CONTINUITY OF THE SOLUTION MAPPINGS TO PARAMETRIC GENERALIZED NON-WEAK VECTOR KY FAN INEQUALITIES

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Abstract. In this paper, by using the nonlinear scalarization method and under some new assumptions, which do not involve any information on the solution set, we establish the continuity of solution mappings of parametric generalized non-weak vector Ky Fan inequality with moving cones. The results are new and improve corresponding ones in the literature. Some examples are given to illustrate our results.

1. Introduction. It is well known that the Ky Fan inequality is a very general mathematical model, which embraces the formats of several disciples, as equilibrium problems of economics, game theory, (vector) optimization problems, (vector) variational inequality problems and so on (see [9, 11]). Since the Ky Fan inequality was introduced in [9], it has been extended and generalized to vector-valued or set-valued mappings. The Ky Fan inequality for a vector valued or set-valued mapping is known as a generalized Ky Fan inequality. In recent years, several researchers have drawn their attention to the semicontinuity of the solution mappings of generalized versions of the parametric Ky Fan inequality. Many results on this direction can be found, e.g., see [1, 2, 5, 6, 8, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24], where the generalized Ky Fan inequalities are called vector equilibrium problems or generalized systems.

Among those papers, we observe that the (linear and nonlinear) scalarization technique is an effective approach to deal with the upper and lower semicontinuity of solution mappings to parametric vector variational inequalities and parametric vector equilibrium problems. By using the linear scalarization method, Cheng and Zhu [8] investigated the upper and lower semicontinuity of the solution mappings to the parametric weak vector variational inequalities in the finite-dimensional spaces. By virtue of the ideas of Cheng and Zhu [8] and based on a theorem of Berge [4] saying that the union of a family of lower semicontinuous set-valued mappings is lower semicontinuous, Gong [12], Chen et al. [6], and Li et al. [13] have extended the lower semicontinuity.

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semicontinuity results of Cheng and Zhu\textsuperscript{8} to parametric weak vector equilibrium problems under a suitable strict cone-monotonicity assumption. Li and Fang\textsuperscript{17}, Peng et al. \textsuperscript{20}, Peng and Chang \textsuperscript{19}, Chen and Huang\textsuperscript{5}, and Wang and Li\textsuperscript{24} have used the linear scalarization approach to improve the lower semicontinuity results in \textsuperscript{6,12,18} by weakening the strict cone-monotonicity assumption.

It is worth noting that the linear scalarization approach to the semicontinuity of solution mappings in \textsuperscript{5,6,8,12,17,18,19,20,24} requires (generalized) cone-convexity or strict cone-monotonicity of the objective functions. To avoid these assumptions, nonlinear scalarization approaches have been used for discussing the stability analysis of generalized Ky Fan inequalities. Namely, Sach\textsuperscript{21} used some nonlinear scalarization functions (generalized versions of Gerstewizt’s function) to discuss the lower semicontinuity of the solution mappings of parametric generalized weak Ky Fan inequalities. The paper \textsuperscript{21} is a path-breaking work on the stability analysis of generalized Ky Fan inequalities. Afterwards, Sach and Tuan \textsuperscript{23} applied the functions to discuss more generalized cases of Ky Fan inequalities, and obtained the upper and lower semicontinuity of the solution mappings to the problems. Very recently, Sach and Minh \textsuperscript{22} also applied the functions defined in \textsuperscript{21} to study the continuity of the solution mappings to parametric generalized non-weak vector Ky Fan inequalities.

However, the results in \textsuperscript{21,22,23} require that the objective functions of the discussed problems have compact or cone-closed values, since the definition and propositions of the nonlinear scalarization function require these assumptions. It may restrict its application scope. Furthermore, the assumptions in \textsuperscript{22,23} involve information about the solution set. Obviously, it is not reasonable from the practical view. To avoid the assumptions involving information about the solution set, in this paper, we establish some new assumptions to obtain the continuity of the solution mappings to parametric generalized non-weak Ky Fan inequalities by using a nonlinear scalarization function defined in \textsuperscript{7}, which is different from the function defined in \textsuperscript{22,23}.

The rest of the paper is organized as follows. In Section 2, we recall some basic concepts and preliminary results. In Section 3, under some suitable assumptions, which do not involve any information about the solution set, we give some sufficient conditions for the continuity of the solution mappings to parametric generalized non-weak Ky Fan inequalities. Meanwhile, some examples are given to illustrate the obtained results.

2. Preliminaries and notations. Throughout this paper, let $T$ and $X$ be Hausdorff topological spaces, and $A_i : T \times K \rightarrow 2^K, i = 0, 1$, be mappings with nonempty values. Let $Y$ be a locally convex topological vector space, $C : T \times X \times X \rightarrow 2^Y$ be a mapping such that each value of $C$ is a proper, closed and convex cone with nonempty interior, and $F : T \times X \times X \rightarrow 2^Y$ be a mapping with nonempty values. For each $t \in T$, we consider the following parametric generalized non-weak vector Ky Fan inequality problems with moving cones:

**Problem** $(P_1^t)$. Find a point $x \in X$ such that $x \in A_0(t,x)$ and for all $\eta \in A_1(t,x)$,

$$F(t,x,\eta) \cap -C(t,x,\eta) \neq \emptyset.$$

**Problem** $(P_2^t)$. Find a point $x \in X$ such that $x \in A_0(t,x)$ and for all $\eta \in A_1(t,x)$,

$$F(t,x,\eta) \subset -C(t,x,\eta).$$
For each \( t \in T \), we denote by \( S^1(t) \) and \( S^2(t) \) the solution set of problems \((P^1_t)\) and \((P^2_t)\), respectively. Throughout this paper, we assume that \( S^1(t) \neq \emptyset \) and \( S^2(t) \neq \emptyset \) for all \( t \in T \). In this paper, we will discuss the continuity of the solution mappings \( S^j(\cdot) \) \((j = 1, 2)\) as set-valued mappings from the set \( T \) to \( X \).

Suppose that \( G : T \to 2^X \) is a set-valued mapping, and \( t \in A \) be given.

**Definition 2.1.** (see [3])

(i): \( G \) is called lower semicontinuous (l.s.c) at \( \bar{t} \) iff for any open set \( V \subseteq X \) with \( V \cap G(\bar{t}) \neq \emptyset \), there exists a neighborhood \( N(\bar{t}) \) of \( \bar{t} \) such that \( G(t) \cap V \neq \emptyset \), for all \( t \in N(\bar{t}) \).

(ii): \( G \) is called upper semicontinuous (u.s.c) at \( \bar{t} \) iff for any open set \( V \subseteq X \) with \( G(\bar{t}) \subseteq V \), there exists a neighborhood \( N(\bar{t}) \) of \( \bar{t} \) such that \( G(t) \subseteq V \), for all \( t \in N(\bar{t}) \).

We say \( G(\cdot) \) is l.s.c (resp. u.s.c) on \( T \), if and only if it is l.s.c (resp. u.s.c) at each \( \bar{t} \in T \). \( G(\cdot) \) is said to be continuous on \( T \) if and only if it is both l.s.c and u.s.c on \( T \).

**Proposition 1.** (see [3, 10])

(i): \( G \) is l.s.c at \( \bar{t} \) and only if for any net \( \{t_\alpha\} \subseteq T \) with \( t_\alpha \to \bar{t} \) and any \( \bar{x} \in G(\bar{t}) \), there exists a net \( \{x_\alpha\} \subseteq G(t_\alpha) \) such that \( x_\alpha \to \bar{x} \).

(ii): If \( G \) has compact values (i.e., \( G(t) \) is a compact set for each \( t \in T \)), then \( G \) is u.s.c at \( \bar{t} \) and only if for any net \( \{t_\alpha\} \subseteq T \) with \( t_\alpha \to \bar{t} \) and any \( \bar{x} \in G(t_\alpha) \), there exist \( \bar{x} \in G(\bar{t}) \) and a subnet \( \{x_\beta\} \) of \( \{x_\alpha\} \) such that \( x_\beta \to \bar{x} \).

**Definition 2.2.** (see [7]) Let \( e : T \times X \times X \to Y \) be a vector-valued mapping and for any \((t, x, \eta) \in T \times X \times X, e(t, x, \eta) \in \text{int} \, C(t, x, \eta) \). The nonlinear scalarization function \( \xi : T \times X \times X \times Y \to \mathbb{R} \) is defined by

\[
\xi(t, x, \eta; z) = \min \{\lambda \in \mathbb{R} : z \in \lambda e(t, x, \eta) - C(t, x, \eta)\}.
\]

**Proposition 2.** (see [7]) The function \( \xi \) defined in Definition 2.2 satisfies the following propositions:

(i): \( \xi(t, x, \eta; z) \leq r \iff z \in r e(t, x, \eta) - C(t, x, \eta) \);

(ii): \( \xi(t, x, \eta; z) > r \iff z \notin r e(t, x, \eta) - C(t, x, \eta) \).

3. Main results. In this section, we discuss the continuity of the solution mappings \( S^j(\cdot) \) \((j = 1, 2)\) as set-valued mappings from the set \( T \) to \( X \).

**Lemma 3.1.** Let \( \hat{\psi} : T \to 2^X \) be a mapping with nonempty values and \( f : T \times X \to \mathbb{R} \) be a function. Let \( \hat{S} : T \to 2^X \) be defined by

\[
\hat{S}(t) := \{x \in \hat{\psi}(t) : \hat{f}(t, x) \leq 0\}
\]

and let \( t_0 \in \text{dom} \hat{S} \). Assume that

(i): \( \hat{\psi} \) is continuous at \( t_0 \) with compact values;

(ii): for each \( x_0 \in \hat{\psi}(t_0) \), if for any nets \( \{x_\alpha\} \) with \( x_\alpha \to x_0 \) and \( \{t_\alpha\} \) with \( t_\alpha \to t_0 \), one has \( \hat{f}(t_0, x_0) > 0 \Rightarrow \exists \alpha_0 \), such that \( f(t_\alpha_0, x_\alpha_0) > 0 \);

(iii): for each \( x_0 \in \hat{\psi}(t_0) \), if for any nets \( \{x_\alpha\} \) with \( x_\alpha \to x_0 \) and \( \{t_\alpha\} \) with \( t_\alpha \to t_0 \), one has \( \hat{f}(t_0, x_0) \leq 0 \Rightarrow \exists \alpha_0 \), such that \( f(t_\alpha_0, x_\alpha_0) \leq 0 \).

Then \( \hat{S}(\cdot) \) is continuous at \( t_0 \).
Proof. Suppose that \( \hat{S}(\cdot) \) is not u.s.c at \( t_0 \). Then there exists an open set \( V \subseteq X \) satisfying \( \hat{S}(t_0) \subseteq V \), nets \( \{t_\alpha\} \) with \( t_\alpha \to t_0 \) and \( x_\alpha \in \hat{S}(t_\alpha) \), such that \( x_\alpha \notin V \), \( \forall \alpha \).

Since \( x_\alpha \in \hat{\psi}(t_\alpha) \) and \( \hat{\psi}(\cdot) \) is u.s.c at \( t_0 \) with compact values on \( T \), by Proposition 1 (ii), there exist \( x_0 \in \hat{\psi}(t_0) \) and a subnet \( \{x_\beta\} \) of \( \{x_\alpha\} \) such that \( x_\beta \to x_0 \). Since \( x_\alpha \notin V \) for all \( \alpha \), \( x_\beta \notin V \) and \( x_0 \notin V \). In particular, \( x_0 \notin \hat{S}(t_0) \), i.e., \( f(t_0, x_0) > 0 \). By assumption (ii), there exists an index \( \beta_0 \) such that \( \hat{f}(t_{\beta_0}, x_{\beta_0}) > 0 \). This contradicts the above condition that \( x_\alpha \in \hat{S}(t_\alpha) \) for all \( \alpha \).

Suppose that \( \hat{S}(\cdot) \) is not l.s.c at \( t_0 \). Then by Proposition 1 (i), there exist a net \( \{t_\alpha\} \) with \( t_\alpha \to t_0 \) and \( x_0 \in \hat{S}(t_0) \) such that for any \( x_\alpha \in \hat{S}(t_\alpha) \), we have \( x_\alpha \not\to x_0 \).

From \( x_0 \in \hat{S}(t_0) \), we have \( x_0 \in \hat{\psi}(t_0) \). As \( \hat{\psi}(\cdot) \) is l.s.c at \( t_0 \), there exists \( \bar{x}_\alpha \in \hat{\psi}(t_\alpha) \) such that \( \bar{x}_\alpha \to x_0 \). By the above contradiction assumption, there must exist subnet \( \{\bar{x}_\beta\} \subseteq \{x_\alpha\} \) such that, \( \forall \beta \) with \( \bar{x}_\beta \not\in \hat{S}(t_\beta) \), i.e.,

\[
\hat{f}(t_\beta, \bar{x}_\beta) > 0. \tag{1}
\]

Since \( x_0 \in \hat{S}(t_0) \), we have \( \hat{f}(t_0, x_0) \leq 0 \). By assumption (iii), there exists an index \( \beta_0 \) such that \( \hat{f}(t_{\beta_0}, \bar{x}_{\beta_0}) \leq 0 \), which contradicts (1).

Remark 1. (i): In [22], Sach and Minh used a key assumption which includes the solution set information to obtain the continuity of set-valued mapping \( \hat{S}(\cdot) \). The main advantage of assumptions (ii) and (iii) in Lemma 3.1 is that they do not require any information on solution set \( \hat{S}(t) \) for each \( t \in T \). Moreover, their results require that the function \( \hat{f} \) is continuous. However, our results do not need the assumption. Example 3.3 below can illustrate this case.

(ii): Obviously, for each \( x_0 \in \hat{\psi}(t_0) \), the assumptions (ii) and (iii) can be ensured by the upper semicontinuity and lower semicontinuity of the real function \( \hat{f} \) at \( (x_0, t_0) \), respectively. Therefore, assumption (ii) is weaker than assumption (ii) in Lemma 4.1 of [23].

(iii): Assumption (iii) of Lemma 3.1 is different from of the assumptions \( \hat{H}_1 \) and \( \hat{H}_2 \) in Lemma 4.2 of [23].

The following example is given to illustrate that the assumption (ii) of Lemma 3.1 is essential.

Example 3.1. Let \( T = [0, 1], X = \mathbb{R} \). For each \( t \in T \), let \( \hat{\psi}(t) = [-1, 1] \). Let \( t_0 = 0 \). For each \((t, x) \in T \times X \), we define:

\[
\hat{f}(t, x) = \begin{cases} x^2, & \text{if } t = t_0, \\ -tx, & \text{if } t \in T \setminus t_0. \end{cases}
\]

It is easy to check that the assumptions (i) and (iii) are satisfied. One has \( \hat{S}(t_0) = \{0\}, \hat{S}(t) = [0, 1], t \in (0, 1) \). Hence \( \hat{S}(\cdot) \) is l.s.c at \( t_0 \), but it is not u.s.c at \( t_0 \). It is equally clear that the assumption (ii) is violated. Indeed, there exist \( x_0 = \frac{1}{2} \) and nets \( \{t_\alpha\} \subset (0, \frac{1}{2}) \) with \( t_\alpha \to t_0 \), \( \{x_\alpha\} \subset (0, 1) \) with \( x_\alpha \to x_0 \), one has \( \hat{f}(t_0, x_0) = x_0^2 = \frac{1}{4} > 0 \). However, for all \( \alpha \), one has \( \hat{f}(t_\alpha, x_\alpha) = -t_\alpha x_\alpha < 0 \). Hence, the assumption (ii) of Lemma 3.1 is essential.

The following example is given to illustrate that the assumption (iii) of Lemma 3.1 is essential.
Example 3.2. Let $T = [0, 1]$, $X = \mathbb{R}$. For each $t \in T$, let $\hat{\psi}(t) = [-1, 1]$. Let $t_0 = 0$. For each $(t, x) \in T \times X$, we define:

$$
\hat{f}(t, x) = \begin{cases}
-x(t + \frac{1}{2}) & \text{if } t = t_0, \\
tx, & \text{if } t \in T \setminus \{t_0\}.
\end{cases}
$$

It is easy to check that the assumptions (i) and (ii) are satisfied. One has $\hat{S}(t_0) = [-1, -\frac{1}{2}] \cup [0, 1]$, $\hat{S}(t) = [0, 1]$, $t \in (0, 1]$. Hence $\hat{S}(\cdot)$ is u.s.c at $t_0$, but is not l.s.c at $t_0$. It is equally clear that the assumption (iii) is violated. Indeed, there exist $x_0 = -\frac{1}{2}$ and nets $\{t_\alpha\} \subset (0, \frac{1}{2})$ with $t_\alpha \to t_0$ and $\{x_\alpha\} \subset (-1, 0)$ with $x_\alpha \to x_0$, one has $\hat{f}(t_0, x_0) = -x_0(x_0 + \frac{1}{2}) = 0$. However, for all $\alpha$, one has $\hat{f}(t_\alpha, x_\alpha) = -t_\alpha x_\alpha > 0$. Thus, the assumption (iii) of Lemma 3.1 is essential.

The example is given to show that Lemma 3.1 is applicable, but Lemma 4.1 of [22] is not applicable.

Example 3.3. Let $T = [0, 1]$, $X = \mathbb{R}$. For each $t \in T$, let $\hat{\psi}(t) = [-1, 1]$. Let $t_0 = 0$. For each $(t, x) \in T \times X$, we define:

$$
\hat{f}(t, x) = \begin{cases}
-x^2, & \text{if } t = t_0, \\
tx^2, & \text{if } t \in T \setminus \{t_0\}.
\end{cases}
$$

It is easy to check that the assumptions (i)-(iii) are satisfied. It follows from a direct computation that $\hat{S}(t) = [-1, 1]$ for each $t \in T$. Since $f$ is not continuity at $(t_0, x)$ for each $x \in [-1, 1] \setminus \{0\}$, Lemma 4.1 of [22] is not applicable here.

From Remark 1 and Lemma 3.1 we can easily get the following corollary.

**Corollary 1.** Let $\hat{\psi} : T \to 2^X$ be a mapping with nonempty values and $\hat{f} : T \times X \to \mathbb{R}$ be a function. Let $\hat{S} : T \to 2^X$ be defined by

$$
\hat{S}(t) := \{x \in \hat{\psi}(t) : \hat{f}(t, x) \leq 0\}
$$

and let $t_0 \in \text{dom}\hat{S}$. Assume that

(i): $\hat{\psi}$ is continuous at $t_0$ with compact values;

(ii): for each $x_0 \in \hat{\psi}(t_0)$, $\hat{f}$ is continuity at $(t_0, x_0)$.

Then $\hat{S}(\cdot)$ is continuous at $t_0$.

For each $(t, x) \in T \times X$, we set

$$
f^1(t, x) = \sup_{\eta \in A_1(t, x)} \inf_{z \in F(t, x, \eta)} \xi(t, x, \eta; z);
$$

and

$$
f^2(t, x) = \sup_{\eta \in A_1(t, x)} \sup_{z \in F(t, x, \eta)} \xi(t, x, \eta; z).
$$

Denote by $\psi(t)$ the fixed points of $A_0(t, \cdot)$:

$$
\psi(t) = \{x \in X : x \in A_0(t, x)\}, \ t \in T.
$$

In the sequel, we always assume that $t_0 \in \text{dom}\hat{S}^j$, $j = 1, 2$.

As a direct consequence of Lemma 3.1 and Proposition 2, we can get the following results on the continuity of $\hat{S}^j(\cdot)$, $j = 1, 2$.

**Theorem 3.2.** Suppose the following conditions are satisfied:

(i): $\hat{\psi}$ is continuous at $t_0$ with compact-values;

(ii): for each $x_0 \in \hat{\psi}(t_0)$, if for any nets $\{x_\alpha\}$ with $x_\alpha \to x_0$ and $\{t_\alpha\}$ with $t_\alpha \to t_0$, one has $f^j(t_0, x_0) > 0 \Rightarrow \exists \alpha_0$, such that $f^j(t_\alpha, x_\alpha) > 0$;
(iii): for each \( x_0 \in \psi(t_0) \), if for any nets \( \{x_\alpha\} \) with \( x_\alpha \to x_0 \) and \( \{t_\alpha\} \) with \( t_\alpha \to t_0 \), one has \( f^1(t_0, x_0) \leq 0 \Rightarrow \exists \alpha_0 \), such that \( f^1(t_{\alpha_0}, x_{\alpha_0}) \leq 0 \);

(iv): \( F \) has compact valued on \( T \times X \times X \).

Then \( S^1(\cdot) \) is continuous at \( t_0 \).

**Proof.** For each \( t \in T \), we prove that the following equality:

\[
S^1(t) = \{ x \in \psi(t) : f^1(t, x) \leq 0 \}.
\]

Indeed, for each \( t \in T \) and \( x \in S^1(t) \), then \( x \in A_0(t, x) \) and for all \( \eta \in A_1(t, x) \), we have

\[
F(t, x, \eta) \cap -C(t, x, \eta) \neq \emptyset.
\]

It implies that \( \inf_{z \in F(t, x, \eta)} \xi(t, x, \eta; z) \leq 0 \) for all \( \eta \in A_1(t, x) \) by Proposition 2 (i), i.e., \( f^1(t, x) \leq 0 \). Since \( \psi(t) = \{ x \in X : x \in A_0(t, x) \} \), \( x \in \psi(t) \). By the arbitrariness of \( x \in S^1(t) \), we have \( S^1(t) \subseteq \{ x \in \psi(t) : f^1(t, x) \leq 0 \} \).

On the other hand, let \( A = \{ x \in \psi(t) : f^1(t, x) \leq 0 \} \). For each \( t \in T \), we need prove \( A \subseteq S^1(t) \). Indeed, for each \( t \in T \) and \( x \in A \), we have \( x \in \psi(t) = \{ x \in X : x \in A_0(t, x) \} \) and

\[
\sup_{\eta \in A_1(t, x)} \inf_{z \in F(t, x, \eta)} \xi(t, x, \eta; z) \leq 0.
\]

Thus \( x \in A_0(t, x) \) and for all \( \eta \in A_1(t, x) \), there exists \( z_0 \in F(t, x, \eta) \) such that \( z_0 \in -C(t, x, \eta) \) by assumption (iv), i.e., \( F(t, x, \eta) \cap -C(t, x, \eta) \neq \emptyset \). Thus, \( x \in S^1(t) \) and (2) holds. By virtue of Lemma 3.1 and (2), we get that \( S^1(\cdot) \) is continuous at \( t_0 \).

Theorem 3.2 may be false if the assumption (iv) is violated.

**Example 3.4.** Let \( T = [0, 1] \), \( X = \mathbb{R} \) and \( Y = \mathbb{R}^2 \). For each \( (t, x, \eta) \in T \times X \times X \), let \( C(t, x, \eta) = \mathbb{R}^2_+ \). For each \( t \in T \), let \( \psi(t) = [-1, 1] \). Let \( t_0 = 0 \). For each \( (t, x, \eta) \in T \times X \times X \), we define:

\[
F(t, x, \eta) = \begin{cases} 
(-1, 0) \times (0, 1), & \text{if } t = t_0, \\
[-1, 0] \times [0, 1], & \text{if } t \in T \setminus t_0.
\end{cases}
\]

It is easy to check that the assumptions (i)-(iii) are satisfied, since \( f^1(t, x) = 0 \), for all \( (t, x) \in T \times X \). Obviously, the assumption (iv) is violated. It follows from a direct computation that \( S^1(t_0) = \emptyset \), and \( S^1(t) = [-1, 1] \) for each \( t \in T \setminus t_0 \). Thus \( S^1(\cdot) \) is not u.s.c. at \( t_0 \).

**Theorem 3.3.** Suppose the following conditions are satisfied:

(i): \( \psi \) is continuous at \( t_0 \) with compact values;

(ii): for each \( x_0 \in \psi(t_0) \), if for any nets \( \{x_\alpha\} \) with \( x_\alpha \to x_0 \) and \( \{t_\alpha\} \) with \( t_\alpha \to t_0 \), one has \( f^1(t_0, x_0) > 0 \Rightarrow \exists \alpha_0 \), such that \( f^1(t_{\alpha_0}, x_{\alpha_0}) > 0 \);

(iii): for each \( x_0 \in \psi(t_0) \), if for any nets \( \{x_\alpha\} \) with \( x_\alpha \to x_0 \) and \( \{t_\alpha\} \) with \( t_\alpha \to t_0 \), one has \( f^2(t_0, x_0) \leq 0 \Rightarrow \exists \alpha_0 \), such that \( f^2(t_{\alpha_0}, x_{\alpha_0}) \leq 0 \).

Then \( S^2(\cdot) \) is continuous at \( t_0 \).

**Proof.** Similarly to the proofs of Theorem 3.2, we have

\[
S^2(t) = \{ x \in \psi(t) : f^2(t, x) \leq 0 \}.
\]

Therefore, by virtue of Lemma 3.1, we get that \( S^2(\cdot) \) is continuous at \( t_0 \).
Example 3.5. Suppose the following conditions are satisfied:

Corollary 3. It is easy to check that the assumptions (i)-(iv) are satisfied. It follows from direct
of Theorems 3.2 and 3.3.

and 3 below, we will give some equivalent assumptions to assumptions (ii) and (iii)
P can be satisfied, without the data of problems (Pj), j = 1, 2. However, our results do not need the assumption.

Assumptions (ii) and (iii) of Theorems 3.2 and 3.3 are blanket assumptions that
can be satisfied, without the data of problems (Pj), j = 1, 2. In Corollaries 2
and 3 below, we will give some equivalent assumptions to assumptions (ii) and (iii)
of Theorems 3.2 and 3.3.

Corollary 2. Suppose the following conditions are satisfied:

(i): \( \psi \) is continuous at \( t_0 \) with compact values;
(ii): for each \( x_0 \in \psi(t_0) \), if there exists \( \eta_0 \in A_1(t_0, x_0) \), such that for any
nets \( \{x_\alpha\} \) with \( x_\alpha \to x_0 \) and \( \{t_\alpha\} \) with \( t_\alpha \to t_0 \), one has \( F(t_0, x_0, \eta_0) \subseteq
Y \setminus C(t_0, x_0, \eta_0) \Rightarrow \exists \alpha_0 \) and \( \eta_{0\alpha} \in A_1(t_\alpha, x_\alpha) \) such that \( F(t_\alpha, x_\alpha, \eta_{0\alpha}) \subseteq
Y \setminus C(t_\alpha, x_\alpha, \eta_{0\alpha}) \);
(iii): for each \( x_0 \in \psi(t_0) \), if for any nets \( \{x_\alpha\} \) with \( x_\alpha \to x_0 \) and \( \{t_\alpha\} \) with
\( t_\alpha \to t_0 \), one has \( F(t_0, x_0, \eta) \cap C(t_0, x_0, \eta) \neq \emptyset \), \( \forall \eta \in A_1(t_0, x_0) \Rightarrow \exists \alpha_0 \),
such that \( F(t_\alpha, x_\alpha, \eta) \cap C(t_\alpha, x_\alpha, \eta) \neq \emptyset \), \( \forall \eta \in A_1(t_\alpha, x_\alpha) \);
(iv): \( F \) has compact valued on \( T \times X \times X \).

Then \( S^1(\cdot) \) is continuous at \( t_0 \).

Proof. By Proposition 2 (i) and (ii), we get the assumptions (ii) and (iii) in Corollary 2 are equivalent to the assumptions (ii) and (iii) in Theorem 3.2 respectively. Therefore, by virtue of Theorem 3.2 we have that \( S^1(\cdot) \) is continuous at \( t_0 \). \hfill \( \square \)

The following example is given to illustrate Corollary 2

Example 3.5. Let \( T = [0, 1] \), \( X = \mathbb{R} \) and \( Y = \mathbb{R}^2 \). For each \( (t, x, \eta) \in T \times X \times X \),
let \( C(t, x, \eta) = \mathbb{R}^+_x \). For each \( (t, x) \in T \times X \), \( A_1(t, x) = [0, 1] \). For each \( t \in T \), let
\( \psi(t) = [-1, 1] \). Let \( t_0 = 0 \). For each \( (t, x, \eta) \in T \times X \times X \), we define:
\[
F(t, x, \eta) = \begin{cases} 
[-1, 1] \times [-1, 1], & \text{if } t = t_0, \\
t \cdot \left[-\frac{1}{2}, \eta\right] \times [-1, 1 + x], & \text{if } t \in T \setminus t_0.
\end{cases}
\]

It is easy to check that the assumptions (i)-(iv) are satisfied. It follows from direct
computation that \( S^1(t) = [-1, 1] \) for each \( t \in T \). Thus \( S^1(\cdot) \) is continuous at \( t_0 \).

Corollary 3. Suppose the following conditions are satisfied:

(i): \( \psi \) is continuous at \( t_0 \) with compact values;
(ii): for each \( x_0 \in \psi(t_0) \), if there exists \( \eta_0 \in A_1(t_0, x_0) \) and \( z_0 \in F(t_0, x_0, \eta_0) \),
such that for any nets \( \{x_\alpha\} \) with \( x_\alpha \to x_0 \) and \( \{t_\alpha\} \) with \( t_\alpha \to t_0 \), one has
\( z_0 \notin -C(t_0, x_0, \eta_0) \Rightarrow \exists \alpha_0 \), \( \eta_{0\alpha} \in A_1(t_\alpha, x_\alpha) \) and \( z_{\alpha_0} \in F(t_\alpha, x_\alpha, \eta_{0\alpha}) \),
such that \( z_{\alpha_0} \notin -C(t_\alpha, x_\alpha, \eta_{0\alpha}) \);}
Example 3.6. Let
$$0 \leq \psi(t_0),$$
if for any nets \( \{x_\alpha\} \) with \( x_\alpha \to x_0 \) and \( \{t_\alpha\} \) with
\( t_\alpha \to t_0 \), one has \( F(t_\alpha, x_\alpha, \eta) \subseteq \mathcal{C}(t_\alpha, x_\alpha, \eta), \forall \eta \in \mathcal{A}(t_\alpha, x_\alpha, \eta) \Rightarrow \exists \alpha_0 \), such that \( F(t_{\alpha_0}, x_{\alpha_0}, \eta) \subseteq \mathcal{C}(t_{\alpha_0}, x_{\alpha_0}, \eta), \forall \eta \in \mathcal{A}(t_{\alpha_0}, x_{\alpha_0}, \eta) \).

Then \( S^2(\cdot) \) is continuous at \( t_0 \).

Proof. The proof is similar to that of Corollary 2.

Next, we give an example to illustrate Corollary 3.

Example 3.6. Let \( T = [0, 1] \), \( X = \mathbb{R} \) and \( Y = \mathbb{R}^2 \). For each \( (t, x, \eta) \in T \times X \times X \), let \( C(t, x, \eta) = \mathbb{R}^2_+ \). For each \( (t, x) \in T \times X \), \( A_1(t, x) = [0, 1] \). For each \( t \in T \), let \( \psi(t) = [-1, 1] \). Let \( t_0 = 0 \). For each \( (t, x, \eta) \in T \times X \times X \), we define:

\[
F(t, x, \eta) = \begin{cases}
(-1, 0) \times (-1, 0), & \text{if } t = t_0, \\
(t, [-1, \eta - 1] \times [-2, x - 1]), & \text{if } t \in T \setminus \{t_0\}.
\end{cases}
\]

It is easy to check that the assumptions (i)-(iii) are satisfied. It follows from a direct computation that \( S^2(t) = [-1, 1] \) for each \( t \in T \). Thus \( S^2(\cdot) \) is continuous at \( t_0 \).

Corollary 4. Let \( c : T \times X \times X \) be the continuous selection of the set-valued mapping \( C(\cdot) \), i.e., \( c \) is continuous and \( c(t, x, \eta) \subseteq C(t, x, \eta) \) for all \( (t, x, \eta) \in T \times X \times X \). Define a set-valued mapping \( W : X \to 2^Y \) by \( W(t, x, \eta) = T \times X \times X \setminus \mathcal{C}(t, x, \eta) \), for \( (t, x, \eta) \in T \times X \times X \). Suppose the following conditions are satisfied:

(i): \( \psi \) is continuous at \( t_0 \) with compact values;

(ii): \( A_1 \) is continuous and compact valued on \( t_0 \times X \);

(iii): \( F \) is continuous and compact valued on \( T \times X \times X \);

(iv): \( W \) is u.s.c on \( T \times X \times X \);

(v): \( C \) is l.s.c on \( T \times X \times X \).

Then \( S^2(\cdot) \) is continuous at \( t_0 \), \( j = 1, 2 \).

Proof. Making use of Theorem 2.1 of [7], the continuity of the function \( \xi \) can be ensured by assumptions (iv) and (v). By assumptions (ii) and (iii), and Propositions 19 and 20 in Chapter 3 of Section 1 of [3], we get the continuity of \( f^j, j = 1, 2 \). Thus \( S^2(\cdot) \) is continuous at \( t_0 \) by Corollary 1.

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