Research Article
Fractional Fourier Transform and Ulam Stability of Fractional Differential Equation with Fractional Caputo-Type Derivative

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In this paper, we study the Ulam-Hyers-Mittag-Leffler stability for a linear fractional order differential equation with a fractional Caputo-type derivative using the fractional Fourier transform. Finally, we provide an enumeration of the chemical reactions of the differential equation.

1. Introduction

Fractional differential equations have more attention in the research area of mathematics, and there has been significant progress in this field. However, this idea is not new and as old as differential equations. The differential equations of fractional order have proved to be valuable tools in modeling multiple phenomena in different areas of science and engineering. Indeed, it has many uses in biology, physics, electromagnetics, mechanics, electrochemistry, etc. [1–3]. Fractional calculus was initiated from a question raised by L’Hospital to Leibnitz, which related to his generalization of meaning of notation \((d^n y/du^n)\) for the derivative of order \(n \in \mathbb{N} = 0, 1, 2, \cdots\), when \(n = 1/2\). In his reply, dated September 30, 1695, Leibnitz wrote to L’Hospital [4], “This is an apparent paradox from which one-day useful consequences will be drawn.” Recently, Ozaktas and Kutay [5] published on this topic, dealing with different characteristics in different ways.

A functional equation is stable if for each approximate answer there is a definite quantity about it. In 1940, the simulation and a hit theory suggested by Ulam [6] prompted the study of stability issues for numerous functional equations. He gave the University of Wisconsin Mathematical Colloquium a long form of talks, presenting a variety of unresolved questions. He raised one of the questions that were connected to the stability of the functional equation: “Give conditions for a linear function near an approximately linear function to exist.” The first result concerning the stability of functional equations was presented by Hyers [7] in 1941. The stability of the form is subsequently referred to as Hyers-Ulam stability. In 1978, the generalization associated with the Hyers theorem given by Rassias [8] makes it possible for the Cauchy difference to be unbounded. In 2004, Jung [9] studied the Hyers-Ulam stability of the differential equations \(\theta(s)p'(s) = p(s)\). Jung [10, 11] continuously published the general setting for Hyers-Ulam stability of first-order linear differential equations. In 2006, Jung [12] concentrated on the Hyers-Ulam stability of an arrangement of differential equations with coefficients through the utilization of a matrix approach. Ponmana Selvan et al. [13] have solved the different types of Ulam stability for the approximate...
solution of a special type of \(m\)-th order linear differential equation with initial and boundary conditions.

Zhang and Li [14] studied the Ulam stabilities of \(m\)-dimensional fractional differential systems with order \(1 < \alpha < 2\) in 2011, and in the same year, Li and Zhang [15] proved the stability of fractional order derivative for differential equations. In 2013, Ibrahim [16] investigated the Ulam-Hyers stability for iterative Cauchy fractional differential equations and Lane-Emden equations. Kalvandi et al. [17], Liu et al. [18], and Vu et al. [19] presented and proved the different types of Ulam-Hyers stability of a linear fractional differential equations.

In 2012, Wang et al. [20] carried out pioneering work on the Ulams-Ulam stability for fractional differential equations with Caputo derivative using a fixed point approach, and in the same year, Wang and Zhou [21] proved the Ulam-Hyers stability of nonlinear impulsive problems for fractional differential equations. Wang et al. [22] investigated the Mittag-Leffler-Ulam-Hyers stability of fractional evolution equations.

In 2020, Uonly et al. [23] studied Ulam stabilities of linear fractional order differential equations in Lizorkin space using the fractional Fourier transform, and in the same year, Hamachukiatkul et al. [24] derived some Ulam-Hyers stability outcomes for fractional differential equations. In the next year, Ganesh et al. [25] derived some Mittag-Leffler-Ulam stability, which makes sure the existence and individuality of an answer for a delay fractional differential equation by using the fractional Fourier transform. In 2022, Ganesh et al. [26] carried out pioneering in the field with the Ulams-Ulam stability for fractional order implicit differential equations with two Caputo derivatives using a fractional Fourier transform.

Motivated and inspired by the above results, in this paper, because of the help of fractional Fourier transform, we would like to investigate the Ulam-Hyers-Mittag-Leffler and Ulam-Hyers-Rassias-Mittag-Leffler stability of linear fractional order differential equations with the fractional Caputo-type derivative of the form:

\[
\left( ^\alpha _{0+}D^\sigma \right)(s) + \eta p(s) = q(s),
\]

where \(q(s)\) is a \(m\)-times continuously differentiable function and \(^\alpha _{0+}D^\sigma\), is the fractional Caputo-type derivative of order \(\sigma \in (m-1, m), m \in \mathbb{N}^\ast\).

2. Preliminaries

The following definitions, theorems, notations, and lemmas will be used to obtain the main objectives of this paper.

Definition 1 (see [27]). The one dimension fractional Fourier transform with rotational angle \(\sigma\) of function \(p(s) \in \mathcal{L}^1(\mathcal{R}^\ast)\) is given by

\[
\mathcal{F}^\sigma p(s)(\omega) = \hat{p}_\sigma(\omega) = \int_{\mathcal{R}} K_\sigma(s, \omega)p(s)ds, \omega \in \mathcal{R},
\]

where the kernel

\[
K_\sigma(s, \omega) = \begin{cases} \mathcal{C}_\omega \left((p^\ast + \omega^\ast) \cot \sigma \right)^{-ip\omega \csc \sigma}, & \text{if } \sigma \neq m\pi, \\ \frac{1}{\sqrt{2\pi}} e^{-ip\omega}, & \text{if } \sigma = \frac{\pi}{2}, \\ \mathcal{C}_\omega = \sqrt{1 - i \cot \frac{\sigma}{2}}, & \text{if } \sigma = m\pi, \end{cases}
\]

As such, the inversion formula of fractional Fourier transform is given by

\[
p(s) = \frac{1}{2\pi} \int_{\mathcal{R}} K_\sigma(s, \omega)\hat{p}_\sigma(\omega)d\omega, \ s \in \mathcal{R},
\]

where the kernel

\[
K_\sigma(s, \omega) = \begin{cases} \mathcal{C}'_\sigma e^{(-ip^\ast + \omega^\ast) \cot \sigma \csc \sigma}, & \text{if } \sigma \neq m\pi, \\ \frac{1}{\sqrt{2\pi}} e^{ip\omega}, & \text{if } \sigma = \frac{\pi}{2}, \\ \mathcal{C}'_\sigma = 2\pi(1 + i \cot \sigma). & \text{if } \sigma = m\pi, \end{cases}
\]

Definition 2. The Mittag-Leffler function is given in the following manner:

\[
E_\sigma(s, \omega) = \sum_{m=0}^\infty \frac{s^m}{\Gamma(\sigma m + 1)}, \quad (\sigma > 0) \quad \text{(One parameter)},
\]

\[
E_{\sigma, \mu}(s, \omega) = \sum_{m=0}^\infty \frac{s^m}{\Gamma(\sigma m + \mu)}, \quad (\sigma > 0, \mu > 0) \quad \text{(Two parameters)}.
\]

where \(\sigma\) and \(\mu\) are nonnegative constant.

Definition 3 (see [28]). The fractional integral operator of order \(s > 0\) of a function \(p \in \mathcal{L}^1(\mathcal{R}^\ast)\) is written as

\[
I_0^\sigma p(s) = \frac{1}{\Gamma(\sigma)} \int_0^s (s-u)^{\sigma-1} p(u)du, \ s > 0,
\]

where \(\Gamma(.)\) is the gamma function and \(\mathbb{R}e > 0\).

Definition 4 (see [28]). The Riemann-Liouville fractional order derivative of \(s > 0, m-1 < \sigma < m, m \in \mathbb{N}\), is written as

\[
\left( ^\sigma DR_{0+}^\mu \right)(s) = \frac{1}{\Gamma(m-\sigma)} \left( \frac{d}{ds} \right)^m \int_0^s (s-u)^{(m-\sigma-1)} p(u)du,
\]

where the function \(p(s)\) is a continuous derivatives up to order \((m-1)\).
Definition 5 (see [28]). The fractional Caputo-type derivative of order \( s > 0, m - 1 < \sigma < m, m \in \mathcal{N} \), is written as

\[
\left( ^c D_0^\sigma p \right)(s) = \frac{1}{\Gamma(m - \sigma)} \int_0^s (s - u)^{(m-\sigma-1)} p^{(m)}(u) du,
\]

where the function \( p(s) \) is a continuous derivatives up to order \( (m - 1) \). Then, let \( s > 0, \sigma \in \mathcal{R}, m - 1 < \sigma < m, m \in \mathcal{N} \). The relation between Caputo and Riemann-Liouville fractional derivative is given by

\[
\left( ^c D_0^\sigma p \right)(s) = (\mathcal{D}_0^\sigma p)(s) - \sum_{k=0}^{m-1} \frac{\Gamma(k+1)}{\Gamma(k-\sigma+1)} p^{(k)}(0),
\]

Definition 6. Equation (1) has Ulam-Hyers-Mittag-Leffler stability, if there exist a continuously differentiable function \( p(s) \) satisfying the inequality

\[
\left| \left( ^c D_0^\sigma p \right)(s) + \eta p(s) - q(s) \right| \leq \varepsilon E_\sigma(s), \forall s > 0,
\]

for every \( \varepsilon > 0 \), there exists a solution \( p_o(s) \) satisfying Equation (1) such that

\[
|p(s) - p_o(s)| \leq \varepsilon E_\sigma(s),
\]

where \( \varepsilon \) is a nonnegative and stability constant.

Definition 7. The considered \( \phi : (0, \infty) \rightarrow (0, \infty) \) is a function. Equation (1) has Ulam-Hyers-Rassias-Mittag-Leffler stability, if there exist a continuously differentiable function \( p(s) \) satisfying the inequality

\[
\left| \left( ^c D_0^\sigma p \right)(s) + \eta p(s) - q(s) \right| \leq \varepsilon \phi(s) E_\sigma(s), \forall s > 0,
\]

for every \( \varepsilon > 0 \), there exists a solution \( p_o(s) \) satisfying Equation (1) such that

\[
|p(s) - p_o(s)| \leq \varepsilon \phi(s) E_\sigma(s),
\]

where \( \varepsilon \) is a nonnegative and stability constant.

3. Main Results

In this section, we will investigate to help of fractional Fourier transform to study the Ulam-Hyers-Mittag-Leffler stability of (1).

Theorem 8. If a function \( p(s) \) satisfies the inequality (11) for every \( \varepsilon > 0 \), there exists a solution \( p_o(s) \) satisfying Equation (1) such that

\[
|p(s) - p_o(s)| \leq \varepsilon E_\sigma(s).
\]

Proof. Let us choose a function \( y(s) \) follow as

\[
y(s) = \left( ^c D_0^\sigma p \right)(s) + \eta p(s) - q(s).
\]

Now,

\[
y(s) = (\mathcal{D}_0^\sigma p)(s) - \sum_{k=0}^{m-1} \frac{s^k-\sigma}{\Gamma(k-\sigma+1)} p^{(k)}(0) + \eta p(s) - q(s), \forall s > 0.
\]

Taking \( \mathcal{F}_\sigma \) (the fractional Fourier transform operator) onto both sides of Equation (17), we have

\[
\mathcal{F}_\sigma \{y(s)\} = \mathcal{F}_\sigma \left\{ \mathcal{D}_0^\sigma p(s) - \sum_{k=0}^{m-1} \frac{s^k-\sigma}{\Gamma(k-\sigma+1)} p^{(k)}(0) + \eta p(s) - q(s) \right\}
\]

where \( p^{(k)}(0) = a_k \), for \( k = 0, 1, \cdots, m - 1 \) and

\[
\mathcal{F}_\sigma \{p(s)\} = \mathcal{F}_\sigma \left\{ \mathcal{D}_0^\sigma \phi(s) E_\sigma(s) \right\}
\]

Setting

\[
p_o(s) = \sum_{k=0}^{m-1} a_k E_\sigma(s) + \int_0^s (s - v)^{-\nu-1} \eta(s - v)^{-\nu-1} q(v) dv.
\]

By using fractional Fourier transform to (20), we have

\[
\mathcal{F}_\sigma \{p_o(s)\} = \frac{e^{i\omega_n s}}{(i\omega_n + \eta)} \sum_{k=0}^{m-1} \frac{a_k}{(i\omega_n + \eta)} + \mathcal{G}_n(\omega).
\]

Hence,

\[
\left( ^c D_0^\sigma p \right)(s) + \eta p(s) = (i\omega_n)^\sigma \mathcal{F}_\sigma \{p(s)\}
\]

\[
= \frac{e^{i\omega_n s}}{(i\omega_n + \eta)} \sum_{k=0}^{m-1} \frac{a_k}{(i\omega_n + \eta)} + \mathcal{G}_n(\omega)
\]

\[
+ \eta \mathcal{F}_\sigma \{p(s) - \mathcal{G}_n(\omega)\} = q(s).
\]

□
Since $\mathcal{F}_\sigma$ is one-to-one operator, $(\mathcal{D}^\alpha_{0^+} p)(t) + \eta p(t) = q(t)$. Now, its follows form (19) and (21) that

$$\mathcal{F}_\sigma \{ p(s) \} - \mathcal{F}_\sigma \{ p_\sigma(s) \} = \mathcal{F}_\sigma \{ y(s) \} \left( \left( (i\omega^{n\sigma}) + \eta \right) \right)^{-1}.$$

(23)

Using the convolution property, we obtain

$$\mathcal{F}_\sigma \{ p(s) - p_\sigma(s) \} = \mathcal{F}_\sigma \{ y(s) \} \ast \frac{1}{((i\omega^{n\sigma}) + \eta)} = y(s) \ast y_\sigma(s).$$

(24)

where $y_\sigma(s) = 1/((i\omega^{n\sigma}) + \eta)$. In view of (13), we have

$$|y(s)| \leq \epsilon E_\sigma(s), \forall s > 0. \quad (25)$$

Now, applying the modules on both sides of Equation (24), we get

$$|p(s) - p_\sigma(s)| = \left| \int_0^s (s-x)^{-\sigma} E_\sigma(\eta(s-v)^\sigma) \ast y(s)dv \right| \leq |y(s)| \int_0^s (s-v)^{-\sigma} E_\sigma(\eta(s-v)^\sigma)dv \leq \epsilon E_\sigma(s) \left| \int_0^s (s-v)^{-\sigma} E_\sigma(\eta(s-v)^\sigma)dv \right| \leq \mathcal{H} E_\sigma(s).$$

where $\mathcal{H} = \left| \int_0^s (s-x)^{-\sigma} E_\sigma(\eta(s-x)^\sigma)dv \right|$. Thus Equation (1) has Ulam-Hyers-Mittag-Leffler stability.

**Corollary 9.** The considered $\phi: (0,\infty) \to (0,\infty)$ is a function. If a function $p(s)$ satisfies the inequality (13), for every $\epsilon > 0$, there exists a solution $p_\sigma(s)$ satisfying Equation (1) such that

$$|p(s) - p_\sigma(s)| \leq \mathcal{H} \phi(s) E_\sigma(s), \forall s > 0. \quad (27)$$

i.e., Equation (1) has Ulam-Hyers-Rassias-Mittag-Leffler stability.

**4. Applications**

In this section, the standard kinetic equation in the chemical reaction that will be used to analyze this experimental data is revealed by the equation as follows: where $\mathcal{L} = \text{xylan}$; $\mathcal{M} = \text{xylene}$; $\mathcal{N} = \text{products of decomposition}$; $r_1 = \text{release rate of sugar}$; $r_2 = \text{decomposition rate of sugar}$. The model is presented in Figure 1.

Material balance for components: $\mathcal{L}$ and $\mathcal{M}$ for the first-order kinetic equation, we get

$$-\frac{dN_\mathcal{M}(s)}{ds} = r_1 N_\mathcal{M}(s),$$

(28)

in which the initial concentration at $s = 0$ is presented by $N_\mathcal{M}(0) = N_\mathcal{M}_0$. Also, we have the same direction for material $\mathcal{N}$:

$$-\frac{dN_\mathcal{N}(s)}{ds} = r_2 N_\mathcal{N}(s) + r_1 N_\mathcal{M}(s),$$

(29)

in which the initial concentration at $s = 0$ is presented by $N_\mathcal{N}(0) = N_\mathcal{N}_0$. Equation (29) can be integrated and, using the provided boundary condition, yields

$$N_\mathcal{N}(s) = N_\mathcal{N}_0 \exp(-r_1 s). \quad (30)$$

Substituting (30) for (29) yields

$$\frac{dN_\mathcal{M}(s)}{ds} + r_2 N_\mathcal{M}(s) = r_1 N_\mathcal{M}(s), \quad (31)$$

in which the initial concentration at $s = 0$ is presented by $N_\mathcal{M}(0) = N_\mathcal{M}_0$. Also, we have the same direction for material $\mathcal{M}$:

Figure 2 shows the solution of Equation (29) for various $r_1$ and $r_2$.

**5. Conclusions**

In this paper, the objective is investigated by using the fractional Fourier transform to study the Ulam-Hyers-Mittag-Leffler stability of linear fractional differential equations. The required outcomes have been achieved by using the fractional Fourier transform. We could reach the suitable approximation value of xylose after a certain period of time, which is crucial for analyzing the kinetic equation in the chemical reaction process.
Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] P. Agarwal and S. Jain, “Further results on fractional calculus of Srivastava polynomials,” Bulletin of Mathematical Analysis and Applications, vol. 3, no. 2, pp. 167–174, 2011.
[2] P. Agarwal, S. Jain, S. Agarwal, and M. Nagpal, “On a new class of integrals involving Bessel functions of the first kind,” Communications in Numerical Analysis, vol. 2014, pp. 1–7, 2014.
[3] P. Agarwal, U. Baltaeva, and Y. Alikulov, “Solvability of the boundary-value problem for a linear loaded integro-differential equation in an infinite three-dimensional domain,” Chaos, Solitons and Fractals, vol. 140, p. 110108, 2020.
[4] S. F. Lacroix, Traité du calcul différentiel et du calcul intégral Tome 3, Traité du calcul différentiel et du calcul intégral, Elsevier, Paris: Courcier, 1819.
[5] H. M. Ozaktas and M. A. Kutay, “The fractional Fourier transform,” in 2001 European Control Conference (ECC), pp. 1477–1483, 2001.
[6] S. M. Ulam, Chapter IV, Problem in Modern Mathematics, Science Editors, Willey, New York, 1960.
[7] D. H. Hyers, “On the stability of the linear functional equation,” Proceedings of the National Academy of Sciences, vol. 27, no. 4, pp. 222–224, 1941.
[8] T. M. Rassias, “On the stability of the linear mapping in Banach spaces,” Proceedings of the American Mathematical Society, vol. 72, no. 2, pp. 297–300, 1978.
[9] S. M. Jung, “Hyers-Ulam stability of linear differential equations of first order,” Applied Mathematics Letters, vol. 17, no. 10, pp. 1135–1140, 2004.
[10] S. M. Jung, “Hyers-Ulam stability of linear differential equations of first order, III,” Journal of Mathematical Analysis and Applications, vol. 311, no. 1, pp. 139–146, 2005.
[11] S. M. Jung, “Hyers-Ulam stability of linear differential equations of first order, II,” Applied Mathematics Letters, vol. 19, no. 9, pp. 854–858, 2006.
[12] S. M. Jung, “Hyers-Ulam stability of a system of first order linear differential equations with constant coefficients,” Journal of Mathematical Analysis and Applications, vol. 320, no. 2, pp. 549–561, 2006.
[13] A. Pommana Selvan, S. Sabarinathan, and A. Selvam, “Approximate solution of the special type differential equation of higher order using Taylor’s series,” Journal of Mathematics and Computer Science, vol. 27, no. 2, pp. 131–141, 2022.
[14] F. Zhang and C. Li, “Stability analysis of fractional differential systems with order lying in (1, 2),” Advances in Difference Equations, vol. 2011, Article ID 213485, 17 pages, 2011.
[15] C. P. Li and F. R. Zhang, “A survey on the stability of fractional differential equations,” The European Physical Journal Special Topics, vol. 193, no. 1, pp. 27–47, 2011.
[16] R. W. Ibrahim, “Stability of fractional differential equations,” International Journal of Mathematics and Computer Science Engineering, vol. 7, no. 3, pp. 212–217, 2013.
[17] V. Kalvandi, N. Eghbali, and J. M. Rassia, “Mittag-Leffler-Hyers-Ulam stability of fractional differential equations of second order,” Journal of Mathematical Extension, vol. 13, pp. 1–15, 2019.
[18] K. Liu, J. Wang, Y. Zhou, and D. O’Regan, “Hyers-Ulam stability and existence of solutions for fractional differential equations with Mittag-Leffler kernel,” Chaos, Solitons & Fractals, vol. 132, p. 109534, 2020.
[19] H. Vu, T. V. An, and N. V. Hoa, “Ulam-Hyers stability of uncertain functional differential equation in fuzzy setting with Caputo-Hadamard fractional derivative concept,” Journal of Intelligent Fuzzy Systems, vol. 38, no. 2, pp. 2245–2259, 2020.
[20] J. Wang, L. Lv, and Y. Zhou, “Ulam stability and data dependence for fractional differential equations with Caputo derivative,” Electronic Journal of Qualitative Theory of Differential Equations, vol. 63, no. 63, pp. 1–10, 2011.
[21] J. R. Wang and Y. Zhou, “Mittag-Leffler-Ulam stabilities of fractional evolution equations,” Applied Mathematics Letters, vol. 25, no. 4, pp. 723–728, 2012.
[22] J. R. Wang, Y. Zhou, and M. Feckan, “Nonlinear impulsive problems for fractional differential equations and Ulam stability,” Computers & Mathematics with Applications, vol. 64, no. 10, pp. 3389–3405, 2012.
[23] B. Unyong, A. Mohanapriya, A. Ganesh et al., “Fractional Fourier transform and stability of fractional differential equation on Lizorkin space,” Advances in Difference Equations, vol. 2020, no. 1, article 578, 23 pages, 2020.
[24] P. Hammachukiattikul, A. Mohanapriya, A. Ganesh et al., “A study on fractional differential equations using the fractional Fourier transform,” Advances in Difference Equations, vol. 2020, no. 1, article 691, 22 pages, 2020.
[25] A. Ganesh, V. Govindan, J. R. Lee, A. Mohanapriya, and C. Park, “Mittag-Leffler-Hyers-Ulam stability of delay fractional differential equation via fractional Fourier transform,” Results in Mathematics, vol. 76, no. 4, pp. 1–7, 2021.
[26] A. Ganesh, S. Deepa, D. Baleanu et al., “Hyers-Ulam-Mittag-Leffler stability of fractional differential equations with two caputo derivative using fractional fourier transform,” AIMS Mathematics, vol. 7, no. 2, pp. 1791–1810, 2022.
[27] A. I. Zayed, “Fractional Fourier transform of generalized functions,” Integral Transforms and Special Functions, vol. 7, no. 3–4, pp. 299–312, 1998.
[28] A. A. Kilbas and J. J. Trujillo, “Differential equations of fractional order: methods results and problem—I,” Applicable Analysis, vol. 78, no. 1-2, pp. 153–192, 2001.