Detecting Extra Dimension by Helium-like Ions

Yu-Xiao Liu ∗, Xin-Hui Zhang †, Yi-Shi Duan

Institute of Theoretical Physics, Lanzhou University, Lanzhou 730000, China

Considering that gravitational force might deviate from Newton’s inverse-square law and become much stronger in small scale, we present a method to detect the possible existence of extra dimensions in the ADD model. By making use of an effective variational wave function, we obtain the nonrelativistic ground energy of a helium atom and its isoelectronic sequence. Based on these results, we calculate gravity correction of the ADD model. Our calculation may provide a rough estimation about the magnitude of the corresponding frequencies which could be measured in later experiments.

Keywords: Extra dimensions; Newton’s inverse-square law; Variational method.

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There are at least two seemingly functional energy scales in nature, the electroweak scale $m_{EW} \sim 10^2$ GeV and the Planck scale $M_{pl} \sim 10^{19}$ GeV, where gravity becomes as strong as the gauge interactions. Over the last two decades, explaining the smallness and radiative stability of the hierarchy $m_{EW}/M_{pl} \sim 10^{-17}$ has been one of the greatest driving forces behind the construction of theories beyond the Standard Model (SM). N. Arkani-Hamed, S. Dimopoulos and G. Dvali proposed an exciting model (ADD model) 1, 2, 3. In the ADD model, a very simple idea is to suppose there are $n$ extra compact spatial dimensions of radius $R$ and to assume that $M_{pl(4+n)}$ is around the scale of $m_{EW}$. The weakness of four dimensional gravitational force is explained as that the origin strong high dimensional one leaks into extra dimensions. If two test masses of mass $m_1$, $m_2$ placed within a distance $r \ll R$, they will fell a gravitational potential dictated by Gauss’s law in $(4+n)$ dimensions

$$V(r) \sim \frac{m_1 m_2}{M_{pl(4+n)}^n r^{n+1}}, \quad (r \ll R)$$  \hspace{1cm} (1)

On the other hand, if the masses are placed at distances $r \gg R$, their gravitational flux lines can not continue to penetrate in the extra dimensions, and the usual $1/r$ potential is obtained

$$V(r) \sim \frac{m_1 m_2}{M_{pl(4+n)}^n R^n r}, \quad (r \gg R)$$  \hspace{1cm} (2)

∗Email: liuyx@lzu.edu.cn
†Corresponding author. Email: zhangxinxh03@lzu.cn
in which the radius of extra dimensions \( R \) is determined by \( M_{pl(4+n)} \) as

\[
R = \frac{1}{\pi} M_{pl(4+n)}^{-\left(\frac{1}{2}\right)} G^{\frac{1}{2}}. \quad (c = 1, \hbar = 1)
\]  

(3)

If \( M_{pl(4+n)} \approx 1\text{TeV} \), one can obtain \( R \approx 10^{-17+\frac{2n}{c}} \) cm. For \( n = 1 \), \( R \approx 10^{12} \) cm implying deviations from Newtonian gravity over solar system distances, so this case is empirically excluded. Particularly, the \( n = 2 \) case implies sub-millimeter extra dimensions, while the experimental conditions for testing Newton’s inverse-square law (ISL) by torsion pendulum method was just about to be available when ADD’s proposal appeared. In fact, for all \( n \geq 2 \), however, the modification of gravity only becomes noticeable at distances smaller that those currently probed by experiments will be performed in the very near future. Therefore, many people have devoted to the search of deviation from ISL as well as extra dimensions during the past few years [14]. Luo and Liu [15] first proposed the idea of detecting the deviation of Newton’s inverse-square law as well as exploring extra dimensions by spectroscopy experiments and took hydrogen-like and muonic atomic system as illustrations. Here we wonder if one can detect the extra dimension by spectroscopic experiments due to a helium or helium-like ions.

From the early days of the quantum mechanics the ground state ionization energy of a helium atom was a benchmark for approximate methods of solving nonrelativistic Schrödinger equation for a few-body system. One of the earliest variational calculations has been performed by Hylleraas [8] in 1929. In 1957, Kinoshita [9] obtained higher order corrections including the Lamb shift calculations confirmed a very good agreement with the best experimental value. We would like to mention here the two most recent calculations. The first is aimed to elaborate an efficient variational method for the many electron atoms. The second is to find an effective and economical way for studying a helium and helium-like two electron atoms. Several techniques for example the finite-element method [10] and the hyperspherical harmonic method [11] have been used for such system, but variational method is the most powerful tool for studying this problem and much work has been carried out [12,13,14,15,16].

In this paper, we adopt a simple effective trial wave function and calculate the variational ground state energy of a helium atom and helium-like ions. Based on this trial wave function and making use of the perturbation method, we obtain the gravity corrections of the ADD model and the contribution to spectroscopy. In these calculations, we use the Mathematic software, which is a tool of symbolic calculation. All the results are exact except for that of the wave function cutoff, which would bring an error but the error can be controlled by setting the precision.

It is known that the non-relativistic Schrödinger equation for two-electron helium-like systems with nuclear charge \( Z \) can be written as (in atomic units)

\[
H \psi(r_1, r_2) = E \psi(r_1, r_2),
\]

(4)

\[
H = \frac{1}{2} p_1^2 + \frac{1}{2} p_2^2 + \frac{1}{M} p^2 - \frac{Z}{r_1} - \frac{Z}{r_2} + \frac{1}{r_{12}},
\]

(5)
where $p_{1,2}$ are the momenta of the electrons, $P = -p_1 - p_2$ is the momentum of the nucleus, $r_1$ and $r_2$ denote the positions of the electrons with respect to the nucleus, and the finite nucleus-to-electron mass ratio is given by $M \equiv m_{\text{nucl}}/m_e$. To solve the above Schrödinger equation for the ground states, we employ variational approach by finding the stationary solutions of the following energy functional

$$E = \min \int \frac{\psi^* H \psi}{\int \psi^* \psi} d\tau,$$

(6)
in which the volume element $d\tau = 8\pi^2 r_1 r_2 dr_1 dr_2 dr_{12}$. In order to perform variational calculation, we need to make a judicious choice for the wave function. To this end we adopt a simply effective variational wave function, which contains a flexible scaling parameter $k$

$$\psi(k_s, k_t, k_u) = e^{-k_s} \sum C_{lmn} \{ (k_t)^{2m} (k_u)^n \},$$

(7)

where $s, t$ and $u$ are called the Hylleraas coordinates and the number of the terms of the polynomial in $\psi$ is denoted as $N$. Under the Hylleraas coordinates, the kinetic energy $K$, the potential energy $P$ and the denominator $W = \int \psi^* \psi d\tau$ in Eq. (6) can be expressed as follows, respectively

$$K = \int_0^\infty ds \int_0^s du \int_{-u}^u dt \pi^2 \left\{ u(s^2 - t^2)^2 \left( \partial_s \psi \right)^2 + (\partial_u \psi)^2 + (\partial_t \psi)^2 \right\} - 2s(t^2 - u^2) \partial_s \psi \partial_u \psi + 2t(s^2 - u^2) \partial_t \psi \partial_u \psi,$$

(9)

$$P = -\int_0^\infty ds \int_0^s du \int_{-u}^u dt \pi^2 (4Zsu - s^2 + t^2) \psi^2,$$

(10)

$$W = \int_0^\infty ds \int_0^s du \int_{-u}^u dt \pi^2 u(s^2 - t^2) \psi^2.$$

(11)

Substituting Eqs. (7), (8) into Eq. (6) and using the variational method, we calculate the ground state energy of a helium atom and helium-like ions, and obtain the approximate wave function $\psi$. For definition and simplicity, in the following calculation, we take $N = 10$.

The above work is performed by a program, which is written by Mathematica Language. In what follows, based on the above calculating results, we investigate the spectroscopy correction from the ADD gravitational potential energy. Even if gravity would become stronger in small scale, it is still very weak compared to electromagnetic force. So it is convenient to treat the gravitational potential as a perturbation to calculate the energy correction of a helium atom and helium-like ions. The correction of the ground state energy is written as

$$\Delta E = 8\pi^2 \int \psi^* \hat{V}(r_1, r_2, r_{12}) \psi r_1 r_2 dr_1 dr_2 dr_{12}.$$  

(12)
The gravitational potential is given by

\[ \hat{V} = \begin{cases} - \sum_{i=1}^{2} \frac{G\left(m_{i}\right)m_{\text{nuc}}}{r_{i}^{n+1}}, & r_{1,2} \ll R \\ - \sum_{i=1}^{2} \frac{G\left(m_{i}\right)m_{\text{nuc}}}{r_{i}} + \frac{G\left(m_{12}\right)m_{\text{nuc}}}{r_{12}}, & r_{1,2} \sim R \\ - \sum_{i=1}^{2} \frac{G\left(m_{i}\right)m_{\text{nuc}}}{r_{i}}, & r_{1,2} \gg R \end{cases} \]  

(13)

We consider the potential for the case of compactification on an n-dimensional torus and assume that the space-time is (4 + n)-dimensional, where the n extra dimensions are compactified on circles with radius R. Then the (4 + n)-dimensional Newton’s constant is expressed as

\[ G_{(4+n)} = \frac{2v_{n}G_{4}}{s_{n}}, \]  

(14)

where \( v_{n} = (2\pi R)^{n} \) is the volume of n torus and \( s_{n} = 2\pi^{\frac{n+1}{2}}/\Gamma\left(\frac{n+1}{2}\right) \) is its surface area. \( (\Gamma(n) = \int_{0}^{\infty} e^{-x}x^{n-1}dx) \). When \( r \sim R \), for the shape of extra dimensions mentioned above, the familiar Yukawa potential is adopted \( \alpha = \frac{8n}{3} \) and \( \lambda = R \).

One can notice that the integral diverges as \( r_{1,2} \rightarrow 0 \). For \( n \geq 2 \), considering the atomic nucleus is not pointlike, we introduce a safe cutoff value \( r_{m} \) with the atomic nucleus size for the lower limit of the integral. A convenient expression of \( r_{m} \) is \( r_{m} = r_{0}A^{\frac{1}{3}} \), where \( A \) is the mass number of the atomic nucleus and \( r_{0} \) is of size \( \sim 10^{-13} \) cm.

If we consider the n-dimensional sphere of radius R as our compactification manifold, the Newton constant is

\[ G_{(4+n)} = 2RG_{4}. \]  

(15)

For the case of large extra dimensions, let us compactify the n dimensions \( y_{\alpha} \) by making the periodic identification \( y_{\alpha} \sim y_{\alpha} + L \). Following exactly the same procedure as Ref. 3, one can obtain

\[ G_{(4+n)} = \frac{4\pi V_{n}G_{4}}{S_{(3+n)}}, \]  

(16)

where \( V_{n} = L^{n} = (2\pi)^{n} \) and \( S_{D} = 2\pi^{\frac{D}{2}}/\Gamma\left(\frac{D}{2}\right) \) is the surface area of the unit sphere in D spatial dimensions. From Eqs. (14, 15, 16), since \( \frac{2v_{n}}{s_{n}} \sim R^{n} \sim \frac{4\pi V_{n}}{S_{(3+n)}}, \) we only take example for an n-dimensional torus.

We apply the above wave function in Eq. (7) to two-electron atomic and ionic systems. In Table 1, we quote the values of the ground state energy for a helium atom and its isoelectronic sequence. For the helium atom we also quote the values of Hartree-Fock (HF) theory\(^{19}\), the 3-parameter Charatzoulas-Knowleswave (CK) function\(^{20}\), the variational perturbation results of Pan\(^{21}\) and the 1078-parameter Pekeris wave function\(^{22}\). For other ions, we compare our calculated values with the variational perturbation results of Pan\(^{21}\) and Aashamar\(^{23}\). The relative errors compared with Pekeris and Aashamar are also given in the last column.

**Table 1.** Nonrelativistic ground energies of a helium atom and helium-like ions calculated with \( \psi \) in Eq. (7) in atomic units and their comparisons with other
references. For the helium atom, the relative error is compared with the value of Pekeris. For the helium-like ions, the relative errors are compared with that of Aashamar.

| System | Wave function | Ground energy | Error(%) |
|--------|---------------|---------------|----------|
| He     | $\psi$        | $-2.90313$    |          |
|        | HF $^{19}$    | $-2.86168$    |          |
|        | CK $^{20}$    | $-2.89007$    |          |
|        | Pan $^{21}$   | $-2.89122$    |          |
|        | Pekeris $^{22}$| $-2.90372$    |          |
| Li$^+$ | $\psi$        | $-7.27906$    |          |
|        | Pan           | $-7.26820$    | 0.0117   |
|        | Aashamar $^{23}$| $-7.27991$    |          |
| Be$^{2+}$ | $\psi$    | $-13.65440$   |          |
|         | Pan           | $-13.64416$   | 0.0086   |
|         | Aashamar      | $-13.65557$   |          |
| B$^{3+}$ | $\psi$        | $-22.02949$   |          |
|         | Pan           | $-22.01973$   | 0.0067   |
|         | Aashamar      | $-22.03097$   |          |
| C$^{4+}$ | $\psi$        | $-32.40438$   |          |
|         | Pan           | $-32.39511$   | 0.0058   |
|         | Aashamar      | $-32.40625$   |          |
| N$^{5+}$ | $\psi$        | $-44.77929$   |          |
|         | Pan           | $-44.77035$   | 0.0048   |
|         | Aashamar      | $-44.78145$   |          |
| O$^{6+}$ | $\psi$        | $-59.15416$   |          |
|         | Pan           | $-59.14554$   | 0.0041   |
|         | Aashamar      | $-59.15660$   |          |

Furthermore, by making use of the above wave function, we calculate the frequencies of the spectrum corresponding to the ADD gravity correction. For a helium atom, we get $\Delta \nu \sim 10^{-6}$Hz with $n = 2$, and $\Delta \nu \sim 10^{-11}$Hz with $n = 3$. Although these corrections are much larger than the one calculated from the exact ISL ($\Delta \nu \sim 10^{-23}$) for a helium atom, they are still too small to be observed in current experiments. So we must find some “large” corrections that cannot be simply explained by the inconsistency of the accurate experimental data and Standard Model theoretical values. We can choose high-Z helium-like systems (such as Ca$^{18+}$ and Pb$^{80+}$) to do the similar calculation. In Table 2, the frequencies of the spectrum for two-electron atom and ions corresponding to the ADD gravity corrections are presented in detail. However, we should point out that for Pb$^{80+}$, since the corresponding Bohr’s radius is smaller than that of a helium atom, the effect of gravity from the nucleus is much more important. Anyway, if we use the
simple perturbation theory, the main correction still comes from the Schrödinger term. Because the masses of Ca and Pb are representative about 40 and 207 times of the mass of proton, the corrections of the frequencies for Ca$^{18+}$, Pb$^{80+}$ are much larger. For Ca$^{18+}$, we have $\Delta \nu \sim 10^{-2}\text{Hz}$ with $n = 2$, and $\Delta \nu \sim 10^{-7}\text{Hz}$ with $n = 3$; For Pb$^{80+}$, $\Delta \nu \sim 10^{0}\text{Hz}$ with $n = 2$ and $\Delta \nu \sim 10^{-5}\text{Hz}$ with $n = 3$, while the corresponding corrections in hydrogen atom are $10^{-8}\text{Hz}$ and $10^{-13}\text{Hz}$ respectively. Also note that the exact ISL, that is, the $n = 0$ case gives a correction as small as $\Delta \nu \sim 10^{-24}\text{Hz}$ for Hydrogen atom and $\Delta \nu \sim 10^{-23}\text{Hz}$ for a helium atom.

| System   | $\Delta \nu(\text{Hz})$ ($n = 2$) | $\Delta \nu(\text{Hz})$ ($n = 3$) |
|----------|----------------------------------|----------------------------------|
| He       | $-1.1 \times 10^{-6}$            | $-1.5 \times 10^{-11}$          |
| Li$^+$   | $-6.7 \times 10^{-6}$            | $-9.9 \times 10^{-11}$          |
| Be$^{2+}$| $-2.1 \times 10^{-5}$            | $-3.2 \times 10^{-10}$          |
| B$^{3+}$ | $-5.0 \times 10^{-5}$            | $-7.9 \times 10^{-10}$          |
| C$^{4+}$ | $-9.5 \times 10^{-5}$            | $-1.5 \times 10^{-9}$           |
| N$^{5+}$ | $-1.8 \times 10^{-4}$            | $-2.9 \times 10^{-9}$           |
| O$^{6+}$ | $-3.0 \times 10^{-4}$            | $-5.0 \times 10^{-9}$           |
| F$^{7+}$ | $-4.9 \times 10^{-4}$            | $-8.5 \times 10^{-9}$           |
| Ca$^{18+}$| $-1.1 \times 10^{-2}$            | $-2.0 \times 10^{-7}$           |
| Pb$^{80+}$| $-3.1 \times 10^{0}$            | $-7.4 \times 10^{-5}$          |

Table 2. The frequencies of the spectrum for two-electron atom and ions corresponding to the ADD gravity corrections

We have referred to many references about experiment values of fine structure interval in helium or helium-like ions, which collected in table 3. Comparing these experiment values to Table 2, we find that even for Pb$^{80+}$, the large shift can not yet been detected because of the precision of the experiment. So, the ADD model can not yet be ruled out in this way. We expect some experiments will be done in this way in the future.

To summarize, we have formulated a calculational scheme for detecting the extra dimension by using the helium and helium-like systems. To perform the variational calculation we have used the variational forms with a flexible parameter $k$ for two-electron correlated wave function taking into account the motion of the nucleus. The results obtained by us for the ground state is quite accurate. Furthermore, based on the results, we illustrate the method numerically by calculating the magnitude of the corresponding frequencies. For Pb$^{80+}$, the frequency derived from the ADD gravity correction is $\Delta \nu \sim 10^{0}\text{Hz}$ with $n = 2$ and $\Delta \nu \sim 10^{-5}\text{Hz}$ with $n = 3$. The corrections might be used to indirectly detect the deviation of ISL down to nanometer scale and to explore the possibility of two or three extra dimensions in
ADD’s model, while current direct gravity tests cannot break through micron scale and go beyond two extra dimensions scenario.

Table 3. Experimental fine structure intervals for $He$, $Li^+$, $Be^{2+}$, $B^{3+}$, $F^{7+}$.

| System | Interval | Experiment | Reference |
|--------|----------|------------|-----------|
| $He$   | $v_{01}$ | 29616943.01(17) kHz | Ref. [24] |
|        | $v_{12}$ | 2291161.13(30) kHz | |
|        | $v_{01}$ | 2961951.66 ± 0.70 kHz | Ref. [25] |
|        | $v_{12}$ | 229175.59 ± 0.51 kHz | |
|        | $v_{01}$ | 29616950.9 ± 0.9 kHz | Ref. [26] |
| $Li^+$ | $v_{01}$ | 155704.2161(30) MHz | Ref. [27] |
|        | $v_{12}$ | −62678.3382(27) MHz | |
| $Be^{2+}$ | $v_{01}$ | 11.5576605(7) cm$^{-1}$ | Ref. [27] |
|         | $v_{12}$ | −14.892209(1) cm$^{-1}$ | |
|         | $v_{01}$ | 11.5586(5) cm$^{-1}$ | Ref. [28] |
|         | $v_{12}$ | −14.8950(4) cm$^{-1}$ | |
| $B^{3+}$ | $v_{01}$ | 16.197573(2) cm$^{-1}$ | Ref. [27] |
|         | $v_{12}$ | −52.661199(4) cm$^{-1}$ | |
|         | $v_{01}$ | 16.203(18) cm$^{-1}$ | Ref. [30] |
|         | $v_{12}$ | −52.660(16) cm$^{-1}$ | |
| $F^{7+}$ | $v_{01}$ | −151.2466(1) cm$^{-1}$ | Ref. [27] |
|         | $v_{12}$ | −957.8487(2) cm$^{-1}$ | |
|         | $v_{12}$ | −957.883(19) cm$^{-1}$ | Ref. [31] |

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