Asset allocation using option-implied moments

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Abstract. This study uses an option-implied distribution as the input in asset allocation. The computation of risk-neutral densities (RND) are based on the Dow Jones Industrial Average (DJIA) index option and its constituents. Since the RNDs estimation does not incorporate risk premium, the conversion of RND into risk-world density (RWD) is required. The RWD is obtained through parametric calibration using the beta distributions. The mean, volatility, and covariance are then calculated to construct the portfolio. The performance of the portfolio is evaluated by using portfolio volatility and Sharpe ratio.

1. Introduction

The moments of asset return such as means, variances, and correlations play the crucial role in asset allocation model. Generally, the historical return data was used to estimate the moments of asset returns but recently the researcher found out that these moments perform poorly in portfolio models [1]. As an alternative, researcher proposed to use a forward-looking information data such as option prices to extract the input for asset allocation model [2-4]. The reason option price is called the forward-looking information because its payoff function relies on the underlying asset price in the future. In addition, option prices can give an overview of the market on how the underlying asset price is evolved [5,6]. It is expected to provide an accurate estimation of the moments and to be used in asset allocation model.

Kostakis, Panigirtzoglou, & Skiadopoulos (KPS) [3] used risk-neutral density (RND) derived from option prices and applied in asset allocation problem. The authors adopted [7] approach in which smoothing spline interpolation is used to extract RND. The RND were converted into risk-world density (RWD) based on [8] in order to incorporate risk premium. KPS used option-implied distribution of asset return to estimate the moments in order to construct a portfolio that consists of one risk-free asset and one risky asset.

This paper aims to extend the strategy developed by KPS [3] to a portfolio of many risky assets. This paper is different than KPS in terms of the calculation of risk-neutral density (RND) and risk-world density (RWD). A fourth-order polynomial interpolation is used as suggested by [9,10] to extract the RND. Also, this paper uses parametric calibration using beta distribution [5,11,12] to convert the RND into RWD. Thus, to the best of our knowledge, there are no study that uses the estimation of moments based on RND using a fourth-order polynomial interpolation and calibrate the RWD using a parametric calibration in asset allocation problem.
This paper is structured as follows: Section 2 describes the data, Section 3 provides the details of the methodology and Section 4 states the asset allocation strategies. Section 5 elaborates the results and the final section is the conclusion.

2. Data
The data can be divided into two groups; option prices data and historical prices data. The option data is obtained from Optiondata.net and the historical prices data is obtained from the DataStream. The data set consists of the stocks listed in Dow Jones Industrial Average (DJIA) index. This paper considers data period from 1/1/2009 until 12/31/2015. London Interbank Offer Rate (LIBOR) is used as the interest rate. Only out-of-the-money (OTM) of call and put options are considered in this study due to liquidity reasons [3,4,9,10,13,14]. In line with [10,15-17] the average of bid-ask quotes is used to represent the closing price of options. This study only considers options with a one-month maturity according to the Chicago Board Options Exchange (CBOE) calendar.

There are several restrictions applied on the data set before the final data set is obtained. Firstly, options with bid or ask quotes equal to zero are eliminated and to ensure that the ask quotes are greater than the bid quotes. Secondly, option prices that violate the arbitrage condition are excluded. Lastly, only options with the lowest delta value less than or equals to 0.25 and the highest delta value equals or greater than 0.75, are used. This condition is to ensure that the extracted RNDs exhibits the true density. The final sample consists of 84 set of options with a one-month maturity.

3. Methodology
3.1. Option-implied risk-neutral density
Price of a call option is given by the discounted value of expected payoff on the maturity date, T with respect to the risk-neutral probability:

\[ (T, K) = e^{-r(T-t)} \int_{S_T}^{K} (S_T - K, 0) f(S_T) dS_T \]  

where \( C \) is the European call price, \( S_T \) is the price of the underlying asset; \( K \) is the strike price; \( r \) is the continuously compounded risk-free rate and \( f(S_T) \) is the RND function. The RND can be obtained by second derivative of equation (1) with respect to strike prices [18]. It can be approximated using following equation:

\[ f(K_n) \approx e^{(r-i)} \frac{C_{n+1} - 2C_n + C_{n-1}}{(\Delta K)^2} \]  

The continuum option prices are needed for RND estimation but it is not existed in the real market. Thus, the researchers propose to extract the RND by applying the interpolation and extrapolation techniques. This paper applied the fourth-order polynomial interpolation [9] to interpolate the option prices. In addition, two pseudo-points are introduced from highest and lowest strike prices to extrapolate the option prices in horizontal manner [7].

The RND estimation based on the interpolation and extrapolation techniques consist of four stages. First, this paper only considered the out-of-the-money (OTM) call and put options. Secondly, the implied volatility of call and put options are calculated using the inverse of Black-Scholes-Merton (BSM) model. It can be calculated using the numerical approach namely, bisection method. Note that, the BSM model is used as a medium to calculate the implied volatility without imposing the assumptions of BSM model. Thirdly, the implied volatilities with respect to strike prices are interpolated by using fourth-order polynomial. Then, 5000 points are evaluated by using implied-volatility function and converted to call prices using the BSM model. Fourthly, the RND estimation is obtained by using equation (2).
3.2. Option-implied risk-world density

Let \( f_0(x) \) and \( F_0(x) \) are the risk-neutral density and cumulative distribution function of \( S_T \), respectively, while \( S_T \) is the price of an underlying asset at maturity, \( T \). A random variable, \( U \) is defined as \( U = F_0(S_T) \) and \( C(u) \) is the calibration function. Generally, the real-world cumulative distribution function, \( F_p(x) \), and the real-world density function, \( f_p(x) \), can be expressed as follows:

\[
F_p(x) = C(F_0(x))
\]

\[
f_p(x) = \frac{dF_p(x)}{dx} = \frac{dC(F_0(x))}{dx} = \frac{dC}{dF_0} \frac{dF_0}{dx} = c(F_0(x))f_0(x)
\]

In this research, the calibration function, \( C(u) \), is based on a standard beta distribution that can be defined in the interval \([0,1]\):

\[
C(u) = \frac{1}{B(\alpha,\beta)} \int_0^u h^{\alpha-1}(1-h)^{\beta-1} dh, \quad \text{where } B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}
\]

Given equation (4) and the calibration function(5), the relationship between RND and RWD can be expressed as follows:

\[
f_p(x) = \frac{F_0(x)^{\alpha-1}(1-F_0(x))^{\beta-1}}{B(\alpha,\beta)} f_0(x)
\]

The advantages of a beta distribution as a calibration function are that it allows to have different shapes and is flexible to correctly specify the RNDs. The price of the underlying asset at the option’s expiration date is used to estimate the parameters of the calibration function. The parameters of \( \alpha \) and \( \beta \) are estimated by using a maximum likelihood estimation (MLE) in which it produces data that is closely resembles actual data. MLE ensures that the parameters obtained have the consistency and optimality properties [19].

Let \( S_i \) be the observed price of an underlying asset at the option’s maturity \( T_i \). The log-likelihood function of \( S_i \) is

\[
\log L(S_i) = \sum_{i=1}^{N} \log \left( f_{p,i}(S_i | \alpha,\beta,\theta) \right)
\]

where \( \theta \) denotes the estimated parameter vector in the RNDs for \( T \).

3.3. Option-implied mean and variance

The option-implied mean and variances is calculated based on RND and RWD estimation. It can be calculated using the following equation:

\[
\mu = \mathbb{E} \left[ \log \left( S_T \right) - \log \left( S_T^r \right) \right] = \int_0^\infty \log(K)f_i(K)dK - \log \left( S_T^r \right)
\]

and

\[
\sigma^2 = \text{Var} \left[ \log \left( S_T \right) - \log \left( S_T^r \right) \right] = \int_0^\infty \left( \log(K) \right)^2 f_i(K)dK - \left( \int_0^\infty \log(K)f_i(K)dK \right)^2
\]

3.4. Option-implied covariance

The option-implied covariance is calculated from the seminal works of [2,20] by combining the historical correlation and option-implied volatility. The variance of a portfolio can be written as
\[
\sigma^2_{P,t} = \sum_{i=1}^{N} w_i^2 \sigma^2_{i,t} + 2 \sum_{i=1}^{N} \sum_{j \neq i}^{N} w_i w_j \sigma_{i,t} \sigma_{j,t} \rho_{ij,t}
\]  
(10)

where \( w_{i,t} \) is the weight, \( N \) is the number of stocks, and \( \sigma_{i,t} \) is the volatility of stock \( i \) at time \( t \). Then, \( \rho_{ij,t} \) represents the pair-wise correlation between stock \( i \) and stock \( j \) and the \( \sigma^2_{P,t} \) is the variance of the portfolio at time \( t \). Assume that the volatilities and the weight are given, the only parameter need to be estimated is the pair-wise correlation among the stocks, \( \hat{\rho}_{ij,t} \). Buss and Vilkov [20] showed that the relationship between historical and expected correlation by single fixed proportions can be represented as follows:

\[
\hat{\rho}_{ij,t} = \rho_{ij,t} - \alpha (1 - \rho_{ij,t})
\]

(11)

The correlation between two stocks cannot be derived from the option prices, Thus, the historical correlation, \( \rho_{ij,t} \), is calculated based on one-year rolling windows of historical asset prices. By substituting equation (11) into equation (10), the fixed proportions, \( \alpha \), can be estimated using the following equation:

\[
\alpha = \frac{\sum_{i=1}^{N} \sum_{j \neq i}^{N} w_i w_j \sigma_{i,t} \sigma_{j,t} \rho_{ij,t}}{\sum_{i=1}^{N} \sum_{j \neq i}^{N} w_i w_j \sigma_{i,t} \sigma_{j,t} (1 - \rho_{ij,t})}
\]

(12)

The value of fixed proportions, \( \alpha \), and the historical correlation, \( \rho_{ij,t} \), is used to calculate the expected correlation, \( \hat{\rho}_{ij,t} \), as in equation (11). According to [2], the implied covariance, \( \Phi \), can be estimated into a diagonal matrix \( G \) of standard deviation and a correlation matrix, \( \psi \), such that

\[
\Phi = G \psi G
\]

(13)

4. Asset allocation strategies

This paper focuses on the minimum-variance strategies [21] since it does not require the estimation of expected return. We impose two strategies of portfolio in which short selling is allowed and short selling is not allowed. For the first strategy (short-selling is allowed), the minimum-variance optimization problem is stated as

\[
\min_w w^T \Phi w \quad \text{s.t.} \quad w_i^T 1 = 1 \quad i = 1,2,...,n
\]

(14)

Then, the second strategy (short-selling is not allowed), the minimum-variance problem is as follows

\[
\min_w w^T \Phi w \quad \text{s.t.} \quad w_i^T 1 = 1 \quad w_i \geq 0 \quad i = 1,2,...,n
\]

(15)

The weight of each stock is assumed to be equal, \( w = 1/N \). The performance of the portfolio is evaluated using two criteria: the portfolio volatility (standard deviation) and the portfolio Sharpe ratio. The Sharpe ratio is calculated as below(16).
Statistical analysis using one-sided \( t \)-test is applied in order to show the significant difference in the portfolio performances based on RND and RWD. The standard deviation and Sharpe ratio of portfolio are used as an input for this analysis. The hypotheses are depicted in Table 1.

### Table 1. Hypotheses for one-sided \( t \)-test.

| \( h_0 \) | \( h_1 \) | \( h_0 \) | \( h_1 \) |
| --- | --- | --- | --- |
| Standard deviation | The standard deviation of a portfolio based on RND and RWD are equal. | The standard deviation of a portfolio based on RWD is higher than that of RND. |
| Sharpe ratio | The Sharpe ratio of a portfolio based on RND and RWD are equal. | The Sharpe ratio of a portfolio based on RWD is higher than that of RND. |

5. **Results and discussions**

In order to determine the weight of the portfolio with a minimum-variance, volatility of the index and stock options and also the correlations with all stocks are needed. The analysis begins with the calculation of RND as explained in subsection 3.1. Then, the RND is calibrated using a beta distribution in order to convert the RND into RWD. This parameter of beta distribution is obtained by using a maximize likelihood and \( \alpha = 1.15776 \) and \( \beta = 0.88026 \) are obtained. The conversion of RND into RWD is by using equation (6). Further, the option-implied mean and volatility of index and stock options for RND and RWD are calculated using equations (8) and (9). Then, the option-implied volatility and the historical correlation is used to estimate the implied covariance as in equation (13). Lastly, the implied covariance is used as an input in the minimum-variance portfolio.

The performance of the minimum-variance with and without short-selling is reported in Table 2. It shows that the average standard deviation and Sharpe ratio of portfolio. Generally, the standard deviation of the portfolio based on RND is lower than that of RWD in both strategies. However, the Sharpe ratio of a portfolio based on RWD gives a higher value as compared to that of RND. Note that the higher is the value of Sharpe ratio, the better is the performance of the portfolio. It can be stated that the portfolio based on RWD performs better compared to that of RND even though it gives high volatilities. For the statistical analysis, we perform one sided \( t \)-test to examine the significant difference between the performances of a portfolio based on RND and RWD. The value in the parentheses from Table 2 shows the \( p \)-values of a one-sided \( t \)-test for lower standard deviation and higher Sharpe ratio. The bold \( p \)-values means rejecting the null hypothesis (as shown in Table 1) and it suggest that there is a significant difference between the performances of the portfolio.

### Table 2. Standard deviation and Sharpe ratio of portfolio.

| Portfolio with short selling | Portfolio without short selling |
| --- | --- |
| **Standard deviation** | **Sharpe ratio** | **Standard deviation** | **Sharpe ratio** |
| Risk-Neutral Density (RND) | 0.0208 | -0.1758 | 0.0413 | -0.080 |
| Risk-World Density (RWD) | 0.0506 | -0.0400 | 0.0983 | -0.0158 |

(0.000) (0.001) (0.000) (0.003)
6. Conclusions
Instead of using historical prices, this paper used the option prices as an alternative to construct the portfolio. The performance of minimum-variance strategies with and without short selling are evaluated using standard deviation and Sharpe ratio of a portfolio. The empirical evidence shows that the portfolio based on the risk-neutral and risk-world densities provide statistical significant differences. In addition, the result shows that the estimation parameter to construct the portfolio from risk-world density performs better than that of a risk-neutral density.

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