Is Nature Generic?

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Abstract
An introductory guide to mathematical cosmology is given focusing on the issue of the genericity of various important results which have been obtained during the last thirty or so years. Some of the unsolved problems along with certain new and potentially powerful methods which may be used for future progress are also given from a unified perspective.

1 Introduction
We live in space and time. For the cosmologist this fact relates to some fundamental and unresolved issues:

- How was our spacetime created?

- What is the shape of our space? Was it always the same? What are the possible ‘admissible’ shapes for our physical space?

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• Was our spacetime so ‘simple’ in the past or more complex? What about in the future?

• What was the structure of the ‘early universe’?

and so on. These issues, when translated into a suitable mathematical language, do in fact drive most of the current research in mathematical and theoretical cosmology.

In this paper we lay the foundations of modern mathematical cosmology in a manner suitable for the nonspecialist or a graduate student who wishes to have some initial orientation in his/her attempts to embark on research in this fascinating field of Science which lies in the interface between Applied Mathematics and Theoretical Physics. We have tried to present (an outline of the elements of) mathematical cosmology from a very broad perspective suitable for many readers and hope that even experts who work in one or more of the many modern branches of cosmology will find here some points of interest.

In the next Section we present the basic principles of cosmological modelling. Modelling the universe presents some new challenges for the applied mathematician or theoretical physicist different from those in other areas of the mathematical modelling of physical phenomena. Section 3 introduces and discusses the fundamental notion of a cosmology. Section 4 gives an overview of the basic unsolved problems of mathematical cosmology and the broad lines of attack that have been and are still being used by different research groups as well as some new and potentially efficient mathematical methods which could powerfully augment the successful treatment of the cosmological problem. Conclusions and future prospects are given in Section 5. Although almost no references are given in the text, the Bibliography presents some very basic items which are meant to serve as a useful entrance to the literature of this vast and truly exciting subject.

2 Principles of cosmological modelling

Modelling the universe, as opposed to that of other physical systems, is an involved and unique process different in nature and scope from other modelling in mathematical physics. There are two basic steps in the process, one we may call the theoretical step (items 1-3 below) and secondly the
observational step (item 4 below). These two steps comprise in turn 3+1
basic features:

1. A (cosmological) spacetime
2. A theory of gravity
3. A collection of matter fields
4. The process of confronting the results of suitable combination(s) and
analyses of 1-3 with the unique observed universe

We can loosely define a *cosmology* as the result of the appropriate combination of the features 1-4 above. The unifying principle that ties the basic features 1-3 together to form what we call a *cosmological model* is the *Action Principle*. Let us first consider in some detail the three most basic constituents of a cosmology.

### 2.1 Spacetimes

There is a basic cosmological hierarchy of spacetimes according to the degree of exact symmetry involved.

- Isotropic (Friedmann-Robertson-Walker) spacetimes
- Homogeneous (Bianchi) nontilted spacetimes
- Homogeneous (Bianchi) tilted spacetimes
- Inhomogeneous $G_2$ spacetimes
- Inhomogeneous $G_1$ spacetimes
- Generic spacetimes

Bianchi is that family of homogeneous but anisotropic spacetimes first classified by the Italian geometer L. Bianchi according to the underlined Lie algebra in nine types $I, \ldots IX$ and two classes $A, B$. This is the most general family of spacetimes for which the Einstein field equations reduced to ordinary differential equations since the space dependence of the metric derivatives are suppressed. Here, $G_i, i = 1, 2$ means the group of symmetries ($i$
indicates the number of Killing vectors) of the underline spacetime manifold. The group $G_2$ is larger than $G_1$ and consequently the spacetimes in the category four above are more symmetric than those in category five. It can be shown that, in a certain sense which can be made precise, spacetimes in a given category above are contained in the next category as special cases. Therefore we have a list of increasing generality (top to bottom) or genericity and the first families in the list may not be considered as realistic candidates for the actual universe as they all contain (or are constructed through the use of) exact symmetries. However, they are very important as toy models as well as simpler cases which may contain the seed of the true dynamics of the more generic (but essentially more difficult to handle mathematically) spaces. The ones without any symmetry are in the last category, generic spacetimes, while those with maximal symmetry are the isotropic spaces. The latter are the most common spacetimes used in cosmology today.

The first five families of spacetimes are basically formed by having a group of transformations acting in some way on the spacetime manifold such that its orbits essentially create the underline point set (the action of the group on the manifold is then called transitive). The dimension of this group as well as the manner it acts on the manifold are responsible for the wide variety of cosmological spacetimes. The simplest ones are the isotropic spacetimes whereas generic spaces are extremely difficult to handle.

## 2.2 Theories of gravity

Since the realization that, under certain assumptions, general relativity leads to singularities and consequently may not correctly or adequately describe the "observed" features of the universe at very small distances or very high energies, there has been an endless process of constructing new theories which incorporate gravity but extend general relativity in many different ways. Here is an incomplete list:

1. General relativity (GR)
2. Higher derivative gravity theories (HDG)
3. Scalar-tensor theories (ST)
4. Supergravity theory
5. String theories
As we shall see, one of the basic problems in cosmology is how to figure out which theory of gravity may be the most suitable for describing the universe at its early stages of evolution. Many cosmologists believe that perhaps some variant of string theory must be the final word, but choosing a gravity theory for such a purpose certainly involves many different and interrelated issues. We consider this problem in more detail below.

The basic method used to construct and compare all these different theories is the Action Principle, familiar from Classical Mechanics we learn as students. This principle forms the basis of modern Theoretical Physics and in fact, all the gravity theories above come out by postulating the Action Principle. HDG theories extend GR by the addition of extra terms in the gravitational action functional (a function defined on the space of metrics), terms which contain higher powers of the curvature invariants. ST theories postulate that the gravitational field is mediated by a scalar field in addition to the spacetime metric, the simplest prototype of this family being the well-known Brans-Dicke theory. This class of gravity theories is a very broad one incorporating in effect many of the string theories as special cases. There has been known for some time that there are certain conformal ‘dualities’ between HDG, ST theories and GR in that these theories are GR in disguise with additional ‘fields’. Dualities have also been recently discovered between different string and supergravity theories and these in turn sometimes are interpreted to imply the existence of a more general theory, which might in some subtle way incorporate all previous ones as special cases, M-theory.

### 2.3 Matter fields

Here too one may easily compose a shopping list of interesting candidates for matter fields which may have played an important role during different epochs in the history of the universe.

- Vacuum
- Fluids
Scalar fields
• Wave maps
• Electromagnetic fields
• Yang-Mills fields
• $n$–form fields
• Spinors

Each one of these different families has its own special role to play in cosmology, but some are definitely more ambiguous than others for different reasons.

With this background, let us now see how modern cosmologists put together spacetimes, gravity theories and matter fields to form the basic ingredient of their subject, a cosmology.

3 Cosmologies

How do we construct a cosmology? Pick up a spacetime from the cosmological hierarchy list, choose a gravity theory and one or more matter fields, tie them together through the Action Principle and try to explain the observed facts in terms of the consequences of the application of the variational principle. The result is called a cosmology. In the form of a symbolic equality,

\[ \text{Cosmology} = \text{Cosmological model(s) + Observations} \]

We shall denote a given family of cosmological models (or a cosmology) with a triplet \{·/·/·\} of the sort \{Spacetime/Gravity theory/Matter field\}. The simplest and best studied (relativistic) cosmology of physical interest is the \{FRW/General Relativity/Fluid\} Cosmology. This actually is the cosmology discussed in many textbooks on the subject under the heading ‘Relativistic Cosmology’.

One may obviously attempt to construct and analyse other cosmologies, based for example on the families:

• FRW/GR/vacuum cosmologies
• FRW/HDG/vacuum cosmologies
• FRW/scalar-tensor/fluid cosmologies
• FRW/string/vacuum cosmologies
• FRW/Brane/scalar cosmologies
• Bianchi/GR/fluid cosmologies
• Bianchi/GR/scalar cosmologies
• Bianchi/scalar-tensor/vacuum cosmologies
• Bianchi/M-theory cosmologies

and so on. How do we study the properties of each one of these cosmologies? There are many questions we can ask, some common to all families and others particular to some family. We have more to say on this in the next Section. The important thing is that the families we construct be mathematically consistent, toy models upon which to base our physical predictions and conclusions for the structure of the universe at different epochs in cosmic history.

A particular issue, connected with the philosophy that there is not one single theory of the universe which would describe it at all times but some cosmologies may be more adapted to some epochs while others not, is the problem of cosmological cohesion, that is to try to connect different cosmological models together to form a consistent frame, a cohesive cosmology, to compare with observations. For example, suppose that an FRW/GR/fluid cosmology is valid after the Planck time onwards and that a Bianchi/M-theory/vacuum cosmology holds well before that time. The cosmological cohesion problem in this case is to connect the physically meaningful solutions of the two cosmology branches into one cohesive cosmology that would describe the entire cosmic history and be compatible with observations and other constraints.

The cohesion problem is one between different cosmologies which have already been studied and their solution spaces are more or less clarified. However, the first step in the study of cosmologies is to single out a particular family and to try to develop a well-defined theory addressing as many issues as one can in the garden of cosmological problems. Some of these are described in the next Section.
4 Cosmological problems

We now translate the questions stated in the Introduction in a more suitable terminology. The result is a number of very broad directions of research currently pursued.

4.1 The singularity problem

This is the ultimate and most important problem that every cosmology has to face. It indicates the true range of validity of any cosmology and of course that of the underlying gravity theory. Its two parts, namely, the existence and the structure/nature of singularities are very different and may play complementary roles in deciding the final fate of any theory of cosmology.

Usually in cosmology the definition of a singularity is taken to mean a place where some physical quantities, for example the spacetime curvature, densities, temperatures of matter fields etc, become infinite or discontinuous there. Hence it is usual that cosmological singularities are connected with either infinities or pole like behaviour. As such, it is not surprising that there is little to be said about their structure or nature using the usual geometric/topological methods. Indeed the singularity theorems in general relativity are geometric existence results about incomplete geodesics the endpoints of which, strikingly, coincide with infinite curvature singularities in most cases but the nature of these singularities is undecided in the general case.

Instead the nature of singularities is commonly tackled via the methods of dynamical systems for particular cosmology families. In the first three families of the basic spacetime hierarchy, we end up typically with ordinary differential equations whereas from the last three categories we find systems of partial differential equations. Using methods borrowed from the qualitative theory of differential equations (theory of dynamical systems) cosmologists have been able to figure out the behaviour of spacetime in the vicinity of a singularity. In the most general case wherein the Einstein equations are reduced to ordinary differential equations, that is the second and third families in the hierarchy, very complex structures can appear in the neighborhood a such singular points. A basic question is whether such structures remain as generic features in the more general cases down the hierarchy or disappear when we consider more general cosmologies as a result of the less and less symmetry imposed.

An special example of the singularity issue in cosmology is the recollapse
problem, that is whether or not all closed (compact, without boundary) cosmologies recollapse to a second singularity. Of course, in general it is very easy to construct examples where closed universes filled with special matter fields do not recollapse, but the question here is, given a cosmology, under what conditions does the subclass of all closed cosmologies recollapse to a singularity in the future. This purely classical problem acquires importance also in the framework of inflationary and quantum cosmology since it is yet to be decided whether or not the universe can recollapse before an inflationary phase is reached (the so-called premature recollapse).

4.2 The problem of cosmic topology.

This problem has two aspects. The observational problem of deciding what the shape of the observed universe is and what would be the consequences of supposing that space has a different topology that the usual one. For example, there are many known examples of different topologies (euclidean, torus) of which can all admit a flat metric. The supposition that the manifold geometry is hyperbolic has become very fashionable and attracts a lot of attention currently. As W. Thurston has put it, ‘it is a wonderful dream to see the topology of the universe some day’. Perhaps the topology of the universe is non-trivial but not very complicated.

The second, theoretical, aspect is more involved with apparently many consequences for different parts of the general cosmological problem, most of which are as yet unclear. It is well-known that, although the Einstein equations evolve only the geometry (that is the spacetime metric) but leave the topology of initial data sets fixed (but arbitrary) during the evolution there are cases, for instance the formation of singularities in the future, where the topology of the initial data set which evolves is expected to be different after some of the space has collapsed. In fact, this issue seems to be related in a subtle way to the fundamental problem of classifying 3-manifolds (in this case the initial data sets). The Einstein flow evolves such data and one would like to know how initially different topologies affect the flow and vice versa.

4.3 The problem of asymptotic states.

The existence or nonexistence of singularities in a particular cosmology, notwithstanding, the issue of providing a detailed description of the dynam-
ical behaviour of cosmological spacetimes at both small and large times in a particular gravity theory with matter fields is a very important one, aiming at establishing a first test of the scope and flavor of any particular candidate cosmology.

One may tackle this problem by finding particular exact solutions that describe special families of models within a given cosmology and this has been in fact the first line of attack in modern cosmological research. However, even if one has succeeded in finding many different solutions of a particular cosmology at hand it is often difficult to combine them into a coherent whole that would indicate the true picture of dynamical possibilities of the given family.

The dynamical systems approach used by several authors in the past is much more promising when we wish to uncover the global structure of the solution space of a given cosmology. An especially important example of an asymptotic problem is the so-called the isotropization problem (or in other contexts a cosmic no-hair conjecture) which aims at examining the possibility of first accepting that at an early stage in cosmic history the universe was in some more complex state described by one of the models down the hierarchy list (eg., Bianchi) and then showing how the present isotropic state is the result of the long term, ‘observed’, dynamical evolution of that less symmetric ‘initial’ era. This is where the central dynamical concepts of trapping sets and attractors may come into full play. What are the attractors of a given cosmology? Is it possible that attractors of one cosmology are related to attractors of another? An answer to this question will clarify to what extend members of one cosmology family belong also to another family and in this sense how different cosmologies are related to one another. A global attractor of a cosmology is defined as one that attracts all neighboring members inside the given cosmology. If one could show how the global attractors of different cosmologies are related, one would have a precise way of deciding which cosmology to pick for specific eras in the cosmic history. Consequently such a result would help to connect apparently different cosmologies.

Another issue is to find whether chaotic, unpredictable dynamical behaviour is a true feature of classical cosmological dynamics. The basic dichotomy of nonlinear dynamics, namely, integrability versus nonintegrability and chaos, is certainly to be found in mathematical cosmology too. Only the simplest cosmologies are translated into two dimensional dynamical systems and most of them are of dimension higher than four. Therefore complex dynamical behaviour is generally to be expected in cosmology and indeed this
has been a subject of considerable research in modern mathematical cosmology. The notion of a cosmological attractor introduced earlier will also play a special role here as dissipative cosmologies are generally expected to have the so-called strange attractors, but their existence may not be easy to unravel except in the case of highly symmetric cosmologies.

An emerging method in recent years to decide whether a given cosmology is integrable, without actually solving the associated differential equations to construct cosmological solutions of physical interest, is based on an intriguing idea of two great mathematicians of the past, S. Kowalevski and P. Painlevé. These people thought that instead of trying to solve the relevant differential equations which describe a given problem, it would be very convenient to decide whether any given system (hamiltonian or not) is integrable if there was a way to merely examine the form of it. The answer appears to lie in the complex plane and the types of singularities the equations can have when analytically continued in the complex time plane. Kowalevski was able to discover a new, as well as recover many of the integrable cases of the so-called Euler-Poisson equations that are associated with the problem of the Lagrange top by analysing these equations when the only movable singularities that the equations can exhibit in the complex plane are poles. This feature is called the Painlevé property and has become very important in recent years in attacking the integrability problem. If this holds, then all solutions lie in a single Riemann sheet. However, much more complicated behaviour can occur when the singularities of the analytically continued system fail to be poles but take the form of movable branch points or the even essential singularities for which the solutions are in general multivalued complex functions. The general integrability conjecture is that if a system has the Painlevé property then it is integrable. Although, not completely proven, this method has been applied with great success in many systems in Mathematical Physics in general and in mathematical cosmology in particular. The reasons why such a method seems apparently to work appears to be connected with algebraic geometry and the theory of elliptic curves. One is therefore hoping that the complicated behaviour seen in the vicinity of the big-bang singularities in many cosmologies could be quantified by using this method and looking at the singularity patterns which the analytically continued solutions of the real-time systems form on approach to the cosmological singularity. This program is still at an infant stage, but it is has the potential to yield interesting results in the coming years.
4.4 Gravity theories and the early universe.

This last problem is a very difficult and basic one to all attempts to construct a realistic, cohesive cosmology. It is evident that the issue of choosing a gravity theory with which to build a cosmology is of paramount importance to cosmological model building. It has been known for many years and regarded as folklore that general relativity cannot be meaningfully extrapolated back to very early times in the history of an expanding universe which is predicted by it. A look at the list of possible alternative gravity theories, however, reveals that none is thus far the unique, problem-free theory. Each time there is some particular theory which is fashionable, M-theory being today’s choice.

One way to decide among a host of possibilities has been to try to get a feeling for how these different cosmologies behave when we ask the same questions. This leads to a picture of dynamical possibilities for the whole set of all conceivable cosmologies and indeed, has been the Holy Grail of modern research in mathematical cosmology. The picture to-date, however, is by no means complete even in General Relativity and the search is continued. Here again we see the need for the full exploitation of the cosmological attractors in an effort to understand the precise relations between different cosmologies.

Although a recurrent theme in this paper has been the fact that we should work among all different possibilities (and this in fact is a basic characteristic in current research in the field), one may think that, for the case where General Relativity is expected to break down, some principle exists that could successfully guide our vision in searching for the ‘right’ theory with which to build a reliable cosmology of the early universe. A principle suitable for such a purpose can be based on the use of the fundamental notion of symmetry. Which cosmology is the most symmetric? This question raises another: What is meant by ”most symmetric”? The notion of a Lie symmetry is a natural one when applied to systems of differential equations. It is related to the fundamental invariant quantities that are preserved during the evolution of the system according to Noether’s theorem. If we could construct the symmetry atlas of any given cosmology we would have gone a long way to answering any given question about the evolution of a cosmology. Such an approach is not difficult to implement and could be done for a great variety of cosmologies. This may prove to be an interesting and fruitful direction of further research in early universe mathematical cosmology in the coming years.
5 Outlook

The strategy is now clear. Study each one of these problems in the framework of every possible cosmology with an effort to decide among the different possibilities for a realistic cosmology.

The cosmological problem is the global problem \textit{par excellence}. In contradistinction with the other, complementary area of modern research in gravitation, namely, asymptotically flat problems, all basic problems in cosmology involve thinking about spacetimes which are nowhere trivial and in this sense the lack of knowledge of initial conditions in cosmology is natural (if only trivial boundary or initial conditions are acceptable!). Asymptotically flat problems on the other hand, being basically local ones, are well-defined mathematically having initial or boundary conditions away from the sources where the spacetime is trivial. But the universe is not asymptotically flat. The Newtonian universe is asymptotically flat, but general relativity introduced the notion of an evolving universe as a whole and therefore did away with asymptotic flatness on a cosmological scale.

The fundamental problems of mathematical cosmology discussed in this paper, namely the singularity problem, the topology problem, the asymptotic problem and the problem of choosing a gravity theory and building a realistic early universe cosmology, frame mathematical cosmology as a separate and important discipline at the interface between Mathematics and Physics and make it an interesting and active branch of Mathematical Physics.

A new direction of research in the singularity problem might consist in using the highly developed theory of singularities of differentiable mappings by Arnol’d and coworkers. Since the usual singularity theorems prove the existence of families of incomplete geodesic curves which in general refocus to form caustics, perhaps a clarification of the nature of these singularities can be attained by their classification through Arnol’d theory. However, the latter is concerned with singularities of a different type namely, those that occur due to the vanishing of certain derivatives and Jacobians rather than infinities or poles. Is it possible that the singularities in general relativity be of the milder type of this sort? The answer to this question is at present unknown.

Much work has been undertaken during the last thirty years or so in the asymptotic problem using the qualitative theory of differential equations. This work can be generalized in at least two directions. Firstly, most of the analyses are concerned with equilibrium solutions and their stability. Bifur-
cation theory may open the way to tackle seemingly unrelated systems as a whole system with parameters, for example, the general Bianchi/GR/vacuum family.

Secondly, the equations of the family: \( G_2/GR/vacuum \) are very similar to those of a spherically symmetric wave map and existence and regularity results for the latter system are known in the literature. The theory of partial differential equations theory has not been used in any systematic way up till now in Mathematical Cosmology. Global information about the solution spaces of some of these inhomogeneous cosmologies may also be obtained by writing them as dynamical systems in infinite dimensions and some work along this lines is now beginning to emerge.

Most of the published literature in the early universe cosmology is mainly concerned with the first two of the six-step spacetime hierarchy given above. It is entirely unknown to what degree important discoveries (a prime example is inflation) that have been made working with ‘low-level’ (i.e., top of the hierarchy) cosmologies (e.g., FRW/GR,HDG,ST,String,Branes/Scalar etc) are justified, i.e., persist as true features of the generic dynamics or are simply artifacts of the high degree of exact symmetry imposed. That is an additional reason why the issue of cosmological attractors in given cosmologies must be faced.

A work that analysed the sixth stage (generic spacetimes) but in the asymptotically flat case in General Relativity is the proof of the global stability of Minkowski space by Christodoulou and Klainerman (Annals of Mathematics Studies, vlm. 41, Princeton University Press, 1993). No result of such generality exists for any cosmology. What could a corresponding analysis in the cosmological case imply (for instance the global stability of the positive curvature FRW spacetime) for the validity of the current cosmological ideas (inflation, attractor properties of the known physically interesting cosmological spacetimes etc) is at present only a matter of conjecture.

Nature is unique, it is not generic. Our attempts to simulate the universe in mathematical and theoretical cosmology will lead to reliable results if and only if they follow from studies of generic cosmologies or show which features of the highly symmetric (and hence unphysical) cosmological models persist and propagate down the hierarchy list so as to become true features of the more general, asymmetric cosmologies. Progress will be made if one finds a way to sidestep the difficulties of analysing the partial differential equations of the inhomogeneous models by showing how the global attractors of different cosmologies are related and picturing more clearly the generic structures
of the cosmological phase space. It is only in this way that the observations showing a homogeneous universe can be justified mathematically and give meaning to our ability to work with models high in the cosmological hierarchy list. On the other hand, if the unique features of Nature cannot be recovered by a sort of generic process the road to understanding will be very long and arduous.

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References

[1] Although we gave no references in the main text of this paper, here is a broad but very short list of references which is meant to indicate useful and/or indispensable places for the prospect mathematical cosmologist. A basic mathematical reference for our subject is the superb two-volume treatise by Y.Choquet-Bruhat et al, *Analysis, Manifolds and Physics*, Volume I: Basics, 2nd Ed. (North-Holland, 1982), Part II, 2nd Ed. (North-Holland, 2000). Each of the following sources discusses one of the fundamental problems of mathematical cosmology.

Singularities
The standard reference is of course,

[2] S.W. Hawking and G.F.R. Ellis, *The large-scale structure of spacetime*, (CUP, 1973).

Singularity theory is discussed in V.I. Arnol’d, *et al*, *Singularities of differentiable mappings*, Volume I, (Birkhauser, 1985). A presentation of parts of Arnol’d’s theory more suitable for physicists is contained in a recent paper by J. Ehlers *et al*, J.Math.Phys. 41 (2000) 3244-3378.

Cosmic topology
For a recent review of the theoretical and observational aspects of cosmic topology see the special issue of Class.Quant.Grav. 15, September 1998, edited by G.D. Starkman.

The theoretical problem has many components. See
[3] Papers by Hosoya, Kodama, Barrow-Kodama in the gr-qc Los Alamos Archives. See also, A.E. Fisher and V. Moncrief, *The reduced hamiltonian of general relativity and the $\sigma$ constant of conformal geometry*, in *Mathematical and Quantum Aspects of Relativity and Cosmology*, S. Cotsakis and G.W. Gibbons, (eds.), Lecture Notes in Physics, 537, (Springer, 1998), pp. 70–101.

**Asymptotic problem**

[4] J. Wainwright and G.F.R. Ellis, *Dynamical systems in cosmology*, (CUP, 1997). The bulk of this beautiful book treats in depth the **Bianchi/GR/Fluid cosmologies**, but some members of the **Inhomogeneous/GR** family are also discussed.

An advanced but excellent discussion of bifurcation theory is given in, V.I. Arnol’d, *Geometrical methods in the theory of ordinary differential equations*, (Springer, 1983).

The elements of analytic structure of dynamical systems and their singularity patterns in the complex plane are beautifully presented in M. Tabor, *Chaos and integrability in nonlinear dynamics*, (Wiley, 1989) ch. 8. There is a recent book on this subject, A. Roy Chowdhuri, *Painlevé analysis and its applications*, (Chapman and Hall/CRC, 2000). For some applications to cosmology see S. Cotsakis and P.G.L. Leach, *Painlevé analysis of the Mixmaster universe*, J.Phys.A27 (1993) 1625-1631; P.G.L. Leach, S. Cotsakis and J. Miritzis), *Symmetries, singularities and integrability in complex dynamics IV: Painlevé integrability of isotropic cosmologies*, Grav.Cosm. 6 (2000) 282-290.

**Gravity theories and the early universe**

For the issue of choosing a gravity theory for building a realistic early universe cosmology no single general reference exists, but research is scattered in virtually every mathematical cosmology paper. We give here a few important recent references to show the flavor of research in a number of different cosmologies.

[5] **FRW/GR cosmologies:** S. Foster, *Scalar field cosmologies and the initial space-time singularity*, gr-qc/9806098.

[6] **FRW/ST cosmologies:** S.J. Kolitch and D.M. Eardley, *Behaviour of the FRW cosmological models in scalar-tensor gravity*, gr-qc/9405016.
[7] **FRW/String cosmologies:** A.P. Billyard, A.A. Coley and J.E. Lidsey, *Qualitative analysis of string cosmologies*, gr-qc/9903095.

[8] **FRW/Brane cosmologies:** J. Khoury, P.J. Steinhardt and D. Waldram, *Inflationary solutions in the brane-world and their geometrical interpretation*, hep-th/0006069.

[9] **FRW/M-theory cosmologies:** A. Lucas, B.A. Ovrut and D Waldram, *Cosmological solutions of Hořava-Witten theory*, hep-th/9806022; A.P. Billyard, A.A. Coley and J.E. Lidsey, *Dynamics of M-theory cosmology*, hep-th/9908102.

[10] **Bianchi/ST cosmologies:** A.A. Coley, *Qualitative properties of scalar-tensor theories of gravity*, astro-ph/9910395.

[11] **Bianchi/String cosmologies:** J.D. Barrow and K.E. Kunze, *Spatially homogeneous string cosmologies*, hep-th/9608043; J.D. Barrow and M.P. Dabrowski, *Is there chaos in low-energy string cosmology?*, hep-th/9711049; A.P. Billyard, A.A. Coley and J.E. Lidsey, *Qualitative analysis of isotropic curvature string cosmologies*, hep-th/9911086.

A guiding ‘principle’ based on the use of symmetries is discussed in

[12] J. Lidsey, Class.Quant.Grav. 13 (1996) 2449-2456.