Dirichlet polynomials and a moment problem

Sameer Chavan1 · Chaman Kumar Sahu1

Received: 21 April 2022 / Accepted: 6 July 2022
© Tusi Mathematical Research Group (TMRG) 2022

Abstract
Consider a linear functional $L$ defined on the space $D[s]$ of Dirichlet polynomials with real coefficients and the set $D_+[s]$ of non-negative elements in $D[s]$. An analogue of the Riesz–Haviland theorem in this context asks: What are all $D_+[s]$-positive linear functionals $L$, which are moment functionals? Since the space $D[s]$, when considered as a subspace of $C([0, \infty), \mathbb{R})$, fails to be an adapted space in the sense of Choquet, the general form of Riesz–Haviland theorem is not applicable in this situation. In an attempt to answer the forgoing question, we arrive at the notion of a moment sequence, which we call the Hausdorff log-moment sequence. Apart from an analogue of the Riesz–Haviland theorem, we show that any Hausdorff log-moment sequence is a linear combination of $\{1, 0, \ldots, \}$ and $\{f(\log(n))\}_{n \geq 1}$ for a completely monotone function $f : [0, \infty) \to [0, \infty)$. Moreover, such an $f$ is uniquely determined by the sequence in question.

Keywords Dirichlet polynomial · Moment functional · Completely monotone function · Bernstein function

Mathematics Subject Classification 44A60 · 26A48 · 30E20 · 30B50

1 Introduction

Let $\mathbb{Z}_+$ denote the set of positive integers and let $\mathbb{R}$ denote the set of real numbers. Let $X$ be a locally compact Hausdorff space. A Radon measure on $X$ is a locally finite, inner regular Borel measure on $X$. The real linear space of continuous
functions from $X$ into $\mathbb{R}$ will be denoted by $C(X, \mathbb{R})$. For a closed subset $K$ of $X$ and a subspace $S$ of $C(X, \mathbb{R})$, let $S^K_+$ denote the cone of all functions in $S$ which are non-negative on $K$. If $K = X$, then we use the notation $S^+_+$ to denote $S^K_+$. For a subset $\mathcal{M}$ of $S$, a linear functional $L$ on $S$ is said to be $\mathcal{M}$-positive if $L(q) \geq 0$ for all $q \in \mathcal{M}$. For a closed subset $K$ of $X$, we say that $L$ is a $K$-moment functional if there exists a positive Radon measure $\nu$ concentrated on $K$ such that

$$L(q) = \int_K q(s)\nu(ds), \quad q \in S.$$  

In this case, we refer to the measure $\nu$ as a representing measure of $L$ (see [4, 16, 18] for the basics of moment theory).

Let $\mathbb{H}_0$ denote the right half plane $\{z \in \mathbb{C} : \Re(z) > 0\}$ in the complex plane $\mathbb{C}$. A Dirichlet polynomial is a function $q : \mathbb{H}_0 \to \mathbb{C}$ given by

$$q(s) = \sum_{n=1}^{k} a_n n^{-s}, \quad s \in \mathbb{H}_0,$$

where $a_n \in \mathbb{C}$ and $k \in \mathbb{Z}_+$. The real linear space of Dirichlet polynomials with real coefficients will be denoted by $\mathcal{D}[s]$. By the uniqueness of the Dirichlet series, $\mathcal{D}[s]$ can be embedded into $C([0, \infty), \mathbb{R})$ via the map $f \mapsto f|_{[0,\infty)}$. Moreover, $\mathcal{D}[s]$ is a unital sub-algebra of $C([0,\infty), \mathbb{R})$ (the reader is referred to [14, Chapter 4] for algebraic properties of Dirichlet polynomials and Dirichlet series). For a closed subset $K$ of $[0, \infty)$, let $\mathcal{D}^K_+[s]$ denote the set of all functions in $\mathcal{D}[s]$ which are non-negative on $K$.

For a sequence $w = \{w_n\}_{n \geq 1}$ of non-negative real numbers, consider the real linear functional $L_w : \mathcal{D}[s] \to \mathbb{R}$ defined by setting $L_w(n^{-s}) = w_n$, $n \geq 1$, and extending linearly to $\mathcal{D}[s]$. If $L_w$ is a $K$-moment functional, then clearly $L_w$ is $\mathcal{D}^K_+[s]$-positive. An analogue of Riesz–Haviland theorem asks for the converse:

**Question 1.1** If $L_w$ is a $\mathcal{D}^K_+[s]$-positive linear functional on $\mathcal{D}[s]$, then whether $L_w$ is a $K$-moment functional?

In case $K$ is bounded, the answer to Question 1.1 is affirmative and this is a consequence of [16, Proposition 1.9]. In general, the answer to Question 1.1 is No (see Theorem 3.1). So the following natural question arises:

**Question 1.2** What are the sequences $w$ for which every $\mathcal{D}^K_+[s]$-positive linear functional $L_w$ on $\mathcal{D}[s]$ is a $K$-moment functional?

To answer Questions 1.1 and 1.2, we must obtain an analogue of Riesz–Haviland theorem (see [16, Theorem 1.12]), which replaces the polynomials by Dirichlet polynomials. One approach to the proof of the Riesz–Haviland theorem, as presented in [16], can be based on the notion of an adapted space (see [16, Definition 1.5]). To see whether or not $\mathcal{D}[s]$ is an adapted space, note that $1 \in \mathcal{D}[s]$ and $\mathcal{D}[s] = \mathcal{D}_+[s] - \mathcal{D}_+[s]$. Since $\mathcal{D}[s]$ is an algebra containing 1,
Dirichlet polynomials and a moment problem  Page 3 of 23  63

is a difference of two Dirichlet series in $D_+[s]$. However, $D[s]$ does not have the crucial property of the existence of the dominating function (see [16, Definition 1.5(iii)]. Indeed, for $f(s) = 1$, there exists no $g \in D_+[s]$ with the following property: For any $\epsilon > 0$, there exists a compact subset $K_\epsilon$ of $[0, \infty)$ such that $|f(s)| \leq \epsilon |g(s)|$ for all $s \in [0, \infty) \setminus K_\epsilon$. It is interesting to note that there exist an $E_+$-positive linear functional $L$ on a linear space $E$ (without the aforementioned property), which is not a moment functional (see [16, Example 1.11] and Corollary 3.2).

2 Hausdorff log-moment sequences

One may answer Question 1.1 by combining [16, Proposition 1.9] with a compactification technique (see [16, Chapter 9]). Since this proof is not relevant to the investigations here, we have relegated it to the appendix. Our ploy is to consider a $[0, 1]$-moment problem, which, after a change of variables (see Lemma 3.1), yields a more general $[0, \infty)$-moment problem than the one discussed in Sect. 1. The solution to the $[0, 1]$-moment problem (which is a simple consequence of [16, Proposition 1.9]) allows us to answer Question 1.1 (see First proof of Theorem 3.1). Note that a particular case of the $[0, \infty)$-moment problem appears in [11, Equation (1.3)] in the study of the multiplier algebra of certain Hilbert spaces of the Dirichlet series (see Example 2.1(a)). This problem also appears in the study of bounded Helson matrices (see [12]). Indeed, it is easily seen from the discussion prior to [12, Theorem 5.1] that if $\nu$ is a representing measure of $L_w$ satisfying $\nu(\{0\}) = 0$, then

$$L_w(n^{-s}) = \int_{(0,1)^\omega} i^{k(n)} \mathcal{B}_s \nu(dt), \quad n \geq 1,$$

where $k(n)$ is the tuple of the exponents of primes appearing in the prime factorization of $n$ and $\mathcal{B}_s \nu$ is the push-forward of $\nu$ by the Bohr lift

$$\mathcal{B}(s) = (p_1^{-s}, p_2^{-s}, \ldots), \quad s \in (0, \infty)$$ (2.1)

with $\{p_1, p_2, \ldots\}$ denoting for the monotone enumeration of the prime numbers (see [12, Section 5] for more details). The $[0, 1]^{\omega}$-moment problem above is a particular instance of the Hausdorff moment problem in an infinite number of variables (see [1, Theorems 5.1 and 5.5] and [9, Theorem 3.8] for variants of Riesz–Haviland theorem for this moment problem). These instances together with the previous discussion motivates the following definition (see [4, Chapter 2] for the notion of the restriction of a Borel measure to a Borel set):

**Definition 2.1** Fix a positive integer $j$. A sequence of non-negative real numbers $w = (w_n)_{n \geq j}$ is called a *Hausdorff log-moment sequence* if there exists a positive Borel measure $\mu$ concentrated on $[0, 1]$ such that the restriction $\mu|_{(0,1]}$ of $\mu$ to $(0, 1]$ is a Radon measure and
\[ w_n = \int_{[0,1]} t^{\log(n)} \mu(dt), \quad n \geq j. \]

We refer to \( \mu \) as a representing measure of \( w \).

**Remark 2.1** For a Hausdorff log-moment sequence \( w = \{w_n\}_{n \geq j} \) with a representing measure \( \mu \), we note the following:

1. if \( j = 1 \), then \( \mu \) is a finite measure with total density equal to \( w_1 \),
2. if \( \mu \) is a representing measure of \( w \) such that \( \mu(\{0\}) \) is finite, then since \( \mu|_{(0,1]} \) is locally finite, \( \mu \) is a \( \sigma \)-finite measure,
3. for every integer \( n \geq j \), \( w_n \leq w_j \) (since \( n \mapsto t^{\log(n)} \) is decreasing for every \( t \in [0,1] \)).

Thus, the sequence \( w \) is decreasing and bounded.

In case \( j \geq 2 \), representing measures of Hausdorff log-moment sequences are not necessarily finite (cf. [11, Section 1]).

**Example 2.1** Let us see some families of Hausdorff log-moment sequences.

(a) For a real number \( \alpha < 0 \), the sequence \( \{\log(n)^{\alpha}\}_{n \geq 2} \) is a Hausdorff log-moment sequence with representing measure \( \mu_\alpha \) equal to

\[
\mu_\alpha(dt) = \frac{1}{\Gamma(-\alpha)} \frac{(-\log(t))^{-1-\alpha}}{t} dt,
\]

where \( \Gamma \) denotes the Gamma function defined on the right half plane \( \mathbb{H}_0 \). To see this, note that \( \mu_\alpha|_{(0,1]} \) is a Radon measure. Moreover, for any positive integer \( n \geq 2 \),

\[
\int_{[0,1]} t^{\log(n)} \mu_\alpha(dt) = \frac{1}{\Gamma(-\alpha)} \int_{[0,\infty)} e^{-s \log(n)} s^{-1-\alpha} ds
= \frac{(\log(n))^{\alpha}}{\Gamma(-\alpha)} \int_{[0,\infty)} e^{-r r^{-1-\alpha}} dr
= (\log(n))^{\alpha}.
\]

(b) For a real number \( \alpha > 0 \), the sequence \( \{\frac{1}{\log(n)+\alpha}\}_{n \geq 1} \) is a Hausdorff log-moment sequence with the representing measure given by \( t^{\alpha-1} dt \). Indeed, for any positive integer \( n \geq 1 \),

\[
\int_{[0,1]} t^{\log(n)} t^{\alpha-1} dt = \frac{t^{\log(n)+\alpha}}{\log(n)} \bigg|_{0}^{1} = \frac{1}{\log(n) + \alpha}.
\]

(c) For a real number \( \alpha \in [0,1] \), the sequence \( \{\alpha^{\log(n)}\}_{n \geq 1} \) is a Hausdorff log-moment sequence with the representing measure equal to the point mass measure \( \delta_\alpha \) at \( \alpha \). Indeed,
In particular, for a non-negative real number \( p \), the sequence \( \{ \frac{1}{n^p} \}_{n \geq 1} \) is a Hausdorff log-moment sequence with the representing measure equal to the atomic measure \( \delta_{e^{-p}} \) with point mass at \( e^{-p} \) (the case in which \( \alpha = e^{-p} \)). Moreover, \( \{1, 0, 0, \ldots\} \) is a Hausdorff log-moment sequence with the representing measure equal to the atomic measure \( \delta_0 \) with point mass at 0 (the case in which \( \alpha = 0 \)). \( \square \)

Our answer to Question 1.1 relies on the following characterization of the Hausdorff log-moment sequences given in terms of the associated linear functional on a certain subspace of the space of continuous functions on \([0, 1]\).

**Proposition 2.1** Let \( \mathbf{w} := \{w_n\}_{n \geq 1} \) be a sequence of non-negative real numbers. Let \( \mathcal{E}[t] \) denote the real linear span of functions \( f_n(t) = t^{\log(n)}, t \in [0, 1], n \geq 1 \). Consider the real linear functional \( R_{\mathbf{w}} : \mathcal{E}[t] \rightarrow \mathbb{R} \) defined by setting \( R_{\mathbf{w}}(f_n) = w_n, n \geq 1 \) and extended linearly to \( \mathcal{E}[t] \).

For a closed subset \( K \) of \([0, 1]\), the following statements are equivalent:

(i) \( R_{\mathbf{w}} \) is \( \mathcal{E}_+^{K}[t] \)-positive,

(ii) \( R_{\mathbf{w}} \) is a \( K \)-moment functional,

(iii) there exists a finite positive Radon measure \( \mu \) concentrated on \( K \) such that

\[
R_{\mathbf{w}}(f_n) = \int_K f_n(t) \mu(dt), \quad n \geq 1,
\]

(2.2)

(iv) \( \mathbf{w} \) is a Hausdorff log-moment sequence with a representing measure concentrated on \( K \).

**Proof** Note that the constant function 1 belongs to \( \mathcal{E}[t] \). Hence, if the linear functional \( R_{\mathbf{w}} \) is \( \mathcal{E}_+^{K}[t] \)-positive, then by [16, Proposition 1.9], there exists a positive Radon measure \( \mu \) concentrated on \( K \) such that

\[
R_{\mathbf{w}}(q) = \int_K q(t) \mu(dt), \quad q \in \mathcal{E}[t].
\]

(2.3)

Thus we obtain the implication (i)\( \Rightarrow \) (ii). Letting \( q = f_n \) in (2.3), we get (2.2) and letting \( n = 1 \) in (2.2), we get \( \mu(K) = w_1 \). This yields the implication (ii)\( \Rightarrow \) (iii). The implication (iii)\( \Rightarrow \) (iv) is trivial. To see the implication (iv)\( \Rightarrow \) (i), note that if \( \mathbf{w} \) is a Hausdorff log-moment sequence, then by the linearity of the integral,

\[
R_{\mathbf{w}}(f) = \int_K f(x) \mu(dx), \quad f \in \mathcal{E}[t].
\]

It now follows that \( R_{\mathbf{w}} \) is \( \mathcal{E}_+^{K}[t] \)-positive. \( \square \)
Proposition 2.1 can be used to prove an analogue of Riesz–Haviland theorem for the space of Dirichlet polynomials. We present two proofs of this analogue. The first one relevant to the study of Hausdorff log-moment sequences is presented in Sect. 3. The second one based on a known compactification technique is given in Appendix.

3 An analogue of the Riesz–Haviland theorem for $\mathcal{D}[s]$

For a subset $\sigma$ of the set $\mathbb{Z}_+$ of positive integers, let $\chi_\sigma$ denote the indicator function of $\sigma$.

The following result characterizes $\mathcal{D}_2^K[s]$-positive linear functionals $L_w$ answering Question 1.1 (cf. [1, Theorem 5.1]).

**Theorem 3.1** Let $w = \{w_n\}_{n \geq 1}$ be a sequence of non-negative real numbers. Consider the real linear functional $L_w : \mathcal{D}[s] \rightarrow \mathbb{R}$ defined by $L_w(n^{-s}) = w_n$, $n \geq 1$, and extended linearly to $\mathcal{D}[s]$. For a closed subset $K$ of $[0, \infty)$, the following statements are equivalent:

(i) $L_w$ is $\mathcal{D}_2^K[s]$-positive,

(ii) there exists a finite positive Radon measure $\nu$ concentrated on $K$ with $\nu(K) \leq w_1$ such that for every $p \in \mathcal{D}[s]$,

$$
L_w(p) = \begin{cases} 
(w_1 - \nu(K)) \lim_{s \to \infty} p(s) + \int_K p(s)\nu(ds) & \text{if } K \text{ is unbounded,} \\
\int_K p(s)\nu(ds) & \text{if } K \text{ is bounded,}
\end{cases}
$$

(iii) there exists a finite positive Radon measure $\nu$ concentrated on $K$ with $\nu(K) \leq w_1$ such that for every positive integer $n \geq 1$,

$$
w_n = \begin{cases} 
(w_1 - \nu(K))\chi_{\{1\}}(n) + \int_K n^{-s}\nu(ds) & \text{if } K \text{ is unbounded,} \\
\int_K n^{-s}\nu(ds) & \text{if } K \text{ is bounded.}
\end{cases}
$$

We do not see any obvious way to derive Theorem 3.1 from [1, Theorem 5.1]. Also, Theorem 3.1 falls short to deduce (via the change of variable using Bohr lift given by (2.1)) the Riesz–Haviland theorem for the moment problem in an infinite number of variables.

In the proof of Theorem 3.1, we need a couple of lemmas. We recall first the notion of an image measure.

Let $(X, \Sigma, \mu)$ be a measure space and $Y$ be a Hausdorff space. Let $\mathcal{B}(Y)$ denote the Borel $\sigma$-algebra of $Y$. If $\phi : X \rightarrow Y$ is a $\Sigma$-measurable function, then $\phi_*\mu$ denotes the push-forward of $\mu$ by $\phi$ or the image of $\mu$ under $\phi$ defined as $\phi_*\mu(\sigma) = \mu(\phi^{-1}(\sigma))$, $\sigma \in \mathcal{B}(Y)$. The reader is referred to [4, Chapter 2.1] for elementary facts pertaining
Lemma 3.1 (Change of variables) Let $\mu$ be a positive Borel measure on $[0, 1]$ and let $\nu$ be a positive Borel measure on $[0, \infty)$. Define functions $\varphi : (0, 1) \to [0, \infty)$ and $\psi : [0, \infty) \to (0, 1)$ by

$$\varphi(t) = -\log(t), \quad t \in (0, 1), \quad \psi(s) = e^{-s}, \quad s \in [0, \infty).$$

Then, for Borel measurable subsets $\widetilde{K}$ and $K$ of $[0, 1]$ and $[0, \infty)$ respectively, the following statements are valid.

(i) If $0 \notin \widetilde{K}$, then for every $\lambda \in [0, \infty)$,

$$\int_{\widetilde{K}} t^\lambda \mu(\text{d}t) = \int_{\varphi(\widetilde{K})} e^{-\lambda s} \varphi^* \mu(\text{d}s).$$

(ii) For every $\lambda \in [0, \infty)$,

$$\int_{K} e^{-\lambda s} \nu(\text{d}s) = \int_{\psi(K)} t^\lambda \psi^* \nu(\text{d}t).$$

Both sides in the above identities could be possibly infinite.

The following lemma provides precise relationship between $D[s]$ (resp. $D^K[s]$, $E[t]$ (resp. $E^K[t]$).

Lemma 3.2 Let $E[t]$ and $E^K[t]$ be as defined in the statement of Proposition 2.1. If $\varphi : (0, 1) \to [0, \infty)$ and $\psi : [0, \infty) \to (0, 1)$ are given by (3.3), then the following statements are valid:

(i) If $q \in E[t]$, then $q \circ \varphi \in D[s]$,

(ii) If $p \in D[s]$, then there exists $q_p \in E[t]$ such that $p = q_p \circ \psi$, where $q_p$ is given by

$$q_p(t) = \begin{cases} \lim_{s \to \infty} p(s) & \text{if } t = 0, \\ p \circ \varphi(t) & \text{if } t \in (0, 1), \end{cases}$$

(3.4)

(iii) If $\widetilde{K}$ is a closed subset of $[0, 1]$ and $q \in E[t]$, then $q \in E^K[t]$ if and only if $q \circ \psi \in D^K_+(s)$, where $K = \psi^{-1}(\widetilde{K}\setminus\set{0})$.

Proof For every integer $n \geq 1$, if $q(t) = t^{\log(n)}$, $t \in [0, 1]$, then $q \circ \psi(s) = n^{-s}$, $s \in [0, \infty)$. The statement (i) is now clear. To see (ii), let $p \in D[s]$ and note that $\lim_{s \to \infty} p(s)$ exists as a real number. Thus $q_p$ given by (3.4) is well-defined. Since $q \circ \psi$ is the identity function on $[0, \infty)$, $p = q_p \circ \psi$. The equivalence in (iii) follows from (i) and the continuity of $q$. \qed
Proof (First proof of Theorem 3.1) In view of [16, Proposition 1.9], we may assume that \( K \) is unbounded. Let \( \varphi : (0, 1) \to [0, \infty) \) and \( \psi : [0, \infty) \to (0, 1) \) be given by (3.3). To see the implication (i)\( \Rightarrow \) (ii), suppose that \( L_{w} \) is a \( D_{+}^{K}[s] \)-positive linear functional satisfying

\[
L_{w}(n^{-s}) = w_{n}, \quad n \geq 1. \tag{3.5}
\]

Let \( \mathcal{E}[t] \) and \( \mathcal{E}_{+}^{K}[t] \) be as defined in the statement of Proposition 2.1. Define \( R_{w} : \mathcal{E}[t] \to \mathbb{R} \) by setting \( R_{w}(\log(n)) = w_{n} \), and extend it linearly to \( \mathcal{E}[t] \). Let \( \tilde{K} = \varphi^{-1}(K) \cup \{0\} \) (see (3.3)). We now check that \( R_{w} \) is \( \mathcal{E}^{K}[t] \)-positive. By Lemma 3.2(iii), if \( q \in \mathcal{E}^{K}[t] \), then \( q \circ \varphi \in D_{+}^{K}[s] \) and

\[
R_{w}(q) \overset{(3.5)}{=} L_{w}(q \circ \psi) \geq 0.
\]

It now follows from Proposition 2.1 that there exists a finite positive Radon measure \( \mu \) concentrated on \( \tilde{K} \) such that

\[
L_{w}(q \circ \psi) = \int_{\tilde{K}} q(t) \mu(dt), \quad q \in \mathcal{E}[t].
\]

By Lemma 3.2(ii), for every \( p \in D[s] \), there is \( q_{p} \in \mathcal{E}[t] \) such that

\[
L_{w}(p) = L_{w}(q_{p} \circ \psi) \overset{(3.4)}{=} \mu(\{0\}) \lim_{s \to \infty} p(s) + \int_{\varphi^{-1}(K)} q_{p}(t) \mu(dt).
\]

By Lemma 3.1(i) and (3.4), for every \( p \in D[s] \),

\[
L_{w}(p) = \mu(\{0\}) \lim_{s \to \infty} p(s) + \int_{K} p(s) \varphi_{*} \mu(ds). \tag{3.6}
\]

Since \( \mu \) is a finite Radon measure, so is \( \varphi_{*} \mu \) (see [4, Proposition 2.1.15]). This completes the verification of (i)\( \Rightarrow \) (ii) with \( v = \varphi_{*} \mu \) provided we check that \( v \) is a finite measure satisfying \( v(K) \leq w_{1} \).

To see this, let \( p(s) = n^{-s}, s \in [0, \infty) \) in (3.6) to obtain

\[
w_{n} = L_{w}(p) = \mu(\{0\}) \chi_{(1)}(n) + \int_{K} n^{-s} \varphi_{*} \mu(ds). \tag{3.7}
\]

This yields

\[
\mu(\{0\}) = w_{1} - \varphi_{*} \mu(K) \tag{3.8}
\]

completing the verification of (i)\( \Rightarrow \) (ii). The implication (ii)\( \Rightarrow \) (iii) is now immediate from (3.7).

To see the implication (iii)\( \Rightarrow \) (i), note that for any \( p \in D[s] \),...
Since \( v(K) \leq w_1 \), both terms on the right hand side of the above equation are non-negative for any \( p \in D_+[s] \). Hence, \( L_w \) is \( D_+[s] \)-positive. \( \square \)

Here is an application of Theorem 3.1 to Hausdorff log-moment sequences.

**Corollary 3.1** Let \( w = \{w_n\}_{n \geq 1} \) be a sequence of non-negative real numbers. Consider the real linear functional \( L_w : D[s] \to \mathbb{R} \) defined by \( L_w(n^{-s}) = w_n, \ n \geq 1 \), and extended linearly to \( D[s] \). Then \( L_w \) is \( D^{[0, \infty)}[s] \)-positive if and only if \( w \) is a Hausdorff log-moment sequence.

**Proof** Applying Theorem 3.1 to the closed set \( K = [0, \infty) \) yields that \( L_w \) is \( D^{[0, \infty)}[s] \)-positive if and only if there exists a finite positive Radon measure \( \mu \) concentrated on \( [0, \infty) \) with \( \mu(K) \leq w_1 \) such that

\[
L_w(p) = (w_1 - v(K)) \lim_{s \to \infty} p(s) + \int_K p(s)v(ds).
\]

The desired equivalence is now immediate from Example 2.1(c) and Lemma 3.1(ii). In this case, one can choose the representing measure of \( w \) to be \( (w_1 - v(K))\delta_0 + \psi_* v \). \( \square \)

As noted in [16, Exercise 1.3.3], the evaluation functional at the point in the Stone–Čech compactification of \( \mathbb{R} \) and not belonging to \( \mathbb{R} \) is a \( C(\mathbb{R}, \mathbb{R})_+ \)-positive functional which is not a \( \mathbb{R} \)-moment functional. This fact together with the second proof of Theorem 3.1 suggests that the perturbation to the integral appearing in (3.1) could be an obstruction in making \( L_w \) a \( K \)-moment functional. This is confirmed by the next result, which also answers Question 1.2.

**Corollary 3.2** Consider the real linear functional \( L_w : D[s] \to \mathbb{R} \) defined by \( L_w(n^{-s}) = w_n, \ n \geq 1 \), and extended linearly to \( D[s] \). For a closed unbounded subset \( K \) of \( [0, \infty) \), assume that \( L_w \) is \( D^K_+\{0\} \)-positive, so that \( L_w \) is given by (3.1) for some finite positive Radon measure \( v \) concentrated on \( K \). Then the following statements are equivalent:

(i) \( L_w \) is a \( K \)-moment functional,
(ii) \( w_1 = v(K) \),
(iii) \( w \) is a Hausdorff log-moment sequence with a representing measure \( \mu \) concentrated on \( \tilde{K} \) such that \( 0 \in \tilde{K} \) and \( \mu(\{0\}) = 0 \), where \( \tilde{K} = \psi(K) \cup \{0\} \) (see (3.3)).

**Proof** (i)\( \Leftrightarrow \) (ii): Let \( L_w \) be a \( K \)-moment functional with a representing measure \( \gamma \). Combining (i) with (3.2), we obtain...
By Lemma 3.1(ii) (with $\psi$ given by (3.3)),

$$(w_1 - \psi(K)) \chi_{(1)}(n) = \int_K n^{-s} \gamma(d\tilde{s}) - \int_K n^{-s} \psi(d\tilde{s}), \quad n \geq 1.$$ 

Let $\delta_0$ denote the atomic measure with point mass at $[0]$ and $\eta$ denote the trivial extension of $\psi_+\gamma - \psi_+\nu$ to $[0, 1]$. Then

$$(w_1 - \nu(K)) \int_{[0,1]} t^{\log(n)} \delta_0(dt) = \int_{[0,1]} t^{\log(n)} \eta(dt), \quad n \geq 1. \quad (3.9)$$

By the Stone–Weierstrass theorem (see [18, Theorem 2.5.5]), the subspace $S = \text{span}_\mathbb{R} \{t^{\log(n)}\}_{n \geq 1}$ is dense in the space $C([0, 1], \mathbb{R})$ of real-valued continuous functions of $[0, 1]$ (as pointed by the referee, this can also be verified using the Münz–Szasz theorem). Hence, the identity (3.9) holds for any $f \in C([0, 1], \mathbb{R})$. If $w_1 \neq \nu(K)$, then by the Riesz–Markov theorem (see [18, Theorem 4.8.8]), we obtain

$$(w_1 - \nu(K))\delta_0 = \eta.$$ 

However, in that case, $\eta$ would be supported only at $[0]$, which is contrary to the definition of $\eta$. This yields $w_1 = \nu(K)$ completing the verification of (i)$\Rightarrow$(ii). Clearly, if $w_1 = \nu(K)$, then by (3.1), $L_w$ is a $K$-moment functional with representing measure $\gamma = \nu$.

(ii) $\Leftrightarrow$ (iii): If $w_1 = \nu(K)$, then by (3.2),

$$w_n = \int_K n^{-s} \nu(ds), \quad n \geq 1.$$ 

Since $0 \notin \psi(K)$ (see (3.3)), one can extend $\psi_+\nu$ trivially to a measure $\mu$ concentrated on $\psi(K) \cup \{0\}$ such that $\mu(\{0\}) = 0$. One may now conclude from Lemma 3.1(ii) that $w$ is a Hausdorff log-moment sequence with representing measure $\mu$. On the other hand, if $w$ is a Hausdorff log-moment sequence with representing measure $\mu$ concentrated on $K$ such that $0 \in K$ and $\mu(\{0\}) = 0$, then by (3.8) (see the first proof of Theorem 3.1), we obtain

$$\mu(\{0\}) = w_1 - \varphi_+\mu(K),$$ 

and hence $w_1 = \varphi_+\mu(K)$. \hfill $\square$

It is well-known that the Riesz–Haviland theorem can be employed to obtain the solution of the multi-dimensional Hausdorff moment problem (see [16, Section 3.2]; for different variants of Riesz–Haviland theorem, see [7, Theorem 1.1], [6, Theorems A and B] and [2, Theorem 2.6]). Thus, in view of Proposition 2.1 and Theorem 3.1, it is natural to look for a solution of the Hausdorff log-moment problem. Although we could not obtain an intrinsic characterization of Hausdorff log-moment sequences (unlike the case of Hausdorff moment sequences; see
[18, Theorem 4.17.4]), it is possible to obtain a handy characterization of these sequences that exploits the theory of completely monotone functions (we would like to draw attention of the reader to [10], which explores a connection between Dirichlet series and completely monotone functions in the context of an approximation problem).

**Theorem 3.2** For a positive integer \( j \), let \( \{w_n\}_{n \geq j} \) be a sequence of non-negative real numbers. Then the following statements are valid:

(i) \( \{w_n\}_{n \geq 1} \) is a Hausdorff log-moment sequence if and only if there exists a unique completely monotone function \( f : [0, \infty) \to [0, \infty) \) such that \( f(0) \leq w_1 \) and

\[
  w_n = (w_1 - f(0)) \chi_{\{1\}}(n) + f(\log(n)), \quad n \geq 1,
\]

where \( \chi_{\{1\}} : \mathbb{Z}_+ \to \mathbb{R} \) denotes the indicator function of \( \{1\} \). If \( \{w_n\}_{n \geq 1} \) is a Hausdorff log-moment sequence with the representing measure \( \mu \), then we may choose \( f(0) \) to be \( w_1 - \mu(\{0\}) \).

(ii) for \( j \geq 2 \), \( \{w_n\}_{n \geq j} \) is a Hausdorff log-moment sequence if and only if there exists a unique completely monotone function \( f : [\log(j), \infty) \to [0, \infty) \) such that

\[
  w_n = f(\log(n)), \quad n \geq j.
\]

If this happens, then the representing measure of \( f \) is equal to \( \varphi, \mu \) (see (3.3)).

It is natural to compare the formulas (3.2) and (3.10), and it is tempting to explore the possibility to deduce the existence part of Theorem 3.2(i) from Theorem 3.1. Unfortunately, the method of proof of Theorem 3.1 does not extend beyond the case \( j = 1 \).

The proof of Theorem 3.2 occupies parts of Sects. 4 and 5. It is worth noting that the proof of the uniqueness part of this theorem relies on Carlson-Fuchs’s uniqueness theorem for holomorphic functions on the right half plane of exponential type (see [5, 8]). In Sect. 5, we also discuss applications of Theorem 3.2 to the Helson matrices and the completely monotone sequences (see Corollaries 5.1 and 5.2).

**4 Basic properties**

In this section, we closely examine Hausdorff log-moment sequences. In particular, we present several algebraic and structural properties of the Hausdorff log-moment sequences. Some of these properties will be used in the proof of Theorem 3.2.

The following proposition provides some algebraic properties of the Hausdorff log-moment sequences:
Proposition 4.1 For a positive integer \( j \), let \( w = \{w_n\}_{n \geq j} \) be a Hausdorff log-moment sequence with a representing measure \( \mu \) and \( v = \{v_n\}_{n \geq j} \) be a Hausdorff log-moment sequence with representing measure \( v \). The following statements are valid:

(i) \( w + v \) is a Hausdorff log-moment sequence with representing measure \( \mu + v \),
(ii) if \( c \geq 0 \) is a real number, then \( cw \) is a Hausdorff log-moment sequence with representing measure \( c\mu \),
(iii) if \( \mu(\{0\}) = 0 \) and \( v(\{0\}) = 0 \), then the point-wise product \( wv \) of \( w \) and \( v \) is a Hausdorff log-moment sequence with representing measure equal to \( f_*(\mu|_{[0,1]} \otimes v|_{[0,1]}) \), where \( f : [0,1] \times [0,1] \rightarrow [0,1] \) is given by \( f(x,y) = xy \) and \( \mu|_{[0,1]} \otimes v|_{[0,1]} \) is a uniquely determined Radon measure concentrated on \( \{0\} \times [0,1] \) such that the restriction \( \mu|_{[0,1]} \otimes v|_{[0,1]} \) to the product \( \sigma \)-algebra coincides with the product measure \( \mu|_{[0,1]} \times v|_{[0,1]} \).

In particular, the set of Hausdorff log-moment sequences forms a convex cone.

Proof We leave the verification of (i) and (ii) to the reader.

To see (iii), assume that \( \mu(\{0\}) = 0 \) and \( v(\{0\}) = 0 \). By Remark 2.1(2), \( \mu \) and \( v \) are \( \sigma \)-finite measures. By [4, Corollary 2.1.11], there exists a uniquely determined Radon measure \( \mu|_{[0,1]} \otimes v|_{[0,1]} \) such that the restriction of \( \mu|_{[0,1]} \otimes v|_{[0,1]} \) to the product \( \sigma \)-algebra coincides with the product measure \( \mu|_{[0,1]} \times v|_{[0,1]} \). Note that for any integer \( n \geq j \) and \( g : [0,1] \rightarrow [0,1] \) given by \( g(x) = x^{\log(n)} \),

\[
w_n v_n = \int_{[0,1]} \int_{[0,1]} (xy)^{\log(n)} \mu(dx)v(dy)
= \int_{[0,1]} \int_{[0,1]} g_{\circ}f(x,y) \mu|_{[0,1]} \otimes v|_{[0,1]} (dy)
\stackrel{(*)}{=} \int_{[0,1] \times [0,1]} g_{\circ}f(x,y) \mu|_{[0,1]} \otimes v|_{[0,1]} (d(x,y))
= \int_{[0,1]} g(x)f_{\circ}(\mu|_{[0,1]} \otimes v|_{[0,1]})(dx) \quad \text{(by change of variables)}
= \int_{[0,1]} x^{\log(n)} f_{\circ}(\mu|_{[0,1]} \otimes v|_{[0,1]})(dx),
\]

where \( (*) \) follows from an analogue of Fubini–Tonelli theorem for Radon measures (see [4, Theorem 2.1.12]). Since \( f_{\circ}(\mu|_{[0,1]} \otimes v|_{[0,1]}|_{[0,1]} \) is a Radon measure (see [4, Proposition 2.1.15]), \( wv \) is a Hausdorff log-moment sequence with representing measure \( f_{\circ}(\mu|_{[0,1]} \times v|_{[0,1]} \).

The following proposition provides some structural properties of Hausdorff log-moment sequences (cf. [18, Theorem 4.17.4]). The reader is referred to [4, 15, 21] for the definitions of positive semi-definite matrix, completely monotone sequence, minimal completely monotone sequence, completely monotone function, Bernstein function and related notions.
Proposition 4.2 For a positive integer \( j \), let \( \{ w_n \}_{n \geq j} \) be a Hausdorff log-moment sequence with a representing measure \( \mu \) concentrated on \([0, 1]\). The following statements are valid:

(i) for every finite subset \( F \) of integers \( n \geq j \), the matrix \( (w_{pq})_{p,q \in F} \) is positive semi-definite,

(ii) if \( k \geq j \) is an integer, then \( \{ w_{km} \}_{m \geq 0} \) is a completely monotone sequence,

(iii) for every integer \( k \geq j \), \( \{ w_{kn} \}_{n \geq 1} \) is a Hausdorff log-moment sequence with representing measure given by \( t^{\log(k)} \mu(dt) \).

Proof

(i) For any complex numbers \( c_1, \ldots, c_n \) and integers \( k_1, \ldots, k_n \) bigger than or equal to \( j \),

\[
\sum_{p,q=1}^{n} c_p \overline{c_q} w_{kpq} = \int_{[0,1]} \sum_{p,q=1}^{n} c_p \overline{c_q} t^{\log(k_p k_q)} \mu(dt) = \int_{[0,1]} \left| \sum_{p=1}^{n} c_p t^{\log(k_p)} \right|^2 \mu(dt),
\]

which is clearly non-negative.

(ii) For integers \( n \geq 0 \) and \( m \geq 1 \),

\[
\sum_{i=0}^{n} (-1)^i \binom{n}{i} w_{km+i} = \int_{[0,1]} \sum_{i=0}^{n} (-1)^i \binom{n}{i} t^{(m+i) \log(k)} \mu(dt) = \int_{[0,1]} t^m t^{\log(k)} (1 - t^{\log(k)})^n \mu(dt),
\]

which is non-negative, as required.

(iii) For any integer \( k \geq j \),

\[
w_{kn} = \int_{[0,1]} t^{\log(kn)} \mu(dt) = \int_{[0,1]} t^{\log(n) \log(k)} \mu(dt), \quad n \geq 1.
\]

Thus \( \{ w_{kn} \}_{n \geq 1} \) is a Hausdorff log-moment sequence with representing measure given by \( t^{\log(k)} \mu(dt) \).

Any Hausdorff log-moment sequence \( \{ w_n \}_{n \geq 1} \) is determinate, that is, its representing measure is unique. Indeed, we have following general fact (see Corollary 5.3 for an improvement).

Proposition 4.3 (Uniqueness) Let \( j \) be a positive integer. If \( \mu \) and \( \nu \) are two finite representing measures for the Hausdorff log-moment sequence \( \{ w_n \}_{n \geq j} \), then \( \mu = \nu \) provided they have the same total mass. In particular, if \( \{ w_n \}_{n \geq 1} \) is a Hausdorff log-moment sequence, then its representing measure is unique.
Proof Let $\mu$ and $\nu$ be two finite representing measures for $\{w_n\}_{n \geq j}$ and assume that $\mu([0,1]) = \nu([0,1]) < \infty$. For a positive integer $n$, consider the function $f_n(t) = t^{\log(n)}$, $t \in [0,1]$. Let $S$ denote the real linear span of $\{f_1, f_j, f_{j+1}, \ldots\}$. Note that

$$\int_{[0,1]} f(t)\mu(dt) = \int_{[0,1]} f(t)\nu(dt), \quad f \in S. \quad (4.1)$$

Since $f_n f_n = f_{nn}$, $m, n \in \mathbb{Z}_+$, $S$ is a real unital sub-algebra of $C([0,1], \mathbb{R})$. Hence, by the Stone–Weierstrass theorem (see [18, Theorem 2.5.5]), $S$ is dense in $C([0,1], \mathbb{R})$. Thus, the identity $(4.1)$ holds for any $f \in C([0,1], \mathbb{R})$. The desired conclusion now follows from the Riesz–Markov theorem (see [18, Theorem 4.8.8]). \(\square\)

The notion of so-called minimal Hausdorff log-moment sequences plays a role in the proof of Theorem 3.2 and this notion is similar to the one that appears in the context of completely monotone sequence (refer to [21, Chapter IV.14]).

Definition 4.1 A Hausdorff log-moment sequence $\mathbf{w} = \{w_n\}_{n \geq 1}$ is said to be minimal if for any $\epsilon > 0$, $\mathbf{w} - \epsilon \chi_{\{1\}}$ is not a Hausdorff log-moment sequence, where $\chi_{\{1\}} : \mathbb{Z}_+ \to \mathbb{R}$ denotes the indicator function of $\{1\}$.

As in the case of completely monotone sequences, the minimal Hausdorff log-moment sequences can be described easily.

We now identify minimal Hausdorff log-moment sequences.

Proposition 4.4 A Hausdorff log-moment sequence $\mathbf{w} = \{w_n\}_{n \geq 1}$ is minimal if and only if the representing measure $\mu$ of $\mathbf{w}$ satisfies $\mu(\{0\}) = 0$.

Proof If $\mathbf{w}$ is a minimal Hausdorff log-moment sequence with a representing measure $\mu$ such that $\mu(\{0\}) > 0$, then

$$w_1 - \mu(\{0\}) = \mu((0,1]), \quad w_n = \int_{(0,1]} t^{\log(n)}\mu(dt), \quad n \geq 2.$$ 

Hence $\mathbf{w} - \mu(\{0\})\chi_{\{1\}}$ is a Hausdorff log-moment sequence with the representing measure $\hat{\mu}$ obtained by extending $\mu$ trivially to $[0,1]$:

$$\hat{\mu}(\sigma) = \mu(\sigma \setminus \{0\}) \quad \text{for every Borel subset } \sigma \text{ of } [0,1]. \quad (4.2)$$

This contradicts our assumption that $\mathbf{w}$ is a minimal Hausdorff log-moment sequence.

Conversely, suppose that $\mathbf{w}$ is a Hausdorff log-moment sequence with a representing measure $\mu$. If there exists $\epsilon > 0$ such that $\mathbf{w}_\epsilon = \mathbf{w} - \epsilon \chi_{\{1\}}$ is a Hausdorff log-moment sequence with a representing measure $\nu$, then by Proposition 4.1 and Example 2.1, $\mathbf{w}_\epsilon + \epsilon \chi_{\{1\}}$ is a Hausdorff log-moment sequence with a representing measure $\nu + \epsilon \delta_0$. By Proposition 4.3, $\mu$ must coincide with $\nu + \epsilon \delta_0$, and hence $\mu(\{0\}) = \nu(\{0\}) + \epsilon > 0$. \(\square\)
Remark 4.1 Proposition 4.4 suggests that for any integer $j \geq 2$, any Hausdorff log-moment sequence $w = \{w_n\}_{n \geq j}$ can be considered as a minimal Hausdorff log-moment sequence. Indeed, if $\mu$ is a representing measure of $w$ and $\tilde{\mu}$ is the trivial extension of $\mu|_{[0,1]}$ (see (4.2)), then since for $n \geq j > 1$, $t^{\log(n)}|_{t=0} = 0$, after replacing $\mu$ by $\tilde{\mu}$ if necessary, we may assume that $\mu(\{0\}) = 0$. We used here the convention that $0 \cdot \infty = 0$.

We need the following corollary in the proof of Theorem 3.2.

Corollary 4.1 Let $w = \{w_n\}_{n \geq 1}$ be a sequence of non-negative real numbers. Then the following statements are valid:

(i) if $w$ is a Hausdorff log-moment sequence with the representing measure $\mu$, then the sequence $w - \mu(\{0\}) \chi_{\{1\}}$ is a minimal Hausdorff log-moment sequence with representing measure $\tilde{\mu}$ given by (4.2),

(ii) if there exists a real number $c \in [0, w_1]$ such that $w - c \chi_{\{1\}}$ is a Hausdorff log-moment sequence with representing measure $\nu$, then $w$ is a Hausdorff log-moment sequence with representing measure $\nu + c\delta_0$, where $\delta_0$ denote the atomic measure with point mass at $\{0\}$.

Proof

(i) Assume that $\{w_n\}_{n \geq 1}$ is a Hausdorff log-moment sequence. By Proposition 2.1, there exists a finite Radon measure $\mu$ on $[0, 1]$ such that $w_n = \int_{[0,1]} t^{\log(n)} \mu(dt)$ for every integer $n \geq 1$. Let $\tilde{\mu}$ be as defined in (4.2). Note that $\tilde{\mu}(\{0\}) = 0$ and $\tilde{\mu}$ is a Radon measure that satisfies

$$w_n = \int_{[0,1]} t^{\log(n)} \mu(dt)$$

$$= \begin{cases} 
\mu(\{0\}) + \int_{[0,1]} t^{\log(n)} \tilde{\mu}(dt) & \text{if } n = 1, \\
\int_{[0,1]} t^{\log(n)} \tilde{\mu}(dt) & \text{if } n \geq 2.
\end{cases}$$

Thus the sequence $w - \mu(\{0\}) \chi_{\{1\}}$ is a Hausdorff log-moment sequence with representing measure $\tilde{\mu}$. Since $\tilde{\mu}(\{0\}) = 0$, by Proposition 4.4, $w - \mu(\{0\}) \chi_{\{1\}}$ is a minimal Hausdorff log-moment sequence.

(ii) Assume that the sequence $w - c \chi_{\{1\}}$ is a Hausdorff log-moment sequence for some real number $c \in [0, w_1]$. Since the sequence $\chi_{\{1\}}$ is also a Hausdorff log-moment sequence with representing measure $\delta_0$ (see Example 2.1(c)), the desired conclusion is now immediate from Proposition 4.1. $\Box$
5 Proof of Theorem 3.2

We need a couple of facts to complete the proof of Theorem 3.2.

Lemma 5.1  For a positive integer \( j \), let \( \{ \omega_n \}_{n \geq j} \) be a Hausdorff log-moment sequence with a representing measure \( \mu \) concentrated on \([0, 1]\). If \( \mu(\{0\}) = 0 \), then there exists a unique completely monotone function \( f : [\log(j), \infty) \to [0, \infty) \) such that

\[
\omega_n = f(\log(n)), \quad n \geq j. \tag{5.1}
\]

If this happens, then the representing measure of \( f \) is equal to \( \varphi_* \mu \) (see (3.3)).

Proof  Assume that \( \mu(\{0\}) = 0 \). By Lemma 3.1(i) (with \( \varphi \) given by (3.3)), we obtain

\[
\omega_n = \int_{[0,\infty)} e^{-\log(n)s} \varphi_* \mu(ds), \quad n \geq j.
\]

Since \( \mu(0,1] \) is a Radon measure, \( \varphi_* \mu \) is a Radon measure concentrated on \([0, \infty) \) (see [4, Proposition 2.1.15]). One may now define the function \( f : [\log(j), \infty) \to [0, \infty) \) by

\[
f(\lambda) = \int_{[0,\infty)} e^{-\lambda s} \varphi_* \mu(ds), \quad \lambda \in [\log(j), \infty).
\]

Thus the representing measure of \( f \) is equal to \( \varphi_* \mu \). Note that \( f(\lambda) \leq \omega_j \) and hence \( f \) is well-defined. Clearly, \( f(\log(n)) = \omega_n \) for every \( n \geq j \). By [21, Theorem IV.12b] and the remark prior to [21, Theorem IV.12c], \( f \) is completely monotone. This completes the proof of the existence of \( f \).

To see the uniqueness of \( f \), for \( k = 1, 2 \), consider the completely monotone function \( f_k : [\log(j), \infty) \to [0, \infty) \) satisfying \( f_k(\log(n)) = \omega_n \) for every integer \( n \geq j \). Fix \( k = 1, 2 \) and let \( \nu_k \) be a positive Radon measure satisfying

\[
f_k(\lambda) = \int_{[0,\infty)} e^{-\lambda s} \nu_k(ds), \quad \lambda \in [\log(j), \infty).
\]

For a real number \( a \), let \( \mathbb{H}_a \) denote the right half plane \( \{ z \in \mathbb{C} : \Re(z) > a \} \). We now define a function \( F_k \) on the closed right half plane \( \overline{\mathbb{H}}_{\log(j)} \) by setting

\[
F_k(z) = \int_{[0,\infty)} e^{-zt} \nu_k(dt), \quad z \in \overline{\mathbb{H}}_{\log(j)}.
\]

Since \( |e^{-zt}| = e^{-\Re(z)t} \leq e^{-\log(j)t} \) for all \( z \in \overline{\mathbb{H}}_{\log(j)} \) and \( t \in [0, \infty) \), by the dominated convergence theorem, \( F_k \) is continuous on \( \overline{\mathbb{H}}_{\log(j)} \). By theorems of Fubini and Morera, it is easily seen that \( F_k \) is holomorphic in \( \overline{\mathbb{H}}_{\log(j)} \) (cf. [3, Proposition 4.1]). Let \( L \) be a holomorphic branch of \( \log \) defined on the right half plane \( \mathbb{H}_j \) and define \( H_k : \mathbb{H}_0 \to \mathbb{C} \) by

\[
H_k(z) = F_k(L(z + j)), \quad z \in \mathbb{H}_0.
\]
Note that $H_k$ is a well-defined holomorphic function. Since $F_k$ is bounded, so is $H_k$. Since $f_k(\log(n)) = w_n$ for every integer $n \geq j$,
\[ H_k(n) = F_k(L(n + j)) = f_k(\log(n + j)) = w_{n+j}, \quad n \geq 0. \]
Clearly, $H_1 - H_2$ is a bounded holomorphic function on $\mathbb{H}_0$ satisfying
\[ (H_1 - H_2)(n) = 0 \text{ for every integer } n \geq 0. \]
Hence, by [5, Corollary 9.5.4], $H_1 - H_2$ is identically zero (cf. [5, Theorem 9.2.1]). It follows that $f_1(\log(x + j)) = f_2(\log(x + j))$ on $[0, \infty)$. Since $x \mapsto \log(x + j)$ is a bijection from $[0, \infty)$ onto $[\log(j), \infty)$, $f_1 = f_2$. This completes the proof of the uniqueness part. \qed

We also need a converse of Lemma 5.1.

**Lemma 5.2** For a positive integer $j$, let $\{w_n\}_{n \geq j}$ be a sequence of non-negative real numbers. If there exists a completely monotone function $f : [\log(j), \infty) \to [0, \infty)$ such that (5.1) holds, then $\{w_n\}_{n \geq j}$ is a Hausdorff log-moment sequence and $\{w_{n+j}\}_{n \geq 0}$ is a minimal completely monotone sequence.

**Proof** Let $\nu$ be a positive Borel measure on $[0, \infty)$ such that
\[ f(\lambda) = \int_{[0, \infty)} e^{-\lambda t} \nu(dt), \quad \lambda \in [\log(j), \infty) \]
and assume that (5.1) holds. By Lemma 3.1(ii) (with $\psi$ given by (3.3)), we obtain
\[ f(\lambda) = \int_{[0, 1]} t^j \psi(\lambda;t) \nu(dt), \quad \lambda \in [\log(j), \infty). \]
It is clear from (5.1) that $\{w_n\}_{n \geq j}$ is a Hausdorff log-moment sequence.
To prove the remaining part, define $g : [0, \infty) \to [0, \infty)$ by
\[ g(x) = \log(x + j), \quad x \in [0, \infty), \]
and note that $g$ is a Bernstein function. Hence, by [15, Theorem 3.6], $f \circ g : [0, \infty) \to [0, \infty)$ is a completely monotone function. It now follows from [21, Theorem IV.14b] that the sequence $\{f \circ g(n)\}_{n \geq 0}$ is a minimal completely monotone sequence. However, $f \circ g(n) = w_{n+j}$ for $n \geq 0$ completing the verification. \qed

**Proof of Theorem 3.2** (i) Assume that $w = \{w_n\}_{n \geq 1}$ is a Hausdorff log-moment sequence with the representing measure $\mu$. By Corollary 4.1(i), the sequence $w - \mu(\{0\}) x_{(1)}$ is a minimal Hausdorff log-moment sequence with a representing measure $\tilde{\mu}$ (see (4.2)). By Lemma 5.1 (with $j = 1$), there exists a unique completely monotone function $f : [0, \infty) \to [0, \infty)$ such that
\[ w_n - \mu(\{0\}) x_{(1)}(n) = f(\log(n)), \quad n \geq 1. \]
Since $w_1 − f(0) = μ(\{0\})$, we have $w_1 ≥ f(0)$, and we obtain the necessity part. To see the sufficiency part, note that by (3.10) and Lemma 5.2, the sequence $w − (w_1 − f(0)) χ_{\{1\}}$ is a Hausdorff log-moment sequence. One may now apply Corollary 4.1(ii).

(ii) This follows from Remark 4.1 and Lemmas 5.1 and 5.2. The remaining part follows from Lemma 5.1.

If $w$ is a Hausdorff log-moment sequence with the completely monotone function $f$ satisfying either (3.10) $(j = 1)$ or (3.11) $(j ≥ 2)$, then we refer to $(w, f)$ as the Dirichlet pair.

**Example 5.1** (Example 2.1 continued) Let us see some examples of Dirichlet pairs $(w, f)$.

(a) For $α < 0$, consider a function $f_α : [\log(2), ∞) → [0, ∞)$ given by $f_α(λ) = λ^α$, $λ ∈ [\log(2), ∞)$. Then $f_α$ is a completely monotone function with the representing measure $v_α(dt) = \frac{\Gamma(−α)}{\Gamma(1)}(dt)$, that is,

$$f_α(λ) = \int_{[0,∞)} e^{−λt}v_α(dt), \quad λ ∈ [\log(2), ∞).$$

If $w_α = \{(\log(n))^α\}_{n≥2}$, then $(w_α, f_α)$ is a Dirichlet pair.

(b) For $α > 0$, consider a function $f_α : [0, ∞) → [0, ∞)$ given by $f_α(λ) = \frac{1}{λ+α}$, $λ ∈ [0, ∞)$. Then $f_α$ is a completely monotone function with the representing measure $v_α(dt) = e^{−λt}(dt)$, that is,

$$f_α(λ) = \int_{[0,∞)} e^{−λt}v_α(dt), \quad λ ∈ [0, ∞).$$

Note that $w_1 = f(0)$. Thus if $w = \{\frac{1}{\log(n)+α}\}_{n≥1}$, then $(w_α, f_α)$ is a Dirichlet pair.

(c) For $α ∈ (0, 1]$, consider a function $f_α : [0, ∞) → [0, ∞)$ given by $f_α(λ) = λ^α$, $λ ∈ [0, ∞)$. Then $f_α$ is a completely monotone function with the representing measure $v_α(dt) = δ_{\{−\log(α)\}}(dr)$, that is,

$$f_α(λ) = \int_{[0,∞)} e^{−λt}v_α(dt), \quad λ ∈ [0, ∞).$$

Note that $w_1 = f(0)$. Thus, if $w_α = \{α^{\log(n)}\}_{n≥1}$, then $(w_α, f_α)$ is a Dirichlet pair. Moreover, $(χ_{\{1\}}, 0)$ is also a Dirichlet pair (the case of $α = 0$).

Let us see an application of Hausdorff log-moment sequences to the theory of Helson matrices. Following [12], we say that a matrix $(a_{m,n})_{m,n=1}^∞$ is a Helson matrix if there exists a sequence $w = \{w_n\}_{n=1}^∞$ such that

$$a_{m,n} = w_{mn}, \quad m, n ≥ 1.$$
In this case, the matrix \( (a_{m,n})_{m,n=1}^{\infty} \) is denoted by \( M(\mathbf{w}) \). If \( \mathbf{w} \) is a Hausdorff log-moment sequence, then by \([12, \text{Theorem 5.1}]\) and the discussion prior to it, the Helson matrix \( M(\mathbf{w}) \) defines a bounded linear operator on \( \ell^2(\mathbb{Z}_+) \) provided \( \mathbf{w} \in \ell^2(\mathbb{Z}_+) \) satisfies

\[
w_n \leq \frac{C}{\sqrt{n \log(n)}}, \quad n \geq 2.
\]

This combined with Theorem 3.2(i) yields the following:

**Corollary 5.1** Assume that \( j = 1 \) and let \((\mathbf{w}, f)\) be a Dirichlet pair such that \( \mathbf{w} \in \ell^2(\mathbb{Z}_+) \) and \( w_1 = f(0) \). If there exists a positive real number \( C \) such that

\[
xf(x) \leq Ce^{-x/2}, \quad x \in [0, \infty),
\]

then the Helson matrix \( M(\mathbf{w}) \) defines a bounded linear operator on \( \ell^2(\mathbb{Z}_+) \).

The foregoing corollary can be used to check that the Helson matrix \( M = [\alpha^{\log(mn)}]_{m,n=1}^{\infty} \) defines a bounded linear operator on \( \ell^2(\mathbb{Z}_+) \) for any \( \alpha \in (0, \frac{1}{\sqrt{e}}) \) (this fact can also be verified directly using the Cauchy–Schwarz inequality). In view of Corollary 5.1, this is immediate from Example 5.1(c) and the fact that the function \( g(\lambda) = \lambda^\alpha e^{\frac{\lambda^2}{2}}, \lambda \in [0, \infty) \) is a bounded function.

As another application of Theorem 3.2, we decipher the relation between Hausdorff log-moment sequences and completely monotone sequences.

**Corollary 5.2** For a positive integer \( j \), let \( \mathbf{w} = \{w_n\}_{n \geq j} \) be a Hausdorff log-moment sequence with a representing measure \( \mu \). Then \( \{w_{n+j}\}_{n \geq 0} \) is a completely monotone sequence. Moreover, the following statements are valid:

\begin{enumerate}[label=(\roman*)]
    
    
    
    (i) if \( j = 1 \), then the representing measure \( \nu \) of the completely monotone sequence \( \{w_{n+1}\}_{n \geq 0} \) is given by

\[
\nu(ds) = \int_{[0,\infty)} \int_{\psi^{-1}(s\setminus\{0\})} \left( \frac{\lambda^{s-1}e^{-t}}{\Gamma(s)} \right) dt \varphi_s \hat{\mu}(ds) + \mu(\{0\}) \delta_0(ds)
\]

for every Borel subset \( \sigma \) of \([0, 1] \), where \( \varphi, \psi \) are as given in Lemma 3.1, and \( \hat{\mu} \) is given by (4.2),

(ii) if \( j \geq 2 \), then \( \{w_{n+j}\}_{n \geq 0} \) is a minimal completely monotone sequence.

\end{enumerate}

**Proof** Assume that \( j = 1 \). By Theorem 3.2(i), there exists a completely monotone function \( f : [0, \infty) \to [0, \infty) \) such that

\[
w_n - \mu(\{0\}) \chi_{\{1\}}(n) = f(\log(n)), \quad n \geq 1.
\]

This combined with Lemma 5.2 yields that \( \{w_{n+1} - \mu(\{0\}) \chi_{\{1\}}(n + 1)\}_{n \geq 0} \) is completely monotone. Since the sequence \( \{\chi_{\{1\}}(n + 1)\}_{n \geq 0} \) is also a completely
monotone sequence, by the fact that the set of completely monotone sequences is a cone (see [4, p. 130]), \{w_{n+1}\}_{n \geq 0} is completely monotone. To see the remaining part, note that by Corollary 4.1(i), the Hausdorff log-moment sequence \(w - \mu(\{0\}) \chi_{\{1\}}\) is a minimal Hausdorff log-moment sequence with the representing measure \(\hat{\mu}\) (see (4.2)). Hence, by Lemma 5.1, the representing measure of \(f\) is given by \(\varphi_s \hat{\mu}\) (see (3.3)). We now apply [17, Theorem 7.2] to \(f\) and the Bernstein function \(g(x) = \log(x+1), x \in [0, \infty)\), to conclude that the representing measure \(\eta\) of the completely monotone function \(f \circ g\) is given by

\[
\eta(\sigma) = \int_{[0, \infty)} \nu_s(\sigma) \varphi_s \hat{\mu}(ds) \quad \text{for every Borel subset } \sigma \text{ of } [0, \infty), \tag{5.4}
\]

where \(\nu_s\) (the so-called convolution semi-group of \(g\)) is the representing measure of the completely monotone function \(e^{-sg}, s \geq 0\). Since \(e^{-sg(x)} = \frac{1}{(x+1)^s}\) for \(x \geq 0\), the measure \(\nu_s\) is the weighted Lebesgue measure given by

\[
\nu_s(dr) = \begin{cases} \frac{e^{-r}}{\Gamma(s)} dr & \text{if } s \in (0, \infty), \\ \delta_0 & \text{if } s = 0. \end{cases} \tag{5.5}
\]

By (5.3) and Lemma 3.1(ii), the representing measure of the completely monotone sequence \(\{w_{n+1} - \mu(\{0\}) \chi_{\{1\}(n+1)}\}_{n \geq 0}\) is \(\varphi_s \hat{\eta}\) (see (4.2)). By Corollary 4.1, the representing measure of \(\{w_{n+1}\}_{n \geq 0}\) is \(\varphi_s \hat{\eta} + \mu(\{0\})\delta_0\). This combined with (5.4) and (5.5) yields (5.2).

If \(j \geq 2\), then by Theorem 3.2(ii) and Lemma 5.2, \(\{w_{n+j}\}_{n \geq 0}\) is a minimal completely monotone sequence. This yields (ii).

**Example 5.2** For a positive integer \(j\), let \(\mathcal{CM}_j\) and \(\mathcal{DM}_j\) denote the cones of completely monotone sequences \(\{a_{n+j}\}_{n \geq 0}\) and Hausdorff log-moment sequences \(\{a_n\}_{n \geq j}\) respectively. By Corollary 5.2, \(\mathcal{DM}_j \subseteq \mathcal{CM}_j\) for every positive integer \(j\). Moreover, this inclusion is strict:

\[
\mathcal{DM}_j \subset \mathcal{CM}_j, \quad j \geq 1.
\]

Indeed, for any integer \(j \geq 1\), \(\{\frac{1}{n+j+1}\}_{n \geq 0}\) is a minimal completely monotone sequence with the representing measure \(x^j dx\). On the other hand, since the matrix \(A = (\frac{1}{pq+1})_{p,q=1}^{j+1}\) has determinant less than 0, by Proposition 4.2(i), \(\{\frac{1}{n+1}\}_{n \geq j}\) is not a Hausdorff log-moment sequence.

We have already seen that Hausdorff log-moment sequence \(\{w_{n}\}_{n \geq 1}\) is determinate (see Proposition 4.3). Surprisingly, for any integer \(j \geq 2\), a Hausdorff log-moment sequence \(\{w_n\}_{n \geq j}\) is almost determinate in the following sense:

**Corollary 5.3** Let \(j \geq 2\) be a positive integer. If \(\{w_n\}_{n \geq j}\) is a Hausdorff log-moment sequence, then the restriction of its representing measure to \((0, 1]\) is uniquely determined.
**Proof** Let \( \{w_n\}_{n \geq j} \) be a Hausdorff log-moment sequence with two representing measures \( \mu \) and \( \nu \).

By Theorem 3.2, there exists a unique completely monotone function \( f : [\log(j), \infty) \to \mathbb{R} \) with representing measure \( \varphi_\mu \) (see (3.3)) such that \( f(\log(n)) = w_n \) for every integer \( n \geq j \). Note that representing measure of a completely monotone function on \([\log(j), \infty)\) is unique. This fact may be derived from Bernstein’s Theorem (see [15, Theorem 1.4]) by considering the completely monotone function on \([0, \infty)\) given by \( s \mapsto f(s + \log(j)) \), \( s \in [0, \infty) \). We may now conclude that \( \varphi_\mu \) and \( \varphi_\nu \) must agree on \([0, \infty)\).

Since \( \varphi_\mu : (0, 1] \to [0, \infty) \) is a bijection, we obtain \( \mu|_{[0,1]} = \nu|_{[0,1]} \) completing the proof.

We conclude the paper with some remarks revealing the relationship between Hausdorff log-moment sequences \( w = \{w_n\}_{n \geq 1} \) and the associated linear functional \( L_w \). Note that the conclusion of Proposition 4.2 can be rephrased in terms of the functional \( L_w \) to derive the following necessary conditions for a sequence \( w \) to be a Hausdorff log-moment sequence.

**Proposition 5.1** If \( \{w_n\}_{n \geq 1} \) is a Hausdorff log-moment sequence, then

\[
L_w(k^{-s}q^2) \geq 0, \quad q \in D[s], \quad k \geq 1. \tag{5.6}
\]

**Proof** To see (5.6), note that by Proposition 4.2(iii), for every integer \( k \geq 1, \{w_{kn}\}_{n \geq 1} \) is a Hausdorff log-moment sequence. Hence, by Proposition 4.2(i), \( (w_{kpq})_{p, q \in F} \) is positive semi-definite for every finite subset \( F \) of \( \mathbb{Z}_+ \). This is easily seen to be equivalent to (5.6) completing the proof.

We do not know whether the conditions (5.6) ensure that the sequence \( w \) is a Hausdorff log-moment sequence (cf. [16, Theorem 3.12] and [12, Theorem 3.3]). In particular, one may ask for a counterpart of [16, Proposition 3.2] for the Dirichlet polynomials (cf. the discussion following [12, Theorem 3.3]). In the unavailability of a required “Positivstellensatz”, it seems desirable to find a counterpart of the method employed in [20, Theorem 2.3] for Dirichlet polynomials. We believe these questions warrant additional attention.

**Appendix: Second proof of Theorem 3.1**

It seems that there could be a direct proof of Theorem 3.1, which resembles a part of the solution of Hausdorff moment problem (see [18, Proposition 4.17.7]). Interestingly, there is an alternate proof of Theorem 3.1 of topological flavour and the idea of the proof has been known in the literature (see, for instance, [16, Proofs of Theorems 9.15 and 9.19] and [13, Proof of Proposition 2.1]).

**Proof** (Second proof of Theorem 3.1) We only verify the implication \((i) \Rightarrow (ii)\). In view of [16, Proposition 1.9], we may assume that \( K \) is unbounded. Assume that \( L_w \) is \( D_+[s] \)-positive. Let \([0, \infty]\) denote the Alexandroff one-point compactification of...
\([0, \infty)\) (which is always Hausdorff) and let \(C([0, \infty], \mathbb{R})\) denote the space of real-valued continuous functions on \([0, \infty]\). Let \(S\) denote the real linear space of functions \(F : [0, \infty] \to \mathbb{R}\) given by

\[
S = \{ F \in C([0, \infty], \mathbb{R}) \mid \text{there exists a } f \in D[s] \text{ such that } F |_{[0, \infty)} = f \}.
\]

Note that for every \(F \in S\), there exists a unique \(f \in D[s]\) such that \(F |_{[0, \infty)} = f\) and \(F(\infty) = \lim_{s \to \infty} f(s)\). Moreover, if \(f \in D[s]\), then \(F\) given by

\[
F(s) = \begin{cases} 
  f(s) & \text{if } s \in [0, \infty), \\
  \lim_{s \to \infty} f(s) & \text{if } s = \infty,
\end{cases}
\]

defines an element in \(S\).

Define a linear functional \(M_w : S \to \mathbb{R}\) by \(M_w(F) = L_w(f)\). Let \(\overline{K}\) denote the closure of \(K\) in \([0, \infty]\). By assumption, \(M_w\) is \(\omega^K[s]\)-positive, and hence by [16, Proposition 1.9], there exists a positive Radon measure \(\nu\) on \(\overline{K}\) such that

\[
M_w(F) = \int_{\overline{K}} F(s) \nu(ds), \quad F \in D^\infty[s].
\]

It follows from the discussion in the previous paragraph that for any \(f \in D[s]\), there exists \(F \in S\) such that

\[
L_w(f) = \int_{\overline{K}} F(s) \nu(ds) = \nu(\{\infty\}) F(\infty) + \int_{K} f(s) \nu(ds) = (w_1 - \nu(K)) \lim_{s \to \infty} f(s) + \int_{K} f(s) \nu(ds),
\]

where we used the equality that \(w_1 = \nu(\overline{K})\) (which can be seen by arguing as in the proof of Theorem 3.1) and fact that \(\overline{K} = K \cup \{\infty\}\) (disjoint union). This completes the proof once we notice that \(\nu\) is a finite measure. \(\square\)

References

1. Alpay, D., Jorgensen, P.E.T., Kimsey, D.P.: Moment problems in an infinite number of variables. Infin. Dimens. Anal. Quantum Probab. Relat. Top. 18, 1550024 (2015)
2. Ambrozie, C.-G.: A Riesz–Haviland type result for truncated moment problems with solutions in \(L_1\). J. Oper. Theory 71, 85–93 (2014)
3. Anand, A., Chavan, S.: A moment problem and joint \(q\)-isometry tuples. Complex Anal. Oper. Theory 11, 785–810 (2017)
4. Berg, C., Christensen, J.P.R., Ressel, P.: Harmonic Analysis on Semigroups. Theory of Positive Definite and Related Functions, Graduate Texts in Mathematics, vol. 100. Springer, New York, x+289 pp (1984)
5. Boas, R.P.: Entire Functions. Academic Press Inc., New York, x+276 pp (1954)
6. Cichoń, D., Stochel, J., Szafraniec, F.H.: Riesz–Haviland criterion for incomplete data. J. Math. Anal. Appl. 380, 94–104 (2011)
7. Curto, R., Fialkow, L.A.: An analogue of the Riesz–Haviland theorem for the truncated moment problem. J. Funct. Anal. 255, 2709–2731 (2008)
8. Fuchs, W.H.J.: A generalization of Carlson’s theorem. J. Lond. Math. Soc. 21, 106–110 (1946)
9. Ghasemi, M., Kuhlmann, S., Marshall, M.: Moment problem in infinitely many variables. Isr. J. Math. 212, 989–1012 (2016)
10. Liu, Y.: Approximation by Dirichlet series with nonnegative coefficients. J. Approx. Theory 112, 226–234 (2001)
11. McCarthy, J.E.: Hilbert spaces of Dirichlet series and their multipliers. Trans. Am. Math. Soc. 356, 881–893 (2004)
12. Perfekt, K., Pushnitski, A.: On Helson matrices: moment problems, non-negativity, boundedness, and finite rank. Proc. Lond. Math. Soc. 116, 101–134 (2018)
13. Putinar, M., Scheiderer, C.: Multivariate moment problems: geometry and indeterminateness. Ann. Sc. Norm. Super. Pisa Cl. Sci. 5, 137–157 (2006)
14. Queffelec, H., Queffelec, M.: Diophantine Approximation and Dirichlet Series, 2nd edn. Texts and Readings in Mathematics, vol. 80. Hindustan Book Agency, New Delhi; Springer, Singapore (2020)
15. Schilling, R., Song, R., Vondraček, Z.: Bernstein Functions, Theory and Applications, 2nd edn. de Gruyter Studies in Mathematics, vol. 37. Walter de Gruyter and Co, Berlin (2012)
16. Schmudgen, K.: The Moment Problem, Graduate Texts in Mathematics, vol. 277. Springer, Cham, xii+535 pp (2017)
17. Sendov, H., Shan, S.: New representation theorems for completely monotone and Bernstein functions with convexity properties on their measures. J. Theor. Probab. 28, 1689–1725 (2015)
18. Simon, B.: Real Analysis. With a 68 page companion booklet. A Comprehensive Course in Analysis, Part 1. American Mathematical Society, Providence, xx+789 pp (2015)
19. Tao, T.: An Introduction to Measure Theory, Graduate Studies in Mathematics, vol. 126. American Mathematical Society, Providence, xvi+206 pp (2011)
20. Vasilescu, F.H.: Moment problems for multi-sequences of operators. J. Math. Anal. Appl. 219, 246–259 (1998)
21. Widder, D.V.: The Laplace Transform, Princeton Mathematical Series, vol. 6. Princeton University Press, Princeton, x+406 pp (1941)