The Massive Multi-flavor Schwinger Model

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Abstract

QED with \( N \) species of massive fermions on a circle of circumference \( L \) is analyzed by bosonization. The problem is reduced to the quantum mechanics of the \( 2N \) fermionic and one gauge field zero modes on the circle, with nontrivial interactions induced by the chiral anomaly and fermions masses. The solution is given for \( N = 2 \) and fermion masses \((m)\) much smaller than the mass of the \( U(1) \) boson with mass \( \mu = \sqrt{2e^2/\pi} \) when all fermions satisfy the same boundary conditions. We show that the two limits \( m \to 0 \) and \( L \to \infty \) fail to commute and that the behavior of the theory critically depends on the value of \( mL |\cos \frac{1}{2} \theta| \) where \( \theta \) is the vacuum angle parameter. When the volume is large \( \mu L \gg 1 \), the fermion condensate \( \langle \bar{\psi} \psi \rangle \) is \(-e^{\gamma} m \mu^2 \cos^4 \frac{1}{2} \theta / 4 \pi^3)^{1/3}\) or \(-2e^{\gamma} m \mu L \cos^2 \frac{1}{2} \theta / \pi^2\) for \( mL(\mu L)^{1/2} |\cos \frac{1}{2} \theta| \gg 1 \) or \( \ll 1 \), respectively. Its correlation function decays algebraically with a critical exponent \( \eta = 1 \) when \( m \cos \frac{1}{2} \theta = 0 \).
The Schwinger model, QED in two dimensions, with $N \geq 2$ species of fermions is distinctly different from that with one flavor \cite{1,7}. For example, Affleck has shown that in the massless fermion case, one massive boson of mass $\mu = \sqrt{Ne^2/\pi}$ and $N - 1$ massless bosons appear, however there is no long range order ($\langle \bar{\psi} \psi \rangle = 0$) in accordance with Coleman’s theorem in a 2-d Lorentz invariant theory \cite{4,8} and correlators of $\bar{\psi} \psi$ show algebraic decay at large distances. Hence the rich vacuum structure of the multi-flavor Schwinger model carries many similarities to 4-dimensional QCD where we are interested in understanding how the effects of quark masses modify the vacuum structure, meson masses, mixing, and the pattern of chiral symmetry breaking.

Years ago Coleman showed \cite{5} that in the presence of small fermion masses $m \ll \mu$ in the $N = 2$ model, the second boson mass has a fractional power dependence on $m$ and the vacuum angle $\theta$: $\mu_2 \propto (m|\cos \frac{1}{2} \theta|)^{2/3}$. This singular dependence poses an intriguing puzzle: how can one get non-analytic dependence in the $m \to 0$ limit where one would expect the validity of a perturbation theory in mass?

Thus there has been growing interest in the Schwinger model, especially when defined on a compact manifold such as a circle or closed interval (a bag) \cite{9-36}. Besides reproducing results in Minkowski spacetime in the infinite volume limit, analyzing the model on a circle has the advantage of being free from infrared divergence and well-defined at every stage of manipulation. Furthermore analytic solutions of the multi-flavor model are extremely useful for comparison with lattice simulations where several flavors are inherent \cite{28,31,37-39}.

In this paper we solve the Schwinger model with two massive fermions on a circle of circumference $L$. We find that the theory sensitively depends on the dimensionless parameter $mL \cos \frac{1}{2} \theta$. In particular, the $m \to 0$ and $L \to \infty$ limits do not commute with each other. This is to be contrasted to the situation in a model with just one fermion, in which a small fermion mass does not alter the structure of the model except for necessitating a $\theta$ vacuum.

In the $SU(2)$ symmetric two flavor case ($m_1 = m_2$), we show that in the large volume $\mu L \gg 1$ limit, the light boson mass $\mu_2 \propto (m|\cos \frac{1}{2} \theta|)^{2/3}$ for $mL(\mu L)^{1/2} |\cos \frac{1}{2} \theta| \gg 1$, \[\mu_2 \propto (m|\cos \frac{1}{2} \theta|)^{2/3}, \quad \text{for } mL(\mu L)^{1/2} |\cos \frac{1}{2} \theta| \gg 1,\]
while it is \( m|\cos \frac{1}{2}\theta| \) for \( mL(\mu L)^{1/2}|\cos \frac{1}{2}\theta| \ll 1 \). In other words physical quantities behave smoothly in the \( m \to 0 \) or \( \theta \to \pm \pi \) limit.

Several authors have given exact solutions for the \( N \geq 2 \) model with massless fermions on various manifolds.\[23, 34, 35\] Yet the importance of the parameter \( mL|\cos \frac{1}{2}\theta| \) has not been stressed in the literature. We adopt the method of abelian bosonization on a circle, generalizing the analysis of the \( N=1 \) case in ref. \[10\]. With \( N \) fermions the problem is eventually reduced to quantum mechanics for the \( 2N+1 \) “zero modes” on the circle, with nontrivial interactions induced by the chiral anomaly and fermions masses. Further reduction is achieved for \( m \ll \mu \). We find that the wave function of the vacuum sensitively depends on fermion masses for \( N \geq 2 \).

The model is given by

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \sum_{a=1}^{N} \overline{\psi}_a \left\{ \gamma^\mu (i\partial_\mu - eA_\mu) - m_a \right\} \psi_a . \tag{1}
\]

Fermion fields obey boundary conditions

\[
\psi_a(t, x + L) = -e^{2\pi i\alpha_a} \psi_a(t, x) . \tag{2}
\]

We suppose that gauge fields are periodic. Note that with \( \alpha_a = 0 \) the system is mathematically equivalent to the Schwinger model on a line at finite temperature \( T = L^{-1} \).

It is possible to choose a gauge in which

\[
A_0(t, x) = -\int_0^L dy G(x-y) j^0_{EM}(t, y) , \quad A_1(t, x) = \frac{\Theta_W(t)}{eL} \tag{3}
\]

where \( j^0_{EM} = e \sum_a \overline{\psi}_a \psi_a \) and \( G(x) \) is the periodic Green’s function on a circle satisfying \( G''(x) = \delta_L(x) - L^{-1} \). \( \Theta_W(t) \) is the non-integrable phase of the Wilson line integral around the circle, and is a physical degree of freedom. In this Coulomb gauge the Hamiltonian is

\[
H = \frac{e^2 L}{2} \Pi_\Theta + \int_0^L dx \sum_a \overline{\psi}_a \left\{ \gamma^1 \left( -i\partial_1 + \frac{\Theta_W}{L} \right) + m_a \right\} \psi_a \\
- \frac{1}{2} \int_0^L dx dy j^0_{EM}(x) G(x-y) j^0_{EM}(y) . \tag{4}
\]

\( \Pi_\Theta \) is the momentum conjugate to \( \Theta_W \): \( \Pi_\Theta = \dot{\Theta}_W/e^2L \), and the anti-symmetrization of fermion operators is understood.
Fermions are bosonized first in the interaction picture defined by free massless fermions. We take \( \gamma^\mu = (\sigma_1, i\sigma_2) \) so that \( \psi^t_a = (\psi^a_+, \psi^a_-) \) satisfies \( (\partial_0 \pm \partial_1)\psi^a_\pm = 0 \), and introduce \( 2N \) sets of bosonic variables \( \{q^a, p^a, c^a_n, c^a_n\} \) satisfying

\[
[q^a_\pm, p^b_\pm] = i\delta^{ab}, \quad [c^a_{\pm,n}, c^{b\dagger}_{\pm,m}] = \delta^{ab}\delta_{nm}
\]

\[
\phi^a_\pm(t, x) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left\{ e^{-2\pi in(t\pm x)/L} + h.c. \right\}
\]

(5)

In terms of these variables the \( \psi^a_\pm(x) \)'s are represented as

\[
\psi^a_\pm(t, x) = \frac{1}{\sqrt{L}} C^a_\pm e^{\pm i(q^a_\pm + 2\pi p^a_\pm (t\pm x)/L)} : e^{\pm i\phi^a_\pm(t, x)} :
\]

(6)

where the Klein factors \( C^a_\pm \) are defined by

\[
C^a_+ = \exp \left\{ i\pi \sum_{b=1}^{a-1} (p^b_+ + p^b_- - 2\alpha_b) \right\}, \quad C^a_- = \exp \left\{ i\pi \sum_{b=1}^{a} (p^b_+ - p^b_-) \right\}
\]

(7)

and \( : : \) indicates normal ordering with respect to \( (c^a_n, c^{a\dagger}_n) \). It is straightforward to show that at equal time

\[
\{\psi^a_\pm(x), \psi^b_\pm(y)^\dagger \} = \delta^{ab} e^{-i\pi(x-y)/L} \cdot e^{2\pi i p^a_\pm (x-y)/L} \delta_L(x-y)
\]

(8)

and that all other anti-commutators vanish. Further

\[
\psi^a_\pm(t, x + L) = -e^{2\pi ip^a_\pm} \psi^a_\pm(t, x) = -e^{2\pi ip^a_\pm} \psi^a_\pm(t, x)
\]

(9)

so that the boundary condition (2) is guaranteed if

\[
e^{2\pi ip^a_\pm} |\text{phys}\rangle = e^{2\pi i\alpha_a} |\text{phys}\rangle.
\]

(10)

With this physical state condition the anticommutation relations (3) are consistent with the boundary condition (2). We note that \( (C^a_\pm \text{ or } C^a_\pm^\dagger) |\text{phys}\rangle = |\text{phys}\rangle \).

The bosonized currents and Hamiltonian are easily deduced by direct substitution of (6). The flavor diagonal currents \( j^\mu_a = \overline{\psi}_a \gamma^\mu \psi^a \) are

\[
j^0_a = -\frac{p^a_+ + p^a_-}{L} - \frac{1}{2\pi} \partial_x (\phi^a_+ + \phi^a_-)
\]

\[
j^1_a = \frac{+p^a_+ + p^a_-}{L} + \frac{1}{2\pi} \partial_t (\phi^a_+ + \phi^a_-) + \frac{\Theta_W}{\pi L}.
\]

(11)
It will prove convenient to rotate to a new basis in the flavor space. Introduce an orthogonal $N$-by-$N$ matrix $v^a_\alpha$, where $v^1_a = 1/\sqrt{N}$; $N$ new fields defined by $\chi_\alpha^a = (4\pi)^{-1/2} v^a_\alpha \phi^a_{\pm}$, and let $\chi_\alpha = \chi_\alpha^+ + \chi_\alpha^-$. Note that $j^\mu_{EM} = e \sum_a j^\mu_a = -\mu \epsilon^{\mu\nu} \partial_\nu \chi_1 + \cdots$ where $\mu^2 = Ne^2/\pi$. The $\chi_1$ field represents the charged part.

Then the Hamiltonian in the Schrödinger picture becomes

$$H = H_0 + H_{osci} + H_{mass} + \text{(constant)}$$

$$H_0 = \frac{e^2L\Pi_0^2}{2} + \frac{\pi}{2} \sum_{a=1}^N \left\{ (p^a_+ - p^a_-)^2 + (p^a_+ + p^a_- + \Theta W)^2 \right\}$$

$$H_{osci} = \int_0^L dx \left\{ N_\mu \Pi_1^2 + \chi_1^2 + \mu^2 \chi_1^2 \right\} + \sum_{\alpha=2}^N N_\alpha \Pi_\alpha^2 + \chi_\alpha^2 \right\}$$

$$H_{mass} = \int_0^L dx \left\{ m_1 M_{11} + m_2 M_{22} + \text{h.c.} \right\} , \quad M_{ab} = \psi^a_\alpha \psi^b_\alpha.$$  (12)

$\Pi_\alpha = \dot{\chi}_\alpha$ is the conjugate momentum to $\chi_\alpha$. $N_\mu[\cdots]$ denotes normal ordering in the Schrödinger picture with respect to a mass $\mu$. In the massless fermion limit we have one massive boson $\chi_1$ and $(N - 1)$ massless bosons $\chi_\alpha$ ($\alpha = 2 \sim N$) irrespective of boundary conditions of fermions. Physical states must satisfy $Q_{EM}|\text{phys}\rangle = 0$ where $Q_{EM} = e \sum_a (-p^a_+ + p^a_-)$.

In the absence of $H_{mass}$, the $2N + 1$ zero modes ($\Theta_W, q^a_\pm$) and the oscillatory modes ($\chi_\alpha$) decouple. Thus our strategy is as follows: we first determine all matrix elements of the total Hamiltonian $H$ in the basis spanned by eigenstates of $H_0 + H_{osci}$, and then diagonalize it in the case $m_\alpha/\mu \ll 1$. As we shall see below, in a large volume the fermion mass term $H_{mass}$ cannot be treated as a small perturbation to $H_0 + H_{osci}$. On the contrary it dominates over $H_0$ and for $N \geq 2$, $H_{mass}$ completely alters the structure of the vacuum.

Let us restrict ourselves henceforth to the case $N = 2$ and $\alpha_a = \alpha$. $H_{osci}$ describes free fields. The boundary condition (10) implies that $q^a_\pm$'s are angular variables so that wave functions can be expanded in a Fourier series in each zero mode: $e^{in_\pm q^a_\pm}$. The relevant eigenstates of $H_0$ are

$$\Phi_s^{(n,r)} = \frac{1}{(2\pi)^2} u_s(\Theta_W + 2\pi n + \pi r + 2\pi \alpha) e^{i(n+\alpha)(q^1_\pm + q^1_-) + i(n+r+\alpha)(q^2_\pm + q^2_-)}$$  (13)

with energy

$$E_s^{(n,r)} = \mu \left( s + \frac{1}{2} \right) + \frac{\pi r^2}{L}.$$  (14)
where \((n, r)\) are integers. \(u_s(x)\) (s = 0, 1, 2, \cdots) is the \(s\)-th eigenfunction in a harmonic oscillator problem. In particular \(u_0(x) = (2/\mu L\pi^2)^{1/4} e^{-x^2/\pi \mu L}\). One need consider only states with \(n_+^a = n_-^a\), since \(n_+^a \neq n_-^a\) gives a higher energy than the corresponding \(n_+^a = n_-^a\) state, and every term in the Hamiltonian, including the fermion mass term, preserves \(n_+^a - n_-^a\). In other words, states with \(n_+^a \neq n_-^a\) decouple from the Hilbert space constructed on the vacuum determined below. The inner product is defined by

\[
\langle \Phi^r_a | \Phi^s_b \rangle = \int_{-\infty}^{\infty} d\Theta_w \int_0^{2\pi} dq^1_+ dq^1_- dq^2_+ dq^2_- \Phi^*_a \Phi_b.
\]

Note

\[
\langle \Phi^r_s | \Phi^{s'}_{s'} \rangle = \delta_{ss'} \delta_{rr'} \delta_{nn'}.
\]

The Hamiltonian has an additional symmetry generated by homotopically non-trivial large gauge transformations which are given by \(A_\mu \rightarrow A_\mu + e^{-1} \partial_\mu \Lambda, \psi_a \rightarrow e^{-i\Lambda} \psi_a\) where \(\Lambda = 2\pi l x/L\) (l integer). They preserve the boundary conditions, inducing transformations \(\Theta_w \rightarrow \Theta_w + 2\pi l\) and \(p^a_\pm \rightarrow p^a_\pm - l\). The \(l=1\) transformation is generated by

\[
U = \exp \left\{ i \left( q^1_+ + q^1_- + q^2_+ + q^2_- + 2\pi \Pi \Theta \right) \right\}
\]

\(UHU^{-1} = H\) (15)

Under such a gauge transformation \(U \Phi^{(n,r)}_s = \Phi^{(n+1,r)}_s\) so that one is naturally led to considering gauge covariant \(\theta\) states

\[
\Phi^r_s(\theta) = \frac{1}{\sqrt{2\pi}} \sum_n e^{in\theta} \Phi^{(n,r)}_s
\]

which satisfy \(\langle \Phi^r_s(\theta) | \Phi^{s'}_{s'}(\theta') \rangle = \delta_{ss'} \delta_{rr'} \delta_{2\pi}(\theta - \theta')\). We shall see below that non-vanishing fermion masses \((m_1, m_2 \neq 0)\) absolutely necessitate the \(\theta\) states.

There are three effects which the fermion mass term \(H_{\text{mass}}\) brings about. Firstly, it induces transitions from one \(\Phi^{(n,r)}_s\) to another \(\Phi^{(n',r')}_{s'}\). Secondly, it gives a mass \(\mu_2\) to the \(\chi_2(x)\) field. Thirdly, it induces interactions among the \(\chi_1\) and \(\chi_2\) fields, and zero modes \(q^a_\pm\). We restrict ourselves to the case \(m_a \ll \mu\). The mass \(\mu_2\) then depends on the vacuum structure and must be determined self-consistently.

It is important to realize that in the zero mode sector \(H_{\text{mass}}\) brings about a significant change in the structure of the ground state. Although \(\Phi^{(n,r)}_{s=0}\) with \(r \neq 0\) has a higher eigenvalue of \(H_0\), it is just \(\pi r^2/L\). Since \(H_{\text{mass}}\) induces transitions among various \((n, r)\) states with typical matrix elements of order \(~ mL(\mu L)^{1/2}\), the ground state of the total
Hamiltonian for a large $L$ becomes a superposition of various $\Phi_0^r(\theta)$ with significant weight for $r \neq 0$.

To see this more clearly, we first introduce the function

$$2\pi \Delta(x; \mu, L) = \pi \sum_{n \neq 0} \frac{e^{-2\pi i nx/L}}{\sqrt{(2\pi n)^2 + (\mu L)^2}}$$

$$= \begin{cases} 
-\frac{1}{2} \ln 2 \left(1 - \cos \frac{2\pi x}{L}\right) & \text{for } \mu = 0 \\
K_0(\mu|x|) - \frac{\pi}{\mu L} + 2 \int_1^\infty du \frac{\cosh \mu Xu}{(e^{\mu Lu} - 1)\sqrt{u^2 - 1}} & \text{for } \mu > 0, |x| < L 
\end{cases}$$

(17)

where $K_0(z)$ is the modified Bessel function. We shall also encounter the quantity

$$B(\mu L) = \exp \left\{ -2\pi [\Delta(0; \mu, L) - \Delta(0; 0, L)] \right\}$$

$$= \frac{\mu L}{4\pi} \exp \left\{ \gamma + \frac{\pi}{\mu L} - 2 \int_1^\infty \frac{du}{(e^{\mu Lu} - 1)\sqrt{u^2 - 1}} \right\}.$$ 

(18)

In the Schrödinger picture we have the following relations for normal ordering of operators

$$N_\mu[e^{i\beta_1(x)}] N_\mu[e^{i\beta_2(0)}] = e^{-\beta_1\beta_2 \Delta(x,\mu L)} N_\mu[e^{i\beta_1(x)}] N_\mu[e^{i\beta_2(0)}]$$

$$N_0[e^{i\beta_1(x)}] N_0[e^{i\beta_2(0)}] = B(\mu L)(\beta_1^2 + \beta_2^2)/4 \pi e^{-\beta_1\beta_2 [\Delta(x,\mu L) - \Delta(x,0,L)]} N_\mu[e^{i\beta_1(x)}] N_\mu[e^{i\beta_2(0)}].$$

(19)

Now consider the operator $M_{ab} = \psi_+^{a*} \psi_+^{b} = \overline{\psi}_a \frac{1}{\pi} (1 + \gamma^5) \psi_b$ introduced above.

$$M_{ab}^S = \text{sign} (a < b) \cdot C_a^b C_+^{a*} \cdot e^{-2\pi i (p_a^a - p_b^b) x/L} \cdot e^{i(q_a^a + q_b^b)}$$

$$\times L^{-1} N_0[e^{i\sqrt{2} \pi \chi_1}] N_0[e^{i\sqrt{2} \pi (a \chi_2 + \epsilon t \chi_2^1 + \epsilon s \chi_2^2 + \epsilon t \chi_2^2)]$$

(20)

where $(\epsilon_1, \epsilon_2) = (+1, -1)$ and sign $(A) = +$ or $-$, if $A$ is true or false, respectively ($S$ stands for Schrödinger picture). In particular, for mass operators $M_{aa}$

$$M_{aa}^S = C_a^{a*} C_+^a \cdot e^{-2\pi i (p_a^a - p_b^b) x/L} \cdot e^{i(q_a^a + q_b^b)} \cdot L^{-1} \overline{B} N_1[e^{i\sqrt{2} \pi \chi_1}] N_2[e^{i\sqrt{2} \pi \epsilon a \chi_2}].$$

(21)

Note $\mu_1 \sim \mu$ for $m/\mu \ll 1$, and we set $\mu_1 = \mu$ in the following. $\mu_2$ is to be determined self-consistently.

It is easy to see that

$$\langle \Phi^{(n', r')}_{s'} | M_{11} | \Phi^{(n, r)}_s \rangle = -L^{-1} \overline{B} U_{s's} \left[ \delta_{n', n+1} \delta_{r', r-1} \right] \delta_{n', n} \delta_{r', r+1}$$

(22)
where \( U_{s's} = \int_{-\infty}^{\infty} dx \, u_{s'}(x + \pi)u_s(x) = (-)^{s'-s}U_{ss'} \) evaluates to

\[
U_{s's}(\mu L) = e^{-\pi/2\mu L} \sum_{p=0}^{s} \frac{(-)^p \sqrt{s'!s!}}{p!(s' - s + p)!(s - p)!} \left( \frac{\pi}{\mu L} \right)^{(s'-s+2p)/2}
\]

for \( s' \geq s \). Notice that for \( \mu L \gg 1 \), \( U_{s's} \) is negligible so that we can safely restrict ourselves to \( s = 0 \) states in constructing the vacuum. Transitions to \( s \geq 1 \) states are very small. For \( \mu L \ll 1 \) the magnitude of \( U_{s's} \) itself is suppressed exponentially \( (\sim e^{-\pi/2\mu L}) \), however the ratio of \( U_{s0} \) \((s \geq 1)\) to \( U_{00} = e^{-\pi/2\mu L} \) becomes large. Since in this paper we are concerned with the large volume physics, we can ignore transitions to \( s \geq 1 \) states, and suppress the index \( s \) henceforth. A more full account incorporating transitions to higher \( s \) states is reserved for future publications. We remark that the effect of \( H_{\text{mass}} \) becomes negligibly small for \( \mu L \ll 1 \), and so most of the qualitative results below would be insensitive to transitions to higher \( s \) states.

Matrix elements of \( H_{\text{mass}} \) take a simple form in a \( \theta-\varphi \) basis defined by

\[
\Phi(\theta; \varphi) = \frac{1}{\sqrt{2\pi}} \sum_r e^{ir\varphi} \Phi_r(\theta) .
\]

They are given by

\[
\langle \Phi(\theta'; \varphi') | H_{\text{mass}} | \Phi(\theta; \varphi) \rangle = -2\bar{m}B e^{-\pi/2\mu L} \cos(\varphi + \bar{\delta}) \delta_{2\pi}(\theta - \theta') \delta_{2\pi}(\varphi - \varphi')
\]

\[
m_1 e^{-i\theta} + m_2 = \bar{m}(\theta) e^{i\bar{\delta}(\theta)} \quad (\bar{m} > 0) .
\]

In the \( SU(2) \) symmetric case \((m_a = m)\)

\[
\bar{\delta}(\theta) = -\frac{\theta}{2} + \pi \text{ floor} \left( \frac{\theta + \pi}{2\pi} \right) , \quad \bar{m}(\theta) = 2m |\cos \frac{1}{2}\theta| \]

where \( \text{floor}(x) \) is the maximum integer which does not exceed \( x \). Notice that \( \bar{\delta}(\theta) \) has discontinuities at \( \theta = \pm\pi, \pm3\pi, \cdots \) where \( \bar{m}(\theta) \) vanishes.

We write the vacuum in the form

\[
|\Phi_{\text{vac}}(\theta) \rangle = \int_0^{2\pi} d\varphi \ f(\varphi + \bar{\delta}) |\Phi(\theta; \varphi) \rangle .
\]

Since \( \pi r^2/L \) in \( H_0 \) acts on \( \Phi(\theta; \varphi) \) as \( -(\pi/L)(\partial^2/\partial \varphi^2) \), the eigenvalue equation \((H_0 + H_{\text{mass}} - E) |\Phi_{\text{vac}}(\theta) \rangle = 0\) leads to

\[
\left( -\frac{d^2}{d\varphi^2} - \kappa \cos \varphi \right) f(\varphi) = \epsilon f(\varphi)
\]
\[ \kappa = \frac{2}{\pi} \bar{m} L B e^{-\pi/2 \mu L} \] (28)

where \( \epsilon = EL/\pi \). This is nothing but the Schrödinger equation in a potential \(-\kappa \cos \varphi\) (the quantum pendulum) whose ground state satisfies \( f(\varphi) = f(\varphi)^* = f(-\varphi) \). Eq. (28) is easily solved for an arbitrary value of \( \kappa \) numerically. Thus \( f(\varphi) \), and therefore the structure of the vacuum \( \Phi_{\text{vac}}(\theta) \), is controlled by \( \kappa \).

In two limits \( f(\varphi; \kappa) \) can be found analytically.

For \( \kappa \ll 1 \), \( \epsilon = -\frac{\kappa^2}{2} + O(\kappa^3) \)
\[ f(\varphi) = 1 + \kappa \cos \varphi - \kappa^2 \left( \frac{1}{4} - \frac{\cos 2\varphi}{8} \right) + O(\kappa^3) \]

For \( \kappa \gg 1 \), \( \epsilon = -\kappa + \sqrt{\frac{\kappa}{2}} \)
\[ f(\varphi) = e^{-2^{-3/2} \kappa^{1/2} \varphi^2} \quad (|\varphi| < \pi). \] (29)

From the definition of \( \kappa \) in (28) we see that the \( m \to 0 \) and \( L \to \infty \) limits do not commute with each other.

For later convenience we introduce
\[ F_r(u) = \int_0^{2\pi} d\varphi f(\varphi + \frac{1}{2}u)^* f(\varphi - \frac{1}{2}u) e^{ir\varphi} / \int_0^{2\pi} d\varphi |f(\varphi)|^2 \] (30)
which satisfies \( F_r(u)^* = F_{-r}(u) = F_r(-u) = F_r(u) \); we denote \( F_r = F_r(0) \) and \( F(u) = F_0(u) \), and recall that \( F_r \) depends implicitly on \( \kappa \) through \( f(\varphi; \kappa) \). Further we write \( \langle M \rangle_\theta = \langle \Phi_{\text{vac}}(\theta)|M|\Phi_{\text{vac}}(\theta) \rangle / \langle \Phi_{\text{vac}}(\theta)|\Phi_{\text{vac}}(\theta) \rangle \).

A useful formula is
\[ \langle e^{i r_1 (q_1^1 + q_1^2) + i r_2 (q_2^1 + q_2^2)} e^{-i u (p_1^1 - p_2^1)} \rangle_\theta = \langle e^{+i u (p_1^2 - p_2^2)} e^{i r_1 (q_1^1 + q_1^2) + i r_2 (q_2^1 + q_2^2)} \rangle_\theta = e^{+i r_1 \delta_1 + i r_2 \delta_2} e^{-\pi (r_1 + r_2)^2 / 2 \mu L} F_{r_1 - r_2}(u) e^{i (r_1 - r_2) u / 2} \]
\[ \tilde{\delta}_1(\theta) = -\theta - \tilde{\delta}(\theta), \quad \tilde{\delta}_2(\theta) = +\tilde{\delta}(\theta) \] (31)
where \( r_1 \) and \( r_2 \) are integers. Then expectation values of \( M_{ab} \)'s are
\[ \langle M_{ab} \rangle_\theta = -\delta_{ab} e^{i \tilde{\delta}(\theta)} L^{-1} B e^{-\pi/2 \mu L} F_1 \] (32)

From (29)
\[ F_1 = \begin{cases} \kappa \exp \left\{ -\frac{1}{2 \sqrt{\kappa}} \right\} & \text{for } \kappa \ll 1 \\ \exp \left\{ -\frac{1}{2 \sqrt{\kappa}} \right\} & \text{for } \kappa \gg 1. \end{cases} \] (33)
Figure 1: \( F_r = F_r(0) \) defined in (30) are plotted as functions of \( \kappa \). Their behavior at \( \kappa \ll 1 \) and \( \kappa \gg 1 \) is given by (33) and (45).

The behavior of \( F_1 \) with \( \kappa \) is depicted in Fig. 1.

We can now determine \( \mu_2 \). Taking the vacuum expectation value in the zero-mode sector, one finds

\[
\langle H_{\text{mass}} \rangle_{\text{zero modes}} = - \int \frac{dx}{L} e^{-\pi/2\mu L} F_1 \sum_{a=1}^2 \left\{ m_a e^{i\delta_a} N_{\mu}[e^{i\sqrt{2}\pi\chi_1}] N_{\mu_2}[e^{i\epsilon_a\sqrt{2}\pi\chi_2}] + \text{h.c.} \right\} \\
= \int \frac{dx}{L} e^{-\pi/2\mu L} F_1 \left\{ -i\sqrt{2}\pi(\sum_a m_a e^{i\delta_a} - \text{h.c.})\chi_1 \\
-2\pi\left(\sum_a \epsilon_a m_a e^{i\delta_a} + \text{h.c.}\right)\chi_1\chi_2 + 2\pi\bar{m}(\chi_1^2 + \chi_2^2) + O(\chi^3) \right\} \\
\]

(34)

In the \( SU(2) \) symmetric case \( m_a e^{i\delta_a} = m e^{-i\theta/2}(-1)^{\text{floor}(\theta+\pi/2)} \) and the \( \chi_1\chi_2 \) term vanishes. The \( \chi_1 \) term is proportional to \( m \sin \frac{1}{2}\theta \) so that a small shift of \( \chi_1 \) field is necessary.
for $\theta \neq 0$, however the correction to $\mu_1$ is minor. $\mu_2$ is determined by

$$
\mu_2^2 = \frac{4\pi}{L} \overline{m} \overline{B} e^{-\pi/2\mu L} F_1 = \frac{2\pi^2}{L^2} \kappa F_1.
$$

(35)

$F_1$ is a function of $\kappa$, and $\kappa$ depends on $\mu_2$. Employing (18), (28), and (33), and classifying cases according to whether $\kappa, \mu L, \mu_2 L$ are large or small one finds

$$
\mu_2 = \begin{cases} 
2\sqrt{2} \overline{m} e^{-\pi/2\mu L} & \text{for } 1 \gg \mu L \gg \mu_2 L \\
2\sqrt{2} \overline{m} \left(\frac{\mu L e^\gamma}{4\pi}\right)^{1/2} & \text{for } \mu L \gg 1 \gg \mu_2 L, \overline{m} L(\mu L)^{1/2} \ll 1 \\
(e^{2\gamma} \overline{m}^2 \mu)^{1/3} & \text{for } \mu L \gg \mu_2 L \gg 1, \overline{m} L(\mu L)^{1/2} \gg 1
\end{cases}
$$

(36)

which reproduces Coleman’s result \cite{5} in Minkowski spacetime when $\overline{m} L(\mu L)^{1/2} \gg 1$. However, if the $\overline{m} \to 0$ limit is taken with fixed $L$, then $\mu_2 = O(\overline{m})$, i.e. $\mu_2$ as a function of $m$ and $\theta$ has a smooth limit at $m \cos \frac{1}{2} \theta = 0$.

Furthermore, combining (32) and (35), one finds

$$
\langle M_{ab} \rangle_{\theta} = -\delta_{ab} e^{i\delta_a(\theta)} \frac{\mu_2^2}{4\pi \overline{m}}. 
$$

(37)

It follows that in the $SU(2)$ symmetric case

$$
\langle \overline{\psi}_a \psi_a \rangle_{\theta} = \begin{cases} 
-\frac{8}{\pi} m e^{-\pi/\mu L} \cos^2 \frac{1}{2} \theta & \text{for } \mu L \ll 1 \\
-\frac{2e^\gamma}{\pi^2} m \mu L \cos^2 \frac{1}{2} \theta & \text{for } \mu L \gg 1 \gg m L(\mu L)^{1/2} \gg \mu_2 L \gg 1 \\
-\left(\frac{e^{2\gamma} \overline{m}^2 \mu^2}{4\pi^3} \cos^4 \frac{1}{2} \theta\right)^{1/3} & \text{for } m L(\mu L)^{1/2} \gg \mu_2 L \gg 1
\end{cases}
$$

(38)

which agrees with Smilga’s estimate of the condensate in Minkowski spacetime \cite{18} when $m L(\mu L)^{1/2} \gg \mu_2 L \gg 1$. The singlet four-fermi operator $M_0 = \psi_+^{1+} \psi_2^+ \psi_2^\dagger \psi_+^\dagger$ is

$$
M_0(x) = e^{i(q_1^1 + q_1^2 + q_2^2)} \frac{1}{L^2} B(\mu L)^2 N_\mu \overline{N}_\mu e^{i\sqrt{8\pi} \chi_1(x)}
$$

(39)

apart from irrelevant operators which act as an identity operator on physical states. Hence

$$
\langle M_0(x) \rangle_{\theta} = e^{-i\theta} \frac{1}{L^2} B(\mu L)^2 e^{-2\pi/\mu L} \\
\sim e^{-i\theta} \left(\frac{\mu e^\gamma}{4\pi}\right)^2 \text{ for } \mu L \gg 1
$$

(40)
which is insensitive to the values of \( m_a \ll \mu \). It is non-vanishing in the massless, infinite volume limit; the associated chiral \( U(1) \) symmetry is broken by the anomaly. We note that the corresponding quantity in the \( N=1 \) case behaves similarly \[14, 15\]:

\[
\langle \bar{\psi} \psi \rangle_\theta = -e^{-i\theta L^{-1}B(\mu L)}e^{-\pi/\mu L} \text{ where } \mu = \sqrt{e^2/\pi}.
\]

We stress that from the definition (25), \( m - (\theta) \) and \( \bar{\delta}(\theta) \) are periodic in \( \theta \) with period \( 2\pi \). Therefore condensates (37) and (40) also are periodic functions of \( \theta \) with period \( 2\pi \).

The phase \( \bar{\delta}(\theta) \) has a discontinuity at \( \theta = \pm \pi \) where \( \mu^2 \) and \( \langle M_{aa} \rangle_\theta \) vanish. In this regard it is important to recognize the non-trivial dependence of the vacuum wave function \( f(\phi) \) on \( \theta \) through \( \kappa \).

When one of fermion masses, \( m_a \), vanishes, the Hamiltonian has an additional symmetry generated by the conserved gauge variant chiral charge

\[
\tilde{Q}_{aa}^5 = \int_0^L dx \tilde{j}_{aa}^{50} = p_1^a + p_2^a , \quad [H, \tilde{Q}_{aa}^5] = 0 . \tag{41}
\]

Consider the two cases: (a) \( m_1 \to 0, m_2 = m \) and (b) \( m_1 = m, m_2 \to 0 \). \( \bar{\delta} = 0 \) and \( -\theta \) in cases (a) and (b), respectively. In both cases \( \bar{m} = m \). Further, \( e^{i\beta\tilde{Q}_{aa}^5} \Phi(\theta, \varphi) = e^{2i\alpha\beta} \Phi(\theta + 2\beta, \varphi + \delta_a \cdot 2\beta) \). Combining these with (27), one finds that

\[
e^{i\beta\tilde{Q}} \Phi_{\text{vac}}(\theta) = e^{2i\alpha\beta} \Phi_{\text{vac}}(\theta + 2\beta) \tag{42}
\]

where \( \tilde{Q} = \tilde{Q}_{11}^5 \) or \( \tilde{Q}_{22}^5 \) in cases (a) or (b), respectively. This establishes the equivalence of all \( \theta \) vacua. In particular, \( E(\theta) \) is \( \theta \)-independent.

It is straightforward to evaluate various correlation functions. For correlators of \( M_{ab} \) with \( a \neq b \) we need, in addition to (13) and (11),

\[
\langle N_0 [e^{i\alpha\chi_{2+}(x)-i\alpha\chi_{2-}(x)+i\beta\chi_{2+}(y)-i\beta\chi_{2-}(y)}] \rangle
= B(\mu_2 L)(\alpha^2 + \beta^2)/4 \pi \ e^{-\alpha\beta \Delta(x-y;\mu_2, L) - \alpha\beta \Delta(x-y;0, L)} \ e^{-\frac{1}{2}(\alpha^2 + \beta^2)h(0;\mu_2, L) - \alpha\beta h(x-y;\mu_2, L)}
\]

\[
h(x; \mu, L) = \frac{1}{2L} \sum_{n \neq 0} \frac{\mu^2}{\omega_n(0)^2 \omega_n(\mu)} \ e^{ip_n x} \tag{43}
\]

Then, to leading order in \( m/\mu \),

\[
\langle M_{aa}(x)M_{aa}(0) \rangle_\theta = D(x; -, -) \ e^{2i\delta_a} \ e^{-2\pi/\mu L} F_2
\]

12
\begin{align}
\langle M_{ab}(x)M_{ba}(0) \rangle_{\theta,a\neq b} &= \begin{cases} 
D(x; -, +) e^{-i\theta} e^{-2\pi/\mu L} & \text{for } \mu \ll 1,
\end{cases} \\
\langle M_{ab}(x)M_{ba}(0) \rangle_{\theta,a\neq b} &= -D(x; -, +) e^{-i\theta} e^{-2\pi/\mu L} F \left( \frac{2\pi x}{L} \right) e^{-2\pi \{ h(0;\mu_2,L) - h(x;\mu_2,L) \}} \\
\langle M_{aa}^\dagger(x)M_{aa}(0) \rangle_{\theta} &= D(x; +, +) \\
\langle M_{ab}^\dagger(x)M_{ab}(0) \rangle_{\theta,a\neq b} &= D(x; +, +) F \left( \frac{2\pi x}{L} \right) e^{-2\pi \{ h(0;\mu_2,L) - h(x;\mu_2,L) \}} \\
\langle M_{aa}^\dagger(x)M_{bb}(0) \rangle_{\theta,a\neq b} &= D(x; +, -) e^{i(\delta_a - \delta_b)} F_2 \\
\text{where } D(x,\sigma_1,\sigma_2) &= \frac{1}{L^{2}} B(\mu L) B(\mu_2 L) e^{2\pi \sigma_1 \Delta(x,\mu, L) + 2\pi \sigma_2 \Delta(x,\mu_2, L)}. \quad (44)
\end{align}

All other correlators vanish. $F_2$ as a function of $\kappa$ is depicted in Fig. 1. In particular

$$F_2 = \begin{cases} 
\frac{3}{8} \kappa^2 \exp \left\{ - \sqrt{2/\kappa} \right\} & \text{for } \kappa \ll 1 \\
\exp \left\{ - \sqrt{2/\kappa} \right\} & \text{for } \kappa \gg 1.
\end{cases} \quad (45)$$

Note that except for the dominant correlators $\langle M_{aa}M_{bb} \rangle_{\theta,a\neq b}$ and $\langle M_{aa}^\dagger M_{aa} \rangle_{\theta}$ all others in (44) depend on the vacuum wave function $f(\varphi)$ and therefore the parameter $\kappa$.

The behavior at large distances depends on $\mu_2$. Recall that

$$2\pi \Delta(x;0,L) \sim - \ln \frac{2\pi x}{L} \quad \text{for } L \gg x$$

$$2\pi \Delta(x;\mu,L) \sim - \frac{\pi}{\mu L} + \sqrt{\frac{\pi}{2\mu x}} e^{-\mu x} \quad \text{for } \mu L \gg \mu x \gg 1$$

$$\sim - \ln \left( \frac{\mu x}{2} e^\gamma \right) \quad \text{for } \mu L \gg 1 \gg \mu x \quad (46)$$

Hence when $\mu_2 = 0$, i.e. $m \cos \frac{1}{2} \theta = 0$,

$$\left[ \begin{array}{c} \langle M_{aa}(x)M_{bb}(0) \rangle_{\theta,a\neq b} \\ \langle M_{aa}^\dagger(x)M_{aa}(0) \rangle_{\theta} \end{array} \right] = \left[ \begin{array}{c} e^{-i\theta} \\ 1 \end{array} \right] \frac{\mu e^\gamma}{8\pi^2 x} \quad \text{for } \mu L \gg \mu x \gg 1. \quad (47)$$

Hence the critical exponent is 1 in accordance with Affleck’s result on $R^1$. On the other hand, when $\mu \gg \mu_2 > 0$, $\langle M_{aa} \rangle_{\theta} \neq 0$ and the correlators decay exponentially:

$$\left[ \begin{array}{c} \langle M_{aa}(x)M_{bb}(0) \rangle_{\theta} - \langle M_{aa} \rangle_{\theta} \langle M_{bb} \rangle_{\theta} \\ \langle M_{aa}^\dagger(x)M_{bb}(0) \rangle_{\theta} - \langle M_{aa}^\dagger \rangle_{\theta} \langle M_{bb} \rangle_{\theta} \end{array} \right]$$

$$\sim e^{i\delta_a + i\delta_b} \frac{\mu \mu_2 e^{2\gamma}}{16\pi^2} \left\{ \sqrt{\frac{\pi}{2\mu x}} e^{-\mu x} + \epsilon_a \epsilon_b \sqrt{\frac{\pi}{2\mu_2 x}} e^{-\mu_2 x} \right\} \quad \text{for } \mu L \gg \mu_2 L \gg \mu_2 x \gg 1. \quad (48)$$

At short distances

$$\langle M_{aa}(x)M_{bb}(0) \rangle_{\theta,a\neq b} = e^{-i\theta} \left( \frac{\mu e^\gamma}{4\pi} \right)^2$$

13
\[
\langle M_{aa}^\dagger(x) M_{aa}(0) \rangle_\theta = \frac{1}{4\pi^2 x^2}
\]

(49)

for \( \mu L \gg 1 \gg \mu x \), \( \mu_2 = 0 \) or \( \mu L \gg \mu_2 L \gg 1 \gg \mu x \gg \mu_2 x \)

For the chiral U(1) condensate \( \langle \rangle \),

\[
\langle M_0^\dagger(x) M_0(0) \rangle_\theta = \frac{1}{L^4} B(\mu L)^4 e^{+8\pi\Delta(x;\mu,L)}
\]

\[
= \begin{cases} 
\left( \frac{\mu e^\gamma}{4\pi} \right)^4 \exp \left( \sqrt{\frac{8\pi}{\mu x}} e^{-\mu x} \right) & \text{for } \mu x \gg 1, \mu L \gg 1, x/L \ll 1 \\
\frac{1}{(2\pi x)^4} & \text{for } \mu x \ll 1, \mu L \gg 1, x/L \ll 1
\end{cases}
\]

(50)

The correlator \( \langle M_0^\dagger(x) M_0(0) \rangle_\theta - |\langle M_0 \rangle_\theta|^2 \) shows an exponential fall-off.

An interesting extension of this work comes from the fact that quantum spin \( \frac{1}{2} \) antiferromagnet chains are almost equivalent to an \( N = 2 \) massless Schwinger model at strong coupling. Our result supports this correspondence, where staggered magnetization corresponds to the chiral condensate. The critical exponent is \( \eta = 1 \), as pointed out by Haldane \cite{Haldane1983} and Affleck \cite{Affleck1983} who converted spin chains to nonlinear sigma models and found that there is no long-range-order in the infinite volume limit. Our \( \chi_2 \) field corresponds to the gapless (spinon) mode in spin chains\cite{Guhr1991}. We also stress that our result is valid for arbitrary \( L \), and thus should be very useful for investigating finite size effects in spin chains.

To conclude, we have analyzed the two-flavor massive Schwinger model on a circle. We have reduced the system to a one-dimensional quantum mechanics problem defined by (23). The approximation is justified for \( \mu L \gg 1 \) but with an arbitrary \( \tilde{m}L \). For \( \mu L < 1 \) excitations to higher \( s \) states need to be taken into account, and we expect the system to reduce to a two-dimensional quantum mechanics problem. Recently Shifman and Smilga \cite{Shifman1998} have proposed that with twisted boundary condition for fermions, excitations with fractional topological number (fractons) become essential. Our analysis should straightforwardly generalize to cases with \( N \) flavors of fermions with arbitrary boundary conditions. We plan to return to these points in separate publications.
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