THE UNITAL Ext-GROUPS AND CLASSIFICATION OF $C^*$-ALGEBRAS

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Abstract. The semigroups of unital extensions of separable $C^*$-algebras come in two flavours: a strong and a weak version. By the unital Ext-groups, we mean the groups of invertible elements in these semigroups. We use the unital Ext-groups to obtain $K$-theoretic classification of both unital and non-unital extensions of $C^*$-algebras, and in particular we obtain a complete $K$-theoretic classification of full extensions of UCT Kirchberg algebras by stable AF algebras.

1. Introduction

Elliott’s programme of classifying nuclear $C^*$-algebras has seen great recent success in the case of finite, simple $C^*$-algebras due to the work of many hands, most prominently by work of Elliott, Gong, Lin, and Niu [GLN15], [EGLN15], as well the Quasidiagonality Theorem of Tikuisis, White, and Winter [TWW17]. This crowning achievement together with the groundbreaking Kirchberg–Phillips classification of purely infinite, simple $C^*$-algebras [Kir94], [Phi00] completes the classification of separable, unital, simple $C^*$-algebras with finite nuclear dimension which satisfy the universal coefficient theorem (UCT).

The main focus of this paper is the classification of non-simple $C^*$-algebras. The non-simple classification is especially convoluted due to the lack of dichotomy between the purely infinite and the stably finite case. A rich class of non-simple $C^*$-algebras failing this dichotomy is the class of graph $C^*$-algebras. Great progress was made recently in [ERRS16], where all unital graph $C^*$-algebras were classified by a $K$-theoretic invariant.

The classification of unital graph $C^*$-algebras was an internal classification result, in the sense that it can only be used to compare objects which are already known to be unital graph $C^*$-algebras. The lack of external classification prevents the result from being applicable in the study of permanence properties for the class of graph $C^*$-algebras. For instance, it is an open problem whether extensions of graph $C^*$-algebras are again graph $C^*$-algebras, subordinate to $K$-theoretic obstructions. The main results of this paper will be used to solve this question for extensions of simple graph $C^*$-algebras in [EGK+18].

The focal point for us is the classification of extensions of classifiable $C^*$-algebras. In seminal work of Rørdam [Rør97], a Weyl–von Neumann–Voiculescu type absorption theorem of Kirchberg was applied to obtain classification of extensions of non-unital UCT Kirchberg algebras. This absorption theorem was generalised by Elliott and Kucerovsky [EK01], thus...

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A UCT Kirchberg algebra is a separable, nuclear, simple, purely infinite $C^*$-algebra satisfying the universal coefficient theorem in KK-theory.
making the techniques of Rørdam applicable for much more general classification results, as explored by Eilers, Restorff, and Ruiz in [ERR09].

These methods relied heavily on the non-unital Ext-group, which is known to be isomorphic to Kasparov’s group $\text{KK}^1$. It is not hard to observe that similar methods should apply to unital extensions if one instead applies the strong unital Ext-group $\text{Ext}^{-1}_u(\mathfrak{A}, \mathfrak{B})$ instead. One difficulty in working with the strong Ext-group is that it is even more sensitive than KK-theory. For instance, let $u \in \mathfrak{A}$ be a unitary. In contrast to KK-theory where $\text{KK}(\text{id}, \mathfrak{A}) = \text{KK}(\text{id}, \mathfrak{B})$, the automorphism on $\text{Ext}^{-1}_u(\mathfrak{A}, \mathfrak{B})$ induced by $\text{id} u$ is not necessarily the identity map. The same phenomena will never happen for the weak Ext-group $\text{Ext}^{-1}_w(\mathfrak{A}, \mathfrak{B})$ as it embeds naturally as a subgroup of $\text{KK}^1(\mathfrak{A}, \mathfrak{B})$.

In [ERR09, Theorem 3.9], all full extensions of non-unital UCT Kirchberg algebras by stable AF algebras are classified by their six-term exact sequences in $K$-theory (with order in $K_0$ of the ideal). We will complete the classification of such extensions obtaining classification in the case where the UCT Kirchberg algebra is unital. This will be divided into two cases: one where the extension algebra is unital, and one where it is non-unital.

In the case of unital extensions, the invariant will be $K^{+\Sigma}_{\text{six}}$, which is the six-term exact sequence in $K$-theory together with order and position of the unit in the $K_0$-groups. The classification is as follows.

**Theorem A.** Let $\varepsilon_i : 0 \to \mathfrak{B}_i \to \mathfrak{E}_i \to \mathfrak{A}_i \to 0$ be unital extensions of $C^*$-algebras for $i = 1, 2$ such that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are UCT Kirchberg algebras, and $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are stable AF algebras. Then $\mathfrak{E}_1 \cong \mathfrak{E}_2$ if and only if $K^{+\Sigma}_{\text{six}}(\varepsilon_1) \cong K^{+\Sigma}_{\text{six}}(\varepsilon_2)$.

Next we turn our attention to non-unital extensions with unital quotients. A unital extension as considered above will always be full, as the Busby map is unital and the quotient is simple. For non-unital extensions it is in general much harder to determine whether they are full or not. However, when mixing sufficient amounts of finiteness and infiniteness, it turns out that fullness is a very natural criterion, witnessed by the existence of a properly infinite, full projection in the extension algebra, see Theorem 6.5.

In [Gab16], examples were given of non-isomorphic full extensions of the Cuntz algebra $\mathcal{O}_2$ by the stabilised CAR algebra $\mathcal{M}_2^\infty \otimes \mathbb{K}$, which had isomorphic six-term exact sequences in $K$-theory with order, scales and units in the $K_0$-groups. This means that one needs a finer invariant to classify non-unital extensions when the quotient is unital.

For this purpose, we introduce an invariant $\tilde{K}^{+\Sigma}_{\text{six}}$ which includes the usual six-term exact sequence of the extension $0 \to \mathfrak{B} \to \mathfrak{E} \xrightarrow{\pi} \mathfrak{A} \to 0$, together with the $K$-theory of the extension $0 \to \mathfrak{B} \to \pi^{-1}(\mathbb{C} 1_{\mathfrak{A}}) \to \mathbb{C} \to 0$. We refer the reader to Section 7 for more details.

**Theorem B.** Let $\varepsilon_i : 0 \to \mathfrak{B}_i \to \mathfrak{E}_i \to \mathfrak{A}_i \to 0$ be full extensions of $C^*$-algebras for $i = 1, 2$ such that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are unital UCT Kirchberg algebras, $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are stable AF algebras. Then $\mathfrak{E}_1 \cong \mathfrak{E}_2$ if and only if $K^{+\Sigma}_{\text{six}}(\varepsilon_1) \cong K^{+\Sigma}_{\text{six}}(\varepsilon_2)$.

In the paper [EGK+18] we will compute the range of the invariant $\tilde{K}^{+\Sigma}_{\text{six}}$ for graph $C^*$-algebras with exactly one non-trivial ideal and for which the non-trivial quotient is unital. This will be used to show that an extension of simple graph $C^*$-algebras is again a graph $C^*$-algebra, provided there are no $K$-theoretic obstructions.
2. Extensions of \( C^* \)-algebras

In this section we recall some well-known definitions and results about extensions of \( C^* \)-algebras. More details can be found in [Di88, Chapter VII].

For a \( C^* \)-algebra \( \mathfrak{B} \), we will denote the multiplier algebra by \( \mathcal{M}(\mathfrak{B}) \), the corona algebra \( \mathcal{M}(\mathfrak{B})/\mathcal{Q}(\mathfrak{B}) \) by \( \mathcal{P}(\mathfrak{B}) \), and the canonical \( * \)-epimorphism from \( \mathcal{M}(\mathfrak{B}) \) to \( \mathcal{P}(\mathfrak{B}) \) by \( \pi_\mathfrak{B} \).

Let \( \mathfrak{A} \) and \( \mathfrak{B} \) be \( C^* \)-algebras. An extension of \( \mathfrak{A} \) by \( \mathfrak{B} \) is a short exact sequence

\[
\varepsilon : 0 \rightarrow \mathfrak{B} \xrightarrow{i} \mathcal{E} \xrightarrow{\pi} \mathfrak{A} \rightarrow 0
\]

of \( C^* \)-algebras. Often we just refer to such a short exact sequence above, as an extension of \( C^* \)-algebras. At times we identify \( \mathfrak{B} \) with its image \( \iota(\mathfrak{B}) \) in \( \mathcal{E} \), which is a two-sided, closed ideal, and at times we identify \( \mathfrak{A} \) with the quotient \( \mathcal{E}/\iota(\mathfrak{B}) \).

To any extension of \( C^* \)-algebras as above, there are induced \( * \)-homomorphisms \( \sigma : \mathfrak{A} \rightarrow \mathcal{M}(\mathfrak{B}) \) and \( \tau : \mathfrak{A} \rightarrow \mathcal{P}(\mathfrak{B}) \), the latter of these called the Busby map (or Busby invariant) of \( \varepsilon \). We sometimes refer to arbitrary \( * \)-homomorphisms \( \mathfrak{A} \rightarrow \mathcal{P}(\mathfrak{B}) \) as Busby maps.

An extension can be recovered up to canonical isomorphism of extensions by its Busby map \( \tau \), as the extension

\[
0 \rightarrow \mathfrak{B} \rightarrow \mathfrak{A} \oplus_{\tau,\pi_\mathfrak{B}} \mathcal{M}(\mathfrak{B}) \rightarrow \mathfrak{A} \rightarrow 0
\]

where

\[
\mathfrak{A} \oplus_{\tau,\pi_\mathfrak{B}} \mathcal{M}(\mathfrak{B}) = \{ a \oplus m \in \mathfrak{A} \oplus \mathcal{M}(\mathfrak{B}) : \tau(a) = \pi_\mathfrak{B}(m) \}
\]

is the pull-back of \( \tau \) and \( \pi_\mathfrak{B} \).

An extension is unital if the extension algebra is unital, or equivalently, if the Busby map is a unital \( * \)-homomorphism.

A (unital) extension \( \varepsilon : 0 \rightarrow \mathfrak{B} \rightarrow \mathcal{E} \xrightarrow{\pi} \mathfrak{A} \rightarrow 0 \) is called trivial (or split) if there is a (unital) \( * \)-homomorphism \( \rho : \mathfrak{A} \rightarrow \mathcal{E} \) such that \( \pi \circ \rho = \operatorname{id}_\mathfrak{A} \). The extension \( \varepsilon \) is called semi-split if there is a (unital) completely positive map \( \eta : \mathfrak{A} \rightarrow \mathcal{E} \) such that \( \pi \circ \eta = \operatorname{id}_\mathfrak{A} \).

Let \( \varepsilon_i : 0 \rightarrow \mathfrak{B} \rightarrow \mathcal{E}_i \rightarrow \mathfrak{A} \rightarrow 0 \) be extensions of \( C^* \)-algebras with Busby maps \( \tau_i \) for \( i = 1, 2 \). We say that \( \varepsilon_1 \) and \( \varepsilon_2 \) are strongly unitary equivalent, written \( \varepsilon_1 \sim_s \varepsilon_2 \), if there exists a unitary \( u \in \mathcal{M}(\mathfrak{B}) \) such that \( \operatorname{Ad} \pi_\mathfrak{B}(u) \circ \tau_1 = \tau_2 \).

By identifying \( \mathcal{E}_i \) with \( \mathfrak{A} \oplus_{\tau_i,\pi_\mathfrak{B}} \mathcal{M}(\mathfrak{B}) \), we obtain the following commutative diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & \mathfrak{B} & \xrightarrow{\varepsilon} & \mathcal{E}_1 & \xrightarrow{\pi} & \mathfrak{A} & \rightarrow & 0 \\
& \searrow & \downarrow \text{Ad } u & \cong & \downarrow \text{Ad}(1_{\mathfrak{A}} \oplus u) & & \searrow & & \\
0 & \rightarrow & \mathfrak{B} & \xrightarrow{\varepsilon} & \mathcal{E}_2 & \xrightarrow{\pi} & \mathfrak{A} & \rightarrow & 0
\end{array}
\]

with exact rows, which shows that \( \operatorname{Ad}(1_{\mathfrak{A}} \oplus u) : \mathcal{E}_1 \xrightarrow{\cong} \mathcal{E}_2 \) is an isomorphism by the five lemma.

Similarly, \( \varepsilon_1 \) and \( \varepsilon_2 \) are weakly unitary equivalent, written \( \varepsilon_1 \sim_w \varepsilon_2 \), if there exists a unitary \( u \in \mathcal{P}(\mathfrak{B}) \) such that \( \operatorname{Ad} u \circ \tau_1 = \tau_2 \).

In contrast to strong unitary equivalence, we cannot in general conclude that the extension algebras \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) are isomorphic from weak unitary equivalence.

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\(^2\)Note that a unital extension being trivial is slightly different from an extension – which happens to be unital – being trivial. In fact, the first requires \( \rho(1_{\mathfrak{A}}) = 1_\mathcal{E} \) which the other does not, and in general these two notions are different.
Definition 2.2. Let \( \mathfrak{A} \) and \( \mathfrak{B} \) be separable \( C^* \)-algebras with \( \mathfrak{B} \) stable. We let

- \( \text{Ext}(\mathfrak{A}, \mathfrak{B}) \) denote the semigroup of extensions of \( \mathfrak{A} \) by \( \mathfrak{B} \) modulo the relation defined by \([e_1] = [e_2]\) if and only if there exist trivial extensions \( f_1, f_2 \) of \( \mathfrak{A} \) by \( \mathfrak{B} \) such that

\[
e_1 \oplus f_1 \sim_w e_2 \oplus f_2,
\]

or equivalently, there exist trivial extensions \( f'_1, f'_2 \) of \( \mathfrak{A} \) by \( \mathfrak{B} \) (which can be taken as \( f'_1 = f_1 + 0 \)) such that

\[
e_1 \oplus f'_1 \sim_s e_2 \oplus f'_2.
\]

Moreover, if \( \mathfrak{A} \) is unital then we let

- \( \text{Ext}_{us}(\mathfrak{A}, \mathfrak{B}) \) denote the semigroup of \textit{unital} extensions of \( \mathfrak{A} \) by \( \mathfrak{B} \) modulo the relation defined by \([e_1]_s = [e_2]_s\) if and only if there exist trivial, \textit{unital} extensions \( f_1, f_2 \) of \( \mathfrak{A} \) by \( \mathfrak{B} \) such that

\[
e_1 \oplus f_1 \sim_s e_2 \oplus f_2.
\]

- \( \text{Ext}_{uw}(\mathfrak{A}, \mathfrak{B}) \) denote the semigroup of \textit{unital} extensions of \( \mathfrak{A} \) by \( \mathfrak{B} \) modulo the relation defined by \([e_1]_w = [e_2]_w\) if and only if there exist trivial, \textit{unital} extensions \( f_1, f_2 \) of \( \mathfrak{A} \) by \( \mathfrak{B} \) such that

\[
e_1 \oplus f_1 \sim_w e_2 \oplus f_2.
\]

If \( \mathfrak{B} \) is not stable, we define \( \text{Ext}_{(us/uw)}(\mathfrak{A}, \mathfrak{B}) := \text{Ext}_{(us/uw)}(\mathfrak{A}, \mathfrak{B} \otimes \mathbb{K}). \)

It is not hard to show that \( \text{Ext}_{(us/uw)}(\mathfrak{A}, \mathfrak{B}) \) is an abelian monoid, and that any trivial (unital) extension induces the zero element. Hence the following makes sense.

Definition 2.3. Let \( \text{Ext}^1(\mathfrak{A}, \mathfrak{B}), \text{Ext}_{us}^1(\mathfrak{A}, \mathfrak{B}) \) and \( \text{Ext}_{uw}^1(\mathfrak{A}, \mathfrak{B}) \) denote the subsemigroups of \( \text{Ext}(\mathfrak{A}, \mathfrak{B}), \text{Ext}_{us}(\mathfrak{A}, \mathfrak{B}) \) and \( \text{Ext}_{uw}(\mathfrak{A}, \mathfrak{B}) \) respectively (whenever these make sense), of elements which have an additive inverse. These subsets are abelian groups.

Remark 2.4 (Semisplit extensions). Let \( \mathfrak{A} \) and \( \mathfrak{B} \) be separable \( C^* \)-algebras with \( \mathfrak{B} \) stable (and \( \mathfrak{A} \) unital). As in \cite{Blackadar98} Section 15.7 it follows that a (unital) extension of \( \mathfrak{A} \) by \( \mathfrak{B} \) induces an element in \( \text{Ext}(\mathfrak{A}, \mathfrak{B}) \) (resp. in either \( \text{Ext}_{us}(\mathfrak{A}, \mathfrak{B}) \) or \( \text{Ext}_{uw}(\mathfrak{A}, \mathfrak{B}) \)) which has an additive inverse, if and only if the extension is semisplit.

In particular, if \( \mathfrak{A} \) is nuclear it follows from the Choi–Effros Lifting Theorem \cite{ChoiEffros76} that

\[
\text{Ext}^1(\mathfrak{A}, \mathfrak{B}) = \text{Ext}(\mathfrak{A}, \mathfrak{B}), \quad \text{Ext}_{us}^1(\mathfrak{A}, \mathfrak{B}) = \text{Ext}_{us}(\mathfrak{A}, \mathfrak{B}), \quad \text{Ext}_{uw}^1(\mathfrak{A}, \mathfrak{B}) = \text{Ext}_{uw}(\mathfrak{A}, \mathfrak{B}).
\]

Definition 2.5 (Pull-back and push-out extensions). Let \( e : 0 \to \mathfrak{B} \to \mathfrak{C} \to \mathfrak{A} \to 0 \) be an extension of \( C^* \)-algebras with Busby map \( \tau \), and let \( \alpha : \mathfrak{C} \to \mathfrak{A} \) be a \( * \)-homomorphism. The pull-back extension \( e \cdot \alpha \) is the extension of \( \mathfrak{C} \) by \( \mathfrak{B} \) with Busby map \( \tau \circ \alpha \).
If $\beta: \mathfrak{B} \to \mathfrak{D}$ is a non-degenerate $*$-homomorphism$^3$ there is an induced unital $*$-homomorphism $\overline{\beta}: \mathcal{D}(\mathfrak{B}) \to \mathcal{D}(\mathfrak{D})$\footnote{A $*$-homomorphism $\beta: \mathfrak{B} \to \mathfrak{D}$ is non-degenerate (or proper) if $\overline{\beta}(\mathfrak{B})\mathfrak{D} = \mathfrak{D}$.} The push-out extension $\beta \cdot e$ is the extension of $\mathfrak{A}$ by $\mathfrak{D}$ with Busby map $\overline{\beta} \circ \tau$.

If $\eta: \mathcal{D}(\mathfrak{B}) \to \mathcal{D}(\mathfrak{D})$ is a $*$-homomorphism, then we let $\eta \cdot e$ denote the extension of $\mathfrak{A}$ by $\mathfrak{D}$ with Busby map $\eta \circ \tau$. In particular, with $\beta$ as above, we have $\beta \cdot e = \overline{\beta} \cdot e$.

With the notation as above, the push-out and pull-back extensions fit into the following commutative diagram with exact rows

$$
\begin{array}{c}
\epsilon \cdot \alpha : & 0 & \rightarrow & \mathfrak{B} & \rightarrow & \mathfrak{E}_\alpha & \rightarrow & \mathfrak{C} & \rightarrow & 0 \\
\epsilon : & 0 & \rightarrow & \mathfrak{B} & \rightarrow & \mathfrak{E} & \rightarrow & \mathfrak{A} & \rightarrow & 0 \\
\beta \cdot \epsilon : & 0 & \rightarrow & \mathfrak{D} & \rightarrow & \mathfrak{E}_\beta & \rightarrow & \mathfrak{A} & \rightarrow & 0.
\end{array}
$$

The top two rows form a pull-back diagram and the bottom two rows form a push-out diagram.

**Remark 2.6** (Functoriality). The pull-back/push-out constructions of extensions turn $\operatorname{Ext}^1_{(\text{us/ww})}(\mathfrak{A}, \mathfrak{B})$ into a bifunctor with respect to (unital) $*$-homomorphisms in the first variable, and non-degenerate $*$-homomorphisms in the second variable.

A fair warning: while any unital $*$-homomorphism $\eta: \mathcal{D}(\mathfrak{B}) \to \mathcal{D}(\mathfrak{D})$ induces a map $\epsilon \mapsto \eta \cdot \epsilon$ which preserves $\sim_w$ (and $\sim_s$ if $\mathfrak{B}$ is stable\footnote{In fact, $\beta$ induces a unital $*$-homomorphism $\mathcal{M}(\beta): \mathcal{M}(\mathfrak{B}) \to \mathcal{M}(\mathfrak{D})$ by $\mathcal{M}(\beta)(m)(\beta(b)d) := \beta(mb)d$ for $m \in \mathcal{M}(\mathfrak{B})$, $b \in \mathfrak{B}$ and $d \in \mathfrak{D}$. This $*$-homomorphism descends to a unital $*$-homomorphism $\overline{\beta}: \mathcal{D}(\mathfrak{B}) \to \mathcal{D}(\mathfrak{D})$.}), it does in general not preserve Cuntz sums. This construction will be crucial in Remark 4.11 where we define $\epsilon_{[u]} = Ad u \cdot \epsilon_0$ for a unitary $u(\mathcal{D}(\mathfrak{B}))$ and a trivial unitial extension $\epsilon_0$.

The following is a celebrated result of Kasparov $\text{[Kas80]}$.

**Theorem 2.7** ($\text{[Kas80]}$). If $\mathfrak{A}$ and $\mathfrak{B}$ are separable $C^*$-algebras, then $\operatorname{Ext}^1(\mathfrak{A}, \mathfrak{B})$ is naturally isomorphic to Kasparov’s group $\mathcal{KK}^1(\mathfrak{A}, \mathfrak{B})$.

**Remark 2.8** (Absorbing extensions). Let $\mathfrak{A}$ and $\mathfrak{B}$ be separable $C^*$-algebras with $\mathfrak{B}$ stable (and $\mathfrak{A}$ unital). A (unital) extension $\epsilon$ of $\mathfrak{A}$ by $\mathfrak{B}$ is called absorbing if $\epsilon \sim_s \epsilon \oplus f$ for any trivial (unital) extension $f$ of $\mathfrak{A}$ by $\mathfrak{B}$\footnote{In fact, if $\mathfrak{B}$ is stable then the unitary group $\mathcal{U}(\mathcal{M}(\mathfrak{B}))$ is connected, and thus a unitary $u \in \mathcal{D}(\mathfrak{B})$ lifts to a unitary in $\mathcal{M}(\mathfrak{B})$ exactly when $u \in \mathcal{U}_0(\mathcal{D}(\mathfrak{B}))$, i.e. the connected component of $1_{\mathcal{D}(\mathfrak{B})}$ in the unitary group. As $\eta(\mathcal{U}_0(\mathcal{D}(\mathfrak{B}))) \subseteq \mathcal{U}_0(\mathcal{D}(\mathfrak{D}))$, and as every unitary in $\mathcal{U}_0(\mathcal{D}(\mathfrak{D}))$ lifts to a unitary in $\mathcal{M}(\mathfrak{D})$, it easily follows that $\eta$ preserves strong unitary equivalence classes of extensions. Just as with triviality, there is a difference between requiring that an extension is absorbing, or that a unital extension is absorbing. Sometimes absorbing unital extensions are said to be unital-absorbing. However, we simply call these absorbing as there is no cause of confusion, since a unital extension can never be absorbing in the general sense (it would have to absorb the extension with zero Busby map, which is never possible).}.
By [Tho01] there always exists an absorbing, trivial (unital) extension \( \epsilon_0 \) of \( \mathfrak{A} \) by \( \mathfrak{B} \). In particular, \( \epsilon \oplus \epsilon_0 \) is absorbing for any (unital) extension \( \epsilon \).

In particular, if \( \epsilon_1 \) and \( \epsilon_2 \) are absorbing extensions of \( \mathfrak{A} \) by \( \mathfrak{B} \) with \([\epsilon_1] = [\epsilon_2]\) in \( \text{Ext}(\mathfrak{A}, \mathfrak{B}) \), then \( \epsilon_1 \sim_s \epsilon_2 \).

Similarly, if \( \epsilon_1 \) and \( \epsilon_2 \) are absorbing unital extensions of \( \mathfrak{A} \) by \( \mathfrak{B} \) with \([\epsilon_1]_w = [\epsilon_2]_w\) in \( \text{Ext}_w(\mathfrak{A}, \mathfrak{B}) \) (resp. \([\epsilon_1]_w = [\epsilon_2]_w\) in \( \text{Ext}_{uw}(\mathfrak{A}, \mathfrak{B}) \)), then \( \epsilon_1 \sim_s \epsilon_2 \) (resp. \( \epsilon_1 \sim_w \epsilon_2 \)).

**Remark 2.9** (Determining absorption). A priori, it seems inconceivable that one could ever determine when an extension is absorbing. However, this was done by Elliott and Kucerovsky in [EK01].

Following [EK01], an extension \( \epsilon : 0 \to \mathfrak{B} \to \mathfrak{E} \to \mathfrak{A} \to 0 \) of separable \( C^* \)-algebras is called purely large if for any \( x \in \mathfrak{E} \setminus \mathfrak{B} \), there exists a stable \( C^* \)-subalgebra \( \mathfrak{D} \subseteq x^{\perp} \mathfrak{B} x \) such that \( \mathfrak{D} \mathfrak{B} \mathfrak{D} = \mathfrak{B} \).

By a remarkable result [EK01 Theorem 6], if \( \epsilon : 0 \to \mathfrak{B} \to \mathfrak{E} \to \mathfrak{A} \to 0 \) is a unital extension of separable \( C^* \)-algebras for which \( \mathfrak{A} \) is nuclear and \( \mathfrak{B} \) is stable, then \( \epsilon \) is absorbing (in the unital sense) if and only if it is purely large. Similar conditions for when non-unital extensions are absorbing were studied in [Gab16].

A separable \( C^* \)-algebra \( \mathfrak{B} \) is said to have the \textit{corona factorisation property} if any full projection \( p \in \mathcal{M}(\mathfrak{B} \otimes \mathbb{K}) \) is equivalent to \( 1_{\mathcal{M}(\mathfrak{B} \otimes \mathbb{K})} \). Many classes of separable \( C^* \)-algebras are known to have the corona factorisation property, e.g. all \( C^* \)-algebras with finite nuclear dimension by [Rob11 Corollary 3.5] (building on the work in [OPR12]). In particular, any AF algebra has the corona factorisation property, as these have nuclear dimension zero.

An extension \( \epsilon \) of \( \mathfrak{A} \) by \( \mathfrak{B} \) with Busby map \( \tau : \mathfrak{A} \to \mathcal{L}(\mathfrak{B}) \) is called full if for every non-zero \( a \in \mathfrak{A} \), \( \tau(a) \) generates all of \( \mathcal{L}(\mathfrak{B}) \) as a two-sided, closed ideal. As observed by Kucerovsky and Ng in [KN06], if \( \epsilon : 0 \to \mathfrak{B} \to \mathfrak{E} \to \mathfrak{A} \to 0 \) is a full extension of separable \( C^* \)-algebras, for which \( \mathfrak{B} \) is stable and has the corona factorisation property, then \( \epsilon \) is purely large.

### 3. \( K \)-theory of unital extensions

The purpose of this section is to collect some results on the \( K \)-theory of extensions of \( C^* \)-algebras, with a main focus on what happens to the unit in the \( K_0 \)-groups under certain operations of unital extensions. While most results in this section are quite elementary and most likely well-known to some experts in the field, we know of no references to these results and have included detailed proofs for completion.

Consider two six-term exact sequences

\[
\begin{array}{cccccc}
x^{(i)} : & H_0^{(i)} & \longrightarrow & L_0^{(i)} & \longrightarrow & G_0^{(i)} \\
& & & & & \\
& G_1^{(i)} & \longleftarrow & L_1^{(i)} & \longleftarrow & H_1^{(i)}
\end{array}
\]

for \( i = 1, 2 \). A homomorphism \((\psi_*, \rho_*, \phi_*): x^{(1)} \to x^{(2)}\) of six-term exact sequences consists of homomorphisms

\[
\phi_* : G_*^{(1)} \to G_*^{(2)}, \quad \psi_* : H_*^{(1)} \to H_*^{(2)}, \quad \rho_* : L_*^{(1)} \to L_*^{(2)}
\]

\footnote{This requires that \( \mathfrak{A} \) and \( \mathfrak{B} \) are separable. Although the definition of absorption makes sense without separability, we stick to this case.}
making the obvious diagram commute.

We may also consider six-term exact sequences with certain distinguished elements, which in our case will always be elements in \( x_i \in L_0^{(i)} \) and \( y_i \in G_0^{(i)} \) for \( i = 1, 2 \), and will correspond to the classes of the units in our \( K_0 \)-groups. If this is the case, we only consider homomorphisms such that \( \rho_0(x_1) = x_2 \) and \( \phi_0(y_1) = y_2 \).

If \( G_s^{(1)} = G_s^{(2)} =: G_s \) and \( H_s^{(1)} = H_s^{(2)} =: H_s \) then we say that \( x^{(1)} \) and \( x^{(2)} \) are congruent, written \( x^{(1)} \equiv x^{(2)} \), if there exists a homomorphism of the form \( (\text{id}_{H_s}, \rho_s, \text{id}_{G_s}) : x^{(1)} \to x^{(2)} \). Note that by the five lemma, this forces \( \rho_s \) to be an isomorphism, but in general many different \( \rho_s \) can implement a congruence.

If any of the groups in the six-term exact sequences contain distinguished elements, we require that our homomorphisms preserve these elements. In particular, when considering congruence with \( x_i \in L_0^{(i)} \) and \( y_i \in G_0^{(i)} = G_0 \) being our distinguished elements, we only consider the case \( y_1 = y_2 \).

**Definition 3.1.** For an extension \( \epsilon : 0 \to \mathcal{B} \to \mathcal{E} \to \mathcal{A} \to 0 \) of (unital) \( \text{C}^* \)-algebras, we let \( K_{\text{six}}(\epsilon) \) (resp. \( K_{\text{six}}^u(\epsilon) \)) denote the six-term exact sequence in \( K \)-theory (resp. with distinguished elements \[1\epsilon] \in K_0(\mathcal{E}) \) and \([1\mathcal{A}] \in K_0(\mathcal{A})\)).

Note that two extensions \( \epsilon \) and \( \eta \) can only have congruent six-term exact sequences, if the two ideals are equal and the two quotients are equal (isomorphisms are not enough for the definition to make sense). So both extensions have to be extensions of \( \mathcal{A} \) by \( \mathcal{B} \) for the definition of congruence to make sense.

The following two lemmas are well-known, but we fill in the proofs for completion.

**Lemma 3.2.** Let \( \epsilon_1 \) and \( \epsilon_2 \) be unital extensions of \( \mathcal{A} \) by \( \mathcal{B} \) which are strongly unitary equivalent. Then \( K_{\text{six}}^u(\epsilon_1) \cong K_{\text{six}}^u(\epsilon_2) \).

**Proof.** If \( u \in \mathcal{M}(\mathcal{B}) \) implements the strong unitary equivalence, then applying \( K \)-theory to the diagram (2.1) and using that \( K_*(\text{Ad } u) = \text{id}_{K_*(\mathcal{B})} : K_*(\mathcal{B}) \to K_*(\mathcal{B}) \), one obtains a congruence \( K_{\text{six}}^u(\epsilon_1) \cong K_{\text{six}}^u(\epsilon_2) \). \( \square \)

**Lemma 3.3.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be \( \text{C}^* \)-algebras with \( \mathcal{A} \) unital and \( \mathcal{B} \) stable. Let \( \epsilon : 0 \to \mathcal{B} \to \mathcal{E} \to \mathcal{A} \to 0 \) be a unital extension, and let \( \epsilon_0 \) be a trivial unital extension of \( \mathcal{A} \) by \( \mathcal{B} \). Then \( K_{\text{six}}^u(\epsilon) \) and \( K_{\text{six}}^u(\epsilon \oplus \epsilon_0) \) are congruent.

**Proof.** Let \( s_1, s_2 \in \mathcal{M}(\mathcal{B}) \) be \( O_2 \)-isometries so that \( \epsilon \oplus \epsilon_0 = \epsilon \oplus s_1, s_2 \epsilon_0 \). Let \( \pi : \mathcal{E} \to \mathcal{A} \) be the quotient map, \( \sigma : \mathcal{E} \to \mathcal{M}(\mathcal{B}) \) be the canonical unital \( * \)-homomorphism, and \( \phi : \mathcal{A} \to \mathcal{M}(\mathcal{B}) \) a unital \( * \)-homomorphism which lifts \( \tau_0 \).

The extension algebra \( \mathcal{F} \) of \( \epsilon \oplus s_1, s_2 \epsilon_0 \) is by definition
\[
\mathcal{F} = \{ a \oplus m \in \mathcal{A} \oplus \mathcal{M}(\mathcal{B}) : \pi_{\mathcal{B}}(s_1) \tau(a) \pi_{\mathcal{B}}(s_1)^* \pi_{\mathcal{B}}(s_2) \tau_0(a) \pi_{\mathcal{B}}(s_2)^* = \pi_{\mathcal{B}}(m) \}.
\]
Define the unital \( * \)-homomorphism \( \Psi : \mathcal{E} \to \mathcal{F} \) by
\[
\Psi(y) = \pi(y) \oplus (s_1 \sigma(y) s_1^* + s_2 \phi(\pi(y)) s_2^*).
\]
This is clearly well-defined, and induces a unital \( * \)-homomorphism of extensions by
\[
\epsilon : 0 \to \mathcal{B} \to \mathcal{E} \to \mathcal{A} \to 0
\]
\[
\epsilon \oplus s_1, s_2 \epsilon_0 : 0 \to \mathcal{B} \to \mathcal{F} \to \mathcal{A} \to 0.
\]
As \((s_1(-s_1^*)_*) = \text{id}_{K_* (\mathfrak{B})} : K_* (\mathfrak{B}) \to K_* (\mathfrak{B})\), applying \(K\)-theory to the above diagram induces a congruence \(K^u_{\text{six}} (\mathfrak{c}) \equiv K^u_{\text{six}} (\mathfrak{c} \oplus s_1, s_2, e_0)\). □

**Corollary 3.4.** Let \(\mathfrak{A}\) and \(\mathfrak{B}\) be separable \(C^*\)-algebras with \(\mathfrak{A}\) unital and \(B\) stable. Suppose that \(\mathfrak{c}_1\) and \(\mathfrak{c}_2\) are unital extensions of \(\mathfrak{A}\) by \(\mathfrak{B}\) for which \([\mathfrak{c}_1] \equiv [\mathfrak{c}_2] \text{ in Ext}_{\text{us}} (\mathfrak{A}, \mathfrak{B})\). Then \(K^u_{\text{six}} (\mathfrak{c}_1) \equiv K^u_{\text{six}} (\mathfrak{c}_2)\).

**Proof.** By definition of \(\text{Ext}_{\text{us}}\), there are trivial, unital extensions \(f_1, f_2\), such that \(\mathfrak{c}_1 \oplus f_1\) and \(\mathfrak{c}_2 \oplus f_2\) are strongly unitarily equivalent. Hence the result follows from Lemmas 3.2 and 3.3 □

**Lemma 3.5.** Let \(\mathfrak{c} : 0 \to \mathfrak{B} \xrightarrow{\tau} \mathfrak{A} \to 0\) be a unital extension \(C^*\)-algebras with boundary map \(\delta_* : K_* (\mathfrak{A}) \to K_{1-*} (\mathfrak{B})\) in \(K\)-theory, let \(u \in \mathcal{Q} (\mathfrak{B})\) be a unitary, and let \(\chi_1 : K_1 (\mathcal{Q} (\mathfrak{B})) \to K_0 (\mathfrak{B})\) denote the index map in \(K\)-theory. Then \(K^u_{\text{six}} (\text{Ad} u \cdot \mathfrak{c})\) (see Definition 2.5) is congruent to

\[
\begin{array}{ccc}
K_0 (\mathfrak{B}) & \xrightarrow{\iota_0} & (K_0 (\mathfrak{E}), [1_\mathfrak{E}] + \iota_0 (\chi_1 ([u]))) \\
\downarrow \delta_1 & & \downarrow \delta_0 \\
K_1 (\mathfrak{A}) & \xrightarrow{\pi_1} & K_1 (\mathfrak{E}) & \xrightarrow{j_1} & K_1 (\mathfrak{B}).
\end{array}
\]

**Proof.** Let \(a \in \mathcal{M} (\mathfrak{B})\) be a lift of \(u\) with \(\|a\| = 1\), and define

\[
v := \begin{pmatrix}
a & 0 \\
(1 - a^* a)^{1/2} & 0
\end{pmatrix} \in M_2 (\mathcal{M} (\mathfrak{B})), \quad v := \begin{pmatrix}
a & 0 \\
(1 - a^* a)^{1/2} & 0
\end{pmatrix} \in M_2 (\mathcal{M} (\mathfrak{B})).
\]

Then \(v\) is a partial isometry for which \(v^* v = 1 \oplus 0\). It is well-known, see e.g. [RLL00, Section 9.2], that

\[
(3.1) \quad \chi_1 ([u]) = [1_{M_2 (\mathfrak{B})} - vv^*] - [0 \oplus 1_{\mathfrak{B}}] \in K_0 (\mathfrak{B}).
\]

Let \(\tau\) denote the Busby map of \(\mathfrak{c}\), and identify \(\mathfrak{E}\) with the pull-back \(\mathfrak{A} \oplus_{\tau, \pi_\mathfrak{B}} \mathcal{M} (\mathfrak{B})\). Define

\[
\mathfrak{E}_2 := \{a \oplus y \in \mathfrak{A} \oplus M_2 (\mathcal{M} (\mathfrak{B})) : (\text{Ad} u \circ \tau (a)) \oplus 0 = M_2 (\pi_\mathfrak{B}) (y) \in M_2 (\mathcal{Q} (\mathfrak{B}))\},
\]

i.e. \(\mathfrak{E}_2\) is the pull-back \(\mathfrak{A} \oplus_{\text{Ad} u \circ \tau, \pi_\mathfrak{B}} \mathcal{M} (\mathfrak{B})\). We obtain an embedding

\[
\text{Ad} (1 \oplus v_c) : \mathfrak{A} \oplus_{\tau, \pi_\mathfrak{B}} \mathcal{M} (\mathfrak{B}) \to \mathfrak{E}_2.
\]

Similarly, identify the extension algebra \(\mathfrak{C}_u\) of \(\text{Ad} u \cdot \mathfrak{c}\) with the pull-back \(\mathfrak{A} \oplus_{\text{Ad} u \circ \tau, \pi_\mathfrak{B}} \mathcal{M} (\mathfrak{B})\). The embedding \(\mathcal{M} (\mathfrak{B}) \to M_2 (\mathcal{M} (\mathfrak{B}))\) into the \((1, 1)\)-corner induces an embedding

\[
\text{id}_\mathfrak{A} \oplus j : \mathfrak{A} \oplus_{\text{Ad} u \circ \tau, \pi_\mathfrak{B}} \mathcal{M} (\mathfrak{B}) \to \mathfrak{E}_2.
\]

We get the following diagram where all rows are short exact sequences and all maps are \(*\)-homomorphisms

\[
\begin{array}{ccc}
0 & \xrightarrow{\iota} & \mathfrak{B} & \xrightarrow{\mathfrak{c}} & \mathfrak{E} & \xrightarrow{\pi} & \mathfrak{A} & \xrightarrow{\iota} & 0 \\
1 & \xrightarrow{\text{Ad} v_c} & M_2 (\mathfrak{B}) & \xrightarrow{j} & \mathfrak{E}_2 & \xrightarrow{id_\mathfrak{A} \oplus j} & \mathfrak{A} & \xrightarrow{0} & 0
\end{array}
\]
Note that \((\text{Ad} \, v_c)_*, j_* : K_*(\mathcal{B}) \to K_*(M_2(\mathcal{B}))\) are the same map, namely the canonical isomorphism. In particular, by considering the induced maps of six-term exact sequences, the five lemma implies that \((\text{id}_\mathfrak{A} \oplus j)_* : K_*(\mathcal{E}_u) \to K_*(\mathcal{E}_2)\) and \(\text{Ad}(1 \oplus v_c)_* : K_*(\mathcal{E}) \to K_*(\mathcal{E}_2)\) are isomorphisms. As \((\text{Ad} \, v_c)_* = j_*\), it follows that
\[
\text{Ad}(1 \oplus v_c)_*^{-1} \circ (\text{id}_\mathfrak{A} \oplus j)_* : K_*(\mathcal{E}_u) \to K_*(\mathcal{E})
\]
duces a congruence \(K_{\text{six}}(\text{Ad} \, u \cdot \mathfrak{e}) \equiv K_{\text{six}}(\mathfrak{e})\) which does not necessarily preserve the class of the unit since \(\text{Ad}(1 \oplus v_c)\) and \(\text{id}_\mathfrak{A} \oplus j\) are not unital maps. Thus it remains to prove that
\[
\text{Ad}(1 \oplus v_c)_*^{-1}((\text{id}_\mathfrak{A} \oplus j)_0([1\mathfrak{e}_u])) = [1\mathfrak{e}] + j_0(\chi_1([u])),
\]
or alternatively, that
\[
\text{Ad}(1 \oplus v_c)_0((1\mathfrak{e} + j_0(\chi_1([u]))) = (\text{id}_\mathfrak{A} \oplus j)_0([1\mathfrak{e}_u])) = [1\mathfrak{A} \oplus (1_{\mathcal{M}(\mathcal{B})} \oplus 0)] \in K_0(\mathcal{E}_2).
\]
Note that the unitisation
\[
\mathcal{E}_2 = \mathcal{E}_2 + \mathbb{C}(0 \mathfrak{A} \oplus (0 \oplus 1_{\mathcal{M}(\mathcal{B})})) \subseteq \mathfrak{A} \oplus M_2(\mathcal{M}(\mathcal{B})).
\]
As \((\text{Ad} \, v_c)_0 = j_0 : K_0(\mathcal{B}) \xrightarrow{\cong} K_0(M_2(\mathcal{B}))\) is the canonical isomorphism, it follows from \((3.1)\) (using that \(1_{\mathfrak{A}} \oplus vv^* \in \mathcal{E}_2\)) that
\[
\text{Ad}(1 \oplus v_c)_0 \circ j_0(\chi_1([u])) = j_0((1\mathfrak{e} - (1\mathfrak{A} \oplus vv^*)) - (0\mathfrak{A} \oplus (0 \oplus 1_{\mathcal{M}(\mathcal{B})})))
\]
\[
= \{1\mathfrak{e} - (1\mathfrak{A} \oplus vv^*)\} - \{1\mathfrak{A} \oplus (1_{\mathcal{M}(\mathcal{B})} \oplus 0)\} - \{1\mathfrak{A} \oplus vv^*\} \in K_0(\mathcal{E}_2).
\]
Clearly
\[
\text{Ad}(1 \oplus v_c)_0([1\mathfrak{e}]) = [1\mathfrak{A} \oplus v_c v_c^*] = [1\mathfrak{A} \oplus vv^*] \in K_0(\mathcal{E}_2),
\]
and combining this with \((3.3)\) yields \((3.2)\).

Recall that if \(L_1, L_2\) and \(G\) are abelian groups and \(\phi_i : L_i \to G\) are homomorphisms, then
\[
L_1 \oplus_{\phi_1, \phi_2} L_2 = \{x_1 \oplus x_2 \in L_1 \oplus L_2 : \phi_1(x_1) = \phi_2(x_2)\}
\]
is the pull-back. When there is no doubt of what the maps \(\phi_i\) are, we simply write \(L_1 \oplus_{\phi_i} L_2\) instead of \(L_1 \oplus_{\phi_1, \phi_2} L_2\).

**Remark 3.6.** Recall that if \(x_i : 0 \to H \xrightarrow{j(i)} L_i \xrightarrow{\pi(i)} G \to 0\) are extensions of abelian groups for \(i = 1, 2\), then their *Baer sum* \(x_1 \oplus x_2\) is the extension given by
\[
0 \to H \xrightarrow{j(1)} L_1 \oplus_{\phi_1, \phi_2} L_2 \xrightarrow{\{(r(1)(x), -r(2)(x)) : x \in H\}} G \to 0.
\]
Addition in the group \(\text{Ext}(G, H)\) is given by the Baer sum.

The following proposition is an explicit formula for computing \(K_{\text{six}}^u(\mathfrak{e}_1 \oplus \mathfrak{e}_2)\) using a similar construction as the Baer sum, when we know that the boundary maps for one of \(\mathfrak{e}_1\) or \(\mathfrak{e}_2\) vanishes.
Proposition 3.7. Let $\epsilon_i : 0 \to \mathcal{B} \xrightarrow{(e_i)} \mathcal{E} \xrightarrow{\pi_i} \mathfrak{A} \to 0$ be unital extensions of $C^*$-algebras for $i = 1, 2$ such that $\mathcal{B}$ is stable. Let $\delta^{(i)}_*: K_*(\mathfrak{A}) \to K_{1-*}(\mathcal{B})$ denote the boundary map of $\epsilon_i$ in $K$-theory for $i = 1, 2$. If $\delta_*^{(2)} = 0$, then $K_{*_{\text{st}}}(\epsilon_1 \oplus \epsilon_2)$ is congruent to

\[
\begin{array}{c}
K_0(\mathcal{B}) \xrightarrow{\iota_0^{(1)}} \left( \frac{K_0(\mathcal{E}_1) \oplus K_0(\mathfrak{A}) K_0(\mathcal{E}_2)}{\{(\iota_1^{(1)}(x), -\iota_0^{(2)}(x)) : x \in K_0(\mathcal{B})\}}, [1_{\mathcal{E}_1}] \oplus [1_{\mathcal{E}_2}] \right) \xrightarrow{\pi_0^{(1)}} (K_0(\mathfrak{A}), [1_{\mathfrak{A}}]) \\
\delta_0^{(1)} \\
K_1(\mathfrak{A}) \xrightarrow{\pi_1^{(1)}} \left( \frac{K_1(\mathcal{E}_1) \oplus K_1(\mathfrak{A}) K_1(\mathcal{E}_2)}{\{(\iota_1^{(1)}(y), -\iota_0^{(2)}(y)) : y \in K_1(\mathcal{B})\}} \right) \xrightarrow{\iota_1^{(1)}} K_1(\mathcal{B}).
\end{array}
\]

The same result also holds in the not necessarily unital case by removing all units from the statement.

Proof. For the not necessarily unital case, one simply ignores any mentioning of units in the argument given.

We fix $\mathcal{O}_2$-isometries $s_1, s_2 \in \mathcal{M}(\mathcal{B})$, and identify $\mathcal{E}_1 \oplus \mathcal{E}_2$ with $\mathcal{E}_1 \oplus s_1, s_2 \mathcal{E}_2$, which we denote as $0 \to \mathcal{B} \to \mathcal{E} \to \mathfrak{A} \to 0$. Construct the pull-back diagram

\[
\begin{array}{c}
\mathcal{B} \xrightarrow{e^{(1)}} \mathcal{B} \\
\downarrow \quad \downarrow e^{(1)} \\
\mathcal{B} \xrightarrow{\pi^{(1)}} \mathcal{E} \xrightarrow{\pi^{(2)}} \mathfrak{A}.
\end{array}
\]

Applying $K$-theory to this diagram, and using that $\delta_*^{(2)} = 0$, one gets the following commutative diagram with exact rows and columns:

\[
\begin{array}{c}
K_0(\mathcal{B}) \xrightarrow{\iota_0^{(1)}} K_0(\mathcal{B}) \\
K_1(\mathcal{E}_1) \xrightarrow{0} K_0(\mathcal{B}) \xrightarrow{\iota_0^{(1)}} K_0(\mathcal{E}_0) \xrightarrow{\pi_1^{(1)}} K_0(\mathcal{E}_1) \xrightarrow{0} K_1(\mathcal{B}) \\
K_1(\mathfrak{A}) \xrightarrow{0} K_0(\mathcal{B}) \xrightarrow{\iota_0^{(2)}} K_0(\mathcal{E}_2) \xrightarrow{\pi_0^{(1)}} K_0(\mathfrak{A}) \xrightarrow{0} K_1(\mathcal{B}).
\end{array}
\]

Hence $K_0(\mathcal{E}_0) \cong K_0(\mathcal{E}_1) \oplus K_0(\mathfrak{A}) K_0(\mathcal{E}_2)$ canonically, and this isomorphism takes $[1_{\mathcal{E}_0}] \in K_0(\mathcal{E}_0)$ to the element $[1_{\mathcal{E}_1}] \oplus [1_{\mathcal{E}_2}] \in K_0(\mathcal{E}_1) \oplus K_0(\mathfrak{A}) K_0(\mathcal{E}_2)$.

The pull-back diagram induces a short exact sequence $\epsilon_0 : 0 \to \mathcal{B} \oplus \mathcal{B} \to \mathcal{E}_0 \to \mathfrak{A} \to 0$ where $\mathcal{B} \oplus 0$ is the “top $\mathcal{B}$” and $0 \oplus \mathcal{B}$ is the “left $\mathcal{B}$” in \[.\] Let $\Phi : \mathcal{B} \oplus \mathcal{B} \to \mathcal{B}$ be the Cuntz sum map $\Phi(b_1 \oplus b_2) = s_1 b_1 s_1^* + s_2 b_2 s_2^*$. We obtain a commutative diagram
with exact rows
\begin{equation}
\begin{array}{ccccccccc}
0 & \xrightarrow{} & \mathcal{B} \oplus \mathcal{B} & \xrightarrow{} & \mathcal{E}_0 & \xrightarrow{} & \mathcal{A} & \xrightarrow{} & 0 \\
\Phi & & & & & & & & \\
0 & \xrightarrow{} & \mathcal{B} & \xrightarrow{} & \mathcal{E} & \xrightarrow{} & \mathcal{A} & \xrightarrow{} & 0,
\end{array}
\end{equation}
for which the \(*\)-homomorphism \(\mathcal{E}_0 \to \mathcal{E}\) is unital. Applying \(K\)-theory to this diagram, and using the canonical identification \(K_0(\mathcal{E}_0) \cong K_0(\mathcal{E}_1) \oplus K_0(\mathcal{E}_2)\) as well as the fact that \(\delta^{(2)}_x = 0\), one obtains the following commutative diagram with exact rows
\[
\begin{array}{cccccc}
K_1(\mathcal{A}) & \xrightarrow{\delta^{(1)}_1 \times 0} & K_0(\mathcal{B}) \oplus K_0(\mathcal{B}) & \xrightarrow{\left(\iota^{(1)}_0, -\iota^{(2)}_0\right)} & K_0(\mathcal{E}_1) \oplus K_0(\mathcal{E}_2) & \xrightarrow{\pi^{(1)}_0} & K_0(\mathcal{A}) & \xrightarrow{\delta^{(1)}_0 \times 0} & K_1(\mathcal{B})^2 \\
\cong & & \text{Sum} & & \text{ker Sum} & & \text{Sum} & & \cong \\
K_1(\mathcal{A}) & \xrightarrow{\delta^{(1)}_1} & K_0(\mathcal{B}) & \xrightarrow{\text{Sum}} & K_0(\mathcal{E}) & \xrightarrow{\text{ker Sum}} & K_0(\mathcal{A}) & \xrightarrow{\delta^{(1)}_0} & K_1(\mathcal{B}).
\end{array}
\]
A diagram chase shows that \(K_0(\mathcal{E}_1) \oplus K_0(\mathcal{E}_2) \to K_0(\mathcal{E})\) is surjective, with kernel \(\{(\iota^{(1)}_0(x), -\iota^{(2)}_0(x)) : x \in K_0(\mathcal{B})\}\). As the map \(\mathcal{E}_0 \to \mathcal{E}\) was unital, \([1_{\mathcal{E}_1}] \oplus [1_{\mathcal{E}_2}]\) is mapped to \([1_{\mathcal{E}}]\). Hence we obtain the following commutative diagram with exact rows
\[
\begin{array}{cccccc}
K_1(\mathcal{A}) & \xrightarrow{\delta^{(1)}_1 \times 0} & K_0(\mathcal{B}) \oplus K_0(\mathcal{B}) & \xrightarrow{\left(\iota^{(1)}_0, -\iota^{(2)}_0\right)} & K_0(\mathcal{E}_1) \oplus K_0(\mathcal{E}_2) & \xrightarrow{\pi^{(1)}_0} & K_0(\mathcal{A}) & \xrightarrow{\delta^{(1)}_0 \times 0} & K_1(\mathcal{B})^2 \\
\cong & & \text{ker Sum} & & \text{ker Sum} & & \text{ker Sum} & & \cong \\
K_1(\mathcal{A}) & \xrightarrow{\delta^{(1)}_1} & K_0(\mathcal{B}) & \xrightarrow{\cong \text{ker Sum}} & K_0(\mathcal{E}) & \xrightarrow{\cong \text{ker Sum}} & K_0(\mathcal{A}) & \xrightarrow{\delta^{(1)}_0} & K_1(\mathcal{B}).
\end{array}
\]
The element \([1_{\mathcal{E}}]\) exactly corresponds to \([1_{\mathcal{E}_1}] \oplus [1_{\mathcal{E}_2}]\) via the above isomorphism. By identifying \(K_0(\mathcal{B})\) with \(\frac{K_0(\mathcal{B}) \oplus K_0(\mathcal{B})}{\text{ker Sum}}\) via the map \(x \mapsto (x, 0)\), one obtains part of the desired congruence. Running the same argument as above where one interchange \(K_0\) and \(K_1\), one obtains the rest of the congruence. \[\square\]

4. A Universal Coefficient Theorem

Recall that a separable \(C^*\)-algebra \(\mathfrak{A}\) satisfies the UCT (in KK-theory) if and only if there is a short exact sequence
\begin{equation}
0 \to \text{Ext}(K_*(\mathfrak{A}), K_*(\mathfrak{B})) \to \text{Ext}^{-1}(\mathfrak{A}, \mathfrak{B}) \xrightarrow{\gamma_{\mathfrak{A}, \mathfrak{B}}} \text{Hom}(K_*(\mathfrak{A}), K_{1-*}(\mathfrak{B})) \to 0
\end{equation}
for every separable \(C^*\)-algebra \(\mathfrak{B}\). Here we made the canonical identification \(KK^1(\mathfrak{A}, \mathfrak{B}) \cong \text{Ext}^{-1}(\mathfrak{A}, \mathfrak{B})\), see Theorem 2.7. In this section we prove a universal coefficient theorem for the unital Ext-groups \(\text{Ext}^{-1}_{un}\) and \(\text{Ext}^{-1}_{un}\). Such UCT’s were stated in \(\text{[Ska84]}\) without a proof, and was proved in \(\text{[Wei15]}\) under the assumption that \(\mathfrak{B}\) has an approximate identity of projections.\footnote{While this isn’t stated explicitly in \(\text{[Wei15]}\) Theorems 4.8 and 4.9, it can be deduced from the proof that \(\mathfrak{B}\) is assumed to have an approximate identity of projections.}

We give a complete proof without this additional assumption and prove that the UCT’s are natural in both variables. Naturality is crucial for our applications, and was not established in \(\text{[Wei15]}\).
**Definition 4.1.** Given abelian groups $K, H$ and an element $h \in H$, we can form the pointed Ext-group of $(H, h)$ by $K$ by considering pointed extensions

$$0 \to K \to (G, g) \xrightarrow{\phi} (H, h) \to 0$$

for which $\phi(g) = h$. The set $\text{Ext}((H, h), K)$ of congruence classes of such extensions form an abelian group as in the classical case with $\text{Ext}(H, K)$, see Remark 3.6.

**Remark 4.2.** There is a homomorphism $K \to \text{Ext}((H, h), K)$ given by

$$k \mapsto [K \to (K \oplus H, k \oplus h) \to (H, h)].$$

The kernel of this map is $\{ \psi(h) : \psi \in \text{Hom}(H, K) \}$. It easily follows that there is a short exact sequence

$$0 \to K/\{ \psi(h) : \psi \in \text{Hom}(H, K) \} \to \text{Ext}((H, h), K) \to \text{Ext}(H, K) \to 0.$$

**Notation 4.3.** For abelian groups $H$ and $K$, and $h \in H$, we let $\text{Hom}((H, h), K)$ denote the subgroup of $\text{Hom}(H, K)$ consisting of homomorphisms $\delta$ for which $\delta(h) = 0$.

**Notation 4.4.** We write $\text{Ext}((K_*(\mathfrak{A}), [1_\mathfrak{A}]), K_*(\mathfrak{B}))$ for the group

$$\text{Ext}((K_0(\mathfrak{A}), [1_\mathfrak{A}]), K_0(\mathfrak{B})) \oplus \text{Ext}((K_1(\mathfrak{A}), K_1(\mathfrak{B}))$$

and $\text{Hom}((K_*(\mathfrak{A}), [1_\mathfrak{A}]), K_{*+1}(\mathfrak{B}))$ for the group

$$\text{Hom}((K_0(\mathfrak{A}), [1_\mathfrak{A}]), K_1(\mathfrak{B})) \oplus \text{Hom}(K_1(\mathfrak{A}), K_0(\mathfrak{B})).$$

**Remark 4.5.** It is easily seen that there is a homomorphism

$$\tilde{\gamma}_{\mathfrak{A}, \mathfrak{B}} : \text{Ext}^{-1}_\mathfrak{A}(\mathfrak{A}, \mathfrak{B}) \to \text{Hom}((K_*(\mathfrak{A}), [1_\mathfrak{A}]), K_{*+1}(\mathfrak{B})),$$

given by mapping $[\epsilon]_s$ to its boundary map in $K$-theory.

Similarly, there is a map

$$\tilde{\kappa}_{\mathfrak{A}, \mathfrak{B}} : \ker \tilde{\gamma}_{\mathfrak{A}, \mathfrak{B}} \to \text{Ext}((K_*(\mathfrak{A}), [1_\mathfrak{A}]), K_*(\mathfrak{B}))$$

given by mapping $[\epsilon]_s$ to its induced six-term exact sequence in $K$-theory with position of the unit. This is well defined since the boundary maps vanish, but a priori it is not obviously a homomorphism (it is a homomorphism by Corollary 4.6 below).

The following is an immediate consequence of Proposition 3.7 and the definition of the sum in the pointed Ext-group.

**Corollary 4.6.** Let $\mathfrak{A}$ and $\mathfrak{B}$ be separable $C^*$-algebras for which $\mathfrak{A}$ is unital. Then the map

$$\tilde{\kappa}_{\mathfrak{A}, \mathfrak{B}} : \ker \tilde{\gamma}_{\mathfrak{A}, \mathfrak{B}} \to \text{Ext}((K_*(\mathfrak{A}), [1_\mathfrak{A}]), K_*(\mathfrak{B}))$$

defined in Remark 4.5 is a homomorphism.

We introduce the following non-standard notation to ease what follows.

**Notation 4.7.** Let $\mathfrak{A}$ be a unital separable $C^*$-algebra, and $\mathfrak{B}$ be a separable $C^*$-algebra. We define

$$\Gamma_{\mathfrak{A}, \mathfrak{B}} := \{ \psi([1_\mathfrak{A}]) : \psi \in \text{Hom}(K_0(\mathfrak{A}), K_0(\mathfrak{B})) \}.$$  

**Remark 4.8.** If $\mathfrak{A}$ and $\mathfrak{B}$ are $C^*$-algebras with $\mathfrak{A}$ unital, then

$$0 \to K_0(\mathfrak{B})/\Gamma_{\mathfrak{A}, \mathfrak{B}} \to \text{Ext}((K_*(\mathfrak{A}), [1_\mathfrak{A}]), K_*(\mathfrak{B})) \to \text{Ext}((K_*(\mathfrak{B}), [1_\mathfrak{B}]), K_*(\mathfrak{B})) \to 0$$

is a short exact sequence by Remark 4.2.
For a unital C*-algebra $\mathfrak{D}$, we let $U(\mathfrak{D})$ denote its unitary group, and let $U_0(\mathfrak{D})$ denote the connected component of $1$ in $U(\mathfrak{D})$. Recall that a unital C*-algebra $\mathfrak{D}$ is $K_1$-surjective (resp. $K_1$-injective) if the canonical homomorphism $U(\mathfrak{D})/U_0(\mathfrak{D}) \to K_1(\mathfrak{D})$ is surjective (resp. injective), and $K_1$-bijective if it is both $K_1$-surjective and $K_1$-injective.

While the following result is well-known to experts, we know of no reference and thus include a proof.

**Proposition 4.9.** If $\mathfrak{B}$ is a stable C*-algebra then the corona algebra $\mathcal{Z}(\mathfrak{B})$ is $K_1$-bijective.

**Proof.** Stability of $\mathfrak{B}$ implies that $\mathcal{Z}(\mathfrak{B})$ is properly infinite and thus $K_1$-surjective by [Cun81]. For $K_1$-injectivity, let $u \in U(\mathcal{Z}(\mathfrak{B}))$ be such that $[u] = 0$ in $K_1(\mathcal{Z}(\mathfrak{B}))$. By [Nis86, Corollary 2.5] the connected stable rank of $\mathfrak{B}$ is at most 2. Consequently the general stable rank$^9$ of $\mathfrak{B}$ is at most 2. By [Nag89, Theorem 2] (which relies on results in [Rie83]) it follows that $u$ lifts to $\tilde{u} \in U(\mathcal{M}(\mathfrak{B}))$. By [CH87] one has $U(\mathcal{M}(\mathfrak{B})) = U_0(\mathcal{M}(\mathfrak{B}))$, and thus $u \in U_0(\mathcal{Z}(\mathfrak{B}))$. Hence $\mathcal{Z}(\mathfrak{B})$ is $K_1$-injective. $\square$

**Remark 4.10.** Let $\mathfrak{A}$ and $\mathfrak{B}$ be separable C*-algebras for which $\mathfrak{A}$ is unital and $\mathfrak{B}$ is stable. For every $x \in K_0(\mathfrak{B}) \cong K_1(\mathcal{Z}(\mathfrak{B}))$ there is an induced semisplit, unital extension $\mathfrak{e}_x$ of $\mathfrak{A}$ by $\mathfrak{B}$ (uniquely determined up to strong unitary equivalence) given as follows: Let $\tau_0 : \mathfrak{A} \to \mathcal{Z}(\mathfrak{B})$ be the Busby map of a trivial, absorbing unital extension [Tho01], and let $u \in U(\mathcal{Z}(\mathfrak{B}))$ be a unitary being mapped to $x$ under the natural isomorphism $K_1(\mathcal{Z}(\mathfrak{B})) \cong \tilde{K}_0(\mathfrak{B})$. Then $\mathfrak{e}_x$ is the extension with Busby map $\Ad u \circ \tau_0$.

As $\tau_0$ is uniquely determined up to strong unitary equivalence, and since $K_1(\mathcal{Z}(\mathfrak{B})) = U(\mathcal{Z}(\mathfrak{B}))/U_0(\mathcal{Z}(\mathfrak{B}))$ by Proposition 1.9 it easily follows that $\mathfrak{e}_x$ is unique up to strong unitary equivalence.

The following elementary lemma will be used frequently.

**Lemma 4.11.** Let $\mathfrak{A}$ and $\mathfrak{B}$ be separable C*-algebras for which $\mathfrak{A}$ is unital and $\mathfrak{B}$ is stable. Let $\mathfrak{e}$ be a unital extension of $\mathfrak{A}$ by $\mathfrak{B}$, and let $u \in U(\mathcal{Z}(\mathfrak{B}))$. Then

$$[\Ad u \cdot \mathfrak{e}]_s = [\mathfrak{e}]_s + [\mathfrak{e}[u]]_s \in \Ext_{us}(A,B).$$

In particular, the map

$$K_0(\mathfrak{B}) \to \Ext_{us}^{-1}(\mathfrak{A},\mathfrak{B}), \quad x \mapsto [\mathfrak{e}_x]_s$$

is a group homomorphism.

**Proof.** Let $s_1, s_2 \in \mathcal{M}(\mathfrak{B})$ be $O_2$-isometries, and let $\oplus$ denote the Cuntz sum induced by this choice of isometries. Then

$$(4.2) \quad \Ad(u \oplus u^*) \circ (\tau_\mathfrak{e} \oplus \tau_{[u]}^*) = \Ad(u \oplus u^*) \circ (\tau_\mathfrak{e} \oplus (\Ad u \circ \tau_0)) = (\Ad u \circ \tau_\mathfrak{e}) \oplus \tau_0$$

where $\tau_0$ is an absorbing, trivial unital extension. As $u \oplus u^*$ lifts to a unitary in $\mathcal{M}(\mathfrak{B})$, the result follows. $\square$

The following is an immediate consequence of Lemma 3.5 applied to the case where $\mathfrak{e}$ is a trivial unital extension.

$^9$Not to be confused with the topological stable rank, which in modern terms is usually just referred to as stable rank.
Corollary 4.12. Let \( \mathfrak{A} \) and \( \mathfrak{B} \) be separable \( C^* \)-algebras for which \( \mathfrak{A} \) is unital and \( \mathfrak{B} \) is stable, and let \( x \in K_0(\mathfrak{B}) \). Then \( \epsilon_x \) induces the element

\[
[0 \to K_0(\mathfrak{B}) \to (K_0(\mathfrak{B}) \oplus K_0(\mathfrak{A}), x \oplus [1_\mathfrak{A}]) \to (K_0(\mathfrak{A}), [1_\mathfrak{A}]) \to 0]
\]
in \( \text{Ext}((K_0(\mathfrak{A}), [1_\mathfrak{A}]), K_0(\mathfrak{B})) \).

Recall that \( \gamma_{\mathfrak{A},\mathfrak{B}} : \text{Ext}^{-1}(\mathfrak{A}, \mathfrak{B}) \to \text{Hom}(K_*(\mathfrak{A}), K_{*-1}(\mathfrak{B})) \) denotes the canonical homomorphism.

Lemma 4.13. Let \( \mathfrak{A} \) be a separable, unital \( C^* \)-algebra satisfying the UCT, and let \( \mathfrak{B} \) be a separable, stable \( C^* \)-algebra. Then there is an exact sequence

\[
0 \to K_0(\mathfrak{B})/\Gamma_{\mathfrak{A},\mathfrak{B}} \to \text{Ext}^{-1}_{\text{us}}(\mathfrak{A}, \mathfrak{B}) \to \text{Ext}^{-1}(\mathfrak{A}, \mathfrak{B}) .
\]
Moreover, the map \( \text{Ext}^{-1}_{\text{us}}(\mathfrak{A}, \mathfrak{B}) \to \text{Ext}^{-1}(\mathfrak{A}, \mathfrak{B}) \) is an isomorphism onto

\[
\gamma_{\mathfrak{A},\mathfrak{B}}^{-1}(\text{Hom}((K_*(\mathfrak{A}), [1_\mathfrak{A}]), K_{*-1}(\mathfrak{B}))) \subseteq \text{Ext}^{-1}(\mathfrak{A}, \mathfrak{B}) .
\]

Proof. By a result of Skandalis [Ska88, Remarque 2.8] (see also [Ska84] or [MT06] for a proof), there is an exact sequence of the form

\[
\begin{array}{ccc}
K_0(\mathfrak{B}) & \longrightarrow & \text{Ext}^{-1}_{\text{us}}(\mathfrak{A}, \mathfrak{B}) \\
\gamma_0 & \downarrow & \downarrow \gamma_1 \\
\text{KK}(\mathfrak{A}, \mathfrak{B}) & \longrightarrow & K_1(\mathfrak{B})
\end{array}
\]

where \( \gamma_1 \) is induced from the unital \( * \)-homomorphism \( \iota : C \to \mathfrak{A} \). It is easily seen that \( \gamma_0 : \text{KK}(\mathfrak{A}, \mathfrak{B}) \to K_0(\mathfrak{B}) \) factors as

\[
\text{KK}(\mathfrak{A}, \mathfrak{B}) \xrightarrow{\gamma_0} \text{Hom}(K_0(\mathfrak{A}), K_0(\mathfrak{B})) \xrightarrow{\text{ev}_{[1_\mathfrak{A}]}} K_0(\mathfrak{B})
\]

where \( \text{ev}_{[1_\mathfrak{A}]} \) is evaluation at \( [1_\mathfrak{A}] \). Similarly, \( \gamma_1 : \text{Ext}^{-1}(\mathfrak{A}, \mathfrak{B}) \to K_1(\mathfrak{B}) \) factors as

\[
\text{Ext}^{-1}(\mathfrak{A}, \mathfrak{B}) \xrightarrow{\gamma_0} \text{Hom}(K_0(\mathfrak{A}), K_1(\mathfrak{B})) \xrightarrow{\text{ev}_{[1_\mathfrak{A}]}} K_1(\mathfrak{B}) .
\]

Since \( \mathfrak{A} \) satisfies the UCT, \( \gamma_0 \) is surjective and thus \( \text{im}(\gamma_0) = \Gamma_{\mathfrak{A},\mathfrak{B}} \). Hence the exact sequence collapses to an exact sequence

\[
0 \to K_0(\mathfrak{B})/\Gamma_{\mathfrak{A},\mathfrak{B}} \to \text{Ext}^{-1}_{\text{us}}(\mathfrak{A}, \mathfrak{B}) \to \text{Ext}^{-1}(\mathfrak{A}, \mathfrak{B})
\]

where the image of \( \text{Ext}^{-1}_{\text{us}}(\mathfrak{A}, \mathfrak{B}) \to \text{Ext}^{-1}(\mathfrak{A}, \mathfrak{B}) \) is \( \ker \gamma_1 \). By the above, it easily follows that \( \ker \gamma_1 = \gamma_{\mathfrak{A},\mathfrak{B}}^{-1}(\text{Hom}((K_*(\mathfrak{A}), [1_\mathfrak{A}]), K_{*-1}(\mathfrak{B}))) \), so we obtain a short exact sequence

\[
0 \to K_0(\mathfrak{B})/\Gamma_{\mathfrak{A},\mathfrak{B}} \to \text{Ext}^{-1}_{\text{us}}(\mathfrak{A}, \mathfrak{B}) \to \gamma_{\mathfrak{A},\mathfrak{B}}^{-1}(\text{Hom}((K_*(\mathfrak{A}), [1_\mathfrak{A}]), K_{*-1}(\mathfrak{B}))) \to 0 .
\]

Using Lemma 4.11 it follows that the quotient \( \text{Ext}^{-1}_{\text{us}}(\mathfrak{A}, \mathfrak{B})/(K_0(\mathfrak{B})/\Gamma_{\mathfrak{A},\mathfrak{B}}) \) is canonically isomorphic to \( \text{Ext}^{-1}_{\text{us}}(\mathfrak{A}, \mathfrak{B})/\Gamma_{\mathfrak{A},\mathfrak{B}} \). Combined with the above short exact sequence it follows that \( \text{Ext}^{-1}_{\text{us}}(\mathfrak{A}, \mathfrak{B}) \to \text{Ext}^{-1}(\mathfrak{A}, \mathfrak{B}) \) is injective and its image is

\[
\gamma_{\mathfrak{A},\mathfrak{B}}^{-1}(\text{Hom}((K_*(\mathfrak{A}), [1_\mathfrak{A}]), K_{*-1}(\mathfrak{B})))
\]
as desired. \( \square \)
We can now assemble the pieces provided by the previous results in this section and obtain the following universal coefficient theorem. This is a minor improvement on the UCT sequences proved by Wei [Wei15, Theorems 4.8 and 4.9], in which the $C^*$-algebra $\mathcal{B}$ was required to have an approximate identity of projections. Also, Wei does not prove that the UCT’s for the unital Ext-groups are natural, which will be important in our applications.

**Theorem 4.14.** Let $\mathfrak{A}$ be a unital, separable $C^*$-algebra satisfying the UCT, and let $\mathcal{B}$ be a separable $C^*$-algebra. There is a commutative diagram

$$
\begin{array}{cccc}
K_0(\mathcal{B})/\Gamma_{\mathfrak{A},\mathcal{B}} & \longrightarrow & K_0(\mathcal{B})/\Gamma_{\mathfrak{A},\mathcal{B}} \\
\text{Ext}(K_*(\mathfrak{A}), [1_\mathfrak{A}], K_*(\mathcal{B})) & \longrightarrow & \text{Ext}_{\text{us}}^{-1}(\mathfrak{A}, \mathcal{B}) & \longrightarrow & \text{Hom}((K_*(\mathfrak{A}), [1_\mathfrak{A}], K_{s+1}(\mathcal{B})) \\
\text{Ext}(K_*(\mathfrak{A}), K_*(\mathcal{B})) & \longrightarrow & \text{Ext}_{\text{uw}}^{-1}(\mathfrak{A}, \mathcal{B}) & \longrightarrow & \text{Hom}((K_*(\mathfrak{A}), [1_\mathfrak{A}], K_{s+1}(\mathcal{B}))
\end{array}
$$

for which all rows and columns are short exact sequences. This diagram is natural with respect to unital $*$-homomorphisms in the first variable, and with respect to non-degenerate $*$-homomorphisms in the second variable.

**Proof.** By replacing $\mathcal{B}$ with $\mathcal{B} \otimes \mathbb{K}$, we may assume that $\mathcal{B}$ is stable.

By Lemma 4.13 and the UCT for $\text{Ext}^{-1}$ (see (1.1)), we obtain a short exact sequence

$$
0 \rightarrow \text{Ext}(K_*(\mathfrak{A}), K_*(\mathcal{B})) \rightarrow \text{Ext}_{\text{us}}^{-1}(\mathfrak{A}, \mathcal{B}) \xrightarrow{\gamma_{\mathfrak{A},\mathcal{B}}} \text{Hom}((K_*(\mathfrak{A}), [1_\mathfrak{A}], K_{s+1}(\mathcal{B})) \rightarrow 0.
$$

The map $\text{Ext}(K_*(\mathfrak{A}), K_*(\mathcal{B})) \rightarrow \ker \gamma_{\mathfrak{A},\mathcal{B}}$ above, which is an isomorphism by exactness, is exactly the inverse of the isomorphism

$$
\kappa_{\mathfrak{A},\mathcal{B}} : \ker \gamma_{\mathfrak{A},\mathcal{B}} \cong \text{Ext}(K_*(\mathfrak{A}), K_*(\mathcal{B}))
$$

given by applying $K$-theory to a given extension (which induce short exact sequences by vanishing of the boundary maps). That $\kappa_{\mathfrak{A},\mathcal{B}}$ is an isomorphism follows from the UCT. The homomorphism

$$
\tilde{\gamma}_{\mathfrak{A},\mathcal{B}} : \text{Ext}_{\text{us}}^{-1}(\mathfrak{A}, \mathcal{B}) \rightarrow \text{Hom}((K_*(\mathfrak{A}), [1_\mathfrak{A}], K_{s+1}(\mathcal{B}))
$$

is the composition of the surjective homomorphisms $\text{Ext}_{\text{us}}^{-1} \rightarrow \text{Ext}_{\text{us}}^{-1}$ and $\gamma_{\mathfrak{A},\mathcal{B}}$ from (4.3), so $\tilde{\gamma}_{\mathfrak{A},\mathcal{B}}$ is surjective. Hence we obtain the commutative diagram

$$
\begin{array}{cccc}
K_0(\mathcal{B})/\Gamma_{\mathfrak{A},\mathcal{B}} & \longrightarrow & K_0(\mathcal{B})/\Gamma_{\mathfrak{A},\mathcal{B}} \\
\ker \tilde{\gamma}_{\mathfrak{A},\mathcal{B}} & \longrightarrow & \text{Ext}_{\text{us}}^{-1}(\mathfrak{A}, \mathcal{B}) & \longrightarrow & \text{Hom}((K_*(\mathfrak{A}), [1_\mathfrak{A}], K_{1-}(\mathcal{B})) \\
\ker \gamma_{\mathfrak{A},\mathcal{B}} & \longrightarrow & \text{Ext}_{\text{uw}}^{-1}(\mathfrak{A}, \mathcal{B}) & \longrightarrow & \text{Hom}((K_*(\mathfrak{A}), [1_\mathfrak{A}], K_{1-}(\mathcal{B}))
\end{array}
$$
for which the rows and columns are short exact sequences. Consider the diagram (4.6)

\[
\begin{array}{ccccccc}
0 & \rightarrow & K_0(\mathcal{B})/\Gamma_{3,\mathcal{B}} & \rightarrow & \ker \gamma_3 & \rightarrow & \ker \gamma_{3,\mathcal{B}} & \rightarrow & 0 \\
\mbox{(\approx)} & \mbox{ker} & \tilde{\gamma}_3 & \mbox{ker} & \gamma_{3,\mathcal{B}} & \mbox{ker} & \gamma_{3,\mathcal{B}} & \rightarrow & 0 \\
0 & \rightarrow & K_0(\mathcal{B})/\Gamma_{3,\mathcal{B}} & \rightarrow & \operatorname{Ext}((K_*(\mathcal{A}), [1_\mathcal{A}]), K_*(\mathcal{B})) & \rightarrow & \operatorname{Ext}(K_* (\mathcal{A}), K_*(\mathcal{B})) & \rightarrow & 0
\end{array}
\]

which has exact rows. The map \( \tilde{\gamma}_{3,\mathcal{B}} \) is a homomorphism by Corollary 4.6, and clearly the right square above commutes. The left square above commutes by Remark 4.2 and Corollary 4.12. Hence \( \tilde{\gamma}_{3,\mathcal{B}} \) is an isomorphism by the five lemma. By gluing together the diagrams (4.5) and (4.6) in the obvious way, we obtain the desired diagram (4.3).

It remains to be shown that the diagram (4.3) is natural in both variable. For verifying this let \( \mathcal{C} \) be separable, unital \( C^* \)-algebra satisfying the UCT, let \( \phi : \mathcal{C} \rightarrow \mathcal{A} \) be a unital \( * \)-homomorphism, let \( \mathcal{D} \) be a separable, stable \( C^* \)-algebra, and let \( \psi : \mathcal{B} \rightarrow \mathcal{D} \) be a non-degenerate \( * \)-homomorphism. We first check that the diagram (4.5) is natural, and then (4.6).

It is well-known that \( \operatorname{Ext}^{-1}_{\mathcal{B}}(\mathcal{A}, \mathcal{B}) \rightarrow \operatorname{Ext}^{-1}_{\mathcal{D}}(\mathcal{A}, \mathcal{B}) \) is natural, and by naturality of six-term exact sequences the maps \( \tilde{\gamma}_{3,\mathcal{B}} \) and \( \gamma_{3,\mathcal{B}} \) are natural.

Again by naturality of six-term exact sequences, it follows that

\[
\phi^*(\ker \tilde{\gamma}_{3,\mathcal{B}}) \subseteq \ker \tilde{\gamma}_{3,\mathcal{D}}, \quad \text{and} \quad \psi^*(\ker \gamma_{3,\mathcal{B}}) \subseteq \ker \gamma_{3,\mathcal{D}}.
\]

Hence the inclusion \( \ker \tilde{\gamma}_{3,\mathcal{B}} \hookrightarrow \operatorname{Ext}^{-1}_{\mathcal{B}}(\mathcal{A}, \mathcal{B}) \) is natural in both variables. Similarly, the inclusion \( \ker \gamma_{3,\mathcal{B}} \hookrightarrow \operatorname{Ext}^{-1}_{\mathcal{D}}(\mathcal{A}, \mathcal{B}) \) and the map \( \ker \tilde{\gamma}_{3,\mathcal{B}} \rightarrow \ker \gamma_{3,\mathcal{B}} \) are natural in both variables. This implies that the diagram (4.5) is natural. Hence it remains to check that the diagram (4.6) is natural.

It is straightforward to verify that the maps in the lower row of (4.6) are natural (this is purely algebraic, and of course uses that \( \phi_0([1_\mathcal{C}]) = [1_\mathcal{A}] \)). We saw above that \( \ker \gamma_{3,\mathcal{B}} \rightarrow \ker \gamma_{3,\mathcal{B}} \) is natural.

We will show that \( \tilde{\gamma}_{3,\mathcal{B}} \) is natural in the first variable. Let \( \epsilon : 0 \rightarrow \mathcal{B} \rightarrow \mathcal{C} \rightarrow \mathcal{A} \rightarrow 0 \) be a unital extension inducing an element in \( \ker \tilde{\gamma}_{3,\mathcal{B}} \), i.e. \( \epsilon \) has vanishing boundary maps in \( K \)-theory. Construct the pull-back diagram

(4.7)

\[
\begin{array}{ccccccc}
\epsilon \cdot \phi : & 0 & \rightarrow & \mathcal{B} & \xrightarrow{\epsilon} & \mathcal{C} & \xrightarrow{\phi} & 0 \\
\epsilon & 0 & \rightarrow & \mathcal{B} & \xrightarrow{\epsilon} & \mathcal{C} & \xrightarrow{\phi} & \mathcal{A} & \xrightarrow{\epsilon} & 0.
\end{array}
\]

As \( \phi \) is a unital map, \( \mathcal{C} \) is unital and the map \( \mathcal{C} \rightarrow \mathcal{A} \) is unital. As \( \phi^*([\epsilon]_s) = [\epsilon \cdot \phi]_s \), we should check that

(4.8)

\[
\tilde{\kappa}_{3,\mathcal{B}} ([\epsilon \cdot \phi]_s) = (\phi^*)_s (\tilde{\kappa}_{3,\mathcal{B}} ([\epsilon]_s)).
\]
Applying $K$-theory to the pull-back diagram (4.7), and using that both $\epsilon$ and $\epsilon \cdot \phi$ have vanishing boundary maps, we obtain the diagram

$$
\begin{array}{cccccc}
\kappa_{\mathfrak{A}, \mathfrak{B}}([\epsilon \cdot \phi]) : & 0 & \longrightarrow & K_*(\mathfrak{B}) & \longrightarrow & (K_*(\mathfrak{C}), [1_{\mathfrak{C}}]) & \longrightarrow & 0 \\
\kappa_{\mathfrak{A}, \mathfrak{B}}([\epsilon]) : & 0 & \longrightarrow & K_*(\mathfrak{B}) & \longrightarrow & (K_*(\mathfrak{C}), [1_{\mathfrak{C}}]) & \longrightarrow & 0.
\end{array}
$$

Since this is a pull-back diagram it follows that (4.8) holds. Hence $\kappa_{\mathfrak{A}, \mathfrak{B}}$ is natural in the first variable. Checking that $\kappa_{\mathfrak{A}, \mathfrak{B}}$ is natural in the second variable, and that $\kappa_{\mathfrak{A}, \mathfrak{B}}$ is natural in both variables, is checked in a similar fashion.

It remains to check that $K_0(\mathfrak{B})/\Gamma_{\mathfrak{A}, \mathfrak{B}} \to \ker \tilde{\gamma}_{\mathfrak{A}, \mathfrak{B}}$ is natural in both variables. For this, fix a unitary in $u \in \mathfrak{U}(\mathfrak{B})$ inducing an arbitrary element in $K_0(\mathfrak{B})$. Let $\epsilon_{\mathfrak{A}, \mathfrak{B}}$ and $\epsilon_{\mathfrak{C}, \mathfrak{B}}$ be absorbing, unital extensions of $\mathfrak{A}$ by $\mathfrak{B}$ and of $\mathfrak{C}$ by $\mathfrak{B}$ respectively. By definition, we have

$$
[u] + \Gamma_{\mathfrak{A}, \mathfrak{B}} \mapsto [\text{Ad } u \cdot \epsilon_{\mathfrak{A}, \mathfrak{B}}], \quad [u] + \Gamma_{\mathfrak{C}, \mathfrak{B}} \mapsto [\text{Ad } u \cdot \epsilon_{\mathfrak{C}, \mathfrak{B}}].
$$

In order to check that $K_0(\mathfrak{B})/\Gamma_{\mathfrak{A}, \mathfrak{B}} \to \ker \tilde{\gamma}_{\mathfrak{A}, \mathfrak{B}}$ is natural in the first variable, we should therefore verify that

$$
\phi^*([\text{Ad } u \cdot \epsilon_{\mathfrak{A}, \mathfrak{B}}]) = [\text{Ad } u \cdot \epsilon_{\mathfrak{C}, \mathfrak{B}}].
$$

This follows easily from Lemma 4.11 since

$$
\phi^*([\text{Ad } u \cdot \epsilon_{\mathfrak{A}, \mathfrak{B}}]) = [\text{Ad } u \cdot (\epsilon_{\mathfrak{A}, \mathfrak{B}} \cdot \phi)] = [\epsilon_{\mathfrak{A}, \mathfrak{B}} \cdot \phi + [\text{Ad } u \cdot \epsilon_{\mathfrak{C}, \mathfrak{B}}],
$$

where we used that $\epsilon_{\mathfrak{A}, \mathfrak{B}} \cdot \phi$ is trivial so that $[\epsilon_{\mathfrak{A}, \mathfrak{B}} \cdot \phi] = 0$. Hence $K_0(\mathfrak{B})/\Gamma_{\mathfrak{A}, \mathfrak{B}} \to \ker \tilde{\gamma}_{\mathfrak{A}, \mathfrak{B}}$ is natural in the first variable. For the second variable, let $\tilde{\psi} : \mathfrak{U}(\mathfrak{B}) \to \mathfrak{U}(\mathfrak{D})$ be the induced $\text{K}$-homomorphism, and let $\epsilon_{\mathfrak{A}, \mathfrak{D}}$ be an absorbing, unital extension of $\mathfrak{A}$ by $\mathfrak{D}$. Note that $\psi([u]) = [\tilde{\psi}(u)]$. As above, we get

$$
\psi_*([\text{Ad } u \cdot \epsilon_{\mathfrak{A}, \mathfrak{B}}]) = [\tilde{\psi} \circ \text{Ad } u \cdot \epsilon_{\mathfrak{A}, \mathfrak{B}}] = [\text{Ad } \tilde{\psi}(u) \cdot (\epsilon_{\mathfrak{C}, \mathfrak{D}})]
$$

by Lemma 4.11, $[\tilde{\psi} \cdot \epsilon_{\mathfrak{A}, \mathfrak{B}}] + [\text{Ad } \tilde{\psi}(u) \cdot \epsilon_{\mathfrak{C}, \mathfrak{D}}] = [\text{Ad } \tilde{\psi}(u) \cdot \epsilon_{\mathfrak{C}, \mathfrak{D}}]$. As $[\text{Ad } \tilde{\psi}(u) \cdot \epsilon_{\mathfrak{C}, \mathfrak{D}}]$ is the image of $\psi_*([u]) + \Gamma_{\mathfrak{A}, \mathfrak{D}}$ via the map $K_0(\mathfrak{B})/\Gamma_{\mathfrak{A}, \mathfrak{B}} \to \ker \tilde{\gamma}_{\mathfrak{A}, \mathfrak{B}}$, it shows that this map is natural in the second variable, thus finishing the proof.

5. Classification of unital extensions

In this section we will apply our universal coefficient theorem to obtain classification results for certain unital extensions of $C^*$-algebras via their six-term exact sequence in $K$-theory.

The main idea is the following: suppose $\epsilon_1$ and $\epsilon_2$ are absorbing, semisplit unital extensions of $\mathfrak{A}$ by $\mathfrak{B}$, and suppose that $[\epsilon_1]_w = [\epsilon_2]_w \in \text{Ext}_w^{-1}(\mathfrak{A}, \mathfrak{B})$. By Theorem 4.14 there is an element $x \in K_0(\mathfrak{B})$ such that $[\epsilon_1] = [\epsilon_2 + \epsilon_x]$, and in particular $\epsilon_1 \cong \epsilon_2 + \epsilon_x$ by absorption. So the goal will be to prove, under certain conditions, that $\epsilon_2 + \epsilon_x \cong \epsilon_2$.

As a technical devise, we introduce the following notation.
Notation 5.1. If $\delta_\ast \in \text{Hom}((K_\ast(\mathcal{A}), [1_\mathcal{A}]), K_{\ast+1}(\mathcal{B}))$, then we define
\[
\Gamma^\delta_{\mathcal{A},\mathcal{B}} := q_{\delta_1}^{-1} \{ \phi([1_\mathcal{A}]) : \phi \in \text{Hom}(\text{ker} \delta_0, \text{coker} \delta_1) \}
\]
where $q_{\delta_1} : K_0(\mathcal{B}) \to \text{coker} \delta_1$ is the canonical epimorphism.

Note that we always have $\Gamma^\delta_{\mathcal{A},\mathcal{B}} = \Gamma^0_{\mathcal{A},\mathcal{B}} \subseteq \Gamma^\delta_{\mathcal{A},\mathcal{B}}$ (see Notation 4.7). The following is essentially [Wei15 Theorem 3.5], but without assuming that $\mathcal{B}$ has an approximate identity of projections.

Lemma 5.2. Let $\epsilon : 0 \to \mathcal{A} \xrightarrow{x} \mathcal{E} \xrightarrow{\pi} \mathcal{A} \to 0$ be a unital extension of separable C*-algebras with $\mathcal{B}$ stable, let $\delta_\ast : K_\ast(\mathcal{A}) \to K_{\ast-1}(\mathcal{B})$ denote the induced boundary map in K-theory, and let $x \in K_0(\mathcal{B})$. Then $K^u_{\text{six}}(\epsilon) \equiv K^u_{\text{six}}(\epsilon \oplus x_\epsilon)$ if and only if $x \in \Gamma^\delta_{\mathcal{A},\mathcal{B}}$.

Proof. By Lemmas 4.11, 3.2, 3.3 and 3.5 $K^u_{\text{six}}(\epsilon \oplus x_\epsilon)$ is congruent to
\[
K_0(\mathcal{B}) \xrightarrow{i_0} (K_0(\mathcal{E}), [1_\mathcal{E}] + \iota_0(x)) \xrightarrow{\pi_0} (K_0(\mathcal{A}), [1_\mathcal{A}])
\]
\[
\delta_1 \downarrow \quad \delta_0
\]
\[
K_1(\mathcal{A}) \leftarrow \pi_1 \quad K_1(\mathcal{E}) \quad \iota_1 \quad K_1(\mathcal{B})
\]
If $x \in \Gamma^\delta_{\mathcal{A},\mathcal{B}}$, then there is a homomorphism $\phi : \ker \delta_0 \to \text{coker} \delta_1$ such that $q_{\delta_1}^{-1} = \phi([1_\mathcal{A}])$. Define $\eta_0 = \text{id}_{K_0(\mathcal{E})} \circ \tau_0 \circ \phi \circ \pi_0 : K_0(\mathcal{E}) \to K_0(\mathcal{E})$, where $\tau_0 : \text{coker} \delta_0 \to K_0(\mathcal{E})$ is the injective homomorphism induced by $\iota_0$. Letting $\eta_1 = \text{id}_{K_1(\mathcal{E})}$ it easily follows that $\eta_* : K_\ast(\mathcal{E}) \to K_\ast(\mathcal{E})$ induces a congruence between $K^u_{\text{six}}(\epsilon)$ and the sequence (5.1).

Now suppose that $K^u_{\text{six}}(\epsilon)$ is congruent to $K^u_{\text{six}}(\epsilon \oplus x_\epsilon)$ which in turn is congruent to the sequence (5.1). There is a homomorphism $\eta_* : K_\ast(\mathcal{E}) \to K_\ast(\mathcal{E})$ such that $\eta_0([1_\mathcal{E}]) = [1_\mathcal{E}] + \iota_0(x)$ and the following diagram with exact rows
\[
K_1(\mathcal{A}) \xrightarrow{\delta_1} K_0(\mathcal{B}) \xrightarrow{i_0} K_0(\mathcal{E}) \xrightarrow{\pi_0} K_0(\mathcal{A}) \xrightarrow{\delta_0} K_1(\mathcal{B})
\]
\[
\eta_0 \quad \eta_1 \quad \eta_0 \quad \eta_1
\]
commutes. By a standard diagram chase, there is a homomorphism $\phi \in \text{Hom}(\ker \delta_0, \text{coker} \delta_1)$ such that $\eta_0 = \text{id}_{K_0(\mathcal{E})} + \tau_0 \circ \phi \circ \pi_0$, where $\tau_0 : \text{coker} \delta_0 \to K_0(\mathcal{E})$ is the map induced by $\iota_0$. Hence
\[
[1_\mathcal{E}] + \iota_0(x) = \eta_0([1_\mathcal{E}]) = [1_\mathcal{E}] + \tau_0 \circ \phi([1_\mathcal{A}]).
\]
Letting $q_{\delta_1} : K_0(\mathcal{B}) \to \text{coker} \delta_1$ denote the quotient map, we get $\tau_0 q_{\delta_1}(x) = \iota_0(x) = \tau_0 \circ \phi([1_\mathcal{A}])$ which implies $q_{\delta_1}(x) = \phi([1_\mathcal{A}])$ since $\tau_0$ is injective. Thus $x \in \Gamma^\delta_{\mathcal{A},\mathcal{B}}$.

Proposition 5.3. Let $\mathcal{A}$ be a separable C*-algebra satisfying the UCT, and let $\alpha \in \text{Aut}(\mathcal{A})$ be an isomorphism such that $K_\ast(\mathcal{A}) = K_\ast(\text{id}_\mathcal{A})$. Then the induced Pimsner–Voiculescu sequence collapses to a short exact sequence
\[
0 \to K_{1-\ast}(\mathcal{A}) \to K_{1-\ast}(\mathcal{A} \rtimes \alpha \mathbb{Z}) \to K_\ast(\mathcal{A}) \to 0,
\]
and the induced element in $\text{Ext}(K_\ast(\mathcal{A}), K_{1-\ast}(\mathcal{A}))$ is mapped to
\[
\text{KK}(\alpha) - \text{KK}(\text{id}_\mathcal{A}) \in \text{KK}(\mathcal{A}, \mathcal{A})
\]
via the map $\text{Ext}(K_\ast(\mathcal{A}), K_{1-\ast}(\mathcal{A})) \to \text{KK}(\mathcal{A}, \mathcal{A})$ from the UCT.
Proof. That the Pimsner–Voiculescu sequence collapses to a short exact sequence is obvious. Let \( \mathfrak{M} := \{ f \in C([0, 1], \mathfrak{A}) : \alpha(f(0)) = f(1) \} \) be the mapping torus of \( \alpha \) and \( \text{id}_\mathfrak{A} \). It is well-known that the extension

\[
0 \to C_0((0, 1), \mathfrak{A}) \to \mathfrak{M} \to \mathfrak{A} \to 0
\]

induces a short exact sequence

\[
0 \to K_{1-\ast}(\mathfrak{A}) \to K_\ast(\mathfrak{M}) \to K_\ast(\mathfrak{A}) \to 0
\]

which represents the element in \( \text{Ext}(K_\ast(\mathfrak{A}), K_{1-\ast}(\mathfrak{A})) \) induced by \( \text{KK}(\alpha) - \text{KK}(\text{id}_\mathfrak{A}) \). By \cite{Bla98} Section 10.4 it follows that this extension is congruent to (5.2). \( \square \)

The following lemma is an immediate consequence of the Elliott–Kucerovsky absorption theorem.

**Lemma 5.5.** Let \( \mathfrak{A} \) and \( \mathfrak{C} \) be separable, unital, nuclear \( C^\ast \)-algebras, with a unital embedding \( \iota : \mathfrak{A} \to \mathfrak{C} \), and let \( \mathfrak{B} \) be a separable stable \( C^\ast \)-algebra. If \( \epsilon \) is an absorbing, unital extension of \( \mathfrak{C} \) by \( \mathfrak{B} \), then \( \epsilon \cdot \iota \) is an absorbing, unital extension of \( \mathfrak{A} \) by \( \mathfrak{B} \).

**Proof.** It follows immediately from the definition of pure largeness that \( \epsilon \cdot \iota \) is also purely large, so the result follows from \cite[Theorem 6]{EK01}. \( \square \)

In the following, we consider

\[
\Gamma^{(0, \delta_1)}_{\mathfrak{A}, \mathfrak{B}} = q^{-1}_\delta\{ \psi(1_{\mathfrak{A}}) : \psi \in \text{Hom}(K_0(\mathfrak{A}), \text{coker}\delta_1) \},
\]

which is a special case of Notation 5.1. Clearly

\[
\Gamma^{(0, \delta_1)}_{\mathfrak{A}, \mathfrak{B}} \subseteq \Gamma^{\delta_1}_{\mathfrak{A}, \mathfrak{B}} \subseteq K_0(\mathfrak{B}).
\]

The following lemma is the main technical tool to obtain our classification of unital extensions. While the conditions on \( \mathfrak{A} \) in the following lemma might look slightly technical, we emphasise that any unital UCT Kirchberg algebra has these properties; \( K_1 \)-surjectivity follows from \cite{Cun81} and the condition on automorphisms follows from the Kirchberg–Phillips theorem \cite{Kir94, Phi00}.

**Lemma 5.5.** Let \( \epsilon : 0 \to \mathfrak{B} \to \mathfrak{C} \to \mathfrak{A} \to 0 \) be a unital extension of separable \( C^\ast \)-algebras with boundary map \( \delta_\ast : K_\ast(\mathfrak{A}) \to K_{1-\ast}(\mathfrak{B}) \) in \( K \)-theory. Suppose that \( \mathfrak{B} \) is stable, and that \( \mathfrak{A} \) is nuclear, \( K_1 \)-surjective, satisfies the UCT, and that for any \( y \in \text{KK}(\mathfrak{A}, \mathfrak{A}) \) for which \( K_\ast(y) = K_\ast(\text{id}_\mathfrak{A}) \), there is an automorphism \( \alpha \in \text{Aut}(\mathfrak{A}) \) such that \( \text{KK}(\alpha) = y \). Then for any \( x \in \Gamma^{(0, \delta_1)}_{\mathfrak{A}, \mathfrak{B}} \) there is an automorphism \( \beta \in \text{Aut}(\mathfrak{A}) \) for which \( K_\ast(\beta) = \text{id}_{K_\ast(\mathfrak{A})} \), and

\[
[\epsilon \cdot \beta]_s = [\epsilon]_s + [\epsilon_2]_s \in \text{Ext}_{\text{us}}(\mathfrak{A}, \mathfrak{B}).
\]

**Proof.** Let \( \epsilon_0 \) be an absorbing, trivial, unital extension \( \epsilon_0 \). Since

\[
[(\epsilon \oplus \epsilon_0) \cdot \beta]_s = [\epsilon \cdot \beta]_s + [\epsilon_0 \cdot \beta]_s = [\epsilon \cdot \beta]_s
\]

for any automorphism \( \beta \in \text{Aut}(\mathfrak{A}) \), it follows that we may replace \( \epsilon \) with \( \epsilon \oplus \epsilon_0 \) without loss of generality, and thus assume that \( \epsilon \) is absorbing.
As \( x \in \Gamma^{(0,\delta_1)}_{\mathfrak{A},\mathfrak{B}} \) we may find a homomorphism \( \psi: K_0(\mathfrak{A}) \to K_0(\mathfrak{B})/\text{im} \delta_1 \), such that 
\( \psi([1\mathfrak{A}]) = x + \text{im} \delta_1 \). Let \( 0 \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} K_0(\mathfrak{A}) \to 0 \) be a free resolution\(^{10}\) As \( F_0 \) and \( F_1 \) are free, we may construct the following commutative diagram with exact rows

\[
\begin{array}{ccccccccc}
0 & \to & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & K_0(\mathfrak{A}) & \to & 0 \\
\psi \downarrow & & \psi_0 \downarrow & & \psi \downarrow & & & & \\
K_1(\mathfrak{A}) & \xrightarrow{\delta_1} & K_0(\mathfrak{B}) & \xrightarrow{\psi} & K_0(\mathfrak{B})/\text{im} \delta_1 & \to & 0.
\end{array}
\]

Letting \( G \) denote the push-out of \( \psi_1 \) and \( f_1 \), we get the following commutative diagram

\[
\begin{array}{ccccccccc}
0 & \to & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & K_0(\mathfrak{A}) & \to & 0 \\
\psi \downarrow & & \psi_0 \downarrow & & \psi \downarrow & & & & \\
0 & \to & K_1(\mathfrak{A}) & \xrightarrow{\delta_1} & G & \rightarrow & K_0(\mathfrak{A}) & \to & 0 \\
\phi \downarrow & & \downarrow & & \psi \downarrow & & & & \\
K_1(\mathfrak{A}) & \xrightarrow{\delta_1} & K_0(\mathfrak{B}) & \xrightarrow{G} & K_0(\mathfrak{B})/\text{im} \delta_1 & \to & 0.
\end{array}
\]

with exact rows. The homomorphism \( \phi: G \to K_0(\mathfrak{B}) \) making the diagram commute, exists by the universal property of push-outs. Let \( x_0 \in \text{Ext}(K_*(\mathfrak{A}), K_{1-*}(\mathfrak{A})) \subseteq \text{KK}(\mathfrak{A}, \mathfrak{A}) \) be such that 
\[ x_0 = [0 \to K_1(\mathfrak{A}) \to G \to K_0(\mathfrak{A}) \to 0] \in \text{Ext}(K_0(\mathfrak{A}), K_1(\mathfrak{A})), \]
and \( x_1 \) is the trivial extension. As \( K_*(x_0) \) is the zero map, it follows from our hypothesis on \( \mathfrak{A} \), that there is an automorphism \( \alpha \in \text{Aut}(\mathfrak{A}) \) such that \( \text{KK}(\alpha) = \text{KK}(\text{id}_A) + x_* \).

Applying Proposition 5.3, the Pimsner–Voiculescu sequence for the \( C^* \)-dynamical system \((\mathfrak{A}, \alpha, Z)\) collapses to a short exact sequence 
\[ 0 \to K_{1-*}(\mathfrak{A}) \xrightarrow{\ell^{1-*}} K_{1-*}(\mathfrak{A} \rtimes_\alpha Z) \to K_*(\mathfrak{A} \rtimes_\alpha Z) \to 0, \]
which exactly induces the element \( x_* \in \text{Ext}(K_*(\mathfrak{A}), K_{1-*}(\mathfrak{A})) \). Here \( \iota: \mathfrak{A} \to \mathfrak{A} \rtimes_\alpha Z \) is the inclusion map. In particular, we may assume that \( K_0(\mathfrak{A} \rtimes_\alpha Z) = K_0(\mathfrak{A}) \oplus K_1(\mathfrak{A}) \), and \( K_1(\mathfrak{A} \rtimes_\alpha Z) = G \), and thus we have a homomorphism
\[ (\delta_0 \oplus 0, \phi): K_*(\mathfrak{A} \rtimes_\alpha Z) \to K_{1-*}(\mathfrak{B}). \]
As \( \iota_*: K_*(\mathfrak{A}) \to K_*(\mathfrak{A} \rtimes_\alpha Z) \) is injective, it induces a surjection
\[ \iota^* : \text{Ext}(K_*(\mathfrak{A} \rtimes_\alpha Z), K_*(\mathfrak{B})) \to \text{Ext}(K_*(\mathfrak{A}), K_*(\mathfrak{B})). \]
As \( \mathfrak{A} \) satisfies the UCT, so does \( \mathfrak{A} \rtimes_\alpha Z \) by [RSS7]. Thus, by Theorem 4.14, we get the following commutative diagram

\[
\begin{array}{ccccccccc}
\text{Ext}(K_*(\mathfrak{A} \rtimes_\alpha Z), K_*(\mathfrak{B})) & \xrightarrow{\iota^*} & \text{Ext}_{uw}(\mathfrak{A} \rtimes_\alpha Z, \mathfrak{B}) & \xrightarrow{\iota^*} & \text{Hom}((K_*(\mathfrak{A} \rtimes_\alpha Z), [1]), K_{1-*}(\mathfrak{B})) \\
\bigg| & & \bigg| & & \bigg| \\
\text{Ext}(K_*(\mathfrak{A}), K_*(\mathfrak{B})) & \xrightarrow{\iota^*} & \text{Ext}_{uw}(\mathfrak{A}, \mathfrak{B}) & \xrightarrow{\iota^*} & \text{Hom}((K_*(\mathfrak{A}), [1]), K_{1-*}(\mathfrak{B}))
\end{array}
\]

\(^{10}\) i.e. a short exact sequence with both \( F_0 \) and \( F_1 \) free abelian groups.
for which the rows are short exact sequences. We may pick \([f']_w \in \text{Ext}_{uw}(\mathfrak{A} \rtimes_{\alpha} \mathbb{Z}, \mathfrak{B})\) which lifts the homomorphism \((\delta_0 \oplus 0, \phi)\). Recall that we identified \(G = K_1(\mathfrak{A} \rtimes_{\alpha} \mathbb{Z})\), so by \((5.3)\), we have
\[
\tau^*(\delta_0 \oplus 0, \phi) = ((\delta_0 \oplus 0) \circ \iota_0, \phi \circ \iota_1) = (\delta_0, \delta_1) = \delta_w.
\]
Thus \(\tau^*([f']_w)\) and \([\epsilon]_w\) induce the same element in \(\text{Hom}\). Thus, by doing a diagram chase in the above diagram (using surjectivity of the left vertical map), there is an element \([f'']_w \in \text{Ext}_{uw}(\mathfrak{A} \rtimes_{\alpha} \mathbb{Z}, \mathfrak{B})\) vanishing in \(\text{Hom}\), such that \(\tau^*([f']_w + [f'']_w) = [\epsilon]_w\). Let \(f\) be an absorbing unital extension of \(\mathfrak{A} \rtimes_{\alpha} \mathbb{Z}\) by \(\mathfrak{B}\) such that \([f]_w = [f']_w + [f'']_w\). Then \([f \cdot \iota]_w = [\epsilon]_w\).

Let \(\tau: \mathfrak{A} \to \mathcal{L}(\mathfrak{B})\) be the Busby map of \(\epsilon\), and \(\eta: \mathfrak{A} \rtimes_{\alpha} \mathbb{Z} \to \mathcal{L}(\mathfrak{B})\) be the Busby map of \(f\). In particular, \(\eta \circ \iota\) is the Busby map of \(f \cdot \iota\). Recall from the beginning of the proof, that we assumed that \(\epsilon\) was absorbing, and by Lemma \((5.4)\) \(f \cdot \iota\) is also absorbing. Thus, as \([f \cdot \iota]_w = [\epsilon]_w\), there is a unitary \(u \in \mathcal{L}(\mathfrak{B})\) such that
\[
\text{Ad} u \circ \tau = \eta \circ \iota.
\]
Let \(w \in \mathfrak{A} \rtimes_{\alpha} \mathbb{Z}\) denote the canonical unitary, so that \(\text{Ad} w \circ \iota = \iota \circ \alpha\). Then
\[
\tau \circ \alpha = \text{Ad} u^* \circ \eta \circ \iota \circ \alpha = \text{Ad} u^* \circ \eta \circ \text{Ad} w \circ \iota = \text{Ad} u^* \circ \text{Ad} \eta(w) \circ \eta \circ \iota = \text{Ad} u^* \circ \text{Ad} \eta(w) \circ \text{Ad} u \circ \tau = \text{Ad}(u^* \eta(w) u) \circ \tau.
\]
Hence it follows from Lemma \((4.11)\) that
\[
[\epsilon \circ \alpha]_s = [\epsilon]_s + [\epsilon u^* \eta(w) u]_s = [\epsilon]_s + [\epsilon \eta(w)]_s \in \text{Ext}_{us}(\mathfrak{A}, \mathfrak{B}).
\]
Recall that \([f]_w = [f']_w + [f'']_w\) where \([f'']_w\) vanishes in \(\text{Hom}\), and \([f']_w\) induces the homomorphism \((\delta_0 \oplus 0, \phi): K_*(\mathfrak{A} \rtimes_{\alpha} \mathbb{Z}) \to K_{1-*}(\mathfrak{B})\). Thus \([f]_w\) also induces the homomorphism \((\delta_0 \oplus 0, \phi)\), so in particular
\[
K_1(\eta) = \phi: K_1(\mathfrak{A} \rtimes_{\alpha} \mathbb{Z}) \to K_1(\mathcal{L}(\mathfrak{B})) = K_0(\mathfrak{B}).
\]
It follows that
\[
[\epsilon \circ \alpha]_s = [\epsilon]_s + [\epsilon \phi([w])]_s \in \text{Ext}_{us}(\mathfrak{A}, \mathfrak{B}).
\]
By commutativity of the lower right square in \((5.3)\), the two compositions
\[
K_1(\mathfrak{A} \rtimes_{\alpha} \mathbb{Z}) \xrightarrow{\phi} K_0(\mathfrak{B}) \to K_0(\mathfrak{B})/\text{im} \delta_1, \quad K_1(\mathfrak{A} \rtimes_{\alpha} \mathbb{Z}) \to K_0(\mathfrak{A}) \xrightarrow{\psi} K_0(\mathfrak{B})/\text{im} \delta_1,
\]
are the same. It is well-known, that \([w]\) is mapped to \([1_{\mathfrak{A}}]\) via the map \(K_1(\mathfrak{A} \rtimes_{\alpha} \mathbb{Z}) \to K_0(\mathfrak{A})\)\(^{11}\) Thus
\[
\phi([w]) + \text{im} \delta_1 = \psi([1_{\mathfrak{A}}]) = x + \text{im} \delta_1,
\]
where \(x \in \Gamma_{\mathfrak{A}, \mathfrak{B}}^{(0, \delta_1)}\) is our given element from the statement of the lemma. As \(\mathfrak{A}\) is \(K_1\)-surjective we may find a unitary \(v \in \mathfrak{A}\) such that
\[
\phi([w]) + \delta_1([v]) = x.
\]
\(^{11}\)The proof of this is identical to the proof showing that the map \(K_1(C(\mathcal{T})) \to K_0(\mathbb{K})\) induced by the usual Toeplitz extension, sends the class of the canonical unitary in \(C(\mathcal{T})\) to \([e_1]_0\).
Let $\beta = \text{Ad} \circ \alpha$ be the induced automorphism on $\mathfrak{A}$. By construction $K_*(\alpha) = \text{id}_{K_*}$. By Lemma 5.2 it follows that

$[e \cdot \beta]_s = [e \cdot \alpha]_s + [e_{\delta_1([v]_1)}]_s$

Cor. 5.3

$= [e]_s + [e_{\phi([w]_1)}]_s + [e_{\delta_1([v]_1)}]_s$

$= [e]_s + [e_{\phi([w]_1)+\delta_1([v]_1)}]_s$

Cor. 5.5

$= [e]_s + [e_x]_s$

as desired.

\[\square\]

**Proposition 5.6.** Let $\mathfrak{A}$ and $\mathfrak{B}$ be separable $C^*$-algebras, with $\mathfrak{A}$ unital, nuclear and satisfying the UCT, and $\mathfrak{B}$ stable. Suppose that $\mathfrak{A}$ is $K_1$-surjective and that for any $y \in KK(\mathfrak{A}, \mathfrak{A})$ for which $K_*(y) = K_*([\mathfrak{A}])$, there is an automorphism $\alpha \in \text{Aut}(\mathfrak{A})$ such that $KK(\alpha) = y$.

Let $\epsilon_1$ and $\epsilon_2$ be unital extensions of $\mathfrak{A}$ by $\mathfrak{B}$ and suppose that

(a) $[\epsilon_1]_w = [\epsilon_2]_w$ in $\text{Ext}_{uw}(\mathfrak{A}, \mathfrak{B})$,

(b) $K^u_{\text{six}}(\epsilon_1) \equiv K^u_{\text{six}}(\epsilon_2)$,

(c) the exponential maps $\delta_0 : K_0(\mathfrak{A}) \to K_{1-*}(\mathfrak{B})$ induced by $\epsilon_1$ and $\epsilon_2$ vanish.

Then there is an automorphism $\beta \in \text{Aut}(\mathfrak{A})$ with $K_*^{\beta}(\mathfrak{A}) = \text{id}_{K_*}([\mathfrak{A}])$ such that $[\epsilon_1 \cdot \beta]_s = [\epsilon_2]_s$ in $\text{Ext}_{uw}(\mathfrak{A}, \mathfrak{B})$.

**Proof.** Let $\delta_* : K_0(\mathfrak{A}) \to K_{1-*}(\mathfrak{B})$ be the connecting maps in the six-term exact sequences of $\epsilon_1$ and $\epsilon_2$, which agree since $K^u_{\text{six}}(\epsilon_1) \equiv K^u_{\text{six}}(\epsilon_2)$. As $[\epsilon_1]_w = [\epsilon_2]_w$, it follows from Theorem 4.14 that there is an $x \in K_0(\mathfrak{B})$ such that $[\epsilon_1 \oplus e_x]_s = [\epsilon_2]_s$ in $\text{Ext}_{uw}(\mathfrak{A}, \mathfrak{B})$. As

$K^u_{\text{six}}(\epsilon_1 \oplus e_x) \overset{\text{Cor. 5.3}}{\equiv} K^u_{\text{six}}(\epsilon_2) \equiv K^u_{\text{six}}(\epsilon_1),$

it follows from Lemma 5.2 that $x \in \Gamma^0_{\mathfrak{A}, \mathfrak{B}}$. As $\delta_0 = 0$, it clearly holds that $\Gamma^0_{\mathfrak{A}, \mathfrak{B}} = \Gamma^{(0, \delta_1)}_{\mathfrak{A}, \mathfrak{B}}$ and thus Lemma 5.5 provides an automorphism $\beta \in \text{Aut}(\mathfrak{A})$ such that

$[\epsilon_1 \cdot \beta]_s = [\epsilon_1]_s + [e_x]_s = [\epsilon_2]_s \in \text{Ext}_{uw}(\mathfrak{A}, \mathfrak{B})$

as wanted.

\[\square\]

**Remark 5.7.** The only thing Condition (c) was used for in Proposition 5.6 was so that $\Gamma^{(0, \delta_1)}_{\mathfrak{A}, \mathfrak{B}} = \Gamma^0_{\mathfrak{A}, \mathfrak{B}}$. Hence one may replace Condition (c) with this more general condition in order to obtain the conclusion of Proposition 5.6.

In particular, Condition (c) in Proposition 5.6 may be replaced by any of the following statements, as these all imply that $\Gamma^{(0, \delta_1)}_{\mathfrak{A}, \mathfrak{B}} = \Gamma^0_{\mathfrak{A}, \mathfrak{B}}$. Proving that (c1)–(c6) imply $\Gamma^{(0, \delta_1)}_{\mathfrak{A}, \mathfrak{B}} = \Gamma^0_{\mathfrak{A}, \mathfrak{B}}$ is left to the reader.

(c1) The class of the unit $[1_\mathfrak{A}]$ vanishes in $K_0(\mathfrak{A})$.
(c2) The exponential map $\delta_0$ is injective.
(c3) The index map $\delta_1$ is surjective.
(c4) $K_0(\mathfrak{A}) \cong \mathbb{Z} \oplus G$, such that $[1_\mathfrak{A}] = (1, g)$ for some $g \in G$.
(c5) $\ker \delta_0$ is a direct summand in $K_0(\mathfrak{A})$.
(c6) $K_0(\mathfrak{A})$ is divisible.
Proposition 5.8. Let \( \epsilon_1: 0 \to \mathcal{B} \to \mathcal{E}_i \to \mathcal{A} \to 0 \) be unital extensions of \( C^* \)-algebras for \( i = 1, 2 \) such that \( \mathcal{A} \) is a unital UCT Kirchberg algebra, and \( \mathcal{B} \) is a stable AF algebra. If \( K^u_{\text{six}}(\epsilon_1) \equiv K^u_{\text{six}}(\epsilon_2) \) then there is an automorphism \( \alpha \in \text{Aut}(\mathcal{A}) \) such that \( \epsilon_1 \) and \( \epsilon_2 \cdot \alpha \) are strongly unitary equivalent.

In particular, if \( K^u_{\text{six}}(\epsilon_1) \equiv K^u_{\text{six}}(\epsilon_2) \) then \( \mathcal{E}_1 \cong \mathcal{E}_2 \)

Proof. We identify \( \text{Ext}(\mathcal{A}, \mathcal{B}) \cong KK^1(\mathcal{A}, \mathcal{B}) \) in the usual way, see Theorem [2.7]. By [ERR09 Theorem 2.3] (which is based on [Cun81 Theorem 3.2]), there exist \( x \in KK(\mathcal{A}, \mathcal{A}) \) and \( y \in KK(\mathcal{B}, \mathcal{B}) \) such that \( K_*(x) = K_*(\text{id}_\mathcal{A}) \), \( K_*(y) = K_*(\text{id}_\mathcal{B}) \), \( [\epsilon_1] \times y = x \times [\epsilon_2] \) in \( KK^1(\mathcal{A}, \mathcal{B}) \). Since \( \mathcal{B} \) is an AF algebra, we have that \( y = KK(\text{id}_\mathcal{B}) \). Thus \( [\epsilon_1] = x \times [\epsilon_2] \). Since \( \mathcal{A} \) is a UCT Kirchberg algebra, by the Kirchberg–Phillips theorem ([Kir94, Phi00]) there exists an isomorphism \( \alpha^1: \mathcal{A} \to \mathcal{A} \) such that \( x = KK(\alpha^1) \). By [Ror97 Proposition 1.1], we get

\[
[\epsilon_2 \cdot (\alpha^1)] = x \times [\epsilon_2] = [\epsilon_1] \in KK^1(\mathcal{A}, \mathcal{B}) \cong \text{Ext}(\mathcal{A}, \mathcal{B})
\]

By Lemma 4.13 it follows that \( [\epsilon_1]_w = [\epsilon_2 \cdot (\alpha^1)]_w \) in \( \text{Ext}_{uw}(\mathcal{A}, \mathcal{B}) \). Now, as \( K_*(\alpha^1) = \text{id}_{K_*(\mathcal{A})} \), it follows that

\[
K^u_{\text{six}}(\epsilon_2 \cdot (\alpha^1)) \cong K^u_{\text{six}}(\epsilon_2) \cong K^u_{\text{six}}(\epsilon_1).
\]

By [Cun81] \( \mathcal{A} \) is \( K \)-surjective, and by the Kirchberg–Phillips theorem (cited above) \( \mathcal{A} \) satisfies the condition in Proposition 5.6 about automorphisms. Hence this proposition produces an automorphism \( \alpha^2 \in \text{Aut}(\mathcal{A}) \) with \( K_*(\alpha^2) = \text{id}_{K_*(\mathcal{A})} \) such that

\[
[\epsilon_1]_s = [\epsilon_2 \cdot \alpha]_s \in \text{Ext}_{us}(\mathcal{A}, \mathcal{B})
\]

where \( \alpha = \alpha^1 \circ \alpha^2 \).

As \( \mathcal{A} \) is simple, unital and nuclear, \( \mathcal{B} \) is stable with the corona factorisation property, and the extensions \( \epsilon_1 \) and \( \epsilon_2 \cdot \alpha \) are unital, it follows that \( \epsilon_1 \) and \( \epsilon_2 \cdot \alpha \) are full and thus absorbing. Hence \( \epsilon_1 \) and \( \epsilon_2 \cdot \alpha \) are strongly unitary equivalent.

The “in particular” part follows since the extension algebra of \( \epsilon_2 \) is isomorphic to the extension algebra of \( \epsilon_2 \cdot \alpha \) and since strong unitary equivalence implies isomorphism of the extension algebras.

By \( K^{+, u}_{\text{six}}(\epsilon) \) we mean the six-term exact sequence in \( K \)-theory with order in all \( K_0 \)-groups. The following is the main classification result of this section and is Theorem 5.9.

Theorem 5.9. Let \( \epsilon_i: 0 \to \mathcal{B}_i \to \mathcal{E}_i \to \mathcal{A}_i \to 0 \) be unital extensions of \( C^* \)-algebras for \( i = 1, 2 \), such that \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) are unital UCT Kirchberg algebras and \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) are stable AF algebras. Then \( \mathcal{E}_1 \cong \mathcal{E}_2 \) if and only if \( K^{+, u}_{\text{six}}(\epsilon_1) \cong K^{+, u}_{\text{six}}(\epsilon_2) \).

Proof. Suppose \( \mathcal{E}_1 \cong \mathcal{E}_2 \). As the extension \( \epsilon_i \) is unital, and as \( \mathcal{A}_i \) is simple, it follows that the extension \( \epsilon_i \) is full. As \( \mathcal{B}_i \) is stable, it therefore follows that \( \mathcal{B}_i \) is the unique maximal ideal in \( \mathcal{E}_i \) for \( i = 1, 2 \). It follows that the extensions \( \epsilon_1 \) and \( \epsilon_2 \) are isomorphic, and thus \( K^{+, u}_{\text{six}}(\epsilon_1) \cong K^{+, u}_{\text{six}}(\epsilon_2) \).

Now suppose that there is an isomorphism \( K^{+, u}_{\text{six}}(\epsilon_1) \cong K^{+, u}_{\text{six}}(\epsilon_2) \) induced by

\[
\phi_*: K_*(\mathcal{A}_1) \cong K_*(\mathcal{A}_2), \quad \psi_*: K_*(\mathcal{B}_1) \cong K_*(\mathcal{B}_2), \quad \rho_*: K_*(\mathcal{E}_1) \cong K_*(\mathcal{E}_2).
\]

\[12\]Clearly \( \mathcal{B}_1 \) is a maximal ideal as the corresponding quotient is simple. If \( \mathcal{J} \subseteq \mathcal{E}_1 \) is a two-sided, closed ideal such that \( \mathcal{J} \nsubseteq \mathcal{B}_1 \), then there is an element \( x \in \mathcal{J} \setminus \mathcal{B}_1 \) inducing a non-zero element in \( \mathcal{A}_1 \). As the extension is full and \( \mathcal{B}_1 \) is stable, it follows that \( x \) induces a full element in \( M(\mathcal{B}_1) \). Hence \( \mathcal{B}_1 x \mathcal{B}_1 = \mathcal{B}_1 \) so \( \mathcal{B}_1 \nsubseteq \mathcal{J} \) and thus \( \mathcal{J} = \mathcal{E}_1 \) by maximality of \( \mathcal{B}_1 \). The same argument works for \( \mathcal{E}_2 \).
By the Kirchberg–Phillips theorem [Kir94, Phi00] we find an isomorphism $\alpha: \mathcal{A}_1 \cong \mathcal{A}_2$ such that $K_*(\alpha) = \phi_*$. Similarly, by Elliott’s classification of AF algebras [Ell76], we find an isomorphism $\beta: \mathcal{B}_1 \cong \mathcal{B}_2$ such that $K_*(\beta) = \psi_*$. We obtain the following commutative diagram

\[
\begin{array}{c}
0 \to \mathcal{B}_1 \to \mathcal{E}_1 \to \mathcal{A}_1 \to 0 \\
\downarrow \cong \downarrow \cong \downarrow \cong \\
0 \to \mathcal{B}_2 \to \mathcal{E}_2' \to \mathcal{A}_1 \to 0 \\
\downarrow \cong \downarrow \cong \\
0 \to \mathcal{B}_2 \to \mathcal{E}_2 \to \mathcal{A}_2 \to 0
\end{array}
\]

which has exact rows. It is easy to see that the map

\[
K_*(\eta_2^{-1}) \circ \rho_* \circ K_*(\eta_1^{-1}): K_*(\mathcal{E}_1') \to K_*(\mathcal{E}_2')
\]

induces a congruence $\mathcal{K}_\text{six}^*(\beta \cdot \epsilon_1) \equiv \mathcal{K}_\text{six}^*(\epsilon_2 \cdot \alpha)$. By Proposition 5.8 it follows that $\mathcal{E}_1' \cong \mathcal{E}_2'$, so it follows that $\mathcal{E}_1 \cong \mathcal{E}_2$. \qed

6. Determining when extensions are full

In this section we characterise when certain extensions are full with a stable ideal. We show that when the ideal is sufficiently finite (e.g. an AF algebra) and the quotient is sufficiently infinite (e.g. a Kirchberg algebra), then this is characterised by the existence of a properly infinite, full projection in the extension algebra.

**Lemma 6.1.** Let $\mathfrak{B}$ be a $\sigma$-unital $C^*$-algebra with stable rank one. Then $\mathfrak{B}$ is stable if and only if there exists a projection $p \in \mathcal{M}(\mathfrak{B})$ which is properly infinite, and which is strictly full, i.e. $\mathfrak{B}p\mathfrak{B} = \mathfrak{B}$.

In particular, if $p \in \mathcal{M}(\mathfrak{B})$ is a strictly full, properly infinite projection, then $p\mathfrak{B}p$ is stable.

**Proof.** If $\mathfrak{B}$ is stable then $1, \mathcal{M}(\mathfrak{B}) \in \mathcal{M}(\mathfrak{B})$ is a strictly full, properly infinite projection.

Conversely, suppose $p \in \mathcal{M}(\mathfrak{B})$ is a strictly full, properly infinite projection. Let $p_1, p_2, \ldots \in \mathcal{M}(\mathfrak{B})$ be a sequence of pairwise orthogonal projections in $\mathcal{M}(\mathfrak{B})$, such that $p_i \leq p$ and $p \sim p_i$ for all $i \in \mathbb{N}$. Then the hereditary $C^*$-subalgebra $\mathfrak{B}_0$ of $\mathfrak{B}$ generated by $p_1, p_2, \ldots$ is isomorphic to $p\mathfrak{B}p \otimes \mathbb{K}$. As $p$ is strictly full it follows that $\mathfrak{B}_0 \subseteq \mathfrak{B}$ is a stable, full, hereditary $C^*$-subalgebra. It is an easy consequence of [PTWW14, Lemma 4.6] that $\mathfrak{B}$ is stable (as any strictly positive element in $\mathfrak{B}_0$ induces a full, properly infinite element in the scale of the Cuntz semigroup of $\mathfrak{B}$).

“In particular” is immediate since $\mathcal{M}(p\mathfrak{B}p) \cong p\mathcal{M}(\mathfrak{B})p$ canonically, and since $p\mathfrak{B}p$ is $\sigma$-unital with stable rank one. \qed

The following is essentially [BRR08, Proposition 2.7].
Lemma 6.2. Let $\mathfrak{A}, \mathfrak{C}$ and $\mathfrak{D}$ be $C^*$-algebras and suppose that $\phi: \mathfrak{A} \to \mathfrak{D}$ and $\pi: \mathfrak{C} \to \mathfrak{D}$ are $*$-homomorphisms for which $\pi$ is surjective. Suppose that $p \in \mathfrak{A}$ and $q \in \mathfrak{C}$ are projections such that $\phi(p) = \pi(q)$ and $\phi(p)\mathfrak{D}\phi(p)$ is $K_1$-injective. If both $p$ and $q$ are properly infinite, then $p \oplus q$ is properly infinite in the pull-back $\mathfrak{A} \oplus_{\phi,\pi} \mathfrak{C}$.

Proof. By replacing $\mathfrak{A}, \mathfrak{C}$ and $\mathfrak{D}$ with $p\mathfrak{A}p$, $q\mathfrak{C}q$ and $\phi(p)\mathfrak{D}\phi(p)$, we may assume that $\mathfrak{A}$, $\mathfrak{C}$ and $\mathfrak{D}$ are unital and properly infinite, that $\phi$ and $\pi$ are unital maps, and that $\mathfrak{D}$ is $K_1$-injective. Under these assumptions, we should show that $\mathfrak{A} \oplus_{\phi,\pi} \mathfrak{C}$ is properly infinite.

The result now follows from [BRR08 Proposition 2.7]. In fact, although said result assumes that both maps are surjective (corresponding in our case to $\phi$ and $\pi$), they only use that one map is surjective. We fill in the proof for completion.

Let $s_1, s_2, s_3 \in \mathfrak{A}$ and $t_1, t_2, t_3 \in \mathfrak{C}$ be isometries with mutually orthogonal range projections. Let

$$v := \sum_{j=1}^{2} \phi(s_j)\pi(t_j)^* \in \mathfrak{D}$$

which is a partial isometry satisfying $\phi(s_j) = v\pi(t_j)$ for $j = 1, 2$. Note that

$$1_{\mathfrak{D}} \sim \phi(s_3s_3^*) \leq 1_{\mathfrak{D}} - vv^*, \quad 1_{\mathfrak{D}} \sim \pi(t_3t_3^*) \leq 1_{\mathfrak{D}} - v^*v.$$

It follows that $1_{\mathfrak{D}} - vv^*$ and $1_{\mathfrak{D}} - v^*v$ are properly infinite and full in $\mathfrak{D}$. By [BRR08 Lemma 2.4(i)] there is a unitary $u \in \mathfrak{D}$ with $[u] = 0 \in K_1(\mathfrak{D})$ such that $v = uv^*v$. As $\mathfrak{D}$ is $K_1$-injective, it follows that $u$ is homotopic to 1, and thus lifts to a unitary $\tilde{u} \in \mathfrak{C}$.

Clearly $\tilde{u}t_1, \tilde{u}t_2 \in \mathfrak{C}$ are isometries with orthogonal range projections, and

$$\pi(\tilde{u}t_j) = u\pi(t_j) = v\pi(t_j) = \phi(s_j)$$

so $s_j \oplus \tilde{u}t_j \in \mathfrak{A} \oplus_{\phi,\pi} \mathfrak{C}$ for $j = 1, 2$ are isometries with orthogonal range projections. Hence $\mathfrak{A} \oplus_{\phi,\pi} \mathfrak{C}$ is properly infinite. □

By the above lemma we deduce the following property about proper infiniteness of projections in purely large extensions (see Remark 2.9).

Proposition 6.3. Let $0 \to \mathfrak{B} \to \mathfrak{C} \to \mathfrak{A} \to 0$ be a purely large extension of separable $C^*$-algebras such that $\mathfrak{B}$ is stable, and suppose that $p \in \mathfrak{C} \setminus \mathfrak{B}$ is a projection. Then $p$ is properly infinite if and only if $p + \mathfrak{B} \in \mathfrak{A}$ is properly infinite.

Proof. “Only if” is trivial. To prove “if”, assume that the image of $p$ in $\mathfrak{A}$ is properly infinite. Let $\tau: \mathfrak{A} \to \mathfrak{B}(\mathfrak{B})$ be the Busby map of the extension. We may identify $\mathfrak{C}$ with the pull-back $\mathfrak{A} \oplus_{\tau,\pi_{\mathfrak{B}}} \mathfrak{M}(\mathfrak{B})$. Let $q \in \mathfrak{M}(\mathfrak{B})$ be the projection induced by $p$. As purely large extensions are full [12], it follows that $q$ is full in $\mathfrak{M}(\mathfrak{B})$. As our given extension is purely large it easily follows that the extension

$$0 \to \mathfrak{B} \to \mathfrak{C} + Cq \to \mathfrak{C} \to 0$$

is purely large. By [Gab16 Proposition 2.7] it follows that $q$ is a properly infinite, full projection in $\mathfrak{M}(\mathfrak{B})$. Hence $q\mathfrak{B}q \cong \mathfrak{B}$ is stable, and thus $\pi_{\mathfrak{B}}(q)\mathfrak{B}(\mathfrak{B})\pi_{\mathfrak{B}}(q) \cong \mathfrak{B}(\mathfrak{B})$ is $K_1$-injective by Proposition 1.9. By Lemma 6.2 $p$ is properly infinite. □

[12] It is easy to see that an extension $\epsilon$ is full if and only if the Cuntz sum $\epsilon \oplus 0$ is full. If $\epsilon$ is properly large, then $\epsilon \oplus 0$ is nuclearly absorbing by [Gab16 Corollary 2.4]. As $\epsilon \oplus 0$ absorbs any full, trivial, weakly nuclear extension (which always exist), it follows that $\epsilon \oplus 0$ and thus also $\epsilon$ is full.
Proposition 6.4. Let $\varepsilon : 0 \to \mathcal{B} \to \mathcal{E} \to \mathfrak{A} \to 0$ be an extension of separable C*-algebras for which $\mathfrak{A}$ is simple and $\mathcal{B}$ has stable rank one and the corona factorisation property. Suppose that there is a projection $p \in \mathcal{E} \setminus \mathcal{B}$ such that $p + \mathcal{B} \subseteq \mathfrak{A}$ is properly infinite. Then $\mathcal{B}$ is stable and $\varepsilon$ is full if and only if $p$ is full and properly infinite in $\mathcal{E}$.

Proof. “Only if” follows from Proposition 6.3 as $\varepsilon$ is purely large by the corona factorisation property. For “if” suppose that $p$ is full and properly infinite. Then $\mathcal{B} = \mathcal{B}p\mathcal{B}$ by fullness of $p$. By Lemma 6.1 it follows that $\mathcal{B}$ is stable, and $\mathcal{B} \cong p\mathcal{B}p$. Hence by [Bro88, Theorem 4.23], $p$ induces a full projection in $\mathcal{M}(\mathcal{B})$. As $\mathfrak{A}$ is simple, and as $p + \mathcal{B} \subseteq \mathfrak{A}$ is mapped to a full projection in $\mathcal{M}(\mathcal{B})$ via the Busby map, it follows that the extension $\varepsilon$ is full. □

The following can be used to characterise when the extensions we wish to classify are full.

Theorem 6.5. Let $\varepsilon : 0 \to \mathcal{B} \to \mathcal{E} \to \mathfrak{A} \to 0$ be an extension of C*-algebras such that $\mathfrak{A}$ is a Kirchberg algebra and $\mathcal{B}$ is an AF algebra. The following are equivalent.

(i) $\mathcal{B}$ is stable and the extension $\varepsilon$ is full,

(ii) $\mathcal{E}$ contains a full, properly infinite projection,

(iii) any projection $p \in \mathcal{E} \setminus \mathcal{B}$ is full and properly infinite (in $\mathcal{E}$).

Proof. (i) $\Rightarrow$ (iii): Suppose that $p \in \mathcal{E} \setminus \mathcal{B}$ is a projection. Fullness of $\varepsilon$ and simplicity of $\mathfrak{A}$ implies that $p$ is full. As $\mathcal{B}$ has the corona factorisation property by virtue of being an AF algebra, it follows from Proposition 6.3 that $p$ is properly infinite.

(iii) $\Rightarrow$ (ii): Let $q \in \mathfrak{A}$ be a non-zero projection. By [BP91, Proposition 3.15], $q$ lifts to a projection $p \in \mathcal{E} \setminus \mathcal{B}$, which is properly infinite and full by assumption.

(ii) $\Rightarrow$ (i): Follows from Proposition 6.4 □

7. Classification of non-unital extensions

In [Gab16 Section 4] an example was given of two non-unital, full extensions $\varepsilon_i : 0 \to \mathcal{B}_i \to \mathcal{E}_i \to \mathfrak{A}_i \to 0$ such that $\mathfrak{A}_i \cong \mathcal{O}_2$, $\mathcal{B}_i \cong M_{2\infty} \otimes \mathbb{K}$, $K_{\text{six}}(\varepsilon_1) \cong K_{\text{six}}(\varepsilon_2)$ (with order, scale and units preserved), but for which $\mathcal{E}_1 \neq \mathcal{E}_2$. In this section we will describe how to obtain classification of such (and more general) extensions. Note that our invariant needs to carry more information than the six-term exact sequence alone.

The following lemma indicates the main trick that will be used to get classification of non-unital extensions with unital quotients. It implies that if one can arrange that the corresponding Busby maps have the same unit, and that the units in the quotients lift to projections, then the classification problem can be reduced to the unital case.

Lemma 7.1. Let $\mathfrak{A}$ and $\mathcal{B}$ be C*-algebras with $\mathfrak{A}$ unital, and let $\tau_i : \mathfrak{A} \to \mathcal{M}(\mathcal{B})$ be (not necessarily unital) Busby maps for $i = 1, 2$. Suppose that $\tau_1(1_\mathfrak{A}) = \tau_2(1_\mathfrak{A})$, and that this projection lifts to a projection $p \in \mathcal{M}(\mathcal{B})$. If the unital extensions

\[
0 \to p\mathcal{B}p \to (1_\mathfrak{A} \oplus p)(\mathfrak{A} \oplus \tau_1, \tau_2 : \mathcal{M}(\mathcal{B}))(1_\mathfrak{A} \oplus p) \to \mathfrak{A} \to 0
\]

for $i = 1, 2$ are strongly unitary equivalent, then so are the extensions induced by $\tau_1$ and $\tau_2$.

Proof. The Busby maps $\tilde{\tau}_i$ of the extensions (7.1) are just the corestrictions of the Busby maps $\tau_i$ to $\tau_i(1_\mathfrak{A})\mathcal{M}(\mathcal{B})\tau_i(1_\mathfrak{A}) \cong \mathcal{M}(p\mathcal{B}p)$ (the canonical isomorphism). By assumption there is a unitary $\tilde{u} \in \mathcal{M}(p\mathcal{B}p)$ such that $\text{Ad} \pi_{p\mathcal{B}p}(\tilde{u}) \circ \tilde{\tau}_1 = \tilde{\tau}_2$. Using the canonical identification
\[ \mathcal{M}(p\mathcal{B}p) \cong p.\mathcal{M}(\mathcal{B})p, \text{ let } u = \tilde{u} + (1_{\mathcal{M}(B)} - p). \] Then \( u \) is a unitary in \( \mathcal{M}(\mathcal{B}) \) satisfying \( \operatorname{Ad}\pi_B(u) \circ \tau_1 = \tau_2. \)

The next goal will be to arrange that \( \tau_1(1_A) = \tau_2(1_A) \in \mathcal{D}(\mathcal{B}) \) by twisting one extension by an automorphism on \( \mathcal{B} \). For this we introduce the following notation.

**Notation 7.2.** Let \( e : 0 \to \mathcal{B} \to \mathcal{E} \to A \to 0 \) be an extension of \( C^\ast \)-algebras where \( A \) is unital, but \( E \) is not necessarily unital. Let \( \mathcal{D}_e := \pi^{-1}(C1_A) \subseteq \mathcal{E}. \)

In the case where \( \mathcal{E} \) is unital, then \( \mathcal{D} = \tilde{\mathcal{B}} \) is the (forced) unitisation of \( \mathcal{B} \).

**Lemma 7.3.** Let \( e_i : 0 \to \mathcal{B}_i \to \mathcal{E}_i \to A_i \to 0 \) be extensions of \( C^\ast \)-algebras for \( i = 1, 2 \) with Busby maps \( \tau_i : \mathcal{A}_i \to \mathcal{D}(\mathcal{B}_i) \). Suppose that \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) are unital, that \( \beta : \mathcal{B}_1 \cong \mathcal{B}_2 \) is an isomorphism, and let \( \tilde{\beta} \in \mathcal{D}(\mathcal{B}_1) \cong \mathcal{D}(\mathcal{B}_2) \) be the induced isomorphism of corona algebras. Then \( \tilde{\beta} \circ \tau_1(1_{\mathcal{A}_1}) = \tau_2(1_{\mathcal{A}_2}) \) if and only if there is a \( \ast \)-homomorphism \( \mu : \mathcal{D}_{e_1} \to \mathcal{D}_{e_2} \) (necessarily unique and necessarily an isomorphism) making the diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathcal{B}_1 & \longrightarrow & \mathcal{D}_{e_1} & \longrightarrow & \mathcal{C} & \longrightarrow & 0 \\
0 & \longrightarrow & \mathcal{B}_2 & \longrightarrow & \mathcal{D}_{e_2} & \longrightarrow & \mathcal{C} & \longrightarrow & 0 \\
\end{array}
\]

commute.

*Proof.* The Busby maps of the extensions in the above diagram are \( \mathcal{C} \ni \lambda \mapsto \lambda \tau_i(1_{\mathcal{A}_i}) \) so the result follows immediately from [ELP99, Theorem 2.2]. \( \Box \)

When considering the ordered \( K \)-theory \( K^+_\ast(\mathcal{A}) = (K^+_0(\mathcal{A}), K_1(\mathcal{A})) \) for unital \( C^\ast \)-algebras \( \mathcal{A} \), we will often add the class of the unit to the invariant

\( K^+_\ast,u(\mathcal{A}) := (K^+_0(\mathcal{A}), [1_{\mathcal{A}}]0, K_1(\mathcal{A})). \)

Alternatively, we may consider the unital embedding \( j : \mathcal{C} \hookrightarrow \mathcal{A} \). This gives an induced diagram

\[
j_* : K^+_\ast(\mathcal{C}) \to K^+_\ast(\mathcal{A}).
\]

This diagram contains the exact same information as \( K^+_\ast,u(\mathcal{A}) \), thus motivating the following construction.

Suppose \( e : 0 \to \mathcal{B} \buildrel \iota \over \to \mathcal{E} \buildrel \pi \over \to \mathcal{A} \to 0 \) is an extension of \( C^\ast \)-algebras for which \( \mathcal{A} \) is unital, but where \( \mathcal{E} \) is not necessarily unital. We assume for convenience that \( 1_A \) lifts to a projection in \( \mathcal{E} \).

Again, we have a unital embedding \( j : \mathcal{C} \hookrightarrow \mathcal{A} \), and we obtain the following pull-back diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathcal{B} & \longrightarrow & \mathcal{D}_\pi & \longrightarrow & \mathcal{C} & \longrightarrow & 0 \\
0 & \longrightarrow & \mathcal{B} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{A} & \longrightarrow & 0,
\end{array}
\]
where \( \mathfrak{D}_\epsilon \) is as in Notation 7.2. Our invariant will be to apply \( K \)-theory with order and scale to this diagram, thus obtaining the following commutative diagram

\[
\begin{array}{c}
\xymatrix{K_0^{+\Sigma}(\mathfrak{B}) \ar[r] \ar[d] & K_0^{+\Sigma}(\mathfrak{D}_\epsilon) \ar[r]^(.65){\jmath_0} & K_0^{+\Sigma}(\mathfrak{C}) \ar[d] \\
K_0^{+\Sigma}(\mathfrak{B}) \ar[r]_(.35){\delta_1} & K_0^{+\Sigma}(\mathfrak{E}) \ar[r]_(.65){\pi_0} & K_0^{+\Sigma}(\mathfrak{A}) \ar[d]_{\delta_0} \\
K_1(\mathfrak{A}) \ar[r]^(.65){\tau_1} & K_1(\mathfrak{E}) \ar[r]^(.65){\iota_1} & K_1(\mathfrak{B}).
\end{array}
\]

We denoted this diagram by \( \tilde{K}_{\text{six}}^{+\Sigma}(\epsilon) \). A homomorphism between such diagrams are defined in the obvious way.

Suppose that \( \epsilon_1 : 0 \to \mathfrak{B}_1 \to \mathfrak{E}_1 \to \mathfrak{A}_1 \to 0 \) are extensions of \( C^* \)-algebras for \( i = 1, 2 \) with \( \mathfrak{A}_i \) unital. Suppose that there is a commutative diagram

\[
\begin{array}{c}
\xymatrix{& \mathfrak{B}_1 \ar[r]^(.55){\beta} & \mathfrak{E}_1 \ar[r]^(.55){\gamma} & \mathfrak{A}_1 \ar[d]^{\alpha} \\
\mathfrak{B}_2 \ar[r]^{\epsilon_2} & \mathfrak{E}_2 \ar[r]^{\epsilon_1} & \mathfrak{A}_2 \ar[r] & 0
\end{array}
\]

where all maps are *-homomorphisms, and \( \alpha \) is unital. Then \( \eta(\mathfrak{D}_{\epsilon_1}) \subseteq \mathfrak{D}_{\epsilon_2} \) and thus it easily follows that \( (\beta, \eta, \alpha) \) induces a homomorphism \( \tilde{K}_{\text{six}}^{+\Sigma}(\epsilon_1) \to \tilde{K}_{\text{six}}^{+\Sigma}(\epsilon_2) \).

In the cases we will be considering below, we assume that \( \mathfrak{A} \) is a unital UCT Kirchberg algebra, \( \mathfrak{B} \) is a stable AF algebra, and \( \mathfrak{E} \) contains a full, properly infinite projection. Hence the order and scale can be ignored in \( K_0(\mathfrak{E}) \) and \( K_0(\mathfrak{A}) \), and the scale of \( K_0(\mathfrak{B}) \) can be ignored when considering \( \tilde{K}_{\text{six}}^{+\Sigma}(\epsilon) \).

We obtain our final classification result which is exactly Theorem 13.

**Theorem 7.4.** Let \( \epsilon_i : 0 \to \mathfrak{B}_i \to \mathfrak{E}_i \to \mathfrak{A}_i \to 0 \) be full extensions of \( C^* \)-algebras for \( i = 1, 2 \), such that \( \mathfrak{A}_1 \) and \( \mathfrak{A}_2 \) are unital UCT Kirchberg algebras, and \( \mathfrak{B}_1 \) and \( \mathfrak{B}_2 \) are stable AF algebras. Then \( \mathfrak{E}_1 \cong \mathfrak{E}_2 \) if and only if \( \tilde{K}_{\text{six}}^{+\Sigma}(\epsilon_1) \cong \tilde{K}_{\text{six}}^{+\Sigma}(\epsilon_2) \).

**Proof.** Suppose \( \mathfrak{E}_1 \cong \mathfrak{E}_2 \). As the extension \( \epsilon_i \) is full, as \( \mathfrak{A}_i \) is simple and \( \mathfrak{B}_i \) is stable, it follows that \( \mathfrak{B}_i \) is the unique maximal ideal in \( \mathfrak{E}_i \) for \( i = 1, 2 \) (see Footnote 12). It follows that the extensions \( \epsilon_1 \) and \( \epsilon_2 \) are isomorphic, and thus \( \tilde{K}_{\text{six}}^{+\Sigma}(\epsilon_1) \cong \tilde{K}_{\text{six}}^{+\Sigma}(\epsilon_2) \).

For the converse, suppose that \( \tilde{K}_{\text{six}}^{+\Sigma}(\epsilon_1) \cong \tilde{K}_{\text{six}}^{+\Sigma}(\epsilon_2) \), and let

\[
\phi_* : K_*^{+\Sigma}(\mathfrak{A}_1) \cong K_*^{+\Sigma}(\mathfrak{A}_2), \quad \psi_* : K_*^{+\Sigma}(\mathfrak{B}_1) \cong K_*^{+\Sigma}(\mathfrak{B}_2), \quad \theta_0 : K_0^{+\Sigma}(\mathfrak{D}_{\epsilon_1}) \cong K_0^{+\Sigma}(\mathfrak{D}_{\epsilon_2})
\]

be a collection of isomorphisms inducing the isomorphism on \( \tilde{K}_{\text{six}}^{+\Sigma} \). We first show that we may assume that \( \mathfrak{A} = \mathfrak{A}_1 = \mathfrak{A}_2, \mathfrak{B} = \mathfrak{B}_1 = \mathfrak{B}_2, \phi_* = \text{id}_{K_*(\mathfrak{A})}, \psi_* = \text{id}_{K_*(\mathfrak{B})} \), that \( \tau_1(1_{\mathfrak{A}}) = \tau_2(1_{\mathfrak{A}}) \), where \( \tau_i \) is the Busby map of \( \epsilon_i \) for \( i = 1, 2 \), and that \( \theta_0 = K_0(\mu) \) where \( \mu : \mathfrak{D}_{\epsilon_1} \to \mathfrak{D}_{\epsilon_2} \) is the isomorphism provided by Lemma 7.3.

By the Kirchberg–Phillips theorem [Kir94, Phi00] we may pick an isomorphism \( \alpha : \mathfrak{A}_1 \cong \mathfrak{A}_2 \) such that \( K_*(\alpha) = \phi_* \).
As $\mathcal{D}_{\mathfrak{e}_1}$ is an extension of two AF algebras, it is itself an AF algebra by [Ell81] Chapter 9. Hence by Elliott’s classification of AF algebras [Ell76] we may pick an isomorphism $\mu: \mathcal{D}_{\mathfrak{e}_1} \cong \mathcal{D}_{\mathfrak{e}_2}$ such that $K_0(\mu) = \theta_0$. In particular, $\mu$ restricts to an isomorphism $\beta: \mathcal{B}_1 \cong \mathcal{B}_2$ satisfying $K_0(\beta) = \psi_0$.

Forming the push-out extension $\beta \cdot \mathfrak{e}_1$ and the pull-back extension $\mathfrak{e}_2 \cdot \alpha$, we obtain a diagram identical to (5.6). By Lemma 7.3 we get

$$\overline{\beta} \circ \tau_1(1_{\mathfrak{e}_1}) = \tau_2(1_{\mathfrak{e}_2}) = \tau_2 \circ \alpha(1_{\mathfrak{e}_1}).$$

Let

$$\mu^{(1)}: \mathcal{D}_{\mathfrak{e}_1} \cong \mathcal{D}_{\beta \cdot \mathfrak{e}_1}, \quad \mu^{(2)}: \mathcal{D}_{\mathfrak{e}_2 \cdot \alpha} \cong \mathcal{D}_{\mathfrak{e}_2}$$

be the induced isomorphisms, i.e. the restriction–corestriction of $\eta^{(1)}$ and $\eta^{(2)}$ respectively.

Now, it follows from (5.6) (by inverting the isomorphisms) that we obtain induced isomorphisms $\overline{K}_{\text{six}}^{+, \Sigma}(\beta \cdot \mathfrak{e}_1) \cong \overline{K}_{\text{six}}^{+, \Sigma}(\mathfrak{e}_1)$ and $\overline{K}_{\text{six}}^{+, \Sigma}(\mathfrak{e}_2) \cong \overline{K}_{\text{six}}^{+, \Sigma}(\mathfrak{e}_2 \cdot \alpha)$. By composing these isomorphisms with the already given isomorphism $\overline{K}_{\text{six}}^{+, \Sigma}(\mathfrak{e}_1) \cong \overline{K}_{\text{six}}^{+, \Sigma}(\mathfrak{e}_2)$, it follows that the compositions

$$K_*(\alpha)^{-1} \circ \phi_\mathfrak{e}_1 \circ K_*(\mathfrak{e}_1)^{-1} = \text{id}_{K_*}(\mathfrak{e}_1), \quad K_*(\mathfrak{e}_2)^{-1} \circ \psi_\mathfrak{e}_2 \circ K_*(\beta)^{-1} = \text{id}_{K_*}(\mathfrak{e}_2)$$

$$K_*(\eta^{(2)})^{-1} \circ \rho_\mathfrak{e}_1 \circ K_*(\eta^{(1)})^{-1}, \quad K_0(\mu^{(2)})^{-1} \circ \theta_0 \circ K_0(\mu^{(1)})^{-1}$$

give rise to an isomorphism $\overline{K}_{\text{six}}^{+, \Sigma}(\beta \cdot \mathfrak{e}_1) \cong \overline{K}_{\text{six}}^{+, \Sigma}(\mathfrak{e}_2 \cdot \alpha)$. Moreover, observe that $\mu^{(0)} := (\mu^{(2)})^{-1} \circ \mu \circ (\mu^{(1)})^{-1}$ is the unique (by Lemma 7.3) $*$-homomorphism making the diagram

$$\begin{array}{cccccc}
0 & \rightarrow & \mathcal{B}_2 & \rightarrow & \mathcal{D}_{\beta \cdot \mathfrak{e}_1} & \rightarrow & \mathbb{C} & \rightarrow & 0 \\
| & & | & & | & & | & & |
0 & \rightarrow & \mathcal{B}_2 & \rightarrow & \mathcal{D}_{\mathfrak{e}_2 \cdot \alpha} & \rightarrow & \mathbb{C} & \rightarrow & 0
\end{array}$$

commute, and that $K_0(\mu^{(0)}) = K_0(\mu^{(2)})^{-1} \circ \theta_0 \circ K_0(\mu^{(1)})^{-1}$.

Therefore, without loss of generality, we may assume that $\mathfrak{A} = \mathfrak{A}_1 = \mathfrak{A}_2$, $\mathfrak{B} = \mathfrak{B}_1 = \mathfrak{B}_2$, $\phi_\mathfrak{e}_1 = \text{id}_{K_*(\mathfrak{e}_1)}$, $\psi_\mathfrak{e}_2 = \text{id}_{K_*(\mathfrak{e}_2)}$ that $\tau_1(1_{\mathfrak{e}_1}) = \tau_2(1_{\mathfrak{e}_2})$, and that $\theta_0 = K_0(\mu)$ where $\mu: \mathcal{D}_{\mathfrak{e}_1} \rightarrow \mathcal{D}_{\mathfrak{e}_2}$ is the map provided by Lemma 7.3 (with $\beta = \text{id}_\mathfrak{B}$).

As $\mathfrak{B}$ has real rank zero and $K_1(\mathfrak{B}) = 0$, the projection $\tau_1(1_{\mathfrak{e}_1}) \in \mathcal{Z}(\mathfrak{B})$ lifts to a projection $p \in \mathcal{M}(\mathfrak{B})$ by [BP91] Corollary 3.16. In particular, by identifying $\mathfrak{E}_1$ with $\mathfrak{A} \oplus_{\tau_1, \pi_\mathfrak{e}_1} \mathcal{M}(\mathfrak{B})$ in the canonical way, $1_{\mathfrak{A}} \oplus p$ defines a projection both in $\mathfrak{E}_1$ and in $\mathfrak{E}_2$ since $\tau_1(1_{\mathfrak{e}_1}) = \tau_2(1_{\mathfrak{e}_2})$. Note that when identifying $\mathcal{D}_{\mathfrak{e}_1}$ and $\mathcal{D}_{\mathfrak{e}_2}$ in a canonical way with a subalgebra of $\mathfrak{A} \oplus \mathcal{M}(\mathfrak{B})$, then $\mu$ is simply the identity map. Hence $\mu(1_{\mathfrak{e}_1} \oplus p) = 1_{\mathfrak{e}_2} \oplus p$. In particular, by commutativity of the diagram

$$K_0(\mathcal{D}_{\mathfrak{e}_1}) \rightarrow K_0(\mathcal{E}_1) \rightarrow K_0(\mathcal{D}_{\mathfrak{e}_2}) \rightarrow K_0(\mathcal{E}_2),$$

which is part of $\overline{K}_{\text{six}}^{+, \Sigma}(\mathfrak{e}_1) \rightarrow \overline{K}_{\text{six}}^{+, \Sigma}(\mathfrak{e}_2)$, it follows that $\rho_0([1_{\mathfrak{e}_1} \oplus p]) = [1_{\mathfrak{e}_2} \oplus p]$. 


By Theorem 6.5 it follows that $1_A \oplus p$ is a full, properly infinite projection in both $E_1$ and $E_2$. Moreover, $pBp$ is a full and stable corner in $B$ by Lemma 6.1. Let

$$\iota_i: pBp \hookrightarrow B, \quad \iota_i: (1_A \oplus p)E_i(1_A \oplus p) \hookrightarrow E_i$$

for $i = 1, 2$ denote the inclusions, which are all inclusions of full, hereditary, $C^*$-subalgebras in separable $C^*$-algebras and thus induce isomorphisms in $K$-theory. Since $\rho_0([1_A \oplus p]) = [1_A \oplus p]$ it follows that the map

$$K_s(\iota_2)^{-1} \circ \rho_s \circ K_s(\iota_1): K_s((1_A \oplus p)E_i(1_A \oplus p)) \to K_s((1_A \oplus p)E_2(1_A \oplus p))$$

induces a congruence $K_{\text{six}}(p\epsilon_1 p) \equiv K_{\text{six}}(p\epsilon_2 p)$, where $p\epsilon_i p$ denotes the unital extension

$$0 \to pBp \to (1_A \oplus p)E_1(1_A \oplus p) \to A \to 0$$

for $i = 1, 2$.

Thus, by Proposition 5.8 there is an automorphism $\alpha \in \text{Aut}(A)$ such that $p\epsilon_1 p$ and $p\epsilon_2 p \cdot \alpha = p(\epsilon_2 \cdot \alpha)p$ are strongly unitary equivalent. By Lemma 7.1 it follows that $\epsilon_1$ and $\epsilon_2 \cdot \alpha$ are strongly unitary equivalent. As the extension algebra of $\epsilon_2 \cdot \alpha$ is isomorphic to $E_2$, it follows that $E_1 \cong E_2$ as desired. \hfill $\square$

**Remark 7.5.** In a future paper [EGK+18], we compute the range of the invariant $K_{\text{six}}^+$ for graph $C^*$-algebras with a unique, non-trivial ideal. This will be used to show that an extension of two simple graph $C^*$-algebras is again a graph $C^*$-algebra, provided there are no $K$-theoretic obstructions.

**References**

[Bla08] B. Blackadar. *K-theory for operator algebras*, volume 5 of *Mathematical Sciences Research Institute Publications*. Cambridge University Press, Cambridge, second edition, 1998.

[BP91] L.G. Brown and G.K. Pedersen. $C^*$-algebras of real rank zero. *J. Funct. Anal.*, 99(1):131–149, 1991.

[Bro88] L.G. Brown. Semicontinuity and multipliers of $C^*$-algebras. *Canad. J. Math.*, 40(4):865–988, 1988.

[BRR08] E. Blanchard, R. Rohde, and M. Rørdam. Properly infinite $C(X)$-algebras and $K_1$-injectivity. *J. Noncommut. Geom.*, 2(3):263–282, 2008.

[CE76] M.D. Choi and E.G. Effros. The completely positive lifting problem for $C^*$-algebras. *Ann. of Math.* (2), 104(3):585–609, 1976.

[CH87] J. Cuntz and N. Higson. Kuiper’s theorem for Hilbert modules. In *Operator algebras and mathematical physics (Iowa City, Iowa, 1985)*, volume 62 of *Contemp. Math.*, pages 429–433. Amer. Math. Soc., Providence, RI, 1987.

[Cun81] J. Cuntz. $K$-theory for certain $C^*$-algebras. *Ann. of Math.* (2), 113(1):181–197, 1981.

[Eff81] E.G. Effros. *Dimensions and $C^*$-algebras*, volume 46 of *CBMS Regional Conference Series in Mathematics*. Conference Board of the Mathematical Sciences, Washington, D.C., 1981.

[EGK+18] S. Eilers, J. Gabe, T. Katsura, E. Ruiz, and M. Tomforde. The extension problem for graph $C^*$-algebras. *In preparation*, 2018.

[EGLN15] G.A. Elliott, G. Gong, H. Lin, and Z. Niu. On the classification of simple $C^*$-algebras with finite decomposition rank, ii. *arXiv:1507.03437v2*, 2015.

[EK01] G.A. Elliott and D. Kucerovsky. An abstract Voiculescu-Brown-Douglas-Fillmore absorption theorem. *Pacific J. Math.*, 198(2):385–409, 2001.

[Eli76] G.A. Elliott. On the classification of inductive limits of sequences of semifinite-dimensional algebras. *J. Algebra*, 38(1):29–44, 1976.

[ELP99] S. Eilers, T.A. Loring, and G.K. Pedersen. Morphisms of extensions of $C^*$-algebras: pushing forward the Busby invariant. *Adv. Math.*, 147(1):74–109, 1999.
S. Eilers, G. Restorff, and E. Ruiz. Classification of extensions of classifiable C\(^*\)-algebras. Adv. Math., 222(6):2153–2172, 2009.

[ERR09] S. Eilers, G. Restorff, and E. Ruiz. The complete classification of unital graph C\(^*\)-algebras: Geometric and strong. Preprint, arXiv:1611.07120, 2016.

[Gab16] J. Gabe. A note on nonunital absorbing extensions. Pacific J. Math., 284(2):383–393, 2016.

[GLN15] G. Gong, H. Lin, and Z. Niu. Classification of finite simple amenable Z-stable C\(^*\)-algebras. arXiv:1501.00135v4, 2015.

[Kas80] G.G. Kasparov. The operator K-functor and extensions of C\(^*\)-algebras. Izv. Akad. Nauk SSSR Ser. Mat., 44(3):571–636, 1980.

[Kir94] E. Kirchberg. The classification of purely infinite C\(^*\)-algebras using Kasparov’s theory. 1994.

[KN06] D. Kucerovsky and P.W. Ng. The corona factorization property and approximate unitary equivalence. Houston J. Math., 32(2):531–550 (electronic), 2006.

[OPR12] E. Ortega, F. Perera, and M. Rørdam. The corona factorization property, stability, and the Cuntz semigroup of a C\(^*\)-algebra. Int. Math. Res. Not. IMRN, (1):34–66, 2012.

[Phi00] N.C. Phillips. A classification theorem for nuclear purely infinite simple C\(^*\)-algebras. Doc. Math., 5:49–114 (electronic), 2000.

[PTWW14] F. Perera, A. Toms, S. White, and W. Winter. The Cuntz semigroup and stability of close C\(^*\)-algebras. Anal. PDE, 7(4):929–952, 2014.

[Rie83] M.A. Rieffel. Dimension and stable rank in the K-theory of C\(^*\)-algebras. Proc. London Math. Soc. (3), 46(2):301–333, 1983.

[RLL00] M. Rørdam, F. Larsen, and N. Laustsen. An introduction to K-theory for C\(^*\)-algebras, volume 49 of London Mathematical Society Student Texts. Cambridge University Press, Cambridge, 2000.

[Rob11] L. Robert. Nuclear dimension and n-comparison. Münster J. Math., 4:65–71, 2011.

[Ror97] M. Rørdam. Classification of extensions of certain C\(^*\)-algebras by their six term exact sequences in K-theory. Math. Ann., 308(1):93–117, 1997.

[Tho01] K. Thomsen. On absorbing extensions. Proc. Amer. Math. Soc., 129(5):1409–1417 (electronic), 2001.

[TWW17] A. Tikuisis, S. White, and W. Winter. Quasidiagonality of nuclear C\(^*\)-algebras. Ann. of Math. (2), 185(1):229–284, 2017.

[Wei15] C. Wei. On the classification of certain unital extensions of C\(^*\)-algebras. Houston J. Math., 41(3):965–991, 2015.

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