Cooper pairs as low-energy excitations in the normal state

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We discuss the normal state of a fermionic system in an idealized pseudogap regime, $k_B T_c \leq k_B T \ll |\Delta| \ll \epsilon_F$. Stable Cooper pairs induce a pseudogap of width $|\Delta|$ in the fermion energy spectrum. Near two dimensions, we find a Bose-like condensation temperature in this predominantly fermionic system.

1. Motivation

A number of studies of high-$T_c$ superconductors show that cuprate superconductors have a short coherence length, comparable to the average distance between doped holes. While this observation casts doubt on the applicability of the BCS model, the limit of local pairs is not a good description of the cuprates either, as the existence of a Fermi surface demonstrates. A more appropriate model of a superconductor with medium-size Cooper pairs should feature both fermionic and bosonic degrees of freedom at low energies.

2. Away from the BCS limit... Where to?

In a Fermi liquid, the propagator of a pair with total momentum $K = 0$ has a cut along the real axis of frequency $\Omega$ (from the continuum of free two-fermion states) and two poles in the unphysical part of the complex $\Omega$ plane. As the temperature is lowered, the poles approach the real axis in the well-known way: $\Omega \sim \pm i(8/\pi) k_B (T - T_c)$. At $T = T_c$, these bound states become stable excitations with zero energy and thus immediately undergo Bose condensation.

To see why stable Cooper pairs in a Fermi liquid necessarily have zero energy, consider the dynamical balance between free fermions and bound states at $K = 0$. Two free fermions with opposite momenta and equal energies $\epsilon$ form a bound state with energy $\Omega = 2\epsilon$ at a rate $\Gamma_+$ proportional to the fermion occupation numbers $n(\epsilon) = (e^{\epsilon/k_B T} + 1)^{-1}$ and the density of states (DOS): $\Gamma_+ \propto D(\epsilon)[n(\epsilon)]^2$. The dissociation rate is $\Gamma_- \propto D(\epsilon)[1 - n(\epsilon)]^2$, with the same prefactor (by virtue of the time-reversal symmetry). The balance $\Gamma_+ = \Gamma_-$ is achieved when $\epsilon = 0$, hence $\Omega = 0$.

Note that $\Gamma_+ - \Gamma_- = 0$ is also when the DOS vanishes in some interval of energies $-|\Delta| < \epsilon < |\Delta|$. In this case, however, a stable bound state can be formed at any real energy in the interval $-2|\Delta| < \Omega < 2|\Delta|$ and the condensation is not imminent. In the BCS model, the origin of the energy gap is scattering of fermions from the condensate of pairs, hence first the pairs condense and then the gap opens up. However, when the pair decay is slow (just above $T_c$), scattering from these long-lived resonances should already start to modify the fermion spectrum. This observation demonstrates that the question “which comes first, the stable pairs or the gap?” has no simple answer beyond the mean-field treatment. Quasistable pairs can produce a pseudogap, in which they might live long enough; if their energy is non zero, they will not form a condensate.

The BCS model is an idealization that is realized when the Fermi sea is robust against the formation of a gap by almost stable Cooper pairs. Note that the BCS approach is justified only a posteriori (the Ginzburg criterion), when the spectrum of its fluctuations is found. It would be most advantageous to have a similarly idealized picture for the opposite limit, when creation of stable pairs precedes their condensation. We also want to stay with a predominantly fermionic system, not a collection of hard-core bosons that arises in the limit of fatally strong attraction be-
tween fermions. Whether this idealization describes any real superconductors (e.g., under-doped cuprates) is a separate question.

Some of the features of such an ideal system can be readily anticipated. The fact that Cooper pairs have non-zero energies and momenta means that each fermion state is no longer coupled to a single hole state (by the scattering process “fermion → pair + hole”), but rather to a continuum of hole states. For this reason, fermions have a finite lifetime and the gap is smeared into a pseudogap. Nevertheless, if the DOS vanishes in the middle of the pseudogap, low-energy Cooper pairs will be almost stable. The thermal energy of a boson must be small compared to the gap width \(|\Delta|\), which itself should not exceed the Fermi energy \(\epsilon_F\). We thus define a pseudogap regime:

\[
k_B T_c \leq k_B T \ll |\Delta| \ll \epsilon_F.
\]

(1)

3. Slowly fluctuating pairing field

The coupling of a fermion to a continuum of holes creates a computational problem. Its physical consequence is broadening of fermion energy levels of order \(k_B T\). If, however, there is another mechanism producing a far stronger broadening, the former effect can be neglected for some purposes. Specifically, we neglect the slight difference between the 4-momenta of the fermion and the hole, which simplifies the problem dramatically. By doing so, we slightly violate the conservation of energy and momenta, which may lead to serious errors in computing transport properties but has little influence on the fermion propagator.

Such broadening can be caused by the pairing field with a fluctuating amplitude \(|\Delta|\) that shifts the energy of a fermion from a bare value \(\epsilon\) to \(\tilde{\epsilon} = \pm (\epsilon^2 + |\Delta|^2)^{1/2}\). Amplitude fluctuations of order \(|\Delta|\) result in the width of order \(|\Delta| \gg kT\) for low-energy fermion levels.

This idea can be illustrated by a model in which fermions are coupled to an external pairing field \(\Delta(r, t)\) obeying Gaussian statistics:

\[
H_{\text{int}}(r, t) = \Delta(r, t)\psi_{\uparrow}^\dagger(r, t)\psi^\dagger_{\downarrow}(r, t) + \text{H.c.}
\]

(2)

When temporal and spatial fluctuations of \(\Delta(r, t)\) are neglected, \(\langle \Delta(r, t)\Delta^*(0, 0) \rangle = |\Delta|^2\), the problem admits an exact solution \([2]\). Low-energy fermion excitations have the linewidth of order \(|\Delta|\). The DOS exhibits a pseudogap of width \(|\Delta|\) and vanishes quadratically at low energies.

4. Cooper pairs in \(2+\varepsilon\) dimensions

The pseudogap regime \([1]\) may be realized in quasi two-dimensional superconductors with moderately weak attraction between fermions, where \(T_c\) is reduced substantially below the BCS value \(T^B_{\text{BCS}} \approx |\Delta|\). (We do not consider here the possibility of a Kosterlitz-Thouless transition.) We have studied the pseudogap regime in \(2+\varepsilon\) dimensions using the self-consistent T-matrix approach \([3]\), which readily permits introduction of a propagator for Cooper pairs \([4]\). The resulting set of diagrams for the fermion propagator is similar to the Gaussian model of the previous section, but without crossing boson lines. A smaller number of diagrams leads to a weaker suppression of the fermion states, \(D(\varepsilon) \propto \varepsilon\) at low energies; otherwise, the DOS has a similar structure.

Inside the pseudogap, long-lived Cooper pairs have the dispersion of a Bogoliubov mode with a mass, which prevents their condensation:

\[
\Omega^2 = \Omega_0^2 + s^2 K^2,
\]

(3)

where \(s^2 = \epsilon_F / m\). The mass term \(\Omega_0\) is determined self-consistently to match the pseudogap width with the average fluctuation of the pairing field. In terms of the fermion density \(n\) and mass \(m\), the resulting condensation temperature is

\[
k_B T_c \sim (\varepsilon \pi / 6)\hbar^2 n/m = (2/3)k_B T_B,
\]

(4)

where \(T_B\) is the condensation temperature of an ideal Bose gas with density \(n/2\) and mass \(2m\).

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