Various Faces of Type IIA Supergravity

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Abstract

We derive a duality–symmetric action for type IIA $D = 10$ supergravity by the Kaluza–Klein dimensional reduction of the duality–symmetric action for $D = 11$ supergravity with the 3–form and 6–form gauge field. We then double the bosonic fields arising as a result of the Kaluza–Klein dimensional reduction and add mass terms to embrace the Romans’s version, so that in its final form the bosonic part of the action contains the dilaton, NS–NS and RR potentials of the standard type IIA supergravity as well as their duals, the corresponding duality relations are deduced directly from the action. We discuss the relation of our approach to the doubled field formalism by Cremmer, Julia, Lü and Pope, complete the extension of this construction to the supersymmetric case and lift it onto the level of the proper duality–symmetric action. We also find a new dual formulation of type IIA $D = 10$ supergravity in which the NS–NS two–form potential is replaced with its six–form counterpart. A truncation of this dual model produces the Chamseddine’s version of $N = 1$, $D = 10$ supergravity.
1 Introduction

A universal, duality–symmetric, formulation of maximal $D = 10$ and $D = 11$ supergravities has proved to be useful for the understanding of many aspects of superstring and M–theory including their symmetry structure and the dynamics of various branes constituting an intrinsic part of these theories. It is well known that superbranes are sources of antisymmetric tensor fields of supergravity multiplets and that higher dimensional superbranes couple to the conventional supergravity tensor fields as well as to their duals (superbrane worldvolume actions describing this coupling have been constructed in [1, 2, 3, 4, 5, 6]). To study interactions of such branes with supergravity backgrounds, and to derive effective brane actions from corresponding supergravities [4], it is therefore desirable to have a formulation of supergravities in which the standard and the dual fields enter the action in a duality–symmetric way. This should be also useful for studying anomalies in M–theory and in the superstring theories in the presence of branes [8–14], and for the analysis of subtleties in duality relations between ordinary and chiral forms [15]. For the discussion of other problems concerning a consistent description of supergravity–superbrane systems see e.g. [16–25].

In [26] a formalism of “doubled fields” related by a twisted self–duality has been developed for describing in a uniform duality invariant way gauge and internal symmetries of maximal supergravities, including $D = 11$, $D = 10$ type IIA and type IIB supergravities, and their dimensionally reduced versions. An interesting (super)algebraic structure underlying this construction has been found. It has allowed one, by introducing a twisted self–duality condition, to represent the equations of motion of dual fields as a Maurer–Cartan zero–curvature equation with the doubled field strengths playing the role of generalized connection forms. The construction of [26] is on the mass shell and involves only the bosonic sector of corresponding supergravities.

The duality–symmetric (doubled field) action for the complete $D = 11$ supergravity has been constructed in [16] and for type IIB $D=10$ supergravity in [27, 28]. The construction of these actions is based on the covariant techniques developed in [29]. The (twisted) self–duality relation arises in such formulations as an equation of motion of the corresponding physical (doubled) fields. However, a duality–symmetric version of type IIA $D = 10$ supergravity is still lacking. A pseudo–action for doubled Ramond–Ramond fields of type IIA supergravity considered in [19] does not produce all the equations of motion, namely the duality relations between the doubled fields. So any modification of the theory, such as a nontrivial self–interaction of fields as in the case of the M5–brane [15], quantum corrections, coupling to other sources etc. would require an appropriate modification of the duality relations which can be hard to guess if they are not yielded by a proper action. Also the democratic formulation of [19] did not involve the dualization of the NS–NS two–form potential which is required for coupling to the NS5–brane.

The aim of this paper is to fill this gap. We obtain the action for type IIA $D = 10$ supergravity by dimensionally reducing the duality–symmetric formulation [16] of $D = 11$ supergravity. We then double the bosonic fields arising as a result of the Kaluza–Klein compactification and add massive terms to embrace the Romans’s version [33], so that in its final form the proper action contains the mass, the dilaton, NS–NS and RR potentials ($m, \phi, B^{(2)}, A^{(1)}, A^{(3)}$) of the conventional type IIA supergravity [30, 31, 32] as well as their duals ($A^{(9)}, A^{(8)}, B^{(6)}, A^{(7)}, A^{(5)}$). This allows one to couple the type IIA supergravity to all Dp–branes, including a topologically massive D2–brane [37]. It is also the most appropriate for coupling to the NS5–brane [6, 13] which carries a $B^{(6)}$ charge and at the same time interacts with the NS–NS field $B^{(2)}$. The potential...
A\(^{(9)}\) dual to the mass parameter is required for coupling type IIA supergravity to domain walls, such as a D8–brane and an O(rientifold)8-brane \([35, 36, 12]\), which have been studied in the context of a higher dimensional and supersymmetric generalization of the Randall–Sundrum Brane World scenario and its promotion to String Theory.

We then show how the actions obtained can be rewritten in a simple “sigma–model” form which produces the supersymmetrized group–theoretical formulation of \([26]\).

As a by–product we also find a new dual formulation of type IIA \(D = 10\) supergravity in which the NS–NS two–form potential is not present. It is replaced with its six–form counterpart. This formulation is characterized by an essentially non–polynomial coupling of the RR one–form potential to the field strengths of the RR three–form potential and of the NS–NS six–form potential \(B^{(6)}\), and as a consequence, by a highly non–linear \(U(1)\) invariance. A truncation of this dual model produces the Chamseddine’s version \([38]\) of \(N = 1, D = 10\) supergravity. Note also that, a superfield formulation of dual \(N = 1, D = 10\) supergravity with both the dilaton and the NS–NS two–form field replaced with their eight–form and six–form counterparts was considered in \([39]\). This formulation can also be obtained by an appropriate truncation of the completely duality–symmetric action for type IIA supergravity considered in this paper.

In what follows, for simplicity, we will focus on the subsector of ten–dimensional type IIA supergravity which does not involve the quartic fermion terms. The reason is that as for any supergravity theory the basic structure of local supersymmetry transformations and their appropriate modifications in the duality–symmetric PST approach \([29]\) can already be deduced at the quadratic level. In the case of \(D=11\) (duality–symmetric) supergravity recovering the quartic fermion terms is reached by the supercovariantization of the action and of local supersymmetry transformations which leaves however intact the general structure obtained without these terms. Moreover, the supercovariantization does not change the PST part of the action and of the local supersymmetry transformations since this part is already constructed out of the supercovariant quantities and, thus, implicitly includes quartic fermionic terms.

“Almost the same” happens with the duality–symmetric version of type IIA supergravity, whose standard formulation was obtained in \([30, 31, 32]\) by the dimensional reduction of the Cremmer–Julia–Scherk \(D = 11\) supergravity \([40]\). Saying “almost the same” we mean that recovering the quartic fermion terms in the standard type IIA supergravity does not only mean the supercovariantization of the action and of the local supersymmetry transformations derived in the absence of these terms, but also requires adding other fermionic terms (see \([32]\) for the discussion of this point). However, as in the case of duality–symmetric \(D = 11\) supergravity, any modifications of local supersymmetry transformations due to the PST approach can already be deduced in the quadratic fermion approximation. Thus the reconstruction of the quartic fermion terms in the duality–symmetric type IIA supergravity can be carried out in the same way as in the usual type IIA supergravity \([30, 31, 32]\).

Following the standard Kaluza–Klein way we shall mainly focus on features of the dimensional reduction and gauge fixing of an auxiliary scalar field (the PST scalar) appearing in the bosonic subsector of the duality–symmetric version of \(D=11\) supergravity \([16]\). This auxiliary field, entering the action in a non–polynomial way, is assumed to be a singlet under the local supersymmetry transformations (see \([29, 28, 41, 16]\) for details). This requires the modification of the local supersymmetry rules for the \(D=11\) gravitino field. After dimensional reduction this results in the modification of local supersymmetry transformations of type IIA gravitini and dilatini. However, as in the case of their eleven–dimensional counterpart, on the shell of duality relations the local supersymmetry transformations of the duality–symmetric type IIA
supergravity coincide with that of the standard version.

Since our starting point is the action for duality–symmetric $D = 11$ supergravity, in Section 2 we briefly discuss the structure of this theory and its relation to the standard $D = 11$ supergravity. Section 3 is devoted to the construction of duality-symmetric type IIA supergravity. In this section we present different but classically equivalent forms of the action, give the analysis of symmetry and dynamical properties of the model, and establish the connection with the standard formulation of type IIA supergravity. In Section 4 we complete our construction by doubling all bosonic fields of the model in a way similar to the formalism of doubled fields [26, 28, 34]. There the completely duality symmetric action for type IIA supergravity is presented (Sec. 4.1) and the doubled field sigma–model representation for a duality symmetric supergravity action is considered (Sec. 4.2). In Section 5 we gauge fix the PST scalar of the $D = 11$ theory in a way to get an action a la Sen and Schwarz [42]. As we shall show, the dimensional reduction of this action, with the PST scalar gauge fixed along the compactified direction, results in a new dual formulation of type IIA supergravity which is the $N = 2$ generalization of [38] and possesses exotic structure of gauge symmetries and of local supersymmetry realized in a non–linear way. Alternatively, this formulation can be obtained from the conventional type IIA supergravity by replacing the NS–NS two–form $B^{(2)}$ with its dual $B^{(6)}$. In Conclusion we discuss the results obtained and in Appendices, for reader’s convenience, we have collected the notation and conventions used throughout the paper, as well as details concerning dimensional reduction and useful identities.

## 2 Duality–symmetric D=11 supergravity

The duality–symmetric action for D=11 supergravity proposed in [16] is

\[
S = \int_{M^{11}} \left[ \hat{R}^{\hat{a}_1\hat{a}_2} \wedge \hat{\Sigma}_{\hat{a}_1\hat{a}_2} + \frac{i}{3!} \hat{\Psi} \wedge \mathcal{D} \left[ \frac{1}{2} (\hat{\omega} + \hat{\omega}) \right] \hat{\Psi} \hat{\Gamma}^{\hat{a}_1\hat{a}_2\hat{a}_3} \wedge \hat{\Sigma}_{\hat{a}_1\hat{a}_2\hat{a}_3} \right] - \int_{M^{11}} \left[ \frac{1}{2} (\hat{C}^{(7)} + \hat{\ast} C^{(4)}) \wedge (\hat{F}^{(4)} + (\hat{F}^{(4)} - \hat{C}^{(4)})) - \frac{1}{2} \hat{F}^{(4)} \wedge \hat{\ast} \hat{F}^{(4)} + \frac{1}{3} \hat{A}^{(3)} \wedge \hat{F}^{(4)} \wedge \hat{F}^{(4)} \right] + \int_{M^{11}} \frac{1}{2} i_v \hat{F}^{(4)} \wedge \hat{\ast} i_v \hat{F}^{(4)},
\]

or in a more symmetric form

\[
S = \int_{M^{11}} \left[ \hat{R}^{\hat{a}_1\hat{a}_2} \wedge \hat{\Sigma}_{\hat{a}_1\hat{a}_2} + \frac{i}{3!} \hat{\Psi} \wedge \mathcal{D} \left[ \frac{1}{2} (\hat{\omega} + \hat{\omega}) \right] \hat{\Psi} \hat{\Gamma}^{\hat{a}_1\hat{a}_2\hat{a}_3} \wedge \hat{\Sigma}_{\hat{a}_1\hat{a}_2\hat{a}_3} \right. \\
\left. - \frac{1}{2} (\hat{C}^{(7)} + \hat{\ast} C^{(4)}) \wedge (\hat{F}^{(4)} - \frac{1}{2} \hat{C}^{(4)}) - \frac{1}{2} (\hat{C}^{(7)} + \hat{\ast} \hat{C}^{(7)}) \wedge (\hat{F}^{(7)} + \frac{1}{2} \hat{C}^{(7)}) \right] + \int_{M^{11}} \frac{1}{4} \hat{F}^{(4)} \wedge \hat{\ast} \hat{F}^{(4)} - \frac{1}{4} \hat{F}^{(7)} \wedge \hat{\ast} \hat{F}^{(7)} \\
+ \frac{1}{4} i_v \hat{F}^{(4)} \wedge \hat{\ast} i_v \hat{F}^{(4)} - \frac{1}{4} i_v \hat{F}^{(7)} \wedge \hat{\ast} i_v \hat{F}^{(7)} + \frac{1}{6} \hat{F}^{(7)} \wedge \hat{F}^{(4)},
\]

where $\hat{\ast}$ is the Hodge operator in $D = 11$ (the hat is put to distinguish it from the $D = 10$ Hodge $\ast$).
Modulo the last term the action (1) is the conventional $D = 11$ supergravity action written in the same notation as in the original paper [40] except for the coefficient (which is one in our conventions and one quarter in “supergravity conventions”) in front of the Einstein-Hilbert term, the first term in (1) and (2), and the coefficient in the definition of the spin connection (one quarter vs. one half of [40], see below). To write the Einstein–Hilbert term and the gravitino kinetic term of the action in the differential form notation it is convenient to introduce a form dual to the wedge product of the vielbeine

$$\hat{\Sigma}_{\hat{a}_1...\hat{a}_n} = \frac{1}{(11 - n)!} \hat{e}_{\hat{a}_1...\hat{a}_{n+1}} \hat{E}^{\hat{a}_{n+1}} \wedge ... \wedge \hat{E}^{\hat{a}_{11}}. \quad (3)$$

Other building blocks of the action are the covariant derivative of the gravitino field

$$\hat{\Psi}^{\hat{a}} = d\hat{X}^{\hat{m}} \hat{\Psi}^{\hat{a}_m}, \quad (4)$$

$$D\hat{\Psi}^{\hat{a}} = d\hat{\Psi}^{\hat{a}} - \hat{\omega}^{\hat{a}}_{\hat{b}} \wedge \hat{\Psi}^{\hat{b}}, \quad (5)$$

the bilinear fermionic terms

$$\hat{C}^{(4)} = -\frac{i}{4} \hat{\bar{\Psi}} \wedge \hat{\Gamma}^{(2)} \wedge \hat{\Psi}, \quad \hat{C}^{(7)} = \frac{i}{4} \hat{\bar{\Psi}} \wedge \hat{\Gamma}^{(5)} \wedge \hat{\Psi}, \quad (6)$$

the supercovariant connection $\hat{\omega}$ determined by

$$d\hat{E}^{\hat{a}} - \hat{E}^{\hat{b}} \wedge \hat{\omega}^{\hat{a}}_{\hat{b}} = \frac{i}{4} \hat{\bar{\Psi}} \hat{\Gamma}^{\hat{a}} \wedge \hat{\Psi},$$

and the field strength

$$\hat{F}^{(4)} = d\hat{A}^{(3)} \quad (8)$$

of the three–form gauge field $\hat{A}^{(3)}$.

The last term of (1) and the corresponding terms in (2) encode the information on duality relations between $\hat{A}^{(3)}$ and a six–form gauge field $\hat{A}^{(6)}$, which can be derived directly from the action (1), and contains the following (anti–)dual combinations of the field strengths

$$\hat{\check{F}}^{(4)} = \hat{F}^{(4)} - \hat{C}^{(4)} - \hat{\ast}(\hat{F}^{(7)} + \hat{C}^{(7)}), \quad (9)$$

$$\hat{\check{F}}^{(7)} = \hat{F}^{(7)} + \hat{C}^{(7)} - \hat{\ast}(\hat{F}^{(4)} - \hat{C}^{(4)}) = -\hat{\ast}\hat{F}^{(4)}, \quad (10)$$

where

$$\hat{F}^{(7)} = d\hat{A}^{(6)} + \hat{A}^{(3)} \wedge \hat{F}^{(4)}. \quad (11)$$

This part of the actions is constructed with the use of the space–like unit vector $\hat{v}_m$ composed of derivatives of the PST scalar $a(x)$ [29]

$$\hat{v}_m = \frac{\partial_m a}{\sqrt{-\partial_a a \hat{g}^{\hat{a}\hat{b}} \partial_m a}} \quad (12)$$

\[1\] The presence in self–dual and duality–symmetric actions of the auxiliary vector field which can be leveled at any direction in space with the use of a local symmetry (see eq. [28] below) is similar to (and is actually a manifestation of) the presence of the unobservable Dirac string in field–theoretical descriptions of monopoles and dyons.
and $i_\hat{v}_a \hat{F}^{(n)}$ is the inner product of $\hat{v}$ with $\hat{F}^{(n)}$ ($n = 4, 7$)

$$i_\hat{v}_a \hat{F}^{(n)} = \frac{1}{(n-1)!} dX^{\hat{m}_1} \cdots dX^{\hat{m}_{n-1}} \hat{v}_a \hat{g}^{\hat{m}_n} \hat{F}^{(n)}_{\hat{m}_1 \cdots \hat{m}_{n-1}} \equiv \frac{1}{(n-1)!} \hat{E}^{\hat{a}_1} \cdots \hat{E}^{\hat{a}_n} \hat{v}_\hat{a}_1 \hat{F}^{(n)}_{\hat{a}_1 \cdots \hat{a}_n}. \quad (13)$$

It is also convenient to introduce the one-form

$$\hat{v} = dX^{\hat{m}} \hat{v}_\hat{m} = \frac{1}{\sqrt{-\partial_i a \hat{g}^{\hat{m} \hat{n}} \partial_\hat{m} a}} da. \quad (14)$$

The action (1) possesses (by construction) $D = 11$ general coordinate and local Lorentz invariance and is also invariant under the following local supersymmetry transformations

$$\delta_\epsilon a = 0, \quad \delta_\epsilon \hat{A}^{(3)} = \frac{i}{2} \hat{\epsilon} \hat{F}^{(2)} \wedge \hat{\Psi}, \quad \delta_\epsilon \hat{A}^{(6)} = -\frac{i}{2} \hat{\epsilon} \hat{F}^{(5)} \wedge \hat{\Psi} + \delta \hat{A}^{(3)} \wedge \hat{A}^{(3)}, \quad \delta_\epsilon \hat{E}^a = -\frac{i}{2} \hat{\epsilon} \hat{F}^{a} \wedge \hat{\Psi}, \quad (15)$$

$$\delta_\epsilon \hat{\Psi} = \mathcal{D}(\hat{\psi}) \epsilon - \frac{i}{3!} \hat{\epsilon} \hat{F}^{(6)} \wedge \mathcal{D}(\hat{\psi}) \hat{\Psi} + 8 \hat{\epsilon} \hat{F}^{(4)} \hat{F}^{(2)} \hat{\Psi} + 4 \hat{\epsilon} \hat{F}^{(4)} \hat{F}^{(2)} \hat{\Psi}$$

$$= \left[ \mathcal{D}(\hat{\psi}) - \frac{1}{3!} \epsilon \epsilon \epsilon (\hat{F}^{(4)} \wedge \hat{F}^{(4)}) - \hat{F}^{(4)} \wedge \hat{F}^{(4)} \right] \epsilon. \quad (16)$$

The variations (15, 16) differ from that of (40) by terms with $i_\hat{v}_a \hat{F}^{(4)}$ and also include the transformation rule for $\hat{A}^{(6)}$ which has been obtained in (16) from the requirement of the super-covariance of $\hat{F}^{(7)} + \hat{C}^{(7)}$.

To derive other symmetries of the action and equations of motion for the gauge fields it is convenient to rewrite the part of the action containing the gauge fields in an equivalent but manifestly duality-symmetric form with respect to $\hat{F}^{(7)}$ and $\hat{F}^{(4)}$

$$S_\hat{A} = \int_{\mathcal{M}^{11}} \left[ \frac{1}{2} \hat{v} \wedge (\hat{F}^{(4)} - \hat{C}^{(4)}) \wedge i_\hat{v} \hat{F}^{(7)} + \frac{1}{2} \hat{v} \wedge \hat{F}^{(7)} + \frac{1}{2} \hat{v} \wedge \hat{F}^{(7)} + \frac{1}{2} \hat{C}^{(7)} \wedge \hat{F}^{(4)} \right]. \quad (17)$$

A general variation of (17) is (see Appendix D)

$$\delta S_\hat{A} = \int_{\mathcal{M}^{11}} [\delta \hat{v} \wedge \hat{v} \wedge i_\hat{v} \hat{F}^{(4)} \wedge i_\hat{v} \hat{F}^{(7)}] + (\hat{v} \wedge i_\hat{v} \hat{F}^{(4)}) \wedge \delta (\hat{F}^{(4)} - \hat{C}^{(4)}) + \frac{1}{2} (\hat{F}^{(7)} + \hat{C}^{(7)}) \wedge \delta (\hat{F}^{(4)} - \hat{C}^{(4)})$$

$$+ (\hat{v} \wedge i_\hat{v} \hat{F}^{(4)}) \wedge \delta (\hat{F}^{(7)} + \hat{C}^{(7)}) - \frac{1}{2} (\hat{F}^{(7)} + \hat{C}^{(7)}) \wedge \delta (\hat{F}^{(4)} + \hat{C}^{(4)}) + \frac{1}{2} \delta \hat{A}^{(3)} \wedge \hat{F}^{(4)}$$

$$- \frac{1}{2} \delta (\hat{C}^{(7)} \wedge \hat{F}^{(4)})$, \quad (18)$$

where we have omitted the total derivative term.

If we are interested only in the variation of the gauge fields, i.e.

$$\delta (\hat{F}^{(4)} - \hat{C}^{(4)}) = d(\delta \hat{A}^{(3)}), \quad \delta (\hat{F}^{(7)} + \hat{C}^{(7)}) = d(\delta \hat{A}^{(6)} - \delta \hat{A}^{(3)} \wedge \hat{A}^{(3)}) + 2 \delta \hat{A}^{(3)} \wedge \hat{F}^{(4)}, \quad (19)$$
and of the PST scalar (12), the general variation (18) reduces to

\[
\delta S_A = - \int_{M^{11}} \left[ (\delta \hat{A}^{(6)} - \delta \hat{A}^{(3)} \wedge \hat{A}^{(3)}) + \frac{\delta a}{\sqrt{-\langle \delta a \rangle^2}} i_{\hat{v}} \hat{F}^{(7)} \wedge d(\hat{v} \wedge i_{\hat{v}} \hat{F}^{(4)}) \\
- \left( \delta \hat{A}^{(3)} + \frac{\delta a}{\sqrt{-\langle \delta a \rangle^2}} i_{\hat{v}} \hat{F}^{(4)} \right) \wedge (d(\hat{v} \wedge i_{\hat{v}} \hat{F}^{(7)}) + 2 \hat{v} \wedge i_{\hat{v}} \hat{F}^{(4)} \wedge \hat{F}^{(4)}) \right].
\] (20)

From (20) it is easy to see that in addition to the conventional gauge transformations

\[
\delta \hat{A}^{(3)} = d\hat{\phi}^{(2)}; \quad \delta \hat{A}^{(6)} = d\hat{\phi}^{(5)} - \hat{\phi}^{(2)} \wedge \hat{F}^{(4)}
\] (21)

the action (11) is invariant under a set of ‘duality related’ transformations [29] (PST symmetries)

\[
\delta \hat{\phi} a = 0, \quad \delta \hat{\phi} \hat{A}^{(3)} = da \wedge \hat{\phi}^{(2)}, \quad \delta \hat{\phi} \hat{A}^{(6)} = da \wedge \hat{\phi}^{(5)} + da \wedge \hat{\phi}^{(2)} \wedge \hat{A}^{(3)},
\] (22)

and

\[
\delta \hat{\phi} a = \Phi(\hat{\phi}), \quad \delta \hat{\phi} \hat{A}^{(3)} = \frac{-\Phi}{\sqrt{-\langle \delta a \rangle^2}} i_{\hat{v}} \hat{F}^{(4)}, \quad \delta \hat{\phi} \hat{A}^{(6)} = \frac{-\Phi}{\sqrt{-\langle \delta a \rangle^2}} i_{\hat{v}} \hat{F}^{(7)} + \delta \hat{\phi} \hat{A}^{(3)} \wedge \hat{A}^{(3)}.
\] (23)

Equations of motion of \( \hat{A}^{(6)} \) and \( \hat{A}^{(3)} \) are

\[
d(\hat{v} \wedge i_{\hat{v}} \hat{F}^{(4)}) = 0; \quad d(\hat{v} \wedge i_{\hat{v}} \hat{F}^{(7)}) + 2 \hat{v} \wedge i_{\hat{v}} \hat{F}^{(4)} \wedge \hat{F}^{(4)} = 0.
\] (24)

The general solution to the equation of motion of \( \hat{A}^{(6)} \) is [29]

\[
\hat{v} \wedge i_{\hat{v}} \hat{F}^{(4)} = da \wedge d\hat{\xi}^{(2)}.
\] (25)

Using the symmetry [22] with \( \hat{\phi}^{(2)} = \hat{\xi}^{(2)} \) one can obtain from [25]

\[
i_{\hat{v}} \hat{F}^{(4)} = 0.
\] (26)

Note that when (26) is satisfied the action (11) and the local supersymmetry transformations (15), (16) coincide with that of [40].

Then, in the same way the equation of motion of \( \hat{A}^{(3)} \) is reduced to

\[
i_{\hat{v}} \hat{F}^{(7)} = 0.
\] (27)

Eqs. (26) and (27) together imply the duality relations between the \( \hat{A}^{(3)} \) and \( \hat{A}^{(6)} \) field strengths

\[
\hat{F}^{(4)} = (\hat{F}^{(4)} - \hat{C}^{(4)}) - \hat{\xi}(\hat{F}^{(7)} + \hat{C}^{(7)}) = 0,
\]
\[
\hat{F}^{(7)} = (\hat{F}^{(7)} + \hat{C}^{(7)}) - \hat{\xi}(\hat{F}^{(4)} - \hat{C}^{(4)}) = 0.
\] (28)

Equation of motion of the PST scalar field \( a(x) \) is satisfied identically as a consequence of the equations of motion (24). This is the Noether identity reflecting the second local PST symmetry (23) which implies the auxiliary (pure gauge) nature of the PST scalar \( a(x) \). The symmetry (23) allows us to fix, for instance, the following gauge

\[
\partial_{\hat{m}} a(x) = \delta_{\hat{m}}^a
\] which breaks the manifest
$D = 11$ Lorentz and general covariance of the theory down to $D = 10$ covariance and establishes the connection with non-covariant approach by Sen and Schwarz [42].

We should also point out that the duality–symmetric version of $D = 11$ supergravity has the following structure

$$S = S_{EH} + S_{\tilde{\Phi}} + S_{\hat{A}},$$

where $S_{EH}$ is the Einstein–Hilbert term, $S_{\tilde{\Phi}}$ is the fermion kinetic term and $S_{\hat{A}}$ is a specific term of the form of eq. (17) which contains the information on the duality relations. In fact, looking ahead, we can claim that modulo quartic fermion terms which can not be included by the supercovariantization of the gauge field strengths any duality-symmetric supergravity can be presented in the form of the action containing the Einstein–Hilbert term, kinetic terms of the fermionic fields and a specific construction a la [43].

To conclude this section let us recall that in the conventional Cremmer–Julia–Scherk formulation of $D = 11$ supergravity [40]

$$S_{CJS} = \int \left[ \hat{R}^\alpha \omega_2 \wedge \hat{\Sigma}_{\alpha} \hat{a}_2 + \frac{i}{3!} \hat{\Psi} \wedge \hat{D}[\frac{1}{2} (\hat{\omega} + \hat{\tilde{\omega}})] \hat{\Psi} \Gamma^\hat{a}_1 \hat{a}_2 \hat{a}_3 \wedge \hat{\Sigma}_{\hat{a}_1} \hat{a}_2 \hat{a}_3 + \frac{1}{2} \hat{F}^{(4)} \wedge \hat{\Psi} \hat{F}^{(4)} \right]$$

the duality conditions arise as a solution of the second order equation of motion of the field $\hat{A}^{(3)}$ (see, e.g., [16, 26])

$$d \left( \hat{\Psi} \left( \hat{F}^{(4)} - \hat{\tilde{C}}^{(4)} \right) - \hat{A}^{(3)} \wedge \hat{\tilde{F}}^{(4)} - \hat{C}^{(7)} \right) = 0.$$  (31)

Eq. (31) implies that the differential form under the external differential is closed, hence in space–time with trivial topology its solution is an exact form $d\hat{A}^{(6)}$,

$$\hat{\Psi} \left( \hat{F}^{(4)} - \hat{\tilde{C}}^{(4)} \right) - \hat{A}^{(3)} \wedge \hat{\tilde{F}}^{(4)} - \hat{C}^{(7)} = d\hat{A}^{(6)},$$

which is an equivalent representation of eq. (28) ($\hat{F}^{(7)} = d\hat{A}^{(6)} + \hat{A}^{(3)} \wedge \hat{\tilde{F}}^{(4)}$, see eq. (11)).

3 Duality–symmetric type IIA $D=10$ supergravity

To get a duality–symmetric version of type IIA supergravity we dimensionally reduce the action a la Kaluza–Klein.

As a first step we separate one spacelike coordinate of the 11-dimensional spacetime,

$$X^{\hat{m}} = (x^m, X^{11}), \quad m = 0, 1, \ldots, 9,$$

and choose the following ansatz for the vielbein and gravitino one–forms (see [30] [31] [32])

$$\hat{E}^a = e^{\frac{1}{2} \phi} E^a = e^{\frac{1}{2} \phi(x)} dx^m E_m^a(x),$$

$$\hat{E}^{11} = e^{-\frac{1}{2} \phi} (dx^{11} + A^{(1)}), \quad A^{(1)} = dx^m A_m(x),$$

$$\hat{\Psi} = e^{\frac{1}{2} \phi} \left( \psi + \frac{1}{12} \Gamma^{(1)} \Gamma^{11} \lambda \right) - \frac{2}{3} e^{-\frac{1}{2} \phi} \lambda (dx^{11} + A^{(1)}),$$

(33)
where \( \phi(x) \) is the dilaton field, \( A_m(x) \) is the ten-dimensional \( U(1) \) gauge field (which from the point of view of string theory is identified with Ramond–Ramond one–form potential),

\[
\psi^a = dx^m \psi_m^a(x),
\]

\( \psi_m^a(x) \) is the ten–dimensional gravitino and \( \lambda(x) \) is a Majorana fermion field which appear as a result of the Kaluza–Klein splitting (34) of the \( D = 11 \) metric and eleven–dimensional gravitino (4).

The \( D = 10 \) decomposition of the gauge fields \( \hat{A}^{(3)} \) and \( \hat{A}^{(6)} \) is

\[
\hat{A}^{(3)} = A^{(3)} - dX^{11} \wedge B^{(2)}, \quad \hat{A}^{(6)} = B^{(6)} + dX^{11} \wedge A^{(5)},
\]

where \( B^{(2)} \) is the type IIA NS–NS gauge field and \( B^{(6)} \) is its dual.

By use of the standard dimensional reduction procedure reviewed in Appendices B and C we get the following conventional part [30 31 32] of the type IIA supergravity action (modulo quartic fermion terms which we shall denote by \( \mathcal{O}(f^4) \))

\[
S_{\text{conven}} = \int_{M^{10}} \left[ -R^{a_1 a_2} \wedge \Sigma_{a_1 a_2} - \frac{i}{3} \bar{\psi} \wedge D\psi \wedge \Gamma^{a_1 a_2 a_3} \Sigma_{a_1 a_2 a_3} - \frac{i}{2} \bar{\lambda} \Gamma^a D\lambda \wedge \Sigma_a \right] + \int_{M^{10}} \left[ \frac{1}{2} d\phi \wedge \ast d\phi - (C^{(9)} - *C^{(1)}) \wedge d\phi \right] - \int_{M^{10}} \left[ \frac{1}{2} e^{-\frac{1}{2} \phi} F^{(2)} \wedge \ast F^{(2)} + (C^{(8)} - e^{-\frac{1}{2} \phi} *C^{(2)}) \wedge F^{(2)} \right] + \int_{M^{10}} \left[ \frac{1}{2} e^{\frac{1}{2} \phi} H^{(3)} \wedge \ast H^{(3)} - (C^{(7)} - e^{\phi} *C^{(3)}) \wedge H^{(3)} \right] - \int_{M^{10}} \left[ \frac{1}{2} e^{-\frac{3}{2} \phi} F^{(4)} \wedge \ast F^{(4)} + (C^{(6)} - e^{-\frac{3}{2} \phi} *C^{(4)}) \wedge F^{(4)} \right] + \int_{M^{10}} B^{(2)} \wedge dA^{(3)} \wedge dA^{(3)} + \mathcal{O}(f^4),
\]

where the field strengths entering the action are defined as follows\(^2\)

\[
F^{(2)} = dA^{(1)}, \quad H^{(3)} = dB^{(2)}, \quad F^{(4)} = dA^{(3)} - H^{(3)} \wedge A^{(1)}
\]

and

\[
C^{(9)} = -\frac{i}{4} \bar{\psi} \Gamma^{a_1 a_2} \lambda \wedge \Sigma_{a_1 a_2}, \quad C^{(1)} = -\frac{i}{2} \bar{\psi} \Gamma^{11} \lambda,
\]

\[
C^{(8)} = \frac{i}{4} e^{-\frac{3}{2} \phi} \bar{\psi} \Gamma^{a_1 a_2 a_3 a_4} \lambda \wedge \Sigma_{a_1 a_2 a_3} \wedge \Sigma_{a_1 a_2 a_3} - \frac{i}{16} e^{-\frac{3}{2} \phi} \bar{\psi} \Gamma^{a_1 a_2 a_3} \lambda \wedge \Sigma_{a_1 a_2 a_3} - \frac{i}{3} e^{-\frac{1}{2} \phi} \bar{\psi} \Gamma^{11} \lambda \wedge \Sigma_{a_1 a_2 a_3} + \frac{15i}{8} e^{-\frac{3}{2} \phi} \bar{\psi} \Gamma^{a_1 a_2} \lambda \wedge \Sigma_{a_1 a_2 a_3} + \frac{i}{24} e^{\frac{1}{2} \phi} \bar{\psi} \Gamma^{11} \lambda \wedge \Sigma_{a_1 a_2 a_3}.
\]

\(^2\)For reader’s convenience we have collected the definition of all gauge field strengths of type IIA supergravity and of their dual in Appendix A.
\[ C^{(3)} = \frac{1}{4} e^{-\frac{1}{2} \phi} \tilde{\psi} \wedge (\Gamma^{(1)}) \Gamma^{11} \wedge \psi + \frac{1}{4} e^{-\frac{1}{2} \phi} \bar{\psi} \wedge \Gamma^{(2)} \lambda, \]
\[ C^{(7)} = \frac{i}{4} e^{\frac{1}{2} \phi} \tilde{\psi} \wedge (\Gamma^{(5)}) \Gamma^{11} \wedge \psi - \frac{i}{4} e^{\frac{1}{2} \phi} \bar{\psi} \wedge \Gamma^{(6)} \Gamma^{11} \lambda, \]
\[ C^{(4)} = -\frac{1}{4} e^{\frac{1}{2} \phi} \tilde{\psi} \wedge (\Gamma^{(2)}) \Gamma^{11} \wedge \psi - \frac{1}{8} e^{\frac{1}{2} \phi} \bar{\psi} \wedge (\Gamma^{(3)}) \Gamma^{11} \lambda + \frac{3}{64} e^{\frac{1}{2} \phi} \bar{\lambda} \Gamma^{(4)} \lambda, \]
\[ C^{(6)} = \frac{i}{4} e^{-\frac{1}{2} \phi} \tilde{\psi} \wedge (\Gamma^{(4)}) \Gamma^{11} \wedge \psi - \frac{i}{8} e^{-\frac{1}{2} \phi} \bar{\psi} \wedge (\Gamma^{(5)}) \Gamma^{11} \lambda + \frac{3i}{64} e^{-\frac{1}{2} \phi} \bar{\lambda} \Gamma^{(6)} \Gamma^{11} \lambda. \]  

For further use let us note that the four fermionic terms \( C^{(n)} \wedge C^{(10-n)}, C^{(1)} \wedge * C^{(1)}, e^{-3/2} \phi C^{(2)} \wedge * C^{(2)}, e^{\phi} C^{(3)} \wedge * C^{(3)} \) and \( e^{-1/2} \phi C^{(4)} \wedge * C^{(4)} \) do not contain dilaton coupling.

In (38) and in what follows we define the field strengths of the NS–NS field \( B^{(2)} \) and of its dual \( B^{(6)} \) as \( H^{(3)} \) and \( H^{(7)} \), to distinguish them from the RR field strengths.

The complete action is
\[ S = S_{\text{conven}} + S_{d.s.}, \]  
where \( S_{d.s.}^{(10)} \) is obtained by the dimensional reduction of the last term in Eq. \( (11) \).

\[ S_{d.s.}^{(11)} = \frac{1}{2} \int_{M^{11}} i_{\hat{v}} \hat{F}^{(4)} \wedge \star i_{\hat{v}} \hat{F}^{(4)} = -\frac{1}{2} \int_{M^{11}} \hat{v} \wedge \hat{F}^{(7)} \wedge i_{\hat{v}} \hat{F}^{(4)}. \]  

In this section, to reduce the field \( \hat{v} \) to ten dimensions we shall assume that it does not depend on the compactified coordinate, i.e. \( \frac{\partial v}{\partial x^m} = 0 \). This implies that
\[ \hat{v} = e^{\frac{1}{2} \phi} v = e^{\frac{1}{2} \phi(x)} dx^m v_m(x), \quad i_{\hat{v}} (dX^{11} + A^{(1)}) = 0, \quad v_m = \frac{\partial_m a}{\sqrt{-\partial_n a \, g^{np} \partial_p a}}, \]  
where we have used the explicit form of the inverse metric
\[ \hat{g}^{mn} = e^{-\frac{1}{2} \phi} \left( g^{mn} - A^m \right) \right), \]  
which follows from (33).

Thus, the reduction of this part of the action results in
\[ S_{d.s.}^{(10)} = \int_{M^{10}} \left[ \frac{1}{2} \hat{v} \wedge \hat{H}^{(7)} \wedge i_{\hat{v}} \hat{H}^{(3)} + \frac{1}{2} \hat{v} \wedge \hat{F}^{(6)} \wedge i_{\hat{v}} \hat{F}^{(4)} \right], \]  
where
\[ \hat{H}^{(3)} = H^{(3)} - C^{(3)} + e^{-\phi} \ast (H^{(7)} + C^{(7)}), \]  
\[ \hat{H}^{(7)} = H^{(7)} + C^{(7)} + e^{\phi} \ast (H^{(3)} - C^{(3)}) = e^{\phi} \ast \hat{H}^{(3)}, \]  
\[ \hat{F}^{(4)} = F^{(4)} - C^{(4)} + e^{\frac{1}{2} \phi} \ast (F^{(6)} + C^{(6)}), \]  
\[ \hat{F}^{(6)} = F^{(6)} + C^{(6)} + e^{-\frac{1}{2} \phi} \ast (F^{(4)} - C^{(4)}) = e^{-\frac{1}{2} \phi} \ast \hat{F}^{(4)} \]  
are intrinsically dual combinations of the field strengths, and
\[ F^{(6)} = dA^{(5)} + A^{(3)} \wedge H^{(3)} - B^{(2)} \wedge dA^{(3)}, \]  
\[ H^{(7)} = dB^{(6)} + A^{(3)} \wedge A^{(1)} - F^{(6)} \wedge A^{(1)} \]  
(47)
arise upon the dimensional reduction of $\hat{F}^7$,

$$\hat{F}^7 = H^7 + F^6 \wedge (dX_{11} + A^{(1)}) .$$  \tag{48}

To summarize, we end up with the following action for duality-symmetric type IIA D=10 supergravity

$$S = \int_{M^{10}} \left[ -R^{a_1 a_2} \wedge \Sigma_{a_1 a_2} - \frac{i}{3!} \bar{\psi} \wedge D\psi \wedge \Gamma^{a_1 a_2 a_3} \Sigma_{a_1 a_2 a_3} - \frac{i}{2} \bar{\lambda} \Gamma^a D\lambda \wedge \Sigma_a \right]$$

$$+ \int_{M^{10}} \left[ \frac{1}{2} d\phi \wedge * d\phi - (C^{(9)} - * C^{(1)}) \wedge d\phi \right]$$

$$- \int_{M^{10}} \left[ \frac{1}{2} e^{-\frac{3}{2} \phi} F^{(2)} \wedge * F^{(2)} + (C^{(8)} - e^{-\frac{3}{2} \phi} * C^{(2)}) \wedge F^{(2)} \right]$$

$$+ \int_{M^{10}} \left[ \frac{1}{2} e^{\frac{3}{2} \phi} H^{(3)} \wedge * H^{(3)} - (C^{(7)} - e^{\frac{3}{2} \phi} * C^{(3)}) \wedge H^{(3)} \right]$$

$$- \int_{M^{10}} \left[ \frac{1}{2} e^{-\frac{3}{2} \phi} F^{(4)} \wedge * F^{(4)} + (C^{(6)} - e^{-\frac{3}{2} \phi} * C^{(4)}) \wedge F^{(4)} \right]$$

$$+ \int_{M^{10}} B^{(2)} \wedge dA^{(3)} \wedge dA^{(3)}$$

$$+ \int_{M^{10}} \left[ \frac{1}{2} \nu \wedge H^{(7)} \wedge i_\nu H^{(3)} + \frac{1}{2} \nu \wedge F^{(6)} \wedge i_\nu F^{(4)} \right] + O(f^4).$$  \tag{49}

and we are ready to discuss its symmetry structure and equations of motion of the gauge fields which follow from this action.

To this end, as in the case of duality–symmetric $D = 11$ supergravity, it is convenient to rewrite the action as follows

$$S = \int_{M^{10}} \left[ -R^{a_1 a_2} \wedge \Sigma_{a_1 a_2} - \frac{i}{3!} \bar{\psi} \wedge D\psi \wedge \Gamma^{a_1 a_2 a_3} \Sigma_{a_1 a_2 a_3} - \frac{i}{2} \bar{\lambda} \Gamma^a D\lambda \wedge \Sigma_a \right]$$

$$+ \int_{M^{10}} \left[ \frac{1}{2} d\phi \wedge * d\phi - (C^{(9)} - * C^{(1)}) \wedge d\phi \right]$$

$$- \int_{M^{10}} \left[ \frac{1}{2} e^{-\frac{3}{2} \phi} F^{(2)} \wedge * F^{(2)} + (C^{(8)} - e^{-\frac{3}{2} \phi} * C^{(2)}) \wedge F^{(2)} \right]$$

$$- \frac{1}{2} \int_{M^{10}} B^{(2)} \wedge dA^{(3)} \wedge dA^{(3)} + \int_{M^{10}} \mathcal{L}_{d.s.}^{(10)} + O(f^4)$$  \tag{50}

with

$$\mathcal{L}_{d.s.}^{(10)} = \frac{1}{2} \nu \wedge (H^{(3)} - C^{(3)}) \wedge i_\nu H^{(7)} - \frac{1}{2} \nu \wedge (F^{(4)} - C^{(4)}) \wedge i_\nu F^{(6)}$$

$$+ \frac{1}{2} \nu \wedge (F^{(6)} + C^{(6)}) \wedge i_\nu F^{(4)} + \frac{1}{2} \nu \wedge (H^{(7)} + C^{(7)}) \wedge i_\nu H^{(3)}$$

$$+ \frac{1}{2} H^{(3)} \wedge C^{(7)} + \frac{1}{2} C^{(3)} \wedge H^{(7)} - \frac{1}{2} C^{(4)} \wedge F^{(6)} - \frac{1}{2} C^{(6)} \wedge F^{(4)} .$$  \tag{51}
The variation of \((51)\) is

\[
\delta L^{(10)}_{d.s.} = -\left[ \delta v \cdot \mathcal{F}^{(4)} \cdot i_v \mathcal{H}^{(6)} + v \cdot i_v \mathcal{F}^{(6)} \cdot \delta (F^{(4)} - C^{(4)}) - \frac{1}{2} (F^{(6)} + C^{(6)}) \cdot \delta (F^{(4)} - C^{(4)}) \right. \\
- v \cdot i_v \mathcal{F}^{(4)} \cdot \delta (F^{(6)} + C^{(6)}) + \frac{\delta v \cdot i_v \mathcal{H}^{(7)} \cdot i_v \mathcal{H}^{(3)}}{2} \\
- \frac{1}{2} (H^{(7)} + C^{(7)}) \cdot \delta (H^{(3)} - C^{(3)}) \right]
\]

modulo a total derivative term. For shortness we omit the wedge product between the differential forms in \((52)\) and in some intermediate formulae below. Since we are interested in the derivation of symmetries and equations of motion of the gauge fields we “freeze” the fermions and deal with the following set of variations of the field strengths

\[
\delta F^{(2)} = d(\delta A^{(1)}) , \quad \delta H^{(3)} = d(\delta B^{(2)}) , \quad \delta F^{(4)} = d(\delta A^{(3)}) - \delta (H^{(3)} \wedge A^{(1)}) , \\
\delta F^{(6)} = d(\delta A^{(5)}) + 2 \delta A^{(3)} \wedge H^{(3)} - 2 \delta B^{(2)} \wedge dA^{(3)} , \\
\delta H^{(7)} = d(\delta B^{(6)}) + 2(\delta A^{(3)} + \delta B^{(2)} \wedge A^{(1)}) \wedge F^{(4)} - \delta A^{(5)} \wedge F^{(2)} - \delta A^{(1)} \wedge F^{(6)} ,
\]

where

\[
\delta A^{(5)} = \delta A^{(5)} + \delta B^{(2)} \wedge A^{(3)} - \delta A^{(3)} \cdot B^{(2)} , \quad \delta B^{(6)} = \delta B^{(6)} - \delta A^{(3)} \wedge A^{(3)} + \delta A^{(5)} \wedge A^{(1)} .
\]

After some algebra one can find that \((52)\) takes the form

\[
\delta L^{(10)}_{d.s.} = -\left( \frac{\delta a}{\sqrt{-(\partial a)^2}} \right) \cdot \left[ i_v \mathcal{H}^{(3)} + \delta B^{(2)} \right] \wedge d(v \wedge i_v \mathcal{H}^{(7)}) \\
+ \left( \frac{\delta a}{\sqrt{-(\partial a)^2}} \right) \cdot \left[ i_v \mathcal{F}^{(4)} + \delta A^{(3)} \right] \wedge d(v \wedge i_v \mathcal{F}^{(6)}) \\
- \left( \frac{\delta a}{\sqrt{-(\partial a)^2}} \right) \cdot \left[ i_v \mathcal{F}^{(6)} + \delta A^{(5)} \right] \wedge d(v \wedge i_v \mathcal{F}^{(4)}) \\
- \left( \frac{\delta a}{\sqrt{-(\partial a)^2}} \right) \cdot \left[ i_v \mathcal{H}^{(7)} + \delta B^{(6)} - \delta A^{(3)} \wedge A^{(3)} + \delta A^{(5)} \wedge A^{(1)} \right] \wedge d(v \wedge i_v \mathcal{H}^{(3)}) \\
+ \delta A^{(1)} \wedge \left[ v \wedge i_v \mathcal{H}^{(3)} \wedge F^{(6)} - v \wedge i_v \mathcal{F}^{(6)} \wedge H^{(3)} + F^{(6)} \wedge H^{(3)} \right] \\
+ \delta B^{(2)} \wedge \left[ d(v \wedge i_v \mathcal{F}^{(6)} \wedge A^{(1)}) + 2v \wedge i_v \mathcal{H}^{(3)} \wedge dA^{(3)} \wedge A^{(1)} - 2v \wedge i_v \mathcal{F}^{(4)} \wedge dA^{(3)} \right] \\
+ \delta A^{(3)} \wedge \left[ 2v \wedge i_v \mathcal{F}^{(4)} \wedge H^{(3)} - 2v \wedge i_v \mathcal{H}^{(3)} \wedge F^{(4)} \right] \\
+ \delta A^{(5)} \wedge v \wedge i_v \mathcal{H}^{(3)} \wedge F^{(2)} + \delta \left( \frac{1}{2} B^{(2)} \wedge dA^{(3)} \wedge dA^{(3)} \right) .
\]

Analyzing this variation we conclude that in addition to the conventional gauge symmetries

\[
\delta A^{(1)} = da^{(0)} , \quad \delta B^{(2)} = da^{(1)} , \quad \delta A^{(3)} = da^{(2)} - B^{(2)} \wedge da^{(0)} ,
\]

12
\[ \delta A^{(5)} = d\alpha^{(4)} - d\alpha^{(2)} \wedge B^{(2)} + d\alpha^{(1)} \wedge A^{(3)} , \quad \delta B^{(6)} = d\alpha^{(5)} - d\alpha^{(2)} \wedge A^{(3)} + d\alpha^{(0)} \wedge A^{(5)} \quad (57) \]

the action \([50]\) (or equivalently \([49]\)) possesses the following set of local PST symmetries

\[ \delta \phi a(x) = 0, \quad \delta \phi A^{(1)} = 0, \quad \delta \phi B^{(2)} = da \wedge \phi^{(1)}, \quad \delta \phi A^{(3)} = da \wedge \phi^{(2)}, \]
\[ \delta \phi A^{(5)} = da \wedge \phi^{(4)}, \quad \delta \phi B^{(6)} = da \wedge \phi^{(5)} + \delta \phi A^{(3)} \wedge A^{(3)} - \delta \phi A^{(5)} \wedge A^{(1)}, \quad (58) \]

and

\[ \delta \phi a(x) = \Phi(x), \quad \delta \phi B^{(2)} = -\frac{\Phi}{\sqrt{-\left(\partial a\right)^2}} i_v H^{(3)}, \quad \delta \phi A^{(3)} = -\frac{\Phi}{\sqrt{-\left(\partial a\right)^2}} i_v F^{(4)}, \]
\[ \delta \phi A^{(5)} = -\frac{\Phi}{\sqrt{-\left(\partial a\right)^2}} i_v F^{(6)}, \quad \delta \phi B^{(6)} = -\frac{\Phi}{\sqrt{-\left(\partial a\right)^2}} i_v H^{(7)} + \delta \phi A^{(3)} \wedge A^{(3)} - \delta \phi A^{(5)} \wedge A^{(1)}, \quad (59) \]

where \(\delta \phi A^{(5)}\) is defined as in Eq. \([51]\).

Varying the rest of the action \([50]\) and having in mind eq. \([56]\) one gets the following equations of motion of the gauge fields

\[ \frac{\delta L}{\delta B^{(6)}} = 0 \implies d(v \wedge i_v H^{(3)}) = 0, \]
\[ \frac{\delta L}{\delta A^{(5)}} = 0 \implies d(v \wedge i_v F^{(4)}) + A^{(1)} \wedge d(v \wedge i_v H^{(3)}) - v \wedge i_v H^{(3)} \wedge F^{(2)} = 0, \]
\[ \frac{\delta L}{\delta A^{(3)}} = 0 \implies d(v \wedge i_v F^{(6)}) + B^{(2)} \wedge d(v \wedge i_v F^{(4)}) + (A^{(3)} + B^{(2)} \wedge A^{(1)}) \wedge d(v \wedge i_v H^{(3)})
+2v \wedge i_v F^{(4)} \wedge H^{(3)} - 2v \wedge i_v H^{(3)} \wedge F^{(4)} = 0, \quad (60) \]
\[ \frac{\delta L}{\delta B^{(2)}} = 0 \implies d(v \wedge i_v H^{(7)}) + A^{(3)} \wedge d(v \wedge i_v F^{(4)}) + A^{(3)} \wedge A^{(1)} \wedge d(v \wedge i_v H^{(3)})
-d(v \wedge i_v F^{(6)} \wedge A^{(1)}) - 2v \wedge i_v H^{(3)} \wedge F^{(4)} \wedge A^{(1)} + 2v \wedge i_v F^{(4)} \wedge F^{(4)} = 0, \]
\[ \frac{\delta L}{\delta A^{(1)}} = 0 \implies d(e^{-\frac{3}{2}\phi} \wedge (F^{(2)} - C^{(2)} + C^{(8)}) + v \wedge i_v H^{(3)} \wedge F^{(6)} - v \wedge i_v F^{(6)} \wedge H^{(3)}
+F^{(6)} \wedge H^{(3)} = 0, \]
\[ \frac{\delta L}{\delta \phi} = 0 \implies d[(d\phi - C^{(1)}) + C^{(9)}] + \frac{3}{4} F^{(2)} \wedge [e^{-\frac{3}{2}\phi} \wedge (F^{(2)} - C^{(2)}) + C^{(8)}]
+ \frac{\delta L_{\phi}}{\delta \phi} = 0, \quad (61) \]

where \(L_{\phi}\) is defined in \([51]\).

Note that these equations are not changed when four-fermion terms are included (cf. \([30]\ [31]\ [32]\).

Applying the same arguments as in the case of duality–symmetric \(D = 11\) supergravity we can reduce the set of eqs. \([60]\) to the duality relations

\[ H^{(3)} = 0, \quad F^{(4)} = 0, \quad F^{(6)} = 0, \quad H^{(7)} = 0. \quad (62) \]
Then, taking into account (62), the equations of motion (61) become

\[ d[e^{-\frac{1}{2}\phi} \ast (F^2 - C^2) + C^8] - [e^{-\frac{1}{2}\phi} \ast (F^4 - C^4) + C^6] \wedge H^3 = 0 , \]
\[ d[(d\phi - C^1) + C^9] + \frac{3}{4} F^2 \wedge [e^{-\frac{1}{2}\phi} \ast (F^2 - C^2) + C^8] \]
\[ + \frac{1}{2} H^3 \wedge [e^{\phi} \ast (H^3 - C^3) + C^7] + \frac{1}{4} F^4 \wedge [e^{-\frac{1}{2}\phi} \ast (F^4 - C^4) + C^6] = 0 . \] (63)

Apparently, the equations (63) coincide with those obtained from the action (67) by varying \(A^{(1)}\) and \(\phi\). Taking the external derivative of the duality–symmetric relations (62) one gets the second order equations of motion of \(B^{(2)}\), \(A^{(3)}\) and their duals \(B^{(6)}\) and \(A^{(5)}\) (see, for instance, [29] 3. Therefore, we conclude that the duality–symmetric action for type IIA \(D = 10\) supergravity is classically equivalent to the conventional action.

The same observation concerns the local supersymmetry transformations. By use of the same procedure as in [30, 31, 32] one can derive the following supersymmetry variations of the fields

\[ \delta_\epsilon a = 0 , \]
\[ \delta_\epsilon \phi = - \frac{i}{2} \epsilon \Gamma^{11} \lambda , \]
\[ \delta_\epsilon A^{(1)} = - \frac{i}{2} \epsilon \phi \Gamma^{11} \psi - \frac{i}{24} e^{\frac{1}{4}\phi} \epsilon \Gamma^{(1)} \lambda , \]
\[ \delta_\epsilon B^{(2)} = \frac{1}{4} e^{-\frac{1}{2}\phi} \epsilon \Gamma^{(2)} \lambda + \frac{1}{2} e^{-\frac{1}{4}\phi} \epsilon \Gamma^{(1)} \Gamma^{11} \wedge \psi , \]
\[ \delta_\epsilon A^{(3)} = \frac{1}{4} e^{\frac{1}{4}\phi} \epsilon \Gamma^{(2)} \psi + \frac{1}{8} e^{\frac{1}{4}\phi} \epsilon \Gamma^{(3)} \Gamma^{11} \lambda - \delta_\epsilon B^{(2)} \wedge A^{(1)} , \]
\[ \delta_\epsilon A^{(5)} = \frac{i}{2} e^{-\frac{1}{4}\phi} \epsilon \Gamma^{(4)} \Gamma^{11} \wedge \psi - \frac{i}{8} e^{-\frac{1}{4}\phi} \epsilon \Gamma^{(5)} \lambda + \delta_\epsilon A^{(3)} \wedge B^{(2)} - \delta_\epsilon B^{(2)} \wedge A^{(3)} , \]
\[ \delta_\epsilon B^{(6)} = - \frac{i}{2} e^{\frac{1}{4}\phi} \epsilon \Gamma^{(5)} \wedge \psi + \frac{i}{4} e^{\frac{1}{4}\phi} \epsilon \Gamma^{(6)} \Gamma^{11} \lambda - \delta_\epsilon A^{(5)} \wedge A^{(1)} + \delta_\epsilon A^{(3)} \wedge (A^{(3)} + B^{(2)} \wedge A^{(1)}) - \delta_\epsilon B^{(2)} \wedge A^{(3)} \wedge A^{(1)} , \]
\[ \delta_\epsilon E^a = - \frac{i}{2} \epsilon \Gamma^{ab} \psi , \]
\[ \delta_\epsilon \psi = \mathcal{D} \epsilon + \frac{1}{24} d \phi + \frac{1}{24} * (d \phi \wedge \ast \Gamma^{(2)}) + \frac{1}{24} * (e^{-\frac{1}{2}\phi} \epsilon E^a F_{ab} \Gamma^{11} \Gamma^{11} \lambda)
\]
\[ + \frac{1}{24} * (e^{\frac{1}{2}\phi} \Gamma^{(6)} \wedge [H^{(3)} + v \wedge i_\nu \mathcal{H}^{(3)}] + e^{-\frac{1}{2}\phi} \Gamma^{(5)} \Gamma^{11} \wedge [F^{(4)} - v \wedge i_\nu \mathcal{F}^{(4)}])
\]
\[ - \frac{2}{3} * (e^{\frac{1}{2}\phi} \Gamma^{(2)} \Gamma^{11} \wedge [H^{(3)} + v \wedge i_\nu \mathcal{H}^{(3)}] + e^{-\frac{1}{2}\phi} \Gamma^{(3)} \wedge \ast [F^{(4)} - v \wedge i_\nu \mathcal{F}^{(4)}]) \epsilon
\]
\[- \frac{1}{12} \Gamma^{(1)} \Gamma^{11} \delta_\epsilon \lambda + \mathcal{O}(\epsilon f^2) , \]
\[ \delta_\epsilon \lambda = \left[ \frac{1}{2} \ast (d \phi \wedge \Gamma^{(1)}) \Gamma^{11} + \frac{3}{8} e^{-\frac{1}{2}\phi} \ast (F^{(2)} \wedge \Gamma^{(2)}) + \frac{1}{4} e^{-\frac{1}{2}\phi} \ast (\Gamma^{(6)} \wedge [F^{(4)} - v \wedge i_\nu \mathcal{F}^{(4)}])
\]
\[ - \frac{i}{2} e^{\frac{1}{2}\phi} \ast \Gamma^{(3)} \wedge [H^{(3)} + v \wedge i_\nu \mathcal{H}^{(3)}]) \epsilon + \mathcal{O}(\epsilon f^2) \right] , \] (64)

\( ^3\)To be precise, in the standard formulation of type IIA supergravity one can derive the relations \(\mathcal{F}^{(6)} = 0\) and \(\mathcal{H}^{(7)} = 0\) by solving formally the equations of motion for \(A^{(3)}\) and \(B^{(2)}\). The relations \(\mathcal{H}^{(3)} = 0\) and \(\mathcal{F}^{(4)} = 0\) can be recovered by taking the Hodge dual of the former two.
where $O(\epsilon f^2)$ stands for terms quadratic in fermionic fields.

One can see that on the shell of the duality relations (62) these transformations coincide with that of the standard type IIA supergravity [30, 31, 32].

4 Completion of the duality–symmetric action

4.1 Dualization of the dilaton and of the KK vector field, and the introduction of the mass term

One can see that neither the action (49) nor (50) possess the structure of (29). To get such a structure let us also double the fields $\phi$ and $A^{(1)}$ by introducing their duals and representing the second order equations of motion of the former as the Bianchi identities for the dual fields [26].

To this end taking into account eqs. (62) one can solve for eqs. (63) in terms of the following dual pairs

$$F^{(1)} = d\phi, \quad F^{(9)} = dA^{(8)} - \frac{3}{4} F^{(8)} \wedge A^{(1)} + \frac{1}{2} B^{(2)} \wedge dB^{(6)} - \frac{1}{4} F^{(6)} \wedge A^{(3)},$$

$$F^{(2)} = dA^{(1)}, \quad F^{(8)} = dA^{(7)} + F^{(6)} \wedge B^{(2)} + B^{(2)} \wedge B^{(2)} \wedge dA^{(3)}.$$  \hspace{1cm} (65)

To include the fermions one should extend the field strengths (65) and (66) as follows

$$F^{(1)} \rightarrow F^{(1)} - C^{(1)}, \quad F^{(2)} \rightarrow F^{(2)} - C^{(2)},$$

$$F^{(8)} \rightarrow F^{(8)} + C^{(8)}, \quad F^{(9)} \rightarrow F^{(9)} + C^{(9)}.$$ \hspace{1cm} (66)

Then, the following intrinsically dual field strengths

$$F^{(1)} = F^{(1)} - C^{(1)} + *(F^{(9)} + C^{(9)}), \quad F^{(9)} = F^{(9)} + C^{(9)} + *(F^{(1)} - C^{(1)}) = *F^{(1)},$$

$$F^{(2)} = F^{(2)} - C^{(2)} + e^{\frac{3}{2}\phi}*(F^{(8)} + C^{(8)}), \quad F^{(8)} = F^{(8)} + C^{(8)} + e^{-\frac{3}{2}\phi}(F^{(2)} - C^{(2)}) = e^{-\frac{3}{2}\phi}*F^{(2)}.$$ \hspace{1cm} (67)

are incorporated into the action as follows

$$S = \int_{M^{10}} \left( -R^{a_{1}a_{2}} \wedge \Sigma_{a_{1}a_{2}} - \frac{i}{3!} \bar{\psi} \wedge D\psi \Gamma^{a_{1}a_{2}a_{3}} \Sigma_{a_{1}a_{2}a_{3}} - \frac{i}{2} \lambda \Gamma^{a} D\lambda \wedge \Sigma_{a} \right)$$

$$+ \frac{1}{2} \sum_{n=1}^{4} \int_{M^{10}} \left( \frac{1}{2^{n+1}} F^{(10-n)} \wedge F^{(n)} - C^{(10-n)} \wedge F^{(n)} - F^{(10-n)} \wedge C^{(n)} \right)$$

$$+ \frac{1}{2} \sum_{n=1}^{4} \left[ i_{\nu} \mathcal{F}^{(10-n)} \wedge \nu \wedge (F^{(n)} - C^{(n)}) + \nu \wedge (F^{(10-n)} + C^{(10-n)}) \wedge i_{\nu} \mathcal{F}^{(n)} \right] + O(f^{4}),$$ \hspace{1cm} (68)

are incorporated into the action as follows

where \( \left\lceil \frac{n+1}{4} \right\rceil \) denotes the integer part of the number \( \frac{n+1}{4} \), and one shall substitute $H^{(3)}$ ($\mathcal{H}^{(3)}$) and $H^{(7)}$ ($\mathcal{H}^{(7)}$) for $F^{(n)}$ ($\mathcal{F}^{(n)}$) with $n = 3, 7$.

This action is the complete duality–symmetric action for type IIA supergravity up to the four–fermion terms and has the characteristic structure of duality–symmetric supergravities.
The variation of this action with respect to the gauge fields leads to the following (PST) gauge fixed set of equations of motion

\[ \mathcal{F}^{(1)} = \mathcal{F}^{(2)} = \mathcal{H}^{(3)} = \mathcal{F}^{(4)} = \mathcal{F}^{(6)} = \mathcal{H}^{(7)} = \mathcal{F}^{(8)} = \mathcal{F}^{(9)} = 0 , \]

which is equivalent to that of the standard formulation.

To extend the action \((69)\) to the Romans’s massive supergravity \((38)\) let us begin with its bosonic sector in the form close to that of Ref. \([36]\), where the contribution from the terms proportional to the mass parameter of new terms in the local supersymmetry transformations is sufficient to completely cancel the contribution from the terms proportional to the mass parameter \(m\) leaving the structure of the four-fermion terms the same as for the case of massless type IIA supergravity.

The action \((71)\) becomes the bosonic action of the standard ‘massless’ type IIA supergravity \((37)\) with the ‘massless’ type IIA supergravity field strengths \((2)\), the ‘mass–extended’ field strengths which are inert under the modified gauge transformations

\[ \delta B^{(2)} = d\alpha^{(1)} , \quad \delta A^{(1)} = d\alpha^{(0)} + m\alpha^{(1)} , \quad \delta A^{(3)} = d\alpha^{(2)} - B^{(2)} \wedge d\alpha^{(0)} - m B^{(2)} \wedge \alpha^{(1)} . \]

The complete massive type IIA supergravity Lagrangian has the following form

\[ \mathcal{L}_m = \mathcal{L}_0(F^{(n)}_m) - \frac{9}{2}e^{-\frac{2}{3}\phi} m^2 \ast 1 - \frac{3}{2}e^{-\frac{2}{3}\phi} m \lambda \Gamma^{a\Gamma} \Sigma_a - \frac{1}{2}e^{-\frac{2}{3}\phi} m \Sigma_{ab} \]

\[ - \frac{5}{4} e^{-\frac{2}{3}\phi} m (\lambda \lambda) \ast 1 + dA^{(3)} \wedge dA^{(3)} \wedge B^{(2)} + \frac{1}{3} m dA^{(3)} \wedge (B^{(2)}) \wedge 3 + \frac{1}{12} m^2 (B^{(2)}) \wedge 5 . \]

Here we have denoted by \(\mathcal{L}_0(F^{(n)}_m) + dA^{(3)} \wedge dA^{(3)} \wedge B^{(2)}\) the Lagrangian for the standard type IIA supergravity \((37)\) with the ‘massless’ type IIA supergravity field strengths \(F^{(n)}\) \((n = 2, 4)\) replaced with \(F^{(n)}_m\) of \((72)\).

The action \((71)\) is invariant under modified gauge transformations \((73)\) (see e.g. \([36]\)) and local supersymmetry transformations

\[ \delta \epsilon E^a = \delta_0 E^a , \quad \delta \epsilon \phi = \delta_0 \phi , \quad \delta \epsilon A^{(n-1)} = \delta_0 A^{(n-1)} \quad (n = 2, 4) , \quad \delta \epsilon B^{(2)} = \delta_0 B^{(2)} , \]

\[ \delta \epsilon \lambda = \delta_0 \lambda (F^{(n)}_m) + \frac{3i}{2} e^{-\frac{2}{3}\phi} m \Gamma^{11} \epsilon , \quad \delta \epsilon \psi = \delta_0 \psi (F^{(n)}_m) - \frac{i}{4} e^{-\frac{2}{3}\phi} m \Gamma^{(1)} \epsilon , \]

where \(\delta_0\) denotes the ‘massless’ type IIA supersymmetry transformations, which can be read off \((69)\), but with the appropriate replacement of the field strengths. We note that the appearance of new terms in the local supersymmetry transformations is sufficient to completely cancel the contribution from the terms proportional to the mass parameter \(m\) leaving the structure of the four-fermion terms the same as for the case of massless type IIA supergravity.

To recover the duality-symmetric structure similar to that of \((39)\) we double the fields introducing their dual partners by presenting the second order equations of motion following from \((71)\).
as the Bianchi identities \[31\]. This procedure leads to the following set of dual field strengths for higher–rank gauge potentials \(A^{(5)}\), \(B^{(6)}\) and \(A^{(7)}\) which are invariant under the modified gauge transformations

\[
F^{(6)}_m = F^{(6)} - \frac{1}{3} m(B^{(2)})^\wedge 3, \quad H^{(7)}_m = H^{(7)} - m A^{(7)} + \frac{1}{3} m A^{(1)} \wedge (B^{(2)})^\wedge 3, \\
F^{(8)}_m = F^{(8)} - \frac{1}{12} m(B^{(2)})^\wedge 4, \quad F^{(9)}_m = F^{(9)} - \frac{5}{4} m A^{(9)} - \frac{1}{2} m B^{(2)} \wedge A^{(7)} + \frac{1}{16} m A^{(1)} \wedge (B^{(2)})^\wedge 4,
\]

(76)

where \(F^{(n+5)}\) \((n = 1, 2, 3, 4)\) are the field strengths defined in \[17\], \(53\), \(66\). To derive \(F^{(9)}_m\), which can be read off the dilaton equation of motion

\[
d \ast d\phi = \frac{1}{4} F^{(4)}_m \wedge F^{(6)}_m + \frac{1}{2} H^{(3)}_m \wedge H^{(7)}_m + \frac{3}{4} F^{(2)}_m \wedge F^{(8)}_m - \frac{45}{4} e^{-\frac{5}{2}\phi} m^2 \ast 1
\]

(77)

we have used that the right hand side of the above equation is a ten–form in ten–dimensional space–time and hence is closed and locally exact. So introducing a nine–form \(A^{(9)}\) defined by

\[
d A^{(9)} = -9 e^{-\frac{5}{2}\phi} m \ast 1 - B^{(2)} \wedge F^{(8)} + \frac{1}{2} (B^{(2)})^\wedge 2 \wedge F^{(6)} + \frac{1}{3} d A^{(3)} \wedge (B^{(2)})^\wedge 3 + \frac{1}{60} m (B^{(2)})^\wedge 5
\]

(78)

one recovers the expression for \(F^{(9)}_m\). We should note that at this stage \(A^{(9)}\) is not a dynamical field but an implicit function of other fields of the theory. It will become a fully fledged field when the mass parameter is promoted to a scalar field \(F^{(0)}\) dual to a field strength of \(A^{(9)}\).

After determining the intrinsically dual combinations of the field strengths

\[
\mathcal{F}^{(1)} = F^{(1)} - C^{(1)} + \ast (F^{(9)}_m + C^{(9)}) , \\
\mathcal{F}^{(9)} = F^{(9)}_m + C^{(9)} + \ast (F^{(1)}_m - C^{(1)}) = \ast \mathcal{F}^{(1)} , \\
\mathcal{H}^{(3)} = H^{(3)}_m - C^{(3)} + e^{-\phi} \ast (H^{(7)}_m + C^{(7)}) , \\
\mathcal{H}^{(7)} = H^{(7)}_m + C^{(7)} + e^{\phi} \ast (H^{(3)}_m - C^{(3)}) = e^{\phi} \ast \mathcal{H}^{(3)}, \\
\mathcal{F}^{(n)} = F^{(n)}_m - C^{(n)} + e^{\frac{(10-2n)}{4}\phi} \ast (F^{(10-n)}_m + C^{(10-n)}) , \quad n = 2, 4 , \\
\mathcal{F}^{(n+5)} = F^{(n+5)}_m + C^{(n+5)} + e^{-\frac{5}{2}\phi} \ast (F^{(5-n)}_m - C^{(5-n)}) = e^{-\frac{5}{2}\phi} \ast \mathcal{F}^{(5-n)} , \\
n = 1, 3,
\]

(79)

the duality-symmetric action for the massive type IIA supergravity has the following form

\[
S = \int_{\mathcal{M}^{10}} \left( - R a_{a_1 a_2} \wedge \Sigma_{a_1 a_2} - \frac{i}{3!} \bar{\psi} \Gamma a_{a_1 a_2 a_3} \Sigma_{a_1 a_2 a_3} - \frac{i}{2} \bar{\lambda} \Gamma^a \mathcal{D} \lambda \wedge \Sigma_a \right) \\
+ \frac{1}{3} \int_{\mathcal{M}^{10}} \left[ \frac{9}{2} e^{-\frac{5}{2}\phi} m^2 \ast 1 + m (C^{(10)} - e^{-\frac{5}{2}\phi} \ast C^{(0)}) \right] \\
+ \frac{1}{3} \int_{\mathcal{M}^{10}} m \left( B^{(2)} \wedge F^{(8)} - \frac{1}{2} (B^{(2)})^\wedge 2 \wedge F^{(6)} - \frac{1}{3} d A^{(3)} \wedge (B^{(2)})^\wedge 3 - \frac{1}{60} m (B^{(2)})^\wedge 5 \right) \\
+ \frac{1}{2} \sum_{n=1}^{4} \int_{\mathcal{M}^{10}} \left( \frac{1}{3(2n+1)} F^{(10-n)}_m \wedge F^{(n)}_m - C^{(10-n)} \wedge F^{(n)}_m - F^{(10-n)}_m \wedge C^{(n)} \right) \\
+ \frac{1}{2} \sum_{n=1}^{4} \int_{\mathcal{M}^{10}} \left[ i_{V} \mathcal{F}^{(10-n)}_m \wedge V \wedge (F^{(n)}_m - C^{(n)}) + V \wedge (F^{(10-n)}_m + C^{(10-n)}) \wedge i_{V} F^{(n)}_m \right] + \mathcal{O}(f^4),
\]

(80)

\footnote{The modified gauge transformations for \(A^{(5)}\), \(B^{(6)}\) and \(A^{(7)}\) can be read off the modified field strengths \[28\].}
where \( \lfloor \frac{n+1}{4} \rfloor \) denotes the integer part of the number \( \frac{n+1}{4} \) and

\[
C^{(0)} = -\frac{5}{4} e^{\frac{3}{2} \phi} \lambda a,
\]

\[
C^{(10)} = \frac{3}{2} e^{-\frac{3}{2} \phi} \lambda a \Gamma^{\alpha \beta} \Sigma_a + \frac{1}{2} e^{\frac{3}{2} \phi} \bar{\psi} \Gamma^{a b} \psi \wedge \Sigma_{a b}.
\] (81)

To extend the action (80) to a complete duality–symmetric form we introduce instead of \( m \) a zero–form field \( F^{(0)} \) and an exact ten–form \( dA^{(9)} \), and rewrite the Lagrangian (74) as follows

\[
\mathcal{L}_m = \mathcal{L}_0(F^{(n)}) - \left[ \frac{9}{2} e^{-\frac{3}{2} \phi} F^{(0)} \wedge F^{(0)} + (C^{(10)} - e^{-\frac{3}{2} \phi} \ast C^{(0)}) \wedge F^{(0)} \right]
\]

\[
+ F^{(0)} \wedge dA^{(9)} + dA^{(3)} \wedge dA^{(3)} \wedge B^{(2)} + \frac{1}{3} F^{(0)} \wedge dA^{(3)} \wedge (B^{(2)})^{\wedge 3} + \frac{1}{20} (F^{(0)})^2 \wedge (B^{(2)})^{\wedge 5}. \] (82)

Note that \( F^{(0)} \) is inert under the local supersymmetry transformations, and the equation of motion of \( A^{(9)} \), \( dF^{(0)} = 0 \), implies that \( F^{(0)} \) is a constant which we choose to be \( m \). This is an example of the mechanism of the dynamical generation of mass and tension of branes various aspects of which have been discussed in the literature [13, 14].

Varying (82) over the new field \( F^{(0)} \) we get the expression for its dual partner \( F^{(10)}_m \), so that together with the dual field strengths (76) the complete set becomes

\[
F_m^{(6)} = F^{(6)} - \frac{1}{3} F^{(0)} (B^{(2)})^{\wedge 3}, \quad H_m^{(7)} = H^{(7)} - F^{(0)} \wedge A^{(7)} + \frac{1}{3} F^{(0)} A^{(1)} \wedge (B^{(2)})^{\wedge 3},
\]

\[
F_m^{(8)} = F^{(8)} - \frac{1}{12} F^{(0)} (B^{(2)})^{\wedge 4}, \quad F_m^{(9)} = F^{(9)} - \frac{5}{4} F^{(0)} A^{(9)} - \frac{1}{2} F^{(0)} B^{(2)} \wedge A^{(7)} + \frac{1}{16} F^{(0)} A^{(1)} \wedge (B^{(2)})^{\wedge 4},
\]

\[
F_m^{(10)} = dA^{(9)} + B^{(2)} \wedge F^{(8)} - \frac{1}{2} (B^{(2)})^{\wedge 2} \wedge F^{(6)} - \frac{1}{3} (B^{(2)})^{\wedge 3} \wedge dA^{(3)} - \frac{1}{60} F^{(0)} (B^{(2)})^{\wedge 5}. \] (83)

After this step one can write the complete duality–symmetric action for the massive type IIA supergravity as follows

\[
S = \int_{\mathcal{M}_{10}} \left[ - R^{a_1 a_2} \wedge \Sigma_{a_1 a_2} - i \bar{\psi} \wedge D \psi \Gamma^{a_1 a_2 a_3} \wedge \Sigma_{a_1 a_2 a_3} - \frac{i}{2} (D \lambda \Lambda a) \right]
\]

\[
- \int_{\mathcal{M}_{10}} \left[ \frac{9}{2} e^{-\frac{3}{2} \phi} F^{(0)} \wedge F^{(0)} + (C^{(10)} - e^{-\frac{3}{2} \phi} \ast C^{(0)}) \wedge F^{(0)} \right]
\]

\[
+ \frac{1}{3} \int_{\mathcal{M}_{10}} \left( F^{(0)} \wedge F^{(10)} - dF^{(0)} \wedge A^{(7)} \wedge B^{(2)} - 4F^{(0)} \wedge dA^{(9)} \right)
\]

\[
+ \frac{1}{2} \sum_{n=1}^{4} \int_{\mathcal{M}_{10}} \left( \frac{1}{3!} [F_m^{(10-n)} \wedge F_m^{(n)} - C^{(10-n)} \wedge F_m^{(n)} - F_m^{(10-n)} \wedge C^{(n)}] \right) + \mathcal{O}(f^4),
\] (84)

where as in the eqs. (69), (80) \( \lfloor \frac{n+1}{4} \rfloor \) denotes the integer part of the number \( \frac{n+1}{4} \). When the equation \( dF^{(0)} = 0 \) is solved, a constant mass is generated \( F^{(0)} = m \) and the duality relation \( F^{(10)} = (F_m^{(10)} + C^{(10)}) + 9 e^{-\frac{3}{2} \phi} (F^{(0)} - \frac{1}{9} C^{(0)}) = 0 \) is taken into account the action (84) reduces to the action (80).
Thus, we have completed our task to construct the duality–symmetric manifestly covariant version of type IIA supergravity. We have presented the action in different but equivalent forms (49), (50) and (69), (80), (84) which serve for different purposes. The action (49) is not manifestly duality–symmetric but its form is close to the standard action for type IIA supergravity, which simplifies the verification of the local supersymmetries. The action (50) is convenient for deriving the duality relations and for carrying out the symmetry analysis. The actions (69), (80) and (84) are manifestly duality–symmetric with respect to the NS–NS and RR fields and their dual. They can be considered as an off–shell and supersymmetric generalization of the democratic formulation of [19] and of the doubled field formalism of [26] [54].

4.2 The sigma model form of the duality–symmetric supergravity

Let us discuss the relation of our construction to that of [26]. In [26] a nice group–theoretical structure behind the duality relations has been found, which is generic for all theories in the doubled field formulation. For simplicity, we shall review this structure with the example of $D = 11$ supergravity, however a corresponding sigma–model form of the duality–symmetric action which we shall present is generic and valid for all doubled field supergravities.

As it has been noticed in [26], because of the presence of the Chern–Simons term the gauge transformations (21) are non–abelian

$$\left[ \delta_{\Lambda_1^{(3)}}, \delta_{\Lambda_2^{(3)}} \right] = \delta_{\Lambda_1^{(6)}}, \quad \left[ \delta_{\Lambda_1^{(3)}}, \delta_{\Lambda_2^{(6)}} \right] = \left[ \delta_{\Lambda_1^{(6)}}, \delta_{\Lambda_2^{(6)}} \right] = 0,$$

(85)

where $\Lambda^{(3)}$ and $\Lambda^{(6)}$ are closed forms, and hence they are locally exact $\Lambda^{(3)} = d\hat{\psi}^{(2)}$, $\Lambda^{(6)} = d\hat{\psi}^{(5)}$. The transformations (85) can be associated with a superalgebra generated by a ‘Grassmann–odd’ generator $t_3$ and a commuting (central charge) generator $t_6$

$$\{t_3, t_3\} = -2t_6, \quad [t_3, t_6] = [t_6, t_6] = 0.$$

(86)

The parity of $t_3$ and $t_6$ is related to the corresponding parity of the differential form potentials $\hat{A}^{(3)}$ and $\hat{A}^{(6)}$, so that, for instance $t_3$ anticommutes with $\hat{A}^{(3)}$ and with the external differential $d$.

An element of the supergroup generated by (85) can be realized exponentially as

$$\hat{A} = e^{t_3 \hat{A}^{(3)}} e^{t_6 \hat{A}^{(6)}}.$$

(87)

Then the Cartan form

$$\mathcal{G} = d\hat{A} A^{-1} = \hat{F}^{(4)} t_3 + \hat{F}^{(7)} t_6$$

(88)

has the field strengths of the gauge fields $\hat{A}^{(3)}$ and $\hat{A}^{(6)}$ as its components.

By construction, the Cartan form identically satisfies the Maurer–Cartan equations (the zero curvature condition)

$$d\mathcal{G} + \mathcal{G} \wedge \mathcal{G} = 0.$$  

(89)

By now in this construction $\hat{F}^{(4)}$ and $\hat{F}^{(7)}$ have been independent field strengths. To impose the duality relation between them in the framework of this algebraic formalism one introduces [26] a pseudo–involution $S$ which interchanges the superalgebra generators $t_3$ and $t_6$

$$St_3 = t_6, \quad St_6 = t_3, \quad S^2 = 1.$$  

(90)
In general, the eigenvalue of $S^2$ on a given generator is the same as the eigenvalue of $\ast^2$ on the associated field strength. Note also that the pseudo–involution does not preserve the superalgebra commutation relations \([SS]\) \[26\].

Using $S$ and the Hodge operator one imposes on \([SS]\) the twisted self–duality condition

\[\ast G = S G,\]  

(91)

which reproduces the duality relations between the field strength components of \([SS]\) in the absence of fermions. When \([91]\) holds the zero curvature condition \([SS]\) amounts to second order equations of motion of $F^{(4)}$ and $F^{(7)}$.

To add fermions we should extend $G$ with the superalgebra valued element $C = -\hat{C}^{(4)} t_3 + \hat{C}^{(7)} t_6$ (where $\hat{C}^{(4)}$ and $\hat{C}^{(7)}$ have been defined in \([I]\))

\[G \rightarrow G + C.\]  

(92)

Then the twisted self–duality condition takes the form

\[\ast (G + C) = S (G + C) \rightarrow (S - \ast)(G + C) = 0,\]  

(93)

and is tantamount to the duality relations \([2S]\).

We are now ready to present a sigma–model action from which the twisted self–duality condition \([93]\) is obtained as an equation of motion

\[S = S_{EH} + S_{\Psi} - Tr \int_{M^{11}} \frac{1}{12} (G + \frac{1}{2} C) \wedge (S - \ast) C - Tr \int_{M^{11}} \left\{ \frac{1}{4} \ast G \wedge G - \frac{1}{12} G \wedge S G - \frac{1}{4} \ast \hat{v}(S - \ast)(G + C) \wedge \hat{v}(S - \ast)(G + C) \right\},\]  

(94)

where $S_{EH}$ and $S_{\Psi}$ stand for the Einstein action and the fermion kinetic terms as in \([29]\), the auxiliary one form $\hat{v}$ has been introduced in \([14]\) and the trace is defined such that

\[Tr(t_3 t_3) = -Tr(t_6 t_6) = -1, \quad Tr(t_3 t_6) = 0.\]  

(95)

Using the definition of $G$ \([SS]\) and of the pseudo involution $S$ \([20]\) one can verify that the action \([94]\) is equivalent to the duality–symmetric action \([2]\). One should only note that because of the anticommutativity of $t_3$ with the odd differential forms the order of the multipliers in \([94]\) is essential.

We should stress that the duality–symmetric action \([94]\) has, actually, a generic form which remains the same also for the doubled field formulations of type IIA and type IIB $D = 10$ supergravities, as well as for lower dimensional supergravities considered in \([23]\). To describe a corresponding supergravity with the action \([94]\) one should introduce the relevant superalgebra \([20]\), which is analogous but much more complicated than \([SS]\), to construct a corresponding group element $A$ and the Cartan form $G$ similar to \([SS]\) and \([SS]\), to define the twisted self–duality condition \([93]\) and insert all these ingredients into the action \([94]\). We thus have extended the construction of \([26]\) to the supersymmetric case and lifting it onto the level of the proper duality–symmetric action.

In the next section we shall obtain a new version of IIA $D=10$ supergravity (without the auxiliary PST scalar) which upon the reduction to N=1 reproduces the six-index photon supergravity by Chamseddine \([8S]\).
5 Gauge fixed version of the duality–symmetric $D = 11$ action and a new exotic formulation of type IIA supergravity

When we reduced the D=11 supergravity action \([1]\) to the type IIA D=10 supergravity action \((49), (50)\) in Section 3 we chose the PST scalar to be independent of the compactified coordinate $X^\perp$ (eq. \((12)\)). Now we shall proceed in a different way. Using the symmetry \((23)\) we identify the scalar $a(x)$ with the coordinate $X^\perp$. This breaks the D=11 general coordinate invariance of the duality–symmetric $D = 11$ supergravity action down to $D = 10$ general coordinate invariance and results in a $D = 11$ supergravity counterpart of the Sen–Schwarz action \([42]\) for duality–symmetric gauge fields. Now if we perform the dimensional reduction of this gauge fixed action along the $X^\perp$ direction, the resulting $D = 10$ action does possess the complete $D = 10$ invariance. In addition to the RR fields $A^{(1)}$, $A^{(3)}$ and the NS–NS field $B^{(2)}$, it also contains the higher form fields $A^{(5)}$ and $B^{(6)}$ dual to $A^{(3)}$ and $B^{(2)}$, but it does not involve the PST scalar.

A peculiar feature of this formulation is that the coupling of the $U(1)$ field $A^{(1)}$ to other fields is non–polynomial and as a result, the local $U(1)$ symmetry and supersymmetry are realized in a nonlinear way.

In the gauge $a(x) = X_{11}^\perp$ we have $\partial_{\hat{m}} a(x) = \delta_{\hat{m}}^{11}$ and $- \partial a(x) \hat{g} \partial a(x) = e^{-\frac{i}{2} \phi} (e^{\frac{i}{2} \phi} - A_{m} g_{m n} A_{n})$. Hence

\[ \hat{v}^{(1)} = \frac{e^{\frac{i}{2} \phi} dX_{11}^\perp}{\sqrt{e^{\frac{i}{2} \phi} - (A^{(1)})^2}} \quad \iff \quad \hat{v}_{\hat{m}} = \left( 0, \frac{e^{\frac{i}{2} \phi}}{\sqrt{e^{\frac{i}{2} \phi} - (A^{(1)})^2}} \right), \]

i.e. only $\hat{v}_{\hat{11}}$ component survives, $\hat{v}_{\hat{m}} = \delta_{\hat{m}}^{11} \hat{v}_{\hat{11}}$. Using \((43)\), one can also check that

\[ \hat{v}_{\hat{m}} = e^{-\frac{i}{2} \phi} \left( \frac{A^{m}}{\sqrt{e^{\frac{i}{2} \phi} - (A^{(1)})^2}}, - \sqrt{e^{\frac{i}{2} \phi} - (A^{(1)})^2} \right), \]

and

\[ i_{\hat{v}} (dX_{11}^\perp + A^{(1)}) = - \frac{e^{\frac{i}{2} \phi}}{\sqrt{e^{\frac{i}{2} \phi} - (A^{(1)})^2}} \]

depend solely on the physical fields $\phi(x)$ and $A^{(1)}$, while $i_{\hat{v}} \hat{E}^{a} = 0$ (for $a = 0, 1, \ldots, 9$). Then the dimensional reduction along the $X^\perp$ direction gives the $D = 10$ supergravity action with the following Lagrangian for the gauge fields

\[ L_{g.f.}^{(10)} = - \frac{1}{2} e^{-\frac{i}{2} \phi} F^{(2)} \wedge \ast F^{(2)} - (C^{(8)} - e^{-\frac{i}{2} \phi} \ast C^{(2)}) \wedge F^{(2)} - \frac{1}{2} [ \hat{F}^{(4)} \wedge \hat{F}^{(6)} - \hat{H}^{(7)} \wedge \hat{H}^{(3)} ] \]
\[ + \frac{1}{2 (A^{(1)})^2 - e^{\frac{i}{2} \phi}} [ \hat{F}^{(4)} \wedge i_{A} (H^{(7)} + \mathcal{F}^{(6)} A^{(1)}) - \hat{H}^{(7)} \wedge i_{A} (\mathcal{F}^{(4)} + H^{(3)} A^{(1)}) ] \]
\[ + \frac{1}{2} C^{(3)} \wedge (dB^{(6)} + A^{(3)} \ast dA^{(3)}) - \frac{1}{2} (C^{(4)} + C^{(3)} \wedge A^{(1)}) \wedge (dA^{(5)} + A^{(3)} \wedge H^{(3)} - B^{(2)} \ast dA^{(3)}) \]
\[ - \frac{1}{2} (C^{(7)} + C^{(6)} \wedge A^{(1)}) \wedge H^{(3)} - \frac{1}{2} C^{(6)} \wedge dA^{(3)}, \]

where

\[ F^{(2)} = dA^{(1)}, \quad \tilde{H}^{(3)} = dB^{(2)} - C^{(3)}, \quad \tilde{H}^{(7)} = dB^{(6)} + A^{(3)} \wedge dA^{(3)} + C^{(7)} + C^{(6)} \wedge A^{(1)}, \]
\[ F^{(4)} = dA^{(3)} - C^{(4)} - C^{(3)} \wedge A^{(1)}, \quad F^{(6)} = F^{(6)} + C^{(6)}, \quad \] (100)

and
\[ H^{(3)} = H^{(3)} + e^{-\phi} * (H^{(1)} - F^{(6)} \wedge A^{(1)}), \quad \mathcal{H}^{(7)} = F^{(6)} \wedge A^{(1)} + e^\phi * H^{(3)} = e^\phi * \mathcal{H}^{(3)}, \]
\[ \mathcal{F}^{(4)} = F^{(4)} - H^{(3)} \wedge A^{(1)} + e^\frac{1}{2} \phi * F^{(6)}, \quad \mathcal{F}^{(6)} = F^{(6)} + e^{-\frac{1}{2} \phi} \ast (F^{(4)} - H^{(3)} \wedge A^{(1)}) = e^{-\frac{1}{2} \phi} \ast \mathcal{F}^{(4)}. \] (101)

Thus, upon the dimensional reduction the gauge fixed \( D = 11 \) supergravity action reduces to the duality–symmetric type IIA D=10 supergravity action given by Eq. (50) with \( L_{g.f.} \) having the form of (99). One could notice that in (99) the Kaluza–Klein vector field \( A^{(1)} \) couples in a direct non–polynomial way to the field strengths of \( A^{(3)} \) and \( B^{(6)} \) and to fermions. However, as we demonstrate below, the action is nevertheless invariant under non–manifest local \( U(1) \) symmetry associated with \( A^{(1)} \).

Note also that in the case under consideration the non–polynomial structure of \( A^{(1)} \) coupling implies the condition
\[ e^{\frac{1}{2} \phi} - A_m g^{mn} A_n \neq 0 \] (102)
which restricts values of the dilaton and of the ‘length’ of the \( U(1) \) gauge field vector. One can also notice that (102) is the \( \hat{g}_{11} \) component of the inverse \( D = 11 \) metric (43). Actually, since the left hand side of (102) is not \( U(1) \) gauge invariant, this condition may impose restrictions on the admissible gauge choices for fixing the \( U(1) \) symmetry. However, it is not the case in the Coulomb gauge in which \( A_0 = 0 \), the l.h.s. of (102) is positive definite (remember that the signature of the metric is \((+, -, \cdots, -)\)) and (when \( A_m = 0 \)) tends to zero only in a non–physical limit \( \langle \phi \rangle \to \infty \). Note also that (102) is always satisfied in a weak field approximation and when the \( U(1) \) transformations are infinitesimal.

Let us consider what happens with the local symmetries (22) and (23). When the gauge fixing condition (96) is imposed the transformations (22) of the \( D = 11 \) action acquire the form
\[ \delta \hat{A}^{(3)} = dX^{11} \wedge \hat{\phi}^{(2)}, \quad \delta \hat{A}^{(6)} = dX^{11} \wedge \hat{\phi}^{(5)} + dX^{11} \wedge \hat{\phi}^{(2)} \wedge \hat{A}^{(3)}, \] (103)
and thus reduce to local symmetries appearing in the Schwarz–Sen formulation. Under the dimensional reduction along the \( X^{11} \) direction the gauge potentials are decomposed as follows
\[ \hat{A}^{(3)} = A^{(3)} - B^{(2)} \wedge dX^{11}, \quad \hat{A}^{(6)} = B^{(6)} - A^{(5)} \wedge dX^{11}. \] (104)

Then the symmetry (103) allows to gauge away the ten–dimensional gauge field potentials \( B^{(2)} \) and \( A^{(5)} \).

As far as the PST symmetry (23) is concerned, although we have used this symmetry to impose the condition \( a(x) = X^{11} \) (96), its combination with the \( U(1) \) gauge transformation (originating in the \( D = 11 \) general coordinate symmetry and, thus, acting also on \( X^{11} \) which preserves (96)
\[ \delta A^{(1)} = d\alpha^{(0)}, \quad \delta X^{11} = -\alpha^{(0)}, \quad \phi(x) = \alpha^{(0)} \] (105)
is still a local \( U(1) \) symmetry of the action (eq. (112) below). Its particular feature is that now also \( A^{(3)} \) and \( B^{(6)} \) nontrivially transformed by this \( U(1) \):
\[ \delta A^{(1)} = d\alpha^{(0)}, \]
where now, since we have gauged away $B^{(2)}$ and $A^{(5)}$,

$$\mathcal{H}^{(3)} = e^{-\phi} \ast (\bar{H}^{(7)} - C^{(6)} A^{(1)}) - C^{(3)}, \quad \mathcal{H}^{(7)} = \bar{H}^{(7)} - C^{(6)} A^{(1)} - e^{\phi} \ast C^{(3)},$$

$$\mathcal{F}^{(4)} = \bar{F}^{(4)} + C^{(3)} A^{(1)} + e^{\frac{1}{2} \phi} \ast C^{(6)}, \quad \mathcal{F}^{(6)} = C^{(6)} + e^{-\frac{3}{2} \phi} \ast (\bar{F}^{(4)} + C^{(3)} A^{(1)})$$

and

$$i_A \mathcal{F}^{(4)} = \frac{1}{3!} dx^{m_3} \land dx^{m_2} \land dx^{m_1} A_n g^{mn} \mathcal{F}^{(4)} \text{ etc.}$$

and the ‘boldface’ forms

$$\mathbf{H}^{(7)} = \mathcal{H}^{(7)} + \frac{1}{e^{\frac{1}{2} \phi} - A^m A_m} (e^\phi \ast \mathcal{F}^{(4)} + i_A \mathcal{H}^{(7)}) \land A^{(1)},$$

and

$$\mathbf{F}^{(4)} = dA^{(3)} + e^{-\phi} \ast \mathbf{H}^{(7)} \land A^{(1)}$$

are field strengths which are invariant under the $U(1)$ transformations of at least on the mass–shell. For instance, $\mathbf{F}^{(4)}$ is invariant only modulo the $B^{(6)}$ field equation of motion

$$ \delta_{U(1)} \mathbf{F}^{(4)} = -\alpha_0 d(e^{-\phi} \ast \mathbf{H}^{(7)}),$$

where $d(e^{-\phi} \ast \mathbf{H}^{(7)}) = 0$ on the mass shell.

As concerns the local supersymmetry transformations, they take the following form

$$\delta_\epsilon E^a = -\frac{i}{2} \epsilon^a \Gamma^a \psi, \quad \delta_\epsilon \phi = -\frac{i}{2} \epsilon \Gamma^{11} \lambda,$$

$$\delta_\epsilon A^{(1)} = -\frac{i}{2} e^{\frac{1}{2} \phi} \epsilon \Gamma^{11} \psi - \frac{i}{24} e^{\frac{1}{2} \phi} \epsilon \Gamma^{(1)} \lambda,$$

$$\delta_\epsilon A^{(3)} = \frac{1}{2} e^{\frac{1}{2} \phi} \epsilon \Gamma^{(2)} \psi + \frac{1}{8} e^{\frac{3}{2} \phi} \epsilon \Gamma^{(3)} \Gamma^{11} \lambda,$$

$$\delta_\epsilon B^{(6)} = -\frac{i}{2} e^{\frac{5}{2} \phi} \epsilon \Gamma^{(5)} \land \psi + \frac{i}{4} e^{\frac{1}{2} \phi} \epsilon \Gamma^{(6)} \Gamma^{11} \lambda + \delta_\epsilon A^{(3)} \land A^{(3)},$$

$$\delta_\epsilon \lambda = \frac{1}{2} \ast (d\phi \land \Gamma^{(1)}) \Gamma^{11} + \frac{3}{8} e^{\frac{3}{2} \phi} \ast (\epsilon \Gamma^{(2)} \land \Gamma^{(2)}),$$

$$\delta_\epsilon (\epsilon^{\frac{1}{2} \phi} \ast \Gamma^{(6)} \land \ast [C^{(6)} - \frac{1}{e^{\frac{1}{2} \phi} - A^{(1)}} i_A (\mathcal{H}^{(7)} + \mathcal{F}^{(6)} \land A^{(1)}) - \mathcal{F}^{(6)}])$$

$$+ \frac{i}{2} e^{-\frac{3}{2} \phi} \ast (\Gamma^{(6)} \land \ast [\bar{H}^{(7)} - C^{(6)} \land A^{(1)} + \frac{1}{e^{\frac{1}{2} \phi} - A^{(1)}} i_A (\mathcal{H}^{(7)} + \mathcal{F}^{(6)} \land A^{(1)} \land A^{(1)} + \mathcal{F}^{(6)} \land A^{(1)})] \epsilon + \mathcal{O}(\epsilon^2))$$

$$+ \frac{i}{2} e^{-\frac{3}{2} \phi} \ast (\Gamma^{(6)} \land \ast [\bar{H}^{(7)} - C^{(6)} \land A^{(1)} + \frac{1}{e^{\frac{1}{2} \phi} - A^{(1)}} i_A (\mathcal{H}^{(7)} + \mathcal{F}^{(6)} \land A^{(1)} \land A^{(1)} + \mathcal{F}^{(6)} \land A^{(1)})] \epsilon + \mathcal{O}(\epsilon^2))$$
\[ \delta \psi = D \epsilon + \frac{1}{24} d \phi + \frac{1}{24} * (d \phi \wedge * \Gamma^{(2)}) + \frac{1}{4} e^{-\frac{2}{3} \phi} \Gamma^{(2)} \Gamma^{11} \]
\[ - \frac{1}{3!} * (e^{-\frac{1}{2} \phi} \Gamma^{(6)} \wedge * [H^{(7)} - C^{(6)} \wedge A^{(1)} + \frac{1}{e^{\frac{2}{3} \phi} - A^{(1)^2}} i A (H^{(7)} + F^{(6)} \wedge A^{(1)}) \wedge A^{(1)} + F^{(6)} \wedge A^{(1)})] ) \]
\[ + e^{\frac{1}{3} \phi} \Gamma^{(5)} \Gamma^{11} \wedge [C^{(6)} - \frac{1}{e^{\frac{2}{3} \phi} - A^{(1)^2}} i A (H^{(7)} + F^{(6)} \wedge A^{(1)}) - F^{(6)} )] \]
\[ + \frac{2i}{3!} * (e^{\frac{1}{3} \phi} \Gamma^{(3)} \wedge [C^{(6)} - \frac{1}{e^{\frac{2}{3} \phi} - A^{(1)^2}} i A (H^{(7)} + F^{(6)} \wedge A^{(1)}) - F^{(6)} )] \]
\[ + e^{-\frac{1}{2} \phi} \Gamma^{(2)} \Gamma^{11} \wedge [H^{(7)} - C^{(6)} \wedge A^{(1)} + \frac{1}{e^{\frac{2}{3} \phi} - A^{(1)^2}} i A (H^{(7)} + F^{(6)} \wedge A^{(1)}) \wedge A^{(1)} + F^{(6)} \wedge A^{(1)})] ] \epsilon \quad (111) \]

with the field strengths defined in (100) and (107).

Consequently, we end up with the following type IIA supergravity action

\[ S = \int_{M^{10}} \left[ - R^{a_1a_2} \wedge \Sigma_{a_1a_2} - \frac{i}{3!} \psi \wedge D \psi \wedge \Gamma^{a_1a_2a_3} \Sigma_{a_1a_2a_3} - \frac{i}{2} \tilde{\lambda} \Gamma^a D \lambda \wedge \Sigma_a \right] \]
\[ + \int_{M^{10}} \left[ \frac{1}{2} d \phi \wedge * d \phi - (C^{(9)} - * C^{(1)}) \wedge d \phi \right] \]
\[ - \int_{M^{10}} \left[ \frac{1}{2} e^{-\frac{2}{3} \phi} F^{(2)} \wedge * F^{(2)} + F^{(2)} \wedge (C^{(9)} - e^{-\frac{2}{3} \phi} * C^{(2)}) \right] \]
\[ - \int_{M^{10}} \left[ \frac{1}{2} e^{-\frac{2}{3} \phi} F^{(4)} \wedge * F^{(4)} + F^{(4)} \wedge (C^{(9)} - e^{-\frac{2}{3} \phi} * (C^{(4)} + \frac{1}{2} C^{(3)} \wedge A^{(1)})) \right] \]
\[ - \int_{M^{10}} \left[ \frac{1}{2} e^{-\phi} H^{(7)} \wedge * H^{(7)} + e^{-\phi} * H^{(7)} \wedge H^{(7)} + \frac{1}{2} C^{(3)} \wedge H^{(7)} \right] \]
\[ + \int_{M^{10}} \frac{1}{2} C^{(3)} \wedge (dB^{(6)} + A^{(3)} \wedge A^{(3)}) + O(f^4) \right] , \]

where, as always in this paper, \( O(f^4) \) stands for quartic fermion terms, and

\[ H^{(7)} = dB^{(6)} + A^{(3)} \wedge A^{(3)} + C^{(7)} - e^\phi \wedge C^{(3)} . \]

We have thus seen that in this new version the \( U(1) \) gauge field potential couples in a non–polynomial way to other fields and, as a consequence, the gauge symmetries and the local supersymmetry are realized in a highly non–linear fashion. In the way in which (112) has been obtained, this is the consequence of the mixture of space–time and PST symmetries caused by their gauge fixing in the self–dual \( D = 11 \) supergravity which gives rise to the action (112) upon dimensional reduction. However, as we shall argue below, the nature of this phenomenon is not in a particular method of dualization but in the presence of the Maxwell potential \( A^{(1)} \) in the field strength \( F^{(4)} \) of the conventional type IIA supergravity.

One may wonder how the action (112) is related to the type IIA supergravity in the form (49) and to the conventional action (47). Firstly, since (112) does not contain the dual RR field \( A^{(5)} \) and its field strength \( F^{(6)} \), to relate (112) to (49) we should get rid of \( F^{(6)} \) also in the latter. Note that this is easy to do since the “bare” potential \( A^{(5)} \) never appears in (49). Hence, we
can use a non–dynamical relation $i_v F^{(4)} = 0$, which is part of the duality–symmetric equations of motion (62), to completely eliminate $F^{(6)}$ from the action (49).

Secondly, in (49) one should also eliminate $B^{(2)}$ and $H^{(3)}$ replacing them with their dual $B^{(6)}$ and $H^{(7)}$. One can, modulo a total derivative, rewrite the Chern–Simons term in (49) such that it will contain $H^{(3)}$ instead of $B^{(2)}$

$$\int_{\mathcal{M}^{10}} B^{(2)} \wedge dA^{(3)} \wedge dA^{(3)} \rightarrow \int_{\mathcal{M}^{10}} H^{(3)} \wedge A^{(3)} \wedge dA^{(3)}.$$ 

Once this is done and when $F^{(6)}$ is eliminated, the action does not contain the “bare” potential $B^{(2)}$ anymore, and its $U(1)$ invariant field strength $H^{(3)} = dB^{(2)}$ can be replaced with the $U(1)$ invariant field strength $H^{(7)}$ (109) by solving the first of the duality relations (62), which now reduces to

$$H^{(3)} = -e^\phi \ast H^{(7)} \quad (113)$$

Substituting (113) into (49) we get the action (112) obtained by an alternative gauge fixing and the dimensional reduction of the duality–symmetric $D = 11$ supergravity.

The same result can also be achieved by a direct dualization of the field strength $H^{(3)}$ in the conventional type IIA supergravity action (37). For this one should regard $H^{(3)}$ as an independent field, add to the action (37) the Lagrange multiplier term $(H^{(3)} - dB^{(2)}) \wedge H_0^{(7)}$ and replace $H^{(3)}$ with $H^{(7)}$ (109) by solving the equations of motion for $H^{(3)}$. Note that the equation of motion of $B^{(2)}$ implies that $H_0^{(7)} = dB^{(6)}$.

So, we should stress that the non–polynomial nature of $A^{(1)}$ coupling in (112) has nothing to do with the PST formulation. It is a result of the transition from the standard type IIA supergravity with the RR field strength $F^{(4)} = dA^{(3)} - H^{(3)} \wedge A^{(1)}$ and the NS–NS two–form field $B^{(2)}$ to the dual formulation with the six–form gauge field $B^{(6)}$. The PST techniques has just allowed us to get this formulation and corresponding gauge and supersymmetry transformations of fields in a relatively simple way.

In conclusion of this section let us discuss the truncation of our model to $N = 1$, $D = 10$ supergravity. To this end in (37) and/or in (112) we should set to zero the gauge fields $A^{(1)}$ and $A^{(3)}$ together with the left–handed gravitino and the right–handed dilatino which implies that

$$\psi_L = 0 \leftrightarrow \psi = \Gamma^{11} \psi, \quad \lambda_R = 0 \leftrightarrow \lambda = -\Gamma^{11} \lambda. \quad (114)$$

After that we arrive at the following action for $N = 1$, $D = 10$ supergravity with the six–index photon instead of $B^{(2)}$ proposed in (38)

$$S = \int_{\mathcal{M}^{10}} \left[ -R^{a_1a_2} \wedge \Sigma_{a_1a_2} - \frac{i}{3!} \bar{\psi} \wedge \mathcal{D} \psi \Gamma^{a_1a_2a_3} \wedge \Sigma_{a_1a_2a_3} - \frac{i}{2} \bar{\lambda} \Gamma^a \mathcal{D} \lambda \wedge \Sigma_a \right]$$

$$+ \int_{\mathcal{M}^{10}} \left[ \frac{1}{2} d\phi \wedge *d\phi - C^{(1)} \ast d\phi + \frac{1}{2} e^{-\phi} dB^{(6)} \wedge *dB^{(6)} + (C^{(3)} - e^{-\phi} \ast C^{(7)}) \wedge dB^{(6)} \right] + \mathcal{O}(f^4),$$

where

$$C^{(1)} = \frac{i}{2} \bar{\psi} \lambda,$$

$$C^{(3)} = \frac{1}{4} e^{-\frac{1}{2} \phi} \bar{\psi} \wedge \Gamma^{(1)} \wedge \psi + \frac{1}{4} e^{-\frac{1}{2} \phi} \bar{\psi} \wedge \Gamma^{(2)} \lambda,$$

Actually, the validity of the duality relation (113) is an implicit proof of the $U(1)$ invariance of $H^{(7)}$ (109).
\[ C^{(7)} = \frac{i}{4} e^{\frac{i}{2} \phi} \bar{\psi} \wedge \Gamma^{(5)} \wedge \psi + \frac{i}{4} e^{\frac{i}{2} \phi} \bar{\psi} \wedge \Gamma^{(6)} \lambda. \]

As in the whole paper we have hidden the quartic fermion terms under the \( O(f^4) \).

Therefore, one of the dual versions of type IIA supergravity considered above is an \( N = 2 \) generalization of \( N = 1, D = 10 \) supergravity by Chamseddine.

## 6 Conclusion

To summarize, we have constructed the duality–symmetric version of type IIA \( D = 10 \) supergravity which in its final form contains in addition to the standard type IIA supergravity bosonic fields also their duals. Although we have not included into consideration the quartic fermion terms this part of the action remains the same as that of the standard type IIA supergravity \[16, 28, 31, 32\]. We have analyzed the symmetry structure of this formulation and its equations of motion, and have established its relation to the conventional type IIA supergravity as well as to the doubled field formalism by Cremmer, Julia, Lü and Pope, which we lifted off–shell, to the level of the covariant actions. We have also obtained a new dual version of type IIA supergravity with the six–form gauge field instead of the NS–NS two–form which is the \( N = 2 \) extension of \( N = 1 \) \( D = 10 \) supergravity by Chamseddine.

Another possible truncation of the duality–symmetric action \[69\] to \( N = 1, D = 10 \) would be to keep, upon solving part of the duality relations, the six–form and the eight–form gauge field. Remember that the latter is dual to the dilaton. In this way one gets the dual version of \( N = 1 \) \( D = 10 \) supergravity whose superfield formulation was considered in \[39\].

One can regard the results of this paper as lifting onto the level of the proper action the on shell constructions of \[26, 34\] and \[19\]. The coupling of this duality–symmetric type IIA supergravity to the Dp–branes and to the NS5–brane can be carried out in a conventional way. Another advantage of our formulation is that the type IIA action is written in a form similar to that of type IIB supergravity \[28\], which allows one to directly verify the T–duality of the whole supersymmetric sectors of these theories.

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Appendix A. Notation and conventions

In the description of dimensional reduction (from D=11 to D=10) we use the hat symbol ` to distinguish the eleven–dimensional quantities (D=11 coordinates, fields, forms and indices), e.g. \(X^\hat{m} = (x^m, X^{11})\). We use letters from the middle of the Latin alphabet for the world indices and from the beginning of the alphabet for the Lorentz indices. We have underlined the world index of the compactified coordinate \(X^{11}\) to distinguish it from the corresponding Lorentz index like that of \(\Gamma^{11}\).

We use the mostly minus signature \(\eta_{ab} = \text{diag}(+ - - \ldots -)\). The antisymmetric \(D\)–dimensional Levi–Civita tensor \(\epsilon^{a_1 \ldots a_D}\) is defined by
\[
\epsilon^{01 \ldots (D-1)} = 1, \quad \epsilon_{01 \ldots (D-1)} = (-1)^{D-1},
\]
so that
\[
\epsilon^{a_1 \ldots a_D} \epsilon_{a_1 \ldots a_D} = (-)^{D-1} D!.
\]

For an arbitrary \(n\)–form we have
\[
F^{(n)} = \frac{1}{n!} dx^{m_n} \wedge \ldots \wedge dx^{m_1} F^{(n)}_{m_1 \ldots m_n} = \frac{1}{n!} E^{a_n} \wedge \ldots \wedge E^{a_1} F^{(n)}_{a_1 \ldots a_n},
\]
and the exterior derivative \(d = dx^m \partial_m\) acts from the right.

The Hodge star operation is defined as follows
\[
(*) F^{(n)} = \frac{\alpha_n}{n!} \epsilon^{a_1 \ldots a_D-n} \epsilon_{a_1 \ldots a_D-n} b_1 \ldots b_n F^{(n)}_{b_1 \ldots b_n},
\]
or, equivalently,
\[
(*) (E^{b_n} \wedge \ldots \wedge E^{b_1}) = \frac{\alpha_n}{(D-n)!} E^{a_{D-n}} \wedge \ldots \wedge E^{a_1} \epsilon^{a_1 \ldots a_{D-n}} b_1 \ldots b_n.
\]
This implies
\[
(*) F^{(n)} = \frac{\alpha_n}{n!(D-n)!} E^{a_{D-n}} \wedge \ldots \wedge E^{a_1} \epsilon^{a_1 \ldots a_{D-n}} b_1 \ldots b_n F^{(n)}_{b_1 \ldots b_n}.
\]

The coefficients \(\alpha_n\) can be fixed to obey
\[
\alpha_n \alpha_{D-n} = (-)^{(D-n)n+(D-1)},
\]
which provides the universal identity
\[
** = 1.
\]

In odd space–time dimensions all \(\alpha_n\) are equal to one, while in even dimensions we have a freedom in fixing their values. For instance, in \(D = 10\) we choose
\[
\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_7 = \alpha_9 = 1, \quad \alpha_6 = \alpha_8 = -1.
\]
Note that (A.5) implies
\[
(*) (dx^{n_n} \wedge \ldots \wedge dx^{n_1}) = \frac{1}{(D-n)!} \sqrt{|g|} \frac{\alpha_n}{\sqrt{|g|}} dx^{m_{D-n}} \wedge \ldots \wedge dx^{m_1} \epsilon_{m_1 \ldots m_{D-n}} n_1 \ldots n_n,
\]
In our notation
\[ dX^{m_1} \land \ldots \land dX^{m_D} = d^D X \ v^{m_1 \ldots m_D}, \]
\[ E^{a_1} \land \ldots \land E^{a_D} = d^D X \ det E \ v^{a_1 \ldots a_D}. \]
and
\[ \int_{\mathcal{M}^D} \ast F^{(n)} \land F^{(n)} = (-)^{\frac{D(D-1)}{2}} \frac{1}{n!2^{D-n}} \int d^D x \sqrt{|g|} F_{m_1 \ldots m_n} \ F^{m_1 \ldots m_n}, \quad (A.11) \]
\[ \int_{\mathcal{M}^D} \omega \land \ast \psi = \int_{\mathcal{M}^D} \psi \land \ast \omega. \quad (A.12) \]

Note also that for any contravariant vector \( V^a \) associated with the one–form \( V^{(1)} = E^a V_a \) and any \( D \)-dimensional form \( \omega^{(n)} \) one can prove the following useful identities
\[ i_V \ast \omega^{(n)} = (-)^{D-n-1} \frac{\alpha_n}{\alpha_{n+1}} \ast (\omega^{(n)} \land V^{(1)}) \],
\[ \ast i_V \omega^{(n)} = (-)^{D-n} \frac{\alpha_{n-1}}{\alpha_n} \ast (\omega^{(n)} \land V^{(1)}) \],
where the contraction is defined by
\[ i_V \omega^{(n)} \equiv \ V^a i_a \omega^{(n)} \equiv \ V^m i_m \omega^{(n)} \],
\[ i_a \omega^{(n)} = \frac{1}{(n-1)!} E^{a_1} \land \ldots \land E^{a_n} \omega^{a_1 \ldots a_{n-1}} \],
\[ i_m \omega^{(n)} = \frac{1}{(n-1)!} dx^{m_1} \land \ldots \land dx^{m_n} \omega^{m_1 \ldots m_{n-1}}. \quad (A.15) \]
In particular, in \( D=10 \) we have
\[ D = 10 : \ast (\ast H^{(7)} \land A^{(1)}) = i_A H^{(7)}, \ast (\ast F^{(4)} \land A^{(1)}) = i_A F^{(4)}. \quad (A.16) \]
One can also check that, in any \( D \),
\[ i_V \left( \omega^{(n)} - \frac{i_V \omega^{(n)} \land V^{(1)}}{V^2} \right) = 0, \quad (A.17) \]
\[ \ast i_V (\omega^{(n)} \land V^{(1)}) = i_V \ast \omega^{(n)} \land V^{(1)}. \quad (A.18) \]

**Gauge field strengths of duality–symmetric type IIA supergravity**

In addition to the gravitational field, the conventional bosonic fields are
\[ \phi(x), \ A^{(1)}(x), \ B^{(2)}(x), \ A^{(3)}(x) \]
and their duals are, respectively
\[ A^{(8)}(x), \ A^{(7)}(x), \ B^{(6)}(x), \ A^{(5)}(x). \]
The field strengths of the dual pairs are
\[ F^{(1)} = d\phi, \quad F^{(9)} = dA^{(8)} - \frac{3}{4} F^{(8)} \land A^{(1)} + \frac{1}{2} B^{(2)} \land dB^{(6)} - \frac{1}{4} F^{(6)} \land A^{(3)}, \]
\[ F^{(2)} = dA^{(1)}, \quad F^{(8)} = dA^{(7)} + F^{(6)} \land B^{(2)} + B^{(2)} \land B^{(2)} \land dA^{(3)}, \]
\[ H^{(3)} = dB^{(2)}, \quad H^{(7)} = dB^{(6)} + A^{(3)} \land dA^{(3)} - F^{(6)} \land A^{(1)}, \]
\[ F^{(4)} = dA^{(3)} - H^{(3)} \land A^{(1)}, \quad F^{(6)} = dA^{(5)} + A^{(3)} \land H^{(3)} - B^{(2)} \land dA^{(3)}. \quad (A.19) \]
Gauge field strengths in Romans’s supergravity

\[ F_{m}^{(1)} = F^{(1)} = d\phi, \quad F_{m}^{(9)} = F^{(9)} - \frac{5}{4} m A^{(9)} - \frac{1}{2} m A^{(7)} \wedge B^{(2)} + \frac{1}{16} m A^{(1)} \wedge (B^{(2)})^{4}, \]
\[ F_{m}^{(2)} = F^{(2)} - m B^{(2)}, \quad F_{m}^{(8)} = F^{(8)} - \frac{1}{12} m (B^{(2)})^{4}, \]
\[ H_{m}^{(3)} = H^{(3)} = dB^{(2)}, \quad H_{m}^{(7)} = H^{(7)} - m A^{(7)} + \frac{1}{3} m A^{(1)} \wedge (B^{(2)})^{3}, \]
\[ F_{m}^{(4)} = F^{(4)} + \frac{1}{7} m (B^{(2)})^{2}, \quad F_{m}^{(6)} = F^{(6)} - \frac{1}{3} m (B^{(2)})^{3}, \]
\[ \text{(A.20)} \]

where \( m \) is the mass parameter and the field strengths \( F^{(n)} \) and \( H^{(n)} \) are defined in eq. \[ \text{(A.19)} \].

When the mass parameter \( m \) is promoted to the field \( F^{(0)}(x) \), one should replace \( m \) with \( F^{(0)}(x) \) in the definition of the field strengths \[ \text{(A.20)} \] and introduce the field strength \( F_{m}^{(10)} \) dual to \( F^{(0)}(x) \)

\[ F_{m}^{(10)} = dA^{(9)} + B^{(2)} \wedge F^{(8)} - \frac{1}{2} (B^{(2)})^{2} \wedge F^{(6)} - \frac{1}{3} (B^{(2)})^{3} \wedge dA^{(3)} - \frac{1}{60} F^{(0)} (B^{(2)})^{5}. \]

Appendix B. Dimensional reduction of Einstein-Hilbert term

To dimensionally reduce the action for \( D = 11 \) supergravity, let us begin with a general representation of the \( D \)-dimensional line element

\[ ds^2 = g_{mn}^{(D)} dx^m \otimes dx^n = e^{2\phi(x)} g_{mn}^{(D-1)} dx^m \otimes dx^n - e^{2\beta \phi} (dx^{11} + A^{(1)}) \otimes (dx^{11} + A^{(1)}), \]
\[ \text{(B.1)} \]

where, as in \( D = 11 \), we have defined the compactified coordinate by \( X^{11} \). This choice corresponds to the following splitting of the \( D \)-dimensional vielbein one–form

\[ \hat{E}^a = e^{\alpha \phi(x)} dx^m E^a_m(x), \quad \hat{E}^{11} = e^{\beta \phi(x)} (dx^{11} + A^{(1)}). \]
\[ \text{(B.2)} \]

The torsion two–form is

\[ \hat{T}^\alpha := d\hat{E}^\alpha - \hat{E}^\beta \wedge \hat{\omega}_\beta^\alpha = \frac{i}{4} \hat{\Psi} \hat{\Gamma}^\alpha \wedge \hat{\Psi}. \]
\[ \text{(B.3)} \]

Splitting the indices and using the following ansatz for the gravitino field

\[ \hat{\Psi} = e^{-\frac{7\alpha + 2}{2} \phi} (\psi + \alpha \Gamma^{(1)} \Gamma^{11} \lambda) + e^{-\frac{7\alpha + 4}{2} \phi} \lambda (dx^{11} + A^{(1)}), \]
\[ \text{(B.4)} \]

we derive the components of the connection one–form defining the curvature two–form \( \hat{R}^{\hat{a}\hat{b}} = d\hat{\omega}^{\hat{a}\hat{b}} - \hat{\omega}^{\hat{c}\hat{d}} \wedge \hat{\omega}_{\hat{c}\hat{d}} \)

\[ \hat{\omega}_{bc}^a = -e^{-\alpha \phi} (\omega_{bc}^a + 2\alpha \partial_b\phi \delta_c^a) + \frac{i}{4} e^{-(\beta + 9\alpha) \phi} (\bar{\psi}_b \Gamma^a \psi_c - 2\alpha \bar{\psi}_b (\Gamma^a \partial_b \psi_c)) - 2\alpha \bar{\psi}_b \bar{\psi}_c \Gamma^{11} \lambda - \alpha^2 \bar{\lambda} \Gamma^a \psi_c \lambda, \]
\[ \hat{\omega}^{11} = -\beta e^{-\alpha \phi} \partial_b \phi - \frac{i}{2} \beta e^{-\alpha \phi} \bar{\psi}_b \Gamma^{11} \lambda, \]
\[ \hat{\omega}^{11}_{ab} = \frac{1}{2} e^{(\beta - 2\alpha) \phi} F^{(2)}_{ab} + \frac{i}{4} e^{-(\beta + 9\alpha) \phi} (\bar{\psi}_a \Gamma^{11} \psi_b + 2\alpha \bar{\psi}_a \Gamma^b \lambda + \alpha^2 \bar{\lambda} (\Gamma_{ab} \Gamma^{11}) \lambda), \]
\[ \text{(B.4)} \]
\[ \hat{\omega}_{11ab} = \hat{\omega}_{ab}^{11} - \frac{i}{2} e^{-\alpha\phi} (\beta \hat{\psi}_a \Gamma_b \lambda + \alpha \beta \hat{\lambda} (\Gamma_{ab} \Gamma^{11}) \lambda), \] (B.5)

where \( F_{ab}^{(2)} \) is the field strength of the KK vector field \( A^{(1)} \).

Since, up to a surface term, the torsion enters the Einstein–Hilbert action only in quadratic combinations (see e.g. [32]) one can neglect it (and, hence, the fermion inputs into the spin connection) in the quadratic fermion approximation.

By use of the Palatini identity [15] (the numerical coefficient \( \Delta \) is equal to zero for \( D=11 \))

\[ \int_{M^D} \frac{1}{(D-2)!} e^{\Delta \phi} \hat{R}^{\hat{\alpha}_1 \hat{\alpha}_2} (\hat{\omega}) \wedge \hat{E}^{\hat{\alpha}_3} \wedge \ldots \wedge \hat{E}^{\hat{\alpha}_D} \epsilon_{\hat{\alpha}_1 \ldots \hat{\alpha}_D} \equiv (-)^D \int d^D x \sqrt{-g} e^{\Delta \phi} \hat{R} \]

\[ = (-)^D \int d^D x \text{ det } \hat{E} e^{\Delta \phi} [\hat{\omega}_{\hat{b}}^{\hat{b}} \hat{\omega}_{\hat{c}}^{\hat{c}} \hat{\phi} + \hat{\omega}_{\hat{a}}^{\hat{b}} \hat{\omega}_{\hat{b}}^{\hat{c}} \hat{\phi} + 2 \Delta \hat{\omega}_{\hat{b}}^{\hat{b}} \hat{\phi}] \] (B.6)

and the expression for the connection coefficients [B.3], after some algebra we arrive at the following intermediate form of the dimensionally reduced Einstein–Hilbert term

\[ \int_{M^D} F_{EH}^{(D)} = (-)^D \int dX^{11} \int d^{D-1} x \ e^{(D-3)\alpha + \beta) \phi} \sqrt{\text{det } g} \{ \omega_{b}^{a} \omega_{c}^{c} \omega_{a} + \omega_{b}^{b} \omega_{c}^{c} \omega_{a}^{a} 
\]

\[ + 2(\alpha(D-3) + \beta) \omega_{b}^{ba} \partial_{a} \phi + (\alpha(D-2) + \beta)^2 (\partial \phi)^2 - \alpha^2(D-2)(\partial \phi)^2 - \beta^2(\partial \phi)^2 
\]

\[ + \frac{1}{4} e^{2(\beta-\alpha)\phi} F_{ab}^{(2)} F^{(2)ab} \}. \] (B.7)

Applying the Palatini identity backwards, we finally obtain

\[ \int_{M^D} \frac{1}{(D-2)!} e^{\Delta \phi} \hat{R}^{\hat{\alpha}_1 \hat{\alpha}_2} (\hat{\omega}) \wedge \hat{E}^{\hat{\alpha}_3} (d) \wedge \ldots \wedge \hat{E}^{\hat{\alpha}_D} (d) \epsilon_{\hat{\alpha}_1 \ldots \hat{\alpha}_D} \]

\[ = (-)^D \int d^{(D-1)} x \ e^{((D-3)\alpha + \beta) \phi} \sqrt{\text{det } g} \{ R + (\alpha(D-2) + \beta)^2 (\partial \phi)^2 
\]

\[ - \alpha^2(D-2)(\partial \phi)^2 - \beta^2(\partial \phi)^2 + \frac{1}{4} e^{2(\beta-\alpha)\phi} F_{ab}^{(2)} F^{(2)ab} \}. \] (B.8)

Note that from the beginning we set the gravitational coupling constant and the compactification radius \( r = \int dX^{11} \) to one.

To get both the \( D \)-dimensional and the \( (D-1) \)-dimensional actions written in the Einstein frame, where the Einstein–Hilbert term does not include an input from dilaton(s), one should assume (see [16])

\[ \alpha^2 = \frac{1}{2(D-2)(D-3)}; \quad \beta = -(D-3)\alpha. \] (B.9)

Indeed, in this case one obtains from (B.8)

\[ \int d^D x \sqrt{-g} R = \int d^{(D-1)} x \ \sqrt{\text{det } g} \{ R - \frac{1}{2} (\partial \phi)^2 + \frac{1}{4} e^{-2(D-2)\alpha \phi} F^{(2)2} \}. \] (B.10)

Recall that there is no dilaton in \( D = 11 \) supergravity multiplet and hence \( \Delta_{D=11} = 0 \). With the choice of \( \alpha = +1/12 \) one gets also the Einstein frame form for the 10–dimensional action,

\[ \int d^{11} x \sqrt{-g} R = \int d^{10} x \ \sqrt{\text{det } g} \{ R - \frac{1}{2} (\partial \phi)^2 + \frac{1}{4} e^{-\frac{5}{6}} F^{(2)2} \}. \] (B.11)
Appendix C. Dimensional reduction of antisymmetric tensor fields

Under dimensional reduction from $D$ to $D - 1$ the $n$-form potential $\hat{A}^{(n)}$ decomposes as

$$\hat{A}^{(n)} = A^{(n)} - A^{(n-1)} \wedge dX^{11},$$  \hspace{1cm} (C.1)

where $X^{11}$ is the compactified dimension, and the field strength $\hat{F}^{(n+1)} = d\hat{A}^{(n)}$ is

$$\hat{F}^{(n+1)} = dA^{(n)} + dA^{(n-1)} \wedge dX^{11} = F^{(n+1)} + F^{(n)} \wedge (dX^{11} + A).$$  \hspace{1cm} (C.2)

By use of Appendix A and the representation for the interval $[B.1]$ one gets the following expression for the dual field strength

$$\hat{*F}^{(n)} = (-) D \frac{\hat{\alpha}_n}{\alpha_n} e^{(D-2n-1)\alpha\phi + \beta\phi} * F^{(n)} \wedge (dX^{11} + A) + (-)^{n-1} D \frac{\hat{\alpha}_n}{\alpha_{n-1}} e^{(D-2n+1)\alpha\phi - \beta\phi} * F^{(n-1)}.$$  \hspace{1cm} (C.3)

In the Einstein frame $[C.3]$ is

$$\hat{*F}^{(n)} = (-) D \frac{\hat{\alpha}_n}{\alpha_n} e^{-2(n-1)\alpha\phi} * F^{(n)} \wedge (dX^{11} + A) + (-)^{n-1} D \frac{\hat{\alpha}_n}{\alpha_{n-1}} e^{2(D-n-1)\alpha\phi} * F^{(n-1)}.$$  \hspace{1cm} (C.4)

Taking into account this relation and separating the part containing $dX^{11}$ we get the gauge field kinetic terms in the form

$$\int_{M^D} \hat{F}^{(n)} \wedge \hat{*F}^{(n)} = (-) D \frac{\hat{\alpha}_n}{\alpha_n} \int_{M^{(D-1)}} e^{(D-2n-1)\alpha\phi + \beta\phi} F^{(n)} \wedge * F^{(n)}$$

$$- \frac{\hat{\alpha}_n}{\alpha_{n-1}} \int_{M^{(D-1)}} e^{(D-2n+1)\alpha\phi - \beta\phi} F^{(n-1)} \wedge * F^{(n-1)} \cdot \int_{M^1} dX^{11}. \hspace{1cm} (C.5)$$

In the Einstein frame, i.e. with $[B.9]$, eq. $(C.5)$ becomes

$$\int_{M^D} \hat{F}^{(n)} \wedge \hat{*F}^{(n)} = (-) D \frac{\hat{\alpha}_n}{\alpha_n} \int_{M^{(D-1)}} e^{-2(n-1)\alpha\phi} F^{(n)} \wedge * F^{(n)}$$

$$- \frac{\hat{\alpha}_n}{\alpha_{n-1}} \int_{M^{(D-1)}} e^{2(D-n-1)\alpha\phi} F^{(n-1)} \wedge * F^{(n-1)} \cdot \int_{M^1} dX^{11}. \hspace{1cm} (C.6)$$

Appendix D. Useful identities

The variational problem for the duality–symmetric part of the supergravity action may be simplified by considering some special identities which hold for any values of space-time dimension $D$ and of the rank $n$ of differential forms. To be precise, let us consider the following variation

$$\delta(v \wedge F^{(n)} \wedge i_v F^{(D-n)})$$  \hspace{1cm} (D.1)

with the one–form $v$ defined in $[D.9]$ and $F^{(D-n)} = F^{(D-n)} - \beta * F^{(n)}$. We have

$$\delta(v \wedge F^{(n)} \wedge i_v F^{(D-n)}) = \delta v \wedge F^{(n)} \wedge i_v F^{(D-n)} + v \wedge \delta F^{(n)} \wedge i_v F^{(D-n)}$$

$$+ v \wedge F^{(n)} \wedge i_{\delta v} F^{(D-n)} + v \wedge F^{(n)} \wedge i_v \delta F^{(D-n)}$$  \hspace{1cm} (D.2)
Consider now the first term of the last line of (D.2)

$$v \wedge F^{(n)} \wedge i_{\delta \nu} F^{(D-n)} = v \wedge F^{(n)} \wedge i_{\delta \nu} F^{(D-n)} - \beta v \wedge F^{(n)} \wedge i_{\delta \nu} F^{(n)}. \quad (D.3)$$

Using the conventions and identities listed in Appendix A, by straightforward calculations we get

$$v \wedge F^{(n)} \wedge i_{\delta \nu} F^{(D-n)} = - (-)^{D(n+1)} \delta v \wedge v \wedge i_{\nu} F^{(n)} \wedge i_{\nu} F^{(D-n)} \quad (D.4)$$

and

$$v \wedge F^{(n)} \wedge i_{\delta \nu} F^{(n)} = - (-)^{D(n+1)} \delta v \wedge v \wedge i_{\nu} F^{(n)} \wedge i_{\nu} F^{(n)}. \quad (D.5)$$

The last term in (D.2) is

$$v \wedge F^{(n)} \wedge i_{\nu} \delta F^{(D-n)} = v \wedge F^{(n)} \wedge i_{\nu} \delta F^{(D-n)} - \beta v \wedge F^{(n)} \wedge i_{\nu} \delta F^{(n)}. \quad (D.6)$$

The first term in the *r.h.s* can be simplified using the identity

$$0 = i_{\nu}(v \wedge F^{(n)} \wedge \delta F^{(D-n)}) = v \wedge F^{(n)} \wedge i_{\nu} \delta F^{(D-n)}$$

$$+ (-)^{D-n} v \wedge i_{\nu} F^{(n)} \wedge \delta F^{(D-n)} - (-)^D F^{(n)} \wedge \delta F^{(D-n)}, \quad (D.7)$$

(remember that $i_{\nu} v = -1$), namely

$$v \wedge F^{(n)} \wedge i_{\nu} \delta F^{(D-n)} = (-)^D F^{(n)} \wedge \delta F^{(D-n)} - (-)^{D-n} v \wedge i_{\nu} F^{(n)} \wedge \delta F^{(D-n)}. \quad (D.8)$$

In view of (D.7), assuming that $\delta g_{mn} = 0$, one gets for the second term of (D.6)

$$v \wedge F^{(n)} \wedge i_{\nu} \delta F^{(D-n)} = - (-)^{D-n} i_{\nu}(v \wedge F^{(n)} \wedge \delta F^{(D-n)}) =$$

$$= - (-)^{D-n} i_{\nu} F^{(n)} \wedge \delta F^{(D-n)} - (-)^{D(n+1)+n} i_{\nu} F^{(n)} \wedge \delta F^{(n)} \quad (D.9)$$

Then, using the identity (A.18), $i_{\nu}(F^{(n)} \wedge v) = i_{\nu} F^{(n)} \wedge v$, one finally obtains

$$v \wedge F^{(n)} \wedge i_{\nu} \delta F^{(n)} = (-)^D \delta v \wedge i_{\nu} F^{(n)} \wedge \delta F^{(n)}, \quad (D.10)$$

which completes the reduction of the variational problem for a PST action to the standard one.

**Appendix E. Gamma–matrix conventions**

We use the following conventions for the Gamma matrices (see, e.g., [47])

$$\{\Gamma^a, \Gamma^b\} = 2\eta^{ab},$$

$$\Gamma_{a_1 \ldots a_k} = 2^m \sum_{l=0}^{\min(j,k)} l! \left( \begin{array}{c} j \\ l \end{array} \right) \left( \begin{array}{c} k \\ l \end{array} \right) \delta_{[b_1}^{a_1} \ldots \delta_{b_l}^{a_l} \Gamma_{a_{j+1} \ldots a_k]}, \quad \left( \begin{array}{c} j \\ l \end{array} \right) = \frac{j!}{l!(j-l)!},$$

$$\Gamma^{(n)} = \frac{1}{n!} F_{a_1 a_2 \ldots a_n} \ldots F_{a_1 a_2 \ldots a_n} \Gamma_{a_1 a_2 \ldots a_n}$$

in any space–time dimension.
In $D = 11$ we define
\[
\Gamma_{\hat{a}_1...\hat{a}_n} = \frac{i}{(11-n)!} (-)^{(11-n)(10-n)} 2 \epsilon_{\hat{a}_1...\hat{a}_n\hat{b}_1...\hat{b}_{11-n}} \Gamma_{\hat{b}_1...\hat{b}_{11-n}}.
\]

In ten space–time dimensions we also have
\[
\Gamma^{11} = -i \Gamma^0 \Gamma^1 ... \Gamma^9, \quad \{\Gamma^a, \Gamma^{11}\} = 0, \quad (\Gamma^{11})^2 = -1,
\]
and
\[
\bar{\psi}_1 \Gamma_{\hat{a}_1...\hat{a}_n} \psi_2 = (-)^n \bar{\psi}_2 \Gamma_{\hat{a}_n...\hat{a}_1} \psi_1,
\]
\[
\bar{\psi}_1 \Gamma_{\hat{a}_1...\hat{a}_n} \Gamma^{11} \psi_2 = -\bar{\psi}_2 \Gamma_{\hat{a}_n...\hat{a}_1} \Gamma^{11} \psi_1
\]
for two Majorana spinors.
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