Precession and interference in the Aharonov-Casher and scalar Aharonov-Bohm effect

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The ideal scalar Aharonov-Bohm (SAB) and Aharonov-Casher (AC) effect involve a magnetic dipole pointing in a certain fixed direction: along a purely time dependent magnetic field in the SAB case and perpendicular to a planar static electric field in the AC case. We extend these effects to arbitrary direction of the magnetic dipole. The precise conditions for having nondispersive precession and interference effects in these generalized set ups are delineated both classically and quantally. Under these conditions the dipole is affected by a nonvanishing torque that causes pure precession around the directions defined by the ideal set ups. It is shown that the precession angles are in the quantal case linearly related to the ideal phase differences, and that the nonideal phase differences are nonlinearly related to the ideal phase differences. It is argued that the latter nonlinearity is due the appearance of a geometric phase associated with the nontrivial spin path. It is further demonstrated that the spatial force vanishes in all cases except in the classical treatment of the nonideal AC set up, where the occurring force has to be compensated by the experimental arrangement. Finally, for a closed space-time loop the local precession effects can be inferred from the interference pattern characterized by the nonideal phase differences and the visibilities. It is argued that this makes it natural to regard SAB and AC as essentially local and nontopological effects.

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I. INTRODUCTION

A. Key discoveries

In the electric and magnetic Aharonov-Bohm (AB) effect a charged particle exhibits a physical effect due to an electromagnetic potential although the electromagnetic field and force vanish along the path of the particle. To each of these, a dual effect exists, the so-called scalar Aharonov-Bohm (SAB) effect and Aharonov-Casher (AC) effect, respectively. In the dual set ups, an electrically neutral particle carrying a nonvanishing magnetic dipole moment exhibits a physical effect due to an electromagnetic field although no force is acting on the particle. Despite this similarity, the physics behind SAB and AC differs crucially from that of the electric and magnetic AB effect. To discuss these differences and to enlighten the extra richness of the dual effects is the major aim of this paper.

In the electric Aharonov-Bohm (EAB) effect (see Fig. 1a), a wave packet carrying electric charge $q$ is split coherently by a beam splitter, each beam passing through a cylindrical metal tube at vanishing electric potential. When the wave packets are well inside the tubes, the beam pair is exposed to a potential difference that varies as a pure function of time. Well before the wave packets leave the tubes, the potential vanishes again. The beams are brought together to interfere coherently at a second beam splitter. This process is essentially captured by the Hamiltonian

$$H = q \varphi(t),$$

where $\varphi$ is a function of time inside each tube. The interference of the wave packets depends on the phase dif-
ference between the two beams
\[ \phi_{\text{EAB}} = -\frac{q}{\hbar} \int_0^\tau \Delta \varphi(t) \, dt, \]  
where \( \Delta \varphi \) is the potential difference between the metal tubes and \( \tau \) is the time interval during which the electric potential is nonvanishing. Thus, there may be a physical effect on the particle although the electric field vanishes and no force is induced on its path by the time dependent potential. The effect is nondispersive, i.e. independent of the particle’s velocity, as long as \( \Delta \varphi(t) \) vanishes before the tail of the wave packets reaches the edges of the tubes.

In the magnetic Aharonov-Bohm (MAB) effect (see Fig. 1b), there is a long thin shielded solenoid perpendicular to the plane of motion and with a current flowing through it so as to produce a magnetic flux \( \Phi \). A beam of particles with charge \( q \) is split coherently into two beams that pass the solenoid on opposite sides and are brought together again to interfere. The solenoid has to be very long to make sure that the magnetic field \( B \) vanishes everywhere outside the solenoid, because the line integral of \( A \) along any closed circuit \( C \) that contains the solenoid equals the magnetic flux \( \Phi \). The Hamiltonian relevant for the MAB case is
\[ H = \frac{1}{2m} (\mathbf{p} - q \mathbf{A})^2, \]  
where \( m \) is the mass of the particle. In this set up the phase difference is given by
\[ \phi_{\text{MAB}} = \frac{q}{\hbar} \oint_C \mathbf{A} \cdot d\mathbf{r} = \frac{q\Phi}{\hbar}, \]
so there may be a physical effect on the particle although the magnetic field vanishes and thus no force is induced along its path. Furthermore, the MAB phase difference is nondispersive as it is independent of the particle’s velocity.

To each of the described above, a dual effect exists: the scalar Aharonov-Bohm (SAB) effect for EAB and the Aharonov-Casher (AC) effect for MAB. Instead of charged particles, the dual effects involve electrically neutral particles carrying a magnetic moment \( \mu \), e.g. neutrons, and the gauge potentials are replaced by magnetic and electric fields. By choosing certain configurations a complete mathematical analogy can be achieved in both cases.

The ideal SAB set up (see Fig. 1c), first suggested by Zeilinger [3], is similar to that of EAB. Instead of the electric potential, there is a spatially uniform magnetic field \( B(t) \) in the \( +z \) direction generated by a solenoid in each arm of the interferometer. The electrically neutral magnetic dipole points in the \( +z \) direction and is split coherently. When the resulting wave packets are well inside the solenoids, the beam pair is exposed to a magnetic field difference that varies as a pure function of time. Well before the wave packets leave the solenoids, the magnetic field falls back to vanish. This process is described by the Hamiltonian
\[ H = -\mu B(t), \]  
where \( B(t) \) is a pure function of time inside each solenoid and \( \mu \) is the magnetic dipole moment. The phase difference between the upper and the lower beam is given by
\[ \phi_{\text{SAB}} = \frac{\mu}{\hbar} \int_0^\tau \Delta B(t) \, dt, \]
where \( \Delta B(t) \) is the magnetic field difference between the two solenoids and \( \tau \) is the time interval during which the magnetic fields are nonvanishing. Again there may be a physical effect although no force is exerted on the particle. The effect is nondispersive as long as the applied particle velocity is such that \( \Delta B(t) \) vanishes before the tail of the wave packets reaches the edges of the solenoids.

Consider again the magnetic AB effect. A solenoid in the \( +z \) direction can be viewed as a line of magnetic dipoles pointing in the \( +z \) direction. In the AC effect [3], the role of the magnetic dipoles and the charged particle is interchanged (see Fig. 1d). The set up is the same as that of the magnetic AB effect, but the solenoid is replaced by a line of charge in the \( +z \) direction and the charged particle is replaced by a particle that carries a magnetic dipole moment \( \mu \). The charged line gives rise to a static \( z \) independent electric field in the \( x-y \) plane. The dipole points in the \( +z \) direction. For this set up, we obtain the Hamiltonian in the \( x-y \) plane as
\[ H = \frac{1}{2m} (\mathbf{p} - \mu \mathbf{A})^2_{x-y}, \]
where \( \mathbf{A} = c^{-2}(-E_y, E_x) \) and \( \mathbf{E} = (E_x, E_y) \) is the electric field of the line of charge and \( c \) is the speed of light. The phase difference is given by
\[ \phi_{\text{AC}} = \frac{\mu}{\hbar} \oint_C \mathbf{A} \cdot d\mathbf{r} = \frac{\mu\lambda}{\hbar c^2}, \]
where \( \lambda \) is charge per unit length and \( C \) is the interferometer loop. Again, there may be a physical effect although no force is exerted on the particle, and the AC phase difference is nondispersive.

Both SAB and AC phase differences have been verified experimentally. Experiments regarding the phase difference for a fixed particle velocity of SAB have been carried out by Allman et al. [6, 7] and regarding AC by Cimino et al. [9, 10] both using neutrons. Badurek et al. [8] examined the nondispersivity of SAB, also with neutrons. Sangster et al. [10] and Görlitz et al. [11] examined the nondispersivity and dependence on the electric field of AC using atom interferometry.

Dowling et al. [12] elaborated two further effects connected to the magnetic AB and the AC effects, related to them by the Maxwell electromagnetic duality relations. The Maxwell dual of MAB is a magnetic monopole in the field of a line of electric dipoles and that of AC is an electric dipole in the field of a line of magnetic monopoles, where the latter is also known as the He-McKellar-Wilkins effect [13, 14].
B. Nonlocality, topology, and nondispersiveness

The deep physical differences of the AB effects and their duals can be illustrated by the debate about nonlocality and topology in this context. Zeilinger [2] observed that a main feature of the AB effects and their duals is that the phase differences are nondispersive, i.e., independent of the particle’s velocity. Ahlman et al. [4, 5, 6] defined a topological effect as one in which a nonvanishing phase difference occurs although no force is exerted on the particle. Peshkin [15] showed that force free motion is a necessary condition for a nondispersive effect.

Peshkin and Lipkin [16] pointed out that a definition of topological effects related only to nondispersivity and force free motion is problematic. To illustrate their point, they suggested that the magnetic field in SAB could be replaced by an optical phase shifter whose uniform index of refraction depends on time while the particle is inside some box in one arm of the interferometer. They claimed that the phase difference in such a set up could in principle be made independent of the particle’s velocity over the experimental range. Thus, a force free effect due to an optical device, which has no deep physical significance, would then be topological regarding the criterion stated in [4, 5, 6].

They analysed the above described effects and came to the conclusion that both the electric and the magnetic AB effect are nonlocal in the sense that there exist no electromagnetic fields along the path of the charged particle and no exchange of physical quantities takes place. They called the AB effects topological in that they require the charged particle to be confined to a multiply connected space-time region and in that there is no objective way to relate a phase shift to any particular place or arm of the interferometer. This is due to the fact that only the differences in phase shifts between different paths is gauge invariant, but no measurable phase shift can be assigned to any place because the integral of an electromagnetic potential along any open space-time path is gauge dependent.

They remarked that in the dual effects there exist electromagnetic fields along the paths of the beams. They claimed that the particle interacts with these fields via angular momentum fluctuations although no force is exerted on it. They concluded that both the SAB and the AC effect are local because there are local interactions and nontopological because the phase shifts depend on the local fields along the paths and can thus be related to a particular place.

To conclude, using the criteria in [16], the AB effects are both nonlocal and topological while the dual effects are local and nontopological. All the effects have in common that they are force free and thus nondispersive. All effects require a closed path to measure the phase difference.

C. Nonideal version of the dual effects

The dual effects have one extra degree of freedom that makes them interesting: the direction of the magnetic dipole [17]. Nevertheless, a thorough analysis of the precession angles and phase differences occurring in the nonideal set ups, where this direction may be arbitrary, is still missing.

Some work have been devoted to the precession of the magnetic moment in the nonideal set ups, but not from this perspective. Han and Koh [18] analysed the AC effect in the rest frame of the magnetic dipole and derived the AC phase from the precession of a magnetic dipole in a magnetic field, but they did not treat the nonideal AC case. Peshkin and Lipkin [16] used a similar point of view to conclude that the dual effects are local and nontopological as mentioned above. They also analyzed SAB in terms of a spin autocorrelation effect related to the dipole precession, and one of the present Authors [19] has recently discussed an analogous correlation effect between the spin and spatial degrees of freedom that arises in the nonideal AC set up.

To discuss and enlighten the extra richness related to the direction of the magnetic dipole in SAB and AC is the major aim of this paper. In particular, we shall examine in detail the conditions under which nondispersive dynamics occur in the generalized set ups. Analyses of both a classical magnetic dipole and a quantal spin-$\frac{1}{2}$ particle with magnetic moment are carried out.

II. CLASSICAL MAGNETIC DIPOLE

A. Hamiltonian

Aharonov and Casher [4] analyzed a situation where an electrically neutral particle exhibits a magnetic AB type effect in the presence of a fixed distribution of charge. In this section we extend their analysis to moving point charges, in order to have a common physical basis for the classical treatment of the AC and SAB effect. We start from the AC Lagrangian describing a system of charged particles, all with charge $q$, interacting with magnetic dipoles, all with magnetic dipole moment $\mu$, (leaving out purely electric-electric terms)

$$L = \frac{1}{2} \sum_k \tilde{m}_k \tilde{v}_k^2 + \frac{1}{2} \sum_l m_l v_l^2 + \sum_{k,l} q A(\tilde{r}_k - r_l) \cdot (\tilde{v}_k - v_l).$$  (9)

Here $\tilde{m}_k, \tilde{r}_k, \tilde{v}_k$ denote the mass, position, and velocity, respectively, of the charged particle $k$, and $m_l, r_l, v_l$ denote the corresponding quantities for dipole $l$. Because of the dependence of $A$ on $(\tilde{r}_k - r_l)$, the system possesses a symmetry when the role of $\tilde{r}_k$ and $r_l$ is reversed.

The vector potential $A$ at $r$ due to a classical magnetic dipole $\mu$ situated at $r$ may be written as (see, e.g., [20].
The electric field at \( r \) due to a particle situated at \( \hat{r}_k \) carrying the charge \( q \) is
\[
E_k \equiv E(\hat{r} - \hat{r}_k) = \frac{q}{4\pi \varepsilon_0} \frac{(\hat{r} - \hat{r}_k)}{|\hat{r} - \hat{r}_k|^3}.
\] (11)

Thus, the Lagrangian of a single magnetic dipole moving in the field of moving charges may be written as
\[
L = \frac{1}{2}mv^2 + \frac{1}{2}\mu \cdot \mathbf{B} + \left( \frac{\mu}{c^2} \times \mathbf{E} \right) \cdot \mathbf{v}.
\] (14)

where we have introduced the total electric field \( \mathbf{E} = \sum_k E_k \).

The corresponding Hamiltonian is obtained by Legendre transforming the variables of the dipole, but treating the velocities of the charged particles as parameters. This yields
\[
H = \frac{1}{2m} \left( p - \frac{\mu}{c^2} \times \mathbf{E} \right)^2 - \mu \cdot \mathbf{B},
\] (15)

where
\[
p = \nabla_\mathbf{v} L = mv + \frac{\mu}{c^2} \times \mathbf{E}.
\] (16)

Note that the only visible classical SAB and AC effects will be the precession of the magnetic dipole, for which any spatial Lorentz force is irrelevant. Therefore, the particle must not necessarily be electrically neutral. The path of such a charged dipole can in principle be controlled by the experimental set up, compensating the occurring forces.

B. Scalar Aharonov-Bohm

From Eq. (15) we see that the Hamiltonian of an electrically neutral magnetic dipole \( \mu \) in a pure magnetic field \( \mathbf{B} \) is given by
\[
H = \frac{p^2}{2m} - \mu \cdot \mathbf{B}.
\] (17)

In the nonideal SAB set up, there is a spatially uniform magnetic field \( \mathbf{B} = B(t)\hat{z} \) and the magnetic dipole can point in any direction. The condition for force free motion in the ideal case is that the magnetic field is nonvanishing only when the particle is well inside the solenoid. This leads to force free motion also in the nonideal case: \( \nabla_x (\mu \cdot B(t)) = 0 \). However, there can be force pairs generating a torque on the magnetic dipole. It can be assumed that \( \mu = \Gamma \mathbf{S} \), where \( \Gamma = \hat{q}/2\mu \). Here \( \hat{q} \) is the charge and \( \mu \) the mass of the particles of the current that is associated with the magnetic dipole moment \( \mu \) and the intrinsic angular momentum \( \mathbf{S} \). The Poisson bracket relations \( \{ S_k, S_l \} = \epsilon_{klm} S_m \) yield
\[
\dot{\mu} = \Gamma \mu \times \mathbf{B},
\] (18)

which reduces to
\[
\dot{\mu} = \Gamma B(t)(\mu_y, -\mu_x, 0)
\] (19)
in the SAB set up. With the initial conditions \( \mu(0) = \mu(\sin \theta, 0, \cos \theta) \) we obtain
\[
\dot{\mu} = \mu(\sin \theta \cos \gamma(t), \sin \theta \sin \gamma(t), \cos \theta),
\]
\[
\gamma(t) = -\Gamma \int_0^t B(t') dt'.
\] (20)

The main point here is that the precession angle \( \gamma(t) \) is independent of the particle’s velocity as the particle does not feel a field gradient under the SAB conditions. This also assures that the angle \( \theta \) with respect to the \( z \) axis remains constant.

C. Aharonov-Casher

From Eq. (15) we see that the Hamiltonian of a magnetic dipole \( \mu \) in a pure electric field \( \mathbf{E} \) is given by
\[
H = \frac{1}{2m} \left( p - \frac{\mu}{c^2} \times \mathbf{E} \right)^2,
\] (21)

where \( m \) is the mass of the dipole. Assuming again \( \mu = \Gamma \mathbf{S} \), the precession and the net force acting on the magnetic dipole can be derived using Hamilton’s equations
\[
\dot{\mu} = -\frac{\Gamma}{c^2} \mu \times (\mathbf{v} \times \mathbf{E})
\] (22)
\[
m\dot{\mathbf{v}} = \mathbf{v} \times (\nabla \times \left( \frac{\mu}{c^2} \times \mathbf{E} \right)) = -\frac{\mathbf{v} \times (\mu \cdot \nabla)\mathbf{E}}{c^2},
\] (23)

where \( \mathbf{v} \) denotes the velocity of the particle. The dipole sees in its own rest frame the magnetic field \( -(1/c^2)(\mathbf{v} \times \mathbf{E}) \), which causes the precession described by Eq. (22). In Eq. (24) we used that \( \nabla \cdot \mathbf{E} \) vanishes where the particle moves. The conditions for the ideal AC effect to be force free and nondispersive are that the particle moves in the \( x - y \) plane and that there is an electric field of the form
\[
\mathbf{E} = E_x(x, y)\hat{x} + E_y(x, y)\hat{y},
\] (24)
which can be generated by a charge distribution \( \rho(x, y) \) that is independent of \( z \) and that is shielded from the dipole’s path. In the nonideal case, we allow for arbitrary direction of the magnetic dipole. With these conditions we obtain a nonvanishing force only in the \( z \) direction. In the classical case, \( v_z = 0 \) is a constraint that has to be fulfilled for a nondispersive effect, thus making it necessary to compensate this force by the experimental set up. Under this constraint, the expression for the precession reduces to

\[
\mathbf{\vec{\mu}} = \Gamma \mathbf{A} \cdot d\mathbf{v}(\mu_y, -\mu_x, 0),
\]

where we defined \( \mathbf{A} = c^{-2}(-E_y, E_x, 0) \). These equations are solved by

\[
\begin{align*}
\gamma(x) &= -\Gamma \int_{x_0}^{x} \mathbf{A} \cdot d\mathbf{r} \\
\end{align*}
\]

with \( \mu(x_0) = \mu(\sin \theta, 0, \cos \theta) \) at the reference point \( x_0 \) in the \( x - y \) plane. Thus, the precession is velocity independent, while the angle \( \theta \) with respect to the \( z \) axis remains constant. The precession is independent of the details of the path since \( \mathbf{\nabla} \times \mathbf{A} \propto \mathbf{\nabla} \times \mathbf{E} \) vanishes where the particle moves. For a closed loop Stokes’ theorem yields the precession angle

\[
\gamma_{AC} = -\Gamma \oint_{C-\delta S} \mathbf{A} \cdot d\mathbf{r} = -\frac{\Gamma}{c^2} \int_{S} (\mathbf{\nabla} \cdot \mathbf{E}) dS = -\frac{\Gamma \lambda}{c^2},
\]

independent of the velocity or the details of the spatial path of the particle. Here \( \lambda \) denotes the linear charge density enclosed by the path \( C \).

III. SPIN - \( \frac{1}{2} \)

A. Hamiltonian

Consider an electrically neutral spin-\( \frac{1}{2} \) particle of rest mass \( m \) carrying a magnetic moment \( \mu \) and moving in an electromagnetic field described by the field tensor \( F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha \), \( A^\alpha \) being the corresponding four-potential. This system obeys the Dirac equation

\[
(i\hbar \gamma^\mu \partial_\mu - mc - \frac{\mu}{2c^2} F_{\alpha\delta} \gamma^{\alpha\delta}) \psi = 0,
\]

which takes the form

\[
(i\hbar \frac{\partial}{\partial t} \psi = \left( \mathbf{\vec{\sigma}} \cdot \mathbf{p} + \beta mc^2 - \beta \mathbf{\mu} \mathbf{\vec{B}} + i\frac{e}{c} \gamma \cdot \mathbf{E} \right) \psi
\]

by rewriting the Dirac matrices as \( (\gamma^0, \gamma^k) = (\beta, \beta \alpha_k) \) and choosing the representation

\[
\begin{align*}
\begin{bmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{bmatrix}, \quad \beta = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}. \end{align*}
\]

Here, \( \sigma_k \) are the standard Pauli matrices defining \( \mathbf{\sigma} = (\sigma_x, \sigma_y, \sigma_z) \), \( I \) is the \( 2 \times 2 \) unit matrix, \( \mathbf{E} \) and \( \mathbf{B} \) denote the electric and the magnetic field, respectively, and we have used that \( \gamma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu] \).

In order to take the nonrelativistic limit it is convenient to express the four-spinor \( \psi \) in terms of the two-spinors \( \varphi \) and \( \chi \)

\[
\psi = e^{-i(mc^2/\hbar)t} \left( \begin{array}{cc} \varphi \\ \chi \end{array} \right)
\]

yielding the coupled equations

\[
\begin{align*}
\hbar \frac{\partial}{\partial t} \varphi &= c \mathbf{\vec{\sigma}} \cdot \mathbf{p} \left( \begin{array}{c} \chi \\ 0 \end{array} \right) - 2mc^2 \left( \begin{array}{c} 0 \\ \chi \end{array} \right) \\
- \mathbf{\mu} \cdot \mathbf{B} \left( \begin{array}{c} \varphi \\ -\varphi \end{array} \right) + i \frac{e}{c} \mathbf{\mu} \cdot \mathbf{E} \left( \begin{array}{c} \chi \\ -\chi \end{array} \right). \end{align*}
\]

In the nonrelativistic regime, the terms \( i\hbar(\partial\varphi/\partial t) \) and \( \mathbf{\mu} \cdot \mathbf{B} \chi \) are negligible compared to the rest energy term \( mc^2 \chi \). Thus, we obtain the approximate expression

\[
\chi = \frac{1}{2mc} \mathbf{\sigma} \cdot (\mathbf{p} - i\frac{\mu}{c^2} \mathbf{E}) \varphi.
\]

Note that \( \chi \) is reduced by \( \sim v/c \ll 1 \) compared to \( \varphi \) in the nonrelativistic regime and can therefore be neglected. We insert Eq. (24) into the upper of Eqs. (22) and obtain the two-spinor equation

\[
\begin{align*}
\hbar \frac{\partial}{\partial t} \varphi &= \frac{1}{2m} \left( \mathbf{\sigma} \cdot \left( \mathbf{p} + i\frac{\mu}{c^2} \mathbf{E} \right) \mathbf{\sigma} \cdot \left( \mathbf{p} - i\frac{\mu}{c^2} \mathbf{E} \right) \right) \varphi
- \mathbf{\mu} \cdot \mathbf{B} \varphi. \end{align*}
\]

Using the identity \( (\mathbf{\sigma} \cdot \mathbf{a})(\mathbf{\sigma} \cdot \mathbf{b}) = \mathbf{a} \cdot \mathbf{b} + i\mathbf{\sigma} \cdot (\mathbf{a} \times \mathbf{b}) \) we finally obtain

\[
\begin{align*}
\hbar \frac{\partial}{\partial t} \varphi &= \left[ \frac{1}{2m} \left( \mathbf{p} - \frac{\mu}{c^2} \times \mathbf{E} \right)^2 - \frac{\mu \hbar}{2mc^2} \mathbf{\nabla} \cdot \mathbf{E} \right. \\
&\quad - \frac{\mu^2 E^2}{2mc^4} - \mathbf{\mu} \cdot \mathbf{B} \left. \right] \varphi,
\end{align*}
\]

which is the Schrödinger equation responsible for the SAB and AC effect. In the following, we use the standard nonrelativistic notation and put \( \varphi \equiv |\Psi\rangle \).

B. Scalar Aharonov-Bohm

In the SAB set up, Eq. (36) reduces to

\[
\hbar \frac{\partial}{\partial t} |\Psi\rangle = \left[ \hat{I} \otimes \frac{\hat{B}^2}{2m} - \mu \hat{\sigma}_z \otimes \mathbf{B}(\mathbf{\hat{x}}, t) \right] |\Psi\rangle.
\]

Here and in the following, tensorial product distinguishes the spatial and the spin degrees of freedom, and hats distinguish operators from their expectation values. The
conditions for the ideal SAB effect to be force free are that the magnetic field is nonvanishing only in a certain spatial region and during a certain time interval $\tau$, when the particle is well inside that region. The time development of $|\Psi\rangle$ is given by

$$\langle x | \Psi(t) \rangle = \exp \left[ - \frac{i}{\hbar} \hat{H} \tau t + \frac{i}{\hbar} \hat{\sigma}_z \otimes \int_0^t B(x, t') dt' \right] \langle x | \Psi_0 \rangle,$$

where $\hat{H} = \frac{\hbar^2}{2m} \nabla \cdot B(x, t)$ does not vanish since there is a gradient at the edge points of the magnetic field region. The SAB conditions ensure that the particle does not feel that gradient. Thus, although the quantum force $\mu \hat{\sigma}_z \otimes \nabla \cdot B(x, t)$ does not vanish everywhere, its expectation value vanishes due to the SAB conditions. Therefore, the particle moves force free and we may effectively write $B(x, t) = B(t) \hat{I}$. This holds for arbitrary direction of the magnetic dipole.

Initially, the state is represented by the product vector $|\Psi_0\rangle = |s_0\rangle \otimes |\psi_0\rangle$, where $|s_0\rangle$ is the spin vector and $|\psi_0\rangle$ is the spatial vector. At a later instant of time, the wave function in position representation reads

$$\langle x | \Psi(t) \rangle = \exp \left[ - \frac{i}{\hbar} \hat{\sigma}_z \gamma(t) \right] |s_0\rangle \psi(x, t), \quad (39)$$

where the spatial part $\psi(x, t) = \langle x | \exp \left[ - (i/\hbar) \hat{H} \tau t \right] |\psi_0\rangle$ is a scalar (spin independent) free particle wave packet that is independent of the magnetic field under the SAB conditions. The operator $\exp[-i\hat{\sigma}_z \gamma/2]$ rotates the spin around the $z$ axis with the angle

$$\gamma(t) = -\frac{\mu}{\hbar} \int_0^t B(t') dt' \quad (40)$$

when the particle moves in the magnetic field.

The precession angle $\gamma$ could be related to the ideal phase difference given by Eq. (18) in the following way: First let the magnetic dipole pass the upper solenoid forward in time and take it back to its starting point through the lower solenoid backward in time, which yields the precession

$$\gamma_{SAB} = -\frac{2\mu}{\hbar} \int_0^\tau B_v(t) dt + \int_\tau^0 \Delta B(t) dt = -2\phi_{SAB}, \quad (41)$$

where $\tau$ is the time interval in which the magnetic field is nonvanishing and the factor 2 is the usual rotation factor for spin-$\frac{1}{2}$. The spin precession effect is nondispersive as the precession angle takes the value $-2\phi_{SAB}$ for all velocities, as long as the SAB conditions are fulfilled. In the ideal SAB case, where $|s_0\rangle$ is an eigenstate of $\hat{\sigma}_z$, there is no precession effect, but the interference pattern is shifted by $\phi_{SAB}$.

### C. Aharonov-Casher

In the AC setup, a neutral spin-$\frac{1}{2}$ particle with a magnetic dipole moment $\mu$ is moving in a plane, say the $x-y$ plane, where a nonvanishing external static electric field $E = E(x)$ that fulfills $\nabla \cdot E = 0$ exists and where the magnetic field vanishes. Thus, Eq. (36) reduces to

$$i\hbar \frac{\partial}{\partial t} |\Psi\rangle = \left[ \frac{1}{2m} \left( \hat{I} \otimes \hat{p} - \frac{\mu c}{\hbar} \hat{\sigma} \otimes \hat{X} \right)^2 \right] |\Psi\rangle,$$

where again the tensorial product distinguishes the spin and spatial degrees of freedom. The ideal AC setup becomes force free and nondispersive if $E_x = 0$, $\partial_z E_y = 0$, and $\hat{p}_z |\Psi\rangle = 0$, where $|\Psi\rangle$ is the total state vector. Under these conditions, Eq. (42) reduces to the Schrödinger equation in the $x-y$ plane

$$i\hbar \frac{\partial}{\partial t} |\Psi\rangle = \frac{1}{2m} \left( \hat{I} \otimes \hat{p} - \mu \sigma_z \otimes \hat{A} \right)^2 |\Psi\rangle,$$

when $\hat{p}$ now is $\hat{p} = (\hat{p}_x, \hat{p}_y)$ and $\hat{A} = e^{-2\hat{\gamma}} \hat{A} = e^{-2\hat{\gamma}} (-\hat{E}_y, \hat{E}_x)$. Note that these conditions are independent of the spin and the resulting Schrödinger equation in the $x-y$ plane should also be valid in the nonideal case. As a result, the motion of the particle under this Hamiltonian is force free, contrary to the classical treatment of the nonideal AC set up. This can be seen by writing $|\Psi\rangle$ as

$$|\Psi\rangle = \hat{U} |\Psi'\rangle, \quad \hat{U} = e^{-2\hat{\gamma} \otimes \hat{\sigma}_z}, \quad \hat{\gamma} = \gamma (x), \quad (44)$$

identifying $\hat{U}$ as the spin rotation operator around the $z$ axis with the local operator $\hat{\gamma}$ that rotates the spin state the angle $\gamma (x)$. Now, the vector $|\Psi'\rangle$ evolves according to the Schrödinger-like equation

$$i\hbar \frac{\partial}{\partial t} |\Psi'\rangle = \hat{H}' |\Psi'\rangle, \quad (45)$$

where the transformed Hamiltonian operator $\hat{H}' = \hat{U} \hat{H} \hat{U}^\dagger$ is given by

$$\hat{H}' = \frac{1}{2m} \left( \hat{I} \otimes \hat{p} - \hat{\sigma}_z \otimes \left[ \frac{\hbar^2}{2} \nabla \gamma + \mu \hat{A} \right] \right)^2. \quad (46)$$

This reduces to the free particle Hamiltonian $\hat{H}_0 = \hat{p}^2/2m$ in the $x-y$ plane if we set

$$\gamma (x) = -\frac{2\mu}{\hbar} \int_x^\infty \hat{A} \cdot \hat{d}x, \quad (47)$$

which can be done because $\partial_x A_y - \partial_y A_x = (1/c^2) \nabla \cdot \hat{E}$ vanishes along the path of the particle. Let us further assume initially that $|\Psi'_0\rangle = |s_0\rangle \otimes |\psi_0\rangle$, where $|s_0\rangle$ is the spin and $|\psi_0\rangle$ is the spatial state vector. The subsequent AC wave function in position representation is given by

$$\langle x | \Psi(t) \rangle = \langle x | \exp \left[ - \frac{i}{\hbar} \hat{\sigma}_z \otimes \gamma (x) \right] |s_0\rangle$$

$$\otimes \exp \left[ - \frac{i}{\hbar} \hat{H}' t \right] |\psi_0\rangle$$

$$= \exp \left[ - \frac{i}{\hbar} \hat{\sigma}_z \gamma (x) \right] |s_0\rangle \psi(x, t) \quad (48)$$

where from now on $\hat{\sigma}_z = 1$ and $|\Psi'\rangle$ is an eigenstate of $\hat{\sigma}_z$.
where \( \psi(x, t) = \langle x| \exp \left[-(i/\hbar)\hat{H}t\right]|\psi_0\rangle \) is a scalar (spin independent) free particle wave packet and we have used that \( \hat{\gamma} \) is a local operator so that \( \langle x|\hat{\gamma}|x'\rangle = f[\gamma(x)]\delta(x' - x) \) for any sufficiently well-behaved function \( f \). Thus, the spin part is precessing the angle \( \gamma(x) \) independent of the path or the velocity of the particle. After the wave packet \( \psi(x, t) \) has traversed a loop enclosing the line charge \( \lambda \), Stokes’ theorem yields the precession angle

\[
\gamma_{AC} = -\frac{2\mu}{\hbar} \oint_{C=\partial S} \mathbf{A} \cdot d\mathbf{r} = \frac{2\mu\lambda}{\hbar c^2\tau_0} = -2\phi_{AC}, \tag{49}
\]

where again the factor 2 is the rotation factor for spin-1/2. In the ideal AC case, where \( \langle s_0|\hat{\sigma}_z|s_0\rangle = +1 \), there is no precession effect, but the interference pattern is shifted by \( \phi_{AC} \).

In the AC set up, the region outside the charge is multiply connected. Under the condition that at any instant of time the wave function does not enclose this charge, the wave function \([35]\) is still single-valued. In interferometry \([7]\), it is reasonable to assume that this condition is fulfilled if the particle’s spatial wave function is a quantal wave packet, giving the macroscopic size of the interferometer.

**D. Phase difference**

In the interferometer set up, both beams are influenced by magnetic fields for SAB and by an electric field in the \( x-y \) plane for AC (see Fig. 2). The resulting phase difference between the state vector \( |\Psi_u\rangle \) that went the upper path and \( |\Psi_d\rangle \) that went the lower path for arbitrary polarization is given by the Pancharatnam connection \([28]\)

\[
\phi = \arg\langle\Psi_d|\Psi_u\rangle. \tag{50}
\]

This phase and the corresponding visibility \( |\langle\Psi_d|\Psi_u\rangle| \) can be measured by adding an additional phase \( \chi \) to one of the beams \([20, 21]\).

\[\begin{align*}
\text{a)} & \quad |\Psi\rangle \quad \text{B}_a \quad \text{D}_1 \\
\text{b)} & \quad |\Psi\rangle \quad \text{B}_d \quad \text{D}_1
\end{align*}\]

**FIG. 2:** Interferometer set ups for the nonideal versions of a) the scalar Aharonov-Bohm and b) the Aharonov-Casher effect. Here \( \mu \) is the magnetic moment vector, \( D_{1,2} \) denote the detectors and \( \chi \) is the extra \( U(1) \) phase shift. Furthermore, \( |\Psi_u, d\rangle \) and \( B_{u, d} \) are the states and the magnetic fields in the upper and lower path, respectively. In b) \( \lambda \) is the line charge density perpendicular to the interferometer plane.

Now, let us assume that the outgoing pair of spatial wave packets both have the same shape after leaving each 50-50 beam splitter and that the free particle spreading is the same in both paths due to symmetry. Under these conditions it follows from Eqs. \([33]\) and \([45]\) that the wave function after the final beam splitter can be written as

\[
\langle x|\Psi \rangle = \frac{1}{2} \left( |s_u\rangle + e^{i\chi}|s_d\rangle \right) \otimes \psi_{D_1} + \frac{1}{2} \left( |s_u\rangle - e^{i\chi}|s_d\rangle \right) \otimes \psi_{D_2}, \tag{51}
\]

where \( \psi_{D_{1,2}} \) correspond to the spatial wave functions at the detectors 1 and 2, respectively, and the phase \( \chi \) was applied to the lower (\( d \)) beam. The spin parts are given by

\[
|s_{u,d}\rangle = \exp \left[-\frac{i}{2}\hat{\sigma}_z\gamma_{u,d}\right]|s_0\rangle, \tag{52}
\]

and we obtain the following detection probabilities for the detectors \( D_1 \) and \( D_2 \):

\[
P_1 = \frac{1}{2} \left[ 1 + \nu \cos(\phi + \chi) \right], \quad P_2 = \frac{1}{2} \left[ 1 - \nu \cos(\phi + \chi) \right]. \tag{54}
\]

The visibility \( \nu \equiv |\langle s_d|s_u\rangle| \) and the phase difference \( \phi \) can be obtained experimentally by varying \( \chi \). Without loss of generality, the initial spin state vector can be written as

\[
|s_0\rangle = \cos \frac{\theta}{2} |\dagger\rangle + \sin \frac{\theta}{2} |\dagger\rangle, \tag{55}
\]

where the spin direction makes an angle \( \theta \) with the \( z \) axis and its angle relative to the \( x \) axis vanishes. For \( \theta \neq \pi/2 \), we obtain the phase difference and visibility

\[
\phi = \arctan \left( \cos \theta \tan \phi_D \right), \quad \nu = \sqrt{1 - \sin^2 \theta \sin^2 \phi_D}, \tag{56, 57}
\]

being nonlinearly related to the ideal phase differences

\[
\phi_D = (\gamma_d - \gamma_u)/2 = \frac{\mu}{\hbar} \int_0^\tau \Delta B dt = \phi_{\text{SAB}} \tag{58}
\]

in the SAB case and

\[
\phi_D = \frac{\mu}{\hbar} \int_{u}^{d} \mathbf{A} \cdot d\mathbf{r} - \frac{\mu}{\hbar} \int_{u}^{d} \mathbf{A} \cdot d\mathbf{r} = \phi_{\text{AC}} \tag{59}
\]

in the AC case. Similarly, for \( \theta = \pi/2 \) the phase shift and visibility become

\[
\phi = \begin{cases} 
0 & \text{for } 0 \leq \phi_D < \pi/2 \\
\text{undefined} & \text{for } \phi_D = \pi/2 \\
\pi & \text{for } \pi/2 < \phi_D \leq \pi
\end{cases}, \tag{60}
\]

\[
\nu = |\cos \phi_D|. \tag{61}
\]
Conversely, it should be noted that by observing the phase difference and visibility in the nonideal case, one may infer the spin precession effect. This can be seen by inverting Eqs. (56) and (57) yielding

\[
\begin{align*}
\tan^2 \phi_D &= \frac{1}{\nu^2 \cos^2 \phi} - 1 \\
\cos^2 \theta &= \nu^2 \frac{1 - \cos^2 \phi}{1 - \nu^2 \cos^2 \phi},
\end{align*}
\]  

which is also consistent with Eqs. (60) and (61). Thus, measuring the interference pattern in fact determines the precession and thereby confirms the total accumulation of local torque around the interferometer loop. This makes it natural to regard also the ideal phase differences \(\phi_{SAB/AC}\) as essentially local quantities originated by the local torque and in turn the dual effects as non-topological. This is in contrast to the view of Aharonov and Reznik [31], who regard the phase shifts accumulated along the beam paths of SAB and AC as nonlocal quantities and suggest a complementarity between these phase shifts and the precession of the dipole moment, and led to further discussion in Refs. [32, 33].

The phase differences in Eqs. (56) and (60) between the beam pair comprise a dynamical contribution

\[\gamma_d = \phi_D \cos \theta\]

that is directly proportional to the projection of the dipole moment onto the direction of the magnetic field in SAB and the axis of the enclosed line charge in AC. In addition, there is a geometrical contribution to the phase shift given by

\[\gamma_g = \phi - \phi_D \cos \theta = -\frac{1}{2} \Omega_{g-c},\]

which is proportional to the geodesically closed solid angle \(\Omega_{g-c}\) defined by the spin path and the shortest geodesic on the Bloch sphere. The phase difference between the upper and the lower beam is directly proportional to the ideal phase shifts under the condition that the geodesic phase vanishes, which happens if \(\theta\) is a multiple of \(\pi\), i.e. the ideal case, where the spin state is an eigenstate of \(\sigma_z\) and no precession occurs. Thus, from this point of view the ideal phase shifts \(\phi_{SAB/AC}\) are dynamical in nature and their nonlinear relation to the observed phase shift \(\phi\) in the nonideal case may be understood from the occurrence of a geometric phase associated with the nontrivial line bundle \(SU(2)/U(1)\) of spin \(-\frac{1}{2}\).

\[\text{IV. CONCLUSIONS}\]

We have extended the scalar Aharonov-Bohm (SAB) effect and the Aharonov-Casher (AC) effect to arbitrary directions of the magnetic dipole. The precise conditions for having nondispersive precession and interference effects in these generalized set ups have been delineated both classically and quantally. Under these conditions the dipole is affected by a nonvanishing torque that rotates its direction on a cone with an angle \(\gamma\) around the directions defined by the ideal set ups. For a closed clockwise space-time interferometer loop, this angle is in the quantal case related to the ideal phase differences as \(\gamma = -2\phi_{SAB/AC}\). The force vanishes in all cases except in the classical treatment of the nonideal AC set up, where the occurring force has to be compensated by the experimental arrangement.

In both SAB and AC, the phase difference that occurs in interferometry is directly proportional to that of the ideal cases only if the initial spin state vector is an eigenstate of \(\sigma_z\). For arbitrary direction of the dipole, the phase differences are nonlinearly related to the ideal phase shifts as \(\phi = \arctan(\cos \theta \tan \phi_{SAB/AC})\) due to the appearance of a geometric phase associated with the nontrivial spin path.

We end by noting that the precession of the dipole in SAB and AC is a local effect that could be obtained classically. In particular, it does not require a closed loop and can therefore be tested in a polarization measurement. Furthermore, for a closed space-time loop as obtained in interferometry, the precession effects can be inferred from the interference pattern characterized by the nonideal phase differences and the visibilities, thereby confirming the local torque. Extending interferometry experiments to spin-polarized particles in nonideal configurations would therefore be pertinent as they would verify SAB and AC as essentially local and non-topological effects.

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[1] Y. Aharonov and D. Bohm, *Phys. Rev.* **115**, 485 (1959).
[2] A. Zeilinger, “Generalized Aharonov-Bohm Experiments with Neutrons”, in *Fundamental Aspects of Quantum Theory*, eds. V. Gorini and A. Frigerio, NATO ASI Series B, Vol. 144, Plenum Press (1986), p. 311.
[3] Y. Aharonov and A. Casher, *Phys. Rev. Lett.* **53**, 319 (1984).
[4] B.E. Allman, A. Cimmino, A.G. Klein, G.I. Opat, H. Kaiser, and S.A. Werner, *Phys. Rev. Lett.* **68**, 2409 (1992).
[5] B.E. Allman, A.Cimmino, A.G. Klein, G.I. Opat, H. Kaiser, and S.A. Werner, Phys. Rev. A 48, 1799 (1993).
[6] B.E. Allman, W.-T. Lee, O.I. Motrunich, and S.A. Werner, Phys. Rev. A 60, 4272 (1999).
[7] A. Cimmino, G.I. Opat, A.G. Klein, H. Kaiser, S.A. Werner, M.Arif, and R. Clothier, Phys. Rev. Lett. 63, 380 (1989).
[8] G. Badurek, H. Weinfurter, R. Gähler, A. Kollmar, S. Wehinger, and A. Zeilinger, Phys. Rev. Lett. 71, 307 (1993).
[9] K. Sangster, E.A. Hinds, S.M. Barnett, E. Riis, and A.G. Sinclair, Phys. Rev. Lett. 71, 3641 (1993).
[10] K. Sangster, E.A. Hinds, S.M. Barnett, E. Riis, and A.G. Sinclair, Phys. Rev. A 51, 1776 (1995).
[11] A. Görßitz, B. Schuh, and A. Weis, Phys. Rev. A 51, R4305 (1995).
[12] J.P. Dowling, C.P. Williams, and J.D. Franson, Phys. Rev. Lett. 83, 2486 (1999).
[13] X.-G. He and B.H.J. McKellar, Phys. Rev. A 47, 3424 (1993).
[14] M. Wilkens, Phys. Rev. Lett. 72, 5 (1994).
[15] M. Peshkin, Found. Phys. 29, 481 (1999).
[16] M. Peshkin and H.J. Lipkin, Phys. Rev. Lett. 74, 2847 (1995).
[17] A.S. Goldhaber, Phys. Rev. Lett. 62, 482 (1989).
[18] Y.D. Han and I.G. Koh, Phys. Lett. A 167, 341 (1992).
[19] E. Sjöqvist, Phys. Lett. A 270, 10 (2000).
[20] J.D. Jackson, Classical Electrodynamics, 2nd ed. (Wiley, New York, 1975).
[21] H. Goldstein, Classical Mechanics, 2nd ed. (Addison-Wesley, Reading, Massachusetts, 1980) p. 232.
[22] W. Pauli, Rev. Mod. Phys. 13, 203 (1941).
[23] L.L. Foldy, Phys. Rev. 87, 688 (1952).
[24] J.D. Bjorken and S.D. Drell, Relativistic Quantum Mechanics (McGraw-Hill, New York, 1964) p. 8.
[25] J. Anandan Phys. Lett. A 138, 347 (1989).
[26] X.-G. He and B.H.J. McKellar, Phys. Lett. B 256, 250 (1991).
[27] C.R. Hagen, Phys. Rev. Lett. 64, 2347 (1990).
[28] S. Pancharatnam, Proc. Indian Acad. Sci. A 44, 247 (1956).
[29] A.G. Wagh and V.C. Rakhecha, Phys. Lett. A 197, 107 (1995).
[30] A.G. Wagh, V.C. Rakhecha, P. Fischer, and A. Ioffe, Phys. Rev. Lett. 81, 1992 (1998).
[31] Y. Aharonov and B. Reznik, Phys. Rev. Lett. 84, 4790 (2000).
[32] P. Hyllus and E. Sjöqvist, Phys. Rev. Lett. (to appear).
[33] Y. Aharonov and B. Reznik, Phys. Rev. Lett. (to appear).