Cuspidal irreducible representations of quaternionic forms of \( p \)-adic classical groups for odd \( p \)

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Abstract

Given a quaternionic form \( G \) of a \( p \)-adic classical group (\( p \) odd) we classify all cuspidal irreducible complex representations of \( G \). It is a straightforward generalization of the classification in the \( p \)-adic classical group case. We prove two theorems: At first: Every irreducible cuspidal representation of \( G \) is induced from a cuspidal type, i.e. from a certain irreducible representation of a compact open subgroup of \( G \), constructed from a \( \beta \)-extension and a cuspidal representation of a finite group. Secondly we show that two intertwining cuspidal types of \( G \) are up to equivalence conjugate under some element of \( G \).

1 Introduction

This work is the third part in a series of three papers, the first two being [28] and [27]. Let \( F \) be a non-Archimedean local field with odd residue characteristic \( p \). The construction and classification of cuspidal irreducible complex representation of the set of rational points \( G(F) \) of a reductive group \( G \) defined over \( F \) has already been successfully studied for general linear groups ([8] Bushnell–Kutzko, [23], [2], [24] Broussous–Secherre–Stevens) and for \( p \)-adic classical groups ([33] Stevens, [17] Kurinczuk–Skodlerack–Stevens). In this paper we are generalizing from \( p \)-adic classical groups to their quaternionic forms. Let us mention [35] Yu, [12], [13] Fintzen and [15] Kim for results over reductive \( p \)-adic groups in general.

We need to introduce notation to describe the result. We fix a skew-field \( D \) of index 2 over \( F \) together with an anti-involution \( (\bar{\cdot}) \) on \( D \) and an \( \epsilon \)-hermitian form \( h : V \times V \to D \) on a finite dimensional \( D \)-vector space \( V \). Let \( G \) be the group of isometries of \( h \). At first we describe the construction of the cuspidal types (imitating the Bushnell–Kutzko–Stevens framework): A cuspidal type is a certain irreducible representation \( \lambda \) of a certain compact open subgroup \( J \) of \( G \). The arithmetic core of \( \lambda \) is given by a skew-semisimple stratum \( \Delta = [\Lambda, n, 0, \beta] \). It provides the following data (see [27] for more information):

- An element \( \beta \) of the Lie algebra of \( G \) which generates over \( F \) a product \( E \) of fields in \( A := \text{End}_D V \). We denote the centralizer of \( \beta \) in \( G \) by \( G_\beta \).
- A self-dual \( \sigma_E \)-\( G \)-lattice sequence \( \Lambda \) of \( V \) which can be interpreted as a point of the Bruhat-Tits building \( \mathfrak{B}(G) \) and as the image of a point \( \Lambda_\beta \) of the Bruhat-Tits building \( \mathfrak{B}(G_\beta) \) under a canonical map (see [23])
\[
j_\beta : \mathfrak{B}(G_\beta) \to \mathfrak{B}(G).
\]
- An integer \( n > 0 \) which is related to the depth of the stratum.
- Compact open groups \( H^1(\beta, \Lambda) \subseteq J^1(\beta, \Lambda) \subseteq J(\beta, \Lambda) \), subgroups of \( G \).
- A set \( C(\Delta) \) of characters of \( H^1(\beta, \Lambda) \). (so-called self-dual semisimple characters)
The representation $\lambda$ consists of two parts:

Part 1 is the arithmetic part: One chooses a self-dual semisimple character $\theta \in \mathbb{C}(\Delta)$, which admits a Heisenberg extension $\eta$ on $J'(\beta, \Lambda)$ (see [7, Section 8] for these extensions) and then constructs a certain extension $\kappa$ of $\eta$ to $J(\beta, \Lambda)$ (see [7, Section 8] for these extensions) and then constructs a certain extension $\kappa$ of $\eta$ to $J(\beta, \Lambda)$. Not every extension is allowed for $\kappa$. For example if $\Lambda_\beta$ corresponds to a vertex in $\mathfrak{B}(G_\beta)$ (which is the case for cuspidal types) we impose that the restriction of $\kappa$ to a pro-$p$-Sylow subgroup of $J$ is intertwined by $G_\beta$.

Part 2 is a representation of a finite group (This is the so called level zero part). Let $k_\mathbb{F}$ be the residue field of $\mathbb{F}$. The group $J(\beta, \Lambda)/\text{regular}$ is the set of $k_\mathbb{F}$-rational point of a reductive group, here denoted by $\mathbb{P}(\Lambda_\beta)$. It is also the reductive quotient of the stabilizer $P(\Lambda_\beta)$ of $\Lambda_\beta$ in $G_\beta$. The pre-image $P(\Lambda_\beta)$ of $\mathbb{P}(\Lambda_\beta)$ (connected component) in $P(\Lambda_\beta)$ is the parahoric subgroup of $G_\beta$ corresponding to $\Lambda_\beta$. we choose an irreducible representation $\rho$ of $P(\Lambda_\beta)(k_\mathbb{F})$ whose restriction to $P(\Lambda_\beta)$ is a direct sum of cuspidal irreducible representations, and we inflate $\rho$ to $J$, still called $\rho$, and define $\lambda := \kappa \otimes \rho$. Then $\lambda$ is called a cuspidal type if $P_0(\Lambda_\beta)$ is a maximal parahoric subgroup in $G_\beta$. (see. Section 6)

Then we obtain the following classification theorem:

**Theorem 1.1** (Main Theorem). (i) Every irreducible cuspidal representation of $G$ is induced by a cuspidal type. (Theorem 8.1)

(ii) If $(\lambda, J)$ is a cuspidal type, then $\text{ind}_J^G \lambda$ is irreducible cuspidal.(Theorem 6.5)

(iii) Two intertwining cuspidal types $(\lambda, J)$ and $(\Lambda', J')$ are conjugate in $G$ of and only if they intertwine in $G$. (Theorem 6.2)

The proof of the theorem needs several steps. We need a quadratic unramified field extension $L/F$ and $G_L := G \otimes L$ with its Bruhat-Tits building $\mathfrak{B}(G_L)$.

(i) At first we show that every irreducible representation of $G$ contains a self-dual semisimple character. (This is the most difficult part of the theory.), see Theorem 3.1. Mainly we use the canonical embedding

$$\mathfrak{B}(G) \to \mathfrak{B}(G_\beta)$$

and Gal$(L/F)$-restriction to the results of [32].

(ii) We generalize the construction of $\beta$-extensions $\kappa$ from the $p$-adic classical to the quaternionic case, see Section 5.

(iii) We prove that a self-dual semisimple character contained in a cuspidal irreducible representation needs to be skew, see Section 6.1.

(iv) We follow the proof in [33] to show the exhaustion part of Theorem 1.1 see Section 9.2. Here we needed to generalize the notion of subordinate decompositions, see Section 7.

(v) For the intertwining implies conjugacy part of Theorem 1.1 we use [27] and follow [18], see Section 10.

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2 Notation

2.1 Semisimple characters

This article is a continuation of [27] and [28] which we call I and II. We mainly follow their notation, but there is a major change, see the remark below, and there are slight changes to adapt the notation to [33]. Let F be a non-archimedean local field of odd residue characteristic p with valuation \( \nu_F : F \to \mathbb{Z} \), valuation ring \( \mathfrak{p}_F \), valuation ideal \( \mathfrak{p}_F \), residue field \( k_F \) and we fix a uniformizer \( \mathfrak{p}_F \) of F. We fix an additive character \( \chi_F \) of F of level 1. We consider a quaternionic form G of a p-adic classical group as in II, i.e. G = U(h) for an \( \epsilon \)-hermitian form

\[ h : V \times V \to (D, (\,)), \]

where D is a skew-field of index 2 and central over F together with an anti-involution \((\,)^* : D \to D\) of D. The ambient general linear group \( \text{Aut}_D(V) \) is denoted by \( \hat{G} \). Let us recall that a stratum has the standard notation \( \Delta = [\Lambda, n, r, \beta] \), i.e. the entries for \( \Delta' \) are \( \Lambda', n', r', \beta' \) and for \( \Delta_i \) are \( \Lambda_i, n_i, r_i \) and \( \beta_i \). A semisimple stratum has a unique decomposition as a direct sum of simple strata: \( \Delta = \oplus_{i \in I} \Delta_i \), in particular it decomposes \( E = F[\beta] \) into a product of fields \( E_i = F[\beta_i] \), provides idempotents via \( 1 = \sum_i 1_i \) and further decompositions

\[ V = \oplus_i V_i, \ A = \text{End}_D(V) = \oplus_{i, j \in I} A^{ij}, \ A^{ij} := \text{Hom}_D(V_i, V_j). \]

We denote by \( C_G(\cdot) \) the centralizer of \( \cdot \) in \( G \), \( B := C_A(\beta) \) decomposes into \( B = \oplus_i B_i \), \( B^i = C_{A^i}(\beta_i) \). We write \( G_\beta \) for \( (A^\beta)^* \). The adjoint anti-involution \( \sigma_\beta \) of \( h \) induces a map on the set of strata \( \Delta \to \Delta^\# \). \( \Delta \) is called self-dual if \( \Delta \) and \( \Delta^\# \) coincide up to a translation of \( \Lambda \), i.e. \( n = n^\#, r = r^\#, \beta = \beta^\# \), and there is an integer \( k \) such that \( \Lambda - k \), which is \( (\Lambda_{j+k})_{j \in \mathbb{Z}} \), is equal to \( \Lambda^\# \). To a semisimple stratum \( \Delta \) is attached a compact open subgroup \( \hat{H}(\Delta) \) of \( \hat{G} \) and a finite set of complex characters \( \hat{C}(\Delta) \) defined on \( \hat{H}(\Delta) \). If \( \Delta \) is self-dual semisimple we define \( C(\Delta) \) as the set of the restriction of the elements of \( \hat{C}(\Delta) \) to \( H(\Delta) := \hat{H}(\Delta) \cap G \). Given a stratum \( \Delta \) we denote by \( \Delta(j-) \) the stratum \( [\Lambda, n, r - j, \beta] \), if \( n - r - j \geq 0 \), for \( j \in \mathbb{Z} \) and analogously we have \( \Delta(j+) \). There is a major change of notation to loc.cit.:

**Remark 2.1.** We make the following convention for the notation. (Caution this is then direct form the notation in loc.cit.) Every object which corresponds to the general linear group \( \hat{G} \) is going to get a \( \hat{\cdot} \) on top. Instead of \( C_G(\Delta) \) in II we write \( C(\Delta) \), and instead of \( \hat{C}(\Delta) \) in I we write \( \hat{\hat{C}}(\Delta) \). Analogously for the groups and characters etc.

2.2 Filtrations

In this section we recall the notion of a Bruhat-Tits building of \( G \) in terms of lattice functions.

To \( G \) and \( \hat{G} \) are attached Bruhat-Tits buildings \( \mathfrak{B}(G) \) and \( \mathfrak{B}(\hat{G}) \) and \( \mathfrak{B}_{\text{red}}(\hat{G}) \), see [34], [4], [5] and [6]. We work with the lattice function model, see [1], [3]. We recall:

**Definition 2.2.** A function \( \Gamma : \mathbb{R} \to V \) is called an \( \sigma_D \)-lattice function if for all real numbers \( t < s \) we have

- \( \Gamma(t) \) is a full \( \sigma_D \)-lattice in \( V \),
- \( \Gamma(t) = \bigcap_{u \in \Gamma} \Gamma(u) \)
- \( \Gamma(t)\mathfrak{p}_D = \Gamma(t + \frac{d}{2}) \),

where \( d \) is the index of \( D \) (In our case of a non-split quaternion algebra \( d = 2 \)).

We further define \( \Gamma(t+) := \bigcup_{t < s} \Gamma(u) \), and we define the set of discontinuity points:

\[ \text{disc}(\Gamma) := \{ s \in \mathbb{R} | \Gamma(s) \neq \Gamma(s+) \} \].

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We can translate $\Gamma$ by a real number $s$: $(\Gamma - s)(t) := \Gamma(t + s)$, and the set of all real translates of $\Gamma$ is called the translation class of $\Gamma$. We denote this class by $[\Gamma]$. The set $\text{Latt}^1_{\mathfrak{D}}(V)$ of all $\mathfrak{D}$-lattice functions for $V$ carries a canonical affine structure, see [1], and there is up to a translation map $\Gamma \mapsto \Gamma - s$ a unique affine $\tilde{G}$-equivariant affine map

$$\iota_{\tilde{G}} : \mathfrak{B}(\tilde{G}) \to \text{Latt}^1_{\mathfrak{D}}(V).$$

Moreover this map is bijective. We write $\text{Latt}^1_{\mathfrak{D}}(V)$ for the set of all translation classes of $\mathfrak{D}$-lattice functions. The map $\iota_{\tilde{G}}$ induces an affine $\tilde{G}$-equivariant bijection

$$\iota_{\tilde{G}, \text{red}} : \mathfrak{B}_{\text{red}}(\tilde{G}) \to \text{Latt}_{\mathfrak{D}}(V)$$

such that the diagram

$$\begin{array}{ccc}
\mathfrak{B}(\tilde{G}) & \longrightarrow & \text{Latt}^1_{\mathfrak{D}}(V) \\
\downarrow & & \downarrow [ ] \\
\mathfrak{B}_{\text{red}}(\tilde{G}) & \longrightarrow & \text{Latt}^1_{\mathfrak{D}}(V)
\end{array}$$

commutes.

We now take the hermitian form $h$ into account. An element $\Gamma \in \text{Latt}^1_{\mathfrak{D}}(V)$ has a dual defined by $\Gamma^\#(t) := \Gamma((-t)^+)$, and we denote with $\text{Latt}^1_{\mathfrak{D}}(V)$ the set of elements of $\text{Latt}^1_{\mathfrak{D}}(V)$ which are self-dual, i.e. $\Gamma^\# = \Gamma$. There is a unique affine $G$-equivariant map

$$\iota_{G} : \mathfrak{B}(G) \to \text{Latt}^1_{h}(V),$$

see [2] 2.12 and [25] 3.5, also cf. [3] 4.2]. Moreover this map is bijective. Thus we can embed $\mathfrak{B}(G)$ into $\mathfrak{B}(\tilde{G})$ and $\mathfrak{B}_{\text{red}}(G)$ via $\text{Latt}^1_{h}(V) \subseteq \text{Latt}^1_{\mathfrak{D}}(V)$.

We need to be very precise in distinguishing between lattice functions and lattice sequences. Note that a point $x \in \mathfrak{B}_{\text{red}}(G)$ has barycentric coordinates with respect to the vertices in a chamber, see [26]. These barycentric coordinates are rational (we say $x$ is rational) if and only if for each pair $s, t \in \text{disc}(\Gamma)(\Gamma \in \iota_{\tilde{G}, \text{red}}(x))$ the difference $s - t$ is rational. In this case there is a minimal positive integer $e$ such that $s - t \in \frac{1}{e}\mathbb{Z}$ for all $s, t \in \text{disc}(\Gamma)$ and we can attach a lattice sequence $\Lambda$ to $\Gamma$ defined via $\Lambda(z) := \Gamma(\frac{z}{e})$, $z \in \mathbb{Z}$. In particular we obtain the translation class attached to $[\Gamma]$,

$$[\Lambda] = \{ \Lambda - k | k \in \mathbb{Z} \}, \quad (\Lambda - k)(l) = \Lambda(l + k), \quad l, k \in \mathbb{Z}.$$

If $x$ is an element of $\mathfrak{B}(G)$ then the above procedure attaches to $x$ a self-dual lattice sequence.

To a point $y \in \mathfrak{B}(G)$ and $x \in \mathfrak{B}(\tilde{G})$ one attaches the following filtrations:

- a Lie algebra filtration constructed by May-Prasad in [21]:
  - $(\mathfrak{g}_y, t)_{t \in \mathbb{R}}, \mathfrak{g}_{y, t} \subseteq \text{Lie}(G)$
  - $(\tilde{\mathfrak{g}}_x, t)_{t \in \mathbb{R}}, \tilde{\mathfrak{g}}_{x, t} \subseteq \text{Lie}(\tilde{G})$

  and

- a filtration of subgroups $(G_y, t)_{t \geq 0}, (\tilde{G}_x, t)_{t \geq 0}$ of $G$ and $\tilde{G}$, respectively, given by $G_{y, t} = G \cap (1 + \mathfrak{g}_{y, t})$ and $\tilde{G}_{x, t} = \tilde{G} \cap (1 + \tilde{\mathfrak{g}}_{x, t})$.

These filtrations can be described in terms of lattice functions and lattice sequences as follows: We take the lattice functions $\Gamma_y = \iota_{G}(y)$ and $\Gamma_x = \iota_{\tilde{G}}(x)$. Then we have

$$\tilde{\mathfrak{g}}_{x, t} = \{ a \in A | a(\Gamma_x(s)) \subseteq \Gamma_x(s + t), \forall s \in \mathbb{R} \}$$

and $\mathfrak{g}_{y, t} = \tilde{\mathfrak{g}}_{y, t} \cap \text{Lie}(G)$, see [19]. If $[x]$ and $[y]$ are rational and $\Lambda$ and $\Lambda'$ are the lattice sequences attached to $x$ and $y$ with $F$-period $e$ and $e'$ then we write

$$\tilde{\Lambda}(\Lambda) := \tilde{\mathfrak{g}}_{x, \frac{e}{e'}}, \quad \tilde{\Lambda}'(\Lambda') := \tilde{\mathfrak{g}}_{y, \frac{e}{e'}}$$
and define the pro-p groups

\[ \tilde{P}^m(\Lambda) := \tilde{G}_{x, \beta}, \quad P^m(\Lambda') := G_{y, \beta}, \quad m \in \mathbb{N}, \]

and the group \( \tilde{P}^0(\Lambda) := \tilde{G}_{x, 0} \) and further the stabilizers and their Lie algebras

\[ \tilde{P}(\Lambda) := \tilde{G}_{x}, \quad P(\Lambda') := G_{y}, \quad \tilde{a}(\Lambda) := \tilde{a}_0(\Lambda), \quad a(\Lambda') := a_0(\Lambda'). \]

We define \( P^0(\Lambda') \) to be the parahoric subgroup of \( G \) with respect to \( \Lambda' \), i.e. in details: The quotient \( P(\Lambda')/P^1(\Lambda') \) is the set of \( k_F \)-rational points of a reductive group \( \tilde{P}(\Lambda') \) defined over \( k_F \). Let \( P(\Lambda')^0 \) be the connected component of \( P(\Lambda') \) containing the identity. We write \( P^0(\Lambda') \) for the preimage of \( P(\Lambda')^0(k_F) \) in \( P(\Lambda') \).

### 2.3 Centralizer

Let \( \Delta \) be a semisimple stratum and \( \tilde{G}_\beta \) be the centralizer of \( \beta \) in \( \tilde{G} \). Assume further that \( \Delta \) is self-dual semisimple and write \( G_\beta \) for the centralizer of \( \beta \) in \( G \). The stratum provides a pair \( (\beta, \Lambda) \) consisting of an element \( \beta \) of the Lie algebra of \( G \) (which generates a product \( E \) of field extensions of \( F \)) and an \( \mathfrak{o}_E - \mathfrak{o}_D \)-lattice sequence \( \Lambda \). We need to attach to them a point of \( \mathfrak{B}(G_\beta) \) \( (\neq \prod_{\epsilon \in I_\beta}, \mathfrak{B}(G_\beta), \text{ cf. } [25, \text{ Section 6}]) \):

\[ \mathfrak{a}_z(\Lambda) \cap \text{End}_{E \otimes \mathcal{D}}(V) = \varnothing_{\epsilon \in I_\beta} \mathfrak{a}_z(\Lambda_\beta^\epsilon) \oplus \mathfrak{a}_z(\Lambda_\beta^\epsilon), \quad z \in \mathbb{Z}. \]

The CLF-property is meant with respect to the canonical embedding of Lie algebras \( \prod_{\epsilon \in I_\beta,} \text{End}_{E \otimes \mathcal{D}} V^\epsilon \hookrightarrow \text{End}_{\mathcal{D}} V \) (no \( I_\beta \), and canonical with respect to \( h \)). We also need the CLF-property on the level of general linear groups:

\[ \tilde{a}_z(\Lambda) \cap \text{End}_{E \otimes \mathcal{D}}(V) = \varnothing_{\epsilon \in I_\beta} \tilde{a}_z(\Lambda_\beta^\epsilon), \quad z \in \mathbb{Z}. \]

The construction of \( (\Lambda_\beta^\epsilon)_{\epsilon \in I_\beta} \) is done in several steps:

1. **By [25 Theorem 7.2]** there exists a \( G_\beta \)-equivariant, affine, injective CLF-map

\[ j_\beta : \mathfrak{B}(G_\beta) \hookrightarrow \mathfrak{B}(G) \]

whose image in terms of lattice functions is the set of self-dual \( \mathfrak{o}_E - \mathfrak{o}_D \)-lattice functions \( \text{Latt}^1_{\mathfrak{a}_E, \mathfrak{o}_E \otimes \mathfrak{o}_D} V \).

2. **We define for \( i \in I \) the skewfields:

\[ \mathcal{D}_\beta^i := \begin{cases} E_i \otimes \mathcal{D}, & \text{if } E_i \text{ has odd degree over } F. \\ E_i, & \text{else} \end{cases} \]

For every \( i \in I \) we have a right-\( \mathcal{D}_\beta^i \)-vector space \( V_\beta^i \) such that \( \text{End}_{\mathcal{D}_\beta^i} V_\beta^i \) is \( E_i \)-algebra isomorphic to \( \text{End}_{E \otimes \mathcal{D}} V^i \), and further we can find for every \( i \in I_\beta \) an \( \epsilon \)-hermitian-\( \mathcal{D}_\beta^i \)-form \( h_\epsilon^i \) on \( V_\beta^i \) such that its adjoint-anti-involution \( \sigma_\epsilon \) coincides with the pullback of the restriction of \( \sigma h_\epsilon \) to \( \mathcal{D}_\beta^i \). We interpret the buildings \( \mathfrak{B}(G_\beta) \) and \( \mathfrak{B}(G) \) in terms of lattice functions using \([2.4]\) and \([2.5]\). Now the construction of \( j_\beta \) in loc. cit. (which mainly uses [1 II.1.1.]) provides a tuple \( (\Gamma_\beta^\epsilon)_{\epsilon \in I_\beta} \) of \( \mathfrak{o}_{\mathcal{D}_\beta^i} \)-lattice functions.

3. **We fix a map [1 II.3.1] and attach an \( \mathfrak{o}_{\mathcal{D}_\beta^i} \)-lattice function \( \Gamma_\beta \) to \( \Gamma \cap V^i \) for \( i \in I \).**

4. **Now, let \( \epsilon \) be the F-period of \( \Lambda \) and \( \Gamma \) be a self-dual lattice function whose attached lattice function is a translate of \( \lambda \). We define the \( \mathfrak{o}_{\mathcal{D}_\beta^i} \)-lattice sequence \( \Lambda_\beta^\epsilon \) via \( \Lambda_\beta^\epsilon(z) := \Gamma_\beta(z) \).**

5. **Both CLF-properties \([2.5]\) and \([2.6]\) are satisfied by the construction given in [25] and [1].**

We write \( \Lambda_\beta \) for \((\Lambda_\beta^\epsilon)_{\epsilon \in I_\beta}\) and we are going to write \( b(\Lambda), \bar{b}(\Lambda), b_z(\Lambda), \bar{b}_z(\Lambda) \) for the intersections of \( \mathfrak{a}(\Lambda), \tilde{a}(\Lambda), \mathfrak{a}_z(\Lambda), \tilde{a}_z(\Lambda) \) with \( B \). Similarly we define \( \tilde{P}(\Lambda_\beta), P(\Lambda_\beta), \) etc. as the intersection with \( \tilde{G}_\beta \) of the respective objects, except \( P^0(\Lambda_\beta) \), which we define as the parahoric subgroup of \( G_\beta \) attached to \( \Lambda_\beta \), in particular \( P^0(\Lambda_\beta) \equiv \prod_{\epsilon \in I_\beta} P^0(\Lambda_\beta^\epsilon) \).
2.4 Extension of scalars

The group $G$ is the set of $F$ rational points of a connected reductive group $G$ defined over $F$. Let $L/F$ be a quadratic unramified extension of $F$. There is a very explicit description in Section 2 of $II$. We are going to reduce some results to the group $G(L)$ which we also call $G \otimes L$. We use a canonical injective map $i_L$ of $\mathfrak{B}(G)$ into $\mathfrak{B}(G \otimes L)$ given by $\Gamma_{i_L(x)} = \Gamma_x$, and the same for the embiant general linear groups: $i_L : \mathfrak{B}(\tilde{G}) \to \mathfrak{B}(\tilde{G} \otimes L)$. If we work over $L$ we give the objects in question the subscript $L$, for example we write $\mathfrak{g}_{L,x}$ and $\mathfrak{g}_{L,x'}$ for the Moy–Prasad filtration of a point $x$ in $\mathfrak{B}(G \otimes L)$. For the definition of semisimple characters of $\tilde{G} \otimes L$ we choose the $\text{Gal}(L/F)$-fixed extension $\psi_L$ of $\psi_F$ given by $\psi_L(x) := \psi_F(\frac{1}{2} \times tr_{L/F}(x))$.

2.5 Intertwining

We recall the notions of intertwining. Suppose we are given a smooth representation $\gamma$ on some copen subgroup $K$ of some totally disconnected locally compact group $H$. For an element $\pi \in \mathfrak{B}(G)$ contained in $\mathfrak{B}(G \otimes L)$ we denote by $I_{\pi}(\gamma)$ the set of intertwining elements of $\gamma$ in $H$.

2.6 Restriction

We recall that, given two locally compact totally disconnected groups $G_1, G_2$ such that $G_2$ is a topological subgroup of $G_1$, we denote by $\text{Res}^{G_1}_{G_2}$ the functor from $\mathfrak{R}(G_1)$ to $\mathfrak{R}(G_2)$ given by restriction from $G_1$ to $G_2$.

3 Exhaustion for semisimple Characters

In this section we prove the following theorem.

**Theorem 3.1** (see [32] 5.1 for the non-quaternionic case). Let $\pi$ be an irreducible representation of $G$. Then there is a self-dual semisimple stratum $\Delta$ with $r = 0$ and an element $\theta$ of $C(\Delta)$ such that $\theta$ is contained in $\pi$.

The proof of this theorem requires several steps which we deduce from *loc.cit.*. We fix an irreducible smooth representation $\pi$ of $G$. The proof of the theorem is done by induction.

(i) In the base case we show the existence of a trivial semisimple character contained in $\pi$.

(ii) The induction for $\theta \in C(\Delta)$ contained in $\pi$ is on the fraction $\frac{r}{e(\Lambda/F)}$.

(iii) For the induction step we need to be able to change lattice sequences: Roughly speaking, given a self-dual semisimple character $\theta \in C(\Delta)$ with positive $r$ and contained in $\pi$ there is a self-dual stratum $\Delta'$ such that $\beta = \beta'$ and $\frac{r'}{e(\Lambda'/F)} < \frac{r}{e(\Lambda/F)}$ and an element $\theta' \in C(\Delta')$ which is contained in $\pi$.

(iv) These steps are not enough, because one has to ensure that the difference between $\frac{r}{e(\Lambda/F)}$ and $\frac{r'}{e(\Lambda'/F)}$ is bounded from below by a positive constant independent of $\Lambda$ and $\Lambda'$.

Recall that the depth of $\pi$ is the infimum of all non-negative real numbers $t$ with a point $x$ in $\mathfrak{B}(G)$ such that the trivial representation of $G_{x,t}$ is contained in $\pi$. Let us recall that the barycentric coordinates of a point $x$ in $\mathfrak{B}(G)$ are the barycentric coordinates of the point with respect to the vertexes of any chamber containing $x$.

**Lemma 3.2** (see [21] 5.3, 7.4). The depth of $\pi$ is attained at a point with rational barycentric coordinates.
This follows from *loc.cit.* because the depth is attained in an optimal point, see *loc.cit.* 7.4.

**Remark 3.3.** One could think that a continuity argument with the function

\[ x \in \mathfrak{B}(G) \mapsto d(\pi, x) := \inf \{ s \geq 0 \mid V^{G_x,x}_s \neq \emptyset \} \]

could lead to Lemma 3.2 but it is unclear if this function is continuous. It is upper-continuous, but maybe not lower continuous. Here the idea in *loc.cit.* of taking optimal points comes into play which form a finite set for a given chamber \( C \).

We prove the upper-continuity of \( d(\pi, \ast) \) in the above remark. It is not needed for what follows in this article.

**Proof.** Take \( x \in \mathfrak{B}(G) \). It corresponds to a self-dual lattice function \( \Gamma_x \) with set disc \( x \) of discontinuity points of \( \Gamma \). We take a CAT(0)-metric \( d(\ast, \ast) \) on \( \mathfrak{B}(G) \) given in [33]. The point \( x \) lies in the interior of a facet \( F_x \). Then for all positive real \( \delta_1 \) there exists a positive real \( \delta_2 \) such that

(i) The ball around \( x \) with radius \( \delta_2 \) does not intersect any facet of lower dimension than \( F_x \).

(ii) For all \( x' \in \mathfrak{B}(G) \) with \( d(x, x') < \delta_2 \) and all \( t \in \text{disc} \( x \) there exists a \( t' \in \text{disc} \( x' \) such that \( |t-t'| < \delta_1 \) and \( \Gamma_x(t) = \Gamma_x(t') \).

(iii) For all \( x' \in \mathfrak{B}(G) \) with \( d(x, x') < \delta_2 \) and all \( t' \in \text{disc} \( x' \) there exists a \( t \in \text{disc} \( x \) such that \( |t-t'| < \delta_1 \) and \( \Gamma_x(t) \geq \Gamma_x(t') \).

Then for every \( x' \in \mathfrak{B}(G) \) and every \( s \geq 0 \) we have

\[ G_{x,s} \supseteq G_{x',(s+2\delta_1)++} \]

In particular these \( x' \) satisfy

\[ d(\pi, x') \leq d(\pi, x) + 2\delta_1, \]

which finishes the proof. \( \square \)

For the proofs of the next lemmas we need some duality for the Moy–Prasad filtrations. Given a subset \( S \) of \( A \) the dual \( S^* \) of \( S \) with respect to \( \psi \) is defined as the subset of \( A \) consisting of all elements \( a \) of \( A \) which satisfy \( \psi A (sa) = 1 \) for all \( s \in S \). (\( \psi A := \psi \circ \text{trd} \)). The main property of this duality is:

**Lemma 3.4.** Let \( x \) be a point of \( \mathfrak{B}(\tilde{G}) \). Then we have \( \tilde{g}_{x,t}^* = \tilde{g}_{x,-t}^* \) for all \( t \in \mathbb{R} \).

**Proof.** We choose a splitting basis \( \{ v_k \}_k \) for \( \Gamma_x \), i.e.

\[ \Gamma_x(t) = \oplus_k v_k p_D^{(d(t-a_k))}. \]

We denote by \( E_{ij} \) the element of \( A \) with kernel \( \oplus_{k+j} v_k D \) which sends \( v_j \) to \( v_i \). They form a \( D \)-left basis of \( A \) which splits \( \tilde{g}_x \), more precisely:

\[ \tilde{g}_{x,t} = \oplus_{ij} p_D^{(d(t+a_j-a_i))} E_{ij}. \]

We now show the assertion of the lemma. The inclusion \( \supseteq \) is obvious. For the other inclusion we first remark that \( \tilde{g}_x^* \) is split by \( (E_{ij})_{ij} \) too:

\[ \tilde{g}_{x,t}^* = \oplus_{ij} p_D^{e_{ij}(t)} E_{ij}, \quad e_{ij}(t) \in \mathbb{Z} \]

Then \( e_{ji}(t) + [d(t+a_j-a_i)] \) is positive and therefore \( e_{ji}(t) > d(-t+a_i-a_j) \) which finishes the proof since \( e_{ji}(t) \) is an integer. \( \square \)

For an element \( \beta \in A \) we define the map \( \tilde{\psi}_\beta : A \to \mathbb{C} \) via \( \tilde{\psi}_\beta(1+a) := \psi_A(\beta a) \). Some restrictions of \( \tilde{\psi}_\beta \) are characters, i.e. multiplicative, as for example in the following case:
Definition 3.5. Let \( \Delta = [A, n, n - 1, \beta] \) be stratum which is not equivalent to a null-stratum. Then we define \( d_\Delta := \deg(\mu(\Delta/F)) \) to be the depth of \( \Delta \). Let \( x \in \mathfrak{B}(\tilde{G}) \) be a point corresponding to \( \Lambda \). The coset of \( \Delta \) in terms of the building is defined as \( \beta + \tilde{g}_{x,-d_\Delta} \cdot \), and if \( \Delta \) is self-dual then we call \( \beta + \tilde{g}_{x,-d_\Delta} \cdot \) its self-dual coset. To \( \Delta \) is attached the character \( \psi_\Delta : G_{\xi,d_\Delta} \to \mathbb{C} \) defined via restriction of \( \psi_\beta \). Note that \( \psi_\Delta \) is trivial on \( G_{\xi,d_\Delta} \). If \( \Delta \) is self-dual we write \( \psi_\Delta \) for the restriction of \( \psi_\Delta \) to \( G_{\xi,d_\Delta} \). We say that \( \pi \) contains \( \Delta \) or the associate coset, if it contains \( \psi_\Delta \).

Proposition 3.6 (see [32] for \( G \otimes L \)). Suppose \( \pi \) has positive depth. Then \( \pi \) contains a self-dual semisimple stratum \( \Delta \) with \( n = r + 1 \) and of the same depth.

For the proposition we need a convexity lemma for semisimple strata. Let us recall: The minimal polynomial \( \mu(\Delta) \) for a stratum \( \Delta = [A, n, n - 1, \beta] \) is the minimal polynomial in \( k_F[X] \) of the residue class \( \eta(\Delta) \) of \( \eta(\Delta) := \lim_{\delta(\Lambda/F)} - \frac{\delta(\Lambda/F)}{\delta(\Lambda/F)} \) modulo \( \tilde{a}_1(\Lambda) \), see I Section 4.2. Further the characteristic polynomial \( \chi(\Delta) \) of \( \Delta \) is the reduced characteristic polynomial of \( \eta(\Delta) \) mod \( \mathfrak{p} \).

Lemma 3.7. Suppose that \( \Delta = [A, n, n - 1, \beta] \) and \( \Delta' = [A', n, n - 1, \beta] \) are strata over \( F \) which are equivalent to semisimple strata and share the characteristic polynomial and the \( F \)-period. Then \( \Delta'' = \left[ \frac{1}{2}A + \frac{1}{2}A', 2n, 2n - 1, \beta \right] \) is equivalent to a semisimple stratum.

In this lemma we have interpreted \( \Lambda \) and \( \Lambda' \) as lattice sequences of period 2.

Proof of Proposition 3.6. The depth of \( \pi \) is rational because \( \pi \) attains its depth at an optimal point of \( \mathfrak{B}(G) \), say \( x \), a point with rational barycentric coordinates by [21]. Then there is an element \( b \) of \( \mathfrak{g}_{x,-d_\Delta} \) such that \( b + \mathfrak{g}_{x,-d_\Delta} \cdot \) is contained in \( \pi \). We are going to show that there is a self-dual semisimple stratum \( \Delta \) with \( r + 1 = n \) whose coset \( b + \mathfrak{g}_{x,-d_\Delta} \cdot \) contains \( \mathfrak{g}_{x,-d_\Delta} \cdot \). Note that semisimple and \( n = r + 1 \) imply that \( \Delta \) is non-null. By [31] 4.2, Proof of 4.4 and [11] 4.7 there is a point \( x_1 \in \mathfrak{B}(G \otimes L) \) with rational barycentric coordinates which satisfies the following property \( \ast \): The coset \( b + \tilde{g}_{x_1,-d_\Delta} \cdot \) is a coset of a semisimple stratum over \( L \), \( b \in \mathfrak{g}_{x_1,-d_\Delta} \) and \( \tilde{g}_{x_1,-d_\Delta} \cdot \) contains \( \tilde{g}_{x_1,-d_\Delta} \cdot \). The Galois group \( \tau \), see [28] 5.5, and the corresponding coset is a coset of a semisimple stratum, by Lemma 3.7. The point \( x_1 '' \) is fixed by \( \tau \) and is therefore of the form \( i_{\tilde{u}}(x'') \) for some \( x'' \in \mathfrak{B}(G) \). Let \( \Delta'' = [A'', n'', n'' - 1, \beta] \) be a self-dual stratum for the coset \( b + \tilde{g}_{x'',-d_\Delta} \cdot \). Note that \( \Delta'' \) is equivalent to a null-stratum by the definition of \( d_\Delta \). An \( \Delta'' \otimes L \) is equivalent to a semisimple stratum and as \( b \in \text{Lie}(G) \) we obtain that \( \Delta'' \) is equivalent to a self-dual semisimple stratum by [14] 54 and [11] 4.7. \( \square \)
The same idea of extension of scalars shows:

**Corollary 3.8** (cf. [31] 4.4). Let $\Delta$ be a self-dual fundamental stratum with $n = r + 1$. Then there is a self-dual semisimple stratum $\Delta'$ with $n' = r' + 1$ such that \( \frac{n}{e(\Lambda/F)} = \frac{n'}{e(\Lambda'/F)} \) and $\beta + a_{-r} \subseteq \beta' + a'_{-r}$.

Apparently we also need the $\tilde{G}$-version, whose proof is similar to the proof of the previous corollary, noting that the proof [31] 4.2 also goes through for non-self-dual strata over $L$.

**Corollary 3.9.** Let $\Delta$ be a fundamental stratum with $n = r + 1$. Then there is a self-dual semisimple stratum $\Delta'$ with $n' = r' + 1$ such that \( \frac{n}{e(\Lambda/F)} = \frac{n'}{e(\Lambda'/F)} \) and $\beta + \tilde{a}_{-r} \subseteq \beta' + \tilde{a}'_{-r}$.

From now on we assume in this section that $\pi$ has positive depth. By Proposition [3.6] there is a self-dual semisimple stratum $[\Lambda, n, n - 1, \beta]$ contained in $\pi$. We formulate the key proposition for the induction step for Theorem $3.4$.

**Proposition 3.10** (see [32] 5.4 over $L$). Given a self-dual semisimple stratum $\Delta$ with positive $r$, an element $\theta \in C(\Delta(1-))$ and an element $c \in a_{-r}$, we suppose that $\theta \psi_c$ is contained in $\pi$. We fix a tame corestriction $s_3$ with respect to $\beta$. Let $\Lambda'$ be a self-dual $\theta$-bi-$L$-lattice sequence, $r'$ a positive integer and $b'$ an element of $\mathfrak{b}_1' \cap \mathfrak{b}_{-r}'$ such that $s_3(c) + \mathfrak{b}_1' - r$ is contained in $b' + \mathfrak{b}_1' - r$. Suppose further that $\frac{r}{e(\Lambda/F)} \leq \frac{r'}{e(\Lambda'/F)}$. Then $\Delta'$ with $\beta' = \beta$ is a self-dual semisimple stratum and there are $\theta' \in C(\Delta'(1-))$ and $c' \in a_{-r}'$, such that $s_3(c')$ is equal to $b'$ and $\theta' \psi_{c'}$ is contained in $\pi$. The element $c'$ can be chosen to vanish if $b' = 0$.

Essentially Proposition 3.10 says that if $\theta \psi_c$ is contained in $\pi$ then one can work in $B$ to find a “better” character $\theta' \psi_{c'}$. We explain the strategy of its proof (taken from [32] 5.4): At first one constructs open compact subgroups $K^1_\ell(\Lambda)$ and $K^2_\ell(\Lambda)$ ($\ell \in \mathbb{N}$) of $P^*(\Lambda)$ via

\[
K^1_\ell(\Lambda) := 1 + \tilde{a}_{(\ell+1)} \cap (\prod_i A^{ij} \oplus \prod_i \tilde{a}_i),
\]

\[
K^2_\ell(\Lambda) := 1 + \tilde{a}_{(\ell+1)} \cap (\prod_i A^{ij} \oplus \prod_i \tilde{a}_i),
\]

and further $\tilde{H}^1(\beta, \Lambda)$ and $\tilde{J}^1(\beta, \Lambda)$ as intersections of $\tilde{H}(\beta, \Lambda)$ and $\tilde{J}(\beta, \Lambda)$ with $K^1_\ell(\Lambda)$. And we get the groups $K^1_\ell, H^1_\ell$ and $J^1_\ell$ if we intersect further down to $G$. We extend $\theta$ to a semisimple character of $H^1(\beta, \Lambda)$ (which we still call $\theta$) and we consider the character $\xi := \theta \psi_c$ on $H^1_\ell(\beta, \Lambda)$. Mutatis mutandis as in [32] 5.7 one shows that $\xi$ is contained in $\pi$. This representation $\xi$ is very helpful for detecting if a certain representation is contained in $\pi$:

**Lemma 3.11** (cf. [8] (8.1.7), [32] 5.8). We granted $r > 0$. Let $\rho$ be an irreducible representation on an open subgroup $U$ of $K^2_\ell(\Lambda)$. Then $\rho$ is a subrepresentation of $\pi$, if its restriction to $U \cap H^1_\ell(\beta, \Lambda)$ contains $\xi$.

By [32] 5.12 we can cut the line in $\mathfrak{B}(G)$ between $\Lambda$ and $\Lambda'$ into segments with cutting points $\Lambda =: \Lambda_1, \Lambda_2, \ldots, \Lambda_s =: \Lambda'$ such that for each index $1 \leq k < s$ we have $P_{\Lambda_{k+1}}(\Lambda_{k+1}) \subseteq K^{s}_{\ell}(\Lambda_{k})$. For the definition of $r_1$, see [32] 5.1. Indeed: One applies loc.cit. to the line in $\mathfrak{B}(\tilde{G} \otimes L)$ and intersects the inclusions down to $\tilde{G}$. They still satisfy $\frac{r}{e(\Lambda/F)} \geq \frac{r}{e(\Lambda_{s-1}/F)}$. By loc.cit. this reduces Proposition 3.10 to the case $s = 2$. Thus we have to prove Case $s = 2$ and Lemma 3.11 to obtain Proposition 3.10.

For the proof of Lemma 3.11 we need:

**Lemma 3.12** (cf. [32] 5.9). Granted $r > 0$, there is a unique irreducible representation $\mu$ of $J^1_\ell$ containing $\xi$ (called the Heisenberg extension of $\xi$ to $J^1_\ell$), because the bi-linear form

\[ k_\xi : J^1_\ell/H^1_\ell \times J^1_\ell/H^1_\ell \to \mathbb{C}^\times, \quad k_\xi(x, y) := \xi([x, y]) = \theta([x, y]) \]

is non-degenerate.

**Proof.** Let $\xi_L$ and $\theta_L$ be the unique $\text{Gal}(L/F)$-fixed extensions to $H^1_{L-1}$ and $H^1_{L+1}$ of $\xi$ and $\theta$, respectively. We have the analogous form $k_{\xi_L}$ on $J^1_{L-1}/H^1_{L-1}$ for $\xi_L$, and this form is non-degenerate by [32] 5.9. Let $\bar{x}$ be in the kernel of $k_\xi$ and let $y$ be an element of $J^1_{L-1}$. Then

\[ k_{\xi_L}(\bar{x}, y) = \theta_L([x, y]) = \theta_L([x, \tau(y)]) = k_{\xi_L}(\bar{x}, \tau(y)) \]
In particular
\[ k_{\xi_L}(\bar{x}, \bar{y})^2 = k_{\xi_L}(\bar{x}, \bar{y} \tau(\bar{y})). \]
We take here the obvious Galois-action on \( J'_{L,1}/H'_{L,1} \). Its fixed point set is \( J'_{L}/H'_{L} \) because the first \( \text{Gal}(L|F) \)-cohomology of \( H'_{L,1} \) trivial, in particular \( \bar{y} \tau(\bar{y}) \) is an element of \( J'_{L}/H'_{L} \). Thus \( k_{\xi_L}(\bar{x}, \bar{y})^2 \) vanishes and therefore \( k_{\xi_L}(\bar{x}, \bar{y}) = 1 \), because it is a \( p \)-th root of unity. Thus \( k_{\xi} \) is non-degenerate. \( \square \)

**Proof of Lemma 3.14.** For the proof we skip the argument \( \Lambda \) in the notation. The proof needs two parts: We show

(i) There exists up to isomorphism only one irreducible representation \( \omega \) of \( K^r \) which contains \( \xi \). In fact we will further obtain that \( \text{ind}_{H^r_{L,1}}^{K^r} \xi \) is a multiple of \( \omega \).

(ii) The restriction of \( \pi \) to \( U \) contains \( \rho \).

Part (ii) follows as in the final argument in the proof of [3] (8.1.8)]. We only have to prove Part (i)

We take the representation \( \mu \) of Lemma 3.12 and prove that the \( \omega := \text{ind}_{H^r_{L,1}}^{K^r} \mu \) is irreducible. Let \( g \) be an element of \( K^r \) which intertwines \( \mu \). Then the \( g \)-intertwining space \( I_g(\mu) \) satisfies the formula:

\[ \text{dim}_C(I_g(\mu) \ast (J^r_{L} : H^r_{L}))^\perp = \text{dim}_C(I_g(\text{ind}_{H^r_{L,1}}^{K^r} \xi)) \]

by [3] (4.1.5)]. The latter cardinality is equal to

\[ \frac{(J^r_{L} : H^r_{L}) \ast (\text{ind}_{H^r_{L,1}}^{K^r} \mu \ast (J^r_{L} : H^r_{L}) \ast (\text{ind}_{H^r_{L,1}}^{K^r} \mu)))}{(H^r_{L} : H^r_{L} \ast \text{ind}_{H^r_{L,1}}^{K^r} \mu) \ast (H^r_{L} : H^r_{L} \ast \text{ind}_{H^r_{L,1}}^{K^r} \mu))} \]

and therefore odd. Thus \( g \) intertwines the Galoisian transfer \( \mu_L \) of \( \mu \) by [30] 2.4, in particular \( g \) is an element of \( J^r_{L,1} \) by [32] 5.9 (See the second part of the proof in loc.cit.. He shows \( I_{K^r_{L,1,2}}(\mu_L) \subset J^r_{L,1} \)). Thus \( g \in J^r_{L,1} \). Therefore \( \omega \) is irreducible and

\[ \text{ind}_{H^r_{L,1}}^{K^r} \xi \equiv \text{ind}_{J^r_{L,1}}^{K^r} \text{ind}_{J^r_{L,1}}^{J^r_{L,1}} \xi \equiv \text{ind}_{J^r_{L,1}}^{K^r} \mu \text{ind}_{J^r_{L,1}}^{(J^r_{L} : H^r_{L}))} \equiv \omega \text{ind}_{(J^r_{L} : H^r_{L}))} \]

which finishes the proof. \( \square \)

To finally prove Proposition 3.10 we need the Cayley map (depending on \( \Lambda \))

\[ \text{Cay} : a_1 \rightarrow \mathbb{P}^1(\Lambda), \text{Cay}(a) := \frac{1 + \frac{a}{2}}{1 - \frac{a}{2}}. \]

It is a bijection.

**Proof of Proposition 3.10.** We only need to consider the case \( s = 2 \) by the above explanation. It follows from the Proof of [32] 5.4 with assumption (H)] (See there the paragraph after assumption (H)). One obtains the inclusions and equations over \( L \), and passing to \( \tau \)-symmetric elements gives the properties over \( D \). Caution: In the proof \( \xi \) and \( \theta \psi_{|H^r(\beta, \Lambda)} \) are called \( \hat{\vartheta} \) and \( \vartheta \), respectively, and the map \( \text{Cay} \) is called \( C \). \( \square \)

The theory of optimal points gives the following lemma.

**Lemma 3.13 (cf [31] 4.3, [21] 6.1).** (i) Let \( \Lambda \) be a lattice sequence of \( D \)-period \( c \) and \( m \) a positive integer such that \( \bar{a} - m \neq \bar{a} - m + 1 \). Then there is a lattice chain \( \Lambda' \) of \( D \)-period \( c' \) and an integer \( m' \) such that \( m' \leq \frac{m}{c} \) and \( \bar{a}_m \leq \bar{a}_m - m' \).

(ii) Let \( \Lambda \) be a self-dual lattice sequence of \( D \)-period \( c \) and \( m \) a positive integer such that \( a_{-m} \neq a_{-m + 1} \). Then there is a self-dual lattice sequence \( \Lambda' \) of \( D \)-period \( c' \) smaller than \( 2 \dim_D V \) and an integer \( m' \) such that \( \frac{m'}{c'} \leq \frac{m}{c} \) and \( a_{-m} \leq a_{-m'} \).
In this lemma D does not play a role, i.e. the proof is the same for F and D. We give here a very simple proof of the above lemma using a different idea than roots.

**Proof.** The second assertion follows directly from the first one. Without loss of generality we can assume that \( \frac{r}{e} \) is smaller than 1. We reformulate the statement.

We consider a point \( x \in B(G) \) and \( t \in ]0, \frac{1}{\pi} [ \). The point \( [x] \) of \( B_{\text{red}}(G) \) lies in the closure of a chamber \( C \). Then there is a midpoint \( [y] \) of a facet of \( C \) such that \( \tilde{g}_{x,-t} \subseteq \tilde{g}_{y,-t} \).

For simplicity we assume \( d = 1 \), i.e. we prove the statement over \( F \). (or one just rescales to get for the \( \tilde{g}_x \) the period 1.) For a lattice \( M \) which occurs in the image of a lattice function \( \Gamma \) corresponding to \( x \) we set \( s_M \) to be the maximum of all real \( s \) such that \( \Gamma(s) = M \). We define the following sequence of real numbers

\[
s_0 := 0, \quad s_j := s_{\Gamma(s_{j-1} - t)}, \quad j \geq 0.
\]

At first we observe that the sequence gets periodic mod \( \mathbb{Z} \), say the period is given by \( s_{j+1}, \ldots, s_{j+c} \). Let \( [y] \in B_{\text{red}}(G) \) be the barycentre of the facet whose vertexes correspond to the homothety classes of the lattices \( \Gamma(s_j) \), \( j = 1, \ldots, c \). Note that these homothety classes differ pairwise. We write \( u \) for \( s_j - s_{j+c} \), in particular \( t \geq \frac{u}{c} \), and let \( \Gamma' \) be a lattice function corresponding to \( y \). Let \( \Gamma'' \) be the lattice function obtained from \( \Gamma \) in deleting all lattices from \( \Gamma \) which are not in the image of \( \Gamma' \), i.e. if \( \Gamma(s) \) does not occur in the image of \( \Gamma' \) then we replace \( \Gamma(s) \) by \( \Gamma((s + v)\ast) \) where \( v \) is the smallest non-negative real number such that \( \Gamma((s + v)\ast) \) is in the image of \( \Gamma' \). Then \( \Gamma'''([s_{j+1}, s_{j+c}]) \) contains exactly \( u \) lattices because there are no repetitions in the period. So \( \tilde{g}_{y,-t} \) contains \( \tilde{g}_{x,-t} \).

Now we are able to finish the proof of Theorem 5.4

**Proof.** The proof is similar to the first part of the argument after the proof of [32, 5.5]. Let \( z \) be the minimal element of \( \frac{1}{\pi} \mathbb{Z} \) \((N := \dim_F V)\) such that there is a self-dual semisimple character \( \theta \in C(\Delta) \) contained in \( \pi \) with \( \frac{r}{c(\Delta_F)} \leq z \). If \( z = 0 \) we are ready. Else if \( z \) is positive then we extend \( \theta \) to \( C(\Delta(1-)) \) and call it again \( \theta \) and there is a \( c \in \alpha_r \) such that \( \theta \psi_c \) is contained in \( \pi \). The element \( c \) can be chosen in \( \prod \Lambda^{ii} \) by loc.cit. 5.2. Let \( s_\beta \) be the tame corestriction with respect to \( \beta \). Then the multi-stratum \([\Lambda_\beta, r, r-1, s_\beta(c)]\) has to be fundamental, i.e. at least one of the strata \([\Lambda_\beta', r, r-1, s_\beta(c)]\) has to be fundamental, by the argument in the proof of loc.cit. 5.5 using Proposition 3.10 and Lemma 3.13.

Note further the latter stratum being fundamental also implies that \( \frac{r}{c(\Delta_F)} \) is an element of \( \frac{1}{\pi} \mathbb{Z} \) by [31, 2.11] (using [23, 3.11]), i.e. \( \frac{r}{c(\Delta_F)} = z \) by the definition of \( z \). We apply [31, 4.4], Corollary 5.8 and Corollary 5.8 and [23, 3.11] to choose for every \( i \in I_0 \cup I_1 \) a semisimple stratum \([\Gamma', r_i, r_i-1, \alpha_i]\), such that

\[
\begin{align*}
(i) & \quad \text{the stratum is self-dual if } i \in I_0, \\
(ii) & \quad s_\beta(c) + \tilde{\alpha}_1 - (\Lambda_\beta') \subseteq \alpha_i + \tilde{\alpha}_1 - (\Gamma') \text{, for all } i \in I_0 \cup I_1, \text{, and} \\
(iii) & \quad \frac{r_i}{e(\Lambda_\alpha|\Lambda_\beta)} \geq \frac{r_i}{e(\Gamma'|\Lambda_\beta)} \text{, for all } i \in I_0 \cup I_1, \text{, with equality if } [\Lambda_\beta', r, r-1, s_\beta(c)] \text{ is fundamental.}
\end{align*}
\]

We take a self-dual \( \sigma_{\alpha_0} - \sigma_{\alpha_1} \)-lattice sequence \( \Lambda' \) such that \( \Lambda'_\beta \) is an affine translate of \( \Gamma' \) for every \( i \in I_0 \), see [27, 5.3]. Note that this affine translation can be chosen such that the scaling constant is bounded by \( N/4 \). We put \( r' := \frac{e(\Delta(1-))}{e(\Delta')} \), and we consider the multi-stratum \([\Lambda_\beta', r', r'-1, s_\beta(c)]\) We have

\[
s_\beta(c) + \tilde{b}_{1-r}(\Lambda) \subseteq s_\beta(c) + \tilde{b}_{1-r}(\Lambda'), \quad s_\beta(c) \in \tilde{b}_{e-r}(\Lambda) \cap \tilde{b}_{e-r}(\Lambda').
\]

Thus, by Proposition 3.10 there is a self-dual semisimple stratum \( \Delta' \) with \( \beta' = \beta \) and a character \( \theta' \in C(\Delta'(1-)) \) and an element \( e' \in \alpha_1' \) such that \( \theta' \psi_{e'} \) is contained in \( \pi \) and \( s_\beta(e') = s_\beta(c) \). Now \( \Delta' \) is semisimple and \([\Lambda_\beta', r', r'-1, s_\beta(c)]\) is equivalent to a semisimple multi-stratum. Then \([\Lambda', n', n'-1, \beta' + e']\) is equivalent to a semisimple stratum by I 4.15. Further the self-duality of the stratum implies that it
is equivalent to a self-dual semisimple stratum, by II 4.7, say $\Delta''$. Then $C(\Delta'') = C(\Delta'(1-))\psi_c$. Thus there is an element $\theta''$ of $C(\Delta'')$ contained in $\pi$ and

$$\frac{r''-1}{e(\Lambda'^{\prime}|F)} = \frac{r'-1}{e(\Lambda'|F)} < \frac{r}{e(\Lambda|F)} = z.$$

Note that on the other hand we could have started with $\theta''$ and therefore $\frac{r''-1}{e(\Lambda'|F)} = z$. A contradiction. □

4 Heisenberg extensions

The study of Heisenberg extensions and their extensions are the technical heart of Bushnell–Kutzko theory, for both: the construction of cuspidal representations and the exhaustion. We will review the results for $G_L := G \otimes L$ and extend them to $G$. In this section we fix a stratum $\Delta = [\Lambda, n, 0, \beta]$. Let $\Lambda'$ be an $\frak{e}_G$-$\frak{e}_D$-lattice sequence which satisfies $\frak{b}(\Lambda') \subseteq \frak{b}(\Lambda)$. Let us recall that we have the following sequence of groups:

$$H^1_\Lambda := H^i(\beta, \Lambda), \quad J^i_\Lambda := J^i(\beta, \Lambda), \quad i \in \mathbb{N}, \quad J_\Lambda := J(\beta, \Lambda), \quad J^0_\Lambda := J^1(\beta, \Lambda)P^0(\Lambda_\beta).$$

and

$$J^i_{\Lambda, L} := J^i_{\Lambda}(L), \quad J^0_{\Lambda, L} := J^1_{\Lambda}(L), \quad J^i_{\Lambda, L} := J^1_{\Lambda}(L).$$

We have similar subgroups $J^i_{\Lambda, L}$ and $H^1_\Lambda$ of $G_L$. We fix a character $\theta \in C(\Delta)$, and let $\theta'$ be the transfer of $\theta$ from $\Lambda$ to $\Lambda'$. We denote the Gal($L/F$)-Glauberman lifts of $\theta$ and $\theta'$ by $\theta_L$ and $\theta'_L$.

At first we recall the Heisenberg extensions for $G_L$.

Proposition 4.1 ([32] 3.29, 3.31, [18] 4.1). (i) There is up to equivalence a unique irreducible representation of $(\eta_{\Lambda L}, J^1_{\Lambda})$ which contains $\theta_L$.

(ii) Let $g$ be an element of $G_L$. The complex dimension of $I_g(\eta_{\Lambda L})$ is at most one, and it is one if and only if $g \in J^1_{\Lambda L}(G)\beta J^1_{\Lambda L}$.

We want to prove its analogue for $G$. At first we need a lemma which allows us to apply Bushnell–Fröhlichs’ work to construct Heisenberg extensions:

Lemma 4.2 (cf. [32] 3.28 for $G_L$). The form

$$k_\theta : J^1_\Lambda/H^1_\Lambda \times J^1_\Lambda/H^1_\Lambda \to \mathbb{C}$$

defined via $k_\theta(x, y) := \theta([x, y])$ is non-degenerate. The pair $(J^1_\Lambda/H^1_\Lambda, k_\theta)$ is a subspace of $(J^1_{\Lambda L}/H^1_{\Lambda L}, k_{\theta_L})$ ($k_{\theta_L}$ similarly defined).

The proof is similar to the proof of Lemma 3.12.

Recall that given an irreducible representation $\gamma$ on some copen pro-$p$-subgroup $K$ of $G_L$ we denote by $gl(\gamma)$ the Gal($L/F$)-Glauberman transfer. It is the unique irreducible representation of $K_{\text{Gal}(L/F)}$ with odd multiplicity in $\gamma$.

Proposition 4.3. (i) There is up to equivalence a unique irreducible representation $\eta_{\Lambda}$ of $J^1_{\Lambda}$ which contains $\theta$. Further $\eta_{\Lambda}$ has degree $(J^1_{\Lambda}:H^1_{\Lambda})^\dag$.

(ii) The representation $\eta_{\Lambda}$ is the Gal($L/F$)-Glauberman transfer of $\eta_{\Lambda L}$ to $J^1_{\Lambda}$.

(iii) Let $g$ be an element of $G$. The complex dimension of $I_g(\eta_{\Lambda})$ is at most one, and it is one if and only if $g \in J^1_{\Lambda L}(G)\beta J^1_{\Lambda L}$.

Proof. We define $\eta_{\Lambda}$ as $gl(\eta_{\Lambda L})$, see Proposition [4.3]. The restriction of $\eta_{\Lambda L}$ to $H^1_\Lambda$ is a multiple of $\theta$. Thus the same is true for $\eta_{\Lambda}$. And thus by [7] 8.1 and Lemma 4.2 up to equivalence $\eta_{\Lambda}$ is the unique
irreducible representation of \( J^1 \) which contains \( \theta \), and further it has the desired degree. An element of \( G_\beta \) intertwines \( \eta_{\lambda L} \) so it intertwines \( \eta \) by \([33] \) 2.4. On the other hand we have

\[
I_G(\eta_{\lambda}) \subseteq I_G(\theta) = J^1_G G_\beta J^1_A,
\]

which finishes the proof of the intertwining formula. The most complicated part is the proof of dimension one of the non-zero intertwining spaces. For this we refer to the proof of \([33] \) 4.1. Note that after taking \( \text{Gal}(L[F]) \) fixed points in the rectangular diagram of \([33] \) 4.1 the rows and columns remain still exact by the additive Hilbert 90. The rest of the proof is mutatis mutandis.

\[\square\]

For the exhaustion the following extensions of \( \eta_{\lambda} \) are the key technical tools. We will emphasize the importance when their application arises. Note that we say that a representation \( (\gamma, K) \) is an extension of a representation \( (\gamma', K) \) if \( K \) is a subgroup of \( \tilde{K} \) and the restriction of \( \gamma \) to \( K \) is equivalent to \( \gamma \).

**Proposition 4.4** (\([33] \) 3.7). Suppose \( \tilde{\alpha}(\Lambda') \subseteq \tilde{\alpha}(\Lambda) \). There is up to equivalence a unique irreducible representation \( (\eta_{\lambda'}, J^1_{\lambda', A}) \) which is an extension of \( (\eta_{\lambda}, J^1_{\lambda, A}) \) such that \( \eta_{\lambda', L} \) and \( \eta_{\lambda} \) induce equivalent irreducible representations on \( P^1(\Lambda') \). Moreover the set of intertwining elements of \( \eta_{\lambda', L} \) in \( G_L \) is \( J^1_{\lambda', A} (G_L)_\beta J^1_{\lambda, A} \). The intertwining spaces \( I_g(\eta_{\lambda', L}) \) have all complex dimension at most one.

**Proposition 4.5**. Suppose \( \tilde{\alpha}(\Lambda') \subseteq \tilde{\alpha}(\Lambda) \). There is up to equivalence a unique irreducible representation \( (\eta_{\lambda}', J^1_{\lambda, A}) \) which is an extension of \( (\eta_{\lambda}, J^1_{\lambda, A}) \) such that \( \eta_{\lambda', L} \) and \( \eta_{\lambda} \) induce equivalent irreducible representations on \( P^1(\Lambda') \). Moreover \( \eta_{\lambda', A} \) is the Gal(\( L[F]\))-Glauberman transfer of \( \eta_{\lambda', L} \) to \( J^1_{\lambda', A} \) and the set of intertwining elements of \( \eta_{\lambda', L} \) in \( G \) is \( J^1_{\lambda', A} G_\beta J^1_{\lambda, A} \). The intertwining spaces \( I_g(\eta_{\lambda', L}) \) have all complex dimension at most one.

**Proof.** We set \( \eta_{\lambda', A} \) to be \( \text{gl}(\eta_{\lambda', L}) \). Now \( \eta_{\lambda', A} \) is the only irreducible representation of \( J^1_{\lambda', A} \) with an odd multiplicity in \( \eta_{\lambda', L} \), and therefore the irreducible constituents of \( \eta_{\lambda', L} \mid_{J^1_{\lambda, A}} \) with odd multiplicity are contained in \( \eta_{\lambda', A} \), i.e. \( \eta_{\lambda} \) is contained in \( \eta_{\lambda', A} \), knowing that the restriction of \( \eta_{\lambda', L} \) to \( J^1_{\lambda, A} \) is equivalent to \( \eta_{\lambda} \). By the trace condition \([33] \) (6) for the Glauberman transfers \( \text{gl}(\eta_{\lambda', L}) \) and \( \text{gl}(\eta_{\lambda}) \) we obtain that both representation have the same degree, i.e. \( \eta_{\lambda', A} \) is an extension of \( \eta_{\lambda} \). As in the proof of Proposition 4.3 we obtain the formula for \( I_G(\eta_{\lambda', A}) \) using Proposition 4.4 instead of Proposition 4.1.

It remains to show the following three statements:

(i) The representations \( \pi := \text{ind}_{\lambda'}^{P^1(\Lambda')} \eta_{\lambda', A} \) and \( \text{ind}_{\lambda'}^{P^1(\Lambda')} \eta_{\lambda} \) are

(a) irreducible, and

(b) equivalent.

(ii) The multiplicity of \( \eta_{\lambda} \) in \( \pi \) is one.

(iii) The intertwining spaces of \( \eta_{\lambda'} \) have at most complex dimension one.

The irreducibility follows from \( \text{I}_{P^1(\Lambda')} (\eta_{\lambda', A}) = J^1_{\lambda', A} \) and \( \text{I}_{P^1(\Lambda')} (\eta_{\lambda}) = J^1_{\lambda} \). The statement about the intertwining spaces follows from Proposition 4.3. For the equivalence note at first that \( \eta_{\lambda', L} \) has multiplicity one in \( \text{ind}_{\lambda'}^{P^1(\Lambda')} \eta_{\lambda', L} \) (because the latter is irreducible), in particular \( \eta_{\lambda', L} \) has odd multiplicity in \( \text{ind}_{\lambda', L}^{P^1(\Lambda')} \eta_{\lambda'} \). Thus

\[
\text{gl}(\text{ind}^{P^1(\Lambda')}_{\lambda, L} \eta_{\lambda'}) \supseteq \text{gl}(\eta_{\lambda', L}) = \eta_{\lambda', A}.
\]

By irreducibility we obtain that \( \text{ind}^{P^1(\Lambda')}_{\lambda', A} \eta_{\lambda'} \) is equivalent to \( \text{gl}((\text{ind}^{P^1(\Lambda')}_{\lambda', L} \eta_{\lambda'})) \). By the same reasoning the latter is also equivalent to \( \text{ind}^{P^1(\Lambda')}_{\lambda', A} \eta_{\lambda'} \). It remains to show the multiplicity assertion: Note that the
set
\[ \text{Hom}_{\mathcal{J}_{\Lambda'}^n} (\eta_{\Lambda'}, \eta_{\Lambda', \Lambda}) \]
is trivial if \( g \notin I_G(\eta_{\Lambda}) \). Thus by Frobenius reciprocity and Mackey theory we have
\[ \text{Hom}_{\mathcal{J}_{\Lambda}^n} (\eta_{\Lambda}, \text{ind}_{\mathcal{J}_{\Lambda'}^n} \eta_{\Lambda', \Lambda}) = \text{Hom}_{\mathcal{J}_{\Lambda}^n} (\eta_{\Lambda}, \eta_{\Lambda}) = \mathbb{C}. \]
This finishes the proof. \( \square \)

We need to show that the definition of \( \eta_{\Lambda', \Lambda} \) only depends on \( \tilde{\mathcal{B}}(\Lambda') \) instead of \( \Lambda' \).

**Proposition 4.6.** Let \( \Lambda'' \) be a self-dual \( \sigma_E - \sigma_D \)-lattice sequence such that \( \tilde{\mathcal{B}}(\Lambda'') = \tilde{\mathcal{B}}(\Lambda') \) and \( \tilde{\mathcal{A}}(\Lambda'') \subseteq \tilde{\mathcal{A}}(\Lambda) \) and suppose \( \tilde{\mathcal{A}}(\Lambda') \subseteq \tilde{\mathcal{A}}(\Lambda) \). Then \( \mathcal{J}_{\Lambda', \Lambda}^1 = \mathcal{J}_{\Lambda''', \Lambda}^1 \) and \( \eta_{\Lambda', \Lambda} \) is equivalent to \( \eta_{\Lambda'', \Lambda} \).

**Proof.** We consider a path of self-dual \( \sigma_D - \sigma_E \)-lattice sequences \( \Lambda' = \Lambda_0, \Lambda_1, \ldots, \Lambda_l = \Lambda'' \) on a segment from \( \Lambda' \) to \( \Lambda'' \) in \( \mathfrak{B}(G) \), such that
\[
\tilde{\mathcal{A}}(\Lambda_i) \cap \tilde{\mathcal{A}}(\Lambda_{i+1}) \in \{ \tilde{\mathcal{A}}(\Lambda_i), \tilde{\mathcal{A}}(\Lambda_{i+1}) \},
\]
for all \( i = 0, 1, \ldots, l-1 \), in particular we have \( \tilde{\mathcal{B}}(\Lambda_i) = \tilde{\mathcal{B}}(\Lambda') \) for all \( i = 0, \ldots, l \). Thus by transitivity it is enough to consider the case \( \tilde{\mathcal{A}}(\Lambda') \supseteq \tilde{\mathcal{A}}(\Lambda'') \). The representations \( \text{ind}_{\mathcal{J}_{\Lambda', \Lambda}^n} \eta_{\Lambda', \Lambda} \) and \( \text{ind}_{\mathcal{J}_{\Lambda''}^{\prime\prime}} \eta_{\Lambda''} \) are equivalent, and thus \( \text{ind}_{\mathcal{J}_{\Lambda', \Lambda}^n} \eta_{\Lambda', \Lambda} \) is equivalent to \( \text{ind}_{\mathcal{J}_{\Lambda''}^{\prime\prime}} \eta_{\Lambda''} \). Now \( \mathcal{J}_{\Lambda', \Lambda}^1 = \mathcal{J}_{\Lambda''', \Lambda}^1 \), \( \mathcal{J}_{\Lambda'}^1 = \mathcal{J}_{\Lambda'''}^1 \), \( \eta_{\Lambda', \Lambda} = \eta_{\Lambda''} \) and
\[
\text{ind}_{\mathcal{J}_{\Lambda', \Lambda}^n} \eta_{\Lambda'} \cong \text{ind}_{\mathcal{J}_{\Lambda''}^{\prime\prime}} \eta_{\Lambda''}
\]
by definition of \( \eta_{\Lambda', \Lambda} \). Thus \( \eta_{\Lambda', \Lambda} \) and \( \eta_{\Lambda'', \Lambda} \) are equivalent by Proposition 4.5. \( \square \)

By last proposition we can now define \( \eta_{\Lambda', \Lambda} \) without assuming \( \tilde{\mathcal{A}}(\Lambda') \subseteq \tilde{\mathcal{A}}(\Lambda) \).

**Definition 4.7.** Granted \( \tilde{\mathcal{B}}(\Lambda') \subseteq \tilde{\mathcal{B}}(\Lambda) \), we define \( (\eta_{\Lambda', \Lambda}, \mathcal{J}_{\Lambda', \Lambda}^1) \) as the representation \( (\eta_{\Lambda'', \Lambda}, \mathcal{J}_{\Lambda''}^{\prime\prime}) \), where \( \Lambda'' \) is a self-dual \( \sigma_E - \sigma_D \)-lattice sequence such that \( \tilde{\mathcal{B}}(\Lambda') = \tilde{\mathcal{B}}(\Lambda'') \) and \( \tilde{\mathcal{A}}(\Lambda'') \subseteq \tilde{\mathcal{A}}(\Lambda) \).

**Corollary 4.8 (cf. [33] 3.8).** Let \( \Lambda'' \) be a further self-dual \( \sigma_E - \sigma_D \)-lattice sequence such that \( \tilde{\mathcal{B}}(\Lambda'') \subseteq \tilde{\mathcal{B}}(\Lambda') \). Then the restriction of \( \eta_{\Lambda'', \Lambda} \) to \( \mathcal{J}_{\Lambda', \Lambda}^1 \) is equivalent to \( \eta_{\Lambda', \Lambda} \).

**Proof.** This follows from [33] Proposition 3.8] and the Glauberman correspondence, indeed
\[
\eta_{\Lambda''', \Lambda} |_{\mathcal{J}_{\Lambda', \Lambda}^1} \cong \eta_{\Lambda'', \Lambda},
\]
by [33] 3.8] and \( g(\eta_{\Lambda''', \Lambda}) = \eta_{\Lambda', \Lambda} \), and thus the latter representation occurs with odd multiplicity in \( \eta_{\Lambda''', \Lambda} |_{\mathcal{J}_{\Lambda', \Lambda}^1} \), thus \( \eta_{\Lambda', \Lambda} \) contains \( \eta_{\Lambda', \Lambda} \) and hence, as they have the same degree, we get the result. \( \square \)

5 \( \beta \)-extension

In this section we generalize \( \beta \)-extensions to \( G \), see [33] Section 4] for the case of \( G \otimes L \). We start with a self-dual semisimple stratum \( \Lambda', n, 0, \beta \). The construction of \( \beta \)-extensions for classical groups is a complicated process (cf. [33] Section 4] for \( G_L \) and [22], [8] 5.2.1 for \( G \)). Let \( \Lambda, \Lambda', \Lambda_M \) be self-dual \( \sigma_E - \sigma_D \)-lattice sequences such that \( \mathcal{B}(\Lambda_M) \) is maximal and \( \tilde{\mathcal{B}}(\Lambda) \subseteq \tilde{\mathcal{B}}(\Lambda') \subseteq \tilde{\mathcal{B}}(\Lambda_M) \).

- At first we construct extensions of \( (\eta_{\Lambda, \Lambda'}, \mathcal{J}_{\Lambda, \Lambda'}^1) \) to \( \mathcal{J}_{\Lambda, \Lambda'} \).
- Secondly, depending on \( \Lambda_M \) we only choose certain of these extensions of \( \eta_{\Lambda} \) to \( \mathcal{J}_{\Lambda} \) as \( \beta \)-extensions.
At first we ignore $\Lambda_M$ and choose a third self-dual $\sigma_F-\sigma_D$-lattice sequence $\Lambda''$ which satisfies $\bar{b}(\Lambda) \subseteq \bar{b}(\Lambda'')$, so we have

$$\bar{b}(\Lambda) \subseteq \bar{b}(\Lambda') \cap \bar{b}(\Lambda'').$$

We put

$$\text{ext}(\Lambda, \Lambda') := \{(\kappa''_{\Lambda'}, J_{\Lambda, \Lambda'}) \mid \kappa'_{\Lambda', \Lambda''} \equiv \eta_{\Lambda, \Lambda'}\},$$

where the subscript $\equiv$ indicates the isomorphism class of the representation in question. We define a map

$$\Psi_{\Lambda, \Lambda', \Lambda''} : \text{ext}(\Lambda, \Lambda') \to \text{ext}(\Lambda, \Lambda'')$$

the by following idea due to Stevens:

- Consider a path of self-dual $\sigma_F-\sigma_D$ lattice sequences

$$\Lambda' = \Lambda_0, \Lambda_1, \cdots, \Lambda_l = \Lambda''$$

such that

$$\bar{a}(\Lambda_i) \cap \bar{a}(\Lambda_{i+1}) \in \{\bar{a}(\Lambda'), \bar{a}(\Lambda'')\}, \quad \bar{b}(\Lambda) \subseteq \bar{b}(\Lambda_i) \cap \bar{b}(\Lambda_{i+1}).$$

for all indexes $i \in \{0, \cdots, l-1\}$.

- Define the maps $\Psi_{\Lambda, \Lambda_i, \Lambda_{i+1}}$ and then put

$$\Psi_{\Lambda, \Lambda', \Lambda''} = \Psi_{\Lambda, \Lambda_{l-1}, \Lambda_l} \circ \Psi_{\Lambda, \Lambda_{l-2}, \Lambda_{l-1}} \circ \cdots \circ \Psi_{\Lambda, \Lambda_0, \Lambda_1}.$$  \hfill (5.3)

### 5.1 The map $\Psi_{\Lambda, \Lambda', \Lambda''}$ in the inclusion case

So at first we assume $\bar{a}(\Lambda) \subseteq \bar{a}(\Lambda') \cap \bar{a}(\Lambda'') \in \{\bar{a}(\Lambda'), \bar{a}(\Lambda'')\}$. Take $\kappa''_{\Lambda} \in \text{ext}(\Lambda, \Lambda')$.

**Lemma 5.4** (cf. [33] 4.3). There a unique $(\kappa''_{\Lambda}, J_{\Lambda, \Lambda''}) \in \text{ext}(\Lambda, \Lambda'')$ such that

$$\text{ind}_{\Lambda, \Lambda'} \kappa''_{\Lambda} \equiv \text{ind}_{J_{\Lambda, \Lambda''}} \kappa''_{\Lambda''}. \hfill (5.5)$$

(Where we define $P_{\Lambda, \Lambda'} := P(\Lambda')P(\Lambda)$, e.g. $P_{\Lambda, \Lambda'} := P(\Lambda')P(\Lambda)$.)

**Proof.** Mutatis mutandis to the proof of [33] Lemma 4.3\] we obtain the assertion for the cases $(\Lambda, \Lambda', \Lambda'') = (\Lambda, \Lambda', \Lambda)$ and $(\Lambda, \Lambda', \Lambda'') = (\Lambda, \Lambda, \Lambda'')$. Then transitivity implies the assertion. \Halmos

We define $\Psi_{\Lambda, \Lambda', \Lambda''}(\kappa''_{\Lambda}) := \kappa''_{\Lambda''}$ using $\kappa''_{\Lambda''}$ from Lemma 5.4. In fact $J_{\Lambda, \Lambda'}$ does only depend on $\bar{b}(\Lambda)$ instead of $\Lambda$, and even more:

**Lemma 5.6.** $\Psi_{\Lambda, \Lambda', \Lambda''} \circ \text{Res}_{J_{\Lambda, \Lambda'}}^{J_{\Lambda, \Lambda''}}$ and $\text{Res}_{J_{\Lambda, \Lambda''}}^{J_{\Lambda, \Lambda'}} \circ \Psi_{\Lambda, \Lambda', \Lambda''}$ coincide if $\bar{b}(\Lambda) \subseteq \bar{b}(\Lambda)$ and $\bar{a}(\Lambda) \subseteq \bar{a}(\Lambda') \cap \bar{a}(\Lambda'')$.

**Proof.** To show this assertion it is enough to consider the case $\bar{a}(\Lambda) \subseteq \bar{a}(\Lambda)$. (For the general case take $\Lambda \in \text{Latt}_{\rhoD}$ with $\bar{a}(\Lambda) \subseteq \bar{a}(\Lambda)$ and $\bar{b}(\Lambda) = \bar{b}(\Lambda)$, and use a segment from $\Lambda$ to $\Lambda$.) We start with $(\kappa', \kappa'')$ satisfying (5.3) for $\Lambda$ instead of $\Lambda$. Then we restrict to $P_{\Lambda, \Lambda}$ and induce to $P_{\Lambda, \Lambda}$ to obtain (5.3) for $\Lambda$. This proves the lemma. \Halmos

By Lemma 5.6 we can define $\Psi_{\Lambda, \Lambda', \Lambda''}$ if $\bar{a}(\Lambda)$ may not be contained in $\bar{a}(\Lambda') \cap \bar{a}(\Lambda'')$. Suppose $\bar{b}(\Lambda) \subseteq \bar{a}(\Lambda') \cap \bar{a}(\Lambda'')$ and choose $\Lambda$ such that $\bar{a}(\Lambda) \subseteq \bar{a}(\Lambda') \cap \bar{a}(\Lambda'')$ and $\bar{b}(\Lambda) = \bar{b}(\Lambda')$. Then define $\Psi_{\Lambda, \Lambda', \Lambda''}$ to be $\Psi_{\Lambda, \Lambda', \Lambda''}$.
5.2 The map $\Psi_{\Lambda,\Lambda',\Lambda''}$ in the general case

We do not require $\tilde{a}(\Lambda') \cap \tilde{a}(\Lambda'') \in \{\tilde{a}(\Lambda'), \tilde{a}(\Lambda'')\}$ here. We choose a path (5.1) of self-dual $\sigma_E-\sigma_D$-lattice sequences and define $\Psi_{\Lambda,\Lambda',\Lambda''}$ as in (5.3). Now one has to prove that this definition is independent of the choice of the path. For that it is enough to consider a triangle of self-dual $\sigma_E-\sigma_D$-lattice sequences $\Lambda_1, \Lambda_2, \Lambda_3$ such that $\tilde{a}(\Lambda_1) \subseteq \tilde{a}(\Lambda_2) \subseteq \tilde{a}(\Lambda_3)$ with $\tilde{b}(\Lambda) \subseteq \tilde{b}(\Lambda_1)$ and show the commutativity

$$\Psi_{\Lambda,\Lambda_2,\Lambda_3} \circ \Psi_{\Lambda,\Lambda_1,\Lambda_2} = \Psi_{\Lambda,\Lambda_1,\Lambda_3}.$$ 

We take an $\sigma_E-\sigma_D$-lattice sequence $\tilde{\Lambda}$ such that $\tilde{a}(\Lambda) \subseteq \tilde{a}(\Lambda_1)$ and $\tilde{b}(\Lambda) = \tilde{b}(\tilde{\Lambda})$. We choose $\kappa_{i,z} \in \text{ext}(\Lambda, \Lambda_i)$, $i = 1, 2, 3$, such that $\Psi_{\Lambda,\Lambda_1,\Lambda_2}(\kappa_{1,z}) = \kappa_{2,z}$ and $\Psi_{\Lambda,\Lambda_1,\Lambda_3}(\kappa_{1,z}) = \kappa_{3,z}$. Then $\Psi_{\Lambda,\Lambda_2,\Lambda_3}(\kappa_{2,z}) = \kappa_{3,z}$ follows from (5.10) and transitivity. This finishes the definition of $\Psi_{\Lambda,\Lambda',\Lambda''}$. We have the following result on intertwining:

**Proposition 5.7** (cf. [33] Lemma 4.3). Suppose $\Psi_{\Lambda,\Lambda',\Lambda''}(\kappa_z') = \kappa''_z$. Then $I_{G_\beta}(\kappa') = I_{G_\beta}(\kappa'')$.

For this proposition we need the following intersection property:

**Lemma 5.8** (cf. [33] 2.6). Let $g$ be an element of $G_\beta$. Then we have

$$(P^1(\Lambda)gP^1(\Lambda)) \cap G_\beta = P^1(\Lambda_\beta)gP^1(\Lambda_\beta). \tag{5.9}$$

**Proof.** At first: The proof of

$$(\tilde{P}^1(\Lambda)g\tilde{P}^1(\Lambda)) \cap \tilde{G}_\beta = \tilde{P}^1(\Lambda_\beta)g\tilde{P}^1(\Lambda_\beta) \tag{5.10}$$

is mutatis mutandis to the proof of [27] 4.8. Now one takes $\tau$-fixed points on both sides of (5.10) to obtain (5.9) by [18] 2.7(i)].

**Proof of Proposition 5.7.** Using the construction of $\Psi_{\Lambda,\Lambda',\Lambda''}$ we can assume without loss of generality that $\tilde{a}(\Lambda) \subseteq \tilde{a}(\Lambda') \subseteq \tilde{a}(\Lambda'')$. Now the proof is as for the second part of [33] 4.3 using Lemma 5.8 instead of [33] 2.6].

5.3 Defining $\beta$-extensions

In this section we fix three self-dual $\sigma_E-\sigma_D$-lattice sequences $\Lambda, \Lambda'$ and $\Lambda_M$, the latter chosen such that $\tilde{b}(\Lambda_M)$ is maximal. We are interested in extensions of $\eta_\Lambda$ to $J_\Lambda$, but not all of them (cf. [33] remark 4.2)). At first we define $\beta$-extensions for $\Lambda_M$.

Let $\Lambda_m$ be an $\sigma_E-\sigma_D$ lattice sequence such that $\tilde{b}(\Lambda_m)$ is minimal and contained in $\tilde{b}(\Lambda_M)$. We define $\beta$-ext$(\Lambda_M)$ as the set of all isomorphism classes of irreducible representations $\kappa$ of $J_{\Lambda_M}$ such that the isomorphism class of $\kappa|_{\Lambda_m,\Lambda_M}$ is an element of ext$(\Lambda_m, \Lambda_M)$.

The following proposition shows that $\beta$-ext$(\Lambda_M)$ is non-empty. In fact we will show a more general result and we therefore assume $\tilde{b}(\Lambda_m) \subseteq \tilde{b}(\Lambda)$. Note that $J_{\Lambda_m,\Lambda}^1$ is a pro-$p$-Sylow subgroup of $J_\Lambda$ and every pro-$p$-Sylow subgroup of $J_\Lambda$ is of such a form, i.e., for an appropriate $\Lambda_m$, and they are all conjugate in $J_\Lambda$.

**Proposition 5.11.** Suppose $\tilde{b}(\Lambda_m) \subseteq \tilde{b}(\Lambda) \subseteq \tilde{b}(\Lambda')$.

(i) There exists an extension $(\kappa, J_{\Lambda})$ of $(\eta_{\Lambda_M,\Lambda}, J_{\Lambda_m,\Lambda})$.

(ii) Let $(\kappa', J_{\Lambda'})$ be an extension of $(\eta_{\Lambda_M,\Lambda'}, J_{\Lambda_m,\Lambda'})$. Then the restriction of $\kappa'$ to $J_{\Lambda,\Lambda'}^1$ is equivalent to $(\eta_{\Lambda',\Lambda'}, J_{\Lambda_m,\Lambda'})$.

(iii) Let $(\kappa, J_{\Lambda_M})$ be an extension of $(\eta_{\Lambda_M}, J_{\Lambda_m}^1)$. Then are equivalent:

(a) $\kappa \in \beta$-ext$(\Lambda_M)$. 

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(b) \( \kappa \) is an extension of \( \eta_{A_M} \) such that for every pro-\( p \)-Sylow subgroup \( S \) of \( J_{\Lambda_M} \) the restriction of \( \kappa \) to \( S \) is intertwined by the whole of \( G_{\beta} \).

For the proof we need a lemma:

**Lemma 5.12** ([8] (5.3.2)(proof)). Let \( K \) be a totally disconnected and locally compact group and let \((\rho_i, W_i), \ i = 1, 2, \) be two smooth representations of \( K \). Suppose \( K_2 \) is a normal open subgroup of \( K \) contained in the kernel of \( \rho_2 \). Suppose that the sets \( \text{End}_K(W_i) \) and \( \text{End}_{K_2}(W_i) \) coincide. Then:

\[
\text{End}_K(W_1 \otimes_C W_2) \cong \text{End}_K(W_1) \otimes_C \text{End}_K(W_2).
\]

In particular if \( K \) is compact and \( \rho_1 \) is irreducible we get:

(i) \( W_1 \otimes W_2 \) is irreducible if and only if \( W_1 \) and \( W_2 \) are irreducible.

(ii) Let \( \rho \) be an irreducible representation of \( K \) such that \( \rho|_{K_2} \cong \rho_1|_{K_2} \). Then there is an irreducible representation \( \rho'_2 \) on \( K \) containing \( K_2 \) in its kernel such that \( \rho \) is equivalent to \( \rho \otimes \rho'_2 \).

**Proof.** Take a basis \( f_i \) of \( \text{End}_C(W_2) \). We have the \( K \)-action on \( \text{End}_C(W_1 \otimes W_2) \) via conjugation: \( k \cdot \Phi := k \cdot \Phi \circ k^{-1} \), where we consider on \( W_1 \otimes W_2 \) the diagonal action of \( K \). Then every element \( \Phi = \sum g_i \otimes f_i \) of \( \text{End}_K(W_1 \otimes_C W_2) \) is fixed by \( K_2 \) and therefore \( g_i \) has to be \( K_2 \)-equivariant and therefore \( K \)-equivariant by assumption. So \( \Phi \) is an element of \( \text{End}_K(W_1 \otimes_C W_2) \cap \text{End}_K(W_1) \otimes_C \text{End}_K(W_2) \). Now a similar argument for \( \Phi \) using a \( C \)-basis of \( \text{End}_K(W_1) \) shows that \( \Phi \) is an element of \( \text{End}_K(W_1) \otimes_C \text{End}_K(W_2) \).

Now (i) follows from semisimplicity of smooth representations of \( K \), and (ii) follows from (i) and Frobenius reciprocity. \( \square \)

**Proof of Proposition 5.11** The existence assertion (i) is proven mutatis mutandis to [33, Theorem 4.1]. Assertion (ii) follows from Corollary 4.8. For (iii) let \( \Lambda_m \) be a self-dual \( \mathfrak{d}_E \)-\( \mathfrak{d}_D \)-lattice sequence such that \( \mathfrak{b}(\Lambda_m) \) is minimal and \( \mathfrak{a}(\Lambda_m) \subseteq \mathfrak{a}(\Lambda_M) \). The representation \( \eta_{\Lambda_m, \Lambda_M} \) is intertwined by the whole of \( G_{\beta} \), by Proposition 4.5 and further the pro-\( p \)-Sylow subgroups of \( J_{\Lambda_M} \) are all conjugate in \( P(\Lambda_{M, \beta}) \). Thus (iii)(a) implies (iii)(b). Suppose (iii)(b) then, by Lemma 5.12, \( \kappa|_{\Lambda_m, \Lambda_M} \) is equivalent to \( \eta_{\Lambda_m, \Lambda_M} \otimes \varphi \) for some inflation \( \varphi \) of a characters of \( J_{\Lambda_m, \Lambda_M} / J_{\Lambda_M}^{1} \) (The latter group is isomorphic to \( P^1(\Lambda_{M, \beta}) / P^1(\Lambda_{M, \beta}) \)). Thus \( \varphi \) is intertwined by the whole of \( G_{\beta} \) and by the analogue of [33, 3.10] we obtain that \( \varphi \) is trivial. \( \square \)

**Corollary 5.13.** The sets \( \text{ext}(\Lambda, \Lambda') \) and \( \beta\text{-ext}(\Lambda_M) \) are not empty.

We can now define \( \beta \)-extensions:

**Definition 5.14** (cf. [33], 4.5, 4.7f). Granted \( \tilde{\mathfrak{b}}(\Lambda) \subseteq \tilde{\mathfrak{b}}(\Lambda_M) \), we call the following set the set of equivalence classes of \( \beta \)-extensions of \( \eta_{\Lambda} \) to \( J_{\Lambda} \) relative to \( \Lambda_M \):

\[
\beta\text{-ext}_{\Lambda_M}(\Lambda) := \{ \psi_{\Lambda, \Lambda_M} \circ \text{Res}_{J_{\Lambda_M}}^{J_{\Lambda}}(\kappa_e) | \kappa_e \in \beta\text{-ext}(\Lambda_M) \}.
\]

An extension of \( \eta_{\Lambda} \) to \( J_{\Lambda} \) is called a \( \beta \)-extension of \( \eta_{\Lambda} \) to \( J_{\Lambda} \) relative to \( \Lambda_M \) if it can be extended to a \( \beta \)-extension of \( \eta_{\Lambda} \) to \( J_{\Lambda} \) relative to \( \Lambda_M \). The set of \( \beta \)-extension of \( \eta_{\Lambda} \) to \( J_{\Lambda} \) is denoted by \( \beta\text{-ext}_{\Lambda_M}^{0}(\Lambda) \).

Suppose now \( \tilde{\mathfrak{b}}(\Lambda'') \cup \tilde{\mathfrak{b}}(\Lambda') \subseteq \tilde{\mathfrak{b}}(\Lambda_M) \).

**Theorem 5.15.** Granted \( \tilde{\mathfrak{b}}(\Lambda') \cup \tilde{\mathfrak{b}}(\Lambda'') \subseteq \tilde{\mathfrak{b}}(\Lambda_M) \), there is a unique map \( \Psi_{\Lambda', \Lambda''}^{0} \) from \( \beta\text{-ext}_{\Lambda_M}^{0}(\Lambda') \) to \( \beta\text{-ext}_{\Lambda_M}^{0}(\Lambda'') \) depending on \( \Lambda_M \) such that

\[
\Psi_{\Lambda', \Lambda''}^{0} \circ \text{Res}_{J_{\Lambda'}}^{J_{\Lambda''}} \circ \psi_{\Lambda', \Lambda''}^{\Lambda} = \text{Res}_{J_{\Lambda'}}^{J_{\Lambda''}} \circ \psi_{\Lambda', \Lambda''}^{\Lambda} \circ \text{Res}_{J_{\Lambda''}}^{J_{\Lambda'}}.
\]

(5.16) on \( \beta\text{-ext}(\Lambda_M) \). The map \( \Psi_{\Lambda', \Lambda''}^{0} \) is bijective.
Proof. At first: A map \( \Psi^0_{\kappa', \kappa''} \) satisfying (5.16) is uniquely determined and surjective by the definition of \( \beta \text{-ext}_{0}^{\kappa'}(\Lambda') \) and \( \beta \text{-ext}_{0}^{\kappa''}((\Lambda'')) \). We have \( \Psi^0_{\kappa', \kappa''} = 1_{\beta \text{-ext}_{0}^{\kappa'}(\Lambda')} \) and if \( \Psi^0_{\kappa', \kappa''} \) and \( \Psi^0_{\kappa', \kappa''} \) satisfy (5.16) then \( \Psi^0_{\kappa', \kappa''} \circ \Psi^0_{\kappa', \kappa''} \) too. Further if \( \Psi^0_{\kappa', \kappa''} \) exists and is bijective then we can take \( (\Psi^0_{\kappa', \kappa''})^{-1} \) as \( \Psi^0_{\kappa', \kappa''} \). Thus we only have to consider the case \( \tilde{a}(\Lambda') \leq \tilde{a}(\Lambda'') \). We define \( \Psi^0_{\kappa', \kappa''} \) in several steps:

- Let \( \kappa_0 \) be a \( \beta \)-extension of \( \eta_{\Lambda'} \) to \( J_{1}^{\kappa_0} \) and let \( \kappa' \) be an arbitrary \( \beta \)-extension of \( \eta_{\Lambda'} \) to \( J_{1}^{\kappa'_0} \) such that the restriction of \( \kappa' \) to \( J_{1}^{\kappa_0} \) is equivalent to \( \kappa_0 \).

- We choose a class \( \kappa_{M, z} \in \beta \text{-ext}(\Lambda_{M}) \) such that \( \Psi_{\kappa', \kappa''}(\text{Res}_{J_{1}^{\kappa_0}} \kappa_{M, z}) = \kappa'_0 \) and we put \( \kappa''_z := \Psi_{\kappa', \kappa''}(\text{Res}_{J_{1}^{\kappa_0}} \kappa_{M, z}) \).

- Then we define:

\[
\Psi^0_{\kappa', \kappa''}(\kappa_z) := \kappa''_z \mid_{\kappa'_0} := \kappa''_0.
\]

We claim that \( \Psi^0_{\kappa', \kappa''} \) is well-defined, i.e. independent of the choices made. Denote

\[
\kappa_z := \Psi^0_{\kappa', \kappa''}(\kappa'_0).
\]

Then we obtain by definition and Lemma (5.6)

\[
\kappa_z = (\Psi_{\kappa', \kappa''}(\text{Res}_{J_{1}^{\kappa_0}} \kappa_{M, z}))(\kappa'_0) = \Psi_{\kappa', \kappa''}(\text{Res}_{J_{1}^{\kappa_0}} \kappa_{M, z}) = (\text{Res}_{J_{1}^{\kappa_0}} \kappa_{M, z})(\text{Res}_{J_{1}^{\kappa_0}} \kappa_{M, z}) = \text{Res}_{J_{1}^{\kappa_0}} \kappa''_z.
\]

Thus (5.3) (with \( \Lambda' = \Lambda \)) is satisfied. We restrict (5.3) to \( P^0_{\kappa', \kappa''} \) (This is \( P^0(\Lambda')P^1(\Lambda') \)) to obtain:

\[
\text{ind}_{P^0_{\kappa', \kappa''}} \kappa'' \mid_{\kappa'_0} \equiv \text{ind}_{P^0_{\kappa', \kappa''}} \kappa''_0.
\]

Both sides are irreducible. This implies that \( \kappa''_0 \) uniquely determines the isomorphism class of the restriction of \( \kappa''_0 \) to \( J_{1}^{\kappa'_0} \), by Frobenius reciprocity, Mackey decomposition and \( I_{\beta \text{-ext}_{0}^{\kappa'}(\Lambda')}(\eta_{\Lambda'}) = I_{\beta \text{-ext}_{0}^{\kappa''}((\Lambda''))} \). Now mutatis mutandis as in the proof of [33, 4.10] one shows that there is only one element of \( \beta \text{-ext}_{0}^{\kappa'(\Lambda''')} \) extending \( \text{Res}_{J_{1}^{\kappa_0} \kappa''_0} \kappa''_0 \). Thus \( \kappa''_0 \) is uniquely determined by \( \kappa''_0 \). This shows that \( \Psi^0_{\kappa', \kappa''} \) is well defined.

On the other hand the restriction of \( \kappa'' \) to \( J_{1}^{0} \) uniquely determines \( \kappa''_0 \), by equation (5.17). Hence the injectivity of \( \Psi^0_{\kappa', \kappa''} \).

\[\Box\]

6 Cuspidal types

In this section we construct cuspidal types for \( G \), similar to [33] for \( G \otimes L \). We make use of the Glauberman correspondence, see [14]. Let \( \theta \in C(\Delta) \) with \( r = 0 \) be a self-dual semisimple characters, and let \( \theta_{\kappa} \) be its Glauberman transfer (with respect to \( \text{Gal}(L/F) \)) into \( C(\Delta \otimes L) \). Let \( \eta_{\kappa} \) be the Heisenberg extension of \( \theta_{\kappa} \) to \( J_{1}^{\kappa} \). (From now on we skip the lattice sequence from the subscript if there is no cause of confusion, e.g. \( J_{1}^{1} := J_{1}^{\kappa_1 \kappa_2} \).) Then the same argument as in [33] Theorem 4.1] shows:

**Proposition 6.1** (cf. [3] (5.2.4) [33, 4.1]). There exists an extension \( \kappa_{L} \) of \( \eta_{L} \) from \( J_{1}^{L} \) to \( J_{L} \) which is normalised by \( \tau \), i.e. \( \kappa_{L}^{\tau} \) is isomorphic to \( \kappa_{L} \), and such that the restriction of \( \kappa_{L} \) to a pro-\( p \)-subgroup is intertwined by \( (G \otimes L)_{\beta} \).

**Proof.** The group \( \text{Gal}(L/F)J_{L} \) normalises \( \eta \). Now the proof is the same as in loc.cit.. 4.1 to show that there is an extension of \( \eta_{L} \) to \( \text{Gal}(L/F)J_{L} \). \[\Box\]
Let $\rho$ be an irreducible complex representation of $P(\Lambda_\beta)$ whose restriction to $P^0(\Lambda_\beta)$ is an inflation of a direct sum of cuspidal irreducible representations of $P(\Lambda_\beta)^0(k_F)$. We call such a $\rho$ a cuspidal inflation w.r.t. $(\Lambda_\beta, \beta)$. Let $\kappa$ be a $\beta$-extension of $\eta$ (with respect to some $\Lambda_M$ with $\mathfrak{b}(\Lambda_M)$ maximal). We call the representation $\lambda := \kappa \otimes \rho$ a cuspidal type of $G$ if

- the parahoric $P^0(\Lambda_\beta)$ is maximal and
- the centre of $G_\beta$ is compact.

**Remark 6.3.** If $\lambda$ is a cuspidal type then the underlying stratum $\Delta$ has to be skew, i.e. the action of $\sigma_h$ on the index set $I$ is trivial, because of the compactness of the centre of $G_\beta$.

We can give a first relation between extensions $\kappa_L$ of $\eta_L$ as in the above Proposition and $\beta$-extensions of $\eta$ in the following way:

**Proposition 6.4.** Let $\kappa_L$ be given as in Proposition 6.1 and let $\kappa$ be a $\beta$-extension of $\eta$ (with respect to some maximal $\mathfrak{b}(\Lambda_M)$). There is a smooth representation $\rho$ of $J/J^1$ such that $\kappa_L|_I = \kappa \otimes \rho$. Moreover we have that $\kappa \otimes \rho$ is irreducible if an only $\rho$ is irreducible.

**Proof.** The restriction of $\kappa_L$ to $J^1$ is a finite multiple of $\eta$ and therefore every irreducible constituent of $\kappa_L|_I$ is of the form $\kappa \otimes \rho'$ for some inflation $\rho'$ of an irreducible representation of $J/J^1$, by Lemma 5.12(ii) This finishes the first part. The remaining part is covered by 5.12(i). \(\Box\)

The main motivation for the definition of a $\beta$-extension is the following theorem:

**Theorem 6.5** (cf. 33 Theorem 6.18]). Let $\lambda$ be a cuspidal type then ind$_G^J \lambda$ is a cuspidal irreducible representation of $G$.

Theorem 6.5 is mutatis mutandis to 33 Theorem 6.18], where the key idea can also be found [8]. We repeat the proof to show that it does only depend on the definition of a $\beta$-extension and 33 1.1.(ii)]. One does not need 33 6.16].

**Proof of Theorem 6.5** Let $\lambda = \kappa \otimes \rho$ be a cuspidal type. Take an irreducible component $(\rho_0, W_{\rho_0})$ of $\rho|_{J^0}$. Then $\kappa|_{J^0} \otimes \rho_0$ is irreducible by Lemma 5.12(i). The restriction of $\lambda$ to $J^0$ is equivalent to a direct sum of $J/J^0$-conjugates of $\kappa \otimes \rho_0$, because $J^0$ is a normal subgroup of $J$. Note that $J/J^0$ is isomorphic to $P(\Lambda_\beta)/P^0(\Lambda_\beta)$, so that the conjugating elements can be taken in $P(\Lambda_\beta)$. Now an element $g \in G$ which intertwines $\lambda$ intertwines $\kappa \otimes \rho_0$ up to $P(\Lambda_\beta)$-conjugation, and it also intertwines $\eta$. So it is an element of $J'G_{\beta}J'$. We can therefore without loss of generality assume $g$ as an element of $G_{\beta}$ which intertwines $\kappa \otimes \rho_0$. Hence as $I_g(\eta)$ is one-dimensional and the restriction of $\rho_0$ to $J^1$ is trivial we obtain that a $g$-intertwiner of $\kappa \otimes \rho_0$, i.e. an non-zero element of $I_g(\kappa \otimes \rho_0)$, has to be a tensor product of endomorphisms $S \in I_g(\eta)$ and $T \in \text{End}_{\mathbb{C}}(W_{\rho_0})$. Now let $Q$ be a pro-p-Sylow subgroup of $J^0$ then $g$ is an element of $I(\kappa_{|Q})$ by the definition of $\beta$-extension. In particular $S \in I_g(\kappa_{|Q})$, because $I_g(\eta)$ is 1-dimensional. Thus $T \in I_g(\rho_0_{|Q})$. In particular $g$ intertwines the restriction of $\rho$ to a pro-p-Sylow subgroup. Thus, by 33 Proposition 1.1.(ii)], $g$ is an element of $P(\Lambda_\beta)$. This finishes the proof. \(\Box\)

7 **Partitions subordinate to a stratum**

In the proof of the exhaustion in 33 the author has to pass to decompositions of $V$ which are so-called exactly subordinates to a skew-semisimple stratum $\Delta$ (with $r = 0$), see 33 Definition 6.5. In our situation of quaternionic forms we need to generalize this approach, because the centralizer of $E_i$ in $\text{End}_F V'$ is not given by the same vector space $V'$, if $\beta_i \neq 0$, i.e. $\text{End}_{E_i \otimes \mathbb{D}} V'$ is isomorphic $\text{End}_{\mathbb{D}_{\beta_i}} V'_{\beta_i}$, see Section (We have $2 \dim_{\mathbb{C}} (V'_{\beta_i}) = \dim_{\mathbb{R}} V'$). We generalize the notion of decompositions of $V$ which are exactly sub-ordinate to a semisimple stratum by certain families of idempotents. (This is indicated by the arguments given in 33 Section 5.) We fix a semisimple stratum $\Delta$ with $r = 0$. 
Definition 7.1. \((\text{i})\) We call a finite tuple of idempotents \((e^{(j)})_j\) of \(B = \text{End}_{\mathbb{B}(\mathbb{D})}(V)\) an \(E \otimes D\)-partition of \(V\) if \(e^{(j)} e^{(k)} = \delta_{jk}\) for all \(j \neq k\) and \(\sum_j e^{(j)} = 1\). An \(E \otimes D\)-partition \((e^{(j)})_j\) is called a subordinate to \(\Delta\) if \(W^{(j)} = e^{(j)} V\) is a splitting of \(\Delta\), or equivalently if \((W^{(j)})_j\) is a splitting of \(\Lambda\), i.e. \(e^{(j)} \in \mathfrak{a}(\Lambda)\) for all \(j\).

\((\text{ii})\) We call an \(E \otimes D\)-partition \((e^{(j)})_j\) of \(V\) properly subordinate to \(\Delta\) if it is subordinate to \(\Delta\) and the residue class \(e^{(j)} + \mathfrak{b}_1(\Lambda)\) in \(\mathfrak{b}(\Lambda)/\mathfrak{b}_1(\Lambda)\) is a central idempotent.

Analogously we have the notion of “being self-dual-subordinate to a stratum”.

Definition 7.2. Suppose \(\Delta\) is a skew semisimple stratum. Let \((e^{(j)})_j\) be an \(E \otimes D\)-partition of \(V\) subordinate to \(\Delta\). The partition \((e^{(j)})_j\) is called self-dual-subordinate to \(\Delta\) if the set of the idempotents \(e^{(j)}\) is \(\sigma_h\)-invariant with at most one fixed point. As in \([33]\) we are then going to use \(\{0, \pm 1, \ldots, \pm m\}\) as the index set, such that \(\sigma_h(e^{(j)}) = e^{(-j)}\) for all \(j\). (We just have added \(e^{(0)} := 0\) if there is no \(\sigma_h\)-fixed idempotent among the \(e^{(j)}\).) An \(E \otimes D\)-partition self-dual-subordinate to \(\Delta\) is called properly self-dual subordinate to \(\Delta\), if the partition is properly subordinate to \(\Delta\). Suppose \((e^{(j)})_j\) is properly self-dual-subordinate to \(\Delta\). We call it exactly subordinate to \(\Delta\) if it cannot be refined by another \(E \otimes D\)-partition of \(V\) properly self-dual-subordinate to \(\Delta\).

These notions of partitions subordinate to a stratum enable Iwahori decompositions as in \([33]\). Suppose \((e^{(j)})_j\) an \(E \otimes D\)-partition of \(V\). Let \(\mathcal{M}\) be the Levi subgroup of \(\mathcal{G}\) defined as:

\[
\mathcal{M} := \mathcal{G} \cap \bigcap_j \text{End}_E(W^{(j)})
\]

\(\mathcal{P}\) be a parabolic subgroup of \(\mathcal{G}\) with Levi \(\mathcal{M}\), and write \(\mathcal{U}_+\) for the radical of \(\mathcal{P}\) and the opposite parabolic \(P^{op}\), respectively. We write \(M, P, U_+, U_-\) for the corresponding intersections with \(G\).

Lemma 7.3 (cf. \([33]\) 5.2, 5.10). Suppose \((e^{(j)})_j\) is subordinate to \(\Delta\).

\((\text{i})\) Then \(H^1(\beta, \Lambda)\) and \(J^1(\beta, \Lambda)\) have Iwahori decompositions with respect to \(\mathcal{U}_-\mathcal{M}\mathcal{U}_+\). If \((e^{(j)})_j\) is properly subordinate to \(\Delta\) then \(H(\beta, \Lambda)\) and \(J(\beta, \Lambda)\) have Iwahori decompositions with respect to \(\mathcal{U}_-\mathcal{M}\mathcal{U}_+\).

\((\text{ii})\) Suppose \(\Delta\) skew-semisimple and that \((e^{(j)})_j\) is self-dual-subordinate to \(\Delta\). Then \(H^1(\beta, \Lambda)\) and \(J^1(\beta, \Lambda)\) have Iwahori decompositions with respect to \(\mathcal{U}_+\mathcal{M}\mathcal{U}_-\). If further \((e^{(j)})_j\) is properly self-dual-subordinate to \(\Delta\) then \(H(\beta, \Lambda)\) and \(J(\beta, \Lambda)\) have Iwahori decompositions with respect to \(\mathcal{U}_+\mathcal{M}\mathcal{U}_-\).

Proof. We just show the first assertion of \((\text{ii})\); the other statements follow similarly. We apply \textit{loc. cit.} to obtain

\[
H^1(\beta, \Lambda) \subseteq H^1(\beta \otimes 1, \Lambda_L) = (H^1(\beta \otimes 1, \Lambda_L) \cap \mathcal{U}_-\mathcal{M}\mathcal{U}_+)(H^1(\beta \otimes 1, \Lambda_L) \cap \mathcal{M}\mathcal{L})(H^1(\beta \otimes 1, \Lambda_L) \cap \mathcal{U}_+\mathcal{L}).
\]

The \(\tau\)-invariance of the three factors and the uniqueness of the Iwahori decomposition (w.r.t. \(\mathcal{U}_-\mathcal{L}\mathcal{M}\mathcal{U}_+\)) gives the result.

Suppose that \(\Delta\) is skew-semisimple and that \((e^{(j)})_j\) is properly self-dual-subordinate to \(\Delta\). Let \((\eta, J^1)\) be the Heisenberg extension of a self-dual semisimple characters \(\theta\) and let \(\kappa\) be a \(\beta\)-extension of \(\eta\). We are going to skip the parameters \(\Lambda\) and \(\beta\) for the sets \(H^1, J^1, J, \text{ etc.}\). As \(\Delta\) is fixed. Analogously to \([33]\) we can introduce representations \((\theta_P), (\eta_P, J^1_P)\) and \((\kappa_P, J^P)\). The corresponding groups are defined via:

\[
J^P_P = (H^1 \cap U_-)(J^1 \cap P) = (H^1 \cap U_-)(J^1 \cap M)(J^1 \cap U_+), \quad J_P = (H^1 \cap U_-)(J \cap P) = (H^1 \cap U_-)(J \cap M)(J \cap U_+).
\]

At first one extends \(\theta\) to a character \(\theta_P\) trivially to \(H^P : = (H^1 \cap U_-)(H^1 \cap M)(J^1 \cap U_+), \) i.e. via

\[
\theta_P(xy) = \theta(x), \quad x \in H^1, \quad y \in (J^1 \cap U_+).
\]

We define \((\eta_P, J^1_P)\) as the natural representation (given by \(\eta\)) on the set of \((J^1 \cap U_+)-\text{fixed point of } \eta\). We define \((\kappa_P, J^P)\) using \(\kappa\).

Then we have the following properties:
Proposition 7.4 (cf. [33] 5.12,5,13). \( \kappa_P \) is an extension of \( \eta_P \) and \( \eta_P \) is the Heisenberg extension of \( \theta_P \) to \( J^1_1 \). Further we have \( \text{ind}^1_{J^1_1} \eta_P \equiv \eta \) and \( \text{ind}^1_{J^1_1} \kappa_P \equiv \kappa \). The restrictions of \( \theta, \eta_P \) and \( \kappa_P \) to \( M \) are tensor-products

\[
\theta|_{H^1_1\cap M} = \otimes_j \theta_j, \quad \eta|_{J^1_1\cap M} = \otimes_j \eta_j, \quad \kappa|_{J^1_1\cap M} = \otimes_j \kappa_j,
\]

where \( \eta_j \) is the Heisenberg extension of the semisimple characters \( \theta_j \).

8 Main theorems for the classification

Given a cuspidal irreducible representation of \( G \) then there is a semisimple character \( \theta \in C(\Lambda,0,\beta) \) contained in \( \pi \). Thus it contains the Heisenberg extension \((\eta,J^1)\) of \( \theta \) and there is an irreducible representation \( \rho \) of \( J/J^1 \) and a \( \beta \)-extension of \( \eta \) such that \( \kappa \otimes \rho \) is contained in \( \pi \). Now one has to prove:

**Theorem 8.1** (Exhaustion). The representation \( \kappa \otimes \rho \) is a cuspidal type. In particular \( \text{ind}^1_G (\kappa \otimes \rho) \equiv \pi \).

Note: The induction assertion is given by Theorem 6.5. The second main theorem is

**Theorem 8.2** (Intertwining implies conjugacy, cf. [10] for \( G_L \)). Suppose \( (\Lambda,J) \) and \( (\Lambda',J') \) are two cuspidal types of \( G \) which intertwine in \( G \) (or equivalently which compactly induce equivalent representations of \( G \)). Then there is an element \( g \in G \) such that \( gJ_0g^{-1} = J' \) and \( \theta\lambda \) is equivalent to \( \theta' \).

9 Exhaustion

9.1 Skew characters and cuspidality

The main statement of this section is the following.

**Theorem 9.1** (cf. [32] 4.1). Let \( \pi \) be a cuspidal irreducible representation of \( G \) and \( \theta \in C(\Delta) \) with \( r = 0 \) be an self-dual semisimple character contained in \( \pi \). Then \( \Delta \) is skew-semisimple, i.e. the adjoint involution of \( h \) acts trivially on the index set of \( \Delta \).

The proposition is an easier analogue of [32] Proposition 4.6. The idea of the proof, see for example [10] 6.6] and [32] Section 4], is an application of the theory of covers in [11].

We start the proof of Theorem 9.1. Suppose for deriving a contradiction that \( \Delta \) is not skew-semisimple.

Consider the decomposition \( V = V_+ \oplus V_0 \oplus V_- \), given by

\[
V_\delta := \oplus_{i \in \Lambda} V^i, \quad \delta \in \{+,0,-\}. \quad (9.2)
\]

Let \( M \) be the Levi subgroup of \( G \) defined over \( F \) given by the stabilizer of the decomposition (9.2). The decomposition also defines unipotent subgroups: \( N_0 \), the unipotent radical of the stabilizer in \( G \) of the flag \( V_+, V_0 \oplus V_0 \oplus V_- \) and the opposite \( N_- \). We have the Iwahori decompositions for \( H^1_1 \) and \( J^1_1 \) with respect to \( N_0MN_0 \), and we write \( H^1_2 \) and \( J^1_2 \) for the obvious intersections (e.g. \( J^1_1 = J^1_1 \cap N_0 \)), \( \delta \in \{\pm,0\} \). Note that every irreducible representation of \( H^1_1J^1_1 \) containing \( \theta \) is a character, because \( (H^1_1J^1_1)/H^1_2 \) is abelian and \( \theta \) admits an extension \( \xi \) to \( H^1_1J^1_1 \), which is trivial on \( J^1_1 \).

**Proposition 9.3.** The group \( J^1_1 \) act transitively on the set of characters of \( H^1_1J^1_1 \) extending \( \theta \).
Proof. At first: the group $K := H^1 J^1_+$ is normalized by $J^1$ because $J^1/H^1$ is abelian. A character extending $\theta$ is contained in $\text{ind}_{H^1}^{J^1} \theta$ and is therefore contained in $\eta$. Thus the action on the set of these characters must be transitive because $\eta$ is irreducible.

For the notion of cover we refer to [11, Section 8]. In fact we take the weaker version where we only want to consider strongly positive elements for the parabolic subgroup $MN_+$, i.e. not for other parabolic subgroups. This is enough for our purposes.

**Proposition 9.4.** The character $\xi$ is a cover of $\xi|_{H^1 \cap \text{M}}$.

Proof. We have to show the existence of a strongly positive element $z$ of the centre of $M$ and with respect to $Q := MN_+$, such that there is an invertible element of the Hecke algebra $H(G, \xi)$ with support in $KzK$. Let $\xi_L$ be the extension of $\theta_L$ to $K_L := H^1_L J^1_L$, trivial on $J^1_L$. By [33, 4.5] there is a strongly positive element $z_L$ of the centre of $M \otimes L$ with respect to $Q \otimes L$ such that $K_L z_L K_L$ supports an invertible element $f_L$ of $H(G_L, \xi_L)$. Then $\tau(z_L)$ is also strongly positive with respect to $Q \otimes L$ and $f_L' := f_L \circ \tau$ is an invertible element of $H(G_L, \xi_L)$. By [11, 7.2(i), 7.1(iii)] we can replace $\xi_L$ by Proposition 9.4 and [11, 7.9(iii)]. A contradiction. This finishes the proof of Theorem 9.1.

9.2 Exhaustion of cuspidal types (Proof of Theorem 8.1)

The proof of exhaustion is mutatis mutandis to [33, see also [20, 3.3] for the final argument. We are going to give the outlook of the proof in this section and refer to the corresponding results in [33]. The referred
statements of Section 6 and 7 are mutatis mutandis valid for the quaternionic case. To start let \( \pi \) be a cuspidal irreducible representation of \( G \). Then, by there exists a skew-semisimple character \( \theta \in C(\Lambda, 0, \beta) \) such that:
\[
\theta \leq \pi. \tag{9.6}
\]
Let \( \Lambda \) be chosen such that \( \tilde{b}(\Lambda) \) is minimal with respect to \( \tilde{b}(A{\Lambda}_M) \). Take any \( \beta \)-extension \( \kappa \) of \( \theta \) (with respect to some maximal \( b(\Lambda_M) \) containing \( \tilde{b}(A{\Lambda}) \)). Then there is an irreducible representation \( \rho \) of \( P^0(\Lambda_\beta) / P^1(\Lambda_\beta) \) such that \( \lambda := \kappa \otimes \rho \) is contained in \( \pi \). Note that \( \rho \) has to be cuspidal by the minimality of \( \tilde{b}(\Lambda) \), see [33, 7.4]. Further by the minimality condition on \( \tilde{b}(\Lambda) \) there is a tuple of idempotents \( (e_j)_{j=m}^{m-1} \) exactly sub-ordinate to \( \Delta = [\Lambda, -\kappa, 0, \beta] \) such that \( P^0(\Lambda_\beta) \otimes_{(e_j)_{j=m}^{m-1}} \) is a maximal parahoric of \( G \). Suppose \( P^0(\Lambda_\beta) \) is not a maximal parahoric of \( G \) (Note that in our quaternionic case the center of \( G \) is compact, i.e. \( \text{SO}(1,1)(F) \) does not occur as a factor of \( G_\beta \)). We then have \( m > 0 \), i.e. \( (e_j)_{j=m}^{m-1} \) has at least 3 idempotents. Let \( M \) be the stabilizer of \( G \) of the decomposition of \( \mathbb{V} \), and let \( U \) be the set of upper unipotent elements of \( G \) with respect to the latter decomposition. We put \( \mathbb{P} = \mathbb{M} \mathbb{U} \). Let \( \lambda \mathbb{P} \) be the natural representation of \( J_\mathbb{P}^0 = H_1^0(J_\mathbb{P}^0 \cap \mathbb{P}) \) on the set of \((U \cap J_\mathbb{P}^0)\)-fixed points of \( \lambda \). We have now two cases to consider:

Case 1: \( \rho \circ \sigma \neq \rho \). In this case Stevens constructs in [33, 7.2.1] a decomposition \( Y_{-1} \oplus Y_0 \oplus Y_1 \) with \( \mathbb{L} \) (the stabilizer of the decomposition) and non-zero \( Y_{-1} \), \( Y_1 \) such that the normalizer of \( \rho(\mathbb{L} \cap P(A_\Lambda)) \) in \( \mathbb{G}_\beta \) is contained in \( \mathbb{M}' \). Then by [33, 6.16] the representation \( \lambda \mathbb{P} \) is a cover of \( \lambda \mathbb{P}[\mathbb{M}' \cap \mathbb{P}] \), see [11, 7.2(ii)].

Case 2: \( \rho \circ \sigma = \rho \). Here let
\[
Y_{-1} := e^{-m} \mathbb{V}, \ Y_1 := e^m \mathbb{V}, \ Y_0 := (1 - e^m - e^{-m}) \mathbb{V}
\]
and \( \mathbb{M}' \), \( \mathbb{P} \) as in Case 1. Here Stevens constructs in [33, 7.2.2] strongly \( (\mathbb{P}, J_\mathbb{P}^0) \)-positive elements of \( \mathbb{H}(\mathbb{G}, \lambda \mathbb{P}) \) of the centre of \( \mathbb{M}' \). Note that the elements \( s_m \) and \( s_m^n \) in Section loc.cit. are automatically in \( G \) because all isometries of \( h \) have reduced norm 1, so one does not need to consider (i) and (ii) in loc.cit. (7.2.2). Thus \( \lambda \mathbb{P} \) is a cover of \( \lambda \mathbb{P}[\mathbb{M}' \cap \mathbb{P}] \).

In either case [11, 7.9(iii)] and the fact that \( \mathbb{M}' \cap G \) is a proper Levi subgroup of \( G \) imply that \( \pi \) is not cuspidal. A contradiction.

### 10 Conjugate cuspidal types (Proof of Theorem 8.2)

In this section we finish the classification of cuspidal irreducible representations of \( G \). Recall from the assumption of Theorem 8.2 that we are given two cuspidal types \( (\lambda, J(\beta, \Lambda)) \) and \( (\lambda', J(\beta', \Lambda')) \) which induce equivalent representations of \( G \). Let us denote the representation \( \text{ind} \mathbb{G} \mathbb{J}^\mathbb{P} \lambda \) by \( \pi \). Let \( \theta \in C(\Lambda, 0, \beta) \) and \( \theta' \in C(\Lambda', 0, \beta') \) be the skew-semisimple characters used for the construction of \( \lambda \) and \( \lambda' \). As \( \theta \) and \( \theta' \) are contained in the irreducible \( \pi \) we obtain that both have to intertwine by an element of \( G \). By [27, 6.10] we can assume without loss of generality \( \beta = \beta' \) and that \( \theta' \) is the transfer of \( \theta \) from \( \Lambda \) to \( \Lambda' \). The proof of [15, Theorem 11.3] is valid for the quaternionic case, see below. We conclude that there is an element \( g \) of \( G \) such that \( g J g^{-1} = J' \) and \( g \lambda = \lambda' \).

Let us outline loc.cit. to show to which their statements and constructions are needed: \( \lambda \) is constructed using an irreducible representation \( \rho \) of \( M(\Lambda_\beta) := \mathbb{P}(\Lambda_\beta)(k_F) \) with cuspidal restriction to \( M(\Lambda_\beta)'^0 := \mathbb{P}(\Lambda_\beta)^0(k_F) \) and a \( (\kappa, J) \)-extension \( (\kappa, J), \lambda = \kappa \otimes \rho \). Analogously we have \( \lambda' = \kappa' \otimes \rho' \) for respective \( \rho' \) and \( \kappa' \).

We now have three pairs of functors:

(i) \( \text{R}_{\kappa} : \mathcal{R}(G) \to \mathcal{R}(M(\Lambda_\beta)) \) and \( \text{I}_{\kappa} : \mathcal{R}(M(\Lambda_\beta)) \to \mathcal{R}(G) \) defined via
\[
\text{I}_{\kappa}(\varphi) := \text{ind}_{\mathbb{G} \mathbb{J}^\mathbb{P} \kappa}^\mathbb{G} \varphi, \ \text{R}_{\kappa}(\sigma) := \text{Hom}_{\mathbb{G} \mathbb{J}^\mathbb{P} \kappa}(\kappa, \sigma)
\]

(ii) \( \text{R}_{\Lambda_\beta} : \mathcal{R}(G_\beta) \to \mathcal{R}(M(\Lambda_\beta)) \) and \( \text{I}_{\Lambda_\beta} : \mathcal{R}(M(\Lambda_\beta)) \to \mathcal{R}(G_\beta) \) defined via
\[
\text{I}_{\Lambda_\beta}(\varphi) := \text{ind}_{\mathbb{G} \mathbb{J}^\mathbb{P} \Lambda_\beta}^\mathbb{G} \varphi, \ \text{R}_{\Lambda_\beta}(\sigma) := \text{Hom}_{\mathbb{G} \mathbb{J}^\mathbb{P} \Lambda_\beta}(1, \sigma)
\]
(iii) \( R^{0}_{\Lambda} : \mathcal{R}(G_{S}) \to \mathcal{R}(M(\Lambda_{S})^{0}) \) and \( I^{0}_{\Lambda} : \mathcal{R}(M(\Lambda_{S})^{0}) \to \mathcal{R}(G_{S}) \) defined via

\[
I^{0}_{\Lambda}(\varphi) := \text{ind}_{\rho}(\Lambda_{S})^{0}(\varphi), \quad R^{0}_{\Lambda}(\sigma) := \text{Hom}_{\rho}(\Lambda_{S})(1, \sigma).
\]

Before we start to explain their proof we want to remark that we use [18] 7.5 which is valid for the case of \( G \), because the key is the exact diagram after [18] Lemma 4.2 which can be obtained for \( G \) by taking \( \text{Gal}(L/F) \)-fixed points of the corresponding diagram for \( G \otimes L \). Now we come to their proof of [18] 11.3. It contains two parts:

Part 1: Without loss of generality we can assume that \( \Lambda \) and \( \Lambda' \) have the same F-period and \( \Lambda(0) = \Lambda(1) \) and \( \Lambda'(0) = \Lambda'(1) \). The first part is to show that \( \Lambda \) and \( \Lambda' \) are \( G_{S} \)-conjugate. It is implied as follows: We have that \( I_{\kappa}(\rho) \) and \( I_{\kappa}(\rho') \) are isomorphic to \( \pi \), in particular isomorphic to each other, and therefore \( R_{\kappa} \circ I_{\kappa}(\rho') \) contains \( \rho \) and therefore is non-zero. Thus \( R_{\Lambda_{S}} \circ I_{\Lambda_{S}}^{0}(\rho') \) is non-zero by [18] 7.5(i)). Let \( \rho^{0} \) be a cuspidal irreducible sub-representation of the restriction of \( \rho' \) to \( M(\Lambda_{S})^{0} \). Then \( R_{\Lambda_{S}} \circ I_{\Lambda_{S}}^{0}(\rho^{0}) \) is non-zero because it contains \( R_{\Lambda_{S}} \circ I_{\Lambda_{S}}^{0}(\rho') \). Thus \( R^{0}_{\Lambda_{S}} \circ I^{0}_{\Lambda_{S}}(\rho^{0}) \) is non-zero and therefore \( P^{0}(\Lambda_{S}) \) and \( P^{0}(\Lambda')_{S} \) are \( G_{S} \)-conjugate by [18] 6.2(ii)). Therefore \( \Lambda_{S} \) and \( \Lambda'_{S} \) are cuspidal irreducible \( S \)-stable maximal parahoric sub-groups of \( G_{S} \). This finishes Part 1 and we can assume \( \Lambda = \Lambda' \) without loss of generality, in particular \( \theta = \theta' \) and we have the same Heisenberg extension.

Part 2 is for showing \( \lambda = \lambda' \). Take a character \( \chi \) of \( M(\Lambda_{E}) \) such that \( \kappa' = \kappa \otimes \chi \). Then we get \( \lambda' = \kappa \otimes (\chi \otimes \rho') \) and we get

\[
(R_{\kappa} \circ I_{\kappa})(\chi \otimes \rho') \cong (R_{\kappa} \circ I_{\kappa})(\rho)
\]

where the left hand side contains \( \chi \otimes \rho' \) and the right hand side is equivalent to \( \rho \) by [18] 7.5(iii)). Thus by irreducibility we obtain the existence of an isomorphism from \( \chi \otimes \rho' \) to \( \rho \), and therefore of an isomorphism from \( \lambda' \) to \( \lambda \).

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