The Caffarelli–Kohn–Nirenberg inequality on metric measure spaces

Received: 20 November 2017 / Accepted: 9 May 2020 / Published online: 2 June 2020

Abstract. In this paper, we prove that if a metric measure space satisfies the volume doubling condition and the Caffarelli–Kohn–Nirenberg inequality with the same exponent \( n(n \geq 2) \), then it has exactly \( n \)-dimensional volume growth. As application, we obtain geometric and topological properties of Alexandrov spaces, Riemannian manifolds and Finsler manifolds which support a Caffarelli–Kohn–Nirenberg inequality.

1. Introduction

The Caffarelli–Kohn–Nirenberg (CKN) inequality is one of the most important and interesting geometric functional inequalities. This inequality was first investigated by Caffarelli, Kohn and Nirenberg in their celebrated work [7], from which the following result originates.

Theorem 1.1. Let \( n \geq 2 \) and \( p, q, r, \alpha, \beta, \gamma, \sigma, a \) be fixed real numbers satisfying:

\[
\begin{align*}
p, q & \geq 1, \quad r > 0, \quad 0 \leq a \leq 1, \\
\frac{1}{p} + \frac{\alpha}{n} & > 0, \quad \frac{1}{q} + \frac{\beta}{n} > 0,
\end{align*}
\]

where

\[
\gamma = a\sigma + (1-a)\beta,
\]

\[
\frac{1}{r} + \frac{\gamma}{n} = a \left( \frac{1}{p} + \frac{\alpha - 1}{n} \right) + (1-a) \left( \frac{1}{q} + \frac{\beta}{n} \right),
\]

and

\[
0 \leq \alpha - \sigma \quad \text{if } a > 0, \quad \text{and}
\]
\[ \alpha - \sigma \leq 1 \quad \text{if } a > 0 \quad \text{and} \quad \frac{1}{p} + \frac{\alpha - 1}{n} = \frac{1}{r} + \frac{\gamma}{n}. \]

Then there exists a positive constant \( C = C(n, p, q, r, \alpha, \beta, \gamma) \) such that the following inequality holds

\[
\left( \int_{\mathbb{R}^n} |x|^{\gamma r} |u|^r \, dx \right)^{\frac{1}{r}} \leq C \left( \int_{\mathbb{R}^n} |x|^{|\alpha p|} |\nabla u|^p \, dx \right)^{\frac{\alpha}{p}} \left( \int_{\mathbb{R}^n} |x|^{|\beta q|} |u|^q \, dx \right)^{\frac{1-q}{q}},
\]

\( \forall u \in C^\infty_0(\mathbb{R}^n). \) (1)

CKN inequality (1) contains many well-known inequalities, such as Sobolev inequality, Hardy–Sobolev inequality, Nash inequality and Gagliardo–Nirenberg inequality. They play an important role in the theory of partial differential equations and have been intensively studied in many settings such as Riemannian manifolds, homogeneous groups, Finsler manifolds and metric measure spaces. We refer the readers to [13,14,16,18,20,27,29,31,33,35,45] for more detailed discussions on this subject.

It is an interesting and nontrivial problem to look for extremal functions and sharp constants for the CKN inequality. There are some well-known important results in this direction. For instance, the sharp constants in the Sobolev inequality were found independently by Aubin [4] and Talenti [41], in the Hardy–Sobolev inequality, the sharp constants were obtained by Lieb [25], where the author applied the symmetrization arguments. Also, for the Gagliardo–Nirenberg inequality, extremal functions and sharp constants were established by Del Pino and Dolbeault [11](see also [10] for a different proof by using mass transportation technique). For the more general CKN inequality, sharp constants and extremal functions were obtained by Lam and Lu in [23], where the authors consider the following change of exponents in (1):

\[ \alpha = -\frac{\mu}{p}, \quad \beta = -\frac{\theta}{q}, \quad \gamma = -\frac{s}{r}, \] \hspace{1cm} (2)

and get the following result:

**Theorem 1.2.** ([23], Theorem 1.2) Let \( n \geq 2 \) and \( p, q, \mu \) be fixed real numbers satisfying

\[ 1 < p < p + \mu < n, \quad 1 \leq q < \frac{p(q-1)}{p-1} < \frac{np}{n-p}, \] \hspace{1cm} (3)

and let \( r, \theta, s \) and \( a \) be given by

\[ r = \frac{p(q-1)}{p-1}, \quad \theta = s = \frac{n\mu}{n-p}, \quad a = \frac{n(q-p)}{(q-1)[np - q(n-p)]}. \] \hspace{1cm} (4)

Then, with \( v = np - q(n-p) \), the sharp constants are given by

\[ C_{opt}(\mathbb{R}^n) = \left( \frac{n-p}{n} \right)^{L} \left( \frac{q-p}{p\sqrt{\pi}} \right)^{a} \left( \frac{pq}{n(q-p)} \right)^{\frac{a}{p}} \left( \frac{v}{pq} \right)^{\frac{1}{r}}. \]
\[
\left( \frac{\Gamma \left( \frac{q-1}{q-p} \right) \Gamma \left( \frac{n}{2} + 1 \right)}{\Gamma \left( \frac{p-1}{p-q-p} \right) \Gamma \left( \frac{n(p-1)}{p} + 1 \right)} \right)^{\frac{a}{n}},
\]

where
\[
L = \frac{1}{r} + \frac{p-1}{p} - \frac{1-a}{q} - \frac{(p-1)(1-a)}{p},
\]
and all extremal functions are of the form
\[
v_\lambda(x) = \left( \lambda + |x|^{-n-p-\mu \frac{p-1}{q-p}} \right)^{-\frac{p-1}{q-p}}, \quad \lambda > 0.
\]

Zhong and Zou [45] considered the same change of exponents (2) into (1) and provided a range of parameters from which extremal functions of CKN inequality can be obtained. In addition, given the proximity between CKN inequality and the resolution of certain variational problems, the authors have been proved several results in the direction of solutions for the following problem:
\[
-\text{div} \left( \frac{|\nabla u|^{p-2} \nabla u}{|x|^{q-t}} \right) = \lambda V(x)|u|^{q-2}u,
\]
where \( V \) is an appropriate function.

In recent years, CKN inequality has also been studied on curved spaces. In [1,2,12,15,24,43,44], the authors consider the study of Riemannian manifolds with non-negative Ricci curvature supporting some of the particular classes of CKN inequality. In particular, in [1,2,12,43,44], the authors obtain metric and topological rigidity results. In the case of CKN inequality type, Xia in [42] considered the case
\[
q = \frac{p(r-1)}{p-1}, \quad 1 < p < r, \quad n-\theta < \left( 1 + \frac{\mu}{p} - \frac{\theta}{p} \right), \quad s = \frac{\mu}{p} + 1 + \frac{\theta(p-1)}{p},
\]
and obtained the extremal functions, which are \( u(x) = \left( \lambda + |x|^{1+\frac{\mu}{p} - \frac{\theta}{p}} \right)^{-\frac{p-1}{r-p}}. \) Furthermore, metrical and topological theorems were obtained.

For metric measure spaces, Kristály and Ohta [18,19] studied metric measure spaces supporting the Gagliardo–Nirenberg inequality and the following class of Caffarelli–Kohn–Nirenberg inequality:
\[
\left( \int_X |u|^p \, dm \right)^\frac{1}{p} \leq K_a \left( \int_X |Du|^2 \, dm \right)^\frac{1}{2}, \quad \forall u \in Lip_0(X),
\]
and obtained that the metric measure space has exactly the \( n \)-dimensional volume growth, which extends do Carmo and Xia’s results [12,44] in the scope of metric spaces, as application, they get some rigidity results on Finsler geometry. On the other hand, Mao in [27,28] has proven that complete noncompact smooth metric measure spaces with nonnegative weighted Ricci curvature on which a functional inequality of some specified type (for instance, the Caffarelli–Kohn–Nirenberg type inequality, the Gagliardo–Nirenberg type inequality, and so on) is satisfied are close
to the Euclidean space with the same dimension. The aforementioned results are attractive and interesting as they allow us to understand the geometry of spaces that supports CKN inequality.

Having this picture in mind, a natural question arises when we ask whether it is possible to extend Kristály, Ohta and Mao’s results in [18,27,28] to the more general class of CKN inequality presented by Lam and Lu [23]. To accomplish such a task, we focus our attention on Ledoux and Xia’s ideas in [24,42–44] and use it to extend the main results present in [18,27,28] to a large class of Caffarelli–Kohn–Nirenberg inequalities, as applications, we obtain several rigidity results on Alexandrov, Riemannian and Finsler geometries.

In order to state the main result of this paper, we consider in a metric space \((X,d, m)\) the Borel measure \(m\) satisfying \(0 < m(U) < \infty\) for all nonempty bounded open sets \(U \subset X\), and for fixed \(x_0 \in X\) and \(C > 0\) we consider the Caffarelli–Kohn–Nirenberg inequality on \((X,d, m)\) of the form

\[
\left( \int_X d(x,x_0)^{r\gamma} |u|^r \, dm(x) \right)^{\frac{1}{r}} \leq C \left( \int_X d(x,x_0)^{\alpha p} |Du|^p \, dm(x) \right)^{\frac{\alpha}{\alpha - a}} \left( \int_X d(x,x_0)^{\beta q} |u|^q \, dm(x) \right)^{\frac{1-a}{\beta q}},
\]

for all \(u \in Lip_0(X)\), where \(Lip_0(X)\) denote the space of Lipschitz functions with compact support and

\[
|Du|(x) := \limsup_{y \to x} \frac{|u(x) - u(y)|}{d(x,y)},
\]

is the local Lipschitz constant of \(u\) at \(x\).

The main result of the paper can be established as follows.

**Theorem 1.3.** Consider \(n, a, p, q, r, s, \mu, \theta\) as in Theorem 1.2. Let \((X,d, m)\) be a proper metric measure space and assume that for some \(x_0 \in X\), \(C \geq \text{Copt}(\mathbb{R}^n)\), \(C_0 \geq 1\), the Caffarelli–Kohn–Nirenberg inequality (5) hold on \(X\) with the following conditions

\[
\frac{m(B_R(x))}{m(B_{\rho}(x))} \leq C_0 \left( \frac{R}{\rho} \right)^{n}, \quad \forall x \in X, \quad 0 < \rho < R,
\]

and

\[
\liminf_{\rho \to 0} \frac{m(B_{\rho}(x_0))}{m_E(\mathbb{B}_{\rho}(0))} = 1,
\]

where \(B_{\rho}(x) := \{y \in X : d(x,y) < \rho\}\), \(\mathbb{B}_{\rho}(0) := \{x \in \mathbb{R}^n : |x| < \rho\}\) and \(m_E\) is the \(n\)-dimensional Lebesgue measure, then we have

\[
m(B_{\rho}(x)) \geq C_0^{-\frac{n}{a}} \left( \text{Copt}(\mathbb{R}^n) \right)^{\frac{n}{a}} m_E(\mathbb{B}_{\rho}(0)), \quad \forall \rho > 0, \; x \in X.
\]
In particular,

\[ C_0^{-1} \left( \frac{C_{opt}(\mathbb{R}^n)}{C} \right) \omega_n^\frac{n}{a} \rho^n \leq m(B_\rho(x_0)) \leq C_0 \omega_n \rho^n, \]

for all \( \rho > 0 \), where \( \omega_n \) denotes the volume of the unit ball in \( \mathbb{R}^n \).

**Remark 1.1.** (a) The choice of constant 1 on the right side of (7) was done for simplicity. In fact, by (6) we have that \( \Lambda_{x_0} = \lim \inf_{\rho \to 0} \frac{m(B_\rho(x_0))}{m_E(\mathbb{R}^n(0))} \) is positive. Then we can normalize the measure \( m \) in order to satisfy (7) whenever \( \Lambda_{x_0} \) is bounded.

(b) As pointed out in ([18], Remark 1.3(2)) if \((X, d, m)\) satisfies the volume doubling condition

\[ m(B_{2\rho}(x)) \leq \Lambda m(B_\rho(x)), \quad \text{for some} \quad \Lambda \geq 1, \quad \text{and all} \quad x \in X, \quad \rho > 0, \]

then it is easy to get that the volume condition (6) is satisfied with, e.g., \( n \geq \log_2 \Lambda \) and \( C = 1 \). Thus, (6) can be interpreted as the volume doubling condition with the explicit exponent \( n \).

In the scope of Riemannian manifolds, we show that the constant in the Caffarelli–Kohn–Nirenberg inequality on complete open Riemannian manifold should be bigger than or equal to the optimal one on the Euclidean space of the same dimension, that is, we have the following.

**Theorem 1.4.** Let \((M^n, g)\) be a complete non-compact Riemannian manifold with volume element \( dv \), distance function \( d(x) = d(x, x_0) \) for fixed point \( x_0 \in M \) and \( n, a, \alpha, \beta, \gamma, p, q, r \) constants as in Theorem 1.1. Suppose that there exists a constant \( C \) such that, for all \( u \in C_0^\infty(M) \)

\[ \left( \int_M d(x)^{\gamma r} |u|^r dv \right)^{\frac{1}{r}} \leq C \left( \int_M d(x)^{\alpha p} |\nabla u|^p dv \right)^{\frac{a}{p}} \left( \int_M d(x)^{\beta q} |u|^q dv \right)^{\frac{1-a}{q}}, \]

then \( C_{opt}(\mathbb{R}^n) \leq C \).

Now, recall first the definition of asymptotically non-negative Ricci curvature.

**Definition 1.1.** A complete open manifold \( M^n \) is said to have asymptotically non-negative Ricci curvature with base point \( x_0 \in M \) if

\[ Ric_{M, g}(x) \geq -(n-1) G(d(x)), \quad \forall x \in M, \]

where \( d(x) \) is the distance function on \( M \) from \( x_0 \) and \( G \in C^1([0, \infty)) \) is a non-negative function satisfying

\[ \int_0^\infty t G(t) dt = b_0 < \infty. \]
In this case, $M^n$ satisfies the following volume growth property (see Corollary 2.17 in [34]):

$$\frac{Vol[B_R(x_0)]}{Vol[B_\rho(x_0)]} \leq e^{(n-1)b_0} \left( \frac{R}{\rho} \right)^n, \quad 0 < \rho < R,$$

which implies easily that $M^n$ has volume doubling property at $x_0$ and

$$Vol[B_R(x_0)] \leq e^{(n-1)b_0} \omega_n R^n, \quad \forall R > 0.$$

Therefore, as a corollary of Theorem 1.3, we have the following.

**Corollary 1.1.** Let $(M^n, g)$ be a complete non-compact Riemannian manifold with Ricci curvature satisfying (9) and suppose that for some positive constant $C > 0$, the following Caffarelli–Kohn–Nirenberg inequality holds

$$\left( \int_M d(x)^{\gamma r} |u|^r dv \right)^{\frac{1}{r}} \leq C \left( \int_M d(x)^{\alpha p} |\nabla u|^p dv \right)^{\frac{\alpha}{p}} \left( \int_M d(x)^{\beta q} |u|^q dv \right)^{\frac{1-\alpha}{q}},$$

$$\forall u \in C_0^\infty(M), (11)$$

then for all $R > 0$ we have

$$e^{-(n-1)b_0} \left( \frac{C_{opt}(\mathbb{R}^n)}{C} \right)^{\frac{n}{\alpha}} \omega_n R^n \leq Vol[B_R(x_0)] \leq e^{(n-1)b_0} \omega_n R^n.$$

A theorem due to Cheeger and Colding [9] states that given an integer $n \geq 2$ there exists a constant $\delta(n) > 0$ such that any $n$-dimensional complete Riemannian manifold with non-negative Ricci curvature and $Vol[B_r(x)] \geq (1 - \delta(n)) \omega_n r^n$ for all $x \in M$ and all $r > 0$ is diffeomorphic to $\mathbb{R}^n$. The Bishop-Gromov comparison theorem tells us that for a complete manifold with non-negative Ricci curvature, the function

$$r \mapsto \frac{Vol[B_r(x)]}{\omega_n r^n}, \quad r \in (0, \infty)$$

is decreasing. Thus the limit

$$\alpha_x = \lim_{r \to +\infty} \frac{Vol[B_r(x)]}{\omega_n r^n},$$

exists. For two fixed points $x, y \in M$ and from the triangle inequality, we obtain $B_r(x) \subset B_{r+d(x, y)}(y)$, consequently

$$\frac{Vol[B_r(x)]}{\omega_n r^n} \leq \frac{Vol[B_{r+d(x, y)}(y)]}{\omega_n r^n} = \frac{Vol[B_{r+d(x, y)}(y)]}{\omega_n (r + d(x, y))^n} \left( \frac{r + d(x, y)}{r} \right)^n.$$

Taking $r \to \infty$, we get

$$\alpha_x \leq \lim_{r \to +\infty} \frac{Vol[B_{r+d(x, y)}(y)]}{\omega_n (r + d(x, y))^n} = \alpha_y,$$  (12)
which shows that $\alpha_x$ is independent of $x$. Thus the condition "\(Vol[B_r(x)] \geq (1 - \delta(n))\omega_n r^n\) for all $x \in M$ and all $r > 0$" in the above Cheeger-Colding’s theorem means that "\(\alpha_x \geq (1 - \delta(n))\) for all $x \in M$" and is equivalent to the fact "\(\alpha_x \geq (1 - \delta(n))\) for some $x \in M$". Hence, combining this Cheeger-Colding’s theorem and Corollary 1.1 we produce the following topological uniqueness theorem.

**Corollary 1.2.** Give an integer $n \geq 2$, there exists $\epsilon = \epsilon(n) > 0$ such that, any complete non-compact Riemannian manifold $(M^n, g)$ with non-negative Ricci curvature in which the following inequality is satisfied

\[
\left( \int_M d(x)^{\gamma r} |u|^r d \nu \right)^{\frac{1}{r}} \leq (C_{opt}(\mathbb{R}^n) + \epsilon)
\]

\[
\left( \int_M d(x)^{\alpha p} |\nabla u|^p d \nu \right)^{\frac{a}{p}} \left( \int_M d(x)^{\beta q} |u|^q d \nu \right)^{\frac{1-a}{q}},
\]

$\forall u \in C_0^\infty(M)$, is diffeomorphic to $\mathbb{R}^n$.

The Bishop comparison theorem [8,36] shows that if a complete Riemannian manifold $(M^n, g)$ has non-negative Ricci curvature, then for all $x \in M$, $Vol[B_R(x)] \leq \omega_n R^n$ and equality holds if, and only if, $B_R(x)$ is isometric to Euclidean ball $\mathbb{B}_R(0)$, so, from Corollary 1.1 jointly with (12), we get the following rigidity theorem.

**Corollary 1.3.** Let $(M^n, g)$ be a complete non-compact Riemannian manifold with non-negative Ricci curvature and suppose that the following Caffarelli–Kohn–Nirenberg inequality holds

\[
\left( \int_M d(x)^{\gamma r} |u|^r d \nu \right)^{\frac{1}{r}} \leq C_{opt}(\mathbb{R}^n)
\]

\[
\left( \int_M d(x)^{\alpha p} |\nabla u|^p d \nu \right)^{\frac{a}{p}} \left( \int_M d(x)^{\beta q} |u|^q d \nu \right)^{\frac{1-a}{q}},
\]

$\forall u \in C_0^\infty(M)$, then $M$ is isometric to Euclidean space $\mathbb{R}^n$.

It has been shown by Zhu [46] that, given $\delta > 0$, there is an $\epsilon(n, \delta)$ such that, if a complete non-compact Riemannian manifold $(M^n, g)$ with sectional curvature $\geq -G(d(x))$, $\int_0^\infty t G(t) dt \leq \epsilon$, and

\[
Vol[B_R(p)] \geq \left( \frac{1}{2} + \delta \right) \omega_n R^n, \quad \forall R > 0,
\]

then the distance function $d = d(x_0, \cdot) : M \to \mathbb{R}$ has no critical points and hence $M$ is diffeomorphic to $\mathbb{R}^n$. Combining this Zhu’s theorem with Corollary 1.1 and (12), we have the following.
Corollary 1.4. Let \((M^n, g)\) be a complete non-compact Riemannian manifold. Fix \(\delta \in (0, \frac{1}{2})\), then there exists an \(\epsilon(n, \delta) > 0\) such that, if the sectional curvature of \(M\) satisfies
\[
K(x) \geq -G(d(x)), \quad \int_0^\infty tG(t)\,dt \leq \epsilon,
\]
and the inequality (5) holds on \(M\) with \(C < (\frac{1}{2} + \delta)^{-\frac{n}{2}} C_{opt}(\mathbb{R}^n)\), then \(M\) is diffeomorphic to Euclidean space \(\mathbb{R}^n\).

It is interesting to know under what kind of conditions a complete metric measure space has finite topological type or is isometric to Euclidean space \(\mathbb{R}^n\). In the context of Alexandrov spaces, as application of Theorem 1.3, we prove the following results.

**Theorem 1.5.** Consider \(n, a, p, q, r, s, \mu, \theta\) as in Theorem 1.2. Let \((X, d)\) be a complete, locally compact non-compact Alexandrov space with non-negative curvature and measure \(\lambda \mathcal{H}^n\), with \(\lambda = \mathcal{H}^n(B_1(o_{x_0}))\), where \(o_{x_0}\) denotes the vertex of the tangent cone \(K_{x_0}X\) at \(x_0\) and \(\mathcal{H}^n\) is the \(n\)-dimensional Hausdorff measure of \(K_{x_0}X\). Suppose that \(X\) supports the CKN inequality with \(C = C_{opt}(\mathbb{R}^n)\) for \(x_0\) above, then \((X, d)\) is isometric to Euclidean space \(\mathbb{R}^n\).

**Theorem 1.6.** Consider \(n, a, p, q, r, s, \mu, \theta\) as in Theorem 1.2, there exists \(\delta = \delta(n) > 0\) such that, any locally compact \(n\)-dimensional complete Alexandrov space \((X, d)\) with curvature \(\geq 0\) and \(n\)-dimensional Hausdorff measure \(\mathcal{H}^n\) satisfying
\[
\liminf_{\rho \to 0} \frac{\mathcal{H}^n(B_\rho(x_0))}{\omega_n \rho^n} = 1,
\]
in which the inequality
\[
\left(\int_X d(x, x_0)^{\gamma r} |u|^r \,d\mathcal{H}^n\right)^{\frac{1}{r}} \leq (C_{opt}(\mathbb{R}^n) + \delta) \left(\int_X d(x, x_0)^{\alpha p} |D u|^p \,d\mathcal{H}^n\right)^{\frac{\alpha}{p}} \left(\int_X d(x, x_0)^{\beta q} |u|^q \,d\mathcal{H}^n\right)^{\frac{1 - \alpha}{q}},
\]
is satisfied for all \(u \in Lip_0(X)\), has finite topological type.

As pointed in [18], on Finsler manifolds with non-negative \(n\)-Ricci curvature, the condition (6) holds with \(C_0 = 1\). In particular, for Finsler manifolds in which a particular class of Caffarelli–Kohn–Nirenberg inequality holds, they get some metric rigidity theorem. Motivated by work [18] we obtain similar results on Finsler manifolds for a class of Caffarelli–Kohn–Nirenberg inequalities given by Theorem 1.3. That is, we have the following.

**Theorem 1.7.** Consider \(n, a, p, q, r, s, \mu, \theta\) as in Theorem 1.2. Let \((M, F)\) be a complete \(n\)-dimensional Finsler manifold. Fix a positive smooth measure on \(M\) and
assume that the n-Ricci curvature $\text{Ric}_n$ of $(M, F, m)$ is non-negative, the sharp Caffarelli–Kohn–Nirenberg inequality (5) holds for some $x_0 \in M$, and in addition
\[
\liminf_{r \to 0} \frac{m(B_r(x))}{\omega_n r^n} = 1,
\]
for all $x \in M$, then the flag curvature of $(M, F)$ is identically zero.

**Theorem 1.8.** Consider $n, a, p, q, r, s, \mu, \theta$ as in Theorem 1.2. Let $(M, F)$ be a complete n-dimensional Berwald space with Busemann-Hausdorff measure $m_{BH}$ and non-negative Ricci curvature. If for some $x_0 \in M$ the sharp Caffarelli–Kohn–Nirenberg inequality (5) holds, then $(M, F)$ is isometric to a Minkowski space.

Finally, in [22] the author defines the concept of large volume growth on Finsler manifolds and asked the following question.

**Question 1.1.** Is a geodesically complete Berwald space $(M, F)$ of non-negative flag curvature and large volume growth diffeomorphic to Euclidean space $\mathbb{R}^n$?

**Remark 1.2.** Kell in [17] gave an affirmative answer to this question (see [17], Corollary 27).

As consequence of this fact, we obtain the following.

**Theorem 1.9.** Consider $n, a, p, q, r, s, \mu, \theta$ as in Theorem 1.2. Let $(M, F, m_{BH})$ be a complete n-dimensional Berwald space with Busemann-Hausdorff measure $m_{BH}$ and non-negative flag curvature. If for some $x_0 \in M$ the Caffarelli–Kohn–Nirenberg inequality (5) holds on $M$ for some constant $C \geq C_{opt}(\mathbb{R}^n)$, then $(M, F)$ is diffeomorphic to Euclidean space $\mathbb{R}^n$.

2. Proof of Theorem 1.3

This section is devoted to the proof of Theorem 1.3. The main idea in our proof goes back to the arguments used by Ledoux and Xia [24,42–44]. The crucial ingredient is the explicit form of extremal functions in the Euclidean spaces. Exploiting this form of extremal functions, we define a new function in $(X, d)$ which depends only on $d$ and then applying the CKN inequality to obtain a differential inequality. Using this differential inequality, we obtain the desired volume comparison theorem.

**Proof of Theorem 1.3.** First, we derive an important ODE associated with the sharp constant $C_{opt}(\mathbb{R}^n)$. Since $\psi_\lambda(x) = (\lambda + |x|^{\frac{n-p-\mu}{p-1}})^{-\frac{p-1}{q-p}}$ is an extremal function in the Euclidean CKN inequality, the following integral identity holds for every $\lambda > 0$:
\[
\left(\int_{\mathbb{R}^n} |x|^\gamma (\lambda + |x|^{\frac{n-p-\mu}{p-1}})^{-\frac{p(\gamma-1)}{q-p}} \ dx\right)^{\frac{1}{\gamma}} = C_{opt}(\mathbb{R}^n) \left(\frac{p(n - p - \mu)}{(n-p)(q-p)}\right)^{\frac{\alpha}{\rho}} \left(\int_{\mathbb{R}^n} |x|^{\beta_0} (\lambda + |x|^{\frac{n-p-\mu}{p-1}})^{\frac{p(n-p-\mu+1)}{n(p-1)-q(p-1)+\mu(1-q)}} \ dx\right)^{\frac{\rho}{\beta_0}} \times \left(\int_{\mathbb{R}^n} |x|^\beta (\lambda + |x|^{\frac{n-p-\mu}{p-1}})^{-\frac{p(\beta-1)}{q-p}} \ dx\right)^{\frac{1-\alpha}{\beta}}.
\]
We introduce the auxiliary function $G : (0, \infty) \to \mathbb{R}$ defined by:

$$
G(\lambda) = \frac{q - p}{r(p - 1) - (q - p)} \int_{\mathbb{R}^n} \frac{|x|^q}{(\lambda + |x|)^{\frac{n - p - \mu}{n - p}} \frac{p}{p - r}} dx. \tag{14}
$$

Then, the identity (13) reduces to

$$
(-G'(\lambda))_{\frac{p}{q - p}} = \tilde{\Gamma} \left( \frac{r(p - 1) - (q - p)}{q - p} G(\lambda) + \lambda G'(\lambda) \right) G(\lambda)^{\frac{p(1 - a)}{aq}}, \tag{15}
$$

where

$$
\tilde{\Gamma} := C_{opt}(\mathbb{R}^n)^\frac{p}{a} \left( \frac{p(n - p - \mu)}{(n - p)(q - p)} \right)^\frac{p(1 - a)}{aq} \left( \frac{r(p - 1) - (q - p)}{q - p} \right)^{\frac{p(1 - a)}{aq}}.
$$

Next, in the scope of metric spaces, consider for each $\lambda > 0$ the sequence of functions $u_{\lambda,k} : X \to \mathbb{R}, k \in \mathbb{N}$ defined by

$$
u_{\lambda,k}(x) := \max\{0, \min\{0, k - d(x, x_0)\} + 1\} \left( \lambda + \max\left\{d(x, x_0), \frac{1}{k}\right\} \right)^{(\frac{p(1 - a)}{aq} - \frac{(p - 1)}{q - p})}.
$$

Note that since $(X, d)$ is proper, the set $supp(u_{\lambda,k}) = \{x \in X : d(x, x_0) \leq k + 1\}$ is compact. Therefore, $u_{\lambda,k} \in Lip_0(X)$ for all $\lambda > 0$ and $k \in \mathbb{N}$. Hence, we can apply $u_{\lambda,k}$ in (5) to get

$$
\left( \int_X d(x, x_0)^{\alpha p} |u_{\lambda,k}|^r d\mathbb{m}(x) \right)^{\frac{1}{r}} \leq C \left( \int_X d(x, x_0)^{\alpha p} |Du_{\lambda,k}|^p d\mathbb{m}(x) \right)^{\frac{a}{p}} \times \left( \int_X d(x, x_0)^{\beta q} |u_{\lambda,k}|^q d\mathbb{m}(x) \right)^{\frac{1-a}{q}}. \tag{16}
$$

Moreover, consider the limit

$$
u_{\lambda}(x) := \lim_{k \to \infty} u_{\lambda,k}(x) = \left( \lambda + d(x, x_0) \frac{n - p - \mu}{n - p} \frac{p}{p - r} \right)^{(\frac{p(1 - a)}{aq} - \frac{(p - 1)}{q - p})}.
$$

It follows from the dominated convergence theorem jointly with inequality (16) that $u_{\lambda}$ also satisfies (5), i.e.,

$$
\left( \int_X d(x, x_0)^{\alpha p} |u_{\lambda}|^r d\mathbb{m}(x) \right)^{\frac{1}{r}} \leq C \left( \int_X d(x, x_0)^{\alpha p} |Du_{\lambda}|^p d\mathbb{m}(x) \right)^{\frac{a}{p}} \times \left( \int_X d(x, x_0)^{\beta q} |u_{\lambda}|^q d\mathbb{m}(x) \right)^{\frac{1-a}{q}}. \tag{17}
$$

The non-smooth chain rule for the local lipschitz constant gives that

$$
|Du_{\lambda}|(x) = -\frac{p(n - p - \mu)}{(q - p)(n - p)} \left( \lambda + d(x, x_0) \frac{n - p - \mu}{n - p} \frac{p}{p - r} \right)^{(\frac{p(1 - a)}{aq} - \frac{(p - 1)}{q - p})}. \tag{18}
$$
\[
\frac{d(x, x_0)^{\frac{n-p(\mu+1)}{(n-p)p-1}}}{|Dd(\cdot, x_0)|}(x).
\]  

Taking into account (17) and (18), we derive that

\[
\left(\int_X d(x, x_0)^{\gamma r} \left(\lambda + d(x, x_0)\right)^{\frac{n-p-\mu}{n-p}} \, \frac{d\mu(x)}{q-p} \right)^{\frac{1}{\gamma r}} \leq C \left(\frac{p(n - p - \mu)}{(n - p)(q - p)}\right)^{\alpha} \left(\int_X d(x, x_0)^{\alpha p + \frac{p(n-p(\mu+1))}{(n-p)(p-1)}} \left(\lambda + d(x, x_0)\right)^{\frac{n-p-\mu}{n-p}} \, \frac{d\mu(x)}{q-p} \right)^{\frac{\alpha}{p}}
\]

(19)

We shall rewrite (19) by means of the function \(F : (0, \infty) \to \mathbb{R}\) defined by:

\[
F(\lambda) = \frac{q - p}{r(p - 1) - (q - p)} \int_0^\infty m\left\{ x : \frac{d(x, x_0)^{\gamma r}}{\left(\lambda + d(x, x_0)\right)^{\frac{n-p-\mu}{n-p}} \frac{\gamma r}{q-p}} > s \right\} ds.
\]  

(20)

By the process of change of variables of the form

\[
s = \frac{h^{\gamma r}}{\left(\lambda + h\right)^{\frac{n-p-\mu}{n-p}} \frac{\gamma r}{q-p}},
\]

we deduce that

\[
F(\lambda) = \frac{q - p}{r(p - 1) - (q - p)} \int_0^\infty m\left\{ x : d(x, x_0) < h \right\} h^{\gamma r - 1} \left[\gamma r \lambda + \left(\frac{pq}{q-p} \frac{n-p-\mu}{n-p} - \gamma r\right) h^{\frac{n-p-\mu}{n-p} \frac{p}{q-p}} \right] dh.
\]

(21)
The hypotheses (6) and (7) imply that \( \mathfrak{m}(B_1(x_0)) \leq Ah^n, \forall h > 0 \), for some positive constant \( A \in \mathbb{R} \), so we derive the following expression

\[
F(\lambda) \leq \frac{(q-p)A}{r(p-1)-(q-p)} \int_0^\infty h^{n+\gamma r-1} \left[ -\gamma r \lambda + \left( \frac{pq}{q-p} \frac{n-p-\mu}{n-p} - \gamma r \right) h^{-\frac{n-p-\mu}{n-p} \frac{p}{q-p}} \right] dh.
\]

On the other hand, from (3) and (4), we can get

\[
n + \gamma r - 1 > -1, \quad n + \gamma r - 1 - \left( \frac{n-p-\mu}{n-p} \right) \frac{pq}{q-p} < -1.
\]

Therefore, \( 0 \leq F(\lambda) < \infty, \forall \lambda > 0 \), and \( F \) is differentiable. Also, we have

\[
F'(\lambda) = -\int_X \frac{d(x, x_0)^\gamma r}{(\lambda + d(x, x_0)^{\frac{n-p-\mu}{n-p} \frac{p}{q-p}})(r(p-1))^\gamma r} d\mathfrak{m}(x). \tag{22}
\]

Now, similarly to (15), we can transform the relation (19) via \( F \) into the following inequality

\[
(-F'(\lambda))^\frac{p}{\overline{\sigma}} \leq \Gamma \left( \frac{r(p-1)-(q-p)}{q-p} F(\lambda) + \lambda F'(\lambda) \right) F(\lambda) \frac{p(1-a)}{aq}, \tag{23}
\]

where

\[
\Gamma := C^\frac{p}{\overline{\sigma}} \left( \frac{p(n-p-\mu)}{(n-p)(q-p)} \right)^p \left( \frac{r(p-1)-(q-p)}{q-p} \right) \frac{p(1-a)}{aq}.
\]

Inspired by (15) and (23), we consider the following ODE:

\[
(-H_0'(\lambda))^\frac{p}{\overline{\sigma}} = \Gamma \left( \frac{r(p-1)-(q-p)}{q-p} H_0(\lambda) + \lambda H_0'(\lambda) \right) H_0(\lambda) \frac{p(1-a)}{aq}. \tag{24}
\]

From (15), one can observe that (24) has the particular solution of the form

\[
H_0(\lambda) = \left( \frac{C_{opt}(\overline{\mathfrak{m}}^n)}{C} \right)^\frac{n}{\overline{\sigma}} G(\lambda). \tag{25}
\]

Now, we claim that if \( F(\lambda_0) < H_0(\lambda_0) \) for some \( \lambda_0 > 0 \), then \( F(\lambda) < H_0(\lambda), \forall \lambda \in (0, \lambda_0] \). Indeed, suppose that there exists some \( \lambda_1 \in (0, \lambda_0) \) such that \( F(\lambda_1) \geq H_0(\lambda_1) \) and set

\[
\lambda_2 := \sup\{ \lambda < \lambda_0; F(\lambda) \geq H_0(\lambda) \}.
\]

Therefore, \( F(\lambda) \leq H_0(\lambda) \) for all \( \lambda \in [\lambda_2, \lambda_0] \), and so, we have from (23) that

\[
(-F'(\lambda))^\frac{p}{\overline{\sigma}} \leq \Gamma \left( \frac{r(p-1)-(q-p)}{q-p} F(\lambda) + \lambda F'(\lambda) \right) F(\lambda) \frac{p(1-a)}{aq}
\]

\[
\leq \Gamma \left( \frac{r(p-1)-(q-p)}{q-p} H_0(\lambda) + \lambda F'(\lambda) \right) H_0(\lambda) \frac{p(1-a)}{aq}. \tag{26}
\]
For each \( \lambda > 0 \), consider the function \( \varphi_\lambda : [0, \infty) \to \mathbb{R} \) defined by
\[
\varphi_\lambda(t) = t^{\frac{p}{aq}} + t \lambda H_0(\lambda) \frac{p(1-a)}{aq}.
\]
Thus, by (24) and (26), we have
\[
\varphi_\lambda(-F'(\lambda)) = (-F'(\lambda))^{\frac{p}{aq}} - \Gamma \lambda F'(\lambda) H_0(\lambda) \frac{p(1-a)}{aq}
\leq \Gamma \left( \frac{r(p-1) - (q-p)}{q-p} \right) H_0(\lambda) H_0(\lambda) \frac{p(1-a)}{aq}
= (-H_0'(\lambda))^{\frac{p}{aq}} - \Gamma \lambda H_0'(\lambda) H_0(\lambda) \frac{p(1-a)}{aq}
= \varphi_\lambda(-H_0'(\lambda)).
\]
For each fixed \( \lambda > 0 \) we can easily notice that \( \varphi_\lambda \) is a non-decreasing function, so we conclude by the above inequality that
\[
-F'(\lambda) \leq -H_0'(\lambda), \quad \forall \lambda \in [\lambda_2, \lambda_0], \tag{27}
\]
consequently,
\[
0 \leq (F - H_0)(\lambda_2) \leq (F - H_0)(\lambda_0) < 0,
\]
which is a contradiction and proves the claim.

Proceeding, by the condition (7) we know that, given \( \epsilon > 0 \), there exists \( \delta > 0 \) such that
\[
h \leq \delta \Rightarrow (1 - \epsilon) m_E(\mathbb{B}_h(0)) \leq m(B_h(x_0)).
\]
It then follows that
\[
F(\lambda) = \frac{q-p}{r(p-1) - (q-p)} \int_0^\infty m(B_h(0)) h^{\gamma r - 1} \left[ -\gamma r \lambda + \left( \frac{pq}{q-p} \frac{n-p-\mu}{n-p} - \gamma r \right) H_0(\lambda) \frac{p(1-a)}{aq} \right] dh
\geq \frac{q-p}{r(p-1) - (q-p)} \int_0^\Delta m_E(\mathbb{B}_h(0)) h^{\gamma r - 1} \left[ -\gamma r \lambda + \left( \frac{pq}{q-p} \frac{n-p-\mu}{n-p} - \gamma r \right) H_0(\lambda) \frac{p(1-a)}{aq} \right] dh
\geq \Theta \int_0^\Delta m_E(\mathbb{B}_h(0)) h^{\gamma r - 1} \left[ -\gamma r + \left( \frac{pq}{q-p} \frac{n-p-\mu}{n-p} - \gamma r \right) H_0(\lambda) \frac{p(1-a)}{aq} \right] dh,
\]
where
\[
\Theta = \frac{(q-p)(1-\epsilon)}{r(p-1) - (q-p)} \lambda^{\frac{(q-p)(p-1)n-pq(p-1)}{p(q-p)}} \text{ and } \Delta = \frac{\delta}{\lambda^{\frac{(n-p)(p-1)}{p(n-p)}}}.
\]
On the other hand, from (14), we have
\[
G(\lambda) = \frac{q-p}{r(p-1) - (q-p)} \int_{\mathbb{R}^n} |x|^{\gamma r} \left( \lambda + |x| \frac{n-p-\mu}{n-p} \right)^{\frac{q(p-1)}{q-p}} d m_E(x),
\]
\[
\frac{q - p}{r(p - 1) - (q - p)} \int_0^\infty m_E(B_h(0)) h^{\gamma r - 1} \left[ -\gamma r + \left( \frac{pq}{q-p} \frac{n-p-\mu}{n-p} - \gamma r \right) s^{\frac{n-p-\mu}{q-p-\mu}} \right] dh
\]
\[
= \frac{\Theta}{1 - \epsilon} \int_0^\infty m_E(B_s(0)) s^{\gamma r - 1} \left[ -\gamma r + \left( \frac{pq}{q-p} \frac{n-p-\mu}{n-p} - \gamma r \right) s^{\frac{n-p-\mu}{q-p-\mu}} \right] \frac{(p-1)}{q} + 1 ds.
\]

Therefore, it is easy to observe that
\[
\frac{F(\lambda)}{G(\lambda)} \geq (1 - \epsilon) \frac{\int_0^\Delta m_E(B_s(0)) s^{\gamma r - 1} \left[ -\gamma r + \left( \frac{pq}{q-p} \frac{n-p-\mu}{n-p} - \gamma r \right) s^{\frac{n-p-\mu}{q-p-\mu}} \right] \frac{(p-1)}{q} + 1 ds}{\int_0^\infty m_E(B_s(0)) s^{\gamma r - 1} \left[ -\gamma r + \left( \frac{pq}{q-p} \frac{n-p-\mu}{n-p} - \gamma r \right) s^{\frac{n-p-\mu}{q-p-\mu}} \right] \frac{(p-1)}{q} + 1 ds},
\]

and from which, one can obtain
\[
\lim_{\lambda \to 0} \frac{F(\lambda)}{G(\lambda)} \geq 1 - \epsilon.
\]

Letting \( \epsilon \to 0 \), we get
\[
\lim_{\lambda \to 0} \frac{F(\lambda)}{G(\lambda)} \geq 1.
\]

Let us separate the proof into two cases.

Case 1: \( C > C_{opt}(\mathbb{R}^n) \). From (25) and (30), it follows that
\[
\lim_{\lambda \to 0} \frac{F(\lambda)}{H_0(\lambda)} = \lim_{\lambda \to 0} \frac{F(\lambda)}{G(\lambda)} \left( \frac{C}{C_{opt}(\mathbb{R}^n)} \right)^{\frac{n}{a}} \geq \left( \frac{C}{C_{opt}(\mathbb{R}^n)} \right)^{\frac{n}{a}} > 1.
\]

The above claim implies that
\[
F(\lambda) \geq H_0(\lambda), \quad \forall \lambda > 0,
\]
that is,
\[
F(\lambda) \geq \left( \frac{C_{opt}(\mathbb{R}^n)}{C} \right)^{\frac{n}{a}} G(\lambda), \quad \forall \lambda > 0.
\]
Consequently, by the expressions (21) and (28) we deduce

\[ \int_0^\infty [m(B_h(x_0)) - d_1 m_E(\mathbb{B}_h(0))] \psi(h) dh \geq 0, \tag{31} \]

where

\[ d_1 := \left( \frac{C_{opt}(\mathbb{R}^n)}{C} \right)^{\frac{n}{r}}, \quad \psi(h) = h^{\gamma r - 1} \left[ -\gamma r \lambda + \left( \frac{pq}{q-p} \frac{n-p}{n-p} - \gamma r \right) h^{\frac{n-p}{n-p} \frac{p}{r+1}} \right]. \]

Now, fix \( x \in X \), since \( B_{R-d(x_0,x)}(x_0) \subset B_R(x) \subset B_{R+d(x_0,x)}(x_0) \) for every \( R > d(x_0,x) \), we deduce from (6) that

\[ C_0 \frac{m(B_R(x))}{m_E(\mathbb{B}_R(0))} \geq \limsup_{R \to \infty} \frac{m(B_R(x))}{m_E(\mathbb{B}_R(0))} = \limsup_{R \to \infty} \frac{m(B_R(x_0))}{m_E(\mathbb{B}_R(0))} =: d_0, \quad \forall \rho > 0. \]

Note that to prove (8) in the case \( C_{opt}(\mathbb{R}^n) < C \) it is sufficient to prove that \( d_1 \leq d_0 \). We argue by contradiction, suppose that \( d_0 < d_1 \), then by definition of \( d_0 \), there exists \( \epsilon_0 > 0 \) such that for some \( h_0 > 0 \),

\[ \frac{m(B_h(x_0))}{m_E(\mathbb{B}_h(0))} \leq d_1 - \epsilon_0, \quad \forall h \geq h_0. \tag{32} \]

It follows from (6) and (7) that

\[ m(B_h(x_0)) \leq C_0 m_E(\mathbb{B}_h(0)). \tag{33} \]

Hence, substituting (32) into (31) and considering inequality (33), we have

\[
\begin{align*}
0 & \leq \int_0^\infty [m(B_h(x_0)) - d_1 m_E(\mathbb{B}_h(0))] \psi(h) dh \\
& \leq \int_0^{h_0} m(B_h(x_0)) \psi(h) dh + (d_1 - \epsilon_0) \int_{h_0}^\infty m_E(\mathbb{B}_h(0)) \psi(h) dh \\
& \quad - d_1 \int_0^\infty m_E(\mathbb{B}_h(0)) \psi(h) dh \\
& \leq C_0 \int_0^{h_0} m_E(\mathbb{B}_h(0)) \psi(h) dh - d_1 \int_0^{h_0} m_E(\mathbb{B}_h(0)) \psi(h) dh \\
& \quad - \epsilon_0 \int_{h_0}^\infty m_E(\mathbb{B}_h(0)) \psi(h) dh \\
& = (C_0 - d_1 + \epsilon_0) \int_0^{h_0} m_E(\mathbb{B}_h(0)) \psi(h) dh - \epsilon_0 \int_0^\infty m_E(\mathbb{B}_h(0)) \psi(h) dh \\
& = (C_0 - d_1 + \epsilon_0) \int_0^{h_0} m_E(\mathbb{B}_h(0)) \psi(h) dh - \epsilon_0 \left( \frac{r(p-1) - (q-p)}{q-p} \right) G(\lambda),
\end{align*}
\tag{34} \]
Since $\lambda \leq (\lambda + h \frac{n-p-\mu}{n-p} \frac{p}{p-1})$, we have that $\frac{1}{(\lambda + h \frac{n-p-\mu}{n-p} \frac{p}{p-1})} \leq \frac{1}{\lambda}$, and then

$$
\int_0^{h_0} h^n \psi(h) dh \\
= \int_0^{h_0} h^{n+\gamma r - 1} \left[ -\gamma r \lambda + \left( \frac{pq}{q-p} \frac{n-p-\mu}{n-p} - \gamma r \right) h^{\frac{n-p-\mu}{n-p} \frac{p}{p-1}} \right] dh \\
\leq \int_0^{h_0} h^{n+\gamma r - 1} \left[ -\gamma r \lambda + \left( \frac{pq}{q-p} \frac{n-p-\mu}{n-p} - \gamma r \right) h^{\frac{n-p-\mu}{n-p} \frac{p}{p-1}} \right] dh \\
= \lambda^{-\frac{q(p-1)-q(1-p)}{q-p}} \int_0^{h_0} h^{n+\gamma r - 1} \left[ -\gamma r \lambda h_0^{n+\gamma r} + \left( \frac{pq}{q-p} \frac{n-p-\mu}{n-p} - \gamma r \right) h_0^{n+\gamma r} \right] dh \\
= \lambda^{-\frac{q(p-1)-q(1-p)}{q-p}} \left[ -\gamma r \lambda h_0^{n+\gamma r} + \left( \frac{pq}{q-p} \frac{n-p-\mu}{n-p} - \gamma r \right) h_0^{n+\gamma r} \right]. 
$$

Substituting (35) and the expression $G(\lambda) = \lambda^{\frac{(q-p)(p-1)-npq(p-1)}{p(q-p)}} G(1)$ into (34), we have

$$
\varepsilon_0 \left( \frac{r(p-1)-(q-p)}{q-p} \right) G(1) \leq \lambda^{q} \left[ -\gamma r \lambda h_0^{n+\gamma r} + \left( \frac{pq}{q-p} \frac{n-p-\mu}{n-p} - \gamma r \right) h_0^{n+\gamma r} \right]. 
$$

where

$$
\eta = -\frac{q(p-1)}{q-p} - 1 = -\frac{(q-p)(p-1)n-pq(p-1)}{p(q-p)}. 
$$

However,

$$
\eta < 0 \quad \text{since} \quad \eta + 1 = -\frac{n(p-1)}{p} < 0.
$$

Then, letting $\lambda \to \infty$ one obtains a contradiction by (36). This completes the proof of Theorem 1.3 in the case $C > C_{opt}(\mathbb{R}^n)$.

Case 2: $C = C_{opt}(\mathbb{R}^n)$. In this case we have for any fixed $\delta > 0$ that

$$
\left( \int_X d(x,x_0)^{\alpha'} |u'|^p d\mathcal{m}(x) \right)^{\frac{1}{p}} \\
\leq (C_{opt}(\mathbb{R}^n) + \delta) \left( \int_X d(x,x_0)^{\alpha p} |Du|^p d\mathcal{m}(x) \right)^{\frac{1}{p}} \left( \int_X d(x,x_0)^{\beta q} |u|^q d\mathcal{m}(x) \right)^{\frac{1}{q}}.
$$

Therefore, from Case 1, we get

$$
\mathcal{m}(B_\rho(x)) \geq C_0^{-1} \left( \frac{C_{opt}(\mathbb{R}^n)}{C_{opt}(\mathbb{R}^n) + \delta} \right)^{\frac{n}{q}} \mathcal{m}_E(B_\rho(0)), \quad \forall \rho > 0, \text{ and } x \in X.
$$

Letting $\delta \to 0$, one obtains that

$$
\mathcal{m}(B_\rho(x)) \geq C_0^{-1} \mathcal{m}_E(B_\rho(0)), \quad \forall \rho > 0, \text{ and } x \in X.
$$

This completes the proof of Theorem 1.3.
3. Proof of Theorem 1.4

Proof. We argue by contradiction, suppose that $C < C_{opt}(\mathbb{R}^n)$ and

$$\left( \int_M d(x)^{\gamma r} |u|^r dv \right)^{\frac{1}{r}} \leq C \left( \int_M d(x)^{\alpha p} |\nabla u|^p dv \right)^{\frac{a}{p}} \left( \int_M d(x)^{\beta q} |u|^q dv \right)^{\frac{1-a}{q}},$$

$\forall u \in C_0^\infty(M)$. \hfill (37)

Given $\epsilon > 0$, there exist a chart $(\Omega, \phi)$ of $M$ at $x_0$ and a $\delta > 0$ such that $\phi(\Omega) = \mathbb{B}_\delta(0), \phi(x_0) = 0$, and that the components $g_{ij}$ of $g$ in this chart satisfy

$$\frac{1}{1 + \epsilon} \delta_{ij} \leq g_{ij} \leq (1 + \epsilon) \delta_{ij}$$

in the sense of bilinear form (see [3]). We claim that by choosing $\epsilon > 0$ small enough we get by (37) that there exist $\delta_0 > 0$ and $C' < C_{opt}(\mathbb{R}^n)$ such that, $\forall f \in C_0^\infty(B_{\delta_0}(0))$

$$\left( \int_{B_{\delta_0}(0)} |x|^\gamma |f|^r dx \right)^{\frac{1}{r}} \leq C' \left( \int_{B_{\delta_0}(0)} |x|^\alpha |\nabla f|^p dx \right)^{\frac{a}{p}} \left( \int_{B_{\delta_0}(0)} |x|^\beta |f|^q dx \right)^{\frac{1-a}{q}}.$$

Indeed, if $f \in C_0^\infty(B_{\delta_0}(0))$, then $u := f \circ \exp_{x_0}^{-1} \in C_0^\infty(\Omega)$. Substituting $u$ into (37) and using the metric estimates (38) we obtain (39), where $C' = (1 + \epsilon)^{\frac{a}{p} + \frac{\alpha(1-a)}{q} + \frac{\beta(1-a)}{2} + \frac{\gamma}{r}} C$. Since $C < C_{opt}(\mathbb{R}^n)$ we know that, if $\epsilon$ is small enough, then $C' < C_{opt}(\mathbb{R}^n)$. This proves our claim.

Let $u \in C_0^\infty(\mathbb{R}^n)$ and set $u_\lambda(x) = u(\lambda x), \lambda > 0$. For $\lambda$ large enough $u_\lambda(x) \in C_0^\infty(B_{\delta_0}(0))$. Substituting $u_\lambda$ into (39), we get

$$\left( \int_{\mathbb{R}^n} |x|^\gamma |u_\lambda|^r dx \right)^{\frac{1}{r}} \leq C' \left( \int_{\mathbb{R}^n} |x|^\alpha |\nabla u_\lambda|^p dx \right)^{\frac{a}{p}} \left( \int_{\mathbb{R}^n} |x|^\beta |u_\lambda|^q dx \right)^{\frac{1-a}{q}}.$$ \hfill (40)

Using a change of variables, we have

$$\left( \int_{\mathbb{R}^n} |x|^\gamma |u_\lambda(x)|^r dx \right)^{\frac{1}{r}} = \lambda^{-\frac{a}{p}} \lambda^{-\gamma} \left( \int_{\mathbb{R}^n} |y|^\gamma |u(y)|^r dy \right)^{\frac{1}{r}},$$

$$\left( \int_{\mathbb{R}^n} |x|^\alpha |\nabla u_\lambda(x)|^p dx \right)^{\frac{a}{p}} = \lambda^{-\frac{a}{p}} \lambda^{\alpha a} \lambda a \left( \int_{\mathbb{R}^n} |y|^\alpha |\nabla u(y)|^p dy \right)^{\frac{a}{p}},$$

and

$$\left( \int_{\mathbb{R}^n} |x|^\beta |u_\lambda(x)|^q dx \right)^{\frac{1-a}{q}} = \lambda^{-\frac{a}{r} - \frac{\alpha(1-a)}{q}} \lambda^{-\beta(1-a)} \left( \int_{\mathbb{R}^n} |y|^\beta |u(y)|^q dy \right)^{\frac{1-a}{q}}.$$\hfill (39)

Combining the above equations with (40), we get

$$\left( \int_{\mathbb{R}^n} |y|^\gamma |u(y)|^r dy \right)^{\frac{1}{r}} \leq C' \lambda^{-\alpha a - \frac{\alpha a}{r} + a - \frac{\alpha(1-a)}{q} + \frac{\beta(1-a)}{p} + \frac{\gamma}{r} + \gamma}$$
\[
\left( \int_{\mathbb{R}^n} |y|^{\alpha p} |\nabla u(y)|^p dy \right)^{\frac{1}{p}} \leq \left( \int_{\mathbb{R}^n} |y|^{\beta q} |u(y)|^q dy \right)^{\frac{1}{q}}.
\]

It follows from the conditions of Theorem 1.1 that
\[-\alpha a - \frac{na}{p} + a - \frac{n(1 - a)}{q} - \beta(1 - a) + \frac{n}{r} + \gamma = 0.\]

Hence,
\[
\left( \int_{\mathbb{R}^n} |y|^\gamma |u(y)|^r dy \right)^{\frac{1}{r}} \leq C'( \int_{\mathbb{R}^n} |y|^{\alpha p} |\nabla u(y)|^p dy \right)^{\frac{a}{p}} \left( \int_{\mathbb{R}^n} |y|^{\beta q} |u(y)|^q dy \right)^{\frac{1-a}{q}}.
\]

This expression contradicts the fact that \( C_{\text{opt}}(\mathbb{R}^n) \) is the best constant for this inequality on \( \mathbb{R}^n \).

4. Proof of Theorems 1.7, 1.8 and 1.9

In this section, we will briefly mention some basic definitions and notions in Finsler geometry. There are many good references in the subject, we refer readers to [5, 38].

4.1. Finsler geometry

**Definition 4.1. (Finslerian Structure)** A Finsler manifold is a pair \((M^n, F)\) consisting of a connected \(C^\infty\) manifold and a continuous function \(F : TM \rightarrow [0, \infty)\) satisfying the following properties

- \(F \in C^\infty(TM - \{0\})\);
- \(F(x, ty) = tF(x, y), \forall t \geq 0\) and \((x, y) \in TM\);
- The \(n \times n\) matrix \((g_{ij}) := \left( \frac{1}{2} F^2 \frac{\partial}{\partial y^i} \frac{\partial}{\partial y^j} \right), y = \sum_{i=1}^{n} y^i \frac{\partial}{\partial x^i}, \) (41)

is positive definite for all \((x, y) \in TM - \{0\} \).

A Finsler manifold \((M, F)\) is called a locally Minkowski space if there exists a certain privileged local coordinate system \((x^i)\) on \(M\) such that in each coordinated neighborhood we have that \(F(x, y)\) depends only on \(y\) and not on \(x\). On the other hand, a Minkowski space consists of a finite dimensional vector space \(V\) and a Minkowski norm which induces a Finsler metric on \(V\) by translation.

We consider on the pull-back bundle \(\pi^*TM\) the Chern connection (see Bao et al. [5], Theorem 2.4.1). The coefficients of the Chern connection are given by

\[
\Gamma^i_{jk}(x, y) = \frac{1}{2} g^{il} \left( \frac{\partial g_{lj}}{\partial x^k} - \frac{\partial g_{lk}}{\partial x^j} + \frac{\partial g_{lj}}{\partial y^r} G^r_k + \frac{\partial g_{lk}}{\partial y^r} G^r_j - \frac{\partial g_{kl}}{\partial y^r} G^r_j \right),
\]
where \( G^i_j = \frac{\partial G^i}{\partial y^j} \) and

\[
G^i(x, y) = \frac{1}{4} g^{ik} \left( 2 \frac{\partial g_{jk}}{\partial x^l} - \frac{\partial g_{jl}}{\partial x^k} \right) y^l y^j.
\]

From the Chern connection, we consider the following space

**Definition 4.2. (Berwaldian Structure)** A Finsler manifold is a Berwald space if the coefficients \( \Gamma^i_{jk}(x, y) \) given by expression (42) in natural coordinates are independent of \( y \).

A geodesic between two points \( x, y \in M \) is a smooth curve \( \tau : [0, 1] \to \mathbb{R} \) of constant speed minimizing the following functional

\[
\sigma \mapsto L_F(\tau) = \int_0^1 F(\tau, \dot{\tau}) \, dt,
\]

and the distance function is given by \( d_F(x_1, x_2) := \inf_{\tau} L_F(\tau) \), where \( \tau \) varies over all smooth curves connecting \( x_1 \) to \( x_2 \). A Finsler manifold \((M, F)\) is said to be complete if any geodesic \( \tau : [0, l] \to M \) can be extended to a geodesic \( \tau : \mathbb{R} \to M \).

Let \( \tau : [0, l] \to M \) be a geodesic with velocity field \( \dot{\tau} \). A vector field \( J \) along \( \tau \) is said to be a *Jacobi field* if it satisfies the equation

\[
D^\dot{\tau} D^\dot{\tau} J + R^\dot{\tau} (J, \dot{\tau}) \dot{\tau} = 0, \tag{43}
\]

where \( D^\dot{\tau} \) is the covariant derivative with reference vector \( \dot{\tau} \), and \( R^\dot{\tau} \) is the curvature tensor (see [5] for details).

For a flag \( P := span\{v, w\} \subset T_x M \), with flag pole \( v \), the *flag curvature* is defined by

\[
K(P, v) := \frac{\langle R^v(w, v)v, w \rangle_v}{F(v)^2 \langle w, w \rangle_v - \langle v, w \rangle_v^2},
\]

where \( \langle , \rangle_v \) denotes the inner product induced by (41). In the Riemannian case the flag curvature reduces to the sectional curvature which depends only on \( P \).

Consider \( v \in T_x M \) with \( F(x, v) = 1 \) and let \( \{e_i\}_{i=1}^n \) with \( e_n = v \) be an orthonormal basis of \((T_x M, \langle , \rangle_v)\). Put \( P_i = span\{e_i, v\} \) for \( 1 \leq i \leq n - 1 \). Then the Ricci curvature of \( v \) is defined by

\[
Ric(v) := \sum_{i=1}^{n-1} K(P_i, v).
\]

For \( c \geq 0 \), we also set \( Ric(cv) := c^2 Ric(v) \).

Motivated by the work of Lott-Villani [26] and Sturm [40] on metric measure spaces, Ohta in [32] introduced the notion of weighted Ricci curvature on Finsler manifolds as follows; consider \( m \) be a positive measure on \((M, F)\), given a unit vector \( v \in T_x M \) extend it to a \( C^\infty \) vector field \( V \) on a neighborhood \( U_x \) of \( x \) such that every integral curve is a geodesic, and decompose \( m \) as \( m = e^{-\psi} \text{vol}_V \) on \( U_x \), where \( \text{vol}_V \) denotes the volume form of the Riemannian structure \( g_V \).
Definition 4.3. (Weighted Ricci Curvature) For \( N \in [n, \infty] \) and a unit vector \( v \in T_x M \) the \( N \)-Ricci curvature \( \mathit{Ric}_N \) is defined by

1. \( \mathit{Ric}_n (v) := \left\{ \begin{array}{ll} \mathit{Ric}(v) + (\psi \circ \sigma)''(0) & \text{if } (\psi \circ \sigma)'(0) = 0; \\ -\infty & \text{otherwise} \end{array} \right. \)
2. \( \mathit{Ric}_N (v) := \mathit{Ric}(v) + (\psi \circ \sigma)''(0) - \frac{(\psi \circ \sigma)'(0)^2}{N-n} \) for \( N \in (n, \infty); \)
3. \( \mathit{Ric}_\infty (v) := \mathit{Ric}(v) + (\psi \circ \sigma)''(0). \)

For \( c \geq 0 \), we also define \( \mathit{Ric}_N (cv) := c^2 \mathit{Ric}_N (v). \)

Inspired by the \( \mathit{Ric}_N \) concept, Ohta in \([32]\) proved the following Bishop-Gromov-type volume comparison theorem.

Theorem 4.1. ([32], Theorem 7.3) Let \( (M, F, m) \) be a complete \( n \)-dimensional Finsler manifold with non-negative \( N \)-Ricci curvature and \( N \in [n, \infty) \). Then we have

\[
\frac{m(B_R(x))}{m(B_\rho(x))} \leq \left( \frac{R}{\rho} \right)^N, \quad \forall x \in M, \quad \text{and} \quad 0 < \rho < R.
\] (44)

Moreover, if equality holds with \( N = n \) for all \( x \in M \) and \( 0 < r < R \), then any Jacobi field \( J \) along a geodesic \( \tau \) has the form \( J(t) = t P(t) \), where \( P \) is a parallel vector field along \( \tau \).

Proof of Theorem 1.7. Since \( (M, F) \) is complete, from the Hopf-Rinow theorem it yields that \( (M, d_F, m) \) is a proper metric measure space. On account of Theorem 4.1, we have that the condition (6) in Theorem 1.3 holds with \( C_0 = 1 \). Moreover

\[
\frac{m(B_R(x))}{\omega_n R^n} \leq \frac{m(B_\rho(x))}{\omega_n \rho^n} = \frac{m(B_\rho(x))}{m(E(B_\rho(0)))}, \quad 0 < \rho < R.
\]

Taking \( \rho \to 0 \), we have from

\[
\liminf_{\rho \to 0} \frac{m(B_\rho(x))}{\omega_n \rho^n} = 1,
\]

that \( m(B_R(x)) \leq \omega_n R^n, \forall x \in M \) and \( R > 0 \).

Here we use (8) to see \( m(B_R(x)) \geq \omega_n R^n, \forall x \in M \) and \( R > 0 \). Thus, \( m(B_R(x)) = m(E(B_R(0))), \forall x \in M \) and \( R > 0 \). By Theorem 4.1, it results that every Jacobi field \( J \) along any geodesic \( \tau \) has the form \( J(t) = t P(t) \), where \( P \) is a parallel vector field along \( \tau \). Hence, from the Jacobi equation (43), we get

\[
\ddot{R}(J, \dot{\tau}) \equiv 0, \text{ so that } K(P, \dot{\tau}) \equiv 0 \quad \text{with } P = \text{span } \{P, \dot{\tau}\}. \]

Due to the arbitrariness of \( \tau \) and \( J \), it turns out that the flag curvature of \((M, F)\) is identically zero.

Proof of Theorem 1.8. Since \( (M, F) \) is a Berwald space, the non-negativity of the Ricci curvature on \((M, F)\) coincides with the non-negativity of the \( n \)-Ricci curvature on \((M, d_F, m_{BH})\), and the Busemann-Hausdorff measure \( m_{BH} \) satisfies the following \( n \)-density assumption

\[
\lim_{\rho \to 0} \frac{m_{BH}(B_\rho(x))}{\omega_n \rho^n} = 1,
\]
see Shen [37] Lemma 5.2. Then, applying Theorem 1.7, we get that the flag curvature of \((M, F)\) is identically zero. On the other hand, every Berwald space with zero flag curvature is necessarily a locally Minkowski space, see [5] section 10.5. Due to the volume identity \(m_B H(B_\rho(x)) = \omega_n \rho^n, \forall x \in M \) and \(\rho > 0\), we have that \((M, F)\) must be isometric to a Minkowski space.

**Proof of Theorem 1.9.** Since \((M, F)\) is complete, by the Hopf-Rinow theorem it yields that \((M, d_F, m)\) is a proper metric measure space. The non-negativity of the flag curvature implies that the Ricci curvature is non-negative, then in the same way as in the proof of Theorem 1.8, we have that the \(n\)-Ricci curvature is non-negative and, by Theorem 4.1, the condition (6) in Theorem 1.3 holds with \(C_0 = 1\). Now, since the Busemann-Hausdorff measure \(m_B H\) satisfies

\[
\lim_{\rho \to 0} \frac{m_B H(B_\rho(x))}{\omega_n \rho^n} = 1,
\]

we have by Theorem 1.3 that

\[
0 < \left(\frac{C_{opt}(\mathbb{R}^n)}{C}\right)^\frac{2}{n} \leq \frac{m_B H(B_\rho(x_0))}{\omega_n \rho^n} \leq 1, \quad \forall \rho > 0.
\]

This inequality implies that \((M, F)\) has large volume growth as defined by Lakzian (see [22] Definition 3.6), then by Remark 1.2, \(M\) is diffeomorphic to Euclidean space \(\mathbb{R}^n\).

**5. Proof of Theorems 1.5 and 1.6**

**5.1. Geometry on Alexandrov spaces**

In this section, by completeness, we define the notion of Alexandrov space. First, let us remember that a length space \((X, d_X)\) is a metric space where the distance function \(d_X\) between two points is given by the infimum of the lengths of all the curves connecting these two points. A triangle in \((X, d_X)\) consists of three points \(x, y, z\) and three minimal geodesics \(xy, xz, yz\). Fix a real number \(\kappa \in \mathbb{R}\), a comparison triangle \(\tilde{\kappa} \tilde{\kappa} \tilde{\kappa} \) is a triangle on the surface of constant curvature \(\kappa\), with the same side lengths. We denote the comparison angle \(\tilde{\kappa} \tilde{\kappa} \tilde{\kappa} \) by \(\tilde{\kappa} \tilde{\kappa} \tilde{\kappa} \). A comparison triangle exists and is unique up to an isometry whenever \(\kappa \leq 0\), or \(\kappa > 0\) and \(d_X(x, y) + d_X(x, z) + d_X(z, y) < \frac{2\pi}{\sqrt{\kappa}}\).

**Definition 5.1.** An length space \(X\) is called an Alexandrov space of curvature \(\geq \kappa\) if any \(x_0 \in X\) has a neighborhood \(U_{x_0}\), such that for any \(x, y, z, w \in U_{x_0}\)

\[
\tilde{\kappa} \tilde{\kappa} \tilde{\kappa} y x z + \tilde{\kappa} \tilde{\kappa} \tilde{\kappa} z x w + \tilde{\kappa} \tilde{\kappa} \tilde{\kappa} w x y \leq 2\pi.
\]

For locally compact spaces this is equivalent to the more familiar Alexandrov-Toponogov distance comparison.
Definition 5.2. (Toponogov-Alexandrov) A locally compact length space $X$ is called an Alexandrov space of curvature $\geq \kappa$ if any $x_0 \in X$ has a neighborhood $U_{x_0}$, such that for any triangle $xyz$ in $U_{x_0}$ and any $y_1 \in x\bar{y}$, $z_1 \in x\bar{z}$, we have $|y_1z_1| \geq |\tilde{y}_1\tilde{z}_1|$, where $\tilde{y}_1$ and $\tilde{z}_1$ are the corresponding points on the sides $\tilde{x}\tilde{y}$ and $\tilde{x}\tilde{z}$ of the comparison triangle $\tilde{x}\tilde{y}\tilde{z}$.

Remark 5.1. If $X$ is complete, the local condition in the above definitions implies a global condition.

Similar to the Bishop–Gromov comparison theorem in Riemannian manifolds, there is an extension for Alexandrov spaces. The next result can be found in [6] (see Theorem 10.6.6).

Theorem 5.1. (Bishop–Gromov Inequality) Let $X$ be an locally compact $n$-dimensional Alexandrov space of curvature $\geq \kappa$. Then for any $x \in X$ the function

$$
\rho \rightarrow \frac{\mathcal{H}^n(B_{\rho}(x))}{V^\kappa_{\rho}}
$$

is not increasing, where $\mathcal{H}^n$ is the n-dimensional Hausdorff measure and $V^\kappa_{\rho}$ is the volume of a ball of radius $\rho$ in the n-dimensional space form $M^\kappa_n$ of constant curvature $\kappa$. i.e., if $R \geq \rho > 0$, then

$$
\frac{\mathcal{H}^n(B_R(x))}{V^\kappa_{R}} \leq \frac{\mathcal{H}^n(B_{\rho}(x))}{V^\kappa_{\rho}}.
$$

Next, Kuwae et al in [21] define the concept of infinitesimal Bishop-Gromov inequality for Alexandrov spaces as follows. For a real number $\kappa$, consider

$$
s_\kappa(\rho) = \begin{cases} 
\frac{\sin(\sqrt{\kappa}\rho)}{\sqrt{\kappa}}, & \text{if } \kappa > 0, \\
\rho, & \text{if } \kappa = 0, \\
\frac{\sinh(\sqrt{\kappa}\rho)}{\sqrt{\kappa}}, & \text{if } \kappa < 0,
\end{cases}
$$

observe that the function $s_\kappa$ is the solution of Jacobi equation $s''_\kappa(\rho) + \kappa s'_\kappa(\rho) = 0$ with initial conditions $s_\kappa(0) = 0$ and $s'_\kappa(0) = 1$. Let $d_{x_0}(x) := d(x_0, x)$, where $x_0, x \in X$ and $d$ is the distance function. For $x_0 \in X$ and $0 < t \leq 1$, we define the set $W_{x_0,t} \subset X$ and the map $\Phi_{x_0,t} : W_{x_0,t} \rightarrow X$ as follows: First, put $\Psi_{x_0,t}(x_0) = x_0 \in W_{x_0,t}$. A point $x(\neq x_0)$ belongs to $W_{x_0,t}$ if, and only if, there exists $y \in X$ such that $x \in x_0y$ and $d_{x_0}(x) : d_{x_0}(y) = t : 1$, where $x_0y$ is a minimal geodesic connecting $x_0$ to $y$. Since a geodesic does not branch on an Alexandrov space, for a given point $x \in W_{x_0,t}$ such a point $y$ is unique and we set $\Psi_{x_0,t}(x) = y$. Now we are in a position to define the notion of infinitesimal Bishop-Gromov inequality.

Definition 5.3. Given real numbers $n \geq 1$ and $\kappa$, we say that the $n$-dimensional Hausdorff measure $\mathcal{H}^n$ satisfies the Bishop-Gromov infinitesimal inequality $BG(\kappa, n)$ if for any $x_0 \in X$ and $t \in (0, 1]$ we have

$$
d(\Psi_{x_0,t} \ast \mathcal{H}^n)(x) \geq \frac{ts_\kappa(t d_{x_0}(x))^{n-1}}{s_\kappa(d_{x_0}(x))^{n-1}} d\mathcal{H}^n(x),
$$

for all $x \in X$ such that $d_{x_0}(x) < \frac{\pi}{\sqrt{\kappa}}$ if $\kappa > 0$, where $\Psi_{x_0,t} \ast \mathcal{H}^n$ is the push-forward of $\mathcal{H}^n$ by $\Psi_{x_0,t}$.
$BG(κ, n)$ is sometimes called the measure contraction property (see [21,32,40]) and is weaker than the curvature dimension(or lower $n$-Ricci curvature) condition $CD((n−1)κ, n)$ introduced by Sturm [40].

In [21], the authors provide the following result.

**Theorem 5.2.** Let $X$ be an $n$-dimensional Alexandrov space of curvature $≥ κ$. Then, the $n$-dimensional Hausdorff measure $H^n$ on $X$ satisfies the infinitesimal Bishop-Gromov condition $BG(κ, n)$.

Let us denote by $Alex^n[κ]$ the class of $n$-dimensional Alexandrov spaces of curvature $≥ κ$. In [30], see Theorem 3.2, the author proved the following.

**Theorem 5.3.** For an integer $n ≥ 2$, let $(X, d) ∈ Alex^n[−κ^2], κ ∈ R$ be a complete non-compact Alexandrov space whose Hausdorff measure $H^n$ satisfies the $BG(0, n)$ condition. There exists an $ε(n, κ) = ε > 0$ such that, if $x ∈ X$

$$H^n(B_ρ(x)) ≥ (1 − ε)ω_nρ^n, \ ∀ ρ > 0,$$

then $(X,d)$ has finite topological type.

**Proof of Theorem 1.5.** Since $X$ is locally compact and complete, we have that closed and bounded subsets of $X$ are compact. Thus $X$ is a proper space. Now, since $X$ has curvature $≥ 0$, we can apply Theorem 5.1 to get

$$\frac{\lambda H^n(B_R(x))}{\lambda H^n(B_ρ(x))} = \frac{H^n(B_R(x))}{H^n(B_ρ(x))} ≤ \frac{R^n}{ρ^n}, \quad x ∈ X, \quad 0 < ρ < R.$$

Thus, the condition (6) of Theorem 1.3 is satisfied with $C_0 = 1$. By Lemma 3.2 of [39], we get

$$\lim_{ρ → 0} \frac{H^n(B_ρ(x_0))}{ρ^n} = H^n(B_1(x_0)).$$

Thus,

$$\liminf_{ρ → 0} \frac{\lambda H^n(B_ρ(x_0))}{ω_nρ^n} = \frac{H^n(B_1(x_0))}{ω_n} = 1.$$

Applying Theorem 1.3, we deduce that

$$\lambda H^n(B_ρ(x_0)) = ω_nρ^n, \quad ∀ ρ > 0$$

and $B_ρ(x_0)$ is isometric to a ball of radius $ρ$ in $R^n$ for any $ρ > 0$, which is equivalent to say that $X$ is isometric to $R^n$ since [6](see Section 10.6). 

**Proof of Theorem 1.6.** Consider $ε > 0$ given by Theorem 5.3. Since the function $φ : [0, 1] → R$ defined by

$$φ(t) := \left( \frac{C_{opt}(R^n)}{C_{opt}(R^n) + t} \right)^{\frac{n}{2}},$$

converges to 1 when $t → 0$, we have that there exists $δ > 0$ such that

$$0 < t ≤ δ \implies 1 − ε ≤ \left( \frac{C_{opt}(R^n)}{C_{opt}(R^n) + t} \right)^{\frac{n}{2}}.$$

Then, applying Theorems 1.3, 5.2 and 5.3 we get the result.
Acknowledgements The authors are very grateful to referees for the valuable suggestions which lead to improvements in the manuscript.

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