CHERN CHARACTERS FOR TWISTED MATRIX FACTORIZATIONS AND THE VANISHING OF THE HIGHER HERBRAND DIFFERENCE

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Abstract. We develop a theory of “ad hoc” Chern characters for twisted matrix factorizations associated to a scheme $X$, a line bundle $L$, and a regular global section $W \in \Gamma(X, L)$.

As an application, we establish the vanishing, in certain cases, of $h^R(M, N)$, the higher Herbrand difference, and $\eta^R(M, N)$, the higher codimensional analogue of Hochster’s theta pairing, where $R$ is a complete intersection of codimension $c$ with isolated singularities and $M$ and $N$ are finitely generated $R$-modules. Specifically, we prove such vanishing if $R = Q/(f_1, \ldots, f_c)$ has only isolated singularities, $Q$ is a smooth $k$-algebra, $k$ is a field of characteristic 0, the $f_i$’s form a regular sequence, and $c \geq 2$.

Such vanishing was previously established in the general characteristic, but graded, setting in [16].

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1. Introduction

This paper concerns invariants of finitely generated modules over an affine complete intersection ring $R$. Such a ring is one that can be written as $R = Q/(f_1, \ldots, f_c)$, where $Q$ is a regular ring and $f_1, \ldots, f_c \in Q$ form a regular sequence of elements. For technical reasons, we will assume $Q$ is a smooth algebra over a field $k$. More precisely, this paper concerns invariants of the singularity category, $D_{sg}(R)$, of $R$. This is the triangulated category defined as the Verdier quotient of $D^b(R)$, the bounded derived category of $R$-modules, by the sub-category of perfect complexes of $R$-modules. More intuitively, the singularity category keeps track of just the “infinite tail ends” of projective resolutions of finitely generated modules. Since $R$ is Gorenstein, $D_{sg}(R)$ is equivalent to $\text{MCM}(R)$, the stable category of maximal Cohen-Macaulay modules over $R$.

Thanks to a Theorem of Orlov [19, 2.1], the singularity category of $R$ is equivalent to the singularity category of the hypersurface $Y$ of $\mathbb{P}_Q^{c-1} = \text{Proj} Q[T_1, \ldots, T_c]$ cut out by the element $W = \sum_i f_i T_i \in \Gamma(\mathbb{P}_Q^{c-1}, \mathcal{O}(1))$. Since the scheme $\mathbb{P}_Q^{c-1}$ is regular $D_{sg}(Y)$ may in turn be given by the homotopy category of “twisted matrix factorizations” associated to the triple $(\mathbb{P}_Q^{c-1}, \mathcal{O}(1), W)$ (see [21], [15], [18], [23], [2]). In general, if $X$ is a scheme (typically smooth over a base field $k$), $\mathcal{L}$ is a line bundle on $X$, and $W \in \Gamma(X, \mathcal{L})$ is regular global section of $\mathcal{L}$, a twisted matrix factorization for $(X, \mathcal{L}, W)$ consists of a pair of locally free coherent sheaves $\mathcal{E}_0, \mathcal{E}_1$ on $X$ and morphisms $d_1 : \mathcal{E}_1 \to \mathcal{E}_0, d_0 : \mathcal{E}_0 \to \mathcal{E}_1 \otimes_{\mathcal{O}_X} \mathcal{L}$ such that each composite $d_0 \circ d_1$ and $(d_1 \otimes \text{id}_{\mathcal{L}}) \circ d_0$ is multiplication by $W$. In the special case where $X = \text{Spec}(Q)$ for a local ring $Q$, so that $\mathcal{L}$ is necessarily the trivial line bundle and $W$ is an element of $Q$, a twisted matrix factorization is just a classical matrix factorization as defined by Eisenbud [10].

Putting these facts together, we get an equivalence

$$D_{sg}(R) \cong hmf(\mathbb{P}_Q^{c-1}, \mathcal{O}(1), \sum_i f_i T_i)$$

between the singularity category of $R$ and the homotopy category of twisted matrix factorizations for $(\mathbb{P}_Q^{c-1}, \mathcal{O}(1), \sum_i f_i T_i)$. The invariants we attach to objects of $D_{sg}(R)$ for a complete intersection ring $R$ are actually defined, more generally, in terms of such twisted matrix factorizations. In detail, for a triple $(X, \mathcal{L}, W)$ as above, let us assume also that $X$ is smooth over a base scheme $S$ and $\mathcal{L}$ is pulled-back from $S$. In the main case of interest, namely $X = \mathbb{P}_Q^{c-1}$ for a smooth $k$-algebra $Q$, the base scheme $S$ is taken to be $\mathbb{P}_k^{c-1}$. In this situation, we define a certain ad hoc Hochschild homology group $\text{HH}_{ad}^{\text{hoch}}(X/S, \mathcal{L}, W)$ associated to $(X, \mathcal{L}, W)$. Moreover, to each twisted matrix factorization $\mathcal{E}$ for $(X, \mathcal{L}, W)$ we attach a class

$$\text{ch}(\mathcal{E}) \in \text{HH}_{ad}^{\text{hoch}}(X/S, \mathcal{L}, W),$$

called the ad hoc Chern character of $\mathcal{E}$. We use the term “ad hoc” since we offer no justification that these invariants are the “correct” objects with these names given by a more theoretical framework. But, if $X$ is affine and $\mathcal{L}$ is the trivial bundle, then our ad hoc Hochschild homology groups and Chern characters coincide with the usual notions found in [9], [8], [22], [7], [25], [17], [26], [20]. In particular, in this case, our definition of the Chern character of a matrix factorization is given by the “Kapustin-Li formula” [14].
When the relative dimension $n$ of $X \to S$ is even, we obtain from $ch(E)$ a “top Chern class”

$$c^{\text{top}}(E) \in H^0(X, \mathcal{J}_W(\frac{\omega}{c}))$$

where $\mathcal{J}_W$ is the “Jacobian sheaf” associated to $W$. By definition, $\mathcal{J}_W$ is the coherent sheaf on $X$ given as the cokernel of the map $\Omega_{X/S}^{-1} \otimes L^{-1} \overset{dW}{\longrightarrow} \Omega_{X/S}^n$ define by multiplication by the element $dW$ of $\Gamma(X, \Omega_{X/S}^1 \otimes_{\mathcal{O}_X} \mathcal{L})$. The top Chern class contains less information than the Chern character in general, but in the affine case with $\mathcal{L}$ trivial and under certain other assumptions, the two invariants coincide.

The main technical result of this paper concerns the vanishing of the top Chern class:

**Theorem 1.1.** Let $k$ be a field of characteristic 0 and $Q$ a smooth $k$-algebra of even dimension. If $f_1, \ldots, f_c \in Q$ is a regular sequence of elements with $c \geq 2$ such that the singular locus of $R = Q/(f_1, \ldots, f_c)$ is zero-dimensional, then

$$c^{\text{top}}(E) = 0 \in H^0(X, \mathcal{J}_W(\frac{\omega}{c}))$$

for all twisted matrix factorizations $E$ of $(\mathbb{P}_{Q}^{c-1}, \mathcal{O}(1), \sum_1 f_i T_i)$.

As an application of this Theorem, we prove the vanishing of the invariants $\eta_c$ and $h_c$ for $c \geq 2$ in certain cases. These invariants are defined for a pair of modules $M$ and $N$ over a complete intersection $R = Q/(f_1, \ldots, f_c)$ having the property that $\text{Ext}^i_R(M, N)$ and $\text{Tor}^i_R(M, N)$ are of finite length for $i \gg 0$. For example, if the non-regular locus of $R$ is zero-dimensional, then every pair of finitely generated modules has this property. In general, for such a pair of modules, the even and odd sequences of lengths of $\text{Ext}^i_R(M, N)$ and $\text{Tor}^i_R(M, N)$ are governed, eventually, by polynomials of degree at most $c - 1$: there exist polynomials

$$p_{ev}(M, N)(t) = a_{c-1}t^{c-1} + \text{lower order terms}$$

$$p_{odd}(M, N)(t) = b_{c-1}t^{c-1} + \text{lower order terms}$$

with integer coefficients such that

$$\text{length}_R \text{Ext}^{2i}_R(M, N) = p_{ev}(M, N)(i)$$

and

$$\text{length}_R \text{Ext}^{2i+1}_R(M, N) = p_{odd}(M, N)(i)$$

for $i \gg 0$, and similarly for the Tor modules. Following Celikbas and Dao [4, 3.3], we define the invariant $h_c(M, N)$ in terms of the leading coefficients of these polynomials:

$$h_c(M, N) := \frac{a_{c-1} - b_{c-1}}{c^{2c}}$$

The invariant $\eta_c(M, N)$ is defined analogously, using Tor modules instead of Ext modules; see [5].

If $c = 1$ (i.e., $R$ is a hypersurface) the invariants $h_1$ and $\eta_1$ coincide (up to a factor of $\frac{1}{2}$) with the Herbrand difference, defined originally by Buchweitz [1], and Hochster’s $\theta$ invariant, defined originally by Hochster [12]. In this case, $\text{Ext}^i_R(M, N)$ and $\text{Tor}^i_R(M, N)$ are finite length modules for $i \gg 0$ and the sequence of their lengths are eventually two periodic, and $h_1$ and $\eta_1$ are determined by the formulas

$$2h^R_1(M, N) = h^R(M, n) = \text{length} \text{Ext}^R_{2i}(M, N) - \text{length} \text{Ext}^R_{2i+1}(M, N), i \gg 0$$

and

$$2\eta^R_1(M, N) = \theta^R(M, n) = \text{length} \text{Tor}^R_{2i}(M, N) - \text{length} \text{Tor}^R_{2i+1}(M, N), i \gg 0.$$
Theorem 1.2. Let $k$ be a field of characteristic 0 and $Q$ a smooth $k$-algebra. If $f_1, \ldots, f_c \in Q$ is a regular sequence of elements with $c \geq 2$ such that the singular locus of $R = Q/(f_1, \ldots, f_c)$ is zero-dimensional, then $h^c_R(M, N) = 0$ and $\eta^c_R(M, N) = 0$ for all finitely generated $R$-modules $M$ and $N$.

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2. Chern classes for generalized matrix factorizations

In this section, we develop ad hoc notions of Hochschild homology and Chern classes for twisted matrix factorizations. The connection with the singularity category for complete intersection rings will be explained more carefully later.

Assumptions 2.1. Throughout this section, we assume

- $S$ is a Noetherian scheme.
- $p : X \to S$ is a smooth morphism of relative dimension $n$.
- $\mathcal{L}_S$ is a locally free coherent sheaf of rank one on $S$. Let $\mathcal{L}_X := p^*\mathcal{L}_S$, the pullback of $\mathcal{L}_S$ along $p$.
- $W \in \Gamma(X, \mathcal{L}_X)$ is a global section of $\mathcal{L}_X$.

Recall that the smoothness assumption means that $p : X \to S$ is flat and of finite type and that $\Omega^1_{X/S}$ is a locally free coherent sheaf on $X$ of rank $n$.

Assumptions 2.2. When discussing ad hoc Hochschild homology and Chern characters (see below), we will also be assuming:

- $p$ is affine and
- $n!$ is invertible in $\Gamma(X, \mathcal{O}_X)$.

It is useful to visualize Assumptions 2.1 by a diagram. Let $q : L_S \to S$ denote the geometric line bundle whose sheaf of sections is $\mathcal{L}_S$; that is, $L_S = \text{Spec}(\bigoplus_{i \geq 0} \mathcal{L}_S^{-i})$. Then we may interpret $W$ as a morphism $X \xrightarrow{W} L_S$ fitting into the commutative triangle

\[
\begin{array}{ccc}
X & \xrightarrow{W} & L_S \\
\downarrow p & & \downarrow q \\
S & \xrightarrow{W} & L_S.
\end{array}
\]

The data $(p : X \to S, \mathcal{L}, W)$ determines a closed subscheme $Y$ of $X$, defined by the vanishing of $W$. More formally, if $z : S \hookrightarrow L_S$ denotes the zero section of the line bundle $L_S$, then $Y$ is defined by the pull-back square

\[
\begin{array}{ccc}
Y & \xrightarrow{W} & X \\
\downarrow & & \downarrow \\
S & \xrightarrow{z} & L_S.
\end{array}
\]

We often suppress the subscripts and write $\mathcal{L}$ to mean $\mathcal{L}_S$ or $\mathcal{L}_X$. For brevity, we write $\mathcal{F}(n)$ to mean $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}_X^{-n}$ if $\mathcal{F}$ is a coherent sheaf or a complex of such on $X$. In our primary application, $\mathcal{L}$ will in fact by the standard very ample line bundle $\mathcal{O}(1)$ on projective space, and so this notation is reasonable in this case.

The case of primary interest for us is described in the following example:
Example 2.3. Suppose $k$ is a field, $Q$ is a smooth $k$-algebra of dimension $n$, and $f_1, \ldots, f_e$ is regular sequence of elements of $Q$. Let
\begin{align*}
X &= \mathbb{P}^{e-1}_Q = \text{Proj} Q[T_1, \ldots, T_e], \\
S &= \mathbb{P}^{e-1}_k = \text{Proj} k[T_1, \ldots, T_e]
\end{align*}
and define
\[ p : X \to S \]
to be the map induced by $k \hookrightarrow Q$. Finally, set $\mathcal{L}_S := \mathcal{O}_S(1)$ and
\[ W := \sum_i f_i T_i \in \Gamma(X, \mathcal{L}_X). \]

Then the triple $(p : X \to S, \mathcal{L}, W)$ satisfies all the hypotheses in Assumptions 2.1 and moreover $p$ is affine. The subscheme cut out by $W$ is
\[ Y = \text{Proj} Q[T_1, \ldots, T_e]/W \]
in this case.

2.1. The Jacobian complex. We construct a complex of locally free sheaves on $X$ that arises from the data $(p : X \to S, \mathcal{L}, W)$. Regarding $W$ as a map $W : X \to L_S$ as above, we obtain an induced map
\[ W^* \Omega^1_{L_S/S} \to \Omega^1_{X/S} \]
on cotangent bundles. There is a canonical isomorphism $\Omega^1_{L_S/S} \cong q^* \mathcal{L}^{-1}$ and hence an isomorphism $\mathcal{L}^{-1}_X \cong W^* \Omega^1_{L_S/S}$. Using this, we obtain a map $\mathcal{L}^{-1}_X \to \Omega^1_{X/S}$, and upon tensoring with $\mathcal{L}$ we arrive at the map:
\[ dW : \mathcal{O}_X \to \Omega^1_{X/S} \otimes \mathcal{O}_X \mathcal{L}_X =: \Omega^1_{X/S}(1). \]

Note that
\[ \Lambda^j_{X/S}(\Omega^1_{X/S} \otimes \mathcal{O}_X \mathcal{L}) \cong \bigoplus_{q} \Omega^q_{X/S}(q) \]
so that we may regard the right-hand side as a sheaf of graded-commutative graded-$\mathcal{O}_X$-algebras. Since $dW$ is a global section of this sheaf lying in degree one, we may form the Jacobian complex with differential given as repeated multiplication by $dW$:
\begin{equation}
\Omega_{dW} := \left( \mathcal{O}_X \xrightarrow{dW} \Omega^1_{X/S}(1) \xrightarrow{dW} \Omega^2_{X/S}(2) \xrightarrow{dW} \cdots \xrightarrow{dW} \Omega^n_{X/S}(n) \right).
\end{equation}

We index this complex so that $\Omega^j_{X/S}(j)$ lies in cohomological degree $j$.

Example 2.5. Let us consider the case where $X$ and $S$ are affine and the line bundle $\mathcal{L}$ is the trivial one. That is, suppose $k$ is a Noetherian ring, $A$ is a smooth $k$-algebra, and $f \in A$, and set $S = \text{Spec}(k)$, $X = \text{Spec}(A)$, $\mathcal{L} = \mathcal{O}_S$, and $W = f \in \Gamma(X, \mathcal{O}_X) = A$. (The map $p : X \to S$ is given by the structural map $k \to A$.)

Then the Jacobian complex is formed by repeated multiplication by the degree one element $df \in \Omega^1_{A/k}$:
\[ \Omega_{df} = \left( A \xrightarrow{df} \Omega^1_{A/k} \xrightarrow{df} \Omega^2_{A/k} \xrightarrow{df} \cdots \xrightarrow{df} \Omega^n_{A/k} \right). \]

If we further specialize to the case $A = k[x_1, \ldots, x_n]$, then we may identify $\Omega^1_{A/k}$ with $A^n$ by using the basis $dx_1, \ldots, dx_n$. Then $\Omega_{df}$ is the $A$-linear dual of the Koszul complex on the sequence $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$ of partial derivatives of $f$. 
If $k$ is a field and $f$ has only isolated singularities, then $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$ form an $A$-regular sequence. In this case, the Jacobian complex has cohomology only in degree $n$ and

$$H^n(\Omega_{df}) \cong \frac{k[x_1, \ldots, x_n]}{(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})} dx_1 \wedge \cdots \wedge dx_n.$$ 

**Example 2.6.** With the notation of Example 2.3 the Jacobian complex is the complex of coherent sheaves on $\mathbb{P}_Q^{-1}$ associated to the following complex of graded modules over the graded ring $Q[\mathcal{I}] := Q[T_1, \ldots, T_c]$:

$$0 \to Q[\mathcal{I}] \xrightarrow{dW} \Omega^1_{Q/k}[\mathcal{I}](1) \xrightarrow{dW} \cdots \xrightarrow{dW} \Omega^n_{Q/k}[\mathcal{I}](n) \to 0$$

Here, $dW = df_1T_1 + \cdots + df_cT_c$, a degree one element in the graded module $\Omega^1_{Q/k}[\mathcal{I}]$.

Further specializing to $Q = k[x_1, \ldots, x_n]$, we have $df_i = \sum_j \frac{\partial f_i}{\partial x_j} dx_j$ and

$$dW = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_c}{\partial x_1} & \cdots & \frac{\partial f_c}{\partial x_n} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_c \end{bmatrix}.$$ 

It is convenient to think of $\Omega_{dW}$ as representing a “family” of complexes of the type occurring in Example 2.5 indexed by the $Q$ points of $\mathbb{P}_Q^{-1}$. That is, given a tuple $\underline{a} = (a_1, \ldots, a_c)$ of elements of $Q$ that generated the unit ideal, we have an associated $Q$-point $i_{\underline{a}} : \text{Spec}(Q) \to \mathbb{P}_Q^{-1}$ of $\mathbb{P}_Q^{-1}$. Pulling back along $i_{\underline{a}}$ (and identifying $i_{\underline{a}}^*\mathcal{O}(1)$ with the trivial bundle in the canonical way) gives the Jacobian complex of Example 2.5 for $(\text{Spec}(Q) \to \text{Spec}(k), \mathcal{O}, \sum_i a_i f_i)$.

We are especially interested in the cokernel (up to a twist) of the last map in the Jacobian complex (2.4), and thus give it its own name:

**Definition 2.7.** The Jacobi sheaf associated to $(p : X \to S, \mathcal{L}, W)$, written $\mathcal{J}(X/S, \mathcal{L}, W)$ or just $\mathcal{J}_W$ for short, is the coherent sheaf on $X$ defined as

$$\mathcal{J}_W = \mathcal{J}(X/S, \mathcal{L}, W) := \text{coker} \left( \Omega_{X/S}^{n-1}(-1) \xrightarrow{dW \wedge -} \Omega^n_{X/S} \right).$$ 

In other words, $\mathcal{J}_W := \mathcal{H}^n(\Omega_{dW})(-n)$, where $\mathcal{H}^n$ denote the cohomology of a complex in the abelian category of coherent sheaves on $X$.

**Example 2.8.** If $S = \text{Spec}(k)$, $k$ is a field, $X = \mathbb{A}^n_k = \text{Spec}(k[x_1, \ldots, x_n])$, $\mathcal{L} = \mathcal{O}_X$ and $f$ a polynomial, then

$$\mathcal{J}_f = \frac{k[x_1, \ldots, x_n]}{(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})} dx_1 \wedge \cdots \wedge dx_n.$$ 

**Example 2.9.** With the notation of 2.3 $\mathcal{J}_W$ is the coherent sheaf associated to the graded $Q[\mathcal{I}]$-module

$$\text{coker} \left( \Omega_{Q/k}^{n-1}[\mathcal{I}](1) \xrightarrow{dW} \Omega^n_{Q/k}[\mathcal{I}] \right).$$

If we further specialize to the case $Q = k[x_1, \ldots, x_n] = k[\underline{x}]$, then using $dx_1, \ldots, dx_n$ as a basis of $\Omega^1_{Q/k}$ gives us that $\mathcal{J}_W$ is isomorphic to the coherent sheaf associated
to
\[ \text{coker} \left( k[x, T](-1)^{\oplus n} \xrightarrow{(\alpha_f^i)_{i,j}} k[x, T] \right) \].

2.2. Matrix factorizations. We recall the theory of “twisted matrix factorizations” from [21] and [2].

Suppose \( X \) is any Noetherian scheme, \( \mathcal{L} \) is a line bundle on \( X \) and \( W \in \Gamma(X, \mathcal{L}) \) any global section of it. (The base scheme \( S \) is not needed for this subsection, and we write \( \mathcal{L} \) for \( \mathcal{L}_X \).) A \textit{twisted matrix factorization} for \((X, \mathcal{L}, W)\) consists of the data \( \mathcal{E} = (\mathcal{E}_0, \mathcal{E}_1, d_0, d_1) \) where \( \mathcal{E}_0 \) and \( \mathcal{E}_1 \) are locally free coherent sheaves on \( X \) and \( d_0 : \mathcal{E}_0 \to \mathcal{E}_1(1) \) and \( d_1 : \mathcal{E}_1 \to \mathcal{E}_0 \) are morphisms such that \( d_0 \circ d_1 \) and \( d_1(1) \circ d_0 \) are each given by multiplication by \( W \). (Recall that \( \mathcal{E}_1(1) \) denotes \( \mathcal{E}_1 \times_{\mathcal{O}_X} \mathcal{L} \).) We visualize \( \mathcal{E} \) as a diagram of the form
\[
\mathcal{E} = \left( \mathcal{E}_0 \xrightarrow{d_0} \mathcal{E}_1 \xrightarrow{d_1} \mathcal{E}_0(1) \right)
\]
or as the “twisted periodic” sequence
\[
\cdots \xrightarrow{d_0(-1)} \mathcal{E}_1(-1) \xrightarrow{d_1(-1)} \mathcal{E}_0 \xrightarrow{d_0} \mathcal{E}_1 \xrightarrow{d_1} \mathcal{E}_0(1) \xrightarrow{d_0(1)} \mathcal{E}_1(1) \xrightarrow{d_1(1)} \cdots.
\]
Note that the composition of any two adjacent arrows in this sequence is multiplication by \( W \) (and hence this is not a complex unless \( W = 0 \)).

A \textit{strict morphism} of matrix factorizations for \((X, \mathcal{L}, W)\), say from \( \mathcal{E} \) to \( \mathcal{E}' = (\mathcal{E}'_0, \mathcal{E}'_1, d'_0, d'_1) \) consists of a pair of morphisms \( \mathcal{E}_0 \to \mathcal{E}'_0 \) and \( \mathcal{E}_1 \to \mathcal{E}'_1 \) such that the evident pair of squares both commute. Matrix factorizations and strict morphisms form an exact category, which we write as \( mf(X, \mathcal{L}, W) \), for which a sequence of morphisms is declared exact if it is so in each degree.

If \( V \in \Gamma(X, \mathcal{L}) \) is another global section, there is a tensor product pairing
\[
- \otimes_{mf} - : mf(X, \mathcal{L}, W) \times mf(X, \mathcal{L}, V) \to mf(X, \mathcal{L}, W + V).
\]
In the special case \( \mathcal{L} = \mathcal{O}_X \) and \( V = W = 0 \), this tensor product is the usual tensor product of \( \mathbb{Z}/2 \)-graded complexes of \( \mathcal{O}_X \)-modules, and the general case is the natural “twisted” generalization of this. We refer the reader to [2 §7] for a more precise definition.

There is also a sort of internal Hom construction. Given \( W, V \in \Gamma(X, \mathcal{L}) \) and objects \( \mathcal{E} \in mf(X, \mathcal{L}, W) \) and \( \mathcal{F} \in mf(X, \mathcal{L}, V) \), there is an object \( \mathcal{H}om_{mf}(\mathcal{E}, \mathcal{F}) \in mf(X, \mathcal{L}, V - W) \). Again, if \( \mathcal{L} = \mathcal{O}_X \) and \( W = V = 0 \), this is the usual internal Hom-complex of a pair of \( \mathbb{Z}/2 \)-graded complexes of \( \mathcal{O}_X \)-modules, and we refer the reader to [2 2.3] for the general definition. If \( V = W \), then \( \mathcal{H}om_{mf}(\mathcal{E}, \mathcal{F}) \) belongs to \( mf(X, \mathcal{L}, 0) \) and hence is actually a complex. In general, we refer to an object of \( mf(X, \mathcal{L}, 0) \) as a \textit{twisted two-periodic complex}, since it is a complex of the form
\[
\cdots \xrightarrow{d_0(-1)} \mathcal{E}_1 \xrightarrow{d_1} \mathcal{E}_0 \xrightarrow{d_0} \mathcal{E}_1(1) \xrightarrow{d_1(1)} \mathcal{E}_0(1) \xrightarrow{d_0(1)} \cdots.
\]
In other words, such an object is equivalent to the data of a (homologically indexed) complex \( \mathcal{E} \) of locally free coherent sheaves and a specified isomorphism \( \mathcal{E}(1) \cong \mathcal{E} \). (We use the convention that \( \mathcal{E}[2]_i = \mathcal{E}_{i-2} \).)

There is also a notion of duality. Given \( \mathcal{E} \in mf(X, \mathcal{L}, W) \), we define \( \mathcal{E}^* \in mf(X, \mathcal{L}, -W) \) as \( \mathcal{H}om_{mf}(\mathcal{E}, \mathcal{O}_X) \) where here \( \mathcal{O}_X \) denotes the object of \( mf(X, \mathcal{L}, 0) \) given by the twisted two-periodic complex
\[
\cdots \to 0 \to \mathcal{O}_X \to 0 \to \mathcal{L} \to 0 \to \mathcal{L} \oplus 2 \to \cdots.
\]
Let \( f \) be the global sections functor \( \Gamma(\cdot) \) defined by taking direct sums of the even and odd terms in a complex. In detail, for objects \( P \) and \( F \),
\[
\text{Hom}_{mf}(E, F) \cong E \otimes_{mf} F
\]
and
\[
\text{Hom}_{mf}(E \otimes_{mf} G, F) \cong \text{Hom}_{mf}(E, \text{Hom}_{mf}(G, F)).
\]

The construction of \( \text{Hom}_{mf}(E, F) \) is such that we have a natural identification
\[
\text{Hom}_{strict}(E, F) = Z_0 \Gamma(X, \text{Hom}_{mf}(E, F)).
\]
Here, \( \Gamma(X, \text{Hom}_{mf}(E, F)) \) is the complex of abelian groups obtained by regarding \( \text{Hom}_{mf}(E, F) \) as an unbounded complex of coherent sheaves on \( X \) and applying the global sections functor \( \Gamma(X, -) \) degree-wise, obtaining a complex of \( \Gamma(X, \mathcal{O}_X) \)-modules, and \( Z_0 \) denotes the cycles lying in degree 0 in this complex.

Let \( \mathcal{P}(X) \) denote the category of complexes of locally free coherent sheaves on \( X \) and let \( \mathcal{P}^b(X) \) denote the full subcategory of bounded complexes. There is a “folding” functor,
\[
\text{Fold} : \mathcal{P}^b(X) \to mf(X, \mathcal{L}, 0),
\]
defined by taking direct sums of the even and odd terms in a complex. In detail, given a complex \((P, d_P) \in \mathcal{P}^b(X)\), let
\[
\text{Fold}(P)_0 = \bigoplus_i P_{2i}(i) \quad \text{and} \quad \text{Fold}(P)_1 = \bigoplus_i P_{2i+1}(i).
\]
The required map \( \text{Fold}(P)_1 \to \text{Fold}(P)_0 \) maps the summand \( P_{2i+1}(i) \) to the summand \( P_{2i}(i) \) via \( d_P(i) \) and \( \text{Fold}(P)_0 \to \text{Fold}(P)_1(1) \) maps \( P_{2i}(i) \) to \( P_{2i-1}(i) \) also via \( d_P(i) \).

There is also an “unfolding” functor
\[
\text{Unfold} : mf(X, \mathcal{L}, 0) \to \mathcal{P}(X)
\]
which forgets the two periodicity. In detail,
\[
\text{Unfold}(\mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{E}_1(1))_{2i} = \mathcal{E}_0(-i) \quad \text{and} \quad \text{Unfold}(\mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{E}_1(1))_{2i+1} = \mathcal{E}_1(-i)
\]
with the evident maps.

These functors are related by the following adjointness properties: Given \( E \in mf(X, \mathcal{L}, 0) \) and \( \mathcal{P} \in \mathcal{P}^b(X) \), there are natural isomorphisms
\[
\text{Hom}_{mf}(X, \mathcal{L}, 0)(\text{Fold} \mathcal{P}, E) \cong \text{Hom}_{\mathcal{P}(X)}(\mathcal{P}, \text{Unfold} E)
\]
and
\[
\text{Hom}_{\mathcal{P}(X)}(\text{Unfold} E, \mathcal{P}) \cong \text{Hom}_{mf}(X, \mathcal{L}, 0)(E, \text{Fold} \mathcal{P}).
\]
(These functors do not technically form an adjoint pair, since Unfold takes values in \( \mathcal{P}(X) \), not \( \mathcal{P}^b(X) \).)

**Example 2.10.** If \( \mathcal{E} \) is a locally free coherent sheaf on \( X \), we write \( \mathcal{E}[0] \) for the object in \( \mathcal{P}(X) \) with \( \mathcal{E} \) in degree 0 and 0’s elsewhere. Then
\[
\text{Fold}(\mathcal{E}[0]) = (0 \to \mathcal{E} \to 0)
\]
indexed so that \( \mathcal{E} \) lies in degree 0. Letting \( \mathcal{E}[j] = \mathcal{E}[0][j] \), we have
\[
\text{Fold}(\mathcal{E}[2i]) = (0 \to \mathcal{E}(i) \to 0)
\]
and
\[
\text{Fold}(\mathcal{E}[2i+1]) = (\cdots \to \mathcal{E}(i-1) \to 0 \to \mathcal{E}(i) \to 0 \to \mathcal{E}(i+1) \to \cdots)
\]
with $E(i)$ in degree 0, and
\[ \text{Fold}(E[2i + 1]) = (E(i) \to 0 \to E(i + 1)) = (\cdots \to E(i - 1) \to 0 \to E(i) \to 0 \to E(i + 1) \to \cdots) \]
with $E(i)$ in homological degree 1.

If $P \in P^b(X)$ is a bounded complex and $E \in mf(X, L, W)$ is a twisted matrix factorization, we define their tensor product to be
\[ P \otimes E := \text{Fold}(P) \otimes_{mf} E \in mf(X, L, W). \]

**Example 2.11.** We will use this construction, in particular, for the complex $P = \Omega^1_{X/S}(1)[-1]$ consisting of $\Omega^1_{X/S}(1)$ concentrated in homological degree $-1$. Since $\Omega^1$ lies in odd degree, the sign conventions give us
\[ \Omega^1_{X/S}(1)[-1] \otimes E = \left( \Omega^1_{X/S} \otimes \mathcal{E}_0 \xrightarrow{-\text{id} \otimes d_0} \Omega^1_{X/S} \otimes \mathcal{E}_1(1) \xrightarrow{-\text{id} \otimes d_1} \Omega^1_{X/S} \otimes \mathcal{E}_0(1) \right). \]

We will also apply this construction to the complex $P := (\mathcal{O}_X \xrightarrow{dW} \Omega^1_{X/S}(1))$ indexed so that $\mathcal{O}_X$ lies in degree 0 to form the matrix factorization
\[ (\mathcal{O}_X \xrightarrow{dW} \Omega^1_{X/S}(1)) \otimes_{\mathcal{O}_X} E \]
which is given explicitly as
\[ \begin{pmatrix} d_1 & 0 \\ dW & -\text{id} \otimes d_0 \end{pmatrix} \begin{pmatrix} d_0 & 0 \\ dW & -\text{id} \otimes d_1 \end{pmatrix} \begin{pmatrix} \mathcal{E}_1(1) \\ \mathcal{E}_0(1) \end{pmatrix}. \]

Building on the identity
\[ \text{Hom}_{\text{strict}}(E, F) = Z_0 \Gamma(X, \mathcal{H}om_{\text{mf}}(E, F)) \]
we obtain a natural notion of homotopy: Two strict morphisms are homotopic if their difference lies in the image of the boundary map in the complex $\Gamma(X, \mathcal{H}om_{\text{mf}}(E, F))$. That is, the group of equivalence classes of strict morphisms up to homotopy from $E$ to $F$ is $H_0 \Gamma(X, \mathcal{H}om_{\text{mf}}(F, E))$. We define $[mf(X, L, W)]_{\text{naive}}$, the “naive homotopy category”, to be the category with the same objects as $mf(X, L, W)$ and with morphisms given by strict maps modulo homotopy; that is,
\[ \text{Hom}_{[mf]_{\text{naive}}}(E, F) := H_0 \Gamma(X, \mathcal{H}om_{\text{mf}}(F, E)). \]
Just as the homotopy category of chain complexes is a triangulated category, so too is the category $[mf(X, L, W)]_{\text{naive}}$; see [21, 1.3] or [3, 2.5] for details.

The reason for the pejorative “naive” in this definition is that this notion of homotopy equivalence does not globalize well: two morphisms can be locally homotopic without being globally so. Equivalently, an objects can be locally contractible with being globally so. To rectify this, we define the homotopy category of matrix factorizations for $(X, L, W)$, written $hmf(X, L, W)$, to be the Verdier quotient of $[mf(X, L, W)]_{\text{naive}}$ by the thick subcategory consisting of objects that are locally contractible. (The notion appears to be originally due to Orlov, but see also [21, 3.13] where this category is referred to as the “derived category of matrix factorizations”.)

We list the properties of $hmf(X, L, W)$ needed in the rest of this paper:
1. There is a functor $mf(X, L, W) \to hmf(X, L, W)$, and it sends objects that are locally contractible to the trivial object.

2. If $\alpha: E \to F$ is strict morphism and it is locally null-homotopic, then $\alpha$ is sent to the 0 map in $hmf(X, L, W)$.

3. When $W = 0$, $hmf(X, L, 0)$ is the Verdier quotient of $mf(X, L, 0)$ obtained by inverting quasi-isomorphisms of twisted two-periodic complexes.

4. At least in certain cases, the hom sets of $hmf(X, L, W)$ admit a more explicit description. In general, for objects $E, F \in mf(X, L, W)$ there is a natural map

$$\text{Hom}_{hmf}(E, F) \to H^0(X, \text{Unfold Hom}_{mf}(E, F))$$

where $H^0(X, -)$ denotes sheaf hyper-cohomology [3, 3.5]. This map is an isomorphism when

(a) $W = 0$ or

(b) $X$ is projective over an affine base scheme and $L = O_X(1)$ is the standard line bundle; see [3, 4.2].

2.3. Connections for matrix factorizations and the Atiyah class. See Appendix [A] for recollections on the notion of a connection for a locally free coherent sheaf on a scheme. Much of the material in this subsection represents a generalization to the non-affine case of constructions found in [20], which were in turn inspired by constructions in [9].

Definition 2.12. With $(p: X \to S, L, W)$ as in Assumptions 2.1, given $E \in mf(X, L, W)$, a connection on $E$ relative to $p$, written as $\nabla_E$ or just $\nabla$, is a pair of connections relative to $p$

$$\nabla_1: E_1 \to \Omega^1_{X/S} \otimes_{O_X} E_1$$

and

$$\nabla_0: E_0 \to \Omega^1_{X/S} \otimes_{O_X} E_0.$$

There is no condition relating $\nabla_1$ and $d_1, d_0$.

Since each $\nabla_i$ is $p^*O_S$-linear and $L$ is pulled back from $S$, we have induced connections

$$\nabla_1(j): E_1(j) \to \Omega^1_{X/S} \otimes_{O_X} E_1(j)$$

and

$$\nabla_0(j): E_0(j) \to \Omega^1_{X/S} \otimes_{O_X} E_0(j)$$

for all $j$.

Example 2.13. Suppose $S = \text{Spec} k$ and $X = \text{Spec} Q$ are affine where $Q$ is a smooth $k$-algebra of dimension $n$, $L = O_X$ so that $W \in Q$, and the components of $E$ are free $Q$-modules. Upon choosing bases, we may represent $E$ as $\left( \begin{array}{c} Q \xrightarrow{A} Q \end{array} \right)$ where $A, B$ are $r \times r$ matrices with entries in $Q$ such that $AB = BA = WI_r$. The choice of basis leads to an associated “trivial” connection

$$\nabla = d: Q^r \to \Omega^1_{Q/k} \otimes Q^r = \left( \Omega^1_{Q/k} \right)^\oplus r$$

given by applying exterior differentiation to the components of a vector.

As noted in Appendix [A] if $p$ is affine, then every vector bundle on $X$ admits a connection relative to $p$ and so every matrix factorization admits a connection in this case.
Definition 2.14. Given a connection $\nabla$ relative to $p$ for a twisted matrix factorization $E \in mf(X, L, W)$, the associated Atiyah class is defined to be the map

$$\text{At}_{E, \nabla} : E \to \Omega^1_{X/S}(1)[-1] \otimes E$$

given as the “commutator” $[\nabla, \delta_E]$. In detail, it is given by the pair of maps

$$\nabla_1 \circ \delta - (\text{id} \otimes \delta) \circ \nabla_0 : E \to \Omega^1 \otimes E$$

and

$$\nabla_0 \circ \delta - (\text{id} \otimes \delta) \circ \nabla_1 : E \to \Omega^1 \otimes E.$$

$\text{At}_{E, \nabla}$ is not a morphism of matrix factorizations in general, but by Lemma A.6 we have that $\text{At}_{E, \nabla}$ is $O_X$-linear.

Example 2.15. Keep the notations and assumptions of Example 2.13, so that $E = (Q \leftarrow A - \rightarrow B Q \leftarrow R)$. Then

$$\Omega^1[-1] \otimes E = \left( \Omega^1_{Q/k} \oplus_{-A} \Omega^1_{Q/k} \right) \oplus B.$$ 

and the map $\text{At} : E \to \Omega^1[-1] \otimes E$ obtained by choosing the trivial connections is represented by the following diagram:

Here $dA$ and $dB$ denote the $r \times r$ matrices with entries in $\Omega^1_{Q/k}$ obtained by applying $d$ entry-wise to $A$ and $B$.

This is not a map of matrix factorizations since the squares do not commute. Indeed, the differences of the compositions around these squares are $(dB)A + A(dA)$ and $(dA)B + A(dB)$. Since $AB = BA = W$, these expression both equal $dW$, a fact which is relevant for the next construction.

Definition 2.16. Given a connection $\nabla$ for a matrix factorization $E$, define

$$\Psi_{E, \nabla} : E \to \left( O_X \xrightarrow{dW} \Omega^1_{X/S}(1) \right) \otimes_{O_X} E$$

as $\text{id}_E + \text{At}_{E, \nabla}$. (Here, $O_X \xrightarrow{dW} \Omega^1_{X/S}(1)$ is the complex with $O_X$ in degree 0.) In other words, $\Psi_{E, \nabla}$ is the morphism whose composition with the canonical projection

$$\left( O_X \xrightarrow{dW} \Omega^1_{X/S}(1) \right) \otimes_{O_X} E \longrightarrow E$$

is the identity and whose composition with the canonical projection

$$\left( O_X \xrightarrow{dW} \Omega^1_{X/S}(1) \right) \otimes_{O_X} E \longrightarrow \Omega^1[-1] \otimes_{O_X} E$$

is $\text{At}_{E, \nabla}$.

Lemma 2.17. The map $\Psi_{E, \nabla}$ is a (strict) morphism of matrix factorizations, and it is independent in the naïve homotopy category $mf(X, L, W)_{\text{naive}}$ (and hence the homotopy category too) of the choice of connection $\nabla$. 
Proof. The proofs found in [26], which deal with the case where $X$ is affine and $Ł$ is trivial, apply nearly verbatim. The homotopy relating $Ψ_{Ł,∇}$ and $Ψ_{Ł,∇'}$ for two different connections $∇$ and $∇'$ on $E$ is given by the map

$$∇ - ∇' : E → Ω^1_{X/S} ⊗ E,$$

which is $O_X$-linear by Lemma A.6. □

Example 2.18. Continuing with Examples 2.13 and 2.15, the map $Ψ$ in this case is represented by the diagram

$$Q^r \xrightarrow{A} Q^r \xrightarrow{B} Q^r$$

$$Ω_{Q/k}^1 \xrightarrow{A} Ω_{Q/k}^1 \xrightarrow{B} Ω_{Q/k}^1.$$ 

This diagram commutes, confirming that $Ψ$ is indeed a morphism of matrix factorizations.

2.4. Ad hoc Hochschild homology. For an integer $j ≥ 0$, we define the twisted two-periodic complex

$$Ω_{dW}^{(j)} := \text{Fold} \left( O_X \xrightarrow{j \cdot dW} Ω^1_{X/S} \xrightarrow{(j-1) \cdot dW} Ω^2_{X/S} \xrightarrow{(j-2) \cdot dW} \cdots \xrightarrow{2 \cdot dW} Ω^j_{X/S} \xrightarrow{(j-1) \cdot dW} Ω^j_{X/S} \right).$$

Explicitly, $Ω_{dW}^{(j)}$ is the twisted periodic complex

$$\cdots → \begin{bmatrix} Ω^1_{X/S} \\ Ω^2_{X/S}(1) \\ Ω^3_{X/S}(2) \\ \vdots \end{bmatrix} \rightarrow \begin{bmatrix} O_X \\ Ω^1_{X/S}(1) \\ Ω^2_{X/S}(2) \\ \vdots \end{bmatrix} \rightarrow \begin{bmatrix} Ω^1_{X/S}(1) \\ Ω^2_{X/S}(2) \\ Ω^3_{X/S}(3) \\ \vdots \end{bmatrix} \rightarrow \cdots$$

with $Ω^i_{X/S}(i)$ in degree 0 and $⊕_{i=0}^{\lfloor \frac{j}{2} \rfloor} Ω^i_{X/S}(i)$ in degree 1.

The coefficients appearing in the maps of $Ω_{dW}^{(j)}$ are necessarily to make the following statement hold true. (We omit its straight-forward proof.)

Lemma 2.19. The pairings

$$Ω^i_{X/S}(i) \otimes O_X Ω^j_{X/S}(l) → Ω^{i+l}_{X/S}(i+l)$$

defined by exterior product induce a morphism

$$Ω_{dW}^{(j)} \otimes_{mf} Ω_{dW}^{(l)} → Ω_{dW}^{(j+l)}.$$ 

in $mf(X, Ł, 0)$. 
If $j!$ is invertible in $\Gamma(X, \mathcal{O}_X)$ (for example, if $X$ is a scheme over a field $k$ with $\text{char}(k) = 0$ or $\text{char}(k) > j$), then there is an isomorphism

$$\Omega_{dW} \cong \text{Fold} \left( \Omega_X \xrightarrow{dW} \Omega_{X/S}(1) \xrightarrow{dW} \cdots \xrightarrow{dW} \Omega_{X/S}(j) \right)$$

given by collection of isomorphisms $\Omega_{X/S}^i(i) \xrightarrow{\alpha^{-1}} \Omega_{X/S}^j(i)$, $0 \leq i \leq j$. In particular, if $n!$ is invertible in $\Gamma(X, \mathcal{O}_X)$, then there is an isomorphism

$$\Omega_{nW} \cong \text{Fold} \Omega_{nW}.$$

(Recall that $n$ is the relative dimension of $X$ of $S$, and hence is the rank of $\Omega_{X/S}^1$. It follows that $\Omega_{X/S}^m = 0$ for $m > n$.)

**Definition 2.20.** Let $p : X \to S, \mathcal{L}, W$ be as in Assumptions 2.1 and assume also that $n!$ is invertible in $\Gamma(X, \mathcal{O}_X)$. We define the (degree 0) **ad hoc Hochschild homology** of $mf(X, \mathcal{L}, W)$ relative to $S$ to be

$$\text{HH}_{ad hoc}^0(X/S, \mathcal{L}, W) := H_0(X, \text{Unfold} \text{ Fold}\Omega_{nW}).$$

As we will see in the next section, $\text{HH}_{ad hoc}^0(X/S, \mathcal{L}, W)$ is the target of what we term the “ad hoc Chern character” of a twisted matrix factorization belonging to $mf(X, \mathcal{L}, W)$.

We use the adjective “ad hoc” for two reasons. The first is that we offer here no justification that this definition deserves to be called “Hochschild homology”. See, however, [22] for the affine case and [20] for the non-affine case with $\mathcal{L} = \mathcal{O}_X$.

The second reason is that whereas the support of every object of $mf(X, \mathcal{L}, W)$ is contained in the singular locus of the subscheme $Y$ cut out by $W$ (as will be justified in the next subsection), the twisted two-periodic complex $\text{Fold}\Omega_{nW}$ is not so supported in general. A more natural definition of Hochshild homology would thus be given by $\mathbb{H}_{Z}^0(\text{Unfold} \text{ Fold}\Omega_{nW})$, where $Z$ is the singular locus of $Y$ and $\mathbb{H}_{Z}$ refers to hypercohomology with supports (i.e., local cohomology). Since this more sensible definition of Hochshild homology is not necessary for our purposes and only adds complications to what we do, we will stick with using $\text{HH}_{ad hoc}^0$.

Recall that the Jacobi sheaf is defined as

$$\mathcal{J}_W = \mathcal{J}(X/S, \mathcal{L}, W) = \text{coker} \left( \Omega_{X/S}^{n-1}(1) \xrightarrow{dW} \Omega_{X/S}^n \right)$$

so that there is a canonical map

$$\Omega_{dW} \to H^n(\mathcal{J}_W)[-n] = \mathcal{J}_W(n)[-n].$$

From it we obtain the map

$$\text{Fold}(\Omega_{dW}) \to \text{Fold}(\mathcal{J}_W(n)[-n]) \cong \text{Fold}(\mathcal{J}_W(\mathfrak{m})).$$

Applying Unfold and using the canonical map $\text{Unfold} \text{ Fold}(\mathcal{J}_W(\mathfrak{m})) \to \mathcal{J}_W(\mathfrak{m})$ results in a map $\text{Unfold} \text{ Fold}\Omega_{dW} \to \mathcal{J}_W(\mathfrak{m})$. Finally, applying $\mathbb{H}_{Z}^0(X, -)$ yields the map

$$\text{HH}_{ad hoc}^0(X/S, \mathcal{L}, W) \to H^0(X, \mathcal{J}_W(\mathfrak{m})).$$

This map will play an important role in the rest of this paper. Let us observe that in certain situations, it is an isomorphism:
Assume \(S = \text{Spec} k\) for a field \(k\), \(X = \text{Spec}(Q)\) for a smooth \(k\)-algebra \(Q\), and hence \(\mathcal{L}_X = \mathcal{O}_X\) and \(W \in Q\). If the morphism of smooth varieties \(W : X \to \mathbb{A}^1\) has only isolated critical points and \(n\) is even, then the canonical map

\[
\text{HH}^0_{ad}^{\text{hoc}}(X/S, W) \to H^0(X, \mathcal{J}_W(z)) = \frac{\Omega^n_{Q/k}}{dW \wedge \Omega^{n-1}_{Q/k}}
\]

is an isomorphism.

**Proof.** These conditions ensure that the complex of \(Q\)-modules \(\Omega \cdot \wedge dW\) is exact except on the far right where it has homology \(\mathcal{J}(n)\). The result follows since \(X\) is affine. \(\square\)

**Example 2.23.** Assume \(S = \text{Spec}(k)\) for a field \(k\), \(X = \mathbb{A}^n_k = \text{Spec} k[x_1, \ldots, x_n]\), and the morphism \(f : \mathbb{A}^n_k \to \mathbb{A}^1_2\) associated to a given polynomial \(f \in k[x_1, \ldots, x_n]\) has only isolated critical points. Then

\[
\text{HH}^0_{ad}^{\text{hoc}}(X/S, f) \cong H^0(X, \mathcal{J}_f(z)) = \frac{\Omega^n_{Q/k}}{df \wedge \Omega^{n-1}_{Q/k}} = \frac{k[x_1, \ldots, x_n]}{(\partial f/\partial x_1, \ldots, \partial f/\partial x_n)} \wedge dx_1 \wedge \cdots \wedge dx_n.
\]

2.5. **Supports.** We fix some notation. For a Noetherian scheme \(Y\), the *non-regular locus of \(Y\)* is

\[
\text{Nonreg}(Y) := \{y \in Y \mid \text{the local ring } \mathcal{O}_{Y,y} \text{ is not a regular local ring}\}.
\]

Under mild additional hypotheses (e.g., \(Y\) is excellent), \(\text{Nonreg}(Y)\) is a closed subset of \(Y\).

Assume \(g : Y \to S\) is a morphism of finite type. Recall that \(g\) is smooth near \(y \in Y\) if and only if it is flat of relative dimension \(n\) near \(y\) and the stalk of \(\Omega^1_{Y/S}\) at \(y\) is a free \(\mathcal{O}_{Y,y}\)-module of rank \(n\). We define *singular locus of \(g : Y \to S\)* to be the subset

\[
\text{Sing}(g) = \text{Sing}(Y/S) = \{y \in Y \mid \text{\(g\) is not smooth near \(y\)}\}.
\]

At least for a flat morphism \(g : Y \to S\) of finite type, the singular locus of \(g\) is a closed subset of \(Y\) by the Jacobi criterion. When \(S = \text{Spec}(k)\) for a field \(k\) and there is no danger of confusion, we write \(\text{Sing}(Y)\) instead of \(\text{Sing}(Y/\text{Spec}(k))\). If \(Y\) is finite type over a field \(k\), then \(\text{Nonreg}(Y) \subseteq \text{Sing}(Y) = \text{Sing}(Y/\text{Spec}(k))\), and equality holds if \(k\) is perfect.

**Proposition 2.24.** Assume \(X\) is a Noetherian scheme, \(\mathcal{L}\) is a line bundle on \(X\), and \(W\) is a regular global section of \(\mathcal{L}\). Let \(Y \subseteq X\) be the closed subscheme cut out by \(W\). Then for every \(E, F \in mf(X, \mathcal{L}, W)\), the twisted two-periodic complex \(\text{Hom}_{mf}(E, F)\) is supported \(\text{Nonreg}(Y)\).

**Proof.** For \(x \in X\), we have

\[
\text{Hom}_{mf}(E, F) \cong \text{Hom}_{mf}(E_x, F_x)
\]

where the hom complex on the right is for the category \(mf(\text{Spec}(\mathcal{O}_{X,x}), \mathcal{L}_x, W_x)\). By choosing a trivialization \(\mathcal{O}_X \cong \mathcal{L}_x\), we may identify this category with \(mf(Q, f)\) for a regular local ring \(Q\) and non-zero-divisor \(f\). The assertion thus becomes that if \(f\) is non-zero-divisor in regular local ring \(Q\) and \(Q/f\) is also regular, then the complex \(\text{Hom}_{mf}(E, F)\) is acyclic for all objects \(E, F \in mf(Q, f)\). This holds since the cohomology modules of the complex \(\text{Hom}_{mf}(E, F)\) are \(\text{Ext}_{Q/f}(\text{coker}(E), \text{coker}(F))\); see Theorem 3.2 below. \(\square\)
Example 2.25. Suppose $k$ is a field, $Q$ is a smooth $k$-algebra, $m$ is a maximal ideal of $Q$, and $f_1, \ldots, f_c \in m$ form a regular sequence such that the non-regular locus of the ring $R := Q/(f_1, \ldots, f_c)$ is $\{m\}$. Let $X = \mathbb{P}^{c-1}_Q = \text{Proj} Q[T_1, \ldots, T_c]$, $\mathcal{L} = \mathcal{O}_X(1)$, and $W = \sum_i f_i T_i$. Then

$$\text{Nonreg}(Y) = \mathbb{P}^{c-1}_{Q/m} \subseteq \mathbb{P}^{c-1}_Q.$$ 

2.6. Chern classes of matrix factorizations. If $E \in mf(X, \mathcal{L}, W)$ is a matrix factorization that admits a connection (for example, if $p$ is affine), we define strict morphisms

$$\Psi^{(j)}_{E, \nabla} : E \to \Omega^{(j)}_{dW} \otimes E, \text{ for } j \geq 0,$$

recursively, by letting

$$\Psi^{(0)} = \text{id}_E, \quad \Psi^{(1)} = \Psi_{E, \nabla},$$

and, for $j \geq 2$, defining $\Psi^{(j)}_{E, \nabla}$ as the composition of

$$E \xrightarrow{\text{id}_E \otimes \Psi^{(j-1)}_{E, \nabla}} \Omega^{(1)}_{dW} \otimes \Omega^{(j-1)}_{dW} \otimes E \xrightarrow{\wedge \otimes \text{id}_E} \Omega^{(j)}_{dW} \otimes E.$$

By Lemma 2.17, $\Psi^{(j)}_{E, \nabla}$ is independent up to homotopy of the choice of $\nabla$.

Example 2.26. Continuing with Examples 2.13, 2.15 and 2.18, the map $\Psi^{(\lambda j)}$ arising from the trivial connection is in degree zero given by

$$j \text{ factors}$$

$$(1 + dA) \cdots (1 + dA)(1 + dB)$$

for $j$ even and by

$$j \text{ factors}$$

$$(1 + dB) \cdots (1 + dA)(1 + dB)$$

for $j$ odd, where the product is taken in the ring $\text{Mat}_{r \times r}(\mathcal{O}_{Q/k})$.

By the duality enjoyed by the category $mf(X, \mathcal{L}, Q)$, we obtain a morphism of objects of $mf(X, \mathcal{L}, 0)$ of the form

$$\hat{\Psi}^{(j)}_E : \mathcal{E}nd_{mf}(E) = E^* \otimes_{mf} E \to \Omega^{(j)}_{dW}.$$ 

Moreover, as a morphism in the category $[mf(X, \mathcal{L}, 0)]_{\text{naive}}$, it is independent of the choice of connection. Take $j = n$ and assume $n!$ is invertible, so that we have an isomorphism

$$\Omega^{(n)}_{dW} \cong \text{Fold} \Omega^{(n)}_{dW}.$$ 

We obtain the morphism

$$\mathcal{E}nd_{mf}(E) \to \text{Fold} \Omega^{(j)}_{dW}$$

in $mf(X, \mathcal{L}, 0)$ and hence a map of unbounded complexes

$$\text{ch} : \text{Unfold} \mathcal{E}nd_{mf}(E) \to \text{Unfold} \text{Fold} \Omega^{(j)}_{dW}. \quad (2.27)$$

The identity morphism on $E$ determines a morphism $\mathcal{O}_X \xrightarrow{id_E} \text{Unfold} \mathcal{E}nd_{mf}(E)$, that is, an element of $H^0(X, \text{Unfold} \mathcal{E}nd_{mf}(E))$.

Definition 2.28. Under Assumptions 2.1 and 2.2, define the ad hoc Chern character of $E \in mf(X, \mathcal{L}, W)$ relative to $p : X \to S$ to be

$$\text{ch}(E) = \text{ch}_E(id_E) \in \text{HH}^{\text{ad hoc}}_0(X/S, \mathcal{L}, W).$$
Recall that when \( n \) is even, we have the map \( \text{HH}^{\text{ad hoc}}_0(X/S, \mathcal{L}, W) \to H^0(X, J_W(\frac{z}{n})) \) defined in (2.21), and hence we obtain an invariant of \( E \) in \( H^0(X, J_W(\frac{z}{n})) \). Since this invariant will be of crucial importance in the rest of this paper, we give it a name:

**Definition 2.29.** Under Assumptions 2.1 and 2.2 and with \( n \) even, define the ad hoc top Chern class of \( E \in mf(X, \mathcal{L}, W) \) relative to \( p : X \to S \) to be the element \( c_{\text{top}}(E) \in H^0(X, J_W(\frac{z}{n})) \) given as the image of \( \text{ch}(E) \) under (2.21).

**Example 2.30.** With the notation and assumptions of Example 2.13, we have \( \text{HH}^{\text{ad hoc}}_0(X/S, \mathcal{L}, W) = \text{HH}^{\text{ad hoc}}_0(\text{Spec}(Q)/\text{Spec}(k), \mathcal{O}, W) = \bigoplus_i \ker \left( \Omega_{Q/k}^{2i} \xrightarrow{dW} \Omega_{Q/k}^{2i+1} \right) \). and, when \( n \) is even, \( H^0(X, J_W(\frac{z}{n})) \) is the last summand:

\[
H^0(X, J_W(\frac{z}{n})) = \frac{\Omega_{Q/k}^n}{dW \wedge \Omega_{Q/k}^{n-1}}.
\]

We have

\[
c_{\text{top}}(E) = \frac{2}{n!} \text{tr}(dA dB \cdots dB).
\]

Here \( dA \) and \( dB \) are \( r \times r \) matrices with entries in \( \Omega^1 - Q/k \) and the product is occurring in the ring \( \text{Mat}_{r \times r}(\Omega_{Q/k}) \), and the map \( \text{tr} : \text{Mat}_{r \times r}(\Omega_{Q/k}) \to \Omega_{Q/k}^n \) is the usual trace map.

Our formula for \( c_{\text{top}}(E) \) coincides with the “Kapustin-Li” formula [14] found in many other places, at least up to a sign: In the work of Polishchuk-Vaintrob [22, Cor 3.2.4], for example, there is an additional a factor of \((-1)^{iR}\) in their formula for the Chern character.

Since the top Chern class of \( E \in mf(X, \mathcal{L}, W) \) is defined as the image of \( \text{id}_E \) under the composition

\[
\text{Unfold} \mathcal{E}nd_{mf}(E) \xrightarrow{\text{ch}_E} \text{Unfold} \text{Fold} \Omega_{dW} \to J_W(\frac{z}{n})
\]

of morphisms of complexes of coherent sheaves on \( X \), the following result is an immediate consequence of Proposition 2.24. Recall that for a quasi-coherent sheaf \( \mathcal{F} \) on \( X \) and a closed subset \( Z \) of \( X \), \( H^0_Z(X, \mathcal{F}) \) denotes the kernel of \( H^0(X, \mathcal{F}) \to H^0(X \setminus Z, \mathcal{F}) \).

**Proposition 2.31.** For any \( E \in mf(X, \mathcal{L}, W) \), its top Chern class is supported on \( \text{Nonreg}(Y) \):

\[
c_{\text{top}}(E) \in H^0_{\text{Nonreg}(Y)}(X, J_W(\frac{z}{n})).
\]

**Remark 2.32.** It is also follows from what we have established so far that the Chern character of any \( E \) is supported on \( \text{Nonreg}(Y) \) in the sense that it lifts canonically to an element of

\[
\text{HH}^0_{\text{Nonreg}(Y)}(X, \text{Unfold} \text{Fold} \Omega_{dW}),
\]

where in general \( \text{HH}^0 \) denote local hyper-cohomology of a complex of coherent sheaves.
2.7. Functorality of the Chern character and the top Chern class. The goal of this subsection is to prove Chern character is functorial in $S$. Throughout, we suppose Assumptions 2.1 and 2.2 hold.

Let $i : S' \to S$ be a morphism of Noetherian schemes and write $\mathcal{L}', X', W'$ for the evident pull-backs: $X' = X \times_S S'$, $\mathcal{L}'_S = i^*(\mathcal{L}_S)$, and $W' = i^*(W)$. The typical application will occur when $i$ is the inclusion of a closed point of $S$.

There is a functor

$$i^* : mf(X, \mathcal{L}, W) \to mf(X', \mathcal{L}', W')$$

induced by pullback along the map $X' \to X$ induced from $i$. There is a canonical isomorphism $i^* \Omega^\cdot dW \cong \Omega^\cdot dW'$ and hence an induced map

$$i^* : HH^\text{ad hoc}_0(X/S, \mathcal{L}, W) \to HH^\text{ad hoc}_0(X'/S', \mathcal{L}', W').$$

**Proposition 2.34.** For $E \in mf(X, \mathcal{L}, W)$, we have

$$i^*(ch(E)) = ch(i^*(E)) \in HH^\text{ad hoc}_0(X'/S', \mathcal{L}', W').$$

**Proof.** Recall that the Chern character of $E$ is determined by a map of matrix factorizations $E \to E \otimes \Omega^\cdot dW$ which is itself determined by a choice of connection $\mathcal{E}_j \to \mathcal{E}_j \otimes \mathcal{O}_X \Omega^1_{X/S}$ for $j = 0, 1$. Since our connections are $\mathcal{O}_S$-linear, we have the pull-back connection

$$i^* \mathcal{E}_j \to i^*(\mathcal{E}_j) \otimes \mathcal{O}_{X'/S'}, \Omega^1_{X'/S'}, j = 0, 1,$$

which we use to define the map

$$i^*E \to i^*E \otimes \Omega^\cdot dW'.$$

Using also that $i^* \text{End}(E) = \text{End}(i^*E)$, it follows that the square

$$\begin{array}{ccc}
\text{End}(E) & \longrightarrow & \text{Fold} \Omega^\cdot dW \\
\downarrow & & \downarrow \\
\mathbb{R}i_* \text{End}(i^*E) & \longrightarrow & \mathbb{R}i_* \text{Fold} \Omega^\cdot dW',
\end{array}$$

commutes. Applying $\mathbb{H}^0(X, \text{Unfold}(-))$ gives the commutative square

$$\begin{array}{ccc}
\mathbb{H}^0(X, \text{Unfold End}(E)) & \longrightarrow & HH^\text{ad hoc}_0(X/S, \mathcal{L}, W) \\
\downarrow_{i^*} & & \downarrow_{i^*} \\
\mathbb{H}^0(X', \text{Unfold End}(i^*E)) & \longrightarrow & HH^\text{ad hoc}_0(X'/S', \mathcal{L}', W').
\end{array}$$

The result follows from the fact that $i^* : \mathbb{H}^0(X, \text{Unfold End}(E)) \to \mathbb{H}^0(X', \text{Unfold End}(i^*E))$ sends $\text{id}_E$ to $\text{id}_{i^*E}$. □

Recall from (2.21) that, when $n$ is even, there is a natural map

$$HH^\text{ad hoc}_0(X/S, \mathcal{L}, W) \to \Gamma(X, \mathcal{J}_{X/S, \mathcal{L}, W}(\bar{z}))$$

Since $i^* \mathcal{J}_{X/S, \mathcal{L}, W} \cong \mathcal{J}_{X'/S', \mathcal{L}', W'}$, we obtain a map

$$i^* : \Gamma(X, \mathcal{J}_{X/S, \mathcal{L}, W}(\bar{z})) \to \Gamma(X', \mathcal{J}_{X'/S', \mathcal{L}', W'}(\bar{z})).$$
Corollary 2.35. When \( n \) is even, for any \( E \in mf(X, \mathcal{L}, W) \), we have
\[
i^*(c_{top}^p(E)) = c_{top}^p(i^*(E)) \in \Gamma(X, \mathcal{J}_{X/S}, \mathcal{L}', W'((\mathbb{Q})))
\]

Proof. This follows from Proposition 2.34 and the fact that
\[
\text{HH}_0^{ad hoc}(X/S, \mathcal{L}, W) \rightarrow i^* \downarrow \Gamma(X, \mathcal{J}_{X/S}, \mathcal{L}, W((\mathbb{Q})))
\]
commutes.

3. Relationship with affine complete intersections

We present some previously known results that relate twisted matrix factorizations with the singularity category of affine complete intersections. We start with some basic background.

For any Noetherian scheme \( X \), we write \( D^b(X) \) for the bounded derived category of coherent sheaves on \( X \) and \( \text{Perf}(X) \) for the full subcategory of perfect complexes — i.e., those complexes of coherent sheaves on \( X \) that are locally quasi-isomorphic to bounded complexes of finitely generated free modules. We define \( D_{sg}(X) \) to be the Verdier quotient \( D^b(X)/\text{Perf}(X) \). For a Noetherian ring \( R \), we write \( D_{sg}(R) \) for \( D_{sg}(\text{Spec}(R)) \).

The category \( D_{sg}(X) \) is triangulated. For a pair of finitely generated \( R \)-modules \( M \) and \( N \), we define their stable Ext-modules as
\[
\widehat{\text{Ext}}^i_R(M, N) := \text{Hom}_{D_{sg}(R)}(M, N[i]),
\]
where on the right we are interpreting \( M \) and \( N[i] \) as determining complexes (and hence objects of \( D_{sg}(R) \)) consisting of one non-zero component, lying in degrees 0 and \(-i\), respectively. Note that since \( D^b(R) \rightarrow D_{sg}(R) \) is a triangulated functor by construction and the usual \( \text{Ext}_R \)-modules are given by \( \text{Hom}_{D^b(R)}(M, N[i]) \), there is a canonical map
\[
\text{Ext}^i_R(M, N) \rightarrow \widehat{\text{Ext}}^i_R(M, N)
\]
that is natural in both variables.

Proposition 3.1. [1, 1.3] For all finitely generated \( R \)-modules \( M \) and \( N \), the natural map
\[
\text{Ext}^i_R(M, N) \rightarrow \widehat{\text{Ext}}^i_R(M, N)
\]
is an isomorphism for \( i \gg 0 \).

3.1. Euler characteristics and the Herbrand difference for hypersurfaces. Throughout this subsection, assume \( Q \) is a regular ring and \( f \in Q \) is a non-zero-divisor, and let \( R = Q/f \). Given an object \( E = (E_1 \xrightarrow{\alpha} E_0) \in mf(Q, f) \), define \( \text{coker}(E) \) to be \( \text{coker}(E_1 \xrightarrow{\alpha} E_0) \). The \( Q \)-module \( \text{coker}(E) \) is annihilated by \( f \) and we will regard it as an \( R \)-module.

Theorem 3.2 (Buchweitz). [1] If \( R = Q/f \) where \( Q \) is a regular ring and \( f \) is a non-zero-divisor, there is an equivalence of triangulated categories
\[
hmf(Q, f) \xrightarrow{\cong} D_{sg}(R)
\]
induced by sending a matrix factorization $E$ to $\text{coker}(E)$, regarded as an object of $\text{D}_{\text{sg}}(R)$.

In particular, we have an isomorphisms

$$\text{Ext}^i_{hmf}(E, F) := \text{Hom}_{hmf}(E, F[i]) \cong \widetilde{\text{Ext}}^i_R(\text{coker}(E), \text{coker}(F)), \quad \text{for all } i,$$

and

$$\text{Ext}^i_{hmf}(E, F) \cong \text{Ext}^i_R(\text{coker}(E), \text{coker}(F)), \quad \text{for } i \gg 0.$$

Since the category $hmf(Q, f)$ is trivial for all $f \in \mathfrak{p}$ such that $R_{\mathfrak{p}}$ is regular, $\text{Hom}^i_{hmf}(E, F)$ is an $R$-module supported on $\text{Nonreg}(R)$, for all $i$. If we assume $\text{Nonreg}(R)$ is a finite set of maximal ideals, then $\text{Hom}^i_{hmf}(E, F)$ has finite length as an $R$-module, allowing us to make the following definition.

**Definition 3.3.** Assume $Q$ is a regular ring and $f \in Q$ is a non-zero-divisor such that $\text{Nonreg}(R)$ is a finite set of maximal ideals, where $R := Q/f$. For $E, F \in mf(Q, f)$, define the Euler characteristic of $(E, F)$ to be

$$\chi(E, E) = \text{length} \text{Hom}^0_{hmf}(E, F) - \text{length} \text{Hom}^1_{hmf}(E, F).$$

In light of Theorem 3.2 if $M$ and $N$ are the cokernels of $E$ and $F$, then

$$\chi(E, F) = h^R(M, N), \quad \text{(3.4)}$$

where the right-hand side is the *Herbrand difference* of the $R$-modules $M$ and $N$, defined as

$$h^R(M, N) = \text{length} \text{Ext}^0_R(M, N) - \text{length} \text{Ext}^1_R(M, N)$$

$$= \text{length} \text{Ext}^2_R(M, N) - \text{length} \text{Ext}^{2i+1}_R(M, N), \quad i \gg 0.$$

**Lemma 3.5.** Suppose $\phi : Q \to Q'$ is a flat ring homomorphism between regular rings, $f$ is a non-zero-divisor in $Q$, and $f' := \phi(f)$ is a non-zero-divisor of $Q'$. Assume $\text{Nonreg}(Q/f) = \{m\}$ and $\text{Nonreg}(Q'/f') = \{m'\}$ for maximal ideals $m, m'$ satisfying the condition that $mQ' = m'-\text{primary}$.

Then, for all $E, F \in mf(Q, f)$,

$$\chi_{mf(Q, f)}(E, F) = \lambda \cdot \chi_{mf(Q, f)}(E \otimes_Q Q', F \otimes_Q Q'),$$

where $\lambda := \text{length}_{Q'}(Q'/mQ')$.

**Proof.** Let $R = Q/f$ and $R' = Q'/f'$. The induced map $R \to R'$ is flat and hence for any pair of $R$-modules $M$ and $N$, we have an isomorphism

$$\text{Ext}^i_R(M, N) \otimes_R R' \cong \text{Ext}^i_{R'}(M \otimes_R R', N \otimes_R R')$$

of $R'$-modules. For $i \gg 0$, $\text{Ext}^i_R(M, N)$ is supported on $\{m\}$ and $\text{Ext}^i_{R'}(M \otimes_R R', N \otimes_R R')$ is supported on $\{m'\}$. It follows that

$$\text{length}_{R'} \text{Ext}^i_{R'}(M \otimes_R R', N \otimes_R R') = \text{length}_R(\text{Ext}^i_R(M, N) \otimes_R R') = \lambda \text{length}_R \text{Ext}^i_R(M, N).$$

Hence $h^R(M', N') = \lambda h^R(M, N)$ and the result follows from (3.4). \qed
3.2. Matrix factorizations for complete intersections. Using a Theorem of Orlov [19 2.1], one may generalize Theorem 3.2 to complete intersection rings. The precise statement is the next Theorem. Versions of it are found in [21, 15, 18, 23]; the one given here is from [2, 2.11].

Theorem 3.6. Assume $Q$ is a regular ring of finite Krull dimension and $f_1, \ldots, f_c$ is a regular sequence of elements of $Q$. Let $R = Q/(f_1, \ldots, f_c)$, define $X = \mathbb{P}^{c-1}_Q$ and set $W = \sum f_i T_i \in \Gamma(X, \mathcal{O}(1))$. There is an equivalence of triangulated categories

$$D_{\text{sg}}(R) \cong \text{hmf}(X, \mathcal{O}(1), W).$$

The isomorphism of Theorem 3.6 has a certain naturality property that we need. Suppose $a_1, \ldots, a_c$ is a sequence of element of $Q$ that generate the unit ideal, and let $i : \text{Spec}(Q) \hookrightarrow X$ be the associated closed immersion. Then we have a functor

$$i^* : \text{hmf}(X, \mathcal{O}(1), W) \rightarrow \text{hmf}(Q, \sum a_i f_i).$$

Technically, the right hand side should be $\text{hmf}(\text{Spec}(Q), i^* \mathcal{O}(1), i^*(W))$, but it is canonically isomorphic to $\text{hmf}(Q, \sum a_i f_i)$ because there is a canonical isomorphism $i^* \mathcal{O}(1) \cong Q$ that sends $i^*(W)$ to $\sum a_i f_i$.

Also, the quotient map $Q/(\sum a_i f_i) \longrightarrow R$ has finite projective dimension, since, locally on $Q$, it is given by modding out by a regular sequence of length $c - 1$. We thus an induced functor

$$\text{res} : D_{\text{sg}}(R) \rightarrow D_{\text{sg}}(Q/\sum a_i f_i)$$
on singularity categories, given by restriction of scalars.

Proposition 3.7. With the notation above, the square

$$D_{\text{sg}}(R) \xrightarrow{\cong} \text{hmf}(\mathbb{P}^{c-1}_Q, \mathcal{O}(1), W)$$

$$\downarrow \text{res} \quad \downarrow i^*$$

$$D_{\text{sg}}(Q/\sum a_i f_i) \xrightarrow{\cong} \text{hmf}(Q, \sum a_i f_i)$$

commutes up to natural isomorphism.

Proof. To prove this, we need to describe the equivalence of Theorem 3.6 explicitly. Let $Y \subseteq \mathbb{P}^{c-1}_Q$ denote the closed subscheme cut out by $W$. Then there are a pair of equivalences

$$(3.8) \quad D_{\text{sg}}(R) \xrightarrow{\cong} D_{\text{sg}}(Y) \xleftarrow{\cong} \text{hmf}(X, \mathcal{O}(1), W),$$

which give the equivalence $D_{\text{sg}}(R) \xrightarrow{\cong} \text{hmf}(X, \mathcal{O}(1), W)$ of the Theorem.

The left-hand equivalence of (3.8) was established by Orlov [19 2.1] (see also [2 A.4]), and it is induced by the functor sending a bounded complex of finitely generated $R$-modules $M$ to $\beta_* \pi^*(M)$, where $\pi : \mathbb{P}^{c-1}_R \rightarrow \text{Spec}(R)$ is the evident projection and $\beta : \mathbb{P}^{c-1}_R \rightarrow Y$ is the evident closed immersion.

The right-hand equivalence of (3.8) is given by [3 6.3], and it sends a twisted matrix factorization $E = (\mathcal{E}_1 \xrightarrow{\alpha} \mathcal{E}_0 \xrightarrow{\beta} \mathcal{E}_1(1))$ to $\text{coker}(\alpha)$ (which may be regarded as a coherent sheaf on $Y$.)
For brevity, set \( g = \sum_i a_i f_i \). The Proposition will follow once we establish that the diagram

\[
\begin{array}{ccc}
D_{sg}(R) & \xrightarrow{\beta \circ \pi^*} & D_{sg}(Y) \\
\cong & & \cong \\
D_{sg}(Q/g) & \xrightarrow{L_j^*} & hm f(Q, g)
\end{array}
\]

commutes up to natural isomorphism, where \( k : \text{Spec}(R) \to \text{Spec}(Q/g) \) is the canonical closed immersion and \( j : \text{Spec}(Q/g) \to Y \) is the restriction of \( i \). (The closed immersion \( j \) is locally a complete intersection and thus of finite flat dimension. It follows that it induces a morphism on singularity categories, which we write \( L_j^* \).)

The right-hand square of (3.9) commutes since for a matrix factorization \( E = (E_1 \to E_0 \to E_1(1)) \) we have

\[ j^* \text{coker}(E_1 \to E_0) \cong \text{coker}(i^*E_1 \to i^*E_0) \]

and

\[ j^* \text{coker}(E_1 \to E_0) \cong L_j^* \text{coker}(E_1 \to E_0). \]

The first isomorphism is evident. The second holds since \( j \) has finite flat dimension and \( \text{coker}(E_1 \to E_0) \) is an "infinite syzygy"; that is, \( \text{coker}(E_1 \to E_0) \) has a right resolution by locally free sheaves on \( Y \), namely

\[ \gamma^*E_1(1) \to \gamma^*E_0(1) \to \gamma^*E_1(2) \to \cdots, \]

where \( \gamma : Y \to X \) is the canonical closed immersion.

To show the left-hand triangle commutes, we consider the Cartesian square of closed immersions

\[
\begin{array}{ccc}
P^{c-1}_R & \xrightarrow{\beta} & Y \\
\downarrow i_R & & \downarrow j \\
\text{Spec}(R) & \xrightarrow{k} & \text{Spec}(Q/g),
\end{array}
\]

where \( i_R \) denote the restriction of \( i \) to \( \text{Spec}(R) \). This square is Tor-independent (i.e., \( \text{Tor}^i_{\mathcal{O}_R}(\beta_*\mathcal{O}_{P^{c-1}_R}, j_*\mathcal{O}_{\text{Spec}(Q/g)}) = 0 \) for \( i > 0 \)), from which it follows that

\[ Lj^* \circ \beta_* = k_* \circ Li_R^*. \]

Since \( Li_R^* \circ \pi^* \) is the identity map, the left-hand triangle of (3.9) commutes. \qed

4. Some needed results on the geometry of complete intersections

Let \( k \) be a field and \( Q \) a smooth \( k \)-algebra of dimension \( n \). Set \( V = \text{Spec}(Q) \), a smooth affine \( k \)-variety. Given \( f_1, \ldots, f_c \in Q \), we have the associated morphism

\[ f := (f_1, \ldots, f_c) : V \to \mathbb{A}^c_k \]

of smooth \( k \)-varieties, induced by the \( k \)-algebra map \( k[t_1, \ldots, t_c] \to Q \) sending \( t_i \) to \( f_i \). Define the Jacobian of \( f \), written \( J_f \), to be the map

\[ J_f : Q^c \to \Omega^1_{Q/k} \]
given by \((df_1, \ldots, df_c)\). (More formally, \(J_f\) is the map \(f^*\Omega^1_{k^c/k} \to \Omega^1_{V/k}\) induced by \(f\), but we identify \(f^*\Omega^1_{k^c/k}\) with \(Q^c\) by using the basis \(dt_1, \ldots, dt_c\) of \(\Omega^1_{k^c/k}\). Also, \(J_f\) is really the dual of what is often called the Jacobian of \(f\).)

For example, if \(Q = k[x_1, \ldots, x_n]\), then \(J_f\) is given by the \(n \times c\) matrix \((\partial f_i / \partial x_j)\), using the basis \(dx_1, \ldots, dx_n\) of \(\Omega^1_{Q/k}\).

For a point \(x \in V\), let \(\kappa(x)\) denote its residue field and define

\[J_f(x) = J_f \otimes_Q \kappa(x) : \kappa(x)^c \to \Omega^1_{Q/k} \otimes \kappa(x) \cong \kappa(x)^n\]

to be the map on finite dimensional \(\kappa(x)\)-vector spaces induced by \(J_f\). Define

\[V_j := \{x \in V \mid \text{rank } J_f(\kappa(x)) \leq j\} \subseteq V\]

so that we have a filtration

\[\emptyset = V_{-1} \subseteq V_0 \subseteq \cdots \subseteq V_c = V\]

Note that the set \(V_{c-1}\) is the singular locus of the map \(f\).

For example, if \(Q = k[x_1, \ldots, x_n]\), then \(V_j\) is defined by the vanishing of the \((j + 1) \times (j + 1)\) minors of the matrix \((\partial f_i / \partial x_j)\).

**Example 4.2.** Let \(k\) be a field of characteristic not equal to 2, let \(Q = k[x_1, \ldots, x_n]\) so that \(V = \mathbb{A}^n_k\), let \(c = 2\), and define \(f_1 = \frac{1}{2}(x_1^2 + \cdots + x_n^2)\) and \(f_2 = \frac{1}{2}(a_1x_1^2 + \cdots + a_nx_n^2)\), where \(a_1, \ldots, a_n \in k\) are such that \(a_i \neq a_j\) for all \(i \neq j\). Thus \(f\) is a morphism between two affine spaces: \(f : \mathbb{A}^n_k \to \mathbb{A}^2_k\). For the usual basis, the Jacobian is the matrix

\[J_f = \begin{bmatrix} x_1 & a_1x_1 \\ x_2 & a_2x_2 \\ \vdots & \vdots \\ x_n & a_nx_n \end{bmatrix}\]

Then \(V_0\) is just the origin in \(V = \mathbb{A}^n_k\) and \(V_1\) is the union of the coordinate axes.

In the previous example, we have \(\dim(V_j) \leq j\) for all \(j < c\). This will turn out to be a necessary assumption for many of our result.

Let us summarize the notations and assumptions we will need in much of the rest of this paper:

**Assumptions 4.3.** Unless otherwise indicated, from now on we assume:

1. \(k\) is field.
2. \(V = \text{Spec}(Q)\) is smooth affine \(k\)-variety of dimension \(n\).
3. \(f_1, \ldots, f_c \in Q\) is a regular sequence of elements. We write
   \[f : V \to \mathbb{A}^c_k\]
   for the flat morphism given by \((f_1, \ldots, f_c)\). (Note that \(c \leq n\).)
4. The singular locus of \(f^{-1}(0) = \text{Spec}(Q/(f_1, \ldots, f_c))\), is zero-dimensional — say \(\text{Sing}(f^{-1}(0)) = \{v_1, \ldots, v_m\}\) where the \(v_i\)'s are closed points of \(V\).
5. \(\dim(V_j) \leq j\) for all \(j < c\), where \(V_j\) is defined in (4.1).

Observe that the singular locus of \(f^{-1}(0)\) may be identified with \(V_{c-1} \cap f^{-1}(0)\).

For the results in this paper, we will be allowed to shrink about the singular locus of \(f^{-1}(0)\). In this situation, if \(\text{char}(k) = 0\), the last assumption in above list is unnecessary, as we now prove:
Lemma 4.4. If all of Assumptions 4.3 hold except possibly (3) and char($k$) = 0, then for a suitably small affine open neighborhood $V'$ of the set $\{v_1, \ldots, v_m\}$, we have $\dim(V'_j) \leq i$ for all $j \leq c - 1$.

Proof. Since the induced map $\text{Sing}(f) \to \mathcal{H}^c$ has finite type, the set

$$B := \{ y \in \text{Sing}(f) \mid \text{Sing}(f) \cap \dim(f^{-1}(f(y))) > 0 \}$$

is a closed subset of $\text{Sing}(f)$ by the upper-semi-continuity of fiber dimensions [EGAIV, 13.1.3]. Since $\text{Sing}(f^{-1}(0)) = f^{-1}(0) \cap \text{Sing}(f)$ is a finite set of closed points, it follows that $\{v_1, \ldots, v_m\} \cap B = \emptyset$. Thus $V' := V \setminus B$ is an open neighborhood of the $v_i$'s and $\text{Sing}(f|_{V'}) = \text{Sing}(f) \cap V' = V'_{c-1}$ is quasi-finite over $\mathcal{H}^c$. Shrinking $V'$ further, we may assume it is affine.

By [11 III.10.6], we have $\dim(f(V'_j)) \leq j$ for all $j$. Since $V'_{c-1} \to f(V'_{c-1})$ is quasi-finite, so is $V'_j \to f(V'_j)$ for all $j \leq c - 1$, and hence $\dim(V'_j) \leq i$ for all $j \leq c - 1$. □

Example 4.5. Suppose char($k$) = $p > 0$, $Q = k[x_1, \ldots, x_c]$, and let $f_1(x) = x_1^p, \ldots, f_c(x) = x_c^p$. Then $J_f : Q^c \to \Omega^1_{Q/k} \cong Q^c$ is the zero map, and yet $k[x_1, \ldots, x_c]/(x_1^p, \ldots, x_c^p)$ has an isolated singularity at $(x_1, \ldots, x_c)$. This shows that the characteristic 0 hypothesis is necessary in Lemma 4.4.

Given the set-up in Assumptions 4.3, we let

$$S = \mathbb{P}^{c-1}_k = \text{Proj} k[T_1, \ldots, T_c]$$

and

$$X = \mathbb{P}^{c-1}_k \times_k V = \text{Proj} Q[T_1, \ldots, T_c],$$

and we define

$$W = f_1T_1 + \cdots + f_cT_c \in \Gamma(X, \mathcal{O}(1)).$$

As before, we have a map

$$dW : \mathcal{O}_X \to \Omega^1_{X/S}(1),$$

or, equivalently, a global section $dW \in \Gamma(X, \Omega^1_{X/S}(1))$. Recall that $\Omega^1_{X/S}(1)$ is the coherent sheaf associated to the graded module $\Omega^1_{Q/k}[T_1, \ldots, T_c](1)$, and $dW$ may be given explicitly as the degree one element

$$dW = df_1T_1 + \cdots + df_cT_c \in \Omega^1_{Q/k}[T_1, \ldots, T_c],$$

or, in other words,

$$dW = J_f : \left[ \begin{array}{c} T_1 \\ T_2 \\ \vdots \\ T_c \end{array} \right].$$

(4.6)

Proposition 4.7. With Assumptions 4.3, $dW$ is a regular section of $\Omega^1_{X/S}(1)$.

Proof. Since $X = \mathbb{P}^{c-1}_k \times_k V$ is smooth of dimension $n + c - 1$ and $\Omega^1_{X/S}$ is locally free of rank $n$, it suffices to prove the subscheme $Z$ cut out by $dW$ has dimension at most $c - 1$. Using (11.6), we see that the fiber of $Z \to V$ over a point $x \in V$ is a linear subscheme of $\mathbb{P}^{c-1}_k$ of dimension $c - 1 - \text{rank}(J_f(x))$. That is, if $x \in V'_j \setminus V'_{c-1}$, then the fiber over $Z \to Y$ over $y$ has dimension $c - 1 - j$. (This includes the case $j = c$ in the sense that the fibers over $V'_c \setminus V'_{c-1}$ are empty.) Since $\dim(V'_j) \leq j$ if $j < c$ by assumption, it follows that $\dim(Z) \leq c - 1$. □
Corollary 4.8. If Assumptions 4.3 hold, then the Jacobian complex $\Omega_{dW}$ is exact everywhere except in the right-most position, and hence determines a resolution of $J_W(n)$ by locally free coherent sheaves.

We will need the following result in the next section.

Proposition 4.9. Under Assumptions 4.3, there is an open dense subset $U$ of $\mathbb{P}^{c-1}_k$ such that for every $k$-rational point $[a_1 : \cdots : a_c] \in U$, the singular locus of the morphism

$$\sum_i a_i f_i : V \to \mathbb{A}^1_k$$

has dimension 0.

Proof. Let $Z$ be the closed subvariety of $\mathbb{P}^{c-1}_k \times_k V$ defined by the vanishing of $dW$ (see (4.6)). The fiber of the projection map $\pi : Z \to \mathbb{P}^{c-1}_k$ over a $k$-rational point $[a_1 : \cdots : a_c]$ may be identified with the singular locus of the morphism $\sum_i a_i f_i : V \to \mathbb{A}^1_k$. The claim is thus that there is a open dense subset $U$ of $\mathbb{P}^{c-1}_k$ over which the fibers of $\pi$ have dimension zero. Such a $U$ exists because dim($Z$) $\leq c-1$, as we established in the proof of Proposition 4.7. ($U$ may be taken to be the complement of $\overline{\pi(B)}$ where $B$ is the closed subset $\{z \in Z | \dim \pi^{-1} \pi(z) \geq 1\}$ of $Z.$) \hfill $\Box$

5. The vanishing of Chern characters

In this section, we prove that the top Chern class of a twisted matrix factorization vanishes in certain situations. We combine this with a Theorem of Polishchuk-Vaintrob to establish the vanishing of $h_c$ and $\eta_c$ under suitable hypotheses.

5.1. The vanishing of the top Chern class. The following theorem forms the key technical result of this paper.

Theorem 5.1. If Assumptions 4.3 hold and in addition $n$ is even, $\text{char}(p) \nmid n!$, and $c \geq 2$, then $c^{op}(E) = 0$ for any $E \in mf(\mathbb{P}^{c-1}_Q, \mathcal{O}(1), \sum_i f_i T_i)$.

Proof. The key point is that, under Assumptions 4.3 with $c \geq 2$, we have

$$H^0_{\mathbb{P}^{c-1}_k \times \{v_1, \ldots, v_m\}}(\mathbb{P}^{c-1}_k \times_k V, J_W(i)) = 0$$

for all $i < n$.

To prove this, since

$$H^0_{\mathbb{P}^{c-1}_k \times \{v_1, \ldots, v_m\}}(\mathbb{P}^{c-1}_k \times_k V, J_W(i)) = \bigoplus_{i=1}^m H^0_{\mathbb{P}^{c-1}_k \times \{v_i\}}(\mathbb{P}^{c-1}_k \times_k V, J_W(i))$$

and since we may replace $V = \text{Spec}(Q)$ by any open neighborhood of $v_i$ without affecting the $Q$-module $H^0_{\mathbb{P}^{c-1}_k \times \{v_i\}}(\mathbb{P}^{c-1}_k \times_k V, J_W(i))$, we may assume $m = 1$ and that there exists a regular sequence of elements $x_1, \ldots, x_n \in \mathfrak{m}$ that generate the maximal ideal of $Q$ corresponding to $v = v_1$.

Write $\mathcal{C} := \mathcal{C}(x_1, \ldots, x_n)$ for the "augmented Cech complex"

$$Q \to \bigoplus_i Q \left[ \frac{1}{x_i} \right] \to \bigoplus_{i,j} Q \left[ \frac{1}{x_i x_j} \right] \to \cdots \to Q \left[ \frac{1}{x_1 \cdots x_n} \right],$$
with $Q$ in cohomological degree 0. Equivalently $C = C(x_1) \otimes Q \cdots \otimes Q C(x_n)$, where $C(x_i) = (Q \rightarrow Q[x_1])$. For any coherent sheaf $F$ on $\mathbb{P}_Q^{c-1}$, we have

$$H^0_{\mathbb{P}_Q^{c-1 \times \{v\}}} (\mathbb{P}_Q^{c-1 \times k} V, F) = \Gamma (\mathbb{P}_Q^{c-1 \times k} V, H^0 (F \otimes Q C)).$$

The complex $C$ is exact in all degrees except in degree $n$, and we set $E = H^n(C)$. Explicitly,

$$E = \frac{Q[x_1/x_n]}{\Sigma_i Q[x_1/x_n]}.$$ 

(The localization of $E$ at $m$ is an injective hull of the residue field, but we do not need this fact.) Thus we have a quasi-isomorphism $C \sim E[-n].$

By Corollary 4.8, there is a quasi-isomorphism

$$\Omega_{\mathcal{W}} (-n)[n] \sim \mathcal{W}.$$

Since $C$ is a complex of flat modules, the map

$$\Omega_{\mathcal{W}} (-n)[n] \otimes Q C \sim \mathcal{W} \otimes Q C$$

is also a quasi-isomorphism. Combining this with the quasi-isomorphism $C \sim E[-n]$ gives an isomorphism in the derived category

$$\Omega_{\mathcal{W}} (-n) \otimes Q E \cong \mathcal{W} \otimes C.$$

For any $i$ we obtain an isomorphism

$$\mathcal{W} (\mathcal{W}(i) \otimes C) \cong \mathcal{W} (\Omega_{\mathcal{W}} (i - n) \otimes Q E)$$

of quasi-coherent sheaves.

We thus obtain from (5.3) the isomorphism

$$(5.4) \quad \Gamma_{\mathbb{P}_Q^{c-1 \times \{v\}}} (\mathbb{P}_k^{c-1 \times k} V, \mathcal{W}(i)) \cong \Gamma (\mathbb{P}_k^{c-1 \times k} V, \mathcal{W} (\Omega_{\mathcal{W}} (i - n) \otimes Q E)).$$

But $\mathcal{W} (\Omega_{\mathcal{W}} (i - n) \otimes Q E)$ is the kernel of

$$\mathcal{O}_{\mathbb{P}_Q^{c-1}} (i - n) \otimes Q E \xrightarrow{\mathcal{W} \otimes -} \Omega_{\mathbb{P}_Q^{c-1 \times k}}^1 (i - n + 1) \otimes Q E.$$

Since $c - 1 \geq 1$, the coherent sheaf $\mathcal{O}_{\mathbb{P}_Q^{c-1}} (i - n)$ has no global section when $i < n$, and hence

$$\Gamma (\mathbb{P}_k^{c-1 \times k} V, \mathcal{W} (\Omega_{\mathcal{W}} (i - n) \otimes Q E)) = 0,$$

which establishes (5.2).

Let $Y$ be the closed subscheme of $X = \mathbb{P}_k^{c-1} \times V$ cut out by $W = \sum_i f_i T_i$. We claim $\text{Sing}(Y/k) \subseteq \mathbb{P}_k^{c-1} \times \{v_1, \ldots, v_m\}$. Indeed, $\text{Sing}(Y/k)$ is defined by the equations $\sum_i d(f_i) T_i = 0$ and $f_i = 0$, $i = 1, \ldots, c$, where $d(f_i) \in \Omega_{Q/k}^1$. The first equation cuts out a subvariety contained in $\mathbb{P}_k^{c-1} \times V_{c-1}$, where, recall, $V_{c-1}$ denotes the singular locus of $f : V \rightarrow \mathbb{A}^c_k$. The equations $f_i = 0$, $i = 1, \ldots, c$, cut out $\mathbb{P}_k^{c-1} \times_k f^{-1}(0)$. But $V_{c-1} \cap f^{-1}(0) = \text{Sing}(f^{-1}(0)) = \{v_1, \ldots, v_m\}$.

Since $\text{Nonreg}(Y) \subseteq \text{Sing}(Y/k)$, we deduce from (5.2) that

$$H^0_{\text{Nonreg}(Y)} (\mathbb{P}_k^{c-1 \times k} V, \mathcal{W}(i)) = 0.$$

The vanishing of $c^{\text{top}}(E)$ now follows from Proposition 2.3.1. \qed

**Corollary 5.5.** If

- $k$ is a field of characteristic 0,
• $Q$ is a smooth $k$-algebra of even dimension,
• $f_1, \ldots, f_c \in Q$ forms a regular sequence of elements with $c \geq 2$, and
• the singular locus of $Q/(f_1, \ldots, f_c)$, is zero-dimensional,

then $c^{top}(E) = 0$ for all $E \in mf(P^{c-1}_Q, \mathcal{O}(1), \sum_i f_i T_i)$.

Proof. Let $\{v_1, \ldots, v_m\}$ be the singular locus of $Q/(f_1, \ldots, f_c)$ and let $V' = \text{Spec}(Q')$ be an affine open neighborhood of it such that $\dim(V') \leq i$ for all $i \leq c - 1$. The existence of such a $V'$ is given by Lemma 4.4. The result follows from the (proof of the) Theorem, since the map

$$H^0_{\mathbb{P}^{c-1}_k \times \{v_1, \ldots, v_m\}}(\mathbb{P}^{c-1}_Q, \mathcal{O}, \sum_i f_i T_i) \to H^0_{\mathbb{P}^{c-1}_k \times \{v_1, \ldots, v_m\}}(\mathbb{P}^{c-1}_Q, \mathcal{O}, \sum_i f_i T_i)$$

is an isomorphism. \qed

5.2. The Polishchuk-Vaintrob Riemann-Roch Theorem. We recall a Theorem of Polishchuk-Vaintrob that relates the Euler characteristic of (affine) hypersurfaces with isolated singularities to Chern characters. This beautiful theorem should be regarded as a form of a “Riemann-Roch” theorem. Since Polishchuk-Vaintrob work in a somewhat different setting that we do, we begin by recalling their notion of a Chern character.

For a field $k$ and integer $n$, define $\hat{Q} := k[[x_1, \ldots, x_n]]$, a power series ring in $n$ variables, and let $f \in \hat{Q}$ be such that the only non-regular point of $\hat{Q}/f$ is its maximal ideal. Define $\hat{\Omega}_{\hat{Q}/k}$ to be the exterior algebra on the free $\hat{Q}$-module

$$\hat{\Omega}^1_{\hat{Q}/k} = \hat{Q} dx_1 \oplus \cdots \oplus \hat{Q} dx_n.$$

In other words,

$$\hat{\Omega}_{\hat{Q}/k} = \Omega_k[x_1, \ldots, x_n]/k \otimes k[x_1, \ldots, x_n] \hat{Q}.$$

Finally, define

$$\hat{\mathcal{J}}(\hat{Q}/k, f) = \text{coker} \left( \hat{\Omega}^{n-1}_{\hat{Q}/k} \xrightarrow{\partial f} \hat{\Omega}^{n}_{\hat{Q}/k} \right).$$

Note that

$$\hat{\mathcal{J}}(\hat{Q}/k, f) = \left\langle \underbrace{\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}}_{r \text{ factors}} \right\rangle dx_1 \wedge \cdots \wedge dx_n.$$

Definition 5.6. With the notation above, given a matrix factorization $E \in mf(\hat{Q}, f)$, upon choosing bases, it may be written as $E = (\hat{Q}^r \xrightarrow{A} \hat{Q}^r)$ for $r \times r$ matrices $A, B$ with entries in $\hat{Q}$. When $n$ is even, we define the Polishchuk-Vaintrob Chern character of $E$ to be

$$\text{ch}^{PV}(E) = (-1)^{(\frac{r}{2})} \text{tr} \left( \frac{1}{n \text{ factors}} dB \wedge \cdots \wedge dB \right) \in \hat{\mathcal{J}}(\hat{Q}/k, f).$$

Theorem 5.7 (Polishchuk-Vaintrob). Assume $k$ is a field of characteristic 0 and $f \in \hat{Q} = k[[x_1, \ldots, x_n]]$ is a power series such that $R := \hat{Q}/f$ has an isolated singularity (i.e., $R_p$ is regular for all $p \neq m$).

If $n$ is odd, then $\chi(E, F) = 0$ for all $E, F \in mf(\hat{Q}, f)$.

If $n$ is even, there is a pairing $\langle -, - \rangle$ on $\hat{\mathcal{J}}(\hat{Q}, f)$, namely the residue pairing, such that for all $E, F \in mf(\hat{Q}, f)$, we have

$$\chi(E, F) = (\text{ch}^{PV}(E), \text{ch}^{PV}(F)).$$
In particular, \( \chi(\mathbb{E}, F) = 0 \) if either \( \text{ch}^{PV}(\mathbb{E}) = 0 \) or \( \text{ch}^{PV}(F) = 0 \).

It is easy to deduce from their Theorem the following slight generalization of it, which we will need to prove Theorem 5.11 below.

**Corollary 5.8.** Assume \( k \) is a field of characteristic 0, \( Q \) is a smooth \( k \)-algebra of dimension \( n \), \( f \in Q \) is a non-zero-divisor such that the singular locus of \( R := Q/f \) is zero-dimensional. Given \( \mathbb{E} \in m f(Q, f) \), if either \( n \) is odd or \( n \) is even and \( c^{\text{top}}(\mathbb{E}) = 0 \) in \( \mathcal{J}(Q/k, f) = \Omega^1_{Q/k}/df \wedge \Omega^{n-1}_{Q/k} \), then

\[
\chi(\mathbb{E}, F) = 0
\]

for all \( F \in mf(Q, f) \).

**Proof.** We start by reducing to the case where \( R \) has just one singular point \( m \) and it is \( k \)-rational. We have

\[
\chi_{mf(Q, f)}(\mathbb{E}, F) = \sum_{m} \chi_{mf(Q, f)}(\mathbb{E}_m, F_m)
\]

where the sum ranges over the singular points of \( R \). If \( \mathbb{T} \) is an algebraic closure of \( k \), then for each such \( m \) we have

\[
\text{Hom}_{m f(Q, f)}^*(\mathbb{E}_m, F_m) \otimes_k \mathbb{T} \cong \text{Hom}_{m f(Q \otimes_k f \otimes 1)}^*(\mathbb{E}_m \otimes_k \mathbb{T}, F_m \otimes_k \mathbb{T})
\]

and hence

\[
\chi_{mf(Q, f)}(\mathbb{E}_m, F_m) = [Q/m : \mathbb{T}] \cdot \chi_{mf(Q, f)}(\mathbb{E}_m, F_m).
\]

It thus suffices to prove the Corollary for a suitably small affine open neighborhood of each maximal ideal \( m \) of \( Q \otimes_k f \) at which \( Q \otimes_k f/f \) is singular.

In this case, a choice of a regular sequence of generators \( x_1, \ldots, x_n \) of the maximal ideal of \( Q_m \) allows us to identify the completion of \( Q \) along \( m \) with \( \hat{Q} = k[[x_1, \ldots, x_n]] \). By Lemma 3.3,

\[
\chi_{mf(Q, f)}(\mathbb{E}, F) = \chi_{mf(\hat{Q}, f)}(\hat{\mathbb{E}}, \hat{F})
\]

where \( \hat{\mathbb{E}} = \mathbb{E} \otimes_Q \hat{Q} \) and \( \hat{F} = \mathbb{F} \otimes_Q \hat{Q} \). Moreover, under the evident map \( \mathcal{J}(Q_m/k, f) \to \mathcal{J}(\hat{Q}/k, f) \), \( c^{\text{top}}(\mathbb{E}) \) is sent to a non-zero multiple of \( \text{ch}^{PV}(\hat{\mathbb{E}}) \) by Example 2.30. The result now follows from the Polishchuk-Vaintrob Theorem. \( \square \)

### 5.3. Vanishing of the higher \( h \) and \( \eta \) Invariants

We apply our vanishing result for \( c^{\text{top}} \) to prove that the invariants \( h_c \) and \( \eta_c \) vanish for codimension \( c \geq 2 \) isolated singularities in characteristic 0, under some mild additional hypotheses.

First, we use the following lemma to prove the two invariants are closely related.

I thank Hailong Dao for providing me with its proof.

**Lemma 5.9** (Dao). Let \( R = Q/(f_1, \ldots, f_c) \) for a regular ring \( Q \) and a regular sequence of elements \( f_1, \ldots, f_c \in Q \) with \( c \geq 1 \). Assume \( M \) and \( N \) are finitely generated \( R \)-modules and that \( M \) is MCM. If \( \text{Tor}_i^R(M, N) \) and \( \text{Ext}^i_R(\text{Hom}_R(M, R), N) \) have finite length for \( i \gg 0 \), then

\[
\eta^i_c(M, N) = (-1)^c h^i_c(\text{Hom}_R(M, R), N).
\]

In particular, if the non-regular locus of \( R \) is zero dimensional and \( h_c \) vanishes for all pairs of finitely generated \( R \)-modules, then so does \( \eta_c \).

**Remark 5.10.** The conditions that \( \text{Tor}_i^R(M, N) \) has finite length for \( i \gg 0 \) and that \( \text{Ext}^i_R(\text{Hom}_R(M, R), N) \) has finite length for \( i \gg 0 \) are equivalent; see [13, 4.2].
Proof. We proceed by induction on $c$. The case $c = 1$ follows from the isomorphism [6 4.2]

$$\Ext^i_R(\Hom_R(M, R), N) \cong \Tor^R_{-i-1}(M, N).$$

Let $S = Q/\langle f_1, \ldots, f_{c-1} \rangle$ so that $R = S/(f_c)$ with $f_c$ a non-zero-divisor of $S$. By [5 4.3] and [4 3.4], $\Tor^S_i(M, N)$ and $\Ext^S_i(\Hom_R(M, R), N)$ have finite length for $i \gg 0$ and

$$\eta^S_{c}(M, N) = \frac{1}{2c} \eta^S_{c-1}(M, N)$$

$$h^R_c(\Hom_R(M, R), N) = \frac{1}{2c} h^S_{c-1}(\Hom_R(M, R), N).$$

Let $S^n \to M$ be a surjection of $S$-modules with kernel $M'$ so that we have the short exact sequence

$$0 \to M' \to S^n \to M \to 0.$$ Observe that $M'$ is an MCM $S$-module. Since $f_c \cdot M = 0$ and $f_c$ is a non-zero-divisor of $S$, we have $\Hom_S(M, S^n) = 0$. Using also the isomorphism $\Hom_R(M, R) \cong \Ext^1_S(M, S)$ of $S$-modules coming from the long exact sequence of Ext modules associated to the sequence $0 \to S \to R \to 0$, we obtain the short exact sequence

$$0 \to S^n \to \Hom_S(M', S) \to \Hom_R(M, R) \to 0.$$ In particular, it follows from these short exact sequences that $\Tor^S_i(M', N)$ and $\Ext^S_i(\Hom_S(M', S), N)$ have finite length for $i \gg 0$. Moreover, since $h^S_{c-1}$ vanishes on free $S$-modules and is additive for short exact sequences, we have

$$h^S_{c-1}(\Hom_S(M', S), N) = h^S_{c-1}(\Hom_R(M, R), N).$$

Using the induction hypotheses, we get

$$h^R_c(\Hom_R(M, R), N) = \frac{1}{2c} h^S_{c-1}(\Hom_S(M', S), N) = \frac{1}{2c} (-1)^{c-1} \eta^S_{c-1}(M', N).$$

Finally, $\eta^S_{c-1}(M', N) = -\eta^S_{c-1}(M, N)$, since $M'$ is a first syzygy of $M$. Combining these equations gives

$$\eta^R_c(M, N) = (-1)^c h^R_{c-1}(\Hom_R(M, R), N).$$

The final assertion holds since $\eta^R_c$ is completely determined by its values on MCM modules. \hfill $\square$

**Theorem 5.11.** Assume

- $k$ is field of characteristic $0$,
- $Q$ is a smooth $k$-algebra,
- $f_1, \ldots, f_c \in Q$ form a regular sequence of elements,
- the singular locus of $R := Q/\langle f_1, \ldots, f_c \rangle$ is zero-dimensional, and
- $c \geq 2$.

Then $h^R_c(M, N) = 0$ and $\eta^R_c(M, N) = 0$ for all finitely generated $R$-modules $M$ and $N$.

**Remark 5.12.** Since $k$ is a perfect field, the hypotheses that $Q$ is a smooth $k$-algebra is equivalent to $Q$ being a finitely generated and regular $k$-algebra. Likewise, $\text{Sing}(R/k) = \text{Nonreg}(R)$. 
Proof. By Lemma 6.3 it suffices to prove \( h^R_c(M, N) = 0 \). The value of this invariant is unchanged if we semi-localize at \( \text{Nonreg}(R) \). So, upon replacing \( \text{Spec}(Q) \) with a suitably small affine open neighborhood of \( \text{Nonreg}(R) \), Lemma 4.4 allows us to assume all of Assumptions 4.3 hold.

Let \( E_M, E_N \in mf(P^{-1}_Q, \mathcal{O}(1), W) \) be twisted matrix factorizations corresponding to the classes of \( M, N \) in \( D_{sg}(R) \) under the equivalence of Theorem 3.6. Now choose \( a_1, \ldots, a_c \in k \) as in Proposition 4.9 so that the singular locus of \( Q/g \) is zero dimensional, where we define \( g := \sum a_i f_i \).

A key fact we use is that \( h^R_c \) is related to the classical Herbrand difference of the hypersurface \( Q/g \): By [4, 3.4], for all finitely generated \( R \)-modules \( M, N \), we have

\[
h^R_c(M, N) = h^{Q/g}_1(M, N) = \frac{1}{2} h^{Q/g}_1(M, N),
\]

and thus it suffices to prove \( h^{Q/g}_1(M, N) = 0 \). By Proposition 3.7 the affine matrix factorizations \( i^*E_M, i^*E_N \in hmf(Q, g) \) represent the classes of \( M, N \) in \( D_{sg}(Q/g) \), where \( i : \text{Spec}(Q) \hookrightarrow \mathbb{P}^{e-1}_Q \) is the closed immersion associated to the \( k \)-rational point \([a_1 : \cdots : a_c] \) of \( \mathbb{P}^{e-1}_k \). It thus follows from (3.4) that

\[
h^{Q/g}_1(M, N) = \chi(i^*E_M, i^*E_N).
\]

By Corollary 5.5 it suffices to prove \( c^{\text{top}}(i^*E_M) = 0 \) in \( \mathcal{J}(Q_m/k) \) (when \( n \) is even). But \( c^{\text{top}}(i^*E_M) = i^*c^{\text{top}}(E_M) \) by Corollary 2.33 and \( c^{\text{top}}(E_M) = 0 \) by Corollary 5.6.

## Appendix A. Relative Connections

We record here some well known facts concerning connections for locally free coherent sheaves. Throughout, \( S \) is a Noetherian, separated scheme and \( p : X \to S \) is a smooth morphism; i.e., \( p \) is separated, flat and of finite type and \( \Omega^1_{X/S} \) is locally free.

**Definition A.1.** For a vector bundle (i.e., locally free coherent sheaf) \( E \) on \( X \), a connection on \( E \) relative to \( p \) is a map of sheaves of abelian groups

\[
\nabla : E \to \Omega^1_{X/S} \otimes_{\mathcal{O}_X} E
\]

on \( X \) satisfying the Leibnitz rule on sections: given an open subset \( U \subseteq X \) and elements \( f \in \Gamma(U, \mathcal{O}_X) \) and \( e \in \Gamma(U, E) \), we have

\[
\nabla(f \cdot e) = df \otimes e + f \nabla(e) \quad \text{in} \quad \Gamma(U, \Omega^1_{X/S} \otimes_{\mathcal{O}_X} E),
\]

where \( \nabla(e) : \mathcal{O}_X \to \Omega^1_{X/S} \otimes_{\mathcal{O}_X} E \) denotes exterior differentiation relative to \( p \).

Note that the hypotheses imply that \( \nabla \) is \( \mathcal{O}_S \)-linear — more precisely, \( p_*(\nabla) : p_*E \to p_* \left( \Omega^1_{X/S} \otimes_{\mathcal{O}_X} E \right) \) is a morphism of quasi-coherent sheaves on \( S \).

### A.1. The classical Atiyah class

Let \( \Delta : X \to X \times_S X \) be the diagonal map, which, since \( X \to S \) is separated, is a closed immersion, and let \( \mathcal{I} \) denote the sheaf of ideals cutting out \( \Delta(X) \). Since \( p \) is smooth, \( \mathcal{I} \) is locally generated by a regular sequence. Recall that \( \mathcal{I}/\mathcal{I}^2 \cong \Delta_* \Omega^1_{X/S} \). Consider the coherent sheaf \( \mathcal{P}_{X/S} := \mathcal{O}_{X \times S} \mathcal{I}/\mathcal{I}^2 \) on \( X \times_S X \). Observe that \( \mathcal{P}_{X/S} \) is supported on \( \Delta(X) \), so that \( (\pi_i)_* \mathcal{P}_{X/S} \) is a coherent sheaf on \( X \), for \( i = 1, 2 \), where \( \pi_i : X \times_S X \to X \) denotes projection onto the \( i \)-th factor.
The two push-forwards \((\pi_i)_*\hat{P}_{X/S}, i = 1, 2\) are canonically isomorphic as sheaves of abelian groups, but have different structures as \(O_X\)-modules. We write \(P_{X/S} = \mathcal{P}\) for the sheaf of abelian groups \((\pi_1)_*\hat{P} = (\pi_2)_*\hat{P}\) regarded as a \(O_X - O_X\)-bimodule where the left \(O_X\)-module structure is given by identifying it with \((\pi_1)_*\hat{P}_{X/S}\) the right \(O_X\)-module structure is given by identifying it with \((\pi_2)_*\hat{P}_{X/S}\).

Locally on an affine open subset \(U = \text{Spec}(Q)\) of \(X\) lying over an affine open subset \(V = \text{Spec}(A)\) of \(S\), we have \(P_{U/V} = (Q \otimes_A Q)/I^2,\) where \(I = \ker(Q \otimes_A Q \rightarrow Q)\) and the left and right \(Q\)-module structures are given in the obvious way.

There is an isomorphism of coherent sheaves on \(X \times X\)

\[
\Delta_* \Omega^1_{X/S} \cong \mathcal{I}/\mathcal{I}^2
\]

given locally on generators by \(dg \mapsto g \otimes 1 - 1 \otimes g\). From this we obtain the short exact sequence

\[
(A.2) \quad 0 \rightarrow \Omega^1_{X/S} \rightarrow \mathcal{P}_{X/S} \rightarrow O_X \rightarrow 0.
\]

This may be thought of as a sequence of \(O_X - O_X\)-bimodules, but for \(\Omega^1_{X/S}\) and \(O_X\) the two structures coincide.

Locally on open subsets \(U\) and \(V\) as above, we have \(\Omega^1_{Q/A} \cong I/I^2,\) and \((A.2)\) takes the form

\[
0 \rightarrow I/I^2 \rightarrow (Q \otimes_A Q)/I^2 \rightarrow Q \rightarrow 0.
\]

Viewing \((A.2)\) as either a sequence of left or right modules, it is a split exact sequence of locally free coherent sheaves on \(X\). For example, a splitting of \(\mathcal{P}_{X/S} \rightarrow O_X\) as right modules may be given as follows: Recall that as a right module, \(\mathcal{P}_{X/S} = (\pi_2)_*\mathcal{P}\) and so a map of right modules \(O_X \rightarrow \mathcal{P}_{X/S}\) is given by a map \(\pi_2^*O_X \rightarrow \mathcal{P}_{X/S}\). Now, \(\pi_2^*O_X = O_{X \times X}\), and the map we use is the canonical surjection. We refer to this splitting as the canonical right splitting of \((A.2)\).

Locally on subsets \(U\) and \(V\) as above, the canonical right splitting of \((Q \otimes_A Q)/I^2 \rightarrow Q\) is given by \(q \mapsto 1 \otimes q\).

Given a locally free coherent sheaf \(\mathcal{E}\) on \(X\), we tensor \((A.2)\) on the right by \(\mathcal{E}\) to obtain the short exact sequence

\[
(A.3) \quad 0 \rightarrow \Omega^1_{X/S} \otimes_{O_X} \mathcal{E} \xrightarrow{\pi} \mathcal{P}_{X/S} \otimes_{O_X} \mathcal{E} \xrightarrow{\pi} \mathcal{E} \rightarrow 0
\]

of \(O_X - O_X\)-bimodules. Taking section on affine open subsets \(U\) and \(V\) as before, letting \(E = \Gamma(U, \mathcal{E})\), this sequence has the form

\[
0 \rightarrow \Omega^1_{Q/A} \otimes Q \rightarrow (Q \otimes_A E)/I^2 : E \rightarrow E \rightarrow 0.
\]

Since \((A.2)\) is split exact as a sequence of right modules and tensor product preserves split exact sequences, \((A.3)\) is split exact as a sequence of right \(O_X\)-modules, and the canonical right splitting of \((A.2)\) determines a canonical right splitting of \((A.3)\), which we write as

\[
\text{can} : \mathcal{E} \rightarrow \mathcal{P}_{X/S} \otimes_{O_X} \mathcal{E}.
\]

The map can is given locally on sections by \(e \mapsto 1 \otimes e\).

In general, \((A.3)\) need not split as a sequence of left modules. Viewed as a sequence of left modules, \((A.3)\) determines an element of

\[
\text{Ext}^1_{O_X}(\mathcal{E}, \Omega^1_{X/S} \otimes_{O_X} \mathcal{E}) \cong H^1(X, \Omega^1_{X/S} \otimes_{O_X} \mathcal{E}_{ndO_X}(\mathcal{E})),
\]

sometimes called the “Atiyah class” of \(\mathcal{E}\) relative to \(p\). To distinguish this class from what we have called the Atiyah class of a matrix factorization in the body of
this paper, we will call this class the classical Atiyah class of the vector bundle \(E\), and we write it as

\[
A_{\text{classical}}^E_{X/S} \in \ext^1_{\mathcal{O}_X}(E, \Omega^1_{X/S}).
\]

The sequence (A.3) splits as a sequence of left modules if and only if \(A_{\text{classical}}^E_{X/S} = 0\).

Lemma A.4. If \(p : X \to S\) is affine, then the classical Atiyah class of any vector bundle \(E\) on \(X\) vanishes, and hence (A.3) splits as a sequence of left modules.

Proof. Since \(p\) is affine, \(p_*\) is exact. Applying \(p_*\) to (A.3) results in a sequence of \(\mathcal{O}_S - \mathcal{O}_S\) bimodules (which are quasi-coherent for both actions). But since \(p \circ \pi_1 = p \circ \pi_2\) these two actions coincide. Moreover, since (A.3) splits as right modules, so does its push-forward along \(p_*\).

It thus suffices to prove the following general fact: If

\[
F := (0 \to F' \to F \to F'' \to 0)
\]

is a short exact sequence of vector bundles on \(X\) such that \(p_*(F)\) splits as a sequence of quasi-coherent sheaves on \(S\), then \(F\) splits. To prove this, observe that \(F\) determines a class in \(H^1(X, \text{Hom}_{\mathcal{O}_X}(F'', F'))\) and it is split if and only if this class vanishes. We may identify \(H^1(X, \text{Hom}_{\mathcal{O}_X}(F'', F'))\) with \(H^1(S, p_* \text{Hom}_{\mathcal{O}_X}(F'', F'))\) since \(p\) is affine. Moreover, the class of \(F \in H^1(S, p_* \text{Hom}_{\mathcal{O}_X}(F'', F'))\) is the image of the class of \(p_*(F) \in H^1(S, \text{Hom}_{\mathcal{O}_X}(p_* F'', p_* F'))\) under the map induced by the canonical map

\[
\text{Hom}_{\mathcal{O}_X}(p_* F'', p_* F') \to p_* \text{Hom}_{\mathcal{O}_X}(F'', F').
\]

But by our assumption the class of \(p_*(F)\) vanishes since \(p_* F\) splits. \(\square\)

A.2. The vanishing of the classical Atiyah class and connections. Suppose \(\sigma : E \to \mathcal{P}_{X/S} \otimes_{\mathcal{O}_X} E\) is a splitting of the map \(\pi\) in (A.3) as left modules and recall can : \(E \to \mathcal{P}_{X/S} \otimes_{\mathcal{O}_X} E\) is the splitting of \(\pi\) as a morphism of right modules given locally by \(e \mapsto 1 \otimes e\). Since \(\sigma\) and can are splittings of the same map regarded as a map of sheaves of abelian groups, the difference \(\sigma - \text{can}\) factors as \(i \circ \nabla_\sigma\) for a unique map of sheaves of abelian groups

\[
\nabla_\sigma : E \to \Omega^1_{X/S} \otimes_{\mathcal{O}_X} \mathcal{P}.
\]

Lemma A.5. The map \(\nabla_\sigma\) is a connection on \(E\) relative to \(p\).

Proof. The property of being a connection may be verified locally, in which case the result is well known.

In more detail, restricting to an affine open \(U = \text{Spec}(Q)\) of \(X\) lying over an affine open \(V = \text{Spec}(A)\) of \(S\), we assume \(E\) is a projective \(Q\)-module and that we are given a splitting \(\sigma\) of the map of left \(Q\)-modules \((Q \otimes_A E)/I^2 \cdot E \to E\). The map \(\nabla_\sigma = (\sigma - \text{can})\) lands in \(I/I^2 \otimes_Q Q = \Omega^1_{Q/A} \otimes_Q E\), and for \(a \in A, e \in E\) we have

\[
\nabla_\sigma(ae) = \sigma(ae) - 1 \otimes ae
\]

\[
= a\sigma(e) - 1 \otimes ae
\]

\[
= a\sigma(e) - a \otimes e + a \otimes e - 1 \otimes ae
\]

\[
= a(\sigma(e) - 1 \otimes e) + (a \otimes 1 - 1 \otimes a) \otimes e
\]

\[
= a\nabla_\sigma(e) + da \otimes e,
\]

since \(da\) is identified with \(a \otimes 1 - 1 \otimes a\) under \(\Omega^1_{Q/A} \cong I/I^2\). \(\square\)
Lemma A.6. Suppose $\mathcal{E}, \mathcal{E}'$ are locally free coherent sheaves on $X$ and $\nabla, \nabla'$ are connections for each relative to $p$. If $g : \mathcal{E} \to \mathcal{E}'$ is a morphisms of coherent sheaves, then the map

$$\nabla' \circ g - (\text{id} \otimes g) \circ \nabla : \mathcal{E} \to \Omega^1_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E}'$$

is a morphism of coherent sheaves.

Proof. Given an open set $U$ and elements $f \in \Gamma(U, \mathcal{O}_X), e \in \Gamma(U, \mathcal{E})$, the displayed map sends $f \cdot e \in \Gamma(U, \mathcal{E})$ to

$$\nabla'(g(fe)) - (\text{id} \otimes g)(df \otimes e - f \nabla(e)) = df \otimes g(e) + f\nabla'(g(e)) - df \otimes g(e) - f(\text{id} \otimes g)(\nabla(e))$$

$$= f \nabla'(g(e)) - f(\text{id} \otimes g)(\nabla(e))$$

$$= f((\nabla' \circ g - (\text{id} \otimes g) \circ \nabla)(e)).$$

Proposition A.7. For a vector bundle $\mathcal{E}$ on $X$, the function $\sigma \mapsto \nabla_{\sigma}$ determines a bijection between the set of splittings of the map $\pi$ in (A.3) as a map of left modules and the set of connections on $\mathcal{E}$ relative to $p$. In particular, $\mathcal{E}$ admits a connection relative to $p$ if and only if $\text{Ar}^{\text{classical}}_{X/S}(\mathcal{E}) = 0$.

Proof. From Lemma A.6 with $g$ being the identity map, the difference of two connections on $\mathcal{E}$ is $\mathcal{O}_X$-linear. By choosing any one splitting $\sigma_0$ of (A.3) and its associated connection $\nabla_0 = \nabla_{\sigma_0}$, the inverse of

$$\sigma \mapsto \nabla_{\sigma}$$

is given by

$$\nabla \mapsto \nabla - \nabla_{\sigma_0} + \sigma_0.$$

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