Entangling strings of neutral atoms in 1D atomic pipeline structures

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We study a chain of neutral atoms with nearest neighbor interaction in a 1D beam splitter configuration, where the longitudinal motion is controlled by a moving optical lattice potential. The dynamics of the atoms crossing the beam splitter maps to a 1D spin model with controllable time dependent parameters, which allows the creation of maximally entangled states of atoms by crossing a quantum phase transition. Furthermore, we show that this system realizes protected quantum memory, and we discuss the implementation of one- and two-qubit gates in this setup.

The recent development of optical and magnetic microtraps allows the confinement of cold atoms in effective 1D “pipeline” geometries, where the transverse quantum motion is frozen out$^1$. Variants of these trap designs promise the realization of beam splitters, and thus atomic interferometry “on a chip”. Usually one envisions that atoms are injected one by one into these pipelines, where the source of cold atoms is provided by a Bose Einstein condensate. Instead we will study below collective beam splitter setups which allows the generation of entangled strings of atoms in 1D trapping configurations with applications in interferometry and quantum computing.

To this end, we assume that the longitudinal motion of the atoms is controlled by storing atoms in a 1D optical lattice potential generated by a standing light laser field. In the transverse direction the particles are confined by a double well potential (see Fig. 1(a)) where we assume that the optical lattice stores exactly one atom per lattice site (i.e. one atom per double well). The preparation of such a Mott insulating state has been reported in a recent experiment, by loading of atoms from a Bose Einstein condensate via a superfluid - Mott insulator quantum phase transition. This setup by itself is an interesting extension of the standard “interferometry on a chip”, as it eliminates collisional shifts since atoms stored on different lattice sites never collide. Furthermore, the atoms are supposed to be initially in the ground state which is a spatial superposition of the particles in the two transverse wells (region (I) of Fig. 1(a)). By moving the lattice we can drag the atomic chain “by hand” across the beam splitter while we increase the distance between the transverse wells adiabatically depending on the position of the atoms (i.e. decrease the tunneling $J^x$ between the wells, see region (II) of Fig. 1(a)). On the other hand, the use of optical lattices allows the engineering of coherent interactions between adjacent atoms (nearest neighbor interaction $W$ in Fig. 1(a)). This can be obtained either by cold collisions and moving optical lattices, or by dipole–dipole coupling of cold heteronuclear molecules. Together with appropriate detection methods like fluorescence imaging these controllable interactions provide us with the tools to generate entanglement of the 1D chain of atoms.

We will study the dynamics of the beam splitter setup indicated in Fig. 1. In particular, we will investigate (i) how to generate a maximally entangled state of atoms. (ii) We will establish the formal equivalence of our model with well-studied models of spin chains. In particular, we will show that the system dynamics is a physical realization of a textbook model of a quantum phase transition with completely controllable (time dependent) parameters. Thus our setup provides an example of engineering a maximally entangled state from a product state via a quantum phase transition. (iii) Finally, the present setup implements the spin analogue of Kitaev’s protected quantum memory, where qubits are represented by Majorana fermions, which provide a stable way to store quantum information due to an excitation gap. Our setup allows to perform single and (collectively enhanced) two qubit operations.
We consider a 1D chain of \( N \) atoms with modes \( |a\rangle \) and \( |b\rangle \) stored in an optical lattice with a lattice constant \( \lambda/2 \) determined by the wave length \( \lambda \) of the laser. The modes correspond either to two spatial modes in a double well structure, where the tunnelling provides a coupling (external beam splitter in Fig. 1), or to two internal atomic states connected via a Raman process (cf. Fig. 1b,c). We suppress hopping of the atoms between adjacent lattice sites by a sufficiently large potential barrier. This leads to an onsite interaction \( U \to \infty \) and we assume to have commensurate filling of one particle per lattice site. Following [2] we derive a Hubbard Hamiltonian

\[
H(t) = 2 \sum_{l=1}^{N-1} W_l(t) \left( a_{l+1}^\dagger a_{l+1} a_l + b_{l+1}^\dagger b_{l+1} b_l \right) - \sum_{l=1}^{N} \left( J_l^x(t) \left( a_l^\dagger b_l + a_l b_l^\dagger \right) + J_l^z(t) \left( a_l^\dagger a_l - b_l^\dagger b_l \right) \right)
\]

Here \( J_l^x \) describes coupling between \( |a\rangle \) and \( |b\rangle \) while the operators \( a_l, b_l \) are bosonic annihilation operators for particles in these modes at site \( l \) with \( [a_l, b_j] = [a_l^\dagger, b_j^\dagger] = 0 \). A term \( J_l^z \) emerges from an additional state dependent superimposed trapped potential. We introduce the spin notation \( x_l^z = a_l^\dagger b_l + a_l b_l^\dagger, \quad x_l^y = a_l^\dagger a_l - b_l^\dagger b_l \) and \( x_l^y = i(a_l^\dagger b_l^\dagger - a_l b_l) \) for which \( n_l = a_l^\dagger a_l + b_l^\dagger b_l \equiv 1 \) are Pauli operators and rewrite the Hamiltonian [1] as

\[
H_S(t) = \sum_{l=1}^{N-1} W_l(t) x_l^z x_{l+1}^z - \sum_{l=1}^{N} J_l(t) \cdot \sigma_l. \quad \text{Thus our setup is formally equivalent to an Ising chain of \( N \) spins in a magnetic field} \quad \mathbf{J} = (J^x, J^y, J^z). \]

**Entanglement via quantum phase transition:** Moving a string of atoms from left to right in the setup of Fig. 1b, or switching the lasers in Fig. 1b,c amounts to a time dependent change of the parameters from the large tunnelling limit \( J_l^x(t = 0) \gg |W_l| \) to small tunnelling \( J_l^x(t = T) \to 0 \). In the following we assume that \( J_l^x \to 0 \) except it is stated differently. In the homogeneous case (i.e. \( J_l^x = J^x, W_l = W \)) the variation of \( J^x \) amounts to crossing the critical point at \( J^x = W \) of a quantum phase transition [3]. Assuming that the atoms are initially prepared in the product state \( |+\rangle \equiv |\uparrow\rangle^z \cdots |\uparrow\rangle^z \) with \( |\uparrow\rangle^z \sim |a\rangle + |b\rangle \) a superposition state of the two modes which is for \( W = 0 \) the (paramagnetic) ground state of \( H_S \). Under adiabatic variation of parameters the system will remain in the ground state and evolve according to \((W < 0)\)

\[
|+\rangle \equiv |\uparrow\rangle^z \cdots |\uparrow\rangle^z \quad \to \quad (|\uparrow\rangle^z \cdots |\uparrow\rangle^z + |\downarrow\rangle^z \cdots |\downarrow\rangle^z) / \sqrt{2}
\]

\[
\equiv \quad (|0\rangle + |1\rangle) / \sqrt{2}, \quad (2)
\]

where the states \( |\uparrow\rangle^z = |a\rangle \) and \( |\downarrow\rangle^z = |b\rangle \) correspond to the atoms being in the upper or lower branch of the beam splitter of Fig. 1. The states \( |0\rangle \) and \( |1\rangle \) are the two degenerate (ferromagnetic) ground states of the \( H_S \) for \( J^x = 0 \) with all atoms in either one of the other arm of the beam splitter (see Fig. 1). Thus the initial product state is transformed to a maximally entangled state via a quantum phase transition. The intuitive physical picture behind [2] is as follows. Consider atoms moving across the beam splitter one by one. The first atom of the string will end up in the state \( |\uparrow\rangle^z + |\downarrow\rangle^z \), and attract the second atom. This leads to a state of the form \( |\uparrow\rangle^z + |\downarrow\rangle^z \). After the last atom has left the interaction zone the maximally entangled state \( |0\rangle + |1\rangle \) has been created.

In the following we discuss the validity of the adiabatic approximation (Eq. (2)) and thus the usefulness of this scheme by studying the scaling of the fidelity \( F = |\langle \psi_{id} \mid \psi(T) \rangle|^2 \) as a function of the length of the string \( N \) and the time variation of \( J^x(t) \) and \( W_l(t) \). Here \( F \) compares the state \( |\psi(T)\rangle \) obtained from a time dependent integration of the Schrödinger equation with the ideal state \( |\psi_{id}\rangle \sim |0\rangle + |1\rangle \). This will be first done numerically, followed by analytical calculations and estimates.

Before entering the time dependent case, we note that for the time independent case the Hamiltonian \( H \) has been studied extensively [8, 15]. For \( J^x = 0 \) it can be fermionized and one obtains \( H_F = \sum_j \epsilon_0 f_j f_j - 1/2 \) with the elementary excitation energies \( \epsilon_0 \) and fermionic annihilation (creation) operators \( f_j, (f_j^\dagger) \). The spectrum for the homogeneous case is shown in Fig. 2. For large \( N \) the spectrum of the elementary excitations is characterized by a gap \( \Delta = 2|W - J^x| \) for the energetically low lying quasi particles with the exception (arising from the free end boundary conditions) that the first excited state becomes degenerate with the ground or vacuum state (here, we do not take into account the second term in \( H_F \), i.e. the vacuum state has zero energy) for \( |W| \gg |J^x| \) (cf. Fig. 2a). For \( J^x = 0 \) the two cat type ground states \( |0\rangle + |1\rangle \) and \( |0\rangle - |1\rangle \) correspond to the vacuum and the first excited state of the fermionized system, respectively.

In Fig. 2 we plot the numerically calculated operation time \( T \) required to perform [2] with a fidelity of \( F = 95\% \) for linearly changing the homogeneous couplings \( J^x(t) \) against \( N \) \((W = \text{const.})\). For \( N > 20 \) we find a (polynomial) scaling of \( T \) for a given infidelity \( 1 - F \approx N^2 \) (cf. Fig. 2b) in agreement with the analytical results below. By optimizing the time dependence of \( J^x(t) \) we can speed up by the entanglement process significantly.

A discussion of the spatially inhomogeneous situation where \( J_l^x \) and \( W_l \) vary as a function of \( l \) corresponding closer to the setup of Fig. 1 is given in Figs. 2c,d. For increasing time the string is moved across a zone of non vanishing \( W_l \) with a maximum \( W^0 \) and a width \( w \). Simultaneously, \( J_l^x(t) \), is decreased from the initial value to \( J_l^x(T) \approx 0 \) over a comparable “width” as \( W \) for all sites. The corresponding instantaneous time dependent energy levels are shown in Fig. 2c. Following the lowest energy curve in this diagram adiabatically from (1) to (2) corresponds to \( |+\rangle \to |0\rangle + |1\rangle \). Fig. 2 shows the infidelity \( 1 - F \) for finite sweeping speed \( v \) against \( N \) for different widths \( w \) of the interaction zone. The infidelity \( 1 - F \) decreases rapidly with increasing \( w \) and scales exponen-
The quantities $A_I \equiv \langle (\sum_{m=1}^{N} f_m^\dagger f_m)^I \rangle = \sum_{m=1}^{N} P(m)|m|^I$. The fidelity $F$ is then given by the solution of a system of $N$ linear equations. An approximate fidelity $F_I$ can be obtained by neglecting the probabilities $P(k)$ with $k > l$. We restrict ourselves to $l = 1, 2$ and find $F_1 = 1 - A_1$ and $F_2 = 1 - (3A_1 - A_2)/2$. The exact fidelity is bounded by these quantities: $F_1 \leq F \leq F_2$. Compared to a calculation in the spin picture which requires the solution of $\sim 2^N$ equations the calculation of $A_1$ and $A_2$ can be done by solving a system of $\sim N^2$ differential equations.

Let us turn to the more technical point of analytically estimating the scaling of the fidelity $F$ when the phase transition point is crossed by linearly changing $J^z = \Theta t + W$ with $\Theta = \text{const}$. First we note that there are no transitions between the ground and the first excited state if they have opposite parity. Close to the phase transition point the energy gap to the remaining excitations $\Delta \approx 0$ and therefore at the time $t = -\tau\ast$ the evolution of the system ceases to be adiabatic and excitations start to be populated. The adiabaticity is restored again at the time $t \sim \tau\ast$, when the gap $\Delta$ becomes sufficiently large to prevent further excitations. Then, the relaxation of the new phase occurs separately within different domains, whose sizes are given by the value $l_0(t\ast)$ of the correlation length at the time $t\ast$. Close to the phase transition $l_0 \sim \Delta^{-1/2}$ and therefore the domain sizes scale like $l_0(t\ast) \sim \Theta^{-1/2}$. The quench through the phase transition point can only be adiabatic if the characteristic size of the domain formed exceeds the size of the system $L \sim N$ and therefore $l_0(t\ast) \gtrsim L$, which gives the scaling condition $\Theta \lesssim W^2/N^2$, or $WT \sim N^2$.

Quantum computing model with protected quantum memory: In the case $W < 0$ the ferromagnetic superposition state is very sensitive to homogeneous distortions of the form $\tilde{J}^z = J^z + \epsilon$ which induce a relative phase shift exp(iN$\int_0^T dt J^z(t)$) scaling with $N$ [16] between the two states $|0\rangle$ and $|1\rangle$ after a time $\tau$. Therefore, in the external beam splitter setup where these two states are spatially separated they can be viewed as two arms of a Heisenberg limited interferometer collectively enhanced by a factor $N$. On the other hand, in the antiferromagnetic case, i.e. for a repulsive interaction $W > 0$, the two degenerate ground states at $J^z = 0$

$$|0\rangle = |\downarrow \cdots \downarrow\rangle^z, \quad |1\rangle = |\uparrow \cdots \uparrow\rangle^z, \quad (3)$$

are closely related to unpaired Majorana fermions which have been considered as candidates for storing quantum information [8, 10]. These states are expected to be insensitive against perturbations since they are separated by a gap of order $W$ from the other states of the system and are only connected via $N$-th order perturbation theory for homogeneous couplings $J_l = J^z$. This yields stability against spin flip errors exponentially increasing with the number of particles in the chain $N$ and is also reflected by the scaling of the energy of the first excited state $\epsilon_1 \sim (J^z/W)^N$ for $J^z < W$ [13]. Furthermore,
completely insensitive to global fluctuations of $\sum W$. Selectively overlapping the wave function. The idea behind the two qubit phase gate is sum-
ble enhanced) two qubit gates and show that our model realizes a quantum computer with protected mem-
 heterogeneous perturbations and can thus be used as qubits.

![FIG. 3: Collectively enhanced interactions between two strings of atoms 1 and 2. a) Antiferromagnetic setup: N/2 particles of each chain interact with strength $W'$ only if they are in states $|0\rangle_1 |1\rangle_2$ or $|1\rangle_1 |0\rangle_2$ yielding a phase gate between the two qubits implemented by those chains. b) Ferromagnetic setup: Entanglement creation between two chains of atoms via interactions $W'$ in the state $|1\rangle_1 |0\rangle_2$. Implementations with optical lattices or atom chips, for instance, offer the scalability of the scheme.](image)

if we assume that $N$ is even the states $|0\rangle$ and $|1\rangle$ are completely insensitive to global fluctuations of $J$ since $\sum_{i} \sigma_i^z |0\rangle = \sum_{i} \sigma_i^z |1\rangle = 0$. Then the two states $|0\rangle$ and $|1\rangle$ constitute a decoherence free subspace for homogeneous perturbations and can thus be used as qubits which store quantum information reliably.

We will now discuss how to implement single and (collectively enhanced) two qubit gates and show that our model realizes a quantum computer with protected memory. The idea behind the two qubit phase gate is sum-
ble enhanced) two qubit gates and show that our model realizes a quantum computer with protected mem-
 heterogeneous perturbations and can thus be used as qubits.

![FIG. 4: Illustration of the Hadamard gate for N = 8 by adiabatically changing $J^x$ and $J^z$ (unprotecting the quantum memory). We follow the lowest two eigenstates (with energies given by the solid curves) transforming as $|0\rangle + |1\rangle \to |0\rangle$ and $|0\rangle - |1\rangle \to |1\rangle$ (up to a dynamical phase) when changing $J^x$, $J^z$ in three steps (1), (2), (3) (followed by turning off $J^x$ in step (4)) as described in the text. Note that if the condition $J^x < W/(N-1)$ is not fulfilled we get unwanted crossings and the first excited state after step (3) will not be of the form $|1\rangle$. The dashed curve shows the third eigenenergy of the system and the inset the path in the $J^x - J^z$ plane.](image)

relative phase $|\phi_1\rangle \to \exp(i \phi_1 |\phi_1\rangle$. The phase $\phi_1$ can be implemented by turning on a trap potential creating a staggered offset of the form $J_{ij} = J^z(-1)^i$ for a time $\tau_1 = \phi_1/2N\tau^2$. The idea behind the Hadamard gate is as follows: At $J^x = J^z = 0$ the states $|0\rangle$, $|1\rangle$ represent a degenerate eigenspace of $H_S$. Turning on the field $J^x$ up to $J^x > W$, thus unprotected the qubit and switching it off when $J^z \neq 0$ will under appropriate conditions induce a rotation in this space. A specific example is illustrated in Fig. 4 (1) at $J^z = 0$ we adiabatically switch on $J^x$ until $J^x > W$ is reached unprotected the qubit, then (2) we increase $J^z$, (3) we return adiabatically to $J^z = 0$, and, finally, (4) switch off $J^x$.

We have shown how to generate maximally entangled states of strings of atoms in 1D pipeline configurations. An extension of this setup allows implementations of a quantum computing model with protected qubits.

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