A PROOF OF FUSION RULES FORMULA

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Abstract. A new proof of the fusion rules formula in the context of vertex operator algebra is given. Some more general relations between the space of intertwining operators and $A(V)$ bimodules are obtained.

1. Introduction

This paper is dedicated to present another proof of the well known fusion rules formula in the context of vertex operator algebra, which was claimed by Zhu in his thesis and first proved by Li [9] using the Verma modules of VOA constructed in [2].

Our method here is different with Li’s, but similar with the one Zhu had used to construct a module over VOA from an simple module over his famous algebra $A(V)$. The idea is that one define first an analogue of n-point correlation functions that verify certain conditions and then use these functions to define the vertex operators $Y$. Just as Zhu had claimed in the joint paper with Frankel [7] that this method would also work in deriving the fusion rules formula. However, they didn’t actually write out the proof of it in [7]. To better understand his method we carry out a proof in this paper, and we can see from our proof that some more general theorems regarding the fusion rules hold true.

We assume that the readers are familiar with the concept of vertex operator algebras (VOA), modules over VOA and the $A(V)$ theory, see [1, 5, 6, 10] for example. The fusion rules formula can be stated as the following theorem:

**Theorem 1.1.** [7] Let $V$ be a VOA with three modules $M^1, M^2, M^3$, where $M^2, M^3$ are irreducible. The space of intertwining operators of type $(M^2, M^3)$ can be identified with

$$\Hom_{A(V)}(A(M^1) \otimes A(V) M^2(0), M^3(0))$$

(1.2)

which was proved to be isomorphic with the space of intertwining operators $I(M^2, M^3)$ by Li [7].
where $h \in \mathbb{C}$ are the conformal weight of modules $M^i$, $i = 1, 2, 3$ respectively. Then as admissible modules, $M^i = \bigoplus_{n=0}^{\infty} M^i(n)$ with $a_m M^i(n) \subseteq M^i(wta - m - 1 + n)$, $\forall a \in V$, $m \in \mathbb{Z}$, $n \in \mathbb{Z}_{\geq 0}$, see [2, 3] for the definition of admissible modules. In [7, 9] it was shown that an intertwining operator $I \in \left( M^3_{M^2} \right)$ can be written as power series

$$I(v, z) = \sum_{n \in \mathbb{Z}} v(n) z^{-n-1} \cdot z^{-h}$$

where $h = h_1 + h_2 - h_3$. Furthermore, if one denote $\deg v = n$ for $v \in M^1(n)$, the action of $v(n)$ satisfies: $v(m)M^2(n) \subseteq M^3(\deg v - m - 1 + n)$, $\forall m \in \mathbb{Z}$, $n \in \mathbb{Z}_{\geq 0}$. Denote $v(\deg v - 1)$ by $o(v)$, then there exists a linear map:

$$\pi : I \left( \begin{array}{c} M^3 \\ M^1, M^2 \end{array} \right) \to (M^3(0)^* \otimes_{A(V)} A(M^1) \otimes_{A(V)} M^2(0))^* \quad (1.3)$$

given by $\pi(I) = f_I$, where $f_I(v'_3 \otimes v_1 \otimes v_2) = \langle v'_3, o(v_1)v_2 \rangle$ for $v \in M^1$, $v'_3 \in M^3(0)^*$, $v_2 \in M^2(0)$. For convenience we will denote $\deg v$ by $\text{wt} v$ in the rest of this paper.

In fact, it is not very hard to prove that this map $\pi$ is injective [9]. However the surjectivity of this map requires a lot of work to carry out. Shortly speaking, it is not quite obvious to see how to rise a linear functional $f$ on $M^3(0)^* \otimes_{A(V)} A(M^1) \otimes_{A(V)} M^2(0)$ to an intertwining operator $I$ that satisfies $f_I = f$.

1.1. Idea of our proof. Given a linear functional $f : M^3(0)^* \otimes_{A(V)} A(M^1) \otimes_{A(V)} M^2(0) \to \mathbb{C}$, we will first define a collection of (n+3)-point rational functions:

$$S(v'_3, (a_1, z_1)\ldots(a_k, z_k)(v, w)(a_{k+1}, z_{k+1})\ldots(a_n, z_n)v_2) \quad (1.4)$$

where $a_i \in V$, $v'_3 \in M^3(0)^*$, $v_2 \in M^2(0)$, $v \in M^1$; $k = 0, 1, \ldots, n$, and show that these functions satisfy certain nice properties. For instance, we will show $S$ doesn’t depend on the position of term $(v, w)$, namely, we can swap $(v, w)$ with any one of the term $(a_i, z_i)$ and still end up with the same function $S$. Then we will use these rational functions $S$ to define the intertwining operator $I$.

We are going to see that $S$ is an analogue of the classical (n+3)-point rational function:

$$(v'_3, Y(a_1, z_1)\ldots I(v, w), \ldots, Y(a_n, z_n)v_2), \quad (1.5)$$

which is the limit of power series $\langle v'_3, Y(a_1, z_1)\ldots I(v, w), \ldots, Y(a_n, z_n)v_2 \rangle$ in n+1 variables $z_i$, $i = 1, 2, \ldots, n$ and $w$ that converges in the domain

$$D = \{(z_1, \ldots, w, \ldots, z_n) \in \mathbb{C}^{n+1} ||z_1| > |z_2| > \ldots > |w| > \ldots > |z_n|\}$$
Recall that the limit of this power series is a rational function in \( z_i, w, z_i - z_j \) and \( z_k - w \) with only possible poles at \( z_i = 0, w = 0, z_i = z_j \) and \( z_k = w \), see [3]. Moreover, this rational function doesn’t depend on place of the term \( I(v, w) \). That is to say, if one swap the terms \( I(v, w) \) and \( Y(a_i, z_i) \) in the power series \( \langle v'_3, Y(a_1, z_1) ... I(v, w), ..., Y(a_n, z_n)v_2 \rangle \) then take the limit, it will end up to be the same rational function as the original rational function \( \langle v'_3, Y(a_1, z_1) ... I(v, w), ..., Y(a_n, z_n)v_2 \rangle \). The only difference between these two power series is the domain of convergence. This property essentially reflects the “locality” of vertex operators.

Note that if we fix the variables \( z_i, w, z_i - z_j \), then the rational function above can be regarded as a map out of \( n + 3 \) vector spaces:

\[
(-, Y(-, z_1) ... I(-, w), ..., Y(-, z_n)) : M^3(0)' \times V \times \ldots \times M^1 \times \ldots V \times M^2(0) \to \mathcal{F},
\]

where \( \mathcal{F} \) is the set of rational functions in \( n+1 \) variables \( z_1, z_2, ..., z_n, w \) with only possible poles at \( z_i = 0, w = 0, z_i = z_j, z_k = w \). This observation justified the name “(n+3)-point function”.

This paper is organized as follows. We define the \( n + 3 \) point functions \( S \) inductively in section 2, and prove that the analogous of locality holds for our function \( S \). In section 3 we follow the method in [10] to extend the first and last inputs spaces of \( S \) from \( M^3(0)' \) and \( M^2(0) \) to the whole modules \( M^3 \) and \( M^2 \) respectively, and carry out a proof of the fusion rules formula. Our proof also derives a more general relation between the space of intertwining operators \( I(M^3_{M^1, M^2}) \) and the space \( (M^3(0)' \otimes_{A(V)} A(M^1) \otimes_{A(V)} M^2(0))' \).

2. THE \( (n+3) \)-POINT FUNCTION \( S \)

The way Zhu constructed a \( V \) module \( M \) out of a simple \( A(V) \) module \( W \) [10] was to define first a \( (n+2) \)-point rational function \( S(v', (a_1, z_1) ... (a_n, z_n)v) \), then use it to construct the vertex operators \( Y_M \). The function \( S \) was defined inductively by the formula:

\[
S(v', (a_1, z_1) (a_2, z_2) ... (a_n, z_n)v) = z^{-wta_1} S(\pi(a_1)v', (a_2, z_2) ... (a_n, z_n)v) \\
+ \sum_{k=2}^{n} \sum_{i \geq 0} F_{wta_1, i}(z_1, z_k) S(v', (a_2, z_2) ... (a_1(i)a_k, z_k) ... (a_n, z_n)v),
\]

where \( F_{wta, i}(z, w) \) is a rational function defined by:

\[
F_{n,i}(z, w) = \frac{z^{-n}}{i!} \left( \frac{d}{dw} \right)^i \frac{w^n}{z - w}
\]

(2.2)
We can view the formula (2.1) as an expansion of $S$ with respect to the left most term $(a_1, z_1)$. This formula could partially give the definition of our $(n+3)$-point function

$$S(v'_3, (a_1, z_1)\ldots(a_k, z_k)(v, w)(a_{k+1}, z_{k+1})\ldots(a_n, z_n)v_2),$$

as long as $(v, w)$ is not at the first place i.e. $k \neq 0$. However, if we wish to define

$$S(v'_3, (v, w)(a_1, z_1)\ldots(a_n, z_n)v_2)$$

under the assumption that all the $(n+2)$ point functions

$$S(v'_3, (a_1, z_1)\ldots(a_k, z_k)(v, w)(a_{k+1}, z_{k+1})\ldots(a_{n-1}, z_{n-1})v_2)$$

are given, then the formula (2.1) would fail to give us a proper definition because the term $(v(i)a_k, z_k)$ would make no sense.

To remedy this situation, we will introduce another recurrent formula that is similar with (2.1) which allows us to expand $S$ with respect to the right most term.

2.1. Formulas that motivated the definition. Let $I \in I\left(M^3_{M^2}, v_3 \in M^3(0)^*, v_2 \in M^2(0), v \in M^1, a \in V$. Following the proof of lemma 2.2.1 in [10], it is easily seen that the 4-point function $(v'_3, Y(a, z)I(v, w)v_2)$ could be reduced to a sum of 3-point functions by the formula:

$$(v'_3, Y(a, z)I(v, w)v_2) = (v'_3o(a), I(v, w)v_2)z^{-wta} + \sum_{i \geq 0} F_{wta,i}(z, w)(v'_3, I(a(i)v, w)v_2),$$

(2.3)

where $v'_3o(a)$ is given by the natural right module action of $A(V)$ on $M^3(0)^*$:

$$\langle v'_3o(a), v_3 \rangle := \langle v'_3, o(a)v_3 \rangle.$$

Similarly, we can derive another recurrent formula for the same rational function but with $Y(a, z)$ on the right, that is, a formula for the rational function $(v'_3, I(v, w)Y(a, z)v_2)$.

Lemma 2.1. As rational functions, we have:

$$(v'_3, I(v, w)Y(a, z)v_2) = (v'_3, I(v, w)o(a)v_2)z^{-wta} + \sum_{i \geq 0} G_{wta,i}(z, w)(v'_3, I(a(i)v, w)v_2),$$

(2.4)

where $G_{wta,i}(z, w)$ is a rational function defined by

$$l_{w,z}(G_{wta,i}(z, w)) = -\sum_{j \geq 0} \binom{wta - 2 - j}{i} w^{wta - j - 2 - i} z^{-wta + 1 + j};$$

$$G_{n,i}(z, w) = \frac{z^{-n+1}}{i!} \left( \frac{d}{dw} \right)^i \left( \frac{w^{n-1}}{z - w} \right).$$

(2.5)
Proof. Note that $a(n)v_2 = 0$ if $wta - n - 1 < 0$, hence as power series:

$$\langle v'_3, I(v, w)Y(a, z)v_2 \rangle = \langle v'_3, I(v, w)o(a)v_2 \rangle z^{-wta} + \sum_{wta-n-1>0} \langle v'_3, I(v, w)a(n)v_2 \rangle z^{-n-1}$$

Recall that by taking residues from Jacobi identity, one has

$$[a(n), I(v, w)] = \sum_{i \geq 0} \binom{n}{i} I(a(i)v, w)w^{n-i},$$

(2.6)

$$\langle v'_3, a(n)u \rangle = \sum_{i \geq 0} \frac{1}{i!}(-1)^i ((L(i)a)(2wta - n - i - 2)v'_3, w),$$

and $(L(i)a)(2wta - n - i - 2)v'_3 \in M^3(-wta + n + 1) = 0$, if $wta - n - 1 > 0$. Thus,

$$\sum_{wta-n-1>0} \langle v'_3, I(v, w)a(n)v_2 \rangle z^{-n-1}$$

$$= \sum_{wta-n-1>0} \langle v'_3, a(n)I(v, w)v_2 \rangle z^{-n-1} - \sum_{wta-n-1>0} \langle v'_3, [a(n), I(v, w)]v_2 \rangle z^{-n-1}$$

$$= - \sum_{wta-n-1>0} \sum_{i \geq 0} \binom{n}{i} \langle v'_3, I(a(i)v, w)v_2 \rangle z^{-n-1}w^{n-i}.$$ 

Make a change of variable $n = wta - 2 - j$ in the last term above, since $wta - n - 1 > 0 \iff wta - 2 - n = j \geq 0$, it becomes

$$- \sum_{j \geq 0} \sum_{i \geq 0} \binom{wta - j - 2}{i} z^{-wta+j+2-1}w^{wta-j-2-i} \langle v'_3, I(a(i)v, w)v_2 \rangle$$

$$= \sum_{i \geq 0} t_{w,z}(G_{wta,i}(z, w)) \langle v'_3, I(a(i)v, w)v_2 \rangle$$

Therefore, as power series,

$$\langle v'_3, I(v, w)Y(a, z) \rangle = \langle v'_3, I(v, w)o(a)v_2 \rangle z^{-wta} + \sum_{i \geq 0} t_{w,z}(G_{wta,i}(z, w)) \langle v'_3, I(a(i)v, w)v_2 \rangle$$

Taking the limit on both sides of this equation yields the formula (2.4). \hfill \Box

In contrast to (2.3), the formula (2.4) can be regarded as expansion with respect to the right most term $Y(a, z)$ in the rational function $(v'_3, I(v, w)Y(a, z)v_2)$. Moreover, we have the following relation of $F$ and $G$:

$$F_{n,i}(z, w) - G_{n,i}(z, w) = \frac{z^{-n}}{i!} \left( \frac{d}{dw} \right)^i \left( \frac{w^n}{z-w} \right) = \frac{z^{-n}}{i!} \left( \frac{d}{dw} \right)^i (-w^{n-1})$$

$$= -(n-1)(n-2)...(n-1-i+1) \frac{1}{i!} z^{-n}w^{n-1-i}$$

$$= - \binom{n-1}{i} z^{-wta} w^{n-1-i}. $$
In particular, we have
\[ F_{wta,i}(z, w) - G_{wta,j}(z, w) = -\left(\frac{wta - 1}{i}\right) z^{-wta} w^{wta - 1 - i}. \] (2.7)

2.2. Construction of 4-point and 5-point functions. From now on, we assume that \( f \) is linear functional on the vector space
\[ M^3(0)^* \otimes_{A(V)} A(M^1) \otimes_{A(V)} M^2(0), \]
\( v_3' \in M^3(0)^* \), \( v \in M^1 \), \( v_2 \in M^2(0) \), and \( a_i \in V \). Denote by \( F(z_1, z_2, ..., z_n, w) \) the vector space of rational functions in variables \( z_i \), \( i = 1, 2, ..., n \) and \( w \) with only possible poles at \( z_i = 0 \), \( w = 0 \), \( z_i = z_j \), \( z_k = w \). We will use the same symbol \( v \) to denote the image \( v + O(M^1) \) of \( v \in M^1 \) in the \( A(V) \) bimodule \( A(M^1) \), and we will denote the degree of \( v \), \( \deg v \) by \( wt_v \).

Motivated by the fact that the left hand sides of (2.3) and (2.4) give rise to the same rational function, we define first a 3-point function \( S_M \) out of \( f \), then use that to construct two 4-point functions \( S_{VM}^L \), \( S_{MV}^R \) in two different ways, and then show that the two 4-point functions so constructed are the same rational function.

**Definition 2.2.** Define \( S_M : M^3(0)^* \times M^1 \times M^2(0) \rightarrow F(w) \) by the formula:
\[ S_M(v_3', (v, w)v_2) := f(v_3' \otimes v \otimes v_2)w^{-wta}, \] (2.8)
where on the right hand side we use the same symbol \( v \) for its image in \( A(M^1) \).

Define \( S_{VM}^L : M^3(0)^* \times V \times M^1 \times M^2(0) \rightarrow F(z, w) \) by the formula:
\[ S_{VM}^L(v_3', (a, z)(v, w)v_2) = S_M(v_3'o(a), (v, w)v_2)z^{-wta} + \sum_{i \geq 0} F_{wta,i}(z, w)S_M(v_3', (a(i)v, w)v_2). \] (2.9)

Finally, define \( S_{MV}^R : M^3(0)^* \times M^1 \times V \times M^2(0) \rightarrow F(z, w) \) by the formula
\[ S_{MV}^R(v_3', (v, w)(a, z)v_2) = S_M(v_3', (v, w)o(a)v_2)z^{-wta} + \sum_{i \geq 0} G_{wta,i}(z, w)S_M(v_3', (a(i)v, w)v_2). \] (2.10)

The upper index \( L \) (resp.\( R \)) in the 4-point functions \( S \) indicate that we use the expansion formulas with respect to the left (resp. right) most term, i.e (2.8) (resp. (2.9)) to define our new \( S \). In what follows, we will denote the 3-point function \( S_M \) simply by \( S \).

**Proposition 2.3.** As rational functions in \( F(z, w) \), we have:
\[ S_{VM}^L(v_3', (a, z)(v, w)v_2) = S_{MV}^R(v_3', (v, w)(a, z)v_2). \]

Therefore, we may use the common symbol \( S \) to denote these two 4-point functions.
Proof. From the definition of $S^L_{VM}$, $S^R_{MV}$ and $S$ we see that:

$$S^L_{VM}(v_3', (a, z)(v, w)v_2) = f(v_3' o(a) \otimes v \otimes v_2)w^{-wta}z^{-wta}$$

$$+ \sum_{i \geq 0} F_{wta,i}(z, w)f(v_3' \otimes a(i)v \otimes v_2)w^{-wta-wtv+i+1}$$

and

$$S^R_{MV}(v_3', (v, w)(a, z)v_2) = f(v_3' \otimes v \otimes o(a)v_2)w^{-wta}z^{-wta}$$

$$+ \sum_{i \geq 0} G_{wta,i}(z, w)f(v_3' \otimes a(i)v \otimes v_2)w^{-wta-wtv+i+1}.$$ 

Combining these two equations with (2.7), we have

$$S^L_{VM}(v_3', (a, z)(v, w)v_2) - S^R_{MV}(v_3', (v, w)(a, z)v_2)$$

$$= f(v_3' o(a) \otimes v \otimes v_2)w^{-wta}z^{-wta} - f(v_3' \otimes v \otimes o(a)v_2)w^{-wta}z^{-wta}$$

$$- \sum_{i \geq 0} \left( \frac{wta - 1}{i} \right) f(v_3' \otimes a(i)v \otimes v_2)w^{-wta-wtv+i+1}z^{-wta}w^{wta-1-i}$$

$$= f(v_3' \otimes (a \ast v - v \ast a) \otimes v_2)w^{-wta}z^{-wta} - \sum_{i \geq 0} \left( \frac{wta - 1}{i} \right) f(v_3' \otimes a(i)v \otimes v_2)z^{-wta}w^{-wtv}$$

Recall that in the $A(V)$ bimodule $A(M^1)$ one has the following formula [10]:

$$a \ast v - v \ast a = \text{Res}_z Y(a, z)v(1 + z)^{wta-1} = \sum_{i \geq 0} \left( \frac{wta - 1}{i} \right) a(i)v.$$ 

Hence we have $S^L_{VM}(v_3', (a, z)(v, w)v_2) - S^R_{MV}(v_3', (v, w)(a, z)v_2) = 0$. \hfill \Box

It follows from the proposition that both of the 4-point functions $S^L_{VM}$ and $S^R_{MV}$ in definition [2.2] give rise to a single 4-point function $S$ which satisfies

$$S(v_3', (a, z)(v, w)v_2) = S(v_3', (v, w)(a, z)v_2), \quad (2.11)$$

and it is defined as a rational function obtained from 3-point function $S$ by the formulas of expansion with respect to either the left most term as in [2.3] or the right most term as in (2.4).

We adopt the same idea as above to define 5-point functions. At first glimpse, there could be in total four of the 5-point functions, namely, $S^L_{VMV}$, $S^R_{VMV}$, $S^R_{MV}$ and $S^R_{MVV}$, where the sub index indicates the position of $M^1$. We can define them using suitable formulas as in (2.3) or (2.4) based on their upper index $L$ and $R$. But in the end we will show that they all give rise to the same rational function no matter what the upper or lower indices they have, just as the case in 4-point functions.
Definition 2.4. The 5-point functions with upper indices \( L \)

\[
S^L_{VMV}(v_3', (a_1, z_1)(v, w)(a_2, z_2)v_2),
\]

\[
S^L_{VVV}(v_3', (a_1, z_1)(a_2, z_2)(v, w)v_2)
\]

are defined by the expansion with respect to the left most term \((a_1, z_1)\), which is given by the common formula:

\[
S(v_3'o(a_1), (v, w)(a_2, z_2)v_2)\ z_1^{-w_{a_1}} + \sum_{j \geq 0} F_{w_{a_1}, j}(z_1, w)S(v_3', (a_1(j)v, w)(a_2, z_2)v_2) \\
+ \sum_{j \geq 0} F_{w_{a_1}, j}(z_1, z_2)S(v_3', (v, w)(a_1(j)a_2, z_2)v_2).
\] (2.12)

The 5-point functions with upper indices \( R \)

\[
S^R_{VMV}(v_3', (a_2, z_2)(v, w)(a_1, z_1)v_2),
\]

\[
S^R_{MVM}(v_3', (v, w)(a_2, z_2)(a_1, z_1)v_2)
\]

are defined by expansion with respect to the right most term \((a_1, z_1)\), which is also given by a common formula

\[
S(v_3', (a_2, z_2)(v, w)o(a_1)v_2)\ z_1^{-w_{a_1}} + \sum_{j \geq 0} G_{w_{a_1}, j}(z_1, w)S(v_3', (a_2, z_2)(a_1(j)v, w)v_2) \\
+ \sum_{j \geq 0} G_{w_{a_1}, j}(z_1, z_2)S(v_3', (a_1(j)a_2, z_2)(v, w)v_2).
\] (2.13)

Note that the function \( S \) in expressions (2.12) and (2.13) above are 4-point functions. In fact, formula (2.12) is just the expansion of \( S^L_{VMV}(v_3', (a_1, z_1)(v, w)(a_2, z_2)v_2) \) with respect to \((a_1, z_1)\). But it follows from (2.11) that:

\[
S(v_3'o(a), (v, w)(a_2, z_2)v_2) = S(v_3'o(a), (a_2, z_2)(v, w)v_2);
\]

\[
S(v_3', (a_1(j)v, w)(a_2, z_2)v_2) = S(v_3', (a_2, z_2)(a_1(j)v, w)v_2);
\]

\[
S(v_3', (a_2, z_2)(a_1(j)v, w)v_2) = S(v_3', (a_1(j)v, w)(a_2, z_2)v_2).
\]

So (2.12) is also an expansion of \( S^L_{VMV}((v_3', (a_1, z_1)(a_2, z_2)(v, w)v_2)) \) with respect to \((a_1, z_1)\), hence it makes sense to define \( S^R_{VMV} \) and \( S^L_{VVV} \) by the same formula, and similar for \( S^R_{MVM}, S^L_{MVM} \).

Now we want to show that all the 5-point functions defined as above are the same rational function, no matter which upper symbols or lower symbols they have, just like property the 4-point functions (2.11). In other words, there is only one \( S(v_3', (a_1, z_1)(v, w)(a_2, z_2)v_2) \), and one can permute the term \((a_1, z_1), (a_2, z_2)\), and \((v, w)\) arbitrarily and still end up with the same function. This is not an easy task, because first we have to show that (2.12) and (2.13) are the same rational function, which is far from being obvious, and furthermore,
observe that by definition 2.4 the function $S_{V^M}(v_3', (a_1, z_1)(v, w)(a_2, z_2)v_2)$ would have two definitions:

$$S^L_{V^M}(v_3', (a_1, z_1)(v, w)(a_2, z_2)v_2) \quad \text{and} \quad S^R_{V^M}(v_3', (a_1, z_1)(v, w)(a_2, z_2)v_2)$$

with the first one defined by expansion with respect to left most term $(a_1, z_1)$ as in (2.12) and second one defined by expansion with respect to right most term $(a_2, z_2)$ as in (2.13).

If all of the 5-point functions in definition 2.4 give rise to one single 5-point function, then these two definition ought to be the same. Nevertheless, we have the following fact:

**Proposition 2.5.** If $S^L_{V^M}(v_3', (a_1, z_1)(v, w)(a_2, z_2)v_2) = S^R_{V^M}(v_3', (a_2, z_2)(v, w)(a_1, z_1)v_2)$ i.e. if (2.12) = (2.13), then it follows that:

$$S^L_{V^M}(v_3', (a_1, z_1)(v, w)(a_2, z_2)v_2) = S^R_{V^M}(v_3', (a_1, z_1)(v, w)(a_2, z_2)v_2)$$

**Proof.** Note that (2.12) is essentially the formula Zhu had used to define his n+1 point function $S$ out of n-point functions, so the formula (2.2.11) in [10] would also hold true in our case, and the proof would be exactly the same, so we won’t write out a proof here.

That formula allows our to swap $(a_1, z_1)$ and $(a_2, z_2)$ in $S^L_{V^M}$:

$$S^L_{V^M}(v_3', (a_1, z_1)(a_2, z_2)(v, w)v_2) = S^L_{V^M}(v_3', (a_2, z_2)(a_1, z_1)(v, w)v_2). \quad (2.14)$$

By assumption of the proposition, definition 2.4 and (2.14), we have:

$$S^L_{V^M}(v_3', (a_1, z_1)(v, w)(a_2, z_2)v_2) = S^L_{V^M}(v_3', (a_1, z_1)(a_2, z_2)(v, w)v_2)$$

$$= S^L_{V^M}(v_3', (a_2, z_2)(a_1, z_1)(v, w)v_2)$$

$$= S^L_{V^M}(v_3', (a_2, z_2)(v, w)(a_1, z_1)v_2)$$

$$= S^R_{V^M}(v_3', (a_1, z_1)(v, w)(a_2, z_2)v_2),$$

which is the equation as proposed. \qed

Next, we are going to show the assumption in proposition 2.5 holds true, namely, the rational function (2.12) is the same with the one in (2.13). We will compute the difference of three summands in those functions separately, then show that the sum of their differences is equal to 0. In order to make our computational work looks more elegant, we introduce symbols $\heartsuit$, $\star$, $\triangle$ to denote the difference of three summands in (2.12) and (2.13), namely:

$$S(v_3' o(a), (v, w)(a_2, z_2)v_2)z_1^{-wta_1} - S(v_3', (a_2, z_2)(v, w)o(a_1)v_2)z_1^{-wta_1} \quad \heartsuit$$

$$\sum_{j \geq 0} F_{wta_1,j}(z_1, w)S(v_3', (a_1(j)v, w)(a_2, z_2)v_2)$$

$$- \sum_{j \geq 0} G_{wta_1,j}(z_1, w)S(v_3', (a_2, z_2)(a_1(j)v, w)v_2) \quad \star$$
\[
\sum_{j \geq 0} F_{wta_1,j}(z_1, z_2) S(v'_3, (v, w)(a_1(j)a_2, z_2)v_2) - \sum_{j \geq 0} G_{wta_1,j}(z_1, z_2) S(v'_3, (a_1(j)a_2, z_2)(v, w)v_2)
\]

Now our goal is to show that \(\bigtriangleup + \bigstar + \bigtriangleup = 0\).

By the property of 4-point functions, we may use formulas \([2.9]\) and \([2.10]\) to expand both of the terms in \(\bigtriangleup\) with respect to \((a_2, z_2)\).

\[
S(v'_3 o(a), (v, w)(a_2, z_2)v_2) z_1^{-wta_1} - S(v'_3, (a_2, z_2)(v, w)o(a_1)v_2) z_1^{-wta_1}
\]

\[
= S(v'_3 o(a_2), (v, w)v_2) z_1^{-wta_1} z_2^{-wta_2} + \sum_{i \geq 0} F_{wta_2,i}(z_2, w) S(v'_3 o(a_1), (a_2(i)v, w)v_2) z_1^{-wta_1}
\]

\[- S(v'_3 o(a_2), (v, w)o(a_1)v_2) z_1^{-wta_1} z_2^{-wta_2} + \sum_{i \geq 0} F_{wta_2,i}(z_2, w) S(v'_3, (a_2(i)v, w)o(a_1)v_2) z_1^{-wta_1}
\]

\[= f(v'_3 \otimes a_1 * a_2 * v \otimes v_2) w^{-wta_2} z_1^{-wta_1} z_2^{-wta_2} - f(v'_3 \otimes a_2 * a_1 \otimes v_2) w^{-wta_2} z_1^{-wta_1} z_2^{-wta_2}
\]

\[+ \sum_{i \geq 0} F_{wta_2,i}(z_2, w) f(v'_3 \otimes (a_1 * (a_2(i)v)) - (a_2(i)v) \otimes v_2) w^{-wta_2} z_1^{-wta_1} z_2^{-wta_2}
\]

For \(\bigstar\) we can apply the formula \([2.9]\) and \([2.11]\) to expand each of the summand in \(\bigstar\) with respect to \((a_2, z_2)\) on the left:

\[
\sum_{j \geq 0} F_{wta_1,j}(z_1, w) S(v'_3, (a_1(j)v, w)(a_2, z_2)v_2) - \sum_{j \geq 0} G_{wta_1,j}(z_1, w) S(v'_3, (a_2, z_2)(a_1(j)v, w)v_2)
\]

\[= \sum_{j \geq 0} F_{wta_1,j}(z_1, w) S(v'_3, o(a_2)(a_1(j)v, w)v_2) z_2^{-wta_2}
\]

\[+ \sum_{j \geq 0} \sum_{i \geq 0} F_{wta_1,j}(z_1, w) F_{wta_2,i}(z_2, w) S(v'_3, (a_2(i)a_1(j)v, w)v_2)
\]

\[- \sum_{j \geq 0} G_{wta_1,j}(z_1, w) S(v'_3, o(a_2)(a_1(j)v, w)v_2) z_2^{-wta_2}
\]

\[- \sum_{j \geq 0} \sum_{i \geq 0} G_{wta_1,j}(z_1, w) F_{wta_2,i}(z_2, w) S(v'_3, (a_2(i)a_1(j)v, w)v_2)
\]

\[= \sum_{j \geq 0} \left(\frac{wta_1 - 1}{j}\right) S(v'_3 o(a_2), (a_1(j)v, w)v_2) z_1^{-wta_1} z_2^{-wta_2} w^{-wta_1-1-j}
\]

\[+ \sum_{j \geq 0} \sum_{i \geq 0} \left(\frac{wta_1 - 1}{j}\right) z_1^{-wta_1} w^{-wta_1-1-j} F_{wta_2,w}(z_2, w) S(v'_3, (a_2(i)a_1(j)v, w)v_2)
\]

\[= \bigstar(1) + \bigstar(2).
\]
Finally, for the function $\triangle$ we expand each of its summand in terms of $(a_1(j)a_2, z_2)$ on the left:

$$
\sum_{j \geq 0} F_{wta_1,j}(z_1, z_2)S(v'_3, (v, w)(a_1(j)a_2, z_2)v_2) - \sum_{j \geq 0} G_{wta_1,j}(z_1, z_2)S(v'_3, (a_1(j)a_2, z_2)(v, w)v_2)
= \sum_{j \geq 0} F_{wta_1,j}(z_1, z_2)S(v'_3 o(a_1(j)a_2), (v, w)v_2)v_2^{-wta_1-wta_2+j+1}
+ \sum_{j \geq 0} \sum_{i \geq 0} F_{wta_1,j}(z_1, z_2)F_{wta_1+wta_2-j-1,i}(z_2, w)S(v'_3, ((a_1(j)a_2)(i)v, w)v_2)
- \sum_{j \geq 0} G_{wta_1,j}(z_1, z_2)S(v'_3 o(a_1(j)a_2), (v, w)v_2)v_2^{-wta_1-wta_2+j+1}
+ \sum_{j \geq 0} \sum_{i \geq 0} G_{wta_1,j}(z_1, z_2)F_{wta_1+wta_2-j-1,i}(z_2, w)S(v'_3, ((a_1(j)a_2)(i)v, w)v_2)
= \sum_{j \geq 0} -\left( wta_1 - 1 \right)_j^{-wta_1-wta_2-j}S(v'_3 o(a_1(j)a_2), (v, w)v_2)v_2^{-wta_1-wta_2+j+1}
+ \sum_{j \geq 0} \sum_{i \geq 0} -\left( wta_1 - 1 \right)_j^{-wta_1-wta_2-j}F_{wta_1+wta_2-j-1,i}(z_2, w)S(v'_3, (a_1(j)a_2)(i)v, w)v_2
= \triangle(1) + \triangle(2).
$$

Now we need to show $\bigcirc(1) + \bigcirc(2) + \bigcirc(3) + \star(1) + \star(2) + \triangle(1) + \triangle(2) = 0$. Since in $A(M^1)$ one has $a * v - v * a = \text{Res}_z Y(a, z)v(1 + z)^{wta-1} = \sum_{j \geq 0} (wta_j^{-1})a(j)v$, we can rewrite $\star(1)$ and $\triangle(1)$ as:

$$
\star(1) = -\sum_{j \geq 0} \left( wta_1 - 1 \right)_j w^{-wta_1-wtv+j+1}z_1^{-wta_1}z_2^{-wta_2}w^{-wta_1-j-1}f(v'_3 o(a_2) \otimes a_1(j)v \otimes v_2)
= -w^{-wta_1-wtv}z_1^{-wta_1}z_2^{-wta_2}f(v'_3 \otimes (a_2 * a_1 * v - a_2 * v * a_1) \otimes v_2);
$$

$$
\triangle(1) = -\sum_{j \geq 0} \left( wta_1 - 1 \right)_j z_1^{-wta_1}z_2^{-wta_2}w^{-wta_2}f(v'_3 o(a_1(j)a_2) \otimes v \otimes v_2)
= -z_1^{-wta_1}z_2^{-wta_2}w^{-wta_2}f(v'_3 \otimes (a_1 * a_2 * v - a_2 * a_1 * v) \otimes v_2)
$$

Now we see that:

$$
\bigcirc(1) + \bigcirc(2) + \star(1) + \triangle(1)
= f(v'_3 \otimes a_1 * a_2 * v \otimes v_2)w^{-wta_1-wta_2} + f(v'_3 \otimes a_2 * a_1 \otimes v_2)w^{-wta_1-wta_2}
- w^{-wta_1-wta_2}f(v'_3 \otimes (a_2 * a_1 * v - a_2 * v * a_1) \otimes v_2)
- z_1^{-wta_1}z_2^{-wta_2}w^{-wta_2}f(v'_3 \otimes (a_1 * a_2 * v - a_2 * a_1 * v) \otimes v_2)
= 0
$$
Moreover, we can also rewrite $\heartsuit(3)$ as

$$\heartsuit(3) = \sum_{i,j \geq 0} \binom{wta_1 - 1}{j} \binom{wta_2 + n}{i} (a_1(j) a_2(i) v - a_2(i) a_1(j) v)$$

Since what we need to show was $\heartsuit + \star + \triangle = 0$, the equation above implies that we only need to show $\heartsuit(3) + \star(2) + \triangle(2) = 0$. We shall use the following lemma to prove this equality.

**Lemma 2.6.** Let $M$ be a $V$ module, $a_1, a_2 \in V$, $v \in M$, $n \in \mathbb{Z}_{\geq 0}$, we have:

$$\sum_{i,j \geq 0} \binom{wta_1 - 1}{j} \binom{wta_2 + n}{i} (a_1(j) a_2(i) v - a_2(i) a_1(j) v)$$

$$= \sum_{i,j \geq 0} \binom{wta_1 - 1}{j} \binom{wta_1 + wta_2 - j - 1 + n}{i} (a_1(j) a_2(i) v)$$

(2.15)

**Proof.** Choose complex variables $z_1, z_2$ in the domain $|z_1| < 1$, $|z_2| < 1$, $|z_1 - z_2| < |1 + z_2|$. By Jacobi identity in the residue form [10], the left hand side of (2.15) can be written as:

$$\operatorname{Res}_{z_1, z_2} \sum_{i,j \geq 0} \binom{wta_1 - 1}{j} \binom{wta_2 + n}{i} z_1^{j} z_2^{i} (Y(a_1, z_1) Y(a_2, z_2) v - Y(a_2, z_2) Y(a_1, z_1) v)$$

$$= \operatorname{Res}_{z_1, z_2} (1 + z_1)^{wta_1 - 1} (1 + z_2)^{wta_2 + n} (Y(a_1, z_1) Y(a_2, z_2) v - Y(a_2, z_2) Y(a_1, z_1) v)$$

$$= \operatorname{Res}_{z_1, z_2} \operatorname{Res}_{z_1 - z_2} (1 + z_2 + (z_1 - z_2))^{wta_1 - 1} (1 + z_2)^{wta_2 + n} (Y(a_1, z_1 - z_2) a_2, z_2) v$$

$$= \operatorname{Res}_{z_1, z_2} \sum_{j \geq 0} \binom{wta_1 - 1}{j} (1 + z_2)^{wta_1 - 1 - j + wta_2 + n} (z_1 - z_2)^{j} Y(Y(a_1, z_1 - z_2) a_2, z_2) v$$

$$= \sum_{i,j \geq 0} \binom{wta_1 - 1}{j} \binom{wta_1 + wta_2 - j - 1 + n}{i} (a_1(j) a_2(i) v),$$

which is equal to the right hand side of (2.15). \qed

We are now left to show the sum of following three terms is equal to 0:

$$\sum_{i,j \geq 0} \binom{wta_1 - 1}{j} z_1^{-wta_1 w-wta_2-wtv+i+1} F_{wta_1, i}(z_2, w) f(v_3 \otimes a_1(i) a_2(i) v \otimes v_2) \quad \heartsuit(3)$$

$$\sum_{i,j \geq 0} - \binom{wta_1 - 1}{j} z_1^{-wta_1 w-wta_1-wtv+i} F_{wta_1, w}(z_2, w) f(v_3 \otimes a_2(i) a_1(i) v \otimes v_2) \quad \star(2)$$

$$\sum_{i,j \geq 0} \binom{wta_1 - 1}{j} z_1^{-wta_1 w-wta_2-j+1} F_{wta_1, j-1}(z_2, w) w^{-wta_1-wta_2+j+1+i-1-wtv} \quad \triangle(2)$$

$$\cdot f(v_3 \otimes (a_1(i) a_2(i) v \otimes v_2)$$
Since the map \( \iota_{z_2,w} \) is injective\[^{53} \), it follows that if we only need to show \( \iota_{z_2,w}(\bigtriangledown(3) + \bigstar(2) + \bigtriangleup(2)) = 0 \). Recall that by \((2.22)\), \( \iota_{z_2,w}(F_{wta_2,i}(z_2, w)) \) can be written as:

\[
\iota_{z_2,w}(F_{wta_2,i}(z_2, w)) = \iota_{z_2,w}(\frac{z_2^{-wta_2}}{i!} (\frac{d}{dw})^i (\frac{w^{-wta_2}}{z - w}))
\]

\[
= \sum_{n \geq 0} \binom{wta_2 + n}{i} w^{wta_2 + n} z^{-wta_2 - n - 1}
\]

Let’s denote the term \( z_i^{wta_2} w^{-wta_2 - n - 1} w^{wta_2 - n - 1} \) by \( \gamma \), then it follows from lemma ?? that

\[
\iota_{z_2,w}(\bigtriangledown(3)) + \iota_{z_2,w}(\bigstar(2))
\]

\[
= \sum_{i,j \geq 0} \binom{wta_1 - 1}{j} \gamma wta_1 w^{-wta_2 - wtv + i + 1} \sum_{n \geq 0} \binom{wta_2 + n}{i} w^{wta_2 + n} z^{-wta_2 - n - 1}
\]

\[
\cdot (f(v_3' \otimes a_1(j)a_2(i)v \otimes v_2) - f(v_3' \otimes a_2(i)a_1(j)v \otimes v_2))
\]

\[
= \sum_{i,j \geq 0} \binom{wta_1 - 1}{j} \binom{wta_2 + n}{i} \gamma \cdot f(v_3' \otimes (a_1(j)a_2(i)v - a_2(i)a_1(j)v) \otimes v_2)
\]

\[
= \sum_{i,j \geq 0} \binom{wta_1 - 1}{j} \binom{wta_1 + wta_2 + n - j - 1}{i} \gamma \cdot f(v_3' \otimes (a_1(j)a_2(i)v \otimes v_2)
\]

\[
= -\iota_{z_2,w}(\bigtriangleup(2))
\]

Now the proof of \((2.12) = (2.13)\) is complete. Putting this together with proposition \((2.5)\) and \((2.14)\), we see that our definition \((2.14)\) for the 5-point functions give rise to one single 5-point function \( S \), which satisfies:

\[
S(v_3', (a_1, z_1)(a_2, z_2)(v, w)v_2) = S(v_3', (a_2, z_2)(a_1, z_1)(v, w)v_2)
\]

\[
= S(v_3', (a_1, z_1)(v, w)(a_2, z_2)v_2) = S(v_3', (a_2, z_2)(v, w)(a_1, z_1)v_2) \quad (2.16)
\]

\[
= S(v_3', (v, w)(a_1, z_1)(a_2, z_2)v_2) = S(v_3', (v, w)(a_2, z_2)(a_1, z_1)v_2)
\]

This is to say that the terms \((a_1, z_1)\), \((a_2, z_2)\) and \((v, w)\) can be permuted arbitrarily within \( S \). Moreover, \( S(v_3', (a_1, z_1)(a_2, z_2)(v, w)v_2) \) can be either defined by expansion with respect to \((a_i, z_i)\), \( i = 1, 2 \) from the left i.e. by formula \((2.2.5)\), or expansion with respect to \((a_i, z_i)\), \( i = 1, 2 \) from the right i.e. by formula \((2.2.6)\).

Note that these properties are also satisfied by the classical 5-point rational function:

\[
(v_3', Y(a_1, z_1)Y(a_2, z_2)I(v, w)v_2),
\]

which is defined to be the limit of series \( \langle v_3', Y(a_1, z_1)Y(a_2, z_2)I(v, w)v_2 \rangle \).

2.3. Construction of \((n+3)\)-point function \( S \). We will define the general \((n+3)\)-point function \( S \) by induction on \( n \). Note that the base cases \( n = 1, 2 \) are achieved in previous
section. Now suppose all the (n+2)-point functions:

\[ S : M^3(0)^* \times V \times \ldots \times M^1 \times \ldots \times V \times M^2(0) \to \mathcal{F}(z_1, \ldots, z_{n-1}, w) \]

are given, and they satisfy the following two properties: The first property is that:

\[
S(v'_3, (a_1, z_1)(a_2, z_2)\ldots(a_{n-1}, z_{n-1})(v, w)v_2) = S(v'_3, (b_1, w_1)(b_2, w_2)\ldots(b_n, w_n)v_2)
\]

(1)

where the finite set \{ \{(b_i, z_i)\}_{i=1}^n = \{(a_1, z_1), \ldots, (a_{n-1}, z_{n-1}), (v, w)\}.

Denote by \( S^L \) (resp. \( S^R \)) the expansion with respect to the left (resp. right) most term in \( S \), see (2.12) (resp. (2.13)). The second property is that for any indices \( i \) and \( j \)

\[
S(v'_3, (b_1, w_1)(b_2, w_2)\ldots(b_n, w_n)v_2) = S^L(v'_3, (a_i, z_i))v_2) = S^R(v'_3, (a_j, z_j)v_2),
\]

(2)

where the symbol \( \square \) in \( S^L \) represents an arbitrary combination of \( (v, w) \), \{ \( (a_k, z_k) \}_{k \neq i} \), while the \( \square \) in \( S^R \) represents an arbitrary combination of \( (v, w) \), \{ \( (a_k, z_k) \}_{k \neq j} \).

Note that these two properties are satisfied by 4 and 5-point functions, see (2.11) and (2.16). With (n+2)-point functions \( S \) in our hand we may define (n+3)-point functions by similar formulas as in definition 2.2 and definition 2.4.

**Definition 2.7.** Assume the number of \( V \) in the subindex of \( S^L_{VV, M^1, \ldots, V} \) and \( S^R_{V, M^1, \ldots, V} \) are both equal to \( n \), and the \( M^1 \) in \( S^L \) can be placed anywhere in between the second and the last place, while the \( M^1 \) in \( S^R \) can be placed anywhere in between the first and the \( n \)th place. We define these functions by the following formulas:

\[
S^L_{VV, M^1, \ldots, V}(v'_3, (a_1, z_1)(v, w)\ldots v_2)
= S(v'_3 a_1, (a_2, z_2)\ldots(a_n, z_n)(v, w)v_2)z_1^{-wta_1}
+ \sum_{k=2}^{n} \sum_{j=0}^{0} F_{wta_1,j}(z_1, z_k) \cdot S(v'_3, (a_2, z_2)\ldots(a_1(j)a_k, z_k)\ldots(v, w)v_2) \quad (2.17)
\]

\[
S^R_{V, M^1, \ldots, V}(v'_3, (v, w)\ldots v_1)
= S(v'_3, (a_2, z_2)\ldots(a_n, z_n)(v, w)a_1, v_2)z_1^{-wta_1}
+ \sum_{k=2}^{n} \sum_{j=0}^{0} G_{wta_1,j}(z_1, z_k) \cdot S(v'_3, (a_2, z_2)\ldots(a_1(j)a_k, z_k)\ldots(v, w)v_2) \quad (2.18)
\]

where the \( S \) on right hand sides of these equations are (n+2)-point functions.

Note that the definition above implies that \( S^L_{VMV, V} = S^L_{VV, M} = \ldots = S^L_{VV, VM} \), which is reasonable because the (n+2)-point functions \( S \) on the right hand side of (2.17) satisfy
property (1), and for the same reason one can expect that \( S_{MV...VV}^R = S_{V...V\overline{V}}^R = \ldots = S_{V...VMV} \). Our next step is to show:

\[
S_{VV...V...V}^L(v'_3, (a_1, z_1)(a_2, z_2)...(v, w)...(a_2, z_2)v_2) = S_{VV...V...V}^R(v'_3, (a_1, z_1)(a_2, z_2)...(a_2, z_2)v_2), \tag{2.19}
\]

which will lead to the conclusion that all of the \((n+3)\)-point functions \(S_{VV...V...V}^L, S_{VV...V...V}^R\) give rise to one single \((n+3)\)-point function \(S\) that satisfies the same properties (1) and (2) as \((n+2)\)-point functions. In order to prove (2.19), we apply a gain formula (2.2.1) in \([10]\) to our situation and get:

\[
S_{VV...V...V}^L(v'_3, (a_1, z_1)(a_2, z_2)...(v, w)...v_2) = S_{VV...V...V}^R(v'_3, (a_2, z_2)(a_1, z_1)...(v, w)...v_2). \tag{2.20}
\]

Since the definition (2.17) of \(S^L\) is essentially the same with (2.2.6) in \([10]\), it is not had to see that the proof of (2.20) would be exactly the same with the corresponding one (2.2.11) in \([10]\), so we omit the proof of it.

**Proposition 2.8.** If \(S_{VV...V...V}^L(v'_3, (a_1, z_1)...v_2) = S_{VV...V...V}^R(v'_3, (a_1, z_1)v_2)\), i.e. if the right hand side of (2.17) is equal to the right hand side of (2.18), then (2.19) follows.

**Proof.** The proof is similar with proposition \(2.5\) By (2.20) and assumption, we have:

\[
S_{VV...V...V}^L(v'_3, (a_1, z_1)...(v, w)...(a_2, z_2)v_2) = S_{VV...V...V}^L(v'_3, (a_1, z_1)(a_2, z_2)...(v, w)...v_2) = S_{VV...V...V}^L(v'_3, (a_2, z_2)(a_1, z_1)...(v, w)...v_2) = S_{VV...V...V}^R(v'_3, (a_1, z_1)...(v, w)...(a_2, z_2)v_2)
\]

as asserted. \(\Box\)

Now we are left to prove that \(S_{VV...V...V}^L(v'_3, (a_1, z_1)...v_2) = S_{VV...V...V}^R(v'_3, (a_1, z_1)v_2)\). We again introduce the following symbols to denote the corresponding terms in right hand side of the difference (2.17) - (2.18).

\[
S(v'_3 o(a_1), (a_2, z_2)...(a_n, z_n)(v, w)v_2)z^{-wta_1} - S(v'_3, (a_2, z_2)...(a_n, z_n)(v, w)o(a_1)v_2)z^{-wta_1} \bigtriangleup
\]

\[
\sum_{j \geq 0} F_{wta_1,j}(z_1, z_2)S(v'_3, (a_1(j)a_2, z_2)...(v, w)v_2) \bigtriangleup
\]

\[
- G_{wta_1,j}(z_1, z_2)S(v'_3, (a_1(j)a_2, z_2)...(v, w)v_2)
\]

\[
\sum_{k=3}^{n} \sum_{j \geq 0} F_{wta_1,j}(z_1, z_k)S(v'_3, (a_2, z_2)...(a_1(j)a_k, z_k)...(v, w)v_2) \bigtriangleup
\]

\[
- \sum_{k=3}^{n} \sum_{j \geq 0} G_{wta_1,j}(z_1, z_k)S(v'_3, (a_2, z_2)...(a_1(j)a_k, z_k)...(v, w)v_2)
\]
Then we need to show that $\sum$ is equal to 0. We expand them with respect to the left most term, then add them up and show that the lemma:

$S_n^\prime \sum$ = $\sum_{k=3}^{n} \sum_{j=0}^{n} \left( w^{ta_1,j} (z_1, w) S(v'_3, (a_2, z_2)...(a_1(j)v, w)v_2) + G_{wta_1,j}(z_1, w) S(v'_3, (a_2, z_2)...(a_1(j)v, w)v_2) \right)$. 

Our strategy is to apply formula (2.17) to each one of the (n+2)-point functions, and (2.7), we have:

$S(v'_3, (a_2, z_2)...(a_1(j)v, w)v_2) z_1^{-wta_1} z_2^{-wta_2} = (\sum_{k=3}^{n} \sum_{j=0}^{n} F_{wta_1,j}(z_2, z_1) S(v'_3, (a_3, z_3)...(a_1(j)a, z_1)...(a_n, z_n)(v, w)v_2) z_1^{-wta_1} + \sum_{i=0}^{n} F_{wta_2,i}(z_2, w) S(v'_3, (a_3, z_3)...(a_n, z_n)(v, w)v_2) z_1^{-wta_1}$

and $S(v'_3, (a_2, z_2)...(a_n, z_n)(v, w)v_2) z_1^{-wta_1}$ can be written as

$S(v'_3, (a_2, z_2)...(a_1(j)v, w)v_2) z_1^{-wta_1} = \sum_{k=3}^{n} \sum_{j=0}^{n} \left( w^{ta_1,j} - 1 \right) z_k^{wta_1-j} S(v'_3, (a_3, z_3)...(a_1(j)a, z_1)...(a_n, z_n)(v, w)v_2)$

we denote the difference of first, second and third terms in (*) and (**) by $\bigtriangledown(1)$, $\bigtriangledown(2)$ and $\bigtriangledown(3)$ respectively. In order to cancel $\bigtriangledown$ with rest of the symbols, we need the following lemma:

**Lemma 2.9.** As (n+1)-point functions, we have:

$S(v'_3 o(a_1), (a_3, z_3)...(a_n, z_n)(v, w)v_2) - S(v'_3, (a_3, z_3)...(a_n, z_n)(v, w)v_2) o(a_1)v_2) = \sum_{k=3}^{n} \sum_{j=0}^{n} \left( w^{ta_1,j} - 1 \right) z_k^{wta_1-j} S(v'_3, (a_3, z_3)...(a_1(j)a, z_1)...(a_n, z_n)(v, w)v_2)$

**Proof.** By the induction hypothesis on (n+2)-point functions and [2,7], we have:

$0 = S(v'_3, (a_1, z_1)(a_3, z_3)...(a_n, z_n)(v, w)v_2) - S(v'_3, (a_3, z_3)...(a_n, z_n)(a_1, z_1)(v, w)v_2) = S(v'_3 o(a_1), (a_3, z_3)...(a_n, z_n)(v, w)v_2) - S(v'_3, (a_3, z_3)...(a_n, z_n)(v, w)v_2) o(a_1)v_2) + \sum_{k=3}^{n} \sum_{j=0}^{n} (F_{wta_1,j}(z_1, z_k) - G_{wta_1,j}(z_1, z_k)) S(v'_3, (a_3, z_3)...(a_1(j)a, z_k)...(a_n, z_n)(v, w)v_2)$
so the lemma follows. □

It follows from the lemma that \( \lozenge(2) \) and \( \lozenge(3) \) can be written as

\[
\sum_{t=3}^{n} \sum_{k=3,k\neq t}^{n} \sum_{i,j \geq 0} F_{wta_{2,i}}(z_2, z_t) \left( \frac{wta_1 - 1}{j} \right) z_1^{-wta_1} z_k^{-wta_1 - 1 - j} \\
\cdot S(v'_3, (a_3, z_3)\ldots(a_1(j)a_k, z_k)\ldots(a_2(i)a_t, z_t)\ldots(a_n, z_n)(v, w)v_2)
\]

\( \lozenge(2) \) \( (21) \)

\[
+ \sum_{t=3}^{n} \sum_{i,j \geq 0} F_{wta_{2,i}}(z_2, z_t) \left( \frac{wta_1 - 1}{j} \right) z_1^{-wta_1} wta_1^{-1 - j} \\
\cdot S(v'_3, (a_3, z_3)\ldots(a_1(j)a_2(i)a_t, z_t)\ldots(a_n, z_n)(v, w)v_2)
\]

\( \lozenge(22) \) \( (22) \)

\[
= \lozenge(21) + \lozenge(22) + \lozenge(23)
\]

\[
\sum_{k=3}^{n} \sum_{i,j \geq 0} F_{wta_{2,i}}(z_2, z_k) \left( \frac{wta_1 - 1}{j} \right) z_1^{-wta_1} z_k^{-wta_1 - 1 - j} \\
\cdot S(v'_3, (a_3, z_3)\ldots(a_1(j)a_k, z_k)\ldots(a_2(i)a_t, z_t)\ldots(a_n, z_n)(v, w)v_2)
\]

\( \lozenge(31) \) \( (31) \)

\[
+ \sum_{i,j \geq 0} F_{wta_{2,i}}(z_2, z_t) \left( \frac{wta_1 - 1}{j} \right) z_1^{-wta_1} wta_1^{-1 - j} \\
\cdot S(v'_3, (a_3, z_3)\ldots(a_n, z_n)(a_1(j)a_2(i)v, w)v_2)
\]

\( \lozenge(32) \) \( (32) \)

\[
= \lozenge(31) + \lozenge(32)
\]

So that \( \lozenge = \lozenge(1) + \lozenge(21) + \lozenge(22) + \lozenge(23) + \lozenge(31) + \lozenge(32) \).

Now by \( (21) \) and the formula for expansion with respect to \( (a_2, z_2) \) on the left \( (21) \), we may express the terms \( \triangle, \blacksquare \) and \( \star \) as follows:
We express $\Delta$ as:

$$
\sum_{j \geq 0} - \left( wta_1 - 1 \right) z_1^{-wta_1} z_2^{wta_2} S(v_3\delta(a_1(j)a_2), (a_3, z_3)(a_n, z_n)(v, w)v_2) 
$$

(1)

$$
+ \sum_{k=3}^{n} \sum_{i,j \geq 0} - \left( wta_1 - 1 \right) z_1^{-wta_1} z_2^{wta_1-1-j} F_{wta_1+wta_2-j-1,i}(z_2, z_k)
\cdot S(v_3', (a_3, z_3)...(a_1(j)a_2)(i)a_k, z_k)...(a_n, z_n)(v, w)v_2)
$$

(2)

$$
+ \sum_{i,j \geq 0} - \left( wta_1 - 1 \right) z_1^{-wta_1} z_2^{wta_1-1-j} F_{wta_1+wta_2-j-1,i}(z_2, w)
\cdot S(v_3', (a_3, z_3)...(a_n, z_n)((a_1(j)a_2)(i)v, w)v_2)
$$

(3)

$$
= \Delta(1) + \Delta(2) + \Delta(3).
$$

We express $\bullet$ as:

$$
\sum_{k=3}^{n} \sum_{j \geq 0} - \left( wta_1 - 1 \right) z_1^{-wta_1} z_k^{wta_1-1-j} S(v_3\delta(a_2), (a_3, z_3)...(a_1(j)a_k, z_k)...(v, w)v_2)z_2^{-wta_2}
$$

(1)

$$
+ \sum_{k=3}^{n} \sum_{j,i \geq 0} - \left( wta_1 - 1 \right) z_1^{-wta_1} z_k^{wta_1-1-j} F_{wta_2,i}(z_2, w)
\cdot S(v_3', (a_3, z_3)...(a_1(j)a_k, z_k)...(a_n, z_n)(a_2(i)v, w)v_2)
$$

(2)

$$
+ \sum_{k=3}^{n} \sum_{j,i \geq 0} \sum_{t=3,t\neq k}^{n} - \left( wta_1 - 1 \right) z_1^{-wta_1} z_k^{wta_1-1-j} F_{wta_2,i}(z_2, z_t)
\cdot S(v_3', (a_3, z_3)...(a_2(i)a_t, z_t)...(a_1(j)a_k, z_k)...(a_n, z_n)(v, w)v_2)
$$

(3)

$$
+ \sum_{k=3}^{n} \sum_{j,i \geq 0} - \left( wta_1 - 1 \right) z_1^{-wta_1} z_k^{wta_1-1-j} F_{wta_2,i}(z_2, z_k)
\cdot S(v_3', (a_3, z_3)...(a_2(i)a_1(j)a_k, z_k)...(a_n, z_n)(v, w)v_2)
$$

(4)

$$
= \bullet(1) + \bullet(2) + \bullet(3) + \bullet(4).
$$
We express ★ as:

\[
\sum_{j \geq 0} \left( \frac{wta_1 - 1}{j} \right) z_1^{-wta_1} w^{wta_1 - 1 - j} S(v'_3 o(a_2), (a_3, z_3) \ldots (a_n, z_n)(a_1(j)v, w)v_2) z_2^{-wta_2}
\]

\[\text{(1)}\]

\[
+ \sum_{t=3}^{n} \sum_{j, i \geq 0} \left( \frac{wta_1 - 1}{j} \right) z_1^{-wta_1} w^{wta_1 - 1 - j} F_{wta-2,i}(z_2, z_k) \cdot S(v'_3, (a_3, z_3) \ldots (a_2(i)a_k, z_k) \ldots (a_n, z_n)(a_1(j)v, w)v_2)
\]

\[\text{(2)}\]

\[
+ \sum_{j, i \geq 0} \left( \frac{wta_1 - 1}{j} \right) z_1^{-wta_1} w^{wta_1 - 1 - j} F_{wta2,i}(z_2, w) \cdot S(v'_3, (a_3, z_3) \ldots (a_n, z_n)(a_2(i)a_1(j)v, w)v_2)
\]

\[\text{(3)}\]

\[
= ★(1) + ★(2) + ★(3).
\]

Now by lemma 2.6 and the formula (2.2) of \( \iota_{z_2, z_i} F_{n,i}(z_2, z_i) \), we have:

\[
\sum_{i, j \geq 0} \left( \frac{wta_1 - 1}{j} \right) F_{wta2,i}(z_2, z_i) a_1(j)a_2(i)a_t + \sum_{i, j \geq 0} \left( \frac{wta_1 - 1}{j} \right) F_{wta2,i}(z_2, z_i) a_1(j)a_2(i)a_t
\]

\[
+ \sum_{i, j \geq 0} \left( \frac{wta_1 - 1}{j} \right) F_{wta2+i}(z_2, z_i) a_1(j)a_2(i)a_t
\]

\[
= 0
\]

and the same equation holds if we replace \( z_i \) by \( w \) and \( a_t \) by \( v \) in the equation above.

Therefore, by looking at the terms we expressed above it is easily seen that \( \heartsuit(22) + \triangle(2) + \bigstar(4) = 0 \) and \( \heartsuit(32) + \triangle(3) + \bigstar(3) = 0 \). Moreover, it is also easy to see that \( \heartsuit(23) + \bigstar(2) = 0, \heartsuit(21) + \bigstar(3) = 0 \) and \( \heartsuit(31) + \bigstar(2) = 0 \). Now it remains to show \( \heartsuit(1) + \triangle(1) + \bigstar(1) + \bigstar(1) = 0 \), or equivalently,

\[
S(v'_3 o(a_1)o(a_2), (a_3, z_3) \ldots (a_n, z_n)(v, w)v_2) - S(v'_3 o(a_2), (a_3, z_3) \ldots (a_n, z_n)(v, w)o(a_1)v_2)
\]

\[
= \sum_{j \geq 0} \left( \frac{wta_1 - 1}{j} \right) S(v'_3 o(a_1(j)a_2), (a_3, z_3) \ldots (a_n, z_n)(v, w)v_2)
\]

\[\text{(2.22)}\]

\[
+ \sum_{k=3}^{n} \sum_{j \geq 0} \left( \frac{wta_1 - 1}{j} \right) z_k^{-wta_1 - 1 - j} S(v'_3 o(a_2), (a_3, z_3) \ldots (a_1(j)a_k, z_k) \ldots (v, w)v_2)
\]

\[
+ \sum_{j \geq 0} \left( \frac{wta_1 - 1}{j} \right) w^{wta1 - 1 - j} S(v'_3 o(a_2), (a_3, z_3) \ldots (a_n, z_n)(a_1(j)v, w)v_2),
\]

but this is a consequence of lemma 2.9. In fact,

\[\text{L.H.S of (2.22)}\]

\[
= S(v'_3 o(a_1)o(a_2), (a_3, z_3) \ldots (a_n, z_n)(v, w)v_2) - S(v'_3 o(a_2)o(a_1), (a_3, z_3) \ldots (a_n, z_n)(v, w)v_2)
\]
\[ + S(v_3' o(a_2) o(a_1), (a_3, z_3)...(a_n, z_n)(v, w)v_2) = S(v_3' o(a_2), (a_3, z_3)...(a_n, z_n)(v, w)o(a_1)v_2). \]

Note that \( S \) is linear with respect to the place \( M^d(0)^* \), so
\[
S(v_3' o(a_1) o(a_2), (a_3, z_3)...(a_n, z_n)(v, w)v_2) = S(v_3' o(a_1) o(a_2)), (a_3, z_3)...(a_n, z_n)(v, w)v_2)
\]
\[
= \sum_{j \geq 0} \left( \frac{wta_1 - 1}{j} \right) S(v_3' o(a_1(j) a_2), (a_3, z_3)...(a_n, z_n)(v, w)v_2),
\]
which is the first term on the right hand side of (2.22). On the other hand, by lemma 2.9
\[
S(v_3' o(a_2) o(a_1), (a_3, z_3)...(a_n, z_n)(v, w)v_2) = S(v_3' o(a_2), (a_3, z_3)...(a_n, z_n)(v, w)o(a_1)v_2)
\]
\[
= \sum_{k=3}^{n} \sum_{j \geq 0} \left( \frac{wta_1 - 1}{j} \right) w^{ta_1 - 1 - j} S(v_3' o(a_2), (a_3, z_3)...(a_1(j) a_k, z_k)...(v, w)v_2)
\]
\[
+ \sum_{j \geq 0} \left( \frac{wta_1 - 1}{j} \right) w^{ta_1 - 1 - j} S(v_3' o(a_2), (a_3, z_3)...(a_n, z_n)(a_1(j) v, w)v_2)
\]
which gives us the last two summands in (2.22). Therefore, (2.22) and hence (2.19) holds true.

Now let’s make a conclusion about what we’ve proved so far. It follows from (2.19), (2.20) and proposition 2.8 that all \((n+3)\)-point functions \( S^L_{V...M...V} \) and \( S^R_{V...M...VV} \) defined by (2.17) and (2.18) give rise to one single \((n+3)\)-point function:
\[
S : M^d(0)^* \times V \times ... \times M \times ... \times V \to \mathcal{F}(z_1, ..., z_n, w), \tag{2.23}
\]
where \( M \) can be placed anywhere in between first \( V \) and last \( V \). This function \( S \) satisfies the following properties. First,
\[
S(v_3, (a_1, z_1)...(a_n, z_n)(v, w)v_2) = S(v_3, (b_1, w_1)...(b_{n+1}, w_{n+1})v_2), \tag{2.24}
\]
where \( \{(b_1, w_1), (b_1, w_1), ..., (b_{n+1}, w_{n+1})\} \) is an arbitrary permutation of the set of pairs \( \{(a_1, z_1), ..., (a_n, z_n), (v, w)\} \). Second, \( S \) satisfies the following recurrent formula:
\[
S(v_3, (b_1, w_1)(b_2, w_2)...(b_{n+1}, w_{n+1})v_2) = S^L(v_3, (a_i, z_i) \square v_2) = S^R(v_3, \square(a_j, z_j)v_2) \tag{2.25}
\]
where the symbol \( \square \) in \( S^L \) represents an arbitrary combination of \((v, w)\), \( \{(a_k, z_k)\}_{k \neq i} \) and \( S^L \) is given by (2.17), which means that one expand in terms of \( (a_i, z_i) \) from the left. Meanwhile, the \( \square \) in \( S^R \) represents an arbitrary combination of \((v, w)\), \( \{(a_k, z_k)\}_{k \neq j} \) and \( S^R \) is given by (2.18), i.e expand in terms of \( (a_j, z_j) \) from the right.

Note that (2.24) and (2.25) are the same properties as (1), (2) of \((n+2)\)-point functions, so the induction step is complete. Therefore, we conclude that for any positive integer \( n \), there exists a \((n+3)\)-point function \( S \) as in (2.23) that satisfies the properties (2.24) and (2.25) (or properties (1) and (2)).
3. Extension of $S$

Following the proof of (2.2.8) and (2.2.9) in [10], it is easy to verify the following formulas for our (n+3)-point function $S[23]$:

\[
S(v'_3, (L(-1)a_1, z_1)\ldots(a_n, z_n)(v, w)v_2) = \frac{d}{dz_1} S(v'_3, (a_1, z_1)\ldots(a_n, z_n)(v, w)v_2)
\]

(3.1)

\[
S(v'_3, (L(-1)v, w)(a_1, z_1)\ldots(a_n, z_n)v_2) = \frac{d}{dw} S(v'_3, (v, w)(a_1, z_1)\ldots(a_n, z_n)v_2);
\]

\[
\int_C S(v'_3, (a_1, z_1)(v, w)\ldots(a_n, z_n)v_2)(z_1 - w)^n dz_1 = S(v'_3, (a_1(k)v, w)\ldots(a_n, z_n)v_2)
\]

(3.2)

\[
\int_C S(v'_3, (a_1, z_1)(a_2, z_2)\ldots(v, w)v_2)(z_1 - z_2)^2 dz_1 = S(v'_3, (a_1(k)a_2, z_2)\ldots(v, w)v_2),
\]

where in the first equation of (3.2) $C$ is a contour of $z_1$ surrounding $w$ with $z_2, \ldots, z_n$ outside of $C$; while in the second equation of (3.2) $C$ is a contour of $z_1$ surrounding $z_2$ with $z_3, \ldots, z_n, w$ outside of $C$.

We only present a proof for the second equation of (3.1) because the proof for rest of them will be exactly the same with corresponding ones in [10], so we omit them.

Use induction on $n$. When $n = 0$, by definition we have:

\[S(v'_3, (L(-1)v, w)v_2) = f(v'_3 \otimes L(-1)v \otimes v_2)w^{-wtv-1}.
\]

But in $A(M^1)$ one has: $L(-1)v + L(0)v \equiv 0 \mod O(M^1)$ [10]. This implies

\[f(v'_3 \otimes L(-1)v \otimes v_2)w^{-wtv-1} = -wtv \cdot f(v'_3 \otimes v \otimes v_2)w^{-wtv-1} = \frac{d}{dw} S(v'_3, (v, w)v_2).
\]

Now assume that formula (3.1) holds true for (n+2)-point functions, then by the properties [224] and (2.2a) we have:

\[
S(v'_3, (L(-1)v, w)(a_1, z_1)\ldots(a_n, z_n)v_2)
\]

\[
= S^L(v'_3, (a_1, z_1)\ldots(a_n, z_n)(L(-1)v, w)v_2)
\]

\[
= S(v'_3o(a_1), (a_2, z_2)\ldots(a_n, z_n)(L(-1)v, w)v_2)z_1^{-wtv}
\]

\[
+ \sum_{k=2}^{n} \sum_{j \geq 0} F_{wtv,a_1,j}(z_1, z_k)S(v'_3, (a_2, z_2)\ldots(a_1(j)a_k, z_k)\ldots(a_n, z_n)(L(-1)v, w)v_2)
\]

\[
+ \sum_{j \geq 0} F_{wtv,a_2,j}(z_1, w)S(v'_3, (a_2, z_2)\ldots(a_n, z_n)(a_1(j)L(-1)v, w)v_2).
\]

Note that we may apply induction hypothesis to the first two terms. Moreover, by some basic communication formulas of VOA that involve $L(-1)$ [8] one has:

\[a_1(j)L(-1)v_2 = L(-1)a_1(j)v_2 - [L(-1), a_1(j)]v_2 = L(-1)a_1(j)v_2 + ja_1(j - 1)v_2.
\]
It follows from the definition of \( F_{n,i}(z_1, w) \) that
\[
\sum_{j \geq 0} F_{wta_2,j}(z_1, w) S(v'_3, (a_2, z_2) \ldots (a_n, z_n)(a_1(j)L(-1)v, w)v_2) \\
= \sum_{j \geq 0} F_{wta_2,j}(z_1, w) \frac{d}{dw} S(v'_3, (a_2, z_2) \ldots (a_n, z_n)(a_1(j)v, w)v_2) \\
+ \sum_{j \geq 1} \frac{z_1^{wta_1}}{(j-1)!} \left( \frac{d}{dw} \right)^j \left( \frac{w^{wta_1}}{z_1 - w} \right) S(v'_3, (a_2, z_2) \ldots (a_n, z_n)(a_1(j-1)v, w)v_2) \\
= \frac{d}{dw} \sum_{j \geq 0} F_{wta_2,j}(z_1, w) S(v'_3, (a_2, z_2) \ldots (a_n, z_n)(a_1(j)v, w)v_2).
\]
This proves \( (3.1) \).

### 3.1. Extension of the last component

Our ultimate goal is to define an intertwining operator \( I(v, w) \) from \( S \), and intuitively we want \( I \) to be given by the equation:
\[
\langle v'_3, I(v, w)v_2 \rangle = S(v'_3, (v, w)v_2).
\]
However in this equation, the input vectors \( v_2 \) and \( v'_3 \) need to be taken over the whole vector space \( M^2 \) and \( M^3 \), not just the bottom level \( M^2(0) \) and \( M^3(0)^* \). Therefore, in order to properly define \( I \), we have to extend the first and last ”point” or input spaces of \( S \) from the bottom level to the whole space and get a new \((n+3)\)-point function:
\[
S : M^3 \times V \times \ldots \times M^1 \times \ldots \times V \times M^2 \to F(z_1, \ldots, z_n, w), \tag{3.3}
\]
then use this extended function to define the intertwining operator \( I \).

We will first extend the last input space of \( S \) from \( M^2(0) \) to \( M^2 \). The method we use here is similar with the way of extension in \([10]\). Let \( M \) be the same vector space as in \([10]\), which is a vector space spanned by symbols:
\[
(b_1, i_1)(b_1, i_2)\ldots(b_m, i_m)v_2 \tag{3.4}
\]
where \( b_i \in V, \ i_k \in \mathbb{Z}, \ v_2 \in M^2(0) \), and \((b, i)\) linear in \( b \). Denote the vector in \( (3.4) \) by \( x \).

Extend \( S \) to \( M^3(0)^* \times V \times \ldots \times M^1 \times \ldots \times V \times M^2 \) by letting:
\[
S(v'_3, (a_1, z_1)\ldots(a_n, v_n)(v, w)x) \\
= \int_{C_1} \ldots \int_{C_m} S(v'_3, (a_1, z_1)\ldots(a_n, v_n)(v, w)(b_1, w_1)\ldots(b_m, w_m)v_2)w_1^{i_1}\ldots w_m^{i_m}dw_1\ldots dw_m, \tag{3.5}
\]
where \( C_k \) is a contour of \( w_k \), \( C_k \) contains \( C_{k+1} \) and \( C_m \) contains \( 0 \); \( z_1, \ldots, z_n \) and \( w \) are outside \( C_1 \). Introduce a gradation on \( M \) by setting
\[
wt((b_1, i_1)(b_1, i_2)\ldots(b_m, i_m)v_2) := \sum_{k=1}^{m} (wtb_k - i_k - 1),
\]
then \( M \) becomes a graded space: \( \bar{M} = \bigoplus_{n \in \mathbb{Z}} \bar{M}(n) \) with \( M^2(0) \subseteq \bar{M}(0) \).
Similar as in [10], we define the radical of $S$ on $\bar{M}$ by
\[
Rad(\bar{M}) := \{ x \in \bar{M} | S(v'_3, (a_1, z_1)\ldots(a_n, z_n)(v, w)x) = 0, \forall a_i \in V, v \in M^1, v_3 \in M^3(0)^* \}. \tag{3.6}
\]

Even through our (n+3)-point function $S$ is not exactly same with the corresponding one in [10], but the recurrent formula (2.17) which defines $S$ are the same, and our $S$ also has intrinsic properties (3.1) and (3.2). So it’s not hard to show that $Rad(\bar{M})$ and the quotient space $M = \bar{M}/Rad(\bar{M})$ share the same properties as the corresponding ones in [10]. In particular, we will show $M = \bar{M}/Rad(\bar{M})$ is an irreducible admissible module with bottom level $M^2(0)$.

**Proposition 3.1.** Let $W$ be the subspace of $\bar{M}$ spanned by the following vectors:

\[
\sum_{i=0}^{\infty} \binom{m}{i} (a(l+i)b, m+n-i)v_2
\]

\[(\sum_{i=0}^{\infty} (-1)^i \binom{l}{i} (a, m+l-i)(b, n+i)v_2 - \sum_{i=0}^{\infty} (-1)^{i+1} \binom{l}{i} (b, n+l-i)(a, m+i)v_2) \tag{3.7}\]

where $a, b \in V, m, n, l \in \mathbb{Z}, v_2 \in M^2(0)$. Then we have $W \subset Rad(\bar{M})$.

**Proof.** Denote the element of (3.7) by $y$. We adopt the following notations in [6]. Let $C_R^i$ be the circle of $w_1, i = 1, 2$ centered at $0$ with radius $R$, and let $C_\epsilon^i(w_2)$ be the circle of $w_1$ centered at $w_2$ with radius $\epsilon$. Choose $R, r, \rho > 0$ so that $R > \rho > r$. In view of (3.5) and (2.24), we have:

\[
S(v'_3, (a_1, z_1)\ldots(a_n, z_n)(v, w)y)
\]

\[
= \int_{C_R^i} \sum_{i=0}^{\infty} \binom{m}{i} S(v'_3, (a_1, z_1)\ldots(a_n, z_n)(v, w)(a(l+i)b, w_2)v_2)w_2^{m+n-i}dw_2
\]

\[-\int_{C_R^i} \int_{C_R^j} \sum_{i=0}^{\infty} (-1)^i \binom{l}{i} S(v'_3, (a_1, z_1)\ldots(a_n, z_n)(v, w)(a, w_1)(b, w_2)v_2)w_1^{m+i}w_2^{n+i}dw_1dw_2
\]

\[+ \int_{C_R^i} \int_{C_R^j} \sum_{i=0}^{\infty} (-1)^{i+1} \binom{l}{i} S(v'_3, (a_1, z_1)\ldots(a_n, z_n)(v, w)(b, w_2)(a, w_1)v_2)w_1^{m+i}w_2^{n+i}dw_1dw_2
\]

\[= \int_{C_R^i} \sum_{i=0}^{\infty} \binom{m}{i} S(v'_3, (a_1, z_1)\ldots(a_n, z_n)(v, w)(a(l+i)b, w_2)v_2)w_2^{m+n-i}dw_2
\]

\[-\int_{C_R^i} \int_{C_R^j} S(v'_3, (a_1, z_1)\ldots(a_n, z_n)(v, w)(a, w_1)(b, w_2)v_2) \cdot \langle (w_1 - w_2)'w_1^m w_2^n dw_1 dw_2
\]

\[+ \int_{C_R^i} \int_{C_R^j} S(v'_3, (a_1, z_1)\ldots(a_n, z_n)(v, w)(b, w_2)(a, w_1)v_2) \cdot \langle (w_2 - w_1)'w_1^m w_2^n dw_1 dw_2
\]
= \int_{C_2^R} \sum_{i=0}^{\infty} \left( \frac{m}{i} \right) S(v_3', (a_1, z_1) \ldots (a_n, z_n)(v, w)(a(l + i)b, w_2)v_2) w_2^{m+n-i} dw_2 \\
- \int_{C_2^R} \int_{C_1^R - C_1^r} S(v_3', (a_1, z_1) \ldots (a_n, z_n)(v, w)(a, w_1)(b, w_2)v_2)(w_1 - w_2)^l w_1^m w_2^n dw_1 dw_2.

Since the only possible poles of the function

\[ S(v_3', (a_1, z_1) \ldots (a_n, z_n)(v, w)(b, w_2)(a, w_1)v_2)(w_1 - w_2)^l w_1^m w_2^n \]

in the area enclosed by \( C_1^R - C_1^r \) are at \( w_1 = w_2 \), so by Cauchy’s integral theorem we have:

\[
\int_{C_2^R} \int_{C_1^R - C_1^r} S(v_3', (a_1, z_1) \ldots (a_n, z_n)(v, w)(a, w_1)(b, w_2)v_2)(w_1 - w_2)^l w_1^m w_2^n dw_1 dw_2 \\
= \int_{C_2^R} \int_{C_1^r(w_2)} S(v_3', (a_1, z_1) \ldots (a_n, z_n)(v, w)(a, w_1)(b, w_2)v_2)(w_1 - w_2)^l w_1^m w_2^n dw_1 dw_2.
\]

On the other hand, we may choose \( \epsilon \) small enough so that when \( w_1 \) is taken from \( C_1^r(w_2) \), one has \( |w_1 - w_2| < |w_2| \). Hence in the integral above, we may write:

\[
(w_1 - w_2)^l w_1^m w_2^n = (w_1 - w_2)^l (w_2 + w_1 - w_2)^m w_2^n = \sum_{i=0}^{\infty} \left( \frac{m}{i} \right) (w_1 - w_2)^l+i w_2^{m+n-i}.
\]

Now by (3.2) we have:

\[
\int_{C_2^R} \int_{C_1^r(w_2)} S(v_3', (a_1, z_1) \ldots (a_n, z_n)(v, w)(a, w_1)(b, w_2)v_2)(w_1 - w_2)^l w_1^m w_2^n dw_1 dw_2 \\
= \int_{C_2^R} \int_{C_1^r(w_2)} \sum_{i=1}^{\infty} \left( \frac{m}{i} \right) S(v_3', (a_1, z_1) \ldots (a_n, z_n)(v, w)(a, w_1)(b, w_2)v_2) \\
\cdot (w_1 - w_2)^l+i w_2^{m+n-i} dw_1 dw_2 \\
= \int_{C_2^R} \sum_{i=1}^{\infty} \left( \frac{m}{i} \right) S(v_3', (a_1, z_1) \ldots (a_n, z_n)(v, w)(a(l + i)b, w_2)v_2) w_2^{m+n-i} dw_1 dw_2,
\]

which implies \( S(v_3', (a_1, z_1) \ldots (a_n, z_n)(v, w)y) = 0 \).

\[ \square \]

**Remark 3.2.** (1) The graph of contours in the proof above looks like:

![Diagram](image-url)
(2) By the definition \([5.5]\) of our extended \(S\), it is easy to see that if we replace \(v_2\) in \([5.7]\) by an arbitrary element \(x \in \tilde{M}\) the conclusion remains valid. That is to say: for any \(x \in \tilde{M}, a, b \in V\) and \(m, n \in \mathbb{Z}\) the element

\[
\sum_{i=0}^{\infty} \binom{m}{i} (a(l + i)b, m + n - i)x
- \left(\sum_{i=0}^{\infty} (-1)^i \binom{l}{i} (a, m + l - i)(b, n + i)x - \sum_{i=0}^{\infty} (-1)^{l+i} \binom{l}{i} (b, n + l - i)(a, m + i)x\right)
\]

(3.8)

belongs to \(\text{Rad}(\tilde{M})\).

Furthermore, we observe the following facts about \(\text{Rad}(\tilde{M})\).

**Lemma 3.3.** (1) If \(x \in \text{Rad}(\tilde{M})\), then \((b, i)x \in \text{Rad}(\tilde{M})\), \(\forall b \in V, i \in \mathbb{Z}\);

(2) \(M^2(0) \cap \text{Rad}(\tilde{M}) = 0\);

(3) If \(n < 0\) one has \(M(n) \subset \text{Rad}(\tilde{M})\).

**Proof.** (1) By definition \([3.5]\) of \(S\) and definition of \(\text{Rad}(\tilde{M})\), we see that

\[
S(v'_3, (a_1, z_1) ... (v, w)(b, i)x) = \int_C S(v'_3, (a_1, z_1) ... (v, w)(b, w_1)x)w_1^1dw_1 = \int_C 0 \cdot w_1^1dw_1 = 0
\]

where \(C\) is a contour of \(w_1\) with \(z_1, ..., z_n, w\) outside. Thus, \((b, i)x \in \text{Rad}(\tilde{M})\).

(2) Suppose there exists some \(v_2 \neq 0\) in \(M^2(0) \cap \text{Rad}(M)\), then by the definition of 4-point functions in section 2.2, we have for any \(a \in V\):

\[
0 = t_{w, z}(S(v'_3, (a, z)(v, w)v_2)) = S(v'_3, (v, w)a(v)v_2)z^{-wta} + \sum_{i \geq 0} t_{w, z}(G_{wta, i}(z, w))S(v'_3, (a(i)v, w)v_2) = f(v'_3 \otimes v \otimes a(v)v_2)z^{-wta}w^{-wtv}
- \sum_{i, j \geq 0} \binom{wta - 2 - j}{i}w^{wtv-j}z^{-wta+1+j}f(v'_3 \otimes a(i)v \otimes v_2).
\]

By comparing the coefficients of \(z^{-wta}w^{-wtv}\) on both sides of this equation, we see that \(f(v'_3 \otimes v \otimes a(v)v_2) = 0\) for any \(a \in V\), \(v \in A(M^1)\), \(v'_3 \in M^3(0)^*\). Moreover, since \(M^2(0)\) is an irreducible \(A(V)\) module we have \(M^2(0) = A(V).v_2 = \text{span}\{o(a)v_2|a \in V\}\). It follows that \(f = 0\), and this is a contradiction.

(3) Let \(x = (b_m, i_m) ... (b_1, i_1)v_2\) with \(\sum_{k=1}^{m}(wtb_k - i_k - 1) < 0\). We use induction on the length \(m\) of \(x\) to show \(x \in \text{Rad}(\tilde{M})\). For base case, let \(x = (b, t)v_2\) with \(wtb - t - 1 < 0\),
then by (3.5) we have
\[ S(v'_3, (a_1, z_1)\ldots(v, w)x) = \int_C S(v'_3, (a_1, z_1)\ldots(v, w)(b, z)v_2)z^dz \]
\[ = \int_C S(v'_3, (a_1, z_1)\ldots(a_n, z_n)(v, w)o(b)v_2)z^dz \]
\[ + \int_C \sum_{k=1}^n \sum_{i \geq 0} G_{wtb,i}(z, z_k)S(v'_3, (a_1, z_1)\ldots(b(i)a_k, z_k)\ldots(v, w)v_2)z^dz \]
\[ + \int_C \sum_{i \geq 0} G_{wtb,i}(z, w)S(v'_3, (a_1, z_1)\ldots(b(i)v, w)v_2)z^dz, \] (3.9)
where \( C \) is a contour of \( z \) surrounding 0 with all other variables lying outside \( C \). In particular, \(|z| < |z_k| \) for any \( k \) and \(|z| < |w| \). Hence for any \( i \geq 0 \) we have:
\[ \int_C G_{wtb,i}(z, z_k)z^idz = \int_C \frac{z^{-wtb+1+t}}{i!} \left( \frac{d}{dz} \right)^i \left( \frac{z^{wtb-1}}{z-z_k} \right) dz = 0, \] (3.10)
since \(-wtb + 1 + t > 0 \) and \( 1/(z-z_k) \) is a sum of positive powers in \( z \) within the contour \( C \). Therefore, since \( t - wt b \) is greater than \(-1 \), all the integrals on the right hand side of (3.9) are equal to 0. The proof for base case is finished.

For general \( z = (b_m, i_m)\ldots(b_1, i_1)v_2 \), by definition (3.5) we have:
\[ S(v'_3, (a_1, z_1)\ldots(v, w)x) \]
\[ = \int_{C_m} \ldots \int_{C_1} S(v'_3, (a_1, z_1)\ldots(v, w)(b_m, w_m)\ldots(b_1, w_1)v_2)w^i_m \ldots w^1_1 dw_1 \ldots dw_m \]
\[ = \int_{C_m} \ldots \int_{C_1} S(v'_3, (a_1, z_1)\ldots(v, w)(b_m, w_m)\ldots o(b_1)v_2)w^i_m \ldots w^{-wtb_1+i_1}_1 dw_1 \ldots dw_m \]
\[ + \int_{C_m} \ldots \int_{C_1} \sum_{k=1}^n \sum_{i \geq 0} G_{wtb,i}(w_1, w_k)S(v'_3, \ldots(b(i)a_k, z_k)\ldots(v, w)v_2)w^i_m \ldots w^1_1 dw_1 \ldots dw_m \]
\[ + \int_{C_m} \ldots \int_{C_1} \sum_{i \geq 0} G_{wtb,i}(w_1, w)S(v'_3, \ldots(b(i)v, w)(b_m, w_m)\ldots v_2)w^i_m \ldots w^1_1 dw_1 \ldots dw_m \]
\[ + \int_{C_m} \ldots \int_{C_1} \sum_{l=2}^m \sum_{i \geq 0} G_{wtb,i}(w_1, w_l)S(v'_3, \ldots(v, w)\ldots (b(i)b_1, w_l)\ldots v_2)w^i_m \ldots w^1_1 dw_1 \ldots dw_m \]
\[ = \diamondsuit + \star + \triangle + \blacksquare, \]
where \( C_1 \) is a contour of \( w_1 \) surrounding 0 with all other variables lying outside. We need to show that the sum of these integrals is equal to 0, i.e. \( \diamondsuit + \star + \triangle + \blacksquare = 0 \).

Case 1. \( wtb_1 - i_1 - 1 < 0 \).

Similar as (3.10), we have for any variable \( z \):
\[ \int_{C_1} G_{wtb,1}(w_1, z)w^i_1 dw_1 = \int_{C_1} \frac{w^{wtb_1+1+i_1}}{i!} \left( \frac{d}{dz} \right)^i \left( \frac{z^{wtb_1-1}}{w_1-z} \right) dw_1 = 0. \]
Thus, \( \star = \bigtriangleup = \bigcirc = 0 \), and we also have \( \heartsuit = 0 \), since \(-wtb_1 + i_1 > -1\).

Case 2. \( wtb_1 - i_1 - 1 > 0 \).

In this case we have \(-wtb_1 + i_1 < -1\), which implies \( \heartsuit = 0 \). Moreover, for any variable \( z \) we have the following evaluation:

\[
\int_{C_1} G_{wtb_1,i} (w_1,z) w_1^{i_1} dw_1 = - \int_{C_1} \sum_{j \geq 0} \left( \frac{wtb_1 - 2 - j}{i} \right) z^{wtb_1 - j - 2 - i} w_1^{i_1 wtb_1 + j + i} dw_1 \\
= \text{Res}_{w_1=0} \left( \sum_{j \geq 0} \left( \frac{wtb_1 - 2 - j}{i} \right) z^{wtb_1 - j - 2 - i} w_1^{i_1 wtb_1 + j + i} \right) \\
= - \left( \frac{i_1}{i} \right) z^{i_1 - i}.
\]

(3.11)

Apply the formula (3.11) to \( \star, \bigtriangleup, \) and \( \bigcirc \), we have:

\[
\star = - \int_{C_m} \int_{C_2} \sum_{k=1}^{n} \sum_{i \geq 0} \left( \frac{i_1}{i} \right) z_1^{i_1 - i} S(v_3', (a_1, z_1)) \ldots (b_1(i)a_k, z_k)(v, w)(b_m, w_m)(b_2, w_2)v_2 \\
= - \sum_{k=1}^{n} \sum_{i \geq 0} \left( \frac{i_1}{i} \right) z_1^{i_1 - i} S(v_3', (a_1, z_1)) \ldots (b_1(i)a_k, z_k)(a_n, z_n)(v, w)y),
\]

where \( y = (b_m, i_m) \ldots (b_2, i_2)v_2 \). Note that \( wt_y = wtx - (wtb_1 - i_1 - 1) < 0 \) and the length of \( y \) is \( m - 1 \), by induction hypothesis, \( S(v_3', (a_1, z_1)) \ldots (b_1(i)a_k, z_k)(a_n, z_n)(v, w)y) = 0 \) for any \( i \), then it follows that \( \bigstar = 0 \). Similarly, we have \( \bigtriangleup = 0 \). Finally,

\[
\bigcirc = \int_{C_m} \int_{C_2} \sum_{l=2}^{m} \sum_{i \geq 0} \left( \frac{i_1}{i} \right) w_1^{i_1 - i} S(v_3', (v, w)(b_1(i)b_l, w_1) \ldots dw_1 \ldots dw_m \\
= \sum_{l=2}^{m} \sum_{i \geq 0} \left( \frac{i_1}{i} \right) S(v_3', (a_1, z_1))(a_n, z_n)(v, w)y),
\]

where \( y_l = (b_m, i_m) \ldots (b_1(i)b_l, i_1 + i_l - i) \ldots (b_2, i_2)v_2 \). Note that

\[
wt(b_1(i)b_l, i_1 + i_l - i) = wt_b + wt_l - i - 1 - i_1 - i_l + i - 1 = wt(b_1, i_l) + wt(b_l, i_l).
\]

Thus, \( wt_y = \sum_{l=1}^{m} wt(b_k, i_k) = wtx < 0 \) but the length of \( y_l \) is \( m - 1 \). By induction, one has \( S(v_3', (a_1, z_1)) \ldots (a_n, z_n)(v, w)y) = 0 \) for any \( l \), which implies \( \bigcirc = 0 \).

Case 3. \( wt_b_1 - i_1 - 1 = 0 \). In this case one has for any variable \( z \):

\[
\int_{C_1} G_{wtb_1,i} (w_1,z) w_1^{i_1} dw_1 = \int_{C_1} \frac{w_1^0}{i!} \left( \frac{d}{dz} \right)^i \left( \frac{z^{wtb_1 - 1}}{w_1 - z} \right) dw_1 = 0.
\]

Hence \( \bigstar = \bigtriangleup = \bigcirc = 0 \). Finally, since \(-wtb_1 + i_1 = -1\), we have:

\[
\heartsuit = \int_{C_m} \int_{C_2} \int_{C_1} S(v_3', (a_1, z_1)) \ldots (v, w)(b_m, w_m) \ldots o(b_1)v_2 w_1^{i_1} dw_1 \ldots dw_m \\
= \int_{C_m} \int_{C_2} S(v_3', (a_1, z_1)) \ldots (v, w)(b_m, w_m) \ldots o(b_1)v_2 w_1^{i_1} w_1^{i_2} dw_2 \ldots dw_m.
\]
where $y = (b_m, i_m) \ldots (b_2, i_2)v_2$ with $wty = wtx < 0$ and length $m - 1$. Again by induction hypothesis, $S(v'_3, (a_1, z_1) \ldots (a_n, z_n)(v, w)y) = 0$, and so $\odot = 0$.

We define the vertex operator $Y_M$ on the quotient space $M = \overline{M}/\text{Rad}(\overline{M})$ as in [10]:

$$Y_M(a, z)(b_1, i_1) \ldots (b_m, i_m)v_2 := \sum_{n \in \mathbb{Z}} (a, n)(b_1, i_1) \ldots (b_m, i_m)v_2 z^{-n-1},$$

(3.12)

where $a \in V$, $(b_1, i_1) \ldots (b_m, i_m)v_2 \in M$, and we use the same notation $(b_1, i_1) \ldots (b_m, i_m)v_2$ for the equivalent class of this element in the quotient space $M$. We may also interpret (3.12) in component form as:

$$a(n). (b_1, i_1) \ldots (b_m, i_m)v_2 = (a, n)(b_1, i_1) \ldots (b_m, i_m)v_2,$$

(3.13)

for all $a \in V$, $n \in \mathbb{Z}$, and $(b_1, i_1) \ldots (b_m, i_m)v_2 \in M$. By part (1) of the lemma 3.3 we see that $a(n). \text{Rad}(\overline{M}) \subseteq \text{Rad}(\overline{M})$ i.e. $Y_M$ is well-defined. Now we claim that for any $x = (b_1, i_1) \ldots (b_m, i_m)v_2 \in M$ one has $1(-1)x = x$ and $1(n)x = 0$ for all $n \neq -1$.

In fact, by formulas (3.13), (2.17) together with the fact that $1(j).a = 0$ for any $a \in V$ and $j \geq 0$, we have:

$$S(v'_3, (a_1, z_1) \ldots (v, w)x)$$

$$= \int_{C_0} \int_{C_m} \ldots \int_{C_1} S(v'_3, (1, w_0)(a_1, z_1) \ldots (v, w)(b_1, w_1) \ldots (b_m, w_m)v_2)$$

$$\cdot w^n w_1^{i_1} \ldots w_m^{i_m} dw_1 \ldots dw_m dw_0$$

$$= \int_{C_0} \int_{C_m} \ldots \int_{C_1} S(v'_3 o(1), (a_1, z_1) \ldots (v, w)(b_1, w_1) \ldots (b_m, w_m)v_2) w^n w_1^{i_1} \ldots w_m^{i_m} dw_1 \ldots dw_m dw_0.$$

We note that $o(1) = 1d$, and

$$\int_{C_0} w_0^n dw_0 = \begin{cases} 0 & \text{if } n \neq -1 \\ 1 & \text{if } n = -1 \end{cases},$$

so our claim follows. Moreover, given $x = (b_1, i_1) \ldots (b_m, i_m)v_2 \in M$ and $a \in V$, we have $wt(a(n).x) = wta - n - 1 + wtx < 0$ if $n > 0$, so by part (3) of the lemma $a(n).x = 0$ if $n$ is large enough. Finally, by proposition 3.4 and (3.8), we have:

$$\sum_{i=0}^{\infty} \binom{m}{i} a(l + i)b(m + n - i)x$$

$$= \sum_{i=0}^{\infty} (-1)^i \binom{l}{i} a(m + l - i)b(n + i)x - \sum_{i=0}^{\infty} (-1)^{l+i} \binom{l}{i} b(n + l - i)a(m + i)x$$

(3.14)

for any $m, n, l \in \mathbb{Z}$; $a, b \in V$ and $x \in M$. Therefore, $Y_M$ satisfies the Jacobi identity and all the other axioms for a weak $V$ module, that is to say, $(M, Y_M)$ is a weak $V$ module.
Furthermore, we can show the following:

**Proposition 3.4.** $M$ has a gradation: $M = \bigoplus_{n \geq 0} M(n)$ with each $M(n)$ an eigenspace of $L(0)$ and $M(0) = M^2(0)$.

**Proof.** Let $M(n)$ be the image of $\bar{M}(n)$ under the quotient map. By lemma 3.3 we have

$$M = \sum_{n \geq 0} M(n), \quad \text{and} \quad M^2(0) \subseteq M(0).$$

In order to show that the subspaces $M(n)$ of $M$ form a direct sum, we first show that $L(0)$ acts semisimply on $M^2(0)$, and to this end we claim first that on $M^2(0)$:

$$a(wta - 1)v_2 = o(a)v_2 \quad (3.15)$$

In fact, it suffices to show that $(a, wta - 1)v_2 - o(a)v_2 \in \text{Rad}(\bar{M})$. By definition we have:

$$S(v'_3, (a_1, z_1)…(a_n, z_n)(v, w)(a, wta - 1)v_2)$$

$$= \int_C S(v'_3, (a_1, z_1)…(a_n, z_n)(v, w)(a, wta)v_2)w^{wta-1}_1dw_1$$

$$= \int_C S(v'_3, (a_1, z_1)…(a_n, z_n)(v, w)wta_1w^{wta-1}_1dw_1$$

$$+ \sum_{k=1}^n \sum_{i \geq 0} \int_C G_{wta,i}(w_1, z_k)S(v'_3, (a_1, z_1)…(a(i)a_k, z_k)…(a_n, z_n)(v, w)v_2)w^{wta-1}_1dw_1$$

$$+ \sum_{i \geq 0} \int_C G_{wta,i}(w_1, w)S(v'_3, (a_1, z_1)…(a_n, z_n)(a(i)v, w)v_2)w^{wta-1}_1dw_1,$$

where $C$ is a contour of $w_1$ surrounding 0 with other variables outside of $C$. Thus for any variable $z$, one has

$$\int_C G_{wta,i}(w_1, z)w^{wta-1}_1dw_1 = \int_C w^{wta-1}_1 \frac{w^{wta+1}_1}{i!} \left( \frac{d}{dz} \right)^i \left( \frac{z^{wta-1}}{w_1 - z} \right)dw_1 = 0,$$

which implies that all the integrals involving $G_{wta,i}$ are equal to 0, while

$$S(v'_3, (a_1, z_1)…(a_n, z_n)(v, w)(a, wta - 1)v_2) = S(v'_3, (a_1, z_1)…(a_n, z_n)(v, w)v_2).$$

Now the proof of (3.15) is complete.

Recall that $[L(0), a(n)] = (wta - n - 1)a(n)$ [8]. Apply (3.15) to $L(0) = \omega(wt\omega - 1)$, we have:

$$[L(0), o(a)]v_2 = [L(0), a(wta - 1)]v_2 = 0.$$

Thus, $L(0)$ is commutative with the action of $A(V)$ on the irreducible module $M^2(0)$. Since $M^2(0)$ is of countable dimension, then by Schur’s lemma $L(0) = \lambda \cdot Id$ on $M^2(0)$.
For any \( x = (b_1, i_1)\ldots(b_m, i_m)v_2 = b_1(i_1)\ldots b_m(i_m)v_2 \in M(n) \), it follows that:

\[
L(0)x = \left( \sum_{k=1}^{m} (wtb_k - k - 1) + \lambda \right)x = (n + \lambda)x.
\]

Therefore, each \( M(n) \) is an eigenspace of \( L(0) \) of eigenvalue \( n + \lambda \) for all \( n \in \mathbb{Z}_{\geq 0} \). This implies that \( M = \bigoplus_{n \geq 0} M(n) \).

Finally, for any \( x = (b_m, i_m)\ldots(b_1, i_1)v_2 \in M(0) \) we use induction on the length \( m \) of \( x \) to show \( x \in M^2(0) \). The base case \( m = 0 \) is clear. Consider general \( m > 0 \), if all \((b_j, i_j)\) have weight 0 then by (3.15) \( x = o(b_m)\ldots o(b_1)v_2 \in M^2(0) \), and so we are done. Otherwise, there must exist some \((b_j, i_j)\) has negative weight because \( wt(x) = 0 \). Without loss of generality, we assume that \( wt(b_m - i_m - 1) < 0 \). Then we may express \( x \) as:

\[
x = (b_{m-1}, i_{m-1})\ldots(b_1, i_1)(b_m, i_m)v_2 + [(b_m, i_m), (b_{m-1}, i_{m-1})\ldots(b_1, i_1)v_2]
\]

\[
= 0 + \sum_{j=1}^{m-1} (b_{m-1}, i_{m-1})\ldots[(b_m, i_m), (b_j, i_j)]\ldots(b_1, i_1)v_2.
\]

Note that each of the term \((b_{m-1}, i_{m-1})\ldots[(b_m, i_m), (b_j, i_j)]\ldots(b_1, i_1)v_2\) in the sum has the same weight as \( x \) but smaller length than \( x \), so by induction hypothesis they all belong to \( M^2(0) \), therefore \( x \in M^2(0) \) and \( M(0) = M^2(0) \).

Next, we show that \( M = M/\text{Rad}(M) \) is in fact an irreducible \( V \) module, and so \( M \) is isomorphic to \( M^2 \). Note that for any \( x \in M \), \( S(v'_3, (a_1, z_1)\ldots(a_n, z_n)(v, w)x) \) is also a rational function in \( z_1, \ldots, z_n, w \), and it has Laurent series expansion

\[
S(v'_3, (a_1, z_1)\ldots(a_n, z_n)(v, w)x) = S(v'_3, (v, w)(a_1, z_1)\ldots(a_n, z_n)x)
\]

\[
= \sum_{i_1, \ldots, i_n \in \mathbb{Z}} \int_{C_1} \ldots \int_{C_n} S(v'_3, (v, w)(a_n, z_n)\ldots(a_1, z_1)x)z_1^{i_1}\ldots z_n^{i_n}dz_1\ldots dz_n z_1^{-i_1-1} \ldots z_n^{-i_n-1}
\]

\[
= \sum_{i_1, \ldots, i_n \in \mathbb{Z}} S(v'_3, (v, w)a_n(i_n)\ldots a_1(i_1)x)z_1^{-i_1-1} \ldots z_n^{-i_n-1}
\]

(3.16)

on the domain \( \mathbb{D} = \{(z_1, \ldots, z_n, w)|w| > |z_n| > \ldots > |z_1| > 0\} \).

**Lemma 3.5.** For any \( y \in M(n) \) with \( n > 0 \) one has: \( S(v'_3, (v, w)y) = 0 \) for all \( v'_3 \in M^3(0)^* \) and \( v \in M^1 \).

**Proof.** It follows from an easy induction that \( y \) can be written as a sum of the terms \((b_m, i_m)\ldots(b_1, i_1)v_2\) with \( wt(b_j - i_j - 1) > 0 \) for all \( j \). Let \( y' = (b_m, i_m)\ldots(b_1, i_1)v_2 \), by definition and properties of the function \( S \), we have

\[
S(v'_3, (v, w)y') = \int_{C_m} \ldots \int_{C_1} S(v'_3, (v, w)(b_m, w_m)\ldots(b_1, w_1)v_2)w_1^{i_1}\ldots w_m^{i_m}dw_1\ldots dw_m
\]
Proof. Proposition 3.6. 

\[ \int_{C_m} \cdots \int_{C_1} S(v_3', (b_1, w_1) (v, w) (b_m, w_m) \cdots (b_2, w_2) v_2) w_1^{i_1} \cdots w_m^{i_m} dw_1 \cdots dw_m \]

\[ = \int_{C_m} \cdots \int_{C_1} S(v_3' o(b_1), (v, w) (b_m, w_m) \cdots (b_2, w_2) v_2) w_1^{-wtb_1+i_1} \cdots w_m^{i_m} dw_1 \cdots dw_m \]

\[ + \int_{C_m} \cdots \int_{C_1} \sum_{i \geq 0} F_{w_1, i} (w_1, w) S(v_3', (b_1 (i) v, w) (b_m, w_m) \cdots (b_2, w_2) v_2) w_1^{i_1} \cdots w_m^{i_m} dw_1 \cdots dw_m \]

\[ + \int_{C_m} \cdots \int_{C_1} \sum_{k=2}^m \sum_{i \geq 0} F_{w_1, i} (w_1, w_k) S(v_3', (v, w) (b_1 (i) b_k, w_k) \cdots (b_2, w_2) v_2) \cdot w_1^{i_1} \cdots w_m^{i_m} dw_1 \cdots dw_m \]

Since \(-wtb_1 + i_1 < -1\), we have for any variable \(z\) (with \(|z| > |w_1|\)) and any \(i \geq 0\)

\[ \int_{C_1} F_{w_1, i} (w_1, z) w_1^{i_1} dw_1 = \int_{C_1} \frac{w_1^{-wtb_1+i_1}}{i!} \left( \frac{d}{dz} \right)^i \left( \frac{z^{wtb_1}}{w_1 - z} \right) dw_1 = 0. \]

It follows that \(S(v_3', (v, w) y')\) and hence \(S(v_3', (v, w) y)\) is equal to 0.

\[ \square \]

Proposition 3.6. \(M = \bar{M} / \text{Rad}(\bar{M})\) is an irreducible \(V\) module with bottom level \(M^2(0)\). Consequently, \(M\) is isomorphic to the irreducible \(V\) module \(M^2\) that we were starting with.

Proof. Let \(N\) be a submodule of \(M\). Since \(M\) is a direct sum of eigenspace for \(L(0)\) and \(L(0).N \subseteq N\), we have \(N = \bigoplus_{n \geq 0} N(n)\). By (3.15), \(o(a) N(0) = a(wta - 1) N(0) \subseteq N(0)\), so \(N(0)\) is a \(A(V)\) submodule of \(M^2(0)\).

Since \(M^2(0)\) is an irreducible \(A(V)\) module, and it generate \(M\) as \(V\) module, we have \(N = M\) if \(N(0) = M^2(0)\). On the other hand, if \(N(0) \neq M^2(0)\) then \(N(0) = 0\), and we claim that in this case \(N = 0\).

In fact, it suffices to show that for any \(x \in N\) we have \(S(v_3', (a_1, z_1) \cdots (a_n, z_n) (v, w) x) = 0\) for any \(a_i \in V\) and \(v \in M^1\). By (3.10) we have on the domain \(\mathbb{D} = \{(z_1, \ldots, z_n, w) ||w| > |z_1| > \ldots > |z_n| > 0\}\) the Laurent series expansion of \(S(v_3', (a_1, z_1) \cdots (a_n, z_n) (v, w) x)\):

\[ S(v_3', (a_1, z_1) \cdots (a_n, z_n) (v, w) x) = \sum_{i_1, \ldots, i_n \in \mathbb{Z}} S(v_3', (v, w) a_n (i_n) \cdots a_1 (i_1) x) z_1^{-i_1} \cdots z_n^{-i_n}. \]

Since \(N\) is a submodule with \(N(0) = 0\), we have \(y = a_n (i_n) \cdots a_1 (i_1) x \in N\) and if \(y \neq 0\) then we have \(wt(y) > 0\). Now the lemma 3.5 tells us that \(S(v_3', (v, w) y) = 0\) for any such \(y\). Therefore, the Laurent series and hence the rational function \(S(v_3', (a_1, z_1) \cdots (a_n, z_n) (v, w) x)\) itself is equal to 0.

\[ \square \]

In conclusion, we proved that the extended function \(S : M^3(0)^* \times V \times \cdots \times M^1 \times \cdots \times V \times M \to \mathcal{F}(z_1, \ldots, z_n, w)\) can factor through \(M = M / \text{Rad}(M) \cong M^2\) and yield a well-defined \((n+3)\)-point function:

\[ S : M^3(0)^* \times V \times \cdots \times M^1 \times \cdots \times V \times M^2 \to \mathcal{F}(z_1, \ldots, z_n, w) \]
This $S$ is defined by formula (3.5). Namely, if $x = b_1(i_1)...b_m(i_m)v_2 \in M^2$ we have
\[
S(v_3; (a_1, z_1)...(a_n, v_n))(v, w)x)
:= \int_{C_1} \ldots \int_{C_m} S(v_3', (a_1, z_1)...(a_n, v_n))(v, w)(b_1, w_1)...(b_m, w_m)v_2)w_1^{i_1}...w_m^{i_m}dw_1...dw_m,
\]
where $C_k$ is a contour of $w_k$, $C_k$ contains $C_{k+1}$ and $C_m$ contains $0$; $z_1,...,z_n$ and $w$ are outside of $C_1$.

**Remark 3.7.** It seems we took a long way to extend the last input component of $S$ from $M^2(0)$ to $M^2$, but it is necessary. Because if we just use the formula (3.17) alone to extend our $S$ and don’t include $\overline{M}$ and $\text{Rad}(\overline{M})$ in our discussion, then we would have to check the well-definedness of $S$. Namely, if $x \in M^2$ has two expressions: $x = b_1(i_1)...b_m(i_m)v_2 = c_1(j_1)...c_k(j_k)v'_2$, we would have to show the right hand side of (3.17) yields the same value on these two different expressions. This is certainly not an easy task because we don’t know how many relations are there in $M^2$.

### 3.2. Extension of the first component.

There is a final preparation step before we can construct an intertwining operator $I$ out of $S$. We need to extend the first input component of $S$ from bottom level $M^3(0)^*$ to the whole irreducible module $M^3$.

Before we formally extend the first component of $S$, let’s make some observations first. These observations are actually the motivation of our definition for the new $S$. Let $M$ be a $V$ module, recall that on the contragradient module $M'[5]$ the action is given by:
\[
\langle Y'(a, z)x', x \rangle = \langle x', Y(e^{zL(1)}(-z^{-2})L(0)^{2}a, z^{-1})x \rangle,
\]
where $x \in M$, $x' \in M'$, and if we compare the coefficients of $z^{-n-1}$ on both sides, it is easy to see that:
\[
\langle a(n)x', x \rangle = \langle x', \sum_{j \geq 0} \frac{1}{j!}(-1)^wta(L(1)^4a)(2wta - n - j - 2)x \rangle.
\]

Let’s denote by $a'(n)$ the term $\sum_{j \geq 0} \frac{1}{j!}(-1)^wta(L(1)^4a)(2wta - n - j - 2)$.

Consider the term $\text{Res}_{w=0}\langle a(n)x', Y(b, w)x \rangle w^m$, we have the following computation:
\[
\text{Res}_{w=0}\langle a(n)x', Y(b, w)x \rangle w^m = \int_{C_2} \langle a(n)x', Y(b, w)x \rangle w^m dw
\]
\[
= \int_{C_2} \int_{C_1} \langle Y'(a, z)x', Y(b, w)x \rangle z^nw^m dz dw
\]
\[
= \int_{C_2} \int_{C_1} \langle x', Y(e^{zL(1)}(-z^{-2})L(0)^{2}a, z^{-1})Y(b, w)x \rangle z^nw^m dz dw,
\]
where $C_1$ is a contour of $z$ surrounding 0 with $w$ outside, and $C_2$ is a contour of $w$ which contains $C_1$. Without loss of generality, we assume $C_1, C_2$ are circles centered at 0 of radius $R_1$, $R_2$ respectively, $R_2 > R_1 > 0$ and $R_2 < 1$.

Let's make a substitution $z \rightarrow 1/z$ in the integral above. Note that the parametrization of $1/z$ is $(1/R_1)e^{-i\theta}$ which gives us clockwise orientation, and $d(1/z) = -(1/z^2)dz$. Thus, if we let $C_1'$ be the circle centered at 0 with radius $1/R_1$ (with counterclockwise orientation), then we have:

$$\text{Res}_{w=0} \langle a(n)x', Y(b, w)x \rangle w^m = \int_{C_2} \int_{C_1} \langle x', Y(e^{zL(1)}(z^{-2})L(0)^2, z^{-1})Y(b, w)x \rangle z^n w^m dzdw$$

$$= \int_{C_2} \int_{C_1} \langle x', Y(e^{z^{-1}L(1)}(-z^2)L(0), z)Y(b, w)x \rangle (-1)z^{-n-2}w^m dzdw$$

$$= \int_{C_2} \int_{C_1'} \langle x', Y(e^{z^{-1}L(1)}(-z^2)L(0), z)Y(b, w)x \rangle z^{-n-2}w^m dzdw,$$

(3.18)

where $C_1'$ is a circle with radius $R_1' = 1/R_1 > 1/R_2 > R_2$ (since $R_2 < 1$), that is, $C_1'$ is a contour of $z$ that contains the contour $C_2$ of the variable $w$.

Therefore, on the domain $\mathbb{D} = \{ w ||w| < 1 \}$ the rational function $(a(n)x', Y(b, w)x)$ has Laurent series expansion:

$$(a(n)x', Y(b, w)x) = \sum_{m \in \mathbb{Z}} \left( \int_{C_2} \int_{C_1} \langle x', Y(e^{z^{-1}L(1)}(-z^2)L(0), z)Y(b, w)x \rangle z^{-n-2}w^m dzdw \right)z^{-m-1},$$

and the right hand side is the Laurent series expansion of the rational function

$$\int_{C_1'} \langle x', Y(e^{z^{-1}L(1)}(-z^2)L(0), z)Y(b, w)x \rangle z^{-n-2}dz$$

on the same domain $\mathbb{D}$. Hence on $\mathbb{D}$ we have the following equality of rational functions:

$$(a(n)x', Y(b, w)x) = \int_{C_1'} \langle x', Y(e^{z^{-1}L(1)}(-z^2)L(0), z)Y(b, w)x \rangle z^{-n-2}dz,$$

(3.19)

where $C_1'$ is a contour of $z$ with $w$ inside.

Moreover, if we replace $a(n)$ in (3.19) by $a'(n)$, then we will get a more elegant formula:

$$(a'(n)x', Y(b, w)x) = \int_{C_1'} \langle x', Y(a, z)Y(b, w)x \rangle z^n dz.$$ 

(3.20)

In fact,

$$(a'(n)x', Y(b, w)x) = \sum_{j \geq 0} \frac{1}{j!} (-1)^{wta}(L(1)^j a)(2wta - n - j - 2)x', Y(b, w)x)$$

$$= \sum_{j \geq 0} \frac{1}{j!} (-1)^{wta} \int_{C_1'} \langle x', Y(e^{z^{-1}L(1)}(-z^2)L(0), z)Y(b, w)x \rangle z^{-2wta+n+j}dz$$

$$= \int_{C_1'} \langle x', Y(e^{z^{-1}L(1)}(-z^2)L(0), e^{zL(1)}(-z^2)L(0), z)Y(b, w)x \rangle z^n$$
We introduce the following two notations which we will use later.

Note that one may restrict the variables $z, D$ the domain $S, D$ $M, b$ where $S$ component of $C$ where $M$ extended function $S$ $b$ to be the vector space spanned by symbols of the following form:

\[
(\mathbf{x}', Y(a, z)Y(b, w)x)z^n dz.
\]

\[
= \int_C \langle (Y(e^{z^{-1}L(1)}e^{-z^{-1}L(1)}a, z)Y(b, w)x)z^n dz.
\]

We introduce the following two notations which we will use later.

\[
\mathcal{F}'(z_1, ..., z_n, w) : = \left\{ f(z_1, ..., z_n, w) \in \mathcal{F}(z_1, ..., z_n, w) \mid |z_i| < 1, \forall i; \ |w| < 1 \right\};
\]

\[
(a, z)' : = (e^{z^{-1}L(1)}(z^2) L(0) a, z).
\]

Note that one may restrict the variables $z_1, ..., z_n, w$ in $S(v_3', (a_1, z_1)...(a_n, z_n) (v, w)x_2)$ to the domain $\mathcal{D} = \{(z_1, ..., z_n, w) \mid |z_i| < 1, \forall i; \ |w| < 1\}$ and get a map:

\[
S : M^3(0)^* \times V \times M^1 \times \times V \times M^2 \rightarrow \mathcal{F}'(z_1, ..., z_n, w)
\]

(3.21)

From now on, we assume that $S$ is the one given in (??), i.e. we restrict the variables of $S$ in the domain $\mathcal{D}$. Motivated by formulas (3.19) and (3.20), we extend the first component $M^3(0)^*$ of $S$ to $M^3'$ in the following way:

Proceed like the extension in section 3.2, we define $\tilde{M}$ to be the vector space spanned by symbols of the following form:

\[
x = (b_1, i_1)...(b_m, i_m)v_3',
\]

where $b_j \in V$, $i_j \in \mathbb{Z}$ and $v_3' \in M^3(0)^*$ and $(b, i)$ is linear in $b$. Next, we extend the first component of $S$ from $M^3(0)^*$ to $\tilde{M}$ by letting:

\[
S((b_1, i_1)...(b_m, i_m)v_3', (a_1, z_1)...(a_n, z_n) (v, w)x_2)
\]

\[
= \int_{C_m} \cdots \int_{C_1} S(v_3', (b_m, w_m)'...(b_1, w_1)'(a_1, z_1)...(a_n, z_n) (v, w)x_2)w_1^{-i_1-2}...w_m^{-i_m-2} dw_1...dw_m,
\]

(3.22)

where $C_k$ is a contour of $w_k$, $C_k$ contains $C_{k-1}$ and $C_1$ contains all variables $z_1, ..., z_n, w$.

Similar as in section 3.2, one can show $\tilde{M}/\text{Rad}(\tilde{M})$ has a natural $V$ module structure defined by $S$, and one can show $\tilde{M}/\text{Rad}(\tilde{M})$ is an irreducible $V$ module isomorphic to $M^3'$. Since $S$ factor through $\tilde{M}/\text{Rad}(\tilde{M}) \cong M^3'$, it follows that we have a well-defined extended function $S$:

\[
S : M^3' \times V \times \times M^1 \times \times V \times M^2 \rightarrow \mathcal{F}'(z_1, ..., z_n, w)
\]

such that $S(b_1(i_1)...b_m(i_m)v_3', (a_1, z_1)...(a_n, z_n) (v, w)x_2)$ is given by (3.22):

\[
S((b_1, i_1)...b_m(i_m)v_3', (a_1, z_1)...(a_n, z_n) (v, w)x_2)
\]

\[
= \int_{C_m} \cdots \int_{C_1} S(v_3', (b_m, w_m)'...(b_1, w_1)'(a_1, z_1)...(a_n, z_n) (v, w)x_2)w_1^{-i_1-2}...w_m^{-i_m-2} dw_1...dw_m.
\]

(3.23)
Moreover, by a similar computation as the proof of formula (3.20), we have:
\[
S(b_1^i(i_1)\cdots b_n^i(i_m)v_3^i, (a_1, z_1)\cdots(a_n, z_n)(v, w)x_2) = \int_{C_m} \int_{C_1} S(v_3, (b_m, w_m)\cdots(b_1, w_1)(a_1, z_1)\cdots(a_n, z_n)(v, w)x_2)w_1^{i_1}\cdots w_m^{i_m}dw_1\cdots dw_m, 
\]
(3.24)
where \(z_1, \ldots, z_n, w\) are inside of \(C_1\).

Note that in the defining formula (3.17) and (3.24) of the extension of \(S\), the terms \((a_i, z_i)\ i = 1, \ldots, n\) and \((v, w)\) in the integrals can be permuted arbitrarily, thanks to (2.24). Therefore, our extended \(S\) also satisfies the permutation invariant property:

\[
S(x_3', (a_1, z_1)\cdots(a_n, z_n)(v, w)x_2) = S(x_3', (b_1, w_1)\cdots(b_n, w_n)(b_{n+1}, w_{n+1})x_2),
\]
(3.25)
where \(x_3' \in M^3\) and \(x_2 \in M^2\), and \(((b_1, w_1), \ldots, (b_{n+1}, w_{n+1}))\) is an arbitrary permutation of \((a_i, z_i)\ i = 1, \ldots, n\) and \((v, w)\).

Finally, it is easy to see from (3.25) that the formulas (3.1) and (3.2) also hold true for our extended \(S : M^3 \times V \times \cdots \times M^1 \times \cdots \times V \times M^2 \rightarrow \mathcal{F}'(z_1, \ldots, z_n, w)\). In other words, we have:

\[
S(x_3', (L(-1)a_1, z_1)\cdots(a_n, z_n)(v, w)x_2) = \frac{d}{dz_1}S(x_3', (a_1, z_1)\cdots(a_n, z_n)(v, w)x_2);
\]
\[
S(x_3', (L(-1)v, w)(a_1, z_1)\cdots(a_n, z_n)x_2) = \frac{d}{dw}S(x_3', (v, w)(a_1, z_1)\cdots(a_n, z_n)x_2).
\]
(3.26)

\[
\int_C S(x_3, (a_1, z_1)(v, w)\cdots(a_n, z_n)x_2)(z_1 - w)^n dz_1 = S(x_3', (a_1(k)v, w)\cdots(a_n, z_n)x_2);
\]
\[
\int_C S(x_3', (a_1, z_1)(a_2, z_2)\cdots(v, w)x_2)(z_1 - z_2)^n dz_1 = S(x_3', (a_1(k)a_2, z_2)\cdots(v, w)x_2).
\]
(3.27)

3.3. Proof fusion rules formula. With \((n+3)\)-point function

\[
S : M^3 \times V \times \cdots \times M^1 \times \cdots \times V \times M^2 \rightarrow \mathcal{F}'(z_1, \ldots, z_n, w)
\]
in our hand we construct an intertwining operator \(I \in I\left(\frac{M^3}{M^1M^2}\right)\) in the following way:

Let \(v \in M^1\), define \(v(n) : M^2 \rightarrow M^3 (= M^{3''})\) by the formula:

\[
\langle x_3', v(n)x_2 \rangle := \int_C S(x_3', (v, w)x_2)w^n dw,
\]
(3.28)
where \(C\) is a contour of \(w\) surrounding 0. Define \(I(v, w)\) by

\[
I(v, w) := \sum_{n \in \mathbb{Z}} v(n)w^{-n-1} \cdot w^{-h},
\]
where \(h = h_1 + h_2 - h_3\) and \(h_i, i = 1, 2, 3\) are the conformal weight of irreducible module \(M^i, i = 1, 2, 3\) correspondingly.
In other words, \( \langle x'_3, I(v, w)x_2 \rangle \) is defined as the Laurent series expansion of \( S(x'_3, (v, w)x_2)w^{-h} \) at \( w = 0 \). For simplicity, we omit the term \( w^{-h} \) in \( I(v, w) \) for the rest of this section and write:

\[
I(v, w) = \sum_{n \in \mathbb{Z}} v(n)w^{-n-1}.
\]

In particular, if we denote by \( (x'_3, I(v, w)x_2) \) the limit of the Laurent series \( \langle x'_3, I(v, w)x_2 \rangle \) in \( |w| < 1 \), then we have:

\[
(x'_3, I(v, w)x_2) = S(x'_3, (v, w)x_2)
\]
as rational functions. Now it follows from (3.26) that

\[
(x'_3, I(L(-1)v, w)x_2) = S(x'_3, (L(-1)v, w)x_2) = \frac{d}{dw}S(x'_3, (v, w)x_2) = \frac{d}{dw}(x'_3, I(v, w)x_2).
\]

Therefore, \( I(L(-1)v, w) = \frac{d}{dw}I(v, w) \). Moreover, \( I(v, w) \) satisfies the Jacobi identity:

**Proposition 3.8.** The operators \( v(n) \) defined in (3.28) satisfies the following:

\[
\sum_{i=0}^{\infty} \binom{m}{i} (a(l+i)v)(m+n-i)x_2
= \sum_{i=0}^{\infty} (-1)^i \binom{l}{i} a(m+l-i)v(n+i)x_2 - \sum_{i=0}^{\infty} (-1)^{l+i} \binom{l}{i} v(n+l-i)a(m+i)x_2
\]

(3.29)

In particular, \( I(v, w) \) is an intertwining operator of the type \( (\frac{M^3}{M^1, M^2}) \) [5].

**Proof.** By (3.28), (3.24) and the definition of contragradient module, we have

\[
\langle x'_3, \sum_{i=0}^{\infty} (-1)^i \binom{l}{i} a(m+l-i)v(n+i)x_2 \rangle
= \sum_{i=0}^{\infty} (-1)^i \binom{l}{i} \langle a'(m+l-i)x'_3, v(n+i)x_2 \rangle
= \sum_{i=0}^{\infty} (-1)^i \binom{l}{i} \int_{C'_1} S(a'(m+l-i)x'_3, (v, w)x_2)w^{n+i}dw
= \sum_{i=0}^{\infty} (-1)^i \binom{l}{i} \int_{C'_1} \int_{C'_2} S(x'_3, (a, z)(v, w)x_2)z^{m+l-i}w^{n+i}dw,
\]

(3.30)

where \( C'_1 \) is a contour of \( w \), \( C'_2 \) is a contour of \( z \) which contains \( C'_1 \).
On the other hand, by (3.17) we have:

\[
\langle x_3', \sum_{i=0}^{\infty} (-1)^{l+i} \frac{l}{i} v(n + l - i) a(m + i)x_2 \rangle
\]

\[
= \sum_{i=0}^{\infty} (-1)^{l+i} \frac{l}{i} \int_{C_1} S(x_3', (v, w)a(m + i)x_2)w^{n+l-i}dw
\]

\[
= \sum_{i=0}^{\infty} (-1)^{l+i} \frac{l}{i} \int_{C_1} \int_{C_2} S(x_3', (v, w)(a, z)x_2)z^{m+i+w^{n+l-i}}dzdw,
\]

where \(C_1\) is a contour of \(w\), \(C_2\) is a contour of \(z\) that is contained in \(C_1\).

We adopt the notations in proposition 3.1, let \(a, z, v, w\) be a circle of variable \(z\) centered at 0 with radius \(\alpha\), and let \(C_\beta^w\) be a circle of variable \(w\) centered at 0 with radius \(\beta\). Choose \(R, r, \rho > 0\) so that \(1 > R > \rho > r\). Combining (3.30), (3.31), (3.25) and (3.27), we have:

\[
\langle x_3', \sum_{i=0}^{\infty} (-1)^{l+i} \frac{l}{i} a(m + l - i)v(n + i)x_2 - \sum_{i=0}^{\infty} (-1)^{l+i} \frac{l}{i} v(n + l - i)a(m + i)x_2 \rangle
\]

\[
= \sum_{i=0}^{\infty} (-1)^i \frac{l}{i} \int_{C_1'} \int_{C_2'} S(x_3', (a, z)(v, w)x_2)z^{m+l-i}w^{n+i}dzdw
\]

\[
- \sum_{i=0}^{\infty} (-1)^{l+i} \frac{l}{i} \int_{C_1'} \int_{C_2'} S(x_3', (v, w)(a, z)x_2)z^{m+i+w^{n+l-i}}dzdw
\]

\[
= \int_{C_1'} \int_{C_2'} S(x_3', (a, z)(v, w)x_2)\ell_{w,z}(z - w)^{l}z^{m}w^{n}dzdw
\]

\[
- \int_{C_1'} \int_{C_2'} S(x_3', (v, w)(a, z)x_2)\ell_{w,z}(z - w)^{l}z^{m}w^{n}dzdw
\]

\[
= \int_{C_1'} \int_{C_2'} S(x_3', (a, z)(v, w)x_2)(z - w)^{l}z^{m}w^{n}dzdw
\]

\[
= \int_{C_1'} \int_{C_2'} S(x_3', (a, z)(v, w)x_2)(z - w)^{l}w^{n}dzdw
\]

\[
= \sum_{i=0}^{\infty} \binom{m}{i} \int_{C_1'} \int_{C_2'} S(x_3', (a, z)(v, w)x_2)(z - w)^{l+i}w^{n+m-i}dzdw
\]

\[
= \sum_{i=0}^{\infty} \binom{m}{i} S(x_3', (a(l + i)v, w)x_2)w^{m+n-i}
\]

\[
= \sum_{i=0}^{\infty} \binom{m}{i} \langle x_3', (a(l + i)v)(m + n - i)x_2 \rangle
\]

(3.32)

The graph of different contours in (3.32) is given as follows:
Since $x_3'$ in (3.32) can be taken arbitrarily, it follows that (3.29) and hence the Jacobi identity of $I(v, w)$ holds.

Therefore, starting with $f \in (M^3(0) \otimes_{A(V)} A(M^1) \otimes_{A(V)} M^2(0))^*$ we can construct a $(n+3)$-point rational function $S$:

$$S : M^3 \times V \times \ldots \times M^1 \times \ldots \times V \times M^2 \rightarrow F'(z_1, \ldots, z_n, w),$$

and using this $S$ we can define an intertwining operator $I = I_f$ of the type $(M^3(M^1, M^2)^*)$.

Finally, we claim that $\pi(I_f) = f$, where

$$\pi : I \begin{pmatrix} M^3 \\ M^1, M^2 \end{pmatrix} \rightarrow (M^3(0)^* \otimes_{A(V)} A(M^1) \otimes_{A(V)} M^2(0))^*$$

is given by $\pi(I)(v'_3 \otimes v \otimes v_2) = \langle v_3', o(v)v_2 \rangle$ as in (1.3), where $v'_3 \in M^3(0)$, $v_2 \in M^2(0)$ and $v \in M^1$.

In fact, by (3.28) and the definition of $S$ we have:

$$\pi(I_f)(v'_3 \otimes v \otimes v_2) = \langle v'_3, v(wtv - 1)v_2 \rangle$$

$$= \int_C S(v'_3, (v, w)v_2)w^{wtv-1}dw$$

$$= \int_C f(v'_3 \otimes v \otimes v_2)w^{-wtv}w^{wtv-1}dw$$

$$= f(v'_3 \otimes v \otimes v_2).$$

(3.33)

This shows $\pi(I_f) = f$. Therefore, the map $\pi$ is surjective. It is shown in [9] that the map $\pi$ is injective provided $M^2$ and $M^3$ are irreducible $V$ modules, hence the map $\pi$ in (1.3) is an isomorphism of vector spaces. Now the proof of fusion rules formula is complete.
3.4. Some Generalizations. Actually, based on our proof of the fusion rules formula we can derive some more facts about $A(V)$ bimodules and fusion rules space $I(M^3_{M^1 M^2})$. Recall that in section 3.2, we’ve proved that $M^2 \cong \hat{M}/\text{Rad}(\hat{M})$. Now if we take any $V$ module $N^2$ with bottom level $N^2(0) = M^2(0)$, then because $M^2 \cong \hat{M}/\text{Rad}(\hat{M})$ is irreducible and the bottom level of which is $M^2(0)$, the universal property guarantees the existence of an epimorphism:

$$N^2 \rightarrow \hat{M}/\text{Rad}(\hat{M}).$$

It follows that the last component of $S$ can be extended to a bigger space $N^2$:

$$S : M^3(0)^* \times V \times \ldots \times M^1 \times \ldots \times V \times N^2 \rightarrow \mathcal{F}'(z_1, \ldots, z_n, w)$$

Similarly, if we take any $V$ module $N^3$ with bottom level $N^3(0) = M^3(0)$, then the first component of $S$ can also be extended to $N^3$, and so we have a well-defined function:

$$S : N^3 \times V \times \ldots \times M^1 \times \ldots \times V \times N^2 \rightarrow \mathcal{F}'(z_1, \ldots, z_n, w). \quad (3.34)$$

Now one can use this new $S$ to construct an intertwining operator $I'_f$ of type $(M^3_{N^2})$ by letting:

$$\langle x'_3, v(n)x_2 \rangle := \int_C S(x'_3, (v, w)x_2)w^n dw$$

as in (3.28), while in this case $x'_3 \in N^3$, $x_2 \in N^2$ instead.

It is easy to see that the proof in section 3.3 goes through for this $I'_f$, since in the proof we’ve only used the properties (3.24), (3.26) and (3.27) of $S$, and these properties are also satisfied by our new $S$ in (3.34). Furthermore, it is also straightforward to see that $\pi(I'_f) = f$ as in (3.33). Thus, we have the following more general theorem:

**Theorem 3.9.** Let $N^2$, $N^3$ be any $V$ modules with bottom level $M^2(0)$, $M^3(0)$ respectively, where $M^2(0)$, $M^3(0)$ are irreducible $A(V)$ modules. Then the map

$$\pi : I(M^3_{M^1 N^2}) \rightarrow (M^3(0)^* \otimes_{A(V)} A(M^1) \otimes_{A(V)} M^2(0))^*$$

given by (1.3) is surjective.

Our proof can also be generalized from $A_n(V)$ perspective. Recall that in $[3]$ a sequence of associative algebras $A_n(V)$, where $n \in \mathbb{Z}_{\geq 0}$ are defined. They satisfy the following:

1. For any $V$ modules $M = \bigoplus_{n=0}^{\infty} M(n)$ the first $n$ level $M(0) \oplus M(1) \oplus \ldots \oplus M(n)$ is a left $A_n(V)$ module with action given by:

$$A_n(V) \rightarrow \text{End} \left( \bigoplus_{i=0}^{n} M(i) \right) : [a] \mapsto o(a) = a(wta - 1).$$

In particular, the bottom level $M(0)$ is a left $A_n(V)$ module for any $n \geq 0$. 

For any $V$ module $M$ there is a well-defined $A_n(V)$ bimodule $A_n(M)$ with the left and right module action $a *_n v$ and $v *_n a$ satisfy:

$$a *_n v - v *_n a = \text{Res}_z Y(a, z)v(1 + z)^{wt a - 1}$$ (3.35)

Now let $g : M^3(0)^* \otimes_{A_n(V)} A_n(M^1) \otimes_{A_n(V)} M^2(0)$ be a linear functional. We may use the same formulas in section 2.2 to define a rational function $S$, i.e. we first define a 3-point function $S_M : M^3(0)^* \times M^1 \times M^2(0) \to \mathcal{F}(w)$ by $S_M(v_3', (v, w)v_2) := g(v_3' \otimes v \otimes v_2)w^{wt v}$ as in (2.8), then use the left and right expansion formula to extend it to a $n$-point function.

Recall that in the proof of the well-definedness and "locality" of $S$ in section 2.2 when it comes to the $A(V)$ bimodule $A(M^1)$, the only essential property of this bimodule we’ve used is the formula:

$$a * v - v * a = \text{Res}_z Y(a, z)v(1 + z)^{wt a - 1}.$$

See proposition 2.8 and the computation in page 11 for example. Since the same formula (3.35) holds for the $A_n(V)$ bimodule $A_n(M)$, all the computations in section 2 go through for the $S$ defined from linear functional $g$, and so the $S$ defined by $g$ is the same $S$ as in (2.23). Therefore, the right hand side of the fusion rules formula theorem 1.1 can be generalized to $A_n(V)$ and $A_n(M^1)$:

**Theorem 3.10.** Let $M^1$, $M^2$ and $M^3$ be three $V$ modules with $M^2$, $M^3$ irreducible, then the map

$$\pi : I \left( \begin{array}{c} N^4 \\ M^1, N^2 \end{array} \right) \to (M^3(0)^* \otimes_{A_n(V)} A_n(M^1) \otimes_{A_n(V)} M^2(0))^*$$

given by $\pi(I)(v_3' \otimes v \otimes v_2) = \langle v_3', o(v)v_2 \rangle$ is a linear isomorphism.

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