EXACT BOUNDARY NULL CONTROLLABILITY FOR A COUPLED SYSTEM OF PLATE EQUATIONS WITH VARIABLE COEFFICIENTS

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Abstract. This paper studies a coupled system of plate equations with variable coefficients, subject to the clamped boundary conditions. By the Riemannian geometry approach, the duality method, the multiplier technique and a compact perturbation method, we establish exact boundary null controllability of the system under verifiable assumptions.

1. Introduction. In this paper, we consider controllability questions for a class of system of coupled partial differential equations, which is comprised of N variable coefficients plates. The mathematical model is described as follows.

Let $\Omega \subset \mathbb{R}^n$ be an (open), bounded domain with (sufficiently) smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$ such that $\Gamma_0 \cap \Gamma_1 = \emptyset$, where both $\Gamma_0$ and $\Gamma_1$ are open and nonempty.

We consider the following system on finite time $(0, T)$:

$$
\begin{cases}
U_{tt} + \mathcal{A}^2 U + BU + EU_t + F \mathcal{A} U = 0 \text{ in } (0, T) \times \Omega, \\
U = U_{\nu_0} = 0 \text{ on } (0, T) \times \Gamma_0, \\
U = V_1, U_{\nu_1} = V_2 \text{ on } (0, T) \times \Gamma_1, \\
U(0) = U_0, U_t(0) = U_1 \text{ in } \Omega,
\end{cases}
$$

(1)

where $U = (u^{(1)}, \ldots, u^{(N)})^T$ denote the vertical displacements of $N$ plates, $\mathbf{0}$ denotes the null vector in $\mathbb{R}^N$, $\mathcal{A} U = \text{div } A(x) \nabla U$, $A(x)$ is a symmetric, positive matrix with smooth elements for each $x \in \mathbb{R}^n$. $B, E$ and $F$ are matrixes of order $N$ with constant elements, which reflects the coupling relations. $V_1 = (v_1^{(1)}, \ldots, v_1^{(N)})^T$ and $V_2 = (v_2^{(1)}, \ldots, v_2^{(N)})^T$ are controls exerted on the partial clamped boundary, $U_{\nu_0} = (u_{\nu_0}^{(1)}, \ldots, u_{\nu_0}^{(N)})^T$, $u_{\nu_0} = \langle A(x) \nabla u, \nu \rangle$. $\text{div}$, $\nabla$ and $\nu$ are the divergence, the gradient and the outside normal along $\Gamma$ in the Euclidean metric, respectively.

The prominent features of our system (1) are as follows: On the one hand, we here consider a coupled system comprising $N$ plate equations with coupling terms below and on the energy level. On the other hand, the model is variable-coefficient, which is more realistic. From the physical point of view, the variable coefficient matrix $A(x)$ reflects the nonhomogeneous material properties of the plate.

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system. Furthermore, the above two characters introduce additional non-trivial and significant complexities to the mathematical analysis. All in all, the problem (1) is of practical and theoretical importance.

With the development of modern control theory and the need of actual applications, an increasing number of researches have been devoted to the control questions on the coupled PDE systems in recent years. See, for instance, the fluid-structure system (like flow-plate [19], wave-beam [1, 2], wave-plate [3, 40]); wave-wave [21, 22, 27]; transmission system (such as wave/wave [6, 25], plate/plate [11, 26], wave/plate [4, 8, 12, 41]); Heat-structure system [5, 16, 42] and so on. Compared with the single systems, the increased complexity of coupled systems require more advanced and efficient analysis tools and estimation techniques, which motivates us to study such problems in this paper.

The Euler-Bernoulli equations, as a kind of classical partial differential equation, are used to describe the vibration of elastic thin plates. Stimulated by the extensive applications in the architectural structures, automobile and aerospace industries, etc. (see [31, 34]), there have been a great many researches on the controllability questions of Euler Bernoulli plates. For the constant coefficient case, we refer the reader to [13, 14, 15, 17, 18, 23, 43], and the references therein. Particularly, in [23], Lions considered the exact controllability of the Euler-Bernoulli model with one control acting through Neumann boundary condition. Later, Lasiecka and Triggiani [17] studied the situation where control acts only on the Dirichlet boundary condition. And in [18], they discussed the exact controllability problem with boundary controls for displacement $u$ and moment $\Delta u$ acting in the Dirichlet boundary conditions. Besides, Horn [13] derived the exact controllability of the Euler-Bernoulli plate with the simply supported boundary condition only via bending moments. For the variable coefficient case, Yao [36] first used the Riemannian geometry approach to give checkable conditions for the exact controllability of two Euler-Bernoulli plates with clamped and hinged boundary conditions respectively. Later, Yang [38] established the exact controllability of an Euler-Bernoulli plate with simply supported boundary condition. In addition, Guo and Zhang [9], Wen et al. [35] showed that the exact controllability of Euler-Bernoulli plate with clamped or hinged boundary conditions under partial boundary Neumann control or Dirichlet control is equivalent to the exponential stability of its closed-loop system under proportional output feedback.

The objective of this paper is to study the controllability properties of coupled system (1) by the Riemannian geometry approach. The Riemannian geometry is a useful tool for the controllability of variable - coefficient systems mainly due to its two virtues: The Bochner technique in Riemann geometry can be used to simplify computation greatly to establish energy estimates, and the curvature theory provides the global information on the existence of an escape vector field which guarantees the exact controllability. For much of the prerequisite of Riemannian geometry involved in this paper, we refer the reader to [37, Chapter 1].

The organization of our paper goes as follows. Section 2 is devoted to presenting our primary results. In Section 3, we use the duality method to find the observability inequality. The proofs of the results are provided in Section 4. In the end, a brief concluding remark is given.

2. Main results. We introduce

$$g = A^{-1}(x) \text{ for } x \in \mathbb{R}^n,$$
as a Riemannian metric on $\mathbb{R}^n$ and consider the couple $(\mathbb{R}^n, g)$ as a Riemannian manifold. Denote by $g = \langle \cdot, \cdot \rangle_g$ the inner product. Then

$$\langle X, Y \rangle_g = \langle A^{-1}(x)X, Y \rangle$$

for $X, Y \in \mathbb{R}^n_x$, $x \in \mathbb{R}^n$,

where $\langle \cdot, \cdot \rangle$ is the Euclidean product of $\mathbb{R}^n$.

We shall give the following assumptions on the geometry of $\Omega$ and part of its boundary $\Gamma_1$.

(A.1) There is an escape vector field $H$ for plate, i.e., there exists a function $b(x)$ on $\Omega$ such that

$$DH(X, X) = b(x)|X|^2 g, \quad \forall X \in \Omega, x \in \Omega,$$

and $b_0 = \min_{x \in \Omega} b(x) > 0,$

(2)

where $D$ is the covariant differential of the metric $g$.

Remark 1. To have a real solution to the control problems for the systems with variable coefficients, it is necessary to give checkable geometric conditions. Escape vector field for plate was introduced by Yao [36] as a verifiable assumption for the controllability of the plate with variable coefficients. It was then extended in [10, 20, 39] and many others. An escape vector field as in (2) always exists locally, but the global existence of such vector fields depends on the Gaussian curvature of $\Omega$. We refer the reader to [36, 37] where some examples of the global existence of such vector fields are given by using the Riemann curvature theory.

(A.2) $\langle H, \nu \rangle > 0$ on $\Gamma_1$, where $H$ is the escape vector field for plate given in assumption (A.1).

Our main result is stated as follow:

Theorem 2.1. Let assumptions (A.1) and (A.2) hold. Then for terminal time $T > 0$ sufficiently large, system (1) is exactly null controllable at time $T$, i.e., for any given initial data $(U_0, U_1) \in (L^2(\Omega))^N \times (H^{-2}(\Omega))^N$, one can find control functions $(V_1, V_2) \in L^2(0, T; L^2(\Gamma_1))^N \times L^2(0, T; L^2(\Gamma_1))^N$ such that the corresponding solution to problem (1) satisfies

$$(U(T), U_t(T)) = (0, 0).$$

(3)

Remark 2. In the case of constant coefficients where $\mathcal{A} = \Delta$, one boundary control for each plate is enough, see [15, 28, 33]. We here add another control for each plate in order to avoid the following uniqueness assumption:

The problem

$$\begin{cases}
\mathcal{A}^2 w = F(w, Dw, D^2w, D^3w) \text{ in } \Omega, \\
w = w_{\nu,\Gamma} = \mathcal{A} w = 0 \text{ or } w = w_{\nu,\Gamma} = (\mathcal{A} w)_{\nu,\Gamma} = 0 \text{ on } \hat{\Gamma}, \\
or w = w_{\nu,\Gamma} = 0 \text{ on } \Gamma \mathcal{A} w = 0 \text{ on } \hat{\Gamma},
\end{cases}$$

(4)

has the unique zero solution, where $\hat{\Gamma}$ is a portion of the boundary $\Gamma$ which is relatively open. We do not know if the uniqueness problem (4) is true or not in the case of variable coefficients, which seems more difficult to handle comparing with the controllability problem we are interested in here.

3. Observability inequality. We denote the state space by

$$\mathcal{H} = (L^2(\Omega))^N \times (H^{-2}(\Omega))^N,$$
and the control space by
\[ \mathcal{U} = L^2(0, T; L^2(\Gamma_1))^N \times L^2(0, T; L^2(\Gamma_1))^N. \]

Define the control-to-initial state map \( P : D(P) \subset \mathcal{U} \to \mathcal{H} \) as
\[ P(V_1, V_2) = (U_t(0), -U(0)), \]
where \((U, U_t)\) is the corresponding solution to system (1) and
\[ D(P) = \{ (V_1, V_2) \in \mathcal{U} : P(V_1, V_2) \in \mathcal{H} \}. \]

\( P \) is a closed, densely defined and unbounded operator. Its adjoint is
\[ P^* : D(P^*) \subset \mathcal{H} \to \mathcal{U} \text{ with } D(P^*) = \{ (W_0, W_1) \in \mathcal{H} : P^*(W_0, W_1) \in \mathcal{U} \}. \]

According to the relation
\[ ((V_1, V_2), P^*(W_0, W_1))_\mathcal{U} = (P(V_1, V_2), (W_0, W_1))_\mathcal{H}, \]
we obtain the dual problem of system (1) as
\[
\begin{cases}
W_{tt} + \mathcal{A}W + B^T W - E^T W_t + F^T \mathcal{A}W = 0 \text{ in } (0, T) \times \Omega, \\
W = W_{\nu,\mathcal{A}} = 0 \text{ on } (0, T) \times \Gamma, \\
W(0) = W_0, W_t(0) = W_1 \text{ in } \Omega.
\end{cases}
\]  

(5)

Moreover, the adjoint operator \( P^* \) is found from the above to be
\[ P^*(W_0, W_1) = (-(\mathcal{A}W)_{\nu,\mathcal{A}}, \mathcal{A}W), \text{ for all } (W_0, W_1) \in D(P^*), \]
where \( W \) is the solution to problem (5).

In order to establish the exact null controllability of system (1), we need to show the surjectivity of operator \( P \). In turn, by duality (similar to a classical functional analysis result, see [7]), it is equivalent to establishing the following inequality, for \( T > 0 \) sufficiently large:
\[ \|P^*(W_0, W_1)\|_{\mathcal{U}}^2 \geq C_T \|W_0, W_1\|_{(H^2(\Omega))^N \times (L^2(\Omega))^N}^2. \]  

(7)

We introduce the energy of system (5) as
\[ 2E(t) = \int_{\Omega} (W_t^2 + (\mathcal{A}W)^2) \, dx, \]
(8)

accordingly, the observability inequality (7) can be rewritten as
\[ \int_{\Sigma_1} \left[ (\mathcal{A}W)^2 + ((\mathcal{A}W)_{\nu,\mathcal{A}})^2 \right] \, d\Sigma \geq C_T E(0), \]
(9)

where \( W \) is a solution to problem (5). The rest of this paper shall be devoted to deriving the above observability inequality.

In what follows, we denote \( C \) or \( C_i \) as any positive constant which may be different from line to line.
4. Proof of Theorem 2.1. Let $A : D(A) \subset (L^2(\Omega))^N \to (L^2(\Omega))^N$ be a linear operator defined by

$$AW = \mathcal{A}^2 W, \quad D(A) = \left\{ W \in (H^4(\Omega))^N : W|_\Gamma = \frac{\partial W}{\partial \nu_{sd}}|_\Gamma = 0 \right\}.$$ 

It is easy to check that $A$ is a densely defined self-adjoint positive operator. Moreover, according to the interpolation results in [24], we are able to identify the following spaces:

$$\begin{aligned}
D(A^\theta) &= (H^{4\theta}(\Omega))^N, \quad 0 < \theta < \frac{1}{8}, \\
D(A^\theta) &= \left\{ W \in (H^{4\theta}(\Omega))^N, W|_\Gamma = 0 \right\}, \quad \frac{1}{8} < \theta < \frac{3}{8}, \\
D(A^\theta) &= \left\{ W \in (H^{4\theta}(\Omega))^N, W|_\Gamma = \frac{\partial W}{\partial \nu_{sd}}|_\Gamma = 0 \right\}, \quad \frac{3}{8} < \theta < 1.
\end{aligned} \quad (10)$$

In particular,

$$A^\frac{1}{2} W = -\mathcal{A} W, \quad D(A^{1/2}) = \left\{ W \in (H^2(\Omega))^N, W|_\Gamma = \frac{\partial W}{\partial \nu_{sd}}|_\Gamma = 0 \right\} = (H^2_0(\Omega))^N.$$ 

For any given $s \in \mathbb{R}$, denote $\mathcal{H}_s = D(A^s)$, particularly, $\mathcal{H}_0 = (L^2(\Omega))^N$. The domain $\mathcal{H}_s$ is endowed with the norm $\| W \|_s = \| A^s W \|_0$, where $\| W \|_0$ is the norm of $\mathcal{H}_0$. $\mathcal{H}_s$ is a Hilbert space, and its dual space with respect to the pivot space $\mathcal{H}_0$ is $\mathcal{H}_{-s}$.

Now, we formulate (5) into an abstract evolution problem as follow:

$$\begin{aligned}
W_{tt} + AW + B^T W - E^T W_t - F^T A^\frac{1}{2} W &= 0, \\
W(0) &= W_0, W_t(0) = W_1.
\end{aligned} \quad (11)$$

Clearly, the problem (11) generates a $C_0$-semigroup in the space $\mathcal{H}_s \times \mathcal{H}_{s-2}$. Furthermore, we have the following result (see [29, Chapter III]).

**Proposition 1.** For any given initial data $(W_0, W_1) \in \mathcal{H}_s \times \mathcal{H}_{s-2}$ with $s \in \mathbb{R}$, the problem (11) admits a unique weak solution $W$ such that

$$W \in C^0([0, +\infty); \mathcal{H}_s) \cap C^1((0, +\infty); \mathcal{H}_{s-2}). \quad (12)$$

Now let $e_m$ be the normalized eigenfunction defined by

$$\begin{aligned}
\mathcal{A}^2 e_m &= \mu_m^2 e_m \text{ in } \Omega, \\
e_m &= (e_m|_{\nu_{sd}}) = 0 \text{ on } \Gamma,
\end{aligned} \quad (13)$$

where the sequence of positive terms $\{ \mu_m \}_{m \geq 1}$ is increasing so that $\mu_m \to +\infty$ as $m \to +\infty$. Clearly, $\{ e_m \}_{m \geq 1}$ is a Hilbert basis in $L^2(\Omega)$. For each $m \geq 1$, we define the subspace $Z_m$ by

$$Z_m = \{ \alpha e_m : \alpha \in \mathbb{R}^N \}. \quad (14)$$

It is easy to check that the subspaces $Z_m(m \geq 1)$ are invariant with respect to the matrices $B^T, E^T$ and $F^T$. And for any given integers $m \neq n$ and any given vectors $\alpha, \beta \in \mathbb{R}^N$, we have

$$\langle \alpha e_m, \beta e_n \rangle_{\mathcal{H}_s} = (\alpha, \beta)(A^\frac{1}{2} e_m, A^\frac{1}{2} e_n)_{L^2(\Omega)} = (\alpha, \beta)\mu_m^2\mu_n^2 (e_m, e_n)_{L^2(\Omega)} = (\alpha, \beta)\delta_{mn}. \quad (15)$$

Accordingly, the subspaces $Z_m(m \geq 1)$ are mutually orthogonal in the Hilbert space $\mathcal{H}_s$ with any given $s \in \mathbb{R}$.
Let $m_0 \geq 1$ be an integer. For all $W \in \bigoplus_{m \geq m_0} Z_m$, by virtue of (13) and (14), we have
\[
\|W\|_{\mathcal{H}_{l+1}} \geq \sqrt{\mu_{m_0}}\|W\|_{\mathcal{H}_l}, \quad \|W\|_{\mathcal{H}_{l+2}} \geq \mu_{m_0}\|W\|_{\mathcal{H}_l},
\]
we will utilize the above relation quite frequently in the following.

We denote by $\bigoplus (Z_m \times Z_m)$ the linear hull of the subspaces $Z_m \times Z_m$ for $m \geq m_0$.

Now we give a useful energy relation for system (5).

**Lemma 4.1.** Let $W$ be the solution to problem (5) with initial data $(W_0, W_1) \in \bigoplus (Z_m \times Z_m)$. Let $\sigma_1, \sigma_2$ and $\sigma_3$ denote the Euclidian norm of matrixes $B, E$ and $F$, respectively. Then we have the following energy estimates:
\[
e^{-Qt} E(0) \leq E(t) \leq e^{Qt} E(0), \quad t \geq 0,
\]
where $Q = \frac{\sigma_1}{\mu_{m_0}} + 2\sigma_2 + \sigma_3$.

**Proof.** After a straightforward computation by using the divergence theorem and the boundary conditions of problem (5), we obtain
\[
E'(t) = \int_{\Omega} (W_t, E^T W_t - B^T W - F^T \nabla W) \, dx.
\]
Using the relation (16), we have
\[
|E'(t)| \leq \left| \int_{\Omega} (W_t, B^T W) \, dx \right| + \left| \int_{\Omega} (W_t, E^T W_t) \, dx \right| + \left| \int_{\Omega} (W_t, E^T \nabla W) \, dx \right|
\]
\[
\leq \frac{\sigma_1}{\mu_{m_0}} \|W\|_{\mathcal{H}_2} \|W_t\|_{\mathcal{H}_0} + \sigma_2 \|W_t\|_{\mathcal{H}_0}^2 + \sigma_3 \|W_t\|_{\mathcal{H}_0} \|\nabla W\|_{\mathcal{H}_0}
\]
\[
\leq \left( \frac{\sigma_1}{\mu_{m_0}} + 2\sigma_2 + \sigma_3 \right) E(t).
\]
Denote $Q \triangleq \frac{\sigma_1}{\mu_{m_0}} + 2\sigma_2 + \sigma_3$, the above inequality gives
\[
-QE(t) \leq E'(t) \leq QE(t).
\]
It follows that the function $e^{Qt} E(t)$ is non-decreasing and the function $e^{-Qt} E(t)$ is non-increasing, which implies (17). This completes the proof of Lemma 4.1. $\square$

Next, we give the following two geometric multiplier identities from [37], which will be used in the sequel.

**Lemma 4.2.** Let $w$ be a sufficiently smooth solution to the problem $w_{tt} + \mathcal{A}^2 w = f$ in $(0, \infty) \times \Omega$, where $f$ is a given function. Let $H$ be a vector field and $p$ be a function on $\overline{\Omega}$. Then the following identities hold:
\[
\int_{\Sigma} \left\{ 2\mathcal{A}(H(w))_{\nu,\nu} - 2H(w)(\mathcal{A}w)_{\nu,\nu} + \|w_t^2 - (\mathcal{A}w)^2\| \langle H, \nu \rangle \right\} \, d\Sigma
\]
\[
= \int_{Q} \left\{ \|w_t^2 - (\mathcal{A}w)^2\| \text{div} H + 4\mathcal{A}w(DH, D^2w)_g + 2\mathcal{A}wP(Dw) - 2fH(w) \right\} \, dQ
\]
\[
+ 2 \left( w_t, H(w) \right)_{T_0},
\]
where $P(Dw) = (\mathcal{A}H)(w) + (2\text{Ric} - D^2\nu_g)(H, Dw)$. $2\nu_g = \log(\det A(x))$ for $x \in \mathbb{R}^n$, and for any vector field $H$ on $\mathbb{R}^n$, $\mathcal{A}H$ is a vector field on $\mathbb{R}^n$ defined by
$\mathcal{A} H = -\triangle_g H + D_{Div} H$, Ric and $\triangle_g$ are the Ricci tensor and Hodge-Laplace operator in the metric $g$, respectively.

And

$$\int_Q \left\{ p \left[ w_t^2 - (\mathcal{A} w)^2 \right] - \mathcal{A} w [2Dp(w) + w\mathcal{A} p] + f_p w \right\} dQ$$

$$= (w_t, pw)^T_0 + \int_\Sigma [pu(\mathcal{A} w)_{\nu,\nu} - \mathcal{A} w (pw)_{\nu,\nu}] d\Sigma.$$

(22)

The following is our main estimate.

**Lemma 4.3.** Let assumptions (A.1) and (A.2) hold. Then there exists an integer $m_0 \geq 1$ and constants $T > 0$ and $C > 0$ independent of initial data, such that the following inequality:

$$\int_{\Sigma_1} \left[ (\mathcal{A} W)^2 + (\mathcal{A} W)^2_{\nu,\nu} \right] d\Sigma \geq CE(0),$$

(23)

holds for all solutions $W$ of problem (5) with initial data $(W_0, W_1) \in \bigoplus_{m \geq m_0} (Z_m \times Z_m)$.

**Proof.** Since $Z_m$ is invariant with respect to $B^T, E^T$ and $F^T$, then for all $(W_0, W_1) \in \bigoplus_{m \geq m_0} (Z_m \times Z_m)$, the corresponding solution $W \in \bigoplus_{m \geq m_0} Z_m$.

First, we write (5) as

$$\begin{cases}
  w_{tt}^{(k)} + \mathcal{A} w^{(k)} + \sum_{i=1}^N b_{ik} w^{(i)} - \sum_{j=1}^N e_{jk} w^{(j)} + \sum_{l=1}^N f_{lk} \mathcal{A} w^{(l)} = 0 \text{ in } (0, T) \times \Omega, \\
  w^{(k)} = w^{(k)}_{\nu,\nu} = 0 \text{ on } (0, T) \times \Gamma, \\
  w^{(k)}(0) = w^{(k)}_0, w^{(k)}(0) = w^{(k)}_1,
\end{cases}$$

(24)

for $k = 1, 2, \ldots, N$, where $b_{ik}, e_{jk}, f_{lk}$ are elements of matrices $B, E$ and $F$, respectively.

Using the boundary conditions of problem (5) in identity (21) with $f = -\sum_{i=1}^N b_{ik} w^{(i)} + \sum_{j=1}^N e_{jk} w^{(j)} - \sum_{l=1}^N f_{lk} \mathcal{A} w^{(l)}$, we obtain

$$\int_\Sigma \left\{ 2\mathcal{A} w^{(k)}(H(w))_{\nu,\nu} - 2H(w^{(k)})(\mathcal{A} w^{(k)})_{\nu,\nu} + \left[ (w^{(k)})^2 - (\mathcal{A} w^{(k)})^2 \right] (H, \nu) \right\} d\Sigma$$

$$= 2 \left( w^{(k)}_1, H(w^{(k)}) \right)_{\Gamma_0} + \int_Q \left\{ \left[ (w^{(k)})^2 - (\mathcal{A} w^{(k)})^2 \right] \text{div} H \right.$$

$$\left. + 4\mathcal{A} w^{(k)}(DH, D^2 w^{(k)})_{g} + 2\mathcal{A} w P(Dw^{(k)}) \right\} dQ$$

$$+ \int_Q \left\{ 2 \left( \sum_{i=1}^N b_{ik} w^{(i)} - \sum_{j=1}^N e_{jk} w^{(j)} + \sum_{l=1}^N f_{lk} \mathcal{A} w^{(l)} \right) H(w^{(k)}) \right\} dQ.$$

(25)

Let $\nu$ be the outside normal along $\Gamma$, $\nu_{\nu} = A(x)\nu$. For $2 \leq i \leq n$, let $e_i$ be the tangential vector fields on $\Gamma$ such that $e_1 = \nu_{\nu}/|\nu_{\nu}|, e_2, \ldots, e_n$ form a unit orthogonal basis of $((\mathbb{R}^n_x, g(x))$ for each $x \in \Gamma$. Noting that the boundary conditions,
Thus, \( H(w^{(k)}) = \langle H, Dw^{(k)} \rangle_g = 0 \) on \( \Gamma \).

Further, for \( 2 \leq i \leq n \),

\[
D_{e_i}Dw^{(k)} = e_i(w^{(k)}) e_1 + \sum_{i=2}^{n} e_i e_i(w^{(k)}) e_i + \sum_{i=1}^{n} e_i(w^{(k)}) D_{e_i} e_i = 0 \text{ on } \Gamma. \tag{26}
\]

Then, \( D^2w^{(k)}(e_j, e_i)|_{\Gamma} = \langle D_{e_i}Dw^{(k)}, e_j \rangle_g |_{\Gamma} = 0 \), for \( 2 \leq i \leq n, 1 \leq j \leq n \). And

\[
(H(w^{(k)}))_{\nu_{\mathcal{d}}} = \nu_{\mathcal{d}}(H(w^{(k)})) = |\nu_{\mathcal{d}}|_g e_1 \langle H, Dw^{(k)} \rangle_g
= |\nu_{\mathcal{d}}|_g \langle H, D_{e_i} Dw^{(k)} \rangle_g = |\nu_{\mathcal{d}}|_g D^2w^{(k)}(H, e_1)
= \langle H, \nu_{\mathcal{d}} \rangle_g D^2w^{(k)}(e_1, e_1) + |\nu_{\mathcal{d}}|_g \sum_{i=2}^{n} \langle H, e_i \rangle_g D^2w^{(k)}(e_1, e_i)
= \langle H, \nu \rangle D^2w^{(k)}(e_1, e_1) = \langle H, \nu \rangle \Delta_gw^{(k)}. \tag{27}
\]

The following identity comes from the identity (3.5) in [37]:

\[
\Delta_gw^{(k)} = \sqrt{\det A(x)} \sum_{i,j=1}^{n} \left( [\det A(x)]^{-\frac{1}{2}} a_{ij}(x) \frac{\partial w^{(k)}}{\partial x_i} \right) x_j
= \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} \left( a_{ij}(x) \frac{\partial w^{(k)}}{\partial x_i} \right)
+ \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial w^{(k)}}{\partial x_i} \sqrt{\det A(x)} \frac{\partial}{\partial x_j} \left( [\det A(x)]^{-\frac{1}{2}} \right)
= \text{div } A(x) \nabla w^{(k)} - \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial w^{(k)}}{\partial x_i} [\det A(x)]^{-1} \frac{\partial [\det A(x)]}{\partial x_j}
= \mathcal{A}w^{(k)} - \frac{1}{2} \langle A(x) \nabla w^{(k)}, \nabla (\log \det A(x)) \rangle
= \mathcal{A}w^{(k)} - \langle Dw^{(k)}, D\nu_g \rangle_g, \tag{28}
\]

where \( 2\nu_g = \log \det A(x) \).

According to the above relations, we have

\[
\text{The left hand side of equality (25)} = \int_{\Sigma} (\mathcal{A}w^{(k)})^2 \langle H, \nu \rangle d\Sigma
\leq C \int_{\Sigma} (\mathcal{A}w^{(k)})^2 d\Sigma. \tag{29}
\]
Let \( \{e_i\}_{1 \leq i \leq n} \) be an orthonormal basis of \( \mathbb{R}_+^n \) in the metric \( g \) for each \( x \in \Omega \). Since
\[
\langle DH, D^2w^{(k)} \rangle_g
= \sum_{i,j=1}^n DH(e_i, e_j) D^2w^{(k)}(e_i, e_j)
= \frac{1}{2} \sum_{i,j=1}^n [DH(e_i, e_j) + DH(e_j, e_i)] D^2w^{(k)}(e_i, e_j)
= \frac{1}{4} \sum_{i,j=1}^n [DH(e_i + e_j, e_i + e_j) - DH(e_i - e_j, e_i - e_j)] D^2w^{(k)}(e_i, e_j)
= \frac{1}{4} \sum_{i,j=1}^n b(x) [e_i + e_j]^2 - |e_i - e_j|^2] D^2w^{(k)}(e_i, e_j) = b(x) \Delta_g w^{(k)},
\]
and using the identity (28), Cauchy inequality and the relation (16), we get
\[
\int_Q 4\mathcal{A} w^{(k)} \langle DH, D^2w^{(k)} \rangle_g dQ
= \int_Q \left[ 4b(x)(\mathcal{A} w^{(k)})^2 - 4b(x)D\nu_g(w^{(k)})\mathcal{A} w^{(k)} \right] dQ
\geq \int_Q \left[ 4b(x)(\mathcal{A} w^{(k)})^2 - 2b(x)(\mathcal{A} w^{(k)})^2 - 2b(x) \left( D\nu_g(w^{(k)}) \right)^2 \right] dQ
\geq \int_Q \left[ 2b(x) - \frac{C \sup_{x \in \Omega} |b(x)|}{\mu_m} \right] (\mathcal{A} w^{(k)})^2 dQ.
\]
Moreover, application of relation (16) yields
\[
\left| \int_Q 2\mathcal{A} w^{(k)} P(Dw^{(k)}) dQ \right|
= \left| \int_Q 2\mathcal{A} w^{(k)} \left[ (\mathcal{A} H)(w^{(k)}) + (2Ric - D^2\nu_g)(H, Dw^{(k)}) \right] dQ \right|
\leq \int_Q \left[ b_0 + \frac{4C}{b_0 \mu_m} \right] (\mathcal{A} w^{(k)})^2 dQ.
\]
Applying the boundary conditions of problem (5) in identity (22) with \( p = \text{div} H - b_0 \) and \( f = - \sum_{i=1}^N b_{ik} w^{(i)} + \sum_{j=1}^N e_{jk} w^{(j)} - \sum_{l=1}^N f_{lk} \mathcal{A} w^{(l)} \), we obtain
\[
\int_Q \left\{ p \left[ (w^{(k)})_t^2 - (\mathcal{A} w^{(k)})^2 \right] - \mathcal{A} w^{(k)} \left[ 2Dp(w^{(k)}) + w^{(k)} \mathcal{A} P \right] \right\} dQ
= \int_Q \left( \sum_{i=1}^N b_{ik} w^{(i)} - \sum_{j=1}^N e_{jk} w^{(j)} + \sum_{l=1}^N f_{lk} \mathcal{A} w^{(l)} \right) pw^{(k)} dQ
+ (w^{(k)}_t, pw^{(k)})_{\Omega}^T.
\]
Using Cauchy inequality and relation (16), we get

$$\left| \int_Q \mathcal{A} w^{(k)} \left[ 2D_p w^{(k)} + w^{(k)} \mathcal{A} p \right] dQ \right|$$

$$\leq \int_Q \left[ \frac{b_0}{4} + \frac{C}{b_0 \mu_{m_0}} + \tilde{C} \frac{1}{b_0 \mu_{m_0}^2} \right] (\mathcal{A} w^{(k)})^2 dQ.$$  \hspace{1cm} (34)

Combining the relations (25),(29) and (31)-(34), we have

$$C_1 \int_{\Sigma_1} (\mathcal{A} w^{(k)})^2 d\Sigma \geq (w^{(k)}, M^{(k)})_{T_0}^T + \frac{b_0}{2} \int_Q \left[ (\mathcal{A} w^{(k)})^2 + (w^{(k)})^2 \right] dQ$$

$$- \int_Q \left[ \frac{C_2 \sup_{x \in \Omega} |b(x)|}{\mu_{m_0}} + \frac{C_3}{b_0 \mu_{m_0}} + \frac{C_4}{b_0 \mu_{m_0}^2} \right] (\mathcal{A} w^{(k)})^2 dQ$$

$$+ \int_Q \left( \sum_{i=1}^N b_i w^{(i)} \right) - \sum_{j=1}^N e_{jk} w^{(j)} + \sum_{l=1}^N f_{ik} \mathcal{A} \mathcal{I}^{(l)} \right) M^{(k)} dQ.$$  \hspace{1cm} (35)

where $M^{(k)} = 2H(w^{(k)}) + pW^{(k)}$.

Taking the summation of (35) with respect to $k = 1, \cdots, N$, we get

$$C_1 \int_{\Sigma_1} (\mathcal{A} W)^2 d\Sigma \geq (W, M)_{L^2(\Omega)}^N T_0 + \int_0^T \int_\Omega (W, BM) dx dt$$

$$- \int_0^T \int_\Omega (W, EM) dx dt + \int_0^T \int_\Omega (\mathcal{A} W, FM) dx dt$$

$$+ \left( b_0 - \frac{C_5}{\mu_{m_0}} - \frac{C_6}{\mu_{m_0}^2} \right) \int_0^T E(t) dt.$$  \hspace{1cm} (36)

Since $H$ is the escape vector field for plate, it follows that

$$\text{div } H = tr DH = \sum_{i=1}^n DH(e_i, e_i) = nb(x).$$  \hspace{1cm} (37)

Then using the above equality and relation (16), we have

$$\| M \|_{H_0} = \| 2H(W) + (\text{div } H - b_0)W \|_{H_0}$$

$$\leq \left( 2 \sup_{x \in \Omega} |H| \sqrt{\mu_{m_0}} + \frac{|nb - b_0|}{\mu_{m_0}} \right) \| W \|_{H_2},$$  \hspace{1cm} (38)

where $C_1 = \frac{2 \sup_{x \in \Omega} |H|}{\sqrt{\mu_{m_0}}} + \frac{|nb - b_0|}{\mu_{m_0}}$.

By Cauchy-Schwarz inequality and relation (38), we estimate the first four terms on the right hand side of equality (35), respectively, as follows:

Since

$$\| W, M \|_{L^2(\Omega)}^N T_0 \leq \| W \|_{H_0} \| M \|_{H_0} \leq C \| W \|_{H_0} \| W \|_{H_2} \leq CE(t),$$  \hspace{1cm} (39)

together with relation (17), it gives

$$\| (W, M) \|_{L^2(\Omega)}^N T_0 \leq C(E(T) + E(0)) \leq C(1 + e^{QT})E(0);$$  \hspace{1cm} (40)
4.3. With initial data \((W_0, W_1)\in \bigoplus_{m \geq m_0} (Z_m \times Z_m)\), in which the unique continuation results for solutions of the second order and fourth order elliptic partial differential equation with variable coefficients are stated in Theorem 2 and Theorem 1, respectively. Let \(\Omega\) be uniformly Lipschitz continuous. If a function \(w\) of class \(C^4\) in \(\Omega\) solves the following problem

\[
\begin{align*}
\sum_{i,j=1}^n a_{ij}(x) \partial_{ij}w(x) + \sum_{i=1}^n b_i(x) \partial_i w(x) + c(x) w(x) &= f(x) \\
\partial_n w|_{\Gamma} &= 0,
\end{align*}
\]

then \(w = 0\) in \(\Omega\).

Combining the relations (36), (40)-(43) and using Lemma 4.1, we derive

\[
\begin{align*}
C_1 \int_{\Sigma_1} (\mathcal{A}W)^2 d\Sigma &
\geq \left[ \left( b_0 - \frac{C_5}{\mu_{m_0}} - \frac{C_6}{\mu_{m_0}^2} - \frac{2\sigma_1 C}{\mu_{m_0}} \right) \frac{1 - e^{-QT}}{Q} - C(1 + e^{QT}) \right] E(0).
\end{align*}
\]

Noting that \(C\) will be sufficiently small and \(\mu_{m_0}\) will be sufficiently large when \(m_0\) is sufficiently large, we obtain inequality (23) for all solutions \(W\) to problem (5) with initial data \((W_0, W_1)\in \bigoplus_{m \geq m_0} (Z_m \times Z_m)\). This concludes the proof of Lemma 4.3. \(\square\)

**Lemma 4.4.** Let \(W\) be the solution to problem (5) with initial data \((W_0, W_1)\in \bigoplus_{m \geq m_0} (Z_m \times Z_m)\), which satisfies additional conditions:

\[
\mathcal{A}W|_{\Sigma_1} = (\mathcal{A}W)_{\nu_{\Sigma_1}}|_{\Sigma_1} = 0.
\]

Then, we have \(W \equiv 0\).

To prove this lemma, we need the following uniqueness result in [37, lemma 3.3], or derived from [30, 32], in which the unique continuation results for solutions of the second order and fourth order elliptic partial differential equation with variable coefficients are stated in Theorem 2 and Theorem 1, respectively.

**Proposition 2.** Let \(\Omega \subset \mathbb{R}^n\) be an open, bounded domain with (sufficiently) smooth boundary \(\Gamma\). Let \(\mathcal{A}w = \text{div} A(x) \nabla w\), where \(A(x)\) is a symmetric, positive matrix whose elements are functions of class \(C^2\) for each \(x \in \Omega\), \(\mathcal{A}\) be a second order elliptic operator. Let the function \(F\) be uniformly Lipschitz continuous. If a function \(w\) of class \(C^4\) in \(\Omega\) solves the following problem

\[
\begin{cases}
\mathcal{A}^2 w = F(w, Dw, D^2 w, D^3 w) \text{ in } \Omega, \\
w = w_{\nu_{\Omega}} = \mathcal{A} w = (\mathcal{A} w)_{\nu_{\Omega}} = 0 \quad \text{on } \Gamma,
\end{cases}
\]

then \(w = 0\) in \(\Omega\).
Proof. We denote by $L(W)$ the lower order terms with respect to the energy $E(t)$, namely,

$$L(W) \equiv O \left( \left\| (W, W_t) \right\|_{C([0,T]; H^{2-\varepsilon}(\Omega)^N \times H^{-\varepsilon}(\Omega)^N)}^2 \right),$$

(47)

where $W$ is the solution to system $(5)$ and $\varepsilon > 0$ is sufficiently small.

Without using the relation (38), we can reestimate (31), (32), (38), (40)-(42) and (43) as follows:

$$\int_Q 4\mathcal{A} w^{(k)}(DH, D^2 w^{(k)})_g dQ \geq \int_Q 2b(x)(\mathcal{A} w^{(k)})^2 dQ - L(w^{(k)});$$

(48)

$$\left| \int_Q 2\mathcal{A} w^{(k)} P(Dw^{(k)}) dQ \right| \leq \epsilon \int_Q (\mathcal{A} w^{(k)})^2 dQ + C_\varepsilon L(w^{(k)});$$

(49)

$$\left| \int_Q \mathcal{A} w^{(k)} \left[ 2Dp(w^{(k)}) + w^{(k)} \mathcal{A} p \right] dQ \right| \leq \epsilon \int_Q (\mathcal{A} w^{(k)})^2 dQ + C_\varepsilon L(w^{(k)}).$$

(50)

$$\|M\|_{\mathcal{H}_0} \leq CL(W);$$

(51)

$$\left| (W_t, M)_{[L^2(\Omega)]^N}^T \right| \leq \epsilon E(0) + \epsilon E(T) + C_\varepsilon L(W);$$

(52)

$$\left| \int_0^T \int_\Omega (W, BM) dxdt \right| \leq CL(W);$$

(53)

$$\left| \int_0^T \int_\Omega (W_t, EM) dxdt \right| \leq \epsilon \int_0^T E(t) dt + C_\varepsilon L(W);$$

(54)

$$\left| \int_0^T \int_\Omega (\mathcal{A} W, FM) dxdt \right| \leq \epsilon \int_0^T E(t) dt + C_\varepsilon L(W).$$

(55)

Accordingly, we obtain

$$\int_{\Sigma_1} (\mathcal{A} W)^2 d\Sigma + C_\varepsilon L(W) \geq (2b_0 - \epsilon) \int_0^T E(t) dt - \epsilon E(0) - \epsilon E(T).$$

(56)

Then there exist positive constants $C_7, C_8$ and $C_9$ such that for $\epsilon$ small enough, we have

$$C_7 \int_{\Sigma_1} \left[ (\mathcal{A} W)^2 + (\mathcal{A} W)^2_{\nu=\alpha} \right] d\Sigma + C_8 L(W) \geq C_9 \int_0^T E(t) dt.$$

(57)

Let $Y = \{ W \in [H^{2,1}(Q)]^N : W$ is a solution to problem $(5)$ with initial data $(W_0, W_1) \in \bigoplus_{m \geq 1} (Z_m \times Z_m)$ satisfying $\mathcal{A} W|_{\Sigma_1} = (\mathcal{A} W)_{\nu=\alpha}|_{\Sigma_1} = 0 \}$. Then

$$Y = \{ 0 \}. $$

(58)

Indeed, from inequality (57), we have

$$C_8 L(W) \geq C_9 \int_0^T E(t) dt \quad \text{for all } W \in Y,$$

which implies that any bounded closed set in $Y \cap [H^{2,1}(Q)]^N$ is compact in $[H^{2,1}(Q)]^N$. Then $Y$ is a finite-dimensional linear space. For any $W \in Y$, we can readily obtain that $W_t \in Y$. Then $\partial_t : Y \rightarrow Y$ is a linear operator. Let $Y \neq \{ 0 \}$, then $\partial_t$ has
at least one eigenvalue \( \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{bmatrix} \). Assume that \( V \neq 0 \) is one of its eigenfunctions, then \( V_t = \Lambda V \). Further, \( V \) is a nonzero solution to the problem

\[
\begin{aligned}
\mathcal{A}^2 V &= (E^T \Lambda - B^T - \Lambda^2) V - F^T \mathcal{A} V \text{ in } \Omega, \\
V &= V_{\nu_{\mathcal{A}}} = \mathcal{A} V = (\mathcal{A} V)_{\nu_{\mathcal{A}}} = 0 \text{ on } \Gamma_1.
\end{aligned}
\]

(59)

By Schur’s theorem, we may assume that \( B, E \) and \( F \) are upper triangular matrices so that the above problem can be rewritten as

\[
\begin{aligned}
\mathcal{A}^2 v^{(k)} &= \sum_{j=1}^{k} e_{jk} \lambda_j v^{(j)} - \sum_{i=1}^{k} b_{ik} v^{(i)} - \lambda_k^2 v^{(k)} - \sum_{l=1}^{k} f_{lk} \mathcal{A} v^{(l)} \text{ on } \Omega, \\
v^{(k)} &= v^{(k)}_{\nu_{\mathcal{A}}} = \mathcal{A} v^{(k)} = (\mathcal{A} v^{(k)})_{\nu_{\mathcal{A}}} = 0 \text{ on } \Gamma_1,
\end{aligned}
\]

(60)

for \( k = 1, 2, \ldots, N \).

However, according to Proposition 4.2, problem (60) only has zero solution, this contradiction shows that (58) holds. The proof of Lemma 4.4 is now completed. \( \square \)

The Completion of the Proof of Theorem 2.1.

Appealing to the ideas originally presented in [22], we shall prove the observability inequality (9) for the initial data \( (W_0, W_1) \in \bigoplus_{m \geq 1} (Z_m \times Z_m) \) by using the following lemma.

**Lemma 4.5.** ([22, Lemma 1.1]) Let \( \mathcal{F} \) be a Hilbert space endowed with the \( p \)-norm. Assume that

\[
\mathcal{F} = \mathcal{N} \oplus \mathcal{L},
\]

(61)

where \( \bigoplus \) denotes the direct sum and \( \mathcal{L} \) is a finite co-dimensional closed subspace in \( \mathcal{F} \). Assume that \( q \) is another norm in \( \mathcal{F} \) such that the projection from \( \mathcal{F} \) into \( \mathcal{N} \) is continuous with respect to the \( q \)-norm. Assume furthermore that

\[
q(y) \leq p(y), \forall y \in \mathcal{L}.
\]

(62)

Then there exists a positive constant \( C > 0 \) such that

\[
q(z) \leq Cp(z), \forall z \in \mathcal{F}.
\]

(63)

For any given \( (W_0, W_1) \in \bigoplus_{m \geq 1} (Z_m \times Z_m) \), we define

\[
p(W_0, W_1) = \sqrt{\int_{\Sigma_1} \left[ (\mathcal{A} W)^2 + (\mathcal{A} W_{\nu_{\mathcal{A}}})^2 \right] d\Sigma},
\]

where \( W \) is the corresponding solution to problem (5). By Lemma 4.4, \( p(\cdot) \) defines well a norm in \( \bigoplus_{m \geq 1} (Z_m \times Z_m) \). Then, we denote by \( \mathcal{F} \) the completion of \( \bigoplus_{m \geq 1} (Z_m \times Z_m) \) with respect to the \( p \)-norm. Clearly, \( \mathcal{F} \) is a Hilbert space. Furthermore, we have \( \mathcal{F} \in \mathcal{H}_2 \times \mathcal{H}_0 \).

We next define \( \mathcal{F} = \mathcal{N} \oplus \mathcal{L} \) with

\[
\mathcal{N} = \bigoplus_{1 \leq m < m_0} (Z_m \times Z_m), \mathcal{L} = \left\{ \bigoplus_{m \geq m_0} (Z_m \times Z_m) \right\}.
\]

(64)
It is easy to see that $N$ is a finite-dimensional subspace and $L$ is a closed subspace in $F$. In particular, the observability inequality (23) can be extended to all initial data $(W_0, W_1)$ in the whole subspace $L$.

Now we introduce the $q$-norm by

$$q(W_0, W_1) = \|(W_0, W_1)\|_{H_2 \times H_0}, \forall (W_0, W_1) \in F.$$  \hfill (65)

By (15), the subspaces $(Z_m \times Z_m)$ are mutually orthogonal in $H_2 \times H_0$ for all $m \geq 1$, then the subspace $N$ is an orthogonal complement of $L$ in $H_2 \times H_0$. In particular, the projection from $F$ into $N$ is continuous with respect to the $q$-norm. On the other hand, since the observability inequality (23) holds for all initial data $(W_0, W_1)$ in the subspace $L$, the condition (62) is verified. Then, applying Lemma 4.5, we obtain the inequality (9) for initial data $(W_0, W_1) \in F$. This completes the proof of Theorem 2.1.

**Remark 3.** As is shown above, we first establish the observability inequality (9) only for the initial data $(W_0, W_1)$ with higher frequencies lying in the sub-linear hull $\bigoplus_{m \geq m_0} (Z_m \times Z_m)$ with an integer $m_0 \geq 1$ large enough. We next extend it to the closure of the whole linear hull $\bigoplus_{m \geq 1} (Z_m \times Z_m)$ by an argument of compact perturbation given in Lemma 4.5.

5. **Concluding remarks.** In this work, we have studied the controllability of a coupled system of $N$ plate equations with variable coefficients. By associating the variable coefficient matrix $A(x)$ with a Riemann metric, we consider the problem on a Riemannian manifold. Using the duality method, we find the corresponding observability inequality (9). After establishing it by virtue of the multiplier technique and a compact perturbation method introduced by Li and Rao [22], we prove that system (1) is exactly null controllable on the space $(L^2(\Omega))^N \times (H^{-2}(\Omega))^N$ by two groups of controls in the space $L^2\left((0, T; L^2(\Gamma_1))^N \times L^2\left((0, T; L^2(\Gamma_1))^N \right.$ acting in partial clamped boundary.

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