POINCARÉ POLYNOMIALS OF HYPERQUOT SCHEMES

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1. INTRODUCTION

In this paper, we find generating functions for the Poincaré polynomials of hyperquot schemes for all partial flag varieties. These generating functions give the Betti numbers of hyperquot schemes, and thus give dimension information for the cohomology ring of every hyperquot scheme.

Let \( F(n; s) \) denote the partial flag variety corresponding to flags of the form:

\[
V_1 \subset V_2 \subset \ldots \subset V_l \subset V = \mathbb{C}^n
\]

with \( \dim V_i = s_i \). The space \( \text{Mor}_d(\mathbb{P}^1, F(n; s)) \) of morphisms from \( \mathbb{P}^1 \) to \( F(n; s) \) of multidegree \( d = (d_1, \ldots, d_l) \) can be viewed as the space of successive quotients of \( V_{\mathbb{P}^1} \) of vector bundles of rank \( r_i \) and degree \( d_i \), where \( r_i := n - s_i \) and \( V_{\mathbb{P}^1} := V \otimes \mathcal{O}_{\mathbb{P}^1} \) is a trivial rank \( n \) vector bundle over \( \mathbb{P}^1 \). Its compactification, the hyperquot scheme which we denote \( HQ_d = HQ_d(F(n; s)) \), parametrizes flat families of successive quotient sheaves of \( V_{\mathbb{P}^1} \) of rank \( r_i \) and degree \( d_i \). It is a generalization of Grothendieck’s Quot scheme \( \mathcal{Q} \).

There has been much interest in compactifications of moduli spaces of maps, for example the stable maps of Kontsevich. Hyperquot schemes are another natural such compactification. Indeed, most of what is known so far about the quantum cohomology of Grassmanians and flag varieties has been obtained by using Quot scheme compactifications. They have been used by Bertram to study Gromov-Witten invariants and a quantum Schubert calculus \[B1\] [B2] for Grassmannians, and by Ciocan-Fontanine and Kim to study Gromov-Witten invariants and the quantum cohomology ring of flag varieties and partial flag varieties \[K\] [C-F1] [C-F2] [FGP], see also \[C1\] [C2].

The paper is organized in the following way:

In section 2, we give some properties of hyperquot schemes, including a description of the Zariski tangent space to \( HQ_d \) at a point.

In section 3, we consider a torus action on the hyperquot scheme. By the theorems of Bialynicki-Birula, the fixed points of this action give a cell decomposition of \( HQ_d \) \[BB1\] [BB2]. The fixed point data is organized to give a generating function for the topological Euler characteristic of \( HQ_d \).

In section 4, we use the Zariski tangent space to \( HQ_d \) at a point as described in section 2 to compute tangent weights at the fixed points. This

Date: November 1, 2018.
gives an implicit formula for the Betti numbers of $HQ_d$. The torus action and techniques are similar to those used by Strømme in the case $l = 1$, where $F(n : s)$ is the Grassmannian $G_s(n)$ and $HQ_d$ is the ordinary Quot scheme $\mathbb{S}$.

In section 3, we reorganize the implicit formula for the Betti numbers in a way that reduces the problem to a purely combinatorial one. In particular, we collect the information into the form of a generating function. Let $\mathcal{P}(X) = \sum_{M} b_{2M}(X)z^M$ denote the Poincaré polynomial of a space $X$. It is classically known that $\mathcal{P}(F(n; s))$ is equal to the following generating function for the Betti numbers of the partial flag variety:

\[
\mathcal{P}(F(n; s)) = \sum_{M} b_{2M}(F)z^M = \frac{\prod_{i=1}^{n}(1 - z^i)}{\prod_{l=1}^{s_1-1} \prod_{j=1}^{s_j-1}(1 - z^i)}
\]

with $s_{l+1} := n$ and $s_0 := 0$.

Defining $f_{i,j}^k := 1 - t_i \cdots t_j z^k$, the main result is:

**Theorem 1.**

\[
\sum_{d_1, \ldots, d_l} \mathcal{P}(HQ_d(F(n; s))) t_1^{d_1} \cdots t_l^{d_l} = \mathcal{P}(F(n; s)) \cdot \prod_{1 \leq i \leq l} \prod_{s_{i-1} < k \leq s_i} \left( \frac{1}{f_{s_j-k}^i} \right) \left( \frac{1}{f_{s_j+k+1}^i} \right)
\]

In section 3, we discuss the special cases of the ordinary Quot scheme and of the hyperquot scheme for complete flags, and provide some specific examples.

2. Hyperquot Schemes

We fix some notation. Let $V = \mathbb{C}^n$ be a complex n-dimensional vector space. Let $F := F(n; s)$ denote the partial flag variety corresponding to flags of the form:

\[
V_1 \subset V_2 \subset \cdots \subset V_l \subset V
\]

where $V_i$ is a complex subspace of dimension $s_i$. We have $s_0 := 0 < s_1 < \ldots < s_l < s_{l+1} := n$. Define $r_i := s_i - s_j$. As a special case, let $F(n) = F(n; 1, 2, \ldots, n - 1)$ denote the complete flag variety, with $HQ_d(F(n))$ the corresponding hyperquot scheme for complete flags. Also note that the Grassmannian parametrizing $r$-dimensional quotients of $V$, is also a special case, $G_r^n(n) = F(n; n - r)$.

For any space $T$, let $V_T$ denote the trivial rank $n$ vector bundle on $T$, i.e. $V_T := V \otimes \mathcal{O}_T$.

Consider a functor $\mathcal{F}_d$ from the category of schemes to the category of sets. For a scheme $T$, $\mathcal{F}_d(T)$ is defined to be the set of flagged quotient sheaves

\[
V_{P_{i,T}} \rightarrow \mathcal{B}_1 \rightarrow \cdots \rightarrow \mathcal{B}_l
\]
with each $\mathcal{B}_i$ flat over $T$ with Hilbert polynomial $\chi(\mathbb{P}^1, (\mathcal{B}_i)_t(m)) = (m + 1)r_i + d_i$ on the fibers of $\pi_T : \mathbb{P}^1 \times T \to T$. This last condition requires that $\mathcal{B}_i$ be of rank $r_i$ and relative degree $d_i$ over $T$, so that for any $t \in T$, $(\mathcal{B}_i)_t$ is of degree $d_i$.

It is proven that the functor $\mathcal{F}_d$ is represented by the projective scheme $\mathcal{H}_{Q_d} = \mathcal{H}_{Q_d}(\mathcal{F}(n; s))$ following the ideas of Grothendieck and Mumford [C-F2 [G] [M]]. It has also been described in a different way by Kim [K], as a closed subscheme of a product of Quot schemes. Kim also proves the following result:

**Theorem 2** (Kim). $\mathcal{H}_{Q_d}(\mathcal{F}(n; s))$ is an irreducible, rational, nonsingular, projective variety of dimension

$$\sum_{i=1}^{l} d_i(s_{i+1} - s_{i-1}) + \dim(\mathcal{F}(n; s))$$

with $s_0 = 0, s_{l+1} = n$ and $\dim(\mathcal{F}(n; s)) = \sum_{i=1}^{l}(s_{i+1} - s_i)s_i$. In particular, the theorem states that $\dim \mathcal{H}_{Q_d}(\mathcal{F}(n)) = 2|d| + \binom{|d|}{2}$ where $|d| = \sum_{i=1}^{l} d_i$.

Associated to $\mathcal{H}_{Q_d}(\mathcal{F}(n; s))$ is a universal sequence of sheaves on $\mathbb{P}^1 \times \mathcal{H}_{Q_d}$ of successive quotients of sheaves, each of which is flat over $\mathcal{H}_{Q_d}$.

$$V_{\mathbb{P}^1 \times \mathcal{H}_{Q_d}} \rightarrow B_1 \rightarrow \cdots \rightarrow B_l.$$ 

Define $A_i$ as the kernel of $V_{\mathbb{P}^1 \times \mathcal{H}_{Q_d}} \rightarrow B_i$. Each $A_i$ is flat over $\mathcal{H}_{Q_d}$, and it is an easy consequence of flatness that each $A_i$ is locally free. Thus, we have the following universal sequence on $\mathbb{P}^1 \times \mathcal{H}_{Q_d}$:

$$A_1 \hookrightarrow A_2 \hookrightarrow \cdots \hookrightarrow A_l \hookrightarrow V_{\mathbb{P}^1 \times \mathcal{H}_{Q_d}} \rightarrow B_1 \rightarrow \cdots \rightarrow B_l.$$  

with $A_i$ of rank $s_i$, $B_i$ of rank $n - s_i$. Denote the inclusion maps by $\gamma_i : A_i \hookrightarrow A_{i+1}$ and the surjections by $\pi_i : B_{i-1} \twoheadrightarrow B_i$ for each $1 \leq i \leq l$. Here, we define $A_{l+1} = B_0 = V_{\mathbb{P}^1 \times \mathcal{H}_{Q_d}}$ and $A_0 = B_{l+1} = 0$. The map $\gamma_i : A_i \hookrightarrow A_{i+1}$ is an inclusion of sheaves, not an inclusion of bundles.

The following proposition, proved by Ciocan-Fontanine following the ideas in Kollar’s work on Hilbert schemes, determines the Zariski tangent space of $\mathcal{H}_{Q_d}$ at a point $[K]$.

**Proposition 1.** Let $x \in \mathcal{H}_{Q_d}$ correspond to successive quotients and sub-sheaves of $V_{\mathbb{P}^1}$:

$$A_1 \hookrightarrow \cdots \hookrightarrow A_l \hookrightarrow V_{\mathbb{P}^1 \times \mathcal{H}_{Q_d}} \rightarrow B_1 \rightarrow \cdots \rightarrow B_l.$$ 

Then we have the following exact sequence:

$$0 \rightarrow (T_{\mathcal{H}_{Q_d}})_x \rightarrow \bigoplus_{i=1}^{l} \text{Hom}(A_i, B_i) \xrightarrow{d} \bigoplus_{i=1}^{l-1} \text{Hom}(A_i, B_{i+1}) \rightarrow 0,$$

where $(T_{\mathcal{H}_{Q_d}})_x$ is the Zariski tangent space to $\mathcal{H}_{Q_d}$ at the point $x$, and $d$ is the restriction of the difference map given by $d(\{\phi_i\}) = \{\pi_{i+1} \circ \phi_i - \phi_{i+1} \circ \gamma_i\}$. 


3. A Torus Action

In this section, we use the torus action introduced by Strømme in Theorem 3.6 of [S]. In the case of the ordinary Quot scheme, we obtain the same description of the fixed points as Strømme, but with slightly different notation. Our description allows us to provide a full description of the fixed point locus of the hyperquot scheme under this torus action.

Consider a maximal (n-dimensional) torus $T$ in $GL(V)$ which acts on $V$ and hence induces an action on subsheaves of $V_{P^1}$. As a $CT$-module, $V$ splits as a direct sum of one-dimensional subspaces, $\oplus_{i=1}^n W_i$. Denote $O_i := W_i \otimes O_{P^1}$.

For $f_k \in H^0(P^1, O_{P^1}(d_k))$, a form of degree $d_k$, let $f_k O_i(-d_k) \hookrightarrow O_i$ denote the sheaf $O_i(-d_k)$ defined by the section $f_k$.

**Lemma 1.** A locally free subsheaf $S \hookrightarrow V_{P^1}$ of rank $s$ and degree $-d$ is fixed by the action of $T$ if and only if it is of the form

$$S = \bigoplus_{k=1}^s f_k O_c_k(-d_k)$$

where $d_k$ are nonnegative integers such that $\sum_k d_k = d$, $1 \leq c_1 < \cdots < c_s \leq n$, and $f_k$ is a homogeneous form in $X$ and $Y$ of degree $d_k$.

**Proof.** Since $T$ acts with different weights on each $O_i$, $S \hookrightarrow V_{P^1}$ is a fixed point of $T$ if and only if $S = \bigoplus_{i=1}^n S_i$ where $S_i := S \cap O_i$. Since $\text{rank}(S) = s$, $S_i \neq \emptyset$ for exactly $s$ such $i$, say $S_{c_k} \neq \emptyset$ for a sequence $1 \leq c_1 < \cdots < c_s \leq n$.

Since we have $S_{c_k} \hookrightarrow O_{c_k}$, we know that we can write $S_{c_k} = O_{c_k}(-d_k)$ for some nonnegative integer $d_k$.

Therefore we have:

$$S = \bigoplus_{c_k} S_{c_k} = \bigoplus_{k=1}^s f_k O_{c_k}(-d_k),$$

where $\deg S_{c_k} = -d_k$ with $\sum_{k=1}^s d_k = d$. Since the inclusion of $S_{c_k}$ is given by some section $f_k$, which is a polynomial of degree $d_k$ in $X$ and $Y$, the lemma is proven.

Let $T'$ be the one-dimensional torus which acts on $H^0(O_{P^1}(1))$ by $X \mapsto tX$ and $Y \mapsto t^{-1}Y$. Then $T'$ acts on $P^1$ and hence on subsheaves of $V_{P^1}$. Thus under the action of the product torus $T \times T'$, the fixed points are subsheaves $S \hookrightarrow V_{P^1}$ fixed by both $T$ and $T'$. A point $(\oplus_{k=1}^s f_k O_{c_k}(-d_k) \hookrightarrow V_{P^1})$ is fixed by $T'$ if and only if each $f_k$ is a monomial in $X$ and $Y$.

Therefore, we have proven:

**Lemma 2.** A locally free subsheaf $S \hookrightarrow V_{P^1}$ of rank $s$ and degree $-d$ is fixed under the action of $T \times T'$ if and only if it is of the form

$$\bigoplus_{k=1}^s X^{a_k} Y^{b_k} O_{c_k}(-a_k - b_k).$$
Here, $O_i$ denotes the $i$th component of the trivial rank $n$ vector bundle $V_{P^1}$, and $(a, b, c)$ are sequences of $s$ nonnegative integers satisfying:

1. $\sum_{k=1}^{s} a_k + b_k = d$
2. $1 \leq c_1 < \cdots < c_s \leq n$

**Remark.** This combinatorial data is equivalent to the fixed point data of Strømme. For an element $(\alpha, \beta, \delta)$ as in \( S \), let $\delta_{c_1} = \cdots \delta_{c_n} = 1$ be the nonzero elements, with $1 \leq c_1 < \cdots < c_{n-r} \leq n$. Then the sequence $(\alpha, \beta, \delta)$ corresponds to the sequence $(a, b, c)$.

### 3.1. A torus action on $\mathcal{H}_d$.

Note that a point of $\mathcal{H}_d$ can be given by successive subsheaves over $P^1$ of $V_{P^1} = \bigoplus_{i=1}^{n} O_i$.

Let $T$ and $T'$ be as above. Since the actions of $T$ and $T'$ extend to actions on $\mathcal{H}_d(P(n; s))$, we have an action of $T \times T'$ on $\mathcal{H}_d$.

Using the same methods as used in section 3, we find the fixed points of $\mathcal{H}_d$ under this action.

A point of $\mathcal{H}_d$ can be given by a sequence of subsheaves

$$A_1 \hookrightarrow \cdots \hookrightarrow A_l \hookrightarrow V_{P^1},$$

where rank $A_i = s_i$ and deg $A_i = -d_i$. Let $A_i$ denote this sequence $\{A_i\}_{i=1}^{l}$. Then $A_i$ is fixed by the action of $T \times T'$ if and only if each $A_i \hookrightarrow V_{P^1}$ is fixed and the inclusions $A_i \hookrightarrow A_{i+1}$ hold. By Proposition 2, $A_i$ is fixed when

$$A_i = \bigoplus_{j=1}^{s_i} A_{i,j} := \bigoplus_{j=1}^{s_i} X^{a_{i,j}} Y^{b_{i,j}} O_{c_{i,j}} (-a_{i,j} - b_{i,j})$$

where $\sum_{1 \leq j \leq s_i} a_{i,j} + b_{i,j} = d_i$ and $1 \leq c_{i,s_i} - 1 < \cdots < c_{i,s_i} \leq n$. Here, we denote by $O(-a - b)$ the line bundle on $P^1$ given by global section $X^a Y^b$.

The inclusion $A_i \hookrightarrow A_{i+1}$ holds under exactly the following conditions:

1. $c_{i,j} = c_{i+1,j}$ whenever $1 \leq j \leq s_i$. 
2. (2) $O_{c_{i,j}} (-a_{i,j} - b_{i,j}) \hookrightarrow O_{c_{i+1,j}} (-a_{i+1,j} - b_{i+1,j})$ is an inclusion of sheaves.

Let $S := S(n; s_1, \ldots, s_l)$ be the subset of $S_n$ consisting of permutations $\sigma \in S_n$ such that if $\sigma(i) < \sigma(i + 1)$ unless $i \in (s_1, \ldots, s_l)$. More explicitly, an element $\sigma \in S$ is such that

$$\sigma(s_i - 1) < \cdots < \sigma(s_i) \text{ for } 1 \leq i \leq l.$$

Therefore every sequence $\{c_{i,j}\}$ corresponds to an element $\sigma \in S$ by $c_{i,j} = \sigma(j)$, and this correspondence is a bijection by the first condition above.

The second condition gives conditions on the sequences of nonnegative integers $a$ and $b$. The inclusion of sheaves $O(-a_{i,j} - b_{i,j}) \hookrightarrow O$ is given by the global section $X^{a_{i,j}} Y^{b_{i,j}}$ and $O(-a_{i+1,j} - b_{i+1,j}) \hookrightarrow O$ is given by the global section $X^{a_{i+1,j}} Y^{b_{i+1,j}}$. Therefore (3) gives an inclusion of subsheaves if and only if that inclusion is given by global sections

$$X^{a_{i,j} - a_{i+1,j}} Y^{b_{i,j} - b_{i+1,j}}.$$
so that the conditions $a_{i,j} - a_{i+1,j} \geq 0$ and $b_{i,j} - b_{i+1,j} \geq 0$ must hold.

Let $P$ be the set of $(a, b, \sigma)$ such that $a$ and $b$ are sequences of nonnegative integers $a_{i,j}$ and $b_{i,j}$ with $1 \leq i \leq l$, $1 \leq j \leq s_i$, and $\sigma \in S(n; s_1, \ldots, s_l)$ which satisfy:

1. $a_{i,j} \geq a_{i+1,j}$
2. $b_{i,j} \geq b_{i+1,j}$
3. $\sum_{j=1}^{s_i} a_{i,j} + b_{i,j} = d_i$

For $(a, b, \sigma) \in P$, define

$$A_{i,j} = \begin{cases} O_{\sigma(j)}(-a_{i,j} - b_{i,j}) & \text{for } 1 \leq i \leq l \text{ and } 1 \leq j \leq s_i \\ 0 & \text{otherwise} \end{cases}$$

This defines sequences of subsheaves $A_{i,j} \hookrightarrow A_{i+1,j}$. Let $A_i = \bigoplus_{j=1}^{s_i} A_{i,j}$. Let $r(a, b, \sigma) \in \mathcal{H}Q_d$ be the associated flag of subsheaves of $V_{P1}$. We have proven:

**Proposition 2.** $r : P \to \mathcal{H}Q_d$ is a bijection onto $\mathcal{H}Q_d^{T \times T'}$.

Let $B_{i,j} := V_{P1}/A_{i,j}$ be the corresponding quotient so that we have the following short exact sequences:

$$0 \to A_{i,j} \to V_{P1} \to B_{i,j} \to 0 \quad (4)$$

Similarly, define the sheaves $B_i = \bigoplus_{j=1}^{s_i} B_{i,j}$.

### 3.2. Euler characteristic.

Under a generic choice of one-dimensional subtorus $\Gamma \subset T \times T'$, $\mathcal{H}Q_d$ has isolated fixed points under the action of $\Gamma$, with $\mathcal{H}Q_d^\Gamma = \mathcal{H}Q_d^{T \times T'}$. We know by Theorem 2 that $\mathcal{H}Q_d(F(n; s))$ is a nonsingular complex projective variety. The odd cohomology of $\mathcal{H}Q_d$ vanishes since the fixed point locus $\mathcal{H}Q_d^\Gamma$ is finite [BB1] [BB2]. Therefore, the Euler characteristic is the number of fixed points. The fixed point data has been collected into the combinatorial data of the set $E$, so that the Euler characteristic is the cardinality of $E$. In this section, we find a generating function for the Euler characteristics of the hyperquot schemes. We prove:

**Theorem 3.**

$$\sum_{d_1 \ldots d_l} \chi(\mathcal{H}Q_d(F(n; s)))t_1^{d_1} \ldots t_l^{d_l} = (\#(S)) \left( \prod_{1 \leq i \leq l} \frac{1}{(1 - t_i \ldots t_l)^{s_i - s_{i-1}}} \right)^2$$

The cardinality of the set of permutations $S = S(n; s_1, \ldots, s_l)$ is:

$$\#(S) = \prod_{i=0}^{l} \left( \frac{n - s_i}{s_{i+1} - s_i} \right)$$

with $s_0 = 0, s_{l+1} = n$.

**Proof.** Let $\alpha_{i,j} = a_{i,j} - a_{i+1,j}$ and $\beta_{i,j} = b_{i,j} - b_{i-1,j}$, where we let $a_{l+1,j} = b_{l+1,j} = 0$. Then $a_{i,j} = \sum_{k=i}^{l} \alpha_{k,j}$ and $b_{i,j} = \sum_{k=i}^{l} \beta_{k,j}$. Note that while the variables $(a, b)$ satisfy various inequalities, the nonnegative integers of
$(\alpha, \beta)$ are independent. Consider the set $P'$ of elements $(\alpha, \beta, \sigma)$ with $\sigma \in S$ satisfying:

$$
\sum_{k=1}^{l} \sum_{j=1}^{s_i} \alpha_{k,j} + \beta_{k,j} = d_i.
$$

Let $P''$ denote the set of pairs $(\alpha, \beta)$ of natural numbers satisfying the linear relations (5). We have constructed a bijection between $E$ and $E'$, so that

$$
\chi(HQ_d(\mathbb{F}(n; s))) = \#(P) = \#(P') = \#(S)\#(P'').
$$

We see that

$$
\left( \prod_{1 \leq i < j \leq l} \frac{1}{(1 - t_i \ldots t_j)^{s_i - s_{i-1}}} \right)^2 = \prod_{1 \leq i \leq l} \prod_{s_{i-1} < k \leq s_i} \sum_{\alpha_{k,j} \in \mathbb{N}} (t_i \ldots t_j)^{\alpha_{k,j}} \sum_{\beta_{k,j} \in \mathbb{N}} (t_i \ldots t_j)^{\beta_{k,j}}
$$

so that each set of natural numbers $(\alpha, \beta)$ satisfying the relations (5) contributes exactly one to the coefficient of $t_1^{d_1} \cdots t_l^{d_l}$. This proves the theorem.

4. An implicit formula for the Betti numbers

Let $N >> 0$ be a large integer. Let $\Gamma \subset T \times T'$ be the one-dimensional subtorus which acts on $O_i$ by $t \cdot v = t^{Ni}v$ and acts on $H^0(\mathbb{P}^1, O(1))$ by $X \mapsto tX, Y \mapsto t^{-1}Y$. It is this $\mathbb{C}^*$ action of $\Gamma$ on $HQ_d(\mathbb{F}(n; s))$ which is used.

In order to find information about the contribution of the various fixed points to the Betti numbers, we must consider our $\mathbb{C}^*$ action of $\Gamma$ on $HQ_d$ more carefully.

Fix the following notation. Consider $\Gamma$ a $\mathbb{C}^*$-action on a scheme $X$. If $x \in X$ is a fixed point of this action, and $E$ is a bundle over $X$, denote by $E^+_\Gamma$ the $\Gamma$-submodule of $E(x)$ where $\Gamma$ acts with positive weights. In particular, the theorems of Bialynicki-Birula state that for isolated fixed points $\{x_i\} \subset X_\Gamma$, there is a cell decomposition of $X$ given by the orbits $X_i$ of $x_i$, with $\dim_{\mathbb{C}} X_i = \dim_{\mathbb{C}} T_X(x_i)^+$. [BB1] [BB2]

We first compute the tangent weights at the fixed points of the $\mathbb{C}^*$ action of $\Gamma$ on $HQ_d$. For $\sigma \in S$, define $\varepsilon^\sigma_{i,j}$ for $1 \leq i, j \leq n$ by $\varepsilon^\sigma_{i,j} = 1$ if $\sigma(i) < \sigma(j)$, $\varepsilon^\sigma_{i,j} = 0$ otherwise. Note that $\varepsilon_{i,j} + \varepsilon_{j,i} = 1$ for $i \neq j$. Define $\varepsilon^\sigma_{i[k,k']}$ for $1 \leq i, k, k' \leq l$ by $\varepsilon^\sigma_{i[k,k']} = \sum s_k \prec j \leq s_{k'} \varepsilon_{i,j}$ and similarly define $\varepsilon^\sigma_{i[k,k']}$ by $\sum s_k \prec i \leq s_{k'} \varepsilon_{i,j}$. If the permutation $\sigma$ is understood, it may be suppressed.
Define a map \( h : P \rightarrow \mathbb{Z} \) by:

\[
(6) \quad h(a, b, \sigma) = \sum_{i=1}^{l} \sum_{k \leq s_i} (a_{i,k} + b_{i,k} + 1) \varepsilon^{\sigma}_{k(i,i+1)} + \sum_{i=1}^{l} \sum_{k \leq s_i} (a_{i,k} + b_{i,k}) \varepsilon^{\sigma}_{i(i,i+1)} + \sum_{i=1}^{l} \sum_{s_i < k \leq s_{i+1}} b_{i,k}
\]

Then we have the following implicit formula for the Betti numbers of the hyperquot scheme:

**Proposition 3.** For any (nonnegative) integer \( M \),

\[
b_{2M}(\mathcal{H}Q_d(F(n; s))) = \text{rank } A_M(\mathcal{H}Q_d) = \#(h^{-1}(M)).
\]

**Proof.** If we show that \( h(a, b, e) = \dim_{\mathbb{C}}(T_{\mathcal{H}Q_d}(r(a, b, \sigma))^+) \), then by applying the theorems of Bialynicki-Birula, we will be done. Let \( \mathcal{A} \) be the sequences of subsheaves associated to \( r(a, b, \sigma) \in \mathcal{H}Q_d \) as in section 3.1. From the short exact sequence on \( \mathcal{H}Q_d \) given in Lemma 1 we have:

\[
0 \rightarrow T_{\mathcal{H}Q_d}(\mathcal{A}) \rightarrow \bigoplus_{i=1}^{l} \text{Hom}(\mathcal{A}_i, \mathcal{B}_i) \rightarrow \bigoplus_{i=1}^{l-1} \text{Hom}(\mathcal{A}_i, \mathcal{B}_{i+1}) \rightarrow 0.
\]

Therefore the tangent weights of \( T_{\mathcal{H}Q_d}(\mathcal{A}) \) are those obtained by removing the weights of the quotient term from those of the middle term. In particular, the positive weights are also be obtained this way. More precisely, we can say that

\[
\dim_{\mathbb{C}}(T_{\mathcal{H}Q_d}(\mathcal{A}))^+ = \dim_{\mathbb{C}} \left( \bigoplus_{i=1}^{l} \text{Hom}(\mathcal{A}_i, \mathcal{B}_i) \right)^+ - \dim_{\mathbb{C}} \left( \bigoplus_{i=1}^{l-1} \text{Hom}(\mathcal{A}_i, \mathcal{B}_{i+1}) \right)^+.
\]

Define maps \( h_{i,j}^k : P \rightarrow \mathbb{Z} \) for \( 0 \leq i \leq j \leq l, k = 1, 2, 3 \) as follows:

\[
(7) \quad h_{i,j}^1(a, b, \sigma) = \sum_{k \leq s_i} (a_{i,k} + b_{i,k} + 1) \varepsilon^{\sigma}_{k(i,l+1)}
\]

\[
= \sum_{k \leq s_j} (a_{j,k} + b_{j,k}) \varepsilon^{\sigma}_{0(i,k)}
\]

\[
= \sum_{k \leq s_i} b_{i,k}
\]

where we allow zero maps when appropriate.

The proposition is then an immediate consequence of the following two lemmas.

**Lemma 3.** For any \( i \leq j \), \( \dim_{\mathbb{C}}(\text{Hom}(\mathcal{A}_i, \mathcal{B}_j))^+ = (h_{i,j}^1 + h_{i,j}^2 + h_{i,j}^3)(a, b, \sigma). \)
Lemma 4.  

\[ h(a, b, \sigma) = \sum_{i=1}^{l} (h_{i,i}^1 + h_{i,i+1}^2 + h_{i,i+1}^3)(a, b, \sigma) - \sum_{i=1}^{l-1} (h_{i,i+1}^1 + h_{i,i+1}^2 + h_{i,i+1}^3)(a, b, \sigma). \]

We first prove Lemma 4.  

As an immediate consequence of the definitions (8), we have:

\[ h_{i,i}^1 - h_{i,i+1}^1 = \sum_{k \leq s_i} (a_{i,k} + b_{i,k} + 1)\varepsilon_{k,(i,i+1]} \]

\[ h_{i,i}^2 - h_{i-1,i}^2 = \sum_{k \leq s_i} (a_{i,k} + b_{i,k})\varepsilon_{(i-1,i),k} \]

\[ h_{i,i}^3 - h_{i-1,i}^3 = \sum_{s_{i-1} < k \leq s_i} b_{i,k} \]

and since \( h_{0,1}^1 = h_{0,1}^2 = h_{0,1}^3 = 0 \), Lemma 4 follows. \( \Box \)

It only remains to prove Lemma 3. We have

\[ \text{Hom}(A_i, B_j) = \bigoplus_{k,m} \text{Hom}(A_{i,k}, B_{j,m}) = \]

\[ \bigoplus_{k \leq s_i, m > s_j} \text{Hom}(O_{\sigma(k)}(-a_{i,k} - b_{i,k}), O_{\sigma(m)}) \]

\[ + \bigoplus_{k \leq s_i, m \leq s_j} \text{Hom}(O_{\sigma(k)}(-a_{i,k} - b_{i,k}), O_{\sigma(m)}/O_{\sigma(m)}(-a_{j,m} - b_{j,m})) \]

We have three situations to consider:

1. \( m > s_j \)

\( \text{Hom}(O_{\sigma(k)}(-a_{i,k} - b_{i,k}), O_{\sigma(m)}) \) is of rank \( a_{i,k} + b_{i,k} + 1 \), with weights the same sign as \( (\sigma(m) - \sigma(k)) \) for \( N \) large enough. The number of \( m > i \) with \( \sigma(m) > \sigma(k) \) is \( i_{m \leq s_i, m \leq s_j} \). In the notation of (3), since \( s_{l+1} = n \), this number is \((a_{i,k} + b_{i,k} + 1)\varepsilon_{k,(j,l+1]}\). This gives the term \( h_{i,j}^1(a, b, \sigma) \) in Lemma 3.

2. \( k \neq m \)

\( \text{Hom}(O_{\sigma(k)}(-a_{i,k} - b_{i,k}), O_{\sigma(m)}/O_{\sigma(m)}(-a_{j,m} - b_{j,m})) \) is of rank \( a_{j,m} + b_{j,m} \), with weights the same sign as \( \sigma(m) - \sigma(k) \). Thus the positive weight contribution is \( \sum_{k \leq s_i} (a_{j,m} + b_{j,m})\varepsilon_{k,m} = (a_{j,m} + b_{j,m})\varepsilon_{(0,l,m)} \), which gives the term \( h_{i,j}^2 \) in Lemma 3.

3. \( k = m \)

\( \text{Hom}(A_{i,k}, B_{j,k}) \) sits inside the long exact sequence induced by the short exact sequence (8):

\[
\begin{align*}
0 & \rightarrow \text{Hom}(A_{i,k}, A_{j,k}) \rightarrow \text{Hom}(A_{i,k}, O_{\sigma(k)}) \\
& \quad \rightarrow \text{Hom}(A_{i,k}, B_{j,k}) \rightarrow H^1(A_{i,k}^* \otimes A_{j,k}) \rightarrow 0
\end{align*}
\]

We have two cases:
(a) \( A_{i,k} \leftrightarrow A_{j,k} \). Here, equation (8) becomes
\[
0 \to \text{Hom}(A_{i,k}, A_{j,k}) \to \text{Hom}(A_{i,k}, O_{\sigma(k)}) \to \text{Hom}(A_{i,k}, B_{i,k}) \to 0.
\]
\( \Gamma \) acts on \( \text{Hom}(A_{i,k}, A_{j,k}) \) by \( t \cdot X^r Y^s = X^{r-(a_i,k-a_j,k)} Y^{-s+(b_i,k-b_j,k)} \) and on \( \text{Hom}(A_{i,k}, O_{\sigma(k)}) \) by \( t \cdot X^r Y^s = X^{r-a_i,k} Y^{-s+b_i,k} \). Thus, the positive part of this piece of \( T_{H,Q_d}(r(a,b,\sigma)) \) has dimension \( b_{j,k} \). This gives the \( h^3_{i,j} \) term.

(b) \( A_{i,k} \not\leftrightarrow A_{j,k} \). From (8) we get
\[
0 \to \text{Hom}(A_{i,k}, O_{\sigma(k)}) \to \text{Hom}(A_{i,k}, B_{j,k}) \to H^1(A_{i,k}^* \otimes A_{j,k}) \to 0.
\]
We have \( H^1(A_{i,k}^* \otimes A_{j,k}) = H^1(\text{O}((a_i,k-a_j,k)+(b_i,k-b_j,k))) \). By Serre duality and the same arguments as in the previous case, the positive contribution of \( \text{Hom}(A_{i,k}, B_{j,k}) \) is \( (b_{j,k} - b_{i,k}) + b_{i,k} = b_{j,k} \). This gives the term \( h^3_{i,j} \).

Therefore we have shown that \( \dim_C(T_{H,Q_d}(A))^+ = h(a,b,\sigma) \), so that the proposition is proved. \( \square \)

5. Poincaré polynomials

We use the implicit formula for the Betti numbers proved in Proposition 3. Rewrite (6) as:
\[
(9) \quad h(a,b,\sigma) = \sum_{i=1}^{l} \sum_{k=1}^{s_i} \varepsilon_{k(i,i+1)} + \sum_{i=1}^{l} \sum_{s_{i-1}<k\leq s_i} b_{i,k}
\]
\[
+ \sum_{i=1}^{l} \sum_{k=1}^{s_i} (a_{i,k} + b_{i,k})(\varepsilon_{k(i,i+1)} + \varepsilon_{(i-1,j)k})
\]

Recall the sequences of independent nonnegative integers \( \alpha \) and \( \beta \) introduced in the proof of Theorem 3 given by \( \alpha_{i,j} = a_{i,j} - a_{i+1,j}, \beta_{i,j} = b_{i,j} - b_{i+1,j} \). We see that \( a_{i,k} = \sum_{j \geq i} \alpha_{j,k} \) and \( b_{i,k} = \sum_{j \geq i} \beta_{j,k} \). Changing to the variables \( (\alpha, \beta, \sigma) \), the middle sum of (6) becomes
\[
\sum_{i \leq j \leq l} \sum_{s_{i-1}<k\leq s_i} (\alpha_{j,k} + \beta_{j,k})(\varepsilon_{k(i,i+1)} + \varepsilon_{(i-1,j)k})
\]
\[
= \sum_{i \leq j \leq l} \sum_{s_{i-1}<k\leq s_i} (\alpha_{j,k} + \beta_{j,k})(\varepsilon_{k(i,i+1)} + \varepsilon_{(i-1,j)k})
\]

Therefore, we can simplify our expressions to get:
\[
(10) \quad H(\alpha,\beta,\sigma) := h(a,b,\sigma) = \sum_{i=1}^{l} \sum_{k=1}^{s_i} \varepsilon_{k(i,i+1)} + \sum_{j=1}^{l} \sum_{k=1}^{s_j} \beta_{j,k}
\]
\[
+ \sum_{i \leq j \leq l} \sum_{s_{i-1}<k\leq s_i} (\alpha_{j,k} + \beta_{j,k})(s_j - s_i + \varepsilon_{k(j,j+1)} + \varepsilon_{(i-1,j)k})
\]

By the definition of \( H \), we have shown
Proposition 4.

\[ b_{2M}(H\mathcal{Q}_d(F(n; s))) = \text{rank } A_M(H\mathcal{Q}_d) = \#(H^{-1}(M)). \]

For \( w \in S \), define \( P(w) \) to be the elements \((\alpha, \beta, \sigma) \in P \) satisfying \( \sigma = w \). Let \( H_w \) denote the restriction of \( H \) to \( F(w) \). Then \( \#(H^{-1}_w(M)) \) counts the number of sequences of natural numbers \((\alpha, \beta)\) satisfying the relation \( H_w(\alpha, \beta) = M \) given by (11) as well as the relations in (9). Since all of these relations are linear in the variables \( \alpha \) and \( \beta \), by the same reasoning as in the proof of Theorem 3, we have the following generating function:

\[
\sum_{M, d_1, \ldots, d_l} \#(H^{-1}_w(M)) t_1^{d_1} \cdots t_l^{d_l} z^M = z^{\sum t_\iota} \sum s_\iota \epsilon_{k_{(\iota, \iota + 1)}} \prod_{1 \leq \iota \leq l} \prod_{s_{\iota - 1} < k \leq s_{\iota}} \frac{1}{f_{\rho_{\iota, \iota, k}} f_{\rho_{\iota, \iota + 1}}}
\]

where we have defined \( f_{\iota, \j} = 1 - \frac{t_\iota t_j z^k}{1 - z} \) and \( \rho_{\iota, \j, k} = s_\j - s_\iota + \epsilon_{k_{(\j, \j + 1)}} + k - s_{\iota - 1} - 1 \).

We are now ready to prove Theorem 1.

By the definitions of \( H \) and \( H_w \), we have

\[
\#(H^{-1}(M)) = \sum_{w \in S} \#(H^{-1}_w(M))
\]

and hence \( b_{2M}(H\mathcal{Q}_d(F(n; s))) = \sum_{w \in S} \#(H^{-1}_w(M)) \). Therefore, by (11), it suffices to prove the following (purely combinatorial) result:

Proposition 5.

\[
\sum_{w \in S(n; s_1, \ldots, s_l)} z^{\sum t_\iota} \sum s_\iota \epsilon_{k_{(\iota, \iota + 1)}} \prod_{1 \leq \iota \leq l} \prod_{s_{\iota - 1} < k \leq s_{\iota}} \frac{1}{f_{\rho_{\iota, \iota, k}} f_{\rho_{\iota, \iota + 1}}}
\]

\[
= \left( \frac{\prod_{i=1}^n (1 - z^i)}{\prod_{j=1}^{l+1} \prod_{s_j - s_j - 1} (1 - z^i)} \right) \prod_{1 \leq \iota \leq l} \prod_{s_{\iota - 1} < k \leq s_{\iota}} \frac{1}{f_{\rho_{\iota, \iota, k}} f_{\rho_{\iota, \iota + 1}}}
\]

We use induction on \( n \) to prove the proposition. For \( n = 1 \), there are two cases:

1. \( s_1 = 0 \). Here, \( S(1; 0) = S_1 = id \). Both sides of the equation are equal to 1, so that the proposition holds.
2. \( s_1 = 1 \). We have \( S(1; 1) = S_1 = id \). Both sides of the equation are equal to \( \frac{1}{(1 - z)(1 - z^2)} \), so again the proposition holds.

The strategy is to break up \( S = S(n; s_1, \ldots, s_l) \) into \( l + 1 \) different permutation groups upon which we can use the inductive hypothesis.

For \( 1 \leq m \leq l + 1 \), let \( S(m) \) denote the subset of \( S \) consisting of permutations \( w \) such that \( w(s_{m-1} + 1) = 1 \). It is clear that \( S = \bigcup_m S(m) \) is a disjoint union. Any \( w \in S(m) \) satisfies: \( \epsilon_{w_{s_{m-1} + 1 + 1}} = 1 \) for \( j \neq s_{m-1} + 1 \) and \( \epsilon_{w_{i, s_{m-1} + 1}} = 0 \) for all \( i \).

For \( w \in S(m) \), let \( w' \in S' := S(n - 1; s'_1, \ldots, s'_l) \) be defined by:
1. \( w'(i) = w(i) - 1 \) for \( i \leq s_{m-1} \)
2. \( w'(j) = w(j + 1) - 1 \) for \( s_{m-1} + 1 < j \).

Let \( \epsilon'_{i,j} = \epsilon_{i,j}^{w'} \) and define \( \epsilon'_{i,k}^{m,k'} \) and \( \rho'_{i,j,k} \) accordingly. Note that \( s'_i = s_i \) for \( i \leq m - 1, s'_j = s_j - 1 \) for \( m \leq j \) and

\[
(12) \sum_{i \leq l} \sum_{k \leq s_i} \epsilon_{k(i,i+1)} = (n - s_m) + \sum_{i \leq l} \sum_{k \leq s_i, k \neq s_{m-1} + 1} \epsilon_{k(i,i+1)}.
\]

We split the left hand sum over \( S \) into sums over \( S(m) \) for \( 1 \leq m \leq l + 1 \). For each \( S(m) \), we can factor out the parts of the product on the left hand side where \( i = m \) and \( k = s_{m-1} + 1 \). Using (12), this gives:

\[
\sum_{w \in S} z^{\sum_{i \leq l} \sum_{k \leq s_i} \epsilon_{k(i,i+1)}} \prod_{1 \leq i \leq j \leq l} \prod_{s_{j-1} < k \leq s_i} \frac{1}{f_{\epsilon_{i,j}^{l,j}}} \frac{1}{f_{\epsilon_{i,j}^{l,j} + 1}^{l,j}}
\]

\[
= \sum_{m=1}^{l+1} z^{n-s_m} \prod_{m \leq j} \frac{1}{f_{m,j}^{l,j}} \frac{1}{f_{m,j}^{l,j} + 1} \left( \sum_{w \in S(m)} \prod_{i \leq j, s_{i-1} < k \leq s'_i} \prod_{s_{j-1} < k \leq s'_i} \frac{1}{f_{\epsilon_{i,j}^{l,j} + 1}^{l,j}} \prod_{m > i \text{ or } j < m} \frac{1}{f_{\epsilon_{i,j}^{l,j} + 1}^{l,j} + 1} \right).
\]

By induction, we may apply the result to \( S(m) \sim S(n-1; s'_1, \ldots, s'_{l+1}) \) to the quantity in parentheses. In particular, by what follows from the proof of Proposition 3 for \( S(m) \), the sum becomes:

\[
\sum_{m=1}^{l+1} z^{n-s_m} \prod_{m \leq j} \frac{1}{f_{m,j}^{l,j}} \frac{1}{f_{m,j}^{l,j} + 1} \left( \prod_{i=1}^{n-1} (1 - z^i) \prod_{j=1}^{l+1} \frac{1}{f_{s_j, s_{j-1}}^{l,j}} \prod_{i \leq m \leq j} \frac{1}{f_{s_{j-1}, s_j}^{l,j}} \prod_{m > i \text{ or } j < m} \frac{1}{f_{s_{j-1}, s_j}^{l,j} + 1} \right).
\]

Since we have:

\[
\frac{\prod_{i=1}^{n} (1 - z^i)}{\prod_{j=1}^{l+1} \prod_{i=1}^{s_{j-1}} (1 - z^i)} = \frac{\prod_{i=1}^{n-1} (1 - z^i)}{\prod_{j=1}^{l+1} \prod_{i=1}^{s_{j-1}-1} (1 - z^i)} \frac{1 - z^n}{1 - z^{s_m - s_{m-1}}},
\]

we can write this sum explicitly as:

\[
\sum_{m=1}^{l+1} z^{n-sm} \frac{\prod_{i=1}^{n} (1 - z^i)}{\prod_{j=1}^{l+1} \prod_{s_j-s_j-1}^{s_j} (1 - z^i)} \cdot \frac{(1 - z^{sm-sm-1})}{(1 - z^n)}
\]

\[
\prod_{m \leq j} \frac{1}{f_{s_{j+1}-s_m}^{m,j}} \cdot \prod_{m \leq j \leq s_{m-1} < k \leq s_{m-1}} \frac{1}{f_{s_{j+1}+1-k}^{m,j}} \cdot \prod_{i \leq m-1} \frac{1}{f_{s_{m-1}-k}^{m,m-1}} \cdot \prod_{i \leq j \neq m \neq m-1} \frac{1}{f_{s_{j+1}+1-k}^{i,j}}
\]

By clearing denominators, the proposition follows once we have proven:

**Lemma 5.**

\[
(1 - z^n) \prod_{1 \leq i < j \leq l} f_{s_{j+1}-s_i}^{i,j} =
\]

\[
\sum_{m=1}^{l+1} z^{n-sm} (1 - z^{sm-sm-1}) \prod_{i \leq m-1} f_{s_{m}-s_{i-1}}^{i,m-1} \prod_{m \leq j} f_{s_{j}-s_{m}}^{m,j} \prod_{i \leq j \neq m} f_{s_{j+1}-s_i}^{i,j}
\]

**Proof.** First, we change our notation so that we work with independent variables \(x_i\) where we define \(x_r = t_1...t_r\), with \(x_0 = 1\). Then for any \(i \leq j\), we have \(t_i...t_j = x_j/x_{i-1}\). Therefore we have

\[
f_{k}^{i,j} = 1 - t_i...t_j z^k = \frac{x_{i-1} - x_j z^k}{x_{i-1}}.
\]

Substituting into both sides of Lemma 5 and multiplying through by \(x_0^{l_0}...x_l^{l_1}\) it suffices to prove the following polynomial identity in the ring \(\mathbb{C}[x_0, ..., x_n, z]\), where we define \(e_{k}^{i,j} = x_i - x_j z^k\):

\[
(1 - z^n) \prod_{1 \leq i < j \leq l} e_{s_{j+1}-s_i}^{i-1,j} =
\]

\[
\sum_{m=1}^{l+1} z^{n-sm} (1 - z^{sm-sm-1}) \prod_{i \leq m-1} e_{s_{m}-s_{i-1}}^{i-1,m-1} \prod_{m \leq j} e_{s_{j}-s_{m}}^{m,j} \prod_{i \leq j \neq m} e_{s_{j+1}-s_i}^{i-1,j}
\]

The polynomial ring \(\mathbb{C}[x_0, ..., x_n, z]\) is a unique factorization domain, and that the left side of (13) completely factored except for the term \((1 - z^n)\). Since the degree of \(z\) matches on both sides, as does the term of \(t_1^0...t_l^0 = 1\), namely \((1 - z^n)\), it is enough to show the right side vanishes with the relation \(e_{s_{j+1}-s_i}^{i-1,j} = 0\) for each \(i \leq j\), i.e. that \((x_{i-1} - x_j z^{s_{j+1}-s_i})\) is a factor of the right side.
Fix $i \leq j$. Then all but two summands on the right vanish. In particular, after some cancellations in the final terms of the two products, and bringing one of the terms to the other side, we have left to show:

\[(14) \quad z^{n-s_i} \prod_{p \leq i-1} e^{p-1,i-1}_{s_{i-p}-s_{p-1}} \prod_{i \leq q} e^{i-1,q}_{s_q-s_{i}} \prod_{p \leq j \quad p \neq i} e^{p-1,j}_{s_{j+1}-s_{p}} \prod_{j+1 \leq q} e^{j,q}_{s_{q+1}-s_{j+1}} = -z^{n-s_{j+1}} \prod_{p \leq j+1} e^{p-1,j}_{s_{j+1}-s_{p}} \prod_{j+1 \leq q} e^{i-1,j}_{s_q-s_{j+1}} \prod_{i \leq q \quad q \neq j} e^{i-1,q}_{s_{q+1}-s_{i}} \prod_{p \leq i-1} e^{p-1,i-1}_{s_{i}-s_{p}} \]

with the relation $e^{i-1,j}_{s_{j+1}-s_{i}} = 0$.

But now, by doing the substitution $x_{i-1} = x_j z^{s_{j+1}-s_i}$, we see that we will have proven Lemma 5 once we show that the polynomial identity (14) holds in the (unique factorization domain) $\mathbb{C}[x_0, \ldots, x_{i-2}, x_i, \ldots, x_n, z]$. We make the following observations about the substitution:

1. $e^{p-1,i-1}_{s_i-s_{p-1}} \mapsto e^{p-1,j}_{s_{j+1}-s_{p-1}}$
2. $e^{p-1,j}_{s_{j+1}-s_p} \mapsto e^{j,j}_{s_{j+1}-s_{p}}$
3. $e^{i-1,q}_{s_q-s_{i}} \mapsto \begin{cases} z^{s_{j+1}-s_i} e^{j,q}_{s_q-s_{j+1}} & \text{when } j + 1 \leq q \\ -z^{s_{j+1}-s_i} e^{j,q}_{s_q-s_{j+1}} & \text{when } q \leq j \end{cases}$
4. $e^{i-1,q}_{s_{q+1}-s_{i}} \mapsto \begin{cases} z^{s_{j+1}-s_i} e^{q,q}_{s_{q+1}-s_{j+1}} & \text{when } j \leq q \\ -z^{s_{q+1}-s_{i}} e^{q,q}_{s_{q+1}-s_{j+1}} & \text{when } q < j \end{cases}$

Substituting into both sides of (14), and using the above properties, we have two completely factored polynomials on each side of the identity. It is easy to check that the degree of $z$ in the two terms match, that the sign matches, and that the factors of the form $e^{i,j}_k$ match exactly.

This concludes the proof of Proposition 5 and Theorem 1.

For any scheme $X$, setting $z = 1$ into the Poincaré polynomial

$$ P(X) = \sum_M (-1)^M b_M(X) z^M $$

gives the Euler characteristic $\chi(X)$. Since odd cohomology of $\mathcal{H}Q_d$ vanishes, this substitution into Theorem 1 provides another proof of Theorem 3.

6. Special cases

6.1. Quot scheme. We can apply Theorem 1 to the ordinary Quot scheme, parametrizing rank $r$ degree $d$ quotients of $V_p$, to get a generating function for the Poincaré polynomials of $\mathcal{H}Q_d(G^r(n))$. 

Theorem 4.
\[
\sum_{d} \mathcal{P}(\mathcal{H}Q_d(G^r(n))) t^d = \\
\mathcal{P}(G^r(n)) \cdot \prod_{i=1}^{n-r} \left( \frac{1}{1 - t z^{i-1}} \right) \left( \frac{1}{1 - t z^{n-i+1}} \right).
\]
where \(\mathcal{P}(G^r(n))\) is the Poincaré polynomial of \(G^r(n)\), which is the following classical generating function for the Betti numbers for the Grassmannian:

\[
\mathcal{P}(G^r(n)) = \sum_M b_{2M}(G^r(n)) z^M = \frac{\prod_{i=1}^{n-r}(1 - z^i)}{\prod_{i=1}^{n-r}(1 - z^i) \prod_{i=1}^{n-r}(1 - z^i)}.
\]

This was the case studied by Strømme, who found implicit formulas for the Betti numbers, which are the same up to notation as those found in this paper. Generators and relations of the Chow ring of \(\mathcal{H}Q_d\) are also given in [S]. However, this set is far from minimal, and is not suited to the study of the Chow rings as the degree \(d\) becomes large.

6.2. Hyperquot scheme for a complete flag variety. Consider the space \(\mathcal{H}Q_d(F^n; s)\) where \(l = n - 1\) and \(s_i = i\), i.e. the space \(\mathcal{H}Q_d(F^n)\).

Fix \(n\). Then we have the following generating function for the Poincaré polynomials of \(\mathcal{H}Q_d(F^n)\) where \(d = (d_1, ..., d_{n-1})\).

Theorem 5.
\[
\sum_{d_1, ..., d_{n-1}} \mathcal{P}(\mathcal{H}Q_d(F(n))) t_1^{d_1} ... t_{n-1}^{d_{n-1}} = \\
\mathcal{P}(F(n)) \cdot \prod_{1 \leq i \leq j \leq n-1} \left( \frac{1}{1 - t_i ... t_j z^{j-i}} \right) \left( \frac{1}{1 - t_i ... t_j z^{j-i+2}} \right).
\]
The classical term \(\mathcal{P}(F(n))\) is given by:

\[
\mathcal{P}(F(n)) = \sum_M b_{2M}(F(n)) z^M = \frac{\prod_{i=1}^{n}(1 - z^i)}{\prod_{j=1}^{n}(1 - z)}.
\]

6.3. Examples.
1. \(F(1; 0) = G^1(1)\) is a point and \(\mathcal{H}Q_d(F(1; 0))\) is a point for \(d = 0\) and empty for \(d > 0\).

\[
\sum_d \mathcal{P}(\mathcal{H}Q_d(F(1; 0))) t^d = 1
\]
which is consistent with the theorem.
2. \(F(1; 1) = G^0(1)\) is a point. \(\mathcal{H}Q_d(F(1; 1))\) parametrizes quotients of \(\mathcal{O}_{\mathbb{P}^1}\) of rank 0 and degree \(d\), which are all of the form \(\mathcal{O} \rightarrow \mathcal{O}_D\), where
\[ D \in \text{Sym}^d \mathbb{P}^1 \cong \mathbb{P}^d. \] Therefore \( \mathcal{H}Q_d(F(1; 1)) \cong \mathbb{P}^d \), giving:
\[
\sum_{d,M} b_{2M}(P^d) t^d z^M = \sum_d t^d \left( \sum_{M \leq d} z^M \right) = \frac{1}{(1 - t)(1 - tz)}.
\]

3. \( \mathcal{H}Q_d(G^{n-1}(n)) \) can be viewed as the space of sheaf injections \( \mathcal{O}(-d) \hookrightarrow \bigoplus_{i=1}^n \mathcal{O} \) up to equivalence. Each inclusion of sheaves is given by \( n \) sections in \( H^0(\mathbb{P}^1, \mathcal{O}(d)) \). Thus, we can view any such inclusion as an element of the vector space \( \bigoplus_{i=1}^n H^0(\mathcal{O}(d)) \) of dimension \( n(d+1) \). Two inclusions are equivalent exactly when they differ by a scalar. Hence, \( \mathcal{H}Q_d \cong \mathbb{P}^{n(d+1)-1} \) so that \( P(\mathcal{H}Q_d) = \sum_{0 \leq M \leq n(d+1)} z^M \).

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