N}} = (nx_n)_{n \in \mathbb{N}}$$

with domain $\mathcal{D}(S) = \{ (x_n)_{n \in \mathbb{N}} \in l^2 : (nx_n)_{n \in \mathbb{N}} \in l^2 \} \subsetneq l^2$. We take any $w \in l^2 \setminus \mathcal{D}(S)$ and define the bounded operator $R$ on $l^2$ by $R x = P_w x$ where $P_w$
is the orthogonal projection onto $w$. Set $T := S + R$. Then $\mathcal{D}(T) = \mathcal{D}(S)$ and $T$ is selfadjoint. On the other hand,

$$x \in \mathcal{D}(S^2) \cap \mathcal{D}(T^2) \quad \implies \quad x \in \mathcal{D}(S^2) \land Tx = Sx + Rx \in \mathcal{D}(T) = \mathcal{D}(S) \quad \implies \quad x \in \mathcal{D}(S^2) \land Rx \in \mathcal{D}(S).$$

Since $\operatorname{Im}(R) \cap \mathcal{D}(S) = \{0\}$ by construction, it follows that $Rx = 0$, hence $x \in \operatorname{span}\{w\}^\perp$. Consequently $\mathcal{D}(S^2) \cap \mathcal{D}(T^2) \subset \operatorname{span}\{w\}^\perp$ and so $\mathcal{D}(S^2) \cap \mathcal{D}(T^2)$ cannot be dense in $l^2$. Therefore

$$\mathcal{D}(S^2) \nsubseteq \mathcal{D}(S^2) \cap \mathcal{D}(T^2) \quad \text{and} \quad \mathcal{D}(T^2) \nsubseteq \mathcal{D}(S^2) \cap \mathcal{D}(T^2)$$

since $S^2$ and $T^2$ are densely defined, and we obtain $\mathcal{D}(T^2) \not\subset \mathcal{D}(S^2)$ as well as $\mathcal{D}(S^2) \not\subset \mathcal{D}(T^2)$. Note that $S$ and $T$ satisfy all conditions of Theorem 7.3, but not (iii) from Theorem 7.1.

One situation, where condition (ii) in Theorem 7.3 is fulfilled, is the case of so-called $p$-subordinate perturbations. The $p$-subordinate perturbations of bisectorial operators, in particular the change of their spectrum, have been studied in [TW14].

**Definition 7.6.** Let $S(X \to X)$, $R(X \to X)$ be linear operators. $R$ is called $p$-subordinate to $S$ with $0 \leq p \leq 1$ if $\mathcal{D}(S) \subset \mathcal{D}(R)$ and there exists $c > 0$ such that

$$\lVert Rx \rVert \leq c\lVert x \rVert^{1-p}\lVert Sx \rVert^p, \quad x \in \mathcal{D}(S).$$

**Corollary 7.7.** Let $S(X \to X)$ be densely defined, almost bisectorial with exponent $\beta > 1/2$ and strictly dichotomous. Let $R$ be $p$-subordinate to $S$ with $p < 2\beta - 1$ and let $T = S + R$. If $i\mathbb{R} \subset \varrho(T)$, then $T$ is strictly dichotomous and almost bisectorial with the same exponent $\beta$. Moreover, if $S$ is bisectorial, then so is $T$.

**Proof.** First note that $\mathcal{D}(T) = \mathcal{D}(S)$ since $\mathcal{D}(S) \subset \mathcal{D}(R)$. Hence condition (iii) in Theorem 7.3 is satisfied and it remains to show that (ii) holds too. Consider $\lambda \in i\mathbb{R}$ and the identity

$$T - \lambda = (I + R(S - \lambda)^{-1})(S - \lambda).$$

Using $p$-subordination and almost bisectoriality, we get

$$\lVert R(S - \lambda)^{-1} \rVert \leq c\lVert (S - \lambda)^{-1} \rVert^{1-p}\lVert S(S - \lambda)^{-1} \rVert^p \leq c\frac{M^{1-p}}{|\lambda|^{(1-p)\beta}}\left(1 + M|\lambda|^{1-\beta}\right)^p \leq \frac{\tilde{c}}{|\lambda|^{\beta-p}}$$

with $M$ as in (33), $\tilde{c} > 0$ appropriate, and $|\lambda|$ large. Note that $p < 2\beta - 1 \leq \beta$. Hence, for $\lambda \in i\mathbb{R}$, $|\lambda|$ large, this implies $\lambda \in \varrho(T)$ and

$$\lVert (T - \lambda)^{-1} \rVert \leq 2\lVert (S - \lambda)^{-1} \rVert \leq \frac{2M}{|\lambda|^\beta}.$$
Consequently,

\[ \|(S - \lambda)^{-1} - (T - \lambda)^{-1}\| \leq \|(T - \lambda)^{-1}\| R(S - \lambda)^{-1} \leq \frac{2\tilde{M}\bar{c}}{|\lambda|^{2\beta - p}}, \]

where \(2\beta - p > 1\).

**Remark 7.8.** In the bisectorial case, the previous result has essentially been obtained in [TW14, Corollary 3.10], with the assumption that \(S\) is sectorially dichotomous and the conclusion that \(T\) is dichotomous, compare Remarks 4.4 and 5.11.

### 8 Examples

#### 8.1 Non-uniqueness of the decomposition \(X = X_+ \oplus X_-\)

In Example 3.3 we saw that the decomposition of a dichotomous operator is not necessarily unique. The following example shows that this is even possible for bisectorial operators on Hilbert spaces.

A linear operator \(S(X \to X)\) on a Hilbert space \(X\) is called *accretive* if

\[ \mathbb{C}_+ \subset \varrho(S) \quad \text{and} \quad \|(S - \lambda)^{-1}\| \leq \frac{1}{|\Re \lambda|}, \quad \Re \lambda < 0. \]

For instance, if \(S\) is the generator of a nilpotent contraction semigroup on \(X\), then \(-S\) is accretive with \(\varrho(S) = \emptyset\).

Every accretive operator \(S\) has a square root \(S^{1/2}\) which is sectorial with any angle \(\theta > \pi/4\), see [Haa06, Proposition 3.1.2]. In particular, \(S^{1/2}\) is bisectorial.

**Example 8.1.** Let \(X\) be a Hilbert space and let \(S(X \to X)\) be an accretive operator with \(\varrho(S) = \emptyset\). Then \(S - \lambda^2 = (S^{1/2} - \lambda)(S^{1/2} + \lambda)\) shows that \(\varrho(S^{1/2}) = \emptyset\) too and, as in Example 3.3, \(S^{1/2}\) is dichotomous with respect to either of the two decompositions

\[ X_+ = X, \quad X_- = \{0\} \quad \text{and} \quad X_+ = \{0\}, \quad X_- = X. \]

Here \(S^{1/2}\) is strictly dichotomous only with respect to the first choice \(X_+ = X, \quad X_- = \{0\}\).

#### 8.2 An invertible bisectorial non-dichotomous operator

McIntosh and Yagi [MY90] gave the following example of an invertible bisectorial operator that is not dichotomous. We only sketch their construction here.

**Example 8.2.** Let \(M > 1\). For every \(m \in \mathbb{N}, \ m \geq 0\), choose \(n \in \mathbb{N}\) such that

\[ \frac{M - 1}{\pi \sqrt{18}} \log \left(\frac{n}{2} + 1\right) \geq m. \]
and \((n + 1) \times (n + 1)\) matrices \(D_m\) and \(B_m\) as follows: \(D_m\) is diagonal with entries 2\(0\), 2\(1\), \ldots , 2\(n\) and \(B_m\) is the Toeplitz matrix
\[
B_m = \frac{M - 1}{\pi} \begin{pmatrix}
  b_0 & b_1 & \ldots & b_n \\
  b_{-1} & \ddots & \ddots & \vdots \\
  \vdots & \ddots & \ddots & b_1 \\
  b_{-n} & \ldots & b_{-1} & b_0
\end{pmatrix}, \quad b_0 = 0, \ b_{\pm j} = \pm \frac{1}{j}, \ j = 1, \ldots , n.
\]

Consider the block diagonal operator \(A\) on \(X = l^2\) given by
\[
A = \begin{pmatrix}
  A_1 & 0 & \cdots & 0 \\
  0 & A_2 & \cdots & 0 \\
  \vdots & \ddots & \ddots & \vdots \\
  0 & \cdots & 0 & A_{n+1}
\end{pmatrix}, \quad A_m = \begin{pmatrix}
  D_m & B_m D_m \\
  0 & -D_m
\end{pmatrix}.
\]

It is then shown in \([MY90]\) that \(\sigma(A) \subset ]-\infty, -1] \cup [1, \infty[\) and \(\| (A - \lambda)^{-1} \| \leq M|\lambda|^{-1}\) for \(\lambda \in \mathbb{R} \setminus \{0\}\). In particular, \(A\) is a bisectorial operator. However, \(A\) is not dichotomous: In fact, the spectral projection \(P_m\) corresponding to the positive eigenvalues of \(A_m\) is
\[
P_m = \begin{pmatrix}
  I & Z_m \\
  0 & 0
\end{pmatrix}
\]
where the \((n + 1) \times (n + 1)\) matrix \(Z_m\) satisfies \(D_m Z_m + Z_m D_m = B_m D_m\) and \(\|Z_m\| \geq m\). Since every dichotomous decomposition \(X = X_+ \oplus X_-\) must contain the eigenspace for \(\lambda \in \mathbb{C}_\pm\) in \(X_\pm\), the projection corresponding porjection \(P_+\) must contain all \(P_m\) and thus is unbounded.

### 8.3 A densely defined operator \(S\) with non-densely defined sectorial \(S|G_\pm\)

In this section we construct a densely defined operator \(S\) whose restriction \(S|G_+\) to the positive spectral subspace is not densely defined. According to Theorem 6.3, this is equivalent to \(M_+ \neq G_+\). Our operator \(S\) will be almost bisectorial, but not bisectorial, and its restriction \(S|G_+\) will be sectorial.

**Example 8.3.** On \(X = C([0,1])\) consider the operator
\[
A_0 f = f', \quad \mathcal{D}(A_0) = \{ f \in C^1([0,1]) : f(0) = 0\}.
\]

Then \(A_0\) is non-densely defined, \(\sigma(A_0) = \emptyset\), and \(A_0\) is accretive. As in Section 8.1 it follows that \(A_0\) has a square root, \(A = A_0^{1/2}\), where \(A\) is sectorial with any angle \(\theta > \pi/4\) and \(\sigma(A) = \emptyset\). Moreover, \(A\) is non-densely defined too (otherwise \(A_0 = A^2\) had to be densely defined). In fact,
\[
\overline{\mathcal{D}(A)} = \overline{\mathcal{D}(A_0)} = \{ f \in X : f(0) = 0\}.
\]
Let \( w \in X \), \( w(t) = 1 \) constant. Hence \( w \in X \setminus \overline{\mathcal{D}(A)} \) and \( X = \overline{\mathcal{D}(A)} \oplus \text{span}\{w\} \).

Fix \( 0 < s < 1/2 \) and consider the rank one operator \( B : \mathcal{D}(B) \subset l^2 \to X \),

\[
B\alpha = \sum_{k=1}^{\infty} k^s \alpha_k \cdot w, \quad \mathcal{D}(B) = \left\{ \alpha = (\alpha_k)_{k=1}^{\infty} \in l^2 : \sum_{k=1}^{\infty} k^s |\alpha_k| < \infty \right\}.
\]

Let \( q \geq 1 \) and let \( C : \mathcal{D}(C) \subset l^2 \to l^2 \),

\[
C\alpha = (-k^q \alpha_k)_{k=1}^{\infty}, \quad \mathcal{D}(C) = \left\{ \alpha = (\alpha_k)_{k=1}^{\infty} \in l^2 : (-k^q \alpha_k)_{k=1}^{\infty} \in l^2 \right\}.
\]

**Lemma 8.4.** The operators \( B \) and \( C \) are densely defined, \( C \) is selfadjoint, \( \sigma(C) = \{-k^q : k \in \mathbb{N}\} \), \( \mathcal{D}(C) \subset \mathcal{D}(B) \), and \( B \) is \( 1/q \)-subordinate to \( C \).

**Proof.** The first assertions are clear. The last assertion follows because for all \( \alpha \in \mathcal{D}(C) \)

\[
\|B\alpha\| \leq \sum_{k=1}^{\infty} k^s |\alpha_k| \leq \left( \sum_{k=1}^{\infty} \frac{1}{k^{2-2s}} \right)^{\frac{1}{2s}} \left( \sum_{k=1}^{\infty} k^{2s} |\alpha_k|^2 \right)^{\frac{1}{2}},
\]

\[
\sum_{k=1}^{\infty} k^{2s} |\alpha_k|^2 \leq \left( \sum_{k=1}^{\infty} |\alpha_k|^2 \right)^{1-\frac{1}{2s}} \left( \sum_{k=1}^{\infty} k^{2q} |\alpha_k|^2 \right)^{\frac{1}{2s}}.
\]

On \( X \times l^2 \) consider the operator

\[
S \left( \begin{pmatrix} f \\ \alpha \end{pmatrix} \right) = \begin{pmatrix} A(f - B\alpha) \\ C\alpha \end{pmatrix}, \quad \mathcal{D}(S) = \left\{ \begin{pmatrix} f \\ \alpha \end{pmatrix} \in X \times \mathcal{D}(C) : f - B\alpha \in \mathcal{D}(A) \right\}.
\]

**Proposition 8.5.** The operator \( S \) has the following properties:

(i) \( S \) is densely defined.

(ii) \( \sigma(S) = \sigma(C) \).

(iii) \( S \) is almost bisectorial.

(iv) \( G_+ = X \times \{0\}, S|G_+ \) is sectorial and \( \mathcal{D}(S|G_+) = \mathcal{D}(A) \times \{0\} \). In particular, \( S|G_+ \) is not densely defined.

**Proof.** For the proof of (i) note that \( \overline{\mathcal{D}(A) \times \{0\}} \subset \overline{\mathcal{D}(S)} \) and \( (w, k^{-s} e_k) \in \mathcal{D}(S) \) where \( (e_k) \) is the standard orthonormal basis in \( l^2 \). This shows \( (w, 0) \in \overline{\mathcal{D}(S)} \), hence \( X \times \{0\} \subset \overline{\mathcal{D}(S)} \). Finally, for every \( x \in \mathcal{D}(C) \), \( (Bx, x) \in \mathcal{D}(S) \) and thus \( (0, x) \in \overline{\mathcal{D}(S)} \), so we showed that \( S \) is densely defined.

(ii) can be shown by direct computation. Moreover, for \( \lambda \in g(S) \),

\[
(S - \lambda)^{-1} \begin{pmatrix} f \\ \alpha \end{pmatrix} = \begin{pmatrix} (A - \lambda)^{-1} f + A(A - \lambda)^{-1} B(C - \lambda)^{-1} \alpha \\ (C - \lambda)^{-1} \alpha \end{pmatrix}. \quad (39)
\]

On \( \mathbb{R} \) the norms \( \|\lambda(A - \lambda)^{-1}\| \) and \( \|A(A - \lambda)^{-1}\| \) as well as the analogous expressions for \( C \) are uniformly bounded. The subordination property of \( B \)
yields the estimate \( \|B(C - \lambda)^{-1}\| \leq c|\lambda|^{-\beta}, \lambda \in i\mathbb{R} \setminus \{0\} \), with \( \beta = 1 - \frac{1}{q} \) and \( c > 0 \). Therefore

\[
\|(S - \lambda)^{-1}\| \leq \frac{M}{|\lambda|^\beta}, \quad \lambda \in i\mathbb{R} \setminus \{0\}, \quad \beta = 1 - \frac{1}{q},
\]

with some \( M > 0 \), which proves (iii).

For the proof of (iv) let us first calculate \( G_+ \). Since for every \( f \in X \),

\[
(S - \lambda)^{-1} \begin{pmatrix} f \\ 0 \end{pmatrix} = \begin{pmatrix} (A - \lambda)^{-1}f \\ 0 \end{pmatrix}
\]

is a bounded analytic function on \( \mathbb{C}_- \), we have \((f, 0) \in G_+ \). On the other hand, if \((f, \alpha) \in G_+ \), then (39) implies that \((C - \lambda)^{-1}\alpha \) has a bounded analytic extension to \( \mathbb{C}_- \), which is possible only if \( \alpha = 0 \). Thus

\[
G_+ = X \times \{0\},
\]

and therefore \( \mathcal{D}(S|G_+) = \mathcal{D}(A) \times \{0\} \). It follows that \( S|G_+ \) is not densely defined and \( S|G_+ \cong A \), so \( S|G_+ \) is sectorial.

### 8.4 A non-densely defined non-sectorial but almost sectorial operator

As a simple example of a non-densely defined operator on a Hilbert space that satisfies an estimate (31) with \( \beta < 1 \) but is not sectorial, we consider now an ordinary differential operator on the Sobolev space \( H^1([0, 1]) \). In a Banach space setting, operators of this form have been considered in [SS86]. A non-densely defined operator which is even sectorial and defined on a non-reflexive Banach space is the operator \( A \) from Example 8.3.

**Example 8.6.** For \( m \geq 1 \) consider the operator \( A_0 \) on \( L^2([0, 1]) \) given by

\[
A_0 f = (-1)^mf^{(2m)},
\]

\[
\mathcal{D}(A_0) = \{ f \in H^{2m}([0, 1]) : f^{(j)}(0) = f^{(j)}(1) = 0, \, j = 0, \ldots, m - 1 \}.
\]

The operator \( A_0 \) is positive selfadjoint. Let \( A \) be the part of \( A_0 \) on the Sobolev space \( H^1([0, 1]) \), i.e.,

\[
Af = A_0 f = (-1)^mf^{(2m)},
\]

\[
\mathcal{D}(A) = \{ f \in H^{2m+1}([0, 1]) : f^{(j)}(0) = f^{(j)}(1) = 0, \, j = 0, \ldots, m - 1 \}.
\]

We easily see that \( A \) is closed and \( \sigma_p(A_0) = \sigma_p(A) \). Moreover, if \( \lambda \in g(A_0) \) then \( A - \lambda \) is bijective with inverse \( (A - \lambda)^{-1} = (A_0 - \lambda)^{-1}|H^1([0, 1]) \) and hence \( \sigma_p(A) = \sigma(A) = \sigma(A_0) \).

Since \( A_0 \) is selfadjoint, it is sectorial with arbitrary small angle \( \theta > 0 \). Let \( g \in L^2([0, 1]) \), \( |\arg \lambda| \geq \theta > 0 \) and set \( f = (A_0 - \lambda)^{-1}g \). Then the identity \( (-1)^mf^{(2m)} = \lambda f + g \) yields

\[
\|f^{(2m)}\|_{L^2} \leq (1 + |\lambda|\|(A_0 - \lambda)^{-1}\|)\|g\|_{L^2} \leq (1 + M)\|g\|_{L^2}
\]

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where $M > 0$ is the constant from the sectoriality estimate (30). Since $|f|_{2m} = \|f\|_{L^2} + \|f^{(2m)}\|_{L^2}$ is an equivalent norm on $H^{2m}([0, 1])$ and $\|(A_0 - \lambda)^{-1}\|$ is uniformly bounded for $|\arg \lambda| \geq \theta$, this implies
\[
\|(A_0 - \lambda)^{-1}g\|_{H^{2m}} \leq c_1 \|g\|_{L^2}, \quad |\arg \lambda| \geq \theta,
\]
with $c_1 > 0$ depending on $\theta > 0$. Using the interpolation inequality $\|f\|_{H^k} \leq c_2 \|f\|_{L^2}^{1-k/n} \|f\|_{H^k}^{k/n}$, we obtain for $g \in H^1([0, 1])$
\[
\|(A_0 - \lambda)^{-1}g\|_{H^1} \leq c_2 \|(A_0 - \lambda)^{-1}g\|_{L^2}^{1-k/n} \|(A_0 - \lambda)^{-1}g\|_{H^{2m}}^{k/n}
\]
\[
\leq \frac{c_3}{|\lambda|^{1-k/n}} \|g\|_{L^2} \leq \frac{c_3}{|\lambda|^{1-k/n}} \|g\|_{H^1}
\]
with $c_3 > 0$ depending on $\theta > 0$. Consequently $A$ is almost sectorial,
\[
\|(A - \lambda)^{-1}\| \leq \frac{c_3}{|\lambda|^{1-k/n}}, \quad |\arg \lambda| \geq \theta, \quad \beta = 1 - \frac{1}{2m}. \quad (40)
\]
Moreover $A$ even has a compact resolvent: indeed our calculations imply that $(A_0 - \lambda)^{-1}$ is a bounded operator from $L^2([0, 1])$ to $H^1([0, 1])$ and the embedding $H^1([0, 1]) \hookrightarrow L^2([0, 1])$ is compact. Finally note that $A$ is not densely defined since the closure of $\mathcal{D}(A)$ in $H^1([0, 1])$ is $H^1_0([0, 1])$. In particular, the estimate (40) cannot be improved to $\beta = 1$, i.e., $A$ is not a sectorial operator because sectorial operators in reflexive spaces are densely defined. On the other hand, (40) can be improved to $\beta = 1 - \frac{1}{2m}$ as indicated in [SS86].

### 8.5 A densely defined operator $S$ with non-densely defined almost sectorial $S|G_\pm$

The following is a variant of Example 8.3 in a Hilbert space setting. Here $S|G_+$ is non-densely defined and almost sectorial.

**Example 8.7.** Let $A$ be the operator from Example 8.6 acting on $X = H^1([0, 1])$. Then $\overline{\mathcal{D}(A)} = H^1_0([0, 1])$. Let $w_1(t) = t$, $w_2(t) = 1 - t$, so that
\[
X = \overline{\mathcal{D}(A)} \oplus \text{span}\{w_1, w_2\}.
\]
For $0 < s < 1/2$ define $B : \mathcal{D}(B) \subset l^2 \to X$ by
\[
B\alpha = \sum_{k=1}^{\infty} (2k)^s \alpha_{2k} \cdot w_1 + \sum_{k=1}^{\infty} (2k - 1)^s \alpha_{2k-1} \cdot w_2,
\]
\[
\mathcal{D}(B) = \left\{ \alpha = (\alpha_k)_{k=1}^{\infty} \in l^2 : \sum_{k=1}^{\infty} k^s |\alpha_k| < \infty \right\}.
\]
As in Example 8.3 consider also the selfadjoint operator $C(\alpha_k) = -(2^s \alpha_k)$ on $l^2$ and then $S : \mathcal{D}(S) \subset X \times l^2 \to X \times l^2$,
\[
S \left( \begin{pmatrix} f \\ \alpha \end{pmatrix} \right) = \begin{pmatrix} A(f - B\alpha) \\ C\alpha \end{pmatrix}, \quad \mathcal{D}(S) = \left\{ \begin{pmatrix} f \\ \alpha \end{pmatrix} \in X \times \mathcal{D}(C) : f - B\alpha \in \mathcal{D}(A) \right\}.
\]
We obtain again that $B$ is $1/q$-subordinate to $C$, that $S$ is densely defined and that $(S - \lambda)^{-1}$ is given by (39) where now $\sigma(S) = \sigma(A) \cup \sigma(C)$. Together with the estimate $\|(A - \lambda)^{-1}\| \leq c|\lambda|^{-\beta}$ on $i\mathbb{R}$, $\beta = 1 - \frac{1}{2m}$, we then arrive at

$$\|(S - \lambda)^{-1}\| \leq \frac{M}{|\lambda|^\gamma}, \quad \lambda \in i\mathbb{R} \setminus \{0\}, \quad \gamma = 1 - \frac{1}{2m} - \frac{1}{q},$$

with some $M > 0$. Theorems 4.1 and 5.6 thus yield the $S$-invariant subspaces $G_{\pm}$. As in Example 8.3, we derive $G_+ = X \times \{0\}$ and $S|G_+ \cong A$, which is not densely defined. Here $S|G_+$ is not sectorial but almost sectorial with $\beta = 1 - \frac{1}{2m}$.

### 8.6 Hamiltonian operator matrices

We can apply our theory to the Hamiltonian operator matrix appearing in systems theory in the case of so-called unbounded control and observation operators. We obtain criteria ensuring that the Hamiltonian is bisectorial and strictly dichotomous. Our setting generalises the ones in [TW14] and [WJZ12] since we do not require a basis of (generalised) eigenvectors of the Hamiltonian and at the same time allow the control and observation operators to map out of the state space.

**Example 8.8.** Let $A$ be a sectorial operator on a Hilbert space $H$ with angle less than $\pi/2$ and $0 \in \rho(A)$. Let $H_1 = \mathcal{D}(A)$ be equipped with the graph norm and $H_{-1}$ the completion of $H$ with respect to the norm $\|A^{-1} \cdot\|$. Let $H^1_{\pm}, H^2_{\pm}$ be the corresponding spaces for $A^*$. Moreover, we consider intermediate spaces in the sense of Lions and Magenes [LM72, Chapter 1],

$$H_s = [H_1, H]_{1-s}, \quad H_{-s} = [H, H_{-1}]_s, \quad s \in [0, 1],$$

Again, $H^1_s$ and $H^2_s$ are defined analogously. In the special case when $A$ is self-adjoint, we obtain the fractional domain spaces $H_s = H^1_s = \mathcal{D}(A^*), s \in [-1, 1]$, compare [WJZ12, §3]. In general however, $H_s \neq H^1_s$. The intermediate spaces yield bounded extensions

$$A : H_{1-s} \rightarrow H_{-s}, \quad A^* : H^1_{1-s} \rightarrow H^1_{-s}, \quad s \in [0, 1].$$

Using the scalar product $(\cdot | \cdot)$ of $H$, we can identify the dual of $H_{-s}$ with $H^d_s$. This means that the scalar product of $H$ extends to a sesquilinear form $(x|y)_{-s,s}, x \in H_{-s}, y \in H^d_s$. Similarly, the dual of $H_s$ is identified with $H^d_{1-s}$. This is also referred to as taking duality with respect to the pivot space $H$, see e.g. [TW09, §§2.9, 2.10].

Let $U, Y$ be Hilbert spaces and consider for some fixed $0 < s < 1/2$ bounded operators $B : U \rightarrow H_{-s}, C : H_s \rightarrow Y$, the control and observation operators, respectively. With the above duality identifications, their adjoints are bounded.

---

1 This is equivalent to taking complex interpolation spaces, see [LM72, Chapter 1, §14]. Note that all involved spaces are Hilbert spaces.
operators $B^* : H^d_s \to U$, $C^* : Y \to H^d_s$. The Hamiltonian is now given by the operator matrix
\[
T = \begin{pmatrix} A & BB^* \\ C^*C & -A^* \end{pmatrix}.
\]
Let $V_s = H_s \times H^d_s$. The Hamiltonian induces the bounded linear mapping $T : V_{1-s} \to V_{-s}$, which we consider, for the moment, as an unbounded operator on $V_{-s}$ with domain $V_{1-s}$. We use the decomposition
\[
T = S + R, \quad S = \begin{pmatrix} A & 0 \\ 0 & -A^* \end{pmatrix}, \quad R = \begin{pmatrix} 0 & BB^* \\ C^*C & 0 \end{pmatrix},
\]
where $S : V_{1-s} \to V_{-s}$ and $R : V_s \to V_{-s}$ are bounded. Let $\lambda \in \mathbb{i} \mathbb{R} \setminus \{0\}$. In the following estimates, $c$ denotes a positive constant which is independent of $\lambda$, but may change from estimate to estimate. By the sectoriality of $A$, we have
\[
\|(A - \lambda)^{-1}\| \leq c|\lambda|^{-1}, \quad \|A(A - \lambda)^{-1}\| \leq 1 + |\lambda|\|(A - \lambda)^{-1}\| \leq c
\]
and the second inequality implies
\[
\|(A - \lambda)^{-1}\|_{H \to H_s} \leq c, \quad \|(A - \lambda)^{-1}\|_{H_{1-s} \to H} \leq c.
\]
Interpolation yields
\[
\|(A - \lambda)^{-1}\|_{H \to H_s} \leq c|\lambda|^{-(1-s)}, \quad \|(A - \lambda)^{-1}\|_{H_{1-s} \to H} \leq c|\lambda|^{-(1-s)},
\]
and then
\[
\|(A - \lambda)^{-1}\|_{H \to H_s} \leq c|\lambda|^{-(1-2s)}.
\]
Since an analogous estimate holds for $(A^* - \lambda)^{-1}$, we obtain
\[
\|(S - \lambda)^{-1}\|_{V_{-s} \to V_s} \leq c|\lambda|^{-(1-2s)}. \quad (41)
\]
Consider the identity
\[
T - \lambda = (I + R(S - \lambda)^{-1})(S - \lambda).
\]
From (41) we obtain
\[
\|R(S - \lambda)^{-1}\|_{V_{-s} \to V_s} \leq \|R\|_{V_{-s} \to V_s}\|(S - \lambda)^{-1}\|_{V_{-s} \to V_s} \leq c|\lambda|^{-(1-2s)}
\]
and since $s < 1/2$ we get that, for $\lambda \in \mathbb{i} \mathbb{R}$ and $|\lambda|$ large, $I + R(S - \lambda)^{-1}$ is an isomorphism on $V_{-s}$; consequently $\lambda \in \sigma(T)$ with
\[
\|(T - \lambda)^{-1}\|_{V_{-s} \to V_s} \leq c\|(S - \lambda)^{-1}\|_{V_{-s} \to V_s} \leq c|\lambda|^{-1}.
\]
Moreover
\[
(T - \lambda)^{-1} - (S - \lambda)^{-1} = -(T - \lambda)^{-1}R(S - \lambda)^{-1}
\]
and hence
\[
\|(T - \lambda)^{-1} - (S - \lambda)^{-1}\|_{V_{-s} \to V_s} \leq c|\lambda|^{-(2-2s)}.
\]
We consider now the part of $T$ in $V$, which we denote again by $T$. Then, similar to the above, the identity

$$T - \lambda = (S - \lambda)(I + (S - \lambda)^{-1}R)$$

and the estimate

$$\|(S - \lambda)^{-1}R\|_{V \to V} \leq c|\lambda|^{-1-2s}$$

yield

$$\|(T - \lambda)^{-1}\|_{V \to V} \leq c|\lambda|^{-1},$$

$$\|(T - \lambda)^{-1} - (S - \lambda)^{-1}\|_{V \to V} \leq c|\lambda|^{-2-2s},$$

for $\lambda \in \mathbb{I}R$, $|\lambda|$ large.

For the rest of this example, we consider $T$ as an operator on $V$, i.e., we take the part of $T$ in $V$. Applying interpolation to the above results, we get that, if $\lambda \in \mathbb{I}R$ and $|\lambda|$ large enough, then $\lambda \in \varrho(T)$ and

$$\|(T - \lambda)^{-1}\| \leq c|\lambda|^{-1}$$

(42)

$$\|(T - \lambda)^{-1} - (S - \lambda)^{-1}\| \leq c|\lambda|^{-2-2s}.$$

(43)

Next we show that $T$ is bisectorial. In view of (42) it suffices to show that $\mathbb{I}R \subset \varrho(T)$. This will be done using the same technique as in [TW14, Lemma 4.5]. Suppose $i \in \mathbb{I}R$ is contained in the approximate point spectrum $\sigma_{\text{app}}(T)$, i.e., there exists a sequence $(x_n, y_n) \in V_{1-s}$ such that $\|x_n\|^2 + \|y_n\|^2 = 1$ and $(T - it)(x_n, y_n) \to 0$ in $V$ as $t \to \infty$. This implies

$$(A - it)x_n|y_n\rangle_{-s,s^*} + (BB^*y_n|y_n\rangle_{-s,s^*} \to 0,$$

$$(C^*Cx_n|x_n\rangle_{-s,s} - ((A^* + it)y_n|x_n\rangle_{-s,s} \to 0.$$

Summing both equations and taking the real part gives us

$$\|B^*y_n\|^2_U + \|C^*x_n\|^2 = (BB^*y_n|y_n\rangle_{-s,s^*} + (C^*Cx_n|x_n\rangle_{-s,s^*} \to 0.$$

(44)

From $(T - it)(x_n, y_n) \to 0$ it follows that $x_n + (A - it)^{-1}BB^*y_n \to 0$ as a limit in $H$. Since $(A - it)^{-1}B$ is bounded as an operator $U \to H$ and $B^*y_n \to 0$ by (44), this yields $x_n \to 0$. Analogously, using $C^*x_n \to 0$, we get $y_n \to 0$, which is a contradiction to $\|x_n\|^2 + \|y_n\|^2 = 1$. Therefore $\sigma_{\text{app}}(T) \cap \mathbb{I}R = \emptyset$. Since $\partial\sigma(T) \subset \sigma_{\text{app}}(T)$ and $\mathbb{I}R \subset \varrho(T)$ we conclude that in fact $\mathbb{I}R \subset \varrho(T)$.

We can now invoke Theorem 7.3 to show the strict dichotomy of $T$. Indeed the assumptions on $A$ imply that $S$ is bisectorial and strictly dichotomous. Moreover, by $\mathbb{I}R \subset \varrho(T)$ and (43) the conditions (i) and (ii) of Theorem 7.3 are satisfied too. Hence if $\mathcal{D}(S) \cap \mathcal{D}(T) = V_1 \cap \mathcal{D}(T) \subset V$ is dense, then $T$ is strictly dichotomous. Note that a typical setting in control theory is $H = L^2(\Omega)$ with $\Omega \subset \mathbb{R}^n$, $A$ is an elliptic differential operator and the control and observation operators act on traces of functions on the boundary of $\Omega$. In this case $\mathcal{D}(S) \neq \mathcal{D}(T)$ but $C^\infty_0(\Omega)^2 \subset \mathcal{D}(S) \cap \mathcal{D}(T)$, i.e., $\mathcal{D}(S) \cap \mathcal{D}(T)$ is in fact dense in $V$. 32
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References

[AZ10] W. Arendt and A. Zamboni. Decomposing and twisting bisectorial operators. Studia Math., 197(3):205–227, 2010.

[BGK86a] H. Bart, I. Gohberg, and M. A. Kaashoek. Wiener-Hopf equations with symbols analytic in a strip. In Constructive methods of Wiener-Hopf factorization, volume 21 of Oper. Theory Adv. Appl., pages 39–74. Birkhäuser, Basel, 1986.

[BGK86b] H. Bart, I. Gohberg, and M. A. Kaashoek. Wiener-Hopf factorization, inverse Fourier transforms and exponentially dichotomous operators. J. Funct. Anal., 68(1):1–42, 1986.

[Con78] J. B. Conway. Functions of one complex variable, volume 11 of Graduate Texts in Mathematics. Springer-Verlag, New York, second edition, 1978.

[DV89] G. Dore and A. Venni. Separation of two (possibly unbounded) components of the spectrum of a linear operator. Integral Equations Operator Theory, 12(4):470–485, 1989.

[Haa06] M. Haase. The functional calculus for sectorial operators, volume 169 of Operator Theory: Advances and Applications. Birkhäuser Verlag, Basel, 2006.

[KVL94] M. A. Kaashoek and S. M. Verduyn Lunel. An integrability condition on the resolvent for hyperbolicity of the semigroup. J. Differential Equations, 112(2):374–406, 1994.

[LM72] J.-L. Lions and E. Magenes. Non-homogeneous boundary value problems and applications. Vol. I. Springer-Verlag, New York, 1972. Translated from the French by P. Kenneth, Die Grundlehren der mathematischen Wissenschaften, Band 181.

[LRvdR02] H. Langer, A. C. M. Ran, and B. A. van de Rotten. Invariant subspaces of infinite dimensional Hamiltonians and solutions of the corresponding Riccati equations. In Linear operators and matrices, volume 130 of Oper. Theory Adv. Appl., pages 235–254. Birkhäuser, Basel, 2002.
[LT01] H. Langer and C. Tretter. Diagonalization of certain block operator matrices and applications to Dirac operators. In Operator theory and analysis (Amsterdam, 1997), volume 122 of Oper. Theory Adv. Appl., pages 331–358. Birkhäuser, Basel, 2001.

[MY90] A. McIntosh and A. Yagi. Operators of type $\omega$ without a bounded $H_\infty$ functional calculus. In Miniconference on Operators in Analysis (Sydney, 1989), volume 24 of Proc. Centre Math. Anal. Austral. Nat. Univ., pages 159–172. Austral. Nat. Univ., Canberra, 1990.

[PS02] F. Periago and B. Straub. A functional calculus for almost sectorial operators and applications to abstract evolution equations. J. Evol. Equ., 2(1):41–68, 2002.

[RvdM04] A. C. M. Ran and C. van der Mee. Perturbation results for exponentially dichotomous operators on general Banach spaces. J. Funct. Anal., 210(1):193–213, 2004.

[SS86] Yu. T. Sil’chenko and P. E. Sobolevski˘ı. Solvability of the Cauchy problem for an evolution equation in a Banach space with a non-densely given operator coefficient which generates a semigroup with a singularity. Sibirsk. Mat. Zh., 27(4):93–104, 214, 1986.

[TW09] M. Tucsnak and G. Weiss. Observation and control for operator semigroups. Birkhäuser Advanced Texts. Birkhäuser Verlag, Basel, 2009.

[TW14] C. Tretter and C. Wyss. Dichotomous Hamiltonians with unbounded entries and solutions of Riccati equations. J. Evol. Equ., 14(1):121–153, 2014.

[vdM08] C. van der Mee. Exponentially dichotomous operators and applications, volume 182 of Operator Theory: Advances and Applications. Birkhäuser Verlag, Basel, 2008. Linear Operators and Linear Systems.

[WJZ12] C. Wyss, B. Jacob, and H. J. Zwart. Hamiltonians and Riccati equations for linear systems with unbounded control and observation operators. SIAM J. Control Optim., 50(3):1518–1547, 2012.