A New Concept of Fixed Point in Metric and Normed Interval Spaces

Hsien-Chung Wu
Department of Mathematics, National Kaohsiung Normal University, Kaohsiung 802, Taiwan; hcuwu@nknucc.nknu.edu.tw
Received: 15 September 2018; Accepted: 22 October 2018; Published: 25 October 2018

Abstract: The main aim of this paper is to propose the concept of so-called near fixed point and establish many types of near fixed point theorems in the set of all bounded and closed intervals in \( \mathbb{R} \).

The concept of null set will be proposed in order to interpret the additive inverse element in the set of all bounded closed intervals. Based on the null set, the concepts of metric interval space and normed interval space are proposed, which are not the conventional metric and normed spaces. The concept of near fixed point is also defined based on the null set. In this case, we shall establish many types of near fixed point theorems in the metric and normed interval spaces.

Keywords: metric interval space; normed interval space; near fixed point; null set; triangle inequality

MSC: 7H10, 54H25

1. Introduction

We denote by \( \mathcal{I} \) the set of all bounded and closed intervals in \( \mathbb{R} \). The element in \( \mathcal{I} \) is denoted by \([a, b]\) where \( a, b \in \mathbb{R} \) with \( a \leq b \). It is clear to see that the real number \( a \in \mathbb{R} \) can also be regarded as an element \([a, a]\) \( \in \mathcal{I} \). The topic of interval analysis has been studied for a long time. The detailed discussion can refer to the monographs [1–5].

Let \( T : \mathcal{I} \rightarrow \mathcal{I} \) be a function from \( \mathcal{I} \) into itself. We say that \([a, b]\) \( \in \mathcal{I} \) is a fixed point if and only if \( T([a, b]) = [a, b] \). The well-known Banach contraction principle presents the fixed point of function \( T \) when \((\mathcal{I}, d)\) is taken to be a metric space. However \((\mathcal{I}, d)\) may not be a metric space for some special kinds of metric \( d \). For example, let us define a nonnegative real-valued function \( d : \mathcal{I} \times \mathcal{I} \rightarrow \mathbb{R}^+ \) by

\[
d([a, b], [c, d]) = |(a + b) - (c + d)|.
\]

Then \((\mathcal{I}, d)\) is not a metric space. In this paper, we shall propose the so-called metric interval space \((\mathcal{I}, d)\) as shown above. The main purpose is to study the so-called near fixed point theorem in metric interval space \((\mathcal{I}, d)\). The concept of near fixed point proposed in this paper is the first attempt for studying the fixed point theory in interval space. Therefore, based on the limited knowledge of author, it seems that there are no appropriate articles that can be cited.

On the other hand, we shall also propose the concept of normed interval space. The Hahn-Banach theorem in normed interval spaces has been studied in Wu [6]. Recall that the (conventional) normed space is based on the vector space by referring to the monographs [7–9]. However, \( \mathcal{I} \) cannot form a real vector space. The main reason is that there is no additive inverse element for each element in \( \mathcal{I} \). Therefore we cannot consider the (conventional) normed space \((\mathcal{I}, \| \cdot \|)\). In this paper, we shall propose the normed interval space \((\mathcal{I}, \| \cdot \|)\) based on the concept of null set, although \( \mathcal{I} \) is not a real vector space. We shall also study the near fixed point theorem in the normed interval space \((\mathcal{I}, \| \cdot \|)\).

In Section 2, since any element in \( \mathcal{I} \) cannot have the additive inverse element, we propose the concept of null set to play the role for interpreting the additive inverse element. In Sections 3 and 4,
we introduce the concepts of metric interval space \((\mathcal{I}, d)\) and normed interval space \((\mathcal{I}, \| \cdot \|)\). We also present many useful properties. In Section 5, we propose the concepts of limit and Cauchy sequences in \((\mathcal{I}, d)\) and \((\mathcal{I}, \| \cdot \|)\). In Sections 6 and 7, based on the concept of null set defined in Section 2, we can define the so-called near fixed point. We also present many kinds of near fixed point theorems in the metric interval space \((\mathcal{I}, d)\) and normed interval space \((\mathcal{I}, \| \cdot \|)\).

2. Interval Space

Let \(\mathcal{I}\) be the set of all closed intervals in \(\mathbb{R}\). The addition is given by
\[
[a, b] \oplus [c, d] = [a + c, b + d],
\]
and the scalar multiplication is given by
\[
k[a, b] = \begin{cases} 
[ka, kb] & \text{if } k \geq 0 \\
[kb, ka] & \text{if } k < 0.
\end{cases}
\]

It is easy to see that \(\mathcal{I}\) is not a (conventional) vector space under the above addition and scalar multiplication. The main reason is that the inverse element does not exist for any non-degenerated closed interval, which will be more clear from the following discussion.

It is clear to see that \([0, 0] \in \mathcal{I}\) is a zero element. However, for any \([a, b] \in \mathcal{I}\), the substraction
\[
[a, b] \ominus [a, b] = [a, b] \oplus (-[a, b]) = [a, b] \oplus [-b, -a] = [a - b, b - a] = [- (b - a), b - a]
\]
is not a zero element. In other words, the inverse element of \([a, b]\) does not exist.

In this paper, instead of considering the zero element, we define the null set as follows
\[
\Omega = \{[a, b] \ominus [a, b] : [a, b] \in \mathcal{I}\}.
\]

It is clear to see that
\[
\Omega = \{-k, k : k \geq 0\}.
\]

We also see that \(\Omega\) is generated by \([-1, 1]\) based on the nonnegative scalar multiplication as shown below
\[
\Omega = \{k[-1, 1] : k \geq 0\}.
\]

In this case, we say that \([-1, 1]\) is a generator of the null set \(\Omega\). Now we have the following observations.

- The distributive law for scalar addition does not hold true in general; that is,
  \[
  (a + \beta) [a, b] \neq a [a, b] \oplus \beta [a, b]
  \]
  for any \([a, b] \in \mathcal{I}\) and \(a, \beta \in \mathbb{R}\).
- The distributive law for positive scalar addition holds true; that is,
  \[
  (a + \beta) [a, b] = a [a, b] \oplus \beta [a, b]
  \]
  for any \([a, b] \in \mathcal{I}\) and \(a, \beta > 0\).
- The distributive law for negative scalar addition holds true; that is,
  \[
  (a + \beta) [a, b] = a [a, b] \oplus \beta [a, b]
  \]
  for any \([a, b] \in \mathcal{I}\) and \(a, \beta < 0\).
For any \([a, b], [c, d], [e, f] \in \mathcal{I}\), we have
\[
[e, f] \ominus ([a, b] \oplus [c, d]) = [e, f] \oplus [a, b] \oplus [c, d] = [e, f] \oplus (-[a, b]) \oplus (-[c, d])
\] (1)

We write \([a, b] \equiv [c, d]\) if and only if there exist \(\omega_1, \omega_2 \in \Omega\) such that
\[
[a, b] + \omega_1 = [c, d] + \omega_2.
\]

It is clear to see that \([a, b] = [c, d]\) implies \([a, b] \equiv [c, d]\) by taking \(\omega_1 = \omega_2 = [0, 0]\). However, the converse does not necessarily hold true. According to the binary relation \(\equiv\), for any \([a, b] \in \mathcal{I}\), we define the class
\[
\langle[a, b]\rangle = \{[c, d] \in \mathcal{I} : [a, b] \equiv [c, d]\}.
\] (2)

The family of all classes \(\langle[a, b]\rangle\) for \([a, b] \in \mathcal{I}\) is denoted by \(\mathcal{I}\).

**Proposition 1.** The binary relation \(\equiv\) is an equivalence relation.

**Proof.** For any \([a, b] \in \mathcal{I}\), since \([0, 0] \in \Omega\), we see that \([a, b] = [a, b]\) implies \([a, b] \equiv [a, b]\), which shows the reflexivity. The symmetry is obvious by the definition of the binary relation \(\equiv\). Regarding the transitivity, for \([a, b] \equiv [c, d]\) and \([c, d] \equiv [e, f]\), we want to claim \([a, b] \equiv [e, f]\). Now we have
\[
[a, b] + \omega_1 = [c, d] + \omega_2 \text{ and } [c, d] + \omega_3 = [e, f] + \omega_4
\]
for some \(\omega_i \in \Omega\) for \(i = 1, \ldots, 4\), which imply
\[
[a, b] + \omega_1 + \omega_3 = [c, d] + \omega_3 + \omega_2 = [e, f] + \omega_4 + \omega_2.
\]

This shows \([a, b] \equiv [e, f]\), since \(\Omega\) is closed under the addition. This completes the proof. \(\square\)

Proposition 1 says that the classes defined in (2) form the equivalence classes. In this case, the family \(\mathcal{I}\) is called the quotient set of \(\mathcal{I}\). We also have that \([c, d] \in \langle[a, b]\rangle\) implies \(\langle[a, b]\rangle = \langle[c, d]\rangle\). In other words, the family of all equivalence classes form a partition of the whole set \(\mathcal{I}\). We also remark that the quotient set \(\mathcal{I}\) is still not a (conventional) vector space. The reason is
\[
(a + b)[a, b] \neq a[a, b] + b[a, b]
\]
for \(a, b < 0\), since \((a + b)[a, b] \neq a[a, b] + b[a, b]\) for \([a, b] \in \mathcal{I}\) and \(a, b < 0\).

### 3. Metric Interval Space

Let \(\mathcal{I}\) be the set of all bounded closed intervals with the null set \(\Omega\). To study the near fixed point, we are going to consider the metric \(d\) defined on \(\mathcal{I} \times \mathcal{I}\).

**Definition 1.** For the nonnegative real-valued function \(d\) defined on \(\mathcal{I} \times \mathcal{I}\), we consider the following conditions:

- (i) \(d([a, b], [c, d]) = 0\) if and only if \([a, b] \equiv [c, d]\) for all \([a, b], [c, d] \in \mathcal{I}\);
- (ii) \(d([a, b], [c, d]) = d([c, d], [a, b])\) for all \([a, b], [c, d] \in \mathcal{I}\);
- (iii) \(d([a, b], [c, d]) \leq d([a, b], [z]) + d([z], [c, d])\) for all \([a, b], [c, d], [z] \in \mathcal{I}\);
- A pair \((\mathcal{I}, d)\) is called a pseudo-metric interval space if and only if \(d\) satisfies conditions (ii) and (iii).
- A pair \((\mathcal{I}, d)\) is called an metric interval space if and only if \(d\) satisfies conditions (i), (ii) and (iii).

We say that \(d\) satisfies the null equalities if and only if the following condition (vi) is satisfied:
(iv) For any \( \omega_1, \omega_2 \in \Omega \) and \([a, b], [c, d] \in \mathcal{I} \), the following three equalities are satisfied:

\[ d([a, b] \oplus \omega_1, [c, d] \oplus \omega_2) = d([a, b], [c, d]), \]

\[ d([a, b] \oplus \omega_1, [c, d]) = d([a, b], [c, d]); \]

\[ d([a, b], [c, d] \oplus \omega_2) = d([a, b], [c, d]). \]

Example 1. Let us define a nonnegative real-valued function \( d : \mathcal{I} \times \mathcal{I} \to \mathbb{R}_+ \) by

\[ d([a, b], [c, d]) = |(a + b) - (c + d)|. \]  

Then \((\mathcal{I}, d)\) is not a (conventional) metric space. However, we are going to claim that \((\mathcal{I}, d)\) is a metric interval space such that \(d\) satisfies the null equality.

(i) We consider the closed intervals \([a, b]\) and \([c, d]\). Then we see that \(a - d \leq b - c\). Therefore,

\[ \text{if } b - c < 0, \text{ then } d([a, b], [c, d]) = |a + b - c - d| \neq 0. \]  

Suppose that

\[ 0 = d([a, b], [c, d]) = |a + b - c - d|. \]

We are going to claim \([a, b] \equiv [c, d]\). From (4), we must have \(b - c \geq 0\). Now we also have \(a + b = c + d\), i.e., \(a + c - d = 2c - b\). It is easy to see that \(a + c - d \leq b + d - c\) and \(2c - b \leq b + d - c\) by using the facts that \(a \leq b\), \(c \leq d\) and \(b \geq c\). Therefore we can form two identical closed intervals

\[ [a + c - d, b + d - c] = [2c - b, b + d - c]. \]

Now the closed intervals \([a + c - d, b + d - c]\) and \([2c - b, b + d - c]\) can be written as

\[ [a + c - d, b + d - c] = [a, b] \oplus [c - d, d - c] \]  

and

\[ [2c - b, b + d - c] = [c, d] \oplus [c - b, b - c]. \]  

Let

\[ \omega_1 = [c - d, d - c] = (d - c)[-1, 1] \in \Omega \]

and

\[ \omega_2 = [c - b, b - c] = (b - c)[-1, 1] \in \Omega. \]

Therefore, from (5) and (6), we obtain

\[ [a, b] \oplus \omega_1 = [c, d] \oplus \omega_2, \]

which shows \([a, b] \equiv [c, d]\), since \(\omega_1, \omega_2 \in \Omega\). Conversely, suppose that \([a, b] \equiv [c, d]\). Then

\[ [a, b] \oplus \omega_1 = [c, d] \oplus \omega_2, \]

where \(\omega_1 = [-k_1, k_1], \omega_2 = [-k_2, k_2] \in \Omega\) for some positive \(k_1\) and \(k_2\). Therefore we have

\[ [a - k_1, b + k_1] = [c - k_2, d + k_2], \]

i.e., \(a - k_1 = c - k_2\) and \(b + k_1 = d + k_2\). Then we obtain

\[ d([a, b], [c, d]) = |(a - c) + (b - d)| = |(k_1 - k_2) + (k_2 - k_1)| = 0. \]

(ii) We have

\[ d([a, b], [c, d]) = |a + b - c - d| = |c + d - a - b| = d([c, d], [a, b]). \]
We say that

\[ d([a, b], [c, d]) = |a + b - c - d| = |(a + b - e - f) + (e + f - c - d)| \]

\[ \leq |a + b - e - f| + |e + f - c - d| 
= d([a, b], [e, f]) + d([e, f], [c, d]). \]

(iv) For any \([a, b], [c, d] \in \mathcal{I}\) and \(k_1, k_2 \in \mathbb{R}_+, i.e., [-k_1, k_1], [-k_2, k_2] \in \Omega\), we have

\[ d([a, b] \oplus [-k_1, k_1], [c, d] \oplus [-k_2, k_2]) = d([a - k_1, b + k_1], [c - k_2, d + k_2]) \]
\[ = |(a - k_1 + b + k_1) - (c - k_2 + d + k_2)| = |(a + b) - (c + d)| = d([a, b], [c, d]). \]

The verification is complete.

4. Normed Interval Space

Different kinds of normed interval spaces are proposed below. We shall also study the near fixed point theorems in the normed interval spaces.

Definition 2. Given a nonnegative real-valued function \(|·|: \mathcal{I} \to \mathbb{R}_+\), we consider the following conditions:

(i) \(|a, b| = |a| \cdot |b|\) for any \([a, b] \in \mathcal{I}\) and \(\alpha \in \mathbb{F}\);
(ii) \(|a, b| \leq |a| + |b|\) for any \([a, b] \in \mathcal{I}\) and \(\alpha \in \mathbb{F}\) with \(\alpha \neq 0\);
(iii) \(|a, b| = 0\) implies \([a, b] \in \Omega\).

We say that \(|·|\) satisfies the null condition when condition (iii) is replaced by \(|a, b| = 0\) if and only if \([a, b] \in \Omega\).

• We say that \((\mathcal{I}, |·|)\) is a pseudo-seminormed interval space if and only if conditions (i) and (ii) are satisfied.
• We say that \((\mathcal{I}, |·|)\) is a seminormed interval space if and only if conditions (i) and (ii) are satisfied.
• We say that \((\mathcal{I}, |·|)\) is a pseudo-normed interval space if and only if conditions (i), (ii) and (iii) are satisfied.
• We say that \((\mathcal{I}, |·|)\) is a normed interval space if and only if conditions (i), (ii) and (iii) are satisfied.

We also consider the following definitions:

• We say that \(|·|\) satisfies the null super-inequality if and only if \(|a, b| \ominus \omega |\geq|| [a, b]|\) for any \([a, b] \in \mathcal{I}\) and \(\omega \in \Omega\).
• We say that \(|·|\) satisfies the null sub-inequality if and only if \(|a, b| \ominus \omega |\leq|| [a, b]|\) for any \([a, b] \in \mathcal{I}\) and \(\omega \in \Omega\).
• We say that \(|·|\) satisfies the null equality if and only if \(|a, b| \ominus \omega |\leq|| [a, b]|\) for any \([a, b] \in \mathcal{I}\) and \(\omega \in \Omega\).

Example 2. Let us define a nonnegative real-valued function \(|·|\) on \(\mathcal{I}\) by

\[ |a, b| = |a + b|. \]

Then \((\mathcal{I}, |·|)\) is a normed interval space such that the norm \(|·|\) satisfies the null equality.

Proposition 2. Let \((\mathcal{I}, |·|)\) be a pseudo-seminormed interval space such that \(|·|\) satisfies the null super-inequality. For any \([a, b], [e, f], [c_1, d_1], \cdots, [c_m, d_m] \in \mathcal{I}\), we have

\[ |[a, b] \ominus [e, f]| \leq |[a, b] \ominus [c_1, d_1]| + |[c_1, d_1] \ominus [c_2, d_2]| + \cdots 
+ |[c_{j-1}, d_{j-1}] \ominus [c_j, d_j]| + \cdots + |[c_m, d_m] \ominus [e, f]|. \]
Proof. Since \([c_j, d_j] \oplus (-[c_j, d_j]) \in \Omega\) for \(j = 1, \cdots, m\), we have

\[
\| [a, b] \oplus [e, f] \| \leq \| [a, b] \oplus (-[e, f]) \oplus [c_1, d_1] \oplus \cdots \oplus [c_m, d_m] \oplus (-[c_1, d_1]) \oplus \cdots \oplus (-[c_m, d_m]) \|
\]

(using the null super-inequality for \(m\) times)

\[
= \| [a, b] \oplus (-[c_1, d_1]) \oplus [c_1, d_1] \oplus (-[c_2, d_2]) \oplus \cdots \oplus [c_m, d_m] \oplus (-[c_1, d_1]) \oplus \cdots \oplus (-[c_m, d_m]) \|
\]

\[
+ [c_m, d_m] \oplus (-[e, f]) \|
\]

\[
\leq \| [a, b] \| \oplus \| [c_1, d_1] \| \| [c_1, d_1] \| + \| [c_2, d_2] \| \| [c_1, d_1] \| + \cdots + \| [c_m, d_m] \| \| [c_1, d_1] \| + \cdots
\]

\[
+ \| [c_m, d_m] \| \| [c, f] \| \| [c_1, d_1] \| (\text{using the triangle inequality}).
\]

This completes the proof. \(\square\)

**Proposition 3.** The following statements hold true.

(i) Let \((\mathcal{I}, \| \|)\) be a pseudo-seminormed interval space such that \(\| \cdot \|\) satisfies the null equality. For any \([a, b], [c, d] \in \mathcal{I}\), if \([a, b] \Omega [c, d]\), then \(\| [a, b] \| = \| [c, d] \|\).

(ii) Let \((\mathcal{I}, \| \|)\) be a pseudo-normed interval space. For any \([a, b], [c, d] \in \mathcal{I}\), \(\| [a, b] \| \oplus [c, d] = 0 \Rightarrow \| [a, b] \| = \| [c, d] \|\).

(iii) Let \((\mathcal{I}, \| \|)\) be a pseudo-seminormed interval space such that \(\| \cdot \|\) satisfies the null super-inequality and null condition. For any \([a, b], [c, d] \in \mathcal{I}\), \([a, b] \Omega [c, d]\) implies \(\| [a, b] \| \oplus [c, d] \| = 0\).

**Proof.** To prove part (i), we see that \([a, b] \Omega [c, d]\) implies \([a, b] \oplus \omega_1 = [c, d] \oplus \omega_2\) for some \(\omega_1, \omega_2 \in \Omega\). Therefore, using the null equality, we have

\[
\| [a, b] \| = \| [a, b] \oplus \omega_1 \| = \| [c, d] \oplus \omega_2 \| = \| [c, d] \|.
\]

To prove part (ii), suppose that \(\| [a, b] \| \oplus [c, d] \| \neq 0\). Then \([a, b] \oplus [c, d] \in \Omega\), i.e., \([a, b] \oplus [c, d] \oplus \omega_1 \in \Omega\) for some \(\omega_1 \in \Omega\). By adding \([c, d] \oplus \omega_1\) on both sides, we have \([a, b] \oplus \omega_2 = [c, d] \oplus \omega_1 \oplus \omega_2 \in \Omega\), which says that \([a, b] \Omega [c, d]\).

To prove part (iii), for \([a, b] \Omega [c, d]\), we have \([a, b] \oplus \omega_1 = [c, d] \oplus \omega_2\) for some \(\omega_1, \omega_2 \in \Omega\). Therefore

\[
[a, b] \oplus [c, d] \oplus \omega_1 = [c, d] \oplus (-[c, d]) \oplus \omega_2 = \omega_3
\]

for some \(\omega_3 \in \Omega\). Using the null super-inequality, null condition and (7), we have

\[
\| [a, b] \oplus [c, d] \| \leq \| [a, b] \oplus [c, d] \oplus \omega_1 \| = \| \omega_3 \| = 0.
\]

This completes the proof. \(\square\)

**5. Cauchy Sequences**

In this section, we are going to introduce the concepts of Cauchy sequences and completeness in the metric interval space and normed interval space.

**5.1. Cauchy Sequences in Metric Interval Space**

We first introduce the concept of limit in the metric interval space.

**Definition 3.** Let \((\mathcal{I}, d)\) be a pseudo-metric interval space. The sequence \(\{[a_n, b_n]\}_{n=1}^{\infty}\) in \(\mathcal{I}\) is said to be convergent if and only if

\[
\lim_{n \to \infty} d([a_n, b_n], [a, b]) = 0 \text{ for some } [a, b] \in \mathcal{I}.
\]

The element \([a, b]\) is called the limit of the sequence \(\{[a_n, b_n]\}_{n=1}^{\infty}\).
Let \( \{[a_n, b_n]\}_{n=1}^{\infty} \) be a sequence in \((I, d)\). If there exist \([a, b], [c, d] \in I\) such that

\[
\lim_{n \to \infty} d([a_n, b_n], [a, b]) = 0 = \lim_{n \to \infty} d([a_n, b_n], [c, d]).
\]

Then, by the triangle inequality (iii) in Definition 1, we have

\[
0 \leq d([a, b], [c, d]) \leq d([a, b], [a_n, b_n]) + d([a_n, b_n], [c, d]) \to 0 + 0 = 0 \text{ as } n \to \infty,
\]

which says that \(d([a, b], [c, d]) = 0\). By condition (i) in Definition 1, we see that \([a, b] = [c, d]\), which also says that \([c, d]\) is in the equivalence class \(\langle [a, b] \rangle\).

**Proposition 4.** Suppose that \(d\) satisfies the null equality (iv) in Definition 1. Let \(\{[a_n, b_n]\}_{n=1}^{\infty}\) be a sequence in \(I\) satisfying \(d([a_n, b_n], [a, b]) \to 0\) as \(n \to \infty\). Then \(d([a_n, b_n], [c, d]) \to 0\) as \(n \to \infty\) for any \([c, d] \in \langle [a, b] \rangle\).

**Proof.** For \([c, d] \in \langle [a, b] \rangle\), we have \([a, b] \oplus \omega_1 = [c, d] \oplus \omega_2\) for some \(\omega_1, \omega_2 \in \Omega\). Using the null equality, we obtain

\[
0 \leq d([a_n, b_n], [c, d]) = d([a_n, b_n], \omega_1 \oplus [c, d]) = d([a_n, b_n], \omega_1 \oplus [a, b])
\]

\[
= d([a_n, b_n], [a, b]) \to 0 \text{ as } n \to \infty.
\]

This completes the proof. \(\Box\)

Inspired by the above result, we propose the following definition.

**Definition 4.** If \(\{[a_n, b_n]\}_{n=1}^{\infty}\) is a sequence in \(I\) satisfying

\[
\lim_{n \to \infty} d([a_n, b_n], [a, b]) = 0
\]

for some \([a, b] \in I\), then the equivalence class \(\langle [a, b] \rangle\) is called the class limit of \(\{[a_n, b_n]\}_{n=1}^{\infty}\). We also write

\[
\lim_{n \to \infty} [a_n, b_n] = \langle [a, b] \rangle \text{ or } [a_n, b_n] \to \langle [a, b] \rangle.
\]

**Proposition 5.** The class limit in the metric interval space \((I, d)\) is unique.

**Proof.** Suppose that the sequence \(\{[a_n, b_n]\}_{n=1}^{\infty}\) is convergent with the class limits \(\langle [a, b] \rangle\) and \(\langle [c, d] \rangle\). Then we have

\[
\lim_{n \to \infty} d([a_n, b_n], [a, b]) = 0 \text{ and } \lim_{n \to \infty} d([a_n, b_n], [c, d]) = 0,
\]

which says that \(d([a, b], [c, d]) = 0\) by referring to (8). Therefore we obtain \([c, d] \in \langle [a, b] \rangle\), i.e., \(\langle [a, b] \rangle = \langle [c, d] \rangle\). This completes the proof. \(\Box\)

**Definition 5.** Let \((I, d)\) be a metric interval space.

- A sequence \(\{[a_n, b_n]\}_{n=1}^{\infty}\) in \(I\) is called a Cauchy sequence if and only if, given any \(\varepsilon > 0\), there exists \(N \in \mathbb{N}\) such that \(d([a_n, b_n], [a_m, b_m]) < \varepsilon\) for all \(n > N\) and \(m > N\).
- A subset \(M\) of \(I\) is said to be complete if and only if every Cauchy sequence in \(M\) is convergent to some element in \(M\).

The following result is not hard to prove.

**Proposition 6.** Every convergent sequence in a metric interval space is a Cauchy sequence.
Example 3. Continued from Example 1, we see that $d$ satisfies the null equality, where the metric $d$ is defined in (3). Now we want to show that this space is also complete. Suppose that $\{[a_n, b_n]\}_{n=1}^{\infty}$ is a Cauchy sequence in the metric interval space $(I, d)$. Then we have

$$d([a_n, b_n], [a_m, b_m]) = |(a_n + b_n) - (a_m + b_m)| < \epsilon$$

for sufficiently large $n$ and $m$. Let $c_n = a_n + b_n$. Then (9) shows that $\{c_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, there exists $c \in \mathbb{R}$ such that $|c_n - c| < \epsilon$ for sufficiently large $n$. Now we can define a closed interval $[a, b]$ such that $a + b = c$. Therefore we have

$$d([a_n, b_n], [a, b]) = |(a_n + b_n) - (a + b)| = |c_n - c| < \epsilon$$

for sufficiently large $n$. This shows that the sequence $\{[a_n, b_n]\}_{n=1}^{\infty}$ is convergent, i.e., the space $(I, d)$ is complete.

5.2. Cauchy Sequences in Normed Interval Space

Let $(I, \| \cdot \|)$ be a pseudo-seminormed interval space. Given a sequence $\{[a_n, b_n]\}_{n=1}^{\infty}$ in $I$, it is clear that

$$\| [a_n, b_n] \oplus [a, b] \| = \| [a, b] \oplus [a_n, b_n] \|.$$  

Therefore the concept of convergence is defined below.

Definition 6. Let $(I, \| \cdot \|)$ be a pseudo-seminormed interval space. A sequence $\{[a_n, b_n]\}_{n=1}^{\infty}$ in $I$ is said to converge to $[a, b] \in I$ if and only if

$$\lim_{n \to \infty} \| [a_n, b_n] \oplus [a, b] \| = \lim_{n \to \infty} \| [a, b] \oplus [a_n, b_n] \| = 0.$$  

Proposition 7. Let $(I, \| \cdot \|)$ be a pseudo-normed interval space with the null set $\Omega$.

(i) Suppose that $\| \cdot \|$ satisfies the null super-inequality. If the sequence $\{[a_n, b_n]\}_{n=1}^{\infty}$ in $(I, \| \cdot \|)$ converges to $[a, b]$ and $[c, d]$ simultaneously, then $([a, b]) = ([c, d])$.

(ii) Suppose that $\| \cdot \|$ satisfies the null equality. If the sequence $\{[a_n, b_n]\}_{n=1}^{\infty}$ in $(I, \| \cdot \|)$ converges to $[a, b] \in I$, then, give any $[c, d] \in ([a, b])$, the sequence $\{[a_n, b_n]\}_{n=1}^{\infty}$ converges to $[c, d]$.

Proof. To prove part (i), we have

$$\lim_{n \to \infty} \| [a_n, b_n] \oplus [a, b] \| = \lim_{n \to \infty} \| [a_n, b_n] \oplus [c, d] \| = 0.$$  

By Proposition 2, we have

$$0 \leq \| [a, b] \oplus [c, d] \| \leq \| [a, b] \oplus [a_n, b_n] \| + \| [a_n, b_n] \oplus [c, d] \| \to 0 + 0 = 0,$$  

which says that $\| [a, b] \oplus [c, d] \| = 0$. By Definition 2, we see that $[a, b] \oplus [c, d] \in \Omega$, i.e., $[a, b] \equiv [c, d]$, which also says that $[c, d]$ is in the equivalence class $\langle [a, b] \rangle$.

To prove part (ii), for any $[c, d] \in ([a, b])$, i.e., $[a, b] \oplus \omega_1 = [c, d] \oplus \omega_2$ for some $\omega_1, \omega_2 \in \Omega$, using the null equality, we have

$$0 \leq \| [a_n, b_n] \oplus [c, d] \| = \| [c, d] \oplus [a_n, b_n] \| = \| \omega_2 \oplus [c, d] \oplus [a_n, b_n] \| = \| \omega_1 \oplus [a, b] \oplus [a_n, b_n] \| = \| [a, b] \oplus [a_n, b_n] \| \to 0.$$  

This completes the proof. \[\square\]

Inspired by part (ii) of Proposition 7, we propose the following concept of limit.
Definition 7. Let \((\mathcal{I}, \parallel \cdot \parallel)\) be a pseudo-seminormed interval space. If the sequence \(\{[a_n, b_n]\}_{n=1}^{\infty}\) in \(\mathcal{I}\) converges to some \([a, b] \in \mathcal{I}\), then the equivalence class \(\langle[a, b]\rangle\) is called the class limit of \(\{[a_n, b_n]\}_{n=1}^{\infty}\). We also write
\[
\lim_{n \to \infty} [a_n, b_n] = \langle[a, b]\rangle \text{ or } [a_n, b_n] \to \langle[a, b]\rangle.
\]
We need to remark that if \(\langle[a, b]\rangle\) is a class limit and \([c, d] \in \langle[a, b]\rangle\) then it is not necessarily that the sequence \(\{[a_n, b_n]\}_{n=1}^{\infty}\) converges to \([c, d]\) unless \(\parallel \cdot \parallel\) satisfies the null equality. In other words, for the class limit \(\langle[a, b]\rangle\), if \(\parallel \cdot \parallel\) satisfies the null equality, then part (ii) of Proposition 7 says that sequence \(\{[a_n, b_n]\}_{n=1}^{\infty}\) converges to \([c, d]\) for any \([c, d] \in \langle[a, b]\rangle\).

Proposition 8. Let \((\mathcal{I}, \parallel \cdot \parallel)\) be a pseudo-normed interval space such that \(\parallel \cdot \parallel\) satisfies the null super-inequality. Then the class limit is unique.

Proof. Suppose that the sequence \(\{[a_n, b_n]\}_{n=1}^{\infty}\) is convergent with the class limits \(\langle[a, b]\rangle\) and \(\langle[c, d]\rangle\). Then, by definition, we have
\[
\lim_{n \to \infty} \parallel [a, b] \ominus [a_n, b_n] \parallel = \lim_{n \to \infty} \parallel [c, d] \ominus [a_n, b_n] \parallel = 0,
\]
which says that \(\parallel [a, b] \ominus [c, d] \parallel = 0\) by referring to (10). By part (ii) of Proposition 3, we have \([a, b] \equiv [c, d]\), i.e., \(\langle[a, b]\rangle = \langle[c, d]\rangle\). This completes the proof. \(\square\)

Definition 8. Let \((\mathcal{I}, \parallel \cdot \parallel)\) be a pseudo-seminormed interval space. A sequence \(\{[a_n, b_n]\}_{n=1}^{\infty}\) in \(\mathcal{I}\) is called a Cauchy sequence if and only if, given any \(\varepsilon > 0\), there exists \(N \in \mathbb{N}\) such that
\[
\parallel [a_n, b_n] \ominus [a_m, b_m] \parallel = \parallel [a_m, b_m] \ominus [a_n, b_n] \parallel < \varepsilon
\]
for \(m, n > N\) with \(m \neq n\). If every Cauchy sequence in \(\mathcal{I}\) is convergent, then we say that \(\mathcal{I}\) is complete.

Proposition 9. Let \((\mathcal{I}, \parallel \cdot \parallel)\) be a pseudo-seminormed interval space such that \(\parallel \cdot \parallel\) satisfies the null super-inequality. Every convergent sequence is a Cauchy sequence.

Proof. If \(\{[a_n, b_n]\}_{n=1}^{\infty}\) is a convergent sequence, then, given any \(\varepsilon > 0\), \(\parallel [a_n, b_n] \ominus [a, b] \parallel = \parallel [a, b] \ominus [a_n, b_n] \parallel < \varepsilon / 2\) for sufficiently large \(n\). Therefore, by Proposition 2, we have
\[
\parallel [a_n, b_n] \ominus [a_m, b_m] \parallel = \parallel [a_m, b_m] \ominus [a_n, b_n] \parallel \\
\leq \parallel [a_m, b_m] \ominus [a, b] \parallel + \parallel [a, b] \ominus [a_n, b_n] \parallel < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\]
for sufficiently large \(n\) and \(m\), which says that \(\{[a_n, b_n]\}_{n=1}^{\infty}\) is a Cauchy sequence. This completes the proof. \(\square\)

Definition 9. Different kinds of Banach interval spaces are defined below.

- Let \((\mathcal{I}, \parallel \cdot \parallel)\) be a pseudo-seminormed interval space. If \(\mathcal{I}\) is complete, then it is called a pseudo-semi-Banach interval space.
- Let \((\mathcal{I}, \parallel \cdot \parallel)\) be a seminormed interval space. If \(\mathcal{I}\) is complete, then it is called an informal semi-Banach interval space.
- Let \((\mathcal{I}, \parallel \cdot \parallel)\) be a pseudo-normed interval space. If \(\mathcal{I}\) is complete, then it is called a pseudo-Banach interval space.
- Let \((\mathcal{I}, \parallel \cdot \parallel)\) be a normed interval space. If \(\mathcal{I}\) is complete, then it is called a Banach interval space.
Example 4. Continued from Example 2, we want to show that \((I, \| \cdot \|)\) is complete. Suppose that \(\{[a_n, b_n]\}\)

is a Cauchy sequence in \((I, \| \cdot \|)\). Then we have

\[
\| [a_n, b_n] \ominus [a_m, b_m] \| = \| [a_n, b_n] \oplus [-b_m, -a_m] \| = \| [a_n - b_m, b_n - a_m] \| = \| (a_n + b_n) - (a_m + b_m) \| < \epsilon
\]

for sufficiently large \(n\) and \(m\). Let \(c_n = a_n + b_n\). Then (11) shows that \(\{c_n\}_{n=1}^{\infty}\) is a Cauchy sequence in \(\mathbb{R}\).

Since \(\mathbb{R}\) is complete, there exists \(c \in \mathbb{R}\) such that \(|c_n - c| < \epsilon\) for sufficiently large \(n\). Now we can define a closed interval \([a, b]\) such that \(a + b = c\). Therefore we have

\[
\| [a_n, b_n] \ominus [a, b] \| = \| [a_n, b_n] \oplus [-b, -a] \| = \| [a_n - b, b_n - a] \| = |(a_n + b_n) - (a + b)| = |c_n - c| < \epsilon
\]

for sufficiently large \(n\). This shows that the sequence \(\{[a_n, b_n]\}\) is convergent, i.e., \((I, \| \cdot \|)\) is a Banach interval space.

6. Near Fixed Point Theorem in Metric Interval Space

Let \(T : I \to I\) be a function from \(I\) into itself. We say that \([a, b] \in I\) is a fixed point if and only if \(T([a, b]) = [a, b]\). The well-known Banach contraction principle presents the fixed point of function \(T\) when \((I, d)\) is taken to be a metric space. Since \((I, d)\) presented in Example 1 is not a metric space, we cannot study the Banach contraction principle on this space \((I, d)\). In other words, we cannot study the fixed point of contractive mappings defined on \((I, d)\) into itself in the conventional way. However, we shall investigate the so-called near fixed point defined below.

Definition 10. Let \(T : I \to I\) be a function defined on \(I\) into itself. A point \([a, b] \in I\) is called a near fixed point of \(T\) if and only if \(T([a, b]) \supseteq [a, b]\).

By definition, we see that \(T([a, b]) \supseteq [a, b]\) if and only if there exist \(k_1, k_2 \in \mathbb{R}_+\) such that one of the following equalities is satisfied:

- \(T([a, b]) \oplus [-k_1, k_1] = [a, b]\);
- \(T([a, b]) = [a, b] \oplus [-k_1, k_1]\);
- \(T([a, b]) \oplus [-k_1, k_1] = [a, b] \oplus [-k_2, k_2]\),

where \([-k_1, k_1]\) and \([-k_2, k_2]\) are in the null set \(\Omega\).

Definition 11. A function \(T : (I, d) \to (I, d)\) is called a metric contraction on \(I\) if and only if there is a real number \(0 < \alpha < 1\) such that

\[
d(T([a, b]), T([c, d])) \leq \alpha \cdot d([a, b], [c, d])
\]

for any \([a, b], [c, d] \in I\).

Given any initial element \([a_0, b_0] \in I\), we define the iterative sequence \(\{[a_n, b_n]\}_{n=1}^{\infty}\) using the function \(T\) as follows:

\[
[a_1, b_1] = T([a_0, b_0]), \quad [a_2, b_2] = T([a_1, b_1]) = T^2([a_0, b_0]), \ldots, [a_n, b_n] = T^n([a_0, b_0]).
\]

Under some suitable conditions, we are going to show that the sequence \(\{[a_n, b_n]\}_{n=1}^{\infty}\) can converge to a near fixed point. If the metric interval space \((I, d)\) is complete, then it is also called a complete metric interval space.
**Theorem 1.** (Near Fixed Point Theorem) Let \((\mathcal{I}, d)\) be a complete metric interval space such that \(d\) satisfies the null equality. Suppose that the function \(T : (\mathcal{I}, d) \rightarrow (\mathcal{I}, d)\) is a metric contraction on \(\mathcal{I}\). Then \(T\) has a near fixed point \([a, b] \in \mathcal{I}\) satisfying \(T([a, b]) \subseteq [a, b]\). Moreover, the near fixed point \([a, b]\) is obtained by the limit

\[
d([a_n, b_n], [a, b]) \to 0 \text{ as } n \to \infty
\]

in which the sequence \([a_n, b_n]_{n=1}^{\infty}\) is generated according to (12). We also have the following properties.

- The uniqueness is in the sense that there is a unique equivalence class \([a, b]\) such that any \([\bar{a}, \bar{b}] \notin [a, b]\) cannot be a near fixed point.
- Each point \([a, b] \in [a, b]\) is also a near fixed point of \(T\) satisfying \(T([a, b]) \subseteq [a, b]\) and \([a, b] = [a, b]\).
- If \([a, b]\) is a near fixed point of \(T\), then \([a, b] \in [a, b]\), i.e., \([a, b] = [a, b]\). Equivalently, if \([a, b]\) and \([a, b]\) are the near fixed points of \(T\), then \([a, b] = [a, b]\).

**Proof.** Given any initial element \([a_0, b_0] \in \mathcal{I}\), we have the iterative sequence \([a_n, b_n]_{n=1}^{\infty}\) according to (12). We are going to show that \([a_n, b_n]_{n=1}^{\infty}\) is a Cauchy sequence. Since \(T\) is a metric contraction on \(\mathcal{I}\), we have

\[
d((a_{m+1}, b_{m+1}), (a_m, b_m)) = d(T([a_m, b_m]), T([a_{m-1}, b_{m-1}]))
\]

\[
\leq \alpha \cdot d([a_m, b_m], [a_{m-1}, b_{m-1}])
\]

\[
= \alpha \cdot d(T([a_{m-1}, b_{m-1}]), T([a_{m-2}, b_{m-2}]))
\]

\[
\leq \alpha^2 \cdot d([a_{m-1}, b_{m-1}], [a_{m-2}, b_{m-2}])
\]

\[
\leq \cdots \leq \alpha^m \cdot d([a_1, b_1], [a_0, b_0]).
\]

For \(n < m\), using the triangle inequality, we obtain

\[
d([a_m, b_m], [a_n, b_n])
\]

\[
\leq d([a_m, b_m], [a_{m-1}, b_{m-1}]) + d([a_{m-1}, b_{m-1}], [a_{m-2}, b_{m-2}]) + \cdots + d([a_{n+1}, b_{n+1}], [a_n, b_n])
\]

\[
\leq (\alpha^{m-1} + \alpha^{m-2} + \cdots + \alpha^n) \cdot d([a_1, b_1], [a_0, b_0])
\]

\[
= \alpha^n \cdot \frac{1 - \alpha^{m-n}}{1 - \alpha} \cdot d([a_1, b_1], [a_0, b_0]).
\]

Since \(0 < \alpha < 1\), we have \(1 - \alpha^{m-n} < 1\) in the numerator, which says that

\[
d([a_m, b_m], [a_n, b_n]) \leq \frac{\alpha^n}{1 - \alpha} \cdot d([a_1, b_1], [a_0, b_0]) \to 0 \text{ as } n \to \infty.
\]

This proves that \([a_n, b_n]_{n=1}^{\infty}\) is a Cauchy sequence.

Since the metric interval space \(\mathcal{I}\) is complete, there exists \([a, b] \in \mathcal{I}\) such that \(d([a_n, b_n], [a, b]) \to 0\), i.e., \([a_n, b_n] \rightarrow [a, b]\) according to Definition 4 and Proposition 5.

Assume further that \(d\) satisfies the null equality. We are going to show that any point \([\bar{a}, \bar{b}] \in [a, b]\) is a near fixed point. Now we have

\[
[\bar{a}, \bar{b}] \oplus \omega_1 = [a, b] \oplus \omega_2 \text{ for some } \omega_1, \omega_2 \in \Omega.
\]
Therefore we obtain
\[
d([\bar{a}, \bar{b}], T([\bar{a}, \bar{b}])) = d([\bar{a}, \bar{b}] \oplus \omega_1, T([\bar{a}, \bar{b}])) \text{ (since } d \text{ satisfies the null equality)}
\]
\[
\leq d([\bar{a}, \bar{b}] \oplus \omega_1, [a_m, b_m]) + d([a_m, b_m], T([\bar{a}, \bar{b}])) \text{ (using the triangle inequality)}
\]
\[
= d([\bar{a}, \bar{b}] \oplus \omega_1, [a_m, b_m]) + d(T([a_m-1, b_m-1]), T([\bar{a}, \bar{b}]))
\]
\[
\leq d([\bar{a}, \bar{b}] \oplus \omega_1, [a_m, b_m]) + \alpha \cdot d([a_m-1, b_m-1], [a, b]) \text{ (using the metric contraction)}
\]
\[
= d([\bar{a}, \bar{b}] \oplus \omega_1, [a_m, b_m]) + \alpha \cdot d([a_m-1, b_m-1], [a, b] \oplus \omega_1) \text{ (since } d \text{ satisfies the null equality)}
\]
\[
= d([\bar{a}, \bar{b}] \oplus \omega_1, [a_m, b_m]) + \alpha \cdot d([a_m-1, b_m-1], [a, b] \oplus \omega_2) \text{ (using (13))}
\]
\[
= d([\bar{a}, \bar{b}], [a_m, b_m]) + \alpha \cdot d([a_m-1, b_m-1], [a, b]) \text{ (since } d \text{ satisfies the null equality)},
\]

which implies \(d([\bar{a}, \bar{b}], T([\bar{a}, \bar{b}])) = 0\) as \(m \to \infty\), i.e., \(T([\bar{a}, \bar{b}]) \sqsubseteq [\bar{a}, \bar{b}]\) for any point \([\bar{a}, \bar{b}] \in ([a, b]).\)

Now assume that there is another near fixed point \([a, b]\) of \(T\) with \([a, b] \notin ([a, b]), \text{i.e.}, [a, b] \not\sqsubseteq T([\bar{a}, \bar{b}]).\) Then
\[
[a, b] \oplus \omega_1 = T([\bar{a}, \bar{b}]) \oplus \omega_2 \text{ and } [a, b] \oplus \omega_2 = T([a, b]) \oplus \omega_3
\]
for some \(\omega_i \in \Omega, i = 1, \ldots, 4\). Since \(T\) is a metric contraction on \(\mathcal{I}\) and \(d\) satisfies the null equality, we obtain
\[
d([a, b], [a, b]) = d([a, b] \oplus \omega_1, [a, b] \oplus \omega_2) = d(T([\bar{a}, \bar{b}]) \oplus \omega_2, T([a, b]) \oplus \omega_3)
\]
\[
= d(T([\bar{a}, \bar{b}]), T([a, b])) \leq \alpha \cdot d([a, b], [a, b]).
\]

Since \(0 < \alpha < 1\), we must have \(d([a, b], [a, b]) = 0\), i.e., \([a, b] \not\sqsubseteq [a, b]\), which contradicts \([a, b] \notin ([a, b]).\) Therefore, any \([a, b] \notin ([a, b])\) cannot be a near fixed point. Equivalently, if \([a, b]\) is a near fixed point of \(T\), then \([a, b] \in ([a, b]).\) This completes the proof. \(\square\)

**Definition 12.** A function \(T: (\mathcal{I}, d) \to (\mathcal{I}, d)\) is called a weakly strict metric contraction on \(\mathcal{I}\) if and only if the following conditions are satisfied:

- \([a, b] \sqsubseteq [c, d], \text{i.e.}, ([a, b]) = ([c, d])\) implies \(d(T([a, b]), T([c, d])) = 0\);
- \([a, b] \not\sqsubseteq [c, d], \text{i.e.}, ([a, b]) \neq ([c, d])\) implies \(d(T([a, b]), T([c, d])) < d([a, b], [c, d]).\)

It is clear that if \(T\) is a metric contraction on \(\mathcal{I}\), then it is also a weakly strict metric contraction on \(\mathcal{I}\).

**Theorem 2.** (Near Fixed Point Theorem) Let \((\mathcal{I}, d)\) be a complete metric interval space. Suppose that the function \(T: (\mathcal{I}, d) \to (\mathcal{I}, d)\) is a weakly strict metric contraction on \(\mathcal{I}\). If \(\{T^n([a_0, b_0])\}_{n=1}^{\infty}\) forms a Cauchy sequence for some \([a_0, b_0] \in \mathcal{I}\), then \(T\) has a near fixed point \([a, b] \in \mathcal{I}\) satisfying \(T([a, b]) \sqsubseteq [a, b]\). Moreover, the near fixed point \([a, b]\) is obtained by the limit
\[
d(T^n([a_0, b_0]), [a, b]) \to 0 \text{ as } n \to \infty.
\]

Assume further that \(d\) satisfies the null equality. Then we also have the following properties.

- The uniqueness is in the sense that there is a unique equivalence class \(([a, b])\) such that any \([a, b] \notin ([a, b])\) cannot be a near fixed point.
- Each point \([a, b] \in ([a, b])\) is also a near fixed point of \(T\) satisfying \(T([a, b]) \sqsubseteq [a, b]\) and \(([a, b]) = ([a, b]).\)
- If \([a, b]\) is a near fixed point of \(T\), then \([a, b] \in ([a, b]), \text{i.e., } ([a, b]) = ([a, b]).\) Equivalently, if \([a, b]\) and \([a, b]\) are the near fixed points of \(T\), then \([a, b] \sqsubseteq [a, b].\)

**Proof.** Since \(\{T^n([a_0, b_0])\}_{n=1}^{\infty}\) is a Cauchy sequence, the completeness says that there exists \([a, b] \in \mathcal{I}\) such that \(d(T^n([a_0, b_0]), [a, b]) \to 0\), i.e., \(T^n([a_0, b_0]) \to ([a, b])\) according to Definition 4 and
Proposition 5. Therefore, given any \( \epsilon > 0 \), there exists an integer \( N \) such that \( d(T^n([a_0, b_0]), [a, b]) < \epsilon \) for \( n \geq N \). Since \( T \) is a weakly strict metric contraction on \( \mathcal{I} \), we consider the following two cases.

- Suppose that \( T^n([a_0, b_0]) \ominus [a, b] \). Then
  \[
  d(T^{n+1}([a_0, b_0]), T([a, b])) = 0 < \epsilon.
  \]

- Suppose that \( T^n([a_0, b_0]) \not\subseteq [a, b] \). Then
  \[
  d(T^{n+1}([a_0, b_0]), T([a, b])) < d(T^n([a_0, b_0]), [a, b]) < \epsilon \text{ for } n \geq N.
  \]

The above two cases say that \( d(T^{n+1}([a_0, b_0]), T([a, b])) \to 0 \). Using the triangle inequality, we obtain
\[
d(T([a, b]), [a, b]) \leq d \left( T([a, b]), T^n([a_0, b_0]) \right) + d \left( T^{n+1}([a_0, b_0]), [a, b] \right) \to 0 \text{ as } n \to \infty,
\]
which says that \( d(T([a, b]), [a, b]) = 0 \), i.e., \( T([a, b]) \equiv [a, b] \). This shows that \([a, b]\) is a near fixed point.

Assume further that \( d \) satisfies the null equality. We are going to claim that each point \([a, b] \in \langle [a, b] \rangle\) is also a near fixed point of \( T \). Since \([a, b] \not\subseteq [a, b], [a, b] \not\subseteq [a, b], [a, b] \not\subseteq [a, b]\), we have \([a, b] \oplus \omega_1 = [a, b] \oplus \omega_2\) for some \( \omega_1, \omega_2 \in \Omega \). Then, using the null equality for \( d \), we obtain
\[
d(T^n([a_0, b_0]), [a, b]) = d(T^n([a_0, b_0]), [a, b] \oplus \omega_1) = d(T^n([a_0, b_0]), [a, b] \oplus \omega_2)
= d(T^n([a_0, b_0]), [a, b]) \to 0 \text{ as } n \to \infty.
\]

We can similarly obtain \( d(T^{n+1}([a_0, b_0]), T([a, b])) \to 0 \text{ as } n \to \infty \). Using the triangle inequality, we have
\[
d([a, b], T([a, b])) \leq d([a, b], T^n([a_0, b_0])) + d(T^{n+1}([a_0, b_0]), T([a, b])) \to 0 \text{ as } n \to \infty,
\]
which says that \( d([a, b], T([a, b])) = 0 \). Therefore we conclude that \( T([a, b]) \equiv [a, b] \) for any \([a, b] \in \langle [a, b] \rangle\).

Suppose that \([a, b] \not\in \langle [a, b] \rangle\) is another near fixed point of \( T \). Then \( T([a, b]) \equiv [a, b] \) and \( \langle [a, b] \rangle \not\equiv \langle [a, b] \rangle \), i.e., \([a, b] \not\subseteq [a, b] \not\subseteq [a, b]\). Then
\[
T([a, b]) \oplus \omega_1 = [a, b] \oplus \omega_2 \text{ and } T([a, b]) \oplus \omega_3 = [a, b] \oplus \omega_4
\]
for some \( \omega_i \in \Omega \) for \( i = 1, 2, 3, 4 \). Therefore we obtain
\[
d([a, b], [a, b]) = d([a, b] \oplus \omega_2, [a, b] \oplus \omega_4) \text{ (since } d \text{ satisfies the null equality)}
= d(T([a, b]) \oplus \omega_1, T([a, b]) \oplus \omega_3) = d(T([a, b]), T([a, b])) \text{ (since } d \text{ satisfies the null equality)}
< d([a, b], [a, b]) \text{ (since } [a, b] \not\equiv [a, b] \text{ and } T \text{ is a weakly strict metric contraction)}.
\]

This contradiction says that \([a, b]\) cannot be a near fixed point of \( T \). Equivalently, if \([a, b]\) is a near fixed point of \( T \), then \([a, b] \in \langle [a, b] \rangle\). This completes the proof. \( \square \)

Now we consider another fixed point theorem based on the weakly uniformly strict metric contraction which was proposed by Meir and Keeler [10]. Under the metric interval space \((\mathcal{I}, d)\), we have \( d([a, b], [c, d]) = 0 \) for \([a, b] \equiv [c, d]\). Therefore we propose the following different definition.

**Definition 13.** A function \( T : (\mathcal{I}, d) \to (\mathcal{I}, d) \) is called a weakly uniformly strict metric contraction on \( \mathcal{I} \) if and only if the following conditions are satisfied:
Remark 1. Given any $[a, b] \supseteq [c, d]$, i.e., $\langle [a, b] \rangle = \langle [c, d] \rangle$ implies $d(T([a, b]), T([c, d])) = 0$.

Lemma 1. Let $T : (\mathcal{I}, d) \to (\mathcal{I}, d)$ be a weakly uniformly strict metric contraction on $\mathcal{I}$, then $T$ is also a weakly strict metric contraction on $\mathcal{I}$ by taking $d = d([a, b], [c, d])$.

Proof. For convenience, we write $T^n([a, b]) = [a_n, b_n]$ for all $n$. Let $\eta_n = d([a_n, b_n], [a_{n+1}, b_{n+1}])$.

• Suppose that $\langle [a_{n-1}, b_{n-1}] \rangle \neq \langle [a_n, b_n] \rangle$. By Remark 1, since $T$ is also a weakly strict metric contraction on $\mathcal{I}$, we have

$$
\eta_n = d([a_n, b_n], [a_{n+1}, b_{n+1}]) = d(T^n([a, b]), T^n([a, b])) < d(T^{n-1}([a, b]), T^n([a, b]))
$$

which says that $\eta_n = 0$. Using the first condition of Definition 13, we also have

$$
\eta_n = d(T^n([a, b]), T^{n+1}([a, b])) = d(T([a_{n-1}, b_{n-1}]), T([a_n, b_n])) = 0 \leq \eta_{n-1}.
$$

The above two cases say that the sequence $\{\eta_n\}_{n=1}^\infty$ is decreasing. We consider the following cases.

• Let $m$ be the first index in the sequence $\{[a_n, b_n]\}_{n=1}^\infty$ such that $\langle [a_{m-1}, b_{m-1}] \rangle = \langle [a_m, b_m] \rangle$. Then we want to claim $\eta_m = \eta_m = \eta_{m-1} = \cdots = 0$. Since $\{a_{m-1}, b_{m-1}\} \supseteq \{a_m, b_m\}$, we have $\eta_{m-1} = d([a_{m-1}, b_{m-1}], [a_m, b_m]) = 0$. Using the first condition of Definition 13, we also have

$$
0 = d(T([a_{m-1}, b_{m-1}]), T([a_m, b_m])) = d(T(T^{m-1}([a, b])), T^m([a, b]))
$$

which says that $[a_m, b_m] \supseteq [a_{m+1}, b_{m+1}]$, i.e., $\langle [a_m, b_m] \rangle = \langle [a_{m+1}, b_{m+1}] \rangle$. Using the similar argument, we can obtain $\eta_{m+1} = 0$ and $\langle [a_{m+1}, b_{m+1}] \rangle = \langle [a_{m+2}, b_{m+2}] \rangle$. Therefore the sequence $\{\eta_n\}_{n=1}^\infty$ is decreasing to zero.

• Suppose that $\langle [a_{m+1}, b_{m+1}] \rangle \neq \langle [a_m, b_m] \rangle$ for all $m \geq 1$. Since the sequence $\{\eta_n\}_{n=1}^\infty$ is decreasing, we assume that $\eta_n \downarrow \epsilon > 0$, i.e., $\eta_n \geq \epsilon > 0$ for all $n$. There exists $\delta > 0$ such that $\epsilon \leq \eta_m < \epsilon + \delta$ for some $m$, i.e.,

$$
\epsilon \leq d([a_m, b_m], [a_{m+1}, b_{m+1}]) < \epsilon + \delta.
$$

By the second condition of Definition 13, we have

$$
\eta_{m+1} = d([a_{m+1}, b_{m+1}], [a_{m+2}, b_{m+2}]) = d(T^{m+1}([a, b]), T^{m+2}([a, b]))
$$

which contradicts $\eta_{m+1} \geq \epsilon$.

This completes the proof. □
Theorem 3. (Near Fixed Point Theorem) Let \((\mathcal{I}, d)\) be a complete metric interval space with the null set \(\Omega\), and let \(T : (\mathcal{I}, d) \rightarrow (\mathcal{I}, d)\) be a weakly uniformly strict metric contraction on \(\mathcal{I}\). Then \(T\) has a near fixed point satisfying \(T([a, b]) \supseteq [a, b]\). Moreover, the near fixed point \([a, b]\) is obtained by the limit

\[
d(T^n([a_0, b_0]), [a, b]) \rightarrow 0 \text{ as } n \rightarrow \infty \text{ for some } [a_0, b_0].
\]

Assume further that \(d\) satisfies the null equality. Then we also have the following properties.

- The uniqueness is in the sense that there is a unique equivalence class \([a, b]\) such that any \([a, b] \notin [a, b]\) cannot be a near fixed point.
- Each point \([a, b] \in [a, b]\) is also a near fixed point of \(T\) satisfying \(T([a, b]) \supseteq [a, b]\) and \([a, b] = [a, b]\).
- If \([a, b]\) is a near fixed point of \(T\), then \([a, b] \in [a, b]\), i.e., \([a, b] = [a, b]\). Equivalently, if \([a, b]\) and \([a, b]\) are the near fixed points of \(T\), then \([a, b] \supseteq [a, b]\).

Proof. According to Theorem 2 and Remark 1, we just need to claim that if \(T\) is a weakly uniformly strict metric contraction, then \(\{T^n([a_0, b_0])\}_{n=1}^\infty = \{[a_n, b_n]\}_{n=1}^\infty\) is a Cauchy sequence for \([a_0, b_0] \in \mathcal{I}\). Suppose that \(\{[a_n, b_n]\}_{n=1}^\infty\) is not a Cauchy sequence. Then there exists \(2\varepsilon > 0\) such that, given any \(N\), there exist \(m, n \geq N\) satisfying \(d([a_m, b_m], [a_n, b_n]) > 2\varepsilon\). Since \(T\) is a weakly uniformly strict metric contraction on \(\mathcal{I}\), there exists \(\delta > 0\) such that

\[
eq d([a, b], [c, d]) < \varepsilon \leq d(T([a, b]), T([c, d])) < \varepsilon \text{ for any } [a, b] \supseteq [c, d].
\]

Let \(\delta' = \min\{\delta, \varepsilon\}\). We are going to claim

\[
eq d([a, b], [c, d]) < \varepsilon + \delta' \implies d(T([a, b]), T([c, d])) < \varepsilon \text{ for any } [a, b] \supseteq [c, d]
\]

(14)

Indeed, if \(\delta' = \delta\), then it is done, and if \(\delta' = \varepsilon\), i.e., \(\varepsilon < \delta\), then \(\varepsilon + \delta' = \varepsilon < \varepsilon + \delta\). This proves the statement (14). Let \(\eta_n = d([a_n, b_n], [a_{n+1}, b_{n+1}])\). Since the sequence \(\{\eta_n\}_{n=1}^\infty\) is decreasing to zero by Lemma 1, we can find \(N\) such that \(\eta_N < \delta' / 3\). For \(n > m \geq N\), we have

\[
d([a_m, b_m], [a_n, b_n]) > 2\varepsilon \geq \varepsilon + \delta',
\]

(15)

which says that \([a_m, b_m] \supseteq [a_n, b_n]\). Since the sequence \(\{\eta_n\}_{n=1}^\infty\) is decreasing by Lemma 1 again, we obtain

\[
d([a_m, b_m], [a_{m+1}, b_{m+1}]) = \eta_m \leq \eta_N < \frac{\delta'}{3} \leq \frac{\varepsilon}{3} < \varepsilon.
\]

(16)

For \(j\) with \(m < j \leq n\), using the triangle inequality, we also have

\[
d([a_m, b_m], [a_{j+1}, b_{j+1}]) \leq d([a_m, b_m], [a_j, b_j]) + d([a_j, b_j], [a_{j+1}, b_{j+1}]).
\]

(17)

We want to show that there exists \(j\) with \(m < j \leq n\) such that \([a_m, b_m] \supseteq [a_j, b_j]\) and

\[
eq d([a_m, b_m], [a_j, b_j]) < \varepsilon + \delta'.
\]

(18)

Let \(\gamma_j = d([a_m, b_m], [a_j, b_j])\) for \(j = m + 1, \cdots, n\). Then (15) and (16) say that

\[
\gamma_{n+1} < \varepsilon \text{ and } \gamma_n > \varepsilon + \delta'.
\]

(19)
Let \( j_0 \) be an index such that

\[
j_0 = \max \left\{ j \in \{m + 1, n\} : \gamma_j \leq \epsilon + \frac{2\delta'}{3} \right\}.
\]

Then, from (19), we see that \( m + 1 \leq j_0 < n \), which says that \( j_0 \) is well-defined. By the definition of \( j_0 \), we also see that \( j_0 + 1 \leq n \) and

\[
\gamma_{j_0 + 1} > \epsilon + \frac{2\delta'}{3},
\]

which also says that \([a_m, b_m] \not\supset [a_{j_0 + 1}, b_{j_0 + 1}]\); otherwise, \( \gamma_{j_0 + 1} = 0 \) that is a contradiction. Therefore, from (20), expression (18) will be sound if we can show that \( \gamma_{j_0 + 1} < \epsilon + \delta' \). Suppose that this is not true, i.e., \( \gamma_{j_0 + 1} \geq \epsilon + \delta' \). We also see that \( \gamma_j \leq \epsilon + \frac{2\delta'}{3} \). Since \( \{\eta_n\}_{n=1}^{\infty} \) is decreasing, from (16) and (17), we have

\[
\frac{\delta'}{3} > \eta_N \geq \eta_{j_0} = d([a_m, b_m], [a_{j_0 + 1}, b_{j_0 + 1}]) \geq \gamma_{j_0 + 1} - \gamma_j \geq \epsilon + \delta' - \epsilon - \frac{2\delta'}{3} = \frac{\delta'}{3}.
\]

This contradiction says that (18) is sound. Since \([a_m, b_m] \not\supset [a_j, b_j]\), using (14), we see that (18) implies

\[
d([a_{m + 1}, b_{m + 1}], [a_{j + 1}, b_{j + 1}]) = d(T([a_{m + 1}, b_{m + 1}]), T([a_j, b_j])) < \epsilon. \tag{21}
\]

Therefore, using the triangle inequality, we obtain

\[
d([a_m, b_m], [a_j, b_j]) \\
\leq d([a_m, b_m], [a_{m + 1}, b_{m + 1}]) + d([a_{m + 1}, b_{m + 1}], [a_{j + 1}, b_{j + 1}]) + d([a_{j + 1}, b_{j + 1}], [a_j, b_j]) \\
< \eta_m + \epsilon + \eta_m \text{ (by (21))} \\
\leq \eta_m + \epsilon + \epsilon < \frac{\delta'}{3} + \epsilon + \frac{\delta'}{3} \text{ (by (16))} \\
= \epsilon + \frac{2\delta'}{3},
\]

which contradicts (18). This contradiction says that every sequence \( \{T^n([a, b])\}_{n=1}^{\infty} = \{[a_n, b_n]\}_{n=1}^{\infty} \) is a Cauchy sequence. This completes the proof. \( \square \)

7. Near Fixed Point Theorems in Banach Interval Space

In this section, we shall study the near fixed point in Banach interval space.

**Definition 14.** Let \((I, \| \cdot \|)\) be a pseudo-seminormed interval space. A function \(T: (I, \| \cdot \|) \rightarrow (I, \| \cdot \|)\) is called a norm contraction on \(I\) if and only if there is a real number \(0 < \alpha < 1\) such that

\[
\| T([a, b]) \odot T([c, d]) \| \leq \alpha \cdot \| [a, b] \odot [c, d] \|
\]

for any \([a, b], [c, d] \in I\).

**Theorem 4.** Let \((I, \| \cdot \|)\) be a Banach interval space with the null set \(\Omega\) such that \(\| \cdot \|\) satisfies the null equality. Suppose that the function \(T: (I, \| \cdot \|) \rightarrow (I, \| \cdot \|)\) is a norm contraction on \(I\). Then \(T\) has a near fixed point \([a, b] \in I\) satisfying \(T([a, b]) \not\supset [a, b]\). Moreover, the near fixed point \([a, b]\) is obtained by the limit

\[
\lim \| [a, b] \odot [a_n, b_n] \| = \| [a_n, b_n] \odot [a, b] \| \rightarrow 0 \text{ as } n \rightarrow \infty.
\]
Proof. Given any initial element \([a_0, b_0] \in I\), we are going to show that \(\{a_n, b_n\}_{n=1}^{\infty}\) is a Cauchy sequence. Since \(T\) is a norm contraction on \(I\), we have

\[
\| [a_{m+1}, b_{m+1}] \ominus [a_m, b_m] \| = \| T([a_m, b_m]) \ominus T([a_{m-1}, b_{m-1}]) \| \\
\leq \alpha \cdot \| [a_m, b_m] \ominus [a_{m-1}, b_{m-1}] \| \\
= \alpha \cdot \| T([a_{m-1}, b_{m-1}]) \ominus T([a_{m-2}, b_{m-2}]) \| \\
\leq \alpha^2 \cdot \| [a_{m-1}, b_{m-1}] \ominus [a_{m-2}, b_{m-2}] \| \\
\leq \cdots \leq \alpha^m \cdot \| [a_1, b_1] \ominus [a_0, b_0] \|. 
\]

For \(n < m\), using Proposition 2, we obtain

\[
\| [a_m, b_m] \ominus [a_n, b_n] \| \leq \| [a_m, b_m] \ominus [a_{m-1}, b_{m-1}] \| + \| [a_{m-1}, b_{m-1}] \ominus [a_{m-2}, b_{m-2}] \| + \cdots \\
+ \| [a_{n+1}, b_{n+1}] \ominus [a_n, b_n] \| \\
\leq (\alpha^{m-1} + \alpha^{m-2} + \cdots + \alpha^n) \cdot \| [a_1, b_1] \ominus [a_0, b_0] \| \\
= \alpha^n \cdot \frac{1 - \alpha^{m-n}}{1 - \alpha} \cdot \| [a_1, b_1] \ominus [a_0, b_0] \|. 
\]

Since \(0 < \alpha < 1\), we have \(1 - \alpha^{m-n} < 1\) in the numerator, which says that

\[
\| [a_m, b_m] \ominus [a_n, b_n] \| \leq \frac{\alpha^n}{1 - \alpha} \cdot \| [a_1, b_1] \ominus [a_0, b_0] \| \to 0 \text{ as } n \to \infty. 
\]

This proves that \(\{a_n, b_n\}_{n=1}^{\infty}\) is a Cauchy sequence. Since \(I\) is complete, there exists \([a, b] \in I\) such that

\[
\| [a, b] \ominus [a_n, b_n] \| \to 0 \text{ as } n \to \infty. 
\]

Assume further that \(\| \cdot \|\) satisfies the null equality. We are going to show that any point \([a, b] \in \langle [a, b] \rangle\) is a near fixed point. Since \([a, b] \ominus \omega_1 = [a, b] \ominus \omega_2\) for some \(\omega_1, \omega_2 \in \Omega\), we have

\[
\| [a, b] \ominus T([a, b]) \| = \| ([a, b] \ominus \omega_1) \ominus T([a, b]) \| \quad \text{(since } \| \cdot \|\) satisfies the null equality)
\[
\leq \| ([a, b] \ominus \omega_1) \ominus [a_m, b_m] \| + \| [a_m, b_m] \ominus T([a, b]) \| \quad \text{(using Proposition 2)}
\]

\[
= \| ([a, b] \ominus \omega_1) \ominus [a_m, b_m] \| + \| T([a_{m-1}, b_{m-1}]) \ominus T([a, b]) \| \\
\leq \| ([a, b] \ominus \omega_1) \ominus [a_m, b_m] \| + \| [a_{m-1}, b_{m-1}] \ominus [a, b] \| \quad \text{(using norm contraction)}
\]

\[
= \| ([a, b] \ominus \omega_1) \ominus [a_m, b_m] \| + \| [a_{m-1}, b_{m-1}] \ominus [a, b] \ominus (-\omega_1) \| \\
\quad \text{(since } \omega_1 \in \Omega \text{ and } \| \cdot \|\) satisfies the null equality)
\]

\[
= \| ([a, b] \ominus \omega_1) \ominus [a_m, b_m] \| + \| [a_{m-1}, b_{m-1}] \ominus [a, b] \ominus \omega_1 \| \\
= \| ([a, b] \ominus \omega_1) \ominus [a_m, b_m] \| + \| [a_{m-1}, b_{m-1}] \ominus ([a, b] \ominus \omega_1) \| \quad \text{(using } (1))
\]

\[
= \| [a, b] \ominus [a_m, b_m] \| + \| [a_{m-1}, b_{m-1}] \ominus ([a, b] \ominus \omega_2) \| \\
\quad \text{(using the null equality and } (1)),
\]

in which the sequence \(\{[a_n, b_n]\}_{n=1}^{\infty}\) is generated according to \((12)\). Assume further that \(\| \cdot \|\) satisfies the null equality. We also have the following properties.

- The uniqueness is in the sense that there is a unique equivalence class \(\langle [a, b] \rangle\) such that any \([a, b] \notin \langle [a, b] \rangle\) cannot be a near fixed point.
- Each point \([a, b] \in \langle [a, b] \rangle\) is also a near fixed point of \(T\) satisfying \(T([a, b]) = [a, b]\) and \(\langle [a, b] \rangle = \langle [a, b] \rangle\).
- If \([a, b]\) is a near fixed point of \(T\), then \([a, b] \in \langle [a, b] \rangle\), i.e., \(\langle [a, b] \rangle = \langle [a, b] \rangle\). Equivalently, if \([a, b]\) and \([\bar{a}, \bar{b}]\) are the near fixed points of \(T\), then \([a, b] = [\bar{a}, \bar{b}]\).
which implies \( \| [a, b] \oplus T([a, b]) \| = 0 \) as \( m \to \infty \). We conclude that \( T([a, b]) \supseteq [a, b] \) for any point \([a, b] \in \langle [a, b] \rangle\) by part (ii) of Proposition 3.

Now assume that there is another near fixed point \([a, b]\) of \( T \) with \([a, b] \not\in \langle [a, b] \rangle\), i.e., \([a, b] \supseteq T([a, b])\). Then
\[
[a, b] \oplus \omega_1 = T([a, b]) \oplus \omega_2
\]
for some \( \omega_1 \in \Omega, i = 1, \cdots, 4 \). We obtain
\[
\| [a, b] \oplus [a, b] \| = \| [a, b] \oplus \omega_1 \| \oplus (\| [a, b] \oplus \omega_2 \| (\text{using the null equality and (1)})
\]
\[
\| T([a, b]) \oplus T([a, b]) \| = \| T([a, b]) \oplus T([a, b]) \| (\text{using the null equality and (1)})
\]
\[
\| [a, b] \oplus [a, b] \| \leq \| [a, b] \oplus [a, b] \| (\text{using the norm contraction}).
\]

Since \( 0 < \alpha < 1 \), we conclude that \( \| [a, b] \oplus [a, b] \| \leq 0 \), i.e., \([a, b] \supseteq [a, b]\), which contradicts \([a, b] \not\in \langle [a, b] \rangle\). Therefore, any \([a, b] \not\in \langle [a, b] \rangle\) cannot be the near fixed point. Equivalently, if \([a, b]\) is a near fixed point of \( T \), then \([a, b] \in \langle [a, b] \rangle\). This completes the proof. \( \square \)

**Definition 15.** Let \((\mathcal{I}, \| \cdot \|)\) be a pseudo-normed interval space. A function \( T : (\mathcal{I}, \| \cdot \|) \to (\mathcal{I}, \| \cdot \|) \) is called a weakly strict norm contraction on \( \mathcal{I} \) if and only if the following conditions are satisfied:

- \([a, b] \supseteq [c, d], \text{ i.e., } \langle [a, b] \rangle = \langle [c, d] \rangle \) implies \( T([a, b]) \supseteq T([c, d]) \) \( \| = 0 \).
- \([a, b] \not\supseteq [c, d], \text{ i.e., } \langle [a, b] \rangle \not\supseteq \langle [c, d] \rangle \) implies \( T([a, b]) \supseteq T([c, d]) \) \( \| \leq 0 \) \( \| [a, b] \oplus [c, d] \| \).

By part (ii) of Proposition 3, we see that if \([a, b] \not\supseteq [c, d]\), then \( \| [a, b] \oplus [c, d] \| \leq 0 \), which says that the weakly strict norm contraction is well-defined. We further assume that \( \| \cdot \| \) satisfies the null super-inequality and null condition. Part (iii) of Proposition 3 says that if \( T \) is a contraction on \( \mathcal{I} \), then it is also a weakly strict contraction on \( \mathcal{I} \).

**Theorem 5.** Let \((\mathcal{I}, \| \cdot \|)\) be a Banach interval space such that \( \| \cdot \| \) satisfies the null super-inequality and null condition. Suppose that the function \( T : (\mathcal{I}, \| \cdot \|) \to (\mathcal{I}, \| \cdot \|) \) is a weakly strict norm contraction on \( \mathcal{I} \). If \( \{T^n([a_0, b_0])\}_{n=1}^\infty \) forms a Cauchy sequence for some \([a_0, b_0] \in \mathcal{I}\), then \( T \) has a near fixed point \([a, b] \in \mathcal{I}\) satisfying \( \langle [a, b] \rangle \supseteq [a, b]\). Moreover, the near fixed point \([a, b]\) is obtained by the limit
\[
\| T^n([a_0, b_0]) \oplus [a, b] \| = \| [a, b] \oplus T^n([a_0, b_0]) \| \to 0 \text{ as } n \to \infty.
\]

Assume further that \( \| \cdot \| \) satisfies the null equality. Then we also have the following properties.

- The uniqueness is in the sense that there is a unique equivalence class \( \langle [a, b] \rangle \) such that any \([a, b] \not\in \langle [a, b] \rangle\) cannot be a near fixed point.
- Each point \([a, b] \in \langle [a, b] \rangle\) is also a near fixed point of \( T \) satisfying \( \langle [a, b] \rangle \supseteq [a, b] \) and \( \langle [a, b] \rangle = \langle [a, b] \rangle \).
- If \([a, b]\) is a near fixed point of \( T \), then \([a, b] \in \langle [a, b] \rangle\), i.e., \( \langle [a, b] \rangle = \langle [a, b] \rangle \). Equivalently, if \([a, b]\) and \([a, b]\) are the near fixed points of \( T \), then \([a, b] \supseteq [a, b]\).

**Proof.** Since \( \{T^n([a_0, b_0])\}_{n=1}^\infty \) is a Cauchy sequence, the completeness says that there exists \([a, b] \in \mathcal{I}\) such that
\[
\| T^n([a_0, b_0]) \oplus [a, b] \| = \| [a, b] \oplus T^n([a_0, b_0]) \| \to 0.
\]

Therefore, given any \( \epsilon > 0 \), there exists an integer \( N \) such that \( \| T^n([a_0, b_0]) \oplus [a, b] \| < \epsilon \) for \( n \geq N \). Since \( T \) is a weakly strict norm contraction on \( \mathcal{I} \), we consider the following two cases.

- Suppose that \( T^n([a_0, b_0]) \supseteq [a, b] \). Then
\[
\| T^{n+1}([a_0, b_0]) \oplus T([a, b]) \| = 0 < \epsilon.
\]
Suppose that $T^n([a_0, b_0]) \not\subseteq [a, b]$. Then

$$\| T^{n+1}([a_0, b_0]) \ominus T([a, b]) \| < \| T^n([a_0, b_0]) \ominus [a, b] \| < \epsilon$$

for $n \geq N$.

The above two cases say that $\| T^{n+1}([a_0, b_0]) \ominus T([a, b]) \| \to 0$. Using Proposition 2, we obtain

$$\| [a, b] \ominus T([a, b]) \| \leq \| [a, b] \ominus T^{n+1}([a_0, b_0]) \| + \| T^{n+1}([a_0, b_0]) \ominus T([a, b]) \| \to 0$$

as $n \to \infty$,

which says that $\| [a, b] \ominus T([a, b]) \| = 0$, i.e., $T([a, b]) \cap [a, b]$ by part (ii) of Proposition 3. This shows that $[a, b]$ is a near fixed point.

Assume further that $\| \cdot \|$ satisfies the null equality. Now we are going to claim that each point $[a, b] \in \langle [a, b] \rangle$ is also a near fixed point of $T$. Since $[a, b] \cap [a, b]$, we have $[a, b] \oplus \omega_1 = [a, b] \oplus \omega_2$ for some $\omega_1, \omega_2 \in \Omega$. Then, using the null equality for $\| \cdot \|$, we obtain

$$\| T^n([a_0, b_0]) \oplus [\bar{a}, \bar{b}] \| = \| [\bar{a}, \bar{b}] \oplus T^n([a_0, b_0]) \| = \| ([\bar{a}, \bar{b}] \oplus \omega_1) \ominus T^n([a_0, b_0]) \|$$

$$= \| ([a, b] \oplus \omega_2) \ominus T^n([a_0, b_0]) \|$$

$$= \| [a, b] \ominus T^n([a_0, b_0]) \| \to 0$$

as $n \to \infty$.

We can similarly obtain $\| T^{n+1}([a_0, b_0]) \ominus T([\bar{a}, \bar{b}]) \| \to 0$ as $n \to \infty$. Using Proposition 2, we have

$$\| [a, b] \ominus T([\bar{a}, \bar{b}]) \| \leq \| [a, b] \ominus T^{n+1}([a_0, b_0]) \| + \| T^{n+1}([a_0, b_0]) \ominus T([\bar{a}, \bar{b}]) \| \to 0$$

as $n \to \infty$,

which says that $\| [a, b] \ominus T([\bar{a}, \bar{b}]) \| = 0$, i.e., $T([\bar{a}, \bar{b}]) \cap [a, b]$ by part (ii) of Proposition 3 for any point $[a, b] \in \langle [a, b] \rangle$.

Suppose that $[a, b] \not\in \langle [a, b] \rangle$ is another near fixed point of $T$. Then $T([\bar{a}, \bar{b}]) \cap [a, b]$ and $\langle [a, b] \rangle \neq \langle [a, b] \rangle$, i.e., $[a, b] \not\in [a, b]$. Then

$$T([\bar{a}, \bar{b}]) \oplus \omega_1 = [a, b] \oplus \omega_2$$

and $T([\bar{a}, \bar{b}]) \oplus \omega_3 = [a, b] \oplus \omega_4$ for some $\omega_1 \in \Omega$ for $i = 1, 2, 3, 4$. Therefore we obtain

$$\| [a, b] \ominus [\bar{a}, \bar{b}] \| = \| ([a, b] \oplus \omega_2) \ominus ([\bar{a}, \bar{b}] \oplus \omega_4) \|$$

(using the null equality and (1))

$$= \| (T([a, b]) \oplus \omega_1) \ominus (T([\bar{a}, \bar{b}]) \oplus \omega_3) \| \| T([a, b]) \ominus T([\bar{a}, \bar{b}]) \|$$

(using the null equality and (1))

$$< \| [a, b] \ominus [\bar{a}, \bar{b}] \|$$

(since $[a, b] \not\in [a, b]$ and $T$ is a weakly strict norm contraction).

This contradiction says that $[a, b]$ cannot be a near fixed point of $T$. Equivalently, if $[a, b]$ is a near fixed point of $T$, then $[a, b] \in \langle [a, b] \rangle$. This completes the proof.

Now we consider another fixed point theorem based on the concept of weakly uniformly strict norm contraction which was proposed by Meir and Keeler [10].

**Definition 16.** Let $(\mathcal{I}, \| \cdot \|)$ be a pseudo-normed interval space with the null set $\Omega$. A function $T : (\mathcal{I}, \| \cdot \|) \to (\mathcal{I}, \| \cdot \|)$ is called a weakly uniformly strict norm contraction on $\mathcal{I}$ if and only if the following conditions are satisfied:

- $[a, b] \in \Omega [c, d]$ i.e., $\langle [a, b] \rangle = \langle [c, d] \rangle$ implies $\| T([a, b]) \ominus T([c, d]) \| = 0$;
• Suppose that \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that
\[
\varepsilon \leq \| [a, b] \cap [c, d] \| < \varepsilon + \delta \implies \| T([a, b]) \cup T([c, d]) \| < \varepsilon
\]
for any \( [a, b] \not= [c, d] \), i.e., \( \langle [a, b] \rangle \not= \langle [c, d] \rangle \).

By part (ii) of Proposition 3, we see that if \( [a, b] \not= [c, d] \), then \( \| [a, b] \cap [c, d] \| \not= 0 \), which says that the weakly uniformly strict norm contraction is well-defined.

**Remark 2.** We observe that if \( T \) is a weakly uniformly strict norm contraction on \( \mathcal{I} \), then \( T \) is also a weakly strict norm contraction on \( \mathcal{I} \).

**Lemma 2.** Let \( (\mathcal{I}, \| \cdot \|) \) be a pseudo-normed interval space with the null set \( \Omega \), and let \( T : (\mathcal{I}, \| \cdot \|) \to (\mathcal{I}, \| \cdot \|) \) be a weakly uniformly strict norm contraction on \( \mathcal{I} \). Then the sequence \( \{ \| T^n([a, b]) \cup T^{n+1}([a, b]) \| \}_{n=1}^\infty \) is decreasing to zero for any \( [a, b] \in \mathcal{I} \).

**Proof.** For convenience, we write \( T^n([a, b]) = [a_n, b_n] \) for all \( n \). Let \( \eta_n = \| [a_n, b_n] \cap [a_{n+1}, b_{n+1}] \| \).

• Suppose that \( \langle [a_{n-1}, b_{n-1}] \rangle \not= \langle [a_n, b_n] \rangle \). By Remark 2, since \( T \) is also a weakly strict norm contraction on \( \mathcal{I} \), we have
\[
\eta_n = \| [a_n, b_n] \cap [a_{n+1}, b_{n+1}] \| = \| T^n([a, b]) \cup T^{n+1}([a, b]) \| < \| T^n([a, b]) \cap T^n([a, b]) \| = \eta_{n-1}.
\]
• Suppose that \( \langle [a_{n-1}, b_{n-1}] \rangle = \langle [a_n, b_n] \rangle \). Then, by the first condition of Definition 16,
\[
\eta_n = \| T^n([a, b]) \cap T^{n+1}([a, b]) \| = \| T([a_{n-1}, b_{n-1}]) \cap T([a_n, b_n]) \| = \eta_{n-1} < 0.
\]

The above two cases say that the sequence \( \{ \eta_n \}_{n=1}^\infty \) is decreasing. We consider the following cases.

• Let \( m \) be the first index in the sequence \( \{ [a_n, b_n] \}_{n=1}^\infty \) such that \( \langle [a_{m-1}, b_{m-1}] \rangle = \langle [a_m, b_m] \rangle \). Then we want to claim \( \eta_{m-1} = \eta_m = \eta_{m+1} = \cdots = 0 \). Since \( [a_{m-1}, b_{m-1}] \supseteq [a_m, b_m] \), we have
\[
\eta_{m-1} = \| [a_{m-1}, b_{m-1}] \cap [a_m, b_m] \| = 0.
\]

Using the first condition of Definition 16, we also have
\[
0 = \| T([a_{m-1}, b_{m-1}]) \cap T([a_m, b_m]) \| = \| T^m([a, b]) \cap T^{m+1}([a, b]) \|
\]
which says that \( [a_m, b_m] \supseteq [a_{m+1}, b_{m+1}] \), i.e., \( \langle [a_m, b_m] \rangle = \langle [a_{m+1}, b_{m+1}] \rangle \). Using the similar arguments, we can obtain \( \eta_{m+1} = 0 \) and \( \langle [a_{m+1}, b_{m+1}] \rangle = \langle [a_{m+2}, b_{m+2}] \rangle \). Therefore the sequence \( \{ \eta_n \}_{n=1}^\infty \) is decreasing to zero.

• Suppose that \( \langle [a_{m+1}, b_{m+1}] \rangle \not= \langle [a_m, b_m] \rangle \) for all \( m \geq 1 \). Since the sequence \( \{ \eta_n \}_{n=1}^\infty \) is decreasing, we assume that \( \eta_n \downarrow \varepsilon > 0 \), i.e., \( \eta_n \geq \varepsilon > 0 \) for all \( n \). There exists \( \delta > 0 \) such that \( \varepsilon \leq \eta_m < \varepsilon + \delta \) for some \( m \), i.e.,
\[
\varepsilon \leq \| [a_m, b_m] \cap [a_{m+1}, b_{m+1}] \| < \varepsilon + \delta.
\]

By the second condition of Definition 16, we have
\[
\eta_{m+1} = \| [a_{m+1}, b_{m+1}] \cap [a_{m+2}, b_{m+2}] \| = \| T^m([a, b]) \cap T^{m+2}([a, b]) \|
\]
which says that \( [a_m, b_m] \supseteq [a_{m+1}, b_{m+1}] \), i.e., \( \langle [a_m, b_m] \rangle = \langle [a_{m+1}, b_{m+1}] \rangle \). Using the similar arguments, we can obtain \( \eta_{m+1} = 0 \) and \( \langle [a_{m+1}, b_{m+1}] \rangle = \langle [a_{m+2}, b_{m+2}] \rangle \). Therefore the sequence \( \{ \eta_n \}_{n=1}^\infty \) is decreasing to zero.
which contradicts \( \eta_{m+1} \geq \varepsilon \).

This completes the proof. \( \square \)

**Theorem 6.** Let \((\mathcal{I}, \| \cdot \|)\) be a Banach interval space with the null set \( \Omega \). Suppose that \( \| \cdot \| \) satisfies the null super-inequality, and that the function \( T : (\mathcal{I}, \| \cdot \|) \to (\mathcal{I}, \| \cdot \|) \) is a weakly uniformly strict norm contraction on \( \mathcal{I} \). Then \( T \) has a near fixed point satisfying \( T([a,b]) \supseteq [a,b] \). Moreover, the near fixed point \([a,b]\) is obtained by the limit

\[
\| T^n([a_0,b_0]) \cap [a,b] \| \to 0 \text{ as } n \to \infty \text{ for some } [a_0,b_0] \in \mathcal{I}.
\]

Assume further that \( \| \cdot \| \) satisfies the null equality. Then we also have the following properties.

- The uniqueness is in the sense that there is a unique equivalence class \( \langle [a,b] \rangle \) such that any \([a,b] \notin \langle [a,b] \rangle\) cannot be a near fixed point.
- Each point \([a,b] \in \langle [a,b] \rangle\) is also a near fixed point of \(T\) satisfying \(T([a,b]) \supseteq [a,b] \text{ and } \langle [a,b] \rangle = \langle [a,b] \rangle\).
- If \([a,b]\) is a near fixed point of \(T\), then \([a,b] \in \langle [a,b] \rangle\), i.e., \(\langle [a,b] \rangle = \langle [a,b] \rangle\). Equivalently, if \([a,b]\) and \([a,b]\) are the near fixed points of \(T\), then \([a,b] \supseteq [a,b]\).

**Proof.** According to Theorem 5 and Remark 2, we just need to claim that if \(T\) is a weakly uniformly strict norm contraction, then \(\{T^n([a_0,b_0])\}_{n=1}^\infty = \{[a_n,b_n]\}_{n=1}^\infty\) forms a Cauchy sequence. Suppose that \(\{[a_n,b_n]\}_{n=1}^\infty\) is not a Cauchy sequence. Then there exists \(2\varepsilon > 0\) such that, given any \(N\), there exist \(n > m \geq N\) satisfying \(\| [a_m,b_m] \cap [a_n,b_n] \| > 2\varepsilon\). Since \(T\) is a weakly uniformly strict norm contraction on \(\mathcal{I}\), there exists \(\delta > 0\) such that

\[
\varepsilon \leq \| [a,b] \cap [c,d] \| < \varepsilon + \delta \implies \| T([a,b]) \cap T([c,d]) \| < \varepsilon \text{ for any } [a,b] \notin [c,d].
\]

Let \(\delta' = \min\{\delta, \varepsilon\}\). We are going to claim

\[
\varepsilon \leq \| [a,b] \cap [c,d] \| < \varepsilon + \delta' \implies \| T([a,b]) \cap T([c,d]) \| < \varepsilon \text{ for any } [a,b] \notin [c,d]. \tag{22}
\]

Indeed, if \(\delta' = \delta\) then it is done, and if \(\delta' = \varepsilon\), i.e., \(\varepsilon < \delta\), then \(\varepsilon + \delta' = \varepsilon + \varepsilon < \varepsilon + \delta\).

Let \(\eta_n = \| [a_n,b_n] \cap [a_{n+1},b_{n+1}] \|\). Since the sequence \(\{\eta_n\}_{n=1}^\infty\) is decreasing to zero by Lemma 2, we can find \(N\) such that \(\eta_N < \delta' / 3\). For \(n > m \geq N\), we have

\[
\| [a_m,b_m] \cap [a_n,b_n] \| > 2\varepsilon \geq \varepsilon + \delta', \tag{23}
\]

which says that \([a_m,b_m] \notin [a_n,b_n]\). Since the sequence \(\{\eta_n\}_{n=1}^\infty\) is decreasing by Lemma 2 again, we obtain

\[
\| [a_m,b_m] \cap [a_{m+1},b_{m+1}] \| = \eta_m \leq \eta_N < \frac{\delta'}{3} \leq \frac{\varepsilon}{3} < \varepsilon. \tag{24}
\]

For \(j\) with \(m < j \leq n\), using Proposition 2, we have

\[
\| [a_m,b_m] \cap [a_{j+1},b_{j+1}] \| \leq \| [a_m,b_m] \cap [a_j,b_j] \| + \| [a_j,b_j] \cap [a_{j+1},b_{j+1}] \|. \tag{25}
\]

We want to show that there exists \(j\) with \(m < j \leq n\) such that \([a_m,b_m] \notin [a_j,b_j]\) and

\[
\varepsilon + \frac{2\delta'}{3} < \| [a_m,b_m] \cap [a_j,b_j] \| < \varepsilon + \delta'. \tag{26}
\]
Let \( \gamma_j = \| [a_m, b_m] \cap [a_j, b_j] \| \) for \( j = m + 1, \ldots, n \). Then (23) and (24) say that \( \gamma_{m+1} < \epsilon \) and \( \gamma_n > \epsilon + \delta' \). Let \( j_0 \) be an index such that

\[
j_0 = \max \left\{ j \in [m+1, n] : \gamma_j \leq \epsilon + \frac{2\delta'}{3} \right\}.
\]

Then we see that \( j_0 < n \), since \( \gamma_n > \epsilon + \delta' \). By the definition of \( j_0 \), we also see that \( j_0 + 1 \leq n \) and

\[
\gamma_{j_0+1} > \epsilon + \frac{2\delta'}{3},
\]

which also says that \( [a_m, b_m] \not\supseteq [a_{j_0+1}, b_{j_0+1}] \); otherwise, \( \gamma_{j_0+1} = 0 \) that is a contradiction. Therefore, from (27), expression (26) will be sound if we can show that \( \gamma_{j_0+1} < \epsilon + \delta' \). Suppose that this is not true, i.e., \( \gamma_{j_0+1} \geq \epsilon + \delta' \). We also see that \( \gamma_j \leq \epsilon + \frac{2\delta'}{3} \). Since \( \{\eta_n\}_{n=1}^{\infty} \) is decreasing, from (24) and (25), we have

\[
\frac{2\delta'}{3} > \eta_N \geq \eta_{j_0} = \| [a_{j_0}, b_{j_0}] \cap [a_{j_0+1}, b_{j_0+1}] \| \geq \gamma_{j_0+1} - \gamma_j \geq \epsilon + \delta' - \epsilon - \frac{2\delta'}{3} = \delta'.
\]

This contradiction says that (26) is sound. Since \( [a_m, b_m] \not\supseteq [a_j, b_j] \), using (22), we see that (26) implies

\[
\| [a_{m+1}, b_{m+1}] \cap [a_{j+1}, b_{j+1}] \| < \| T([a_m, b_m]) \cap T([a_j, b_j]) \| < \epsilon.
\]

Therefore we obtain

\[
\| [a_m, b_m] \cap [a_j, b_j] \| < \| [a_m, b_m] \cap [a_{m+1}, b_{m+1}] \| + \| [a_{m+1}, b_{m+1}] \cap [a_{j+1}, b_{j+1}] \| (\text{by Proposition 2})
\]
\[
< \eta_m + \epsilon + \delta' \quad \text{(by (28))}
\]
\[
\leq \eta_m + \epsilon + \frac{2\delta'}{3} \quad \text{(by (24))}
\]
\[
= \epsilon + \frac{2\delta'}{3},
\]

which contradicts (26). This contradiction says that the sequence \( \{T^n([a, b])\}_{n=1}^{\infty} = \{[a_n, b_n]\}_{n=1}^{\infty} \) is a Cauchy sequence. This completes the proof. \( \square \)

8. Conclusions

Owing to the set of all bounded and closed intervals, \( \mathbb{R} \) cannot form a real vector space, and the concept of null set is proposed for the purpose of endowing a norm to the set of all bounded and closed intervals in \( \mathbb{R} \). Although the (conventional) metric space is not necessarily a real vector space, we also endow a metric to the set of all bounded and closed intervals in \( \mathbb{R} \) based on the concept of null set, which is then called a metric interval space and is different from the conventional metric space.

Since we do not consider the conventional normed and metric space in this paper, the concept of so-called near fixed point is proposed based on the concept of null set. The main contribution of this paper is to establish many types of near fixed point theorems in metric interval space and normed interval space. We also remark that the conventional fixed point theorems cannot be established in metric interval space and normed interval space, since they are not the conventional normed spaces.

**Funding:** This research received no external funding.

**Conflicts of Interest:** The author declares no conflict of interest.
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