Quantum evolution of Schwarzschild-de Sitter (Nariai) black holes

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abstract

We calculate the one-loop effective action for conformal matter (scalars, spinors and vectors) on spherically symmetric background. Such effective action (in large \(N\) approximation and expansion on curvature) is used to study quantum aspects of Schwarzschild-de Sitter black holes (SdS BHs) in nearly degenerated limit (Nariai BH). We show that for all types of above matter SdS BHs may evaporate or anti-evaporate in accordance with recent observation by Bousso and Hawking for minimal scalars. Some remarks about energy flow for SdS BHs in regime of evaporation or anti-evaporation are also done. Study of no boundary condition shows that this condition supports anti-evaporation for nucleated BHs (at least in frames of our approximation). That indicates to the possibility that some pair created cosmological BHs may not only evaporate but also anti-evaporate. Hence, cosmological primordial BHs may survive much longer than it is expected.

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1 Introduction

In the absence of consistent quantum gravity, the natural way to take into account quantum effects in the early Universe or in black holes (BHs) is to consider matter quantum field theory (say, some GUT) in curved background. The study of quantum GUTs in curved space-time (see [1] for a review) shows the existence of beautiful phenomenon – asymptotic conformal invariance (see original works [2] or book [1], for a review). According to it, there exists large class of asymptotically free GUTs which tend to conformally invariant free theory at high curvature or at high energies (i.e. in the vicinity of BH or in the early Universe). Hence, for above background one can describe GUT as the collection of free conformal fields. If one knows the effective action of such system one can apply it for investigation of quantum evolution of strongly gravitating objects.

In recent work [3] the quantum evolution of Schwarzschild-de Sitter (Nariai) BHs has been studied for Einstein gravity with $N$ minimal quantum scalars. Large $N$ and $s$-waves approximation has been used in such an investigation. The possibility of quantum anti-evaporation of such BHs (in addition to well-known evaporation process [4]) has been discovered. In the paper [3] another model (of quantum conformal scalars with Einstein gravity) has been considered in a better approach to effective action (large $N$ approximation, partial expansion on curvature and partial $s$-waves reduction). The possibility of Schwarzschild-de Sitter (SdS) BHs anti-evaporation has been confirmed as well in the model of ref. [5].

Having in mind above remarks on the representation of some GUT in the vicinity of BHs as collection of free conformal fields, we continue to study quantum dynamics of SdS BHs. We start from Einstein gravity with quantum conformal matter ($N$ scalars, $N_1$ vectors and $N_{1/2}$ fermions). Working in large $N$ approximation (where only matter quantum effects are dominant) we also use partial derivative expansion of EA (without $s$-waves reduction). As a main qualitative result we find that extreme SdS (Nariai) BHs may indeed evaporate as well as anti-evaporate. We also try to answer the question: Can the no-boundary Hartle-Hawking condition be consistent with anti-evaporation? This question may be really important for estimation of primordial BHs creation ([3] and refs. therein) (and their existence in the present Universe) as SdS BHs actually may appear through such a process.
2 Effective Action for Conformal Matter

We first derive the effective action for conformally invariant matter (for a general review of effective action in curved space, see [1]). Let us start from Einstein gravity with \( N \) conformal scalars \( \chi_i \), \( N_1 \) vectors \( A_\mu \) and \( N_{1/2} \) Dirac spinors \( \psi_i \).

\[
S = -\frac{1}{16\pi G} \int d^4x \sqrt{-g(4)} \left\{ R(4) - 2\Lambda \right\} + \int d^4x \sqrt{-g(4)} \left\{ \frac{1}{2} \sum_{i=1}^N (g^\alpha_\beta \partial_\alpha \chi_i \partial_\beta \chi_i + \frac{1}{6} R(4) \chi_i^2) - \frac{1}{4} \sum_{j=1}^{N_1} F_{j\mu\nu} F^{j\mu\nu} + \sum_{k=1}^{N_{1/2}} \bar{\psi}_k D /\psi_k \right\}.
\]

The convenient choice for the spherically symmetric space-time is the following:

\[
ds^2 = f(\phi) \left\{ f^{-1}(\phi) g_{\mu\nu} dx^\mu dx^\nu + r_0^2 d\Omega \right\}.
\]

where \( \mu, \nu = 0, 1 \), \( g_{\mu\nu} \) and \( f(\phi) \) depend only from \( x^0, x^1 \) and \( r_0^2 \) is the constant.

Let us start the calculation of effective action due to conformal matter on the background (2). In the calculation of effective action, we present effective action as : \( \Gamma = \Gamma_{\text{ind}} + \Gamma_{[1, g_{\mu\nu}(4)]} \) where \( \Gamma_{\text{ind}} = \Gamma[f, g_{\mu\nu}(4)] - \Gamma[1, g_{\mu\nu}(4)] \) is conformal anomaly induced action which is quite well-known [7], \( g_{\mu\nu}(4) \) is metric (2) without multiplier in front of it, i.e., \( g_{\mu\nu}(4) \) corresponds to

\[
d s^2 = \left[ \tilde{g}_{\mu\nu} dx^\mu dx^\nu + r_0^2 d\Omega \right], \quad \tilde{g}_{\mu\nu} \equiv f^{-1}(\phi) g_{\mu\nu}.
\]

The conformal anomaly for above matter is well-known

\[
T = b \left( F + \frac{2}{3} \Box R \right) + b' G + b'' \Box R
\]

where \( b = \frac{(N+6N_1/2+12N_{1/2})}{120(4\pi)^2} \), \( b' = \frac{(N+11N_1/2+62N_{1/2})}{360(4\pi)^2} \), \( b'' = 0 \) but in principle, \( b'' \) may be changed by the finite renormalization of local counterterm in gravitational effective action, \( F \) is the square of Weyl tensor, \( G \) is Gauss-Bonnet invariant.

Conformal anomaly induced effective action \( \Gamma_{\text{ind}} \) may be written as follows [7]:

\[
W = b \int d^4x \sqrt{-g} \sigma F + b' \int d^4x \sqrt{-g} \left\{ 2\Box^2 + 4R^{\mu\nu} \nabla_\mu \nabla_\nu \right\}
\]
where $\sigma = \frac{1}{2} \ln f(\phi)$, and $\sigma$-independent terms are dropped. All 4-dimensional quantities (curvatures, covariant derivatives) in Eq.(5) should be calculated on the metric (3). (We did not write subscript (4) for them.) Note that after calculation of (5) on the metric (3), we will get effectively two-dimensional gravitational theory.

In the next step, we are going to calculate $\Gamma[1, g_{\mu\nu}^{(4)}]$. This term corresponds to the conformally invariant part of effective action. In this calculation, we may apply Schwinger-De Witt (SDW) type expansion of effective action with zeta-regularization[8] (or other ultraviolet regularization). This expansion represents the expansion on powers of curvature invariants. Note that we add such EA to Einstein gravity action. Hence, two first terms of the SDW expansion (cosmological and linear curvature terms) may be dropped as they only lead to finite renormalization of Hilbert-Einstein action (redefinition of cosmological and gravitational coupling constants). Then the leading (curvature quadratic term of this expansion) may be read off (see [1]) as follows:

$$\Gamma[1, g_{\mu\nu}^{(4)}] = \int d^4x \sqrt{-g} \left\{ \left[ b F + b' G + \frac{2b}{3} R \right] \ln \frac{R}{\mu^2} \right\} + O(R^3) \quad (6)$$

where $\mu$ is mass-dimensional constant parameter, all the quantities are calculated on the background (3). The condition of application of above expansion is $|R| < R^2$ (curvature is nearly constant), in this case we may limit by only few first terms.

3 Quantum Dynamics on Spherical Background

We now consider to solve the equations of motion obtained from the above effective Lagrangians $S + \Gamma$. In the following, we use $\tilde{g}_{\mu\nu}$ and $\sigma$ as a set of independent variables and we write $\tilde{g}_{\mu\nu}$ as $g_{\mu\nu}$ if there is no confusion.
\[ \Gamma_{\text{ind}} \] (\( W \) in Eq.(5)) is rewritten after the reduction to 2 dimensions as follows:

\[
\frac{\Gamma_{\text{ind}}}{4\pi} = \frac{b r_0^2}{3} \int d^2x \sqrt{-g} \left( (R^{(2)} + R_\Omega)^2 + \frac{2}{3} R_\Omega R^{(2)} + \frac{1}{3} R^{(2)}_\Omega \right) \sigma
\]

\[
+ b' r_0^2 \int d^2x \sqrt{-g} \left\{ \sigma \left( 2\Box^2 + 4 R^{(2)\mu\nu} \nabla_\mu \nabla_\nu - \frac{4}{3} (R^{(2)} + R_\Omega) \Box \right) \right.
\]

\[
+ \frac{2}{3} (\nabla^\mu R^{(2)}) \nabla_\mu \sigma + \left( 2 R_\Omega R^{(2)} - \frac{2}{3} \Box R^{(2)} \right) \sigma \}
\]

\[
- \frac{1}{12} \left\{ b'' + \frac{2}{3} (b + b') \right\} r_0^2 \int d^2x \sqrt{-g} \times \left\{ \left( R^{(2)} + R_\Omega - 6 \Box \sigma - 6 \nabla^\mu \sigma \nabla_\mu \sigma \right)^2 - \left( R^{(2)} + R_\Omega \right)^2 \right\} \] (7)

Here \( R_\Omega = \frac{2}{r_0^2} \) is scalar curvature of \( S^2 \) with the unit radius. The suffix “\((2)\)” expresses the quantity in 2 dimensions but we abbreviate it if there is no any confusion. We also note that in two dimensions the Riemann tensor \( R_{\mu\nu\sigma\rho} \) and \( R_{\mu\nu} \) are expressed via the scalar curvature \( R \) and the metric tensor \( g_{\mu\nu} \) as:

\[ R_{\mu\nu\sigma\rho} = \frac{1}{2} (g_{\mu\sigma} g_{\nu\rho} - g_{\mu\rho} g_{\nu\sigma}) R \] and \[ R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R. \]

Let us derive the equations of motion with account of quantum corrections from above effective action. In the following, we work in the conformal gauge:

\[ g_{\pm\pm} = -\frac{1}{2} e^{2\rho}, \quad g_{\pm\mp} = 0 \] after considering the variation of the effective action \( \Gamma + S \) with respect to \( g_{\mu\nu} \) and \( \sigma \). Note that the tensor \( g_{\mu\nu} \) under consideration is the product of the original metric tensor and the \( \sigma \)-function \( e^{-2\rho} \), the equations given by the variations of \( g_{\mu\nu} \) are the combinations of the equations given by the variation of the original metric and \( \sigma \)-equation.

It often happens that we can drop the terms linear in \( \sigma \) in (7). In particular, one can redefine the corresponding source term as it is in the case of IR sector of 4D QG [9]. In the following, we only consider this case. Then the variation of \( S + \Gamma_{\text{ind}} + \Gamma[1, g_{\mu\nu}^{(4)}] \) with respect to \( g^{\pm\pm} \) is given by

\[
0 = \frac{1}{4\pi} \frac{\delta(S + \Gamma_{\text{ind}} + \Gamma[1, g_{\mu\nu}^{(4)}])}{\delta g^{\pm\pm}}
\]

\[
= -\frac{r_0^2}{16\pi G} e^{2\rho + 2\sigma} \left( (\partial_\pm \sigma)^2 - \partial_\pm^2 \sigma + 2 \partial_\pm \sigma \partial_\pm \rho \right) + b r_0^2 \left[ 8 e^{2\rho} \partial_\pm \sigma \partial_\pm (e^{-2\rho} \partial_+ \partial_- \sigma) \right.
\]

\[
-8 \sigma \partial_\pm^2 \sigma \partial_+ \partial_- \rho + \frac{2}{3} e^{2\rho} \partial_+ \sigma \partial_\pm \{ R_4 \sigma \} + \frac{8}{3} e^{2\rho} \sigma \partial_\pm R_4 \partial_\pm \sigma \]
The variations with respect to $\rho$ are given by

$$0 = -\frac{1}{4\pi} \frac{\delta(S + \Gamma_{\text{ind}} + \Gamma[1, g_{\mu\nu}^{(4)}])}{\delta \rho}$$

$$= -\frac{r_0^2}{16\pi G} \left[ 4\partial_+\partial_- e^{2\rho} - 2e^{2\rho+4\sigma} \Lambda + \frac{4}{r_0^2} e^{2\rho+2\sigma} \right] + b'r_0^2 \left\{ -32(\partial_+\partial_- \sigma)^2 e^{-2\rho} - \frac{128}{3} e^{-2\rho} \partial_+\partial_- (\sigma \partial_+\partial_- \sigma) + \frac{64}{3} \partial_+\partial_- e^{-2\rho} \sigma \partial_+\partial_- \sigma \right\}$$

$$- \frac{16}{3} \left\{ -2\partial_+\partial_- \sigma e^{-2\rho} \partial_+\partial_- \rho + \partial_+\partial_- (\partial_+\partial_- \sigma e^{-2\rho}) \right\}$$

$$- \left\{ b'' + \frac{2}{3} (b + b') \right\} r_0^2 \left[ 16\partial_+\partial_- \left\{ e^{-2\rho} (\partial_+\partial_- \sigma + \partial_+\partial_- \sigma) \right\} - 48e^{-2\rho} (\partial_+\partial_- \sigma + \partial_+\partial_- \sigma)^2 \right]$$

Here $R_4 \equiv 8e^{-2\rho} \partial_+\partial_- \rho + \frac{2}{r_0^2}$. Usually the equation given by $g^{++}$ or $g^{--}$ can be regarded as the constraint equation with respect to the initial or boundary conditions. The equations obtained here are, however, combinations of the constraint and $\sigma$-equation of the motion since the tensor $g_{\mu\nu}$ under consideration is the product of the original metric tensor and the $\sigma$-function $e^{-2\sigma}$.
quantum corrected equations of motion for the system under discussion. Note that now the real 4 dimensional metric is given by:

\[ ds^2 = -e^{2\rho} dx^+ dx^- + r_0^2 e^{2\sigma} d\Omega. \]

Above equations give the complete system of quantum corrected equations of motion for the system under discussion.

The variations with respect to \( \sigma \) may be found as

\[
0 = \frac{1}{4\pi} \frac{\delta(S + \Gamma_{ind} + \Gamma[1,g^{(4)}_{\mu\nu}])}{\delta \sigma}
\]

\[
= -\frac{b_0^2}{16\pi G} \left[ 8e^{2\rho} \left\{ 3\partial_+ \partial_- \sigma + 3\partial_+ \sigma \partial_- + \partial_+ \partial_- \rho \right\} - 4e^{2\rho+4\sigma} \Lambda + \frac{4}{r_0^2} e^{2\rho+2\sigma} \right]
\]

\[
+ b'_0^2 \left[ 32\partial_+ \partial_- (e^{-2\rho} \partial_+ \sigma) + \frac{64}{3} \left( e^{-2\rho} \partial_+ \partial_- \rho \partial_+ \partial_- \sigma + \partial_+ \partial_- (\sigma e^{-2\rho} \partial_+ \partial_- \rho) \right) \right]
\]

\[
+ \frac{32}{3r_0^2} \partial_+ \partial_- + \frac{2}{3} \left\{ \partial_+ R_4 \partial_- \sigma + \partial_- R_4 \partial_+ \sigma \right\}
\]

\[
- \left\{ b'' + \frac{2}{3} (b + b') \right\} r_0^2 \left[ 2\partial_+ \partial_- R_4 - 2 \left\{ \partial_+ R_4 \partial_- \sigma + \partial_- R_4 \partial_+ \sigma \right\} \right]
\]

\[
+ 48\partial_+ \partial_- \left\{ e^{-2\rho} (\partial_+ \partial_- \sigma + \partial_+ \sigma \partial_- \sigma) \right\} - 48\partial_+ \left\{ e^{-2\rho} \sigma (\partial_+ \partial_- \sigma + \partial_+ \sigma \partial_- \sigma) \right\}
\]

\[
- 48\partial_- \left\{ e^{-2\rho} \partial_+ \sigma (\partial_+ \partial_- \sigma + \partial_+ \sigma \partial_- \sigma) \right\}
\]

(10)
4 Evolution of Schwarzschild-de Sitter Black Holes due to Quantum Conformal Matter Back-Reaction

We now consider the Schwarzschild-de Sitter family of black holes and its nearly degenerated case, so-called Nariai solution [10]. We here follow the work by Bousso-Hawking [3]. The Schwarzschild type black hole solution in de Sitter space has two horizons, one is the usual event horizon and another one is the cosmological horizon, which is proper one in de Sitter space. The Nariai solution is given by a limit of the Schwarzschild-de Sitter black hole where two horizons coincide with each other. In the limit, the two horizons have the same temperature since the temperature is proportional to the inverse root of the horizon area. Therefore two horizons are in the thermal equilibrium in the limit. We are now interesting in the instability of the Nariai limit. Near the limit the temperature of the event horizon is higher than that of the cosmological one since the area of the event horizon is smaller than that of the cosmological horizon. This implies that there would be a thermal flow from the event horizon to the cosmological one. This means the system would become instable and the black hole would evaporate. We also have to note that above cosmological black holes may naturally appear through quantum pair creation [11, 6] which may occur in the inflationary universe [12].

In the Nariai limit, the space-time has the topology of $S^1 \times S^2$ and the metric is given by

$$ds^2 = \frac{1}{\Lambda} \left( \sin^2 \chi d\psi^2 - d\chi^2 - d\Omega \right).$$  \hspace{1cm} (11)

Here coordinate $\chi$ has a period $\pi$. If we change the coordinates variables by

$$r = \ln \tan \frac{\chi}{2}, \quad t = \frac{\psi}{4},$$  \hspace{1cm} (12)

we obtain

$$ds^2 = \frac{1}{\Lambda \cosh^2 r} \left( -dt^2 + dr^2 \right) + \frac{1}{\Lambda} d\Omega.$$  \hspace{1cm} (13)

This form corresponds to the conformal gauge in two dimensions. Note that the transformation (12) has one to one correspondence between $(\psi, \chi)$ and $(t, r)$ if we restrict $\chi$ by $0 \leq \chi < \pi$ ($r$ runs from $-\infty$ to $+\infty$).
Now we solve the equations of motion. Since the Nariai solution is characterized by the constant $\phi$ (or $\sigma$), we now assume that $\sigma$ is a constant even when including quantum correction

$$\sigma = \sigma_0 \quad \text{(constant)} .$$

(14)

We also first consider static solutions and replace $\partial_{\pm}$ by $\pm \frac{1}{2} \partial_r$. Then we find that the total constraint equation obtained by (8) is trivially satisfied.

Assuming a solution is given by a constant 2d scalar curvature

$$R = -2e^{-2\rho} \partial_r^2 \rho = R_0 \quad \text{(constant)} ,$$

(15)

the equations of motions given by (9) (variation over $\rho$) and (10) (variation over $\sigma$) become the following two algebraic equations

$$0 = -\frac{R_0^2}{8\pi G} \left( -\Lambda e^{4\sigma_0} + \frac{2}{r_0^2} e^{2\sigma_0} \right) + \frac{1}{r_0^2} \left\{ b \left( -\frac{R_0^2}{3} + \frac{4}{3r_0^4} \right) \ln \left( \frac{R_0 + \frac{2}{r_0}}{\mu^2} \right) 
\right. 
\left. - \left\{ b \left( \frac{R_0^2}{3} + \frac{4}{3r_0^4} \right) + \frac{8}{r_0^2} \left( b + b' \right) \right\} R_0 \right\} \frac{R_0}{R_0 + \frac{2}{r_0}} \right\}$$

(16)

$$0 = R_0 - 4\Lambda e^{2\sigma_0} + \frac{4}{r_0^2} .$$

(17)

The above equations can be solved with respect to $\sigma_0$ and $R_0$ in general, although it is difficult to get the explicit expression.

Equation (15) can be integrated to be

$$e^{2\rho} = e^{2\rho_0} \equiv \frac{2C}{R_0} \cdot \frac{1}{\cosh^2 \left( \frac{1}{r\sqrt{C}} \right)} .$$

(18)

Here $C$ is a constant of integration.

We now consider the perturbation around the Nariai type solution (14) and (18)

$$\rho = \rho_0 + \epsilon R(t,r) , \quad \sigma = \sigma_0 + \epsilon S(t,r) .$$

(19)

Here $\epsilon$ is a infinitesimally small parameter. Then we obtain the following linearized $\rho$ and $\sigma$ equations

$$0 = 4\pi b' r_0^2 \left\{ -\frac{16}{3} \frac{R_0 \sigma_0 \Delta S}{C} + \frac{32}{3} \frac{R_0 \sigma_0}{C} \Delta \left( \Delta S \right) \right\} - 4\pi (b + b') r_0^2 \frac{16 R_0}{3} \frac{\Delta (\Delta S)}{C}$$

(16)
and $R_4$ becomes constant $R_4 = R_0 + \frac{4}{r_0}$. The equations (20) and (21) can be solved by assuming that $R$ and $S$ are given by the eigenfunctions of $\Delta$:

$$R(t, r) = P f_A(t, r), \quad S(t, r) = Q f_A(t, r), \quad \Delta f_A(t, r) = A f_A(t, r).$$

Note that $\Delta$ can be regarded as the Laplacian on the two dimensional hyperboloid and the explicit expression for the eigenfunctions is given later. Using (23), we can rewrite (20) and (21) (using (17)) as follows:

$$0 = \left\{ \frac{64\pi b'\sigma_0 R_0}{3} \left( -A + 2A^2 \frac{3}{C} \right) - \frac{64\pi(b + b')R_0 A^2}{3C} - \frac{1}{4\pi G} e^{2\sigma_0} (2A - C) \right\} Q + \left\{ 4\pi \left\{ \frac{2C}{R_0} \left( -2R_0 + 4 \frac{R_0^2}{3r_0^3} \right) \right\} - \frac{b}{R_4} \left( -R_0^2 + 4 \frac{R_0^2}{3r_0^3} \right) \right\} R_0 + \Delta \left( -2R_0 R + 4 \frac{R_0}{C} \Delta R \right) \right\} CR_0$$

Here

$$\Delta = \cosh^2 \left( r\sqrt{C} \right) \partial_+ \partial_-$$

and $\Delta$ becomes constant $\Delta = \cosh^2 \left( r\sqrt{C} \right) \partial_+ \partial_-$. The equations (20) and (21) can be solved by assuming that $R$ and $S$ are given by the eigenfunctions of $\Delta$:

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\[
- \frac{2}{r_0^2} \left( \frac{b}{3} + b' \right) \left\{ \frac{4R_0}{R_4} - \frac{2R_0^2}{R_4^2} \right\} \times \left( -2R_0 + \frac{4R_0}{C} A \right) \\
+ \left\{ \frac{8b}{3} \ln \left( \frac{R_4}{\mu^2} \right) + \frac{32bR_0}{3R_4} + \left( \frac{R_0^2}{360} + \frac{1}{90r_0^3} \right) \frac{4}{R_4^2} + \frac{16}{r_0^2R_4} \left( \frac{b}{3} + b' \right) \right\} \\
\times \left( -2R_0 A + \frac{4R_0}{C} A^2 \right) - \frac{32bR_0}{3CR_4} \left( -2R_0 A^2 + \frac{4R_0}{C} A^3 \right) \\
+ \left\{ b \left( -\frac{4R_0}{3} + \frac{16}{3r_0^3R_0} \right) \ln \left( \frac{R_4}{\mu^2} \right) - \frac{4bR_0^2}{3R_4} \\
- \frac{8b}{3r_0^2R_4} - \frac{8}{r_0^2} \left( \frac{b}{3} + b' \right) \frac{2R_0^2}{R_4} \right\} C \right] - \frac{C}{8\pi G} e^{2\sigma_0} \left( -1 + \frac{4}{r_0^2R_0} \right) \right\} P \\
eq M_Q(A)Q + M_P(A)P \\
0 = \left\{ 64\pi b' R_0 \left( \frac{1}{3} A + \frac{1}{C} A^2 \right) + \frac{128\pi}{3r_0^2}b' A - 64\pi (b + b') \frac{R_0}{C} A^2 \\
- \frac{e^{2\sigma_0}}{4\pi G} \left( 6A - C - \frac{4C}{r_0^2R_0} \right) \right\} Q \\
+ \left\{ \frac{64\pi b' \sigma_0 R_0}{3} \left( -A + \frac{2}{C} A^2 \right) - \frac{e^{2\sigma_0}}{4\pi G} (2A - C) \right\} P \\
eq N_Q(A)Q + N_P(A)P . \tag{24}
\]

In order that the above two algebraic equations have non-trivial solutions for \( P \) and \( Q \), \( A \) should satisfy the following equation

\[
0 = F(A) \equiv M_Q(A)N_P(A) - M_P(A)N_Q(A) . \tag{25}
\]

Before analyzing Eqs.\((16), (17) \) and \((25)\), we now briefly discuss how (anti-)evaporation is described. As in \((3)\), we consider the following function as an eigenfunction of \( \Delta \) in \((23)\):

\[
f_A(t, r) = \cosh t\alpha \sqrt{C} \cosh \alpha r \sqrt{C} , \quad A \equiv \frac{\alpha(\alpha - 1)C}{4} . \tag{26}
\]

Note that there is one to one correspondence between \( A \) and \( \alpha \) if we restrict \( A > 0 \) and \( \alpha < 0 \). Any linear combination of two solutions is a solution. The perturbative equations of motion \((24)\) are always linear differential equations. The horizon is given by the condition

\[
\nabla \sigma \cdot \nabla \sigma = 0 . \tag{27}
\]
Substituting (26) into (27), we find the horizon is given by \( r = \alpha t \). Therefore on the horizon, we obtain \( S(t, r(t)) = Q \cosh^{1+\alpha} \alpha \sqrt{C} \). This tells that the system is unstable if there is a solution \( 0 > \alpha > -1 \), i.e., \( 0 < A < \frac{C}{2} \).

On the other hand, the perturbation becomes stable if there is a solution where \( \alpha < -1 \), i.e., \( A > \frac{C}{2} \). The radius of the horizon \( r_h \) is given by \( r_h = e^\sigma = e^{\sigma_0 + \epsilon S(t, r(t))} \). Let the initial perturbation is negative \( Q < 0 \). Then the radius shrinks monotonically, i.e., the black hole evaporates in case of \( 0 > \alpha > -1 \). On the other hand, the radius increases in time and approaches to the Nariai limit asymptotically \( S(t, r(t)) \rightarrow Q e^{(1+\alpha)\mu \sqrt{C}} \) in case of \( \alpha < -1 \). The latter case corresponds to the anti-evaporation of black hole observed by Bousso and Hawking [3].

We should be more careful in the case of \( A = \frac{C}{2} \). When \( A = \frac{C}{2} \), \( f_A(r, t) \) are, in general, given by

\[
 f_A(r, t) = \left\{ \frac{\cosh \left( (t + a) \sqrt{C} \right)}{\cosh \left( r \sqrt{C} \right)} + \sinh \left( b \sqrt{C} \right) \tanh \left( r \sqrt{C} \right) \right\}.
\] (28)

Then the condition (27) gives \( t + a = \mp (r - b) \). Therefore on the horizon, we obtain \( S(t, r(t)) = Q \cosh b \). This is a constant, that is, there does not occur evaporation nor anti-evaporation. The radius of the horizon does not develop in time. In classical case, (25) has the following form

\[
 0 = \left\{ \frac{1}{4\pi G} e^{2\sigma_0} (2A - C) \right\}^2 - \frac{C}{8\pi G} e^{2\sigma_0} \times \frac{e^{2\sigma_0}}{4\pi G} \left( 6A - C - \frac{4C}{r_0^2 R_0} \right).
\] (29)

Since \( R_0 = 4r_0^2 \) in the classical case, the solution of (29) is given by \( A = \frac{C}{2} \). Therefore the horizon does not develop in time and the black hole does not evaporate nor anti-evaporate. The result is, of course, consistent with that of ([3]).

In general, it is very difficult to analyze Eqs. (16), (17) and (25). The equations, however, become simple in the limit of \( r_0 \rightarrow \infty \). Note that \( r_0 \), the radius of \( S_2 \) is a free parameter and SDW type expansion in Eq. (6) becomes exact in the limit since \( R \sim O(r_0^{-2}) \). In order to consider the limit, we now redefine \( r_0 \) and \( \sigma_0 \) as follows

\[
 R_0 = \frac{1}{r_0^2} H_0, \quad \sigma_0 = -\ln(\mu r_0) + s_0.
\] (30)
Then (16) is rewritten as follows:

\[ 0 = H_0 - \frac{4\Lambda}{\mu^2} e^{2s_0} + 4 . \] (31)

Substituting (31) into (17), we obtain

\[ (-H_0^2 + 4) \ln(\mu r_0) + \mathcal{O}(1) = 0 . \] (32)

This tells

\[ H_0 = \pm 2 + \mathcal{O} \left( (\ln(\mu r_0))^{-1} \right) . \] (33)

Since we can expect that \( H_0 \sim 2 \) would correspond to the classical limit where \( H_0 = 4 \), we only consider the case of \( H_0 \sim 2 \). Then using (31), we find

\[ e^{2s_0} = \frac{3\mu^2}{2\Lambda} . \] (34)

Then the metric in the quantum Nariai-type solution has the following form:

\[ ds^2 = \frac{3C'}{2\Lambda \cosh^2 \left( \frac{r}{\sqrt{C'}} \right)} (-dt^2 + dr^2) + \frac{3C'}{2\Lambda} d\Omega \] (35)

Substituting (33) and (34) into (25), we obtain

\[ 0 = F(A) \]
\[ = \frac{1}{r_0^4} \left\{ (\ln(\mu r_0))^2 \left( \frac{128\pi b'}{3} \right)^2 \left( -A + \frac{2A^2}{C'} \right)^2 + \mathcal{O}(1) \right\} . \] (36)

Eq. (36) tells that \( A = 0, \frac{C'}{2} \). When \( A = 0, S \) is constant and there does not occur evaporation nor anti-evaporation. As discussed before, the horizon does not develop in time when \( A = \frac{C'}{2} \), either. The result, however, might be an artifact in the limit of \( r_0 \to \infty \). In order to consider physically reliable results, we start from the next order of \((\ln(\mu r_0))^{-1}\). Then using (16) and (17), we find

\[ H_0 = 2 + (\ln(\mu r_0))^{-1} \left( \frac{2b + 3b'}{b} + \frac{9}{512\pi^2 bG\Lambda} \right) \]
\[ \sigma_0 = -\ln(\mu r_0) + \frac{1}{2} \ln \left( \frac{3\mu^2}{2\Lambda} \right) \]
\[ + (\ln(\mu r_0))^{-1} \frac{\mu^2}{8\Lambda} \left( \frac{2b + 3b'}{b} + \frac{9}{512\pi^2 bG\Lambda} \right) . \] (37)
Since we are now interesting in the problem of anti-evaporation, we consider
the solution of $A \sim C^2$. The solution $A \sim 0$ would correspond to usual
evaporation. Assuming $A = C^2 + (\ln(\mu r_0))^{-1} a_1$ and substituting (37) into
(25), we find

$$a_1 = 0, \quad a_1 = -\frac{(b + b')C}{8b}$$

(38)

In the first solution, the horizon does not develop in time again and we would
need the analysis of the higher order of $(\ln(\mu r_0))^{-1}$. An important thing is
the second solution is positive when $2N + 7N_{1/2} > 26N_1$. When $a_1$ is positive,
$A > C^2$, i.e., there occurs anti-evaporation. Let us consider $SU(5)$ group with
$N_s$ scalar multiplets and $N_f$ fermion multiplets in the adjoint representation
of gauge group. Then, above relation looks as $2N_s + 7N_f > 26$. We see
that for $SU(5)$ GUT with three spinor multiplets and three scalar multiplets
it is expected anti-evaporation for SdS BH. Similarly one can estimate the
chances for anti-evaporation in the arbitrary GUT under discussion. On
the other hand, when $2N + 7N_{1/2} < 26N_1$, there occurs evaporation. For
example for above GUT with two spinor multiplets and two scalar multiplets
we expect that matter quantum effects induce the evaporation of SdS BH.
This result would be non-perturbative and exact in the leading order of $1/N$
expansion. Of course, $A$ goes to $C^2$ in the limit of $r_0 \rightarrow +\infty$, when the SDW
type expansion becomes exact, and the black hole does not evaporate nor
anti-evaporate in the limit. If there is some external perturbation, which
gives effectively finite $r_0$, however, there may occur the anti-evaporation.

5 Energy Flow and No Boundary Condition

We now briefly discuss Hawking radiation. Since the space-time which
we are now considering is not asymptotically Minkowski, we will consider the
radial component in the flow of the energy $T_{tr} = T_{++} - T_{--}$.

Usually Einstein equation can be written as

$$\frac{1}{16\pi G} \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = T_{\mu\nu}^c + T_{\mu\nu}^q . \quad (39)$$

Here $T_{\mu\nu}^c$ is the classical part of the matter energy momentum tensor, which
vanishes in the case under consideration, and $T_{\mu\nu}^q$ is the quantum part, which
we are now interested in. Comparing (39) with (8), we find

$$T_{q}^{\pm} = \frac{r_0^2}{32\pi G} e^{2\sigma} \left( (\partial_\pm \sigma)^2 - \partial^2_\pm \sigma + 2\partial_\pm \sigma \partial_\pm \rho \right).$$

(40)

Substituting the solution of the perturbation (26), we find

$$T_{tr}^q = -\frac{3\alpha(\alpha + 1)\epsilon}{32\pi G\Lambda} \sinh (t\alpha \sqrt{C}) \sinh (r\sqrt{C}) \cosh^{\alpha - 1} (r\sqrt{C}) + O(\epsilon^2).$$

(41)

$T_{tr}^q$ is positive when $0 > \alpha > -1$ and negative when $\alpha < -1$, i.e., there is a flow from the event horizon to the cosmological horizon when $0 > \alpha > -1$. The direction of the flow is changing when $\alpha < -1$. They exactly correspond to evaporation ($0 > \alpha > -1$) and anti-evaporation ($\alpha < -1$).

Let us turn now to the study of no boundary condition in the evolution of black holes. As is known, the cosmological BH does not appear after the gravitational collapse of star since the background space-time is not de Sitter space but flat Minkowski space. Bousso and Hawking, however, conjectured that the cosmological black holes could be pair created by the quantum process in an inflationary universe since the universe is similar to de Sitter space. They have also shown that no boundary condition [13] determines the fate of the black holes and they should always evaporate (for minimal scalars). Now we make a similar analysis for conformal matter. In our case, the analytic continuation to Euclidean space-time is given by replacing $t = i\tau$. By the further changing the variables $\tau$ and $r$ by

$$u = \tau \sqrt{C}, \quad \sin u = \frac{1}{\cosh r\sqrt{C}},$$

(42)

the metric in the quantum Nariai type solution corresponding to (35) has the following form:

$$ds^2 = \frac{3}{2\Lambda} \left( du^2 + \sin^2 u dv^2 + d\Omega \right).$$

(43)

The metric (43) tells that 4 dimensional Euclidean space-time can be regarded as the product of two two-sphere $S^2 \times S^2$. In the Euclidean signature, the operator $\frac{4}{C} \Delta$ becomes the Laplacian on the unit two-sphere $S^2$. The nucleation of the black hole is described by cutting two sphere at $u = \pi$ and joining to it a Lorentzian $1 + 1$-dimensional de Sitter hyperboloid [3] by
analytically continuing $u$ by $u = \frac{\pi}{2} + i \hat{t}$ and regarding $\hat{t}$ as the time coordinate. In the Euclidean signature, the eigenfunction $f_A(t, r)$ in (26) is not single-valued unless $\alpha$ is an integer. Therefore $f_A(t, r)$ is not adequate when we discuss the nucleation of the black holes. Instead of $f_A(t, r)$, we consider the following eigengfunction $\tilde{f}_A(t, r)$ in the Euclidean signature:

$$\tilde{f}_A(u, v) = f_0 \cos v \cdot P^1_\nu(\cos u), \quad A = \frac{\nu(\nu + 1)}{4} C. \quad (44)$$

Here $f_0$ is a constant and $P^1_\nu(x)$ is given by the associated Legendre function:

$$P^1_\nu(x) = \sin \pi \nu (1 - x^2)^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{\Gamma(1 - \nu + n) \Gamma(2 + \nu + n)}{(n + 1)!n!} \left( \frac{1 - x}{2} \right)^n. \quad (45)$$

We can assume $\nu \geq 0$ without any loss of generality. Note that $\tilde{f}(u, v)$ vanishes at the south pole $u = 0$ since $P^1_\nu(x = \cos u = 1) = 0$. Therefore $\tilde{f}(u, v)$ is single-valued on the hemisphere and does not conflict with the no boundary condition [13]. We should also note that $\tilde{f}(u, v)$ is not real when analytically continuing $u$ by $u = \frac{\pi}{2} + i \hat{t}$ into Lorentzian signature but, as in [3], we can make $\tilde{f}(u, v)$ to be real in the late Lorentzian time (large $\hat{t}$) by the suitable choice of the constant $f_0$ since

$$P^1_\nu(\cos u) = P^1_\nu(-i \sinh \hat{t}) \sim \frac{\Gamma \left( \nu + \frac{1}{2} \right)}{\sqrt{\pi} \Gamma(\nu)} (-i)^\nu e^{i\hat{t}} \text{ when } \hat{t} \to +\infty. \quad (46)$$

Since $A > \frac{C}{2}$ when $\nu > 1$, we can expect that there would occur anti-evaporation in the pair-created black holes. In order to confirm it, we consider the behavior of the horizon in the late Lorentzian time (large $\hat{t}$). By using (46), we find that the condition of the horizon (27) gives $\cos v \propto e^{-i\hat{t}}$. Therefore we find on the horizon $\dot{f}(v(\hat{t}), \hat{t}) \propto e^{(\nu - 1)\hat{t}}$. This tells, as in the case of the previous section, the perturbation is stable when $A > \frac{C}{2}$ ($\nu > 1$) and instable when $A < \frac{C}{2}$ ($0 < \nu < 1$). The previous analysis in (38) implies that there is a solution of $A > \frac{C}{2}$. Therefore the anti-evaporation (stable mode) can occur even in the nucleated black holes and some of black holes do not evaporate and can survive. This result is different from that of Bousso and Hawking, who claimed that the pair created cosmological black holes most probably evaporate. (Note however that they considered other type of matter-minimal scalars, while we deal with conformal matter). In any case, this question deserves further deep investigation.
6 Discussion

In summary, we studied large $N$ effective action for conformal matter on spherically symmetric background. Application of this effective action to the investigation of quantum evolution of SdS BHs shows that such BHs may evaporate or anti-evaporate in nearly degenerated limit. No boundary condition is shown to be consistent with anti-evaporation of SdS BHs. Other boundary conditions may be discussed in the same way in generalization of this work. Some remarks about energy flow in regime of evaporation or anti-evaporation are also given.

Let us compare now our results with ones of the previous papers [3, 5], where similar questions have been investigated. In the work by Bousso and Hawking, they treated the quantum effects of 4D minimal scalars using $s$-wave and large $N$ approximations. In our previous paper [5], the scale $(\sigma)$ dependent part of the effective action is given by the large $N$ trace anomaly induced effective action (without using $s$-wave approximation) but the scale independent part is determined using $s$-wave approximation, i.e., spherical reduction. The work [5] deals with conformal quantum scalars only. In the present work, we used the effective action whose scale dependent part is given by large $N$ trace anomaly as in the previous paper [5] but whose scale independent part is given by the Schwinger-De Witt type expansion, which is essentialy the power series expansion on the curvature invariants corresponding to the rescaled metric. Since the curvature in the Nariai limit and the perturbation around it is almost constant, the rescaled scalar curvature is always $O(r_0^{-2})$. This tells that the Schwinger-De Witt type expansion given in this paper would become exact in the limit of $r_0 \to +\infty$. Therefore the analysis given here using $r_0 \to +\infty$ limit would be also exact. Moreover, the results of present work are given for arbitrary conformal matter (scalars, spinors and vectors). In all these works, it is found the same qualitative result — the possibility that SdS (Nariai) BH may anti-evaporate. As we see from the estimation above such anti-evaporation may be quite general for many GUTs. Moreover, pair created (primordial) BHs may anti-evaporate due to conformal quantum matter effects when applying no boundary condition.

As very interesting generalization of above work, it could be helpful to understand if anti-evaporation is specific feature of SdS BHs or it may be realized also for other BHs with multiply horizons. In order to clarify this
issue we studied Reissner-Nordstrøm(RN)-de Sitter BHs where preliminary investigation shows also the possibility of anti-evaporation due to quantum effects. We hope to report on this in near future.

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Appendix

A Stringy Presentation of 4D Einstein-scalar Theory

Starting from Einstein gravity with $N$ minimal scalars,

$$S = -\frac{1}{16\pi G} \int d^4x \sqrt{-g(4)} \left[ (R^{(4)} - 2\Lambda) - \frac{16\pi G}{2} \sum_{a=1}^{N} g_{(4)}^{\alpha\beta} \partial_\alpha \chi_a \partial_\beta \chi_a \right]. \quad (47)$$

one can consider the spherically symmetric space-time:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu + f(\phi) d\Omega. \quad (48)$$

Reducing the action (47) for the metric (48), we will get

$$\frac{S}{4\pi} = -\frac{1}{16\pi G} \int d^2x \sqrt{-g} \left\{ f(R - 2\Lambda) + 2 + 2 \left( \nabla^\mu f^{1/2} \right) \left( \nabla_\mu f^{1/2} \right) - \frac{16\pi G}{2} \sum_{a=1}^{N} f(\phi) \nabla^a \chi_a \nabla_a \chi_a \right\}. \quad (49)$$

The reduced action belongs to the class of actions described by

$$S = \int d^2x \sqrt{-g} \left\{ C(\phi) R + V(\phi) + \frac{1}{2} Z(\phi) g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} f(\phi) \sum_{a=1}^{N} \nabla^a \chi_a \nabla_a \chi_a \right\}. \quad (50)$$
where from (49) we get
\[
C(\phi) = -\frac{f(\phi)}{4G}, \quad V(\phi) = -\frac{1}{4G} (2 - 2\Lambda f(\phi)),
\]
\[
Z(\phi) = -\frac{1}{4G} \frac{f'^2}{f}, \quad \tilde{f}(\phi) = -4\pi f(\phi).
\] (51)

Working in the conformal gauge \( g_{\mu\nu} = e^{2\sigma} \bar{g}_{\mu\nu} \), we may present the action (50) as a sigma model
\[
S = \int d^2 x \sqrt{-g} \left[ \frac{1}{2} G_{ij}(X) \bar{g}^{\mu\nu} \partial_\mu X^i \partial_\nu X^j + R \Phi(X) + T(X) \right]
\] (52)
where
\[
X^i = \{ \phi, \sigma, \chi_a \}, \quad \Phi(X) = C(\phi), \quad T(X) = V(\phi) e^{2\sigma},
\]
\[
G_{ij} = \begin{pmatrix} Z(\phi) & 2C'(\phi) & 0 \\ 2C'(\phi) & 0 & 0 \\ 0 & 0 & -f(\phi) \end{pmatrix}.
\] (53)
Thus, we presented reduced 4D Einstein-scalar theory as sigma-model. Similarly one can present 4D Einstein-conformal scalar reduced theory
\[
S = \int d^2 x \sqrt{-g} \left\{ C(\phi) R + V(\phi) + \frac{1}{2} Z(\phi) g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} \tilde{f}(\phi) \sum_{a=1}^N \left( \nabla^2 \chi_a \nabla \chi_a + \frac{1}{6} \chi_a^2 \right) \right\}. \]
\] (54)
in a form like (52) with slightly changed metric \( G_{ij} \) and \( \Phi, T \):
\[
\Phi(X) = C(\phi) - \frac{1}{2} \tilde{f}(\phi), \quad T(X) = V(\phi) e^{2\sigma},
\]
\[
G_{ij} = \begin{pmatrix} Z(\phi) & 2C'(\phi) & -\frac{1}{3} \tilde{f}'(\phi) \sum_{a=1}^N \chi_a^2 \\ 2C'(\phi) & 0 & -\frac{2}{3} \tilde{f}(\phi) \chi_a \\ -\frac{1}{3} \tilde{f}'(\phi) \sum_{a=1}^N \chi_a^2 & -\frac{2}{3} \tilde{f}(\phi) \chi_a & -f(\phi) \end{pmatrix}.
\] (55)

One can show (see [14]) that off-shell effective action in stringy parametrization (52) is different from the one calculated in dilatonic gravity (50) in covariant gauge. However, on-shell all such effective actions coincide as it should be (see [14]). The main qualitative result of this Appendix is that one can study quantum evolution of black holes using also sigma-model approach.
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