NAMBU JONA-LASINIO LIKE MODELS
AND
THE LOW ENERGY EFFECTIVE ACTION OF QCD

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Abstract
We present a derivation of the low energy effective action of an extended Nambu Jona-Lasinio (ENJL) model to $O(p^4)$ in the chiral counting. Two alternative scenarios are considered on how the ENJL model could originate as a low energy approximation to QCD. The low energy effective Lagrangian we derive includes the usual pseudoscalar Goldstone modes, as well as the lower scalar, vector and axial-vector degrees of freedom. By taking appropriate limits, we recover most of the effective low-energy models discussed in the literature; in particular the gauged Yang-Mills vector Lagrangian, the Georgi-Manohar constituent quark-meson model, and the QCD effective action approach model. Another property of the ensuing effective Lagrangian is that it incorporates most of the short-distance relations which follow from QCD. (We derive these relations in the presence of all possible gluonic interactions to leading order in the $1/N_c$-expansion.) Finally the numerical predictions are compared to the experimental values of the low energy parameters.
1 INTRODUCTION.

The original model of Nambu and Jona-Lasinio [1] is based on an analogy with superconductivity, and it was proposed as a dynamical model of the strong interactions between nucleons and pions. In this model the pions appear as the massless composite bosons associated with the dynamical spontaneous breakdown of the chiral symmetry of the initial Lagrangian. This pioneering work has had an enormous impact in the development of modern particle physics and the standard model in particular.

The idea that in QCD with three light flavours $u$, $d$ and $s$, a mechanism “somewhat like” the one originally proposed by Nambu and Jona-Lasinio may be at the origin of the dynamics responsible for the spontaneous chiral symmetry breaking of the flavour group $SU(3)_L \times SU(3)_R$ to $SU(3)_V$ is an appealing one. Several authors have investigated this assumption with various claims of success in reproducing the features of low energy hadron phenomenology. (Refs. [2] to [24].) From a theoretical point of view, however, very little is known at present on how an effective four-fermion interaction à la Nambu Jona-Lasinio could be triggered within the framework of QCD. Two alternative mechanisms can be envisaged. One may first consider the formal integration over gluon fields in the path integral formulation of the generating functional for the Green’s functions of quark-currents, and then truncate the resulting non-local quark effective action to its lowest dimensional interactions by invoking some “local approximation.” Alternatively, one may consider a renormalization à la Wilson where the quark and gluon degrees of freedom have been integrated out down to a scale $\Lambda_\chi$ of the order of the spontaneous chiral symmetry breaking scale. The new effective QCD-Lagrangian will indeed have local four-fermion couplings normalized to the $\Lambda_\chi$-cut-off scale. The assumption then is that higher dimension operators play “no fundamental role” in the description of the low energy physics. Within this alternative, one is still left with a functional integral over the low frequency modes of the quark and gluon fields.

The QCD effective action approach proposed in Ref. [25] and applied successfully to non-leptonic $\Delta S = 1$ decays in Ref. [26], as well as to the calculation of the imaginary part (terms of $O(p^6)$) of the QCD effective action [27], and to the pion electromagnetic mass-difference [28], falls in the second alternative described above. As we shall later discuss (see also Ref. [29]) the constituent chiral quark mass term

$$- M_Q(\bar{q}_L U^\dagger q_R + \bar{q}_R U q_L), \quad (1)$$

which in the approach of Ref. [25] is added to the usual QCD-Lagrangian, is equivalent to the mean-field approximation of a Nambu Jona-Lasinio mechanism triggered
by a four quark operator of the type $\sum_{a,b} (\bar{q}_a^R q_b^L)(\bar{q}_b^L q_a^R)$, where $a, b$ denote SU(3) flavour indices and summation over colour degrees of freedom within each bracket is understood. Here

$$q_L \equiv \frac{1}{2}(1 - \gamma_5)q(x) \quad \text{and} \quad q_R \equiv \frac{1}{2}(1 + \gamma_5)q(x),$$

with $\bar{q}$ the flavour triplet of Dirac spinors ($\bar{q}(x) = u^\dagger(x)\gamma^0$).

The $3 \times 3$ matrix $U$ in (1) is the unitary matrix which collects the Goldstone modes, i.e., the pseudoscalar degrees of freedom ($\pi$, $K$ and $\eta$) of the hadronic spectrum. Under chiral $SU(3)_L \times SU(3)_R$ transformations $U \rightarrow g_R U g_L^\dagger$ and (1) is therefore invariant. The mass parameter $M_Q$ provides a regulator of the infra-red behaviour of the low energy effective action when the quark fields are integrated out in the presence of a gluonic field background. The term in (1) was proposed in Ref. [25] as a phenomenological parametrization of spontaneous chiral symmetry breaking.

The purpose of this article is to report on a systematic study of the low energy effective action of the extended Nambu Jona-Lasinio model (ENJL) which, at intermediate energies below or of the order of a cut-off scale $\Lambda_{\chi}$, is expected to be a good effective realization of the standard QCD Lagrangian $L_{QCD}$. The Lagrangian in question is

$$L_{QCD} \rightarrow L_{QCD} + L_{N,JL}^{S,P} + L_{N,JL}^{V,A} + O\left(\frac{1}{\Lambda_{\chi}^4}\right),$$

with

$$L_{N,JL}^{S,P} = \frac{8\pi^2 G_S (\Lambda_{\chi})}{N_c \Lambda_{\chi}^2} \sum_{a,b} (\bar{q}_a^R q_b^L)(\bar{q}_b^L q_a^R)$$

and

$$L_{N,JL}^{V,A} = -\frac{8\pi^2 G_V (\Lambda_{\chi})}{N_c \Lambda_{\chi}^2} \sum_{a,b} \left[(\bar{q}_a^L \gamma^\mu q_b^L)(\bar{q}_b^L \gamma_{\mu}^\dagger q_a^L) + (L \rightarrow R)\right].$$

The couplings $G_S$ and $G_V$ are dimensionless quantities. In principle they are calculable functions of the $\Lambda_{\chi}$-cut-off scale. In practice, the calculation requires knowledge of the non-perturbative behaviour of QCD, and we shall take $G_S$ and $G_V$ as independent unknown constants. In choosing the forms (1) and (2) of the four-quark operators, we have only kept those couplings which are allowed by the symmetries of the original QCD-Lagrangian, and which are leading in the $1/N_c$-expansion, where $N_c$ denotes the number of colours. With one inverse power of $N_c$ pulled-out; $G_S$ and $G_V$ are constants of $O(1)$ in the large $N_c$ limit.
It is instructive to figure out how a four-fermion interaction like the one above arises already at short distances in perturbative QCD. This is illustrated in Fig. 1. The regulator replacement

\[ \frac{1}{Q^2} \rightarrow \int_0^{1/\Lambda^2} d\tau e^{-\tau Q^2} \]  

(7)
in the gluon propagator between the two interaction vertices in Fig. 1a leads to a local effective four-quark interaction (\( g_s \) is the colour coupling constant)

\[ \frac{1}{\Lambda^2} g_s^2 \sum_a (\bar{q}(x) \gamma^\mu \lambda^{(a)} q(x))(\bar{q}(x) \gamma_\mu \lambda^{(a)} q(x)) \]  

(8)
as illustrated in Fig. 1b, which, using Fierz identities and neglecting subleading terms in \( 1/N_c \) reproduces the form of the interaction terms in 5 and 6 with

\[ G_S = 4G_V = \frac{g_s^2}{4\pi^2} N_c. \]  

(9)

This perturbative estimate of \( G_S \) and \( G_V \) is however only valid at \( \Lambda \) scales sensibly larger than the \( \Lambda_\chi \)-scale at which spontaneous chiral symmetry breaking takes place. In other words a reliable calculation of \( G_S \) and \( G_V \) at realistic scales of \( O(\Lambda_\chi) \) necessarily involves non-perturbative dynamics.

The \( \Lambda_\chi \) index in \( L_{QCD}^{\Lambda_\chi} \) means that only the low frequency modes of the quark and gluon fields are to be considered. Of course, at the level where these low frequency gluonic effects are ignored, we are practically led to the first alternative we described above, where the assumption is that “all the relevant” gluonic effects for low energy physics can be absorbed in the new couplings \( G_S \) and \( G_V \). We shall discuss how the two alternatives compare when confronted to phenomenological predictions.

The paper is organized as follows. In section 2 we describe the general framework of QCD Green’s functions at low energies. In section 3 the form of the low energy effective Lagrangians is given and its parameters derived from experiment. The next section contains the discussion of spontaneous chiral symmetry breaking in the extended Nambu Jona-Lasinio model including gluonic interactions. Sections 5 and 6 are the main part of this work. There we derive, respectively, the low energy effective action in the two alternatives described above. We pay special attention to the existence of relations between the low energy constants in the effective action that are independent of the input parameters of the ENJL model. Some of these relations remain true even after the inclusion of gluonic interactions. In section 7 we describe how this approach encompasses most other models in the literature as various limits of parameters. In section 8 we give our numerical results using the formulas we derived earlier and section 9 contains our main conclusions. A number
of technical remarks regarding the derivation of the low energy effective action are collected in the appendices.

2 QCD GREEN’S FUNCTIONS AT LOW ENERGIES.

In QCD with three light flavours u,d and s, the generating functional for the Green’s functions of vector, axial-vector, scalar and pseudoscalar quark currents is defined by the vacuum-to-vacuum transition amplitude

\[ e^{i\Gamma(v,a,s,p)} = \left\langle 0 \left| T e^{i \int d^4 x L_{QCD}(x)} \right| 0 \right\rangle, \]

with \( L_{QCD}(x) \) the QCD Lagrangian in the presence of external vector \( v_\mu(x) \), axial-vector \( a_\mu(x) \), scalar \( s(x) \) and pseudoscalar \( p(x) \) field sources; i.e.,

\[ L_{QCD}(x) = L_{QCD}^0 + \bar{q} \gamma^\mu (v_\mu + \gamma_5 a_\mu) q - \bar{q} (s - i \gamma_5 p) q, \]

where \( q \) denotes the flavour triplet of Dirac spinors in (3);

\[ L_{QCD}^0 = -\frac{1}{4} \sum_{a=1}^{8} G_{\mu \nu}^{(a)} G^{(a) \mu \nu} + i \bar{q} \gamma^\mu \left( \partial_\mu + i G_\mu \right) q, \]

and

\[ G_\mu \equiv g_s \sum_{a=1}^{N_c^2-1} \frac{\chi^{(a)}}{2} G^{(a)}_\mu(x) \]

is the gluon field matrix in the fundamental \( SU(N_c = 3) \) colour representation, with \( G^{(a)}_{\mu \nu} \) the gluon field strength tensor

\[ G^{(a)}_{\mu \nu} = \partial_\mu G^{(a)}_\nu - \partial_\nu G^{(a)}_\mu - g_s f_{abc} G^{(b)}_\mu G^{(c)}_\nu, \]

and \( g_s \) the colour coupling constant \( (\alpha_s = g_s^2/4\pi) \). The external field sources are hermitian \( 3 \times 3 \) matrices in flavour space and are colour singlets. The matrix field \( s(x) \) contains in particular the quark mass matrix \( \mathcal{M} = \text{diag}(m_u, m_d, m_s) \), i.e., \( s(x) = \mathcal{M} + \cdots \). The other external matrix fields are traceless.

The Lagrangian \( L_{QCD}(x) \) is invariant under local chiral \( SU(3)_L \times SU(3)_R \) transformations; i.e., with \( g_L, g_R \in SU(3)_L \times SU(3)_R \), and \( q_{L,R}(x) \) defined as in (3)

\[ q_L \rightarrow g_L(x)q_L \quad \text{and} \quad q_R \rightarrow g_R(x)q_R, \]

\[ l_\mu \equiv v_\mu - a_\mu \rightarrow g_L l_\mu g_L^\dagger + i g_L \partial_\mu g_L^\dagger, \]
\[ r_\mu \equiv v_\mu + a_\mu \to g_R r_\mu g_R^\dagger + ig_R \partial_\mu g_R^\dagger, \]  
(17)

and

\[ s + ip \to g_R (s + ip) g_L^\dagger. \]  
(18)

In fact, the Lagrangian \( \mathcal{L}_{QCD}^0(x) \) is formally invariant under global \( U(3)_L \times U(3)_R \) transformations. However, because of the \( U(1)_A \) anomaly this symmetry is broken from \( U(3)_L \times U(3)_R \) to \( SU(3)_L \times SU(3)_R \times U(1)_V \). Since we shall restrict ourselves to states with zero baryon number the \( U(1)_V \) factor plays no role. We shall also work in the large \( N_c \)-limit and restrict ourselves to physics at leading order in the \( 1/N_c \)-expansion. In this limit the \( U(1)_A \)-anomaly effects are absent.

It is convenient to use a path integral representation for the generating functional \( \Gamma(v, a, s, p) \),

\[
e^{i\Gamma(v,a,s,p)} = \frac{1}{Z} \int \mathcal{D}G \mathcal{D}\bar{q} \mathcal{D}q \exp \left( i \int d^4x \mathcal{L}_{QCD}(q, \bar{q}, G; v, a, s, p) \right)
\]

\[
= \int \mathcal{D}G \exp \left( -i \int d^4x \frac{1}{4} G^{(a)}_{\mu \nu} G^{(a)\mu \nu} \right) \mathcal{D}\bar{q} \mathcal{D}q \exp \left( i \int d^4x \bar{q} i D q \right),
\]  
(19)

where \( D \) denotes the Dirac operator

\[ D = \gamma^\mu (\partial_\mu + iG_\mu) - i\gamma^\mu (v_\mu + \gamma_5 a_\mu) + i(s - i\gamma_5 p). \]  
(20)

The normalization factor \( Z \) is fixed so that \( \Gamma(0,0,0,0) = 0 \). The generating functional \( \Gamma(v, a, s, p) \) is not invariant under local chiral transformations due to the existence of anomalies in the fermionic determinant. The structure of the anomalous piece in \( \Gamma \) is known however from the work of Bardeen \[30\] and Wess and Zumino \[31\].

In QCD, the \( SU(3)_L \times SU(3)_R \) symmetry in flavour space is expected to be spontaneously broken down to \( SU(3)_V \). In the limit of large \( N_c \), this has been proven to be the case under rather reasonable assumptions \[32\]. Numerical simulations of lattice regularized QCD also support this assumption \[33\]. According to Goldstone’s theorem, there appears then an octet of massless pseudoscalar particles (\( \pi, K, \eta \)). The fields of these particles can be conveniently collected in a \( 3 \times 3 \) unitary matrix \( U(\Phi) \) with \( \det U = 1 \). Under local chiral transformations

\[ U(x) \to g_R U(x) g_L^\dagger. \]  
(21)

Whenever necessary, a useful parametrization for \( U(\Phi) \), which we shall adopt, is
\[ U(\Phi) = \exp \left( -i \sqrt{2} \frac{\Phi(x)}{f_0} \right) \]  

(22)

where \( f_0 \simeq f_\pi = 93.2 \text{ MeV} \) and \( \lambda \) are Gell-Mann’s \( SU(3) \) matrices with \( tr\lambda_a\lambda_b = 2\delta_{ab} \)

\[ \Phi(x) = \frac{\lambda}{\sqrt{2}} \Phi(x) = \left( \begin{array}{ccc} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & \frac{-\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & K^0 & -2\frac{\eta}{\sqrt{6}} \end{array} \right). \]  

(23)

The 0− octet \( \Phi(x) \) is the ground state of the QCD hadronic spectrum. There is a mass-gap from the ground state to the first massive multiplets with 1−, 1+ and 0+ quantum numbers. The basic idea of the effective chiral Lagrangian approach is that, in order to describe physics of the strong interactions at low energies, it may prove more convenient to replace QCD by an effective field theory which directly involves the pseudoscalar 0− octet fields; and, perhaps, the fields of the first massive multiplets 1−, 1+ and 0+ as well.

The chiral symmetry of the underlying QCD theory implies that \( \Gamma(v,a,s,p) \) in eq. (19) admits a low energy representation

\[ e^{i\Gamma(v,a,s,p)} = \frac{1}{Z} \int DUe^{i\int d^4xL_{\text{eff}}(U,v,a,s,p)} \]

where the fields \( S(x), V_\mu(x) \) and \( A_\mu(x) \) are those associated with the lowest massive scalar, vector and axial-vector particle states of the hadronic spectrum. Both \( L_{\text{eff}}^R \) and \( L_{\text{eff}} \) are local Lagrangians which contain in principle an infinite number of terms. The hope is that, for sufficiently small energies as compared to the spontaneous chiral symmetry breaking scale \( \Lambda_\chi \), the restriction of \( L_{\text{eff}}^R \) and/or \( L_{\text{eff}} \) to a few terms with the lowest chiral dimension should provide a sufficiently accurate description of the low energy physics. The success of this approach at the phenomenological level is by now confirmed by many examples. Our aim here is to derive the effective Lagrangians \( L_{\text{eff}}^R \) and \( L_{\text{eff}} \) which follow from the Nambu Jona-Lasinio cut-off version of QCD described in the introduction.

3 LOW ENERGY EFFECTIVE LAGRANGIANS.

The purpose of this section is to summarize briefly what is known at present about the low energy mesonic Lagrangians \( L_{\text{eff}}^R \) and \( L_{\text{eff}} \) from the chiral invariance properties of \( L_{QCD} \) alone.

*For recent reviews see e.g. Refs. [34] to [36].
The terms in $L_{\text{eff}}$ with the lowest chiral dimension, i.e., $O(p^2)$ are

$$L^{(2)}_{\text{eff}} = \frac{1}{4} f_0^2 \{ \text{tr} D_\mu U D^\mu U^\dagger + \text{tr} (\chi U^\dagger + U^\dagger \chi) \}$$

(25)

where $D_\mu$ denotes the covariant derivative

$$D_\mu U = \partial_\mu U - i (v_\mu + a_\mu) U + i U (v_\mu - a_\mu)$$

(26)

$$\chi = 2 B_0 (s(x) + i p(x)).$$

(27)

The constants $f_0$ and $B_0$ are not fixed by chiral symmetry requirements. The constant $f_0$ can be obtained from $\pi \to \mu \nu$ decay, and it is the same which appears in the normalization of the pseudoscalar field matrix $\Phi(x)$ in (23), i.e.,

$$f_0 \approx f_\pi = 93.2 \text{ MeV}.$$  

(28)

The constant $B_0$ is related to the vacuum expectation value

$$<0|\bar{q}q|0>_{q=u,d,s} = - f_0^2 B_0 (1 + O(M)).$$  

(29)

The terms in $L_{\text{eff}}$ of $O(p^4)$ are also known. They have been classified by Gasser and Leutwyler [37]:

$$L^{(4)}(x) = L_1 (\text{tr} D_\mu U^\dagger D^\mu U)^2 + L_2 \text{tr} (D_\mu U^\dagger D_\nu U) \text{tr} (D^\mu U D^\nu U)$$

$$+ L_3 \text{tr} (D_\mu U^\dagger D^\mu U D_\nu U^\dagger D^\nu U) + L_4 \text{tr} (D_\mu U^\dagger D^\mu U) \text{tr} (\chi^\dagger U + U^\dagger \chi)$$

$$+ L_5 \text{tr} [D_\mu U^\dagger D^\mu U (\chi^\dagger U + U^\dagger \chi)] + L_6 [\text{tr} (\chi^\dagger U + U^\dagger \chi)]^2$$

$$+ L_7 [\text{tr} (\chi U - U^\dagger \chi)]^2 + L_8 \text{tr} (\chi^\dagger U \chi^\dagger U + \chi U^\dagger \chi U^\dagger)$$

$$- i L_9 \text{tr} (F^\mu_{\nu} D_\mu U D_\nu U^\dagger + F^\mu_{\nu} D_\mu U^\dagger D_\nu U) + L_{10} \text{tr} (U^\dagger F^\mu_{\nu} U F_L^{\mu\nu})$$

$$+ H_1 \text{tr} (F^\mu_{\nu} F_R^{\mu\nu} + F_L^{\mu\nu} F_L^{\mu\nu}) + H_2 \text{tr} (\chi^\dagger \chi),$$

where $F_L^{\mu\nu}$ and $F_R^{\mu\nu}$ are the external field-strength tensors

$$F_L^{\mu\nu} = \partial_\mu l_\nu - \partial_\nu l_\mu - i [l_\mu, l_\nu]$$

(31)

$$F_R^{\mu\nu} = \partial_\mu r_\nu - \partial_\nu r_\mu - i [r_\mu, r_\nu]$$

(32)
associated with the external left ($l_\mu$) and right ($r_\mu$) field sources

$$l_\mu = v_\mu - a_\mu, \quad r_\mu = v_\mu + a_\mu. \quad (33)$$

The constants $L_i$ and $H_i$ are again not fixed by chiral symmetry requirements. The $L_i$’s were phenomenologically determined in Ref. [37]. Since then, $L_{1,2,3}$ have been fixed more accurately using data from $K_{l4}$ [38]. The phenomenological values of the $L_i$’s which will be relevant for a comparison with our calculations, at a renormalization scale $\mu = M_\rho = 770 \text{MeV}$, are collected in the first column in Table 1.

By contrast to $\mathcal{L}_{\text{eff}}$, which only has pseudoscalar fields as physical degrees of freedom, the Lagrangian $\mathcal{L}_{\text{eff}}^R$ involves chiral couplings of fields of massive $1^-, 1^+$ and $0^+$ states, to the Goldstone fields. The general method to construct these couplings was described long time ago in Ref. [39]. An explicit construction of the couplings for $1^-, 1^+$ and $0^+$ fields can be found in Ref. [40]. As discussed in Ref. [41], the choice of fields to describe chiral invariant couplings involving spin-1 particles is not unique and, when the vector modes are integrated out, leads to ambiguities in the context of chiral perturbation theory to $O(p^4)$ and higher. As shown in [41], these ambiguities are, however, removed when consistency with the short-distance behaviour of QCD is incorporated. The effective Lagrangian which we shall choose here to describe vector couplings corresponds to the so called model II in Ref. [41].

The wanted ingredient for a non-linear representation of the chiral $SU(3)_L \times SU(3)_R \equiv G$ group when dealing with matter fields is the compensating $SU(3)_V$ transformation $h(\Phi, g_{L,R})$ which appears under the action of the chiral group $G$ on the coset representative $\xi(\Phi)$ of the $G/SU(3)_V$ manifold, i.e.,

$$\xi(\Phi) \rightarrow g_R \xi(\Phi) h^\dagger(\Phi, g_{L,R}) = h(\Phi, g_{L,R}) \xi(\Phi) g^\dagger_L \quad (34)$$

where $\xi(\Phi)\xi(\Phi) = U$ in the chosen gauge. This defines the $3 \times 3$ matrix representation of the induced $SU(3)_V$ transformation. Denoting the various matter $SU(3)_V$-multiplets by $R$ (octet) and $R_1$ (singlets), the non-linear realization of $G$ is given by

$$R \rightarrow h(\Phi, g_{L,R}) R h^\dagger(\Phi, g_{L,R}) \quad (35)$$

$$R_1 \rightarrow R_1, \quad (36)$$

with the usual matrix notation for the octet

$$R = \frac{1}{\sqrt{2}} \sum_{i=1}^{8} \lambda^{(i)} R^{(i)}. \quad (37)$$
The vector field matrix $V^\mu(x)$ representing the $SU(3)_V$-octet of $1^-$ particles; the axial-vector field matrix $A^\mu(x)$ representing $SU(3)_V$-octet of $1^+$ particles; and the scalar field matrix $S(x)$ representing $SU(3)_V$-octet of $0^+$ particles are chosen to transform like $R$ in eq. (35), i.e.,

$$V_\mu \rightarrow h V_\mu h^\dagger; \quad A_\mu \rightarrow h A_\mu h^\dagger; \quad S \rightarrow h S h^\dagger.$$  

(38)

The procedure now to construct the lowest order chiral Lagrangian $\mathcal{L}_{\text{eff}}^R$ is to write down all possible invariant couplings to first non-trivial order in the chiral expansion which are linear in the $R$-fields and to add of course the corresponding invariant kinetic couplings. It is convenient for this purpose to set first the list of possible tensor structures involving the $R$-fields which transform like $R$ in eq. (35) under the action of the chiral group $G$. Since the non-linear realization of $G$ on the octet field $R$ is local, one is led to define a covariant derivative

$$d_\mu R = \partial_\mu R + [\Gamma_\mu, R]$$  

(39)

with a connection

$$\Gamma_\mu = \frac{1}{2} \{ \xi^\dagger [\partial_\mu - i(v_\mu + a_\mu)] \xi + \xi [\partial_\mu - i(v_\mu - a_\mu)] \xi^\dagger \}$$  

(40)

ensuring the transformation property

$$d_\mu R \rightarrow h d_\mu R h^\dagger.$$  

(41)

We can then define vector and axial-vector field strength tensors

$$V^\mu_\nu = d^\mu V_\nu - d_\nu V^\mu \quad \text{and} \quad A^\mu_\nu = d^\mu A_\nu - d_\nu A^\mu$$  

(42)

which also transform like $R$, i.e.,

$$V^\mu_\nu \rightarrow h V^\mu_\nu h^\dagger \quad \text{and} \quad A^\mu_\nu \rightarrow h A^\mu_\nu h^\dagger.$$  

(43)

There is a complementary list of terms one can construct with the coset representative $\xi(\Phi)$ and which transform homogeneously; i.e., like $R$ in (35). If we restrict ourselves to terms of $O(p^2)$ at most, here is the list:

$$\xi_\mu = i \{ \xi^\dagger [\partial_\mu - i(v_\mu + a_\mu)] \xi - \xi [\partial_\mu - i(v_\mu - a_\mu)] \xi^\dagger \} = i \xi^\dagger D_\mu U \xi^\dagger = \xi^\dagger_{\mu\iota},$$  

(44)

$$\xi_\mu \xi_\nu \quad \text{and} \quad d_\mu \xi_\nu.$$  

(45)
\[ \chi_\pm = \xi^\dagger \chi \xi \pm \xi \chi^\dagger \xi, \]  
\[ f_{\mu\nu}^\pm = \xi F_{L\mu\nu} \xi^\dagger \pm \xi^\dagger F_{R\mu\nu} \xi. \]  
Notice that \( \Gamma_\mu \) in (40) does not transform homogeneously, but rather like an \( SU(3)_V \) Yang-Mills field, i.e.,
\[ \Gamma_\mu \rightarrow h \Gamma_\mu h^\dagger + h \partial_\mu h^\dagger. \]

The most general Lagrangian \( \mathcal{L}_{eff}^R \) to lowest non-trivial order in the chiral expansion is then obtained by adding to \( \mathcal{L}_{eff}^{(2)} \) in eq.(25) the scalar Lagrangian
\[ \mathcal{L}^S = \frac{1}{2} \text{tr} \left( d_\mu S d^\mu S - M_S^2 S^2 \right) + c_m \text{tr} \left( S \chi^\dagger \right) + c_d \text{tr} \left( S \xi^\dagger \xi \right); \]  
the vector Lagrangian
\[ \mathcal{L}^V = -\frac{1}{4} \text{tr} \left( V_{\mu\nu} V^{\mu\nu} - 2 M_V^2 V_{\mu} V^\mu \right) - \frac{1}{2 \sqrt{2}} \left[ f_V \text{tr} \left( V_{\mu\nu} f^{(+)}_{\mu\nu} \right) + i g_V \text{tr} \left( V_{\mu\nu} \left[ \xi^\mu, \xi^\nu \right] \right) \right] + \cdots, \]  
and the axial-vector Lagrangian
\[ \mathcal{L}^A = -\frac{1}{4} \text{tr} \left( A_{\mu\nu} A^{\mu\nu} - 2 M_A^2 A_{\mu} A^\mu \right) - \frac{1}{2 \sqrt{2}} f_A \text{tr} \left( A_{\mu\nu} f^{(-)}_{\mu\nu} \right) + \cdots, \]  
The dots in \( \mathcal{L}^V \) and \( \mathcal{L}^A \) stand for other \( O(p^3) \) couplings which involve the vector field \( V^\mu \) and axial-vector field \( A^\mu \) instead of the field-strength tensors \( V_{\mu\nu} \) and \( A_{\mu\nu} \). They have been classified in Ref. [41]. As discussed there, they play no role in the determination of the \( O(p^4) \) \( L_i \) couplings when the vector and axial-vector fields are integrated out.

The masses \( M_V, M_S \) and \( M_A \) and the coupling constants \( c_m, c_d, f_V, g_V \) and \( f_A \) are not fixed by chiral symmetry requirements. They can be determined phenomenologically as it was done in Ref. [40]. Since later on we shall only calculate masses and couplings in the chiral limit, we identify \( M_V, M_S \) and \( M_A \) to those of non-strange particles of the corresponding multiplets, i.e.,
\[ M_V = M_\rho = 770 \text{ MeV}; \quad M_S = M_{a_0} = 983 \text{ MeV} \]  
and
\[ M_A = M_{a_1} = 1260 \pm 30 \text{ MeV}. \]
The couplings $f_V$ and $g_V$ can be then determined from the decay $\rho^0 \to e^+ e^-$ and $\rho \to \pi\pi$ respectively, with the result

$$|f_V| = 0.20 \text{ and } |g_V| = 0.090.$$  \hspace{1cm} (54)

The decay $a_1 \to \pi\gamma$ fixes the coupling $f_A$ to

$$|f_A| = 0.097 \pm 0.022,$$  \hspace{1cm} (55)

where the error is due to the experimental error in the determination of the partial width [12]. $\Gamma(a_1 \to \pi\gamma) = (640 \pm 246)\text{keV}$. For the scalar couplings $c_m$ and $c_d$, the decay rate $a_0 \to \eta\pi$ only fixes the linear combination [10]

$$|c_d + \frac{2m_\pi^2}{M_{a_0}^2 - m_\eta^2 - m_\pi^2} c_m| = (34.3 \pm 3.3)\text{MeV}.$$  \hspace{1cm} (56)

In confronting these results to theoretical predictions, one should keep in mind that they have not been corrected for the effects of chiral loop contributions.

4 SPONTANEOUS CHIRAL SYMMETRY BREAKING A LA NAMBU JONA-LASINIO.

Following the standard procedure of introducing auxiliary fields, one can rearrange the Nambu Jona-Lasinio cut-off version of the QCD Lagrangian in an equivalent Lagrangian which is only quadratic in the quark fields. For this purpose we shall introduce three complex $3 \times 3$ auxiliary field matrices $M(x)$, $L_\mu(x)$ and $R_\mu(x)$; the so called collective field variables, which under the chiral group $G$ transform as

$$M \to g_R Mg_L$$  \hspace{1cm} (57)

$$L_\mu \to g_L L_\mu g_L^\dagger \text{ and } R_\mu \to g_R R_\mu g_R^\dagger.$$  \hspace{1cm} (58)

We can then write the following identities:

$$\exp i \int d^4x {\mathcal L}^{S,P}_{NJL}(x) =$$

$$\int {\mathcal D}M \exp i \int d^4x \left\{ - (\bar{q}_L M^\dagger q_R + \text{h.c.}) - \frac{N_c A_X^2}{8\pi^2 G_S(A_X)} tr(M^\dagger M) \right\};$$  \hspace{1cm} (59)

and

$$\exp i \int d^4x {\mathcal L}^{V,A}_{NJL}(x) =$$
\[
\int D\mu D\nu \exp i \int d^4x \left\{ \bar{q}_L \gamma^\mu L_\mu q_L + \frac{N_c \Lambda^2}{8\pi^2 G_V(\Lambda)} \left[ \frac{1}{4} tr L^\mu L_\mu + (L \rightarrow R) \right] \right\},
\]

where \( L^{S,P}_{NJL}(x) \) and \( L^{V,A}_{NJL}(x) \) are the four-fermion Lagrangians in (3) and (4).

By polar decomposition

\[
M = U \tilde{H} = \xi H \xi,
\]

with \( U \) unitary and \( \tilde{H} \) (and \( H \)) hermitian. From the transformation laws of \( M \) and \( \xi \) in eqs. (57) and (34), it follows that \( H \) transforms homogeneously, i.e.,

\[
H \rightarrow h(\Phi, g_{L,R}) H h^\dagger(\Phi, g_{L,R}).
\]

The path integral measure in eq. (59) can then also be written as

\[
\exp i \int d^4x L^{S,P}_{NJL}(x) =
\]

\[
\int D\xi DH \exp i \int d^4x \left\{ -\left( \bar{q}_L \xi^\dagger H \xi q_R + \bar{q}_R \xi H \xi q_L \right) - \frac{N_c \Lambda^2}{8\pi^2 G_S(\Lambda)} tr H^2 \right\}. \tag{63}
\]

We are interested in the effective action \( \Gamma_{eff}(H, \xi, L_\mu, R_\mu; v, a, s, p) \) defined in terms of the new auxiliary fields \( H, \xi, L_\mu, R_\mu \); and in the presence of the external field sources \( v_\mu, a_\mu, s \) and \( p \), i.e.,

\[
e^{i \Gamma_{eff}(H, \xi, L_\mu, R_\mu; v, a, s, p)} = \frac{1}{Z} \int DG_\mu \exp \left( -i \int d^4x \frac{1}{4} G^{(a)}_{\mu\nu} G^{(a)\mu\nu} \right) \times \]

\[
\exp i \int d^4x \left\{ \frac{N_c \Lambda^2}{8\pi^2 G_V(\Lambda)} \frac{1}{4} \left[ tr (L^\mu L_\mu) + tr (R^\mu R_\mu) \right] - \frac{N_c \Lambda^2}{8\pi^2 G_S(\Lambda)} tr H^2 \right\} \times \]

\[
\int D\bar{q} Dq \exp i \int d^4x \left\{ \bar{q} D_{QCD} q + \bar{q}_L \gamma^\mu L_\mu q_L + \bar{q}_R \gamma^\mu R_\mu q_R - \left( \bar{q}_L \xi^\dagger H \xi q_R + \bar{q}_R \xi H \xi q_L \right) \right\},
\]

with \( D_{QCD} \) the QCD Dirac operator

\[
D_{QCD} = \gamma^\mu (\partial_\mu + i G_\mu) - i \gamma^\mu (v_\mu + \gamma_5 a_\mu) + i (s - i \gamma_5 p).
\]

The integrand is now quadratic in the fermion fields, and the path integral over the quark fields is the determinant of the full Dirac-operator in the Lagrangian (see appendix for technical details).

Here, we are looking for translational invariant solutions which minimize the effective action, i.e.,

\[
\frac{\delta \Gamma_{eff}(H, \ldots)}{\delta H} \bigg|_{L_\mu=R_\mu=0, \xi=1, H=\langle H \rangle; v_\mu=a_\mu=s=p=0} = 0 \tag{66}
\]
where \(< H > = \text{diag}(M_u, M_d, M_s)\). The minimum is reached when all the eigenvalues of \(< H >\) are equal, i.e.,

\[ < H > = M_Q 1 \]  \hspace{1cm} (67)

and the minimum condition leads to the so called gap equation

\[ Tr(x|D_E^{-1}|x)|_{L_\mu=R_\mu=0,\xi=1, H=M_Q; \psi_\mu=a_\mu=s=p=0} = -4M_Q \frac{N_c \Lambda^2}{16\pi^2 G_S(\Lambda)} \int d^4x, \]  \hspace{1cm} (68)

where \(D_E^{-1}\) denotes the full Dirac-operator in euclidean space-time. The trace in the l.h.s. gives the formal evaluation of the vacuum expectation value of the quark-bilinear

\[ < \bar{\psi}\psi > \equiv < \bar{u}u > = < \bar{d}d > = < \bar{s}s >. \]  \hspace{1cm} (69)

In the large \(N_c\)-limit, and with neglect of the gluonic couplings in \(\mathcal{L}_{QCD}^A\), this trace can be evaluated in the cut-off theory, with the result (see appendix)

\[ < \bar{\psi}\psi > = -\frac{N_c}{16\pi^2} 4M_Q^3 \Gamma(-1, \frac{M_Q^2}{\Lambda^2}) \]  \hspace{1cm} (70)

where \(\Gamma(-1, x)\) denotes the incomplete gamma function

\[ \Gamma(n-2, x = \frac{M_Q^2}{\Lambda^2}) = \int_{\frac{M_Q^2}{\Lambda^2}}^{\infty} \frac{dz}{z} e^{-z} z^{n-2}; \quad n = 1, 2, 3, \ldots. \]  \hspace{1cm} (71)

The corresponding gap equation results in the constraint

\[ \frac{M_Q}{G_S(\Lambda)} = M_Q \left\{ \exp(-\frac{M_Q^2}{\Lambda^2}) - \frac{M_Q^2}{\Lambda^2} \Gamma(0, \frac{M_Q^2}{\Lambda^2}) \right\}. \]  \hspace{1cm} (72)

This is the same solution as the one from the Schwinger-Dyson equation which in a diagrammatic notation is written in Fig. 2. In terms of conventional Feynman diagrams, the set of diagrams which are summed in the leading large-\(N_c\) approximation are chains of fermion bubbles as indicated in Fig. 3a; as well as trees of chains, like in Fig. 3b; but not loops of chains, as in Fig. 4. Loops of chains are next to leading in the \(1/N_c\)-expansion.

Equations (71) and (72) show the existence of two-phases with regards to chiral symmetry. The unbroken phase corresponds to

\[ M_Q = 0 \Rightarrow < \bar{\psi}\psi > = 0 \]  \hspace{1cm} (73)

The broken phase corresponds to the possibility that as we decrease the ultraviolet cut-off \(\Lambda\) down to \(\Lambda_\chi\) the coupling \(G_S(\Lambda)\) increases allowing for solutions to eq. (72)
with \( M_Q > 0 \) and therefore \( \bar{\psi}\psi \neq 0 \) and negative. In this phase, the hermitian auxiliary field \( H(x) \) develops a nonvanishing vacuum expectation value, which is at the origin of a constituent chiral quark mass term

\[
-M_Q(\bar{q}_L U^\dagger q_R + \bar{q}_R U q_L)
\]  

(74)
in the effective Lagrangian. This is precisely the term which in the approach of Ref. [25] was incorporated as a phenomenological parametrization of spontaneous chiral symmetry breaking.

Gluonic interactions due to fluctuations below the cut-off scale \( \Lambda_\chi \) can be incorporated phenomenologically as proposed in Ref. [25]; i.e., keeping the contributions from the vacuum expectation values of gluon fields which are leading in the \( 1/N_c \)-expansion. This leads to correction terms in eqs. (70) and (72) with the result [13]:

\[
<\bar{\psi}\psi> = -\frac{N_c}{16\pi^2}4M_Q^3\Gamma(-1, \frac{M_Q^2}{\Lambda_\chi^2})
\]

\[
-\frac{1}{12}\frac{\alpha_s G G}{M_Q^2}\Gamma(1, \frac{M_Q^2}{\Lambda_\chi^2}) - \frac{1}{360}\frac{\alpha_s}{\pi}g_S \frac{G G G}{M_Q^3}\Gamma(2, \frac{M_Q^2}{\Lambda_\chi^2}) + \ldots,
\]

(75)

where the gluon condensates are understood as averages over frequency modes below the cut-off scale \( \Lambda_\chi \).

These gluonic corrections can be reabsorbed by an appropriate change of the \( \Lambda_\chi \)-scale and a redefinition of the coupling constant \( G_S \). For this particular case, the two alternative mechanisms described in the introduction are therefore equivalent.

5 THE LOW ENERGY EFFECTIVE ACTION OF THE NAMBU JONA-LASINIO CUT-OFF VERSION OF QCD.

5.1 The Mean Field Approximation.

We shall first discuss a particular case of \( \Gamma_{\text{eff}}(H, \xi, L_\mu, R_\mu; v, a, s, p) \) as defined in eq. (54). It is the case corresponding to the mean field approximation, where

\[
H(x) = <H> = M_Q 1,
\]

(76)

and where we set

\[
L_\mu = R_\mu = 0.
\]

(77)

The effective action \( \Gamma_{\text{eff}}(M_Q, \xi, 0, 0; v, a, s, p) \) coincides then with the one calculated in Ref. [23], except that the regularization of the UV-behaviour is different. In
Ref. [25], the regularization which is used is the $\zeta$-function regularization. In a cut-off theory, like the one we have now, we have a physical UV-cut-off $\Lambda_\chi$; and the regularization must explicitly exhibit this $\Lambda_\chi$-dependence. In the calculations reported here we have used a proper-time regularization (see appendix for details). The results, to a first approximation where low frequency gluonic terms are ignored, are as follows:

$$f_0^2 = \frac{N_c}{16\pi^2} 4 M_Q^2 \Gamma(0, \frac{M_Q^2}{\Lambda_\chi^2})$$

and

$$f_0^2 B_0 = - < \bar{\psi} \psi > = \frac{N_c}{16\pi^2} 4 M_Q^2 \Gamma(-1, \frac{M_Q^2}{\Lambda_\chi^2})$$

for the lowest $O(p^2)$-couplings of the low energy effective Lagrangian in (25).

For the $O(p^4)$-couplings which exist in the chiral limit we find

$$L_2 = 2L_1 = \frac{N_c}{16\pi^2} \frac{1}{12} \Gamma(2, \frac{M_Q^2}{\Lambda_\chi^2}),$$

$$L_3 = \frac{N_c}{16\pi^2} \frac{1}{6} \left[ \Gamma(1, \frac{M_Q^2}{\Lambda_\chi^2}) - 2\Gamma(2, \frac{M_Q^2}{\Lambda_\chi^2}) \right]$$

for the four derivative terms; and

$$L_9 = \frac{N_c}{16\pi^2} \frac{1}{3} \Gamma(1, \frac{M_Q^2}{\Lambda_\chi^2}),$$

$$L_{10} = - \frac{N_c}{16\pi^2} \frac{1}{6} \Gamma(1, \frac{M_Q^2}{\Lambda_\chi^2})$$

for the two couplings involving external fields. If one lets $M_Q^2 \Lambda_\chi^2 \to 0$, then $\Gamma(n,0) = \Gamma(n) = (n-1)!$ for $n \geq 1$, and these results coincide with those previously obtained in Refs. [13], [14], [15], [25], and [44] to [47].

When terms proportional to the quark mass matrix $\mathcal{M}$ are kept, there appear four new $L_1$-couplings (see eq. (31)). With

$$\rho = \frac{M_Q}{|B_0|} = \frac{M_Q f_0^2}{| < \bar{\psi} \psi > |},$$

the results we find for these new couplings are

$$L_4 = 0$$
\begin{align*}
L_5 &= \frac{N_c}{16\pi^2} \frac{\rho}{2} \left[ \Gamma(0, \frac{M_Q^2}{\Lambda^2_\chi}) - \Gamma(1, \frac{M_Q^2}{\Lambda^2_\chi}) \right] \\
L_6 &= 0 \\
L_7 &= \frac{N_c}{16\pi^2} \frac{1}{12} \left[ -\rho \Gamma(0, \frac{M_Q^2}{\Lambda^2_\chi}) + \frac{1}{6} \Gamma(1, \frac{M_Q^2}{\Lambda^2_\chi}) \right] \\
L_8 &= -\frac{N_c}{16\pi^2} \frac{1}{24} \left[ 6\rho(\rho - 1) \Gamma(0, \frac{M_Q^2}{\Lambda^2_\chi}) + \Gamma(1, \frac{M_Q^2}{\Lambda^2_\chi}) \right].
\end{align*}

If we identify \( \Gamma(0, \frac{M_Q^2}{\Lambda^2_\chi}) \equiv \log \frac{\mu^2}{M_Q^2} \), and take the limit \( \Gamma(n \geq 1, \frac{M_Q^2}{\Lambda^2_\chi} \to 0) \) these results coincide then with those obtained in Ref. [25]. (Notice that \( \rho \) is twice the parameter \( x \) of Ref. [25].)

The fact that \( L_4 = L_6 = 0 \) and \( L_2 = 2L_1 \), is more general than the model calculations we are discussing. As first noticed by Gasser and Leutwyler [37], these are properties of the large \( N_c \) limit. The contribution we find for \( L_7 \) is in fact non-leading in the \( 1/N_c \)-expansion. The result above is entirely due to the use of the lowest order equations of motion (see the erratum to Ref. [25]). In the presence of the \( U_A(1) \)-anomaly, \( L_7 \) picks up a contribution from the \( \eta' \)-pole and becomes \( O(N_c^2) \), [37].

Finally, we shall also give the results for the \( H_1 \) and \( H_2 \) coupling constants of terms which only involve external fields:

\begin{align*}
H_1 &= -\frac{N_c}{16\pi^2} \frac{1}{12} \left[ 2\Gamma(0, \frac{M_Q^2}{\Lambda^2_\chi}) - \Gamma(1, \frac{M_Q^2}{\Lambda^2_\chi}) \right] \\
H_2 &= \frac{N_c}{16\pi^2} \frac{1}{12} \left[ 6\rho^2 \Gamma(-1, \frac{M_Q^2}{\Lambda^2_\chi}) - 6\rho(\rho + 1) \Gamma(0, \frac{M_Q^2}{\Lambda^2_\chi}) + \Gamma(1, \frac{M_Q^2}{\Lambda^2_\chi}) \right].
\end{align*}

### 5.2 Beyond the Mean Field Approximation.

In full generality,

\[ H(x) = M_Q1 + \sigma(x); \]

and the effective action \( \Gamma_{\text{eff}}(H, \xi, L_\mu, R_\mu; v, a, s, p) \) has a non-trivial dependence on the auxiliary field variables \( \sigma(x) \), \( L_\mu(x) \) and \( R_\mu(x) \). It is convenient to trade the auxiliary left and right vector field variables \( L_\mu \) and \( R_\mu \) which were introduced in eq. (90), by the new vector fields

\[ W^\pm_\mu = \xi L_\mu \xi^\dagger \pm \xi^\dagger R_\mu \xi. \]
From the transformation properties in eqs. (34) and (58) it follows that $W^\pm_\mu$ transform homogeneously; i.e.,

$$W^\pm_\mu \rightarrow h(\Phi, g)W^\pm_\mu h^\dagger(\Phi, g).$$

We also find it convenient to rewrite the effective action in eq. (64) in a basis of constituent chiral quark fields

$$Q = Q_L + Q_R$$

and

$$\bar{Q} = \bar{Q}_L + \bar{Q}_R$$

where

$$Q_L = \xi q_L, \quad \bar{Q}_L = \bar{q}_L \xi^\dagger; \quad Q_R = \xi^\dagger q_R, \quad \bar{Q}_R = \bar{q}_R \xi,$$

which under the chiral group $G$, transform like

$$Q \rightarrow h(\Phi, g)Q$$

and

$$\bar{Q} \rightarrow \bar{Q} h(\Phi, g)^\dagger.$$  

In this basis, the linear terms (in the auxiliary field variables) in the r.h.s. of eq. (64) become

$$\bar{q}_L \gamma^\mu L_\mu q_L + \bar{q}_R \gamma^\mu R_\mu q_R - \left(\bar{q}_L \xi^\dagger H \xi^\dagger q_R + \bar{q}_R \xi H \xi q_L\right)$$

$$\rightarrow \bar{Q} \left(-H + \frac{1}{2} \gamma^\mu W^+_\mu + \frac{1}{2} \gamma^\mu \gamma_5 W^-_\mu\right) Q.$$  

The effective action $\Gamma_{eff}(M_Q, \xi, \sigma, W^\pm_\mu; v, a, s, p)$ in terms of the new auxiliary field variables, and in euclidean space is then

$$e^{\Gamma_{eff}(M_Q, \xi, \sigma, W^\pm_\mu; v, a, s, p) =}$$

$$\exp\left(-\int d^4x \left\{ \frac{N_c \Lambda^2}{8\pi^2 G_S(\Lambda)} tr H^2 + \frac{N_c \Lambda^2}{16\pi^2 G_V(\Lambda)} \frac{1}{4} tr (W^+_\mu W^+_\mu + W^-_\mu W^-_\mu) \right\} \right) \times$$

$$\frac{1}{Z} \int \mathcal{D}G_\mu \exp\left(-\int d^4x \frac{1}{4} G^{(a)}_\mu G^{(a)}_\mu \right) \int \mathcal{D}\bar{Q} \mathcal{D}Q \exp \int d^4x \bar{Q} \mathcal{D}_E Q$$

where $\mathcal{D}_E$ denotes the euclidean Dirac operator

$$\mathcal{D}_E = \gamma_\mu \nabla_\mu - \frac{1}{2}(\Sigma - \gamma_5 \Delta) - H(x)$$

with $\nabla_\mu$, the covariant derivative

$$\nabla_\mu = \partial_\mu + iG_\mu + \Gamma_\mu - \frac{i}{2} \gamma_5 (\xi_\mu - W^{(-)}_\mu) - \frac{i}{2} W^{(+)\mu}$$

and

From the transformation properties in eqs. (34) and (58) it follows that $W^\pm_\mu$ transform homogeneously; i.e.,

$$W^\pm_\mu \rightarrow h(\Phi, g)W^\pm_\mu h^\dagger(\Phi, g).$$

We also find it convenient to rewrite the effective action in eq. (64) in a basis of constituent chiral quark fields

$$Q = Q_L + Q_R$$

and

$$\bar{Q} = \bar{Q}_L + \bar{Q}_R$$

where

$$Q_L = \xi q_L, \quad \bar{Q}_L = \bar{q}_L \xi^\dagger; \quad Q_R = \xi^\dagger q_R, \quad \bar{Q}_R = \bar{q}_R \xi,$$

which under the chiral group $G$, transform like

$$Q \rightarrow h(\Phi, g)Q$$

and

$$\bar{Q} \rightarrow \bar{Q} h(\Phi, g)^\dagger.$$  

In this basis, the linear terms (in the auxiliary field variables) in the r.h.s. of eq. (64) become

$$\bar{q}_L \gamma^\mu L_\mu q_L + \bar{q}_R \gamma^\mu R_\mu q_R - \left(\bar{q}_L \xi^\dagger H \xi^\dagger q_R + \bar{q}_R \xi H \xi q_L\right)$$

$$\rightarrow \bar{Q} \left(-H + \frac{1}{2} \gamma^\mu W^+_\mu + \frac{1}{2} \gamma^\mu \gamma_5 W^-_\mu\right) Q.$$  

The effective action $\Gamma_{eff}(M_Q, \xi, \sigma, W^\pm_\mu; v, a, s, p)$ in terms of the new auxiliary field variables, and in euclidean space is then

$$e^{\Gamma_{eff}(M_Q, \xi, \sigma, W^\pm_\mu; v, a, s, p) =}$$

$$\exp\left(-\int d^4x \left\{ \frac{N_c \Lambda^2}{8\pi^2 G_S(\Lambda)} tr H^2 + \frac{N_c \Lambda^2}{16\pi^2 G_V(\Lambda)} \frac{1}{4} tr (W^+_\mu W^+_\mu + W^-_\mu W^-_\mu) \right\} \right) \times$$

$$\frac{1}{Z} \int \mathcal{D}G_\mu \exp\left(-\int d^4x \frac{1}{4} G^{(a)}_\mu G^{(a)}_\mu \right) \int \mathcal{D}\bar{Q} \mathcal{D}Q \exp \int d^4x \bar{Q} \mathcal{D}_E Q$$

where $\mathcal{D}_E$ denotes the euclidean Dirac operator

$$\mathcal{D}_E = \gamma_\mu \nabla_\mu - \frac{1}{2}(\Sigma - \gamma_5 \Delta) - H(x)$$

with $\nabla_\mu$, the covariant derivative

$$\nabla_\mu = \partial_\mu + iG_\mu + \Gamma_\mu - \frac{i}{2} \gamma_5 (\xi_\mu - W^{(-)}_\mu) - \frac{i}{2} W^{(+)\mu}$$

and
The quantities $G_\mu$, $\Gamma_\mu$ and $\xi_\mu$ are those defined in eqs. (13), (40) and (44).

At this stage, it is worth pointing out a formal symmetry which is useful to check explicit calculations. We can redefine the external vector-field sources via

$$l_\mu \rightarrow l'_\mu = l_\mu + L_\mu$$  \hspace{1cm} (105)

$$r_\mu \rightarrow r'_\mu = r_\mu + R_\mu$$  \hspace{1cm} (106)

and

$$\mathcal{M} \rightarrow \mathcal{M}'(x) = \mathcal{M} + \xi(x)\sigma.$$  \hspace{1cm} (107)

The Dirac operator $D_E$ in eq. (101), when reexpressed in terms of the “primed” external fields reads

$$D_E = \gamma_\mu(\partial_\mu + A_\mu) + M$$  \hspace{1cm} (108)

with

$$A_\mu = ig_\mu + \Gamma'_\mu - \frac{i}{2}\gamma_5\xi'_\mu \quad \text{and} \quad M = \frac{1}{2}(\Sigma' - \gamma_5\Delta') - M_Q,$$  \hspace{1cm} (109)

and where $\Gamma'_\mu$, $\xi'_\mu$, $\Sigma'$ and $\Delta'$ are the same as in eqs. (103), (104) and (104) with $l_\mu \rightarrow l'_\mu$, $r_\mu \rightarrow r'_\mu$ and $\mathcal{M} \rightarrow \mathcal{M}'$. Formally, this is the same Dirac operator as the one corresponding to the “mean field approximation” which we discussed in the previous paragraph. In practice, it means that once we have evaluated the formal effective action

$$\exp \Gamma_{eff}(A_\mu, M) = \int D\bar{Q}DQ \exp \int d^4x \bar{Q}D_EQ = \det D_E$$  \hspace{1cm} (110)

we can easily get the new terms involving the new auxiliary fields $L_\mu$, $R_\mu$ and $\sigma$ by doing the appropriate shifts. The formal evaluation of $\Gamma_{eff}(A_\mu, M)$ to $O(p^4)$ in the chiral expansion has been made by several authors, Refs. [44] to [47]. We reproduce the results in the appendix.
5.3 The constant $g_A$ and resonance masses.

When computing the effective action $\Gamma_{\text{eff}}(A_\mu, M)$ in eq. (110) there appears a mixing term proportional to $\text{tr} \xi W^{(-)\mu}$. More precisely, one finds a quadratic form in $\xi_\mu$ and $W^{(-)}_\mu$ (in Minkowski space-time):

$$\Gamma = \alpha < W^{(-)\mu} W^{(-)\mu} > + \beta < \xi_\mu W^{(-)\mu} > + \gamma < \xi_\mu \xi_\mu >$$  \hspace{1cm} (111)

with

$$\alpha = \frac{N_c}{16\pi^2} \left( \frac{1}{4} \frac{\Lambda^2}{G_V} + M_Q^2 \frac{\Gamma(0, M_Q^2 \Lambda^2)}{\Lambda^2} \right)$$  \hspace{1cm} (112)

$$\beta = -2 \frac{N_c}{16\pi^2} \Gamma(0, M_Q^2 / \Lambda^2) M_Q^2$$ \hspace{1cm} (113)

The field redefinition

$$W^{(-)\mu} \rightarrow \hat{W}^{(-)}_\mu + (1 - g_A) \xi_\mu$$  \hspace{1cm} (114)

with

$$g_A = 1 + \frac{\beta}{2\alpha}$$  \hspace{1cm} (115)

diagonalizes the quadratic form. There is a very interesting physical effect due to this diagonalization, which is that it redefines the coupling of the constituent chiral quarks to the pseudoscalars. Indeed, the covariant derivative in eq. (102) becomes

$$\nabla_\mu = \partial_\mu + iG_\mu + \Gamma_\mu - \frac{i}{2} \gamma_5 (g_A \xi_\mu - \hat{W}^{(-)}_\mu) - \frac{i}{2} W^{(+)}_\mu.$$  \hspace{1cm} (116)

We shall later come back to $g_A$ and its possible identification with the $g_A$-coupling constant of the constituent chiral quark model of Manohar and Georgi [48].

In the calculation of $\Gamma_{\text{eff}}(A_\mu, M)$ we also encounter kinetic like terms for the fields $\hat{W}^{(-)}_\mu$ and $W^{(+)}_\mu$:

$$- \frac{N_c}{16\pi^2} \frac{1}{3} \Gamma(0, M_Q^2 \Lambda^2) \frac{1}{4} \text{tr}(\partial_\mu W^{(+)}_\nu - \partial_\nu W^{(+)}_\mu)(\partial^\mu W^{(+)} - \partial^\nu W^{(+)}),$$  \hspace{1cm} (117)

and

$$- \frac{N_c}{16\pi^2} \frac{1}{3} \gamma(0, M_Q^2 \Lambda^2) - \Gamma(1, M_Q^2 \Lambda^2) \frac{1}{4} \text{tr}(\partial_\mu \hat{W}^{(-)}_\nu - \partial_\nu \hat{W}^{(-)}_\mu)(\partial^\mu \hat{W}^{(-)} - \partial^\nu \hat{W}^{(-)}) \hspace{1cm} (118)$$
Comparison with the standard vector and axial-vector kinetic terms in eqs. (50) and (51), requires a scale redefinition of the fields $W^{(+)}_\mu$ and $\hat{W}^{(-)}_\mu$ to obtain the correct kinetic couplings, i.e.,

$$V_\mu = \lambda_V W^{(+)}_\mu \quad A_\mu = \lambda_A \hat{W}^{(-)}_\mu,$$

with

$$\lambda_V^2 = \frac{N_c}{16\pi^2} \frac{1}{3} \Gamma(0, \frac{M_Q^2}{\Lambda^2})$$

and

$$\lambda_A^2 = \frac{N_c}{16\pi^2} \frac{1}{3} \left[ \Gamma(0, \frac{M_Q^2}{\Lambda^2}) - \Gamma(1, \frac{M_Q^2}{\Lambda^2}) \right].$$

This scale redefinition gives rise to mass terms (in Minkowski space-time)

$$\frac{1}{2} M_V^2 tr(V_\mu V^\mu) + \frac{1}{2} M_A^2 tr(A_\mu A^\mu)$$

with

$$M_V^2 = \frac{2\alpha + \beta}{\lambda_V^2} \quad \text{and} \quad M_A^2 = \frac{2\alpha}{\lambda_A^2}.$$  

The same comparison between the calculated kinetic and mass terms in the scalar sector with the standard scalar Lagrangian in eq. (19), requires the scale redefinition

$$S(x) = \lambda_S \sigma(x),$$

with

$$\lambda_S^2 = \frac{N_c}{16\pi^2} \frac{2}{3} \left[ 3\Gamma(0, \frac{M_Q^2}{\Lambda^2}) - 2\Gamma(1, \frac{M_Q^2}{\Lambda^2}) \right].$$

The scalar mass is then

$$M_S^2 = \frac{N_c}{16\pi^2} \frac{8M_Q^2}{\lambda_S^2} \Gamma(0, \frac{M_Q^2}{\Lambda^2}).$$

### 5.4 The couplings of the $L_{eff}^R$-Lagrangian.

The Lagrangian in question is the one we have written in section 3 in eqs. (49), (50) and (51), based on chiral symmetry requirements alone. These requirements did not fix, however, the masses and the interaction couplings with the pseudoscalar fields and external fields. The results for the masses which we now find in the extended Nambu Jona-Lasinio model are given by eqs. (123) and (126) in the previous
subsection. These are the results in the limit where low frequency gluonic interactions in \( \mathcal{L}_{QCD}^{\Lambda} \) in eq. (4) are neglected, i.e., the results corresponding to the first alternative scenario we discussed in the introduction. For the other coupling constants, and also in the limit where low frequency gluonic interactions are neglected, the results are the following:

\[
\frac{1}{4} f_{\pi}^2 = \frac{N_c}{16\pi^2} M_Q^2 g_A \Gamma(0, \frac{M_Q^2}{\Lambda^2}),
\]

(127)

instead of the mean field approximation result in eq. (78); \( \dagger \)

\[
f_V = \sqrt{2} \lambda_V, \quad f_A = \sqrt{2} g_A \lambda_A;
\]

(128)

\[
g_V = \frac{N_c}{16\pi^2} \frac{\sqrt{2}}{\lambda_V} \frac{\sqrt{2}}{6} \left[ (1 - g_A^2) \Gamma(0, \frac{M_Q^2}{\Lambda^2}) + 2g_A^2 \Gamma(1, \frac{M_Q^2}{\Lambda^2}) \right]
\]

(129)

for the vector and axial-vector coupling constants in (50) and (51); and

\[
c_m = \frac{N_c}{16\pi^2} \frac{M_Q}{\lambda_s} \rho \left[ \Gamma(-1, \frac{M_Q^2}{\Lambda^2}) - 2\Gamma(0, \frac{M_Q^2}{\Lambda^2}) \right],
\]

(130)

\[
c_d = \frac{N_c}{16\pi^2} \frac{M_Q}{\lambda_s} 2g_A^2 \left[ \Gamma(0, \frac{M_Q^2}{\Lambda^2}) - \Gamma(1, \frac{M_Q^2}{\Lambda^2}) \right]
\]

(131)

for the scalar coupling constants in (49).

There are a series of interesting relations between these results, which we collect below:

\[
M_V^2 = \frac{3}{2} \frac{\Lambda^2}{G_V(\Lambda^2)} \frac{1}{\Gamma(0, \frac{M_Q^2}{\Lambda^2})},
\]

(132)

\[
M_A^2 \left\{ 1 - \frac{\Gamma(1, \frac{M_Q^2}{\Lambda^2})}{\Gamma(0, \frac{M_Q^2}{\Lambda^2})} \right\} = M_V^2 + 6M_Q^2,
\]

(133)

\[
g_A = 1 + \frac{\beta}{2\alpha} = \frac{f_V^2 M_V^2}{f_A^2 M_A^2} g_A^2,
\]

(134)

with the two solutions

\[
g_A = 0 \quad \text{and} \quad g_A = \frac{f_V^2 M_V^2}{f_A^2 M_A^2},
\]

(135)

and

\[
f_V^2 M_V^2 = f_A^2 M_A^2 + f_{\pi}^2.
\]

(136)

\( \dagger \)This implicitly changes the value of \( B_0 \) via eq. (78).
The last relation is the 1st Weinberg sum rule [49]. Using this sum rule and the second solution for $g_A$, we also have that

$$g_A = 1 - \frac{f_\pi^2}{f_V^2 M_V^2}.$$  \hspace{1cm} (137)

We find therefore that $g_A < 1$. As we shall discuss in section 6, the two relations in eqs. (136) and (137) remain valid in the presence of gluonic interactions; i.e., the gluonic corrections do modify the explicit form of the calculation we have made of $f_\pi, f_V, M_V$ and $g_A$, but they do it in such a way that eqs. (136) and (137) remain unchanged.

5.5 The coupling constants $L_i$’s, $H_1$ and $H_2$ beyond the mean field approximation.

These coupling constants are now modified because no longer we have $g_A = 1$. With the short-hand notation

$$x = \frac{M_0^2}{\Lambda_\chi^2},$$  \hspace{1cm} (138)

the analytic expressions we find from the quark-loop integration are the following:

$$L_2 = 2 L_1 = \frac{N_c}{16 \pi^2} \frac{1}{24} \left[(1 - g_A^2)^2 \Gamma(0, x) + 4 g_A^2 (1 - g_A^2) \Gamma(1, x) + 2 g_A^4 \Gamma(2, x)\right],$$  \hspace{1cm} (139)

$$\tilde{L}_3 = \frac{N_c}{16 \pi^2} \frac{1}{24} \left[-3(1 - g_A^2)^2 \Gamma(0, x) + 4 \left(g_A^4 - 3 g_A^2 (1 - g_A^2)\right) \Gamma(1, x) - 8 g_A^4 \Gamma(2, x)\right],$$  \hspace{1cm} (140)

$$L_4 = 0,$$  \hspace{1cm} (141)

$$\tilde{L}_5 = \frac{N_c}{16 \pi^2} \frac{1}{24} \left[\frac{\rho}{12} [\Gamma(0, x) - \Gamma(1, x)]\right],$$  \hspace{1cm} (142)

$$L_6 = 0,$$  \hspace{1cm} (143)

$$L_7 = O(N_c^2),$$  \hspace{1cm} (144)

$$\tilde{L}_8 = -\frac{N_c}{16 \pi^2} \frac{1}{24} \left[6 \rho (\rho - g_A) \Gamma(0, x) + g_A^2 \Gamma(1, x)\right],$$  \hspace{1cm} (145)

$$L_9 = \frac{N_c}{16 \pi^2} \frac{1}{6} \left[(1 - g_A^2) \Gamma(0, x) + 2 g_A^2 \Gamma(1, x)\right],$$  \hspace{1cm} (146)

$$L_{10} = -\frac{N_c}{16 \pi^2} \frac{1}{6} \left[(1 - g_A^2) \Gamma(0, x) + 3 g_A^2 \Gamma(1, x)\right],$$  \hspace{1cm} (147)

$$H_1 = -\frac{N_c}{16 \pi^2} \frac{1}{12} \left[(1 + g_A^2) \Gamma(0, x) - g_A^2 \Gamma(1, x)\right],$$  \hspace{1cm} (148)

$$\tilde{H}_2 = \frac{N_c}{16 \pi^2} \frac{1}{12} \left[6 \rho^2 \Gamma(-1, x) - 6 \rho (\rho + g_A) \Gamma(0, x) + g_A^2 \Gamma(1, x)\right].$$  \hspace{1cm} (149)
Three of the $L_i$-couplings ($i = 3, 5$ and $8$) as well as $H_2$ receive explicit contributions from the integration of scalar fields. This is why we write $L_i = \tilde{L}_i + L_i^S$, $i = 3, 5, 8$; $H_2 = \tilde{H}_2 + H_2^S$ with $\tilde{L}_i$, $\tilde{H}_2$ the contribution from the quark-loop and $L_i^S$, $H_2^S$ from the scalar field. The results for $L_1$, $L_2$ and $\tilde{L}_3$ agree with those of Ref. \cite{50} where these couplings were obtained by integrating out the constituent quark fields in the model of Manohar and Georgi \cite{48}. At the level where possible gluonic corrections are neglected, the two calculations are formally equivalent. There also exists a recent calculation of the $L_i$-couplings in the literature within the framework of an extended Nambu Jona-Lasinio model as we are discussing here. Our results for $L_4$ to $L_{10}$ agree with those of Ref. \cite{23}.

We note that between these results for the $L_i$’s, $H_1$ and the results for couplings and masses of the vector and axial-vector Lagrangians which we obtained before there are the following interesting relations:

$$L_9 = \frac{1}{2} f_V g_V,$$

$$L_{10} = -\frac{1}{4} (f_V^2 - f_A^2) \text{ and } 2 H_1 = -\frac{1}{4} (f_V^2 + f_A^2).$$

(150) (151)

As we shall see in the next section, these relations, like those in eqs. (136) and (137), are also valid in the presence of gluonic interactions. The alerted reader will recognize that these relations are precisely the QCD short-distance constraints which, as discussed in Ref. \cite{41}, are required to remove the ambiguities in the context of chiral perturbation theory to $O(p^4)$ when vector and axial-vector degrees of freedom are integrated out. They are the relations which follow from demanding consistency between the low energy effective action of vector and axial-vector mesons and the QCD short-distance behaviour of two-point functions and three-point functions. It is rather remarkable that the simple ENJL model we have been discussing incorporates these constraints automatically.

There is a further constraint which was also invoked in Ref. \cite{41}. It has to do with the asymptotic behaviour of the elastic meson-meson scattering, which in QCD is expected to satisfy the Froissart bound \cite{51}. If that is the case, the authors of Ref. \cite{41} concluded that, besides the constraints already discussed, one also must have

$$L_1 = \frac{1}{8} g_V^2 ; \quad L_2 = 2L_1 ; \quad L_3 = -6L_1.$$  

(152)

As already mentioned, the second constraint is a property of QCD in the large $N_c$-limit. The first and third constraint however are highly non-trivial. We observe that, to the extent that $O(N_c g_A^4)$ terms can be neglected, these constraints are then
also satisfied in the ENJL model we are considering. Indeed, it follows from eqs. (139), (140) and (129) that

\[ \tilde{L}_3 + 6L_1 = \frac{N_c}{16\pi^2} g_A^4 \frac{1}{12} [2\Gamma(1, x) - \Gamma(2, x)], \]  
\[ 8L_1 - g_V^2 = \frac{N_c}{16\pi^2} g_A^4 \frac{1}{3} \Gamma(2, x) - 2\frac{\Gamma(1, x)^2}{\Gamma(0, x)}. \]  

When the massive scalar field is integrated out [40], there is a further contribution to the constants \( L_3, L_5, L_8 \) and \( H_2 \) with the results:

\[ L^S_3 = \frac{c_d^2}{2M_S^2} = \frac{N_c}{16\pi^2} \frac{1}{4} g_A^4 \frac{1}{\Gamma(0, x)} [\Gamma(0, x) - \Gamma(1, x)]^2, \]  
\[ L^S_5 = \frac{c_m c_d}{M_S^2} = \frac{N_c}{16\pi^2} \frac{1}{4} \rho g_A^2 \frac{1}{\Gamma(0, x)} [\Gamma(-1, x) - 2\Gamma(0, x)] [\Gamma(0, x) - \Gamma(1, x)], \]  
\[ L^S_8 = \frac{c_m^2}{2M_S^2} = \frac{N_c}{16\pi^2} \frac{1}{16} \rho^2 \frac{1}{\Gamma(0, x)} [\Gamma(-1, x) - 2\Gamma(0, x)]^2, \]  
\[ H^S_2 = 2L^S_8. \]  

Our result for \( L^S_5 \) disagrees with the one found in Ref. [23]. Also, contrary to what is found in Ref. [23], there is no contribution from scalar exchange to \( L_2 \).

It is interesting to point out that \( \tilde{L}_5, L^S_5 \) and \( \tilde{L}_8, L^S_8 \) each depend explicitly on the parameter \( \rho \). This dependence however, disappears in the sums

\[ L_5 = \tilde{L}_5 + L^S_5 = \frac{N_c}{16\pi^2} \frac{1}{4} g_A^4 [\Gamma(0, x) - \Gamma(1, x)]; \]  
\[ L_8 = \tilde{L}_8 + L^S_8 = \frac{1}{4} f_\pi^2 \frac{g_A}{16M_Q^2} - \frac{N_c}{16\pi^2} \frac{1}{24} g_A^4 \Gamma(1, x). \]  

6 THE LOW ENERGY EFFECTIVE ACTION IN THE PRESENCE OF GLUONIC INTERACTIONS.

The purpose of this section is to explore more in detail the second alternative which we described in the introduction wherewith the four quark operator terms in eqs. (5) and (6) are viewed as the leading result of a first step renormalization à la Wilson, once the quark and gluon degrees of freedom have been integrated out down to a scale \( \Lambda_\chi \). Within this alternative, one is still left with a fermionic determinant which has to be evaluated in the presence of gluonic interactions due to fluctuations below the \( \Lambda_\chi \)-scale. The net effect of these long distance gluonic interactions is to modify the various incomplete gamma functions \( \Gamma(n, x = \frac{M_Q^2}{\Lambda_\chi^2}) \) which modulate the calculation of the fermionic determinant in the previous sections, into new (a
priori incalculable) constants. We examine first, how many independent unknown constants can appear at most. Then, following the approach developed in Ref. [25], we shall proceed to an approximate calculation of the new constants to order $\alpha_S N_c$.

6.1 Book-keeping of (a priori) unknown constants.

The calculation of the effective action in the previous sections, has been organized as a power series in proper time (see the appendix for details). This is the origin of the integrals of the type

$$\int_{1/\Lambda^2}^{\infty} \frac{d\tau}{\tau} \frac{1}{16\pi^2 \tau^2} e^{-\tau M_Q^2} = \frac{1}{16\pi^2} \frac{1}{(M_Q^2)^{n-2}} \int_{M_Q^2/\Lambda^2}^{\infty} \frac{dz}{z} e^{-z} z^{n-2} = \frac{1}{16\pi^2} (M_Q^2)^{n-2} \Gamma(n-2, x = M_Q^2/\Lambda^2); \quad n = 1, 2, 3, \ldots . \quad (161)$$

In the presence of a gluonic background, each term in the effective action which originates on a fixed power of the proper time expansion of the heat kernel, becomes now modulated by an infinite series in powers of colour singlet gauge invariant combinations of gluon field operators. Eventually, we have to take the statistical gluonic average over each of these series. In practice, each different average becomes an unknown constant. If we limit ourselves to terms in the effective action to $O(p^4)$ at most, there can only appear a finite number of these unknown constants. We can make their book-keeping by tracing back all the possible different types of terms which can appear.

In order to proceed further with this book-keeping, it is convenient to rewrite the operator $E$ (see eq. (243) in the appendix) in the following short-hand notation

$$E = S + \gamma_{\mu} V_{\mu} + \sigma_{\mu\nu} R_{\mu\nu} . \quad (162)$$

Clearly,

$$S \equiv \frac{1}{4} \Sigma^2 - M_Q^2 - \frac{1}{4} \Delta^2 - \frac{1}{8} \gamma_5 [\Sigma', \Delta'], \quad (163)$$

$$V_{\mu} \equiv \frac{i}{4} \gamma_5 \{ \xi_{\mu}, \Sigma' - \gamma_5 \Delta' \} + \frac{1}{2} d_{\mu}' (\Sigma' - \gamma_5 \Delta'), \quad (164)$$

$$R_{\mu\nu} \equiv -\frac{i}{2} R_{\mu\nu}'. \quad (165)$$

We now observe that terms with only one power of $S$, when calculated in the limit $\alpha_S N_c \to 0$ can only appear modulated by the factor $\Gamma(-1, x)$, since they necessarily come from $tr E$ which is in $H_1(x, x)$ (see eq. (250)). For these terms, the net effect
of the gluonic interactions will be to renormalize the factor $\Gamma(-1, x)$ into a new constant

$$\Gamma(-1, x) \rightarrow \Gamma(-1, x)(1 + \gamma_{-1}), \quad (166)$$

with $\gamma_{-1}$ an unknown functional of gluonic averages. This explains the meaning of eq. (167). The meaning of the others, eqs. (168) to (171), is similar. E.g., the terms in eq. (168) proportional to $1 + \gamma_{01}$ all come from the $V_\mu V_\mu$ in the $E^2$ part of $H_2$. In the limit $\alpha_s N_c \rightarrow 0$ they are modulated by $\Gamma(0, x)$. All these terms will be modulated by the same new unknown factor when including gluonic effects. These we have absorbed in the free coefficient $\gamma_{01}$.

The net effect of the gluonic interactions is to modulate the various terms in $\Gamma^{(i)}_{eff}$, $i = 1, 2, 3, 4$, as given in appendix E, equations (261) to (263), in the following way:

$$\Gamma^{(1)}_{eff} = \frac{N_c}{16\pi^2} \Gamma(-1, x)(1 + \gamma_{-1}) tr \left[ 2 \left( \frac{1}{4} \Sigma^2 - M_Q^2 \right) \right]; \quad (167)$$

$$\Gamma^{(2)}_{eff} = \frac{N_c}{16\pi^2} \Gamma(0, x) tr \left[ \frac{1}{2} (1 + \gamma_{01}) \{ \xi^\mu, \Sigma' \}^2 + \frac{i}{4} (1 + \gamma_{01}) \{ \xi'_\mu, \Sigma' \} d'^\mu \Delta' 
+ (1 + \gamma_{02}) \frac{1}{4} \Sigma'^2 - M_Q^2 \right]^2 + \frac{1}{4} (1 + \gamma_{01}) d'_\mu \Sigma' d'^\mu \Sigma' 
- \frac{1}{12} (1 + \gamma_{03}) \left( f^{(+)}_{\mu\nu} f^{(+)}_{\mu\nu} + f^{(-)}_{\mu\nu} f^{(-)}_{\mu\nu} \right) \right]; \quad (168)$$

$$\Gamma^{(3)}_{eff} = \frac{N_c}{16\pi^2} \Gamma(1, x) tr \left[ -\frac{1}{16} (1 + \gamma_{11}) \left( \frac{1}{4} \Sigma^2 - M_Q^2 \right) \{ \xi'_\mu, \Sigma' \}^2 - \frac{i}{2} (1 + \gamma_{12}) f^{(+)}_{\mu\nu} \xi'^\mu \xi'^\nu 
+ \frac{1}{6} (1 + \gamma_{13}) \left( (d'_\mu \xi'_\nu)^2 + \frac{1}{2} (\xi'_\mu, \xi'_\nu, \xi'^\mu, \xi'^\nu) + \xi'_\mu \xi'^\nu \xi'_\nu \xi'^\mu \right) \right] - \frac{1}{6} (1 + \gamma_{14}) d'_\mu \Sigma' d'^\mu \Sigma' \right], \quad (169)$$

with

$$\quad (d'_\mu \xi'_\nu)^2 = -\frac{1}{4} \left[ \xi'_\mu, \xi'_\nu \right] \left[ \xi'^\mu, \xi'^\nu \right] + i f^{(+)}_{\mu\nu} \xi'^\mu \xi'^\nu + \frac{1}{2} f^{(-)}_{\mu\nu} f^{(-)}_{\mu\nu} 
+ (d'_\mu \xi'_\nu)^2 + \text{total derivative terms}; \quad (170)$$

$$\Gamma^{(4)}_{eff} = \frac{N_c}{16\pi^2} \Gamma(2, x) tr \left[ \frac{1}{12} (1 + \gamma_{21}) \xi'_\mu \xi'_\nu \xi'^\mu \xi'^\nu - \frac{1}{6} (1 + \gamma_{22}) \xi'_\mu \xi'^\mu \xi'_\nu \xi'^\nu \right]. \quad (171)$$

In the presence of gluonic interactions, there appear then 10 unknown constants: $\gamma_{-1}$; $\gamma_{01}$; $\gamma_{02}$; $\gamma_{03}$; $\gamma_{11}$; $\gamma_{12}$; $\gamma_{13}$; $\gamma_{14}$; $\gamma_{21}$; $\gamma_{22}$. To these, we have to add the original $G_S$ and $G_V$ constants, as well as the scale $\Lambda_\chi$. However, as already mentioned in section
4, the unknown constant \((1 + \gamma_{-1})\) in eq. (167) can be traded by an appropriate change of the scale \(\Lambda_x\),

\[
\Gamma(-1, \bar{x}) = \Gamma(-1, x)\{1 + \gamma_{-1}\}; \quad \bar{x} = \frac{M_Q^2}{\Lambda_x^2},
\]

and a renormalization of the constant \(G_S\),

\[
G_S \to \tilde{G}_S = \frac{\Lambda_x^2}{\Lambda^2}G_S.
\]

Altogether, we then have 12 (a priori unknown) theoretical constants and one scale \(\Lambda_x\). They determine 18 non-trivial physical couplings (in the large \(N_c\)-limit) of the low energy QCD effective Lagrangian:

\[
< \bar{\psi} \psi >, f_\pi, L_1, L_3, L_5, L_8, L_9, L_{10}, H_1, H_2, f_V, f_A, g_V, c_m, c_d, M_S, M_V \text{ and } M_A.
\]

In full generality, the results are a follows:

\[
< \bar{\psi} \psi > = -\frac{N_c}{16\pi^2} \frac{1}{4} M_Q^3 \Gamma_{-1}(1 + \gamma_{-1}).
\]

\[
\frac{1}{4} f^2_\pi = \frac{N_c}{16\pi^2} M_Q^2 g_A \Gamma_0(1 + \gamma_{01}).
\]

\[
L_2 = 2L_1 = \frac{N_c}{16\pi^2} \frac{1}{24} \times \left[(1 - g_A^2)\Gamma_0(1 + \gamma_{03}) + 4g_A^2(1 - g_A^2)\Gamma_1(1 + \frac{3}{2}\gamma_{12} - \frac{1}{2}\gamma_{13}) + 2g_A^4\Gamma_2(1 + \gamma_{21})\right].
\]

\[
\tilde{L}_3 = \frac{N_c}{16\pi^2} \frac{1}{24} \left[-3(1 - g_A^2)^2\Gamma_0(1 + \gamma_{03}) + 4g_A^4\Gamma_1(1 + \gamma_{13})
\]

\[-12g_A^2(1 - g_A^2)\Gamma_1(1 + \frac{3}{2}\gamma_{12} - \frac{1}{2}\gamma_{13}) - 8g_A^4\Gamma_2(1 + \frac{1}{2}(\gamma_{21} + \gamma_{22}))\right]
\]

\[
L_3^S = \frac{c_A^2}{2M_S^2}.
\]

\[
L_5 = \frac{N_c}{16\pi^2} \frac{1}{4} g_A^3 \frac{1 + \gamma_{01}}{1 + \gamma_{02}} \left[\Gamma_0(1 + \gamma_{01}) - \Gamma_1(1 + \gamma_{11})\right].
\]

\[
L_8 = \frac{N_c}{16\pi^2} \left[\frac{1}{16} \frac{1 + \gamma_{01}}{1 + \gamma_{02}} - \frac{1}{24} \frac{\Gamma_1(1 + \gamma_{13})}{\Gamma_0(1 + \gamma_{01})}\right] g_A^2 \Gamma_0(1 + \gamma_{01}).
\]

\[
L_9 = \frac{N_c}{16\pi^2} \frac{1}{6} \left[(1 - g_A^2)\Gamma_0(1 + \gamma_{03}) + 2g_A^2\Gamma_1(1 + \frac{3}{2}\gamma_{12} - \frac{1}{2}\gamma_{13})\right].
\]
\[
L_{10} = -\frac{N_c}{16\pi^2} \frac{1}{6} \left[ (1 - g_A^2) \Gamma_0(1 + \gamma_{03}) + g_A^2 \Gamma_1(1 + \gamma_{13}) \right].
\]

\[
H_1 = -\frac{N_c}{16\pi^2} \frac{1}{12} \left[ (1 + g_A^2) \Gamma_0(1 + \gamma_{03}) - g_A^2 \Gamma_1(1 + \gamma_{13}) \right].
\]

\[
\dot{H}_2 = \frac{N_c}{16\pi^2} \frac{1}{12} \left[ 6\rho^2 \Gamma_{-1}(1 + \gamma_{-1}) - 6\rho^2 \Gamma_0(1 + \gamma_{02}) - 6\rho g_A \Gamma_0(1 + \gamma_{01}) + g_A^2 \Gamma_1(1 + \gamma_{13}) \right],
\]

\[
H_2^S = \frac{c_m^2}{M_S^2}
\]

\[
f_V = \sqrt{2}\lambda_V
\]

and

\[
f_A = \sqrt{2}g_A\lambda_A,
\]

with

\[
\lambda_V^2 = \frac{N_c}{16\pi^2} \frac{1}{3} \Gamma_0(1 + \gamma_{03});
\]

and

\[
\lambda_A^2 = \frac{N_c}{16\pi^2} \frac{1}{3} \left[ \Gamma_0(1 + \gamma_{03}) - \Gamma_1(1 + \gamma_{13}) \right].
\]

\[
g_V = \frac{N_c}{16\pi^2} \frac{1}{\lambda_V} \frac{\sqrt{2}}{6} \left[ (1 - g_A^2) \Gamma_0(1 + \gamma_{03}) + 2g_A^2 \Gamma_1(1 + \frac{3}{2} \gamma_{12} - \frac{1}{2} \gamma_{13}) \right].
\]

\[
c_m = \frac{N_c}{16\pi^2} \frac{M_Q}{\lambda_S} \rho \left[ \Gamma_{-1}(1 + \gamma_{-1}) - 2\Gamma_0(1 + \gamma_{02}) \right],
\]

\[
c_d = \frac{N_c}{16\pi^2} \frac{M_Q}{\lambda_S} 2g_A^2 \left[ \Gamma_0(1 + \gamma_{01}) - \Gamma_1(1 + \gamma_{11}) \right],
\]

with

\[
\lambda_S^2 = \frac{N_c}{16\pi^2} \frac{2}{3} \left[ 3\Gamma_0(1 + \gamma_{01}) - 2\Gamma_1(1 + \gamma_{14}) \right].
\]

\[
M_S^2 = \frac{N_c}{16\pi^2} \frac{8M_Q^2}{\lambda_S^2} \Gamma_0(1 + \gamma_{02}).
\]

\[
M_V^2 = \frac{3}{2} \frac{A^2}{G_{V\overline{A}}} \frac{1}{\Gamma(0, x)(1 + \gamma_{03})}.
\]

\[
M_A^2 \left\{ 1 - \frac{\Gamma(1, x)(1 + \gamma_{13})}{\Gamma(0, x)(1 + \gamma_{03})} \right\} = M_V^2 + 6M_Q^2 \frac{1 + \gamma_{01}}{1 + \gamma_{03}}.
\]

There exist relations among the physical couplings above which are independent of the unknown gluonic constants. They are clean tests of the basic assumption
that the low energy effective action of QCD follows from an ENJL Lagrangian of
the type considered here. The relations are

\[ f_V^2 M_V^2 - f_A^2 M_A^2 = f_\pi^2 \quad (1^{st} \text{ Weinberg sum rule}), \]
\[ L_9 = \frac{1}{2} f_V g_v, \]
\[ L_{10} = -\frac{1}{4} f_V^2 + \frac{1}{4} f_A^2, \]
\[ 2H_1 = -\frac{1}{4} f_V^2 - \frac{1}{4} f_A^2, \]

and

\[ \frac{H_2 + 2L_8}{2L_5} = \frac{c_m}{c_d}. \]

The first four relations have already been discussed in the previous section. The combination of couplings in the r.h.s. of eq.(201) is the one which appears in the context of non leptonic weak interactions, when one considers weak decays like \( K \to \pi H \) (light Higgs) \[52\]. In fact, from the low energy theorem derived in \[37\] it follows that

\[ \frac{H_2 + 2L_8}{2L_5} = \frac{1}{4} \frac{\langle 0 | \bar{s}s | 0 \rangle - \langle 0 | \bar{u}u | 0 \rangle}{f_K / f_\pi - 1}. \]

Experimentally

\[ f_K / f_\pi - 1 = 0.22 \pm 0.01. \]

Unfortunately, the numerator in the r.h.s. of (202) is poorly known. If we vary the ratio

\[ \frac{\langle \bar{s}s \rangle}{\langle \bar{u}u \rangle} - 1 \quad \text{from} \quad -0.1 \quad \text{to} \quad -0.2, \]

as suggested by the authors of ref. \[52\], then eq.(201) leads to the estimate

\[ c_m / c_d = -1.1 \times 10^{-1} \quad \text{to} \quad -2.3 \times 10^{-1}. \]

With this estimate incorporated in eq.(56), we are led to the conclusion that

\[ |c_d| \simeq 34 \text{MeV}. \]

In the version corresponding to the 1\textsuperscript{st} alternative, the results for \( c_m \) and \( c_d \) are those in eqs.(130) and (131). We observe that in this case \( c_m / c_d \) comes out always positive for reasonable values of \( M_Q^2 / \Lambda^2 \).
6.2 Gluonic correction to $O(\alpha_s N_c)$.

We can make an estimate of the ten constants $\gamma_{-1}$; $\gamma_{01}$, $\gamma_{02}$, $\gamma_{03}$; $\gamma_{11}$, $\gamma_{12}$, $\gamma_{13}$, $\gamma_{14}$; $\gamma_{21}$ and $\gamma_{22}$ by keeping only the leading contribution which involves the gluon vacuum condensate $\frac{\langle \alpha_s GG \rangle}{M_Q^4}$ as was done in Ref. [25]. The relevant dimensionless parameter is

$$g = \frac{\pi^2 \langle \alpha_s GG \rangle}{6N_c M_Q^4}. \quad (207)$$

Notice that in the large-$N_c$ limit $g$ is a parameter of $O(1)$. One should also keep in mind that the gluon average in (207) is the one corresponding to fluctuations below the $\Lambda_\chi$-scale. The relation of $g$ to the conventional gluon condensate which appears in the QCD sum rules [53] is rather unclear. We are forced to consider $g$ as a free parameter. Up to order $O(\alpha_s N_c)$, this is the only unknown quantity which appears, and we can express all the $\gamma$’s in terms of $g$. We find:

$$\gamma_{-1} = \frac{\Gamma(1, x)}{\Gamma(-1, x)} 2g; \quad (208)$$

$$\begin{pmatrix} \gamma_{01} \\ \gamma_{02} \\ \gamma_{03} \end{pmatrix} = \begin{pmatrix} \Gamma(2, x) \\ \Gamma(0, x) \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -3/5 \end{pmatrix} g; \quad (209)$$

$$\begin{pmatrix} \gamma_{11} \\ \gamma_{12} \\ \gamma_{13} \\ \gamma_{14} \end{pmatrix} = \begin{pmatrix} \Gamma(3, x) \\ \Gamma(1, x) \end{pmatrix} \begin{pmatrix} 1 \\ 1/5 \\ 3/5 \\ 9/5 \end{pmatrix} g; \quad (210)$$

$$\begin{pmatrix} \gamma_{21} \\ \gamma_{22} \end{pmatrix} = \begin{pmatrix} \Gamma(4, x) \\ \Gamma(2, x) \end{pmatrix} \begin{pmatrix} 0 \\ 2/5 \end{pmatrix} g. \quad (211)$$

Notice that the combination $\frac{3}{2} \gamma_{12} - \frac{1}{2} \gamma_{13}$ entering some of the $L_i$’s coupling constants is zero. This is the reason why in Ref. [25] it was found that in the limit $g_A \to 1$, $L_2$ and $L_9$ have no gluon correction of $O(\alpha_s N_c)$.

To this approximation, we have then reduced the theoretical parameters to three unknown constants

$$G_S, G_V \text{ and } g;$$

and the scale $\Lambda_\chi$. 
7 COMPARISON WITH OTHER MODELS.

It is interesting that practically all the models of low energy QCD discussed in the literature can be obtained as some limit of the ENJL model developed in the previous sections. Here, we wish to show this for some of them.

As already mentioned, the QCD effective action approach proposed in Ref. [25] is obtained in the limit where

\[ G_V \to 0 \text{ and } < H > = M_Q \]

Only \( M_Q \) and \( \Lambda_\chi \), which in Ref. [23] is taken as an ultraviolet cut-off, with \( M_Q/\Lambda_\chi \to 0 \) whenever the limit exists, are left as basic parameters. In this case \( g_A = 1 \).

The comparison with the model of Manohar and Georgi [48] is more subtle. These authors suggest as an effective Lagrangian of QCD at energies below the chiral symmetry breaking scale, the Lagrangian (in our notation):

\[
\mathcal{L}_{G-M} = i\bar{Q}\gamma^\mu (\partial_\mu + iG_\mu + \Gamma_\mu)Q + \frac{i}{2}g_A\bar{Q}\gamma_5\gamma^\mu \xi_\mu Q - M_Q\bar{Q}Q + 1 \]

\[ f \frac{1}{4} tr U \partial^\mu U - \frac{1}{4} tr \sum_a G^{(a)}_{\mu\nu} G^{(a)\mu\nu} + \cdots , \] (212)

Couplings to external fields are not discussed in [48]. The dots in (212) stand for terms of higher chiral dimension which in practice are ignored. In the limit \( g_A = 1, f \to 0 \) and with the external fields \( l_\mu = r_\mu = 0 \), this Lagrangian coincides with the one proposed in Ref. [25]. We have already discussed how the mass term \( M_Q\bar{Q}Q \) originates in the Nambu Jona-Lasinio approach. What is the origin of the explicit kinetic term of Goldstone fields above; i.e., the coupling \( f \)? As we have seen in section 5, the integration over constituent quarks in a Lagrangian like the one above produces such a kinetic term with the result as shown in eq.(127) (with gluons ignored). Remember also that integration of the vector and axial-vector mesons, as well as the scalars in the Nambu Jona-Lasinio approach did not contribute to the kinetic term of Goldstone fields. On the other hand the \( g_A \)-coupling, with \( g_A < 1 \), appears naturally in the Nambu Jona-Lasinio approach. It originates in the mixing of the primitive \( G_S \) and \( G_V \) four-fermion couplings; and the “prediction” for \( g_A \) is

\[
g_A = 1 - \frac{f^2_\pi}{f^2 V M_V^2} . \]

(213)

In conclusion we find that the Lagrangian of Georgi and Manohar can be understood within the framework of a Nambu Jona-Lasinio mechanism, provided \( f \) in (212) is very small (perhaps coming from having integrated out the baryon’s degrees of freedom).
Models like the one presented in Ref. ([54]) are less well defined and therefore more difficult to compare with. They have to do more with what we call the 1st alternative and the way to reach a local approximation to the "gluonless" effective action.

The hidden gauge vector meson model of Bando, Kugo and Yamawaki [55] is a chiral effective Lagrangian of vector fields coupled to Goldstone fields which, as discussed in [41], automatically satisfies the QCD short-distance constraints necessary to remove ambiguities when the vector fields are integrated out. The model however has extra symmetries, like

$$g_V = \frac{1}{2}f_V ,$$  \hspace{1cm} (214)

and

$$L_2 = 2L_1 = \frac{1}{4}g_V^2 ,$$  \hspace{1cm} (215)

and

$$L_3 = -3L_2$$  \hspace{1cm} (216)

which are not automatically implemented in the Nambu Jona-Lasinio approach that we have been discussing. As we shall see however, our numerical predictions satisfy rather closely these relations.

The most economical phenomenological effective chiral Lagrangian with vector and axial-vector fields is the one proposed in Ref. [41], where

$$f_V = \sqrt{2} \frac{f_\pi}{M_V} ; \quad g_V = \frac{1}{\sqrt{2}} \frac{f_\pi}{M_V} ; \quad f_A = \frac{f_\pi}{M_A} ; \quad M_A = \sqrt{2}M_V .$$  \hspace{1cm} (217)

It predicts the five $0(p^4)$-couplings which exist in the chiral limit as follows:

$$L_3 = -3L_2 = -6L_1 = -\frac{3}{4}L_9 = L_{10} = -\frac{3}{8}f_V^2 .$$  \hspace{1cm} (218)

It also satisfies the first and second Weinberg sum rules:

$$f_\pi^2 + f_A^2 M_A^2 = f_V^2 M_V^2$$  \hspace{1cm} (219)

and

$$f_V^2 M_V^4 = f_A^2 M_A^4 .$$  \hspace{1cm} (220)

From eqs. (176) to (182) it appears that a very similar result emerges in the limit where $g_A \to 0$:

$$L_3 = -3L_2 = -6L_1 = -\frac{3}{4}L_9 = \frac{3}{4}L_{10} = -\frac{3}{16}f_V^2 .$$  \hspace{1cm} (221)

In this limit, we also find the relation

$$g_V = \frac{1}{2}f_V .$$  \hspace{1cm} (222)
These are in fact the relations derived in Ref. [41] when the contribution from the axial vector mesons is removed.

As we have seen in sections 5 and 6 the first Weinberg sum rule also follows from the Nambu Jona-Lasinio model we have been considering. The second Weinberg sum rule, as given in eq. (220), is however only satisfied if one arbitrarily requires

\[ g_A = \frac{M_V^2}{M_A^2} \]  

or, equivalently,

\[ g_A^2 = \frac{f^2_A}{f^2_V}. \]  

8 DISCUSSION OF NUMERICAL RESULTS.

In the ENJL model, we have three input parameters:

\[ G_S, G_V \text{ and } \Lambda_\chi. \]  

(225)

The gap equation introduces a constituent chiral quark mass parameter \( M_Q \), and the ratio

\[ x = \frac{M_Q^2}{\Lambda_\chi^2} \]  

(226)

is constrained to satisfy the equation

\[ \frac{1}{G_S} = x\Gamma(-1, x)(1 + \gamma_{-1}). \]  

(227)

Once \( x \) is fixed, the constants \( g_A \) and \( G_V \) are related by the equation

\[ g_A = \frac{1}{1 + 4G_V x\Gamma(0, x)(1 + \gamma_0)} \]  

(228)

Therefore, we can trade \( G_S \) and \( G_V \) by \( x \) and \( g_A \); but we need an observable to fix the scale \( \Lambda_\chi \). This is the scale which determines the \( \rho \) mass in eq. (197), i.e.,

\[ \Lambda_\chi^2 = \frac{2}{3} M_V^2 G_V \Gamma(0, \frac{M_Q^2}{\Lambda_\chi^2})(1 + \gamma_{03}). \]  

(229)

There are various ways one can proceed. We find it useful to fix as input variables the values of \( M_Q, \Lambda_\chi \) and \( g_A \). Then we have predictions for

\[ f_\pi^2, <\bar{\psi}\psi>, M_S, M_V \text{ and } M_A \]  

(230)

(231)
f_V, g_V and f_A; and c_m and c_d

(232)

and the $O(p^4)$ couplings:

\begin{align*}
L_i, \ i = 1, 2, \ldots, 10 \text{ and } H_1, \ H_2.
\end{align*}

(233)

In principle we can also calculate any higher $O(p^6)$ coupling which may become of interest. So far, we have fixed twenty-two parameters. Eighteen of them are experimentally known.

In the first column of Table 1 we have listed the experimental values of the parameters which we consider. In comparing with the predictions of the ENJL model it should be kept in mind that the relations (197) to (199) are satisfied by the model while relations (214) to (216) only have numerically small corrections. These relations are rather well satisfied by the experimental values and thus constitute a large part of the numerical success of the model.

We have also used the predictions leading in $1/N_c$ so we have $L_1 = L_2/2, L_4 = L_6 = 0$ and we do not consider $L_7$ since this is given mainly by the $\eta'$-contribution [37]. In evaluating the predictions given in Table 1 we have used the full expressions for the incomplete gamma functions and the numerical value of the $\gamma_{ij}$ in terms of $g$ given in eqs. (208) to (211).

The first column of errors in Table 1 shows the experimental ones. The second column gives the errors we have used for the fits. When no error is indicated in this column, it means that we never use the corresponding parameter for fitting. This is the case for $<\bar{q}q>$ which is quadratically divergent in the cut-off and which is not very well known experimentally. This is also the case for $c_m$ which depends on $<\bar{q}q>$. Fit 1 corresponds to a least squares fit with the maximal set of parameters and requiring $g \geq 0$. Fit 2 corresponds to a fit where only $f_\pi$ and the $L_i$ are used as input in the fit while fit 3 has as additional input the vector and scalar mass. The next column, fit 4, is the one where we require $g_A = 1$; i.e., we start with a model without the vector four-quark interaction. Here there are no explicit vector (axial) degrees of freedom so those have been dropped in this case. This fit includes all parameters except $M_V, M_A, f_V, g_V$ and $f_A$. Finally, fit 5 is the fit to all data keeping the gluonic parameter $g$ fixed at a value of 0.5. The main difference with fit 1 is a decrease in the value of $M_Q$. The value of $\Lambda_{\chi}$ changes very little.

The typical variation of the results with respect to each of the 4 input parameters can be judged from the results in table 2. Here we have changed each of the input parameters by the error quoted by the least squares fitting routine (MINUIT). Typically, this changes the overall fit by about one standard deviation with respect to the input errors quoted in table 1 (third column). The results of fit 2 were used
as the standard. In this fit it was the value of $f_\pi$ and $L_9$ that constrained the input parameters most.

The expected value for the parameter $g$ if we take typical values from, e.g., QCD sum rules is of $O(1)$. None of the fits here really makes a qualitative difference between a $g$ of about 0.5 to 0. Numerically we can thus not decide between the two alternatives mentioned in the introduction. This can be easily seen by comparing fit 1 and fit 5 in table 1.

In all cases acceptable predictions for all relevant parameters are possible. The scalar sector parameters tend all to be a bit on the low side; but so is the constituent quark mass. The predictions for the $L_i$’s are reasonably stable versus a variation of the input parameters. For $L_5$ and $L_8$ this is a major improvement as compared to the predictions of the mean field approximation [25].

9 CONCLUSIONS.

In this paper we have determined the low energy effective action of an ENJL cut-off version of QCD to $O(p^4)$. We have done this in two alternative scenarios. One where the ENJL model is a local approximation to the QCD effective action where the gluons have been fully integrated over. In the other scenario we envisaged the four-fermion operators as the result of integrating over both quark and gluon degrees of freedom down to a scale $\Lambda_\chi$, and then performing a local expansion of that effective action. In this way we have avoided possible problems of double-counting of contributions. For example, the value of $L_9$ can be well reproduced within a simple quark model approach [25]; and also as coming from integrating out the vector mesons [40]. By studying this within the context of a well defined model where both degrees of freedom are present we have clarified this ambiguity.

The ENJL model incorporates in various limits most of the effective low-energy models discussed in the literature. In particular the gauged Yang-Mills vector Lagrangian [55], the Georgi-Manohar effective quark-meson model [48] and the QCD effective action approach model [25]. It also incorporates most of the short distance relations which are expected in QCD [51]. The derivation of these relations has been done including all possible gluonic interactions to leading order in the $1/N_c$-expansion. The relations are valid for all models where the effective quark meson Lagrangian can be written in the form of eq. (108). This is one of the main results of this paper.

The other short distance relations derived using the Froissart bound and the second Weinberg sum rule [11] are not explicitly satisfied by the model. The deviations are however suppressed by terms of order $O(g^4_A)$, so they can be recovered in an appropriate limit. For reasonable values of the input parameters these deviations
are very small.

The numerical predictions in terms of only 4 input parameters are very successful. The constituent quark mass is, however, somewhat smaller than expected. Both alternatives turn out to work rather well. It is only when the scalar mass is required to be close to its measured value that gluonic corrections become important.

In conclusion, the extended Nambu Jona-Lasinio model incorporates a surprisingly large amount of information of the short distance structure of QCD. This is probably the main reason for the successful predictions of this and related models.

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Table 1: Experimental values and predictions of the ENJL model for the various low energy parameters discussed in the text. All dimensionful quantities are in MeV. The difference between the predictions is explained in the text.

| Parameter     | exp. value | exp. error | fit 1 | fit 2 | fit 3 | fit 4 | fit 5 |
|---------------|------------|------------|-------|-------|-------|-------|-------|
| $f_\pi$       | 86(*)      | 10         | 89    | 86    | 86    | 87    | 83    |
| $\sqrt{- <\bar q q>}$ | 194(*) | 8(§)       |       |       |       |       |       |
| $10^3 \cdot L_2$ | 1.2       | 0.4        | 0.5   | 1.7   | 1.6   | 1.6   | 1.6   |
| $10^3 \cdot L_3$ | -3.6      | 1.3        | 1.3   | -4.2  | -4.1  | -4.4  | -5.3  | -4.7  |
| $10^3 \cdot L_5$ | 1.4       | 0.5        | 0.5   | 1.6   | 1.5   | 1.1   | 1.7   |
| $10^3 \cdot L_8$ | 0.9       | 0.3        | 0.5   | 0.8   | 0.8   | 0.7   | 1.1   |
| $10^3 \cdot L_9$ | 6.9       | 0.7        | 0.7   | 7.1   | 6.7   | 6.6   | 5.8   |
| $10^3 \cdot L_{10}$ | -5.5      | 0.7        | 0.7   | -5.9  | -5.5  | -5.8  | -5.1  |
| $10^3 \cdot H_1$ | -         | -          | -     | -4.7  | -4.4  | -4.0  | -2.4  |
| $10^3 \cdot H_2$ | -         | -          | -     | -0.3  | -0.3  | -0.4  | -0.5  |
| $M_V$         | 768.3      | 0.5        | 100   | 811   | 830   | 831   | -     | 802   |
| $M_A$         | 1260       | 30         | 300   | 1331  | 1376  | 1609  | -     | 1610  |
| $f_V$         | 0.20(*     | 0.02       | 0.18  | 0.17  | 0.17  | -     | 0.18  |
| $g_V$         | 0.090(*)   | 0.009      | 0.081 | 0.079 | 0.079 | -     | 0.080 |
| $f_A$         | 0.097      | 0.022(*)   | 0.022 | 0.083 | 0.080 | 0.068 | -     | 0.072 |
| $M_S$         | 983.3      | 2.6        | 200   | 617   | 620   | 709   | 989   | 657   |
| $c_m$         | -          | -          | -     | 20    | 18    | 20    | 24    |
| $c_d$         | 34(*)      | 10         | 21    | 21    | 18    | 23    |
| $x$           |            |            | 0.052 | 0.063 | 0.057 | 0.089 | 0.035 |
| $g_A$         |            |            | 0.61  | 0.62  | 0.62  | 1.0   | 0.66  |
| $M_Q$         | 265        | 263        | 246   | 199   | 204   |
| $g$           |            |            | 0.0   | 0.0   | 0.25  | 0.58  | 0.5   |

(*) This corresponds to $f_0$ which is the value of $f_\pi$ in the chiral limit.
(§) See Ref. 56.
(*) In addition to the experimental error, the chiral loop corrections to these parameters have not been calculated.
Table 2: Experimental values and predictions for these quantities in the ENJL model. The input parameters are chosen such that the typical variation with input parameters can be seen. Set 1 is the standard, corresponding to fit 2 in Table III.

|          | exp. | set 1 | set 2 | set 3 | set 4 | set 5 |
|----------|------|-------|-------|-------|-------|-------|
| $f_\pi$  | 86   | 86    | 81    | 91    | 98    | 97    |
| $\sqrt{-\langle \bar{q} q \rangle}$ | 194  | 260   | 236   | 260   | 296   | 267   |
| $10^3 \cdot L_2$ | 1.2  | 1.6   | 1.5   | 1.6   | 1.6   | 1.5   |
| $10^3 \cdot L_3$ | -3.6 | -4.1  | -4.0  | -3.6  | -4.1  | -4.5  |
| $10^3 \cdot L_5$ | 1.4  | 1.5   | 1.3   | 2.1   | 1.5   | 0.7   |
| $10^3 \cdot L_8$ | 0.9  | 0.8   | 0.7   | 0.9   | 0.7   | 0.6   |
| $10^3 \cdot L_9$ | 6.9  | 6.7   | 6.2   | 6.6   | 6.7   | 6.0   |
| $10^3 \cdot L_{10}$ | -5.5 | -5.5  | -5.1  | -5.1  | -5.5  | -5.7  |
| $10^3 \cdot H_1$ | -    | -4.4  | -3.9  | -4.6  | -4.3  | -3.1  |
| $10^3 \cdot H_2$ | -    | -0.3  | 0.4   | 0.7   | 0.5   | 0.1   |
| $M_V$    | 768.3| 830   | 828   | 984   | 945   | 1016  |
| $M_A$    | 1260 | 1376  | 1420  | 1540  | 1567  | 3662  |
| $f_V$    | 0.20 | 0.17  | 0.16  | 0.17  | 0.17  | 0.15  |
| $g_V$    | 0.090| 0.079 | 0.077 | 0.078 | 0.079 | 0.077 |
| $f_A$    | 0.097| 0.080 | 0.074 | 0.090 | 0.080 | 0.034 |
| $M_S$    | 983.3| 620   | 630   | 620   | 706   | 1102  |
| $c_m$    | -    | 18    | 16    | 20    | 21    | 26    |
| $c_d$    | 34   | 21    | 18    | 26    | 23    | 16    |
| $x$      | 0.063| 0.08  |       |       |       |       |
| $g_A$    | 0.62 |       | 0.7   |       |       |       |
| $M_Q$    | 263  |       |       | 300   |       |       |
| $g$      | 0.0  |       |       | 0.6   |       |       |
APPENDIX

A Euclidean conventions.

To go to Euclidean space, we adopt the usual prescription

\[ x^\mu \equiv (x^0, x^i) = (-i\bar{x}^0, \bar{x}^i) \] (234)

\[ \partial_\mu \equiv (\partial_0, \partial_i) = (i\bar{\partial}_0, \bar{\partial}_i) \] (235)

For arbitrary 4-vectors \( a_\mu, b_\mu \), one has then

\[ a_\mu b_\mu = -\bar{a}_\mu \bar{b}_\mu. \] (236)

It is convenient to work with hermitian gamma matrices with positive metric. We take

\[ \tilde{\gamma}_\mu \equiv -i\bar{\gamma}_\mu, \] (237)

i.e., \( \tilde{\gamma}_0 = -\gamma_0 \) and \( \tilde{\gamma}_i = -i\gamma_i \), which have the required properties

\[ \tilde{\gamma}_\mu^+ = \tilde{\gamma}_\mu \quad ; \quad \{\tilde{\gamma}_\mu, \tilde{\gamma}_\nu\} = 2\delta_{\mu\nu}. \] (238)

Other useful relations are

\[ \gamma_5 \equiv -i\gamma_0\gamma_1\gamma_2\gamma_3 = \bar{\gamma}_0\bar{\gamma}_1\bar{\gamma}_2\bar{\gamma}_3 = \bar{\gamma}_0\bar{\gamma}_1\bar{\gamma}_2\bar{\gamma}_3 = \gamma_5^+ \] (239)

\[ \bar{\sigma}_{\mu\nu} = \frac{i}{2}[\bar{\gamma}_\mu, \bar{\gamma}_\nu] = -\frac{i}{2}[\gamma_\mu, \gamma_\nu]. \] (240)

In the following of this appendix we will omit the bars in the Euclidean quantities.

B Covariant derivatives.

We shall next recall the Seeley-DeWitt coefficients of the Heat kernel expansion of the operator

\[ \mathcal{D}_E^+ \mathcal{D}_E - M_Q^2 \equiv -\nabla_\mu \nabla_\mu + E, \] (241)

with \( \nabla_\mu \) the full covariant derivative in eq. (102) reexpressed in terms of the “primed” external fields,

\[ \nabla_\mu = \partial_\mu + iG_\mu + \Gamma_\mu - \frac{i}{2}\gamma_5 \xi'_\mu. \] (242)
\[ \mathcal{D}_E \] the euclidean Dirac operator defined in eq. (108), and \( E \) the quantity (25):

\[
E = \frac{i}{4} \tilde{\gamma}_\mu \gamma_5 \{ \xi_\mu', \Sigma' - \gamma_5 \Delta' \} - \frac{i}{2} \sigma_{\mu\nu} R'_{\mu\nu} + \frac{1}{4} \Sigma'^2 - M_Q^2 \\
- \frac{1}{4} \Delta'^2 - \frac{1}{8} \gamma_5 [\Sigma', \Delta'] + \frac{1}{2} \tilde{\gamma}_\mu d'_\mu (\Sigma' - \gamma_5 \Delta').
\]

(243)

We recall that

\[
\Gamma'_\mu = \frac{1}{2} \{ \xi_\mu [\partial_\mu - i r'_\mu] \xi + \xi [\partial_\mu - i l'_\mu] \xi^\dagger \} = \Gamma_\mu - \frac{i}{2} W^{(+)\mu},
\]

(244)

\[
\xi'_\mu = i \{ \xi^\dagger [\partial_\mu - i r'_\mu] \xi - \xi [\partial_\mu - i l'_\mu] \xi^\dagger \} = \xi_\mu - W^{(-)\mu},
\]

(245)

\[
\Sigma' = \xi^\dagger \mathcal{M}' \xi^\dagger + \xi \mathcal{M}'^\dagger \xi = \Sigma + 2 \sigma,
\]

(246)

\[
\Delta' = \xi^\dagger \mathcal{M}' \xi^\dagger - \xi \mathcal{M}'^\dagger \xi = \Delta.
\]

(247)

where \( l'_\mu, r'_\mu \) and \( \mathcal{M}' \) were defined in eqs. (105) to (107).

\( d'_\mu \) is the covariant derivative with respect to the \( \Gamma'_\mu \)-connection, i.e.,

\[
d'_\mu A \equiv \partial_\mu A + [\Gamma'_\mu, A],
\]

(248)

\( R'_{\mu\nu} \) is the full strength tensor, i.e.,

\[
R'_{\mu\nu} = [\nabla_\mu, \nabla_\nu] = i G_{\mu\nu} - \frac{i}{2} f'_{\mu\nu},
\]

(249)

with

\[
G_{\mu\nu} = \partial_\mu G_\nu - \partial_\nu G_\mu + i [G_\mu, G_\nu],
\]

(250)

and

\[
- \frac{i}{2} f'_{\mu\nu} = - \frac{i}{2} f_{\mu\nu} - \frac{1}{4} \gamma_5 (\{ \xi_\mu, W_\nu \} + \{ W_\mu, \xi_\nu \}) - \frac{1}{4} [W_\mu, W_\nu] - \frac{i}{2} W_{\mu\nu},
\]

(251)

where

\[
W_\mu = W^{(+)\mu} - \gamma_5 W^{(-)\mu},
\]

(252)

\[
f_{\mu\nu} = f^{(+)\mu\nu} - \gamma_5 f^{(-)\mu\nu},
\]

(253)

\[
W_{\mu\nu} = W^{(+)\mu\nu} - \gamma_5 W^{(-)\mu\nu}.
\]

(254)

Here \( f^{(\pm)\mu\nu} \) and \( W^{(\pm)\mu\nu} \) are respectively defined as in eqs. (14) and (12).
C  Seeley-DeWitt coefficients.

The Heat kernel expansion of the operator $D^\dagger E D_E - M_Q^2$ in eq. (241) defines the Seeley-DeWitt coefficients $H_n(x, x)$, which were used in ref. [25] up to $n = 3$. Here we need them up to $n = 6$. In the literature, they have been computed up to $n = 5$. Fortunately, $\gamma_{21}$ and $\gamma_{22}$ at the order $\alpha_s N_c$ were calculated in Ref. [25] by other methods, and we have taken their results. All the other quantities can be computed using only the $H_n$’s up to $n = 5$.

In what follows, the $H_n$’s are defined up to a total derivative and a circular permutation. See ref [57] and [58]. Terms that do not contribute to our results are dropped.

\[ H_0(x, x) = 1, \quad (255) \]

\[ H_1(x, x) = -E, \quad (256) \]

\[ H_2(x, x) = \frac{1}{2} [E^2 + \frac{1}{6} R_{\mu\nu} R_{\mu\nu}], \quad (257) \]

\[ H_3(x, x) = -\frac{1}{6} [E^3 - \frac{1}{2} E \nabla_\mu \nabla_\mu E + \frac{1}{2} E R_{\mu\nu} R_{\mu\nu}], \quad (258) \]

\[ H_4(x, x) = \frac{1}{24} [E^4 - E^2 \nabla_\mu \nabla_\mu E \]
\[ + \frac{1}{5} (E R_{\mu\nu} E R_{\mu\nu} + 4 E^2 R_{\mu\nu} R_{\mu\nu}) - \frac{4}{15} \nabla_\rho \nabla_\rho ER_{\mu\nu} R_{\mu\nu} \]
\[ + \frac{17}{210} R_{\mu\nu} R_{\rho\sigma} R_{\rho\sigma} R_{\mu\nu} + \frac{2}{35} R_{\mu\nu} R_{\rho\sigma} R_{\mu\nu} R_{\rho\sigma} \]
\[ + \frac{1}{105} R_{\mu\nu} R_{\rho\sigma} R_{\rho\sigma} R_{\mu\nu} + \frac{1}{420} R_{\mu\nu} R_{\rho\sigma} R_{\mu\nu} R_{\rho\sigma}], \quad (259) \]

\[ H_5(x, x) = -\frac{1}{120} [E^5 - 2 E^3 \nabla_\mu \nabla_\mu E \]

\[ -E^2 \nabla_\mu E \nabla_\mu E + \frac{2}{3} E^2 R_{\mu\nu} E R_{\mu\nu} + E^3 R_{\mu\nu} R_{\mu\nu} + \frac{2}{3} E R_{\mu\nu} \nabla_\mu E \nabla_\nu E - \frac{8}{7} E \nabla_\rho E \nabla_\rho E R_{\mu\nu} R_{\mu\nu} - \frac{4}{21} R_{\mu\nu} E \nabla_\rho E \nabla_\rho E R_{\mu\nu} \]
\[ -\frac{10}{21} \nabla_\mu E \nabla_\nu E R_{\rho\rho} R_{\mu\nu} + \frac{2}{21} \nabla_\mu E \nabla_\nu E R_{\mu\nu} R_{\rho\rho} \]
\[ + \frac{2}{21} \nabla_\mu R_{\mu\nu} E \nabla_\rho E R_{\rho\nu} - \frac{11}{21} \nabla_\mu E \nabla_\nu E R_{\mu\nu} R_{\rho\rho} + \frac{1}{42} \nabla_\mu R_{\mu\rho} E \nabla_\mu E R_{\nu\rho}. \]  

(260)

D Effective actions.

\[ \Gamma_{\text{eff}}^{(1)} = \frac{N_c}{16\pi^2} \Gamma(-1, x) \left[ 2 \left( \frac{1}{4} \Sigma^2 - M_Q^2 \right) \right], \]  

(261)

\[ \Gamma_{\text{eff}}^{(2)} = \frac{N_c}{16\pi^2} \Gamma(0, x) \left[ \frac{1}{2} \{ \xi', \Sigma' \}^2 + \frac{i}{4} \{ \xi', \Sigma' \} d^{\mu'} \Delta' - \left( \frac{1}{4} \Sigma'^2 - M_Q^2 \right)^2 \right. \]
\[ \left. + \frac{1}{12} d^{\mu'} \Sigma' d^{\mu'} \Sigma' - \frac{1}{12} \left( (f^{(+)}')_{\mu\nu} (f^{(+)})^{\mu\nu} + (f^{(-)}')_{\mu\nu} (f^{(-)})^{\mu\nu} \right) \right], \]  

(262)

\[ \Gamma_{\text{eff}}^{(3)} = \frac{N_c}{16\pi^2} \Gamma(1, x) \left[ \frac{1}{16} \left( \frac{1}{4} \Sigma^2 - M_Q^2 \right) \{ \xi', \Sigma' \}^2 \right. \]
\[ \left. - \frac{i}{2} f^{(+)}_{\mu\nu} \xi' \Sigma' - \frac{1}{2} \left( (d'_{\mu} \xi')^2 + \frac{1}{2} \{ \xi', \xi' \} d^{\mu} \Sigma' - \frac{1}{4} (d'_{\mu} \xi')^2 + \cdots \right) \right], \]  

(263)

with

\[ (d'_{\mu} \xi')^2 = -\frac{1}{4} \{ \xi', \xi' \} \{ \xi', \xi' \} + i f^{(+)}_{\mu\nu} \xi' \Sigma' + \frac{1}{2} f^{(+)}_{\mu\nu} f^{(-)}_{\mu\nu} + (d'_{\mu} \xi')^2 + \cdots, \]  

(264)

\[ \Gamma_{\text{eff}}^{(4)} = \frac{N_c}{16\pi^2} \Gamma(2, x) \left[ \frac{1}{12} \xi' \xi' \xi' \Sigma' + \frac{1}{4} \xi' \Sigma' \xi' \xi' \right]. \]  

(265)

When the shifts described in appendix B and diagonalization of quadratic terms \((\sigma \rightarrow \sigma + M_Q \text{ and } W_{\mu}^- \rightarrow W_{\mu}^- + (1 - g_A) \xi_\mu)\) are performed we are left with the effective action in terms of the \(0^-\), \(1^-\), \(1^+\) and \(0^+\) fields.

D.1 Effective action for pseudoscalars.

\[ \Gamma_{\text{eff}}^{(2)} = \frac{N_c}{16\pi^2} \Gamma(0, x) < M_Q^2 g_A \xi_\mu \xi_\mu + M_Q g_A^2 \xi_\mu \xi_\mu \Sigma + M_Q i g_A \xi_\mu d^{\mu} \Delta \]
\[ + \frac{1}{48} (1 - g_A^2) \{ \xi_\mu, \xi_\nu \} \{ \xi_\alpha, \xi_\alpha \} - \frac{i}{6} (1 - g_A^2) f^{(+)}_{\mu\nu} \xi_\mu \xi_\nu \]
\[ - \frac{1}{12} f^{(+)}_{\mu\nu} f^{(+)}_{\mu\nu} - \frac{1}{12} g_A^2 f^{(-)}_{\mu\nu} f^{(-)}_{\mu\nu} > \]  

(266)
\[ \Gamma_{\text{eff}}^{(3)} = \frac{N_c}{16\pi^2} \Gamma(1, x) < M_Q g_A^2 \xi_{\mu}^{\ast} \xi_{\nu} \sum - \frac{i}{2} g_A^2 f^{(+)}_{\mu \nu} \xi_{\mu}^{\ast} \xi_{\nu} + \frac{1}{8} g_A^2 (1 - g_A^2)[\xi_{\mu}^{\ast} \xi_{\nu}] [\xi_{\mu}^{\ast} \xi_{\nu}] \\
+ \frac{1}{6} g_A^2 \left( \frac{1}{4} [\xi_{\mu}^{\ast} \xi_{\nu}][\xi_{\mu}^{\ast} \xi_{\nu}] + \frac{1}{2} f^{(-)\mu \nu} f^{(-)\mu \nu} + i f^{(+)}_{\mu \nu} \xi_{\mu}^{\ast} \xi_{\nu} \\
+ (d_{\mu} \xi_{\mu})^2 + \frac{1}{2} g_A^2 (\xi_{\mu}^{\ast} \xi_{\nu} \xi_{\mu}^{\ast} \xi_{\nu} + \xi_{\mu} \xi_{\nu} \xi_{\mu}^{\ast} \xi_{\nu}) \right) > \] (267)

\[ \Gamma_{\text{eff}}^{(4)} = \frac{N_c}{16\pi^2} \Gamma(2, x) < \frac{1}{12} g_A^4 \xi_{\mu}^{\ast} \xi_{\nu} \xi_{\mu}^{\ast} \xi_{\nu} - \frac{1}{6} g_A^4 \xi_{\mu}^{\ast} \xi_{\nu} \xi_{\mu}^{\ast} \xi_{\nu} > \] (268)

**D.2 effective action for vectors.**

\[ \Gamma_{\text{eff}}^{(2)} = \frac{N_c}{16\pi^2} \Gamma(0, x) < -\frac{1}{12} W^{(+)}_{\mu \nu} W^{(+)}_{\mu \nu} - \frac{1}{6} W^{(+)}_{\mu \nu} f^{(+)}_{\mu \nu} > \]

\[ \Gamma_{\text{eff}}^{(3)} = \frac{N_c}{16\pi^2} \Gamma(1, x) < -\frac{i}{12} (1 - g_A^2) W^{(+)}_{\mu \nu} [\xi_{\mu}^{\ast} \xi_{\nu}] > \] (269)

**D.3 effective action for axial-vectors.**

\[ \Gamma_{\text{eff}}^{(2)} = \frac{N_c}{16\pi^2} \Gamma(0, x) < M_Q^2 W^{(-)}_{\mu \nu} W^{(-)}_{\mu \nu} - \frac{1}{12} W^{(-)}_{\mu \nu} W^{(-)}_{\mu \nu} - \frac{1}{6} g_A W^{(-)}_{\mu \nu} f^{(-)}_{\mu \nu} > \]

\[ \Gamma_{\text{eff}}^{(3)} = \frac{N_c}{16\pi^2} \Gamma(1, x) < \frac{1}{12} W^{(-)}_{\mu \nu} W^{(-)}_{\mu \nu} + \frac{1}{6} g_A W^{(-)}_{\mu \nu} f^{(-)}_{\mu \nu} > \] (270)

**D.4 effective action for scalars.**

\[ \Gamma_{\text{eff}}^{(1)} = \frac{N_c}{16\pi^2} \Gamma(-1, x) < 2(\sigma + M_Q)^2 - 2 M_Q^2 + 2 M_Q \Sigma + 2 \sigma \Sigma > \] (271)

\[ \Gamma_{\text{eff}}^{(2)} = \frac{N_c}{16\pi^2} \Gamma(0, x) < d_{\mu}^{\ast} d_{\mu}^{\ast} \sigma + 2 M_Q g_A^2 \xi_{\mu}^{\ast} \xi_{\mu}^{\ast} \sigma - 4 M_Q^2 (\sigma^2 + \sigma \Sigma) > \]

\[ \Gamma_{\text{eff}}^{(3)} = \frac{N_c}{16\pi^2} \Gamma(1, x) < -2 M_Q g_A^2 \xi_{\mu}^{\ast} \xi_{\mu}^{\ast} \sigma - \frac{2}{3} d_{\mu}^{\ast} d_{\mu}^{\ast} \sigma > \] (272)
FIGURE CAPTIONS

Figure 1:
(a) Conventional one-gluon exchange between two quark vertices in QCD.
(b) Local effective four-quark interaction which emerges from (a) with the replacement in eq. (7).

Figure 2:
Schwinger-Dyson equation for the quark propagator which leads to the gap equation in eq. (72).

Figure 3:
In the leading large-$N_c$ approximation, the diagrams which are summed are chains of fermion “bubbles” as in (a), as well as trees of chains as in (b)

Figure 4:
An example of a loop of chains, which is next-to-leading order in the $1/N_c$-expansion.
\( q(x) \rightarrow G \rightarrow q(x) \)

\( q(y) \rightarrow G \rightarrow q(y) \)

\( \Rightarrow \)

\( q(x) \rightarrow q(x) \)

\( q(x) \rightarrow q(x) \)

(a) \hspace{1cm} (b)

Fig. 1

---

\[ \quad = \quad + \]

Fig. 2
Fig. 3

Fig. 4
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