A probabilistic study of the kinetic Fokker–Planck equation in cylindrical domains

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Abstract. We consider classical solutions to the kinetic Fokker–Planck equation on a bounded domain \( \mathcal{O} \subset \mathbb{R}^d \) in position, and we obtain a probabilistic representation of the solutions using the Langevin diffusion process with absorbing boundary conditions on the boundary of the phase-space cylindrical domain \( D = \mathcal{O} \times \mathbb{R}^d \). Furthermore, a Harnack inequality, as well as a maximum principle, are provided on \( D \) for solutions to this kinetic Fokker–Planck equation, together with the existence of a smooth transition density for the associated absorbed Langevin process. This transition density is shown to satisfy an explicit Gaussian upper-bound. Finally, the continuity and positivity of this transition density at the boundary of \( D \) are also studied. All these results are in particular crucial to study the behavior of the Langevin diffusion process when it is trapped in a metastable state defined in terms of positions.

1. Introduction and motivation

In statistical physics, the evolution of a molecular system at a given temperature is typically modeled by the Langevin process:

\[
\begin{aligned}
\mathinner{dq_t} &= M^{-1} p_t \, dt, \\
\mathinner{dp_t} &= F(q_t) \, dt - \gamma M^{-1} p_t \, dt + \sqrt{2 \gamma \beta^{-1}} \, dB_t,
\end{aligned}
\]

where \( d = 3N \) for a number \( N \) of particles, \((q_t, p_t) \in \mathbb{R}^d \times \mathbb{R}^d\) denotes the respective vectors of positions and momenta of the particles, \( M \in \mathbb{R}^{d \times d} \) is the mass matrix, \( F : \mathbb{R}^d \to \mathbb{R}^d \) is the force acting on the particles, \( \gamma > 0 \) is the friction parameter, and \( \beta^{-1} = k_B T \) with \( k_B \) the Boltzmann constant and \( T \) the temperature of the system. Such dynamics are used in particular to compute thermodynamic and dynamic quantities, with numerous applications in biology, chemistry and materials science. In practice, the system remains trapped for very long times in subsets of the phase space, called metastable states, see for example [29, Sects. 6.3 and 6.4]. Typically, these states are defined in terms of positions only, and are thus cylinders of the form \( D = \mathcal{O} \times \mathbb{R}^d \) with positions living in an open set \( \mathcal{O} \) of \( \mathbb{R}^d \), and momenta in \( \mathbb{R}^d \).

Mathematics Subject Classification: 82C31, 35B50, 35B65, 60H10

Keywords: Langevin process, Kinetic Fokker-Planck equation, Transition density, Harnack inequality, Maximum principle, Gaussian upper-bound.
In order to understand the behavior of the stochastic process in such a metastable state, it is important to study the Langevin diffusion with absorbing boundary conditions when leaving $D$. This is for example useful to define the quasi-stationary distribution which can be seen as a “local stationary distribution” within the metastable state, see [27,34] and [33, Chapter 4]. This distribution is the cornerstone of the so-called accelerated dynamics algorithms to sample metastable processes over long times, see for example [26,32]. Studying this process is also important to identify the stationary distribution of the entry and exit points of the process in $D$, see [33, Chapter 5], which can then be employed to build unbiased estimators of the mean transition time between metastable states [4].

However, if the Langevin process is very much used in practice, a complete theory for the related kinetic Fokker–Planck equation, with boundary conditions, has yet to be established. In the literature, weak solutions in a domain have been studied in [2,7,17,31], as well as classical solutions in [21] for the case $d = 1$ with $F = 0$ and $\gamma = 0$, later extended for $d = 2, 3$ in [20]. The main issue for the study of solutions with boundary conditions lies in the fact that, unlike the elliptic case, solutions exhibit a loss of regularity close to a subset of the boundary called singular set, see [20,21] for more details. This work consists in an extension of the framework of classical solutions for a domain in the multi-dimensional case for the kinetic Fokker–Planck operator related to the Langevin process.

The objective of this work is to provide an ensemble of crucial properties on the absorbed Langevin process and the related kinetic Fokker–Planck equation. In particular, we will obtain:

(i) a Feynman-Kac type formula to represent probabilistically the classical solution to a partial differential equation associated with the Langevin process on a cylindrical domain, which is usually called the kinetic Fokker–Planck equation in the partial differential equation literature;

(ii) a Harnack inequality as well as a maximum principle for this partial differential equation;

(iii) the existence of a smooth transition density for the absorbed process, continuous up to the boundary, which satisfies an explicit Gaussian upper bound.

As will be explained below, such results are standard for elliptic diffusions (over-damped Langevin process), but were not proven for the Langevin process (which is not elliptic but only hypoelliptic). The non-ellipticity requires in particular a careful treatment of the boundary conditions (determining precisely the set of exit points). The proofs rely on a combination of tools from stochastic analysis (in particular a parametrix method, inspired by [24]) and analysis of partial differential equations (in particular a generalization of [17]).

Outline In Sect. 2, we give the main results, which are then proven in the subsequent sections. More precisely, Sect. 3 is devoted to the proof of the existence of a classical solution to the kinetic Fokker–Planck equation, as well as its probabilistic representation. Section 4 gives the proof of the Harnack inequality and the maximum
principle. In Sect. 5, we provide the proofs of the existence of a smooth transition density of the absorbed Langevin process as well as Gaussian upper bounds on the latter. Finally, we prove in Sect. 6 the continuity of this transition density up to the boundary of \( D \), using a so-called adjoint process and time-reversibility arguments. The proofs of intermediate or technical results are postponed to several Appendix sections.

**Notation** Let us conclude this introductory section with some notation that will be used in the following. We denote by \( x = (q, p) \) generic elements of \( \mathbb{R}^{2d} \). The Euclidean norm is denoted by \(| \cdot |\), indifferently on \( \mathbb{R}^d \) and on \( \mathbb{R}^{2d} \), and the scalar product between vectors \( \xi \) and \( \zeta \) of \( \mathbb{R}^d \) or \( \mathbb{R}^{2d} \) is denoted by \( \xi \cdot \zeta \). The open ball centered at \( \xi \) with radius \( \rho \) is denoted by \( B(\xi, \rho) \). The distance between a point \( \xi \) (resp. a subset \( A \)) and a subset \( B \) is denoted, and defined, by \( d(\xi, B) := \inf_{\zeta \in B} |\xi - \zeta| \) (resp. \( d(A, B) := \inf_{\xi \in A, \zeta \in B} |\xi - \zeta| \)).

For a subset \( A \) of \( \mathbb{R}^d \) or \( \mathbb{R}^{2d} \), we denote by:

(i) \( \overline{A} \) the closure of \( A \), \( \partial A \) its boundary and \( A^c \) its complement,
(ii) \( \mathcal{B}(A) \) the Borel \( \sigma \)-algebra on \( A \),
(iii) \( |A| \) the Lebesgue measure of \( A \) (if \( A \) is measurable).

For a subset \( A \) of \( \mathbb{R}^d, \mathbb{R}^{2d}, \mathbb{R}_+^* \times \mathbb{R}^{2d} \) or \( \mathbb{R}_+^* \times \mathbb{R}^{2d} \times \mathbb{R}^{2d} \), we denote by:

(i) for \( 1 \leq p \leq \infty \), \( \text{L}^p(A) \) the set of \( \text{L}^p \) scalar-valued functions on \( A \) and \( \| \cdot \|_{\text{L}^p(A)} \) the associated norm,
(ii) \( \mathcal{C}(A) \) (resp. \( \mathcal{C}^b(A) \)) the set of scalar-valued continuous (resp. continuous and bounded) functions on \( A \),
(iii) for \( 1 \leq k \leq \infty \), \( \mathcal{C}^k(A) \) (resp. \( \mathcal{C}^k_c(A) \)) the set of scalar-valued \( \mathcal{C}^k \) (resp. \( \mathcal{C}^k \) with compact support) functions on \( A \),
(iv) if \( A \subset \mathbb{R}^d \) or \( \mathbb{R}^{2d}, \mathcal{C}^{1,2}(\mathbb{R}_+^* \times A) \) the set of scalar-valued functions \( u(t, \xi) \) on \( \mathbb{R}_+^* \times A \) such that \( u, \partial_t u, \nabla_\xi u \) and \( \nabla_\xi^2 u \) exist and are continuous on \( \mathbb{R}_+^* \times A \).

When we work with vector-valued functions, we use such notations as \( \mathcal{C}^{\infty}(\mathbb{R}^d, \mathbb{R}^d) \) or \( \mathcal{C}([0, T], \mathbb{R}^{2d}) \). For bounded functions \( \phi \), we shall also use the notation \( \| \phi \|_\infty \) as a shorthand for the \( L^\infty \) norm.

For \( a, b \in \mathbb{R} \), we use the notation \( a \wedge b = \min(a, b) \) and \( a \vee b = \max(a, b) \). We write \( \mathbb{N} = \{0, 1, 2, \ldots\} \) and \( \mathbb{N}^* = \{1, 2, \ldots\} \). Integer intervals are denoted by \([a, b]\).

2. Main results

This section presents the main results we obtained. As a motivation, we first recall in Sect. 2.1 some well-known results for parabolic equations and overdamped Langevin processes, which we then extend to our hypoelliptic and degenerate framework: the existence of a classical solution to the kinetic Fokker–Planck equation, as well as its probabilistic representation using the absorbed Langevin process in Sect. 2.2; the existence of a transition density and Gaussian upper bounds for the Langevin process (without absorption) in Sect. 2.3; the existence of a transition density which is smooth
in the domain and continuous up to the boundary, as well as Gaussian upper bounds for the absorbed Langevin process in Sect. 2.4.

2.1. Parabolic equations and the overdamped Langevin process

As an introduction to our results, we briefly review standard material on the probabilistic interpretation, and a few properties of the associated diffusion process, of Initial-Boundary Value Problems for parabolic equations on bounded domains. The prototypical example of such a problem writes

\[
\begin{align*}
    \partial_t \overline{u}(t, q) &= \mathcal{L} \overline{u}(t, q), \quad t > 0, \quad q \in \mathcal{O}, \\
    \overline{u}(0, q) &= \overline{f}(q), \quad q \in \mathcal{O}, \\
    \overline{u}(t, q) &= \overline{g}(q), \quad t > 0, \quad q \in \partial \mathcal{O},
\end{align*}
\]

(2)

where \( \mathcal{L} \) is the second-order differential operator

\[
\mathcal{L} = F \cdot \nabla + \frac{\sigma^2}{2} \Delta
\]

for some vector field \( F : \mathbb{R}^d \rightarrow \mathbb{R}^d \) and \( \sigma > 0 \); \( \mathcal{O} \) is an open, regular and bounded subset of \( \mathbb{R}^d \); \( \overline{f} : \mathcal{O} \rightarrow \mathbb{R}, \overline{g} : \partial \mathcal{O} \rightarrow \mathbb{R} \) are given initial and boundary conditions. Under smoothness as well as compatibility assumptions on the data \( \overline{f} \) and \( \overline{g} \), the following standard reasoning can be followed:

1. weak solutions \( \overline{u} \) can be constructed by variational approach;
2. parabolic regularization implies that weak solutions are actually smooth;
3. the smoothness of \( \overline{u} \) allows to apply the Itô formula to obtain the probabilistic representation

\[
\overline{u}(t, q) = \mathbb{E} \left[ \mathbf{1}_{\tau_{\partial}^q \geq t} \overline{f}(\overline{q}^q_t) + \mathbf{1}_{\tau_{\partial}^q \leq t} \overline{g}(\overline{q}^q_{\tau_{\partial}^q}) \right],
\]

(4)

where \( (\overline{q}^q_t)_{t \geq 0} \) is the so-called overdamped Langevin process, defined by the stochastic differential equation

\[
\begin{align*}
    \mathrm{d}\overline{q}^q_t &= F(\overline{q}^q_t)\mathrm{d}t + \sigma \mathrm{d}B_t, \\
    \overline{q}^q_0 &= q,
\end{align*}
\]

(5)

and

\[
\tau_{\partial}^q := \inf\{t > 0 : \overline{q}^q_t \notin \mathcal{O}\}.
\]

This representation implies in particular the uniqueness of classical solutions to (2).

We refer for example to Evans [12, Sect. 7.1] for the first two results, and Friedman [13–15] for the last result. These references also present a Harnack inequality and a maximum principle for (2). In addition, the following facts are closely related with the probabilistic representation formula (4):
(i) for any $q \in \mathcal{O}$, the nonnegative measure \( \overline{P}_t^\mathcal{O}(q, \cdot) := \mathbb{P}(\tau_{q} > t) \) has a smooth density \( p_{t}^\mathcal{O}(q, q') \) with respect to the Lebesgue measure on \( \mathcal{O} \); (ii) this transition density satisfies the backward and forward Kolmogorov equations
\[
\partial_t \overline{P}_t^\mathcal{O}(q, q') = \mathcal{L}_q \overline{P}_t^\mathcal{O}(q, q'), \quad \partial_t \overline{P}_t^\mathcal{O}(q, q') = \mathcal{L}_q^* \overline{P}_t^\mathcal{O}(q, q'),
\]
where \( \mathcal{L}_q^* \) is the formal \( L^2(d\mu) \) adjoint of \( \mathcal{L}_q \) and the subscripts \( q, q' \) in the notation \( \mathcal{L}_q, \mathcal{L}_q^* \) indicate the variable on which the operator acts; (iii) for all \( t > 0 \), the function \( \overline{P}_t^\mathcal{O} \) is positive on \( \mathcal{O} \times \mathcal{O} \) and has a continuous extension to \( \overline{\mathcal{O}} \times \overline{\mathcal{O}} \) which vanishes on \( \partial(\mathcal{O} \times \mathcal{O}) \).

The aim of this work is to obtain similar results for the Langevin process (1) rather than the overdamped Langevin process (5). A technical tool on which several of our results crucially relies is the fact that the transition density of the Langevin process is bounded from above by an explicit Gaussian transition density (see Theorem 2.19 below). This fact is a natural extension of Baldi’s results [3, Thorme 4.2] for the overdamped Langevin process, based on the so-called parametrix method.

2.2. Kinetic Fokker–Planck equation and Langevin process

2.2.1. The kinetic Fokker–Planck equation

From now on, we fix \( \gamma \in \mathbb{R} \) and \( \sigma > 0 \), and let \( F : \mathbb{R}^d \to \mathbb{R}^d \) be a vector field satisfying the following

**Assumption (F1).** \( F \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R}^d) \).

The kinetic Fokker–Planck operator \( \mathcal{L}_{F, \gamma, \sigma} \), simply denoted by \( \mathcal{L} \) when there is no ambiguity, writes for \( (q, p) \in \mathbb{R}^d \times \mathbb{R}^d \),
\[
\mathcal{L} = \mathcal{L}_{F, \gamma, \sigma} = p \cdot \nabla q + F(q) \cdot \nabla p - \gamma p \cdot \nabla p + \frac{\sigma^2}{2} \Delta p.
\]

The operator \( \mathcal{L} \) is the infinitesimal generator of the Langevin process (1), in which we consider the mass to be identity without loss of generality (see the change of variables in [28, Equation (3.117)]). As explained in the introduction, in the case \( \gamma > 0 \) and \( \sigma^2 = 2 \gamma \beta^{-1} \) with \( \beta^{-1} = k_B T \) this process is used to describe the behavior of particles moving in a thermal bath at temperature \( T \) and a rate \( \gamma \) and subject to the force field \( F \). Let us emphasize that in the following, we consider the general case \( \gamma \in \mathbb{R} \) and \( \sigma > 0 \) not necessarily related to \( \gamma \).

Let \( \mathcal{O} \subset \mathbb{R}^d \) satisfy

**Assumption (O1).** \( \mathcal{O} \) is open, \( \mathcal{C}^2 \) and bounded, and consider the following cylindrical domain of \( \mathbb{R}^{2d} \):
\[
D := \mathcal{O} \times \mathbb{R}^d.
\]
This is the natural phase space domain of the Langevin process absorbed when leaving the set of positions in $\mathcal{O}$. For $q \in \partial \mathcal{O}$, let $n(q) \in \mathbb{R}^d$ be the unitary outward normal vector to $\mathcal{O}$ at $q \in \partial \mathcal{O}$. Let us introduce the following partition of $\partial D$:

$$
\Gamma^0 = \{(q, p) \in \partial \mathcal{O} \times \mathbb{R}^d : p \cdot n(q) = 0\},
$$

$$
\Gamma^+ = \{(q, p) \in \partial \mathcal{O} \times \mathbb{R}^d : p \cdot n(q) > 0\},
$$

$$
\Gamma^- = \{(q, p) \in \partial \mathcal{O} \times \mathbb{R}^d : p \cdot n(q) < 0\}.
$$

The kinetic Fokker–Planck equation on the domain $D$ with initial condition $f$ and boundary condition $g$ is the Initial-Boundary Value Problem

$$
\begin{cases}
\partial_t u(t, x) = Lu(t, x) & t > 0, \ x \in D, \\
u(0, x) = f(x) & x \in D, \\
u(t, x) = g(x) & t > 0, \ x \in \Gamma^+.
\end{cases} \tag{7}
$$

Notice that, in contrast with the Initial-Boundary Value Problem (2) associated with the overdamped Langevin process (5), the boundary condition only applies on the subset $\Gamma^+$ of the boundary $\partial D$. We refer the reader to [15, Chapter 11] for a study of boundary conditions for diffusions degenerating at the boundary of the domain $D$. Even if these results do not apply in our case, since the diffusion is degenerated everywhere, they still provide useful intuition on the behavior of the stochastic process at the boundary.

**Remark 2.1.** Our convention to call the Initial-Boundary Value Problem (2) ‘kinetic Fokker-Planck equation’ is the same as that in the work by Armstrong and Mourrat [2]. As we shall see in Theorem 2.10 below, in the homogeneous case $g = 0$ the solution $u$ describes the evolution of the semigroup of the Langevin process absorbed at $\partial D$.

In the literature, it seems more standard to reserve this denomination for the dual equation describing the evolution of the law of the Langevin process absorbed at $\partial D$ [7,17,20,21]. This equation writes

$$
\begin{cases}
\partial_t v(t, x) = L^* v(t, x) & t > 0, \ x \in D, \\
v(0, x) = f(x) & x \in D, \\
v(t, x) = 0 & t > 0, \ x \in \Gamma^-,
\end{cases} \tag{8}
$$

where $f$ now denotes the initial distribution of the process and $L^*$ is the formal adjoint of $L$ in $L^2(dx)$, given by

$$
L^* = - p \cdot \nabla_q - F(q) \cdot \nabla_p + \gamma \text{div}_p(p) + \frac{\sigma^2}{2} \Delta_p. \tag{9}
$$

The boundary condition is set on $\Gamma^-$ because, in this probabilistic interpretation, $v$ denotes the density of a system of particles which are allowed to escape the domain $D$ but cannot re-enter it.
We also note that, in the case where \( F = -\nabla V, \gamma > 0 \) and \( \sigma = \sqrt{2\gamma\beta^{-1}} \), the denomination ‘kinetic Fokker–Planck equation’ may refer to the variant of (8) in which the operator \( \mathcal{L}^* \) in (8) is replaced by
\[
\mathcal{L}^\dagger := -p \cdot \nabla_q + \nabla V(q) \cdot \nabla p - \gamma p \cdot \nabla V(q) - \gamma \beta^{-1} \Delta p,
\]
and which describes the evolution of the density of the Langevin process with respect to the invariant measure \( \exp(-\beta(V(q) + \frac{|p|^2}{2})) \) rather than the Lebesgue measure. See for instance [10] and the discussion in [37, Sect. 7].

In any case, there is a natural link between the solutions to (7) and (8), as it is easily checked that if \( u \) solves (7) with operator \( \mathcal{L} = \mathcal{L}_{F,\gamma,\sigma} \) and homogeneous boundary condition \( g = 0 \), then the function \( v \) defined by
\[
v(t, (q, p)) := e^{-d\gamma t} u(t, (q, -p))
\]
solves (8) with operator \( \mathcal{L}^* = \mathcal{L}_{F,\gamma,\sigma}^* \). Therefore, our results for the equation (7) also apply to (8).

For the sake of clarity, let us make precise the standard notions of solutions we will need in the following.

**Definition 2.2 (Classical solutions).** A function \( u \) is a classical solution to (7) if \( u \in C^1(\mathbb{R}^*_+ \times D) \cap C((\mathbb{R}^*_+ \times (D \cup \Gamma^+)) \setminus \{(0) \times \Gamma^+\}) \) and \( u \) satisfies (7).

Notice that the regularity on \( u \) in this definition is required for the boundary value and initial condition in (7) to hold in a classical sense. We will also use the notion of distributional solutions to
\[
\partial_t u = \mathcal{L} u \quad \text{on } \mathbb{R}^*_+ \times D,
\]
without neither initial condition nor boundary value.

**Definition 2.3 (Distributional solutions).** A distribution \( u \) on \( \mathbb{R}^*_+ \times D \) is a distributional solution of (10) if for all \( \Phi \in \mathcal{C}_c^\infty(\mathbb{R}^*_+ \times D) \),
\[
\int\int_{\mathbb{R}^*_+ \times D} u(t, x) \left( \partial_t \Phi(t, x) + \mathcal{L}^* \Phi(t, x) \right) dt dx = 0,
\]
where the operator \( \mathcal{L}^* \) is defined in (9).

A distributional solution to (10) differs from a classical solution to (7) in two ways: interior regularity, and boundary regularity (required to define boundary and initial conditions in (7)). On the one hand, additional regularity is necessary to properly define the initial and boundary values of distributional solutions, see for example the works [2,7,21,31]. On the other hand, concerning interior regularity, it is actually known that distributional solutions of (10) are \( \mathcal{C}^\infty(\mathbb{R}^*_+ \times D) \) by hypoellipticity. Let us recall these standard results, see [19].
Definition 2.4 (Hypoellipticity). A differential operator $\mathcal{G}$ is said to be hypoelliptic on an open set $A \subset \mathbb{R}^d$, $\mathbb{R}^{2d}$ or $\mathbb{R}_+^d \times \mathbb{R}^{2d}$ if for all $f \in C^\infty(A)$ and $u$ a distributional solution to $\mathcal{G}u = f$ on $A$ then $u \in C^\infty(A)$.

It is well known that under Assumption (F1) the operators $L$ and $L^*$ (resp. $\partial_t - L$ and $\partial_t - L^*$) are hypoelliptic on $D$ (resp. on $\mathbb{R}_+^d \times D$), see for example [29, Section 2.3.1] and references therein.

2.2.2. Probabilistic representation of classical solution

In this work, we are interested in the existence and uniqueness of classical solutions of (7), see Theorem 2.10 below. Besides we will show that this solution admits a probabilistic representation in terms of the Langevin process $(X_x = (q_t^x, p_t^x))_{t \geq 0}$, described by its position $q_t^x$ and velocity $p_t^x$ at time $t$ and defined by the following SDE:

\begin{align}
\begin{cases}
\mathrm{d}q_t^x &= \mathrm{d}p_t^x, \\
\mathrm{d}p_t^x &= F(q_t^x)\mathrm{d}t - \gamma p_t^x\mathrm{d}t + \sigma \mathrm{d}B_t, \\
(q_0^x, p_0^x) &= x.
\end{cases}
\end{align}

Let $\tau_0^x$ be the first exit time from $D$ of the process $(X_t^x)_{t \geq 0}$, i.e.

$$\tau_0^x = \inf\{t > 0 : X_t^x \notin D\} = \inf\{t > 0 : q_t^x \notin \mathcal{O}\}.$$

Under Assumption (F1), the drift coefficient $(q, p) \mapsto (p, F(q) - \gamma p)$ in (11) is locally Lipschitz continuous on $\mathbb{R}^{2d}$, therefore (11) admits a unique strong solution $(X_t^x)_{t \geq 0}$, which is \textit{a priori} only defined up to some explosion time $\tau_\infty^x$ by [22, Theorem IV.3.1]. Under the additional Assumption (O1), this drift coefficient is globally Lipschitz continuous on $D$, and thus the process exits the set $D$ before the explosion time almost surely, so that the solution $(X_t^x)_{t \geq 0}$ is at least well-defined until $\tau_0^x$. Since observing the process only up to the time $\tau_0^x$ amounts to imposing an absorbing boundary condition on $\partial D$, this justifies the following definition.

Definition 2.5 (Absorbed Langevin process). Under Assumptions (F1) and (O1), the process $(X_t^x)_{0 \leq t \leq \tau_0^x}$ is called the absorbed Langevin process.

Since $(X_t^x)_{0 \leq t \leq \tau_0^x}$ is a solution to the SDE (11), it is a continuous-time Markov process with almost surely continuous sample paths. Besides, since the coefficients in (11) are locally bounded on $\mathbb{R}^{2d}$, then $(X_t^x)_{0 \leq t \leq \tau_0^x}$ satisfies the strong Markov property, see [23, Theorem 4.20 p. 322].

Remark 2.6. Friedman’s uniqueness result [14, Theorem 5.2.1.] ensures that the trajectories $(X_t^x)_{0 \leq t \leq \tau_0^x}$ do not depend on the values of $F$ outside of $\mathcal{O}$. Therefore, whenever we are interested in quantities which only depend on the absorbed Langevin process, there is no loss of generality in modifying $F$ outside of $\mathcal{O}$ so that it satisfies the following strengthening of Assumption (F1):

...
**Assumption (F2).** $F \in C^\infty(\mathbb{R}^d, \mathbb{R}^d)$ and $F$ is bounded and globally Lipschitz continuous on $\mathbb{R}^d$.

Under Assumption (F2), the drift coefficient $(q, p) \mapsto (p, F(q) - \gamma p)$ in (11) is globally Lipschitz continuous, with a Lipschitz constant which we shall denote by $C_{\text{Lip}}$, and therefore the strong solution $(X_t^x)_{t \geq 0}$ (without absorbing boundary condition) is defined globally in time.

In order to describe the probabilistic representation of the classical solution to (7), we first state a trajectorial result on the solution $(q_t^x, p_t^x)_{t \geq 0}$ for $x \in \mathbb{R}^d \setminus \Gamma_0$.

We prove that, almost surely, the process $(q_t^x, p_t^x)_{t \geq 0}$ does not reach the set $\Gamma_0$ in finite time. In other words, the set $\Gamma_0$ is non attainable in the sense of [15, Chapter 11.1].

**Proposition 2.7** (Non-attainability of $\Gamma_0$). Under Assumptions (O1) and (F2), for all $x \in \mathbb{R}^d \setminus \Gamma_0$,

\[ P \left( \exists t > 0 : X_t^x \in \Gamma_0 \right) = 0. \]  

Using this non-attainability result we are able to characterize more precisely the exit event from $D$ in the following proposition.

**Proposition 2.8** (Attainability of the boundary). Let Assumptions (O1) and (F1) hold.

(i) If $x \in \Gamma^+ \cup \Gamma_0$, then for all $\epsilon > 0$, $(q_t^x, p_t^x)_{t \geq 0}$ visits $\overline{D}$ almost surely on $[0, \epsilon]$, i.e.

\[ P \left( \exists t \in [0, \epsilon] : q_t^x \in \overline{D} \right) = 1. \]  

In particular, $\tau^x_\partial = 0$ almost surely.

(ii) If $x \in D \cup \Gamma^-$, then $\tau^x_\partial > 0$ almost surely and one has

\[ P \left( p_{\tau^x_\partial} \cdot n(q_{\tau^x_\partial}) \leq 0, \tau^x_\partial < \infty \right) = 0. \]  

**Remark 2.9.** One can actually prove that for all $x \in D \cup \Gamma^-$, $\tau^x_\partial < \infty$ almost surely (since it admits exponential moments see [27, Remark 2.20]). Therefore, the equality (13) can be written equivalently: for all $x \in D \cup \Gamma^-$, $P(X_{\tau^x_\partial} \in \Gamma^-) = 1$.

Proposition 2.8 implies that for all $t \geq 0$, almost surely, if $\tau^x_\partial > t$ then $X_t^x \in D \cup \Gamma^-$, and if $\tau^x_\partial \leq t$ then $X_{\tau^x_\partial} \in \Gamma^+ \cup \Gamma^0$. This ensures that the definition of the function $u$ in Equation (14) below is legitimate.

We are now in position to state the main result of this section, namely the existence of a unique classical solution to the kinetic Fokker–Planck equation (7), and its probabilistic representation.

**Theorem 2.10** (Classical solution and probabilistic representation for the kinetic Fokker–Planck equation (7)). Under Assumptions (O1) and (F1), let $f \in C^b(D \cup \Gamma^-)$ and $g \in C^b(\Gamma^+ \cup \Gamma^0)$, and define the function $u$ on $\mathbb{R}_+ \times \overline{D}$ by

\[ u : (t, x) \mapsto \mathbb{E} \left[ 1_{\tau^x_\partial > t} f(X_t^x) + 1_{\tau^x_\partial \leq t} g(X_{\tau^x_\partial}^x) \right]. \]  

Then we have the following results:
(i) Initial and boundary values: the function $u$ satisfies
\[ u(0, x) = \begin{cases} f(x) & \text{if } x \in D \cup \Gamma^-, \\ g(x) & \text{if } x \in \Gamma^+ \cup \Gamma^0, \end{cases} \]
and
\[ \forall t > 0, \forall x \in \Gamma^+ \cup \Gamma^0, \quad u(t, x) = g(x). \]

(ii) Continuity: $u \in C^b((\mathbb{R}_+ \times \bar{D}) \setminus ([0] \times (\Gamma^+ \cup \Gamma^0)))$, and if $f$ and $g$ satisfy the compatibility condition
\[ x \in \bar{D} \mapsto 1_{x \in D \cup \Gamma^-} f(x) + 1_{x \in \Gamma^+ \cup \Gamma^0} g(x) \in C^b(\bar{D}), \]
then $u \in C^b(\mathbb{R}_+ \times \bar{D})$.

(iii) Interior regularity: $u \in C^\infty(\mathbb{R}_+^* \times D)$ and, for all $t > 0$, $x \in D$,
\[ \partial_t u(t, x) = Lu(t, x). \]

The three items (i), (ii) and (iii) show that $u$ defined by (14) is a classical solution to (7) in the sense of Definition 2.2. We also have the following uniqueness result for classical solutions to (7):

(iv) Let $v$ be a classical solution to (7) in the sense of Definition 2.2. If, for all $T > 0$, $v$ is bounded on the set $[0, T] \times D$, then $v(t, x) = u(t, x)$ for all $(t, x) \in (\mathbb{R}_+ \times (D \cup \Gamma^+)) \setminus ([0] \times \Gamma^+)$.

Proposition 2.8 and Theorem 2.10 are proven in Sect. 3. The proof essentially follows the same three-step structure as for the probabilistic representation formula (4) of solutions to the Initial-Boundary Value Problem (2): we construct weak (actually, distributional) solutions to (16) by parabolic approximation, use the hypoellipticity of $\partial_t - L$ to obtain the smoothness of such solutions and apply the Itô formula to identify the solution with $u$ defined by (14). We mention here that, regarding the first step, a variational approach to (7), closer to the spirit of the proof outlined in Sect. 2.1 than our parabolic approximation argument, was recently developed by Armstrong and Mourrat [2].

Remark 2.11. (Extension to bounded and measurable functions) Let $f : D \cup \Gamma^- \to \mathbb{R}$ be measurable and bounded, and take $g \equiv 0$. Using an elementary regularization argument, which can be rigorously justified with the help of Theorem 2.20 and Corollary 2.22 stated below, it is easy to check that the function $u$ defined by (14) remains a distributional solution of (16) on $\mathbb{R}_+^* \times D$ and therefore, by hypoellipticity, still satisfies Assertion (iii).

Remark 2.12. It is easy to check that, using the same proofs, these results also hold for a time-dependent boundary condition $g(t, x) \in C^b(\mathbb{R}_+ \times (\Gamma^+ \cup \Gamma^0))$, replacing (14) by $u(t, x) = \mathbb{E}[1_{\tau_{\Gamma^+} > t} f(X^x_t) + 1_{\tau_{\Gamma^0} \leq t} g(t - \tau_{\Gamma^0}, X^x_{\tau_{\Gamma^0}})]$. We stick to a time-homogeneous boundary conditions for the ease of notation.
Remark 2.13. Note that the compatibility condition (15) is necessary to ensure the continuity of the solution at $[0] \times \overline{D}$. Furthermore, it is known that even for smooth and compatible boundary and initial conditions, one cannot expect the solution to be smooth at the boundary $\partial D$: it has indeed been shown in the one-dimensional case ($d = 1$) that the solution is only expected to be Hölder-continuous on the singular set $\Gamma^0 = \{(q, p) \in \partial \mathcal{O} \times \mathbb{R}^d : p \cdot n(q) = 0\}$, and not differentiable, see [21].

2.2.3. Maximum principle and Harnack inequality

As an immediate consequence of Theorem 2.10, under Assumptions (O1) and (F1), if $f \geq 0$ on $D$ and $g \geq 0$ on $\Gamma^+$ then it follows that any solution $v$ of (7) which satisfies the conditions of item (iv) in Theorem 2.10 is such that $v \geq 0$ on $\mathbb{R}^+ \times \overline{D}$.

We now state stronger forms of this maximum principle, as well as a Harnack inequality, under the following supplementary assumption on the domain $\mathcal{O}$.

**Assumption (O2).** The set $\mathcal{O}$ is connected.

**Theorem 2.14** (Maximum principle). Let Assumptions (F1), (O1) and (O2) hold. Let $u \in C^{1,2}(\mathbb{R}^+ \times D)$ such that $\partial_t u - Lu \leq 0$ on $\mathbb{R}^+ \times D$.

(i) Assume that $u \in C^b((\mathbb{R}^+ \times (D \cup \Gamma^+)) \setminus ([0] \times \Gamma^+))$, then

$$\sup_{\mathbb{R}^+ \times D} u(t, x) = \sup_{(t=0] \times D) \cup ([0] \times \Gamma^+)} u(t, x).$$

(ii) Assume that $u$ reaches a maximum at a point $(t_0, x_0) \in \mathbb{R}^+ \times D$, then

$$\forall t \leq t_0, \quad \forall x \in D, \quad u(t, x) = u(t_0, x_0).$$

Theorem 2.14 is proven in Sect. 4.2. Let us conclude this section by stating a Harnack inequality. In the literature, a variant of the Harnack inequality was obtained in the stationary case for hypoelliptic operators, see [6]. In [17], the authors prove a Harnack inequality satisfied by distributional solutions to $\partial_t u = Lu$ in sufficiently small domains. Here, we extend their result on a general compact set of $D$ in the following theorem. The proof uses in particular the concept of Harnack chains from [1].

**Theorem 2.15** (Harnack inequality). Let Assumptions (F1), (O1) and (O2) hold. For any compact set $K \subset D$, $\epsilon > 0$ and $T > 0$, there exists a constant $C_{K, \epsilon, T} > 0$ such that for any non-negative distributional solution $u$ of $\partial_t u = Lu$ on $\mathbb{R}^+ \times D$ (in the sense of Definition 2.3), for all $t \geq \epsilon$,

$$\sup_{x \in K} u(t, x) \leq C_{K, \epsilon, T} \inf_{x \in K} u(t + T, x).$$

Theorem 2.15 is proven in Sect. 4.1.

**Remark 2.16.** For a given compact set $K \subset \mathbb{R}^d$, one can find an open set $\mathcal{O}$ satisfying Assumptions (O1) and (O2) such that $K \subset \mathcal{O} \times \mathbb{R}^d$. Therefore, Theorem 2.15 implies
the following statement: under Assumption (F1), for any compact set $K \subset \mathbb{R}^d$, $\epsilon > 0$ and $T > 0$, there exists a constant $C_{K,\epsilon,T} > 0$ such that, for all $t \geq \epsilon$, the Harnack inequality (18) holds for any non-negative distributional solution $u$ of $\partial_t u = L u$ on the whole space $\mathbb{R}_+^* \times \mathbb{R}^d$.

2.3. Kolmogorov equations and Gaussian bounds for the Langevin process

In this section, we consider the Langevin process (11) without absorbing boundary condition. We recall that under Assumption (F2), for all $x \in \mathbb{R}^d$, the equation (11) admits a unique strong global solution $(X_t^x)_{t \geq 0}$ on $\mathbb{R}^d$. Let us introduce the associated transition kernel $P_t$:

$$
\forall t > 0, \quad \forall x \in \mathbb{R}^d, \quad \forall A \in \mathcal{B}(\mathbb{R}^d), \quad P_t(x, A) := \mathbb{P}(X^x_t \in A).
$$

The following standard properties of $P_t(x, \cdot)$ are for example proven in [35, Corollary 7.2] (see also Equations (153) and (155) there).

**Proposition 2.17** (Kolmogorov equations for the Langevin process). Under Assumption (F2), there exists a function $(t, x, y) \mapsto p_t(x, y) \in C^\infty(\mathbb{R}_+^* \times \mathbb{R}^d \times \mathbb{R}^d)$ (19) such that for all $t > 0$, $x \in \mathbb{R}^d$ and $A \in \mathcal{B}(\mathbb{R}^d)$,

$$
P_t(x, A) = \int_A p_t(x, y) dy.
$$

Moreover, this transition density satisfies the backward and forward Kolmogorov equations:

(i) $(t, x) \mapsto p_t(x, y)$ satisfies $\partial_t p = L_x^* p$ on $\mathbb{R}_+^* \times \mathbb{R}^d$,

(ii) $(t, y) \mapsto p_t(x, y)$ satisfies $\partial_t p = L_y^* p$ on $\mathbb{R}_+^* \times \mathbb{R}^d$.

The subscripts in $L_x$ and $L_y^*$ indicate on which variables the differential operators apply. We will also need the following immediate corollary of Proposition 2.17.

**Corollary 2.18** (Atoms of $\tau^x_\partial$). Under the assumptions of Proposition 2.17, for all $x \in \overline{D}$, for all $t > 0$,

$$
\mathbb{P}(\tau^x_\partial = t) \leq \mathbb{P}(q_t^x \in \partial \mathcal{O}) = 0.
$$

Theorem 2.19 below states that the transition density $p_t(x, y)$ admits an explicit Gaussian upper bound. This has already been proven in [24] in the case $\gamma = 0$, using the parametrix method, if Assumption (F2) holds. In this case, the drift coefficient $(q, p) \mapsto F(q)$ for the velocity-related SDE in (11) is indeed globally Lipschitz continuous and bounded, and thus satisfies the hypothesis required in [24]. If $\gamma \neq 0$ the drift coefficient $(q, p) \mapsto F(q) - \gamma p$ is not bounded in $\mathbb{R}^d$, but we adapt the idea of the parametrix method to obtain a Gaussian upper bound in this case, see Section 5.1.
**Theorem 2.19** (Gaussian upper bound on $p_t$). Under Assumption (F2), the transition density $p_t(x, y)$ of the Langevin process $(X_t^x)_{t \geq 0}$ satisfying (11) is such that for all $\alpha \in (0, 1)$, there exists $c_\alpha > 0$, depending only on $\alpha$, such that for all $T > 0$ and $t \in (0, T]$, for all $x, y \in \mathbb{R}^{2d}$,

$$p_t(x, y) \leq \frac{1}{\alpha^d} \sum_{j=0}^{\infty} \left( \|F\|_\infty c_\alpha (1 + \sqrt{\gamma_- T})^j \sigma^j \right) \hat{p}_{\alpha}^j(x, y),$$

(20)

where $\gamma_- = \max(-\gamma, 0)$ is the negative part of $\gamma \in \mathbb{R}$, $\Gamma$ is the Gamma function and $\hat{p}_{\alpha}^j(x, y)$ is the transition density of the Gaussian process with infinitesimal generator $\mathcal{L}_{0, \gamma, \sigma / \sqrt{\alpha}}$ defined in (6), see also Equations (75)–(79) below for explicit formulas.

2.4. Kolmogorov equations for the absorbed Langevin process

Let us define the transition kernel $P_t^D$ for the absorbed Langevin process $(X_t^x)_{0 \leq t \leq \tau_\delta^x}$:

$$\forall t \geq 0, \quad \forall x \in \overline{D}, \quad \forall A \in \mathcal{B}(D), \quad P_t^D(x, A) := \mathbb{P}(X_t^x \in A, \tau_\delta^x > t).$$

(21)

It is easy to see that for any $t \geq 0, x \in \overline{D}$ and $A \in \mathcal{B}(D)$,

$$P_t^D(x, A) \leq P_t(x, A).$$

(22)

The next theorem is the equivalent of Proposition 2.17 for the transition kernel $P_t^D(x, \cdot)$.

**Theorem 2.20** (Kolmogorov equations for the absorbed Langevin process). Under Assumptions (O1) and (F1), there exists a function

$$(t, x, y) \mapsto p_t^D(x, y) \in C^\infty(\mathbb{R}_+^* \times D \times D) \cap C(\mathbb{R}_+^* \times \overline{D} \times \overline{D})$$

such that for all $t > 0, x \in \overline{D}$ and $A \in \mathcal{B}(D)$,

$$P_t^D(x, A) = \int_A p_t^D(x, y) dy.$$

Moreover, this transition density $p_t^D$ satisfies the backward and forward Kolmogorov equations:

(i) $(t, x) \mapsto p_t^D(x, y)$ satisfies $\partial_t p_t^D = \mathcal{L}_x p_t^D$ on $\mathbb{R}_+^* \times D$,

(ii) $(t, y) \mapsto p_t^D(x, y)$ satisfies $\partial_t p_t^D = \mathcal{L}^*_y p_t^D$ on $\mathbb{R}_+^* \times D$.

Finally, for all $t > 0, (x, y) \in \overline{D} \times \overline{D}$,

(i) $p_t^D(x, y) = 0$ if $x \in \Gamma^+ \cup \Gamma^0$ or $y \in \Gamma^- \cup \Gamma^0$,

(ii) under Assumption (O2), $p_t^D(x, y) > 0$ if $x \notin \Gamma^+ \cup \Gamma^0$ and $y \notin \Gamma^- \cup \Gamma^0$.

**Remark 2.21.** The positivity of the transition density $p_t^D$ is obtained using the Harnack inequality from Theorem 2.15. In addition, as explained above, if the unabsorbed Langevin process is defined globally in time, the existence of a smooth transition
density $p_t$ satisfying the Kolmogorov equations follows from [35, Corollary 7.2]. Using Remark 2.16, one can then apply the Harnack inequality to ensure the positivity of $p_t$ on $\mathbb{R}^{2d} \times \mathbb{R}^{2d}$. This extends the positivity result from [18, Corollary 3.3] to non-polynomial coefficients.

The proof of the existence of a transition density $p^D_t(x, y)$ which is smooth on $\mathbb{R}^*_+ \times D \times D$ and satisfies the backward and forward Kolmogorov equations will be done in Propositions 5.4 and 5.5, in Sect. 5.2. The results on the continuity at the boundary as well as on the positivity of $p^D_t$ are stated in Theorem 6.3. They crucially rely on the introduction and study of a so-called adjoint process to (11), which is carried out in Sect. 6.

Under Assumptions (O1) and (F1), the transition density $p^D_t$ only depends on the values of $F$ inside $O$, see Remark 2.6. Therefore, up to a modification of $F$ outside of $O$ so that $F$ satisfies (F2), the following useful corollary of Theorem 2.20, the inequality (22) and Theorem 2.19 can be deduced.

**Corollary 2.22** (Gaussian upper bound on $p^D_t$). Under Assumptions (O1) and (F1), $p^D_t(x, y)$ satisfies the Gaussian upper bound of Theorem 2.19, in which the quantity $\|F\|_{\infty}$ is replaced by $\|F\|_{L^\infty(D)}$.

### 3. The initial-boundary value problem

This section is devoted to the proof of Theorem 2.10. In this theorem, Assertion (i) is an immediate consequence of Proposition 2.8 which is proven in Section 3.4. Assertions (iv), (ii) and (iii) are, respectively, proven in Sects. 3.1, 3.2 and 3.3. Finally, Sect. 3.4 is devoted to the proofs of Propositions 2.7 and 2.8.

All the results in this section are proven under Assumptions (O1) and (F2). Since the final statement of Theorem 2.10 only depends on the values of $F$ in $O$ (see Remark 2.6), this statement remains valid if Assumption (F2) is replaced by Assumption (F1).

Before proceeding, we introduce the following notation: we let $d_{\partial}$ be the Euclidean distance function to the boundary $\partial O$ from a point in $O$, i.e.

$$d_{\partial} : q \in \mathbb{R}^d \mapsto \begin{cases} d(q, \partial O) & \text{if } q \in O, \\ 0 & \text{if } q \notin O, \end{cases}$$

and $d_{\overline{O}}$ be the Euclidean distance to the compact set $\overline{O}$,

$$d_{\overline{O}} : q \in \mathbb{R}^d \mapsto d(q, \overline{O}).$$

These distance functions are 1-Lipschitz continuous.

#### 3.1. Uniqueness: proof of Assertion (iv) in Theorem 2.10

Assertion (iv) in Theorem 2.10 follows from the application of the Itô formula. Although the general argument is well-known, we detail its application here in order to
emphasize two specificities of our framework: the fact that the domain \( D \) is unbounded, and the fact that the kinetic nature of the Langevin process \((X_t^x)_{t \geq 0}\) makes boundary conditions on \( v \) only necessary on the subset \( \Gamma^+ \) of \( \partial D \).

**Proof of Assertion (iv) in Theorem 2.10.** Let \( g \in C^b(\Gamma^+ \cup \Gamma^0) \) and \( f \in C^b(D \cup \Gamma^-) \). Let \( v \) be a solution of (7), satisfying the conditions of Assertion (iv) in Theorem 2.10.

Let \( x \in D \). Let \((X_t^x = (q_t^x, p_t^x))_{t \geq 0}\) be the strong solution of (11) on \( \mathbb{R}^2 \). For \( k > 0 \), let \( V_k \) be the following open and bounded subset of \( D \)

\[
V_k := \left\{ (q, p) \in D : |p| < k, d_\partial(q) > \frac{1}{k} \right\}.
\]

(25)

Let us choose \( k \) large enough so that \( x \in V_k \). Let \( \tau_{V_k}^x \) be the following stopping time:

\[
\tau_{V_k}^x = \inf\{ t > 0 : X_t^x \notin V_k \}.
\]

(26)

Let \( t > 0 \) and \( s \in [0, t) \). Since \( v \in C^{1,2}(\mathbb{R}^n_+ \times D) \), Itô’s formula applied to the process \( (v(t-r, X_r^x))_{0 \leq r \leq s} \) between 0 and \( s \wedge \tau_{V_k}^x \) yields: almost surely, for \( s \in [0, t) \)

\[
v(t - s \wedge \tau_{V_k}^x, X_{s \wedge \tau_{V_k}^x}^x) = v(t, x) + \sigma \int_0^{s \wedge \tau_{V_k}^x} \nabla_p v(t - r, X_r^x) \cdot dB_r,
\]

since \( \partial_t v - \mathcal{L} v = 0 \) on \( \mathbb{R}^n_+ \times D \). Besides, since \( \nabla_p v \) is continuous on the compact set \([t-s, t] \times V_k\), hence bounded on \((t-s, t) \times V_k\), the stochastic integral in the right-hand side is a martingale and its expectation vanishes. Therefore

\[
v(t, x) = \mathbb{E}\left[ v(t - s \wedge \tau_{V_k}^x, X_{s \wedge \tau_{V_k}^x}^x) \right] = \mathbb{E}\left[ 1_{\tau_{V_k}^x > s} v(t - s, X_s^x) + 1_{\tau_{V_k}^x \leq s} v(t - \tau_{V_k}^x, X_{\tau_{V_k}^x}^x) \right].
\]

(26)

Now we would like to let \( k \rightarrow \infty \) and then \( s \rightarrow t \) in (26).

First let us prove the following limit

\[
\lim_{k \rightarrow \infty} \tau_{V_k}^x = \tau_0^x \quad \text{almost surely.}
\]

The sequence \( (\tau_{V_k}^x)_{k \geq 1} \) is an increasing sequence of random variables, therefore it converges almost surely to \( \sup_{k \geq 1} \tau_{V_k}^x \). Besides, using the continuity of the trajectories of \((X_t^x)_{t \geq 0}\), one gets, for all \( r > 0 \),
\[
\left\{ \sup_{k \geq 1} \tau^x_{k \ell} > r \right\} = \left\{ \exists k \geq 1 : \tau^x_{k \ell} > r \right\} \\
= \left\{ \exists k \geq 1 : \sup_{u \in [0,r]} |p^x_u| < k, \ \inf_{u \in [0,r]} d_\theta(q^x_u) > \frac{1}{k} \right\} \\
= \left\{ \sup_{u \in [0,r]} |p^x_u| < \infty, \ \inf_{u \in [0,r]} d_\theta(q^x_u) > 0 \right\} \\
= \left\{ \sup_{u \in [0,r]} |p^x_u| < \infty, \ \tau^\ell_{\theta} > r \right\}.
\]

For all \( r > 0 \), we have that, almost surely, \( \sup_{u \in [0,r]} |p^x_u| < \infty \). Therefore, \( \sup_{k \geq 1} \tau^x_{k \ell} > r \) if and only if \( \tau^\ell_{\theta} > r \), that is to say \( \sup_{k \geq 1} \tau^x_{k \ell} = \tau^\ell_{\theta} \) almost surely. As a result, one gets \( \lim_{k \to \infty} \tau^x_{k \ell} = \tau^\ell_{\theta} \) almost surely. Consequently, since \( k \mapsto \tau^x_{k \ell} \) is increasing and \( t \mapsto \mathbb{1}_{t > s} \) is left-continuous, one has almost surely that for all \( s > 0 \),

\[
\mathbb{1}_{\tau^x_{k \ell} > r} \xrightarrow{k \to \infty} \mathbb{1}_{\tau^\ell_{\theta} > s}.
\]

Second, notice that \( X^x_{\tau^\ell_{\theta}} \in \Gamma^+ \) almost surely on the event \( \{ \tau^\ell_{\theta} \leq s \} \) by Proposition 2.8 since \( x \in D \). Consequently, since \( v \in C(\mathbb{R}^*_+ \times (D \cup \Gamma^+)) \) and \( v = g \) on \( \mathbb{R}^*_+ \times \Gamma^+ \)

\[
\mathbb{1}_{\tau^x_{k \ell} \leq s} v(t - \tau^x_{k \ell}, X^x_{\tau^x_{k \ell}}) \xrightarrow{k \to \infty} \mathbb{1}_{\tau^\ell_{\theta} \leq s} g(X^x_{\tau^\ell_{\theta}}) \quad \text{almost surely.}
\]

We now use (27) and (28) to apply the dominated convergence theorem to (26) when \( k \) goes to infinity, using the fact that \( v \) is assumed to be bounded on \([0, t] \times D \). Therefore, one gets for \( x \in D \) and \( s \in [0, t) \),

\[
v(t, x) = \mathbb{E} \left[ \mathbb{1}_{\tau^\ell_{\theta} > s} v(t - s, X^x_s) \right] + \mathbb{E} \left[ \mathbb{1}_{\tau^\ell_{\theta} \leq s} g(X^x_{\tau^\ell_{\theta}}) \right].
\]

Finally, let us consider the limit \( s \to t \) in (29). Notice that

\[
\mathbb{1}_{\tau^\ell_{\theta} > s} \xrightarrow{s \to t} \mathbb{1}_{\tau^\ell_{\theta} > t} \quad \text{almost surely,}
\]

using Corollary 2.18 (which follows from Proposition 2.17, which holds independently from the results proven in this section, as will become clear in Sect. 3.4). Therefore, the second term in the right-hand side of the equality (29) satisfies by dominated convergence,

\[
\mathbb{E} \left[ \mathbb{1}_{\tau^\ell_{\theta} \leq t} g(X^x_{\tau^\ell_{\theta}}) \right] \xrightarrow{s \to t} \mathbb{E} \left[ \mathbb{1}_{\tau^\ell_{\theta} \leq t} g(X^x_{\tau^\ell_{\theta}}) \right].
\]

Moreover the continuity of the trajectories of \((X^x_s)_{s \geq 0}\) and the continuity of \( v \) on \( \mathbb{R}^*_+ \times D \) ensure that

\[
\mathbb{1}_{\tau^\ell_{\theta} > s} v(t - s, X^x_s) \xrightarrow{s \to t} \mathbb{1}_{\tau^\ell_{\theta} > t} v(0, X^x_t) = \mathbb{1}_{\tau^\ell_{\theta} > t} f(X^x_t) \quad \text{almost surely.}
\]
Finally, taking the limit \( s \to t \) in (29), the dominated convergence theorem ensures that
\[
v(t, x) = \mathbb{E} \left[ 1_{\tau_{x}^{t} > t} f(X_{t}^{x}) + 1_{\tau_{x}^{t} \leq t} g(X_{t}^{x}) \right]
\]
for all \( t > 0, x \in D \). This concludes the proof of Assertion (iv) in Theorem 2.10 using the continuity of \( v \) in \((\mathbb{R}_{+} \times (D \cup \Gamma^{+})) \setminus \{(0) \times \Gamma^{+}\} \). \( \square \)

### 3.2. Continuity: proof of Assertion (ii) in Theorem 2.10

The proof of Assertion (ii) in Theorem 2.10 relies on the following lemmas. We recall that under Assumption (F2), we denote by \( C_{\text{Lip}} \) the Lipschitz constant of the drift of (11).

**Lemma 3.1** (Gronwall Lemma). **Under Assumption (F2), for all \( t \geq 0, for all x, y \in \mathbb{R}^{2d}, one has**
\[
\sup_{s \in [0, t]} |X_{x}^{t} - X_{y}^{t}| \leq |x - y| e^{C_{\text{Lip}} t} \text{ almost surely.} \tag{30}
\]

**Besides, for \((t_0, x_0) \in \mathbb{R}_{+} \times \mathbb{R}^{2d} we have**
\[
X_{x}^{t} \underset{(t, x) \to (t_0, x_0)}{\longrightarrow} X_{t_0}^{x_0} \text{ almost surely.} \tag{31}
\]

The estimate (30) follows from a standard application of the Gronwall Lemma which we do not detail here. The joint continuity statement (31) is then a straightforward consequence of the continuity of the trajectories of \((X_{x}^{t})_{t \geq 0}\).

**Lemma 3.2** (Continuity of the exit event indicator). **Under Assumptions (O1) and (F2), let \((t, x) \in (\mathbb{R}_{+} \times \overline{D}) \setminus (\{0\} \times (\Gamma^{+} \cup \Gamma^{0}))\). Let \((t_n, x_n)_{n \geq 1} be a sequence of \( \mathbb{R}_{+} \times \overline{D} \) converging towards \((t, x)\). Then one has**
\[
1_{\tau_{x}^{t} > t_n} \underset{n \to \infty}{\longrightarrow} 1_{\tau_{x}^{t} > t} \text{ almost surely.} \tag{32}
\]

**Proof.** We prove the convergence (32) on the events \( \{\tau_{x}^{t} < t\}, \{\tau_{x}^{t} = t\} \) and \( \{\tau_{x}^{t} > t\} \), separately.

**Step 1.** Let us start by proving (32) on the event \( \{\tau_{x}^{t} < t\} \) (necessarily \( t > 0 \)). Let \( \epsilon \in (0, t - \tau_{x}^{t}) \). If \( x \in \Gamma^{+} \cup \Gamma^{0} \), then by Proposition 2.8, \( \tau_{x}^{t} = 0 \) and there exists \( s \in (0, \epsilon) \) such that \( d_{\overline{\sigma}}(q_{x}^{s}) > 0 \). If \( x \in D \cup \Gamma^{+} \), then \( X_{x}^{t} \overset{\text{a.s.}}{\longrightarrow} \Gamma^{+} \) almost surely by Proposition 2.8. By the strong Markov property and Proposition 2.8, there exists again \( s \in (0, \epsilon) \) such that \( d_{\overline{\sigma}}(q_{x}^{s}) > 0 \).

Since \( x_n \overset{n \to \infty}{\longrightarrow} x \), there exists \( N_1 \geq 1 \) such that for all \( n \geq N_1 \), \( |x_n - x| \leq \frac{d_{\overline{\sigma}}(q_{x}^{s}) e^{-C_{\text{Lip}}(\tau_{x}^{t} + \epsilon)}}{2} \). As a result using Lemma 3.1 along with the fact that the distance function \( d_{\overline{\sigma}} \) is 1-Lipschitz continuous, it follows that for all \( n \geq N_1 \)
\[
d_{\overline{\sigma}}(q_{x}^{s}) \geq \frac{d_{\overline{\sigma}}(q_{x}^{s})}{2} > 0.
\]
Therefore, we have $\tau_\theta^{x_n} < s \leq \tau_\theta^x + \epsilon$. In addition, the convergence $t_n \rightarrow t$ implies that there exists $N_2 \geq 1$ such that for all $n \geq N_2$, one has $t_n \geq \tau_\theta^x + \epsilon$, since $\tau_\theta^x + \epsilon < t$. As a result for $n \geq \max(N_1, N_2)$,

$$\tau_\theta^{x_n} < \tau_\theta^x + \epsilon \leq t_n.$$ 

Hence the convergence (32) on the event $\{\tau_\theta^x < t\}$.

**Step 2.** Let us consider the event $\{\tau_\theta^x = t\}$. For $t = 0$ and $x \in D \cup \Gamma^-$, $\mathbb{P}(\tau_\theta^x = 0) = 0$ by Proposition 2.8. Moreover for $t > 0$, $\mathbb{P}(\tau_\theta^x = t) = 0$ by Corollary 2.18. As a result, it is not necessary to prove the convergence (32) on the event $\{\tau_\theta^x = t\}$ as the latter is negligible.

**Step 3.** Finally, it only remains to prove the convergence (32) on the event $\{\tau_\theta^x > t\}$. Let $t' := \frac{1}{2}(\tau_\theta^x + t)$. Since $t_n \rightarrow t$, there exists $N_1 \geq 1$ such that for $n \geq N_1$, $t_n \leq t'$. On the one hand, if $x \in D$, then by the continuity of the trajectories of $(q_s^x)_{s \in [0, t']}$, one has $\inf_{s \in [0, t']} d_\theta(q_s^x) > 0$. By Lemma 3.1 and the fact that the distance function $d_\theta$ is 1-Lipschitz continuous, there exists $N_2 \geq 1$ such that for $n \geq N_2$,

$$\inf_{s \in [0, t']} d_\theta(q_s^x) > 0,$$

which yields $\tau_\theta^{x_n} > t'$. As a result, for $n \geq \max(N_1, N_2)$,

$$\tau_\theta^{x_n} > t' \geq t_n.$$ 

On the other hand, if $x \in \partial D$, then necessarily $x \in \Gamma^-$, otherwise $\tau_\theta^x = 0$ by Proposition 2.8. Then for all $k \geq 1$,

$$\inf_{s \in \left[\frac{1}{k}, t'\right]} d_\theta(q_s^x) > 0.$$ 

Using Lemma 3.1 again, we get that there exists $M_k \geq 1$ such that for $n \geq M_k$, $\tau_\theta^{x_n} > t'$ or $\tau_\theta^{x_n} \leq \frac{1}{k}$. Assume that there exists an unbounded sequence $(n_k)_{k \geq 1}$ such that $\tau_\theta^{x_{n_k}} \leq \frac{1}{k}$. Then $x \in \Gamma^+ \cup \Gamma^0$ since $X_{\tau_\theta^{x_{n_k}}}^x \in \Gamma^+$ and $X_{\tau_\theta^{x_{n_k}}}^x \rightarrow x$ by Lemma 3.1, which is in contradiction with the fact that $x \in \Gamma^-$. As a result there exists $N_2 \geq 1$ such that for all $n \geq N_2$, $\tau_\theta^{x_n} > t'$. Hence, for $n \geq \max(N_1, N_2)$, $\tau_\theta^{x_n} > t_n$. This concludes the proof of the convergence (32) on the event $\{\tau_\theta^x > t\}$. \hfill \Box

**Remark 3.3.** Notice that the convergence (32) cannot be satisfied for $(t, x) \in \{0\} \times (\Gamma^+ \cup \Gamma^0)$. Indeed, if $x \in \Gamma^+ \cup \Gamma^0$ and $(x_n)_{n \geq 1}$ is a sequence of elements of $D$ which converges towards $x$, then by Proposition 2.8 we have $\mathbb{1}_{\tau_\theta^{x_n} > 0} = 1$ almost surely while $\mathbb{1}_{\tau_\theta^x > 0} = 0$ almost surely.

**Remark 3.4.** Let us take $t_n = t > 0$ for all $n \geq 1$ in Lemma 3.2. Then we get that for any sequence $(x_n)_{n \geq 1}$ of elements of $\overline{D}$ converging to some $x \in \overline{D}$, $\mathbb{1}_{\tau_\theta^{x_n} > t}$ converges
almost surely to \( \mathbb{1}_{t^*_0 > t} \). Using the monotonicity of the functions \( t \mapsto \mathbb{1}_{t^*_0 > t} \) and \( t \mapsto \mathbb{1}_{t^*_0 > t} \), we deduce that almost surely, for any \( t > 0 \) such that \( t \neq \tau^i_0, \mathbb{1}_{t^*_0 > t} \) converges almost surely to \( \mathbb{1}_{t^*_0 > t} \). Integrating in time, we conclude that \( \tau^i_0 \) converges almost surely to \( \tau^* \).

We are now in position to prove Assertion (ii) in Theorem 2.10.

**Proof of Assertion (ii) in Theorem 2.10.** The proof is divided into two steps. In the first step we show that \( u \) is continuous on \((\mathbb{R}^+ \times \overline{D}) \setminus \{(0) \times (\Gamma^+ \cup \Gamma^0)\}\), and in the second step we show that if \( f \) and \( g \) satisfy the compatibility condition (15) then \( u \) is continuous on \( \mathbb{R}^+ \times \overline{D} \).

**Step 1** Let \((t, x) \in (\mathbb{R}^+ \times \overline{D}) \setminus \{(0) \times (\Gamma^+ \cup \Gamma^0)\}\). Let \((t_n, x_n)_{n \geq 1} \) be a sequence in \( \mathbb{R}^+ \times \overline{D} \) converging to \((t, x)\). Let us prove that

\[
  u(t_n, x_n) \xrightarrow{n \to \infty} u(t, x). \tag{33}
\]

To this aim, let us study the difference \(|u(t_n, x_n) - u(t, x)|\). It follows from the expression (14) of \( u \) and the triangle inequality that

\[
  |u(t_n, x_n) - u(t, x)| \leq \mathbb{E} \left[ |f(X^{x_n}_{t_n}) \mathbb{1}_{t^*_0 > t_n} - f(X^x_t) \mathbb{1}_{t^*_0 > t}| \right] + \mathbb{E} \left[ |g(X^{x_n}_{t_n}) \mathbb{1}_{t^*_0 > t} - g(X^x_t) \mathbb{1}_{t^*_0 > t}| \right]. \tag{34}
\]

Let us start with the first term in the right-hand side of the inequality above. We have that

\[
  \left| f(X^{x_n}_{t_n}) \mathbb{1}_{t^*_0 > t_n} - f(X^x_t) \mathbb{1}_{t^*_0 > t} \right| = \left| \mathbb{1}_{t^*_0 > t_n} \mathbb{1}_{t^*_0 > t_n} f(X^{x_n}_{t_n}) - f(X^x_t) \mathbb{1}_{t^*_0 > t_n} \right| + \left| f(X^x_t) \mathbb{1}_{t^*_0 > t_n} - f(X^x_t) \mathbb{1}_{t^*_0 > t} \right|
\]

\[
  \leq \mathbb{1}_{t^*_0 > t_n} \mathbb{1}_{t^*_0 > t_n} \mathbb{1}_{t^*_0 > t_n} \left| f(X^{x_n}_{t_n}) - f(X^x_t) \right| + \|f\| \mathbb{1}_{t^*_0 > t_n} \mathbb{1}_{t^*_0 > t_n} \mathbb{1}_{t^*_0 > t_n} \left| 1_{t^*_0 > t_n} - 1_{t^*_0 > t} \right|,
\]

since \( 1_{t^*_0 > t_n} 1_{t^*_0 > t_n} 1_{t^*_0 > t_n} = |1_{t^*_0 > t_n} - 1_{t^*_0 > t}| \). By Lemmas 3.1 and 3.2, since \( f \in C^b(\overline{D} \cup \Gamma^-) \), it follows by the dominated convergence theorem that

\[
  \mathbb{E}[|f(X^{x_n}_{t_n}) \mathbb{1}_{t^*_0 > t_n} - f(X^x_t) \mathbb{1}_{t^*_0 > t}|] \xrightarrow{n \to \infty} 0.
\]

Let us now consider the second term in the right-hand side of the inequality (34). We have that

\[
  \left| g(X^{x_n}_{t_n}) \mathbb{1}_{t^*_0 > t} - g(X^x_t) \mathbb{1}_{t^*_0 > t} \right| = \left| g(X^{x_n}_{t_n}) - g(X^x_t) \mathbb{1}_{t^*_0 > t_n} \right| + \left| g(X^x_t) \mathbb{1}_{t^*_0 > t_n} - g(X^x_t) \mathbb{1}_{t^*_0 > t} \right|
\]

\[
  \leq \mathbb{1}_{t^*_0 > t_n} \mathbb{1}_{t^*_0 > t_n} \left| g(X^{x_n}_{t_n}) - g(X^x_t) \right| + \|g\| \mathbb{1}_{t^*_0 > t_n} \mathbb{1}_{t^*_0 > t_n} \left| 1_{t^*_0 > t_n} - 1_{t^*_0 > t} \right|,
\]

so that we deduce again from Lemmas 3.1 and 3.2 along with Remark 3.4 and the dominated convergence theorem, since \( g \in C^b(\Gamma^+ \cup \Gamma^0) \), that

\[
  \mathbb{E} \left[ \left| g(X^{x_n}_{t_n}) \mathbb{1}_{t^*_0 > t_n} - g(X^x_t) \mathbb{1}_{t^*_0 > t} \right| \right] \xrightarrow{n \to \infty} 0.
\]
which completes the proof of (33).

**Step 2** Assume now that \( f \) and \( g \) satisfy the compatibility condition (15). Let \( x \in \Gamma^+ \cup \Gamma^0 \) and \((t_n, x_n)_{n \geq 1}\) be a sequence in \( \mathbb{R}_+ \times D \) converging to \((0, x)\). Let us prove that

\[
\lim_{n \to \infty} u(t_n, x_n) = u(0, x) = g(x).
\]

We have that

\[
|u(t_n, x_n) - g(x)| \leq \mathbb{E}\left[ |f(X_{t_n}^{x_n}) - g(x)| \mathbb{1}_{\tau_\partial^{x_n} > t_n} \right] + \mathbb{E}\left[ |g(X_{t_n}^{x_n}) - g(x)| \mathbb{1}_{\tau_\partial^{x_n} \leq t_n} \right]
\]

It follows from the compatibility condition (15) and Lemma 3.1 that \( \tau_\partial^{x_n} \to t_n \) almost surely. Therefore, using the dominated convergence theorem, the first term in the right-hand side of the inequality above converges to 0. Furthermore, on the event \( \{ \tau_\partial^{x_n} \leq t_n \} \), it follows by Lemma 3.1 that

\[
|X_{\tau_\partial^{x_n}}^{x_n} - x| \leq \left| X_{\tau_\partial^{x_n}}^{x_n} - X_{\tau_\partial^{x_n}}^{X_{t_n}^{x_n}} \right| + \left| X_{\tau_\partial^{x_n}}^{X_{t_n}^{x_n}} - x \right|
\]

It follows from the compatibility condition (15) and Lemma 3.1 that \( \tau_\partial^{x_n} \to t_n \) almost surely. Therefore, using the dominated convergence theorem, the first term in the right-hand side of the inequality above converges to 0. Furthermore, on the event \( \{ \tau_\partial^{x_n} \leq t_n \} \), it follows by Lemma 3.1 that

\[
\lim_{n \to \infty} |X_{\tau_\partial^{x_n}}^{x_n} - x| = 0 \quad \text{a.s.}
\]

since \( \tau_\partial^{x_n} \leq t_n \to 0 \). As a result, \( \mathbb{E}\left[ |g(X_{\tau_\partial^{x_n}}^{x_n}) - g(x)| \mathbb{1}_{\tau_\partial^{x_n} \leq t_n} \right] \to 0 \) since \( g \in \mathcal{C}^b(\Gamma^+ \cup \Gamma^0) \). Hence \( u(t_n, x_n) \to g(x) \). This concludes the proof of Assertion (ii) in Theorem 2.10.

\[ \square \]

3.3. Interior regularity: proof of Assertion (iii) in Theorem 2.10

The link between functions of the form of \( u \) defined by (14) and parabolic problems of the form (16) is standard for uniformly elliptic operators in bounded domains with compatible initial and boundary conditions, see for instance [14, Chapter 6]. In order to extend this link to the degenerate operator \( L \), we proceed by approximating (16) by the uniformly elliptic problem

\[
\partial_t u_\varepsilon = \mathcal{L} u_\varepsilon + \varepsilon \Delta_q u_\varepsilon.
\]

Let \( \varepsilon > 0 \) and \((\tilde{B}_t)_{t \geq 0}\) be a \( d \)-dimensional Brownian motion independent of \((B_t)_{t \geq 0}\). Under Assumption (F2), for all \( x \in D \) we denote by \((X_{t}^{\cdot, \varepsilon}, p_{t}^{\cdot, \varepsilon}))_{t \geq 0}\) the strong solution of

\[
\begin{align*}
\mathrm{d}q_{t}^{x, \varepsilon} &= p_{t}^{x, \varepsilon} \, \mathrm{d}t + \sqrt{2 \varepsilon} \, \mathrm{d}\tilde{B}_t, \\
\mathrm{d}p_{t}^{x, \varepsilon} &= F(q_{t}^{x, \varepsilon}) \, \mathrm{d}t - \gamma p_{t}^{x, \varepsilon} \, \mathrm{d}t + \sigma \, \mathrm{d}B_t,
\end{align*}
\]

(36)
Let \( \tau_{x,\epsilon} = \inf\{ t > 0 : X^x_t \notin D \} \) be the first exit time from \( D \) of the process \((X^x_t)_{t \geq 0}\).

We first assume that the functions \( f \) and \( g \) satisfy the compatibility condition (15), define the function \( h \in C^b(D) \) by

\[
\begin{align*}
    h(x) &= 1_{x \in D} f(x) + 1_{D^c} g(x),
\end{align*}
\]

and state the following two lemmas.

**Lemma 3.5.** (Perturbed problem) Under Assumptions (O1) and (F2), let \( \epsilon > 0 \) and let \( f \in C^b(D \cup \Gamma^-), g \in C^b(\Gamma^+ \cup \Gamma^0) \) satisfy (15). Let \( h \in C^b(D) \) be defined by (37). The function \( u^\epsilon \) on \( \mathbb{R}^+ \times D \) defined by

\[
\begin{align*}
    u^\epsilon(t, x) := \mathbb{E}\left[ 1_{\tau_{x,\epsilon} > t} h(X^x_t) + 1_{\tau_{x,\epsilon} \leq t} g(X^x_t) \right],
\end{align*}
\]

satisfies (35) in the sense of distributions on \( \mathbb{R}^+ \times D \).

**Lemma 3.6.** (Convergence) Under the assumptions of Lemma 3.5, for all \( t > 0 \) and \( x \in D \),

\[
\begin{align*}
    u^\epsilon(t, x) \xrightarrow{\epsilon \to 0} u(t, x).
\end{align*}
\]

Before proving Lemmas 3.5 and 3.6, let us conclude the proof of Assertion (iii) in Theorem 2.10 using these results. Under the assumption that \( f \) and \( g \) satisfy the compatibility condition (15), it is immediate, using the result of Lemma 3.6, to obtain that \( u \) solves (16) in the sense of distributions, by passing to the limit \( \epsilon \to 0 \) in the weak formulation of the partial differential equation and using the fact that \( \|u^\epsilon\|_{L^\infty(D)} \leq \|h\|_{L^\infty(D)} \).

If \( f \) and \( g \) do not satisfy the compatibility condition (15), one can use the following approximation argument to conclude. First, we note that since \( g \) is continuous on the closed set \( \Gamma^+ \cup \Gamma^0 \), there exists a function \( \tilde{g} \in C^b(D) \) which coincides with \( g \) on \( \Gamma^+ \cup \Gamma^0 \) by Tietze-Urysohn’s extension theorem [9, Theorem 4.5.1]. For any \( k \geq 1 \) and \( x \in D \cup \Gamma^- \), let us now set

\[
    \tilde{f}_k(x) = (1 - \psi_k(x)) f(x) + \psi_k(x) \tilde{g}(x),
\]

where \( \psi_k : D \to [0, 1] \) is a continuous function such that

\[
    \psi_k(x) = \begin{cases} 
        1 & \text{if } x \in \Gamma^+ \cup \Gamma^0, \\
        0 & \text{if } d(x, \Gamma^+ \cup \Gamma^0) \geq 1/k.
    \end{cases}
\]

Then \( \tilde{f}_k \) and \( g \) satisfy the compatibility condition (15), so that the argument above shows that the function \( \tilde{u}_k \) defined by

\[
\begin{align*}
    \tilde{u}_k(t, x) := \mathbb{E}\left[ 1_{\tau_{x,\epsilon} > t} \tilde{f}_k(X^x_t) + 1_{\tau_{x,\epsilon} \leq t} g(X^x_t) \right],
\end{align*}
\]
solves (16) in the distributional sense. On the other hand, \( \tilde{f}_k(x) \) converges to \( f(x) \) for all \( x \in D \cup \Gamma \) when \( k \to +\infty \), which by the dominated convergence theorem implies that \( \tilde{u}_k(t, x) \) converges to \( u(t, x) \) and therefore shows that \( u \) is a distributional solution to (16), also in the case when \( f \) and \( g \) do not satisfy the compatibility condition (15).

It finally follows from the hypoellipticity of the operator \( \partial_t - \mathcal{L} \) that \( u \) is actually in \( C^\infty (\mathbb{R}_+^+ \times D) \), which completes the proof of Assertion (iii) in Theorem 2.10.

Let us now conclude this section by proving the two Lemmas 3.5 and 3.6.

**Proof of Lemma 3.5.** The result is standard for bounded domains, but \( D \) is not bounded.

We thus use an approximation argument. Let \( \{\tilde{V}_k\}_{k \geq 1} \) be a sequence of \( C^2 \) bounded open subsets of \( D \) such that:

- (i) for all \( k \geq 1 \), \( \tilde{V}_k \subset D \cap \{(q, p) \in \mathbb{R}^{2d} : |p| \leq k\} \),
- (ii) for all \( k \geq 1 \), \( \tilde{V}_k \subset \tilde{V}_{k+1} \),
- (iii) \( \bigcup_{k \geq 1} \tilde{V}_k = D \).

For \( \epsilon > 0 \), let \( \tau^{x,\epsilon}_{\tilde{V}_k} \) be the following stopping time:

\[
\tau^{x,\epsilon}_{\tilde{V}_k} = \inf\{t > 0 : X_t^{x,\epsilon} \notin \tilde{V}_k\}.
\]

Let \( T > 0 \). Consider the following Initial-Boundary Value Problem,

\[
\begin{aligned}
\partial_t v_{k,\epsilon}(t, x) &= \mathcal{L}v_{k,\epsilon}(t, x) + \epsilon \Delta q v_{k,\epsilon}(t, x), \quad t \in (0, T], \quad x \in \tilde{V}_k, \\
v_{k,\epsilon}(0, x) &= h_{\mid_{\tilde{V}_k}}(x), \quad x \in \tilde{V}_k, \\
v_{k,\epsilon}(t, x) &= h_{\mid_{\partial \tilde{V}_k}}(x), \quad t \in (0, T], \quad x \in \partial \tilde{V}_k.
\end{aligned}
\]

(39)

By [14, Chapter 6, Theorem 5.2] there exists a unique classical solution \( v_{k,\epsilon} \) in \( C^2((0, T] \times \tilde{V}_k) \cap C^b((0, T] \times \tilde{V}_k) \) of (39). Furthermore, the solution can be written as follows: for all \( t > 0 \) and \( x \in D \)

\[
v_{k,\epsilon}(t, x) = \mathbb{E}\left[ \mathbb{1}_{\tau_{\tilde{V}_k}^{x,\epsilon} > t} h_{\mid_{\tilde{V}_k}}(X_t^{x,\epsilon}) + \mathbb{1}_{\tau_{\tilde{V}_k}^{x,\epsilon} \leq t} h_{\mid_{\partial \tilde{V}_k}} \left( X_t^{x,\epsilon} \right) \right].
\]

Moreover when \( k \) goes to infinity one has (following the proof of Assertion (iv) in Theorem 2.10, see Section 3.1):

\[
v_{k,\epsilon}(t, x) \underset{k \to \infty}{\longrightarrow} u_\epsilon(t, x).
\]

(40)

Therefore, since \( v_{k,\epsilon} \) is a classical solution of (39) it is also a solution in the sense of distributions of \( \partial_t v_{k,\epsilon} = \mathcal{L}v_{k,\epsilon} + \epsilon \Delta q v_{k,\epsilon} \) on \( (0, T) \times \tilde{V}_k \). But then, since \( T \) is arbitrary, \( u_\epsilon \) is also a solution in the sense of distributions of \( \partial_t u_\epsilon = \mathcal{L}u_\epsilon \) on \( \mathbb{R}_+^+ \times D \). Indeed, for \( \Phi \in C^\infty_c(\mathbb{R}_+^+ \times D) \), there exists \( k_0 > 0 \) and \( T_0 > 0 \) such that \( \text{supp}(\Phi) \subset (0, T_0] \times \tilde{V}_{k_0} \).

As a result, for all \( k > k_0 \) and \( T > T_0 \),

\[
\iint_{\mathbb{R}_+^+ \times D} v_{k,\epsilon}(t, x) \left( \partial_t \Phi(t, x) + \mathcal{L}^* \Phi(t, x) + \epsilon \Delta q \Phi(t, x) \right) \, dr \, dx = 0.
\]

The proof is then easily completed, using (40) and the dominated convergence theorem. \( \square \)
Proof of Lemma 3.6. An application of Gronwall’s Lemma, as in the proof of Lemma 3.1,
shows that, almost surely,
\[
\sup_{s \in [0, t]} |X_s^{x, \epsilon} - X_s^x| \leq \sqrt{2 \epsilon} \sup_{s \in [0, t]} |\tilde{B}_s| e^{C_{\text{Lip}} t}
\]  
(41)
where $C_{\text{Lip}}$ is the Lipschitz constant of the drift of (11). In particular, for all $t \geq 0,$
$X_t^{x, \epsilon} \longrightarrow X_t^x$ almost surely.

Let us now consider the difference between $u_\epsilon(t, x)$ and $u(t, x)$ for $t > 0, x \in D.$
Using the same triangle inequality as in the proof of Assertion (ii) of Theorem 2.10
(see Sect. 3.2), one has
\[
|u_\epsilon(t, x) - u(t, x)| \leq E \left[ \mathbb{1}_{\tau_\partial^x \epsilon > t, \tau_\partial^x > t} \left| h|_{\partial D}(X_t^{x, \epsilon}) - h|_{\partial D}(X_t^x) \right| \right] \\
+ 2\|h\|_{\infty} E \left[ \mathbb{1}_{\tau_\partial^x \epsilon > t, \tau_\partial^x > t} \left| h|_{\partial D}(X_t^{x, \epsilon}) - h|_{\partial D}(X_t^x) \right| \right].
\]
Using (41) and the fact that $h|_{\partial D} \in C^b(D),$ it follows from the dominated convergence
theorem that the first term in the right-hand side of the inequality converges to 0
as $\epsilon$ goes to 0. Besides, remember that $P(\tau_\partial^x = t) = 0$ for $x \in D$ and $t > 0$ by
Corollary 2.18. As a result if one can prove that for all $x \in D, t > 0,$
\[
\mathbb{1}_{\tau_\partial^x \epsilon > t} \longrightarrow \mathbb{1}_{\tau_\partial^x > t}, \text{ almost surely on the events } \{\tau_\partial^x < t\} \text{ and } \{\tau_\partial^x > t\},
\]  
(42)
and
\[
\tau_\partial^{x, \epsilon} \longrightarrow \tau_\partial^x \text{ almost surely on the event } \{\tau_\partial^x < t\},
\]  
(43)
then using (41), the fact that $h|_{\partial D} \in C^b(D)$ and the continuity of the trajectories
of $(X_t^x)_{t \geq 0},$ the convergence of $u_\epsilon(t, x)$ towards $u(t, x)$ follows from the dominated
convergence theorem, and the proof is complete.

Let us now prove the two convergences (42) and (43).

Step 1 Consider first the convergence (42) on the event $\{\tau_\partial^x > t\}.$ By the continuity of
the trajectories of $(q_s^x)_{s \geq 0},$
\[
\epsilon_0 := \inf_{0 \leq s \leq t} d_\partial(q_s^x) > 0.
\]
Let $S_t := \sup_{0 \leq s \leq t} |\tilde{B}_s|.$ For $\epsilon \leq \frac{\epsilon_0^2}{8S_t} e^{-2C_{\text{Lip}} t}$ (which is positive since $S_t < \infty$
almost surely), one has by (41):
\[
\sup_{0 \leq s \leq t} |q_s^{x, \epsilon} - q_s^x| \leq \sup_{0 \leq s \leq t} |X_s^{x, \epsilon} - X_s^x| \leq \frac{\epsilon_0}{2}.
\]
Hence, since $d_\partial$ is 1-Lipschitz continuous, for $\epsilon \leq \frac{\epsilon_0^2}{8S_\partial^2} e^{-2C_{\text{Lip}}t}$

$$\inf_{0 \leq s \leq t} d_\partial(q_{x_s}^{x,\epsilon}) \geq \frac{\epsilon_0}{2} > 0,$$

which implies $\mathbb{1}_{\tau_{\partial}^{x,\epsilon} > t} = 1$ and (42) thus holds on the event $\{\tau_{\partial}^{x} > t\}$.

**Step 2** Let us now prove the convergences (42) and (43) on the event $\{\tau_{\partial}^{x} < t\}$. Since $x \in D$, by Proposition 2.8 one has $(q_{\tau_{\partial}^{x}}^x, p_{\tau_{\partial}^{x}}^x) \in \Gamma^+$ almost surely. Let $0 < \eta < (t - \tau_{\partial}^{x}) \wedge \tau_{\partial}^{x}$. The strong Markov property along with Proposition 2.8 ensure that there exists almost surely $t_0 \in (\tau_{\partial}^{x}, \tau_{\partial}^{x} + \eta)$ such that

$$\epsilon_1 := d_\partial(q_{t_0}^x) > 0.$$

Besides, the continuity of the trajectories of $(q_s^x)_{s \geq 0}$ ensures that

$$\epsilon_2 := \inf_{0 \leq s \leq \tau_{\partial}^{x} - \eta} d_\partial(q_s^x) > 0.$$

As a result, for $\epsilon \leq \frac{\epsilon_1^2 \wedge \epsilon_2^2}{8S_{\partial}^2} e^{-2C_{\text{Lip}}t}$,

$$\sup_{0 \leq s \leq t} |q_{s}^{x,\epsilon} - q_{s}^{x}| \leq \sup_{0 \leq s \leq t} |X_{s}^{x,\epsilon} - X_{s}^{x}| \leq \frac{\epsilon_1 \wedge \epsilon_2}{2}.$$

Hence, since $d_\partial$ is 1-Lipschitz continuous,

$$d_\partial(q_{t_0}^{x,\epsilon}) \geq \frac{\epsilon_1}{2} > 0,$$

and since $d_\partial$ is 1-Lipschitz continuous as well, one has

$$\inf_{0 \leq s \leq \tau_{\partial}^{x} - \eta} d_\partial(q_s^{x,\epsilon}) \geq \frac{\epsilon_2}{2} > 0.$$

Therefore, for $\epsilon$ small enough,

$$|\tau_{\partial}^{x,\epsilon} - \tau_{\partial}^{x}| \leq \eta \quad \text{and in particular} \quad \tau_{\partial}^{x,\epsilon} \leq \tau_{\partial}^{x} + \eta < t.$$

Consequently, the convergences (42) and (43) hold on the event $\{\tau_{\partial}^{x} < t\}$. \qed

### 3.4. Proof of Propositions 2.7 and 2.8

We conclude Sect. 3 with the proofs of Propositions 2.7 and 2.8, which are the cornerstones of all the previous results. In Sect. 3.4.1, we deduce from a simple geometric argument that Assertion (i) in Proposition 2.8 holds for $x \in \Gamma^+$, and that, taking Proposition 2.7 for granted, Assertion (ii) in Proposition 2.8 holds. The proof of the remaining statements, namely Proposition 2.7 and Assertion (i) in Proposition 2.8 for $x \in \Gamma^0$, both rely on a preliminary reduction to a Gaussian process, thanks to the Girsanov theorem, which is detailed in Sect. 3.4.2. Last, we complete the proof of Assertion (i) in Proposition 2.8 in Sect. 3.4.3 and we provide the proof of Proposition 2.7 in Sect. 3.4.4.
3.4.1. The interior and exterior sphere conditions

The first part of the proof of Proposition 2.8 relies on the following geometric property of the set \( \mathcal{O} \) which is standard.

**Proposition 3.7.** (Uniform interior and exterior sphere conditions) *Under Assumption (O1), there exists \( \rho > 0 \) such that for any \( q \in \partial \mathcal{O} \), there exist two points \( q_{\text{int}} \in \mathcal{O} \) and \( q_{\text{ext}} \in \overline{\mathcal{O}}^c \) such that the open Euclidean balls \( B(q_{\text{int}}, \rho) \) and \( B(q_{\text{ext}}, \rho) \) satisfy*

\[
B(q_{\text{int}}, \rho) \subset \mathcal{O}, \quad B(q_{\text{ext}}, \rho) \subset \overline{\mathcal{O}}^c, \quad \overline{B(q_{\text{int}}, \rho)} \cap \mathcal{O}^c = \overline{B(q_{\text{ext}}, \rho)} \cap \overline{\mathcal{O}} = \{q\}.
\]

Let us now detail the application of Proposition 3.7 to the proof of Proposition 2.8.

For \( x = (q, p) \in \Gamma^+ \), let \( q_{\text{ext}} \in \overline{\mathcal{O}}^c \) be given by the exterior sphere condition. Necessarily, the vectors \( q_{\text{ext}} - q \) and \( n(q) \) are colinear. On the other hand, for \( t \to 0 \), \( (q_{\text{int}}^x - q) \cdot n(q) \sim t p \cdot n(q) > 0 \), which then implies that \( |q_{\text{int}}^x - q_{\text{ext}}^x|^2 = \rho^2 - 2 \rho t p \cdot n(q) + o(t) \) so that \( q_{\text{int}}^x \in B(q_{\text{ext}}, \rho) \subset \overline{\mathcal{O}}^c \) for \( t \) small enough, and therefore (12) holds.

With similar arguments, the interior sphere condition shows that if \( x \in \Gamma^- \), then \( \tau_{\partial}^x > 0 \) almost surely. Moreover, it is obvious that if \( x \in D \), then \( \tau_{\partial}^x > 0 \) almost surely.

Finally, if \( x \in D \cup \Gamma^- \), then on the event \( \tau_{\partial}^x \leq T \) one necessarily has \( X_{t\tau_{\partial}^x} \in \Gamma^+ \cup \Gamma^0 \) almost surely, which rewrites:

\[
\forall T > 0, \quad \forall x \in D \cup \Gamma^-, \quad \mathbb{P}\left(p_{t\tau_{\partial}^x}^x \cdot n(q_{t\tau_{\partial}^x}^x) < 0, \tau_{\partial}^x \leq T\right) = 0. \tag{44}
\]

Therefore, taking Proposition 2.7 for granted, we obtain Assertion (ii) in Proposition 2.8.

3.4.2. Reduction to a Gaussian process

Proposition 2.7 and Assertion (i) in Proposition 2.8 rely on the following preliminary result.

**Lemma 3.8.** (Girsanov Theorem) *Let Assumption (F2) hold. Let \( x \in \mathbb{R}^{2d} \) and let \( (\tilde{q}_t^x, \tilde{p}_t^x)_{t \geq 0} \) be the strong solution on \( \mathbb{R}^{2d} \) of*

\[
\begin{cases}
\text{d}\tilde{q}_t^x = \tilde{p}_t^x \text{d}t, \\
\text{d}\tilde{p}_t^x = \sigma \text{d}B_t, \\
(\tilde{q}_0^x, \tilde{p}_0^x) = x.
\end{cases} \tag{45}
\]

*For \( T \geq 0 \), the laws of \( (\tilde{q}_t^x, \tilde{p}_t^x)_{t \in [0, T]} \) and \( (q_t^x, p_t^x)_{t \in [0, T]} \) are equivalent in the space of sample paths \( \mathcal{C}([0, T], \mathbb{R}^{2d}) \), i.e. for all Borel sets \( A \) of \( \mathcal{C}([0, T], \mathbb{R}^{2d}) \),

\[
\mathbb{P}((\tilde{q}_t^x, \tilde{p}_t^x)_{t \in [0, T]} \in A) = 0 \quad \text{if and only if} \quad \mathbb{P}((q_t^x, p_t^x)_{t \in [0, T]} \in A) = 0.
\]

**Proof.** Let \( x = (q, p) \in \mathbb{R}^{2d} \). Equation (45) admits a unique global in time strong solution \( (\tilde{q}_t^x, \tilde{p}_t^x)_{t \geq 0} \) on \( \mathbb{R}^{2d} \) since its coefficients are globally Lipschitz continuous. For \( T \geq 0 \), let us define,

\[
Z_T^x = F(\tilde{q}_T^x) - \gamma \tilde{p}_T^x,
\]
and
\[ E^x_T = \exp \left( \int_0^T Z^x_s \cdot dB_s - \frac{1}{2} \int_0^T |Z^x_s|^2 ds \right). \]

It is clear that \( E^x_T \) is \( \mathcal{F}_T \)-measurable. Let us show that for all \( T \geq 0 \), \( \mathbb{E}[E^x_T] = 1 \). According to \cite[Theorem 1.1 p. 152]{14}, this equality is satisfied if there exists \( \mu > 0 \) such that
\[
\sup_{s \in [0,T]} \mathbb{E}[\exp(\mu|Z^x_s|^2)] < \infty,
\]
which we now prove. Since \( F \) satisfies Assumption (F2), it follows that for \( s \in [0, T] \),
\[
|Z^x_s|^2 = |F(\tilde{q}^x_s) - \gamma \tilde{p}^x_s|^2 \leq 2\|F\|_\infty^2 + 2\gamma^2|\tilde{p}^x_s|^2.
\]
In addition, \( \tilde{p}^x \sim \mathcal{N}_d(p, \sigma^2 I_d) \). Let \( G \sim \mathcal{N}_d(0, I_d) \), we get for \( s \in [0, T] \),
\[
\mathbb{E}[\exp(\mu|Z^x_s|^2)] \leq \exp(2\mu\|F\|_\infty^2)\mathbb{E}(\exp(2\mu\gamma^2|\tilde{p}^x_s|^2)) = \exp(2\mu\|F\|_\infty^2)\mathbb{E}(\exp(2\mu\gamma^2|p + \sigma \sqrt{s} G|^2)) \leq \exp(2\mu\|F\|_\infty^2 + 4\mu\gamma^2|p|^2)\mathbb{E}(\exp(4\mu\gamma^2\sigma^2 T|G|^2)).
\]
Moreover, \( \mathbb{E}(\exp(4\mu\gamma^2\sigma^2 T|G|^2)) < \infty \) for sufficiently small \( \mu \).

This result allows us to define the probability measure \( Q_T \) on \( \mathcal{F}_T \) by \( dQ_T = E^x_T dP|x_T \). Since \( E^x_T > 0 \), \( P|x_T \)-a.s., the measures \( P|x_T \) and \( Q_T \) are equivalent. Besides, by the Girsanov Theorem \cite[Theorem 1.1 p. 152]{14} the process
\[
\left( \tilde{B}_s := B_s - \int_0^s Z^x_r dr \right)_{0 \leq s \leq T}
\]
is a \( (\mathcal{F}_s)_{s \in [0,T]} \)-Brownian motion under the probability \( Q_T \). As a result, the process \( (\tilde{X}^x_s, \tilde{B}_s)_{s \in [0,T]} \) satisfies (11) on the probability space \((\Omega, \mathcal{F}, (\mathcal{F}_s)_{s \in [0,T]}, Q_T)\). On the other hand, the pathwise uniqueness for (11) implies the uniqueness in distribution by Yamada Watanabe’s theorem, so that the law of \((\tilde{q}^x_s, \tilde{p}^x_s)_{s \in [0,T]} \) under \( Q_T \) is the law of \((q^x_s, p^x_s)_{s \in [0,T]} \) under \( P \), whence the final result.

We are now in position to complete the proof of Proposition 2.8 and to detail the proof of Proposition 2.7. By Lemma 3.8 it is sufficient to prove both statements for the process \((\tilde{X}^x_t)_{t \geq 0}\) defined in (45), and for which we introduce the notation \( \bar{\tau}^x_0 := \inf\{t > 0 : \tilde{X}^x_t \notin D\} \).

3.4.3. Proof of Assertion (i) in Proposition 2.8

Proof of Assertion (i) in Proposition 2.8. The case \( x \in \Gamma^+ \) has been addressed in Sect. 3.4.1. It remains now to prove (12) for \( x = (q, p) \in \Gamma^0 \). One has from (45) that for all \( t \geq 0 \),
\[
\tilde{p}^x_t = p + \sigma B_t \quad \text{and thus} \quad \tilde{q}^x_t = q + pt + \sigma \int_0^t B_s \, ds.
\]
The idea of the proof is to reduce the problem to the case of a flat boundary by a change of variable, and then to use known results for 1-d integrated Brownian motion, see [25].

Let \((e_1, \ldots, e_d)\) be the canonical basis of \(\mathbb{R}^d\). Since \(O\) is a bounded \(C^2\) set of \(\mathbb{R}^d\), by [5, Theorem 2.1.2] there exists an open neighborhood \(U\) of \(q\) and a \(C^2\)-diffeomorphism \(\phi : (-1, 1)^d \rightarrow U\) satisfying \(\phi(0) = q\) and \(O \cap U = \phi([-1, 1]^d : y \cdot e_d < 0)\) and \(\partial O \cap U = \phi([-1, 1)^d : y \cdot e_d = 0]\).

Moreover, \(n(q) \in \mathbb{R}^d\) is the unique vector such that
\[
n(q) \in \text{Span}(d_0\phi(e_1), \ldots, d_0\phi(e_{d-1})), \quad |n(q)| = 1, \quad \text{and} \quad d_0\phi(e_d) \cdot n(q) > 0, \quad (46)
\]
where \(d_0\phi\) is the differential at \(0 \in \mathbb{R}^d\) of \(\phi\).

Now let \(K\) be a compact set included in \(U\) such that \(q \in \hat{K}\). Let \(\tilde{\tau}^x_K := \inf\{t > 0 : \tilde{q}^x_t \notin K\}\) be the first exit time of \(K\) for \((\tilde{q}^x_t)_{t \geq 0}\), then \(\tilde{\tau}^x_K > 0\) almost surely by continuity of the trajectories of \((\tilde{q}^x_t)_{t \geq 0}\).

For \(t \leq \tilde{\tau}^x_K\) we have
\[
\phi^{-1}(\tilde{q}^x_t) = \phi^{-1}(q) + \int_0^t d\tilde{q}^x_s(\phi^{-1})(p + \sigma B_s) \, ds. \quad (47)
\]

Since \(\phi^{-1}\) is a \(C^2\)-diffeomorphism from \(U\) to \((-1, 1)^d\), then \(y \in K \subset U \mapsto d_y(\phi^{-1})\) is \(C^1\) on the compact set \(K\). In particular it is Lipschitz continuous with some Lipschitz constant \(k\). As a result, since for \(t \geq 0\), \(\tilde{q}^x_t = q + \int_0^t \tilde{p}^x_s ds\), then for all \(t \in [0, \tilde{\tau}^x_K]\) and \(z \in \mathbb{R}^d\),
\[
|d\tilde{q}^x_t(\phi^{-1})(z) - d_q(\phi^{-1})(z)| \leq k|\tilde{q}^x_t - q||z| = k\left|tp + \sigma \int_0^t B_s ds\right||z|,
\]
\[
\leq kt \left(|p| + \sigma \sup_{s \in [0, t]} |B_s|\right) |z|. \quad (48)
\]

Hence we have from (47) and (48)
\[
\left|\phi^{-1}(\tilde{q}^x_t) - \int_0^t d_q(\phi^{-1})(p + \sigma B_s) ds\right| \leq kt^2 \left(|p| + \sigma \sup_{s \in [0, t]} |B_s|\right)^2.
\]

Therefore
\[
\phi^{-1}(\tilde{q}^x_t) \cdot e_d - td_q(\phi^{-1})(p) \cdot e_d - \sigma d_q(\phi^{-1})\left(\int_0^t B_s ds\right) \cdot e_d \leq kt^2 \left(|p| + \sigma \sup_{s \in [0, t]} |B_s|\right)^2. \quad (49)
\]

Let us now prove that, since \(x \in \Gamma^0\),
\[
d_q(\phi^{-1})(p) \cdot e_d = 0. \quad (50)
\]
Since $\phi$ is a $C^1$-diffeomorphism from $(-1, 1)^d$ to $U$ with $U$ a neighborhood of $q$ and $\phi(0) = q$, then $d_0(\phi)$ is invertible with inverse satisfying

$$(d_0(\phi))^{-1} = d_q(\phi^{-1})$$

In particular, the family $(d_0(\phi)(e_1), \ldots, d_0(\phi)(e_d))$ is a basis of $\mathbb{R}^d$. Let us now decompose the vector $p$ in this basis:

$$p = \sum_{j=1}^d p_j d_0(\phi)(e_j).$$

Using (46) and the fact that $p \cdot n(q) = 0$ since $x \in \Gamma_0$, we get $p_d = 0$. As a result,

$$d_q(\phi^{-1})(p) \cdot e_d = d_q(\phi^{-1}) \left( \sum_{j=1}^d p_j d_0(\phi)(e_j) \right) \cdot e_d = \sum_{j=1}^d p_j (d_0(\phi))^{-1} d_0(\phi)(e_j) \cdot e_d = p_d = 0.$$

This concludes the proof of (50).

Now notice that

$$d_q(\phi^{-1}) \left( \int_0^t B_s ds \right) \cdot e_d = \int_0^t B_s \cdot d_0(\phi)^{-T}(e_d) ds,$$

where $d_0(\phi)^{-T}$ is the transpose matrix of $d_0(\phi)^{-1}$. Moreover, $|d_0(\phi)^{-T}(e_d)| > 0$, since $d_0(\phi)^{-T}$ is also invertible. Using (50) and (51) in (49), one gets

$$\left| \phi^{-1}(\tilde{q}_t^x) \cdot e_d - \sigma \int_0^t B_s \cdot d_0(\phi)^{-T}(e_d) ds \right| \leq k t^2 \left( |p| + \sigma \sup_{s \in [0, t]} |B_s| \right)^2. \quad (52)$$

Let us define the process $(\hat{B}_s)_{s \in [0, t]}$ by

$$\forall s \in [0, t], \quad \hat{B}_s := B_s \cdot \frac{d_0(\phi)^{-T}(e_d)}{|d_0(\phi)^{-T}(e_d)|}.$$ 

It is clearly a one-dimensional Brownian motion on $[0, t]$. Then (52) rewrites

$$\left| \phi^{-1}(\tilde{q}_t^x) \cdot e_d - \sigma |d_0(\phi)^{-T}(e_d)| \int_0^t \hat{B}_s ds \right| \leq k t^2 \left( |p| + \sigma \sup_{s \in [0, t]} |B_s| \right)^2.$$ 

The law of the iterated logarithm for the integrated Brownian motion (see [25, Theorem 1]) provides us with the following asymptotic result:

$$\limsup_{t \to 0} \frac{\int_0^t \hat{B}_s ds}{\sqrt{\frac{2}{3} t^\frac{3}{2} \log \log(1/t)}} = 1 \quad \text{almost surely.}$$
For $t > 0$, let $\Psi(t) = \sqrt{\frac{2}{3} t^3} \log \log (1/t)$, then
\[
\left| \frac{\phi^{-1}(\tilde{q}^x_t) \cdot e_d}{\Psi(t)} - \sigma |d_0(\phi)^{-T}(e_d)| \int_0^t \frac{\partial \tilde{B}_s}{\Psi(t)} ds \right| \leq k t^2 \left( |p| + \sigma \sup_{s \in [0, t]} |B_s| \right)^2.
\]
Therefore, almost surely,
\[
\limsup_{t \to 0} \frac{\phi^{-1}(\tilde{q}^x_t) \cdot e_d}{\Psi(t)} = \sigma |d_0(\phi)^{-T}(e_d)| > 0.
\]
As a result, the process $(\tilde{q}^x_t)_{t \geq 0}$ visits $U \cap \overline{O}$ infinitely often for times close to 0. This implies in particular that $\tilde{\tau}^x_0 = 0$ almost surely. \hfill $\square$

### 3.4.4. Proof of Proposition 2.7

We now address the proof of Proposition 2.7. For $x \in \mathbb{R}^d$, let $\tilde{\tau}^x_0 := \inf \{ t > 0 : (\tilde{q}^x_t, \tilde{p}^x_t) \in \Gamma^0 \}$ and let us show here that for all $x \in \mathbb{R}^d \setminus \Gamma^0$,
\[
\mathbb{P}(\tilde{\tau}^x_0 < \infty) = 0,
\]
which is equivalent to
\[
\forall T > 0, \quad \mathbb{P}(\tilde{\tau}^x_0 \leq T) = 0. \tag{53}
\]
The idea of the proof is the following. If one replaces the random time $\tilde{\tau}^x_0$ by a deterministic time $t \leq T$, and denote by $\tilde{n}$ some continuous extension of the normal vector $n$ in a neighborhood of $\partial O$, then using the fact that $\tilde{p}^x_t$ has a nondegenerate Gaussian conditional distribution given $\tilde{q}^x_t$, allows us to write
\[
\mathbb{P} \left( \tilde{p}^x_t \cdot \tilde{n}(\tilde{q}^x_t) = 0 \right) = \mathbb{E} \left[ \mathbb{P} \left( \tilde{p}^x_t \cdot \tilde{n}(\tilde{q}^x_t) = 0 | \tilde{q}^x_t \right) \right] = 0.
\]
Our proof therefore relies on the approximation of $\tilde{\tau}^x_0$ by a grid of deterministic times and exploits the fact that while assuming that such a time $t$ is close to $\tilde{\tau}^x_0$ makes the distribution of $\tilde{q}^x_t$ quite singular, it leaves ‘enough randomness’ in the distribution of $\tilde{p}^x_t$ for quantities of the form $\mathbb{P}(\tilde{p}^x_{\tilde{\tau}^x_0} \cdot n(\tilde{q}^x_{\tilde{\tau}^x_0}) = 0, \tilde{\tau}^x_0 \simeq t)$ to be sufficiently small.

**Proof of Proposition 2.7.** Let $x = (q, p) \in \mathbb{R}^{2d} \setminus \Gamma^0$. As explained above, the objective is to prove (53).

Let $\alpha \in (0, 1/2)$. Since $(\tilde{p}^x_t)_{0 \leq t \leq T}$ is a Brownian motion, one has that
\[
\sup_{0 \leq t \leq T} |\tilde{p}^x_t| < \infty, \quad \sup_{0 \leq s < t \leq T} \frac{|\tilde{p}^x_t - \tilde{p}^x_s|}{|t - s|^\alpha} < \infty \quad \text{almost surely.}
\]
Let $\epsilon > 0$ and let us choose $M$ large enough so that
\[
\mathbb{P} \left( \sup_{0 \leq t \leq T} |\tilde{p}^x_t| > M \right) \leq \epsilon, \quad \mathbb{P} \left( \sup_{0 \leq s < t \leq T} \frac{|\tilde{p}^x_t - \tilde{p}^x_s|}{|t - s|^\alpha} > M \right) \leq \epsilon. \tag{54}
\]
Therefore,
\[
P(\tilde{\tau}_0^x \leq T) = P\left( \tilde{p}_0^x \cdot n(\tilde{q}_0^x) = 0, \tilde{\tau}_0^x \leq T \right)
\]
\[
\leq P\left( \tilde{p}_0^x \cdot n(\tilde{q}_0^x) = 0, \tilde{\tau}_0^x \leq T, \sup_{0 \leq t \leq T} |\tilde{p}_t^x| \leq M, \sup_{0 \leq s < t \leq T} \frac{|\tilde{p}_t^x - \tilde{p}_s^x|}{|t-s|^\alpha} \leq M \right) + 2\epsilon.
\]

Let us now consider the first term in the right-hand side of the inequality above.

**Step 1** Let \( N \in \mathbb{N}^* \). We divide the interval \((0, T]\) into \( N \) intervals \((t_k, t_{k+1}]\) with \( t_k := k\eta_N \) and \( \eta_N := \frac{T}{N} \). As a result, since \( \tilde{\tau}_0^x > 0 \) almost surely, because \( x \) belongs to the open set \( \mathbb{R}^{2d} \setminus \Gamma^0 \),
\[
P\left( \tilde{p}_0^x \cdot n(\tilde{q}_0^x) = 0, \tilde{\tau}_0^x \leq T, \sup_{0 \leq t \leq T} |\tilde{p}_t^x| \leq M, \sup_{0 \leq s < t \leq T} \frac{|\tilde{p}_t^x - \tilde{p}_s^x|}{|t-s|^\alpha} \leq M \right)
\]
\[
= \sum_{k=0}^{N-1} P\left( \tilde{p}_0^x \cdot n(\tilde{q}_0^x) = 0, \tilde{\tau}_0^x \in (t_k, t_{k+1}], \sup_{0 \leq t \leq T} |\tilde{p}_t^x| \leq M, \sup_{0 \leq s < t \leq T} \frac{|\tilde{p}_t^x - \tilde{p}_s^x|}{|t-s|^\alpha} \leq M \right).
\]

Let us denote by \( \overline{d}_\beta \) the signed Euclidean distance to the boundary \( \partial \mathcal{O} \), i.e.
\[
\overline{d}_\beta : q \in \mathbb{R}^d \mapsto \begin{cases} d(q, \partial \mathcal{O}) & \text{if } q \in \mathcal{O}, \\ -d(q, \partial \mathcal{O}) & \text{if } q \not\in \mathcal{O}, \end{cases}
\]
so that \( d_\beta \) is the positive part of \( \overline{d}_\beta \), and the function \( \overline{d}_\beta \) is 1-Lipschitz continuous.

On the event
\[
\mathcal{A}_{k,M} := \left\{ \tilde{p}_0^x \cdot n(\tilde{q}_0^x) = 0, \tilde{\tau}_0^x \in (t_k, t_{k+1}], \sup_{0 \leq t \leq T} |\tilde{p}_t^x| \leq M, \sup_{0 \leq s < t \leq T} \frac{|\tilde{p}_t^x - \tilde{p}_s^x|}{|t-s|^\alpha} \leq M \right\},
\]
we have
\[
|\tilde{q}_{0_k}^x - \tilde{q}_{0_k}^x| = \left| \int_{t_k}^{\tilde{\tau}_0^x} \tilde{p}_t^x \, dt \right| \leq M(\tilde{\tau}_0^x - t_k) \leq M\eta_N.
\]

Thus,
\[
|\overline{d}_\beta(\tilde{q}_{0_k}^x)| \leq |\tilde{q}_{0_k}^x - \tilde{q}_{0_k}^x| \leq M\eta_N.
\]

For \( \mu > 0 \), let
\[
\overline{\mathcal{O}}_\mu := \{ q \in \mathbb{R}^d : |\overline{d}_\beta(q)| \leq \mu \}.
\]

Since the bounded open set \( \mathcal{O} \) is \( C^2 \) there exists a constant \( \mu > 0 \) such that the signed distance \( \overline{d}_\beta(q) \) to \( \partial \mathcal{O} \) is \( C^2 \) on the set \( \overline{\mathcal{O}}_\mu \) according to [16, Lemma 14.16]. Moreover \( \overline{d}_\beta(q) \) satisfies the following eikonal equation
\[
\begin{cases}
|\nabla \overline{d}_\beta(q)| = 1 & \text{for } q \in \overline{\mathcal{O}}_\mu, \\
\nabla \overline{d}_\beta(q) = -n(q) & \text{for } q \in \partial \mathcal{O}.
\end{cases}
\]
Let us now choose \( N \) large enough so that \( M \eta_N = M \frac{L_N}{N} \leq \mu \). As a result, since \( \tilde{q}^x_{t_k} \in \partial \mu \) on \( A_{k,M} \),

\[
\left| \tilde{p}^x_{t_0} \cdot \nabla \tilde{d}_\theta(\tilde{q}^x_{t_0}) \right| \leq \left| \left( \tilde{p}^x_{t_0} - \tilde{p}^x_{t_0} \right) \cdot \nabla \tilde{d}_\theta(\tilde{q}^x_{t_0}) \right| + \left| \tilde{p}^x_{t_0} \cdot (\nabla \tilde{d}_\theta(\tilde{q}^x_{t_0}) - \nabla \tilde{d}_\theta(\tilde{q}^x_{t_0})) \right|
\]

\[
\leq \left| \tilde{p}^x_{t_0} - \tilde{p}^x_{t_0} \right| + M \left| \nabla \tilde{d}_\theta(\tilde{q}^x_{t_k}) - \nabla \tilde{d}_\theta(\tilde{q}^x_{t_0}) \right|
\]

\[
\leq M \eta_N^a + M^2 K \eta_N
\]

with \( K \) the Lipschitz constant of \( \nabla \tilde{d}_\theta \) on the compact set \( \partial \mu \) since \( \tilde{d}_\theta \) is \( C^2 \) on \( \partial \mu \).

Defining \( M_1 := M + M^2 K \), one gets for \( N \) large enough so that \( \eta_N \leq 1 \)

\[
\left| \tilde{p}^x_{t_0} \cdot \nabla \tilde{d}_\theta(\tilde{q}^x_{t_k}) \right| \leq M_1 \eta_N^a.
\]

This yields that

\[
\mathbb{P} \left( \tilde{p}^x_{t_0} \cdot n(\tilde{q}^x_{t_0}) = 0, \tilde{t}_0 \leq T, \sup_{0 \leq t \leq T} |\tilde{p}^x_t| \leq M, \sup_{0 \leq s < t \leq T} \left| \frac{\tilde{p}^x_t - \tilde{p}^x_s}{t - s} \right| \leq M \right) \leq \sum_{k=0}^{N-1} \mathbb{P} \left( \left| \tilde{p}^x_{t_k} \cdot \nabla \tilde{d}_\theta(\tilde{q}^x_{t_k}) \right| \leq M_1 \eta_N^a, |\tilde{d}_\theta(\tilde{q}^x_{t_k})| \leq M \eta_N \right).
\]

(57)

Let \( k_0 := \left\lceil \frac{4M}{|p \cdot \tilde{n}(q)|} \right\rceil \). For \( k \in \llbracket 0, k_0 - 1 \rrbracket \), the summand in (57) vanishes when \( N \) goes to infinity since \( t_k = k \frac{T}{N} \leq (k_0 - 1) \frac{T}{N} \to 0 \) and either \( |\tilde{d}_\theta(q)| > 0 \) (if \( q \notin \partial O \)) or \( |p \cdot \nabla \tilde{d}_\theta(q)| > 0 \) (if \( q \in \partial O \), because \( (q, p) \notin \Gamma^0 \)).

**Step 2** Let us now prove that for \( k \in \llbracket k_0 + 1, N - 1 \rrbracket \), the summand in (57) is of order \( \eta_N^{1+a} \).

It is easy to check that \((\tilde{q}^x_t, \tilde{p}^x_t)\) is a Gaussian vector in \( \mathbb{R}^{2d} \) with law

\[
\left( \tilde{q}^x_t, \tilde{p}^x_t \right) \sim \mathcal{N}_{2d} \left( \begin{pmatrix} q + tp \\ p \end{pmatrix}, \begin{pmatrix} \frac{\sigma^2 t^3}{3} I_d & \frac{\sigma^2 t^2}{2} I_d \\ \frac{\sigma^2 t^2}{2} I_d & \sigma^2 t I_d \end{pmatrix} \right),
\]

(58)

where \( I_d \) denotes the identity matrix of \( \mathbb{R}^d \). In particular,

\[
\tilde{q}^x_t \sim \mathcal{N}_d \left( q + tp, \frac{\sigma^2 t^3}{3} I_d \right)
\]

with a density denoted by \( f_t \) for \( t > 0 \). We can also compute the conditional law of \( \tilde{p}^x_t \) knowing \( \tilde{q}^x_t \) using [11, Prop 3.13]. It is given by

\[
\mathcal{N}_d \left( p + \frac{3}{2t} (\tilde{q}^x_t - q - tp), \frac{\sigma^2 t}{4} I_d \right),
\]

where
with a density denoted by \( h_t(\cdot | \tilde{q}_t^x) \). As a result, using the fact that \( |\nabla \tilde{d}_\theta| = 1 \) on \( \overline{O}_\mu \), (see (56)), the conditional law of \( \tilde{p}_t^x \cdot \nabla \tilde{d}_\theta(\tilde{q}_t^x) \) knowing \( \tilde{q}_t^x \), when \( \tilde{q}_t^x \in \overline{O}_\mu \), is given by

\[
\mathcal{N}\left( \left( p + \frac{3}{2t}(\tilde{q}_t^x - q - tp) \right) \cdot \nabla \tilde{d}_\theta(\tilde{q}_t^x), \frac{\sigma^2 t}{4} \right)
\]

with a density denoted by \( g_t(\cdot | \tilde{q}_t^x) \).

As a consequence, for \( k \in [k_0, N - 1] \),

\[
\mathbb{P}\left( \left| \tilde{p}_k^x \cdot \nabla \tilde{d}_\theta(\tilde{q}_k^x) \right| \leq M_1 \eta_N, \left| \tilde{d}_\theta(\tilde{q}_k^x) \right| \leq M \eta_N \right)
\]

\[
= \int_{q' \in \mathbb{R}^d} \mathbb{1}_{|\tilde{d}_\theta(q')| \leq M \eta_N} f_k(q') \left( \int_{y \in \mathbb{R}} \mathbb{1}_{|y| \leq M_1 \eta_N} g_k(y|q')dy \right) dq'.
\]

Let \( m_k(q') := (p + 3(\cdot | q - t_k p) \cdot \nabla \tilde{d}_\theta(q') \) then for \( q' \in \overline{O}_\mu \),

\[
\int_{y \in \mathbb{R}} \mathbb{1}_{|y| \leq M_1 \eta_N} g_k(y|q')dy = \int_{y \in \mathbb{R}} \mathbb{1}_{|y| \leq M_1 \eta_N} e^{-\frac{(y-m_k(q'))^2}{\sigma^2}} \frac{1}{\sqrt{2\pi \sigma^2}} dy \leq \frac{2\sqrt{2} M_1 \eta_N}{\sqrt{\pi \sigma^2} t_k} \leq \frac{1}{\sqrt{\pi \sigma^2} t_k}
\]

Hence

\[
\mathbb{P}\left( \left| \tilde{p}_k^x \cdot \nabla \tilde{d}_\theta(\tilde{q}_k^x) \right| \leq M_1 \eta_N, \left| \tilde{d}_\theta(\tilde{q}_k^x) \right| \leq M \eta_N \right)
\]

\[
\leq \int_{q' \in \mathbb{R}^d} \mathbb{1}_{|\tilde{d}_\theta(q')| \leq M \eta_N} \frac{2\sqrt{2} M_1 \eta_N}{\sqrt{\pi \sigma^2} t_k} \left( \frac{3}{2\pi \sigma^2 t_k} \right) e^{-\frac{3|q' - q - t_k p|^2}{2\sigma^2 t_k}} dq'
\]

\[
\leq \left( \frac{3}{2\pi \sigma^2} \right)^{\frac{d}{2}} \frac{2\sqrt{2} M_1 \eta_N}{\sqrt{\pi \sigma^2} t_k} \int_{\mathbb{R}^d} e^{-\frac{3|q' - q - t_k p|^2}{2\sigma^2 t_k}} dq'. \tag{59}
\]

Let us now prove that the integrand is bounded by a constant independent of \( k \).

Case (a) Assume that \( q \notin \partial O \), then \( |\tilde{d}_\theta(q)| > 0 \) and there exists \( \mu_1 > 0 \) such that for any \( q' \in \overline{O}_\mu_1 \),

\[
|q' - q| \geq \sqrt{\frac{2}{3}} |\tilde{d}_\theta(q)|.
\]

Let us pick \( N \) large enough so that

\[
M \eta_N \leq \min(\mu, \mu_1). \tag{60}
\]

Let \( C_2 := \sup_{q \in O} |q - q'|. \) For \( C_2 |p| t_k \leq \frac{3|\tilde{d}_\theta(q)|^2}{6} \),

\[
-|q' - q - t_k p|^2 = -|q' - q|^2 - t_k^2 |p|^2 + 2t_k (q' - q) \cdot p \leq -|q' - q|^2 + 2t_k C_2 |p|
\]

\[
\leq -\frac{|\tilde{d}_\theta(q)|^2}{3}
\]
and
\[
\begin{aligned}
\frac{3|q' - q - t_k p|^2}{2\pi^{d+1}} \leq e^{-\frac{3|q|^2}{2\pi^{d+1}}}.
\end{aligned}
\]

Moreover, if \( C_2 |p| t_k > \frac{d_0(q)}{6} \) (necessarily \(|p| \neq 0\)),
\[
\begin{aligned}
\frac{e^{-\frac{3|q' - q - t_k p|^2}{2\pi^{d+1}}}}{\frac{3d+1}{t_k^2}} \leq \frac{1}{\left( \frac{d_0(q)^2}{6C_2|p|} \right)^{\frac{3d+1}{2}}}.
\end{aligned}
\]

Besides, the function \( t > 0 \mapsto e^{-\frac{d_0(q)^2}{2\pi^{d+1}}} + \frac{1}{\left( \frac{d_0(q)^2}{6C_2|p|} \right)^{\frac{3d+1}{2}}} \) is bounded by a constant \( C_3 \) which depends only on \( q, p \) and \( d \).

**Case (b)** Assume that \( q \in \partial \mathcal{O} \), then necessarily \(|p \cdot n(q)| > 0\) since \((q, p) \notin \Gamma^0\). By the right continuity in \( 0 \) of \( s \mapsto p \cdot \nabla d_0(q + sp) \), there exists \( \beta > 0 \) such that for all \( s \in [0, \beta] \), \(|p \cdot \nabla d_0(q + sp)| \geq \frac{|p \cdot n(q)|}{2} \) (and \( p \cdot \nabla d_0(q + sp) \) has constant sign on \([0, \beta])\).

Assume that \( t_k \leq \beta \). One has that
\[
\begin{aligned}
\bar{d}_0(q + t_k p) = \int_0^{t_k} p \cdot \nabla \bar{d}_0(q + sp) ds.
\end{aligned}
\]

Therefore, since the integrand \( p \cdot \nabla \bar{d}_0(q + sp) \) has constant sign on \([0, t_k]\),
\[
|\bar{d}_0(q + t_k p)| \geq t_k \frac{|p \cdot n(q)|}{2}.
\]

As a result, for \( q' \in \mathcal{O}_{M\eta_N} \) since \( \bar{d}_0 \) is 1–Lipschitz continuous,
\[
\begin{aligned}
|q' - q - t_k p| &\geq |\bar{d}_0(q + t_k p) - \bar{d}_0(q')| \\
&\geq t_k \frac{|p \cdot \nabla \bar{d}_0(q)|}{2} - M\eta_N \\
&= t_k \frac{|p \cdot \nabla \bar{d}_0(q)|}{2} \left( 1 - \frac{\eta_N}{t_k} \frac{2M}{|p \cdot \nabla \bar{d}_0(q)|} \right).
\end{aligned}
\]

Besides, since \( k \geq k_0 \),
\[
\begin{aligned}
\frac{\eta_N}{t_k} \frac{2M}{|p \cdot \nabla \bar{d}_0(q)|} &\leq 1 \frac{2M}{k \frac{|p \cdot \nabla \bar{d}_0(q)|}} \leq \frac{1}{k_0} \frac{2M}{|p \cdot \nabla \bar{d}_0(q)|} \leq \frac{1}{2},
\end{aligned}
\]

(62)

(63)
by definition of $k_0$. Therefore, for $t_k \leq \beta$,
\[
|q' - q - t_k p| \geq t_k \frac{|p \cdot \nabla d_\beta(q)|}{4},
\]
which ensures that the integrand in (59) is smaller than $e^{-\frac{3|p \cdot \nabla d_\beta(q)|^2}{32\sigma^2 t_k}}$ which is smaller than a constant $C_4 > 0$ only depending on $q$, $p$ and $d$. On the other hand, if $t_k \geq \beta$, the integrand (59) also admits a constant upper-bound independent of $k$.

As a result, one gets that $q' \in \mathcal{O}_{M\eta N} \mapsto e^{-\frac{3|q' - q - t_k p|^2}{2\sigma^2 t_k^2}}$ is bounded by $C_5 := C_3 \wedge C_4$, which is independent of $k$.

Last, using Weyl’s tube formula [38], one gets that there exists $C_6 > 0$ only depending on $\mathcal{O}$ such that
\[
\int_{\mathcal{O}_{M\eta N}} dq' \leq C_6 M\eta N.
\]
As a consequence,
\[
\mathbb{P}\left(|\tilde{p}_{t_0}^x \cdot \nabla d_\beta(\tilde{q}_{t_0}^x)| \leq M_1 \eta N, |\nabla d_\beta(\tilde{q}_{t_0}^x)| \leq M \eta N \right) \leq \left(\frac{3}{2\pi \sigma^2}\right)^d \frac{2\sqrt{2}M_1 \eta N}{\sqrt{\pi \sigma^2}} C_5 C_6 M \eta N
\]
which is independent of $k$.

**Step 3.** Finally summing over all $k$ one gets from (57) and (64), for $N$ large enough:
\[
\mathbb{P}\left(\tilde{p}_{t_0}^x \cdot n(\tilde{q}_{t_0}^x) = 0, \tilde{r}_0^x \leq T, \sup_{0 \leq t \leq T} |\tilde{p}_t^x| \leq M, \sup_{0 \leq s < t \leq T} |\tilde{p}_t^x - \tilde{p}_s^x| \leq M \right)
\]
\[
\leq \sum_{k=0}^{k_0-1} \mathbb{P}\left(\tilde{p}_{t_k}^x \cdot \nabla d_\beta(\tilde{q}_{t_k}^x) \leq M_1 \eta N, |\nabla d_\beta(\tilde{q}_{t_k}^x)| \leq M \eta N \right)
\]
\[
\xrightarrow{N \to \infty} 0
\]
\[
+ \frac{N}{N - k_0} \left(\frac{3}{2\pi \sigma^2}\right)^d \frac{2\sqrt{2}M_1}{\sqrt{\pi \sigma^2}} C_5 C_6 M T \eta N, \text{ not depending on } N
\]

Letting $\eta N \xrightarrow{N \to \infty} 0$, we get
\[
\mathbb{P}\left(\tilde{p}_{t_0}^x \cdot n(\tilde{q}_{t_0}^x) = 0, \tilde{r}_0^x \leq T, \sup_{0 \leq t \leq T} |\tilde{p}_t^x| \leq M, \sup_{0 \leq s < t \leq T} |\tilde{p}_t^x - \tilde{p}_s^x| \leq M \right) = 0.
Thus, for all $\epsilon > 0$,

$$
\mathbb{P}\left( \tilde{\eta}_{x_0}^x \cdot n(\tilde{q}_{x_0}^x) = 0, \tilde{z}_0^x \leq T \right) \leq 2\epsilon
$$

which concludes the proof of (53).

\[\Box\]

4. Harnack inequality and Maximum principle

This section is devoted to the proof of the Harnack inequality stated in Theorem 2.15 and of the maximum principle of Theorem 2.14. The proofs are respectively detailed in Sections 4.1 and 4.2.

4.1. Proof of Theorem 2.15

A Harnack inequality for weak solutions of (16) was already proven in [17]. It says that for every point $(t_0, x_0) \in \mathbb{R}_+^* \times D$ there exist $T > 0$, two small disjoint cylinders $Q^+, Q^- \subset D$ close to $x_0$ and a constant $C > 0$ such that for any non-negative distributional solution $u$ of $\partial_t u = \mathcal{L} u$, we have for all $t_0 \geq 0$,

$$
sup_{x \in Q^-} u(t_0, x) \leq C \inf_{x \in Q^+} u(t_0 + T, x).
$$

Adapting a chaining argument from [1] with a suitable chaining function, we extend this inequality to any compact set $K$ of $D$ to obtain the result (18). In order to prepare the chaining argument, we first introduce some notation. For all $x, y \in D$ and $T, M, \delta > 0$, we denote by $\mathcal{H}_{T,x,y,M,\delta}$ the set of $C^1$ and piecewise $C^2$ paths $\phi : [0, T] \to \mathcal{O}$ such that

$$(\phi(0), \dot{\phi}(0)) = x, \quad (\phi(T), \dot{\phi}(T)) = y, \quad \sup_{s \in [0,T]} (|\dot{\phi}| + |\ddot{\phi}|)(s) \leq M, \quad \inf_{s \in [0,T]} d_3(\phi(s)) > \delta.$$ 

**Lemma 4.1.** There exists a universal constant $C > 1$ such that for all $\Delta > 0$, $x = (q, p), y = (q', p') \in \mathbb{R}^{2d}$, there exists $\phi \in C^2([0, \Delta], \mathbb{R}^d)$ such that

- (i) $\phi(0), \dot{\phi}(0) = x$ and $(\phi(\Delta), \dot{\phi}(\Delta)) = y$,
- (ii) $\sup_{t \in [0,\Delta]} |\phi(t) - q| \leq C (|q' - q| + \Delta |p' - p| + \Delta |p|)$,
- (iii) $\sup_{t \in [0,\Delta]} |\dot{\phi}(t)| \leq \frac{C}{\Delta^2} (|q' - q| + \Delta |p' - p| + \Delta |p|)$,
- (iv) $\sup_{t \in [0,\Delta]} |\ddot{\phi}(t)| \leq \frac{C}{\Delta^3} (|q' - q| + \Delta |p' - p| + \Delta |p|)$.

**Proof.** Let $\Delta > 0$. For all $t \in [0, \Delta]$, we define

$$
\phi(t) := q + (q' - q) \left(3 \frac{t^2}{\Delta^2} - 2 \frac{t^3}{\Delta^3} \right) + \Delta (p' - p) \left( \frac{t^3}{\Delta^3} - \frac{t^2}{\Delta^2} \right) + \Delta p \left( \frac{t}{\Delta} + 2 \frac{t^3}{\Delta^3} - 3 \frac{t^2}{\Delta^2} \right).
$$

It is easy to see that $\phi$ satisfies the conditions above. \[\Box\]
We recall that under Assumption (O1), the set $O$ satisfies the uniform interior sphere condition and denote by $\rho > 0$ the associated radius, given by Proposition 3.7. The following lemma is proven in Appendix A.

**Lemma 4.2.** (Admissible paths) Under Assumptions (O1) and (O2), let $K \subset D$ be a compact set and $\delta_K := d(K, \partial D) \land \rho$. For all $T > 0$, there exists $M_{K,T} > 0$ such that for all $x, y \in K$ the set $\mathcal{H}_{T,x,y,M_{K,T},\delta_K/2}$ is nonempty.

Let us now prove the Harnack inequality in Theorem 2.15.

**Proof of Theorem 2.15.** Let $K \subset D$ be a compact set. Let $T > 0$ and let $u$ be a non-negative distributional solution of $\partial_t u - Lu = 0$ on $\mathbb{R}_+^* \times D$. The proof is divided into two steps. In the first step, we introduce the necessary background in order to apply the Harnack inequality from [17]. In the second step, we detail the chaining argument, based on Lemma 4.2, which allows us to obtain the Harnack inequality of Theorem 2.15.

**Step 1** Let $M_{K,T} > 0$ and $\delta_K > 0$ be the constants given in Lemma 4.2 and let us define the constant $r_{K,T} := \sqrt{\frac{\delta_K}{1 + M_{K,T}}} \land \frac{1}{2}$. Let $r \in (0, r_{K,T}]$. Let us define

$$D_{K,T,r} = \{(t, q, p) \in \mathbb{R}_+^* \times D : t > r^2, d_0(q) > \delta_K/2, |p| \leq M_{K,T}\}.$$ 

Notice that $(r^2, \infty) \times K \subset D_{K,T,r}$. Let $Q$ be the following unit box

$$Q := \{(t, q, p) \in \mathbb{R} \times \mathbb{R}^d : t \in (-1, 0], |q| < 1, |p| < 1\}.$$

For all $z_0 = (t_0, q_0, p_0) \in D_{K,T,r}$, let us define the following function on $Q$

$$h_{r,z_0} : (t, q, p) \mapsto (r^2t + t_0, q_0 - r^2tp_0 + r^3q, p_0 - rp).$$

Notice that for all $z_0 \in D_{K,T,r}$ and $(t, q, p) \in Q$,

$$-r^2tp_0 + r^3q| \leq M_{K,T}r_{K,T}^2 + r_{K,T}^3 < r_{K,T}(1 + M_{K,T}) \leq \delta_K,$$

since $r_{K,T} \in (0, 1)$. As a result, $h_{r,z_0}$ is a function on $Q$ with values in $\mathbb{R}_+^* \times D$.

Since $\partial_t - L$ is a hypoelliptic operator on $\mathbb{R}_+^* \times D$ it follows that $u$ is in $C^\infty(\mathbb{R}_+^* \times D)$.

Let us now define the following smooth function

$$u_{r,z_0} := u \circ h_{r,z_0}$$

on $Q$. It satisfies

$$\partial_t u_{r,z_0} = -p \cdot \nabla q u_{r,z_0} + \gamma(rp_0 - r^2p) \cdot \nabla p u_{r,z_0} - rF(q_0 - r^2tp_0 + r^3q) \cdot \nabla p u_{r,z_0}$$

$$+ \frac{\sigma^2}{2} \Delta p u_{r,z_0}.$$

Besides for $(t, q, p) \in Q$,

$$|\gamma(rp_0 - r^2p) - rF(q_0 - r^2tp_0 + r^3q)| \leq |\gamma| (M_{K,T} + \delta_K) + \|F\|_{L^\infty(O)}$$
which is a constant depending only on the compact $K$, $T$ and the coefficients of $\mathcal{L}$. As a result, Theorem 4 in [17] ensures the existence of constants $C_{K,T} > 1$ and $R_{K,T}, \Delta_{K,T} \in (0,1)$ (which do not depend on $r \in (0, r_{K,T})$ or $z_0$) such that
\[
\Delta_{K,T} + R_{K,T}^2 < 1
\]
and
\[
\sup_{(t,q,p) \in Q_{K,T}^+} u_{r,z_0}(t,q,p) \leq C_{K,T} \inf_{(t,q,p) \in Q_{K,T}^+} u_{r,z_0}(t,q,p) \quad (65)
\]
where
\[
Q_{K,T}^+ := \{(t,q,p) : t \in (-R_{K,T}^2,0], |q| < R_{K,T}^3, |p| < R_{K,T} \} \subset Q,
\]
\[
Q_{K,T}^- := \{(t,q,p) : t \in (-R_{K,T}^2 - \Delta_{K,T}, -\Delta_{K,T}], |q| < R_{K,T}^3, |p| < R_{K,T} \} \subset Q.
\]
Introducing the notation $Q_{K,T,r,z_0}^\pm = h_{r,z_0}(Q_{K,T}^\pm)$, (65) rewrites, for all $r \in (0,r_{K,T})$ and $z_0 \in D_{K,T,r}$,
\[
\sup_{(t,q,p) \in Q_{K,T,r,z_0}^-} u(t,q,p) \leq C_{K,T} \inf_{(t,q,p) \in Q_{K,T,r,z_0}^+} u(t,q,p). \quad (66)
\]

**Step 2** Let $\epsilon > 0$. Let us choose $r_{K,T}^\epsilon$ satisfying

(i) $0 < r_{K,T}^\epsilon \leq r_{K,T}$,

(ii) $r_{K,T}^\epsilon < \frac{2R_{K,T}^2}{M_{K,T} \left( \Delta_{K,T} + \frac{R_{K,T}^2}{2} \right)} \land \frac{R_{K,T}}{M_{K,T} \left( \Delta_{K,T} + \frac{R_{K,T}^2}{2} \right)} \land \sqrt{1 - \Delta_{K,T} - \frac{R_{K,T}^2}{2}}$,

(iii) the quantity $\alpha_{K,T}^\epsilon := (r_{K,T}^\epsilon)^2 \left( \Delta_{K,T} + \frac{R_{K,T}^2}{2} \right)$ is such that $n_{K,T}^\epsilon := \frac{T}{\alpha_{K,T}^\epsilon} \in \mathbb{N}$.

Let $t \geq T + \epsilon$. Let $(x,y)$ be two arbitrary points in the compact set $K$. Let $\phi \in \mathcal{H}_{T,x,y,M_{K,T},\delta_{K}/2}$ (which exists by Lemma 4.2) and let
\[
\Phi : s \in [0,T] \mapsto \left( \begin{array}{c} \phi(s) \\ \phi(s) \\ t-s \end{array} \right) \in \mathbb{R}^{2d+1}.
\]

Now let $(s_j^\epsilon)_{0 \leq j \leq n_{K,T}^\epsilon}$ be the sequence defined by $s_j^\epsilon := j\alpha_{K,T}^\epsilon$ for $0 \leq j \leq n_{K,T}^\epsilon$.

Let us show that $\Phi(s_j^\epsilon) \in D_{K,T,r_{K,T}^\epsilon}$ for all $0 \leq j \leq n_{K,T}^\epsilon - 1$. Indeed, one has
\[
t - s_{n_{K,T}^\epsilon - 1} = t - (n_{K,T}^\epsilon - 1)\alpha_{K,T}^\epsilon = t - T + (r_{K,T}^\epsilon)^2 \left( \Delta_{K,T} + \frac{R_{K,T}^2}{2} \right) > (r_{K,T}^\epsilon)^2,
\]
since $(r_{K,T}^\epsilon)^2 < \frac{\epsilon}{1 - \Delta_{K,T} - \frac{R_{K,T}^2}{2}}$. The rest follows from the definition of $\mathcal{H}_{T,x,y,M_{K,T},\delta_{K}/2}$.

Hence, (66) is satisfied for $r = r_{K,T}^\epsilon$ and $z_0 = \Phi(s_j^\epsilon)$ for all $0 \leq j \leq n_{K,T}^\epsilon - 1$, i.e.
\[
\sup_{(t,q,p) \in Q_{K,T,r_{K,T}^\epsilon}^-} u(t,q,p) \leq C_{K,T} \inf_{(t,q,p) \in Q_{K,T,r_{K,T}^\epsilon}^+} u(t,q,p).
\]

Let us now prove that for every $0 \leq j \leq n_{K,T}^\epsilon - 1$, $\Phi(s_{j+1}^\epsilon) \in Q_{K,T,r_{K,T}^\epsilon}^-(\Phi(s_j^\epsilon))$. Let
(i) \( \hat{t}_j := -\frac{\alpha^e_{K,T}}{(r^e_{K,T})^2} = -\Delta_{K,T} - \frac{R^2_{K,T}}{2}, \)

(ii) \( \hat{q}_j := \frac{1}{r^e_{K,T}} \left( \phi(s^e_{j+1}) - \phi(s^e_j) - \alpha^e_{K,T} \dot{\phi}(s^e_j) \right), \)

(iii) \( \hat{p}_j := \frac{1}{r^e_{K,T}} \left( \dot{\phi}(s^e_j) - \dot{\phi}(s^e_{j+1}) \right). \)

Then it only remains to prove that \((\hat{t}_j, \hat{q}_j, \hat{p}_j) \in Q^-_{K,T}\) for every \(0 \leq j \leq n^e_{K,T} - 1\), i.e. that

\[
hr^e_{K,T} \Phi(s^e_j) (\hat{t}_j, \hat{q}_j, \hat{p}_j) = \Phi(s^e_{j+1}).
\]

First, concerning \(\hat{t}_j\), it is clear by definition of \(\alpha^e_{K,T}\) that

\[
-\Delta_{K,T} - R^2_{K,T} < -\frac{\alpha^e_{K,T}}{(r^e_{K,T})^2} \leq -\Delta_{K,T}.
\]

Second, for \(\hat{q}_j\),

\[
\left| \phi(s^e_{j+1}) - \phi(s^e_j) - \alpha^e_{K,T} \dot{\phi}(s^e_j) \right| = \left| \int_{s^e_j}^{s^e_{j+1}} \left( \dot{\phi}(\eta) - \dot{\phi}(s^e_j) \right) \, d\eta \right|
\leq \int_{s^e_j}^{s^e_{j+1}} \int_{s^e_j}^{\eta} |\dddot{\phi}(\mu)| \, d\mu \, d\eta
\leq MK_{K,T} \int_{s^e_j}^{s^e_{j+1}} (\eta - s^e_j) \, d\eta \leq M_{K,T} \frac{(\alpha^e_{K,T})^2}{2}
\]
and since

\[
r^e_{K,T} < \frac{2R^3_{K,T}}{M_{K,T} \left( \Delta_{K,T} + \frac{R^2_{K,T}}{2} \right)^2},
\]
we have

\[
(r^e_{K,T})^4 \left( \Delta_{K,T} + \frac{R^2_{K,T}}{2} \right)^2 < 2(r^e_{K,T})^3 R^3_{K,T},
\]
and therefore

\[
M_{K,T} \frac{(\alpha^e_{K,T})^2}{2} < (r^e_{K,T})^3 R^3_{K,T}.
\]

Third, for \(\hat{p}_j\),

\[
\left| \dot{\phi}(s^e_{j+1}) - \dot{\phi}(s^e_j) \right| \leq \int_{s^e_j}^{s^e_{j+1}} |\dddot{\phi}(\eta)| \, d\eta \leq M_{K,T} \alpha^e_{K,T}
\]
and the assumption that
\[ r_{K,T}^\epsilon < \frac{R_{K,T}}{M_{K,T} \left( \Delta_{K,T} + \frac{R_{K,T}^2}{2} \right)} \]
ensures that
\[ M_{K,T} a_{K,T}^\epsilon < r_{K,T}^\epsilon R_{K,T}. \]
Hence \( \Phi(s_{j+1}^\epsilon) \in Q_{K,T}^{-}(r_{K,T}^\epsilon \Phi(s_j^\epsilon)) \).

Finally, one gets for \( 0 \leq j \leq n_{K,T}^\epsilon - 1 \)
\[ u(\Phi(s_{j+1}^\epsilon)) \leq \sup_{Q_{K,T}^{-}} u \leq C_{K,T} \inf_{Q_{K,T}^{-}} u \leq C_{K,T} u(\Phi(s_j^\epsilon)) \]
which yields by iterating,
\[ u(\Phi(T)) = u(t - T, y) \leq C_{K,T}^{n_{K,T}^\epsilon} u(\Phi(0)) = C_{K,T}^{n_{K,T}^\epsilon} u(t, x) \]
where \( C_{K,T}^{n_{K,T}^\epsilon} \) does not depend on \( (t, x, y) \) but only on the compact set \( K \) and \( T, \epsilon \). As a result, we have for all \( t \geq T + \epsilon \),
\[ \sup_{x \in K} u(t - T, x) \leq C_{K,T}^{n_{K,T}^\epsilon} \inf_{x \in K} u(t, x) \]
which concludes the proof. \( \Box \)

4.2. Proof of Theorem 2.14

In order to prove the maximum principle stated in Theorem 2.14, we need the following lemma.

**Lemma 4.3 (Irreducibility).** Let Assumptions (F1), (O1) and (O2) hold. Let \( A \) be an open subset of \( D \), then
\[ \forall x \in D, \quad \forall t > 0, \quad \forall s \in (0, t), \quad \mathbb{P}(X_s^x \in A, \tau_{x}^\epsilon > t) > 0. \]

**Proof.** Let \( x \in D, \ t > 0, \ s \in (0, t) \). Let \( A \) be an open subset of \( D \), the Markov property at time \( s \) ensures that
\[ \mathbb{P}(X_s^x \in A, \tau_{x}^\epsilon > t) = \mathbb{E} \left[ 1_{X_s^x \in A, \tau_{x}^\epsilon > s} \mathbb{P}(\tau_{x}^\epsilon > t - s) \big| y = X_s^x \right]. \]

By Theorem 2.20, which is proven in Sections 5 and 6, the kernel \( P_s^D(x, \cdot) \) defined in (21) admits a positive density function \( p_t^D(x, \cdot) \). Therefore, \( \mathbb{P}(X_s^x \in A, \tau_{x}^\epsilon > s) > 0 \).

Again, by the positivity of \( p_t^D(s, \cdot) \) in Theorem 2.20, on the event \( \{ X_s^x \in A, \tau_{x}^\epsilon > s \} \), one has that \( \mathbb{P}(\tau_{x}^\epsilon > t - s) | y = X_s^x > 0 \) almost surely. This concludes the proof using the Markov property stated above. \( \Box \)
Let us now prove Theorem 2.14.

**Proof of Theorem 2.14.** Let \( x \in D \). Let \((X^x_t, (q^x_t, p^x_t)))_{t \geq 0}\) be the strong solution of (11) on \( \mathbb{R}^{2d} \). For \( k > 0 \), let \( V_k \) be the following open and bounded subset of \( D \)

\[
V_k := \left\{(q, p) \in D : \left|p\right| < k, d_\theta(q) > \frac{1}{k}\right\}.
\]

Let \( \tau_{V_k}^x \) be the following stopping time:

\[
\tau_{V_k}^x = \inf\{t > 0 : X^x_t \notin V_k\}.
\]

Let \( t > 0 \) and \( s \in [0, t) \). Since \( u \in C^{1,2}(\mathbb{R}^n_+ \times D) \), Itô’s formula applied to the process \((u(t - r, X^x_r))_{0 \leq r \leq s}\) between 0 and \( \tau_{V_k}^x \) yields: almost surely, for \( s \in [0, t) \),

\[
u(t, x) = \mathbb{E}\left[\mathbb{1}_{\tau_{V_k}^x > s} u(t - s, X^x_s)\right] + \mathbb{E}\left[\mathbb{1}_{\tau_{V_k}^x \leq s} u(t - \tau_{V_k}^x, X^x_{\tau_{V_k}^x})\right] + \mathbb{E}\left[\int_0^{s \wedge \tau_{V_k}^x} (\partial_t u(t - r, X^x_r) - \mathcal{L} u(t - r, X^x_r)) \, dr\right].
\]  

(67)

**Step 1** Let us prove Assertion (i) in Theorem 2.14 using (67). It follows from (67) and the inequality \( \partial_t u - \mathcal{L} u \leq 0 \) on \( \mathbb{R}^n_+ \times D \), that

\[
u(t, x) \leq \mathbb{E}\left[\mathbb{1}_{\tau_{V_k}^x > t} u(0, X^x_t)\right] + \mathbb{E}\left[\mathbb{1}_{\tau_{V_k}^x \leq t} u(t - \tau_{V_k}^x, X^x_{\tau_{V_k}^x})\right].
\]

By assumption, \( u \in C^b((\mathbb{R}^n_+ \times \overline{D}) \setminus \{(0)_n \times (\Gamma^+ \cup \Gamma^0))\). Therefore, following the same reasoning as in the proof of Assertion (iv) of Theorem 2.10 in Section 3.1, one obtains by letting \( s \to t \) and \( k \to \infty \) that

\[
u(t, x) \leq \mathbb{E}\left[\mathbb{1}_{\tau_{\tilde{\theta}}^x > t} u(0, X^x_t)\right] + \mathbb{E}\left[\mathbb{1}_{\tau_{\tilde{\theta}}^x \leq t} u(t - \tau_{\tilde{\theta}}^x, X^x_{\tau_{\tilde{\theta}}^x})\right].
\]

Since \( X^x_{\tau_{\tilde{\theta}}} \in \Gamma^+ \) almost surely by Proposition 2.8, the inequality above immediately yields Assertion (i).

**Step 2** We now prove Assertion (ii). Applying the equality (67) for \((t, x) = (t_0, x_0)\) and subtracting \( u(t_0, x_0) \), we obtain that for all \( s \in [0, t_0) \),

\[
0 = \mathbb{E}\left[\mathbb{1}_{\tau_{V_k}^x > t_0} (u(t_0 - s, X^x_{t_0}) - u(t_0, x_0)\right] + \mathbb{E}\left[\mathbb{1}_{\tau_{V_k}^x \leq t_0} (u(t_0 - \tau_{V_k}^x, X^x_{\tau_{V_k}^x}) - u(t_0, x_0)\right] + \mathbb{E}\left[\int_0^{s \wedge \tau_{V_k}^x} (\partial_t u(t_0 - r, X^x_{t_0}) - \mathcal{L} u(t_0 - r, X^x_{t_0})) \, dr\right].
\]

Using the fact that \( u(t_0, x_0) = \|u\|_\infty \) and that \( \partial_t u - \mathcal{L} u \leq 0 \) on \( \mathbb{R}^n_+ \times D \), it follows that, necessarily, for all \( k > 0 \) and \( s \in [0, t_0) \), (since \( \mathbb{1}_{\tau_{V_k}^x > t_0} \leq \mathbb{1}_{\tau_{V_k}^x > s} \))

\[
\mathbb{E}\left[\mathbb{1}_{\tau_{V_k}^x > t_0} (u(t_0 - s, X^x_{t_0}) - u(t_0, x_0)\right] = 0.
\]
Taking $k \to \infty$ as in the proof of Assertion (iv) of Theorem 2.10, one obtains that for all $s \in [0, t_0)$,

$$
\mathbb{E}\left[ \mathbb{1}_{\tau_{x_0}^0 > t_0} \left( u(t_0 - s, X_{x_0^0}^s) - u(t_0, x_0) \right) \right] = 0. \tag{68}
$$

Assume now that Assertion (ii) is not satisfied, then there exist $c > 0, s_0 \in (0, t_0)$ and an open subset $A$ of $D$ such that for all $y \in A$, $u(t_0 - s_0, y) - u(t_0, x_0) \leq -c$. Therefore,

$$
\mathbb{E}\left[ \mathbb{1}_{\tau_{y_0}^0 > t_0, X_{y_0}^0 \in A} \left( u(t_0 - s_0, X_{y_0}^s) - u(t_0, x_0) \right) \right] \leq -c \mathbb{P}(\tau_{y_0}^0 > t_0, X_{y_0}^s \in A) < 0,
$$

by Lemma 4.3. Moreover,

$$
\mathbb{E}\left[ \mathbb{1}_{\tau_{y_0}^0 > t_0} \left( u(t_0 - s_0, X_{y_0}^s) - u(t_0, x_0) \right) \right] \leq \mathbb{E}\left[ \mathbb{1}_{\tau_{y_0}^0 > t_0, X_{y_0}^s \in A} \left( u(t_0 - s_0, X_{y_0}^s) - u(t_0, x_0) \right) \right] < 0,
$$

which is in contradiction with (68), hence Assertion (ii). \qed

5. Gaussian upper bound and existence of a smooth transition density for the absorbed Langevin process

The proof of the Gaussian upper bound stated in Theorem 2.19 is provided in Sect. 5.1. Section 5.2 is devoted to the proof of the existence of a smooth transition density for the absorbed Langevin process from Definition 2.5, and the fact that this density satisfies the backward and forward Kolmogorov equations. This yields the first part of Theorem 2.20, the boundary continuity will be proven in Sect. 6. Sect. 5.3 is devoted to the study of some preliminary boundary continuity properties of the transition density for the absorbed Langevin process (11) which will be useful in Sect. 6.

5.1. Gaussian upper bound for the Langevin process in $\mathbb{R}^d$

The purpose of this section is to provide a Gaussian upper bound satisfied by the transition density $p_t(x, y)$ of the process $(X_t^x = (q_t^x, p_t^x))_{t \geq 0}$ defined by (11) under Assumption (F2). We do not consider absorption in this section.

For $x = (q, p) \in \mathbb{R}^d$, let $(\hat{X}_t^x = (\hat{q}_t^x, \hat{p}_t^x))_{t \geq 0}$ be the strong solution on $\mathbb{R}^d$ of the following SDE

$$
\begin{align*}
\frac{d\hat{q}_t^x}{dt} &= \hat{p}_t^x dt, \\
\frac{d\hat{p}_t^x}{dt} &= -\gamma \hat{p}_t^x dt + \sigma dB_t, \\
(\hat{q}_0^x, \hat{p}_0^x) &= x,
\end{align*}
$$

(69)
with infinitesimal generator \( \widehat{L} := L_{0, \gamma, \sigma} \). Let \( \Phi_1, \Phi_2 \) be the following positive continuous functions on \( \mathbb{R} \):

\[
\Phi_1 : \rho \in \mathbb{R} \mapsto \begin{cases} \frac{1-e^{-\rho}}{\rho} & \text{if } \rho \neq 0, \\ 1 & \text{if } \rho = 0, \end{cases}
\]

\[
\Phi_2 : \rho \in \mathbb{R} \mapsto \begin{cases} \frac{2\rho - 3 + 4e^{-\rho} - e^{-2\rho}}{2\rho} & \text{if } \rho \neq 0, \\ 1 & \text{if } \rho = 0. \end{cases}
\]

The process \((\widehat{q}_t^x, \widehat{p}_t^x)_{t \geq 0}\) is Gaussian and for all \( t \geq 0 \), the vector \((\widehat{q}_t^x, \widehat{p}_t^x)\) admits the following law

\[
\left( \begin{array}{c} \widehat{q}_t^x \\ \widehat{p}_t^x \end{array} \right) \sim \mathcal{N}_{2d} \left( \left( \begin{array}{c} m_q^x(t) \\ m_p^x(t) \end{array} \right), C(t) \right),
\]

where the mean vector is

\[
m_q^x(t) := q + tp \Phi_1(\gamma t), \quad m_p^x(t) := pe^{-\gamma t},
\]

and the covariance matrix is

\[
C(t) := \begin{pmatrix} c_{qq}(t)I_d & c_{qp}(t)I_d \\ c_{qp}(t)I_d & c_{pp}(t)I_d \end{pmatrix},
\]

where \( I_d \) is the identity matrix in \( \mathbb{R}^{d \times d} \) and

\[
c_{qq}(t) := \frac{\sigma^2 t^3}{3} \Phi_2(\gamma t), \quad c_{qp}(t) := \frac{\sigma^2 t^2}{2} \Phi_1(\gamma t)^2, \quad c_{pp}(t) := \sigma^2 t \Phi_1(2\gamma t).
\]

The determinant of the covariance matrix \( C(t) \) is \( \det(C(t)) = (\frac{\sigma^4 t^4}{12 \phi(\gamma t)})^d \) where \( \phi \) is the positive continuous function defined by

\[
\phi : \rho \in \mathbb{R} \mapsto 4\Phi_2(\rho) \Phi_1(2\rho) - 3\Phi_1(\rho)^4 = \begin{cases} \frac{6(1-e^{-\rho})}{\rho^3} \left[-2 + \rho + (2 + \rho)e^{-\rho}\right] & \text{if } \rho \neq 0, \\ \frac{6(1-\rho)}{\rho^3} & \text{if } \rho = 0. \end{cases}
\]

As a result, one can easily obtain an explicit expression of the transition density \( \widehat{p}_t((q, p), (q', p')) \) of the process \((\widehat{q}_t^x, \widehat{p}_t^x)_{t \geq 0}\): for \( t > 0 \), \((q, p), (q', p') \in \mathbb{R}^{2d} \),

\[
\widehat{p}_t((q, p), (q', p')) := \frac{1}{\sqrt{(2\pi)^{2d} \left( \frac{\sigma^4 t^4}{12 \phi(\gamma t)} \right)^d}} e^{-\frac{\delta x(t)C^{-1}(t)\delta x(t)}{2}}
\]

where

\[
\delta x(t) := \begin{pmatrix} \delta q(t) \\ \delta p(t) \end{pmatrix} := \begin{pmatrix} q' - m_q^x(t) \\ p' - m_p^x(t) \end{pmatrix}, \quad C^{-1}(t) = \frac{1}{\frac{\sigma^4 t^4}{12 \phi(\gamma t)}} \begin{pmatrix} c_{pp}(t)I_d & -c_{qp}(t)I_d \\ -c_{qp}(t)I_d & c_{qq}(t)I_d \end{pmatrix}.
\]
We now give a useful rewriting of $\delta x(t) \cdot C^{-1}(t)\delta x(t)$ as a sum of squares, inspired by [36, Equation (2.5)], using an additional positive continuous function on $\mathbb{R}$:

$$
\Phi_3 : \rho \in \mathbb{R} \mapsto \begin{cases} 
\frac{2(1-\Phi_1(\rho))}{\rho} & \text{if } \rho \neq 0, \\
1 & \text{if } \rho = 0.
\end{cases}
$$

(77)

The proof of the following lemma is detailed in Appendix B.

**Lemma 5.1** (Covariance decomposition). For all $t > 0$, $\delta x = \begin{pmatrix} \delta q \\ \delta p \end{pmatrix} \in \mathbb{R}^{2d}$,

$$
\delta x \cdot C^{-1}(t)\delta x = \frac{1}{\sigma^2 t} |\Pi_1\delta x|^2 + \frac{12}{\sigma^2 t^3 \phi(\gamma t)} |\Pi_2(t)\delta x|^2,
$$

where $\Pi_1 := \begin{pmatrix} \gamma I_d & I_d \end{pmatrix} \in \mathbb{R}^{d \times 2d}$ and $\Pi_2(t) := \begin{pmatrix} \Phi_1(\gamma t)I_d & -\frac{1}{2}\Phi_3(\gamma t)I_d \end{pmatrix} \in \mathbb{R}^{d \times 2d}$. 

Now let $\alpha \in (0, 1]$. For $x = (q, p), y = (q', p') \in \mathbb{R}^{2d}$ and $t > 0$, let $\hat{p}_t^{(\alpha)}(x, y)$ be the transition density of the process $(\alpha^{-1/2}X_t^\alpha)_{t \geq 0}$, with infinitesimal generator $L_{0, \gamma, \sigma/\sqrt{\alpha}}$, i.e.

$$
\hat{p}_t^{(\alpha)}((q, p), (q', p')) := \sqrt{\alpha^{2d}} \hat{p}_t(\sqrt{\alpha}(q, p), \sqrt{\alpha}(q', p')).
$$

(79)

Let us state the following useful properties which are also proven in Appendix B.

**Lemma 5.2** (Transition density properties). The transition densities $\hat{p}_t$ and $\hat{p}_t^{(\alpha)}$ satisfy:

(i) For all $t > 0$, and $x = (q, p), y = (q', p') \in \mathbb{R}^{2d}$, (using the notation (76))

$$
\hat{p}_t(x, y) = \frac{1}{\sqrt{\alpha^{2d}}} e^{-\frac{1}{2\alpha} \int_0^t \delta x(t) \cdot C^{-1}(t)\delta x(t)} \hat{p}_t^{(\alpha)}(x, y).
$$

(80)

(ii) Chapman-Kolmogorov relation: For all $t > 0$, for all $u \in (0, t)$ and $x, y \in \mathbb{R}^{2d}$,

$$
\int_{\mathbb{R}^{2d}} \hat{p}_u^{(\alpha)}(x, z) \hat{p}_t^{(\alpha)}(z, y) dz = \hat{p}_t^{(\alpha)}(x, y).
$$

(81)

(iii) For all $t > 0$, $\varphi \in C_b(\mathbb{R}_+ \times \mathbb{R}^{2d})$ and $y, x_0, y_0 \in \mathbb{R}^{2d}$,

$$
\int_{\mathbb{R}^{2d}} \hat{p}_t^{(\alpha)}(x, y) dx = e^{d\gamma t}
$$

and

$$
\int_{\mathbb{R}^{2d}} \hat{p}_t^{(\alpha)}(x, y) \varphi(t, x) dx \bigg|_{(t,y) \rightarrow (0, y_0)} \varphi(0, y_0), \quad \int_{\mathbb{R}^{2d}} \hat{p}_t^{(\alpha)}(x, y) \varphi(t, y) dy \bigg|_{(t,x) \rightarrow (0, x_0)} \varphi(0, x_0).
$$

(82, 83)
(iv) For all $\alpha \in (0, 1)$, there exists $c_\alpha > 0$ depending only on $\alpha$ such that for all $t > 0$, $x, y \in \mathbb{R}^{2d}$,

$$|\nabla_p \hat{p}_t(x, y)| \leq \frac{c_\alpha (1 + \sqrt{\gamma - t})}{\sqrt{\sigma^2 t}} \hat{p}_t(x, y), \quad (84)$$

where $\gamma_-$ is the negative part of $\gamma \in \mathbb{R}$.

We are now in position to prove Theorem 2.19.

**Proof of Theorem 2.19.** The idea is to first establish a mild formulation of the difference between the two transition densities $p_t(x, y)$ and $\hat{p}_t(x, y)$, adapting the reasoning from [24]. Secondly, iterating the obtained equality, one obtains, following the steps of [24], an expression of the difference between $p_t(x, y)$ and $\hat{p}_t(x, y)$ in the form of a series, which then yields the Gaussian upper bound stated in Theorem 2.19.

**Step I** Let us first obtain the mild formulation linking $p_t(x, y)$ and $\hat{p}_t(x, y)$. Let $T > 0$ and $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$. Then the function

$$\Phi : (t, (q, p)) \in [0, T) \times \mathbb{R}^{2d} \mapsto \int_{\mathbb{R}^{2d}} \hat{p}_{T-t}((q, p), y)\varphi(y)dy,$$

is in $\mathcal{C}^\infty([0, T) \times \mathbb{R}^{2d})$ by the Lebesgue differentiation theorem. Besides, it satisfies $\partial_t \Phi + \mathcal{L} \Phi = 0$ using the backward Kolmogorov equation satisfied by $\hat{p}_t(x, y)$ (see Proposition 2.17). As a result, the Itô formula ensures that for all $x \in \mathbb{R}^{2d}$, $t \in [0, T)$,

$$\Phi(t, X_t) = \Phi(0, x) + \int_0^t (\mathcal{L} - \mathcal{L})\Phi(u, X_u^x)du + \sigma \int_0^t \nabla_p \Phi(u, X_u^x) \cdot dB_u. \quad (85)$$

Besides, one has for $u \in [0, t]$, $(q, p) \in \mathbb{R}^{2d}$,

$$\nabla_p \Phi(u, (q, p)) = \int_{\mathbb{R}^{2d}} \nabla_p \hat{p}_{T-u}((q, p), y)\varphi(y)dy.$$

Let $\alpha \in (0, 1)$. It follows from Lemma 5.2 that there exist $C_1, C_2 > 0$ depending only on $\alpha, \sigma, \gamma, T$ such that for all $t \in [0, T)$, $u \in [0, t]$, $(q, p)$ and $y$ in $\mathbb{R}^{2d}$,

$$|\nabla_p \hat{p}_{T-u}((q, p), y)| \leq \frac{C_1}{\sqrt{T-u}} \hat{p}^{(\alpha)}_{T-u}((q, p), y) \leq \frac{C_2}{(T-u)^{2d+1/2}} \varphi(T-u)^d/2. \quad (86)$$

Therefore $\nabla_p \Phi$ is bounded on $[0, t] \times \mathbb{R}^{2d}$ and the integrand of the last term in the right-hand side of the equality (85) is bounded, which implies that its expectation vanishes.
Furthermore, using the Fubini-Tonnelli theorem along with (86), one gets
\[
\mathbb{E} \left( \int_{\mathbb{R}^{2d}} \left| F(q_x^t) \right| \left| \nabla_p \hat{p}_{T-t}(X_x^t, y) \right| |\varphi(y)| dy \right) \\
\leq C_1 \|\varphi\|_{\infty} \|F\|_{\infty} \sqrt{T-u} \mathbb{E} \left( \int_{\mathbb{R}^{2d}} \hat{p}_{T-u}(X_u^x, y) dy \right),
\]
which is integrable on [0, T). Consequently,
\[
\mathbb{E} \left( \int_{\mathbb{R}^{2d}} \hat{p}_{T-t}(X_t^x, y) \varphi(y) dy \right) = \int_{\mathbb{R}^{2d}} \hat{p}_T(x, y) \varphi(y) dy + \int_0^t \int_{\mathbb{R}^{2d}} \mathbb{E} \left( F(q_x^t) \cdot \nabla_p \hat{p}_{T-u}(X_u^x, y) \varphi(y) \right) dy du.
\]

It follows from Lemma 5.2 that \( \int_{\mathbb{R}^{2d}} \hat{p}_{T-t}(X_t^x, y) \varphi(y) dy \) converges almost surely to \( \varphi(X_T^t) \) when \( t \) converges to \( T \). By considering the limit \( t \to T \) (using the dominated convergence theorem in the term in the left-hand side of (88)), one obtains from (88) and (87) that for all \( x \in \mathbb{R}^{2d} \),
\[
\int_{\mathbb{R}^{2d}} p_T(x, y) \varphi(y) dy = \mathbb{E} \left( \varphi(X_T^x) \right)
= \int_{\mathbb{R}^{2d}} \hat{p}_T(x, y) \varphi(y) dy + \int_0^T \int_{\mathbb{R}^{2d}} \mathbb{E} \left( F(q_x^t) \cdot \nabla_p \hat{p}_{T-u}(X_u^x, y) \varphi(y) \right) dy du
= \int_{\mathbb{R}^{2d}} \hat{p}_T(x, y) \varphi(y) dy + \int_0^T \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} p_u(x, (q, p)) F(q) \cdot \nabla_p \hat{p}_{T-u}((q, p), y) dy dq dp dy du.
\]
Since this is satisfied for all \( T > 0 \) and \( \varphi \in C^\infty_c(\mathbb{R}^d) \), then by continuity of the transition density, one obtains that for all \( T > 0 \) and \( x, y \in \mathbb{R}^{2d} \),
\[
p_t(x, y) - \hat{p}_t(x, y) = \int_0^t \int_{\mathbb{R}^{2d}} p_u(x, (q, p)) F(q) \cdot \nabla_p \hat{p}_{T-u}((q, p), y) dy dq dp du
\]
which is a mild formulation of the Fokker–Planck equation associated with (11).

In order to rewrite this mild formulation, let us define the following kernel \( H \) for \( t > 0 \), \( (q, p) \) and \( y \) in \( \mathbb{R}^{2d} \),
\[
H(t, (q, p), y) := F(q) \cdot \nabla_p \hat{p}_t((q, p), y).
\]
For \( t > 0 \), \( x, y \) in \( \mathbb{R}^{2d} \), let us define \( p \otimes H(t, x, y) \) by
\[
(p \otimes H)(t, x, y) = \int_0^t \int_{\mathbb{R}^{2d}} p_u(x, z) H(t-u, z, y) dz du.
\]
The mild formulation can thus be rewritten: for all \( t > 0 \) and \( x, y \in \mathbb{R}^{2d} \),

\[
p_t(x, y) - \widehat{p}_t(x, y) = p \otimes H(t, x, y).
\] (90)

We notice that \((p \otimes H) \otimes H = p \otimes (H \otimes H)\), which allows us to define univocally

\[
H^{(k)} = H \otimes \cdots \otimes H.
\]

Besides, iterating \( r \) times the equality (90) we get

\[
p_t(x, y) = \widehat{p}_t(x, y) + \sum_{j=1}^{r} \widehat{p} \otimes H^{(j)}(t, x, y) + p \otimes H^{(r+1)}(t, x, y).
\] (91)

**Step 2** Let us prove that the series \( \sum_{j=1}^{\infty} \widehat{p} \otimes H^{(j)}(t, x, y) \) converges by getting upper bounds on \( \widehat{p} \otimes H^{(j)} \) for \( j \geq 1 \). Let \( \alpha \in (0, 1) \). By Lemma 5.2, there exists \( c_\alpha > 0 \) such that for all \( t > 0, x, y \in \mathbb{R}^{2d} \), \(|H(t, x, y)| \leq \|F\|_\infty c_\alpha (1 + \sqrt{\gamma - t})^{n-\alpha}(x, y)\). Therefore, for a fixed \( T > 0 \), for all \( t \in (0, T] \) and \( x, y \in \mathbb{R}^{2d} \),

\[
|H(t, x, y)| \leq \frac{C_3 c_\alpha}{\sqrt{t}} p_t^\alpha(x, y) \text{ where } C_3 := \|F\|_\infty c_\alpha (1 + \sqrt{\gamma - T}) \tag{92}
\]

Besides, for \( u \in (0, t) \) and \( t \in (0, T] \), \( x, z, y \in \mathbb{R}^{2d} \), one has from (92), since \( \widehat{p}_t(x, y) \leq \alpha^{-d} p_t^\alpha(x, y) \) (from (80)), that

\[
|\widehat{p}_u(x, z)H(t-u, z, y)| \leq \frac{C_3 c_\alpha}{\alpha^d p_t} \frac{p_t^\alpha(z, y)}{\sqrt{t-u}}.
\]

For \( m, n > 0 \), let \( B(m, n) := \int_0^1 u^{m-1}(1-u)^{n-1}du = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \), with \( \Gamma \) the Gamma function. Therefore, for \( t \in (0, T] \), \( x, y \in \mathbb{R}^{2d} \), by the Chapman-Kolmogorov relation (81),

\[
|\widehat{p} \otimes H(t, x, y)| = \left| \int_0^t \int_{\mathbb{R}^{2d}} \widehat{p}_u(x, z)H(t-u, z, y)dzdu \right|
\]

\[
\leq \frac{C_3 c_\alpha}{\alpha^d p_t} p_t^\alpha(x, y) t^{\frac{1}{2}} B \left( 1, \frac{1}{2} \right).
\]

By induction, for all \( j \geq 1 \),

\[
|\widehat{p} \otimes H^{(j)}(t, x, y)| \leq \frac{C_3^j}{\alpha^d p_t} p_t^\alpha(x, y) t^{\frac{j}{2}} \prod_{l=1}^{j} B \left( \frac{l + 1}{2}, \frac{1}{2} \right).
\] (93)

Consequently, since \( \prod_{l=1}^{j} B \left( \frac{l+1}{2}, \frac{1}{2} \right) = \frac{\sqrt{\pi}^j}{\Gamma(\frac{j+1}{2})} \), it is easy to see from the Stirling formula that for all \( t > 0, x, y \in \mathbb{R}^{2d} \), the series \( \sum_{j=1}^{\infty} \widehat{p} \otimes H^{(j)}(t, x, y) \) converges absolutely.

**Step 3** Let us now prove that \( p \otimes H^{(r+1)}(t, x, y) \xrightarrow{r \to \infty} 0 \) for all \( t \in (0, T] \), \( x, y \in \mathbb{R}^{2d} \).

By (92) and the Chapman-Kolmogorov relation (81), we have for all \( t \in (0, T] \), \( x, y \in \mathbb{R}^{2d} \)
\[ |H \otimes H(t, x, y)| \leq C_3^2 B \left( \frac{1}{2}, \frac{1}{2} \right) \hat{p}_{t}^{(\alpha)}(x, y). \]

By induction, for all \( j \geq 2 \), for all \( t \in (0, T] \), \( x, y \in \mathbb{R}^{2d} \),
\[ |H^{(j)}(t, x, y)| \leq C_3^j \hat{p}_{t}^{(\alpha)}(x, y) t^{\frac{j-1}{2}} \frac{\sqrt{\pi}^{j-1}}{\Gamma(\frac{j+1}{2})}. \]  

(94)

As a consequence,
\[ |\rho \otimes H^{(r+1)}(t, x, y)| \leq C_3^{r+1} \frac{\sqrt{\pi}^{r+1}}{\Gamma(\frac{r+3}{2})} \int_0^t \int_{\mathbb{R}^{2d}} p_u(x, z) \hat{p}_{t-u}^{(\alpha)}(z, y) (t - u) \frac{r-1}{2} \, dz \, du. \]

From the expression of \( \hat{p}_{t-u}^{(\alpha)}(z, y) \) it follows that there exists \( C_4 > 0 \) depending only on \( \alpha, \gamma, \sigma \) such that
\[ \hat{p}_{t-u}^{(\alpha)}(z, y) (t - u) \frac{r-1}{2} \leq \frac{C_4 (t - u) \frac{r-1}{2}}{(t - u)^{2d} \phi(t - u)} \frac{r}{2}. \]

Let us choose \( r \geq 4d + 1 \). Since \( \phi \) is positive and continuous then it is bounded from below for \( s \in (-|\gamma| T, |\gamma| T) \), therefore there exists \( C_5 > 0 \) depending on \( \alpha, \gamma, \sigma, d \) and \( T \) such that for all \( t \in (0, T), r \geq 4d + 1, u \in (0, t) \) and \( z, y \in \mathbb{R}^{2d} \),
\[ \hat{p}_{t-u}^{(\alpha)}(z, y) (t - u) \frac{r-1}{2} \leq C_5. \]

Therefore, for all \( t \in (0, T], x, y \in \mathbb{R}^{2d} \),
\[ |\rho \otimes H^{(r+1)}(t, x, y)| \leq C_3^{r+1} C_5 T B \left( \frac{1}{2}, \frac{1}{2} \right) B \left( 1, \frac{1}{2} \right) \cdots B \left( \frac{r}{2}, \frac{1}{2} \right) \rightarrow 0. \]

\textbf{Step 4} As a result, using the results of Step 2 and 3 in the equality (91) we get for all \( t \in (0, T], x, y \in \mathbb{R}^{2d} \),
\[ p_t(x, y) - \hat{p}_t(x, y) = \sum_{j=1}^{\infty} \hat{p}_t \otimes H^{(j)}(t, x, y). \]

From the formula defining \( C_3 \) in (92) and (93), the inequality (20) follows. \( \square \)

\textbf{Remark 5.3.} In view of the proof of Theorem 2.19, it is clear that (20) holds for \( F \) only bounded (dropping the assumptions that \( F \) is \( C^\infty \) and globally Lipschitz continuous in Assumption (F2)), as soon as there exists a weak solution to (11). Gaussian upper bounds for the Langevin process thus hold under slightly more general assumptions than those originally stated in [24].
5.2. Existence of a smooth transition density for the absorbed Langevin process

**Proposition 5.4** (Existence of a measurable transition density). Under Assumption (F1), there exists a measurable function 

$$(t, x, y) \in \mathbb{R}^*_+ \times D \times D \mapsto p^D_t(x, y)$$

such that for all $t > 0$ and $x \in D$, the kernel $P^D_t(x, \cdot)$, defined in (21), has the density $p^D_t(x, \cdot)$ with respect to the Lebesgue measure on $D$.

The proof of the proposition above is detailed in Appendix B. Let us now prove that this transition density $p^D_t(x, y)$ is smooth on $\mathbb{R}^*_+ \times D \times D$. This will be managed by first showing that it is a distributional solution of the backward and forward Kolmogorov equations. The smoothness of $p^D_t$ will then follow from the hypoellipticity of the differential operators $\partial_t - L$ and $\partial_t - L^*$, see Definition 2.4. This scheme of proof is inspired from [30, Sect. 3.5]. Notice that Proposition 5.4 only defines the transition density on $\mathbb{R}^*_+ \times D \times D$. The extension to a continuous function on $\mathbb{R}^*_+ \times D \times D$ will be done in Sect. 6 (see Theorem 6.3).

**Proposition 5.5** (Kolmogorov equations). Under Assumption (F1), the transition density $(t, x, y) \mapsto p^D_t(x, y)$ is a $C^\infty(\mathbb{R}^*_+ \times D)$ function. Besides it satisfies the backward and forward Kolmogorov equations:

(i) $(t, x) \mapsto p^D_t(x, y)$ is a solution of $\partial_t p^D = L_x p^D$ on $\mathbb{R}^*_+ \times D$,

(ii) $(t, y) \mapsto p^D_t(x, y)$ is a solution of $\partial_t p^D = L^*_y p^D$ on $\mathbb{R}^*_+ \times D$.

**Proof.** Let $\Phi \in C^\infty_c(\mathbb{R}^*_+ \times D)$. Notice that $\Phi$ can be extended by zero to a $C^\infty(\mathbb{R}^*_+ \times \partial D)$ function.

Let $(X^x_t = (q^x_t, p^x_t))_{t \geq 0}$ be the process satisfying (11). Using Itô’s formula, one gets for all $x \in D$ and $t > 0$,

$$\Phi(t, X^x_t) = \Phi(0, x) + \int_0^t \left[ \partial_s \Phi(s, X^x_s) + L \Phi(s, X^x_s) \right] ds + \sigma \int_0^t \nabla_p \Phi(s, X^x_s) \cdot dB_s.$$

Thus,

$$\Phi(\tau^x_0 \land t, X^x_{\tau^x_0 \land t}) = \int_0^t 1_{t \geq s} \left[ \partial_s \Phi(s, X^x_s) + L \Phi(s, X^x_s) \right] ds + \sigma \int_0^t \nabla_p \Phi(s, X^x_s) \cdot dB_s.$$

As a result, the stochastic integral in the right-hand side is a martingale. Taking the expectation, we get

$$\mathbb{E} \left[ \Phi(\tau^x_0 \land t, X^x_{\tau^x_0 \land t}) \right] = \int_0^t \mathbb{E} \left[ 1_{t \geq s} \left( \partial_s \Phi(s, X^x_s) + L \Phi(s, X^x_s) \right) \right] ds.$$
Since \( X^x_{\tau^+_0} \in \partial D \) and \( \Phi \) vanishes on \( \mathbb{R}_+ \times \partial D \),
\[
E \left[ \Phi(\tau^+_0 \wedge t, X^x_{\tau^+_0}) \right] = E[1_{\tau^+_0 > t} \Phi(t, X^x_t) + 1_{\tau^+_0 \leq t} \Phi(\tau^+_0, X^x_{\tau^+_0})] = 0
\]
Thus
\[
E \left[ 1_{\tau^+_0 > t} \Phi(t, X^x_t) \right] = \int_0^t E \left[ 1_{\tau^+_0 > s} (\partial_t \Phi(s, X^x_s) + \mathcal{L} \Phi(s, X^x_s)) \right] ds.
\]
For \( t \) large enough since \( \Phi \) has a compact support on \( \mathbb{R}^*_+ \times D \), the left-hand side in the equality above is zero. Therefore,
\[
\int \int_{\mathbb{R}^*_+ \times D} (\partial_t \Phi(s, y) + \mathcal{L} \Phi(s, y)) p^D_s(x, y) ds dy = 0.
\]
As a result for all \( x \in D \),
\[(t, y) \in \mathbb{R}^*_+ \times D \mapsto p^D_t(x, y)\]
is a distributional solution of
\[
\partial_t p^D_t = \mathcal{L}^*_x p^D_t
\]
on \( \mathbb{R}^*_+ \times D \). Since the operator \( \partial_t - \mathcal{L}^* \) is hypoelliptic one has that
\[(t, y) \in \mathbb{R}^*_+ \times D \mapsto p^D_t(x, y) \in C^\infty(\mathbb{R}^*_+ \times D),\]
which proves the forward Kolmogorov equation.

We now address the backward Kolmogorov equation. Let \( \Phi_1 \in C_c^\infty(\mathbb{R}^*_+ \times D), \Phi_2 \in C_c^\infty(D) \), and let us define the function \( \Phi \) as follows: for all \((t, x, y) \in \mathbb{R}^*_+ \times D \times D,\)
\[
\Phi(t, x, y) = \Phi_1(t, x) \Phi_2(y).
\]
Let us compute the following integral
\[
I = \int \int \int_{\mathbb{R}^*_+ \times D \times D} p^D_t(x, y) \left( \partial_t \Phi(t, x, y) + \mathcal{L}^*_x \Phi(t, x, y) \right) dt dx dy.
\]
On the one hand, since \((t, x) \mapsto \int_D p^D_t(x, y) \Phi_2(y) dy = E[1_{\tau^+_0 > t} \Phi_2(X^x_{\tau^+_0})] \) is a solution of \( \partial_t u = \mathcal{L}_x u \) by Theorem 2.10, then \( I = 0 \). On the other hand, it follows from Fubini’s theorem that
\[
\int_D \Phi_2(y) \left( \int \int_{\mathbb{R}^*_+ \times D} p^D_t(x, y) \left( \partial_t \Phi_1(t, x) + \mathcal{L}^*_x \Phi_1(t, x) \right) dt dx \right) dy = 0.
\]
Since \( y \in D \mapsto \int_{\mathbb{R}_+^* \times D} p_t^D (x, y) \left( \partial_t \Phi_1(t, x) + L_x^* \Phi_1(t, x) \right) \, dx \, dt \in L^1_{\text{loc}}(D) \) (since \( \Phi_1 \in C^\infty_c(\mathbb{R}_+^* \times D) \)), this ensures that for almost every \( y \in D \),

\[
\int_{\mathbb{R}_+^* \times D} p_t^D (x, y) \left( \partial_t \Phi_1(t, x) + L_x^* \Phi_1(t, x) \right) \, dx \, dt = 0.
\]

Using the continuity of \( y \in D \mapsto p_t^D (x, y) \) from (95), the equality above remains true for all \( y \in D \). Thus, for all \( y \in D \), \( (t, x) \mapsto p_t^D (x, y) \) is a distributional solution of the backward Kolmogorov equation

\[
\partial_t p_t^D = L_x p_t^D
\]

on \( \mathbb{R}_+^* \times D \).

Consequently, the hypoellipticity of \( \partial_t - L \) on the open set \( \mathbb{R}_+^* \times D \) ensures that for all \( y \in D \)

\[
(t, x) \in \mathbb{R}_+^* \times D \mapsto p_t^D (x, y) \in C^\infty(\mathbb{R}_+^* \times D).
\]

Therefore, using (95), it follows that

\[
(t, x, y) \in \mathbb{R}_+^* \times D \times D \mapsto p_t^D (x, y) \in C^\infty(\mathbb{R}_+^* \times D \times D),
\]

which concludes the proof of Proposition 5.5. \( \square \)

Corollary 2.22 shows that the Gaussian upper bound on the transition density \( p_t \) immediately transfers to the transition density \( p_t^D \). In fact, in the next lemma, we show that the latter also satisfies a mild formulation of the form

\[
p_t^D - \widehat{p}_t = p^D \otimes H^D,
\]

(96)

for some kernel \( H^D \), and compute estimates on this kernel to obtain an asymptotic expansion of \( p_t^D \) in compact sets of \( D \). This lemma will be useful in Sect. 6.

**Lemma 5.6.** (Local asymptotic expansion around \( t = 0 \)) Under Assumption (F1), the density \( p_t^D \) is such that for all compact sets \( K \subset D \), \( T > 0 \) and \( \alpha \in (0, 1) \), there exists \( C > 0 \) such that for all \( x, y \in K \) and \( t \in (0, T) \),

\[
\left| p_t^D (x, y) - \widehat{p}_t (x, y) \right| \leq C \sqrt{t} \widehat{p}_t^{(\alpha)} (x, y).
\]

(97)

**Proof.** Since the density \( p_t^D \) only depends on the values of \( F \) in \( \mathcal{O} \) (see Remark 2.6), we can assume that \( F \) satisfies Assumption (F2) for the sake of simplicity. The first step of the proof consists in establishing the mild formulation (96). In contrast with the proof of Theorem 2.19, where a mild formulation of the forward Kolmogorov equation satisfied by \( p_t \) is established, the absorbing boundary condition makes the use of the Itô formula inappropriate. We adopt a different approach, inspired from [24, Proposition 2.2].
Let $T > 0$ and $K \subset D$ be a compact set. Let $x = (q, p)$, $y = (q', p') \in K$ and $t \in (0, T]$. Let us define $\varphi \in \mathcal{C}^\infty_c(D)$ such that
\[
0 \leq \varphi(z) \leq 1 \text{ for all } z \in D, \text{ and } \varphi(z) = 1 \text{ for all } z \in K.
\]
(98)
Let us define the function $h_t$ as follows:
\[
h_t : u \in (0, t) \mapsto \int_D p^D_u(x, z)\hat{p}_{t-u}(z, y)\varphi(z)dz.
\]
Let us identify the limits of $h_t(u)$ when $u \to 0$ and $u \to t$. First we have that
\[
h_t(u) = \mathbb{E}\left[\hat{p}_{t-u}(X^T_u, y)\varphi(X^T_u)1_{t_{\hat{p}} > u}\right] \underset{u \to 0}{\longrightarrow} \hat{p}_t(x, y)\varphi(x) = \hat{p}_t(x, y),
\]
by the dominated convergence theorem using Lemma 3.2 and the continuity and boundedness of $\hat{p}_t(\cdot, y)\varphi(\cdot)$ when $s$ is close to $t$. Second, it follows from the convergence (83) in Lemma 5.2 and the boundedness and continuity of the product $p^D_u(x, \cdot)\varphi(\cdot)$ when $u$ is close to $t$ that (remember that $y \in K$)
\[
h_t(u) \underset{u \to t}{\longrightarrow} p^D_t(x, y)\varphi(y) = p^D_t(x, y).
\]
Therefore, using the fact that $h_t \in \mathcal{C}^1((0, t))$, we have that
\[
p^D_t(x, y) - \hat{p}_t(x, y) = \int_0^t \frac{dh_t}{du}(u)du
\]
\[
= \int_0^t \int_D \left(\partial_u[p^D_u(x, z)\hat{p}_{t-u}(z, y)] + p^D_u(x, z)\partial_u[\hat{p}_{t-u}(z, y)]\right)\varphi(z)dzdu,
\]
by the Lebesgue differentiation theorem as $p^D_t(x, y)$, $\hat{p}_t(x, y)$ are smooth on $\mathbb{R}^d \times D \times D$ and $\varphi \in \mathcal{C}^\infty_c(D)$.

Recall that we denote by $\hat{\mathcal{L}} = \mathcal{L}_{0, \gamma, \sigma}$ the infinitesimal generator of $(\hat{X}^T_t)$ $t \geq 0$. Since $\partial_t p^D_t(x, y) = \mathcal{L}^* p^D_t(x, y)$ (see Theorem 2.20) and $\partial_t \hat{p}_t(x, y) = \hat{\mathcal{L}} \hat{p}_t(x, y)$ (see Proposition 2.17), one has (using the notation $z = (q'', p'') \in \mathbb{R}^{2d}$)
\[
p^D_t(x, y) - \hat{p}_t(x, y)
\]
\[
= \int_0^t \int_D \left(\mathcal{L}^*_z p^D_u(x, z)\hat{p}_{t-u}(z, y) - p^D_u(x, z)\hat{\mathcal{L}}_z \hat{p}_{t-u}(z, y)\right)\varphi(z)dzdu
\]
\[
= \int_0^t \int_D p^D_u(x, z)\left(\mathcal{L}_z (\hat{p}_{t-u}(z, y))\varphi(z) - \hat{\mathcal{L}}_z (\hat{p}_{t-u}(z, y))\varphi(z)\right)dzdu
\]
\[
= \int_0^t \int_D p^D_u(x, z)\left[\left(\mathcal{L} - \hat{\mathcal{L}}\right)(\hat{p}_{t-u}(z, y))\varphi(z) + \sigma^2\nabla p''(\hat{p}_{t-u}(z, y)) \cdot \nabla p''(\varphi(z))\right]dzdu
\]
\[
+ \int_0^t \int_D p^D_u(x, z)\hat{\mathcal{L}}_z \varphi(z)dzdu,
\]
which is the claimed mild formulation (96). We have $\mathcal{L} - \hat{\mathcal{L}} = F(q'') \cdot \nabla p''$. Furthermore, $\varphi \in \mathcal{C}^\infty_c(D)$, therefore its gradient is bounded on $D$ and $\mathcal{L}\varphi \in \mathcal{C}^\infty_c(D)$. Besides,
it follows from (84) in Lemma 5.2 that for any \( \alpha \in (0, 1) \) there exists \( C_1 > 0 \) such that for all \( t \in (0, T], u \in [0, t) \) and \((q'', p'')\), \( y \in \mathbb{R}^{2d}\),

\[
|\nabla_{p''} \hat{p}_{t-u}((q'', p''), y)| \leq \frac{C_1}{\sqrt{t-u}} \hat{p}_{t-u}^{(\alpha)}((q'', p''), y).
\]

In addition, from (80), \( \hat{p}_t(x, y) \leq \alpha^{-d} \hat{p}_t^{(\alpha)}(x, y) \) for all \( t > 0, x, y \in \mathbb{R}^{2d} \). Consequently, under Assumption (F2), there exists a constant \( C_K > 0 \) such that

\[
\left| p_t^D(x, y) - \hat{p}_t(x, y) \right| \leq C_K \int_0^t \int_D p_u^D(x, z) \frac{\hat{p}_t^{(\alpha)}(z, y)}{\sqrt{t-u}} dz du.
\]

Furthermore, by Corollary 2.22 there exists \( C_2 > 0 \) such that for all \( u \in (0, t), t \in (0, T), p_u^D(x, y) \leq p_u(x, y) \leq C_2 \hat{p}_u^{(\alpha)}(x, y) \). Hence the existence of \( C_K' > 0 \) such that for all \( t \in (0, T) \) and \( x, y \in K \),

\[
\left| p_t^D(x, y) - \hat{p}_t(x, y) \right| \leq C_K' \int_0^t \int_D \hat{p}_u^{(\alpha)}(x, z) \frac{\hat{p}_t^{(\alpha)}(z, y)}{\sqrt{t-u}} dz du \\
\leq C_K' \sqrt{p_t^{(\alpha)}(x, y)} \int_0^1 \frac{ds}{\sqrt{1-s}},
\]

since \( \hat{p}_t^{(\alpha)} \) satisfies the Chapman-Kolmogorov relation (81) in Lemma 5.2. This concludes the proof of (97). \( \square \)

5.3. Boundary behavior of the transition density

The purpose of this subsection is to study the behavior of \( p_t^D(x, y) \) at the boundaries \((t, x) \in \mathbb{R}_+^* \times (\Gamma^+ \cup \Gamma^0) \) and \((t, y) \in \mathbb{R}_+^* \times \Gamma^- \) (see Proposition 5.7 below). This result will be useful for the proof of Theorem 6.2, which will then allow to complete the proof of Theorem 2.20.

**Proposition 5.7** (Boundary limits). Let Assumptions (O1) and (F1) hold. Let \( t_0 > 0, x_0 \in \Gamma^+ \cup \Gamma^0 \) and \( y_0 \in \Gamma^- \). Let \((t_n, x_n, y_n)_{n \geq 1}\) be a sequence of points in \( \mathbb{R}_+^* \times D \times D \) converging towards \((t_0, x_0, y_0)\), then one has the following convergences:

(i) For all \( y \in D, p_t^D(x_n, y) \longrightarrow 0 \).
(ii) For all \( x \in D, p_t^D(x, y_n) \longrightarrow 0 \).

The proof of this proposition relies partly on the following lemma which is proven in Appendix B.

**Lemma 5.8.** Let \( y_0 \in \mathbb{R}^{2d} \), \( M > 0 \) and \( \alpha \in (0, 1) \). There exist \( C_0 > 0, \mu > 0, \delta_0 > 0 \) such that for all \( s \in (0, \delta_0], (q', p') \in B(y_0, M/6) \) and \((q, p) \in \mathbb{R}^{2d} \) satisfying \(|p - p'| \geq M/3\),

\[
\hat{p}_s^{(\alpha)}((q, p), (q', p')) \leq C_0 \exp(-\mu/s).
\]  (99)
Proof of Proposition 5.7. Since the density $p_t^D$ only depends on the values of $F$ in $O$ (see Remark 2.6), we can assume that $F$ satisfies Assumption (F2) for the sake of simplicity.

Both proofs of (i) and (ii) rely on the elementary remark that, for any $t > 0$ and $x \in D$, since the function $y \mapsto p_t^D(x, y)$ is continuous on $D$, we have for any $y \in D$,

$$p_t^D(x, y) = \lim_{h \to 0} \frac{p_t^D(x, D \cap B(y, h))}{|B(y, h)|}. \quad (100)$$

Notice that, here and in the sequel, we take the intersection of $B(y, h)$ with $D$ because $p_t^D(x, \cdot)$ is defined as a measure on $B(D)$.

Proof of (i). Let $t_0 > 0$, $x_0 \in \Gamma^+ \cup \Gamma^0$. Let $(t_n, x_n)_{n \geq 1}$ be a sequence of points in $\mathbb{R}_+^* \times D$ converging towards $(t_0, x_0)$. Let $N \geq 1$ be such that, for any $n \geq N$, $t_0/2 \leq t_n \leq 3t_0/2$. For any $n \geq N$, $h > 0$ and $y \in D$, the Markov property shows that

$$p_{t_n/2}^D(x_n, D \cap B(y, h)) = \mathbb{E}\left[\mathbb{1}_{\tau_{\partial}^x_{t_n} > t_n/2} p_{t_n/2}^D(X_{t_n/2}^x, D \cap B(y, h))\right].$$

Besides, by Corollary 2.22, there exists a constant $C \geq 0$ which depends on $t_0$ such that for any $n \geq N$, the transition density $p_{t_n/2}^D$ is uniformly bounded on $D \times D$ by $C$, therefore

$$\frac{p_{t_n/2}^D(x_n, D \cap B(y, h))}{|B(y, h)|} \leq C \mathbb{P}(\tau_{\partial}^x_{t_n} > t_n/2).$$

The right-hand side no longer depends on $h$ and vanishes when $n \to +\infty$ by Lemma 3.2 and Proposition 2.8, therefore by (100) we get Assertion (i).

Remark 5.9. The proof shows that the convergence of Assertion (i) is actually uniform in $y$, that is to say $\sup_{y \in D} p_{t_n}^D(x_n, y) \to 0$.

Proof of (ii). The proof of (ii) needs more work. Let $x \in D$, $t_0 > 0$ and $y_0 = (q_0, p_0) \in \Gamma^-$. Let $(t_n, y_n)_{n \geq 1}$, with $y_n := (q_n, p_n)$, be a sequence of points in $\mathbb{R}_+^* \times D$ converging towards $(t_0, y_0)$. In order to prove the convergence $p_{t_n}^D(x, y_n) \to 0$, it is enough by (100) to prove the following double limit

$$\lim_{n \to \infty} \lim_{h \to 0} \frac{p_{t_n}^D(x, D \cap B(y_n, h))}{|B(y_n, h)|} = 0. \quad (101)$$

Let us define for $0 \leq r \leq t$ the following modulus of continuity

$$Z_{r, t}^x := \sup_{r \leq s \leq t} |p_s^x - p_t^x|.$$ 

For two constants $\delta \in (0, t_0/2]$ and $M > 0$ to be fixed later on, let us rewrite the numerator in (101) as follows: for $n$ sufficiently large so that $t_n \geq t_0/2$ (and thus
\[ t_n - \delta \geq 0, \]
\[
P_{t_n}^D(x, D \cap B(y_n, h)) = \mathbb{P}( (q_{t_n}^x, p_{t_n}^x) \in B(y_n, h), \tau_{t_n}^x > t_n )
\]
\[
= \mathbb{P}( (q_{t_n}^x, p_{t_n}^x) \in B(y_n, h), Z_{t_n-\delta, t_n}^x \leq M, \tau_{t_n}^x > t_n ) + \mathbb{P}( (q_{t_n}^x, p_{t_n}^x) \in B(y_n, h), Z_{t_n-\delta, t_n}^x > M, \tau_{t_n}^x > t_n ).
\] (102)

The idea of the proof of (101) relies on the decomposition in (102) and is divided into two steps. In Step 1, we consider the probability corresponding to the first term in the right-hand side of the equality (102). We show that there is a value of \( M \) and a \( \delta_1 \in (0, t_0/2] \) such that, for all \( \delta \in (0, \delta_1] \), there exist \( N_1 \geq 1 \) and \( h_1 > 0 \) such that for any \( n \geq N_1 \) and \( h \leq h_1 \), the event \( \{ (q_{t_n}^x, p_{t_n}^x) \in B(y_n, h), Z_{t_n-\delta, t_n}^x \leq M, \tau_{t_n}^x > t_n \} \) has probability 0. Indeed, for \( n \) large and \( h \) small the event \( \{ (q_{t_n}^x, p_{t_n}^x) \in B(y_n, h) \} \) implies that \( (q_{t_n}^x, p_{t_n}^x) \) is “close” to \( y_0 \in \Gamma^- \), which is a boundary point with inward velocity. Therefore, using our control on the modulus of continuity of the velocity, we can prove the existence of a time \( s \in (0, t_n) \) such that \( (q_s^x, p_s^x) \) is outside of \( D \), which contradicts the fact that \( \tau_{t_n}^x > t_n \).

In Step 2, we consider the second term in the right-hand side of the equality (102), divided by \( |B(y_n, h)| \). For the value of \( M \) determined in Step 1, we show the existence of \( \delta_2 \in (0, t_0/2] \) and \( C, \mu > 0 \) such that, for all \( \delta \in (0, \delta_2] \), there exist \( N_2 \geq 1 \) and \( h_2 > 0 \) such that, for any \( n \geq N_2 \) and \( h \leq h_2 \),
\[
\mathbb{P}( (q_{t_n}^x, p_{t_n}^x) \in B(y_n, h), Z_{t_n-\delta, t_n}^x > M ) / |B(y_n, h)| \leq C e^{-\mu \delta / \tau_{t_n}^x}.
\] (103)

As a result, the two steps yield the following inequality
\[
\limsup_{n \to \infty} \limsup_{h \to 0} \frac{P_{t_n}^D(x, D \cap B(y_n, h))}{|B(y_n, h)|} \leq C e^{-\mu \delta / \tau_{t_n}^x}
\]
for any \( \delta \in (0, \delta_1 \wedge \delta_2] \), then taking \( \delta \to 0 \) we are able to conclude the proof of (101).

**Step 1** Let us prove here that one can fix \( M > 0 \) and choose \( \delta > 0 \) small enough such that the first term in the right-hand side of the equality (102) vanishes for \( n \) sufficiently large and \( h \) sufficiently small. By Proposition 3.7, for \( \eta > 0 \), there exists \( c_0 := c_0(\eta) > 0 \) such that if \( q \in \mathbb{R}^d \) satisfies \( (q - q_0) \cdot n(q_0) \geq \eta |q - q_0| \) and \( |q - q_0| \leq c_0 \) then \( q \notin \mathcal{O} \). Now let \( M := -\frac{p_0 \cdot n(q_0)}{\beta} \) which is positive because \( y_0 = (q_0, p_0) \in \Gamma^- \). Let \( \eta := \frac{M}{\sum_{q_0} M_{(p_0)}} \) and \( c_0 := c_0(\eta) \) as defined above.

Let \( \delta_1 := \frac{t_0}{2} \wedge \frac{c_0}{\sum_{q_0} M_{(p_0)}} \). We fix \( \delta \in (0, \delta_1] \) and define \( h_1 := \frac{\delta M}{2(1 + \delta)} \). Remembering that \( t_n \xrightarrow{n \to \infty} t_0 \) and \( y_n \xrightarrow{n \to \infty} y_0 \) we can choose \( N_1 \geq 1 \) such that for \( n \geq N_1 \),
\[
t_n \in [t_0/2, 3t_0/2] \quad \text{and} \quad y_n = (q_n, p_n) \in B \left( y_0, \frac{\delta M}{2(1 + \delta)} \right).
\]

Let \( n \geq N_1 \) and \( h \in (0, h_1) \). Notice that
\[
\mathbb{P}( (q_{t_n}^x, p_{t_n}^x) \in B(y_n, h), Z_{t_n-\delta, t_n}^x \leq M, \tau_{t_n}^x > t_n ) \leq \mathbb{P}( (q_{t_n}^x, p_{t_n}^x) \in B(y_n, h), Z_{t_n-\delta, t_n}^x \leq M, q_{t_n}^x \notin \mathcal{O} ).
\]
Therefore, the first term in (102) vanishes if one can prove that \( q_{t_n - \delta}^x \not\in \mathcal{O} \) on the event \( \{(q_{t_n}^x, p_{t_n}^x) \in B(y_n, h), Z_{t_n - \delta, t_n}^x \leq M\} \). By (11), one has that

\[
q_{t_n}^x = q_{t_n - \delta}^x + \int_{t_n - \delta}^{t_n} p_s^x \, ds.
\]

Therefore,

\[
q_{t_n - \delta}^x = q_{t_n - \delta}^x - \delta p_{t_n}^x - \int_{t_n - \delta}^{t_n} (p_s^x - p_{t_n}^x) \, ds.
\]

Let

\[
v_{t_n}^x := q_{t_n - \delta}^x - \left( q_0 - \delta p_0 - \int_{t_n - \delta}^{t_n} (p_s^x - p_{t_n}^x) \, ds \right) = q_{t_n}^x - q_0 - \delta \left( p_{t_n}^x - p_0 \right).
\]

As a result, on the event \( \{(q_{t_n}^x, p_{t_n}^x) \in B(y_n, h)\} \), the triangle inequality ensures that

\[
\begin{align*}
|v_{t_n}^x| &\leq |q_{t_n}^x - q_n| + |q_n - q_0| + \delta |p_{t_n}^x - p_n| + \delta |p_n - p_0| \\
&\leq |X_{t_n}^x - y_n|(1 + \delta) + |y_n - y_0|(1 + \delta) \\
&\leq h(1 + \delta) + \frac{\delta M}{2(1 + \delta)}(1 + \delta) \leq \delta M,
\end{align*}
\]

by definition of \( N_1 \) and \( h_1 \). Consequently, we have on the event \( \{(q_{t_n}^x, p_{t_n}^x) \in B(y_n, h), Z_{t_n - \delta, t_n}^x \leq M\} \),

\[
(q_{t_n - \delta}^x - q_0) \cdot n(q_0) = -\delta p_0 \cdot n(q_0) + \int_{t_n - \delta}^{t_n} (p_s^x - p_{t_n}^x) \cdot n(q_0) \, ds + v_{t_n}^x \cdot n(q_0) \geq \delta M.
\]

Furthermore, on the event \( \{(q_{t_n}^x, p_{t_n}^x) \in B(y_n, h), Z_{t_n - \delta, t_n}^x \leq M\} \),

\[
|q_{t_n - \delta}^x - q_0| = -\delta p_0 - \int_{t_n - \delta}^{t_n} (p_s^x - p_{t_n}^x) \, ds + v_{t_n}^x \leq \delta(\|p_0\| + 2M).
\]

As a result, since \( (q_{t_n - \delta}^x - q_0) \cdot n(q_0) \geq \delta M \geq \eta|q_{t_n - \delta}^x - q_0| \) and \( |q_{t_n - \delta}^x - q_0| \leq \delta(\|p_0\| + 2M) \leq c_0 \), the exterior sphere condition ensures that \( q_{t_n - \delta}^x \not\in \mathcal{O} \).

**Step 2** Let \( M > 0 \) be defined as in **Step 1** We fix a value of \( \alpha \in (0, 1) \), let \( C_0, \mu, \delta_0 > 0 \) be given by Lemma 5.8 and define \( \delta_2 := \delta_0 \wedge (t_0/2) \). We now let \( \delta \in (0, \delta_2) \), define \( N_2 \geq 1 \) be such that for any \( n \geq N_2 \), \( |y_n - y_0| \leq M/12 \), and finally set \( h_2 := M/12 \).

Let \( n \geq N_2, h \in (0, h_2) \) and define the following stopping time

\[
\tau_n^{(\delta)} := \inf\{s \geq t_n - \delta : |p_s^x - p_0| \geq M/2\}.
On the event \{\{(q_{tn}^x, p_{tn}^x) \in B(y_n, h), Z_{tn-\delta, tn}^x > M\}\}, one has by the triangle inequality
\[
\sup_{tn-\delta \leq s \leq tn} |p_s^x - p_0| \geq \sup_{tn-\delta \leq s \leq tn} |p_s^x - p_{tn}^x| - |p_{tn}^x - p_n| - |p_n - p_0| \\
\geq M - h - \frac{M}{12} \geq \frac{5M}{6} > \frac{M}{2},
\]
by the definitions of \(N_2\) and \(h_2\). Therefore, \(\tau_n^{(\delta)} \leq t_n\) and
\[
\mathbb{P}(\{(q_{tn}^x, p_{tn}^x) \in B(y_n, h), Z_{tn-\delta, tn}^x > M\}) \leq \mathbb{P}(\{(q_{tn}^x, p_{tn}^x) \in B(y_n, h), \tau_n^{(\delta)} < t_n\}),
\]
since \(\mathbb{P}(\tau_n^{(\delta)} = t_n) \leq \mathbb{P}(|p_{tn}^x - p_0| = M/2) = 0\) because \(p_{tn}^x\) admits a density on \(\mathbb{R}^d\) with respect to the Lebesgue measure by Proposition 2.17.

Therefore, applying the strong Markov property at \(\tau_n^{(\delta)}\), one has
\[
\mathbb{P}(\{(q_{tn}^x, p_{tn}^x) \in B(y_n, h), Z_{tn-\delta, tn}^x > M\})
\leq \mathbb{E} \left[ \mathbb{I}_{\tau_n^{(\delta)} < t_n} \frac{\mathbb{P}\left(\{(q_{tn-r}^x, p_{tn-r}^x) \in B(y_n, h)\}\right)}{\mathbb{P}(\tau_n^{(\delta)})} \Bigg|_{z=\left(\{q_{tn}^x, p_{tn}^x\}, r=\tau_n^{(\delta)}\), y'=y'} \right]
\]
(104)
\[
\text{Let } s_n := t_n - \tau_n^{(\delta)}. \text{ On the event } \{\tau_n^{(\delta)} < t_n\}, \text{ one has } s_n \in (0, \delta] \text{ and}
\]
\[
\mathbb{P}\left(\{(q_{tn-r}^x, p_{tn-r}^x) \in B(y_n, h)\}, r=\tau_n^{(\delta)}\right) = \int_{B(y_n, h)} p_{tn}((q_{tn}^x, p_{tn}^x), y') dy'.
\]
(105)

Besides, since \(s_n \leq \delta\) and \(\delta \leq t_0/2\), one has by Theorem 2.19 that there exists \(C' > 0\) depending only on \(\alpha\) and \(t_0\), but not on \(n\), such that for any \(y' \in B(y_n, h)\),
\[
p_{sn}(\{(q_{tn}^x, p_{tn}^x), y'\}) \leq C' \mathbb{P}_{y_n}^{(\alpha)}((q_{tn}^x, p_{tn}^x), y').
\]
(106)

It now follows from the definition of \(N_2\) and \(h_2\) that \(B(y_n, h) \subset B(y_0, M/6)\) and, from the continuity of the trajectories of \((p_t^x)_{t \geq 0}\), one has almost surely that \(|p_{tn}^x - p_0| \geq M/2\) so that for any \(y' \in B(y_n, h)\),
\[
|p_{tn}^x - p'| \geq |p_{tn}^x - p_0| - |p_0 - p'| \geq M/2 - M/6 = M/3.
\]

These estimates allow to apply Lemma 5.8 and deduce that, on the event \(\{\tau_n^{(\delta)} < t_n\}\),
\[
\mathbb{P}_{y_n}^{(\alpha)}((q_{tn}^x, p_{tn}^x), y') \leq C_0 \exp(-\mu/s_n) \leq C_0 \exp(-\mu/\delta),
\]
which, combined with (104–106), concludes to (103). \(\square\)
Remark 5.10. A formal conditioning argument shows that, for \( x, y \in D \),

\[
p_{D}^{T}(x, y) = \lim_{h \to 0} \frac{P(X_{t}^{x} \in B(y, h), \tau_{\partial}^{x} > t)}{|B(y, h)|}
= \lim_{h \to 0} \frac{P(\tau_{\partial}^{x} > t | X_{t}^{x} \in B(y, h))P(X_{t}^{x} \in B(y, h))}{|B(y, h)|}
= P(\tau_{\partial}^{x} > t | X_{t}^{x} = y)p_{t}(x, y),
\]

so that Proposition 5.7 should amount to studying the limiting behavior, when \( x \) or \( y \) respectively approach \( \Gamma^{+} \) or \( \Gamma^{-} \), of the probability that the diffusion bridge associated with (11) between \( x \) and \( y \) remains in \( D \). With this interpretation at hand, both convergence results (i) and (ii) become very intuitive, and they seem to be the time-reversal statement of each other — a point which will be clarified with the introduction of the adjoint process, and the proof of the reversibility relation (112), in the next section.

Our proof of Proposition 5.7, and in particular of (ii), can be related to the work on diffusion bridges of Chaumont and Uribe-Bravo in [8], where they study formal characterizations of the \( h \to 0 \) limit of such an expression as \( P(\tau_{\partial}^{x} > t | X_{t}^{x} \in B(y, h)) \).

6. Reversibility and boundary continuity

In this section we define the “adjoint” Langevin process, which is later shown to be closely related to the Langevin process, through a reversibility result linking both transition densities of the respective absorbed processes. This result is useful for being able to describe precisely the boundary behavior of \( p_{D}^{T} \) and thereby complete the proof of Theorem 2.20.

6.1. Adjoint process

Let \( x = (q, p) \in \mathbb{R}^{2d} \). Let us call the “adjoint” Langevin process the diffusion process \((\tilde{X}_{t}^{x} = (\tilde{q}_{t}^{x}, \tilde{p}_{t}^{x})_{t \geq 0} \) with infinitesimal generator \( \tilde{L} := L^{*} - d\gamma \) (see (9) for the definition of \( L^{*} \)), satisfying the following SDE:

\[
\begin{align*}
    d\tilde{q}_{t}^{x} &= -\tilde{p}_{t}^{x} dt, \\
    d\tilde{p}_{t}^{x} &= -F(\tilde{q}_{t}^{x}) dt + \gamma \tilde{p}_{t}^{x} dt + \sigma dB_{t}, \\
    (\tilde{q}_{0}^{x}, \tilde{p}_{0}^{x}) &= x.
\end{align*}
\]

Let \( \tilde{\tau}_{\partial}^{x} \) be the first exit time from \( D \) of \((\tilde{X}_{t}^{x})_{t \geq 0} \), i.e.

\[
\tilde{\tau}_{\partial}^{x} = \inf\{t > 0 : \tilde{X}_{t}^{x} \notin D\}.
\]
Let $y := (q, -p)$. Let us now define the process $(\tilde{X}^{\circ,y}_t = (\tilde{q}^{\circ,y}_t, \tilde{p}^{\circ,y}_t))_{t \geq 0} := (\tilde{q}^{x}_t, -\tilde{p}^{x}_t)_{t \geq 0}$, it is easy to see that it satisfies the following SDE

$$
\begin{align*}
\begin{cases}
\, d\tilde{q}^{\circ,y}_t = \tilde{p}^{\circ,y}_t \, dt, \\
\, d\tilde{p}^{\circ,y}_t = F(\tilde{q}^{\circ,y}_t) \, dt + \gamma \tilde{p}^{\circ,y}_t \, dt + \sigma \, dB^0_t,
\end{cases}
\end{align*}
$$

(108)

where $(B^0_t)_{t \geq 0} = (-B_t)_{t \geq 0}$ is a Brownian motion on $\mathbb{R}^d$. Its infinitesimal generator therefore writes

$$
\tilde{\mathcal{L}}^\circ := \mathcal{L}_{F,-\gamma,\sigma}
$$

with the notation of (6). Hence all the results proven in the previous sections apply to $(\tilde{X}^{\circ,y}_t)_{t \geq 0}$ as well. Furthermore, $\tilde{X}^{\circ,y}_t$ and $\tilde{X}^x_t$ share the same first exit time from $D$, i.e.

$$
\tau^\circ_t := \inf \{ t > 0 : \tilde{X}^{\circ,y}_t \notin D \} = \tau^x_t \quad \text{almost surely.}
$$

Let us now write and prove the equivalent of Theorem 2.10 for the process $(\tilde{X}^x_t)_{t \geq 0}$.

By Proposition 2.8 applied to $(\tilde{X}^{\circ,y}_t)_{t \geq 0}$, we have, for all $t \geq 0$, almost surely, if $\tilde{\tau}^x_\theta > t$ then $\tilde{X}^x_t \in D \cup \Gamma^+$, and if $\tilde{\tau}^x_\theta \leq t$ then $\tilde{X}^x_{\tilde{\tau}^x_\theta} \in \Gamma^- \cup \Gamma^0$. This ensures that the definition of the function $\tilde{u}$ in Equation (109) below is legitimate.

**Proposition 6.1** (Classical solution and probabilistic representation for the adjoint kinetic Fokker–Planck equation). Under Assumptions (O1) and (F1), let $\tilde{f} \in C^b(D \cup \Gamma^+)$ and $\tilde{g} \in C^b(\Gamma^- \cup \Gamma^0)$, and define the function $\tilde{u}$ on $\mathbb{R}^+ \times \overline{D}$ by

$$
\tilde{u} : (t, x) \mapsto \mathbb{E} \left[ 1_{\tilde{\tau}^x_\theta > t} \tilde{f}(\tilde{X}^x_t) + 1_{\tilde{\tau}^x_\theta \leq t} \tilde{g}(\tilde{X}^x_{\tilde{\tau}^x_\theta}) \right].
$$

(109)

Then we have the following results:

(i) **Initial and boundary values:** the function $\tilde{u}$ satisfies

$$
\tilde{u}(0, x) = \begin{cases}
\tilde{f}(x) & \text{if } x \in D \cup \Gamma^+,
\tilde{g}(x) & \text{if } x \in \Gamma^- \cup \Gamma^0,
\end{cases}
$$

and

$$
\forall t > 0, \quad \forall x \in \Gamma^- \cup \Gamma^0, \quad \tilde{u}(t, x) = \tilde{g}(x).
$$

(ii) **Continuity:** $\tilde{u} \in C^b((\mathbb{R}^+ \times \overline{D}) \setminus ([0] \times (\Gamma^- \cup \Gamma^0)))$, and if $\tilde{f}$ and $\tilde{g}$ satisfy the compatibility condition

$$
x \in \overline{D} \mapsto 1_{x \in D \cup \Gamma^+} \tilde{f}(x) + 1_{x \in \Gamma^- \cup \Gamma^0} \tilde{g}(x) \in C^b(\overline{D}),
$$

(110)

then $\tilde{u} \in C^b(\mathbb{R}^+ \times \overline{D})$. 

(iii) **Interior regularity:** \( \tilde{u} \in C^\infty(\mathbb{R}_+^* \times D) \) and, for all \( t > 0, x \in D \),

\[
\partial_t \tilde{u}(t, x) = \hat{L}\tilde{u}(t, x).
\] (111)

(iv) **Uniqueness:** let \( \tilde{v} \) be a classical solution, in the sense of Definition 2.2, to the Initial-Boundary Value Problem

\[
\begin{align*}
\partial_t \tilde{v} &= \hat{L}\tilde{v} & t > 0, x \in D, \\
\tilde{v}(0, x) &= f(x) & x \in D, \\
\tilde{v}(t, x) &= \tilde{g}(x) & t > 0, x \in \Gamma^-.
\end{align*}
\]

If, for all \( T > 0, \tilde{v} \) is bounded on the set \([0, T] \times D \), then \( \tilde{v}(t, x) = \tilde{u}(t, x) \) for all \((t, x) \in (\mathbb{R}_+ \times (D \cup \Gamma^-)) \setminus \{(0) \times \Gamma^-\} \).

**Proof.** Let \( \tilde{f} \in C^b(D \cup \Gamma^+) \) and \( \tilde{g} \in C^b(\Gamma^- \cup \Gamma^0) \). Let \( \tilde{f}^\infty, \tilde{g}^\infty \) be defined by

\[
\tilde{f}^\infty(q, p) = \tilde{f}(q, -p), \quad \tilde{g}^\infty(q, p) = \tilde{g}(q, -p).
\]

It is easy to see that \( \tilde{f}^\infty \in C^b(D \cup \Gamma^-) \) and \( \tilde{g}^\infty \in C^b(\Gamma^+ \cup \Gamma^0) \). Using the process \((\tilde{X}^\infty_t, x)_{t \geq 0}\) defined in (108), the function \( \tilde{u} \) defined in (109) also writes for \((q, p) \in D\) as follows:

\[
\tilde{u}(t, (q, p)) = \mathbb{E}\left[ \mathbb{I}_{\tilde{\tau}^\infty_q, (q, p) > t} \tilde{f}^\infty(\tilde{X}^\infty_t, (q, -p)) + \mathbb{I}_{\tilde{\tau}^\infty_q, (q, -p) \leq t} \tilde{g}^\infty(\tilde{X}^\infty_t, (q, -p)) \right].
\]

Let us define \( \tilde{u}^\infty \) for \( t \geq 0, (q, p) \in \overline{D} \) by \( \tilde{u}^\infty(t, (q, p)) := \tilde{u}(t, (q, -p)) \). Then, \( \tilde{u}^\infty \) satisfies all the assertions of Theorem 2.10 for the kinetic Fokker–Planck equation

\[
\begin{align*}
\partial_t \tilde{u}^\infty(t, x) &= \hat{L}\tilde{u}^\infty(t, x) & t > 0, x \in D, \\
\tilde{u}^\infty(0, x) &= \tilde{f}(x) & x \in D, \\
\tilde{u}^\infty(t, x) &= \tilde{g}(x) & t > 0, x \in \Gamma^+.
\end{align*}
\]

Therefore, \( \tilde{u} \) as defined in (109) satisfies all the assertions of Proposition 6.1. \( \square \)

Let us define the transition kernel \( \tilde{P}^D_t \) for the absorbed adjoint process \((\tilde{X}^x_t, x)_{0 \leq t \leq \tilde{\tau}^\infty_q}\):

\[
\forall t \geq 0, \ \forall x \in D, \ \forall A \in \mathcal{B}(D), \quad \tilde{P}^D_t(x, A) := \mathbb{P}(\tilde{X}^x_t \in A, \tilde{\tau}^\infty_q > t).
\]

In the next theorem, we show that this kernel admits a transition density \( \tilde{p}^D_t \) which satisfies a simple reversibility relation with \( p^D_t \).

**Theorem 6.2** (Reversibility). Let Assumptions \((O1)\) and \((F1)\) hold. For all \( t > 0, x, y \in D \), let us define

\[
\tilde{p}^D_t(x, y) = e^{-d_{yt}} p^D_t(y, x).
\] (112)

For any \( t > 0 \in D \) and \( A \in \mathcal{B}(D) \),

\[
\tilde{P}^D_t(x, A) = \int_A \tilde{p}^D_t(x, y) dy.
\]
Proof. Let $\varphi \in C^c_\infty(D)$. Let $\tilde{u} : \mathbb{R}_+ \times \overline{D} \to \mathbb{R}$ be defined by

$$
\tilde{u}(t, x) := \mathbb{E} \left[ \mathbb{1}_{\tau^*_n > t} \varphi(\tilde{X}^*_n) \right].
$$

By Assertion (ii) in Proposition 6.1, this function is continuous on $\mathbb{R}_+ \times \overline{D}$. Let us define the function $\tilde{v}$ on $\mathbb{R}_+ \times \overline{D}$ by

$$
\tilde{v}(t, x) = \begin{cases} 
    e^{-d\gamma t} \int_D p^D_t(y, x)\varphi(y)dy & \text{if } t > 0 \text{ and } x \in D, \\
    \varphi(x) & \text{if } (t, x) \in (\mathbb{R}_+ \times \overline{D}) \setminus (\mathbb{R}_+^+ \times D).
\end{cases}
$$

Let us prove that $\tilde{v}(t, x) = \tilde{u}(t, x)$ for $t > 0$ and $x \in D$, which will ensure (112).

In this purpose, we use the uniqueness result of Assertion (iv) in Proposition 6.1. By Definition 2.2, we need to check that:

(i) $(t, x) \mapsto \tilde{v}(t, x) \in C^{1,2}(\mathbb{R}_+^+ \times D)$ and $\tilde{v}$ satisfies $\partial_t \tilde{v} = \tilde{L} \tilde{v}$,

(ii) $(t, x) \mapsto \tilde{v}(t, x) \in C((\mathbb{R}_+ \times (D \cup \Gamma^-)) \setminus ([0] \times \Gamma^-)), \tilde{v}(0, \cdot) = \varphi$ on $D$ and, for $t > 0$, $\tilde{v}(t, \cdot) = 0$ on $\Gamma^-$,

(iii) $\forall T > 0, \sup_{t \in [0, T], x \in D} |\tilde{v}(t, x)| < \infty$.

Since $\varphi$ has a compact support in $D$ and, by Proposition 5.5, $p^D$ is $C^\infty$ on $\mathbb{R}_+^+ \times D \times D$ and satisfies $\partial_t p^D(x, y) = \mathcal{L}_x p^D(x, y)$, we deduce that $\tilde{v}$ is $C^{1,2}$ on $\mathbb{R}_+^+ \times D$ and satisfies $\partial_t \tilde{v} = (\mathcal{L}^* - d\gamma)\tilde{v} = \tilde{L} \tilde{v}$.

Let $t > 0$ and $x \in \Gamma^-$. By the definition of $\tilde{v}$, we have $\tilde{v}(t, x) = \varphi(x) = 0$ since $\varphi$ has a compact support in $D$. On the other hand, if $(t_n, x_n)_{n \geq 1}$ is a sequence of elements of $(\mathbb{R}_+ \times (D \cup \Gamma^-)) \setminus ([0] \times \Gamma^-)$ which converge to $(t, x)$, then it follows from Assertion (ii) in Proposition 5.7, Remark 2.22 and the dominated convergence theorem that $\tilde{v}(t_n, x_n)$ converges to 0.

Similarly, if $x \in D$ then it follows from the definition of $\tilde{v}$ that $\tilde{v}(0, x) = \varphi(x)$. Now let $(t_n, x_n)_{n \geq 1}$ be a sequence of elements of $(\mathbb{R}_+ \times (D \cup \Gamma^-)) \setminus ([0] \times \Gamma^-)$ which converge to $(0, x)$, and let us check that $\tilde{v}(t_n, x_n)$ converges to $\varphi(x)$. We first remark that if $t_n = 0$ then $\tilde{v}(t_n, x_n) = \varphi(x_n)$, so that along the subsequence $\{n \geq 1 : t_n = 0\}$, the claimed convergence is immediate. Therefore, we may now assume that $t_n > 0$ for any $n \geq 1$. Let $K \subset D$ be a compact set which contains the support of $\varphi$ and an open ball centered at $x$. There exists $N_1 \geq 1$ such that for all $n \geq N_1, x_n \in K$. Moreover, there exists $N_2 \geq 1$ such that for $n \geq N_2, t_n \in (0, 1]$ since $t_n \to 0$. Therefore, by Lemma 5.6, there exist a constant $C > 0$ and $\alpha \in (0, 1)$ such that for all $y \in K$ and $n \geq N_1 \vee N_2$,

$$
\left| p^D_{t_n}(y, x_n) - \hat{p}_{t_n}(y, x_n) \right| \leq C \sqrt{t_n} p_{t_n}^{(\alpha)}(y, x_n).
$$

Consequently, since $\varphi = 0$ outside $K$,

$$
\left| \tilde{v}(t_n, x_n) - e^{-d\gamma t_n} \int_D \hat{p}_{t_n}(y, x_n)\varphi(y)dy \right| \leq C \sqrt{t_n} e^{-d\gamma t_n} \int_D \hat{p}_{t_n}^{(\alpha)}(y, x_n)\varphi(y)dy \leq C \|\varphi\|_\infty \sqrt{t_n} e^{-d\gamma t_n} \int_{\mathbb{R}^d} \hat{p}_{t_n}^{(\alpha)}(y, x_n)dy.
$$
By Lemma 5.2, one has that
\[ \int_{\mathbb{R}^{2d}} \hat{p}^{(\alpha)}_{\nu}(y, x_n)dy = e^{d\gamma t_n}, \quad \int_{D} \hat{p}_{\nu}(y, x_n)\varphi(y)dy \xrightarrow{n\to\infty} \varphi(x). \]

Therefore, \( \tilde{v}(t_n, x_n) \xrightarrow{n\to\infty} \varphi(x) \).

We finally fix \( T > 0 \) and show that \( \sup_{t \in [0, T], x \in D} |\tilde{v}(t, x)| < \infty \). Again, Corollary 2.22 ensures the existence of \( C' > 0 \) such that for all \( t \in (0, T], x \in D \),
\[
|\tilde{v}(t, x)| = e^{-d\gamma t} \left| \int_{D} p^{D}_{t}(y, x)\varphi(y)dy \right| \\
\leq C'\|\varphi\|_\infty e^{-d\gamma t} \int_{\mathbb{R}^{2d}} \hat{p}^{(\alpha)}_{\nu}(y, x)dy \\
\leq C'\|\varphi\|_\infty,
\]
using Lemma 5.2, which concludes the proof. \( \square \)

6.2. Completion of the proof of Theorem 2.20

Let us now conclude this section with results on the boundary continuity of the density \( p^{D}_{t}(x, y) \) for a fixed \( t > 0 \). The proof relies on Theorem 6.2, and this result will complete the proof of Theorem 2.20. It also completes the results of Proposition 5.7 since we consider the continuity with respect to the three variables \( (t, x, y) \) at the same time and extend the limit with respect to \( y \) going to a point in \( \Gamma^0 \).

**Theorem 6.3** (Boundary continuity). Under Assumptions (O1) and (F1), the transition density \( p^{D} \) can be extended to a function in \( C(\mathbb{R}^*_+ \times \overline{D} \times \overline{D}) \) which satisfies for all \( t > 0 \):

(i) \( p^{D}_{t}(x, y) = 0 \) if \( x \in \Gamma^+ \cup \Gamma^0 \) or if \( y \in \Gamma^- \cup \Gamma^0 \),

(ii) if Assumption (O2) holds, \( p^{D}_{t}(x, y) > 0 \) for all \( x \notin \Gamma^+ \cup \Gamma^0 \) and \( y \notin \Gamma^- \cup \Gamma^0 \).

**Proof.** Step 1. We first study the behavior of \( p^{D}_{t}(x, y) \) when \( x \) and \( y \) approach \( \partial D \), and show that the function \( p^{D} \) can be continuously extended on \( \mathbb{R}^*_+ \times \overline{D} \times \overline{D} \). Let \( t_0 > 0 \), \( x_0 \in \overline{D} \) and \( y_0 \in \overline{D} \). Let \( (t_n, x_n, y_n)_{n \geq 1} \) be a sequence of points in \( \mathbb{R}^*_+ \times D \times D \) converging towards \( (t_0, x_0, y_0) \).

In the next three cases, we show that \( p^{D}_{t_0}(x_n, y_n) \) has a limit which does not depend on the sequence \( (t_n, x_n, y_n)_{n \geq 1} \). If \( x_0, y_0 \in D \) then by Proposition 5.5, this limit coincides with \( p^{D}_{t_0}(x_0, y_0) \). Otherwise, we denote this limit by \( p^{D}_{t_0}(x_0, y_0) \), which thereby defines a continuous function \( p^{D} \) on \( \mathbb{R}^*_+ \times \overline{D} \times \overline{D} \).

**Case 1** Assume that \( x_0 \in \Gamma^+ \cup \Gamma^0 \). By Assertion (i) in Proposition 5.7 and Remark 5.9, we immediately get
\[
p^{D}_{t_0}(x_n, y_n) \xrightarrow{n \to \infty} 0,
\]
and therefore set \( p_t^D(x_0, y_0) = 0 \).

**Case 2** Assume that \( y_0 = (q_0, p_0) \in \Gamma^- \cup \Gamma^0 \). For any \( (q, p) \in \mathbb{R}^{2d} \), let us define \( \diamond (q, p) \) := \( (q, -p) \). From the definition of the process \( (\tilde{X}_t^{\diamond})_{t \geq 0} = (\diamond X_t^{\diamond})_{t \geq 0} \), we deduce that the absorbed version of the latter possesses a transition density \( \tilde{p}^D_t \) which satisfies

\[
\tilde{p}^D_t (x, y) = \tilde{p}^D_t (\diamond x, \diamond y).
\]

As a consequence, using Theorem 6.2 we rewrite

\[
p_t^n (x_n, y_n) = e^{d y^n_t} \tilde{p}^D_t (y_n, x_n) = e^{d y^n_t} \tilde{p}^D_t (\diamond y_n, \diamond x_n).
\]

On the one hand, \( \diamond y_n \to \diamond y_0 \in \Gamma^+ \cup \Gamma^0 \). On the other hand, the process \( (\tilde{X}_t^{\diamond})_{t \geq 0} \) has infinitesimal generator \( \tilde{L}^\diamond = L_{F, -\gamma, \sigma} \), and therefore Assertion (i) in Proposition 5.7 and Remark 5.9 apply to show that

\[
\sup_{x \in D} \tilde{p}^D_t (\diamond y_n, x) \to 0,
\]

from which we deduce that \( p_t^n (x_n, y_n) \to 0 \) as \( n \to \infty \) and set \( p_{t_0}^D (x_0, y_0) = 0 \).

**Case 3** Assume that \( x_0 \in D \cup \Gamma^- \) and \( y_0 = (q_0, p_0) \in D \cup \Gamma^+ \). For \( h > 0 \), by the Markov property,

\[
\forall 0 \leq s < t, \quad \forall x, y \in D,
\]

\[
\frac{p_t^n (x, D \cap B(y, h))}{|B(y, h)|} = E \left[ 1_{t_3^+ > s} \frac{p_{t-s}^D (X^x_s, D \cap B(y, h))}{|B(y, h)|} \right].
\]

Using the Gaussian upper-bound from Corollary 2.22 and the dominated convergence theorem when \( h \to 0 \), we obtain from the equality above the following Chapman-Kolmogorov relation:

\[
\forall 0 \leq s < t, \quad \forall x, y \in D,
\]

\[
p_t^n (x, y) = E \left[ 1_{t_3^+ > s} p_{t-s}^D (X^x_s, y) \right]. \quad (113)
\]

By (113) applied with \( s = t_n/3 \), one has that

\[
p_t^n (x_n, y_n) = \int_D p_{t_n/3}^D (x_n, z) p_{t_n/3}^D (z, y_n) dz. \quad (114)
\]

Let us prove now the convergence of both integrands in (114). Using Theorem 6.2 and (113) again, one has for all \( z \in D \),

\[
p_{t_n/3}^D (z, y_n) = e^{2d y^n_t/3} \tilde{p}_{t_n/3}^D (y_n, z)
\]

\[
= e^{2d y^n_t/3} E \left[ 1_{t_3 > t_n/3} \tilde{p}_{t_n/3}^D (X_{t_n/3}^{y_n}, z) \right]
\]

\[
= e^{2d y^n_t/3} E \left[ 1_{t_3 > t_n/3} \tilde{p}_{t_n/3}^D (X_{t_n/3}^{\diamond y_n}, z) \right].
\]
By construction, $\hat{p}_t^{\infty,D}$ is continuous on $\mathbb{R}_+^* \times D \times D$ and is the transition density of the process $(\hat{X}_t^{\infty,x})_{t \geq 0}$ with infinitesimal generator $\hat{L}^\infty = L_{F,-}\gamma,\sigma$. Therefore, Lemma 3.2 and Corollary 2.22 apply here and ensure, using the dominated convergence theorem, that for $z \in D$,

$$
p_{2t_n/3}^D(z, y_n) \xrightarrow{n \to \infty} h_t^{(1)}(z) := e^{2d\gamma t_0/3}E\left[1_{t_0^{\infty,\gamma} \to t_0/3} \hat{p}_{t_0/3}^{\infty,D}(\hat{X}_{t_0/3}^{\infty,\gamma}, \cdot, \cdot)\right]. \quad (115)
$$

Furthermore, considering now the first integrand in (114), for all $z \in D$,

$$
p_{t_n/3}^D(x_n, z) = E\left[1_{t_n^{x_n} \to t_n/6} p_{t_n/6}^D(X_{t_n/6}^{x_n}, z)\right] \xrightarrow{n \to \infty} h_t^{(2)}(z) := E\left[1_{t_0^{x_0} \to t_0/6} p_{t_0/6}^D(X_{t_0/6}^{x_0}, z)\right]. \quad (116)
$$

using the continuity of $p^D$ and Lemma 3.2. It remains to prove that the integral (114) converges to the integral $\int_D h_t^{(1)}(z)h_t^{(2)}(z)dz$.

Since the term $p_{2t_n/3}^D(z, y_n)$ is bounded by a constant depending only on $t_n$ by Corollary 2.22 and since $t_n \xrightarrow{n \to \infty} t_0 > 0$ the associated constant can easily be obtained independent of $n$. In order to use the dominated convergence theorem to the product of both integrands in (114), it remains to obtain a bound on $p_{t_n/3}^D(x_n, z)$ in $L^1(D)$ which is independent of $n$. This follows from Lemma 6.4 below since $(t_n, x_n)$ converges to $(t_0, x_0) \in \mathbb{R}_+^* \times \mathbb{D}$, therefore the sequence $(t_n, x_n)_{n \geq 1}$ stays in some compact set $K$ of $\mathbb{R}_+^* \times \mathbb{D}$. We thus obtain a limit independent on the sequence $(t_n, x_n, y_n)_{n \geq 1}$. By Proposition 5.5, if $x_0, y_0 \in D$, it coincides with $p_{t_0}^D(x_0, y_0)$, otherwise we denote it by $p_{t_0}^D(x_0, y_0)$.

**Step 2** Let us now work under Assumption (O2) and first prove that for all $t > 0$ and $x, y \in D$,

$$
p_t^D(x, y) > 0. \quad (117)
$$

Let us argue by contradiction. Assume there exists $t_0 > 0$ and $x_0, y_0 \in D$ such that $p_{t_0}^D(x_0, y_0) = 0$. Let us introduce

$$
\Psi : (t, y) \in \mathbb{R}_+^* \times D \mapsto e^{-d\gamma t} p_t^D(x_0, \cdot, y).
$$

It follows from the forward Kolmogorov equation satisfied by $p_t^D(x_0, \cdot)$ (see Proposition 5.5), that on $\mathbb{R}_+^* \times D$,

$$
\partial_t \Psi = \hat{L}^\infty \Psi.
$$

Using the Harnack inequality stated in Theorem 2.15, we have that for any $K \subset D$ compact set and $t \in (0, t_0)$ there exists $C > 0$ such that

$$
\sup_{y \in K} \Psi(t, y) \leq C \inf_{y \in K} \Psi(t_0, y).
$$
In particular, it yields that for all \( y \in D, t \in (0, t_0) \), \( \Psi(t, y) = 0 \). As a result, for all \( t \in (0, t_0) \) and \( y \in D \), \( p_t^D(x_0, y) = 0 \). Integrating over \( y \in D \), it follows that for all \( t \in (0, t_0) \),

\[
P(\tau_{x_0}^y > t) = 0.
\]

Besides, it follows from Lemma 3.2 and Proposition 2.8 that \( P(\tau_{x_0}^y > t) \longrightarrow 1 \). This is contradiction with the equality above, and this thus concludes the proof of (117).

**Step 3** It remains to extend the result of **Step 2** to show that \( p_t^D(x, y) > 0 \) for \( t > 0 \), \( x \in D \cup \Gamma^- \) and \( y \in D \cup \Gamma^+ \). In this purpose, we first show that, for any \( x_0 \in D \cup \Gamma^- \), for all \( t > 0 \),

\[
P(\tau_{x_0}^y > t) > 0.
\]  

(118)

Indeed, using again Lemma 3.2 and Proposition 2.8, there exists necessarily \( s \in (0, t) \) such that \( P(\tau_{x_0}^y > s) > 0 \). As a result, the Markov property at time \( s \) ensures that

\[
P(\tau_{x_0}^y > t) = \mathbb{E} \left[ 1_{\tau_{x_0}^y > s} P(\tau_{x_0}^y > t - s) \right]_{z = x_s^{x_0}} > 0
\]

by (117), which yields (118).

We now recall from **Case 3** in **Step 1** that for \( t > 0 \), \( x \in D \cup \Gamma^- \) and \( y \in D \cup \Gamma^+ \),

\[
p_t^D(x, y) = \int_D h_1^{(1)}(z)h_2^{(2)}(z)dz,
\]

where \( h_1, h_2 \) are defined in (115) and (116). By **Step 2** applied to \( \tilde{p}_{t/3}^{\circ, D} \), for all \( z \in D \) we have \( \tilde{p}_{t/3}^{\circ, D}(\tilde{X}_{t/3}^{\circ, y}, oz) > 0 \) on the event \( (\tau_{x_0}^y > t/3) \). And by (118) applied to \( (\tilde{X}_{t}^{\circ, y})_{t \geq 0} \), this event has positive probability, so that \( h_1^{(1)}(z) > 0 \). Similarly, one has \( h_2^{(2)}(z) > 0 \). Therefore, we conclude that \( p_t^D(x, y) > 0 \).

Let us now state and prove Lemma 6.4.

**Lemma 6.4** (Transition density domination). Let Assumptions (O1) and (F1) hold. Let \( U \) be a compact set of \( \mathbb{R}^*_+ \times D \). There exist a constant \( C > 0 \) and a function \( h \in L^1(D) \) such that for all \( (t, x) \in U \) and \( y \in D \),

\[
p_t^D(x, y) \leq Ch(y).
\]

**Proof.** Let \( U \) be a compact set of \( \mathbb{R}^*_+ \times D \) and let \( (t_0, x_0) \) be a fixed element of \( U \). Let \( (t, x) \in U \), by Corollary 2.22, for any \( \alpha \in (0, 1) \), there exists \( C_1 > 0 \) such that

\[
p_t^D(x, y) \leq C_1 \tilde{p}_t^{(\alpha)}(x, y).
\]

Besides, by Proposition 2.17 and the equality (79), for all \( y \in D \), the function \( (t, x) \in \mathbb{R}^*_+ \times \mathbb{R}^{2d} \mapsto \tilde{p}_t^{(\alpha)}(x, y) \) satisfies

\[
\partial_t \tilde{p}_t^{(\alpha)} = \mathcal{L}_{0, y, \sigma/\sqrt{\alpha}} \tilde{p}_t^{(\alpha)}.
\]
As a result, the Harnack inequality in Theorem 2.15 along with Remark 2.16, applied on the compact set $U$, ensure the existence of $C_2 > 0$ only depending on $U$ such that

$$\hat{p}_t^{(\alpha)}(x, y) \leq C_2 \hat{p}_{t+1}^{(\alpha)}(x_0, y).$$

Finally, one has

$$p_t^D(x, y) \leq C_1 C_2 \hat{p}_{t+1}^{(\alpha)}(x_0, y), \quad (119)$$

where we eliminated the dependence with respect to the variable $x$. It remains to eliminate the dependence with respect to $t$ on the time variable. Consider now the expression of $\hat{p}^{(\alpha)}$ following from (79) and (75). Using Lemma 5.1, especially the first term in the right-hand side of the equality (78), one has for $x_0 = (q_0, p_0)$ and $y = (q', p') \in D$,

$$\hat{p}_{t+1}^{(\alpha)}((q_0, p_0), (q', p')) \leq \frac{\sqrt{\alpha}^{2d}}{\sqrt{(2\pi)^{2d}} \frac{(t+1)^4}{12}} e^{-\frac{\alpha}{\sigma^2(t+1)} |y(q'-q_0)+p'-p_0|^2}. $$

Hence, since $(t, x) \in U$, $t$ is bounded from above and below by positive constants. Therefore, reinjecting into (119) there exist $C_3 > 0$ and $\beta > 0$, which do not depend on $(t, x)$, such that for any $y = (q', p') \in D$,

$$p_t^D(x, y) \leq C_3 e^{-\beta |y(q'-q_0)+p'-p_0|^2},$$

which is an integrable function of $y$. \hfill \Box

To complete the proof of Theorem 2.20, it remains to check that, for any $t > 0$, the extension of $p_t^D$ constructed in Theorem 6.3 remains the density of the kernel $P_t^D(x, \cdot)$. This is already the case for $x \in D$ by Proposition 5.4, and we prove this fact for $x \in \partial D$ in the next proposition.

**Proposition 6.5** (Identification of the transition density on $\partial D$). Under Assumptions $(O1)$ and $(F1)$, for all $t > 0$, $x \in \partial D$ and $A \in \mathcal{B}(D)$,

$$P_t^D(x, A) = \int_A p_t^D(x, y) dy.$$ 

**Proof.** Let $t_0 > 0$, $x_0 = (q_0, p_0) \in \partial D$ and $(t_n, x_n)_{n \geq 1}$ be a sequence of points in $\mathbb{R}_+ \times D$ converging to $(t_0, x_0)$. Let us show that for any open set $A \subset D$, $P_{t_n}^D(x_n, A)$ admits two limits when $n \to \infty$ which are $P_{t_0}^D(x_0, A)$ and $\int_A P_{t_0}^D(x_0, y) dy$. This yields the desired equality $P_{t_0}^D(x_0, A) = \int_A P_{t_0}^D(x_0, y) dy$.

Let $A$ be an open subset of $D$. The limit $P_{t_n}^D(x_n, A) \to P_{t_0}^D(x_0, A)$ is a straightforward consequence of Lemmas 3.1 and 3.2, which ensure that

$$P(X_{t_n}^{x_n} \in A, \tau_{\partial}^{x_n} > t_n) \to P(X_{t_0}^{x_0} \in A, \tau_{\partial}^{x_0} > t_0),$$

$$P(X_{t_n}^{x_n} \in A, \tau_{\partial}^{x_n} \leq t_n) \to P(X_{t_0}^{x_0} \in A, \tau_{\partial}^{x_0} \leq t_0),$$

$$P(X_{t_n}^{x_n} \in A, \tau_{\partial}^{x_n} = t_n) \to P(X_{t_0}^{x_0} \in A, \tau_{\partial}^{x_0} = t_0).$$
using the dominated convergence theorem since $P(X_{t_0}^{x_0} \in \partial A, \tau_{x_0}^{x_0} > t_0) = 0$.

Let us now prove that $P_t^n(x_n, A) = \int_A p_t^n(x_n, y)dy \longrightarrow \int_A p_t^D(x_0, y)dy$. In order to do that we apply the dominated convergence theorem on the integrand of $\int_A p_t^n(x_n, y)dy$ which requires an upper bound of $p_t^n(x_n, y)$ independent of $n$ for $n$ large enough, and integrable on $D$. Such an upper bound is provided in Lemma 6.4 if for all $n \geq 1, (t_n, x_n)$ is in a compact of $\mathbb{R}_+^* \times \overline{D}$ which is the case since $t_n \longrightarrow t_0 > 0$ and $x_n \longrightarrow x_0 \in \overline{D}$.

**Remark 6.6. (Strong Feller property)** Let $(t_n, x_n)_{n \geq 1}$ be a sequence in $\mathbb{R}_+^* \times \overline{D}$ converging to $(t, x) \in \mathbb{R}_+^* \times \overline{D}$. By the previous construction, it follows that for any $y \in D, p_t^n(x_n, y) \longrightarrow p_t^D(x, y)$. Furthermore, the proof of Proposition 6.5 shows that

$$\int_D p_t^n(x_n, y)dy \longrightarrow \int_D p_t^D(x, y)dy.$$  

These two convergences guarantee by Scheffé’s lemma that

$$\int_D \left| p_t^n(x_n, y) - p_t^D(x, y) \right| dy \longrightarrow 0.$$  

As a consequence, the semigroup $(P_t^D)_{t \geq 0}$ defined on $L^\infty(\overline{D})$ by

$$\forall t \geq 0, \forall x \in \overline{D}, \quad P_t^D f(x) := \mathbb{E} \left[ f(X_t^x) 1_{\tau_{x}^{x_0} \leq t} \right]$$

satisfies the strong Feller property that $P_t^D f \in C^b(\overline{D})$ for any $t > 0$ and $f \in L^\infty(\overline{D})$.

**Acknowledgements**

M. Ramil is supported by the Région Ile-de-France through a PhD fellowship of the Domaine d’Intérêt Majeur (DIM) Math Innov. This work also benefited from the support of the projects ANR EFI (ANR-17-CE40-0030) and ANR QuAMProcs (ANR-19-CE40-0010) from the French National Research Agency. Finally, T. Lelièvre is partially funded by the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement No 810367), project EMC2.

**Data Availability Statement** Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

**Publisher’s Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.
A. Proof of Lemma 4.2

Proof. Let $K \subset D$ be a compact set and $\delta_K > 0$ be defined accordingly. Let $k := \sup_{(q, p) \in K} |p|$, $K' := \{(q, p) \in D : |p| \leq k, d_0(q) \geq \delta_K\}$ and $M_K := k + 1$. By Proposition 3.7 and Assumption (O2), $K'$ is a connected compact subset of $D$, and $K \subset K'$.

Let $T > 0$ and $\epsilon := \delta_K / 2$. Let $C > 1$ be the constant from Lemma 4.1. The compact set $K'$ can be covered by $N \geq 1$ closed balls $B(z_1, \frac{\epsilon}{2C}), \ldots, B(z_N, \frac{\epsilon}{2C})$ included in $D$ with $z_i = (q_i, p_i) \in K'$ for all $i \in [1, N]$. We can take $N$ large enough so that $\Delta := \frac{T}{N+1} \in (0, \frac{\epsilon}{2M_K C} \wedge 1)$. We can now build a graph $G$ with $N$ vertices corresponding to the points $(z_i)_{1 \leq i \leq N}$, and for every $i, j \in [1, N]$, we link $z_i$ to $z_j$ if $|z_j - z_i| \leq \frac{\epsilon}{4C}$. Since the set $K'$ is connected, then so is the graph $G$.

Furthermore, for all $i, j \in [1, N]$ which are adjacent in $G$, by Lemma 4.1, there exists a path $\phi_{i,j} \in C^2([0, \Delta], \mathbb{R}^d)$ such that

(i) $(\phi_{i,j}(0), \phi_{i,j}(\Delta)) = (z_i, z_j)$

(ii) $\sup_{t \in [0, \Delta]} |\phi_{i,j}(t) - q_i| \leq C(|z_j - z_i| + \Delta) + \Delta |p_i| \leq C \left( \frac{\epsilon}{4C} + k \frac{\epsilon}{2M_K C} \right) < \epsilon$

(iii) $\sup_{t \in [0, \Delta]} \left( |\phi_{i,j}(t)| + |\dot{\phi}_{i,j}(t)| \right) \leq C \left( \frac{1}{\Delta} + 1 \right) (|z_j - z_i| + \Delta) + \Delta |p_i| \leq \epsilon \left( \frac{1}{\Delta} + \frac{2}{\Delta^2} \right) \leq \frac{2\epsilon}{\Delta^2}$.

Since $z_i \in K'$, the second condition ensures that, for all $t \in [0, \Delta]$, $\phi_{i,j}(t)$ remains at a distance from $\partial O$ strictly larger than $\delta_K - \epsilon \geq \delta_K / 2$.

Now let $x, y \in K'$. By the previous cover, there exist $i_0, i_N \in [1, N]$ such that

$$|x - z_{i_0}| \leq \frac{\epsilon}{8C}, \quad |y - z_{i_N}| \leq \frac{\epsilon}{8C}.$$

Using Lemma 4.1 again, we construct $\psi_0, \psi_N : [0, \Delta] \to O$ respectively joining $x$ to $z_{i_0}$ and $z_{i_N}$ to $y$ and such that $\sup_{t \in [0, \Delta]} (|\psi_0(t)| + |\dot{\psi}_0(t)|) \leq \frac{\delta_K}{\Delta}$ and $\sup_{t \in [0, \Delta]} (|\psi_N(t)| + |\dot{\psi}_N(t)|) \leq \frac{\delta_K}{\Delta}$. The connectedness of the graph $G$ ensures the existence of $i_1, \ldots, i_{N-1} \in [1, N]$ such that for all $j \in [1, N]$, $|z_{i_j} - z_{i_{j-1}}| \leq \frac{\epsilon}{4C}$. If the path obtained on the graph $G$ is smaller than $N$ then we can complete with loops around the same point. It is important here that this path in the graph have exactly $N - 1$ vertices, because in the end we aim at constructing a trajectory $\phi$ by piecing together trajectories $\phi_{i_j, i_{j+1}}$, $j = 0, \ldots, N$, of length $\Delta = T/(N + 1)$ and we want the final trajectory $\phi$ to have exact length $T$. Let us now define the path function $\phi$ on $[0, T]$ as follows:

$$\phi(t) = \begin{cases} 
\psi_0(t) & \text{if } t \in [0, \Delta], \\
\phi_{i_j, i_{j+1}}(t - j \Delta) & \text{if } t \in [j \Delta, (j + 1)\Delta], \\
\psi_N(T - T + \Delta) & \text{if } t \in [T - \Delta, T],
\end{cases}$$

then it is easy to see that $\phi \in \mathcal{H}_{T, x, y, \delta_K, \delta_K / 2}$. Since $K'$ contains the compact set $K$ this concludes the proof for the compact set $K$. \qed
B. Proof of Lemmas 5.1, 5.2, 5.8 and Proposition 5.4

Proof of Lemma 5.1. On the one hand, from (73) and (76), easy computations show that
\[
\delta x \cdot C^{-1}(t) \delta x = \frac{1}{\sigma^2 t^3} \Phi(\gamma t) \left[ \Phi_1(2\gamma t)|\delta q|^2 - \Phi_1(\gamma t)^2 \delta q \cdot t \delta p + \frac{1}{3} \Phi_2(\gamma t)|t \delta p|^2 \right].
\] (120)

On the other hand,
\[
\frac{1}{\sigma^2 t^3} |\Pi_1 \delta x|^2 + \frac{12}{\sigma^2 t^3} \Phi(\gamma t) |\Pi_2(t) \delta x|^2
\]
\[
= \frac{1}{\sigma^2 t^3} \Phi(\gamma t) \left[ \left( \frac{\gamma t}{12} \Phi(\gamma t) + \Phi_1(\gamma t)^2 \right) |\delta q|^2
\]
\[
+ \left( \frac{\gamma t}{6} \Phi(\gamma t) - \Phi_1(\gamma t) \Phi_3(\gamma t) \right) \delta q \cdot t \delta p
\]
\[
+ \left( \frac{1}{12} \Phi(\gamma t) + \frac{1}{4} \Phi_3(\gamma t)^2 \right) |t \delta p|^2 \right],
\]
so that the claimed expression follows from the identities
\[
\frac{\rho^2}{12} \phi(\rho) + \Phi_1(\rho)^2 = \Phi_1(2\rho), \quad \frac{\rho}{6} \phi(\rho) - \Phi_1(\rho) \Phi_3(\rho) = -\Phi_1(\rho)^2,
\]
\[
\frac{1}{12} \phi(\rho) + \frac{1}{4} \Phi_3(\rho)^2 = \frac{1}{3} \Phi_2(\rho),
\]
for all \(\rho \in \mathbb{R}\).

We now move on to the proof of Lemma 5.2.

Proof of Lemma 5.2. The equality (80) easily follows from the formulas defining \(\hat{p}_t\) and \(\hat{p}_t^{(\alpha)}\). Moreover, since \(\hat{p}_t^{(\alpha)}(x, y)\) is the transition density of the process \((\alpha^{-1/2} \hat{X}_t(\sqrt{\alpha} t)^t_{t \geq 0})\), the Chapman-Kolmogorov relation (81) follows from the Markov property. Let us prove (82). For \(\alpha \in (0, 1]\),
\[
\int_{\mathbb{R}^2d} \hat{p}_t^{(\alpha)}(x, y) dx = \int_{\mathbb{R}^2d} \frac{\sqrt{\alpha^2 d}}{\sqrt{(2\pi)^{2d} \det(C(t))}} e^{-\frac{1}{2} \delta x(t) \cdot C^{-1}(t) \delta x(t)} dx,
\]
where \(\delta x(t)\) and \(C^{-1}(t)\) are defined in (76). Let us define the matrix \(M(t)\) as follows:
\[
M(t) := \begin{pmatrix}
I_d & t \Phi_1(\gamma t) I_d \\
0_d & e^{-\gamma t} I_d
\end{pmatrix}
\]
so that for \(x = (q, p), y = (q', p')\) one has \(M(t)x = \begin{pmatrix} m_x(t) \\ m_p(t) \end{pmatrix}\). Therefore, \(\delta x(t) = y - M(t)x\). As a result, making the following change of variables,
\[
x \in \mathbb{R}^{2d} \mapsto z := y - M(t)x
\]
one has $dz = e^{-d\gamma t}dx$ and one obtains
\[
\int_{\mathbb{R}^d} \rho_t^{(\alpha)}(x, y)dx = e^{d\gamma t} \int_{\mathbb{R}^d} \frac{\sqrt{2d}}{(2\pi)^{d/2} \det(C(t))} e^{-\frac{1}{2}z^T C^{-1}(t)z} dz = e^{d\gamma t}.
\]

Let us now prove the first convergence in (83). Using the same change of variables as above, one obtains that
\[
\int_{\mathbb{R}^d} \rho_t^{(\alpha)}(x, y)\varphi(t, x)dx = e^{d\gamma t} E \left[ \varphi(t, M^{-1}(t)(y - Z(t))) \right],
\]
where $Z(t) \sim \mathcal{N}_{2d}(0, C(t)) \xrightarrow{L} 0$, since $C(t) \xrightarrow{t \to 0} 0$. Therefore, since $M(t) \xrightarrow{t \to 0} I_{2d}$, one has by Slutsky’s theorem that
\[
\left( t, M^{-1}(t)(y - Z(t)) \right) \xrightarrow{L} (0, y_0)
\]
which yields the first convergence in (83) using (121) and the dominated convergence theorem. The second convergence follows easily with a similar change of variables.

Let us finally prove (84). By (75), (76) along with Lemma 5.1 we have
\[
\nabla_p \rho_t((q, p), (q', p')) \xrightarrow{\hat{\rho}_t} - \frac{1}{2} \left( \frac{2}{\sigma^2 t} \nabla_p (\Pi_1 \delta x(t)) \Pi_1 \delta x(t) + \frac{24}{\sigma^2 t^3} \phi(\gamma t) \nabla_p (\Pi_2(t) \delta x(t)) \Pi_2(t) \delta x(t) \right).
\]

Since $\nabla_p \Pi_1 \delta x(t) = -(\gamma t \Phi_1(\gamma t) + e^{-\gamma t})I_d = -I_d$, the first term in the right-hand side of the equality (122) multiplied by $\hat{\rho}_t$ satisfies (using (80), and again Lemma 5.1 in the first inequality)
\[
\left| - \frac{1}{2} \frac{2}{\sigma^2 t} \nabla_p (\Pi_1 \delta x(t)) \Pi_1 \delta x(t) \right| \hat{\rho}_t((q, p), (q', p'))
\]
\[
= \frac{1}{\sigma^2 t} |\Pi_1 \delta x(t)| \hat{\rho}_t((q, p), (q', p'))
\]
\[
= \frac{1}{\sqrt{2d} \sigma^2 t} |\Pi_1 \delta x(t)| e^{-\frac{1}{2} \frac{\gamma^2}{\sigma^2 t} \delta x(t) C^{-1}(t) \delta x(t)} \hat{\rho}_t^{(\alpha)}((q, p), (q', p'))
\]
\[
\leq \frac{1}{\sqrt{2d} \sigma^2 t} \sup_{\theta \geq 0} \theta e^{-\frac{1}{2} \frac{\gamma^2}{\sigma^2 t} \delta x(t) C^{-1}(t) \delta x(t)} \hat{\rho}_t^{(\alpha)}((q, p), (q', p'))
\]
\[
\leq \sup_{\theta \geq 0} \theta e^{-\frac{1}{2} \frac{\gamma^2}{\sigma^2 t} \delta x(t) C^{-1}(t) \delta x(t)} \hat{\rho}_t^{(\alpha)}((q, p), (q', p')).
\]

Let us now estimate the second term in the right-hand side of the equality (122) multiplied by $\hat{\rho}_t$. Since $\nabla_p \Pi_2(t) \delta x(t) = (-t \Phi_1(\gamma t)^2 + \frac{\gamma}{2} \Phi_3(\gamma t)e^{-\gamma t})I_d$, we have
(using the same reasoning as above)

\[
- \frac{1}{2} \frac{24}{\sigma^2 t^3 \phi(\gamma t)} \nabla_p \left( \Pi_2(t) \delta x(t) \right) \Pi_2(t) \delta x(t) \left| \right. \bar{P}_t((q, p), (q', p')) \\
= \frac{12t}{\sigma^2 t^3 \phi(\gamma t)} \left| -\Phi_1(\gamma t)^2 + \frac{1}{2} \Phi_3(\gamma t) e^{-\gamma t} \right| \left| \Pi_2(t) \delta x(t) \right| \left| \bar{P}_t((q, p), (q', p')) \right|
\]

\[
\leq \frac{\sqrt{12t}}{\alpha^2 d \sigma^2 t^3 \phi(\gamma t)} \left| -\Phi_1(\gamma t)^2 + \frac{1}{2} \Phi_3(\gamma t) e^{-\gamma t} \right| \left( \left| \Pi_2(t) \delta x(t) \right| \right) \frac{1}{\sqrt{\phi(\gamma t)}} \left( \sup_{\theta \geq 0} \theta e^{-\frac{1}{2} \theta^2} \right) \bar{P}_t^{(\alpha)}((q, p), (q', p')).
\]

(124)

Let us now study the behavior of \( \frac{|-\Phi_1(\rho)^2 + \frac{1}{2} \Phi_3(\rho) e^{-\rho}|}{\sqrt{\phi(\rho)}} \) for \( \rho \in \mathbb{R} \). We have that

\[
\frac{|-\Phi_1(\rho)^2 + \frac{1}{2} \Phi_3(\rho) e^{-\rho}|}{\sqrt{\phi(\rho)}} \begin{cases} \sim \frac{1}{\rho \to \infty} \frac{1}{\sqrt{6} \rho}, \\
\to \frac{1}{\rho \to 0} \frac{1}{2}, \\
\sim \frac{\sqrt{\rho}}{\rho \to -\infty} \frac{1}{\sqrt{6}}. 
\end{cases}
\]

Therefore there exists a universal constant \( c > 0 \) such that for all \( \rho \in \mathbb{R} \)

\[
\frac{|-\Phi_1(\rho)^2 + \frac{1}{2} \Phi_3(\rho) e^{-\rho}|}{\sqrt{\phi(\rho)}} \leq c(1 + \sqrt{\rho_-}),
\]

where \( \rho_- \) is the negative part of \( \rho \). As a result, it follows from (122), (123) and (124) that there exists a constant \( c_\alpha > 0 \) depending only on \( \alpha \in (0, 1) \) such that for all \( t > 0 \) and \( x, y \in \mathbb{R}^{2d} \),

\[
\left| \nabla_p \bar{P}_t((q, p), (q’, p')) \right| \leq \frac{c_\alpha}{\sigma^2 t} (1 + \sqrt{t y_-}) \bar{P}_t^{(\alpha)}((q, p), (q’, p'))
\]

which concludes the proof of (84).

Consider now the proof of Proposition 5.4.

Proof of Proposition 5.4. For all \( t > 0 \) and \( x \in D \), it follows from (22) that the measure \( P_t^D(x, \cdot) \) is absolutely continuous with respect to the measure \( P_t(x, \cdot) \). Since, by Proposition 2.17, the latter measure is absolutely continuous with respect to the Lebesgue measure, by the Radon-Nikodym theorem, we deduce that \( P_t^D(x, \cdot) \) possesses a density \( q_t^D(x, \cdot) \) with respect to the Lebesgue measure on \( D \). We now study
the joint measurability of the mapping \((t, x, y) \mapsto q_t^D(x, y)\); more precisely, we construct a measurable function \((t, x, y) \mapsto p_t^D(x, y)\) such that, for all \(t > 0\) and \(x \in D\), \(p_t^D(x, y) = q_t^D(x, y)\), dy-almost everywhere on \(D\).

For all \(r > 0\), it follows from Proposition 2.17 and Lemmas 3.1 and 3.2 that the function

\[
\varphi_r : (t, x, y) \in \mathbb{R}_+^* \times D \times D \mapsto \frac{P_t^D(x, B(y, r) \cap D)}{|B(y, r)|} = \frac{\mathbb{P}(|X^x_t - y| < r, \tau_0^x > t)}{|B(y, r)|}
\]

is continuous. Let \((r_q)_{q \geq 1}\) be a sequence of positive real numbers decreasing towards 0. By definition, for any \(t > 0\) and \(x \in D\), the density \(q_t^D(x, \cdot)\) is integrable on \(D\). As a result, the Lebesgue differentiation theorem states that almost every \(y \in D\) is a Lebesgue point, hence

\[
\forall t > 0, \ \forall x \in D, \quad \varphi_{r_q}(t, x, y) \xrightarrow[q \to \infty]{} q_t^D(x, y) \text{ dy-almost everywhere on } D.
\]

As a consequence, \(q_t^D(x, y)\) coincides, dy-almost everywhere, with the measurable function

\[
p_t^D(x, y) := \limsup_{q \to \infty} \varphi_{r_q}(t, x, y),
\]

which completes the proof. \(\square\)

Last, we show the proof of Lemma 5.8.

**Proof of Lemma 5.8.** Let \(y_0 = (q_0, p_0) \in \mathbb{R}^{2d}\), \(M > 0\) and \(\alpha \in (0, 1)\). Let \(\delta_0 > 0\) be small enough for the assertion

\[
\forall s \in (0, \delta_0], \quad \left(\frac{M}{6} + |p_0|\right) |e^{\gamma s} - 1| \leq \frac{M}{12}
\]

(125)

to hold.

Let \((q', p') \in B(y_0, M/6)\) and \((q, p) \in \mathbb{R}^{2d}\) satisfying \(|p - p'| \geq M/3\). For \(s \in (0, \delta_0]\), we consider the transition density \(\widehat{P}_s^{(\alpha)}((q, p), (q', p'))\) defined in (79).

One has that

\[
\widehat{P}_s^{(\alpha)}((q, p), (q', p')) = \frac{\sqrt{2\pi}^{2d}}{\sqrt{2\pi}^{2d} \left(\frac{\alpha^4 \phi(\gamma s)}{12}\right)^d} e^{-\frac{a^2}{2\phi(\gamma s)} |\gamma \delta q + \delta p|^2 - \frac{\alpha^2}{\sigma^2 \phi(\gamma s)} |\Phi_1(\gamma s)\delta q - \frac{\delta}{2} \Phi_3(\gamma s)\delta p|^2},
\]

(126)

where

\[
\left(\begin{array}{c}
\delta q \\
\delta p
\end{array}\right) = \left(\begin{array}{c}
q' - q - s \Phi_1(\gamma s) p \\
p' - pe^{-\gamma s} - \Phi_3(\gamma s) p
\end{array}\right).
\]
Let us start by introducing some notations. Let \( m := 1 + |\gamma|\delta_0 + \sup_{|\rho| \leq |\gamma|\delta_0} |\rho|/2 \Phi_3(\rho) \) and \( a := M e^{-|\gamma|\delta_0}/4m \). Let us prove that necessarily
\[
|\gamma \delta q + \delta p| \geq a \quad \text{or} \quad |\Phi_1(\gamma s)\delta q - \frac{s}{2} \Phi_3(\gamma s)\delta p| \geq as, \tag{127}
\]
then reinjecting this statement onto the expression (126) of \( \hat{p}_s^{(\alpha)}((q, p), (q', p')) \) we will be able to obtain (99).

Assume now that
\[
|\gamma \delta q + \delta p| < a \quad \text{and} \quad |\Phi_1(\gamma s)\delta q - \frac{s}{2} \Phi_3(\gamma s)\delta p| < as,
\]
we will prove that \( |p' - p| < M/3 \), thus contradicting the initial assumption on \((q, p)\) and \((q', p')\).

Using the triangle inequality, since \( \Phi_1(\rho) + \frac{s}{2} \Phi_3(\rho) = 1 \) for all \( \rho \in \mathbb{R} \), one has that
\[
\delta q = \delta q \left( \Phi_1(\gamma s) + \frac{\gamma s}{2} \Phi_3(\gamma s) \right) \leq |\delta q \Phi_1(\gamma s) - \frac{s}{2} \Phi_3(\gamma s)\delta p| + \frac{s}{2} \Phi_3(\gamma s)|\gamma \delta q + \delta p| < a \left( s + \frac{s}{2} \Phi_3(\gamma s) \right).
\]
As a result,
\[
|\delta p| \leq |\gamma \delta q + \delta p| + |\gamma| |\delta q| < a \left( 1 + |\gamma|s + \frac{|\gamma|s}{2} \Phi_3(\gamma s) \right).
\]
Since \( s \leq \delta_0 \), one obtains that \( |\delta p| < am \). Therefore \( |\delta p| < M e^{-|\gamma|\delta_0}/4 \). Furthermore, by the triangle inequality, for \((q', p') \in B(\gamma_0, M/6)\),
\[
|p' - p| \leq e^{\gamma s} \left| p' - pe^{-\gamma s} \right| + \left| p' - p_0 \right| e^{\gamma s} - 1 + \left| p_0 \right| e^{\gamma s} - 1 \\
= |\delta p| \\
< e^{\gamma s} \frac{Me^{-|\gamma|\delta_0}}{4} + \frac{M}{6} \left| e^{\gamma s} - 1 \right| \leq \frac{M}{4} + \frac{M}{12} = \frac{M}{3},
\]
by (125) since \( s \leq \delta_0 \), hence (127).

Reinjecting the inequality (127) into (126), we get
\[
\hat{p}_s^{(\alpha)}((q, p), (q', p')) \leq \frac{\sqrt{\alpha^{2d}}}{\sqrt{(2\pi)^{2d}(\frac{\alpha^d}{12}\phi(\gamma s))^d}} \exp \left( -\frac{1}{s} \min \left( \frac{a^2\alpha}{2\sigma^2}, \frac{6a^2\alpha}{\sigma^2\phi(\gamma s)} \right) \right),
\]
and using the fact that $\phi(\gamma s)$ is a positive and bounded continuous function for $s \in [-|\gamma|\delta_0, |\gamma|\delta_0]$, it follows that there exist some constants $C_0 \geq 0$ and $\mu > 0$, which only depend on $\gamma, \sigma, M, \alpha$ and $\delta_0$, such that for any $s \in (0, \delta_0]$,

$$\hat{P}_s^{(\alpha)}((q, p), (q', p')) \leq C_0 e^{-\mu/s},$$

which completes the proof. □

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Accepted: 10 March 2022