Reflective Abstraction in Mathematics Learning

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Abstract. The aim of this research was on how the students’ reflective abstraction worked in mathematics learning. This research was qualitative research with case study research design. In Mathematics learning, before the concept of learning material is related to another concept of learning material, the students are expected to have good Reflective Abstraction capability. In Mathematics learning, if a student has good Reflective Abstraction capability, the student will not find difficulty in Mathematics learning which are the difficulty in comprehending the material learned, the difficulty in comprehending and doing tasks given by the teachers, and the difficulty in learning other Mathematics learning materials. The result of this study were, on the process of Mathematics learning, students’ reflective abstraction was needed especially when the students had to solve mathematics problems. The students’ reflective abstraction level might be different between one and another depending on the level of problems and the concept comprehension that the students had.

1. Introduction

Concepts in mathematics learning are interconnected one another. The mastery of prior mathematics learning materials will determine the understanding of next learning materials. The understanding of prior concept is not only for the mastery of next mathematical concept problem, but also for the mastery of other content fields \cite{1}. The ability of students to alter, frame, or attribute a pre-existing conception in order to project and to create structure in the new situation is called Reflective Abstraction \cite{2,3}. Reflective Abstraction, in Piaget’s view, is the highest form of thought and a basis in mathematical thinking \cite{4}.

Reflective abstraction encourages the development of thinking that begins with action and students will observe the results of actions performed that results a mathematical logic experience. Therefore, a reflective abstraction is very important \cite{5}. Since the mathematics learning concept of prior material will be related to the concept of matter afterwards, then a student is expected to have a good Reflective Abstraction ability. In mathematics learning, if a student has good reflective abstraction ability, then students will not find difficulty in learning mathematics such as the difficulty in understanding the material being studied, the difficulty in understanding and solving problems given by the teacher, and the difficulty in learning other mathematics materials. Therefore, this article discusses about how the students’ reflective abstraction worked in mathematics learning.

According to the Indonesian dictionary, abstraction is the process of preparing something that is
intangible or shaped [6]. Abstraction is an activity in forming a mathematical concept that involves interconnected structures or mathematical objects grouped by important properties and relationships [3,7]. In abstraction activities, students are introduced to the important elements related to concepts to be constructed. By knowing important elements and relationships among elements, students are expected to develop mathematical concepts. Students build their self-concepts based on previous material concepts and important elements in building new concepts guided by teachers. The teachers in this case still provide scaffolding to students to avoid mistakes in understanding the process of forming new materials.

Reflective abstraction is the ability of students to transform, incorporate or associate pre-existing conceptions to project and to create structures in new situations [2,3]. Reflective Abstraction in Piaget’s is the highest form of thought and the foundation in mathematical [4]. According to Piaget's mathematical logic knowledge derived from students 'reflective abstraction results, abstractions are developed based on students’ reflections on their respective activities without their awareness [8]. Beth and Piaget developed the idea of a reflective abstraction to describe the process of forming students' abstract mathematical logic [4]. Of the three abstractions, reflective abstraction is the highest thought among other abstractions. Reflective abstraction encourages the development of thinking that begins with action and students will observe the results of actions performed on something that produces a mathematical logic experience. Therefore, reflective abstraction is very important [5]. According to Piaget, a reflective abstraction is more important than the abstraction aspect, since reflective abstraction is the essence of mathematical thinking [9]. Reflective Abstraction is more important than other abstractions; it is not only produce abstract knowledge of mathematical logic but also produce students' mathematical concepts, since the mathematical concepts of students are not the result of the empirical approach, but the result of the student's reflective abstraction [10]. Students who have experience in associating, combining, projecting and creating a concept will have more understanding and memory than students given the concept directly by the teacher. By linking the materials, students have to have an understanding about related concepts so that students understand whether the related concepts can be really related to the concepts to be constructed. By combining, the students must understand the interrelationships among some of the concepts that have been linked and will be incorporated into new concepts. Through the process of linking and combining and understanding the relationship between related concepts, then the students are able to create a new concept. Such process is called Reflective abstractions.

During the process of mathematics learning, the ability of reflective abstraction should be owned and involved by each student, because the students who associate, combine, and create new concepts based on related concepts will have better understanding of mathematics than students who know the concept of direct mathematics given by teachers. Students who experience and have the ability to reflect abstractions have sharper memories based on concepts that have been linked between each other, combined and created; students will not easily forget the concepts for a long period of time.

In mathematics learning, students who have reflective abstraction ability can be differentiated into several levels as revealed by Cifarelli that there are four levels of abstraction process [3]. There are recognition, representation, structural abstraction, and structural awareness. Recognition is the stage where the student's activity is identifying the problem at hand and to relate to the issues related to the problem. Students associate new problems with the old ones and then connect them. In the beginning level, students must have understood the related concepts to combine each of those concepts. Representation is the stage where students use the help of diagrams, tables, or others to assist students in solving the problems at hand. In solving related problems, the students usually need help to understand the materials easily in the process of solving the problem. Structural Abstraction is the stage where students are able to make abstractions and representations of settlement activities. In this level, students are able to solve problems without using any help and are able to represent the settlement activities that they have already known and understood. Structural Awareness is the stage where the student is able to anticipate the results without having to complete all the settlement methods. This level is the highest level of reflective abstraction, students who are already at this level, will easily solve math problems in
any form, without following all the structures in solving math problems. From the above levels, it is found the Reflective Abstraction indicators as presented in table 1.

**Table 1. Indicators of Reflective Abstraction’s Level**

| No | Level of Reflective Abstraction | Indicators |
|----|---------------------------------|------------|
| 1  | Introduction                     | a) Identifying the current problems  
|    |                                  | b) Understanding the current problems  
|    |                                  | c) Recall related concepts to understand the problems  
|    |                                  | d) linking concepts relating to problems encountered |
| 2  | Representation                   | a) States the results on the process of introduction into a form that can help in the form of tables, figure, mathematical symbols, mathematical variables.  
|    |                                  | b) Presents into a mathematical model  
|    |                                  | c) Devise a method to solve the problem using a pre-designed help  
|    |                                  | d) Running a settlement plan |
| 3  | Structural Abstraction           | a) Analyse related concepts to answer and solve problems without using mathematical help as it is at level two  
|    |                                  | b) Using several methods to solve the problem |
| 4  | Structural Awareness             | a) Solve problems without difficulty  
|    |                                  | b) Solve the problem without requiring a thorough completion step  
|    |                                  | c) Can solve all math problems |

A student at the time of mathematics learning takes place and is at the level of abstraction which can be seen and connected with indicators of each level.

2. **Methods**

This research was qualitative research with case study research design. The focus of this research was on how the students’ reflective abstraction worked in mathematics learning. This research involved 31 students of SMA N 1 Pangkalpinang class XI MIA 6. The students were given three problems during the mathematics learning process, then the students’ answer sheets were analyzed qualitatively and triangulated to know how reflective abstraction of students in mathematics learning worked.

3. **Result and Discussion**

The results of this study were presented in the following table:

**Table 2. Result of student’s reflective abstraction test.**

| No | Name  | First Problem | Levels of Student’s Second Problem | Third Problem |
|----|-------|---------------|-----------------------------------|--------------|
| 1  | R S   | Representation| Structural Awareness               |              |
| 2  | R W   | Recognition   | Structural Abstraction             |              |
| 3  | R F S | Recognition   | Representation                     |              |
| 4  | M A S | Representation| Structural Abstraction             |              |
| No | Name | First Problem | Second Problem | Third Problem |
|----|------|---------------|----------------|--------------|
| 5. | E A A | Structural Awareness | Structural Awareness | Structural Awareness |
| 6. | F V | Structural Awareness | Structural Awareness | Structural Abstraction |
| 7. | H N | Structural Awareness | Structural Awareness | Structural Awareness |
| 8. | A 0 | Structural Awareness | Structural Awareness | Structural Abstraction |
| 9. | M A R | Recognition | Recognition | Recognition |
| 10. | S R D | Recognition | Structural Awareness | Structural Awareness |
| 11. | H A | Recognition | Structural Awareness | Structural Awareness |
| 12. | A R K | Recognition | Structural Awareness | Structural Awareness |
| 13. | M P J | Structural Awareness | Structural Awareness | Structural Abstraction |
| 14. | R A T | Structural Awareness | Structural Awareness | Structural Abstraction |
| 15. | S A | Recognition | Structural Abstraction | Structural Abstraction |
| 16. | G S L | Recognition | Recognition | Structural Abstraction |
| 17. | N D P | Recognition | Recognition | Structural Abstraction |
| 18. | S H P | Recognition | Recognition | Recognition |
| 19. | M | Recognition | Structural Awareness | Recognition |
| 20. | E W L | Recognition | Recognition | Recognition |
| 21. | A P S | Structural Awareness | Structural Awareness | Structural Awareness |
| 22. | P S | Recognition | Structural Awareness | Recognition |
| 23. | J S | Recognition | Recognition | Recognition |
| 24. | M A Y | Recognition | Recognition | Recognition |
| 25. | M G R | Structural Awareness | Structural Awareness | Structural Awareness |
| 26. | A M S | Recognition | Structural Abstraction | Structural Awareness |
| 27. | H N D A | Recognition | Structural Awareness | Recognition |
| 28. | A H | Recognition | Structural Abstraction | Structural Abstraction |
| 29. | R A | Structural Awareness | Structural Awareness | Recognition |
| 30. | A B | Representation | Structural Awareness | Representation |
| 31. | H S | Representation | Structural Awareness | Representation |
From the table above, it could be seen that when students dealt with Mathematical problems in Mathematics learning process, each student had different reflective abstraction ability level to project new structures based on the results of combining and linking the previous concepts (reflective abstraction) depend on mathematical problems that they had. In the first problem, the level of students’ reflective abstraction that appeared was at the first and second level, namely recognition and representation. In the second problem, the the student reflective abstraction were at the first, third and fourth levels, namely recognition, structural abstraction and structural awareness. In the third problem, the level of students’ reflective abstraction that appeared were all levels, namely, recognition, representation, structural abstraction and structural awareness.

The first mathematics problem was the most difficult one compared with others that were given to students so that when the students solved mathematics problems with their reflective abstraction ability, they had difficulty to do it due to less aware of the previous related concepts. Thus, the level of reflective abstraction that appeared were the first and the second levels. The students could identify the presented problems, however, it was found several students who did not understand about how to identify the problems proven by the mistakes that they did in understanding problems, and the students did not correlate some related concepts to understand the problems. In solving the problems, some students used some help by making mathematics model, yet it did not work well to solve the first mathematics problem. Thus, no one was able to reach the third and the fourth reflective abstraction levels. The second Mathematics problem was the one in medium level. In the process of solving the problem, the students’ reflective abstraction was on the first level, the third level and the fourth level. On the second problem, most students were on the fourth level in which the students solved the problem without solving all solving methods. However, some students were even at the first level only because they could only understand the problem and were not able to relate some concepts to solve the problems. The third mathematics problem was the easiest one given to the students. The levels of students’ reflection abstraction in solving the problem were at all reflective abstraction levels. The students were able to relate some concepts in projecting new concept. In short, the students could solve the problem easily.

4. Conclusion
On the process of Mathematics learning, students’ reflective abstraction was needed especially when the students had to solve mathematics problems. The students’ reflective abstraction level might be different between one and another depending on the level of problems and the concept comprehension that the students had. The students’ reflective abstraction level that was on the first level only caused by their incapability to relate the mathematics problem with the previous concepts that they know. The students’ reflective abstraction level that was on the second level only caused by their incapability to solve mathematics problem after they used some help that they made. On both the third and the fourth level of reflective abstraction, the students were able to solve mathematics problem in which on the third level the students used some help and on the fourth level the students were able to solve the problems without using all solving methods. On the third level, problem solving taken was the one with complete solving method in which if the students put the answer in written script, they would seem comprehending the problem well and were able to correlate between on mathematical concepts and others, while the fourth level students were able to solve the problem without using complete solving method. However, the four levels of students’ reflective abstraction could be ensured by students' perfect understanding when asked verbally about solving the problems they face.

5. References
[1] Claessens, A and Engel, M 2013 How Important Is Where You Start? Early Mathematics Knowledge and Later School Success Teacher Collage Record 115 1-29
[2] Nutchey, D., Grant, E., & Cooper, T. (t.thn.). Operationalising Constructivist Theory Using Popper's Three Worlds. QUT
[3] Wiryanto 2014 Level - Level Abstraksi Dalam Pemecahan Masalah Matematika Jurnal Pendidikan Elektro 3(3) 569 - 578.
[4] I Cetin and Dubinsky E 2017 Reflective Abstraction in Computational Thinking Journal of Mathematical Behavior 47 70-80
[5] Cooley, L 2002 Writing in Calculus and Reflective Abstraction Journal of Mathematical Behavior 21 255 – 282
[6] Indonesia, K. B 2017 Kamus Besar Bahasa Indonesia (KBBI)
[7] Walida, S E and Fuady, A 2017 Level Abstraksi Refleksi Mahasiswa dalam Pemecahan Masalah Matematika Jurnal Pendidikan Matematika (JPM) 3(1) 41-48
[8] Simon A M 2016 Explicating Mathematical Concept and Mathematical Conception as Theoretical Constructs for Mathematics Education Research CrossMark Educ Stud Math
[9] Dubinsky, E 2002 Reflective Abstraction in Advanced Mathematical Thinking Springer
[10] Simon M A, Placa N and Avitzur A 2016 Participatory and Anticipatory Stages of Mathematical Concept Learning: Further Empirical and Theoretical Development Journal For Research in Mathematics Education 47(1) 63-93