Bruck decomposition for endomorphisms of quasigroups

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Abstract

In 1944, R. H. Bruck has described a very general construction method which he called the extension of a set by a quasigroup. We use it to construct a class of examples for LF-quasigroups in which the image of the map $e(x) = x \setminus x$ is a group. More generally, we consider the variety of quasigroups which is defined by the property that the map $e$ is an endomorphism and its subvariety where the image of the map $e$ is a group. We characterize quasigroups belonging to these varieties using their Bruck decomposition with respect to the map $e$.

2000 MSC: 20N05

1 Introduction

A binary algebra $(Q, \cdot)$ with multiplication $(x, y) \mapsto x \cdot y$ is called a quasigroup if the equations $a \cdot y = b$ and $x \cdot a = b$ have precisely one solution in $Q$ which we denote by $y = a \setminus b$ and $x = b / a$. The element $1_l(a) = a / a$ (resp., $1_r(a) = a \setminus a$) is the left (resp., the right) local unit element of the element $a$. If the left (right) local unit elements coincide for all elements of $(Q, \cdot)$, then the element $1_l = 1_l(a)$ (resp., $1_r = 1_r(a)$) is called the left (resp., right) unit element of $(Q, \cdot)$. If a quasigroup $(Q, \cdot)$ has both left and right unit elements, then they coincide $1 = 1_l = 1_r$; in this case $(Q, \cdot)$ is called a loop.

In 1944, R. H. Bruck has described a very general construction method which he called the extension of a set by a quasigroup (cf. [3, 4]). Epimorphisms of quasigroups in general cannot be described by cosets of a normal subquasigroup, but only by congruence relations in the sense of universal algebra. Bruck’s construction takes this into account giving a manageable description of quasigroup epimorphisms. In this note we discuss this method for endomorphisms of quasigroups.

A quasigroup $(Q, \cdot)$ is called an LF-quasigroup if the identity $x \cdot yz = xy \cdot (x \setminus x \cdot z)$ holds in $Q$. In his book [1], Belousov initiated a systematic study of LF-quasigroups using isotopisms. Recently progress has been made in this topic (cf. [7, 8, 9]). It is known that in an LF-quasigroup the map $e(x) = x \setminus x$ is an endomorphism, which we call the left deviation. In this situation Bruck’s theory is available. We use it to construct a class of examples for LF-quasigroups $Q$ in which $e(Q)$ is a group.

More generally, we consider the variety $\mathcal{D}_l$ of quasigroups which is defined by the property that the left deviation is an endomorphism and its subvariety $a\mathcal{D}_l$ where the image of the left deviation is a group. We characterize quasigroups belonging to these varieties using their Bruck decomposition with respect to their left deviation.
2 The Bruck decomposition of a quasigroup

In Bruck’s papers [3, Theorem 10A, pp. 166–168] and [4, pp. 778–779], a principal construction for quasigroups is given. Let \((E, \cdot, \backslash, /)\) be a quasigroup, let \(T\) be a set, and let \(\{\nabla_{a,b}; a,b \in E\}\) be a family of multiplications on \(T\). Define on the set \(Q = T \times E\) a multiplication by

\[(\alpha, a) \circ (\beta, b) = (\alpha \nabla_{a,b} \beta, a \circ b), \quad (\alpha, a), (\beta, b) \in T \times E\] (2.1)

Then \((Q, \circ)\) is a quasigroup if and only if for any \(a, b \in E\) the multiplication \(\nabla_{a,b}\) on \(T\) defines a quasigroup \(T_{a,b} = (T, \nabla_{a,b})\). In this case we call \(B = (E, T, (\nabla_{a,b})(a,b \in E))\) a Bruck system and put \(Q(B) = Q\). Obviously, the projection \(((\alpha, a) \mapsto a) : Q(B) \rightarrow E\) is an epimorphism of quasigroups (cf. [6, pp. 35–36]). We call this epimorphism the canonical epimorphism for \(B\).

Conversely, let \((Q, \circ)\) and \((E, \cdot)\) be quasigroups and let \(\pi : Q \rightarrow E\) be an epimorphism. For the inverse images \(T_a = \pi^{-1}(a), a \in E\), one has

\[T_a \circ T_b = T_{ab}\]

hence the set \(\{T_a, a \in E\}\) forms a quasigroup \(E'\) isomorphic to \((E, \cdot)\). Using a transversal for the partition \(\{T_a, a \in E\}\) of the set \(Q\), one can identify the inverse images \(T_a, a \in E\), with a subset \(T \subset Q\) and the set \(Q\) with the cartesian product \(T \times E\). The multiplication in \(Q\) can be written in the form (2.1), where \(\alpha, \beta \in T, a, b \in E\). This means that we have obtained a Bruck system \(B_a = (E, T, (\nabla_{a,b})(a \in E))\) for which \(Q(B_a)\) is isomorphic to \(Q\) and which has \(\pi\) as canonical epimorphism. We call this representation of \((Q, \circ)\) a Bruck decomposition of \((Q, \circ)\) with respect to \(\pi : Q \rightarrow E\).

Let \(\eta : Q \rightarrow Q\) be an endomorphism. We consider the set \(E\) of the congruence classes \(T_x = \eta^{-1}(x), x \in \eta(Q)\). The multiplication \(T_x \star T_y = T_{xy}\) defines a quasigroup \((E, \star)\) such that the mapping \(\pi : (x \mapsto T_x) : Q \rightarrow E\) is an epimorphism. We put \(\iota = \eta \circ \pi^{-1} = (T_x \mapsto \eta(x)) : E \rightarrow Q\). Then \(\iota\) is an injective homomorphism (see [5, Theorem 6.12, p. 50]).

We consider the Bruck decomposition \(Q = T \times E\) with respect to the epimorphism \(\pi\). For \(a \in E = \{T_x, x \in \eta(Q)\}\) one has \(\iota(a) = (\gamma(a), g(a))\), where \(\gamma(a) \in T, g(a) \in E\). The maps \(\gamma : E \rightarrow T\) and \(g : E \rightarrow E\) satisfy

\[\gamma(ab), g(ab) = \gamma(ab) = \iota(ab) = \iota(a) \iota(b) = (\gamma(a), g(a))(\gamma(b), g(b))\]

\[=(\gamma(a) \nabla_{g(a),g(b)} \gamma(b), g(a)g(b))\]

It follows that \(g : E \rightarrow E\) is an endomorphism and

\[\gamma(ab) = (\gamma(a)) \nabla_{g(a),g(b)} \gamma(b)\] (2.2)

holds for all \(a, b \in E\). We call the structure described here the Bruck decomposition of the quasigroup \((Q, \circ)\) with respect to the endomorphism \(\eta : Q \rightarrow Q\).

If the quasigroup \(Q\) is a loop and \(\eta : Q \rightarrow Q\) is an endomorphism, then \(K = \eta^{-1}(1)\) is a normal subloop of \(Q\). In this situation \(Q\) is a semidirect product of \(K\) and \(\eta(Q)\) if and only if \(\eta = \eta^2\) holds. We describe the Bruck decomposition with respect to an idempotent endomorphism for arbitrary quasigroups.

**Proposition 2.1.** An endomorphism \(\eta\) of a quasigroup \(Q\) is idempotent if and only if in the Bruck decomposition with respect to \(\eta\) the maps \(\gamma : E \rightarrow T\) and \(g : E \rightarrow E\) satisfy

\[g^2 = g\quad \text{and} \quad \gamma \circ g = \gamma\]

i.e., if and only if the endomorphism \(g : E \rightarrow E\) is idempotent and the map \(\gamma : E \rightarrow T\) factors over the congruence relation defined by \(g\) on \(E\).
Proof. Since \( \eta(\alpha, a) = \iota(\pi(\alpha, a)) = \iota(a) = (\gamma(a), g(a)) \) the assertion follows from

\[
\eta(\eta(\alpha, a)) = \eta(\gamma(a), g(a)) = \gamma(g(a), g(g(a)))
\]

\( \Box \)

3 The left deviation of a quasigroup

For a quasigroup \((Q, -, \backslash, /)\) we call the map \( e = (x \mapsto x\backslash x) : Q \to Q \) the \textit{left deviation}. As mentioned in the preliminaries, the left deviation of \( x \in Q \) is the local right unit element of \( x \). In a Bruck decomposition \( Q = T \times E \) with respect to an epimorphism \( Q \to E \), the deviation is \( \epsilon(a, a) = (\alpha \backslash a, a \backslash a) \), where \( a \backslash a \) is computed in the quasigroup \( E \) and \( \alpha \backslash a \) is computed in the quasigroup \( T_{a,a} = (T, \nabla_{a,a}) \). Obviously, the quasigroups in which the \textit{left deviation is an endomorphism form a variety} \( \mathcal{D}_l \) of quasigroups. For \((Q, -, \backslash, /) \in \mathcal{D}_l \) we consider the Bruck decomposition \( Q = T \times E \) with respect to the left deviation. In this case \( \epsilon(a, a) = (\gamma(a), g(a)) \) and hence \( g(a) = a \backslash a \) (computed in \( E \)) and

\[
\alpha \nabla_{a,g(a)}(a) = \alpha
\]

(3.1)

Theorem 3.1. A quasigroup \( Q \) belongs to the variety \( \mathcal{D}_l \) if and only if there exists a Bruck system \( \mathcal{B} = (E, T, (\nabla_{a,b})_{a,b \in E}) \) satisfying

(i) \( Q \cong Q(\mathcal{B}) \),

(ii) for any \( a \in E \) the quasigroup \( T_{a,a} = (T, \nabla_{a,a}) \) has a right unit element, denoted by \( \epsilon(a) \),

(iii) the map \((a \mapsto (\epsilon(a), a \backslash a)) : E \to Q(\mathcal{B}) \) is a homomorphism.

In this case the left deviation of \( Q(\mathcal{B}) \) is the map

\[ e = ((a, a) \mapsto (\epsilon(a), a \backslash a)) : Q(\mathcal{B}) \to Q(\mathcal{B}) \]

Proof. Assume first that \( Q \) belongs to \( \mathcal{D}_l \) and consider the Bruck decomposition with respect to the left deviation \( e(x) = x \backslash x \) of \( Q \). Then \( E \) is isomorphic to the subquasigroup \( e(Q) \) of \( Q \). Putting \( e = \gamma \) the assertion (ii) follows from equation (3.1) and the assertion (iii) follows from equation (2.2).

Conversely, if \( \mathcal{B} = (E, T, (\nabla_{a,b})_{a,b \in E}) \) is a Bruck system satisfying (i), then

\[
e(\alpha, a) = (\alpha, a) \backslash (\alpha, a) = (\alpha', a \backslash a)
\]

where \( \alpha = \alpha \nabla_{a,a} a' \). From (ii) it follows that \( \alpha' = \epsilon(a) \) and the deviation satisfies \( e(\alpha, a) = (\epsilon(a), a \backslash a) \). Hence we obtain from (iii) that the quasigroup \( Q \) belongs to the variety \( \mathcal{D}_l \). \( \Box \)

Corollary 3.2. Let \( \mathcal{B} = (E, T, (\nabla_{a,b})_{a,b \in E}) \) be a Bruck system satisfying the conditions (ii) and (iii) of the previous theorem. Then \( \mathcal{B} \) is a Bruck decomposition of the quasigroup \( Q(\mathcal{B}) \) with respect to the left deviation of \( Q(\mathcal{B}) \) if and only if the homomorphism

\[ e = ((a, a) \mapsto (\epsilon(a), a \backslash a)) : E \to Q(\mathcal{B}) \]

is injective.

Example 3.3. Let \((E, \cdot), T^{(1)} = (T, \circ)\), and \((T^{(2)} = (T, \star)\) be quasigroups such that the following properties are satisfied:

(a) \( E \) is a \( \mathcal{D}_l \)-quasigroup,
In this case one has

Proof.

Example 4.2.

According to Theorem 4.1 the multiplication \((a, a) \circ (\beta, b) = (\alpha \vee_{a, a} \beta, ab)\) on the set \(T \times E\) is an \(\mathcal{D}_1\)-quasigroup. For \((E, \cdot)\) and \((T, \circ)\) one can take groups having \(\epsilon : E \to T\) as a group homomorphism. The decomposition \(Q = T \times E\) is a Bruck decomposition with respect to the left deviation of \(Q\) if and only if the homomorphism \(\epsilon : E \to T\) is injective.
5 LF-quasigroups

It is known that the LF-quasigroups form a subvariety of \( \mathcal{D}_l \) (cf. [2, p. 108] and [9, Lemma 2.1]). We will now give examples of LF-quasigroups even belonging to the variety \( \mathfrak{a}\mathcal{D}_l \). Let \( E, T \) be groups and let \( \epsilon : E \rightarrow T \) be a homomorphism. Put

\[
\alpha \nabla_{a,b} \beta = \alpha \cdot \epsilon(a)^{-1} \cdot \beta
\]

for all \( a, b \in E \). Then every \( T_{a,b} = (T, \nabla_{a,b}) \) is a group (with unit element \( \epsilon(a) \)) which is isotopic and hence isomorphic to the group \( T \).

As in the previous example the multiplication \( (\alpha, a) \circ (\beta, b) = (\alpha \nabla_{a,b} \beta, ab) \) on the set \( T \times E \) defines an \( \mathfrak{a}\mathcal{D}_l \)-quasigroup (with left unit element \( (1, 1) \)) in which the left deviation is given by \( \epsilon(\alpha, a) = (\epsilon(a), 1) \) (Theorem 4.1).

**Theorem 5.1.** Let \( E, T \) be groups and let \( \epsilon : E \rightarrow T \) be a homomorphism. The set \( Q = T \times E \) equipped with multiplication \( (\alpha, a) \circ (\beta, b) = (\alpha \cdot \epsilon(a)^{-1} \cdot \beta, ab) \) is an LF-quasigroup with left unit element \( (1, 1) \) satisfying the left inverse property.

**Proof.** An easy calculation shows

\[
(\alpha, a) \circ ((\beta, b) \circ (\gamma, c)) = (\alpha \cdot \epsilon(a)^{-1} \cdot \beta \cdot \epsilon(b)^{-1} \cdot \gamma, abc)
\]

On the other hand,

\[
((\alpha, a) \circ (\beta, b)) \circ ((\epsilon(a), 1) \circ (\gamma, c)) = (\alpha \cdot \epsilon(a)^{-1} \cdot \beta, ab) \circ (\epsilon(a) \cdot \gamma, c) = (\alpha \cdot \epsilon(a)^{-1} \cdot \beta \cdot \epsilon(ab)^{-1} \cdot \epsilon(a) \cdot \gamma, abc)
\]

Hence \( (Q, \circ) \) is an LF-quasigroup in which \( (1, 1) \) is the left unit element. The left inverse of an element \( (\alpha, a) \) is given by \( (\epsilon(a) \cdot \alpha^{-1} \cdot \epsilon(a)^{-1}, a^{-1}) \). Indeed one has \( (\epsilon(a) \cdot \alpha^{-1} \cdot \epsilon(a)^{-1}, a^{-1}) \circ ((\alpha, a) \circ (\beta, b)) = (\beta, b) \).

The decomposition \( Q = T \times E \) is a Bruck decomposition with respect to the left deviation of \( Q \) if and only if the homomorphism \( \epsilon : E \rightarrow T \) is injective.

We note that in accordance with [9, Theorem 4.1] the quasigroup \( Q = T \times E \) is isotopic to the direct product of the groups \( T \) and \( E \); the isotopism is given by the triple \( (\phi, \text{id}, \text{id}) \), where \( \phi : (\alpha, a) \mapsto (\alpha \cdot \epsilon(a)^{-1}, a) \).

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Received February 15, 2009
Revised June 21, 2009