Censored Tail Regression: New Evidence on Tax and Wealth Inequality from Forbes 400

Ji Hyung Lee∗ Yuya Sasaki† Alexis Akira Toda‡ Yulong Wang§

Abstract

Data used to study wealth inequality are often bottom-censored. In this light, we propose a novel tail regression method to estimate conditional tail index models with censored data. Unlike existing methods, our proposed method enjoys (i) no parametric assumption on the underlying distribution, (ii) robustness against censoring and dependence among order statistics, and (iii) validity under time-series dependence of macroeconomic control variables. Applying it to Forbes 400 data, which is bottom-censored at the 400th order statistic, we find that the maximum marginal income tax rates are significantly associated with wealth inequality.

Keywords: Pareto exponent, censored tail regression, wealth inequality.

JEL Codes: C13, D31, H24.

∗ Ji Hyung Lee: jihyung@illinois.edu. Department of Economics, University of Illinois, 214 David Knley Hall, 1407 West Gregory Drive, Urbana, IL 61801, USA
† Yuya Sasaki: yuya.sasaki@vanderbilt.edu. Department of Economics, Vanderbilt University, VU Station B #351819, 2301 Vanderbilt Place, Nashville, TN 37235-1819, USA
‡ Alexis Akira Toda: atoda@ucsd.edu. Department of Economics, University of California San Diego, 9500 Gilman Dr, #0508, La Jolla, CA 92093-0508, USA
§ Yulong Wang: ywang402@syr.edu. Department of Economics, Syracuse University, 110 Eggers Hall, Syracuse, NY 13244-1020, USA
1 Introduction

Wealth inequality and its relationship with income tax rates are central issues in public policy debates. Since the distribution of wealth, denoted by $Y_i$, exhibits a Pareto tail in data,\(^1\) the Pareto exponent has been the parameter of interest as a measure of inequality. A large theoretical literature has developed quantitative macroeconomic models that generate and explain Pareto tails in wealth.\(^2\) This literature shows that the Pareto exponent can be expressed as a function of macroeconomic fundamentals, denoted by $X$. However, a formal econometric method of estimation and inference for the effects of the fundamentals $X$ on the Pareto exponent, or more generally, the tail index of the distribution of $Y$, is arguably missing. The main contribution of this paper is to develop such an econometric method.

Consider a repeated cross section $\{Y_{it} : i = 1, \ldots, n, t = 1, \ldots, T\}$ of the values of wealth, where $i$ indexes individuals and $t$ indexes calendar years, along with a time series $\{X_t\}_{t=1}^T$ of macroeconomic fundamentals. Our paper is strongly motivated by the Forbes 400 data set that is particularly suitable for studying tail inequality. It lists the values of the worth for the 400 wealthiest individuals in each year, i.e., the 400 largest order statistics of $\{Y_{it} : i = 1, \ldots, n\}$ at each $t$. In comparison, other common used data sets typically do not provide tail observations due to confidentiality concerns. Featured studies using this data set include Klass et al. (2006), Kaplan and Rauh (2013a,b), Capehart (2014), Gărleanu and Panageas (2017), and Gomez (2021).

Although the Forbes data have been explored in the above literature, rigorous econometric model and method of estimation and inference for such data sets have been missing. Researchers typically model the extreme values of wealth in the Forbes data as a random sample drawn from a Pareto distribution to conduct the maximum likelihood estimation (MLE; e.g., Hill, 1975) or through the log-log rank-size regression (e.g., Gabaix and Ibragimov, 2011). However, these approaches suffer from two issues in our context. First, they fail to account for the constructed correlations among the extreme order statistics, and therefore the estimated Pareto exponent and

\(^1\)See, for instance, Pareto (1896, 1897), Klass, Biham, Levy, Malcai, and Solomon (2006), Nirei and Souma (2007), and Vermeulen (2018), among others. See Gabaix (2009); Ibragimov, Ibragimov, and Walden (2015); Gabaix (2016) for a comprehensive review of this literature.

\(^2\)See Benhabib, Bisin, and Zhu (2011), Toda (2014, 2019), Toda and Walsh (2015), Gabaix, Lasry, Lions, and Moll (2016), Nirei and Aoki (2016), Aoki and Nirei (2017), Cao and Luo (2017), Jones and Kim (2018), Benhabib, Bisin, and Luo (2019), Ma, Stachurski, and Toda (2020), and de Vries and Toda (2021), among many others.
its standard error can be misleading. Second, the large sample theory assumes the data to be uncensored, whereas the Forbes data are truncated from below at the 400th order statistic. We develop a new method in this paper to resolve these issues with the existing methods.

For a fixed positive integer $k \geq 3$, our proposed method uses the largest $k$ order statistics $\{Y_{(i)t} : i = 1, \ldots, k\}$ from $\{Y_{it} : i = 1, \ldots, n\}$ along with macro explanatory variables $X_t$ at each $t$. While the underlying sample size $n$ (reflecting the U.S. population) is assumed to diverge to infinity, we allow $k$ to be fixed, unlike the aforementioned existing approaches that require $k$ to diverge. Thus, our approach models the data generating process where observations are bottom-censored at a fixed number $k = 400$ as in the Forbes data, resolving one of the two issues with existing approaches mentioned above. We approximate the conditional joint distribution of the $k$ order statistics $\{Y_{(i)t} : i = 1, \ldots, k\}$ (after a location and scale normalization) given $X_t$ with a known distribution based on extreme value theory (EVT) in the limit as $n \to \infty$ for each $t$. Our limiting conditional joint distribution accounts for constructed statistical dependence among the $k$ order statistics, resolving the other issue with the existing MLE approaches mentioned above. Even after allowing for such statistical dependence, our limiting distribution can still be characterized solely by the tail index $\xi$ that governs the tail heaviness of the distribution of $Y_{it}$. Modeling $\xi$ as a function $\xi(X_t)$ of macro control variables $X_t$, we conduct the MLE of $\xi(\cdot)$ and perform statistical inference and counterfactual analysis.

Compared with the existing methods, our MLE has three key advantages. First, our MLE does not require any parametric assumption on the distribution of $Y_{it}$, which is unknown \emph{a priori}. Instead, we only assume the distribution of $Y_{it}$ satisfies EVT and hence allow for many commonly used distributions. Accordingly, we focus on the tail index in EVT instead of the Pareto exponent since the latter is only defined when $Y_{it}$ is exactly Pareto.\footnote{For example, the tail index of the Pareto distribution with exponent $\alpha$ is $1/\alpha$, and that of the Student-t distribution with $\nu$ degrees of freedom is $1/\nu$.} EVT guarantees that the conditional joint distribution of $\{Y_{(i)t} : i = 1, \ldots, k\}$ given $X_t$ can be asymptotically approximated by a known distribution indexed by the tail index $\xi = \xi(X_t)$ robustly across a wide class of underlying distributions. To the best of our knowledge, this paper is the first to establish such a parametric conditional density approximation for a nonparametric class of distributions. We summarize this main auxiliary result as Lemma 1 in the
appendix, which may be of independent interest to econometricians. Note that the existing nonparametric methods (e.g., Martins-Filho, Yao, and Torero, 2015, 2018) model tail index to be constant in controls. Our MLE complements these methods by allowing the tail index, as well as the location and scale, to vary with $X_t$.

Second, to estimate the tail index with a cross-sectional sample $\{Y_1, Y_2, \ldots, Y_n\}$, the existing methods typically take the largest $k$ order statistics and treat them as independent draws from the Pareto distribution. See, for example, Hill (1975), Smith (1987a), Gabaix and Ibragimov (2011), Hill (2011, 2015), and de Haan and Ferreira (2006, Ch. 3). The independence holds asymptotically under the so-called increasing-$k$ asymptotics where $k \to \infty$ and $k/n \to 0$. In our Forbes data set, however, $k$ is fixed at 400 for every year $t$. This feature violates the conventional requirement of $k \to \infty$ and hence compromises the finite sample performance of these methods (e.g., Toda and Wang, 2021; Wang and Xiao, 2021). To solve this issue, our method relies on the fixed-$k$ asymptotic framework in which $k$ is fixed while $n$ diverges. Although Müller and Wang (2017) and Sasaki and Wang (2021a,b) use the fixed-$k$ asymptotics for inference, none of these existing papers develops the MLE, and hence the current paper is the first in the literature propose the MLE under the fixed-$k$ asymptotics.

Third, we consider sampling processes that have not been considered in the literature. The existing methods (e.g., Wang and Tsai, 2009; Wang and Li, 2013) for analyzing effects of predictors $X$ on the tail index of the distribution of $Y$ assume independent sampling of $(Y, X)$. In our application, on the other hand, we need to allow for a time-series dependence of $\{X_t\}_{t=1}^T$ for the nature of our macroeconomic fundamentals $X_t$. In this light, we advance the existing literature by accommodating time-series dependence in the proposed tail regression method. Moreover, in our application, we observe multiple extremal order statistics $\{Y_{(i)}\}_{i=1}^k$ for each time $t$, while $X_t$ is common across all $i$ for each $t$. This peculiar sampling process relevant to our application is not handled by any of the existing methods listed above, but is accommodated by our proposed method.

Using the proposed econometric method, we discover that the tail index or the Pareto exponent of the wealth distribution of the richest is significantly associated with the maximum marginal income tax rate. Moreover, we further analyze counterfactual levels of economic (in)equality that would result under counterfactual policies that set various levels of the maximum marginal income tax rate. Under low levels of the top tax rate, such as 30–40% as is the case with the United States today, we find
that the density of wealthy population entails Pareto exponents of 1.5–1.8 regardless of the current macroeconomic states. On the other hand, under high levels of the top tax rate, such as 60% as was the case with the United States before 1980, we find that the density of wealthy population would entail Pareto exponents of around 2.1 regardless of the current macroeconomic states.

To explain these empirical findings, we present a simple dynamic general equilibrium model that features entrepreneurs that are subject to capital income risk and workers. We derive the equilibrium wealth Pareto exponent in closed-form and show that it depends only on idiosyncratic volatility, bankruptcy rate, and the tax rate, and that it is increasing in the tax rate, consistent with our empirical findings. We also derive the welfare of workers and entrepreneurs as a function of the tax rate and show that the worker welfare is decreasing in the tax rate (because high tax leads to low capital stock and wage), whereas the entrepreneur welfare is single-peaked (because high tax leads to low capital stock but provides insurance against capital income risk through loss deductions).

**Organization**  The econometric method and theory are presented in Sections 2 and 3. This is followed by a data description and an empirical analysis in Sections 4 and 5, respectively. Section 6 contains a macroeconomic model, and Section 7 concludes. Mathematical proofs, additional theoretical and estimation results as well as simulations are presented in the Appendix.

## 2 The censored tail regression

In this section, we propose a tail regression to model the dependence of the tail index (the reciprocal of the Pareto exponent) on controls, and develop a method for estimating its model parameters along with their standard errors. Consequently, we may conduct inference on the partial effect of the top marginal tax rate on the tail index and inference on counterfactual Pareto exponents under alternative levels of the top marginal tax rate that are considered by policy makers.

Consider a repeated cross section \( \{ Y_{it} \in \mathbb{R} : i = 1, \ldots, n, t = 1, \ldots, T \} \) of the values \( Y_{it} \) of wealth, where \( i \) indexes individuals and \( t \) indexes calendar years. For our data, the year 1982 is set as \( t = 1 \) and there are \( T = 36 \) periods until 2017. This repeated cross section may be considered to consist of the universe of all the individuals
in the population, but our econometric method to be presented below requires us to observe only the top $k$ values of wealth in each year $t$ for some predetermined natural number $k$. For our sample of Forbes 400, we can set $k$ to be any natural number that is smaller than or equal to 400 (but at least 3 for methodological reasons). As such, we only need to observe the top $k$ observations of $Y_{it}$, and do not need to observe $Y_{it}$ for the rest of the population.\footnote{For now we suppose that $Y_{it}$ are observed without errors. However, as we show in Appendix B, our estimation and inference method is valid under measurement errors, which are likely to arise in the Forbes 400 data set.}

In addition to the repeated cross section of the values of wealth, we also observe a time series $\{X_t \in \mathbb{R}^p : t = 1, \ldots, T\}$ of a vector consisting of policy factors and macroeconomic indices such that $X_t$ may affect the right tail of the cross sectional distribution of $\{Y_{it} \in \mathbb{R} : i = 1, \ldots, n\}$. For our empirical question, $X_t$ includes the $\Delta t$-year averages of the maximum marginal income tax rate, idiosyncratic volatility, and bankruptcy rate for various values of $\Delta t \in \{3, 5, 7, 9\}$.\footnote{This choice of controls is motivated by the economic model to be presented in Section 6.} In this setup, we are interested in the right tail behavior of the conditional distribution $F_{Y_{it}|X_t}$ of $Y_{it}$ given $X_t$. Specifically, when $F_{Y_{it}|X_t=x}$ is approximately Pareto in the right tail, the Pareto exponent should be considered as a function of $x$, that is, $\alpha = \alpha(x)$. This feature tells us how the policy and macroeconomic predictors $X_t$ may (or may not) explain the heaviness of the right tail of the wealth distribution.

For notational simplicity, we introduce the reciprocal of the Pareto exponent $\xi(x) := 1/\alpha(x)$, where $\xi(x)$ is referred to as the tail index (conditional on $X_t = x$). We model the tail index $\xi(x)$ as

$$\xi(x) = \Lambda(x^\top \theta_0) \quad \text{for some true parameter } \theta_0 \in \Theta \subset \mathbb{R}^p, \quad (1)$$

where $\Lambda(\cdot)$ is some link function chosen by the econometrician. We primarily use the standard logistic link in our analysis. By construction, we have $\alpha(X_t) = 1/\xi(X_t) > 1$, which is consistent with the theoretical formula (24) to be presented in Section 6, as well as our empirical findings. Within this framework, our problem of learning about the tail index $\xi(x)$ as a function of $x$ boils down to learning about the parameter vector $\theta_0$. To this goal, we propose a method to estimate $\theta_0$. With an estimate $\hat{\theta}$ of $\theta_0$ along with its standard errors, we can conduct statistical inference about which of the policy and macroeconomic factors $X_t$ explain the heaviness of the right tail of the
wealth distribution. Furthermore, we can also make predictions about the right tail behavior \( \hat{\xi}(x) = \Lambda(x^\top \hat{\theta}_0) \) of the wealth distribution under counterfactual values of \( x \), such as counterfactual policies of maximum marginal income tax rates.

Not surprisingly, a na"ıve regression analysis will not estimate \( \theta_0 \) for this non-standard formulation of the tail regression. We therefore propose the following econometric method of estimation and inference about \( \theta_0 \).

**Algorithm 1 (Censored Tail Regression).** Let \( Y_{it} \) denote the level of wealth of individual \( i \) in year \( t \). Let \( X_t \) denote the vector of controls in year \( t \).

1. Sort \( \{Y_{1t}, \ldots, Y_{nt}\} \) in descending order as \( Y_{(1)t} \geq Y_{(2)t} \geq \cdots \geq Y_{(nt)t} \) for each year \( t \) and take the largest \( k \geq 3 \) order statistics \( Y_t = (Y_{(1)t}, Y_{(2)t}, \ldots, Y_{(k)t})^\top \).

2. Form the self-normalized statistics

\[
Y_t^* = \left( 1, \frac{Y_{(2)t} - Y_{(k)t}}{Y_{(1)t} - Y_{(k)t}}, \ldots, \frac{Y_{(k-1)t} - Y_{(k)t}}{Y_{(1)t} - Y_{(k)t}}, 0 \right).
\] (2)

3. Letting

\[
f_{V^*|\Lambda(X_t^\top \theta)}(1, v_2^*, \ldots, v_{k-1}^*, 0) = \Gamma(k) \int_0^\infty s^{k-2} \exp \left( - \left( 1 + \frac{1}{\Lambda(X_t^\top \theta)} \right) \sum_{j=1}^k \log \left( 1 + \Lambda(X_t^\top \theta)v_j^*s \right) \right) ds,
\] (3)

where \( \Gamma(\cdot) \) denotes the Gamma function,\(^6\) estimate \( \theta_0 \) by the maximum likelihood estimator (MLE) \( \hat{\theta} = \arg \min_{\theta \in \Theta} S_T(\theta) \), where

\[
S_T(\theta) = -\frac{1}{T} \sum_{t=1}^T \log f_{V^*|\Lambda(X_t^\top \theta)}(Y_t^*).
\] (4)

---

\(^6\)This density involves an integral, which can be computed as follows. We first consider the change of variables \( s \to Q_{Pa(\alpha)}(t) \), where \( Q_{Pa(\alpha)}(t) \) denotes the quantile function of the Pareto distribution with exponent \( \alpha_{GQ} \). By doing so, the integral is over \([0, 1]\) and can be computed by Gauss-Legendre quadrature. In our simulation and empirical studies, we use \( \alpha_{GQ} = 1/10 \) in the change of variables. The number of sample points in the Gauss-Legendre quadrature is set to 1000 if \( \xi \leq 0.1 \) and 500 otherwise.
4. Estimate the standard errors of \( \hat{\theta} \) by \( \text{diag} \left( T \hat{\mathcal{I}}(\theta_0) \right)^{-1/2} \), where

\[
\hat{\mathcal{I}}(\theta_0) = \frac{1}{T} \sum_{t=1}^{T} \left. \frac{\partial \log f_{V^*|\Lambda(X_t^\top \theta_0)}}{\partial \theta} \frac{\partial \log f_{V^*|\Lambda(X_t^\top \theta_0)}}{\partial \theta^\top} \right|_{\theta = \hat{\theta}}.
\]

Formal asymptotic statistical theories that guarantee that this proposed method works is presented in Section 3 – see Theorem 1 in particular.

Once the estimate \( \hat{\theta} \) of the parameter vector \( \theta_0 \) is obtained, we can estimate the marginal effect \( \partial \xi(X_t)/\partial X_{jt} = \theta_0j \Lambda'(X_t^\top \theta_0) \) of the \( j \)-th predictor \( X_{jt} \) (e.g., the maximum marginal tax rate) on the tail index \( \xi(X_t^\top \theta_0) \) in year \( t \) by the plug-in counterpart

\[
\partial \xi(X_t)/\partial X_{jt} = \hat{\theta}_j \Lambda'(X_t^\top \hat{\theta}).
\] (5)

Its standard error can be obtained by the delta method provided that the link function \( \Lambda \) is twice continuously differentiable in a neighborhood of \( X_t^\top \theta_0 \) – see Corollary 1 in Section 3. Similarly, letting \( X_t^{\text{cf}} \) denote a counterfactual of \( X_t \) (e.g., by replacing the actual maximum marginal tax rate by a counterfactual rate under consideration by the policy maker), we can estimate the counterfactual value of the tail index \( \xi(X_t^{\text{cf}}) \) by

\[
\hat{\xi}(X_t^{\text{cf}}) = \Lambda((X_t^{\text{cf}})^\top \hat{\theta}).
\] (6)

Its standard error can be obtained, again, by the delta method provided that the link function \( \Lambda \) is continuously differentiable in a neighborhood of \( (X_t^{\text{cf}})^\top \theta_0 \) – see Corollary 2 in Section 3. When one is interested in the Pareto exponent as opposed to the tail index, its counterfactual value \( \alpha(X_t^{\text{cf}}) = 1/\xi(X_t^{\text{cf}}) \) can be estimated by

\[
\hat{\alpha}(X_t^{\text{cf}}) = 1/\Lambda((X_t^{\text{cf}})^\top \hat{\theta}).
\] (7)

See Corollary 3 in Section 3 for its standard error.

### 3 Econometric theory of the tail regression

In this section, we present the asymptotic theory to guarantee that the method proposed in Section 2 works in large sample. Letting \( \alpha(x) = 1/\xi(x) \), we assume that our data set conforms with the following four sets of conditions.
**Condition 1.** \( X_t \) is strictly stationary martingale difference sequence. Conditional on \( X_t \), \( \{Y_{it}\}_{i=1}^n \) are IID across \( i \).

**Condition 2.** \( F_{Y|X=x}(y) \) satisfies

\[
1 - F_{Y|X=x}(y) = c(x)y^{-\alpha(x)}(1 + d(x)(y)^{-\beta(x)} + r(x, y)),
\]

where \( c(\cdot) > 0 \) and \( d(\cdot) \) are uniformly bounded between 0 and \( \infty \) and continuously differentiable with uniformly bounded derivatives, \( \alpha(\cdot) > 0 \) and \( \beta(\cdot) > 0 \) are continuously differentiable functions, and \( r(x, y) \) is continuously differentiable with bounded derivatives with respect to both \( x \) and \( y \) with

\[
\limsup_{y \to \infty} \sup_x |r(x, y)y^{\beta(x)}| \to 0.
\]

**Condition 3.** \( \theta_0 \) is in the interior of \( \Theta \), a compact subset of \( \mathbb{R}^p \), and \( X_t \) has a compact support.

**Condition 4.** \( n \to \infty \) and \( T \to \infty \). In addition, \( T^{1/2} \sup_x n^{-\beta(x)/\alpha(x)} \to 0 \).

We provide some discussions about these conditions. **Condition 1** assumes that conditional on the macroeconomic indicators \( X_t = x \), individual wealth is randomly drawn from the population wealth distribution \( F_{Y|X=x}(\cdot) \). **Condition 2** assumes that such conditional wealth distribution has an approximate Pareto tail with the approximation error characterized by the second-order parameter \( \beta(x) \) and the remainder function \( r(x, y) \). The unconditional version of this condition has been commonly used in the statistics literature to study tail features: see, for example, de Haan and Ferreira (2006, Ch. 2). This condition is mild and satisfied by many commonly used distributions, including, for example, joint normal and joint Student-t distributions. See the discussion in Sasaki and Wang (2021b). **Condition 3** requires that the true parameter is not on the boundary of the parameter space and the covariate has a compact support. The compact support assumption is sufficient but not necessary. It is imposed to simplify the proof and plausibly satisfied by our data set, especially in our sampling periods. **Condition 4** assumes a large \( n \) and large \( T \) asymptotic framework and requires that the magnitude of \( T \) is not too large relative to \( n \). Since \( n \) is gigantic in practice, we find through Monte Carlo studies that \( T = 36 \) as in our Forbes 400 data set is sufficient to guarantee the good performance of our estimator. See Appendix D ahead for more details.
Recall that in Algorithm 1 we sort \( \{Y_{1t}, \ldots, Y_{nt}\} \) in descending order as \( Y_{(1)t} \geq Y_{(2)t} \geq \cdots \geq Y_{(nt)t} \) and take the largest \( k \) order statistics \( Y_t = (Y_{(1)t}, Y_{(2)t}, \ldots, Y_{(k)t})^T \). By Conditions 1 and 2, extreme value theory and the continuous mapping theorem imply that there exist sequences of constants \( a_n \) and \( b_n \) such that for each \( t \) and any fixed \( k \),

\[
\frac{Y_t - b_n}{a_n} \Rightarrow V_t \quad \text{as} \quad n \to \infty ,
\]

where \( V_t = (V_{1t}, V_{2t}, \ldots, V_{kt})^T \) is jointly EV distributed with PDF

\[
f_{V_t}(v_1, \ldots, v_k) = G_{\xi}(v_k) \prod_{j=1}^{k} g_{\xi}(v_j)/G_{\xi}(v_j)
\]

with \( g_{\xi}(v) = \frac{\partial G_{\xi}(v)}{\partial v} \) and

\[
G_{\xi}(v) = \begin{cases} 
\exp(-\frac{(1 + \xi v)^{-1/\xi}}{1 + \xi v}) & \text{if } \xi \neq 0 \text{ and } 1 + \xi v \geq 0, \\
\exp(-e^{-v}) & \text{if } \xi = 0 \text{ and } v \in \mathbb{R}.
\end{cases}
\]

The tail index parameter \( \xi \) characterizes the tail shape of \( Y_{it} \), and it is a function \( \xi(X_t) = \Lambda(X_t^T \theta_0) \) of \( X_t \).

The constants \( a_n \) and \( b_n \) depend on the unknown distribution of \( Y_t \) and are difficult to estimate. This is why we consider the self-normalized statistics \( (2) \) to eliminate them. By the variable transformation, we obtain

\[
Y_t^* | X_t = \left( \frac{Y_{(2)t} - Y_{(k)t}}{Y_{(1)t} - Y_{(k)t}}, \ldots, \frac{Y_{(k-1)t} - Y_{(k)t}}{Y_{(1)t} - Y_{(k)t}}, 0 \right) | X_t
\]

\[
\Rightarrow V_t^* := \left( \frac{V_{2t} - V_{kt}}{V_{1t} - V_{kt}}, \ldots, \frac{V_{k-1,t} - V_{kt}}{V_{1t} - V_{kt}}, 0 \right),
\]

where the PDF \( f_{V^*_t|\xi(X_t)} \) of the non-degenerate coordinates of \( V^* \) can be calculated as in (3).

Define the negative log-transformed likelihood function as in (4). Minimizing \( S_T(\theta) \) over \( \theta \) results in the approximate MLE \( \hat{\theta} = \arg \min_{\theta \in \Theta} S_T(\theta) \). The following theorem establishes the asymptotic normality of \( \hat{\theta} \).

**Theorem 1.** Suppose Conditions 1-4 hold. Then as \( T \to \infty \) and \( n \to \infty \), for any fixed \( k \),

\[
T^{1/2}(\hat{\theta} - \theta_0) \Rightarrow \mathcal{N}\left(0, \mathcal{I}(\theta_0)^{-1}\right),
\]
where
\[
\mathcal{I}(\theta_0) = \mathbb{E} \left[ \frac{\partial \log f_{V|\Lambda(X^\top \theta_0)}}{\partial \theta} \frac{\partial \log f_{V|\Lambda(X^\top \theta_0)}}{\partial \theta^\top} \right]
\]
denotes the Fisher information matrix.

Theorem 1 shows that the econometric method proposed in Section 2 is guaranteed
to work in large sample. Applying this theorem with the delta method to the marginal
effect (5), we obtain the following corollary.

**Corollary 1.** Suppose Conditions 1-4 hold. If \( \Lambda \) is twice continuously differentiable,
then as \( T \to \infty \) and \( n \to \infty \), for any fixed \( k \),
\[
T^{1/2} \left( \frac{\partial \hat{\xi}(x)/\partial x_j - \partial \xi(x)/\partial x_j}{\partial \xi(x)/\partial x_j} \right) \Rightarrow \mathcal{N} \left( 0, \Xi_j(x)^\top \mathcal{I}(\theta_0)^{-1} \Xi_j(x) \right),
\]
where \( \Xi_j(x) = x\theta_0 \Lambda''(x^\top \theta_0) + e_j \Lambda'(x^\top \theta_0) \) with \( e_j \) denoting the \( j \)-th standard unit vector.

Similarly, applying the theorem with the delta method to the counterfactual level (6),
we obtain the following corollary.

**Corollary 2.** Suppose Conditions 1-4 hold. If \( \Lambda \) is continuously differentiable, then
as \( T \to \infty \) and \( n \to \infty \), for any fixed \( k \),
\[
T^{1/2} \left( \frac{\partial \hat{\xi}(x)/\partial x_j - \partial \xi(x)/\partial x_j}{\partial \xi(x)/\partial x_j} \right) \Rightarrow \mathcal{N} \left( 0, \Xi(x)^\top \mathcal{I}(\theta_0)^{-1} \Xi(x) \right),
\]
where \( \Xi(x) = x\Lambda'(x^\top \theta_0) \).

Likewise, for the counterfactual value (7) of the Pareto exponent, we obtain the
following corollary.

**Corollary 3.** Suppose Conditions 1-4 hold. If \( \Lambda \) is continuously differentiable, then
as \( T \to \infty \) and \( n \to \infty \), for any fixed \( k \),
\[
T^{1/2} \left( \frac{\partial \hat{\alpha}(x)/\partial x_j - \partial \alpha(x)/\partial x_j}{\partial \alpha(x)/\partial x_j} \right) \Rightarrow \mathcal{N} \left( 0, A(x)^\top \mathcal{I}(\theta_0)^{-1} A(x) \right),
\]
where \( A(x) = x\Lambda'(x^\top \theta_0)/\Lambda(x^\top \theta_0)^2 \).
4 Data and preliminary analysis

In this section, we introduce the data set and provide some preliminary analysis. First, Forbes 400 provides the list of individuals ranked in the top 400 in the United States in terms of the wealth, along with the value of the wealth for each individual in the list. These values are calculated by the staff in Forbes Magazine who “[...] pored over hundreds of Securities Exchange Commission documents, court records, probate records, federal financial disclosures and Web and print stories.”\(^7\) We use the same sample of Forbes 400 as the one that has also been used in the literature mentioned in the literature review in Section 1. This sample consists of repeated cross sections of 400 individuals for 36 years from 1982 to 2017. We use the raw values provided in data set without inflating or deflating them, because this data set will be used only for the purpose of estimating tail indices and Pareto exponents, which are invariant from scales. Table 1 presents a partial list of the raw values in millions of U.S. dollars.

| Year | Rank | 1   | 2   | 3   | 4   | 5   | 10  | 20  | 50  | 100 | 200 | 300 | 400 |
|------|------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1985 | 2800 | 1800| 1500| 1400| 1300| 1000| 875 | 600 | 360 | 233 | 183 | 150 |
| 1990 | 5600 | 3343| 2870| 2650| 2500| 2000| 1300| 730 | 450 | 340 | 260 |
| 1995 | 14800| 11800| 6700| 6100| 4800| 4300| 3000| 1800| 900 | 600 | 435 | 340 |
| 2000 | 63000| 58000| 36000| 28000| 26000| 17000| 10000| 4700| 2600| 1500| 980 | 725 |
| 2005 | 51000| 40000| 22500| 18000| 17000| 15400| 10000| 4200| 2500| 1600| 1200| 900 |
| 2010 | 54000| 45000| 27000| 24000| 21500| 18000| 12400| 5300| 3200| 2000| 1400| 1000|
| 2015 | 76000| 62000| 47500| 47000| 41000| 33300| 23400| 9000| 5000| 3300| 2300| 1700|

Table 1: The values of the wealth in millions of U.S. dollars at the ranks 1, 2, 3, 4, 5, 10, 20, 50, 100, 200, 300 and 400 in Forbes 400 in 1985, 1990, 1995, 2000, 2005, 2010, and 2015.

Second, we use the data about the top marginal income tax rates available from Tax Policy Center, Urban Institute & Brookings Institution.\(^8\) This tax data is available from 1913 to 2020, thus sufficiently covering the period 1982–2017 of our Forbes 400 sample. Figure 1 shows the time series of the top marginal income tax rate in the United States. The rates were high at 70% or above until 1980, followed by a rapid decline during the 1980s down to 28% in 1990. The rates have eventually fluctuated

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\(^7\) We thank Matthieu Gomez for sharing this data set.

\(^8\) [https://www.taxpolicycenter.org/statistics/historical-highest-marginal-income-tax-rates](https://www.taxpolicycenter.org/statistics/historical-highest-marginal-income-tax-rates)
within a narrow range of 30–40% after 1990.

Combining these two data sets in the aforementioned event study of the 1980s, we investigate the association between the top marginal income tax rate and the right tail of the distribution of wealth. Since wealth is cumulative, a relevant measure of tax rate is the average of lagged tax rates as opposed to the contemporaneous rate. Figure 2 illustrates the time series of the 10 year average of the top marginal income tax rate in dashed lines. For preliminary data analysis in the current section, we characterize the right tail of the distribution of wealth by estimating the Pareto exponents. Figure 2 illustrates the time series of the estimated Pareto exponents in solid lines, where the left panel uses Hill’s (1975) estimator and the right panel uses Gabaix and Ibragimov’s (2011) estimator.

Observe that the series of the estimated Pareto exponents follow similar trajectories to that of the top marginal income tax rate. Namely, both the Pareto exponents and the tax rates were at high levels in the early 1980s, followed by a decade of rapid declines. These series suggest strong positive correlations between the top marginal income tax rate and the wealth Pareto exponent (negative correlations with the heaviness of the right tail of the wealth distribution). The scatter plots along with fitted lines in Figure 3 highlight this strong relationship. The data points in the left (respectively, right) panel of Figure 3 correspond to the series in the left (respectively, right) panel of Figure 2. The qualitative patterns characterizing the positive correlations are the same between the two alternative estimators of the Pareto exponents.
Figure 2: Time series of the top marginal income tax rate (10 year average) and the estimated Pareto exponents for the period between 1982 and 2017.

Figure 3: Scatter plots of the estimated Pareto exponents against the top marginal income tax rate (10 year average) for the period between 1982 and 2017.

5 Estimation results

In this section, we present and discuss the results of estimating the tail regression model (1). We begin with formally confirming the heuristic findings from the informal bivariate analysis from Section 4 by first running the simple tail regression (1) on only the top marginal income tax rate (and the constant) as \( x \). Since the wealth is cumulative, we define \( x \) as the average of the \( \Delta t \) lags of the top marginal income tax rates for each of \( \Delta t \in \{3, 5, 7, 9\} \). Columns (I)-(IV) in Table 2 show estimates of \( \theta_0 \) under this tail regression model specification for various \( \Delta t \). Observe that the
estimates of the coefficient of the top marginal income tax rate range quite narrowly from $-1.84$ to $-1.71$ and are significantly different from zero robustly across different time horizons $\Delta t \in \{3, 5, 7, 9\}$. These results imply that the top marginal income tax rate $x$ has significantly negative effects on the tail index $\xi(x)$ of the top wealth distribution. Recalling that the tail index is the reciprocal of the Pareto exponent, we formally confirm that an increase in the top marginal income tax rate contributes to reducing the density of wealthy population.

| (I)  | (II) | (III) | (IV) | (V)  | (VI) | (VII) | (VIII) |
|------|------|-------|------|------|------|-------|--------|
| Top Tax Rate | -1.71*** | -1.84*** | -1.76*** | -1.74*** | -1.62* | -1.84** | -1.54* | -1.58** |
|          | (0.84)  | (0.69)  | (0.62)  | (0.57)  | (0.87) | (0.74) | (0.79) | (0.63) |
| Volatility | 0.48    | 0.75    | 1.70    | 1.60    | (0.91) | (1.08) | (1.49) | (2.02) |
| Bankruptcy Rate | 11.77  | 0.06    | 8.14    | 0.03    | (23.63) | (17.95) | (36.73) | (15.14) |
| Constant   | 1.01    | 1.07    | 1.05    | 1.06    | 0.74   | 0.86   | 0.42   | 0.54   |
|          | (0.33)  | (0.28)  | (0.26)  | (0.24)  | (0.46) | (0.47) | (0.70) | (0.69) |
| Years Averaged ($\Delta t$) | 3       | 5       | 7       | 9       | 3       | 5       | 7       | 9       |

Table 2: Estimation of $\theta_0$ based on the MLE with linear specifications. ***p<0.01, **p<0.05, *p<0.10 (except for the constant). Standard errors are reported in parentheses.

We next evaluate the robustness of this observation with richer multivariate specifications of the tail regression model (1) controlling for macroeconomic indicators. Specifically, motivated by the economic model presented in Section 6, we include the following additional controls to $x$: the averages of the idiosyncratic volatility and bankruptcy rate for the $\Delta t$-year horizon for each of $\Delta t \in \{3, 5, 7, 9\}$. To construct the idiosyncratic volatility series, we follow the method of Bekaert, Hodrick, and Zhang (2012). Using NYSE stock return data from CRSP (Center for Research in Security Prices), we calculate the daily individual excess returns for 1,837 companies from NYSE composite. The daily individual residual series is then constructed from Fama-French 3-factor (MKT-RF, SMB, HML)\(^9\) regressions. The sample variance of the residuals of $j$-th company, $\sigma^2(u_{j,y})$, for the corresponding year provides annualized idiosyncratic variance of each company. The weighted average $\sigma_y^2 = \sum_{j=1}^{N} w_{j,y} \sigma^2(u_{j,y})$ provides the aggregate idiosyncratic variance, where the weight $w_{j,y}$ is the fraction

\(^9\)https://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html
of the company’s yearly average equity over yearly total market equity. We then use the aggregate idiosyncratic volatility $\sigma_y$ as a regressor. We compute the annual bankruptcy rate as the yield spread of long-term AAA corporate bonds. Specifically, we obtain the monthly Moody’s Seasoned AAA Corporate Bond Yield\(^{10}\) and 20-Year Treasury Constant Maturity Rate,\(^{11}\) compute the yield spread as their difference, and convert to annual frequency by taking the average over each year. The $\Delta t$-year averages for $\Delta t \in \{3, 5, 7, 9\}$ are used for the empirical analysis. Figure 4 shows the time series of these additional covariates.

![Figure 4: Time series of additional covariates.](image)

(a) Aggregate idiosyncratic volatility. (b) Bankruptcy rate.

Columns (V)–(VIII) in Table 2 show estimates of $\theta_0$ under the multivariate tail regression model specification for various $\Delta t$. Focusing on the coefficient of the top marginal income tax rate, we observe that the estimates are still close to those previously obtained without the additional covariates, and now range from $-1.84$ to $-1.54$. These results further support the robustness of the aforementioned role that the top marginal income tax rate plays in determining the top wealth distribution. Moreover, additional robustness check results with more control variables are presented in Appendix C for readability.

When a policy maker revises the income tax schedule, a natural question is whether this policy instrument has any interactions (such as complementarity) with the macroeconomic state. To address this question, we next run the multivariate tail regression (1) with interactions between the top marginal income tax rate and the
two controls, namely the idiosyncratic volatility and bankruptcy rate. Table 3 summarizes results across various combinations of interactions between the top tax rate and the two macroeconomic controls. It turns out from these estimation results that the effects of the top marginal income tax rates on top wealth inequality are largely through the main effects, and not significantly through the interaction or complementary effects with the macroeconomic indicators. These results suggest that a policy maker can target desired shapes of the top wealth distribution without accounting for the current macroeconomic states, as far as the two macroeconomic indicators under our consideration are concerned.

|                  | (II)   | (VI)   | (IX)   | (X)    | (XI)   |
|------------------|--------|--------|--------|--------|--------|
| Top Tax Rate     | -1.84*** | -1.84** | -2.46** | -1.89** | -2.45** |
|                  | (0.69)  | (0.74)  | (1.03)  | (0.69)  | (1.04)  |
| Volatility       | 0.75    |        |        |        |        |
|                  | (1.08)  |        |        |        |        |
| Bankruptcy Rate  | 0.06    |        |        |        |        |
|                  | (17.95) |        |        |        |        |
| Volatility × Tax Rate | 2.30   | 2.11   |        |        |        |
|                  | (2.85)  | (2.92)  |        |        |        |
| Bankruptcy Rate × Tax Rate | 49.54 | 40.14   |        |        |        |
|                  | (75.86) | (75.70) |        |        |        |
| Constant         | 1.07    | 0.86   | 1.06   | 0.93   | 0.95   |
|                  | (0.28)  | (0.47)  | (0.28)  | (0.35)  | (0.35)  |
| Years Averaged   | 5       | 5      | 5      | 5      | 5      |

Table 3: Estimation of $\theta_0$ based on the MLE with nonlinear specifications. ***p<0.01, **p<0.05, *p<0.10 (except for the constant). Standard errors are reported in parentheses.

Our discussions thus far are based on the estimation results for $\theta_0$ in the tail regression model (1). A feature of this approach is that this parameter $\theta_0$ in the nonlinear single index model does not directly quantify the marginal effects of the top marginal income tax rate as a policy instrument. The true marginal effects may depend not only on the parameter $\theta_0$, but also on the current state $x$ of the controls including the level of the top marginal income tax rate that varies over time. To address this issue, we next use the estimator (5) to directly measure the marginal effects for the actual policy and macroeconomic states $x$ of each of the years, 1985, 1990, 1995, 2000, 2005, 2010, and 2015. Table 4 shows the estimated marginal effects of the top marginal income tax rate for each of the eight tail regression specifications (I)–(VIII). Despite the widely time-varying levels of $x$ and the nonlinearity in the tail
regression model (1), the marginal effects remain fairly stable across time for each tail regression specification. Specifically, increasing the top marginal income tax rate by 1 percentage point (0.01) leads to a reduction of the right tail index of the wealth distribution approximately by around 0.004 in any year (i.e., under any values $x$ of the policy and macroeconomic indicators $X_t$ in the displayed years).

| Year | Top Tax Rate | (I)  | (II)  | (III) | (IV)  | (V)  | (VI) | (VII) | (VIII) |
|------|--------------|------|-------|-------|-------|------|------|-------|--------|
| 1985 | 0.54         | -0.43* | -0.46** | -0.44*** | -0.43*** | -0.41* | -0.46** | -0.39* | -0.40** |
|      |              | (0.22) | (0.18) | (0.16) | (0.14) | (0.23) | (0.19) | (0.20) | (0.16) |
| 1990 | 0.35         | -0.40* | -0.44** | -0.43** | -0.42*** | -0.38* | -0.44** | -0.38* | -0.39** |
|      |              | (0.22) | (0.19) | (0.17) | (0.16) | (0.23) | (0.20) | (0.21) | (0.17) |
| 1995 | 0.36         | -0.42* | -0.44** | -0.42** | -0.41*** | -0.40* | -0.44** | -0.37* | -0.38** |
|      |              | (0.23) | (0.19) | (0.17) | (0.15) | (0.23) | (0.20) | (0.21) | (0.17) |
| 2000 | 0.40         | -0.41* | -0.45** | -0.43** | -0.42*** | -0.39* | -0.44** | -0.37* | -0.38** |
|      |              | (0.23) | (0.19) | (0.17) | (0.16) | (0.24) | (0.20) | (0.21) | (0.17) |
| 2005 | 0.37         | -0.41* | -0.44** | -0.42** | -0.42*** | -0.39* | -0.44** | -0.36* | -0.38** |
|      |              | (0.23) | (0.19) | (0.17) | (0.15) | (0.23) | (0.20) | (0.21) | (0.17) |
| 2010 | 0.35         | -0.41* | -0.44** | -0.42** | -0.41*** | -0.38* | -0.43** | -0.37* | -0.38** |
|      |              | (0.23) | (0.19) | (0.17) | (0.15) | (0.23) | (0.20) | (0.21) | (0.17) |
| 2015 | 0.38         | -0.42* | -0.44** | -0.42** | -0.42*** | -0.40* | -0.45** | -0.37* | -0.38** |
|      |              | (0.23) | (0.19) | (0.17) | (0.15) | (0.23) | (0.20) | (0.21) | (0.17) |

Table 4: Estimation of the marginal effects of the maximum marginal tax rate on the tail index. ***p<0.01, **p<0.05, *p<0.10. Standard errors are reported in parentheses. The tax rate indicates the 5-year average of the maximum marginal tax rates.

An important objective of economic research is to advise policy makers on concrete levels of policy instrument (top marginal tax income rate) to achieve specific objectives. To this goal, we estimate the counterfactual values of the tail index $\xi(X_t^{cf})$ under alternative values of $X_t^{cf}$ under consideration by using the estimator (6). To define the counterfactual $X_t^{cf}$, we consider alternative tax rates from the list \{0.30, 0.40, 0.50, 0.60, 0.70\} while we set the the other controls, namely the idiosyncratic volatility and bankruptcy rate, to the actual values in each of the years 1985, 1990, 1995, 2000, 2005, 2010, and 2015. Table 5 shows estimates of the actual and counterfactual tail index values for each of the alternative counterfactual policies. Observe that the counterfactual tail index would range from 0.61 to 0.64 under the counterfactual marginal income tax rate of 0.30, while it would range from 0.43 to 0.46 under the counterfactual marginal income tax rate of 0.70. Each of these ranges is stable across different years. This table thus provides a guideline for a policy maker.
in choosing appropriate levels of the top marginal income tax rate for given goals of the density of wealthy population.

| Year | Actual Tax Rate | Actual Tail Index | Counterfactual Tax Rate | Counterfactual Tail Index |
|------|-----------------|-------------------|-------------------------|--------------------------|
| 1985 | 0.54            | 0.52              | 0.30                    | 0.58                     |
|      | (0.03)          |                   | (0.02)                  | (0.02)                   |
| 1990 | 0.35            | 0.60              | 0.30                    | 0.58                     |
|      | (0.02)          |                   | (0.02)                  | (0.02)                   |
| 1995 | 0.36            | 0.60              | 0.30                    | 0.58                     |
|      | (0.02)          |                   | (0.02)                  | (0.02)                   |
| 2000 | 0.40            | 0.60              | 0.30                    | 0.58                     |
|      | (0.02)          |                   | (0.02)                  | (0.02)                   |
| 2005 | 0.37            | 0.60              | 0.30                    | 0.58                     |
|      | (0.01)          |                   | (0.02)                  | (0.02)                   |
| 2010 | 0.35            | 0.61              | 0.30                    | 0.57                     |
|      | (0.02)          |                   | (0.03)                  | (0.03)                   |
| 2015 | 0.38            | 0.58              | 0.30                    | 0.57                     |
|      | (0.03)          |                   | (0.03)                  | (0.03)                   |

Table 5: Estimation of the actual and counterfactual tail indices under Model (VI). Standard errors are reported in parentheses. The actual and counterfactual tax rate indicates the 5-year average of the maximum marginal tax rates.

The tail regression (1) is modeled in terms of the tail index $\xi(x)$ as the response variable because the tail index can be conveniently restricted to the interval $(0, 1)$, consistently with the range of the link function $\Lambda$ in (1). That said, its reciprocal, namely the Pareto exponent $\alpha = \alpha(x) = 1/\xi(x)$, is the measure of tail thickness that has been used more often in the economics literature. To translate the counterfactual tail index values reported in Table 5 into this more familiar measure, we use the estimator (7) and report the counterfactual Pareto exponents in Table 6. The maximum level, 0.70, of the counterfactual marginal income tax rate under our consideration used to be the steady policy state for long prior to the 1980s in the United States, while our Forbes 400 data are available only since 1982. Our extrapolated estimates of the counterfactual Pareto exponent, ranging from 2.17 to 2.31, thus recovers the density of the wealthy population in the United States under the high-income-tax regime before the 1980s that is not directly available from our data. These estimates also suggest the density of wealthy population that our economy would achieve if the maximum marginal income tax rate were to be raised again back to old steady level of 0.70.
Table 6: Estimation of the actual and counterfactual Pareto exponents under Model (VI). Standard errors are reported in parentheses. The actual and counterfactual tax rate indicates the 5-year average of the maximum marginal tax rates.

Finally, we conduct out-of-sample counterfactual predictions for the Pareto exponents in year 2020 using the controls that are available up to year 2019. The top row in Table 7 shows estimates of the Pareto exponents in 2020 under each of the alternative counterfactual policies of the maximum marginal income tax rates from the list \{0.20, 0.30, \ldots, 0.70, 0.80\}. As expected, a higher marginal tax rate will increase the Pareto exponent, implying a thinner tail and hence less inequality. Observe that the counterfactual maximum marginal income tax rate of 0.60 or above would yield the Pareto exponent above 2 (in terms of the point estimates), allowing for finite second moments of the wealth distribution.

To conduct a formal test of this statement, however, we should also pay attention to the standard errors as well as the point estimates. Using these estimates and the associated standard errors, we conduct the one-sided tests of the null hypothesis of the finite $r$-th moment of the wealth distribution for $r \in \{1, 2, 3\}$. The bottom parts of Table 7 show the test results. We mean by “$< \infty$” that we fail to reject the null hypothesis of $\alpha(X_{it}^{cf}) \geq r$ at 5% significance level, while we mean by “$= \infty$” that we reject it. The p-values are reported in parentheses. The test results suggest that the first moment of the wealth distribution is finite at any counterfactual tax rate, but the second and higher moments are infinite at lower tax rates as is the case with the United States today. In order to reduce the wealth inequality, the tax rate
| Counterfactual Tax Rate | 0.20 | 0.30 | 0.40 | 0.50 | 0.60 | 0.70 | 0.80 |
|-------------------------|------|------|------|------|------|------|------|
| Pareto Exponent         | 1.52 | 1.63 | 1.75 | 1.91 | 2.09 | 2.31 | 2.57 |
|                         | (0.09)| (0.08)| (0.09)| (0.13)| (0.22)| (0.34)| (0.51) |
| Test of Finite Moments: |      |      |      |      |      |      |      |
| First Moment            | $< \infty$ | $< \infty$ | $< \infty$ | $< \infty$ | $< \infty$ | $< \infty$ | $< \infty$ |
|                         | (1.00) | (1.00) | (1.00) | (1.00) | (1.00) | (1.00) | (1.00) |
| Second Moment           | $= \infty$ | $= \infty$ | $= \infty$ | $< \infty$ | $< \infty$ | $< \infty$ | $< \infty$ |
|                         | (0.00) | (0.00) | (0.00) | (0.24) | (0.66) | (0.82) | (0.87) |
| Third Moment            | $= \infty$ | $= \infty$ | $= \infty$ | $= \infty$ | $= \infty$ | $= \infty$ | $< \infty$ |
|                         | (0.00) | (0.00) | (0.00) | (0.00) | (0.00) | (0.02) | (0.21) |

Table 7: Extrapolated estimates of the counterfactual Pareto exponents in 2020 under Model (VI) and tests of finite first, second, and third moments of the wealth distribution. Standard errors are reported in parentheses below the estimates. P values of the tests of infinite moments are reported in parentheses below the test results. The counterfactual tax rate indicates the 5-year average of the maximum marginal tax rates.

needs to be raised to 0.60 for finite second moment and 0.80 or above for finite third moment.

6 A model of tax rate and Pareto exponent

In the previous two sections, we have documented positive effects of the marginal tax rate on the wealth Pareto exponent. In this section, we present a minimal stylized theoretical model that sheds light on these effects.

6.1 Model description

Because a large fraction of Forbes 400 individuals are self-made entrepreneurs,\textsuperscript{12} we model the top of the wealth distribution as consisting of entrepreneurs. We consider an economy consisting of two agent types, entrepreneurs and workers, with mass 1 and $L$, respectively. Time is continuous and is denoted by $t \in [0, \infty)$. Workers inelastically supply one unit of labor and earn wage $\omega_t$, which they consume entirely.\textsuperscript{13}

\textsuperscript{12}According to Forbes, about 70% of individuals in the list are self-made: https://www.forbes.com/sites/jonathanponciano/2020/09/08/self-made-score.

\textsuperscript{13}This hand-to-mouth assumption is for simplicity only and can be microfounded by supposing that workers are impatient and cannot borrow.
Entrepreneurs operate their own firms using capital and labor as inputs and trading risk-free bonds in zero net supply, pay taxes on profits, and go bankrupt at a Poisson rate \( \lambda > 0 \). The budget constraint of a typical entrepreneur is

\[
dw_t = (1 - \tau_t)((F(k_t, l_t) - \omega_l l_t) \, dt + \sigma k_t \, dB_t + r_t b_t \, dt) - c_t \, dt. \tag{10}
\]

Here \( k_t, l_t \) are capital and labor inputs and \( F \) is the production function including capital depreciation; \( \tau_t \in [0, 1) \) is the marginal income tax rate; \( B_t \) is a standard Brownian motion assumed to be independent across entrepreneurs and \( \sigma > 0 \) is the idiosyncratic volatility; \( r_t \) is the interest rate including the risk premium for bankruptcy and \( b_t \) is bond holdings; \( c_t \) is consumption rate; and \( w_t = k_t + b_t \) is net worth. When entrepreneurs go bankrupt, they exit the economy and are replaced by new entrepreneurs. Upon exit, capital is recycled back to the economy, and the initial endowment of a new entrepreneur equals the average capital of exiting entrepreneurs. The tax revenue is used in a way that does not affect the behavior of entrepreneurs (e.g., wasted or redistributed to workers).

Entrepreneurs have the continuous-time analog of the Epstein-Zin preferences with discount rate \( \beta \), relative risk aversion \( \gamma \), and elasticity of intertemporal substitution equal to 1. More precisely, the continuation utility \( U_t \) satisfies

\[
U_t = \mathbb{E}_t \int_t^{\infty} h(c_s, U_s) \, ds, \tag{11}
\]

where \( c_t \) is the consumption rate at time \( t \) and

\[
h(c, v) = \beta v \left( (1 - \gamma) \log c - \log[(1 - \gamma)v] \right) \tag{12}
\]

is the intertemporal aggregator. (See Duffie and Epstein (1992) for technical details.) The discount rate \( \beta \) should be understood to incorporate the agent’s preference for current over future consumption as well as their awareness of the risk of bankruptcy. The objective of an entrepreneur is to maximize the recursive utility (11) subject to the budget constraint (10) and nonnegativity constraints \( k_t, l_t, c_t \geq 0 \), taking as given the paths of interest rate and wage \( \{(r_t, \omega_t)\}_{t \in [0, \infty)} \).

Throughout the rest of this section, we maintain the following assumption.

**Assumption 1.** The production function \( F(k, l) \) is homogeneous of degree 1. Letting \( f(k) := F(k, 1) \) be the production function with unit labor, \( f : (0, \infty) \to \mathbb{R} \) is twice
continuously differentiable and satisfies \( f'' < 0 \), \( f'(0) = \infty \), and \( f'(< \infty) \leq 0 \).

A typical example satisfying Assumption 1 is the Cobb-Douglas production function with constant depreciation

\[
F(k, l) = Ak^\rho l^{1-\rho} - \delta k,
\]

where \( A > 0 \) is the productivity, \( \rho \in (0, 1) \) is the capital share, and \( \delta \geq 0 \) is the depreciation rate.

### 6.2 Entrepreneur’s problem

In this section we solve the entrepreneur’s problem. We suppress the time subscript to simplify the notation. Since labor \( l \) enters only the budget constraint (10), given the wage \( \omega \) and capital holdings, it is obvious that the entrepreneur chooses \( l \) to solve

\[
\max_{l \geq 0} [F(k, l) - \omega l] = \max_{l \geq 0} [lf(k/l) - \omega l],
\]

where we have used the homogeneity of \( F \) and the definition of \( f \). Using \( f'' < 0 \), it is straightforward to show that the objective function in the right-hand side of (14) is strictly concave. The first-order condition with respect to \( l \) is

\[
f(y) - yf'(y) = \omega,
\]

where \( y = k/l \) is the capital-labor ratio. Since \( (f(y) - yf'(y))' = -yf''(y) > 0 \), the value of \( y \) (hence \( l \)) satisfying (15) is unique. Since the wage \( \omega \) and the capital-labor ratio \( y \) have a one-to-one relationship, in the subsequent discussion we use \( y \) as an endogenous variable and recover \( \omega \) using the first-order condition (15). Given capital \( k \) and capital-labor ratio \( y = k/l \), we have

\[
F(k, l) - \omega l = l(f(k/l) - \omega) = l(f(y) - \omega) = lyf'(y) = kf'(y),
\]

where we have used the first-order condition (15). Therefore the budget constraint (10) reduces to

\[
dw = [(1 - \tau)(f'(y)\theta + r(1 - \theta)) - m]w \, dt + (1 - \tau)\theta w \, dB,
\]

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where $\theta := k/w \geq 0$ is the fraction of wealth invested in capital and $m := c/w$ is the marginal propensity to consume. The following proposition characterizes the solution to the entrepreneur’s problem.

**Proposition 1.** Consider an entrepreneur facing expected return $\mu_t = f'(y_t)$, tax rate $\tau_t$, and volatility $\sigma$. Then the solution to the entrepreneur’s problem is

$$\theta_t = \max\left\{ \frac{\mu_t - r_t}{(1 - \tau_t) \gamma \sigma^2}, 0 \right\},$$

(17a)

$$m_t = \beta.$$  

(17b)

**Proof.** Special case of Proposition 4 in Appendix A.

### 6.3 Equilibrium

We now characterize the equilibrium and the stationary wealth distribution. Loosely speaking, given the path of tax rates $\{\tau_t\}_{t \in [0, \infty)}$ and the initial condition (wealth distribution), a sequential equilibrium is defined by paths of interest rate and wage $\{(r_t, \omega_t)\}_{t \in [0, \infty)}$, consumption-portfolio rules of entrepreneurs, and wealth distributions such that entrepreneurs maximize utility, the bond and labor markets clear, and the wealth distribution is consistent with the behavior of entrepreneurs and the entry/exit mechanism.

Due to our simplifying assumption of unit elasticity, the equilibrium analysis is quite tractable. First, note that the optimal behavior of entrepreneurs is characterized as in Proposition 1. Because entrepreneurs hold identical portfolios $\theta$ in (17a), the bond is in zero net supply, and only entrepreneurs trade bonds, the bond market clearing condition implies

$$1 - \theta = 0 \iff \theta = 1.$$  

(18)

By the first-order condition of the profit maximization (15), all entrepreneurs choose the same capital-labor ratio $y = k/l$. Because aggregate labor supply is fixed at $L$, letting $K$ be the aggregate capital, the labor market clearing condition implies

$$y = k/l = K/L.$$  

(19)

Next, substituting (17b), (18), and (19) into the budget constraint (16) and using
$b = 0$, we obtain the individual wealth (capital) dynamics

$$dk = gk \, dt + vk \, dB := [(1 - \tau)f'(K/L) - \beta]k \, dt + (1 - \tau)\sigma k \, dB. \quad (20)$$

Because the capital of exiting entrepreneurs is recycled back to the economy as the initial capital of entering entrepreneurs and the investment risk $dB$ is independent across entrepreneurs, aggregating (20) we obtain the law of motion for aggregate capital

$$dK = [(1 - \tau)f'(K/L) - \beta]K \, dt. \quad (21)$$

In the steady state ($\tau$ and $K$ are constant), (21) implies that aggregate capital satisfies

$$(1 - \tau)f'(K/L) - \beta = 0 \iff f'(K/L) = \frac{\beta}{1 - \tau}. \quad (22)$$

Because the wealth dynamics (20) of entrepreneurs is a geometric Brownian motion (when $K$ is constant), and entrepreneurs enter and exit at Poisson rate $\lambda$, it follows from the result of Reed (2001) that the stationary wealth distribution is double Pareto, which obeys the power law in both the upper and lower tails.\footnote{This “double power law” is empirically observed for income as documented by Reed (2001) and Toda (2012). More generally, Beare and Toda (2022) (discrete-time) and Beare, Seo, and Toda (2021) (continuous-time) show that a random Markov-modulated multiplicative process stopped at an exponentially-distributed time exhibits Pareto upper and lower tails. Although we focus on the iid case for simplicity, it is straightforward to allow for Markov modulation in our model and apply their results to characterize the Pareto exponent.} According to Reed (2001, Equation (A.4)), the Pareto exponent $\alpha$ is the positive root of the quadratic equation

$$\frac{v^2}{2} z^2 + \left(g - \frac{v^2}{2}\right) z - \lambda = 0. \quad (23)$$

We therefore obtain the following proposition.

**Proposition 2.** The stationary wealth distribution of entrepreneurs is double Pareto. The Pareto exponent is given by

$$\alpha = \frac{1}{2} \left(1 + \sqrt{1 + \frac{8\lambda}{(1 - \tau)^2\sigma^2}}\right), \quad (24)$$

which is increasing in the marginal tax rate $\tau$.

**Proof.** Since in the steady state we have $g = 0$ and $v = (1 - \tau)\sigma$ in (20), solving the
quadratic equation (23) for the positive root, we obtain (24). Since $\tau \in [0,1)$ and all parameters are positive, $\alpha$ is increasing in $\tau$.

What is remarkable about Proposition 2 is that the theoretical Pareto exponent $\alpha$ takes the simple form (24), which depends only on the idiosyncratic volatility, bankruptcy rate, and the tax rate. In particular, it is independent from preference parameters (discount rate, risk aversion, etc.), the production function, and aggregate labor supply. As a numerical illustration, we set $\sigma = 0.2633$ and $\lambda = 0.0078$, which are the average values in our data set used in Section 5. Figure 5 shows the theoretical Pareto exponent computed from (24) when we vary the marginal tax rate in the range between 30% and 70%, together with the empirical estimates obtained in Figure 3. Although the model is quite stylized, the theoretical Pareto exponent is similar to the empirical estimates.

![Figure 5: Marginal tax rate and Pareto exponent.](image)

(a) Hill (1975) estimator.  
(b) Gabaix and Ibragimov (2011) estimator.

### 6.4 Taxation and welfare

So far we know from Proposition 2 that within the model, a higher tax rate unambiguously reduces wealth inequality. However, inequality itself may not be a relevant policy target. We now use the model to study the welfare implications of taxation.

Because the economy consists of two agent types, instead of considering a particular social welfare function, we study the welfare of entrepreneurs and workers separately. For entrepreneurs, according to Proposition 4, the value function of an
entrepreneur with wealth $w$ takes the form $J(w) = \frac{1}{1-\gamma} (aw)^{1-\gamma}$ for some constant $a > 0$ that depends on exogenous parameters. Because an entrepreneur has initial capital $K$, we convert the value function into units of consumption and define the welfare criterion by $W_E = aK$. For workers, because they are hand-to-mouth, the welfare criterion is consumption. If the tax revenue is wasted, then the worker welfare equals the wage and $W_W^0 := \omega$. If the tax revenue is fully redistributed to workers, then worker welfare is $W_W^1 := \omega + \mathcal{T}/L$, where $\mathcal{T}$ is the aggregate tax revenue. The following proposition characterizes these objects.

**Proposition 3.** Let

$$K = K(\tau) := (f')^{-1} \left( \frac{\beta}{1-\tau} \right) L$$

(25)

be the steady state aggregate capital determined by (22). Then the welfare of workers and entrepreneurs as well as the aggregate tax revenue are given by

$$W_W^0 = f(K/L) - (K/L)f'(K/L),$$

(26a)

$$W_W^1 = f(K/L) - \beta K/L,$$

(26b)

$$W_E = \beta e^{-\frac{1}{\rho \sigma^2}(1-\tau)^2 \gamma \sigma^2} K,$$

(26c)

$$\mathcal{T} = \tau K f'(K/L) = (f'(K/L) - \beta) K.$$  

(26d)

The worker welfare with/without redistribution $W_W^1, W_W^0$ are both strictly decreasing in the tax rate $\tau$. If in addition the production function is Cobb-Douglas as in (13), then the entrepreneur welfare $W_E$ and tax revenue $\mathcal{T}$ are both single-peaked (initially increasing and then decreasing) in $\tau$.

The intuition for why the worker welfare is decreasing in the tax rate is straightforward: when the tax rate is high, the steady state capital is low due to the concavity of the production function (see (22)), and hence the wage is low. The intuition for why the entrepreneur welfare is non-monotonic is due to the trade-off between risk and wealth level. A higher tax rate provides insurance against capital income risk because losses can be deducted, which improves welfare as we can see from the first term in (26c). However, a higher tax rate also prevents capital accumulation and hence wealth, as we can see from (22) and the second term in (26c). Consequently, entrepreneurs prefer an intermediate tax rate.

As a numerical illustration, we set $\beta = 0.05$, $\gamma = 2$, $\rho = 0.38$, and $\delta = 0.08$, which are all standard values. We also set $A = 1$ (normalization) and $L = 9$ (so
90% of agents are workers), although these two parameters are unimportant due to the homogeneity of the production function (different choices of \((A, L)\) only shift the quantities in (26) by some multiplicative factors independent of \(\tau\)). Figure 6 shows the worker welfare (with/without redistribution), entrepreneur welfare, and the tax revenue. Consistent with Proposition 3, the worker welfare in Figure 6a is decreasing in the tax rate \(\tau\), although with redistribution the welfare is nearly flat for tax rates below 40%. The entrepreneur welfare in Figure 6b is greatly affected by the tax rate and peaks at \(\tau = 0.444\), which is only slightly higher than the current top marginal income tax rate (37%). The tax revenue in Figure 6c is single-peaked, consistent with Proposition 3 and the Laffer curve.

Figure 6: Effects of tax rate on welfare and tax revenue.
7 Summary and discussions

We use Forbes 400 data along with historical data on tax rates and macroeconomic indicators to study the effects of the maximum marginal income tax rate on wealth inequality. To analyze these effects, we propose a novel censored tail regression model and an econometric method to analyze this model. Using these data sets and the econometric model, we find that a higher maximum tax rate induces a lower tail index, or equivalently, a higher Pareto exponent of the wealth distribution. Setting the maximum tax rate to 30–40%, as is the case with U.S. today, leads to the Pareto exponent of 1.5–1.8. Counterfactually setting the maximum tax rate to 80% as suggested by Piketty (2014) would achieve the Pareto exponent of 2.6. Finally, we develop a simple economic model which explains these empirical findings about the effects of the marginal tax rate on wealth inequality. We find that the worker welfare is strictly decreasing in the tax rate even when the tax revenue is redistributed, and the entrepreneur-optimal tax rate is slightly above 40%.

We close this paper with a list of limitations and directions for future research. First, although Feldstein (1995) describe the Tax Reform Act of 1986 leading to a decline in the tax rates as a “natural experiment,” there may be other latent factors on the wealth accumulation process which may have simultaneously changed during the 1980s. If this is the case, our analysis that exploits the changes in the tax rates during this period may be potentially subject to omitted variables biases. To at least partially account for this issue, we ran additional tail regressions including macroeconomic indicators to control for some latent factors and obtain similar estimates to our baseline estimates – see Appendix C. That being said, one direction of future research to overcome this issue is to develop a new method of tail regression that can exploit exogenous variations (e.g., of an instrument) and to find such an instrument in data. Second, given the nature of the event analysis with the tax decline perceived as a single-time event, difference-in-difference type analysis may be useful as well. This will require a control group (e.g., countries with no tax decline in the 1980s) as well as a treatment group (e.g., United States). But this approach introduces another difficulty, such as the requirement of a common trend assumption for identification, which is hardly plausible in cross-country analyses. An additional difficulty with this approach is that extreme wealth data are not available for many other countries than the United States. In this light, another direction of future research is to develop an
econometric method for event study with more plausible assumptions (than the standard common trend assumptions) and to develop an extreme wealth value database across countries. Third, while we focus on the common income tax rates, this aspect of the tax information only incompletely explains the wealth accumulation process. For instance, we do not use information about heterogeneous tax rates across states, as the Forbes 400 data do not contain information about the identity of all the listed individuals and thus their geographical attributes. We also do not account for the possibility of retaining earnings through the company and not having to pay taxes every period (Acemoglu, Manera, and Restrepo, 2020). To at least partially account for the this possibility, we consider including corporate tax rates in the additional regressions run in Appendix C. That said, an important direction for future research is to acquire more detailed tax information about the entrepreneurs listed in the extreme order database.

Appendix

Appendix A contains mathematical proofs. Appendix B contains an additional theoretical result about robustness to measurement error. Appendix C contains additional empirical estimation results. Appendix D contains simulation studies.

A Mathematical proofs

A.1 Proof of Section 3 results

We first establish a lemma about the Hellinger distance between generic densities $f$ and $g$. Define the Hellinger distance between $f$ and $g$ by

$$H(f, g) = \left( \int \left( f(t)^{1/2} - g^{1/2}(t) \right)^2 dt \right)^{1/2}.$$ 

**Lemma 1.** Conditional on $X_t = x$ for any $x$, the Hellinger distance between $f_{Y_t^*|X_t=x}$ and $f_{V_{t_i}^*|\xi_t = \Lambda(\alpha \tau \theta_0)}$ satisfies

$$H \left( f_{Y_t^*|X_t=x}, f_{V_{t_i}^*|\xi_t = \Lambda(\alpha \tau \theta_0)} \right) = O \left( n^{-\beta(x)/\alpha(x)} \right) + O \left( 1/n \right)$$

(27)
for any fixed \( k \) as \( n \to \infty \).

**Proof of Lemma 1.** Since this proof is conditional on \( X_t = x \), we therefore omit the subscript \( t \) for notational simplicity. Also denote \( \xi_0 = \Lambda (x^\top \theta_0) \).

Denote \( f_{a_n}^{-1}(Y - b_n)|X=x \) as the density of \( a_n^{-1}(Y - b_n) \) conditional on \( X = x \), and similarly \( f_{V|\xi=\Lambda(x^\top \theta_0)} \) the density of \( V \) conditional on \( X = x \). Following Smith (1987b), we define the following objects, all of which are conditional on \( X = x \). Let

\[
\phi(v) = \frac{1 - F_Y|X=x(v)}{f_Y|X=x(v)}, \quad h_\beta(v) = -\frac{v^{-\beta(x)} - 1}{\beta(x)},
\]

\[
H_\beta(v, \eta) = \frac{h_\beta(1 + v\eta) - \beta(x)h_1(1 + v\eta) - (1 - \beta(x)) \log(1 + v\eta)}{-\beta(x)(1 - \beta(x))\eta^3},
\]

\[
b_n = Q_Y|X=x(1 - 1/n),
\]

\[
\xi_n = \partial \phi(v)/\partial v|v=b_n,
\]

\[
r_n = n^{-\beta(x)/\alpha(x)},
\]

\[
\psi_0(v) = (1 + \xi v)^{-1/\xi},
\]

\[
\psi_n(v; c) = (1 + \xi_n v)^{-1/\xi_n} (1 + cH_\beta(v, \xi_n)),
\]

\[
g_n(y) = \prod_{i=1}^k (-\psi'_n(y_i; cr_n)) \exp(-\psi_n(y_k; cr_n))
\]

for \( y_1 \geq \cdots \geq y_k \) and \( 1 + y_i\xi_n > 0 \) for each \( i = 1, \ldots, k \). Let \( \psi'_n \) denote the derivative with respect to the first argument. Recall the notation \( y = (y_1, \ldots, y_k) \).

Using Theorem 3.6 of Smith (1987b), we have that

\[
H \left( f_{a_n}^{-1}(Y_t - b_n)|X_t=x; g_n \right) = o \left( n^{-\beta(x)/\alpha(x)} \right) + O(1/n). \tag{28}
\]

Then the proof is complete by the following two steps. First, we show that

\[
H \left( g_n, f_{V|\xi=\Lambda(x^\top \theta_0)} \right) = O \left( n^{-\beta(x)/\alpha(x)} \right), \tag{29}
\]

which yields that

\[
H \left( f_{a_n}^{-1}(Y_t - b_n)|X_t=x; f_{V|\xi=\Lambda(x^\top \theta_0)} \right) = O \left( n^{-\beta(x)/\alpha(x)} \right) + O(1/n) \tag{30}
\]

by the triangle inequality. Second, we show that the densities of the self-normalized statistics \( Y^* \) and \( V^* \) also share the same convergence rate as desired in (27).
To establish (29), note that conditional on $X = x$, for any compact set $[s_1, s_2]$ that is strictly within (max\{-1/ξ_n, −1/ξ_0\}, ∞),

$$
\sup_{v \in [s_1, s_2]} \left| \frac{\psi'_n(v; cr_n)}{\psi'_0(v)} \right| \leq \sup_{v \in [s_1, s_2]} \left| \frac{(1 + ξ_n v)^{-1/ξ_n - 1} (1 + cr_n H_β(v, ξ_n))}{(1 + ξ v)^{-1/ξ - 1}} \right| + \sup_{v \in [s_1, s_2]} r_n \left| \frac{c (1 + ξ_n v)^{-1/ξ_n} \frac{\partial H_β(v, ξ_n)}{\partial v}}{(1 + ξ v)^{-1/ξ - 1}} \right|
$$

\leq 1 + O(ξ_n − ξ) + O(r_n)

(31)

= 1 + O(r_n)

(32)

where (31) follows the mean value expansion and (32) follows from the fact that $O(ξ_n − ξ) = O\left(b_n^{β(x)}\right) = O\left(n^{−β(x)/α(x)}\right) = O(r_n)$; see, for example, Smith (1987b, p. 7).

Similarly, we have that

$$
\sup_{v \in [s_1, s_2]} \exp \left( -\psi_n(v; cr_n) + \psi_0(v) \right)
$$

\[= \sup_{v \in [s_1, s_2]} \exp \left( - (1 + ξ_n v)^{-1/ξ_n} (1 + cr_n H_β(v, ξ_n)) + (1 + ξ v)^{-1/ξ} \right)\]

= $\exp \left( O(r_n) \right)$ = $1 + O(r_n)$.

Therefore,

$$
\int_{s_1}^{s_2} \left| g_n(y) - f_{V|ξ=Λ(x^τθ_0)}(y) \right| dy
$$

\[= \int_{s_1}^{s_2} \prod_{i=1}^{k} \frac{\psi'_n(y_i; cr_n)}{\psi'_0(y_i)} \exp \left( -\psi_n(y_i; cr_n) + \psi_0(y_i) \right) - 1 \left| f_{V|ξ=Λ(x^τθ_0)}(y) \right| dy \] (33)

\[= O(r_n).\]

Once we extend the bound (33) to allow $1 + ξ s_1 \to 0$ and $s_2 \to ∞$, it is then sufficient for (29) since the bound in total variation implies the bound in Hellinger distance. The extension with $s_2 \to ∞$ is straightforward since $f_{V|ξ=Λ(x^τθ_0)}(y)$ decays exponentially as $1 + ξ y_k \to ∞$. Now consider the case with $1 + ξ s_1 \to 0$. As $1 + ξ v \to 0$, we have
that
\[
\frac{\psi_n'(v; cr_n)}{\psi_0'(v)} - 1 \asymp r_n \log(1 + \xi v) \quad \text{and} \\
\exp (-\psi_n(v; cr_n) + \psi_0(v)) - 1 \asymp r_n (1 + \xi v)^{-1/\xi - 1},
\]
where we use \(\asymp\) to denote equivalence in the order of magnitude. Then the facts (e.g., Smith (1987b, p. 17)) that
\[
\int_{-1/\xi}^{\infty} z^{-1/\xi - 1} \psi_0'(v) \exp(-\psi_0(v)) \, dv < \infty
\]
yield the bound in (33) for \(1 + \xi s_1 \rightarrow 0\).

To establish (27), consider the change of variables \(y \rightarrow (y^*, z, y_k)\) defined by

\[
z = y_1 - y_k \quad \text{and} \quad y^*_i = \frac{y_i - y_k}{z} \quad \text{for} \quad i = 2, \ldots, k - 1.
\]

Then
\[
f_{Y^*|\xi = \Lambda(x^T \theta_0)}(y^*) = \int \int \int_{0}^{\infty} z^{k-2} f_{a^{-1}_n(Y - b_n)|X = x} (y_k + z, y^*_2 z + y_k, \ldots, y^*_k z + y_k) \, dz \, dy_k,
\]
\[
f_{V^*|\xi = \Lambda(x^T \theta_0)}(y^*) = \int \int \int_{0}^{\infty} z^{k-2} f_{V|\xi = \Lambda(x^T \theta_0)} (y_k + z, y^*_2 z + y_k, \ldots, y^*_k z + y_k) \, dz \, dy_k.
\]

We can now complete the proof using
\[
\int \left| f_{Y^*|\xi = \Lambda(x^T \theta_0)}(y^*) - f_{V^*|\xi = \Lambda(x^T \theta_0)}(y^*) \right| \, dy^*
\]
\[
\leq \int \int \int_{0}^{\infty} z^{k-2} \left| f_{a^{-1}_n(Y - b_n)|X_i = x} (y_k + z, y^*_2 z + y_k, \ldots, y^*_k z + y_k) \right. \\
\quad - \left. f_{V|\xi = \Lambda(x^T \theta_0)} (y_k + z, y^*_2 z + y_k, \ldots, y^*_k z + y_k) \right| \, dz \, dy_k \, dy^*
\]
\[
= \int \left| f_{a^{-1}_n(Y - b_n)|X_i = x}(y) - f_{V|\xi = \Lambda(x^T \theta_0)}(y) \right| \, dy = O(r_n) + O(n^{-1}),
\]
where the last equation follows from (30).

Using Lemma 1, we are now ready to prove Theorem 1.

Proof of Theorem 1. Using Taylor expansion and the fact that \(f_{V^*|\Lambda(\cdot)}\) is continuously differentiable, we have that for some value \(\dot{\theta}\) that lies on the line segment connecting
θ₀ and \( \hat{\theta} \),
\[
\hat{\theta} - \theta_0 = -\left( T^{-1} \sum_{t=1}^{T} \frac{\partial^2 \log f_{V^\ast_{\Lambda(X^t_\theta)}(Y^*_t)}}{\partial \theta \partial \theta^t} \right)^{-1} \times \left( T^{-1} \sum_{t=1}^{T} \frac{\partial \log f_{V^\ast_{\Lambda(X^t_\theta_0)}(Y^*_t)}}{\partial \theta} \right)
\]
\[=: -Q_{T1}^{-1} \times Q_{T2}.\]

In the following three steps, we establish that (i) \( Q_{T1} \overset{p}{\to} \mathcal{I}(\theta_0) \) and (ii) \( T^{1/2} Q_{T2} \Rightarrow \mathcal{N}(0, \mathcal{I}(\theta_0)). \)

To simplify notations, we present the case where \( X_t \) and \( \theta_0 \) are both scalars. Denote
\[
\hat{f}_{V^\ast_{\Lambda(X^t_\theta)}(y^*_t)}(y^*_t) = \frac{\partial}{\partial \theta} f_{V^\ast_{\Lambda(X^t_\theta)}(y^*_t)} \quad \text{and} \quad \hat{f}^\ast_{V^\ast_{\Lambda(X^t_\theta)}(y^*_t)}(y^*_t) = \frac{\partial}{\partial \theta} \hat{f}_{V^\ast_{\Lambda(X^t_\theta)}(y^*_t)}. \]

For part (i), note that \( \hat{f}^\ast_{V^\ast_{\Lambda(X^t_\theta)}(y^*_t)}(y^*_t) \) and \( \hat{f}^\ast_{V^\ast_{\Lambda(X^t_\theta)}(y^*_t)}(y^*_t) \) are both bounded from above, say by some constant \( C \), uniformly over \( \theta, y^*_t, \) and \( X_t \) given Condition 3.

Therefore, the standard uniform law of large numbers for martingale sequences implies that
\[
\sup_{\theta} \left| T^{-1} \sum_{t=1}^{T} \frac{\partial^2 \log f_{V^\ast_{\Lambda(X^t_\theta)}(Y^*_t)}}{\partial \theta^2} - \mathbb{E} \left[ \frac{\partial^2 \log f_{V^\ast_{\Lambda(X^t_\theta)}(Y^*_t)}}{\partial \theta^2} \right] \right| = o_p(1). \]

Furthermore, we have that
\[
- \mathbb{E} \left[ \frac{\partial^2 \log f_{V^\ast_{\Lambda(X^t_\theta_0)}(Y^*_t)}}{\partial \theta^2} \right] - \mathcal{I}(\theta_0) \]
\[= - \mathbb{E}_{X_t} \left[ \mathbb{E}_{Y^*_t} \left[ \frac{\partial^2 \log f_{V^\ast_{\Lambda(X^t_\theta_0)}(Y^*_t)}}{\partial \theta^2} \Big| X_t \right] \right] - \mathcal{I}(\theta_0) \]
\[\leq \mathbb{E}_{X_t} \left[ \int \frac{\partial^2 \log f_{V^\ast_{\Lambda(X^t_\theta_0)}(Y^*_t)}}{\partial \theta^2} \left( f_{Y^*_t|X_t}(y^*_t) - f_{V^\ast_{Y^*_t|X_t}}(y^*_t) \right) dy^*_t \right] \]
\[+ \mathbb{E}_{X_t} \left[ \int \frac{\partial^2 \log f_{V^\ast_{\Lambda(X^t_\theta_0)}(Y^*_t)}}{\partial \theta^2} f_{V^\ast_{Y^*_t|X_t}}(y^*_t) dy^*_t \right] - \mathcal{I}(\theta_0) \]
\[= 0 \]
\[
\leq C \mathbb{E}_{X_t} \left[ \int \left| f_{Y^*_t|X_t}(y^*_t) - f_{V^\ast_{Y^*_t|X_t}}(y^*_t) \right| dy^*_t \right] = O(r_n) = o(1), \]

where the last equality follows from the proof of Lemma 1.

Similarly, we have that
\[
\mathbb{E}[Q_{T2}] \]
\[
\begin{align*}
\mathbb{E}_X \left[ \mathbb{E}_{Y_i^*} \left[ \left( \frac{\partial \log f_{v^*|\Lambda(X_i^T \theta_0)}(Y_i^*)}{\partial \theta} \right)^2 \right] \right] \\
= \mathbb{E}_X \left[ \left( \frac{\partial \log f_{v^*|\Lambda(X_i^T \theta_0)}(Y_i^*)}{\partial \theta} \right)^2 \right] \\
= \mathbb{E}_X \left[ \int \frac{\hat{f}_{v^*|\Lambda(X_i^T \theta_0)}(y^*)}{f_{v^*|\Lambda(X_i^T \theta_0)}(y^*)} f_{Y_i^*|X_i}(y^*) \, dy^* \right] \\
= \mathbb{E}_X \left[ \int \frac{\hat{f}_{v^*|\Lambda(X_i^T \theta_0)}(y^*)}{f_{v^*|\Lambda(X_i^T \theta_0)}(y^*)} (f_{Y_i^*|X_i}(y^*) - f_{v^*|X_i}(y^*)) \, dy^* + \int \frac{\hat{f}_{v^*|\Lambda(X_i^T \theta_0)}(y^*)}{f_{v^*|\Lambda(X_i^T \theta_0)}(y^*)} \, dy^* \right] \\
= \mathbb{E}_X \left[ \left( \int \frac{\hat{f}_{v^*|\Lambda(X_i^T \theta_0)}(y^*)}{f_{v^*|\Lambda(X_i^T \theta_0)}(y^*)} \, dy^* \right)^2 \left( \int (f_{Y_i^*|X_i}(y^*) - f_{v^*|X_i}(y^*))^2 \, dy^* \right)^{1/2} \right] \\
\leq C \mathbb{E}_X \left[ n^{-\beta(X_i)/\alpha(X_i)} + n^{-1} \right] \\
\leq C \sup_x n^{-\beta(x)/\alpha(x)} = o(1), 
\end{align*}
\]

where (34) follows from the fact that \( \int \hat{f}_{v^*|\Lambda(X_i^T \theta_0)}(y^*) \, dy^* = 0 \), as implied by the Leibniz rule; (35) follows from Cauchy-Schwarz inequality; (36) follows from Lemma 1; and (37) is from Condition 4. Then part (i) is established by combining the above results.

For part (ii), we can establish the asymptotic normality by showing that \( T^{1/2} \mathbb{E} [Q_{T2}] \to 0 \) as \( n \to \infty \) and \( T \to \infty \) and \( \text{Var} [T^{1/2}Q_{T2}] \to \mathcal{I}(\theta_0) \). First, the above derivation in part (i) and Condition 4 imply that

\[
T^{1/2} \mathbb{E} [Q_{T2}] \leq T^{1/2} C \sup_x n^{-\beta(x)/\alpha(x)} = o(1).
\]

Then, given the martingale assumption, we have

\[
\text{Var} \left[ T^{1/2}Q_{T2} \right] = T \mathbb{E} \left[ Q_{T2}^2 \right] + o(1)
\]

\[
= \mathbb{E}_X \left[ \mathbb{E}_{Y_i^*} \left[ \left( \frac{\partial \log f_{v^*|\Lambda(X_i^T \theta_0)}(Y_i^*)}{\partial \theta} \right)^2 \right] \right]
\]

\[
= \mathbb{E}_X \left[ \int \left( \frac{\hat{f}_{v^*|\Lambda(X_i^T \theta_0)}(y^*)}{f_{v^*|\Lambda(X_i^T \theta_0)}(y^*)} \right)^2 \left( f_{Y_i^*|X_i}(y) - f_{v^*|X_i}(y^*) \right) \, dy^* \right]
\]

35
\[
+ \mathbb{E}_t \left[ \int \left( \frac{\dot{v}^*|_{\Lambda(X_t^t, \theta_0)}(y^*)}{f^*|_{\Lambda(X_t^t, \theta_0)}(y^*)} \right)^2 f^*|_{\Lambda(X_t^t, \theta_0)}(y^*) \, dy^* \right] \\
= \mathbb{E}_t \left[ \int \left( \frac{\dot{v}^*|_{\Lambda(X_t^t, \theta_0)}(y^*)}{f^*|_{\Lambda(X_t^t, \theta_0)}(y^*)} \right)^2 \left( f_{Y_t^*|X_t}(y^*) - f^*|_{\Lambda(X_t^t, \theta_0)}(y^*) \right) \, dy^* \right] + \mathcal{I}(\theta_0) \\
= o(1) + \mathcal{I}(\theta_0),
\]

where the last line above follows the proof of from Lemma 1 and

\[
\left| \mathbb{E}_t \left[ \int \left( \frac{\dot{v}^*|_{\Lambda(X_t^t, \theta_0)}(y^*)}{f^*|_{\Lambda(X_t^t, \theta_0)}(y^*)} \right)^2 \left( f_{Y_t^*|X_t}(y^*) - f^*|_{\Lambda(X_t^t, \theta_0)}(y^*) \right) \, dy^* \right] \right| \\
\leq \mathbb{E}_t \left[ \sup_{y^*} \left( \frac{\dot{v}^*|_{\Lambda(X_t^t, \theta_0)}(y^*)}{f^*|_{\Lambda(X_t^t, \theta_0)}(y^*)} \right)^2 \int \left( f_{Y_t^*|X_t}(y^*) - f^*|_{\Lambda(X_t^t, \theta_0)}(y^*) \right) \, dy^* \right] \\
\leq C \mathbb{E}_t \left[ \int \left| f_{Y_t^*|X_t}(y^*) - f^*|_{\Lambda(X_t^t, \theta_0)}(y^*) \right| \, dy^* \right] = o(1).
\]

The proof is complete by Slutsky’s theorem. \qed

### A.2 Proof of Section 6 results

We solve the optimal decision problem of an entrepreneur when the elasticity of intertemporal substitution takes an arbitrary value \( \varepsilon > 0 \). In this case the aggregator is

\[
h(\varepsilon, v) = \beta \left[ (1 - \gamma) v \right]^{\frac{1}{1-\gamma}} e^{1-1/\varepsilon} - \left[ (1 - \gamma) v \right]^{\frac{1-1/\varepsilon}{1-1/\varepsilon}}. \tag{38}
\]

The expression (12) is a special case by taking the limit of (38) as \( \varepsilon \to 1 \).

The following proposition generalizes Proposition 1.

**Proposition 4.** Consider an entrepreneur facing expected return \( \mu_t = f'(y_t) \), tax rate \( \tau_t \), and volatility \( \sigma \). Suppose that a solution to the individual optimization problem exists and let \( J(t, w) \) be the value function (the maximized utility in (11)). Then the solution is given by

\[
J(t, w) = \frac{1}{1 - \gamma} a_t^{1-\gamma} w^{1-\gamma}, \tag{39a}
\]
\[ \theta_t = \frac{(\mu_t - r_t)_+}{(1 - \tau_t)\gamma \sigma^2}, \quad (39b) \]
\[ m_t = \beta^\varepsilon a_t^{1-\varepsilon}, \quad (39c) \]

where \( x_+ = \max\{x, 0\} \) and the coefficient of the value function \( a_t > 0 \) satisfies the ordinary differential equation (ODE)

\[ \frac{\dot{a}_t}{a_t} = -\left( (1 - \tau_t)r_t + 2^{-1}(\varepsilon - \beta^{-1-\varepsilon}a_t^{1-\varepsilon}) \right) \quad (\varepsilon \neq 1) \]
\[ (1 - \log \beta + \log a_t). \quad (\varepsilon = 1) \quad (40) \]

In steady state (\( r_t = r, \mu_t = \mu, \) and \( \tau_t = \tau \) are constant), the optimal portfolio, consumption, and the coefficient of the value function are

\[ \theta = \frac{(\mu - r)_+}{(1 - \tau)\gamma \sigma^2}, \quad \text{(41a)} \]
\[ m = \varepsilon \beta + (1 - \varepsilon) \left( (1 - \tau)r + 2^{-1}(\mu - r)_+ \right), \quad \text{(41b)} \]
\[ a = \begin{cases} \beta^{-1-\varepsilon} \left[ \varepsilon \beta + (1 - \varepsilon) \left( (1 - \tau)r + 2^{-1}(\mu - r)_+ \right) \right] \frac{1}{1-\varepsilon}, & (\varepsilon \neq 1) \\ \beta \exp \left( \frac{1}{\beta} \left( (1 - \tau)r + 2^{-1}(\mu - r)_+ \right) - 1 \right), & (\varepsilon = 1) \end{cases} \quad (41c) \]

**Proof.** We can solve the individual optimization problem using stochastic control (Fleming and Soner, 2006). For notational simplicity, suppress the time subscript \( t \). Using the budget constraint (16), the Hamilton-Jacobi-Bellman (HJB) equation becomes

\[ 0 = \max_{m, \theta \geq 0} \left[ h(mw, J_t) + J_t + J_w(\tilde{r} + \tilde{\alpha}\theta - m)w + \frac{1}{2} J_{ww}(\tilde{\sigma}\theta w)^2 \right], \quad (42) \]

where \( h \) is given by (38), \( \tilde{r} = (1 - \tau)r, \tilde{\alpha} = (1 - \tau)(\mu - r), \) and \( \tilde{\sigma} = (1 - \tau)\sigma. \)

Assume \( \varepsilon \neq 1 \). Due to homogeneity, the value function takes the form \( J(t, w) = \frac{1}{1-\gamma}a_t^{1-\gamma}w^{1-\gamma} \), where \( a_t > 0 \). Fixing the state variables \( t, w, \) let

\[ H(m, \theta) = h(mw, J_t) + J_t + J_w(\tilde{r} + \tilde{\alpha}\theta - m)w + \frac{1}{2} J_{ww}(\tilde{\sigma}\theta w)^2 \]

be the expression inside the bracket of (42). Since \( J_{ww} = -\gamma a_t^{1-\gamma}w^{1-\gamma} < 0, \) \( F \) is a concave quadratic function of \( \theta \). Therefore it takes the global maximum over \( \theta \in \mathbb{R} \)
\[ 0 = \frac{\partial H}{\partial \theta} = J_w \tilde{\alpha} w + J_{ww} \tilde{\sigma}^2 w^2 \theta \quad \iff \quad \theta = \frac{\tilde{\alpha}}{2 \gamma \tilde{\sigma}^2} = \frac{\mu - r_t}{(1 - \tau) \gamma \tilde{\sigma}^2}. \]

Since \( \theta \geq 0 \), (39b) is the solution.

Since by (12) \( h(c, v) \) is concave in \( c \), \( H \) is concave in \( m \). Therefore it takes the maximum over \( m \geq 0 \) when

\[ 0 = \frac{\partial H}{\partial m} = \beta [(1 - \gamma) J]^{1/1 - \gamma} (mw)^{1/\varepsilon} w - J_w w \quad \iff \quad m = \beta \varepsilon a_1^{1-\varepsilon} w, \]

which is (39c).

To derive an ODE for \( a \), we use the HJB equation (42). Since \( J \) is proportional to \( w^{1-\gamma} \), \( H \) is homogeneous of degree \( (1 - \gamma) \) in \( w \). Therefore without loss of generality we may set \( w = 1 \). In this case, using the first-order condition with respect to \( m \),

\[ [(1 - \gamma) J]^{1/1 - \gamma} = a, \quad J_w = a^{1-\gamma}, \quad \text{and} \quad J_{ww} = -\gamma a^{1-\gamma}, \]

we can simplify (42) as

\[ 0 = \frac{1}{1 - 1/\varepsilon} J_w m - \frac{\beta}{1 - 1/\varepsilon} a^{1-\gamma} + a^{-\gamma} \dot{a} + \left( \tilde{r} + \tilde{\alpha} \theta - \frac{1}{2} \gamma \tilde{\sigma}^2 \theta^2 \right) a^{1-\gamma} - J_w m. \quad (43) \]

Since \( J_w m = a^{1-\gamma} \beta \varepsilon a^{1-\varepsilon} = \beta \varepsilon a^{2-\varepsilon-\gamma} \), multiplying both sides of (43) by \( (1 - \varepsilon) a^\gamma \), we obtain

\[ 0 = -\beta \varepsilon a^{2-\varepsilon} + \left[ \varepsilon \beta + (1 - \varepsilon) \left( \tilde{r} + \tilde{\alpha} \theta - \frac{1}{2} \gamma \tilde{\sigma}^2 \theta^2 \right) \right] a + (1 - \varepsilon) \dot{a}, \]

which is equivalent to (40) if \( \varepsilon \neq 1 \). If \( \varepsilon = 1 \), letting \( \varepsilon \to 1 \) in the formula (40) with \( \varepsilon \neq 1 \) and using l'Hôpital's rule, we obtain the expression for \( \varepsilon = 1 \).

**Proof of Proposition 3.** Let \( y = K/L \) be the capital-labor ratio. Then (26a) immediately follows from \( W_W^0 = \omega \) and (15). (26d) follows from aggregating the budget constraint (16) and using (22). Using (26a) and (26d), we obtain

\[ W_W^1 = f(y) - y f'(y) + (f'(y) - \beta) y = f(y) - \beta y, \]

which is (26b).

Since by Assumption 1 we have \( f'' < 0 \), by (22) the capital-labor ratio \( y = K/L \)
is strictly decreasing in \( \tau \). Then

\[
\frac{\partial W^0_y}{\partial y} = (f(y) - yf'(y))' = -yf''(y) > 0,
\]

so \( W^0_y \) is strictly decreasing in \( \tau \). Furthermore, using (22), we obtain

\[
\frac{\partial W^1_y}{\partial y} = f'(y) - \beta = \frac{\beta}{1 - \tau} - \beta = \frac{\beta \tau}{1 - \tau} > 0,
\]

so \( W^1_y \) is also strictly decreasing in \( \tau \).

To show (26c), note that by (22) the expected return satisfies

\[
\mu = f'(y) = f'(K/L) = \frac{\beta}{1 - \tau}.
\]

Combining (17a), (18), and (44), the equilibrium interest rate must satisfy

\[
r = \frac{\beta}{1 - \tau} - (1 - \tau)\gamma \sigma^2.
\]

Combining \( W_E = aK \), (41c), (17a), and (45), it follows that

\[
W_E = \beta \exp \left( \frac{1}{\beta} \left( \beta - (1 - \tau)2\gamma \sigma^2 + \frac{1}{2} (1 - \tau)^2 \gamma \sigma^2 \right) - 1 \right) K = \beta e^{\frac{1}{\beta} \left( 1 - \tau \right)^2 \gamma \sigma^2} K,
\]

which is (26c).

Now suppose the production function is Cobb-Douglas as in (13). Then (22) implies

\[
\frac{\beta}{1 - \tau} = f'(K/L) = A\rho(K/L)^{\rho - 1} - \delta \iff K(\tau) = L \left[ \frac{1}{A\rho} \left( \frac{\beta}{1 - \tau} + \delta \right) \right]^\frac{1}{\rho - 1}.
\]

Therefore the logarithm of entrepreneur welfare becomes

\[
\log W_E = -\frac{1}{2\beta} (1 - \tau)^2 \gamma \sigma^2 + \frac{1}{\rho - 1} \log \left( \frac{\beta}{1 - \tau} + \delta \right) + \text{constant}.
\]

Letting \( x = \frac{1}{1 - \tau} \) and ignoring the constant term, the above expression becomes

\[
g(x) := -\frac{\gamma \sigma^2}{2\beta x^2} + \frac{1}{\rho - 1} \log(\beta x + \delta).
\]
Taking the derivative with respect to $x$, we obtain
\[
g'(x) = \frac{\gamma \sigma^2}{\beta x^3} + \frac{1}{\rho - 1} \frac{\beta}{\beta x + \delta} = \frac{1}{x} \left( \frac{\gamma \sigma^2}{\beta x^2} + \frac{1}{\rho - 1} \frac{\beta x}{\beta x + \delta} \right) =: \frac{h(x)}{x}.
\]
Since $\rho \in (0, 1)$, clearly $h$ is strictly decreasing in $x$. Furthermore, $h(0) = \infty$ and $h(\infty) = \frac{1}{\rho - 1} < 0$. Therefore $g$ is initially increasing and then decreasing in $x$, and because $x = \frac{1}{1-\tau}$ is monotonic in $\tau$, the entrepreneur welfare $W_E$ is single-peaked in $\tau$. Using (13) and (26d), the tax revenue becomes $T = A\rho K^\rho L^{1-\rho} - \delta K$, which is strictly concave (single-peaked) in $K$. Because $K(\tau)$ in (25) is strictly decreasing in $\tau$, it follows that $T$ is single-peaked in $\tau$. \hfill \Box

\section*{B Robustness to measurement error}

In this section, we show that our asymptotic theory is robust to measurement error, which may arise in the Forbes 400 data set. In particular, we aim to show that the self-normalized statistic $Y^*_t$ has the same limit as before even when some random error is added. We suppress the subscript $t$ and $X_t$ without loss of generality since such robustness is established to hold conditional on $X_t$ for any $t$.

To fix the idea, let $\varepsilon_i$ be some random measurement error, which is assumed to be independent from the true wealth $Y_i$ and also independent across $i$, though not necessarily identically distributed across $i$. The econometrician observes the composite $\tilde{Y}_i = Y_i + \varepsilon_i$ and select the largest $k$ order statistics
\[
\tilde{Y} = (\tilde{Y}_{(1)}, \tilde{Y}_{(2)}, \ldots, \tilde{Y}_{(k)}).
\]
The following proposition shows that $\tilde{Y}$ has the same limit as $Y$ in (8).

\textbf{Proposition 5.} Suppose (i) $Y_i$ is IID with CDF satisfying the Pareto-type tail condition as in Condition 2 with tail index $\xi > 0$, and (ii) $\varepsilon_i$ is independent from $Y_i$ and satisfies $\sup_i |\varepsilon_i| = o_p(n^\xi)$. Then there exist sequences of constants $a_n$ and $b_n$ such that for any fixed $k$,
\[
\frac{\tilde{Y} - b_n}{a_n} \Rightarrow V
\]
as $n \to \infty$, where $V$ is defined as in (8).

\textbf{Proof.} The proof is analogous to that of Lemma 1 in Sasaki and Wang (2021a). By
Corollary 1.2.4 and Remark 1.2.7 in de Haan and Ferreira (2006), the constants \(a_n\) and \(b_n\) can be chosen as \(a_n = Q_Y (1 - 1/n) = O(n^{\xi})\) and \(b_n = 0\), where \(Q_Y(\cdot)\) denotes the quantile function of \(Y\).

Now, let \(I = (I_1, \ldots, I_k) \in \{1, \ldots, n\}^k\) be the \(k\) random indices such that \(Y_{(j)} = Y_{I_j}, j = 1, \ldots, k\), and let \(\tilde{I}\) be the corresponding indices such that \(\tilde{Y}_{(j)} = \tilde{Y}_{I_j}\). Then, the convergence of \(\tilde{Y}\) follows from (8) once we establish \(\left|\tilde{Y}_{I_j} - Y_{I_j}\right| = o_p(a_n)\) for \(j = 1, \ldots, k\). We present the case of \(k = 1\), but the argument for a general \(k\) is similar. Recall that \(\varepsilon_i = \tilde{Y}_i - Y_i\), and Condition (ii) in the proposition implies that \(\sup_i |\varepsilon_i| = o_p(a_n)\).

Given this result, we have that, on one hand,

\[
\tilde{Y}_{I_j} = \max_i \{Y_i + \varepsilon_i\} \leq Y_I + \sup_i |\varepsilon_i| = Y_I + o_p(a_n).
\]

On the other hand,

\[
\tilde{Y}_I = \max_i \{Y_i + \varepsilon_i\} \geq \max_i \{Y_i + \min_i \varepsilon_i\}
\geq Y_I + \min_i \varepsilon_i \geq Y_I - \sup_i |\varepsilon_i| = Y_I - o_p(a_n).
\]

Therefore \(\left|\tilde{Y}_{I_j} - Y_{I_j}\right| = o_p(a_n)\) holds.

Proposition 5 establishes the robustness to measurement error, which is asymptotically dominated. Note that this result allows for non-identically distributed errors \(\varepsilon_i\) as long as its maximum magnitude is smaller than that of the true wealth.

C Additional empirical results

This section provides some additional empirical results for robustness check. The econometric method is the same as in Section 5.

First, in the macroeconomic model, the Pareto tail is only derived under the equilibrium. To shed some light on the transition period, we repeat our analysis by dropping some transition years and report the results in Table 8. In particular, the top income tax rate declines substantially in late 80’s. To capture this feature, we delete the observations in five consecutive years starting at 1985, 86, 87, and 88 and run the MLE under Model (IV). The results are robust in the sense that the coefficient
associated with top tax rate remains significant. The estimates are between -1.41 and -1.81 when the observations from different years are excluded.

| Excluded Years | None | 1985-1989 | 1986-1990 | 1987-1991 | 1988-1992 |
|----------------|------|-----------|-----------|-----------|-----------|
| Top Tax Rate   | -1.84** (0.74) | -1.60** (0.81) | -1.41* (0.82) | -1.81** (0.76) |
| Volatility     | 0.75 (1.08) | 1.65 (1.19) | 1.58 (1.19) | 1.46 (1.18) |
| Bankruptcy Rate| 0.06 (17.95) | 9.02 (30.28) | 8.50 (29.89) | 0.15 (16.16) |
| Constant       | 0.86 (0.47) | 0.41 (0.56) | 0.37 (0.57) | 0.64 (0.50) |
| Years Averaged | 5 | 5 | 5 | 5 |

Table 8: Estimation of $\theta_0$ based on the MLE under Model (VI) with some years excluded. ***p < 0.01, **p < 0.05, *p < 0.10 (except for the constant). Standard errors are reported in parentheses.

Next, we examine the robustness of our previous results with more control variables, following the literature on wealth inequality. See Alvaredo, Atkinson, Piketty, and Saez (2013), Benhabib and Bisin (2018), and Benhabib et al. (2019) for example. In particular, we include housing returns, stock returns, and the top capital gain tax rate as additional controls. We collect data on housing values from United States Census Bureau, the data on stock returns from Amit Goyal’s webpage, and the data on top capital gain tax rate from tax foundation’s website. For $\Delta t$-year horizon returns, we construct the continuously compounded $\Delta t$-year returns $r_t(\Delta t) = \log P_{t+\Delta t} - \log P_t$, where $P_t$ is the corresponding housing price or S&P500 index adjusted for the real term. As in Section 5, we also use the five-year average of housing returns, stock returns, and volatility of these controls.

Table 9 presents the estimation results, which can be summarized as follows. First, the top income tax rate is still significant with a very stable magnitude. Second, all the additional control variables are not significant. Third, although not reported, we find that the top corporate tax rate is strongly correlated with the top income tax

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15https://www.census.gov/construction/nrs/historicaldata/index.html
16See http://www.hec.unil.ch/agoyal/ for the original data and their description.
17https://taxfoundation.org/federal-capital-gains-tax-collections-historical-data
18See Table 24 at https://www.irs.gov/statistics/soi-tax-stats-historical-data-tables.
rate (the correlation coefficient is approximately 0.9). Including both taxes leads to potential multi-collinearity. Therefore, our results should be interpreted as composite effects of the overall tax rate, not necessarily a particular tax rate.

|                  | (VI)  | (XII) | (XIII) | (XIV) | (XV)  |
|------------------|-------|-------|--------|-------|-------|
| Top Tax Rate     | -1.84** | -1.85*** | -1.69** | -1.80** | -1.70** |
|                  | (0.74) | (0.69) | (0.78) | (0.73) | (0.71) |
| Volatility       | 0.75  | 1.62  | 0.61   | 1.57  |
|                  | (1.08) | (1.27) | (1.12) | (1.27) |
| Bankruptcy Rate  | 0.06  | 11.47 | 0.08   | 0.08  |
|                  | (17.95) | (31.09) | (17.63) | (18.02) |
| Stock Return     | 0.15  | 0.28  | 0.38   |
|                  | (0.16) | (0.19) | (0.21) |
| Housing Return   | 0.09  | 0.06  | 0.00   |
|                  | (0.40) | (0.42) | (0.42) |
| Top Capital Gain Tax Rate | -0.61 | -1.75 |
|                  | 1.32  | (1.47) |
| Constant         | 0.86  | 1.03  | 0.39   | 1.02  | 0.88  |
|                  | (0.47) | (0.28) | (0.59) | (0.60) | (0.61) |
| Years Averaged   | 5     | 5     | 5      | 5     | 5     |

Table 9: Estimation of $\theta_0$ based on the MLE with additional variables. ***p<0.01, **p<0.05, *p<0.10 (except for the constant). Standard errors are reported in parentheses.

D Simulation studies

In this section, we evaluate the finite sample properties of the newly proposed MLE through simulations. The data generating process (DGP) is designed as follows:

$X_t = \rho X_{t-1} + \sqrt{1 - \rho^2} u_t$ where $u_t$ is IID standard normal. We set $\rho = 0.5$. Conditional on $X_t = x$, $Y_{it}$ for $i = 1, \ldots, n$ are randomly generated from the following three distributions with tail index $\xi = \Lambda(0.5 + x\beta_0)$ and $\beta_0 = 1$: (i) Pareto distribution with minimum size 1, (ii) absolute value of the Student-$t$ distribution, and (iii) double Pareto-lognormal distribution (dPIN). The double Pareto-lognormal distribution is the product of independent double Pareto and lognormal variables such that

$$Y = \exp(\mu + \sigma Z_1 + Z_2 \xi - Z_3),$$

where $Z_1, Z_2, Z_3$ are independent and $Z_1 \sim \mathcal{N}(0, 1)$ and $Z_2, Z_3 \sim \text{Exp}(1)$. For parameter values, we set $\mu = 0$ and $\sigma = 0.5$, which are typical for income data.
We set $T = 36$ as in the Forbes data set and experiment with $n = (10^4, 10^5, 10^6)$. The number of top order statistics is $k = (100, 200, 400)$, all of which are small relative to $n$. The numbers are based on $M = 1000$ simulations.

Table 10 shows the following summary statistics of the estimates of $\beta_0$: (i) Bias: $\frac{1}{M} \sum_{m=1}^{M} (\hat{\beta}_m - \beta_0)$, where $m$ indexes simulations and $M = 1000$. (ii) RMSE: root mean squared error defined by $\sqrt{\frac{1}{M} \sum_{m=1}^{M} (\hat{\beta}_m - \beta_0)^2}$. (iii) Coverage: the fraction of simulations for which the true value $\beta_0 = 1$ falls into the 95% confidence interval. (iv) Length: the average length of confidence intervals across simulations.

| DGP | Pareto | $|t|$ | dPIN |
|-----|--------|------|------|
| $k$ | 100    | 200  | 400  |
| $n$ | Bias   |       |      |
| $10^4$ | 0.01  | 0.01 | 0.01 |
| $10^5$ | 0.01  | 0.01 | 0.00 |
| $10^6$ | 0.02  | 0.01 | 0.00 |
| $n$ | RMSE   |       |      |
| $10^4$ | 0.19  | 0.13 | 0.09 |
| $10^5$ | 0.19  | 0.12 | 0.09 |
| $10^6$ | 0.18  | 0.13 | 0.09 |
| $n$ | Coverage |       |      |
| $10^4$ | 0.95  | 0.95 | 0.95 |
| $10^5$ | 0.95  | 0.96 | 0.95 |
| $10^6$ | 0.96  | 0.95 | 0.95 |
| $n$ | Length |       |      |
| $10^4$ | 0.72  | 0.49 | 0.35 |
| $10^5$ | 0.72  | 0.50 | 0.35 |
| $10^6$ | 0.72  | 0.50 | 0.35 |

Table 10: Finite sample properties of the proposed MLE. Based on 1000 simulation draws. See the main text for more details.

We can make a few observations from Table 10. First, when the underlying distribution is exactly Pareto, the bias of the MLE is very small as shown in the first three columns, and the coverage probability is also close to the nominal level 95%. Second, however, when the underlying distribution is not Pareto, any parametric estimator that relies on the Pareto tail assumption would suffer from the misspecification bias. In our case, this bias is reflected in the Student-t and dPIN case with $n = 10^4$. Such bias becomes negligible eventually when $n$ is sufficiently large, as seen in the rows with $n = 10^5$ and $10^6$, which are still smaller in magnitudes than the total population.
in the U.S. Therefore, we believe that our estimator as well as the extreme value approximation (9) perform well. Finally, the randomness of our estimator does not decrease with $n$ significantly, which is not surprising. A larger $n$ only improves the extreme value approximation, while a larger $T$ will substantially reduce the RMSE. Again because of the potentially large $n$, we are essentially facing the parametric problem where $T$ draws are selected from the extreme value distribution (9), and therefore a $T$ as small as 36 still produces informative and significant estimates.

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