POSTNIKOV TOWERS, $k$-INVARIANTS AND OBSTRUCTION THEORY FOR DG CATEGORIES

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Abstract. By inspiring ourselves in Drinfeld’s DG quotient, we develop Postnikov towers, $k$-invariants and an obstruction theory for dg categories. As an application, we obtain the following ‘rigidification’ theorem: let $A$ be a homologically connective dg category and $F_0 : B \to \mathbb{H}_0(A)$ a dg functor to its homotopy category. If the family $\{\omega_n(F_n)\}_{n \geq 0}$ of obstruction classes vanishes, then a lift $F : B \to A$ for $F_0$ exists.

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1. Introduction

A differential graded (dg) category is a category enriched in the category of complexes of modules over some commutative base ring $R$. Dg categories provide a framework for ‘homological geometry’ and for non-commutative algebraic geometry in the sense of Bondal, Drinfeld, Kapranov, Kontsevich, Toën, Van den Bergh, ... [1] [2] [6] [7] [11] [12] [13] [22]. They are considered as (enriched) derived categories of quasi-coherent sheaves on a hypothetical non-commutative space (see Keller’s ICM-talk survey [10]).

In [19], the homotopy theory of dg categories was constructed. This theory has allowed several developments such as: the creation by Toën of a derived Morita theory [22]; the construction of a category of ‘non-commutative motives’ [19]; the first conceptual characterization [20] of Quillen-Waldhausen’s $K$-theory [16] [24] since its definition in the early 70’s. ...

Key words and phrases. Dg category, Postnikov tower, $k$-invariants, obstruction theory, non-commutative algebraic geometry.

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In this article, we develop new ‘ingredients’ in this homotopy theory: Postnikov towers, $k$-invariants and an obstruction theory for homologically connective dg categories.

**Homologically connective dg categories:** A dg category $\mathcal{A}$ is homologically connective if for all objects $x, y \in \mathcal{A}$, the homology $R$-modules $H_i(\mathcal{A}(x, y))$ are zero for $i < 0$. Our motivation comes from non-abelian Hodge theory (see [17] [18] [23]):

**Example 1.1.** We can associate to a $C^{\infty}$-manifold $M$ its (homologically connective) dg category $T_{DR}(M)$ of flat vector bundles on $M$. For two flat bundles $V$ and $W$, $T_{DR}(M)(V, W)$ is the complex of smooth forms on $M$ with coefficients on the vector bundle of morphisms from $V$ to $W$. Although the homotopy category $\mathcal{H}^0(T_{DR}(M))$ is equivalent, by the Riemann-Hilbert correspondence, to the category of finite dimensional linear representations (up to homotopy) of the fundamental group of $M$, the dg category $T_{DR}(M)$ carries much more information: for two flat bundles $V$ and $W$, corresponding to two local systems $L_1$ and $L_2$, the homology group $H_i(T_{DR}(M)(V, W))$ is isomorphic to the Ext-group $\text{Ext}_i(L_1, L_2)$, computed in the category of abelian sheaves over $M$.

We can also associate to a complex manifold $M$ its (homologically connective) dg category $T_{Dol}(M)$ of holomorphic complex vector bundles on $M$. For two holomorphic bundles $V$ and $W$, $T_{Dol}(M)(V, W)$ is the Dolbeault complex with coefficients in the vector bundle of morphisms from $V$ to $W$. Although the homotopy category $\mathcal{H}^0(T_{Dol}(M))$ is equivalent to the full subcategory of the bounded coherent derived category of $M$, whose objects are the holomorphic vector bundles, the dg category $T_{Dol}(M)$ carries much more information: for two holomorphic vector bundles $V$ and $W$, the homology group $H_i(T_{Dol}(M)(V, W))$ is isomorphic to the Ext-group $\text{Ext}_i^O(V, W)$ calculated in the category of holomorphic coherent sheaves. For instance, if $1$ if the trivial vector bundle of rank $1$ and $V$ is any holomorphic vector bundle

$$H_i(T_{Dol}(M)(1, V)) \cong H_i(X, V),$$

where $H_i(X, V)$ is the $i$th Dolbeault homology group of $V$.

The purpose of this article is to develop a general ‘technology’ that allow us to characterize precisely which are the obstructions appearing when one tries to lift ‘information’ from the homotopy category to the differential graded one.

**Postnikov towers:** A Postnikov tower $(A_n)_{n \geq 0}$ for a homologically connective dg category $\mathcal{A}$ is a commutative diagram in the category $\text{dgcat}$ of dg categories

such that:

A) The dg functor $P_n : \mathcal{A} \longrightarrow A_n$ satisfies the following conditions:
A1) for all objects \(x, y \in \mathcal{A}\), the induced map on the homology \(R\)-modules

\[ H_i(\mathcal{A}(x, y)) \xrightarrow{\sim} H_i(\mathcal{A}_n(P_n x, P_n y)) \]

is an isomorphism for \(i \leq n\) and

A2) the dg functor \(P_n\) induces an equivalence of categories \(H_0(\mathcal{A}) \xrightarrow{\sim} H_0(\mathcal{A}_n)\).

B) For all objects \(x, y \in \mathcal{A}_n\), the homology \(R\)-modules \(H_i(\mathcal{A}_n(x, y))\) are zero for \(i > n\).

By inspiring ourselves in Drinfeld's description of the Hom complexes in his DG quotient (see [6, 3.1]), we construct in section 4.2 a Big (functorial) Postnikov model \(P(\mathcal{A})\) for \(\mathcal{A}\). We then use it to prove the following 'uniqueness' theorem.

**Theorem 1.2.** [4, 17] Given two objects in the category Post(\(\mathcal{A}\)) of Postnikov towers for \(\mathcal{A}\), there exists a zig-zag of weak equivalences relating the two.

For many purposes, a dg category \(\mathcal{A}\) can be replaced by any of its Postnikov sections \(\mathcal{A}_n\). For example if one is only interested in its homotopy category \(H_0(\mathcal{A})\) or if one is only interested in its homology \(R\)-modules in a finite range of dimensions.

On the other hand, using a small Postnikov model \(P(\mathcal{A})\) for \(\mathcal{A}\) (see 4.1), we prove that the full homotopy type of \(\mathcal{A}\) can be recovered from any of its Postnikov towers by a homotopy limit procedure (see proposition 4.19).

**\(k\)-invariants:** Having seen how to decompose a homologically connective dg category \(\mathcal{A}\) into its Postnikov sections \(\mathcal{A}_n\), for \(n \geq 0\), we consider the inverse problem of building a Postnikov tower for \(\mathcal{A}\), starting with \(\mathcal{A}_0\) and inductively constructing \(\mathcal{A}_{n+1}\) from \(\mathcal{A}_n\). In order to solve this problem, we construct (see 5.9) a dg functor

\[ \gamma_n : P_n(\mathcal{A}) \rightarrow \mathbb{P}_n(\mathcal{A}) \ltimes H_{n+1}(\mathcal{A})[n+2], \]

from the \(n\)th Big Postnikov section of \(\mathcal{A}\) to a square zero extension (see 5.7) of \(\mathbb{P}_n(\mathcal{A})\). The image of \(\gamma_n\) in the homotopy category \(Ho(\text{dgcat} \downarrow \mathbb{P}_n(\mathcal{A}))\) of dg categories over \(\mathbb{P}_n(\mathcal{A})\) is called the \(n\)th \(k\)-invariant \(\alpha_n(\mathcal{A})\) of \(\mathcal{A}\) (see 5.12). We show that \(\alpha_n(\mathcal{A})\) corresponds to a derived derivation of \(\mathbb{P}_n(\mathcal{A})\) with values in the \(\mathbb{P}_n(\mathcal{A})\)-\(\mathbb{P}_n(\mathcal{A})\)-bimodule \(H_{n+1}(\mathcal{A})[n+2]\) (see 5.13).

Then we prove our main theorem, which shows how the full homotopy type of \(P_{n+1}(\mathcal{A})\) in \(\text{dgcat}\) can be entirely recovered from \(\alpha_n(\mathcal{A})\).

**Theorem 1.3.** [5, 16] We have a homotopy fiber sequence

\[ P_{n+1}(\mathcal{A}) \rightarrow P_n(\mathcal{A}) \xrightarrow{\gamma_n} \mathbb{P}_n(\mathcal{A}) \ltimes H_{n+1}(\mathcal{A})[n+2] \]

in \(Ho(\text{dgcat} \downarrow \mathbb{P}_n(\mathcal{A}))\).

**Obstruction theory:** By inspiring ourselves in the examples appearing in non-abelian Hodge theory, we formulate the following general ‘rigidification’ problem:

Let \(\mathcal{A}\) be an homologically connective dg category and \(F_0 : \mathcal{B} \rightarrow H_0(\mathcal{A})\) a dg functor with values in its homotopy category, with \(\mathcal{B}\) a cofibrant dg category. Is there a lift \(F : \mathcal{B} \rightarrow \mathcal{A}\) making the diagram

\[ \begin{array}{ccc} \mathcal{B} & \xrightarrow{F_0} & H_0(\mathcal{A}) \\ \downarrow & & \downarrow \gamma_{\leq 0} \\ \mathcal{A} \end{array} \]

commute?
Intuitively the dg functor $F_0$ represents the ‘up-to-homotopy’ information that one would like to rigidify, i.e. lift to the dg category $A$.

In order to solve this problem, we consider a Postnikov tower for $A$ (e.g. its Big Postnikov model)

\[
\begin{array}{c}
B \xrightarrow{F_0} H_0(A) \simeq P_0(A)
\end{array}
\]

and we try to lift $F_0$ to dg functors $F_n : B \to P_n(A)$ for $n = 1, 2 \ldots$ in succession. The image of the composed dg functor

\[
B \xrightarrow{F_n} P_n(A) \xrightarrow{\gamma_n} P_n(A) \times H_{n+1}(A)[n + 2]
\]

in the homotopy category $\text{Ho}(\text{dgcat} \downarrow P_n(A))$ is called the obstruction class $\omega_n(F_n)$ of $F_n$ (see 6.2). We interpret it as a derived derivation of $B$ with values in a $B$-$B$-bimodule (see 6.3).

We then prove that, if at each stage of the inductive process of constructing lifts $F_n : B \to P_n(A)$, the obstruction class $\omega_n(F_n)$ vanishes, then a lift $F : B \to A$ for $F_0$ exists.

**Theorem 1.4.** If the family $\{\omega_n(F_n)\}_{n \geq 0}$ of obstruction classes vanishes, then the ‘rigidification’ problem has a solution.

2. Acknowledgements

It is a great pleasure to thank Carlos Simpson for motivating conversations and Gustavo Granja for several important comments on an older version of this article. I would like also to thank the Laboratoire J.-A. Dieudonné at Nice-France for his hospitality, where some of this work was carried out.

3. Preliminaries

In what follows, $R$ will denote a commutative ring with unit. The tensor product $\otimes$ will denote the tensor product over $R$. Let $Ch$ be the category of complexes of $R$-modules and $Ch_{\geq 0}$ the full subcategory of positively graded complexes (we consider homological notation, i.e. the differential decreases the degree). Recall from [9, 2.3.11], that $Ch$ carries a projective model structure, whose weak equivalences are the quasi-isomorphisms and whose fibrations are the degreewise surjective maps.

We denote by $\text{dgcat}$ the category of small dg categories, see [6] [10] [19].

**Definition 3.1.** Let $A$ be a small dg category.

- the opposite dg category $A^{op}$ of $A$ has the same objects as $A$ and its complexes on morphisms are defined by $A^{op}(x, y) = A(y, x)$.
- a $A$-$A$-bimodule $M$ is a dg functor $M : A^{op} \otimes A \to Ch$. 
Recall from [19, 1.8] that \( \text{dgcat} \) carries a cofibrantly generated Quillen model structure whose weak equivalences are defined as follows:

**Definition 3.2.** A dg functor \( F : \mathcal{A} \longrightarrow \mathcal{B} \) is a quasi-equivalence if:

(i) for all objects \( x, y \in \mathcal{A} \), the induced morphism

\[
F(x, y) : \mathcal{A}(x, y) \xrightarrow{\sim} \mathcal{B}(Fx, Fy)
\]

is a quasi-isomorphism in \( \text{Ch} \) and

(ii) the induced functor \( H_0(F) : H_0(\mathcal{A}) \xrightarrow{\sim} H_0(\mathcal{B}) \) is an equivalence of categories.

**Remark 3.3.** Notice that if condition (i) is verified, condition (ii) is equivalent to:

(ii)' the induced functor

\[
H_0(F) : H_0(\mathcal{A}) \xrightarrow{\sim} H_0(\mathcal{B})
\]

is essentially surjective.

Let us now recall from [19, 1.13], the following characterization of the fibrations in \( \text{dgcat} \).

**Proposition 3.4.** A dg functor \( F : \mathcal{A} \longrightarrow \mathcal{B} \) is a fibration if and only if:

F1) for all objects \( x, y \in \mathcal{A} \), the induced morphism

\[
F(x, y) : \mathcal{A}(x, y) \longrightarrow \mathcal{B}(Fx, Fy)
\]

is a fibration in \( \text{Ch} \) and

F2) for every object \( a_1 \in \mathcal{A} \) and every morphism \( v \in \mathcal{B}(F(a_1), b) \) which becomes invertible in \( H_0(\mathcal{B}) \), there exists a morphism \( u \in \mathcal{A}(a_1, a_2) \) such that \( F(u) = v \) and which become invertible in \( H_0(\mathcal{A}) \).

**Remark 3.5.** Since the terminal object in \( \text{dgcat} \) is the zero category 0 (one object and trivial dg algebra of endomorphisms), every object in \( \text{dgcat} \) is fibrant.

**Corollary 3.6.** Let \( F : \mathcal{A} \longrightarrow \mathcal{B} \) be a dg functor such that:

- induces a surjective map on the set of objects,
- for all objects \( x, y \in \mathcal{A} \), the induced morphism

\[
F(x, y) : \mathcal{A}(x, y) \longrightarrow \mathcal{B}(Fx, Fy)
\]

is a fibration in \( \text{Ch} \) and

- the induced functor

\[
H_0(F) : H_0(\mathcal{A}) \longrightarrow H_0(\mathcal{B})
\]

is an equivalence of categories.

Then \( F \) is a fibration in \( \text{dgcat} \).

**Definition 3.7.** Let \( \mathcal{A} \) be a small dg category.

- We say that \( \mathcal{A} \) is homologically connective if for all objects \( x, y \in \mathcal{A} \), the homology R-modules \( H_i(\mathcal{A}(x, y)) \) are zero for \( i < 0 \).
- We say that \( \mathcal{A} \) is positively graded if for all objects \( x, y \in \mathcal{A} \), the R-modules \( \mathcal{A}(x, y)_i \) are zero for \( i < 0 \).

**Notation 3.8.** We denote by \( \text{dgcat}_{\geq 0} \) the category of small positively graded dg categories.
Recall from [21, 4.16] that we have an adjunction

\[ \begin{array}{c}
\text{dgcat} \\
\tau \geq 0 \\
\downarrow \\
\downarrow \\
\text{dgcat} \geq 0 \\
\end{array} \]

where \( \tau \geq 0 \) denotes the ‘intelligent’ truncation functor.

\textbf{Remark 3.9.} Notice that for a homologically connective dg category \( A \), the co-unit of the previous adjunction, furnishes us a natural quasi-equivalence

\[ \eta_A : \tau \geq 0(A) \sim A, \]

which induces the identity map on set of objects. This (functorial) procedure will allow us to extended several constructions from positively graded to homologically connective dg categories.

We finish these preliminaries with some homotopical algebra results and the notion of lax monoidal functor. Let \( M \) be a Quillen model category and \( X \) an object of \( M \).

\textbf{Notation 3.10.} We denote by \( M \downarrow X \) the category of objects of \( M \) over \( X \), see [8, 7.6.2]. Notice that its terminal object is the identity morphism on \( X \).

\textbf{Remark 3.11.} Recall from [8, 7.6.5] that \( M \downarrow X \) carries a natural Quillen model structure induced by the one on \( M \). In particular an object \( Y \rightarrow X \) in \( M \downarrow X \) is cofibrant if and only if \( Y \) is cofibrant in \( M \) and is fibrant if and only if the morphism \( Y \rightarrow X \) is a fibration in \( M \). Notice also that if \( f : X \rightarrow X' \) is a morphism in \( M \), we have a Quillen adjunction

\[ \begin{array}{c}
M \downarrow X \\
f \\
\downarrow \\
f' \\
M \downarrow X', \\
\end{array} \]

where \( f' \) associates to an object \( Y \rightarrow X' \) in \( M \downarrow X' \) the object \( X \times Y \rightarrow X \) in \( M \downarrow X \) and \( f \) associates to an object \( Z \rightarrow X \) in \( M \downarrow X \) the object \( Z \rightarrow X \rightarrow X' \) in \( M \downarrow X' \). We have also a natural forgetful functor

\[ U : M \downarrow X \rightarrow M, \]

which preserves cofibrations, fibrations and weak equivalences. This implies that \( U \) descends to the homotopy categories \( U : \text{Ho}(M \downarrow X) \rightarrow \text{Ho}(M) \) and so we obtain the following lemma.

\textbf{Lemma 3.12.} Let \( f \) and \( f' \) be two morphisms in \( M \downarrow X \). If they become equal in \( \text{Ho}(M \downarrow X) \), then \( U(f) \) and \( U(f') \) become equal in \( \text{Ho}(M) \).

\textbf{Lemma 3.13.} Let \( M \) be a Quillen model category. Suppose we have a (non-commutative) diagram

\[ \begin{array}{c}
X \\
\downarrow p \\
Y, \\
Z \rightarrow Y , \\
\end{array} \]

\[ f' \rightarrow X, \]

\[ p \rightarrow Y, \]

\[ f \rightarrow Z, \]

\[ \text{Lemma 3.13.} \]

\[ \text{Let } M \text{ be a Quillen model category. Suppose we have a (non-commutative) diagram} \]

\[ \begin{array}{c}
X \\
\downarrow p \\
Y , \\
Z \rightarrow Y , \\
\end{array} \]

\[ f' \rightarrow X, \]

\[ p \rightarrow Y , \\
Z \rightarrow Y, \]

\[ f \rightarrow Z, \]

\[ \text{Lemma 3.13.} \]

\[ \text{Let } M \text{ be a Quillen model category. Suppose we have a (non-commutative) diagram} \]

\[ \begin{array}{c}
X \\
\downarrow p \\
Y, \\
Z \rightarrow Y, \\
\end{array} \]

\[ f' \rightarrow X, \]

\[ p \rightarrow Y, \]

\[ f \rightarrow Z, \]

\[ \text{Lemma 3.13.} \]
where \( Z \) is cofibrant, \( Y \) is fibrant, \( p \) is a fibration in \( \mathcal{M} \) and the composition \( p \circ f' \) becomes equal to \( f \) in the homotopy category \( \text{Ho}(\mathcal{M}) \). Then, there exists a lift \( \tilde{f} : Z \rightarrow X \) of \( f \) which makes the diagram commute.

**Proof.** Notice that since \( Z \) is cofibrant and \( Y \) is fibrant, the composition \( p \circ f' \) becomes equal to \( f \) in \( \text{Ho}(\mathcal{M}) \), if and only if \( p \circ f' \) and \( f \) are left homotopic. This allow us to construct a (solid) commutative square

\[
\begin{array}{ccc}
Z & \xrightarrow{f'} & X \\
\downarrow{i_0} & \searrow{\sim} & \downarrow{p} \\
I(Z) & \xrightarrow{\sim} & Y,
\end{array}
\]

where \( I(Z) \) is a cylinder object for \( Z \) and \( H \) is an homotopy between \( p \circ f' \) and \( f \). Finally, \( p \) has the right lifting property with respect to \( i_0 \) and so we obtain a desired morphism

\[
\tilde{f} : Z \xrightarrow{i_1} I(Z) \xrightarrow{H} X,
\]

such that \( p \circ \tilde{f} = f \).

**Definition 3.14.** Let \((\mathcal{C}, - \otimes -, \mathbb{I}_C)\) and \((\mathcal{D}, - \wedge -, \mathbb{I}_D)\) be two symmetric monoidal categories. A lax monoidal functor is a functor \( F : \mathcal{C} \rightarrow \mathcal{D} \) equipped with:

- a morphism \( \eta : \mathbb{I}_D \rightarrow F(\mathbb{I}_C) \) and
- natural morphisms

\[
\psi_{X,Y} : F(X) \wedge F(Y) \rightarrow F(X \otimes Y), \quad X, Y \in \mathcal{C}
\]

which are coherently associative and unital (see diagrams 6.27 and 6.28 in [3]).

A lax monoidal functor is strong monoidal if the morphisms \( \eta \) and \( \psi_{X,Y} \) are isomorphisms.

Throughout this article the adjunctions are displayed vertically with the left, resp. right, adjoint on the left side, resp. right side.

### 4. Postnikov towers

In this chapter, we construct (functorial) Postnikov towers for homologically connective dg categories. We prove that they are ‘essentially’ unique (see theorem [4.17]) and that the full homotopy type of a homologically connective dg category can be recovered from any of its Postnikov towers (see proposition [4.19]).
Definition 4.1. A Postnikov tower \((A_n)_{n \geq 0}\) for a positively graded dg category \(A\) is a commutative diagram in \(\text{dgcat}\)

\[
\begin{array}{cccccl}
\vdots & & A_2 & & A_1 & & A_0 \\
P_3 & & & & P_1 & & P_0 \\
& & A_n & & A_{n+1} & & A_{n+2} \\
\end{array}
\]

such that:

A) The dg functor \(P_n : A \to A_n\) satisfies the following conditions:
   A1) for all objects \(x, y \in A\), the induced map on the homology \(R\)-modules
   \[H_i(A(x, y)) \to H_i(A_n(P_n x, P_n y))\]
   is an isomorphism for \(i \leq n\) and
   A2) it induces an equivalence of categories \(H_0(A) \sim \to H_0(A_n)\).

B) For all objects \(x, y \in A_n\), the homology \(R\)-modules \(H_i(A_n(x, y))\) are zero for \(i > n\).

The dg functor \(P_n : A \to A_n\) is called the \(n\)th Postnikov section of \(A\).

Remark 4.2. By the 2 out of 3 property, the dg functors \(A_{n+1} \to A_n\) induce an equivalence of categories \(H_0(A_{n+1}) \sim \to H_0(A_n)\).

Definition 4.3. A morphism \(M : (A_n)_{n \geq 0} \to (A'_n)_{n \geq 0}\) between two Postnikov towers for \(A\) is a family of dg functors \(M_n : A_n \to A'_n\) which makes the obvious diagrams commute.

Notation 4.4. We denote by Post\((A)\) the category of Postnikov towers for \(A\).

Remark 4.5. Let \(M : (A_n)_{n \geq 0} \to (A'_n)_{n \geq 0}\) be a morphism between Postnikov towers for \(A\). By the 2 out of 3 property, its Postnikov sections \(M_n : A_n \sim \to A'_n\) are all quasi-equivalences.

Remark 4.6. Observe that in a Postnikov tower \((A_n)_{n \geq 0}\) for \(A\), we can replace each dg functor \(A_{n+1} \to A_n\) by a fibration \(F(A_{n+1}) \to F(A_n)\), starting with \(A_1 \to A_0\) and then going upward. For the inductive step, we factor the composition \(A_{n+1} \to A_n \sim \to F(A_n)\) by a trivial cofibration followed by a fibration.
\[ F(\mathcal{A}_{n+1}) \rightarrow F(\mathcal{A}_n) \, . \] We obtain then a morphism \((\mathcal{A}_n)_{n \geq 0} \rightarrow F(\mathcal{A}_n)_{n \geq 0}\) between Postnikov towers

\[ \vdots \rightarrow \mathcal{A}_2 \rightarrow \sim \rightarrow F(\mathcal{A}_2) \rightarrow \vdots \]

\[ \mathcal{A}_1 \rightarrow \sim \rightarrow F(\mathcal{A}_1) \rightarrow \mathcal{A}_0 \rightarrow \sim \rightarrow F(\mathcal{A}_0) \rightarrow \vdots \]

**Definition 4.7.** Let \( \mathcal{A} \) be a homologically connective dg category. By a Postnikov tower for \( \mathcal{A} \), we mean a Postnikov tower for \( \tau_{\geq 0}(\mathcal{A}) \), see remark 3.9.

We now present two functorial Postnikov tower models.

4.1. **Small model.** Let \( n \geq 0 \). Consider the ‘intelligent’ truncation functor

\[ \tau_{\leq n} : Ch_{\geq 0} \rightarrow Ch_{\geq 0} \]

which associates to a complex

\[ M_\bullet : \quad 0 \leftarrow M_0 \leftarrow \cdots \leftarrow M_{n-1} \leftarrow M_n \leftarrow M_{n+1} \leftarrow \cdots \]

its ‘intelligent’ truncation

\[ \tau_{\leq n}(M_\bullet) : \quad 0 \leftarrow M_0 \leftarrow \cdots \leftarrow M_{n-1} \leftarrow M_n/\text{Im}(M_{n+1}) \leftarrow 0 \leftarrow \cdots \]

Notice that when \( n \) varies, we obtain the following natural tower of complexes

\[ \vdots \rightarrow \tau_{\leq 2}(M_\bullet) \rightarrow \tau_{\leq 1}(M_\bullet) \rightarrow \tau_{\leq 0}(M_\bullet) \rightarrow \vdots \]

Moreover each vertical map is a fibration and the induced map on the homology \( R \)-modules

\[ H_i(M_\bullet) \rightarrow H_i(\tau_{\leq n}(M_\bullet)) \]

is an isomorphism for \( i \leq n \). Notice also that the homology \( R \)-modules \( H_i(\tau_{\leq n}(M_\bullet)) \) are zero for \( i > n \).

Now, let \( \mathcal{A} \) be a positively graded dg category. Since for every \( n \geq 0 \), the truncation functor \( \tau_{\leq n} \) is lax monoidal (see 3.14), the above remarks imply the following:
if we apply the ‘intelligent’ truncation functors to each complex of morphisms of $\mathcal{A}$, we obtain a Postnikov tower

\[
\begin{array}{c}
\vdots \\
\tau_{\leq 2}(\mathcal{A}) \\
\tau_{\leq 1}(\mathcal{A}) \\
\tau_{\leq 0}(\mathcal{A}) \\
\mathcal{A}
\end{array}
\]

for $\mathcal{A}$. Moreover, by construction, all the dg functors in the diagram induce the identity map on the set of objects. Notice also that since the morphisms of complexes

\[
\tau_{\leq n+1}(M_\bullet) \longrightarrow \tau_{\leq n}(M_\bullet)
\]

are fibrations, remark 4.12 and corollary 5.6 imply that the dg functors

\[
\tau_{\leq n+1}(\mathcal{A}) \longrightarrow \tau_{\leq n}(\mathcal{A})
\]

are fibrations in $\text{dgcat}$.

Notation 4.8. We denote by $P(\mathcal{A})$ the small Postnikov model obtained. In particular $P_n(\mathcal{A})$ denotes the dg category $\tau_{\leq n}(\mathcal{A})$.

4.2. Big model. We start by recalling from [19, 1.3] same generating cofibrations for the Quillen model structure on $\text{dgcat}$.

**Definition 4.9.** For $n \in \mathbb{Z}$, let $S_n$ be the complex $R[n]$ (with $R$ concentrated in degree $n$) and let $D_{n+1}$ be the mapping cone on the identity of $S_n$. We denote by $1_n$ the element of degree $n$ in $S_n$, which corresponds to the unit of $R$. Let $C(n)$ be the dg category with two objects $1$ and $2$ such that $C(n)(1,1) = R$, $C(n)(2,2) = R$, $C(n)(2,1) = 0$, $C(n)(1,2) = S_n$ and composition given by multiplication. We denote by $D(n+1)$ the dg category with two objects $3$ and $4$ such that $P(n)(3,3) = R$, $P(n)(4,4) = R$, $P(n)(4,3) = 0$, $P(n)(3,4) = D_{n+1}$ and with composition given by multiplication. Finally, let $S(n)$ be the dg functor from $C(n)$ to $D(n+1)$ that sends $1$ to $3$, $2$ to $4$ and $S_n$ to $D_{n+1}$ by the identity on $R$ in degree $n$.

**Lemma 4.10.** Let $\mathcal{A}$ be a small dg category and $n \geq 0$. Suppose that the dg functor $\mathcal{A} \longrightarrow 0$ (where $0$ denotes the terminal object in $\text{dgcat}$) has the right lifting property with respect to the set $\{S(m) \mid m > n\}$. Then for all objects $x, y \in \mathcal{A}$, the homology $R$-modules $H_i(\mathcal{A}(x,y))$ are zero for $i > n$.

**Proof.** This follows easily from the above definitions.

**Lemma 4.11.** Let $\pi : M_\bullet \longrightarrow N_\bullet$ be a fibration in $Ch$ and $n+1 > 0$. If the induced map on the homology $R$-modules

\[
H_i(M_\bullet) \sim\rightarrow H_i(N_\bullet)
\]
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is an isomorphism for \( i > n + 1 \), then \( \pi \) has the right lifting property with respect to the set \( \{ S_m \to D_{m+1} \mid m > n + 1 \} \), see definition 4.9.

Proof. Consider the short exact sequence of complexes

\[
0 \to K \to M \to N \to 0,
\]

where \( K \) denotes the kernel of \( \pi \). Notice that in the induced long exact sequence on homology, the isomorphisms \( \cdots \to H_{m+1}(M) \to H_{m+1}(N) \to H_m(K) \to H_m(N) \to \cdots \), imply that \( H_m(K) = 0 \). Now a simple diagram chasing argument (see [9, 2.3.5]) allow us to conclude the proof.

Corollary 4.12. Let \( F : A \to B \) be a dg functor such that for all objects \( x, y \in A \), the induced morphism

\[
F(x, y) : A(x, y) \to B(Fx, Fy)
\]

in \( Ch \) satisfies the conditions of lemma 4.11. Then \( F \) has the right lifting property with respect to the elements of the set \( \{ S(m) \mid m > n + 1 \} \).

Now, let \( A \) be a positively graded dg category. For each \( n \geq 0 \), apply the small object argument (see 10.5.14) to the dg functor \( A \to 0 \), using the set \( \{ S(m) \mid m > n \} \) of generating cofibrations (see 4.9). We obtain the following factorization

\[
A \to P_n(A) \to 0,
\]

where the dg functor \( P_n \) is obtained by an infinite composition of pushouts along the elements of the set \( \{ S(m) \mid m > n \} \). Notice that the small object argument furnishes us natural dg functors \( P_{n+1}(A) \to P_n(A) \) making the following diagram commutative. Moreover, by construction, all the dg functors in the diagram induce the identity map on the set of objects.

Proposition 4.13. The above construction is a Postnikov tower for \( A \).

Proof. We verify the conditions of definition 4.1.
A1) Since $P_n(A)$ is obtained by an infinite composition of pushouts along the elements of the set $\{S(m) \mid m > n\}$ and the homology functors commute with infinite compositions, it is enough to prove the following: let $\mathcal{B}$ be a positively graded dg category and consider the following pushout ($m > n$)

\[
\begin{array}{ccc}
\mathcal{C}(m) & \xrightarrow{T} & \mathcal{B} \\
S(m) \downarrow & & \downarrow \\
\mathcal{D}(m + 1) & \xrightarrow{\partial} & \mathcal{B}
\end{array}
\]

in $\text{dgcat}$. We need to show that $\tilde{\mathcal{B}}$ is also positively graded and that for all objects $x, y \in \mathcal{B}$, the induced map on the homology $R$-modules

$$H_i(\mathcal{B}(x, y)) \xrightarrow{\sim} H_i(\tilde{\mathcal{B}}(x, y))$$

is an isomorphism for $i \leq n$.

Observe that, as in Drinfeld’s description of the Hom complexes in his DG quotient [6, 3.1], we have an isomorphism of graded $R$-modules (but not an isomorphism of complexes)

$$\tilde{\mathcal{B}}(x, y) \xrightarrow{\sim} \bigoplus_{l=0}^{\infty} \tilde{\mathcal{B}}^l(x, y),$$

where $\tilde{\mathcal{B}}^l(x, y)$ is by definition the graded $R$-module

$$\mathcal{B}(T(2), y) \otimes R[m + 1] \otimes \cdots \otimes \mathcal{B}(T(2), T(1)) \otimes R[m + 1] \otimes \mathcal{B}(x, T(1)).$$

The differential of an element

$$\underbrace{g_{n+1} \cdot h \cdots g_2 \cdot h \cdot g_1}_{l \text{ factors } h} \in \tilde{\mathcal{B}}^l(x, y)$$

is equal to

$$d(g_{n+1} \cdot h \cdots g_2 \cdot h \cdot g_1 + (-1)^{l(g_{n+1})} g_{n+1} \cdot d(h) \cdots g_2 \cdot h \cdot g_1 + \cdots),$$

where $d(h) \in \mathcal{B}(T(1), T(2))$ corresponds to the image of $1_m \in S_m$ (see 4.9) under the dg functor $T$. This implies that, for every $j \geq 0$, the sum

$$\bigoplus_{l \geq 0} ^{j} \tilde{\mathcal{B}}^l(x, y) \hookrightarrow \tilde{\mathcal{B}}(x, y)$$

is a subcomplex and so we obtain an exhaustive filtration of $\tilde{\mathcal{B}}(x, y)$. Observe that since $m > n$ and $\mathcal{B}$ is positively graded the natural inclusion

$$\mathcal{B}(x, y) = \tilde{\mathcal{B}}^0(x, y) \hookrightarrow \tilde{\mathcal{B}}(x, y)$$

induces isomorphisms

$$\mathcal{B}(x, y)_i \xrightarrow{\sim} \tilde{\mathcal{B}}(x, y)_i$$

for $i \leq n + 1$ and so an isomorphism

$$\tau_{\leq n} \mathcal{B}(x, y) \xrightarrow{\sim} \tau_{\leq n} \tilde{\mathcal{B}}(x, y)$$
between the truncated complexes. We conclude that \( \widetilde{B} \) is positively graded and that the induced map on the homology \( R \)-modules

\[
H_i(B(x, y)) \xrightarrow{\sim} H_i(\widetilde{B}(x, y))
\]

is an isomorphism for \( i \leq n \).

A2) By condition A1), for all objects \( x, y \in A \), the induced map on the homology \( R \)-modules

\[
H_0(A(x, y)) \xrightarrow{\sim} H_0(P_n(A)(x, y))
\]

is an isomorphism. Since the dg functor \( P_n : A \to A_n \) induces the identity map on the set of objects, we conclude that the induced functor \( H_0(A) \xrightarrow{\sim} H_0(A_n) \) is an equivalence of categories.

B) By construction, the dg functor \( P_n : A \to A_n \) has the right lifting property with respect to the set \( \{ S(m) \mid m > n \} \). This implies, by lemma 4.10, that for all objects \( x, y \in A \) the homology \( R \)-modules \( H_i(P_n(A)(x, y)) \) are zero for \( i > n \).

\[ \sqrt{2} \]

**Notation 4.14.** We denote by \( P(A) \) the Big Postnikov model thus obtained.

### 4.3. Uniqueness and homotopy type.

**Proposition 4.15.** Let \( (A_n)_{n \geq 0} \) be a Postnikov tower for a homologically connective dg category \( A \), where all the dg functors \( A_{n+1} \to A_n \) are fibrations. Then there exists a morphism

\[
M : P(A) \to (A_n)_{n \geq 0}
\]

between Postnikov towers.

**Proof.** We will construct \( M \) recursively, starting with the case \( n = 0 \) and then going upwards.

\( (n = 0) \) Notice that the small object argument allows us to construct inductively a dg functor \( M_0 : P_0(A) \to A_0 \) as follows:

**step:** suppose we have the following (solid) diagram \((i \geq 0, P_0(A)^i = A)\)

\[
\begin{array}{ccc}
\coprod_{m > 0} C(m) & \xrightarrow{p_0(A)^i} & P_0(A)^i \\
\coprod_{m > 0} C(m) & \xrightarrow{p_0(A)^i} & P_0(A)^{i+1}
\end{array}
\]

Recall from that we denote by \( 1_m \) the cycle of degree \( m \) in \( S_m \) (and so in \( C(m)(1, 2) \)) which corresponds to the unit of \( R \). Since \( A_0 \) satisfies condition (B) of definition 4.1 we can choose a bounding chain \( b \) in \( A_0 \) for each cycle \( T_i(1_m), m > 0 \) (i.e. \( db = T_i(1_m) \)). These choices give rise to a dg functor \( M_0^{i+1} \) which makes the above diagram commute.

By passing to the colimit on \( i \), we obtain our desired dg functor

\[
M_0 = \text{colim}_i M_0^i : P_0(A) = \text{colim}_i P_0(A)^i \to A_0.
\]
(n ⇒ n + 1) Suppose we have a dg functor $M_n : P_n(A) → A_n$ between the nth Postnikov sections. We will construct a 'lift' $M_{n+1}$ which makes the square

\[
\begin{array}{c}
P_{n+1}(A) \xrightarrow{M_{n+1}} A_{n+1} \\
\downarrow \quad \downarrow \\
P_n(A) \xrightarrow{M_n} A_n
\end{array}
\]

commutative. Our argument is also an inductive one:

**step:** suppose we have the following (solid) diagram ($i ≥ 0, P_{n+1}(A)^0 = A$)

\[
\begin{array}{c}
\coprod_{m > n+1} C(m) \xrightarrow{M_{n+1}} P_{n+1}(A)^i \xrightarrow{M_{n+1}^i} A_{n+1} \\
\downarrow \quad \downarrow \quad \downarrow \\
\coprod_{m > n+1} D(m+1) \xrightarrow{D(m+1)} P_{n+1}(A)^{i+1} \xrightarrow{M_{n+1}^{i+1}} A_{n+1}
\end{array}
\]

Notice that the left (solid) square appears in the construction of $P_n(A)^{i+1}$. This implies that the dg functor $M_{n+1}^{i+1} : P_n(A)^{i+1} → A_n$ restricts to a dg functor $M_{n+1}^{i+1}$, which makes the right square commutative. Now, observe that the dg functor $A_{n+1} → A_n$ satisfies the conditions of corollary 4.12 and so it has the right lifting property with respect to the elements of the set $\{S(m) | m > n\}$. This implies that there exists an induced dg functor $M_{n+1}^{i+1}$ which makes the above diagram commute.

By passing to the colimit on $i$, we obtain our desired morphism

\[
M_{n+1} = \text{colim}_i M_{n+1}^{i+1} : P_n(A)^{i+1} → A_n
\]

The proof is now finished.

**Remark 4.16.** Since in the small Postnikov model $\mathbb{P}(A)$ for $A$, the dg functors

\[
\tau_{≤ n+1}(A) \xrightarrow{\tau_{≤ n}} A_n
\]

are fibrations, proposition [4.12] implies the existence of a morphism

\[
M : P(A) → \mathbb{P}(A)
\]

from the Big to the small Postnikov model. Moreover, the bounding chains in $\mathbb{P}(A)$ used in the construction of $M$ are all trivial and so this morphism is well-defined. Notice also that for $n ≥ 0$, the dg functor $M_n$ satisfies all the conditions of corollary [3.7] and so it is a fibration in $\text{dgcat}$.

We now prove that Postnikov towers are ‘essentially’ unique.

**Theorem 4.17.** Let $A$ be a homologically connective dg category. Given two objects in $\text{Post}(A)$ (see [4.4]), there exists a zig-zag of weak equivalences (see [4.7]) relating the two.
Proof. Let \((A_n)_{n \geq 0}\) and \((A'_n)_{n \geq 0}\) two Postnikov towers for \(A\). By remark 4.6 we can construct morphisms in Post\((A)\)

\[
(A_n)_{n \geq 0} \sim F(A_n)_{n \geq 0} \quad (A'_n)_{n \geq 0} \sim F(A'_n)_{n \geq 0},
\]

such that the dg functors

\[
F(A_{n+1}) \to F(A_n) \quad F(A'_{n+1}) \to F(A'_n)
\]

are fibrations in \(dgcat\). Moreover, by proposition 4.15 we can also construct morphisms as follows

\[
P(A) \sim F(A_n)_{n \geq 0} \quad P(A) \sim F(A'_n)_{n \geq 0}.
\]

We obtain finally, the following zig-zag

\[
(A_n)_{n \geq 0} \sim F(A_n)_{n \geq 0} \sim P(A) \sim F(A'_n)_{n \geq 0} \sim (A'_n)_{n \geq 0}
\]

of weak equivalences in Post\((A)\).

Remark 4.18. Notice that by theorem 4.17, the classifying space \([8, 14]\) of Post\((A)\) has a single connected component.

We now show how the full homotopy type of a homologically connective dg category can be recovered from any of its Postnikov towers.

**Proposition 4.19.** Let \(A\) be a homologically connective dg category and \((A_n)_{n \geq 0}\) a Postnikov tower for \(A\). Then the natural dg functor

\[
A \to \lim_n A_n
\]

is a quasi-equivalence.

Proof. Notice that theorem 4.17 and remark 4.5 imply that the homotopy limit of any Postnikov tower for \(A\) is well defined up to quasi-equivalence. We can then consider the small Postnikov model \(P(A)\) for \(A\). Since every object in \(dgcat\) is fibrant (see 3.5) and the dg functors

\[
\tau_{\leq n+1}(A) \to \tau_{\leq n}(A)
\]

are fibrations in \(dgcat\), we have a natural quasi-equivalence

\[
\lim_n \tau_{\leq n}(A) \sim \holim_n \tau_{\leq n}(A).
\]

By construction of limits in \(dgcat\), we conclude that the natural dg functor

\[
A \sim \lim_n P_n(A)
\]

is an isomorphism.

\(\surd\)
5. $k$-INVARIENTS

In this chapter we construct $k$-invariants for homologically connective dg categories (see definitions 5.12 and 5.14). We show that these invariants correspond to derived derivations with values in a certain bimodule (see 5.13). Then we prove our main theorem (5.16), which shows how the full homotopy type of the $n + 1$ Postnikov section of an homologically connective dg category $A$ can be recovered from the $n$th $k$-invariant of $A$. For constructions of $k$-invariants in the context of spectral algebra see [4] [5] [14]. Let us start with some general constructions.

Definition 5.1. Let $A$ be a small dg category and $M$ a $A$-$A$-bimodule (see 3.1).

The square zero extension $A \ltimes M$ of $A$ by $M$ is the dg category defined as follows:
its objects are those of $A$ and for objects $x,y \in A$ we have
$$A \ltimes M(x,y) := A(x,y) \oplus M(x,y).$$
The composition in $A \ltimes M$ is defined using the composition on $A$, the above bimodule structure and by imposing that the composition between $M$-factors is zero.

Remark 5.2. Notice that $A$ is a (non-full) dg subcategory of $A \ltimes M$ and that we have a natural projection dg functor
$$A \ltimes M \rightarrow A,$$
which is clearly a fibration in $dgcat$, see proposition 3.4.

Definition 5.3.
- A derivation of $A$ with values in a $A$-$A$-bimodule $M$ is a morphism in $dgcat \downarrow A$ (see 3.10) from $A$ to $A \ltimes M$, or equivalently a section of the natural projection dg functor $A \ltimes M \rightarrow A$.
- A derived derivation of $A$ with values in a $A$-$A$-bimodule $M$ is a morphism in the homotopy category $Ho(dgcat \downarrow A)$ (see 3.11) from $A$ to $A \ltimes M$.

Notation 5.4. We denote by $\text{Der}(A,M)$ (resp. $R\text{Der}(A,M)$) the set of derivations (resp. derived derivations) of $A$ with values in $M$. The (derived) derivation obtained by considering $A$ as a dg subcategory of $A \ltimes M$ is called the trivial one.

Remark 5.5. Notice that if $A$ is a $R$-algebra $A$ (i.e. $A$ has only one object and its endomorphisms $R$-algebra is $A$), the notion of derivation coincides with the classical one, i.e. a $R$-linear map $D : A \rightarrow M$ which satisfies the Leibniz relation
$$D(ab) = a(Db) + (Da)b \quad a,b \in A.$$

Proposition 5.6. Let $F : A \rightarrow B$ be an object in $dgcat \downarrow B$ and $M$ a $B$-$B$-bimodule. Then the set $Ho(dgcat \downarrow B)(A,B \ltimes M)$ is naturally isomorphic to the set of derived derivations $R\text{Der}(A,F^*(M))$ of $A$ with values in the $A$-$A$-bimodule $F^*(M)$ obtained by restricting $M$ along $F$.

Proof. Recall from remark 5.11 the (derived) Quillen adjunction
$$\begin{array}{ccc}
Ho(dgcat \downarrow A) & \xrightarrow{F} & \text{R}F^* \\
\xrightarrow{\Phi} & & \\
Ho(dgcat \downarrow B).
\end{array}$$
Notice that we have the following pull-back square

\[
\begin{array}{ccc}
\mathcal{A} \times F^*(M) & \xrightarrow{F \times 1_d} & B \times M \\
\downarrow r & & \downarrow \\
\mathcal{A} & \xrightarrow{F} & B,
\end{array}
\]

which shows us that the image of \( B \times M \) under the functor \( \mathbb{RF}^i \) is isomorphic to \( \mathcal{A} \times F^*(M) \). Moreover the image of \( \mathcal{A} \) under the functor \( F_1 \) is isomorphic to the object \( F : \mathcal{A} \to B \) in \( \text{Ho}(dg\text{cat} \downarrow B) \) and so by adjunction we obtain the desired isomorphism.

We now define the dg categories which play the same role as the Eilenberg-Mac Lane spaces in the classical theory of \( k \)-invariants.

**Definition 5.7.** Let \( \mathcal{A} \) be a positively graded dg category and \( n \geq 0 \). Consider the following bimodule:

\[
\text{Ho}(\mathcal{A})^\text{op} \otimes \text{Ho}(\mathcal{A}) \rightarrow \text{Ch} \rightarrow \text{Ho}(\mathcal{A})^\text{op} \otimes \text{Ho}(\mathcal{A})\]

where the complex \( \text{Ho}(\mathcal{A})(x,y)[n+2] \) is simply the \( R \)-module \( \text{Ho}(\mathcal{A})(x,y)[n+2] \) concentrated in degree \( n+2 \). Notice that the natural projection dg functor \( \mathbb{P}_n(\mathcal{A}) \rightarrow \mathbb{P}_0(\mathcal{A}) = \text{Ho}(\mathcal{A}) \) endow \( \mathbb{P}_n(\mathcal{A})[n+2] \) with a structure of \( \mathbb{P}_n(\mathcal{A}) \) \( \mathbb{P}_n(\mathcal{A}) \)-bimodule. Finally, we denote by \( \mathcal{A} \triangleright \mathbb{P}_n(\mathcal{A})[n+2] \) the square zero extension obtained \( \text{5.10} \) using this bimodule structure.

**Remark 5.8.** Notice that by remark \( \text{5.2} \) \( \mathbb{P}_n(\mathcal{A}) \) is a dg subcategory of \( \mathbb{P}_n(\mathcal{A}) \triangleright \mathbb{P}_n(\mathcal{A})[n+2] \) and we have a natural projection dg functor

\[
\mathbb{P}_n(\mathcal{A}) \times \mathbb{P}_n(\mathcal{A})[n+2] \rightarrow \mathbb{P}_n(\mathcal{A}).
\]

**Definition 5.9.** Let

\[
\gamma_n : \mathbb{P}_n(\mathcal{A}) \rightarrow \mathbb{P}_n(\mathcal{A}) \times \mathbb{P}_n(\mathcal{A})[n+2]
\]

be the natural dg functor obtained by modifying the dg functor \( M_n : \mathbb{P}_n(\mathcal{A}) \rightarrow \mathbb{P}_n(\mathcal{A}) \) (see \( \text{4.10} \)) as follows:

\[
\begin{array}{ccc}
\prod_{m > n} \mathcal{C}(m) - \mathcal{P}_n(\mathcal{A}) & \xrightarrow{T_i} & \mathcal{P}_n(\mathcal{A})[n+1] \\
\downarrow \gamma_n & & \downarrow \\
\prod_{m > n} \mathcal{C}(m) - \mathcal{P}_n(\mathcal{A}) & \xrightarrow{T_i} & \mathcal{P}_n(\mathcal{A})[n+1].
\end{array}
\]

For every cycle \( T_i(1_m), m > n + 1 \) choose 0 as a bounding chain in \( \mathbb{P}_n(\mathcal{A}) \) (and so in \( \mathbb{P}_n(\mathcal{A}) \times \mathbb{P}_n(\mathcal{A})[n+2] \)), as in the case of the dg functor \( M_n \). Now, let \( T_i(1_{n+1}) \in \mathbb{P}_n(\mathcal{A})[n+1] \) be a cycle of degree \( n+1 \). Since \( m > n \), the description of the complexes of morphisms in \( \mathbb{P}_n(\mathcal{A}) \) (see proof of proposition \( \text{4.13} \)) implies that we have natural isomorphisms

\[
\mathcal{A}(T_i(1), T_i(2)) \sim \mathcal{A}(T_i(1), T_i(2)) \rightarrow \mathcal{P}_n(\mathcal{A})[n+1]
\]
Theorem 5.16. We have a homotopy fiber sequence

\[ P_{n+1}(A) \rightarrow P_n(A) \xrightarrow{\gamma_n} P_n(A) \times H_{n+1}(A)[n+2] \]

in \( \text{Ho}(\text{dgcat} \downarrow P_n(A)) \).

Remark 5.10. Notice that by construction, the dg functor \( \gamma_n \) satisfies all the conditions of corollary 3.6 and so it is a fibration in \( \text{dgcat} \). Moreover for \( n \geq 0 \), we have the following commutative diagram in \( \text{dgcat} \):

\[
\begin{array}{ccc}
P_n(A) & \xrightarrow{\gamma_n} & P_n(A) \times H_{n+1}(A)[n+2] \\
& \sim & \\
& \downarrow M_n & \\
P_n(A) & & \\
\end{array}
\]

Notation 5.11. We denote by \( \text{dgcat} \downarrow P_n(A) \) the category of objects in \( \text{dgcat} \) over \( P_n(A) \), see notation 3.10.

Definition 5.12. Let \( A \) be a positively graded dg category and \( n \geq 0 \). Its \( n \)th \( k \)-invariant \( \alpha_n(A) \) is by definition the image of the dg functor \( \gamma_n \) in the homotopy category \( \text{Ho}(\text{dgcat} \downarrow P_n(A)) \), see remark 5.10.

Remark 5.13. Since the dg functor \( M_n : P_n(A) \rightarrow P_n(A) \) is a quasi-equivalence, we have an isomorphism between

\[ \text{Ho}(\text{dgcat} \downarrow P_n(A))(P_n(A), P_n(A) \times H_{n+1}(A)[n+2]) \]

and

\[ \text{Ho}(\text{dgcat} \downarrow P_n(A))(P_n(A), P_n(A) \times H_{n+1}(A)[n+2]) \]

which implies that \( \alpha_n(A) \) corresponds to a derived derivation of \( P_n(A) \) with values in the \( P_n(A)\text{-}P_n(A) \)-bimodule \( H_{n+1}(A)[n+2] \), see definition 5.10.

Definition 5.14. Let \( A \) be a homologically connective dg category. Its \( n \)th \( k \)-invariant \( \alpha_n(A) \) is by definition the \( n \)th \( k \)-invariant of \( \tau_{\geq 0}(A) \), see remark 5.10.

Remark 5.15. Notice that although the category \( \text{dgcat} \downarrow P_n(A) \) is not pointed (the initial and terminal objects are not isomorphic), there is a natural morphism (in \( \text{dgcat} \downarrow P_n(A) \)) from its terminal object \( P_n(A) \) to \( P_n(A) \times H_{n+1}(A)[n+2] \) (see remark 5.8).

We now show how the full homotopy type of \( P_{n+1}(A) \) in \( \text{dgcat} \) can be entirely recovered from the \( n \)th \( k \)-invariant \( \alpha_n(A) \).

Theorem 5.16. We have a homotopy fiber sequence

\[ P_{n+1}(A) \rightarrow P_n(A) \xrightarrow{\gamma_n} P_n(A) \times H_{n+1}(A)[n+2] \]

for \( j \leq n+1 \). We can then choose for bounding chain for \( T_j(T_{n+1}) \) its homology class in \( H_{n+1}(A(T_j(T_{n+1}), T_i(2))) \). These choices give rise to a dg functor \( \gamma_{n+1} \) which makes the above diagram commute.

By passing to the colimit on \( i \), we obtain our desired dg functor

\[ \gamma_n = \colim_i \gamma_i^n : P_n(A) = \colim_i P_n(A)^i \rightarrow P_n(A) \times H_{n+1}(A)[n+2]. \]
Proof. We need to show that $P_{n+1}(A)$ is quasi-equivalent in $\text{dgcat}$ to the homotopy pullback of the diagram

$$
\begin{array}{c}
P_n(A) \\
| \\
\gamma_n \downarrow \\
\downarrow \\
P_{n+1}(A) \times H_{n+1}(A)[n+2].
\end{array}
$$

Since $\gamma_n$ is a fibration (see 5.10) and every dg category is fibrant (see 3.5), the homotopy pullback and the pullback are quasi-equivalent. Notice that we have the following commutative diagram

$$
\begin{array}{c}
P_n(A) \\
| \\
\gamma_n \downarrow \\
\downarrow \\
P_n(A) \times H_{n+1}(A)[n+2].
\end{array}
$$

This diagram gives rise to the following factorization

$$
\begin{array}{c}
\mathcal{A} \\
| \\
\phi \downarrow \\
\downarrow \\
\mathcal{W},
\end{array}
$$

where $\theta$ and $\phi$ are the induced dg functors to the pullback $\mathcal{W}$. We need to show that $\theta$ is a quasi-equivalence. By construction of limits in $\text{dgcat}$, all the dg functors in the previous diagrams induce the identity map on the set of objects and so it is enough to prove that for all objects $x, y \in P_{n+1}(A)$, the morphism of complexes

$$
\theta(x, y) : P_{n+1}(A)(x, y) \longrightarrow \mathcal{W}(x, y)
$$

is a quasi-isomorphism. Let us denote by

$$
0 \leftarrow M_0 \leftarrow M_1 \leftarrow \cdots \leftarrow M_n \leftarrow M_{n+1} \leftarrow M_{n+2} \leftarrow M_{n+3} \leftarrow \cdots
$$

the complex $A(x, y)$. Notice that by construction of $P_n(A)$ (see 4.13), the complex $P_n(A)(x, y)$ is of the following shape

$$
0 \leftarrow \widetilde{M}_0 \leftarrow \widetilde{M}_1 \leftarrow \cdots \leftarrow \widetilde{M}_n \leftarrow \widetilde{M}_{n+1} \leftarrow \widetilde{M}_{n+2} \leftarrow \cdots.
$$
The complex $\mathcal{W}(x,y)$ identifies then with the pullback of the following diagram:

\[
\begin{array}{ccc}
M_{n+3} & \to & 0 \\
\downarrow & & \downarrow \\
M_{n+2} & \to & H_{n+1}(\mathcal{A}(x,y)) \\
\downarrow & & \downarrow \\
M_{n+1} & \to & 0 \\
\downarrow & & \downarrow \\
M_n & \to & M_n/\text{Im}(M_{n+1}) \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
M_1 & \to & M_1 \\
\downarrow & & \downarrow \\
M_0 & \to & M_0 \\
0 & & 0
\end{array}
\]

The above diagram allow us to conclude that $H_j\mathcal{W}(x,y) = 0$ for $j \geq n+2$ and that the induced map

\[
H_j(P_{n+1}(\mathcal{A})(x,y)) \simto H_j\mathcal{W}(x,y)
\]

is an isomorphism for $j \neq n+1$.

We now prove that the induced map

\[
H_{n+1}\mathcal{A}(x,y) \simto H_{n+1}\mathcal{W}(x,y)
\]

is an isomorphism. Notice that this implies (by the 2 out of 3 property) that $\theta(x,y)$ is a quasi-isomorphism. In order to prove this, we start by observing that in $\text{dgcat}$, pullbacks commute with filtered colimits. Since $P_n(\mathcal{A})$ is constructed as a filtered colimit and the homology functor $H_{n+1}(-)$ preserves filtered colimits it is then enough to prove the following:

\textit{start:} consider the following pullback square

\[
\begin{array}{ccc}
A & \to & P_n(\mathcal{A}) \\
\downarrow & \searrow & \downarrow \\
A & \to & P_n(\mathcal{A}) \ltimes H_{n+1}(\mathcal{A})[n+2].
\end{array}
\]
step: consider the commutative diagram \((i \geq 0, P_n(A)^0 = \mathcal{A})\)

\[
\begin{array}{ccccccc}
\mathcal{C}(n+1) & \xrightarrow{T} & P_n(A)^i & \xrightarrow{\gamma_i(T)} & \mathbb{P}_n(A) \times H_{n+1}(A)[n+2] \\
\downarrow S(n+1) & & \downarrow \gamma_i(T) & & \\
\mathcal{D}(n+2) & \xrightarrow{} & P_n(A)^i
\end{array}
\]

used in the construction of the natural dg functor \(\gamma_n\) (see 5.9), and suppose that
the induced dg functor from \(\mathcal{A}\) to the pullback

\[
\begin{array}{ccccccc}
\mathcal{W}_i(T) & \xrightarrow{r} & \mathbb{P}_n(A) \\
\downarrow & & \downarrow \\
P_n(A)^i & \xrightarrow{\gamma_i(T)} & \mathbb{P}_n(A) \times H_{n+1}(A)[n+2]
\end{array}
\]

induces an isomorphism

\[
H_{n+1}(\mathcal{A}, x, y) \xrightarrow{\sim} H_{n+1}(\mathcal{W}_i(T)(x, y))
\]

We need to show that the induced dg functor from \(\mathcal{W}_i(T)\) to the pullback

\[
\begin{array}{ccccccc}
\mathcal{W}_i(T) & \xrightarrow{r} & \mathbb{P}_n(A) \\
\downarrow & & \downarrow \\
P_n(A)^i & \xrightarrow{\gamma_i(T)} & \mathbb{P}_n(A) \times H_{n+1}(A)[n+2]
\end{array}
\]

induces an isomorphism

\[
H_{n+1}(\mathcal{W}_i(T)(x, y)) \xrightarrow{\sim} H_{n+1}(\mathcal{W}_i(T)(x, y))
\]

Recall that for all objects \(x, y \in P_n(A)^i\), we have an isomorphism of graded \(R\)-modules

\[
\mathcal{P}_n(A)^i(x, y) \xrightarrow{\sim} \bigoplus_{l=0}^{\infty} \mathcal{P}_n(A)^i(x, y),
\]

where \(\mathcal{P}_n(A)^i(x, y)\) is the graded \(R\)-module

\[
P_n(A)^i(T(2), y) \otimes R[n+2] \otimes \cdots \otimes P_n(A)^i(T(2), T(1)) \otimes R[n+2] \otimes P_n(A)^i(x, T(1)).
\]

The differential of an element

\[
g_{n+1} \cdot h \cdots g_2 \cdot h \cdot g_1 \in \mathcal{P}_n(A)^i(x, y)
\]

is equal to

\[
d(g_{n+1}) \cdot h \cdots g_2 \cdot h \cdot g_1 + (-1)^{\lfloor g_{n+1} \rfloor \cdot \lfloor g_{n+1} \rfloor} d(h) \cdots g_2 \cdot h \cdot g_1 + \cdots,
\]

where \(d(h) \in P_n(A)^i(T(1), T(2))\) corresponds to the image of \(1_{n+1} \in S_{n+1}\) (see 4.9)
under the dg functor \(T\). This description show us that the unique elements in
\( \widehat{\mathcal{W}}_1(T)(x, y) \), which eventually ‘destroy’ the \((n+1)\)-homology of the complex \( \mathcal{W}_1(T)(x, y) \) belong to the graded ring module 

\[
P_n(A)_{n+1}(x, y) = P_n(A)^1(T(2), y) \otimes R[n + 2] \otimes P_n(A)^1(x, T(1)).
\]

We now show that if \( g_2 \cdot h \cdot g_1 \) is an (homogeneous) element of degree \( n + 2 \) in 

\[
P_n(A)_{n+1}(x, y),
\]

whose differential 

\[
g_2 \cdot d(h) \cdot g_1 \in (P_n(A)^1(x, y))_{n+1} \cong (\mathcal{W}_1(T)(x, y))_{n+1}
\]

is non-trivial in the homology ring module \( H_{n+1}(\mathcal{W}_1(T)(x, y)) \), then the element \( g_2 \cdot h \cdot g_1 \) does not belong to \( \mathcal{W}_1(T)(x, y) \). By hypothesis we have an induced isomorphism 

\[
H_{n+1}(A(x, y)) \cong H_{n+1}(\mathcal{W}_1(T)(x, y))
\]

and so by definition \( \simeq \) the image of \( g_2 \cdot h \cdot g_1 \) under the dg functor \( \gamma(T) \) corresponds precisely to this non-trivial element in the homology ring module \( H_{n+1}(A(x, y)) \). This implies that \( g_2 \cdot h \cdot g_1 \) does not belong to the pullback complex \( \mathcal{W}_1(T)(x, y) \) and so we conclude that we have an induced isomorphism 

\[
H_{n+1}(\mathcal{W}_1(T)(x, y)) \cong H_{n+1}(\mathcal{W}_1(T)(x, y)).
\]

Finally, by and infinite composition procedure, we obtain the pullback \( \mathcal{W} \). Since the homology functor \( H_{n+1}(\mathcal{W}) \) commutes with filtered colimits, the induced map 

\[
H_{n+1}(A(x, y)) \cong H_{n+1}(\mathcal{W}(x, y))
\]

is an isomorphism and so we conclude that 

\[
\theta(x, y) : P_{n+1}(A)(x, y) \cong \mathcal{W}(x, y)
\]

is a quasi-isomorphism. This proves the theorem. \( \square \)

6. Obstruction theory

In this chapter, we develop an obstruction theory for dg categories. Our motivation comes from the examples appearing in non-abelian Hodge theory (see \( \simeq \)). We formulate the following general ‘rigidification’ problem.

The ‘rigidification’ problem: Let \( A \) be a positively graded dg category and \( F_0 : B \to H_0(A) \) a dg functor with values in its homotopy category, with \( B \) a cofibrant dg category. Is there a lift \( F : B \to A \) making the diagram

\[
\begin{array}{ccc}
B & \xrightarrow{F} & A \\
\downarrow F_0 \uparrow \tau_{\leq 0} & & \downarrow \tau_{\geq 0} \\
H_0(A) & &
\end{array}
\]

commute?

Intuitively the dg functor \( F_0 \) represents the ‘up-to-homotopy’ information that one would like to rigidify, i.e. lift to the dg category \( A \).

Remark 6.1. Notice that if \( A \) is a homologically connective dg category, we have a zig-zag of dg functors

\[
\begin{array}{ccc}
A & \xleftarrow{\sim} & \tau_{\geq 0}(A) & \xrightarrow{\tau_{\leq 0}} & H_0(A).
\end{array}
\]
In this situation we search for a lift \( B \to A \) which factors through \( \tau_{\geq 0}(A) \).

In order to solve this problem we consider the following notion: let \( A \) be a positively graded dg category and recall from section 4.2 its Big Postnikov model

\[
P_0(A) \to P_1(A) \to P_2(A) \to \cdots \]

**Definition 6.2.** Let \( F : B \to P_n(A) \) be a dg functor. Its obstruction class \( \omega_n(F) \) is the image of the composed dg functor (see 5.9)

\[
B \xrightarrow{F} P_n(A) \xrightarrow{\gamma_n} P_n(A) \ltimes H_{n+1}(A)[n+2]
\]

in the homotopy category \( \text{Ho}(\text{dgcat} \downarrow P_n(A)) \), see remark 5.10.

We say that the obstruction class \( \omega_n(F) \) vanishes if it factors through the canonical morphism

\[
P_n(A) \to P_n(A) \ltimes H_{n+1}(A)[n+2]
\]

in \( \text{dgcat} \downarrow P_n(A) \), see remark 5.15.

**Remark 6.3.** Consider the composed dg functor \( B \to P_n(A) \) as an object in \( \text{dgcat} \downarrow P_n(A) \). By proposition 5.6 the set

\[
\text{Ho}(\text{dgcat} \downarrow P_n(A))(B, P_n(A) \ltimes H_{n+1}(A)[n+2])
\]

is naturally isomorphic to the set

\[
\mathbb{R}\text{Der}(B, (M_n \circ F)^*H_{n+1}(A)[n+2])
\]

of derived derivations of \( B \) with values in \((M_n \circ F)^*H_{n+1}(A)[n+2]\). This implies that the obstruction class \( \omega_n(F) \) of \( F \) corresponds to a derived derivation of \( B \) with values in the \( B-B \)-bimodule \((M_n \circ F)^*H_{n+1}(A)[n+2]\). Moreover by the above isomorphism, the obstruction class \( \omega_n(F) \) of \( F \) vanishes if and only if the associated derived derivation of \( B \) is the trivial one, see notation 5.11.

**Proposition 6.4.** Let \( B \) be a cofibrant dg category. If two dg functors \( F_1, F_2 : B \to P_n(A) \) become equal in the homotopy category \( \text{Ho}(\text{dgcat})(B, P_n(A)) \), they give rise to isomorphic obstruction classes. In particular \( \omega_n(F_1) \) vanishes if and only if \( \omega_n(F_2) \) vanishes.

**Proof.** Notice that since every object in \( \text{dgcat} \) is fibrant (see 8.4) and \( B \) is cofibrant, two dg functors \( F_1 \) and \( F_2 \) become equal in \( \text{Ho}(\text{dgcat})(B, P_n(A)) \) if and only if they
are left homotopic. We can then construct the following diagram

\[
\begin{array}{cccccc}
B & \xrightarrow{F_1} & P_n(A) & \rightarrow & P_n(A) \\
\downarrow & & \downarrow & & \downarrow \\
I(B) & \xrightarrow{H} & P_n(A) & \rightarrow & P_n(A) \\
\downarrow & & \downarrow & & \downarrow \\
B & \xrightarrow{F_2} & P_n(A) & \rightarrow & P_n(A) \\
\end{array}
\]

where \( I(B) \) is a cylinder object for \( B \) and \( i_0 \) and \( i_1 \) are quasi-equivalences. Observe that the previous diagram gives rise to a zig-zag of weak equivalences in \( \text{dgcat} \downarrow P_n(A) \) between \( \omega_n(F_1) \) and \( \omega_n(F_2) \), which implies that the obstruction classes are isomorphic. In particular \( \omega_n(F_1) \) vanishes if and only if so does \( \omega_n(F_2) \).

Let us return to our ‘rigidification’ problem: let \( A \) be a positively graded dg category and \( F : B \rightarrow H_0(A) \) a dg functor, with \( B \) a cofibrant dg category. Consider the diagram

\[
\begin{array}{cccccc}
P_2(A) & \xrightarrow{M_2} & P_2(A) \\
\downarrow & & \downarrow \\
P_1(A) & \xrightarrow{M_1} & P_1(A) \\
\downarrow & & \downarrow \\
P_0(A) & \xrightarrow{M_0} & P_0(A) = H_0(A) & \rightarrow & B \\
\end{array}
\]

where the left (resp. right) column is the Big (resp. small) Postnikov model for \( A \) and the morphism between the two is the one of remark 4.16. Our strategy will be to try to lift \( F_0 : B \rightarrow H_0(A) \) to dg functors \( F_n : B \rightarrow P_n(A) \) for \( n = 1, 2, \ldots \) in succession. If we are able to find all these lifts, there will be no difficulty in constructing the desired lift

\[
F = \lim_n F_n : B \rightarrow A \simeq \lim_n P_n(A) .
\]

For the inductive step, we have a commutative (solid) diagram as follows \((n \geq 0)\)

\[
\begin{array}{cccccc}
P_{n+1}(A) & \xrightarrow{M_{n+1}} & P_{n+1}(A) \\
\downarrow & & \downarrow \\
P_n(A) & \xrightarrow{M_n} & P_n(A) \\
\downarrow & & \downarrow \\
P_n(A) & \xrightarrow{F_n} & B \\
\end{array}
\]
Since $\mathcal{B}$ is cofibrant and $M_n$ is a trivial fibration, there exists a lift $\widetilde{F}_n$ of $F_n$ such that $M_n \circ \widetilde{F}_n = F_n$. Moreover since $M_n$ is a quasi-equivalence, any two such lifts become equal in $\text{Ho}(\text{dgcat})(\mathcal{B}, P_n(A))$ and so by proposition 6.4, they give rise to isomorphic obstruction classes. In what follows, we denote by $\omega_n(F_n)$ the obstruction class of $\widetilde{F}_n$.

**Proposition 6.5.** A lift $F_{n+1}$ of $F_n$, making the diagram

\[
\begin{array}{ccc}
\mathbb{P}_{n+1}(A) & \xrightarrow{F_{n+1}} & \mathbb{P}_n(A) \\
\downarrow & & \downarrow \phi_n \\
\mathbb{P}_n(A) & \xleftarrow{F_n} & \mathcal{B}
\end{array}
\]

commute, exists if and only if the obstruction class $\omega_n(F_n)$ vanishes (see 6.3).

**Proof.** Let us suppose first that $\omega_n(F_n)$ vanishes. Recall from theorem 5.10 that we have an homotopy fiber sequence

\[
P_{n+1}(A) \longrightarrow P_n(A) \xrightarrow{\gamma_n} \mathbb{P}_n(A) \times H_{n+1}(A)[n+2]
\]

in $\text{Ho}(\text{dgcat} \downarrow \mathbb{P}_n(A))$. By hypothesis, the obstruction class $\omega_n(F_n)$ vanishes and so the choice of a homotopy in $\text{dgcat} \downarrow \mathbb{P}_n(A)$ between $\gamma_n \circ F_n$ and

\[
\mathcal{B} \longrightarrow \mathbb{P}_n(A) \longrightarrow \mathbb{P}_n(A) \times H_{n+1}(A)[n+2] \quad (\text{see 5.15})
\]

induces a morphism in $\text{Ho}(\text{dgcat} \downarrow \mathbb{P}_n(A))(\mathcal{B}, P_{n+1}(A))$. Since $\mathcal{B}$ is cofibrant (and $P_{n+1}(A)$ is fibrant) in $\text{dgcat} \downarrow \mathbb{P}_n(A)$ (see 3.11), we can represent this morphism by a dg functor $\psi : \mathcal{B} \longrightarrow P_{n+1}(A)$. Moreover, by lemma 3.12, any two such representatives become equal in $\text{Ho}(\text{dgcat})(\mathcal{B}, P_{n+1}(A))$. This implies that $F_n$ and the composition

\[
\begin{array}{ccc}
\mathcal{B} & \xrightarrow{M_{n+1} \circ \psi} & \mathbb{P}_{n+1}(A) \\
& & \xrightarrow{\phi_n} \mathbb{P}_n(A)
\end{array}
\]

becomes equal in $\text{Ho}(\text{dgcat})(\mathcal{B}, \mathbb{P}_n(A))$. Finally, by lemma 3.13, we conclude that there exists a desired lift $F_{n+1}$ as in the proposition.

Let us now prove the converse. Suppose we have a lift $F_{n+1}$ of $F_n$ as in the proposition. Since $\mathcal{B}$ is cofibrant and $M_{n+1}$ is a trivial fibration there exists a lift $\widetilde{F}_{n+1}$ of $F_{n+1}$ such that $M_{n+1} \circ \widetilde{F}_{n+1} = F_{n+1}$. Observe that $\widetilde{F}_n$ and the composition

\[
\begin{array}{ccc}
\mathcal{B} & \xrightarrow{\widetilde{F}_{n+1}} & P_{n+1}(A) \\
& & \xrightarrow{\phi_n} P_n(A)
\end{array}
\]

becomes equal in $\text{Ho}(\text{dgcat})(\mathcal{B}, P_n(A))$. This implies, by theorem 5.10 and proposition 6.4 that the obstruction class $\omega_n(F_n)$ vanishes.

Thus if it happens that at each stage of the inductive process of constructing lifts $F_n : \mathcal{B} \longrightarrow \mathbb{P}_n(A)$, the obstruction class $\omega_n(F_n)$ vanishes, then the ‘rigidification’ problem has a solution.

**Theorem 6.6.** Let $\mathcal{A}$ be a positively graded dg category and $F_0 : \mathcal{B} \longrightarrow \text{Ho}(\mathcal{A})$ a dg functor, with $\mathcal{B}$ a cofibrant dg category. If the family $\{\omega_n(F_n)\}_{n \geq 0}$ of obstruction
classes vanishes, then there exists a lift $F : \mathcal{B} \to \mathcal{A}$ of $F_0$, making the diagram

\[
\begin{array}{ccc}
\mathcal{B} & \xrightarrow{F_0} & \mathcal{H}_0(\mathcal{A}) \\
\downarrow & & \\
\mathcal{A} & \xrightarrow{F} & \mathcal{B} \\
\end{array}
\]

commute.

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