A SHORT PROOF OF GAMAS’S THEOREM

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Abstract. If \( \chi^\lambda \) is the irreducible character of \( S_n \) corresponding to the partition \( \lambda \) of \( n \) then we may symmetrize a tensor \( v_1 \otimes \cdots \otimes v_n \) by \( \chi^\lambda \). Gamas’s theorem states that the result is not zero if and only if we can partition the set \( \{ v_i \} \) into linearly independent sets whose sizes are the parts of the transpose of \( \lambda \). We give a short and self-contained proof of this fact.

1. Introduction

Let \( \lambda \) be a partition of a positive integer \( n \) and let \( \chi^\lambda \) be the irreducible character of the symmetric group \( S_n \) corresponding to \( \lambda \). There is a right action of \( S_n \) on \( V^\otimes n \), where \( V \) is a finite-dimensional complex vector space, by permuting tensor positions. Let \( T_\lambda \) be the endomorphism of \( V^\otimes n \) given by

\[
(v_1 \otimes \cdots \otimes v_n)T_\lambda = \frac{\chi^\lambda(1)}{n!} \sum_{\sigma \in S_n} \chi_\lambda(\sigma)v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}.
\]

Our goal is to prove the following result of Carlos Gamas [3].

**Theorem 1** (Gamas’s Theorem). Let \( v_1, \ldots, v_n \) be vectors in \( V \). Then

\[
(v_1 \otimes \cdots \otimes v_n)T_\lambda \neq 0
\]

if and only if it is possible to partition the set \( \{ v_i \} \) into linearly independent sets whose sizes are the parts of the transpose of \( \lambda \).

If \( \{ v_1, \ldots, v_n \} \) is a collection of vectors satisfying the condition of the theorem we will say that it satisfies “Gamas’s Condition for \( \lambda \)”. The theorem is a generalization of the well known fact that the exterior product of a set of vectors is nonzero if and only the set of vectors is linearly independent.

In addition to Gamas’s proof of this result there was a second one given by Pate in 1990 [4] using results he obtained in [5]. The benefit of our proof is that it is self-contained and short. It relies on standard facts from the representation theory of \( GL(V) \), namely, Schur-Weyl duality and the Pieri Rule. We refer to Fulton and Harris’s book [2] for the needed background and notation.

2. Preliminaries and Proof

Let \( V \) be a finite dimensional complex vector space. The general linear group \( GL(V) \) acts diagonally on \( V^\otimes n \). Let \( w \in V^\otimes n \) be any tensor. Define \( G(w) \) to be the \( GL(V) \)-module spanned by

\[
GL(V)w = \{ g \cdot w : g \in GL(V) \}\]

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We are interested in which irreducible \(GL(V)\)-modules appear in \(G(w)\). Since \(G(w) \subset V^{\otimes n}\) is a polynomial representation, the isomorphism type of the irreducible \(GL(V)\)-modules which can appear in \(G(w)\) are indexed by partitions \(\lambda\) of \(n\) with at most \(\dim V\) parts. If \(\lambda\) is such a partition, we will say that \(\lambda\) appears in \(G(w)\) if this module has a highest weight vector of weight \(\lambda\) (see [2 Chapter 15]). We will write \(\ell(\lambda)\) for the number of parts of \(\lambda\).

**Proposition 2.** If \(\lambda\) is a partition of \(n\), then \(\lambda\) appears in \(G(w)\) if and only if \(wT_\lambda \neq 0\).

*Proof.* Note that \(\lambda\) appears in \(G(w)\) if and only if the projection of \(G(w)\) onto its \(\lambda\)-th isotypic component is not zero. By Schur-Weyl duality (see [2 Lemma 6.22]) \(T_\lambda\) is this projector, since it is the projector of \(\mathbb{C}S_n\) onto its \(\lambda\)-th isotypic component. Since \(T_\lambda\) commutes with the \(GL(V)\) action, this isotypic component is zero if and only if \(G(wT_\lambda) = 0\), which happens if and only if \(wT_\lambda = 0\).

The following corollary is immediate from Proposition 2.

**Corollary 3.** Suppose that \(W\) is a subspace of \(V\) and \(w \in W^{\otimes n} \subset V^{\otimes n}\). The shape \(\lambda\) appears in \(\text{span} GL(V)w\) if and only if it appears in \(\text{span} GL(W)w\).

*Proof of Gamas’s Theorem.* Assume that the vectors \(\{v_1, \ldots, v_n\}\) span \(V\), as we may by Corollary 3. Suppose that \(\{v_i\}\) satisfy Gamas’s condition for \(\lambda\). We prove the result by induction on \(n + \ell(\lambda)\). Write \(v^{\otimes} \) for the tensor \(v_1 \otimes \cdots \otimes v_n\).

If \(\lambda\) has one part \(\chi^\lambda\) is the trivial character and \(v^{\otimes} T_\lambda\) is a scalar multiple of the fully symmetrized tensor \(v_1 \cdots v_n\) in \(\text{Sym}^n(V)\). This is not zero since none of the \(v_i\) are zero.

If \(\ell(\lambda) < \dim V\) let \(A \in \text{End}(V)\) be a generic projection to a subspace \(W \subset V\) with dimension equal to the length of \(\lambda\). Since \(A\) is generic, the collection \(\{Av_1, \ldots, Av_n\}\) still satisfies Gamas’s condition for \(\lambda\). It follows by induction that \(\lambda\) appears in the span of \(GL(W)(A \cdot v^{\otimes})\) and hence it also appears in \(G(A \cdot v^{\otimes})\). Since \(A\) is a limit of elements of \(GL(V)\) we have \(G(A \cdot v^{\otimes}) \subset G(v^{\otimes})\) and hence \(\lambda\) appears in \(G(v^{\otimes})\).

If \(\ell(\lambda) = \dim V\) then we consider a Young tableau of shape \(\lambda\) whose columns index independent subsets of the set \(v = \{v_1, \ldots, v_n\}\). Let \(B\) be the set of numbers in the first column of the tableau. The map

\[
b_B = \sum_{\sigma \in S_B} (-1)^{c} \sigma \in \mathbb{C}S_n = \text{End}_{GL(V)}(V^{\otimes n})
\]

is a map of \(GL(V)\)-modules and hence we have a surjection of \(GL(V)\)-modules \(G(v^{\otimes}) \twoheadrightarrow G(v^{\otimes} b_B)\). Without loss of generality, we write \(B = \{1, \ldots, k\}\) where \(k = \dim V\), so that

\[
G(v^{\otimes} b_B) = \det_V \otimes G(v_{k+1} \otimes \cdots \otimes v_n).
\]

Here \(\det_V\) is the one dimensional representation \(g \mapsto \det(g)\) of \(GL(V)\). For example, if \(\dim V = 2\) and \(B = \{1, 2\}\) then

\[
(v_1 \otimes v_2 \otimes v_3 \otimes v_4 \otimes v_5) b_B = (v_1 \otimes v_2 - v_2 \otimes v_1) \otimes v_3 \otimes v_4 \otimes v_5
\]

Since \(v_1\) and \(v_2\) are a basis of \(V\), we see that \(g \in GL(V)\) acts by its determinant on the exterior power \(\wedge^2 V\) and hence on \(v_1 \wedge v_2 = v_1 \otimes v_2 - v_2 \otimes v_1\).

Denote by \(\lambda^-\) the shape obtained from \(\lambda\) by removing the first column. Then \(\{v_{k+1}, \ldots, v_n\}\) satisfies Gamas’s condition for \(\lambda^-\). By induction we know that \(\lambda^-\) appears in \(G(v_{k+1} \otimes \cdots \otimes v_n)\). By Pieri’s Rule (see [2 Equation 6.9]) it follows that \(\lambda\) appears in

\[
\det_V \otimes G(v_{k+1} \otimes \cdots \otimes v_n)
\]

and, hence, in \(G(v^{\otimes})\) since it appears in its homomorphic image \(G(v^{\otimes} b_B)\). This completes the more difficult implication of Gamas’s Theorem.
Although our proof of the converse was essentially known to Pate \cite{4}, we include it to keep this paper self-contained. We will need the standard construction of the irreducible $GL(V)$ and $\mathbb{C}S_n$ modules via Young symmetrizers. To this end let $T$ be a tableau of shape $\lambda$, $a_T$ its row symmetrizer, and $b_T$ its column antisymmetrizer. These are given by

$$\sum_{\sigma \in \text{Row}(T)} \sigma, \quad \sum_{\sigma \in \text{Col}(T)} \text{sign}(\sigma)\sigma,$$

respectively. For example, using cycle notation for permutations in $S_n$, if $T = \begin{array}{ccc} 2 & 3 & 4 \\ 1 & 5 \end{array}$ then $b_T = (1 - (12))(1 - (35))$ while $a_T = (1 + (23) + (24) + (34) + (234) + (243))(1 + (15))$.

A product $b_T a_T$ is called a Young symmetrizer and the right ideal in $\mathbb{C}S_n$ generated by a Young symmetrizer is an irreducible $\mathbb{C}S_n$-module with character $\chi_\lambda$ while the image of $b_T a_T$ on $V \otimes n$ is zero, or irreducible with highest weight $\lambda$ (see \cite{2} Chapters 4 and 15). It is clear that $v \otimes b_T$ is not zero if and only if the sets of vectors indexed by the columns of the tableau $T$ are linearly independent.

It follows from Schur-Weyl duality that if $\lambda$ appears in $G(v \otimes c)$ then there is an element $c \in \text{End}_{GL(V)}(V \otimes n) = \mathbb{C}S_n$ such that $G(v \otimes c)$ equals the irreducible $GL(V)$-module $V \otimes n b_T a_T$. It then follows that the right $\mathbb{C}S_n$-module generated by $v \otimes c$ is isomorphic to both $c \mathbb{C}S_n$ and $b_T a_T \mathbb{C}S_n$, in particular the latter two modules are isomorphic. We conclude that $c$ can be written as a sum $c = \sum_{\sigma \in \mathbb{C}S_n} x_\sigma \sigma b_T a_T$, $x_\sigma \in \mathbb{C}$, and hence one of these terms $x_\sigma \sigma b_T a_T$ applied to $v \otimes$ is not zero. Finally, since $v \otimes \sigma b_T$ is not zero Gamas’s Condition holds for $\lambda$, the shape of $T$. □

Define a sequence of integers $\rho_i$ by the condition that

$$\rho_1 + \cdots + \rho_k$$

is the size of the largest union of $k$ linearly independent subsets of $\{v_i\}$. The sequence $\rho$ is called the rank partition of $\{v_i\}$ and was introduced by Dias da Silva in \cite{1}. In our language, Dias da Silva proved the following strengthening of Gamas’s Theorem.

**Theorem 4** (Dias da Silva). The partition $\lambda$ appears in $G(v \otimes)$ if and only if $\lambda$ is larger (in dominance order) than the transposed rank partition of $\{v_i\}$.

The extent to which one can further predict the irreducible $GL(V)$-decomposition of $G(v \otimes)$ is the subject of the author’s Ph.D. thesis.

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