TIME-DEPENDENT SCATTERING THEORY ON MANIFOLDS

K. ITO AND E. SKIBSTED

Abstract. This is the third and the last paper in a series of papers on spectral and scattering theory for the Schrödinger operator on a manifold possessing an escape function, for example a manifold with asymptotically Euclidean and/or hyperbolic ends. Here we discuss the time-dependent scattering theory. A long-range perturbation is allowed, and scattering by obstacles, possibly non-smooth and/or unbounded in a certain way, is included in the theory. We also resolve a conjecture by Hempel–Post–Weder on cross-ends transmissions between two or more ends, formulated in a time-dependent manner.

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1. Introduction

Let \((M, g)\) be a connected Riemannian manifold. In this paper we study time-dependent scattering theory for the geometric Schrödinger operator

\[ H = H_0 + V; \quad H_0 = -\frac{1}{2} \Delta = \frac{1}{2} p_i g^{ij} p_j, \quad p_i = -i \partial_i, \]

on the Hilbert space \( \mathcal{H} = L^2(M) \). The potential \( V \) is real-valued and bounded, and the self-adjointness of \( H \) is realized by the Dirichlet boundary condition. For previous time-dependent short-range scattering theories on manifolds we refer to

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[Co, II, IN, IS1]. For a review of scattering by an unbounded obstacle in $\mathbb{R}^d$ as studied in [Co, II] we refer to [Ya2].

Our theory includes a generalization of asymptotic completeness on the Euclidean space stated in terms of a “time-dilated” comparison dynamics by Yafaev [Ya1]. In its simplest form given for $H = -\frac{1}{2} \Delta$ on the Euclidean space $\mathbb{R}^d$ it amounts to the comparison (using spherical coordinates $(r, \sigma)$)

$$
(e^{\pm itH} f)(r, \sigma) \approx r^{(1-d)/2} e^{\pm ir^2/(2t)} t^{-1/2} \phi_{\pm}(r/t, \sigma) \quad \text{for} \quad t \to \infty,
$$

and showing a unitary correspondence $f \leftrightarrow \phi_{\pm}$. The virtue of this type of comparison dynamics is its simplicity and, more importantly, that it can be generalized to manifolds without invoking micro-local analysis (for example the Fourier transform). It was used in [IS1] to obtain the asymptotic completeness on a manifold with asymptotically Euclidean ends of short-range type by a time-dependent method. In this paper we develop a time-dependent theory applicable to both asymptotically Euclidean and hyperbolic ends of long-range type, and moreover we show a relationship between the time-dependent and stationary wave operators, the latter of which been constructed in [IS3]. More precisely, let $F_{\pm}$ be the future/past stationary wave operators of [IS3] and $W_{\pm}$ be the time-dependent operators of this paper. Then we show that

$$
F_{\pm} = (W_{\pm})^*.
$$

These are relationships for general long-range models. However under additional conditions more specialized wave operators of ‘Dollard type’ are constructed by a time-dilated comparison dynamics modeled after Dollard’s construction on the Euclidean space [Do]. Deducing from (1.1) we then obtain similar results for the simplified Dollard type operators. Moreover yet similar formulas are obtained under short-range conditions. In particular the latter formulas complement the purely time-dependent theory of [IS1]. We use the Cook–Kuroda method to show the existence of the time-dependent wave operators in the most general setting. Asymptotic completeness is then a consequence of (1.1) (and its simplified versions), see [II] for a similar application of stationary scattering theory in the Euclidean context.

Another result of this paper is a resolution of a conjecture of [HPW] on cross-end transmissions on a manifold with two or more ends. We give a time-dependent formulation in a strong form, see Corollary 2.2. Again the result is a consequence of the stationary theory [IS3]. In fact we believe that stationary theory is essential not only for our proof but for any conceivable proof. This is in contrast to asymptotic completeness of time-dependent wave operators. Although the stationary theory is important for our procedure of proof, the highly ‘symmetric’ arguments of [IS1] would plausibly extend in a modified form to provide a purely time-dependent proof of asymptotic completeness. However this is not pursued by us partly because in general such procedure seems to require other conditions (and whence would not yield a strict improvement) compared to the approach given in this paper.

1.1. Setting and review. Our paper is a direct continuation of [IS2, IS3], and we start by recalling the setting and various results from there. This section exhibits only a minimal review, and we refer to [IS2, Subsection 1.2] for several examples of manifolds satisfying the abstract conditions appearing below.
1.2. Basic framework. We assume an end structure on $M$ in a somewhat disguised form.

**Condition 1.1.** Let $(M, g)$ be a connected Riemannian manifold of dimension $d \geq 1$. There exist a function $r \in C^\infty(M)$ with image $r(M) = [1, \infty)$ and constants $c > 0$ and $r_0 \geq 2$ such that:

1. The gradient vector field $\omega = \text{grad} r \in X(M)$ is forward complete in the sense that the forward integral curve $(x(t))_{t \geq 0}$ of $\omega$ is defined for any initial point $x = x(0) \in M$.

2. The bound $|dr| = |\omega| \geq c$ holds on $\{x \in M \mid r(x) > r_0/2\}$.

Under Condition 1.1 each component of the subset $E = \{x \in M \mid r(x) > r_0\}$ is called an end of $M$, and, along with Condition 1.2 below, the function $r$ may model a distance function there. We note that by Condition 1.1 (2) and the implicit function theorem the $r$-spheres

$$S_R = \{x \in M \mid r(x) = R\}; \quad R > r_0/2,$$

are submanifolds of $M$. We will later in this section introduce corresponding spherical coordinates on $E$.

Let us impose more conditions on the geometry of $E$ in terms of the radius function $r$. Choose $\chi \in C^\infty(\mathbb{R})$ such that

$$\chi(t) = \begin{cases} 1 & \text{for } t \leq 1, \\ 0 & \text{for } t \geq 2, \end{cases} \quad \chi \geq 0, \quad \chi' \leq 0, \quad \sqrt{1 - \chi} \in C^\infty, \quad (1.2)$$

and set

$$\eta = 1 - \chi(2r/r_0), \quad \tilde{\eta} = |dr|^{-2} \eta = |dr|^{-2}(1 - \chi(2r/r_0)). \quad (1.3)$$

We introduce a “radial” differential operator $A$:

$$A = \text{Re} p^r = \frac{1}{2}(p^r + (p^r)^*) \quad (1.4)$$

and the “spherical” tensor $\ell$ and the associated differential operator $L$:

$$\ell = g - \tilde{\eta} dr \otimes dr, \quad L = p_i^* \ell^{ij} p_j. \quad (1.5)$$

In the spherical coordinates, the tensor $\ell$ may be identified with the pull-back of $g$ to the $r$-spheres. We call $L$ the spherical part of $-\Delta$. If $|dr| = 1$ then $-L$ acts as the Laplace–Beltrami operator on $S_r$ (in general as a kind of perturbation of this operator). We remark that the tensor $\ell$ clearly satisfies

$$0 \leq \ell \leq g, \quad \ell^{ij}(\nabla r)_i = (1 - \eta)dr, \quad (1.6)$$

where the first bounds of (1.6) are understood as quadratic form estimates on the fibers of the tangent bundle of $M$.

Let us recall a local expression of the Levi–Civita connection $\nabla$: If we denote the Christoffel symbol by $\Gamma^k_{ij} = \frac{1}{2}g^{kl}(\partial_i g_{lj} + \partial_j g_{li} - \partial_l g_{ij})$, then for any smooth function $f$ on $M$

$$(\nabla f)_i = (\nabla_i f) = (df)_i = \partial_i f, \quad (\nabla^2 f)_{ij} = \partial_i \partial_j f - \Gamma^k_{ij} \partial_k f. \quad (1.7)$$

Note that $\nabla^2 f$ is the geometric Hessian of $f$. 

**Condition 1.2.** There exist constants $\tau, C > 0$ such that globally on $M$

\[
|\nabla|dr|^2| \leq Cr^{-1-\tau/2}, \quad |\ast\nabla|dr|^2| \leq Cr^{-1-\tau/2}, \\
|\nabla^k r| \leq C \quad \text{for } k \in \{1, 2\}, \quad |\ast\nabla|\Delta r| \leq Cr^{-1-\tau/2}. \tag{1.8a}
\]

In addition, there exists $\sigma' > 0$ such that for all $R > r_0/2$, and as quadratic forms on fibers of the tangent bundle of $S_R$,

\[
R \ast r^* \nabla^2 r \geq \frac{1}{2} \sigma'|dr|^2 \ast r^* g,
\tag{1.8b}
\]

where $r^*: S_R \rightarrow M$ is the inclusion map.

The second bound of (1.8a) is not necessary for the results of [IS2], but it is for [IS3] and this paper. We note that Condition 1.2 and the identity

\[
(\nabla^2 r)^{ij}(\nabla r)_j = \frac{1}{2}(\nabla|dr|^2)^i
\tag{1.9}
\]

was used in [IS2] to obtain the more practical version of (1.8b): For any $\sigma \in (0, \sigma')$ and $\tau$ as in Condition 1.2 there exists $C > 0$ such that globally on $M$

\[
r(\nabla^2 r - \frac{1}{2} \tilde{\eta}^2 (\nabla'|dr|^2)dr \otimes dr) \geq \frac{1}{2} \sigma |dr|^2 \ell - Cr^{-\tau} g. \tag{1.10}
\]

If $|dr| = 1$ for $r > r_0/2$ then (1.10) is fulfilled with $\sigma = \sigma'$ and any $\tau$.

Next we introduce an effective potential:

\[
q = V + \frac{1}{8} \tilde{\eta} [(\Delta r)^2 + 2\nabla^r \Delta r]. \tag{1.11}
\]

Here we remark that

\[
H = \frac{1}{2} A \tilde{\eta} A + \frac{1}{2} L + q + \frac{1}{8} (\nabla^r \tilde{\eta})(\Delta r). \tag{1.12}
\]

**Condition 1.3.** There exists a splitting by real-valued functions:

\[
q = q_1 + q_2; \quad q_1 \in C^1(M) \cap L^\infty(M), \quad q_2 \in L^\infty(M),
\]

such that for some $\rho', C > 0$ the following bounds hold globally on $M$:

\[
|\nabla^r q_1| \leq Cr^{-1-\rho'}, \quad |q_2| \leq Cr^{-1-\rho'}.
\tag{1.13}
\]

Now let us explain the self-adjoint realizations of $H$ and $H_0$. Since $(M, g)$ can be incomplete, the operators $H$ and $H_0$ are not necessarily essentially self-adjoint on $C_c^\infty(M)$. We realize $H_0$ as a self-adjoint operator by imposing the Dirichlet boundary condition, i.e. $H_0$ is the unique self-adjoint operator associated with the closure of the quadratic form

\[
\langle H_0 \rangle_\psi = \langle \psi, -\frac{1}{2} \Delta \psi \rangle, \quad \psi \in C_c^\infty(M).
\]

We denote the form closure and the self-adjoint realization by the same symbol $H_0$. Define the associated Sobolev spaces $\mathcal{H}^s$ by

\[
\mathcal{H}^s = (H_0 + 1)^{-s/2} \mathcal{H}, \quad s \in \mathbb{R}. \tag{1.14}
\]

Then $H_0$ may be understood as a closed quadratic form on $Q(H_0) = \mathcal{H}^1$. Equivalently, $H_0$ makes sense also as a bounded operator $\mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$, whose action coincides with that for distributions. By the definition of the Friedrichs extension the self-adjoint realization of $H_0$ is the restriction of such distributional $H_0$: $\mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ to the domain:

\[
\mathcal{D}(H_0) = \{ \psi \in \mathcal{H}^1 \mid H_0 \psi \in \mathcal{H} \} \subseteq \mathcal{H}.
\]
Since $V$ is bounded and self-adjoint by Conditions 1.1–1.3, we can realize the self-adjoint operator $H = H_0 + V$ simply as

$$H = H_0 + V, \quad \mathcal{D}(H) = \mathcal{D}(H_0).$$

In contrast to (1.14) we introduce the Hilbert spaces $\mathcal{H}_s$ and $\mathcal{H}_{s\pm}$ with configuration weights:

$$\mathcal{H}_s = r^{-s} \mathcal{H}, \quad \mathcal{H}_{s+} = \bigcup_{s' > s} \mathcal{H}_{s'}, \quad \mathcal{H}_{s-} = \bigcap_{s' < s} \mathcal{H}_{s'}, \quad s \in \mathbb{R}.$$ 

We consider the $r$-balls $B_R = \{r(x) < R\}$ and the characteristic functions

$$F_\nu = F(B_{R_{\nu+1}} \setminus B_{R_{\nu}}), \quad R_\nu = 2^\nu, \quad \nu \geq 0,$$

(1.15)

where $F(\Omega) = 1_\Omega$ is used for the characteristic function of a subset $\Omega \subseteq M$. Define the associated Besov spaces $B$ and $B^*$ by

$$B = \{\psi \in L^2_{\text{loc}}(M) \mid \|\psi\|_B < \infty\}, \quad \|\psi\|_B = \sum_{\nu=0}^{\infty} R_\nu^{1/2} \|F_\nu \psi\|_{\mathcal{H}},$$

$$B^* = \{\psi \in L^2_{\text{loc}}(M) \mid \|\psi\|_{B^*} < \infty\}, \quad \|\psi\|_{B^*} = \sup_{\nu \geq 0} R_\nu^{-1/2} \|F_\nu \psi\|_{\mathcal{H}},$$

(1.16)

respectively. We also define $B^*_0$ to be the closure of $C^\infty_c(M)$ in $B^*$. Recall the nesting:

$$\mathcal{H}_{1/2-} \subsetneq B \subsetneq \mathcal{H}_{1/2} \subsetneq \mathcal{H} \subsetneq \mathcal{H}_{-1/2} \subsetneq B^*_0 \subsetneq B^* \subsetneq \mathcal{H}_{-1/2-}.$$ 

Using the function $\chi \in C^\infty(\mathbb{R})$ of (1.2), define $\chi_n, \bar{\chi}_n, \chi_{m,n} \in C^\infty(M)$ for $n > m \geq 0$ by

$$\chi_n = \chi(r/R_n), \quad \bar{\chi}_n = 1 - \chi_n, \quad \chi_{m,n} = \bar{\chi}_m \chi_n.$$ 

(1.17)

Let us introduce an auxiliary space:

$$\mathcal{N} = \{\psi \in L^2_{\text{loc}}(M) \mid \chi_n \psi \in \mathcal{H}^1 \text{ for all } n \geq 0\}.$$ 

This is a space of functions that intuitively satisfy the Dirichlet boundary condition, although possibly with infinite $\mathcal{H}^1$-norm on $M$. Note that under Conditions 1.1–1.3 the manifold $M$ may be, e.g., a half-space in the Euclidean space (see [IS2, Subsection 1.2]), and there could be a ‘boundary’ even for large $r$, which in our framework appears ‘invisible’ from inside $M$ (see Remark 1.13 for some elaboration). Recall a similar interpretation of the space $\mathcal{H}^1$.

1.3. Review of results from [IS2]. Now we gather and review the main results from [IS2],

Our first theorem is Rellich’s theorem, the absence of $B^*_0$-eigenfunctions with eigenvalues above a certain “critical energy” $\lambda_0 \in \mathbb{R}$ defined by

$$\lambda_0 = \limsup_{r \to \infty} q_1 = \lim_{R \to \infty} \left(\sup_{r(x) \geq R} \{q_1(x) \mid r(x) \geq R\}\right).$$

(1.18)

For the Euclidean and the hyperbolic spaces and many other examples the critical energy $\lambda_0$ can be computed explicitly, and the essential spectrum is given by $\sigma_{\text{ess}}(H) = [\lambda_0, \infty)$. The latter is usually seen in terms of Weyl sequences, see [Ku].

**Theorem 1.4.** Suppose Conditions 1.1–1.3, and let $\lambda > \lambda_0$. If a function $\phi \in L^2_{\text{loc}}(M)$ satisfies that

(1) $(H - \lambda)\phi = 0$ in the distributional sense,
(2) $\bar{\chi}_m \phi \in \mathcal{N} \cap B^*_r$ for all $m \geq 0$ large enough, then $\phi = 0$ in $M$.

Next we discuss the limiting absorption principle and the radiation condition related to the resolvent $R(z) = (H - z)^{-1}$. We state a locally uniform bound for the resolvent as a map: $B \to B^*$. For that we need a compactness condition.

**Condition 1.5.** In addition to Conditions 1.1–1.3, there exists an open subset $\mathcal{I} \subseteq (\lambda_0, \infty)$ such that for any $n \geq 0$ and compact interval $I \subseteq \mathcal{I}$ the mapping

$$\chi_n P_H(I) : \mathcal{H} \to \mathcal{H}$$

is compact, where $P_H(I)$ denotes the spectral projection onto $I$ for $H$.

Due to Rellich’s compact embedding theorem, “boundedness” of $r$-balls provides a criterion for Condition 1.5: If each $r$-ball $B_r$, $R \geq 1$, is isometric to a bounded subset of a complete manifold, Condition 1.5 is satisfied for $\mathcal{I} = (\lambda_0, \infty)$.

We fix any $\sigma \in (0, \sigma')$ and then large enough $C > 0$ in agreement with (1.10), and introduce the positive quadratic form

$$h := \nabla^2 r - \frac{1}{2}\eta^2 (\nabla r^2) dr \otimes dr + 2Cr^{-1-\tau}g \geq \frac{1}{2}\sigma r^{-1}|dr|^2 \ell + Cr^{-1-\tau}g.$$ 

For any subset $I \subseteq \mathcal{I}$ we denote

$$I_\pm = \{ z = \lambda \pm i\Gamma \in \mathbb{C} | \lambda \in I, \Gamma \in (0,1) \},$$

respectively. We also use the notation $\langle T \rangle_\phi = \langle \phi, T \phi \rangle$.

**Theorem 1.6.** Suppose Condition 1.5 and let $I \subseteq \mathcal{I}$ be a compact interval. Then there exists $C > 0$ such that for any $\phi = R(z)\psi$ with $z \in I_\pm$ and $\psi \in B$

$$\| \phi \|_{B^*} + \| p^\tau \phi \|_{B^*} + \| p^\tau h^{1/2} \phi \|_{p}^{1/2} + \| H_0 \phi \|_{B^*} \leq C \| \psi \|_B. \quad (1.19)$$

In our theory the Besov boundedness (1.19) does not immediately imply the limiting absorption principle, and for the latter we need also radiation condition bounds implied by minor additional regularity conditions.

**Condition 1.7.** In addition to Condition 1.5 there exist splittings $q_1 = q_{11} + q_{12}$ and $q_2 = q_{21} + q_{22}$ by real-valued functions

$$q_{11} \in C^2(M) \cap L^\infty(M), \quad q_{12}, q_{21} \in C^1(M) \cap L^\infty(M), \quad q_{22} \in L^\infty(M)$$

and constants $\rho, C > 0$ such that for $k = 0, 1$

$$|\nabla^r q_{11}| \leq Cr^{-(1+\rho/2)/2}, \quad |\ell^* \nabla q_{11}| \leq Cr^{-1-\rho/2}, \quad |d\nabla^r q_{11}| \leq Cr^{-1-\rho/2},$$

$$|dq_{12}| \leq Cr^{-1-\rho/2}, \quad |(\nabla^r)^k q_{21}| \leq Cr^{-k-\rho}, \quad q_{21} \nabla q_{11} \leq Cr^{-1-\rho},$$

|q_{22}| \leq Cr^{-1-\rho/2|}.

Our radiation condition bounds are stated in terms of the distributional radial differential operator $A$ defined in (1.4) and an asymptotic complex phase $a$ given below. Pick a smooth decreasing function $r_\lambda \geq 2r_0$ of $\lambda > \lambda_0$ such that

$$\lambda + \lambda_0 - 2q_1 \geq 0 \text{ for } r \geq r_\lambda/2, \quad (1.20)$$

and that $r_\lambda = 2r_0$ for all $\lambda$ large enough. Then we set

$$\eta_\lambda = 1 - \chi(2r/r_\lambda),$$
and for \( z = \lambda \pm i\Gamma \in I \cup I_\pm \)
\[
\begin{align*}
    b &= \eta_1 |dr| \sqrt{2(z - q_1)}, \\
    a &= b \pm \frac{1}{2} \eta_1 (p^* q_1) / (z - q_1),
\end{align*}
\]
respectively, where the branch of square root is chosen such that \( \text{Re} \sqrt{w} > 0 \) for \( w \in \mathbb{C} \setminus (-\infty, 0] \). Note that for \( z \in I \) there are two values of \( a \) which could be denoted \( a_\pm \). For convenience we prefer to use the shorter notation. Note also that the phase \( a \) of (1.21b) is an approximate solution to the radial Riccati equation
\[
\pm p^* a + a^2 - 2 |dr|^2 (z - q_1) = 0
\]
in the sense that it makes the quantity on the left-hand side of (1.22) small for large \( r \geq 1 \). The quantity \( b \) of (1.21a) alone already gives an approximate solution to the same equation, however with the second term of (1.21b) a better approximation is obtained. Set
\[
\beta_c = \frac{1}{2} \min \{ \sigma, \tau, \rho \}.
\]
Here and henceforth we consider \( \sigma \in (0, \sigma') \) as a fixed parameter.

**Theorem 1.8.** Suppose Condition 1.7, and let \( I \subseteq I \) be a compact interval. Then for all \( \beta \in [0, \beta_c) \) there exists \( C > 0 \) such that for any \( \phi = R(z) \psi \) with \( \psi \in r^{-\beta} B \) and \( z \in I_\pm \)
\[
\| r^\beta (A \mp a) \phi \|_B^* + \langle p^*_i r^{2\beta} h^{ij} p_j \rangle_\phi^{1/2} \leq C \| r^\beta \psi \|_B,
\]
respectively. The limiting absorption principle reads.

**Corollary 1.9.** Suppose Condition 1.7, and let \( I \subseteq I \) be a compact interval. For any \( s > 1/2 \) and \( \epsilon \in (0, \min\{(2s - 1)/(2s + 1), \beta_c/(\beta_c + 1)\}) \) there exists \( C > 0 \) such that for \( k = 0, 1 \) and any \( z, z' \in I_\pm \) or \( z, z' \in I_\mp \)
\[
\| p^k R(z) - p^k R(z') \|_{B(H_s, H_{-s})} \leq C |z - z'|^\epsilon.
\]
In particular, the operators \( p^k R(z) \), \( k = 0, 1 \), attain uniform limits as \( I_\pm \ni z \to \lambda \in I \) in the norm topology of \( B(H_s, H_{-s}) \), say denoted
\[
p^k R(\lambda \pm i0) := \lim_{I_\pm \ni z \to \lambda} p^k R(z), \quad \lambda \in I,
\]
respectively. These limits \( p^k R(\lambda \pm i0) \in B(B, B^*) \), and \( R(\lambda \pm i0) : B \to N \cap B^* \).

Given the limiting resolvents \( R(\lambda \pm i0) \) the radiation condition bounds for real spectral parameters follow directly from Theorem 1.8.

**Corollary 1.10.** Suppose Condition 1.7, and let \( I \subseteq I \) be a compact interval. Then for all \( \beta \in [0, \beta_c) \) there exists \( C > 0 \) such that for any \( \phi = R(z) \psi \) with \( \psi \in r^{-\beta} B \) and \( \lambda \in I \)
\[
\| r^\beta (A \mp a) \phi \|_B^* + \langle p^*_i r^{2\beta} h^{ij} p_j \rangle_\phi^{1/2} \leq C \| r^\beta \psi \|_B,
\]
respectively.
For the Euclidean and the hyperbolic spaces without potential $V$ we can assume $\beta_\epsilon \geq 1 - \epsilon$ for any (small) $\epsilon > 0$ (in fact since (1.10) is fulfilled with $\sigma = \sigma'$ in these cases we can assume $\beta_\epsilon \geq 1$).

As another application of the radiation condition bounds we have characterized the limiting resolvents $R(\lambda \pm i0)$. For the Euclidean space such characterization is usually referred to as the Sommerfeld uniqueness result.

**Corollary 1.11.** Suppose Condition 1.7, and let $\lambda \in \mathcal{I}$, $\phi \in L^2_{\text{loc}}(M)$ and $\psi \in r^{-\beta}B$ with $\beta \in [0, \beta_\epsilon)$. Then $\phi = R(\lambda \pm i0)\psi$ holds if and only if both of the following conditions hold:

(i) $(H - \lambda)\phi = \psi$ in the distributional sense.

(ii) $\phi \in \mathcal{N} \cap r^{-\beta}B^*$ and $(A \mp a)\phi \in r^{-\beta}B_0^*$.

1.4. **Extended framework.** Let $r \geq r_0$, $d\mathcal{A}_r$ be the naturally induced measure on $S_r$ and

$$\mathcal{G}_r = L^2(S_r, d\mathcal{A}_r); \quad d\mathcal{A}_r = |\nu|^{-1}d\mathcal{A}_r. \quad (1.28)$$

Recall the co-area formula implying that for all integrable functions $\phi$ supported in $E$

$$\int_E \phi(x)(\det g(x))^{1/2} \, dx = \int_{r_0}^{\infty} dr \int_{S_r} \phi \, d\mathcal{A}_r, \quad (1.29)$$

in particular that for square integrable functions

$$\|1_E\phi\| = \int_{r_0}^{\infty} \|\phi|_{S_r}\|_{L_2}^2 \, dr.$$ 

We can describe the measure $d\mathcal{A}_r$ in some details using the following condition.

**Condition 1.12.** Let $(M, g)$ be the manifold, $r$ be the function and $c$ and $r_0$ be the constants of Condition 1.1. Let $M_0 = \{x \in M | r(x) > r_0/2\}$. There exists a Riemannian manifold $(M^\infty, g^\infty)$ of dimension $d$ in which the manifold $(M_0, g)$ is isometrically embedded. There exists an extension $r^\infty \in C^\infty(M^\infty)$ of the restriction $r|_{M_0}$ such that the extended vector field $\omega^\infty := \text{grad } r^\infty$ is complete in $M^\infty$ and $|\omega^\infty| \geq c$ on $\{x \in M^\infty | r^\infty(x) > r_0/2\}$. Let $\tilde{\omega}^\infty = \tilde{\gamma}^\infty \omega^\infty$ be the complete vector field defined with $\tilde{\gamma}^\infty = |\omega^\infty|^{-2}(1 - \chi(2r^\infty/r_0))$, and let $\tilde{\gamma}^\infty(t, \cdot) = \exp(t\tilde{\omega}^\infty)$ denote the corresponding flow. Then

$$\forall \sigma \in S : \{\tilde{\gamma}^\infty(t, \sigma) | t \geq 0\} \cap M \neq \emptyset,$$

where $S = S^\infty_{r_0} = \{x \in M^\infty | r^\infty(x) = r_0\}$.

**Remark 1.13.** The reader might prefer to think about $(M^\infty, g^\infty)$ and $r^\infty$ as given from the outset. Then $M_0 \subseteq M^\infty$ would be a subset invariant under the forward flow of the vector field $\omega^\infty$. However since almost all of our conditions are needed for $M$ only (actually (1.34) is the only quantitative exception) we have pursued the given presentation. For many examples, cf. [IS2, Subsection 1.2], the vector field $\omega$ of Condition 1.1 is forward as well as backward complete (i.e. complete) and we can take $(M^\infty, g^\infty, r^\infty) = (M, g, r)$. The typical origin for non-backward completeness for a sub-manifold $M \subseteq M'$, $M$ open in $M'$, is ‘crossing’ of integral curves of $\omega$ at the boundary $\partial M \subseteq M'$.

We note

$$\forall \sigma \in S \forall t \geq 0 : \quad r^\infty(\tilde{\gamma}^\infty(t, \sigma)) = r_0 + t, \quad (1.30)$$
and that any \( x \in E^{\text{ex}} := \{ x \in M^{\text{ex}} \mid r^{\text{ex}}(x) > r_0 \} \) has spherical coordinates defined as

\[
(r, \sigma) = (r^{\text{ex}}(x), \tilde{y}^{\text{ex}}(r_0 - r^{\text{ex}}(x), x)) \in (r_0, \infty) \times S.
\]

In particular any \( x \in E \) has spherical coordinates defined this way. Mimicking the constructions (1.28) we introduce

\[
\mathcal{G} = L^2(S, d\tilde{A}), \quad d\tilde{A} = d\tilde{A}^{\text{ex}} = |\omega^{\text{ex}}|^{-1}d\tilde{A}^{\text{ex}},
\]

in terms of the naturally induced measure \( d\tilde{A}^{\text{ex}} \). Now, indeed in spherical coordinates

\[
d\tilde{A}_r = \exp \left( \int_{r_0}^r (\text{div} \tilde{\omega}^{\text{ex}})(s, \sigma) \, ds \right) d\tilde{A} \quad \text{for} \quad x = \tilde{y}^{\text{ex}}(r - r_0, \sigma) \in S_r.
\]

This leads to the isometrical embedding \( \mathcal{G}_r \subseteq \mathcal{G}, \ r \geq r_0, \) given by mapping \( \mathcal{G}_r \ni \xi_r \to \xi^{\text{ex}} \in \mathcal{G} \) where

\[
\xi^{\text{ex}}(\sigma) = \begin{cases} 
\exp \left( \int_{r_0}^r \frac{1}{2}(\text{div} \tilde{\omega}^{\text{ex}})(s, \sigma) \, ds \right) \xi_r(x) & \text{for} \ x = \tilde{y}^{\text{ex}}(r - r_0, \sigma) \in S_r, \\
0 & \text{otherwise}
\end{cases}
\]

(1.32)

The formula (1.32) can be understood in terms of (a group of) translations on the extended Hilbert space \( \mathcal{H}^{\text{ex}} = L^2(M^{\text{ex}}, g^{\text{ex}}) \). We introduce the normalized extended radial translation \( \tilde{T}^{\text{ex}}(\tau) : \mathcal{H}^{\text{ex}} \to \mathcal{H}^{\text{ex}}, \ \tau \in \mathbb{R}, \) in terms of the self-adjoint operator

\[
\tilde{A} = A^{\text{ex}} = \text{Re} \left( -i\nabla_{\tilde{\omega}^{\text{ex}}} \right)
\]

by \( \tilde{T}^{\text{ex}}(\tau) = e^{i\tau \tilde{A}}. \) Then (1.32) is naturally rewritten as \( \xi^{\text{ex}} = e^{i(r-r_0)\tilde{A}}\xi_r \) since for \( \psi \in \mathcal{H}^{\text{ex}} \) and \( x \in M^{\text{ex}} \)

\[
(\tilde{T}^{\text{ex}}(\tau)\psi)(x) = \exp \left( \int_0^\tau \frac{1}{2}(\text{div} \tilde{\omega}^{\text{ex}})(\tilde{y}^{\text{ex}}(t, x)) \, dt \right) \psi(\tilde{y}^{\text{ex}}(\tau, x)).
\]

We also note that the relation \( x = \tilde{y}^{\text{ex}}(r - r_0, \sigma) \) of (1.32) naturally defines an embedding \( S_r \subseteq S \) given as the map \( S_r \ni x \to \sigma \in S \). We shall sometimes slightly abuse notation and write \( \sigma \in S_r \), leaving it to the reader to decide from the context whether \( \sigma \) should be thought of as a point in the subset \( S_r \) of \( M \) or rather as a point in the image of this map.

1.5. Review of results from [IS3]. We need additional assumptions. The following one suffices for constructing the distorted Fourier transform.

**Condition 1.14.** Along with Condition 1.12, Condition 1.7 holds with

\[
2\beta_\varepsilon = \min\{\sigma, \tau, \rho\} > 1.
\]

(1.33)

In addition, the functions \( \tilde{b} = \tilde{b}(\lambda, x) \) and \( q_1(x) \) have real \( C^1 \)-extensions to \( I \times M^{\text{ex}} \) and \( M^{\text{ex}} \), respectively, say denoted by \( \tilde{b}^{\text{ex}} \) and \( q_1^{\text{ex}} \) (or for short by \( \tilde{b} \) and \( q_1 \) again), and the following bound holds uniformly in the spherical coordinates on \( E \) and locally uniformly in \( \lambda \in I \):

\[
\sup_{r_0 \leq r \leq r_1} \left| \ell^* \nabla_f \int_{2\beta_\varepsilon}^r \tilde{b}^{\text{ex}}(s, \sigma) \, ds \right| \leq C r^{-1/2}.
\]

(1.34)
Theorem 1.15. Suppose Condition 1.14. Then for any \( \psi \in H_{1+} \) and \( r \geq r_0 \) we introduce a function \( \xi(r) \in G \) using the mapping (1.32), omitting here (and often henceforth) the superscript ‘ex’:

\[
\xi(r)(\sigma) = \exp \left( \int_{r_0}^{r} \left( \mp i \dot{b} + \frac{1}{2} \text{div} \dot{\omega} \right)(s, \sigma) \, ds \right) [\sqrt{b}R(\lambda \pm i0)\psi](r, \sigma),
\]

(1.35)

(and = 0 for \( \sigma \notin S_r \)) or, alternatively,

\[
\xi(r) = e^{i(r-r_0)(\lambda \pm i0)} [\sqrt{b}R(\lambda \pm i0)\psi]|_{S_r}, = e^{i(r-r_0)(\lambda \pm i0)} [\sqrt{b}R(\lambda \pm i0)\psi]|_{S_r}.
\]

The \( G \)-valued function \( \xi \) has a limit for \( r \to \infty \) allowing us to define the “distorted Fourier transform” by

\[
F^\pm(\lambda)\psi = G^-\lim_{r \to \infty} \xi(r); \quad \psi \in H_{1+}.
\]

(1.37)

Theorem 1.15. Suppose Condition 1.14. Then for any \( \psi \in H_{1+} \) there exist the limits (1.37). The maps \( I \ni \lambda \mapsto F^\pm(\lambda)\psi \in G \) are continuous. Moreover, putting \( \delta(H - \lambda) = \pi^{-1} \text{Im} R(\lambda + i0) \),

\[
\|F^\pm(\lambda)\psi\|^2 = 2\pi \langle \psi, \delta(H - \lambda)\psi \rangle.
\]

(1.38)

By definition the function \( F^\pm(\lambda)\psi \in G = L^2(S, d\lambda) \), and we note that our construction of \( F^\pm(\lambda)\psi \) is non-canonical primarily due to the freedom in choosing \( G \). In fact for \( M^{ex} = M \) the only non-canonical feature comes from the dependence of \( r_0 \) (determining \( G \) in that case), while in general there is an additional freedom in choosing extended functions.

Due to (1.38) the operators \( F^\pm(\lambda) \) extend as continuous operators \( B \to G \), and for any \( \psi \in B \) the maps \( F^\pm(\cdot)\psi \in G \) are continuous. In Proposition 1.16 stated below we give a formula for these extensions.

Introduce

\[
\mathcal{H}_I = P_H(I)\mathcal{H}, \quad \mathcal{H}_I = L^2(I, (2\pi)^{-1} \, d\lambda; G), \quad (1.39)
\]

set \( H_I = HP_H(I) \) and let \( M_\lambda \) be the operator of multiplication by \( \lambda \) on \( \mathcal{H}_I \). We define

\[
F^\pm = \int_I \oplus F^\pm(\lambda) \, d\lambda : B \to C(I; G).
\]

These operators can be extended to proper spaces which is stated as the first part of the following result.

Proposition 1.16. Suppose Condition 1.14. The operators \( F^\pm \) considered as maps \( B \cap \mathcal{H}_I \to \mathcal{H}_I \) extend uniquely to isometries \( \mathcal{H}_I \to \mathcal{H}_I \). These extensions obey \( F^\pm \mathcal{H}_I \subseteq M_\lambda F^\pm \). Moreover for any \( \psi \in B \) the vectors \( F^\pm(\lambda)\psi \) are given as averaged limits. More precisely introducing for any such \( \psi \) the integral \( \int_R^{2R} \xi(r) \, dr := R^{-1} \int_R^{2R} \xi(r) \, dr \), these vectors are given as

\[
F^\pm(\lambda)\psi = G^-\lim_{R \to \infty} \int_R^{2R} \xi(r) \, dr
\]

(1.40)
and the limits (1.40) are attained locally uniformly in $\lambda \in \mathcal{I}$.

The above extended isometries $F^\pm: \mathcal{H}_\mathcal{I} \to \tilde{\mathcal{H}}_\mathcal{I}$ are actually unitary under an additional condition, and for this reason we call them the Fourier transforms associated with $H_\mathcal{I}$. This condition consists of two alternatives. The first one is a partial strengthening of Condition 1.14. The other one is primarily a set of bounds on higher order derivatives of various quantities defined on $M$.

For simplicity for any smooth function $f$ on $M$ let us set

$$
\nabla' f = \nabla f - (\nabla f)\nabla r, \quad \nabla'^2 f = \nabla^2 f - (\nabla f)\nabla^2 r.
$$

(1.41)

In $E$ the spherical parts of $\nabla' f$ and $\nabla'^2 f$, i.e. $\ell^* (\nabla' f)$ and $\ell^* (\nabla'^2 f)$, coincide with first and second order derivatives computed by the Levi–Civita connections on the $r$-spheres $S_r$ associated with the induced Riemannian metrics $g_r := \iota^*_r g$.

**Condition 1.17.** In addition to Condition 1.14 with the extension $\tilde{b}^{\text{ex}}$ being $C^2$ one of the following properties holds:

1. $\min\{\sigma, \tau, \rho\} > 2.$
2. The restriction $q_{1|E}$ belongs to $C^2(E)$, and there exists $C > 0$ such that

$$
|\ell^* \ell^* \ell^k (\nabla^3 r)_{ij}| \leq C r^{-1-\tau/2},
$$

(1.43a)

$$
|\ell^* \ell^* (\nabla^2 q_1)_{ij}| \leq C r^{-1-\rho},
$$

(1.43b)

and

$$
|\ell^* \ell^j (\nabla^2 |dr|^2)_{ij}| \leq C r^{-1-\tau}, \quad |\ell^* \ell^j (\nabla^2 \nabla' |dr|^2)_{ij}| \leq C r^{-1-\tau},
$$

(1.43c)

$$
|\ell^* \ell^j (\nabla^2 \Delta r)_{ij}| \leq C r^{-1-\tau}.
$$

(1.43d)

We also remark that the additional smoothness condition on $\tilde{b}^{\text{ex}}$ in the case of (1) is needed only in the proof of Lemma 3.7 (actually it is only the smoothness in $\lambda$ that is used there).

**Theorem 1.18.** Suppose Condition 1.17. Then the operators $F^\pm: \mathcal{H}_\mathcal{I} \to \tilde{\mathcal{H}}_\mathcal{I}$ are unitarily diagonalizing transforms for $H_\mathcal{I}$, that is, they are unitary and

$$
F^\pm H_\mathcal{I} = M_\lambda F^\pm,
$$

respectively.

Under Condition 1.17 and for any $\lambda \in \mathcal{I}$ the scattering matrix $S(\lambda): \mathcal{G} \to \mathcal{G}$ is defined by the identity

$$
F^+(\lambda) \psi = S(\lambda) F^-(\lambda) \psi; \quad \psi \in B.
$$

(1.44)

It follows from [IS3] that $C^\infty_c(S) \subseteq \text{Ran} F^\pm(\lambda)$, and hence, with Theorem 1.15, Proposition 1.16 and a density argument, $S(\cdot)$ is a well-defined strongly continuous unitary operator. We state below a characterization of the generalized eigenfunctions in $\mathcal{N} \cap B^*$, i.e. the elements of

$$
\mathcal{E}_\lambda := \{ \phi \in \mathcal{N} \cap B^* \mid (H - \lambda) \phi = 0 \}.
$$
Due to Theorem 1.4 these eigenfunctions may be called *minimum*. We introduce for any $\xi \in \mathcal{G}$ purely outgoing/incoming approximate generalized eigenfunctions $\phi^\pm[\xi] = \phi^\pm_\mp[\xi] \in B^*$ in terms of the spherical coordinates by

$$\phi^\pm[\xi](r, \sigma) = \eta[2^2\int |(\lambda - q_1)|^{-1/4}\exp\left(\int_{r_0}^r \left(\pm ib - \frac{1}{2}\div \omega\right)(s, \sigma) ds\right)\xi(\sigma). (1.45)$$

Of course these quantities are well-defined independently of all the estimates of Conditions 1.14 and 1.17. We remark that formulas like (1.45) in the context of Schrödinger operators are referred to as (zeroth order) WKB-approximations.

**Theorem 1.19.** Suppose Condition 1.17. Then for any $\lambda \in \mathcal{I}$ the following assertions hold.

(i) For any one of $\xi_\pm \in \mathcal{G}$ or $\phi \in \mathcal{E}_\lambda$ the two other quantities in $\{\xi_-, \xi_+, \phi\}$ uniquely exist such that

$$\phi - \phi^+[\xi_+] + \phi^-[\xi_-] \in B^*_0. (1.46a)$$

(ii) The correspondences in (1.46a) are given by the formulas (recall (1.36))

$$\phi = iF^\pm(\lambda)^*\xi_\pm, \quad \xi_\pm = S(\lambda)\xi_\mp, (1.46b)$$

$$\xi_\pm = 2^{-1} G \lim_{R \to \infty} \int_R e^{i(r-r_0)(\Delta^+ \mp b^\pm)} [b^{-1/2}(A \pm b)^2]_{\mathcal{S}_0} dr. (1.46c)$$

In particular the wave matrices $F^\pm(\lambda)^* : \mathcal{G} \to \mathcal{E}_\lambda$ are linear isomorphisms.

(iii) The wave matrices $F^\pm(\lambda)^* : \mathcal{G} \to \mathcal{E}_\lambda (\subseteq B^*)$ are bi-continuous. In fact

$$2\|\xi_\pm\|^2_\mathcal{G} = \lim_{R \to \infty} R^{-1} \int_{B^*_0 \setminus B_R} \|b^{1/2}\phi\|^2 (\det g)^{1/2} dx. (1.46d)$$

(iv) The operators $F^\pm(\lambda) : B \to \mathcal{G}$ and $\delta(H - \lambda) : B \to \mathcal{E}_\lambda$ are onto.

Finally we give an application of our results to channel scattering theory addressed, but treated very differently, in [HPW]. Suppose $M^{ex}$ has $N \geq 2$ number of ends, i.e. $E^{ex} = \{x \in M^{ex} | r^{ex}(x) > r_0\}$ has $N \geq 2$ components $E_i$, $i = 1, \ldots, N$. (Note that this implies that $E_i \cap M$, $i = 1, \ldots, N$, are the components of $E = E^{ex} \cap M$.) Then the Hilbert space $\mathcal{G}$ splits as

$$\mathcal{G} = \mathcal{G}_1 \oplus \cdots \oplus \mathcal{G}_N; \quad \mathcal{G}_i = L^2(S_i), \quad S_i = S \cap \overline{E_i},$$

and, accordingly, the scattering matrix $S(\lambda)$ has a matrix representation

$$S(\lambda) = (S_{ij}(\lambda))_{1 \leq i, j \leq N}, \quad S_{ij}(\lambda) \in \mathcal{B}(\mathcal{G}_j, \mathcal{G}_i).$$

**Corollary 1.20.** Suppose Condition 1.17, and that $E^{ex}$ has $N$ number of ends. For any $\lambda \in \mathcal{I}$ let the scattering matrix $S(\lambda)$ be decomposed into components as above. Then the off-diagonal components, $S_{ij}(\lambda)$ with $i \neq j$, are one-to-one mappings.

We note that Corollary 1.20 may be seen as a stationary solution to conjectures of [HPW], see [HPW, Remark 5.7]. We shall develop the time-dependent version of this result, more directly addressing conjectures of [HPW].

2. MAIN RESULTS

We present our results deferring the main parts of the proofs to Section 3.
2.1. General results for long-range models. According to the arguments of Subsection 1.5 we introduce the following free comparison dynamics defined on $\tilde{\mathcal{H}}_I$ in (1.39), employing the purely outgoing/incoming approximate generalized eigenfunctions (1.45): For any $t \geq 0$ and $h \in C^1_c(I \times S) \subseteq \tilde{\mathcal{H}}_I$ we let

$$U^\pm(t)h = (\pm 2\pi i)^{-1} \int_I e^{\mp i\lambda \phi^\pm_h(\lambda, \cdot)} d\lambda.$$  \hspace{1cm} (2.1)

In Lemma 3.2 we will see that $U^\pm(t)h \in \mathcal{H}$ for any $t \geq 0$ and $h \in C^1_c(I \times S)$. A consequence of Lemma 3.3 is that these dynamics are asymptotically isometric, i.e. $\lim_{t \to \infty} \|U^\pm(t)h\|_{\mathcal{H}} = \|h\|_{\tilde{\mathcal{H}}_I}$. We also note the time reversal invariance property $U^+(t)\bar{h} = U^-(t)h$.

**Theorem 2.1.** Suppose Condition 1.17. Then for any $h \in C^2_c(I \times S)$ there exist the limits

$$W^\pm h := \lim_{t \to \infty} e^{\pm itH}U^\pm(t)h \text{ in } \mathcal{H}.\hspace{1cm} (2.2)$$

These limits $W^\pm$ extend uniquely to unitary operators $W^\pm: \tilde{\mathcal{H}}_I \to \mathcal{H}_I$, and

$$F^\pm = (W^\pm)^*,\hspace{1cm} (2.3)$$

respectively. Whence the wave operators $W^\pm$ are complete on $\mathcal{I}$.

In the proof of Theorem 2.1 we will use a more simplified free dynamics, which comes about as the leading term of (2.1) extracted by the stationary phase argument, see Lemma 3.3. We also note that for so-called short-range and Dollard type models there are further simplified dynamics for which the analogues of Theorem 2.1 hold. This will be discussed later in this section.

Now let us apply Theorem 2.1 to one aspect of channel scattering theory. In the $N$-ends setting as in Corollary 1.20 let us consider the orthogonal projections

$$P^\pm_i = W^\pm_1 \mathbb{1}_{I \times S_i}(W^\pm)^*; \hspace{0.5cm} i = 1, \ldots, N.\hspace{1cm} (2.4)$$

Clearly these projections non-trivially resolve the space $\mathcal{H}_I$:

$$P^\pm_i \neq 0 \text{ for all } i, \hspace{0.5cm} P^\pm_i P^\pm_j = 0 \text{ for all } i \neq j, \hspace{0.5cm} P^\pm_1 + \cdots + P^\pm_N = I \text{ on } \mathcal{H}_I.$$ 

It is not difficult to see by the stationary phase method (or by using (3.4c), (3.10) and (3.13) directly) that these projections have dynamical representations

$$P^\pm_i = \lim_{t \to \infty} e^{\pm itH}1_{E_i \cap M} e^{\mp itH} P_H(I),$$

and hence we can interpret that $P^\pm_i$ describe the initial/final state components of wave packets that go to/come from the $i$-th end $E_i \cap M$, respectively. Then the following corollary says that any wave packet coming from one end, say the $j$-th end, always enters all the others.

**Corollary 2.2.** Suppose the conditions of Corollary 1.20, and let $P^\pm_i, i = 1, \ldots, N$, be defined by (2.4). For any $i, j \in \{1, \ldots, N\}$ with $i \neq j$ and for any nonzero $\psi \in P^-_j \mathcal{H}_I$ one has $P^+_i \psi \neq 0$. 
Proof. Let \( i \neq j \), and suppose \( 0 \neq \psi \in P_j^{-} \mathcal{H}_I \). By Theorem 2.1 we have
\[
P_i^+ \psi = P_i^+ P_j^\dagger \psi \\
= W^+ 1_{\Sigma_r} F^+ W^{-} 1_{\Sigma_r} (W^-)^* \psi \\
= W^+ 1_{\Sigma_r} \psi (W^-)^* \psi \\
= W^+ S_{ij} (W^-)^* \psi \\
= W^+ S_{ij} (W^-)^* \psi ,
\]
from which we can deduce that \( P_i^+ \psi \neq 0 \) thanks to Corollary 1.20.
\[\square\]

In the above discussion and proof we identified ‘dynamical transmission’ and ‘spectral transmission’. In the literature a similar identification has been shown for reflection for 1-dim Schrödinger operators, see \[LPPZ\] for a recent contribution.

2.2. Short-range and Dollard classes of perturbations. The conditions in Section 1.1 are satisfied for the free Euclidean and hyperbolic spaces, and hence also for their perturbations to some extent. In fact perturbations of long-range types are allowed in Section 1.1, and the results so far are the most general ones holding for such long-range models. However in certain more restrictive settings it is possible to show versions of Theorem 2.1 with considerably simpler comparison dynamics and phase modifiers.

Here let us precisely formulate the notions of short-range and Dollard types of the effective potential \( q \) assuming Condition 1.14. We will discuss the corresponding simplifications in the following subsections.

We introduce the following quantity using spherical coordinates
\[
\lambda_0(\sigma) = \limsup_{r \to \infty} q_1(r, \sigma); \quad \sigma \in S.
\]
Note that \( \lambda_0(\sigma) \leq \lambda_0 \) for all \( \sigma \in S \).

**Definition 2.3.** The effective potential \( q = q_1 + q_2 \) of Condition 1.3 is said to be of short-range type, if there exist \( \epsilon, C > 0 \) such that uniformly in the spherical coordinates on \( E \)
\[
|q_1(r, \sigma) - \lambda_0(\sigma)| \leq C r^{-1-\epsilon}. \tag{2.5a}
\]

Note the following consequence of \( q \) being of short-range type: Locally uniformly in \( \lambda \in \mathcal{I} \)
\[
\lim_{R \to \infty} \sup_{\sigma \in S_R} \int_{R}^{\infty} |\tilde{b}_{st} - \tilde{b}|(s, \sigma) \, ds = 0; \tag{2.5b}
\]
here
\[
\tilde{b}_{st}(r, \sigma) := |dr(r, \sigma)|^{-1} \sqrt{2(\lambda - \lambda_0(\sigma))} \quad \text{for } r > r_0/2.
\]

Upon replacing \( dr \) by \( d\text{r}^{\text{ex}} \) we shall use this definition on \( E^{\text{ex}} \) as well.

The effective potential \( q \) not necessarily of short-range type is said to be of long-range type. The manifold \((M, g)\) is said to be of short-range or long-range type if \( q \) is of short-range or long-range type when \( V = 0 \), respectively.

**Definition 2.4.** The effective potential \( q = q_1 + q_2 \) of Condition 1.3 is said to be of Dollard type, if there exist \( \epsilon, C > 0 \) such that uniformly in the spherical coordinates on \( E \)
\[
|q_1(r, \sigma) - \lambda_0(\sigma)| \leq C r^{-(1+\epsilon)/2}. \tag{2.6a}
\]
Note the following consequence of \( q \) being of Dollard type: Locally uniformly in \( \lambda \in I \)
\[
\lim_{R \to \infty} \sup_{\sigma \in S_R} \int_R^\infty |\tilde{b}_{do} - \tilde{b}|(s, \sigma) \, ds = 0, \tag{2.6b}
\]
where
\[
\tilde{b}_{do}(r, \sigma) = \tilde{b}_{sr}(r, \sigma) \left( 1 - \frac{1}{2} \frac{q_1(r, \sigma) - \lambda_{0}(\sigma)}{\lambda - \lambda_{0}(\sigma)} \right) \quad \text{for } r > r_0.
\]
Upon replacing \( q_1 \) by \( q^e_1 \) (the latter function introduced in Condition 1.14) we shall use this definition on \( E^e \) as well.

Clearly a \( q \) of short-range type is also of Dollard type. The Dollard type is a particular long-range type for which certain approximations work. We will elaborate on this in Corollary 2.10 and Theorem 2.11. In the Schrödinger operator literature this is usually accredited Dollard, [Do].

We remark that, if \( M^e = M \), obviously (2.5b) and (2.6b) follow from (2.5a) and (2.6a), respectively.

2.3. Simplification for short-range models. Here we discuss simplifications of the main result, Theorem 2.1, for a short-range model. At the end of the subsection we also discuss a comparison with the setting of [IS1].

2.3.1. Simplification. Assume that the effective potential \( q \) is of short-range type. For simplicity we also assume
\[
|d r^e| = 1 \quad \text{for } r > r_0/2. \tag{2.7}
\]
Then we set
\[
b_{sr}(\sigma) = \tilde{b}_{sr}(\sigma) = \sqrt{2(\lambda - \lambda_{0}(\sigma))},
\]
and for any \( \xi \in \mathcal{G} \)
\[
\phi^e_{sr}[\xi](r, \sigma) = \eta(r)b_{sr}(\sigma)^{-1/2}\exp\left( \pm ib_{sr}(\sigma)(r - r_0) - \frac{1}{2} \int_{r_0}^r \Delta r(s, \sigma) \, ds \right)\xi(\sigma), \tag{2.8}
\]
\( \text{cf. (1.21a) and (1.45). Note that } b_{sr} \text{ is the zeroth order approximation of } b \text{ as } r \to \infty. \)

We first present a simplified version of the distorted Fourier transforms.

**Corollary 2.5.** In addition to Condition 1.14, suppose that \( q \) is of short-range type, and that (2.7) holds.

1. For any \( \lambda \in \mathcal{I} \) and \( \psi \in B \) there exist \( F^e_{sr}(\lambda)\psi \in \mathcal{G} \) such that
\[
R(\lambda \pm i0)\psi - \phi^e_{sr}[F^e_{sr}(\lambda)\psi] \in B^e_0.
\]
In addition, \( F^e_{sr}(\lambda)\psi \) are separately continuous in \( \lambda \in \mathcal{I} \) and \( \psi \in \mathcal{G} \).

2. The operators
\[
F_{sr}^e = \int_{\mathcal{I}} \oplus F^e_{sr}(\lambda) \, d\lambda
\]
considered as mappings \( B \cap \mathcal{H}_{\mathcal{I}} \to \tilde{\mathcal{H}}_{\mathcal{I}} \) extend uniquely to isometric operators \( \mathcal{H}_{\mathcal{I}} \to \tilde{\mathcal{H}}_{\mathcal{I}} \).

3. The above \( F_{sr}^e \) are related to \( F^e \) as follows:
\[
F_{sr}^e = e^{\mp i\theta} F^e; \quad \theta(\lambda, \sigma) = \int_{r_0}^\infty (b_{sr} - b)(s, \sigma) \, ds.
\]
Proof. The assertions are immediate consequences of the assumptions and our previous results, that is more precisely, those reviewed in Section 1.1 and [IS3, Lemma 3.13].

A simplified free dynamics for a short-range model may be constructed just by replacing $\phi^\pm$ by $\phi_{sr}^\pm$ in (2.1), but here let us go one step forward. According to the stationary phase argument, cf. Lemma 3.3, we should for $h \in \mathcal{H}_I$ and $t > 0$ study

$$U_{sr}^\pm(t)h(r, \sigma) = (2\pi)^{-1/2} e^{\mp 3i\pi/4} 1_{[r_0, \infty)}(r) e^{\pm i K_{sr} t} e^{-\int_0^t (\Delta r)(s, \sigma)/2 \, ds} \cdot \left( \frac{r-r_0}{t} \right)^{1/2} h \left( \frac{(r-r_0)^2}{2t^2} + \lambda_0(\sigma), \sigma \right),$$

where

$$K_{sr} = \frac{(r-r_0)^2}{2(t-t_0)}.$$  

Note that $U_{sr}^\pm(t)$ is a contraction, i.e. $\|U_{sr}^\pm(t)h\|_{\mathcal{H}} \leq \|h\|_{\mathcal{H}_I}$, and that the dynamics $U_{sr}^\pm(\cdot)$ are asymptotically isometric, i.e. $\lim_{t \to \infty} \|U_{sr}^\pm(t)h\|_{\mathcal{H}} = \|h\|_{\mathcal{H}_I}$. In any case Theorem 2.1 simplifies as follows.

**Theorem 2.6.** In addition to Condition 1.17, suppose that $q$ is of short-range type, and that (2.7) holds. For all $h \in \mathcal{H}_I$ there exist the limits

$$W_{sr}^\pm h := \lim_{t \to \infty} e^{\mp i t H} U_{sr}^\pm(t)h \quad \text{in } \mathcal{H}.$$  

These limits $W_{sr}^\pm$ are unitary operators $W_{sr}^\pm : \mathcal{H}_I \to \mathcal{H}_I$, and

$$F_{sr}^\pm = (W_{sr}^\pm)^*,$$

respectively. Whence the wave operators $W_{sr}^\pm$ are complete on $I$.

We note that the standard hyperbolic space fits into this framework with a positive constant $\lambda_0(\sigma) \equiv \lambda_0 > 0$, and the Euclidean space does so with the zero constant $\lambda_0(\sigma) \equiv 0$.

2.3.2. **Comparison with [IS1].** Next we furthermore assume $\lambda_0(\sigma) \equiv 0$ and compare Theorem 2.6 with the main result of [IS1] on existence and completeness of wave operators on asymptotically Euclidean manifolds.

We begin with a comparison of the settings. For reference let us quote below the conditions of [IS1] in a form suitable for comparison.

**Condition 2.7.** Let $(M, g)$ be a connected and complete Riemannian manifold, and let $V \in L^\infty(M)$ be real-valued. There exist a function $r \in C^\infty(M)$ with image $r(M) = [1, \infty)$ and constants $\delta, \kappa, \eta, C > 0$ and $r_0 \geq 2$ such that:

1. The $r$-balls $\{ x \in M \mid r(x) < R \}$, $R > 0$, are relatively compact in $M$.
2. The following relations hold for $r(x) \geq r_0/2$:

$$|dr| = 1, \quad R^* \nabla^2 r \geq \frac{1}{2}(1 + \delta) R^* g.$$

3. The following estimates hold globally on $M$ for $\alpha = 0, 1$:

$$r \geq 1, \quad |\nabla^\alpha \Delta r| \leq C r^{-1/2-\alpha-\kappa}, \quad |V| \leq C r^{-1-\eta}.$$  

(2.12)

Condition 2.7 above would seem quite different from the setting of [IS1], but obviously Condition 2.7 follows from Conditions 1.1–1.4 of [IS1]. On the other hand, Condition 2.7 constitutes what is used in the proofs of [IS1], and consequently the
results of [IS1] remain valid under Condition 2.7. Hence we may regard Condition 2.7 exactly as the setting of [IS1].

The setting of this paper may be considered slightly more restrictive than that of [IS1]. In fact, we have the following lemma (with obvious proof).

**Lemma 2.8.** Suppose Condition 2.7 with a splitting \( V = V_1 + V_2 \) such that
\[
|V_1| \leq Cr^{-1-n}, \quad |∇^r V_1| \leq Cr^{-2-n}, \quad |V_2| \leq Cr^{-3/2-n}.
\]
Then Condition 1.14 is satisfied for
\[
q_{11} = 0, \quad q_{12} = 0, \quad q_{21} = V_1 + \frac{1}{8}η(Δr)^2, \quad q_{22} = V_2 + \frac{1}{4}η∇^r Δr.
\]
In addition, if (1.43a) and (1.43d) hold, then Condition 1.17 is also satisfied.

If \( λ_0(σ) ≡ 0 \) the simplified comparison dynamics \( U_{sr}^±(t) \) defined by (2.9) coincides with the comparison dynamics of [IS1]. This observation will allow us to employ the existence of the wave operator \( Ω_+ \) from [IS1] to deduce the assertions of Theorem 2.6 without imposing the stronger Condition 1.17. In particular this shows a stationary representation of \( Ω_+ \). Whence we conclude the following alternative version of Theorem 2.6.

**Theorem 2.9.** Suppose Condition 2.7 and the splitting assumption of Lemma 2.8. For all \( h ∈ \tilde{H}_I \) there exist the limits
\[
W_{sr}^± h := \lim_{t→∞} e^{±iH}U_{sr}^±(t)h \quad \text{in } H.
\]
These limits \( W_{sr}^± \) are unitary operators \( W_{sr}^± : \tilde{H}_I → H_I \), and
\[
F_{sr}^± = (W_{sr}^±)^*.
\]
Since Condition 1.17 is missing, \( F_{sr}^± : H_I → \tilde{H}_I \) can be constructed only as isometries by the arguments of [IS2, IS3]. However, we can deduce unitarity of \( F_{sr}^± \) from (2.14) without Condition 1.17.

### 2.4. Simplification for Dollard type models

Similarly to the short-range case we can simplify our main result Theorem 2.1 for the Dollard case too. Here we assume that \( q \) is of Dollard type and also that (2.7) holds. Then we set for \( r > r_0 \)
\[
b_{do}(r, σ) = \hat{b}_{do}(r, σ) = \sqrt{2(λ - λ_0(σ)) + [λ_0(σ) - q_1(r, σ)]} / \sqrt{2(λ - λ_0(σ))},
\]
and for \( ξ ∈ G \)
\[
φ_{do}^±[ξ](r, σ) = η_λ(r)b_{do}(σ)^{-1/2}\exp\left(\int_{r_0}^{r} (±ib_{do} - \frac{1}{4}Δr)(s, σ) ds\right)ξ(σ),
\]
Note that \( b_{do} \) is the first order approximation of \( b \) as \( r → ∞ \). By mimicking the proof of Corollary 2.5 we obtain the following result.

**Corollary 2.10.** In addition to Condition 1.14, suppose that \( q \) is of Dollard type, and that (2.7) holds.

1. For any \( λ ∈ I \) and \( ψ ∈ B \) there exist \( F_{do}^±(λ)ψ ∈ G \) such that
\[
R(λ ± i0)ψ - φ_{do}^±[F_{do}^±(λ)ψ] ∈ B_0^*.
\]
In addition, \( F_{do}^±(λ)ψ \) are separately continuous in \( λ ∈ I \) and \( ψ ∈ G \).
(2) The operators

\[ F^\pm_{\text{do}} = \int_{\mathcal{I}} \oplus F^\pm_{\text{do}}(\lambda) \, d\lambda \]

considered as mappings \( B \cap \mathcal{H}_I \to \tilde{\mathcal{H}}_I \) extend uniquely to isometric operators \( \mathcal{H}_I \to \tilde{\mathcal{H}}_I \).

(3) The above \( F^\pm_{\text{do}} \) are related to \( F^\pm \) as follows:

\[ F^\pm_{\text{do}} = e^{\mp i\Theta} F^\pm; \quad \Theta(\lambda, \sigma) = \int_{r_0}^{\infty} \left( b_{\text{do}}(s, \sigma) \right) \, ds. \]

Next we set for \( h \in \tilde{\mathcal{H}}_I \)

\[ U^\pm_{\text{do}}(t)h(r, \sigma) = (2\pi)^{-1/2} e^{\mp 3\pi i/4} \left( \frac{t}{t(r_0)} \right) \frac{1}{2} h \left( \frac{(r-r_0)^2}{2t^2} + \lambda_0(\sigma) \right), \]

where

\[ K_{\text{do}} = \frac{(r-r_0)^2}{2t} - t\lambda_0(\sigma) - \frac{t}{(r-r_0)} \int_{r_0}^{r} \left( q_1(s, \cdot) - \lambda_0(\sigma) \right) \, ds. \]

As for the short-range type dynamics the Dollard type dynamics are asymptotically isometric families of contractions.

**Theorem 2.11.** Suppose Condition 1.17, \( q \) is of Dollard type, and that (2.7) holds. For all \( h \in \tilde{\mathcal{H}}_I \) there exist the limits

\[ W^\pm_{\text{do}} := \lim_{t \to \infty} e^{\pm itH} U^\pm_{\text{do}}(t)h \quad \text{in} \ \mathcal{H}. \]

These limits \( W^\pm_{\text{do}} \) are unitary operators \( W^\pm_{\text{do}} : \tilde{\mathcal{H}}_I \to \mathcal{H}_I \), and

\[ F^\pm_{\text{do}} = (W^\pm_{\text{do}})^*, \]

respectively. Whence the wave operators \( W^\pm_{\text{do}} \) are complete on \( \mathcal{I} \).

Similarly to the previous subsection, the standard hyperbolic space and the Euclidean space fit also into this framework with a positive constant \( \lambda_0(\sigma) \equiv \lambda_0 > 0 \) and the zero constant \( \lambda_0(\sigma) \equiv 0 \), respectively.

### 3. Proofs

**3.1. Leading asymptotics of comparison dynamics.** Here we study basic properties of the comparison dynamics \( U^\pm(t) \) defined by (2.1). The main result of this subsection is Lemma 3.3, which extracts the leading asymptotics of \( U^\pm(t)h \) as \( t \to \infty \). Actually, the integrand of (2.1) has an oscillatory factor with phase

\[ \Theta_\pm = \pm \Theta; \quad \Theta(\lambda, t, r, \sigma) = \int_{r_0}^{r} \tilde{b}_\lambda(s, \sigma) \, ds - t\lambda, \]

and we are going to employ the one-dimensional stationary phase argument; see [II] for a somewhat related analysis in higher dimensions. Throughout this subsection we assume Condition 1.14, and all proofs are given only for the upper sign, since this is sufficient due to time reversal invariance.
For technical reasons it turns out to be appropriate to use the stationary phase method with a modified phase depending on $h$. We will choose a constant $r_1 \geq r_0$ depending on $\text{supp } h$ and consider a stationary point of the function

$$
\Theta_1(\lambda, t, r, \sigma) = \int_{r_1}^{r} \tilde{b}_\lambda(s, \sigma) \, ds - t\lambda,
$$

(3.2)

instead of the function $\Theta$ of (3.1). We study such stationary point in the following lemma in which $h$ at most enters in a disguised form: The parameters $\lambda_1 > \lambda_0$ and $D \subseteq S$ in the lemma will later be chosen by the requirement $\text{supp } h \subseteq (\lambda_1, \infty) \times D$ after having fixed $h \in C^1_c(\mathcal{I} \times S)$.

**Lemma 3.1.** Let $\lambda_1 > \lambda_0$ and $D \subseteq S$ be a relatively compact open subset. Fix $r_1 \geq r_0$ such that

$$
[r_1, \infty) \times \overline{D} \subseteq E, \quad r_1 \geq r_{\lambda_1} = \sup_{\lambda \geq \lambda_1} r_\lambda,
$$

(3.3a)

where $r_\lambda$ is defined in agreement with (1.20). With $\Theta_1$ given by (3.2) in terms of this $r_1$ we set

$$
\Omega_c = \{(t, r, \sigma) \in (0, \infty) \times (r_1, \infty) \times D \mid \partial_\lambda \Theta_1(\lambda_1, t, r, \sigma) > 0\}.
$$

(3.3b)

For any $(t, r, \sigma) \in \Omega_c$ there exists a unique solution $\lambda_c > \lambda_1$ to the equation

$$
(\partial_\lambda \Theta_1)(\lambda_c, t, r, \sigma) = 0.
$$

(3.4a)

This solution $\lambda_c = \lambda_c(t, r, \sigma)$ is $C^1$ (more generally $C^k$ if $q_1$ is $C^k$) and satisfies

$$
\partial_t \lambda_c + b_\lambda \partial_r \lambda_c = 0, \quad \partial_\sigma \lambda_c > 0.
$$

(3.4b)

In addition, there exist constants $c, C > 0$ such that for any $(t, r, \sigma) \in \Omega_c$

$$
c(r - r_1)^2/t^2 \leq \lambda_c(t, r, \sigma) - \lambda_0 \leq C(r - r_1)^2/t^2.
$$

(3.4c)

Moreover, letting

$$
K_1(t, r, \sigma) = \Theta_1(\lambda_c(t, r, \sigma), t, r, \sigma) \quad \text{for } (t, r, \sigma) \in \Omega_c,
$$

(3.5a)

the following identities hold:

$$
\partial_t K_1 = -\lambda_c, \quad \partial_r K_1 = \tilde{b}_{\lambda_c}.
$$

(3.5b)

In particular $K_1$ is a solution to the Hamilton–Jacobi equation

$$
\partial_t K_1 + \frac{1}{2}(|\partial_r K_1|^2 + q_1) = 0.
$$

(3.5c)

**Proof.** We first note that thanks to (3.3a) we have the following expressions when $\lambda \geq \lambda_1$ and $(r, \sigma) \in [r_1, \infty) \times \overline{D}$:

$$
\Theta_1(\lambda, t, r, \sigma) = \int_{r_1}^{r} |dr|^{-1}[2(\lambda - q_1)]^{1/2}(s, \sigma) \, ds - t\lambda,
$$

(3.6a)

$$
\partial_\lambda \Theta_1(\lambda, t, r, \sigma) = \int_{r_1}^{r} |dr|^{-1}[2(\lambda - q_1)]^{-1/2}(s, \sigma) \, ds - t,
$$

(3.6b)

$$
\partial_\lambda^2 \Theta_1(\lambda, t, r, \sigma) = -\int_{r_1}^{r} |dr|^{-1}[2(\lambda - q_1)]^{-3/2} \, ds.
$$

(3.6c)

For any fixed $(t, r, \sigma) \in \Omega_c$ the quantity (3.6b) is positive for $\lambda = \lambda_1$, monotonically decreasing in $\lambda > \lambda_1$, and takes negative values for large $\lambda > \lambda_1$. Hence there exists a unique solution $\lambda_c = \lambda_c(t, r, \sigma)$ to (3.4a), and by the implicit function theorem it is $C^1$. 
Noting the expression (3.6b), we can differentiate (3.4a), and obtain formulas

\begin{align*}
\partial_t \lambda_c &= -\left(\int_{r_1}^r |dr|^{-1}[2(\lambda_c - q_1)]^{-3/2} \, ds\right)^{-1}, \\
\partial_r \lambda_c &= |dr|^{-1}[2(\lambda_c - q_1)]^{-1/2} \left(\int_{r_1}^r |dr|^{-1}[2(\lambda_c - q_1)]^{-3/2} \, ds\right)^{-1},
\end{align*}

(3.7)

which verifies (3.4b). The bounds in (3.4c) are verified easily by inserting \( \lambda = \lambda_c \) in (3.6b) (taken equal to zero) and estimating the integral using (1.20). (In fact by this argument the upper bound of (3.4c) holds with \( C = 1 \).)

We can verify the formulas in (3.5b) by differentiating the definition (3.5a) and using (3.4a) and (3.6a). Obviously (3.5c) is a consequence of (3.5b). \( \square \)

We can now show a basic property of \( U^\pm(t)h \) defined by (2.1).

**Lemma 3.2.** For each \( t \geq 0 \) and \( h \in C^1_c(\mathcal{I} \times S) \) the functions \( U^\pm(t)h \) belong to \( \mathcal{H} \). Moreover \( U^\pm(\cdot)h \) is a continuous \( \mathcal{H} \)-valued function.

**Proof.** Let \( h \in C^1_c(\mathcal{I} \times S) \) and \( T > 0 \) be given. It suffices to show that the function 
\( [0, T] \ni t \to U^+(t)h \in \mathcal{H} \) is well-defined and continuous.

We pick a number \( \lambda_1 > \lambda_0 \) and a relatively compact open subset \( D \subseteq S \) such that
\[
supp h \subseteq (\lambda_1, \infty) \times D.
\]
(3.8)

We fix \( r_1 \geq r_0 \) satisfying (3.3a) in agreement with Lemma 3.1 with input given by the above \( \lambda_1 \) and \( D \).

Let us write for short, using (3.2),
\[
U^+(t)h(r, \sigma) = \int_{\mathcal{I}} e^{i\Theta r(t, r, \sigma)} \Phi(\lambda, r, \sigma) \, d\lambda,
\]
(3.9)

where
\[
\Phi(\lambda, r, \sigma) = (2\pi i)^{-1} \eta_{\lambda}(r)[2|dr|^2(\lambda - q_1)]^{-1/4} e^{-\int_0^r \text{div} \tilde{\omega}(s, \sigma)/2 \, ds} \left(e^{i(\Theta - \Theta_1)}h\right)(\lambda, \sigma).
\]

Note that indeed \( \Theta - \Theta_1 \) is independent of \((t, r)\), and we omit these variables:
\[
(\Theta - \Theta_1)(\lambda, \sigma) = \int_{r_0}^{r_1} \tilde{b}_{\lambda}(s, \sigma) \, ds.
\]

We also note that \( \Phi(\lambda, \cdot) \) belongs to \( B^s \) uniformly in \( \lambda \in \mathcal{I} \). With an extra decay factor \( r^{-\delta}, \delta > 1/2 \), obviously we obtain a vector in \( \mathcal{H} \). In particular \( \chi_n U^+(t)h \) for any \( n \), and this leads us to consider \( \bar{\chi}_n U^+(t)h \) only. We choose and fix \( n \) so large that \( R_n = 2^n > r_1 \) and that for some \( c, C > 0 \) the bounds
\[
\partial_t \Theta_1 \geq c(|\lambda| + 1)^{-1/2} r, \quad |\partial^2_\lambda \Theta_1| \leq C r
\]
hold for \( \lambda \geq \lambda_1, t \leq T, r \geq R_n \) and \( \sigma \in \overline{D} \). These bounds are immediate from the formulas (3.6b) and (3.6c). We insert \( e^{i\Theta_1} = (i\partial_\lambda \Theta_1)^{-1} \partial_\lambda e^{i\Theta_1} \) in (3.9) and do a single integration by parts, which is legitimate since \( e^{i(\Theta - \Theta_1)}h \in C^1 \). The bounds provide an extra decay factor \( r^{-1} \) for \( \chi_n U^+(t)h \). Hence we have shown that \( U^+(t)h \in \mathcal{H} \) for \( t \leq T \).

The continuity statement follows from the resulting representation of \( U^\pm(t)h \) after doing the integration by parts; we omit the details. \( \square \)
Now letting \( h \in C^1_c(I \times S) \) be given we aim at extracting the leading term of \( U^\pm(t)h \) as \( t \to \infty \). As in the proof of Lemma 3.2 we can take a number \( \lambda_1 > \lambda_0 \) and a relatively compact open subset \( D \subseteq S \) such that (3.8) is fulfilled. Again we fix \( r_1 \geq r_0 \) satisfying (3.3a) in agreement with Lemma 3.1 with input given by the above \( \lambda_1 \) and \( D \). Let \( \lambda_c = \lambda_c(t, r, \sigma) \) be the solution to (3.4a). Then we set

\[
\Omega_c(t) = \{(r, \sigma) \mid (t, r, \sigma) \in \Omega_c \}; t > 0,
\]

and

\[
U_0^\pm(t)h(r, \sigma) = (2\pi)^{-1/2}e^{\mp3i\pi/4}1_{\Omega_c(t)}(r, \sigma)e^{\mp iK(t, \sigma, \sigma)}e^{-\int_{r_0}^r(\text{div} \hat{\omega})(s, \sigma)/2 \, ds} \cdot (\partial_r \lambda_c(t, r, \sigma))^{1/2}h(\lambda_c(t, r, \sigma), \sigma),
\]

where

\[
K(t, r, \sigma) = \Theta(\lambda_c(t, r, \sigma), t, r, \sigma).
\]

The factor \( 1_{\Omega_c(t)} \) is essentially redundant, since the support of the factor \( h(\lambda_c(t, \cdot), \cdot) \) is contained in \( \Omega_c(t) \) which in turn is an easy consequence of (3.4c). In fact it is not difficult to show using (3.4c) and (3.7) that \( U_0^\pm(t)h \in C^1_t(M) \) for any \( t > 0 \). We also note that the right-hand side of (3.10) depends on a choice of parameters \( \lambda_1 \), \( D \) and \( r_1 \) which possibly have a non-linear dependence of \( h \). Hence the operator-like notation \( U_0^+(t) \) is somewhat abuse of notation, however we prefer to use it for simplicity.

**Lemma 3.3.** Under the above assumptions \( U_0^+(\cdot)h \) are continuously differentiable \( \mathcal{H} \)-valued functions in \( t > 0 \), and satisfy that for all \( t > 0 \)

\[
\|U_0^\pm(t)h\|_{\mathcal{H}} = \|h\|_{\mathcal{H}}, \tag{3.11}
\]

and

\[
\frac{d}{dt}U_0^\pm(t)h = -iG^\pm(t)U_0^\pm(t)h; \quad G^\pm(t) = \text{Re}\left(\hat{b}_{\lambda_c} A\right) \mp \frac{1}{2}[\partial_r^2 + q_1], \tag{3.12}
\]

respectively. Moreover,

\[
U^\pm(t)h = U_0^\pm(t)h + O_\mathcal{H}(t^{-1/8}) \quad \text{as } t \to \infty. \tag{3.13}
\]

**Proof.**\textit{ Step I.} Since \( \partial_r \partial_\sigma \Theta_1(\lambda_1, t, r, \sigma) = b_{\lambda_1}^{-1} > 0 \) on \( \Omega_c \) we see that \( \lambda_c \) as a function of \( r \) only is defined on a half-axis, say \( (r_c(t, \sigma), \infty) \). In fact we know from (3.7) that \( \lambda_c(t, \cdot, \sigma) \) is increasing. Thanks to (3.4c) this function tends to \( \infty \) as \( r \to \infty \). The left end point \( r = r_c(t, \sigma) \) is the biggest solution to the equation

\[
\partial_r \Theta_1(\lambda_1, t, r, \sigma) = 0,
\]

and it is easy to see that

\[
\lambda_c(t, r, \sigma) \to \lambda_1 \text{ for } r \searrow r_c(t, \sigma).
\]

Due to these remarks the norm identity (3.11) follows by first writing the square of the left-hand side as an integral in the spherical coordinates \( (r, \sigma) \) and then changing to the variables \( (\lambda, \sigma) = (\lambda_c(t, r, \sigma), \sigma) \). Obviously this also verifies that \( U_0^+(\cdot)h \) is \( \mathcal{H} \)-valued.

**Step II.** Next, we show (3.12), which in particular implies the continuous differentiability of \( U_0^+(\cdot)h \). To compute the derivative as (3.12) let us write (3.10) as

\[
U_0^+(\cdot)h = (2\pi)^{-1/2}e^{-3i\pi/4}1_{\Omega_c(t)}e^{iK_1}e^{-\int_{r_0}^r(\text{div} \hat{\omega})(s, \sigma)/2 \, ds} \partial_r \lambda_c(1/2)(e^{i(\Theta - \Theta_1)}h)(\lambda_c, \sigma). \tag{3.14}
\]
By (3.5b) and (3.5c) we have

\[ \partial_t e^{iK_1} = -i\lambda_c e^{iK_1} \]
\[ = -i\left( \frac{1}{2} |dr|^2 \hat{b}_{\lambda_c}^2 + q_1 \right) e^{iK_1} \]
\[ = -i\left( \hat{b}_{\lambda_c} p^r - \frac{1}{2} |dr|^2 \hat{b}_{\lambda_c}^2 + q_1 \right) e^{iK_1}. \quad (3.15) \]

Next, using (3.4b), we can compute

\[ \partial_t (\partial_r \lambda_c)^{1/2} = \frac{1}{2}(\partial_r \partial_t \lambda_c)(\partial_r \lambda_c)^{-1/2} \]
\[ = -\frac{1}{2}(\partial_r |dr|^2 \hat{b}_{\lambda_c} \partial_r \lambda_c)(\partial_r \lambda_c)^{-1/2} \]
\[ = -|dr|^2 \hat{b}_{\lambda_c} \partial_r (\partial_r \lambda_c)^{1/2} - \frac{1}{2}(\partial_r |dr|^2 \hat{b}_{\lambda_c}^2) (\partial_r \lambda_c)^{1/2} \]
\[ = -i\hat{b}_{\lambda_c} p^r (\partial_r \lambda_c)^{1/2} - \frac{1}{2} \left[ \text{div}(|dr|^2 \hat{b}_{\lambda_c} \hat{\omega}) - |dr|^2 \hat{b}_{\lambda_c} (\text{div} \hat{\omega}) \right] (\partial_r \lambda_c)^{1/2} \]
\[ = -i\hat{b}_{\lambda_c} p^r (\partial_r \lambda_c)^{1/2} - i\frac{1}{2} \left[ (p^r)^* \hat{b}_{\lambda_c} \right] (\partial_r \lambda_c)^{1/2} + \frac{1}{2} |dr|^2 \hat{b}_{\lambda_c} (\text{div} \hat{\omega}) (\partial_r \lambda_c)^{1/2}, \]

so that

\[ \partial_t e^{-\int_0^r (\text{div} \hat{\omega})/2} d\psi (\partial_r \lambda_c)^{1/2} = -i\left[ \hat{b}_{\lambda_c} p^r + \frac{1}{2} \left( (p^r)^* \hat{b}_{\lambda_c} \right) \right] (\partial_r \lambda_c)^{1/2} e^{-\int_0^r (\text{div} \hat{\omega})/2} d\psi. \quad (3.16) \]

Finally by (3.4b) again we have

\[ \partial_t (e^{i(\Theta - \Theta_1)} h)(\lambda_c, \sigma) = (\partial_r \lambda_c)(\partial_r e^{i(\Theta - \Theta_1)} h)(\lambda_c, \sigma) \]
\[ = -i\left( \hat{b}_{\lambda_c} \partial_r \lambda_c \right)(\partial_r e^{i(\Theta - \Theta_1)} h)(\lambda_c, \sigma) \]
\[ = -i\hat{b}_{\lambda_c} p^r (e^{i(\Theta - \Theta_1)} h)(\lambda_c, \sigma). \quad (3.17) \]

Using (3.15), (3.16), (3.17) and the product rule we obtain

\[ G^+ = \text{Re}(\hat{b}_{\lambda_c} p^r) - \frac{1}{2} |dr|^2 \hat{b}_{\lambda_c}^2 + q_1, \]

and hence (3.12) follows.

**Step III.** It remains to show (3.13). We are going to apply the stationary phase method, and we start by doing some cut-offs. Using again (3.9) we note (as before) that \( \Phi(\lambda, \cdot) \) belongs to \( B^* \) uniformly in \( \lambda \in \mathcal{I} \) and that with an extra decay factor \( r^{-1/2-\epsilon}, \epsilon > 0 \), we obtain a vector in \( \mathcal{H} \). For any \( M > m > 0 \) we can split the integral as

\[ U^+(t)h = 1_{(m,M)}(r/t)1_{\Omega_c(t)}(\cdot) \int_{\mathcal{I}} e^{i\Theta_1(\lambda,t)} \Phi(\lambda, \cdot) d\lambda \]
\[ + 1_{(0,m]}(r/t)1_{(r_1,\infty)}(r) \int_{\mathcal{I}} e^{i\Theta_1(\lambda,t)} \Phi(\lambda, \cdot) d\lambda \]
\[ + 1_{(0,m]}(r/t)1_{(r_0,r_1)}(r) \int_{\mathcal{I}} e^{i\Theta_1(\lambda,t)} \Phi(\lambda, \cdot) d\lambda \]
\[ + 1_{(M,\infty)}(r/t) \int_{\mathcal{I}} e^{i\Theta_1(\lambda,t)} \Phi(\lambda, \cdot) d\lambda \]
\[ + 1_{(m,M)}(r/t)1_{\Omega_c(t)}(\cdot) \int_{\mathcal{I}} e^{i\Theta_1(\lambda,t)} \Phi(\lambda, \cdot) d\lambda. \quad (3.18) \]
Using the expression (3.6b) we can pick $m > 0$ small enough such that for some $c_1 > 0$ the following bound holds for all large $t$ uniformly in $r > r_1$, $r/t \leq m$ and $(\lambda, \sigma) \in \text{supp}\ h$:

$$\partial_\lambda \Theta_1(\lambda, t, r, \sigma) \leq -c_1 t. \tag{3.19}$$

With (3.19) and (3.6c) we can treat the second term on the right-hand side of (3.18) by an integration parts, yielding that

$$\int_{I} e^{i \Theta_1(\lambda, t, \cdot)} \Phi(\lambda, \cdot) d\lambda \leq C_1 t^{-1/2}.$$

The third term is treated similarly by using the phase function $-\lambda t$ instead of $\Theta_1$, yielding the same bound.

Similarly by taking $M > 0$ large enough (this part is very similar to the proof of Lemma 3.2), we can bound the fourth term of (3.18) as

$$\int_{I} e^{i \Theta_1(\lambda, t, \cdot)} \Phi(\lambda, \cdot) d\lambda \leq C_2 t^{-1/2}.$$

Next, let us consider the fifth term. By (3.3b), (3.6c) and (3.8) it follows that for large $t > 0$ and on $\text{supp}\ (1_{(m, M)}(r/t)1_{\Omega_{c}(t)^c}(\cdot)\Phi(\cdot))$

$$\partial_\lambda \Theta_1(\lambda, t, r, \sigma) \leq \partial_\lambda \Theta_1(\lambda, t, r, \sigma) - \partial_\lambda \Theta_1(\lambda_1, t, r, \sigma)$$

$$\leq -c_2 (r - r_1)$$

$$\leq -c_3 t.$$

Whence by an integration by parts we obtain the bound $O(t^{-1/2})$ again:

$$\int_{I} e^{i \Theta_1(\lambda, t, \cdot)} \Phi(\lambda, \cdot) d\lambda \leq C_3 t^{-1/2}.$$

**Step IV.** It remains to compare the first term of (3.18) with (3.14). Let us look at the phase in (3.18) in more detail. By a Taylor expansion we have, assuming the front factor $1_{(m, M)}(r/t)1_{\Omega_{c}(t)}(r, \sigma) = 1$,

$$\Theta_1(\lambda, t, x) = K_1(t, x) + \frac{1}{2} (\partial^2_\lambda \Theta_1)(\lambda_c, x)(\lambda - \lambda_c)^2 + \Xi(\lambda, \lambda_c, x)(\lambda - \lambda_c)^3$$

with

$$\Xi(\lambda, \lambda_c, x) = \frac{1}{2} \int_0^1 (1 - \kappa)^2 (\partial^2_\lambda \Theta_1)(\lambda_c + \kappa(\lambda - \lambda_c), x) d\kappa.$$

Here, since $\partial^k_\lambda \Theta_1$ is independent of $t$ for $k = 2, 3, \ldots$, we have omitted the $t$ variable for short. Then we further decompose the first term of (3.18) (assuming still
where recall that \( \Phi(\lambda, H) \) hence \( t \) this yields an extra decay factor 2nd term of (3.20) we use an integration by parts, noting 

\[
\int_I e^{i\Theta_1(\lambda, t, \cdot)} \Phi(\lambda, \cdot) \, d\lambda = e^{iK_1(t, \cdot)} \Phi(\lambda_c, \cdot) \int_{\mathbb{R}} e^{i(\partial^2_\lambda \Theta_1)(\lambda, c)}(\lambda - \lambda_c)^2/2 \, d\lambda 
- e^{iK_1(t, \cdot)} \Phi(\lambda_c, \cdot) \int_{\mathbb{R} \setminus I_c} e^{i(\partial^2_\lambda \Theta_1)(\lambda, c)}(\lambda - \lambda_c)^2/2 \, d\lambda 
+ \int_{I_c} e^{i\Theta_1(\lambda, \cdot)} \Phi(\lambda, \cdot) \, d\lambda 
+ e^{iK_1(t, \cdot)} \int_{I_c} e^{i(\partial^2_\lambda \Theta_1)(\lambda, c)}(\lambda - \lambda_c)^2/2 \left[e^{i\Xi(\lambda, c)}(\lambda - \lambda_c)^3 \Phi(\lambda, \cdot) - \Phi(\lambda_c, \cdot)\right] \, d\lambda
\]

(3.20)

where \( I_c = I_c(t, x) = [\lambda_c(t, x) - t^{-\delta}, \lambda_c(t, x) + t^{-\delta}], \delta \in (0, 1/2) \). Since by 3.6c and 3.7 

\[
(\partial^2_\lambda \Theta_1)(\lambda, c, \cdot) = - \int_{I_c} |dr|^{-1} [2(\lambda_c - q_1)]^{-3/2}(s, \sigma) \, ds = -(b_{\lambda_c} \partial_{\lambda_c})^{-1},
\]

e we can do the Gaussian integration in the first term of (3.20) (see for example [Hö, Theorem 7.6.1]) and obtain using (3.4c) that 1\(_{(m,M)}\)(\(r/t\)) \(1_{\Omega_c(t)}(r, \sigma) = 1\) as 

\[
1_{(m,M)}(r/t)1_{\Omega_c(t)}(r, \sigma) = 1
\]

Hence we are left with bounding the three other terms of (3.20). To bound the second term of (3.20) we use an integration by parts, noting

\[
\left| (\partial^2_\lambda \Theta_1)(\lambda_c, \cdot)(\lambda - \lambda_c) \right| \geq c_{\delta, 1}rt^{-\delta}
\]

This yields an extra decay factor \( t^{-1+\delta} \) along with the cut-off \( 1_{(m,M)}(r/t) \). We also recall that \( \Phi(\lambda, \cdot) \) belongs to \( B^* \) uniformly in \( \lambda \in \mathcal{I} \), so that with \( 1_{(m,M)}(r/t) \)

\[
\|1_{(m,M)}(r/t)1_{\Omega_c(t)}(\cdot)e^{iK_1(t, \cdot)}\Phi(\lambda_c, \cdot)\|_{\mathcal{H}} \leq C_2t^{1/2}.
\]

Hence

\[
\|1_{(m,M)}(r/t)1_{\Omega_c(t)}(\cdot)J_1(t, \cdot)|_{\mathcal{H}} \leq C_{\delta, 1}t^{-1/2+\delta}.
\]

(3.21)

To bound the third term of (3.20) let us implement yet another integration by parts:

\[
J_3(t, \cdot) = \left[(i\partial_\lambda \Theta_1)^{-1}e^{i\Theta_1}\Phi\right]|_{\lambda=\lambda_c-t^{-\delta}} - \left[(i\partial_\lambda \Theta_1)^{-1}e^{i\Theta_1}\Phi\right]|_{\lambda=\lambda_c+t^{-\delta}} 
- \int_{\mathbb{R} \setminus I_c} e^{i\Theta_1(\lambda, \cdot)} \partial_{\lambda_c}[(i\partial_\lambda \Theta_1)^{-1}\Phi(\lambda, \cdot)] \, d\lambda.
\]

(3.22)

Here, including the cut-off \( 1_{(m,M)}(r/t)1_{\Omega_c(t)}(\cdot) \) and taking note of the integration region \( \mathcal{I} \setminus I_c \), we have

\[
|\partial_\lambda \Theta_1| \geq c_{\delta, 2}rt^{-\delta} \geq c_{\delta, 3}t^{1-\delta},
\]

so that the contributions from the boundary terms in (3.22) are estimated as

\[
\|1_{(m,M)}(r/t)1_{\Omega_c(t)}(\cdot)[i\partial_\lambda \Theta_1)^{-1}e^{i\Theta_1}\Phi]|_{\lambda=\lambda_c+t^{-\delta}}\|_{\mathcal{H}} \leq C_{\delta, 2}t^{-1/2+\delta}.
\]
Moreover, by the product rule, a part of the integral in (3.22) estimated as
\[ \left\| 1_{(m,M)}(r/t)1_{\Omega_{\ell}(t)}(\cdot) \int_{\Gamma \setminus L_{\epsilon}} e^{i\Theta(\lambda, \cdot)} (i\partial_{\lambda} \Theta_1)^{-1} \partial_{\lambda} \Phi(\lambda, \cdot) \, d\lambda \right\|_{L^2} \leq C\delta_{\delta, \ell} t^{-1/2+\delta}. \]
The other contribution in (3.22) comes from differentiating the factor \((i\partial_{\lambda} \Theta_1)^{-1}\), and we can then use
\[ \left\| \partial_{\lambda}(\partial_{\lambda} \Theta_1)^{-1} \right\| \leq C\delta_{\delta, \ell} t^{-1+2\delta}. \]
Since \(\delta < 1/2\) the right-hand side is decaying. The bound leads to the estimate \(O_{\mathcal{H}}(t^{-1/2+\delta})\), however by repeated integrations by parts, we can estimate this contribution in (3.22) as \(O_{\mathcal{H}}(t^{-1/2+\delta})\). To sum up we obtain
\[ \left\| 1_{(m,M)}(r/t)1_{\Omega_{\ell}(t)}(\cdot) J_{\delta}(t, \cdot) \right\|_{L^2} \leq C\delta_{\delta, \ell} t^{-1/2+\delta}. \] (3.23)
Finally for the fourth term of (3.20) we write
\[ e^{i\Theta(\lambda, \cdot)} (\lambda - \lambda_{\epsilon})^2/2 = \frac{d}{d\lambda} \int_{\lambda_{\epsilon}}^{\lambda} e^{i\Theta(\lambda, \cdot)} (\lambda' - \lambda_{\epsilon})^2/2 \, d\lambda', \]
and perform one integration by parts. Using the van der Corput Lemma, cf. [St, p. 332], we then obtain
\[ \left\| 1_{(m,M)}(r/t)1_{\Omega_{\ell}(t)}(\cdot) J_{4}(t, \cdot) \right\|_{L^2} \leq C\delta_{\delta, \ell} t^{1-3\delta}. \] (3.24)
The bounds (3.21), (3.23) and (3.24) are optimized by taking \(\delta = 3/8\). Hence we obtain the desired asymptotics (3.13). \(\square\)

3.2. Separation of radial and angular variables. In this subsection we quote results from [IS3] concerning properties of the tensor \(\ell\). The statements are given in a self-contained manner; we refer to [IS3, Section 2] for proofs.

It is clear from (1.29) that we naturally have the identification
\[ L^2(E) \cong L^2([r_0, \infty)_r; G), \quad \langle \psi, \phi \rangle_{L^2(E)} = \int_{r_0}^{\infty} \langle \psi, \phi \rangle_{G} \, dr. \]
Such a decomposition holds also for the Riemannian metric, and hence for the Laplace–Beltrami operator.

Lemma 3.4. Suppose Condition 1.1. Then in the spherical coordinates \((r, \sigma) = (r, \sigma^2, \ldots, \sigma^d)\) in \(E\) one has
\[ g = |dr|^{-2} \, dr \otimes dr + g_{\alpha\beta} \, d\sigma^{\alpha} \otimes d\sigma^{\beta}, \]
where the Greek indices run over \(2, \ldots, d\). In particular, by the definition (1.5), the tensor \(\ell\) coincides with the spherical part of \(g\):
\[ \ell = g_{\alpha\beta} \, d\sigma^{\alpha} \otimes d\sigma^{\beta} \quad \text{on } E, \]
and the operator \(L\) can be identified with a direct sum:
\[ L \cong \int_{r_0}^{\infty} \oplus L_r \, dr \quad \text{as quadratic forms on } C_c^\infty(E), \] (3.25)
where \(L_r\) is the Laplace–Beltrami operator on \(S_r\) with respect to the induced metric \(g_r := \iota_r^* g\) and the (non-Riemannian) density \(d\mathcal{A}_r\), i.e.,
\[ L_r = p_r^* g_r^{\alpha\beta} p_r; \quad p_r^* = |dr|(\det g_r)^{-1/2} p_r |dr|^{-1}(\det g_r)^{1/2}. \] (3.26)
Denote the spherical part of the derivative of \( p \) by \( p' \), or \( p' = -i\nabla' \), cf. (1.41). The operator \( p' \) is well-defined on \( C^1(S_r) \) as well as on \( C^1(M) \), and we do not distinguish them. It is clear from (3.26) that for any \( \xi, \zeta \in C_c^\infty(S_r) \)

\[
\langle \zeta, L_r \xi \rangle_{G_r} = \int_{S_r} g_r^{ij}(p_i \zeta)(p_j \xi) \, d\tilde{A}_r = \langle p' \xi, p' \eta \rangle_{G_r}.
\]

We can at this point use local coordinates of \( S \) to define and implement the integration, since in any case clearly the radially derivative \( \partial_r \) does not enter. Hence in what follows we may consider \( L_r \) as a self-adjoint operator on \( G_r \) defined by the Friedrichs extension of (3.26) from \( C_c^\infty(S_r) \subseteq G_r \). Then by an approximation argument it follows that for any \( \phi \in H^1 \) the restriction \( \phi|_{S_r} \in D(L_r^{1/2}) = D(p') \) for almost every \( r \geq r_0 \). In fact, we have for all \( r \geq r_0 \)

\[
\int_{r_0}^r \| p' \phi|_{S_r} \|^2_{G_r} \, ds = \int_{B_r \setminus B_{r_0}} \ell^{ij}(p_i \phi)(p_j \phi)(\det g)^{1/2} \, dx \leq \| \phi \|^2_{H^1}.
\]

The following formula will be useful when we compute and estimate the second spherical derivative \( L \).

**Lemma 3.5.** For any \( f \in C^2(M) \), if one abbreviates \( \tilde{y} = \tilde{y}(t, \cdot) \), then

\[
L[f(\tilde{y})] = -\ell^{ij}(\partial_i \tilde{y}^\alpha)(\partial_j \tilde{y}^\beta)(\nabla^2 f)_{\alpha \beta}(\tilde{y}) + (L\tilde{y})^\alpha(\partial_{\alpha} f)(\tilde{y}),
\]

where \( \nabla^2 f \) is defined in (1.41), and

\[
L\tilde{y}^\alpha = -\ell^{ij}\left[ \partial_i \partial_j \tilde{y}^\alpha - \Gamma^k_{ij} \partial_k \tilde{y}^\alpha + \Gamma^\alpha_{k \beta} \partial_k \tilde{y}^\beta(\partial_j \tilde{y}^\gamma)(\nabla^2 r)_{\beta \gamma} \right]
\]

\[
- \tilde{\eta} \ell^{ij}(\nabla r)^{\alpha \beta}(\partial_i \tilde{y}^\gamma)(\partial_j \tilde{y}^\gamma)(\nabla^2 r)_{\beta \gamma}
\]

\[
+ \left[ (\nabla' \tilde{y})(\nabla r)^{\alpha \beta} + \tilde{\eta}(\Delta r)(\nabla r)^{\alpha \beta} + \frac{1}{2} \tilde{\eta}(\nabla |dr|^2)^{\alpha \beta} \right] \partial_j \tilde{y}^\alpha.
\]

Here the Roman and the Greek indices are those concerning \( x \) and \( \tilde{y} = \tilde{y}(t, x) \), respectively, differently from those in Lemma 3.4. In addition, in the spherical coordinates in \( E \) the first term on the right-hand side of (3.27) does not contain an \( r \)-derivative of \( f \), and neither does the second term.

Lemma 3.5 motivates us to estimate \( \partial_j \tilde{y}^\alpha \) and \( L\tilde{y} \). We can estimate the former quantity through the push-forward \( \ell^*_s(t, x) \) of \( \ell(x) \) under the map \( \tilde{y}(t, \cdot) \), defined by

\[
\ell^*_s(t, x) = \left( \ell^{ij}(x)[\partial_i \tilde{y}^\alpha(t, x)][\partial_j \tilde{y}^\beta(t, x)] \right)_{\alpha \beta}.
\]

On the other hand, introducing a “backwards hitting time” for \( x \in E \) by

\[
r_{\text{bht}}(x) = \sup\{ s \leq r(x) - r_0 \mid \tilde{y}(-s, x) \in M \},
\]

for any \( x \in E \) and \( t \in (-r_{\text{bht}}(x), 0] \) the quantity \( (L\tilde{y})^\alpha(t, x)_{\alpha=1,\ldots,d} \) defines a tangent vector at \( \tilde{y}(t, x) \in E \). It is in fact tangent to the \( r \)-sphere \( S_{\tilde{y}(t, x)} = S_{r(x) + t} \) due to \( L\tilde{y} = 0 \).

**Lemma 3.6.** Suppose Conditions 1.1 and 1.2 (and \( \sigma \in (0, \sigma') \)). Then for all \( x \in E \) and \( t \in (-r_{\text{bht}}(x), 0] \)

\[
\ell^*_s(t, x) \leq (d - 1)\left[ (r(x) + t)/r(x) \right]^\sigma \ell(\tilde{y}(t, x))
\]

(3.29)
as quadratic forms on the fibers of the cotangent bundle. In spherical coordinates the estimate (3.29) reads: For any \( r > s \geq r_0 \) and \( \sigma \in S_{s} \subseteq S \)
\[
\ell(r, \sigma) \leq (d-1)(s/r)^{\alpha^*} \ell(s, \sigma).
\] (3.30)

If in addition (1.43a) is fulfilled, then there exists \( C > 0 \) such that uniformly in \( x \in E \) and \( t \in (-r^{\beta_{h}(x)} \cdot 0) \)
\[
|L\tilde{y}(t, x)| \leq C [(r(x) + t)^{1/2} / r(x)]^{\min(\sigma, \tau)}.
\] (3.31)

3.3. Proof for general long-range model. In this subsection we show Theorem 2.1.

**Lemma 3.7.** Suppose Condition 1.17. Then for any \( h \in C_{c}^{2}(\mathcal{I} \times S) \) there exist the limits (2.2).

*Proof.* We shall employ the Cook–Kuroda method and Lemma 3.3. Due to time reversal invariance it suffices to consider the upper sign. Let \( h \in C_{c}^{2}(\mathcal{I} \times S) \) be given. We divide the proof into three steps. In the third step we treat the two cases of Condition 1.17.

**Step I.** We prepare for applying the Cook–Kuroda method by introducing an energy cut-off \( \tilde{\chi}_{n}(H + C) \). Pick \( C > 0 \) such that \( H + C \geq 0 \), and introduce \( U_{0}^{+}(t) h \) by (3.10). Here we are going to show that
\[
\sup_{t \geq 1} \left\| \tilde{\chi}_{n}(H + C) U_{0}^{+}(t) h \right\|_{\mathcal{H}} \rightarrow 0 \text{ as } n \rightarrow \infty.
\] (3.32)

By the Chebyshev type inequality we have
\[
\left\| \tilde{\chi}_{n}(H + C) U_{0}^{+}(t) h \right\|_{\mathcal{H}}^{2} \leq R_{0}^{-1} (H + C) U_{0}^{+}(t) h,
\]
and hence in order to prove (3.32) it suffices to show that
\[
\sup_{t \geq 1} \left\| pU_{0}^{+}(t) h \right\|_{\mathcal{H}}^{2} < \infty.
\] (3.33)

To prove (3.33) let us write (3.10) for short as
\[
U_{0}^{+}(t) h(r, \sigma) = e^{Y(t, r, \sigma)} (\partial_{r} \lambda_{c}(t, r, \sigma))^{1/2} Z(\lambda_{c}(t, r, \sigma), \sigma),
\] (3.34)
where \( Y \in C^{k}(\Omega_{c}) \) (according to \( q_{1} \in C^{k} \)) and \( Z \in C_{c}^{2}((\lambda_{1}, \infty) \times D), \overline{D} \subseteq S_{r_{1}} \), are defined by
\[
Y(t, r, \sigma) = iK_{1}(t, r, \sigma) - \frac{1}{2} \int_{r_{1}}^{r} (\text{div } \tilde{\omega})(s, \sigma) \, ds,
\]
\[
Z(\lambda, \sigma) = (2\pi)^{-1/2} e^{-3\pi/4} 1_{\Omega_{c}}(t) (r, \sigma)e^{\int_{t_{0}}^{t} [\tilde{b}(s, \sigma) - \text{div } \tilde{\omega}(s, \sigma)] / 2} \, ds \lambda(\sigma).
\]

From this representation we obtain using (3.5b) and (3.7) that
\[
\sup_{t \geq 1} \left\| pU_{0}^{+}(t) h \right\|_{\mathcal{H}}^{2} < \infty.
\]

Therefore it suffices to show that
\[
\sup_{t \geq 1} \left\| pU_{0}^{+}(t) h \right\|_{\mathcal{H}}^{2} < \infty,
\]
but here for later use we are going to show a sharper estimate for any \( \beta \in (0, \beta_{c}) \)
\[
\left\| pU_{0}^{+}(t) h \right\|_{\mathcal{H}} \leq C_{\beta} t^{-\beta}
\] (3.35)
We recall that \( p' = -i \nabla' \) is the spherical part of the derivative of \( p \), cf. (1.41). Omitting variables, we can differentiate (3.34):

\[
p'U_0^+ h = (p'Y)U_0^+ h + \frac{1}{2}(p' \ln \partial_c \lambda_c)U_0^+ h + e^Y (\partial_r \partial_c \lambda_c)^{1/2}(p' \lambda_c)(\partial_c Z)|_{\lambda=\lambda_c} + e^Y (\partial_r \partial_c \lambda_c)^{1/2}(p'Z)|_{\lambda=\lambda_c}.
\]  

(3.36)

As for the first to third terms of (3.36), if we note the isometry properties, cf. (3.11) and similar identity holding for \( (\partial_c \lambda_c)^{1/2}(\partial_c Z)|_{\lambda=\lambda_c} \) due to a change of variables, it suffices to show that

\[
\sup_{t \geq 1, (t, r, \sigma) \in \Omega_c} t^3 (|p'Y| + |p' \ln \partial_c \lambda_c| + |p' \lambda_c|) < \infty.
\]  

(3.37)

We compute using (3.4a), (3.5a) and (3.6b)

\[
p'Y(t, r, \sigma) = \left( \int_{r_1}^r p' \left( \tilde{b} - \frac{i}{2} \text{div} \tilde{\omega} \right)(s, \sigma) \right)_{|\lambda=\lambda_c(t, r, \sigma)}
\]

so that by (3.30) and (3.4c) for \((t, r, \sigma) \in \Omega_c\)

\[
l^{2\beta} |p'Y(t, r, \sigma)|^2 = \int_{r_1}^r \| \tilde{b} - \frac{i}{2} \text{div} \tilde{\omega} \|_{|\lambda=\lambda_c(t, r, \sigma)} ds \leq C_1 l^{2\beta} \left( \int_{r_1}^r \| \tilde{b} - \frac{i}{2} \text{div} \tilde{\omega} \|_{|\lambda=\lambda_c(t, r, \sigma)} ds \right)^2 \leq C_2 l^{2\beta} r^{-2\beta} \leq C_3.
\]

Similarly, since we have

\[
p' \lambda_c(t, r, \sigma) = \left( \int_{r_1}^r b - \frac{i}{2} b \tilde{b} \right)(s, \sigma) \right)_{|\lambda=\lambda_c(t, r, \sigma)}
\]

(3.39)

it follows that

\[
\sup_{(t, r, \sigma) \in \Omega_c} t^{\beta+1} |p' \lambda_c(t, r, \sigma)| \leq C_4.
\]

(3.40)

We also compute using (3.7)

\[
p' \ln \partial_r \lambda_c = \left[ p' \left( \int_{r_1}^r |dr|^2 b^{-3} \right) + \left( \int_{r_1}^r b \left( \int_{r_1}^r |dr|^2 b^{-3} \right) \right) \right]|_{\lambda=\lambda_c}
\]

(3.41)

so by estimating similarly again we obtain

\[
\sup_{(t, r, \sigma) \in \Omega_c} t^{\beta+1} |p' \ln \partial_r \lambda_c| \leq C_5.
\]

(3.42)

We have shown that the first to third terms of (3.36) satisfy the desired estimate. The fourth term of (3.36) can be bounded similarly using (3.30) and a change of variables. Hence we obtain (3.35), and in particular (3.33).

Step II. Here we reduce the proof of the lemma to the following estimate involving \( L \) where \( \chi = \chi_N(H + C) \):

\[
\int_1^{\infty} \| \chi LU_0^+(t) h \|_{H^1} dt < \infty.
\]

(3.43)
Due to (3.13) and (3.32) the lemma follows if we can show that for any large \( n \geq 1 \)
\[
\int_1^\infty \| \chi \frac{d}{dt} e^{tH} U_0^+ (t)h \|_{\mathcal{H}} \, dt < \infty.
\]
Whence in turn it suffices to show that
\[
\int_1^\infty \| \chi (H - G^+ (t)) U_0^+ (t)h \|_{\mathcal{H}} \, dt < \infty.
\] (3.44)

By the expressions (1.12) and (3.12) we can write
\[
H - G^+ (t) = \frac{1}{2} (A - b_\lambda c) \tilde{\eta}(A - b_\lambda c) + \frac{1}{2} L + q_2 + \frac{i}{4} (\nabla \tilde{\eta})(\Delta r).
\]
This leads to the formula
\[
(H - G^+ (t)) 1_{\Omega_c(t)} (r, \sigma) e^{Y(t,r,\sigma)} = 1_{\Omega_c(t)} (r, \sigma) e^{Y(t,r,\sigma)} \left( \frac{1}{2} p_r |dr|^2 p_r + \frac{1}{2} L + q_2 \right).
\]
Whence, if we can verify (3.43) and
\[
\int_1^\infty \| \chi e^{Y(t,\cdot)} p_r |dr|^2 p_r (\partial_r \lambda_c(t, \cdot))^{1/2} Z(\lambda(t, \cdot), \cdot) \|_{\mathcal{H}} \, dt < \infty,
\] (3.45)
then (3.44) follows and the proof is done.

Now let us prove (3.45). We use the product rule and (3.7) to compute
\[
p_r |dr|^2 p_r (\partial_r \lambda_c(t, \cdot))^{1/2} Z(\lambda(t, \cdot), \cdot).
\]
Only the term
\[
-|dr|^2 \frac{\partial^2}{\partial r^2} (\partial_r \lambda_c(t, \cdot))^{1/2} Z(\lambda(t, \cdot), \cdot)
\]
needs examination. In turn, when we expand it further, only a single term might not contribute in agreement with (3.45). This is a term that contains a second order derivative of \( q_{12} \). Explicitly it may be expressed as
\[
-i \frac{1}{2} |dr|^4 b_{\lambda=\lambda_c} p_r (\partial_r \lambda_c(t, \cdot))^{1/2} Z(\lambda(t, \cdot), \cdot)
\]
that possibly do not seem to agree with (3.45). However since \( \partial_r q_{12} = O(r^{-1-\rho/2}) \) we can pull the operator \( p_r \) to the left in the corresponding time-integral, bound it with the factor \( \chi \) (note that indeed \( \chi p_r \) is bounded) and then use that \( t^{-1-\rho/2} \) is integrable. Hence (3.45) is verified, and the proof of the lemma reduces to (3.43).

**Step III.** Finally we verify the estimate (3.43) by using Condition 1.17.

First we assume (1) of Condition 1.17. To prove the estimate (3.43) it suffices to show
\[
\int_1^\infty \| p' U_0^+ (t)h \|_{\mathcal{H}} \, dt < \infty.
\]
However, this bound is easily verified by noting that (3.35) is valid for some \( \beta > 1 \) in this case.
Hence for the rest of the proof we assume (2) of Condition 1.17. Using the fibration (3.25) of $L$ in spherical coordinates and the notation as in (3.34), we can write

$$L_r U_0^+ h = (L_r Y)^r U_0^+ h + \frac{1}{2}(L_r \ln \partial_r \lambda_c) U_0^+ h + e^Y (\partial_r \lambda_c)^{1/2} (L_r \lambda_c (\partial_r Z)|_{\lambda = \lambda_c} + |p' Y|^2 U_0^+ h + \ell^\lambda_j (p_j Y) (p_j \ln \partial_r \lambda_c) U_0^+ h$$

$$+ 2 \ell^\lambda_j (p_j Y) (p_j \lambda_c) e^Y (\partial_r \lambda_c)^{1/2} (\partial_r Z)|_{\lambda = \lambda_c}$$

$$+ 2 \ell^\lambda_j (p_j Y) e^Y (\partial_r \lambda_c)^{1/2} (p_j \lambda_c) |_{\lambda = \lambda_c} + \frac{1}{2} |p' \ln \partial_r \lambda_c|^2 U_0^+ h$$

$$(3.46)$$

$$+ \ell^\lambda_j (p_j \ln \partial_r \lambda_c) e^Y (\partial_r \lambda_c)^{1/2} (p_j \lambda_c) (\partial_r Z)|_{\lambda = \lambda_c}$$

$$e^Y (\partial_r \lambda_c)^{1/2} |p' \lambda_c|^2 (\partial_r^2 Z)|_{\lambda = \lambda_c}$$

$$+ 2 \ell^\lambda_j e^Y (\partial_r \lambda_c)^{1/2} (p_j \lambda_c) (\partial_r p_j Z)|_{\lambda = \lambda_c},$$

cf. (3.36). Similarly to Step II, we can show that the fifth to thirteenth terms of (3.46) are $O(t^{-2\beta})$ for any $\beta \in (0, \beta_c)$. Here we in particular implemented (3.37) and the isometric properties due to a change of variables. To bound the first term of (3.46) we first compute by (3.4a), (3.5a), (3.6b) and (2.37) that

$$L_r Y(t, r, \sigma) = \left( L_r \int_{r_1}^r (\tilde{b} - \frac{1}{2} \text{div } \tilde{\omega}) (s, \sigma) ds \right)_{|\lambda = \lambda_c (t, r, \sigma)}$$

$$\left( (\ell^\lambda_j (r, \sigma) \int_{r_1}^r \left[ \nabla^2 (\tilde{b} - \frac{1}{2} \text{div } \tilde{\omega}) \right]_{ij} (s, \sigma) ds \right)_{|\lambda = \lambda_c (t, r, \sigma)}$$

$$+ \left( \int_{r_1}^r (L \tilde{y})^i (s - r, r, \sigma) \left[ \partial_i (\tilde{b} - \frac{1}{2} \text{div } \tilde{\omega}) \right] (s, \sigma) ds \right)_{|\lambda = \lambda_c (t, r, \sigma)}.$$

Then by (3.30), (3.31) and (2) of Condition 1.17 we obtain that the first term of (3.46) is $O(t^{-2\beta})$ for any $\beta \in (0, \beta_c)$. The second to fourth terms of (3.46) are treated basically in the same manner as the first. We first use the formula (3.27) with $t = 0$ to reduce the estimate to those of

$$\nabla' \ln \partial_r \lambda_c, \nabla'^2 \ln \partial_r \lambda_c, \nabla' \lambda_c, \nabla'^2 \lambda_c, (\nabla' Z)|_{\lambda = \lambda_c}, (\nabla'^2 Z)|_{\lambda = \lambda_c}.$$

The first order derivatives are already estimated in Step II. The second order derivatives are computed, e.g., from (3.39) and (3.41), and then estimated similarly. Hence we can conclude that $L_r U_0^+ h$ is $O(t^{-2\beta})$ for any $\beta \in (0, \beta_c)$, and this in particular implies the integrability (3.43). \hfill $\Box$

Next we extend the wave operator to a bounded operator $W^\pm : \mathcal{H}_I \rightarrow \mathcal{H}_I$ and show (2.3).

**Lemma 3.8.** Suppose Condition 1.14 and that there exist the limits (2.2) for any $h \in C^2_c(T \times S)$. Then $W^\pm$ extend to isometric operators $W^\pm : \mathcal{H}_I \rightarrow \mathcal{H}_I$, and (2.3) holds. In particular $W^\pm : \mathcal{H}_I \rightarrow \mathcal{H}_I$ are unitary.

**Proof.** Again only the upper sign is considered. Due to (3.11) and (3.13) the operator $W^+$ is isometric. The property (2.3) (to be shown) implies the unitarity. Hence it suffices to show that $W^+$ maps into $\mathcal{H}_I$ and that (2.3) is fulfilled.
We shall proceed partially following [HS, Appendix A]. For any $h \in C_c^2(I \times S)$ and $\psi \in H_{1+} \cap H^1$

$$2\pi i \langle \psi, W^+ h \rangle = 2\pi i \lim_{\epsilon \downarrow 0} \int_0^\infty e^{-it} \langle \psi, e^{itH} \bar{\chi}_n U^+(t)h \rangle \, dt$$

$$= \lim_{\epsilon \downarrow 0} \int_0^\infty \left\langle e^{-it(H-i\epsilon)} \psi, \int_I e^{-it\lambda} \bar{\chi}_n \phi^+_\lambda [h(\lambda)] \, d\lambda \right\rangle \, dt ;$$

here the cut-off function $\bar{\chi}_n$ is chosen with $n$ big enough (possibly depending on $h$) to assure the property $\bar{\chi}_n \phi^+_\lambda [h(\lambda)] \in \mathcal{N}$ for all $\lambda$. Moreover, here and below, we use the $L^2$-pairing of the spaces $\mathcal{H}_s$ and $\mathcal{H}_{-s}$ for any real $s$ (denoted by $\langle \cdot, \cdot \rangle$ as the inner product). Note that by commutation we may bound

$$\int_0^\infty \| r e^{-it(H-i\epsilon)} \psi \| \, dt < \infty ;$$

so that we may insert a configuration cut-off and obtain

$$2\pi i \langle \psi, W^+ h \rangle = \lim_{\epsilon \downarrow 0} \lim_{m \to \infty} \int_I \langle \psi, \int_0^\infty e^{it(H+i\epsilon)} \int_I e^{-it\lambda} \bar{\chi}_n \phi^+_\lambda [h(\lambda)] \, d\lambda \rangle \, d\lambda \, dt \rangle .$$

Now we can change the order of integration and then compute, abbreviating $\phi_\epsilon = R(\lambda + i\epsilon) \psi$,

$$2\pi i \langle \psi, W^+ h \rangle$$

$$= \lim_{\epsilon \downarrow 0} \lim_{m \to \infty} \int_I \langle \psi, \int_0^\infty e^{it(H+i\epsilon)} \chi_{n,m} \phi^+_\lambda [h(\lambda)] \, dt \rangle \, d\lambda$$

$$= \lim_{\epsilon \downarrow 0} \lim_{m \to \infty} \int_I \langle \psi, i\epsilon R(\lambda - i\epsilon) \chi_{n,m} \phi^+_\lambda [h(\lambda)] \rangle \, d\lambda$$

$$= \lim_{\epsilon \downarrow 0} \lim_{m \to \infty} \int_I \left( \langle \psi, \chi_{n,m} \phi^+_\lambda [h(\lambda)] \rangle - \langle \psi, R(\lambda - i\epsilon)(H - \lambda) \chi_{n,m} \phi^+_\lambda [h(\lambda)] \rangle \right) \, d\lambda$$

$$= \int_I \langle \psi, \bar{\chi}_n \phi^+_\lambda [h(\lambda)] \rangle \, d\lambda - \lim_{\epsilon \downarrow 0} \lim_{m \to \infty} \int_I \langle \phi_\epsilon, (H - \lambda) \chi_{n,m} \phi^+_\lambda [h(\lambda)] \rangle \, d\lambda .$$

We shall compute the double limit by an integration by parts procedure first rewriting the integrand by using the spherical decomposition formula [IS3, (2.8a)] (as in the proof of [IS3, Theorem 1.14]). A commutation error disappears when taking the $m$-limit and whence, more precisely,

$$\lim_{m \to \infty} 2 \int_I \langle \phi_\epsilon, (H - \lambda) \chi_{n,m} \phi^+_\lambda [h(\lambda)] \rangle \, d\lambda = \lim_{m \to \infty} \int_I (T_1 + T_2 + T_3) \, d\lambda ,$$

where (for some $\kappa > 1$ and $p'$ being the covariant derivative on $S$)

$$T_1 = \langle (A + \bar{a}) \phi_\epsilon, \chi_{n,m} \bar{\eta} (A - a) \bar{\chi}_n \phi^+_\lambda [h(\lambda)] \rangle ,$$

$$T_2 = \int_{r_0}^\infty \chi_m (r) \langle p' \phi_\epsilon, p' \bar{\chi}_n \phi^+_\lambda [h(\lambda)] \rangle \, d\lambda ,$$

$$T_3 = \int_{r_0}^\infty \chi_m (r) \langle \phi_\epsilon, O(r^{-\kappa}) \bar{\chi}_n \phi^+_\lambda [h(\lambda)] \rangle \, d\lambda .$$

We can compute the $m$-limit from this representation, however since we need to compute a double limit it turns out to be better to compute the $\epsilon$-limit first. This change of order is legitimate since the $m$-limit exists uniformly in small $\epsilon$ due to Theorem 1.8 and (3.29) (at this point Condition 1.14 is crucial). By taking the
\( \epsilon \)-limit inside the \( \lambda \)-integrals we can then compute the \( m \)-limit for each term after picking up (for \( T_1 \)) the factor \( \chi_m' \) (i.e. the \( r \)-derivative of \( \chi_m \)) from a commutation (precisely as in the proof of [IS3, Theorem 1.14]). Now the corresponding term does contribute in the \( m \)-limit and we obtain the following formula, abbreviating \( \phi = R(\lambda + i0) \psi \),

\[
- \lim_{\epsilon \downarrow 0} \lim_{m \to \infty} \int_I \langle \phi_\epsilon, (H - \lambda) \chi_{n,m} \phi_\lambda^+ [h(\lambda)] \rangle \, d\lambda = \int_I \left( i(F^+(\lambda) \psi, h(\lambda))_G - \langle \psi, \chi_n \phi_\lambda^+ [h(\lambda)] \rangle \right) \, d\lambda.
\]

By combining this with the previous computation we observe a cancellation and conclude that

\[
2\pi i \langle \psi, W^+ h \rangle = \int_I i(1 - f(\lambda)) \langle F^+(\lambda) \psi, h(\lambda) \rangle_\mathcal{G} \, d\lambda.
\]  

(3.47)

Let us consider \( \psi = f(H) \tilde{\psi} \) where \( f \in C_c^\infty(\mathbb{R}) \) and \( \tilde{\psi} \in \mathcal{H}_{1+} \cap \mathcal{H}^1 \). We obtain from (3.47) that

\[
2\pi i \langle (I - f(H)) \tilde{\psi}, W^+ h \rangle = \int_I i(1 - f(\lambda)) \langle F^+(\lambda) \tilde{\psi}, h(\lambda) \rangle_\mathcal{G} \, d\lambda.
\]

Clearly if \( f \) is chosen such that \( f(\lambda) h(\lambda, \cdot) = h(\lambda, \cdot) \) for all \( \lambda \) the right-hand side vanishes and whence by density we conclude that \( W^+ \) maps into \( \mathcal{H}_I \) and extends to an isometry \( W^+ : \mathcal{H}_I \to \mathcal{H}_I \). Moreover by choosing \( f \in C_c^\infty(I) \) and \( \tilde{\psi} \in \mathcal{H}_{1+} \) and then using \( \psi = f(H) \tilde{\psi} \) in (3.47) it follows (again by density) that \( (W^+)^* \psi = F^+ \psi \) for all \( \psi \in \mathcal{H}_I \) showing (2.3).

Theorem 2.1 is a direct consequence of Lemmas 3.7 and 3.8. In fact using only Lemma 3.8 we obtain a version of Theorem 1.18 without need for Condition 1.17.

**Corollary 3.9.** Suppose Condition 1.14 and that there exist the limits (2.2) for any \( h \in C^2_c(I \times S) \). Then the operators \( F^\pm : \mathcal{H}_I \to \mathcal{H}_I \) are unitary diagonalizing transformations.

**Proof.** We know that \( F^+ \) is an isometry, and hence it suffices to show that \( F^+ \) is onto. But since \( W^+ \) is an isometry the conclusion follows from (2.3). \( \Box \)

### 3.4. Proofs for short-range and Dollard type models.

In this subsection we prove Theorems 2.6 and 2.9 stated in Subsection 2.3 under the short-range assumption and Theorem 2.11 stated in Subsection 2.4 under the Dollard type assumption.

Let us begin with the proof of Theorem 2.11 since the short-range model can be treated easily using a bound from the proof. It suffices to do the upper case due to time reversal invariance.

**Proof of Theorem 2.11.** Let \( \theta = \theta(\sigma, \lambda) \) be the real-valued function appearing in Corollary 2.10. Fix any \( h \in C^2_c(I \times S) \) and corresponding quantities \( \lambda_1, D \) and \( r_1 \) (determining \( U^+_0(t)h \) by (3.10)). In order to prove the existence of the limit (2.16) for this \( h \) it suffices to show that

\[
\| U^+_{do}(t) \hat{h} - U^+_0(t)h \| \to 0 \quad \text{as} \quad t \to \infty; \quad h := e^{i\theta} \hat{h},
\]  

(3.48)
cf. Theorem 2.1 and Lemma 3.3. By the properties of \( h \) we can find \( \lambda_2 > \lambda_1 \) such that \( \lambda \leq \lambda_2 \) on \( \text{supp} \, h \). Then, recalling the quantities of (3.10), we let

\[
\Omega_c^2(t) = \{(r, t) \in \Omega_c(t) \mid \lambda_c \leq \lambda_2\}.
\]

First we note

\[
\|U_0^+(t)h\| = \|h\| \geq \|U_0^+(t)\tilde{h}\|,
\]

reducing the proof of (3.48) to showing

\[
\|1_{\Omega_c^2(t)}U_0^+(t)\tilde{h} - U_0^+(t)\tilde{h}\| \to 0 \text{ as } t \to \infty. \tag{3.49}
\]

Hence let us prove (3.49). Obviously we need to ‘compare’ the expressions (3.10) and (2.15) with the factor \( e^{i\theta} \). By using (2.6b) and (3.4c) we obtain that

\[
\sup_{(r, \sigma) \in \Omega_c^2(t)} \int_r^\infty \left( b_{\lambda_c}(s, \sigma) - b_{\text{do}, \lambda_c}(s, \sigma) \right) ds = o(t^0) .
\]

Whence it suffices for (3.49) to show the following asymptotics uniformly in \( (r, \sigma) \in \Omega_c^2(t) \):

\[
\begin{align*}
\lambda_c &= \lambda_0(\sigma) + \frac{1}{2} \frac{(r-r_0)^2}{t^2} + o(t^0), \\
\partial_r \lambda_c &= (r-r_0)/t^2 + o(t^{-1}), \\
K_{\text{do}} &= \int_{r_0}^r b_{\text{do}, \lambda_c}(s, \sigma) ds - \lambda_c t + o(t^0).
\end{align*}
\]

Now to prove these asymptotics we do a Taylor expansion of the integrand of (3.6b) to find a continuous function \( f_1(\lambda, \sigma) \) (\( \lambda \) being close to \( \lambda_c \)) such that

\[
\partial_\lambda \Theta_1(\lambda, t, r, \sigma) = b_{sr}^{-1}(r-r_1) + \frac{1}{2} b_{sr}^{-3} \int_{r_1}^r Q ds + f_1 + O(t^{-\epsilon}) - t,
\]

where \( \epsilon \in (0, 1) \) is a constant satisfying (2.6a) and

\[
Q = Q(x) = 2(q_1^x - \lambda_0(\sigma)).
\]

Alternatively, we can rewrite for some continuous function \( f_2(\lambda, \sigma) \)

\[
\partial_\lambda \Theta_1(\lambda, t, r, \sigma) = b_{sr}^{-1}(r-r_0) + \frac{1}{2} b_{sr}^{-3} \int_{r_0}^r Q ds + f_2 + O(t^{-\epsilon}) - t.
\]

Let us substitute \( \lambda = \lambda_c(t, r, \sigma) \) and abbreviate \( f_2 = f_2(\lambda_c, \sigma) \) and \( b_{sr} = b_{sr}(\lambda_c, \sigma) \).

This leads to

\[
b_{sr} = \frac{r-r_0}{t} + \frac{1}{2} \frac{t}{(r-r_0)^2} \int_{r_0}^r Q ds + \frac{(r-r_0)^2}{t^2} f_2 + O(t^{-1-\epsilon}), \tag{3.50}
\]

and consequently

\[
\lambda_c = \lambda_0(\sigma) + \frac{1}{2} \frac{(r-r_0)^2}{t^2} + \frac{1}{2} \frac{1}{(r-r_0)^2} \int_{r_0}^r Q ds + \frac{(r-r_0)^2}{t^2} f_2 + O(t^{-1-\epsilon}). \tag{3.51}
\]

In particular

\[
\lambda_c = \lambda_0(\sigma) + \frac{1}{2} \frac{(r-r_0)^2}{t^2} + O(t^{-(1+\epsilon)/2}).
\]

In turn using

\[
\partial_r \lambda_c = \left( b_{\lambda_c}(r, \sigma) \int_{r_1}^r b_{\lambda_c}(s, \sigma)^{-3} ds \right)^{-1},
\]
we obtain
\[ \partial_r \lambda_c = \frac{(r - r_0)}{t^2} + O(t^{-(3+\epsilon)/2}). \]

Finally we compute using (3.50) and (3.51)
\[
K_{do} - \int_{r_0}^{r} b_{do, \lambda_c}(s, \sigma) \, ds + \lambda_c t \\
= K_{do} - (r - r_0) \left( \frac{(r - r_0)}{t} + \frac{1}{2} \frac{t}{(r-r_0)^2} \int_{r_0}^{r} Q \, ds + \frac{(r-r_0)}{t^2} f_2 \right) + \frac{1}{2} \frac{t}{(r-r_0)^2} \int_{r_0}^{r} Q \, ds \\
+ \left( \lambda_0(\sigma) + \frac{1}{2} \frac{(r-r_0)}{t^2} + \frac{1}{2} \frac{1}{(r-r_0)} \int_{r_0}^{r} Q \, ds + \frac{(r-r_0)}{t^2} f_2 \right) t + O(t^{-\epsilon}) \\
= O(t^{-\epsilon}).
\]

Whence we have shown that (3.49) holds.

The rest of the assertions is clear from (2.16), (3.48), Theorem 2.1 and Corollary 2.10. □

Proof of Theorem 2.6. As we already noted \( q \) is of Dollard type since it is of short-range type. Let \( \theta = \theta(\sigma, \lambda) \) be the real-valued function appearing in Corollary 2.5. Similarly to the proof of Theorem 2.11, in order to prove the existence of the limit (2.10) it suffices to show that for all \( h \in C_c^2(I \times S) \)
\[
\| U_{\kappa}^+(t) \tilde{h} - U_{\kappa}^+(t) h \| \to 0 \quad \text{as} \quad t \to \infty; \quad h := e^{i\theta} \tilde{h}. \quad (3.52)
\]

If we compare the expressions (2.9) and (2.15), then we can see that (3.52) follows from (3.48). Here we note that the functions \( \theta \) in (3.48) and (3.52) are chosen differently as in Corollary 2.10 and Corollary 2.5, respectively. Hence we are done with (2.10).

The rest of the assertions is clear from (2.10), (3.52), Theorem 2.1 and Corollary 2.5. □

We above proof is short since we could use the proof of Theorem 2.11, more precisely (3.48). A different and possibly more appealing procedure would be show (3.52) directly by mimicking the proof of Theorem 2.11 for the short-range setting.

Proof of Theorem 2.9. Under Condition 2.7 the existence of the limits (2.13) follows by [IS1]. Then by Lemma 2.8 and (3.52) the assumptions of Lemma 3.8 and Corollary 3.9 are fulfilled. Now we can argue as at the end of the proof of of Theorem 2.6 and complete the proof of the theorem. □

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