Critical edge behavior in the perturbed Laguerre ensemble and the Painlevé V transcendent

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Abstract

In this paper, we consider the perturbed Laguerre unitary ensemble described by the weight function of

\[ w(x, t) = (x + t)^\lambda x^\alpha e^{-x} \]

with \( x \geq 0, \ t > 0, \ \alpha > 0, \ \alpha + \lambda + 1 > 0 \). The Deift-Zhou nonlinear steepest descent approach is used to analyze the limit of the eigenvalue correlation kernel. It was found that under the double scaling \( s = 4nt, \ n \to \infty, \ t \to 0 \) such that \( s \) is positive and finite, at the hard edge, the limiting kernel can be described by the \( \varphi \)-function related to a third-order nonlinear differential equation, which is equivalent to a particular Painlevé V (shorted as \( P_V \)) transcendent via a simple transformation. Moreover, this \( P_V \) transcendent is equivalent to a general Painlevé III transcendent. For large \( s \), the \( P_V \) kernel reduces to the Bessel kernel \( J_{\alpha + \lambda} \). For small \( s \), the \( P_V \) kernel reduces to another Bessel kernel \( J_\alpha \). At the soft edge, the limiting kernel is the Airy kernel as the classical Laguerre weight.

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1 Introduction and statement of results

In random matrix theory, the unitary random matrix ensemble on the space of $n \times n$ positive definite Hermitian matrices, described by the following measure,

$$Z_n^{-1} e^{\text{Tr} \log w(M)} dM, \quad dM = \prod_{j=1}^{n} dM_{jj} \prod_{1 \leq \ell < k \leq n} d\text{Re}M_{\ell k} d\text{Im}M_{\ell k},$$

where the normalization constant $Z_n$ ensures the above measure is a probability measure, and $w(x)$ is the weight function on $\mathcal{L} \subset \mathbb{R}$.

The correlation kernel in the following as an important object is investigated in random matrix theory \[\{17, 21, 27, 36\},

$$K_n(x, y) = (w(x))^{\frac{1}{2}} (w(y))^{\frac{1}{2}} \sum_{j=0}^{n-1} \frac{\pi_j(x) \pi_j(y)}{h_j}, \quad (1.1)$$

where

$$\int_{\mathcal{L}} \pi_n(x) \pi_m(x) w(x) dx = h_n \delta_{nm}, \quad (1.2)$$

and $\pi_n(x)$ is the monic orthogonal polynomials associated with the weight $w(x)$ on $\mathcal{L}$.

It is interesting to study the local eigenvalue behavior by characterizing the kernel in the large $n$ limit with a suitable scale. For instance, in the case of Laguerre Unitary Ensemble (LUE), see \[\{6, 27, 37\}], the limiting density of eigenvalue for a fixed $x$, is known as a type of Marčenko-Pastur law \[35\] as follows

$$\mu(x) = \lim_{n \to \infty} 4K_n(4nx, 4nx) = \frac{2}{\pi} \sqrt{\frac{1-x}{x}}, \quad 0 < x < 1. \quad (1.3)$$

In previous works \[\{26, 43, 44\}], at the hard edge of the eigenvalue density, the limiting kernel is known as the Bessel Kernel

$$J_\beta(x, y) := \frac{J_\beta(\sqrt{x}) \sqrt{y} J_\beta(\sqrt{y}) - \sqrt{x} J_\beta'(\sqrt{x}) J_\beta(\sqrt{y})}{2(x - y)},$$

and at the soft edge of the eigenvalue density, the limiting kernel is known as the Airy Kernel

$$A(x, y) := \frac{Ai(x) Ai'(y) - Ai'(x) Ai(y)}{x - y}.$$

Since the Deift-Zhou nonlinear steepest descent approach applied in random matrix theory, some new properties has been identified in random matrix theory. For example, the
limiting kernels in the bulk of the spectrum is usually described by the sine kernel, which is well-known as universality phenomenon (see [17, 18, 20, 32, 45]). Using this powerful method, it has been found that some limiting kernels are related to Painlevé equations. In a particular double scaling scheme, the limiting kernel involves Painlevé I equation in [14]. An appropriate double scaling limit of the kernel relates to the Painlevé II equation, see [4, 12, 13]. A limiting kernel relates to the Painlevé II equation and $4 \times 4$ RH problem discussed in the Hermitian two matrix model, see [15]. An $\alpha$-generalized Airy kernel expressed by a solution of a Painlevé XXXIV equation in [29]. A Painlevé III equation involves the double scaling limit of the kernel, and this $\text{P}_{\text{III}}$ kernel translates to Bessel kernel and Airy kernel in certain conditions [46].

Moreover, a general Jacobi unitary ensemble described by the weight function $w(x) = (1 - x)^\alpha (1 + x)^\beta h(x)$, $\alpha > -1$, $\beta > -1$, where $h(x)$ is positive and analytic on $[-1, 1]$. The university at the hard edge is studied in [32, 34]. Under certain double scaling scheme, the limiting kernel can be expressed in terms of a solution of Painlevé V function and this $\text{P}_{\text{V}}$ kernel degenerates to Bessel kernels with different conditions, see [47]. A limiting kernel in terms of the hypergeometric functions investigated in [16]. A type of general Bessel kernel derives from a scaling limit of kernel at the hard edge and its explicit integration formula is given in [33].

In this paper, we consider the following weight function

$$w(x) := w(x, t) = (x + t)^\lambda x^\alpha e^{-x}, \quad x \geq 0, \ t > 0, \ \alpha > 0, \ \alpha + \lambda + 1 > 0. \quad (1.4)$$

When $t = 0$, or $\lambda = 0$, the weight degenerates to the classical Laguerre weight. A more general weight of this type arises from Multiple-Input-Multiple-Output (MIMO) wireless communication system. It is interesting that the free parameter $\lambda$ “generates” the Shannon capacity, and the moment generating function (MGF) can be expressed by the ration of Hankel determinants [9] for fine $n$, where the Hankel determinant involves a particular Painlevé V equation. The double scaling limit of the Hankel determinant described by another particular $\text{P}_{\text{V}}$ equation which is equivalent to a particular $\text{P}_{\text{III}}$, see [10]. For more information related to wireless communication, we refer [2, 7] and references therein. Moreover, a special case of (1.4) as $w(x, t) = (x + t)^a x^2 e^{-x}, \ x > 0, \ a > -1$ appeared in the study of the smallest eigenvalue at the hard edge of the Laguerre unitary ensemble, see [25].
The weight (1.4) can be seen as $x^{\lambda+\alpha}e^{\frac{4\alpha}{s}x}$. Up to an essential singular point, this is the singularly perturbed Laguerre weight considered in [8], and they obtained a connection with \( P_{III} \) for finite \( n \). Heuristically, this maybe seen as follows (note \( s = 4nt \)),

\[(t+x)^\lambda x^\alpha e^{-x} = x^{\alpha+\lambda} \left( (1 + s/4nx)^{\frac{4\alpha}{s}} \right)^{\frac{4\alpha}{s}x} e^{-x} \to x^{\lambda+\alpha} e^{\frac{4\alpha}{s}x}, \quad n \to \infty.\]

By the Riemann-Hilbert approach, the double scaling limit of the kernel associated with this singular weight presents as a \( P_{III} \) kernel, see [46], with physical background provided in [38] and for a type of singularly perturbed Gaussian weight relevant study (for Hermite) see [5].

It is interesting to investigate the double scaling limit of the kernel related with the weight (1.4) on \((0, \infty)\). We adapt Deift-Zhou nonlinear steepest descent method to investigate the limiting behavior of the kernel in this paper.

For preparation, we recall that a Lax pair in the following is given by Kapaev and Hubert [31] and note that this Lax pair and its Painlevé V equation do not contain in [24], with a change of variables \( \lambda \) and \( x \) in [31] as \( \xi \) and \( s \), respectively, which will help us to derive \( P_{V} \) transcendent directly in our situation.

**Proposition 1.** *(Kapaev and Hubert [31])* The Lax pair for \( \Psi \) is given by

\[
\Psi_\xi \Psi^{-1} = A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \frac{1}{\xi} \begin{pmatrix} a & b \\ c & -a \end{pmatrix} + \frac{1}{\xi - s} \begin{pmatrix} p & q \\ r & -p \end{pmatrix},
\]

(1.5)

\[
\Psi_s \Psi^{-1} = U = -\frac{1}{\xi - s} \begin{pmatrix} p & q \\ r & -p \end{pmatrix},
\]

(1.6)

where \( a, b, c, p, q, \) and \( r \) are dependent on \( s \) and \( \rho, \mu \) and \( \upsilon \) are constant. The compatibility condition for the system (1.3) and (1.4) is equivalent to the system of equations

\[b = (a^2 - \mu^2)\frac{y-1}{\rho}, \quad c = -\frac{\rho}{y-1}, \quad p = \frac{s}{2y-1} - ay, \quad q = -\left( p^2 - \upsilon^2 \right)\frac{y-1}{\rho y}, \quad r = \frac{\rho y}{y-1},\]

where the unknown function \( a(s) \) satisfies the differential equation,

\[a_s + a \frac{ys}{y-1} = \frac{s}{4y(y-1)^2} + \frac{1}{s} \left( \mu^2 y - \upsilon^2 \right),\]

and the function \( y(s) \) satisfies the special \( P_{V}(2\mu^2, -2\upsilon^2, 2\rho, 0) \) as follows

\[y_{ss} = \left( \frac{1}{2y} + \frac{1}{y-1} \right) y_s^2 - \frac{ys}{s} + \frac{2(y-1)^2}{s^2} \left( \mu^2 y - \upsilon^2 \right) + \frac{2\rho}{s} y.\]

(1.7)

Moreover, the above \( P_{V} \) equation is equivalent to the general \( P_{III} \) equation [28].
In our case, the following Lax pair for $\Phi(\xi, s)$ and an auxiliary function $r(s)$ is a solution of a third-order nonlinear differential equation associated with a particular Painlevé $V$ transcendent.

### 1.1 Lax pair, a third-order nonlinear differential equation and $P_V$

**Proposition 2.** The Lax pair for $\Phi(\xi, s)$ is given by

$$
\Phi_{\xi}(\xi, s) = \left( A_0(s) + \frac{A_1(s)}{\xi} + \frac{A_2(s)}{\xi - s} \right) \Phi(\xi, s),
$$

$$
\Phi_s(\xi, s) = \frac{B_2(s)}{\xi - s} \Phi(\xi, s),
$$

where

$$
A_0(s) = \frac{1}{2} \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix}, \quad A_1(s) = \begin{pmatrix} \frac{1}{4} + \frac{i}{2} r(s) + q'(s) & -i \left( \frac{1}{4} + r'(s) \right) \\ i(-q(s) + t'(s)) & \frac{1}{4} - \frac{i}{2} r(s) - q'(s) \end{pmatrix}, \quad A_2(s) = -B_2(s),
$$

$$
B_2(s) = \begin{pmatrix} q'(s) & -ir'(s) \\ it'(s) & -q(s) \end{pmatrix},
$$

and $q(s)$, $q'(s)$ and $t'(s)$ in terms of $r(s)$ and its derivatives,

$$
q' = -sr'' + r'r - \frac{1}{2} r',
$$

$$
t' = \lambda^2 - (2sr'' - 2r'r + r')^2
$$

$$
q = \frac{-8s^2 r'' + r'(4r(r-1)(2r'+1) + 2r' + 4\lambda^2 - 4\alpha^2 + 1) - 2\lambda^2}{8r'(2r'+1)},
$$

with $\alpha > 0$, $\alpha + \lambda + 1 > 0$.

**Remark 1.** It’s easy to check that the Lax pair \((1.5), (1.6)\) for $\Psi$ matches the Lax pair \((1.8), (1.9)\) for $\Phi$, in the sense of the following linear transformation

$$
\Phi = \begin{pmatrix} 0 & 1 \\ i & 0 \end{pmatrix} \Psi.
$$

From the above equation, we can identify that $\mu^2 = \frac{\alpha^2}{4}$, $v^2 = \frac{\lambda^2}{4}$, and $\rho = \frac{1}{4}$, and especially the unknown function $a(s) = \frac{1}{4} + \frac{i}{2} r(s) - q'(s)$. Then, we can identify all other quantities in the Lax pair can be expressed in terms of the auxiliary function $r(s)$ and its derivatives.

Although the $P_V$ transcendent in \((1.7)\) is equivalent to the general $P_{\text{III}}$ transcendent \([28]\), but there is no algebraic gauge transformation from the Lax pair for $P_V$ equation to the Lax pair for $P_{\text{III}}$ equation which has only two irregular singular points of zero and infinity.
Proposition 3. If \( r(s) \) satisfies the Lax pair (1.8) and (1.9) for \( \Phi(\xi, s) \), then it also satisfies a third-order nonlinear differential equation,

\[
8s^2r'(2r' + 1)r'' - 4s^2(1 + 4r')r'' + 8sr'(2r' + 1)r'' - 4sr^2(2r' + 1)^2 + \lambda^2(2r' + 1)^2 - 4\alpha^2r'^2 = 0,
\]

where \( \alpha > 0 \), \( \alpha + \lambda + 1 > 0 \), and \( r(s) \) satisfies the following boundary conditions,

\[
r(0) = \frac{1 - 4(\alpha + \lambda)^2}{8}, \quad \text{and} \quad r(s) = -\lambda s^{-\frac{1}{2}} + \mathcal{O}(s^{-1}), \quad s \to \infty.
\]

If \( \alpha + \lambda > 0 \) and \( s \to 0 \), then

\[
r'(0) = -\frac{\lambda}{2(\alpha + \lambda)}.
\]

Proof. With the aid of the equations (1.12) and (1.14) which derive from the compatibility condition of the Lax pair (1.8) and (1.9) for \( \Phi(\xi, s) \), one takes derivative of (1.14) and a combination of (1.12), then one obtains the third-order nonlinear differential equation (1.15). The boundary conditions of \( r(s) \) in (1.16) derives from (4.114), (4.145) for \( s \to 0 \) and \( s \to \infty \), respectively. Moreover, (4.116) gives (1.17).

Remark 2. The third-order equation (1.13) is an integrable equation. The integration of the equation (1.12) presents as

\[
q(s) = -sr'(s) + \frac{r(s)^2 + r(s)}{2} + c_1,
\]

where \( c_1 \) is an integration constant. From the boundary condition (4.114), it follows that \( c_1 = 0 \) in this situation. By the equivalence of \( q(s) \) in (1.14) and (1.18), then one finds \( r(s) \) also satisfies another second-order nonlinear differential equation,

\[
s^2r'' - 2sr'^2 + \left(2r + 2c_1 - s - \frac{1}{4}\right)r'^2 + \left(r + \frac{\alpha^2}{2} + c_1 - \frac{1}{8} - \frac{\lambda^2}{2}\right)r' - \frac{\lambda^2}{4} = 0,
\]

it can also be rewritten as follows

\[
\frac{s^2r''}{r'(2r' + 1)} - \frac{2sr'^2}{2r' + 1} - \frac{sr'}{2r' + 1} + \frac{\alpha^2 - \lambda^2}{2(2r' + 1)} - \frac{\lambda^2}{4r'(2r' + 1)} + r = \frac{1}{8} - c_1,
\]

and one takes derivative with respective to \( s \) on both sides of the above equation, then one obtains the third-order equation (1.17). Taking the inverse procedure, the third-order equation in (1.15) is indeed an integrable equation.
Furthermore, inserting the equation (1.18) into (2.44), then one finds \( r(s) \) satisfies the following equation

\[
8s^2 r''r''' - 4s^2 r''^2 + 8sr'r'' - 16sr'^3 + (8r - 4s - 1 + 8c_1) r'^2 + \lambda^2 = 0. \tag{1.21}
\]

Then the third-order equation (1.15) is the sum of \(-8r'(s)\) times of the equation (1.19) and \(2r'(s) + 1\) multiples the equation (1.21).

**Proposition 4.** Let

\[
r'(s) = \frac{y(s)}{2(1 - y(s))}, \tag{1.22}
\]

then \( y(s) \) satisfies the following \( PV \left( \frac{\alpha^2}{2}, \frac{\lambda^2}{2}, \frac{1}{2}, 0 \right) \) transcendent,

\[
y''(s) = \left( \frac{1}{2y(s)} + \frac{1}{y(s) - 1} \right) y'^2(s) - \frac{y'(s)}{s} + \frac{(y(s) - 1)^2}{2s^2} \left( \alpha^2 y(s) - \frac{\lambda^2}{y(s)} \right) + \frac{y(s)}{2s}, \tag{1.23}
\]

where \( \alpha > 0, \alpha + \lambda + 1 > 0, \) and \( y(s) \) satisfies the boundary conditions

\[
y(0) = -\frac{\lambda}{\alpha}, \text{ and } y(s) = -\lambda s^{-\frac{1}{2}} + O\left(s^{-1}\right), \ s \to \infty. \tag{1.24}
\]

**Proof.** Inserting (1.22) into the third-order nonlinear differential equation (1.15), one finds the \( PV \) transcendent in (1.23). A combination of (1.22) and (1.16) yields the initial data (1.24) for \( y(s) \). The above \( PV \) equation is also derived by the analysis of the single-user MIMO system [10], but it is not from the Riemann-Hilbert approach.

### 1.2 Main results

We denote the kernel (1.1) associated with the weight (1.4) as \( K_n(x, y; t) \) in the rest of this paper. By two scaling steps, \( K_n(x, y; t) \) scaled as \( 4nK_n(4nx, 4ny; t) \), and the “coordinates” \( x \) and \( y \) are re-scaled as \( x = \frac{u}{16n^2}, \ y = \frac{v}{16n^2} \). After that, the limiting kernel is described by the \( \varphi^- \) functions which involve the above \( PV \) equation (1.23), also known as \( PV \) kernel.

**Theorem 1.** Let \( s = 4nt, \ n \to \infty, \) and \( t \to 0^+ \), such that \( s \) is positive and finite, then

\[
\lim_{n \to \infty} \frac{1}{4n} K_n\left( \frac{u}{4n}; \frac{v}{4n}; \frac{s}{4n} \right) = \frac{\varphi_1(-v, s)\varphi_2(-u, s) - \varphi_1(-u, s)\varphi_2(-v, s)}{i2\pi(u - v)}, \tag{1.25}
\]
which is uniform for \( u, v \in (0, \infty) \), \( \varphi_k(\xi, s), k = 1, 2 \) satisfy the following second order differential equation,

\[
\varphi''(\xi) + \left( \frac{1}{\xi} + \frac{1}{\xi - s} - \frac{1}{\xi - s - 2sr'} \right) \varphi'(\xi) + W(\xi, s)\varphi(\xi) = 0, \tag{1.26}
\]

where \( \xi \in \tilde{\Omega}_3 \), see Figure 1, \( \varphi'(\xi) \) and \( r' \) denote \( d\varphi(\xi, s)/d\xi \), \( dr(s)/ds \), respectively, \( r'(s) \) involves \( P_V \) (see (1.22) and (1.23)) and \( W(\xi, s) \) is given by

\[
W(\xi, s) = -\frac{(s + 2sr' - \xi)[4\xi(\xi - s)r'' + 2((s - \xi)(\alpha^2 + \xi) + \lambda^2\xi)r' + \lambda^2\xi]}{8\xi^2(\xi - s)^2r'(1 + 2r')(s + 2sr' - \xi)}
+ \frac{4s^2(s - \xi)\xi r''[r''(s - \xi + 2sr') + 2r'(1 + 2r')]}{8\xi^2(\xi - s)^2r'(1 + 2r')(s + 2sr' - \xi)},
\]

and \( \alpha > 0, \alpha + \lambda + 1 > 0 \). If \( s = 0 \) and \( \alpha + \lambda > 0 \), then the equation (1.26) degenerates to a modified Bessel differential equation

\[
\varphi''(\xi) + \frac{1}{\xi}\varphi'(\xi) - \left( \frac{1}{4\xi} + \frac{(\alpha + \lambda)^2}{4\xi^2} \right) \varphi(\xi) = 0. \tag{1.27}
\]

The \( P_V \) kernel (1.25) degenerates to the Bessel kernel \( J_{\alpha+\lambda} \) as \( s \to 0 \).

**Theorem 2.** Let \( s = 4nt, n \to \infty, \) and \( t \to 0^+ \), such that \( s \to 0^+ \), then the following limiting kernel can be identified as the Bessel kernel \( J_{\alpha+\lambda} \),

\[
\lim_{n \to \infty} \frac{1}{4n}K_n\left(\frac{u}{4n}, \frac{v}{4n}, \frac{s}{4n}\right) = J_{\alpha+\lambda}(u, v) = \frac{J_{\alpha+\lambda}(u)vJ'_{\alpha+\lambda}(v) - J_{\alpha+\lambda}(v)uJ'_{\alpha+\lambda}(u)}{2(u - v)}, \tag{1.28}
\]

where \( \alpha > 0, \alpha + \lambda + 1 > 0 \). The scaling limit of the kernel is independent of \( s \) and uniform for \( u, v \in (0, \infty) \).

The \( P_V \) kernel (1.25) degenerates to the Bessel kernel \( J_{\alpha} \) as \( s \to \infty \).

**Theorem 3.** Let \( s = 4nt, t \in (0, c], c > 0, n \to \infty, \) and \( t \to c^- \), such that \( s \to \infty \), then the following limiting kernel is explicit as the Bessel kernel \( J_{\alpha} \),

\[
\lim_{n \to \infty} \frac{1}{4n}K_n\left(\frac{u}{4n}, \frac{v}{4n}, \frac{s}{4n}\right) = J_{\alpha}(u, v) = \frac{J_{\alpha}(u)vJ'_{\alpha}(v) - J_{\alpha}(v)uJ'_{\alpha}(u)}{2(u - v)}, \tag{1.29}
\]

where \( \alpha > 0, \alpha + \lambda + 1 > 0 \). The scaling limit of the kernel is independent of \( s \) and uniform for \( u, v \in (0, \infty) \).
The limiting behavior of the kernel at the soft edge. Firstly, $K_n(x, y; t)$ scaled as $4nK_n(4nx, 4ny; t)$. Secondly, $x$ and $y$ scaled as $x = 1 + (2n)^{-\frac{2}{3}}u$, $y = 1 + (2n)^{-\frac{2}{3}}v$. After that the limiting kernel reads as the following Airy kernel.

**Theorem 4.** The kernel (1.1) associated with the weight (1.4) denotes as $K_n(x, y; t)$. If $x$ and $y$ scaled as $x = 4n + 2(2n)^{\frac{1}{3}}u$, $y = 4n + 2(2n)^{\frac{1}{3}}v$, then the limiting kernel is the Airy kernel $A$,

$$
\lim_{n \to \infty} 2(2n)^{\frac{1}{3}}K_n(4n + 2(2n)^{\frac{1}{3}}u, 4n + 2(2n)^{\frac{1}{3}}v; t) = A(u, v) = \frac{Ai(u)Ai'(v) - Ai(v)Ai'(u)}{u - v},
$$

which is uniform for $u, v$ in compact subsets of $(-\infty, 0)$ and $t \in (0, \infty)$.

The remainder of this paper is organized as follows. In Sect.2, we propose a model Riemann-Hilbert problem associated to $\Phi(\xi, s)$, and derive its Lax pair by proving the Proposition 2. We prove the solvability of the RH problem for $\Phi(\xi, s)$ via a vanishing lemma. In Sect.3 we apply the Deift-Zhou nonlinear steepest descent method to analyze the RH problem for orthogonal polynomials with respect to the weight function (1.4) and prove Theorem 1. Sect.4 focuses on the reduction of the $P_V$ kernel to two different Bessel kernels as $s \to \infty$ and $s \to 0^+$, respectively, and the proof of Theorem 2 and Theorem 3. Sect.5 is to show that the limiting kernel is the Airy kernel at the hard edge and gives the proof of Theorem 4.

We claim that the following Pauli matrices $\sigma_1$, $\sigma_3$ and two auxiliary matrices $\sigma_-$, $\sigma_+$ have been used in this paper,

$$
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
$$

## 2 A model RH problem and its solvability

A model RH problem in the following for $\Phi(\xi, s)$ and it will benefit for the steepest descent analysis.

(a) $\Phi(\xi, s)$ is analytic in $\mathbb{C} \setminus \bigcup_{j=1}^{3}\hat{\Sigma}_j \cup (0, s)$, illustrated in Figure 1.
(b) \( \Phi(\xi, s) \) fulfills the jump relation
\[
\Phi_+(\xi, s) = \Phi_-(\xi, s) \begin{cases} 
e^{-i\lambda\pi}\sigma_3, & \xi \in (0, s), \\
\begin{pmatrix} 1 & 0 \\ e^{i\pi(\lambda+\alpha)} & 1 \end{pmatrix}, & \xi \in \hat{\Sigma}_1, \\
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \xi \in \hat{\Sigma}_2, \\
\begin{pmatrix} 1 & 0 \\ e^{-i\pi(\lambda+\alpha)} & 1 \end{pmatrix}, & \xi \in \hat{\Sigma}_3. \end{cases}
\] (2.31)

(c) For \( \xi \to \infty \), the asymptotic behavior of \( \Phi(\xi, s) \) is given by
\[
\Phi(\xi, s) = \left( I + \frac{C_1(s)}{\xi} + \mathcal{O}\left(\frac{1}{\xi^2}\right) \right) \xi^{-\frac{i}{2}\sigma_3} \frac{I + i\sigma_1}{\sqrt{2}} e^{\pm\xi\sigma_3},
\] (2.32)
where \( C_1(s) \) only dependents on \( s \) and \( \arg \xi \in (-\pi, \pi) \).

(d) For \( \xi \to 0 \), the asymptotic behavior of \( \Phi(\xi, s) \) in four sectors \( \hat{\Omega}_j, j = 1, \ldots, 4 \), are given by
\[
\Phi(\xi, s) = Q_1(s) \left( I + \mathcal{O}(\xi) \right) \xi^{\frac{i}{2}\sigma_3} \begin{cases} 
e^{-i\lambda\pi}\frac{\sigma_3}{2}, & \xi \in \hat{\Omega}_1, \\
\begin{pmatrix} 1 & 0 \\ -e^{i\pi(\lambda+\alpha)} & 1 \end{pmatrix}, & \xi \in \hat{\Omega}_2, \\
\begin{pmatrix} 1 & 0 \\ e^{-i\pi(\lambda+\alpha)} & 1 \end{pmatrix}, & \xi \in \hat{\Omega}_3, \\
e^{-i\lambda\pi}\frac{\sigma_3}{2}, & \xi \in \hat{\Omega}_4, \end{cases}
\] (2.33)
where four sectors \( \hat{\Omega}_j, j = 1, \ldots, 4 \), illustrated in Figure 1, and \( Q_1(s) \) only depends on \( s \), such that \( \det(Q_1(s)) = 1 \).

(e) For \( \xi \to s \), the asymptotic behavior \( \Phi(\xi, s) \) is given by
\[
\Phi(\xi, s) = Q_2(s) \left( I + \mathcal{O}(\xi - s) \right) (\xi - s)^{\frac{i}{2}\sigma_3},
\] (2.34)
where \( Q_2(s) \) only depends on \( s \), such that \( \det(Q_2(s)) = 1 \), and \( \arg(\xi - s) \in (-\pi, \pi) \).
Figure 1. Contours $\mathbb{C} \setminus \cup_{j=1}^{3} \widehat{\Sigma}_{j} \cup (0,s)$, and regions $\widehat{\Omega}_{j}, j = 1, \ldots, 4$.

The proof of Proposition 2, we follow similar line in [24] to derive the Lax pair (1.8) and (1.9) for $\Phi(\xi,s)$.

Proof. From the constant jumps in (2.31) of $\Phi(\xi,s)$, therefore det $(\Phi(\xi,s))$ is an entire function. From (2.32), det $(\Phi(\xi,s)) = 1 + O(1/\xi)$ is uniform for $\xi \to \infty$ in $\mathbb{C} \setminus \cup_{j=1}^{3} \widehat{\Sigma}_{j} \cup (0,s)$. Hence, by Liouville’s theorem, one finds det $(\Phi(\xi,s)) = 1$. So, tr$C_{1}(s) = 0$ is valid. We suppose

$$C_{1}(s) = \begin{pmatrix} q(s) & -ir(s) \\ it(s) & -q(s) \end{pmatrix}. \quad (2.35)$$

Both rational matrix functions $\Phi_{\xi}\Phi^{-1}$ and $\Phi_{\xi}\Phi^{-1}$ are analytic in $\xi$ plane, with two possible simple poles 0 and $s$, since all the jumps in (2.31) of the RH problem for $\Phi(\xi,s)$ are constant matrices.

The rational function $\Phi_{\xi}\Phi^{-1}$ has a removable singularity at $\xi = \infty$, and two possible simple poles at $\xi = 0$, $\xi = s$, respectively, by the asymptotic behaviors (2.32), (2.33) and (2.34) of $\Phi(\xi,s)$. This leads to

$$\Phi_{\xi}\Phi^{-1} = A_{0}(s) + \frac{A_{1}(s)}{\xi} + \frac{A_{2}(s)}{\xi - s}. \quad (2.36)$$

The asymptotic behavior of the LHS of (2.36) at $\xi = \infty$ derives from (2.32) and expanding the RHS of (2.36) at $\xi = \infty$. Comparing the constant terms and the coefficients of $\xi^{-1}$ on both sides, then one finds,

$$A_{0} = \frac{i}{2}\sigma_{-}, \quad A_{1} + A_{2} = -\frac{1}{4}\sigma_{3} + \frac{i}{2}(C_{1}\sigma_{-} - \sigma_{-}C_{1}) - \frac{i}{2}\sigma_{+}. \quad (2.37)$$
A calculation of the LHS of (2.36) with (2.33) and comparing the coefficients of $\xi^{-1}$ on both sides as $\xi \to 0$, one has,

$$A_1 = \frac{\alpha}{2} Q_1(s) \sigma_3 Q_1^{-1}(s), \quad \det(A_1) = -\frac{\alpha^2}{4}. \quad (2.38)$$

Similarly, a calculation of the LHS of (2.36) with (2.34) and comparing the coefficients of $(\xi - s)^{-1}$ on both sides as $\xi \to s$, one finds,

$$A_2 = \frac{\lambda}{2} Q_2(s) \sigma_3 Q_2^{-1}(s), \quad \det(A_2) = -\frac{\lambda^2}{4}. \quad (2.39)$$

Moreover, the rational function $\Phi_s \Phi^{-1}$ has a removable singularity at $\xi = \infty$, and a simple pole at $\xi = s$. From the asymptotic behaviors (2.32) and (2.34) of $\Phi(\xi, s)$, this provides

$$\Phi_s \Phi^{-1} = B_0(s) + \frac{B_2(s)}{\xi - s}. \quad (2.40)$$

Similarly, substituting (2.34) into the rational function $\Phi_{\xi} \Phi^{-1}$, and comparing the coefficient of $(\xi - s)^{-1}$ on both sides of (2.40) as $\xi \to s$, then one finds,

$$B_2 = \frac{\lambda}{2} Q_2(s) \sigma_3 Q_2^{-1}(s), \quad \det(B_2) = -\frac{\lambda^2}{4}. \quad (2.41)$$

Inserting (2.32) into the rational function $\Phi_{\xi} \Phi^{-1}$, and comparing the constant terms and the coefficients of $\xi^{-1}$ on both sides of (2.40), one has, $B_0(s) = 0$ and $B_2(s) = C'_1(s)$, furthermore, with the equations (2.35), (2.37), (2.39) and (2.41), then (1.8), (1.9), (1.10) and (1.11) are valid.

From the mixed derivatives $\Phi_{\xi s} = \Phi_{s \xi}$, which leads to the compatibility condition

$$A_s - B_\xi = BA - AB, \quad (2.42)$$

where

$$A = A_0(s) + \frac{A_1(s)}{\xi} + \frac{A_2(s)}{\xi - s}, \quad B = \frac{B_2(s)}{\xi - s}. $$

After some calculation and simplify, the compatibility condition (2.42) leads to

$$s A'_1(s) = A_1(s) B_2(s) - B_2(s) A_1(s), \quad (2.43)$$

where $A_1(s), B_2(s)$ are in (1.10) and (1.11). Rewriting the equation (2.43), $\det(A_1) = -\frac{\alpha^2}{4}$ (see (2.38)) and $\det(B_2) = -\frac{\lambda^2}{4}$ (see (2.41)) in terms of the auxiliary functions $q(s), r(s), t(s)$
and their derivatives, after some calculations, one gets,

\[ q''(s) = \frac{t'(s)}{2s} + \frac{r'(s)q(s)}{s} - \frac{r'(s)}{2}, \]  \hspace{1cm} (2.44)

\[ r''(s) = -\frac{r'(s)}{2s} + \frac{r'(s)r(s)}{s} - \frac{q'(s)}{s}, \]  \hspace{1cm} (2.45)

\[ q'(s)^2 + r'(s)t'(s) = \frac{\lambda^2}{4}, \]  \hspace{1cm} (2.46)

\[ \left(q'(s) + \frac{1}{2}r(s) - \frac{1}{4}\right)^2 + \left(\frac{1}{2} + r'(s)\right)\left(t'(s) - q(s)\right) = \frac{\alpha^2}{4}. \]  \hspace{1cm} (2.47)

Although the above system has four equations with three unknown functions, it is solvable. In fact, the first equation is equivalent to a suitable combination of the rest three equations (2.45)-(2.47). With the aid of (2.46), the first equation (2.44) is equivalent to

\[ q''(s) = \frac{\lambda^2 - 4q'(s)^2}{8sr'(s)} - \frac{r'(s)}{2} + \frac{q(s)r'(s)}{s}. \]  \hspace{1cm} (2.48)

Substituting (2.45) and (2.46) into the derivative of the fourth equation (2.47), then one finds the following equation,

\[ \frac{r''(s)}{r'(s)} \left(q''(s) - \frac{\lambda^2 - 4q'(s)^2}{8sr'(s)} + \frac{r'(s)}{2} - \frac{q(s)r'(s)}{s}\right) = 0, \]  \hspace{1cm} (2.49)

which is equivalent to the equation (2.48), so the first equation (2.44) in the system can be neglected. From (2.45), (2.46) and (2.47), one finds that \( q'(s), t'(s) \) and \( q(s) \) in terms of \( r(s) \) and its derivatives, these are explicit in (1.12), (1.13) and (1.14). \( \square \)

**Solvability of the Riemann-Hilbert problem for \( \Phi(\xi, s) \)**

In general case, the solvability of a RH problem derives from the triviality of its homogeneous RH problem, namely the vanishing lemma. With the aid of the Cauchy operator, a RH problem turns out to be a Fredholm singular equation, for examples [17, 20, 23, 24, 29]. The vanishing lemma follows that the null space is trivial, and with the Fredholm alternative theorem, imply that the index of the Fredholm singular equation is zero and the RH problem is solvable. A summary contained in [24] and references therein. Some examples and details see also [11, 30, 47]. Here, a vanishing lemma proved for the homogeneous RH problem \( \tilde{\Phi}(\xi, s) \).
Lemma 1. Suppose that \( \tilde{\Phi}(\xi, s) \) satisfies the jump conditions (2.31) and the boundary conditions (2.33), (2.34) of the RH problem for \( \Phi(\xi, s) \), with the asymptotic behavior (2.32) at infinity replaced by

\[
\tilde{\Phi}(\xi, s) = O \left( \frac{1}{\xi} \right) \xi^{-\frac{i\sigma_3}{2}} \frac{I + i\sigma_1}{\sqrt{2}} e^{\sqrt{\xi} \sigma_3}, \quad \text{as} \quad \xi \to \infty. \tag{2.50}
\]

If \( s \in (0, \infty) \), then it follows that the solution is trivial, \( \tilde{\Phi}(\xi, s) \equiv 0 \).

Proof. In order to eliminate the exponential factor at infinity and remove the jumps on \( \hat{\Sigma}_1 \) and \( \hat{\Sigma}_3 \), let \( \tilde{\Phi}_1(\xi, s) \) be defined as follows

\[
\tilde{\Phi}_1(\xi, s) = \begin{cases} 
\tilde{\Phi}(\xi, s)e^{-\sqrt{\xi}\sigma_3}, & \xi \in \hat{\Omega}_1 \cup \hat{\Omega}_4, \\
\tilde{\Phi}(\xi, s)e^{-\sqrt{\xi}\sigma_3} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, & \xi \in \hat{\Omega}_2, \\
\tilde{\Phi}(\xi, s)e^{-\sqrt{\xi}\sigma_3} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, & \xi \in \hat{\Omega}_3,
\end{cases} \tag{2.51}
\]

where the sectors \( \hat{\Omega}_j, j = 1, \ldots, 4 \), illustrated in Figure 1, \( \arg \xi \in (-\pi, \pi) \).

Then \( \tilde{\Phi}_1(\xi, s) \) fulfills the following RH conditions:

(a) \( \tilde{\Phi}_1(\xi, s) \) is analytic in \( \xi \in \mathbb{C} \setminus \hat{\Sigma}_2 \cup (0, s) \),

(b) \( \tilde{\Phi}_1(\xi, s) \) satisfies the following jump condition

\[
(\tilde{\Phi}_1)_+(\xi, s) = (\tilde{\Phi}_1)_-(\xi, s) \begin{pmatrix} e^{i\pi\sigma_3} \xi^{-\frac{i\sigma_3}{2}} & \mathbf{0} \\
\mathbf{0} & e^{i\pi\sigma_3} \xi^{-\frac{i\sigma_3}{2}} \end{pmatrix}, \quad \xi \in \hat{\Sigma}_2, \tag{2.52}
\]

where \( \sqrt{\xi} = i \sqrt{|\xi|} \) for \( \xi \in \hat{\Sigma}_2 \cup (0, s) \).

(c) The asymptotic behavior of \( \tilde{\Phi}_1(\xi, s) \) at \( \xi = \infty \) is

\[
\tilde{\Phi}_1(\xi, s) = O \left( \xi^{-\frac{3}{4}} \right), \quad \arg \xi \in (-\pi, \pi). \tag{2.53}
\]

(d) The asymptotic behavior of \( \tilde{\Phi}_1(\xi, s) \) at \( \xi = 0 \) is

\[
\tilde{\Phi}_1(\xi, s) = O(1) (\xi - s) \tilde{\Phi}_1^{\sigma_3} \xi^{\frac{3}{2} \sigma_3}. \tag{2.54}
\]
(e) The asymptotic behavior of $\tilde{\Phi}_1(\xi, s)$ at $\xi = s$ is

$$\tilde{\Phi}_1(\xi, s) = O(1) (\xi - s)^{i \frac{\pi}{2}}.$$

In order to translate the oscillation terms in diagonal to off-diagonal, another transformation applied in the following,

$$\tilde{\Phi}_2(\xi, s) = \left\{ \begin{array}{ll}
\tilde{\Phi}_1(\xi, s) \left( \begin{array}{cc}
0 & -1 \\
1 & 0 
\end{array} \right), & \text{Im} \xi > 0, \\
\tilde{\Phi}_1(\xi, s), & \text{Im} \xi < 0.
\end{array} \right. \quad (2.55)$$

Then $\tilde{\Phi}_2(\xi, s)$ satisfies the RH problem as follows

(a) $\tilde{\Phi}_2(\xi, s)$ is analytic in $\mathbb{C} \setminus \mathbb{R}$.

(b) $\tilde{\Phi}_2(\xi, s)$ satisfies the following jump condition

$$\left( \tilde{\Phi}_2 \right)_{+}(\xi, s) = (\tilde{\Phi}_2)_{-}(\xi, s) \left\{ \begin{array}{cl}
\left( \begin{array}{cc}
1 & -e^{-2\sqrt{\xi} e^{i(\alpha + \lambda)\pi}} \\
e^{2\sqrt{\xi} e^{-i(\alpha + \lambda)\pi}} & 0 
\end{array} \right), & \xi \in (-\infty, 0), \\
\left( \begin{array}{cc}
0 & -e^{i\lambda \pi} \\
e^{-i\pi} & 0 
\end{array} \right), & \xi \in (0, s), \\
\left( \begin{array}{cc}
0 & -1 \\
1 & 0 
\end{array} \right), & \xi \in (s, +\infty).
\end{array} \right. \quad (2.56)$$

(c) The behavior of $\tilde{\Phi}_2(\xi, s)$ at infinity is the same as (2.53).

(d) The behavior of $\tilde{\Phi}_2(\xi, s)$ at $\xi = 0$ is

$$\tilde{\Phi}_2(\xi, s) = O(1) (\xi - s)^{i \frac{\pi}{2}} \xi^{i \frac{\sigma_3}{2} \xi^{\pi}} \left\{ \begin{array}{ll}
\left( \begin{array}{cc}
0 & -1 \\
1 & 0 
\end{array} \right), & \text{Im} \xi > 0, \\
I, & \text{Im} \xi < 0.
\end{array} \right. \quad (2.57)$$

(e) The behavior of $\tilde{\Phi}_2(\xi, s)$ at $\xi = s$,

$$\tilde{\Phi}_2(\xi, s) = O(1) (\xi - s)^{i \frac{\pi}{2}} \xi^{i \frac{\sigma_3}{2} \xi^{\pi}} \left\{ \begin{array}{ll}
\left( \begin{array}{cc}
0 & -1 \\
1 & 0 
\end{array} \right), & \text{Im} \xi > 0, \\
I, & \text{Im} \xi < 0.
\end{array} \right. \quad (2.58)$$
For later convenience, an auxiliary matrix function defined as follows

\[ H(\xi) = \Phi_2(\xi, s) \left( \Phi_2(\xi, s) \right)^*, \quad \xi \notin \mathbb{R}, \]

where \( Z^* \) denotes the Hermitian conjugate of \( Z \). With the asymptotic behavior of \( \Phi_2(\xi, s) \) at \( \xi = \infty \) \((2.53)\), \( \xi = 0 \) \((2.57)\), and \( \xi = s \) \((2.58)\), one finds,

\[ H(\xi) = O \left( \xi^{-\frac{3}{2}} \right), \quad \xi \rightarrow \infty; \quad H(\xi) = O(1), \quad \xi \rightarrow 0; \quad H(\xi) = O(1), \quad \xi \rightarrow s. \]

Then, \( H_+(\xi) \) is integrable over the real axis, by Cauchy’s formula, one has

\[ \int_{\mathbb{R}} H_+(\xi) d\xi = 0. \quad (2.59) \]

From \((2.56)\), the sum of equation \((2.59)\) and its Hermitian conjugate, then one finds,

\[ \int_{-\infty}^{0} (\Phi_2^-(\xi, s)) \left( \begin{array}{cc} 2 & 0 \\ 0 & 0 \end{array} \right) (\Phi_2^-)^* (\xi, s) d\xi = \int_{-\infty}^{0} \left( |((\Phi_2)_{11})^-|^2 \right) \left( |((\Phi_2)_{21})^-|^2 \right) d\xi = 0. \]

From the above equation, for \( \xi \in (-\infty, 0) \), the first column of \( (\Phi_2^-)(\xi, s) \) vanishes. By the jump condition \((2.56)\) on the interval \((-\infty, 0)\), the second column of \( (\Phi_2^+)(\xi, s) \) also vanishes.

With the aid of the theorem due to Carlson in \[41\], we prove the rest entries of \( \Phi_2(\xi, s) \) also vanish. We follow similar line in \[46\], to construct a auxiliary function \( G_j(\xi), j = 1, 2 \), in the following,

\[ G_j(\xi) = \begin{cases} (\Phi_2(\xi, s))_{j1}, & \text{Im}\xi > 0, \\ (\Phi_2(\xi, s))_{j2}, & \text{Im}\xi < 0, \end{cases} \]

then \( G_j \) is analytic in \( \mathbb{C} \setminus (-\infty, s) \). By \((2.56)\), then \( G_j(\xi)(j = 1, 2) \) satisfy the jump condition

\[ G_{j+}(\xi) = G_{j-}(\xi) \begin{cases} e^{2\sqrt{\xi}-i(\alpha+\lambda)\pi}, & \xi \in (-\infty, 0), \\ e^{-i\lambda\pi}, & \xi \in (0, s). \end{cases} \]

We extend scalar function \( G_j(\xi) \) in the following for \( \xi \in (-\infty, 0) \),

\[ \hat{G}_j(\xi) = \begin{cases} G_j(e^{-i2\pi} \xi) e^{2\sqrt{\xi} - i(\alpha+\lambda)\pi}, & \pi \leq \arg\xi < 2\pi, \\ G_j(e^{i2\pi} \xi) e^{2\sqrt{\xi} e^{i(\alpha+\lambda)\pi}}, & -2\pi < \arg\xi \leq -\pi, \end{cases} \]
then \( \hat{G}_j(\xi) \) is analytic in a large sector \(-2\pi < \arg \xi < 2\pi \) and \( \xi \notin [0,s] \).

Let
\[
h_j(\xi) = \hat{G}_j((\xi + s + 1)^4), \quad \text{for } \Re \xi \geq 0,
\]
hence \( h_k(\xi) \) is analytic in \( \Re \xi > 0 \), continuous and bounded in \( \Re \xi \geq 0 \), and fulfills
\[
|h_j(\xi)| = O\left(e^{-|\xi|^2}\right), \quad \text{for } \arg \xi = \pm \pi \text{ and } |\xi| \to \infty,
\]
by Carlson’s theorem, one gets \( h_j(\xi) \equiv 0, j = 1,2 \) for \( \Re \xi \geq 0 \). Tracing back steps, \( (\tilde{\Phi}_2)_+(\xi,s) \) vanishes identically for \( \xi \in \mathbb{C} \setminus \mathbb{R} \). Then, \( \tilde{\Phi}(\xi,s) \equiv 0 \) by (2.51) and (2.55). This proves the vanishing lemma.

**Remark 3.** Note that the “free” parameter \( \lambda \) is valid for \( s > 0 \), and \( \alpha > 0 \) is also necessary. One checks by substituting (2.54) into (2.52) for \( \xi \in \hat{\Sigma}_2 \), and \( \xi \to 0 \), then one has
\[
(\xi)^{\lambda} = \begin{pmatrix}
e^{-2\sqrt{\xi}} & (|\xi| + s)^\lambda |\xi|^\alpha \\
0 & e^{2\sqrt{\xi}}
\end{pmatrix}, \quad (2.60)
\]
where \( O(1) \) is a bounded, and with determinant 1. From factors \((|\xi| + s)^\lambda \) and \(|\xi|^\alpha \) in the above matrix, the solvability of \( \tilde{\Phi}(\xi,s) \) confined for \( s > 0 \) and \( \alpha > 0 \).

### 3 Deift-Zhou steepest descent analysis

With the well-known relation presented in [22], the orthogonal polynomials with respect to the weight (1.4) can be characterized by a RH problem for \( Y \). We adopt the powerful Deift-Zhou steepest descent analysis method (or Riemann-Hilbert method) in [19], see also [18, 20], to analyze the RH problem for \( Y \). Following the standard process, we obtain a series of invertible transformations \( Y \to T \to S \to R \), at last, the matrix function \( R \) is close to the identity matrix. After that, taking a list of inverse transformations, then the kernel (1.1) associated with the weight (1.4) can be represented by the uniform asymptotics of the orthogonal polynomials in the complex plane for large \( n \).

#### 3.1 Riemann-Hilbert problem for \( Y \)

The orthogonal polynomials with respect to the weight function \( w(x,t) \) in (1.4) are described by the following \( 2 \times 2 \) matrix valued function \( Y(z) \).
(a) \( Y(z) \) is analytic for \( z \in \mathbb{C} \setminus [0, \infty) \).
(b) \( Y(z) \) satisfies the jump condition
\[
Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w(x, t) \\ 0 & 1 \end{pmatrix}, \quad \text{for} \quad x \in (0, +\infty),
\]
where \( w(x, t) \) is given by (1.4).
(c) The asymptotic behavior of \( Y(z) \) at infinity is
\[
Y(z) = \left( I + \mathcal{O}(z^{-1}) \right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}, \quad z \to \infty.
\]
(d) The asymptotic behavior of \( Y(z) \) at the origin is
\[
Y(z) = \begin{pmatrix} O(1) & O(1) \\ O(1) & O(1) \end{pmatrix}, \quad z \to 0.
\]

By the work of Fokas, Its and Kitaev [22], the above RH problem for \( Y(z) \) has a unique solution,
\[
Y(z) = \begin{pmatrix} \pi_n(z) \\ -2\pi i \gamma_2 \pi_{n-1}(z) \end{pmatrix} \begin{pmatrix} \frac{1}{2\pi i} \int_s^\infty \frac{\pi_n(s)w(s)}{s-z} ds \\ -\gamma_2 \int_s^\infty \frac{\pi_{n-1}(s)w(s)}{s-z} ds \end{pmatrix}, \quad (3.61)
\]
where \( \pi_n(z) \) is the monic polynomial, and \( P_n(z) = \gamma_n \pi_n(z) \) is the orthonormal polynomial with respect to the weight \( w(x, t) \), see (1.4).

### 3.2 Riemann-Hilbert problem for \( T \)

In order to normalize the matrix function \( Y(z) \) at infinity, we rescale the variable and introduce the first transformation \( Y \to T \), and \( T \) is defined as follows
\[
T(z) = (4n)^{-\left(n+\frac{\alpha+\lambda}{2}\right)}e^{-\frac{n}{2}\lambda}Y(4nz)e^{-n(g(z)-\frac{\ell}{2})\lambda} (4n)^{\frac{\alpha+\lambda}{2}}, \quad (3.62)
\]
for \( z \in \mathbb{C} \setminus [0, +\infty) \), where \( \ell = -2(1 + 2\ln 4) \) is the Euler-Laguerre constant.

Then \( T(z) \) solves the following RH problem,
(a) \( T(z) \) is analytic for \( z \in \mathbb{C} \setminus [0, \infty) \).
(b) \( T(z) \) satisfies the following jump condition,
\[
T_+(x) = T_-(x) \begin{pmatrix} e^{n(g_-(x)-g_+(x))}x^\alpha \\ 0 \end{pmatrix} \begin{pmatrix} x + \frac{\gamma}{4n}\lambda \quad e^{n(g_+(x)+g_-(x)-4x-\ell)} \\ 0 \end{pmatrix}, \quad x \in (0, +\infty). \quad (3.63)
\]
(c) The asymptotic behavior of $T(z)$ at infinity is
\[ T(z) = I + O(z^{-1}), \quad z \to \infty. \]  
(3.64)

(d) The asymptotic behavior of $T(z)$ at the origin is
\[ T(z) = \begin{pmatrix} O(1) & O(1) \\ O(1) & O(1) \end{pmatrix}, \quad z \to 0. \]  
(3.65)

For the classical Laguerre weight $x^\alpha e^{-4x}$, $x > 0$, $\alpha > -1$, the equilibrium measure is given by
\[ \mu(x) = \frac{2}{\pi} \sqrt{\frac{1-x}{x}}, \quad 0 < x < 1, \text{ see } (1.3), \text{ which is independent of } \alpha, \text{ see } [45, 40]. \]

From (3.72), we can construct $g(z)$ with the equilibrium measure of the classical Laguerre weight for large $n$ and achieve the normalization of $Y(z)$ at infinity. Similar method see [46]. We define several auxiliary functions and refer [40].

\[ g(z) = \int _0^1 \log(z - x)\mu(x)dx, \]  
(3.66)

where $\arg(z - x) \in (-\pi, \pi)$, and

\[ \phi(z) = 2 \int_0^z \sqrt{\frac{s-1}{s}} ds, \quad z \in \mathbb{C} \setminus [0, +\infty), \]  
(3.67)

where $\arg z \in (0, 2\pi)$.

From the definition of $g(z)$ and $\phi(z)$, the following properties are easily to check,

\[ g_+(x) = g_-(x), \quad x \in (1, +\infty), \]  
(3.68)

\[ g_+(x) + g_-(x) - 4x - \ell = -2\phi(x), \quad x \in (1, +\infty), \]  
(3.69)

and

\[ g_+(x) - g_-(x) = 2\pi i - 2\phi_+(x) = 2\pi i + 2\phi_-(x), x \in (0, 1), \]  
(3.70)

\[ g_+(x) + g_-(x) - 4x - \ell = 0, \quad x \in (0, 1), \]  
(3.71)

where $\ell$ is the Euler-Lagrange constant. With the aid of the above properties of $g(z)$ and $\phi(z)$, see (3.68), (3.69), (3.70) and (3.71), then the jump conditions for $T$ in (3.72) can be expressed in terms of $\phi$ and simplify as follows

\[ T_+(x) = T_-(x) \begin{cases} 
\left( e^{2n\phi_+(x)} \begin{pmatrix} x^\alpha \left(x + \frac{t}{2n}\right)^\lambda \\ 0 \\ 1 \begin{pmatrix} x^\alpha \left(x + \frac{t}{2n}\right)^\lambda \\ 0 \end{pmatrix} e^{-2n\phi(x)} \right), & x \in (0, 1), \\
\left( e^{2n\phi_-(x)} \begin{pmatrix} 0 \\ x^\alpha \left(x + \frac{t}{2n}\right)^\lambda \\ 0 \end{pmatrix} \right), & x \in (1, +\infty). 
\end{cases} \]  
(3.72)
In order to remove the oscillation diagonal entries in the above jump matrix for \( x \in (0, 1) \), the contour can be deformed as an opening lens with the matrix factorization. We define the following piecewise analytic function \( S(z) \).

### 3.3 Riemann-Hilbert problem for \( S \)

We introduce the second transformation \( T \to S \), and \( S(z) \) is defined as

\[
S(z) = \begin{cases} 
T(z), & \text{for } z \text{ outside the lens shaped region,} \\
T(z) \begin{pmatrix} -z^{-\alpha}(z + \frac{t}{4n})^{-\lambda} e^{2n\phi(z)} & 0 \\ 1 & 1 \end{pmatrix}, & \text{for } z \text{ in the upper lens region,} \\
T(z) \begin{pmatrix} z^{-\alpha}(z + \frac{t}{4n})^{-\lambda} e^{2n\phi(z)} & 0 \\ 1 & 1 \end{pmatrix}, & \text{for } z \text{ in the lower lens region,}
\end{cases}
\]  

(3.73)

where \( \text{arg } z \in (-\pi, \pi) \).

Combining the conditions (3.72), (3.64) and (3.65) of the RH problem for \( T \) and the definition (3.73) of \( S \), then \( S \) satisfies the following RH problem,

\( a \) \( S(z) \) is analytic in \( \mathbb{C} \setminus \bigcup_{k=1}^{3} \Sigma_k \cup (1, \infty) \), illustrated in Fig.2.

\( b \) \( S_+(z) = S_-(z) J_S \) for \( z \in \bigcup_{k=1}^{3} \Sigma_k \cup (1, \infty) \), where the jump \( J_S \) is

\[
J_S(z) = \begin{cases} 
\begin{pmatrix} -z^{-\alpha}(z + \frac{t}{4n})^{-\lambda} e^{2n\phi(z)} & 0 \\ 1 & 1 \end{pmatrix}, & \text{for } z \in \Sigma_1 \cup \Sigma_3, \\
\begin{pmatrix} 0 & x^\alpha \left( x + \frac{t}{4n} \right) \lambda \\ -x^{-\alpha} \left( x + \frac{t}{4n} \right)^{-\lambda} & 0 \end{pmatrix}, & \text{for } z = x \in \Sigma_2 = (0, 1), \\
\begin{pmatrix} 1 & x^\alpha \left( x + \frac{t}{4n} \right) \lambda e^{2n\phi(x)} \\ 0 & 1 \end{pmatrix}, & \text{for } z = x \in (1, +\infty).
\end{cases}
\]  

(3.74)

\( c \) The asymptotic behavior at infinity is

\[ S(z) = I + \mathcal{O}(z^{-1}), \text{ as } z \to \infty. \]
The asymptotic behavior at the origin is

\[ S(z) = O(1) \]

\[ \begin{cases} 
I, & \text{outside the lens region,} \\
\left( -z^{-\alpha} \left( z + \frac{t}{4n} \right)^{-\lambda} e^{2n\phi(z)} 0 \right), & \text{in the upper lens region,} \\
\left( -z^{-\alpha} \left( z + \frac{t}{4n} \right)^{-\lambda} e^{2n\phi(z)} 0 \right), & \text{in the lower lens region.}
\end{cases} \] \hspace{1cm} (3.75)

\[ \text{(Figure 2. The contour } \bigcup_{k=1}^3 \Sigma_k \cup (1,\infty) \text{ for } S(z)). \]

\[ 3.4 \text{ Global parametrix} \]

For \( n \to \infty \), due to the exponentially small term in (3.74), the jump matrix \( J_S \) on \( \Sigma_1, \Sigma_3 \) and \( (1,\infty) \) converges to the identity matrix very quickly. Moreover, \( J_S \) on \( \Sigma_2 \), the items \( (x + t/4n)^{\pm\lambda} = x^{\pm\lambda} + O(n^{-1}), \ x > 0 \), and \( t \in (0,c], \ 0 < c < \infty \), as \( n \to \infty \). Then \( S(z) \) can be approximated by a solution of the following RH problem for \( P^{(\infty)}(z) \).

(a) \( P^{(\infty)}(z) \) is analytic in \( \mathbb{C} \setminus [0,1] \).

(b) \( P^{(\infty)}(z) \) satisfies the jump condition,

\[ P_+^{(\infty)}(x) = P_-^{(\infty)}(x) \begin{pmatrix} 0 & x^{\alpha+\lambda} \\ -x^{-\alpha+\lambda} & 0 \end{pmatrix}, \ \text{for } x \in \Sigma_2 = (0,1). \] \hspace{1cm} (3.76)

(c) The asymptotic behavior at infinity is

\[ P^{(\infty)}(z) = I + O\left( z^{-1} \right), \ z \to \infty. \]
In the spirit of \cite{17} and \cite{34}, the unique solution to the above RH problem can be constructed as follows

\[ P^{(\infty)}(z) = D(\infty)^{\sigma_3} M^{-1} a^{-\sigma_3}(z) M D^{\sigma_3}(z), \text{ for } z \in \mathbb{C} \setminus [0,1], \]  

(3.77)

where \( M = (I + i\sigma_1)/\sqrt{2} \), \( a(z) = ((z-1)/z)^{\frac{1}{4}} \), \( D(\infty) = 2^{-(\alpha+\lambda)} \), \( D(z) = (\sqrt{z}/(\sqrt{z}+\sqrt{z-1}))^{-(\alpha+\lambda)} \) is the Szegö function, such that \( D_+(x)D_-(x) = x^{\alpha+\lambda} \) for \( x \in (0,1) \). The branches in the above are chosen as \( \text{arg } z \in (-\pi,\pi) \) and \( \text{arg}(z-1) \in (-\pi,\pi) \).

Then

\[ S(z)P^{(\infty)}(z)^{-1} = I + \mathcal{O}(n^{-1}), \quad n \to \infty, \]  

(3.78)

where the error term is uniform for \( z \) away from the end points 0, 1 and \( t \in (0,c] \), \( c \) is a positive and finite constant. Due to the factor \( a(z) \) in (3.77) with fourth root singularities at \( z = 0, 1 \), the jump matrices \( S(z)P^{(\infty)}(z) \) are not uniformly close to the unit matrix in the neighborhood of 0 and 1. So, it’s need to construct the local parametrices in the neighborhoods of these points.

**Remark 4.** Let

\[ N(z) = P^{(\infty)}(z) \left( \frac{z + \frac{1}{4n}}{z} \right)^{-\frac{\lambda}{2}\sigma_3}, \]  

(3.79)

where \( \text{arg } z \in (-\pi,\pi) \), \( \text{arg}(z + t/4n) \in (-\pi,\pi) \), and \( P^{(\infty)}(z) \) is given by (3.77). Then \( N(z) \) satisfies the same RH problem for \( P^{(\infty)}(z) \) and the jump condition (3.76) replaced by

\[ N_+(x) = N_-(x) \left\{ \begin{array}{ll}
             e^{i\lambda\pi\sigma_3}, & x \in (-\frac{1}{4n},0), \\
             \left( \begin{array}{cc}
             0 & x^\alpha \left( x + \frac{1}{4n} \right)^\lambda \\
             -x^{-\alpha} \left( x + \frac{1}{4n} \right)^{-\lambda} & 0
            \end{array} \right), & x \in (0,1).
            \end{array} \right. \]  

(3.80)

Moreover, from (3.77) and (3.79), one finds,

\[ P^{(\infty)}(z)N(z) = I + \mathcal{O}(n^{-1}), \quad z \in \mathbb{C} \setminus \{U(0,r) \cup U(1,r)\}, \text{ and } r > t/4n, \]  

(3.81)

where the error term is uniform for \( z \in \mathbb{C} \setminus \{U(0,r) \cup U(1,r)\} \), \( r > t/4n, t \in (0,c] \) and \( c \) is a positive constant.
3.5 Local parametrix \(P^{(0)}(z)\) at \(z = 0\)

We construct the local parametrix \(P^{(0)}(z)\) in the neighborhood \(U(0, r) = \{z \in \mathbb{C} : |z| < r\}\) for small \(r\), and \(r > t/4n\). The parametrix \(P^{(0)}(z)\) satisfies a RH problem as follows

(a) \(P^{(0)}(z)\) is analytic in \(U(0, r) \setminus \bigcup_{k=1}^{3} \Sigma_k\), see Figure 2.

(b) \(P^{(0)}(z)\) has the same jump conditions with \(S(z)\) on \(U(0, r) \cap \Sigma_k, k = 1, 2, 3\), see \[3.71\].

(c) For \(z \in \partial U(0, r)\), \(P^{(0)}(z)\) satisfies the following matching condition,

\[
P^{(0)}(z)P^{(\infty)^{-1}}(z) = I + O\left(n^{-\frac{1}{2}}\right), \quad n \to \infty.
\]  \[3.82\]

(d) The asymptotic behavior of \(P^{(0)}(z)\) at \(z = 0\) is the same as \(S(z)\) in \[3.75\].

In order to convert the jump of the \(P^{(0)}(z)\) to constant jump, a transformation is defined as

\[
\tilde{P}^{(0)}(z) = P^{(0)}(z) e^{-n\phi(z)\sigma_3} (-z)^{\frac{\lambda \sigma_3}{2}} \left(\frac{z + \frac{t}{4n}}{-z}\right)^{\frac{\lambda \sigma_3}{2}}, \quad z \in U(0, r) \setminus \Sigma_k, \quad k = 1, 2, 3,
\]  \[3.83\]

where \(\arg(-z) \in (-\pi, \pi)\), \(\arg(-z + t/4n) \in (-\pi, \pi)\), and \(\phi(z)\) is given by \[3.67\]. Here, we set that \(\arg(-z) = \mp\pi\) on the positive and negative side of \(\Sigma_2\), respectively, and \(\arg(-z + t/4n) = \mp\pi\) as \(z\) is form above and below of \((-t/4n, 0)\), respectively.

Obviously, \(\tilde{P}^{(0)}(z)\) satisfies the following RH problem.

(a) \(\tilde{P}^{(0)}(z)\) is analytic for \(z \in U(0, r) \setminus \bigcup_{k=1}^{3} \gamma_k \cup (-\frac{t}{4n}, 0)\), see Figure 3.

(b) \(\tilde{P}^{(0)}(z)\) fulfills the following constant jumps,

\[
\tilde{P}^{(0)}_+(z) = \tilde{P}^{(0)}_-(z) \begin{cases} 
  e^{-i\lambda_2 \sigma_3}, & z \in (-\frac{t}{4n}, 0), \\
  \left(\begin{array}{cc} 1 & 0 \\ e^{-i(\lambda+\alpha)\pi} & 1 \end{array}\right), & z \in \gamma_3 \cap U(0, r), \\
  \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right), & z \in \gamma_2 \cap U(0, r), \\
  \left(\begin{array}{cc} 1 & 0 \\ e^{i(\lambda+\alpha)\pi} & 1 \end{array}\right), & z \in \gamma_1 \cap U(0, r).
\end{cases}
\]  \[3.84\]

(c) The asymptotic behavior of \(\tilde{P}^{(0)}(z)\) at the origin in the following four domains \(\Omega_k, k = 1, \ldots, 4\), illustrated in Figure 3,
\[ (c_1) \quad z \in \Omega_1, \]
\[ \tilde{P}^{(0)}(z) = \mathcal{O}(1)e^{-n\phi(z)\sigma_3}(-z)^\frac{\sigma_3}{2}(-\frac{z + t}{4n})^\frac{\sigma_3}{2}, \quad z \to 0, \]

\[ (c_2) \quad z \in \Omega_2, \]
\[ \tilde{P}^{(0)}(z) = \mathcal{O}(1)e^{-n\phi(z)\sigma_3}(-z)^\frac{\sigma_3}{2}(-\frac{z + t}{4n})^\frac{\sigma_3}{2}\left(\begin{array}{cc}
1 & 0 \\
e^{i(\lambda + \alpha)\pi} & 1
\end{array}\right), \quad z \to 0, \]

\[ (c_3) \quad z \in \Omega_3, \]
\[ \tilde{P}^{(0)}(z) = \mathcal{O}(1)e^{-n\phi(z)\sigma_3}(-z)^\frac{\sigma_3}{2}(-\frac{z + t}{4n})^\frac{\sigma_3}{2}\left(\begin{array}{cc}
1 & 0 \\
e^{-i(\alpha + \lambda)\pi} & 1
\end{array}\right), \quad z \to 0, \]

\[ (c_4) \quad z \in \Omega_4, \]
\[ \tilde{P}^{(0)}(z) = \mathcal{O}(1)e^{-n\phi(z)\sigma_3}(-z)^\frac{\sigma_3}{2}(-\frac{z + t}{4n})^\frac{\sigma_3}{2}, \quad z \to 0. \]

(d) The behavior of \( \tilde{P}^{(0)}(z) \) at \( z = -\frac{t}{4n} \) is
\[ \tilde{P}^{(0)}(z) = \mathcal{O}(1)(-\frac{z + t}{4n})^\frac{\sigma_3}{2}. \]

Figure 3. Counters for the RH problem of \( \tilde{P}^{(0)}(z) \) in the \( z \) plane

From (3.67), \( \xi \) is defined as
\[ \xi = n^2 \phi^2(z) = -16n^2 z \left(1 - \frac{z}{3} - \frac{z^2}{45} + \mathcal{O}(z^3) \right), \quad (3.85) \]
Here, we introduce the last transformation $S$, the Riemann-Hilbert problem for presented in Sect. 5 below. For $z \in \mathbb{C} \setminus \{U(0, r) \cup U(1, r) \cup \Sigma_S\}$, we have that $E(z)$ is given by

$$E(z) = N(z)e^{\frac{4i\sigma_3}{3}}(-z)^{\alpha\frac{4\pi}{3}\sigma_3} \left(-\frac{z + \frac{i}{40}}{-z}\right)^{\frac{i\sigma_3}{2}} \frac{I - i\sigma_1}{\sqrt{2}} (n^2\phi^2(z))^{\frac{i\sigma_3}{2}},$$

where $N(z)$ is given by (3.79), and (3.85) and (2.32), one can verify the matching condition (3.82), see also (3.84) and the matching condition (3.82), $\tilde{E}$ and (2.32), E(z) is given by

$$\tilde{E}(z) = E(z)\Phi(n^2\phi^2(z), 4nt)e^{-\frac{4}{3}n\sigma_3},$$

where the factor $e^{-\frac{4}{3}n\sigma_3}$ compensates for the reversed orientation of $\xi \sim -16n^2z$, see (3.85) and (2.32). For $z \in U(0, r)$, since $E(z)$ is analytic in $U(0, r) \setminus (-t/4n, 0) \cup \gamma_2$. It derives directly that $E_+(x) = E_-(x)$ for $x \in (-t/4n, 0) \cup \gamma_2$ from (3.77), (3.79) and (3.87), and $E(z)$ has a weak singularity at origin.

### 3.6 Local parametrix $P^{(1)}(z)$ at $z = 1$

For $z \in U(1, r)$, $U(1, r) = \{z : |z - 1| < r\}$, and $r$ is sufficiently small and positive, then the local parametric $P^{(1)}(z)$ satisfies the following RH problem.

(a) $P^{(1)}(z)$ is analytic in $U(1, r) \setminus \Sigma_S$, $\Sigma_S$ denotes the contour of $S$, see Figure 2.

(b) $P^{(1)}(z)$ shares the jump conditions of $S(z)$ in $U(1, r)$, see (3.74).

(c) For $z \in \partial U(1, r)$, $P^{(1)}(z)$ satisfies the matching condition as follows

$$P^{(1)}(z)P^{(\infty)-1}(z) = I + O\left(n^{-1}\right), \text{ as } n \to \infty. \quad (3.88)$$

In fact, $P^{(1)}(z)$ can be expressed in terms of the Airy function and its derivatives, the detail presented in Sect. 5 below.

### 3.7 Riemann-Hilbert problem for $R$

Here, we introduce the last transformation $S \to R$, and $R(z)$ is defined as follows

$$R(z) = \begin{cases} 
S(z)P^{(\infty)-1}(z), & z \in \mathbb{C} \setminus \{U(0, r) \cup U(1, r) \cup \Sigma_S\}, \\
S(z)P^{(0)-1}(z), & z \in U(0, r) \setminus \Sigma_S, \\
S(z)P^{(1)-1}(z), & z \in U(1, r) \setminus \Sigma_S. 
\end{cases} \quad (3.89)$$
Hence, $R(z)$ satisfies the following RH problem.

(a) $R(z)$ is analytic in $\mathbb{C} \setminus \Sigma_R$, see Figure 4.

(b) $R(z)$ satisfies the jump conditions

$$R_+(z) = R_-(z) \begin{cases} P^{(\infty)}(z)J_S(z)P^{(\infty)-1}(z), & z \in \Sigma_R \setminus \{\partial U(0, r) \cup \partial U(1, r)\}, \\ P^{(0)}(z)P^{(\infty)-1}(z), & z \in \partial U(0, r), \\ P^{(1)}(z)P^{(\infty)-1}(z), & z \in \partial U(1, r), \end{cases} \tag{3.90}$$

where $J_S$ denotes the jump matrices in (3.74).

(c) For $z \to \infty$, $R(z) = I + O\left(\frac{n}{z}\right)$.

By (3.74), (3.90) and the matching conditions on the boundaries (3.78), (3.82), (3.88) and $\phi(z)$ in (3.67), one finds,

$$J_R(z) = \begin{cases} I + O\left(n^{-\frac{1}{2}}\right), & z \in \partial U(0, r) \cup \partial U(1, r), \\ I + O\left(n^{-1}\right), & z \in \tilde{\Sigma}_2, \\ I + O\left(e^{-cn}\right), & z \in \Sigma_R \setminus \{\partial U(0, r) \cup \partial U(1, r) \cup \tilde{\Sigma}_2\}, \end{cases} \tag{3.91}$$

where $c$ is a positive constant, and the error term is uniform for $z \in \Sigma_R$. Then one finds,

$$\|J_R(z) - I\|_{L^2 \cap L^\infty(\Sigma_R)} = O\left(n^{-\frac{1}{2}}\right). \tag{3.92}$$

Furthermore,

$$R(z) = I + \frac{1}{i2\pi} \oint_{\Sigma_R} \frac{R_-(\tau)(J_R(\tau) - I)}{\tau - z} d\tau, \quad z \notin \Sigma_R, \quad \tag{3.93}$$

by the method and procedure of norm estimation of Cauchy operator as show in [17, 18], one has,

$$R(z) = I + O\left(n^{-\frac{1}{2}}\right), \tag{3.94}$$

where the error term is uniform for $z \in \mathbb{C}$.

**Remark 5.** In the double scaling process, let $n \to \infty$, $t \to 0$, and $s = 4nt$ such that $s \in (0, \infty)$. In fact $t$ can be taken in the interval $(0, c]$ with $c$ is finite and positive, for $s \to \infty$, then the error term (3.91) uniformly $O\left(n^{-\frac{1}{2}}\right)$ for $z \in \mathbb{C}$. This can be verified from calculations, see (4.152).
It now completes the nonlinear steepest descent analysis of $Y \rightarrow T \rightarrow S \rightarrow R$. By a list of inverse transformations, we will show the large-$n$ asymptotic behavior of the kernel in the below sections.

\[ \Sigma_3 \]
\[ \Sigma_2 \]
\[ \Sigma_1 \]

Figure 4. The contour for $R(z)$.

3.8 Proof of Theorem 1

In this subsection, we prove that the Painlevé V Kernel at the hard edge.

Proof. The kernel in (1.1) associated with the perturbed Laguerre weight $w(x, t)$ in (1.4) denoted as $K_n(x, y; t)$ and by the Christoffel–Darboux formula [42], it has a simple closed form as follows

\[
K_n(x, y; t) = (w(x, t))^{\frac{1}{2}} (w(y, t))^{\frac{1}{2}} \frac{\pi_n(x)\pi_{n-1}(y) - \pi_n(y)\pi_{n-1}(x)}{(x-y)h_{n-1}},
\]

where $h_{n-1}$ is the square of the $L^2$ norm, see [1.2]. The above kernel can also expressed by $Y(z)$ in (3.61),

\[
K_n(x, y; t) = \frac{(w(x, t)w(y, t))^{\frac{1}{2}}}{i2\pi(x-y)} (Y_+^{-1}(y)Y_+(x))_{21}.
\]

From the first transform (3.62) and (1.3), the support of the equilibrium measure is $[0, 4n]$ and $x, y \in [0, 4n]$ in the kernel (3.96). In order to normalize the support of the equilibrium measure $[0, 4n]$ as $[0, 1]$, so we consider the following re-scaled kernel,

\[
4nK_n(4nx, 4ny; t) = \frac{(w(4nx, t)w(4ny, t))^{\frac{1}{2}}}{i2\pi(x-y)} (Y_+^{-1}(4ny)Y_+(4nx))_{21}
\]

\[
= \frac{1}{i2\pi(x-y)} \begin{pmatrix} 0 & 1 \end{pmatrix} (w(4ny, t))^{\frac{1}{2}} Y_+^{-1}(4ny)Y_+(4nx)w(4nx, t)^{\frac{1}{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]
Taking the inverse transformations from $Y$ to $R$, with the aid of (3.62), (3.73), (3.83) and (3.86), one finds,

$$Y_+(4nx)u^2(4nx, t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = M_0 R(x) E(x) \Phi_- (n^2 \phi^2(x), s) e^{i\pi (\alpha + \lambda - 1) \sigma_3} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad (3.98)$$

and

$$0 \\ 1 \end{pmatrix} w^2(4ny) Y_+^{-1}(4ny, t) = \begin{pmatrix} (-1) & 1 \\ 1 & (-1) \end{pmatrix} e^{-i\pi (\alpha + \lambda - 1) \sigma_3} \Phi_-^{-1} (n^2 \phi^2(y), s) E^{-1}(y) R^{-1}(y) M_0^{-1}, \quad (3.99)$$

where $M_0 = (-1)^n(4n)(n + \frac{\alpha + \lambda}{2})^{\sigma_3} e^{\frac{i\pi}{2} \sigma_3}$, $s = 4nt$ is finite, and one has been used the fact

$$e^{n(g_+(x) + \phi_+(x) - \frac{\alpha}{2})} = e^{2nx + in\pi} = (-1)^n e^{2nx},$$

which follows from (3.70) and (3.71).

Let

$$\begin{pmatrix} \phi_1 (n^2 \phi^2(x), s) \\ \phi_2 (n^2 \phi^2(x), s) \end{pmatrix} = \Phi_- (n^2 \phi^2(x), s) e^{i\pi (\alpha + \lambda - 1) \sigma_3} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad (3.100)$$

by (3.97), (3.98) and (3.99), then the kernel can be rewritten as

$$4n K_n(4nx, 4ny; s) = \frac{\begin{pmatrix} -\phi_2 (n^2 \phi^2(y), s) \\ \phi_1 (n^2 \phi^2(y), s) \end{pmatrix}^T E^{-1}(y) R^{-1}(y) R(x) \begin{pmatrix} \phi_1 (n^2 \phi^2(x), s) \\ \phi_2 (n^2 \phi^2(x), s) \end{pmatrix}}{i2\pi(x-y)}, \quad (3.101)$$

where $Z^T$ represents the transpose of $Z$. By (3.85), the variable $x$, and $y$ rescaled as $x = \frac{u}{16n^2}, \quad y = \frac{v}{16n^2}$, and $u, v \in (0, +\infty)$, then we have,

$$n^2 \phi^2(x) = -u \left(1 + O \left(n^{-2}\right)\right), \quad n^2 \phi^2(y) = -v \left(1 + O \left(n^{-2}\right)\right). \quad (3.102)$$

While $\varphi_k (\xi, s), \quad \xi \in (-\infty, 0), \quad k = 1, 2$, which can be extended as analytic functions, so one finds,

$$\varphi_k (n^2 \phi^2(x), s) = \varphi_k (-u, s) + O \left(n^{-2}\right), \quad \varphi_k (n^2 \phi^2(y), s) = \varphi_k (-v, s) + O \left(n^{-2}\right), \quad k = 1, 2. \quad (3.103)$$

Further more, $E(z)$ is analytic in $U(0, r)$, one finds,

$$E^{-1}(y) E(x) = I + O \left(x - y\right) = I + O \left(n^{-2}\right), \quad (3.104)$$
and \( R(z) \) is also analytic in \( U(0, r) \), and

\[
R^{-1}(y)R(x) = I + \mathcal{O}(x - y) = I + \mathcal{O}(n^{-2}). \tag{3.105}
\]

Substituting (3.102), (3.103), (3.104) and (3.105) into (3.101), then the re-scaled kernel for large \( n \) as follows

\[
\frac{1}{4n}K_n\left(\frac{u}{4n}, \frac{v}{4n}; s\right) = \frac{\varphi_1(-v, s)\varphi_2(-u, s) - \varphi_1(-u, s)\varphi_2(-v, s)}{i2\pi(u - v)} + \mathcal{O}\left(\frac{1}{n^2}\right) \tag{3.106},
\]

where the error term is uniform for \( u, v \in (0, +\infty) \) and \( s \in (0, +\infty) \). As \( n \to \infty \), then (1.25) follows from (3.106).

By the unique solution of the RH problem (2.31), (2.32), (2.33) and (2.34) for \( \Phi(\xi, s) \) in the \( \xi \) plane, we extend (3.100) as follows

\[
\begin{pmatrix}
\varphi_1(\xi, s) \\
\varphi_2(\xi, s)
\end{pmatrix} = \Phi(\xi, s)e^{i\pi(\alpha + \lambda - 1)\frac{2}{\sigma_3}}\begin{pmatrix}1 \\ 1\end{pmatrix}, \tag{3.107}
\]

where \( \xi \in \hat{\Omega}_3 \), see Figure 1. Substituting (3.107) into the Lax pair (1.8) and (1.9) for \( \Phi(\xi, s) \), and eliminate \( \varphi_1(\xi, s) \) or \( \varphi_2(\xi, s) \), then it follows the second order differential equation in (1.26). By the initial conditions of \( r(0) \) and \( r'(0) \) are in (1.16), (1.17), respectively, and set \( s = 0 \) in (1.26), then the differential equation reduces to (1.27).

\[\square\]

## 4 Painlevé V kernel to Bessel kernels as \( s \to 0^+ \) and \( s \to \infty \)

In this section, we show that the Painlevé V kernel (1.25) translates to the Bessel kernel \( J_{\alpha+\lambda} \) and the Bessel kernel \( J_{\alpha} \) as \( s \to 0^+ \) and \( s \to \infty \), respectively.

### 4.1 Painlevé V kernel to the Bessel kernel \( J_{\alpha+\lambda} \) as \( s \to 0^+ \)

If \( s \to 0 \), the factor \((\xi - s)\lambda\) merges \( \xi^\alpha \) as \( \xi^{\alpha+\lambda} \) in the RH problem for \( \Phi(\xi, s) \), see (2.31) - (2.34), and the jump on the interval \((0, s)\) vanishes. Hence, the RH problem for \( \Phi(\xi, s) \) can be approximated by the following limiting RH problem for \( \Phi_0(\xi, s) \).

(a) \( \Phi_0(\xi) \) is analytic in \( \mathbb{C} \setminus \bigcup_{j=1}^{3} \tilde{\Sigma}'_j \), and the contour see Figure 5.
(b) \( \Phi_0(\xi) \) fulfills the jump relation

\[
(\Phi_0)_+(\xi) = (\Phi_0)_-(\xi) = \begin{cases}
(0 \ 1 \\
1 \ 0)
& , \xi \in \tilde{\Sigma}'_1, \\
(e^{\pi(\lambda+\alpha)i} \ 0 \\
0 \ e^{-\pi(\lambda+\alpha)i})
& , \xi \in \tilde{\Sigma}'_2, \\
(1 \ 0)
& , \xi \in \tilde{\Sigma}'_3.
\end{cases}
\text{(4.108)}
\]

(c) For \( \xi \to \infty \),

\[
\Phi_0(\xi) = \left(I + \mathcal{O}\left(\frac{1}{\xi}\right)\right)\xi^{\frac{1}{4}\sigma_3}I + \frac{i\sigma_1}{\sqrt{2}}e^{i\sqrt{2}\sigma_3},
\text{(4.109)}
\]

where the argument takes as \( \arg \xi \in (-\pi, \pi) \).

The modified Bessel functions are used to construct the local parametrix at origin, we refer \cite{34}, more information about the modified Bessel function can be found in \cite{3, 39}. The unique solution of the RH problem (4.108)-(4.109) for \( \Phi_0(\xi) \) can be constructed in terms of
the modified Bessel functions \( I_\nu(z) \) and \( K_\nu(z) \) as follows

\[
\Phi_0(\xi) = M_0 \begin{pmatrix}
I_{\alpha+\lambda} \left( \xi^{\frac{2}{\lambda}} \right) & i\frac{1}{\pi} K_{\alpha+\lambda} \left( \xi^{\frac{2}{\lambda}} \right) \\
i\pi \xi^{\frac{2}{\lambda}} I'_{\alpha+\lambda} \left( \xi^{\frac{2}{\lambda}} \right) & -\xi^{\frac{2}{\lambda}} K'_{\alpha+\lambda} \left( \xi^{\frac{2}{\lambda}} \right)
\end{pmatrix}, \quad \xi \in \hat{\Omega}_1 \cup \hat{\Omega}_4',
\]

(4.110)

where the constant matrix \( M_0 \) is given by

\[
M_0 = \left( (4(\alpha + \lambda)^2 + 3) i\sigma / 8 + i \right) \pi \frac{\xi^{2\sigma}}{}.
\]  

(4.111)

From (4.110), \( \Phi_0(\xi) \) has the following asymptotic behavior at infinity,

\[
\Phi_0(\xi) = \left( I + C_0 \xi^{-1} + O \left( \xi^{-\frac{3}{2}} \right) \right) \xi^{-\frac{\lambda}{2}} I \frac{\xi^{\sigma}}{\sqrt{2}} e^{i\pi \sigma}, \quad \xi \rightarrow \infty,
\]  

(4.112)

where

\[
C_0 = \frac{4(\alpha + \lambda)^2 - 1}{128} \left( \frac{4(\alpha + \lambda)^2 - 9}{4(\alpha + \lambda)^2 - 13} \right) i16 \left( \frac{9 - 4(\alpha + \lambda)^2}{9} \right).
\]  

(4.113)

These can be verified by the following asymptotic behavior of the modified Bessel functions [1], see also [3, 39],

\[
K_\nu(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z} \left( 1 + \sum_{j=1}^{\infty} \frac{(-1)^j(v + 1/2)j}{(2z)^j j!} \right), \quad |\arg z| < \frac{3\pi}{2},
\]

\[
I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}} \sum_{j=0}^{\infty} \frac{(v + 1/2)j}{(2z)^j j!}, \quad -\frac{\pi}{2} < \arg z < \frac{3\pi}{2},
\]

where

\[
(a)_n = a(a + 1)(a + 2) \ldots (a + n - 1), \quad (a)_0 = 1.
\]

Remark 6. Comparing (2.32) with (4.112), we note that \( C_1(s) \big|_{s=0} = C_0 \) provides the initial values of functions \( r(s), q(s), \) and \( t(s) \) in the Lax pair (1.8)-(1.9), then one finds,

\[
r(0) = \frac{1 - 4(\alpha + \lambda)^2}{8}, \quad q(0) = \frac{(4(\alpha + \lambda)^2 - 1)(4(\alpha + \lambda)^2 - 9)}{128},
\]  

(4.114)
\[
t(t) = \frac{(4(\alpha + \lambda)^2 - 1)(4(\alpha + \lambda)^2 - 9)(4(\alpha + \lambda)^2 - 13)}{1536}.
\]

(4.115)

If \( \alpha + \lambda > 0 \) and \( \alpha + \lambda \notin \mathbb{N} \), substituting the initial data (4.114) into (1.14) and let \( s = 0 \), then one finds,
\[
r'(0) = -\frac{\lambda}{2(\alpha + \lambda)}.
\]

(4.116)

We construct the local parametrix \( \Phi_1(\xi, s) \) in \( U(0, \varepsilon) \) with a fixed \( \varepsilon > 0 \), to approximate \( \Phi(\xi, s) \) and match \( \Phi_0(\xi) \) on the boundary \( \partial U(0, \varepsilon) \). By the different behaviors of the modified Bessel functions for \( \alpha + \lambda \notin \mathbb{N} \) and \( \alpha + \lambda \in \mathbb{N} \), we consider two cases separately.

For \( \alpha + \lambda \notin \mathbb{N} \), we split the modified Bessel matrix function in (4.110) as follows
\[
\begin{pmatrix}
I_{\alpha+\lambda}(\xi^{1/2}) & \frac{i\pi}{2} K_{\alpha+\lambda}(\xi^{1/2}) \\
i\pi\xi^{1/2} I'_{\alpha+\lambda}(\xi^{1/2}) & -\frac{i\pi}{2} K'_{\alpha+\lambda}(\xi^{1/2})
\end{pmatrix} = G(\xi) \xi^{\frac{\alpha+\lambda}{2} \sigma_3} \begin{pmatrix} 1 & 1 \\
\frac{1}{i \sin((\alpha+\lambda)\pi)} & 1 \end{pmatrix},
\]
(4.117)

where
\[
G(\xi) = \begin{pmatrix}
\xi^{-\frac{\alpha+\lambda}{2}} I_{\alpha+\lambda}(\xi^{1/2}) & \frac{i}{2} \sin((\alpha+\lambda)\pi) \xi^{\frac{\alpha+\lambda}{2}} I_{-(\alpha+\lambda)}(\xi^{1/2}) \\
i\pi\xi^{1-(\alpha+\lambda)/2} I'_{\alpha+\lambda}(\xi^{1/2}) & -\frac{i\pi}{2} \sin((\alpha+\lambda)\pi) \xi^{1+(\alpha+\lambda)/2} I'_{-(\alpha+\lambda)}(\xi^{1/2})
\end{pmatrix},
\]
(4.118)

and \( G(\xi) \) is an entire matrix function, one checks with the formulas,
\[
K_{\nu} = \frac{\pi (I_{-\nu}(z) - I_{\nu}(z))}{2 \sin(\nu \pi)}, \quad I_{\nu}(z) = (\frac{z}{2})^\nu \sum_{j=0}^{\infty} \frac{1}{\Gamma(j+1) \Gamma(j+\nu+1)} \left( \frac{z^2}{4} \right)^j, \quad \text{for } \nu \notin \mathbb{N}.
\]

With the aid of the decomposition in (4.117), we construct \( \Phi_1(\xi, s) \) in \( U(0, \varepsilon) \) in the following and \( s \ll \varepsilon \),
\[
\Phi_1(\xi, s) = G(\xi) \begin{pmatrix} 1 & f(\xi) \\
0 & 1 \end{pmatrix} \xi^{\frac{\alpha+\lambda}{2} \sigma_3} \begin{pmatrix} 1 & s \\
\xi \end{pmatrix} \frac{1}{2} \sigma_3 J,
\]
(4.119)

where \( \arg \xi \in (-\pi, \pi) \), \( G(\xi) \) is an entire matrix function, see (4.118), and the jump matrix \( J \) is given by
\[
J = \begin{cases}
I, & \xi \in \tilde{\Omega}_1 \cup \tilde{\Omega}_4, \\
\begin{pmatrix} 1 & 0 \\
e^{i(\lambda+\alpha)\pi} & 1 \end{pmatrix}, & \xi \in \tilde{\Omega}_2, \\
\begin{pmatrix} 1 & 0 \\
e^{-i(\lambda+\alpha)\pi} & 1 \end{pmatrix}, & \xi \in \tilde{\Omega}_3.
\end{cases}
\]
(4.120)
and the sectors $\hat{\Omega}_j, j = 1, \ldots, 4$ are described in Fig. 1.

Form the above expression of $\Phi_1(\xi, s)$ in (4.119), then $\Phi_1(\xi, s)$ satisfies the following RH problem,

(a) $\Phi_1(\xi, s)$ is analytic in $U(0, \varepsilon) \setminus \bigcup_{j=1}^{4} \hat{\Sigma}_j \cup (0, s)$, the contour see Figure 1.

(b) $\Phi_1(\xi, s)$ satisfies the same jump conditions in $U(0, \varepsilon)$ as $\Phi(\xi, s)$ in (2.31).

(c) For $\xi \to 0$, $\Phi_1(\xi, s)$ keeps the same asymptotic behavior of $\Phi(\xi, s)$ in (2.33).

(d) For $\xi \to s$, $\Phi_1(\xi, s)$ also has the same asymptotic behavior as $\Phi(\xi, s)$ in (2.34).

(e) For $\xi \in \partial U(0, \varepsilon)$ and $s \to 0$, $\Phi_1(\xi, s)$ and $\Phi_0(\xi)$ satisfies the following matching condition,

\[ \Phi_1(\xi, s) \Phi_0(\xi)^{-1} = I + O(s) + O(s^{\alpha+\lambda+1}), \quad \alpha + \lambda \notin \mathbb{N}, \quad \alpha + \lambda + 1 > 0, \]  

(4.121) and

\[ \Phi_1(\xi, s) \Phi_0(\xi)^{-1} = I + O(s) + O\left(s^{\alpha+\lambda+1} \log s\right), \quad \alpha + \lambda \in \mathbb{N}, \quad \alpha + \lambda + 1 > 0. \]  

(4.122)

Now we construct the scalar function $f(\xi)$ in (4.119). The scalar function $f(\xi)$ is analytic in $U(0, \varepsilon) \setminus (-\varepsilon, 0)$ and satisfies the following jump relation,

\[ f_+(\xi) - f_-(\xi) = |\xi|^\alpha |\xi - s|^{\lambda}, \quad \xi \in (-\varepsilon, 0), \]  

(4.123) and it is convenient to define $f(\xi)$ as follows

\[ f(\xi) = -\frac{1}{4\pi \sin (\alpha + \lambda)\pi} \int_\Gamma \frac{z^\alpha(z - s)^\lambda}{z - \xi} \, dz, \quad \xi \in U(0, \varepsilon) \setminus (-\varepsilon, 0), \ \arg \xi \in (-\pi, \pi), \]  

(4.124) where the integration contour $\Gamma$ is defined as the upper and lower line segments of $[-\varepsilon_2, 0]$ and the circle $|z| = \varepsilon_2, \varepsilon_2 > \varepsilon > 0$, see Figure 6.

Moreover, to give an estimation of $f(\xi)$ with the restrictions of $s < \varepsilon << \varepsilon_1 = |\xi| \leq \varepsilon < \varepsilon_2$. Furthermore, in order to use the Plemelj-Sokhotski formula, we split the original integration contour $\Gamma$ as three parts, see Figure 6, the first part is a closed integration path $\Gamma'$ which consists of the circle $|z| = \varepsilon_2$, the upper and lower line segments of $[-\varepsilon_2, -\varepsilon]$, and the circle $|z| = \epsilon$, the second part is the circle $|z| = \epsilon$, the third part is the upper and lower
line segments of $[-\epsilon, 0]$. Then the explicit expression rewrites as follows

$$f(\xi) = -\frac{1}{4\pi \sin (\alpha + \lambda)\pi} \int_{\Gamma'} \frac{z^\alpha(z-s)^\lambda}{z-\xi} dz + \frac{1}{4\pi \sin (\alpha + \lambda)\pi} \int_{|z|=\epsilon} \frac{z^\alpha(z-s)^\lambda}{z-\xi} dz$$

$$- \frac{1}{4\pi \sin (\alpha + \lambda)\pi} \left( \int_{-\epsilon}^{0} \left( \frac{z^\alpha(z-s)^\lambda}{z-\xi} \right)_+ dz + \int_{0}^{-\epsilon} \left( \frac{z^\alpha(z-s)^\lambda}{z-\xi} \right)_- dz \right)$$

$$= \frac{\xi^\alpha(\xi-s)^\lambda}{i2\sin (\alpha + \lambda)\pi} + \frac{i}{4\pi \sin (\alpha + \lambda)\pi} \lim_{\epsilon' \to 0} \int_{-\pi+\epsilon'}^{\pi-\epsilon'} \frac{(ee^{i\theta})^{\alpha+1}(ee^{i\theta}-s)^\lambda}{ee^{i\theta}-\xi} d\theta$$

$$+ \frac{1}{i2\pi} \int_{-\epsilon}^{0} \frac{|z|^\alpha|z-s|^\lambda}{z-\xi} dz. \quad (4.125)$$

After some calculations, and set $\epsilon = 2s$, one has

$$f(\xi) = \frac{\xi^{\alpha+\lambda}}{i2\sin (\alpha + \lambda)\pi} \left( 1 + \mathcal{O} \left( \frac{s}{\epsilon_1} \right) + \mathcal{O} \left( \left( \frac{s}{\epsilon_1} \right)^{\alpha+\lambda+1} \right) \right), \quad 2s = \epsilon \ll |\xi| = \epsilon_1 \leq \epsilon, \quad (4.126)$$

valid for $\alpha + \lambda + 1 > 0$, where we have used the following estimations,

$$\left| \lim_{\epsilon' \to 0} \int_{-\pi+\epsilon'}^{\pi-\epsilon'} \frac{(ee^{i\theta})^{\alpha+1}(ee^{i\theta}-s)^\lambda}{ee^{i\theta}-\xi} d\theta \right| \leq \frac{2\pi\epsilon^{\alpha+1}(\epsilon+s)^\lambda}{\epsilon_1-\epsilon},$$

$$\left| \int_{-\epsilon}^{0} \frac{|z|^\alpha|z-s|^\lambda}{z-\xi} dz \right| \leq \frac{\epsilon^{\alpha+1}(\epsilon+s)^\lambda}{\epsilon_1-\epsilon}.$$
Remark 7. In order to compare the results of the situation of the singularly perturbed Laguerre weight in [46], we take the same integration contour of (5.11) therein as the integration contour \( \Gamma \) in (4.124). They are different, for example the lower bound \( \delta \) satisfies \( s/\delta = O(1) \), and takes \( \delta = s \) in [46], but in our case the lower bound \( \epsilon > s \) and set \( \epsilon = 2s \).

It is interesting to investigate the behavior of \( \Phi_1(\xi, s) \) on the boundary \( \partial U(0, \epsilon) \). Substituting (4.126) into (4.119) and set \( \varepsilon_1 = \varepsilon \), then one finds

\[
\Phi_1(\xi, s)J^{-1} = G(\xi) \left( \begin{array}{c} \frac{\xi^{\alpha+\lambda}}{2} \left( 1 + \mathcal{O}\left( \frac{s}{\xi} \right) + \mathcal{O}\left( \left( \frac{s}{\xi} \right)^{\alpha+\lambda+1} \right) \right) \xi^{\alpha+\lambda} \left( 1 - \frac{s}{\xi} \right)^{\frac{1}{2}} \sigma \end{array} \right),
\]

(4.127)

if \( s = 0 \), the right hand side of (4.127) is the same as (4.117).

From (4.119), (4.126) and \( \Phi_0(\xi) \) in (4.108), one follows the matching condition (4.127).

For \( \alpha + \lambda \in \mathbb{N} \), then the modified Bessel matrix function has a factorization as follows

\[
\left( \begin{array}{c} I_{\alpha+\lambda}\left( \xi^\frac{1}{2} \right) \\ i\pi \xi^\frac{1}{2} I'_{\alpha+\lambda}\left( \xi^\frac{1}{2} \right) \end{array} \right) = \tilde{G}(\xi) \xi^{\frac{1}{2}} \sigma \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \frac{\log \left( \frac{\xi}{2} \right)}{\pi},
\]

(4.128)

where

\[
\tilde{G}(\xi) = \left( \begin{array}{c} \xi^{-\frac{1}{2}} I_{\alpha+\lambda}\left( \xi^\frac{1}{2} \right) \\ i\pi \xi^{-\frac{1}{2}} I'_{\alpha+\lambda}\left( \xi^\frac{1}{2} \right) \end{array} \right) \left( \begin{array}{c} \frac{i(-1)^{\alpha+\lambda+1}}{\pi} \xi^{\alpha+\lambda} \sigma \xi^{\alpha+\lambda} \left( \xi^\frac{1}{2} \right) \\ \frac{(-1)^{\alpha+\lambda+1}}{2\pi} \xi^{\alpha+\lambda} \sigma \left( \xi^\frac{1}{2} \right) \end{array} \right),
\]

and \( \tilde{G}(\xi) \) is an entire matrix function. One verifies with the following formulas,

\[
K_v = \frac{\pi}{2} \lim_{\nu \to m} \frac{I_{-v}(z) - I_v(z)}{\sin \nu \pi} = \frac{(-1)^m}{2} \left( \frac{\partial}{\partial \nu} I_v(z) \right)_{\nu=m} - \frac{\partial}{\partial \nu} I_v(z)_{\nu=m}, \quad v = m \in \mathbb{N},
\]

where

\[
I_{-m}(z) = \left. \frac{\partial}{\partial \nu} I_\nu(z) \right|_{\nu=-m} = \left( \frac{z}{2} \right)^{-m} \sum_{j=0}^{m-1} \frac{(-1)^{m-j} (m-j-1)!}{\Gamma(j+1)} \left( \frac{z^2}{4} \right)^j + \left( \frac{z}{2} \right)^m \sum_{j=0}^{\infty} \frac{- \log \left( \frac{z}{2} \right) + \psi(j+1)}{\Gamma(j+1) \Gamma(j+m+1)} \left( \frac{z^2}{4} \right)^j,
\]

\[
\frac{\partial}{\partial \nu} I_\nu(z) \bigg|_{\nu=m} = \left( \frac{z}{2} \right)^m \sum_{j=0}^{\infty} \frac{\log \left( \frac{z}{2} \right) - \psi(j+m+1)}{\Gamma(j+1) \Gamma(j+m+1)} \left( \frac{z^2}{4} \right)^j.
\]
and

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$ 

With $f(\xi)$ in (4.119) replaced by $\tilde{f}(\xi)$, which satisfies (4.123), then one can define

$$\tilde{f}(\xi) = \frac{(-1)^{\alpha+\lambda+1}}{2\pi^2} \int_{\Gamma} \frac{z^\alpha (z-s)^\lambda \log \left( \frac{\sqrt{z}}{\sqrt{s}} \right)}{z-\xi} \, dz, \quad \xi \in U(0,\varepsilon) \setminus (-\varepsilon,0), \quad \text{arg} \, \xi \in (-\pi,\pi),$$

where the integration contour $\Gamma$ keeps the same as in (4.124).

For $\xi \in \partial U(0,\varepsilon)$ and $s \to 0$, the matching condition (4.122) can be verified with the similar steps of (4.125)-(4.126), one gets

$$\tilde{f}(\xi) = \frac{(-1)^{\alpha+\lambda}}{i\pi} \xi^{\alpha+\lambda} \log \frac{\sqrt{\xi}}{2} \left( 1 + \mathcal{O} \left( \frac{s}{\varepsilon_1} \right) + \mathcal{O} \left( \left( \frac{s}{\varepsilon_1} \right)^{\alpha+\lambda+1} \log s \right) \right), \quad (4.130)$$

where $2s = \varepsilon \ll |\xi| = \varepsilon_1 \leq \varepsilon$.

At last, we define

$$R_0(\xi, s) = \begin{cases} 
\Phi(\xi, s)\Phi_0^{-1}(\xi), & \xi \in \mathbb{C} \setminus U(0,\varepsilon), \\
\Phi(\xi, s)\Phi_1^{-1}(\xi, s), & \xi \in U(0,\varepsilon).
\end{cases} \quad (4.131)$$

Then the matrix function $R_0(\xi, s)$ fulfills the following properties,

(a) $R_0(\xi, s)$ is analytic in $\xi \in \mathbb{C} \setminus \partial U(0,\varepsilon)$.

(b) From (4.119) and (4.130), then $R_0(\xi, s)$ satisfies the following jump on the counter clockwise circle $\partial U(0,\varepsilon)$,

$$(R_0)_+(\xi, s) = (R_0)_-(\xi, s) \begin{cases} 
I + \mathcal{O}(h_1(s)), & \alpha + \lambda \notin \mathbb{N}, \\
I + \mathcal{O}(h_2(s)), & \alpha + \lambda \in \mathbb{N},
\end{cases} \quad (4.132)$$

where

$$h_1(s) = \begin{cases} 
s, & \alpha + \lambda > 0, \\
\alpha + \lambda + \lambda, & -1 < \alpha + \lambda < 0,
\end{cases} \quad \text{and} \quad h_2(s) = \begin{cases} 
s, & \alpha + \lambda > 0, \\
\alpha + \lambda + \lambda \log s, & -1 < \alpha + \lambda < 0.
\end{cases} \quad (4.133)$$

With a similar argument as (3.91), (3.92), (3.93) and (3.94) in section 3.7, we have
(c) The uniformly asymptotic behavior of $R_0(\xi, s)$ for bounded $\xi$, $s < \epsilon \ll |\xi| = \varepsilon_1 \leq \varepsilon$ and $s \to 0$,

$$R_0(\xi, s) = \begin{cases} I + \mathcal{O}(h_1(s)), & \alpha + \lambda \notin \mathbb{N}, \\ I + \mathcal{O}(h_2(s)), & \alpha + \lambda \in \mathbb{N}, \end{cases}$$

(4.134)

where $h_1(s)$ and $h_2(s)$ are given by (4.133).

(d) The uniformly asymptotic behavior of $R_0(\xi, s)$ for $\xi \to \infty$ and $s \to 0$,

$$R_0(\xi, s) = \begin{cases} I + \mathcal{O}(h_1(s)\xi^{-1}), & \alpha + \lambda \notin \mathbb{N}, \\ I + \mathcal{O}(h_2(s)\xi^{-1}), & \alpha + \lambda \in \mathbb{N}. \end{cases}$$

(4.135)

It completes the nonlinear steepest descent analysis for $\Phi(\xi, s)$ as $s \to 0$.

**Proof of Theorem 2**

*Proof.* Combining (3.100)-(3.101), (4.110), (4.134)-(4.135) and set $\xi = n^2\phi^2(z)$ in (4.110) for $\xi \in \hat{\Omega}_3$, then

$$\begin{pmatrix} \varphi_1(\xi, s) \\ \varphi_2(\xi, s) \end{pmatrix} = \Phi_-(\xi, s)e^{i\pi(\alpha+\lambda-1)/2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= R_0(\xi, s)M_0 \begin{pmatrix} I_{\alpha+\lambda} \left( \xi^{\frac{1}{2}} \right) & i\frac{1}{\pi}K_{\alpha+\lambda} \left( \xi^{\frac{1}{2}} \right) \\ i\pi\xi^{\frac{1}{2}}I'_{\alpha+\lambda} \left( \xi^{\frac{1}{2}} \right) & -\xi^{\frac{1}{2}}K'_{\alpha+\lambda} \left( \xi^{\frac{1}{2}} \right) \end{pmatrix} \begin{pmatrix} e^{\frac{i\pi}{2}(\lambda+\alpha-1)} \\ 0 \end{pmatrix}$$

$$= R_0(\xi, s)M_0 \begin{pmatrix} -iJ_{\alpha+\lambda} \left( |\xi|^{\frac{1}{2}} \right) \\ \pi|\xi|^{\frac{1}{2}}J'_{\alpha+\lambda} \left( |\xi|^{\frac{1}{2}} \right) \end{pmatrix},$$

(4.136)

where $M_0$ and $R_0(\xi, s)$ are given by (4.111) and (4.134), respectively, and it has been used the facts that for $\arg z \in (-\pi, \pi/2]$, $e^{\frac{i\pi}{2}}I_\beta(ze^{-\frac{i\pi}{2}}) = J_\beta(z)$, see [39].

Let $\xi = n^2\phi^2(x)$, $x = \frac{u}{\text{Im}z}$, then

$$\begin{pmatrix} \varphi_1(n^2\phi^2(x), s) \\ \varphi_2(n^2\phi^2(x), s) \end{pmatrix} = \begin{cases} (I + \mathcal{O}(h_1(s)))M_0 \begin{pmatrix} -iJ_{\alpha+\lambda}(u) \\ \pi|\xi|^{\frac{1}{2}}J'_{\alpha+\lambda}(u) \end{pmatrix} + \mathcal{O} \left( \frac{1}{n^{\pi}} \right), & \alpha + \lambda \notin \mathbb{N}, \\ (I + \mathcal{O}(h_2(s)))M_0 \begin{pmatrix} -iJ_{\alpha+\lambda}(u) \\ \pi|\xi|^{\frac{1}{2}}J'_{\alpha+\lambda}(u) \end{pmatrix} + \mathcal{O} \left( \frac{1}{n^{\pi}} \right), & \alpha + \lambda \in \mathbb{N}, \end{cases}$$

(4.137)
where $h_1(s)$ and $h_2(s)$ are given by (4.133). Moreover, inserting (4.137) and similarly estimations for $\xi = n^2 \phi^2(y)$ and $y = \frac{x}{16n^3}$ into (3.101), then
\[
\frac{1}{4n}K_n\left(\frac{u}{4n}, \frac{v}{4n}; s\right) = \frac{\varphi_1(-v, s)\varphi_2(-u, s) - \varphi_1(-u, s)\varphi_2(-v, s)}{i2\pi(u - v)} + \mathcal{O}\left(\frac{1}{n^2}\right)
\]
\[
= \begin{cases}
\frac{J_{\alpha + \lambda}(u)J_{\alpha + \lambda}(v) - J_{\alpha + \lambda}(v)uJ_{\alpha + \lambda}(u)}{2(u-v)} + \mathcal{O}(h_1(s)) + \mathcal{O}\left(\frac{1}{n}\right), & \alpha + \lambda \notin \mathbb{N}, \\
\frac{J_{\alpha + \lambda}(u)J_{\alpha + \lambda}(v) - J_{\alpha + \lambda}(v)uJ_{\alpha + \lambda}(u)}{2(u-v)} + \mathcal{O}(h_2(s)) + \mathcal{O}\left(\frac{1}{n}\right), & \alpha + \lambda \in \mathbb{N}.
\end{cases}
\] (4.138)

For $n \to \infty$, (4.138) follows (1.28).

### 4.2 Painlevé V kernel to the Bessel kernel $J_\alpha$ as $s \to \infty$

We start with the RH problem (2.31)-(2.34) for $\Phi(\xi, s)$, let $\xi = s\hat{z}$, and define
\[
A(\hat{z}, s) := s^\frac{1}{2}\sigma_3 \Phi(s\hat{z}, s)e^{-\sqrt{s^2\sigma_3}},
\] (4.139)
then $A(\hat{z}, s)$ satisfies the following RH problem.

(a) $A(\hat{z}, s)$ is analytic in $\mathbb{C} \setminus \bigcup_{j=1}^{3} \Sigma_j' \cup (0, 1)$, see Figure 7.

(b) $A(\hat{z}, s)$ fulfills the jump relation and $\arg \hat{z} \in (-\pi, \pi)$,
\[
A_+(\hat{z}, s) = A_-(\hat{z}, s)
\]
\[
= \begin{cases}
\begin{pmatrix} e^{\lambda \pi \sigma_3} & 1 \\ \frac{1}{e^{i(\lambda+\alpha)\pi-2\sqrt{s^2}}} & 0 \end{pmatrix}, & \hat{z} \in \Sigma_1', \\
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \hat{z} \in \Sigma_2', \\
\begin{pmatrix} 1 & 0 \\ e^{-i(\lambda+\alpha)\pi-2\sqrt{s^2}} & 1 \end{pmatrix}, & \hat{z} \in \Sigma_3'.
\end{cases}
\] (4.140)

(c) As $\hat{z} \to \infty$,
\[
A(\hat{z}, s) = \left(I + \mathcal{O}\left(\frac{1}{\hat{z}}\right)\right)\hat{z}^{-\frac{1}{2}\sigma_3}I + i\sigma_1 \sqrt{2}, \quad \arg \hat{z} \in (-\pi, \pi).
\] (4.141)

(d) As $\hat{z} \to 0$,
\[
A(z, s) = Q_3(s)\left(I + \mathcal{O}(\hat{z})\right)\hat{z}^{\frac{3}{2}\sigma_3}, \quad \arg \hat{z} \in (-\pi, \pi).
\]

(e) As $\hat{z} \to 1$,
\[
A(\hat{z}, s) = \mathcal{O}(1)(\hat{z} - 1)^{\frac{1}{2}\sigma_3}, \quad \arg (\hat{z} - 1) \in (-\pi, \pi).
\]
Fig.7. Contours of $\bigcup_{j=1}^{3} \Sigma'_j \cup (0,1)$.

From the jump condition of (4.140), as $s \to \infty$ and $\hat{z}$ away from the origin, then the jumps on $\Sigma'_{1}$ and $\Sigma'_{3}$ tend to identical matrices. So, $A(\hat{z},s)$ can be approximated by $B(\hat{z})$ which independents of $s$, and $B(\hat{z})$ satisfies the following RH problem.

(a) $B(\hat{z})$ is analytic in $\mathbb{C} \setminus (-\infty,1)$.

(b) $B(\hat{z})$ satisfies the jump condition

$$B_+ (\hat{z}) = B_- (\hat{z}) \begin{cases} 
 e^{i\lambda \pi \sigma_3}, & \hat{z} \in (0,1), \\
 \begin{pmatrix} 0 & 1 \\
 -1 & 0 
\end{pmatrix}, & \hat{z} \in (-\infty,0),
\end{cases}$$

where $\arg \xi \in (-\pi,\pi)$.

(c) For $\hat{z} \to \infty$,

$$B(\hat{z}) = \left( I + \mathcal{O} \left( \frac{1}{\hat{z}} \right) \right) \hat{z}^{-\frac{1}{4}\sigma_3} \frac{I + i\sigma_1}{\sqrt{2}}, \ \arg \xi \in (-\pi,\pi).$$

(d) For $\hat{z} = 1$,

$$B(\hat{z}) = \mathcal{O}(1)(\hat{z} - 1)^{\frac{i}{2}\sigma_3}, \ \arg \xi \in (-\pi,\pi).$$
By the Szegö function, $B(\hat{z})$ can be constructed as follows

$$B(\hat{z}) = \left( \begin{array}{cc} 1 & 0 \\ i\lambda & 1 \end{array} \right) \hat{z}^{-\frac{i}{2}\sigma_3} \frac{I + i\sigma_1}{\sqrt{2}} \left( \exp \left( \frac{\lambda\sqrt{\hat{z}}}{2\pi} \int_{-\infty}^{0} \log \frac{\hat{z} - 1}{\sqrt{-x}} \frac{1}{\sqrt{-x} - \hat{z}} dx \right) \right)^{\sigma_3} \left( \frac{\hat{z} - 1}{\hat{z}} \right)^{\frac{i}{2}\sigma_3}$$

$$= \left( \begin{array}{cc} 1 & 0 \\ i\lambda & 1 \end{array} \right) \hat{z}^{-\frac{i}{2}\sigma_3} \frac{I + i\sigma_1}{\sqrt{2}} \left( \exp \left( \frac{\lambda\sqrt{\hat{z}}}{2} \int_{0}^{1} \frac{1}{\sqrt{x} - \hat{z}} dx \right) \right)^{\sigma_3}$$

$$= \left( \begin{array}{cc} 1 & 0 \\ i\lambda & 1 \end{array} \right) \hat{z}^{-\frac{i}{2}\sigma_3} \frac{I + i\sigma_1}{\sqrt{2}} \left( \frac{\hat{z} - 1}{(\sqrt{\hat{z}} + 1)^2} \right)^{\frac{i}{2}\sigma_3}, \quad \arg \hat{z} \in (-\pi, \pi).$$ (4.142)

**Remark 8.** From (2.32), (2.33) and (4.139), it follows that

$$A(\hat{z}, s) \frac{I + i\sigma_1}{\sqrt{2}} \hat{z}^{\frac{i}{2}\sigma_3} = I + \left( \frac{q(s)s^{-1}}{it(s)s^{-\frac{3}{2}}} - ir(s)s^{-\frac{3}{2}} - q(s)s^{-1} \right) \frac{1}{\hat{z}} + \mathcal{O} \left( \frac{1}{\hat{z}^2} \right).$$ (4.143)

By $A(\hat{z}, s)$ can be approximated by $B(\hat{z})$ for large $s$ and (4.142), then one finds,

$$B(\hat{z}) \frac{I + i\sigma_1}{\sqrt{2}} \hat{z}^{\frac{i}{2}\sigma_3} = I + \left( \frac{i\lambda^2/2}{it(\lambda^2 - 1)/3} - i\lambda - \lambda^2/2 \right) \frac{1}{\hat{z}} + \mathcal{O} \left( \frac{1}{\hat{z}^2} \right).$$ (4.144)

From (4.143) and (4.144), one obtains the initial data for $q(s)$, $r(s)$ and $t(s)$,

$$r(s) = -\lambda s^{-\frac{1}{2}} + \mathcal{O} \left( s^{-1} \right), \quad q(s) = \frac{\lambda^2}{2} s + \mathcal{O} \left( s^{\frac{1}{2}} \right), \quad t(s) = \frac{\lambda(\lambda^2 - 1)}{3} s^{\frac{3}{2}} + \mathcal{O} \left( s \right), \quad s \to \infty.$$ (4.145)

As $\hat{z} \to 0$, $B(\hat{z})$ can not be applied to approximate $A(\hat{z}, s)$, and the jumps on $\Sigma_1'$ and $\Sigma_3'$ in (4.140) may have oscillation entries. Hence, it is need to construct a local parametrix $B_0(\hat{z}, s)$ in the disk $|\hat{z}| < 1$, which satisfies the jump in (4.140) of $A(\hat{z}, s)$, and matches the following condition.

$$B(\hat{z})B_0^{-1}(\hat{z}, s) = I + \mathcal{O} \left( s^{-\frac{1}{2}} \right), \quad |\hat{z}| = 1, \quad s \to \infty.$$ (4.146)

We construct the matrix function $B_0(\hat{z}, s)$ as follows

$$B_0(\hat{z}, s) = E_{0L}(\hat{z})F(\hat{z}, s) \begin{cases} e^{-\frac{\sqrt{-1}\lambda s}{2}\sigma_3}, & 0 < \arg \hat{z} < \pi, \\ e^{-\frac{\sqrt{-1}\lambda s}{2}\sigma_3}, & -\pi < \arg \hat{z} < 0, \end{cases}$$ (4.147)

where $E_{0L}(\hat{z})$ is analytic in $U(0, 1)$. $F(\hat{z}, s)$ fulfills the following RH problem.

(a) $F(\hat{z}, s)$ is analytic in $\mathbb{C} \setminus \bigcup_{j=1}^{11} \Sigma_j$, see Fig. 8.
(b) $F(\hat{z}, s)$ satisfies the jump condition as follows

$$F_{+}(\hat{z}, s) = F_{-}(\hat{z}, s) \begin{cases} 
\begin{pmatrix} 1 & 0 \\ e^{i\alpha\pi} & 1 \end{pmatrix}, & \hat{z} \in \Sigma_{I}, \\
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \hat{z} \in \Sigma_{II}, \\
\begin{pmatrix} 1 & 0 \\ e^{-i\alpha\pi} & 1 \end{pmatrix}, & \hat{z} \in \Sigma_{III}, 
\end{cases}$$

where $\arg \hat{z} \in (-\pi, \pi)$.

(c) As $\hat{z} \to \infty$,

$$F(\hat{z}, s) = \left( I + O\left(\frac{1}{\hat{z}}\right) \right) \hat{z}^{-\frac{1}{2}\sigma_3} I + i\sigma_1 \sqrt{2} e^{\sqrt{s_2}\sigma_3}, \text{ arg } \hat{z} \in (-\pi, \pi).$$

(d) As $\hat{z} \to 0$,

$$F(\hat{z}, s) = O(1) \hat{z}^{\frac{1}{2}\sigma_3}, \text{ arg } \hat{z} \in (-\pi, \pi).$$

Figure 8. Contours $\bigcup_{j=I}^{III} \Sigma_j$, and regions $\Omega_j, \ j = I, II, III$. 

41
By the modified Bessel functions, \( F(\hat{z}, s) \) can be constructed as follows

\[
F(\hat{z}, s) = M_1 \begin{cases} 
\left( I_\alpha \left( \sqrt{s\hat{z}} \right) - i\pi I'_\alpha \left( \sqrt{s\hat{z}} \right) \right), & \hat{z} \in \Omega_I, \\
\left( I_\alpha \left( \sqrt{s\hat{z}} \right) - i\pi K'_\alpha \left( \sqrt{s\hat{z}} \right) \right) \left( \frac{1}{e^{-i\alpha}} 0 \right), & \hat{z} \in \Omega_{II}, \quad (4.148) \\
\left( I_\alpha \left( \sqrt{s\hat{z}} \right) - i\pi K'_\alpha \left( \sqrt{s\hat{z}} \right) \right) \left( 0 e^{-i\alpha} 1 \right), & \hat{z} \in \Omega_{III},
\end{cases}
\]

where the regions \( \Omega_j, j = I, II, III \), illustrated in Fig.8, \( M_1 \) only dependents on \( s \),

\[
M_1 = \left( i (4\alpha^2 + 3) s^{-\frac{1}{2}} \sigma_-/8 + I \right) s^{\frac{1}{2}\sigma_3 \pi^\frac{1}{2}\sigma_3}. \quad (4.149)
\]

With the matching condition \( B_0(\hat{z}, s) \sim B(\hat{z}) \) for \( \hat{z} \in \partial U(0,1) \) and the expression of \( B(\hat{z}) \) in (4.142), then the matrix function \( E_{0L}(\hat{z}) \) is given by

\[
E_{0L}(\hat{z}) = B(\hat{z}) \begin{cases} 
e^{-i\frac{\pi}{2} \sigma_3 \frac{L}{\sqrt{2}} \hat{z}^{-\frac{1}{2} \sigma_3}}, & \arg \hat{z} \in (0, \pi), \\
\left( e^{-i\frac{\pi}{2} \sigma_3 \frac{L}{\sqrt{2}} \hat{z}^{-\frac{1}{2} \sigma_3}}, & \arg \hat{z} \in (\pi, 0). \quad (4.150)
\end{cases}
\]

One checks that \( E_{0L}(\hat{z}) \) has no jump on the real axis and is analytic in the unite disk. \( E_{0L}(\hat{z}) \) has square root singularities at \( \hat{z} = 0 \), hence, these singularities are weak and removable.

From (4.147), (4.148) and (4.150), it follows the matching condition (4.146) which is uniform for bounded \( \hat{z} \).

We define

\[
R_{0L}(\hat{z}, s) = \begin{cases} 
A(\hat{z}, s)B^{-1}(\hat{z}), & |\hat{z}| > 1, \\
A(\hat{z}, s)B_0^{-1}(\hat{z}, s), & |\hat{z}| < 1. \quad (4.151)
\end{cases}
\]

\( R_{0L}(\hat{z}, s) \) is a piecewise analytic function in \( \mathbb{C} \setminus \Sigma_{R_{0L}} \), where the contour \( \Sigma_{R_{0L}} \) contains the counter clockwise unite circle, parts of \( \Sigma_I \) and \( \Sigma_{III} \) (see Figure 8) for \( |\hat{z}| > 1 \), then one has

\[
J_{R_{0L}}(\hat{z}, s) = \begin{cases} 
I + \mathcal{O} \left( s^{-\frac{1}{2}} \right), & |\hat{z}| = 1, \\
I + \mathcal{O} \left( e^{-c\sqrt{\sigma}} \right), & \hat{z} \in \{ \hat{z} \in \Sigma_I \cap \Sigma_{III} : |\hat{z}| > 1 \}, \quad (4.152)
\end{cases}
\]

where \( c \) is a positive constant.
With a similar argument in section 3.7, one finds,

\[ R_{0L}(\hat{z}, s) = I + \mathcal{O}(s^{-\frac{1}{2}}), \]  

where the error term is uniform for bounded \( \hat{z} \in (0, \infty) \) and \( t \in (0, c] \), \( c \) is a finite and positive constant.

**Proof of Theorem 3**

Proof. With the help of (3.100), (4.139), (4.148) and (4.153), one finds,

\[
\begin{pmatrix}
\varphi_1(\xi, s) \\
\varphi_2(\xi, s)
\end{pmatrix}
= \Phi_-(\xi, s)e^{i\pi(\alpha+\lambda-1)/2}\sigma_3
\begin{pmatrix}
1 \\
1
\end{pmatrix}
\]

\[ = s^{-\frac{1}{2}\sigma_3} \left( I + \mathcal{O}(s^{-\frac{1}{2}}) \right) E_{0L}(\hat{z}) M_1 \left( \frac{-iJ_\alpha(|\xi|^2)}{\pi|\xi|^2 J'_\alpha(|\xi|^2)} \right), \]  

where \( M_1, E_{0L}(\hat{z}) \) are given by (4.149) and (4.150), respectively.

Set \( \xi = n^2\phi^2(x), \ x = \frac{u}{16n^2} \) and \( \xi = n^2\phi^2(y), \ y = \frac{v}{16n^2} \) in (4.154), and inserting these equations into (3.101), hence

\[
\frac{1}{4n}K_n\left( \frac{u}{4n}, \frac{v}{4n} ; s \right) = \frac{\varphi_1(-v, s)\varphi_2(-u, s) - \varphi_1(-u, s)\varphi_2(-v, s)}{i2\pi(u-v)} + \mathcal{O}\left( \frac{1}{n^2} \right) + \mathcal{O}\left( s^{-\frac{1}{2}} \right) + \mathcal{O}\left( \frac{1}{n^2} \right) \quad s \to \infty, \ n \to \infty,
\]

where the error term is uniform for \( u, v \in (0, \infty) \) and \( t \in (0, c], \) \( c \) is a positive and fixed constant. Then (1.29) derives from (4.155).

\[ \square \]

### 5 Airy kernel at the soft edge

**The construction of \( P^{(1)}(z) \) in the neighbourhood of \( z = 1. \)**

For convenience, we define

\[ \hat{\mu}(z) := \frac{2}{i\pi} \sqrt{\frac{z-1}{z}}, \quad z \in \mathbb{C} \setminus [0, 1], \]

and \( \hat{\mu}(z) \) such that \( \hat{\mu}_+(x) = -\hat{\mu}_-(x) = \mu(x), \ x \in (0, 1), \) where \( \mu(x) \) is given by (1.3).
We define an auxiliary function \(\hat{\phi}(z)\) as follows

\[
\hat{\phi}(z) = -i\pi \int_1^z \hat{\mu}(y)dy = -2 \int_1^z \sqrt{y-1} dy, \quad z \in \mathbb{C} \setminus (-\infty, 1].
\]  

(5.156)

Note that \(\hat{\phi}_+(z)\) and \(\hat{\phi}_-(z)\) purely imaginary on \((0, 1)\) and

\[
g_+(x) - g_-(x) = 2\hat{\phi}_+(x) = -2\hat{\phi}_-(x), \quad x \in (0, 1),
\]

(5.157)

\[
\hat{\phi}_+(x) - \hat{\phi}_-(x) = i2\pi, \quad x \in (-\infty, 0),
\]

\[
2\hat{\phi}(x) = 2g(x) - 4x - \ell_n, \quad x \in [1, +\infty).
\]

(5.158)

where \(g(z)\) is given by (3.66), \(\ell_n = -2 - 4 \log 2\). The above facts can also be found in (40).

By (5.157), (5.158), then (3.72) rewrite as

\[
T_+(x) = T_-(x) \begin{cases}
    \begin{pmatrix}
        e^{-2n\hat{\phi}_+(x)} & x^\alpha \left(x + \frac{4}{4n}\right)^\lambda \\
        0 & e^{-2n\hat{\phi}_-(x)}
    \end{pmatrix}, & x \in (0, 1), \\
    \begin{pmatrix}
        1 & x^\alpha \left(x + \frac{4}{4n}\right)^\lambda e^{2n\hat{\phi}(x)} \\
        0 & 1
    \end{pmatrix}, & x \in (1, +\infty).
\end{cases}
\]

(5.159)

(S) \(S(z)\) is defined as (3.73) and replaced \(\phi_+(x), \phi_-(x)\) by \(\hat{\phi}_+(x), \hat{\phi}_-(x)\), respectively. Then \(S(z)\) satisfies the following RH problem.

\(S_a\) \(S(z)\) is analytic in \(\mathbb{C} \setminus \{\bigcup_{k=1}^3 \Sigma_k\} \cup (1, \infty)\), see Figure 2.

\(S_b\) \(S_+(z) = S_-(z)J_S\) for \(z \in \{\bigcup_{k=1}^3 \Sigma_k\} \cup (1, \infty)\) and the jump \(J_S\) is given by

\[
J_S(z) = \begin{cases}
    \begin{pmatrix}
        0 & z^{-\alpha} \left(z + \frac{4}{4n}\right)^{-\lambda} e^{-2n\hat{\phi}(z)} \\
        -z^{-\alpha} \left(z + \frac{4}{4n}\right)^{-\lambda} & 0
    \end{pmatrix}, & \text{for } z \in \Sigma_1 \cup \Sigma_3,
\end{cases}
\]

(5.160)

where \(\arg \xi \in (-\pi, \pi)\).

\(S_c\) The asymptotic behavior at infinity is

\[
S(z) = I + \mathcal{O}(z^{-1}).
\]

44
The asymptotic behavior at $z = 1$ is

$$S(z) = \begin{pmatrix} O(1) & O(1) \\ O(1) & O(1) \end{pmatrix}. \quad (5.161)$$

We seek the local parametrix $P^{(1)}(z)$ in the neighborhood $U(1, r) = \{ z \in \mathbb{C} : |z - 1| < r \}$ for small $r > 0$. Moreover, $P^{(1)}(z)$ satisfies the following RH problem.

(a) $P^{(1)}(z)$ is analytic in $U(1, r) \setminus \{ \bigcup_{k=1}^{3} \Sigma_k \cup (1, +\infty) \}$, see Figure 2.

(b) $P^{(1)}(z)$ has the same jump condition as $S(z)$, on $U(1, r) \cap \{ \bigcup_{k=1}^{3} \Sigma_k \cup (1, +\infty) \}$, see (5.160).

(c) For $z \in \partial U(1, r)$, $P^{(1)}(z)$ satisfies the following matching condition,

$$P^{(1)}(z) P^{(\infty)-1}(z) = I + \mathcal{O}(n^{-1}), \quad n \to \infty. \quad (5.162)$$

(d) The asymptotic behavior of $P^{(1)}(z)$ at $z = 1$ is the same as $S(z)$ in (5.161).

Taking the following transformation to constant jump matrices.

$$\tilde{P}^{(1)}(z) = E_1^{-1}(z) P^{(1)}(z) e^{n\tilde{\phi}(z)\sigma_3} (z + t/4n)^{\frac{1}{2}\sigma_3} z^{\frac{2}{\sigma_3}}, \quad (5.163)$$

where $E_1(z)$ is an invertible analytic matrix function in $U(1, r)$. $\tilde{P}^{(1)}(z)$ fulfills the following RH problem.

(a) $\tilde{P}^{(1)}(z)$ is analytic in $U(1, r) \setminus \{ \bigcup_{k=1}^{3} \Sigma_k \cup (1, +\infty) \}$.

(b) $\tilde{P}^{(1)}(z)$ satisfies the following jump condition,

$$\tilde{P}_+^{(1)}(z) = \tilde{P}_-^{(1)}(z) \begin{cases} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & \text{for } z \in U(1, r) \cap \{ \Sigma_1 \cup \Sigma_3 \}, \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \text{for } z = x \in U(1, r) \cap (0, 1), \\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, & \text{for } z = x \in U(1, r) \cap (1, +\infty). \end{cases} \quad (5.164)$$

It is well-known that the Airy function has been successful applied to construct the local parametrix in [17, 20], see also [45]. Some more information, see [39]. With the jump condition of $\tilde{P}^{(1)}(z)$ in (5.161), it is natural to consider the following RH problem for $\tilde{P}^{(1)}(z)$ which can be constructed by the Airy function and its derivatives.

(a) $\tilde{P}^{(1)}(z)$ is analytic in $z \in \mathbb{C} \setminus \bigcup_{k=1}^{4} \Sigma_k^\prime$, see Figure 9.
(b) $\tilde{P}^{(1)}(z)$ satisfies the jump condition

$$\tilde{P}^{(1)}_+(z) = \tilde{P}^{(1)}_-(z)$$

$$\begin{cases}
  \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & z \in \Sigma'_1 \cup \Sigma'_3, \\
  \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in \Sigma'_2, \\
  \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, & z \in \Sigma'_4,
\end{cases}$$

(c) For $z \to \infty$,

$$\tilde{P}^{(1)}(z) = z^{-\frac{i}{4}\sigma_3} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \left( I + \mathcal{O}(z^{-\frac{3}{2}}) \right) e^{-\frac{i\pi}{4}\sigma_3} e^{-\frac{3}{2}z^{\frac{3}{2}}\sigma_3}, \quad (5.165)$$

where $\arg \xi \in (-\pi, \pi)$.

Fig.9. Contours $\mathbb{C} \setminus \bigcup_{j=1}^4 \Sigma''_j$, and regions $\Omega''_j, j = 1, \ldots, 4$.

A solution to the above RH problem for $\tilde{P}^{(1)}(z)$ can be constructed by the Airy function
and its derivatives as follows

\[
\begin{align*}
\tilde{P}^{(1)}(z) &= M_2 \\
&= \left\{ \begin{array}{ll}
\begin{pmatrix}
Ai(z) & Ai(ω^2 z) \\
Ai'(z) & ω^2 Ai'(ω^2 z)
\end{pmatrix} e^{-i\frac{π}{8}σ_3}, & z \in Ω_1'',
\end{array}
\begin{array}{ll}
\begin{pmatrix}
Ai(z) & Ai(ω^2 z) \\
Ai'(z) & ω^2 Ai'(ω^2 z)
\end{pmatrix} e^{-i\frac{π}{8}σ_3} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, & z \in Ω_2'',
\end{array}
\begin{array}{ll}
\begin{pmatrix}
Ai(z) & -ω^2 Ai(ωz) \\
Ai'(z) & -Ai'(ωz)
\end{pmatrix} e^{-i\frac{π}{8}σ_3} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & z \in Ω_3'',
\end{array}
\begin{array}{ll}
\begin{pmatrix}
Ai(z) & -ω^2 Ai(ωz) \\
Ai'(z) & -Ai'(ωz)
\end{pmatrix} e^{-i\frac{π}{8}σ_3}, & z \in Ω_4''.
\end{array}
\end{align*}
\]

(5.166)

where \(M_2 = \sqrt{2π}e^{-i\frac{π}{8}}\), the regions \(Ω_j'', j = 1, \ldots, 4\) illustrated in Figure 9.

We construct \(\tilde{P}^{(1)}(z)\) by defining \(\tilde{P}^{(1)}(z) := \tilde{P}^{(1)}(g_n(z))\), where \(g_n(z) : U(1, r) \to g_n(U(1, r))\) with \(g_n(1) = 0\), is an appropriate biholomorphic map. To matching the exponential term in (5.163) with the asymptotic expansion of \(\tilde{P}^{(1)}(z)\) in (5.165), let

\[
-\frac{2}{3}g_n^2(z) = n\tilde{ϕ}(z), \quad z \in U(1, r),
\]

(5.167)

by (5.156), it rewrites as

\[
g_n(z) = \left(3n \int_1^x \sqrt{\frac{x-1}{x}} dx\right)^{\frac{2}{3}},
\]

(5.168)

and its Maclaurin expansion at \(z = 1\),

\[
g_n(z) = (2n)^\frac{2}{3}(z-1) \left(1 - \frac{1}{5}(z-1) + \frac{17}{157}(z-1)^2 + \mathcal{O}((z-1)^3)\right).
\]

(5.169)

By (5.162), (5.163), (5.165) and (5.167), then the explicit \(E_1(z)\) and \(P^{(1)}(z)\) are given by

\[
P^{(1)}(z) = E_1(z)\tilde{P}^{(1)}(g_n(z))e^{-n\tilde{ϕ}(z)}(z + t/4n)^{-\frac{1}{4}σ_3} z^{-\frac{3}{4}σ_3},
\]

(5.170)

and

\[
E_1(z) = P^{(∞)}(z)z^{\frac{1}{4}σ_3}(z + t/4n)^{\frac{1}{4}σ_3} e^{i\frac{π}{4}σ_3} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} g_n^{\frac{1}{4}σ_3}(z),
\]

(5.171)

where \(P^{(∞)}(z)\), \(\tilde{P}^{(1)}(z)\) are given by (3.77) and (5.166), respectively.

By (3.77), (5.170) and (5.171), the matching condition (5.162) is easy to verify.
For small $r$, we define
\[
R_1(z, t) = \begin{cases} 
S(z, t)P^{(\infty)-1}(z), & |z - 1| > r, \\
S(z, t)P^{(1)-1}(z, t), & |z - 1| < r.
\end{cases}
\] (5.172)

With $P^{(\infty)}(z)$, $P^{(1)}(z)$ are in (3.77), (5.170), respectively, after some calculations, one has,
\[
J_{R_1}(z) = \begin{cases} 
I + \mathcal{O}(n^{-1}), & z \in \Sigma'' \cap U(1, r), \\
I + \mathcal{O}(n^{-1}), & z \in \partial U(1, r), \\
I + \mathcal{O}(e^{-cn}), & z \in U(1, r) \cap \Sigma'' \cap \Sigma'' \cap \Sigma'' \cap \Sigma'' \cap \Sigma'',
\end{cases}
\] (5.173)

where $c$ is a positive constant. By a similar argument in section 3.7, we have,
\[
R_1(z) = I + \mathcal{O}(n^{-1}),
\] (5.174)

where the error term is uniform for bounded $z$ and $t$ in compact subsets of $(0, \infty)$.

**Proof of Theorem 4**

*Proof.* From (3.62), (3.73) and (5.172), one takes the inverse transformations from $Y$ to $R_1$, and combining with (5.170), (5.171) and (5.158), then the explicit expression of $Y_+(x)$ in the interval $(1, 1 + r)$ is given by
\[
Y_+(4nx)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = (4n)^{(n + \frac{\alpha + \lambda}{2})}e^{-\frac{nt}{2}}R_1(x)E_1(x)\tilde{P}^{(1)}(g_n(x)) (w(4nx, t))^{-\frac{1}{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]
\[
= (4n)^{(n + \frac{\alpha + \lambda}{2})}e^{-\frac{nt}{2}}R_1(x)E_1(x)\tilde{P}^{(1)}(g_n(x)) \begin{pmatrix} 1 \\ 0 \end{pmatrix} (w(4nx, t))^{-\frac{1}{2}},
\] (5.175)

which is also can rewrite as
\[
Y_+(4nx)(w(4nx, t))^\frac{1}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (4n)^{(n + \frac{\alpha + \lambda}{2})}e^{-\frac{nt}{2}}R_1(x)E_1(x)\tilde{P}^{(1)}(g_n(x)) \begin{pmatrix} 1 \\ 0 \end{pmatrix},
\] (5.176)

where $\tilde{P}^{(1)}(z)$, $g_n(z)$ are given by (5.166), (5.168), respectively.

With the aid of (5.176), the equation (3.97) can be rewritten as
\[
4nK_n(4nx, 4ny; t) = \frac{(0 \hspace{10pt} 1) \tilde{P}^{(1)-1}(g_n(y)) E^{-1}(y) R_1^{-1}(y) R_1(x) E(x) \tilde{P}^{(1)}(g_n(x)) \begin{pmatrix} 1 \\ 0 \end{pmatrix}}{i2\pi(x - y)}. \tag{5.177}
\]
From (5.169) and $z \in \Omega''$, see Figure 9, variables $x$ and $y$ re-scaled as $x = 1 + (2n)^{-\frac{k}{3}}u$, $y = 1 + (2n)^{-\frac{k}{3}}v$, where $u, v \in (-\infty, 0)$. $E_1(z)$ is analytic and bounded in $U(1, r)$ for small $r$ and its explicit expression given by (5.171), after some calculations, one finds,

$$E_1^{-1}(y)E_1^{-1}(x) = I + \mathcal{O}(x - y) = I + (u - v)\mathcal{O}\left(n^{-\frac{2}{3}}\right).$$  \hspace{1cm} (5.178)

From (5.174), it follows that

$$R_1^{-1}(y)R_1(x) = I + (x - y)\mathcal{O}\left(n^{-1}\right) = I + (u - v)\mathcal{O}\left(n^{-\frac{2}{3}}\right).$$  \hspace{1cm} (5.179)

With (5.169) and $x = 1 + (2n)^{-\frac{k}{3}}u$, one finds, $g_n(x) = u\left(1 - \frac{1}{5}(2n)^{-\frac{k}{3}}u + \mathcal{O}\left(n^{-\frac{4}{3}}\right)\right)$, and

$$Ai\left(g_n(x)\right) = Ai(u) + \mathcal{O}\left(n^{-\frac{2}{3}}\right),$$  \hspace{1cm} (5.180)

similar estimations are also valid for $g_n(y)$ and $Ai\left(g_n(y)\right)$.

The above asymptotic expansions (5.178), (5.179) and (5.180) are uniform for $u$ and $v$ in compact subsets of $(-\infty, 0)$. Inserting (5.166), (5.178), (5.179), and (5.180) into (5.177), one finds,

$$2(2n)^{\frac{1}{3}}K_n(4n + 2(2n)^{\frac{1}{3}}u, 4n + 2(2n)^{\frac{1}{3}}v; t) = \frac{Ai(u)Ai'(v) - Ai(v)Ai'(u)}{u - v} + \mathcal{O}\left(n^{-\frac{2}{3}}\right),$$  \hspace{1cm} (5.181)

and this asymptotic expansion is uniform for $u$, $v$ and $t$ in compact subsets of $(-\infty, 0)$. Then (1.30) derives from (5.181).

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