RECURRENT SOLUTIONS OF STOCHASTIC DIFFERENTIAL EQUATIONS WITH NON-CONSTANT DIFFUSION COEFFICIENTS WHICH OBEY THE LAW OF THE ITERATED LOGARITHM

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Abstract. By using a change of scale and space, we study a class of stochastic differential equations (SDEs) whose solutions are drift–perturbed and exhibit behaviour analogous to standard Brownian motion including to the Law of the Iterated Logarithm (LIL). Sufficient conditions ensuring that these processes obey the LIL are given.

1. Introduction

The Law of the Iterated Logarithm is one of the most important results on the asymptotic behavior of one–dimensional standard Brownian motion:

$$\limsup_{t \to \infty} \frac{|B(t)|}{\sqrt{2t \log \log t}} = 1, \quad \text{a.s.}$$

Classical work on iterated logarithm–type results, as well as associated lower bounds on the growth of transient processes date back to Dvoretzky and Erdős [5]. There is an interesting literature on iterated logarithm results and the growth of lower envelopes for self-similar Markov processes (cf. e.g., Rivero [12], Chaumont and Pardo [4]) which exploit a Lamperti representation [9], processes conditioned to remain positive (cf. Hambly et al. [6]), and diffusion processes with special structure (cf. e.g. Bass and Kumagi [3]).

In contrast to these papers the analysis here is inspired by work of Motoo [11] on iterated logarithm results for Brownian motions in finite dimensions, in which the asymptotic behaviour is determined by means of time change arguments to reduce the process under study to a stationary one.

2. Main Result

2.1. Preliminaries. Throughout the paper, we use $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t \geq 0}, \mathbb{P})$ to denote a complete filtered probability space. The set of non-negative real numbers is denoted by $\mathbb{R}^+$. Let $L^1([a, b]; \mathbb{R})$ be the family of Borel measurable functions $h : [a, b] \to \mathbb{R}$ such that $\int_a^b |h(x)|dx < \infty$. The abbreviation a.s. stands for almost surely.

Throughout the paper, we assume that both the drift and the diffusion coefficients of SDE being studied satisfies the local Lipschitz condition. If an autonomous
scalar SDE has drift coefficient \( f(\cdot) \) and non-degenerate diffusion coefficient \( g(\cdot) \), then a scale function and speed measure of solution of this SDE are defined as

\[
s_c(x) = \int_c^\infty e^{-2 \int_x^\infty \frac{1}{s'(x)} \, dy} \, dx, \quad m(dx) = \frac{2dx}{s'(x)g^2(x)}, \quad c, x \in (0, \infty)
\]  

(2.1)

respectively. These functions help us to determine the recurrence and stationarity of a process on \((0, \infty)\) (cf.[8]). Moreover, Feller’s test for explosions (cf.[8]) allows us to examine whether a process will never escape from its state space in finite time. This in turn relies on whether

\( v(0+) = v(\infty-) = \infty \)

or not, where \( v \) is defined as

\[
v_c(x) = \int_c^\infty s_c'(y) \int_c^y \frac{2dz}{s_c'(z)g^2(z)} \, dy, \quad c, x \in (0, \infty).
\]  

(2.2)

As mentioned in the introduction, Motoo’s Theorem is an important tool in determining the largest deviations for any stationary or asymptotically stationary processes, we state it here for future use.

**Theorem 2.1.** (Motoo) Let \( X \) be the unique continuous real valued process satisfying the following equation

\[
dX(t) = f(X(t)) \, dt + g(X(t)) \, dB(t), \quad t \geq 0,
\]

with \( X(0) = x_0 \). Let \( s \) and \( m \) be the scale function and speed measure of \( X \) as defined in (2.1), and let \( h : (0, \infty) \to (0, \infty) \) be an increasing function. If \( X \) is recurrent on \((0, \infty)\) (or \([0, \infty)\) in the case when \( 0 \) is an instantaneous reflecting point) and \( m(0, \infty) < \infty \), then

\[
\mathbb{P} \left[ \limsup_{t \to \infty} \frac{X(t)}{h(t)} \geq 1 \right] = 1 \text{ or } 0
\]

according to whether

\[
\int_0^\infty \frac{1}{s'(h(t))} \, dt = \infty \quad \text{or} \quad \int_0^\infty \frac{1}{s'(h(t))} \, dt < \infty, \quad \text{for some } t_0 > 0.
\]

2.2. The main result. Let \( g : \mathbb{R} \to \mathbb{R} \) be locally Lipschitz continuous and suppose that \( f : [0, \infty) \times \mathbb{R} \to \mathbb{R} \) is locally Lipschitz continuous. Suppose that

There exists \( \rho > 0 \) such that \( \sup_{(x,t) \in [0, \infty) \times \mathbb{R}} x f(x, t) \leq \rho \), \hspace{1cm} (2.3)

There exists \( \mu > -1/2 \) such that \( \mu = \inf_{(x,t) \in [0, \infty) \times \mathbb{R}} \frac{x f(x, t)}{g^2(x)} \). \hspace{1cm} (2.4)

Suppose also that \( g : \mathbb{R} \to \mathbb{R} \) satisfies

\[
\forall x \in \mathbb{R}, \quad g(x) \neq 0, \quad \lim_{|x| \to \infty} g(x) = \sigma \in \mathbb{R}/\{0\}.
\]  

(2.5)

Then there exists a unique continuous adapted process \( X \) which obeys

\[
dX(t) = f(X(t), t) \, dt + g(X(t)) \, dB(t), \quad t \geq 0, \quad X(0) = x_0
\]  

(2.6)

(see e.g., [10]).

**Theorem 2.2.** Suppose that \( f : [0, \infty) \times \mathbb{R} \to \mathbb{R} \) is locally Lipschitz continuous. Suppose \( g : \mathbb{R} \to \mathbb{R} \) is locally Lipschitz, even and satisfies (2.3), and that \( f \) and \( g \) obey (2.4). Then there is a unique continuous adapted process satisfying (2.3). If moreover \( f \) and \( g \) obey (2.4), then \( X \) obeys

\[
\limsup_{t \to \infty} \frac{|X(t)|}{\sqrt{2t \log \log t}} = |\sigma|, \quad a.s.
\]  

(2.7)
In a recent paper [2], we established (2.7) for the solution of (2.6) under the conditions (2.3), (2.5). In the condition (2.3) it was presumed that $\mu > 1/2$. This restriction forces $|X(t)| \to \infty$ as $t \to \infty$ a.s. Therefore we have shown here that the asymptotic growth implied by (2.7) also holds in some cases where the diffusion coefficient is asymptotically constant and the solution can be recurrent rather than transient.

In the case when $g$ is constant with $g(x) = \sigma$ (so that $g$ obviously obeys (2.5)), $f$ obeys (2.3) and (2.4) (with the last condition written as $\inf_{(x,t) \in \mathbb{R} \times (0,\infty)} xf(x,t) > -\sigma^2/2$), it was proven in [2] that the solution of (2.6) obeys (2.7). In this paper, we have managed to extend the result to the case where the diffusion coefficient is only asymptotically constant.

3. Proof of Theorem 2.2

The result is proven by establishing that the solution $X$ obeys

$$\limsup_{t \to \infty} \frac{|X(t)|}{\sqrt{2t \log \log t}} \leq |\sigma|, \text{ a.s.} \quad (3.1)$$

and then that it obeys

$$\limsup_{t \to \infty} \frac{|X(t)|}{\sqrt{2t \log \log t}} \geq |\sigma|, \text{ a.s.} \quad (3.2)$$

The proof of (3.1) was given in [2], but is included here in order to introduce notation used in the proof of (3.2).

3.1. Proof of (3.1). Without loss of generality, we can choose $\rho > \sigma^2/2$. By Itô’s rule

$$X^2(t) = x_0^2 + \int_0^t 2X(s)f(X(s), s) + g^2(X(s)) \, ds + \int_0^t 2X(s)g(X(s)) \, dB(s), \quad \text{a.s.}$$

Let $Z(t) = X^2(t)$ so that $|X(t)| = \sqrt{Z(t)}$ for all $t \geq 0$. Define $\gamma(x) = x/|x|$ for $x \neq 0$ and $\gamma(0) = 1$. Then $\gamma^2(x) = 1$ for all $x \in \mathbb{R}$. Also define

$$W(t) = \int_0^t \gamma(X(s)) \, dB(s), \quad \text{a.s.}$$

Then $W$ is a $\mathcal{F}^B$-adapted Brownian motion. If $Z(t) \neq 0$

$$2\sqrt{Z(t)}g(\sqrt{Z(t)})\gamma(X(t)) = 2|X(t)||g(X(t))|\gamma(X(t)) = 2|X(t)|g(X(t))\gamma(X(t)) = 2X(t)g(X(t))\gamma^2(X(t)) = 2X(t)g(X(t)).$$

If $Z(t) = 0$ then $2\sqrt{Z(t)}g(\sqrt{Z(t)})\gamma(X(t)) = 0 = 2X(t)g(X(t))$. Hence,

$$Z(t) = x_0^2 + \int_0^t 2X(s)f(X(s), s) + g^2(\sqrt{Z(s)}) \, ds + \int_0^t 2\sqrt{Z(s)}g(\sqrt{Z(s)}) \, dW(s),$$

and so with $\beta(t) := 2X(t)f(X(t), t) - \rho \leq 0$ for $t \geq 0$, we have

$$Z(t) = x_0^2 + \int_0^t \beta(s) + 2\rho + g^2(\sqrt{Z(s)}) \, ds + \int_0^t 2\sqrt{Z(s)}g(\sqrt{Z(s)}) \, dW(s), \quad \text{a.s.} \quad (3.3)$$

Note also that $\beta$ is $\mathcal{F}^B$-adapted. Define $Z_u(t)$ by

$$Z_u(t) = 1 + x_0^2 + \int_0^t 2\rho + g^2(\sqrt{|Z_u(s)|}) \, ds + \int_0^t 2\sqrt{|Z_u(s)|}g(\sqrt{|Z_u(s)|}) \, dW(s), \quad \text{a.s.} \quad (3.4)$$

Since $W$ is a $\mathcal{F}^B$-Brownian motion, $Z_u$ is $\mathcal{F}^B$-adapted. Notice that

$$dZ_u(t) = \left\{2\rho + g^2(\sqrt{|Z_u(t)|})\right\} \, dt + 2\sqrt{|Z_u(t)|}g(\sqrt{|Z_u(t)|}) \, dW(t).$$
Define the process \( X_u \) by
\[
dX_u(t) = \frac{\mu}{X_u(t)} \, dt + g(X_u(t)) \, dB(t), \quad t \geq 0
\]
where \( X_u(0) = \sqrt{1 + |\rho|^2} \). It is easy to check that the scale function of \( X_u \) satisfies \( \sigma_{X_u}(\infty) < \infty \) and \( \sigma_{X_u}(0) = \infty \). Thus \( \mathbb{P}[\lim_{t \to \infty} X_u(t) = \infty] = 1 \). Moreover \( v_{X_u}(\infty) = v_{X_u}(0) = \infty \), which implies that \( \mathbb{P}[\lim_{t \to \infty} X_u(t) > 0; \forall 0 < t < \infty] = 1 \). Then \( Z_u = \frac{X_u}{v_{X_u}} > 0 \) obeys \( \{2\} \). Therefore we have
\[
dZ_u(t) = \left\{2\rho + g^2(\sqrt{Z_u(t)})\right\} \, dt + 2\sqrt{Z_u(t)} g(\sqrt{Z_u(t)}) \, dW(t),
\]
and so by the Ikeda-Watanabe comparison theorem [2] Chapter VI, Theorem 1.1 we have \( Z_u(t) \geq Z(t) \) for all \( t \geq 0 \) a.s. and so \( X_u^2(t) \leq X_u^2(t) \) for all \( t \geq 0 \) a.s. By a result in [2], it is known that \( X_u \) obeys
\[
\limsup_{t \to \infty} \frac{X_u(t)}{\sqrt{2t \log \log t}} = |\sigma|, \quad \text{a.s.,}
\]
which implies \( \{3\} \).

3.2. Proof of \( \{3.2\} \). Define
\[
\beta_2(t) = 2 \left( \frac{X(t) f(X(t), t)}{g^2(X(t))} - \mu \right) g^2(X(t)), \quad t \geq 0.
\]
Then \( \beta_2 \) is \( \mathcal{F}^B \)-adapted and \( \beta_2(t) \geq 0 \) for all \( t \geq 0 \). Moreover
\[
2X(t) f(X(t), t) + g^2(X(t)) = \beta_2(t) + (2\mu + 1) g^2(\sqrt{Z(t)}), \quad t \geq 0.
\]
Therefore from \( \{3.3\} \) we obtain
\[
Z(t) = x_0^2 + \int_0^t \left\{ \beta_2(s) + (2\mu + 1) g^2(\sqrt{Z(s)}) \right\} \, ds + \int_0^t 2\sqrt{Z(s)} g(\sqrt{Z(s)}) \, dW(s), \tag{3.5}
\]
Define \( Z_L(t) \) by
\[
Z_L(t) = x_0^2 + \int_0^t (2\mu + 1) g^2(\sqrt{Z_L(s)}) \, ds + \int_0^t 2\sqrt{Z_L(s)} g(\sqrt{Z_L(s)}) \, dW(s), \quad \text{a.s.} \tag{3.6}
\]
Define
\[
\theta(t) = \int_0^t g^2(\sqrt{Z_L(s)}) \, ds, \quad t \geq 0. \tag{3.7}
\]
Then \( \theta \) is increasing and let \( \tau = \theta^{-1} \). Define \( \tilde{Z}_L(t) = Z_L(\tau(t)) \) for \( t \geq 0 \). Let \( \mathcal{G}(t) = \mathcal{F}^B(\tau(t)) \) for all \( t \geq 0 \) and define
\[
M(t) = \int_0^t 2\sqrt{Z_L(s)} g(\sqrt{Z_L(s)}) \, dW(s), \quad t \geq 0.
\]
Then \( M \) is a \( \mathcal{G}(t) \)-martingale. Clearly
\[
\langle M \rangle(t) = \int_0^t 4|Z_L(s)| g^2(\sqrt{Z_L(s)}) \, ds = \int_0^t 4|\tilde{Z}_L(u)| \, du.
\]
Thus, by Theorem 3.4.2 in [3], there is an extension of \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})\) of \((\Omega, \mathcal{F}, \mathbb{P})\) on which is defined a one-dimensional Brownian motion \( \tilde{W} = \{\tilde{W}(t); \tilde{\mathcal{G}}(t); 0 \leq t < +\infty\} \) such that
\[
M(t) = \int_0^t 2\sqrt{Z_L(s)} \, d\tilde{W}(s), \quad \tilde{\mathbb{P}}\text{-a.s.} \tag{3.8}
\]
The filtration \( \mathcal{G}(t) \) in the extended space is such that \( \tilde{Z}_L \) is \( \mathcal{G}(t) \)-adapted. Thus by (3.11) and (3.3) we have
\[
\tilde{Z}_L(t) = x_0^2 + \int_0^t (2\mu + 1) \, ds + \int_0^t 2\sqrt{\tilde{Z}_L(s)} \, d\tilde{W}(s), \quad t \geq 0. \tag{3.9}
\]
Since \( \mu > -1/2 \), it now follows that \( \tilde{Z}_L(t) \geq 0 \) for all \( t \geq 0 \) a.s. Therefore \( Z_L(t) \geq 0 \) for all \( t \geq 0 \) a.s. Hence \( Z_L \) obeys
\[
dZ_L(t) = (2\mu + 1)g^2(\sqrt{Z_L(t)}) \, dt + 2\sqrt{Z_L(t)}g(\sqrt{Z_L(t)}) \, dW(t), \quad t \geq 0 \text{ a.s.}
\]
and so by (3.5) and the Ikeda–Watanabe comparison theorem [7, Chapter VI, Theorem 1.1], we have \( Z(t) \geq Z_L(t) \) for all \( t \geq 0 \) a.s. Hence \( X^2(\tau(t)) \geq Z_L(\tau(t)) = \tilde{Z}_L(t) \) for all \( t \geq 0 \) a.s. If we define \( Z_L(t) = \tilde{Z}_L(e^t - 1) \) for \( t \geq 0 \), then there is another Brownian motion \( W \) such that
\[
d\tilde{Z}_L(t) = (2\mu + 1)e^t \, dt + 2\sqrt{\tilde{Z}_L(t)e^{t/2}} \, d\tilde{W}(t), \quad t \geq 0.
\]
Define
\[
U(t) = e^{-t} \tilde{Z}_L(e^t - 1) = e^{-t} \tilde{Z}_L(t), \quad t \geq 0. \tag{3.10}
\]
Then \( U(0) = x_0^2 \) and
\[
dU(t) = ((2\mu + 1) - U(t)) \, dt + 2\sqrt{U(t)} \, dW(t), \quad t \geq 0.
\]
By theorem 2.1 it follows that
\[
\limsup_{t \to \infty} \frac{U(t)}{2 \log t} = 1, \quad \text{a.s.}
\]
Using the connection between \( U \) and \( \tilde{Z}_L \) we obtain
\[
\limsup_{t \to \infty} \frac{\tilde{Z}_L(t)}{2t \log t} = 1, \quad \text{a.s.}
\]
Therefore
\[
\limsup_{t \to \infty} \frac{X^2(\tau(t))}{2t \log t} \geq \limsup_{t \to \infty} \frac{\tilde{Z}_L(t)}{2t \log t} = 1, \quad \text{a.s.}
\]
Since \( \theta = \tau^{-1} \) and \( \theta(t) \to \infty \) as \( t \to \infty \) we have
\[
\limsup_{t \to \infty} \frac{X^2(t)}{2\theta(t) \log \theta(t)} \geq 1, \quad \text{a.s.} \tag{3.11}
\]
By (2.4), \( g^2(x) \to \sigma^2 > 0 \) as \( |x| \to \infty \). Since \( g^2 \) is continuous and \( g(x) \neq 0 \) for all \( x \in \mathbb{R} \) (by assumption (2.3)), it follows that there exist \( K_1^2 > 0 \) and \( K_2^2 \in [K_1^2, \infty) \) such that \( 0 < K_1^2 \leq g^2(x) \leq K_2^2 \) for all \( x \in \mathbb{R} \). Therefore
\[
0 < K_1^2 t \leq \theta(t) \leq K_2^2 t, \quad \text{for all } t \geq 0 \text{ a.s.,} \tag{3.12}
\]
which implies
\[
\lim_{t \to \infty} \frac{\log \log \theta(t)}{\log t} = 1, \quad \text{a.s.}
\]
Using (3.11) now yields
\[
\limsup_{t \to \infty} \frac{X^2(t)}{2\theta(t) \log \theta(t)} \geq 1, \quad \text{a.s.} \tag{3.13}
\]
The rest of the proof is devoted to showing that
\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t g^2(\sqrt{Z_L(s)}) \, ds = \sigma^2, \quad \text{a.s.} \tag{3.14}
\]
which together with (3.13) yields (3.12).

We prove (3.14) in four steps:
Step A. If $U$ is the process defined in (3.10), then for every $c > 0$
\[\int_{1}^{\infty} \tilde{P}[U(s) \leq ce^{-s}] \, ds < +\infty \] (3.15)

Step B. If $U$ is the process defined in (3.10), then (3.15) implies
\[\lim_{t \to \infty} \frac{1}{t} \int_{1}^{t} e^{s} I(U(s) \leq ce^{-s}) \, ds = 0, \quad \text{a.s.} \] (3.16)

Step C. (3.10) implies
\[\lim_{t \to \infty} \frac{1}{t} \int_{1}^{t} I(Z_{L}(s) \leq c) \, ds = 0, \quad \text{a.s.} \] (3.17)

Step D. If $Z_{L}$ obeys (3.17), then it also obeys (3.14).

The proof of Steps A–D are given in the next four subsections, which completes the proof of (3.2).

3.3. Proof of Step A, i.e., (3.15). Clearly by the definition of $U$ and $\tilde{Z}_{L}$ we have
\[\int_{1}^{\infty} \tilde{P}[U(s) \leq ce^{-s}] \, ds = \int_{c-1}^{\infty} \frac{1}{1 + t} \tilde{P}[\tilde{Z}_{L}(t) \leq c] \, dt. \] (3.18)

Define the modified Bessel function with index $\nu \in \mathbb{R}$ by
\[I_{\nu}(x) = \left(\frac{x}{2}\right)^{\nu} \sum_{n=0}^{\infty} \frac{(x/2)^{2n}}{n! \Gamma(\nu + n + 1)}, \quad x \geq 0. \] (3.19)

Clearly
\[\lim_{x \to +0^{+}} \frac{I_{\nu}(x)}{x^{\nu}} = \left(\frac{1}{2}\right)^{\nu} \frac{1}{\Gamma(\nu + 1)}. \] (3.20)

Define $\delta = 2\mu + 1 > 0$ and $\nu(\delta) = \delta/2 - 1$. Since $\tilde{Z}_{L}$ obeys (3.9) we have
\[\tilde{P}[\tilde{Z}_{L}(t) \leq c] = \int_{0}^{c} q_{t}(\tilde{x}_{0}^{2}, y) \, dy, \] (3.21)

where
\[q_{t}(\tilde{x}_{0}^{2}, y) = \begin{cases} \frac{1}{\sqrt{2\pi}} \left(\frac{y}{\tilde{x}_{0}^{2}}\right)^{\nu(\delta)/2} \exp \left(\frac{-y^{2} + y}{\tilde{x}_{0}^{2}}\right) I_{\nu(\delta)}(|\tilde{x}_{0}|\sqrt{2}/t), & \tilde{x}_{0} \neq 0, \\ \frac{1}{\Gamma(\delta/2)} y^{\delta/2 - 1} \exp \left(\frac{-y}{\tilde{x}_{0}^{2}}\right), & \tilde{x}_{0} = 0. \end{cases} \] (3.22)

See e.g., [13, Ch. XI, Corollary 1.4] and [24, Chap. IV., Example 8.3], noting that there is a missing factor of $1/t$ in the first formula in [13, Ch. XI, Corollary 1.4].

We now estimate $q_{t}(\tilde{x}_{0}^{2}, y)$, and use this to prove that $\tilde{P}[\tilde{Z}_{L}(t) \leq c]$ tends to zero sufficiently quickly as $t \to \infty$. This guarantees that the integral on the righthand side of (3.18) is finite and hence that (3.15) holds.

In the case that $\tilde{x}_{0} = 0$, since $\delta > 0$ we have
\[\tilde{P}[\tilde{Z}_{L}(t) \leq c] = \frac{1}{(2t)^{\delta/2}} \frac{1}{\Gamma(\delta/2)} \int_{0}^{c} y^{\delta/2 - 1} \exp \left(\frac{-y}{2t}\right) \, dy \leq \frac{1}{(2t)^{\delta/2}} \frac{1}{\Gamma(\delta/2)} \int_{0}^{c} y^{\delta/2 - 1} \, dy = \frac{1}{(2t)^{\delta/2}} \frac{1}{\Gamma(\delta/2)} c^{\delta/2}. \]

Inserting this estimate into (3.18) and using the fact that $\delta > 0$ implies (3.15).

In the case when $\tilde{x}_{0} \neq 0$, by (3.20) there exists $\tilde{x}_{3} > 0$ such that
\[I_{\nu(\delta)}(x) < 2 \left(\frac{1}{2}\right)^{\nu(\delta)} \frac{1}{\Gamma(\nu(\delta) + 1)} \cdot x^{\nu(\delta)}, \quad x < \tilde{x}_{3}. \]
Define \( t_{\delta,c} = |x_0|\sqrt{c}/\bar{x}_\delta \). Then for \( 0 \leq y \leq c \) and \( t > t_{\delta,c} \) we have \( |x_0|\sqrt{\gamma}/t < \bar{x}_\delta \). 

Thus

\[ q_t(x_0^2, y) \leq \left( \frac{1}{2} \right)^{\delta/2 - 1} \frac{1}{\Gamma(\delta/2)} \frac{1}{t^{\delta/2}} \frac{1}{\bar{x}_\delta^{\delta/2}}, \quad t > t_{\delta,c}, \ y \in [0,c]. \]

Therefore for \( t \geq t_{\delta,c} \) the last estimate yields

\[ \mathbb{P}[\tilde{Z}_L(t) \leq c] = \int_0^c q_t(x_0^2, y) \, dy \leq \left( \frac{1}{2} \right)^{\delta/2 - 1} \frac{1}{\Gamma(\delta/2)} \int_0^c \frac{1}{t^{\delta/2}} \frac{1}{\bar{x}_\delta^{\delta/2}} \, dy \]

\[ = \left( \frac{1}{2} \right)^{\delta/2 - 1} \frac{1}{\Gamma(\delta/2)} \frac{1}{\bar{x}_\delta^{\delta/2}} \cdot \left( \frac{\delta}{2} \right) ^{\delta/2}. \]

Inserting this estimate into (3.21) we have

\[ \mathbb{P}[U(t) \leq ce^{-t}] \]

by (3.21) we have

\[ \mathbb{P}[\tilde{Z}_L(e^t - 1) \leq ce^{-t}] = \mathbb{P}[\tilde{Z}_L(e^t - 1) \leq c] = \int_0^c q_{e^t-1}(x_0^2, y) \, dy \]

where \( q \) is given by (3.22). Therefore \( \pi^*_c \in C([1,\infty),(0,\infty)) \) and by (3.23) we have \( \pi^*_c \in L^1([1,\infty),(0,\infty)) \). Next define

\[ F_c(t) = \frac{1}{e^t} \int_1^t e^s I_{\{U(s) \leq ce^{-s}\}} \, ds, \quad t \geq 1. \]

Then \( \lim_{t \to \infty} F_c(t) = 0 \) a.s. implies (3.10).

Since \( \mathbb{E}[F_c(t)] = e^{-t} \int_t^1 e^s \pi^*_c(s) \, ds \) and \( \pi^*_c \in L^1([1,\infty),(0,\infty)) \), we have that \( \int_1^\infty \mathbb{E}[F_c(t)] \, dt < +\infty. \) Since \( \pi^*_c \in C([1,\infty),(0,\infty)) \), we have that \( \mathbb{E}[F_c(\cdot)] \in \mathcal{O}([1,\infty),(0,\infty)) \). Then by [1] Lemma 2.3, there exists a deterministic and increasing sequence \( (a_n(c))_{n \geq 0} \) with \( a_0 = 1 \) and \( a_n(c) \to \infty \) as \( n \to \infty \) such that

\[ \sum_{n=0}^\infty \mathbb{E}[F_c(a_n(c))] < +\infty. \]

Next, define \( G_c(n) = \int_{a_n(c)}^{a_{n+1}(c)} I_{\{U(s) \leq ce^{-s}\}} \, ds \). Then \( \mathbb{E}[G_c(n)] = \int_{a_n(c)}^{a_{n+1}(c)} \pi^*_c(s) \, ds \). Since \( \pi^*_c \in L^1([1,\infty),(0,\infty)) \) we have

\[ \sum_{n=0}^\infty \mathbb{E}[G_c(n)] = \int_1^\infty \pi^*_c(s) \, ds < +\infty. \]

Now let \( t \in [a_n(c),a_{n+1}(c)] \). Then

\[ F_c(t) = e^{-(t-a_n(c))} F_c(a_n(c)) + \int_{a_n(c)}^t e^{-(t-s)} I_{\{U(s) \leq ce^{-s}\}} \, ds \]

\[ \leq F_c(a_n(c)) + G_c(n). \]

Therefore by (3.23) and (3.24) we have

\[ \sum_{n=0}^\infty \mathbb{E} \left[ \sup_{a_n(c) \leq t \leq a_{n+1}(c)} F_c(t) \right] \leq \sum_{n=0}^\infty \mathbb{E}[F_c(a_n(c))] + \int_1^\infty \pi^*_c(s) \, ds < +\infty \]

Therefore

\[ \sum_{n=0}^\infty \mathbb{E} \left[ \sup_{a_n(c) \leq t \leq a_{n+1}(c)} F_c(t) \right] = 0 \text{ a.s.} \]

and so \( \lim_{n \to \infty} \sup_{a_n(c) \leq t \leq a_{n+1}(c)} F_c(t) = 0 \) a.s. Hence \( F_c(t) \to 0 \) as \( t \to \infty \) a.s.
3.5. Proof of Step C i.e., \((3.17)\). We now show that \((3.16)\) implies \((3.17)\). By the definition of \(\theta\) in \((3.7)\), we have that \(\theta'(t) = \theta'(\sqrt{L_L(t)})\) for \(t > 0\). Recall moreover \(0 < K_1^2 < \theta'(x) < K_2^2\) for \(x \in \mathbb{R}\). Therefore

\[
\frac{1}{t} \int_1^t I_{\{Z_L(s) \leq c\}} \, ds = \frac{1}{t} \int_1^t I_{\{\tilde{Z}_L(\theta(s)) \leq c\}} \, ds = \frac{1}{t} \int_{\theta(1)}^{\theta(t)} I_{\{\tilde{Z}_L(\theta) \leq c\}} \frac{1}{\theta'(\theta)} \, d\theta(u)
\]

where we used \((3.12)\) at the last step. Hence

\[
\limsup_{t \to \infty} \frac{1}{t} \int_1^t I_{\{Z_L(s) \leq c\}} \, ds \leq \limsup_{t \to \infty} \frac{1}{t} \int_0^t I_{\{\tilde{Z}_L(s) \leq c\}} \, ds.
\] (3.25)

Now using the connection between \(\tilde{Z}_L\) and \(U\) we have

\[
\frac{1}{t} \int_0^t I_{\{\tilde{Z}_L(\theta) \leq c\}} \, d\theta = \frac{1}{t} \int_0^{\log(1+t)} I_{\{U(s) \leq ec^{-\epsilon}\}} e^s \, ds.
\]

Using this identity and \((3.25)\) we have

\[
\limsup_{t \to \infty} \frac{1}{t} \int_1^t I_{\{Z_L(s) \leq c\}} \, ds \leq \limsup_{t \to \infty} \frac{1}{t} \int_0^{\log(1+t)} I_{\{U(s) \leq ec^{-\epsilon}\}} e^s \, ds
\]

\[
= \frac{K_2^2}{K_1^2} \limsup_{t \to \infty} \frac{1}{t} \int_0^t I_{\{U(s) \leq ec^{-\epsilon}\}} e^s \, ds.
\]

Therefore, as \((3.16)\) holds, we have \((3.17)\).

3.6. Proof of Step D i.e., \((3.14)\). We have that \((3.17)\) holds i.e., for each \(c > 0\) there exists an a.s. event \(\Omega_c\) such that

\[
\Omega_c := \left\{ \omega : \lim_{t \to \infty} \frac{1}{t} \int_0^t I_{\{Z_L(s,\omega) \leq c\}} \, ds = 0 \right\}.
\]

Moreover, for \(\omega \in \Omega_c\) we also have

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t I_{\{Z_L(s,\omega) > c\}} \, ds = 1.
\] (3.26)

For every \(\epsilon \in (0, 1)\), there exists \(c(\epsilon) > 0\) such that \(\sigma^2(1 - \epsilon) < \sigma^2(1 + \epsilon)\) for all \(x > \sqrt{c(\epsilon)}\). Therefore

\[
\sigma^2(1 - \epsilon)I_{\{Z_L(s) > c(\epsilon)\}} \leq g^2(\sqrt{Z_L(s)})I_{\{Z_L(s) > c(\epsilon)\}} \leq \sigma^2(1 + \epsilon)I_{\{Z_L(s) > c(\epsilon)\}}.
\]

Therefore

\[
\sigma^2(1 - \epsilon)\frac{1}{t} \int_0^t I_{\{Z_L(s) > c(\epsilon)\}} \, ds \leq \frac{1}{t} \int_0^t g^2(\sqrt{Z_L(s)})I_{\{Z_L(s) > c(\epsilon)\}} \, ds
\]

\[
\leq \sigma^2(1 + \epsilon)\frac{1}{t} \int_0^t I_{\{Z_L(s) > c(\epsilon)\}} \, ds.
\] (3.27)
Now, for $\omega \in \Omega_\epsilon(c)$, by using (3.29) and the lefthand member of (3.27), we have
\[
\liminf_{t \to \infty} \frac{1}{t} \int_0^t g^2(\sqrt{Z_L(s)}) \, ds \geq \liminf_{t \to \infty} \frac{1}{t} \int_0^t g^2(\sqrt{Z_L(s)}) I_{\{Z_L(s) > c\}} \, ds \\
\geq \sigma^2(1 - \epsilon) \liminf_{t \to \infty} \frac{1}{t} \int_0^t I_{\{Z_L(s) > c\}} \, ds \\
= \sigma^2(1 - \epsilon).
\]
Therefore with $\Omega_\epsilon^1 = \cap_{t \in (0,1)} \cap_{c} \Omega_\epsilon(c)$ we have that $\Omega_\epsilon^1$ is an almost sure event and moreover
\[
\liminf_{t \to \infty} \frac{1}{t} \int_0^t g^2(\sqrt{Z_L(s)}) \, ds \geq \sigma^2, \quad \text{a.s. on } \Omega_\epsilon^1. \tag{3.28}
\]
To obtain an upper bound, first note that $g^2(x) \leq K_2^2$ implies
\[
\frac{1}{t} \int_0^t g^2(\sqrt{Z_L(s)}) I_{\{Z_L(s) \leq c\}} \, ds \leq K_2^2 \frac{1}{t} \int_0^t I_{\{Z_L(s) \leq c\}} \, ds.
\]
Therefore
\[
\limsup_{t \to \infty} \frac{1}{t} \int_0^t g^2(\sqrt{Z_L(s)}) I_{\{Z_L(s) \leq c\}} \, ds = 0, \quad \text{a.s. on } \Omega_\epsilon(c). \tag{3.29}
\]
Since
\[
\frac{1}{t} \int_0^t g^2(\sqrt{Z_L(s)}) \, ds \\
= \frac{1}{t} \int_0^t g^2(\sqrt{Z_L(s)}) I_{\{Z_L(s) \leq c\}} \, ds + \frac{1}{t} \int_0^t g^2(\sqrt{Z_L(s)}) I_{\{Z_L(s) > c\}} \, ds,
\]
by combining (3.29) with the righthand member of (3.27) we get
\[
\limsup_{t \to \infty} \frac{1}{t} \int_0^t g^2(\sqrt{Z_L(s)}) \, ds \leq \sigma^2(1 + \epsilon) \quad \text{a.s. on } \Omega_\epsilon(c).
\]
Therefore with $\Omega_\epsilon^1$ as the almost sure event defined above we have
\[
\liminf_{t \to \infty} \frac{1}{t} \int_0^t g^2(\sqrt{Z_L(s)}) \, ds \leq \sigma^2, \quad \text{a.s. on } \Omega_\epsilon^1. \tag{3.30}
\]
Combining (3.28) and (3.30) yields (3.14) as required.

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