Time-periodic solution to nonhomogeneous isentropic compressible Euler equations with time-periodic boundary conditions

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Abstract: In this paper, we study one-dimensional nonhomogeneous isentropic compressible Euler equations with time-periodic boundary conditions. With the aid of the energy methods, we prove the existence and uniqueness of the time-periodic supersonic solutions after some certain time.

Keywords: Isentropic compressible Euler equations, nonhomogeneous term, time-periodic boundary conditions, time-periodic supersonic solutions

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1 Introduction

In this paper, we study the initial-boundary value problem for one dimensional isentropic compressible Euler system with nonhomogeneous term

\[
\begin{cases} 
\rho_t + (\rho u)_x = 0, \\
(\rho u)_t + (\rho u^2 + p)_x = \alpha \rho u, \\
\rho(0, x) = \rho_0,\ u(0, x) = u_0, \\
\rho(t, 0) = \rho_l(t),\ u(t, 0) = u_l(t)
\end{cases}
\]

(1.1)

in the domain \((t, x) \in [0, \infty) \times [0, L]\) with a given constant \(L > 0\), where \(\rho \geq 0\) and \(u \in \mathbb{R}\) represent the density and velocity of the fluid, respectively. \(p\) presents the pressure function, which is given by

\[p = a \rho^\gamma,\ a > 0,\ \gamma > 1\]

(1.2)

for the isentropic polytropic gas. Here the initial data \(\rho_0 > 0\) and \(u_0 > 0\) are two constants, the boundary conditions \(\rho_l(t),\ u_l(t)\) are periodic functions with

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a period \( P > 0 \) and satisfy the corresponding compatibility conditions, i.e.

\[
\rho_l(t) = \rho_l(t + P), \quad u_l(t) = u_l(t + P),
\]  

and

\[
\rho_l(0) = \rho, \quad u_l(0) = u.
\]  

The term \( \alpha \rho u \) in the second equation of (1.1) is the so-called damping (acceleration) effect on the fluid for \( \alpha < 0 \) (\( \alpha > 0 \)). Here \( \alpha \) denote the external force coefficient.

We consider the inflow problem for the one-dimensional isentropic compressible Euler equation with source term \( \alpha(t) \rho u \). We focus on the conditions, imposed on the external force coefficient \( \alpha(t) \), that the Euler equation can trigger a time periodic solution by a time periodic boundary condition \((\rho_l(t), u_l(t))\). We consider this problem for two reasons: In one hand, from a physical point of view, the force produced by the wall of (constant-area, convergent, or divergent) duct \[21\] can be regarded as some source term added to the Euler system. On the other hand, Yuan in \[25\] considered a similar problem for Euler equation without source term, we want to extend the results to a more general model.

Moreover, since the initial data is a constant which is independent of space variable \( x \), the boundary data \( \rho(0, t), u(0, t) \) is only functions of time. A natural consideration is whether the system have a stable smooth solution, which is also independent of space \( x \), for later time?

There are many important progresses on the studies of the time-periodic solutions of the partial differential equations, for instance the viscous fluids equations \[1, 11, 13, 14, 17\] and the hyperbolic conservation laws \[6, 19, 23, 24\]. All of the works mentioned above discuss the time-periodic solutions which are driven by the time-periodic external forces or the piston motion. For the case of the time-periodic boundary condition, there are fewer works on the time-periodic solutions of the hyperbolic conservation laws. Yuan considered the time-periodic solution for the isentropic compressible Euler equations (i.e. \( \alpha = 0 \)) triggered by time-periodic supersonic boundary condition in \[25\]. For the quasilinear hyperbolic system

\[
\partial_t u + A(u) \partial_x u = 0, \quad (t, x) \in \mathbb{R} \times [0, L]
\]

with a more general time-periodic boundary conditions, Qu studied the existence and stability of the time-periodic solutions around a small neighborhood of \( u \equiv 0 \) in \[20\]. Intensive literatures have investigated the isentropic compressible Euler system with source terms. We refer to \[3, 22\] and the references therein for the results on the formation of singularity, \[2, 5, 9, 10, 26\] for the existence and large time behavior of weak solutions, and \[3, 4, 7, 8, 12, 16, 18\] for the asymptotic behavior of smooth solutions, etc.

We assume the coefficient \( \alpha = \alpha(t) = \alpha(t + P) \), which belongs to \( C^2 \) and satisfies

\[
0 \leq \int_0^t \alpha(s) ds < +\infty, \quad \forall t \in [0, \infty),
\]  

(1.5)
\[ \int_0^P \alpha(t) dt = 0. \quad (1.6) \]

Obviously, \( \alpha(t) \equiv 0 \) satisfy (1.5) and (1.6). In this case, the equations (1.1) turns into the usual isentropic compressible Euler equations. Yuan proved the existence of the time-periodic supersonic solution derived by the periodic boundary condition (1.1) in [25], where the initial data is assumed near a supersonic constant state \((\rho, u)\) with

\[
u > \varepsilon := \sqrt{\gamma \rho \frac{\gamma - 1}{2}}.
\quad (1.7)
\]

Moreover, the existence of time-periodic weak solution with small initial data around \((\rho, u)\) is also considered in [25].

In this paper, we consider the one dimensional isentropic compressible Euler equations with the time-periodic external force coefficient \(\alpha(t)\), and prove the existence of a time-periodic supersonic solution induced by the time-periodic boundary condition. Unlike the constant supersonic state \((\rho, u)\), here we consider the time-periodic background supersonic solution \((\rho, e^{\int_0^t \alpha(s) ds} u)\) and prove that the problem (1.1)-(1.6) has a time-periodic supersonic solution after some certain time. Our precise results are sated below.

**Theorem 1.1.** There exist positive constants \(\varepsilon_0, T_0, C_0\), such that if

\[
\|\rho_l - \rho\|_{H^2([0, P])} + \|u_l - e^{\int_0^t \alpha(s) ds} u\|_{H^2([0, P])} \leq \varepsilon
\]

for any \(\varepsilon \leq \varepsilon_0\), then the initial-boundary value problem (1.1) admits a unique solution \((\rho, u) \in C^1([0, +\infty) \times [0, L])\), which satisfies

\[
\rho(t + P, x) = \rho(t, x), u(t + P, x) = u(t, x), \quad \forall t > T_0, x \in [0, L],
\quad (1.9)
\]

\[
\|\rho - \rho\|_{C^1([0, \infty) \times [0, L])} + \|u - e^{\int_0^t \alpha(s) ds} u\|_{C^1([0, \infty) \times [0, L])} \leq C_0 \varepsilon.
\quad (1.10)
\]

There are a few remarks in order.

**Remark 1.1.** The conditions (1.5) and (1.6), imposed on the external force coefficient \(\alpha(t)\), are proposed to ensure the background solution \((\rho, e^{\int_0^t \alpha(s) ds} u)\) is a periodic supersonic solution.

As for the assumption on \(\alpha(t)\) in (1.5), namely

\[
\int_0^t \alpha(s) ds > 0,
\]

can be extended to

\[
\int_0^t \alpha(s) ds > \ln \varepsilon - \ln u,
\]

which is imposed to guarantee the background solution is supersonic everywhere.
Remark 1.2. The conditions (1.5) and (1.6) are only sufficient conditions to ensure the existence of time-periodic solution. To see this, we assume \( \alpha \equiv \text{const.} \neq 0 \) which do not satisfy these two conditions. However, the periodic boundary condition can trigger time-periodic solution, too. We will deal with this problem in the following works.

Remark 1.3. Supersonic is an essential assumption which implies that all characteristics propagate forward in both space and time. In particular, this implies that the effects of any nonlinear interaction at \((x_0, t_0)\) are confined to the region \( x > x_0, t > t_0 \). Supersonic condition plays a very important role in our proof. For example, in the case of exchanging \( x \) and \( t \) in (2.5), we need \( \lambda_1 \) and \( \lambda_2 \) are not zero in all \((t, x) \in [0, +\infty) \times [0, L]\); in the proof of Theorem 4.2, the positive definite matrix \( \Lambda \) is significant. However, these two things can be ensured by the supersonic condition.

Remark 1.4. The results can be extend to higher estimate by using a standard \( H^s \) energy estimates together with a finite speed of propagation/domain of dependence argument, here we omit the detail.

Remark 1.5. Using a similar method, we can also consider the problem with nonlinear friction \( \alpha(t) \rho u |u|^{\theta} \), for \( \theta > 0 \), here omit the detail.

The rest of the paper is organized as follows. In Section 2, we give some basic facts for the Euler equations. In Section 3, we show that the smooth solution to the initial-value problem (1.1) is time-periodic after some certain time when it is a small perturbation of the supersonic background state. In Section 4, we give the proof of Theorem 1.1 by the aid of two solutions of the two initial(-boundary) value problems stated in Theorem 4.1 and Theorem 4.2.

2 Preliminary and Formulation

We first introduce some basic facts for system (1.1). The eigenvalues are

\[
\lambda_1 = u - c = u - \sqrt{\alpha \gamma \rho^{\frac{\gamma - 1}{2}}} , \quad \lambda_2 = u + c = u + \sqrt{\alpha \gamma \rho^{\frac{\gamma - 1}{2}}} \tag{2.1}
\]

and the corresponding right eigenvectors are

\[
\vec{r}_1 = \frac{1}{\sqrt{\rho^2 + c^2}}(\rho, -c)^T , \quad \vec{r}_2 = \frac{1}{\sqrt{\rho^2 + c^2}}(\rho, c)^T .
\]

With the aid of the Riemann invariants

\[
r = \frac{1}{2} (u - \frac{2}{\gamma - 1} c) , \quad s = \frac{1}{2} (u + \frac{2}{\gamma - 1} c) , \tag{2.2}
\]
the problem (1.1) can be rewritten as follows

\[\begin{align*}
  r_t + \lambda_1 (r, s) r_x &= \frac{\alpha(t) (r + s)}{2}, \\
  s_t + \lambda_2 (r, s) s_x &= \frac{\alpha(t) (r + s)}{2}, \\
  r(0, x) &= r, \quad s(0, x) = \bar{s}, \\
  r(t, 0) &= r_l(t), \quad s(t, 0) = s_l(t)
\end{align*}\]

(2.3)

with

\[\begin{align*}
  r &= \frac{1}{2} u - \frac{\sqrt{\alpha \gamma}}{\gamma - 1} \rho^\frac{\gamma-1}{2}, \\
  s &= \frac{1}{2} u + \frac{\sqrt{\alpha \gamma}}{\gamma - 1} \rho^\frac{\gamma-1}{2}, \\
  r_l(t) &= \frac{1}{2} u_l(t) - \frac{\sqrt{\alpha \gamma}}{\gamma - 1} \rho_l^\frac{\gamma-1}{2}(t), \\
  s_l(t) &= \frac{1}{2} u_l(t) + \frac{\sqrt{\alpha \gamma}}{\gamma - 1} \rho_l^\frac{\gamma-1}{2}(t).
\end{align*}\]

(2.4)

After exchanging the roles of \(t\) and \(x\) in (2.3), we consider the following Cauchy problem

\[\begin{align*}
  r_x + \frac{1}{\lambda_1} r_t &= \frac{\alpha(t) (r + s)}{2 \lambda_1}, \\
  s_x + \frac{1}{\lambda_2} s_t &= \frac{\alpha(t) (r + s)}{2 \lambda_2}, \\
  r(t, 0) &= r_r(t), \quad t > 0, \\
  r(t, 0) &= r_\alpha(t), \quad t \leq 0, \\
  s(t, 0) &= s_r(t), \quad t > 0, \\
  s(t, 0) &= s_\alpha(t), \quad t \leq 0.
\end{align*}\]

(2.5)

Here

\[\begin{align*}
  r_\alpha(t) &= \frac{1}{2} e^{\int_0^t \alpha(s) ds} u - \frac{1}{\gamma - 1} \rho_{\alpha}^\frac{\gamma-1}{2}, \\
  s_\alpha(t) &= \frac{1}{2} e^{\int_0^t \alpha(s) ds} u + \frac{1}{\gamma - 1} \rho_{\alpha}^\frac{\gamma-1}{2},
\end{align*}\]

(2.6)

which can be looked as a background periodic solution of problem (2.5) satisfying

\[r'_\alpha(t) = \frac{\alpha(t)}{2} (r_\alpha + s_\alpha), \quad s'_\alpha(t) = \frac{\alpha(t)}{2} (r_\alpha + s_\alpha).
\]

3 Periodic Solution

In this Section, we will prove the supersonic smooth solution \((r, s)(t, x)\), satisfying the a-priori estimate

\[\|r - r_\alpha\|_{C^1(R \times [0, L])} + \|s - s_\alpha\|_{C^1(R \times [0, L])} \leq \varepsilon\]

(3.1)

for some sufficiently small \(\varepsilon > 0\), is a time-periodic function when \(t\) is big enough. To this end, we set

\[m = \left( \begin{array}{c}
  r(t, x) - r_\alpha(t) \\
  s(t, x) - s_\alpha(t)
\end{array} \right),\]
which satisfies that
\[
\begin{cases}
  m_x + \Lambda(t, x) m_t = \frac{\alpha(t)}{2} \Lambda(t, x) \begin{pmatrix}
    r - r_\alpha + s - s_\alpha \\
    r - r_\alpha + s - s_\alpha
  \end{pmatrix}, \\
  m(t, 0) = \begin{cases}
    (r(t) - r_\alpha(t), s(t) - s_\alpha(t))^T, & t > 0, \\
    0, & t \leq 0,
  \end{cases}
\end{cases}
\] (3.2)

where
\[
\Lambda(t, x) = \begin{pmatrix}
  \frac{1}{\lambda_1(r(t, x), s(t, x))} & 0 \\
  0 & \frac{1}{\lambda_2(r(t, x), s(t, x))}
\end{pmatrix}.
\] (3.3)

From (3.1), for small \( \varepsilon > 0 \), we deduce that the flow is still supersonic and
\[
\lambda_0 = \inf_{t \geq 0, x \in [0, L]} \lambda_1(r(t, x), s(t, x)) > 0.
\] (3.4)

We next prove that
\[
\begin{align*}
  r(t + P, x) - r_\alpha(t + P) &= r(t, x) - r_\alpha(t), \\
  s(t + P, x) - s_\alpha(t + P) &= s(t, x) - s_\alpha(t),
\end{align*}
\] (3.5)

for any \( t \geq T_0 := \frac{L}{\lambda_0} \) and \( x \in [0, L] \). Then we can conclude \( r(t, x) \) and \( s(t, x) \) are time-periodic functions.

Write
\[
V(t, x) = m(t + P, x) - m(t, x).
\]

After a straightforward computation, we have from (3.2) that
\[
\begin{cases}
  V_x + \Lambda(t, x) V_t = G(t, x), \\
  V(t, 0) = \begin{cases}
    (r(t + P) - r_\alpha(t + P), s(t + P) - s_\alpha(t + P))^T, & P \leq t \leq 0, \\
    0, & t > 0, \text{ or } t < -P,
  \end{cases}
\end{cases}
\] (3.6)

where
\[
G(t, x) = \frac{\alpha(t)}{2} \Lambda(t + P, x) \begin{pmatrix}
  r(t + P, x) + s(t + P, x) \\
  r(t + P, x) + s(t + P, x)
\end{pmatrix} - \frac{\alpha(t)}{2} \Lambda(t, x) \begin{pmatrix}
  r(t, x) + s(t, x) \\
  r(t, x) + s(t, x)
\end{pmatrix} - (\Lambda(t + P, x) - \Lambda(t, x)) \begin{pmatrix}
  r_\alpha'(t + P) \\
  s_\alpha'(t + P)
\end{pmatrix} - (\Lambda(t + P, x) - \Lambda(t, x)) m_t(t + P, x).
\]

In above calculations, we have used the facts that
\[
\alpha(t + P) = \alpha(t), \text{ and } \int_0^{t+P} \alpha(s) ds = \int_0^t \alpha(s) ds.
\]
By (1.5), (1.6) and (3.1) with $\varepsilon > 0$ small enough, we have for any $t \in [0, \infty)$ and $x \in [0, L]$ that

\begin{align*}
|m_t(t + P, x)| &\leq C_2 \varepsilon, \quad (3.7) \\
|\Lambda(r(t, x), s(t, x))| + |\Lambda_s(r(t, x), s(t, x))| &\leq C_2, \quad (3.8) \\
|r(t + P, x) + s(t + P, x)| + |r'_\alpha(t + P)| + |s'_\alpha(t + P)| &\leq C_2, \quad (3.9) \\
|\Lambda(t + P, x) - \Lambda(t, x)| &\leq |\Lambda_r||r(t + P, x) - r(t, x)| \\
&+ |\Lambda_s||s(t + P, x) - s(t, x)| \leq C_2|V(t, x)|, \quad (3.10)
\end{align*}

where

\begin{align*}
\Lambda_r &= \begin{pmatrix}
-\frac{1}{\lambda_1^2} \frac{\gamma + 1}{2} & 0 \\
0 & -\frac{1}{\lambda_2^2} \frac{3 - \gamma}{2}
\end{pmatrix}, & \Lambda_s &= \begin{pmatrix}
-\frac{1}{\lambda_1^2} \frac{\gamma - 3}{2} & 0 \\
0 & -\frac{1}{\lambda_2^2} \frac{\gamma + 1}{2}
\end{pmatrix},
\end{align*}

and $C_2 > 0$ only depends on $\rho, u, \gamma$ and $L$. Then we obtain from (3.7)-(3.10) that

\begin{align*}
|G(t, x)| &= \frac{\alpha(t)}{2} [\Lambda(t + P, x) - \Lambda(t, x)] \begin{pmatrix}
 r(t + P, x) + s(t + P, x) \\
 r(t + P, x) + s(t + P, x)
\end{pmatrix} \\
&+ \frac{\alpha(t)}{2} \Lambda_r(t, x) \begin{pmatrix}
 r(t + P, x) - r(t, x) + s(t + P, x) - s(t, x) \\
 r(t + P, x) - r(t, x) + s(t + P, x) - s(t, x)
\end{pmatrix} \\
&- (\Lambda(t + P, x) - \Lambda(t, x)) \begin{pmatrix}
 r'_\alpha(t + P) \\
 s'_\alpha(t + P)
\end{pmatrix} \\
&- (\Lambda(t + P, x) - \Lambda(t, x))m_t(t + P, x)| \\
&\leq C_3|V(t, x)|, \quad (3.11)
\end{align*}

where $C_3 > 0$ only depends on $\rho, u, \gamma$ and $L$.

Fixing a point $(t', x')$ with $T > T_0$ and $0 < x' < L$, we draw the slow and fast characteristic curves $\Gamma_1 : t = t_1(x)$ and $\Gamma_2 : t = t_2(x)$

\begin{align*}
\frac{dt_1}{dx} &= \frac{1}{\lambda_1(r(t_1, x), s(t_1, x))}, \quad t_1(x') = t', \\
\frac{dt_2}{dx} &= \frac{1}{\lambda_2(r(t_2, x), s(t_2, x))}, \quad t_2(x') = t',
\end{align*}

which intersect the t-axis. Noting that $\Gamma_1$ is below $\Gamma_2$ as shown in the figure below.
For any $x \in [0, x')$, we set
\[
I(x) = \frac{1}{2} \int_{t_1(x)}^{t_2(x)} |V(t, x)|^2 dt.
\] (3.12)

Due to $t' > T_0 := \frac{L}{\lambda_0}$ and the definition of $\lambda_0$, we have $(t_1(0), t_2(0)) \subset (0, +\infty)$. Then by (3.6), $V(t, 0) \equiv 0$, therefore
\[
I(0) = 0.
\] (3.13)

Taking derivative on $I(x)$ with regard to $x$, it holds that
\[
I'(x) = \int_{t_1(x)}^{t_2(x)} V(t, x)^T V_x(t, x) dt + \frac{1}{2} |V(t_2(x), x)|^2 \frac{1}{\lambda_2(t_2(x), x)}
\[
- \frac{1}{2} |V(t_1(x), x)|^2 \frac{1}{\lambda_1(t_1(x), x)}
\leq - \int_{t_1(x)}^{t_2(x)} V(t, x)^T \Lambda(t, x) V(t, x) dt + \int_{t_1(x)}^{t_2(x)} V(t, x)^T G(t, x) dt
\[
+ \frac{1}{2} V(t, x)^T \Lambda(t, x) V(t, x) |_{t=t_2(x)} - V(t, x)^T \Lambda(t, x) V(t, x) |_{t=t_1(x)}
\leq - \frac{1}{2} \int_{t_1(x)}^{t_2(x)} \left( (V(t, x)^T \Lambda(t, x) V(t, x))_t - V(t, x)^T \Lambda_t(t, x) V(t, x) \right) dt
\[
+ \int_{t_1(x)}^{t_2(x)} V(t, x)^T G(t, x) dt + \frac{1}{2} V(t, x)^T \Lambda(t, x) V(t, x) |_{t=t_2(x)} - V(t, x)^T \Lambda(t, x) V(t, x) |_{t=t_1(x)}
\leq \frac{1}{2} \int_{t_1(x)}^{t_2(x)} V(t, x)^T \Lambda_t(t, x) V(t, x) dt + \int_{t_1(x)}^{t_2(x)} V(t, x)^T G(t, x) dt
\leq (\frac{C_2}{2} + C_3) I(x).
\]
In the last inequality above, we have used (3.8) and (3.11). Using the Gronwall’s inequality, we can get from (3.13) that $I(x) \equiv 0$. Then it follows from the continuity of $I(x)$ that $I(x') = 0$ which implies that $V(t', x') = 0$.

Because of the arbitrary of $(t', x')$, we have $V(t, x) \equiv 0$, $\forall t > T_0$, $x \in [0, L]$.

Thus, we prove (3.5). Since $\alpha(t)$ and $\int_0^t \alpha(s)ds$ are periodic functions with a period $P > 0$, then we get from (3.5) that $r(t)$ and $s(t)$ are also periodic functions with a period $P > 0$. Namely,

$$r(t + P, x) = r(t, x), \quad s(t + P, x) = s(t, x), \quad \forall t > T_0, x \in [0, L].$$

4 Existence of Solutions

In this Section, we prove the existence of the time-periodic solution $m = (m_1, m_2)^T := (r - r_\alpha, s - s_\alpha)^T$ for $(t, x) \in [0, +\infty) \times [0, L]$ to the following Cauchy problem

$$\begin{cases}
  m_x + \Lambda m_t = \frac{\alpha(t)}{2} \Lambda \begin{pmatrix}
    r - r_\alpha + s - s_\alpha \\
    r - r_\alpha + s - s_\alpha
  \end{pmatrix}, \\
  m(t, 0) = m_0(t) = (r_l(t) - r_\alpha(t), s_l(t) - s_\alpha(t))^T.
\end{cases}$$

(4.1)

Firstly, for the smooth solution $m$ to the Cauchy problem (4.1) in $(t, x) \in [T_0, T_0 + P] \times [0, L]$, we will establish an a-priori estimate

$$\sup_{x \in [0, L]} \|m\|_{H^2([T_0, T_0 + P])} < \delta$$

(4.2)

for some constant $\delta > 0$ small enough. Denoting

$$\begin{pmatrix}
  r - r_\alpha + s - s_\alpha \\
  r - r_\alpha + s - s_\alpha
\end{pmatrix} = \begin{pmatrix}
  1 & 1 \\
  1 & 1
\end{pmatrix} \begin{pmatrix}
  r - r_\alpha \\
  s - s_\alpha
\end{pmatrix} = Am,$

and letting $\hat{\Lambda} = \hat{\Lambda}(t, x) = \Lambda A$, then (4.1) has the form

$$m_x + \hat{\Lambda} m_t = \frac{\alpha(t)}{2} \hat{\Lambda} m.$$  

(4.3)

Without losing generality, we let the period $P = 1$ for simplicity in the rest of the paper. Now, we have the following Lemma:

**Lemma 4.1.** For sufficiently small $\delta > 0$, it holds that

$$\lambda_1(t, x) \geq \frac{1}{2} (\mu - \varsigma), \quad \lambda_2(t, x) \geq \frac{1}{2} (\mu + \varsigma)$$

(4.4)

for any $(t, x) \in [T_0, T_0 + 1] \times [0, L]$. 

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and (4.2) we know where we have used the integration by parts in the second identity. From Lemma 4.1 \( \lambda \) that 

\[
\lambda(t, x) = \frac{\gamma + 1}{2}(r - r_0) - \frac{\gamma - 3}{2}(s - s_0) + u_0 - \xi
\]

\[
\geq u_0 - \xi - \frac{\gamma + 1 + |\gamma - 3|}{2}m
\]

\[
\geq u - \xi - \frac{\gamma + 1 + |\gamma - 3|}{2}K\delta,
\]

(4.5)

where we have used \( \int_0^t \alpha(s)ds \geq 0 \) and the following embedding inequality

\[
\sup_{t \in [T_0, T_0 + 1], x \in [0, L]} |m| \leq K_1 \sup_{x \in [0, L]} \|m(x)\|_{H^2([T_0, T_0 + 1])} < K_1 \delta
\]

(4.6)

for some constant \( K_1 > 0 \). By choosing \( \delta \leq \frac{u - \xi}{(\gamma + 1 + |\gamma - 3|)K_1} \), we obtain from (4.5) that \( \lambda \geq \frac{1}{2}(u - \xi) \). The second inequality in (4.4) can be obtained in a similar way, the details are omitted here. The proof of Lemma 4.1 is completed.

**Proposition 4.1.** Let \( m(t, x) \) be a smooth time-periodic solution to the Cauchy problem (3.2) satisfying (4.2). Then it holds that

\[
\frac{d}{dx}\|m\|^2_{H^2([T_0, T_0 + 1])} \leq C\|m\|^2_{H^2([T_0, T_0 + 1])}
\]

(4.7)

for some constant \( C > 0 \).

**Proof** Multiplying (4.3) by \( m^T \) and integrating it from \( T_0 \) to \( T_0 + 1 \), we have

\[
\frac{d}{dx}\|m\|^2_{L^2} = \int_{T_0}^{T_0 + 1} \left( -m^T \Lambda m + \frac{\alpha(t)}{2} m^T \tilde{\Lambda} m \right) dt
\]

\[
= \int_{T_0}^{T_0 + 1} \left( \frac{1}{2} m^T \Lambda m + \frac{\alpha(t)}{2} m^T \tilde{\Lambda} m \right) dt
\]

\[
\leq \frac{1}{2} \sup_{t \in [T_0, T_0 + 1]} |\Lambda_t| + \sup_{x \in [0, L]} |\alpha(t)||\tilde{\Lambda}|\|m\|^2_{L^2},
\]

(4.8)

where we have used the integration by parts in the second identity. From Lemma 4.1 and (4.2) we know

\[
\sup_{t \in [T_0, T_0 + 1]} |\tilde{\Lambda}| \leq \sup_{t \in [T_0, T_0 + 1]} \frac{1}{\Lambda_1} \leq C_2 := 2(u - \xi)^{-1},
\]

(4.9)

\[
\sup_{x \in [0, L]} |\Lambda_t| \leq \sup_{t \in [T_0, T_0 + 1]} \Lambda_1^{-2} \left( \frac{\gamma + 1 + |\gamma - 3|}{2} |m_t| + \frac{C_3}{2} e^{C_3(T_0 + 1)\gamma} \right)
\]

\[
\leq C_2 \left( \frac{\gamma + 1 + |\gamma - 3|}{2} K_2 \sup_{x \in [0, L]} \|m\|_{H^2([T_0, T_0 + 1])} + \frac{C_3}{2} e^{C_3(T_0 + 1)\gamma} \right)
\]

(4.10)
Taking the derivative \( \partial_t \) on both sides of (4.3), multiplying it by \( m_t^T \) and integrating it from \( T_0 \) to \( T_0 + 1 \), it holds that

\[
\frac{d}{dx} \frac{1}{2} \| m_t \|_{L^2}^2 \leq - \int_{T_0}^{T_0+1} \frac{\alpha'(t)}{2} m_t^T \tilde{\Lambda} m_t dt + \int_{T_0}^{T_0+1} \frac{\alpha(t)}{2} m_t^T \tilde{\Lambda} m_t dt
\]

\[
+ \int_{T_0}^{T_0+1} \frac{\alpha'(t)}{2} m_t^T \tilde{\Lambda} m_t dt + \frac{\alpha(t)}{2} m_t^T \tilde{\Lambda} m_t dt
\]

\[
= \int_{T_0}^{T_0+1} \frac{\alpha'(t)}{2} m_t^T \tilde{\Lambda} m_t dt + \frac{\alpha(t)}{2} m_t^T \tilde{\Lambda} m_t dt
\]

\[
\leq \left( \frac{1}{2} C_4 + \frac{1}{2} C_2 C_3 \right) \| m_t \|_{L^2}^2 + \frac{1}{2} \left( C_2 C_3 + C_3 C_4 \right) \int_{T_0}^{T_0+1} |m_t| |m| dt
\]

\[
\leq \frac{1}{2} C_6 \| m_t \|_{L^2}^2 + \frac{1}{2} C_7 \| m_t \|_{L^2}^2,
\]

where we have used the integration by parts in the second identity, (4.9)-(4.10), and the fact \( \sup_{t \in [T_0, T_0+1]} |\Lambda_t| = \sup_{t \in [T_0, T_0+1]} |\Lambda_t| \).

Similarly, taking the derivative \( \partial^2_t \) on both sides of (4.3), multiplying it by \( m_{tt}^T \) and integrating it from \( T_0 \) to \( T_0 + 1 \), we have

\[
\frac{d}{dx} \frac{1}{2} \| m_{tt} \|_{L^2}^2 = - \int_{T_0}^{T_0+1} m_{tt}^T \Lambda(m_t) dt + \int_{T_0}^{T_0+1} m_{tt}^T \tilde{\Lambda} m_t dt
\]

\[
+ \alpha(t) m_{tt}^T \tilde{\Lambda} m_t + \frac{\alpha'(t)}{2} m_{tt}^T \tilde{\Lambda} m_t + \frac{\alpha''(t)}{2} m_{tt}^T \Lambda m + \alpha(t) m_{tt}^T \Lambda m
\]

\[
- m_{tt}^T \Lambda m dt + \frac{\alpha(t)}{2} m_{tt}^T \tilde{\Lambda} m dt
\]

\[
= \int_{T_0}^{T_0+1} \left( - \frac{3}{2} m_{tt}^T \Lambda m + \frac{\alpha(t)}{2} m_{tt}^T \Lambda m + \alpha(t) m_{tt}^T \Lambda m \right) dt
\]

\[
+ \int_{T_0}^{T_0+1} \frac{\alpha'(t)}{2} m_{tt}^T \tilde{\Lambda} m dt + \frac{\alpha(t)}{2} m_{tt}^T \tilde{\Lambda} m dt
\]

where \( C_3 = \max_{t \geq 0} \{ |\alpha(t)|, |\alpha(t)|, |\alpha'(t)|, |\alpha''(t)| \} \), \( K_2 > 0 \) are constants. Denote \( C_5 = C_4 + C_2 C_3 \), then (4.8) turns into

\[
\frac{d}{dx} \frac{1}{2} \| m \|_{L^2}^2 \leq \frac{1}{2} C_5 \| m \|_{L^2}^2.
\]
\[ + \alpha'(t)m_t^T \Lambda_t m + \alpha''(t) m_t^T \Lambda_m + \alpha'(t)m_t^T \Lambda_t m \]
\[ - m_t^T \Lambda_t m_t + \alpha(t) \frac{\alpha}{2} m_t^T \Lambda_t m dt \]
\[ \leq \left( \frac{3}{2} C_4 + \frac{5}{4} C_2 C_3 + C_3 C_4 \right) \| m_t \|^2_L + \frac{1}{2} (C_3 C_4 + C_2 C_3) \| m_t \|^2_L + C_3 C_t C_4 + \frac{1}{4} C_2 C_3 \| m_t \|^2_L + \int_{T_0}^{T_0+1} \left( \frac{\alpha(t)}{2} m_t^T \Lambda_m m | \right)
\[ + | - m_t^T \Lambda_t m_t |) dt. \]

Since \( \tilde{\Lambda}_t = \tilde{\Lambda}_1 t + \tilde{\Lambda}_2 t + \tilde{\Lambda}_3 t \), with
\[ \tilde{\Lambda}_1 t = \begin{pmatrix} 2 \lambda_2^{-3} (\lambda_2)^2 & 2 \lambda_2^{-3} (\lambda_2)^2 \\ 2 \lambda_1^{-3} (\lambda_1)^2 & 2 \lambda_1^{-3} (\lambda_1)^2 \end{pmatrix}, \]
\[ \tilde{\Lambda}_2 t = \begin{pmatrix} -\lambda_2^{-2} \frac{\alpha'(t)}{2} e_{\alpha(s)} ds \frac{u}{k} & -\lambda_2^{-2} \frac{\alpha'(t)}{2} e_{\alpha(s)} ds \frac{u}{k} \\ -\lambda_1^{-2} \frac{\alpha'(t)}{2} e_{\alpha(s)} ds \frac{u}{k} & -\lambda_1^{-2} \frac{\alpha'(t)}{2} e_{\alpha(s)} ds \frac{u}{k} \end{pmatrix}, \]
\[ \tilde{\Lambda}_3 t = \begin{pmatrix} -\lambda_2^{-2} \frac{3-\gamma}{2} m_{2_t} + \frac{\gamma+1}{2} m_{1_t} & -\lambda_2^{-2} \frac{3-\gamma}{2} m_{2_t} + \frac{\gamma+1}{2} m_{1_t} \\ -\lambda_1^{-2} \frac{3-\gamma}{2} m_{2_t} - \frac{\gamma+1}{2} m_{1_t} & -\lambda_1^{-2} \frac{3-\gamma}{2} m_{2_t} - \frac{\gamma+1}{2} m_{1_t} \end{pmatrix}, \]
we have from Lemma 4.1, 4.6 and 4.10 that
\[ \int_{T_0}^{T_0+1} \frac{\alpha(t)}{2} m_t^T \Lambda_t m dt \leq \frac{1}{2} C_3 \int_{T_0}^{T_0+1} \| m_t \| \| m_t \| dt \]
\[ \leq \frac{1}{2} C_3 \int_{T_0}^{T_0+1} \| m_t \| \| \Lambda_t \| \| m \| dt \]
\[ + \frac{1}{2} C_3 \int_{T_0}^{T_0+1} \| m_t \| \| \Lambda_t \| \| m \| dt \]
\[ \leq \frac{1}{2} C_3 \sup_{t \in [T_0, T_0+1]} (| \Lambda_{1_t} | + | \Lambda_{2_t} |) \int_{T_0}^{T_0+1} \| m_t \| \| m \| dt \]
\[ + \frac{1}{2} C_3 \sup_{t \in [T_0, T_0+1]} \| m \| \int_{T_0}^{T_0+1} \| m_t \| \| \Lambda_t \| \| m \| dt \]
\[ \leq \frac{1}{2} C_3 C_8 \int_{T_0}^{T_0+1} \| m_t \| \| m \| dt + \frac{1}{2} C_3 C_8 K_1 \int_{T_0}^{T_0+1} \| m_t \|^2 dt \]
\[ \leq \frac{1}{4} C_3 C_8 \int_{T_0}^{T_0+1} \| m_t \|^2 dt + \int_{T_0}^{T_0+1} \| m \|^2 dt \]
\[ + \frac{1}{2} C_3 C_9 K_1 \delta \int_{T_0}^{T_0+1} |m_{tt}|^2 dt \]
\[ = \left( \frac{1}{4} C_3 C_8 + \frac{1}{2} C_3 C_9 K_1 \delta \right) \|m_{tt}\|_{L^2}^2 + \frac{1}{4} C_3 C_8 \|m\|_{L^2}^2, \]
\[ (4.14) \]

where
\[ C_8 = 16(u - \xi)^{-3} C_4^2 + 2C_3^2(u - \xi)^{-2} \varepsilon C_3^{(T_0+1)} u, \]
\[ C_9 = 4\left( \frac{\gamma + 1}{2} + \frac{\gamma - 3}{2} \right) (u - \xi)^{-2}. \]
\[ (4.15) \]

By the similar arguments as above, we can get the estimate of \( \int_{T_0}^{T_0+1} |m_{tt}^T \Lambda_{tt} m_t| dt \) as follows
\[ \int_{T_0}^{T_0+1} |m_{tt}^T \Lambda_{tt} m_t| dt \leq \left( \frac{1}{2} C_8 + C_9 K_2 \delta \right) \|m_{tt}\|_{L^2}^2 + \frac{1}{2} C_8 \|m_t\|_{L^2}^2, \]
\[ (4.17) \]

where \( C_8 \) and \( C_9 \) are defined by (4.15) and (4.16) respectively. The details are omitted here.

Substituting (4.14) and (4.17) into (4.13), we have
\[ \frac{d}{dx} \left( \frac{1}{2} \|m_{tt}\|_{L^2}^2 \leq \frac{1}{2} C_{10} \|m_{tt}\|_{L^2}^2 + \frac{1}{2} C_{11} \|m_t\|_{L^2}^2 + \frac{1}{2} C_{12} \|m\|_{L^2}^2, \right. \]
\[ (4.18) \]

where
\[ C_{10} = \left( \frac{1}{2} C_3 C_8 + C_3 C_9 K_1 \delta + C_8 + 2C_3 K_2 \delta + 3C_4 + \frac{5}{2} C_2 C_3 + 2C_3 C_4 \right), \]
\[ C_{11} = (C_8 + C_2 C_3 + C_3 C_4), \quad C_{12} = \left( \frac{1}{2} C_3 C_8 + \frac{1}{2} C_2 C_3 + C_3 C_4 \right). \]

Combining (4.11), (4.12) and (4.18), it holds that
\[ \frac{d}{dx} \left( \|m\|_{L^2}^2 + \|m_t\|_{L^2}^2 + \|m_{tt}\|_{L^2}^2 \right) \]
\[ \leq \left( C_5 + C_7 + C_{12} \right) \|m\|_{L^2}^2 + \left( C_6 + C_{11} \right) \|m_t\|_{L^2}^2 + C_{10} \|m_{tt}\|_{L^2}^2 \]
\[ \leq C_{13} \left( \|m\|_{L^2}^2 + \|m_t\|_{L^2}^2 + \|m_{tt}\|_{L^2}^2 \right), \]

where \( C_{13} = \max\{ C_5, C_7 + C_{12}, C_6 + C_{11}, C_{10} \} \). Thus, we get (4.7). The proof of Proposition 4.1 is completed.

**Theorem 4.1.** There are positive constants \( \varepsilon_0 \) and \( C_0 \), such that if
\[ \|m_0\|_{H^2([T_0, T_0+1])} \leq \varepsilon \]
for any \( \varepsilon \in (0, \varepsilon_0) \), then there is a unique time-periodic solution \( m \in C([0, L]; H^2([T_0, T_0+1])) \cap C^1([0, L]; H^1([T_0, T_0+1])) \) to the Cauchy problem (4.1), which satisfies
\[ \sup_{x \in [0, L]} \|m\|_{H^2([T_0, T_0+1])} + \sup_{x \in [0, L]} \|\partial_x m\|_{H^1([T_0, T_0+1])} \leq C_0 \varepsilon. \]
\[ (4.19) \]
Proof. We can refer to [15] for the local existence and uniqueness of the $C^1$ solution to the Cauchy problem (4.1).

Applying the Gronwall’s inequality to (4.7), we have

$$\sup_{x \in [0, L]} \| m_t(t, x) \|_{H^2([T_0, T_0 + 1])} \leq \| m_0 \|_{H^2([T_0, T_0 + 1])} e^{CL}. \quad (4.20)$$

By choosing $\epsilon_0 = e^{-\frac{CL}{C}} \delta$, if $\| m_0 \|_{H^2([T_0, T_0 + 1])} \leq \epsilon$ for any $\epsilon \in (0, \epsilon_0)$, we have from (4.20) that

$$\sup_{x \in [0, L]} \| m \|_{H^2([T_0, T_0 + 1])} < \delta, \quad (4.21)$$

which verifies (4.2). With the help of (4.7) and (4.21), we can easily check that $m \in C([0, L]; H^2([T_0, T_0 + 1]));$ (4.22)

Furthermore, from (4.3), Lemma 4.1 and (4.22), we can get

$$m(x) \in C([0, L]; H^1([T_0, T_0 + 1])); \quad (4.23)$$

for some constant $C > 0$.

By the aid of the uniform a-priori estimates (4.21) and (4.23) and the standard continuity arguments, we can extend the local solution in $x \in [0, L]$ to obtain the time-periodic solution to the Cauchy problem (4.1), which belongs to $C([0, L]; H^2([T_0, T_0 + 1])); \cap C^1([0, L]; H^1([T_0, T_0 + 1]));$ and satisfies (4.19).

The proof of Theorem 4.1 is completed.

Let $T_1 > T_0$. We next consider the following initial boundary value problem for $m(t, x) = (m_1(t, x), m_2(t, x))^T := (r(t, x) - r_\alpha(t), s(t, x) - s_\alpha(t))^T$ with $(t, x) \in [0, T_1] \times [0, L]

\[
\begin{aligned}
&m_x + \Lambda m_t = \frac{\alpha(t)}{2} \Lambda \begin{pmatrix} r - r_\alpha + s - s_\alpha \\ r - r_\alpha + s - s_\alpha \end{pmatrix}, \\
m(t, 0) = m_0(t) = (r_\alpha(t) - r_\alpha(t), s_\alpha(t) - s_\alpha(t))^T, \\
m(0, x) = 0.
\end{aligned}
\]

(4.24)

For the smooth solution to the system (4.24), it holds that

$m(0, x) = m(t, 0) = m_0(0, x), \quad x \in [0, L].$

Noticing the positive definite of $\Lambda$ and applying the similar arguments to those in the proof of Theorem 4.1 we have the following result.

**Theorem 4.2.** There exist positive constants $\epsilon_1$, $C_1$ and $T_1 > T_0$, such that if

$$\| m_0 \|_{H^2([0, T_1])} \leq \epsilon$$
for any $\varepsilon \in (0, \varepsilon_1)$, then there is a unique solution $m \in C([0,L]; H^2([0,T])) \cap C^1([0,L]; H^1([0,T]))$ to the initial boundary value problem (4.24), which satisfies
\begin{align}
\sup_{x \in [0,L]} \|m\|_{H^2([0,T])} + \sup_{x \in [0,L]} \|m_x\|_{H^1([0,T])} &\leq C_1 \varepsilon. \tag{4.25}
\end{align}

Now, it is time for us to give the proof of Theorem 1.1:

**Proof of Theorem 1.1** Making an extension in time with a period $P$ of the solution given in Theorem 4.1 from $[T_0, T_0 + P]$ to $[T_0, \infty)$, and putting together with it and the solution proved in Theorem 4.2, we obtain a unique solution $m$ to the Cauchy problem (3.2), which satisfies (1.9).

By the regularities of the solutions stated in Theorem 4.1 and Theorem 4.2 and applying the Sobolev embeddings $H^1(I) \hookrightarrow C(I)$ and $H^2(I) \hookrightarrow C^1(I)$ with $I = [T_0, T_0 + P]$ or $[0, T_1]$, we have $m \in C^1([0,\infty) \times [0,L])$. Furthermore, from (4.19) and (4.25), we can obtain (1.10). The proof of Theorem 1.1 is completed.

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