IDENTIFICATION OF MATCHING COMPLEMENTARITIES: A GEOMETRIC VIEWPOINT

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Abstract. We provide a geometric formulation of the problem of identification of the matching surplus function and we show how the estimation problem can be solved by the introduction of a generalized entropy function over the set of matchings.

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1. Setting

We consider the Becker model of the marriage market as a bipartite matching game with transferable utility. Let \( \mathcal{X} \) and \( \mathcal{Y} \) be finite sets of “types” of men and women where \( |\mathcal{X}| = d_x \) and \( |\mathcal{Y}| = d_y \). Assume that the number of men and women is equal, and that the number of men of type \( x \) (resp. of women of type \( y \)) is \( p_x \) (resp. \( q_y \)). We normalize the total number of men and women to one, that is we set \( \sum_{x \in \mathcal{X}} p_x = 1 \) and \( \sum_{y \in \mathcal{Y}} q_y = 1 \). Let \( \Phi_{xy} \geq 0 \) be the...
joint surplus (to be split endogenously across the pair) from matching a man of type $x$ and a woman of type $y$. For the clarity of exposition we do not allow for unmatched individuals.

Recall that under transferable utility, in the Shapley-Shubik model, the stable matching also maximizes the total surplus

$$\sum_{x,y} \mu_{xy} \Phi_{xy}$$

over $\mu \in \mathcal{M}$ the set of matchings, defined by

$$\mathcal{M} = \left\{ \mu : \mu_{xy} \geq 0, \sum_y \mu_{xy} = p_x, \sum_x \mu_{xy} = q_y \right\},$$

where $\mu_{xy}$ is interpreted as the number of $(x, y)$ pairs, which is allowed to be a fractional number.

Note that the equations defining $\mathcal{M}$ have $d_x + d_y - 1$ degrees of redundancy, hence the dimension of $\mathcal{M}$ is $d_x d_y - d_x - d_y + 1 = (d_x - 1)(d_y - 1)$.

Further, if $\mu$ and $\tilde{\mu}$ are in $\mathcal{M}$, then for $t \in [0, 1]$, $(t\mu + (1-t) \tilde{\mu})$ is also in $\mathcal{M}$. Finally, $\mathcal{M}$ is obviously bounded in $\mathbb{R}^{d_x d_y}$. Hence:

**Claim 1.** The set of matchings $\mathcal{M}$ is a compact convex set of $\mathbb{R}^{d_x d_y}$.

### 2. Identification

One observes a matching $\hat{\mu} \in \mathcal{M}$ and one wonders whether $\hat{\mu}$ is rationalizable, i.e. whether there exists some surplus function $\Phi$ such that $\hat{\mu}$ is the optimal matching in the problem with surplus $\Phi$, that is

$$\hat{\mu} \in \arg \max_{\mu \in \mathcal{M}} \sum_{x,y} \mu_{xy} \Phi_{xy}.$$  

As it is classically the case in revealed preference analysis, some restrictions on $\Phi$ are needed in order to have a meaningful definition. Indeed, the null surplus function $\Phi_{xy} = 0$ always trivially rationalizes any matching; similarly, $\Phi_{xy} = f_x + g_y$ which also rationalizes
any $\mu$ as the value of the total surplus evaluated at $\mu$ is $\sum_x p_x f_x + \sum_y q_y g_y$ irrespective of $\mu \in \mathcal{M}$. Hence in order to have some empirical bite, we need to impose

$$\arg \max_{\mu \in \mathcal{M}} \sum_{x,y} \mu_{xy} \phi_{xy} \neq \mathcal{M}.$$ 

Let $\mathbf{S}$ be the set of $\Phi$ such that $\Phi_{xy}$ does not coincides with $f_x + g_y$ for some vectors $(f_x)$ and $(g_y)$. We shall thus seek $\Phi$ in $\mathbf{S}$. The following assertion characterizes $\Phi$ in dimension two.

Claim 2. Assume $d_x = d_y = 2$. Then $\mathbf{S}$ is the set of $(\Phi_{xy})$ such that $\Phi_{11} + \Phi_{22} \neq \Phi_{12} + \Phi_{21}$.

The previous considerations lead to the following definition:

Definition 1. $\hat{\mu} \in \mathcal{M}$ is rationalizable if there is $\hat{\phi} \in \mathbf{S}$ such that

$$\hat{\mu} \in \arg \max_{\mu \in \mathcal{M}} \sum_{x,y} \mu_{xy} \hat{\phi}_{xy}.$$ (1)

Introducing $W_0$ the indirect surplus function, defined as

$$W_0 (\Phi) = \max_{\mu \in \mathcal{M}} \langle \mu, \Phi \rangle$$ (2)

where the product $\langle \mu, \Phi \rangle$ is defined as

$$\langle \mu, \Phi \rangle = \sum_{xy} \mu_{xy} \Phi_{xy},$$ (3)

condition (1) is equivalent, by the Envelope theorem, to

$$\hat{\mu} \in \partial W_0 (\hat{\phi})$$

where $\partial W_0 (\Phi)$ denotes the subgradient of $W_0$ at $\Phi$. See the Appendix for some basic results on convex analysis. In the terminology of convex analysis, $W_0$ is the support function of set $\mathcal{M}$, a geometric property which we shall develop in the next paragraph.

The following remark is obvious.
Claim 3. $W_0$ is positive homogenous of degree one, hence for $t > 0$, one has

\[ W_0(t\Phi) = tW_0(\Phi) \]  (4)

\[ \partial W_0(t\Phi) = \partial W_0(\Phi). \]  (5)

3. Geometry

The following result provides the geometric interpretation of rationalizability. Formula (1) means that for $\hat{\mu}$ to be rationalizable, it needs to maximize a linear functional over the compact convex set $M$. As it is well known, a necessary and sufficient for this to hold is that $\hat{\mu}$ should belong to the boundary of $M$.

Theorem 1. The following three conditions are equivalent:

(i) $\hat{\mu}$ is rationalizable,

(ii) $\hat{\mu}$ lies on $\overline{M\setminus M^{\text{int}}}$, the boundary of $M$,

(iii) There is $\hat{\Phi} \in S$ such that

\[ \hat{\mu} \in \partial W_0\left(\hat{\Phi}\right). \]  (6)

This theorem is illustrated in Figure 1. While the equivalence between part (ii), of geometric kind and part (iii), of analytic nature follows from standard convex analysis, the insight of this result is to connect this to the economic notion of rationalizability (i), of revealed preference flavour. This result provides a geometric understanding of revealed preference analysis in matching models with transferable utility. See Echenique et al. (2012).

Geometrically, this means that the matchings that are rationalizable lie on the boundary of $M$. We give a very simple example of a $\hat{\mu}$ which is rationalizable.

Example 1. Assume $d_x = d_y = 2$ and consider matrix

\[ \hat{\mu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

then any $\hat{\Phi}$ such that $\hat{\Phi}_{11} + \hat{\Phi}_{22} > \hat{\Phi}_{12} + \hat{\Phi}_{21}$ rationalizes $\hat{\mu}$. 
Figure 1. Geometric view of rationalizability. In order for matching $\mu$ to be rationalized by surplus function $\Phi$, $\mu$ need to lie on the geometric frontier of $\mathcal{M}$.

We now give a very simple example of a $\hat{\mu}$ which not is rationalizable, i.e. where $\hat{\mu}$ is in the strict interior of $\mathcal{M}$.

Example 2. Assume $d_x = d_y = 2$ and consider matrix

$$\hat{\mu} = \begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{pmatrix}.$$  

This matrix is equal to $0.7 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 0.3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Hence for a production function $\Phi$, we get

$$\sum_{xy} \hat{\mu}_{xy} \Phi_{xy} = 0.7 (\Phi_{11} + \Phi_{22}) + 0.3 (\Phi_{12} + \Phi_{21})$$

Hence it cannot be rationalized by a production function $\Phi$ unless $\Phi_{11} + \Phi_{22} = \Phi_{12} + \Phi_{21}$.

But in that case, set $a_1 = \Phi_{11}$, $b_1 = 0$, $a_2 = \Phi_{21}$, and $b_2 = \Phi_{12} - \Phi_{11}$, thus $\Phi_{ij} = a_i + b_j$ - which contradicts $\Phi \in \mathbf{S}$. Therefore $\hat{\mu}$ cannot be rationalized.
Example 3. As another example, consider $p \otimes q$ defined by $(p \otimes q)_{xy} = p_x q_y$. Clearly, $p \otimes q \in \mathcal{M}$; intuitively this matching corresponds to matching randomly men and women, so that the characteristics of the partner are independent. This matching cannot be rationalized as it lies in the strict interior of $\mathcal{M}$. Indeed, $p \otimes q$ is the barycenter of the full set $\mathcal{M}$.

4. Entropy

In practice, it is almost never the case that a matching $\hat{\mu}$ observed in the population is rationalizable. This is understandable using the geometric interpretation provided above: the locus of matchings that are rationalizable being the frontier of a convex set, it is “small” with respect to the set of matchings that are not rationalizable, which is the strict interior of this same convex set.

Mathematically speaking, we are looking for a solution $\Phi \in \mathcal{S}$ satisfying

$$\hat{\mu} \in \partial \mathcal{W}_0(\Phi).$$

(7)

If $\mathcal{W}_0$ was “well behaved,” more precisely if $\mathcal{W}_0$ was strictly convex and continuously differentiable, then the gradient $\nabla \mathcal{W}_0$ would exist and be invertible with inverse $\nabla \mathcal{W}_0^*$, where $\mathcal{W}_0^*$ is the convex conjugate of $\mathcal{W}_0$. Then relation (7) would imply $\Phi = \nabla \mathcal{W}_0^*(\hat{\mu})$. But $\mathcal{W}_0$ is not strictly convex, so this approach does not work, and in fact relation (7) has no solution. Geometrically, it is quite clear why. As remarked above, the image of $\partial \mathcal{W}_0$ is included in the frontier of $\mathcal{M}$, hence if $\hat{\mu}$ does not lie on the geometric frontier of $\mathcal{M}$, then relation (7) cannot possibly have a solution.

In order to be able to estimate $\Phi$ based on the observation of $\hat{\mu}$, most of the literature following the seminal paper of Choo and Siow (2005) introduce heterogeneities in matching surpluses. Without trying to be exhaustive, let us mention Fox (2010, 2011), Galichon and Salanié (2010, 2012), Decker et al. (2012), Chiappori et al. (2012). As argued in Galichon and Salanié (2012), this consists in essence in introducing a generalized entropy function $\mathcal{I}(\mu)$ which is strictly convex, and which is such that

$$\mathcal{I}(\mu) = +\infty \text{ if } \mu \notin \mathcal{M},$$
such that $I$ is differentiable on $M^{\text{int}}$ the interior of $M$, with, for all $\mu \in M^{\text{int}}$,
\[
\nabla I(\mu) \in S,
\]
and such $\hat{\Phi}$ is identified by
\[
\hat{\Phi} = \nabla I(\hat{\mu}).
\]

Noting that (8) is the first order condition to the following optimization program
\[
W_I(\Phi) = \max_{\mu \in M} \langle \mu, \Phi \rangle - I(\mu)
\]
which, as argued in Galichon and Salanié (2010, 2012), can be interpreted in some cases as the social welfare of a matching model with unobserved heterogeneity.

**Example 4.** Recall the definition $(p \otimes q)_{xy} = p_x p_y$, and remember that $p \otimes q$ is never on the frontier of $M$, hence never rationalizable. When $\hat{\mu}$ is not rationalizable either, one may consider the smallest $t$ such that $p \otimes q + t (\hat{\mu} - p \otimes q)$ is rationalizable. This number exists and is finite because the halfline which starts from $p \otimes q$ through $\hat{\mu}$ must cross the frontier of $M$, which is a convex and compact set. Letting $t^*$ be the corresponding value of $t$, and $\mu^* = p \otimes q + t^* (\hat{\mu} - p \otimes q)$, there exists by definition an element $\hat{\Phi} \in S \setminus \{0\}$ such that $\mu^* \in \partial W_0(\hat{\Phi})$, where $W_0$ is as in (2). Note that if $\hat{\mu}$ is rationalizable, then $t^* = 1$ and $\mu^* = \mu$. See Figure 2.

This construction can be expressed in terms of $I$. Letting
\[
I(\hat{\mu}) = \begin{cases} 
- \max_{t \geq 1} \{ t : p \otimes q + t (\hat{\mu} - p \otimes q) \in M \} & \text{if } \hat{\mu} \in M \\
+ \infty & \text{else}
\end{cases}
\]
so that $I(\hat{\mu})$ can be formulated as a max-min problem, that is, for $\hat{\mu} \in M$,
\[
I(\hat{\mu}) = - \max_{t \geq 1} \min_{\Phi \in S} \{ t + W_0(\Phi) - \langle \Phi, p \otimes q + t (\hat{\mu} - p \otimes q) \rangle \}.
\]

Because the objective function is convex in $\Phi$ and linear in $t$, this problem has a saddle-point which will be denoted $(\Phi^*, t^*)$. Let $\mu^* = p \otimes q + t^* (\hat{\mu} - p \otimes q)$. By optimality with respect to $\Phi$, $\mu^* \in \partial W_0(\Phi^*)$, thus $\Phi^*$ rationalizes $\mu^*$. By the envelope theorem
\[
t^* \Phi^* = \nabla I(\hat{\mu}),
\]
thus we take
\[ \hat{\Phi} = t^* \Phi^*. \]

and \( \mu^* \) is the matching which is on the halfline which starts from \( p \otimes q \) through \( \hat{\mu} \) and which is rationalizable.

**Figure 2.** Geometric illustration of Example 4. \( \hat{\mu} \) is not rationalizable, but it is associated to some proximate \( \mu^* \) on the boundary of \( M \), which is itself rationalized by \( \hat{\Phi} \).

**Example 5.** In the Choo and Siow (2005) model, the surplus function is \( \Phi_{ij} = \Phi(x,y) + \varepsilon_{iy} + \eta_{jx} \) where \( \varepsilon_{iy} \) and \( \eta_{jx} \) are iid extreme value type I random variables. Choo and Siow use this model nonparametrically identifies \( \Phi \). Galichon and Salanié (2010) show that this model leads to the following specification of \( I \):

\[
I(\mu) = \sum_{xy} \mu_{xy} \log \mu_{xy} \text{ if } \mu \in M \\
= +\infty \text{ else.} \tag{12}
\]
Example 6. Galichon and Salanié (2012) argue that the model of Choo and Siow actually extends in the case where the matching surplus function in the presence of heterogeneities between man $i$ of type $x$ and woman $j$ of type $y$ is $\Phi_{ij} = \Phi(x, y) + \varepsilon_{ixy} + \eta_{jxy}$, and letting $G_x(U) = \mathbb{E}[\max_y (U_{xy} + \varepsilon_{ixy})]$ and $H_y(V) = \mathbb{E}[\max_x (V_{xy} + \eta_{jxy})]$ be the ex-ante indirect utilities of respectively the man of type $x$ and the woman of type $y$, and letting $G^*$ and $H^*$ their respective convex conjugate transforms, that is

$$G^*_x(\mu|x) = \sup_{U_{xy}} \{ \sum_y \mu_{y|x} U_{xy} - G_x(U) \} \text{ if } \sum_y \mu_{y|x} = 1$$

$$= +\infty \text{ else,}$$

and

$$H^*_y(\mu|y) = \sup_{V_{xy}} \{ \sum_x \mu_{x|y} V_{xy} - H_y(V) \} \text{ if } \sum_x \mu_{x|y} = 1$$

$$= +\infty \text{ else.}$$

Then $I(\mu)$ is given by

$$I(\mu) = \sum_x p_x G^*_x(\mu|x) + \sum_y q_y H^*_y(\mu|y). \quad (13)$$

which coincides with (12) in the case studied by Choo and Siow, hence the term “generalized entropy”. As an important consequence, this paves the way to the continuous generalization of the Choo and Siow model. See Dupuy and Galichon (2012), and Bojilov and Galichon (2013).

Example 7. Applying this setting, Galichon and Salanié (2012, Example 3) assume that $X$ and $Y$ are finite subsets of $\mathbb{R}$, and that $\varepsilon_{ixy} = e_{ix}$ while $\eta_{jxy} = f_{jx}$ where $e_i$ and $f_j$ are drawn from $\mathcal{U}([0, 1])$ distributions. In this case the utility shocks are perfectly correlated across alternatives, in sharp contrast with Example 1, where they are independent. Then, letting $Q^\mu_{Y|X=x}$ be the conditional quantile of $Y$ conditional on $X = x$ under distribution $\mu$, one has

$$I(\mu) = \sum_x p_x \int_0^1 Q^\mu_{Y|X=x}(t) \, dt + \sum_y q_y \int_0^1 Q^\mu_{X|Y=y}(t) \, dt.$$
Facts from Convex Analysis

The definitions below are included for completeness and the reader is referred to Ekeland and Temam (1976) for a thorough exposition of the topic.

Take any set \( Y \subset \mathbb{R}^d \); then the convex hull of \( Y \) is the set of points in \( \mathbb{R}^d \) that are convex combinations of points in \( Y \). We usually focus on its closure, the closed convex hull, denoted \( cch(Y) \).

The support function \( S_Y \) of \( Y \) is defined as

\[
S_Y(x) = \sup_{y \in Y} x \cdot y
\]

for any \( x \) in \( Y \), where \( x \cdot y \) denotes the standard scalar product. It is a convex function, and it is homogeneous of degree one. Moreover, \( S_Y = S_{cch(Y)} \) where \( cch(Y) \) is the closed convex hull of \( Y \), and \( \partial S_Y(0) = cch(Y) \).

A point in \( Y \) is an boundary point if it belongs in the closure of \( Y \), but not in its interior.

Now let \( u \) be a convex, continuous function defined on \( \mathbb{R}^d \). Then the gradient \( \nabla u \) of \( u \) is well-defined almost everywhere and locally bounded. If \( u \) is differentiable at \( x \), then

\[
u(x') \geq u(x) + \nabla u(x) \cdot (x' - x)
\]

for all \( x' \in \mathbb{R}^d \). Moreover, if \( u \) is also differentiable at \( x' \), then

\[
(\nabla u(x) - \nabla u(x')) \cdot (x - x') \geq 0.
\]

When \( u \) is not differentiable in \( x \), it is still subdifferentiable in the following sense. We define \( \partial u(x) \) as

\[
\partial u(x) = \left\{ y \in \mathbb{R}^d : \forall x' \in \mathbb{R}^d, u(x') \geq u(x) + y \cdot (x' - x) \right\}.
\]

Then \( \partial u(x) \) is not empty, and it reduces to a single element if and only if \( u \) is differentiable at \( x \); in that case \( \partial u(x) = \{ \nabla u(x) \} \).
Given a convex function \( u \) defined on a convex subset of \( \mathbb{R}^d \), one defines its **convex conjugate** as

\[
u^*(y) = \sup_{x \in \mathbb{R}^d} \left\{ x \cdot y - u(x) \right\}.
\]

One has \( y \in \partial u(x) \) if and only if \( x \in \partial u^*(y) \) if and only if \( u(x) + u^*(y) = x \cdot y \).

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