ON THE COMPUTATION OF MANIN PRODUCTS FOR OPERADS

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Abstract. In the theory of binary quadratic operads, the white and black products of operads (called Manin products) play an important role. Given two such operads, the computation of either of their Manin products is a routine task. We present and describe a computer program that helps to compute white and black Manin products of binary quadratic operads. The same utility allows to find the Koszul-dual operads. In particular, we compute the white product of the operads Lie and As (governing the varieties of Lie and associative algebras, respectively). It turns out that the resulting operad is magmatic, i.e., defines the variety of all algebras with one bilinear operation.

1. Introduction

In a series recent works related to Leibniz algebras, Rota—Baxter operators, and dendriform algebras, the important role of the white and black products of operads (called Manin products [1], they are denoted by ◦ and •, respectively) has been established. As a general reference in the operad theory one may apply to [2]. However, most of examples that can be found in the literature deal with white products $P \circ \text{Perm}$, $P \circ \text{ComTrias}$, and black products $P \bullet \text{PreLie}$, $P \bullet \text{PostLie}$. Here $P$ is a binary quadratic operad, the definitions of Perm, ComTrias, PreLie, and PostLie can be found in the reference preprint [3]. Another class of operads has been considered in [5], where white products also coincide with Hadamard products.

These examples of products are “degenerate” in the sense that for both Perm and ComTrias the white products $P \circ \text{Perm}$, $P \circ \text{ComTrias}$ coincide with the Hadamard products $P \otimes \text{Perm}$, $P \otimes \text{ComTrias}$ [4]. Since the white and black Manin products of binary quadratic operads are dual with respect to Koszul duality of operads, i.e., $(P \bullet Q)^! = P^! \circ Q^!$, and PreLie = Perm, PostLie = ComTrias, we have $P \bullet \text{PreLie} = (P^! \otimes \text{Perm})^!$, $P \bullet \text{PostLie} = (P^! \otimes \text{ComTrias})^!$.

The computation of a Manin product may require a lot of computations even in the simplest cases. In a recent paper [6], the following question has been stated: What is the black product of Com and As? (Equivalently, this is the dual to the white product of Lie and As.)

In this note, we describe in elementary terms the process of computation of Manin products for binary quadratic operads with no nontrivial unary operators. We also describe an utility called manin designed to perform the corresponding computations over the field of rational numbers. The source code of the utility is available at http://math.nsc.ru/LBRT/a1/pavelsk/manin2.zip. It is designed

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It is convenient to identify the basic elements of $E$ and the space of relations degrees 2 or 3. To define such an operad, we need a space of (binary) operations $P$. A binary quadratic operad corresponds to a variety whose defining identities have degrees 2 or 3. To define such an operad, we need a space of (binary) operations $E$ and the space of relations $R$, see \[1\] for details.

The space of multilinear terms of degree 3 is identified with $E(3) = kS_3 \otimes kS_2 (E \otimes E)$ where the action of $(12) \in S_2$ on $E \otimes E$ is given by $\text{id} \otimes (12)$.

If $\mu$ is an element of an $S_2$-module $E$, representing binary operation $(x_1, x_2) \mapsto x_1 \ast x_2$ then $\mu^{(12)}$ corresponds to $(x_1, x_2) \mapsto x_2 \ast x_1$.

If $\mu_1$ and $\mu_2$ represent two binary operations $(x_1, x_2) \mapsto x_1 \ast_k x_2$ ($k = 1, 2$) then $\mu_1 \otimes \mu_2 \in E \otimes E$ corresponds to the following function: $(x_1, x_2, x_3) \mapsto (x_1 \ast_k x_2) \ast_1 x_3$.

It is convenient to identify the basic elements of $E(3)$ with appropriate monomials in formal variables $x_1, x_2, x_3$ as stated in the table below.

| $1 \otimes kS_2 (\mu_1 \otimes \mu_2)$ | $1 \otimes kS_2 (\mu_1 \otimes \mu_2^{(12)})$ | $1 \otimes kS_2 (\mu_1^{(12)} \otimes \mu_2)$ | $(13) \otimes kS_2 (\mu_1 \otimes \mu_2^{(12)})$ | $(13) \otimes kS_2 (\mu_1^{(12)} \otimes \mu_2)$ | $(13) \otimes kS_2 (\mu_1^{(12)} \otimes \mu_2^{(12)})$ |
|--------------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|
| $(x_1 \ast_2 x_2) \ast_1 x_3$      | $(x_2 \ast_2 x_1) \ast_1 x_3$  | $x_3 \ast_1 (x_1 \ast_2 x_2)$  | $(x_3 \ast_2 x_2) \ast_1 x_1$  | $(x_2 \ast_2 x_3) \ast_1 x_1$  | $(x_1 \ast_2 x_3) \ast_1 x_2$  |
| $x_3 \ast_1 (x_1 \ast_2 x_2)$    | $x_3 \ast_1 (x_2 \ast_2 x_1)$  | $x_1 \ast_1 (x_3 \ast_2 x_2)$  | $x_1 \ast_1 (x_2 \ast_2 x_3)$  | $x_2 \ast_1 (x_1 \ast_2 x_3)$  | $x_2 \ast_1 (x_3 \ast_2 x_1)$  |

Suppose $P_1$ and $P_2$ are two binary quadratic operads, and $P_1(n) = 1$, $i = 1, 2$. Then $P_i = P(E_i, R_i)$, where $E_i$ are the spaces of binary operations considered as $S_2$-modules, $R_i$ are the spaces of quadratic relations. Assume $\text{dim} E_i < \infty$.

Recall that the Hadamard product $P = P_1 \otimes P_2$ is given by the rule $P(n) = P_1(n) \otimes P_2(n)$, $n \geq 1$, and the composition maps are expanded on $P(n)$ in the componentwise way. It the same way, the structure of an $S_n$-module is defined on $P(n)$: A permutation $\sigma \in S_n$ acts on $P_1(n) \otimes P_2(n)$ as $\sigma \otimes \sigma$.

By definition \[1\], the white product $P_1 \circ P_2$ is the sub-operad of $P_1 \otimes P_2$ generated by the space of operations $E = E_1 \otimes E_2$. It is known to be a binary quadratic operad. To compute the space of relations, consider the map

$$\tau : E(3) \rightarrow E_1(3) \otimes E_2(3)$$

defined by

$$\tau : \sigma \otimes kS_2 ((a_1 \otimes b_1) \otimes (a_2 \otimes b_2)) \mapsto (\sigma \otimes kS_2 (a_1 \otimes a_2)) \otimes (\sigma \otimes kS_2 (b_1 \otimes b_2)),$$
$a_i \in E_1$, $b_i \in E_2$, $\sigma \in S_3$. This is a well-defined $S_3$-linear map. Obviously, $\tau$ is injective. Denote by $D(E_1, E_2)$ the image of $\tau$.

Since $\mathcal{P}_i(3) = E_i(3)/R_i$ for $i = 1, 2$, the desired space of relations (a subspace in $E(3)$) is exactly the kernel of

$$E(3) \xrightarrow{\tau_1 \otimes \tau_2} E_1(3) \otimes E_2(3) \xrightarrow{\tau_1 \otimes \tau_2} \mathcal{P}_1(3) \otimes \mathcal{P}_2(3),$$

where $\tau_i : E_i(3) \to \mathcal{P}_i(3)$ are the natural epimorphisms. The kernel of $\tau_1 \otimes \tau_2$ is equal to $R_1 \otimes E_2(3) + E_1(3) \otimes R_2$. It remains to find the intersection of $\text{Ker} (\tau_1 \otimes \tau_2)$ with $D(E_1, E_2)$ and apply $\tau^{-1}$ to get $R$—the space of relations, defining $\mathcal{P}_1 \circ \mathcal{P}_2$.

For every finite-dimensional $S_n$-module $M$, let $M^e$ stand for the dual space of $M$ considered as an $S_n$-module with respect to $\text{sgn}$-twisted action: $(f^e, e) = - (f, e^e)$, $f \in M^e, e \in E, \sigma \in S_n$.

Recall that if $\mathcal{P} = \mathcal{P}(E, R), R \subseteq E(3)$, then the Koszul dual operad $\mathcal{P}^!$ is defined as $\mathcal{P}(E^e, R^e)$, where $E^e$ is the dual space to $E$ endowed with $\text{sgn}$-twisted $S_2$-action and $R^e$ is the subspace of $E^e(3) \cong E(3)^e$ orthogonal to $R$.

To get the black product of two binary quadratic operads $\mathcal{P}_1$ and $\mathcal{P}_2$, it is enough to compute

$$\mathcal{P}_1 \bullet \mathcal{P}_2 = (\mathcal{P}_1 \circ \mathcal{P}_2)^!.$$

3. Description of the program

3.1. Presenting initial data. Each input operad $\mathcal{P}(E, R)$ is described in a separate file (e.g., the operad of Leibniz algebras—in the file leib). The description consists of several parts. The first line of the file contains the number $n = \dim E$. The next $n$ lines present the action of $(12) \in S_2$ on $E$: Each $i$th line $x_{i1} x_{i2} \ldots x_{in}$ consists of coordinates of $e_i^{(12)} = x_{i1} e_1 + x_{i2} e_2 + \cdots + x_{in} e_n$. Below, the number of defining relation should be stated in a separate line (followed by a comment, e.g., what an operad is defined by this file). After that, the list of relations $R$ comes. They are presented as integer coordinate rows in the following standard basis of $E(3)$:

$$a_1 = \text{id} \otimes (e_1 \otimes e_1), \ldots, a_n = \text{id} \otimes (e_1 \otimes e_n),$$

$$a_{n+1} = \text{id} \otimes (e_2 \otimes e_1), \ldots, a_{2n} = \text{id} \otimes (e_2 \otimes e_n),$$

$$a_{n(n-1)+1} = \text{id} \otimes (e_n \otimes e_1), \ldots, a_{n^2} = \text{id} \otimes (e_n \otimes e_n),$$

$$a_{n^2+1} = (13) \otimes (e_1 \otimes e_1), \ldots, a_{2n^2} = (13) \otimes (e_n \otimes e_n),$$

$$a_{2n^2+1} = (23) \otimes (e_1 \otimes e_1), \ldots, a_{3n^2} = (23) \otimes (e_n \otimes e_n).$$

Here we identify the space $E(3) = kS_3 \otimes S_2 (E \otimes E)$ with a sum of three copies of $E \otimes E$, i.e., $E(3) \cong V_3 \otimes E \otimes E$, where $V_3$ is formally spanned by id, $(13), (23) \in S_3$.

The rest of the file can be used as a notebook, the program does not read these data.

For example, let us state the content of the file as describing the operad governing associative algebras.

```
2 % e_1 = x_1 x_2, e_2 = x_2 x_1
0 1
1 0
6 % ident. of associative algebra
1 0 0 0 0 0 0 -1 0 0 0 0
```
\[ \begin{align*}
0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 
\end{align*} \]

\[ a_1 = e \circ e, \ a_2 = e \circ e^{\{12\}} \circ e, \ a_3 = e^{\{12\}} \circ e, \ a_4 = e^{12} \circ e^{12} \]

\[ a_5 = (13) a_1, \ a_6 = (13) a_2, \ a_7 = (13) a_3, \ a_8 = (13) a_4 \]

\[ a_9 = (23) a_1, \ a_{10} = (23) a_2, \ a_{11} = (23) a_3, \ a_{12} = (23) a_4 \]

3.2. Usage of the utility. To compute the white or black product of operads described in files file1 and file2, type

\[ \text{manin w file1 file2} \]

or

\[ \text{manin b file1 file2} \]

respectively. To compute the Koszul-dual operad $\mathcal{P}!$ to an operad $\mathcal{P}$ described in file1, type

\[ \text{manin d file1} \]

The output is written into two files: result and result.amx. The first one contains a description of the resulting operad $\mathcal{P}(E, R)$ in the same form as the input files do, i.e., after a minor editing (and, possibly, commenting) it can be used as an input file. The second one contains the description in a “human-readable” AMS-TeX format. To write down the identities one should assign binary operations to basic vectors of $E(3)$ and rewrite the relations in terms of these operations (as in the table stated above). Note that the final form of identities highly depends on this assignment.

3.3. Overview of the units. The main program manin.pas uses three units: lspace, dynarr, and shmidt.

The unit lspace contains the definitions of main types of data: Vector and Space. Vectors are presented as lists of integers, spaces—as lists of vectors. Also, in the unit lspace the main arithmetic operations with vectors and spaces are defined (sum, tensor product, intersection), as well as input-output routines.

For intersection, the Hauss reduction method is implemented: If $V_1$ is a $\mathbb{Q}$-linear span of vectors $a_1, \ldots, a_m \in \mathbb{Z}^n$, $V_2$—of $b_1, \ldots, b_l \in \mathbb{Z}^m$ then the basis of $V_1 \cap V_2$ can be found as follows. Consider a matrix of size $(k + l) \times (2n)$ given by

\[
\begin{pmatrix}
- a_1 & - a_1 & - & - \\
\cdots & \cdots & - & - \\
- a_m & - a_m & - & - \\
- b_1 & - 0 & - & - \\
\cdots & \cdots & - b_l & - 0 & -
\end{pmatrix}
\]

and apply elementary transformations of rows to make it a trapezoid. All vectors remaining in the right half of the table, opposite to zero vectors in the left half, span the intersection $V_1 \cap V_2$.

The unit dynarr is just a description of two-dimensional dynamic arrays and procedures allowing to convert a list of vectors to an array and converse.

In the unit shmidt, the orthogonalization procedure (Gram—Schmidt process) is implemented. We use this process to compute the orthogonal complements. Namely, in order to find the orthogonal component of a vector $v \in \mathbb{Z}^n$ relative to
a subspace \( V \) spanned by \( a_1, \ldots, a_m \in \mathbb{Z}^n \), we first make the vectors \( a_1, \ldots, a_m \) pairwise mutually orthogonal and then compute \( v_1 = \langle a_1, a_1 \rangle v - \langle a_1, v \rangle a_1 \), \( v_2 = \langle a_2, a_2 \rangle v - \langle a_2, v_1 \rangle a_2 \), and so on (cancellation of coefficients is applied on each step). Thus, to find an orthogonal complement \( V^\perp \) for \( V \subseteq \mathbb{Q}^n \), we find orthogonal components of all standard basic vectors \( e_1, \ldots, e_n \) relative to \( V \) and then apply Hauss reduction process.

The Hauss reduction method as well as the Gram—Schmidt process are encoded in such a way that both input and output lists of vectors have integer coordinates.

4. Examples

4.1. White product of Lie and Perm. The white product of the operads governing Lie and Perm algebras is known to be the operad of Leibniz algebras \[7\]. Entering

\[
\text{manin w lie perm}
\]

we obtain the following relations (written in result.amx; here we have just replaced the \texttt{AMSP-TeX} commands \texttt{\pmatrix} and \texttt{\endpmatrix} with the corresponding \texttt{\LaTeX} environment):

\[
\begin{pmatrix}
0 & -1 \\
-1 & 0
\end{pmatrix}
\]

Space of operations \( E \): \( a_1, \ldots, a_2 \)

\( S_2 \) acts by:

\[
\begin{pmatrix}
0 & -1 \\
-1 & 0
\end{pmatrix}
\]

Relations:
\[
\begin{align*}
+1(23) \otimes_{S_2} (a_1 \otimes a_1) & - 1(23) \otimes_{S_2} (a_1 \otimes a_2) \\
+1(13) \otimes_{S_2} (a_1 \otimes a_1) & - 1(13) \otimes_{S_2} (a_1 \otimes a_2) \\
-1(id) \otimes_{S_2} (a_1 \otimes a_1) & + 1(13) \otimes_{S_2} (a_2 \otimes a_2) + 1(23) \otimes_{S_2} (a_2 \otimes a_1) \\
+1(id) \otimes_{S_2} (a_2 \otimes a_1) & - 1(13) \otimes_{S_2} (a_2 \otimes a_2) - 1(23) \otimes_{S_2} (a_2 \otimes a_1) \\
+1(id) \otimes_{S_2} (a_2 \otimes a_2) & - 1(13) \otimes_{S_2} (a_1 \otimes a_2) - 1(23) \otimes_{S_2} (a_2 \otimes a_2)
\end{align*}
\]

Now, let us interpret \( a_1 \) as the operation \([x_1x_2]\). Then, according to the obtained \( S_2 \)-action on \( E \), \( a_2 = -a_1^{(12)} \), i.e., \( a_2 \) should be interpreted as \([-x_2x_1]\). Thus the six relations above turn into

\[
\begin{align*}
[[x_1x_3]x_2] + [[x_3x_1]x_2], \\
[[x_3x_2]x_1] + [[x_2x_3]x_1], \\
-[[x_1x_2]x_3] + [x_1[x_2x_3]] - [x_2[x_1x_3]], \\
-[[x_3x_1]x_2] + [x_1[x_3x_2]] + [[x_3x_1]x_2], \\
-[[x_2x_1]x_3] - [x_1[x_2x_3]] + [x_2[x_1x_3]], \\
[[x_3x_2]x_1] + [[x_2x_3]x_1] - [x_2[x_3x_1]].
\end{align*}
\]

These are corollaries of the only identity \([xy]z - [y[xz]] - [[xy]z], \) the left Leibniz identity.

4.2. The black product of PreLie and As. The operad of dendriform algebras \[8\] is known to be the black product of operads governing the varieties of pre-Lie and associative algebras (see also \[9\]). The command

\[
\text{manin b prelie as}
\]
generates the following output:
Space of operations $E$: $a_1, \ldots, a_4$

$S_2$ acts by:

$$
\begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 \\
0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix}
$$

Relations:

$$
\begin{array}{l}
-1(id) \otimes_{S_2} (a_4 \otimes a_4) + 1(13) \otimes_{S_2} (a_1 \otimes a_1) - 1(13) \otimes_{S_2} (a_1 \otimes a_3) \\
+1(id) \otimes_{S_2} (a_3 \otimes a_3) - 1(13) \otimes_{S_2} (a_2 \otimes a_2) + 1(13) \otimes_{S_2} (a_2 \otimes a_4) \\
-1(id) \otimes_{S_2} (a_4 \otimes a_2) + 1(13) \otimes_{S_2} (a_3 \otimes a_1) \\
+1(id) \otimes_{S_2} (a_2 \otimes a_2) - 1(id) \otimes_{S_2} (a_2 \otimes a_4) - 1(13) \otimes_{S_2} (a_3 \otimes a_3) \\
+1(id) \otimes_{S_2} (a_3 \otimes a_1) - 1(13) \otimes_{S_2} (a_4 \otimes a_2) \\
+1(id) \otimes_{S_2} (a_1 \otimes a_1) - 1(id) \otimes_{S_2} (a_1 \otimes a_3) - 1(13) \otimes_{S_2} (a_4 \otimes a_4) \\
+1(13) \otimes_{S_2} (a_4 \otimes a_1) + 1(23) \otimes_{S_2} (a_1 \otimes a_1) - 1(23) \otimes_{S_2} (a_1 \otimes a_3) \\
-1(id) \otimes_{S_2} (a_4 \otimes a_1) - 1(23) \otimes_{S_2} (a_1 \otimes a_2) + 1(23) \otimes_{S_2} (a_1 \otimes a_4) \\
+1(id) \otimes_{S_2} (a_3 \otimes a_2) + 1(23) \otimes_{S_2} (a_2 \otimes a_1) - 1(23) \otimes_{S_2} (a_2 \otimes a_3) \\
+1(13) \otimes_{S_2} (a_3 \otimes a_2) + 1(23) \otimes_{S_2} (a_2 \otimes a_2) - 1(23) \otimes_{S_2} (a_2 \otimes a_4) \\
+1(13) \otimes_{S_2} (a_4 \otimes a_3) + 1(23) \otimes_{S_2} (a_3 \otimes a_1) \\
-1(id) \otimes_{S_2} (a_2 \otimes a_1) + 1(id) \otimes_{S_2} (a_2 \otimes a_3) - 1(23) \otimes_{S_2} (a_3 \otimes a_2) \\
-1(13) \otimes_{S_2} (a_2 \otimes a_1) + 1(13) \otimes_{S_2} (a_2 \otimes a_3) + 1(23) \otimes_{S_2} (a_3 \otimes a_3) \\
-1(id) \otimes_{S_2} (a_4 \otimes a_3) + 1(23) \otimes_{S_2} (a_3 \otimes a_4) \\
-1(id) \otimes_{S_2} (a_1 \otimes a_2) + 1(id) \otimes_{S_2} (a_1 \otimes a_4) - 1(23) \otimes_{S_2} (a_4 \otimes a_1) \\
+1(13) \otimes_{S_2} (a_3 \otimes a_4) + 1(23) \otimes_{S_2} (a_4 \otimes a_2) \\
+1(id) \otimes_{S_2} (a_3 \otimes a_4) - 1(23) \otimes_{S_2} (a_4 \otimes a_3) \\
+1(13) \otimes_{S_2} (a_1 \otimes a_2) - 1(13) \otimes_{S_2} (a_1 \otimes a_4) - 1(23) \otimes_{S_2} (a_4 \otimes a_4)
\end{array}
$$

The 18 relations above split into three orbits with respect to the action of $S_3$.

The representatives of these orbits are:

$$
\begin{array}{l}
-\text{id} \otimes_{S_2} (a_4 \otimes a_2) + (13) \otimes_{S_2} (a_3 \otimes a_1), \\
-\text{id} \otimes_{S_2} (a_4 \otimes a_4) + (13) \otimes_{S_2} (a_1 \otimes a_1) - (13) \otimes_{S_2} (a_1 \otimes a_3), \\
\text{id} \otimes_{S_2} (a_3 \otimes a_2) - (13) \otimes_{S_2} (a_2 \otimes a_2) + (13) \otimes_{S_2} (a_2 \otimes a_4).
\end{array}
$$

Let us interpret $x_1 \succ x_2$ as $a_1$ and $x_2 \prec x_1$—as $a_2$. Then $a_3$ corresponds to $-x_1 \prec x_2$ and $a_4$—to $-x_2 \succ x_1$. Hence, the defining identities of PreLie + As are:

$$
\begin{array}{l}
x_3 \succ (x_2 \prec x_1) - (x_3 \succ x_2) \prec x_1, \\
(x_3 \succ x_2) \succ x_1 + (x_3 < x_2) \succ x_1 - x_3 \succ (x_2 \succ x_1), \\
x_1 \prec (x_2 < x_3) + x_1 < (x_2 \succ x_3) - (x_1 \prec x_2) \prec x_3.
\end{array}
$$

4.3. **Black product of Com and As.** The command

```python
manin b as comm
```

generates the following output:

Space of operations $E$: $a_1, \ldots, a_2$

$S_2$ acts by:

$$
\begin{pmatrix}
0 & -1 \\
-1 & 0
\end{pmatrix}
$$

Relations:

$$
\begin{array}{l}
+1(id) \otimes_{S_2} (a_1 \otimes a_1) \\
+1(id) \otimes_{S_2} (a_1 \otimes a_2)
\end{array}
$$
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This is clear that such an operad corresponds to the variety of 3-nilpotent algebras. As a corollary (which is also easy to check by means of manin w lie as), the white product Lie ◦ As is the magmatic algebra.

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