Stochastic 2D Navier-Stokes equations on time-dependent domains

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Abstract: We establish the existence and uniqueness of solutions to stochastic 2D Navier-Stokes equations in a time-dependent domain driven by Brownian motion. A martingale solution is constructed through domain transformation and appropriate Galerkin approximations on time-dependent spaces. The probabilistic strong solution follows from the pathwise uniqueness and the Yamada-Watanable theorem.

Keywords: Stochastic Navier-Stokes equations; time-dependent domain; tightness; Yamada-Watanabe theorem.

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1 Introduction

Fix \( T > 0 \) and let \( \mathcal{O}_T = \bigcup_{t \in [0,T]} \mathcal{D}(t) \times \{t\} \) be a non-cylindrical space-time domain, each \( \mathcal{D}(t), t \in [0,T], \) being a bounded open domain in \( \mathbb{R}^2 \) with smooth boundary \( \Gamma(t) := \partial \mathcal{D}(t). \) Let \( \bar{\Gamma}_T = \bigcup_{t \in [0,T]} \Gamma(t) \times \{t\} \) represent the boundary of \( \mathcal{O}_T, \) and set \( \bar{\mathcal{O}}_T = \mathcal{O}_T \cup \bar{\Gamma}_T. \)

Consider 2D stochastic Navier-Stokes equation with the Dirichlet boundary conditions:

\[
du(t) - \mu \Delta u(t) dt + (u(t) \cdot \nabla) u(t) dt + \nabla p(t) dt = f(t) dt + \sigma(t) dW(t), \quad x \in \mathcal{D}(t), t \in (0, T],
\]

\[
\text{div} \ u(t) = 0, \quad x \in \mathcal{D}(t), t \in [0, T],
\]

\[
u(t) = 0, \quad x \in \Gamma(t), t \in [0, T],
\]

\[
u(x, 0) = \nu_0(x), \quad x \in \mathcal{D}(0),
\]

where \( \nu(x, t) = (\nu_1(x, t), \nu_2(x, t)) : \bar{\mathcal{O}}_T \mapsto \mathbb{R}^2 \) and \( p(x, t) : \bar{\mathcal{O}}_T \mapsto \mathbb{R} \) are the unknown velocity field and pressure of the fluid, respectively; the constant \( \mu > 0 \) is the coefficient of viscosity, without loss of generality, here we take \( \mu = 1; \) \( f(x, t) : \bar{\mathcal{O}}_T \mapsto \mathbb{R}^2 \) is the external force and

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\( \mathbf{u}_0(x) : \mathcal{D}(0) \rightarrow \mathbb{R}^2 \) is the initial velocity; \( \{W(t), t \in [0, T]\} \) is a one-dimensional Brownian motion defined on a stochastic basis \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})\), where \((\mathcal{F}_t)_{0 \leq t \leq T}\) satisfies the usual conditions. For the precise conditions on \( \mathcal{O}_T, \mathbf{f}, \mathbf{u}_0, \) and \( \mathbf{\sigma} \), we refer the reader to Section 2 and Section 3.

The purpose of this paper is to establish the existence and uniqueness of the solution to equation (1.1).

The Navier-Stokes equation is an important model for atmospheric and ocean dynamics, water flow, and other viscous flow. It has been extensively studied by many authors, mainly on a fixed domain which is independent of time. For deterministic 2D Navier-Stokes equations, we refer readers to Hopf [13], Ladyzhenskaya [15], Leray [16], Lions and Prodi [17], Temam [24], and references therein.

Since the seminal work [2] by Bensoussan and Temam, a great number of papers have been devoted to the subject of stochastic Navier-Stokes equations. It is still an ongoing very active research area. We mention some of the relevant works. The well-posedness of stochastic Navier-Stokes equations were studied by Flandoli and Gatarek [9], Menaldi and Sritharan [20], Liu and Röckner [18]. Large deviations for 2D stochastic Navier-Stokes equations were proved by Chang [5], Sritharan and Sundar [22], Wang, Zhai and Zhang [26]. Ergodicity of 2D Navier-Stokes equations with stochastic forcing were studied by Flandoli and Maslowski [8], Hairer and Mattingly [11].

When modeling fluid, the area usually changes with time. It is therefore important to consider Navier-Stokes equations in a time-varying domain. There are a few papers on this topic in the deterministic setting. Applying the penalty methods, Fujita and Sauer [10] proved the existence and uniqueness of the solution to Navier-Stokes equations in domains with the boundaries covered by finite number of \( C^3 \)-class simple closed surfaces. In Bock [3] and Inoue-Wakimoto [14], the authors transformed Navier-Stokes equations on time-dependent domain into nonlinear partial differential equations on a cylindrical domain. The equations in a time-dependent domain with Neumann boundary conditions were considered by Filo and Zaušková [7]. Some related free boundary problems were considered by Bae [1], Shibata and Shimizu [23].

Taking into account the influence of unknown external forces, which may be caused by environmental noises in both mathematical and physical senses, in this paper we consider 2D stochastic Navier-Stokes equations in a time-dependent domain. To the best of our knowledge, this is the first result of this kind. The moving boundary of the domain raises serious difficulties, for example, we can’t solve the equation in the usual setting of a Gelfand triple like in the case of a fixed domain, the Itô formula widely used in the literature is also not available as the state space of the solution changes with time. Due to the lack of the Itô formula, the local monotonicity arguments used in the literature for stochastic Navier-Stokes equations fail to work. In this paper, we use finite dimensional approximations. The main difficulty lies in the proof of the tightness of the family of the laws of the finite dimensional approximations because the state space of the solution changes with time. To overcome this, we need to find a suitable criterion to characterize the compact subsets of the state space of the solution. We first obtain the existence of the probabilistic weak solution and then prove the pathwise uniqueness of the solutions. The probabilistic strong solutions are obtained thanks to the Yamada-Watanabe theorem.

The paper is divided into four sections. In Section 2, we introduce the time-dependent domain and its transformation. In Section 3, we lay out the setup and state the main result. Section 4 is devoted to the proofs of the main results.
2 Transformation of domain

In Euclidean space $\mathbb{R}^n$, we write $x = (x^1, x^2, \ldots, x^n)$ as the points of the space, and $x^T$ means the transposition of $x$. We will make the following assumptions about the region $\mathcal{O}_T$. Assume that there exists a bounded open domain $\mathcal{D} \subset \mathbb{R}^2$ satisfying the following condition:

(A1) Let $\mathcal{O}_T = \mathcal{D} \times [0, T]$, there exists a level-preserving $C^\infty$-diffeomorphism $L$ from $\overline{\mathcal{O}_T}$ (the closure of $\mathcal{O}_T$) to $\overline{\mathcal{O}_T}$ (the closure of $\mathcal{O}_T$), i.e.,

$$(y, s) = L(x, t) = (y^1(x, t), y^2(x, t), t), \quad (x, t) \in \mathcal{O}_T, \ (y, s) \in \overline{\mathcal{O}_T},$$

and

$$\det M(x, t) \equiv J(t)^{-1} > 0, \ (x, t) \in \overline{\mathcal{O}_T},$$

here

$$M(x, t) = \begin{bmatrix} \partial y^1(x, t) / \partial x^1 & \partial y^2(x, t) / \partial x^1 \\ \partial y^1(x, t) / \partial x^2 & \partial y^2(x, t) / \partial x^2 \end{bmatrix}, \ (x, t) \in \overline{\mathcal{O}_T}. \quad (2.1)$$

Let $L^{-1}$ be the inverse of $L$, which means that $L^{-1} : \overline{\mathcal{O}_T} \mapsto \overline{\mathcal{O}_T}$, and for every $(y, s) \in \overline{\mathcal{O}_T}$ and $(x, t) \in \overline{\mathcal{O}_T}$ with $(y, s) = L(x, t)$,

$$L^{-1}(y, s) = L^{-1}L(x, t) = (x, t).$$

Note that the time variable remains unchanged during the transformation, i.e., $s(x, t) = t$. For each function $\Gamma : \overline{\mathcal{O}_T} \mapsto \mathbb{R}^2$, $\tilde{\Gamma}$ will always mean a function mapping $\overline{\mathcal{O}_T}$ into $\mathbb{R}^2$ obtained by the transformation

$$\tilde{\Gamma}(y, s) = \Gamma(L^{-1}(y, s))M(L^{-1}(y, s)). \quad (2.2)$$

Conversely, we can recover the function $\Gamma$ from $\tilde{\Gamma}$ by setting

$$\Gamma(x, t) = \tilde{\Gamma}(L(x, t))M(x, t)^{-1}.$$

In the following, we will set

$$\tilde{u}(y, s) = u(L^{-1}(y, s))M(L^{-1}(y, s)),$$

$$\tilde{u}_0(y, s) = u_0(L^{-1}(y, s))M(L^{-1}(y, s)),$$

$$\tilde{f}(y, s) = f(L^{-1}(y, s))M(L^{-1}(y, s)),$$

$$\tilde{\sigma}(y, s) = \sigma(L^{-1}(y, s))M(L^{-1}(y, s)).$$

Note that, under this transformation, we have

**Lemma 2.1** $\text{div} \ \Gamma(x, t) = 0, \ \forall (x, t) \in \mathcal{O}_T$ if and only if $\text{div} \ \tilde{\Gamma}(y, s) = 0, \ \forall (y, s) \in \overline{\mathcal{O}_T}$.

**Proof** Indeed, for each $(y, s) \in \overline{\mathcal{O}_T}$ and $(x, t) \in \mathcal{O}_T$ such that $(y, s) = L(x, t)$, we recall

$$\tilde{\Gamma}(y, s) = \Gamma(x, t)M(x, t).$$

If we set

$$K = \frac{\partial(x^1, x^2)}{\partial(y^1, y^2)} = \begin{bmatrix} \partial x^1 / \partial y^1 & \partial x^1 / \partial y^2 \\ \partial x^2 / \partial y^1 & \partial x^2 / \partial y^2 \end{bmatrix}, \quad (2.4)$$

then

$$\text{div} \ \Gamma(x, t) = \text{div} \ \tilde{\Gamma}(y, s) = \text{tr} \ (K^{-1} J(t)^{-1} K) \equiv 0.$$
then
\[
KM^T = \left[\begin{array}{c}
\sum_{k=1}^{2} (\partial x^1/\partial y^k)(\partial y^k/\partial x^1) \\
\sum_{k=1}^{2} (\partial x^2/\partial y^k)(\partial y^k/\partial x^1)
\end{array}\right] 
\left[\begin{array}{c}
\sum_{k=1}^{2} (\partial x^1/\partial y^k)(\partial y^k/\partial x^2) \\
\sum_{k=1}^{2} (\partial x^2/\partial y^k)(\partial y^k/\partial x^2)
\end{array}\right] = \left[\begin{array}{cc}
\partial x^1/\partial x^1 & \partial x^1/\partial x^2 \\
\partial x^2/\partial x^1 & \partial x^2/\partial x^2
\end{array}\right] = I.
\]

This means that \(M^T\) is the inverse of \(K\), i.e.,
\[
M^T = K^{-1} = \frac{1}{\det K} \left[\begin{array}{cc}
\partial x^2/\partial y^2 & -\partial x^1/\partial y^2 \\
-\partial x^2/\partial y^1 & \partial x^1/\partial y^1
\end{array}\right],
\]
where
\[
\det K = \frac{1}{\det M} = J(t).
\]
The equation (2.5) illustrates that
\[
M = (J(t)^{-1}) \left[\begin{array}{cc}
\partial x^2/\partial y^2 & -\partial x^2/\partial y^1 \\
-\partial x^1/\partial y^2 & \partial x^1/\partial y^1
\end{array}\right],
\]
namely,
\[
\begin{align*}
\partial y^1/\partial x^1 &= (J(t)^{-1}) \partial x^2/\partial y^2, \\
\partial y^2/\partial x^1 &= -(J(t)^{-1}) \partial x^2/\partial y^1, \\
\partial y^1/\partial x^2 &= -(J(t)^{-1}) \partial x^1/\partial y^2, \\
\partial y^2/\partial x^2 &= (J(t)^{-1}) \partial x^1/\partial y^1.
\end{align*}
\]

Apply equation (2.6) in the following calculation to obtain
\[
\text{div } \tilde{\Gamma}(y, s) = \sum_{j=1}^{2} \partial \tilde{\Gamma}^j(y, s)/\partial y^j
\]
\[
= \sum_{j=1}^{2} \sum_{k=1}^{2} \left[ (\partial(\partial y^j/\partial x^k))/\partial y^j \right] \Gamma^k(x, t) + (\partial y^j/\partial x^k)(\partial \Gamma^k(x, t))/\partial y^j
\]
\[
= \sum_{j=1}^{2} \sum_{k=1}^{2} (\partial y^j/\partial x^k)(\partial \Gamma^k(x, t))/\partial y^j
\]
\[
+ \left[ \Gamma^1(x, t) \left\{ \partial (J(t)^{-1})(\partial x^2/\partial y^2) \right\}/\partial y^1 \right] - \Gamma^1(x, t) \left\{ \partial (J(t)^{-1})(\partial x^2/\partial y^1) \right\}/\partial y^2
\]
\[
+ \left[ \Gamma^2(x, t) \left\{ \partial (J(t)^{-1})(\partial x^1/\partial y^2) \right\}/\partial y^1 \right] - \Gamma^2(x, t) \left\{ \partial (J(t)^{-1})(\partial x^1/\partial y^1) \right\}/\partial y^2
\]
\[
= \sum_{k=1}^{2} \partial \Gamma^k(x, t)/\partial x^k = \text{div } \Gamma(x, t).
\]

The proof of Lemma 2.1 is complete.

Set \(\tilde{\rho}(y, s) = p(L^{-1}(y, s))\). The transformation \(\tilde{\rho}\) and the above lemma imply that (1.1) can be transformed into the following problem on \(\tilde{O}_T\):
\[
d\tilde{u}(s) - F\tilde{u}(s)ds + G\tilde{u}(s)ds + N\tilde{u}(s)ds + \nabla_x \tilde{\rho}(s)ds
\]
\[
= \tilde{f}(s)ds + \tilde{\sigma}(s)dW(s), \quad y \in \tilde{D}, s \in (0, T],
\]
\[
(2.7)
\]
\[
\text{div } \mathbf{u}(s) = 0, \quad y \in \tilde{D}, s \in [0, T],
\]
\[
\mathbf{u}(s) = 0, \quad y \in \partial \tilde{D}, s \in [0, T],
\]
\[
\mathbf{u}(y, 0) = \mathbf{u}_0(y), \quad y \in \tilde{D},
\]

where, for \( i = 1, 2, \)

\[
(F\tilde{u})^i = \sum_{j=1}^{2} \sum_{k=1}^{2} h^{jk}_i \nabla_j \nabla_k \tilde{u}^i,
\]
\[
(G\tilde{u})^i = \sum_{j=1}^{2} (\partial y^j / \partial t) \nabla_j \tilde{u}^i + \sum_{j=1}^{2} \sum_{k=1}^{2} (\partial y^j / \partial x^k)(\partial^2 x^k / \partial s \partial y^j) \tilde{u}^i,
\]
\[
(\mathcal{N}\tilde{u})^i = \sum_{j=1}^{2} \tilde{u}^j \nabla_j \tilde{u}^i,
\]
\[
(\nabla h\tilde{p})^i = \sum_{j=1}^{2} h^{ij} (\partial \tilde{p} / \partial y^j),
\]

and, for \( i, j \in \{1, 2\}, \)

\[
h^{ij} = \sum_{k=1}^{2} (\partial y^i / \partial x^k)(\partial y^j / \partial x^k),
\]
\[
h_{ij} = \sum_{k=1}^{2} (\partial x^k / \partial y^i)(\partial x^k / \partial y^j),
\]
\[
\nabla_j \tilde{u}^i = \partial \tilde{u}^i / \partial y^j + \sum_{k=1}^{2} \Phi^{ij}_k \tilde{u}^k,
\]
\[
\nabla_j \nabla_k \tilde{u} = \partial (\nabla_k \tilde{u}^i) / \partial y^j + \sum_{l=1}^{2} \Phi^{ij}_l \nabla_l \tilde{u}^i - \sum_{l=1}^{2} \Phi^{kij}_l \nabla_l \tilde{u}^i,
\]
\[
2\Phi^{ij}_k = \sum_{l=1}^{2} h^{kl}(\partial h_{il} / \partial y^j + \partial h_{jl} / \partial y^i - \partial h_{ij} / \partial y^l)
\]
\[
= 2 \sum_{l=1}^{2} (\partial y^k / \partial x^l)(\partial^2 x^l / \partial y^j \partial y^i).
\]

**Remark 2.1** Set \( H_1 = (h^{ij})_{2 \times 2} \) and \( H_2 = (h_{ij})_{2 \times 2}. \) The condition (A1) implies that

\[
H_1 = M^T M, \quad H_1^{-1} = H_2, \quad \text{det } H_2 = J(t)^2.
\]

### 3 Statement of the main result

In this section we introduce the precise definition of solutions and state the main result. We begin with some notations. Let \( C_0^\infty(\tilde{D}) \) be the space of all \( \mathbb{R} \)-valued \( C^\infty \) functions on \( \tilde{D} \) with compact supports. We denote by \( L^2(\tilde{D}) \) and \( H^1_0(\tilde{D}) \) the closure of \( C_0^\infty(\tilde{D}) \) under the following norms:

\[
\| u \|_{L^2(\tilde{D})} = \left( \int_{\tilde{D}} |u(y)|^2 \, dy \right)^{1/2},
\]  

(3.1)
∥u∥_{\mathbb{H}_0^1(\tilde{D})} = \left( \int_{\tilde{D}} |\nabla u(y)|^2 dy \right)^{1/2}. \quad (3.2)

Introduce the spaces

\begin{align*}
\mathcal{C} &= \{ u \in (C_0^\infty(\tilde{D}))^2 : \text{div } u = 0 \}, \\
\tilde{V} &= \text{the closure of } \mathcal{C} \text{ in } (\mathbb{H}_0^1(\tilde{D}))^2, \\
\tilde{H} &= \text{the closure of } \mathcal{C} \text{ in } (L^2(\tilde{D}))^2.
\end{align*}

For fixed \( t \in [0, T] \), similarly we can define the spaces \( H_t \) and \( V_t \) on the domain \( \mathcal{D}(t) \):

\begin{align*}
\mathcal{C}_t &= \{ u \in (C_0^\infty(\mathcal{D}(t)))^2 : \text{div } u = 0 \}, \\
\tilde{V}_t &= \text{the closure of } \mathcal{C}_t \text{ in } (\mathbb{H}_0^1(\mathcal{D}(t)))^2, \\
\tilde{H}_t &= \text{the closure of } \mathcal{C}_t \text{ in } (L^2(\mathcal{D}(t)))^2.
\end{align*}

The corresponding inner products on \( H_t \) and \( V_t \) are defined as

\begin{align*}
(u, v)_t &= \int_{\mathcal{D}(t)} u(x)v^T(x)dx - \sum_{i=1}^2 \int_{\mathcal{D}(t)} u^i(x)v^i(x)dx, \\
(\nabla u, \nabla v)_t &= \sum_{i=1}^2 \sum_{j=1}^2 \int_{\mathcal{D}(t)} (\partial u^i/\partial x^j)(\partial v^i/\partial x^j)dx. \quad (3.3)
\end{align*}

For every \( t \in [0, T] \), \( \tilde{H} \) is a Hilbert space with the inner product

\begin{align*}
\langle \tilde{u}, \tilde{v} \rangle_t &= \int_{\tilde{D}} \tilde{u}(y)H_2(y, t)\tilde{v}^T(y)J(t)dy, \quad (3.5)
\end{align*}

i.e.

\begin{align*}
\langle \tilde{u}, \tilde{v} \rangle_t &= \sum_{i=1}^2 \sum_{j=1}^2 \int_{\tilde{D}} h_{ij}(y, t)\tilde{u}^i(y)\tilde{v}^j(y)J(t)dy. \quad (3.6)
\end{align*}

Similarly, we can define the inner product on \( \tilde{V} \) as follows

\begin{align*}
\langle \nabla_h \tilde{u}, \nabla_h \tilde{v} \rangle_t &= \sum_{i=1}^2 \sum_{j=1}^2 \sum_{k=1}^2 \sum_{l=1}^2 \int_{\tilde{D}} h_{ij}(y, t)h_{kl}(y, t)\nabla_h \tilde{u}^i(y)\nabla_h \tilde{v}^j(y)J(t)dy, \quad t \in [0, T]. \quad (3.7)
\end{align*}

**Remark 3.1** \( \tilde{H} \) and \( \tilde{V} \) are usually equipped with the following inner products, respectively:

\begin{align*}
\langle \tilde{u}, \tilde{v} \rangle_{\tilde{H}} &= \int_{\tilde{D}} \tilde{u}(y)\tilde{v}^T(y)dy, \quad (3.8) \\
\langle \tilde{u}, \tilde{v} \rangle_{\tilde{V}} &= \int_{\tilde{D}} \nabla \tilde{u}(y)(\nabla \tilde{v})^T(y)dy. \quad (3.9)
\end{align*}

The condition (A1) implies that the norms induced by the above two inner products are equivalent to that induced by (3.5) and (3.7), respectively.
After a change of variable, we see that
\[
(u, v)_t = \langle \tilde{u}, \tilde{v} \rangle_t, \quad \text{and} \quad (\nabla u, \nabla v)_t = \langle \nabla_h \tilde{u}, \nabla_h \tilde{v} \rangle_t, \quad \forall t \in [0, T].
\] (3.10)
The following notations will be used.
\[
|\tilde{u}|_t = \langle \tilde{u}, \tilde{u} \rangle_t^{1/2}, \quad \text{for each} \quad \tilde{u} \in \tilde{H};
\]
\[
|\nabla_h \tilde{u}|_t = \langle \nabla_h \tilde{u}, \nabla_h \tilde{u} \rangle_t^{1/2}, \quad \text{for each} \quad \tilde{u} \in \tilde{V};
\]
\[
\|u\|_t = \langle u, u \rangle_t^{1/2}, \quad \text{for each} \quad u \in H_t;
\]
\[
\|\nabla u\|_t = \langle \nabla u, \nabla u \rangle_t^{1/2}, \quad \text{for each} \quad u \in V_t.
\]
Identifying \((\tilde{H}, \langle \cdot, \cdot \rangle_{\tilde{H}})\) with its dual space \(\tilde{H}^* = (\tilde{H}, \langle \cdot, \cdot \rangle_{\tilde{H}})\), we denote by \(\tilde{V}^*\) the dual spaces of \(\tilde{V}\). The corresponding norm in \(\tilde{V}^*\) is given by
\[
\|\tilde{u}\|_{\tilde{V}^*} = \sup_{\tilde{v} \in \tilde{V}, \|\tilde{v}\| \leq 1} \langle \tilde{u}, \tilde{v} \rangle_{\tilde{H}}.
\] (3.11)
The spaces \(L^2([0, T]; \tilde{V})\) and \(L^2([0, T]; \tilde{V}^*)\) are defined as
\[
L^2([0, T]; \tilde{V}) = \left\{ v = \left\{ v(t) \in \tilde{V}, t \in [0, T] \right\} \mid \int_0^T \|v(t)\|^2 dt < \infty \right\},
\]
\[
L^2([0, T]; \tilde{V}^*) = \left\{ v = \left\{ v(t) \in \tilde{V}^*, t \in [0, T] \right\} \mid \int_0^T \|v(t)\|_{\tilde{V}^*}^2 dt < \infty \right\}.
\]
\(C([0, T]; \tilde{H})\) is the space of continuous functions from the closed interval \([0, T]\) to the Hilbert space \((H, \langle \cdot, \cdot \rangle_{\tilde{H}})\).

For each \(t \in [0, T]\), identifying \((\tilde{H}, \langle \cdot, \cdot \rangle_{\tilde{H}})\) with its dual space \(\tilde{H}_t^* = (\tilde{H}, \langle \cdot, \cdot \rangle_{\tilde{H}})\), we denote by \(\tilde{V}_t^*\) the dual spaces of \(\tilde{V}_t\). The corresponding norm in \(\tilde{V}_t^*\) is given by
\[
|\tilde{u}|_{t}^* = \sup_{|\nabla_h \tilde{v}|_{t} \leq 1, \tilde{v} \in \tilde{V}} \langle \tilde{u}, \tilde{v} \rangle_{t}.
\] (3.12)
Similarly for each \(t \in [0, T]\), identifying \((H_t, \langle \cdot, \cdot \rangle_t)\) with its dual space \(H_t^* = (H_t, \langle \cdot, \cdot \rangle_t)\), we denote by \(V_t^*\) the dual spaces of \(V_t\). The corresponding norm in \(V_t^*\) is given by
\[
\|u\|_{t}^* = \sup_{|\nabla \tilde{v}|_{t} \leq 1, \tilde{v} \in V_t} \langle u, \tilde{v} \rangle_{t}.
\] (3.13)
For fixed \(t \in [0, T]\), we denote by \(\pi_t\) the orthogonal projection from \(L^2(D_t; \mathbb{R}^2)\) to \(H_t\), and define \(A_t, B_t, b_t\) as follows: for any \(u, v, w \in V_t\),
\[
A_t : V_t \mapsto V_t^*, \quad A_t u = -\pi_t \Delta u;
\]
\[
B_t : V_t \times V_t \mapsto V_t^*, \quad B_t(u, v) = \pi_t (u \cdot \nabla v), \quad B_t(u) = B_t(u, u);
\]
\[
b_t(u, v, w) = \langle B_t(u, v), w \rangle_{V_t} = \sum_{i,j=1}^{2} \int_{D(t)} u^i \partial_i v^j w^j \ dx.
\]
We have the estimates for \(b_t(u, v, w)\) (see [24] Lemma 3.4 in Chapter III):
\[
|b_t(u, v, w)| \leq C_1 \|u\|_{t}^{1/2} \|\nabla u\|_{t}^{1/2} \|\nabla v\|_{t} \|w\|_{t}^{1/2} \|\nabla w\|_{t}^{1/2}, \quad u, v, w \in V_t.
\] (3.14)
Taking into account the property: \( b_t(u, v, w) = -b_t(u, w, v) \), we have
\[
| v^*_t \langle B_t(u) - B_t(v), u - v \rangle_{V_t} | \leq \frac{1}{2} \| \nabla(u - v) \|_t^2 + C_2 \| u - v \|_t^2 \| v \|_{L^4(D(t); \mathbb{R}^2)}^4, \quad (3.15)
\]
\[
v^*_t \langle A_t u - A_t v + B_t(u) - B_t(v), u - v \rangle_{V_t} + C_2 \| u - v \|_t^2 \| v \|_{L^4(D(t); \mathbb{R}^2)}^4 \geq \frac{1}{2} \| \nabla(u - v) \|_t^2. \quad (3.16)
\]
Similarly, we denote
\[
N(\tilde{u}, \tilde{v}) = \sum_{j=1}^2 \tilde{u}^j \nabla_j \tilde{v}, \quad N(\tilde{u}) = N(\tilde{u}, \tilde{u}),
\]
where \( \nabla_j \) and \( N(\tilde{u}) \) are consistent with the notations in (2.7).

The spaces \( L^2([0, T]; V_t^*) \), \( L^p([0, T]; H_t) \), and \( L^\infty([0, T]; H_t) \) are defined as
\[
L^2([0, T]; V_t^*) = \left\{ v = \left\{ v(t) \in V_t^*, t \in [0, T] \right\} \mid \int_0^T \| v(t) \|_{V_t}^2 dt < \infty \right\},
\]
\[
L^p([0, T]; H_t) = \left\{ v = \left\{ v(t) \in H_t, t \in [0, T] \right\} \mid \int_0^T \| v(t) \|_{H_t}^p dt < \infty \right\},
\]
\[
L^\infty([0, T]; H_t) = \left\{ v = \left\{ v(t) \in H_t, t \in [0, T] \right\} \mid \text{ess sup}_{t \in [0, T]} \| v(t) \|_t < \infty \right\}.
\]

Now we define the solution of (1.1).

**Definition 3.1** Let \( p > 2 \). For \( u_0 \in H_0, f \in L^2([0, T]; V_t^*) \) and \( \sigma \in L^p([0, T]; H_t) \), we call a stochastic process \( u \) a solution of (1.1), if
(i) \( u \in L^\infty([0, T]; H_t) \cap L^2([0, T]; V_t) \), \( \mathbb{P} \)-a.s.;
(ii) \( u \) is \((\mathcal{F}_t)_{0 \leq t \leq T}\)-adapted;
(iii) for all \( \varphi \in \mathcal{C}_\sigma(\mathcal{O}_T) = \{ v \in (C^\infty(\mathcal{O}_T))^2 \mid \text{div } v = 0, \ v(T) = 0 \},
\]
\[
\int_0^T (u(s), \varphi'(s))_s ds + \int_0^T (\nabla u(s), \nabla \varphi(s))_s ds - \int_0^T ((u(s) \cdot \nabla) \varphi(s), u(s))_s ds \leq (u_0, \varphi(0))_0 + \int_0^T (f(s), \varphi(s))_s ds + \int_0^T (\varphi(s), \varphi(s))_s d\mathcal{W}(s). \quad (3.17)
\]

**Remark 3.2** Set \( \mathcal{C}_\sigma(\tilde{\mathcal{O}}_T) = \{ \tilde{\varphi} \in (C^\infty(\tilde{\mathcal{O}}_T))^2 \mid \text{div } \tilde{\varphi} = 0, \ \tilde{\varphi}(T) = 0 \}. \) Let
\[
\mathcal{C}_1 = \{ \tilde{v} \mid v \in \mathcal{C}_\sigma(\mathcal{O}_T), \ \tilde{v} (y, s) = v(L^{-1}(y, s)) M(L^{-1}(y, s)), \ (y, s) \in \tilde{\mathcal{O}}_T \},
\]
and
\[
\mathcal{C}_2 = \{ \chi \mid \tilde{\chi} \in \mathcal{C}_\sigma(\tilde{\mathcal{O}}_T), \ \chi(x, t) = \tilde{\chi}(L(x, t)) M(x, t)^{-1}, \ (x, t) \in \mathcal{O}_T \}.
\]
The condition (A1), (2.7) and Lemma 2.1 imply that
\[
\mathcal{C}_1 = \mathcal{C}_\sigma(\tilde{\mathcal{O}}_T) \quad \text{and} \quad \mathcal{C}_2 = \mathcal{C}_\sigma(\mathcal{O}_T).
\]
The above fact implies that (3.7) is equivalent to that \( \tilde{u}(y, s) = u(L^{-1}(y, s)) M(L^{-1}(y, s)), \ (y, s) \in \tilde{\mathcal{O}}_T \) satisfies the following identity: for any \( \tilde{\varphi} \in \mathcal{C}_\sigma(\tilde{\mathcal{O}}_T),
\]
\[
\int_0^T \langle \tilde{u}(s), \tilde{\varphi}'(s) \rangle_s ds - \int_0^T \langle \tilde{u}(s), \tilde{G}(\tilde{\varphi}(s))_s \rangle_s ds + \int_0^T \langle \nabla h \tilde{u}(s), \nabla \tilde{\varphi}(s) \rangle_s ds + \int_0^T \langle \tilde{N}(\tilde{u}(s)), \tilde{\varphi}(s) \rangle_s ds \leq (\tilde{u}_0, \tilde{\varphi}(0))_0 + \int_0^T \langle \tilde{f}(s), \tilde{\varphi}(s) \rangle_s ds + \int_0^T \langle \tilde{\sigma}(s), \tilde{\varphi}(s) \rangle_s d\mathcal{W}(s). \quad (3.18)
\]
Now we are in a position to state the main result.

**Theorem 3.1** Let \( p > 2 \). For \( u_0 \in H_0, f \in L^2([0, T]; V'_t) \) and \( \sigma \in L^p([0, T]; H_1) \), there exists a unique solution to the stochastic Navier-Stokes equation \((1.1)\).

## 4 Proof of Theorem 3.1

We will use Galerkin approximations to prove Theorem 3.1. Let \( \{e_n\}_{n \in \mathbb{N}} \) be a sequence of linearly independent elements in the space \( C \) which is total in \( V \). For \( s \in [0, T] \), let \( \{\tilde{w}_n(s)\}_{n \in \mathbb{N}} \) be its Schmidt orthogonalization with respect to the inner product \((4.5)\). Note that

\[
\{\tilde{w}_n(s)\}_{n \in \mathbb{N}, s \in [0, T]} \text{ is smooth in } (y, s), \text{ and } \forall i, j \in \mathbb{N}, \langle \tilde{w}_i(s), \tilde{w}_j(s) \rangle_s = \delta_{i,j}, s \in [0, T]. \tag{4.1}
\]

The following result is taken from Lemma 2.7 in [21].

**Lemma 4.1** If \( \tilde{w} \in L^2([0, T]; \tilde{V}) \) and \( \tilde{w}' \in L^2([0, T]; \tilde{V}^*) \), then \( \tilde{w} \in C([0, T]; \tilde{H}) \) and

\[
\frac{d|\tilde{w}(s)|^2_s}{ds} = 2\langle \tilde{w}'(s) + G\tilde{w}(s), \tilde{w}(s) \rangle_s, \quad s \in [0, T], \tag{4.2}
\]

where \( G \) was defined as in Section 2.

Combining the above lemma with \((4.1)\), we have

**Lemma 4.2** For any \( i, j \in \mathbb{N} \),

\[
0 = \frac{d\langle \tilde{w}_i(s), \tilde{w}_j(s) \rangle_s}{ds} = \langle \tilde{w}'_i(s) + G\tilde{w}_i(s), \tilde{w}_j(s) \rangle_s + \langle \tilde{w}'_j(s) + G\tilde{w}_j(s), \tilde{w}_i(s) \rangle_s, \quad s \in [0, T]. \tag{4.3}
\]

**Proof** \((4.1)\) obviously implies the first equality in \((4.3)\). Now replace \( \tilde{w} \) in Lemma 4.1 by \( \tilde{w}_i + \tilde{w}_j \) and \( \tilde{w}_i - \tilde{w}_j \), respectively, to get

\[
\frac{d|\tilde{w}_i(s) + \tilde{w}_j(s)|^2_s}{ds} = 2\langle \tilde{w}'_i(s) + \tilde{w}'_j(s) + G(\tilde{w}_i(s) + \tilde{w}_j(s)), \tilde{w}_i(s) + \tilde{w}_j(s) \rangle_s, \quad \tag{4.4}
\]

and

\[
\frac{d|\tilde{w}_i(s) - \tilde{w}_j(s)|^2_s}{ds} = 2\langle \tilde{w}'_i(s) - \tilde{w}'_j(s) + G(\tilde{w}_i(s) - \tilde{w}_j(s)), \tilde{w}_i(s) - \tilde{w}_j(s) \rangle_s. \tag{4.5}
\]

Using the fact that

\[
4\langle \tilde{w}_i(s), \tilde{w}_j(s) \rangle_s = |\tilde{w}_i(s) + \tilde{w}_j(s)|^2_s - |\tilde{w}_i(s) - \tilde{w}_j(s)|^2_s
\]

we have

\[
\frac{d4\langle \tilde{w}_i(s), \tilde{w}_j(s) \rangle_s}{ds} = 2\langle \tilde{w}'_i(s) + \tilde{w}'_j(s) + G\tilde{w}_i(s) + G\tilde{w}_j(s), \tilde{w}_i(s) + \tilde{w}_j(s) \rangle_s

\]

\[
-2\langle \tilde{w}'_i(s) - \tilde{w}'_j(s) + G\tilde{w}_i(s) - G\tilde{w}_j(s), \tilde{w}_i(s) - \tilde{w}_j(s) \rangle_s
\]

\[
= 4\langle \tilde{w}'_i(s) + G\tilde{w}_i(s), \tilde{w}_j(s) \rangle_s + 4\langle \tilde{w}'_j(s) + G\tilde{w}_j(s), \tilde{w}_i(s) \rangle_s.
\]
The proof of Lemma 4.2 is complete.

For each \( m \in \mathbb{N} \), we define an approximate process \( \tilde{u}_m \) as follows:

\[
\tilde{u}_m(s) = \sum_{j=1}^{m} g_{jm}(s) \tilde{w}_j(s), \quad s \in [0, T],
\]

where \( g_{jm} \), \( 1 \leq j \leq m \) are real-valued stochastic processes on \([0, T]\) determined by the equations:

\[
\langle d\tilde{u}_m(s), \tilde{w}_j(s) \rangle_s = \langle F\tilde{u}_m(s), \tilde{w}_j(s) \rangle_s \, ds - \langle G\tilde{u}_m(s), \tilde{w}_j(s) \rangle_s \, ds - \langle N(\tilde{u}_m(s)), \tilde{w}_j(s) \rangle_s \, ds
\]

\[
+ \langle \tilde{f}(s), \tilde{w}_j(s) \rangle_s \, ds + \langle \tilde{\sigma}(s), \tilde{w}_j(s) \rangle_s \, dW(s), \quad j = 1, 2, \ldots, m,
\]

with initial data

\[
\tilde{u}_m(0) = \sum_{j=1}^{m} g_{jm}(0) \tilde{w}_j(0),
\]

where

\[
g_{jm}(0) = \langle \tilde{u}_0, \tilde{w}_j(0) \rangle_0, \quad j = 1, 2, \ldots, m.
\]

Since \( F \) and \( G \) are linear operators, for \( j = 1, 2, \ldots, m \),

\[
\langle F\tilde{u}_m(s), \tilde{w}_j(s) \rangle_s = \sum_{k=1}^{m} g_{km}(s) \langle F\tilde{w}_k(s), \tilde{w}_j(s) \rangle_s,
\]

\[
\langle G\tilde{u}_m(s), \tilde{w}_j(s) \rangle_s = \sum_{k=1}^{m} g_{km}(s) \langle G\tilde{w}_k(s), \tilde{w}_j(s) \rangle_s.
\]

And for the bilinear operator \( \mathcal{N} \), we have, for \( j = 1, 2, \ldots, m \),

\[
\langle \mathcal{N}(\tilde{u}_m(s)), \tilde{w}_j(s) \rangle_s = \langle \mathcal{N}(\tilde{u}_m(s), \tilde{u}_m(s)), \tilde{w}_j(s) \rangle_s
\]

\[
= \left\langle \sum_{i=1}^{2} \left( \sum_{k=1}^{m} g_{km}(s) \tilde{w}_k(s) \right)^i \nabla_i \left( \sum_{l=1}^{m} g_{lm}(s) \tilde{w}_l(s) \right), \tilde{w}_j(s) \right\rangle_s
\]

\[
= \sum_{k=1}^{m} \sum_{l=1}^{m} g_{km}(s) g_{lm}(s) \left\langle \sum_{i=1}^{2} \left( \tilde{w}_k(s) \right)^i \nabla_i \left( \tilde{w}_l(s) \right), \tilde{w}_j(s) \right\rangle_s
\]

\[
= \sum_{k=1}^{m} \sum_{l=1}^{m} g_{km}(s) g_{lm}(s) \langle \mathcal{N}(\tilde{w}_k(s), \tilde{w}_l(s)), \tilde{w}_j(s) \rangle_s.
\]

Thus, equivalently \( g_{jm}, 1 \leq j \leq m \) solve the following system of stochastic differential equations on \([0, T]\):

\[
dg_{jm}(s) + \sum_{k=1}^{m} g_{km}(s) \langle \tilde{w}_k'(s), \tilde{w}_j(s) \rangle_s \, ds - \sum_{k=1}^{m} g_{km}(s) \langle F\tilde{w}_k(s), \tilde{w}_j(s) \rangle_s \, ds
\]

\[
+ \sum_{k=1}^{m} g_{km}(s) \langle G\tilde{w}_k(s), \tilde{w}_j(s) \rangle_s \, ds + \sum_{k=1}^{m} \sum_{l=1}^{m} g_{km}(s) g_{lm}(s) \langle \mathcal{N}(\tilde{w}_k(s), \tilde{w}_l(s)), \tilde{w}_j(s) \rangle_s \, ds
\]

\[
= \langle \tilde{f}(s), \tilde{w}_j(s) \rangle_s \, ds + \langle \tilde{\sigma}(s), \tilde{w}_j(s) \rangle_s \, dW(s), \quad j = 1, 2, \ldots, m,
\]
with the initial condition
\[ g_{jm}(0) = \langle \tilde{u}_0, \tilde{w}_j(0) \rangle_0, \quad j = 1, 2, \cdots, m. \]

**Lemma 4.3** There exists a unique solution \( g_{jm}, 1 \leq j \leq m \) to equation (4.8).

**Proof** For \( k, l = 1, 2, \cdots, m \), set
\[ a_{jk}(s) = \left\langle \tilde{w}'_k(s), \tilde{w}_j(s) \right\rangle_s - \left\langle F\tilde{w}_k(s), \tilde{w}_j(s) \right\rangle_s + \left\langle G\tilde{w}_k(s), \tilde{w}_j(s) \right\rangle_s, \]
\[ a_{jkl}(s) = \left\langle N(\tilde{w}_k(s), \tilde{w}_l(s)), \tilde{w}_j(s) \right\rangle_s. \]

The equation (4.8) can be expressed as
\[
dg_{jm}(s) + \sum_{k=1}^{m} a_{jk}(s)g_{km}(s)ds + \sum_{k=1}^{m} \sum_{l=1}^{m} a_{jkl}(s)g_{km}(s)g_{lm}(s)ds
= \left\langle \tilde{f}(s), \tilde{w}_j(s) \right\rangle_s ds + \left\langle \tilde{\sigma}(s), \tilde{w}_j(s) \right\rangle_s dW(s), \quad j = 1, 2, \cdots, m. \quad (4.9) \]

For \( y = (y^1, y^2, \cdots, y^m) \in \mathbb{R}^m \), since the coefficients \( a_{jk}(s) y^k, a_{jkl}(s) y^k y^l \) are locally Lipschitz, equation (4.9) admits a unique local solution. The global existence and uniqueness of the solution \( g_{jm}, 1 \leq j \leq m \) follow from the a priori estimates proved later in Lemma 4.4 below for \( \tilde{u}_m \).

The proof of Lemma 4.3 is complete.

To obtain the estimate for \( \tilde{u}_m \), we need the following chain rule.

**Proposition 4.1** We have the following chain rule for \( \tilde{u}_m \):
\[
d|\tilde{u}_m(s)|^2 = 2 \left\langle F\tilde{u}_m(s), \tilde{u}_m(s) \right\rangle_s ds - 2 \left\langle N(\tilde{u}_m(s)), \tilde{u}_m(s) \right\rangle_s ds
+ 2 \left\langle \tilde{f}(s), \tilde{u}_m(s) \right\rangle_s ds + 2 \left\langle \tilde{\sigma}(s), \tilde{u}_m(s) \right\rangle_s dW(s) + |\tilde{\sigma}_m(s)|^2 ds. \quad (4.10) \]

**Proof** Recall (4.8). By the Itô formula,
\[
\sum_{j=1}^{m} dg_{jm}^2(s) = 2 \sum_{j=1}^{m} g_{jm}(s)dg_{jm}(s) + \sum_{j=1}^{m} |\left\langle \tilde{\sigma}(s), \tilde{w}_j(s) \right\rangle_s|^2 ds
\]
\[
= -2 \sum_{j=1}^{m} \sum_{k=1}^{m} g_{km}(s)g_{jm}(s) \left\langle \tilde{w}'_k(s), \tilde{w}_j(s) \right\rangle_s ds
+ 2 \sum_{j=1}^{m} \sum_{k=1}^{m} g_{km}(s)g_{jm}(s) \left\langle F\tilde{w}_k(s), \tilde{w}_j(s) \right\rangle_s ds
- 2 \sum_{j=1}^{m} \sum_{k=1}^{m} g_{km}(s)g_{jm}(s) \left\langle G\tilde{w}_k(s), \tilde{w}_j(s) \right\rangle_s ds
- 2 \sum_{j=1}^{m} \sum_{k=1}^{m} \sum_{l=1}^{m} g_{km}(s)g_{jm}(s)g_{lm}(s) \left\langle N(\tilde{w}_k(s), \tilde{w}_l(s)), \tilde{w}_j(s) \right\rangle_s ds
+ 2 \sum_{j=1}^{m} \left\langle \tilde{f}(s), \tilde{w}_j(s) \right\rangle_s g_{jm}(s)ds + 2 \sum_{j=1}^{m} \left\langle \tilde{\sigma}(s), \tilde{w}_j(s) \right\rangle_s g_{jm}(s)dW(s). \]
In the last equality of (4.11), we have used (4.6), and the facts that it follows from (4.11) that

\[ 2 \sum_{j=1}^{m} g_{km}(s) g_{jm}(s) \langle \tilde{w}'_k(s), \tilde{w}_j(s) \rangle_s \, ds \]

\[ - 2 \sum_{j=1}^{m} g_{km}(s) g_{jm}(s) \langle G \tilde{w}_k(s), \tilde{w}_j(s) \rangle_s \, ds \]

\[ + 2 \left( \sum_{k=1}^{m} g_{km}(s) F \tilde{w}_k(s), \sum_{j=1}^{m} g_{jm}(s) \tilde{w}_j(s) \right) \, ds \]

\[ - 2 \sum_{k=1}^{m} \sum_{l=1}^{m} g_{km}(s) g_{lm} \mathcal{N}(\tilde{w}_k(s), \tilde{w}_l(s)), \sum_{j=1}^{m} g_{jm}(s) \tilde{w}_j(s) \rangle_s \, ds \]

\[ + 2 \left( \tilde{f}(s), \sum_{j=1}^{m} g_{jm}(s) \tilde{w}_j(s) \right) \, ds + 2 \left( \tilde{\sigma}(s), \sum_{j=1}^{m} g_{jm}(s) \tilde{w}_k(s) \right) \, dW(s) \]

\[ + \sum_{j=1}^{m} | \langle \tilde{\sigma}(s), \tilde{w}_j(s) \rangle_s |^2 \, ds \]

\[ = - 2 \sum_{j=1}^{m} g_{km}(s) g_{jm}(s) \langle \tilde{w}'_k(s), \tilde{w}_j(s) \rangle_s \, ds \]

\[ - 2 \sum_{j=1}^{m} g_{km}(s) g_{jm}(s) \langle G \tilde{w}_k(s), \tilde{w}_j(s) \rangle_s \, ds \]

\[ + 2 \langle F \tilde{u}_m(s), \tilde{u}_m(s) \rangle_s \, ds - 2 \langle \mathcal{N}(\tilde{u}_m(s)), \tilde{u}_m(s) \rangle_s \, ds + 2 \left( \tilde{f}(s), \tilde{u}_m(s) \right) \, ds \]

\[ + 2 \langle \tilde{\sigma}(s), \tilde{u}_m(s) \rangle \, dW(s) + | \tilde{\sigma}_m(s) |^2 \, ds. \]  

In the last equality of (4.11), we have used (4.6), and the facts that \( F \) is linear operator and \( \mathcal{N} \) is a bilinear operator.

Observing that, (4.6), (4.1) and Lemma 4.2 imply that

\[ | \tilde{u}_m(s) |^2_s = \sum_{j=1}^{m} g_{jm}^2(s), \]

and

\[ 2 \sum_{j=1}^{m} \sum_{k=1}^{m} g_{km}(s) g_{jm}(s) \langle \tilde{w}'_k(s), \tilde{w}_j(s) \rangle_s \, ds + 2 \sum_{j=1}^{m} \sum_{k=1}^{m} g_{km}(s) g_{jm}(s) \langle G \tilde{w}_k(s), \tilde{w}_j(s) \rangle_s \, ds = 0, \]

it follows from (4.11) that

\[ d | \tilde{u}_m(s) |^2_s = 2 \langle F \tilde{u}_m(s), \tilde{u}_m(s) \rangle_s \, ds - 2 \langle \mathcal{N}(\tilde{u}_m(s)), \tilde{u}_m(s) \rangle_s \, ds \]

\[ + 2 \left( \tilde{f}(s), \tilde{u}_m(s) \right) \, ds + 2 \langle \tilde{\sigma}(s), \tilde{u}_m(s) \rangle \, dW(s) + | \tilde{\sigma}_m(s) |^2 \, ds. \]

The proof of Proposition 4.1 is complete.

Recall

\[ u_m(x, t) = \tilde{u}_m(L(x, t)) M(x, t)^{-1}. \]

Next result provides a uniform bound for the family \( \{ u_m, m \geq 1 \} \).
Lemma 4.4 There exists a constant $M$, independent of $m$, such that

$$
\mathbb{E} \sup_{0 \leq t \leq T} \|u_m(t)\|_t^2 + \mathbb{E} \int_0^T \|\nabla u_m(t)\|_t^2 \, dt \leq M.
$$

(4.12)

Proof By a change of variable, we see that

$$
\langle F\tilde{u}_m(t), \tilde{u}_m(t) \rangle_t \, dt = (\Delta u_m(t), u_m(t))_t \, dt = -\|\nabla u_m(t)\|_t^2 \, dt,
$$

(3.10)

and (3.10) implies

$$
|\tilde{u}_m(t)|^2_t = \|u_m(t)\|_t^2_t \text{ and } |\tilde{\sigma}_m(t)|^2_t = \|\sigma_m(t)\|_t^2_t.
$$

Equation (4.10) becomes

$$
d\|u_m(t)\|_t^2 + 2\|\nabla u_m(t)\|_t^2 \, dt = 2(f(t), u_m(t))_t \, dt + 2(\sigma_m(t), u_m(t))_t \, dW(t) + \|\sigma_m(t)\|_t^2 \, dt.
$$

(4.13)

Hence, we have

$$
d\|u_m(t)\|_t^2 + 2\|\nabla u_m(t)\|_t^2 \, dt \leq \|f(t)\|_t^2 \, dt + \|\nabla u_m(t)\|_t^2 \, dt + 2(\sigma_m(t), u_m(t))_t \, dW(t) + \|\sigma_m(t)\|_t^2 \, dt.
$$

(4.14)

By the Burkholder-Davis-Gundy inequality, there exists some constant $C_4$ such that

$$
\mathbb{E} \left\{ \sup_{0 \leq t \leq T} \left( \int_0^T (\sigma(s), u_m(s))_s \, dW(s) \right) \right\} \leq C_4 \mathbb{E} \left\{ \left( \int_0^T (\sigma(s), u_m(s))^2_s \, ds \right)^{1/2} \right\}.
$$

It follows from (4.14) that

$$
\mathbb{E} \sup_{0 \leq t \leq T} \|u_m(t)\|_t^2 + 2\mathbb{E} \int_0^T \|\nabla u_m(t)\|_t^2 \, dt
$$

\begin{align*}
\leq & \|u_m(0)\|_0^2 + \mathbb{E} \int_0^T \|f(t)\|_t^2 \, dt + \mathbb{E} \int_0^T \|\nabla u_m(t)\|_t^2 \, dt + \mathbb{E} \int_0^T \|\sigma_m(t)\|_t^2 \, dt \\
& + C_4 \mathbb{E} \left\{ \left( \int_0^T (\sigma_m(s), u_m(s))^2_s \, ds \right)^{1/2} \right\},
\end{align*}

\begin{align*}
\leq & \|u_m(0)\|_0^2 + \mathbb{E} \int_0^T \|f(t)\|_t^2 \, dt + \mathbb{E} \int_0^T \|\nabla u_m(t)\|_t^2 \, dt + \mathbb{E} \int_0^T \|\sigma_m(t)\|_t^2 \, dt \\
& + C_4 \mathbb{E} \sup_{0 \leq t \leq T} \|u_m(t)\|_t \left( \int_0^T \|\sigma_m(s)\|_s^2 \, ds \right)^{1/2} \\
\leq & \|u_m(0)\|_0^2 + \mathbb{E} \int_0^T \|f(t)\|_t^2 \, dt + \mathbb{E} \int_0^T \|\nabla u_m(t)\|_t^2 \, dt + \mathbb{E} \int_0^T \|\sigma_m(t)\|_t^2 \, dt \\
& + \frac{1}{2} \mathbb{E} \sup_{0 \leq t \leq T} \|u_m(t)\|_t^2_t + C_4 \mathbb{E} \int_0^T \|\sigma_m(s)\|_s^2 \, ds.
\end{align*}

Re-arranging the terms, we get

$$
\mathbb{E} \sup_{0 \leq t \leq T} \|u_m(t)\|_t^2 + 2\mathbb{E} \int_0^T \|\nabla u_m(t)\|_t^2 \, dt \leq C_6.
$$

(4.15)
The proof of Lemma 4.4 is complete.

To obtain a martingale solution to the stochastic Navier-Stokes equation (1.1), we will prove that the family of laws \( \{ \mathcal{L}(u_m), m \geq 1 \} \) is tight in \( L^2([0,T]; H_t) \). Let

\[
\mathbf{w}_j(x,t) := \overline{w}_j(L(x,t))M(x,t)^{-1}, \quad j \geq 1.
\]

By (3.10) and (4.1),

The following lemma will be used.

**Lemma 4.5** (Lemma 2.5 in [21]) For each \( \epsilon > 0 \), there exists a positive integer \( N = N_\epsilon \) independent of \( t \in [0,T] \) such that for any \( \mathbf{v} \in V_i \), we have

\[
\|\mathbf{v}\|_t^2 \leq \sum_{j=1}^{N} (\mathbf{v}, \mathbf{w}_j(t))_t^2 + \epsilon \|\nabla \mathbf{v}\|_t^2.
\]

Next we are going to introduce a family of precompact subsets of \( L^2([0,T]; H_t) \). For \( i \geq 1 \), \( J_i \) denotes a family of equicontinuous functions on \([0,T]\). Recall that \( J_i \) is said to be equicontinuous on \([0,T]\) if for \( \forall \epsilon > 0 \), \( \exists \delta > 0 \) such that \( \forall s,t \in [0,T] \) with \( |s-t| < \delta, |h(t)-h(s)| < \epsilon \), \( \forall h_t \in J_i \). For \( N > 0 \), set \( K_{N,J_i} = \bigcap_{i=1}^{\infty} K_{N,J_i} \), where

\[
K_{N,J_i} = \left\{ g \in L^\infty([0,T]; H_t) \cap L^2([0,T]; V_i) : \sup_{0 \leq t \leq T} \|g(t)\|_t \leq N, \int_0^T \|\nabla g(t)\|_t^2 \, dt \leq N, \right\}
\]

\[
g_i = \left\{ g_i(t) = (g(t), \mathbf{w}_i(t))_t, t \in [0,T] \right\} \in J_i \right\}.
\]

**Proposition 4.2** \( K_{N,J} \) is precompact in \( L^2([0,T]; H_t) \).

**Proof** For each sequence \( \{\mathbf{z}_m\}_{m \geq 1} \in K_{N,J} \), set \( \rho_{m,j}(t) = (\mathbf{z}_m(t), \mathbf{w}_j(t))_t \) for all \( j \geq 1 \). By the definition of \( K_{N,J} \), \( \{\rho_{m,j}(t), t \in [0,T]\}_{m \geq 1} \) is equicontinuous. Noting that (4.1) and (4.10) imply that \( \mathbf{w}_j \) is smooth in \((x,t)\), there exists a constant \( M_j \) such that

\[
|\mathbf{w}_j(x,t)| \leq M_j, |\nabla \mathbf{w}_j(x,t)| \leq M_j, \quad \frac{\partial}{\partial t} |\mathbf{w}_j(x,t)| \leq M_j, \quad \forall x \in \mathcal{D}(t), t \in [0,T],
\]

which implies

\[
|\rho_{m,j}(t)| \leq \|\mathbf{z}_m(t)\|_t M_j |\mathcal{D}(t)|^{1/2} \leq M_{J,N},
\]

where \( |\mathcal{D}(t)| \) denotes the volume of \( \mathcal{D}(t) \), i.e. \( |\mathcal{D}(t)| = \int_{\mathcal{D}(t)} 1 \, dx \). The last inequality is obtained from the facts that \( \sup_{m \geq 1} \sup_{0 \leq t \leq T} \|\mathbf{z}_m(t)\|_t \leq N \) and the condition (A1).

Applying the Arzelà-Ascoli theorem and a diagonalization argument, we can choose a subsequence \( \{m_k, k \geq 1\} \) such that \( \{\rho_{m_k,j}(t), t \in [0,T]\}, k \geq 1 \) is a Cauchy sequence in \( C([0,T]; \mathbb{R}) \) for each fixed \( j \).

From Lemma 4.3 for any \( \epsilon > 0 \), we have

\[
\int_0^T \|\mathbf{z}_{m_k}(t)-\mathbf{z}_m(t)\|_t^2 \, dt \leq \sum_{j=1}^{N} \int_0^T |\rho_{m_k,j}(t) - \rho_{m_l,j}(t)|_t^2 \, dt + 2\epsilon \sup_{m \geq 1} \int_0^T \|\nabla \mathbf{z}_m(t)\|_t^2 \, dt.
\]

Let \( k, l \to \infty \) to get

\[
\limsup_{k,l \to \infty} \int_0^T \|\mathbf{z}_{m_k}(t)-\mathbf{z}_m(t)\|_t^2 \, dt \leq 2\epsilon N.
\]
Since $\varepsilon$ is arbitrary, we see that $\{z_{m_k}\}_{k \geq 1}$ is a Cauchy sequence in $L^2([0, T]; H_t)$, completing the proof.

Recall that $\{u_m\}_{m=1}^\infty$ satisfy the equation: for $1 \leq j \leq m$ and $t \in [0, T]$,

$$
(u_m(t), w_j(t))_t - (u_m(0), w_j(0))_0 + \int_0^t \langle \nabla u_m(s), \nabla w_j(s) \rangle ds - \int_0^t \langle f(s), w_j(s) \rangle ds
$$

$$
+ \int_0^t b_s(u_m(s), u_m(s), w_j(s)) ds - \int_0^t (u_m(s), w_j'(s)) ds = \int_0^t (\sigma(s), w_j(s)) dW(s).
$$

\( (4.21) \)

**Proposition 4.3** \( \mathcal{L}(u_m), m \geq 1 \) is tight in $L^2([0, T]; H_t)$.

**Proof** Let $K_{N,j}$ be defined as in Proposition 4.2. We just need to prove: for $\forall \varepsilon > 0$, there exist $J_s, i \in \mathbb{N}$ and $N$ such that $\mathbb{P}(u_m \in K_{N,j}) \geq 1 - \varepsilon$.

Set

$$
X_j(t) = \int_0^t (\sigma(s), w_j(s)) dW(s).
$$

Recall $p > 2$ in Theorem 3.1. By the Burkholder-Davis-Gundy inequality, the Hölder inequality and (4.19), for every $n \geq 1$ and $0 \leq s \leq t \leq T$,

$$
\mathbb{E}|X_j(t) - X_j(s)|^{2n} \leq \mathbb{E}\left| \int_s^t (\sigma(l), w_j(l)) dW(l) \right|^{2n}
$$

$$
\leq C_n \left( \int_s^t (\sigma(l), w_j(l))^2 dl \right)^n
$$

$$
\leq C_{J,n}|t - s|^{\frac{n(p-2)}{p}} \left( \int_0^T \|\sigma(l)\|_p^p dl \right)^{\frac{2n}{p}}
$$

$$
\leq C'|t - s|^{\frac{n(p-2)}{p}}.
$$

The condition that $\sigma \in L^p([0, T]; H_t)$ has been used.

Letting $n = \frac{4p}{p-2}$ and applying the Garsia lemma (see Corollary 1.2 in [25]), there exists a random variable $Y_j$ such that with probability one, for all $0 \leq s \leq t \leq T$,

$$
|X_j(t, \omega) - X_j(s, \omega)| \leq Y_j(\omega)|t - s|^{\frac{p-2}{2}},
$$

where

$$
\mathbb{E}(Y_j^{\frac{4p}{p-2}}) < \infty.
$$

It follows from (3.14), (4.19) and (4.21) that, for $0 \leq s \leq t \leq T$,

$$
\| (u_m(t), w_j(t))_t - (u_m(s), w_j(s))_s \|
$$

$$
\leq \int_s^t \| (\nabla u_m(l), \nabla w_j(l))_t \|_d l + \int_s^t \| b_l(u_m(l), u_m(l), w_j(l)) \|_d l + \int_s^t \| v_l \langle f(l), w_j(l) \rangle \|_d l
$$

$$
+ \int_s^t \| (u_m(l), w_j'(l))_t \|_d l + |X_j(t) - X_j(s)|
$$

$$
\leq C_J \left( \int_s^t \| \nabla u_m(l) \|_d l + \int_s^t \| u_m(l) \|_d l \| \nabla u_m(l) \|_d l + \int_s^t \| u_m(l) \|_d l + \int_s^t \| f(l) \|_d l \right)
$$

$$
+ |X_j(t) - X_j(s)|
$$
\[
\begin{align*}
&\leq C_J \left( (t-s)^{1/2} \left( 1 + \sup_{0 \leq t \leq T} \| u_m(t) \|_t \right) \left( \int_0^T \| \nabla u_m(t) \|^2 dt \right)^{1/2} + (t-s) \sup_{0 \leq t \leq T} \| u_m(t) \|_t \right) \\
&\quad + (t-s)^{1/2} \left( \int_0^T \| f(t) \|^2 dt \right)^{1/2}) + Y_j |t-s|^{p/2}, \quad (4.23)
\end{align*}
\]

where \( C_J \) depends on \( M_j \) in \( [119] \) and \( \sup_{t \in [0,T]} |D(t)| \), and is independent on \( m, s, t \). For \( N > 0 \) and \( q_j > 0 \), define
\[
J_N^J = \left\{ g \in C([0,T];\mathbb{R}); \ |g(t) - g(s)| \leq C_J \left( (t-s)^{1/2} \left( 1 + N \right) N^{1/2} + (t-s) N \right) \\
+ (t-s)^{1/2} \left\{ \int_0^T \| f(t) \|^2 dt \right\}^{1/2}) + q_j |t-s|^{p/2} \right\}.
\]

Obviously \( J_N^J \) is a class of equicontinuous functions. Now define the relatively compact subset \( K_{N,J} = \bigcap_{i=1}^{\infty} K_{N,J}^i \) as in Proposition 4.2 with \( J_i \) replaced by \( J_N^J \).

Recall \((4.12)\) that
\[
\sup_{m \geq 1} \left( \mathbb{E} \sup_{0 \leq t \leq T} \| u_m(t) \|^2_t + \mathbb{E} \int_0^T \| \nabla u_m(t) \|^2 dt \right) \leq M < \infty.
\]

If we denote
\[
A_N^m = \{ \omega : \sup_{t \in [0,T]} \| u_m(t, \omega) \|_t \leq N \}, \\
B_N^m = \{ \omega : \int_0^T \| \nabla u_m(t, \omega) \|^2 dt \leq N \},
\]
then
\[
\mathbb{P} \left( (A_N^m \cap B_N^m)^c \right) \leq \mathbb{P} \left( (A_N^m)^c \right) + \mathbb{P} \left( (B_N^m)^c \right) \leq M \left( \frac{1}{N^2} + \frac{1}{N^4} \right).
\]

Given \( \varepsilon > 0 \). We can choose \( N \) sufficiently large such that
\[
\mathbb{P} \left( (A_N^m \cap B_N^m)^c \right) \leq \frac{\varepsilon}{2}. \quad (4.24)
\]

For \( i \geq 1 \), define \( D_i = \{ \omega : Y_i(\omega) \leq q_i \} \). We can take \( q_i \) sufficiently large such that
\[
\mathbb{P}(D_i^c) \leq \frac{\mathbb{E}[Y_i^{2p/2}]}{q_i^{p/2}} \leq \frac{\varepsilon}{2^{i+1}}.
\]

Consequently we have
\[
\mathbb{P} \left( ((A_N^m \cap B_N^m) \cap (\bigcap_{i \geq 1} D_i))^c \right) \leq \frac{\varepsilon}{2} + \left( \frac{\varepsilon}{2^2} + \cdots + \frac{\varepsilon}{2^{i+1}} + \cdots \right) \leq \varepsilon.
\]

Using the fact that
\[
A_N^m \cap B_N^m \cap (\bigcap_{i \geq 1} D_i) \subset \{ u_m \in K_{N,J} \}
\]
we deduce that
\[
\mathbb{P}(u_m \in K_{N,J}) \geq 1 - \varepsilon,
\]
proving the tightness.
Proof of Theorem 3.1

Proof We have proved that \( \mathcal{L}(u_m), m \geq 1 \) is tight in \( L^2([0, T]; H_1) \). There exists a subsequence of \( \{u_m, m \geq 1\} \), still denoted by \( \{u_m, m \geq 1\} \), such that \( \mathcal{L}(u_m), m \geq 1 \) converges weakly in \( L^2([0, T]; H_1) \). By the generalisation of the Skorohod representation theorem (Theorem C.1 in Appendix C in [3]), there exists a probability space \((\Omega^*, \mathbb{F}^*, \mathbb{P}^*)\) and a sequence of \( L^2([0, T]; H_1)\)-valued random variables \( \{u_m^*, m \geq 1\} \) and \( u^* \), and \( C([0, T]; \mathbb{R})\)-valued random variable \( W^* \) such that \( \mathbb{P}^* \circ (u_m, W^*)^{-1} = \mathbb{P} \circ (u_m, W)^{-1} \) and that \( u_m^* \to u^* \) in \( L^2([0, T]; H_1) \) \( \mathbb{P}^* \)-a.s. By (1.12), we also have

\[
\mathbb{P}^* \sup_{0 \leq t \leq T} \|u_m^*(t)\|_t^2 + \mathbb{E}^* \int_0^T \|\nabla u_m^*(t)\|_t^2 dt \leq M.
\]

Hence,

\[
u^* \in L^2([0, T]; V) \cap L^\infty([0, T]; H_1), \quad \mathbb{P}^*-a.s.
\]

and \( u_m^* \to u^* \) weakly in \( L^2(\Omega \times [0, T]; V) \). From the equation (1.7) satisfied by \( \tilde{u}_m \) and Remark 3.2 we see that for \( v \in \mathcal{C}_\sigma(O_T) = \{v \in C_0^\infty(O_T) | \text{div } v = 0, v(T) = 0\} \),

\[
(du_m^*(s), v(s)) + (\nabla u_m^*(s), \nabla v(s)) + (B_s(u_m^*(s)), v(s)) ds + (\sigma^*(s), v(s))_s dW^*(s).
\]

Intergrating by parts, we have

\[
-\int_0^T (u_m^*(s), v'(s))_s ds + \int_0^T (\nabla u_m^*(s), \nabla v(s))_s ds + \int_0^T (B_s(u_m^*(s)), v(s))_s ds = (u_0^*, v(0))_0 + \int_0^T (f(s), v(s))_s ds + \int_0^T (\sigma^*(s), v(s))_s dW^*(s). \quad (4.25)
\]

Since \( v \in \mathcal{C}_\sigma(O_T) \) and \( m \to \infty u_m^* = u^* \) in \( L^2([0, T]; H_1), \mathbb{P}^*-a.s. \),

\[
\left| \int_0^T (B_s(u_m^*(s)), v(s))_s ds - \int_0^T (B_s(u^*(s)), v(s))_s ds \right| \\
\leq \int_0^T \left| (B_s(u_m^*(s)), v(s)) - (B_s(u^*(s)), v(s)) \right|_s ds \\
+ \int_0^T \left| (B_s(u_m^*(s)), v(s)) - (B_s(u^*(s)), v(s)) \right|_s ds \\
\leq C_v \left( \int_0^T \|u_m^*(s)\|_s^2 ds \right)^{1/2} \left( \int_0^T \|u_m^*(s) - u^*(s)\|_s^2 ds \right)^{1/2} \\
+ C_v \left( \int_0^T \|u^*(s)\|_s^2 ds \right)^{1/2} \left( \int_0^T \|u_m^*(s) - u^*(s)\|_s^2 ds \right)^{1/2} \to 0, \text{ as } m \to \infty, \mathbb{P}^*-a.s.
\]

Here \( C_v \) depends on \( \sup_{(x,s) \in O_T} \left( \|\partial v(x,s) / \partial x\| + \|\partial v(x,s) / \partial s\| \right) \).

Let \( m \to \infty \) in (4.25) to obtain

\[
-\int_0^T (u^*(s), v'(s))_s ds + \int_0^T (\nabla u^*(s), \nabla v(s))_s ds + \int_0^T (B_s(u^*(s)), v(s))_s ds = (u_0^*, v(0))_0 + \int_0^T (f(s), v(s))_s ds + \int_0^T (\sigma^*(s), v(s))_s dW^*(s).
\]
This shows that \( u^* \) is a martingale solution to the stochastic Navier-Stokes equation (1.1), completing the proof of the existence of a martingale solution.

Next we will consider the uniqueness of the solution. Suppose there are two solutions of (1.1), denoted by \( u_1 \) and \( u_2 \), i.e., \( u_1 \) and \( u_2 \) satisfy Definition 3.1 with \( u \) replaced by \( u_1 \) and \( u_2 \), respectively. In particular, we have

\[
du_1(t) - \Delta u_1(t)dt + (u_1(t) \cdot \nabla)u_1(t)dt + \nabla p(t)dt = f(t)dt + \sigma(t)dW(t)
\]

and

\[
du_2(t) - \Delta u_2(t)dt + (u_2(t) \cdot \nabla)u_2(t)dt + \nabla p(t)dt = f(t)dt + \sigma(t)dW(t)
\]

with \( u_1(0) = u_2(0) = u_0 \). Setting \( z(t) = u_1(t) - u_2(t) \), then \( z(t) \) solves the deterministic equation

\[
\partial_t z(t) - \Delta z(t) = B_t(u_2(t)) - B_t(u_1(t)),
\]

\[
z(0) = 0.
\]

Equivalently,

\[
\partial_t \bar{z}(t) + G\bar{z}(t) - F\bar{z}(t) = \mathcal{N}(\bar{u}_2(t)) - \mathcal{N}(\bar{u}_1(t)),
\]

\[
\bar{z}(0) = 0.
\]

By Lemma 4.1,

\[
\frac{d}{dt}|\bar{z}(t)|^2_t = 2 \langle \bar{z}'(t) + G\bar{z}(t), \bar{z}(t) \rangle_t,
\]

inserting (4.26) into the above equation, we get that

\[
\frac{1}{2} \frac{d}{dt}|\bar{z}(t)|^2_t + |\nabla_h \bar{z}(t)|^2_t = \langle \mathcal{N}(\bar{u}_2(t)) - \mathcal{N}(\bar{u}_1(t)), \bar{z}(t) \rangle_t.
\]

For the term on the right, by (4.14) and a change of variable, we have the following estimate

\[
|\langle \mathcal{N}(\bar{u}_1(t)) - \mathcal{N}(\bar{u}_2(t)), \bar{z}(t) \rangle_t| = |b_t(z(t), u_2(t), z(t))| \\
\quad \leq C_1|\bar{z}(t)|_t|\nabla_h \bar{z}(t)|_t|\nabla_h \bar{u}_2(t)|_t \\
\quad \leq |\nabla_h \bar{z}(t)|^2_t + C_7|\bar{z}(t)|^2_t|\nabla_h \bar{u}_2(t)|^2_t.
\]

Combined with (4.27), we have

\[
\frac{d}{dt}|\bar{z}(t)|^2_t \leq 2C_7|\bar{z}(t)|^2_t|\nabla_h \bar{u}_2(t)|^2_t.
\]

Integrate on \([0, t]\) to obtain

\[
|\bar{z}(t)|^2_t \leq \int_0^t 2C_7|\bar{z}(s)|^2_s|\nabla_h \bar{u}_2(s)|^2_s ds.
\]

Applying Gronwall’s lemma, we obtain \( \bar{z} = 0 \), proving the pathwise uniqueness. As the consequence of the Yamada-Watanable theorem, we have proved the existence of a unique strong solution in the probabilistic sense.

The proof of Theorem 3.1 is complete.

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