Sonine Transform Associated to the Dunkl Kernel on the Real Line

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Abstract. We consider the Dunkl intertwining operator $V_\alpha$ and its dual $\dagger V_\alpha$, we define and study the Dunkl Sonine operator and its dual on $\mathbb{R}$. Next, we introduce complex powers of the Dunkl Laplacian $\Delta_\alpha$ and establish inversion formulas for the Dunkl Sonine operator $S_{\alpha,\beta}$ and its dual $\dagger S_{\alpha,\beta}$. Also, we give a Plancherel formula for the operator $\dagger S_{\alpha,\beta}$.

Key words: Dunkl intertwining operator; Dunkl transform; Dunkl Sonine transform; complex powers of the Dunkl Laplacian

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1 Introduction

In this paper, we consider the Dunkl operator $\Lambda_\alpha$, $\alpha > -1/2$, associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$. The operators were in general dimension introduced by Dunkl in [2] in connection with a generalization of the classical theory of spherical harmonics; they play a major role in various fields of mathematics [3, 4, 5] and also in physical applications [6].

The Dunkl analysis with respect to $\alpha \geq -1/2$ concerns the Dunkl operator $\Lambda_\alpha$, the Dunkl transform $\mathcal{F}_\alpha$ and the Dunkl convolution $*_\alpha$ on $\mathbb{R}$. In the limit case ($\alpha = -1/2$); $\Lambda_\alpha$, $\mathcal{F}_\alpha$ and $*_\alpha$ agree with the operator $d/dx$, the Fourier transform and the standard convolution respectively.

First, we study the Dunkl Sonine operator $S_{\alpha,\beta}$, $\beta > \alpha$:

$$S_{\alpha,\beta}(f)(x) := \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \alpha)\Gamma(\alpha + 1)} \int_{-1}^{1} f(xt)(1 - t^2)^{\beta - \alpha - 1}(1 + t)|t|^{2\alpha + 1}dt,$$

and its dual $\dagger S_{\alpha,\beta}$ connected with these operators. Next, we establish for them the same results as those given in [8, 14] for the Radon transform and its dual; and in [9] for the spherical mean operator and its dual on $\mathbb{R}$. Especially:

- We define and study the complex powers for the Dunkl Laplacian $\Delta_\alpha = \Lambda_\alpha^2$.
- We give inversion formulas for $S_{\alpha,\beta}$ and $\dagger S_{\alpha,\beta}$ associated with integro-differential and integro-differential-difference operators when applied to some Lizorkin spaces of functions (see [3, 4, 13]).
- We establish a Plancherel formula for the operator $\dagger S_{\alpha,\beta}$.

The content of this work is the following. In Section 2 we recall some results about the Dunkl operators. In particular, we give some properties of the operators $S_{\alpha,\beta}$ and $\dagger S_{\alpha,\beta}$.

This paper is a contribution to the Special Issue on Dunkl Operators and Related Topics. The full collection is available at [http://www.emis.de/journals/SIGMA/Dunkloperators.html](http://www.emis.de/journals/SIGMA/Dunkloperators.html)
In Section 3, we consider the tempered distribution $|x|^{\lambda}$ for $\lambda \in \mathbb{C}\setminus\{-(\ell + 1), \ell \in \mathbb{N}\}$ defined by
\[
\langle |x|^{\lambda}, \varphi \rangle := \int_{\mathbb{R}} |x|^{\lambda} \varphi(x) dx.
\]
Also we study the complex powers of the Dunkl Laplacian $(-\Delta_{\alpha})^{\lambda}$, for some complex number $\lambda$. In the classical case when $\alpha = -1/2$, the complex powers of the usual Laplacian are given in [16].

In Section 4, we give the following inversion formulas:
\[
g = S_{\alpha, \beta}K_1(tS_{\alpha, \beta})(g), \quad f = (tS_{\alpha, \beta})K_2S_{\alpha, \beta}(f),
\]
where
\[
K_1(f) = \frac{c_\beta}{c_\alpha} (-\Delta_\alpha)^{\beta-\alpha} f, \quad K_2(f) = \frac{c_\beta}{c_\alpha} (-\Delta_\beta)^{\beta-\alpha} f \quad \text{and} \quad c_\alpha = \frac{1}{[2^{\alpha+1}\Gamma(\alpha + 1)]^2}.
\]
Next, we give the following Plancherel formula for the operator $tS_{\alpha, \beta}$:
\[
\int_{\mathbb{R}} |f(x)|^2 |x|^{2\beta+1} dx = \int_{\mathbb{R}} |K_3(tS_{\alpha, \beta}(f))(y)|^2 |x|^{2\alpha+1} dy,
\]
where
\[
K_3(f) = \sqrt{\frac{c_\beta}{c_\alpha}} (-\Delta_{\alpha})^{(\beta-\alpha)/2} f.
\]

## 2 The Dunkl intertwining operator and its dual

We consider the Dunkl operator $\Lambda_\alpha$, $\alpha \geq -1/2$, associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$:
\[
\Lambda_\alpha f(x) := \frac{d}{dx} f(x) + \frac{2\alpha + 1}{x} \left[ \frac{f(x) - f(-x)}{2} \right].
\]

For $\alpha \geq -1/2$ and $\lambda \in \mathbb{C}$, the initial problem:
\[
\Lambda_\alpha f(x) = \lambda f(x), \quad f(0) = 1,
\]
has a unique analytic solution $E_\alpha(\lambda x)$ called Dunkl kernel [8, 15] given by
\[
E_\alpha(\lambda x) = \Im_\alpha(\lambda x) + \frac{\lambda x}{2(\alpha + 1)} \Im_{\alpha+1}(\lambda x),
\]
where
\[
\Im_\alpha(\lambda x) := \Gamma(\alpha + 1) \sum_{n=0}^{\infty} \frac{(\lambda x/2)^{2n}}{n! \Gamma(n + \alpha + 1)}.
\]
is the modified spherical Bessel function of order $\alpha$.

Notice that in the case $\alpha = -1/2$, we have
\[
\Lambda_{-1/2} = d/dx \quad \text{and} \quad E_{-1/2}(\lambda x) = e^{\lambda x}.
\]

For $\lambda \in \mathbb{C}$ and $x \in \mathbb{R}$, the Dunkl kernel $E_\alpha$ has the following Bochner-type representation (see [8, 11]):
\[
E_\alpha(\lambda x) = a_\alpha \int_{-1}^{1} e^{\lambda xt} (1 - t^2)^{\alpha-1/2} (1 + t) dt,
\]

where $a_\alpha$ is a constant depending on $\alpha$.
where
\[ a_\alpha = \frac{\Gamma(\alpha + 1)}{\sqrt{\pi} \Gamma(\alpha + 1/2)}. \]
which can be written as:
\[
E_\alpha(\lambda x) = a_\alpha \text{sgn}(x) |x|^{-(\alpha+1)} \int_{-|x|}^{|x|} e^{\lambda y} (x^2 - y^2)^{-\alpha/2}(x+y)dy, \quad x \neq 0,
\]
\[ E_\alpha(0) = 1. \]

We notice that, the Dunkl kernel \( E_\alpha (\lambda x) \) can be also expanded in a power series [10] in the form:
\[
E_\alpha (\lambda x) = \sum_{n=0}^{\infty} \frac{(\lambda x)^n}{b_n(\alpha)},
\]
where
\[
b_{2n}(\alpha) = \frac{2^{2n}n!}{\Gamma(\alpha + 1)} \Gamma(n + \alpha + 1), \quad b_{2n+1}(\alpha) = 2(\alpha + 1)b_{2n}(\alpha + 1).
\]

Let \( \alpha > -1/2 \) and we define the Dunkl intertwining operator \( V_\alpha \) on \( \mathcal{E}(\mathbb{R}) \) (the space of \( C^\infty \)-functions on \( \mathbb{R} \)), by
\[
V_\alpha(f)(x) := a_\alpha \int_{-1}^{1} f(xt) (1 - t^2)^{\alpha - 1/2}(1+t)dt,
\]
which can be written as:
\[
V_\alpha(f)(x) = a_\alpha \text{sgn}(x) |x|^{-(\alpha+1)} \int_{-|x|}^{|x|} f(y) (x^2 - y^2)^{-\alpha/2}(x+y)dy, \quad x \neq 0,
\]
\[ V_\alpha(f)(0) = f(0). \]

**Remark 1.** For \( \alpha > -1/2 \), we have
\[ E_\alpha(\lambda.) = V_\alpha(e^{\lambda.}), \quad \lambda \in \mathbb{C}. \]

**Proposition 1** (see [10], Theorem 6.3). The operator \( V_\alpha \) is a topological automorphism of \( \mathcal{E}(\mathbb{R}) \), and satisfies the transmutation relation:
\[
\Lambda_\alpha(V_\alpha(f)) = V_\alpha \left( \frac{d}{dx} f \right), \quad f \in \mathcal{E}(\mathbb{R}).
\]

Let \( \alpha > -1/2 \) and we define the dual Dunkl intertwining operator \( tV_\alpha \) on \( \mathcal{S}(\mathbb{R}) \) (the Schwartz space on \( \mathbb{R} \)), by
\[
tV_\alpha(f)(x) := a_\alpha \int_{|y| \geq |x|} \text{sgn}(y) (y^2 - x^2)^{-\alpha/2}(x+y)f(y)dy,
\]
which can be written as:
\[
tV_\alpha(f)(x) = a_\alpha \text{sgn}(x) |x|^{2\alpha+1} \int_{|t| \geq 1} \text{sgn}(t) (t^2 - 1)^{-\alpha/2}(1+t)f(xt)dt.
\]
Proposition 2 (see [19], Theorems 3.2, 3.3).

(i) The operator \( tV_\alpha \) is a topological automorphism of \( \mathcal{S}(\mathbb{R}) \), and satisfies the transmutation relation:

\[
tV_\alpha(\Lambda_\alpha f) = \frac{d}{dx} (tV_\alpha(f)), \quad f \in \mathcal{S}(\mathbb{R}).
\]

(ii) For all \( f \in \mathcal{E}(\mathbb{R}) \) and \( g \in \mathcal{S}(\mathbb{R}) \), we have

\[
\int_\mathbb{R} V_\alpha(f(x))g(x)|x|^{2\alpha+1}dx = \int_\mathbb{R} f(x) tV_\alpha(g(x))dx.
\]

Remark 2 (see [15]).

(i) For \( \alpha > -\frac{1}{2} \) and \( f \in \mathcal{E}(\mathbb{R}) \), we can write

\[
V_\alpha(f)(x) = \mathcal{R}_\alpha(f_e)(|x|) + \frac{1}{x} \mathcal{R}_\alpha(Mf_o)(|x|),
\]

where

\[
f_e(x) = \frac{1}{2} (f(x) + f(-x)), \quad f_o(x) = \frac{1}{2} (f(x) - f(-x)), \quad Mf_o(x) = xf_o(x),
\]

and \( \mathcal{R}_\alpha \) is the Riemann–Liouville transform (see [17], page 75) given by

\[
\mathcal{R}_\alpha(f_e)(x) := 2a_\alpha \int_0^1 f_e(xt)(1 - t^2)^{\alpha-1/2}dt, \quad x \geq 0.
\]

Thus, we obtain

\[
V_\alpha^{-1}(f)(x) = \mathcal{R}_\alpha^{-1}(f_e)(|x|) + \frac{1}{x} \mathcal{R}_\alpha^{-1}(Mf_o)(|x|).
\]

Therefore (see also [20], Proposition 2.2), we get

\[
V_\alpha^{-1}(f_e)(x) = d_\alpha \frac{d}{dx} \left( \frac{d}{xdx} \right)^r \left\{ x^{2r+1} \int_0^1 f_e(xt)(1 - t^2)^{r-\alpha-1/2} t^{2\alpha+1}dt \right\},
\]

\[
V_\alpha^{-1}(f_o)(x) = d_\alpha \left( \frac{d}{xdx} \right)^{r+1} \left\{ x^{2r+2} \int_0^1 f_o(xt)(1 - t^2)^{r-\alpha-1/2} t^{2\alpha+2}dt \right\},
\]

where \( r = \lfloor \alpha + 1/2 \rfloor \) denote the integer part of \( \alpha + 1/2 \), and \( d_\alpha = \frac{2^{\alpha-\pi}}{1(\alpha+1)!}. \)

(ii) For \( \alpha > -\frac{1}{2} \) and \( f \in \mathcal{S}(\mathbb{R}) \), we can write

\[
{^tV}_\alpha(f)(x) = W_\alpha(f_e)(|x|) + xW_\alpha(M^{-1}f_o)(|x|),
\]

where

\[
M^{-1}f_o(x) = \frac{1}{2x}(f(x) - f(-x)),
\]

and \( W_\alpha \) is the Weyl integral transform (see [17], page 85) given by

\[
W_\alpha(f_e)(x) := 2a_\alpha x^{2\alpha+1} \int_1^\infty f_e(xt)(t^2 - 1)^{\alpha-1/2}tdt, \quad x \geq 0.
\]

Thus, we obtain

\[
({^tV}_\alpha)^{-1}f(x) = W_\alpha^{-1}(f_e)(|x|) + xW_\alpha^{-1}(M^{-1}f_o)(|x|).
\]
The Dunkl kernel gives rise to an integral transform, called Dunkl transform on $\mathbb{R}$, which was introduced by Dunkl in [4], where already many basic properties were established. Dunkl’s results were completed and extended later on by de Jeu in [5].

The Dunkl transform of a function $f \in S(\mathbb{R})$, is given by
\[
F_\alpha(f)(\lambda) := \int_\mathbb{R} E_\alpha(-i\lambda x) f(x) |x|^{2\alpha+1} dx, \quad \lambda \in \mathbb{R}.
\]

We notice that $F_{-1/2}$ agrees with the Fourier transform $F$ that is given by:
\[
F(f)(\lambda) := \int_\mathbb{R} e^{-i\lambda x} f(x) dx, \quad \lambda \in \mathbb{R}.
\]

**Proposition 3** (see [5]).

(i) For all $f \in S(\mathbb{R})$, we have
\[
F_\alpha(\Lambda_\alpha f)(\lambda) = i\lambda F_\alpha(f)(\lambda), \quad \lambda \in \mathbb{R},
\]
where $\Lambda_\alpha$ is the Dunkl operator given by (1).

(ii) $F_\alpha$ possesses on $S(\mathbb{R})$ the following decomposition:
\[
F_\alpha(f) = F \circ t V_\alpha(f), \quad f \in S(\mathbb{R}).
\]

(iii) $F_\alpha$ is a topological automorphism of $S(\mathbb{R})$, and for $f \in S(\mathbb{R})$ we have
\[
f(x) = c_\alpha \int_\mathbb{R} E_\alpha(i\lambda x) F_\alpha(f)(\lambda) |\lambda|^{2\alpha+1} d\lambda,
\]
where
\[
c_\alpha = \frac{1}{[2\alpha+1 \Gamma(\alpha+1)]^2}.
\]

(iv) The normalized Dunkl transform $\sqrt{c_\alpha} F_\alpha$ extends uniquely to an isometric isomorphism of $L^2(\mathbb{R}, |x|^{2\alpha+1} dx)$ onto itself. In particular,
\[
\int_\mathbb{R} |f(x)|^2 |x|^{2\alpha+1} dx = c_\alpha \int_\mathbb{R} |F_\alpha(f)(\lambda)|^2 |\lambda|^{2\alpha+1} d\lambda.
\]

For $T \in S'(\mathbb{R})$, we define the Dunkl transform $F_\alpha(T)$ of $T$, by
\[
\langle F_\alpha(T), \varphi \rangle := \langle T, F_\alpha(\varphi) \rangle, \quad \varphi \in S(\mathbb{R}).
\]

Thus the transform $F_\alpha$ extends to a topological automorphism on $S'(\mathbb{R})$.

In [19], the author defines:

- The Dunkl translation operators $\tau_x$, $x \in \mathbb{R}$, on $E(\mathbb{R})$, by
\[
\tau_x f(y) := (V_\alpha)_x \otimes (V_\alpha)_y [ (V_\alpha)^{-1}(f)(x + y) ], \quad y \in \mathbb{R}.
\]

These operators satisfy for $x, y \in \mathbb{R}$ and $\lambda \in \mathbb{C}$ the following properties:
\[
E_\alpha(\lambda x) E_\alpha(\lambda y) = \tau_x (E_\alpha(\lambda))(y), \quad \text{and}
\]
\[
F_\alpha(\tau_x f)(\lambda) = E_k(i\lambda x) F_\alpha(f)(\lambda), \quad f \in S(\mathbb{R}).
\]
Proposition 4 (see [11]). If $f \in C(\mathbb{R})$ (the space of continuous functions on $\mathbb{R}$) and $x, y \in \mathbb{R}$ such that $(x, y) \neq (0, 0)$, then

$$
\tau_x f(y) = a_\alpha \int_0^\pi \left[ f_\epsilon((x, y)\theta) + f_0((x, y)\theta) \frac{x + y}{(x, y)\theta} \right] \left[ 1 - \text{sgn}(xy) \cos \theta \right] \sin^{2\alpha} \theta d\theta,
$$

$$
\tau_x f(z) = \frac{1}{2}(f(z) + f(-z)), \quad f_0(z) = \frac{1}{2}(f(z) - f(-z)),
$$

$$(x, y)\theta = \sqrt{x^2 + y^2 - 2|xy| \cos \theta}.
$$

- The Dunkl convolution product $*_\alpha$ of two functions $f$ and $g$ in $\mathcal{S}(\mathbb{R})$, by

$$
f*\alpha g(x) := \int_\mathbb{R} \tau_x f(-y) g(y) |y|^{2\alpha+1} dy, \quad x \in \mathbb{R}.
$$

This convolution is associative, commutative in $\mathcal{S}(\mathbb{R})$ and satisfies (see [19, Theorem 7.2]):

$$
\mathcal{F}_\alpha(f*_\alpha g) = \mathcal{F}_\alpha(f)\mathcal{F}_\alpha(g).
$$

For $T \in \mathcal{S}'(\mathbb{R})$ and $f \in \mathcal{S}(\mathbb{R})$, we define the Dunkl convolution product $T*_\alpha f$, by

$$
T*_\alpha f(x) := (T(y), \tau_x f(-y)), \quad x \in \mathbb{R}.
$$

Note that $*_{-1/2}$ agrees with the standard convolution $*$:

$$
T*f(x) := (T(y), f(x-y)).
$$

3 The Dunkl Sonine transform

In this section we study the Dunkl Sonine transform, which also studied by Y. Xu on polynomials in [20]. For thus we consider the following identity, which is a consequence of Xu’s result when we extend the result of Lemma 2.1 on $\mathcal{E}(\mathbb{R})$.

Proposition 5. Let $\alpha, \beta \in ]-1/2, \infty[,$ such that $\beta > \alpha$. Then

$$
E_\beta(\lambda x) = a_{\alpha, \beta} \int_{-1}^1 E_\alpha(\lambda x t)(1 - t^2)^{\alpha-1}(1 + t)|t|^{2\alpha+1} dt,
$$

where

$$
a_{\alpha, \beta} = \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \alpha)\Gamma(\alpha + 1)}.
$$

Proof. From [2], we have

$$
\int_{-1}^1 E_\alpha(\lambda x t)(1 - t^2)^{\beta-\alpha-1}(1 + t)|t|^{2\alpha+1} dt = \sum_{n=0}^\infty \frac{(\lambda x)^n}{b_n(\alpha)} I_n(\alpha, \beta),
$$

where

$$
I_n(\alpha, \beta) = \int_{-1}^1 t^n (1 - t^2)^{\beta-\alpha-1}(1 + t)|t|^{2\alpha+1} dt,
$$

or

$$
I_{2n}(\alpha, \beta) = 2 \int_{0}^1 (1 - t^2)^{\beta-\alpha-1}t^{2n+2\alpha+1} dt = \int_{0}^1 (1 - y)^{\beta-\alpha-1}y^{n+\alpha} dy.
$$
Proposition 6. 

\[
\frac{\Gamma(\beta - \alpha)\Gamma(n + \alpha + 1)}{\Gamma(n + \beta + 1)},
\]

and

\[
I_{2n+1}(\alpha, \beta) = 2 \int_0^1 (1 - t^2)^{\beta - \alpha - 1} t^{2n+2\alpha+3} dt = I_{2n}(\alpha + 1, \beta + 1).
\]

Thus

\[
\int_{-1}^1 E_\alpha(\lambda x)(1 - t^2)^{\beta - \alpha - 1}(1 + t)|t|^{2\alpha+1} dt = \frac{\Gamma(\beta - \alpha)\Gamma(\alpha + 1)}{\Gamma(\beta + 1)} E_\beta(\lambda x),
\]

which gives the desired result. ■

Remark 3. We can write the formula (5) by the following

\[
E_\beta(\lambda x) = a_{\alpha, \beta} \sgn(x) |x|^{-(2\beta+1)} \int_{-|x|}^{+|x|} E_\alpha(\lambda y)(x^2 - y^2)^{\beta - \alpha - 1} (x + y)|y|^{2\alpha+1} dy, \quad x \neq 0.
\]

Definition 1. Let \( \alpha, \beta \in ]-1/2, \infty[, \) such that \( \beta > \alpha \). We define the Dunkl Sonine transform \( S_{\alpha, \beta} \) on \( E(\mathbb{R}) \), by

\[
S_{\alpha, \beta}(f)(x) := a_{\alpha, \beta} \int_{-1}^1 f(xt)(1 - t^2)^{\beta - \alpha - 1}(1 + t)|t|^{2\alpha+1} dt,
\]

which can be written as:

\[
S_{\alpha, \beta}(f)(x) = a_{\alpha, \beta} \sgn(x) |x|^{-(2\beta+1)} \int_{-|x|}^{+|x|} f(y)(x^2 - y^2)^{\beta - \alpha - 1} (x + y)|y|^{2\alpha+1} dy, \quad x \neq 0,
\]

\[
S_{\alpha, \beta}(f)(0) = f(0).
\]

Remark 4. For \( \alpha, \beta \in ]-1/2, \infty[, \) such that \( \beta > \alpha \), we have

\[
E_\beta(\lambda) = S_{\alpha, \beta}(E_\alpha(\lambda)), \quad \lambda \in \mathbb{C}. \tag{6}
\]

Definition 2. Let \( \alpha, \beta \in ]-1/2, \infty[, \) such that \( \beta > \alpha \). We define the dual Dunkl Sonine transform \( {}^tS_{\alpha, \beta} \) on \( S(\mathbb{R}) \), by

\[
{}^tS_{\alpha, \beta}(f)(x) := a_{\alpha, \beta} \int_{|y| \geq |x|} \sgn(y)(y^2 - x^2)^{\beta - \alpha - 1}(x + y)f(y)dy,
\]

which can be written as:

\[
{}^tS_{\alpha, \beta}(f)(x) = a_{\alpha, \beta} \sgn(x)|x|^{2(\beta - \alpha)} \int_{|t| \geq 1} \sgn(t)(t^2 - 1)^{\beta - \alpha - 1}(t + 1)f(xt)dt.
\]

Proposition 6. 

(i) For all \( f \in E(\mathbb{R}) \) and \( g \in S(\mathbb{R}) \), we have

\[
\int_{\mathbb{R}} S_{\alpha, \beta}(f)(x)g(x)|x|^{2\beta+1} dx = \int_{\mathbb{R}} f(x){}^tS_{\alpha, \beta}(g)(x)|x|^{2\alpha+1} dx.
\]

(ii) \( F_\beta \) possesses on \( S(\mathbb{R}) \) the following decomposition:

\[
F_\beta(f) = F_{\alpha} \circ {}^tS_{\alpha, \beta}(f), \quad f \in S(\mathbb{R}).
\]
Proof. Part (i) follows from Definition 1 by Fubini’s theorem. Then part (ii) follows from (i) and 3 by taking \( f = E_\alpha(-i\lambda) \). ■

In [20, Lemma 2.1] Y. Xu proves the identity \( S_{\alpha,\beta} = V_\beta \circ V_\alpha^{-1} \) on polynomials. As the intertwiner is a homeomorphism on \( \mathcal{E}(\mathbb{R}) \) and polynomials are dense in \( \mathcal{E}(\mathbb{R}) \), this gives the identity also on \( \mathcal{E}(\mathbb{R}) \). In the following we give a second method to prove this identity.

Theorem 1.

(i) The operator \( ^tS_{\alpha,\beta} \) is a topological automorphism of \( S(\mathbb{R}) \), and satisfies the following relations:

\[
^tS_{\alpha,\beta}(f) = (^tV_\alpha)^{-1} \circ ^tV_\beta(f), \quad f \in S(\mathbb{R}),
\]

\[
^tS_{\alpha,\beta}(\Lambda_\beta f) = \Lambda_\alpha(^tS_{\alpha,\beta}(f)), \quad f \in S(\mathbb{R}).
\]

(ii) The operator \( S_{\alpha,\beta} \) is a topological automorphism of \( \mathcal{E}(\mathbb{R}) \), and satisfies the following relations:

\[
S_{\alpha,\beta}(f) = V_\beta \circ V_\alpha^{-1}(f), \quad f \in \mathcal{E}(\mathbb{R}),
\]

\[
\Lambda_\beta(S_{\alpha,\beta}(f)) = S_{\alpha,\beta}(\Lambda_\alpha f), \quad f \in \mathcal{E}(\mathbb{R}).
\]

Proof. (i) From Proposition 6 (ii), we have

\[
^tS_{\alpha,\beta}(f) = (\mathcal{F}_\alpha)^{-1} \circ \mathcal{F}_\beta(f).
\]

Using Proposition 3 (ii), we obtain

\[
^tS_{\alpha,\beta}(f) = (^tV_\alpha)^{-1} \circ ^tV_\beta(f), \quad f \in S(\mathbb{R}).
\]

Thus from Proposition 2 (i),

\[
^tS_{\alpha,\beta}(\Lambda_\beta f) = (^tV_\alpha)^{-1} \circ ^tV_\beta(\Lambda_\beta f) = (^tV_\alpha)^{-1}\left(\frac{d}{dx}^tV_\beta(f)\right).
\]

Using the fact that

\[
^tV_\alpha(\Lambda_\alpha f) = \frac{d}{dx}(^tV_\alpha(f)) \iff \Lambda_\alpha(^tV_\alpha)^{-1}(f) = (^tV_\alpha)^{-1}\left(\frac{d}{dx}f\right),
\]

we obtain

\[
^tS_{\alpha,\beta}(\Lambda_\beta f) = \Lambda_\alpha(^tV_\alpha)^{-1}(^tV_\beta(f)) = \Lambda_\alpha(^tS_{\alpha,\beta}(f)).
\]

(ii) From Proposition 2 (ii), we have

\[
\int_{\mathbb{R}} f(x)^tV_\beta(g)(x)dx = \int_{\mathbb{R}} V_\beta(f)(x)g(x)|x|^{2\beta+1}dx.
\]

On other hand, from [5], Proposition 2 (ii) and Proposition 6 (i) we have

\[
\int_{\mathbb{R}} f(x)^tV_\beta(g)(x)dx = \int_{\mathbb{R}} f(x)^tV_\alpha \circ ^tS_{\alpha,\beta}(g)(x)dx = \int_{\mathbb{R}} V_\alpha(f)(x)^tS_{\alpha,\beta}(g)(x)|x|^{2\alpha+1}dx
\]

\[
= \int_{\mathbb{R}} S_{\alpha,\beta} \circ V_\alpha(f)(x)g(x)|x|^{2\beta+1}dx.
\]

Then

\[
S_{\alpha,\beta} \circ V_\alpha(f) = V_\beta(f).
\]
Hence from Proposition \[1\]
\[
\Lambda_\beta(S_{\alpha,\beta}(f)) = \Lambda_\beta V_\beta(V_\alpha^{-1}(f)) = V_\beta \left( \frac{d}{dx} V_\alpha^{-1}(f) \right).
\]
Using the fact that
\[
\Lambda_\alpha(V_\alpha(f)) = V_\alpha \left( \frac{d}{dx} f \right) \iff V_\alpha^{-1}(\Lambda_\alpha f) = \frac{d}{dx} V_\alpha^{-1}(f),
\]
we obtain
\[
\Lambda_\beta(S_{\alpha,\beta}(f)) = V_\beta \circ V_\alpha^{-1}(\Lambda_\alpha f) = S_{\alpha,\beta}(\Lambda_\alpha f),
\]
which completes the proof of the theorem. ■

4 Complex powers of $\Delta_\alpha$

For $\lambda \in \mathbb{C}$, Re($\lambda$) $>$ $-1$, we denote by $|x|^\lambda$ the tempered distribution defined by
\[
\langle |x|^\lambda, \varphi \rangle := \int_{\mathbb{R}} |x|^\lambda \varphi(x) dx, \quad \varphi \in \mathcal{S}(\mathbb{R}). \tag{9}
\]
We write
\[
\langle |x|^\lambda, \varphi \rangle = \int_0^\infty x^\lambda [\varphi(x) + \varphi(-x)] dx, \quad \varphi \in \mathcal{S}(\mathbb{R}),
\]
then from \[1\], we obtain the following result.

**Lemma 1.** Let $\varphi \in \mathcal{S}(\mathbb{R})$. The mapping $g : \lambda \rightarrow \langle |x|^\lambda, \varphi \rangle$ is complex-valued function and has an analytic extension to $\mathbb{C}\setminus\{-(2\alpha + 2\ell + 2), \ell \in \mathbb{N}\}$, with simple poles $-(2\ell + 1), \ell \in \mathbb{N}$ and
\[
\text{Res}(g, -1 - 2\ell) = 2\varphi^{(2\ell)}(0) \frac{\Gamma(\alpha + 1)\Gamma(2\alpha + \lambda + 1)}{\Gamma(-\lambda/2)}.
\]

**Proposition 7.** Let $\varphi \in \mathcal{S}(\mathbb{R})$.

(i) The function $\lambda \rightarrow \langle |x|^{\lambda+2\alpha+1}, \varphi \rangle$ is analytic on $\mathbb{C}\setminus\{-(2\alpha + 2\ell + 2), \ell \in \mathbb{N}\}$, with simple poles $-(2\alpha + 2\ell + 2), \ell \in \mathbb{N}$.

(ii) The function $\lambda \rightarrow \frac{2^{2\alpha+\lambda+2}\Gamma(\alpha+1)\Gamma(\lambda+1)\Gamma(2\alpha+\lambda+2)}{\Gamma(-\lambda/2)} \langle |x|^{-(\lambda+1)}, \varphi \rangle$ is analytic on $\mathbb{C}\setminus\{-(2\alpha + 2\ell + 2), \ell \in \mathbb{N}\}$, with simple poles $-(2\alpha + 2\ell + 2), \ell \in \mathbb{N}$.

(iii) For $\lambda \in \mathbb{C}\setminus\{-(2\alpha + 2\ell + 2), \ell \in \mathbb{N}\}$ we have
\[
\mathcal{F}_\alpha(|x|^{\lambda+2\alpha+1}) = \frac{2^{2\alpha+\lambda+2}\Gamma(\alpha+1)\Gamma(2\alpha+\lambda+2)}{\Gamma(-\lambda/2)} |x|^{-(\lambda+1)}, \quad \text{in} \ S'-\text{sense}.
\]

(iv) For $\lambda \in \mathbb{C}\setminus\{-(2\alpha + 2\ell + 2), \ell \in \mathbb{N}\}$ we have
\[
|x|^{\lambda+2\alpha+1} = \frac{2^\lambda \Gamma(\frac{2\alpha+\lambda+2}{2})}{\Gamma(\alpha+1)\Gamma(-\lambda/2)} \mathcal{F}_\alpha(|x|^{-(\lambda+1)}), \quad \text{in} \ S'-\text{sense}.
Proof. (i) Follows directly from Lemma 1.

(ii) From [7] pages 2 and 8 the function $\lambda \to \Gamma(\frac{2\alpha+\lambda+2}{2})$ has an analytic extension to $\mathbb{C}\setminus\{-(2\alpha+2\ell+2), \ell \in \mathbb{N}\}$, with simple poles $-(2\alpha+2\ell+2), \ell \in \mathbb{N}$, and the function $\lambda \to \frac{1}{\Gamma(-\lambda/2)}$ has zeros $2\ell, \ell \in \mathbb{N}$. Thus from Lemma 1 we see that

$$
\lambda \to \frac{2^{\alpha+\lambda+2}\Gamma(\alpha+1)\Gamma(\frac{2\alpha+\lambda+2}{2})}{\Gamma(-\lambda/2)}(\lambda+1, \varphi)
$$

is analytic on $\mathbb{C}\setminus\{-(2\alpha+2\ell+2), \ell \in \mathbb{N}\}$, with simple poles $-(2\alpha+2\ell+2), \ell \in \mathbb{N}$.

(iii) Let determine the value of $\mathcal{F}_\alpha(|x|^{\lambda+2\alpha+1})$ in the $\mathcal{S}'$-sense. We put $\psi_t(x) := e^{-tx^2}, t > 0$. Then $\psi_t \in \mathcal{S}(\mathbb{R})$, and from [12]:

$$
\mathcal{F}_\alpha(\psi_t)(x) = \Gamma(\alpha+1)e^{-(\alpha+1)}e^{-x^2/4t}, \quad x \in \mathbb{R}.
$$

Furthermore, for $\varphi \in \mathcal{S}(\mathbb{R})$ we have

$$
\int_{\mathbb{R}} \mathcal{F}_\alpha(\varphi)(x)\psi_t(x)|x|^{2\alpha+1}dx = \Gamma(\alpha+1)\int_{\mathbb{R}} \varphi(x)t^{-\alpha-1}e^{-x^2/4t}|x|^{2\alpha+1}dx.
$$

Multiplying both sides by $t^{-\lambda/2-1}$ and integrating over $(0, \infty)$, we obtain for $\text{Re}(\lambda) \in \mathbb{R}$, $0$:

$$
\int_{\mathbb{R}} \mathcal{F}_\alpha(\varphi)(x)|x|^{\lambda+2\alpha+1}dx = \frac{2^{\alpha+\lambda+2}\Gamma(\alpha+1)\Gamma(\frac{2\alpha+\lambda+2}{2})}{\Gamma(-\lambda/2)}\int_{\mathbb{R}} \varphi(x)|x|^{-(\lambda+1)}dx.
$$

This and from [3] we get for $\text{Re}(\lambda) \in \mathbb{R}$, $0$:

$$
\mathcal{F}_\alpha(|x|^{\lambda+2\alpha+1}) = \frac{2^{\alpha+\lambda+2}\Gamma(\alpha+1)\Gamma(\frac{2\alpha+\lambda+2}{2})}{\Gamma(-\lambda/2)}|x|^{-(\lambda+1)}.
$$

The result follows by analytic continuation.

(iv) From (iii) we have

$$
|x|^{\lambda+2\alpha+1} = \frac{2^{\alpha+\lambda+2}\Gamma(\alpha+1)\Gamma(\frac{2\alpha+\lambda+2}{2})}{\Gamma(-\lambda/2)}\mathcal{F}_\alpha^{-1}(|x|^{-(\lambda+1)}).
$$

Using the fact that

$$
\langle \mathcal{F}_\alpha^{-1}(|x|^{-(\lambda+1)}), \varphi \rangle = \langle |x|^{-(\lambda+1)}, \mathcal{F}_\alpha^{-1}(\varphi) \rangle, \quad \varphi \in \mathcal{S}(\mathbb{R})
$$

By applying [3] and Proposition 3 (iii), we obtain

$$
\langle \mathcal{F}_\alpha^{-1}(|x|^{-(\lambda+1)}), \varphi \rangle = c_\alpha \int_{\mathbb{R}} |x|^{-(\lambda+1)}\mathcal{F}_\alpha(\varphi)(-x)dx, \quad \varphi \in \mathcal{S}(\mathbb{R})
$$

Then

$$
\mathcal{F}_\alpha^{-1}(|x|^{-(\lambda+1)}) = c_\alpha \mathcal{F}_\alpha(|x|^{-(\lambda+1)}),
$$

which gives the result.

Definition 3. For $\lambda \in \mathbb{C}\setminus\{-(\alpha+\ell+1), \ell \in \mathbb{N}\}$, the complex powers of the Dunkl Laplacian $\Delta_\alpha$ are defined for $f \in \mathcal{S}(\mathbb{R})$ by

$$
(-\Delta_\alpha)^\lambda f(x) := \frac{2^{\alpha+\lambda+1}\Gamma(\alpha+\lambda+1)}{\Gamma(\alpha+1)\Gamma(-\lambda)}|x|^{-(2\lambda+1)}*_\alpha f(x),
$$

where $*_\alpha$ is the Dunkl convolution product given by [10].
In the next part of this section we use Definition 3 and Proposition 7 (iv) to establish the following result:

\[ \mathcal{F}_\alpha((-\Delta_\alpha)^\lambda f)(x) = |x|^{2\lambda} \mathcal{F}_\alpha(f)(x). \]

**Proposition 8.** For \( \lambda \in \mathbb{C}\{-(\alpha + \ell + 1), \ell \in \mathbb{N}\} \) and \( f \in \mathcal{S}(\mathbb{R}) \),

\[ (-\Delta_\alpha)^\lambda f(x) = b_\alpha(\lambda) \int_\mathbb{R} \left[ \int_0^\pi \frac{(1 + \text{sgn}(xy) \cos \theta)}{(x,y)_{\theta}^{2(\lambda+\alpha)+1}} \sin^{2\alpha} \theta d\theta \right] f(y) |y|^{2\alpha+1} dy, \]

where

\[ b_\alpha(\lambda) = \frac{2^{2\lambda} \Gamma(\alpha + \lambda + 1)}{\sqrt{\pi} \Gamma(\alpha + 1/2) \Gamma(-\lambda)}, \quad (x,y)_{\theta} = \sqrt{x^2 + y^2 - 2|xy| \cos \theta}. \]

**Proof.** From Definition 3, (4) and (9), we have

\[ (-\Delta_\alpha)^\lambda f(x) = \frac{2^{2\lambda} \Gamma(\alpha + \lambda + 1)}{\Gamma(\alpha + 1) \Gamma(-\lambda)} \langle |y|^{-(\lambda+1)}, \tau_x f(-y) \rangle \]

\[ = \frac{2^{2\lambda} \Gamma(\alpha + \lambda + 1)}{\Gamma(\alpha + 1) \Gamma(-\lambda)} \int_\mathbb{R} |y|^{-2(\lambda+\alpha+1)} \tau_x f(-y) |y|^{2\alpha+1} dy. \]

So

\[ (-\Delta_\alpha)^\lambda f(x) = \int_\mathbb{R} \tau_x (|y|^{-2(\lambda+\alpha+1)}(-y)f(y)) |y|^{2\alpha+1} dy. \]

Then the result follows from Proposition 4. \( \blacksquare \)

**Note 1.** We denote by

- \( \Psi \) the subspace of \( \mathcal{S}(\mathbb{R}) \) consisting of functions \( f \), such that \( f^{(k)}(0) = 0, \quad \forall k \in \mathbb{N} \).

- \( \Phi_\alpha \) the subspace of \( \mathcal{S}(\mathbb{R}) \) consisting of functions \( f \), such that \( \int_\mathbb{R} f(y) y^k |y|^{2\alpha+1} dy = 0, \quad \forall k \in \mathbb{N} \).

The spaces \( \Psi \) and \( \Phi_{-1/2} \) are well-known in the literature as Lizorkin spaces (see [1, 9, 13]).

**Lemma 2** (see [1]). The multiplication operator \( M_\lambda : f \to |x|^\lambda f, \lambda \in \mathbb{C}, \) is a topological automorphism of \( \Psi \). Its inverse operator is \( (M_\lambda)^{-1} = M_{-\lambda} \).

**Theorem 2.**

(i) The Dunkl transform \( \mathcal{F}_\alpha \) is a topological isomorphism from \( \Phi_\alpha \) onto \( \Psi \).

(ii) The operator \( S_{\alpha,\beta} \) is a topological isomorphism from \( \Phi_\beta \) onto \( \Phi_\alpha \).

(iii) For \( \lambda \in \mathbb{C}\{-(\alpha + \ell + 1), \ell \in \mathbb{N}\} \) and \( f \in \Phi_\alpha \), the function \( (-\Delta_\alpha)^\lambda f \) belongs to \( \Phi_\alpha \), and

\[ \mathcal{F}_\alpha((-\Delta_\alpha)^\lambda f)(x) = |x|^{2\lambda} \mathcal{F}_\alpha(f)(x). \] (10)
Then by Lemma 2 and (i)

\[ (S_5) \text{ Inversion formulas for } \alpha, \beta \]

We define the operators \( f \in S \)

On the other hand from (3), \( \Phi \)

Hence \( F \)

Conversely, let \( g \in \Psi \). Since \( F \) is a topological automorphism of \( S(\mathbb{R}) \). There exists \( f \in S(\mathbb{R}) \), such that \( F(f) = g \). Thus

\[ g^{(k)}(0) = (-i)^k \frac{k!}{b_k(\alpha)} \int_{\mathbb{R}} f(x) x^k |x|^{2\alpha + 1} dy = 0, \quad \forall k \in \mathbb{N}. \]

So \( f \in \Phi \) and \( F(f) = g \).

(ii) follows directly from (i) and (7).

(iii) Similarly to the standard convolution if \( f \in S(\mathbb{R}) \) and \( S \in S'(\mathbb{R}) \), then \( S^\ast \in E(\mathbb{R}) \) and \( T_{|x|^{2\alpha + 1}} S^\ast f \in S'(\mathbb{R}) \). Moreover

\[ F(T_{|x|^{2\alpha + 1}} S^\ast f) = F(f) F(S). \]

Let \( f \in \Phi \) and \( \lambda \in \mathbb{C} \setminus \{-\alpha + \lambda + 1, \lambda \in \mathbb{N}\} \). Consequently, from Definition 3, Proposition 7 (iv) and (9) we have

\[ F(T_{|x|^{2\alpha + 1}} (-\Delta)^\lambda f) = |x|^{2\lambda + 2\alpha + 1} F(f) = T_{|x|^{2\lambda + 2\alpha + 1}} F(f). \]

On the other hand from (8),

\[ F(T_{|x|^{2\alpha + 1}} (-\Delta)^\lambda f) = T_{|x|^{2\alpha + 1}} F((-\Delta)^\lambda f). \]

From (11) and (12), we obtain

\[ F((-\Delta)^\lambda f) = |x|^{2\lambda} F(f). \]

Then by Lemma 2 and (i) we deduce that \((-\Delta)^\lambda f \in \Phi \).

5 Inversion formulas for \( S_{\alpha,\beta} \) and \( ^t S_{\alpha,\beta} \)

In this section, we establish inversion formulas for the Dunkl Sonine transform and its dual.

Definition 4. We define the operators \( K_1, K_2 \) and \( K_3 \), by

\[ K_1(f) := c_\beta c_\alpha \langle |\lambda|^{2(\beta - \alpha)} F_\alpha(f) \rangle = c_\beta c_\alpha (-\Delta)^{\beta - \alpha} f, \quad f \in \Phi_\alpha, \]

\[ K_2(f) := c_\beta c_\alpha \langle |\lambda|^{2(\beta - \alpha)} F_\beta(f) \rangle = c_\beta c_\alpha (-\Delta)^{\beta - \alpha} f, \quad f \in \Phi_\beta, \]

\[ K_3(f) := \sqrt{c_\beta c_\alpha} \langle |\lambda|^{\beta - \alpha} F_\alpha(f) \rangle = \sqrt{c_\beta c_\alpha} (-\Delta)^{\beta - \alpha} f, \quad f \in \Phi_\alpha. \]

Lemma 3. For all \( g \in \Phi_\beta \), we have

\[ K_1(\mathcal{S}_{\alpha,\beta}^t g) = (\mathcal{S}_{\alpha,\beta}^t) K_2(g). \]

Proof. Let \( g \in \Phi_\beta \). Using Proposition 6 (ii),

\[ K_1(\mathcal{S}_{\alpha,\beta}^t g) = c_\beta c_\alpha \langle |\lambda|^{2(\beta - \alpha)} F_\alpha(g) \rangle = (\mathcal{S}_{\alpha,\beta}^t) K_2(g). \]
Theorem 3. (i) Inversion formulas: For all \( f \in \Phi_\alpha \) and \( g \in \Phi_\beta \), we have the inversions formulas:

(a) \( g = S_{\alpha, \beta}K_1(\text{t}S_{\alpha, \beta})(g) \),  
(b) \( f = (\text{t}S_{\alpha, \beta})K_2S_{\alpha, \beta}(f) \).

(ii) Plancherel formula: For all \( f \in \Phi_\beta \) we have

\[
\int_{\mathbb{R}} |f(x)|^2 |x|^{2\beta+1} \, dx = \int_{\mathbb{R}} |K_3(\text{t}S_{\alpha, \beta}(f))(x)|^2 |x|^{2\alpha+1} \, dx.
\]

Proof. (i) Let \( g \in \Phi_\beta \). From Proposition 3 (iii), (6) and Proposition 6 (ii), we obtain

\[
g = c_\beta \int_{\mathbb{R}} S_{\alpha, \beta}(E_\alpha(i\lambda))(\mathcal{F}_\beta(g)(\lambda))|\lambda|^{2\beta+1} \, d\lambda
\]

\[
= c_\beta S_{\alpha, \beta} \left[ \int_{\mathbb{R}} E_\alpha(i\lambda) \mathcal{F}_\alpha \circ \text{t}S_{\alpha, \beta}(g)(\lambda)|\lambda|^{2\beta+1} \, d\lambda \right]
\]

\[
= \frac{c_\beta}{c_\alpha} S_{\alpha, \beta} \left[ \mathcal{F}_\alpha^{-1}(|\lambda|^{2(\beta-\alpha)} \mathcal{F}_\alpha \circ \text{t}S_{\alpha, \beta}(g)) \right].
\]

Thus

\[ g = S_{\alpha, \beta}K_1(\text{t}S_{\alpha, \beta})(g), \quad g \in \Phi_\beta. \]

From the previous relation and (13), we deduce the relation:

\[ f = (\text{t}S_{\alpha, \beta})K_2S_{\alpha, \beta}(f), \quad f \in \Phi_\alpha. \]

(ii) Let \( f \in \Phi_\beta \). From Proposition 3 (iv) and Proposition 6 (ii), we deduce that

\[
\int_{\mathbb{R}} |f(x)|^2 |x|^{2\beta+1} \, dx = c_\beta \int_{\mathbb{R}} |\lambda|^{\beta-\alpha} \mathcal{F}_\alpha(\text{t}S_{\alpha, \beta}(f))(\lambda)|^2 |\lambda|^{2\alpha+1} \, d\lambda.
\]

Thus we obtain

\[
\int_{\mathbb{R}} |f(x)|^2 |x|^{2\beta+1} \, dx = c_\alpha \int_{\mathbb{R}} |\mathcal{F}_\alpha(K_3(\text{t}S_{\alpha, \beta}(f)))(\lambda)|^2 |\lambda|^{2\alpha+1} \, d\lambda.
\]

Then the result follows from this identity by applying Proposition 3 (iv). \( \blacksquare \)

Remark 5. Let \( f \in \Phi_\alpha \) and \( g \in \Phi_\beta \). By writing (a) and (b) respectively for the functions \( S_{\alpha, \beta}(f) \) and \( \text{t}S_{\alpha, \beta}(g) \), we obtain

(c) \( f = K_1(\text{t}S_{\alpha, \beta})S_{\alpha, \beta}(f) \),  
(d) \( g = K_2S_{\alpha, \beta}(\text{t}S_{\alpha, \beta})(g) \).

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