REAL ZEROS OF RANDOM DIRICHLET SERIES

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Abstract. Let $F(\sigma)$ be the random Dirichlet series $F(\sigma) = \sum_{p \in \mathcal{P}} \frac{X_p}{p^\sigma}$, where $\mathcal{P}$ is an increasing sequence of positive real numbers and $(X_p)_{p \in \mathcal{P}}$ is a sequence of i.i.d. random variables with $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = 1/2$. We prove that, for certain conditions on $\mathcal{P}$, if $\sum_{p \in \mathcal{P}} \frac{1}{p} < \infty$ then with positive probability $F(\sigma)$ has no real zeros while if $\sum_{p \in \mathcal{P}} \frac{1}{p} = \infty$, almost surely $F(\sigma)$ has an infinite number of real zeros.

1. Introduction.

A Dirichlet series is an infinite sum of the form $F(\sigma) := \sum_{p \in \mathcal{P}} \frac{X_p}{p^\sigma}$, where $\mathcal{P}$ is an increasing sequence of positive real numbers and $(X_p)_{p \in \mathcal{P}}$ is any sequence of complex numbers. If $F(\sigma)$ converges then $F(s)$ converges for all $s \in \mathbb{C}$ with real part greater than $\sigma$ (see [4] Theorem 1.1). The abscissa of convergence of a Dirichlet series is the smallest number $\sigma_c$ for which $F(\sigma)$ converges for all $\sigma > \sigma_c$.

The problem of finding the zeros of a Dirichlet series is classical in Analytic Number Theory. For instance, the Riemann hypothesis states that the zeros of the analytic continuation of the Riemann zeta function $\zeta(\sigma) := \sum_{k=1}^{\infty} \frac{1}{k^\sigma}$ in the half plane $\{\sigma + it \in \mathbb{C} : \sigma > 0\}$ all have real part equal to $1/2$. This analytic continuation can be described in terms of a convergent Dirichlet series – The Dirichlet $\eta$-function $\eta(s) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^s}$ satisfies $\eta(s) = (1 - 2^{1-s})\zeta(s)$, for all complex $s$ with positive real part. Thus, to find zeros of $\eta(s)$ for $0 < \text{Re}(s) < 1$ is the same as finding non-trivial zeros of $\zeta$.

In this paper we are interested in the real zeros of the random Dirichlet series $F(\sigma) := \sum_{p \in \mathcal{P}} \frac{X_p}{p^\sigma}$, where the coefficients $(X_p)_{p \in \mathcal{P}}$ are random and $\mathcal{P}$ satisfies:

\[(P1) \quad \mathcal{P} \cap [0, 1) = \emptyset,\]
\[(P2) \quad \sum_{p \in \mathcal{P}} \frac{1}{p^\sigma} \text{ has abcissa of convergence } \sigma_c = 1.\]

For instance, $\mathcal{P}$ can be the set of the natural numbers. The conditions $(P1 - P2)$ imply, in particular, that the series $\sum_{p \in \mathcal{P}} \frac{1}{p^\sigma}$ converges for each $\sigma > 1/2$. Therefore,
if \((X_p)_{p \in \mathcal{P}}\) is a sequence of i.i.d. random variables with \(\mathbb{E}X_p = 0\) and \(\mathbb{E}X_p^2 = 1\), then, by the Kolmogorov one-series Theorem, the series \(F(\sigma) = \sum_{p \in \mathcal{P}} \frac{X_p}{p^\sigma}\) has a.s. absissa of convergence \(\sigma_c = 1/2\). Moreover, the function of one complex variable \(\sigma + it \mapsto F(\sigma + it)\) is a.s. an analytic function in the half plane \(\{\sigma + it \in \mathbb{C} : \sigma > 1/2\}\).

In the case \(X_p = \pm 1\) with equal probability, the line \(\sigma = \sigma_c\) is a natural boundary for \(F(\sigma + it)\), see [2] (pg. 44 Theorem 4).

Our main result states:

**Theorem 1.1.** Assume that \(\mathcal{P}\) satisfies P1-P2 and let \((X_p)_{p \in \mathcal{P}}\) be i.i.d and such that \(\mathbb{P}(X_p = 1) = \mathbb{P}(X_p = -1) = 1/2\). Let \(F(\sigma) = \sum_{p \in \mathcal{P}} \frac{X_p}{p^\sigma}\).

i. If \(\sum_{p \in \mathcal{P}} \frac{1}{p} < \infty\), then with positive probability \(F\) has no real zeros;

ii. If \(\sum_{p \in \mathcal{P}} \frac{1}{p} = \infty\), then a.s. \(F\) has an infinite number of real zeros.

It follows as corollary to the proof of item i. that in the case \(\sum_{p \in \mathcal{P}} \frac{1}{p} = \infty\), with positive probability \(F(\sigma)\) has no zeros in the interval \([1/2 + \delta, \infty)\), for fixed \(\delta > 0\).

Since a Dirichlet series \(F(s) = \sum_{p \in \mathcal{P}} \frac{X_p}{p^s}\) is a random analytic function, it can be viewed as a random Taylor series \(\sum_{k=0}^\infty Y_k(s-a)^k\), where \(a > \sigma_c\) and \((Y_k)_{k \in \mathbb{N}}\) are random and dependent random variables. The case of random Taylor series and random polynomials where \((Y_k)_{k \in \mathbb{N}}\) are i.i.d. has been widely studied in the literature, for an historical background we refer to [3] and [5] and the references therein.

2. **Preliminaries**

2.1. **Notation.** We employ both \(f(x) = O(g(x))\) and Vinogradov’s \(f(x) \ll g(x)\) to mean that there exists a constant \(c > 0\) such that \(|f(x)| \leq c|g(x)|\) for all sufficiently large \(x\), or when \(x\) is sufficiently close to a certain real number \(y\). For \(\sigma \in \mathbb{R}\), \(H_\sigma\) denotes the half plane \(\{z \in \mathbb{C} : \text{Re}(z) > \sigma\}\). The indicator function of a set \(S\) is denoted by \(1_S(s)\) and it is equal to 1 if \(s \in S\), or equal to 0 otherwise. We let \(\pi(x)\) to denote the counting function of \(\mathcal{P}\):

\[
\pi(x) := |\{p \leq x : p \in \mathcal{P}\}|.
\]

2.2. **The Mellin transform for Dirichlet series.** In what follows \(\mathcal{P} = \{p_1 < p_2 < ...\}\) is a set of non-negative real numbers satisfying P1-P2 above. A generic element of \(\mathcal{P}\) is denoted by \(p\), and we employ \(\sum_{p \leq x}\) to denote \(\sum_{p \in \mathcal{P} : p \leq x}\). Let \(A(x) = \sum_{p \leq x} X_p\) and \(F(s) = \sum_{p \in \mathcal{P}} \frac{X_p}{p^s}\). Let \(\sigma_c > 0\) be the absissa of convergence
of $F(\sigma)$. Then $F$ can be represented as the Mellin transform of the function $A(x)$ (see, for instance, Theorem 1.3 of [4]):

\begin{equation}
F(s) = s \int_1^\infty A(x) \frac{dx}{x^{1+s}}, \text{ for all } s \in \mathbb{H}_{\sigma_c}.
\end{equation}

In particular, we can state:

**Lemma 2.1.** Let $F(s) = \sum_{p \in \mathbb{P}} \frac{X_p}{p^s}$ be such that $F(1/2)$ is convergent. Then for each $\sigma \geq 1/2$ and all $\epsilon > 0$, for all $U > 1$:

$$
F(\sigma + \epsilon) = \sum_{p \leq U} \frac{X_p}{p^{\sigma + \epsilon}} + O \left( U^{-\epsilon} \sup_{x > U} \left| \sum_{U < p \leq x} \frac{X_p}{p^\sigma} \right| \right),
$$

where the implied constant in the $O(\cdot)$ term above can be taken to be 1.

**Proof.** Put $A(x) = \sum_{p \leq x} \frac{1_{(U, \infty)}(p) X_p}{p^\sigma}$. By (1) it follows that

$$
\sum_{p > U} \frac{X_p p^{-\sigma}}{p^\epsilon} = \epsilon \int_1^\infty A(x) \frac{dx}{x^{1+\epsilon}} = \epsilon \int_U^\infty \left( \sum_{U < p \leq x} \frac{X_p}{p^\sigma} \right) \frac{dx}{x^{1+\epsilon}}
\ll \sup_{x > U} \left| \sum_{U < p \leq x} \frac{X_p}{p^\sigma} \right| \int_U^\infty \epsilon \frac{dx}{x^{1+\epsilon}} = U^{-\epsilon} \sup_{x > U} \left| \sum_{U < p \leq x} \frac{X_p}{p^\sigma} \right|.
\]

\]
fixed number to be chosen later, $A_U$ is the event in which $X_p = 1$ for all $p \leq U$ and $B_U$ is the event in which

$$\sup_{x > U} \left| \sum_{U < p \leq x} \frac{X_p}{\sqrt{p}} \right| < \frac{1}{10}.$$ 

We claim that for sufficiently large $U$ on the event $A_U \cap B_U$ the function $F(s) = \sum_{p \in \mathcal{P}} \frac{X_p}{p^s}$ does not vanish for all $s \geq 1/2$. Further for sufficiently large $U$ we will show that $\mathbb{P}(A_U \cap B_U) > 0$.

On the event $A_U \cap B_U$ we have by lemma 2.1 that

$$F(1/2 + \epsilon) \geq \sum_{p \leq U} \frac{1}{p^{1/2+\epsilon}} - \frac{1}{10U^\epsilon} \geq \pi(U) \frac{1}{U^{1/2+\epsilon}} - \frac{1}{10U^\epsilon},$$

where $\pi(U) = \#\{p \leq U : p \in \mathcal{P}\}$. We claim that for each $\delta > 0$ we have that

$$\limsup_{U \to \infty} \frac{\pi(U)}{U^{1-\delta}} = \infty.$$

In fact, this is a consequence from P2: For any $\delta > 0$ the series diverges $\sum_{p \in \mathcal{P}} \frac{1}{p^{1-\delta}} = \infty$. To show that this is true we argue by contraposition: Assume that for some fixed $\delta > 0$ $\limsup_{U \to \infty} \frac{\pi(U)}{U^{1-\delta}} < \infty$ and hence that there exists a constant $c > 0$ such that for all $U > 0$, $\pi(U) \leq cU^{1-\delta}$. In that case we have for $0 < \epsilon < \delta$

$$\sum_{p \leq U} \frac{1}{p^{1-\delta}} = \int_1^U \frac{d\pi(x)}{x^{1-\epsilon}} = \pi(U) \frac{1}{U^{1-\epsilon}} - \pi(1) + (1 - \epsilon) \int_1^U \frac{\pi(x)}{x^{2-\epsilon}} dx$$

$$\leq \frac{cU^{1-\delta}}{U^{1-\epsilon}} + 1 + (1 - \epsilon) \int_1^U \frac{cx^{1-\delta}}{x^{2-\epsilon}} dx \ll 1 + \int_1^U \frac{1}{x^{1+(\delta-\epsilon)}} dx \ll 1,$$

and hence that the series $\sum_{p \in \mathcal{P}} \frac{1}{p^{1-\delta}}$ converges. Therefore, we showed that $\limsup_{U \to \infty} \frac{\pi(U)}{U^{1-\delta}} < \infty$ implies that $\sum_{p \in \mathcal{P}} \frac{1}{p^{1-\epsilon}}$ has abscissa of convergence $\sigma_c \leq 1 - \delta$.

Now we may select arbitrarily large values of $U > 1$ for which $\pi(U) \geq U^{1-1/4}$ and $\sum_{p \leq U} \frac{1}{\sqrt{p}} > \frac{1}{10}$, and hence, by (4), for all $\epsilon > 0$ we obtain that

$$F(1/2 + \epsilon) \geq \frac{U^{1-1/4}}{U^{1/2+\epsilon}} - \frac{1}{10U^\epsilon} = \frac{1}{U^\epsilon} \left( U^{1/4} - \frac{1}{10} \right) > 0.$$ 

This proves that on the event $A_U \cap B_U$ we have that $F(s) \neq 0$ for all $s \in [1/2, \infty)$.}

Observe that $A_U$ and $B_U$ are independent and $A_U$ has probability $\frac{1}{2^{10U^\epsilon}} > 0$. Now we will show that the complementary event $B^c_U$ has small probability. Indeed,
by applying the Levy’s maximal inequality and the Hoeffding’s inequality, we obtain:

$$
P(B_U) = \lim_{n \to \infty} P\left( \max_{U < x \leq n} \left| \sum_{U < p \leq x} \frac{X_p}{\sqrt{p}} \right| \geq \frac{1}{10} \right) \leq 3 \lim_{n \to \infty} \max_{U < x \leq n} P\left( \left| \sum_{U < p \leq x} \frac{X_p}{\sqrt{p}} \right| \geq \frac{1}{30} \right)
$$

$$
\leq 6 \lim_{n \to \infty} \max_{U < x \leq n} P\left( \sum_{U < p \leq x} \frac{X_p}{\sqrt{p}} \geq \frac{1}{30} \right) \leq 6 \lim_{n \to \infty} \exp\left( -\frac{1}{2} \cdot \frac{30^2}{\sum_{p > U} \frac{1}{p^2}} \right).
$$

Since $\sum_{p \in \mathcal{P}} \frac{1}{p}$ is convergent, the tail $\sum_{p > U} \frac{1}{p}$ converges to 0 as $U \to \infty$. Therefore, for sufficiently large $U$ we can make $P(B_U) < 1/2$.

Now we are going to prove Theorem 1.1 part ii. We present two different proofs.

In the first proof we assume that the counting function of $\mathcal{P}$

$$
\pi(x) \ll \frac{x}{\log x},
$$

In this case, for instance, $\mathcal{P}$ can be the set of prime numbers. In this proof we show that, for $\sigma$ close to $1/2$, the infinite sum $\sum_{p \in \mathcal{P}} \frac{X_p}{p^\sigma}$ can be approximated by the partial sum $\sum_{p \leq y} \frac{X_p}{\sqrt{p}}$ for a suitable choice of $y$ (Lemma 3.1). Then we show that these partial sums change sign for an infinite number of $y$, and hence, $F(\sigma) = \sum_{p \in \mathcal{P}} \frac{X_p}{p^\sigma}$ changes sign for an infinite number of $\sigma \to 1/2^+$.

The case in which $\mathcal{P}$ is the set of natural numbers, the infinite sum $\sum_{p \in \mathcal{P}} \frac{X_p}{p^\sigma}$ can not be approximated by the finite sum $\sum_{p \leq y} \frac{X_p}{\sqrt{p}}$, i.e, Lemma 3.1 fails in this case. Thus, our approach is different in the general case. First we show (Lemma 3.3) that $\sum_{p \in \mathcal{P}} \frac{1}{p} = \infty$ implies that

$$
\frac{\sum_{p \in \mathcal{P}} \frac{X_p}{p^\sigma}}{\sqrt{\sum_{p \in \mathcal{P}} \frac{1}{p^2}}} \to_d \mathcal{N}(0, 1), \quad \text{as } \sigma \to 1/2^+,
$$

and second, for each $L > 0$, the event

$$
\limsup_{\sigma \to \frac{1}{2}^+} \frac{\sum_{p \in \mathcal{P}} \frac{X_p}{p^\sigma}}{\sqrt{\sum_{p \in \mathcal{P}} \frac{1}{p^2}}} \geq L
$$

is a tail event, and by (6), it has positive probability. Similarly,

$$
\liminf_{\sigma \to \frac{1}{2}^+} \frac{\sum_{p \in \mathcal{P}} \frac{X_p}{p^\sigma}}{\sqrt{\sum_{p \in \mathcal{P}} \frac{1}{p^2}}} \leq -L
$$

also is a tail event and has positive probability. Thus, by the Kolmogorov $0-1$ Law, with probability 1, $\sum_{p \in \mathcal{P}} \frac{X_p}{p^\sigma}$ changes sign for an infinite number of $\sigma \to 1/2^+$. 
3.1. Proof of Theorem 1.1 (ii) in the case $\pi(x) \ll \frac{x}{\log x}$.

**Lemma 3.1.** Assume that $\mathcal{P}$ satisfies P1-P2 and that $\sum_{p \in \mathcal{P}} \frac{1}{p} = \infty$. Further, assume that $\pi(x) \ll \frac{x}{\log x}$. Let $\sigma > 1/2$ and $y = \exp((2\sigma - 1)^{-1}) \geq 10$. Then there is a constant $d > 0$ such that for all $\lambda > 0$

$$\mathbb{P}\left(\left| \sum_{p \in \mathcal{P}} \frac{X_p}{p^\sigma} - \sum_{p \leq y} \frac{X_p}{\sqrt{p}} \right| \geq 2\lambda \right) \leq 4 \exp(-d\lambda^2).$$

**Proof.** If $|a + b| \geq 2\lambda$ then either $|a| \geq \lambda$ or $|b| \geq \lambda$. This fact combined with the Hoeffding’s inequality allows us to bound:

$$\mathbb{P}\left(\left| \sum_{p \in \mathcal{P}} \frac{X_p}{p^\sigma} - \sum_{p \leq y} \frac{X_p}{\sqrt{p}} \right| \geq 2\lambda \right) \leq \mathbb{P}\left(\left| \sum_{p \leq y} \frac{X_p}{p^\sigma} - \sum_{p \leq y} \frac{1}{\sqrt{p}} \right| \geq \lambda \right)$$

$$+ \mathbb{P}\left(\left| \sum_{p > y} \frac{X_p}{p^\sigma} \right| \geq \lambda \right) \leq \exp\left(-\frac{\lambda^2}{2V_y}\right) + \exp\left(-\frac{\lambda^2}{2U_y}\right),$$

where $V_y = \sum_{p \leq y} \left(\frac{1}{p^{\sigma}} - \frac{1}{\sqrt{p}}\right)^2$ and $U_y = \sum_{p > y} \frac{1}{p^{\sigma}}$. To complete the proof we only need to estimate these quantities. By the mean value theorem

$$\frac{1}{p^\sigma} - \frac{1}{\sqrt{p}} = (\sigma - 1/2)\frac{\log p}{p^{\sigma}},$$

for some $\theta = \theta(p, \sigma) \in [1/2, \sigma]$. Therefore

$$V_y \leq (\sigma - 1/2)^2 \sum_{p \leq y} \frac{\log^2 p}{p^{\sigma}} = (\sigma - 1/2)^2 \int_{1^-}^{y} \frac{\log^2 t}{t^{2\sigma}} d\pi(t)$$

$$= (\sigma - 1/2)^2 \left(\frac{\pi(y) \log^2 y}{y} - \int_{1^-}^{y} \pi(t) \frac{2 \log t - \log^2 t}{t^2} dt\right)$$

$$\ll (\sigma - 1/2)^2 \left(\log y + \int_{1^-}^{y} \frac{\log t}{t} dt\right) \ll (\sigma - 1/2)^2 \log^2 y.$$

$$U_y = \int_{y}^{\infty} \frac{d\pi(t)}{t^{2\sigma}} = -\frac{\pi(y)}{y^{2\sigma}} - \int_{y}^{\infty} \frac{-2\sigma \pi(t)}{t^{2\sigma+1}} dt$$

$$\leq \frac{1}{y^{2\sigma-1} \log y} + 2\sigma \int_{y}^{\infty} \frac{1}{t^{2\sigma} \log t} dt \ll \frac{1}{y^{2\sigma-1} \log y} + \frac{2\sigma}{(2\sigma - 1)y^{2\sigma-1} \log y}$$

$$\ll (2\sigma - 1)y^{2\sigma-1} \log y.$$ 

In particular, the choice $y = \exp((2\sigma - 1)^{-1})$ implies that both variances $V_y$ and $U_y$ are $O(1)$. □
The simple random walk $S_n = \sum_{k=1}^{n} X_n$ where $(X_n)_{n \in \mathbb{N}}$ is i.i.d with $X_1 = \pm 1$ with probability $1/2$ each, satisfies a.s. $\limsup_{n \to \infty} S_n = \infty$ and $\liminf_{n \to \infty} S_n = -\infty$. We follow the same line of reasoning as in the proof of this result ([6] pg. 381, Theorem 2) to prove:

**Lemma 3.2.** Assume that $\sum_{p \in P} \frac{1}{p} = \infty$. Let $y_k$ be an increasing sequence of positive real numbers such that $\lim y_k = \infty$. Then it a.s. holds that:

\[
\limsup_{k \to \infty} \frac{\sum_{p \leq y_k} X_p}{\sqrt{\sum_{p \leq y_k} \frac{1}{p}}} = \infty,
\]

\[
\liminf_{k \to \infty} \frac{\sum_{p \leq y_k} X_p}{\sqrt{\sum_{p \leq y_k} \frac{1}{p}}} = -\infty.
\]

**Proof.** We begin by observing that $(X_p/\sqrt{p})_{p \in P}$ is a sequence of independent and symmetric random variables that are uniformly bounded by 1. It follows that

\[
\lim_{y \to \infty} \text{Var} \left( \sum_{p \leq y} \frac{X_p}{\sqrt{p}} \right) = \lim_{y \to \infty} \sum_{p \leq y} \frac{1}{p} = \infty,
\]

and hence this sequence satisfies the Lindenberg condition. By the Central Limit Theorem it follows that for each fixed $L > 0$ there exists a $\delta > 0$ such that for sufficiently large $y > 0$

\[
P\left( \sum_{p \leq y} \frac{X_p}{\sqrt{p}} \geq L \sqrt{\sum_{p \leq y} \frac{1}{p}} \right) = P\left( \sum_{p \leq y} \frac{X_p}{\sqrt{p}} \leq -L \sqrt{\sum_{p \leq y} \frac{1}{p}} \right) \geq \delta.
\]

Next observe that the event in which $\limsup_{k \to \infty} \frac{\sum_{p \leq y_k} X_p}{\sqrt{\sum_{p \leq y_k} \frac{1}{p}}} \geq L$ is a tail event, and hence by the Kolmogorov zero or one law it has either probability zero or one. Since

\[
P\left( \sum_{p \leq y_k} \frac{X_p}{\sqrt{p}} \geq L \sqrt{\sum_{p \leq y_k} \frac{1}{p}} \text{ for infinitely many } k \right) = \lim_{n \to \infty} P\left( \bigcup_{k=n}^{\infty} \left[ \sum_{p \leq y_k} \frac{X_p}{\sqrt{p}} \geq L \sqrt{\sum_{p \leq y_k} \frac{1}{p}} \right] \right) \geq \delta,
\]

it follows that for each fixed $L > 0$ $\limsup_{k \to \infty} \frac{\sum_{p \leq y_k} X_p}{\sqrt{\sum_{p \leq y_k} \frac{1}{p}}} \geq L$, a.s. Similarly, we can conclude that for each fixed $L > 0$ $\liminf_{k \to \infty} \frac{\sum_{p \leq y_k} X_p}{\sqrt{\sum_{p \leq y_k} \frac{1}{p}}} \leq -L$, a.s. \hfill \(\square\)

**Proof of item ii.** Take $\lambda = \lambda(y) = \sqrt{\sum_{p \leq y} \frac{1}{p}}$ in Lemma 3.1 and let $y = \exp((2\sigma - 1)^{-1})$. Since $\lim_{y \to \infty} \lambda(y) = \infty$, it follows that there is a subsequence $y_k \to \infty$ for
which $\sum_{k=1}^{\infty} \exp(-d\lambda^2(y_k)) < \infty$ and hence, by the Borel-Cantelli Lemma, it a.s. holds that

$$\limsup_{k \to \infty} \frac{\left| \sum_{p \in P} \frac{X_p}{p^{\sigma_k}} - \sum_{p \leq y_k} \frac{X_p}{\sqrt{p}} \right|}{\sqrt{\sum_{p \leq y_k} \frac{1}{p}}} \leq 2,$$

where $y_k = \exp((2\sigma_k - 1)^{-1})$. This combined with Lemma 3.2 gives a.s.

$$\limsup_{\sigma \to 1/2^+} \sum_{p \in P} \frac{X_p}{p^{\sigma}} \geq \limsup_{k \to \infty} \frac{\left| \sum_{p \leq y_k} \frac{X_p}{\sqrt{p}} - \sum_{p \leq y_k} \frac{X_p}{\sqrt{p}} \right|}{\sqrt{\sum_{p \leq y_k} \frac{1}{p}}} \geq \limsup_{k \to \infty} \left( \frac{\sum_{p \leq y_k} \frac{X_p}{\sqrt{p}}}{\sqrt{\sum_{p \leq y_k} \frac{1}{p}}} - 3 \right) = \infty.$$

Similarly, we conclude that $\liminf_{\sigma \to 1/2^+} \sum_{p \in P} \frac{X_p}{p^{\sigma}} = -\infty$, a.s. Since $F(\sigma)$ is a.s. analytic, it follows that there is an infinite number of $\sigma > 1/2$ for which $F(\sigma) = 0$. \hfill $\square$

3.2. Proof of Theorem 1.1 (ii), the general case. The following Lemma is an adaptation of [1], Theorem 1.2:

**Lemma 3.3.** Assume that $\mathcal{P}$ satisfies $P1-P2$ and that $\sum_{p \in \mathcal{P}} \frac{1}{p} = \infty$. Then

$$\tag{7} \frac{\sum_{p \in \mathcal{P}} \frac{X_p}{p^{\sigma}}}{\sqrt{\sum_{p \in \mathcal{P}} \frac{1}{p^{2\sigma}}}} \to_d N(0, 1), \text{ as } \sigma \to \frac{1}{2}^+. $$

**Proof.** Let $V(\sigma) = \sqrt{\sum_{p \in \mathcal{P}} \frac{1}{p^{2\sigma}}}$. Observe that $V(\sigma) \to \infty$ as $\sigma \to 1/2^+$. For each fixed $y > 0$

$$\liminf_{\sigma \to 1/2^+} \sum_{p \leq y} \frac{1}{p^{2\sigma}} \geq \lim_{\sigma \to 1/2^+} \sum_{p \leq y} \frac{1}{p^{2\sigma}} = \sum_{p \leq y} \frac{1}{p}.$$

Thus, by making $y \to \infty$ in the equation above, we obtain the desired claim.

For each fixed $\sigma > 1/2$, by the Kolmogorov one series Theorem, we have that $\sum_{p \leq y} X_p/p^{\sigma}$ converges almost surely as $y \to \infty$. Since $(X_p)_{p \in \mathcal{P}}$ are independent, by the
dominated convergence theorem:

\[
\varphi_\sigma(t) := \mathbb{E}\exp\left(\frac{it}{V(\sigma)} \sum_{p \in \mathcal{P}} \frac{X_p}{p^\sigma}\right) = \lim_{y \to \infty} \mathbb{E}\exp\left(\frac{it}{V(\sigma)} \sum_{p \leq y} \frac{X_p}{p^\sigma}\right) = \prod_{p \in \mathcal{P}} \cos\left(\frac{t}{V(\sigma)p^\sigma}\right).
\]

We will show that for each fixed \( t \in \mathbb{R} \), \( \varphi_\sigma(t) \to \exp(-t^2/2) \) as \( \sigma \to 1/2^+ \). Observe that \( \varphi_\sigma(t) = \varphi_\sigma(-t) \), so we may assume \( t \geq 0 \). Thus, for each fixed \( t \geq 0 \) we may choose \( \sigma > 1/2 \) such that \( 0 \leq \frac{t}{V(\sigma)p^\sigma} \leq \frac{1}{100} \) and \( 0 \leq 1 - \cos\left(\frac{t}{V(\sigma)p^\sigma}\right) \leq \frac{1}{100} \), for all \( p \in \mathcal{P} \).

For \( |x| \leq 1/100 \), we have that \( \log(1-x) = -x + O(x^2) \) and \( \cos(x) = 1 - \frac{x^2}{2} + O(x^4) \). Further, \( 1 - \cos(x) = 2\sin^2(x/2) \leq \frac{x^2}{2} \). Thus, we have:

\[
\log \varphi_\sigma(t) = \sum_{p \in \mathcal{P}} \log \cos\left(\frac{t}{V(\sigma)p^\sigma}\right) = \sum_{p \in \mathcal{P}} \log \left(1 - \left(1 - \cos\left(\frac{t}{V(\sigma)p^\sigma}\right)\right)\right) = - \sum_{p \in \mathcal{P}} \left(1 - \cos\left(\frac{t}{V(\sigma)p^\sigma}\right)\right) + \sum_{p \in \mathcal{P}} O\left(1 - \cos\left(\frac{t}{V(\sigma)p^\sigma}\right)^2\right) = - \sum_{p \in \mathcal{P}} \left(\frac{t^2}{2V^2(\sigma)p^{2\sigma}} + O\left(\frac{t^4}{V^4(\sigma)p^{4\sigma}}\right)\right) + \sum_{p \in \mathcal{P}} O\left(\frac{t^4}{V^4(\sigma)p^{4\sigma}}\right) = - \frac{t^2}{2} \sum_{p \in \mathcal{P}} \frac{1}{p^{2\sigma}} + \sum_{p \in \mathcal{P}} O\left(\frac{t^4}{V^4(\sigma)p^{2}}\right) = - \frac{t^2}{2} + O\left(\frac{t^4}{V^4(\sigma)}\right).
\]

We conclude that \( \varphi_\sigma(t) \to \exp(-t^2/2) \) as \( \sigma \to 1/2^+ \).

\( \square \)

**Proof of item ii.** Let \( V(\sigma) \) be as in the proof of Lemma 3.3. Since \( V(\sigma) \to \infty \) as \( \sigma \to 1/2^+ \), we have, for each fixed \( y > 0 \)

\[
\limsup_{\sigma \to 1/2^+} \frac{1}{V(\sigma)} \sum_{p \leq y} \frac{X_p}{p^\sigma} = \limsup_{\sigma \to 1/2^+} \frac{1}{V(\sigma)} \sum_{p \leq y} \frac{X_p}{p^\sigma}.
\]

Thus, for each fixed \( L > 0 \)

\[
\limsup_{\sigma \to 1/2^+} \frac{1}{V(\sigma)} \sum_{p \leq y} \frac{X_p}{p^\sigma} \geq L
\]

is a tail event. By Lemma 3.3, \( \frac{1}{V(\sigma)} \sum_{p \leq y} \frac{X_p}{p^\sigma} \to_d \mathcal{N}(0,1) \), as \( \sigma \to 1/2^+ \). Thus, this tail event has positive probability (see the proof of Lemma 3.2). By the Kolmogorov
zero or one Law, a.s.:
\[
\limsup_{\sigma \to 1/2^+} \frac{1}{V(\sigma)} \sum_{p \in P} \frac{X_p}{p^\sigma} = \infty.
\]
Similarly, a.s.:
\[
\liminf_{\sigma \to 1/2^+} \frac{1}{V(\sigma)} \sum_{p \in P} \frac{X_p}{p^\sigma} = -\infty.
\]
Since \( F(\sigma) = \sum_{p \in P} \frac{X_p}{p^\sigma} \) is a.s. an analytic function, with probability 1 we have that \( F(\sigma) = 0 \) for an infinite number of \( \sigma \to 1/2^+ \).

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