A Suggested Answer To Wallstrom’s Criticism: Zitterbewegung Stochastic Mechanics I

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Abstract

Wallstrom’s criticism of existing formulations of stochastic mechanics is that they fail to derive the empirical predictions of orthodox quantum mechanics because they require an ad hoc quantization condition on the postulated velocity potential, $S$, in order to derive Schrödinger wave functions. We propose an answer to this criticism by modifying the Nelson-Yasue formulation of non-relativistic stochastic mechanics for a spinless particle with the following hypothesis: a spinless Nelson-Yasue particle of rest mass $m$ continuously undergoes a driven steady-state oscillation of ‘zitterbewegung’ (zbw) frequency, $\omega_c = (1/\hbar) mc^2$, in its instantaneous mean forward (and backward) translational rest frame. With this hypothesis we show that, in the lab frame, $S$ arises from imposing the constraint of conservative diffusions on the time-symmetrized steady-state phase of the zbw particle, satisfies the required quantization condition, and evolves in time by the Hamilton-Jacobi-Madelung equations (when generalized to describe a statistical ensemble of zbw particles). From the mathematical equivalence of Schrödinger’s equation with the Hamilton-Jacobi-Madelung equations plus the quantization condition, Schrödinger wave functions for a spinless particle (in and excluding external fields) are thereby recovered. We also apply this ‘zitterbewegung stochastic mechanics’ (ZSM) to the case of a central potential and show that it predicts angular momentum quantization. This paper sets the foundation for Part II, which will (primarily) work out the many-particle version of ZSM.

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1 Introduction

Since its introduction by Fényes in 1952 [1], the goal of the stochastic mechanics research program has been to derive quantum theory from a classical-like statistical mechanics of particles undergoing Brownian motion. Towards this end, non-relativistic and relativistic models of stochastic mechanics have been constructed for both spin-0 particles [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32] and spin-1/2 particles [33, 22, 34, 35, 31]. A non-relativistic theory of single-time and multi-time measurements has also been developed [36, 37, 38, 39, 40], as have extensions of non-relativistic stochastic mechanics to finite temperature and non-equilibrium open systems [5, 41, 42, 19, 43]. Field theoretic generalizations also exist, for the cases of scalar fields [44, 45, 46, 10, 47], Maxwell fields [48, 49], vector-meson fields [50], the linearized gravitational field [51], coupling to dissipative environments [45, 52], non-Abelian gauge theory [9], bosonic string theory [53], M-theory [25], and background-independent quantum gravity [20]. However, Wallstrom [54, 55] pointed out that extant formulations of stochastic mechanics ultimately fail to derive quantum mechanics because they require an “ad hoc” quantization condition on the postulated velocity potential, S, in order to recover single-valued Schrödinger wave functions. Moreover, this criticism appears to generalize to the field-theoretic and quantum gravitational versions of stochastic mechanics developed before, during, and after Wallstrom’s publications, insofar as they require analogous quantization conditions and don’t seem to give non-circular justifications for them.
Since Wallstrom, sporadic attempts have been made to answer his criticism \cite{56, 55, 57, 58, 59, 60, 61, 62}. However, in our view, all these attempts are either problematic or limited in their applicability to stochastic mechanics (the follow up paper, Part II, will give a discussion). Nevertheless, if a convincing answer can be found, stochastic mechanics may once again be viewed as a viable research program, and one that (in our view) offers elegant solutions to many of the foundational problems with quantum mechanics. As examples, stochastic mechanics would provide:

1. an unambiguous solution to the quantum measurement problem (the local beables of the theory on which measurement outcomes depend are point masses with definite trajectories at all times) \cite{37, 38, 39, 40};
2. a novel and unambiguous physical interpretation of the wave function (it is epistemic in the sense of being defined from field variables describing a fictitious ensemble of point masses undergoing conservative diffusions; and it has ontic properties in the specific sense that the evolutions of said variables are constrained by beables over and above the point masses) \cite{4, 55, 63};
3. an explanation for why the position basis is preferred in decoherence theory (the form of the Schrödinger Hamiltonian is a consequence of the particle diffusion process happening in position space) \cite{57, 19}; and
4. a justification for the symmetry postulates for wave functions of identical particles (they arise from natural symmetry conditions on the particle trajectories, with the possibility of parastatistics being excluded) \cite{4, 37, 17}.

In this connection, it is worth mentioning that some of the aforementioned virtues of stochastic mechanics, such as (1) and (4), are shared by de Broglie-Bohm theories \cite{64, 65, 66, 67, 68, 69, 70}; conversely, virtually all of the technical results obtained from de Broglie-Bohm theories can be directly imported into stochastic mechanics (basically because stochastic mechanics contains the dynamical equations of de Broglie-Bohm theories as a subset).

This being said, stochastic mechanics (if viable) has a notably significant difference from the ‘standard’ approaches to interpreting or reformulating or replacing the quantum formalism in a realist way that solves the measurement problem, those being many-worlds theories \cite{71, 72, 73}, de Broglie-Bohm theories \cite{64, 65, 66, 67, 68, 69}, and dynamical collapse theories \cite{74, 75, 76}. In all these approaches, the wave function is interpreted as fundamental and ontic (or as some kind of physical law \cite{77, 68, 78}), and the Schrödinger equation (or some nonlinear modification of it) is taken as a dynamical law. So if stochastic mechanics succeeds in deriving the Schrödinger equation and wave function, it constitutes (arguably) the first example of a measurement-problem-free ontological reconstruction of quantum mechanics in which the wave function could be considered (in a well-defined sense) as genuinely derived and epistemic, and the Schrödinger evolution as phenomenological rather than law-like\footnote{The recent “Many-Interacting-Worlds” (MIW) theory of Hall, Deckert, and Wiseman \cite{79}, shares some of these features in that it recovers the Schrödinger wave function as an effective, mean-field description of a large number of real classical worlds interacting through a non-classical (quantum) force. On the other hand, it seems that their approach is also subject to Wallstrom’s criticism in that they also have to assume the quantization condition (or something like it) on the dynamics of their classical worlds. Similar comments apply to the “Prodigal QM” theory of Sebens \cite{80}.}

Thus stochastic mechanics would (if viable) constitute a counterexample to an implicit assumption that motivates the aforementioned standard approaches

\footnote{Similarly, the “Trace Dynamics” theory of Steven Adler \cite{81, 76, 82, 76} aims to derive the quantum formalism as an approximation to the thermodynamic limit of a statistical mechanical description of Grassmannian matrices living on space-time. However, Trace Dynamics requires certain ad hoc assumptions, namely that the state-vector in the thermodynamic description has a norm-preserving nonlinear stochastic evolution. Such an assumption is ad hoc because it seems to have no justification from within the assumptions of Trace Dynamics, whereas it presumably should have such a justification in order to sustain the claim that Trace Dynamics derives the quantum formalism in a certain approximation. (This view is also espoused by Bassi et al. in \cite{76}.) In this sense, it seems fair to say that the norm-preserving assumption is to Trace Dynamics what the quantization condition is to (extant formulations of) stochastic mechanics.}
that the wave function and Schrödinger equation must be part of the fundamental ontology (or laws) and dynamical laws, respectively, in order to have a realist alternative to standard quantum theory that solves the measurement problem, is empirically adequate, and has a coherent physical/ontological interpretation.

It is also noteworthy that, as a dynamical theory of particle motion in which probabilities play no fundamental role, stochastic mechanics shares with de Broglie-Bohm theories the ability to justify the “quantum equilibrium” density $|\psi|^2$ from typicality arguments [83] and from dynamical relaxation of non-equilibrium densities to future equilibrium [37, 15, 16, 18]. As a result, stochastic mechanics can, on its own terms, be regarded as a more general physical theory that contains quantum mechanics as a fixed point - and outside this fixed point, it admits the possibility of non-equilibrium physics, e.g., measurements more precise than the uncertainty principle allows and superluminal signaling [4, 84, 85, 86]. We will also argue in Part II [63] that quantum non-equilibrium states are more plausibly motivated in stochastic mechanics than in deterministic de Broglie-Bohm theories.

For all these reasons and more, it seems worthwhile to consider whether the central obstacle for the stochastic mechanics research program - Wallstrom’s criticism - can be surmounted. The objective of this series of papers is to suggest how non-relativistic stochastic mechanics for spinless particles can be modified to provide a non-ad-hoc physical justification for the required quantization condition on $S$, and thereby recover all and only the single-valued wave functions of non-relativistic quantum mechanics. In this paper, we propose to modify the Nelson-Yasue formulation [4, 7] of non-relativistic stochastic mechanics for a spinless particle with the following hypothesis: a spinless particle of rest mass, $m$, bounded to a harmonic potential of natural frequency, $\omega_c = (1/\hbar) mc^2$, and immersed in Nelson’s hypothetical ether medium (appropriately modified in its properties), undergoes a driven steady-state oscillation of ‘zitterbewegung’ (zbw) frequency, $\omega_c$, in its instantaneous mean forward (and backward) translational rest frame. With this hypothesis we show that, in the lab frame, the stochastic mechanical velocity potential, $S$, arises from imposing the constraint of conservative diffusions on the time-symmetrized steady-state phase of the zbw particle, implies the needed quantization condition, and evolves by the stochastically derived Hamilton-Jacobi-Madelung equations (when generalized to describe a statistical ensemble of zbw particles). This modification of Nelson-Yasue stochastic mechanics (NYSM), which we term ‘zitterbewegung stochastic mechanics’ (ZSM), then allows us to derive the single-valued wave functions of non-relativistic quantum mechanics for a spinless particle. The problem of justifying the quantization condition is thereby reduced to justifying the zitterbewegung hypothesis. Accordingly, it is among the tasks of Part II to argue that the hypothesis can be justified in terms of physical/dynamical models and can be plausibly generalized to particles with spin as well as relativistic particles and fields.

The outline of this paper is as follows. In section 2, we give a concise review of the formal derivation of the Schrödinger equation from NYSM for a single, spinless particle in an external scalar potential. (Such a review will be useful for the reader who is unfamiliar with NYSM, and essential for following the logic and presentation of the arguments later in the paper.) In section 3, we review the Wallstrom criticism. In section 4, we introduce a classical model of a spinless zitterbewegung particle which implies the quantization condition for the phase of its oscillation, excluding and including interactions with external fields. In each case, we extend the model to a classical Hamilton-Jacobi statistical mechanics involving a Gibbsian ensemble of such particles, with the purpose of making as clear as possible the physical assumptions of the model in a well-established classical physics framework that has conceptual and mathematical similarities to stochastic mechanics. In
section 5, we construct a Nelson-Yasue stochastic mechanics for the zitterbewegung particle (ZSM), excluding and including field interactions. In this way we derive one-particle Schrödinger equations with single-valued wave functions that have (generally) multi-valued phases, and use the hydrogen-like atom as a worked example.

This paper lays the foundation for Part II, where we will: (1) develop the (non-trivial) many-particle cases of ZSM, (2) explicate the beables of ZSM, (3) assess the plausibility and generalizability of the zitterbewegung hypothesis, and (4) compare ZSM to other proposed answers to Wallstrom’s criticism.

2 Nelson-Yasue Stochastic Mechanics

In Edward Nelson’s non-relativistic stochastic mechanics\[2, 3, 4\], it is first hypothesized that the vacuum is pervaded by a homogeneous and isotropic “ether” fluid with classical stochastic fluctuations of uniform character.\[2\] To ensure that observers in the ether can’t distinguish absolute rest from uniform motion, it is further hypothesized that the interaction of a point mass with the ether is a frictionless diffusion process.\[3\] Accordingly, a point particle of mass $m$ within this frictionless ether will in general have its position 3-vector $q(t)$ constantly undergoing diffusive motion with drift, as modeled by the first-order stochastic differential equation,

$$d q(t) = b(q(t), t) dt + dW(t). \quad (1)$$

The vector $b(q(t), t)$ is the deterministic “mean forward” drift velocity of the particle, and $W(t)$ is the Wiener process modeling the effect of the particle’s interaction with the fluctuating ether.

The Wiener increment, $dW(t)$, is assumed to be Gaussian with zero mean, independent of $dq(s)$ for $s \leq t$, and with covariance,

$$E_t [dW_i(t) dW_j(t)] = 2\nu \delta_{ij} dt, \quad (2)$$

where $E_t$ denotes the conditional expectation at time $t$.

Note that although Equations (1-2) are formally the same as those used for the kinematical description of classical Brownian motion in the Einstein-Smoluchowski (ES) theory, the physical context is different; the ES theory uses (1-2) to model the Brownian motion of macroscopic particles in a classical fluid in the large friction limit \[5\], whereas Nelson uses (1-2) to model frictionless stochastic motion (i.e., “conservative diffusions” \[4\]) for elementary particles interacting with a fluctuating ether fluid that permeates the vacuum.

In this connection, it is further hypothesized that the magnitude of the diffusion coefficient $\nu$ is proportional to the reduced Planck’s constant, and inversely proportional to the particle mass $m$ so that

$$\nu = \frac{\hbar}{2m}, \quad (3)$$

The microscopic constituents of this ether are left unspecified by Nelson; however, he suggests by tentative dimensional arguments relating to the choice of diffusion constant in Eq. (3) (namely, that we can write $\hbar = e^2/\alpha c$, where $\alpha$ is the fine-structure constant and $e$ the elementary charge) that it may have an electromagnetic origin \[4\].

Nelson points out \[4\] that this frictionless diffusion process is an example of “conservative diffusions”, or diffusions in which the ensemble-averaged energy of the particle is conserved in time (for a time-independent external potential). In other words, on the (ensemble) average, there is no net transfer of energy between the particle and the fluctuating ether, in contrast to classical Brownian diffusions which are fundamentally dissipative in character.
In addition to (1), the particle’s trajectory $q(t)$ can also satisfy the time-reversed equation

$$dq(t) = b_*(q(t), t)dt + dW_*(t),$$

(4)

where $b_*(q(t), t)$ is the mean backward drift velocity, and $dW_*(t) = dW(-t)$ is the backward Wiener process. The $dW_*(t)$ has all the properties of $dW(t)$, except that it is independent of $dq(s)$ for $s \geq t$. With these conditions on $dW(t)$ and $dW_*(t)$, (1) and (4) respectively define forward and backward Markov processes on $\mathbb{R}^3$.

The forwards and backwards transition probabilities defined by (1) and (4), respectively, should be understood, in some sense, as ontic probabilities [87, 88]. (Generally speaking, ‘ontic probabilities’ can be understood as probabilities about objective physical properties of the $N$-particle system, as opposed to ‘epistemic probabilities’ which are about our ignorance of objective physical properties of the $N$-particle system.) Just how ‘ontic’ these transition probabilities should be is an open question. One possibility is that these transition probabilities should be viewed as phenomenologically modeling complicated deterministic interactions of a massive particle (or particles) with the fluctuating ether, in analogy with how equations such as (1) and (4) are used in the ES to phenomenologically model the complicated deterministic interactions of a macroscopic particle immersed in a fluctuating classical fluid of finite temperature [34]. Another possibility is that the fluctuations of the ether are irreducibly stochastic, and this irreducible stochasticity is ‘transferred’ to a particle immersed in and interacting with the ether. We prefer the former possibility, but acknowledge that the latter possibility is also viable.  

Associated to the trajectory $q(t)$ is the probability density $\rho(q, t) = n(q, t)/N$, where $n(q, t)$ is the number of particles per unit volume and $N$ is the total number of particles in a definite region of space. Corresponding to (1) and (4), then, are the forward and backward Fokker-Planck equations,

$$\frac{\partial \rho(q, t)}{\partial t} = -\nabla \cdot [b(q, t)\rho(q, t)] + \frac{\hbar}{2m} \nabla^2 \rho(q, t),$$

(5)

and

$$\frac{\partial \rho(q, t)}{\partial t} = -\nabla \cdot [b_*(q, t)\rho(q, t)] - \frac{\hbar}{2m} \nabla^2 \rho(q, t),$$

(6)

where we require that $\rho(q, t)$ satisfies the normalization condition,

$$\int \rho_0(q)d^3q = 1.$$  

We emphasize that, in contrast to the transition probabilities defined by (1) and (4), the probability distributions satisfying (5) and (6) are epistemic distributions in the sense that they are distributions over a Gibbsian ensemble of identical systems (i.e., the distributions reflect our ignorance of the actual positions of the particles). Nevertheless, for an epistemic distribution satisfying (5) or (6) at time $t$, its subsequent evolution will be determined by the ontic transition probabilities.

Concerning whether or not the forward and backwards transition probabilities should be understood as ‘objective’ (i.e., as chances governed by natural law) versus ‘subjective’ (i.e., encoding our expectations or degrees of belief) [49, 51, 52], this seems to depend on whether the transition probabilities are merely phenomenological (in which case they would seem to be subjective) or reflect irreducible stochasticity in the ether (in which case they would seem to be objective). Our preference for viewing the transition probabilities as phenomenological seems to commit us to the subjective view, but the objective view also seems viable (the objective view is taken by Bacciagaluppi in [51, 53]). It is worth noting that, under the objective view, the backwards transition probabilities can be regarded as being just as objective/law-like as the forwards transition probabilities (but see [59] for a different view).
so that the distribution at later times will partly come to reflect ontic features of the $N$-particle system, and may asymptotically become independent of the initial distribution. Of course, the asymptotic distribution would still be epistemic in the sense of encoding our ignorance of the actual particle positions, even though it would be determined by the ontic features of the system.

A frictionless (hence energy-conserving or conservative) diffusion process such as Nelson’s should have a time-symmetric probability density evolution. The Fokker-Planck equations (5-6), on the other hand, describe time-asymmetric evolutions in opposite time directions. The reason is that, given all possible solutions to (1), one can define as many forward processes as there are possible initial distributions satisfying (5); likewise, given all possible solutions to (4), one can define as many backward processes as there are possible ‘initial’ distributions satisfying (6). Consequently, the forward and backward processes are both underdetermined, and neither (1) nor (4) has a well-defined time-reversal. We must therefore restrict the diffusion process to simultaneous solutions of (5) and (6).

Note that the sum of (5) and (6) gives the continuity equation

$$\frac{\partial \rho(q,t)}{\partial t} = -\nabla \cdot [v(q,t)\rho(q,t)], \quad (8)$$

where

$$v(q,t) := \frac{1}{2} [b(q,t) + b_*(q,t)] \quad (9)$$

is called the “current velocity” field. As it stands, this current velocity field could have vorticity. But if vorticity is allowed, then the time-reversal operation on (5.8) will change the orientation of the curl, thus distinguishing time directions. So we impose

$$v(q,t) = \nabla S(q,t)/m \quad (10)$$

or that the current velocity field is irrotational. Accordingly, (8) becomes

$$\frac{\partial \rho(q,t)}{\partial t} = -\nabla \cdot \left[ \frac{\nabla S(q,t)}{m} \rho(q,t) \right], \quad (11)$$

a time-reversal invariant evolution equation for the single-time density $\rho(q,t)$.

Physically speaking, the $S$ function in (10-11) has the interpretation of a velocity potential connected with a Gibbsian ensemble of fictitious, non-interacting, identical particles with density $\rho(q,t)$, where each particle in the ensemble differs from the other in its initial position (hence the dependence of $S$ on the generalized coordinate $q$) and initial irrotational velocity given by (10). It is thereby analogous to the $S$ function in the Hamilton-Jacobi formulation of classical statistical mechanics for a single point particle.

Note also that subtracting (5) from (6) yields equality on the right hand side of

$$u(q,t) := \frac{1}{2} [b(q,t) - b_*(q,t)] = \frac{\hbar}{2m} \nabla \rho(q,t)/\rho(q,t), \quad (12)$$

Note 5: I thank Guido Bacciagaluppi for emphasizing this point.

Note 6: Of course, one can still add to $\nabla S$ a solenoidal vector field of any magnitude and, upon insertion into (8), recover the same continuity equation. But the assumption of only irrotational flow velocity is the simplest one, and as we already mentioned, it follows from the requirement of time symmetry for the $\rho(q,t)$ of the diffusion process.
where \( u(q,t) \) is called the “osmotic velocity” field (because it has the same dependence on the density as the velocity acquired by a classical Brownian particle in equilibrium with respect to an external force, in the ES theory \( \text{[2, 3, 4]} \)).

As a consequence of (9), (10), and (12), we have that \( b = v + u \) and \( b_s = v - u \), which when inserted back into (5) and (6), respectively, reduce both Fokker-Planck equations to the time-reversal invariant continuity equation (11). So the combination of (9), (10), and (12) fixes \( \rho \) as the common, single-time, ‘equilibrium’ probability density (in analogy with a thermal equilibrium density) for solutions of (1) and (4), even though it is a time-dependent density.

In our view, the physical meaning of (12) has been misconstrued by some researchers \( \text{[102, 103, 104]} \) to imply that \( \rho \) must be interpreted as the physical cause of the osmotic velocity of Nelson’s particle. We want to stress that this is not the case, and that such an interpretation would be logically and physically inconsistent with the definition of \( \rho \) as a probability density. Instead, Nelson physically motivates his osmotic velocity by analogy with the osmotic velocity in the ES theory \( \text{[2, 3]} \) - essentially, he postulates the presence of an external (i.e., not sourced by the particle) potential, \( U(q,t) \), which couples to the particle via some coupling constant, \( \mu \), such that \( R(q(t),t) = \mu U(q(t),t) \) defines a ‘potential momentum’ for the particle. \( \text{[1]} \) (Hereafter we shall permit ourselves to refer to \( U(q,t) \) and \( R(q,t) \) interchangeably as the ‘osmotic potential’.) When \( U(q,t) \) is spatially varying, it imparts to the particle a momentum, \( \nabla R(q(t),t) \), which is then counter-balanced by the ether fluid’s osmotic impulse pressure, \( (\hbar/2m) \nabla \ln[n(q,t)] \), which leads to the equilibrium condition \( \nabla R/m = (\hbar/2m) \nabla \rho/\rho \) (using \( \rho = n/N \)), which implies that \( \rho \) depends on \( R \) as \( \rho = e^{2R/\hbar} \) for all times. Hence, the physical cause of \( u \) is \( R \) (or technically \( U \)), and (12) is just a mathematically equivalent and convenient rewriting of this relation.

So far our discussion has been restricted to the first-order stochastic differential equations for Nelson’s particle, and the associated Fokker-Planck evolutions. In order to discuss the second-order dynamics for Nelson’s particle, we must first motivate Nelson’s analogues of the Ornstein-Uhlenbeck mean derivatives. In the Itô calculus, the mean forward and backward derivatives of a solution \( q(t) \) satisfying (1) and (4) are respectively defined as

\[
D_q(q,t) = \lim_{\Delta t \to 0^+} \mathbb{E}_t \left[ \frac{q(t + \Delta t) - q(t)}{\Delta t} \right],
\]

and

\[
D_\cdot_q(q,t) = \lim_{\Delta t \to 0^+} \mathbb{E}_t \left[ \frac{q(t) - q(t - \Delta t)}{\Delta t} \right].
\]

Because \( dW(t) \) and \( dW_s(t) \) are Gaussian with zero mean, it follows that \( D_q(q(t),t) = b(q(t),t) \) and \( D_\cdot_q(q(t),t) = b_s(q(t),t) \). To compute the second mean derivative, \( D_b(q(t),t) \) (or \( D_\cdot b(q(t),t) \)), we must expand \( b \) in a Taylor series up to terms of order two in \( dq(t) \):

\[
db(q(t),t) = \frac{\partial b(q(t),t)}{\partial t} dt + dq(t) \cdot \nabla b(q(t),t) + \frac{1}{2} \sum_{i,j} dq_i(t) dq_j(t) \frac{\partial^2 b(q(t),t)}{\partial q_i \partial q_j} + \ldots
\]

\( \text{[1]} \) It should be emphasized that \( U(q,t) \) is not defined over an ensemble of systems, but is a real physical field on 3-space analogous to the classical external potential, \( V(q,t) \), that causes the osmotic velocity of a Brownian particle in the ES theory. Nelson does not specify whether \( U(q,t) \) is sourced by the ether or is an independently existing field on space-time, nor does he specify whether the coupling \( \mu \) corresponds to any of the fundamental force interactions of the Standard Model. These elements of his theory are phenomenological hypotheses that presumably should be made more precise in a ‘deeper’ extension of stochastic mechanics. Nonetheless, as we will see in Part II, the many-particle extension of stochastic mechanics puts additional constraints on how the osmotic potential should be understood.
From (1), we can replace \( dx_i(t) \) by \( dW_i(t) \) in the last term, and when taking the conditional expectation in (13), we can replace \( dq(t) \cdot \nabla b(q(t), t) \) by \( b(q(t), t) \cdot \nabla b(q(t), t) \) since \( dW(t) \) is independent of \( q(t) \) and has mean 0. Using (2-3), we then obtain

\[
Db(q(t), t) = \left[ \frac{\partial}{\partial t} + b(q(t), t) \cdot \nabla + \frac{\hbar}{2m} \nabla^2 \right] b(q(t), t),
\]

and likewise

\[
D_b b_*(q(t), t) = \left[ \frac{\partial}{\partial t} + b_*(q(t), t) \cdot \nabla - \frac{\hbar}{2m} \nabla^2 \right] b_*(q(t), t).
\]

Using (16-17), along with Newton’s 2nd law, Nelson wanted to construct an expression for the ‘mean acceleration’ of the particle consistent with the principle of time-symmetry. He proposed

\[
ma(q(t), t) = \frac{m}{2} [D_b D + D D_b] q(t) = -\nabla V(q(t))|_{q=q(t)}.
\]

Physically, this equation says that the mean acceleration Nelson’s particle feels in the presence of an external (conservative) force is the equal-weighted average of its mean forward drift \( b \) transported backwards in time, with its mean backward drift \( b_* \) transported forwards in time. It is this peculiar mean dynamics that preserves the time-symmetry of Nelson’s diffusion process.

Of course, other time-symmetric mean accelerations are possible. For example, \((1/2)|D^2 + D_2^2|q(t)\), or any weighted average of this with (18). So it may be asked: what other physical principles (if any) privilege Nelson’s choice? As first shown by Yasue \([7,8]\) and later adopted by Nelson \([4]\), a physically well-motivated stochastic variational principle can give (18). Consider the ensemble-averaged, time-symmetric mean action

\[
J = E \left[ \int_{t_i}^{t_f} \left\{ \frac{1}{2} \frac{1}{2} m b(q(t), t)^2 + \frac{1}{2} m b_*(q(t), t)^2 - V(q(t), t) \right\} \right] dt
\]

\[
= E \left[ \int_{t_i}^{t_f} \left\{ \frac{1}{2} m v^2 + \frac{1}{2} m u^2 - V \right\} \right] dt.
\]

In other words, for a particle in a (possibly) time-dependent potential \( V \), undergoing the Markov processes given by (1) and (4) with the restriction to simultaneous solutions of the Fokker-Planck equations via (9), (10), and (12), a time-symmetric mean Lagrangian can be defined by averaging together the mean Lagrangians associated with the forward and backward processes. The ensemble averaged action obtained from this time-symmetric mean Lagrangian then corresponds to (19), where \( E[...] \) denotes the absolute expectation. Upon imposing the conservative diffusion condition through the variational principle,

\[
J = E \left[ \int_{t_i}^{t_f} \left\{ \frac{1}{2} m v^2 + \frac{1}{2} m u^2 - V \right\} \right] dt = extremal,
\]

a straightforward computation (see Appendix A) shows that this implies (18) as the equation of motion. If, instead, we had allowed the mean kinetic energy terms in (19) to not be positive-definite and used the alternative time-symmetric mean kinetic energy, \((1/2)m b b_* = (1/2)m(v^2 - u^2)\), then

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8 Another widely used stochastic variational principle is the one due to Guerra and Morato [11]. We don’t use their approach because it entails an S function that’s globally single-valued, which excludes the possibility of systems with angular momentum [34] and therefore will not be applicable to our proposed answer to Wallstrom’s criticism.
it can be shown \[100, 103, 113\] that imposing (20) would give the alternative time-symmetric mean acceleration involving the derivatives \([D^2 + D^2]\). \[\sqrt{D}\] So Nelson’s mean acceleration choice is justified by the principle of time-symmetry and the natural physical requirement that all the contributions to the mean kinetic energy of the Nelsonian particle should be positive-definite.

By applying the mean derivatives in (18) to \(q(t)\), using that \(b = v + u\) and \(b_\ast = v - u\), and removing the dependence of the mean acceleration on the actual particle trajectory \(q(t)\) so that \(a(q(t), t)\) gets replaced by the mean acceleration field \(a(q, t)\), a straightforward computation gives

\[
ma(q, t) = m \left[ \frac{\partial (v(q, t))}{\partial t} + v(q, t) \cdot \nabla v(q, t) - u(q, t) \cdot \nabla u(q, t) - \frac{\hbar}{2m} \nabla^2 u(q, t) \right]
\]

\[
= \nabla \left[ \frac{\partial S(q, t)}{\partial t} + \frac{(\nabla S(q, t))^2}{2m} - \frac{\hbar^2}{2m} \frac{\nabla^2 \sqrt{\rho(q, t)}}{\sqrt{\rho(q, t)}} \right] = -\nabla V(q, t).
\]

(21)

The mean acceleration field \(a(q, t)\) describes the possible mean accelerations of the actual particle given all of the possible spatial locations that the actual particle can occupy at time \(t\). In other words, \(a(q, t)\) is the mean acceleration field connected with the set of fictitious particles forming the Gibbsian ensemble that reflects our ignorance of the actual trajectory \(q(t)\) \[64\]. Integrating both sides of (21), and setting the arbitrary integration constants equal to zero, we then obtain the Quantum Hamilton-Jacobi equation,

\[
-\frac{\partial S(q, t)}{\partial t} = \frac{(\nabla S(q, t))^2}{2m} + V(q, t) - \frac{\hbar^2}{2m} \frac{\nabla^2 \sqrt{\rho(q, t)}}{\sqrt{\rho(q, t)}},
\]

(22)

which describes the total energy field over the possible positions of the actual point mass, and upon evaluation at \(q = q(t)\), the total energy of the point mass along its actual trajectory.

Although the last term on the right hand side of (22) is often called the “quantum potential”, we note that it arises here from the osmotic kinetic energy term in (19). So the quantum potential must be physically understood in stochastic mechanics as a kinetic energy field (which hereafter we prefer to call the ‘quantum kinetic’ for accuracy of meaning) arising from the osmotic velocity field.

The pair of nonlinear equations coupling the evolution of \(\rho\) and \(S\), as given by (11) and (22), are generally known as the Hamilton-Jacobi-Madelung (HJM) equations, and can be formally identified with the imaginary and real parts of the Schrödinger equation under polar decomposition \[106, 64\]. Therefore, (11) and (22) can be formally rewritten as the Schrödinger equation,

\[
\imath \hbar \frac{\partial \psi(q, t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi(q, t) + V(q, t)\psi(q, t),
\]

(23)

where \(\psi(q, t) = \sqrt{\rho(q, t)} e^{iS(q, t)/\hbar}\). In contrast to other ontological formulations of quantum mechanics, this wave function must be interpreted as an epistemic field in the sense that it encodes information about the possible position and momenta states that the actual particle can occupy at any instant, since it is defined in terms of the ensemble variables \(\rho\) and \(S\). \[4\] Although the treatment here did not include coupling to electromagnetic potentials, it is straightforward to do so \[4\] (see also Appendix A).

\[9\] Additionally, Davidson \[13\] showed that by defining a Lagrangian of the form \[(1/2)\mu [(1/2)(b^2 + b_\ast^2) - \beta/\hbar (b - b_\ast)^2]\], where \(\beta\) is a constant, the resulting equation of motion is also equivalent to the usual Schrödinger equation, provided that the diffusion coefficient \(\nu = (1/\sqrt{1 - \beta^2/\hbar})\). We can see, however, that our criterion of restricting the kinetic energy terms in the Lagrangian to only terms that are positive-definite, excludes Davidson’s choice of Lagrangian too.

\[10\] Though it may not be obvious here, this interpretation of the Nelson-Yasue wave function is not undermined by
3 Wallstrom’s Criticism

In the previous section, we referred to the correspondence between the HJM equations and (23) as only formal because we had not considered the boundary conditions that must be imposed on solutions of the Schrödinger equation and the HJM equations, respectively, in order for mathematical equivalence to be established. In standard quantum mechanics, it is well-known that physical wave functions satisfying the Schrödinger equation are required to be single-valued. For the HJM equations, it was not specified in the Nelson-Yasue derivation whether \( S \) is assumed to be single-valued, arbitrarily multi-valued, or multi-valued in accordance with a quantization condition. Wallstrom [64, 55] showed that for all existing formulations of stochastic mechanics, all these possible conditions on \( S \) are problematic in one way or another.

If \( S \) is constrained to be single-valued, then stochastic mechanical theories exclude single-valued Schrödinger wave functions with angular momentum. This is so because single-valued wave functions with angular momentum have phase factors of the form \( \exp(\pm i \mathcal{M} \phi) \), which implies that \( S(\phi) = m \hbar \phi \). By contrast, if \( S \) is assumed to be arbitrarily multi-valued, they produce all the single-valued wave functions of the Schrödinger equation, along with infinitely many multi-valued ‘wave functions’, which smoothly interpolate between the single-valued wave functions. This can be seen by comparing solutions of the Schrödinger and HJM equations for a two-dimensional central potential, \( V(r) \) [54]. The Schrödinger equation with \( V(r) \) has single-valued wave functions of the form \( \psi_m(r, \phi) = R_m(r) \exp(\pm i m \phi) \), where \( \psi_m(r, \phi) = \psi_m(r, \phi + 2\pi n) \), implying that \( m \) is an integer. For the HJM equations, however, the solutions \( \rho_m = |R_m(r)|^2 \) and \( \psi_m = (m \hbar \mu/mr) \phi \) don’t require \( m \) to be integral. To see this, consider the effective central potential, \( V_\phi(r) = V(r) + a/r^2 \), where \( a \) is a positive real constant. For this potential, consider the Schrödinger equation with stationary solution \( \psi_\phi(r, \phi) = R_\phi(r) \exp(\pm i \mathcal{M} \phi) \), where \( m = 1 \) and radial component corresponding to the ground state solution of the radial equation. This wave function yields osmotic and current velocities, \( \mathbf{u}_a \) and \( \mathbf{v}_a \), which satisfy (11) and (21) with the potential \( V_a \):

\[
0 = \frac{\partial \rho_a}{\partial t} = -\nabla \cdot (\mathbf{v}_a \rho_a),
\]

\[
0 = \frac{\partial \mathbf{v}_a}{\partial t} = -\nabla \left( V + \frac{a}{r^2} \right) - \mathbf{v}_a \cdot \nabla \mathbf{v}_a + \mathbf{u}_a \cdot \nabla \mathbf{u}_a + \frac{\hbar^2}{2m} \nabla^2 \mathbf{u}_a. \tag{25}
\]

Using \( \mathbf{v}_a = (m \hbar \mu/mr) \phi \) and \( \mathbf{u}_a \cdot \nabla \mathbf{v}_a = \nabla \left[ m \mathbf{v}_a^2 / 2 \right] \), we can then rewrite (25) as

\[
0 = -\nabla V - \nabla \left( \frac{a}{r^2} + \frac{1}{2} m \mathbf{v}_a^2 \right) + \mathbf{u}_a \cdot \nabla \mathbf{u}_a + \frac{\hbar^2}{2m} \nabla^2 \mathbf{u}_a
\]

\[
= -\nabla V - \frac{m}{2} \nabla \left( \frac{2ma}{\hbar^2} + 1 \right) \mathbf{v}_a^2 + \mathbf{u}_a \cdot \nabla \mathbf{u}_a + \frac{\hbar^2}{2m} \nabla^2 \mathbf{u}_a. \tag{26}
\]

This gives us \( \mathbf{v}_a = \mathbf{v}_a \sqrt{2ma/\hbar^2 + 1} \) and \( \mathbf{u}_a = \mathbf{u}_a \). Note that since \( a \) is a constant that can take any positive real value, \( \mathbf{v}_a \) is not quantized, and yet it is a solution of the HJM equations. By contrast, in the quantum mechanical version of this problem, we would have \( V_a(r) = V(r) + m^2 / 2r^2 \), where \( m = \sqrt{2ma/\hbar^2 + 1} \) would be integral due to the single-valuedness condition on \( \psi_m \). In other words, the Pusey-Barrett-Rudolph theorem [117]. Whereas this theorem assumes factorizability/separability of the “ontic state space”, the ontic osmotic potential, \( U \), which is encoded in the amplitude of the wave function via \( R \) and plays a role in the particle dynamics via (21), is in general not separable when extended to the \( N \)-particle case (as will be shown in Part II [63]).
the $v_a$ and $u_a$ in stochastic mechanics only correspond to a single-valued wave function when $a$ is an integer, and this is true of all systems of two dimensions or higher. Equivalently, we may say that the HJM equations predict a continuum of energy and momentum states for the particle, which smoothly interpolate between the quantized energy and momentum eigenvalues predicted by the quantum mechanical case.\footnote{Before Wallstrom’s critiques, it was pointed out by Albeverio and Hoegh-Krohn \cite{108} as well as Goldstein \cite{37} that, for the cases of stationary bound states with nodal surfaces that separate the manifold of diffusion into disjoint components, Nelson’s equations (the HJM equations and his stochastic differential equations) contain more solutions than Schrödinger’s equation. In addition, Goldstein \cite{37} was the first to point out that solutions exist to the HJM equations which don’t correspond to any single-valued solution of the Schrödinger equation, for the case of a multiply-connected configuration space. Nevertheless, Wallstrom’s example of extraneous solutions is of a more general nature, as it applies to a simply-connected space where the diffusion process is not separated into disjoint components.}

The only condition on $S$ (and hence the current velocity $v_a$) that allows stochastic mechanics to recover all and only the single-valued wave functions of the Schrödinger equation is the condition that the change in $S$ around any closed loop $L$ in space (with time held constant) is equal to an integer multiple of Planck’s constant, \footnote{Wallstrom notes that Takabayasi \cite{106} was first to recognize the necessity of this quantization condition and suggests [private communication] that priority of credit for this discovery should go to him \cite{55}. However, it seems that Takabayasi only recognized this issue in the context of Bohm’s 1952 hidden-variables theory, even though Fényes proposed the first formulation of stochastic mechanics that same year \cite{1}. Wallstrom appears to have been the first in the literature to recognize and discuss the full extent of this inequivalence in the context of stochastic mechanical theories.} or

$$\oint_L dS = \oint_L \nabla S \cdot dq = nh. \quad (27)$$

But this condition is arbitrary, Wallstrom argued, as there’s no reason in stochastic mechanics why the change in $S$ along $L$ should be constrained to an integer multiple of $h$. Indeed, assuming this condition amounts to assuming that wave functions are single-valued, which amounts to assuming that the solution space of the Nelson-Yasue stochastic mechanical equations is equivalent to the solution space of the quantum mechanical Schrödinger equation. Such an assumption cannot be made, however, in a theory purporting to \textit{derive} the Schrödinger equation of quantum mechanics.

These arguments notwithstanding, one might question whether the requirement of single-valued wave functions in quantum mechanics is any less arbitrary than imposing (27) in stochastic mechanics. This is not the case. The single-valuedness condition, as usually motivated, is a consequence of imposing two completely natural boundary conditions on solutions of (23): (a) that the solutions satisfy the linear superposition principle \cite{109, 54}, and (b) that $|\psi|^2$ can be physically interpreted as a probability density \cite{110, 111, 112}. Condition (a) is natural to the single-valuedness requirement because of the linearity of the Schrödinger equation, and condition (b) is natural to it because a probability density is, by definition, a single-valued function on its sample space. Moreover, it can be shown that if (a) doesn’t hold then (b) doesn’t hold for any linear superposition of two or more solutions. To illustrate this, consider the free particle Schrödinger equation on the unit circle, $S^1$.\footnote{Henneberger et al. \cite{113} argue that the single-valuedness condition on wave functions is strictly a consequence of the linear superposition principle. However, this nuance is inessential to our arguments.}

$$-\frac{\hbar^2}{2m} \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} = E\psi. \quad (28)$$

The un-normalized wave function satisfying this equation is of the form $\psi(\theta) = Ne^{ik\theta}$, where $E$ is the energy of the particle, $\hbar$ is Planck’s constant, $m$ is the mass of the particle, and $k$ is the wavenumber. The normalization constant $N$ can be found by requiring that $\int_{\text{S}^1} |\psi|^2 d\theta = 1$. For the case of a free particle on a circle, the wave function is given by $\psi(\theta) = Ne^{ik\theta}$, where $k = \frac{nh}{\hbar}$. This shows that the single-valuedness condition is equivalent to the quantization condition.
\[ k = \frac{\pi}{\sqrt{2mE}}. \] For this wave function to satisfy (b), \( k \) (and hence the energy \( E \)) can take any positive value among the real numbers since obviously \( |\psi|^2 = N^2 \). Consider now a superposition of the form \( \psi_s(\theta) = N(e^{i k_1 \theta} + e^{i k_2 \theta}) \), which leads to the density

\[ |\psi_s|^2 = 2N^2 (1 + \cos [(k_1 - k_2)\theta]). \quad (29) \]

If \( k_1 \) and \( k_2 \) are allowed to take non-integer values, then \((k_1 - k_2)\) can also take non-integer values, and the density formed from the superposition can be multi-valued, thereby violating (b). Condition (a) will also be violated since, although a single wave function in the superposition satisfies (b), the superposition does not; so the set of wave functions of the form \( \psi(\theta) = Ne^{ik\theta} \), where \( k \) can take non-integer values, does not form a linear space. If, however, \( k_1 \) and \( k_2 \) are integers, then \((k_1 - k_2)\) is an integer, and conditions (a) and (b) will be satisfied since \(|\psi_s|^2\) will always be single-valued. Correspondingly, it follows that the energy and momentum of the particle on the unit circle will be quantized with

\[ e^{i2\pi \frac{\pi}{\sqrt{2mE}}} = 1 = e^{i2\pi n} \] yielding

\[ E_n = \frac{p^2}{2mr^2} = \frac{n^2\hbar^2}{2mr^2}, \] where \( n \) is an integer.

The wave functions constructed from stochastic mechanics will therefore satisfy only (b) if \( S \) is arbitrarily multi-valued, while they will satisfy (a) and (b) together only when (27) is imposed. But as previously mentioned, (27) is ad hoc in stochastic mechanics, and assuming it to obtain only single-valued wave functions is logically circular if the objective of stochastic mechanics is to derive quantum mechanics. The challenge then is to find a physically plausible justification for (27) strictly within the assumptions of existing formulations of stochastic mechanics, or otherwise some new formulation. Accordingly, we shall now begin the development of our proposed justification through a reformulation of Nelson-Yasue stochastic mechanics (NYSM).

### 4 Classical Model of Constrained Zitterbewegung Motion

Here we develop a classical model of a particle of mass \( m \) constrained in its rest frame to undergo a simple harmonic oscillation of (electron) Compton frequency, and show that it gives rise to a quantization condition equivalent to (27). Our model motivates the quantization condition from essentially the same physical arguments used by de Broglie in his “phase-wave” model [114, 115] and by Bohm in his subquantum field-theoretic models [116, 117]. However, it differs from both de Broglie’s model and Bohm’s models in that we do not need to refer to fictitious “phase waves”, nor assume that our particle is some localized distribution of a (hypothetical) fluctuating subquantum field [110], nor assume a non-denumerable infinity of “local clocks” at each point in space-time [117]. We start by developing the free particle case, extend it to a classical Hamilton-Jacobi (HJ) statistical mechanical description, and repeat these steps with the inclusion of interactions with external fields.

The purpose of this section is three-fold: (i) to explicitly show, without the added conceptual complications of stochastic mechanics, the basic physical assumptions underlying our particle model; (ii) to show how our model can be consistently generalized to include interactions with external fields; (iii) to show, using a well-established formulation of classical statistical mechanics that has conceptual and mathematical similarities to stochastic mechanics, how our model can be consistently generalized to a statistical ensemble description (which will also be necessary in the stochastic mechanical case), and how doing so gives a quantization condition equivalent to (27) for a ‘classical’ wave function satisfying a nonlinear Schrödinger equation. No attempt will be made here to suggest a physical/dynamical model for the zitterbewegung motion. A framework for a
physical model is given in section 5, while a discussion of possible physical models is reserved for Part II.

4.1 One free particle

Suppose that a classical particle of rest mass $m$ is rheonomically constrained to undergo a periodic process with constant angular frequency, $\omega_0$, about some fixed point in 3-space, $q_0$, in a Lorentz frame where the particle has translational velocity $v = dq_0/dt = 0$. The exact nature of this process is not important for the argument that follows, as long as it is periodic. For example, this process could be an oscillation or (if the particle is spinning) a rotation. But since we are considering the spinless case, we will take the periodic process to be some kind of oscillation. The constancy of $\omega_0$ implies that the oscillation is simply harmonic with phase $\theta = \omega_0 t_0 + \phi$. Although the assumption of simple harmonic motion implies that $\theta$ is a continuous function of the particle’s position, in the translational rest frame, it must be the case that the phase change $\delta \theta$ at any fixed instant $t_0$ will be zero for some translational displacement $\delta q_0$. Otherwise, such a displacement would define a preferred direction in space given by $\nabla \theta(q_0)$. Hence, in the translational rest frame, we can write

$$\delta \theta = \omega_0 \delta t_0,$$

(30)

where $\delta t_0$ is the change in proper time.

If we Lorentz transform to the lab frame where the particle has constant translational velocity, $v$, and undergoes a displacement $\delta q(t)$ in $\delta t$, then $\delta t_0 = \gamma (\delta t - v \cdot \delta q(t)/c^2)$ and (30) becomes

$$\delta \theta(q(t), t) = \omega_0 \gamma \left( \delta t - \frac{v \cdot \delta q(t)}{c^2} \right),$$

(31)

where $\gamma = 1/\sqrt{1 - v^2/c^2}$. Recalling that for a relativistic free particle we have $E = \gamma mc^2$ and $p = \gamma mv$, (31) can be equivalently expressed as

$$\delta \theta(q(t), t) = \frac{\omega_0 m c^2}{E} \left( E \delta t - p \cdot \delta q(t) \right).$$

(32)

Suppose now that the oscillating particle is physically or virtually displaced around a closed loop $L$ (i.e., a continuous, non-self-intersecting loop that is otherwise arbitrary) in which both position and time can vary. The consistency of the model requires that the accumulated phase change be given by

$$\oint_L \delta \theta(q(t), t) = \frac{\omega_0 m c^2}{E} \int_L (E \delta t - p \cdot \delta q(t)) = 2\pi n,$$

(33)

where $n$ is an integer. This follows from the assumption that the oscillation is simply harmonic in the particle’s rest frame, which makes $\theta$ in the lab frame a single-valued function of $q(t)$ (up to an additive integer multiple of $2\pi$). Indeed, if (33) were not true, we would contradict our hypothesis that the oscillating particle has a well-defined phase at each point along its space-time trajectory.

\footnote{Because we permit a virtual displacement where time changes, we cannot use the definition of a virtual displacement often found in textbooks \cite{118,119} (which assumes time is fixed under the displacement). Instead, we use the more refined definition of virtual displacements proposed by Ray & Shamanna \cite{120}, namely that a virtual displacement is the difference between any two (unequal) “allowed displacements”, or $\delta q_k = d q_k - d q'_k$, where $k = 1, 2, \ldots, N$, and an allowed displacement is defined as $d q_k = v_k dt$, where $v_k$ are the “virtual velocities”, or the velocities allowed by the mechanical constraints of a given system.}
If we further make the ‘zitterbewegung’ (zbw) hypothesis that \( m = m_e = 9.11 \times 10^{-28} \text{g} \) and \( \omega_0 / m_e c^2 = 1 / \hbar \) so that \( \omega_0 = \omega_c = 7.77 \times 10^{20} \text{rad/s} \), which is the electron Compton angular frequency, then we can define \( \theta := -\frac{1}{\hbar} \theta \) and (33) can be rewritten as

\[
\int_L \delta S(q(t),t) = \int_L (p \cdot \delta q(t) - E \delta t) = nh. \tag{34}
\]

Finally, for the special case of loop integrals in which time is held fixed (\( \delta t = 0 \)), (34) reduces to

\[
\int_L p \cdot \delta q(t) = nh, \tag{35}
\]

which we may observe is formally identical to the Bohr-Sommerfeld-Wilson quantization condition.

By integrating (32) and using the Legendre transformation, it can be shown that the phase of the free zbw particle is, equivalently, its relativistic action up to an additive constant, or \( S(q(t),t) = p \cdot q(t) - Et - \hbar \phi = -mc^2 \int_L^t dt' / \gamma + C \), where \( \phi \) is the initial phase constant. Recognizing also that \( p = h\gamma \omega_c v / c^2 = h\gamma k \) and \( E = h\gamma \omega_c \), the translational 3-velocity of the particle can be obtained from \( S(q(t),t) \) as \( v = (1/\gamma m) \nabla S(q(t)) |_{q=q(t)} \), and the total relativistic energy as \( E = -\partial_t S(q,t) |_{q=q(t)} \).

It follows then that \( S(q(t),t) \) is a solution of the classical relativistic Hamilton-Jacobi equation,

\[
-\partial_t S(q,t) |_{q=q(t)} = \sqrt{m^2 c^4 + (\nabla S(q,t))^2 c^2} \bigg|_{q=q(t)}. \tag{36}
\]

In the non-relativistic limit, \( v \ll c, S(q(t),t) \approx m v \cdot q(t) - \left( mc^2 + \frac{mv^2}{2} \right) t - h\phi \), and (36) becomes

\[
-\partial_t S(q,t) |_{q=q(t)} = \frac{(\nabla S(q,t))^2}{2m} \bigg|_{q=q(t)} + mc^2, \tag{37}
\]

where \( v = (1/m) \nabla S |_{q=q(t)} = (1/m) \hbar k \) and satisfies the trivial classical Newtonian equation

\[
ma = \left( \frac{\partial}{\partial t} + v \cdot \nabla \right) S = 0. \tag{38}
\]

We find then that, in the non-relativistic limit, the oscillation frequency of the zbw particle has two parts - a low frequency oscillation, \( \omega_{dB} = \hbar k^2 / 2m \), which modulates the high frequency oscillation \( \omega_c \).

Evidently (37) has the form of the non-relativistic dispersion relation \( E = \hbar^2 k^2 / 2m + mc^2 \), which naively suggests that one can obtain the free-particle Schrödinger equation for a plane wave by introducing operators \( \hat{p} = -i\hbar \nabla \) and \( \hat{E} = i\hbar \partial_t \) such that \( \hat{p} \psi = \hbar k \psi, \hat{E} \psi = \hbar \omega_c \psi \), and \( i\hbar \partial_t \psi = - (\hbar^2 / 2m) \nabla^2 \psi \) for \( \psi(q,t) = A e^{i(p \cdot q - Et) / \hbar} \). However, there is no physical wave for such a plane wave to be identified with in our model. Such a plane wave and Schrödinger equation are nothing more than abstract, mathematically equivalent re-writings of the zbw particle energy equation (37). On the other hand, as we will see next, a non-linear Schrödinger equation that describes the dynamical evolution of a statistical ensemble of identical zbw particles is derivable from the classical HJ description of the ensemble.

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\( ^{16} \)The proof is as follows. From \( L = -mc^2 / \gamma \), the Legendre transform gives \( E = p \cdot v - L = \gamma mv^2 + mc^2 / \gamma = \gamma mc^2 \) and \( L = p \cdot v - E \). So for the free zbw particle, \( S = \int L dt + C = \int (p \cdot v - E) dt + C = \int (p \cdot dq - Edt) + C = p \cdot q - Et + C \) (absorbing the integration constants arising from \( dq \) and \( dt \) into \( C \)).
4.2 Classical Hamilton-Jacobi statistical mechanics for one free particle

Suppose that the actual position and momentum of a zbw particle, \((q(t), p(t))\), are unknown. Then we must resort to the description of a classical (i.e., Gibbsian) statistical ensemble of fictitious, identical, non-interacting zbw particles \([44]\), which differ from each other only by virtue of their initial positions, velocities, and (possibly) phases. (Consideration of this in the classical context will be helpful for seeing how our model can be incorporated into stochastic mechanics.) In terms of the zbw phase, this change in description corresponds to replacing \(\delta S(q(t), t)\) by \(dS(q, t) = p(q, t) \cdot dq - E(q, t)dt\), which we obtained from replacing \(q(t)\) by \(q\), where \(q\) labels a possible position in 3-D space that the actual zbw particle could occupy at time \(t\). Integrating \(dS(q, t)\) then gives \(S(q, t) = \int p(q, t) dq - \int E(q, t) dt + C\), where \(C = \hbar \phi\) is just the initial phase constant. So \(S(q, t)\) is a phase field connected with the ensemble, \(p(q, t) = \nabla S(q, t)\) is the corresponding translational momentum field, and \(E(q, t) = -\partial_t S(q, t)\) is the total energy field. Note that, for any initial \(q\) and \(t\), the constant \(\phi\) can be given any value on the interval \([0, 2\pi]\); i.e., the initial phase constant associated with any member of the ensemble can be freely specified on that interval. (Of course, this phase constant does not affect the momentum field or the total energy field, as these fields are obtained from space-time derivatives of the phase field. Thus there are many phase fields corresponding to a unique momentum field and total energy field."

Now, in the specific case of the free zbw particle, \(p = \text{const}\) and \(E = \text{const}\) for each member of the ensemble. So the infinitesimal phase change connected with the ensemble is just \(dS(q, t) = p \cdot dq - Edt\), yielding \(S(q, t) = p \cdot q - Et + C\) upon integration.

With this phase field in hand, we can now construct a classical HJ statistical mechanics for our zbw particle. Essentially, \(S(q, t)\) and \(\nabla S(q, t)\) will respectively satisfy the classical Hamilton-Jacobi equation,

\[-\partial_t S(q, t) = \left(\frac{\nabla S(q, t)}{2m}\right)^2 + mc^2,\]

and the trivial classical Newtonian equation,

\[ma(q, t) = \left(\frac{\partial}{\partial t} + v(q, t) \cdot \nabla\right) \nabla S(q, t) = 0.\]

If we now suppose that the density of ensemble particles per unit volume in an element \(d^3q\) surrounding the point \(q\) at time \(t\) is given by the function \(\rho(q, t) \geq 0\), which satisfies the normalization condition \(\int \rho_0(q) d^3q = 1\), then it is straightforward to show \([44]\) that \(\rho(q, t)\) evolves in time by the continuity equation

\[\frac{\partial \rho(q, t)}{\partial t} = -\nabla \cdot \left[\frac{\nabla S(q, t)}{m} \rho(q, t)\right].\]

Accordingly, \(\rho(q, t)\) carries the interpretation of the probability density for the actual zbw particle position \(q(t)\). And since \(S(q, t)\) is a field over the possible positions that the actual zbw particle can occupy at time \(t\), where for each possible position the actual zbw particle’s phase will satisfy the relation (35), \(S(q, t)\) will be a single-valued function of \(q\) and \(t\) (up to an additive integer multiple of \(2\pi\)) and satisfy

\[\oint_L dS(q, t) = \oint_L \nabla S(q, t) \cdot dq = nh.\]

The use of exact differentials in (42) indicates that the loop integral is now an integral of the momentum field along any closed mathematical loop in 3-space with time held constant; that is,
a closed loop around which the actual particle with momentum $p$ could potentially be displaced, starting from any possible position $q$ it can occupy at fixed time $t$. This tells us that the circulation of the momentum field is quantized, in contrast to an ordinary classical statistical mechanical ensemble for which the momentum field circulation need not satisfy (42).

Finally, we can combine (39) and (41) into the nonlinear Schrödinger equation [97, 98, 64, 99, 100, 101],

$$i\hbar \frac{\partial \psi(q, t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi(q, t) + \frac{\hbar^2}{2m} |\psi(q, t)|^2 \psi(q, t) + mc^2 \psi(q, t),$$

with general solution $\psi(q, t) = \sqrt{\rho_0(q - v_0 t)} e^{iS(q, t)/\hbar}$, which is single-valued because of (42). (Note that $C$ will contribute a global phase factor, $e^{iC/\hbar}$, which cancels out from both sides.) As an example of a specific solution, the complex phase $e^{iS/\hbar}$ takes the form of a plane-wave, $S = p \cdot q - Et + \hbar \phi$, while the initial probability density, $\rho_0$, can take the form of a Gaussian that propagates with fixed profile and speed $v$ (in contrast to a Gaussian density in free particle quantum mechanics, which disperses over time).

We have thereby shown that extending our free $zbw$ particle model to a classical HJ statistical mechanics allows us to derive a nonlinear Schrödinger equation with single-valued wave functions. Next we will incorporate interactions of the $zbw$ particle with external fields.

4.3 One particle interacting with external fields

To describe the interaction of our $zbw$ particle with fields, let us reconsider the change in the $zbw$ phase in the rest frame. In terms of the rest energy of the $zbw$ particle, we can rewrite (30) as

$$\delta \theta = \omega_c \delta t_0 = \frac{1}{\hbar} \left( mc^2 \right) \delta t_0.$$  (44)

Any additional contribution to the energy of the particle, such as from a weak external gravitational field (e.g. the Earth’s gravitational field) coupling to the particle’s mass via $\Phi_g = g \cdot q$, will then modify (44) as

$$\delta \theta = (\omega_c + \kappa(q)) \big|_{q=q_0} \delta t_0 = \frac{1}{\hbar} \left( mc^2 + m \Phi_g(q) \right) \big|_{q=q_0} \delta t_0,$$  (45)

where $\kappa = \omega_c \Phi_g/c^2$. In other words, the gravitational field shifts the $zbw$ frequency in the rest frame by a very small amount. For example, if $|g| = 10^3 cm/s^2$ and is in the $\hat{z}$ direction, and we take $|q| = 100 cm$, then $\kappa \approx \omega_c \times 10^{-16}$. Here we have approximated the point at which the $zbw$ particle interacts with the external gravitational field to be just its equilibrium position, $q_0$, because the displacement $|q| \gg \lambda_c$, allowing us to approximate the interaction with the mass as point-like.  

In addition, we could allow the $zbw$ particle to carry charge $e$ (so that it now becomes a classical charged oscillator, subject to the hypothetical constraint that it does not radiate electromagnetic energy in its rest frame, or the constraint that the oscillation of the charge is radially symmetric so that there is no net energy radiated [121, 122, 123], or constrained to correspond to one of the non-spherically-symmetric charge distributions considered by Bohm and Weinstein [124] for which the retarded self-fields cause the charge distribution to oscillate at a fixed frequency without radiating) which couples to an external (and possibly space-time varying) electric field such that

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17This appears to be the same assumption made by de Broglie for his equivalent model, although he never explicitly says so. Bohm, to the best of our knowledge, never extended his models to include field interactions.
\[ \Phi_e = \mathbf{E}(q, t) \cdot q, \text{ where } q \text{ is the displacement vector in some arbitrary direction from the field source. Here again we can make the point-like approximation, as in laboratory experiments the displacement of a particle from a field source is typically on the centimeter scale, making } |q| \gg \lambda_c. \]

Then

\[ \delta \theta = (\omega_c + \kappa(q_0) + \varepsilon(q_0, t_0)) \delta t_0 = \frac{1}{\hbar} (mc^2 + m\Phi_g(q_0) + e\Phi_e(q_0, t_0)) \delta t_0, \]

where \( \varepsilon = \omega_c (e/mc^2) \Phi_e \). Assuming \( \mathbf{E} \) has an experimental value of \( \sim 10^5 \text{V/cm} \approx 0.03 \text{stV/cm} \), which is the upper limit laboratory field strength that can be produced in Stark effect experiments \[125\], and \( |q| = 1 \text{cm} \), then \( \varepsilon \approx \omega_c \times 10^{-5} \), which is also a very small shift.

If we now transform to the laboratory frame where the \( zbw \) particle has nonzero but variable translational velocity, (46) becomes

\[ \delta \theta(q(t), t) = \left[ (\omega_{AB} + \kappa(q) + \varepsilon(q)) \gamma \left( \delta t - \frac{v_0(q, t) \cdot \delta q}{c^2} \right) \right]_{q=q(t)} \]

\[ = \frac{1}{\hbar} \left[ (\gamma mc^2 + \gamma m\Phi_g(q) + e\Phi_e(q, t)) \delta t \right. \]

\[ - (\gamma mc^2 + \gamma m\Phi_g(q) + e\Phi_e(q, t)) \left. \frac{v_0(q, t) \cdot \delta q}{c^2} \right]_{q=q(t)} \]

\[ = \frac{1}{\hbar} (E(q(t), t) \delta t - p(q(t), t) \cdot \delta q(t)), \]

where \( E = \gamma mc^2 + \gamma m\Phi_g + e\Phi_e \) and \( p = mv = (\gamma mc^2 + \gamma m\Phi_g + e\Phi_e) \left( \frac{v_0/c^2}{c^2} \right) \). (Note that the term \( e\Phi_e \) is unaffected by the Lorentz transformation because it doesn’t involve the particle’s rest mass.) Here \( \varepsilon_{AB} \) is that of a free particle, while \( \varepsilon \) is the adjusted velocity due to the presence of external potentials. In this moving frame, we can also have the \( zbw \) particle couple to an external magnetic vector potential \([13\) such that \( \mathbf{v} \rightarrow \mathbf{v}' = \mathbf{v} + e\mathbf{A}_{ext}/\gamma mc \) (and \( \gamma \) depends on \( v \)). Although the physical influence of the fields now allows the \( \omega \) and \( \mathbf{k} \) of the particle to vary as a function of position and time, the phase of the oscillation is still a well-defined function of the particle’s space-time location; so if we displace the oscillating particle around a closed loop, the phase change is still given by

\[ \oint_L \delta \theta(q(t), t) = \frac{1}{\hbar} \oint_L (E(q(t), t) \delta t - p'(q(t), t) \cdot \delta q(t)) = 2\pi n, \]

or

\[ \oint_L \delta S(q(t), t) = \oint_L \left( p'(q(t), t) \cdot \delta q(t) - E(q(t), t) \delta t \right) = nh. \]

For the special case of a loop in which time is held fixed, we then have

\[ \oint_L \nabla S(q, t)|_{q=q(t)} = \oint_L p'(q(t), t) \cdot \delta q(t) = nh, \]

or

\[ \oint_L m\mathbf{v}(q(t), t) \cdot \delta \mathbf{q}(t) = nh - \frac{e}{c} \oint_L \mathbf{A}_{ext}(q(t), t) \cdot \delta \mathbf{q}(t), \]

\[ ^{13\text{ We could of course also include a gravitational vector potential, but for simplicity we’ll just stick with the magnetic version.}} \]
where the last term on the right hand side of (51) is, by Stokes’ theorem, the magnetic flux enclosed by the loop.

We can also integrate (47) and rewrite in terms of \( S(q(t), t) \) to obtain

\[
S(q(t), t) = \int_{q_i(t_i)}^{q(t)} p'(q(s), s) \cdot dq(s) - \int_{t_i}^{t} E(q(s), s) ds - \hbar \phi,
\]

(52)

where \( \phi \) is the initial phase constant and (52) is equivalent (up to an additive constant) to the relativistic action of a particle in the presence of external fields.

As before, the translational kinetic 3-velocity of the particle can be obtained from \( S(q(t), t) \) as \( v(q(t), t) = p(q(t), t)/\gamma m = (1/\gamma m) \nabla S(q(t), t) \big|_{q=q(t)} \), and the total relativistic energy as \( E(q(t), t) = -\partial_t S(q(t), t) |_{q=q(t)} \).

It then follows that \( S(q(t), t) \) is a solution of the classical relativistic Hamilton-Jacobi equation

\[
- \partial_t S(q, t) |_{q=q(t)} = \sqrt{m^2 c^4 + \left( \nabla S(q, t) - \frac{e}{c} A_{ext}(q, t) \right)^2} c^2 |_{q=q(t)} + \gamma m \Phi_g(q(t)) + e \Phi_e(q(t), t).
\]

(53)

When \( v \ll c \),

\[
S(q(t), t) \approx \int_{q_i(t_i)}^{q(t)} m v'(q(s), s) \cdot dq(s) - \int_{t_i}^{t} \left( mc^2 + \frac{1}{2m} \left[ p(q(s), s) - \frac{e}{c} A_{ext}(q(s), s) \right]^2 + m \Phi_g(q(s)) + e \Phi_e(q(s), s) \right) ds - \hbar \phi,
\]

and (53) becomes

\[
- \partial_t S(q, t) |_{q=q(t)} = \frac{\left( \nabla S(q, t) - \frac{e}{c} A_{ext}(q, t) \right)^2}{2m} |_{q=q(t)} + mc^2 + m \Phi_g(q(t)) + e \Phi_e(q(t), t),
\]

(54)

with \( v(q(t), t) = (1/m) \nabla S(q(t), t) |_{q=q(t)} \) and satisfies the classical Newtonian equation of motion,

\[
ma(q(t), t) = \left( \frac{\partial}{\partial t} + v(q(t), t) \cdot \nabla \right) \left[ \nabla S(q, t) - \frac{e}{c} A_{ext}(q, t) \right] |_{q=q(t)}
= -\nabla [m \Phi_g(q(t)) + e \Phi_e(q(t), t)] |_{q=q(t)} - \frac{e}{c} \frac{\partial A_{ext}(q, t)}{\partial t} |_{q=q(t)} + \frac{e}{c} v(q(t), t) \times B_{ext}(q(t), t).
\]

(56)

Incidentally, if we choose \( \Phi_e \) as the Coulomb potential for the hydrogen atom and set \( B_{ext} = 0 \), then our model is empirically equivalent to the Bohr model of the hydrogen atom (the demonstration of this can be found in Appendix B). As in the previous section, we now want to extend our model to a classical HJ statistical mechanics.

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19The proof is as follows. From \( L = -mc^2/\gamma - \gamma m \Phi_g - e \Phi_e + e \Phi_e \cdot A_{ext} \), the Legendre transform gives \( E = p' \cdot v - L = \gamma mc^2 + mc^2/\gamma + \gamma m \Phi_g + e \Phi_e = \gamma mc^2 + \gamma m \Phi_g + e \Phi_e \) and \( L = p' \cdot v - E \). So, \( S = \int L dt + C = \int (p' \cdot dq - Edt) + C \).
4.4 Classical Hamilton-Jacobi statistical mechanics for one particle interacting with external fields

Suppose now that, in the lab frame with \( v \ll c \), we do not know the actual position \( q(t) \) of the \( zbw \) particle. Then the phase (54) becomes the phase field

\[
S(q, t) = \int_{q(t_i)}^{q(t_f)} m v'(q(s), s) \cdot dq(s)_{q(t)=q} \left. \right|_{q(t_i)}^{q(t_f)} - \int_{t_i}^{t_f} \left( m c^2 + \frac{1}{2m} \left[ p(q(s), s) - \frac{e}{c} A_{\text{ext}}(q(s), s) \right]^2 + m \Phi_g(q(s)) + e \Phi_e(q(s), s) \right) \, ds_{q(t)=q} - \hbar \phi. \tag{57}
\]

To obtain the equations of motion for \( S(q, t) \) and \( v(q, t) \) we will apply the classical analogue of Yasue’s variational principle, in anticipation of the method we will use for constructing ZSM.

First we introduce the ensemble-averaged action/phase functional (inputting limits between initial and final states),

\[
J = E \left[ \int_{t_i}^{t_f} m v'(q(t)) \cdot dq(t) - \int_{t_i}^{t_f} \left( m c^2 + \frac{1}{2m} \left[ p - \frac{e}{c} A_{\text{ext}} \right]^2 + m \Phi_g + e \Phi_e \right) \, dt - \hbar \phi \right] \tag{58}
\]

where the equated expressions are related by the usual Legendre transformation. Imposing the variational constraint,

\[
J = \text{extremal} \tag{59}
\]

a straightforward computation exactly along the lines of that in Appendix A yields (56), which, upon replacing \( q(t) \) by \( q \), corresponds to the classical Newtonian equation,

\[
ma(q, t) = \left( \frac{\partial}{\partial t} + v(q, t) \cdot \nabla \right) \left[ \nabla S(q, t) - \frac{e}{c} A_{\text{ext}}(q, t) \right] = - \nabla \left[ m \Phi_g(q) + e \Phi_e(q, t) \right] - \frac{e}{c} \frac{\partial A_{\text{ext}}(q, t)}{\partial t} + \frac{e}{c} v(q, t) \times B_{\text{ext}}(q, t), \tag{60}
\]

where \( v(q, t) = (1/m) \nabla S(q, t) - e A_{\text{ext}}(q, t)/mc \) corresponds to the kinetic velocity field. By integrating both sides and setting the integration constant equal to the rest mass, we then obtain the classical Hamilton-Jacobi equation for (57),

\[
- \partial_t S(q, t) = \frac{\left( \nabla S(q, t) - \frac{e}{c} A_{\text{ext}}(q, t) \right)^2}{2m} + m c^2 + m \Phi_g(q) + e \Phi_e(q, t). \tag{61}
\]

Because the momentum field couples to the vector potential, it can be readily shown that \( \rho(q, t) \) now evolves by the modified continuity equation

\[
\frac{\partial \rho(q, t)}{\partial t} = - \nabla \cdot \left( \frac{\nabla S(q, t) - e A_{\text{ext}}(q, t)}{m} \right) \rho(q, t), \tag{62}
\]

which preserves the normalization, \( \int \rho_0(q) d^3 q = 1 \). As before, \( S(q, t) \) is a field over the possible positions that the actual \( zbw \) particle can occupy at time \( t \). Since for each possible position the
actual *zbw* particle’s phase will satisfy the relation (50), $S(q, t)$ will be a single-valued function of $q$ and $t$ (up to an additive integer multiple of $2\pi$) and

$$\int_L \nabla S(q, t) \cdot dq = n\hbar. \quad (63)$$

Finally, we can combined (61) and (62) into the nonlinear Schrödinger equation,

$$i\hbar \frac{\partial \psi}{\partial t} = \left[ -i\hbar \nabla - \frac{e}{c} A_{ext} \right]^{2} \psi + \frac{\hbar^{2}}{2m} \nabla^{2} |\psi| \psi + m\Phi_{g} \psi + e\Phi_{e} \psi + mc^{2} \psi, \quad (64)$$

with wave function $\psi(q, t) = \sqrt{\rho(q, t)} e^{iS(q, t)/\hbar}$, which is single-valued because of (63). (Again, $C$ will contribute a global phase $e^{iC/\hbar}$ which drops out.)

5 Zitterbewegung Stochastic Mechanics

We are now ready to extend the classical *zbw* model developed in section 4 to Nelson-Yasue stochastic mechanics for all the same cases. In doing so, we will show how this ‘zitterbewegung stochastic mechanics’ (ZSM) avoids the Wallstrom criticism and explain the ‘quantum-classical correspondence’ between the ZSM equations and the classical HJ statistical mechanical equations. We will also apply ZSM to the central potential problem considered by Wallstrom, to demonstrate how angular momentum quantization emerges and therefore that the solution space of ZSM’s HJM equations is equivalent to the solution space of the quantum mechanical Schrödinger equation.

5.1 One free particle

As in NYSM, we take as our starting point that a particle of rest mass $m$ is immersed in Nelson’s hypothesized ether and has a 3-space coordinate $q(t)$ undergoes a frictionless diffusion process according to the stochastic differential equations,

$$dq(t) = b(q(t), t)dt + dW(t), \quad (65)$$

for the forward-time direction, and

$$dq(t) = b_{*}(q(t), t)dt + dW_{*}(t), \quad (66)$$

for the backward-time direction. As in NYSM, $dW$ is the Wiener process satisfying $E_{t}[dW] = 0$ and $E_{t}[dW^{2}] = (\hbar/m) dt$. Now, in order to incorporate the *zbw* oscillation as a property of the particle, we must amend Nelson’s original phenomenological hypotheses about his ether and particle with the following additional hypotheses of phenomenological character:\footnote{Meaning, we will follow Nelson’s approach of provisionally not offering an explicit physical model of the ether, and de Broglie-Bohm’s approach of provisionally not offering an explicit physical model for the *zbw* particle, beyond the hypothetical characteristics listed here. However, these characteristics should be regarded as general constraints on any future physical model of Nelson’s ether, the *zbw* particle, and the dynamical coupling between the two.}

1. Nelson’s ether is not only a stochastically fluctuating medium in space-time, but an oscillating medium with a spectrum of angular frequencies superposed at each point in 3-space. More precisely, we imagine the ether as a continuous (or effectively continuous) medium composed...
of a countably infinite number of fluctuating, stationary, spherical waves\textsuperscript{21} superposed at each point in space, with each wave having a different (constant) angular frequency, $\omega_k^0$, where $k$ denotes the $k$-th ether mode. (If we assume an upper frequency cut-off for our modes as the inverse Planck time, this will imply an upper bound on the Compton frequency of an elementary particle immersed in the ether, as we will see from hypothesis 3 below.) The relative phases between the modes are taken to be random so that each mode is effectively uncorrelated with every other mode.

2. The particle of rest mass $m$, located in its instantaneous mean forward translational rest frame (IMFTRF), i.e., the frame in which $D\mathbf{q}(t) = b(\mathbf{q}(t), t) = 0$, at some point $\mathbf{q}_0$, is bounded to a harmonic oscillator potential with fixed natural frequency $\omega_0 = \omega_c = (1/\hbar) mc^2$. In keeping with the phenomenological approach of ZSM and the approach taken by de Broglie and Bohm with their $zbw$ models, we need not specify the precise physical nature of this harmonic oscillator potential. This task is left for a future physical model of the ZSM particle.

3. The particle’s center of mass, as a result of being immersed in the ether, undergoes an approximately frictionless translational Brownian motion (due to the homogeneous and isotropic ether fluctuations that couple to the particle by possibly electromagnetic, gravitational, or some other means), as already described by (65-66); and, in its IMFTRF, undergoes a driven oscillation about $\mathbf{q}_0$ by coupling to a narrow band of ether modes that resonantly peaks around the particle’s natural frequency. However, in order that the oscillation of the particle doesn’t become unbounded in its kinetic energy, there must be some mechanism by which the particle dissipates energy back into the ether modes so that, on the average, a steady-state equilibrium regime is reached for the oscillation. That is to say, on some hypothetical characteristic short time-scale, $\tau$, the average energy absorbed from the driven oscillation by the resonant ether modes equals the average energy dissipated back to the ether by the particle. We note that the average, in the present sense, would be over the random phases of the ether modes. (Here we are taking inspiration from stochastic electrodynamics \textsuperscript{126, 127}, where it has been shown that a classical charged harmonic oscillator immersed in a classical electromagnetic zero-point field has a steady-state regime where the phase-averaged power absorbed by the oscillator balances the phase-averaged power radiated by the oscillator back to the zero-point field, yielding a steady-state oscillation at the natural frequency of the oscillator \textsuperscript{126, 127, 128, 129, 130, 131}. However, in keeping with our phenomenological approach, we will not propose a specific mechanism for this energy exchange in ZSM, only provisionally assume that it occurs somehow.) Accordingly, we suppose that, in this steady-state regime, the particle undergoes a steady-state $zbw$ oscillation of angular frequency $\omega_c$ about $\mathbf{q}_0$ in its IMFTRF, as characterized by the ‘fluctuation-dissipation’ relation, $\langle H \rangle_{\text{steady-state}} = \hbar \omega_c = mc^2$, where $\langle H \rangle_{\text{steady-state}}$ is the conserved random-phase-average energy associated with the steady-state oscillation.

It follows then that, in the IMFTRF, the mean forward steady-state $zbw$ phase change is given by

\begin{equation}
\delta \bar{\theta}_0+ := \omega_c \delta t_0 = \frac{mc^2}{\hbar} \delta t_0,
\end{equation}

\textsuperscript{21}These ether waves could be fundamentally continuous field variables or perhaps collective modes arising from non-linear coupling between (hypothetical) discrete constituents of the ether. Both possibilities are logically compatible with what follows.
and the cumulative forward steady-state $zbw$ phase, obtained from the indefinite integral of (67), is

$$\theta_{0+} = \omega_c t_0 + \phi = \frac{mc^2}{\hbar} t_0 + \phi_+, \quad (68)$$

where $\phi_+$ is the initial (forward) phase constant.

The reason for starting our analysis with the IMFTRF goes back to the fact that, before constraining the diffusion process to simultaneous solutions of the forward and backward Fokker-Planck equations associated to (65-66), neither the forward nor the backward stochastic differential equations (65-66) have well-defined time reversals. So the forward and backward stochastic differential equations describe independent, time-asymmetric diffusion processes in opposite time directions, and we must start by considering the steady-state $zbw$ phase in each time direction separately. We chose to start with the more intuitive forward time direction.

For the $zbw$ particle in the instantaneous mean backward translational rest frame (IMBTRF), i.e., the frame defined by $D, q(t) = b_\ast(q(t), t) = 0$, its mean backward steady-state $zbw$ phase change is given by

$$\delta \theta_{0-} := -\omega_c \delta t_0 = -\frac{mc^2}{\hbar} \delta t_0, \quad (69)$$

and

$$\bar{\theta}_{0-} = (-\omega_c t_0) + \phi = \left(-\frac{mc^2}{\hbar} t_0\right) + \phi_-. \quad (70)$$

Note that, in the above construction, both the diffusion coefficient $\nu = \hbar/2m$ and the (reduced) $zbw$ period $T_c = 1/\omega_c = \hbar/mc^2$ are scaled by $\hbar$. This is consistent with our hypothesis that the ether is the common physical cause of both the frictionless diffusion process and the steady-state $zbw$ oscillation. Had we not proposed Nelson’s ether as the physical cause of the steady-state $zbw$ oscillation as well as the frictionless diffusion process, the occurrence of $\hbar$ in both of these particle properties would be inexplicable and compromising for the plausibility of our proposed modification of NYSM.

It should be stressed here that it is not possible to talk of the $zbw$ phase in a rest frame other than the IMFTRF or IMBTRF of the $zbw$ particle, as we cannot transform to a frame in which $dq(t)/dt = 0$, since such an expression is undefined for the (non-differentiable) Wiener process.

Now suppose we Lorentz transform back to the lab frame. For the forward time direction, this corresponds to a boost of (67) by $-b(q(t), t)$ where $b(q(t), t) \neq 0$. Approximating the transformation for non-relativistic velocities so that $\gamma = 1/\sqrt{1-B^2/c^2} \approx 1 + b^2/2c^2$, the forward steady-state $zbw$ phase change (67) becomes

$$\delta \bar{\theta}_+(q(t), t) := \frac{\omega_c}{mc^2} E_t[Dq(t)] \delta t - mDq(t) \cdot (Dq(t)) \delta t$$

$$= \frac{\omega_c}{mc^2} E_t [E_+ Dq(t)] \delta t = \frac{mc^2}{\hbar} t_0 + \phi_+,$$  

$$\quad (71)$$

where

$$E_+(Dq(t)) = mc^2 + \frac{1}{2}m |Dq(t)|^2 = mc^2 + \frac{1}{2}mb^2,$$  

$$\quad (72)$$

neglecting the momentum term proportional to $b^3/c^2$, and where $\delta q_+(t)$ in (71) corresponds to the physical, translational, mean forward displacement of the $zbw$ particle, defined by

$$\delta q_+(t) = [Dq(t)] \delta t = b(q(t), t) \delta t.$$  

$$\quad (73)$$
The first line on the right hand side of (71) is the straightforward stochastic generalization of the Lorentz-transformed classical \( zbw \) phase (just as Yasue’s mean action functional (19) is the straightforward stochastic generalization of the ordinary action functional in classical mechanics [8]) for non-relativistic velocities. Note, however, that the conditional expectation \( E_t[...] \) in (71) is redundant since the right hand side of (71) involves terms depending only on the mean forward velocity \( Dq(t) = b(q(t), t) \), where \( D \) already involves taking a conditional expectation (see the definitions (13) and (14) in section 2). However, in the more general case of a \( zbw \) particle in an external potential \( V_{ext} \), a case we will consider in the next section, the conditional expectation cannot be dropped since there will be an external-potential-dependent term in \( E_0 \) that will depend directly on \( q(t) \) via \( V_{ext}(q(t)) \). The expectation will also be useful for giving a natural connection between the integral of the time-symmetrized analogue of (71) (which we will introduce shortly) and Yasue’s mean action functional, as we will show later in this section.

For the backward time direction, the Lorentz transformation to the lab frame corresponds to a boost of (69) by \(-b_\ast(q(t), t)\) where \( b_\ast(q(t), t) \neq 0 \). Then the backward steady-state \( zbw \) phase change (69) becomes

\[
\delta \tilde{\theta}_-(q(t), t) := \frac{\omega c}{mc^2} E_t[-E_-(D_\ast q(t)) \cdot (D_\ast q(t)) \cdot \delta t] = \frac{\omega c}{mc^2} E_t[-E_0 \delta t + m b_\ast(q(t), t) \cdot \delta q_\ast(t)],
\]

where

\[
E_-(D_\ast q(t)) = mc^2 + \frac{1}{2} m |D_\ast q(t)|^2 = mc^2 + \frac{1}{2} m b_\ast^2,
\]

and where \( \delta q_\ast(t) \) in (74) corresponds to the physical, translational, mean backward displacement of the \( zbw \) particle, defined by

\[
\delta q_\ast(t) = [D_\ast q(t)] \delta t = b_\ast(q(t), t) \delta t.
\]

(Notice that \( \delta q_\ast(t) \) and \( \delta q_\ast(t) \) are not equal in general since \( \delta q_\ast(t) - \delta q_\ast(t) = (b - b_\ast) \delta t \neq 0 \) in general.) Since, at this stage, the forward and backward steady-state \( zbw \) phase changes, (71) and (74), are independent of one another, each must equal \( 2\pi n \) when integrated along a closed loop \( L \) in which both time and position change. Otherwise we will contradict our hypothesis that, up to this point, the \( zbw \) particle has a well-defined steady-state phase at each point along its mean space-time trajectory in the forward or backward time direction.

In the lab frame, the forward and backward stochastic differential equations for the \( zbw \) particle’s translational motion are as before

\[
dq(t) = b(q(t), t) dt + dW(t),
\]

and

\[
dq(t) = b_\ast(q(t), t) dt + dW_\ast(t),
\]

with corresponding Fokker-Planck equations

\[
\frac{\partial \rho(q, t)}{\partial t} = -\nabla \cdot [b(q, t) \rho(q, t)] + \frac{\hbar}{2m} \nabla^2 \rho(q, t),
\]

and

\[
\frac{\partial \rho(q, t)}{\partial t} = -\nabla \cdot [b_\ast(q, t) \rho(q, t)] - \frac{\hbar}{2m} \nabla^2 \rho(q, t).
\]
Restricting the diffusion process to simultaneous solutions of (79) and (80) via

\[ \mathbf{v} := \frac{1}{2} [\mathbf{b} + \mathbf{b}_*] = \frac{\nabla S(\mathbf{q}, t)}{m} \]  

(81)

and

\[ \mathbf{u} := \frac{1}{2} [\mathbf{b} - \mathbf{b}_*] = \frac{\hbar}{2m} \frac{\nabla \rho(\mathbf{q}, t)}{\rho(\mathbf{q}, t)} \]  

(82)

reduces the forward and backward Fokker-Planck equations to

\[ \frac{\partial \rho(\mathbf{q}, t)}{\partial t} = - \nabla \cdot \left[ \nabla S(\mathbf{q}, t) \frac{\rho(\mathbf{q}, t)}{m} \right], \]  

(83)

with \( \mathbf{b} = \mathbf{v} + \mathbf{u} \) and \( \mathbf{b}_* = \mathbf{v} - \mathbf{u} \). We also follow Nelson in postulating the presence of an external osmotic potential \( \bar{U}(\mathbf{q}, t) \) which couples to the zbw particle as \( R(\mathbf{q}, t) = \mu \bar{U}(\mathbf{q}, t) \), and by the same reasoning discussed in section 2, imparts an osmotic velocity \( \nabla R/m = (\hbar/2m) \nabla \rho/\rho \). We then have \( \rho = e^{2R/\hbar} \) for all times.

To obtain the 2nd-order time-symmetric mean dynamics for the translational motion of the zbw particle, we will use the variational principle of Yasue. To do this, we must first define the time-symmetric steady-state phase change of the zbw particle in the lab frame, via a symmetric combination of the forward and backward steady-state phase changes (71) and (74). This is natural to do since (71) and (74) correspond to the same frame (the lab frame), and since (71) and (74) are no longer independent of one another as a result of the constraints (81-82). Taking the difference between (74) and (71), we obtain (replacing \( \delta \rightarrow dt \), hence \( \delta \mathbf{q}_{\pm}(t) \rightarrow d\mathbf{q}_{\pm}(t) \))

\[
\begin{align*}
d\bar{\theta}(\mathbf{q}(t), t) &:= \frac{1}{2} \left[ d\bar{\theta}_+(\mathbf{q}(t), t) - d\bar{\theta}_-(\mathbf{q}(t), t) \right] \\
&= \frac{\omega_c}{mc^2} E_t \left[ E(D\mathbf{q}(t), D_\star \mathbf{q}(t)) dt - \frac{m}{2} \left( \mathbf{b}(\mathbf{q}(t), t) \cdot d\mathbf{q}_+(t) + \mathbf{b}_*(\mathbf{q}(t), t) \cdot d\mathbf{q}_-(t) \right) \right] \\
&= \frac{\omega_c}{mc^2} E_t \left[ \frac{m}{2} \left( \mathbf{b} \cdot \frac{d\mathbf{q}_+(t)}{dt} + \mathbf{b}_* \cdot \frac{d\mathbf{q}_-(t)}{dt} \right) \right] \\
&= \frac{\omega_c}{mc^2} E_t \left[ \left( E - \frac{m}{2} \left( \mathbf{b}^2 + \mathbf{b}_*^2 \right) \right) dt \right] \\
&= \frac{\omega_c}{mc^2} E_t \left[ (E - (m \mathbf{v} \cdot \mathbf{v} + m \mathbf{u} \cdot \mathbf{u})) dt \right] \\
&= \frac{\omega_c}{mc^2} E_t \left[ \left( mc^2 - \frac{1}{2} m \mathbf{v}^2 - \frac{1}{2} m \mathbf{u}^2 \right) dt \right],
\end{align*}
\]

where, from (72) and (75), we have defined

\[ E = \frac{1}{2} (E_+ + E_-) = mc^2 + \frac{1}{2} \left[ \frac{1}{2} m \mathbf{b}^2 + \frac{1}{2} m \mathbf{b}^2_\star \right] = mc^2 + \frac{1}{2} m \mathbf{v}^2 + \frac{1}{2} m \mathbf{u}^2, \]  

(85)

and where we have used (73) and (76) in (84).

It is important to note that because \( \bar{\theta}_+ \) and \( \bar{\theta}_- \) are no longer independent of one another, it is no longer the case that \( \delta \bar{\theta}_+ \) and \( \delta \bar{\theta}_- \) will each equal \( 2\pi n \) when integrated along a closed loop \( L \).
in which both time and position change. However, the consistency of our theory does require that \( \delta \theta = 2\pi n \), otherwise we would contradict our hypothesis that the \( zbw \) particle, after restricting to simultaneous solutions of (79) an (80), has a well-defined and unique steady-state phase at each 3-space location it can occupy at each time, regardless of time-direction. Note also that, without the constraints (81-82), we would always have \( \delta \theta_+ = 2\pi n \) and \( \delta \theta_- = 2\pi n \), hence \( \delta \theta = 0 \).

In other words, a time-symmetrized “phase” defined from the subtractive combination of phase-principal function as \( \theta \) an additive integer multiple of the constraints (81-82), we would always have symmetric, steady-state phase at a time \( t \) of the integral are both required to be finite quantities here \([132]\). We note that (86) is formally stochastic calculus, since the integral of the conditional expectation and the conditional expectation identical to the \( \delta W/\delta \theta \) variation in which both time and position change. However, the consistency of our theory does require that \( \theta \).

Now, from the last line of (84), we can integrate and define the time-symmetric steady-state phase function as

\[
I(q(t), t) = -h\theta(q(t), t) \equiv E \left[ \int_{t_1}^{t_2} \left( \frac{1}{2} m\dot{v}^2 + \frac{1}{2} m\dot{u}^2 - mc^2 \right) dt' - h\phi |q(t)\right],
\]

where the expectation on the right hand side is now conditional on the Nelsonian path \( q(t) \). (Note that the interchangeability of the expectation and the time integral follows from Fubini’s theorem in stochastic calculus, since the integral of the conditional expectation and the conditional expectation of the integral are both required to be finite quantities here \([132]\).) We note that (86) is formally identical to the \( W \) function introduced by Yasue in \([5]\), and from which Yasue shows that the variation \( \delta W/\delta q(t) \) implies the current velocity relation (81) with \( W \) in place of \( S \). The latter result also applies to (86), given the formal identicality between \( I \) and \( W \), however we will use a different approach to connect \( \nabla I \) with the current velocity (81). Also, whereas Yasue’s \( W \) function isn’t constrained to satisfy \( \delta L/\delta \phi \), (86) does satisfy \( \delta I = nh \since it is explicitly defined in terms of the phase function \( \theta \).

By a slight modification of (86), we can also define the steady-state phase-action functional

\[
J := I = E \left[ \int_{t_1}^{t_2} \left( \frac{1}{2} m\dot{v}^2 + \frac{1}{2} m\dot{u}^2 - mc^2 \right) dt' - h\phi \right],
\]

where \( \phi \) is the initial phase constant, and where (87) differs from (86) by the end-point at \( t_f \) being fixed and \( E[...] \) being the absolute expectation. It is easily seen that (87) is just Yasue’s time-symmetric ensemble-averaged action functional, Eq. (19) in section 2, with \( V = 0 \), inclusion of the rest-energy term \( -mc^2 \), and inclusion of the initial phase constant \( \phi \).

Note, also, that from the second to last line of (84), we can obtain the cumulative, time-symmetric, steady-state phase at a time \( t \) as

\[
\bar{\theta}(q(t), t) = \frac{\omega_e}{mc^2} E \left[ \int_{t_1}^{t_2} (E - (m\cdot v + m\cdot u)) dt' |q(t)\right] + \phi
\]

\[
= \frac{\omega_e}{mc^2} E \left[ \int_{t_1}^{t_2} ((E - m\cdot u) - m\cdot v) dt' |q(t)\right] + \phi
\]

\[
= \frac{\omega_e}{mc^2} E \left[ \int_{t_1}^{t_2} (H - m\cdot v) dt' |q(t)\right] + \phi
\]

\[
= \frac{\omega_e}{mc^2} E \left[ \int_{t_1}^{t_2} \left( H - \frac{m}{4} (Dq(t') + D^*q(t')) \cdot (D + D^*) q(t') \right) dt' |q(t)\right] + \phi
\]

\[
= \frac{\omega_e}{mc^2} E \left[ \int_{t_1}^{t_2} H dt' - \int_{q(t_1)}^{q(t)} \frac{m}{2} (Dq(t') + D^*q(t')) \cdot Dq(t') |q(t)\right] + \phi,
\]

26
where
\[ H := E - m u \cdot u = mc^2 + \frac{1}{2}mv^2 - \frac{1}{2}m\dot{u}^2, \] (89)
and where we have used the fact that 0.5(D + D\ell)q(t) = (\partial_t + v \cdot \nabla)q(t), and v(q(t), t) = (\partial_t + v \cdot \nabla)q(t) =: Dq(t)/Dt, and and Dq(t) = (Dq(t)/Dt) dt. Now, given an integral curve Q(t) of the current velocity/momentum field, i.e., a solution of
\[ m dQ(t)/dt = mv(Q(t), t) = p(Q(t), t) = \nabla S(q, t)|_{q=Q(t)}, \] (90)
and given that \( \bar{\theta}(q, t) = \bar{\theta}|_{q(t)=q} \) is a field on 3-space representing the possible phases that the actual zbw particle could have at a point q at time t (up to addition of a constant), we can also evaluate \( \bar{\theta}(q, t) \) with respect to \( Q(t) \), which allows us to drop the conditional expectation (since \( Q(t) \) is deterministic) to obtain
\[ \bar{\theta}(Q(t), t) = \frac{\omega_c}{mc^2} \int_{t_i}^t \left[ H - mv(Q(t'), t') \cdot \frac{dQ(t')}{dt'} \right] dt' + \phi \]
\[ = \frac{\omega_c}{mc^2} \int_{t_i}^t Hdt' - \int_{Q(t_i)}^{Q(t)} p \cdot dQ(t') + \phi. \] (91)

Here (91) corresponds to the time-symmetrized steady-state phase of the zbw particle in the lab frame, evaluated along the zbw particle’s ‘time-symmetric mean trajectory’, where the time-symmetric mean trajectory corresponds to an integral curve of the current velocity field, i.e., (90).

That the time-symmetric mean trajectories should correspond to integral curves of the current velocity field can be seen from the fact that the single-time probability density \( \rho(q, t) \), after imposing (81-82), is a solution of the continuity equation (83), from which it follows that the possible mean trajectories of the zbw particle are the flow lines of the probability current \( \rho v \), i.e., the solutions of (90) for all possible initial conditions \( Q(0) \).

Now, taking the total differential of the left hand side of (91) gives
\[ d\bar{\theta} = \nabla \bar{\theta}|_{q=Q(t)} dQ(t) + \partial_t \bar{\theta}|_{q=Q(t)} dt. \] (92)
This allows us to identify
\[ p(Q(t), t) = -\left( \frac{mc^2}{\omega_c} \right) \nabla \bar{\theta}|_{q=Q(t)} = \nabla S|_{q=Q(t)}, \] (93)
where we have used (92) along with (91) and (90). Thus the current velocity of the zbw particle can be identified with the gradient of the zbw particle’s time-symmetrized steady-state phase with respect to the location of the zbw particle at time t in the lab frame, given the assumption that the current velocity is integrable, i.e., given (81) and (90). Accordingly, the S function can be identified with (91). In addition, (92) along with (91) relates the H function to \( \bar{\theta} \) (hence S) by
\[ H(Q(t)) = \left( \frac{mc^2}{\omega_c} \right) \partial_t \bar{\theta}|_{q=Q(t)} = -\partial_t S|_{q=Q(t)}. \] (94)
From (94), (93), and (91), it follows that

$$S(Q(t), t) = \int_{Q(t_i)}^{Q(t)} \mathbf{p} \cdot d\mathbf{Q}(t') - \int_{t_i}^{t} H dt' - h\phi$$

$$= \int_{t_i}^{t} \left[ \frac{1}{2} m \mathbf{v}(Q(t'), t')^2 + \frac{1}{2} m \mathbf{u}(Q(t'), t')^2 - mc^2 \right] dt' - h\phi = I(Q(t), t), \tag{95}$$

and

$$\oint dS(Q(t), t) = \left( -\frac{m^2 c^2}{\omega_c} \right) \oint L \delta(q(t) - t) = \oint L \left[ \mathbf{p} \cdot \delta(Q(t) - H\delta t) = nh. \tag{96} \right.$$

We will use these last two expressions for later comparisons.

As an aside, let us recall that after restricting the forward and backward diffusions to simultaneous solutions of (79-80), we had \( \mathbf{b} = \mathbf{v} + \mathbf{u} \) and \( \mathbf{b}_s = \mathbf{v} - \mathbf{u} \). So the IMFTRF and the IMBTRF will not coincide since for \( \mathbf{b} = \mathbf{v} + \mathbf{u} = 0 \) it will not generally be the case that \( \mathbf{b}_s = \mathbf{v} - \mathbf{u} = 0 \). Nevertheless, we can define an instantaneous mean (time-)symmetric rest frame (IMSTRF) as the frame in which \( \mathbf{b} + \mathbf{b}_s = 2\mathbf{v} = 0 \). In the IMSTRF, (88) or (91) or (95) reduces to

$$\dot{\theta} = (\omega_c/mc^2) \left( (mc^2 - \frac{1}{2}m\mathbf{u}^2) t + \phi \right),$$

since \( \mathbf{v} = 0 \) and \( \partial_t \rho = 0 \). This shows that the kinetic energy term due to the osmotic velocity contributes a tiny shift to the steady-state \( zbw \) phase (88) or (91) or (95) in the IMSTRF (since, in the non-relativistic regime, \( \mathbf{u}^2/c^2 \ll 1 \)).

Returning now to (87), the imposition of the conservative-diffusions constraint implies extremality of (87), which further implies (see Appendix A) Nelson’s mean acceleration equation,

$$ma(q(t), t) = \frac{m}{2} \left[ D_s D + DD_\star \right] q(t) = 0. \tag{97}$$

Computing the derivatives in (97), and using that \( \mathbf{b} = \mathbf{v} + \mathbf{u} \) and \( \mathbf{b}_s = \mathbf{v} - \mathbf{u} \), we obtain

$$\begin{align*}
ma(q(t), t) &= m \left[ \frac{\partial \mathbf{v}(q(t))}{\partial t} + \mathbf{v}(q(t)) \cdot \nabla \mathbf{v}(q(t)) - \mathbf{u}(q(t)) \cdot \nabla \mathbf{u}(q(t)) - \frac{\hbar}{2m} \nabla^2 \mathbf{u}(q(t)) \right]_{q=q(t)} \\
&= \nabla \left[ \frac{\partial S(q(t))}{\partial t} + \left( \nabla S(q(t)) \right)^2 - \frac{\hbar}{2m} \frac{\nabla^2 \sqrt{\rho(q(t))}}{\sqrt{\rho(q(t))}} \right]_{q=q(t)} = 0. \tag{98} \end{align*}$$

Integrating both sides of (98) gives the total translational energy of the \( zbw \) particle along the stochastic trajectory \( q(t) \):

$$E(q(t), t) = -\frac{\partial S(q(t))}{\partial t} \bigg|_{q=q(t)} = mc^2 + \frac{\left( \nabla S(q(t)) \right)^2}{2m} \bigg|_{q=q(t)} - \frac{\hbar^2}{2m} \frac{\nabla^2 \sqrt{\rho(q(t))}}{\sqrt{\rho(q(t))}} \bigg|_{q=q(t)}, \tag{99}$$

where we have set the integration constant equal to the \( zbw \) particle’s rest energy. Alternatively, we can again consider integral curves of the current velocity/momentum field, but where now the integral curves are obtained from solutions of

$$m \frac{d^2 Q(t)}{dt^2} = m (\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}) \bigg|_{Q=q(t)} = -\nabla \left( -\frac{\hbar^2}{2m} \frac{\nabla^2 \sqrt{\rho(q(t))}}{\sqrt{\rho(q(t))}} \right) \bigg|_{Q=q(t)}, \tag{100}$$

i.e., the mean acceleration equation (98), rewritten so that only the \( \mathbf{v} \)-dependent terms are kept on the left hand side. Then we can replace \( q(t) \) in (99) with \( Q(t) \) to obtain the total translational energy.
associated with the \( zbw \) particle’s time-symmetric mean trajectory, i.e., \( \hat{E}(Q(t), t) \). Moreover, we can express the solution of (99) in terms of \( Q(t) \), thereby obtaining

\[
S(Q(t), t) = \int_{Q(t_{i})}^{Q(t)} p \cdot dQ(t') - \int_{t_{i}}^{t} \hat{E}dt' - \hbar \phi
\]

\[
= \int_{t_{i}}^{t} \left[ \frac{1}{2} m v(Q(t'), t')^2 - \frac{\hbar^2}{2m} \frac{\nabla^2 \sqrt{\rho(Q(t'), t')}}{\sqrt{\rho(Q(t'), t')}} - mc^2 \right] dt' - \hbar \phi
\]

(101)

We identify (101) as the conservative-diffusion-constrained, time-symmetrized, steady-state phase (action) of the \( zbw \) particle in the lab frame, evaluated along an integral curve \( Q(t) \) obtained from (100).

Notice that the last line of (101) differs from the last line of (95) only by addition of the term involving \( \nabla \cdot u \). (The equality between the last two lines of (101) follows from the well-known fact that the quantum kinetic can be decomposed as \( -\hbar^2/2m) \rho^{-1/2} \nabla^2 \rho^{1/2} = 0.5mu^2 - (h^2/4m) \rho^{-1} \nabla^2 \rho \)

and by the product rule, \( 0.5mu^2 - (h^2/4m) \rho^{-1} \nabla^2 \rho = -0.5mu^2 - m(h/2m) \nabla \cdot u \).

Notice also that the equation of motion for (101) differs from the equation of motion for the classical \( zbw \) particle phase by the presence of the quantum kinetic entering into (98-99). The two phases might appear to be connected by the ‘classical limit’ \( (h/2m) \to 0 \), but this is only a formal connection since such a limit corresponds to deleting the presence of the ether, thereby also deleting the physical mechanism that we hypothesize to cause the \( zbw \) particle to oscillate at its Compton frequency. The physically realistic ‘classical limit’ for (98-99) corresponds to situations where the quantum kinetic and quantum force are negligible. Such situations will arise (as in the dBB theory) whenever the center of mass of a system of particles is sufficiently large and environmental decoherence is appreciable [133, 134, 69, 70].

Inasmuch as (101) is a well-defined phase function of the \( zbw \) particle’s time-symmetric mean trajectory \( Q(t) \) in the lab frame (because it was derived from applying the variational principle to (87), the latter of which was defined in terms of (84), which we argued must satisfy \( \int_{L} \delta \theta = 2\pi n \)), if we integrate \( \delta S(Q(t), t) \) around a closed loop \( L \) in which time and position may change, we will have

\[
\oint_{L} \delta S(Q(t), t) = \oint_{L} \left[ p \cdot \delta Q(t) - \hat{E} \delta t \right] = nh,
\]

(102)

and for a special loop in which time is held fixed,

\[
\oint_{L} \nabla S|_{Q=Q(t)} \cdot \delta Q(t) = \oint_{L} p \cdot \delta Q(t) = nh.
\]

(103)

Otherwise, we would contradict our hypothesis that the \( zbw \) particle still has a well-defined, time-symmetrized, steady-state phase at each 3-space location it can occupy along a mean trajectory \( Q(t) \) in either time direction, after the constraint of conservative diffusions has been imposed. (Notice that (102) differs from (96) by \( \hat{E} \) replacing \( H \), and that \( \hat{E} - H = -(\hbar/2) \nabla \cdot u \). If we also consider the time-symmetrized steady-state phase field, \( S(q, t) \), which is a field over the possible locations of the actual \( zbw \) particle (as described in section 4.2), then by applying the same physical reasoning above to each possible initial position that the \( zbw \) particle can occupy, it follows that
the net change of the phase field along any mathematical loop in space (with time held fixed) will be
\[
\int_L dS(q, t) = \int_L \mathbf{p} \cdot d\mathbf{q} = nh. \tag{104}
\]
(The justification for (104) where \( \rho = 0 \) is discussed in section 5.2, since such “nodal points” commonly arise in the presence of bound states.)

The total energy field \( E(q, t) \) will correspondingly be given by (99) when \( Q(t) \) is replaced by \( q \). So with (104), (99), and (83), we can construct the 1-particle Schrödinger equation,
\[
i\hbar \frac{\partial \psi(q, t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi(q, t) + mc^2 \psi(q, t), \tag{105}
\]
where \( \psi(q, t) = \sqrt{\rho(q, t)} e^{iS(q, t)/\hbar} \) is a single-valued wave function as a result of (104). As in the classical case, the constant \( C = \hbar \phi \) will contribute a global phase factor \( e^{iC/\hbar} \) which cancels out from both sides of (105). We thereby have a formulation of free-particle ZSM that recovers the usual free-particle Schrödinger equation.

### 5.2 One particle interacting with external fields

Suppose again that the particle undergoes a steady-state \( zbw \) oscillation in the IMFTRF, but now carries charge \( e \) so that it is a classical charged harmonic oscillator of some type (subject again to the hypothetical constraint of no electromagnetic radiation emitted when there is no translational motion; or the constraint that the oscillation of the charge is radially symmetric so that there is no net energy radiated; or, if the ether turns out to be electromagnetic in nature as Nelson suggested [4], then that the steady-state \( zbw \) oscillation is due to a balancing between the random-phase-averaged electromagnetic energy absorbed from the charged harmonic oscillator’s driven oscillation, and the random-phase-averaged electromagnetic energy radiated back to the ether, much like in stochastic electrodynamics [126, 127, 128, 129, 130, 131]). Then, in the presence of an external electric potential \( \Phi_e(q_0(t_0), t_0) = E_{ext}(\mathbf{q}_0(t_0), t_0) \cdot \mathbf{q}_0(t_0) \), where \( \mathbf{q}_0(t_0) \) is the positional displacement of the \( zbw \) particle in some arbitrary direction from the field source (again making the point-like approximation for \( |\mathbf{q}_0| \gg \lambda_e \) and satisfies the forward stochastic differential equation (77) with \( b = 0 \), the \( zbw \) phase change in this IMFTRF is shifted by

\[
\delta \theta_{t_0} = E_t \left[ (\omega_c + \varepsilon(q_0(t_0), t_0)) \delta t_0 \right] = \frac{1}{\hbar} \left( mc^2 \delta t_0 + E_t \left[ e \Phi_e(q_0(t_0), t_0) \delta t_0 \right] \right), \tag{106}
\]

where \( \varepsilon(q_0(t_0), t_0) = \omega_c \, (e/mc^2) \, \Phi_e(q_0(t_0), t_0) \). Direct integration gives

\[
\bar{\theta}_{t_0} = E \left[ \int_{t_0}^{t_0} (\omega_c + \varepsilon(q_0(t_0), t_0)) dt_0 | q_0(t_0) \right] = \frac{1}{\hbar} \left( mc^2 t_0 + E \left[ e \int_{t_0}^{t_0} \Phi_e(q_0(t_0), t_0') dt_0' | q_0(t_0) \right] \right) + \phi. \tag{107}
\]

In the IMBTRF,
\[
\delta \bar{\theta}_{t_0} = -E_t \left[ (\omega_c + \varepsilon(q_0(t_0), t_0)) \delta t_0 \right] = -\frac{1}{\hbar} \left( mc^2 \delta t_0 + E_t \left[ e \Phi_e(q_0(t_0), t_0) \delta t_0 \right] \right). \tag{108}
\]
Direct integration gives
\[
\delta \theta_0^- = -E \left[ \int_{t_0}^{t_f} (\omega_c + \varepsilon(q(t'_0), t'_0)) \, dt'_0 \big| q(t_0) \right]
\]
\[= -\frac{1}{\hbar} \left( mc^2 t_0 + E \left[ \int_{t_0}^{t_f} \Phi_c(q(t'_0), t'_0) \, dt'_0 \big| q(t_0) \right] \right) + \phi. \tag{109}\]

Now suppose we Lorentz transform back to the lab frame. For the forward time direction, this corresponds to a boost of (106) by \(-\mathbf{b}(q(t), t)\) where \(\mathbf{b}(q(t), t) \neq 0\). Approximating the transformation for non-relativistic velocities so that \(\gamma = 1/\sqrt{1 - b^2/c^2} \approx 1 + b^2/2c^2\), (106) becomes
\[
\delta \theta_+(q(t), t) = \frac{\omega_c}{mc^2} E_t [E_+(q(t), Dq(t), t) \delta t - m\mathbf{b}(q(t), t) \cdot \delta \mathbf{q}_+(t)], \tag{110}\]
where
\[
E_+(q(t), Dq(t), t) = mc^2 + \frac{1}{2} m\mathbf{b}^2 + e\Phi_c, \tag{111}\]

neglecting the momentum term proportional to \(b^3/c^2\). Again we take \(\delta \mathbf{q}_+(t)\) to correspond to (73). For the backward time direction, we have a boost of (108) by \(-\mathbf{b}_*(q(t), t)\) where \(\mathbf{b}_*(q(t), t) \neq 0\), hence
\[
\delta \theta_-(q(t), t) = \frac{\omega_c}{mc^2} E_t [-E_-(q(t), D_*(q(t), t) \delta t + m\mathbf{b}_*(q(t), t) \cdot \delta \mathbf{q}_-(t)], \tag{112}\]
where
\[
E_-(q(t), D_*(q(t), t) = mc^2 + \frac{1}{2} m\mathbf{b}_*^2 + e\Phi_c. \tag{113}\]

Again we take \(\delta \mathbf{q}_-(t)\) to correspond to (76).

As in the free particle case, at this stage, the forward and backward steady-state \(zbw\) phase changes, (110) and (112), are independent of one another. So both (110) and (112) must equal \(2\pi n\) when integrated along a closed loop \(L\) in which both time and position change. Otherwise we will contradict our hypothesis that, up to this point, the \(zbw\) particle has a well-defined mean forward or backward steady-state phase at each point along its mean forward or backward space-time trajectory.

In the lab frame, the forward and backward stochastic differential equations for the translational motion are once again
\[
d\mathbf{q}(t) = \mathbf{b}(\mathbf{q}(t), t) + d\mathbf{W}(t), \tag{114}\]
and
\[
d\mathbf{q}(t) = \mathbf{b}_*(\mathbf{q}(t), t) + d\mathbf{W}_*(t), \tag{115}\]
with corresponding Fokker-Planck equations
\[
\frac{\partial \rho(\mathbf{q}, t)}{\partial t} = -\nabla \cdot [\mathbf{b}(\mathbf{q}, t) \rho(\mathbf{q}, t)] + \frac{\hbar}{2m} \nabla^2 \rho(\mathbf{q}, t), \tag{116}\]
and
\[
\frac{\partial \rho(\mathbf{q}, t)}{\partial t} = -\nabla \cdot [\mathbf{b}_*(\mathbf{q}, t) \rho(\mathbf{q}, t)] - \frac{\hbar}{2m} \nabla^2 \rho(\mathbf{q}, t). \tag{117}\]

Let us now suppose that an external magnetic field \(\mathbf{B}_{ext}(\mathbf{q}, t) = \nabla \times \mathbf{A}_{ext}(\mathbf{q}, t)\) is also present. Then, restricting ourselves to simultaneous solutions of (116-117) via
\[
\mathbf{v} := \frac{1}{2} [\mathbf{b} + \mathbf{b}_*] = \frac{\nabla S}{m} - \frac{e}{mc} \mathbf{A}_{ext} \tag{118}\]
and

$$u := \frac{1}{2} [b - b_*] = \frac{\hbar}{2m} \nabla \rho$$  \hfill (119)

entails (116-117) reduce to

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \left( \left( \frac{\nabla S}{m} - \frac{e}{mc} A_{ext} \right) \rho \right).$$  \hfill (120)

We can then write $b' = v' + u$ and $b'_* = v' - u$, where we recall that $v' = v + (e/mc)A_{ext}$, implying $b = b' - (e/mc)A_{ext}$ and $b_* = b'_* - (e/mc)A_{ext}$. Once again the osmotic potential $R(q, t) = \mu U(q, t)$ impacts to the particle an osmotic velocity $\nabla R/m = (\hbar/2m) \nabla \rho/\rho$ (see section 2), implying $\rho = e^{2R/h}$ for all times.

As in the free particle case, we can obtain the 2nd-order time-symmetric mean dynamics from Yasue’s variational principle.

Since (110) and (112) correspond to the same (lab) frame and are no longer independent because of (118-119), it is natural to define the time-symmetric steady-state $zbw$ particle phase in the lab frame by taking the difference between (110) and (112) (under the replacements $2^*$), implying $\rho$ of (118-119), it is natural to define the time-symmetric steady-state $Y$ as Yasue’s variational principle.

As in the free particle case, the consistency of our theory requires that the time-symmetrized steady-state $zbw$ phase change of the $zbw$ particle in the lab frame, (121), satisfies $\Phi_e \delta \theta = 2\pi n$. 

\begin{align*}
\begin{align*}
\frac{d\theta(q(t), t)}{dt} &:= \frac{1}{2} [d\theta_+(q(t), t) - d\theta_-(q(t), t)] \\
&= \frac{\omega_e}{mc^2} E_t \left[ E(q(t), Dq(t), D*, q(t), t)dt - \frac{m}{2} \left( b'(q(t), t) \cdot d\rho_+(t) + b'_*(q(t), t) \cdot d\rho_-(t) \right) \right] + \phi \\
&= \frac{\omega_e}{mc^2} E_t \left[ E - \frac{m}{2} \left( b' \cdot \frac{d\rho_+(t)}{dt} + b'_* \cdot \frac{d\rho_-(t)}{dt} \right) \right] + \phi \\
&= \frac{\omega_e}{mc^2} E_t \left[ \left( E - \frac{m}{2} \left( b' \cdot \frac{d\rho_+(t)}{dt} + b'_* \cdot \frac{d\rho_-(t)}{dt} \right) \right) \right] + \phi \\
&= \frac{\omega_e}{mc^2} E_t \left[ \left( E - \frac{m}{2} \left( b' + b'_* \cdot b_+ \right) \right) \right] + \phi \\
&= \frac{\omega_e}{mc^2} E_t \left[ \left( E - \frac{m}{2} \left( b^2 + \frac{e}{mc} b \cdot A_{ext} + b'_* + \frac{e}{mc} b_* \cdot A_{ext} \right) \right) \right] + \phi \\
&= \frac{\omega_e}{mc^2} E_t \left[ \left( E - \frac{m}{2} \left( b^2 + \frac{e}{mc} b \cdot A_{ext} + b'_* + \frac{e}{mc} b_* \cdot A_{ext} \right) \right) \right] + \phi \\
&= \frac{\omega_e}{mc^2} E_t \left[ \left( E - (mv \cdot v + mu \cdot u) \right) \right] + \phi \\
&= \frac{\omega_e}{mc^2} E_t \left[ \left( mc^2 + e\Phi_e - \frac{1}{2} mv^2 - \frac{1}{2} mu^2 - \frac{e}{c} v \cdot A_{ext} \right) \right] + \phi.
\end{align*}
\end{align*}

where, using (111) and (13), along with the constraints (118) and (119), we have defined

$$E(q(t), Dq(t), D*, q(t), t) = mc^2 + \frac{1}{2} \left[ \frac{1}{2} mb^2 + \frac{1}{2} m b'_* \right] + e\Phi_e$$

\begin{align*}
\begin{align*}
E(q(t), Dq(t), D*, q(t), t) &:= mc^2 + \frac{1}{2} \left[ \frac{1}{2} mb^2 + \frac{1}{2} m b'_* \right] + e\Phi_e. \\
&= mc^2 + \frac{1}{2} mv^2 + \frac{1}{2} mu^2 + e\Phi_e.
\end{align*}
\end{align*}

As in the free particle case, the consistency of our theory requires that the time-symmetrized steady-state $zbw$ phase change of the $zbw$ particle in the lab frame, (121), satisfies $\Phi_e \delta \theta = 2\pi n$. 

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Otherwise we would contradict our hypothesis that the \( zbw \) particle, under the time-symmetric constraints (118-119), has a well-defined and unique steady-state phase at each 3-space location it can occupy at each time, regardless of time direction.

Using the integral of (121) in the definition of the steady-state phase-principal function

\[
I = -\frac{mc^2}{\omega_c} \tilde{\theta} = E \left[ \int_{t_i}^{t_f} \left( \frac{1}{2} m \mathbf{v}^2 + \frac{1}{2} m \mathbf{u}^2 + \frac{e}{c} \mathbf{v} \cdot \mathbf{A}_{ext} - mc^2 - e\Phi_e \right) dt' | \mathbf{q}(t) \right] - \hbar \phi,
\]

we can define the steady-state phase-action functional as

\[
J = I_{sf} = E \left[ \int_{t_i}^{t_f} \left( \frac{1}{2} m \mathbf{v}^2 + \frac{1}{2} m \mathbf{u}^2 + \frac{e}{c} \mathbf{v} \cdot \mathbf{A}_{ext} - mc^2 - e\Phi_e \right) dt' \right] - \hbar \phi.
\]

Equation (124) is just Yasue’s mean action functional, Eq. (145) in Appendix A, but with the inclusion of the rest-energy term \(-mc^2\) and the time-symmetrized initial phase constant \(\phi\).

Note, also, that from the second to last line of (121), we can write the cumulative, time-symmetric, steady-state phase at a time \(t\) as

\[
\tilde{\theta}(\mathbf{q}(t), t) = \frac{\omega_c}{mc^2} E \left[ \int_{t_i}^{t} \left( E - (m \mathbf{v} \cdot \mathbf{v} + m \mathbf{u} \cdot \mathbf{u}) - \frac{e}{c} \mathbf{v} \cdot \mathbf{A}_{ext} \right) dt' | \mathbf{q}(t) \right] + \phi
\]

(125)

where

\[
H := E - m \mathbf{u} \cdot \mathbf{u} = mc^2 + \frac{1}{2} m \mathbf{v}^2 - \frac{1}{2} m \mathbf{u}^2 + E\Phi_e.
\]

Now, given an integral curve \(\mathbf{Q}(t)\) obtained from

\[
m \frac{d\mathbf{Q}(t)}{dt} = p(\mathbf{Q}(t), t) = \nabla S(\mathbf{q}(t))_{\mathbf{q}=\mathbf{Q}(t)} - \frac{e}{c} \mathbf{A}_{ext}(\mathbf{Q}(t), t),
\]

we can replace (125) with

\[
\tilde{\theta}(\mathbf{Q}(t), t) = \frac{\omega_c}{mc^2} \int_{t_i}^{t} \left( H - m \mathbf{v} \cdot \frac{d\mathbf{Q}(t')}{dt'} - \frac{e}{c} \frac{d\mathbf{Q}(t')}{dt'} \cdot \mathbf{A}_{ext}(\mathbf{Q}(t'), t') \right) dt' + \phi
\]

(126)

\[
= \frac{\omega_c}{mc^2} \left[ \int_{t_i}^{t} H dt' - \int_{\mathbf{Q}(t_i)}^{\mathbf{Q}(t)} \left( p + \frac{e}{c} \mathbf{A}_{ext} \right) \cdot d\mathbf{Q}(t') \right] + \phi.
\]

The total differential of the left hand side of (128) gives

\[
d\tilde{\theta} = \nabla \tilde{\theta}|_{\mathbf{q}=\mathbf{Q}(t)} d\mathbf{Q}(t) + \partial_t \tilde{\theta}|_{\mathbf{q}=\mathbf{Q}(t)} dt.
\]
Hence,
\[ \mathbf{p}(\mathbf{Q}(t), t) + \frac{e}{c} \mathbf{A}_{\text{ext}}(\mathbf{Q}(t), t) = -\left( \frac{mc^2}{\omega_e} \right) \nabla \tilde{\theta}|_{\mathbf{q} = \mathbf{Q}(t)} = \nabla S|_{\mathbf{q} = \mathbf{Q}(t)}. \] (130)

Thus the current velocity, plus the correction due to the external vector potential, corresponds the gradient of the zbw particle’s time-symmetrized steady-state phase at the location of the zbw particle, and \( S \) can again be identified with the time-symmetrized steady-state action/phase function of the zbw particle in the lab frame. Along with
\[ H(\mathbf{Q}(t), t) = \left( \frac{mc^2}{\omega_e} \right) \partial_t \tilde{\theta}|_{\mathbf{q} = \mathbf{Q}(t)} = -\partial_t S|_{\mathbf{q} = \mathbf{Q}(t)}, \] (131)

it follows that
\[ S(\mathbf{Q}(t), t) = \int_{t_i}^{t} \left( \mathbf{p} + \frac{e}{c} \mathbf{A}_{\text{ext}} \right) \cdot d\mathbf{Q}(t') - \int_{t_i}^{t} H dt' - \hbar \phi \]
\[ = \int_{t_i}^{t} \left[ \frac{1}{2} m \dot{\mathbf{v}}^2 + \frac{1}{2} m \mathbf{u}^2 + \frac{e}{c} \mathbf{v} \cdot \mathbf{A}_{\text{ext}} - mc^2 - e\Phi_e \right] dt' - \hbar \phi = I(\mathbf{Q}(t), t), \] (132)
and
\[ \oint_L \delta S(\mathbf{Q}(t), t) = \left( -\frac{mc^2}{\omega_e} \right) \oint_L \delta \tilde{\theta}(\mathbf{q}(t), t) = \oint_L \left[ \mathbf{p}' \cdot \delta \mathbf{Q}(t) - H \delta t \right] = nh. \] (133)

Recall that after restricting the forward and backward diffusions to simultaneous solutions of (116-117), we have \( \mathbf{b} = \mathbf{v} + \mathbf{u} \) and \( \mathbf{b}_s = \mathbf{v} - \mathbf{u} \). So the IMFTRF and the IMBTRF will not coincide since, for \( \mathbf{b} = \mathbf{v} + \mathbf{u} = 0 \), it will generally not be the case that \( \mathbf{b}_s = \mathbf{v} - \mathbf{u} = 0 \). This motivates defining an instantaneous mean (time-)symmetric rest frame (IMSTRF) as the frame in which \( \mathbf{b} + \mathbf{b}_s = 2 \mathbf{v} = 0 \). In the IMSTRF, (128) reduces to \( \tilde{\theta} = (\omega_c/mc^2) \left[ mc^2 - \frac{1}{2} m \mathbf{u}^2 \right] t + \int_{t_i}^{t} e\Phi_e(\mathbf{Q}_0, t') dt' \right] + \phi \), since \( \mathbf{v} = 0 \) and \( \partial_t \rho = 0 \). So the external potential contributes a tiny shift to the time-symmetrized steady-state zbw phase in the IMSTRF, along with the kinetic energy term involving the osmotic velocity.

Applying the conservative diffusion constraint to the steady-state phase/action functional (124), we recover the mean acceleration equation
\[ ma(\mathbf{q}(t), t) = \frac{m}{2} [\mathcal{D} + \mathcal{D}_s] \mathbf{q}(t) = e \left[ -\frac{1}{c} \frac{\partial \mathbf{A}_{\text{ext}}}{\partial t} - \nabla \Phi_e + \frac{\mathbf{v}}{c} \times \mathbf{B}_{\text{ext}} \right]|_{\mathbf{q} = \mathbf{q}(t)}. \] (134)

Applying the mean derivatives in (133), we find
\[ ma(\mathbf{q}(t), t) = m \left[ \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} - \mathbf{u} \cdot \nabla \mathbf{u} - \frac{\hbar}{2m} \nabla^2 \mathbf{u} \right]|_{\mathbf{q} = \mathbf{q}(t)} = e \left[ -\frac{1}{c} \frac{\partial \mathbf{A}_{\text{ext}}}{\partial t} - \nabla \Phi_e + \frac{\mathbf{v}}{c} \times \mathbf{B}_{\text{ext}} \right]|_{\mathbf{q} = \mathbf{q}(t)}. \] (135)

Integrating both sides gives
\[ \dot{E}(\mathbf{q}(t), t) = -\frac{\partial S(\mathbf{q}, t)}{\partial t}|_{\mathbf{q} = \mathbf{q}(t)} = mc^2 + \left[ \frac{(\nabla S - \frac{e}{c} \mathbf{A}_{\text{ext}})^2}{2m} + e\Phi_e - \frac{\hbar^2}{2m} \nabla^2 \sqrt{\rho} \right]|_{\mathbf{q} = \mathbf{q}(t)}. \] (136)
where we have fixed the integration constant equal to the particle rest energy. Alternatively, we can again consider integral curves of the current velocity/momentum field, but where now the integral curves are obtained from solutions of

$$m \frac{d^2 Q(t)}{dt^2} = m \left( \partial_t v + v \cdot \nabla v \right)_{q=Q(t)}$$

$$= -\nabla \left( -\frac{\hbar^2}{2m} \frac{\nabla^2 \sqrt{\rho(q,t)}}{\sqrt{\rho(q,t)}} \right)_{q=Q(t)} + e \left[ -\frac{1}{c} \partial_t A_{ext} - \nabla \Phi_e + \frac{v}{c} \times B_{ext} \right]_{q=Q(t)},$$

(137)

i.e., the mean acceleration equation (98), rewritten so that only the $v$-dependent terms are kept on the left hand side. Then we can replace $q(t)$ in (136) with $Q(t)$ to obtain $\tilde{E}(Q(t),t)$. The corresponding general solution, i.e., the time-symmetrized steady-state phase/action of the $zbw$ particle in the lab frame, after having imposed the conservative diffusion constraint on (124), is of the form

$$S(Q(t),t) = \int_{Q(t)}^{Q(t)} \left( p + \frac{e}{c} A_{ext} \right) \cdot dQ(t') - \int_{t_i}^{t} \tilde{E}dt' - \hbar \phi$$

$$= \int_{t_i}^{t} \left[ \frac{1}{2} m v^2 - \left( -\frac{\hbar^2}{2m} \frac{\nabla^2 \sqrt{\rho}}{\sqrt{\rho}} \right) + \frac{e}{c} v \cdot A_{ext} - mc^2 - e\Phi_e \right] dt' - \hbar \phi$$

$$= \int_{t_i}^{t} \left[ \frac{1}{2} m v^2 + \frac{1}{2} m u^2 + \frac{\hbar}{2} \nabla \cdot u + \frac{e}{c} v \cdot A_{ext} - mc^2 - e\Phi_e \right] dt' - \hbar \phi. \quad (138)$$

Notice that the last line of (138) differs from the last line of (132) only by addition of the term involving $\nabla \cdot u$.

As also in the free particle case, the equation of motion for (138) differs from the equation of motion for the classical $zbw$ particle phase by the presence of the quantum kinetic in (135-136). Our earlier discussion of the quantum-classical correspondence applies here as well.

Insofar as (138) is a well-defined phase function, if we integrate $\delta S(Q(t),t)$ around a closed loop $L$ in which time and position may change, we will have

$$\oint_L \delta S(Q(t),t) = \oint_L \left[ p' \cdot \delta Q(t) - \tilde{E}\delta t \right] = nh,$$  

(139)

and for a special loop in which time is held fixed,

$$\oint_L \delta S(Q(t)) = \oint_L \nabla S_{q=Q(t)} \cdot \delta Q(t) = \oint_L p' \cdot \delta Q(t) = nh. \quad (140)$$

Considering also the $zbw$ phase field $S(q,t)$, which we recall is a field over the possible locations of the actual $zbw$ particle, and applying the same physical reasoning above to each possible initial position that the $zbw$ particle can occupy, it follows that the net phase change along any mathematical loop in space (with time held fixed) will be given by

$$\oint_L \nabla S \cdot dq = \oint_L p' \cdot dq = nh. \quad (141)$$

The corresponding total energy field $E(q,t)$ is given by (136) when $Q(t)$ is replaced by $q$. From (141), (136), and (120), we can then construct the 1-particle Schrödinger equation in external fields.
as

$$i\hbar \frac{\partial \psi}{\partial t} = \left[ -i\hbar \nabla - \frac{eA_{\text{ext}}}{2m} \right]^2 \psi + e\Phi e \psi + mc^2 \psi,$$

where \( \psi(q, t) = \sqrt{\rho(q, t)} e^{iS(q(t))/\hbar} \) is a single-valued wave function as a consequence of (141).

At this point, it is worth observing an important difference between the (time-symmetrized steady-state \( zbw \)) phase field evolving by (136) and the classical \( zbw \) phase field evolving by Eq. (61) in section 4.4. In the former case, the nonlinear coupling to the density \( \rho \) via the quantum kinetic implies that, at nodal points (i.e., where \( \rho = \psi = 0 \)), such as found in excited states of the hydrogen atom or quantum harmonic oscillator, the phase field develops a singularity where both \( v = \nabla S \) and \( u = (\hbar/2m) \nabla \ln \rho \) diverge. Moreover, (141) implies that the phase field along a closed loop \( L \) undergoes a discontinuous jump of magnitude \( nh \) if the loop happens to cross a nodal point. Neither of these observations are inconsistent with our hypothesis that the steady-state phase of the actual \( zbw \) particle is a well-defined function of the actual particle’s mean space-time trajectory (or any mean space-time trajectory it can potentially realize), since it can be readily shown that the particle’s actual (mean or stochastic) trajectory never hits a nodal point [2, 135, 136, 64].

Indeed, if the phase field would not undergo the discontinuous jump at a nodal point, then this would imply that there are mean trajectories near nodes for which the actual particle does not have a well-defined mean phase, thereby contradicting our hypothesis. By contrast, for the classical \( zbw \) phase field, there is no reason for it to be undefined at nodal regions since there is no nonlinear coupling to the (inverse of the) probability density. Rather, the fact that the classical phase field also satisfies a condition of the form (141) implies that it changes discontinuously across a discontinuity in the external potential, \( V \), and takes discrete values for changes along a closed loop \( L \) encircling the discontinuity in \( V \) (as demonstrated for the hydrogen-like atom in Appendix B).

We thus have a formulation of ZSM in external fields that avoids the Wallstrom criticism and is ready to be applied to the central potential example considered in section 3.

### 5.3 The central potential revisited

With ZSM in hand, we can now return to the central potential example considered by Wallstrom, and show how ZSM gives the same result as quantum mechanics.

For the effective central potential, \( V_a(r) = V(r) + a/r^2 \), we found that the HJM equations implied \( v_a'' = v_a \sqrt{2ma/\hbar^2} + 1 \) and \( u_a'' = u_a \), where \( v_a = (\hbar/mr) \dot{\varphi} \). The problem in standard NYSM was that the constant \( a \) could take any positive real value, making \( v_a'' \) not quantized. By contrast, in the quantum mechanical version, \( m = \sqrt{2ma/\hbar^2} + 1 \) would be integral due to the single-valuedness condition on \( \psi_m \).

In the ZSM version of this problem, the \( zbw \) phase field in the lab frame, \( S_a = \hbar \varphi \), satisfies

$$\oint \frac{dS_a}{d\varphi} d\varphi = \oint \hbar d\varphi = m\hbar,$$

where \( A \) is the area enclosed by the phase field.
as a consequence of the reasoning used in section 5.2. Accordingly, for the effective $zbw$ phase field, $S'_a = \hbar \sqrt{2ma}/h^2 + 1\varphi = \hbar \varphi'$, we will also have
\[
\oint \hbar \sqrt{2ma} + 1d\varphi = \oint \hbar d\varphi' = mh,
\]
where $m = \sqrt{2ma}/h^2 + 1$ is integral. So ZSM predicts quantized energy-momentum in the central potential case, in accordance with quantum mechanics.

6 Conclusion

To answer Wallstrom’s criticism, we first developed a classical $zbw$ model (based on the earlier models of de Broglie and Bohm) which implies a quantization condition reminiscent of the Bohr-Sommerfeld-Wilson condition. We did this excluding and including interactions with external fields, and formulated the classical Hamilton-Jacobi statistical mechanics of each case. We then extended this model to Nelson-Yasue stochastic mechanics - which we termed zitterbewegung stochastic mechanics (ZSM) - and showed, using the same two cases, that it allows us to recover the Schrödinger equation for single-valued wave functions with (in general) multi-valued phases. Finally, we showed that ZSM works for the concrete case of a two-dimensional central potential.

In Part II, our approach will be generalized to the case of many $zbw$ particles, excluding and including (external and inter-particle) field interactions, the latter of which turns out to be a non-trivial task. We will also: (i) elaborate on the beables of ZSM, (ii) assess the plausibility and generalizability of the $zbw$ hypothesis, and (iii) compare ZSM to other (previously) proposed answers to Wallstrom’s criticism.

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A Proof of the Stochastic Variational Principle

Following Yasue’s presentation [3], let $q'(t) = q(t) + \delta q(t)$ be a variation of the sample path $q(t)$, with end-point constraints $\delta q(t_i) = \delta q(t_f) = 0$. Let us also assume, for the sake of generality, that the particle has charge $e$ and couples to the external magnetic vector potential, $A_{ext}(q(t), t)$, as well as the external electric scalar potential, $\Phi_e(q(t), t)$. Then the condition
\[
J = E \left[ \int_{t_i}^{t_f} \left\{ \frac{1}{2} \left[ m \left( Dq(t) \right)^2 + \frac{1}{2} m (Dq(t))^2 \right] + \frac{c}{e} A_{ext}(q(t), t) \cdot \mathbf{v}(q(t), t) - e\Phi_e(q(t), t) \right\} dt \right]
\]

\[
= E \left[ \int_{t_i}^{t_f} \left\{ \frac{1}{2} \left[ m \left( Dq(t) \right)^2 + \frac{1}{2} m (Dq(t))^2 \right] + \frac{c}{e} A_{ext} \cdot \mathbf{v} - c\Phi_e \right\} dt \right] = extremal,
\]

(145)
where \( E[\ldots] \) is the absolute expectation, is equivalent to the variation,

\[
\delta J(q) = J(q') - J(q),
\]

up to first order in \( ||\delta q(t)|| \). So (146) gives

\[
\delta J = E\left[ \int_{t_i}^{t_f} \left\{ \frac{1}{2} m (Dq(t) \cdot D\delta q(t) + D_\alpha q(t) \cdot D_\alpha \delta q(t)) \right\} \right.
\]

\[
+ c A_{ext} \cdot \frac{1}{2} (D\delta q(t) + D_\alpha \delta q(t)) + \frac{c}{e} (\delta q(t) \cdot \nabla A_{ext}) \left. \frac{1}{2} (Dq(t) + D_\alpha q(t)) - e\nabla \Phi_e \cdot \delta q(t) \right) |_{q=q(t)} dt, \]

(147)

where we note that \( v = \frac{1}{2} (D + D_\alpha) q(t) \) and is constrained by Eq. (10). Now, observing that for an arbitrary function, \( f(q(t), t) \), we have

\[
E\left[ \int_{t_i}^{t_f} |f(q(t), t)D\delta q(t)| dt \right] = -E\left[ \int_{t_i}^{t_f} |\delta q(t)D_\alpha f(q(t), t)| dt \right],
\]

(148)

and

\[
E\left[ \int_{t_i}^{t_f} |f(q(t), t)D_\alpha \delta q(t)| dt \right] = -E\left[ \int d^3q \int_{t_i}^{t_f} |\delta q(t)D_\alpha f(q(t), t)| dt \right],
\]

(149)

and

\[
\frac{1}{2} (D + D_\alpha) f(q(t), t) = \left\{ \frac{\partial}{\partial t} + \frac{1}{2} [Dq(t) + D_\alpha q(t)] \cdot \nabla \right\} f(q, t)|_{q=q(t)},
\]

(150)

we then obtain

\[
\delta J = E\left[ \int_{t_i}^{t_f} \left\{ \frac{m}{2} [D_\alpha D + DD_\alpha] q(t) \right. \right.
\]

\[
\left. \left. - \frac{e}{c} \nabla \times (\nabla \times A_{ext}) + \frac{e}{c} \frac{\partial A_{ext}}{\partial t} + e \nabla \Phi_e \right\} \right. |_{q=q(t)} \delta q(t)| dt \left. \right] + \vartheta(||\delta q||).
\]

(151)

From the variational constraint (145-46), it follows that the first-order variation of \( J \) must be zero for arbitrary sample-wise variation \( \delta q(t) \). Moreover, since the expectation is a positive linear functional, we will have the equation of motion

\[
\frac{m}{2} [D_\alpha D + DD_\alpha] q(t) = -e \left[ \nabla \Phi_e + \frac{1}{c} \frac{\partial A_{ext}}{\partial t} \right] |_{q=q(t)} + \frac{e}{c} \nabla \times (\nabla \times A_{ext}) |_{q=q(t)}
\]

(152)

for each time \( t \in [t_i, t_f] \) with probability one.

### B Classical Zitterbewegung in the Central Potential

Suppose that the non-relativistic zwv particle in the lab frame is moving in a circular orbit about some central potential, \( V(r) \), where \( r \) is the radius of the orbit. In this case, for the spherical coordinates \((r, \alpha, \beta)\), \( r \) is fixed, \( \alpha \) is varies with time, and \( \beta \) has the constant value \( \pi/2 \), giving
translational velocities \( v_r = \dot{r} = 0 \) (and we require \( \dot{r} = 0 \)), \( v_\alpha = r\dot{\alpha} \), and \( v_\beta = r\dot{\beta}\sin \alpha = 0 \). The \( v \ll c \) approximated \( zbw \) phase change in the lab frame is then

\[
\delta\theta(\alpha(t), t) = (\omega_c + \omega_\alpha + \kappa(r)) \delta t - \frac{v_\alpha r \delta \alpha(t)}{c^2}
\]

\[
= \frac{\omega_c}{mc^2} \left[ \left( mc^2 + \frac{p_\alpha^2}{2mr^2} + V(r) \right) \delta t - v_\alpha r \delta \alpha(t) \right],
\]

(153)

where \( p_\alpha = mr^2\dot{\alpha} \). Because the total energy of the system is constant, integrating this gives

\[
\theta = \frac{\omega_c}{mc^2} \left[ \left( mc^2 + \frac{p_\alpha^2}{2mr^2} + V(r) \right) t - p_\alpha \alpha(t) \right] + C,
\]

(154)
or

\[
S = p_\alpha \alpha(t) - \left( mc^2 + \frac{p_\alpha^2}{2mr^2} + V(r) \right) t + C
\]

\[
= p_\alpha \alpha(t) - Et + C.
\]

(155)

Incidentally, we could have also obtained (155) by starting with the non-relativistic Lagrangian

\[
L(\alpha(t), t) = \frac{1}{2} mr^2 \dot{\alpha}(t)^2 - V(r) - mc^2,
\]

(156)

and using the Legendre transformation,

\[
E = p_\alpha \dot{\alpha} - L = \frac{p_\alpha^2}{2mr^2} + V(r) + mc^2,
\]

(157)
to get

\[
S = \int L \, dt + C = \int (p_\alpha \dot{\alpha} - E) \, dt + C = p_\alpha \alpha - Et + C.
\]

(158)

Clearly (155) satisfies the classical Hamilton-Jacobi equation

\[
-\frac{\partial S}{\partial t} = \frac{1}{2mr^2} \left( \frac{\partial S}{\partial \alpha} \right)^2 + V(r),
\]

(159)

where \( -\partial S/\partial t = E \) and \( \partial S/\partial \alpha = p_\alpha = L_\alpha \), the latter being the constant angular momentum of the particle in the \( \hat{z} \)-direction.

Because the \( zbw \) oscillation is simply harmonic and the phase is a well-defined function of the particle position, the change in \( S \) will now be quantized upon fixed time integration around a closed (circular) orbit \( L \). In other words, we will have

\[
\oint_L p_\alpha \delta \alpha = 2\pi mv_\alpha r = nh,
\]

(160)
or

\[
L_\alpha = mv_\alpha r = nh,
\]

(161)

where \( n \) is an integer. From (161) and the force balance equation (assuming a Coulomb force), \( mv_\alpha^2/r = (1/4\pi\epsilon_0)c^2/r^2 \), it follows that the radius is quantized as

\[
r_n = \frac{4\pi\epsilon_0 n^2}{mc^2} a^2,
\]

(162)
where for \( n = 1 \), (162) gives the Bohr radius. Inserting (162) into the force balance equation and recognizing that \( E = V/2 \), we then obtain the quantized energy states

\[
E_n = \frac{E_1}{n^2},
\]  

(163)

where \( E_1 = -\frac{e^2}{8\pi\varepsilon_0 r_1} = -13.6eV \) is precisely the magnitude of the ground state energy of the Bohr hydrogen atom.

We wish to emphasize that, whereas Bohr simply assumed a condition equivalent to (160) in order to stabilize the electron’s circular orbit in the classical hydrogen atom, we obtained (160) just from the zitterbewegung hypothesis in the particle’s instantaneous translational rest frame combined with the usual Lorentz transformation. In other words, in Bohr’s model, (160) is imposed ad hoc while in our model it arises as a direct consequence of a relativistic \((zbw)\) constraint on the particle’s motion.

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