Noisy matrix completion: understanding statistical guarantees for convex relaxation via nonconvex optimization

Abstract—Despite the remarkable theory of convex relaxation for low-rank matrix completion in the last ten years, its statistical guarantee in the noisy case remains highly suboptimal, falling short of explaining its practical efficacy. We make significant progress by establishing the near-optimal estimation error of convex relaxation vis-à-vis random noise in a wide range of noise levels. A novel ingredient in our analysis is bridging convex relaxation with the seemingly distinct nonconvex Burer–Monteiro approach.

I. INTRODUCTION

Suppose we are interested in a large low-rank data matrix, but only get to observe a highly incomplete subset of its entries. Can we hope to estimate the underlying data matrix in a reliable manner? This problem, often dubbed as low-rank matrix completion, spans a diverse array of science and engineering applications and has inspired a flurry of research activities in the past decade. Recent years have witnessed the development of tractable algorithms that come with theoretical comparisons to the noiseless setting. When $E_{ij}$ vanishes for all $(i, j) \in \Omega$, the solution to (3) is known to be faithful (i.e. the estimation error becomes zero) even under near-minimal sample complexity.

By contrast, the performance of convex relaxation remains largely unclear when it comes to more practically relevant noisy scenarios. To begin with, the stability of an equivalent variant of (3) against noise was first studied by Candès and Plan [6]. The estimation error $\|Z_{\text{cvx}} - M^\star\|_F$ derived therein, of the solution $Z_{\text{cvx}}$ to (3), is significantly larger than the oracle lower bound. This does not explain well the effectiveness of (3) in practice. In fact, the numerical experiments reported in [6] already indicated that the performance of convex relaxation is far better than their theoretical bounds. In order to improve the stability guarantees, several variants of (3) have been put forward, most notably by Negahban and Wainwright [10] and Koltchinskii et al. [11]; see Section I-D for more details. Nevertheless, the stability analysis in [10], [11] could often be suboptimal when the magnitudes of the noise are not sufficiently large; in fact, their error bounds do not vanish as the size of the noise approaches zero.

All of these give rise to the following natural yet challenging questions: Where does the convex program (3) stand in terms of its stability vis-à-vis additive noise? Can we establish a theoretical statistical guarantee that matches its practical effectiveness? Is it capable of accommodating a wider (and more practical) range of noise levels?

B. A detour: nonconvex optimization

While the focus of the current paper is convex relaxation, we take a moment to discuss a seemingly distinct algorithmic paradigm: nonconvex optimization, which turns out to be remarkably helpful in understanding convex relaxation. Inspired by the Burer–Monteiro approach [12], the nonconvex scheme starts by representing the rank-$r$ decision matrix (or parameters) $Z$ as $Z = XY^\top$ via low-rank factors $X, Y \in \mathbb{R}^{n \times r}$, and proceeds by solving the following nonconvex (regularized) least-squares problem directly [13].

$$\min_{X,Y} \frac{1}{2} \sum_{(i,j) \in \Omega} \left[ (XY^\top)_{ij} - M_{ij} \right]^2 + \text{reg}(X, Y).$$

(4)

Here, $\text{reg}(\cdot, \cdot)$ denotes a certain regularization term that promotes additional structural properties.

To see its intimate connection [14], [15] with the convex program (3), we make the following observation: if the solution to (3) has rank $r$, then it must coincide with the solution to

$$\min_{X,Y} \frac{1}{2} \sum_{(i,j) \in \Omega} \left[ (XY^\top)_{ij} - M_{ij} \right]^2 + \lambda_{\text{reg}} \frac{1}{2} \|X\|_F^2 + \lambda_{\text{reg}} \frac{1}{2} \|Y\|_F^2.$$

(5)
This can be easily verified by recognizing the elementary fact that
\[
\|Z\|_F = \inf_{X,Y \in \mathbb{R}^{n \times r}} \left\{ \frac{1}{2} \|X\|_F^2 + \frac{1}{2} \|Y\|_F^2 \right\} \tag{6}
\]
for any rank-$r$ matrix $Z$ \cite{23, 24}. Note, however, that it is very challenging to predict when the key assumption in establishing this connection — namely, the rank-$r$ assumption of the solution to (3) — can possibly hold.

Despite the nonconvexity of (3), simple first-order optimization methods — in conjunction with proper initialization — are often effective in solving (4). Partial examples include gradient descent on manifold \cite{13, 17, 18}, gradient descent \cite{19, 20}, and projected gradient descent \cite{21, 22}. Apart from their practical efficiency, the nonconvex optimization approach is also appealing in theory. To begin with, algorithms tailored to (4) often enable exact recovery in the noiseless setting. Perhaps more importantly, for a wide range of noise settings, the nonconvex approach achieves appealing estimation accuracy \cite{20, 21}, which could be significantly better than those bounds derived for convex relaxation discussed earlier. See \cite{5, 23} for a summary of recent results. Such intriguing statistical guarantees motivate us to take a closer inspection of the underlying connection between the two contrasting algorithmic frameworks.

\textbf{C. Empirical evidence: convex and nonconvex solutions are close}

In order to obtain a better sense of the relationships between convex and nonconvex approaches, we begin by comparing the estimates returned by the two approaches via numerical experiments. Fix $n = 1000$ and $r = 5$. We generate $M^* = X^*Y^*\top$, where $X^*, Y^* \in \mathbb{R}^{n \times r}$ are random orthonormal matrices. Each entry of $M^*$ is observed with probability $p = 0.2$ independently, and then corrupted by an independent Gaussian noise $E_{ij} \sim \mathcal{N}(0, \sigma^2)$. Throughout the experiments, we set $\lambda = 5\sigma \sqrt{mp}$. The convex program (3) is solved by the proximal gradient method \cite{24}, whereas we attempt solving the nonconvex formulation (5) by gradient descent with spectral initialization (see \cite{23} for details). Let $Z_{\text{cvx}}$ (resp. $Z_{\text{ncvx}} = X_{\text{ncvx}}Y_{\text{ncvx}}\top$) be the solution returned by the convex program (3) (resp. the nonconvex program (5)). Figure 1 displays the relative estimation errors of both methods and $\|Z_{\text{cvx}} - M^*\|_F / \|M^*\|_F$ as well as the relative distance $\|Z_{\text{cvx}} - Z_{\text{ncvx}}\|_F / \|M^*\|_F$ between the two estimates. The results are averaged over 20 independent trials.

Interestingly, the distance between the convex and the nonconvex solutions seems extremely small (e.g. $\|Z_{\text{cvx}} - Z_{\text{ncvx}}\|_F / \|M^*\|_F$ is typically below $10^{-7}$); in comparison, the relative estimation errors of both $Z_{\text{cvx}}$ and $Z_{\text{ncvx}}$ are substantially larger. In other words, the estimate returned by the nonconvex approach serves as a remarkably accurate approximation of the convex solution. Given that the nonconvex approach is often guaranteed to achieve intriguing statistical guarantees vis-à-vis random noise \cite{20}, this suggests that the convex program is equally stable — a phenomenon that was not captured by prior theory \cite{6}. Can we leverage existing theory for the nonconvex scheme to improve the statistical analysis of the convex relaxation approach?

\textbf{D. Models and main results}

The numerical experiments reported in Section C suggest an alternative route for analyzing convex relaxation for noisy matrix completion. If one can formally justify the proximity between the convex and the nonconvex solutions, then it is possible to propagate the appealing stability guarantees from the nonconvex scheme to the convex approach. As it turns out, this simple idea leads to significantly enhanced statistical guarantees for the convex program (3), which we formally present in this subsection.

Before proceeding, we introduce a few model assumptions that play a crucial role in our theory.

\textbf{Assumption 1}

\begin{enumerate}
\item[(a)] \textbf{(Random sampling)} Each index $(i, j)$ belongs to the index set $\Omega$ independently with probability $p$.
\item[(b)] \textbf{(Random noise)} The noise matrix $E = [E_{ij}]_{1 \leq i,j \leq n}$ is composed of i.i.d. sub-Gaussian random variables with sub-Gaussian norm at most $\sigma > 0$, i.e. $\|E_{ij}\|_2 \leq \sigma$ (see \cite{25} Definition 5.7).
\end{enumerate}

In addition, let $M^* = U^*\Sigma^*V^*\top$ be the singular value decomposition (SVD) of $M^*$, where $U^*, V^* \in \mathbb{R}^{n \times r}$ consist of orthonormal columns and $\Sigma^* = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r) \in \mathbb{R}^{r \times r}$ is a diagonal matrix obeying $\sigma_{\text{max}} \triangleq \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r \triangleq \sigma_{\text{min}}$. Denote by $\kappa \triangleq \sigma_{\text{max}} / \sigma_{\text{min}}$ the condition number of $M^*$. We impose the following incoherence condition on $M^*$, which is known to be crucial for reliable recovery of $M^*$ \cite{2, 9}.

\textbf{Definition 1.} A rank-$r$ matrix $M^* \in \mathbb{R}^{n \times n}$ with SVD $M^* = U^*\Sigma^*V^*\top$ is said to be $\mu$-incoherent if
\[
\max\{\|U\|_{2,\infty}, \|V\|_{2,\infty}\} \leq \sqrt{\mu r / n}.
\]
Here, $\|U\|_{2,\infty}$ denotes the largest $\ell_2$ norm of all rows of a matrix $U$.

With these in place, we are ready to present our improved statistical guarantees for convex relaxation.

\textbf{Theorem 1.} Let $M^*$ be rank-$r$ and $\mu$-incoherent with a condition number $\kappa$. Suppose Assumption 1 holds and take $\lambda = C_\lambda \sigma \sqrt{mp}$ in (3) for some large enough constant $C_\lambda > 0$. Assume the sample size obeys $n^2 p \geq C_\lambda \mu \kappa^2 r^2 n \log^3 n$ for some sufficiently large constant $C > 0$, and the noise satisfies $\sigma \sqrt{\frac{p}{n}} \leq \frac{\sigma_{\text{min}}}{\sqrt{\log n}}$ for some sufficiently small constant $\epsilon > 0$. Then with probability exceeding $1 - O(n^{-3})$,

1) Any minimizer $Z_{\text{cvx}}$ of (3) obeys
\[
\|Z_{\text{cvx}} - M^*\|_F \lesssim \frac{\kappa \sigma}{\sigma_{\text{min}}} \sqrt{\frac{m}{p}} \|M^*\|_F, \tag{7a}
\]
\[
\|Z_{\text{cvx}} - M^*\|_\infty \lesssim \frac{\kappa^2 \mu r}{\sigma_{\text{min}}} \sqrt{\frac{n \log n}{p}} \|M^*\|_\infty. \tag{7b}
\]

1Here and throughout, $f(n) \leq g(n)$ or $f(n) = O(g(n))$ means $\lim_{n \to \infty} |f(n)| / |g(n)| \leq C$ for some constant $C > 0$, $f(n) \asymp g(n)$ means $C_1 \leq \lim_{n \to \infty} |f(n)| / |g(n)| \leq C_2$ for some constants $C_1, C_2 > 0$, $\|\cdot\|_\infty$ denotes the entrywise $\ell_\infty$ norm and $\|\cdot\|$ is the spectral norm.
\[ \| \mathbf{Z}_{cvx} - M^* \| \lesssim \frac{\sigma}{\sigma_{\text{min}}} \sqrt{\frac{n}{p}} \| M^* \|; \quad (7c) \]

2) Letting \( \mathbf{Z}_{cvx, r} \triangleq \text{arg min}_{\mathbf{Z}, \text{rank}(\mathbf{Z}) \leq r} \| \mathbf{Z} - \mathbf{Z}_{cvx} \|_F \) be the best rank-\( r \) approximation of \( \mathbf{Z}_{cvx} \), we have
\[
\| \mathbf{Z}_{cvx, r} - \mathbf{Z}_{cvx} \|_F \leq \frac{1}{n^3} \frac{\sigma}{\sigma_{\text{min}}} \sqrt{\frac{n}{p}} \| M^* \|, \quad (8)
\]
and the error bounds in (7) continue to hold if \( \mathbf{Z}_{cvx} \) is replaced by \( \mathbf{Z}_{cvx, r} \).

**Remark 1.** The factor \( 1/n^3 \) in (8) can be replaced by \( 1/n^r \) for an arbitrarily large fixed constant \( c > 0 \).

Several implications of Theorem 1 follow immediately. The discussions below concentrate on the case where \( r, \mu \) and \( \kappa \) are all \( O(1) \), under the noise model assumption.

- **Improved stability guarantees.** Our results reveal that the Euclidean error of any convex optimizer \( \mathbf{Z}_{cvx} \) of (3) obeys
\[
\| \mathbf{Z}_{cvx} - M^* \|_F \lesssim \sigma \sqrt{n/p}, \quad (9)
\]
implying that the performance of convex relaxation degrades gracefully as the signal-to-noise ratio decreases. This result matches the minimax lower bound derived in [10]. This range of noise obeying \( \sigma \lesssim \| M^* \|_\infty \), which improves upon prior art in the following aspects:

- Candès and Plan [6] provided a stability guarantee in the presence of arbitrary bounded noise. When applied to the random noise model assumed here, their results yield \( \| \mathbf{Z}_{cvx} - M^* \|_F \lesssim \sigma n^{1/2} \), which could be \( O(\sqrt{n^2p}) \) times more conservative than our bound (9).

- Koltchinskii et al. [11] proposed to replace \( \sum_{(i,j) \in \Omega} (Z_{ij} - M_{ij})^2 \) in (3) with \( \sum_{(i,j)} (Z_{ij} - \frac{1}{2} M_{ij})^2 \), where \( M_{ij} \) is set to 0 for any unobserved entry (i.e. those with \( (i, j) \notin \Omega \)). This variant effectively performs singular value thresholding on a rescaled zero-padded data matrix. Under our conditions, their results read (up to some logarithmic factor)
\[
\| \hat{Z} - M^* \|_F \lesssim \max \{ \sigma, \| M^* \|_\infty \} \sqrt{n/p}, \quad (10)
\]
where \( \hat{Z} \) is the estimate returned by their algorithm. This becomes suboptimal when \( \sigma \ll \| M^* \|_\infty \) — a highly relevant regime covered by our analysis.

- Negahban and Wainwright [10] proposed to enforce an extra constraint \( \| Z \|_\infty \leq \alpha \) when solving (3), in order to explicitly control the spikiness of the estimate. When applied to our model, their error bound is the same as (10) (modulo some log factor), which also becomes increasingly looser as \( \sigma \) decreases. In addition, the choice of \( \alpha \) may add unwanted variations in practice.

- **Nearly low-rank structure of the convex solution.** In light of (8), the optimizer of the convex program (3) is almost, if not exactly, rank-\( r \). When the true rank \( r \) is known a priori, it is not uncommon for practitioners to return the rank-\( r \) approximation of \( \mathbf{Z}_{cvx} \). Our theorem formally justifies that there is no loss of statistical accuracy — measured in terms of either \( \| \cdot \|_F, \| \cdot \|_1 \), or \( \| \cdot \|_\infty \) — when performing the rank-\( r \) projection operation.

- **Entrywise and spectral norm error control.** Moving beyond the Euclidean loss, our theory uncovers that the estimation errors of the convex optimizer are fairly spread out across all entries, thus implying near-optimal entrywise error control. This is a stronger form of error bounds, as an optimal Euclidean estimation accuracy alone does not preclude the possibility of the estimation errors being spiky and localized. Furthermore, the spectral norm error of the convex optimizer is also well-controlled. Figure 2 displays the relative estimation errors in both the \( \ell_\infty \) norm and the spectral norm, under the same setting as in Figure 1. As can be seen, both forms of estimation errors scale linearly with the noise level, corroborating our theory.

- **Statistical guarantees for fast iterative optimization methods.** Various iterative algorithms have been developed to solve the nuclear norm regularized least-squares problem (3) up to an arbitrarily prescribed accuracy, examples including SVT (or proximal gradient methods) [25], FPC [27], SOFT-IMPUTE [14], FISTA [28], [29], to name just a few. Our theory immediately provides statistical guarantees for these algorithms. More specifically, any point \( Z \) with \( g(Z) \leq g(Z_{cvx}) + \varepsilon \) (where \( g(\cdot) \) is defined in (3)) enjoys the same error bounds as in (7) (with \( \mathbf{Z}_{cvx} \) replaced by \( Z \) in (7)), provided that \( \varepsilon > 0 \) is sufficiently small. In other words, when these convex optimization algorithms converge w.r.t. the objective value, they are guaranteed to return a statistically reliable estimate.

Finally, we remark that our results are likely suboptimal when \( r \) and \( \kappa \) are allowed to be scaled with \( n \).

### II. Strategy and novelty

In this section, we introduce the strategy for proving our main theorem. Informally, the main technical difficulty stems from the lack of closed-form expressions for the primal solution to (3), which in turn makes it difficult to construct a dual certificate. This is in stark contrast to the noiseless setting, where one clearly anticipates the ground truth \( M^* \) to be the primal solution; in fact, this is precisely why the analysis for the noisy case is significantly more challenging. Our strategy, as we shall detail below, mainly entails invoking an iterative nonconvex algorithm to “approximate” such a primal solution.

Before concluding, we introduce a few more notations. Let \( \mathcal{P}_{\Omega}(\cdot) : \mathbb{R}^{n \times n} \mapsto \mathbb{R}^{n \times n} \) represent the projection onto the subspace of matrices supported on \( \Omega \), namely,
\[
[\mathcal{P}_{\Omega}(\mathbf{Z})]_{ij} = \begin{cases} 
Z_{ij}, & \text{for } (i,j) \in \Omega \\
0, & \text{otherwise}
\end{cases} \quad (11)
\]
for any matrix \( Z \in \mathbb{R}^{n \times n} \). For a rank-\( r \) matrix \( M \) with singular value decomposition \( U \Sigma V^\top \), denote by \( T \) its tangent space, i.e.

\[
T = \left\{ U A^\top + B V^\top \mid A, B \in \mathbb{R}^{n \times r} \right\}.
\]

(12)

Correspondingly, let \( \mathcal{P}_T(\cdot) \) be the orthogonal projection onto the subspace \( T \), that is,

\[
\mathcal{P}_T(Z) = U U^\top Z + Z V V^\top - U U^\top Z V V^\top
\]

(13)

for any matrix \( Z \in \mathbb{R}^{n \times n} \). In addition, let \( T^\perp \) and \( \mathcal{P}_{T^\perp}(\cdot) \) denote the orthogonal complement of \( T \) and the projection onto \( T^\perp \), respectively. With regards to the ground truth, we denote

\[
X^* = U^*(\Sigma^*)^{1/2} \quad \text{and} \quad Y^* = V^*(\Sigma^*)^{1/2}.
\]

(14)

The nonconvex problem \( \mathcal{P}(\Sigma) \) is equivalent to minimizing

\[
f(X, Y) \triangleq \frac{1}{2p} \| \mathcal{P}_\Omega(XY^\top - M) \|^2_2 + \frac{\lambda_2}{2p} \| X \|^2_2 + \frac{\lambda_3}{2p} \| Y \|^2_2.
\]

(15)

where we have inserted an extra factor \( 1/p \) (compared to \( \mathcal{P}(\Sigma) \)) to simplify the presentation of the analysis later on. Due to the space limit, all the proofs are omitted. Check the arxiv version for details.

A. Exact duality

In order to analyze the convex program \( \mathcal{P}(\Sigma) \), it is natural to start with the first-order optimality condition. Specifically, suppose that \( Z \in \mathbb{R}^{n \times n} \) is a (primal) solution to \( \mathcal{P}(\Sigma) \) with SVD \( Z = U \Sigma V^\top \). As before, let \( T \) be the tangent space of \( Z \), and let \( T^\perp \) be the orthogonal complement of \( T \). Then the first-order optimality condition for \( \mathcal{P}(\Sigma) \) reads: there exists a matrix \( W \in T^\perp \) (called a dual certificate) such that

\[
\begin{alignat}{2}
\frac{1}{\lambda} \mathcal{P}_\Omega(M - Z) &= UV^\top + W; \\
\| W \| &\leq 1.
\end{alignat}
\]

(16a)

(16b)

This condition is not only necessary to certify the optimality of \( Z \), but also “almost sufficient” in guaranteeing the uniqueness of the solution \( Z \). The challenge then boils down to identifying such a primal-dual pair \((Z, W)\) satisfying the optimality condition \( \mathcal{P}(\Sigma) \). For the noise-free case, the primal solution is clearly \( Z = M^* \); if exact recovery is to be expected; the dual certificate can then be either constructed exactly by the least-squares solution to a certain underdetermined linear system \( \mathcal{P}(\Sigma) \), or produced approximately via a clever golvening scheme pioneered by Gross \( \mathcal{P}(\Sigma) \). For the noisy case, however, it is often difficult to hypothesize on the primal solution \( Z \), as it depends on the random noise in a complicated way. In fact, the lack of a suitable guess of \( Z \) (and hence \( W \)) was the major hurdle that prior works faced when carrying out the duality analysis.

B. A candidate primal solution via nonconvex optimization

Motivated by the numerical experiment in Section \( \mathcal{P}(\Sigma) \), we propose to examine whether the optimizer of the nonconvex problem \( \mathcal{P}(\Sigma) \) stays close to the solution to the convex program \( \mathcal{P}(\Sigma) \). Towards this, suppose that \( X, Y \in \mathbb{R}^{n \times r} \) form a critical point of \( \mathcal{P}(\Sigma) \) with \( \text{rank}(X) = \text{rank}(Y) = r \). Then the first-order condition reads

\[
\begin{alignat}{2}
\frac{1}{\lambda} \mathcal{P}_\Omega(M - XY^\top) &= X; \\
\frac{1}{\lambda} \left[ \mathcal{P}_\Omega(M - XY^\top) \right]^\top &= Y.
\end{alignat}
\]

(17a)

(17b)

To develop some intuition about the connection between \( \mathcal{P}(\Sigma) \) and \( \mathcal{P}(\Sigma) \), let us take a look at the case with \( r = 1 \). Denote \( X = x \) and \( Y = y \) and assume that the two rank-1 factors are “balanced”, namely, \( \| x \|_2 = \| y \|_2 \neq 0 \). It then follows from \( \mathcal{P}(\Sigma) \) that \( \lambda^{-1} \mathcal{P}_\Omega(M - x y^\top) \) has a singular value 1, whose corresponding left and right singular vectors are \( x/\| x \|_2 \) and \( y/\| y \|_2 \), respectively. In other words, one can express

\[
\frac{1}{\lambda} \mathcal{P}_\Omega(M - x y^\top) = \frac{1}{\| x \|_2 \| y \|_2} x y^\top + W,
\]

(18)

where \( W \) is orthogonal to the tangent space of \( x y^\top \); this precisely the condition \( \mathcal{P}(\Sigma) \). It remains to argue that \( \mathcal{P}(\Sigma) \) is valid as well. Towards this end, the first-order condition \( \mathcal{P}(\Sigma) \) alone is insufficient, as there might be non-global critical points (e.g. saddle points) that are unable to approximate the convex solution well. Fortunately, as long as the candidate \( x y^\top \) is not far away from the ground truth \( M^* \), one can guarantee \( W \| \leq 1 \) as required in \( \mathcal{P}(\Sigma) \).

The above informal argument about the link between the convex and the nonconvex problems can be rigorized. To begin with, we introduce the following conditions on the regularization parameter \( \lambda \).

**Condition 1 (Regularization parameter).** The regularization parameter \( \lambda \) satisfies

\[
\begin{alignat}{2}
1 \quad &\text{(Relative to noise)} \quad \| \mathcal{P}_\Omega(E) \| < \lambda/8; \\
2 \quad &\text{(Relative to nonconvex solution)} \quad \| \mathcal{P}_\Omega(XY^\top - M^*) - p(XY^\top - M^*) \| < \lambda/8.
\end{alignat}
\]

**Remark.** Condition \( \mathcal{P}(\Sigma) \) requires that the regularization parameter \( \lambda \) should dominate a certain norm of the noise, as well as of the deviation of \( XY^\top - M^* \) from its mean \( p(XY^\top - M^*) \); as will be seen shortly, the latter condition can be met when \( (X, Y) \) is sufficiently close to \( (X^*, Y^*) \).

With the above condition in place, the following result demonstrates that a critical point \( (X, Y) \) of the nonconvex problem \( \mathcal{P}(\Sigma) \) readily translates to the unique minimizer of the convex program \( \mathcal{P}(\Sigma) \).

**Lemma 1 (Exact nonconvex vs. convex optimizers).** Suppose that \( (X, Y) \) is a critical point of \( \mathcal{P}(\Sigma) \) satisfying \( \text{rank}(X) = \text{rank}(Y) = r \), and the sampling operator \( \mathcal{P}_\Omega \) is injective when restricted to the elements of the tangent space \( T \) of \( XY^\top \), namely,

\[
\mathcal{P}_\Omega(H) = 0 \iff H = 0, \quad \text{for all } H \in T.
\]

(19)

Under Condition \( \mathcal{P}(\Sigma) \), the point \( Z \triangleq XY^\top \) is the unique minimizer of \( \mathcal{P}(\Sigma) \).

In order to apply Lemma \( \mathcal{P}(\Sigma) \) one needs to locate a critical point of \( \mathcal{P}(\Sigma) \) that is sufficiently close to the truth, for which one natural candidate is the global optimizer of \( \mathcal{P}(\Sigma) \). The caveat, however, is the lack of theory characterizing directly the properties of the optimizer of \( \mathcal{P}(\Sigma) \). Instead, what is available in prior theory is the characterization of some iterative sequence (e.g. gradient descent iterates) aimed at solving \( \mathcal{P}(\Sigma) \). It is unclear from prior theory whether the iterative algorithm under study (e.g. gradient descent) converges to the global optimizer in the presence of noise. This leads to technical difficulty in justifying the proximity between the nonconvex optimizer and the convex solution via Lemma \( \mathcal{P}(\Sigma) \).

C. Approximate nonconvex optimizers

Fortunately, perfect knowledge of the nonconvex optimizer is not pivotal. Instead, an approximate solution to the nonconvex problem \( \mathcal{P}(\Sigma) \) (or equivalently \( \mathcal{P}(\Sigma) \)) suffices to serve as a reasonably tight approximation of the convex solution. More precisely, we desire two factors \( (X, Y) \) that result in nearly zero (rather than exactly zero) gradients:

\[
\nabla_X f(X, Y) \approx 0 \quad \text{and} \quad \nabla_Y f(X, Y) \approx 0,
\]

\[(17)\]
where $f(\cdot, \cdot)$ is the nonconvex objective function as defined in (13). This relaxes the condition discussed in Lemma 1 (which only applies to critical points of $f$ as opposed to approximate critical points). As it turns out, such points can be found via gradient descent tailored to $f$. The sufficiency of the zero-gradient condition is made possible by slightly strengthening the injectivity assumption (19), which is stated below.

**Condition 2 (Injectivity).** Let $T$ be the tangent space of $XY^\top$. There is a quantity $C_{inj} > 0$ such that

$$p^{-1} \|P_{\Omega}(H)\|_F^2 \geq C_{inj} \|H\|_F^2,$$

for all $H \in T$. (20)

The following lemma states quantitatively how an approximate nonconvex optimizer serves as an excellent proxy of the convex solution.

**Lemma 2 (Approximate nonconvex vs. convex optimizers).** Suppose that $(X, Y)$ obeys

$$\|\nabla f(X, Y)\|_F \leq \epsilon \sqrt{C_{inj} \|X\|_F^2 \|Y\|_F^2},$$

for some sufficiently small constant $\epsilon > 0$. Further assume that any singular value of $X$ and $Y$ lies in $(\sqrt{\sigma_{min}/2}, \sqrt{2\sigma_{max}}]$. Then under Conditions 1-2 any minimizer $Z_{con}$ of (3) satisfies

$$\|XY^\top - Z_{con}\|_F \leq \frac{\epsilon}{C_{inj} \sqrt{\sigma_{min}}} \|\nabla f(X, Y)\|_F.$$ (22)

**Remark 3.** In fact, this lemma continues to hold if $Z_{con}$ is replaced by any $Z$ obeying $g(Z) \leq g(XY^\top)$, where $g(\cdot)$ is the objective function defined in (3) and $X$ and $Y$ are low-rank factors obeying conditions of Lemma 2. This is important in providing statistical guarantees for iterative methods like SVT [26], FPC [27], SOFT-IMPUTE [14], FISTA [28], etc. More to be specific, suppose that $(X, Y)$ results in an approximate optimizer of (3), namely, $g(XY^\top) = g(Z_{con}) + \epsilon$ for some sufficiently small $\epsilon > 0$. Then for any $Z$ obeying $g(Z) \leq g(XY^\top) = g(Z_{con}) + \epsilon$, one has

$$\|XY^\top - Z\|_F \leq \frac{\epsilon}{C_{inj} \sqrt{\sigma_{min}}} \|\nabla f(X, Y)\|_F.$$ (23)

As a result, as long as the above-mentioned algorithms converge in terms of the objective value, they must return a solution obeying (23), which is exceedingly close to $XY^\top$ if $\|\nabla f(X, Y)\|_F$ is small.

It is clear from Lemma 2 that, as the size of the gradient $\nabla f(X, Y)$ gets smaller, the nonconvex estimate $XY^\top$ becomes an increasingly tighter approximation of any convex optimizer of (3), which is consistent with Lemma 1. In contrast to Lemma 1 due to the lack of strong convexity, a nonconvex estimate with a near-zero gradient does not imply the uniqueness of the optimizer of the convex program (3); rather, it indicates that any minimizer of (3) lies within a sufficiently small neighborhood surrounding $XY^\top$ (cf. (22)).

**D. Construction of an approximate nonconvex optimizer**

So far, Lemmas 1 and 2 are both deterministic results based on Condition 1. As we will see soon, under Assumption 1 we can derive simpler conditions that — with high probability — guarantee Condition 1. We start with Condition 1.a.

**Lemma 3.** Suppose $n^2p \geq Cn \log^2 n$ for some sufficiently large constant $C > 0$. Then with probability at least $1 - O(n^{-10})$, one has $\|P_{\Omega}(E)\|_F \leq \sigma \sqrt{mp}$. As a result, Condition 1 holds (i.e. $\|P_{\Omega}(E)\|_F \leq \lambda/8$) as long as $\lambda = C_\lambda \sigma \sqrt{mp}$ for some sufficiently large constant $C_\lambda > 0$.

Turning attention to Condition 1.b and Condition 2 we have the following lemma.

**Lemma 4.** Under the assumptions of Theorem 1 with probability exceeding $1 - O(n^{-10})$ we have

$$\|P_{\Omega}(XY^\top - M^*)\|_F \leq C_{\lambda} \sigma \sqrt{mp},$$

for all $H \in T$ hold simultaneously for all $(X, Y)$ obeying

$$\max \left\{ \|X - X^\star\|_{2, \infty}, \|Y - Y^\star\|_{2, \infty} \right\} \geq \frac{C_{\lambda} \sigma}{p \sigma_{min}} \left( \frac{\sqrt{n \log n} + \lambda}{p \sigma_{min}} \right) \max \left\{ \|X^\star\|_{2, \infty}, \|Y^\star\|_{2, \infty} \right\}.$$

Here, $T$ denotes the tangent space of $XY^\top$, and $C_{\lambda} > 0$ is some absolute constant.

This lemma is a uniform result, namely, the bounds hold irrespective of the statistical dependency between $(X, Y)$ and $\Omega$. As a consequence, to demonstrate the proximity between the convex and nonconvex solutions (cf. (22)), it remains to identify a point $(X, Y)$ with vanishingly small gradient (cf. (21)) that is sufficiently close to the truth (cf. (24)).

As we already alluded to previously, a simple gradient descent algorithm aimed at solving the nonconvex problem (5) might help us produce an approximate nonconvex optimizer. This procedure is summarized in Algorithm 1. Our hope is this: when initialized at the ground truth and run for sufficiently many iterations, the GD trajectory produced by Algorithm 1 will contain at least one approximate stationary point of (5) with the desired properties (21) and (22). We shall note that Algorithm 1 is not practical since it starts from the ground truth $(X^\star, Y^\star)$; this is an auxiliary step mainly to simplify the theoretical analysis.

**Initialization:** $X^0 = X^\star; Y^0 = Y^\star$.

**Gradient updates:** for $t = 0, 1, \ldots, k - 1$ do

$$X^{t+1} = X^t - \eta \nabla f(X^t, Y^t);$$

$$Y^{t+1} = Y^t - \eta \nabla Y f(X^t, Y^t).$$

Here, $\eta > 0$ is the step size.

**E. Properties of the nonconvex iterates**

In this subsection, we shall build upon the literature on nonconvex low-rank matrix completion to justify that the estimates returned by Algorithm 1 satisfy the requirement stated in (22). Our theory will be largely established upon the leave-one-out strategy introduced by Ma et al. [20], which is an effective analysis technique to control the $\ell_2, \infty$-error of the estimates. This strategy has recently been extended by Chen et al. [20] to the more general rectangular case with an improved sample complexity bound.

Before continuing, we introduce several useful notations. Notice that the matrix product of $X^\top$ and $Y^\top$ is invariant under global orthonormal transformation, namely, for any orthonormal matrix $R \in \mathbb{R}^{r \times r}$ one has $X^\top R(Y^\top R)^\top = X^\top Y^\top$. Viewed in this light, we shall consider distance metrics modulo global rotation. In particular, the theory relies heavily on a specific global rotation matrix defined as follows

$$H^\top = \arg \min_{R \in O(r \times r)} \{ \|X^\top R - X^\star\|_F^2 + \|Y^\top R - Y^\star\|_F^2 \}^{1/2},$$

where $O(r \times r)$ is the set of $r \times r$ orthonormal matrices.
We are now ready to present the performance guarantees for Algorithm II-D.

**Lemma 5 (Quality of the nonconvex estimates).** Instantiate the notation and hypotheses of Theorem [2] With probability at least $1 - O(n^{-3})$, the iterates $\{X^t, Y^t\}_{0 \leq t \leq t_0}$ of Algorithm II-D satisfy

\[
\max \left\{ \|X^t H^t - X^t\|_F, \|Y^t H^t - Y^t\|_F \right\} 
\leq C_F \left( \frac{\sigma}{\sigma_{\text{min}}} \sqrt{\frac{n}{p}} + \frac{\lambda}{p \sigma_{\text{min}}} \right) \|X^*\|_F,
\]

\[
\max \left\{ \|X^t H^t - X^t\|_F, \|Y^t H^t - Y^t\|_F \right\} 
\leq C_{op} \left( \frac{\sigma}{\sigma_{\text{min}}} \sqrt{\frac{n}{p}} + \frac{\lambda}{p \sigma_{\text{min}}} \right) \|X^*\|_F,
\]

\[
\max \left\{ \|X^t H^t - X^t\|_{2,\infty}, \|Y^t H^t - Y^t\|_{2,\infty} \right\} 
\leq C_{op} \left( \frac{\sigma}{\sigma_{\text{min}}} \sqrt{\frac{n \log n}{p}} + \frac{\lambda}{p \sigma_{\text{min}}} \right) \max \left\{ \|X^*\|_{2,\infty}, \|Y^*\|_{2,\infty} \right\},
\]

where $C_F, C_{op}, C_{\infty} > 0$ are some absolute constants, provided that $\eta \leq 1/n(\sigma_\text{min} \sigma_{\text{max}})$ and that $t_0 = n^{18}$.

This lemma reveals that for a polynomially large number of iterations, all iterates of the gradient descent sequence — when initialized at the ground truth — remain fairly close to the true low-rank factors. This holds in terms of the estimation errors measured by the Frobenius norm, the spectral norm, and the $\ell_2,\infty$ norm. In particular, the proximity in terms of the $\ell_2,\infty$ norm error plays a pivotal role in implementing our analysis strategy (particularly Lemmas [4] described previously. In addition, this lemma (cf. [28]) guarantees the existence of a small-gradient point within this sequence $\{(X^t, Y^t)\}_{0 \leq t \leq t_0}$, a somewhat straightforward property of GD tailored to smooth problems [31]. This in turn enables us to invoke Lemma [2].

Let $t_0 = \arg \min_{0 \leq t \leq t_0} \|\nabla f(X^t, Y^t)\|_F$ and take $(X_{\text{conv}}, Y_{\text{conv}}) = (X^t H^t, Y^t H^t)$ (cf. [26]). It is straightforward to verify Theorem [1].

### III. CONCLUSION

We provide an improved statistical analysis for the natural convex program [3], without enforcing additional spikiness constraint. Our theoretical analysis uncovers an intriguing connection between convex relaxation and nonconvex optimization, which we believe is applicable to many other problems beyond matrix completion.

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