ON THE INFINITE-DIMENSIONAL HIDDEN SYMMETRIES.
II. \textit{q}_R\text{-CONFORMAL MODULAR FUNCTORS.}

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Abstract. The article is devoted to the \textit{q}_R\text{-conformal modular functors, which be-
ing "deformations" of the conformal modular functors (the projective representations
of the category Train(Diff_+(S^1)), the train of the group Diff_+(S^1) of all orientation
preserving diffeomorphisms of a circle) in the class of all projective modular func-
tors (the projective representations of the category Train(PSL(2, \mathbb{R})), the train of the
projective group PSL(2, \mathbb{R})), may be regarded as their “Berezin quantizations”.

This paper being the continuation of the first part [1] belongs to the series of
articles supplemental to [2], and also lies in lines of the general ideology exposed
in the review [3]. The main purpose of the activity, which has its origin and
motivation presumably in the author’s applied researches [4] on the interactively
controlled systems (i.e. the controlled systems, in which the control is coupled with
unknown or incompletely known feedbacks), is to explicate the essentially infinite-
dimensional aspects of the hidden symmetries, which appear in the representation
theory of the finite dimensional Lie algebras and related algebraic structures. The
relations between the control and the representation theories will be discussed in
[5]. The present series is organized as a sequence of topics, which illustrate this
basic idea on the simple and tame examples without superfluous difficulties and
details as well as in the series [1] but from a bit more geometric point of view.

On the other hand this concrete article is placed at the crossing of two very dif-
ferent ideologies of hidden symmetries (however, the ideological differences may be
rather subtle in practice). The used version of the first was developed by G.Segal [6],
M.Kontsevich, K.Gawedzki [7], M.Atiyah, G.Moore and N.Seiberg [8], Yu.A.Neretin
[9; and refs wherein], E.Witten [10] and others [11]. This ideology, which was for-
mulated most clearly in purely mathematical fashion by G.Segal (pioneered the
considered version), E.Witten, M.Atiyah and to a certain extent by Yu.A.Neretin,
may be characterized as a formal search of hidden symmetries on the abstract level
and is related to the direct problems of representation theory. This ideology un-
derlies the approaches of J.Mickellson and D.P.Zhelobenko [12; and refs wherein]
to the representation theory of reductive Lie algebras and partially penetrates the
researches of V.G. Drinfeld and his numerous successors on Hopf and quasi-Hopf algebras. The second manifested itself in the research activity of many specialists in mathematical physics (e.g. in the Leningrad mathematical school’s activity on the quantum inverse scattering method [13; and refs wherein], or in the V.P. Maslov’s group investigations on nonlinear classical and quantum brackets [14,15; and refs wherein], the used below more formally mathematical version was elaborated by the author [3]. The ideology may be characterized as a search of hidden symmetries on the concrete level and is related to the inverse problems of representation theory.

I hope that the presented crossing of two very different ideologies but treating one subject will be rather interesting and crucial for the understanding of various recently formed “new looks” in the representation theory.

**Topic Three: Projective modular functors and related objects**

3.1. The projective semigroups Mantle(PSL(2, \(\mathbb{R}\))) and Voile(PSL(2, \(\mathbb{R}\))): the mantle and veil (voile) of the projective group PSL(2, \(\mathbb{R}\)) [16:App.C.1].

The projective semigroup Mantle(PSL(2, \(\mathbb{R}\))), the mantle of the projective group PSL(2, \(\mathbb{R}\)), can be realized in one of the following two ways. **Realization 1.** The elements of the semigroup Mantle(PSL(2, \(\mathbb{R}\))) are linear-fractional mappings \(f : z \mapsto (az + b)(cz + d)^{-1}\) such that \(f(D_+) \subset D_+, f' \neq 0.\) Multiplication of elements is a composition of mappings. **Realization 2.** The elements of the semigroup Mantle(PSL(2, \(\mathbb{R}\))) are domains \(K\) in \(\mathbb{C}\) that are homeomorphic to a annulus and for which the components \(\partial K_+\) and \(\partial K_-\) of the boundary \(\partial K\) are circles in \(\mathbb{C}\) with linear-fractional parametrization. The inner circle \(\partial K_+\) is parametrized in such a way that as one passes around it the domain is on the right (such parametrization is called ingoing), while for the outer circle \(\partial K_-\) the domain is on the left (outgoing parametrization). Two domains are said to be equivalent if there exists a linear-fractional automorphism of \(\hat{\mathbb{C}}\) that carries one of them into the other, with allowance for the parametrization. Multiplication of elements is their glueing along the parametrized boundaries.

The equivalence of the constructions is established as follows. If \(f\) is a linear-fractional mapping from \(D_+\) to \(D_+\), then as domain \(K\) one may consider the annulus \(D_+ \setminus \mathring{D_+}\), whose outer boundary has the standard parametrization, while the parametrization of the inner boundary is given by the mapping \(f\). Conversely, if \(K\) is an arbitrary domain that satisfies the required conditions, then there exists a unique linear-fractional automorphism of \(\hat{\mathbb{C}}\) that maps \(K\) to an annulus whose outer boundary is the unit circle with the standard parametrization. The required mapping \(f\) is determined by the parametrization of the inner component of the boundary.

The projective semigroup Mantle(PSL(2, \(\mathbb{R}\))), the mantle of the projective group PSL(2, \(\mathbb{R}\)) is a partial complexification of it (cf.[17] and also [18]) and has the complex dimension 3. An arbitrary finite-dimensional representation or infinite-dimensional representation with highest weight of the projective group PSL(2, \(\mathbb{R}\)) can be extended by holomorphicity to a representation of its mantle. This situation does not change if one consider projective representations of both objects or linear representations of their universal covers. Thus, the theory of representations of the projective group is, in the words of Yu.A., Neretin, the theory of representations of a larger semigroup that is “invisible to the unaided eye” – its mantle, the projective semigroup Mantle(PSL(2, \(\mathbb{R}\))). This reformulation of the theory of representations
of $\text{PSL}(2, \mathbb{R})$, which appears little more than tautologous (though providing us with new interesting formulas or interpretations), has great interest in that it stimulates one to look for further hidden structures “invisible to the unaided eye”, for which the theory of representations is richer than the theory of representations of the original projective group. In particular, it would desirable if the theory of the irreducible representations of this hidden structure automatically includes, in addition to the theory of the irreducible representations of the projective group, the corresponding Clebsch-Gordan coefficient calculus. The existence of such a structure was indicated for a long time by the presence of a certain similarity at the level of concrete formulas between the various objects of the theory of irreducible representations and the Clebsch-Gordan coefficient calculus. As an example, we can give the similarity of the matrix elements of the irreducible representations and the corresponding Clebsch-Gordan coefficients (see e.g.[19]).

As first step, we note that the construction of the semigroup $\text{Mantle}(\text{PSL}(2, \mathbb{R}))$ can be generalized to arbitrary Riemann surfaces. The semigroup $\text{Voile}(\text{PSL}(2, \mathbb{R}))$ that is then obtained is called the \textit{veil (voie)} of the projective group $\text{PSL}(2, \mathbb{R})$. The elements of the projective semigroup $\text{Voile}(\text{PSL}(2, \mathbb{R}))$ are triples $(R, f_+, f_-)$, where $R$ is a Riemann surface with the fixed projective structure, and $f_+: D_+ \hookrightarrow R$, $f_-: D_- \hookrightarrow R$ ($D_+ = \{z: |z| \leq 1\}$, $D_- = \{z: |z| \geq 1\}$) are holomorphic imbeddings of the complex disks $D_+$ and $D_-$ into $R$ (with allowance for the projective structure) with nonintersecting images. Multiplication of elements of the semigroup is a sewing and so is defined in the similar way as multiplication in Realization 2 of the projective semigroup $\text{Mantle}(\text{PSL}(2, \mathbb{R}))$.

The semigroup $\text{Voile}(\text{PSL}(2, \mathbb{R}))$ is a $\mathbb{Z}_+$-graded infinite-dimensional semigroup. the grading is specified by the genus of the Riemann surface. Formally, one is able to construct the noncommutative Grothendieck group $\Gamma(\text{Voile}(\text{PSL}(2, \mathbb{R})))$ of the semigroup $\text{Voile}(\text{PSL}(2, \mathbb{R}))$. The Grothendieck group $\Gamma(\text{Voile}(\text{PSL}(2, \mathbb{R})))$ is an infinite dimensional group, however, its structure was not investigated.

The theory of representations of the projective semigroup $\text{Voile}(\text{PSL}(2, \mathbb{R}))$, the veil of the projective group $\text{PSL}(2, \mathbb{R})$, is much richer than the theory of representations of its mantle. The number of representations depends on the topology introduced on the semigroup. Among all topologies, the most interesting are the following two: the ordinary topology on components of fixed grading which discretely distinguishes the components, and a topology that takes into account possible continuous changes of genus. Problems of the theory of representations of the semigroup $\text{Voile}(\text{PSL}(2, \mathbb{R}))$ as well as its Grothendieck group $\Gamma(\text{Voile}(\text{PSL}(2, \mathbb{R})))$ are very important but completely unexplored.

Some remarks should be added. The projective semigroup $\text{Voile}(\text{PSL}(2, \mathbb{R}))$, the veil of the projective group $\text{PSL}(2, \mathbb{R})$, can be regarded as a “fluctuating” exponential of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ that describes the process of evolution with continuous creation and annihilation of “virtual particles”. From this point of view, it is very interesting to study fractal (corresponding to nonperturbative effects) generalizations of the veil $\text{Voile}(\text{PSL}(2, \mathbb{R}))$ of the projective group $\text{PSL}(2, \mathbb{R})$, the elements of which can have infinite genus (cf.[20]). The corresponding completions of the Grothendieck group $\Gamma(\text{Voile}(\text{PSL}(2, \mathbb{R})))$ are also of interest; probably, some of such completions coincide with certain versions of one of the infinite dimensional classical groups. Note that the nonperturbative effects are essential for the quantum theory for a self-interacting string field [21; and refs wherein].
3.2. The manifold of projective trinions $\text{Trinion}(\text{PSL}(2, \mathbb{R}))$ and its representations. Projective vertices [16:App.C.2].

Our next step will be to adapt the ideology of trinions [8] to the discussed case.

A **projective trinion** is a quadruplet $(R, \partial R^1_+, \partial R^2_+, \partial R_-)$, where $R$ is a Riemann surface of genus $0$ ($R \subset \mathbb{C}$) equipped with a projective structure and with a boundary whose components $\partial R^1_+$, $\partial R^2_+$, $\partial R_-$ are homeomorphic to the circle $S^1$ with ingoing linear-fractional parametrization defined on $\partial R^1_+$ and $\partial R^2_+$ and outgoing linear-fractional parametrization defined on $\partial R_-$ (the existence of such parametrizations means that the real projective structures on the components of the boundary, which are the restrictions of the complex projective structure on the surface, are canonical).

The manifold of projective trinions $\text{Trinion}(\text{PSL}(2, \mathbb{R}))$ has the complex dimension $6$. The Lie group $G = \text{PSL}(2, \mathbb{R}) \times \text{PSL}(2, \mathbb{R}) \times \text{PSL}(2, \mathbb{R})$ acts on the set of trinions $\text{Trinion}(\text{PSL}(2, \mathbb{R}))$, and the corresponding action of the Lie algebra $\mathfrak{g}^C = \mathfrak{sl}(2, \mathbb{C}) + \mathfrak{sl}(2, \mathbb{C}) + \mathfrak{sl}(2, \mathbb{C})$ is transitive. The stabilizer of the trinion in the Lie algebra $\mathfrak{g}^C$ is the subalgebra of holomorphic vector fields that admit extension to the complete trinion. Note that the action of the Lie algebra $\mathfrak{g}^C$ on the manifold $\text{Trinion}(\text{PSL}(2, \mathbb{R}))$ of projective trinions can be extended to the action of the semigroup $\text{Mantle}(\text{PSL}(2, \mathbb{R})) \times \text{Mantle}(\text{PSL}(2, \mathbb{R})) \times \text{Mantle}(\text{PSL}(2, \mathbb{R}))$ with two copies of the projective semigroup $\text{Mantle}(\text{PSL}(2, \mathbb{R}))$, the mantle of the projective group $\text{PSL}(2, \mathbb{R})$, acting from the right, and one from the left. At the same time, for all $\text{Mantle}(\text{PSL}(2, \mathbb{R}))$-integrable Verma modules $V_{h_1}, V_{h_2}, V_{h_3}$ over the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ there exists no more than one projective representation of the manifold of projective trinions $\text{Trinion}(\text{PSL}(2, \mathbb{R}))$ (for the definition of a representation of a homogeneous space based on the concept of a Mackey imprimitivity system, see [22,23]) in the projective space $\mathbf{P} (\text{Hom}(V_{h_1} \otimes V_{h_2}; V_{h_3}))$ consistent with the action of the projective semigroup $\text{Mantle}(\text{PSL}(2, \mathbb{R}))$, the mantle of the projective group $\text{PSL}(2, \mathbb{R})$, in these modules. Note that one may consider as linear as projective representations of the $\text{Mantle}(\text{PSL}(2, \mathbb{R}))$, in the least case one should consider the universal cover of the manifold $\text{Trinion}(\text{PSL}(2, \mathbb{R}))$.

We now define the operation of **inserting a vertex** into an element of the mantle $\text{Mantle}(\text{PSL}(2, \mathbb{R}))$. Let $g = (R, \partial R^1_+, \partial R^2_+, \partial R_-)$ be a projective trinion, and $V_{h_1}, V_{h_2}, V_{h_3}$ be three integrable Verma modules. We denote by $A_g : V_{h_1} \otimes V_{h_2} \to V_{h_3}$ the operator (defined up to a factor) corresponding to the projective trinion $g$ in the projective representation of the manifold $\text{Trinion}(\text{PSL}(2, \mathbb{R}))$ of projective trinions in $\mathbf{P} (\text{Hom}(V_{h_1} \otimes V_{h_2}; V_{h_3}))$. Let $v$ be the highest vector in the Verma module $V_{h_1}$, $k = (K, \partial K_+, \partial K_-)$ be an element of the projective semigroup $\text{Mantle}(\text{PSL}(2, \mathbb{R}))$, the mantle of the projective group $\text{PSL}(2, \mathbb{R})$, where $K = R \cup_{\partial R^1_+} D$, $\partial K_+ = \partial R^2_+$, $\partial K_- = \partial R_-$ ($D$ is the disk bounded by $\partial R^1_+$), and $u$ be an arbitrary point in $D$. Insertion of a vertex in the element $k$ means specification of an operator $A_k(u; v)$, defined up to a factor, in $\text{Hom}(V_{h_2}; V_{h_3})$ as the limit of the family of operators $A_g$ as $D \mapsto u$.

By means of the vertex insertion operation, we can define a projective vertex. We consider an arbitrary element $k$ of the projective semigroup $\text{Mantle}(\text{PSL}(2, \mathbb{R}))$, the mantle of the projective group $\text{PSL}(2, \mathbb{R})$, with the vertex $v$ of the weight $\mu$ that is inserted at the point $u$ and acts as the operator $A_k(u; v)$ from the Verma module $V_h$ to the Verma module $V_g$. A **projective vertex** $V_\mu(u; v)$ ($u \in S^1$) is the limit of the family of operators $A_k(u; v)$ as $\partial K_+, \partial K_- \mapsto S^1$ (together with the
parametrizations). The operator field $V_\mu(u;v)$ obtained by means of this construction may be extended holomorphically with respect to $u$. The result is just the $\mathfrak{sl}(2, \mathbb{C})$--primary field of weight $\mu$ from the Verma module $V_h$ to the Verma module $V_g$. The definition and explicit formulas for the $\mathfrak{sl}(2, \mathbb{C})$--primary fields in the Verma modules over the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ may be found in [24]. Briefly, the $\mathfrak{sl}(2, \mathbb{C})$--primary fields of weight (spin) $\mu$ may be characterized as generating functions for the tensor operators, which transform as analytical $\mu$--differentials on a circle $S^1$ (perhaps, with non-trivial monodromy).

3.3. Polynions, their representations, and (derived) QPFT-operator algebras [16:App.C.3;24].

A projective polynion of degree $n$ is a data $(R, \partial R_+, \partial R^2_+, \ldots \partial R^{n+1}_+, \partial R_-)$, where $R$ is a Riemann surface of genus $0$ ($R \subset \mathbb{C}$) that is equipped with a projective structure and has a boundary whose components $\partial R_+, \partial R^2_+, \ldots \partial R^{n+1}_+, \partial R_-$ are homeomorphic to the circle $S^1$ with ingoing linear-fractional parametrization defined on $\partial R_+$, $\partial R^2_+$, $\ldots \partial R^{n+1}_+$, and outgoing linear-fractional parametrization on $\partial R_-$. 

On the set of projective polynions $\text{Polynion}(\text{PSL}(2, \mathbb{R}))$ there are defined the sewing operations

$$s : \text{Polynion}(\text{PSL}(2, \mathbb{R})) \times \text{Polynion}(\text{PSL}(2, \mathbb{R})) \mapsto \text{Polynion}(\text{PSL}(2, \mathbb{R})), $$

which are consistent with the grading

$$s : \text{Polynion}_n(\text{PSL}(2, \mathbb{R})) \times \text{Polynion}_m(\text{PSL}(2, \mathbb{R})) \mapsto \text{Polynion}_{n+m}(\text{PSL}(2, \mathbb{R})).$$

The manifold $\text{Polynion}_n(\text{PSL}(2, \mathbb{R}))$ has the complex dimension $3(n + 1)$. We have

$$\text{Polynion}_0(\text{PSL}(2, \mathbb{R})) \simeq \text{Mantle}(\text{PSL}(2, \mathbb{R})), $$

$$\text{Polynion}_1(\text{PSL}(2, \mathbb{R})) \simeq \text{Trinion}(\text{PSL}(2, \mathbb{R})), $$

the polynions of degree greater than 1 being represented as compositions of trinions. The Lie group $[\text{PSL}(2, \mathbb{R})]^{n+2}$ acts on $\text{Polynion}_n(\text{PSL}(2, \mathbb{R}))$, and the corresponding action of the Lie algebra $\mathfrak{g}^\mathbb{C} = (n + 2) \mathfrak{sl}(2, \mathbb{C})$ is transitive. The stabilizer of a polynion in the Lie algebra $\mathfrak{g}^\mathbb{C} = (n + 2) \mathfrak{sl}(2, \mathbb{C})$ is the subalgebra of holomorphic vector fields that admit extension to the complete polynion. Note that the action of $\mathfrak{g}^\mathbb{C}$ on $\text{Polynion}_n(\text{PSL}(2, \mathbb{R}))$ can be exponentiated to the action of $[\text{Mantle}(\text{PSL}(2, \mathbb{R}))]^{n+2}$ with $n + 1$ copies of the projective semigroup $\text{Mantle}(\text{PSL}(2, \mathbb{R}))$, the mantle of the projective group $\text{PSL}(2, \mathbb{R})$, acting from the right, and one from the left.

The (projective) representation of polynions is a family of representations $\pi_n$ of the homogeneous manifolds $\text{Polynion}_n(\text{PSL}(2, \mathbb{R}))$

$$\pi_n : \text{Polynion}_n(\text{PSL}(2, \mathbb{R})) \mapsto \mathbf{P}(\text{Hom}(V^{\otimes(n+1)}; V))$$

such that the diagram

$$\begin{array}{ccc}
\text{Polynion}_n(\text{PSL}(2, \mathbb{R})) \times \text{Polynion}_m(\text{PSL}(2, \mathbb{R})) & \xrightarrow{s} & \text{Polynion}_{n+m}(\text{PSL}(2, \mathbb{R})) \\
\downarrow \pi_n \times \pi_m & & \downarrow \pi_{n+m} \\
\mathbf{P}(\text{Hom}(V^{\otimes(n+1)}; V)) \times \mathbf{P}(\text{Hom}(V^{\otimes(m+1)}; V)) & \mapsto & \mathbf{P}(\text{Hom}(V^{\otimes(n+m+1)}; V))
\end{array}$$
where the lower arrow is the contraction operation, is commutative.

Let us now consider the infinitesimal counterparts of the representations of projective polynions [25;16:§1] (see also [26] and refs wherein). Some preliminary general concepts are necessary.

The operator algebra of a quantum field theory (QFT-operator algebra) is a pair \((H, T_{ij}(x))\) \((x \text{ belongs to } \mathbb{R}^n \text{ or to } \mathbb{C}^n)\), where \(h\) is a linear space, and \(t_{ij}^k(x)\) is a \(H\)-valued tensor field that satisfies \(t_{im}^l(x)t_{jk}^m(y) = \delta_{ij}t_{lk}^m(x, y)\). One may introduce the operators \(l_x(e_i)\) \((e_i \text{ is an element of a basis in the space } H)\): \(l_x(e_i)e_j = t_{ij}^k(x)e_k\). These operators satisfy the identities \(l_x(e_i)l_y(e_j) = t_{ij}^k(x-y)e_k\) \((the \text{ operator product expansion})\) and \(l_x(e_i)l_y(e_j) = l_yl_x(e_i)e_j\) \((the \text{ duality})\). Also one may define the multiplication operation \(m_x\), which depends on the parameter \(x\), in the space \(H\): \(m_x(a, b) = l_x(a)b\). For this operation the identity of smeared associativity \(m_x(a, m_y(b, c)) = m_y(m_x-y(a, b), c)\) holds. The operators \(l_x\) are the operators of multiplication from the left in the obtained algebra, and \(t_{ij}^k(x)\) are the structural functions. Such definition of the QFT-operator algebras is an axiomatization of the well-known operator product expansions in quantum field theory.

In a QFT-operator algebra with unit there is defined the operator \(L\): \(L = i\frac{d}{dx}(l_x(1))|_{x=0}\). This operator generates the infinitesimal translations \([L, l_x(a)] = \frac{d}{dx}l_x(a)\). In what follows, we shall assume that the variable \(x\) ranges over the complex plane and that the tensor field \(t_{ij}^k(x)\) is analytic.

Let us now describe an object, which is an infinitesimal counterpart of the representations of polynions. A QFT-operator algebra \((H, t_{ij}^k(u))\) is called a \((derived)\) QPFT-operator algebra \((and, \text{ by some authors, quasi-vertex algebra})\) iff

1. the linear space \(H\) can be decomposed into a direct sum or a direct integral of Verma modules \(V_\alpha\) over the Lie algebra \(\mathfrak{sl}(2, \mathbb{C})\) with the highest vectors \(v_\alpha\) and the highest weight \(h_\alpha\);
2. the operator fields \(l_u(v_\alpha)\) are \(\mathfrak{sl}(2, \mathbb{C})\)-primary fields of weight \(h_\alpha\), in other words, on commutation with the generators of the Lie algebra \(\mathfrak{sl}(2, \mathbb{C})\) they transform in accordance with a tensor law as \(h_\alpha\)-differentials:

\[
[L_\alpha, l_u(v_\alpha)] = (-u)^k(u\partial_u + (k + 1)h_\alpha)l_u(v_\alpha), \quad k = -1, 0, 1;
\]

3. the following derivative rule of generation of descendents holds:

\[
[L_{-1}, l_u(f)] = \frac{d}{du}l_u(f) = l_u(L_{-1}f).
\]

Let us formulate now the main result establishing a connection between the representations of polynions and (derived) QPFT-operator algebras. If the representation space \(V\) of the polynions is decomposed into a direct sum or a direct integral of the Verma modules over the Lie algebra \(\mathfrak{sl}(2, \mathbb{C})\), then the transition to projective vertices in the representation of polynions defines the structure of a \((derived)\) QPFT-operator algebra in the representation space. In general, the converse is not true; not every (derived) QPFT-operator algebra can be integrated to a representation of polynions (in the same way that not every representation of a Lie algebra can be integrated to a representation of the corresponding Lie group). One may consider the relations between (derived) QPFT-operator algebras and representations of polynions as analogous to ones between Lie algebras and Lie groups [16:App.C].
**Remark: Local projective field algebras.** The method of smoothing (that means the transition from the operators representing elements of a group to their integrals representing the group algebra) is very effective in the representation theory (see e.g. [22, 27]). The analogous procedure may be applied to the QFT-operator algebras [28] (one should use the smoothing by a “vertex position” $x$). In the case of the QPFT-operator algebras the result may be defined axiomatically; the obtained object is called a local projective field algebra [26], it may be considered as a “vertex analogue” of the group algebra. Note that one may construct an analogue of the group algebra (“polynomial algebra”) by an integration (“smoothing”) of the operators representing polynomials.

### 3.4. The projective category $\text{Train}(\text{PSL}(2, \mathbb{R}))$, the train of the projective group $\text{PSL}(2, \mathbb{R})$, and the projective modular functor [16:App.C.4].

The projective category $\text{Train}(\text{PSL}(2, \mathbb{R}))$, the train of the projective group $\text{PSL}(2, \mathbb{R})$, is a category whose objects $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \ldots$ are finite ordered sets, morphisms in $\text{Mor}(\mathfrak{A}, \mathfrak{B})$ are the data $(R, \partial R^1, \ldots \partial R^n, n_+ = \#\mathfrak{A}, n_- = \#\mathfrak{B})$, where $R$ is a Riemann surface equipped with projective structure and possessing a boundary whose components $\partial R^i_+, \partial R^i_-$ are homeomorphic to the circle $S^1$ with ingoing linear-fractional parametrization on $\partial R^1_+, \ldots \partial R^n_+, n_+ = \#\mathfrak{A}$, and outgoing linear-fractional parametrization on $\partial R^1_-, \ldots \partial R^n_-, n_- = \#\mathfrak{B}$. Composition of morphisms is the sewing operation $s$.

On the set $\text{Mor}(\mathfrak{A}, \mathfrak{B})$ there acts the Lie group $\text{PSL}(2, \mathbb{R})^{\#\mathfrak{A} + \#\mathfrak{B}}$, but in contrast to polynomials the corresponding action of the Lie algebra $\mathfrak{g}^C = (\#\mathfrak{A} + \#\mathfrak{B}) \text{sl}(2, \mathbb{C})$ is not transitive (this being due to the presence of moduli of Riemann surfaces of nonvanishing genus). The stabilizer of a morphism in the Lie algebra $\mathfrak{g}^C$ is the subalgebra of holomorphic vector fields that admit an extension to the geometrical image of the morphism. The action $\mathfrak{g}^C$ on the set of morphisms can be exponentiated to the action of the semigroup $\text{Mantle}(\text{PSL}(2, \mathbb{R}))^{\#\mathfrak{A} + \#\mathfrak{B}}$, with $\#\mathfrak{A}$ copies of the projective semigroup $\text{Mantle}(\text{PSL}(2, \mathbb{R}))$, the mantle of the projective group, acting from the right, and $\#\mathfrak{B}$ from the left. Although the action of the Lie algebra $\mathfrak{g}^C$ on the set of morphisms is not transitive, it is possible to define the concept of a representation of the family of morphisms as a continuous family of representations of the orbits of this action.

A (projective) representation of the projective category $\text{Train}(\text{PSL}(2, \mathbb{R}))$, the train of the projective group $\text{PSL}(2, \mathbb{R})$ (*projective modular functor*), is a set of representations $\pi_{\mathfrak{A} \mathfrak{B}} : \text{Mor}(\mathfrak{A}, \mathfrak{B}) \mapsto \mathbb{P}(\text{Hom}(V^\otimes \#\mathfrak{A}, V^\otimes \#\mathfrak{B}))$ such that

- the diagram
  
  $\begin{array}{ccc}
  \text{Mor}(\mathfrak{A}, \mathfrak{B}) \times \text{Mor}(\mathfrak{B}, \mathfrak{C}) & \xrightarrow{s} & \text{Mor}(\mathfrak{A}, \mathfrak{C}) \\
  \downarrow \pi_{\mathfrak{A} \mathfrak{B}} \times \pi_{\mathfrak{B} \mathfrak{C}} & & \downarrow \pi_{\mathfrak{A} \mathfrak{C}} \\
  \mathbb{P}(\text{Hom}(V^\otimes \#\mathfrak{A}, V^\otimes \#\mathfrak{B})) \times \mathbb{P}(\text{Hom}(V^\otimes \#\mathfrak{B}, V^\otimes \#\mathfrak{C})) & \xrightarrow{\pi_{\mathfrak{A} \mathfrak{B}} \times \pi_{\mathfrak{B} \mathfrak{C}}} & \mathbb{P}(\text{Hom}(V^\otimes \#\mathfrak{A}, V^\otimes \#\mathfrak{C})),
  \end{array}$

  where the lower arrow is the contraction operation, is commutative;
- if $\mathfrak{A} = \mathfrak{A}_1 \sqcup \mathfrak{A}_2$, $\mathfrak{B} = \mathfrak{B}_1 \sqcup \mathfrak{B}_2$, $R = R_1 \sqcup R_2$, $R_i \in \text{Mor}(\mathfrak{A}_i, \mathfrak{B}_i)$, then $\pi_{\mathfrak{A} \mathfrak{B}}(R) = \pi_{\mathfrak{A}_1 \mathfrak{B}_1}(R_1) \times \pi_{\mathfrak{A}_2 \mathfrak{B}_2}(R_2)$.

Every projective modular functor corresponds to some representation of polynomials, since polinions are a special case of morphisms in the projective category...
Train(PSL(2, ℜ)), the train of the projective group PSL(2, ℜ). In general, the converse is not true – not every representation of polynomials can be extended to a projective modular functor. Indeed, the projective semigroup Voile(PSL(2, ℜ)), the veil of the projective group PSL(2, ℜ), is identified with the semigroup of all endomorphisms of object of cardinality 1 in the projective category Train(PSL(2, ℜ)), and this is the “topological” obstruction to an extension of representations of polynomials to projective modular functors. Thus, among the structures of the theory of representations in the quantum projective field theory (derived QPFT-operator algebras, representations of projective polynomials, projective modular functors) the last [the representations of the projective category Train(PSL(2, ℜ)), the train of the projective group PSL(2, ℜ)] form the smallest class (under the condition that the representation spaces are sums of Verma modules over the Lie algebra $\mathfrak{sl}(2, ℂ)$), and the first [the derivative QPFT-operator algebras or, equivalently, the QPFT-operator algebras] form the largest class.

Remark: “Pseudogroup” variations on the theme of “Train”. Let $N$ be any finite set. Let us consider some lattice $P$ of subsets of $N$ ($P$ is closed under intersections and unions). One may construct a generalization $\text{Train}(PSL(2, ℜ), P)$ of the projective category $\text{Train}(PSL(2, ℜ))$, which objects belong to $P$. Morphisms are defined in the same manner as for $\text{Train}(PSL(2, ℜ))$. The category $\text{Train}(PSL(2, ℜ), P)$ is supplied with a natural structure of a topologized category.

Certainly, one may consider the manifold $\tilde{N} = N \times S^1$ with the projective structure instead of $N$. Some elements of the category $\text{Train}(PSL(2, ℜ), P)$ will form a pseudogroup of projective transformations of $\tilde{N}$. Though other elements of $\text{Train}(PSL(2, ℜ), P)$ do not constitute a pseudogroup, its categorical properties are analogous to ones of ordinary pseudogroups and may be straightforwardly axiomatized. Some examples of analogous (“pseudogroup”) categories were considered by Yu.A.Neretin [9]. The representations of such “pseudogroup” categories may be naturally defined in the same universal manner. Note that the representations of smooth pseudogroups of transformations appear in the framework of the asymptotic quantization [15:Ch.4]. The algorithm of asymptotic quantization uses the representations “mod $\hbar$” ($\hbar$ is a parameter). The infinitesimal counterparts of such representations, the asymptotic representations of the pseudoalgebras of vector fields “mod $\hbar$”, are a partial case of the general $\mathfrak{A}$–projective representations of the topic 10 of series [2].

Remark: In all constructions above the real projective structures on the boundaries of the suitable Riemann surfaces were canonical so that the components of the boundaries admitted linear-fractional parametrizations. One may generalize the situation omitting this condition.

3.5. Conclusions.

Thus, a general scheme of the reconstruction of hidden objects related to the Lie groups is briefly exposed above on the simplest example of the projective group PSL(2, ℜ). However, it may be evidently generalized to other finite-dimensional semisimple Lie groups with changes in minor details. In the next topic we shall discuss how it is adapted to the infinite-dimensional group Diff$_+(S^1)$ of all orientation preserving diffeomorphisms of a circle $S^1$ (or its central extension Vir, the Virasoro-Bott group). The exposition of the general scheme above lacks a lot of interesting details (such as QPFT-operator crossing-algebras, which are the infin-
itesimal counterparts of projective modular functors, and many related structures or the projective Krichever-Novikov functors, cf.[21]) for the simplicity and clarity. The more detailed exposition of the scheme should be found in [21]. The relation of the reconstructed algebraic objects to the quantum group foundations of the self-interacting string field theory is also described in the article [21], which contains a necessary bibliography. Here we mark only that the (renormalized) version of the projective Krichever-Novikov functor is based on the concept of a sewing of noncommutative coverings of the Riemann surfaces [29], which are realized by means of sheaves of the local projective field algebras. The original version of the Krichever-Novikov construction is adapted to the conformal case [30] and has a deal with operator product expansions on the Riemann surfaces instead of local field algebras.

Note that our scheme is ideologically the same as one of G.Segal, E.Witten and M.Atiyah but differs from the essentially less known general scheme of Yu.A.Neretin, who systematically avoids, neglects or reduces any topological effects in the reconstruction of the hidden objects (at least, such effects appear only episodically in his articles and in my opinion their appearing is motivated presumably by influences of the other authors such as G.Segal, M.Kontsevich, M.Atiyah, I.M.Krichver and S.P.Novikov, G.Moore or E.Witten), and in such a way describes some very interesting purely algebraic or analytic phenomena [9; and refs wherein]. Note that the topological aspects of an analysis of hidden symmetries in the representation theory connect the least with the bordism theory (indeed, the veil Voile(PSL(2,\( \mathbb{R} \))) of the projective group PSL(2,\( \mathbb{R} \)) is just the semigroup of all bordisms of a circle supplied with the canonical projective structure).

Here it is convenient to formulate some open problems related to the material of this topic.

Problems:

- To generalize the scheme to the simplest nonlinear objects such as the Racah-Wigner algebra or the higher \( \mathfrak{u} \)-algebras for \( \mathfrak{sl}(2,\mathbb{C}) \) [2;3;§1] and the Sklyanin algebra [31;3;§3.1]. It is especially interesting to define the “fluctuating exponents” for the nonlinear objects similar to the Racah-Wigner algebras. Such construction should be considered in a general context of the nonlinear geometric algebra [32] and quantization of nonlinear Poisson brackets [15].
- To generalize the scheme to the isotopic pairs [33;3;§2.3], especially to R-matrix ones.
- To generalize the scheme to the “quantum \( \mathfrak{sl}(2,\mathbb{C}) \)” [34] and, perhaps only partially, to other “nonlinear \( \mathfrak{sl}(2,\mathbb{C}) \)” [35].

**Topic Four: Conformal modular functors and related objects**

The objects of this topic are infinite-dimensional analogs of ones discussed above, and so they constitute a subject of the representation theory of the infinite dimensional Lie groups, Lie algebras and related structures. Note that the procedures, which were almost trivial and tautological for the finite-dimensional case, become very profound in its infinite-dimensional counterpart. For instance, the construction of mantles of the infinite-dimensional groups is a part of a general ideology of G.I.Olshanskiĭ [36] of the semigroup approach to the representation theory of such groups.
4.1. The Lie algebra $\mathfrak{Vect}(S^1)$ of vector fields on a circle, the group $\text{Diff}_+(S^1)$ of diffeomorphisms or a circle, the Virasoro algebra $\text{vir}$, the Virasoro-Bott group $\text{Vir}$ and the Neretin semigroup $\text{Ner}$.

Let $\text{Diff}(S^1)$ denote the group of diffeomorphisms of the unit circle $S^1$. The group manifold $\text{Diff}(S^1)$ splits into two connected components, the subgroup $\text{Diff}_+(S^1)$ and the coset $\text{Diff}_-(S^1)$. The diffeomorphisms in $\text{Diff}_+(S^1)$ preserve the orientation on the circle $S^1$ and those in $\text{Diff}_-(S^1)$ reverse it.

The Lie algebra of $\text{Diff}_+(S^1)$ can be identified with the linear space $\mathfrak{vect}(S^1)$ of smooth vector fields on the circle equipped with the commutator

$$[v(t)d/dt, u(t)d/dt] = (v(t)u'(t) - v'(t)u(t))d/dt.$$ 

In the basis $s_n = \sin(nt)d/dt, c_n = \cos(nt)d/dt, h = d/dt$ the commutation relations have the form

$$[s_n, s_m] = 0.5((m-n)s_{m+n} + \text{sgn}(n-m)(n+m)s_{|n-m|}),$$
$$[c_n, c_m] = 0.5((m-n)s_{n+m} + \text{sgn}(n-m)(n+m)s_{|n-m|}),$$
$$[s_n, c_m] = 0.5((m-n)c_{n+m} - (n+m)c_{|n-m|}) - n\delta_{nm}h,$$
$$[h, s_n] = nc_n, \quad [h, c_n] = ns_n.$$

The generators $h, s_n, c_n$ form a Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$ for each $n$.

The complexification of the Lie algebra $\mathfrak{vect}(S^1)$ will be denoted by $\mathbb{C}\mathfrak{vect}(S^1)$. It is convenient to choose the basis $e_k = ie^{ikt}d/dt$ in $\mathbb{C}\mathfrak{vect}(S^1)$. The commutation relations of the algebra $\mathbb{C}\mathfrak{vect}(S^1)$ have the following form

$$[e_j, e_k] = (j - k)e_{j+k}$$

in the basis $e_k$. The generators $e_n, e_{-n}, e_0$ form a Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{C})$ for each $n$.

In 1968 I.M.Gelfand and D.B.Fuchs discovered [37] that $\mathfrak{vect}(S^1)$ possesses a nontrivial central extension. The corresponding 2-cocycle is $c(u, v) = \int v'(t)du'(t)$ or, equivalently, $c(u, v) = \frac{u''(t_0)u'(t_0) - v''(t_0)v'(t_0)}{u''(t_0) - v''(t_0)}$ (see [38]). This central extension was independently discovered by M.Virasoro [39] and named after him. Let us denote the Virasoro algebra by $\text{vir}$. Its complexification, which is also called the \textit{Virasoro algebra}, will be denoted $\mathbb{C}\text{vir}$. As a vector space $\text{vir}$ is generated by the vectors $e_k$ and the central element $c$. The commutation relations have the form

$$[e_j, e_k] = (j - k)e_{j+k} + \delta(j + k)\frac{j^2 - j}{12}c.$$

The imbeddings of the Lie algebras $\mathfrak{sl}(2, \mathbb{R})$ and $\mathfrak{sl}(2, \mathbb{C})$ into $\mathfrak{vect}(S^1)$ and $\mathbb{C}\mathfrak{vect}(S^1)$ may be lifted to the imbeddings of these Lie algebras into $\text{vir}$ and $\mathbb{C}\text{vir}$.

The infinite-dimensional group $\text{Vir}$ corresponding to the algebra $\text{vir}$ is a central extension of the group $\text{Diff}(S^1)$. The corresponding 2-cocycle was calculated by R.Bott [40] so the group $\text{Vir}$ is called the \textit{Virasoro-Bott group}. The imbeddings of the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ into $\mathfrak{vect}(S^1)$ or $\text{vir}$ are exponentiated to the imbeddings of the $n$-coverings of the projective group $\text{PSL}(2, \mathbb{R})$ into $\text{Diff}_+(S^1)$ and $\text{Vir}$ ($n$ labels the imbedding).
In 1968 in the cited paper I.M.Gelfand and D.B.Fuchs computed completely the cohomology of $\text{Vect}(S^1)$ and, thus, discovered a non-trivial 3-cocycle of this Lie algebra. It may be written as $c(u, v, w) = \det(B(t_0))$, where

$$B(t) = \begin{bmatrix} u'(t) & v'(t) & w'(t) \\ u''(t) & v''(t) & w''(t) \\ u'''(t) & v'''(t) & w'''(t) \end{bmatrix},$$

or as $c(u, v, w) = \int \det(B(t))dt$ [38]. This cocycle defines [41.§1.1] a nonassociative deformation of the Virasoro-Bott group, which was called the Gelfand-Fuchs loop and denoted by $Gf$.

There are no groups corresponding to the Lie algebras $\mathbb{C}\text{Vect}(S^1)$ or $\mathbb{C}\text{vir}$, but one can consider the following construction due to Yu.A.Neretin, M.L.Kontsevich and G.Segal. Let us denote by $\mathcal{L}\text{Diff}^\mathbb{C}_+(S^1)$ the set of all analytic mappings $g : S^1 \rightarrow \mathbb{C} \setminus \{0\}$ such that $g(S^1)$ is a Jordan curve surrounding zero, the orientations of $S^1$ and $g(S^1)$ are the same, and $g'(e^{i\theta})$ is everywhere different from zero. $\mathcal{L}\text{Diff}^\mathbb{C}_+(S^1)$ is a local group. Let $\mathcal{L}\text{Ner} \subset \mathcal{L}\text{Diff}^\mathbb{C}_+(S^1)$ be the local subsemigroup of mappings $g$ such that $|g(e^{i\theta})| < 1$. As it was shown by Yu.A.Neretin in 1987, the structure of local semigroup on $\mathcal{L}\text{Ner}$ extends to the structure of global semigroup Ner. There exist at least two constructions of the semigroup Ner.

The first construction (Yu.A.Neretin). An element of Ner is a formal product $p \cdot A(t) \cdot q$ (**), where $p, q \in \text{Diff}_+(S^1)$, $p(1) = 1$, $t > 0$, $A(t) : \mathbb{C} \mapsto \mathbb{C}$, $A(t)z = e^{-t}z$. To define multiplication in Ner, one must describe the rule used to transform the formal product $A(s) \cdot p \cdot A(t)$ to the form (**).

**A.** Let $t$ be so small that the diffeomorphism $p$ extends holomorphically to the annulus $e^{-t} \leq |z| \leq 1$. Then the product $g = A(s)pA(t)$ is well defined. Let $K$ be the domain bounded by $S^1$ and $g(S^1)$. Let $Q$ be the canonical conformal mapping of $K$ onto the annulus $e^{-t} \leq z \leq 1$, normalized by the condition $Q(1)$. Then $g = p' \cdot A(t) \cdot q'$, where $p' = Q^{-1}|_{S^1}$ and $q'$ is determined by the identity $A(s) \cdot p \cdot A(t) = p' \cdot A(t') \cdot q'$.

**B.** For an arbitrary $t$ there exists a suitable $n$ such that the product

$$A(s) \cdot p \cdot A(t) = (\ldots (A(s) \cdot p \cdot A(t/n))A(t/n) \ldots)A(t/n)$$

(***)

can be calculated. It can be shown that the product does not depend on the choice of the representation (**) and is associative.

The second construction (M.L.Kontsevich and G.Segal). An element $g$ of the semigroup Ner is a triple $(K, p, q)$, where $K$ is a Riemann surface with boundary $\partial K$ such that $K$ is biholomorphically equivalent to the annulus and $p, q : S^1 \mapsto \partial K$ are fixed parametrizations of the components of $\partial K$. The mapping $p$ realizes the ingoing parametrization whereas $q$ realizes the outgoing parametrization. Two elements $g_i = (K_i, p_i, q_i)$, $i = 1, 2$, are equivalent if there exists a conformal mapping $R : K_1 \mapsto K_2$ such that $p_2 = Rp_1$, and $q_2 = Rq_1$. The product of two elements $g_1$ and $g_2$ is the element $g_3 = (K_3, p_3, q_3)$, where

$$K_3 = K_1 \sqcup_{q_1(e^{it} = p_2(e^{it})} K_2,$$

$p_3 = p_1$, and $q_3 = q_2$. 


The Neretin semigroup Ner is called the mantle of the group Diff_+(S^1) of diffeomorphisms of a circle and is denoted by Mantle(Diff_+(S^1)) The Neretin semigroup Ner possesses a central extension, which was discovered by Yu.A.Neretin in 1989. This extension is compatible with the Virasoro-Bott extension Vir of the group Diff_+(S^1) and is a mantle Mantle(Vir) of the group Vir. The imbeddings of the n-coverings of PSL(2,ℝ) into the groups Diff_+(S^1) and Vir may be extended to the holomorphic imbeddings of the mantles of such coverings (which are coverings of Mantle(PSL(2,ℝ))) into the mantles of the infinite-dimensional groups Diff_+(S^1) and Vir. The first imbedding (n = 1) may be used for the construction of a certain enlargement of the semigroup Mantle(Diff_+(S^1)), namely, Mantle(Diff_+(S^1)) × Mantle(PSL(2,ℝ)) PSL(2,ℂ).

Note that in the infinite-dimensional case each projective representation of the group Diff_+(S^1) is linearized over the universal covering of its central extension Vir whereas in the finite-dimensional case each projective representation of the group PSL(2,ℝ) is linearized over its own universal covering. The same situation is for the mantles.

The further interesting information on the objects of this paragraph and their relation to the infinite-dimensional geometry should be found in the papers [42,43].

4.2. The semigroup Voile(Diff_+(S^1)) (the veil of the group Diff_+(S^1)), the manifolds Trinion(Diff_+(S^1)) of (conformal) trinions and Polynion(Diff_+(S^1)) of (conformal) polynomials. The representations of polynomials and QCFT-operator algebras [21].

The construction of the semigroup Mantle(Diff_+(S^1)) can be generalized to arbitrary Riemann surfaces. The semigroup Voile(Diff_+(S^1)) that is then obtained is called the veil (voile) of the group Diff_+(S^1)). The elements of the semigroup Voile(Diff_+(S^1)) are triples (R, f_+, f_-), where R is a Riemann surface, and f_+ : D_+ ↪ R, f_- : D_- ↪ R (D_+ = {z : |z| ≤ 1}, D_- = {z : |z| ≥ 1}) are holomorphic imbeddings of the complex disks D_+ and D_- into R with nonintersecting images. Multiplication of elements of the semigroup is a sewing. The semigroup Voile(Diff_+(S^1)) is a ℤ+-graded infinite-dimensional semigroup. The grading is specified by the genus of the Riemann surface. The semigroup Voile(Diff_+(S^1)), the veil of the group Diff_+(S^1)), can be regarded as a “fluctuating” exponential of the Lie algebra Vect(S^1) that describes the process of evolution with continuous creation and annihilation of “virtual particles”. From this point of view, it is very interesting to study fractal (corresponding to nonperturbative effects) generalizations of the veil Voile(Diff_+(S^1)) of the projective group Diff_+(S^1)), the element of which can have infinite genus; note once more that the nonperturbative effects in this situation are essential for the quantum theory for a self-interacting string field [21; and refs wherein].

Let us briefly repeat the construction of trinions [8].

A (conformal) trinion is a quadruplet (R, ∂R^1_+, ∂R^2_+, ∂R_-), where R is a Riemann surface of genus 0 (R ⊂ ℂ) with a boundary whose components ∂R^1_+, ∂R^2_+, ∂R_- are homeomorphic to the circle S^1 with ingoing parametrization defined on ∂R^1_+ and ∂R^2_+ and outgoing parametrization defined on ∂R_-.

The situation differs from one described in Topic 3 by the lack of projective structure and the related conditions on the ingoing and outgoing parametrizations.

The manifold of conformal trinions Trinion(Diff_+(S^1)) is an infinite-dimensional complex manifold. The Lie group G = Diff_+(S^1) × Diff_+(S^1) × Diff_+(S^1) acts on the
set of trinions $\text{Trinion}(\text{Diff}_+(S^1))$, and the corresponding action of the Lie algebra $\mathfrak{g}^C = \mathbb{C}\text{Vect}(S^1) + \mathbb{C}\text{Vect}(S^1) + \mathbb{C}\text{Vect}(S^1)$ is transitive. The stabilizer of the trinion in the Lie algebra $\mathfrak{g}^C$ is the subalgebra of all holomorphic vector fields that admit extension to the complete trinion. Note that the action of the Lie algebra $\mathfrak{g}^C$ on the manifold $\text{Trinion}(\text{Diff}_+(S^1))$ of conformal trinions can be extended to the action of the semigroup $\text{Mantle}(\text{Diff}_+(S^1)) \times \text{Mantle}(\text{Diff}_+(S^1)) \times \text{Mantle}(\text{Diff}_+(S^1))$ with two copies of the semigroup $\text{Mantle}(\text{Diff}_+(S^1))$, the mantle of the group $\text{Diff}_+(S^1)$, acting from the right, and one from the left. At the same time, for all Verma modules $V_{h_{1,c}, V_{h_{2,c},, V_{h_{3,c}}}$ over the Lie algebra $\mathbb{C}\text{vir}$ [44] (here $h_i$ are the extremal weights and $c$ is the central charge), which are integrable to the projective representations of the semigroup $\text{Mantle}(\text{Diff}_+(S^1))$ there exists no more than one projective representation of the manifold of conformal trinions $\text{Trinion}(\text{Diff}_+(S^1))$ in the projective space $\mathbf{P}(\text{Hom}(V_{h_{1,c}, V_{h_{2,c},, V_{h_{3,c}}})$ consistent with the action of the semigroup $\text{Mantle}(\text{Diff}_+(S^1))$ in these modules. One can also consider the universal cover of the manifold $\text{Trinion}(\text{Diff}_+(S^1))$.

The operation of the vertex insertion and the (conformal) vertices themselves can be defined in a way similar to one specified in the Topic 3. A (conformal) polynomial of degree $n$ is a data $(R, \partial R^1_+, \partial R^2_+, \ldots \partial R^n_+, \partial R_-)$, where $R$ is a Riemann surface of genus 0 ($R \subset \mathbb{C}$) that has a boundary whose components $\partial R^1_+, \partial R^2_+, \ldots \partial R^n_+, \partial R_-$ are homeomorphic to the circle $S^1$ with ingoing parametrization defined on $\partial R^1_+, \partial R^2_+, \ldots \partial R^n_+$, and outgoing parametrization on $\partial R_-$. The situation differs from one described in Topic 3 by the lack of projective structure and the related conditions on the ingoing and outgoing parametrizations.

On the set of conformal polynomials $\text{Polynion}(\text{Diff}_+(S^1))$ there are defined the sewing operations

$$s : \text{Polynion}(\text{Diff}_+(S^1)) \times \text{Polynion}(\text{Diff}_+(S^1)) \mapsto \text{Polynion}(\text{Diff}_+(S^1)),$$

which are consistent with the grading

$$s : \text{Polynion}_n(\text{Diff}_+(S^1)) \times \text{Polynion}_m(\text{Diff}_+(S^1)) \mapsto \text{Polynion}_{n+m}(\text{Diff}_+(S^1)).$$

The manifold $\text{Polynion}_n(\text{Diff}_+(S^1))$ is an infinite-dimensional complex manifold. We have

$$\text{Polynion}_0(\text{Diff}_+(S^1)) \simeq \text{Mantle}(\text{Diff}_+(S^1)),$$

$$\text{Polynion}_1(\text{Diff}_+(S^1)) \simeq \text{Trinion}(\text{Diff}_+(S^1)),$$

the polynomials of degree greater than 1 being represented as compositions of trinions. The Lie group $[\text{Diff}_+(S^1)]^{n+2}$ acts on $\text{Polynion}_n(\text{Diff}_+(S^1))$, and the corresponding action of the Lie algebra $\mathfrak{g}^C = (n + 2)\mathbb{C}\text{Vect}(S^1)$ is transitive. The stabilizer of a polynomial in the Lie algebra $\mathfrak{g}^C = (n + 2)\mathbb{C}\text{Vect}(S^1)$ is the subalgebra of all holomorphic vector fields that admit an extension to the complete polynomial. Note that the action of $\mathfrak{g}^C$ on $\text{Polynion}_n(\text{Diff}_+(S^1))$ can be exponentiated to the action of $[\text{Mantle}(\text{Diff}_+(S^1))]^{n+2}$ with $n + 1$ copies of the projective semigroup $\text{Mantle}(\text{Diff}_+(S^1))$, the mantle of the group $\text{Diff}_+(S^1)$, acting from the right, and one from the left.

The (projective) representation of polynomials is a family of representations $\pi_n$ of the homogeneous manifolds $\text{Polynion}_n(\text{Diff}_+(S^1))$

$$\pi_n : \text{Polynion}_n(\text{Diff}_+(S^1)) \mapsto \mathbf{P}(\text{Hom}(V^{\otimes(n+1)}; V))$$
such that the diagram
\[
\begin{array}{ccc}
\text{Polynion}_n(\text{Diff}_+(S^1)) \times \text{Polynion}_m(\text{Diff}_+(S^1)) & \longrightarrow & \text{Polynion}_{n+m}(\text{Diff}_+(S^1)) \\
\downarrow \pi_n \times \pi_m & & \downarrow \pi_{n+m} \\
P(\text{Hom}(V^\otimes (n+1); V)) \times P(\text{Hom}(V^\otimes (m+1); V)) & \longrightarrow & P(\text{Hom}(V^\otimes (n+m+1); V))
\end{array}
\]
where the lower arrow is the contraction operation, is commutative.

The extremal vector $T$ of the weight 2 in the QPFT-operator algebra is called a conformal stress-energy tensor, if $T(u) := l_u(T) = \sum L_k(-u)^{k-2}$, and operators $L_k$ generate the Virasoro algebra: $[L_i, L_j] = (i - j)L_{i+j} + \frac{i^3-j^3}{12}c \cdot I$. In view of results of [45] the QPFT-operator algebras with the conformal stress-energy tensor are just the operator algebras of the quantum conformal field theory (QCFT-operator algebras) in sense of the papers [46,29].

A connection between the representations of polynions and QCFT-operator algebras was established in the article [21]. If the representation space $V$ of the polynions is decomposed into a direct sum or a direct integral of the Verma modules over the Lie algebra $\text{Cvir}$, then the transition to (conformal) vertices in the representation of polynions defines the structure of a QCFT-operator algebra in the representation space. In general, the converse is not true; not every QCFT-operator algebra can be integrated to a representation of polynions (in the same way that not every representation of a Lie algebra can be integrated to the representation of the corresponding Lie group). One may consider the relations between QCFT-operator algebras and representations of polynions as analogous to ones between Lie algebras and Lie groups.

Remarks:

- There is a subtle distinction between the derived QPFT-operator algebras, which were defined above, and the QPFT-operator algebras, which were considered in [46,29]. The fact that such distinction is subtle indeed and may be neglected was explicated in the article [25].
- Some subclasses of the QCFT-operator algebras were considered in [47; and refs wherein] under the title of “vertex operator algebras” and “vertex algebras” (see [21]).
- The local projective field algebras for QCFT-operator algebras (the local conformal field algebras) were defined and explored in articles [48]. Their geometric interpretation (as structural rings of the noncommutative coverings of Riemann surfaces) was given in [29]. The relations between the local conformal and projective field algebras [the “projective reduction”] were analyzed in [49] (see also [29]).
- One may consider the projective and conformal antitrinions and antipolynions, which are received from trinions and polynions by the changing of the ingoing parametrizations to the outgoing ones and vice versa, as well as their representations in the way analogous to one described above (see [21]).

4.3. The category $\text{Train}(\text{Diff}_+(S^1))$, the train of the group $\text{Diff}_+(S^1)$, and the (conformal) modular functor [21].

The category $\text{Train}(\text{Diff}_+(S^1))$, the train of the group $\text{Diff}_+(S^1)$, is a category whose objects $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, ... are finite ordered sets, morphisms in $\text{Mor}(\mathfrak{A}, \mathfrak{B})$ are
the data \((R, \partial R^1_+, \partial R^1_-, \partial R^{n_+}_+, \partial R^{n_+}_-, n_+ = \#\mathfrak{A}, n_- = \#\mathfrak{B})\), where \(R\) is a Riemann surface possessing a boundary whose components \(\partial R^1_+, \partial R^1_-, \partial R^{n_+}_+, \partial R^{n_+}_-\) are homeomorphic to the circle \(S^1\) with ingoing parametrization on \(\partial R^1_+, \partial R^{n_+}_+\), \(n_+ = \#\mathfrak{A}\), and outgoing parametrization on \(\partial R^1_-, \partial R^{n_+}_-\), \(n_- = \#\mathfrak{B}\). Composition of morphisms is the sewing operation \(s\).

The situation differs from one described in Topic 3 by the lack of projective structure and the related conditions on the ingoing and outgoing parametrizations.

On the set \(\text{Mor}(\mathfrak{A}, \mathfrak{B})\) there acts the Lie group \([\text{Diff}_+(S^1)]^{\#\mathfrak{A} + \#\mathfrak{B}}\), but in contrast to polynomials the corresponding action of the Lie algebra \(\mathfrak{g}^C = (\#\mathfrak{A} + \#\mathfrak{B})\mathfrak{C}_{\text{vir}}\) is not transitive (this being due to the presence of moduli of Riemann surfaces of nonvanishing genus). The stabilizer of a morphism in the Lie algebra \(\mathfrak{g}^C\) is the subalgebra of all holomorphic vector fields that admit an extension to the geometrical image of the morphism. The action \(\mathfrak{g}^C\) on the set of morphisms can be exponentiated to the action of the semigroup \([\text{Mantle}(\text{Diff}_+(S^1))]^{\#\mathfrak{A} + \#\mathfrak{B}}\), with \(\#\mathfrak{A}\) copies of the semigroup Mantle(\text{Diff}_+(S^1)) acting from the right, and \(\#\mathfrak{B}\) from the left. Although the action of the Lie algebra \(\mathfrak{g}^C\) on the set of morphisms is not transitive, it is possible to define the concept of a representation of the family of morphisms as a continuous family of representations of the orbits of this action.

A (projective) representation of the category \(\text{Train}(\text{Diff}_+(S^1))\), the train of the group \(\text{Diff}_+(S^1)\) ((conformal) modular functor), is a set of representations \(\pi_{\mathfrak{A}\mathfrak{B}} : \text{Mor}(\mathfrak{A}, \mathfrak{B}) \mapsto \text{P}(\text{Hom}(V^{\otimes \#\mathfrak{A}}, V^{\otimes \#\mathfrak{B}}))\) such that

- the diagram

\[
\begin{array}{ccc}
\text{Mor}(\mathfrak{A}, \mathfrak{B}) \times \text{Mor}(\mathfrak{B}, \mathfrak{C}) & \xrightarrow{s} & \text{Mor}(\mathfrak{A}, \mathfrak{C}) \\
\downarrow \pi_{\mathfrak{A}\mathfrak{B}} \times \pi_{\mathfrak{B}\mathfrak{C}} & & \downarrow \pi_{\mathfrak{A}\mathfrak{C}} \\
\text{P}(\text{Hom}(V^{\otimes \#\mathfrak{A}}, V^{\otimes \#\mathfrak{B}})) \times \text{P}(\text{Hom}(V^{\otimes \#\mathfrak{B}}, V^{\otimes \#\mathfrak{C}})) & \longrightarrow & \text{P}(\text{Hom}(V^{\otimes \#\mathfrak{A}}, V^{\otimes \#\mathfrak{C}})),
\end{array}
\]

where the lower arrow is the contraction operation, is commutative;

- if \(\mathfrak{A} = \mathfrak{A}_1 \sqcup \mathfrak{A}_2, \mathfrak{B} = \mathfrak{B}_1 \sqcup \mathfrak{B}_2, R = R_1 \sqcup R_2, R_i \in \text{Mor}(\mathfrak{A}_i, \mathfrak{B}_i)\), then

\(\pi_{\mathfrak{A}\mathfrak{B}}(R) = \pi_{\mathfrak{A}_1\mathfrak{B}_1}(R_1) \times \pi_{\mathfrak{A}_2\mathfrak{B}_2}(R_2)\).

Every conformal modular functor corresponds to some representation of polynomials, since polynomials are a special case of morphisms in the projective category \(\text{Train}(\text{Diff}_+(S^1))\), the train of the group \(\text{Diff}_+(S^1)\). In general, the converse is not true – not every representation of polynomials can be extended to a conformal modular functor. Indeed, the semigroup \(\text{Voile}(\text{Diff}_+(S^1))\), the veil of the group \(\text{Diff}_+(S^1)\), is identified with the semigroup of all endomorphisms of object of cardinality 1 in the category \(\text{Train}(\text{Diff}_+(S^1))\), and this is the “topological” obstruction to an extension of representations of polynomials to projective modular functors. Thus, among the structures of the theory of representations in the quantum conformal field theory [QFT-operator algebras, representations of (conformal) polynomials, (conformal) modular functors] the last [the representations of the category \(\text{Train}(\text{Diff}_+(S^1))\), the train of the group \(\text{Diff}_+(S^1)\)] form the smallest class (under the condition that the representation spaces are sums of Verma modules over the Lie algebra \(\mathfrak{C}_{\text{vir}}\)), and the first [the QCFT-operator algebras] form the largest class.

Note that each conformal modular functor is a projective modular functor as well as each (projective) representation of (conformal) polynomials is a (projective) representation of the projective polynomials.
Let us now plunge the infinite dimensional picture of [1] into the framework of two preceding topics. However, a preliminary lemma is necessary. In this topic the representation means the projective representation.

**Lemma.**

A. The QPFT-operator algebra $\mathfrak{V}$ generated by the $\mathfrak{sl}(2, \mathbb{C})$–primary fields of non-negative integral spins $n$ (with the field of spin 2 as $q_R$–conformal stress-energy tensor) [26] in the Verma module $V_h$ of the extremal weight $h$ determines the representation of projective polynomials in the space $\mathfrak{V} = \oplus_n V_n$.

B. The representation of projective polynomials in the space $\mathfrak{V}$ is naturally extended to the projective modular functor.

The first statement of the lemma is almost evident. To prove the second statement one should mark that the QPFT-operator algebra $\mathfrak{V}$ may be supplied with the operators $r_u(\phi)$ of the multiplication from the right: $r_u(\phi)\psi = \ell_u(\psi)\phi$. Such operators generate the QPFT-operator algebra and commute with the operators $l_u(\phi)$ of the initial QPFT-operator algebra $\mathfrak{V}$. Both structures of QPFT-operator algebras (of vertices $l_u(\phi)$ and co-vertices [21] $r_u(\phi)$) supply the space $\mathfrak{V}$ with the structure of the QPFT-operator crossing-algebra, from which the projective modular functor may be restored [21].

Let us put $\mathfrak{W} = \mathfrak{V} \oplus V_h$ ($V_h$ – unitarizable Verma module). The space $\mathfrak{W}$ is supplied with the structure of the QPFT-operator algebra, which is an abelian extension of $\mathfrak{V}$.

**Corollary.** The QPFT-operator algebra $\mathfrak{W}$ determines the representation of projective polynomials in the space $\mathfrak{W}$, which is an extension of the representation of projective polynomials in $\mathfrak{V}$.

Let $\pi$ be a $\mathcal{K}$–pseudorepresentations of the semigroup $\text{Mantle}(\text{Diff}_+(S^1))$ in the space $V$ [1] (here $\mathcal{K}$ denotes the class of the compact operators in $V$). The class $\mathcal{K}$ includes the class $\mathcal{HS}$ of all Hilbert-Schmidt operators so the $\mathcal{HS}$–pseudorepresentations of [1] are automatically $\mathcal{K}$–pseudorepresentations.

The projective modular functor in $\mathfrak{W}$ define a representation of the projective semigroup $\text{Voile}(\text{PSL}(2, \mathbb{R}))$ in this space.

**Theorem A.**

The representation of the projective semigroup $\text{Voile}(\text{PSL}(2, \mathbb{R}))$ in the space $\mathfrak{W}$ may be extended to the $\mathcal{K}$–pseudorepresentation of the semigroup $\text{Voile}(\text{Diff}_+(S^1))$ compatible with the $\mathcal{K}$–representation of the semigroup $\text{Mantle}(\text{Diff}_+(S^1))$.

For any linear operator $A$ from $\text{Hom}(V^\otimes n; V)$ let us define the operators $F^{(i)}_x(A)$ from $\text{End}(V)$ ($1 \leq i \leq n$, $x \in V^{\otimes (n-1)}$), which are received from $A$ by the substitution of $x$ instead of all arguments except the $i$-th argument. The linear operator $A$ from $\text{Hom}(V^\otimes n; V)$ will be called polycompact if it is a linear combination of the operators $A^{(i)}$ such that the operators $F^{(i)}_x(A)$ are compact for all $x$. The (projective) $\mathcal{PK}$–pseudorepresentation of (conformal) polynomials is a family of representations $\pi_n$ of the homogeneous manifolds $\text{Polynomial}_n(\text{Diff}_+(S^1))$ up to the polycompact operators compatible with $\pi$:

$$\pi_n : \text{Polynomial}_n(\text{Diff}_+(S^1)) \mapsto \mathcal{P}(\text{Hom}^{\mathcal{B}/\mathcal{P}_K}(V^\otimes (n+1); V)),$$
where \( \text{Hom}_{B/P_K} \) is the quotient of the space of all bounded operators by the subspace of all polycompact operators, such that the diagram

\[
\begin{array}{ccc}
\text{Polynion}_n(\text{Diff}_+(S^1)) \times \text{Polynion}_m(\text{Diff}_+(S^1)) & \longrightarrow & \text{Polynion}_{n+m}(\text{Diff}_+(S^1)) \\
\downarrow \pi_n \times \pi_m & & \downarrow \pi_{n+m}
\end{array}
\]

\[
P(\text{Hom}_{B/P_K}(V^\otimes(n+1); V)) \times P(\text{Hom}_{B/P_K}(V^\otimes(m+1); V)) \longrightarrow P(\text{Hom}_{B/P_K}(V^\otimes(n+m+1); V))
\]

where the lower arrow is the contraction operation, is commutative.

**Theorem B.**

The representation of the projective polynomials in the space \( \mathfrak{V} \) may be extended to the \( PK \)-pseudorepresentation of the conformal polynomials compatible with the \( K \)-pseudorepresentation of the semigroup \( \text{Mantle}(\text{Diff}_+(S^1)) \) in this space.

**Corollary.** The representation of the projective polynomials in the space \( \widetilde{\mathfrak{V}} \) of the corollary to lemma may be extended to the \( PK \)-pseudorepresentation of the conformal polynomials compatible with the \( K \)-pseudorepresentation of the semigroup \( \text{Mantle}(\text{Diff}_+(S^1)) \) in \( \mathfrak{V} \).

The construction of the \( PK \)-pseudorepresentations of polynomials may be reformulated for the antipolynomials.

For any the linear operator \( A \) from \( \text{Hom}(V^\otimes n, V^\otimes m) \) let us denote by \( G_y^{(j)}(A) \) (\( 1 \leq j \leq m, \ y \in V^\otimes(m-1) \)) the operator from \( \text{Hom}(V^\otimes n, V) \), which is received from \( A \) by the pairing of the image of \( A \) with \( y \) by all variables except the \( j \)-th one. Such linear operator \( A \) will be called polycompact if it is a linear combination of the operators \( A^{(j)} \) such that \( G_y^{(j)}(A^{(j)}) \) are polycompact operators uniformly by \( y \) from any bounded subset of \( V^\otimes(m-1) \).

The (projective) semi-\( PK \)-pseudorepresentation of the category \( \text{Train}(\text{Diff}_+(S^1)) \), the train of the group \( \text{Diff}_+(S^1) \), is a set of representations \( \pi_{\mathfrak{A}\mathfrak{B}} : \text{Mor}(\mathfrak{A}, \mathfrak{B}) \rightarrow P(\text{Hom}_{B/P_K}(V^\otimes\#\mathfrak{A}, V^\otimes\#\mathfrak{B})) \) up to the polycompact operators such that

- the set \( \pi_{\mathfrak{A}\mathfrak{B}} \) being restricted to \( \text{Train}(\text{PSL}(2, \mathbb{R})) \) realizes the projective modular functor;
- the diagram

\[
\begin{array}{ccc}
\text{Mor}(\mathfrak{A}, \mathfrak{B}) \times \text{Mor}(\mathfrak{B}, \mathfrak{C}) & \longrightarrow & \text{Mor}(\mathfrak{A}, \mathfrak{C}) \\
\downarrow \pi_{\mathfrak{A}\mathfrak{B}} \times \pi_{\mathfrak{B}\mathfrak{C}} & & \downarrow \pi_{\mathfrak{A}\mathfrak{C}}
\end{array}
\]

\[
P(\text{Hom}_{B/P_K}(V^\otimes\#\mathfrak{A}, V^\otimes\#\mathfrak{B})) \times P(\text{Hom}_{B/P_K}(V^\otimes\#\mathfrak{B}, V^\otimes\#\mathfrak{C})) \longrightarrow P(\text{Hom}_{B/P_K}(V^\otimes\#\mathfrak{A}, V^\otimes\#\mathfrak{C}))
\]

where the lower arrow is the contraction operation, is commutative for any two morphisms \( f_1 \in \text{Mor}(\mathfrak{A}, \mathfrak{B}), \ f_2 \in \text{Mor}(\mathfrak{B}, \mathfrak{C}) \) such that \( g(f_1 \circ f_2) = g(f_1) + g(f_2) \), where \( g(f) \) is the genus of the geometric image of the morphism \( f \);
- if \( \mathfrak{A} = \mathfrak{A}_1 \sqcup \mathfrak{A}_2, \ \mathfrak{B} = \mathfrak{B}_1 \sqcup \mathfrak{B}_2, \ R = R_1 \sqcup R_2, \ R_i \in \text{Mor}(\mathfrak{A}_i, \mathfrak{B}_i) \), then
  \[ \pi_{\mathfrak{A}\mathfrak{B}}(R) = \pi_{\mathfrak{A}_1\mathfrak{B}_1}(R_1) \times \pi_{\mathfrak{A}_2\mathfrak{B}_2}(R_2). \]

**Theorem C.**

The projective modular functor in the space \( \mathfrak{V} \) may be extended to the semi-\( PK \)-pseudorepresentation of the category \( \text{Train}(\text{Diff}_+(S^1)) \) compatible with the \( PK \)-pseudorepresentations of the conformal polynomials and antipolynomials as well as with the \( K \)-pseudorepresentation of the semigroup \( \text{Voile}(\text{Diff}_+(S^1)) \).
The semi-$\mathcal{PK}$-pseudorepresentation of the category $\text{Train}(\text{Diff}_+(S^1))$ will be called the $q_R$-conformal modular functor. It may be considered as the “Berezin quantization” of the conformal modular functor in view of the original construction of the $q_R$-conformal symmetries and QPFT-operator algebra $\mathfrak{Y}$ from the Lobachevskii $C^{\ast}$-algebra, the Berezin quantization of the Lobachevskii plane [26] (see also [4]); here $q_R = \frac{1}{2h-1}$ is the quantization parameter.

Problems:

- To investigate the asymptotic behaviour of the $\mathcal{PK}$-pseudorepresentations of conformal polynomials and $q_R$-conformal modular functors in the space $\mathfrak{Y}$ if as $h$ tends to $\infty$ (or $q_R$ tends to zero).
- To explore the possible relations between $q_R$-conformal modular functors and quantizations of Riemann surfaces in sense of S.Klimek and A.Lesniewski [50].

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