Reworking The Antonsen-Bormann Idea

To cite this article: S G Kamath 2012 J. Phys.: Conf. Ser. 343 012051

View the article online for updates and enhancements.

Related content
- Spontaneous Excitation of an Accelerated Detector Interacting with a Massive Scalar Field and Unruh Effect
  Zhou Yue-Bing, Yu Hong-Wei and Zhu Zhi-Ying
- Alice and Bob in an expanding spacetime
  Helder Alexander, Gustavo de Souza, Paul Mansfield et al.
- CCDD model from quantum particle creation: constraints on dark matter mass
  J.F. Jesus and S.H. Pereira
Reworking The Antonsen-Bormann Idea

S.G.Kamath*
Department of Mathematics, Indian Institute of Technology Madras
Chennai 600 036, India
*e-mail: kamath@iitm.ac.in

Abstract: The Antonsen – Bormann idea was originally proposed by these authors for the computation of the heat kernel in curved space; it was also used by the author recently with the same objective but for the Lagrangian density for a real massive scalar field in 2 + 1 dimensional curved space. It is now reworked here with a different purpose – namely, to determine the zeta function for the said model using the Schwinger operator expansion.

PACS: 04.60.Kz, 04.62.+v,11.10.Kk

Introduction

This is the third of a three part paper [1, 2] dealing with the Lagrangian density

\[ L = \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2} m^2 \varphi^2 \]  

(1)

for a real massive scalar field in 2 + 1 dimensional curved space. To motivate this paper we consider the operator

\[ B = -\partial^\mu (g_{\mu\nu} \partial^\nu) - m^2 \]  

(2)

and work with the stationary solutions of the Einstein field equations given by Deser et al.[3] and Clement[4] to define the metric \( g^{\mu\nu} \); in detail

\[ g^{00} = 1 - \frac{\lambda^2}{r^2}, 
  g^{01} = -\frac{\lambda^2}{r^2}, 
  g^{02} = \frac{\lambda^2}{r^2}, 
  g^{11} = -1, 
  g^{12} = 0, 
  g^{22} = -1 \]  

(3a)

and \( g_{\mu\nu} \) is

\[ g_{00} = 1, 
  g_{01} = -\frac{\lambda^2}{r^2}, 
  g_{02} = \frac{\lambda^2}{r^2}, 
  g_{11} = -1 + \left(\frac{\lambda^2}{r^2}\right)^2, 
  g_{12} = -\left(\frac{\lambda^2}{r^2}\right), 
  g_{22} = -1 + \left(\frac{\lambda^2}{r^2}\right)^2 \]  

(3b)

with \( r = |\vec{r}| \), \( 2\pi \lambda = \kappa J \), and \( \kappa = 8\pi G \); \( G \) being the gravitational constant and \( J = |J| \) being the spin of the massless particle located at the origin (see eq.(20) in Clement[4]).

The effort in Ref.2 was to the use of the Antonsen-Bormann idea [5,6] for the heat kernel \( G(x,x';\sigma) \) in planar curved space, it being the solution of

\[ B G(x,x';\sigma) = -i \frac{\partial G}{\partial \sigma} \]  

(4)

Published under licence by IOP Publishing Ltd
with $G(x, x'; \sigma \to 0) = -i \delta^{(3)}(x - x')$; by introducing the vierbeins $e^a_a$ defined by $g^{ab} = \eta^{ab} e^a_a e^b_b$ with $\eta^{ab} = \text{diag}(1, -1, -1)$ one first reworks $B$ as

$$B = -\eta^{ab} \partial_a \partial_b - m^2 - e^m_m \partial_m (e^a_a) \partial^a$$

(5)

Writing $G(x, x'; \sigma)$ in terms of the flat space heat kernel $G_0(x, x'; \sigma)$ as

$$G(x, x'; \sigma) = G_0 e^{-\frac{1}{2} \int e^m_m \partial_m (e^a_a) dx^a e^{-\sigma}}$$

(6)

with $(\eta^{ab} \partial_a \partial_b + m^2) G_0 = i \frac{\partial g_0}{\partial \sigma}$ and $G_0(x, x; \sigma \to 0) = -i \delta^{(3)}(x - x')$, one then obtains as in Ref.5

$$\partial^a \partial_a T - \partial^a T \partial_a T + f + 2 \frac{\partial^a g_0}{\partial \sigma} \partial_a T = i \frac{\partial g_0}{\partial \sigma}$$

(7)

with $f = \frac{1}{4} (e^m_m \partial_m (e^a_a))^2 + \frac{1}{2} \partial^m (e^m_m \partial_m (e^a_a))^2$. Since $G_0 = (4\pi \sigma)^{-3/2} e^{-\frac{|x - x'|^2}{4\sigma} - im^2 \sigma}$ the ratio in (7) works to $-i \frac{(x - x')^a}{\sigma} \partial_a T$. With $T$ taken as

$$T = \frac{\tau_{-1}}{\sigma} + \sum_{k=0}^{\infty} \tau_k \sigma^k$$

and not as in eq.(24) in Ref.5 to meet the twin requirements of the boundary condition on the heat kernels $G$ and $G_0$ as $\sigma \to 0$ thus setting $\tau_0 = -\frac{1}{2} \int e^m_m \partial_n (e^a_a) dx^m$ and retaining the extra term $-i \frac{(x - x')^a}{\sigma} \partial_a T$ one gets $\tau_{-1}$ as a solution of

$$\partial^a \partial_a \tau_{-1} - 2 \partial^a \tau_0 \partial_a \tau_{-1} - i (x - x')^a \partial_a \tau_0 = 0$$

(8)

with the $\tau_k$ for $k \geq 1$ being likewise obtained[2] through the solution of similar coupled partial differential equations. This latter aspect is clearly a disincentive to the above program of obtaining the heat kernel $G(x, x'; \sigma)$; but we believe the idea [5,6] itself is too good to ignore and it is therefore appreciated below from a different point of view.

**The reworking**

To elaborate, let’s return to the operator $B$ as given in eq.(5) and write it as

$$B = H_0 + H_1, H_0 \equiv -\eta^{\mu \nu} \partial_\mu \partial_\nu - m^2, \ H_1 \equiv -e^m_m \partial_m (e^n_n) \partial^n$$

(9)

We now present two sets of vierbeins $e^\mu_a$ and $e^a_\mu$ that are easily obtained from eqs.(3) above using $g^{ab} = \eta^{ab} e^a_a e^b_b$ and $g_{ab} = \eta_{ab} e^a_a e^b_b$, together with the respective operator $H_1$ for each set:
\[ e_0^0 = 1, e_1^0 = -\frac{1}{r}, e_2^0 = 0 \quad e_0^0 = 0, e_1^0 = i, e_2^0 = 0 \]

**Set 1:**
\[ e_0^\alpha = e_1^\alpha = 0, e_2^\alpha = \frac{x^\alpha}{r} \quad e_0^\alpha = -i, e_1^\alpha = i \frac{\lambda y^\alpha}{r^2}, e_2^\alpha = -i \frac{\lambda x^\alpha}{r^2} \]
\[ e_0^2 = 0, e_1^2 = \frac{x^2}{r}, e_2^2 = \frac{y^2}{r} \quad e_0^2 = 0, e_1^2 = 0, e_2^2 = 1 \]
\[ H_I = -\frac{1}{r^3} \{ x(\lambda + iy)p_1 + (\lambda y - ix^2)p_2 \} \]  
(10a)

\[ e_0^\mu = e_1^\mu = 0, e_2^\mu = 1 \quad e_0^\mu = -i, e_1^\mu = i \frac{\lambda y}{r^2}, e_2^\mu = -i \frac{\lambda x}{r^2} \]
\[ e_0^2 = 0, e_1^2 = -1, e_2^2 = 0 \quad e_0^2 = 0, e_1^2 = -\frac{1}{\sqrt{2}}, e_2^2 = -\frac{1}{\sqrt{2}} \]
\[ H_I = -\frac{\lambda}{r^4} \left\{ (y^2 - x^2)p_1 - 2xy p_2 \right\} \]  
(10b)

with \( p_I = -i \partial_I \), \( i = 1, 2 \) in eqs.(10). Note that eq.(10a) is linear in \( \lambda \) while the operator \( H_I \) in (10b) is a multiple of \( \lambda \); this difference between the two operators in eqs.(10) will be decisive in the sequel.

**The Schwinger Expansion**

To take up the subject of this paper, we adopt the method of operator regularization – a perturbative expansion introduced by Schwinger [7] to compute amplitudes in quantum field theory in the context of background field quantization. Specifically, we shall use the method of operator regularization [8,9] to attempt a calculation of the \( \zeta(s) \) function associated with the Lagrangian density in eq.(1). To this end we recall the definition of the \( \zeta \) function for the operator \( B \) following McKeon and Sherry [8]

\[ \zeta(s) = \frac{1}{(s)(s)} \int_0^\infty dt \ t^{s-1} \ tr \ e^{-tB} \]  
(11)

with the functional trace being computed in momentum space after Schwinger [7]. For convenience we shall work in Euclidean space below and rewrite eq.(9) as \( B = C_0 + C_1 \) with

\[ C_0 = \vec{p}^2 - m^2 \quad C_1 = \vec{p}^2 + H_I \]  
(12)

Since the metric \( g^{\mu\nu} \) in eqs.(3) is time independent it is obvious that

\[ e^{-(C_0+C_1)t} = e^{-C_0t} e^{-C_1t} \]  
(13)

The Schwinger expansion [8] is now applied to the second exponential in (13) to get,
\[ e^{-t} c_1 = e^{-t} p^2 + (-t) \int_0^1 d \theta e^{-t (1-\theta)p^2} H_1 e^{-t \theta p^2} + (-t)^2 \int_0^1 d \theta \int_0^1 d \theta' e^{-t (1-\theta)p^2} H_1 e^{-t \theta p^2} H_1 e^{-t \theta' p^2} \]
\[ + (-t)^3 \int_0^1 d \theta \int_0^1 d \theta' \int_0^1 d \theta'' e^{-t (1-\theta)p^2} H_1 e^{-t \theta p^2} H_1 e^{-t \theta' p^2} H_1 e^{-t \theta'' p^2} \]
\[ + (-t)^4 \int_0^1 d \theta \int_0^1 d \theta' \int_0^1 d \theta'' \int_0^1 d \theta''' e^{-t (1-\theta)p^2} H_1 e^{-t \theta p^2} H_1 e^{-t \theta' p^2} H_1 e^{-t \theta'' p^2} H_1 e^{-t \theta''' p^2} \]
\[ \times e^{-t} u_4 u_2 (1-u_3)p^2 H_1 e^{-t} u_4 u_2 u_3 p^2 + \ldots \] (14)

The scope of this paper is limited to the third order term in \(-t\) above with the higher order terms being dealt with in more detail elsewhere. An obvious advantage from the use of (10b) over (10a) in (14) is that it becomes a power series in \(\lambda\) and this favours the use of eq.(10b) for the calculation done below.

Writing the first order term in (14) as
\[ (-t) \int_0^1 d \theta e^{-t (1-\theta)p^2} \langle p | H_1 | p \rangle e^{-t \theta p^2} \] (15)

The matrix element in (15) now becomes,
\[ \langle p | H_1 | p \rangle = -\lambda \int \frac{1}{(q_1^2 + q_2^2)} \left( (q_2^2 - q_1^2) p_1 - 2q_1 q_2 p_2 \right) \] (16)

with \(q_1\) and \(q_2\) being the two components of the position vector \(\vec{q}\) and \(\int\) being a symbol for \(\int \frac{d^2 q}{(2\pi)^2}\). On integration the first and second terms in (16) yield zero by symmetry ; thus the first order term in (14) is zero. The second order term in (14) is given by
\[ (-t)^2 \int_0^1 d \theta \int_0^1 d \theta' e^{-t (1-\theta)p^2} \int \langle p | H_1 | r \rangle e^{-t \theta (1-u_1) p^2} \langle r | H_1 | p \rangle e^{-t (1-u_1) \theta p^2} \] (17)

with \(\vec{r}\) being a two component momentum vector and \(\int\) shorthand for \(\int d^2 r\). As in (16), one gets
\[ \langle p | H_1 | r \rangle = -\lambda \int \frac{e^{-i(p-r) \cdot \vec{q}}}{(q_1^2 + q_2^2)} \left( (q_2^2 - q_1^2) r_1 - 2q_1 q_2 r_2 \right) = \frac{\lambda}{4\pi} \left\{ r_1 - 2 \left( \frac{(p_2-r_2)(p_1-r_1 p_2)}{(\vec{r}-\vec{p})^2} \right) \right\} \] (18)

Therefore
\[ \langle r | H_1 | p \rangle = -\frac{\lambda}{4\pi} \left\{ r_1 - 2 \left( \frac{(r_2-p_2)(r_2-p_1-p_1 p_2)}{(\vec{r}-\vec{p})^2} \right) \right\} \] (19)

With eqs. (18) and (19), (17) is easily determined from the three apparently nonzero momentum integrals:
\[ J_0 \equiv (2) \int r \frac{(r_2-p_2)(r_2-p_1-p_1 p_2)}{(\vec{r}-\vec{p})^2} e^{-ut(1-u_1)p^2} \] , \( J_1 \equiv (2) \int p \frac{(p_2-r_2)(p_1-r_1 p_2)}{(\vec{r}-\vec{p})^2} e^{-ut(1-u_1)p^2} \) (20)
and

\[ J_2 \equiv (-2)^2 \int \frac{(r_3-p_2)(r_2-p_1-r_1p_2)}{(\beta - \beta')^2} e^{-u(1-u_1)^2} \frac{(p_2-r_2)(p_2r_1-p_1r_2)}{(\beta - \beta')^2} \]  

(21)

The integrals in (20) yield

\[ J_0 + J_1 = (-2) \frac{\pi}{2} \int_0^\infty da \frac{e^{-ap^2}}{(a+z)^3} \left[ a(p_1^2 - p_2^2) + u(p_1^2 + p_2^2) \right] \]  

(22)

with \( a = x + z \), \( z = ut(1 - u_1) \) and \( \beta = \frac{ax}{a+z} \). Likewise one gets from (21)

\[ J_2 = (-2)^2 \frac{\pi}{2} \int_0^\infty da \frac{e^{-ap^2}}{u^4} \left[ \beta^2 z^2 p_2^2 + \frac{1}{2} u(3p_1^2 + p_2^2) \right] \]  

(23)

On doing the integration one gets

\[ J_0 + J_1 + J_2 = \frac{\pi}{2} (p_1^2 - p_2^2) e^{-ap^2} \left\{ (1 + e^{ap^2}) + \frac{2z}{a} (1 - e^{ap^2}) \right\} \]  

(24)

with \( a = z^2 \beta^2 \). Note that eq.(22) is antisymmetric to the exchange of \( p_1 \) and \( p_2 \) and will be further multiplied by \( e^{-ap^2} \) where \( c = t(1 - u(1 - u_1)) \) as seen from eq.(17). This will alter (24) to

\[ J_0 + J_1 + J_2 = \frac{\pi}{2} (p_1^2 - p_2^2) e^{-ap^2} \left\{ (1 + e^{ap^2}) + \frac{2z}{a} (1 - e^{ap^2}) \right\} \]  

(24a)

To this order therefore,

\[ e^{-t \cdot C_1} = e^{-t^2 \cdot p^2} + (-t)^2 (\frac{\lambda}{4\pi})^2 \pi (p_1^2 - p_2^2) e^{-t \cdot \beta^2} \int_0^1 du \int_0^1 du' \frac{1}{a} \left\{ (1 + e^{ap^2}) + \frac{2z}{a} (1 - e^{ap^2}) \right\} \]

\[ = e^{-t \cdot \beta^2} \left\{ 1 + (\frac{\lambda}{4\pi})^2 \pi (p_1^2 - p_2^2) \int_0^1 du \int_0^1 du' \frac{1}{a} \left\{ (1 + e^{ap^2}) + \frac{2z}{a} (1 - e^{ap^2}) \right\} \right\} \]  

(25)

with \( a = z^2 \beta^2 \), \( z = ut(1 - u_1) \).

The third order term: Following (17) it can be written as

\[ (-t)^3 \int_0^1 u_2 du \int_0^1 u_1 du_1 \int_0^1 u_2 du_2 e^{-(1-u_1)p^2} \int (p|H_1|r)e^{-x \cdot r^2(r|H_1|q)e^{-z \cdot q^2 (q|H_1|p)e^{-t_1 u_1 u_2 \beta^2}} \]  

(26)

with the integration now understood as \( \int d^2r d^2q \), \( x \equiv tu(1 - u_1) \) and \( z \equiv tuu_1(1 - u_2) \). Using eqs.(18) and (19) the product of the matrix elements now becomes

\[ \left\{ -\frac{\lambda}{4\pi} \right\}^3 \left\{ r_1 - 2 (p_2r_1)(p_2r_1 - p_1r_2) \right\} \left\{ q_1 - 2 (r_2q_1)(r_2q_1 - q_1r_2) \right\} \left\{ p_1 - 2 (q_2p_1)(q_2p_1 - p_1q_2) \right\} \]  

(27)

of which only the following four will be apparently non-zero.
\[
K_0 = \left( -\frac{\lambda}{4\pi} \right)^3 (-2)^2 r_1 \frac{(r_2 - q_2)(r_2 q_1 - r_1 q_2)}{(\sqrt{q} - \sqrt{q})^2} e^{-x r^2} \frac{(q_2 - p_2)(q_2 p_1 - q_1 p_2)}{(q - p)^2} e^{-z q^2}
\]

\[
K_1 = \left( -\frac{\lambda}{4\pi} \right)^3 (-2)^2 q_1 \frac{(p_2 - r_2)(p_2 r_1 - p_1 r_2)}{(\sqrt{p} - \sqrt{p})^2} e^{-x r^2} \frac{(q_2 - p_2)(q_2 p_1 - q_1 p_2)}{(q - p)^2} e^{-z q^2}
\]

\[
K_2 = \left( -\frac{\lambda}{4\pi} \right)^3 (-2)^2 p_1 \frac{(p_2 - r_2)(p_2 r_1 - p_1 r_2)}{(\sqrt{p} - \sqrt{p})^2} e^{-x r^2} \frac{(q_2 - r_2)(q_2 q_1 - r_1 q_2)}{(q - q)^2} e^{-z q^2}
\]

\[
K_3 = \left( -\frac{\lambda}{4\pi} \right)^3 (-2)^3 \frac{(p_2 - r_2)(p_2 r_1 - p_1 r_2)}{(\sqrt{p} - \sqrt{p})^2} \frac{(r_2 - q_2)(r_2 q_1 - r_1 q_2)}{(r - q)^2} e^{-x r^2} \frac{(q_2 - p_2)(q_2 p_1 - q_1 p_2)}{(q - p)^2} e^{-z q^2}
\]

We begin with \( K_0 \). On integrating over \( \mathbf{r} \) one gets

\[
K_0 = \left( -\frac{\lambda}{4\pi} \right)^3 (-2)^2 \frac{\pi}{2} \int_0^\infty da \frac{e^{-\beta a^2}}{(a + x)^3} \left[ a(q_1^2 - q_2^2) + (a + x)q_2^2 \right] \frac{(q_2 - p_2)(q_2 p_1 - q_1 p_2)}{(q - p)^2} e^{-z q^2}
\]

\[
= \left( -\frac{\lambda}{4\pi} \right)^3 (-2)^2 \frac{\pi}{2} \int_0^\infty da \frac{e^{-\beta a^2 + \beta x^2}}{(a + x)^3} \frac{e^{\beta a^2}}{(a + x)^3} e^{-z q^2}
\]

with \( T_1 = p_1(6\mu^2 p_3 + w(3 - 4\mu p_3^2)) \), \( T_2 = p_1(2\mu^2 (p_1^2 - 2p_3^2) + w(1 + 2\mu p_3^2)) \), \( w = \beta + \mu + z \) and \( \sigma = \frac{\beta + z}{w} \), \( \beta \equiv \frac{ax}{a+x} \). The integration over \( \mu \) yields,

\[
K_0 = \left( -\frac{\lambda}{4\pi} \right)^3 (-2)^2 \frac{\pi}{2} \int_0^\infty da \frac{e^{-\beta a^2 + \beta x^2}}{(a + x)^3} \frac{e^{\beta a^2}}{(a + x)^3} e^{-z q^2}
\]

With \( G_1 = \left\{ \frac{1}{a} 2p_2^2 + 3(p_1^2 - 3p_2^2) \frac{1}{a^2 p_2^2} (-1 + e^{a/b}) \right\} \),

\[
G_2 = \left\{ \frac{1}{a} 2p_2^2 + 3(p_1^2 - 3p_2^2) \frac{1}{a^2 p_2^2} (-1 + e^{a/b}) \right\}
\]

and \( a = (\beta + x)^2 \). Similarly,

\[
K_1 = \left( -\frac{\lambda}{4\pi} \right)^3 (-2)^2 \frac{\pi}{2} \int_0^\infty da \frac{e^{-\beta a^2}}{(a + x)^3} \frac{p_1 q_1}{q - q_2} \frac{(q_2 - p_2)(q_2 p_1 - q_1 p_2)}{(q - p)^2} e^{-z q^2}
\]

\[
= \left( -\frac{\lambda}{4\pi} \right)^3 (-2)^2 \frac{\pi}{2} \frac{p_1 e^{-\beta a^2}}{(a + x)^3} \frac{p_1}{q - p_2} \frac{(p_2 - r_2)(p_2 q_1 - r_1 q_2)}{(q - q)^2} e^{-z q^2}
\]

with \( b = x^2 \beta^2 \) and \( c = z^2 \beta^2 \).

Likewise, with \( \beta \equiv \frac{ax}{a+x} \)

\[
K_2 = \left( -\frac{\lambda}{4\pi} \right)^3 (-2)^2 \frac{\pi}{2} \int_0^\infty da \frac{e^{-\beta a^2 + \beta x^2}}{(a + x)^3} \frac{p_1 r_1}{p_2 r_2} \frac{(p_2 - r_2)(p_2 q_1 - r_1 q_2)}{(q - q)^2} e^{-z q^2}
\]

\[
= \left( -\frac{\lambda}{4\pi} \right)^3 (-2)^2 \frac{\pi}{2} \frac{p_1}{x} (S_1 + S_2)
\]

With
\[ S_1 = \int_0^\infty d\lambda \ e^{-b/w} \left\{ \frac{1}{b}(p_1^2 - p_2^2) e^{b/w} - \frac{1}{b}w(p_1^2 - p_2^2)(1 - e^{b/w}) \right\} \quad w = \lambda + x + z \quad b = w^2 \bar{p}^2 \]  

\[ S_2 = \int_0^\infty d\lambda \ e^{-c/u} \left\{ \frac{1}{c}(p_1^2 - p_2^2) e^{c/u} - \frac{1}{c}u(p_1^2 - p_2^2)(1 - e^{c/u}) \right\} \quad u = \lambda + x \quad c = u^2 \bar{p}^2 \]  

Finally,

\[ K_3 = \left( -\frac{\lambda}{4\pi} \right)^3(-2)^3 \frac{\pi}{2} \int_0^\infty \int_0^\infty d\beta e^{\frac{w^2/2}{t^2}} \{ t^3r_2p_2 \hat{r} \cdot \hat{p} + t^2B_2 + tB_1 - 8\alpha\beta w_2^2(r_1p_2 - r_2p_1)^2 \} \frac{(p_2 - r_2)(p_2r_1 - p_1r_2)}{(\bar{p} - \bar{r})^2} e^{-\lambda(\beta + x)^2/2 - \beta \bar{p}^2} \]  

with \( \bar{w} = \alpha \hat{r} + \beta \hat{p} \) and \( t = \alpha + \beta + z \) and

\[ B_1 = r_2p_2(w_1^2 + w_2^2) + 3w_2(p_2r_1 - p_1r_2)(\beta p_1 - \alpha r_1) + 2\alpha \beta w_2(r_2 + p_2)(p_2r_1 - p_1r_2)^2 \]

\[ B_2 = \frac{1}{2}(3r_1p_1 + r_2p_2) - 2\alpha \beta r_2p_2(p_2r_1 - p_1r_2)^2 - w_2(r_2 + p_2)\hat{r} \cdot \hat{p} \]

\( (p_2 - r_2)(p_2r_1 - p_1r_2) \) (\( r_1p_2 - r_2p_1 \))

Integrating (35) over \( \alpha \) gives

\[ K_3 \equiv (-\frac{\lambda}{4\pi})^3(-2)^3 \frac{\pi}{2} \int_0^\infty d\beta \sum_{j=0}^\infty C_je^{-\lambda(\beta + x)^2/2} \frac{(p_2 - r_2)(p_2r_1 - p_1r_2)}{(\bar{p} - \bar{r})^2} e^{-\lambda(\beta + x)^2/2} \]

with each of the \( C_j \) given below.

We now take up the integration over \( \hat{r} \) with \( C_0 \) as an example, it being given by

\[ C_0 = \int \frac{r_1(r_2 - p_2) a + 6\beta r_2^2 a^2}{K^2} \left\{ \frac{1}{K^2} \left\{ 2r_2p_2 \hat{r} \cdot \hat{p} + 2k(r_2p_2 + 3r_1p_1) + 2p_2p_1(r_2 - r_1) a + \beta x(r_2 \hat{p}^2 - p_1p_2 a - \frac{3}{2}r_2 \hat{p}) \right\} \right\} e^{|k|} \]

with \( f = [\beta \hat{p} - k\hat{r}]^2 \), \( k = \beta + z \) and \( a = r_1p_2 - r_2p_1 \);

defining

\[ I_0 \equiv \int d^2r \ C_0 \ e^{-\lambda(\beta + x)^2/2} \frac{(p_2 - r_2)(p_2r_1 - p_1r_2)}{(\bar{p} - \bar{r})^2} e^{-\lambda(\beta + x)^2/2} \]

we obtain

\[ I_0 \equiv \int \frac{\mu^3}{2} \int_0^\mu d\nu \int_0^\nu \frac{d\nu}{\bar{p}^2} e^{\mu^2\nu^2/2} \left\{ (-\mu^2 + \nu\mu) e^{-\mu^2(\beta + x)^2/2} + \frac{\pi}{2} \int_0^\mu d\nu \int_0^\nu \frac{d\nu}{\bar{p}^2} e^{\mu^2\nu^2/2} A_2 e^{-\nu^2[\beta(\mu + x)^2]} \right\} \]

with \( l = (\beta + z)(\mu(\beta + z) + 1) + \nu + x \), \( m = \mu(\beta + z) + 1 + \nu \), \( n = \beta(\beta + z) + \nu \) and

\[ b = \mu(\beta + z)^2 + \nu + x, \]

\[ A_0 = p_1 \left[ p_2^2 \hat{p}^2 m^3 + l \left( \frac{m}{2} r_2^2 - 2m^2p_2^2 \right) + l^2(m^2p_2^2 + 2p_2^2) \right] \]

\[ A_1 = \frac{\mu}{\pi} p_1 \left( 3m^2p_2^2 + 2m^2p_2^2 + l^2 \left( p_2^2 + \frac{2}{2} p_2^2 - 2m^2p_2^2 \right) \right) \]
we define

\[
A_2 = p_1 \frac{1}{k^2} \left[ \beta^2 \left( 2n^2 p^2 \bar{p}^2 + b p^2 [1 - n(2 p^2 + p_1^2)] \right) + bk \left( \frac{1}{2} n(3 p_1^2 - p_2^2) + b p^2 \right) \right] \\
+ p_1 \frac{1}{k^2} \beta^2 \left( n^2 p^2 \bar{p}^2 + b \left[ \frac{1}{2} (3 p_1^2 + p_2^2) - 2 n p_2^2 \bar{p}^2 \right] \right) + \frac{\beta z b}{b} \left( \frac{3}{\beta} n^2 p^2 + \left( 3 - \frac{1}{4} n p_2^2 \right) - b p_2^2 \right)
\]

Similarly, with

\[
C_3 = \frac{1}{f^2} \left[ \frac{1}{2} (3 r_1 p_1 + r_2 p_2) + \beta (p_2 \bar{r} \cdot \bar{p}(r_2 - p_2) - p_2 (p_1 + r_1) a - 2 a^2) + 2 \beta^2 a^2 p_2 (p_2 - 7 r_2) \right] \\
+ e^{f/k} \left[ - \frac{1}{2} (3 r_1 p_1 + r_2 p_2) + \beta (-2 a^2 - a p_2 r_1 + r_2 p_2 \bar{p}^2) + 2 \beta^2 a^2 p_2 (p_2 + r_2) \right]
\]

we define

\[
I_1 \equiv \int d^2 r \ C_1 e^{-x^2 - \beta \left( \bar{r} \cdot \bar{p} \right) \left( p^2 - 2 \beta r_2 \right)} e^{-x \bar{r}^2}
\]

(37)

to get

\[
I_1 = \\
\frac{\pi}{2} \int_0^\infty \mu \ d\mu \int_0^\infty \frac{e^{m^2 \bar{p}^2 / \mu}}{i^3} p_1 (B_0 + \beta B_1 + 2 \beta^2 B_2) e^{-(\beta (\mu + 1) + \nu) \bar{p}^2} + \frac{\pi}{2} \int_0^\infty \mu \ d\mu \int_0^\infty \frac{e^{m^2 \bar{p}^2 / \mu}}{b^3} p_1 (B_3) e^{-\nu + \beta (\mu + 1) \bar{p}^2}
\]

with

\[
B_0 = \frac{1}{2} m (3 p_1^2 - p_2^2) + lp_2^2
\]

\[
B_1 = \frac{1}{i} (3 m^2 p_1^2 \bar{p}^2 - l [3 p_1^2 - p_2^2 + 2 m p_2^2 \bar{p}^2] - l^2 m^2 p_2^2 \bar{p}^2)
\]

\[
B_2 = 3 \frac{2 \beta}{i^2} (5 m^4 p_1^2 \bar{p}^2 - 7 l m \bar{p}^2 + 4 l^2 \bar{p}^2)
\]

\[
B_3 = - \left( \frac{1}{2} n (3 p_1^2 - p_2^2) + b p_2^2 \right) + \beta \left( \frac{3}{\beta} n^2 p_1^2 \bar{p}^2 - [3 p_1^2 + p_2^2 + 2 n p_2^2 \bar{p}^2] - b^2 p_2^2 \bar{p}^2 \right) + \frac{6}{b} \beta^2 n p_2^2 \bar{p}^2
\]

Continuing, we use

\[
C_2 = \\
\frac{k}{r^2} \left[ (ar_1 (3 r_2 - p_2) + r_2 (p_2 - \bar{r} \cdot \bar{p}) + r_1 r_2 p_1 p_2 + \beta a^2 (20 r_1^2 - 2 r_2 p_2)) + e^{f/k} (r_2 + p_2) (ar_1 - p_2 \bar{r} \cdot \bar{p}) - 2 \beta a^2 r_2 p_2] \right]
\]

\[
C_3 = \frac{\beta^2}{r^2} \left( r_2 p_2 \bar{r}^2 + 6 p_1 p_2 a \right) (-1 + e^{f/k}) + 16 \beta a^2 p_2^2 (1 + 2 e^{f/k})
\]

\[
C_4 = \frac{\beta k}{r^2} \left( (4 r_2 p_2 \bar{r} \cdot \bar{p} - 6 a^2 + 4 \beta p_2 (r_2 + p_2) a^2) (1 - e^{f/k}) - 32 \beta r_2 p_2 a^2 (2 + e^{f/k}) \right)
\]
\[ C_5 = \frac{k^2}{f^3} \left\{ (2r_2^2 \hat{p} - 4r_1 r_2 a + 48\beta a^2 r_2^2) - 4\beta r_2(p_2 + r_2)a^2(-1 + e^{f/k}) \right\} \]

\[ C_6 = -\beta^2 \frac{p_2}{f^2k} e^{f/k}(-4p_1a + 2p_2^2 \cdot \hat{p} + 8\beta a^2 p_2^2) \]

\[ C_7 = -48\beta \frac{k^3}{f^4} a^2 r_2^2 \left( -1 + e^{f/k} \right) \]

\[ C_8 = 48 \frac{k^3}{f^4} a^2 p_2^2 \left( 1 - e^{f/k} \right) \]

\[ C_9 = -96\beta^2 \frac{k^2}{f^4} r_2 p_2 a^2 \left( 1 - e^{f/k} \right) \]

to define for all \( k \geq 2 \)

\[ I_k = \int d^2r \ C_k \ e^{-x r^2 - \beta (\hat{r} - \hat{p})^2 \frac{(p_2 - r_2)(p_2 r_1 - p_1 r_2)}{\hat{p} - \hat{r}}^2} e^{-x r^2} \quad (38) \]

The calculation of the integrals involved in each of the terms above is tedious and will be reported elsewhere; simultaneously an effort will be made to present the result of the calculation in a manner so as to appear as an extension of eq.(25) to the third order in \( \lambda \).

To conclude, we have used the Schwinger expansion [7] in this paper to rework the Antonsen-Bormann idea[5,6]to obtain the \( \zeta(s) \) function to second order in the gravitational constant \( G \) for the Lagrangian density (1) in \( 2 + 1 \) dimensional curved space, the metric for the latter being defined by the stationary solutions[3,4] of the Einstein field equations.

Acknowledgements

A preliminary version of this work was presented at FFP11 that was held in Paris, France from July 6 – 9, 2010 and this was improved upon at the presentation at QTS7 that was held at Prague from Aug 7 – 13, 2011.I thank the respective organizers for their generous invitation to the two conferences and am grateful to the Indian Institute of Technology Madras for their financial support. I thank K.P.Deepesh, Ravi Shankar and G.Krishna Kumar for their generous assistance in the preparation of this manuscript. While working on this paper I have had useful correspondence with D.Broadhurst of The Open University, Milton Keynes, U.K , S.Laporta of INFN, Bologna,Italy and Ivan Gonzalez of the Universidad Santa Maria, Valparaiso, Chile and thank them for it.
References

1. Kamath, S G 2009 “An exact calculation of the Casimir energy in two planar models” in Frontiers of Physics – 2009, AIP Conference Proceedings edited by Swee-Ping Chia, vol.1150 pp.402-406.

2. Kamath S G 2009 “A derivation of the scalar propagator in a planar model in curved space”, in Frontiers of Fundamental and Computational Physics – 2009, AIP Conference Proceedings edited by J.G.Hartnett and P.C.Abbott, vol.1246 pp.174-177.

3. Deser S, Jackiw R and ’tHooft G 1984, Ann.Phys.120 220

4. Clement G. 1985 Int.J.Theor.Phys.24 267

5. Antonsen F and Bormann K 1996 “Propagators in Curved Space ”Preprint hep-th/9608141v1.

6. Bormann K and Antonsen F 1995 in Proceedings of the 3rd Alexander Friedmann International Seminar,St.Petersburg 1995;Preprint hep-th/9608142v1.

7. Schwinger J 1951 Phys.Rev.82 664

8. McKeon D G C and Sherry T N 1987 Phys.Rev.D35 3854

9. See Kamath S G 1999 Mod.Phys.Lett.A14 1391 for an exact evaluation of the scale and conformal anomalies for the Landau problem with this method.