Stationary and oscillatory bound states of dissipative solitons created by the third-order dispersion

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Experimental and theoretical studies of temporal solitons in fiber lasers is a vast area of fundamental and applied research [1]-[3], which suggests generic paradigms of stable dynamics of dissipative solitons to many other fields, in optics and beyond [4, 5]. In addition to isolated solitons, experiments and theoretical (analytical and numerical) investigations of models based on complex Ginzburg-Landau equations (CGLEs) reveal the existence of stable bound complexes of dissipative fiber solitons, starting from the prediction [6] and first experimental observation [10]. The latter subject has drawn a great deal of interest in the course of the last two decades [7, 9–11].

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The fundamental model of lossless nonlinear fibers based on the cubic nonlinear Schrödinger equation (NLSE) does not admit solutions in the form of bound states. The simplest possibility to produce them is to include the third-order dispersion (TOD), represented by the linear term with the third derivative. It is well known that this modification of the NLSE gives rise to bound states, which are, however, structurally unstable [12, 13]. The TOD term is a relevant one in fiber-optic settings when the pump wavelength is close to the zero-GVD (group-velocity dispersion) point. While dealing with fiber lasers, it is also imperative that the TOD term is a relevant one in fiber-optic settings when the pump wavelength is close to the zero-GVD (group-velocity dispersion) point. 

The form and stability of isolated bright dissipative solitons in the CQ-CGLE with the TOD term was investigated too, but to a lesser degree [26–28]. A possibility of the existence of the soliton bound states in this model is a natural extension of the analysis, with obvious perspectives for experimental realization and applications. To address this possibility, we adopt the scaled CQ-CGLE for complex amplitude \( u(z,\tau) \) of the electromagnetic field, where \( z \) and \( \tau \) are the propagation distance and reduced time [1]:

\[
\frac{\partial u}{\partial z} = (i\delta - \beta_2) \frac{\partial^2 u}{\partial \tau^2} - |u|^2 u - i \left( \varepsilon - \alpha_1 |u|^2 + \alpha_2 |u|^4 \right) u + i\beta_3 \frac{\partial^3 u}{\partial \tau^3}.
\]

Here \( \beta_2 \) and \( \beta_3 \) are, respectively, the usual second-order GVD and TOD coefficients (\( \beta_2 > 0 \) corresponds to the anomalous GVD), \( \delta \geq 0 \) is the spectral-filtering parameter (dispersive losses), the effective Kerr coefficient is normalized to be 1, and positive constants \( \varepsilon, \alpha_1 \), and \( \alpha_2 \) account for the linear loss, cubic gain, and quintic loss, respectively.

First, to forecast the existence of bound states of dissipative solitons, it is relevant to analyze the structure of their tails, which overlap with cores of neighboring solitons, giving rise to the effective interaction potential, \( U(T) \), where \( T \) is the temporal separation between the cores. Local minima of the potential, if any, predict values of \( T \) for stationary bound states [6]. The solution for the decaying tail of a dissipative soliton with a propagation constant, \( k \), is looked for as

\[
u(z,\tau) = u_0 \exp \left( ikz - (\chi + i\omega) |\tau| - i\Omega \tau \right),
\]

where \( 1/\chi > 0 \) is the temporal width of the soliton, \( \omega \) determines oscillations of the tails on both sides of the soliton, and \( \Omega \) is an overall frequency shift. Assuming \( \delta, \varepsilon, \alpha_1 \), and \( \alpha_2 \) to be small perturbations, one can solve the resulting cubic equations for real constants \( \chi, \omega, \) and \( \Omega \):

\[
\chi \approx \sqrt{\frac{k}{\beta_2}}, \quad \omega \approx -\frac{\delta k - \varepsilon \beta_2}{2\beta_2^2 \sqrt{k}}, \quad \Omega \approx -\frac{\beta_3 k}{2\beta_2^2}.
\]

where \( \beta_2 k > 0 \) is assumed. Note that Eq. (3) makes it possible to predict, in the present approximation, the group velocity induced by the TOD term, as per Eq. (3):

\[
v \equiv \frac{d\tau}{dz} \approx d\Omega/dk \approx -\left(2\beta_2^2\right)^{-1} \beta_3.
\]

An essential peculiarity of the tail solution (2) is the combination of \( \tau \) and \( |\tau| \) in the phase of Eq. (2). Then, following
a standard method [6] for constructing the effective interaction potential, $U(T, \phi)$, for the pair of solitons with phase shift $\phi$ and temporal separation $T$, one obtains

$$U(T, \phi) = U_0 \exp(i\phi - \chi T) \cos(\omega T) \cos(\Omega T).$$

Equation (5) gives rise to the two sets of the soliton separation distances,

$$T_m^{(\omega)} = (\pi/2\omega)(1 + 2m) ; T_n^{(\Omega)} = (\pi/2\Omega)(1 + 2n),$$

with integer $m$ and $n$, corresponding to potential minima and hence to stationary bound states. Note that $\Omega \sim \beta_3$ in Eq. (3) implies that TOD introduces a new family of bound states that are absent in the well studied no-TOD case.

Numerical results are produced here for fixed parameters

$$\beta_2 = 0.1, \delta = 0.01, \varepsilon = 0.01, \alpha_1 = 0.06, \alpha_2 = 0.006,$$

while varying the TOD coefficient, $\beta_3$, as broader numerical data demonstrate that this parameter set adequately represents the generic situation, the deviation from Eq. (7) producing inconspicuous changes in the results (see details below). In terms of physical parameters, these values correspond, roughly, to fiber solitons with temporal width $\sim 1$ ps.

The numerical solution of Eq. (1) reveals, along with single solitons (see Fig. 1), their bound states of two types, viz., static ones with $z$-independent separation $T$ (Fig. 2), and dynamical bound states, with $T$ oscillating in $z$, as shown in Fig. 3.

**Fig. 1.** (a) A typical profile of a stable dissipative solitons, produced by the numerical solution of Eq. (1), for parameters given by Eq. (7) and $\beta_3 = 0.01$. (b) The solitons’ velocity vs. $\beta_3$, the dashed line showing the prediction given by Eq. (4).

**Fig. 2.** Stable static bound states: (a) a two-soliton one at $\beta_3 = 0.005$; (b) and (c) three- and four-soliton complexes found at $\beta_3 = 0.0065$. Here and below, other parameters are fixed as per Eq. (7).

In Fig. 1(a), the soliton’s profile $|u(\tau)|$, traveling at constant velocity $dt/dz = 0.687$, is shown for $\beta_3 = 0.01$. The front side of the pulse features an oscillating tail, which is reasonably well approximated by Eq. (5), while the trailing tail decays monotonously. Figure 1(b) shows the velocity $v$ as a function of $\beta_3$, and its approximation provided by Eq. (4), which does not take into account nonlinear and dissipative terms in Eq. (1), but nevertheless produces a reasonable approximation. For complexes of bound solitons, dependence $v(\beta_3)$ is very similar to that displayed in Figure 1(b). Solitons bound in static states always have equal amplitudes, while in the oscillatory state the relative difference between instantaneous amplitudes may be $\sim 10\%$.

Figure 3 shows that the two-soliton dynamical bound states exhibit both periodic and chaotic oscillations. Generally, the increase of the TOD parameter, $\beta_3$, leads to the transition from static bound states to regularly oscillating ones, and further to ones with random internal oscillations. Further, Fig. 3(c) demonstrates quasiperiodic oscillations observed at still larger $\beta_3$, with a superposition of long-period large-amplitude oscillations and fast small-amplitude vibrations of the two-soliton states.

Results produced by the systematic simulations of the two-soliton bound states are summarized in Fig. 4(a), which shows the separation between solitons in stable bound states as a function of the TOD coefficient, $\beta_3$. In this plot, a single dot marks a separation between the core in a static bound state of two solitons. Each oscillatory state, periodic, chaotic, or quasiperiodic one, is represented by a vertical segment which covers an interval of values of the separation covered by the oscillations. In broad white gaps, no stable static or dynamical bound states were produced by the simulations. It is worthy to note that the presence of the TOD is indeed necessary for the existence of the bound states represented in Fig. 4, as their families emerge at finite values of $\beta_3$.

A noteworthy finding observed in Fig. 4(a) is the multistability of the static bound states with different values of the tempo-
ral separation between the paired solitons. In accordance with the equidistant spectra of the separation, predicted by Eq. (3), differences between values of the separation in the coexisting states are approximately equal. In the case shown in Fig. 4, six distinct families of the stationary states are found, demonstrating bistability and tristability: two or three different stable bound states may coexist at given values of parameters. On the other hand, stationary bound states never coexist with dynamical ones.

Another essential feature seen in Fig. 4(a) is that, while the static states corresponding to higher values of the separation terminate with the increase of $\beta_3$, the branch corresponding to the minimum separation undergoes a Hopf bifurcation, which replaces the stable fixed point by an emerging limit cycle [29], at $\beta_3 \approx 0.009$. The subsequent transition from regular to chaotic oscillations may be understood as a bifurcation which transforms the limit cycle into a strange attractor [29].

The existence and stability of the bound states reported here most essentially depends on the TOD coefficient, $\beta_3$. The dependence on other parameters was explored too, showing that the bound states are weakly sensitive to their variations. For instance, the variation of the linear-loss parameter $\epsilon$ in Eq. (1) by a factor of $\pm 10$ changes the value of $\beta_3$ at the boundary between the static and oscillatory states only by $\pm 25\%$.

The presence of stable paired states of two dissipative solitons suggests looking for multi-soliton complexes, which were also observed experimentally in fiber lasers [30, 31]. Indeed, in the same parameter region where multistable two-soliton states are found, three- and four-soliton complexes are present too, as shown in Fig. 2(b,c), with equal separations between adjacent solitons. Further, Figs. 4 demonstrates that the multi-soliton states are multistable too, and the one with the smallest separation between the bound solitons, similarly its two-soliton counterpart, ends by a Hopf bifurcation, which leads to a robust complex with periodic oscillations of the separation between the bound solitons.

In conclusion, by means of an analytical approximation and systematic numerical calculations, we have demonstrated that the addition of the TOD term to the standard model of fiber-laser cavities, based on the CGLE with the CQ nonlinearity, which is an essential ingredient of the model near the zero-dispersion point, creates families of stable two-, three-, and four-soliton bound states, which do not exist in the absence of TOD. The bound states may be stationary and oscillatory, including robust bound pairs of two dissipative solitons with the chaotically or quasi-periodically oscillating separation between them. A noteworthy finding is the multistability of stationary two- and multi-soliton bound states, which correspond to different separations between the solitons.

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