Extended bodies moving on geodesic trajectories

Sajal Mukherjee1,2 · Georgios Lukes-Gerakopoulos2 · Rajesh Kumble Nayak1,3

Received: 14 February 2021 / Accepted: 25 August 2022 / Published online: 24 September 2022 © The Author(s), under exclusive licence to Springer Science+Business Media, LLC, part of Springer Nature 2022

Abstract
This work investigates whether an extended test body obeying the Mathisson–Papapetrou–Dixon equations under the Ohashi–Kyrian–Semerák spin supplementary condition can follow geodesic trajectories in curved spacetimes. In particular, we explore what are the requirements under which pole-dipole and pole-dipole-quadrupole approximated bodies moving in the Schwarzschild or Kerr spacetimes can follow equatorial geodesic trajectories. We do this exploration thoroughly in the pole-dipole case, while we focus just on particular trajectories in the pole-dipole-quadrupole case. Using the Ohashi–Kyrian–Semerák spin supplementary condition to fix the center of the mass of a pole-dipole body has the advantage that the hidden momentum is eliminated. This allows the four-velocity to be parallel to the four-momentum, which provides a convenient framework for our investigation. We discuss how this feature can be recovered at a pole-dipole-quadrupole approximation and what are the consequences.

Keywords Mathisson–Papapetrou–Dixon equations · Particle dynamics · Black holes

Contents
1 Introduction ............................................. 2
2 The pole-dipole-quadrupole approximation and the Ohashi–Kyrian–Semerák spin supplementary condition .................................................. 4

Sajal Mukherjee
mukherjee@asu.cas.cz

Georgios Lukes-Gerakopoulos
gglukes@gmail.com

Rajesh Kumble Nayak
rajesh@iiserkol.ac.in

1 Department of Physical Sciences, IISER-Kolkata, Mohanpur 741246, India
2 Astronomical Institute of the Czech Academy of Sciences, Bocni II 1401/1a, 14100 Prague, Czech Republic
3 Center of Excellence in Space Sciences India, IISER-Kolkata, Mohanpur, India
1 Introduction

The motion of an extended test body has long and extensive history starting with the seminal works of Mathisson [1], Papapetrou [2, 3] and Dixon [4–7] up to more recent treatments [8–23]. In the framework of the Mathisson–Papapetrou–Dixon equations, the extended body is often viewed as a point particle endowed with multipole moments, since practically the whole body is represented by its center of mass called centroid with a set of multipole moments defined around it. Moreover, in the test particle limit, we ignore any back-reaction on the background and we assume that the geometry remains unaltered even if the particle is endowed with a finite size. In fact, the only corrections that one anticipates in the Mathisson–Papapetrou–Dixon framework originate from the non-vanishing coupling between the curvature and the higher order multipole moments than the mass (monopole), i.e., dipole, quadrupole and higher multipoles of the body. Due to this coupling, the particle, in general, experiences an acceleration making it deviate from a geodesic trajectory.

There is a useful analogy between the electromagnetic field and the gravitational field helping us to understand the orbital dynamics of a test body. Even though the corresponding theories are intrinsically different, numerous useful analogies between these fields have been found over the years [24–26]. In the case of an extended body, this analogy was first explored by Wald, who proved that the force on a magnetic dipole in a electromagnetic field is analogous to the force experienced by a spinning particle in a weak gravitational field [27]. Later the analogy between the spin precession of a gyroscope in a gravitational field and the precession of a magnetic dipole was discussed in [28–30]. In recent years, this analogy have been further explored in [31, 32] and the connection between the motion of a spinning particle with its electromagnetic counterpart has been emphasized [33].

Our work focuses on a counterexample to the aforementioned analogy. Namely, contrary to the electromagnetic field case, where a moving magnetic dipole can never experience a vanishing force, in a gravitational field, a spinning particle may follow a geodesic trajectory [33]. Such trajectories may appear as special cases, but their existence suggests an interesting twist in the analogy. Hence, in the present work, we aim to investigate under which conditions such trajectories can be obtained. Such purely academic investigations have been already undertaken in the Mathisson–Papapetrou–Dixon pole-dipole approximation [33–35]. In particular, in Refs. [34, 35] these
trajectories have been studied in the Schwarzschild spacetime and it has been argued that in the case of a radially in-falling spinning particle, the spin curvature coupling identically vanishes. In Ref. [33], the existence of geodesic trajectories for a spinning particle has been considered in Kerr and Kerr-dS geometry showing that the circular geodesic in Kerr-dS can have a zero curvature coupling while the same is not true for Kerr. Hence, the existence of this special type of trajectories appears to depend on the background geometry.

In our analysis, we explore the existence of geodesic trajectories followed by a body with dipole and quadrupole moments in the Schwarzschild and Kerr spacetimes. Our study is restricted on the equatorial plane; in the case of Schwarzschild background this is imposed by the background’s geometry, while in the case of a Kerr background this is imposed by the complexity of treating generic off-equatorial orbits. In both backgrounds, we investigate two types of bodies: the pole-dipole body, in which we ignore all the higher moments than the dipole one, and the pole-dipole-quadrupole one, in which we ignore all the higher moments than the quadrupole one. By searching for the existence of geodesic orbits followed by extended bodies, we are not only able to find the existence of such orbits in the pole-dipole approximation, but also exclude all the other possibilities. In the pole-dipole-quadrupole approximation, we investigate only the geodesic trajectories which we found to be compatible with the pole-dipole body. For these geodesic trajectories we couldn’t find a viable solution for a pole-dipole-(spin induced) quadrupole body.

When dealing with Mathisson–Papapetrou–Dixon equations, a discussion about a spin supplementary condition is unavoidable [3, 36–39]. Different spin supplementary conditions can lead to different worldlines of the same physical body [8, 20, 21]. This is intimately related with the fact that in relativity, the location of the center of mass of an extended spinning body is frame-dependent and not defined uniquely [20, 40]. In the present work, we use the Ohashi–Kyrian–Semerák spin supplementary condition [9, 18, 41] because in the pole-dipole approximation under this spin supplementary condition the four-momentum and the four-velocity of the body are parallel, which is a characteristic feature of geodesic orbits not necessarily holding for the Mathisson–Papapetrou–Dixon equations. Actually, this feature does not have to hold for Ohashi–Kyrian–Semerák when higher multipoles than the dipole are included into the extended bodies’ approximation. However, the authors in [20] have suggested that enforcing an extra condition involving torque can allow this feature to be recovered in the pole-dipole-quadrupole approximation framework. Following this suggestion we elaborate on this condition and discuss its consequences.

The manuscript is organized as follows: Sect. 2 introduces the equations of motion of a pole-dipole-quadrupole particle, provides a detailed discussion about the Ohashi–Kyrian–Semerák spin supplementary condition and briefs the framework of the analysis that follows. The specific calculations regarding the extended bodies on geodesic orbits are carried out in Sect. 3 for Schwarzschild and for Kerr in Sect. 4. The article concludes in Sect. 5.

Notation and Conventions: In the paper, we have used the \((-, +, +, +)\) metric signature and set the speed of light, \(c\), and gravitational constant \(G\), as \(c = 1 = G\). The “\([\ ]\)” describes the standard antisymmetric expansion given by \(A[^{\alpha \beta}] = \frac{1}{2} (A^\alpha B^\beta - \)
A covariant derivative is denoted as $\nabla_\mu$. Note that a partial derivative with respect to variable $z$ is denoted as $\partial_z$. A projection of a tensor on a tetrad field frame $e^\nu_\mu$, where $(\nu)$ is the tetrad index and $\nu$ is the tensor index, is denoted by putting a parenthesis around the index of the tensor, e.g. $X(\nu) = e^\nu_\mu X^\mu$.

2 The pole-dipole-quadrupole approximation and the Ohashi–Kyrian–Semerák spin supplementary condition

The Mathisson–Papapetrou–Dixon equations of a body in the pole-dipole-quadrupole approximation \[42, 43\], when only gravitational interactions are considered, read

$$
\dot{P}_\mu = -\frac{1}{2} R^\mu_\nu\rho_\beta U^\nu S^{\alpha\beta} - \frac{1}{6} J^{\alpha\beta\gamma\delta} \nabla_\mu R_{\alpha\beta\gamma\delta},
$$

$$
\dot{S}_{\mu\nu} = 2 P^{[\mu \nu]} U^{|\nu|} + \frac{4}{3} J^{\alpha\beta\gamma\delta} R_{\gamma\alpha\beta},
$$

where $R^\mu_\nu\rho_\beta$ is the Riemann tensor, $S^{\alpha\beta}$ is the spin tensor, $J^{\alpha\beta\gamma\delta}$ is the quadrupole tensor, $P^\mu$ is the four-momentum and $U^\mu$ is the four-velocity, while the “dot” defines a covariant derivative $\nabla_\mu$ with respect to the proper time. Note that Mathisson–Papapetrou–Dixon evolution equations describe only the evolution of the momentum and the spin, while the quadrupole moment is determined by the matter structure of the body.

Let us now briefly discuss the multipoles of the body in the pole-dipole-quadrupole approximation. The mass of the body corresponding to the monopole can be defined as $m = -U^\mu P_\mu$ or as $m^2 = -P^\mu P_\mu$. These two masses are in general not equal, but it is interesting to notice that contractions between the vectors $(P^\mu - m U^\mu)$, $U^\mu$, $P^\mu$ lead to the relations

$$
(P^\mu - m U^\mu)(P_\mu - m U_\mu) = -\mu^2 + m^2,
$$

$$
(P^\mu - m U^\mu) P_\mu = -\mu^2 + m^2, \quad (P^\mu - m U^\mu) U_\mu = 0,
$$

which imply that if $\mu^2 = m^2$, then $P^\mu = m U^\mu$, since $(P^\mu - m U^\mu)$ cannot be a null-vector \[9\]. The spin tensor is associated with the dipole moment of the body, and provides a nonzero contribution to the acceleration due to the coupling with the background Riemann tensor. An identical phenomenon occurs when the particle is endowed with the quadrupole moment. In general, the quadrupole tensor $J^{\alpha\beta\gamma\delta}$ has a more complicated and nontrivial physical description than the spin tensor \[44–47\]. The total quadrupole moment is composed from different parts, i.e. the mass quadrupole $Q^{\alpha\beta}$, the flow quadrupole $\Pi^{\alpha\beta\gamma}$ and the stress quadrupole $\tau^{\alpha\beta\gamma\delta}$ \[44\]. Following the conventions introduced in \[44\], the quadrupole components can be written as

$$
\tau^{\alpha\beta\gamma\delta} = J^{\alpha\beta\gamma\sigma} (h^\nu_\sigma)^\delta, \quad \Pi^{\alpha\beta\gamma} = -J^{\alpha\sigma\beta\delta} (h^\nu_\delta)^\gamma \nu_\sigma, \quad Q^{\alpha\beta} = \frac{4}{3} J^{\alpha\gamma\beta\delta} \nu_\gamma \nu_\delta,
$$

$\odot$ Springer
for a generic time-like vector $V^\mu$ satisfying $V^\mu V_\mu = -1$ [46], where $(h V)^{\delta}_\sigma$ is a projection operator defined as

\[(h V)^{\delta}_\sigma = \delta^\delta_\sigma + V^\delta V_\sigma.\]  
\[\text{(5)}\]

This operator projects any vector on a local rest frame orthogonal to the timelike vector field $V^\alpha$. The quadrupole components are spatial with respect to the reference vector $V^\mu$ [46], i.e.

\[V^\alpha \tau^{\alpha\beta\gamma\delta} = V^\mu \Pi^{\alpha\beta\gamma} = V^\alpha Q^{\alpha\beta} = 0.\]  
\[\text{(6)}\]

In this decomposition framework the quadrupole moment reads [48]

\[J^{\alpha\beta\gamma\delta} = \tau^{\alpha\beta\gamma\delta} - 3 V^\alpha Q^{\beta}\gamma \delta - V^\beta \Pi^{\beta\gamma\delta} - V^\gamma \Pi^{\alpha\beta}.\]  
\[\text{(7)}\]

The presence of a dipole and quadrupole moment has an impact on the relation between the four-momentum $P^\mu$ and the four-velocity $U^\mu$. Namely, these four-vectors are not, in general, parallel as they are for the geodesic motion. In fact, $P^\mu$ can be split in a part parallel $P^\mu_\parallel$ and a part orthogonal $P^\mu_{\text{hid}}$ to $U^\mu$ [20], i.e.

\[P^\mu = P^\mu_\parallel + P^\mu_{\text{hid}},\]  
\[\text{(8)}\]

where the orthogonal part is called hidden momentum and is defined as

\[P^\mu_{\text{hid}} = (h U)^{\mu}_\nu P^\nu,\]  
\[\text{(9)}\]

which reduces the parallel part to

\[P^\mu_\parallel = m U^\mu.\]  
\[\text{(10)}\]

The dependence of the hidden momentum on the multipole moments can be seen, by contracting Eq. (2) with $U_\nu$, i.e.

\[P^\mu_{\text{hid}} = (K^\mu_\nu - \dot{S}^\mu_\nu) U_\nu,\]  
\[\text{(11)}\]

where the second-rank anti-symmetric tensor $K^\mu_\nu = \frac{4}{3} J^{\alpha\beta\gamma\delta} [\alpha R^\gamma_{\nu\alpha\beta}]$ has been defined. There is, however, a way to eliminate the quadrupole hidden momentum in some cases as we discuss in the following section.

### 2.1 The Ohashi–Kyrian–Semerák spin supplementary condition

The Mathisson–Papapetrou–Dixon equations are not sufficient to close the system, a spin supplementary condition is needed to fix the centre of mass, i.e. the centroid. In
general a spin supplementary condition can be written as

\[ V_{\mu} S^{\mu\nu} = 0. \]  \hspace{1cm} (12)

As was already mentioned, in our analysis the Ohashi–Kyrian–Semerák spin supplementary condition shall be employed. According to this spin supplementary condition [9], a vector \( V^\mu = w^\mu \), i.e. \( S^{\mu\nu} w_\nu = 0 \), is chosen such that \( \dot{w}^\mu = 0 \) and \( w^\mu w_\mu = -1 \). This implies that \( \ddot{S}^{\mu\nu} w_\mu = 0 \), \( \dddot{S}^{\mu\nu} w_\mu = 0 \) and all the contractions of \( w^\mu \) with the higher covariant derivatives of spin are equal to zero as well. Using this fact the contraction of Eq. (2) with \( w_\mu \) leads to

\[ P^\mu = \frac{1}{-w_\nu U^\nu} [(-P^\gamma w_\gamma) U^\mu + K^{\mu\delta} w_\delta]. \]  \hspace{1cm} (13)

It is obvious from Eq. (13) that the hidden momentum vanishes for Ohashi–Kyrian–Semerák spin supplementary condition in the pole-dipole approximation allowing the four-velocity and the four-momentum to be parallel [9]. At pole-dipole approximation relation Eq. (13) when contracted with \( U^\mu \) suggests that

\[ m = P^\nu w_\nu. \]  \hspace{1cm} (14)

In the original work Ref. [9], the fact that \( P^\mu = m \ U^\mu \) holds for Ohashi–Kyrian–Semerák spin supplementary condition in the pole-dipole approximation has been achieved by following a different way. In particular, in Ref. [9] successive contractions of Eq. (2)

\[ P^\mu \ddot{S}^{\mu\nu} w_\nu = (m^2 - \mu^2) P^\kappa w_\kappa = 0, \]  \hspace{1cm} (15)

\[ U^\mu \ddot{S}^{\mu\nu} w_\nu = (m^2 - \mu^2) U^\kappa w_\kappa = 0, \]  \hspace{1cm} (16)

led to \( \mu^2 = m^2 \), since \( \ddot{S}^{\nu\kappa} w_\kappa = 0 \) and \( w^\mu \), \( U^\mu \), \( P^\mu \) are time-like vectors. Having found that \( \mu^2 = m^2 \), the relations in Eq. (3) imply that \( P^\mu = m \ U^\mu \), i.e. that the hidden momentum vanishes. The feature of a vanishing hidden momentum in the pole-dipole approximation is what makes Ohashi–Kyrian–Semerák a very interesting spin supplementary condition. However, even if Ohashi–Kyrian–Semerák spin supplementary condition holds for any multipole approximation, the vanishing hidden momentum feature does not. Keeping this in mind we discuss under which condition, this feature can hold in the pole-dipole-quadrupole approximation.

In [20], it has been shown that by using the hidden momentum definition on Eq. (13) leads to

\[ P^\mu_{\text{hid}} = -\frac{1}{w_\nu U^\nu} (U^\mu U_\kappa + \delta^\mu_\kappa) K^{\kappa\gamma} w_\gamma. \]  \hspace{1cm} (17)
Hence, to eliminate the hidden momentum, one needs such $w^\mu$ that
\begin{equation}
(\mathcal{U}^\mu \mathcal{U}_\kappa + \delta^\mu_\kappa) K^{\kappa\gamma} w_\gamma = 0. \tag{18}
\end{equation}
Note, however, that the above condition after it is contracted with $w_\mu$ leads to
\begin{equation}
w_\mu \mathcal{U}^\mu \mathcal{U}_\sigma K^{\sigma\gamma} w_\gamma = 0. \tag{19}
\end{equation}
Since both $w_\mu$ and $\mathcal{U}^\mu$ are time-like vectors $w_\mu \mathcal{U}^\mu \neq 0$, then $\mathcal{U}_\sigma K^{\sigma\gamma} w_\gamma = 0$, which in turn implies because of Eq. (18) that
\begin{equation}
K^{\mu\gamma} w_\gamma = 0. \tag{20}
\end{equation}
Hence, the latter is actually the condition for the Ohashi–Kyrian–Semerák spin supplementary condition in the pole-dipole-quadrupole approximation leading to a vanishing hidden momentum.

We can crosscheck this condition with the successive contractions of Eq. (2) in the pole-dipole-quadrupole approximations. These contractions read
\begin{align}
P^\mu \dot{S}_{\mu\nu} \dot{S}^{\nu\kappa} w_\kappa &= (m^2 - \mu^2) P^\kappa w_\kappa + K^{\kappa\sigma} K_{\sigma\mu} P^\mu w_\kappa + K_{\sigma\mu} \mathcal{U}^\sigma P^\mu P^\kappa w_\kappa \\
&\quad + K^{\kappa\sigma} w_\kappa (\mu^2 \mathcal{U}_\sigma - mP_\sigma) = 0, \tag{21}
\end{align}
and once the condition Eq. (20) is applied, we are led to
\begin{equation}
m^2 - \mu^2 + K_{\sigma\mu} \mathcal{U}^\sigma P^\mu = 0, \tag{23}
\end{equation}
since the contraction between time-like vectors cannot be zero. Contracting Eq. (2) with $\mathcal{U}_\sigma P_\mu$ and taking into account Eq. (23) results in
\begin{equation}
\dot{S}^{\sigma\mu} \mathcal{U}_\sigma P_\mu = 0. \tag{24}
\end{equation}
The latter has to hold for any four-velocity and four-momentum, hence $P^\mu \parallel \mathcal{U}^\mu$. This in turn implies through Eq. (23) that $\mu^2 = m^2$. The approximation independent relations Eq. (3) are consistent with this result, since it implies that if $\mu^2 = m^2$, then $P^\mu = m \mathcal{U}^\mu$.

Let us explore some further consequences of the condition Eq. (20):

- Since $P^\mu \parallel \mathcal{U}^\mu$, Eq. (2) is reduced to
  \begin{equation}
  \dot{S}^{\mu\nu} = K^{\mu\nu}. \tag{25}
  \end{equation}
- Under condition Eqs. (20) and (13) implies that Eq. (14) holds also for the pole-dipole-quadrupole approximation.
Since $\dot{w}^\mu = 0$, then it should hold that $\dot{K}^{\mu\nu} w_\mu = 0$, $\ddot{K}^{\mu\nu} w_\mu = 0$ and all the contractions of $w^\mu$ with the higher covariant derivatives of $K^{\mu\nu}$ are equal to zero as well, as it holds for the spin tensor. This implies a coupling of the evolution of the quadrupole moment with the derivatives of the Riemann tensor.

Note that contrary to the Mathisson–Pirani and Tulczyjew–Dixon spin supplementary conditions, where the reference vectors are specific and have physical interpretation, for the Ohashi–Kyrian–Semeráč spin supplementary condition there is a freedom of how to choose the reference vector, which should allow condition Eq. (20) to hold for some cases. In Sects. 3.2 and 4.2 we verify for a specific quadrupole component, orbital setups and specific $w^\mu$ choices that this condition is satisfied. However, it should be stressed that these cases appear to be the exception of the rule, i.e. we do not expect condition Eq. (20) to hold in general.

Having chosen a framework for which $P^\mu = m U^\mu$ holds, it is somehow intuitive to assume at first that $\dot{m} = 0$, but this is not in general the case. In particular, contracting Eq. (1) with $U_\mu$ and taking into account that $\dot{P}^\mu = \dot{m} U^\mu + m \dot{U}^\mu$, leads to

$$\dot{m} = \frac{1}{6} J^{\alpha\beta\gamma\delta} U_\mu \nabla^\mu R_{\alpha\beta\gamma\delta}. \quad (27)$$

Hence, only if the rhs of Eq. (27) is zero, $m$ is a conserved quantity. In this case, since $m = \mu$, $\mu$ is conserved as well. If the mass is a constant of motion, then Eq. (26) reduces to $\dot{P}^\mu = m \dot{U}^\mu$ and the proportionality of $P^\mu$ and $U^\mu$ holds for every higher derivatives of these vectors. Another way to address the conservation of the mass issue is to take the derivative of Eq. (14), this leads to

$$\dot{m} = \frac{(\dot{P}^\nu - m \dot{U}^\nu) w_\nu}{U^\nu w_\nu}. \quad (28)$$

In order for the above expression to be zero, either $\dot{P}^\nu = m \dot{U}^\nu$ or the vector $\dot{P}^\nu - m \dot{U}^\nu$ is spacelike. We have no reason to assume a priori either.

The spin measure

$$S^2 = \frac{1}{2} S^{\mu\nu} S_{\mu\nu}, \quad (29)$$

is not a conserved quantity as well, since

$$\dot{S}^2 = S_{\mu\nu} K^{\mu\nu}, \quad (30)$$

where Eq. (25) has been employed. Contrary, to the pole-dipole approximation case, for which the conservation of the mass and the spin measure depends solely on the choice of a spin supplementary condition [8], in the pole-dipole-quadrupole approximation...
the conservation of these quantities depends on the background geometry, see, e.g., Eqs. (27) and (30). Actually, a background symmetry expressed by a Killing vector field $\xi^\mu$ provides a constant of motion \[44\]

$$ C(\xi) = P^\mu \xi_\mu + \frac{1}{2} S^{\nu\mu} \nabla_\nu \xi_\mu. \quad (31) $$

Note that the quadrupole moment imparts no contribution to the conserved quantities $C(\xi)$.

### 2.2 Kerr spacetime

Our work concerns Schwarzschild and Kerr spacetimes. The Kerr black hole spacetime is described by an elegant multipole structure \[49–52\], which depends only on the mass $M$ of the black hole and the Kerr parameter $a$, i.e. its spin. The line element of a Kerr spacetime in the Boyer-Lindquist coordinates \{t, r, \(\theta\), \(\phi\)\} reads

$$ ds^2 = g_{tt} dt^2 + 2g_{t\phi} dt d\phi + g_{\phi\phi} d\phi^2 + g_{rr} dr^2 + g_{\theta\theta} d\theta^2, $$

where the metric coefficients are

$$ g_{tt} = -\left( 1 - \frac{2Mr}{\Sigma} \right) , \quad g_{t\phi} = -\frac{2aMr \sin^2 \theta}{\Sigma} , \quad g_{\phi\phi} = \frac{(\varpi^2 - a^2 \Delta \sin^2 \theta) \sin^2 \theta}{\Sigma} , $$

$$ g_{\theta\theta} = \Sigma , \quad g_{rr} = \frac{\Sigma}{\Delta}, \quad (32) $$

with

$$ \Sigma = r^2 + a^2 \cos^2 \theta , \quad \Delta = \varpi^2 - 2Mr , \quad \varpi^2 = r^2 + a^2. \quad (33) $$

If we set $a = 0$ in the above metric elements, we get the Schwarzschild spacetime.

In the Kerr spacetime, we can construct an orthonormal tetrad basis, which can simplify the computations significantly \[53\]. The explicit expressions for this tetrad field are given below

$$ e^{(0)}_\mu = \left( \sqrt{\frac{\Delta}{\Sigma}}, 0, 0, -a \sin^2 \theta \sqrt{\frac{\Delta}{\Sigma}} \right) , \quad e^{(1)}_\mu = \left( 0, \sqrt{\frac{\Sigma}{\Delta}}, 0, 0 \right) , $$

$$ e^{(2)}_\mu = \left( 0, 0, \sqrt{\Sigma}, 0, 0 \right) , \quad e^{(3)}_\mu = \left( \frac{-a \sin \theta}{\sqrt{\Sigma}}, 0, 0, \frac{r^2 + a^2}{\sqrt{\Sigma}} \sin \theta \right). \quad (34) $$

The metric tensor is related with the tetrad field through the relation $g_{\mu\nu} = e^{(\alpha)}_\mu e_{(\alpha)\nu}$. 

\[\copyright\] Springer
The Kerr spacetime consists of a timelike Killing field $\xi^\mu_{(t)}$ that gives through Eq. (31) the energy

$$C_t = - \left( P_t + \frac{1}{2} S^{\mu\nu} \partial_\mu g_{t\nu} \right),$$  
(35)

and a space-like one $\xi^\mu_{(\phi)}$ that gives through Eq. (31) a component of the total angular momentum

$$C_\phi = \left( P_\phi + \frac{1}{2} S^{\mu\nu} \partial_\mu g_{\phi\nu} \right).$$  
(36)

for an extended test body.

### 2.3 The geodesic limit

Having introduced and discussed the framework of the Mathisson–Papapetrou–Dixon equations in the pole-dipole-quadrupole approximation under the Ohashi–Kyrian–Semerák spin supplementary condition, let us now focus on the special cases when such an extended body follows a geodesic trajectory. For a body to follow a geodesic orbit, we set

$$\dot{U}^\mu = 0.$$  
(37)

Note, that even if the latter holds, if $\dot{m} \neq 0$ and/or $\dot{P}_h \neq 0$, then $\dot{P}_I \neq 0$. We have showed that if $w^\mu$ is chosen such that Eq. (20) holds, then $P^\mu_h = 0$, so the only obstacle for $\dot{U}^\mu = 0$ to imply $\dot{P}_I = 0$ is that one needs $\dot{m} = 0$.

In the special case that $\dot{m} = 0$, given that the particle is following a geodesic on a Kerr spacetime, for which it holds that $U_t$ and $U_\phi$ are constants of motion, then $P_t$ and $P_\phi$ are constants as well. This in turn implies, via Eqs. (35) and (36), that $\frac{1}{2} S^{\mu\nu} \partial_\mu g_{t\nu}$ and $\frac{1}{2} S^{\mu\nu} \partial_\mu g_{\phi\nu}$ are conserved too. Therefore, to sum up:

$$E_g = - P_t = \text{constant}, \quad E_s = - \frac{1}{2} S^{\mu\nu} \partial_\mu g_{t\nu} = \text{constant},$$

$$L_z = P_\phi = \text{constant}, \quad J_s = \frac{1}{2} S^{\mu\nu} \partial_\mu g_{\phi\nu} = \text{constant}.$$  
(38)

Note that the contributions from the dipole moment are only contained in the terms $E_s$ and $J_s$.

With this, we finish our discussion on the underlying formalism and the key equations. In the following sections, we employ the above discussed framework in some examples to illustrate various possibilities related to the geodesic limit [Eq. (37)] of an extended body. In these scenarios, whenever it is needed to consider the quadrupole

---

1 A similar phenomenon also appears in electrodynamics, and for details, we refer our readers to Refs. [20, 21, 23, 33].
moment, we restrict the analysis to the mass quadrupole moment, for which the quadrupole tensor Eq. (7) reduces to

\[ J^\alpha\beta\gamma\delta = -3w^{[\alpha}Q^{\beta]\gamma\delta]w^{\beta}\]

\[ = -\frac{3}{4}(w^\alpha Q^\beta w^\delta - w^\beta Q^\alpha w^\delta - w^\alpha Q^\beta w^\gamma + w^\beta Q^\alpha w^\gamma), \quad (39) \]

and in particular to the spin-induced quadrupole model case, for which

\[ Q^\beta\gamma = C_S^2 S^\alpha_{\alpha}S^{\alpha\gamma}, \quad (40) \]

where \( C_S^2 \) is a constant depending on the internal structure of the body [43].

3 Geodesic trajectories of an extended body in the Schwarzschild background

We start our examples with the Schwarzschild spacetime, for which the non-vanishing metric coefficient Eq. (32) reduce to

\[ g_{tt} = -f = -(1 - 2M/r), \quad g_{\phi\phi} = r^2 \sin^2 \theta, \quad g_{\theta\theta} = r^2, \quad g_{rr} = f^{-1}. \quad (41) \]

3.1 Pole-dipole body

The equations of motion for a pole-dipole test body are given by Eqs. (1) and (2) when \( J^\alpha\beta\gamma\delta = 0 \). In order to have an extended body following a geodesic trajectory, we need to ensure that both Mathisson–Papapetrou–Dixon and geodesic equations are simultaneously satisfied. For a geodesic orbit in Schwarzschild spacetime we have

\[ \dot{U}_t = -\tilde{E}_g, \quad \dot{U}_r = \pm \left\{ \tilde{E}_g^2 - f \left( 1 + \frac{\tilde{L}_z^2}{r^2} \right) \right\}^{1/2} f^{-1}, \]

\[ \dot{U}_\theta = 0, \quad \dot{U}_\phi = \tilde{L}_z, \quad (42) \]

where \( \tilde{E}_g \) and \( \tilde{L}_z \) denote the conserved specific energy and orbital angular momentum. Without loss of generality, because of the spherical symmetry, we have set for simplicity \( U^\theta = 0 \) to constrain the motion on the equatorial plane.

Since, for a pole-dipole body under Ohashi–Kyrian–Semerák spin supplementary condition \( \dot{m} = 0 \), Eq. (37) is equivalent to \( \dot{P}^\mu = 0 \), we arrive to

\[ \dot{P}^t = \frac{M}{r^3} \left\{ 2S^{t\tau}U_t - S^{t\phi}U_\phi \right\} = 0, \quad \dot{P}^\tau = -\frac{M}{r^3} \left\{ 2S^{t\tau}U_t - 2S^{\phi\tau}U_\phi \right\} = 0, \]

\[ \dot{P}^\phi = \frac{M}{r^3} \left\{ -2S^{\phi\phi}U_\phi - S^{\phi\tau}U_t - S^{\phi\tau}U_\tau \right\} = 0, \quad \dot{P}^\phi = -\frac{M}{r^3} \left\{ S^{\phi\phi}U_\phi + S^{\phi\tau}U_\tau \right\} = 0. \quad (43) \]
Rearranging the above equations leads to

\[ S^{tr} = \frac{S^{t\phi} L_z}{2U_r}, \quad S^{t\theta} = -\frac{S^{\theta r} U_r + 2S^{\phi\theta} L_z}{E_g}, \quad S^{t\phi} = -\frac{S^{\phi r} U_r}{E_g}. \]  

(44)

There is no fourth equation, because from Eq. (43) only three of the four equations are linearly independent. Having constrained the motion of a spinning body on the equatorial plane one would expect that \( S^{\theta \mu} = 0 \) [54], however, when the spinning body is following a geodesic the spin should not be able to affect the motion, since the spin curvature coupling is annihilated Eq. (43), hence the \( S^{\theta \mu} \) components are not necessarily zero. Taking into account Eqs. (35), (36), (38) and (44), the conserved quantities can be rewritten as

\[ C_t = -P_t - \frac{L_z MS_t}{2r^2 U_r}, \quad E_s = -\frac{L_z MS_t}{2r^2 U_r}, \quad C_{\phi} = P_{\phi} - rS^{\phi r}. \quad J_s = -rS^{\phi r}. \]

(45)

Since the Schwarzschild spacetime is spherical symmetric, there are two extra constants of motion

\[ C_x = -\sin \phi P_\theta - \cos \phi \cot \theta P_\phi + r^2 \cos \phi \sin^2 \theta S^{\phi \theta} + r \sin \phi S^{\phi r} + r \cos \phi \sin \theta \cos \theta S^{\phi r}, \]

(46)

\[ C_y = \cos \phi P_\theta - \sin \phi \cot \theta P_\phi + r^2 \sin \phi \sin^2 \theta S^{\phi \theta} - r \cos \phi S^{\phi r} + r \sin \phi \sin \theta \cos \theta S^{\phi r}, \]

(47)

which on the equatorial plane for the geodesic motion reduce to

\[ C_x = r^2 \cos \phi S^{\phi \theta} + r \sin \phi S^{\phi r}, \]

\[ C_y = r^2 \sin \phi S^{\phi \theta} - r \cos \phi S^{\phi r}. \]

(48)

Having enforced geodesic motion, let us now see what are the consequences on the conserved quantities \( E_s, J_s, C_x \) and \( C_y \) involving spin contributions. The conservation of \( E_s \) implies that

\[ \frac{dE_s}{d\tau} = -\frac{ML_z}{2} \left\{ \frac{1}{r^2 U_r} \frac{dS^{t\phi}}{d\tau} - \frac{2S^{t\phi} U^{r}}{r^3 U_r} - \frac{S^{t\phi}}{r^2 U_r^2} \frac{dU_r}{d\tau} \right\} = 0, \]

(49)

which can be expressed by using the parallel transport of the spin tensor evolution and the geodesic equation as follows

\[ \frac{ML_z}{2} \left\{ \frac{1}{r^2 U_r} \left[ -\left( \Gamma^t_{\alpha \beta} S^{\alpha \phi} U^{\beta} + \Gamma^\phi_{\alpha \beta} S^{t \alpha} U^{\beta} \right) \right] - \frac{2S^{t\phi}}{r^3 g_{rr}} - \frac{S^{t\phi}}{r^2 U_r^2} \left( \Gamma^\alpha_{\beta \rho} U^{\beta} U_{\alpha} \right) \right\} = 0. \]

(50)
By expanding the above equation, we reach
\[
\frac{ML_z S^{\phi r}}{4(r-2M)^2 r^5 \tilde{E}_g U_r} \left\{ L_z^2 (r^2 - 8M r + 12M^2) - 2r^2 (r^2 - 6M r) (\tilde{E}_g^2 - 1) + 16M^2 r^2 \right\} = 0.
\] (51)

For an equatorial orbit the above equation can be zero if $S^{\phi r} = 0$ or $L_z = 0$, or the expression in the curl brackets is zero. The latter would imply that a spinning body can follow circular an equatorial geodesic orbit.

Let us breakdown these three cases to see whether they are possible and where do they lead to.

- If we assume that there is a circular orbit, then, since $U_r = 0$, Eq. (44) implies that $S^{t\phi} = 0$. By substituting this in the $\dot{S}^{t\phi} = 0$ equation, we find that $E = L_z = 0$ is needed. However, such a solution is inconsistent with the $U^\nu U_\nu = -1$ constraint. Hence, a spinning body under Ohashi–Kyrian–Semerák spin supplementary condition moving in the Schwarzschild background cannot follow a geodesic circular orbit.

- If we assume that $S^{\phi r} = 0$ and $L_z \neq 0$, then Eq. (44) implies that both $S^{t\nu}$ and $S^{t\phi}$ vanish as well. This leads in turn from Eq. (45) that $E_s = J_s = 0$. If we combine the expansions of the $\dot{S}^{t\mu} = 0$ equations with the middle relation of Eq. (44), then we arrive to $L_z (S^{t\nu} U^\phi + S^{\phi\phi} U^\nu) = 0$, which is true if $S^{t\mu} = 0$, but then the body is left without spin. If the spin tensor components are not zero, then after some further manipulations of the $\dot{S}^{t\mu} = 0$ equations, during which we take into account that $S^{t\nu} U^\phi + S^{\phi\phi} U^\nu = 0$, we end up with $\frac{dU^\nu}{d\tau} = -\frac{L_z^2}{r^3}$. This result is inconsistent with the geodesic equations. Hence, $S^{\phi r} = 0$ option cannot provide a geodesic orbit for a spinning body under Ohashi–Kyrian–Semerák spin supplementary condition.

- For $L_z = 0$, Eq. (45) implies that $E_s = 0$ and Eq. (44) implies that $S^{t\nu} = 0$. It can be shown
  - by expressing $\dot{S}^{\theta\nu} = 0$ in terms of Christoffel symbols that $r S^{t\theta} = K_r$ is constant.
  - by expressing $\dot{S}^{\phi\phi} = 0$ in terms of Christoffel symbols that $r^2 S^{\phi\phi} = K_\phi$ is constant.
  - from Eq. (48) that
    \[
    C_x = \cos \phi K_\phi - \sin \phi K_r,
    C_y = \sin \phi K_\phi + \cos \phi K_r.
    \] (52)

    Hence, for the radial motion during which $\phi$ is constant, $K_r$ and $K_\phi$ are recast expressions of $C_x$ and $C_y$.
  - by expressing $\dot{S}^{t\phi} = 0$ in terms of Christoffel symbols that $r^2 S^{t\phi} = C_x$ is constant, which is just reproducing $J_s$.
  - from the middle relation of Eq. (44) that $S^{\theta t} = -(K_r U_r) / (r \tilde{E}_g).$
by expressing $\dot{S}^r$, $\dot{S}^\theta$, $\dot{S}^\phi$ and $\dot{S}^t$ in terms of Christoffel symbols that they just trivially vanish.

Note that $J_s$, $C_x$ and $C_y$ reemerged naturally, so they are consistent with the radial geodesic motion of a spinning body. Moreover, since $S^\phi_t$ can also be expressed from the third relation in Eq. (43), the spin tensor takes the following form

$$\begin{pmatrix}
0 & S^r & S^\theta & S^\phi \\
-S^r & 0 & S^r & S^r \\
-S^\theta & -S^\phi & 0 & S^\phi \\
-S^\phi & -S^r & -S^\theta & 0
\end{pmatrix}$$

which shows that after fixing the initial conditions the spin tensor is just function of $r$, since $U_r(r, \tilde{E}_g, J_s)$ [Eq. (42)]. In conclusion, $L_z = 0$ corresponds to a consistent solution of the Mathisson–Papapetrou–Dixon equations, which shows a spinning body falling radially into a Schwarzschild black hole following a geodesic trajectory. An interesting outcome of the above calculation is that the condition $S^\alpha\beta U_\beta = 0$, which is the Mathisson–Pirani spin supplementary condition, naturally emerges from Eq. (44). Furthermore, since for Ohashi–Kyrian–Semerákov spin supplementary condition it holds that $P_\mu || U^\mu$, we have that the centroids of the Ohashi–Kyrian–Semerákov, Mathisson–Pirani and Tulczyjew–Dixon ($S^{\mu\nu} P_\mu = 0$) spin supplementary conditions coincide. This fact was already noticed in [21], and the corresponding trajectory is known in the literature [20, 21, 33]. The novel result is that the radial motion is the only geodesic solution for a spinning body under Ohashi–Kyrian–Semerákov spin supplementary condition in the Schwarzschild spacetime.

### 3.2 Pole-dipole-quadrupole body

Let us now examine whether a pole-dipole quadrupole body under Ohashi–Kyrian–Semerákov spin supplementary condition can follow a geodesic trajectory. Since the geodesic motion on Schwarzschild background is equatorial, i.e. $U_\theta = 0$, the parallel transport of the $\theta$ component of the vector $w^\mu$ reduces to

$$\frac{d w^\theta}{d \tau} + \frac{U^r}{r} w^\theta = 0,$$

which translates to $r w^\theta = \text{constant}$. We assume for simplicity that this constant is zero, hence $w^\theta = 0$.

Our investigation focuses on a solely spin induced quadrupole moment given by Eq. (40). Since in Sect. 3.1 we have shown that only a radially moving pole-dipole body
in a Schwarzschild background can follow a geodesic orbit, it would be interesting to see if this is also the case when the spin induced quadrupole moment is added to the model. Hence, we study only the radial geodesic Eq. (42) and check whether it can be a solution to the Mathisson–Papapetrou–Dixon equations, as in the pole-dipole case Sect. 3.1. On this radial orbit, \( w_\phi \) follows an equation similar to Eq. (54), which allows us to assume again for simplicity that \( w_\phi = 0 \). The other components of the reference vector \( w_\mu \) read

\[
\frac{dw_t}{d\tau} = \frac{M}{r^2} \left( U_t w_t - U_t w_r \right),
\]

\[
\frac{dw_r}{d\tau} = -M \left[ \frac{U_t w_t}{(r - 2M)^2} + \frac{U_r w_r}{r^2} \right].
\] (55)

Note that Eqs. (42) and (55) lead to \( \frac{d}{d\tau} (U_t w_r + U_r w_t) = 0 \), which implies that along the radial geodesic trajectory \( -U_t w_r + U_r w_t = -C_r = \text{constant} \).

From the spin supplementary condition equations and by using \( w_\theta = w_\phi = 0 \), (56)

we obtain

\[
S^{tr} = 0, \quad S^{t\theta} = -S^{t\theta} w_t / w_r, \quad S^{r\phi} = -S^{t\phi} w_t / w_r.
\] (57)

When Eqs. (56) and (57) are implemented on \( K^{\mu\nu} \) leads to the following non-diagonal components of the tensor:

\[
K^{tr} = 0, \quad K^{t\theta} = \frac{3C S^2 M S^{r\phi} S^{r\phi} w_r}{r w_t}, \quad K^{t\phi} = \frac{-3C S^2 MS^{r\theta} S^{r\phi} w_r}{r w_t},
\]

\[
K^{r\theta} = \frac{-3C S^2 MS^{r\phi} S^{r\phi}}{r}, \quad K^{r\phi} = \frac{3C S^2 MS^{r\phi} S^{r\phi}}{r}.
\] (58)

Using the above expressions, it is straightforward to see that all the four components of \( K^{\mu\nu} w_\nu \) identically vanish. Hence, the hidden momentum vanishes, i.e. \( P^\mu = mU^\mu \), on a radial trajectory of the under investigation extended body.

Having verified that the hidden momentum vanishes, we can now investigate Eq. (1), which can be rewritten as

\[
mU^\mu + \frac{1}{2} R_{\nu\alpha\beta}^\mu U^\nu S^{\alpha\beta} + \frac{1}{6} J^{\alpha\beta\gamma\delta} \nabla^\mu R_{\alpha\beta\gamma\delta} = 0,
\] (59)

since for a geodesic orbit \( \dot{U}^\mu = 0 \).

**Time component:** For the time component \( \mu = t \), Eq. (59) reduces to \( \dot{m} = 0 \), since using Eqs. (56) and (57) and being on a radial geodesic leads to \( R_{\nu\alpha\beta}^t U^\nu S^{\alpha\beta} = 0 \),
\[ J^{\alpha \beta \gamma \delta} \nabla^\ell R_{\alpha \beta \gamma \delta} = 0 \] and \( U^\mu \neq 0 \). When we expand Eq. (27) and recall that \( w^\mu w_\mu = -1 \), we obtain

\[
\dot{m} = \frac{3MC_S^2(r - 2M)U_r}{2r^3w_r^2} \left\{ \left[ (S^{t\theta})^2 + (S^{t\phi})^2 \right] - 2r^2w_r^2(S^{\theta\phi})^2 \right\} = 0. \tag{60}
\]

This implies that \((S^{t\theta})^2 + (S^{t\phi})^2 = 2r^2w_r^2(S^{\theta\phi})^2 \). For the remaining three components of Eq. (59) we take into account that \( \dot{m} = 0 \).

**Radial component:** For the radial component \( \mu = r \), Eq. (59) reads

\[
\frac{3C_S^2M(r - 2M)}{2r^3w_r^2} \left\{ \left[ (S^{t\theta})^2 + (S^{t\phi})^2 \right] - 2r^2w_r^2(S^{\theta\phi})^2 \right\} = 0. \tag{61}
\]

Note that Eqs. (61) and (60) are referring to the same set of constraints on the spin tensor.

**Polar component:** For \( \mu = \theta \) component, Eq. (59) provides the relation

\[
MS^{t\theta}C_r \frac{C_r}{rw_r} + 3C_S^2MS^{t\phi}S^{\theta\phi}w_t = 0. \tag{62}
\]

**Azimuthal component:** For \( \mu = \phi \) component Eq. (59) provides the relation

\[
MS^{t\phi}C_r \frac{C_r}{rw_r} - 3C_S^2MS^{t\theta}S^{\theta\phi}w_t = 0. \tag{63}
\]

In Eqs. (62) and (63) there are both the dipole and the quadrupole contribution assuming that \( C_r \neq 0 \) and \( C_S^2 \neq 0 \). Having done this assumption, Eq. (63) can be written as

\[
S^{t\phi} = \frac{3C_S^2S^{t\theta}S^{\theta\phi}w_t w_r}{C_r}. \tag{64}
\]

Plugging this expression in Eq. (62), we obtain

\[
S^{t\theta} \left[ \left( \frac{C_r}{rw_r} \right)^2 + 3C_S^2S^{\theta\phi}w_t \right] = 0.
\]

The latter leads to \( S^{t\theta} = 0 \), since we have assumed that \( C_r \neq 0 \). In turn, Eq. (64) leads to \( S^{t\phi} = 0 \) and Eq. (61) to \( S^{\theta\phi} = 0 \). Moreover, from Eq. (57), we notice that all the other spin components vanish as well, i.e., \( S^{tr} = S^{t\theta} = S^{t\phi} = 0 \). Hence, the body could not be spinning.

If \( C_r = 0 \), then Eqs. (62) and (63) are simultaneously satisfied if \( S^{\theta\phi} = 0 \), which because of Eqs. (61) and (57) finally leads again to \( S^{\mu\nu} = 0 \). Note that the \( C_r = 0 \) case implies that \( U^\mu = w^\mu \). In order to show this, recall that both \( U^\mu \) and \( w^\mu \) are timelike future pointing unit vectors and they have only time and radial components.
Hence, $C_r = 0$ leads to $w_r^2 = U_r^2$, which in turn implies that $C_r = 0 \iff U^{[\mu} w^{\nu]} = 0$ gives $w^{\mu} = U^{\mu}$. Therefore, for $C_r = 0$ our discussion includes also the Mathisson–Pirani spin supplementary condition and the Tulczyjew–Dixon spin supplementary condition.

With $w^\theta = w^\phi = 0$, we are led to the conclusion that a pole-dipole-(spin induced) quadrupole body under Ohashi–Kyrian–Semerák spin supplementary condition cannot follow a radial geodesic trajectory. Note that we have not examined the cases for which $w^\theta = \text{constant}_1/r$ and $w^\phi = \text{constant}_2/r$. However, for continuity reasons with respect to constant $i$ ($i = 1, 2$), we would not expect this to change our conclusion.

### 4 Geodesic trajectories of an extended body in the Kerr background

The results we present for the Kerr background in this section reduce for $a = 0$ to the Schwarzschild case of Sect. 3. However, we find that since the Schwarzschild case is simpler than Kerr, one can gain more insight into the results discussed in our work. Therefore, we have presented the Schwarzschild case first, before discussing the more complicated Kerr case.

#### 4.1 Pole-dipole body

When dealing with the pole-dipole approximation in Kerr, we have found that the Mathisson–Papapetrou–Dixon equations simplify significantly when the tetrad field Eq. (34) is employed. However, this simplification is not sufficient to allow us to deal with generic orbits. Hence, we have confined our study on the equatorial plane by taking advantage of the fact that since $U(\theta) = e^{(2)}_2 U^\theta$ when $U^\theta = 0$, then $U(\theta) = 0$ as well. Since $\dot{m} = 0$ in the pole-dipole approximation, assuming geodesic motion $U(\nu) = 0$ implies that Eq. (1) reduces to

\[
\begin{align*}
\dot{P}^{(t)} &= \frac{M}{r^3} \left\{ 2S^{(t)(r)} U^{(r)} - S^{(t)(\phi)} U^{(\phi)} \right\} = 0, \\
\dot{P}^{(r)} &= \frac{M}{r^3} \left\{ 2S^{(t)(r)} U^{(t)} - S^{(r)(\phi)} U^{(\phi)} \right\} = 0, \\
\dot{P}^{(\theta)} &= \frac{M}{r^3} \left\{ 2S^{(\theta)(\phi)} U^{(\phi)} + S^{(r)(\theta)} U^{(r)} - S^{(t)(\theta)} U^{(t)} \right\} = 0, \\
\dot{P}^{(\phi)} &= \frac{M}{r^3} \left\{ S^{(r)(\phi)} U^{(r)} - S^{(t)(\phi)} U^{(t)} \right\} = 0,
\end{align*}
\]

(65)

From the first three expressions of above set of equations we get:

\[
\begin{align*}
S^{(t)(r)} &= \frac{S^{(t)(\phi)} U^{(\phi)}}{2U^{(r)}}, & S^{(t)(\phi)} &= \frac{S^{(r)(\phi)} U^{(r)}}{U^{(t)}}, \\
S^{(t)(\theta)} &= \frac{S^{(r)(\phi)} U^{(r)} + 2S^{(\theta)(\phi)} U^{(\phi)}}{U^{(t)}},
\end{align*}
\]

(66)
which is similar to that of the Schwarzschild case [Eq. (44)]. As in that case the fourth equation is trivially satisfied because of the second relation of Eq. (66).

In the tetrad field frame [Eq. (34)], the conserved quantities given in Eq. (38) read

$$E_s = -\frac{MS^{(t)(\phi)}U^{(\phi)}}{2r^2U^{(r)}}, \quad J_s = -\frac{\sqrt{\Delta}}{r}S^{(r)(\phi)}. \quad (67)$$

As in Sect. 3.1, we will check the consequences of having enforced the geodesic motion on the extended body by checking the above conserved quantities $E_s$, $J_s$. Let us start from

$$\frac{dE_s}{d\tau} = \frac{3MS^{(r)(\phi)}U^{(r)}U^{(\phi)}}{2r^4(U^{(t)})^2} \left\{ 2U^{(t)}\sqrt{\Delta} + aU^{(\phi)} \right\} = 0, \quad (68)$$

which implies the following cases:

- If $U^{(r)} = 0$, then the body has to follow circular geodesic orbits. By using Eq. (66), this leads to $S^{(t)(r)} = S^{(t)(\phi)} = 0$. If we now expand the expression $\dot{S}^{(t)(\phi)} = 0$, we get $(a^2 - Mr)U^{(t)} + a\sqrt{\Delta}U^{(\phi)} = 0$. Substituting this expression in the timelike condition, i.e., $[U^{(t)}]^2 - [U^{(\phi)}]^2 = 1$, we obtain, $[U^{(t)}]^2 = \frac{a^2\Delta}{r^2(a^2 - M^2)}$. Since $\Delta/(a^2 - M^2)$ is always negative outside the event horizon for a Kerr black hole, the case $U^{(r)} = 0$ does not hold.

- In the case that the quantity in the bracket is equal to zero, since $U^{(t)}$, $U^{(\phi)}$ are functions of the radius and the constants $E_g$, $L_z$, the body should be on a circular orbit of constant radius. A circular orbit implies again that $U^{(r)} = 0$. Hence, this case cannot hold.

- For the case that $S^{(r)(\phi)} = 0$, we get $S^{(t)(r)} = S^{(t)(\phi)} = 0$ from Eq. (66). However, if we differentiate the third expression in Eq. (66) with respect to $\tau$, we arrive at

$$\frac{3aU^{(\phi)}}{r^2} \left\{ aS^{(r)(\theta)}U^{(t)} - S^{(t)(\theta)}U^{(r)} \right\} + \sqrt{\Delta} \left\{ S^{(r)(\phi)}U^{(\phi)} - S^{(\phi)(\phi)}U^{(r)} \right\} = 0, \quad (69)$$

by using the fact that $\dot{S}^{(\mu)(\nu)} = 0$, $\dot{U}^{(\mu)} = 0$ and their expansions into Christoffel symbols and the spin connection. By expanding $\dot{S}^{(t)(\phi)} = 0$, one obtains $rS^{(t)(\phi)} = \text{constant}$. Combining the latter result, Eqs. (69) and (66) and the fact that $U^{(t)}$, $U^{(r)}$, $U^{(\phi)}$ are functions of the radius and the constants $E_g$, $L_z$ allows us to express all the non-vanishing spin components in terms of the radial coordinate $r$ and the constants $E_g$, $L_z$. However, plunging these expression of the spin components into $\dot{S}^{(t)(\phi)} = 0$ one runs into inconsistencies with the geodesic equation, on that ground, this case is excluded.

- If $U^{(\phi)} = 0$, then it is immediately implied that $L_z = aE_g$. From Eq. (67) we see that $E_s = 0$, while from Eq. (66) we get $S^{(t)(r)} = 0$. Moreover, it can be shown that
the expansion of $\dot{S}^{(r)}(\phi) = 0$ leads to $\sqrt{\Delta}S^{(r)}(\phi)/r = \text{constant}$, which is actually $J$. Interestingly, this expression can also be written as $S^{(r)}(\phi)/U^{(t)} = \text{constant}$, as we have $U^{(t)} = E_g r / \sqrt{\Delta}$.

– the expansion of $\dot{S}^{(t)}(\phi) = 0$ leads to $S^{(t)}(\phi)/U^{(r)} = \text{constant}$, which is in agreement with the middle relation in Eq. (66) when we take into account that $S^{(r)}(\phi)/U^{(t)} = \text{constant}$.

– $\dot{S}^{(r)}(\phi) = 0$ is trivially satisfied.

– by using the relations $\dot{S}^{(\theta)(\mu)} = 0$, we arrive at the following expressions:

$$\frac{d}{d\tau} \left( \frac{S^{(r)}(\theta)}{U^{(t)}} \right) = -\left( a/r^2 \right) S^{(\theta)(\phi)}$$,

$$\frac{d}{d\tau} \left( \frac{S^{(t)}(\theta)}{U^{(r)}} \right) = -\left( a/r^2 \right) S^{(\theta)(\phi)}$$,

$$\frac{dS^{(\theta)(\phi)}}{d\tau} = \left( a/r^2 \right) \left( S^{(r)}(\theta)/U^{(t)} \right)$$.  \( \tag{70} \)

It is interesting to note that the subtraction of the first two evolution equations in Eq. (70) leads to a conserved quantity, which in fact can be retrieved by the right relation of Eq. (66) for $U^{(\phi)} = 0$. This quantity reads

$$S^{(r)}(\theta)U^{(r)} - S^{(t)}(\theta)U^{(t)} = 0.$$

Note that in fact Eq. (66) for $U^{(\phi)} = 0$ implies that the Mathisson–Pirani spin supplementary condition holds along with Ohashi–Kyrian–Semerák, which in turn implies that Tulczyjew–Dixon spin supplementary condition is satisfied as well. To conclude, we have not only found a solution providing geodesic orbits for a spinning particle in Kerr background, but we have also shown that this solution is the only possible for equatorial orbits in Kerr under Ohashi–Kyrian–Semerák spin supplementary condition.

### 4.2 Pole-dipole-quadrupole body

Let us study now a pole-dipole-quadrupole body with spin induced quadrupole in a Kerr spacetime by following a similar route as in the Schwarzschild BH case Sect. 3.2. We assume that the particle follows the geodesic trajectory that we found in Sect. 4.1 to be compatible with a pole-dipole body. For this trajectory holds that $L_z = a E_g$ and the respective four-velocity reads:

$$U_t = -\tilde{E}_g, \quad U_r = r^2 U^t \Delta^{-1} = \pm r \left\{ \tilde{E}_g^2 r^2 - \Delta \right\}^{1/2} \Delta^{-1}, \quad U_\theta = 0, \quad U_\phi = \tilde{L}_z.$$  \( \tag{71} \)

Regarding the reference vector $w^\mu$, note that, as in the Schwarzschild case, for the equatorial motion in Kerr it holds that $w^\theta r = \text{constant}$, which allows us for simplicity to choose $w^\theta = 0$, which implies also $w^{(\theta)} = 0$. By expanding $\dot{w}_\mu = 0$ for the other
components of the reference vector along the under investigation geodesic trajectory we get:

\[
\frac{d w_t}{d \tau} = \frac{M}{r^2 \Delta} \left\{ \mathcal{U}^r \left[ (r^2 + a^2) w_t + a w_\phi \right] - M r^2 \mathcal{U}_t w^r \right\}, \tag{72}
\]

\[
\frac{d w_r}{d \tau} = \frac{1}{\Delta^2} \left[ r(a^2 - M r) \mathcal{U}^r w^r + \mathcal{U}_t \left( M(a^2 - r^2) + a(M - r) w_\phi \right) \right]. \tag{73}
\]

\[
\frac{d w_\phi}{d \tau} = \frac{1}{r^2 \Delta} \left\{ \mathcal{U}^r \left[ -a M(a^2 + 3r^2) w_t + (r^3 - 2Mr^2 - a^2 M) w_\phi \right] + a r^2 (r + M) \mathcal{U}_t w^r \right\}, \tag{74}
\]

A simple combination between the \(t\) and \(\phi\) component reveals that:

\[
a(r + M) \frac{d w_t}{d \tau} + \frac{d w_\phi}{d \tau} = \frac{M}{r} \mathcal{U}^r \left( w_\phi + aw_t \right). \tag{75}
\]

It is possible to rewrite the above relation into a more convenient form as:

\[
\frac{M}{r} (w_\phi + aw_t (1 + r/M)) = C_{rl} = \text{constant}. \tag{76}
\]

Note that in the tetrad field formulation, we have

\[
w(\phi) = w(r) = \frac{1}{r} (w_\phi + aw_t). \tag{77}
\]

We will discuss the implications of the above two relations below. For this discussion we need also the relation

\[
[w^r]^2 = \frac{1}{r^2 M^2} \left\{ \left[ (M^2 - a^2) w_t^2 + 2a C_{rl} w_t - (M^2 + C_{rl}^2) \right] r^2 + 2Mr[C_{rl}^2 + M^2 - a C_{rl} w_t] - a^2 M^2 \right\}, \tag{78}
\]

which comes from the fact that \(w^\mu w_\mu = -1\) and Eq. (76).

Our first aim is to check whether it is possible to have a vanishing hidden momentum in this setup. We start with the Ohashi–Kyrian–Semerákov spin supplementary condition, which introduces the following relations

\[
S^{(i)(r)} = -S^{(r)(\phi)} \frac{w(\phi)}{w(r)}, \quad S^{(i)(\phi)} = (S^{(r)(\phi)} \frac{w(r)}{w(r)} - S^{(\phi)(\phi)} \frac{w(\phi)}{w(r)}).
\]

\[
S^{(i)(\phi)} = S^{(r)(\phi)} \frac{w(r)}{w(r)}. \tag{79}
\]
Using the above expressions, we expand $K^{(\omega)(\beta)} w(\beta)$, and arrive at

$$K^{(t)(\beta)} w(\beta) = \frac{3MC^2 w(\phi)}{r^3 w(t)} \left( S^{(r)(\theta)} S^{(\theta)(\phi)} w^{(r)} + (S^{(r)(\theta)} - S^{(\theta)(\phi)}) w(\phi) \right),$$

$$K^{(r)(\beta)} w(\beta) = \frac{3MC^2 w(\phi)}{r^3 [w(t)]^2} \left( S^{(r)(\theta)} [w(t)]^2 + S^{(r)(\theta)} w^{(r)} w(\phi) \right),$$

$$K^{(\theta)(\beta)} w(\beta) = \frac{3MC^2 S^{(r)(\phi)} w(\phi)}{r^3 [w(t)]^2} \left( S^{(r)(\theta)} ([w(t)]^2 - [w^{(r)}]^2) + S^{(\theta)(\phi)} w^{(r)} w(\phi) \right),$$

$$K^{(\phi)(\beta)} w(\beta) = -\frac{3C^2 M w(\phi)}{r^2 [w(t)]^2} \left( [S^{(\theta)(\phi)}]^2 [w(t)]^2 + [S^{(r)(\phi)}]^2 ([w(t)]^2 - [w^{(r)}]^2) \right).$$

(80)

The above relations identically become zero if we chose $w^{(\phi)} = 0$. For $w^{(\phi)} = 0$, Eqs. (76) and (77) lead to $a w_t = C_{r\ell}$ and $u_{\phi} = -C_{r\ell}$. Note, that for $a = 0$ we get back to the Schwarzschild case, for which we have chosen $w_{\phi} = 0$. Moreover, for the above choices of $w_t$ and $w_{\phi}$, Eq. (78) reduces to

$$[w^{(r)}]^2 = \frac{C^2_{r\ell}}{a^2} - \frac{\Delta}{r^2}. \quad (81)$$

When we compare $w^f$ with $U^f$ from Eq. (71), we see that by choosing $C_{r\ell} = -a \tilde{E}_g = -\tilde{L}_z$, they become the same. This implies that $w^{\mu} = U^{\mu}$, i.e. the Ohashi–Kyrian–Smeráček spin supplementary condition and the Mathisson–Pirani spin supplementary condition centroids coincide. Since we have a vanishing hidden momentum also the Tulczyjew–Dixon spin supplementary condition coincides with the other two.

We stress that $C_{r\ell} = -a \tilde{E}_g$ is just an initial condition choice for the $w^{\mu}$. Namely, if we assume that initially $w^{\mu}$ is equal to $U^{\mu}$, and after that it follows its own evolution equation, i.e., $\dot{w}^{\mu} = 0$, we found that both $d w_t / d \tau$ and $d w_{\phi} / d \tau$ identically vanish. This would essentially mean that $w_t$ and $w_{\phi}$ remain constant throughout the evolution. Actually, if $w^{a}$ is initially set to be equal to $U^{a}$, it will remain equal to $U^{a}$ for the entire evolution. We confirmed this also numerically. Hence, setting $w^{(\phi)} = 0$ and $C_{r\ell} = -a \tilde{E}_g$ is equivalent to setting initially $w^{\mu} = U^{\mu}$. We can now state that $w^{(\phi)} = w^{(\theta)} = 0$ is a sufficient condition to have a vanishing hidden momentum in the present setup, as it is in Sect. 3.2 for the Schwarzschild case.

With the issue of choosing the vector $w^{\mu}$ settled, we now get back to the Mathisson–Papapetrou–Dixon equations. Having a vanishing hidden momentum allows us to use again Eq. (59), but now for the Kerr case. By using the tetrad field to express Eq. (59), we analyze the equations component after component.

**Time component:** For the $t$ component one immediately obtains $R^{(t)}_{(\omega)(\beta)(\gamma)(\delta)} U^{\mu} S^{(\nu)(\gamma)}$ $= 0$, and we get

$$\dot{m} U^{(t)} = -(1/6) J^{(\omega)(\beta)(\gamma)(\delta)} \nabla^{\mu} R_{(\omega)(\beta)(\gamma)(\delta)}.$$ 

The expressions for the quadrupole term and $\dot{m}$ are given below:
\[
\frac{1}{6} J^{(\alpha)(\beta)(\gamma)(\delta)} \nabla^{(t)} R_{(\alpha)(\beta)(\gamma)(\delta)} = -\frac{3aMCs^2}{r^5} S^{(r)(\theta)} S^{(\theta)(\phi)},
\]
\[
\dot{m} = \frac{3Cs^2M\sqrt{\Delta}}{2r^5[w(t)]^2} \left\{ \mathcal{U}^{(r)} \left( ([S^{(r)(\theta)}]^2 + [S^{(r)(\phi)}]^2] - 2[S^{(\theta)(\phi)}][w(t)]^2) \right) \right\}.
\] (82)

It is easy to notice that apart from the tetrad notation, the expression for \(\dot{m}\) remains identical to the Schwarzschild case, as shown in Eq. (60).

**Polar component:** For the \(\theta\)-component, we have \(\mathcal{U}^{(\theta)} = 0\), and it simply reduces to
\[
\frac{3MCs^2\sqrt{\Delta}S^{(r)(\phi)}S^{(\theta)(\phi)}}{r^5} = 0,
\] (83)
and we end up with \(S^{(r)(\phi)} = 0\) or \(S^{(\theta)(\phi)} = 0\).

**Azimuthal component:** For the \(\phi\)-component, we obtain
\[
\frac{3MCs^2\sqrt{\Delta}S^{(r)(\theta)}S^{(\theta)(\phi)}}{r^5} = 0,
\] (84)
and here, we either have \(S^{(r)(\theta)} = 0\) or \(S^{(\theta)(\phi)} = 0\). Note that to satisfy both the \(\theta\) and \(\phi\) components, we must have \(S^{(\theta)(\phi)} = 0\), and \(S^{(r)(\theta)}\) and/or \(S^{(r)(\phi)} = 0\). However, if we substitute \(S^{(\theta)(\phi)} = 0\) into the first expression of Eq. (82), it vanishes. With this, we arrive at \(\dot{m} = 0\), which translates into \(S^{(r)(\theta)} = S^{(r)(\phi)} = 0\). So, all the components of the spin tensor identically vanish. This concludes that the geodesic solution found in Sect. 4.1 cannot hold when the body has a spin-induced quadrupole moment.

### 5 Conclusions

In this work, we took advantage of the vanishing hidden momentum property of the Ohashi–Kyrian–Semerák spin supplementary condition [9, 20], which allows the four-momentum to be parallel to the four-velocity, to find common solutions between geodesic and Mathisson–Papapetrou–Dixon equations in Schwarzschild and Kerr. Namely, we showed that in the Schwarzschild spacetime, the only possible solution for a pole-dipole body to follow a geodesic trajectory is the radial motion. It should be stressed, that this solution holds not only for the Ohashi–Kyrian–Semerák condition, but for the Mathisson–Pirani and Tulczyjew–Dixon spin supplementary conditions as well, which implies that for the radial trajectories the three centroids coincide. However, this does not imply that either the Tulczyjew–Dixon or the Mathisson–Pirani condition can not have any other geodesic solution apart from the radial one. This is yet to be confirmed.

On the equatorial plane of Kerr spacetime, we established that for a pole-dipole body under Ohashi–Kyrian–Semerák spin supplementary condition, there is a unique solution allowing geodesic trajectories to exist. For these trajectories there is a special relation between the geodesic orbital angular momentum \(L_z\) and the geodesic energy \(E_g\), which reads \(L_z = aE_g\). It has been shown [55] that these geodesic trajectories in Kerr have similar features with the radial geodesics \((L_z = 0)\) found in the...
Schwarzschild background. The solution we found is also valid for the Mathisson–Pirani and Tulczyjew–Dixon spin supplementary conditions, but the uniqueness has not been proven for these spin supplementary conditions in our work.

In the case of a pole-dipole-quadrupole body, we discussed the condition \( K^{\mu \nu} w_\nu = 0 \) under which the hidden momentum can vanish for the Ohashi–Kyrian–Semerák spin supplementary condition as in the pole-dipole case. For the specific cases examined in this work, we employed solely spin-induced quadrupole moment. By considering a particular choice of the reference vector \( w^\mu \), we confirmed that \( K^{\mu \nu} w_\nu = 0 \) is valid on a radial trajectory in the Schwarzschild spacetime and on an equatorial orbit in Kerr for which \( L_z = a E_g \) holds. But, we speculate that in general, for a pole-dipole-(spin induced) quadrupole body under this supplementary condition, the hidden momentum may not vanish. Finally, we demonstrated that in the above two examples the extended body under Ohashi–Kyrian–Semerák spin supplementary condition cannot follow a geodesic trajectory.

Acknowledgements S.M. wish to acknowledge Narayan Banerjee for some useful interactive sessions at the earlier stage of this work. The authors would like to express their gratitude to L. Filipe O. Costa for the clarification he has provided and for pointing out some important consequences of the present work. They would also like to thank the Inter-University Centre for Astronomy and Astrophysics (IUCAA), Pune, India, where parts of this work were carried out during academic visits. The authors wish to thank the referees for their constructive criticism on the article. S.M. and G.L.-G. have been supported by the fellowship Lumina Quaeruntur No. LQ100032102 of the Czech Academy of Sciences.

Data Availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

References

1. Mathisson, M.: Neue mechanik materieller systemes. Acta Phys. Polon. 6, 163–2900 (1937)
2. Papapetrou, A.: Spinning test particles in general relativity. 1. Proc. Roy. Soc. Lond. A 209, 248–258 (1951). https://doi.org/10.1098/rspa.1951.0200
3. Corinaldesi, E., Papapetrou, E.: Spinning test-particles in general relativity. ii. In: Proceedings R. Soc. Lond. A, vol. 209, pp. 259–268, The Royal Society (1951)
4. Dixon, W.: A covariant multipole formalism for extended test bodies in general relativity. II Nuovo Cimento (1955-1965) 34(2), 317–339 (1964)
5. Dixon, W.G.: Dynamics of extended bodies in general relativity. I. Momentum and angular momentum. Proc. R. Soc. London Ser. A 314(1519), 499–527 (1970). https://doi.org/10.1098/rspa.1970.0020
6. Dixon, W.G.: Dynamics of extended bodies in general relativity-ii. Moments of the charge-current vector. Proc. R. Soc. London. A. Math. Phys. Sci. 319(1539), 509–547 (1970)
7. Dixon, W.G.: Dynamics of extended bodies in general relativity iii. Equations of motion. Philos. Trans. R. Soc. London. Ser. A. Math. Phys. Sci. 277(1264), 59–119 (1974)
8. Semerák, O.: Spinning test particles in a Kerr field. 1. Mon. Not. Roy. Astron. Soc. 308, 863–875 (1999). https://doi.org/10.1046/j.1365-8711.1999.02754.x
9. Kyrian, K., Semerák, O.: Spinning test particles in a Kerr field. Mon. Not. Roy. Astron. Soc. 382, 1922 (2007). https://doi.org/10.1111/j.1365-2966.2007.12502.x
10. Saijo, M., Maeda, K.-I., Shibata, M., Mino, Y.: Gravitational waves from a spinning particle plunging into a Kerr black hole. Phys. Rev. D 58, 064005 (1998). https://doi.org/10.1103/PhysRevD.58.064005
11. Hartl, M.D.: A Survey of spinning test particle orbits in Kerr space-time. Phys. Rev. D 67, 104023 (2003). https://doi.org/10.1103/PhysRevD.67.104023. arXiv:gr-qc/0302103 [gr-qc]
12. Barausse, E., Buonanno, A.: An Improved effective-one-body Hamiltonian for spinning black-hole binaries. Phys. Rev. D 81, 084024 (2010). https://doi.org/10.1103/PhysRevD.81.084024. arXiv:0912.3517 [gr-qc]
13. Taracchini, A., et al.: Effective-one-body model for black-hole binaries with generic mass ratios and spins. Phys. Rev. D 89(6), 061502 (2014). https://doi.org/10.1103/PhysRevD.89.061502. arXiv:1311.2544 [gr-qc]

14. Jefremov, P.I., Tsupko, O.Yu., Bisnovatyi-Kogan, G.S.: Innermost stable circular orbits of spinning test particles in Schwarzschild and Kerr space-times. Phys. Rev. D 91(12), 124030 (2015). https://doi.org/10.1103/PhysRevD.91.124030. arXiv:1503.07060 [gr-qc]

15. Mukherjee, S.: Periastron shift for a spinning test particle around naked singularities. Phys. Rev. D 97(12), 124006 (2018). https://doi.org/10.1103/PhysRevD.97.124006

16. Mukherjee, S., Rajesh Nayak, K.: Off-equatorial stable circular orbits for spinning particles. Phys. Rev. D 98(8), 084023 (2018). https://doi.org/10.1103/PhysRevD.98.084023. arXiv:1804.06070 [gr-qc]

17. Mukherjee, S.: Collisional Penrose process with spinning particles. Phys. Lett. B 778, 54–59 (2018). https://doi.org/10.1016/j.physletb.2018.01.003

18. Semerák, O., Sramek, M.: Spinning particles in vacuum spacetimes of different curvature types. Phys. Rev. D 92(6), 064032 (2015). https://doi.org/10.1103/PhysRevD.92.064032. arXiv:1505.01069 [gr-qc]

19. Deriglazov, A.A., Guzmán Ramírez, W.: Frame-dragging effect in the field of non rotating body due to unit gravimagnetic moment. Phys. Lett. B 779, 210–213 (2018). https://doi.org/10.1016/j.physletb.2018.01.063. arXiv:1802.08079 [gr-qc]

20. Costa, L.F.O., Natário, J.: Center of mass, spin supplementary conditions, and the momentum of spinning particles. Fund. Theor. Phys. 179, 215–258 (2015). https://doi.org/10.1007/978-3-319-18355-0_6. arXiv:1410.6443 [gr-qc]

21. Costa, L.F.O., Lukes-Gerakopoulos, G., Semerák, O.: Spinning particles in general relativity: momentum-velocity relation for the Mathisson-Pirani spin condition. Phys. Rev. D 97(8), 084023 (2018). https://doi.org/10.1103/PhysRevD.97.084023. arXiv:1712.07281 [gr-qc]

22. Lukes-Gerakopoulos, G., Seyrich, J., Kunst, D.: Investigating spinning test particles: spin supplementary conditions and the Hamiltonian formalism. Phys. Rev. D 81, 104012 (2010). https://doi.org/10.1103/PhysRevD.81.104012

23. Gralla, S.E., Harte, A.I., Wald, R.M.: Bobbing and kicks in electromagnetism and gravity. Phys. Rev. D 81(6), 104012 (2000). https://doi.org/10.1103/PhysRevD.81.104012

24. Harris, E.G.: Analogy between general relativity and electromagnetism for slowly moving particles in weak gravitational fields. Am. J. Phys. 59(5), 421–425 (1991)

25. Ciufolini, I., Wheeler, J.A.: Gravitation and inertia, vol. 101. Princeton University Press, Princeton (1995)

26. Ruggiero, M.L., Tartaglia, A.: Gravitomagnetic effects. Nuovo Cim. B 117, 743–768 (2002). arXiv:gr-qc/0207065

27. Wald, R.M.: Gravitational spin interaction. Phys. Rev. D 6, 406–413 (1972). https://doi.org/10.1103/PhysRevD.6.406

28. Mashhoon, B., Hehl, F.W., Theiss, D.S.: On the Gravitational effects of rotating masses - The Thirring–Lense Papers. Gen. Rel. Grav. 16, 711–750 (1984). https://doi.org/10.1007/BF00762913

29. Jantzen, R.T., Carini, P., Bini, D.: The Many faces of gravitoelectromagnetism. Ann. Phys. 215, 1–50 (1992). https://doi.org/10.1016/0003-4916(92)90297-Y. arXiv:gr-qc/0106043

30. Filipe Costa, L., Herdeiro, C.A.R.: A Gravito-electromagnetic analogy based on tidal tensors. Phys. Rev. D 78, 024021 (2008). https://doi.org/10.1103/PhysRevD.78.024021. arXiv:gr-qc/0612140

31. Natario, J.: Quasi-Maxwell interpretation of the spin-curvevature coupling. Gen. Rel. Grav. 39, 1477–1487 (2007). https://doi.org/10.1007/s10714-007-0474-7. arXiv:gr-qc/0701067

32. Costa, L.F.O., Natario, J.: Gravitio-electromagnetic analogies. Gen. Rel. Grav. 46, 1792 (2014). https://doi.org/10.1007/s10714-014-1792-1. arXiv:1207.0465 [gr-qc]

33. Hojman, S.A., Asenjo, F.A.: Spinning particles coupled to gravity and the validity of the universality of free fall. Class. Quant. Grav. 34(11), 115011 (2017). https://doi.org/10.1088/1361-6382/aa6ca2. arXiv:1610.08719 [gr-qc]

34. White, A., Raine, D., Dampier, M.: Radial infall of a spinning particle into a schwarzschild black hole. Classic. Quant. Grav. 17(18), 3681 (2000)

35. Pirani, F.A.E.: On the Physical significance of the Riemann tensor. Acta Phys. Polon. 15, 389–405 (1956). https://doi.org/10.1007/s10714-009-0787-9
37. Pirani, F.A.E.: On the Physical significance of the Riemann tensor. Gen. Rel. Grav. 41, 1215 (2009)
38. Tulczyjew, W.: Motion of multipole particles in general relativity theory. Acta Phys. Pol. 18, 393 (1959)
39. Newton, T.D., Wigner, E.P.: Localized states for elementary systems. Rev. Mod. Phys. 21, 400–406 (1949). https://doi.org/10.1103/RevModPhys.21.400
40. Steinhoff, J.: Spin gauge symmetry in the action principle for classical relativistic particles, arXiv:1501.04951 [gr-qc]
41. Ohashi, A.: Multipole particle in relativity. Phys. Rev. D 68(4), 044009 (2003). https://doi.org/10.1103/PhysRevD.68.044009. arXiv:gr-qc/0306062 [gr-qc]
42. Steinhoff, J., Puetzfeld, D.: Multipolar equations of motion for extended test bodies in General Relativity. Phys. Rev. D 81, 044019 (2010). https://doi.org/10.1103/PhysRevD.81.044019. arXiv:0909.3756 [gr-qc]
43. Steinhoff, J., Puetzfeld, D.: Influence of internal structure on the motion of test bodies in extreme mass ratio situations. Phys. Rev. D 86, 044033 (2012). https://doi.org/10.1103/PhysRevD.86.044033. arXiv:1205.2842 [gr-qc]
44. Ehlers, J., Rudolph, E.: Dynamics of extended bodies in general relativity center-of-mass description and quasirigidity. Gen. Relat. Gravit. 8(3), 197–217 (1977). https://doi.org/10.1007/BF00763547
45. Bini, D., Fortini, P., Geralico, A., Ortolan, A.: Quadrupole effects on the motion of extended bodies in Kerr spacetime. Class. Quant. Grav. 25, 125007 (2008). https://doi.org/10.1088/0264-9381/25/12/125007. arXiv:0906.2842 [gr-qc]
46. Bini, D., Geralico, A.: Deviation of quadrupolar bodies from geodesic motion in a Kerr spacetime. Phys. Rev. D 89(4), 044013 (2014). https://doi.org/10.1103/PhysRevD.89.044013. arXiv:1311.7512 [gr-qc]
47. Bini, D., Geralico, A.: Extended bodies in a Kerr spacetime: exploring the role of a general quadrupole tensor. Class. Quant. Grav. 31, 075024 (2014). https://doi.org/10.1088/0264-9381/31/7/075024. arXiv:1408.5484 [gr-qc]
48. Harte, A.I.: Extended-body motion in black hole spacetimes: What is possible? Phys. Rev. D 102(12), 124075 (2020). https://doi.org/10.1103/PhysRevD.102.124075. arXiv:2011.00110 [gr-qc]
49. Geroch, R.P.: Multipole moments. I. Flat space. J. Math. Phys. 11, 1955–1961 (1970). https://doi.org/10.1063/1.1665348
50. Geroch, R.P.: Multipole moments. II. Curved space. J. Math. Phys. 11, 2580–2588 (1970). https://doi.org/10.1063/1.1665427
51. Hansen, R.: Multipole moments of stationary space-times. J. Math. Phys. 15, 46–52 (1974). https://doi.org/10.1063/1.1666501
52. Thorne, K.: Multipole expansions of gravitational radiation. Rev. Mod. Phys. 52, 299–339 (1980). https://doi.org/10.1103/RevModPhys.52.299
53. Carter, B.: Hamilton–Jacobi and schrodinger separable solutions of Einstein’s equations. Commun. Math. Phys. 10(4), 280–310 (1968). https://doi.org/10.1007/BF03399503
54. Harms, E., Lukes-Gerakopoulos, G., Bernuzzi, S., Nagar, A.: Spinning test body orbiting around a Schwarzschild black hole: circular dynamics and gravitational-wave fluxes. Phys. Rev. D 94(10), 104010 (2016). https://doi.org/10.1103/PhysRevD.94.104010. arXiv:1609.00356 [gr-qc]
55. Chandrasekhar, S.: The mathematical theory of black holes, vol. 69. Oxford University Press, Oxford (1998)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.