ALL THE EXACT SOLUTIONS OF GENERALIZED CALOGERO-SUTHERLAND MODELS

Nobuyoshi Ohta\(^1\)

Department of Physics, Osaka University

Toyonaka, Osaka 560, Japan

Abstract

A collective field method is extended to obtain all the explicit solutions of the generalized Calogero-Sutherland models that are characterized by the roots of all the classical groups, including the solutions corresponding to spinor representations for \(B_N\) and \(D_N\) cases.

Key words: exact solutions, Calogere-Sutherland models, collective field method, \(W_N\) algebra, singular vectors, spinor solutions, and conformal field theory.

\(^{1}\)e-mail address: ohta@phys.wani.osaka-u.ac.jp
1 Introduction

Recently there has been much interest in the Calogero-Sutherland (CS) models, [1, 2] which describe one-dimensional many-body quantum systems with inverse-square long-range interactions. The reason for the interest is that the models play significant roles in diverse subjects such as fractional statistics, [3, 4, 5, 6, 7] quantum Hall effect [8, 9, 10] and $W_\infty$ algebra. [11] The original models have a structure related to the classical group $A_N$.

Among many variants of the CS models, [3] a class of models have been known to be exactly solvable and show interesting behaviors similar to the original ones. [12] In particular the so-called CS model of $BC_N$-type (hereafter referred to as $BC_N$-CS model) is the most general one with $N$ interacting particles. By setting various coupling constants to zero, we can reduce the model to all other models of $B_N$, $C_N$ and $D_N$ types. These models are known to be relevant to one-dimensional physics with boundaries.

It was Stanley and Macdonald [13] who found that the solutions for $A_N$ type are expressed by Jack symmetric polynomials and studied their properties. The explicit formulae for the wave functions for these models have recently been obtained by the use of collective field method [14, 15] and conformal field theory technique by Awata et al. [16, 17] They showed that the Hamiltonian can be expressed in terms of Virasoro and $W_M$ generators of positive modes and hence the solutions can be represented as $W_M$ singular vectors, whose explicit forms are given by integral representations using free bosons. (Here the notation is slightly changed from that in ref. [16]; $M$ is an arbitrary integer ($\geq 2$) which characterizes the $W$ algebra used in the construction of the solutions and is independent of the number of the particles $N$.) Unfortunately the wave functions were not known for the generalized $BC_N$-CS models except for the ground states, [12, 18] though the energy eigenvalues have been obtained for both ground and excited states. [18, 19] These wave functions are important for examining various properties of the models, like correlation functions.

In a previous paper [20], we have given a systematic method to construct the wave functions for excited states in these generalized models by extending the collective field method. The method was applied to those solutions common to the CS models of all types.
of the classical groups. These solutions are the only ones to the models characterized by
the root systems of the $C_N$ and $BC_N$ groups. However, it turns out that they do not
exhaust the whole solutions for the generalized systems; there are additional solutions
in the models of $B_N$ and $D_N$ types corresponding to the spinor representations in these
groups. In this paper, we will derive all these solutions, namely wave functions and
energy eigenvalues, explicitly by extending our previous results using the collective field
method and clarify the whole structure of the solutions.

The paper is organized as follows. In order to establish our notations and conven-
tions, we summarize in § 2 the $BC_N$-CS models characterized by the root systems of the
classical groups and the free field realization of the $W_M$ algebra which is necessary for
our subsequent discussions. In § 3, our method is explained and applied to the $BC_N$-
and $C_N$-CS models. In § 4, we discuss the solutions for the $B_N$-CS models. In § 5, the
solutions for the $D_N$-CS models are obtained. Section 6 is devoted to discussions.

2 Preliminaries

In this section, we summarize our notations and conventions which will be used in the
following discussions.

2.1 Generalized CS models

It has been known for some time [12] that there exist a class of models that are character-
ized by the root systems of the classical groups and are exactly solvable. The Hamiltonian
is given by

$$H_{GCS} = -\sum_{i=1}^{N} \frac{1}{2} \frac{\partial^2}{\partial q_i^2} + \frac{1}{2} \left( \frac{\pi}{L} \right)^2 \sum_{\vec{\alpha} \in R_+} \frac{\mu_\alpha (\mu_\alpha + 2\mu_2) - 1 |\vec{\alpha}|^2}{\sin^2 \frac{\pi}{L} (\vec{\alpha} \cdot \vec{q})},$$

(2.1)

where $R_+$ stands for positive roots of the classical group under consideration and the
coupling constants $\mu_\alpha$ are equal for the roots of the same length. The most general
model is the one with all the roots in $B_N$ and $C_N$ algebras. This is the $BC_N$-CS model
we are going to discuss.
We first introduce the variables

\[ x_j \equiv \exp \left( \frac{2\pi i q_j}{L} \right); \quad D_i \equiv x_i \frac{\partial}{\partial x_i}. \]  

(2.2)

Using these variables, the Hamiltonian (2.1) is cast into

\[ H_{GCS}(x_i; \beta, \gamma, \delta) = \frac{1}{2} \left( \frac{2\pi}{L} \right)^2 \left[ \sum_{i=1}^{N} D_i^2 - 2\beta(\beta - 1) \sum_{i,j=1}^{N} \frac{x_i x_j}{(x_i - x_j)^2} + \frac{x_i x_j^{-1}}{(x_i - x_j^{-1})^2} \right] \]

\[ - \sum_{i=1}^{N} \left( \gamma(\gamma + 2\delta - 1) \frac{x_i}{(x_i - 1)^2} + 4\delta(\delta - 1) \frac{x_i^2}{(x_i^2 - 1)^2} \right), \]  

(2.3)

where we have used \( \beta, \gamma, \delta \) for coupling constants. We note that putting \( \gamma = 0 \) reduces the model to \( C_N \)-type, \( \delta = 0 \) to \( B_N \)-type, and finally \( \gamma = \delta = 0 \) to \( D_N \)-type. We also refer to these models as \( C_N \)-, \( B_N \)- and \( D_N \)-CS models, respectively.

The ground state wave function and energy are given by \[ \Delta_{GCS} = \prod_{i=1}^{N} \left( \sin \frac{\pi}{L} q_i \right)^{\gamma} \left( \sin \frac{2\pi}{L} q_i \right)^{\delta} \prod_{i<j}^{N} \left( \sin \frac{\pi}{L} (q_i - q_j) \sin \frac{\pi}{L} (q_i + q_j) \right)^{\beta} \]

\[ \simeq \prod_{i=1}^{N} x_i^{-\beta(N-1)-\gamma/2-\delta}(x_i - 1)^{\gamma(x_i^2 - 1)\delta} \prod_{i<j}^{N} (x_i - x_j)^{\beta(x_i x_j - 1)^\beta} \]

\[ E_{0}^{GCS} = \sum_{i=1}^{N} \left[ \frac{\gamma}{2} + \delta + \beta(N - i) \right]^2. \]  

(2.4)

Now our eigenvalue problem

\[ H_{GCS} \Delta_{GCS} \Phi^{GCS} = E_{GCS} \Delta_{GCS} \Phi^{GCS}, \]  

(2.5)

reduces to

\[ H_{eff} \Phi^{GCS} = E_{eff} \Phi^{GCS}; \]

\[ E_{GCS} = \frac{1}{2} \left( \frac{2\pi}{L} \right)^2 \left[ E_{0}^{GCS} + E_{eff} \right], \]  

(2.6)

where the effective Hamiltonian \( H_{eff} \) acting on the function \( \Phi^{GCS}(x) \) is given by \[ H_{eff}(x_i : \beta, \gamma, \delta) = \sum_{i=1}^{N} D_i^2 + \beta \sum_{i,j=1}^{N} \frac{x_i + x_j}{x_i - x_j} (D_i - D_j) + \frac{x_i + x_j^{-1}}{x_i - x_j^{-1}} (D_i + D_j) \]

\[ + \sum_{i=1}^{N} \left( \gamma \frac{x_i + 1}{x_i - 1} + 2\delta \frac{x_i + x_i^{-1}}{x_i - x_i^{-1}} \right) D_i. \]  

(2.7)
We express this Hamiltonian by free bosons and relate it to the free boson representation of the $W_M$ algebra. For this purpose, let us next summarize relevant results in the free boson representation of this algebra. Since this is described in detail elsewhere, [21, 16, 20] we will be very brief.

### 2.2 $W_M$ algebra

Let $\vec{e}_i \ (i = 1, \cdots, M)$ stand for an orthonormal basis ($\vec{e}_i \cdot \vec{e}_j = \delta_{ij}$) for $A_M$ algebra. We define the weights of the vector representation $\vec{h}_i$, the simple roots $\vec{\alpha}_a \ (a = 1, \cdots, M-1)$ and the fundamental weights $\vec{\Lambda}_a$ by

\[
\vec{h}_i = \vec{e}_i - \frac{1}{M} \sum_{j=1}^{M} \vec{e}_j, \quad \vec{\alpha}_a = \vec{h}_a - \vec{h}_{a+1}, \quad \vec{\Lambda}_a = \sum_{i=1}^{a} \vec{h}_i,
\]

\[
\vec{\alpha}_a \cdot \vec{\alpha}_b \equiv A_{ab} = 2\delta_{a,b} - \delta_{a,b+1} - \delta_{a,b-1}, \quad \vec{\alpha}_a \cdot \vec{\Lambda}_b \equiv A_{a}^{b} = \delta_{a}^{b}.
\] (2.8)

We then introduce $M-1$ free bosons

\[
\vec{\phi}(z) = \sum_{a=1}^{M-1} \phi^a(z) \vec{\Lambda}_a = \sum_{a=1}^{M-1} \phi^a(z) \vec{\alpha}_a.
\] (2.9)

They have the mode expansion

\[
\vec{\phi}(z) = \vec{q} + \vec{a}_0 \ln z - \sum_{n \neq 0} \frac{1}{n} \vec{a}_n z^{-n},
\] (2.10)

with the commutation relations

\[
[a^a_n, a^b_m] = A^{ab} n \delta_{n+m,0}, \quad [a^a_0, q^b] = A^{ab},
\] (2.11)

The boson Fock space is generated by the oscillators of negative modes on the highest weight state

\[
|\vec{\lambda}\rangle = e^{\vec{\lambda} \cdot \vec{q}} |\vec{0}\rangle; \quad \vec{a}_n |\vec{0}\rangle = 0 \ (n \geq 0).
\] (2.12)

We define $\langle \vec{\lambda}' | \vec{\lambda} \rangle$ similarly with the inner product $\langle \vec{\lambda} | \vec{\lambda}' \rangle = \delta_{\vec{\lambda},\vec{\lambda}'}$.

We need only the spin 2 and 3 generators of the $W_M$ algebra, which are given by [21]

\[
T(z) \equiv \sum_n L_n z^{-n-2} = \frac{1}{2} (\partial \vec{\phi}(z))^2 + \vec{\alpha}_0 \cdot \partial^2 \vec{\phi},
\]
\[ W(z) \equiv \sum_n W_n z^{-n-3} \]
\[ = \sum_{a=1}^{M-1} (\partial \phi_a(z))^2 (\partial \phi_{a+1}(z) - \partial \phi_{a-1}(z)) \]
\[ + \alpha_0 \sum_{a,b=1}^{M-1} (1-a) A^{ab} \partial \phi_a(z) \partial^2 \phi_b(z) + \alpha_0^2 \sum_{a=1}^{M-1} (1-a) \partial^3 \phi_a(z), \quad (2.13) \]

where
\[ \alpha_0 = \sqrt{\beta} - \frac{1}{\sqrt{\beta}}, \]
\[ \bar{\rho} = \sum_{a=1}^{M-1} \bar{\Lambda}_a; \quad (\bar{\rho})^2 = \frac{1}{12} M(M^2 - 1). \quad (2.14) \]

The highest weight states of the \( W_M \) algebra are created from the vacuum by the vertex operator as \(|\bar{\lambda}\rangle = :e^{\bar{\lambda} \phi(0)} : |0\rangle\), whose conformal weight \( h(\bar{\lambda}) \) and \( W_0 \)-eigenvalue \( w(\bar{\lambda}) \) are
\[ h(\bar{\lambda}) = \frac{1}{2} \left[ (\bar{\lambda} - \alpha_0 \bar{\rho})^2 - \alpha_0^2 (\bar{\rho})^2 \right], \]
\[ w(\bar{\lambda}) = \sum_{a=1}^{M-1} \left[ \lambda_a^2 (\lambda_{a+1} - \lambda_{a-1}) + \alpha_0 (2(a-1) \lambda_a + (1-2a) \lambda_{a+1}) + (1-2a) \lambda_{a+1} \right]. \quad (2.15) \]

Another formula which will be useful is
\[ h(\tilde{\lambda}_{\vec{r}, \vec{s}}) - \sqrt{\beta} \sum_{a=1}^{M-1} r^a s^a = h(\tilde{\lambda}_{\vec{r}, \vec{s}}) + \sum_{a=1}^{M-1} r^a s^a. \quad (2.16) \]

We can define singular vectors \(|\vec{r}, \vec{s}\rangle\) at level \( \sum_{a=1}^{M-1} r^a s^a \) with the highest weight \(|\tilde{\lambda}_{\vec{r}, \vec{s}}\rangle\), where \( \tilde{\lambda}_{\vec{r}, \vec{s}} \) is defined by
\[ \tilde{\lambda}_{\vec{r}, \vec{s}}^+ = \sum_{a=1}^{M-1} \left[ (1+r^a-r^{a+1}) \sqrt{\beta} - (1+s^a) \sqrt{\beta} \right] \bar{\Lambda}_a, \]
\[ \tilde{\lambda}_{\vec{r}, \vec{s}}^- = \sum_{a=1}^{M-1} \left[ (1+r^a) \sqrt{\beta} - (1+s^a-s^{a+1}) \sqrt{\beta} \right] \bar{\Lambda}_a. \quad (2.17) \]

The explicit forms of the singular vectors are known in an integral form using free bosons, but will not be needed in our following discussions. Suffice it to say that they are annihilated by Virasoro \( L_n \) and \( W_n \) generators of positive modes and correspond to the following Young diagrams parameterized by the numbers of boxes in each row, \( \lambda = (\lambda_1, \cdots, \lambda_M) \), \( \lambda_1 \geq \cdots \geq \lambda_M \geq 0 \):
We can read off the relation between $\lambda$ and $\vec{r}, \vec{s}$ from this diagram.

3 Exact solutions for $BC_N$- and $C_N$-CS models

In this section, we begin with the brief description of the exact solutions for the models of $BC_N(\gamma \neq 0)$ and $C_N(\gamma = 0)$ types with $\delta \neq 0$. These are also common solutions to all the CS models if we set the coupling constants to zero appropriately. This was the main result in our previous paper, [20] but will be heavily used in our following construction of all exact solutions.

First note that our system (2.7) has the reflection invariance under $x_i \rightarrow x_{i-1}$ for each $i$ in addition to the permutation symmetry under $x_i \leftrightarrow x_j$. In fact, the solutions can be given by the symmetric power sums

$$p_n = \sum_{i=1}^{N} (x_i^n + x_i^{-n}).$$  \hspace{1cm} (3.1)

It is known in mathematical literature [22] that the representation ring for $BC_N$ and $C_N$ systems is isomorphic to the ring generated by these symmetric power sums. Hence all the solutions for these systems, which correspond to representations in the algebras, can be obtained by using these functions (3.1).

In terms of (3.1), we can express our effective Hamiltonian. This Hamiltonian is then mapped into oscillator representation by

$$|f\rangle \mapsto f(x) \equiv \langle \vec{\lambda}|C_{\beta}|f\rangle$$

$$C_{\beta} \equiv \exp \left( \sqrt{\beta} \sum_{n>0} \frac{1}{n} a_{n,1} p_n \right),$$ \hspace{1cm} (3.2)

which gives the following correspondence between the oscillators and the power sums (3.1):

$$\sqrt{\beta} p_n \leftrightarrow a_{n}^1; \quad \frac{n}{\sqrt{\beta}} \frac{\partial}{\partial p_n} \leftrightarrow a_{n,1}.$$ \hspace{1cm} (3.3)
The Hamiltonian is further rewritten using the Virasoro generator $L_n$ of positive modes and $W_0$. The result of this series of manipulations is \[ \hat{H}_{\text{eff}} = \hat{H}' + \sum_{n>0} \hat{H}_n + \sum_{a>1} \sum_{n>0} a_n^a (\cdots) + \sqrt{\beta} \sum_{n>0} \left( \frac{2}{\sqrt{N}} a_{-n}^1 L_n - 2 a_{n,1} L_n \right) + 2 \sum_{n>0} \{ \gamma L_n + (2 \delta - \beta) L_{2n} \}, \tag{3.4} \]

where

\[
\begin{align*}
\hat{H}' &= \sum_{n>0} \tilde{a}_n \cdot \tilde{a}_n \left( 2N \beta - 1 + \gamma + 2 \delta - 2 \sqrt{\beta} a_{0,1} \right) + \sqrt{\beta} (W_0 - W_{0,\text{zero}}), \\
\hat{H}_n &= 2 \gamma \sum_{a=1}^{M-1} \sum_{m=1}^{n-1} a_{n-m,a}(a_{m,a+1} - a_{m,a}) + \frac{2 \gamma}{\sqrt{\beta}} a_{n,1} \left\{ (n+1)(\beta-1) + N \beta - \sqrt{\beta} a_{0,1}^1 \right\} \\
&\quad + 2 \frac{2 \gamma}{\sqrt{\beta}} \sum_{a=2}^{M-1} a_{n,a} \left\{ (n+1)(\beta - 1) - \sqrt{\beta} a_{0,a}^a \right\} \\
&\quad + 2 \frac{2 \gamma}{\sqrt{\beta}} \sum_{a=1}^{M-1} \sum_{m=1}^{n-1} a_{n,1}(a_{n-m,a} - a_{n-m,a+1} a_{m,a}) \\
&\quad + 2(\beta - 2 \delta) \sum_{a=1}^{M-1} \sum_{m=1}^{n-1} a_{2n-m,a}(a_{m,a} - a_{m,a+1}) \\
&\quad - 2(a_{n,1})^2 \left\{ (n+1)(\beta - 1) + N \beta - \sqrt{\beta} a_{0,1}^1 \right\} \\
&\quad - 2 \sum_{a=2}^{M-1} a_{n,1} a_{n,a} \left\{ (n+1)(\beta - 1) - \sqrt{\beta} a_{0,a}^a \right\} \\
&\quad - \frac{2}{\sqrt{\beta}} \left[ (\beta - 2 \delta) a_{2n,1} \left\{ (2n+1)(\beta - 1) + N \beta - \sqrt{\beta} a_{0,1}^1 \right\} + \beta a_{2n,1} \right] \\
&\quad - \frac{2}{\sqrt{\beta}}(\beta - 2 \delta) \sum_{a=2}^{M-1} a_{2n,a} \left\{ (2n+1)(\beta - 1) - \sqrt{\beta} a_{0,a}^a \right\}. \tag{3.5} \end{align*}
\]

Here caret on the Hamiltonian implies that it is expressed in terms of oscillators and $W_{0,\text{zero}}$ in $\hat{H}'$ is the zero mode part of $W_0$. The third term involving $a_{n,a}^a$ ($a > 1$, $n > 0$) in (3.4) vanishes after multiplying by $\langle \tilde{\lambda} | C_\beta$ and may be disregarded in the following. An important observation is that $\hat{H}'$ is the sum of number operators and $W_M$ zero mode and also that $\hat{H}_n$ consist of annihilation operators only. It is at this point that the Virasoro generators $L_n$ and $W_0$ are necessary; they are used to put cubic terms involving $a_{n,a}^a$ ($n > 0$) in (3.4) into the form combined with $L_n$.

To construct our eigenstates of the Hamiltonian $\hat{H}_{\text{eff}}$, we take singular vectors at the level $\sum_{a=1}^{M-1} r^a s^a$. Since these are annihilated by Virasoro generators $L_n$ of positive
modes, only the first two terms in (3.4) are relevant to our problem. These are already eigenstates of $\hat{H}'$ with the eigenvalue $E_\lambda$

\[
E_\lambda = \left[h \left( \vec{\lambda}_{\vec{r}, \vec{s}} - \sqrt{\beta} \sum_{a=1}^{M-1} r^a \bar{\alpha}^a \right) - h \left( \vec{\lambda}_{\vec{r}, \vec{s}} \right) \right] \\
\times \left[ 2N\beta - 1 + \gamma + 2\delta - 2 \left( \beta r_1 - s_1 + \sqrt{\beta\alpha_0}\beta_1 \right) \right] \\
+ \sqrt{\beta} \left[ w \left( \vec{\lambda}_{\vec{r}, \vec{s}} - \sqrt{\beta} \sum_{a=1}^{M-1} r^a \bar{\alpha}^a \right) - w \left( \vec{\lambda}_{\vec{r}, \vec{s}} \right) \right] \\
= \sum_{a=1}^{M-1} r^a s^a s^a + 2 \sum_{a,b=1, a>b}^{M-1} r^a s^a s^b \\
+ \sum_{a=1}^{M-1} r^a s^a \left( 2N\beta - \beta + \gamma + 2\delta - \beta r^a \right), \\
= \sum_{i=1}^{N} \left[ \lambda_i^2 + 2 \left\{ \beta(N - i) + \frac{\gamma}{2} + \delta \right\} \lambda_i \right]. \tag{3.6}
\]

Here use has been made of eqs. (2.15) and (2.16) in deriving the second equality, and of the relation between $\lambda$ and $\vec{r}, \vec{s}$ obtained from the Young diagram in getting the third equality.

It is noted in ref. [20] that applying $\hat{H}_n$ on the singular vectors produces only states at the lower levels, and that the excitation energy is given by the eigenvalue given in (3.6); $E_{e\text{ff}} = E_\lambda$. Thus the eigenstates of our system can be written as

\[
|\Phi_{GCS}^{\lambda} \rangle = |J_\lambda \rangle + \sum_{\mu < \lambda} C_\mu |J_\mu \rangle, \tag{3.7}
\]

where $|J_\lambda \rangle$ is the oscillator representation of the Jack polynomials for the $A_N$ case (or the $W_M$ singular vectors) with the coefficients $C_\mu$ to be determined from the highest state $|J_\lambda \rangle$ by the application of $\hat{H}_n$:

\[
\langle J_\nu | \sum_{n>0} \hat{H}_n |J_\lambda \rangle + \sum_{\mu < \lambda} C_\mu \langle J_\nu | \sum_{n>0} \hat{H}_n |J_\mu \rangle \\
= C_\nu (E_\lambda - E_\nu), \quad (\nu < \lambda), \tag{3.8}
\]

The inner products are easily evaluated by using the oscillator algebra. A systematic algorithm was given in ref. [20] how to solve this master equation (3.8) to determine the coefficients $C_\mu$ successively starting from $|J_\lambda \rangle$. In this way the oscillator representation
for our system can be determined from the exact solution for $A_N$ case, which are given by singular vectors of the $W_M$ algebra (but modified to be reflection invariant).

The actual eigenstates in terms of the symmetric power sums (3.1) can then be read off from the explicit expression in terms of the boson oscillators by the rule (3.3). For our later convenience, let us denote the solutions thus constructed as

$$\Phi^\text{GCS}_\lambda(x_i; \beta, \gamma, \delta).$$

(3.9)

The total energy is obtained from eqs. (2.4) and (3.6) as

$$E_0^\text{GCS} + E_\lambda = \sum_{i=1}^{N} \left[ \lambda_i + \beta(N - i) + \frac{\gamma}{2} + \delta \right]^2.$$

(3.10)

A simple example of the solutions is

$$p_1 + 2N\gamma \over 2\beta(N - 1) + \gamma + 2\delta + 1,$$

(3.11)

with the excitation energy $E_\lambda = 2\beta(N - 1) + \gamma + 2\delta + 1$.

These general solutions will play important roles in our construction of all solutions for other systems.

4 Exact solutions for $B_N$-CS models ($\delta = 0$)

In this section, we will derive exact solutions for $B_N$-CS models. The solutions described in the preceding section are, of course, solutions for the $B_N$-CS models if we put $\delta = 0$. However, in addition to these solutions, it turns out that there are other solutions corresponding to the spinor representations for $B_N$-CS models.

Now the Hamiltonian in question is

$$H^\text{eff}_B(x_i; \beta, \gamma) = H^\text{eff}_B(x_i; \beta, \gamma, \delta = 0).$$

(4.1)

It has been known [22] that the representation ring for the $B_N$-CS model is isomorphic to the ring generated by (3.1) and

$$\Delta_B \equiv \prod_{i=1}^{N} \left( \sqrt{x_i} + \frac{1}{\sqrt{x_i}} \right).$$

(4.2)
This is like a “spin field” in string theory and produces solutions corresponding to spinor representations. Obviously the solutions involving odd powers of $\Delta_B$ do not mix with those without $\Delta_B$. Since the square of this function can be expressed in terms of the power sums (3.1), the solutions are divided into two different classes.

(i) Solutions of first class

The first class is those that can be expressed solely by power sums (3.1). These are the solutions $\Phi^GCS_{\lambda}(x; \beta, \gamma, 0)$ already given in (3.9) and the energy eigenvalues are given by

$$\sum_{i=1}^{N} \left[ \lambda_i + \beta(N - i) + \frac{\gamma}{2} \right]^2.$$  \hspace{1cm} (4.3)

(ii) Solutions of second class

The second class is those that contain one $\Delta_B$. We will refer to this class of solutions as spinor solutions. We now show how to derive these spinor solutions in our approach.

First by applying our effective Hamiltonian (4.1), we see that $\Delta_B$ is an eigenstate:

$$H_{eff}^B(x; \beta, \gamma)\Delta_B = E_{B, spinor}^0 \Delta_B,$$

$$E_{B, spinor}^0 = N \left[ \frac{1}{2} + \beta(N - 1) + \gamma \right].$$  \hspace{1cm} (4.4)

Our new solutions are expressed as

$$\Phi^GCS_{B} = \Delta_B \Psi^\text{spinor}_B.$$  \hspace{1cm} (4.5)

The effective Hamiltonian acting on the wave function $\Psi^\text{spinor}_B$ is then derived as

$$H^eff_{\text{spinor}}(x; \beta, \gamma) \equiv (\Delta_B)^{-1}H^eff_B(x; \beta, \gamma)\Delta_B - E_{B, spinor}^0$$

$$= H^eff_B(x; \beta, \gamma) + \sum_{i=1}^{N} \frac{x_i - 1}{x_i + 1} D_i$$

$$= H^eff(x; \beta, \gamma - 1, \delta = 1).$$  \hspace{1cm} (4.6)

We thus see that our problem is reduced to the eigenvalue problem for the first class of solutions with the value of $\gamma$ and $\delta$ shifted to $\gamma - 1$ and 1, respectively. Namely our problem goes back to the solutions of $BC_N$-type with nonzero $\delta$. Fortunately this is already solved in the previous section, and hence our solution is given by (4.3) with

$$\Psi^\text{spinor}_B = \Phi^GCS_{\lambda}(x; \beta, \gamma - 1, 1),$$  \hspace{1cm} (4.7)
in terms of the solution in (3.9). The eigenvalues for these solutions are given by

\[
E = E_{0}^{\text{GCS}}|_{\delta=0} + E_{0,\text{spinor}}^{B} + E_{\lambda|_{\gamma\to\gamma-1,\delta=1}}
\]

\[
= \sum_{i=1}^{N} \left[ \lambda_{i} + \beta(N-i) + \gamma \frac{1}{2} + \frac{1}{2} \right]. \tag{4.8}
\]

Thus the eigenenergy for this case is obtained from that (4.3) of the solution of the first class just by shifting \(\lambda_{i}\) by \(\frac{1}{2}\). This is the reflection of the fact that this second class of solutions correspond to spinor representation of the classical groups \(B_{N}\).

A simple example of this class of solutions is obtained from (3.11) as

\[
\left( p_{1} + \frac{2N(\gamma-1)}{2\beta(N-1)+\gamma+2} \right) \Delta_{B}. \tag{4.9}
\]

5 Exact solutions for \(D_{N}\)-CS models (\(\gamma = \delta = 0\))

We now turn to the solutions for \(D_{N}\)-CS models. There are several complications in this model. Some of the solutions for \(D_{N}\) case are very similar to those for \(B_{N}\). The difference arises because there are two distinct classes of spinor representations for \(D_{N}\), and hence two classes of solutions corresponding to these in addition to the solutions in §3. Not only those, we also have further additional solutions.

Our Hamiltonian is

\[
H_{D}^{\text{eff}}(x_{i};\beta) = H_{D}^{\text{eff}}(x_{i};\beta,\gamma = 0, \delta = 0). \tag{5.1}
\]

The representation ring for the \(D_{N}\)-CS model is isomorphic \([22]\) to the ring generated by (3.1) and

\[
\Delta^{\pm} = \prod_{i=1}^{N} \left( \sqrt{x_{i}} \pm \frac{1}{\sqrt{x_{i}}} \right). \tag{5.2}
\]

In a different basis

\[
\Delta^{+} \pm \Delta^{-} = \sum_{\epsilon_{1}\epsilon_{2} \cdots \epsilon_{N} = \pm 1} 2x_{1}^{\epsilon_{1}/2}x_{2}^{\epsilon_{2}/2} \cdots x_{N}^{\epsilon_{N}/2},
\]

\[
(\epsilon_{i} = \pm 1; i = 1, \cdots, N), \tag{5.3}
\]

we see that these correspond to the spinor representations of opposite chirality.
Again the solutions involving odd powers of $\Delta^\pm$ do not mix with the solutions without $\Delta^\pm$. Since the square of these functions can be expressed in terms of the power sums (3.1), our exact solutions fall into several different classes.

(i) Solutions of first class

The first class is again those that can be expressed solely by power sums (3.1). These are the solutions with $\gamma = \delta = 0$ already given in § 3:

$$\Phi_{\lambda}^{GCS}(x_i; \beta, 0, 0); \quad E = \sum_{i=1}^{N} [\lambda_i + \beta(N - i)]^2. \quad (5.4)$$

(ii) Solutions of second class

The second class is those that contain $\Delta^\pm$. This class is further divided into three classes since the solutions involving $\Delta^\pm$ do not mix with each other and hence they are divided into those with $\Delta^\pm$ and the product of these. (Even powers of these functions are not independent and do not produce new solutions.) These can be constructed in exactly the same way as the spinor solutions for $B_N$.

First by applying our effective Hamiltonian (5.1), we see that $\Delta^\pm$ is an eigenstate with the same eigenvalue:

$$H_{D, \pm}^{eff} \Delta^\pm = E_{0, \text{spinor}}^{D, \Delta^\pm},$$

$$E_{0, \text{spinor}}^{D} = \frac{N}{2} \left[ \frac{1}{2} + \beta(N - 1) \right]. \quad (5.5)$$

Our new solutions corresponding to spinor representations are expressed as

$$\Phi_D^{GCS, \pm} = \Delta^\pm \Psi_{D, \text{spinor}}^\pm. \quad (5.6)$$

The effective Hamiltonian acting on the wave function $\Psi_{D, \text{spinor}}^\pm$ is then derived as

$$H_{D, \pm}^{eff} \equiv (\Delta^\pm)^{-1} H_{D}^{eff} \Delta^\pm - E_{0, \text{spinor}}^{D}$$

$$= H_{D}^{eff} + \sum_{i=1}^{N} \frac{x_i \mp 1}{x_i \pm 1} D_i. \quad (5.7)$$

We thus see that our problem is again reduced to the eigenvalue problems for the first class of solutions, with the value of $\gamma$ and $\delta$ shifted as

$$\begin{cases} 
\gamma = 0 \rightarrow -1 \\
\delta = 0 \rightarrow 1
\end{cases} \text{ for } \Delta^+, \quad (5.8)$$
\[
\begin{aligned}
\left\{
\begin{array}{l}
\gamma = 0 \rightarrow 1 \\
\delta = 0 \rightarrow 0
\end{array}
\right. 
\text{for } \Delta^-.
\end{aligned}
\] (5.8)

The first one \(\Psi^{+}_{D,\text{spinor}}\) gives the same solution (4.7) as the \(B_N\) case with energy eigenvalues (4.8), in which we should put \(\gamma = 0\). The second one is also already solved in § 3, and hence the solutions are given by (5.6) with

\[
\Psi^{+}_{D,\text{spinor}} = \Phi^{GCS}_\lambda(x_i; \beta, -1, 1), \\
\Psi^{-}_{D,\text{spinor}} = \Phi^{GCS}_\lambda(x_i; \beta, 1, 0).
\] (5.9)

The eigenvalues for the solutions \(\Phi^{GCS,-}_{D}\) are given by

\[
E = E^{GCS}_{\gamma=\delta=0} + E^{D,\text{spinor}}_{\gamma=\delta=0} + E^{D}_{\lambda=1,\delta=0}
\]

\[
= \sum_{i=1}^{N} \left[ \lambda_i + \beta(N - i) + \frac{1}{2} \right]^2,
\] (5.10)

which are degenerate with those for \(\Phi^{GCS,+}_{D}\). Thus the energy eigenvalues for this case are again obtained from those of the solutions of the first class just by shifting \(\lambda_i\) by \(\frac{1}{2}\).

Simple examples again from eq. (3.11) are

\[
\left( p_1 + \frac{N}{\beta(N-1)+1} \right) \Delta^\pm.
\] (5.11)

These examples of the solutions for \(N = 2\) were pointed out to the author by D. Serban.

(iii) Solutions of third class

The last class of solutions have the form

\[
\Phi^{GCS}_{D} = \Delta^+ \Delta^- \Psi_{D}.
\] (5.12)

Again \(\Delta^+ \Delta^-\) is an eigenstate of \(H^\text{eff}_{D}\) with eigenvalue

\[
E^D_0 = N[1 + \beta(N - 1)].
\] (5.13)

The effective Hamiltonian acting on the wave function \(\Psi_{D}\) is then derived as

\[
H^\text{eff}_{D} \equiv (\Delta^+ \Delta^-)^{-1} H^\text{eff}_{D} \Delta^+ \Delta^- - E_0^D
\]

\[
= H^\text{eff}_{D} + 2 \sum_{i=1}^{N} \frac{x_i + x_i^{-1}}{x_i - x_i^{-1}} D_i
\]

\[
= H^\text{eff}_{D}(x_i; \beta, \gamma = 0, \delta = 1).
\] (5.14)
We thus see that our problem is again reduced to the eigenvalue problems for the first class of solutions, with the value of $\delta$ shifted as $\delta = 0 \rightarrow 1$, resulting in the $BC_N$ case. The solutions are thus

$$\Psi_D = \Phi_{\lambda}^{GCS}(x_i; \beta, 0, 1). \quad (5.15)$$

The energy eigenvalues for these solutions are given by

$$E = E_0^{GCS}|_{\gamma=\delta=0} + E_0^{D} + E_{\lambda}|_{\gamma=0, \delta=1}$$

$$= \sum_{i=1}^{N} [\lambda_i + \beta(N - i) + 1]^2. \quad (5.16)$$

Though these are similar to those for the first class of solutions with the value of $\lambda_i$ shifted by 1, the symmetry of the wave functions is different; they are anti-symmetric under the reflection.

This exhausts the exact solutions for the $D_N$-CS models.

6 Conclusions and discussions

In this paper, we have first given a systematic algorithm to compute eigenfunctions for excited states for the most general $BC_N$-CS models. It is remarkable that these can be easily obtained from those for the $A_N$ case (but modified to be reflection invariant), which are nothing but singular vectors of the $W_M$ algebra. These are the only solutions for the systems of $BC_N$ type with nonzero coupling constants $\gamma$ and $\delta$ and of $C_N$ type with $\gamma = 0, \delta \neq 0$.

When the coupling constant $\delta = 0$, the model reduces to the $B_N$-CS model, which has additional solutions corresponding to the unique spinor representation of the group. If $\gamma$ is further set to zero, the model reduces to $D_N$-CS models, which has three additional classes of solutions, two of which correspond to the two distinct spinor representations of the group. We have been able to derive all these solutions thanks to the isomorphism between the ‘polynomial’ ring and representations of classical groups. These additional solutions can be actually expressed by using the universal solutions described in § 3 and a kind of “spin fields”. In this sense, the solutions discovered in ref. are the fundamental ones.
There are several interesting extensions of the present work. For example, it should be straightforward to apply our method to CS models based on the exceptional groups, which are also known to be exactly solvable. [12] Another immediate problem is whether this method can be applied to the models involving elliptic functions. It is also known [23] that there is a supersymmetric extension of these models. The exact solvability of the extended models remains unchanged, but this class of models may have deeper connection with the recent exact solutions of four-dimensional $N = 2$ super-Yang-Mills theory. [24]

Another possible direction of investigation is the connection of the present approach and the critical behavior of the models. [25] It was shown that the critical behavior of these models are governed by $c = 1$ conformal field theory. It would be interesting to examine what is the relation between $W_M$ algebra in our formulation and the conformal field theory description with $c = 1$.

We hope that our present investigation motivates these studies and help to shed some light on these problems.

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