Vertices in all minimum paired-dominating sets of block graphs

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Abstract Let \(G = (V, E)\) be a simple graph without isolated vertices. A set \(S \subseteq V\) is a paired-dominating set if every vertex in \(V - S\) has at least one neighbor in \(S\) and the subgraph induced by \(S\) contains a perfect matching. In this paper, we present a linear-time algorithm to determine whether a given vertex in a block graph is contained in all its minimum paired-dominating sets.

Keywords: Algorithm; Block graph; Domination; Paired-domination; Tree.

2000 Mathematics Subject Classification: 05C69; 05C85; 68R10

1 Introduction

Let \(G = (V, E)\) be a simple graph without isolated vertices. The distance between \(u\) and \(v\) in \(G\), denoted by \(d_G(u, v)\), is the minimum length of a path between \(u\) and \(v\) in \(G\). For a vertex \(v \in V\), the neighborhood of \(v\) in \(G\) is defined as \(N_G(v) = \{u \in V \mid uv \in E\}\) and the closed neighborhood is defined as \(N_G[v] = N_G(v) \cup \{v\}\). The degree of \(v\), denoted by \(d_G(v)\), is defined as \(|N_G(v)|\). We use \(d(u, v)\) for \(d_G(u, v)\), \(N(v)\) for \(N_G(v)\), \(N[v]\) for \(N_G[v]\) and \(d(v)\) for \(d_G(v)\) if there is no ambiguity. For a subset \(S\) of \(V\), the subgraph of \(G\) induced by the vertices in \(S\) is denoted by \(G[S]\) and \(G - S\) denote the subgraph induced by \(V - S\). A matching in a graph \(G\) is a set of pairwise nonadjacent edges in \(G\). A perfect matching \(M\) in \(G\) is a matching such that every vertex of \(G\) is incident to an edge of \(M\). Some other notations and terminology not introduced in here can be found in [1].

*Supported in part by National Natural Science Foundation of China (Nos. 60673048 and 10471044) and Shanghai Leading Academic Discipline Project (No. B407).

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Domination and its variations in graphs have been extensively studied \[2, 3\]. A set \(S \subseteq V\) is a paired-dominating set of \(G\), denoted PDS, if every vertex in \(V - S\) has at least one neighbor in \(S\) and the induced subgraph \(G[S]\) has a perfect matching \(M\). Two vertices joined by an edge of \(M\) are said to be paired in \(S\). The paired-domination number, denoted by \(\gamma_{pr}(G)\), is the minimum cardinality of a PDS. A paired-dominating set of cardinality \(\gamma_{pr}(G)\) is called a \(\gamma_{pr}(G)\)-set. The paired-domination was introduced by Haynes and Slater \[4, 5\]. There are many results on this problem \[6, 7, 8, 9, 10, 11\].

The study of characterizing vertices contained in all various kinds of minimum dominating set, such as dominating set, total dominating set and paired-dominating set, has received considerable attention (see \[12\], \[13\], \[14\]). Those results are all restricted in trees. In this paper, we will extend the result in \[14\] to block graphs, which contain trees as its subclass. In fact, we give a linear-time algorithm to determine whether a given vertex in a block graph is contained in all its minimum paired-dominating sets. If changing the pruning rules and judgement rules in our algorithm, our method is also available to determine whether a given vertex is contained in all minimum (total) dominating sets of a block graph.

2 Pruning block graphs

Let \(G = (V, E)\) be a simple graph. A vertex \(v\) is a cut-vertex if deleting \(v\) and all edges incident to it increases the number of connected components. A block of \(G\) is a maximal connected subgraph of \(G\) without cut-vertices. A block graph is a connected graph whose blocks are complete graphs. If every block is \(K_2\), then it is a tree.

Let \(G = (V, E)\) be a block graph. As we know, every block graph not isomorphic to complete graph has at least two end blocks, which are blocks with only one cut-vertex. A vertex in \(G\) is a leaf if its degree is one. If a vertex is adjacent to a leaf, then we call it a support vertex.

Lemma 1 \[14\] Let \(T\) be a tree of order at least three. If \(u\) is a leaf in \(T\), then there exists a \(\gamma_{pr}(T)\)-set not containing \(u\).

For block graphs, we have the following generalized result. The proof is almost same as that of Lemma 1 so it is omitted.

Lemma 2 Let \(G\) be a block graph of order at least three. If \(u\) is not a cut-vertex of \(G\), then there exists a \(\gamma_{pr}(G)\)-set not containing \(u\).

If \(G\) is a block graph with order two, then every vertex is contained in the only minimum paired-dominating set. If \(G\) is a complete graph with order at least three, no vertex of \(G\) is contained in all minimum paired-dominating sets. Thus, in here, we assume that the block graph \(G\) with at least one cut-vertex. Let \(r\) be the given vertex in \(G\) and we want to determine
whether \( r \) is contained in every \( \gamma_{pr}(G) \)-set. By Lemma 2 it is enough to assume that \( r \) is a cut-vertex of \( G \).

Our idea is to prune the original graph \( G \) into a small block graph \( \tilde{G} \) such that the given vertex \( r \) is contained in all minimum paired-dominating sets of \( G \) if and only if it is contained in all minimum paired-dominating sets of \( \tilde{G} \). To do this, we first need a vertex ordering and follow this ordering we can prune the original graph. For a vertex \( v \in V(G) \) and a block \( B \), the distance of \( v \) and \( B \), denoted by \( d(v,B) \), is defined as the maximum of \( d(u,v) \) for \( u \in V(B) \). We say a block \( B \) is farthest from \( v \) if \( d(v,B) \) is maximum over all blocks. Note that \( B \) is an end block if \( B \) is farthest from \( r \). To find the vertex ordering, in here, we need to define a vertex ordering connected operation. Let \( S = x_1, x_2, \cdots, x_s \) be a vertex ordering and \( T = u_1, u_2, \cdots, u_t \) be another vertex ordering. We use \( S + T \) to denote a new vertex ordering \( x_1, x_2, \cdots, x_s, u_1, u_2, \cdots, u_t \). Beginning with a block farthest from \( r \) and working recursively inward, we can find a vertex order \( v_1, v_2, \cdots, v_n \) as follows.

**Procedure VO**

\( S = \emptyset \); (\( S \) is a vertex ordering.)

Let \( r \) be a cut-vertex of \( G \);

While \( (G \neq \emptyset) \) do

If \((G \) is a complete graph) then

\[ S = S + u_1, u_2, \cdots, u_a; \]
\[ G = G - \{u_1, u_2, \cdots, u_a\}; \]

else

\[ S = S + u_1, u_2, \cdots, u_b; \]
\[ G = G - \{u_1, u_2, \cdots, u_b\}; \]

endif

dendo

Output \( S \).

Let \( v_1, v_2, \cdots, v_n = r \) be the vertex ordering of a block graph \( G \) which is obtained by procedure VO. We define the following notations:

(a) \( F_G(v_i) = v_j, j = \max \{k \mid v_i v_k \in E, k > i\} \). \( v_j \) is called the father of \( v_i \) and \( v_i \) is a child of \( v_j \). Obviously, \( v_j \) must be a cut-vertex in \( G \). We use \( F(v_i) \) for \( F_G(v_i) \) if there is no ambiguity.

(b) \( C_G(v_i) = \{v_j \mid F_G(v_j) = v_i\} \).

(c) For a block graph \( G \), we define a rooted tree \( T(G) \), whose vertex set is \( V(G) \), and \( uv \) is an edge of \( T(G) \) if and only if \( F_G(u) = v \). The root of \( T(G) \) is \( r \). Moreover let \( T_v \) be a subtree of \( T(G) \) rooted at \( v \). Every vertex in \( T_v \) except \( v \) is a descendant of \( v \). For a vertex \( v \in V(G) \), \( D_G(v) \) denotes the vertex set consisting of the descendants of \( v \) in \( T(G) \) and \( D_G[v] = D_G(v) \cup \{v\} \).
That is, $D_G[v] = V(T_v)$.

Except the vertex ordering, we also need a labeling function $l(v) : V \rightarrow \{\emptyset, r_1, r_2\}$ of each vertex $v$ to help us to determine which vertices can be pruned. At first, $l(v) = \emptyset$ for every vertex $v \in V$.

The following procedure can prune a big block graph $G$ into a small block graph $\tilde{G}$ such that $r$ is contained in all minimum paired-dominating sets of $G$ if and only if $r$ is contained in all minimum paired-dominating sets of $\tilde{G}$.

**Procedure PRUNE.** Prune a given block graph into a small block graph.

**Input** A block graph with at least one cut-vertex and a vertex ordering $v_1, v_2, \ldots, v_n$ obtained by procedure VO. For every vertex $v$, $l(v) = \emptyset$.

**Output** A smaller block graph.

**Method**

\[ S = \emptyset; \]
For $i = 1$ to $n - 1$ do
\begin{align*}
& \text{If } (v_i \not\in S) \text{ then } \\
& \quad \text{If } (l(v_i) = \emptyset \text{ and there is no child } v \text{ such that } l(v) = r_1 \text{ or } l(v) = r_2) \text{ then } \\
& \qquad l(F(v_i)) = r_1; \\
& \quad \text{else if } (v_i \text{ satisfies the conditions of Lemma 3 or Lemma 4 or Lemma 5}) \text{ then } \\
& \qquad G = G - D_G[v_i]; \\
& \quad \quad \text{If } (d(v_i) = 2 \text{ and } |V(B_1)| = |V(B_2)| = 2 \text{ and } C_G(F(v_i)) = \{v_i\}) \text{ then } \\
& \qquad \quad S = S \cup \{F(v_i)\}; \\
& \quad \quad \text{endif}
\end{align*}
else if $(v_i \text{ satisfies the conditions of Lemma 6})$ then
\begin{align*}
& \quad G = G - (D_G(v_i) - V(B')) \quad \text{where } B' \text{ is same as } B' \text{ in Lemma 6} \\
& \quad \text{else if } (v_i \text{ satisfies the conditions of Lemma 7 or Lemma 8}) \text{ then } \\
& \quad \quad G = G - (D_G(v_i) - D_G[u]), \text{ where } u \text{ is same as } u \text{ in Lemma 7 and Lemma 8} \\
& \quad \quad \quad l(v_i) = l(u) = r_2; \quad (*) \\
& \quad \quad \text{If } (d(v_i, r) = 2 \text{ and } |V(B_1)| = |V(B_2)| = 2 \text{ and } C_G(F(v_i)) = \{v_i\}) \text{ then } \\
& \quad \quad \quad \text{Where } B_1 \text{ and } B_2 \text{ are same as those in Lemma 8} \\
& \quad \quad \quad S = S \cup \{F(v_i)\}; \\
& \quad \quad \text{endif}
\end{align*}
\end{align*}
\text{endif}
\end{align*}
\text{endfor}

Output $G$.  

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Next, we will prove the correctness of procedure PRUNE. Let $G_i$ be a subgraph of the original graph $G$ after $v_i$ is considered and $G_0 = G$. It is clear that $G_i$ is a block graph for every $1 \leq i \leq n - 1$. We define that $C_i(v) = C_G(v) \cap V(G_i)$, $D_i(v) = D_G(v) \cap V(G_i)$ and $D_i[v] = D_G[v] \cap V(G_i)$ for $0 \leq i \leq n - 1$. Note that at the $i$-th loop, the pruning vertices, for example say $D_G[v_i]$, are $D_{i-1}[v_i]$ as $G$ is updated at each step, i.e., $G = G_{i-1}$ at this time. It is enough to prove that $r$ is contained in all $\gamma_{pr}(G_{i-1})$-set if and only if $r$ is contained in all $\gamma_{pr}(G_i)$-set for $1 \leq i \leq n - 1$. If $G_i = G_{i-1}$ for some $i$, then it is obviously true. When $v_i$ is considered, let $R_j = \{v \mid v \in V(G_{i-1}) \text{ and } l(v) = r_j\}$ for $j = 1, 2$.

**Lemma 3** When $v_i$ is a considering vertex such that $d(r, v_i) \geq 3$. If $l(v_i) = \emptyset$, $(R_1 \cup R_2) \cap C_{i-1}(v_i) \neq \emptyset$ and $G_{i-1}[R_1 \cap C_{i-1}(v_i)]$ has a perfect matching, then $r$ is contained in all $\gamma_{pr}(G_{i-1})$-set if and only if $r$ is contained in all $\gamma_{pr}(G_i)$-set, where $G_i = G_{i-1} - D_{i-1}[v_i]$.

**Proof** Let $D_1 = R_1 \cap C_{i-1}(v_i)$, $D_2 = R_2 \cap D_{i-1}(v_i)$ and $D = D_1 \cup D_2$. In details, $D_1 = \{u_1, u_2, \ldots, u_a\}$ and $D_2 = \{x_1, y_1, \ldots, x_b, y_b\}$, where $x_jy_j \in E$ and $F(y_j) = x_j$ for $1 \leq j \leq b$ for $1 \leq j \leq b$ (see the line indicated (*) in the procedure PRUNE.) Then we obtain the following claim.

**Claim 1** $\gamma_{pr}(G_{i-1}) = \gamma_{pr}(G_i) + |D|$.

**Proof** Any $\gamma_{pr}(G_i)$-set can be extended to a PDS of $G_{i-1}$ by adding $D$. Thus $\gamma_{pr}(G_{i-1}) \leq \gamma_{pr}(G_i) + |D|$. For converse, let $S$ be a $\gamma_{pr}(G_{i-1})$-set. If $y_j \notin S$, then $|D_{i-1}(y_j) \cap S| \geq 2$ and $S - D_{i-1}(y_j) \cup \{y_j, z_j\}$, where $z_j$ is a child of $y_j$, is also a $\gamma_{pr}(G_{i-1})$-set. Thus we may assume $y_j \in S$ and $w_j$ be its paired vertex. If $x_j \notin S$, then $S - \{w_j\} \cup \{x_j\}$ is also a $\gamma_{pr}(G_{i-1})$-set. If $x_j \in S$ and $w_j \neq x_j$, let $x'_j$ is the paired vertex of $x_j$. Then $x'_j = v_i$, otherwise $S - \{w_j, x'_j\}$ is a smaller PDS of $G_{i-1}$. It is a contradiction. If $N(v_i) \subseteq S$, then $S - \{v_i, w_j\}$ is a smaller PDS of $G_{i-1}$. Thus there is a neighbor $v'_i$ of $v_i$ such that $v'_i \notin S$. In this case, $S - CC \cup \{v_i, v'_i\}$ is also a $\gamma_{pr}(G_{i-1})$-set. Therefore, we may assume that $D_2 \subseteq S$ and every vertex in $D_2$ is paired with another vertex in $D_2$.

With the similar argument, we may assume that $D_1 \subseteq S$. Let $u'_j$ is the paired vertex of $u_j$ and $CC = \{u'_j \mid u'_j \notin D_1\}$. If $CC = \emptyset$, then we do nothing. If $CC \neq \emptyset$ and $v_i \notin CC$, then $S - CC$ is a smaller PDS of $G_{i-1}$, a contradiction. Thus we assume $v_i \in CC$ and it is paired with $u_1$. Since $G_{i-1}[D_1]$ has a perfect matching, there must be a vertex in $D_1$, say $u_2$, such that $u_2 \in CC$. If $N(v_i) \subseteq S$, then $S - \{u_2, v_i\}$ is a smaller PDS of $G_{i-1}$. Thus there exists a neighbor $v'_i$ of $v_i$ such that $v'_i \notin S$. In this case, $S - CC \cup \{v_i, v'_i\}$ is a $\gamma_{pr}(G_{i-1})$-set. Up to now, we may assume that $D \subseteq S$ and every vertex in $D$ is paired with another vertex in $D$.

If $v_i \notin S$, then $S - D$ is a PDS of $G_i$. Thus $\gamma_{pr}(G_i) \leq |S| - |D| = \gamma_{pr}(G_{i-1}) - |D|$. Therefore $\gamma_{pr}(G_{i-1}) = \gamma_{pr}(G_i) + |D|$. If $v_i \in S$, let $v'_i$ be its paired vertex. If $v'_i \in C_{i-1}(v_i)$, then there exists a neighbor $v''_i$ of $v_i$ such that $v''_i \notin S \cup C_{i-1}(v_i)$. Otherwise $S - \{v_i, v'_i\}$ is a smaller PDS of
Thus $S - \{v'_i\} \cup \{v''_i\}$ is a $\gamma_{pr}(G_{i-1})$-set. So we assume that $v'_i \not\in C_{i-1}(v_i)$. If $N(v'_i) \subseteq S$, then $S - \{v_i, v'_i\}$ is a smaller PDS of $G_{i-1}$. Thus there is a neighbor $v''_i$ of $v'_i$ such that $v''_i \not\in S$, in this case, $S - \{v_i\} \cup \{v''_i\}$ is a $\gamma_{pr}(G_{i-1})$-set not containing $v_i$. We may assume $S$ is such a $\gamma_{pr}(G_{i-1})$-set. Then $S - D$ is a PDS of $G_i$. Thus $\gamma_{pr}(G_i) \leq \vert S \vert - \vert D \vert = \gamma_{pr}(G_{i-1}) - \vert D \vert$. Therefore $\gamma_{pr}(G_{i-1}) = \gamma_{pr}(G_i) + \vert D \vert$. □

If there is a $\gamma_{pr}(G_i)$-set $S'$ such that $r \not\in S'$, then let $S = S' \cup D$. By Claim 1 $S$ is a $\gamma_{pr}(G_{i-1})$-set and $r \not\in S$. Therefore, if $r$ is contained in all $\gamma_{pr}(G_{i-1})$-set, then $r$ is contained in all $\gamma_{pr}(G_i)$-set.

For converse, let $S$ be an arbitrary $\gamma_{pr}(G_{i-1})$-set and $PD = S \cap D_{i-1}[v_i]$. 

Claim 2 $\vert D \vert \leq \vert PD \vert \leq \vert D \vert + 2$.

Proof It is obvious that $\vert PD \vert \geq \vert D \vert$. Next, we prove $\vert PD \vert \leq \vert D \vert + 2$. Let $v'_i$ be the father of $v_i$, i.e., $F(v_i) = v'_i$, and $B$ is a block of $G_{i-1}$ containing $v_i$ and $v'_i$. We discuss it according to the order of $B$.

Case 1: $V(B) = \{v_i, v'_i\}$

If $\vert PD \vert \geq \vert D \vert + 4$ and $\vert PD \vert$ is even, then $v'_i, v''_i \not\in S$, where $v''_i$ is the father of $v'_i$. Otherwise, $S - PD \cup D$ is a smaller PDS of $G_{i-1}$. However, $S - PD \cup D \cup \{v'_i, v''_i\}$ is also a smaller PDS of $G_{i-1}$, a contradiction. If $\vert PD \vert \geq \vert D \vert + 3$ and $\vert PD \vert$ is odd, then $v_i$ and $v'_i$ are paired in $S$. If $N(v'_i) \subseteq S$, then $S - PD - \{v'_i\} \cup D$ is a smaller PDS of $G_{i-1}$. Thus there is a neighbor $w$ of $v'_i$ such that $w \not\in S$, then $S - PD \cup D \cup \{w\}$ is also a smaller PDS of $G_{i-1}$. It is a contradiction.

Case 2: $V(B) \neq \{v_i, v'_i\}$

Let $w$ be another vertex in $V(B)$. If $\vert PD \vert \geq \vert D \vert + 4$ and $\vert PD \vert$ is even, then $w, v'_i \not\in S$, then $S - PD \cup D \cup \{v'_i, w\}$ is a smaller PDS of $G_{i-1}$. If $\vert PD \vert \geq \vert D \vert + 3$ and $\vert PD \vert$ is odd, then $v_i \in S$. If $w$ is the paired vertex of $v_i$, then there exists a neighbor $w'$ of $w$ such that $w' \not\in S$. However, $S - PD \cup D \cup \{w'\}$ is a smaller PDS of $G_{i-1}$. It is a contradiction. If $v'_i$ is the paired vertex of $v_i$, with the same argument to Case 1, we can also get a contradiction. □

By Claim 2 we have $\vert D \vert \leq \vert PD \vert \leq \vert D \vert + 2$. We discuss the following cases according to the size of $PD$.

Case 1: $\vert PD \vert = \vert D \vert + 2$

In this case, $(N(v_i) \cap V(G_i)) \cap S = \emptyset$. If $\vert N(v_i) \cap V(G_i) \vert \geq 2$, then let $S' = S - PD \cup \{w', w''\}$, where $w', w'' \in N(v_i) \cap V(G_i)$. By claim 1 $S'$ is a $\gamma_{pr}(G_i)$-set. Then $r \in S'$. Since $d(v_i, r) \geq 3$, then $r \in S$. If $\vert N(v_i) \cap V(G_i) \vert = 1$, then $F(v_i), F(F(v_i)) \not\subseteq S$, then let $S' = S - PD \cup \{F(v_i), F(F(v_i))\}$. By Claim 1 $S'$ is a $\gamma_{pr}(G_i)$-set. Then $r \in S'$. Since $d(v_i, r) \geq 3$, then $r \in S$.

Case 2: $\vert PD \vert = \vert D \vert + 1$
In this case, \( v_i \in S \), let \( \tilde{v} \) be its paired vertex. If \( N(\tilde{v}) \subseteq S \), then \( S - PD - \{\tilde{v}\} \cup D \) is a smaller PDS of \( G_{i-1} \). Thus there is a neighbor \( w \) of \( \tilde{v} \) such that \( w \not\in S \). Let \( S' = S - PD \cup \{w\} \). by Claim \(1\) \( S' \) is a \( \gamma_{pr} (G_i) \)-set. Then \( r \in S' \). Since \( d(v_i, r) \geq 3 \), then \( r \in S \).

**Case 3:** \( |PD| = |D| \)

In this case, let \( S' = S - PD \). Then by Claim \(1\) \( S' \) is a \( \gamma_{pr} (G_i) \)-set. Then \( r \in S' \). Thus \( r \in S \). □

**Lemma 4** When \( v_i \) is a considering vertex such that \( d(r, v_i) = 2 \). Let \( B_1 \) be the block containing \( v_i \) and \( F(v_i) \), and let \( B_2 \) be the block containing \( F(v_i) \) and \( r \). Suppose \( l(v_i) = \emptyset \), \( (R_1 \cup R_2) \cap C_{i-1}(v_i) \neq \emptyset \) and \( G_{i-1}[R_1 \cap C_{i-1}(v_i)] \) has a perfect matching. If \( G_{i-1} \) satisfies one of the following conditions:

1. \( |V(B_1)| \geq 3 \);
2. \( |V(B_1)| = 2 \) and \( C_{i-1}(F(v_i)) \neq \{v_i\} \);
3. \( |V(B_1)| = 2 \), \( C_{i-1}(F(v_i)) = \{v_i\} \) and \( |V(B_2)| \geq 3 \).

Then \( r \) is contained in all \( \gamma_{pr} (G_{i-1}) \)-set if and only if \( r \) is contained in all \( \gamma_{pr} (G_i) \)-set, where \( G_i = G_{i-1} - D_{i-1}[v_i] \).

**Proof** We still use the notations in Lemma \(3\). With the same argument to Claim \(1\), \( \gamma_{pr} (G_{i-1}) = \gamma_{pr} (G_i) + |D| \).

If there is a \( \gamma_{pr} (G_i) \)-set \( S' \) such that \( r \notin S' \), then let \( S = S' \cup D \). Thus \( S \) is a \( \gamma_{pr} (G_{i-1}) \)-set and \( r \notin S \). Therefore, if \( r \) is contained in all \( \gamma_{pr} (G_{i-1}) \)-set, then \( r \) is contained in all \( \gamma_{pr} (G_i) \)-set.

For converse, let \( S \) be an arbitrary \( \gamma_{pr} (G_{i-1}) \)-set and \( PD = S \cap D_{i-1}[v_i] \). With the similar argument to Claim \(2\), \( |D| \leq |PD| \leq |D| + 2 \). We discuss the following case according to the size of \( PD \).

**Case 1:** \( |PD| = |D| + 2 \)

If \( |V(B_1)| \geq 3 \), then let \( w \) be a vertex in \( V(B_1) \) other than \( v_i \) and \( F(v_i) \). Then \( w, F(v_i) \notin S \). Let \( S' = S - PD \cup \{w, F(v_i)\} \). Then \( S' \) is a \( \gamma_{pr} (G_i) \)-set. Since any new added vertex is not \( r \), then \( r \in S \). If \( |V(B_1)| = 2 \) and \( C_{i-1}(F(v_i)) \neq \{v_i\} \), let \( w \) be a child of \( F(v_i) \) other than \( v_i \). It is obvious that \( r, F(v_i) \notin S \). If \( w \notin S \), then \( S' = S - PD \cup \{F(v_i), w\} \) is a \( \gamma_{pr} (G_i) \)-set. If \( w \in S \) and \( w' \) is its paired vertex, then there is a neighbor \( w'' \) of \( w' \) such that \( w'' \notin S \). Then \( S' = S - PD \cup \{F(v_i), w''\} \) is a \( \gamma_{pr} (G_i) \)-set. Thus \( r \notin S' \). It contradicts that \( r \) is contained in all \( \gamma_{pr} (G_i) \)-set. If \( |V(B_1)| = 2 \), \( C_{i-1}(F(v_i)) = \{v_i\} \) and \( |V(B_2)| \geq 3 \), let \( w \) be a vertex in \( V(B_2) \) other than \( F(v_i) \) and \( r \). Then \( r, F(v_i), w \) and \( S = \emptyset \). Let \( S' = S - PD \cup \{w, F(v_i)\} \). Then \( S' \) is a \( \gamma_{pr} (G_i) \)-set. However, \( r \notin S' \). It contradicts that \( r \) is contained in all \( \gamma_{pr} (G_i) \)-set.

**Case 2:** \( |PD| = |D| + 1 \)

In this case, \( v_i \in S \). Let \( v_i' \) be the paired vertex of \( v_i \), then \( v_i' \in V(B_1) \). Suppose \( |V(B_1)| \geq 3 \). If \( v_i' \notin F(v_i) \) and \( F(v_i) \notin S \), then \( S' = S - PD \cup \{F(v_i)\} \) is a \( \gamma_{pr} (G_i) \)-set. If \( v_i' \neq F(v_i) \) and \( F(v_i) \in S \), then \( v_i' \) is a cut-vertex of \( G_{i-1} \). Otherwise, \( S - PD - \{v_i'\} \cup D \) is a smaller PDS
of $G_{i-1}$. It is impossible that $C_{i-1}(v_i') \subseteq S$. Thus there is a child $w$ of $v_i'$ such that $w \not\in S$. $S' = S - PD \cup \{w\}$ is a $\gamma_{pr}(G_i)$-set. If $v_i' = F(v_i)$, let $w$ be a vertex in $V(B_1)$ other than $v_i$ and $F(v_i)$. If $w \not\in S$, then $S' = S - PD \cup \{w\}$ is a $\gamma_{pr}(G_i)$-set. If $w \in S$, then $w$ is a cut-vertex. If its paired vertex $w' \in C_{i-1}(w)$, then there is a neighbor $w''$ of $w'$ such that $w'' \not\in S$. $S' = S - PD \cup \{w''\}$ is a $\gamma_{pr}(G_i)$-set. If $w \in S$ and its paired vertex $w' \in V(B_1)$, then $w'$ is also a cut-vertex. It is impossible that $C_{i-1}(w) \subseteq S$, i.e., there is a child $w''$ of $w$ such that $w'' \not\in S$. $S' = S - PD \cup \{w''\}$ is a $\gamma_{pr}(G_i)$-set. Then we can not prune $V$.

Proof We still use the notations in Lemma 3. With the same argument to Claim 1 $\gamma_{pr}(G_{i-1}) = \gamma_{pr}(G_i) + |D|$.

If $d(v_i, r) = 2$, $|V(B_1)| = 2$, $C_{i-1}(F(v_i)) = \{v_i\}$, $|V(B_2)| = 2$ and $v_i$ satisfies other conditions in Lemma 4 then we can not prune $G_{i-1}$. We call $B_2$ the first kind of TYPE-1 block containing $r$.

Lemma 5 When $v_i$ is a considering vertex such that $d(r, v_i) = 1$. Let $B$ be the block containing $v_i$ and $r$. Suppose $l(v_i) = \emptyset$, $(R_1 \cup R_2) \cap C_{i-1}(v_i) \neq \emptyset$ and $G_{i-1}[R_1 \cap C_{i-1}(v_i)]$ has a perfect matching. If $|V(B)| \geq 4$ or $|V(B)| = 3$ and every vertex in $V(B)$ is cut-vertex, then $r$ is contained in all $\gamma_{pr}(G_{i-1})$-set if and only if $r$ is contained in all $\gamma_{pr}(G_i)$-set, where $G_i = G_{i-1} - D_{i-1}[v_i]$.

Proof We still use the notations in Lemma 3. With the same argument to Claim 1 $\gamma_{pr}(G_{i-1}) = \gamma_{pr}(G_i) + |D|$.

If there is a $\gamma_{pr}(G_i)$-set $S'$ such that $r \not\in S'$, then let $S = S' \cup D$. Thus $S$ is a $\gamma_{pr}(G_{i-1})$-set and $r \not\in S$. Therefore, if $r$ is contained in all $\gamma_{pr}(G_{i-1})$-set, then $r$ is contained in all $\gamma_{pr}(G_i)$-set.

For converse, let $S$ be an arbitrary $\gamma_{pr}(G_{i-1})$-set and $PD = S \cap D_{i-1}[v_i]$. With the similar argument to Claim 2 $|D| \leq |PD| \leq |D| + 2$.

Suppose $|PD| = |D| + 2$, then $N(v_i) \cap V(B) \cap S = \emptyset$. Otherwise, $S - PD \cup D$ is a smaller PDS of $G_{i-1}$. Thus $r \not\in S$. If $|V(B)| \geq 4$, let $w_1$ and $w_2$ be two vertices other than $v_i$ and
In this case, \( S' = S - PD \cup \{w_1, w_2\} \) is a \( \gamma_{pr}(G_i) \)-set. However, \( r \notin S' \). It contradicts that \( r \) is contained in all \( \gamma_{pr}(G_{i-1}) \)-set. If \( |V(B)| = 3 \) and every vertex in \( V(B) \) is cut-vertex, let \( w \) be another vertex in \( V(B) \) other than \( v_i \) and \( r \). If there is a child \( w_1 \) of \( w \) such that \( w_1 \notin S \), then \( S - PD \cup \{w, w_1\} \) is a \( \gamma_{pr}(G_i) \)-set not containing \( r \). It is also a contradiction.

Otherwise, take any child of \( w \), say \( w_1 \). Suppose \( w_2 \) is the paired vertex of \( w_1 \). If \( N(w_2) \subseteq S \), then \( S - PD - \{w_2\} \cup D \) is a smaller PDS of \( G_i \). Thus there is a neighbor \( w_3 \) of \( w_2 \) such that \( w_3 \notin S \). Then \( S' = S - PD \cup \{w, w_3\} \) is a \( \gamma_{pr}(G_i) \)-set not containing \( r \). It is still a contradiction.

Suppose \( |PD| = |D| + 1 \), then \( v_i \in S \). If \( r \) is paired with \( v_i \), then we have done. If \( |V(B)| \geq 4 \), let \( w_1 \) and \( w_2 \) are two vertices other than \( v_i \) and \( r \). We assume \( w_1 \) is the paired vertex of \( v_i \). If \( w_2 \notin S \), then \( S' = S - PD \cup \{w_2\} \) is a \( \gamma_{pr}(G_i) \)-set. If \( w_2 \in S \), let \( w_3 \) be its paired vertex. If \( w_2 \) is not a cut-vertex, then \( S - PD - \{w_2\} \cup D \) is a smaller PDS of \( \gamma_{pr}(G_i) \)-set. Thus \( w_2 \) is a cut-vertex. If \( w_3 \in C_{i-1}(w_2) \), then there is a neighbor \( w_4 \) of \( w_3 \) such that \( w_4 \notin S \). \( S' = S - PD \cup \{w_4\} \) is a \( \gamma_{pr}(G_i) \)-set. If \( w_3 \in V(B) \), then \( w_3 \) is also a cut-vertex and there is a child \( w_4 \) of \( w_3 \) such that \( w_4 \notin S \). \( S' = S - PD \cup \{w_4\} \) is a \( \gamma_{pr}(G_i) \)-set. If \( |V(B)| = 3 \) and every vertex in \( V(B) \) is cut-vertex, let \( w \) be another vertex in \( V(B) \) other than \( v_i \) and \( r \).

In this case, \( w \) is the paired vertex of \( v_i \). If there is a child \( w_1 \) of \( w \) such that \( w_1 \notin S \), then \( S' = S - PD \cup \{w_1\} \) is a \( \gamma_{pr}(G_i) \)-set. Otherwise, take any child of \( w \), say \( w_1 \), and \( w_2 \) is its paired vertex. If \( N(w_2) \subseteq S \), then \( S - PD - \{w_2\} \cup D \) is a smaller PDS of \( G_{i-1} \). Thus there is a neighbor \( w_3 \) of \( w_2 \) such that \( w_3 \notin S \). Then \( S' = S - PD \cup \{w_3\} \) is a \( \gamma_{pr}(G_i) \)-set. In any case, \( r \in S' \). However, any new added vertex is not \( r \). Thus \( r \in S \).

If \( |PD| = |D| \), then \( S' = S - PD \) is a \( \gamma_{pr}(G_i) \)-set. Thus \( r \in S \) due to \( r \in S' \). □

If \( d(v_i, r) = 1 \), \( |V(B)| = 3 \), there is a vertex in \( V(B) \) which is not cut-vertex and \( v_i \) satisfies other conditions in Lemma 5 then we can not prune \( G_{i-1} \). We call \( B \) the second kind of TYPE-1 block containing \( r \). If \( d(v_i, r) = 1 \), \( |V(B)| = 2 \) and \( v_i \) satisfies other conditions in Lemma 5 we call \( B \) the first kind of TYPE-2 block containing \( r \).

**Lemma 6** When \( v_i \) is a considering vertex such that \( l(v_i) = r_1 \). Let \( B' \) is an end block containing \( v_i \) in \( G_{i-1} \). If \( G_{i-1}[R_1 \cap C_{i-1}(v_i)] \) has a perfect matching, then \( r \) is contained in all \( \gamma_{pr}(G_{i-1}) \)-set if and only if \( r \) is contained in all \( \gamma_{pr}(G_i) \)-set, where \( G_i = G_{i-1} - (D_{i-1}(v_i) - V(B')) \).

**Proof** Let \( D_1 = R_1 \cap C_{i-1}(v_i) \), \( D_2 = R_2 \cap D_{i-1}(v_i) \) and \( D = D_1 \cup D_2 \). Similar to Claim 1 we obtain the following claim.

**Claim 3** \( \gamma_{pr}(G_{i-1}) = \gamma_{pr}(G_i) + |D| \).

If there is a \( \gamma_{pr}(G_i) \)-set \( S' \) such that \( r \notin S' \), then let \( S = S' \cup D \). By Claim 3 \( S \) is a \( \gamma_{pr}(G_{i-1}) \)-set and \( r \notin S \). Therefore, if \( r \) is contained in all \( \gamma_{pr}(G_{i-1}) \)-set, then \( r \) is contained in all \( \gamma_{pr}(G_i) \)-set.
For converse, let \( S \) be an arbitrary \( \gamma_{pr}(G_{i-1}) \)-set and \( PD = S \cap (D_{i-1}(v_i) - V(B')) \).

**Claim 4** \(|D| \leq |PD| \leq |D| + 1.\)

**Proof** If \( |PD| \geq |D| + 2 \) and \(|PD|\) is even. Since \(|V(B') \cap S| \geq 1\), then either \( v_i \in S \) or \( y \in S \), where \( y \in V(B') - \{v_i\} \). \( S - PD \cup D \) is a smaller PDS of \( G_{i-1} \). It is a contradiction.

If \(|PD| \geq |D| + 3 \) and \(|PD|\) is odd. In this case, \( v_i \in S \) and its paired vertex \( v \in C_{i-1}(v_i) - V(B') \). Let \( x \in V(B') - \{v_i\} \), then \( x \notin S \). \( S - PD \cup D \cup \{x\} \) is a smaller PDS of \( G_{i-1} \). It is a contradiction. \( \square \)

If \(|PD| = |D| + 1 \), then \( v_i \in S \) and its paired vertex \( v \in C_{i-1}(v_i) - V(B') \). Let \( S' = S - PD \cup \{x\} \), where \( x \in V(B') - \{v_i\} \). By Claim 3 \( S' \) is a \( \gamma_{pr}(G_i) \)-set. Then \( r \in S' \). Since \( x \neq r \), \( r \in S \).

If \(|PD| = |D| \). Since \( V(B') \cap S \neq \emptyset \), Thus \( S' = S - PD \) is a PDS of \( G_i \). By Claim 3 \( S' \) is also a \( \gamma_{pr}(G_i) \)-set. Thus \( r \in S \) due to \( r \in S' \). \( \square \)

**Lemma 7** When \( v_i \) is a considering vertex such that \( d(r, v_i) \geq 3 \) and \( G_{i-1}[R_1 \cap C_{i-1}(v_i)] \) has not a perfect matching, let \( M \) be the maximum matching in \( G_{i-1}[R_1 \cap C_{i-1}(v_i)] \) and \( u \in (R_1 \cap C_{i-1}(v_i)) - V(M) \). Then \( r \) is contained in all \( \gamma_{pr}(G_{i-1}) \)-set if and only if \( r \) is contained in all \( \gamma_{pr}(G_i) \)-set, where \( G_i = G_{i-1} - (D_{i-1}(v_i) - D_{i-1}[u]) \).

**Proof** Let \( D_1 = R_1 \cap C_{i-1}(v_i) \) and \( D_2 = R_2 \cap D_{i-1}(v_i) \). Take one child of each vertex in \( D_1 - V(M) - \{u\} \) to construct vertex set \( D'_1 \). \( D = D_1 \cup D_2 \cup D'_1 - \{u\} \). Then we obtain the following claim.

**Claim 5** \( \gamma_{pr}(G_{i-1}) = \gamma_{pr}(G_i) + |D|. \)

**Proof** Any \( \gamma_{pr}(G_i) \)-set can be extended to a PDS of \( G_{i-1} \) by adding \( D \). Thus \( \gamma_{pr}(G_{i-1}) \leq \gamma_{pr}(G_i) + |D| \).

For converse, let \( S \) be a \( \gamma_{pr}(G_{i-1}) \)-set. With the same argument to Claim 4 \( D_2 \subset S \) and every vertex in \( D_2 \) is paired with another vertex in \( D_2 \). Moreover, we may assume \( D_1 \subset S \). Let \( CC = \{x \mid x \notin D_1, x \text{ is paired with one vertex in } D_1\} \). Since \( M \) is a maximum matching of \( G_{i-1}[D_1] \), thus \( |CC| \leq |D_1| - |V(M)| = |D'_1| + 1 \). If \( v_i \notin S \), then \( S - CC \cup D'_1 \cup \{v_i\} \) is also a \( \gamma_{pr}(G_{i-1}) \)-set. If \( v_i \in S \) and \( v_i \) is paired with one vertex in \( D_1 \), then \( S - CC \cup D'_1 \cup \{v_i\} \) is also a \( \gamma_{pr}(G_{i-1}) \)-set. If \( v_i \in S \) and \( v_i \) is not paired with any vertex in \( D_1 \), let \( v \) be its paired vertex. Then \( v \notin C_{i-1}(v_i) \), otherwise, \( S - CC - \{v\} \cup D'_1 \) is also a \( \gamma_{pr}(G_{i-1}) \)-set. Thus \( v \in V(B) \), where \( B \) is a block containing \( v_i \) and \( F(v_i) \). If \( N(v) \subseteq S \), then \( S - CC - \{v\} \cup D'_1 \) is also a \( \gamma_{pr}(G_{i-1}) \)-set. Therefore, we may assume \( D_1 \cup D'_1 \cup \{v_i\} \subseteq S \) and they are paired each other. Since \( u \) is the paired vertex of \( v_i \), \( S - D \) is a PDS of \( G_i \). Therefore, \( \gamma_{pr}(G_i) \leq |S - D| = |S| - |D| = \gamma_{pr}(G_{i-1}) - |D| \).
So $\gamma_{pr}(G_{i-1}) = \gamma_{pr}(G_i) + |D|$. □

If there is a $\gamma_{pr}(G_i)$-set $S'$ such that $r \notin S'$, then let $S = S' \cup D$ if $u \in S'$ or $v_i \in S'$ and otherwise, let $S = S' - D_{i-1}[u] \cup \{u, v_i\} \cup D$. By claim 5 $S$ is a $\gamma_{pr}(G_{i-1})$-set and $r \notin S$. Therefore, if $r$ is contained in all $\gamma_{pr}(G_{i-1})$-set, then $r$ is contained in all $\gamma_{pr}(G_i)$-set.

For converse, let $S$ be an arbitrary $\gamma_{pr}(G_{i-1})$-set and $PD = (D_{i-1}(v_i) - D_{i-1}[u]) \cap S$. We obtain the following claim.

**Claim 6** $|D| \leq |PD| \leq |D| + 1$

**Proof** It is obvious that $|PD| \geq |D|$. Suppose $|PD| \geq |D| + 2$ and $|PD|$ is even. If $v_i \in S$, then $S - PD \cup D$ is a smaller PDS of $G_{i-1}$. If $v_i \notin S$, then $S - D_{i-1}[v_i] \cup D \cup \{v_i, u\}$ is a smaller PDS of $G_{i-1}$. It is a contradiction. Suppose $|PD| \geq |D| + 3$ and $|PD|$ is odd. In this case, one of vertices $v_i, u$ is in $S$ such that its paired vertex is in $D_{i-1}(v_i) - D_{i-1}[u]$. If $v_i$ is such a vertex, then $|D_{i-1}[v_i] \cap S| \geq |D| + 4$. $S - D_{i-1}[v_i] \cup D \cup \{u, v_i\}$ is a smaller PDS of $G_{i-1}$. It is a contradiction. If $u$ is such a vertex and $v_i \notin S$, then $S - PD \cup D \cup \{v_i\}$ is a smaller PDS of $G_{i-1}$. It is also a contradiction. If $u$ is such a vertex and $v_i \in S$, then the paired vertex of $v_i$ is not a child of $v_i$. Let $v$ be its paired vertex. If $N(v) \subset S$, then $S - PD \cup \{v\} \cup D$ is a smaller PDS of $G_{i-1}$. Thus there is a neighbor $v'$ of $v$ such that $v' \notin S$. However, $S - PD \cup D \cup \{v'\}$ is also a smaller PDS of $G_{i-1}$. It is also a contradiction. □

Suppose $|PD| = |D| + 1$. If $v_i \in S$ and its paired vertex is in $D_{i-1}(v_i) - D_{i-1}[u]$, then $|D_{i-1}[v_i] \cap S| \geq |D| + 2$. Let $S' = S - D_{i-1}[v_i] \cup \{u, v_i\}$. By Claim 5 $S'$ is a $\gamma_{pr}(G_i)$-set. If $u \in S$ and its paired vertex is in $D_{i-1}(v_i) - D_{i-1}[u]$. If $v_i \notin S$, then $S' = S - PD \cup \{v_i\}$ is a $\gamma_{pr}(G_i)$-set by Claim 5. If $v_i \in S$, let $v$ be its paired vertex. Then $v \in V(G_i)$ and there is a neighbor $v'$ of $v$ such that $v' \notin S$. $S' = S - PD \cup D \cup \{v'\}$ is a $\gamma_{pr}(G_i)$-set. In any case, $r \in S'$. Since $d(r, v_i) \geq 3$, any new added vertex is not $r$. thus $r \in S$.

Suppose $|PD| = |D|$. If $v_i \notin S$, then $S' = S - D_{i-1}[v_i] \cup \{v_i, u\}$ is a $\gamma_{pr}(G_i)$-set. If $v_i \in S$, then $S' = S - PD$ is a $\gamma_{pr}(G_i)$-set. In any case, $r \in S'$. Since $d(v_i, r) \geq 3$, then $r \in S$. □

Similar to Lemma 3 and Lemma 5 we can obtain the following lemma. The detail of the proof is omitted in here.

**Lemma 8** When $v_i$ is a considering vertex such that $d(r, v_i) = 2$ and $G_{i-1}[R_1 \cap C_{i-1}(v_i)]$ has not a perfect matching, let $M$ be the maximum matching in $G_{i-1}[R_1 \cap C_{i-1}(v_i)]$ and $u \in R_1 \cap C_{i-1}(v_i) - V(M)$. Let $B_1$ be a block containing $v_i$ and $F(v_i)$ and $B_2$ be a block containing $F(v_i)$ and $F(F(v_i))$ if exists. If $G_{i-1}$ satisfies one of the following conditions:

1. $d(v_i, r) = 2$ and $|V(B_1)| \geq 3$;
2. $d(v_i, r) = 2$, $|V(B_1)| = 2$ and $C_{i-1}(F(v_i)) \neq \{v_i\}$;
(3) $d(v_i, r) = 2$, $|V(B_1)| = 2$, $C_{i-1}(F(v_i)) = \{v_i\}$ and $|V(B_2)| \geq 3$;
(4) $d(v_i, r) = 1$ and $|V(B_2)| \geq 4$;
(5) $d(v_i, r) = 1$, $|V(B_2)| = 3$ and every vertex in $V(B_2)$ is cut-vertex.

Then $r$ is contained in all $\gamma_{pr}(G_{i-1})$-set if and only if $r$ is contained in all $\gamma_{pr}(G_i)$-set, where $G_i = G_{i-1} - (D_{i-1}(v_i) - D_{i-1}(u))$.

If $d(v_i, r) = 2$, $|V(B_1)| = 2$, $C_{i-1}(F(v_i)) = \{v_i\}$, $|V(B_2)| = 2$ and $v_i$ satisfies other conditions in Lemma 8, then we can not prune $G_{i-1}$. We call $B_2$ the first kind of TYPE-3 block containing $r$. If $d(v_i, r) = 1$, $|V(B_2)| = 3$ and there is a vertex in $V(B_2)$ which is not cut-vertex and $v_i$ satisfies other conditions in Lemma 8, then we still can not prune $G_{i-1}$. We call $B_2$ the second kind of TYPE-3 block containing $r$. If $d(v_i, r) = 1$, $|V(B_2)| = 2$ and $v_i$ satisfies other conditions in Lemma 8 we call $B_2$ the second kind of TYPE-2 block containing $r$.

Summarizing the above lemmas, we have

**Theorem 1** Let $G$ be a block graph with at least one cut-vertex and let $\hat{G}$ be the output of procedure $\text{PRUNE}$. Then $r$ is contained in all minimum paired-dominating sets of $G$ if and only if $r$ is contained in all minimum paired-dominating sets of $\hat{G}$.

3 Algorithm

In this section, we will give some judgement rules to determine whether $r$ is contained in all minimum paired-dominating sets of $\hat{G}$, where $\hat{G}$ is the output of procedure $\text{PRUNE}$. Let $\hat{R}_j = \{v \mid v \in V(\hat{G}) \text{ and } l(v) = r_j\}$ for $j = 1, 2$. For $v \in V(\hat{G})$, define $C_{\hat{G}}(v) = C_G(v) \cap V(\hat{G})$, $D_{\hat{G}}(v) = D_G(v) \cap V(\hat{G})$ and $D_{\hat{G}}[v] = D_G[v] \cap V(\hat{G})$.

According to lemmas in section 2, we can divide blocks containing $r$ in $\hat{G}$ into the following categories (suppose $B$ is a block containing $r$ in $\hat{G}$). Some examples of each category are shown in Fig. 1.):

- $L_1 = \{B \mid B$ is an end block with $|V(B)| = 2\}$;
- $L_2 = \{B \mid B$ is an end block with $|V(B)| \geq 3\}$;
- $L_3 = \{B \mid B$ is a TYPE-1 block\};
- $L_4 = \{B \mid B$ is a TYPE-2 block\};
- $L_5 = \{B \mid B$ is a TYPE-3 block\};
- $L_6 = \{B \mid |\hat{R}_1 \cap (V(B) - \{r\})| \text{ is odd and } \hat{R}_2 \cap V(B) = \emptyset\}$;
- $L_7 = \{B \mid |\hat{R}_1 \cap (V(B) - \{r\})| \neq 0 \text{ is even and } \hat{R}_2 \cap V(B) = \emptyset\}$;
- $L_8 = \{B \mid |\hat{R}_1 \cap (V(B) - \{r\})| \text{ is odd and } \hat{R}_2 \cap V(B) \neq \emptyset\}$;
- $L_9 = \{B \mid |\hat{R}_1 \cap (V(B) - \{r\})| \text{ is even and } \hat{R}_2 \cap V(B) \neq \emptyset\}$.
In order to simply the proof of judgement rules, we define $D(B)$ for any block $B \in \bigcup_{i=3}^{9} L_i$ as follows:

1. If $B \in L_3$, then $|V(B)| = 2$ or $|V(B)| = 3$ and there is a vertex in $V(B)$ that is not cut-vertex. If $|V(B)| = 2$, then $B$ is the first kind. Let $u$ be the child of $r$ in $V(B)$ and $v$ be the child of $u$. If $|V(B)| = 3$, then $B$ is the second kind. Let $v$ be the child of $r$ in $V(B)$ and $v$ is a cut-vertex. In any case, $\tilde{G}[\tilde{R}_1 \cap C_{\tilde{G}}(v)]$ has a perfect matching. $D(B) = (\tilde{R}_1 \cap C_{\tilde{G}}(v)) \cup (\tilde{R}_2 \cap D_{\tilde{G}}(v))$.

2. If $B \in L_4$, then $|V(B)| = 2$. Let $v$ be the child of $r$ in $V(B)$. If $B$ is the first kind, then $\tilde{G}[\tilde{R}_1 \cap C_{\tilde{G}}(v)]$ has a perfect matching. Let $D(B) = (\tilde{R}_1 \cap C_{\tilde{G}}(v)) \cup (\tilde{R}_2 \cap D_{\tilde{G}}(v))$. Otherwise, let $M$ be the maximum matching in $\tilde{G}[\tilde{R}_1 \cap C_{\tilde{G}}(v)]$. Take one child of each vertex in $(\tilde{R}_1 \cap C_{\tilde{G}}(v)) - V(M) - \{w\}$ to construct $D'$, where $w \in \tilde{R}_1 \cap C_{\tilde{G}}(v) - V(M)$. $D(B) = (\tilde{R}_1 \cap C_{\tilde{G}}(v)) \cup D' \cup (\tilde{R}_2 \cap D_{\tilde{G}}(v)) \cup \{v, w\}$.

3. If $B \in L_5$, then $|V(B)| = 2$ or $|V(B)| = 3$ and there is a vertex in $V(B)$ that is not cut-vertex. If $B$ is the first kind, let $u$ be the child of $r$ in $V(B)$ and $v$ be the child of $u$. If $B$ is the second kind, let $v$ be the child of $r$ in $V(B)$ and $v$ is a cut-vertex. In any case, $\tilde{G}[\tilde{R}_1 \cap C_{\tilde{G}}(v)]$ has not a perfect matching. $D(B)$ is defined same as the second kind of (2).

4. If $B \in L_6 \cup L_8$, let $CC = \bigcup_{v \in V(B)} D_{\tilde{G}}[v]$. $D(B) = ((\tilde{R}_1 \cup \tilde{R}_2) \cap CC) \cup \{w\}$, where $w$ is a child of some vertex in $\tilde{R}_1 \cap CC$.

5. If $B \in L_7 \cup L_9$, let $CC = \bigcup_{v \in V(B)} D_{\tilde{G}}[v]$. $D(B) = (\tilde{R}_1 \cup \tilde{R}_2) \cap CC$. 

Fig. 1. Some examples of nine categories of blocks containing $r$ in $\tilde{G}$.
Lemma 9 Let $\tilde{G}$ be a output of procedure PRUNE, then $r$ is contained in all minimum paired-dominating sets of $\tilde{G}$ if and only if $\tilde{G}$ satisfies one of the following conditions:

1. $|L_1| \geq 1$;
2. $|L_1| = 0$ and $|L_2| \geq 2$;
3. $|L_1| = 0$, $|L_2| = 1$ and $|L_3 \cup L_6 \cup L_8| \geq 1$;
4. $|L_1| = 0$, $|L_2| = 0$ and $|L_3| \geq 2$;
5. $|L_1| = 0$, $|L_2| = 0$, $|L_3| = 1$ and $|L_6 \cup L_8| \geq 1$.

Proof If $|L_1| \geq 1$, then $r$ is a support vertex in $\tilde{G}$, and hence $r$ is contained in all minimum paired-dominating sets of $\tilde{G}$. Thus in the following discussion, we assume $|L_1| = 0$.

**Case 1:** $|L_2| \geq 2$

In this case, $r$ is contained in at least two end block with order at least three, say $B_1$ and $B_2$ are two such blocks. Let $S$ be an arbitrary $\gamma_{pr}(\tilde{G})$-set. If $r \notin S$, then $|V(B_i) \cap S| \geq 2$ for $i = 1, 2$. Then $S - V(B_1) - V(B_2) \cup \{r, x\}$, where $x$ is a vertex in $V(B_1) - \{r\}$, is a smaller PDS of $\tilde{G}$, a contradiction. Thus $r \in S$.

**Case 2:** $|L_2| = 1$ and $|L_3 \cup L_6 \cup L_8| \geq 1$

Let $B' \in L_2$ and $S$ be an arbitrary $\gamma_{pr}(\tilde{G})$-set not containing $r$. It is obvious $|V(B') \cap S| \geq 2$. If $|L_3| \geq 1$, let $B \in L_3$. If $B$ is the first kind, let $u$ be a child of $r$ in $V(B)$. Since $r \notin S$, $|D_{\tilde{G}}[u] \cap S| \geq 2 + |D(B)|$. However, $S - D_{\tilde{G}}[u] - V(B') \cup D(B) \cup \{r, u\}$ is a smaller PDS of $\tilde{G}$. If $B$ is the second kind, let $w$ be a vertex in $V(B)$ which is not cut-vertex and $u$ be another vertex. Since $r \notin S$, $|(D_{\tilde{G}}[u] \cup \{w\}) \cap S| \geq |D(B)| + 2$. Then $S - D_{\tilde{G}}[u] - V(B') - \{w\} \cup D(B) \cup \{r, u\}$ is a smaller PDS of $\tilde{G}$, a contradiction. Thus $r \in S$.

If $|L_6 \cup L_8| \geq 1$, let $B \in L_6 \cup L_8$. $CC = \bigcup_{v \in V(B)} D_{\tilde{G}}[v]$. Since $r \notin S$, $|CC \cap S| \geq |D(B)|$. However, $S - CC - V(B') \cup D(B) \cup \{r\} - \{w\}$, where $w \in D(B)$ and $l(w) = \emptyset$, is a smaller PDS of $\tilde{G}$, a contradiction. Thus $r \in S$.

**Case 3:** $|L_2| = 1$ and $|L_3 \cup L_6 \cup L_8| = 0$

Let $B' \in L_2$ and $y, z \in V(B') - \{r\}$. Since $r$ is a cut-vertex, So $L_4 \cup L_5 \cup L_7 \cup L_9 \neq \emptyset$. Let $S'$ be a vertex set by collecting $D(B)$ for any $B \in L_4 \cup L_5 \cup L_7 \cup L_9$. It is obvious that $S' \cup \{y, z\}$ is a $\gamma_{pr}(\tilde{G})$-set. However, $r \notin S$.

**Case 4:** $|L_2| = 0$ and $|L_3| \geq 2$

Let $B_1, B_2 \in L_3$ and $S$ be an arbitrary $\gamma_{pr}(\tilde{G})$-set. Suppose $r \notin S$. For $B_j$ $(j = 1, 2)$, let $CC_j = \bigcup_{v \in V(B_j)} D_{\tilde{G}}[v]$. Since $r \notin S$, $|CC_j \cap S| \geq |D(B_j)|$ for $j = 1, 2$. However, $S - CC_1 - CC_2 - D(B_1) \cup D(B_2) \cup \{r, u\}$, where $u$ is a child of $r$ in $V(B_1)$, is a smaller PDS of $\tilde{G}$, a contradiction. Thus $r \in S$.

**Case 5:** $|L_2| = 0$, $|L_3| = 1$ and $|L_6 \cup L_8| \geq 1$

Let $B_1 \in L_3$ and $B_2 \in L_6 \cup L_8$. Suppose $S$ be an arbitrary $\gamma_{pr}(\tilde{G})$-set and $r \notin S$. For $B_j$ $(j = 1, 2)$,
let $CC_j = \bigcup_{v \in V(B_j)} D_G[v]$. Since $r \notin S$, $|CC_1 \cap S| \geq |D(B_1)| + 2$ and $|CC_2 \cap S| \geq |D(B_2)|$. However, $S - CC_1 - CC_2 \cup D(B_1) \cup D(B_2) \cup \{r\} - \{w\}$, where $w \in D(B_2)$ and $l(w) = \emptyset$, is a smaller PDS of $\tilde{G}$, a contradiction. Thus $r \in S$.

**Case 6:** $|L_2| = 0$, $|L_3| = 1$ and $|L_6 \cup L_8| = 0$

Let $B \in L_3$. If $B$ is the first kind, let $u$ be the child of $r$ in $V(B)$ and $v$ be the child of $u$. If $B$ is the second kind, let $\{u, v\} = V(B) - \{r\}$. Let $S'$ be a vertex set by collecting $D(B^*)$ for any $B^* \in L_4 \cup L_5 \cup L_7 \cup L_9$. Let $S = S' \cup D(B) \cup \{u, v\}$. Then it is obvious $S$ is a $\gamma_{pr}(\tilde{G})$-set. However, $r \notin S$.

**Case 7:** $|L_2| = |L_3| = 0$

Let $B$ be any block containing $r$, then $B \in L_4 \cup L_5 \cup L_6 \cup L_7 \cup L_8 \cup L_9$. Let $S'$ be a vertex set by collecting $D(B)$ for any $B \in L_4 \cup L_5 \cup L_6 \cup L_7 \cup L_8 \cup L_9$. If $L_6 \cup L_7 \cup L_8 \cup L_9 \neq \emptyset$, then $S'$ is a $\gamma_{pr}$-set of $\tilde{G}$. However, $r \notin S'$. Thus we may assume $L_6 \cup L_7 \cup L_8 \cup L_9 = \emptyset$. Then $B \in L_4 \cup L_5$. If there is a block $B \in L_4 \cup L_5$ which is the second kind of TYPE-2 or TYPE-3 block, then $S'$ is still a $\gamma_{pr}$-set of $\tilde{G}$. Thus we may assume that $B \in L_4 \cup L_5$ and $B$ is the first kind of TYPE-2 or TYPE-3 block. If there is a block $B \in L_5$, let $u$ be the child of $r$ in $V(B)$ and $v$ is the child of $u$. Let $w$ be the paired vertex in $D(B)$ and $w'$ be the child of $w$. Then $S = S' \cup \{u, w'\}$ is a $\gamma_{pr}(\tilde{G})$-set of $\tilde{G}$. However, $r \notin S$. Then $B \in L_4$ for any block $B$ and $B$ is the first kind of TYPE-2 block. Let $v$ be the child of $r$ in $V(B)$. If there is a child $w$ of $v$ such that $l(w) = r_1$. Let $w'$ be the child of $w$. Then $S = S' \cup \{v, w'\}$ is a $\gamma_{pr}(\tilde{G})$-set not containing $r$. Thus we may assume every child $w$ of $v$ satisfies $l(w) = r_2$. Let $w'$ be the child of $w$ such that $l(w') = r_2$ and let $w''$ be the child of $w'$. Take $S = S' \cup \{v, w''\}$. It is obvious that $S$ is a $\gamma_{pr}(\tilde{G})$-set not containing $r$.  

Now we are ready to present the algorithm to determine whether $r$ is contained in all minimum paired-dominating sets of $G$.

**Algorithm VIAMPDS.** Determine whether the cut-vertex $r$ of a block graph $G$ is contained in all minimum paired-dominating sets of $G$

**Input.** A block graph $G$ with at least one cut-vertex and a cut-vertex $r$. The vertex ordering obtained by procedure VO.

**Output.** True or False

**Method**

Let $\tilde{G}$ be the output of procedure PRUNE with input $G$.

Let $L_1 = \{B \mid B$ is an end block with $|V(B)| = 2\}$;

$L_2 = \{B \mid B$ is an end block with $|V(B)| \geq 3\}$;

$L_3 = \{B \mid B$ is a TYPE-1 block\};

$L_6 = \{B \mid B$ is a block such that $|\tilde{R}_1 \cap (V(B) - \{r\})|$ is odd and $\tilde{R}_2 \cap V(B) = \emptyset\}$;
\[ L_8 = \{ B \mid B \text{ is a block such that } |\tilde{R}_1 \cap (V(B) - \{r\})| \text{ is odd and } \tilde{R}_2 \cap V(B) \neq \emptyset \}. \]

\((B \text{ is a block containing } r)\)

If \(|L_1| \geq 1\) then
- Return True;
else if \(|L_2| \geq 2\) then
- Return True;
else if \(|L_2| = 1 \text{ and } |L_3 \cup L_6 \cup L_8| \geq 1\) then
- Return True;
else if \(|L_2| = 0 \text{ and } |L_3| \geq 2\) then
- Return True;
else if \(|L_2| = 0 \text{ and } |L_3| = 1 \text{ and } |L_6 \cup L_8| \geq 1\) then
- Return True;
else
- Return False;
endif
end

**Theorem 2** Algorithm VIAMPDS can determine whether the give cut-vertex of a block graph \(G\) with at least one cut-vertex is contained in all minimum paired-dominating sets in linear-time \(O(n + m)\), where \(n = |V(G)|\) and \(m = |E(G)|\).

**Proof** By Theorem 1, \(r\) is contained in all minimum paired-dominating sets of \(G\) if and only if \(r\) is contained in all minimum paired-dominating sets of \(\tilde{G}\), where \(\tilde{G}\) is the output of procedure PRUNE with input \(G\). Moreover, by Lemma 9 the judgement rules in algorithm VIAMPDS can determine whether \(r\) is contained in all minimum paired-dominating sets of \(\tilde{G}\). On the other hand, every vertex and edge is used in a constant times in algorithm VIAMPDS. Thus the theorem follows. \(\square\)

**4 Conclusion**

In this paper, we give a linear-time algorithm VIAMPDS to determine whether the given vertex is contained in all minimum paired-dominating sets of a block graph. Furthermore, the algorithm VIAMPDS can be used to determine the set of vertices contained in all minimum paired-dominating sets of a blocks graph in polynomial time. Finally, we would like to point out that if changing the pruning rules and judgement rules, our method is also available to determine whether a given vertex is contained in all minimum (total) dominating sets of a block graph.
References

[1] D.B. West, Introduction to Graph Theory, 2nd ed., Prentice Hall, Inc., NJ, 2001.

[2] T.W. Haynes, S.T. Hedetniemi and P.J. Slater (eds), Fundamentals of Domination in Graphs, New York, Marcel Dekker 1998.

[3] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater (eds), Domination in Graphs: Advanced Topics, New York, Marcel Dekker 1998.

[4] T.W. Haynes, P.J. Slater, Paired-domination and the paired-domatic number, Congr. Numer. 109(1995), 65-72.

[5] T.W. Haynes and P.J. Slater, Paired-domination in graphs, Networks 32(1998), 199-206.

[6] M.A. Henning, Graphs with large paired-domination number, J. Comb. Optim. 13(2007), 61-78.

[7] H. Qiao, L.Y. Kang, M. Caedei, D.Z. Du, Paired-domination of trees, J.Global Optim. 25(2003), 43-54.

[8] L. Kang, M.Y. Sohn, T.C.E. Cheng, Paired-domination in inflated graphs, Theoretical Comp. Sci. 320(2004), 485-494.

[9] O. Favaron, M.A. Henning, Paired-Domination in claw-free Cubic Graphs, Graphs and Comb. 20(2004), 447-456.

[10] P. Dorbec, S. Gravier, M.A. Henning, Paired-domination in generalized claw-free graphs, J. Comb. Optim. 14(2007), 1-7.

[11] L. Chen, C. Lu, Z. Zeng, Labelling algorithms for paired-domination problems in block and interval graphs, to appear in J. Comb. Optim.

[12] C.M. Mynhardt, Vertices contained in every minimum dominating set of a tree, J. Graph Theory, 31(1999), 163-177.

[13] E.J. Cockayne, M.A. Henning, C.M. Mynhardt, Vertices contained in all or in no minimum total dominating set of a tree, Disc. Math. 260(2003), 37-44.

[14] M.A. Henning, M.D. Plummer, Vertices contained in all or in no minimum paired-dominating set of a tree, J. Comb. Optim. 10(2005), 283-294.