Abstract. We study the global decay properties of solutions to the linear wave equation in 1+3 dimensions on time-dependent, weakly asymptotically flat space-times. Assuming non-trapping of null geodesics and a local energy decay estimate, we prove that sufficiently regular solutions to this equation have bounded conformal energy. As an application we also show a conformal energy estimate with vector fields applied to the solution as well as a global $L^\infty$ decay bound in terms of a weighted norm on initial data. Our results reduce the problem of establishing the pointwise decay rate $t^{-\frac{3}{2}}$ in the interior and $t^{-1}$ along outgoing light cones for solutions to the wave equation in these backgrounds to simply proving that the local energy decay estimate holds.

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1. Introduction

Let \((\mathcal{M}, g)\) be a 4-dimensional, smooth, asymptotically flat, Lorentzian manifold. Assume \(\mathcal{M}\) is of the form \(\mathbb{R} \times \mathbb{R}^3\) and that there exist global coordinates \((t, x')\) such that the level sets of \(t\) are uniformly space-like. In this work we study the dispersive properties of solutions to the wave equation:

\[
\square_g \phi = F(t, x), \quad (\phi, \partial_0 \phi)(0, x) = (\phi_0, \phi_1), \quad (\phi_0, \phi_1) \in C^\infty_c(\mathbb{R}^3),
\]

where \(\square_g = |\det g^{-\frac{1}{2}} \partial_0 g^{\alpha \beta} \det g^{\frac{1}{2}} \partial_\beta|\) in local coordinates. One can think of problem (1) with \(F = N(\phi, \nabla \phi)\) and \(N\) a non-linear function as a toy model for many important systems of hyperbolic PDE including: Maxwell-Klein-Gordon, Yang-Mills, and Wave Maps. A necessary first step in understanding the stability properties of these problems is to prove that smooth solutions launched from sufficiently small, smooth initial data can be extended for all time. One particularly fruitful strategy for proving this type of result is to show that solutions to the linear wave equation propagating in a fixed background, evolving from \(C^\infty_c\) data, have pointwise rates similar to those in flat space. Provided that the method of proof is robust enough, one can often leverage the linear bound into a small-data global existence result for some perturbative non-linear problems. In this work we focus on one such method for proving \(L^\infty\) linear decay: the vector field method first introduced by S. Klainerman in [10]. In the case of \(1+3\) dimensions, this \(L^2\)-based method has a proven record of success in dealing with many small-data semilinear and quasilinear problems – at least when the metric \(g\) is uniformly close to Minkowski and the nonlinearities have special structure. The most well-known demonstration the power of this method is the proof of the global non-linear stability of the Minkowski space by Christodoulou-Klainerman [11]. Since the appearance of that work there has been a concerted effort to extend the vector field method to backgrounds far from Minkowski space. In many cases these efforts have been successful and have yielded small data global existence results for non-perturbative non-linear problems such as: Minkowski space with obstacles [20], exterior Kerr with \(|a| \ll M\) (20, 24) and time-dependent, inhomogeneous media ([38, 59])

In light of this connection with nonlinear stability results, the problem of pointwise decay via vector field method for solutions to the linear wave equation on curved, asymptotically flat backgrounds has been intensely studied. In the last few years the main focus of research activity has been on the Schwarzschild and Kerr metrics. This is due to the fact that it is widely believed that this type of strategy has the best chance of proving the non-linear stability of the (exterior) subextremal Kerr family. For cutting-edge results for the linear problem in black hole space-times we mention, without being exhaustive, the work of Dafermos-Rodnianski-Shlapentokh-Rothman [14] and Metcalfe-Tataru-Tohaneanu [28] (see also [1] for the case of Maxwell field). In our work we will turn away from black hole space-times to focus on a different, but related, problem for which vector field methods have not yet been developed: pointwise linear decay for solutions to (1) on radiating, non-trapping space-times. The motivation to look at this problem comes from the fact that these space-times provide a model for the far exterior portion of a dynamic perturbation of a body emitting gravitational waves. Heuristically speaking, one can think of gravitational waves as local disturbances to the space-time geometry propagating along (characteristic) null hypersurfaces. Their presence perturbs the metric \(g\) and leads it to decay towards flat space in the null outgoing region at slower rates than the Schwarzschild or Kerr space-times. This, in turn, precludes us from just repeating the proofs of the vector field methods that are available near Minkowski space and forces us to produce a new method for dealing with these weak asymptotics. In short, weak decay for the metric takes away the classical VF proofs of \(L^\infty\) decay for solutions to (1).

In order to quantitatively define our weak decay for the metric we take the work of Klainerman-Nicolo [18] as a starting point. In that work, the authors use a double null foliation to derive a hierarchy of decay for all connection coefficients and curvature components near null infinity for radiating space-times satisfying the Einstein vacuum equations. In our work, instead of relying on a null frame, we assume the existence a coordinate system such that metric coefficients, after subtracting the Minkowski metric, obey symbol bounds in the null outgoing region which include those of [15] as a special case (see assumption 1.2 below). We chose a coordinate-dependent condition mostly to simplify matters since the extensive geometric computations associated to null frames are unnecessary for this problem. We also mention that our decay conditions in the null outgoing region are general enough to include, as examples, the space-times of Lindblad-Rodnianski [21] and Bieri [6]. On the other hand, within the domain of dependence of a compact set, our metric is allowed
to be time-dependent and may remain far from Minkowski space for all time as long as a local energy
decay estimate and non-trapping of null geodesics hold. For space-times satisfying all these assumptions,
we are able to produce a novel vector field method yielding weighted $L^2$ and pointwise decay bounds
that are analogous to what is available near Minkowski space via classical vector field method. Furthermore, the
norms we impose on the initial data are suitable for non-linear applications – a topic which we will explore
in subsequent work. We now give a detailed description of the space-times we work with.

1.1. Decay Assumptions on the Metric. There are mainly two regions that need to be considered
separately: a sufficiently large compact set and its exterior. In general what happens outside of a compact
set only needs a detailed description where $r \sim t$. In view of this, we make the following:

**Definition 1.1.** Let $r = \sum_{i=1}^{3} \sqrt{x^i r^2}$. Define the Wave Zone to be the set \( \{(t,x)| \frac{2}{3} t < r < \frac{3}{2} t \} \) and the
Interior Region to be the wedge \( \{(t,x)| r < \frac{1}{2} t \} \).

We let $i, j = 1, 2, 3$ and $\alpha, \beta = 0, 1, 2, 3$ and make use of the Einstein summation convention throughout.
The first condition on the metric is the following:

**Assumption 1.2** (Existence of normalized coordinates). There exists $u(t,x^i) \in C^\infty(M)$ with $du \neq 0$
satisfying the following conditions:

i) $u = t - r$ on the set $\{t > 1\} \cup \{r < \frac{1}{2} t\} \cup \{r > 2 t\}$.

ii) (Asymptotics). On the set $\{t > 1\} \cup \{ \frac{1}{2} t < r < 2 t\}$ we have $u \approx t - r$ in the following sense:

\[
|\partial_t^{\nu} (\partial_h u - 1, \partial_t u + \omega^{i})| \lesssim r^{-\delta - |\nu|} \gamma^{-|\nu|} t_0^{-|\nu|},
\]

where we have set:

\[
\tau_- = \langle u \rangle, \quad \tau_+ = C + u + 2r, \quad \tau_0 = \tau_- \tau_+^{-1},
\]

and $C$ is chosen large enough so that $\tau_+ > 1$ in $\{t \geq 0\}$. In particular $\tau_+ \approx \langle t + r \rangle$ and $\tau_0 \approx \langle t - r / (t + r) \rangle$.

iii) (Renormalization). There exists $\delta, \gamma > 0$ such that for any multi-index $J$ the inverse metric coefficients $g^{\alpha \beta}$ in $(u,x^i)$ coordinates satisfy:

\[
\begin{align*}
|\partial_u^{\alpha} \partial_x^{\beta} (g^{ij} - \delta^{ij})| & \lesssim r^{-k - |\nu|} \gamma_k^{-k} \tau_{0}^{-k} (\tau_- / r)^{-k} \\
|\partial_u^{\alpha} \partial_x^{\beta} (d^2 g^{ui} + \omega^{i})| & \lesssim r^{-k - |\nu|} \gamma_k^{-k} \tau_{0}^{-k} (\tau_- / r)^{-k} \\
|\partial_u^{\alpha} \partial_x^{\beta} (g^{ui} - \omega^{i} \omega^{j} g^{uj})| & \lesssim r^{-k - |\nu|} \gamma_k^{-k} \tau_{0}^{-k} (\tau_- / r)^{-k} \\
|\partial_u^{\alpha} \partial_x^{\beta} (\omega^{i} )| & \lesssim r^{-k - |\nu|} \gamma_k^{-k} \tau_{0}^{-k} (\tau_- / r)^{-k}
\end{align*}
\]

where $d = |\det(g_{\alpha \beta})|$ is in $(u,x^i)$ coordinates, $\partial_x$ are the derivatives corresponding to $x^i$, and $\omega^i = x^i / r$.

In the sequel we also refer to the $(u,x^i)$ coordinates as Bondi Coordinates.

**Remark 1.3** (Decay outside the wave zone). Let $g^{\alpha \beta}$ be in $(t,x^i)$ coordinates. In the interior region a
computation using the chain rule yields:

\[
|\partial_t^{\nu} \partial_x^{\beta} (g^{\alpha \beta} - m^{\alpha \beta})| \lesssim \langle r \rangle^{-k - |\nu| - \delta} \langle t / r \rangle^{-\gamma_k},
\]

with $m^{-1} = m = \text{diag}(-1,1,1,1)$. By a similar reasoning inside \( \{ \frac{1}{2} t < r < \frac{3}{2} t \} \) \cup \( \{ \frac{3}{2} t < r \} \) we have:

\[
|\partial_t^{\nu} \partial_x^{\beta} (g^{\alpha \beta} - m^{\alpha \beta})| \lesssim \langle t + r \rangle^{-k - |\nu| - \delta}.
\]

In practice we use \( \frac{3}{2} t \) within the wave zone and we often switch to \( \frac{1}{2} t \) for the interior and to \( \frac{1}{2} t \) outside these two regions.

**Remark 1.4**. This hierarchy of decay for different components of the metric along outgoing null directions
is consistent with the radiation of gravitational waves and with the peeling estimates in \( \mathbb{H} \). These decay
rates are also consistent with the metrics constructed in the stability of Minkowski space in wave coordinates
\( \mathbb{H} \). Note, in particular, that the ‘shear’ terms $g^{ui} - \omega^i \omega^j g^{uj}$ decay only slightly faster than a solution to
the wave equation on any fixed hypersurface $u = \text{const}$.

**Remark 1.5**. (Examples). In Schwarzschild one can define $r^* = r + 2M \log \frac{r}{2M} - 1$ and let $u = t - r^*$. The
metric in $(u,x^i)$ coordinates then satisfies assumption \( \mathbb{H} \) in the wave zone with $\delta = 1$. The subextremal
Kerr family also satisfies this assumption in the wave zone (see the upcoming work \( \mathbb{H} \) for details).
Remark 1.6. Estimates (2) and (3) also hold if we trade \( u \) for \( \tilde{u} = \chi u + (1 - \chi)(t - r) \), where \( \chi \equiv 1 \) when \( \tau_0 \leq c \) for any \( 0 < c < 1 \) with bounds \(|(\tau_0^\perp \partial)\tilde{\chi}| \leq 1 \). Thus the assumption that \( u \equiv t - r \) in the set \( \{ t > 1 \} \cup \{ r < 4 t \} \cup \{ r > 2 t \} \) is not the weakest condition we can impose. We chose this condition mostly to simplify matters dealing with estimates in the interior region. However, the reader should keep in mind that one only needs to use the exact form of \( u \) in a narrow wedge \(| t - r | \ll r \).

Remark 1.7. By estimates (4) the \( u = \text{const} \) hypersurfaces are approximately null. We also note that, by construction, the \( \partial_x \) derivatives are tangential to these hypersurfaces.

Remark 1.8. Inside sets of the form \( r \leq R_0 \) with \( R_0 > 0 \) our assumptions allow for \( g \) to be a large perturbation of the Minkowski metric as \( t \to +\infty \). In particular, the metric \( g \) does not have to converge to a stationary metric.

1.2. Non-trapping and Local Energy Decay. Next we introduce our non-trapping and local energy decay assumptions. These are the two key ingredients that allow us to handle large deformation errors inside regions of the form \( r \leq R_0 \) without losing regularity. Let us start by making the non-trapping assumption precise:

Assumption 1.9 (Quantitative non-trapping for null geodesics). Let \( R_0 > 0 \) and \( \gamma(s) \) be a forward, affinely parametrized null geodesic satisfying \( \gamma(0) \in \{ r \leq R_0 \} \), \( \gamma s \equiv 1 \), and \( \dot{\gamma} |_{s=0} = 1 \). For any such \( \gamma \), there exists a uniform constant \( C = C(R_0, g) \) such that \( \gamma(s) \in \{ r > R_0 \} \) for all \( s \geq C \).

Next we define the Local Energy Decay (LED) norms:

\[
\| \phi \|_{LE_0[t_0, t_1]} = \sup_i 2^{-\frac{1}{4} i} \| \chi(r) \approx 2^i \phi \|_{L^2_t L^2_r [t_0, t_1]},
\]
\[
\| \phi \|_{LE[t_0, t_1]} = \sup_i \left( 2^{-\frac{1}{4} i} \| \chi(r) \approx 2^i \nabla \phi \|_{L^2_t L^2_r [t_0, t_1]} + 2^{-\frac{1}{4} i} \| \chi(r) \approx 2^i \phi \|_{L^4_t L^2_r [t_0, t_1]} \right) = \| \langle r \rangle^{-\frac{1}{2}} (\nabla \phi, (\langle r \rangle)^{-1} \phi) \|_{L^\infty_t L^2_r [t_0, t_1]},
\]
\[
\| F \|_{LE^\star [t_0, t_1]} = \sum_i 2^{\frac{1}{4} i} \| \chi(r) \approx 2^i F \|_{L^2_t L^2_r [t_0, t_1]} = \| \langle r \rangle^{\frac{1}{2}} F \|_{L^2_r [t_0, t_1]}.
\]

We use these norms to state the final decay assumption on the metric:

Assumption 1.10 (Local energy decay estimate). For all values \( 0 \leq t_0 < t_1 \) the evolution (1) satisfies:

\[
(6) \quad \| \phi \|_{LE[t_0, t_1]} \lesssim \| \nabla \phi(t_0) \|_{L^2_r} + \| \Box_g \phi \|_{LE^\star [t_0, t_1]}.
\]

Non-trapping of null geodesics is a necessary condition for (6) to hold in this form. If there’s trapping, the work ofRalston [32] and some additional geometric optics considerations can be used to show that the LED estimate, if it holds at all, must lose derivatives in a neighborhood of the trapping set. In the non-trapping case, estimates such as (6) date back to work of Morawetz [30] and are known to hold in a variety of settings. In the case of Minkowski space, Keel-Smith-Sogge proved a limiting version of this estimate [15] (see also [36]). For uniformly small, time-dependent perturbations of Minkowski, Alinhac [2] and Metcalfe-Tataru [27] both established this result. The work of Bony-Hafer [10] extended the validity of this estimate to the case of large, stationary, non-trapping metrics of the form \( ds^2 = -dt^2 + h_{ij}(x)dx^idx^j \) with \( h \) Riemannian (see also [33]). As the Schwarzschild and Kerr solutions have trapped null geodesics, this work will not apply to full perturbations of such space-times. However, a suitable modification in the upcoming work [31] will do so.

Since the LED estimate is generally expected to hold for a large class of space-times, it is a natural assumption to include in our problem. In particular, the LED estimate should hold for time-dependent, non-trapping, asymptotically flat space-times satisfying a smallness condition for \( \partial \tilde{g} \) and it should also hold (with loss of derivatives) for the domain of outer communications of a small time-dependent perturbation of the sub-extremal Kerr family. In the non-trapping, time-dependent case, some work is already under way to prove this result (upcoming work of Sterbenz-Tataru). In the trapping case, the work [28] already established the result for fast decaying perturbations of Kerr with \( |a| \ll M \). Since we are interested in global decay bounds, we believe the reasons above provide adequate justification to separate the proof of (6) from our results here. In view of this, we will take the estimate as given and focus on developing a precise understanding of the asymptotic properties of the solution via vector fields.
1.3. Weighted Norms. We define the Weighted LED norms by:

$$\| \phi \|_{LE^a[t_0,t_1]} = \| \sum_{i,j} \phi_{ij,m} \|_{LE[t_0,t_1]}, \quad \| \phi \|_{LE^a[t_0,t_1]} = \| \phi \|_{LE^a[t_0,t_1]},$$

with analogous definitions for $\| F \|_{LE^a[t_0,t_1]}$. Next we set up the fixed-time and null energies that we will use in the sequel. Define the vector $\nabla \phi = (\partial_0 \phi, ..., \partial_d \phi)$ where $\partial_0, ..., \partial_d$ denotes any basis which can be written as a bounded linear combination of $(t, x)$ coordinate derivatives. For $t \geq 0$ we define the fixed-time Conformal Energy:

$$\| \phi(t) \|_{CE} = \| \tau_+ \tau_0 \nabla \phi(t) \|_{L^2_t} + \| \tau_+ (\nabla \phi(t), r^{-1} \partial_t \phi(t)) \|_{L^2_t}.$$

For a smooth, positive weight function $\Omega$ we also have the Conjugated Conformal Energy:

$$(7) \quad \| \phi(t) \|_{\alpha CE} = \| \tau_+ \tau_0 \Omega^{-1} \nabla (\Omega \phi(t)) \|_{L^2_t} + \| \tau_+ \Omega^{-1} \nabla (\Omega \phi(t)) \|_{L^2_t}.$$

In this work we only use the conformal weights $\Omega = (r)$, $\Omega = \tau_+ \tau_0$. We set up the notation:

$$\| \cdot \|_{CE} = \| \cdot \|_{\alpha CE}, \quad \| \cdot \|_{\alpha CE} = \| \cdot \|_{\alpha CE}.$$

Note that $\| \cdot \|_{CE}$ and $\| \cdot \|_{\alpha CE}$ are fixed-time norms and do not contain null energies. To introduce these we define the scale of spaces:

$$\| \phi \|_{NLE[t_0,t_1]} = \sup_{i,j} 2^{-\frac{i}{2}} \| \chi_{(r \geq t)} \|_{L^2_t} + \| \phi \|_{CE[t_0,t_1]} + \| \phi \|_{\alpha CE[t_0,t_1]}.$$

Our (weighted) null energies are defined to be:

$$\| \phi \|_{CH[t_0,t_1]} = \| (r)^{\frac{1}{2}} \Omega^{-\frac{1}{2}} \partial_t (\Omega \phi) \|_{NLE^{\frac{1}{2}}, [t_0,t_1]} + \| \phi \|_{CH[t_0,t_1]} + \| \phi \|_{\alpha CE[t_0,t_1]}.$$

Using the norms above as building blocks we then define the $S$ norm:

$$\| \phi \|_{S[t_0,t_1]} = \| \phi \|_{CE[t_0,t_1]} + \| \phi \|_{\alpha CE[t_0,t_1]}.$$

Associated to this is the source term norm $N$:

$$\| F \|_{N[t_0,t_1]} = \| (\chi_{r < \frac{1}{2} t} + \chi_{r \geq 2 t}) F \|_{LE^{\frac{1}{2}}, [t_0,t_1]} + \| \phi \|_{\alpha CE[t_0,t_1]}.$$

1.4. Vector Fields and Associated Norms. We now define versions of the previous norms which incorporate additional decay. These will be stated in terms of modifications of the usual Lorentz vector fields which are adapted to the null geometry of our space-time as dictated by the function $u(t,x)$. First we define the Lie algebra in $(t, x)$ coordinates:

$$(8) \quad \mathcal{L} = \{ T = (u_t)^{-1} \partial_t, S = (u_t)^{-1}(u - ru_x) \partial_x + r \partial_t, \Omega_{ij} = \Omega_{ij,m} - (u_t)^{-1} \Omega_{ij,m} \partial_x \partial_t \}.$$

where $\Omega_{ij,m} = x^i \partial_j - x^j \partial_i$. With $\Gamma \in \mathcal{L}$ we define the higher order norms to be:

$$\| \phi \|_{LE^1[t_0,t_1]} = \sum_{|J| \leq 1} \| \Gamma^J \phi \|_{LE[t_0,t_1]} + \| \phi(t) \|_{CE^1[t_0,t_1]} = \sum_{|J| \leq 1} \| \Gamma^J \phi \|_{CE[t_0,t_1]}.$$

with analogous definitions for the weighted $LE$ and $CH$ norms respectively. Using these as building blocks we define the higher order $S$ norm by:

$$\| \phi \|_{S^1[t_0,t_1]} = \| \phi \|_{CH^1[t_0,t_1]} + \| \phi \|_{\alpha CE^1[t_0,t_1]}.$$

Associated to this is the higher-order source term norm:

$$\| F \|_{N^1[t_0,t_1]} = \| (\chi_{r < \frac{1}{2} t} + \chi_{r \geq 2 t}) F \|_{LE^{\frac{1}{2}}, [t_0,t_1]} + \| \chi_{r < \frac{1}{2} t} \|_{NLE^{\frac{1}{2}}, [t_0,t_1]} + \| \tau_+^{\frac{1}{2}} \nabla F \|_{N[t_0,t_1]}.$$

Finally, we have the following initial data spaces which will applied to $\nabla \phi(0)$:

$$\| f \|_{H^k} = \sum_{|J| \leq k} \| \langle r \rangle^{|a|} \nabla^{|J|} f \|_{L^2}.$$
1.5. Statement of the Main Results.

**Theorem 1.11** (Main Theorem). Assume that $\mathcal{M}$ is of the form $\mathbb{R} \times \mathbb{R}^3$ and that the metric $g$ satisfies the decay assumption with $0 < \gamma < \delta$, the non-trapping assumption and the local energy decay assumption. We then have the following uniform estimates for all $T > 0$:

**I) (Conformal Energy Estimate)**

(9) \[
\sup_{0 \leq t \leq T} \| \phi(t) \|_{CE} + \| \phi \|_{L^\infty_S[0,T]} \leq \| \nabla \phi(0) \|_{H^{1,1}_0} + \| \Box_g \phi \|_{L^1 N[0,T]}.
\]

**II) (Conformal Energy Estimate With Vector fields)**

(10) \[
\sup_{0 \leq t \leq T} \| \phi(t) \|_{CE}, \Box_g \phi \|_{L^\infty_S[0,T]} \leq \| \nabla \phi(0) \|_{H^{1,1}_0} + \| \Box_g \phi \|_{L^1 N[0,T]}.
\]

**III) (Global Pointwise Decay)**

(11) \[
\| \tau^\frac{1}{2} \tau^{\frac{1}{2}} \phi \|_{L^\infty[0,T]} \leq \| \nabla \phi(0) \|_{H^{1,1}_0} + \| \Box_g \phi \|_{L^1 N[0,T]}.
\]

Here $\ell^p_B[0,T]$ denotes the norm $\| f \|_{\ell^p_B[0,T]} = \sum_{j \geq 0} \| \chi_j(0) f \|_{B}[0,T]$ where $\chi_j(t)$ is a series of dyadic cutoffs on $t \approx 2^k$ covering $[0,T]$, with the obvious modification for $p = \infty$.

One can think of this as a conditional linear stability result for the wave equation in non-trapping backgrounds satisfying our weak asymptotics. In other words: for such backgrounds, as long as the LED estimate (6) holds, then the pointwise decay rates on establishes an radiation field asymptotics for weakly decaying metrics with a full asymptotic expansion. The work \[39\] previous result are those mentioned above: \[38\], \[39\] and \[28\] (see also \[5\] for a non-vector field proof of non-linear applications.

M can deviate significantly from being a true optical function for ($u$) and from the work of S. Yang above, is that we commute once with the scaling vector field $S$. As mentioned in that work, it is a commonly held belief that one needs to have $t \partial_t g = O(1)$ in the set $r \leq 1$ for all time in order to use a scaling vector field as a commutator. We show here that in fact $|t \partial_t g| \lesssim t^{1-\gamma}$ will suffice and that therefore the classical methods involving commuting the scaling vector field into conformal energy still apply in this general setting. Commuting with $S$ is desirable since it leads to the higher interior decay rate of $t^{-\frac{3}{2}}$. Since this rate of decay is integrable in time, we hope that it is useful for some non-linear applications.

For pointwise decay via vector fields on large, time-dependent, asymptotically flat space-times the only previous result are those mentioned above: \[38\], \[39\] and \[28\] (see also \[5\] for a non-vector field proof of radiation field asymptotics for weakly decaying metrics with a full asymptotic expansion). The work \[39\] establishes an $L^\infty$ decay rate of $(t + r)^{-1}$ for compactly supported metrics satisfying a non-sharp version of \[0\] for large perturbations. One of the main differences with our results is that $\partial_t g$ only needs to be small in the interior leading to more general metrics. However, the $L^\infty$ decay proved in that work is weaker than ours in the interior and the metric equals Minkowski in the exterior. As an application of his method the author...
also shows a small data global existence result for semilinear equations satisfying the null condition. In the
more recent work [28] the same author proves a pointwise decay rate of \( \langle r \rangle^{-\frac{k}{2}} (t - r)^{-\frac{k}{2}} \) for time-dependent
metrics which are uniformly close to Minkowski and decay weakly in the null outgoing region. The main
difference with our work again is that both the interior decay and the wave zone decay achieved for the
solution are weaker. Additionally, the outgoing conditions assumed for the metric are inhomogeneous and
demand more decay on the undifferentiated terms \( g^{\mu \nu} \) as well as \( \Omega_{ij} g^{\alpha \beta} \). On the other hand, we point out
that once again the assumptions on the metric in the interior are slightly more general than ours and that
global existence for quasilinear equations satisfying a null condition is again shown as an application. Lastly,
in [28] the authors prove a decay rate of \( (t + r)^{-1} (t - r)^{-2} \) (Price Law) for non-trapping space-times with
t\( t \)\( \partial_t g = O(1) \). The authors assume a sharp LED estimate with norms similar to ours as well as wave zone
decay rates \( |\partial^k (g - m)| \lesssim \langle r \rangle^{-1-k} \) – which are more restrictive than ours. Despite the fact that a lot of
decay is achieved for the linear problem the norms for the source term \( \Box_g \phi \) involved in getting that decay
do not allow for applications to non-linear problems. However, we mention that [28] also proves the result
for the black hole case.

1.7. Organization of the Paper. In section 2.1 we recall the standard energy formalism for the wave
equation. In section 2.2 we set up and prove a generalization of the conformal method of Lindblad-Sterbenz
[22]. This method is a general framework for proving weighted energy estimates arising from asymptotically
conformal Killing vector fields in curved space-times. This framework is central to our work since it is the
foundation upon which our exterior proof of conformal energy is built. In a curved space-time there’s three
advantages to using this method versus the classical proof of conformal energy\(^1\). Firstly, since the identities
are already in divergence form we avoid having to perform several integrations by parts in order to take
advantage of special cancellations for the boundary terms. Secondly, this method is robust enough to prove
other useful weighted energy estimates such as the fractional conformal energy bounds we see in [22]. Lastly,
the method is capable of handling the weak decay of our metrics in the wave zone. To the best of the author’s
knowledge, no other method has proved capable of proving conformal energy bounds with these types of
conditions.

In the case of Minkowski space, which is the only case covered in [22], this method is motivated by the
observation that the Morawetz vector field \( K_{\text{mink}} = (t^2 + r^2) \partial_t + 2tr \partial_r \) is conformal Killing. Therefore it is
desirable to understand how the energy formalism for multipliers changes under conformal maps: \( g \to \Omega^{-2} g \).
The choices \( ^I \Omega = r \) and \( ^H \Omega = t^2 - r^2 \) make \( K_{\text{mink}} \) a Killing field in these backgrounds. Since the deformation
errors vanish, it is a simple matter to then use the energy formalism corresponding to the conformal metrics
to prove two conjugated Morawetz estimates which, together, combine to yield the conformal energy bound.
To extend this method to curved space-times we once again look at the conformal wave equation and use it
to develop a general formalism for multipliers. Inspired by the Minkowski case, we choose smooth positive
weights \( ^I \Omega = \langle r \rangle \), \( ^H \Omega = r - r_{\text{null}} \) which asymptotically behave like \( r \) and \( t^2 - r^2 \). We use these in combination
with a modified Morawetz vector field \( K_0 = u^2 \partial_u + 2(u + r) \partial_r \) which is asymptotically Killing with respect
to these conformal backgrounds. Given this input, the generalized Lindblad-Sterbenz machinery established
in our work effectively reduces the bulk of the proof of [9] to a multiplier bound modulo error terms. It
is important to use both of these weights in our method since \( ^I \Omega \) degenerates where \( r \sim 0 \) but \( ^H \Omega \) is
well-behaved there and (locally) controls the bulk of the conformal energy. In the wave zone the opposite
behavior takes place and it is the weight \( ^I \Omega \) that is responsible for the bound on conformal energy here.

The preceding method requires a positive-definite energy density. This is addressed in section 2.3 by
proving a general result stating that non-trapping, plus smallness of \( \partial_i g \), plus asymptotic flatness implies
that the vector field \( \partial_i \) is uniformly timelike. For our types of metrics this implies that \( \partial_i \) is uniformly
timelike in the asymptotic region \( t \gg 1 \). The proof of this fact is by contradiction: if \( \partial_i g \) is small and \( \partial_i \)
is close to null then the inner product \( \langle \partial_i, \partial_i \rangle \approx 0 \) and is almost conserved along the null geodesic flow.
However, by the non-trapping condition and asymptotic flatness \( \langle \partial_i, \partial_i \rangle \approx -1 \) in the far exterior – a fact
which contradicts the previous claim. In section 3 we derive all the identities for error terms in Bondi
coordinates as well the corresponding asymptotic bounds. In section 4 we reduce to proving our results in
the asymptotic region \( t \gg 1 \) and set up some notation for absorbing small errors there. In section 5 we use of

\(^1\) See [17] for the classical proof of the conformal energy in a curved background close to Minkowski.
our Lindblad-Sterbenz machinery to prove (9). The proof of the error bounds for this estimate rely on three
main building blocks: firstly, an upgrade of (9) to a $t$-weighted LED estimate in the interior which is used to
cal control all large deformation tensor errors supported within this region. Secondly, the fact that $\partial_0$ is timelike
plus the dominant energy condition gives us a coercive bound for the weighted energies we wish to control.
Thirdly, in the wave zone the $K_0$ field is set-up so that the deformation tensors yield better space-time errors
compared to the standard Minkowski Morawetz field $K_{\text{mink}}$. In short, the Lindblad-Sterbenz formalism
coupled with the hierarchy of decay (3) suffices to control these error terms.

In section 6 we prove the higher conformal bound (10). We do this by commuting the equation once
with the Lie algebra $L$ — in particular we avoid the use of the Lorentz boosts $\Omega_0i = t\partial_i - x^i\partial_t$. In the wave
zone the desired estimates follow from the fact that the modified vector fields have favorable errors that
work well with the renormalization (3). In the interior the main problem is that (4) implies $|t\partial_0g| \lesssim t^{1-\gamma}$,
thus commuting with the scaling vector field is non-trivial. We fix this in stages: we start by proving some
commutator estimates. After applying Hardy estimates to the ensuing lower order terms, the main errors
arising from commuting with $S$ consist of $T$-weighted $L^2$ terms with two derivatives supported inside $r < ct$
with $c \ll \frac{1}{t}$. We control these by proving a $T$-weighted LED estimate with vector fields and use it to trade
$t\partial_t\phi$ for $S\phi$ plus small errors. This leaves only terms with two spatial derivatives $t\nabla^2r\phi$ to be bounded.

Thanks to the global weighted $L^2$ elliptic estimate (136), we are able to trade two space derivatives for the
elliptic part of the wave operator $P = g|\tau_p\phi|^2 \partial_p g\partial_p \phi$. We then trade $tP$ for $\nabla\phi_t + t\Omega_0\phi + t\nabla\partial_0\phi$. This
method is somewhat reminiscent of the work of Klainerman-Sideris [19] and relies crucially on the global
weighted $L^2$ elliptic estimate (136) which is shown in the appendix. In this procedure it is convenient to use
the $L^2$ norm (instead of the dyadic LED norms) when commuting with $S, \Omega_{ij}$ because we ultimately need
to resort to an $L^2$ Hardy estimate in order to deal with the lower order terms. Only then can we apply
the elliptic estimate (136) to close the argument outlined above. Additionally, using the $L^2$ norms for these
terms is advantageous since it also sets up the estimates so we can re-use them in the proof of the global
pointwise decay in section 6 which follows by a similar type of argument.

1.8. Basic Notation. The following notation will be used in the sequel:
We denote $A \lesssim B$ (resp. “$A \ll B$”; “$A \approx B$”) if $A \leq CB$ for some fixed $C > 0$ which may change from
line to line (resp. $A \leq \epsilon B$ for a small $\epsilon$; both $A \lesssim B$ and $B \lesssim A$).

By default, any norms involving a range for the $t$ variable have $t \in [t_0, t_1]$ with $0 \leq t_0 < t_1$.

Given norms $\| \cdot \|_A$, $\| \cdot \|_B$ and a weight $\omega$, the notation $\omega A \lesssim B$ means $\| \omega f \|_A \lesssim \| f \|_B$.

The notation $m = m^{-1} = \text{diag}(-1, 1, 1, 1)$ denotes the Minkowski metric in $(t, x)$ coordinates.
The notation $\eta^{-1}$ denotes the inverse Minkowski metric in $(u_{\text{mink}}, x^i)$ coordinates with $u = u_{\text{mink}} = t - r$:

$$\eta^{uu} = 0, \quad \eta^{ui} = -\omega^i, \quad \eta^{ij} = \delta^{ij}, \quad \text{where } \omega^i = \frac{x^i}{r}.$$ 

Let $\tilde{\partial}_i$ denote coordinate derivatives for the $x^i$ in Bondi coordinates. We also denote $\tilde{\partial}_r = \omega^i \tilde{\partial}_i$,
$\Omega_{ij} := \{\Omega_{ij}\}$ for $i < j$ denotes the modified rotations. $\Omega > 0$ will denote a smooth conformal weight.

The conformal weights are $\bar{t}\Omega = \langle r \rangle$; $\bar{\Omega} = \tau_0 - \tau_+$ and their corresponding rescaled solutions are: $\psi = \Omega \phi$, $\bar{t}\psi = \bar{t}\Omega \phi$, and $\bar{\Omega} \psi = \bar{\Omega} \Omega \phi$, respectively.

$D$ will denote the Levi-Civita connection corresponding to the metric $g$.

$N$ will denote the future-directed unit normal vector to the level sets $t = \text{const}$. That is:

$$N^\alpha = \frac{-g^{\alpha\beta}\partial_\beta t}{(g^{\alpha\beta}\partial_\alpha t)^{1/2}}.$$ 

2. Preliminary Setup

2.1. Energy Formalism. Here we recall the basic energy setup for vector fields multipliers and commutators
for problem (1).

2.1.1. Vector Field Multipliers. Define the Energy-Momentum Tensor associated to $(g, \phi)$ by:

$$T_{\alpha\beta} = \partial_\alpha \phi \partial_\beta \phi - \frac{1}{2} g_{\alpha\beta} \partial^\mu \phi \partial_\mu \phi.$$
$T_{\alpha \beta}$ is related to the wave operator $\Box g \phi$ by the identity $D^\alpha T_{\alpha \beta} = \Box g \phi \cdot \partial_\beta \phi$, with $D$ denoting the Levi-Civita connection. Given a smooth vector field $X$ we define the 1-form $(X)P_\alpha = T_{\alpha \beta}X^\beta$. Taking the divergence of this we arrive at the well-known formula:

$$D^\alpha (X) P_\alpha = \Box g \phi \cdot X \phi + \frac{1}{2} (X) \pi^\alpha \beta T_{\alpha \beta},\tag{13}$$

with $L_X g_{\alpha \beta} = (X) \pi_{\alpha \beta}$. The symmetric 2-tensor $(X) \pi$ is the Deformation Tensor of $g$ with respect to $X$ and measures the change of $g$ under the flow generated by $X$. Integrating (13) over the time slab $\{(t,x)\colon t_0 \leq t \leq t_1\}$ and using Stokes' theorem we get the Multiplier Identity:

$$\int_{t_0}^{t_1} (X) P_\alpha N^\alpha d^2 dx = \int_{t_0}^{t_1} \int_{\mathbb{R}^3} (\Box g \phi \cdot X \phi + \frac{1}{2} (X) \pi^\alpha \beta T_{\alpha \beta}) \, dV_g,$$

where $dV_g = d^2 dt dx$ and $N$ is the vector defined in (12). The integrand on the left hand side is the Energy Density associated to $X$ through the foliation by spacelike hypersurfaces $t = \text{const}$. We also recall that $T_{\alpha \beta}$ satisfies the Dominant Energy Condition: For any two timelike, future-directed vector fields $X,Y$ the energy momentum tensor $T_{\alpha \beta}$ satisfies $|\nabla \phi|^2 \lesssim T(X,Y)$.

2.1.2. Formulas for Commutators and Multipliers. Given a vector field $X$, we define the Normalized Deformation Tensor of $X$ to be: $(X) \tilde{\pi} = (X) \pi - \frac{1}{2} g \cdot \text{trace} ((X) \pi)$. This tensor is present in some of the most important formulas dealing with vector fields for the wave equation.

Lemma 2.1 (Basic formulas involving $(X) \tilde{\pi}$). Let $\phi$ be a smooth function and $X$ a vector field. The following identities hold:

\begin{align*}
(15a) \quad (X) \tilde{\pi}^\alpha \beta &= -d^\gamma (d^\gamma g^{\alpha \beta}) - g^{\alpha \gamma} \partial_\gamma X^\beta + g^{\beta \gamma} \partial_\gamma X^\alpha,
(15b) \quad [\Box_g, X] \phi &= D_\alpha (X) \tilde{\pi}^\alpha \beta D^\beta \phi + (D_\alpha (X) \gamma) \Box g \phi, \tag{15b}
(15c) \quad D^\alpha (X) P_\alpha &= \Box_g \phi \cdot X \phi + \frac{1}{2} (X) \tilde{\pi}^\alpha \beta \partial_\alpha \phi \partial_\beta \phi, \tag{15c}
\end{align*}

where (15a) is computed in local coordinates.

Proof. We'll prove each of these formulas separately.

Part 1: (The identity (15a)) In local coordinates we have the well-known formula:

$$\Box g \phi \cdot X \phi + g^{\alpha \gamma} \partial_\gamma X^\beta + g^{\beta \gamma} \partial_\gamma X^\alpha.\tag{16}$$

Subtracting the expression: $\frac{1}{2} g^{\alpha \beta} \text{trace} ((X) \pi) = g^{\alpha \beta} d^\gamma \partial_\gamma (X^\beta d^\gamma)$ from both sides gives the result.

Part 2: (The identity (15b)). In local coordinates we have the following well-known formula (see 3):

$$[\Box_g, X] \phi = (X) \tilde{\pi}^\alpha \beta D^2 \phi + (D_\alpha (X) \pi^{\alpha \beta}) \partial_\beta \phi - \frac{1}{2} g^{\alpha \beta} \text{trace} ((X) \pi) \partial_\alpha \phi.$$

Using the definition of $(X) \tilde{\pi}$:

$$(X) \tilde{\pi}^\alpha \beta D^2 \phi = (X) \tilde{\pi}^\alpha \beta D^2 \phi + \frac{1}{2} (\text{trace} (X) \pi) \Box g \phi, \quad (D_\alpha (X) \pi^{\alpha \beta}) \partial_\beta \phi = (D_\alpha (X) \pi^{\alpha \beta}) \partial_\beta \phi + \frac{1}{2} \partial_\alpha (\text{trace} (X) \pi) \partial_\beta \phi.$$

Identity (15b) follows by combining these two statements.

Part 3: (The identity (15c)). Combining (13) with the identity:

$$T_{\alpha \beta}^{(X) (X) \pi^{\alpha \beta}} = \partial_\alpha \phi \partial_\beta \phi (X) \pi^{\alpha \beta} - \frac{1}{2} g^{\alpha \beta} \partial_\alpha \phi \partial_\beta \phi \cdot \text{trace} ((X) \pi) = \partial_\alpha \phi \partial_\beta \phi (X) \tilde{\pi}^{\alpha \beta},$$

gives the result. □

2.2. Conformal Changes for Vector Field Multipliers. As mentioned in the introduction, our goal in this section is to record how all the formulas associated with the vector field multiplier method change under conformal deformations of the metric.
2.2.1. *The Conformal Wave Equation.* Let $g_{\alpha\beta}$ be a Lorentzian metric on an $1 + 3$ dimensional space-time. We consider a conformally equivalent metric $\tilde{g}_{\alpha\beta}$ where $\tilde{\Omega}^2 \tilde{g} = g$ for some smooth weight function $\Omega > 0$. Let $\tilde{D}$ denote the Levi-Civita connection of $\tilde{g}$ and $\Box_{\tilde{g}} = \tilde{D}^\alpha \tilde{D}_\alpha$ the corresponding wave operator. We then have the following standard formula from geometry:

**Lemma 2.2** (Conformal wave equation). Let $\psi = \Omega \phi$ and $\Box_g \phi = F$. Then, for the wave operator $\Box_{\tilde{g}}$ of the conformal metric $\tilde{g} = \Omega^{-2} g$ we have:

\begin{equation}
\Box_{\tilde{g}} \psi + V \psi = G = \Omega^2 F , \quad V = \Omega^2 \Box_g \Omega^{-1} .
\end{equation}

**Proof.** Since $\Box_g = \Omega^2 (\Box_{\tilde{g}} - 2g^{\alpha\beta} \partial_\alpha \ln \Omega \partial_\beta)$, rescaling $\Box_{\tilde{g}} \psi = \Box_g \Omega \phi$ yields:

\begin{equation}
\Box_{\tilde{g}} \Omega = \Omega^2 \Box_g \Omega^{-1} + \Omega^{-1} \Box_g \Omega^2 + 2 \partial_\alpha (\Omega^2) \partial^\alpha \Omega^{-1} = \Omega^2 \Box_g \Omega^{-1} + 2 \Box_g \Omega - 2 \partial_\alpha (\ln \Omega) \partial^\alpha (\ln \Omega) .
\end{equation}

Therefore it suffices to show $-\Box_{\tilde{g}} \Omega = \Omega^2 \Box_g \Omega^{-1} - 2 \partial_\alpha (\ln \Omega) \partial^\alpha (\ln \Omega)$. But this follows directly by:

\[ \Box_{\tilde{g}} \Omega = \Omega^2 \Box_g \Omega^{-1} - 2 \partial_\alpha (\ln \Omega) \partial^\alpha (\ln \Omega) . \]

\[ \Box_{\tilde{g}} \Omega = \Omega^2 \Box_g \Omega^{-1} - 2 \partial_\alpha (\ln \Omega) \partial^\alpha (\ln \Omega) . \]

2.2.2. *Conformal Vector Field Multipliers.* Let $\chi(t, x)$ a smooth cutoff function and $\psi = \Omega \phi$. Using equation (17) we define the *Conformal Energy-Momentum Tensor* for $(\psi, \chi)$ by:

\begin{equation}
\tilde{T}^X_{\alpha\beta} = \partial_\alpha \psi \partial_\beta \psi - \frac{1}{2} \tilde{g}_{\alpha\beta} (\tilde{g}^{\mu\nu} \partial_\mu \psi \partial_\nu \psi - \chi V \psi^2) , \quad V = \Omega^2 \Box_g \Omega^{-1} .
\end{equation}

This satisfies the divergence law:

\begin{equation}
\tilde{D}^\alpha \tilde{T}^X_{\alpha\beta} = ((\chi - 1) V \psi + \Omega^2 F) \partial_\beta \psi + \frac{1}{2} \partial_\beta (\chi V) \psi^2 .
\end{equation}

Define the *Conformal Energy Density* for $(X, \Omega)$ by $(X) \tilde{P}^X_\alpha = \tilde{T}^X_{\alpha\beta} X^\beta$. We have the following identities:

**Lemma 2.3** (Conformal multiplier identity). Let $X$ be a vector field, $\chi(t, x)$ be a smooth cutoff function and $\Box_g \phi = F$. Then, with respect to the conformal metric $\Omega^2 \tilde{g}_{\alpha\beta} = g_{\alpha\beta}$, one has the multiplier identity:

\begin{equation}
\int_{t_0}^{t_1} \int_{\mathbb{R}^3} (X) \tilde{P}^X_\alpha N^\alpha \Omega^{-2} \, d^4 x - \int_{t_0}^{t_1} \int_{\mathbb{R}^3} (X) \tilde{P}^X_\alpha N^\alpha \Omega^{-2} \, d^4 x = \int_{t_0}^{t_1} \int_{\mathbb{R}^3} \tilde{D}^\alpha (X) \tilde{P}^X_\alpha \Omega^{-4} \, dV_\tilde{g} ,
\end{equation}

where $dV_\tilde{g} = dt^4 d^3 x$ and $N$ given by (12). Additionally, for the divergence on the RHS we have the identity:

\begin{equation}
\tilde{D}^\alpha (X) \tilde{P}^X_\alpha \Omega^{-4} = F \cdot \Omega^{-1} X \psi + \Omega^{-2} ((\Omega, X) A^\alpha \partial_\alpha \psi \partial_\beta \psi + (\Omega, X) B^\alpha \phi^2 + (\Omega, X) C^\alpha \phi \Omega^{-1} X \psi,
\end{equation}

with:

\begin{equation}
(\Omega, X) A = \frac{1}{2} (\omega + 2X \ln (\Omega) g^{-1} - 1), \quad (\Omega, X) B = \frac{1}{2} \Omega^{-2} (X (\chi V) - \text{trace}(A) \chi V), \quad (\Omega, X) C = \Omega^{-2} (\chi - 1) V .
\end{equation}

On the RHS of the last two lines above all contractions are computed with respect to $g$.

**Proof.** Identity (21) follows immediately by integrating $\tilde{D}^\alpha (X) \tilde{P}^X_\alpha$ with respect to the volume form $dV_\tilde{g} = \sqrt{|g|} dt dx$ over the set $t_0 \leq t \leq t_1$ and applying Stokes’ theorem. It remains to compute the divergence $\tilde{D}^\alpha (X) \tilde{P}^X_\alpha$. Using formula (15) we have:

\begin{equation}
\tilde{D}^\alpha (X) \tilde{P}^X_\alpha = \frac{1}{2} (\chi \tilde{g})^\alpha \partial_\alpha \psi \partial_\beta \psi - \frac{1}{3} \text{trace} (L_X \tilde{g}) V \psi^2 + ((\chi - 1) V \psi + \Omega^2 F) X \psi + \frac{1}{2} X (\chi V) \psi^2 ,
\end{equation}

where all the contractions are computed with respect to $\tilde{g}$. To compute the first two RHS terms we use the identities:

\begin{equation}
L_X \tilde{g} = \Omega^{-2} (L_X g - 2X \ln (\Omega) g) , \quad \text{so} \quad L_X \tilde{g} = \Omega^{-2} (L_X g + 2X \ln (\Omega) g) ,
\end{equation}

Substituting the last line into RHS (24) gives us (22) and (23). □
Remark 2.4. Choosing $\Omega = 1$ and $\chi \equiv 0$ we recover the classical multiplier identities in section 2.1. In the Minkowski space, choosing $g = m$, $\chi \equiv 0$ and weights $i^g \Omega = r, \overline{i^g \Omega} = i^2 - r^2$ gives $L_X \overline{g} \equiv 0$ in line (24) and $V \equiv 0$ in line (17). Thus we recover the conformal energy formalism used in the work of Lindblad-Sterbenz [22].

Remark 2.5 (Notation). The quantity $\tilde{P}_{\alpha} N^\alpha$ will denote $\tilde{P}_{\alpha} N^\alpha$ with $\chi \equiv 0$.

2.3. No Superradiance. In order to produce the coercive bound $|\nabla \psi|^2 \lesssim (\partial_s) P_{\alpha} N^\alpha$ inside the set $r \leq 1$, the conformal multiplier method above requires the vector field $\partial_t$ to be uniformly timelike. Since $(g - m)$ could remain large inside this set as $t \to +\infty$, we have no a priori reason to expect this condition to hold. To address this issue, we will prove below that given some mild conditions on the metric $g$, the vector field $\partial_t$ is uniformly timelike everywhere. Let's start with some preliminary lemmas:

Lemma 2.6 (Coercive bound for energy on null-geodesics). Let $T$ be a past-directed, uniformly timelike vector field and $L$ be a future-directed null vector field with $L^\alpha = g^{\alpha \beta} \xi_\beta$. The following uniform bound holds:

$$\langle T, L \rangle \approx \|\xi\|_{\ell^2(\mathbb{R}^4)}.$$

Proof. It suffices to show $\langle T, L \rangle \gtrsim \|\xi\|_{\ell^2(\mathbb{R}^4)}$. Since $T$ is uniformly time-like we can construct a (local) system of coordinates $\{\partial_t, \partial_i\}$ such that:

$$\begin{align*}
\partial_0 &= T, & g_{00} &= -1, & g_{0i} &= 0, & g_{ij} Y^i Y^j &\approx \delta_{ij} Y^i Y^j = \|Y\|_{\ell^2(\mathbb{R}^3)}.
\end{align*}$$

By our hypotheses we have $\langle T, L \rangle > 0$. Therefore, in this system of coordinates $\langle T, L \rangle = g^{03} \xi_3 = -\xi_0 > 0$. Since $L$ is null we have:

$$g^{00}(\xi_0)^2 + g^{ij} \xi_i \xi_j = 0 \Rightarrow -\xi_0 = (g^{ij} \xi_i \xi_j)^{\frac{1}{2}} \approx (\sum_i \xi_i^2)^{\frac{1}{2}}.$$

The lemma follows.

Lemma 2.7 (Exponential bounds for null-geodesic coefficients). Let $(\mathcal{M}, g_{\alpha \beta})$ be asymptotically flat and suppose the vector field $N$, given by (12), is uniformly timelike and future-directed. Additionally, assume that $g_{\alpha \beta}$ satisfies the quantitative non-trapping assumption (19). Let $\gamma = \xi$ be an outgoing, future-directed null-geodesic given in $(t, x^i)$ coordinates by $\xi$ with affine parameter $s$ satisfying $\gamma|_{s=0} = 1$ with $\gamma(s) \equiv 1$, $\gamma(0) \in \{|x| \leq R_0\}$. Then, there exists a constant $A(R_0) > 0$ such that for all $t > 0$:

$$\|\xi(0)\|_{\ell^2(\mathbb{R}^4)} A^{-1} \lesssim \|\xi(t)\|_{\ell^2(\mathbb{R}^4)} \lesssim \|\xi(0)\|_{\ell^2(\mathbb{R}^4)} A.$$

Proof. Let $\lambda > 0$ and $\|\xi(0)\|_{\ell^2(\mathbb{R}^4)} = 1$ and $s$ be an affine parameter. By assumption (19), there exists $C_\lambda > 0$ such that for all $s > C_\lambda, \gamma(s) \in \{|g - m| < \lambda\}$. Therefore, by choosing $\lambda$ small it suffices to prove the result for the range $s \in [0, C_\lambda]$. For this we will use $(t, x^i)$ coordinates. We claim that there exists a constant $k(R_0) > 0$ such that for all $t > 0$:

$$\|\xi(t)\|_{\ell^2(\mathbb{R}^4)} \exp(-kt) \lesssim \|\xi(t)\|_{\ell^2(\mathbb{R}^4)} \lesssim \|\xi(0)\|_{\ell^2(\mathbb{R}^4)} \exp(kt).$$

To prove the claim we consider the Hamiltonian formulation for the geodesic flow. Let $p(x, \xi) = g^{\alpha \beta} \xi_\alpha \xi_\beta$ be the principal symbol for $\square_g$. The Hamiltonian flow obeys the equations:

$$\frac{dx^\alpha}{ds} = \partial_{\xi^\alpha} p, \quad \frac{d\xi^\alpha}{ds} = -\partial_{x^\alpha} p.$$

Since $-N$ is uniformly timelike, past-directed and $L^\alpha = g^{\alpha \beta} \xi_\beta$ is null, line (20) implies:

$$\frac{dt}{ds} = \partial_{\xi^0} p = 2g^{03} \xi_3 = 2\langle -N, L \rangle \approx \|\xi\|_{\ell^2(\mathbb{R}^4)}.$$

By chain rule: $\frac{dt}{d\xi} = (2\langle -N, L \rangle)^{-1} \approx \|\xi\|_{\ell^2(\mathbb{R}^4)}^{-1}$. This leads to:

$$\begin{align*}
\frac{dx^0}{dt} &= 1, \quad \frac{dx^i}{dt} = \frac{dx^i}{ds} \frac{ds}{dt} = \frac{\partial_{\xi^i} p}{2g^{03} \xi_3} = O(1), \quad \frac{d\xi^\alpha}{dt} = -\partial_{x^\alpha} p \approx O(\|\xi\|_{\ell^2(\mathbb{R}^4)}). 
\end{align*}$$
Therefore the $x^\alpha$ change with constant speed. For the frequency $\xi_\alpha$, since the last statement holds for all $\alpha$, there exists a $k > 0$ sufficiently large so that:

$$-k \| \xi \|_{L^1(\mathbb{R}^4)} \leq \frac{d}{dt} \| \xi \|_{L^1(\mathbb{R}^4)} \leq k \| \xi \|_{L^1(\mathbb{R}^4)} .$$

Integrating these bounds finishes the proof of the claim. Combining with our initial remarks yields the lemma. □

**Lemma 2.8.** Let $(\mathcal{M}, g)$ satisfy all the hypotheses of Lemma 2.7. Choose $R_0 > 0$ and denote $\xi, \lambda, A(R_0)$ as above. Then, for $\lambda$ sufficiently small, there exists $C_1 > 0$ such that in the exterior region $\{|g - m| < \lambda\}$:

$$|\langle \partial_t, \xi \rangle| > C_1 A^{-1} > 0 .$$

**Proof.** Since $\xi$ is a null vector, choosing $\lambda$ sufficiently small and using asymptotic flatness and non-trapping gives us $\xi = (\partial_t + \omega^j \partial_j + O(\lambda)) \| \xi \|_{L^1(\mathbb{R}^4)}$. By lemma 2.7 this implies $|\langle \partial_t, \xi \rangle| \geq C_1 \| \xi \|_{L^1(\mathbb{R}^4)} \geq C_1 A^{-1}$. □

**Remark 2.9.** In general the quantity $C_1 A^{-1} \leq 1$ due to the exponential nature of $A$.

Here’s the main result of this section:

**Proposition 2.10** (uniformly timelike). Let $(\mathcal{M}, g)$ satisfy all the hypotheses of Lemma 2.7 together with $|L_{\partial_t} g| < \epsilon$. Choose $R_0 > 0$ and denote $A, C_1 > 0$ as above. Then there exists $\epsilon_{\text{trap}}(A) > 0$ sufficiently small such that for all $0 < \epsilon < \epsilon_{\text{trap}}$ the bound $|\langle \partial_t, \partial_\xi \rangle| \leq -\frac{1}{2} C_1 A^{-1}$ holds everywhere.

**Proof.** Let $\epsilon > 0$ and $|L_{\partial_t} g| < \epsilon$ and assume for a contradiction that there exists a $p \in \mathcal{M}$ such that $|\langle \partial_t, \partial_\xi \rangle_p| < \frac{1}{6} C_1 A^{-1}$. This implies that we can find a null vector $\xi \in T_p \mathcal{M}$ such that the difference $Y = \partial_t - \xi$ is small enough to satisfy $\| Y \|_{L^1(\mathbb{R}^4)} < \frac{1}{6} C_1 (\| g \|_{L^\infty})^{-1}$. Let $\gamma(t)$ be the unique, affinely parametrized forward null geodesic with $\gamma(0) = \xi \in T_p \mathcal{M}$ and $\gamma'(0) = 1$. For sufficiently small $t \in (-\tau_0, \tau_0)$ we can define a smooth 1-parameter family of curves $\gamma_t(s)$ with $\gamma_0(s) = \gamma(s)$. Let $\frac{\partial}{\partial s} = \xi$, then along the null geodesic $\gamma$ we have:

$$\xi \langle \partial_\xi, \xi \rangle = \langle \nabla_\xi \partial_\xi, \xi \rangle + \langle \partial_\xi, \nabla_\xi \xi \rangle = -\langle \partial_\xi, \xi \rangle \xi + \langle \nabla_\partial_\xi, \xi \rangle = \frac{1}{2} L_{\partial_t} g(\xi, \xi) .$$

Integrating this along $\gamma$ from $s = 0$ to $s = s_1 > 0$:

$$\langle \partial_\xi, \xi \rangle_{s=s_1} = \langle \partial_\xi, \xi \rangle_{s=0} + \frac{1}{2} \int_0^{s_1} L_{\partial_t} g(\xi, \xi) \, ds .$$

Since $\xi(0) = \xi$ and $\| Y \|_{L^1(\mathbb{R}^4)} < \frac{1}{6} C_1 (\| g \|_{L^\infty})^{-1}$, the first term on the RHS (30) satisfies:

$$|\langle \partial_\xi, \xi \rangle_{s=0}| \leq |\langle \partial_\xi, \partial_\xi \rangle_p| + |\langle \partial_t, Y \rangle_p| \leq \frac{2}{3} C_1 A^{-1} .$$

Combining this with line (30) and using lemma 2.7 together with the hypotheses yields the almost conservation law:

$$|\langle \partial_\xi, \xi \rangle_{s_1}| \leq \frac{2}{3} C_1 A^{-1} + \frac{1}{2} A^2 \, \epsilon \cdot s_1 .$$

Choosing: $\epsilon_{\text{trap}} < \frac{2}{3} C_1 A^{-1} (A^2 \cdot 2 C_\lambda)^{-1}$ contradicts (29) when $s_1 = 2 C_\lambda$. □

This proposition leads to two important consequences:

**Corollary 2.11.** Let $(\mathcal{M}, g)$ satisfy the conditions of the main theorem. Then:

1) There exists a $T_{\text{trap}} > 0$ such that $\partial_t$ is uniformly timelike for all $t \in [T_{\text{trap}}, +\infty)$.

2) In $(t, x^i)$ coordinates, the operator $P(t, x, \nabla_x) = \partial_t h^{ij} \partial_j$ with $h^{ij} = d^2 g^{ij}$ is uniformly elliptic. Furthermore, inside the region $\{ r < \frac{1}{2} t \cap [T_{\text{trap}}, +\infty) \}, P(t, x, \nabla_x)$ satisfies the uniform estimate:

$$|P(t, x, \nabla_x)\phi| \leq |\partial_t \phi| + |\partial_x \phi| + |r|^{-\delta} |\partial_t \nabla_x \phi| + (r)^{-1-\delta} (|\partial_\phi| + (r/t)^{\gamma} |\nabla_x \phi|) ,$$

with $h^{ij} = \delta^{ij} + O(|r|^{-\delta})$. 


Proof. **Part 1:** (Statement for \( \partial_t \)) Choose \( T_{ntrap} \gg 1 \) sufficiently large so that \( \langle T_{ntrap} \rangle^{-\gamma} C < \frac{1}{\text{me}_{ntrap}} \) holds, with \( C \) the maximum of the implicit constants in estimates \( (3) \). In the interior this immediately yields \( |\mathcal{L}_{\partial_t} g| < \epsilon \) with \( \epsilon = \frac{\gamma}{\text{me}_{ntrap}} \). In the exterior this bound follows by \( (3) \) and the fact that \( \gamma < \delta \). An application of proposition \( (2.10) \) then yields the result.

**Part 2:** (Ellipticity) By Cramer’s rule we have \( \det(M^0) = (\partial_t, \partial_t) \det(g^{-1}) \) with \( M^0 = (g^{ij}) \) the 3 \( \times \) 3 matrix arising from eliminating the first row and column from the matrix coefficient of \( g^{-1} \). By part I we have \( g_{00} < C < 0 \) and thus \( \det(g^{-1}) < C < 0 \) therefore it follows that \( \det(M^0) \) is always positive and has uniform lower bounds. Since \( M^0 \) is a 3 \( \times \) 3 matrix and \( g \) has Lorentzian signature it follows that \( M^0 \) has three positive eigenvectors. This proves the first claim. For \( (32) \) we note that:

\[
(33) \quad \Box g = g^{00} \partial_t^2 + d^{-2} \partial_t g^{0i} d^2 \partial_t + d^{-2} \partial_t g^{0i} d^2 \partial_t + P .
\]

This, together with the asymptotic form \( (4) \) of the metric inside \( r < \frac{1}{2} t \), combine to give the result. \( \square \)

3. Bondi Coordinates

3.1. **Algebraic Formulas Involving Bondi Coordinates.** The vector fields \( X \in L \) written in \( (u, x^i) \) coordinates have a very simple form:

**Lemma 3.1** (Lie algebra property). In \( (u, x^i) \) coordinates the vector fields defined in \( (8) \) are given by:

\[
(34) \quad L = \{T = \partial_u, S = u \partial_u + r \bar{\partial}_r , \Omega_{ij} = x^i \bar{\partial}_j - x^j \bar{\partial}_i \},
\]

and \( L \) forms a Lie algebra on \( M \).

**Proof.** By chain rule:

\[
(35) \quad \partial_u = (u_i)^{-1} \partial_i , \quad \bar{\partial}_i = \partial_i - (u_i)^{-1} u_i \partial_i , \quad \bar{\partial}_r = \partial_r - (u_i)^{-1} u_i \partial_i .
\]

Line \( (2) \) implies \( u_t \approx 1 \) everywhere; thus, the equations in \( (35) \) are also valid everywhere. Applying this to \( (8) \) yields \( (34) \). From this we can compute the commutators to be:

\[
(36) \quad [\partial_u, S] = \partial_u , \quad [\partial_u, \Omega_{ij}] = 0 , \quad [S, \Omega_{ij}] = 0 , \quad [\Omega_{ij}, \Omega_{kl}] = -\delta_{ik} \Omega_{jl} .
\]

The Lie algebra property follows. \( \square \)

**Remark 3.2.** Since \( u_t \approx 1 \) the first identity in \( (35) \) implies that \( \partial_t \approx \partial_u \). This will be used often in the sequel.

Next we compute the key quantities from lemmas \( (2.1) \) and \( (2.3) \) of the previous section.

**Lemma 3.3** (Formulas for deformation tensors). Let \( \Omega \) be a smooth function and \( X \) be a vector field in Bondi coordinates. We have the following formula for the contravariant tensor \( 2^{(\Omega, X)} A \) in line \( (23) \):

\[
(37) \quad (X) \pi + 2X \ln(\Omega) g^{-1} = -d^{-\frac{2}{3}} (\mathcal{L}_X \eta^{-1} + (\partial_u X^u + \bar{\partial}_r X^r + \bar{\partial}_i \mathcal{X}^i + 2(r^{-1} X^r - X \ln \Omega)) \eta^{-1}) + R_X .
\]

where \( \mathcal{X}^i = X^i - \omega^i \omega_j X^j \) denotes the angular portion of \( X \) and \( X^r = \omega_i X^i \) the radial portion. The remainder tensor \( R \) is given by the covariant formula:

\[
(38) \quad R_X = -d^{-\frac{2}{3}} (\mathcal{L}_X (d^2 g^{-1} - \eta^{-1}) + (\partial_u X^u + \bar{\partial}_r X^r + \bar{\partial}_i \mathcal{X}^i + 2(r^{-1} X^r - X \ln \Omega)) (d^2 g^{-1} - \eta^{-1})) .
\]

**Proof.** We start with the identity:

\[
(39) \quad \bar{\partial}_r X^r = 2r^{-1} X^r + \partial_u X^u + \bar{\partial}_r X^r + \bar{\partial}_i \mathcal{X}^i .
\]

Applying this to formula \( (15a) \) in Bondi coordinates:

\[
(40) \quad (X) \pi + 2X \ln(\Omega) g^{-1} = -d^{-\frac{2}{3}} (\mathcal{L}_X (d^2 g^{-1}) + (\partial_u X^u + \bar{\partial}_r X^r + \bar{\partial}_i \mathcal{X}^i + 2(r^{-1} X^r - X \ln \Omega)) (d^2 g^{-1}) .
\]

Adding and subtracting \( d^{-\frac{2}{3}} \eta^{-1} \) on the last line gives \( (37) \) and \( (38) \). \( \square \)
Lemma 3.4 (Formulas for commutators). Let $d = |\det(g_{\alpha\beta})|$ and $X = X^\alpha \partial_\alpha$ be in Bondi coordinates. When $X \in \{\partial_\alpha, \Omega_{ij}\}$ we have the formula:

\begin{equation}
[\square_g, X] = D_\alpha R^{\alpha\beta}_X D_\beta + \frac{1}{2} X \ln(d) \square_g ,
\end{equation}

where:

\begin{equation}
R_X = -d^{-\frac{1}{2}} \mathcal{L}_X (d^{\frac{1}{2}} g^{-1} - \eta^{-1}) .
\end{equation}

On the other hand, when $X = S$ we have:

\begin{equation}
[\square_g, S] = D_\alpha R^{\alpha\beta}_S D_\beta + \frac{1}{2} (4 + S \ln(d)) \square_g ,
\end{equation}

where $R_S$ is given by formula:

\begin{equation}
R_S = -d^{-\frac{1}{2}} (\mathcal{L}_X (d^{\frac{1}{2}} g^{-1} - \eta^{-1}) + (\partial_\alpha X^u + \tilde{\partial}_\alpha X^r + \tilde{\partial}_r X^\tau) (d^{\frac{1}{2}} g^{-1} - \eta^{-1})) .
\end{equation}

Proof. When $\Omega = 1$ formula \((47)\) can be rewritten as:

\begin{equation}(x)\pi = -d^{-\frac{1}{2}} (\mathcal{L}_X \eta^{-1} + (\partial_\alpha X^\alpha) \eta^{-1} + \mathcal{L}_X (d^{\frac{1}{2}} g^{-1} - \eta^{-1}) + (\partial_\alpha X^\alpha) (d^{\frac{1}{2}} g^{-1} - \eta^{-1})) .
\end{equation}

For each $X \in \{\partial_\alpha, \Omega_{ij}\}$ we have $\mathcal{L}_X \eta^{-1} = 0$ and $\partial_\alpha X^u + \tilde{\partial}_\alpha X^i = 0$. Applying identity \(14\)a gives us \(11\). For $X$ and \(S\) one can compute:

$$
\mathcal{L}_S \eta^{-1} + (\partial_\alpha S^u + \tilde{\partial}_\alpha S^r + \tilde{\partial}_r S^\tau) \eta^{-1} = 0 , \quad r^{-1} S^r = 1 , \quad \partial_\alpha S^u + \tilde{\partial}_\alpha S^i = 4 .
$$

Substituting this together with \(39\) into \(45\) then applying \(15\)a finishes the proof of \(13\). \(\square\)

3.2. Asymptotic Estimates Involving Bondi Coordinates. Our first task here is to compute the decay rates for the Lie derivatives $\mathcal{L}_X g$ with $X \in L$.

Lemma 3.5 (Basic Lie derivative estimates). Let $X = X^\alpha \partial_\alpha$ be in Bondi coordinates.

I) Suppose that $X$ satisfies the symbol-type bounds:

\begin{equation}
| (\tau_\alpha \partial_\alpha)^k (r \tilde{\partial}_x)^j X^u | \lesssim \tau_- , \quad | (\tau_\alpha \partial_\alpha)^k (r \tilde{\partial}_x)^j X^i | \lesssim \langle r \rangle ,
\end{equation}

and obeys the conditions:

\begin{equation}
\tilde{\partial}_r X^u = \partial_\alpha X^i = \tilde{\partial}_r r^{-1} (X^i - \omega^i \omega_j X^j) \equiv 0 .
\end{equation}

Let $R^{\alpha\beta}$ be a contravariant two tensor satisfying the bounds:

\begin{equationa}
| \partial_\alpha \tilde{\partial}_\beta (\tilde{R}^{ij}) | \lesssim \langle r \rangle^{-k - |\alpha| - \delta} \tau_0^{-\frac{1}{2} - k} (\tau_+/\langle r \rangle)^{1 - \gamma k} ,
\end{equationa}

\begin{equationb}
| \partial_\alpha \tilde{\partial}_\beta (\tilde{R}^{ij} - \omega^i \omega^j X\tau^{\alpha\beta} ) | \lesssim \langle r \rangle^{-k - |\alpha| - \delta} \tau_0^{-\frac{1}{2} - k} (\tau_+/\langle r \rangle)^{1 - \gamma k} ,
\end{equationb}

with similar estimates for $R^{\alpha\beta}$. Then, the Lie derivative $\mathcal{L}_X R$ satisfies the bounds \(15\) with the exponent $-\gamma k$ above replaced by $1 - \gamma (1 + k)$.

II) Alternatively, if we substitute the condition $\partial_\alpha X^u = 0$ with $\partial_\alpha (X^i - \omega^i X^j) = 0$ and keep the rest of \(10\) and \(17\) the same, then the result of part I holds with the first bound on line \(15a\) replaced by:

\begin{equation}
| \partial_\alpha \tilde{\partial}_\beta (\omega^i \omega^j X\tau^{\alpha\beta} ) | \lesssim \langle r \rangle^{-k - |\alpha| - \delta} \tau_0^{-\frac{1}{2} - k} (\tau_+/\langle r \rangle)^{1 - \gamma (1 + k)} ,
\end{equation}

\begin{equation}
| \partial_\alpha \tilde{\partial}_\beta (\omega^i \omega^j X\tau^{\alpha\beta} ) | \lesssim \langle r \rangle^{-k - |\alpha| - \delta} \tau_0^{-\frac{1}{2} - k} (\tau_+/\langle r \rangle)^{1 - \gamma (1 + k)} .
\end{equation}

III) Let $X = \partial_\alpha$ and suppose $R$ satisfies \(15\). Then \((r) \tau_0 \mathcal{L}_X R$ satisfies \(15\) as well.

Proof of Lemma 3.5 Part 1: \(The \mathcal{R} \ bounds \ involving \ condition \ (10)\). We begin with the proof of estimates \(18\) for $\mathcal{L}_X R$ assuming conditions \(16\) and \(17\) or the alternative listed in item II above. The formula for the Lie derivative is $\mathcal{L}_X R^{\alpha\beta} = X (R^{\alpha\beta}) - \partial_\alpha (X^\alpha) R^{\beta \gamma} - \partial_\beta (X^\beta) R^{\alpha \gamma}$. We check each component:

Case 1: \(The uu \ component\) Here we have:

$$
\mathcal{L}_X R^{uu} = X (R^{uu}) - 2 \partial_\alpha (X^u) R^{uu} - \tilde{\partial}_\alpha (X^u) (R^{uu}) - \tilde{\partial}_\alpha (X^u) (R^{uu}) .
$$

Since $\tilde{\partial}_r X^u = 0$, the second estimate on line \(18\) for $\mathcal{L}_X R^{uu}$ is immediate from the estimates \(10\)–\(18\).
Case 2: (The $u_i$ and $iu$ components) By symmetry of the estimates on lines (48a) and (48b) it suffices to treat the $ui$ case. We have:

$$\mathcal{L}_\nu R^{ui} = X(R^{ui}) - \partial_u (X^u) R^{ui} - \partial_u (X^i) R^{iu} - \tilde{\partial}_j(X^u) R^{ij} - \tilde{\partial}_j(X^i) R^{uj}.$$  

Using estimates (46)–(48b) we get a symbol bound on the order of $\langle r \rangle^{-\delta} \tau_0^{-\frac{1}{2}} (\tau_+ / r)^{1-\gamma}$ for this term. In addition one sees that for all parts of the above formula save the expression $X(\omega^i R^{ur}) - \tilde{\partial}_u (X^i) R^{ur}$ the bound is on the order of $\langle r \rangle^{-\delta} \tau_0 (\tau_+ / r)^{1-\gamma}$. To see the improvement for $\mathcal{L}_\nu R^{ui} - \omega^j \omega_j \mathcal{L}_\nu R^{ui}$ we note that the worst term is absent once one subtracts off the radial part since:

$$\langle r \rangle^{-\delta} \tau_0^{-\frac{1}{2}} (\tau_+ / r)^{1-\gamma} \text{ where we used (47) for the last identity above.}$$

Case 3a: (The $ij$ components assuming $\partial_u X^i = 0$) Here we have:

$$\mathcal{L}_\nu R^{ij} = X(R^{ij}) - \tilde{\partial}_k (X^i) R^{kj} - \tilde{\partial}_k (X^j) R^{ik}.$$  

The first bound on line (48a) for $\mathcal{L}_\nu R^{ij}$ follows by multiplying together (46) and (48a).

Case 3b: (The $ij$ components assuming $\partial_u X^i \neq 0$) In this case we are still assuming $\partial_u X^i - \omega^j \omega_j X^i = 0$.

Therefore:

$$\mathcal{L}_\nu R^{ij} = X(R^{ij}) - \omega^j \partial_u (X^i) R^{uj} - \omega^j \partial_u (X^i) R^{iu} - \tilde{\partial}_k (X^i) R^{kj} - \tilde{\partial}_k (X^j) R^{ik}.$$  

By (46)–(48b) this has a symbol bound of order $\langle r \rangle^{-\delta} \tau_0^{-\frac{1}{2}} (\tau_+ / r)^{1-\gamma}$. On the other hand all but the second and third terms above yield a bound of order $\langle r \rangle^{-\delta} (\tau_+ / r)^{1-\gamma}$. Subtracting the radial part yields:

$$\mathcal{L}_\nu R^{ij} - \omega^j \omega_i \mathcal{L}_\nu R^{rr} = -\omega^j \partial_u (X^i) (R^{uj} - \omega^j R^{ur}) - \omega^j \partial_u (X^i) (R^{iu} - \omega^i R^{ru}) + O(\langle r \rangle^{-\delta} (\tau_+ / r)^{1-\gamma}).$$  

By line (48b) we have $\langle r \rangle^{-\delta} (\tau_+ / r)^{1-\gamma}$ symbol bounds for the first two terms on the RHS above as well.

Part 2: (Estimates involving $\partial_u$) Fix a dyadic region $r \tau_0 \approx 2^k$. Then if $X = \partial_u$ the vector field $2^k X$ satisfies conditions (46) and all of (47). From the above calculations one immediately has all of (48) for $r \tau_0 \mathcal{L}_\nu \mathcal{R}$.

Along a similar vein we can derive the following set of bounds which will be needed when employing the conformal multiplier method.

**Lemma 3.6.** Let $g_{\alpha\beta}$ be in Bondi coordinates.

1. (Estimates for the determinant) Let $d = |g| = |\det(g_{\alpha\beta})|$ be computed in Bondi coordinates $(u, x^i)$. We have the symbol bounds:

   $$|\langle r \rangle \tau_0 \partial_u k^k (\langle r \rangle \tilde{\partial}_x) (d^2 - 1)| \lesssim \langle r \rangle^{-\delta}.$$  

2. (Asymptotics of the conformal potential) Let $l \Omega = \langle r \rangle$, $h \Omega = \tau_- \tau_+$ and $l V = \Omega^3 \Box_{\eta} \Omega^{-1}$. The conformal potentials $l V$, $h V$ satisfy the following symbol bounds:

   $$|\langle r \rangle \tau_0 \partial_u k^k (\langle r \rangle \tilde{\partial}_x) l(V)| \lesssim \langle r \rangle^{-\delta} \tau_0^{-\frac{1}{2}}, \quad |\langle r \rangle \tau_0 \partial_u k^k (\langle r \rangle \tilde{\partial}_x) h(V)| \lesssim \langle r \rangle^{-1-\delta} (\tau_- \tau_+)^{\frac{3}{2}}.$$  

**Proof.** Part 1: (Determinant bounds) Follows from estimates (49) since the determinant is a continuous function of the metric components $g_{\alpha\beta}$. Part 2: (Potential bounds) Let $\mathcal{R} = d^\frac{2}{3} g - \eta$ and write the wave operator in Bondi coordinates as $\Box_{\eta} = -d^2 \Box_{\eta} + d^2 \tilde{\partial}_u R^{\alpha\beta} \partial_{\beta}$, where $\Box_{\eta}$ is the Minkowski wave equation in Bondi coordinates. Expanding $\Box_{\eta}$ yields:

$$\Box_{\eta} = -2 \partial_u \tilde{\partial}_r + \tilde{\partial}_r^2 - 2 r^{-1} \partial_u + 2 r^{-1} \tilde{\partial}_r + r^{-2} \sum_{i<j} (\Omega_{ij})^2.$$  

Using this identity we can compute:

$$l^l \Omega^3 \Box_{\eta} l(\Omega^{-1}) = -3 \langle r \rangle^{-2}, \quad h^3 \Omega^3 \Box_{\eta} h(\Omega^{-1}) = -4 \tau_-^{-1} r_+^{-1} + (u/r) O(\tau_-^{-1} r_+^{-1}).$$  

Thus:

$$l V = -d^2 \Box_{\eta} R^{ur} + r \tilde{\partial}_i R^{ir} - 2 R^{rr} + O(\langle r \rangle^{-2}),$$  

$$h V = -d^2 \Box_{\eta} R^{\alpha\beta} \partial_{\beta} h(\Omega) - 2 R^{\alpha\beta} (\partial_\alpha h(\Omega)) (\partial_{\beta} h(\Omega)) - 4 \tau_-^{-1} r_+^{-1} + (u/r) O(\tau_-^{-1} r_+^{-1}).$$
The estimate for $IV$ in (52) follows from this and the fact that $R$ satisfies the estimates (48) by assumption 1.2. For $HV$ the two worst cases are when $\alpha = \beta = u$ and $\alpha = u, \beta = r$. A quick count of the weights in these two cases and an application of (51) and (48) gives us the result.

Lemma 3.7 (Formulas for boundary terms). Let $X^r$ and $Y^u$ be positive and set $X = X^r \tilde{\partial}_r$, $Y = Y^u \partial_u$. Then there exist a constant $C > 0$ such that the following pointwise estimates for boundary terms on the divergence identity (21) hold:

\[
(55) \quad (X)\barP^\alpha\, N^\alpha \Omega^{-2} \gtrsim X|\Omega^{-1}\bar\nabla_x(\Omega\phi)|^2 - C(X^r(r)^{-\delta_1^2})|\Omega^{-1}\nabla(\Omega\phi)|^2 - C(|IV|X^r)\Omega^{-2}\phi^2, \\
(56) \quad (Y)\barP^\alpha\, N^\alpha \Omega^{-2} \gtrsim Y^u|\Omega^{-1}\nabla(\Omega\phi)|^2 - C(|IV|Y^u)\Omega^{-2}\phi^2.
\]

To prove estimate (55) we will need the following elementary result:

Lemma 3.8 (Comparison with a null frame). Let $X$ and $Y_A$, $A = 1, 2$ be vector fields satisfying $\sup_x \{|X^\alpha| \approx 1$ and $\sup_{A} |Y^\alpha_A| \approx 1$ for their components. Suppose that there exists $\omega > 0$ such that $\langle X, X \rangle = O(\omega^2)$, $\langle X, Y_A \rangle = O(\omega)$, and in addition $\langle Y_A, Y_B \rangle - \delta_{AB} \ll 1$. Then there exists a null frame $\{L, L_e, e_A\}$ with $\langle L, L \rangle = (L, e_A) = (L, L_e) = 0$, $\langle L, L \rangle = -1$, and $\langle e_A, e_B \rangle = \delta_{AB}$, and coefficients $c_A^A$ and $c_B^B$ for $A, B = 1, 2$, and $\gamma$, such that:

\[
X = \theta L + c_A^X e_A + \gamma L_e, \quad Y_A = c_A^B e_B, \quad \text{where } \theta = O(\omega^2), \quad \text{and } c_A^X = O(\omega), \quad \text{and } |c_A^X - \gamma| \ll 1.
\]

Proof. Choose $e_A$ to be an orthonormal basis for the space-like two plane spanned by $Y_A$ and let $c_B^B$ be the change of basis. Then $|c_B^B - \delta^A_B| \ll 1$. Let $L, L_e$ be the two null generators over span $e_A$ with $\langle L, L_e \rangle = -1$. We have: $X = \theta L + c_A^X e_A + \gamma L$ for some set of coefficients $\theta$, $c_A^X$, $\gamma$. From $\langle X, Y_A \rangle = O(\omega)$ we have $(X, e_A) = O(\omega)$ and so $c_A^X = O(\omega)$. Thus $\langle X, X \rangle = -2\theta^2 + O(\omega^2)$ and so $\theta = O(\omega^2)$ follows from $\langle X, X \rangle = O(\omega^2)$.

Proof of (57). By choosing $C$ sufficiently large it suffices to prove the result in the wave zone $t \approx r$ for large values of $t$. Consider the basis $\{\tilde{\partial}_r, Y_A\}$ where $Y_A$ is a (local) euclidean ONB on the spheres $r = const, u = const$. We now check the hypotheses of the preceding lemma. Since the metric $g$ is asymptotically flat $|\langle Y_A, Y_B \rangle - \delta_{AB} | \ll 1$. On the other hand by the asymptotic formulas (3) and Cramer’s rule we have $\langle \tilde{\partial}_r, \tilde{\partial}_r \rangle = O(r^{-\delta_1^2})$ and $\langle \tilde{\partial}_r, Y_A \rangle = O(r^{-\delta_1})$. Additionally, inside the wave zone assumption 1.2 implies $|\langle \partial_r, L \rangle | = |\theta| \approx 1$. An application of the previous lemma then gives us, with $\omega = \langle r \rangle^{-\delta_1}$:

\[
\tilde{\partial}_r = L + O((r)^{-\delta_1}) \bar\nabla_x + O((r)^{-\delta_1^2}) \nabla,
\]

where $L$ is (outgoing) null and $\bar\nabla_x$ denotes derivatives tangent to $u = const, r = const$ which also lie in the null plane generated by $L$. Let $\bar{T}$, $\psi$ be as in line (19) with $\chi \equiv 0$. Since $N$ is uniformly timelike and future-directed we have:

\[
\bar{T}(L, N) \approx |L\psi|^2 + |\bar\nabla_x \psi|^2, \quad |\bar{T}(\bar{\nabla}_x, N)| \lesssim |\bar{\nabla}_x \psi| \cdot |\nabla \psi| + |g^{\alpha\beta} \partial_\alpha \psi \partial_\beta \psi|, \quad |\bar{T}(\nabla, N)| \lesssim |\nabla \psi|^2.
\]

Using the bounds (3) and Young’s inequality with $C > 0$:

\[
\langle r \rangle^{-\delta_1} |g^{\alpha\beta} \partial_\alpha \psi \partial_\beta \psi| \lesssim C^{-1}|\bar{\nabla}_x \psi|^2 + C \langle r \rangle^{-\delta_1^2} |\nabla \psi|^2.
\]

Choosing $C \ll 1$, applying the last three inequalities and absorbing the small error term:

\[
\bar{T}(\tilde{\partial}_r, N) \gtrsim |\bar{\nabla}_x \psi|^2 + O((r)^{-\delta_1^2}) |\nabla \psi|^2.
\]

Adding the undifferentiated terms to this and using the definition of $\bar{T}^\chi$ finishes the proof.

Proof of (58). The vector field $N$ is uniformly timelike and future-directed by our assumptions. The vector field $\partial_u$ is uniformly timelike and future-directed by proposition 2.10. The proof of (58) then follows from $\partial_u \approx \partial_t$ and the dominant energy condition.

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4. Additional Notation and Preliminary Reduction

In this section we reduce to the asymptotic region where \( t \gg 1 \). We now set up some notation which, in the sequel, will be used to absorb small errors inside this region.

**Definition 4.1 (Description of \( \mu, \epsilon, T^* \) and \( I^* \)).** We make the following definitions:

a) Let \( 0 < \mu < \frac{1}{2} \) be sufficiently small so that \( \mu \cdot \sup_{\gamma} c_j \ll \frac{1}{2} \) where \( c_j \) are the implicit constants in all the estimates in Lemma 6.7 and lines (133), (134).

b) Choose \( 0 < \epsilon \ll \min\{\gamma, \mu\} \) satisfying the following property: for any estimate in the sequel, of the form \( A \leq C_0 (B + cA) \) with absolute constant \( C_0 > 0 \), the number \( \epsilon \) is small enough that we can absorb \( C_0 \epsilon A \) on the LHS to yield the bound \( A \leq 2C_0 B \).

c) Let \( T^*(\epsilon) > T_{\text{trap}} \gg 1 \) be sufficiently large so that the following holds:

\[
0 < (\epsilon T^*)^{-\frac{n}{2n+2}} < \epsilon.
\]

d) Let \( I^* = [T^*, \infty) \) with \( T^* \) as above.

The constant \( \epsilon \) depends only on \( \{\| g \|_{C^2}, \gamma, \delta, \mu\} \) and on the implicit constants in the assumptions of the main theorem. In principle, we can choose an explicit \( \epsilon \) satisfying the property above. However, as we only use this constant to close a finite number of estimates below, it is neither necessary nor particularly useful to keep track of its size. We also note that \( \epsilon \) will usually arise from a small gain in \( t^\alpha \) power in our estimates, with the only exception being the small interior wedge in the proof of estimate (76)). On the other hand, the purpose of \( T^* \) is to give us an explicit lower bound for \( t \) which help us produce \( \epsilon \) via (59) when \( t \gg 1 \).

In order to take advantage of the setup above we now reduce to the asymptotic region \( t \in I^* \). In the sequel it suffices to show (9), (10), and (11) hold for all \( t \in I^* \) with constants that do not depend on \( t \). This is a straightforward consequence of local energy estimates.

**Lemma 4.2 (Reduction to the asymptotic region \( t \in I^* \)).** Estimates (9), (10), and (11) hold for all \( t \in [0, T^*) \).

**Proof.** By local energy estimates both (9) and (10) follow in the range \( t \in [0, T^*) \) with constants that depend on \( T^* \). Estimate (11) follows similarly after an application of the \( L^\infty - L^2 \) Sobolev embedding. \( \square \)

5. Conformal Energy Estimate

In this section we prove the conformal energy estimate (9). This will form the basis for the higher regularity bound (10) as well as the global \( L^\infty \) decay (11). In the asymptotic region \( t \in I^* \) the conformal energy estimate will follow from:

**Theorem 5.1.** Assume all the hypotheses of the main theorem. Then, for all \( [t_0, t_1] \subset I^* \) the following estimates hold:

I) \((T\text{-weighted \textit{LED} estimate in timelike regions})\)

\[
\| \chi_{r<\frac{1}{2}t} \phi \|_{\epsilon^r L^E_1} \lesssim \| \chi_{r<\frac{1}{4}t} \Box g \phi \|_{\epsilon^r N} + \sup_{t_0 \leq t \leq t_1} \| \phi(t) \|_{CE}.
\]

II) \((\text{Uniform boundedness})\)

\[
\sup_{t_0 \leq t \leq t_1} \| \nabla \phi(t) \|_{L^2_x} + \| \tilde{\nabla}_x \phi \|_{NLE^{\alpha}, \frac{1}{2}} + \| \phi \|_{LE} \lesssim \| \nabla \phi(t_0) \|_{L^2_x} + \| \Box g \phi \|_{LE^*}.
\]

III) \((\text{Conformal energy estimate with interior error})\) For any \( 0 < c < 1 \):

\[
\sup_{t_0 \leq t \leq t_1} \| \phi(t) \|_{CE} + \| \phi \|_{CH} \lesssim \| \phi(t_0) \|_{CE} + c^{-1} \| \Box g \phi \|_{\epsilon^{r} N} + c \| \chi_{r<\frac{1}{4}t} \phi \|_{\epsilon^r L^E_1}.
\]

Let’s show how the conformal energy estimate follows from this:

**Proof of (9).** By definition (11) we can choose a \( c \) small enough satisfying \( \epsilon \ll c \), yet smaller than the implicit constants in estimates (60) - (62). Taking an appropriate linear combination of (60) and (62):

\[
\sup_{t_0 \leq t \leq t_1} \| \phi(t) \|_{CE} + \| \phi \|_{CH} + c \| \chi_{r<\frac{1}{4}t} \phi \|_{\epsilon^r L^E_1} \lesssim \| \phi(t_0) \|_{CE} + c^{-1} \| \Box g \phi \|_{\epsilon^{r} N} + c \sup_{t_0 \leq t \leq t_1} \| \phi(t) \|_{CE}.
\]
Since $c \ll 1$ we can bootstrap the last term above on to the LHS and close the estimate. Inside $[0,T^+]$ the result follows from Lemma (1.10).

The proof of the T-weighted LED estimate is modular and does not depend on anything other than assumption (1.10). Let’s prove this estimate right now.

**Proof of estimate (61).** We apply the LED bound (60) to $2^k \chi_k(t)\chi_{r<\frac{1}{2}t}\phi$ where $\chi_k$ are a series of dyadic cutoffs supported where $t \approx 2^k$ and $\chi_k(t_0) = 0$. This gives us:

$$\tau_+ \chi_k \chi_{r<\frac{1}{2}t} \frac{\langle \nabla \phi, \langle \rangle^k \rangle}{\langle \rangle^k} \leq \|
abla \chi_k \chi_{r<\frac{1}{2}t} \frac{\langle \nabla \phi, \langle \rangle^k \rangle}{\langle \rangle^k} \phi \|_{LE^\ast}.$$  

where $\chi$ denotes cutoffs with slightly larger support. For the first RHS term and for fixed $N, C > 0$ there exists an $M = M(C, N) > 0$ and a uniform implicit constant such that:

$$\| \tau_+ \chi_k \chi_{r<\frac{1}{2}t} \frac{\langle \nabla \phi, \langle \rangle^k \rangle}{\langle \rangle^k} \phi \|_{LE^\ast} \leq M \| \tau_+ \chi_k \chi_{r<\frac{1}{2}t} \frac{\langle \nabla \phi, \langle \rangle^k \rangle}{\langle \rangle^k} \phi \|_{L^2}$$  

$$+ \sum_{j<k-C} 2^{j-k} \| \tau_+ \chi_k \chi_{r<\frac{1}{2}t} \frac{\langle \nabla \phi, \langle \rangle^k \rangle}{\langle \rangle^k} \phi \|_{L^2}.$$  

Plugging this into RHS (61) and taking $\ell_1$ for the resulting bound over a collection of finitely overlapping $\chi_k(t)$, and choosing $C$ large enough to absorb the $LE_0$ error from the last line above, we have:

$$\| \tau_+ \chi_k \chi_{r<\frac{1}{2}t} \frac{\langle \nabla \phi, \langle \rangle^k \rangle}{\langle \rangle^k} \phi \|_{\ell_1} \leq \| \tau_+ \chi_k \chi_{r<\frac{1}{2}t} \frac{\langle \nabla \phi, \langle \rangle^k \rangle}{\langle \rangle^k} \phi \|_{\ell_1 L^2} + \| \tau_+ \chi_k \chi_{r<\frac{1}{2}t} \frac{\langle \nabla \phi, \langle \rangle^k \rangle}{\langle \rangle^k} \phi \|_{L^2}.$$  

Choosing $N = \frac{1}{2}$, using the support property and the definition of the norms yields:

$$\| \tau_+ \chi_k \chi_{r<\frac{1}{2}t} \frac{\langle \nabla \phi, \langle \rangle^k \rangle}{\langle \rangle^k} \phi \|_{\ell_1 L^2} \leq \sup_{t_0 \leq t \leq t_1} \| \phi(t) \|_{CE}.$$  

This finishes the proof.

The rest of this section is devoted to the proof of the estimates (61) and (62). This will be done over the course of the next three subsections.

5.1. **Some Preliminary Estimates.** Here we establish a number of technical estimates needed in the proof of Theorem (5.2.1). Each argument below is self-contained.

**Lemma 5.2 (Hardy estimates).** For test functions $\phi$ we have the following fixed-time bound:

$$\| r^{a-1} \phi(t) \|_{L^2} \leq \frac{2}{2a+1} \| r^a \partial_r \phi(t) \|_{L^2}, \quad 0 < a < \infty.$$  

Additionally let $\chi_r < 2^k$, $\chi_u < 2^k$ be smooth cutoff functions supported on the sets $\langle r \rangle < 2^k$, $\langle u \rangle < 2^k$ respectively. For all $t \in T^*$ one has the fixed-time bounds:

$$\chi_r < 2^k \tau_+ \phi(t) \|_{L^2} \leq \chi_r \partial_r \phi(t) \|_{L^2} + \| \phi(t) \|_{CE} + \| \nabla \phi(t) \|_{L^2}.$$  

$$\chi_u < 2^k \phi(t) \|_{L^2} \leq \chi_u \partial_r \phi(t) \|_{L^2} + \| \phi(t) \|_{CE} + \| \nabla \phi(t) \|_{L^2}.$$  

**Proof of estimate (61).** For a fixed value of the angular variable we have the integral identity:

$$2a+1 \int_0^\infty r^{2a+1} \phi \partial_r \phi \, dr = -2 \int_0^\infty r^{2a+1} \phi \partial_r \phi \, dr.$$  

As long as $2a+1 > 0$ estimate (61) follows from integration of this identity in the angular variable and Cauchy-Schwarz.

**Proof of (67).** We apply (60) with $a = 0$ to $\chi_r < 2^r \tau_+ \phi \|_{L^2}$. Since $t \approx \tau_+ \tau_0$ inside this set we get:

$$\| \chi_r < 2^r \tau_+ \phi(t) \|_{L^2} \leq \| \chi_r < 2^r \tau_+ \phi(t) \|_{L^2} + \| \chi_r < 2^r \tau_+ \phi(t) \|_{L^2}.$$  

Within this region we have $\| \chi_r < 2^r \tau_+ \phi(t) \|_{L^2} \leq 2(\tau_+ \tau_0)^{-1}$. Combining this with Young’s inequality:

$$\| \tau_+ \tau_0 \partial_r \phi \|_{L^2} \leq 24(\tau_+ \tau_0)^{-1} \partial_r \phi(t) \|_{L^2} + \phi^2(t).$$
Applying this inequality to the first term on the RHS of (70) yields:

\[(72) \quad \| \chi_{(r)<2}(\tau+\tau_0 \partial_r \phi)(t) \|_{L^2_r} \lesssim \| \chi_{(r)<2} \tau_0 (H_O)^{-1} \partial_r (H_O \phi)(t) \|_{L^2_t} + \| \chi_{(r)<2} \phi(t) \|_{L^2_t}.\]

For the last term on RHS of (72) we use the support property followed by (66) with \(a = 0\) to get:

\[\| \chi_{(r)<2} \phi(t) \|_{L^2_r} \lesssim \| \chi_{(r)<2} r^{-1} \phi(t) \|_{L^2_t} \lesssim \| \nabla \phi \|_{L^2_t} + \| \chi_{(r)<2} \tau^{-1} \phi \|_{L^2_t}.\]

This finishes the proof of (71). \(\square\)

**Proof of (68).** For a fixed value of the angular variable we have the integral identity:

\[(73) \quad - \int_0^\infty u_r \phi^2 \chi_{(u)<2} r^2 dr = \int_0^\infty (2u \partial_r \phi \cdot \phi \chi_{(u)<2} + 2ur^{-1} \phi^2 \chi_{(u)<2} + \phi^2 \chi_{(u)<2} r^2 dr) = \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3.\]

Integrating (73) in the angular variable and using (2) on the LHS above gives us:

\[(74) \quad (1 + O(r^{-\delta})) \int_{S^2} \int_0^\infty \phi^2 \chi_{(u)<2} r^2 dr d\omega = - \int_{S^2} \int_0^\infty u_r \phi^2 \chi_{(u)<2} r^2 dr d\omega.\]

Since \(t \gg 1\) and \(t \approx r\) holds inside the set \(\{\langle u \rangle < 2\} \cap I^*\), it follows that \(r \gg 1\) here as well. Thus we may absorb the \(O(r^{-\delta})\) term as a bootstrap error. Next we take absolute values on RHS of (73) and bound each term separately. For \(\int |\mathcal{E}_1|\, d\omega\) we go back to Bondi derivatives via \(\partial_r = \partial_r + u_r \partial_u\), then apply \(|(t \Omega)^{-1} \partial_r (t \Omega)| \lesssim (r)^{-1}\) to conjugate by \(t \Omega\). Using Young’s inequality then gives us, with \(0 < c \ll 1\):

\[
\int_{S^2} |\mathcal{E}_1|\, d\omega \lesssim c^{-1} \left( \| \chi_{(u)<2} \tau_0 (t \Omega)^{-1} (t \partial_u (t \Omega \phi), \tilde{\partial}(t \Omega \phi)) \|_{L^2_t} + \| \chi_{(u)<2} r^{-1} \tau_0 \phi \|_{L^2_t}^2 + \| \nabla \phi \|_{L^2_t}^2 + \| \chi_{(u)<2} \phi \|_{L^2_t}^2 \right).
\]

The last term above can be bootstrapped to LHS of (74) by choosing \(c\) sufficiently small. For the next-to-last term on the RHS above we use \(\tau_0 = O(1)\) inside \(\langle u \rangle < 2\) followed by (66) with \(a = 0\). This gives us:

\[
\int_{S^2} |\mathcal{E}_2|\, d\omega \lesssim c^{-1} \left( \| \chi_{(u)<2} r^{-1} \tau_0 \phi \|_{L^2_t}^2 + \| \chi_{(u)<2} \phi \|_{L^2_t}^2 \right).
\]

Choosing \(c\) sufficiently small allows us to bootstrap the small error term to LHS of (74). Since the term \(\int |\mathcal{E}_3|\, d\omega\) is acceptable as part of the RHS, combining the last few lines and taking square roots establishes estimate (68). \(\square\)

**Lemma 5.3 (Estimates for undifferentiated boundary terms).** Let \(\chi_{(u)<2}(r/t)\) be a smooth cutoff supported on the wedge \(r < \frac{1}{4} t\) and let \(IV, HIV\) be as in lemma 76. For test functions \(\phi\) and for all values \(t \in I^*\) one has the fixed-time bounds:

\[(75) \quad \| r_+ \chi_{r<ct}(r/t)^{-2} \cdot IV \phi^2(t) \|_{L^2_t} \lesssim \| \phi(t) \|_{u_C^2}^2 + \epsilon \cdot \| \phi(t) \|_{u_C^2},\]

\[(76) \quad \| r^2 \chi_{r<ct}(r/t)^{-2} \cdot HIV \phi^2(t) \|_{L^2_t} \lesssim \epsilon \cdot \| \phi(t) \|_{u_C^2}.\]

**Proof.** **Part 1:** (Proof of (75)) Define \(c_6 = \frac{1}{96}\) and let \(\chi_{r<ct^\delta}(r/t)\) be a smooth cutoff to the region \(r < c_6 t\) with \(\chi_{r<ct^\delta} = r^{-1} \chi_{r<ct^\delta}\). We do separate proofs for the regions \(r < c_6 t\) and \(\frac{ct^\delta}{2} t < r < \frac{c}{2} t\). Substituting \(a = -\delta/2\) into estimate (66) gives us \(C = 2/(1 - \delta)\). Using this together with the bound (72) for \(IV\) yields:

\[(77) \quad \| r_+ \chi_{r<ct} (r/t)^{-2} \cdot IV \phi^2(t) \|_{L^2_t} \lesssim \| \chi_{r<ct} t^{r_+} (r/t)^{-2} \phi \|_{L^2_t}^2 \leq \frac{2}{1 - \delta} \left( \| \chi_{r<ct} \tau_0 t^{r_+} \tau_0 \phi \|_{L^2_t} + C \| \chi_{r<ct} t^{r_+} (r/t)^{-2} \phi \|_{L^2_t}^2 \right) = \mathcal{E}_1 + \mathcal{E}_2.\]

For \(\mathcal{E}_1\) we multiply line (71) by \(\chi_{r<ct} t^{r_+} \) and integrate to get:

\[
\mathcal{E}_1 \lesssim \| \chi_{r<ct} t^{r_+} (r/t)^{-2} \tau_0 (r/t)^{-1} \partial_r (r/t)^{-1} \phi \|_{L^2_t} \lesssim \| \chi_{r<ct} t^{r_+} \partial_r (r/t)^{-1} \phi \|_{L^2_t}^2 + \| \tau_0 (r/t)^{-1} \partial_r (r/t)^{-1} \phi \|_{L^2_t}^2.
\]

For \(\mathcal{E}_2\) we multiply line (71) by \(\chi_{r<ct} t^{r_+} \phi \) and integrate to get:

\[
\mathcal{E}_2 \lesssim \| t \chi_{r<ct^\delta} (r/t)^{-2} \tau_0 (r/t)^{-1} \partial_r (r/t)^{-1} \phi \|_{L^2_t} \lesssim \| \chi_{r<ct^\delta} t^{r_+} (r/t)^{-2} \phi \|_{L^2_t}^2 + \frac{1}{2} \| \chi_{r<ct^\delta} t^{r_+} (r/t)^{-2} \phi \|_{L^2_t}^2.
\]
where in the last line we used $r/t < c_δ$. Since we have carefully kept track of the constants in the preceding steps, our last term above can be bootstrapped onto the second term of (74). For $Ε_2$ we note that the support is where $r \sim c_δt$, so we can use (55) to get $r^{-\frac{3}{2}} \lesssim (c_δt)^{-\frac{3}{2}} \lesssim ε$ which implies:

$$Ε_2 \leq τ^{-\frac{3}{2}} \| r^{-1} t φ \|^2_2 \lesssim ε \| φ(t) \|^2_2.$$

This proves (75) inside $r < c_δt$. In the complement we apply (52) plus the estimate (55) to get $r^{-\frac{3}{2}} \lesssim (c_δt)^{-\frac{3}{2}} \lesssim ε$ and finish the proof.

**Part 2**: (Proof of (76)) we do separate proofs for the regions $r < et$ and $et/2 < r < t/2$. Inside $r < et$ we use the bound (52) for $u$, estimate, absorbing the small error term on the LHS, then adding (80) to get:

$$\| r^{-1} τ \partial_r φ \|_2 \lesssim \| (r/t)^{\frac{3}{2}} τ^{-1} r^{-1} τ^{-1} t φ \|^2_2 \lesssim ε \| φ \|^2_2.$$

For the term supported where $et/2 < r < t/2$ the bound follows by using (52), then (55) to produce $r^{-\frac{3}{2}} \lesssim (et)^{-\frac{3}{2}} \lesssim ε$.

**Lemma 5.4** (Conjugation removal). For all $t \in I^*$ we have the fixed-time bounds:

$$(78) \quad \| φ(t) \|_C \approx \| φ(t) \|_{C_E} + \| φ(t) \|_{u_C} + \| ∇ φ(t) \|_L^2.$$

**Proof.** It suffices to show $\| φ(t) \|_C \lesssim \| φ(t) \|_{C_E} + \| φ(t) \|_{u_C} + \| ∇ φ(t) \|_L^2$. This reduces to proving:

$$(79) \quad \| r \tau^{-1} τ \partial_r φ \|_2 \lesssim \| r^{-1} τ \partial_r φ \|_2 \lesssim \sum_{j=1}^\infty \| r \tau^{-1} \partial \partial_j \partial_r φ \|_2 \lesssim \| r \tau^{-1} τ \partial_r φ \|_2 + \| ∇ φ(t) \|_L^2.$$

To prove (79) we start with the identities:

$$(81) \quad t \Omega^{-1} ∂_r t φ = ∂_r φ + (r^{-2}) r φ, \quad t \Omega^{-1} ∂_r t φ = ∂_r φ + 2 t^{-1} φ.$$

We multiply the first identity by $τ_+ r$ then square. Applying Young’s inequality with $c \ll 1$ to the resulting estimate, absorbing the small error term on the LHS, then adding $|r \tau_0 ∂_r φ|_2^2$ yields:

$$|r \tau_+ ∂_r φ|^2 + |r \tau_0 ∂_r φ|^2 \lesssim |r \tau^{-1} τ \partial_r φ|^2 + |r \tau_+ τ^{-1} τ \partial_r φ|^2 + |r^{-1} τ \partial_r φ|^2.$$

Integrating the last line gives us (79). Estimate (80) will follow directly from the claim:

$$(82) \quad \| r^{-1} τ \partial_r φ(t) \|_2 + \| φ(t) \|_2 + \| r^{-1} τ \partial_r φ(t) \|_2 \lesssim RHS \| S0 \|.$$

To prove the claim we bound each of the terms in the LHS above from left-to-right. For the first term we subtract the two identities in (51) then multiply by $τ_+$ to get:

$$(83) \quad \tau_+ (t \Omega^{-1} ∂_r t φ - t \Omega^{-1} ∂_r t φ) = - (t^{-2} r (C + u) φ + 2 t^{-2} φ, \quad \text{where we have used } τ_+ = C + u + 2 r. \text{ Next observe that } (2r^{-1}) \lesssim (2r^{-1}) \lesssim r \text{ holds in the set where } 1 \leq r \text{. Therefore re-arranging } (83), \text{ squaring and using Young’s inequality yields:}$$

$$(84) \quad |r^{-1} τ \partial_r φ|^2 \lesssim |r \tau^{-1} τ \partial_r φ|^2 + |r \tau^{-1} τ \partial_r φ|^2 + |r^{-1} τ \partial_r φ|^2.$$

which is valid for $1 \leq r$. Multiplying this by a smooth cutoff $χ_{(r)/2}$, integrating, and using the Hardy estimate (55) for the undifferentiated term then adding (77) to the result gives us $\| r^{-1} τ \partial_r φ(t) \|^2_2 \lesssim RHS \| S0 \|$. This bounds the first term on LHS (82). To control $φ^2$ in $L^2$ we use the linear combination $2 \partial_u - ∂_r$. This satisfies the properties: $2(τ_) = 0, 2(τ_r) = 2, 2(τ_u) = -1$. Collectively these imply:

$$(85) \quad u(t \Omega^{-1} ∂_r t φ - t \Omega^{-1} ∂_r t φ) = 2(1 - \langle u \rangle_2) φ + (1 - r^{-2}) r^{-1} u φ \approx (u + 2 r) r^{-1} φ - (2 \langle u \rangle_2 + r^{-2} r^{-1} u) φ.$$

Rearranging the first equation above, squaring and using Young’s inequality gives us:

$$(86) \quad |(1 - \langle u \rangle_2) φ|^2 \lesssim u^2 (|t \Omega^{-1} ∂_r t φ|^2 + |t \Omega^{-1} ∂_r t φ|^2) + (1 - r^{-2}) r^{-1} τ \partial_r φ.$$

Next we apply (55) to the last term on the RHS (85). Multiplying the resulting bound by $χ_{(u)/1}$, then integrating and using (55) for the undifferentiated term gives us control of $\| φ(t) \|^2_2$ within the region $\langle u \rangle > 1$. 

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Adding \( \delta \) to this result then gives us \( \| \phi(t) \|_{L^2}^2 \lesssim RHS \). Finally, the result for \( \| r^{-1} \tau_r \phi(t) \|_{L^2}^2 \lesssim RHS \) follows by using the last equation in \( \delta \), integrating and applying the previous estimates. \( \square \)

5.2. Core Multiplier Estimates. In this section we list and prove two multiplier bounds which will be the core constituents of estimates \( \| \partial \| \delta \) and \( \| \partial \| \delta \).

**Proposition 5.5** (Output of \( \partial_u \)). For any interval \( [t_0, t_1] \subset I^* \) we have the uniform estimate:

(87) \[
\sup_{t_0 \leq t \leq t_1} \| \nabla \phi(t) \|_{L^2}^2 + \| \nabla_x \phi \|_{NLE, -\frac{1}{2}} \lesssim \| \chi_{\Delta_t < r} r^{-\frac{1}{2}} (\nabla \phi, \tau_0^{-\frac{1}{2}} \Delta_x \phi) \|_{L^2}^2 + \| \chi_{\Delta_t < r} r^{-\frac{1}{2}} (\nabla \phi, \tau_0^{-\frac{1}{2}} \Delta_x \phi) \|_{L^2}^2 + \| \nabla \phi(t_0) \|_{L^2}^2 + \sup_j \iint_{t_0 \leq t \leq t_1} \nabla \phi \cdot \nabla \phi \, dv_g \],
\]

where the vector fields \( Y_j = q_j(u) \partial_u \) are indexed by \( j \in \mathbb{Z} \) with uniform bounds \( |\partial_u q_j| \lesssim \tau^{-1} \).

**Proposition 5.6** (Output of \( K_0 \)). Let \( [t_0, t_1] \subset I^* \). For each of the weights \( \Omega = \Omega_1, \Omega_2, \Omega_3 \) have the estimate:

(88) \[
\sup_{t_0 \leq t \leq t_1} \| \phi(t) \|_{NCE} + \| (r) \frac{1}{2} \Omega^{-1} \Delta_t (\Omega \phi) \|_{NLE, -\frac{1}{2}} \lesssim \| \chi_{\Delta_t < r} r^{-\frac{1}{2}} \tau_+ \Omega^{-1} (\tau_0 \nabla (\Omega \phi), \nabla_x (\Omega \phi), r^{-1} \phi) \|_{L^2}^2 + \| \chi_{\Delta_t < r} r^{-\frac{1}{2}} \tau_+ \Omega^{-1} (\Omega \nabla \phi) \|_{L^2}^2 + \sup_{t_0 \leq t \leq t_1} \| \nabla \phi(t_0) \|_{NCE} + \sup_j \iint_{t_0 \leq t \leq t_1} \phi \cdot \Omega^{-1} (\tau_+ \chi \Omega \phi) \, dv_g \],
\]

where \( X_j = q_j(u) K_0 \) are indexed by \( j \in \mathbb{Z} \), with \( K_0 \) given by the formula \( K_0 = \tau_2 \partial_u + 2(u + r) \bar{r} \partial_r \), and where \( q_j \) has the uniform bounds \( |\partial_u q_j| \lesssim \tau^{-1} \).

**Proof of Proposition 5.5.** This is a classical multiplier calculation using formulas \( \| \partial \| \delta \) and \( \| \partial \| \delta \). For convenience we employ these in the form of Lemma \( \| \partial \| \delta \) with \( \Omega \equiv 1 \) and \( \chi \equiv 0 \) in formulas \( \| \partial \| \delta \). We define the multiplier vector fields \( Y_j = (1 + \chi_{<j}(u)) \partial_u \), which are indexed by \( j \in \mathbb{Z} \) and where \( \chi_{<j} \) is a non-negative, uniformly bounded, and monotone decreasing function with \( \chi_{<j} = -2^{-j} \chi_j \) supported where \( \langle u \rangle \approx 2^j \).

**Step 1:** (Output of the \( Y_j \) \( A^{\alpha \beta} \partial_u \delta \partial_u \delta \) contraction) We compute using polar Bondi coordinates \( (u, r, x^A) \). Since \( Y^r \equiv 0 \), identities \( \| \partial \| \delta \) and \( \| \partial \| \delta \) get us:

\[
d^{2}(Y_j) R^{\alpha \beta} = 2^{-j} \chi_j(u) (\eta^{\alpha \beta} - \eta^{\alpha u} \delta^u - \eta^{\beta u} \delta^u) + R^{\alpha \beta} \cdot
\]

where:

\[
R^{\alpha \beta} = -L_{Y_j} (d^2 y^{-1} - \eta^{-1}) + 2^{-j} \chi_j(u) (d^2 y^{-1} - \eta^{-1}) \cdot
\]

On a dyadic scale \( \langle u \rangle \approx 2^j \) the vector field \( 2Y \) satisfies all of the assumptions \( \| \partial \| \delta \). By the results of part I of Lemma \( \| \partial \| \delta \) \( R^{\alpha \beta} \) satisfies the pointwise bounds:

\[
|\langle R^{\alpha \beta} \rangle | \lesssim (r)^{-\delta} (\tau_+ / r)^{-\gamma} \cdot
\]

Thus a little bit of additional calculation involving the previous formulas shows that \( A^{\alpha \beta} \) satisfies the pointwise bounds:

\[
|\langle Y_j \rangle A^{\alpha \beta} | \lesssim (r)^{-\delta} (\tau_+ / r)^{-\gamma} \cdot
\]

where \( \delta^{\alpha \beta} \) denotes the standard inverse metric on \( S^2 \). Integrating the resulting contraction over the time slab \( [t_0, t_1] \) and taking sup \( j \) gives us, with suitable constants \( 0 < c \ll C \):

(89) \[
c \| \nabla_x \phi \|_{NLE, -\frac{1}{2}} \leq C \| \chi_{\Delta_t < r} r^{-\frac{1}{2}} (\nabla \phi, \tau_0^{-\frac{1}{2}} \Delta_x \phi) \|_{L^2}^2 + C \| \chi_{\Delta_t < r} r^{-\frac{1}{2}} (\nabla \phi, \tau_0^{-\frac{1}{2}} \Delta_x \phi) \|_{L^2}^2 + \sup_j \iint_{t_0 \leq t \leq t_1} (Y_j) A^{\alpha \beta} \partial_u \delta \partial_u \delta \, dv_g \cdot
\]
Step 2: (Output of the boundary terms) Proposition 2.10 implies that the vector fields $Y_j$ are uniformly timelike for all $j \in \mathbb{Z}$. Thus, we have for the boundary terms on line (21):

$$\sup_j \left( \int_{t_0}^{t_1} \langle Y_j \rangle P_\alpha N^\alpha \, d^2x - \int_{t_0}^{t_1} \langle Y_j \rangle P_\alpha N^\alpha \, d^2x \right) \lesssim \| \nabla \phi(t_0) \|^2_{L^2} - \| \nabla \phi(t_1) \|^2_{L^2}.$$

Combining this with (59) taking sup in $t$ then square roots gives the result.

\[ \square \]

Proof of Proposition 5.6: Here we use the conformal multiplier setup of Lemma 2.2 with weights $\Omega = t\Omega, \Omega$. For $j \in \mathbb{Z}$ we define the multiplier vector fields:

$$X_j = (1 + \chi_{<j}(u))K_0, \quad K_0 = \tau^2 \partial_\nu + 2(u + r)\tilde{\partial}_r,$$

where the $X_j$ are the same as in the previous proof. Set $\chi = \chi(-\frac{1}{t} \frac{\delta}{\delta t})(r/t)$ which smoothly cuts off on $r < \frac{1}{t}$. Using the divergence identity (21) we need to estimate each term on lines (52), (58) as well as the boundary errors on (55) and (50). We do this for each term separately:

Step 1: (Output of the $A^\alpha\beta$ contraction) Once again we compute using polar Bondi coordinates $(u, r, x^A)$. The key property of $K_0$ is that $\mathcal{L}_{K_0} \eta^{-1} + (\partial_u K^0_0 + \tilde{\partial}_r K^0_0) \eta^{-1} = 0$. Therefore, from (55) and (50) we have:

$$\langle \bar{\mathcal{L}}_{X_j} g - 2X_j \ln \Omega \rangle \eta^{-1} - t \chi_j(u)(\tau^2 \eta^{-1} - \eta^u \eta^0 K^0_0 - \eta^\beta u K^\beta_0) + 2(1 + \chi_{<j}(u))(r^{-1} K^0_0 - K_0 \ln \Omega))\eta^{-1} + \mathcal{R},$$

where,

$$\langle \mathcal{R} \rangle = -\mathcal{L}_{X_j} (d^2 g - \eta^{-1}) + \left[ \tau^2 \bar{\chi}_j(u) - (1 + X_{<j}(u))(4(u + r) + (2r^{-1} K^0_0 - K_0 \ln \Omega)) \right](d^2 g - \eta^{-1}).$$

For the term $r^{-1} K^0_0 - K_0 \ln \Omega$ one can compute:

$$\mathcal{R} = -\mathcal{L}_{X_j} (d^2 g - \eta^{-1}) + \left[ \tau^2 \bar{\chi}_j(u) - (1 + X_{<j}(u))(4(u + r) + (2r^{-1} K^0_0 - K_0 \ln \Omega)) \right](d^2 g - \eta^{-1}).$$

These terms are $O(\tau_+ / \langle r \rangle^{-2})$ and $O(1)$, respectively; this allows us to treat them as lower order errors below. To bound $\mathcal{R}$ we use the fact that on a dyadic scale $\tau_+ \approx 2^k$ the vector field $2^{-k} K_0$ satisfies the symbol bounds (10) and all the conditions on line (17) except we have $\partial_u X^r \neq 0$. Therefore we are in case II of lemma 8.3 and so we have estimates (48) with the modification (49) and (50). In particular thanks to (3) the error $\mathcal{R}$ satisfies:

$$|\mathcal{R}| \lesssim \langle r \rangle^{-1} \tau_+ t_0^{-\frac{1}{2}}(\tau_+/\langle r \rangle)^{1-\gamma}, \quad |(r \mathcal{R}^{rA}, r^2 \mathcal{R}^{AB})| \lesssim \langle r \rangle^{-1} \tau_+ t_0^{-\frac{1}{2}}(\tau_+/\langle r \rangle)^{1-\gamma},
$$

$$|\mathcal{R}^{ru}| \lesssim \langle r \rangle^{-1} \tau_+ t_0^{-\frac{1}{2}}(\tau_+/\langle r \rangle)^{1-\gamma}, \quad |(r \mathcal{R}^{ruA})| \lesssim \langle r \rangle^{-1} \tau_+ t_0^{-\frac{1}{2}}(\tau_+/\langle r \rangle)^{1-\gamma},
$$

Combining the last few lines we get:

$$|A^{ru}| \lesssim \langle r \rangle^{-1} \tau_+ t_0^{-\frac{1}{2}}(\tau_+/\langle r \rangle)^{1-\gamma}, \quad A^{ru} \gtrsim \tau_+^{\frac{1}{2}} \tau_1 \chi_j(u) + O(\langle r \rangle^{-1} \tau_+ t_0^{-\frac{1}{2}}(\tau_+/\langle r \rangle)^{1-\gamma}),
$$

$$r^2 A^{AB} \gtrsim \tau_1 \chi_j(u) + O(\langle r \rangle^{-1} \tau_+ t_0^{-\frac{1}{2}}(\tau_+/\langle r \rangle)^{1-\gamma}), \quad |A^{ru}| \lesssim \langle r \rangle^{-1} \tau_+ t_0^{-\frac{1}{2}}(\tau_+/\langle r \rangle)^{1-\gamma},
$$

where $\delta^{AB}$ again denotes the standard inverse metric on $S^2$. Integrating the resulting contraction over the time slab $[t_0, t_1]$ and taking sup $\tilde{\partial}_r$ then gives us, with suitable constants $0 < \epsilon < C$:

$$\langle r \rangle^{\frac{1}{2}} \Omega^{-1} \tilde{\nabla} \phi, (\Omega \phi) \rangle \lesssim C \chi_{4^{-1} t_+} \langle r \rangle^{-1} \tau_+ t_0^{-\frac{1}{2}}(\tau_+/\langle r \rangle)^{1-\gamma}, \quad \langle r \rangle^{-1} \Omega^{-1} \tilde{\nabla} \phi, (\Omega \phi) \rangle \lesssim C \chi_{4^{-1} t_+} \langle r \rangle^{-1} \tau_+ t_0^{-\frac{1}{2}}(\tau_+/\langle r \rangle)^{1-\gamma},$$

$$+ C \| \chi_{t_0} \langle r \rangle^{\frac{1}{2}} \langle r \rangle^{-1} \tau_+ t_0^{-\frac{1}{2}}(\tau_+/\langle r \rangle)^{1-\gamma} \|_{L^2} + \langle r \rangle^{-1} \Omega^{-1} \tilde{\nabla} \phi, (\Omega \phi) \rangle \lesssim \Omega^{-1} \tilde{\nabla} \phi, (\Omega \phi) \rangle \lesssim C \chi_{4^{-1} t_+} \langle r \rangle^{-1} \tau_+ t_0^{-\frac{1}{2}}(\tau_+/\langle r \rangle)^{1-\gamma},$$

$$\lesssim \langle r \rangle^{-1} \tau_+ t_0^{-\frac{1}{2}}(\tau_+/\langle r \rangle)^{1-\gamma}, \quad \langle r \rangle^{-1} \Omega^{-1} \tilde{\nabla} \phi, (\Omega \phi) \rangle \lesssim C \chi_{4^{-1} t_+} \langle r \rangle^{-1} \tau_+ t_0^{-\frac{1}{2}}(\tau_+/\langle r \rangle)^{1-\gamma},$$

$$\lesssim \langle r \rangle^{-1} \tau_+ t_0^{-\frac{1}{2}}(\tau_+/\langle r \rangle)^{1-\gamma}, \quad \langle r \rangle^{-1} \Omega^{-1} \tilde{\nabla} \phi, (\Omega \phi) \rangle \lesssim C \chi_{4^{-1} t_+} \langle r \rangle^{-1} \tau_+ t_0^{-\frac{1}{2}}(\tau_+/\langle r \rangle)^{1-\gamma},$$

where we have used $\| \Omega^{-1} \nabla (\Omega \phi) \| \lesssim \| \nabla \phi \| + (\langle r \rangle^{-1} \phi \| - \langle r \rangle^{-1} \phi \| L^2 \lesssim NLE.$

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Step 2: (Output of the $B^\alpha$ term) Let $\tilde{\chi}$ be an auxiliary smooth cutoff function supported inside $r < \frac{1}{8}t$.
Multiplying the bounds we derived for $A$ in the last paragraph times the bounds \[[52]\] for the conformal potentials $IV$, $IIV$ and dividing by $I\Omega^2$ and $II\Omega^2$ we get:

$$|tB^\alpha| \lesssim \tau_+^{2-\gamma}(r)^{-3-\delta-\gamma}\tilde{\chi}(r/t), \quad |t\mu B^\alpha| \lesssim \tau_+^{2-\gamma}(\tau_+)^{-\frac{1}{2}}(r)^{-2-\delta+\gamma}\tilde{\chi}(r/t).$$

Integrating these bounds:

\[(94) \quad \begin{aligned}
\int_{t_0}^{t_1} |B^\alpha| \phi^2 \, dV_g & \lesssim \|\chi_{r<\tilde{\phi}(\tau_+)^{\frac{1}{2}}}(r)^{-\frac{1}{2}+\frac{1}{2}}\phi\|_{L^2}^2.
\end{aligned}
\]

Step 3: (Output of the $C^\alpha$ term) Using \[[52]\] and the fact that $C^\alpha$ is supported where $r > \frac{1}{8}t$:

$$|tC^\alpha| \lesssim \langle r \rangle^{-2-\delta}\Omega^{-1}K_0(\Omega\phi), \quad |t\mu C^\alpha| \lesssim \langle r \rangle^{-1-\delta}(\tau_+)^{-\frac{1}{2}}\Omega^{-1}K_0(\Omega\phi).$$

Integrating these bounds and using Young’s inequality with $0 < c_0 < 1$:

\[(95) \quad \begin{aligned}
\int_{t_0}^{t_1} |C^\alpha\Omega^{-1}K_0(\Omega\phi)| \, dV_g & \lesssim \|\chi_{\tilde{\phi}(\tau_+)^{\frac{1}{2}}<r<\tau_+^{\frac{1}{2}}}(r)^{-\frac{1}{2}}\phi\|_{L^2} + \|\chi_{\tilde{\phi}(\tau_+)^{\frac{1}{2}}<r<\tau_+^{\frac{1}{2}}}\partial_\tau(\Omega\phi)\|_{L^1} \\
& \lesssim c_0^{-1}\|\chi_{\tilde{\phi}(\tau_+)^{\frac{1}{2}}<r<\tau_+^{\frac{1}{2}}}r^{-\frac{1}{2}}\tau_+^{\frac{1}{2}}\Omega^{-1}\nabla(\Omega\phi)(r^{-\frac{1}{2}}\phi)\|_{L^2} + c_0\|\langle r \rangle^{\frac{1}{2}}\Omega^{-1}\tilde{\phi}(\Omega\phi)\|_{NLE}^2.
\end{aligned}
\]

Step 4: (Output of the boundary term) Adding estimates \[[55] + 56]:

\[(96) \quad \begin{aligned}
(\xi)^\alpha N^\alpha\Omega^{-2} & \geq \tau_+^2 - CK_0^\alpha\langle r \rangle^{-\delta}r_0^{\frac{1}{2}} \Omega^{-1}\Omega^{-2}V\phi^2(t) + K_0^\alpha\Omega^{-1}\tilde{\phi}(\Omega\phi)^2 - C(\Omega V)(K_0^\alpha + K_0^\tau)\Omega^{-2}\phi^2.
\end{aligned}
\]

We claim that the following two bounds hold in the asymptotic region $I^*$:

\[(97) \quad \begin{aligned}
\tau_+^2 - CK_0^\alpha\langle r \rangle^{-\delta}r_0^{\frac{1}{2}} & \geq |\tau_+\tau_0|^2, \quad K_0^\alpha \geq 2tr.
\end{aligned}
\]

To prove this we let $0 < c < 1$ and treat the regions $r < ct$ and $ct < r$ separately.

Case 1: (Inside $r < ct$) Since $\tau_0 \approx 1$ the first estimate in \[[97]\] reduces to proving $(t-r)^2 + C_0 tr \geq |\tau_+\tau_0|^2$ for some constant $C_0 > 0$. This clearly holds by choosing $c$ sufficiently small. Since $u = t-r$ here, the second bound holds trivially.

Case 2: (Exterior region $ct < r$) The first bound in \[[97]\] holds since $K_0^\alpha \tau_0^2 \approx \tau_+^2$ and the term containing the gain $r^{-\delta}$ can be turned into a small bootstrap error. The second bound only needs to be proved inside the wave zone. Integrating estimate \[[52]\] yields $K_0^\alpha = 2tr + o(r^{2-\delta})$ so we can bootstrap the term with the gain $r^{-\delta}$ here. This finishes the proof of \[[97]\].

Integrating \[[95] + 96] and using line \[[97]\] gives us:

\[\begin{aligned}
\sup_j \left( \int_{t_0}^{t_1} (\xi)^\alpha N^\alpha\Omega^{-2} \, dr^2 \right) - \int_{t_1}^{t_1} (\xi)^\alpha N^\alpha\Omega^{-2} \, dr^2 \right) \lesssim \sup_{t_0 \leq t \leq t_1} \|\tau_+^2 \Omega^{-1}\Omega^{-2}\Omega V\phi^2(t)\|_{L^1} \\
+ \|\phi(t_0)\|_{H^{\alpha}}^2 - \|\phi(t_1)\|_{H^{\alpha}}^2.
\end{aligned}\]

We add \[[51] + 54 + 55 + 98] and choose $c_0$ sufficiently small to bootstrap the $NLE$ term on RHS \[[55]\]. Re-arranging terms, taking sup in $t$ then square roots finishes the proof of \[[55]\].

\[\square\]

Remark 5.7. We require the full decay for $g^{\alpha\alpha} - \omega_j\omega_j g^{\alpha\alpha}$ and $g^{\alpha\alpha}$ given by \[[53]\] in order to produce estimate \[[97]\]. If we assume weaker decay for these components, estimate \[[55]\] would be also be weaker and the terms containing the gain $r^{-\delta}$ in the exterior could no longer be bootstrapped. In other words, the exterior proof of \[[97]\] above would no longer work.

5.3. Proof of the Main Estimates. We now prove parts II and III of Theorem 5.1.

Proof of Proposition 5.4. We bound each of the error terms on RHS \[[54].

Step 1: (Bounding the source term) By Young’s inequality:

\[(99) \quad \begin{aligned}
\sup_j \left| \int_{t_0 \leq t \leq t_1} \square g^\alpha \phi \cdot Y_j \phi \, dV_g \right|^\frac{1}{2} & \lesssim c^{-1}\|\square g^\alpha\phi\|_{LE^*} + c\|\phi\|_{LE}.
\end{aligned}\]
Step 2: (Bounding the spacetime error terms) To control $\nabla \phi$ in the exterior we use $r^{-\frac{\delta}{2}} \lesssim \epsilon$ together with $r^{-\frac{\gamma}{2}} L^2 \leq NLE$. For $\tau_0^{r^{\frac{\gamma}{2}}} \nabla_x \phi$ in the exterior we use $r^{-\frac{\delta}{2}} \lesssim \epsilon$ together with $r^{-\frac{\gamma}{2}} \tau_0^{r^{\frac{\gamma}{2}}} L^2 \leq NLE$. For the interior we use $\gamma < \delta$ so that $r^{-\frac{\gamma}{2} + \frac{\gamma}{2}} L^2 \leq LE$ and concatenate this with $t^{-\frac{\gamma}{2}} \lesssim \epsilon$. This yields:

$$\tag{100} \| \chi_{\frac{1}{2}t<r} r^{-\frac{\gamma}{2}} (\nabla \phi, \nabla_x \phi) \|_{L^2} + \| \chi_{r<t}^{t_1} (r^{\frac{\gamma}{2}} \tau_0^{r^{\frac{\gamma}{2}}} \phi, r^{\frac{\gamma}{2}} \tau_0^{r^{\frac{\gamma}{2}}} \phi) \|_{L^2} \lesssim \epsilon \left( \| \nabla \phi \|_{NLE} + \| \phi \|_{LE} \right).$$

We use this and (99) on the RHS [87]. Adding estimate (3) to the resulting bound and choosing $c \ll 1$ allows us to bootstrap $c \| \phi \|_{LE}$ present on RHS [88]. Bootstrapping all the terms containing $\epsilon$ finishes the proof.

Proof of Proposition 5.6. We work on each error term on RHS [88].

Step 1: (Bounding the source term) Let $c_1, c_2 \ll 1$ be small positive constants. We bound the timelike and null/space-like regions separately. In the exterior we use Young's inequality together with $\ell_1^1 \rightarrow \ell_1^1 \times \ell_1^\infty$ Holder in time to get:

$$\tag{101} \sup_j \| \chi_{\frac{1}{2}t<r} \phi \|_{\Omega^{-1} X_1(\Omega \phi)} \lesssim \| \chi_{\frac{1}{2}t<r} (r^{\frac{\gamma}{2}} \tau_0^{r^{\frac{\gamma}{2}}} \phi, \nabla_x \phi, \nabla_x \phi) \|_{L^2} + \| \chi_{r<t}^{t_1} (r^{\frac{\gamma}{2}} \tau_0^{r^{\frac{\gamma}{2}}} \phi, \nabla_x \phi) \|_{L^2} \lesssim c_1^{-1} \| \phi \|_{\ell_1^t N^2} + c \left( \sup_{t_0 \leq t_1} \| \phi(t) \|_{\ell_1^t CE} + \| \phi(t) \|_{\ell_1^t \nabla \phi, \nabla_x \phi} \|_{L^2} \right),$$

where we applied $\chi_{\frac{1}{2}t<r} \phi \|_{\Omega^{-1} \phi} \lesssim \| \phi \|_{CE}$ on the last line. In the interior we use a similar argument in addition to $\| \Omega^{-1} \nabla \phi \| \lesssim \| \nabla \phi \| + \| r^{-1} \phi \|$ to remove the conjugation and get:

$$\sup_j \| \chi_{r<t} \phi \|_{\Omega^{-1} \phi} \|_{L^2} \lesssim c^{-1} \| \phi \|_{\ell_1^t N^2} + c \| \chi_{r<t} \phi \|_{\ell_1^t LE^1}.$$

Step 2: (Bounding the space-time errors) In the interior and the exterior we use a similar proof to (100) with the important caveat that we now must split the interior gain $\tau_0^{r^{\frac{\gamma}{2}}}$: half goes to produce $\tau_0^{r^{\frac{\gamma}{2}}}$ $\lesssim \epsilon$ while the other half is used to sum the dyadic pieces in $t$. A bit of additional computation then yields:

$$\tag{102} \| \chi_{\frac{1}{2}t<r} r^{-\frac{\gamma}{2}} \tau_0^{r^{\frac{\gamma}{2}}} \Omega^{-1} \nabla \phi \|_{L^2} + \| \chi_{r<t} \phi \|_{\Omega^{-1} \nabla \phi} \|_{L^2} \lesssim \epsilon \left( \sup_{t_0 \leq t_1} \| \phi(t) \|_{CE} + \sup_{t_0 \leq t_1} \| \phi(t) \|_{\ell_1^t \nabla \phi} \right).$$

Step 3: (Bounding the undifferentiated boundary errors) We apply the bounds (75) and (70) for each undifferentiated boundary error. Next we choose $c_1 \ll 1$ small in step 1 above so we can bootstrap the small errors in (101). This, together with steps 1 and 2 above yields the following two bounds for the conjugated energies $^1 CE$ and $^2 CE$:

$$\tag{103} \sup_{t_0 \leq t \leq t_1} \| \phi(t) \|_{^1 CE} + \| \phi(t) \|_{\Omega^{-1} \nabla \phi} \|_{\Omega^{-1} \nabla \phi} \|_{L^2} \lesssim \sup_{t_0 \leq t \leq t_1} \| \phi(t) \|_{CE} + \epsilon \sup_{t_0 \leq t \leq t_1} \| \phi(t) \|_{CE} + \| \phi(t) \|_{\nabla \phi} \|_{\Omega^{-1} \nabla \phi} \|_{L^2} \lesssim \epsilon \sup_{t_0 \leq t \leq t_1} \| \phi(t) \|_{CE} + \| \phi(t) \|_{\nabla \phi} \|_{\Omega^{-1} \nabla \phi} \|_{L^2} \lesssim \epsilon \sup_{t_0 \leq t \leq t_1} \| \phi(t) \|_{CE} + \| \phi(t) \|_{\nabla \phi} \|_{\Omega^{-1} \nabla \phi} \|_{L^2} \lesssim \epsilon \sup_{t_0 \leq t \leq t_1} \| \phi(t) \|_{CE} + \| \phi(t) \|_{\nabla \phi} \|_{\Omega^{-1} \nabla \phi} \|_{L^2} \lesssim \epsilon \sup_{t_0 \leq t \leq t_1} \| \phi(t) \|_{CE} + \| \phi(t) \|_{\nabla \phi} \|_{\Omega^{-1} \nabla \phi} \|_{L^2}.$$
6. Commutators

In this section we commute the equation once with the Lie algebra \( L = \{ \partial_u, S, \Omega_{ij} \} \) and use our conformal energy estimate \((9)\) to produce the higher order bound \((10)\). By Lemma 4.2 it suffices to prove our estimate for the asymptotic region \( t \in I^* \). Inside \( I^* \) the estimate \((10)\) will follow from:

**Theorem 6.1.** Assume the hypotheses of the main theorem as well as estimate \((9)\). Then for all \([t_0, t_1] \subset I^*\) we have the following uniform bounds:

I) (CE estimate for \( \partial_u \phi \) with error)

\[
\sup_{t_0 \leq t \leq t_1} || \partial_u \phi(t) ||_{CE} + || \partial_u \phi ||_{\ell^\infty S} \lesssim \epsilon \left( \sup_{t_0 \leq t \leq t_1} || \phi(t) ||_{CE_1} + || \chi_{\tau < \frac{1}{4} \tau} || \phi ||_{\ell^\infty L^{E_1}} \right) + || \partial_u \phi(t_0) ||_{CE} + \| \nabla (\Box \phi) \|_{\ell^1_N}.
\]

II) (CE estimate for \( S \phi \) and \( \Omega_{ij} \phi \) with error)

\[
\sup_{t_0 \leq t \leq t_1} \sum_{\Gamma = S, \Omega_{ij}} || \Gamma \phi(t) ||_{CE} + || \phi \|_{\ell^\infty S} \lesssim \epsilon \left( \sup_{t_0 \leq t \leq t_1} || \phi(t) ||_{CE_1} + || \chi_{\tau < \frac{1}{4} \tau} \phi \|_{\ell^\infty L^{E_1}} \right) + || \phi(t_0) ||_{CE_1} + \| \Box \phi \|_{\ell^1_N}.
\]

Let’s show how the conformal energy estimate with vector fields follows from Theorem 6.1.

**Proof of estimate** \((10)\). Adding \((105) + (106) + (9)\) yields:

\[
\sup_{t_0 \leq t \leq t_1} \| \phi(t) \|_{CE_1} + \| \phi \|_{\ell^\infty S} \lesssim \epsilon \left( \sup_{t_0 \leq t \leq t_1} || \phi(t) ||_{CE_1} + || \chi_{\tau < \frac{1}{4} \tau} \phi \|_{\ell^\infty L^{E_1}} \right) + || \phi(t_0) ||_{CE_1} + \| \Box \phi \|_{\ell^1_N}.
\]

We bootstrap the small error terms and close the estimate. Inside \([0, T^*]\), lemma \(4.2\) gives us the analogous bound. A combination of these two estimates then yields \((10)\).

The rest of this section is devoted to proving Theorem 6.1.

6.1. Some Preliminary Estimates. Our goal in this first part is to establish some commutator bounds for all vector fields \( \Gamma \in \mathbb{L} \). We start with the following:

**Lemma 6.2.** (Pointwise bounds) Let \( \Gamma \in \mathbb{L} \) be in rectangular Bondi coordinates \((u, x^r)\). We have the following uniform estimate:

\[
\sum_{\Gamma = S, \Omega_{ij}} \| [\Box \phi, \Gamma] \phi \| \lesssim \langle r \rangle^{-2 - \delta} |\langle \tau_+ / \langle r \rangle \rangle|^{1 - \gamma} \left( \sum_{1 \leq k + |j| \leq 2} \tau_0^{-\frac{1}{2}} \langle \langle r \rangle \tau_0 \partial_u k \rangle \langle \langle r \rangle \partial_x \rangle^j \phi \right) + \| \langle \langle r \rangle \tau_0 \partial_u \rangle \phi \| + \| \langle \langle r \rangle \partial_x \rangle \phi \| + \| \Box \phi \|.
\]

In the case of \([\Box \phi, \partial_u] \) the same estimate holds with the exponent \(1 - \gamma\) above replaced with \(-\gamma\).

**Proof.** By \((11), (12), \) and \((13)\) it suffices to bound the quantities:

\[
\mathcal{E}(u, \alpha) = \partial_u d^{\frac{1}{2}} R^\alpha_{\gamma \nu} \partial_{\nu} \phi, \quad \mathcal{E}(u, u) = \partial_u d^{\frac{1}{2}} R^\alpha_{\gamma \nu} \partial_{\nu} \phi, \quad \mathcal{E}(i, j) = \partial_i d^{\frac{1}{2}} R^\alpha_{\gamma \nu} \partial_{\nu} \phi, \quad \mathcal{E}(s) = d^{-\frac{1}{2}} \Gamma (d^{\frac{1}{2}}),
\]

where:

\[
d^{\frac{1}{2}} R^\alpha_{\gamma \nu} = -\mathcal{L}_V (d^{\frac{1}{2}} g^{-1} - \eta^{-1}), \quad d^{\frac{1}{2}} R_S = -\mathcal{L}_V (d^{\frac{1}{2}} g^{-1} - \eta^{-1} - (\partial_u S^u + \tilde{\partial}_x S^x + \tilde{\partial}_{\gamma} S^\gamma) (d^{\frac{1}{2}} g^{-1} - \eta^{-1})),
\]

To bound these quantities we use part I of lemma \(3.5\) together with estimate \((9)\) for the \(\det (g_{\alpha \beta})\). This yields:

\[
|\mathcal{E}(u, \alpha)| \lesssim \langle r \rangle^{-\delta} \langle \tau_+ / \langle r \rangle \rangle^{1 - \gamma} \left( \tau_0^{-\frac{1}{2}} \tilde{\partial}_x \partial_u \phi + \tau_0^2 |\partial_x^2 \phi| + \langle \langle r \rangle \rangle^{-\delta} \tau_0^{-\frac{1}{2}} |\partial_u \phi| + \langle \langle r \rangle \rangle^{-1} \tau_0^{-\frac{1}{2}} |\partial_x \phi| \right),
\]

\[
|\mathcal{E}(i, j)| \lesssim \langle r \rangle^{-\delta} \langle \tau_+ / \langle r \rangle \rangle^{1 - \gamma} (|\partial_x^2 \phi| + \langle \langle r \rangle \rangle^{-1} |\partial_x \phi|), \quad |\mathcal{E}(s)| \lesssim \langle r \rangle^{-\delta} \langle \tau_+ / \langle r \rangle \rangle^{1 - \gamma} \Box \phi.
\]

In the case of \(\Gamma = \partial_u\) we use part III of lemma \(3.6\) which yields the improvement in the interior by the same argument. \(\square\)
Remark 6.3. A quick computation using (3) shows that the estimate derived in lemma 6.2 for the vector field $\partial_u$ is missing a weight $\tau^{-1}_-$ in the exterior. This omission comes from the fact that there’s no real improvement in the estimates below if we were to include this weight. This is a consequence of our weak decay for the $g^{ij}$ components which would, even in that case, still force us to treat $\partial_u$ at the same level of decay as $S$ and $\Omega_{ij}$ in the exterior.

Remark 6.4. From this point on we will often make use of the parameter $\mu \ll 1$ satisfying the smallness condition in definition 4.1a.

In the region $\{ \mu t < r \} \cap I^*$ we control all weighted combinations of two derivatives in RHS 107 via:

**Lemma 6.5 (Exterior Klainerman-Sideris identity).** Let $\Gamma \in \mathbb{L}$ be in rectangular Bondi coordinates $(u, x^i)$. In the exterior region $\{ \mu t < r \} \cap I^*$ one has the following uniform estimate:

$$
\sum_{1 \leq k + |J| \leq 2} \left| (r \tau_0 \partial_u)^k (r \tilde{\partial}_x)^J \phi \right| \leq \sum_{k + |J| = 1, |J| \leq 1} \left| (r \tau_0 \partial_u)^k (r \tilde{\partial}_x)^J \phi \right| + r^2 \tau_0 |\Box_g \phi|.
$$

**Proof.** Let $\mathcal{R} = d^2 g - \eta$. Inside $\{ \mu t < r \} \cap I^*$ the estimates (3) imply the uniform bound:

$$
r^2 \tau_0 |\partial_u \mathcal{R} \phi| \lesssim r^{-\delta} \sum_{1 \leq k + |J| \leq 2} \left| (r \tau_0 \partial_u)^k (r \tilde{\partial}_x)^J \phi \right|,
$$

where all quantities are computed with respect to Bondi coordinates. By the support property and estimate 59 we have $r^{-\delta} \leq (\mu t)^{-\delta} \leq \epsilon$. Thus, we may replace $\Box_g$ by the Minkowski wave operator $\Box_\eta$ in 108.

Next, we have the two identities:

$$
r^2 \tau_+^{-1} \partial_\eta S = r^2 u \tau_+^{-1} \partial_u \partial_\eta + r^3 \tau_+^{-1} (\partial \eta)^2 + r^2 \tau_+^{-1} \partial_\eta,
$$

$$
\frac{1}{2} r^2 \tau_+^{-1} \Box_\eta = -r^2 u \tau_+^{-1} \partial_u \partial_\eta + \frac{1}{2} r^2 u \tau_+^{-1} (\partial \eta)^2 - ru \tau_+^{-1} \partial_u + ru \tau_+^{-1} \partial_\eta + \frac{1}{2} r^2 \tau_+^{-1} \sum_{i < j} (\Omega_{ij})^2,
$$

where we used 53 on the second line. Adding the two operators on the LHS above applied to $\phi$ yields:

$$
\left| (r \tilde{\partial}_x)^2 \phi \right| \lesssim \sum_{k + |J| = 1, |J| \leq 1} \left| (r \tau_0 \partial_u)^k (r \tilde{\partial}_x)^J \phi \right| + r^2 \tau_0 |\Box_\eta \phi|.
$$

In addition to this we have the inequality:

$$
|\Box_g (r \tau_0 \partial_u)^2 \phi| + |(r \tau_0 \partial_u)(r \tilde{\partial}_x) \phi| \lesssim |(r \tilde{\partial}_x)^2 \phi| + \sum_{k + |J| = 1} \left| (r \tau_0 \partial_u)^k (r \tilde{\partial}_x)^J \phi \right| + \sum_{k + |J| = 1} \left| (r \tau_0 \partial_u)^k (r \tilde{\partial}_x)^J \partial_u \phi \right|.
$$

All other combinations of derivatives on LHS 108 are controlled by the sum on RHS 108.

As a consequence of these two lemmas we get:

**Proposition 6.6 (Global commutator bounds).** The following uniform estimate holds on $I^*$:

$$
\sum_{\Gamma = S, \Omega_{ij}} \left\| [\Box_g, \Gamma] \phi \right\|_{L^2} \lesssim \left\| \langle r \rangle^{-\frac{1}{2}} r \tau_+^{-2} \chi_{r < \mu t} (\nabla^2 \phi, \langle r \rangle^{-1} \nabla \phi) \right\|_{L^2} + \epsilon \sup_{t_0 \leq t \leq t_1} \left\| \phi(t) \right\|_{C^1} + \left\| \Box_g \phi \right\|_{L^1}.
$$

with the following (interior) improvement in the case of $\partial_u$:

$$
\sum_{\Gamma = S, \Omega_{ij}} \left\| [\Box_g, \partial_u] \phi \right\|_{L^2} \lesssim \left\| \chi_{r < \mu t} \langle r \rangle^{-\frac{1}{2}} r \tau_+^{-2} \chi_{r < \mu t} (\nabla^2 \phi, \langle r \rangle^{-1} \nabla \phi) \right\|_{L^2} + \epsilon \sup_{t_0 \leq t \leq t_1} \left\| \phi(t) \right\|_{C^1} + \left\| \nabla (\Box_g \phi) \right\|_{L^1}.
$$

**Proof of Estimate (109).** We treat the regions $r < \mu t$ and $\mu t < r$ separately.

**Case 1:** ($\text{Inside } r < \mu t$) We multiply estimate 107 by $2^{2/2k} \chi_{r \approx 2s} \chi_{r \approx 2s}$, integrate, then use $\gamma < \delta$ together with the inclusion $\langle r \rangle^{-\delta} r \tau_+^{-2} \chi_{r < \mu t} \Box_g \phi \lesssim 2^2 \langle r \rangle^{-\frac{1}{2}} r \tau_+^{-2} \chi_{r < \mu t} (\nabla^2 \phi, \langle r \rangle^{-1} \nabla \phi) \right\|_{L^2}$. For the source term we apply estimate 66 with $a = 1/2$. Combining all this:

$$
\sum_{\Gamma = S, \Omega_{ij}} \left\| \chi_{r < \mu t} [\Box_g, \Gamma] \phi \right\|_{L^2} \lesssim \left\| \langle r \rangle^{-\frac{1}{2}} r \tau_+^{-2} \chi_{r < \mu t} (\nabla^2 \phi, \langle r \rangle^{-1} \nabla \phi) \right\|_{L^2} + \left\| \chi_{r < \mu t} \Box_g \phi \right\|_{L^1}.
$$
In the exterior region $\mu t < r$ we multiply estimate (107) by $(r)^{\frac{\gamma}{2}} \tau_+ \tau_0 \chi_{\tau \approx 2t} \chi_{(r) \approx 2} \chi_{(\mu) \approx 2}$. Squaring, integrating and using the support property together with $(110)$, \( \square \) estimate (112) to establish the necessary inequalities in this case. In light of this discussion, the ellipticity with cutoffs. To resolve this issue, we will use a Klainerman-Sideris-type bound together with the elliptic estimate (108) to integrate in time since $t^{\frac{\gamma}{2}} \approx r^2$. Taking \( \sup \), then integrating the \( t \) weight finishes the proof of (109).

**Proof of Estimate (110).** This follows the same exact proof as the previous estimate. The improvement in the interior is a direct result of using the version of (107) with the better weight $\sqrt{\tau} \sqrt{\tau} \in L^2$ for the interior terms. □

### 6.2. Interior $L^2$-estimates for Two Derivatives

In order to establish our main estimates, the major obstacle that remains is to bound the error terms supported inside $r < \mu t$ on RHS(109) and RHS(110). This will be achieved in the next lemma by establishing some weighted $L^2$ bounds for \( \| \chi_{r < \mu t} (r)^{\frac{\gamma}{2}} \tau_+ \nabla^2 \phi \|_{L^2} \) with the exponent “a” depending on the vector field $\Gamma$. In the case $\Gamma = \partial_n$, we have $a = 1 - \gamma/2$ and it will suffice to commute with cutoffs and use some standard elliptic estimates. For the cases $\Gamma = S, \Omega_{ij}$, the $\tau_+^{\gamma}$ weight comes with an exponent of $a = 2 - \gamma/2$, a number which no longer allows us to commute with cutoffs. To resolve this issue, we will use a Klainerman-Sideris-type bound together with the elliptic estimate (109) to establish the necessary inequalities in this case. In light of this discussion, the ellipticity of the operator $P(t, x, \nabla_x)$ (see Corollary 2.11) is of fundamental importance for the proof of the next lemma. Consequently, all computations are in \((t, x^1)\) coordinates here.

**Lemma 6.7** (Klainerman-Sideris-type estimates for the interior). The following uniform bounds hold inside the region $I^{*}$:

1. (Bounds for $\nabla \partial_t \phi$)
   - \( \| \chi_{r < \mu t} (r)^{\frac{\gamma}{2}} \tau_+ \nabla \partial_t \phi \|_{\ell^2 L^2} \lesssim \epsilon \| \chi_{r < \frac{\mu t}{2}} \partial_t \phi \|_{\ell^2 \leq 1} \).
   - \( \| \chi_{r < \mu t} (r)^{-\frac{1}{2}} \tau_+ \nabla \partial_t \phi \|_{\ell^2 L^2} \lesssim \epsilon \| \chi_{r < \frac{\mu t}{2}} \partial_t \phi \|_{\ell^2 \leq 1} + \mu \| \chi_{r < \mu t} (r)^{-\frac{1}{2}} \tau_+ \nabla^2 \phi \|_{\ell^2 L^2} \).

2. (Bounds for $\nabla^2 \phi$)
   - \( \| \chi_{r < \mu t} (r)^{\frac{\gamma}{2}} \tau_+ \nabla^2 \phi \|_{\ell^2 L^2} \lesssim \epsilon \left( \| \chi_{r < \frac{\mu t}{2}} \partial_t \phi \|_{\ell^2 \leq 1} + \| \chi_{r < \frac{\mu t}{2}} \phi \|_{\ell^2 \leq 1} \right) + \| \nabla (\partial_t \phi) \|_{N} \).
   - \( \| \chi_{r < \mu t} (r)^{-\frac{1}{2}} \tau_+ \nabla^2 \phi \|_{\ell^2 L^2} \lesssim \epsilon \left( \sup_{t_0 \leq t < t_1} \| \phi(t) \|_{CE_1} + \| \chi_{r < \frac{\mu t}{2}} \phi \|_{\ell^2 \leq 1} \right) + \| \partial_t \phi \|_{N} \).

3. (Bounds for $\nabla^2 \phi$)
   - \( \| \chi_{r < \mu t} (r)^{-\frac{1}{2}} \tau_+ \nabla^2 \phi \|_{\ell^2 L^2} \lesssim \epsilon \left( \| \chi_{r < \frac{\mu t}{2}} \partial_t \phi \|_{\ell^2 \leq 1} + \| \chi_{r < \frac{\mu t}{2}} \phi \|_{\ell^2 \leq 1} \right) + \| \nabla (\partial_t \phi) \|_{N} \).
   - \( \| \chi_{r < \mu t} (r)^{-\frac{1}{2}} \tau_+ \nabla^2 \phi \|_{\ell^2 L^2} \lesssim \epsilon \left( \sup_{t_0 \leq t < t_1} \| \phi(t) \|_{CE_1} + \| \chi_{r < \frac{\mu t}{2}} \phi \|_{\ell^2 \leq 1} \right) + \| \partial_t \phi \|_{N} \).

**Proof of Lemma Part 1:** (Bounds for $\nabla \partial_t \phi$) For estimate (112) we split the gain $t^{-\frac{\gamma}{2}}$: half goes to $t^{-\frac{\gamma}{2}} \lesssim \epsilon$ and the other half is used in the inclusion $t^{-\frac{\gamma}{2}} \lesssim \epsilon^2 L^2 \subseteq \ell^2 L^2$. The bound follows by using the definition of the norms and the support property. For estimate (113) we start with the pointwise inequality:

\[|\partial_t \phi| \lesssim |t^{-1} \partial_t S \phi| + (r/t) |t_0 \partial_t \phi|, \]

which is valid inside $r < \frac{1}{2} t$. Applying this to $\partial_t \phi$ and $\nabla x \phi$ yields, respectively:

\[|\partial_t^2 \phi| \lesssim |t^{-1} \partial_t S \phi| + |t^{-1} \partial_t \phi| + (r/t) |t_0 \partial_t \phi|, \]
\[|\partial_t \nabla x \phi| \lesssim |t^{-1} \nabla x S \phi| + |t^{-1} \nabla x \phi| + (r/t) |t_0 \partial_t \nabla x \phi|. \]
Multiplying these two bounds by \( \langle r \rangle^{-\frac{1}{2} + \frac{\mu}{r} + \frac{2}{3} L^2} \) \( \chi_{r<\mu t} \), squaring, integrating and using the support property \( (r/t) < \mu \) we get:

\[
\begin{align*}
(119) \quad & \| \chi_{r<\mu t} \langle r \rangle^{-\frac{1}{2} + \frac{\mu}{r} + \frac{2}{3} L^2} \partial_t \phi \|_{L^2} \lesssim \epsilon \| \chi_{r<\mu t} \|_{L^2} + \mu \| \chi_{r<\mu t} \langle r \rangle^{-\frac{1}{2} + \frac{\mu}{r} + \frac{2}{3} L^2} \nabla_x \partial_t \phi \|_{L^2}, \\
(120) \quad & \| \chi_{r<\mu t} \langle r \rangle^{-\frac{1}{2} + \frac{\mu}{r} + \frac{2}{3} L^2} \nabla_x \partial_t \phi \|_{L^2} \lesssim \epsilon \| \chi_{r<\mu t} \|_{L^2} + \mu \| \chi_{r<\mu t} \langle r \rangle^{-\frac{1}{2} + \frac{\mu}{r} + \frac{2}{3} L^2} \nabla_x \phi \|_{L^2},
\end{align*}
\]

where we have split the gain \( \frac{\mu}{r} - \frac{2}{3} L^2 \) as in the previous proof to produce the \( \epsilon \cdot \ell^2 \| L^2 \) terms. By definition \( 4.1.a \), the constant \( \mu \) is small enough that we can add (119) + (120), bootstrap the term \( \nabla_x \partial_t \phi \) and get estimate (113).

**Part 2: (Bounds for \( \nabla_x^2 \phi \))** For estimate (114) we let \( P(t,x,\nabla_x) \) be as in corollary 4.1II. By that result, the operator \( P \) is uniformly elliptic in \( (t,x') \) coordinates. Thus, commuting with \( \nabla_x^2 \) and using standard elliptic estimates:

\[
\| \chi_{r<\mu t} \langle r \rangle^{-\frac{1}{2} + \frac{\mu}{r} + \frac{2}{3} L^2} \nabla_x^2 \phi \|_{L^2} \lesssim \| \chi_{r<\mu t} \langle r \rangle^{-\frac{1}{2} + \frac{\mu}{r} + \frac{2}{3} L^2} P \|_{L^2} + \| \chi_{r<\mu t} \langle r \rangle^{-\frac{1}{2} + \frac{\mu}{r} + \frac{2}{3} L^2} (r^{-1} \nabla_x, r^{-2} \phi) \|_{L^2} \lesssim \| \chi_{r<\mu t} \|_{L^2} + \| \chi_{r<\mu t} \phi \|_{L^2},
\]

where \( \mu^{-1} \) comes from terms where derivatives land on the cutoff \( \chi_{r<\mu t} \). Note also that we used \( \mu^{-1} \cdot \frac{\mu}{r} L^2 \lesssim \epsilon \) on the last line which follows since definition (14) implies \( \epsilon \ll \mu \). Applying estimate (122) to the first term on the RHS above we get:

\[
\begin{align*}
(121) \quad & \| \chi_{r<\mu t} \langle r \rangle^{-\frac{1}{2} + \frac{\mu}{r} + \frac{2}{3} L^2} P \|_{L^2} \lesssim \| \chi_{r<\mu t} \langle r \rangle^{-\frac{1}{2} + \frac{\mu}{r} + \frac{2}{3} L^2} \nabla_x \phi \|_{L^2} + \| \chi_{r<\mu t} \langle r \rangle^{-\frac{1}{2} + \frac{\mu}{r} + \frac{2}{3} L^2} (r^{-1} \nabla_x, r^{-2} \phi) \|_{L^2} + \epsilon \| \chi_{r<\mu t} \|_{L^2} + \| \chi_{r<\mu t} \phi \|_{L^2},
\end{align*}
\]

on the last line we used (115) and the gain \( \frac{\mu}{r} - \frac{2}{3} L^2 \) to put the \( \nabla_x \phi \) term in the form above. For all other terms we split \( t^{-\frac{1}{2}} \) as we did before and use the definition of the norms to finish the proof of (114).

For estimate (115), since the exponent of \( \langle r \rangle^{-\frac{1}{2} + \frac{\mu}{r} + \frac{2}{3} L^2} \) is above the threshold \( a = \frac{1}{2} \), we can apply the weighted \( L^2 \) estimate (130). This introduces a term supported where \( \mu r \) we need to control. After an application of (32) we get:

\[
\begin{align*}
(122) \quad & \| \chi_{r<\mu t} \langle r \rangle^{-\frac{1}{2} + \frac{\mu}{r} + \frac{2}{3} L^2} \nabla_x \phi \|_{L^2} \lesssim \| \chi_{r<\mu t} \langle r \rangle^{-\frac{1}{2} + \frac{\mu}{r} + \frac{2}{3} L^2} \nabla_x \partial_t \phi \|_{L^2} + \| \chi_{r<\mu t} \langle r \rangle^{-\frac{1}{2} + \frac{\mu}{r} + \frac{2}{3} L^2} \nabla_x (\nabla_x \phi) \|_{L^2}.
\end{align*}
\]

Let’s start by bounding all the terms supported inside \( r < \mu \). For the source term we use the Hardy estimate (16):

\[
\| \chi_{r<\mu t} \langle r \rangle^{-\frac{1}{2} + \frac{\mu}{r} + \frac{2}{3} L^2} \nabla_x \phi \|_{L^2} \lesssim \| \chi_{r<\mu t} \langle r \rangle^{-\frac{1}{2} + \frac{\mu}{r} + \frac{2}{3} L^2} \nabla_x \phi \|_{L^2}.
\]

For combinations of the form \( \nabla \partial_t \phi \) we apply estimate (113). For \( \langle r/t \rangle \gamma \partial_x \phi \) we use \( \langle r/t \rangle^{-\gamma} \lesssim 1 \) and \( t^{-\gamma} \lesssim \epsilon \) to get:

\[
\| \chi_{r<\mu t} \langle r \rangle^{-\frac{1}{2} + \frac{\mu}{r} + \frac{2}{3} L^2} \nabla_x \phi \|_{L^2} \lesssim \epsilon \| \chi_{r<\mu t} \langle r \rangle^{-\frac{1}{2} + \frac{\mu}{r} + \frac{2}{3} L^2} \nabla_x \phi \|_{L^2},
\]

and bootstrap this term to LHS (121). For the \( \partial_t \phi \) term we multiply (112) \times \langle r \rangle^{-\frac{1}{2} + \frac{\mu}{r} + \frac{2}{3} L^2} \chi_{r<\mu t} \), square, integrate, and use \( t^{-\frac{1}{2}} \lesssim \epsilon \) together with the inclusion \( t^{-\frac{1}{2}} \| \nabla_x \partial_t \phi \|_{L^2} \lesssim \ell \| \nabla_x \partial_t \phi \|_{L^2} \) to get:

\[
\begin{align*}
(123) \quad & \| \chi_{r<\mu t} \langle r \rangle^{-\frac{1}{2} + \frac{\mu}{r} + \frac{2}{3} L^2} \partial_t \phi \|_{L^2} \lesssim \| \chi_{r<\mu t} \langle r \rangle^{-\frac{1}{2} + \frac{\mu}{r} + \frac{2}{3} L^2} \nabla_x \phi \|_{L^2} + \epsilon \| \chi_{r<\mu t} \|_{L^2},
\end{align*}
\]

This takes care of all the terms supported inside \( r < \mu \) on the last two lines of (121). For the term supported where \( \mu r < r \), we go back to Bondi derivatives via \( \partial_t = \partial_t + u_i \partial_i \) and apply the gradient bounds (2):

\[
\begin{align*}
& \| \chi_{r<\mu t} \langle r \rangle^{-\frac{1}{2} + \frac{\mu}{r} + \frac{2}{3} L^2} \partial_t \phi \|_{L^2} \lesssim \| \chi_{r<\mu t} \langle r \rangle^{-\frac{1}{2} + \frac{\mu}{r} + \frac{2}{3} L^2} \nabla_x \partial_t \phi \|_{L^2} + \epsilon \| \phi \|_{CE_1} \lesssim \epsilon \| \phi \|_{CE_1} + \| \nabla_x \phi \|_{N_1},
\end{align*}
\]

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the last inequality follows from using the Klainerman-Sideris identity \((108)\) on the first term and using the gain \(r^{-\frac{3}{2}}\) as we’ve done above.

**Part 3:** (Bounds for \(\nabla^2_\phi\)) Estimate \((110)\) follows by adding \((112) + (113)\) and using the fact that \(\mu\) a ensures that \(\mu\) is small enough to bootstrap the errors. Estimate \((114)\) follows by adding \((113) + (115)\) and again using the fact that \(\mu\) was chosen small enough to bootstrap the ensuing errors.

6.3. **Proof of the Main Estimates.** We now prove Theorem 6.1.

**Proof.** Proof of \((105)\) We commute the equation with the vector field \(\partial_u\) then use estimate \((9)\) and apply the commutator bound \((110)\) to get:

\[
\sup_{t_0 \leq t \leq t_1} \| \partial_u \phi(t) \|_{CE} + \| \partial_u \phi \|_{\ell^\infty S} \lesssim \| \chi_{r < \mu} \langle r \rangle^{-\frac{3}{2}} \tau_+^{1 - \frac{3}{2}} \langle \nabla^2 \phi \rangle \langle r \rangle^{-1} \nabla \phi \|_{\ell^\infty L^2} + \epsilon \sup_{t_0 \leq t \leq t_1} \| \phi(t) \|_{CE} + \| \partial_u \phi(t_0) \|_{CE} + \| \nabla (\square_g \phi) \|_{\ell^1_t N}.
\]

For the interior terms with one derivative we drop the weight \(\langle r \rangle^{-1}\) and use \(t^{-\frac{3}{2}} \lesssim \epsilon\) together with the inclusion \(\langle r \rangle^{-\frac{3}{2}} t^{-\frac{3}{2}} \ell^\infty L^2 \subseteq \ell^{\infty} L^2\). For the interior terms with two derivatives we apply \((110)\) directly. Using these two bounds in succession on the first term on RHS \((123)\) then yields:

\[
\| \chi_{r < \mu} \langle r \rangle^{-\frac{3}{2}} \tau_+^{1 - \frac{3}{2}} \langle \nabla^2 \phi \rangle \langle r \rangle^{-1} \nabla \phi \|_{\ell^\infty L^2} \lesssim \epsilon \left( \| \chi_{r < \mu} \partial_t \phi \|_{\ell^\infty LE} + \| \chi_{r < \mu} \phi \|_{\ell^\infty LE} \right).
\]

Bootstrapping the first interior error term above to LHS \((123)\) finishes the proof.

**Proof.** Proof of \((106)\) We commute the equation with the vector fields \(S, \Omega_{ij}\), use \((9)\), then apply \((109)\) to the commutator to get:

\[
\sup_{t_0 \leq t \leq t_1} \sum_{\Gamma = S, \Omega_{ij}} \| \Gamma \phi(t) \|_{CE} + \| \Gamma \phi \|_{\ell^\infty S} \lesssim \| \langle r \rangle^{-\frac{3}{2}} \tau_+^{2 - \frac{3}{2}} \chi_{r < \mu} \langle \nabla^2 \phi \rangle \langle r \rangle^{-1} \nabla \phi \|_{L^2} + \epsilon \sup_{t_0 \leq t \leq t_1} \| \phi(t) \|_{CE} + \sum_{\Gamma = S, \Omega_{ij}} \| \Gamma \phi(t_0) \|_{CE} + \| \square_g \phi \|_{\ell^1_t N_1}.
\]

We multiply all the interior error terms above by \(\langle r \rangle^{\frac{3}{2}}\). For the terms with only one derivative the exponent of \(\langle r \rangle^3\) is now above the \(a = 3/2\) threshold. This allows us to use the Hardy estimate \((60)\) and get:

\[
\| \langle r \rangle^{-\frac{3}{2}} \tau_+^{2 - \frac{3}{2}} \chi_{r < \mu} \nabla \phi \|_{L^2} \lesssim \| \langle r \rangle^{-\frac{3}{2}} \tau_+^{2 - \frac{3}{2}} \chi_{r < \mu} \nabla_x \nabla \phi \|_{L^2}.
\]

All interior terms on RHS \((124)\) are now of the form LHS \((117)\). An application of this estimate finishes the proof.

7. **Global \(L^\infty\) Decay**

In this section we prove the pointwise bound \((111)\). Let’s begin by showing some preliminary estimates.

**Lemma 7.1** (Preliminary estimates). For test functions \(\phi\) the following uniform bounds hold:

1) (Global \(L^\infty\)-estimate)

\[
\| \langle r \rangle^{\frac{3}{2}} \tau_0^{\frac{3}{2}} \phi \|_{L^\infty} \lesssim \sum_{k + |J| \leq 2} \| \langle \langle r \rangle \tau_0 \phi \rangle^k (\langle r \rangle \partial_x )^J \phi \|_{L^2}.
\]

2) (Interior \(L^\infty\) estimate) Assume additionally \(\phi\) is supported in \(r < \frac{1}{2} t\). Then one has:

\[
\| \phi \|_{L^\infty} \lesssim \| \langle r \rangle^{\frac{3}{2}} \langle \nabla^2 \phi \rangle \langle r \rangle^{-1} \nabla \phi \|_{\ell^\infty L^2}.
\]

3) (Average to uniform bounds via scalings) For all \(a \in \mathbb{R}\) and \(T > 0\):

\[
\sup_{0 \leq t \leq T} \| \tau_+^a \phi(t) \|_{L^2} \lesssim \| r^a \phi(0) \|_{L^2} + \| \tau_+^{-\frac{3}{2} a} (\phi, r \nabla \phi, S \phi) \|_{L^2[0,T]}.
\]

4) (Trace inequality)

\[
\sup_{0 \leq t \leq T} \| \tau_+^{\frac{3}{2}} \tau^\frac{3}{2}_0 \phi(t) \|_{L^2} \lesssim \| S f \|_{L^2[0,T]} + \| f \|_{L^2[0,T]}.
\]
Proof. Step 1: (Proof of (126)) Inside the set $r \leq 1$ this follows from the standard $L^\infty - L^2$ Sobolev estimate. For the complement, it suffices to consider the region $\frac{1}{2} < r < \frac{3}{2}$ as the remainder is easier to handle because we have $u = t - r$ there. Using dyadic cutoffs we may assume $\phi$ is supported where $r_- \approx 2^k$ and $r \approx 2^l$. Using angular sector cutoffs in the $x$ variable we may further assume without loss of generality that $\phi$ is supported in a $\frac{\pi}{4}$ wedge about the $x^1$ axis. Now introduce new variables on $t = \text{const}$:

$$y^1 = 2^{-k}u, \quad y^2 = 2^{-j}x^2, \quad y^3 = 2^{-j}x^3.$$ 

There exists vector fields $e_a$, $\alpha = 0, 1, 2, 3$ such that $\partial_y^\alpha \big|_{t=\text{const}} = \sum_a c_a^\alpha e_a$ where $c_a^\alpha$ are uniformly bounded and such that:

$$2^{\frac{1}{2}k+j} \| e_1^f \phi \|_{L^2(dy)} \lesssim \sum_{k+j \leq 2} \| (r \tau_0 \partial_u)^k (r \partial_x)^j \phi \|_{L^2(dx)}.$$ 

Estimate (126) follows from this last line by concatenating the Sobolev embeddings.

Step 2: (Proof of (127)) Once again by the $L^\infty - L^2$ Sobolev estimate it suffices to prove the result outside $r \leq 1$. By the support property we have $\tau_0 \approx 1$, therefore applying estimate (126) to $r^{-\frac{1}{2}} \chi_{r \approx 2}\phi$, taking $\sup_k$ and using the fact that $\partial_u, \partial_x$ are bounded linear combinations of $\partial_i, \partial_y$ derivatives yields the claim.

Step 3: (Proof of (128)) Integrating the time derivative of $(r + t)^{2a} \phi^2$ over $0 < t < T$ we have:

$$(T + r)^{2a} \phi^2(T) = r^{2a} \phi^2(0) + 2a \int_0^T (T + r)^{2a-1} \phi^2 dt + 2 \int_0^T (T + r)^{2a} \phi \frac{\partial \phi}{\partial t} dt.$$ 

Using Cauchy-Schwarz on the last RHS term and integrating in $x$ over $0 < r < \infty$ we have:

$$\| \tau_+^a \phi(T) \|_{L^2} \lesssim \| r^n \phi(0) \|_{L^2_x} + \| \tau_+^{a-\frac{1}{2}} (\phi, r \nabla \phi, S \phi) \|_{L^2_x[0,T]}.$$ 

This yields (128).

Step 4: (Proof of (129)) Follows by integrating along the integral curves of $S$ and applying Cauchy-Schwarz. □

Now we demonstrate the main $L^\infty$ bound.

Proof of estimate (11). We estimate the timelike and null/space like regions separately. Let $\mu$ be as in definition (11).a.

Step 1: (Estimate for $r > \mu$) Applying (126) to $\chi_{r > \mu} \phi$ followed by (108):

$$\| \chi_{r > \mu} r^{\frac{1}{2}} \tau_0^\frac{1}{2} \phi(t) \|_{L^\infty} \lesssim \sum_{k+j \leq 2} \| (r \tau_0 \partial_u)^k (r \partial_x)^j \phi(t) \|_{L^2} \lesssim \| \phi(t) \|_{CE_\ell} + \| r^2 \tau_0 \partial_y \phi(t) \|_{\ell_1^2 L^2}.$$ 

We take $\sup_t$ then apply the bound (10) to control $\| \sup_{0 \leq t \leq T} \| \cdot \|_{CE_\ell}$. For the source term we use the trace estimate (129).

Step 2: (Estimate for $r < \mu$) Let $\chi_{r < \mu}$ be a smooth cutoff to the region $r < \mu$. Applying estimate (127) to $\chi_{r < \mu} \tau_+^a \phi$ we get:

$$\| \chi_{r < \mu} r^{\frac{1}{2}} \tau_0^\frac{1}{2} \phi(t) \|_{L^\infty} \lesssim \sup_{0 \leq t \leq T} \| \chi_{r < \mu} r^{\frac{1}{2}} \phi(t) \|_{L^\infty} \lesssim \| \chi_{r < \mu} \phi(t) \|_{H^\infty \ell_1^2},$$ 

where $\chi_{r < \mu}$ is a smooth cutoff with slightly larger support. For the undifferentiated terms we apply (128) with $\alpha = 3/2$ to $\langle r \rangle^{-\frac{3}{2}} \chi_{r < \mu} \phi$ to produce:

$$\| \chi_{r < \mu} r^{\frac{1}{2}} \phi(t) \|_{L^2} \lesssim \| \phi(t) \|_{CE_\ell} + \| \chi_{r < \mu} \phi(t) \|_{\ell_1^1 L^2_1}.$$ 

In remains for us to control the rest of the terms on RHS (130). We apply the Hardy estimate (66) with $\alpha = \frac{3}{2}$ for terms with one derivative. Thus it suffices to estimate the terms with two derivatives. For this we claim:

$$\| \chi_{r < \mu} r^{\frac{1}{2}} \phi(t) \|_{L^2} \lesssim \| \phi(t) \|_{CE_\ell} + \| \tau_0^\frac{1}{2} (r \partial_y \phi(t)) \|_{L^2}.$$ 

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By using a similar proof to (117), this estimate can be further reduced to proving:

\[(133) \quad \| \tilde{\chi}_{\tau < \mu} r^2(t) \phi(t) \|_{L^2_x} \lesssim \| \phi(t) \|_{C^\infty_x} + \| \chi_{\tau < \mu} r^2(t) \phi(t) \|_{L^2_x} ,\]

\[(134) \quad \| \chi_{\tau < \mu} r^2(t) \phi(t) \|_{L^2_x} \lesssim \| \phi(t) \|_{C^\infty_x} + \| \phi(t) \|_{C^\infty_x} + \| \tilde{\chi}_{\tau < \mu} r^2(t) \phi(t) \|_{L^2_x} + \| \chi_{\tau < \mu} r^2(t) \phi(t) \|_{L^2_x} .\]

Estimate (133) follows by a slightly simpler version of the proof of (132) and using the support property. Likewise, estimate (134) follows by applying the elliptic estimate (136) with \( a = \frac{1}{2} \) then following a similar argument to (115) and using the support property. Taking an appropriate linear combination of (133) and (134) then using the smallness of \( \mu \) to bootstrap errors proves (133). To get (11) from this, we take sup in \( t \) and use the bound (10) to control \( \sup_{0 \leq t \leq T} \| \cdot \|_{C^\infty_x} + \| \chi_{\tau < \mu} \phi \|_{L^\infty_x L^1_t} \) along with the trace estimate (129) for the source term. This finishes the proof. 

\[\square\]

8. Appendix: Weighted \( L^2 \)-Elliptic Estimates

Our main result in this section is the following:

**Theorem 8.1** (Global elliptic estimate). Let the operator \( P(t,x,\nabla_x) = \partial_t h^{ij} \partial_j \) with \( h^{ij} = d^{ij} g^{ij} \) be uniformly elliptic. Suppose that for all \( t > 0 \), the \( h^{ij} \) satisfy the uniform bounds:

\[(135) \quad \| \langle r \rangle^{\alpha} (h^{ij} - \delta^{ij}) \| \lesssim \langle r \rangle^{-\delta} .\]

Then, for all \( t > 0 \) the operator \( P \) satisfies the fixed-time estimates:

\[(136) \quad \| \langle r \rangle^a \nabla_x^2 \phi(t) \|_{L^2_x} + \| \langle r \rangle^{a-1} \nabla_x \phi(t) \|_{L^2_x} \lesssim \| \langle r \rangle^a P \phi(t) \|_{L^2_x} , \quad -\frac{1}{2} < a < \frac{3}{2} ;\]

where the implicit constants are independent of \( t \).

**Proof.** Let \( \Delta, \Delta^{-1} \) denote the standard 3D Laplacian and its inverse, respectively. Write \( P \phi = F \). We approximately solve for \( F \) in terms of a Neumann series:

\[\tilde{\phi} = \Delta^{-1} \sum_{i=0}^k R^i F , \quad R = I - P \Delta^{-1} .\]

Then we have: \( P(\tilde{\phi} - \phi) = R^{k+1} F \). Therefore, setting \( L_{x,a}^{2} \) for the norms on line (136) it suffices to show:

\[(137) \quad \nabla^2 \Delta^{-1} : L_{x,a}^{2} \to L_{x,a}^{2} , \quad \langle r \rangle^{-1} \nabla \Delta^{-1} : L_{x,a}^{2} \to L_{x,a}^{2} , \quad R : L_{x,a}^{2} \to L_{x,a}^{2+\delta} ,\]

for the range \( \frac{1}{2} < a < a + \delta < \frac{3}{2} \), followed by the non-perturbative estimate:

\[(138) \quad \| \langle r \rangle^a \nabla_x^2 \phi(t) \|_{L^2_x} + \| \nabla_x \phi(t) \|_{L^2_x} \lesssim \| \langle r \rangle^a P \phi(t) \|_{L^2_x} .\]

\[\square\]

**Proof of (137).** We decompose into dyadic scales \( |x| \sim 2^i , |y| \sim 2^j \) with \( |x| \leq 1 \) when \( i = 0 \) since the weights are non-singular. **Step 1:** \((\nabla^2 \Delta^{-1} : L_{x,a}^{2} \to L_{x,a}^{2} \text{ is bounded})\) To establish this it suffices to show:

\[(139) \quad \sum_{i,j} \int \int_{|x| \sim 2^i , |y| \sim 2^j} F(x) K_1(x-y) G(y) \, dxdy \lesssim \| F \|_{L^{2,a}} \| G \|_{L^{2,a}} ,\]

where \( K_1 \) is the convolution kernel for \( \nabla^2 \Delta^{-1} \). We break up the proof into cases:

**Case 1:** \(|i-j| = O(1)| \) The operator defined above is a singular integral operator. In this case the weights \( 2^{-a} \approx 1 \) and \( 2^{-ai} \approx 1 \) balance since they are both approximately of size one. By Cauchy-Schwarz:

\[\sum_{i+j = O(1)} \int \int_{|x| \sim 2^i , |y| \sim 2^j} F(x) K_1(x-y) G(y) \, dxdy \lesssim \sum_{i,j} \| \chi_i F \|_{L^{2,a}} \| \chi_j G \|_{L^{2,a}} ,\]

with \( \chi_i, \chi_j \) smooth cutoff functions supported where \( |x| \sim 2^i , |y| \sim 2^j \), respectively.

**Case 2:** \(|i > j + c| \) We now have \( K_1(x-y) = O(|x|^{-3}) \) and since convolution with an \( L^1 \) function is a bounded operator in any \( L^p \) space with \( p \geq 1 \):

\[\sum_{i>j+c} \int \int_{|x| \sim 2^i , |y| \sim 2^j} F(x) K_1(x-y) G(y) \, dxdy \lesssim \sum_{i>j+c} 2^{-(i-j)/2} 2^{a(i-j)} \| \chi_i F \|_{L^{2,a}} \| \chi_j G \|_{L^{2,a}} ,\]

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which forces $-\frac{3}{2} + a \leq 0$.

**Case 3:** ($j > i + c$) by switching the roles of $x$ and $y$ and using the same argument as case 2:

\[
\sum_{j > i + c} \left| \iint_{|x| \sim 2^i, |y| \sim 2^j} F(x)K_1(x - y)G(y) \, dxdy \right| \lesssim \sum_{j > i + c} 2^{\frac{3}{2}(i-j)} \cdot 2^{a(i-j)} \| \chi_i F \|_{L^2} \| \chi_j G \|_{L^{2,a}} ,
\]

which is convergent for $-\frac{3}{2} + a < 0$. This proves (139).

**Step 2:** ($\langle x \rangle^{-1} \nabla \Delta^{-1} : L^2_a \rightarrow L^2_a$ is bounded) We again aim to show:

\[
\sum_{i,j} \left| \iint_{|x| \sim 2^i, |y| \sim 2^j} F(x)K_2(x - y)G(y) \, dxdy \right| \lesssim \| F \|_{L^2} \| G \|_{L^{2,a}} ,
\]

with the kernel $K_2(x, y)$ the convolution kernel for $\langle r \rangle^{-1} \nabla \Delta^{-1}$.

**Case 1:** ($|i - j| = O(1)$) Here we have $K_2(x, y) = O(\langle x \rangle^{-1} |x - y|^{-2})$. By the Hardy-Littlewood-Sobolev inequality:

\[
\sum_{i,j = O(1)} \left| \iint_{|x| \sim 2^i, |y| \sim 2^j} F(x)K_2(x - y)G(y) \, dxdy \right| \lesssim \sum_{i,j = O(1)} \left| \iint_{|x| \sim 2^i, |y| \sim 2^j} \frac{F(x)G(y)}{|x| |y|^2} \, dxdy \right|
\]

\[
\lesssim \sum_{i,j} 2^{-i} \cdot 2^{\frac{3}{2}i} \cdot 2^{\frac{1}{2}j} \| \chi_i F \|_{L^2} \| \chi_j G \|_{L^2} \lesssim \sum_{i,j} \| \chi_i F \|_{L^2} \| \chi_j G \|_{L^{2,a}} ,
\]

where we’ve used $2^{-i} \approx 2^{\frac{3}{2}i} \cdot 2^{\frac{1}{2}j} \approx 1$ on the last line.

**Case 2:** ($i > j + c$) Here we have $|K_2(x, y)| = O(\langle x \rangle^{-1} |x|^{-2})$. Therefore:

\[
\sum_{i > j + c} \left| \iint_{|x| \sim 2^i, |y| \sim 2^j} F(x)K_2(x - y)G(y) \, dxdy \right| \lesssim \sum_{i > j + c} 2^{-\frac{1}{2}i} \cdot 2^{-i} \cdot 2^{a(i-j)} \| \chi_i F \|_{L^2} \| \chi_j G \|_{L^{2,a}} ,
\]

and since $i > j + c$ the extra $2^{-\frac{1}{2}i}$ helps us get $-\frac{3}{2} + a < 0$ once again.

**Case 3:** ($j > i + c$) Here we have $|K_2(x, y)| = O(\langle x \rangle^{-1} |y|^{-2})$. Hence:

\[
\sum_{j > i + c} \left| \iint_{|x| \sim 2^i, |y| \sim 2^j} F(x)K_2(x - y)G(y) \, dxdy \right| \lesssim \sum_{j > i + c} 2^{\frac{3}{2}i} \cdot 2^{i-j} \cdot 2^{a(i-j)} \| \chi_i F \|_{L^2} \| \chi_j G \|_{L^{2,a}} ,
\]

and since $j > i + c$ the extra $2^{\frac{3}{2}i}$ gives us the restriction $-\frac{1}{2} - a < 0$.

**Step 3:** ($R : L^2_a \rightarrow L^2_{a+\delta}$ is bounded) We use the expansion $P - \Delta = \partial_i (h^{ij} - \delta^{ij}) \partial_j$ and by (133) the coefficients obey the decay bounds $| (h^{ij} - \delta^{ij}) | \lesssim \langle r \rangle^{-\delta}$ and $| \partial_r (h^{ij} - \delta^{ij}) | \lesssim \langle r \rangle^{-1-\delta}$. This observation together with the results above finish the proof.

**Proof of (133).** Let $D$ denote the Levi-Civita connection for $h$ and let $dV_h$ be its volume form. We have the estimate:

\[
\int_{\mathbb{R}^3} D^j \phi D_i \phi \, dV_h \lesssim \int_{\mathbb{R}^3} \langle r \rangle^{2} |P \phi|^2 \, dV_h ,
\]

which follows from Green’s identity $-\int_{\mathbb{R}^3} D^j \phi D_i \phi \, dV_h = \int_{\mathbb{R}^3} P \phi \cdot \phi \, dV_h$ by taking absolute value, applying Young’s inequality and using the Hardy estimate $\int_{\mathbb{R}^3} |r^{-1} \phi|^2 \, dV_h \lesssim \int_{\mathbb{R}^3} |D \phi|^2 \, dV_h$. To prove the estimate for two derivatives we integrate by parts twice then take absolute value, apply (133) together with the Hardy estimate (111) and Young’s inequality to produce:

\[
\int_{\mathbb{R}^3} \langle r \rangle^{2} (D_i D_j \phi)(D^i D^j \phi) \, dV_h \lesssim \int_{\mathbb{R}^3} \langle r \rangle^{2} |P \phi|^2 + \langle r \rangle |P \phi||D \phi| + \langle r \rangle^{-1-\delta} |\phi||D \phi| + D_i \phi D^i \phi) \, dV_h
\]

\[
\lesssim \int_{\mathbb{R}^3} \langle r \rangle^{2} |P \phi|^2 \, dV_h .
\]

This finishes the proof of (133).
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