On Exponential Type Entire Functions without Zeros in the Open Lower Half-plane

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Abstract. We obtain sufficient conditions for an exponential type entire function not to have zeros in the open lower half-plane. An exact inequality containing the real and imaginary parts of such functions and their derivatives restricted to the real axis is deduced.

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1 Introduction. Formulation of the Main Results

In this paper, we study entire functions that do not have zeros in the open lower half-plane, \( \text{Im} z < 0 \). The results applied to algebraic polynomials give the known Hermite–Biller theorem. An extension of this theorem to arbitrary entire functions was carried out in works of M. G. Krein, B. Ya. Levin, and N. N. Meiman, for more details see, for example, [7, 8, 11, 4].

An entire function \( f \) is called an exponential type entire function (in [7], such functions are called finite exponent functions), if there exist numbers \( A > 0, B > 0 \) such that \( |f(z)| \leq Ae^{B|z|} \) holds for all \( z \in \mathbb{C} \). The exact lower bound of such numbers \( B \) is denoted by \( \sigma(f) \geq 0 \) and is called a type of the function \( f \). Denote by \( E_{\sigma} \), \( \sigma \geq 0 \), the class of entire functions \( f \) of exponential type \( \sigma(f) \leq \sigma \).

The purpose of this work is to prove Theorems 1.1, 1.2, 1.3, and 1.4.
Theorem 1.1. Let the following conditions hold: 1) \( \omega(z) = P(z) + iQ(z) \), where \( P, Q \) are real functions of the class \( E_\sigma \), \( \sigma > 0 \), and \( \omega(x) = o(x), \ x \to \pm \infty \) on the real axis; 2) the following inequality holds for some \( \tau \in \mathbb{R} \):

\[
E(x) := P(x) \cos(\sigma x + \tau) + Q(x) \sin(\sigma x + \tau) \geq 0, \quad x \in \mathbb{R}. \quad (1.1)
\]

Let \( d(x) := P(x)Q'(x) - P'(x)Q(x) \). Then the following holds.

1) For all \( x \in \mathbb{R} \),

\[
4\sigma d(x) \geq \left\{ (\sigma P(x) + Q'(x)) \sin(\sigma x + \tau) + (P'(x) - \sigma Q(x)) \cos(\sigma x + \tau) \right\}^2. \quad (1.2)
\]

2) The following conditions are equivalent:

i) inequality (1.2) becomes equality;

ii) for some \( c \geq 0, \beta \in \mathbb{R} \), we have the identity \( E(x) \equiv c \sin^2(\sigma x + \tau + \beta) \);

iii) for some \( \beta \in \mathbb{R} \) and all \( k \in \mathbb{Z} \), we have \( E\left(\frac{k\pi - \beta + \tau}{\sigma}\right) = 0 \);

iv) for some \( c \geq 0 \) and \( \beta, \gamma \in \mathbb{R} \), we have

\[
P(x) \equiv c \sin \beta \sin(\sigma x + \tau + \beta) + \gamma \sin(\sigma x + \tau),
\]

\[
Q(x) \equiv c \cos \beta \sin(\sigma x + \tau + \beta) - \gamma \cos(\sigma x + \tau). \quad (1.3)
\]

In this case, \( d(x) \equiv \gamma^2 \sigma \).

3) If for any \( \alpha \in \mathbb{R} \) and \( c \geq 0 \), \( E(x) \not\equiv c \sin^2(\sigma x + \tau + \alpha) \), then inequality (1.2) becomes equality for some \( x = x_0 \in \mathbb{R} \iff E(x_0) = 0 \).

For an entire function \( \omega(z) = P(z) + iQ(z) \), where \( P(z) \) and \( Q(z) \) are real entire functions, set \( \overline{\omega}(z) := P(z) - iQ(z) \). This is an entire function obtained from \( \omega(z) \) by replacing all the coefficients in its expansion in the powers \( \{z^{n-1}\}_{n \in \mathbb{N}} \) with their complex conjugates. It is clear that \( \overline{\omega}(z) = \overline{\omega(z)} \).

Definition 1.1. An entire function \( \omega(z) \) is called a class \( HB \) function if it does not have zeros in the closed lower half-plane \( \text{Im} z \leq 0 \), and \( \left| \frac{\omega(z)}{\overline{\omega(z)}} \right| < 1 \) if \( \text{Im} z > 0 \).

\(^1\text{We call a function real if it takes real values on the real axis.}\)
Definition 1.2. An entire function $\omega(z)$ is called a class $\mathcal{HB}$ function if it does not have zeros in the open lower half-plane $\text{Im} \ z < 0$, and $|\frac{\omega(z)}{\overline{\omega(z)}}| \leq 1$ if $\text{Im} \ z > 0$.

Equality in Definition 1.2 can hold only if $\omega(z)$ is a real function up to a constant factor. Such functions are called trivial class $\mathcal{HB}$ functions.

It is clear that $\omega \in \mathcal{HB} \iff$ the function $\omega \in \overline{\mathcal{HB}}$ does not have real zeros and is not trivial.

Theorem 1.2. Let conditions of Theorem 1.1 be satisfied and assume that the function $\omega$ is not real up to a constant factor. Then we have the following.

1) $\omega \in \mathcal{HB}$. If the function $\omega$ has real zeros, then they are simple.

2) $d(x_0) = 0$ for some $x_0 \in \mathbb{R} \iff \omega(x_0) = 0$. If a number $x_0 \in \mathbb{R}$ is a zero of the function $\omega$, then the number $x_0$ is a zero of the function $d$ of multiplicity 2.

3) For all $n \in \mathbb{N}$, $\omega^{(n)} \in \mathcal{HB}$.

In Corollary 4.1 we give examples where conditions of Theorems 1.1 and 1.2 hold. Note that Theorems 1.1 and 1.2 cease to hold if the condition $\omega(x) = o(x), x \to \pm \infty$, is replaced with the condition $\omega(x) = O(x), x \to \pm \infty$, see Remark 3.2.

For a given function $\mu$ that has bounded variation on the segment $[0, \sigma], \sigma > 0$, we will consider the following entire functions:

$$F(z) := \int_{0}^{\sigma} e^{itz} d\mu(t), \quad G(z) := \int_{0}^{\sigma} \cos zt \, d\mu(t),$$

$$H(z) := \int_{0}^{\sigma} \sin zt \, d\mu(t),$$

$$\Delta(z) := G(z)H'(z) - G'(z)H(z),$$

$$h_\alpha(z) := G(z) \cos \alpha - H(z) \sin \alpha,$$

$$C(z) := -\int_{0}^{\sigma} \cos zt \, d\mu(\sigma - t), \quad S(z) := -\int_{0}^{\sigma} \sin zt \, d\mu(\sigma - t).$$

(1.4)

(1.5)

(1.6)

It is clear that the function $\mu$ can always be regarded as left continuous in every point of the interval $(0, \sigma)$. If $F(z) \neq ce^{iaz}$, then the function $F$ has infinitely many zeros, see for example [6]. A function of type (1.4) is used for solving many problems in analysis, for example, in spectral
theory, in the theory of differential-difference equations, in the theory of positive definite functions, in the Fourier analysis, etc.; see for example [6, 9, 13, 17, 19]. The distribution of zeros of such functions was studied in the works of Hardy [5], Polya [12], Titchmarsh [18], Cartwright [2, 3], Sedlitskii [14, 15], and others. In [12], the case where the function \( \mu \) is absolutely continuous and \( d\mu(t) = g(t)dt \) was studied, where the function \( g \) is integrable, positive, and does not decrease on the interval \( (0, \sigma) \). It was proved there that, in this case, all zeros of the function \( F \) lie in the closed upper half-plane \( \text{Im} \, z \geq 0 \), and if \( g \) is not piecewise constant with uniformly distributed nodes, then the function \( F \) does not have real zeros. These results of Polya were extended and made more precise in the works of the author [20, 21].

**Theorem 1.3.** Let \( \mu \) be a real function with bounded variation on the segment \([0, \sigma]\), \( C(x) \geq 0 \) for all \( x \in \mathbb{R} \), and \( F(z) \neq 0 \). Then the function \( F \in \mathcal{HB} \) is not trivial. All real zeros of the function \( F \), if there are any, are simple.

**Theorem 1.4.** Let \( \mu \) be a real function with bounded variation on the segment \([0, \sigma]\), \( S(x) \geq 0 \) for all \( x > 0 \), and \( F(z) \neq 0 \). Then all real zeros distinct from 0 of the function \( F \), if there are any, are simple, and if the number \( x = 0 \) is a zero of the function \( F \), then its multiplicity does not exceed 2. Moreover, in such a case, \( \mu(\sigma - 0) \geq \mu(0) \) and the following holds.

1) If \( F(0) \in (\infty, 0) \cup [\mu(\sigma - 0) - \mu(0), +\infty) \), then \( F \in \mathcal{HB} \), and the function is not trivial.

2) If \( F(0) \in (0, \mu(\sigma - 0) - \mu(0)) \), then the function \( F \) has exactly one zero in the lower half-plane \( \text{Im} \, z < 0 \), and it is pure imaginary.

In Section 5, we give examples where conditions of Theorems 1.3 and 1.4 hold. Polya’s case is contained in statement 1 of Theorem 1.4 (see Example 5.1 in Section 5). Note that Theorem 1.4 realizes two cases (if the quantity \( \mu(\sigma) \) is changed, then the values of \( S(x) \) and \( \mu(\sigma - 0) \) do not change, and the value \( F(0) = \mu(\sigma) - \mu(0) \) can be made arbitrary). Also note that \( S(x) \geq 0 \) for all \( x > 0 \) \( \iff \mu(t) - \mu(0) \equiv f(\sigma - t) \) for \( 0 \leq t < \sigma \), where \( f \) is an even function that is positive definite and continuous on \( \mathbb{R} \), and equal to zero for \( |t| \geq \sigma \) (Lemma 2.2). A relation between class \( \mathcal{HB} \) functions of type (1.4) and positive definite functions is contained in Proposition 5.1 (see Section 5).
2 Auxiliary Propositions

2.1 Functions (1.4), (1.5), and (1.6).

We have the following:

\[ F(z) ≜ G(z) + iH(z), \]

\[ F(z)e^{-iσz} ≜ - \int_0^α e^{-izt} dμ(σ - t) ≜ C(z) - iS(z), \]  \hspace{1cm} (2.1)

\[ G(z) ≜ C(z) \cos σz + S(z) \sin σz, \]

\[ H(z) ≜ C(z) \sin σz - S(z) \cos σz, \]

\[ G(z) \cos(σz + τ) + H(z) \sin(σz + τ) ≜ C(z) \cos τ - S(z) \sin τ, \]  \hspace{1cm} (2.2)

\[ h_α(z) ≜ C(z) \cos(σz + α) + S(z) \sin(σz + α), \]

\[ h_α(z)h_β'(z) - h_α'(z)h_β(z) ≜ Δ(z) \sin(α - β), \]  \hspace{1cm} (2.3)

\[ Δ(x) ≜ C'(x)S(x) - C(x)S'(x) + σ \left( C^2(x) + S^2(x) \right). \]  \hspace{1cm} (2.4)

These identities can be directly obtained from (1.4), (1.5), and (1.6).

Lemma 2.1. 1) i) \( G(z) ≜ 0 \iff F(z) ≜ 0 \). ii) \( H(z) ≜ 0 \iff F(z) ≜ c \). iii) \( C(z) ≜ 0 \iff F(z) ≜ 0 \). iv) \( S(z) ≜ 0 \iff F(z) ≜ ce^{iαz} \). v) \( h_α(z) ≜ 0 \) for some \( α \iff G(z) \cos α ≜ H(z) \sin α ≜ 0 \).

2) The function \( F \) is real up to a constant factor \( \iff F(z) ≜ c \).

3) \( x^n(C(x) \cos τ - S(x) \sin τ) ≜ c \sin^2(σx + τ + α) \) for some \( τ, α, c ∈ \mathbb{C}, n ∈ \mathbb{Z} \iff C(x) \cos τ = S(x) \sin τ ≜ 0 \iff C(x) \cos τ \equiv S(x) \sin τ ≜ 0 \).

4) If \( F(z) ≜ ce^{i(α+β)z} \) for some \( α, β ∈ \mathbb{R}, c ≠ 0 \), then \( β = 0 \) and \( α ∈ [0, σ] \).

5) If \( F(z) ≜ ce^{iαz} \) for some \( c ≠ 0, α ∈ [0, σ] \), and for all \( x > 0 \) the inequality \( C(x) \cos τ - S(x) \sin τ ≥ 0 \) holds, then \( α = σ \).

6) If \( F(z) ≠ 0 \) and \( C(x) \cos τ - S(x) \sin τ ≥ 0 \) for \( x > 0 \), then the function \( F \) is not real up to a constant factor, and \( h_α(z) ≠ 0 \).

7) If for some \( a, b, c, d, e ∈ \mathbb{R} \) and any \( t > e \), the inequality \( f(t) := a \sin^2(t + b) + c \cos t + d \sin t ≥ 0 \) holds, then \( c = d = 0 \) and \( a ≥ 0 \).

8) If \( μ \) is real and \( F(0) ≠ 0 \), then \( F(z) ≠ cR(z)(z + iξ) \), where \( ξ ∈ \mathbb{R} \) and \( R(z) \) is a real entire function.

Proof. Let us prove statement 1).

i) If \( G(z) ≜ 0 \), then \( ∫_0^α t^{2p}dμ(t) = 0 \) for all \( p ∈ \mathbb{Z}^+ := \mathbb{N} \cup \{0\} \). By Muntz’s theorem, the system of powers \( \{t^{2p}\}_{p∈\mathbb{Z}^+} \) is dense in \( C[0, 1] \). Hence, \( ∫_0^α f(t)dμ(t) = 0 \) for any function \( f ∈ C[0, 1] \). Thus, \( F(z) ≜ 0 \). Conversely, it follows from the identity \( F(z) = G(z) + iH(z) ≜ 0 \), since \( G \) is even and \( H \) is odd, that \( G(z) ≜ H(z) ≜ 0 \).
ii) If \( H(z) \equiv 0 \), then \( \int_0^\sigma t^{2p+1}d\mu(t) = 0 \), \( p \in \mathbb{Z}_+ \). Then \( \int_0^\sigma f(t) \, t \, d\mu(t) = 0 \) for any function \( f \in C[0, 1] \). Hence, \( F'(z) \equiv 0 \), and so \( F(z) \equiv c \). Conversely, if \( F(z) \equiv c \), then \( 2iH(z) \equiv F(z) - F(-z) \equiv 0 \).

Statements iii) and iv) follow from i), ii), and (1.1). Statement v) follows at once from (1.5) if we recall that \( G \) is even and \( H \) is odd functions.

Let us prove statement 2). Let a function \( F \) be real up to a constant factor. Without loss of generality, we can assume that the function \( F \) is real. Let \( \mu(t) = \mu_1(t) + i\mu_2(t) \), where \( \mu_1(t), \mu_2(t) \) are real functions with bounded variation on the segment \([0, \sigma]\). Then

\[
\text{Im}(F(x)) = \int_0^\sigma \sin xt \, d\mu_1(t) + \int_0^\sigma \cos xt \, d\mu_2(t) \equiv 0, \quad x \in \mathbb{R},
\]

and, hence,

\[
\int_0^\sigma \cos xt \, d\mu_1(t) \equiv \int_0^\sigma \sin xt \, d\mu_2(t) \equiv 0.
\]

It follows from statement 1) that

\[
\int_0^\sigma e^{izt} \, d\mu_1(t) \equiv c \quad \text{and} \quad \int_0^\sigma e^{izt} \, d\mu_2(t) \equiv 0.
\]

Hence, \( F(z) \equiv c \). The converse is clear.

Now we prove statement 3). If the indicated identity holds, then \( c = 0 \); otherwise the left-hand side contains an entire function of exponential type \( \leq \sigma \), and the type of the function in the right-hand side is precisely \( 2\sigma \). So, \( C(x) \cos \tau - S(x) \sin \tau \equiv 0 \), which is equivalent to two identities,

\[
C(x) \cos \tau \equiv S(x) \sin \tau \equiv 0.
\]

Let us prove 4). If \( \beta \neq 0 \), then the right-hand side of the identity is unbounded on \( \mathbb{R} \), whereas the left-hand side is bounded. Hence, \( \beta = 0 \) and \( F(z) \equiv ce^{iax}, \ c \neq 0 \). If \( \alpha > \sigma \), then the function in the right-hand side of the identity has type greater than that of the function in the left-hand side. If \( \alpha < 0 \), then \( F(iy) = \int_0^\sigma e^{-yt} \, d\mu(t) \equiv ce^{-\alpha y} \). The left-hand side, as \( y \to +\infty \), is bounded and the right-hand side is not. Hence, \( \alpha \in [0, \sigma] \).

Now, let us prove 5). In this case, \( C(x) \cos \tau - S(x) \sin \tau = c \cos((\sigma - \alpha)x + \tau) \geq 0 \) for \( x > 0 \), where \( c 
eq 0, \alpha \in [0, \sigma] \). If \( \alpha < \sigma \), then \( \exists x_0 > 0 : c \cos((\sigma - \alpha)x_0 + \tau) \neq 0 \). Then, for \( x = x_0 \) and \( x = x_0 + \frac{\pi}{\sigma-\alpha} \), the left-hand side of the inequality takes values of different sign. Thus, \( \alpha = \sigma \).

Let us prove 6). If a function \( F \) is a constant, up to a constant factor, then \( F(z) \equiv c \), and \( c \neq 0 \), which can not happen (see statement 5 with \( \alpha = 0 \)). If \( h_\alpha(z) \equiv 0 \), then \( G(z) \cos \alpha \equiv H(z) \sin \alpha \equiv 0 \) and, hence, \( F(z) \equiv c, \ c \neq 0 \), which is a contradiction.

Now we prove 7). Let \( t_k := -b + k\pi, \ k \in \mathbb{Z} \). Then, for all \( k > k_0 \), we have \( f(t_k) = (-1)^k(c \cos b - d \sin b) \geq 0 \) and, hence, \( c \cos b - d \sin b = 0 \). Thus, \( f(t_k) = 0 \) for all \( k > k_0 \), and so, \( f'(t_k) = (-1)^k(c \sin b + d \cos b) = 0 \). Consequently, \( c = d = 0 \), which means that \( a \geq 0 \).

Let us prove 8). Without loss of generality, we can assume that the function \( \mu \) is left continuous in every point of the interval \((0, \sigma)\). Assume that \( F(z) = cR(z)(z + i\xi) \), where \( \xi \in \mathbb{R} \) and \( R(z) \) is a real entire function. Since \( F(0) = ic\xi R(0) \in \mathbb{R} \setminus \{0\} \), we can assume that \( c = -i \)
and $\xi \neq 0$. Then $\xi R(z) = \int_0^\sigma \cos z t \, d\mu(t)$ and $z R(z) = -\int_0^\sigma \sin z t \, d\mu(t) = z \int_0^\sigma \cos z t (\mu(t) - \mu(\sigma)) \, dt = z \int_0^\sigma \cos z t \, d\mu_1(t)$, where $\mu_1(t) = \int_t^\sigma (\mu(u) - \mu(\sigma)) \, du$. Hence, if $t \in [0, \sigma]$, we have $\mu(t) - \mu(\sigma) \equiv \xi \int_0^\sigma (\mu(u) - \mu(\sigma)) \, du$, which implies that $\mu \in C^1[0, \sigma]$ and $\mu(t) - \mu(\sigma) \equiv c_1 e^{\xi t}$. Consequently, $\mu(t) - \mu(\sigma) \equiv 0$ and $F(z) \equiv 0$, which contradicts the condition. \qed

A function $f : \mathbb{R} \to \mathbb{C}$ is called positive definite on $\mathbb{R}$ if, for any $n \in \mathbb{N}$, $\{x_k\}_{k=1}^n \subset \mathbb{R}$, and $\{c_k\}_{k=1}^n \subset \mathbb{C}$, the inequality $\sum_{k,j=1}^n c_k \bar{c}_j f(x_k - x_j) \geq 0$ holds. For such functions, $|f(x)| \leq f(0)$, $x \in \mathbb{R}$, and continuity at zero is equivalent to continuity on $\mathbb{R}$. By Bokhner–Hinchin theorem, a function $f$ is positive definite and continuous on $\mathbb{R}$ if and only if $f(x) = \int_{-\infty}^{+\infty} e^{-ixu} \, d\nu(u)$, where $\nu$ is a nonnegative, finite Borel measure on $\mathbb{R}$. If $f \in C(\mathbb{R}) \cap L(\mathbb{R})$, then positive definiteness of a function $f$ is equivalent to its Fourier transform being nonnegative, that is, $\hat{f}(x) := \int_{-\infty}^{+\infty} f(u) e^{-ixu} \, du \geq 0$, $x \in \mathbb{R}$, and in this case, $\hat{f} \in L(\mathbb{R})$, see [16, Ch. I, §1, Corollary 1.26]).

**Lemma 2.2.** Let $\mu$ be a real function of bounded variation on the segment $[0, \sigma]$, left continuous in every point of the interval $(0, \sigma)$. Then the following holds.

1) $S(x) \geq 0$ for all $x > 0 \iff \mu(t) - \mu(0) \equiv f(\sigma - t)$ for $0 \leq t < \sigma$, where $f$ is an even function that is positive definite and continuous on $\mathbb{R}$, $f(t) = 0$, $|t| \geq \sigma$. In this case, $f(0) = \mu(\sigma - 0) - \mu(0) \geq 0$ and $\mu(\sigma - 0) - \mu(0) = 0 \iff S(x) \equiv 0 \iff F(z) \equiv c e^{ix \sigma}, c \in \mathbb{R}$.

2) If $S(x) \geq 0$ for all $x > 0$ and $F(0) \leq 0$, then $H'(0) \leq 0$. In this case, $F'(0) = 0 \iff H'(0) = 0 \iff F(0) = 0$ and $f_0^x f(t) \, dt = 0$, where $f$ is the corresponding function in statement 1).

3) If $S(x) \geq 0$ for all $x > 0$ and $F(0) \geq \mu(\sigma - 0) - \mu(0)$, then $H'(0) \geq 0$. In this case, $F'(0) = 0 \iff H'(0) = 0 \iff F(z) \equiv 0$.

**Proof.** Let us prove statement 1). It follows from [16] and the integration by parts formula that $S(x) = xK(x)$, where $K(x) := \int_0^\sigma \cos t x (\mu(\sigma - t) - \mu(0)) \, dt$. Let us first assume that $S(x) \geq 0$ for all $x > 0$. The function $2K(x)$ is a Fourier transform of a finite function, integrable on $\mathbb{R}$, $\mu((\sigma - |t|)_+ - \mu(0))$, which is bounded in a neighborhood of zero. Since $K(x) \geq 0$ for all $x \in \mathbb{R}$, we have $K \in L(\mathbb{R})$, see for example [19], and hence, for almost all $t \in \mathbb{R}$, one can apply the inverse transform formula, see for example [19], [16] [4],

$$\mu((\sigma - |t|)_+) - \mu(0) = \frac{1}{\pi} \int_{-\infty}^{+\infty} e^{ixt} K(x) \, dx =: f(t).$$
The function \( f \) in the right-hand side of the latter identity is even, continuous, and positive definite on \( \mathbb{R} \). The left-hand side equals 0 for \(|t| \geq \sigma\).

Hence, continuity of the function \( f \) implies that \( f(t) = 0 \) for all \(|t| \geq \sigma\). Since the function \( \mu \) is left continuous in every point of the interval \((0, \sigma)\), the above identity implies that the function \( \mu \) is continuous in every point of this interval, and \( \mu(0 + 0) = \mu(0) = f(\sigma) = 0 \). Hence, \( \mu \in C[0, \sigma] \) and \( \mu(t) - \mu(0) \equiv f(\sigma - t) \) for \( 0 \leq t < \sigma \). The converse is obvious. The first part of statement 1) is proved. The second part follows from the inequality \( |f(x)| \leq f(0), x \in \mathbb{R}, \) and Lemma 2.1 (statement 1).

Statement 2) follows at once from the identities \( F'(0) = iH'(0), F(0) = \mu(\sigma) - \mu(0), H'(0) = \int_0^\sigma t \, d\mu(t) = \sigma F(0) - \int_0^\sigma f(\sigma - t) \, dt, \) and the inequality \( \int_0^\sigma f(t) \, dt \geq 0. \)

Statement 3) immediately follows from the inequalities \( |f(t)| \leq f(0) \) and \( H'(0) = \int_0^\sigma (F(0) - f(t)) \, dt \geq \int_0^\sigma (f(0) - f(t)) \, dt \geq 0. \) If \( H'(0) = 0, \) then \( F(0) = f(0) \equiv f(t) \) for \( t \in [0, \sigma] \) and, hence, \( f(0) = 0, S(x) \equiv 0, F(z) \equiv ce^{iz}, \) where \( c = F(0) = 0. \)

**Lemma 2.3.** Let \( f, g \) be functions positive definite on \( \mathbb{R} \) from the class \( C(\mathbb{R}) \cap L(\mathbb{R}), \) and \( f(x) \neq 0, g(x) \neq 0 \) on \( \mathbb{R}. \) If the function \( f \) is finite, then 1) \( \int_{-\infty}^{+\infty} g(x)f(x) \, dx > 0; \) 2) for all \( \alpha > 0 \) and \( \beta \in \mathbb{R}, |\beta| \leq \alpha, \) the following inequality holds: \( \int_{-\infty}^{+\infty} e^{-\alpha|x|}(1 - \beta|x|)f(x) \, dx > 0. \)

**Proof.** Since \( \hat{f}(t) \geq 0 \) and \( \hat{g}(t) \geq 0 \) for all \( t \in \mathbb{R}, \) it follows that \( \hat{f}, \hat{g} \in L(\mathbb{R}) \cap C(\mathbb{R}). \) The multiplication formula gives that

\[
\int_{-\infty}^{+\infty} g(x)f(x) \, dx = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{g}(-t)\hat{f}(t) \, dt \geq 0.
\]

If the integral equals 0, then \( \hat{g}(-t)\hat{f}(t) \equiv 0 \) on \( \mathbb{R}. \) Since \( g(x) \neq 0, \) we have \( \hat{g}(-t) \neq 0 \) on some interval \((a, b), a < b. \) Hence, \( \hat{f}(t) = 0 \) on \((a, b), \) and, since \( f \) is entire, \( \hat{f}(t) \equiv 0 \) on \( \mathbb{R}. \) Hence, \( f(x) \equiv 0 \) on \( \mathbb{R}, \) which contradicts the condition. The first inequality is proved. The second inequality follows from the first one, if we take \( g(x) := e^{-\alpha|x|}(1 - \beta|x|). \) Then, for the indicated values of the parameters, \( g \in C(\mathbb{R}) \cap L(\mathbb{R}) \) and \( \hat{g}(t) = \frac{2[(\alpha-\beta)\alpha^2+(\alpha+\beta)t^2]}{(\alpha^2+t^2)^2} = 0 \) for all \( t \in \mathbb{R}. \)

**Lemma 2.4.** Let \( \nu \) be a function of bounded variation on a segment \([0, \sigma]. \) Then \( \lim_{t \to +\infty} \int_0^\sigma e^{-\alpha u} \, d\nu(u) = \nu(0) - \nu(0). \)

**Proof.** Let first the function \( \nu \) be right continuous in the point \( t = 0. \) Then for any \( \varepsilon \in (0, \sigma) \) and \( t > 0, \) we have \( \left| \int_0^\sigma e^{-\alpha u} \, d\nu(u) \right| \leq V_0^\sigma \varepsilon + e^{-\varepsilon t}V_0^\sigma. \)

Here \( V_0^\sigma \) is the variation of the function \( \nu \) on the segment \([0, t]. \) By passing
to the limit, we get that \( \limsup_{t \to +\infty} |\int_0^\sigma e^{-tu}d\nu(u)| \leq V_0^\varepsilon. \) Since \( \nu \) is right continuous in the point \( t = 0 \), we have \( \lim_{t \to +0} V_0^\varepsilon = 0. \) In this case, the lemma is proved. In the general case, we assume that \( \nu_1(0) := \nu(+0) \) and \( \nu_1(t) := \nu(t) \) for \( 0 < t \leq \sigma. \) It is clear that \( \nu_1 \) is a function of bounded variation on the segment \([0, \sigma]\) and is right continuous in the point \( t = 0. \) Then \( \int_0^\sigma e^{-tu}d\nu(u) = \int_0^\sigma e^{-tu}d\nu_1(u) + \nu(0) + \nu(+0) - \nu(0) \) for \( t \to +\infty. \)

2.2 Statements about Functions in the Class \( \overline{HB} \)

The following properties were proved in [7, Ch. VII].

1) If \( \omega(z) \in \overline{HB}, \) then the common zeros, if they exist, of the functions \( \omega(z) \) and \( \overline{\omega}(z) \) are real.

2) Trivial functions in the class \( \overline{HB} \) do not have zeros in \( \mathbb{C} \setminus \mathbb{R}. \)

3) Let a function \( \omega(z) \) be nontrivial. Then \( \omega(z) \in \overline{HB} \iff \omega(z) = R(z)\omega_1(z), \) where \( R(z) \) is a real entire function that does not have zeros in \( \mathbb{C} \setminus \mathbb{R}, \) and \( \omega_1(z) \in HB. \)

4) If a sequence of functions \( \omega_n(z) \in \overline{HB} \) converges to a function \( \omega(z) \neq 0 \) on every compact subset of \( \mathbb{C}, \) then \( \omega(z) \in \overline{HB}. \)

For a function \( \omega(z) = P(z) + iQ(z), \) where \( P(z) \) and \( Q(z) \) are real entire functions, set \( d(z) := P(z)Q'(z) - P'(z)Q(z) \) and \( H_\alpha(z) := P(z) \times \cos \alpha - Q(z) \sin \alpha. \) If the function \( \omega(z) \) is real, up to a constant factor, then it is clear that \( d(x) = 0. \)

**Theorem A** (B. Ya. Levin [7, Ch. VII, Th. 4] and N. N. Mejman [4, Ch. IV, Th. 15]). \( \omega(z) \in HB \iff 1) \) the functions \( P(z) \) and \( Q(z) \) do not have common zeros and, for any \( \mu, \nu \in \mathbb{R}, |\mu| + |\nu| \neq 0, \) the function \( \mu P(z) + \nu Q(z) \) does not have zeros in \( \mathbb{C} \setminus \mathbb{R}; 2) \) for some \( x_0 \in \mathbb{R}, \) we have \( d(x_0) > 0. \) Moreover, if conditions 1) and 2) hold, then \( d(x) > 0 \) holds for any \( x \in \mathbb{R}. \)

**Theorem B** (B. Ya. Levin [7, Ch. VII, Th. 4']). Let a function \( \omega(z) \) be nontrivial. Then \( \omega(z) \in \overline{HB} \iff 1) \) for any \( \mu, \nu \in \mathbb{R}, |\mu| + |\nu| \neq 0, \) the function \( \mu P(z) + \nu Q(z) \) does not have zeros in \( \mathbb{C} \setminus \mathbb{R}; 2) \) for some \( x_0 \in \mathbb{R}, \) we have \( d(x_0) > 0. \) Moreover, if conditions 1) and 2) are satisfied, then the inequality \( d(x) \geq 0 \) holds for any \( x \in \mathbb{R}. \)

The following proposition immediately follows from Theorems A and B.
Proposition 2.1. Let a function $\omega(z) \in \mathcal{HB}$ be nontrivial. Then

1) $d(x) \geq 0$, $x \in \mathbb{R}$.

2) $d(x_0) = 0$ for some $x_0 \in \mathbb{R} \iff \omega(x_0) = 0 \iff P(x_0) = Q(x_0) = 0$. If a number $x_0 \in \mathbb{R}$ is a zero of multiplicity $p$ for the function $\omega$, then $x_0$ is a zero of the function $d$ of multiplicity $2p$.

3) For all $\alpha \in \mathbb{R}$, the function $H_\alpha$ is real and does not have zeros in $\mathbb{C} \setminus \mathbb{R}$. If a number $x_0 \in \mathbb{R}$ is a zero of the function $H_\alpha$ of multiplicity $q$, then $q \leq p + 1$, where $p$ is the multiplicity of the zero $x_0$ for the function $\omega$ ($p = 0$, if $\omega(x_0) \neq 0$). If the function $\omega$ does not have real zeros, then all zeros of the function $H_\alpha$, if it has any, are simple.

Proof. By property 3, $\omega(z) = R(z)\omega_1(z)$, where $R(z)$ is a real entire function that does not have zeros in $\mathbb{C} \setminus \mathbb{R}$, and $\omega_1(z) \in \mathcal{HB}$. Then $\omega(x_0) = 0$ for some $x_0 \in \mathbb{R} \iff R(x_0) = 0$ and, in this case, multiplicities of the zero $x_0$ for $\omega(z)$ and $R(z)$ coincide. If $\omega_1(z) = P_1(z) + iQ_1(z)$, where $P_1(z)$ and $Q_1(z)$ are real entire functions, then, by Theorem A, $d_1(x) := P_1(x)Q_1'(x) - P_1'(x)Q_1(x) > 0$, $x \in \mathbb{R}$. It is clear that $P(z) = R(z)P_1(z)$ and $Q(z) = R(z)Q_1(z)$. Then $d(x) = R^2(x)d_1(x)$. Hence, $d(x_0) = 0$ for some $x_0 \in \mathbb{R} \iff R(x_0) = 0$ and, in this case, the multiplicity of the zero $x_0$ of the function $d$ is two times greater than the multiplicity of the zero $x_0$ of the function $R$. Statements 1) and 2) are proved.

To prove statement 3) it should be noted that, by Theorem A, the function $H_{1,\alpha}(z) := P_1(z)\cos \alpha - Q_1(z)\sin \alpha$ does not have zeros in $\mathbb{C} \setminus \mathbb{R}$ for all $\alpha \in \mathbb{R}$, and all its real zeros, if there are any, are simple. This follows from the identity $H_{1,\alpha}(x)H'_{1,\beta}(x) - H'_{1,\alpha}(x)H_{1,\beta}(x) \equiv d_1(x)\sin(\alpha - \beta)$. It remains to make a use of $H_\alpha(x) = R(x)H_{1,\alpha}(x)$. \qed

An indicator of growth of an exponential type function is defined by $h_f(\varphi) := \limsup_{r \to +\infty} \frac{\ln|f(re^{i\varphi})|}{r}$, $\varphi \in \mathbb{R}$.

Definition 2.1. A function $\omega(z)$ is called a class $P$ function if it is an exponential type function, does not have zeros in the open lower half-plane $\text{Im} z < 0$ and $2d_\omega := h_\omega(-\frac{\pi}{2}) - h_\omega\left(\frac{\pi}{2}\right) \geq 0$ (the quantity $d_\omega$ is called a defect of the function $\omega$).

Theorem C (B. Ya. Levin [7, Ch. VII, Lemma 1]). $\omega(z) \in P \iff \omega(z) \in \overline{\mathcal{HB}}$ and $\omega(z)$ is an exponential type entire function.

Since $\overline{\omega(z)} \equiv \overline{\omega(z)}$, it is clear that the product of two class $\mathcal{HB}$ functions is also a class $\mathcal{HB}$ function, that is, $\mathcal{HB} \cdot \mathcal{HB} \subset \mathcal{HB}$. Similarly, $\overline{\mathcal{HB}} \cdot \overline{\mathcal{HB}} \subset \overline{\mathcal{HB}}$. It follows from Theorem C that $P \cdot P \subset P$, too. The
function classes $HB$ and $P$ were introduced and studied, correspondingly, by M. G. Krein and B. Ya. Levin. The given Definition [4.1] is due to N. N. Meiman

Let $\mu(t)$ be a function with bounded variation on the segment $[a, b]$, $a < b$, which is left continuous in every point of the interval $(a, b)$, and $\omega(z) := \int_a^b e^{itz}d\mu(t)$.

The following results of such functions are contained in the monograph [3, Ch. I]; they allow to easily determine the defect. Let $[a_1, b_1]$ be the smallest segment that is contained in $[a, b]$ and possessing the following property: the function $\mu(t)$ is constant on $[a, a_1]$ and $(b_1, b]$. If there is no such intervals $[a, a_1]$ or $(b_1, b]$, then we take $a_1 = a$ or $b_1 = b$, correspondingly. If $a_1 = b_1$, then $\omega(z) \equiv ce^{i\alpha z}$. If $a_1 < b_1$, then $\omega(z) = \int_{a_1}^{b_1} e^{itz}d\mu_1(t)$, where the function $\mu_1(t)$ coincides with $\mu(t)$ for $a_1 \leq t < b_1$ and $\mu_1(b_1) := \mu(b)$. In this case, see [3, Ch. I, §4.3], the function $\omega(z)$ has infinitely many zeros and $h_\omega \left( \pm \frac{\pi}{2} \right) = b_1$, $h_\omega \left( \frac{\pi}{2} \right) = -a_1$ and, hence, $2d_\omega = b_1 + a_1$. Moreover, the upper limit in the definition of the growth indicator, for almost all $\varphi \in \mathbb{R}$, is equal to the limit.

**Proposition 2.2.** Let $F(z) := \int_0^\sigma e^{itz}d\mu(t)$, where $\mu(t)$ is a function with bounded variation on the segment $[0, \sigma]$, $\sigma > 0$. Then,

1) if $F(z) \not\equiv ce^{i\alpha z}$, $\alpha \in [0, \sigma]$, then the function $F$ has infinitely many zeros;

2) $F \in \overline{HB} \iff F$ does not have zeros in the open lower half-plane $\text{Im } z < 0$.

**Proof.** Without loss of generality, we can assume that the function $\mu(t)$ is left continuous in every point of the interval $(0, \sigma)$. Statement 1) has been considered above. Necessity of 2) is immediate. Let us prove sufficiency of 2). Assume that the function $F$ does not have zeros in the open lower half-plane $\text{Im } z < 0$. If $F(z) \equiv ce^{i\alpha z}$, then $c \neq 0$ and $\alpha \in [0, \sigma]$, by Lemma [2.4], statement 4). In this case, it is easy to check that $F \in \overline{HB}$, and if $\alpha > 0$, then $F \in HB$. If $F(z) \not\equiv ce^{i\alpha z}$, then, by the above, $0 \leq a_1 < b_1 \leq \sigma$, $2d_\omega = b_1 + a_1 > 0$ and, hence, $F \in P$. By Theorem [4] $F \in \overline{HB}$. \hfill \Box

### 2.3 An Interpolation Formula

Denote by $B_m^\sigma$, $m \in \mathbb{Z}_+$ := $\mathbb{N} \cup \{0\}$, the class of functions $f \in E^\sigma$, for which $f(x) = o(x^m)$, $x \to \pm \infty$, on the real axis.

By $S_\sigma, \sigma > 0$, denote the class of sine-type functions, that is, the set of functions $F \in E^\sigma$ that satisfy the following conditions: a) $h_F(\pm \frac{\pi}{2}) = \sigma$; b) all zeros $\lambda_k$ of the function $F$ are simple and satisfy the condition $\inf_{k \neq n} |\lambda_k - \lambda_n| = 2\delta > 0$; c) all zeros are located in a strip parallel to
the real axis, that is, $\sup_k |\text{Im} \lambda_k| = H < \infty$; d) there exist constants $C_k, h \in \mathbb{R}$, such that $0 < C_1 \leq |F(x + ih)| \leq C_2 < \infty, x \in \mathbb{R}$.

If $F \in S_\sigma$, then $\sigma(F) = \sigma > 0$ and $F$ has infinitely many zeros both in the left half-plane $\text{Re} z \leq 0$ and in the right half-plane $\text{Re} z \geq 0$. Zeros of the function $F \in S_\sigma$ are always indexed in the increasing order of their real parts, that is, $\text{Re} \lambda_k \leq \text{Re} \lambda_{k+1}, k \in \mathbb{Z}$. The function $F(z) := \sin(\sigma z + \alpha)$ is an example of such a function.

**Theorem D** ([21, Lemma 1]). Let $F \in S_\sigma, \sigma > 0$, and $\{\lambda_k\}$ be a sequence of all its zeros. Then for any $m \in \mathbb{Z}_+, f \in B^n_\sigma, \tau \in \mathbb{C}$, and $z \in \mathbb{C}, z \neq \lambda_k + \tau$, we have

$$\frac{d^n}{du^n} \left\{ \frac{f(u)}{F(u - \tau)} \right\} \bigg|_{u = z} = -m! \lim_{n \to \infty} \sum_{|\lambda_k| < n} \frac{f(\lambda_k + \tau)}{F'(\lambda_k)(\lambda_k + \tau - z)^{m+1}}.$$ 

Note that, for a smaller class of functions $f$, this formula is well-known, see details in [21, § 1].

3 Proof of Theorems 1.1 and 1.2

**Proof of Theorem 1.1.** If $f$ is an entire function of exponential type $\leq \sigma, \sigma > 0$, and $f(x) = o(x), x \to \pm \infty$, then the following interpolation formula holds for any $\alpha$ and $x$:

$$\sigma f(x) \cos(\sigma x + \alpha) - f'(x) \sin(\sigma x + \alpha)$$

$$= \sigma \lim_{n \to +\infty} \sum_{k=-n}^{n} \frac{\sin^2(\sigma x + \alpha)}{\sigma x + \alpha - k\pi} \cdot (-1)^k f\left(\frac{k\pi - \alpha}{\sigma}\right). \quad (3.1)$$

This follows from Theorem D for $F(z) := \sin(\sigma z + \alpha), \lambda_k = \frac{k\pi - \alpha}{\sigma}, m = 1, \tau = 0, z = x$.

Apply formula (3.1) to the function $f(x) := P\left(x - \frac{\tau}{\sigma}\right) \cos \alpha - Q\left(x - \frac{\pi}{\sigma}\right) \sin \alpha, \alpha \in \mathbb{R}$. Since $(-1)^k f\left(\frac{k\pi - \alpha}{\sigma}\right) = E\left(\frac{k\pi - \alpha - \tau}{\sigma}\right) \geq 0$ for all $k \in \mathbb{Z}$, we have

$$\sigma \left(P\left(x - \frac{\tau}{\sigma}\right) \cos \alpha - Q\left(x - \frac{\tau}{\sigma}\right) \sin \alpha\right) \cos(\sigma x + \alpha)$$

$$- \left(P'\left(x - \frac{\tau}{\sigma}\right) \cos \alpha - Q'\left(x - \frac{\tau}{\sigma}\right) \sin \alpha\right) \sin(\sigma x + \alpha) =$$

$$\sigma \sum_{k=-\infty}^{+\infty} \frac{\sin^2(\sigma x + \alpha)}{\frac{\sigma x + \alpha - k\pi}{\sigma}} \cdot E\left(\frac{k\pi - \alpha - \tau}{\sigma}\right) \geq 0, \quad \alpha, x \in \mathbb{R}. \quad (3.2)$$
Consider first the case $\tau = 0$. Then
\[
\sigma(P(x) \cos \alpha - Q(x) \sin \alpha) \cos(\sigma x + \alpha) - (P'(x) \cos \alpha - Q'(x) \sin \alpha) \sin(\sigma x + \alpha) \geq 0, \quad \alpha, x \in \mathbb{R}. \quad (3.3)
\]
Assume that the identities $E\left(\frac{k\pi - \beta}{\sigma}\right) = 0$ hold for some $\beta \in \mathbb{R}$ and all $k \in \mathbb{Z}$. Then inequality $\text{(3.3)}$ becomes identity in $x \in \mathbb{R}$ for $\alpha = \beta$ and, hence, $P(x) \cos \beta - Q(x) \sin \beta \equiv \gamma \sin(\sigma x + \beta)$, $x \in \mathbb{R}$, for some constant $\gamma \in \mathbb{R}$. Let, for example, $\cos \beta \neq 0$. Then expressing $P$ in terms of $Q$ and substituting it into (1.1) for $E$ we get the identity $E(x) \cos \beta \equiv f_1(x) \sin(\sigma x + \beta)$, where $f_1(x) := \gamma \cos \sigma x + Q(x)$. Since $E(x) \geq 0$ for all $x \in \mathbb{R}$, all real zeros of the function $E$ have even multiplicity. Hence, $f_1\left(\frac{k\pi - \beta}{\sigma}\right) = 0$ for all $k \in \mathbb{Z}$. Applying formula (3.1) to the function $f_1$, we get the identity $\gamma \cos \sigma x + Q(x) \equiv c_1 \sin(\sigma x + \beta)$ for some constant $c_1 \in \mathbb{R}$. Setting $c := \frac{c_1}{\cos \beta}$ we get identities (1.3). In a similar way, we can consider the case $\sin \beta \neq 0$. One can directly check that inequality (1.2) becomes identity if (1.3) holds and, in this case, we get $d(x) \equiv \gamma^2 \sigma$.

Assume now that, for all $\alpha \in \mathbb{R}$, $E(x) \not\equiv \cos^2(\sigma x + \alpha)$. Then for any $\alpha \in \mathbb{R}$ there exists $k_0 \in \mathbb{Z}$ such that $E\left(\frac{k_0\pi - \alpha}{\sigma}\right) > 0$. In this case, inequality (3.3) is strict for all $x \neq \frac{k\pi - \alpha}{\sigma}$, $k \in \mathbb{Z}$. Hence,
\[
\text{inequality (3.3) becomes identity for some } x = x_0 \in \mathbb{R} \quad \text{and} \quad \alpha = \alpha_0 \in \mathbb{R} \iff x_0 = \frac{k_0\pi - \alpha_0}{\sigma} \quad \text{and} \quad E(x_0) = 0 \quad \text{for some} \quad k_0 \in \mathbb{Z}. \quad (3.4)
\]

Let
\[
A_1(x) := \sigma P(x) \cos \sigma x - P'(x) \sin \sigma x, \\
A_2(x) := \sigma Q(x) \sin \sigma x + Q'(x) \cos \sigma x, \\
A_3(x) := \sigma (P(x) \sin \sigma x + Q(x) \cos \sigma x) + P'(x) \cos \sigma x - Q'(x) \sin \sigma x.
\]
Then inequality (3.3) is equivalent to the inequality
\[
A_1(x) + A_2(x) + (A_1(x) - A_2(x)) \cos 2\alpha - A_3(x) \sin 2\alpha \geq 0, \quad \alpha, x \in \mathbb{R}. \quad (3.5)
\]
Inequality (3.5) with two parameters is equivalent to the following inequality with one parameter:
\[
\sqrt{(A_1(x) - A_2(x))^2 + A_3^2(x)} \leq A_1(x) + A_2(x), \quad x \in \mathbb{R}. \quad (3.6)
\]
Here, see (3.3).

inequality (3.6) becomes identity for some $x = x_0 \in \mathbb{R} \iff$

inequality (3.3) becomes identity for $x = x_0 \in \mathbb{R}$ and some $\alpha = \alpha_0 \in \mathbb{R} \iff E(x_0) = 0$. 

Since \( A_1(x) \geq 0 \) for all \( x \in \mathbb{R} \) (this is inequality (3.3) with \( \alpha = 0 \)) and \( A_2(x) \geq 0 \) for all \( x \in \mathbb{R} \) (this is inequality (3.5) with \( \alpha = \frac{\pi}{2} \)), inequality (3.6) is equivalent to the inequality

\[
A_3^2(x) \leq 4A_1(x)A_2(x), \quad x \in \mathbb{R}.
\] (3.7)

Inequality (3.7) is equivalent to inequality (1.2). This is implied by the following identity:

\[
A_3^2(x) - \left\{ (\sigma P(x) + Q'(x)) \sin \sigma x + (P'(x) - \sigma Q(x)) \cos \sigma x \right\}^2
\equiv 4A_1(x)A_2(x) - 4\sigma (P(x)Q'(x) - P'(x)Q(x)) .
\]

Hence, if \( \tau = 0 \), statements 1), 2) iii) \( \Rightarrow \) iv) \( \Rightarrow \) i) and 3) are proved. The general case is reduced to the previous one by considering the functions \( P_1(x) := P(x - \frac{\tau}{2}) \) and \( Q_1(x) := Q(x - \frac{\tau}{2}) \). Then \( E_1(x) := P_1(x) \cos \sigma x + Q_1(x) \sin \sigma x = E(x - \frac{\tau}{2}) \geq 0, x \in \mathbb{R} \).

Let us prove the remaining statements in 2). The implication ii) \( \Rightarrow \) iii) is clear. Let inequality (1.2) become an identity. Assume that, for any \( \beta \in \mathbb{R} \) and \( c \geq 0 \), \( E(x) \not\equiv c \sin^2(\sigma x + \tau + \beta) \) and, hence, \( E(x) \not\equiv 0 \). But it follows from statement 3) that \( E(x) \equiv 0 \). This contradiction proves the implication i) \( \Rightarrow \) ii). Theorem 1.1 is proved. \( \square \)

Remark 3.1. It can be seen from the above proof that the following converse proposition also holds. Let \( \omega(z) = P(z) + iQ(z) \), where \( P, Q \) are real functions of the class \( E_\alpha \), \( \sigma > 0 \), and \( \omega(x) = o(x) \), \( x \to \pm \infty \), on the real axis. If for some \( \tau \in \mathbb{R} \) and all \( x \in \mathbb{R} \) the identity (1.2) holds and, moreover, the inequalities \( A_1(x) := \sigma P(x) \cos(\sigma x + \tau) - P'(x) \sin(\sigma x + \tau) \geq 0 \) and \( A_2(x) := \sigma Q(x) \sin(\sigma x + \tau) + Q'(x) \cos(\sigma x + \tau) \geq 0 \) hold for all \( x \in \mathbb{R} \), then the inequality \( E(x) := P(x) \cos(\sigma x + \tau) + Q(x) \sin(\sigma x + \tau) \geq 0 \) holds for all \( x \in \mathbb{R} \) also holds.

Proposition 3.1. Let the conditions of Theorem 1.1 be satisfied and \( H_\alpha(z) := P(z) \cos \alpha - Q(z) \sin \alpha \). Then the following conditions are equivalent:

i) the function \( \omega \) is real up to a constant factor;

ii) \( d(x) \equiv 0 \);

iii) for some \( c \geq 0 \), \( \beta \in \mathbb{R} \), the identity \( \omega(x) \equiv ce^{i(\frac{\pi}{2} - \beta)} \sin(\sigma x + \tau + \beta) \) holds;

iv) for some \( \alpha \in \mathbb{R} \), we have \( H_\alpha(x) \equiv 0 \).

Proof. Let us prove the implication i) \( \Rightarrow \) ii). Let \( \omega(z) \equiv e^{i\beta} \omega_0(z) \), where \( \omega_0 \) is real and \( \beta \in \mathbb{R} \). Then \( P(x) = \omega_0(x) \cos \beta \), \( Q(x) = \omega_0(x) \sin \beta \) and, clearly, \( d(x) \equiv 0 \).
ii) ⇒ iii). Let \(d(x) \equiv 0\). Then inequality (1.2) becomes identity. It follows from Theorem 1.1 that, for some \(c \geq 0\) and \(\gamma \in \mathbb{R}\), identities (1.3) hold and, moreover, \(d(x) \equiv \gamma^2 \sigma\). Hence, \(\gamma = 0\) and so \(\omega(x) \equiv ce^{i(\frac{x}{\pi} - \beta)} \sin(\sigma x + \tau + \beta)\).

iii) ⇒ iv). Let the identity \(\omega(x) \equiv ce^{i(\frac{x}{\pi} - \beta)} \sin(\sigma x + \tau + \beta)\) hold for some \(c \geq 0\), \(\beta \in \mathbb{R}\). Then \(P(x) = c \sin \beta \sin(\sigma x + \tau + \beta), Q(x) = c \cos \beta \sin(\sigma x + \tau + \beta)\) and, hence, \(H_\alpha(x) = c \sin(\sigma x + \tau + \beta)(\sin \beta \cos \alpha - \cos \beta \sin \alpha) \equiv 0\) for \(\alpha = \beta\).

iv) ⇒ i). Let, for some \(\alpha \in \mathbb{R}\), \(H_\alpha(x) \equiv 0\). Then \(P(x) \cos \alpha - Q(x) \sin \alpha \equiv 0\). Hence, either \(Q(x) \equiv \lambda P(x)\), or \(P(x) \equiv \lambda Q(x)\) for some \(\lambda \in \mathbb{R}\). In any case, \(\omega\) is a real function up to a constant factor. Proposition 3.1 is proved.

Proposition 3.2. Let the conditions of Theorem 1.1 be satisfied, \(H_\alpha(z) := P(z) \cos \alpha - Q(z) \sin \alpha\), and assume that the function \(\omega\) is not real up to a constant multiple. Then we have the following.

i) \(H_\alpha\), for any \(\alpha \in \mathbb{R}\), is a real function of the class \(E_\sigma\), \(H_\alpha \not\equiv 0\), \(H_\alpha(x) = o(x), x \to \pm \infty\), on the real axis, and \((-1)^pH_\alpha(\frac{p\pi - \alpha - \tau}{\sigma}) = E(\frac{p\pi - \alpha - \tau}{\sigma}) \geq 0, p \in \mathbb{Z}\).

ii) For any \(\alpha \in \mathbb{R}\), the function \(H_\alpha\) has infinitely many zeros and all of them are real, \(xH_\alpha(x) \neq o(1), x \to \pm \infty\), on the real axis. In every interval \(I_p := (\lambda_{p-1}, \lambda_p)\), where \(\lambda_p = \lambda_p(\alpha) := \frac{p\pi - \alpha - \tau}{\sigma}\), the function \(H_\alpha\) can have only one zero, and it is simple. Moreover, if \(x_0 \in I_p\) and \(H_\alpha(x_0) = 0\), then \((-1)^pH_\alpha'(x_0) > 0\). If, for some \(p \in \mathbb{Z}\), the number \(\lambda_p\) is a zero of the function \(H_\alpha\), then its multiplicity does not exceed 2, and one of the intervals \(I_p\) or \(I_{p+1}\) does not contain zeros of the function \(H_\alpha\). If the number \(\lambda_p\) is a zero of the function \(H_\alpha\) of multiplicity 2, then \((-1)^pH_\alpha^{(2)}(\lambda_p) < 0\) and \((-1)^pH_\alpha(x) < 0\) for \(x \in I_p \cup I_{p+1}\), and the numbers \(\lambda_{p-1}\) and \(\lambda_{p+1}\) could only be simple zeros.

iii) We have \(\omega \in HB\).

iv) If the function \(\omega\) has real zeros, then they are simple.

v) \(d(x_0) = 0\) for some \(x_0 \in \mathbb{R}\) if and only if \(\omega(x_0) = 0\) if and only if \(P(x_0) = Q(x_0) = 0\). If a number \(x_0 \in \mathbb{R}\) is a zero of the function \(\omega\), then the number \(x_0\) is a zero of the function \(d\) of multiplicity 2.

vi) If a number \(x_0 \in \mathbb{R}\) is a multiplicity 2 zero of the function \(H_\alpha\), then \(\omega(x_0) = 0\).

vii) If \(E(x) > 0, x \in \mathbb{R}\), then \(\omega \in HB\), and all zeros of the function \(H_\alpha\), \(\alpha \in \mathbb{R}\), are simple.
Proof. Assume that the function $\omega$ is not real up to a constant factor. Then $H_\alpha(x) \neq 0$ for any $\alpha \in \mathbb{R}$ (Proposition 3.1). The remaining part of statement i) is clear. Statement ii) follows from statement i) and Theorem [E].

Theorem E [21]. Theorem 1 for $F(z) := \sin(\sigma z + \beta)$, $\beta \in \mathbb{R}$, $\lambda_k = \frac{k \sigma - \beta}{\sigma}$, $n = 0$ [3]. Let a function $f$ satisfy the following conditions: a) $f$ is a real entire function of exponential type $\leq \sigma$, $\sigma > 0$, $f \neq 0$, and $f(x) = o(x)$, $x \to \pm \infty$ on the real axis; b) for some $\beta \in \mathbb{R}$ and $\forall k \in \mathbb{Z}$, we have $(-1)^k f(\lambda_k) \geq 0$, where $\lambda_k := \frac{k \sigma - \beta}{\sigma}$. Then the following holds.

1) In every interval $I_p := (\lambda_{p-1}, \lambda_p)$, $p \in \mathbb{Z}$, there may exist only one zero of the function $f$ and if there is one, then it is simple. Moreover, if $x_0 \in I_p$ and $f(x_0) = 0$, then $(-1)^p f'(x_0) > 0$.\footnote{For a smaller class of functions $f$, this theorem was proved in author’s work [20].}

2) $xf(x) \neq o(1)$, $x \to \pm \infty$.

3) The function $f$ has only real zeros.

4) If for some $p \in \mathbb{Z}$, the number $\lambda_p$ is a zero of the function $f$, then its multiplicity is not greater than 2, and one of the intervals $I_p$ or $I_{p+1}$ does not contain zeros of the function $f$. If the number $\lambda_p$ is a zero of multiplicity 2, then $(-1)^p f^{(2)}(\lambda_p) < 0$ and $(-1)^p f(x) < 0$ for $x \in I_p \cup I_{p+1}$, and the numbers $\lambda_{p-1}$ and $\lambda_{p+1}$ can only be simple zeros.

Let us prove statement iii). It follows from inequality (122) and Proposition [3.1] that $d(x_0) > 0$ for some $x_0 \in \mathbb{R}$. Statement ii) implies that, for any $\alpha \in \mathbb{R}$, the function $H_\alpha$ does not have zeros in $\mathbb{C} \setminus \mathbb{R}$. Since, by the condition, the function $\omega$ is not trivial, by Theorem [B], $\omega \in \mathbb{H}$.

Let us prove statement iv). Assume that for some $x_0 \in \mathbb{R}$, we have $\omega(x_0) = \omega'(x_0) = 0$. Then $P(x_0) = Q(x_0) = P'(x_0) = Q'(x_0) = 0$. Thus, for any $\alpha \in \mathbb{R}$, the number $x = x_0$ is a zero of the function $H_\alpha$, and its multiplicity is not less than 2. Statement ii) implies that $x_0 \in \{\frac{p \sigma - \alpha - \tau}{\sigma} : p \in \mathbb{Z}\} \cap \{\frac{p \sigma - \delta - \tau}{\sigma} : p \in \mathbb{Z}\}$, $\alpha, \delta \in \mathbb{R}$, however, for $\alpha = \delta - \frac{\tau}{2}$, this intersection is empty.

Let us prove statements v) and vi). As was proved in iii), $\omega \in \mathbb{H}$. But then, we can apply Proposition [2.1] and use statement iv).

Now we prove vii). It is clear that all real zeros of the function $\omega$ are zeros of the function $E$. Hence, if $E(x) > 0$, $x \in \mathbb{R}$, then the function $\omega$ does not have real zeros, and all zeros of the function $H_\alpha$, $\alpha \in \mathbb{R}$, are real, see statement ii), and simple, see Proposition [3.2].
Proposition 3.3. Assume that conditions of Theorem 1.1 hold. Then, for any \( n \in \mathbb{N} \), the function \( \omega^{(n)}(z) = P^{(n)}(z) + i Q^{(n)}(z) \) satisfies conditions of Theorem 1.1 for \( \tau_n = \tau + \frac{\pi n}{2} \), that is, \( P^{(n)}, Q^{(n)} \) are real functions of class \( E_\sigma \), \( \omega^{(n)}(x) = o(x) \), \( x \to \pm \infty \), on the real axis, and

\[
E_n(x) := P^{(n)}(x) \cos \left( \sigma x + \tau + \frac{\pi n}{2} \right) + Q^{(n)}(x) \sin \left( \sigma x + \tau + \frac{\pi n}{2} \right) \geq 0, \quad x \in \mathbb{R}.
\]

Moreover,

i) \( E_n(x_0) = 0 \) for some \( x_0 \in \mathbb{R} \) \( \iff \) \( E(x) \equiv c \sin^2 \left( \sigma x + \frac{\pi n}{2} - \sigma x_0 \right) \) for some \( c \geq 0 \). In this case, inequality (1.2) becomes identity for \( \omega^{(n)}(z) \), and \( d_n(x) := P^{(n)}(x)Q^{(n+1)}(x) - P^{(n+1)}(x)Q^{(n)}(x) \equiv \gamma^2 \sigma^{2n+1} \), where \( \gamma \) is in (1.3) with \( \beta = \frac{\pi n}{2} - \tau - \sigma x_0 \).

ii) \( \omega^{(n)} \) is a real function up to a constant factor \( \iff \omega \) is real up to a constant factor.

Proof. Consider the case \( n = 1 \). It was proved in [21 §1] that \( P', Q' \in E_\sigma \) and \( \omega'(x) = o(x) \), \( x \to \pm \infty \). If we take \( \alpha = \frac{\pi}{2} - \sigma x \) in (3.2) and then replace \( x \) with \( x + \frac{\pi}{2} \), we get the inequality

\[
E_1(x) = \sigma \sum_{k=-\infty}^{+\infty} \frac{E \left( \frac{k\pi - \frac{\pi}{2} + \sigma x}{\sigma} \right)}{\left( \frac{\pi}{2} - k\pi \right)^2} \geq 0, \quad x \in \mathbb{R}.
\]

It follows at once from this inequality and Theorem 1.1 statement 2) that i) holds.

Let us prove ii). Let \( \omega' \) be real up to a constant factor. It follows from Proposition 3.1 applied to \( \omega' \) that, for some \( c \geq 0 \), \( \beta \in \mathbb{R} \), we have the identity \( \omega'(x) \equiv ce^{i\left( \frac{\pi}{2} - \beta \right)} \sin(\sigma x + \tau + \frac{\pi}{2} + \beta) \) and, hence, \( \omega(x) \equiv c^{-1}e^{i\left( \frac{\pi}{2} - \beta \right)} \sin(\sigma x + \tau + \beta) + A + iB \), where \( A, B \in \mathbb{R} \). Thus, \( E(x) \equiv c^{-1} \sin^2(\sigma x + \tau + \beta) + A \cos(\sigma x + \tau) + B \sin(\sigma x + \tau) \geq 0 \), \( x \in \mathbb{R} \). Lemma 2.1 statement 7 gives that \( A = B = 0 \) and, hence, by Proposition 3.1 used for \( \omega, \omega \) is real up to a constant. The converse is clear.

Proposition 3.3 has been proved for \( n = 1 \). The general case is proved by induction on \( n \in \mathbb{N} \).

Proof of Theorem 1.2 Statements 1) and 2) are proved in Proposition 3.2.

Let us prove 3). Assume that a function \( \omega \) is not real up to a constant. It follows from Propositions 3.3 and 3.2 that \( \omega^{(n)} \in \mathbb{H}B \). Let us show
that the function \( \omega^{(n)} \) does not have real zeros. If for any \( \alpha \in \mathbb{R} \) and 
\( c \geq 0, E(x) \neq c \sin^2(\sigma x + \tau + \alpha) \), we have (see Proposition 3.3) that 
\( E_n(x) > 0, x \in \mathbb{R} \), and, hence (see Proposition 3.2 vii applied to \( \omega^{(n)} \)), the function \( \omega^{(n)} \) does not have real zeros.

Let, for some \( c \geq 0, \beta \in \mathbb{R} \), the identity \( E(x) \equiv c \sin^2(\sigma x + \tau + \beta) \) hold. Then, for some \( \gamma \in \mathbb{R} \) we have (1.3). Assume that \( \omega^{(n)}(x_0) = 0 \) for some \( x_0 \in \mathbb{R} \). Then

\[
P^{(n)}(x_0) = c\sigma^n \sin \beta \sin \left( \sigma x_0 + \tau + \beta + \frac{\pi n}{2} \right) \\
+ \gamma \sigma^n \sin \left( \sigma x_0 + \tau + \frac{\pi n}{2} \right) = 0,
\]

\[
Q^{(n)}(x_0) = c\sigma^n \cos \beta \sin \left( \sigma x_0 + \tau + \beta + \frac{\pi n}{2} \right) \\
- \gamma \sigma^n \cos \left( \sigma x_0 + \tau + \frac{\pi n}{2} \right) = 0.
\]

Hence,

\[
P^{(n)}(x_0) \cos(\sigma x_0 + \tau + \frac{\pi n}{2}) + Q^{(n)}(x_0) \sin(\sigma x_0 + \tau + \frac{\pi n}{2}) \\
= c\sigma^n \sin^2(\sigma x_0 + \tau + \beta + \frac{\pi n}{2}) = 0,
\]

and so \( \gamma = 0 \). Then \( \omega \) is real up to a constant factor, which contradicts the condition. Statement 3) is proved.

\[\square\]

**Remark 3.2.** If conditions of Theorem 1.1 are fulfilled and \( \omega(z) \neq 0 \), it follows from Propositions 3.1 and 3.2 ii) that \( x \omega(x) \neq o(1), x \to \pm \infty \), on the real axis. Moreover, Theorems 1.1 and 1.2 cease to hold if the condition \( \omega(x) = o(x), x \to \pm \infty \), is replaced with the condition

\( \omega(x) = O(x), x \to \pm \infty \). This is easily seen by considering the function

\( \omega(z) := \sin z + az \cos z + \frac{i}{2} (az \sin z + 1 - \cos z) \), where \( -1 < a < -\frac{1}{2} \).

It is clear that \( \omega(z) = O(x) \) for \( x \to \pm \infty \), and inequality (1.1) holds for \( \tau = -\frac{\pi}{2}, \sigma = 1 \); in this case, \( E(x) = 1 - \cos x \). It is easy to check that \( d(x) = a^2 x^2 + ax \sin x + (a + 1)(1 - \cos x) \). Since \( d(x) \sim a^2 x^2 \) for \( x \to \pm \infty \) and \( d(x) \sim x^2 (a + 1)(a + \frac{1}{2}) \) for \( x \to 0 \), the function \( d \) does not preserve the sign on the real axis and, hence, \( \omega(z) \not\in \overline{BB} \); in the opposite case, Theorem 1.3 implies that \( d(x) \geq 0, x \in \mathbb{R} \). In the case

Theorem 1.2 is applied, we can consider the function \( \omega(z) := z F(z) \), where the function \( F \) is of the form (1.4) and satisfies the conditions of Theorem 1.4 (statement 2). Clearly, \( \omega(x) = O(x) \) for \( x \to \pm \infty \), and inequality (1.1) holds for \( \tau = -\frac{\pi}{2} \) (in this case, \( E(x) = xS(x) \geq 0, x \in \mathbb{R} \)), but \( \omega(z) \not\in \overline{BB} \), since the function \( F \) has one zero in the open lower half-plane.
4 Proofs of Theorems 1.3 and 1.4

In this section, we consider the functions defined by (1.4), (1.5), (1.6), or (4.4).

**Corollary 4.1.** Let \( \mu \) be a real function with bounded variation on the segment \([0, \sigma]\), and let one of the four conditions hold, (1.1) (4.2), (1.3), or (4.4).

\[
C(x) \geq 0, \quad x \in \mathbb{R} \quad \text{and} \quad n = 0, \quad \tau = 0, \quad (4.1)
\]

\[
S(x) \geq 0, \quad x > 0, \quad F(x) = o(1), \quad x \to \pm \infty \quad \text{and} \quad n = 1, \quad \tau = -\frac{\pi}{2}, \quad (4.2)
\]

\[
S(x) \geq 0, \quad x > 0, \quad F(0) = 0 \quad \text{and} \quad n = -1, \quad \tau = -\frac{\pi}{2}, \quad (4.3)
\]

\[
\exists \tau_0 \in \mathbb{R} : C(x) \cos \tau_0 - S(x) \sin \tau_0 \geq 0, \quad x \in \mathbb{R} \quad \text{and} \quad n = 0, \quad \tau = \pm \tau_0. \quad (4.4)
\]

Let \( \omega(x) := z^n F(z) \equiv P(z) + iQ(z) \), where \( P(z) \equiv z^n G(z) \), \( Q(z) \equiv z^n H(z) \). Then we have the following.

1) The function \( \omega \) satisfies the conditions of Theorem 1.1, that is, 1) \( P, Q \) are real functions of class \( E_\sigma \) and \( \omega(x) = o(x), \ x \to \pm \infty, \ \text{on the real axis}; \ 2) \) for a corresponding value of \( \tau \), \( E(x) := P(x) \cos(\sigma x + \tau) + Q(x) \sin(\sigma x + \tau) \equiv x^n (C(x) \cos \tau - S(x) \sin \tau) \geq 0, \ x \in \mathbb{R}, \ \text{and, hence, for all} \ x \in \mathbb{R}, \)

\[
4\sigma d(x) \geq x^{2n-2} D(x), \quad (4.5)
\]

where

\[
D(x) := \{(2\sigma x S(x) + xC'(x) + nC(x)) \cos \tau + (2\sigma x C(x) - xS'(x) - nS(x)) \sin \tau\}^2
\]

and \( d(x) := P(x) Q'(x) - P'(x) Q(x) \equiv x^{2n} \Delta(x) \).

2) Inequality (4.5) becomes identity \( \iff E(x) \equiv 0 \iff C(x) \cos \tau \equiv S(x) \sin \tau \equiv 0 \). In this case, \( F(z) \equiv \Delta(z) \equiv 0 \) if \( \cos \tau \neq 0 \), and \( F(z) \equiv c e^{i\sigma z}, \ \Delta(z) \equiv c^2 \sigma, \ c \in \mathbb{R} \) if \( \cos \tau = 0 \).

3) Inequality (4.5) becomes identity for some \( x = x_0 \in \mathbb{R} \iff E(x_0) = 0 \).

4) If \( F(z) \not\equiv 0 \), then the function \( \omega \) is not real up to a constant multiple and, hence, Theorem 1.2 and Proposition 3.2 hold for the functions \( \omega, P, Q, E, d, \ \text{and} \ H_\sigma(z) := P(z) \cos \alpha - Q(z) \sin \alpha \equiv z^n h_\sigma(z) \).

5) If condition (4.1) is satisfied and, in addition, \( F(z) \not\equiv 0, \ S(x) \geq 0 \ for \ x \not> 0, \ F(0) > 0 \), then the function \( F \) does not have real zeros.

6) If condition (4.2) is satisfied and, additionally, \( F(z) \not\equiv 0 \), then \( F'(0) > 0, \ H'(0) > 0, \ and \ \Delta(0) > 0 \).
7) If condition (4.4) is satisfied and, additionally, \( F(z) \neq 0, \sin \tau_0 \neq 0, \) then the function \( F \) does not have real zeros.

Proof. Statement 1) follows at once from Theorem 1.1 statement 1, if identities (2.2) and (2.3) are used. The first part in statement 2) immediately follows from Theorem 1.1 statement 2, and Lemma 2.1 statement 3. The second part of this statement follows from the first one and Lemma 2.1 statement 1. Statement 3) follows from statement 2) and Theorem 1.1 statement 3. Statement 4) immediately follows from Lemma 2.1 statement 6.

Let us now prove statement 5). If \( x_0 \in \mathbb{R} \) and \( F(x_0) = 0 \), then \( x_0 \neq 0 \) and \( C(x_0) = S(x_0) = 0 \). Hence, \( C'(x_0) = S'(x_0) = 0 \) and so, \( F'(x_0) = 0 \), which contradicts statement iv) in Proposition 3.2 about simplicity of real roots of the function \( \omega(z) \equiv F(z) \). Statement 5) is proved.

Now we prove statement 6). Let us first apply statement ii) of Proposition 3.2 for \( \alpha = 0 \). Then \( \lambda_\alpha = \frac{p\pi}{\sigma} \). The number \( x_0 = 0 \in I_0 = (-\frac{\pi}{2}, \frac{\pi}{2}) \) and it is a zero of the function \( H_0(x) = xG(x) \). Hence, \( H_0'(0) > 0 \). It remains to use that \( H_0'(0) = G(0) = F(0) \). Now apply statement ii) of Proposition 3.2 for \( \alpha = -\frac{p}{2} \). Then \( \lambda_\alpha = \frac{(p+1)\pi}{\sigma} \). The number \( \lambda_{-1} = 0 \) is a zero of the function \( H_{-\frac{\pi}{2}}(x) = xH(x) \) of multiplicity not less than 2. Thus, \( H_{-\frac{\pi}{2}}'(0) > 0 \). It remains to use that \( H_{-\frac{\pi}{2}}''(0) = 2H'(0) \) and \( \Delta(0) = G(0)H'(0) \). Statement 6) is proved.

Now, consider statement 7). Since \( C(x) \) is even and \( S(x) \) is odd, inequality (4.4) is equivalent to the inequality \( |S(x)\sin \tau_0| \leq C(x)\cos \tau_0 \), \( x \in \mathbb{R} \). If \( x_0 \in \mathbb{R} \) and \( F(x_0) = 0 \), then \( C(x_0) = S(x_0) = 0 \) and \( \cos \tau_0 \neq 0 \); otherwise \( S(x) \equiv 0 \) and \( F(z) \equiv ce^{iz}, c \neq 0 \), but this function does not have zeros. Hence, \( C'(x_0) = S'(x_0) = 0 \) and, so, \( F'(x_0) = 0 \), which contradicts statement iv) of Proposition 3.2 on simplicity of real roots of the function \( \omega(z) \equiv F(z) \). □

Proof of Theorem 1.3. Theorem 1.3 immediately follows from Corollary 4.1 statement 4). Let us give another proof that the function \( F \) does not have zeros in the open lower half-plane. Let \( z = x + iy, x \in \mathbb{R}, \)

---

4The inequality \( H'(0) > 0 \) also follows from Lemma 2.2.
\[ y < 0, \text{ and } h(t) := -\frac{y}{\pi(y^2 + t^2)}, \ t \in \mathbb{R}. \text{ Then, see (2.7),} \]

\[ F(z)e^{-i\sigma z} = -\int_0^\sigma e^{-ixu}e^{yu} \, d\mu(\sigma - u) \]

\[ = -\int_0^\sigma e^{-ixu} \left( \int_{-\infty}^{+\infty} e^{itu} h(t) \, dt \right) \, d\mu(\sigma - u) \]

\[ = -\int_{-\infty}^{+\infty} h(t) \left( \int_0^\sigma e^{i(t-x)u} \, d\mu(\sigma - u) \right) \, dt \]

\[ = \int_{-\infty}^{+\infty} h(t+x)(C(t) + iS(t)) \, dt. \]

If \( C(x) \geq 0 \) for all \( x \in \mathbb{R} \) and \( F(z) \neq 0 \), then \( C(x) \neq 0 \) and, hence,

\[ \Re \left( F(z)e^{-i\sigma z} \right) = -\int_{-\infty}^{+\infty} \frac{y}{\pi(y^2 + (t+x)^2)} \cdot C(t) \, dt > 0, \quad \Im z < 0. \]

**Proof of Theorem 1.4.** Let \( S(x) \geq 0 \) for all \( x > 0 \). Then \( \mu(\sigma - 0) \geq \mu(0) \), by Lemma 2.2 and \( \mu(\sigma - 0) - \mu(0) = 0 \iff S(x) \equiv 0 \iff F(z) \equiv ce^{i\sigma z}, \ c \in \mathbb{R}. \) In the case under consideration, \( F(z) \neq 0 \). Hence, if \( S(x) \equiv 0 \), then the function \( F \) does not have zeros. Let \( S(x) \neq 0 \). Then, see the proof of Theorem 1.3, for \( z = x + iy, x \in \mathbb{R}, y < 0, \) and \( h(t) := -\frac{y}{\pi(y^2 + t^2)}, \ t \in \mathbb{R}, \) we have

\[ \Im \left( F(z)e^{-i\sigma z} \right) = \int_0^{+\infty} (h(x+t) - h(x-t))S(t) \, dt = \]

\[ \int_0^{+\infty} \frac{4xyt}{\pi(y^2 + (t+x)^2)(y^2 + (t-x)^2)} \cdot S(t) \, dt \neq 0, \quad \Im z < 0, \ \Re z \neq 0. \]

If \( x < 0 \) or \( x > 0 \), then the latter integral is positive or negative, respectively. Hence, in the half-plane \( \Im z < 0 \), the function \( F \) does not have
zeros for Re $z \neq 0$. Consider the case Re $z = 0$. If $f$ is a function from Lemma 2.2, then

$$g(y) := F(iy)e^{\sigma y} = -\int_0^\sigma e^{uy} d\mu(\sigma - u) = F(0) + y \int_0^\sigma e^{yu} f(u) du,$$

$$g'(y) = \int_0^\sigma e^{yu}(1 + yu) f(u) du.$$

It follows from Lemma 2.4 that $g(-\infty) := \lim_{y \to -\infty} g(y) = \mu(\sigma) - \mu(\sigma - 0) = F(0) - (\mu(\sigma - 0) - \mu(0))$. If $y < 0$, then Lemma 2.3 ($\beta = \alpha = -y > 0$) implies that $g'(y) > 0$ and, hence, the function $g$ is strictly increasing on $(-\infty, 0]$. Hence, $g(-\infty) < g(y) < g(0) = F(0)$ for all $y < 0$. This inequality, Proposition 2.2 and Lemma 2.1 statement 6) yield statement 1). Statement 2) follows since the function $g$ is strictly monotone on $(-\infty, 0]$.

Let us now prove the statement on multiplicity of real roots of the function $F$. If for some $x_0 \in \mathbb{R}$, $x_0 \neq 0$, the identities $F(x_0) = F'(x_0) = 0$ hold, it would follow from (2.1) that the numbers $z(x)$ are zeros of the functions $G$, $H$, $C$, $S$, and their multiplicities are not less than 2. The number $\alpha \in \mathbb{R}$ is chosen in such a way that $\frac{\sigma - \alpha + \frac{\pi}{2}}{\sigma} \neq \pm x_0$ for all $p \in \mathbb{Z}$, which is equivalent to the inequality $\cos(\alpha \pm \sigma x_0) \neq 0$. It follows from (2.4) and Lemma 2.1 statement 6) that the function $f(z) := \frac{z h_\alpha(z)}{(z^2 - x_0^2)^2}$ satisfies the conditions of Theorem E (for $\beta = \alpha - \frac{\pi}{2}$) and, hence, $xf(x) \neq o(1)$, $x \to \pm \infty$, which is clearly not the case.

Let $F(0) = 0$. It follows from (2.4) and Lemma 2.1 statement 6) that the function $f(z) := h_\alpha(z)$ satisfies the conditions of Theorem E (for $\beta = -\frac{\pi}{2}$) and the number $x_0 = 0 \in I_0 = (-\frac{\pi}{2}, \frac{\pi}{2})$ is a zero of $f$. Hence, $f'(0) > 0$. It remains to take into account that $F''(0) = h_0''(0) = 2f'(0)$. Theorem 1.4 is proved.

**Corollary 4.2.** Let $\mu$ be a real function with bounded variation on the interval $[0, \sigma]$.

1) If $F(z) \neq 0$ and one of the two conditions holds, $C(x) \geq 0$ for $x \in \mathbb{R}$ or $S(x) \geq 0$ for $x > 0$, $F(0) \in (-\infty, 0] \cup [\mu(\sigma - 0) - \mu(0), +\infty)$, then we have the following.

i) $\Delta(x) \geq 0$, $x \in \mathbb{R}$, and $\Delta(x_0) = 0$ for some $x_0 \in \mathbb{R} \iff F(x_0) = 0$. If a number $x_0 \in \mathbb{R}$ is a zero of multiplicity $p$ of the function $F$, then the number $x_0$ is a zero of the function $\Delta$ of multiplicity $2p$.

ii) For all $\alpha \in \mathbb{R}$, the function $h_\alpha$ has an infinite number of zeros and all of them are real. If the number $x_0 \in \mathbb{R}$ is a zero of the function $h_\alpha$ of
1) The function $h_\alpha$ is real and, for all $p \in \mathbb{Z}$, we have the inequalities $(-1)^p H_\alpha (\lambda_p) = E(\lambda_p) \geq 0$, where $\lambda_p = \lambda_p (\alpha) := \frac{p \pi - \alpha + \tau}{\pi}$, $H_\alpha (x) := x h_\alpha (x)$ and $E(x) := x S(x)$. Moreover, $h_\alpha (x) \neq 0$, $x \geq 0$, and the function $h_\alpha$ has infinitely many real zeros.

2) If $F(0) \in (-\infty, 0] \cup [\mu (\sigma - 0) - \mu (0), +\infty)$, then the function $h_\alpha$ does not have zeros in $\mathbb{C} \setminus \mathbb{R}$.

3) If $F(0) \in (0, \mu (\sigma - 0) - \mu (0))$, then, for some $\alpha \in \mathbb{R}$, the function $h_\alpha$ has complex (not real) zeros.

Proof. Let us prove 1). It is clear that $h_\alpha$ is real, and (2.4) shows that the needed inequalities hold. This immediately implies that the function $h_\alpha$ is real and has infinitely many real zeros. This follows from the inequality, see (2.4),

$$(-1)^p h_\alpha \left( \frac{p \pi - \alpha + \tau}{\sigma} \right) = C \left( \frac{p \pi - \alpha + \tau}{\sigma} \right) \cos \tau - S \left( \frac{p \pi - \alpha + \tau}{\sigma} \right) \sin \tau \geq 0,$$

which holds for all integers $p \geq \frac{\alpha + \tau}{\pi}$. Here $\tau = 0$ or $\tau = -\frac{\pi}{2}$ in the first or the second case, correspondingly. It follows from Theorems 1.3, 1.4 that the function $F \in \mathbb{H}\mathbb{B}$ is not trivial. So, we need to use Proposition 2.1.

Let us prove statement 2). The function $\omega(z) := \frac{F(z)}{z + \xi}$ is an entire exponential type function that does not have zeros in the open half-plane $\text{Im } z < 0$, and its deficiency is $d_\omega = d_F > 0$. Theorem C implies that $\omega \in \mathbb{H}\mathbb{B}$, and Lemma 2.4 statement 8, shows that $\omega$ is not trivial. Hence, we can apply Proposition 2.1 to the function $\omega$. It should be taken into account that all real zeros of the function $F$, if the exist, are simple, Theorem 1.4, $(x^2 + \xi^2) d(x) \equiv \Delta_\xi (x)$ and $\omega (x_0) = 0$ for some $x_0 \in \mathbb{R} \iff F(x_0) = 0$. \qed

**Proposition 4.1.** Let $\mu$ be a real function with bounded variation on the segment $[0, \sigma]$, $S(x) \geq 0$ for all $x > 0$ and $F(z) \neq 0$. Let $\alpha \in \mathbb{R}$. Then we have the following.

1) The function $h_\alpha$ is real and, for all $p \in \mathbb{Z}$, we have the inequalities $(-1)^p H_\alpha (\lambda_p) = E(\lambda_p) \geq 0$, where $\lambda_p = \lambda_p (\alpha) := \frac{p \pi - \alpha + \tau}{\pi}$, $H_\alpha (x) := x h_\alpha (x)$ and $E(x) := x S(x)$. Moreover, $h_\alpha (x) \neq 0$, $x \geq 0$, and the function $h_\alpha$ has infinitely many real zeros.

2) If $F(0) \in (-\infty, 0] \cup [\mu (\sigma - 0) - \mu (0), +\infty)$, then the function $h_\alpha$ does not have zeros in $\mathbb{C} \setminus \mathbb{R}$.

3) If $F(0) \in (0, \mu (\sigma - 0) - \mu (0))$, then, for some $\alpha \in \mathbb{R}$, the function $h_\alpha$ has complex (not real) zeros.

Proof. Let us prove 1). It is clear that $h_\alpha$ is real, and (2.4) shows that the needed inequalities hold. This immediately implies that the function
\( h_\alpha \) has infinitely many real zeros. Lemma 2.1 statement 6) implies that \( h_\alpha(x) \not\equiv 0 \). If \( x^2 h_\alpha(x) = o(1) \) for \( x \to \pm \infty \), then the function \( H_\alpha \) satisfies the conditions of Theorem E and, hence, \( xH_\alpha(x) = x^2 h_\alpha(x) \not\equiv o(1), x \to \pm \infty \), which contradicts the assumption.

Statement 2) is contained in Corollary 4.2.

Let us prove 3). Assume that for all \( \alpha \in \mathbb{R} \), the function \( h_\alpha \) does not have zeros in \( \mathbb{C} \setminus \mathbb{R} \). Since \( F(z) \not\equiv 0 \), by Lemma 2.1, statement 6), the function \( F \) is not real up to a constant multiple. Hence, \( d(x) \not\equiv 0 \) and, so, \( d(x_0) \not\equiv 0 \) for some \( x_0 \in \mathbb{R} \). If \( d(x_0) > 0 \), then it follows from Theorem E that \( F \in \mathcal{T}\mathcal{T}\mathcal{B} \) and, hence, the function \( F \) does not have zeros in the open half-plane \( \text{Im } z < 0 \), which contradicts Theorem 1.4 statement 2). Hence, \( d(x_0) < 0 \). Then, by Theorem E it follows that \( F(z) = F(\bar{z}) \in \mathcal{T}\mathcal{T}\mathcal{B} \) and, hence, the function \( F \) does not have zeros in the open upper half-plane \( \text{Im } z > 0 \). Thus, all zeros of the function \( F \), save for one (see statement 2 in Theorem 1.4), are real and there is an infinite number of them. In this case, see [14, Corollary 1], if for some \( \delta \in (0, \sigma) \) there exists the limit

\[
\lim_{x \to +\infty} \left| \frac{\int_0^\delta e^{-xu} d\mu(u)}{\int_0^{\sigma-x} e^{-xu} d\mu(u)} \right| = a,
\]

then \( a = 1 \). In the case under consideration, this limit exists and equals \( \left| \frac{\mu(\sigma) - \mu(0)}{\mu(\sigma - 0) - \mu(0)} \right| \), which follows from Lemma 2.4 and inequality \( F(0) = \mu(\sigma) - \mu(0) \not\equiv \mu(\sigma - 0) - \mu(0) \). Lemma 2.2 statement 1) implies that \( \mu(+0) = \mu(0) \). Hence, \( a = 0 \). This contradiction proves statement 3). 

5 Examples

**Example 5.1** (See also [12, 20, 21]). Let a function \( \mu \) be absolutely continuous on \([0, \sigma] \), that is, \( d\mu(t) = g(t)dt \), where \( g \in L(0, \sigma) \). Assume that the function \( g \) is nonnegative, nondecreasing, and \( g(t) \not\equiv 0 \) on \((0, \sigma) \). It is known that, in the considered case, \( S(x) = \int_0^\sigma g(\sigma - t) \sin xt \, dt \geq 0 \) for all \( x > 0 \). The following proof of the inequality is due to R. M. Trigub. For an arbitrary fixed \( x > 0 \), set \( G(u) := 0 \) for \( u > \sigma x \) and \( G(u) := g(\sigma - \frac{u}{x}) \) for \( 0 \leq u \leq \sigma x \). Then the function \( G \) is nonnegative, does not increase on \((0, +\infty) \). It is clear that for all \( p \in \mathbb{Z}_+ \) and \( u \in [2p\pi, 2(p+1)\pi] \),
we have \( G_p(u) := (G(u) - G(2p\pi + \pi)) \sin u \geq 0 \). Thus
\[
xS(x) = \int_0^{\sigma x} g\left(\sigma - \frac{u}{x}\right) \sin u \, du = \int_0^{+\infty} G(u) \sin u \, du = \sum_{p=0}^{+\infty} \frac{2(p+1)\pi}{2p\pi} \int G_p(u) \, du \geq 0.
\]

In this case, conditions (4.22) are satisfied and \( F(z) \neq 0 \). Hence, all zeros of the function \( F \) lie in the closed half-plane \( \mathop{\text{Im}} z \geq 0 \), and zeros of \( F' \) belong to the open half-plane \( \mathop{\text{Im}} z > 0 \). If \( S(x) > 0 \) for all \( x > 0 \), then it is clear that the function \( F \) does not have real zeros. From the latter inequality, it immediately follows that \( S(x_0) = 0 \) for some \( x_0 > 0 \) \iff for some \( \beta \in [0, \sigma) \) the function \( g \) is piece-wise constant on \( (\beta, \sigma) \) with equidistant nodes, that is, the interval \( (\beta, \sigma) \) can be subdivided into a finite number of intervals of equal length \( d > 0 \) such that the function \( g \) is constant on each of them, and \( g(t) \equiv 0 \) on \( (0, \beta) \) if \( \beta > 0 \); here we can always assume that \( g(\beta - 0) > 0 \). Let \( S(x_0) = 0 \) for some \( x_0 > 0 \). Then
\[
F(z) = \frac{e^{iz} - 1}{iz} \cdot e^{i\beta z} F_1(z), \quad \text{where} \quad F_1(z) := \sum_{p=1}^{m} c_p e^{i(p-1)z}
\]
and \( m = \frac{\sigma - \beta}{d} \in \mathbb{N} \), \( 0 < c_1 \leq \cdots \leq c_m \).

In this case, the function \( F \) has an infinite number of real zeros \( z_k = \frac{2\pi k}{d} \), \( k \in \mathbb{Z} \), \( k \neq 0 \). Since all zeros of the function \( F \) lie in the half-plane \( \mathop{\text{Im}} z \geq 0 \), we see that, for \( m \geq 2 \), all zeros of the function \( F_1 \) lie on a finite number of the lines \( \mathop{\text{Im}} z = c \geq 0 \), the number of which is not greater than \( m - 1 \), and each of them contains infinitely many zeros of \( F_1 \), and its real zeros, if they exist, are simple. This is equivalent to that all zeros of the algebraic polynomial \( P(w) := c_1 + c_2 w + \cdots + c_m w^{m-1} \) lie in the closed disk \( |w| \leq 1 \), and if there are zeros on the circle \( |w| = 1 \), then they are simple. This is a well-known fact.

**Example 5.2.** Let \( F(z) := \sum_{k=0}^{m} c_k e^{i\lambda_k z} \neq 0 \), where \( c_k \in \mathbb{R} \) and \( 0 = \lambda_0 < \lambda_1 < \cdots < \lambda_m = \sigma \). Then \( F(z) = \int_0^{\sigma} e^{itz} d\mu(t) \), where \( \mu \) is a step function that has jumps in the points \( t = \lambda_k \). In this case, \( C(x) = \sum_{k=0}^{m} c_k \cos(\lambda_m - \lambda_k)x \). Let \( C(x) \geq 0 \) for all \( x \in \mathbb{R} \). Then conditions of Corollaries 4.1 and 4.2 are satisfied and, hence, the function \( F \) does not have zeros in the half-plane \( \mathop{\text{Im}} z < 0 \), and inequality (4.5), in this case,
becomes

\[ 4\lambda_m \sum_{k,j=0}^{m} c_k c_j \lambda_j \cos(\lambda_k - \lambda_j)x \geq \left( \sum_{k=0}^{m} c_k (\lambda_m + \lambda_k) \sin(\lambda_m - \lambda_k)x \right)^2, \]

\[ x \in \mathbb{R}. \]

This inequality becomes equality for some \( x = x_0 \in \mathbb{R} \iff C(x_0) = 0. \)

If \( f \) is an even, continuous function, positive definite on \( \mathbb{R} \), and \( f(x) = 0 \) for \( |x| \geq m \), then \( f(0) + 2 \sum_{k=1}^{m} f(k) \cos kx = \sum_{k=-m}^{m} f(k) e^{ikx} \geq 0, \)

\( x \in \mathbb{R}, \) for a proof of this statement due to R. M. Trigub, see [22]. If we take \( \lambda_k = k \) for \( 0 \leq k \leq m \) and \( c_k = f(m - k), \) \( 0 \leq k < m, \) \( c_m = \frac{f(0)}{2}, \)

then \( C(x) \geq 0 \) for all \( x \in \mathbb{R}. \) One can take, for example, the function \( f(x) = (1 - (\frac{|x|}{m})^\delta)^\delta, \) where \( 0 < \lambda \leq 1, \) \( \delta \geq 1. \)

**Example 5.3.** Let a function \( \mu \) be real, absolutely continuous on \([0,\sigma],\)

and \( d\mu(t) = g(t)dt, \) where \( g \in C[0,\sigma], \) \( g(0) = 0, \) \( g(\sigma) > 0, \) and the function \( g((\sigma - |t|)_+) \) is positive definite on \( \mathbb{R}. \) Then \( C(x) = \int_{0}^{\sigma} g(\sigma - t) \cos xt \, dt \geq 0 \) for all \( x \in \mathbb{R}. \) In this case, conditions of Corollaries 4.1 and 4.2 are satisfied.

The following proposition gives a relation between functions of the class \( \mathcal{PHB} \) of the form \([14]\) and positive definite function.

**Proposition 5.1.** Let \( g \in L(0,\sigma) \) and be real, and an even function \( h \) be defined by the identities \( h(x) := 0 \) for \( |x| \geq \sigma \) and \( h(x) := \int_{|x|}^{\sigma} (2u - |x|) g(u) g(u - |x|) \, du, \) \( |x| < \sigma. \) Assume that the function \( F(z) := \int_{0}^{\sigma} e^{izt} g(t) \, dt \) does not have zeros in the lower half-plane \( \text{Im} z < 0. \)

Then the following holds.

1) \( h \in L(\mathbb{R}), \) and the function \( F \in \mathcal{PHB} \) is not trivial.

2) The Fourier transform satisfies \( \hat{h}(x) \geq 0 \) for all \( x \in \mathbb{R} \iff F(x_0) = 0. \) If a number \( x_0 \in \mathbb{R} \) is zero of the function \( F \) of multiplicity \( p, \) then \( x_0 \) is a zero of function \( \hat{h} \) of multiplicity \( 2p. \)

3) If, additionally, \( g \in L_2(0,\sigma), \) then the function \( h \) is continuous and positive definite on \( \mathbb{R}. \)

**Proof.** Let us prove 1). If \( g \) is continued by zero to \((\sigma, +\infty),\) is easy to show that

\[ 2h(x) = \int_{-\infty}^{+\infty} g(|u|) g(|x-u|) (x-u) \left( \text{sign}(x-u) - \text{sign} u \right) \, du. \]
Since the convolution of two functions in \( L(\mathbb{R}) \) is a function in \( L(\mathbb{R}) \), we have \( h \in L(\mathbb{R}) \). Using the connection between Fourier transform and convolution, we get the identity \( \hat{h}(x) = 2\Delta(x) \). Here the function \( \Delta \) is defined by (1.3) and (1.5) in which \( d\mu(t) = g(t)dt \). If \( F \) does not have zeros in the half-plane \( \text{Im} z < 0 \), it follows from Proposition 2.2 that \( F \in \mathcal{H} \). Statement 2 in Lemma 2.1 shows that \( F \) is not trivial for, otherwise, \( F(z) \equiv F(+\infty) = 0 \) that contradicts the condition.

Statement 2) follows from statement 1) and Proposition 2.1.

Let us prove 3). If, in addition, \( g \in L_2(0, \sigma) \), then it is clear that \( h \in C(\mathbb{R}) \). As was proved, \( h \in L(\mathbb{R}) \) and \( \hat{h}(x) \geq 0 \) for all \( x \in \mathbb{R} \). Hence, the function \( h \) is positive definite on \( \mathbb{R} \).

Example 5.4. As an example, consider the function \( g(t) := g_{\mu,\nu}(t) = t^{\mu-1}(1-t^2)^{\nu-1} \) in Proposition 5.1 for \( \sigma = 1 \). If \( \mu \geq 1, 0 < \nu \leq 1 \), and \((\mu, \nu) \neq (1, 1)\), then \( F(0) > 0 \) and \( S(x) > 0 \) for all \( x > 0 \), see, Example 5.1 and hence the function \( F \) does not real zeros. Hence, \( \hat{h}_{\mu,\nu}(x) > 0 \) for all \( x \in \mathbb{R} \). This shows, see [23, identity (44)], that for the indicated \( \mu \) and \( \nu \), the function \( f(x) = x^{-\mu}(1+x^2)^{-\nu} \) is completely monotone on \((0, +\infty)\), that is, \((-1)^nf^{(n)}(x) > 0 \) for all \( n \in \mathbb{Z}_+ \) and \( x > 0 \), which is the main result in [10].

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