A Note on Weyl invariance in gravity and the Wess-Zumino functional *

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Abstract

It is shown that the explicit calculation of the Wess-Zumino functional pertaining to the breaking term of the Weyl symmetry for the Einstein-Hilbert action allows to restore the Weyl symmetry by introducing the extra dilaton field as Goldstone field. Adding the Wess-Zumino counter-term to the Einstein-Hilbert action reproduces the usual Weyl invariant action used in standard literature. Further consideration might confer to the Einstein-Hilbert action a new status.

Keywords: Gauge field theory, Weyl symmetry, BRST symmetry, Wess-Zumino functional, compensating field, dilaton.

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* Dedicated to the memory of Raymond Stora (1930-2015).
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1 Introduction and motivation

It is fair to say that the conformal Weyl symmetry which induces a local rescaling of a metric $g_{\mu\nu}(x) \mapsto \Omega^2(x) g_{\mu\nu}(x)$ is still a fascinating local symmetry. As such, it can be considered on the same footing as a gauge symmetry, a standpoint we shall adopt in the paper.

Quite recently G. ’t Hooft in [1] put an emphasis on the use of what he called the local conformal symmetry, and the rôle of a dilaton field with only renormalizable interactions, in particular when gravity couples to matter. Some years before, in [2] Weyl symmetry is shown to be related to the origin of mass. As ’t Hooft says, local conformal symmetry is not well understood yet. Let us add in particular, its relationship with scale (or dilation) symmetry $x^\mu \mapsto \lambda x^\mu$ which confers the canonical dimension to a field. This canonical dimension can be reinterpreted as a weight when fields are geometrically considered as densities.

Both [2, 1] used the conformal Weyl invariant local functional in gravitation in $n$-dimensional spacetime

$$S(g, \sigma) = \frac{1}{2} \int \sqrt{|\det g|} \sigma^{-n} \left( (n-2)(n-1)||\nabla \sigma||^2 + R(g) \sigma^2 \right) d^n x$$

where $||\nabla \sigma||^2 = g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma$ is the Riemannian scalar product. The Weyl invariance is achieved when beside the Weyl rescaling of the metric $g$, the scalar field $\sigma$ transforms according to $\sigma \mapsto \Omega \sigma$. The latter is a Weyl compensating field (or dilaton) in order to restore Weyl invariance of the theory while maintaining the locality principle for the functional in the fields $g$ and $\sigma$.

Former approaches showed that non-Weyl invariant theories (as the Einstein-Hilbert action for gravity is) can be turned into Weyl invariant ones by using a Weyl compensating scalar field. According to the existing literature on the subject, it would appear that this functional action was considered at the beginning of the 1970’s by B. Zumino [3, in particular, reference 13 therein] and S. Deser [4], independently. This was around the same period of the occurrence of the so-called Wess-Zumino term [5]. Surprisingly enough, no relationship with the latter was put into evidence at that time and subsequently [6, formula (14)], according to the best of our knowledge.

Relying onto the standpoint of locality principle in QFT and considering the metric $g$ as an external gravitational field, we shall adopt the attitude to consider the breaking of the Weyl rescalings (conformal transformations) of the Einstein-Hilbert action as an “anomaly”. If it turns out to be the case, by making use of the BRST techniques, namely, it is a BRST 1-cocycle from which the Wess-Zumino term can be constructed. Recall that the latter integrates the anomalous term and can be used as counter-term to restore the symmetry. The price to pay usually is to add in the theory a new Goldstone field which carries a non linear transformation law.

This legitimately raises the natural question whether the introduction of the above dilaton field $\sigma$ as compensating field pertaining to the Weyl conformal symmetry, and accordingly, the modification of the Einstein-Hilbert action into the action (1), stems from the usual generic construction of a Wess-Zumino term (or action or functional). This issue will be addressed in the present paper whose purpose is to provide a positive answer to this somewhat conceptual issue.

The construction of the Wess-Zumino functional as in [5], [7], [8, formula (4.33)], [9] and [10], see p. 164 is well-known for usual gauge theories. That is, when the gauge anomaly is of polynomial type, i.e. constructed through the so-called descent equations stemming from an invariant polynomial given by a characteristic class of the underlying principle fiber bundle. What about the construction of a Wess-Zumino functional in the case where the anomaly is not “polynomial”, namely, there is no $Q$ coming from the Chern-Simons transgression formula? For instance, diffeomorphism anomalies, Weyl anomaly fall into this category. The generic construction given in [11, 12] takes fully into account both the cases. Let us call that construction the “Stora construction”\(^1\). We shall apply it in the context of local conformal symmetry in gravity. In order to achieve this goal, the general BRST treatment of anomalies will be used.

The paper is organised as follows. Section 2 will serve to fix the notation. In section 3, we shall show that the breaking term inferred by the Einstein-Hilbert action falls into the usual algebraic BRST cohomology. In section 4 the corresponding Wess-Zumino term will be computed explicitly. The paper is closed by some concluding remarks and open questions. Two appendices are devoted to the construction of the Wess-Zumino action according to Raymond Stora.

\(^1\)Historically and according to the best of our knowledge, a preliminary proof of the Stora construction was given as an appendix in [13, appendix F] along the Stora’s ideas and used for integrating the 2-D Weyl anomaly in [14]. The proof was completed in [12] by Stora himself.
2 Weyl rescalings

Let $M$ be a $n$-dimensional smooth manifold and consider $\text{Met}(M)$ the infinite dimensional space of (pseudo-)
Riemannian metrics on $M$.

A Weyl rescaling is a mapping on $\text{Met}(M)$ defined by $g(x) \mapsto \varphi(x)g(x)$ where $\varphi \in C^\infty(M,\mathbb{R}_+^*)$ is a smooth
positive function defined on $M$. It is also named a conformal change of the metric $g$. Such transformations
yield an abelian transformation group acting on the space $\text{Met}(M)$.

We shall parametrize for convenience $\varphi(x) = \Omega^2(x) = e^{2\delta_\varphi(x)}$ so that the Weyl rescaling of a metric reads

$$g \rightarrow \bar{g} = \Omega^2 g,$$

$$\delta_{\text{Weyl}} g =: \delta \varphi = 2\varphi g$$

for respectively, a finite local conformal change of the metric $g$, or infinitesimally linearized version. Notice
that $[\delta \varphi, \delta \varphi] = \delta_{[\varphi,\varphi]} \equiv 0$ due to the abelian feature of the Lie algebra $(\mathbb{R},+)$ of the Weyl gauge group.

Accordingly, the scalar curvature $R(g)$ transforms under a conformal rescaling as (see e.g. [15, § 3.9])

$$R(\bar{g}) = e^{-2\delta_\varphi} (R(g) + 2(n-1)\Delta \varphi - (n-1)(n-2)||\nabla \varphi||^2)$$

or under an infinitesimal Weyl transformation as

$$\delta_{\text{Weyl}} R(g) = 2(n-1)\Delta \varphi - 2\varphi R(g)$$

where the Laplacian acting on scalar functions is given by (see e.g. [15, § 2.7])

$$\Delta = -\nabla_\mu \nabla^\mu = \frac{-1}{\sqrt{|\det g|}} \partial_\mu \left( g^{\mu\nu} \sqrt{|\det g|} \partial_\nu \right).$$

If one considers the quotient space $\text{Met}(M)/C^\infty(M,\mathbb{R}_+^*)$, this is the infinite dimensional space of the
conformal classes $[g]$ of the metrics. Morally, a Weyl (scale) invariant physical theory is supposed to depend
only on the conformal classes of metrics. Considering the local Weyl rescalings as a gauge group, one can pass
to the quotient space $\text{Met}(M) \xrightarrow{\text{Weyl}} \text{Met}(M)/C^\infty(M,\mathbb{R}_+^*)$. However, as Raymond Stora used to insist on,
owing to the locality principle, the (gauge) Weyl invariant action functional must depend on a representative
of the conformal class, with the inherent problem of the ambiguity on the choice of this representative, say $g$. This amounts one to working on the space $\Gamma_\infty(\text{Met}(M))$ of local functionals in the metric. However, in gauge theory, observables are functions on orbit space, or equivalently, gauge invariant functions on field
space (which is here $\text{Met}(M)$) [16].

For the local conformal symmetry the corresponding (classical) Ward identity acting on local functionals reads

$$\int_M d^n x \delta_{\text{Weyl}} g_{\mu\nu}(x) \frac{\delta \Gamma[g]}{\delta g_{\mu\nu}(x)} = \int_M d^n x 2\varphi(x) g_{\mu\nu}(x) \frac{\delta \Gamma[g]}{\delta g_{\mu\nu}(x)} = 0.$$  

(4)

Let us now consider the Einstein-Hilbert action which is such a local functional of $g$

$$S_{\text{EH}}(g) = \int_M \text{vol}(g) R(g) = \int_M d^n x \sqrt{|\det g|} R(g)$$

(5)

where vol($g$) is the volume form on $M$ and $R(g)$ is the scalar curvature, both associated to the metric $g$.

The next step will be the study of the behavior of the Einstein-Hilbert action under local conformal
rescalings.

3 Another Weyl “anomaly”

As is well known, the Einstein-Hilbert gravitation theory yields a spontaneous breaking of the conformal Weyl
symmetry. We shall study this breaking of conformal symmetry in the framework of the BRST differential
algebra [17, 18]. Thus, turning the Weyl parameter $\varphi$ into the Faddeev-Popov ghost field, still denoted by $\varphi$
with $\varphi^2 = 0$ (abelian Lie algebra), the corresponding Slavnov operation $s$ acting on the field generators—see
e.g. [19, 20]— is defined by

$$sg = 2\varphi g,$$

$$s\varphi = 0,$$

with $s^2 = 0.$  

(6)
In particular, one has
\[ s\sqrt{|\det g|} = u\phi \sqrt{|\det g|}, \quad sR(g) = 2(n-1)\Delta \phi - 2\phi R(g). \] (7)

This Slavnov operation \( s \) will act on the functional space \( \Gamma_{\text{loc}}(\text{Met}(M)) \). A direct computation shows that the variation of the Einstein-Hilbert action under Weyl transformations is infinitesimally given by
\[ sS_{\text{EH}}(g) = A(\phi, g) \] (8)

with
\[ A(\phi, g) = \int_M d^n x \sqrt{|\det g|} \left( (n-2)\phi R(g) + 2(n-1)\Delta \phi \right). \] (9)

This means that the Ward identity (4) is already broken at the classical level by the Einstein-Hilbert action. By inspection, the functional \( A(\phi, g) \) is local in \( g \) and linear in the ghost argument \( \phi \), and it turns out to fulfill the celebrated Wess-Zumino consistency condition \([5, 18]\) in its BRST formulation, namely,
\[ sA(\phi, g) = 0 \] (10)
as it can be checked explicitly. In the course of the computation, an integration by parts must be performed in order to get an integrand of the type \( ||\nabla \phi||^2 \) which vanishes by Faddeev-Popov argument.

Hence, since (10) expresses a 1-cocycle condition, one can analogously treat \( A(\phi, g) \) as an “anomaly” even if it is not a quantum breaking of the Weyl symmetry. Here, it is rather a geometrical breaking. However, one might speculate that the Einstein-Hilbert action could be considered as a local “vacuum functional” depending on an external gravitational field \( g \). The latter ought to come from a field theory coupled to a gravitational field, similarly to the approach followed in \([20, 21]\) for the bosonic string in the 2D case, where by a gauge fixing condition the gravitational field becomes external. Let us also mention that the Einstein-Hilbert action may be seen as resulting from the spectral action principle initiated in \([22]\) together with the Standard Model Lagrangian (at least at the tree level).

The main issue now is to restore the local Weyl conformal symmetry by proceeding along the line used in gauge theory to reabsorb the anomaly.

4 The Wess-Zumino functional

The reader is recalled that the Wess-Zumino functional is defined to integrate the anomaly, see e.g. \([10, 12]\) for a BRST treatment of this issue, that is (according to our situation at hand)
\[ s\Gamma_{\text{WZ}} = -A(\phi, g). \] (11)

In fact, the anomaly is trivialized at the cost of introducing an extra field with values in the Lie algebra of the gauge group. Two appendices give a detailed account on Raymond Stora’s ideas on the construction of the Wess-Zumino action as an important item in gauge theory.\(^2\)

In our case, the anomaly does not seem to be a “polynomial anomaly” (namely of Adler-Bardeen type). A priori, there is no Chern-Simons transgression formula. However, as said in section 1, there is still a way to construct the Wess-Zumino functional \([12]\).

To this end, take a 1-parameter subgroup for \( t \in [0, 1] \), \( \gamma_t = e^{-t \tau} \) in the Weyl gauge group from the identity element \( \gamma_0 = 1 \) (constant function) to the positive function \( \gamma_1 = e^{-\tau} =: \gamma \). Accordingly, its action on a given metric \( g \) defines a path in the conformal class \([g]\) of the metric \( g \) (namely, in the Weyl gauge orbit of the latter)
\[ g_t = \gamma_t^2 g = e^{-2t\tau} g, \quad \text{with} \ g_0 = g \text{ and } g_1 = \gamma^2 g = e^{-2\tau} g. \] (12)

Consider the pull-back to the interpolating family \( \gamma_t \) of the Maurer-Cartan form on the gauge group associated to the Weyl group
\[ \omega_t = \gamma_t^{-1} d_t \gamma_t = -2\tau dt \] (13)

\(^2\)The minus sign in formula (11) is introduced for convenience, contrary to the general construction given in Appendix A.
which is found to be no longer depending on the $t$-parameter. Following \cite{10, 11} one extends the Slavnov operation $s$ to the added scalar field $\tau$ by requiring

$$sg_1 = s(e^{-2\tau}g) = 0 \Rightarrow s\tau = \phi. \quad (14)$$

It is like a gauge fixing on the conformal component of the metric field. Rather, it might be viewed as a change of variables within the field space which $g$ and $\tau$ belong to \cite{23}. Hence, the field $\tau$ carries 1 as Weyl conformal weight and will play the rôles of dilaton as we shall see.

Raymond Stora \cite{12} defines the Wess-Zumino functional, $\Gamma_{WZ}$, by integrating over the interpolating family the anomalous term according to

$$\Gamma_{WZ}(\tau, g) = \int_0^1 A(\omega_t, g_t) = \int_0^1 dt A(-2\tau, g_t) \quad (15)$$

upon using (13). One has to perform the integration over $t$ of the functional

$$A(-2\tau, g_t) = -2 \int_M d^n x \sqrt{|\det g_t|} \left((n-2)\tau R(g_t) + 2(n-1)\Delta_\tau \right).$$

The overall factor 2 in the r.h.s. comes from the linearity of $A$ in its first argument. Upon using (3) and (12) and after some algebra, one gets the expression

$$\Gamma_{WZ}(\tau, g) = -2 \int_M d^n x \sqrt{|\det g|} \left(\frac{1}{n-2}\right) \left( (n-2)\tau R(g) + 2(n-1)\Delta_\tau \right) + 2(n-2)(n-1)||\nabla r||^2 - \tau \Delta t$$

Performing the integration over $t$, the corresponding Wess-Zumino functional reads

$$\Gamma_{WZ}(\tau, g) = 2 \int_M d^n x \sqrt{|\det g|} \left(\frac{1}{n-2}\right) \left( R(g) - 2(n-1)\Delta_\tau - (n-2)(n-1)||\nabla r||^2 - R(g) \right). \quad (16)$$

Upon defining $\sigma = e^\tau$ (= $1/\gamma$) (the inverse element to $\gamma$) with $s\sigma = \phi\sigma$, as a compensating field of conformal weight one, the Wess-Zumino action can be rewritten as

$$\Gamma_{WZ}(\tau, g) = 2 \int_M d^n x \left(\frac{1}{n-2}\right) \left( R(g) - 2(n-1)\Delta_\tau - (n-2)(n-1)||\nabla r||^2 - \sqrt{|\det g|} R(g) \right)$$

and next by using once more the Weyl transformation (3) for $g \rightarrow \sigma^{-2}g$, one is led to

$$\Gamma_{WZ}(\tau, g) = 2 \int_M d^n x \left(\frac{1}{n-2}\right) \left( R(g) - 2(n-1)\Delta_\tau - (n-2)(n-1)||\nabla r||^2 - \sqrt{|\det g|} R(g) \right)$$

which is nothing but the difference

$$\frac{1}{2} \Gamma_{WZ}(\tau, g) = S_{\text{eh}}(\sigma^{-2}g) - S_{\text{eh}}(g). \quad (17)$$

Since the rôles of the counter-term $\Gamma_{WZ}$ is to cancel the anomaly, one thus has by construction

$$s S_{\text{eh}}(\sigma^{-2}g) = s \left( \frac{1}{2} \Gamma_{WZ}(\tau, g) + S_{\text{eh}}(g) \right) = 0 \quad (18)$$

which is consistent actually with the constraint (14). Finally, this yields the Weyl invariant local functional

$$S_{\text{eh}}(\sigma^{-2}g) = S_{\text{eh}}(g) + \frac{1}{2} \Gamma_{WZ}(\tau, g) = \int_M d^n x \sqrt{|\det(\sigma^{-2}g)|} R(\sigma^{-2}g) \quad (19)$$

Remembering that $\sigma = e^\tau$, one can check that this formula, up to an integration by parts and up to an overall factor $-\frac{1}{2}$, is nothing but the Weyl invariant action (1) given in \cite{3, 4} and used in \cite{2, 1}.

At this stage some comments are in order.

\footnote{This means, at the finite level $\sigma \rightarrow \bar{\sigma} = \Omega \sigma$.}
5 Comments and outlook

On the one hand, since $s(\sigma^{-2}g) = 0$, it turns out to be obvious that $s S_{\text{EH}}(\sigma^{-2}g) = 0$. But, on the other hand, the Weyl invariance has been restored by mimicking a construction coming from Lagrangian gauge theory, by adding the Wess-Zumino counter-term to the well-known Einstein-Hilbert action and thus introducing the so-called compensating field $\sigma$ in order to be consistent with the locality principle. In this respect, and to parallel some Raymond Stora’s viewpoints (see section 2) this rises the following question: Does the Weyl invariant combination $\sigma^{-2}g$ gives a substitute to parametrize the conformal class $[g]$ of the metric $g$ compatible with the locality principle? And to “mirror” [1]: Does the construction of the Wess-Zumino term provides a canonical way to isolate the dilaton component of the metric?

Moreover, according to [3, p.464] an action which is invariant under both Einstein and Weyl symmetries is invariant in the Minkowski flat limit under the 15-dimensional conformal group. One is led to make contact also with [19, 20], and one may also remark that if it would be possible to set $\sigma = (\det g)^{1/2n}$ by gauge fixing or as an equation of motion, then $s = \phi$, since $s(\det g) = 2n\phi(\det g)$. Accordingly, $\sigma^{-2}g = (\det g)^{-1/n}g$ and $\det(\sigma^{-2}g) = 1$ so that $\sigma^{-2}g = \hat{g}$, the associated unimodular matrix to the metric $g$, will also serve as a representative of the conformal class $[g]$ of the metric $g$. It ought to be useful to investigate more in that direction.

Going back to the construction of the Wess-Zumino action, in particular the rôle of the interpolating family, formula (17) can be simply recast as

$$\frac{1}{2} \Gamma_{WZ}(\tau, g) = \int_0^1 dt S_{\text{EH}}(\sigma^{-2}\tau g) = \int_0^1 \frac{\partial S_{\text{EH}}(\sigma^{-2}\tau g)}{\partial \tau} dt = \int_0^1 dt S_{\text{EH}}(g_t).$$  \hspace{1cm} (20)

Not only the integration over the family can be explicitly performed for computing the Wess-Zumino action, but it highlights the Einstein-Hilbert action since one may write

$$A(\omega_t, g_t) = 2 d_t S_{\text{EH}}(g_t),$$  \hspace{1cm} (21)

along a path in the gauge orbit given by the conformal class $[g]$ of the metric $g$. Following Stora’s tricks [12], at the algebraic level (here, the Lie algebra is abelian) the Maurer-Cartan equation $d_t \omega_t = 0$, together with $d_t g_t = -d_M \omega_t$ ($d_M$ is the de Rham differential on spacetime $M$), yields a differential algebra which is similar to the BRS one given in (6). The Wess-Zumino consistency condition leads to $d_t A(\omega_t, g_t) = 0$. Since $d_t^2 = 0$, equation (21) indicates that the Wess-Zumino term turns out to be independent of the interpolating family up to smooth deformations and depends only on the bounds.

Thus $S_{\text{EH}}(g_t)$ interpolates along a family of conformally related metrics to the metric $g$. That is within a fiber of $\text{Met}(M)$ with respect to the conformal symmetry. This might indicate that the Einstein-Hilbert action plays a special role in the construction. If one was able to pass to the quotient space $\text{Met}(M)/C^\infty(M, \mathbb{R}^*_+)$, as configuration space, the “genuine” physical theory ought to depend only on the conformal classes $[g]$ of the metrics $g$. More investigation deserves to be performed in this matter.

Acknowledgments

One of us (SL) has got the great honor of being the last Raymond Stora’s PhD student and would like, with this paper, to pay tribute to the memory of Raymond.

The authors apologize for references that have been implicitly quoted in the bibliography.

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\[4\text{Even if this sounds quite simple at first sight, it is grounded on strong QFT principles. Recall that the construction presented in the paper does not appear to have been done in standard literature.}\]
Appendix A  The Stora construction of the Wess-Zumino action

In [8, 9], [7, Section III] or [10, 25, 11] the construction of the Wess-Zumino functional was mainly performed in the case where the gauge anomaly was of polynomial type, i.e. constructed through the so-called descent equations stemming from an invariant polynomial coming from a characteristic class of the underlying principle fiber bundle. What about the construction of a Wess-Zumino functional in the case where the anomaly does not come from a polynomial, namely there is no local functional coming from the Chern-Simons transgression formula. For instance, diffeomorphism anomalies, Weyl anomaly or presently, the “anomalous” term obtained from the Einstein-Hilbert action as a spontaneous breaking of the conformal Weyl invariance, they all fall into this latter type.

In this appendix, we would like to report on a more general construction which deals such a situation, even more, with any generic situation, for any Lie algebra Lie G and any representation space.

Historically and according to the best of our knowledge, a preliminary construction was explicitly given in [13, see appendix F] or [12]. It could be viewed as a most formalized version of [9, see pages 485 and 486] and exploits results in [11, section II and appendix]. Latter, Raymond Stora guessed an homotopy after a thorough discussion at CERN-TH in the mid of the 1990’s with one of us (SL) and evoked by Raymond Stora himself in a seminar given at CPT (Marseilles) in November 1995.

Let us give a short account on this construction mainly recorded in [26] in order to bring this nice construction out of the shadows as a part of the wide legacy left by Raymond Stora.

To start with, let us denote the gauge group by $G = \{ g : U \to G \}$ the set of local maps with values in a compact, simple symmetry group G. It carries a group law inherited from that of G. As field representation spaces for G, one can distinguish

- when elements of G are considered as fields, one has the so-called gauge group $G = \{ \gamma \in G, \gamma^g = g^{-1}g \}$, which is compatible with the group law in G.
- the gauge transformation of gauge potentials with the usual action of G given by the right action $a\gamma = \gamma^{-1}a\gamma + \gamma^{-1}d\gamma$.

Consider a local consistency anomaly as an element in $H^1(\text{Lie } G, \Gamma_{\text{loc}}(a))$

$$A(c, a) = \int_M \Delta(c, a),$$

fulfilling $sA(c, a) = 0$ (A.1)

where $a$ is a Yang-Mills like potential and $c$ the Faddeev-Popov ghost associated to the Lie algebra Lie G of the gauge group G. It may be recalled that $\Delta(c, a)$ is a differential polynomial in $a$ (i.e. local in a in the QFT sense), top form on $M$ which is linear in $a$. The Wess-Zumino consistency condition $sA(c, a) = 0$ (local 1-cocycle) yields [7, 27]

$$s\Delta(c, a) + d\Delta'(c, a) = 0$$

(A.2)

where $d$ is the de Rham differential on $M$.

For instance, in a Yang-Mills theory, the Slavnov operation $s$ is explicitly given by

$$sa = -D_a c = -dc - [a, c], \quad sc = -\frac{1}{2}[c, c], \quad s^2 = 0$$

(A.3)

where $[\cdot, \cdot]$ is the graded Lie algebra bracket. Recall that $c \in (\text{Lie } G)^* \otimes \text{Lie } G$ and is the generator of cochains on Lie G [11] or [28, section 6.10]).

Theorem. [5] Given an anomaly $A(c, a)$ with $sA(c, a) = 0$, one can construct a functional on the field space of gauge fields $a$ and on field $u \in \mathcal{G}$

$$\Gamma_{\text{wz}}(u, a) = \int_0^1 A(\gamma_t^{-1}d\gamma_t, a\gamma) = \int_a^u A(\gamma^{-1}\delta\gamma, a\gamma)$$

(A.4)

where $\{ \gamma_t \}$ is a 1-dimensional family in G, for $t \in [0, 1]$ and $\gamma_0 = e$ and $\gamma_1 = u$. If $s$ is extended on $u$ by $su = -cu$ such that $s(u^n) = 0$ is secured, then

$$s\Gamma_{\text{wz}}(u, a) = A(c, a).$$

(A.5)

$\delta e$ is the identity of G and denotes the constant field $x \mapsto e_G$. 

\[\end{raw_text}\]
The Wess-Zumino trick [5, 8] is to use a seesaw mechanism between left and right actions by extending the Slavnov operation $s$ on the gauge group $G$ by $su = -cu$, where $u$ is considered as a field, in order to guarantee $s(u) = 0$. This is, in the BRST language, the infinitesimal version of the gauge invariance of a composite field

$$a^u := \text{Ad}(u^{-1})a + u^{-1}du$$

under the gauge transformations of both elementary fields $a$ and $u$ (the latter are considered as belonging to representation spaces of $G$ — a right group action) according to

$$a \rightarrow a^\gamma = \text{Ad}(\gamma^{-1})a + \gamma^{-1}d\gamma, \quad u \rightarrow u^\gamma = \gamma^{-1}u \quad \Rightarrow (a^u)^\gamma = (a^\gamma)^u = (a^\gamma)^{-1}u = a^u \quad (A.6)$$

where $\gamma$ is a gauge group element, $\gamma \in G$. Let us stress that the composite field $a^u$ must not be considered as a gauge transformation of $a$ because $a^u$ is no longer a connection owing to the fact that $u$ carries a non-linear transformation law $u^\gamma = \gamma^{-1}u$ which is different from the required transformation law $\gamma^{\gamma} = \gamma^{-1}\gamma^\gamma$ for a genuine gauge group element considered as a field in the ‘adjoint’ representation, (as recalled above (A.1)). In fact, $a^u$ may be interpreted as a change of variable in the field space of the $(a, u)$’s; see discussion in [23, 29, see in particular section 2 in both of those references] on what we called the dressing field method, a construction which goes back to Dirac [30] and which, in turn, enters in the construction of the Wess-Zumino functional.

The Stora construction. Given a family $\{\gamma_t\}$ in $G$, $0 \leq t \leq 1$, with $\gamma_0 = e$ and $\gamma_1 = u$, its action on $a$ gives a family $\{a_t := a^\gamma_t\}$, interpolating from $a_0 = a$ to $a_1 = a^u$. In the field space, consider the interpolating family $\{u_t, a_t\}$ defined by $(a_t)^u = a^u$ for all $t$. This constraint implies $u_t = \gamma_t^{-1}u$ (in the case of a transitive action of $G$), that is $u_t$ is a family of dressing fields with $u_0 = u$ and $u_1 = e$.

Requiring the invariance of $(a_t)^u$ under the gauge transformations given in (A.6) infers

$$(a_t)^u_t = (a^\gamma)^{-1}u = (a_t^{\gamma^{-1}})^\gamma = (a_t^{\gamma^{-1}})^{\gamma_t}u_t = (a_t^{\gamma^{-1}})^{\gamma_t}u_t.$$ 

This shows the gauge invariance of $(a_t)^u$ under the following gauge transformations on the family

$$a_t \rightarrow a_t^{\gamma_t^{-1} \gamma_t}, \quad u_t \rightarrow (\gamma_t^{-1} \gamma_t)^{-1}u_t.$$ 

It is worthwhile to notice that $\gamma_t^{-1} \gamma_t$ is a family which stays within the gauge group $G$ as a field space for the ‘adjoint’ representation. Let us turn to the infinitesimal version of the latter in a BRST language. To sum up, one has the family in field space

$$u_t = \gamma_t^{-1}u, \quad a_t = a^{\gamma_t}, \quad c_t = \gamma_t^{-1}c \gamma_t$$

(the latter is the adjoint action of the family $\gamma_t$ on $(\text{Lie } G)^*$) and it can be checked that

$$su_t = -c_t u_t, \quad sa_t = -dc_t - c_t a_t - a_c c_t, \quad sc_t = -\frac{1}{2}[c_t, c_t], \quad s^2 = 0. \quad (A.7)$$

Along some ideas given in [10], the Stora’s trick is to introduced a homotopy for $s$ on the family through an even derivation $k_t$ defined as

$$k_t a_t = k_t a_t = 0, \quad k_t c_t = -d_t u_t u_t^{-1} - \gamma_t^{-1}d_t \gamma_t \quad (A.8)$$

in order to satisfies

$$k_t s - s k_t = d_t, \quad s^2 = d_t^2 = s^2 = 0, \quad \text{where } d_t \text{ is an antiderivation along the 1-parameter family induced by } \gamma_t.$$

Since the differential algebra (A.7) is similar to the BRST algebra (A.3), one has the consistency condition (A.1) along the whole family, namely, $sA(c_t, a_t) = 0$, or $s\Delta(c_t, a_t) + d\Delta'(c_t, a_t) = 0$. Therefore, by (A.8), one gets

$$d_t A(c_t, a_t) = -s k_t A(c_t, a_t) = -sA(\gamma_t^{-1}d_t \gamma_t, a^{\gamma_t})$$

where on the r.h.s. the integrand used in the Wess-Zumino-Stora formula (A.4) occurs. Integration in $t$ yields

$$\int_0^1 d_t A(c_t, a_t) = A(c_1, a_1) - A(c_0, a_0) = A(u^{-1}c u, a^u) - A(c, a) = -\int_0^1 s A(\gamma_t^{-1}d_t \gamma_t, a^{\gamma_t}) \quad (A.9)$$
At this stage, some care is required, because \( s \) acts on the upper integration bound \( u \) also of the interpolating family \( \{ \gamma_t \} \). Hence, \( s \) does not commute with the integration. The latter integration can be rewritten as

\[
- \int_0^1 sA(\gamma_t^{-1} dt \gamma_t, a^{\gamma_t}) = - \int_0^u sA(\gamma_t^{-1} \delta \gamma_t, a^{\gamma_t}) = - s \int_u^1 A(\gamma_t^{-1} \delta \gamma_t, a^{\gamma_t}) + s \int_0^u A(\gamma_t^{-1} \delta \gamma_t, a^{\gamma_t})
\]

where in the last term, only the upper bound is varied. Raymond Stora used to rewrite the latter as

\[
s \int_0^u A(\gamma_t^{-1} \delta \gamma_t, a^{\gamma_t}) = \left( \int_0^{u+su} - \int_0^u \right) A(\gamma_t^{-1} \delta \gamma_t, a^{\gamma_t})
\]

in order to consider the difference between the integration of the anomaly along two different paths in \( G \), thanks to the possibility\(^6\) to smoothly deform \( \{ \gamma_t \} \) from \( e \) to \( u \) into a path from \( e \) to \( u + su = u - cu \) for a variation \( \delta \gamma_t = -tc \gamma_t \). Then, using (B.10) of the Corollary in Appendix B where a more detailed construction is given, one gets

\[
s \int_0^1 A(\gamma_t^{-1} dt \gamma_t, a^{\gamma_t}) = -A(\gamma_1^{-1} \delta \gamma_1, a^{\gamma_1}) = A(u^{-1} cu, a^u).
\]

Collecting all the various terms, formula (A.9) reduces to

\[
s \int_0^1 A(\gamma_t^{-1} dt \gamma_t, a^{\gamma_t}) = A(c, a)
\]

a result which achieves the proof.

**Appendix B**  More about deformation of interpolating families

Let \( \delta \) denote the differential on the gauge group \( G \) and the Maurer-Cartan form on \( G \) at \( \gamma \) reads \( \gamma^{-1} \delta \gamma \)\(^7\). Following [10] and [11, see appendix], if

\[
a^{\gamma} = \gamma^{-1} a\gamma + \gamma^{-1} d\gamma
\]

is the finite gauge transformed (obtained by right action) of the gauge field \( a \) and since \( \delta \) acts on the gauge group element \( \gamma \) only, taking into account the Maurer-Cartan equation, one has the algebra

\[
\delta(a^{\gamma}) = -d(\gamma^{-1} \delta \gamma) - [a^{\gamma}, \gamma^{-1} \delta \gamma],
\]

\[
\delta(\gamma^{-1} \delta \gamma) = -\frac{1}{2}[\gamma^{-1} \delta \gamma, \gamma^{-1} \delta \gamma],
\]

\[
\delta^2 = 0. \quad (B.1)
\]

Note that this differential algebra (B.1) is isomorphic to the BRS algebra given in (A.3). This yields an homomorphism from \( H^\ast(G, \Gamma_{\text{loc}}(a)) \) to \( H^2_\ast(G, \Gamma_{\text{loc}}(a)) \) [11, 27]. This shifts cohomological issues on \( \text{Lie} G \) to the ones of left invariant forms on \( G \) with values in \( \Gamma_{\text{loc}}(a) \). For instance, the replacements

\[
s \to \delta

c \to \gamma^{-1} \delta \gamma

a \to a^{\gamma}
\]

The formula on the r.h.s. shows that one is led to work with differential forms on \( M \times G \) [32]. In particular, 

\[
A = \int_M \Delta \text{ is of degree 1 and is linear in } \gamma^{-1} \delta \gamma \in T^* G \otimes \text{Lie} G.
\]

By exploiting further this correspondence, and thus the Wess-Zumino consistency condition on the 1-cocycle \( A \), one can show the following important result. Consider an interpolating family \( \{ \gamma_t \} \), \( 0 \leq t \leq 1 \) in \( G \) from \( \gamma_0 = e \) to \( \gamma_1 = g \). Restricting the 1-cocycle \( A \) to this interpolating family, one gets by pulling-back the consistency condition (B.2) on \([0, 1]\)

\[
dt A(\gamma_t^{-1} dt \gamma_t, a^{\gamma_t}) = 0.
\]

\(^6\)This is at least possible in the connected component to the identity \( e \) of \( G \), or if the topology of \( G \) is suitable for a vanishing fundamental group \( \pi_1(G) = 0 \), see [7].

\(^7\)This field is the Faddev-Popov ghost according to the Zumino standpoint; see e.g. [31].
The Wess-Zumino action is a particular example of functionals over paths in $\mathcal{G}$ and given by
\[
\int_{0}^{1} A(\gamma_t^{-1} d\gamma_t, a^{\gamma_t}) = \int_{M \times [0,1]} (\tilde{\gamma}^{-1} \delta \tilde{\gamma}, a^{\tilde{\gamma}})
\]
by lifting the situation to paths $\tilde{\gamma} : M \times [0,1] \to G$, [10, p.164]. In particular, it makes sense to integrate the 1-cochain $A = \int_{M} \Delta$ which is linear in $\gamma^{-1} \delta \gamma$.

We address now the issue of the behavior of such functionals under smooth deformations of the interpolating family $\{\gamma_t\}$ (as a 1-dimensional submanifold § of $\mathcal{G}$). Deforming the family $\{\gamma_t\}$ amounts to defining a map
\[
\tilde{\gamma} : [0,1] \times [0,1] \to \mathcal{G}, \quad (t, \tau) \mapsto \tilde{\gamma}(t, \tau) = \gamma_{t, \tau} \quad \text{with } \gamma_{t, \tau=0} = \gamma_t.
\] (B.3)
The smooth deformation $\tau \mapsto \gamma_{t, \tau}$ defines, for each $t$, a curve in $\mathcal{G}$ passing through $\gamma_t$ at $\tau = 0$ with velocity $\frac{\partial}{\partial \tau} (\gamma_{t, \tau})|_{\tau=0}$. This gives rise to a $t$-dependent family of tangent vectors (as it will be explicitly seen later on) defines along the family $\{\gamma_t\}$ by
\[
X_{\gamma_t} = \frac{\partial}{\partial \tau} (\gamma_{t, \tau})|_{\tau=0} \in T_{\gamma_t} \mathcal{G}
\]
where $T_{\gamma_t} \mathcal{G}$ is the tangent space to $\mathcal{G}$ at the point $\gamma_t$. We are interested in computing the variation of the smooth map
\[
\tau \mapsto \int_{0}^{1} A(\gamma_{t, \tau}^{-1} d\gamma_{t, \tau}, a^{\gamma_{t, \tau}})
\]
namely, one has indeed to compute one of the derivatives
\[
either \frac{\partial}{\partial \tau} \left( \int_{0}^{1} A(\gamma_{t, \tau}^{-1} d\gamma_{t, \tau}, a^{\gamma_{t, \tau}}) \right)_{\tau=0} \quad \text{or} \quad \frac{\partial}{\partial \tau} \left( \int_{0}^{1} \Delta(\gamma_{t, \tau}^{-1} d\gamma_{t, \tau}, a^{\gamma_{t, \tau}}) \right)_{\tau=0}.
\] (B.4)
The derivative on the r.h.s. allows to work easier with differential forms on $M \times \mathcal{G}$. In this respect, interesting developments might be found in [24, p.192ff.]. By smoothness, one has
\[
\frac{\partial}{\partial \tau} \left( \int_{0}^{1} \Delta(\gamma_{t, \tau}^{-1} d\gamma_{t, \tau}, a^{\gamma_{t, \tau}}) \right)_{\tau=0} = \int_{0}^{1} \frac{\partial}{\partial \tau} \left( \Delta(\gamma_{t, \tau}^{-1} d\gamma_{t, \tau}, a^{\gamma_{t, \tau}}) \right)_{|\tau=0}.
\]
It is useful to work at the level of generators of the differential algebra, see e.g. [32]. One can check, by virtue of (B.1) and by $(\delta \gamma)(X_{\gamma_t}) = X_{\gamma_t}$, that
\[
\frac{\partial}{\partial \tau} \left( a^{\gamma_{t, \tau}} \right)_{\tau=0} = \frac{\partial}{\partial \tau} \left( \gamma_{t, \tau}^{-1} a \gamma_{t, \tau} + \gamma_{t, \tau}^{-1} d\gamma_{t, \tau} \right)_{|\tau=0} = (\delta(a^{\gamma_t}))(X_{\gamma_t}),
\]
\[
\frac{\partial}{\partial \tau} \left( \gamma_{t, \tau}^{-1} d\gamma_{t, \tau} \right)_{\tau=0} = d_t \left( \gamma_{t, \tau}^{-1} (\delta \gamma)(X_{\gamma_t}) \right) + [\gamma_{t, \tau}^{-1} d\gamma_{t, \tau}, \gamma_{t, \tau}^{-1} (\delta \gamma)(X_{\gamma_t})].
\] (B.5)
Dropping out the tangent vectors $X_{\gamma_t}$ yields that the l.h.s. derivative in (B.4) comes down to computing
\[
\delta \int_{0}^{1} A(\gamma_{t, \tau}^{-1} d\gamma_{t, \tau}, a^{\gamma_{t, \tau}}) = \int_{0}^{1} \delta A(\gamma_{t, \tau}^{-1} d\gamma_{t, \tau}, a^{\gamma_{t, \tau}})
\] (B.6)
and (B.5) leads to
\[
\delta (a^{\gamma_t}) = -d (\gamma_{t, \tau}^{-1} \delta \gamma_t) \quad \text{[a^{\gamma_t}, \gamma_{t, \tau}^{-1} \delta \gamma_t]}, \quad \delta (\gamma_{t, \tau}^{-1} d\gamma_{t, \tau}) = -d_t (\gamma_{t, \tau}^{-1} \delta \gamma_t) - [\gamma_{t, \tau}^{-1} \delta \gamma_t, \gamma_{t, \tau}^{-1} d\gamma_{t, \tau}]
\]
for any variation $\{\delta \gamma_t\}$ of the family. Moreover, one has also (morally, it corresponds to the pull-back on $[0,1] \times [0,1]$ – see [24])
\[
(d_t + \delta) (a^{\gamma_t}) = -d (\gamma_{t, \tau}^{-1} (d_t + \delta) \gamma_t) \quad \text{[a^{\gamma_t}, \gamma_{t, \tau}^{-1} (d_t + \delta) \gamma_t]}
\]
"Raymond Stora had in mind that it was possible to extend the deformation to any submanifold in $\mathcal{G}$."
together with the Maurer-Cartan equation
\[(d_t + \delta) (\gamma_t^{-1}(d_t + \delta) \gamma_t) + \frac{1}{2} [\gamma_t^{-1}(d_t + \delta) \gamma_t, \gamma_t^{-1}(d_t + \delta) \gamma_t] = 0.\]

The last two equations reproduce a differential algebra similar to the BRS algebra (A.3) and accordingly the cocycle condition (A.1) on $A$ yields
\[(d_t + \delta) A(\gamma_t^{-1}(d_t + \delta) \gamma_t, a^{\tau}) = 0.\]  

(B.7)

By polarization, one gets the important condition
\[\delta A(\gamma_t^{-1} d_t \gamma_t, a^{\tau}) + d_t A(\gamma_t^{-1} \delta \gamma_t, a^{\tau}) = 0.\]

which once inserted into (B.6) gives \[^9\]
\[\delta \int_0^1 A(\gamma_t^{-1} d_t \gamma_t, a^{\tau}) = -\int_0^1 d_t A(\gamma_t^{-1} \delta \gamma_t, a^{\tau}) = A(\gamma_0^{-1} \delta \gamma_0, a^{\tau}) - A(\gamma_1^{-1} \delta \gamma_1, a^{\tau}).\]  

(B.8)

This shows that the variation depends on the integration limits only. One has thus proved the

**Lemma** (Stora). *An infinitesimal deformation of the interpolation depends only of the variation at the ends of the path:*

\[\delta \int_0^1 A(\gamma_t^{-1} d_t \gamma_t, a^{\tau}) = A(\gamma_0^{-1} \delta \gamma_0, a^{\tau}) - A(\gamma_1^{-1} \delta \gamma_1, a^{\tau}).\]  

(B.9)

In particular, if $\delta \gamma_0 = \delta \gamma_1 = 0$, this result shows the independence on the choice of the interpolating family $\gamma_t$ between $e$ and $g$ for computing $\int_0^1 A(\gamma_t^{-1} d_t \gamma_t, a^{\tau})$. If the topology of $G$ is suitable, e.g. the fundamental group $\pi_1(G) = 0$, the consistency condition (B.2) can be recast as a coboundary condition on $G$, and by Stokes theorem
\[\int_S \delta A(\gamma^{-1} \delta \gamma, a^{\tau}) = 0 = \oint_{\text{loop}} A(\gamma^{-1} \delta \gamma, a^{\tau})\]
for $S$ a surface in $G$ enclosed by a loop passing through $e$ and $g$. This result is to some extent the infinitesimal version of the group cohomology introduced in [9].

The deformation of the family $\{\gamma_t\}$ considered in Appendix A with one end kept fixed, is achieved by choosing for (B.3) the left action, $\gamma_{t,\tau} = e^{-\tau \chi} \gamma_t$, (as a smooth homotopy) with $\chi \in \text{Lie} \ G \cong T_e \ G$. One readily checks that this choice for $\gamma_{t,\tau}$ gives, on the one hand, $\gamma_{0,\tau} = \gamma_0 = e$ for $t = 0$ and for any $\tau$, and, on the other hand, $\gamma_{1,\tau} = e^{-\tau \chi} \gamma_1$ for $t = 1$. With such a smooth deformation, the induced vector field along the family $\{\gamma_t\} \subset G$ is given by
\[X_{\gamma_t} = \frac{\partial}{\partial \tau} (\gamma_{t,\tau})|_{\tau=0} = -t T_e R_{\gamma_1} \chi\]
where $T_e R_{\gamma_1}$ is the tangent map of the right translation. The evaluation of the Maurer-Cartan form gives
\[(\gamma_t^{-1} \delta \gamma_t)(X_{\gamma_t}) = -t T_{e,\gamma_1} L_{-\gamma_1^{-1}} T_e R_{\gamma_1} \chi = -t T_e (L_{-\gamma_1^{-1}} \circ R_{\gamma_1}) \chi = -t \gamma_1^{-1} \chi \gamma_t\]
which reflects (up to a sign) the right-equivariance of the Maurer-Cartan form on $G$: $(\gamma^{-1} \delta \gamma)(T_e R_{\gamma} \chi) = \gamma^{-1} \chi \gamma$ for $\chi \in \text{Lie} \ G$. One has $\delta \gamma_0 = 0$ and $(\delta \gamma_1)(X_{\gamma_1}) = -\chi \gamma_1$. Using the algebraic definition of the Faddeev-Popov ghost [11], as the "Lie $G$-valued generator of $(\text{Lie} G)^*$", namely, $c(\chi) = \chi$, one can write $\delta \gamma_1 = -c \gamma_1$. Combining this construction with the above Lemma, and with a slight abuse of notation, one has the

**Corollary** (Stora). *For an infinitesimal deformation $\{\delta \gamma_t = -tc \gamma_t\}$ of the interpolating family $\{\gamma_t\}$ from $\gamma_0 = e$ to $\gamma_1$, one has the variation*

\[\delta \int_0^1 A(\gamma_t^{-1} d_t \gamma_t, a^{\tau}) = A(\gamma_1^{-1} c \gamma_1, a^{\tau}).\]  

(B.10)

\[^9\text{Adapting to our context the approach given in [24, \S 6.70, 6.71] corresponds to}\]
\[\delta \int_{[0,1]} A(\gamma_t^{-1} d_t \gamma_t, a^{\tau}) = \int_{[0,1]} (d_t + \delta) A(\gamma_t^{-1} (d_t + \delta) \gamma_t, a^{\tau}) - \int_{[0,1]} d_t A(\gamma_t^{-1} \delta \gamma_t, a^{\tau}) = -\int_{[0,1]} A(\gamma_t^{-1} \delta \gamma_t, a^{\tau})\]
and due to (B.7), only the integral on the boundary contributes in the r.h.s.
References

[1] G. ’t Hooft. Singularities, horizons, firewalls, and local conformal symmetry. In 2nd Karl Schwarzschild Meeting on Gravitational Physics (KSM 2015) Frankfurt am Main, Germany, July 20-24, 2015, 2015.

[2] A. R. Gover, A. Shaukat, and A. Waldron. Weyl Invariance and the Origins of Mass. Phys. Lett., B675:93–97, 2009.

[3] B. Zumino. Effective Lagrangians and Broken Symmetries. In Lectures on Elementary Particles and Quantum Field Theory, volume 2, pages 437–500. Brandeis Univ., 1970.

[4] S. Deser. Scale invariance and gravitational coupling. Annals Phys., 59:248–253, 1970.

[5] J. Wess and B. Zumino. Consequences of anomalous Ward identities. Phys. Lett. B37, page 95, 1971.

[6] J. Wess. Tensor and scalar dominance of the energy-momentum tensor. In R. Gatto, editor, Advanced school of physics, informal meeting on outlook for broken conformal symmetry in elementary particle physics: scale and conformal symmetry in hadron physics, Wiley-Interscience Publication, pages 31–41. John Wiley and Sons, 1973.

[7] R. Stora. Algebraic structure and topological origin of chiral anomalies. In G. ’t Hooft and et al., editors, Progress in Gauge Field Theory, Cargèse 1983, NATO ASI Ser.B, Vol.115. Plenum Press, 1984.

[8] B. Zumino. Chiral anomalies and differential geometry. In B. S. DeWitt and R. Stora, editors, Relativity, groups and topology II, Les Houches, Session XL, pages 1291–1322, B. V., 1984. Elsevier Science Publishers.

[9] B. Zumino. Cohomology of gauge groups: cocycles and Schwinger terms. Nucl. Phys., B253:477–493, 1985.

[10] J. Mañes, R. Stora, and B. Zumino. Algebraic study of chiral anomalies. Comm. Math. Phys., 102:157–174, 1985.

[11] R. Stora. Algebraic structure of chiral anomalies. In J. Abad, M. Asorey, and A. Cruz, editors, New perspectives in quantum field theories. Preprint LAPP-TH-143, World Scientific, 1986. Lectures given at 16th GIFT Seminar on Theoretical Physics, Jaca, Spain, Jun 3-8, 1985.

[12] R. Stora. Differential algebras in Lagrangean Field Theory. ETHZ Lectures. Unpublished notes., 1993.

[13] S. Lazzarini. Sur les modèles conformes lagrangiens bidimensionnels (On two dimensional Lagrangian conformal models). PhD thesis, April 1990. Université de Savoie and LAPP-TH. In French.

[14] M. Knecht, S. Lazzarini, and F. Thuillier. Shifting the Weyl anomaly to the chirally split diffeomorphism anomaly in two-dimensions. Phys. Lett., B251:279–283, 1990.

[15] S. Goldberg. Curvature and Homology. Dover Publications, 1962.

[16] R. Stora. The Slavnov symmetry, cousins and descendants. In BRS symmetry. Proceedings, International Symposium on the Occasion of its 20th Anniversary, Kyoto, Japan, September 18-22, 1995, pages 1–16, 1995.

[17] C. Becchi, A. Rouet, and R. Stora. Renormalization of Gauge Theories. Annals Phys., 98:287–321, 1976.

[18] R. Stora. Continuum gauge theories. In M. Lévy and P. Mitter, editors, New Developments in Quantum Field Theory and Statistical Mechanics, Cargèse 1976, NATO ASI Ser.B, Vol.26. Plenum Press, 1977.

[19] G. Bandelloni, C. Becchi, A. Blasi, and R. Collina. Local approach to dilatation invariance. Nucl. Phys., B197:347, 1982.

[20] L. Baulieu, C. Becchi, and R. Stora. On the Covariant Quantization of the Free Bosonic String. Phys. Lett., B180:55, 1986.

[21] C.M. Becchi. On the covariant quantization of the free string: the conformal structure. Nucl. Phys., B304:513, 1988.
[22] Ali H. Chamseddine and Alain Connes. The Spectral action principle. *Commun. Math. Phys.*, 186:731–750, 1997.

[23] C. Fournel, J. François, S. Lazzarini, and T. Masson. Gauge invariant composite fields out of connections, with examples. *Int. J. Geom. Meth. Mod. Phys.*, 11(1):1450016, 2014.

[24] P. Iglesias-Zemmour. *Diffeology*, volume 185 of *Mathematical Surveys and Monographs*. AMS, 2013.

[25] J. L. Manes. Differential Geometric Construction of the Gauged Wess-Zumino Action. *Nucl. Phys.*, B250:369–384, 1985.

[26] R. Stora, 2005. Private communication.

[27] M. Dubois-Violette, M. Talon, and C. M. Viallet. BRS Algebras: Analysis of the Consistency Equations in Gauge Theory. *Commun. Math. Phys.*, 102:105, 1985.

[28] J. A. De Azcarraga and J. M. Izquierdo. *Lie Groups, Lie Algebras, Cohomology and some Applications in Physics*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1995.

[29] J. François, S. Lazzarini, and T. Masson. Residual Weyl symmetry out of conformal geometry and its BRST structure. *JHEP*, 09:195, 2015. doi:10.1007/JHEP09(2015)195.

[30] P. A. M. Dirac. Gauge-invariant formulation of Quantum Electrodynamics. *Canad. J. Phys.*, 33:650–660, 1955.

[31] R. A. Bertlmann. *Anomalies In Quantum Field Theory*, volume 91 of *International Series of Monographs on Physics*. Oxford University Press, 1996.

[32] M. Dubois-Violette. Structure algébrique des anomalies et cohomologie de B.R.S. (in French). In Y. Choquet-Bruhat, B. Coll, R.Kerner, and A. Lichnérowicz, editors, *Géométrie et Physique*, pages 84–101. Hermann, 1987. Journées relativistes de Marseille-Luminy–April 1985.