Growth Estimates on Positive Solutions of the Equation $\Delta u + Ku^{\frac{n+2}{n-2}} = 0$ in $\mathbb{R}^n$

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Abstract

We construct unbounded positive $C^2$-solutions of the equation $\Delta u + Ku^{\frac{n+2}{n-2}} = 0$ in $\mathbb{R}^n$ (equipped with Euclidean metric $g_0$) such that $K$ is bounded between two positive numbers in $\mathbb{R}^n$, the conformal metric $g = u^{4/(n-2)}g_0$ is complete, and the volume growth of $g$ can be arbitrarily fast or reasonably slow according to the constructions. By imposing natural conditions on $u$, we obtain growth estimate on the $L^{2n/(n-2)}$-norm of the solution and show that it has slow decay.

1. Introduction

In this article we derive $L^p$-estimates on positive solutions of the conformal scalar curvature equation

$$\Delta u + Ku^{\frac{n+2}{n-2}} = 0 \quad \text{in} \quad \mathbb{R}^n,$$

where $n \geq 3$ is an integer, $\Delta$ the standard Laplacian on $\mathbb{R}^n$, $K$ a smooth function. Equation (1.1) relates the scalar curvature of the conformal metric $g = u^{4/(n-2)}g_0$ to $4K(n-1)/(n-2)$, where $g_0$ is Euclidean metric \[10\]. It is assumed throughout this note that

$$0 < a^2 \leq K(x) \leq b^2 \quad \text{for large} \quad |x|$$

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and for some positive constants $a$ and $b$. The following estimates are known for any positive smooth solution $u$ of equation (1.1) with condition (1.2).

\begin{equation}
\int_{S^{n-1}} u(r, \theta) \, d\theta \leq C_1 r^{\frac{2-n}{2}},
\end{equation}

\begin{equation}
\int_{B_o(r)} u^{\frac{n+2}{n-2}}(x) \, dx \leq C_2 r^{\frac{n-2}{2}}
\end{equation}

for large $r$ and for some positive constants $C_1$ and $C_2$ depending on $u$ (see, for example, [11]). Here $B_o(r)$ is the ball with center at the origin and radius $r$, and $S^{n-1}$ is the unit sphere in $\mathbb{R}^n$. We seek to obtain higher order estimates of the forms

\begin{equation}
\int_{S^{n-1}} u^p(r, \theta) \, d\theta \leq C_3 r^{\frac{(2-n)p}{2}}, \quad p > 1;
\end{equation}

\begin{equation}
\int_{B_o(r)} u^q(x) \, dx \leq \begin{cases} C_4 r^{n-(n-2)q/2} & \text{if } q > \frac{n+2}{n-2}, q \neq \frac{2n}{n-2}; \\ C_5 \ln r & \text{if } q = \frac{2n}{n-2}, \end{cases}
\end{equation}

for large $r$, where $C_3$, $C_4$ and $C_5$ are positive constants. The above estimates are based on the slow decay of $u$, that is,

\begin{equation}
|u(x)| \leq C_6 |x|^{(2-n)/2} \quad \text{for large } |x|,
\end{equation}

where $C_6$ is a positive constant.

Our first observation is that, in general, (1.5), (1.6) or (1.7) do not hold. Taliaferro [13] shows that positive solution of (1.1) outside a ball in $\mathbb{R}^n$ with condition (1.2) may not have slow decay. We modify the construction in [13] to obtain positive $C^2$-solutions of (1.1) in $\mathbb{R}^n$ with $K$ bounded between two positive numbers in $\mathbb{R}^n$, such that the conformal metric $g = u^{4/(n-2)} g_o$ is complete and the volume growth of $(\mathbb{R}^n, g)$ can be arbitrarily fast or reasonably slow with respect to the constructions. This suggests that the geometric structure of complete manifolds of bounded positive scalar curvature could be very complicated (cf. [9]).

It is observed in [8] that if estimate (1.5) holds for some number $p \geq 2n/(n-2)$, then $u$ has slow decay and hence (1.5) and (1.6) hold for all $p, q > 1$. The integral in estimate (1.6) is the volume growth of $(\mathbb{R}^n, g)$ when $q = 2n/(n-2)$. In order to obtain (1.5) and (1.6) for large $p$ and $q$, additional conditions on $K$ or $u$ are required. By using a novel version of the moving plane method, Chen-Lin ([2] [3] and [4]) and Lin [12] examine, among other things, slow decay of $u$ under the condition

\begin{equation}
0 < \frac{C_7}{|x|^{1+\alpha}} \leq |\nabla K(x)| \leq \frac{C_8}{|x|^{1+\alpha}} \quad \text{for large } |x|.
\end{equation}
and for some positive constants $\alpha$, $C_7$ and $C_8$.

To gain better understanding on $u$, consider the case when $K$ is equal to a positive constant, say $K = n(n - 2)/4$, outside a compact subset of $\mathbb{R}^n$. We express $u$ as an associated function on the cylinder $\mathbb{R} \times S^{n-1}$ by letting

$$
(1.9) \quad v(s, \theta) = |x|^\frac{n-2}{2} u(x), \quad \text{where} \quad |x| = e^s \quad \text{and} \quad \theta = x/|x| \in S^{n-1}.
$$

Then $v$ satisfies the equation

$$
(1.10) \quad \frac{\partial^2 v}{\partial s^2} + \Delta_\theta v - \frac{(n - 2)^2}{4} v + K v^\frac{n+2}{n-2} = 0 \quad \text{in} \quad \mathbb{R} \times S^{n-1},
$$

where $\Delta_\theta$ is the standard Laplacian on $S^{n-1}$. Here $K$ is interpreted as a function on $\mathbb{R} \times S^{n-1}$ such that $(s, \theta) \mapsto K(e^s, \theta)$ for $s \in \mathbb{R}$ and $\theta \in S^{n-1}$. By a result of Caffarelli, Gidas and Spruck [1], with improvements by Korevaar, Mazzeo, Pacard and Schoen [8], either $g$ can be realized as a smooth metric on $S^n$ (in this case $u$ is said to have fast decay), or

$$
(1.11) \quad v(s, \theta) = v_\varepsilon(s + T) \left[ 1 + O(e^{-\kappa s}) \right] \quad \text{for large} \quad s, \quad \theta \in S^{n-1}
$$

and for some constants $\kappa > 0$ and $T \in \mathbb{R}$. Here $v_\varepsilon$, $\varepsilon \in (0, [(n - 2)/n]^{(n-2)/4}]$, is one of a one-parameter family of positive solutions of the O.D.E.

$$
(1.12) \quad v'' - \frac{(n - 2)^2}{4} v + \frac{n(n - 2)}{4} v^{\frac{n+2}{n-2}} = 0 \quad \text{in} \quad \mathbb{R},
$$

and $\varepsilon = \min_{t \in \mathbb{R}} v(t)$ is referred as the necksize of the solution [8]. As O.D.E. (1.12) is autonomous, $|v'|$ is bounded in $\mathbb{R}$. Furthermore, the Pohozaev number

$$
(1.13) \quad P(u) = \lim_{r \to +\infty} P(u, r) \quad \text{where} \quad P(u, r) = \frac{n - 2}{2n} \int_{B_o(r)} \nabla K(x) u^{\frac{2n}{n-2}}(x) \, dx
$$

is a negative number [8]. When $K$ may not be a constant outside a compact subset of $\mathbb{R}^n$, we have the following results.

**Theorem A.** Let $u$ be a positive smooth solution of equation (1.1) with condition (1.2), and $v$ given by (1.9). Assume that there exist positive constants $C_9$ and $C_{10}$ such that

$$
(1.14) \quad \int_{S^{n-1}} \left( \frac{\partial v}{\partial s} \right)^2 (s, \theta) \, d\theta \leq C_9 + C_{10} \int_{S^{n-1}} v^2(s, \theta) \, d\theta
$$

for large $s$. If $P(u, r) \geq -\delta^2$ for large $r$ and for a positive constant $\delta$, then

$$
(1.15) \quad \int_{B_o(r)} u^{\frac{2n}{n-2}} \, dx \leq C' \ln r \quad \text{and} \quad \int_{B_o(r)} |\nabla u|^2 \, dx \leq C'' \ln r
$$
for large \( r \) and for some positive constants \( C' \) and \( C'' \).

**Theorem B.** Assumed that there exist positive constants \( C_{11} \) and \( C_{12} \) such that

\[
\frac{\partial v}{\partial s}(s, \theta) \leq C_{11} + C_{12} v(s, \theta) \quad \text{for large } s \text{ and } \theta \in S^{n-1}.
\]

If \( P(u, r) \geq -\delta^2 \) for large \( r \) and for a positive constant \( \delta \), then

\[
\int_{S^{n-1}} u^{2n/(n-2)}(r, \theta) \, d\theta \leq C r^{-n}
\]

for large \( s \) and for some positive constant \( C \). Moreover, \( u \) has slow decay.

We prove theorems A and B in section 4. Lower bounds on \( P(u, r) \) are obtained in section 3, and examples are constructed in section 2. We use \( c, C, C_1, C_2, ... \) to denote positive constants, which may be different from section to section.

## 2. Examples

We begin with a construction of positive \( C^2 \)-solution \( u \) of equation (1.1) with \( K \) bounded between two positive constants in \( \mathbb{R}^n \), such that \( u \) is unbounded from above in \( \mathbb{R}^n \) (and hence does not have slow decay), and the conformal metric \( g \) is complete. Throughout this note \( n \geq 3 \) is an integer. Let

\[
\bar{u} (r, \lambda) = \alpha_n \left( \frac{\lambda}{\lambda^2 + r^2} \right)^{(n-2)/2} \quad \text{for } r \geq 0 \text{ and } \lambda > 0,
\]

where \( \alpha_n = [n(n-2)]^{(n-2)/4} \), and

\[
u_o(x) = \bar{u} (|x|, 1) = \frac{\alpha_n}{(1 + |x|^2)^{(n-2)/2}} \quad \text{for } x \in \mathbb{R}^n.
\]

Let \( \{\varepsilon_k\}_{k=1}^{\infty} \subset (0, 1) \) be a sequence of decreasing numbers such that

\[
\sum_{k=1}^{\infty} \varepsilon_k = 1,
\]

\( \{r_k\}_{k=1}^{\infty} \) a sequence of positive numbers such that \( r_1 \geq 1 \), \( r_{k+1} - r_k \geq 1 \) for \( k = 1, 2, ... \), and \( \{M_k\} \) a sequence of positive numbers such that \( M_k \to +\infty \) as \( k \to +\infty \). For \( x^{1,k} := (r_k, 0, ..., 0) \in \mathbb{R}^n \), \( k = 1, 2, ... \), there exist positive numbers \( \lambda_k, k = 1, 2, ... \), such that

\[
u_k(x) := \bar{u} (|x - x^{1,k}|, \lambda_k) \quad \text{for } x \in \mathbb{R}^n
\]
satisfies
\begin{equation}
\Delta u_k + u_k^{\frac{n+2}{2}} = 0 \quad \text{in } \mathbb{R}^n,
\end{equation}

\begin{equation}
u_k(x) \leq \varepsilon_k u_o(x) \quad \text{and} \quad |\nabla u_k(x)| < \varepsilon_k \quad \text{for } |x - x^{1,k}| \geq \frac{1}{4}, \quad \text{and}
\end{equation}

\begin{equation}u_k(x^{1,k}) = \alpha_n \lambda_k^{(2-n)/2} \geq M_k
\end{equation}

for \( k = 1, 2, \ldots \). Using (2.3) and (2.6), it follows as in [13] that \( \sum \infty \) \( u_k \) converges uniformly on compact subsets of \( \mathbb{R}^n \) to a positive \( C^2 \)-function. For a positive number \( b \), let
\begin{equation}
\tilde{u}_b(x) = (|x|^{2} + b^2)^{(2-n)/4} \quad \text{for } x \in \mathbb{R}^n.
\end{equation}

We have
\begin{equation}
\Delta \tilde{u}_b + K_b u_b^{\frac{n+2}{2}} = 0 \quad \text{in } \mathbb{R}^n,
\end{equation}

where
\begin{equation}
K_b(x) = \frac{n(n-2)}{2} \left( 1 - \frac{n + 2}{2n} \frac{|x|^2}{|x|^2 + b^2} \right) \quad \text{for } x \in \mathbb{R}^n.
\end{equation}

In particular
\begin{equation}
\frac{n(n-2)^2}{4n} \leq K_b(x) \leq \frac{n(n-2)}{2} \quad \text{for } x \in \mathbb{R}^n.
\end{equation}

Let
\begin{equation}
u(x) = \tilde{u}_b(x) + \sum_{k=0}^{\infty} u_k(x) \quad \text{for } x \in \mathbb{R}^n.
\end{equation}

It follows from (2.5), (2.9) and (2.11) that
\begin{equation}
-\Delta u(x) = \left[ K_b(x)\tilde{u}_b^{\frac{n+2}{2}}(x) + \sum_{k=0}^{\infty} u_k^{\frac{n+2}{2}}(x) \right] \leq \frac{n(n-2)}{2} u^{\frac{n+2}{2}}(x)
\end{equation}

for \( x \in \mathbb{R}^n \). Assume that \( x \in B_{x^{k'}}(1/4) \) for some positive integer \( k' \). Using (2.3), (2.6) and the inequality \((a+b)^p \leq 2^{p-1}(a^p + b^p)\) for \( a, b \geq 0 \) and \( p \geq 1 \), we have
\begin{equation}u^{\frac{n+2}{2}}(x) = \left[ \tilde{u}_b(x) + u_{k'}(x) + \sum_{k \neq k'} u_k(x) \right]^{\frac{n+2}{2}} \leq \left[ \tilde{u}_b(x) + u_{k'}(x) + 2u_o(x) \right]^{\frac{n+2}{2}}
\end{equation}

\begin{equation}
\leq c_1 \left[ \tilde{u}_b(x) + u_{k'}(x) + u_o^{\frac{n+2}{2}}(x) \right] \leq -c_2 \Delta u(x),
\end{equation}
where $c_1$ and $c_2$ are positive constants depending on $n$ only. Similar estimate holds for $x \notin B_{x_k}(1/4)$ for $k' = 1, 2, \ldots$, if we choose $c_2$ to be large enough, which depends on $n$ only. Thus $u$ satisfies the equation $\Delta u + Ku^{(n+2)/(n-2)} = 0$ in $\mathbb{R}^n$, where

$$K(x) = [-\Delta u(x)]\left[u^{\frac{n+2}{n-2}}(x)\right]^{-1} \quad \text{for } x \in \mathbb{R}^n$$

is a continuous function which is bounded in $\mathbb{R}^n$ between two positive constants by (2.13) and (2.14). (2.7) shows that $u$ is not bounded from above in $\mathbb{R}^n$. The conformal metric $u^{4/(n-2)}g_o$ is complete because

$$(2.15) \quad u^{4/(n-2)}(x) \geq u_{b}^{4/(n-2)}(x) \geq (1/2)|x|^{-2}$$

for large $|x|$. Let

$$(2.16) \quad V_n := \omega_n \int_0^{\infty} \left(\frac{\lambda \sqrt{n(n-2)}}{\lambda^2 + r^2}\right)^n r^{n-1} dr = \omega_n \int_0^{\infty} \left(\frac{\sqrt{n(n-2)}}{1 + t^2}\right)^n t^{n-1} dt$$

for $\lambda > 0$, where $t = \lambda^{-1} r$ and $\omega_n$ is the volume of the unit sphere in $\mathbb{R}^n$. By choosing $r_k$ suitably far from each other, together with (2.16) and the fact that the first integral in (2.16) concentrates more on a neighborhood of 0 for smaller $\lambda$, we have

$$(2.17) \quad \int_{B_o(r)} u^{\frac{2n}{n-2}}(x) \, dx \leq C_2 \ln r$$

for large $r$ and for a positive constant $C_2$.

Next, given a function $\phi : [0, \infty) \to [0, \infty)$, we construct a positive $C^2$-solution $u$ of equation (1.1) with $K$ bounded between two positive constants in $\mathbb{R}^n$, such that the conformal metric $g = u^{4/(n-2)}g_o$ is complete and

$$(2.18) \quad \int_{B_o(r)} u^{\frac{2n}{n-2}}(x) \, dx \geq \phi(r) \quad \text{for } r > 2.$$ 

Without loss of generality, we may assume that $\phi$ is increasing and $\phi(0) \geq 10V_n$. For $k = 1, 2, \ldots$, let $N_k$ be a positive integer such that

$$(2.19) \quad N_k \geq 2V_n^{-1} \phi(k+2) \quad \text{for } k = 1, 2, \ldots.$$ 

Let $\{\epsilon_k\}_{k=1}^{\infty} \subset (0, 1)$ be a sequence of decreasing numbers such that

$$(2.20) \quad \sum_{k=1}^{\infty} N_k \epsilon_k \leq 1.$$ 

Let $\theta_k = 2\pi/N_k$. Let

$$(2.21) \quad x_{k,j} = (k \sin(j \theta_k), k \cos(j \theta_k), 0, \ldots, 0) \in \mathbb{R}^n \quad \text{for } j = 1, 2, \ldots, N_k,$$
and
\( u_{k,j}(x) = \bar{u}(|x - x_{k,j}|, \lambda_k) \) for \( x \in \mathbb{R}^n \) and \( j = 1, 2, \ldots, N_k \).

We choose \( \lambda_k \) to be small so that
\[
(2.23) \quad u_{k,j}(x) \leq \epsilon_k u_o(x) \quad \text{and} \quad |\nabla u_{k,j}(x)| < \epsilon_k \quad \text{for} \quad |x - x_{k,j}| \geq \pi/(10N_k),
\]
and
\[
(2.24) \quad \int_{B_{x_{k,j}}(\pi/(10N_k))} u_{k,j}^{\frac{2n}{n-2}}(x) \, dx \geq \frac{V_n}{2} \quad \text{for} \quad j = 1, 2, \ldots, N_k,
\]
where \( B_{x_{k,j}}(\pi/(10N_k)) \) is the ball with center at \( x_{k,j} \) and radius equal to \( \pi/(10N_k) \).

(2.24) is possible because, when \( \lambda \) is smaller, the first integral in (2.16) concentrates more on a neighborhood of the origin. It follows from (2.20) and (2.23) that the series
\[
\sum_{k=1}^{\infty} \sum_{j=1}^{N_k} u_{k,j}
\]
converges uniformly on compact subsets of \( \mathbb{R}^n \) to a positive \( C^2 \)-function. Let
\[
u = \bar{u}_b + u_o + \sum_{k=1}^{\infty} \sum_{j=1}^{N_k} u_{k,j} \quad \text{in} \quad \mathbb{R}^n.
\]
As above, we have \( \Delta u + K u^{(n+2)/(n-2)} = 0 \) in \( \mathbb{R}^n \), where \( K \) is a continuous function on \( \mathbb{R}^n \) that is bounded between two positive constants. For any \( r > 2 \), let \( k \) be the integer such that \( k + 1 \leq r < k + 2 \). By (2.19) we have
\[
\phi(r) \leq \phi(k + 2) \leq \frac{V_n N_k}{2} \leq \sum_{j=1}^{N_k} \int_{B_{x_{k,j}}(\pi/(10N_k))} u_{k,j}^{\frac{2n}{n-2}}(x) \, dx
\]
\[
\leq \int_{B_o(k+1)} u^{\frac{2n}{n-2}}(x) \, dx \quad \leq \int_{B_o(r)} u^{\frac{2n}{n-2}}(x) \, dx.
\]

3. Estimates on \( P(u, r) \)

Let \( P(u, r) \) be given by (1.13) in the introduction. The Pohozaev identity (see, for example, [7]) states that
\[
(3.1) \quad P(u, r) = \int_{S_r} \left[ r \left( \frac{\partial u}{\partial r} \right)^2 - \frac{r}{2} |\nabla u|^2 + \frac{n-2}{2n} rKu^{\frac{2n}{n-2}} + \frac{n-2}{2} u \frac{\partial u}{\partial r} \right] \, dS
\]
for \( r > 0 \), where \( S_r = \partial B_o(r) \) is the sphere of radius \( r \).
Theorem 3.2. Let $u$ be a positive smooth solution of equation (1.1) with condition (1.2). Assume that $u$ is bounded from above in $\mathbb{R}^n$ and

\begin{equation}
\frac{\partial K}{\partial r}(x) \geq -\frac{C_1}{|x|^{(n+2)/2} (\ln |x|)^{1+\epsilon}}
\end{equation}

for large $|x|$ and for some positive constants $C_1$ and $\epsilon$. Then $P(u, r) \geq -\delta^2$ for large $r$ and for a positive constant $\delta$.

Proof. Fixing a large number $R$ and using (1.4) we have

\begin{align*}
\int_{B_{\Lambda}(m+1)R \setminus B_{\Lambda}(mR)} r \frac{\partial K}{\partial r}(x) u^{\frac{2n}{n-2}}(x) \, dx &\geq - \frac{C_1}{(mR)^\frac{n}{2} [\ln (mR)]^{1+\epsilon}} \int_{B_{\Lambda}(m+1)R \setminus B_{\Lambda}(mR)} u^{\frac{2n}{n-2}}(x) \, dx \\
&\geq - \frac{C_2}{(mR)^\frac{n}{2} [\ln (mR)]^{1+\epsilon}} \int_{B_{\Lambda}(m+1)R \setminus B_{\Lambda}(mR)} u^{\frac{n+2}{n-2}}(x) \, dx \\
&\geq - \frac{C_3[(m+1)R]^{\frac{n-2}{2}}}{(mR)^\frac{n}{2} [\ln (mR)]^{1+\epsilon}} \geq - \frac{C_4}{m (\ln m)^{1+\epsilon}}
\end{align*}

for any positive integer $m$ larger than 1, where $r = |x|$. Here $C_2$, $C_3$ and $C_4$ are positive constants. As the series

\[ \sum_{m=2}^{\infty} \frac{1}{m (\ln m)^{1+\epsilon}} \]

converges, we conclude that there exists a positive constant $\delta$ such that $P(u, r) \geq -\delta^2$ for large $r$. \qed

Theorem 3.4. Let $u$ be a positive smooth solution of equation (1.1) with condition (1.2). Assume that there exists a positive constant $c$ such that

\begin{equation}
\frac{\partial K}{\partial r}(r, \theta) \geq -\frac{c}{r^2} \quad \text{for large } r \text{ and } \theta \in S^{n-1}.
\end{equation}

If there exist positive constants $C$ and $\lambda \in (0, 1)$ such that

\begin{equation}
\int_{S^{n-1}} \left(\frac{\partial v}{\partial s}\right)^{\frac{2n}{n-2}}(s, \theta) \, d\theta \leq C \, e^{\lambda s}
\end{equation}

for large $s$, then $P(u, r) \geq -\delta^2$ for large $r$ and for a positive constant $\delta$. 8
Proof. For a positive number \( \varepsilon > 0 \) such that \( \varepsilon + \lambda < 1 \), using Young's inequality we have

\[
\begin{align*}
\frac{d}{dr} \left( \int_{S_r} r^\varepsilon u^{\frac{2n}{n-2}}(r, \theta) \, dS \right) &= \frac{d}{dr} \left( \int_{S^{n-1}} r^{n-1+\varepsilon} u^{\frac{2n}{n-2}}(r, \theta) \, d\theta \right) \\
&= \frac{n-1 + \varepsilon}{r} \int_{S^{n-1}} r^{n-1+\varepsilon} u^{\frac{2n}{n-2}}(r, \theta) \, d\theta + \frac{2n}{n-2} \int_{S^{n-1}} \frac{u^{\frac{n+2}{2}}}{u^{\frac{n+2}{2}}(r, \theta)} \frac{\partial u}{\partial r}(r, \theta) \, r^{n-1+\varepsilon} \, d\theta \\
&= -1 + \varepsilon \int_{S^{n-1}} r^{n-1+\varepsilon} u^{\frac{2n}{n-2}}(r, \theta) \, d\theta \\
&\quad + \frac{2n}{n-2} \int_{S^{n-1}} \left( \frac{\partial^2 u}{\partial r^2} + \frac{n-2}{2} \frac{u}{r^2} \right)(r, \theta) \, r^{n-1+\varepsilon} \, d\theta \\
&\leq \frac{C_5}{r^{2-\varepsilon}} \int_{S^{n-1}} \left( r^{\frac{2n}{n-2}} \left( \frac{\partial^2 u}{\partial s^2} + \frac{n-2}{2} \right)(s, \theta) \right) d\theta \\
&\quad \leq \frac{C_6}{r^{2-\lambda-\varepsilon}}
\end{align*}
\]
for large \( r \), where \( r = e^s \) and \( C_5 \) and \( C_6 \) are positive constants. It follows that there exists a positive constant \( C_7 \) such that

\[
\int_{S_r} r^\varepsilon u^{\frac{2n}{n-2}} \, dS \leq C_7 \quad \text{or} \quad \int_{S_r} u^{\frac{2n}{n-2}} \, dS \leq C_7 \, r^{-\varepsilon}
\]
for large \( r \). For a fixed large number \( R_o \), we have

\[
\frac{2n}{n-2} P(u, R) = \int_{B_o(R)} r \frac{\partial K}{\partial r} u^{\frac{2n}{n-2}} \, dx \geq -C_8 - C_9 \int_{R_o}^{R} \int_{S_r} u^{\frac{2n}{n-2}} \, dS \, dr \geq -C_{10}
\]
for large \( R \) with \( R_o < R \). Here \( C_8, C_9 \) and \( C_{10} \) are positive constants. \( \square \)

4. Proofs of Theorem A and B

Proof of Theorem A. Let

\[
w(s) = \frac{1}{2} \int_{S^{n-1}} v^2(s, \theta) \, d\theta \quad \text{for } s \in \mathbb{R},
\]
where \( v \) is defined in (1.9). Using equation (1.10) we have

\[
\begin{align*}
w''(s) &= \int_{S^{n-1}} \left( \frac{\partial v}{\partial s} \right)^2(s, \theta) \, d\theta + \int_{S^{n-1}} \left| \nabla_{\theta} v(s, \theta) \right|^2 d\theta \\
&\quad + \left( \frac{n-2}{2} \right)^2 \int_{S^{n-1}} v^2(s, \theta) \, d\theta - \int_{S^{n-1}} K(e^s, \theta) v^{\frac{2n}{n-2}}(s, \theta) \, d\theta
\end{align*}
\]
for $s \in \mathbb{R}$. The Pohozaev identity can be expressed as

$$
(4.3) \quad 2P(u, e^s) = \int_{S^{n-1}} \left( \frac{\partial v}{\partial s} \right)^2 (s, \theta) d\theta - \int_{S^{n-1}} |\nabla v(s, \theta)|^2 d\theta
$$

$$
- \left( \frac{n-2}{2} \right)^2 \int_{S^{n-1}} v^2(s, \theta) d\theta + \frac{n-2}{n} \int_{S^{n-1}} K(e^s, \theta) v^{\frac{2n}{n-2}}(s, \theta) d\theta
$$

for $s \in \mathbb{R}$. It follows from (1.2), (1.4), (4.2) and (4.3) that

$$
(4.4) \quad w''(s) \leq C_1 + C_2 \int_{S^{n-1}} v^2(s, \theta) d\theta - \frac{2a^2}{n} \int_{S^{n-1}} v^{\frac{2n}{n-2}}(s, \theta) d\theta
$$

for large $s$, where $C_1$ and $C_2$ are positive constants. Applying Young’s inequality we obtain

$$
(4.5) \quad - \frac{2a^2}{n} \int_{S^{n-1}} v^{\frac{2n}{n-2}}(s, \theta) d\theta \leq C_3 - C_4 \int_{S^{n-1}} v^2(s, \theta) d\theta
$$

for large $s$, where $C_3$ and $C_4$ are positive constants. Furthermore, by choosing $C_3$ to be large, we can take $C_4$ to be large as well. Hence there exists a positive constant $C_5$ such that

$$
(4.6) \quad w''(s) \leq C_5 - w(s) \quad \text{for large } s.
$$

From (4.6) it is easy to see that $w(s)$ is uniformly bounded from above for large $s$. To prove this assertion, assume that there is a large $s'$ such that $w(s') \geq C_5 + 1$. (4.6) implies that $w''(s') \leq -1$. Let $s_o$ be a number larger than $s'$ such that $w(s_o) < C_5 + 1$ and $w'(s_o) \leq 0$. If $w(s) < C_5 + 1$ for all $s > s_o$, then we are done. Assume that $s_1$ is the smallest number larger than $s_o$ such that $w(s_1) = C_5 + 1$. We claim that

$$
(4.7) \quad D := w'(s_1) < 2(C_5 + 1).
$$

Let $\bar{s} \in (s_o, s_1)$ be the largest number such that $w'(\bar{s}) = D/2$. As $w'' \leq C_5$ on $(s_o, s_1)$, we have $s_1 - \bar{s} \geq D/(2C_5)$. On the other hand, $w' \geq D/2$ on $(\bar{s}, s_1)$. Therefore we have

$$
C_5 + 1 \geq w(s_1) - w(\bar{s}) \geq \frac{D}{2C_5} \cdot \frac{D}{2} \Rightarrow D^2 \leq 4C_5(C_5 + 1).
$$

Hence we have (4.7). From $s_1$, $w(s)$ can become no larger than $(C_5 + 1) + [2(C_5 + 1)]^2$ before $w'(s)$ becomes negative again. Hence we conclude that $w(s)$ is uniformly bounded from above for large $s$. 

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From Pohozaev identity (3.1) we obtain

\[(4.8)\]
\[\int_{S_r} r |\nabla u|^2 dS = 2 \int_{S_r} \left[ r \left( \frac{\partial u}{\partial r} \right)^2 + \frac{n-2}{2n} r K u^{\frac{2n}{n-2}} + \frac{n-2}{2} u \frac{\partial u}{\partial r} \right] dS - 2P(u, r) \]

for \( r > 0 \). We have

\[(4.9)\]
\[\int_{S_r} r \left( \frac{\partial u}{\partial r} \right)^2 dS = \int_{S_r} \left[ \frac{\partial u}{\partial r} + \frac{n-2}{2} u \right]^2 dS - \left( \frac{n-2}{2} \right)^2 \int_{S_r} \frac{u^2}{r} dS \]

for \( r > 0 \). Using (1.14) and the fact that \( w \) is bounded from above we obtain

\[(4.10)\]
\[- \int_{S_r} u \frac{\partial u}{\partial r} dS = - \int_{S_r} u \left[ \frac{\partial u}{\partial r} + \frac{n-2}{2} u \right] dS + \frac{n-2}{2} \int_{S_r} \frac{u^2}{r} dS \leq \int_{S^{n-1}} \left( \frac{\partial u}{\partial s} \right)^2 (s, \theta) d\theta + \frac{n}{2} \int_{S^{n-1}} v^2 (s, \theta) d\theta \leq C_6 \]

for large \( r \) and for a positive constant \( C_6 \), where \( r = e^s \). It follows from (4.8), (4.9) and (4.10) that

\[(4.11)\]
\[\int_{S_r} |\nabla u|^2 dS \leq \frac{C_7}{r} + \frac{n-2}{n} \int_{S_r} K u^{\frac{2n}{n-2}} dS \]

for large \( r \), where \( C_7 \) is a positive constant. Therefore we obtain

\[(4.11)\]
\[\int_{B_o(r)} |\nabla u|^2 dx \leq C_8 \ln r + \frac{n-2}{n} \int_{B_o(r)} K u^{\frac{2n}{n-2}} dx \]

for large \( r \) and for a positive constant \( C_8 \geq C_7 \). On the other hand we have

\[\int_{B_o(r)} K u^{\frac{2n}{n-2}} dx = \int_{B_o(r)} u (-\Delta u) dx = \int_{B_o(r)} |\nabla u|^2 dx - \int_{S_r} u \frac{\partial u}{\partial r} dS \leq C_8 \ln r + \frac{n-2}{n} \int_{B_o(r)} K u^{\frac{2n}{n-2}} dx + C_6 \]

for large \( r \), where we use (4.10). Hence there exists a positive constant \( C_9 \) such that

\[\int_{B_o(r)} K u^{\frac{2n}{n-2}} dx \leq C_9 \ln r \]

for large \( r \). If \( u \in L^{2n/(n-2)}(\mathbb{R}^n) \), then clearly we have the first inequality in (1.15). Assume that \( u \notin L^{2n/(n-2)}(\mathbb{R}^n) \). Using (1.2) we have

\[\int_{B_o(r)} u^{\frac{2n}{n-2}} dx \leq \frac{2}{a_2} \int_{B_o(r)} K u^{\frac{2n}{n-2}} dx \leq \frac{2C_9}{a_2} \ln r \]
for large $r$. Hence we have the first inequality in (1.15). The second inequality follows from (4.11). \qed

**Proof of Theorem B.** From the proof of theorem A we have

\begin{equation}
\int_{S_r} \frac{u^2(x)}{r} dS = \int_{S^{n-1}} v^2(s, \theta) d\theta = 2 w(s) \leq C_{10}
\end{equation}

for large $r$, where $r = |x| = e^s$ and $C_{10}$ is a positive constant. By using (1.16) and (4.12) we also have

\begin{equation}
\int_{S_r} r \left[ \frac{\partial u}{\partial r} + \frac{n - 2 u}{2} \right]^2 dS = \int_{S^{n-1}} r^n \left[ \frac{\partial u}{\partial r} + \frac{n - 2 u}{2} \right]^2 d\theta = \int_{S^{n-1}} \left( \frac{\partial v}{\partial s} \right)^2 d\theta \leq C_{11}
\end{equation}

for large $r$, where $C_{11}$ is a positive constant. It follows from Pohozaev identity (3.1) that

\begin{equation}
\int_{S_r} r | \nabla u |^2 dS \leq C_{12} + \frac{n - 2}{n} \int_{S_r} r K u^{\frac{n}{n-2}} dS + 2 \int_{S_r} r \left[ \frac{\partial u}{\partial r} + \frac{n - 2 u}{2} \right]^2 dS \geq -(n - 2) \int_{S_r} u \left[ \frac{\partial u}{\partial r} + \frac{n - 2 u}{2} \right] dS
\end{equation}

\begin{equation}
\leq C_{13} + \frac{n - 2}{n} \int_{S_r} r K u^{\frac{n}{n-2}} dS + C_{14} \int_{S_r} r \left[ \frac{\partial u}{\partial r} + \frac{n - 2 u}{2} \right]^2 dS
\end{equation}

\begin{equation}
+ C_{15} \int_{S_r} \frac{u^2}{r} dS \leq C_{16} + \frac{n - 2}{n} \int_{S_r} r K u^{\frac{n}{n-2}} dS
\end{equation}

for large $r$, where we use (4.12) and (4.13). Here $C_{12}$, $C_{13}$, $C_{14}$, $C_{15}$ and $C_{16}$ are positive constants. (4.14) implies that there exists a positive constant $C_{17}$ such that

\begin{equation}
\int_{B_{r}(R)} r | \nabla u |^2 dx \leq C_{17} R + \frac{n - 2}{n} \int_{B_{r}(R)} r K u^{\frac{n}{n-2}} dx
\end{equation}

for large $R$. We have

\begin{equation}
\int_{B_{r}(R)} r K u^{\frac{n}{n-2}} dx = \int_{B_{r}(R)} (ru)(-\Delta u) dx = \int_{B_{r}(R)} r | \nabla u |^2 dx + \int_{B_{r}(R)} u \frac{\partial u}{\partial r} dx - R \int_{S_R} u \frac{\partial u}{\partial r} dS
\end{equation}

for $R > 0$. Using (4.13) we obtain

\begin{equation}
\int_{B_{r}(R)} u \frac{\partial u}{\partial r} dx = \int_{B_{r}(R)} u \left[ \frac{\partial u}{\partial r} + \frac{n - 2 u}{2} \right] dx - \frac{n - 2}{2} \int_{B_{r}(R)} \frac{u^2}{r} dx \leq C_{18} \int_{B_{r}(R)} r \left[ \frac{\partial u}{\partial r} + \frac{n - 2 u}{2} \right] dx \leq C_{19} R
\end{equation}
for large $R$, where $C_{18}$ and $C_{19}$ are positive constants. As in (4.10) we have

$$ (4.18) \quad \left| R \int_{S_R} u \frac{\partial u}{\partial r} \, dS \right| \leq C_{20} R $$

for large $R$, where $C_{20}$ is a positive constant. From (4.15), (4.16), (4.17) and (4.18) we obtain

$$ (4.19) \quad \frac{2}{n} \int_{B_o(R)} r K u^{\frac{2n}{n-2}} \, dx \leq C_{21} R $$

for large $R$, where $C_{21}$ is a positive constant. Using (1.16) we have

$$ \frac{d}{dr} \left( \int_{S_r} r^2 u^{\frac{2n}{n-2}} \, dS \right) = \frac{d}{dr} \left( \int_{S_{n-1}} r^{n+1} u^{\frac{2n}{n-2}} \, d\theta \right) $$

$$ = (n + 1) \int_{S_{n-1}} r^n u^{\frac{2n}{n-2}} \, d\theta + \frac{2n}{n - 2} \int_{S_{n-1}} r^{n+1} u^{\frac{2n}{n-2}} \frac{\partial u}{\partial r} \, d\theta $$

$$ = \int_{S_{n-1}} r^n u^{\frac{2n}{n-2}} \, d\theta + \frac{2n}{n - 2} \int_{S_{n-1}} \left( r^{n+2} u^{\frac{2n}{n-2}} \right) \left\{ r \frac{n}{2} \left[ \frac{\partial u}{\partial r} + \frac{n - 2 u}{r} \right] \right\} \, d\theta $$

$$ \leq C_{22} \int_{S_{n-1}} r^n u^{\frac{2n}{n-2}} \, d\theta + C_{23} \int_{S_{n-1}} \left\{ r \frac{n}{2} \left[ \frac{\partial u}{\partial r} + \frac{n - 2 u}{r} \right] \right\} \frac{2n}{n-2} \, d\theta $$

$$ = C_{22} \int_{S_{n-1}} r^n u^{\frac{2n}{n-2}} \, d\theta + C_{23} \int_{S_{n-1}} \left| \frac{\partial u}{\partial S} \right| \frac{2n}{n-2} \, d\theta \leq C_{24} \int_{S_{n-1}} r^n u^{\frac{2n}{n-2}} \, d\theta + C_{25} $$

for large $r$, and for some positive constants $C_{22}, C_{23}, C_{24}$ and $C_{25}$, where $r = e^t$. Hence

$$ (4.20) \quad \int_{S_R} R^2 u^{\frac{2n}{n-2}} \, dS = \int_0^R \left( \int_{S_t} t^2 u^{\frac{2n}{n-2}} \, dS \right) ' \, dt \leq C_{26} + \int_0^R \left( \int_{S_t} t^2 u^{\frac{2n}{n-2}} \, dS \right)' \, dt $$

$$ \leq C_{26} + C_{24} \int_{r_o}^R \int_{S_t} t u^{\frac{2n}{n-2}} \, dS \, dt + C_{25} (R - r_o) $$

$$ \leq C_{27} R + C_{28} \int_{B_o(R)} r u^{\frac{2n}{n-2}} \, dx $$

for $R$ and $r_o$ large, with $R > r_o$. Here $C_{26}, C_{27}$ and $C_{28}$ are positive constants. Consider the case when $u \notin L^{2n/(n-2)}(\mathbb{R}^n)$. It follows from (1.2) and (4.19) that

$$ (4.21) \quad \int_{B_o(R)} r u^{\frac{2n}{n-2}} \, dx \leq C_{29} \int_{B_o(R)} r K u^{\frac{2n}{n-2}} \, dx \leq C_{30} R $$

for large $R$ and for some positive constants $C_{29}$ and $C_{30}$. Clearly we have

$$ (4.22) \quad \int_{B_o(R)} r u^{\frac{2n}{n-2}} \, dx \leq C_{31} R $$
for large $R$ and for some positive constants $C_{31}$ if $u \in L^{2n/(n-2)}(\mathbb{R}^n)$. From (4.20), (4.21) and (4.22) we have (1.17). By the results in [11] (see also [6]), we obtain slow decay (1.7) as well.

\[
\square
\]

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