Problem of optimal control for bilinear systems with endpoint constraint

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**ABSTRACT**

In this work, we will investigate the question of optimal control for bilinear systems with constrained endpoint. The optimal control will be characterised through a set of unconstrained minimisation problems that approximate the former. Then a class of bilinear systems for which the optimal control can be expressed as a time-varying feedback law will be identified. Finally, applications to parabolic and hyperbolic partial differential equations are provided.

1. Introduction and the problem statement

Linear systems are usually preferable when approximating nonlinear dynamical processes for their simplicity. However, there are many other practical situations for which bilinear models are more appropriate (see Beauchard, 2011; Bradley & Lenhart, 1994; El Alami, 1986; Khapalov, 2010; Mohler & Khapalov, 2000; Wei & Pearson, 1978 and the references therein). In general, a problem of control aims to achieve a certain degree of performance for the system at hand using suitable control laws among available options. If this is indeed feasible, then one usually aims to achieve this performance while optimising a certain criterion. A problem of optimal control is an optimisation problem on a reasonable set described by dynamic constraints. As an interesting example, the question of describing the best control among those that allow to reach a desired state with minimal cost or energy. Such problems arise in various applications, such as the optimisation of hydrothermal systems and non-smooth modelling in mechanics and engineering, etc. (see e.g. Ball, 1984; Bayón et al., 2006, 2014; Lopes, 2009; Polyakova et al., 1996).

The problem of optimal control for bilinear and semilinear systems with unconstrained endpoint has been treated by many authors (see Aronna et al., 2018; Bradley & Lenhart, 1994; Cannarsa & Frankowska, 1992; El Alami, 1986; Li & Yong, 1995; Liang, 1999; Zerrik & El Boukhari, 2018, 2019; Zerrik & Kabouss, 2017). The question of optimal control with endpoint constraint has been treated in Frankowska and Lü (2021) for Mayer type cost functional. Also in the context of linear and semilinear systems with additive controls. The problem of quadratic optimal control with endpoint constraint has been studied by Fattorini and Fattorini (1999), Li and Yong (1995) and the references therein. The main goal of this paper is to study the optimal control problem with a fixed endpoint state for a quadratic cost function. In the case of a bounded set of admissible controls, we will characterise the optimal control either for exactly or approximately attainable states. This problem can be formulated as an optimisation problem with endpoint constraint, which can also be approximated by a set of problems without endpoint constraint. Moreover, if the steering control is scalar valued, then the optimal control can be expressed as a time-varying feedback law.

Let us consider the following system

\[
\begin{align*}
    y(t) &= Ay(t) + B(u(t), y(t)) \\
    y(0) &= y_0 \in X
\end{align*}
\]

where

- \(A : D(A) \subset X \mapsto X\) is the infinitesimal generator of a linear \(C_0\)-semigroup \(S(t)\) on a real separable Hilbert space \(X\) whose inner product and corresponding norm are denoted respectively by \(\langle \cdot, \cdot \rangle\) and \(\| \cdot \|\),
- \(u \in L^2(0, T; U)\), where \(U\) is a real separable Hilbert space equipped with inner product \(\langle \cdot, \cdot \rangle_U\) and corresponding norm \(\| \cdot \|_U\), and \(y\) is the corresponding mild solution to the control \(u\).
- \(B : U \times X \mapsto X\) is a bounded bilinear operator.

Let us now consider the following assumptions:

- \((a)\) For all \(y \in X\) the mapping \(u \mapsto B(u, y)\) is compact,
- \((b)\) \(A\) is the infinitesimal generator of a linear compact \(C_0\)-semigroup \(S(t)\) (i.e. \(S(t)\) is compact operator for all \(t > 0\)).

Note that assumption \((a)\) is systematically satisfied for \(U = \mathbb{R}\).

The quadratic cost function \(J\) to be minimised is defined by

\[
J(u) = \int_0^T \|y(t)\|^2 \, dt + r \int_0^T \|u(t)\|^2_U \, dt.
\]
Here, \( r > 0 \) and \( u \) belongs to the set of admissible controls
\[
U_{ad} = \{ u \in V \mid y(T) = y_d \},
\]
where \( V \) is a nonempty closed convex subset of \( L^2(0, T; U) \) and \( y_d \in X \) is the desired state.

The optimal control problem may be stated as follows
\[
(P) \quad \min_{u \in U_{ad}} J(u).
\]

In order to solve the problem \( (P) \), let us introduce the following auxiliary cost function
\[
J_\varepsilon(u) = \|y(T) - y_d\|^2 + \varepsilon J(u),
\]
where \( \varepsilon > 0 \), and let us consider the following optimal control problem
\[
(P_\varepsilon) \quad \min_{u \in V} J_\varepsilon(u).
\]

This paper is organised as follows: In Section 2, we will first provide a solution to the auxiliary problem \( (P_\varepsilon) \). This result is then applied to build a solution of the problem \( (P) \). We will further provide sufficient conditions on the operators \( A \) and \( B \) under which the solution of the problem \( (P) \) can be expressed as a time-varying feedback law. Section 3 is devoted to examples and simulations.

## 2. Characterisation of the optimal control

### 2.1 Preliminaries

First, let us recall that for all \( u \in L^2(0, T; U) \), the system (1) has a unique mild solution \( y \) corresponding to \( u \), that is continuous from \([0, T]\) into \( X \) and verifies the following variation of constants formula (see e.g. Li & Yong, 1995, p. 66):
\[
y(t) = S(t)y_0 + \int_0^T S(t-s)B(u(s), y(s)) \, ds,
\]
and let us recall the notion of attainability.

**Definition 2.1:**

- A target state \( y_d \in X \) is approximately attainable for the system (1), if for all \( \varepsilon > 0 \) there exists \( u_\varepsilon \in V \) such that \( \| y_{u_\varepsilon}(T) - y_d \| < \varepsilon \).
- A target state \( y_d \in X \) is exactly attainable for the system (1), if there exists \( u \in V \) such that \( y_u(T) = y_d \).

The following lemma provides a continuity property of the solution \( y \) with respect to the control \( u \). The proof of this Lemma has a similar argument with Theorem 3.6 in Ball et al. (1982).

**Lemma 2.2:** If one of the assumptions (a) or (b) hold, then for any sequence \( (u_n) \subset L^2(0, T; U) \) such that \( u_n \rightharpoonup u \) in \( L^2(0, T; U) \), we have
\[
\lim_{n \to +\infty} \sup_{0 \leq t \leq T} \| y_{u_n}(t) - y(t) \| = 0,
\]
where \( y_{u_n} \) and \( y \) are the mild solutions of the system (1) respectively corresponding to \( u_n \) and \( u \).

### 2.2 Optimal control for the problem \( (P_\varepsilon) \)

The following result discusses the existence of the optimal control related to the auxiliary problem \( (P_\varepsilon) \).

**Theorem 2.3:** Let one of the assumptions (a) or (b) hold. Then

- If \( V = \{ u \in L^2(0, T; U) \mid \| u \| \leq M \} \) for some \( M > 0 \), then there exists an optimal control \( u^* \) for the problem \( (P_\varepsilon) \), which satisfies the following formula:
\[
u^*(t) = -\left( \frac{\| B(y^*(t), y^*(t)) \| \phi(t)}{M} + \varepsilon \right)^{-1} \times B(y^*(t), y^*(t)) \phi(t),
\]
where \( y^* \in C(0, T; X) \) is the mild solution of the system (1) corresponding to \( u^* \), \( (B(\cdot, y^*(\cdot)))^* : X \mapsto U \) is the adjoint of the operator \( B(\cdot, y^*(\cdot)) \) and \( \phi \) is the mild solution of the following adjoint system
\[
\begin{cases}
\phi(t) = -A^*\phi(t) - B^*(u^*(t), \phi(t)) - 2\varepsilon y(t) \\
\phi(T) = 2(y(T) - y_d)
\end{cases}
\]

- Suppose that \( V = L^2(0, T; U) \), then the control defined by
\[
u^*(t) = -\frac{1}{\varepsilon} (B(\cdot, y^*(\cdot))^* \phi(t)
\]
is an optimal solution of the problem \( (P_\varepsilon) \), where \( \phi \) is the mild solution of the adjoint system (3).

**Proof:** First let us show the existence of a solution of the problem \( (P_\varepsilon) \).

Since the set \( \{ J_\varepsilon(u) \mid u \in V \} \subset \mathbb{R}^+ \) is not empty and bounded from below, it admits a lower bound \( J^* \). Let \( (u_n) \subset \mathbb{R}^+ \) be a minimizing sequence such that \( J_\varepsilon(u_n) \rightarrow J^* \), then the sequence \( (J_\varepsilon(u_n)) \) is bounded. Since
\[
\frac{r\varepsilon}{2} \int_0^T \| u_n(t) \|^2 \, dt \leq J_\varepsilon(u_n),
\]
the sequence \( (u_n) \) is also bounded, so it admits a subsequence still denoted by \( (u_n) \), which weakly converges to \( u^* \in V \).

Let \( y_n \) and \( y^* \) be the solutions of (1) corresponding to \( u_n \) and \( u^* \), respectively.

From Lemma 2.2 we have
\[
\lim_{n \to +\infty} \| y_n(t) - y^*(t) \| = 0, \quad \forall t \in [0, T].
\]

By the dominated convergence theorem we get
\[
\int_0^T \| y^*(t) \|^2 \, dt = \lim_{n \to +\infty} \int_0^T \| y_n(t) \|^2 \, dt.
\]

Since \( R : u \mapsto \int_0^T \| u(t) \|^2 \, dt \) is convex and lower semi-continuous with respect to the weak topology, we have (see Corollary III.8 of Brezis, 1983)
\[
R(u^*) \leq \liminf_{n \to +\infty} R(u_n).
\]

Combining the formulas (4), (5) and (6) we deduce that
\[
J_\varepsilon(u^*) = \| y^*(T) - y_d \|^2 + \varepsilon \int_0^T \| y(t) \|^2 \, dt
\]
\[
\begin{align*}
&+ \frac{\varepsilon r}{2} \int_0^T \|u^*(t)\| U^2 \, dt \\
&\leq \liminf_{n \to +\infty} \|y_n(T) - y_d\|^2 + \varepsilon \liminf_{n \to +\infty} \int_0^T \|y_n(t)\|^2 \, dt \\
&+ \frac{\varepsilon r}{2} \liminf_{n \to +\infty} \int_0^T \|u_n(t)\|^2 \, dt \\
&\leq \liminf_{n \to +\infty} J_e(u_n) \\
&= f^*.
\end{align*}
\]

We conclude that \( J_e(u^*) = f^* \) and so \( u^* \) is a solution of the problem \( (P_e) \).

Let us proceed to the characterisation of the optimal control. According to Ahmed and Xiang (1994) and Zerrik and El Boukhari (2018, 2019), the mapping \( u \to y \) is Frechét differentiable and the derivative at \( u \) is \( L^2(0, T; U) \) for a given \( h \) is \( L^2(0, T; U) \), denoted by \( z_n \), is the mild solution for the following system

\[
\begin{align*}
&\dot{z}_n(t) = A z_n(t) + B(u(t), z_n(t)) + B(h(t), y(t)) \\
&z_n(0) = 0
\end{align*}
\]

Since the mappings \( y \mapsto \|y\|^2_{L^2(0, T; X)} \) and \( u \mapsto \|u\|^2_{L^2(0, T; U)} \) are Frechét differentiable, the cost function \( J_e(u) \) is Frechét differentiable as well, and the derivative is given by

\[
D_u J_e(u, h) = (J'_e(u, h))_{L^2(0, T; X)} \times L^2(0, T; U)
\]

\[
= \langle y(T) - y_d, z_n(T) \rangle + \varepsilon \int_0^T \langle 2y(t), z_n(t) \rangle \, dt
\]

\[
+ \varepsilon \int_0^T \langle u(t), h(t) \rangle \, dt.
\]

Let \( f \) be the mild solution of the adjoint system (3), whose existence and uniqueness can be deduced from Li and Yong (1995, p. 66), via the change of variables given by \( q(t) = \phi(T - t) \).

Let \( A_n = nA(nI - A)^{-1} \) be the Yosida approximation of the operator \( A \) for a sufficiently large. Then the operator \( A_n \) is linear bounded and such that \( A_n x \to Ax \), as \( n \to +\infty \) for all \( x \in D(A) \) (see Chapter 2 paragraph 3 in Engel & Nagel, 2001).

Let \( u \in L^2(0, T; U) \) and let \( y_n \) and \( \phi_n \) be the respective solutions to (1) and (3) with \( A_n \) instead of \( A \). The introduction of the adjoint equation allows us to express the derivative of the state with respect to the control \( u \). We deduce that

\[
\int_0^T \langle 2y_n(t), z_{hn}(t) \rangle \, dt = \int_0^T (-\phi_n(t) - A_n^* \phi_n(t) - B^*(u(t), \phi_n(t)), z_{hn}(t)) \, dt
\]

\[
= -\int_0^T \langle \phi_n(t), z_{hn}(t) \rangle + \langle \phi_n(t), A_n z_{hn}(t) \rangle \, dt
\]

\[
+ B(u(t), z_{hn}(t)) \, dt
\]

\[
= \int_0^T \langle \phi_n(t), z_{hn}(t) \rangle + \langle \phi_n(t), \dot{z}_{hn}(t) \rangle \, dt
\]

\[
+ \int_0^T \langle \phi_n(t), B(h(t), y_n(t)) \rangle \, dt
\]

\[
= -\langle \phi_n(T), z_{hn}(T) \rangle - \langle \phi_n(0), z_{hn}(0) \rangle
\]

\[
+ \int_0^T \langle \phi_n(t), B(h(t), y_n(t)) \rangle \, dt.
\]

Moreover, since \( \phi_n(T) = 2(y_n(T) - y_d) \) and \( z_{hn}(0) = 0 \), we conclude that

\[
\int_0^T (2\langle y_n(t), z_{hn}(t) \rangle \, dt = -2\langle y(T) - y_d, z_{hn}(T) \rangle
\]

\[
+ \int_0^T \langle (B(\cdot, y_n(t))^* \phi_n(t), h(t)) \rangle \, dt.
\]

Using the fact that \( \phi_n \to \phi, z_{hn} \to z_h \) and \( y_n \to y \) where \( n \to +\infty \), we obtain

\[
\int_0^T (2\langle y(t), z_h(t) \rangle \, dt = -2\langle y(T) - y_d, z_h(T) \rangle
\]

\[
+ \int_0^T \langle (B(\cdot, y(t))^* \phi(t), h(t)) \rangle \, dt.
\]

Consequently, we have the following estimate derivative of \( J_e \).

\[
J'_e(u)(t) = B(\cdot, y^*(t))^* \phi(t) + re(u(t)).
\]

1. The case \( V = L^2(0, T; U) \). Let \( u^* \) be an optimal control solution of the problem \( (P_e) \), then we have \( J'_e(u^*) = 0 \). So we conclude that

\[
u^*(t) = \frac{1}{\varepsilon r} B(\cdot, y^*(t))^* \phi(t),
\]

where \( f^* \) is the mild solution of the adjoint system (3).

2. The case \( V = L^2(0, T; U) \) \( \|u\|_{L^2(0, T; U)} \leq M \).

Let \( u^* \) be an optimal control solution of the problem \( (P_e) \). If \( u^* \) is in the topological interior of \( V \) (i.e. \( \|u^*\|_{L^2(0, T; U)} < M \)), then we have \( J'_e(u)(t) = 0 \), so from (8) we deduce that

\[
u^*(t) = \frac{1}{\varepsilon r} B(\cdot, y^*(t))^* \phi(t).
\]

Otherwise (i.e. \( \|u^*\|_{L^2(0, T; U)} = M \)), we can distinguish two cases, if \( J'_e(u^*) = 0 \) then the control is given by (9) and if \( J'_e(u^*) \neq 0 \) then, we proceed as follows:

Let

\[
\nu_1(t) = \frac{1}{M} u^*(t) \quad \text{and} \quad \nu_2(t) = \frac{1}{\|J'_e(u^*)\|_{L^2(0, T; U)}} J'_e(u^*)(t).
\]

We will show that \( \nu_1 = \nu_2 \).

For all \( u \in V \) we have

\[
\langle \nu_1, u \rangle_{L^2(0, T; U)} \leq \|\nu_1\|_{L^2(0, T; U)} \|u\|_{L^2(0, T; U)} \leq M \quad \text{and} \quad \langle \nu_1, u^* \rangle_{L^2(0, T; U)} = M.
\]

Hence,

\[
\forall u \in V, \quad \langle \nu_1, u \rangle_{L^2(0, T; U)} \leq \langle \nu_1, u^* \rangle_{L^2(0, T; U)}.
\]

Moreover, the fact that \( V \) is convex, implies

\[
\forall u \in V, \quad \forall \lambda \in [0, 1], \quad u^* + \lambda (u - u^*) \in V.
\]

Then since \( u^* \) is a solution of the problem \( (P_e) \), we derive for all \( \lambda \in [0, 1] \) and \( u \in V \)

\[
J_e(u^*) \leq J_e(u^* + \lambda (u - u^*))
\]
\[
\begin{align*}
&= I_v(u^*) + \langle J'_v(u^*), \lambda(u - u^*) \rangle_{L^2(0, T; U)} \\
&\quad + \lambda \|u^* - u\|_{L^2(0, T; U)} \theta(\lambda \|u^* - u\|_{L^2(0, T; U)}),
\end{align*}
\]

where \( \theta \) is a function such that
\[
\lim_{\lambda \to 0^+} \theta(\lambda \|u^* - u\|_{L^2(0, T; U)}) = 0.
\]

From (11) and (12) it comes
\[
\langle J'_v(u^*), u \rangle_{L^2(0, T; U)} \geq \langle J'_v(u^*), u^* \rangle_{L^2(0, T; U)}.
\]

So, we arrive at
\[
\forall u \in V, \quad \langle v_2, u \rangle_{L^2(0, T; U)} \leq \langle v_2, u^* \rangle_{L^2(0, T; U)}.
\]

Taking into account that \( \sup_{u \in V} \langle v_2, u \rangle_{L^2(0, T; U)} = M \), we get from (10) that \( \langle v_2, u^* \rangle_{L^2(0, T; U)} = M \) and
\[
\frac{1}{2} \langle v_1 + v_2, u \rangle_{L^2(0, T; U)} = \frac{1}{2} \langle v_1, u^* \rangle_{L^2(0, T; U)} + \frac{1}{2} \langle v_2, u^* \rangle_{L^2(0, T; U)} = M,
\]

thus
\[
\|\frac{1}{2} (v_1 + v_2)\|_{L^2(0, T; U)} \geq 1.
\]

Using the above arguments, we get
\[
\|v_1 + v_2\|_{L^2(0, T; U)} = \|v_1\|_{L^2(0, T; U)} + \|v_2\|_{L^2(0, T; U)},
\]

and that \( v_1 = v_2 \).

Furthermore, we have
\[
\frac{1}{M} u^*(t) = -\frac{1}{\|J'_v(u^*)\|_{L^2(0, T; U)}} J'_v(u^*)(t).
\]

According to (9) and (13) we have
\[
u^*(t) = -\frac{1}{M} J'_v(u^*)(t) + \mathcal{B}(\cdot, y^*(t))^\phi(t),
\]

where
\[
J'_v(u)(t) = \epsilon ru(t) + \mathcal{B}(\cdot, y^*(t))^\phi(t).
\]

This achieves the proof of Theorem 2.3.

### 2.3 Sequential characterisation of the solution of the problem (P)

In the sequel, we consider a decreasing sequence \( (\epsilon_n) \) such that \( \epsilon_n \to 0 \) with corresponding sequence of controls \( (u^*_n) \) solutions of problems \( (P_n) \).

**Theorem 2.4:** Assume that \( V \) is a bounded subset of \( L^2(0, T; U) \) and let \( y_d \) be an approximately attainable state by a control from \( V \). Then the problem (P) possesses a solution. Moreover any weak limit value of \( (u^*_n) \) in \( L^2(0, T, U) \) is a solution of (P).

**Proof:** Since \( V \) is bounded, we deduce that the sequence \( (u^*_n) \) is bounded, so it admits a weakly converging subsequence, denoted by \( (u^*_n) \) as well. Let \( u^* \) be a weak limit value of \( (u^*_n) \) in \( V \).

The remainder of the proof is divided into three steps

**Step 1.** \( y_d \) is exactly attainable or, equivalently, \( U_{ad} \neq \emptyset \)

Let us consider the following problem
\[
\begin{cases}
\min \|y_u(T) - y_d\|_2^2 \\ u \in V.
\end{cases}
\]

The set \( \{\|y_u(T) - y_d\|_2^2 \in V \} \subset \mathbb{R}^+ \) is not empty and bounded from below, so it admits a lower bound \( I_d \).

Let \( (v_n)_{n \in \mathbb{N}} \) be a minimising sequence such that \( \|y_{v_n}(T) - y_d\|^2 \to -\infty \).

Since \( V \) is bounded, we deduce that the sequence \( (v_n) \) is bounded, so it admits a weakly converging subsequence to \( v \in V \) still denoted by \( (v_n) \).

By Lemma 2.2, we have for all \( t \in [0, T] \)
\[
\lim_{n \to +\infty} \|y_{v_n}(t) - y_v(t)\| = 0
\]

then, we conclude that
\[
\|y_v(T) - y_d\|^2 = \lim_{n \to +\infty} \|y_{v_n}(T) - y_d\|^2 = I_d = \min_{u \in V} \|y_u(T) - y_d\|^2
\]

So the control \( v \) is a solution of the problem (14).

Since the system (1) is approximately attainable, we have
\[
\forall \varepsilon > 0, \quad \exists v_\varepsilon \in V : \|y_{v_\varepsilon}(T) - y_d\| \leq \varepsilon
\]

According to (15) and (16), we get
\[
\forall \varepsilon > 0, \quad \exists v_\varepsilon \in V, \quad \|y_{v_\varepsilon}(T) - y_d\| \leq \|y_{v_n}(T) - y_d\| \leq \varepsilon
\]

So we conclude that \( \|y_{v_\varepsilon}(T) - y_d\| = 0 \) and hence \( v \in U_{ad} \).

**Step 2.** \( \forall v \in U_{ad}, \ I(u^*) \leq I(v) \)

Taking into account that \( u^*_n \) is a solution of the problem \( (P_n) \) and \( y^*_n \) is the corresponding solution of the system (1), we get for all \( v \in U_{ad} \)
\[
I_{e_n}(u^*_n) = \|y^*_n(T) - y_d\|^2 + \epsilon_n J(u^*_n) \leq J_{e_n}(v)
\]

from which, it comes
\[
\epsilon_n J(u^*_n) \leq J_{e_n}(v) - \|y^*_n(T) - y_d\|^2 \leq \epsilon_n I(v)
\]

So we find
\[
J(u^*_n) \leq I(v) \quad \text{for all } v \in U_{ad}
\]

Let \( y^* \) be the solution of system (1) corresponding to \( u^* \).
Since $u_n \to u^*$ in $L^2(0, T; U)$, we have by Lemma 2.2
\[
\lim_{n \to +\infty} \|y_n^*(t) - y^*(t)\|^2 = 0, \quad \forall t \in [0, T].
\] (18)

The norm $\| \cdot \|$ is lower semi-continuous for the weak topology, it follows that for all $t \geq 0$ we have
\[
\|y^*(t)\|^2 = \liminf_{n \to +\infty} \|y_n^*(t)\|^2.
\]

Applying Fatou’s lemma we get
\[
\int_0^T \|y^*(t)\|^2 \, dt = \liminf_{n \to +\infty} \int_0^T \|y_n^*(t)\|^2 \, dt.
\] (19)

The function $R$ is lower semi-continuous for the weak topology and convex, it follows from Brezis (1983) that
\[
R(u^*) \leq \liminf_{n \to +\infty} R(u_n^*).
\] (20)

By the inequalities (19) and (20) we deduce that
\[
J(u^*) \leq \liminf_{n \to +\infty} J(u_n^*).
\] (21)

Combining (17) and (21) we deduce that
\[
J(u^*) \leq J(v).
\]

**Step 3.** $u^* \in U_{ad}$.

According to the inequality (17), we deduce that $J(u_n^*)$ is bounded and
\[
\lim_{n \to +\infty} \|y_n^*(T) - y_d\|^2 = \lim_{n \to +\infty} J_{e_n}(u_n^*) \leq \lim_{n \to +\infty} J_{e_n}(v) = \|y_v(T) - y_d\|^2 = 0.
\]

Then, taking into account the formula (18), we derive via the continuity of the norm that
\[
\|y_v^*(T) - y_d\| = \lim_{n \to +\infty} \|y_n^*(T) - y_d\| \leq \|y_v(T) - y_d\| = 0.
\]

Consequently, $y_v^*(T) = y_d$ and the control $u^*$ is a solution of problem (P).

**Theorem 2.5:** If $U_{ad} \neq \emptyset$, then there exists a solution $u^*$ of the problem (P). Furthermore, any weak limit value of the solution $(u_n^*)$ of $(P_{e_n})$ in $L^2(0, T; U)$ is a solution of (P).

**Proof:** Let $v \in U_{ad}$. Then keeping in mind that $u_n^*$ is a solution of the problem $(P_{e_n})$ corresponding to $e_n$, we can see that
\[
J_{e_n}(u_n^*) \leq J_{e_n}(v) = e_n J(v).
\]

It follows that
\[
e_n J(u_n^*) = J_{e_n}(u_n^*) - \|y_n^*(T) - y_d\|^2 \leq J_{e_n}(u_n^*) \leq e_n J(v).
\]

Using the definition of the cost function $J$ stated in (2), the last equality gives
\[
r \int_0^T \|u_n^*(t)\|^2 \, dt \leq J(u_n^*) \leq J(v).
\] (22)

We deduce that the sequence $(u_n^*)$ is bounded, so it admits a weakly converging subsequence in $V$, also denoted by $(u_n^*)$. Let $u^*$ be a weak limit value of $(u_n^*)$ in $V$ and let $y^*$ be the solution of system (1) corresponding to $u^*$.

Since $u_n \to u^*$ in $L^2(0, T; U)$, we have by Lemma 2.2
\[
\lim_{n \to +\infty} \|y_n^*(t) - y^*(t)\|^2 = 0, \quad \forall t \in [0, T].
\]

Similarly to the proof of Theorem 2.4 we can show that
\[
J(u^*) \leq J(v).
\]

According to the inequality (22), we deduce that $J(u_n^*)$ is bounded and
\[
\lim_{n \to +\infty} J_{e_n}(u_n^*) = \lim_{n \to +\infty} \|y_n^*(T) - y_d\|^2 \leq \|y_v(T) - y_d\|^2.
\]

Hence
\[
\lim_{n \to +\infty} \|y_n^*(T) - y_d\| = \|y_v(T) - y_d\| \leq \|y_v(T) - y_d\| = 0.
\]

We conclude that $u^* \in U_{ad}$.

**2.4 Optimal feedback control**

In this part, we will try to express the optimal control $u^*$ of the problem (P) as a time-varying feedback law for the class of commutative bilinear systems with scalar control (El Ali, 1986; Wei & Pearson, 1978).

Assume that $U = \mathbb{R}$, then we can write the system (1) as follows
\[
\begin{align*}
\dot{y}(t) &= Ay(t) + u(t)By(t) \\
y(0) &= y_0 \in X
\end{align*}
\]

where $A : D(A) \subset X \mapsto X$ is the infinitesimal generator of a linear $C_0$-semigroup $S(t)$, $B$ is a bounded linear operator and $u \in V := L^2(0, T)$.

**Theorem 2.6:** Assume that $A$ and $B$ commute with each other and that $U_{ad} \neq \emptyset$. Let $v \in U_{ad}$ and let $y_0 \in X$ be such that $S(T)y_0 \notin \text{Ker}(B)$. Then, for any solution $u^*$ of the problem (P), we have the following formula
\[
u^*(t) = \frac{1}{T} \int_0^T v(s) \, ds + \frac{2}{T^r} \int_0^T \int_0^T (y^*(s), By^*(s)) \, ds \, d\alpha
\]
\[
- \frac{2}{T} \int_0^T (y^*(s), By^*(s)) \, ds.
\]

**Proof:** Let us consider the system (1) in the time horizon $[0, T]$, and let $A_k = kA(kI - A)^{-1}$ be the Yosida approximation of the operator $A$. Let $y_k$ and $\phi_k$ be the respective solutions to (1) and (3) with $A_k$ instead of $A$. For $u \in L^2(0, T)$, since $A_k$ is bounded, we have $y_k, \phi_k \in H^1(0, T; X)$ and
\[
\langle \phi_k(t), By_k(t) \rangle + \langle \phi_k(t), B\phi_k(t) \rangle
\]
\[
= \langle -A_k^2 \phi_k(t) - u(t)B\phi_k(t) - 2e_y(t), By_k(t) \rangle
\]
\[
+ \langle B^* \phi_k(t), A_ky_k(t) + u(t)By_k(t) \rangle
\]
\[
= \langle \phi_k(t), B\phi_k(t) \rangle - A_ky_k(t) - 2e_y(t), By_k(t) \rangle.
\]

Thus
\[
\langle \phi_k(t), By_k(t) \rangle + \langle \phi_k(t), B\phi_k(t) \rangle
\]

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Let \( \phi_k \rightarrow \phi \) and \( y_k \rightarrow y \) strongly (see Proposition 5.4 Chapter 2 in Li & Yong, 1995), we obtain by letting \( k \rightarrow +\infty \)

\[
(\phi(t), By(t)) = 2(y(T) - y_d, By(T)) + 2\epsilon(y(s), By_k(s))\ ds.
\]

So, by Theorem 2.3, we conclude that the solution of the problem \( (P_{\epsilon_n}) \) corresponding to \( \epsilon_n \), is given by

\[
u_n^*(t) = -\frac{1}{\epsilon_n r} \langle \phi_n(t), By_n^*(t) \rangle = -\frac{2}{\epsilon_n r} (y_n^*(T) - y_d, By_n^*(T)) - \frac{2}{r} \int_t^T (y_n^*(s), By_n^*(s))\ ds.
\]

Let \( \nu \in U_{ad} \). By Theorem 2.5, any limit value \( u^* \) of \( u_n^* \) in \( L^2(0, T) \) is a solution of the problem \( (P) \).

Since \( A \) and \( B \) commute, we have the following formulas

\[
y_n(t) = S(t) \exp \left( B \int_0^t \nu(s)\ ds \right) y_0,
\]

and

\[
y^*_n(t) = S(t) \exp \left( B \int_0^t u^*(s)\ ds \right) y_0.
\]

Using the fact that \( \nu, u^* \in U_{ad} \) and \( \lim_{n \to +\infty} y_n^*(T) = y_d \), we obtain

\[
\lim_{n \to +\infty} y_n^*(T) = y_u^*(T) = y_v(T) = y_d.
\]

Hence

\[
\lim_{n \to +\infty} S(T) \exp \left( B \int_0^T u_n^*(t)\ dt \right) y_0 = S(T) \exp \left( B \int_0^T \nu(t)\ dt \right) y_0 = S(T) \exp \left( B \int_0^T u^*(t)\ dt \right) y_0.
\]

The fact that \( A \) and \( B \) commute we can write

\[
\lim_{n \to +\infty} \exp \left( B \int_0^T u_n^*(t)\ dt \right) S(T)y_0 = \exp \left( B \int_0^T \nu(t)\ dt \right) S(T)y_0
\]

where \( [B, A_k] := BA_k - A_kB \).

Integrating (23) over \( [t, T] \), we get

\[
\langle \phi_k(t), By_k(t) \rangle = 2(y_k(T) - y_d, By_k(T)) - \int_t^T (\langle \phi_k(s), [B, A_k]y_k(s) \rangle - 2\epsilon(y_k(s), By_k(s)))\ ds.
\]

From the assumption \( S(T)y_0 \notin \ker(B) \) and using the spectral mapping theorem in Lemma 3.13 p. 19 in Engel and Nagel (2001), we deduce from the last relation that

\[
\lim_{n \to +\infty} \int_0^T u_n^*(t)\ dt = \int_0^T \nu(t)\ dt = \int_0^T u^*(t)\ dt.
\]

Moreover, we deduce from the formula (24), that

\[
\lim_{n \to +\infty} \int_0^T u_n^*(t)\ dt = \lim_{n \to +\infty} \int_0^T \left( \frac{2}{\epsilon_n r} (y_n^*(T) - y_d, By_n^*(T)) - \frac{2}{r} \int_t^T (y_n^*(s), By_n^*(s))\ ds \right)\ dt = \lim_{n \to +\infty} \frac{2T}{\epsilon_n r} (y_n^*(T) - y_d, By_n^*(T)) - \frac{2}{r} \int_0^T \int_t^T (y^*(s), By^*(s))\ ds\ dt
\]

from which we derive

\[
\lim_{n \to +\infty} -\frac{2T}{\epsilon_n r} (y_n^*(T) - y_d, By_n^*(T)) = \int_0^T \nu(t)\ dt + \frac{2}{r} \int_0^T \int_t^T (y^*(s), By^*(s))\ ds\ dt. \tag{25}
\]

By (24) and (25) we deduce that \( u_n^*(t) \to u^*(t) \) for all \( t \in [0, T] \) and

\[
\lim_{n \to +\infty} u_n^*(t) = \lim_{n \to +\infty} -\frac{2}{\epsilon_n r} (y_n^*(T) - y_d, By_n^*(T)) - \frac{2}{r} \int_t^T (y_n^*(s), By_n^*(s))\ ds \to \frac{1}{T} \int_0^T \nu(s)\ ds + \frac{2}{Tr} \int_0^T \int_t^T (y^*(s), By^*(s))\ ds\ ds\ \alpha = \int_t^T (y^*(s), By^*(s))\ ds\ ds\ \alpha = u^*(t).
\]

Finally we get

\[
uu(t) = \frac{1}{T} \int_0^T \nu(s)\ ds + \frac{2}{Tr} \int_0^T \int_t^T (y^*(s), By^*(s))\ ds\ d\alpha - \frac{2}{r} \int_0^T (y^*(s), By^*(s))\ ds.
\]
Remark 2.1: In the case where $S(t_1)$ is one to one for some $t_1 > 0$ and $y_0 \notin \text{Ker}(B)$, the assumption $S(T)y_0 \notin \text{Ker}(B)$ in Theorem 6 is satisfied.

3. Examples

3.1 Wave equation

Let us consider the following wave equation

$$\begin{align*}
\frac{\partial^2}{\partial t^2}z(t, x) &= \Delta z(t, x) + u(t, x)z(t, x), & t \in [0, T] \\
\frac{\partial}{\partial t}z(t, x) &= z(t, 1), & x \in \Omega = (0, 1) \\
\frac{\partial}{\partial t}z(0, x) &= z_0(x), & x \in \Omega.
\end{align*}$$

where

- $u \in L^2(0, T, L^2(\Omega))$,
- $T > 4 \max_{x \in \Omega} |x - x_0|$ for some $x_0 \in \mathbb{R} \setminus [0, 1]$,
- the desired state $z_d \in H^1(\Omega) \cap H^2(\Omega)$ is such that $\Delta z_d^e 1_{(z_d \neq 0)} \in L^\infty(\Omega)$, where $1_{(z_d \neq 0)}$ indicates the characteristic function of the set $\{x \in \Omega : z_d(x) \neq 0\}$.

This system has the form of the system (1) if we take

$$\begin{align*}
\frac{\partial}{\partial t}y(t, x) &= \Delta y(t, x) + u(t, x)y(t, x), & \text{in } \Omega = (0, 1) \\
y(t, 0) &= y(t, 1), & \text{on } (0, T) \\
y(0) &= y_0 & \text{in } \Omega.
\end{align*}$$

where $\Omega = (0, 1)$ and $u \in L^2(0, T, U)$ is a control function.

Case 1. Distributed control ($U = L^2(\Omega)$)

Assume that $y_0, y_d \in L^2(\Omega)$ are such that

- for a.e. $x \in \Omega$, $y_d(x) \geq 0$,
- for a.e. $x \in \Omega$, $y_0(x) = 0$ if and only if $y_d(x) = 0$,
- $a := \ln(\frac{T}{y_d})1_{(y_d \neq 0)} \in L^\infty(\Omega)$, where $1_{(y_d \neq 0)}$ indicates the characteristic function of the set $\{x \in \Omega : y_d(x) \neq 0\}$,
- $\Delta y_d^e 1_{(y_d \neq 0)} \in L^\infty(\Omega)$,
- $|y_d| > 0$ a.e. on some nonempty open subset $O$ of $\Omega$.

According to Theorem 2 in Ouzahra (2016), there is a time $T$ for which $y_d$ is exactly attainable for the system (26) using a control $v \in L^2(0, T, L^2(\Omega))$, so $U_{ad} \neq \emptyset$. As example of $y_d$ and $y_0$ satisfying the above condition we refer to Example 8 in Ouzahra (2016) Then, according to Theorem 2.5, there exists a control $u^*$ which guarantees the exact attainability of $y_d$ at time $T$, and is solution of the following problem

$$\begin{align*}
\min_{u} J(u) = \int_0^T \left( \frac{1}{2} \|z(t)\|^2_{H^1_0(\Omega)} + \frac{1}{2} \|\frac{\partial}{\partial t}z(t)\|^2_{L^2(\Omega)} \right) dt \\
\text{subject to } u(t) \in U_{ad} = \{u \in L^2(0, T, L^2(\Omega)) : z(t) = z_d \},
\end{align*}$$

Remark 3.1:

- The optimal control of the bilinear wave equation has been considered in Aronna et al. (2018), Liang (1999), and Zerrik and El Boukhari (2018) in the context of unconstrained endpoint.
- As an example of a target state $y_d$ satisfying the above assumptions we can take $y_d(x) = \sin(\pi x)$. Moreover, the optimal control problem provides a better control that allows us to achieve the same state reached by any other control $v$ (i.e. $y_u(T) = y_d(T)$).

3.2 Heat equation

In this part we study the optimal exact attainability for the reaction-diffusion equation.

Let us consider the following system

$$\begin{align*}
\frac{\partial}{\partial t}y(t, x) &= \Delta y(t, x) + u(t, x)y(t, x), & \text{in } \Omega = (0, 1) \\
y(t, 0) &= y(t, 1), & \text{on } (0, T) \\
y(0) &= y_0 & \text{in } \Omega.
\end{align*}$$

where $\Omega = (0, 1)$ and $u \in L^2(0, T, U)$ is a control function.

Case 2. Scalar control ($U = \mathbb{R}$)

Here, we have $u(t, x) = u(t) \in \mathbb{R}$.

Assume that $y_0, y_d \in L^2(\Omega)$ are such that $y_d = \lambda S(T)y_0$ with $\lambda > 1$ and $y_0 > 0$, a.e in $\Omega$. According to Theorem II 4 and Remark 4 in Ouzahra (2021), $y_d$ is exactly attainable for the system (26) using the control $v(t) = \frac{\lambda^2 - 1}{\lambda T} y_d \in L^2(0, T, \mathbb{R})$, so $U_{ad} \neq \emptyset$.

By Theorem 2.6, there exists a feedback control $u^* \in L^2(0, T, \mathbb{R})$ which guarantees the exact attainability of $y_d$ at
time $T$, and is solution of the problem ($P$) with $U_{ad} = \{ u \in L^2(0, T, \mathbb{R}) \mid y^s(T) = y_d \}$, and satisfies the following formula
\[
u^s(t) = \frac{1}{T} \ln(\lambda) + \frac{2}{Tr} \int_0^T \int_0^T \|y^s(s)\|^2 ds \, d\alpha - \frac{2}{r} \int_0^T \|y^s(s)\|^2 ds.
\]

### 3.3 Transport equation

Let us consider the following transport problem
\[
\begin{align*}
\frac{\partial}{\partial t} y(t, x) &= -\frac{\partial}{\partial x} y(t, x) + u(t) y(t, x), & t \in (0, T), \ x \in \Omega = (0, +\infty) \\
y(t, 0) &= y_0(x), & t \in (0, T) \\
y(0, x) &= y_0(x), & x \in \Omega
\end{align*}
\]  

where $u \in L^2(0, T)$. Here the operator $A = -\frac{\partial}{\partial x}$ with the domain $D(A) = H^1_0(\Omega)$ generates a $C_0$-semi-group of isometries $S(t)$ in $X = L^2(\Omega)$. Below, we will develop numerical simulation for the example (28). For this end, we take $r = 2, T = 9, y_0 = x \exp(-x)$ and
\[
y_d(x) = \begin{cases} 0, & \text{if } x \leq 9 \\ (x - 9) \exp(9 - x), & \text{if } x \geq 9 \end{cases}
\]
then the state $y_v$, corresponding to the control $v = 0$ is given by $y_v(T) = S(T) y_0 = y_d$. So $v = 0 \in U_{ad} = \{ u \in L^2(0, T) \mid y^s(T) = y_d \}$, and $S(T) y_0 \notin \ker(B) = \{0\}$.

By Theorem 2.6, there exists a feedback control $u^* \in L^2(0, T)$ which guarantees the exact attainability of $y_d$ at time $T$. Moreover $u^*$ is the solution of the problem ($P$) and satisfies the following formula
\[
u^s(t) = \frac{1}{T} \int_0^T \int_\alpha^T \|y^s(s)\|^2 ds \, d\alpha - \int_0^T \|y^s(s)\|^2 ds. \tag{29}
\]
For the numerical approach, we can characterise the optimal control under the form of the below algorithm:

- **Step 1.** Choose an initial control $u_0$, the precision $\epsilon$ and the step $dt$
- **Step 2.** Compute $y_n$, solution of the system (1) and

$$P_n(t) = \frac{1}{T} \int_0^T \int_0^T \|y_n^*(s)\|^2 \, ds \, d\alpha - \int_1^T \|y_n^*(s)\|^2 \, ds$$

- **Step 3.** Compute $u_{n+1}$ by $u_{n+1} = P_n$
- **Step 4.** while $\|u_{n+1} - u_n\| > \epsilon : n = n + 1$, go to step 2.

In Figure 1, we compare numerically the two controls $u^*$ (see Figure 2) and $v = 0$ in term of the state at the finite time $T = 9$. Moreover, we find $J(u^*) = 1.2442$ and $J(v) = 2.25 \approx 2(f(u^*))$

We observe that the desired state is exactly attainable either by using the optimal control $u^*$ or the control $v = 0$. However, the control $u^*$ leads to a lower cost than the zero control.

**Remark 3.2:** Unlike the case of linear systems, the uniqueness of the optimal control of the quadratic cost (2) is not guaranteed in general when dealing with bilinear systems, which is due to the lack of convexity of the state w.r.t control. For instance, if we assume that $r = 0$ and that $A = B$ is a skew-adjoint matrix, we can see that the cost function is constant so we have an infinity of optimal controls. However, in the case of the quadratic cost function $J(u) = \int_0^T u^2(t) \, dt$, the uniqueness of the optimal control is assured by the strict convexity of the cost $J$ (see Wei & Pearson, 1978). Moreover, in the case of a cost function $J$ of the form (2), one can prove the uniqueness of the optimal bilinear control under some constraint relaxing $T$ and $y_0$ (Bradley & Lenhart, 1994; Zerrik & El Boukhari, 2018, 2019).

**4. Conclusion**

In this work, we studied the question of quadratic optimal control with endpoint constraint for bilinear systems. The optimal control is characterised via a set of unconstrained minimisation problems, then it is expressed as a time varying feedback for commutative bilinear systems. The obtained results are applied to parabolic and hyperbolic partial differential equations. As an interesting continuation of the present work, one can consider the same questions for unbounded control operators, such as the case of the Fokker Planck equation (Aronna & Tröltzsch, 2021).

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