Is the Universe Flat?

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Abstract

Geometry of the universe has always intrigued mathematicians and cosmologists. Recent results from the Wilkinson Microwave Anisotropy Project (WMAP) indicate that the visible universe is incredibly flat. This apparent flatness could be due to the fact that only a small part of the universe is visible, thus indicating that the geometry of the universe is still an open and an interesting problem in cosmology. Assuming a profound connection between Friedmann, Robertson, Walker (FRW) geometries and thermodynamics, we construct a parameter free exploratory model that allows us to predict the geometry of the universe by thermodynamic arguments. The key parameters in this model are the concept of global equation of state and the concept of gravitational temperature. By comparing the equal time expansion of the Green function for the massless conformal scalar field in background FRW geometry with the thermal Green function in Minkowski space-time, we define the gravitational temperature. We also give the protocol for determining the global equation of state for a given local equation of state. Using a local equation of state that covers a wide range of physically acceptable cases, \( P = \alpha \rho, \alpha > 0 \), and within the context of FRW thermodynamics, we predict that the geometry of the visible universe is Lobachevskian (open), albeit being very close to flat. This is consistent with the WMAP data, which indicates that the universe may deviate from flatness by as much as 1%. We also discuss the self-consistency of this suggestive model along with its possible connections with nonextensive thermodynamics and black hole thermodynamics.
I. INTRODUCTION

A profound connection between geometry and thermodynamics was first hinted in 1970s by the works of Bekenstein [1] and Hawking [2] with the black hole thermodynamics. Later, the discovery of black hole radiation by Hawking established the physical link between the area of the horizon with entropy and the surface gravity of the black hole with temperature. However, despite the success of black hole thermodynamics, the relation between general theory of relativity and thermodynamics remains to be established for more general metrics.

In 1995, assuming extensivity of entropy, Jacobson [3] obtained Einstein’s field equations from the proportionality of entropy and the area of the horizon, and the relation

$$\delta Q = TdS,$$

(1)

where $Q$ is heat, $S$ is entropy, and $T$ is the temperature. Jacobson’s treatment hinges on the fact that causal horizons hide information, hence should be associated with entropy, where heat is defined as the energy that flows through the causal horizon that continues to interact with the outside gravitationally. Recently, assuming that the holographic principle holds, and building on the works of Jacobson [3] and Padmanabhan [4], Verlinde [5] proposed a framework for gravity as an entropic force via the assumption that holographic principal holds. On the nonextensive thermodynamics side, recently Tsallis and Cirto [6] argued that gravitating systems should be associated with an appropriate nonextensive generalization of the additive expression

$$S = k_B \ln W.$$

(2)

The fact that horizons hide information motivates their association with entropy, but due to the universality of gravity and the principle of equivalence, we can also expect a more general relation between gravity and entropy, independent of the existence of a horizon. This follows from the fact that all one can say about the source of a gravitational field is its total energy-momentum distribution. Since there are many different ways to generate the same energy-momentum distribution, the gravitational field itself embodies a certain amount of information. Even though classically the number of possible sources that could produce a given gravitational field is infinity, we expect quantum mechanics to give a finite number.

On the thermodynamic side of this captivating connection between gravitation and thermodynamics, we revisit the suggestive model introduced in 1986 as the Friedmann ther-
modynamics [7-10] and give new derivations of its key components like the gravitational temperature and the global equation of state. By comparing the equal time expansion of the Green function of the massless conformal scalar field in background FRW geometry with the thermal Green function in Minkowski space-time, we give a new derivation of the gravitational temperature. Using this temperature within the context of Friedmann thermodynamics, which we also reestablish, we argue that the geometry of the visible universe is Lobachevskian. In this regard, the lack of any measurable curvature in current observations means that the universe must be much larger than what is visible to us. We also discuss limitations and the logical consistency of this model, along with its connections with nonextensive thermodynamics.

II. GEOMETRY AND EINSTEIN COSMOLOGY IN A NUTSHELL

The large scale homogeneity and isotropy of the universe suggest that the universe at large can be described by one of the FRW models, which are represented by the line element

\[ ds^2 = g_{\mu\nu}dx^\mu dx^\nu, \quad \mu, \nu = 0, 1, 2, 3, \]

\[ = dt^2 - e^{g(t)} \left[ \frac{1}{1 - k\frac{r^2}{R_0}} dr^2 + r^2 d^2 \Omega \right], \quad k = 0, \pm 1, \quad (3) \]

where \( g_{\mu\nu} \) is the FRW metric that gives a complete description of the local space-time geometry. In FRW models the spatial universe is described by the constant time slices of the line element. For \( k = 0 \) the universe is flat and the geometry is Euclidean, for \( k = 1 \) the geometry is Gaussian and the universe is the three dimensional surface of a four dimensional hypersphere with time dependent radius \( R(t) = R_0 e^{g(t)/2} \), for \( k = -1 \) the geometry is Lobachevskian, where the universe can be considered as the surface of a hypersphere with imaginary radius \( R(t) = i R_0 e^{g(t)/2} \). The FRW models are in general time dependent, where \( R_0 e^{g(t)/2} \) is the scale factor. We will take the initial conditions so that \( R_0 \) is the present radius of the universe. One of the trademarks of a given geometry is the sum of the interior angles of triangles, which for the Euclidean geometry is always equal to \( \pi \) and greater than \( \pi \) for the Gaussian, and less than \( \pi \) and for the Lobachevskian geometries.

Even though geometry is a directly observable feature of the universe through angle, distance, area, etc. measurements, cosmic scales make this rather impractical. On the other
hand, Einstein’s field equations:

\[ G^{\mu\nu} = -8\pi T^{\mu\nu}, \quad c = G = 1, \]  

which relate the space-time geometry to the matter content of the universe, offer a much more practical alternative. The left hand side of the Einstein’s field equations, \( G^{\mu\nu} \), is called the Einstein tensor, which is a purely geometric quantity entirely composed of the metric tensor and its first two derivatives. The right hand side, \( T^{\mu\nu} \), is the energy-momentum tensor, which describes the matter content of the universe. In this work, we treat the cosmological constant as a part of the energy-momentum tensor, hence do not write it explicitly.

For perfect fluids the energy-momentum tensor is given as

\[ T^{\mu\nu} = (P_0 + \rho_0)U^\mu U^\nu - P_0 g^{\mu\nu}, \]  

where in comoving coordinates we set \( U^1 = U^2 = U^3 = 0 \). With the FRW metric [Eq. (3)], the field equations reduce to two equations in three unknowns, \( P_0, \rho_0, R \), as

\[ \ddot{R}(t) = -4\pi(\rho_0 + 3P_0)\frac{R}{3}, \]

\[ \left( \frac{3}{8\pi R^2} \right) k = \left( \rho_0 - \frac{3H(t)^2}{8\pi} \right). \]  

In these equations, \( P_0 \) is the pressure, \( \rho_0 \) is the mass density and \( H(t) = \frac{\dot{R}}{R} \) is the Hubble parameter of the universe. Supplemented with the necessary information about the matter content of the universe, that is, an equation of state, \( P_0 = P_0(\rho_0) \), the above system of equations can be solved for a given geometry, that is, for a given value of \( k \), to yield the solution as \( \{ P_0(t), \rho_0(t), R(t) \} \).

Except for exotic forms of matter \( (\rho_0 + 3P_0) \) is positive, hence the first field equation [Eq. (6)] indicates that the universe is decelerating. Since \( k \) is a constant, we can set the values of the quantities in Equation (7) to their current values, thereby reducing the problem of determining the geometry of the universe to observing the difference between the current density and the critical density, that is, the sign of

\[ k = \frac{\rho_0(t_{\text{now}}) - \rho_c}{|\rho_0(t_{\text{now}}) - \rho_c|}, \]  

where, \( \rho_c = \frac{3H_0^2}{8\pi} \) is called the critical density and \( H_0 \) is the Hubble constant \( H(t_{\text{now}}) \). Now
the geometry of the universe is

\[ k = \begin{cases} 
1 & \text{for } \rho(0) > \rho_c ; \text{ Gaussian} \\
0 & \text{for } \rho(0) = \rho_c ; \text{ Euclidean} \\
-1 & \text{for } \rho(0) < \rho_c ; \text{ Lobachevskian} 
\end{cases} \]  

(9)

From the redshift measurements of galaxies, the current value of the Hubble parameter indicates that \( \rho_c \) is around \( 10^{-29} \text{gm/cc} \). On the other hand, the dynamic mass measurements indicate that the amount of luminous matter plus the dark matter in the universe only adds up to around 30% of the critical density, which from Equation (9) indicates that we live in a Lobachevskian universe. Just when all such data pointed to a Lobachevskian universe, the WMAP satellite measurements of the anisotropy of the cosmic microwave background radiation surprised cosmologists by yielding a \( k \) value very close to zero. This also meant that the current density of the universe is very close to the critical density, thus indicating that almost 70% of the matter content of the universe is yet to be accounted for.

Another surprise came with the discovery of the acceleration of the universe, which allowed us to interpret this missing 70% in terms of some exotic form of matter called the dark energy. Even though the physics of dark energy is still unknown, we can say that it has positive mass and responds to gravity with repulsion. This naturally demands a radical change in our understanding of the contents of the universe. Considering that a flat universe demands a precisely tuned current density to the critical density, which is a tall order for any observation, it is only fair to say that the geometry of the universe will still continue to captivate cosmologists for years to come. In fact, the recent data from the WMAP indicate that the total density differs from the critical density by as much as 1% or so [11].

Since FRW models with different \( k \) values correspond to different spatial distributions of galaxies, we address this problem by investigating the connection between geometry and thermodynamics. In thermodynamics, a particular crystal structure, that is, a particular spatial distribution of atoms, becomes the stable form of matter with respect to the entropy criteria. In this regard, we think that it may be possible to predict the geometry of the universe via some thermodynamic arguments. A profound connection between geometry and thermodynamics was first hinted by the black hole thermodynamics [1,2,12]. We now further the suggestive (toy) model we introduced in 1986 as the Friedmann thermodynamics and clarify some of its critical concepts [7-10], which has some definite predictions about
the geometry of the universe. In Section III, we introduce the concept of global equation of state, which plays a central role in this model. In Section IV, we give a derivation of the gravitational temperature for the FRW geometries. In Section V, we introduce the local equation of state concept and discuss the assumptions involved. In Section VI, for a given local equation of state, we discuss how to find the global equation of state and then how to determine the geometry of the universe. In Section VII, using a local equation of state that covers a wide range of physically acceptable cases, we argue that with respect to the Friedmann thermodynamics, the geometry of the universe is Lobachevskian. Finally, in Section VIII, we discuss our results.

III. THE GLOBAL EQUATION OF STATE

Einstein once said that the left hand side of the field equations is as solid as the rock of Gibraltar but the right hand side is like a house built from a deck of cards [Eq. (4)]. The left hand side of the Einstein’s field equations, $G^{\mu\nu}$, is a unique divergenceless tensor, entirely composed of the metric tensor and its first two derivatives. On the other hand, the right hand side is the energy-momentum tensor, $T^{\mu\nu}$, which represents the matter content of the universe. The problem is due to the fact that matter not only affects geometry but also gets affected by it, hence $T^{\mu\nu}$ is also a function of the unknown, that is, the metric tensor. Unlike the special theory of relativity, where one can obtain the energy-momentum tensor in a moving frame from the rest frame expression by a Lorentz transformation, it is not possible to generate the curved space-time energy-momentum tensor, $T^{\mu\nu}(g^{\mu\nu})$, from its flat (Minkowski) space-time expression by a general coordinate transformation. In flat space-time, that is, in a free fall frame, the perfect fluid energy-momentum tensor is written as

$$T_0^{\mu\nu} = (P_0 + \rho_0)u^\mu u^\nu - P_0 \eta^{\mu\nu}, \quad (10)$$

where $P_0$ and $\rho_0$ are the pressure and the density distributions, respectively. In principle, there are infinitely many expressions that one could write for the curved space-time energy momentum tensor. Of course, all of them reducing to Equation (10) in the limit $g^{\mu\nu} \to \eta^{\mu\nu}$. For the energy-momentum tensor in curved space-time, the common practice is to replace the Minkowski metric in $T_0^{\mu\nu}$ with the general curved space-time metric, $g^{\mu\nu}$, and express $P_0$
and \( \rho_0 \) in curved space-time coordinates to write

\[
T^{\mu\nu} = (P_0 + \rho_0)u^\mu u^{\nu} - P_0 g^{\mu\nu}.
\]  

(11)

Since \( P_0 \) and \( \rho_0 \) are defined \textit{locally}, their values at a given point in curved space-time coordinates do not change, thus the equation of state in curved space-time is still taken as the flat space-time equation of state, that is, as \( P_0 = P_0(\rho_0) \).

Friedmann thermodynamics is based on the observation that different crystal structures of matter correspond to different spatial distributions of atoms over lattices with discrete symmetry properties. At a given temperature, what defines the stable phase is the crystal structure that has the lowest Gibbs energy. Analogously, the FRW models correspond to different spatial distributions of galaxies, where the space over which the galaxies are distributed is curved over a hyperspace with continuous but nevertheless with distinct symmetry properties called the Bianchi symmetries.

Phase transitions are due to collective behavior of matter, hence they are governed by the entropy criteria rather than the energy. For example, tin has two allotropic forms at normal pressure as white and gray tin, where white tin exists in stable form at 298 K, a form that should be unstable according to the energy criteria [13]. Similarly, we can expect to determine the geometry of the universe via thermodynamic arguments, but the conventional energy-momentum tensor [Eq. (11)] with a given flat space-time equation of state, \( P_0 = P_0(\rho_0) \), yields isentropic (or isothermal) models that all have the same Gibbs energy. In other words, with the conventional approach, all FRW models have the same Gibbs energy, \( \rho_0 = \rho_0(P_0) \), which leaves \( k \) as a parameter that can only be determined by observation.

In 1902 Gibbs [14] pointed out that systems with long range interactions like gravity, are in principle beyond the scope of standard (Boltzmann-Gibbs) statistical mechanics. In line with Gibbs, we now argue that thermodynamic systems that are large enough for the effects of curvature to be important, are endowed with a \textit{gravitational temperature} and a \textit{gravitational entropy}, and define the curved space-time energy-momentum tensor as

\[
T^{\mu\nu} = (P + \rho)u^\mu u^{\nu} - Pg^{\mu\nu},
\]  

(12)

where \( P(\rho, T) \) and \( \rho(P,T) \) are the \textit{global pressure} and \textit{density} distributions, respectively, and \( T \) is the \textit{gravitational temperature}. In this approach, the vacuum field equations remain
intact, but the right hand side of the field equations is modified profoundly so that the effect of curvature on the energy-momentum tensor is included not only explicitly through the replacement $\eta^{\mu\nu} \rightarrow g^{\mu\nu}$, but also implicitly through the global equation of state $P = P(\rho, T)$. As we will show, compared to other modifications of the Einstein’s field equations, this is a rather intricate change that allows us to predict the geometry of the universe. Note that the new energy-momentum tensor, as it should, in the Minkowski limit, $g^{\mu\nu} \rightarrow \eta^{\mu\nu}$, reduces to $T^{\mu\nu}_0$, where $P \rightarrow P_0$ and $\rho \rightarrow \rho_0$.

IV. GRAVITATIONAL TEMPERATURE

To define a gravitational temperature for the FRW geometries, we use the massless conformal scalar field in curved background FRW geometry. The corresponding vacuum Wightman functions, $G^{(+)\beta}_{\{0\}}(x, x') = \langle 0 | \phi(x)\phi(x') | 0 \rangle$, in static FRW universes are given as [15]

$$G^{(+)\beta}_{\{0\}}(x, x') = -\frac{1}{4\pi^2 \Delta t^2 - \Delta \rho^2},$$

$$G^{(+)\beta}_{\{-1\}}(x, x') = \frac{\Delta \chi}{4\pi^2 R^2 \sinh(\Delta \chi)} \left[ \Delta \chi^2 - (\Delta \eta - i\epsilon)^2 \right], \quad \sinh \chi = \frac{r}{R}, \quad \chi \in [0, \infty],$$

where $R$ is the constant scale factor and $\eta$ is the conformal time.

To find the gravitational temperature, we use the thermal Green function of the massless conformal scalar field obeying Maxwell-Boltzmann statistics, which can be written as an infinite sum over imaginary time as [15]

$$G^{(1)}_{\beta}(x, x') = \sum_{n=-\infty}^{\infty} G^{(1)}(t + in\beta, t', \tau', \tau), \quad \beta = \frac{1}{T}, \quad k_B = \hbar = 1,$$

where $G^{(1)} = \langle 0 | \{\phi(x), \phi(x')\} | 0 \rangle$ is the zero temperature Green function.

Using the zero temperature Green function of the massless conformal scalar field in Minkowski space-time:

$$M G^{(1)}(x, x') = -\left( \frac{2}{4\pi^2} \right) \frac{1}{\Delta t^2 - \Delta \rho^2}, \quad \Delta t = t - t', \quad \Delta \rho = r - r',$$

we write the corresponding thermal Green function as

$$M G^{(1)}_{\beta}(x, x') = \frac{1}{2\pi^2} \sum_{n=-\infty}^{\infty} \frac{1}{\langle \Delta t + in\beta \rangle^2 - \Delta \rho^2}.$$
Following Mamaev and Trunov [16], we now write the equal time, \( \Delta t = 0 \), expansion of \( MG^{(1)}_\beta(x, x') \) in powers of \( \Delta \rho \) as

\[
MG^{(1)}_\beta(x, x') \simeq \frac{1}{2\pi^2 \Delta \rho^2} + \frac{T^2}{6} + 0(\Delta \rho^2). \tag{19}
\]

We also write the equal time expansion of the Green function \( G^{(1)}_k = 2 \text{Re} G^{(+)}_k \), for the massless conformal scalar field in static FRW universe with \( k = 1 \) as

\[
G^{(1)}_k \simeq \frac{1}{2\pi^2 \Delta \rho^2} + \frac{1}{3(8\pi^2)R^2} + 0 \left( (\Delta \rho/R)^2 \right), \quad \Delta \rho = R \Delta \chi, \tag{20}
\]

which when compared with Equation (19) yields the gravitational temperature:

\[
T = \frac{1}{2\pi R}, \quad k = 1. \tag{21}
\]

This is in agreement with the effective gravitational temperature deduced from the Casimir effect calculations [8].

Similarly, for \( k = -1 \), we write the equal time expansion for the massless conformal scalar field as

\[
G^{(1)}_{(-1)} \simeq \frac{1}{2\pi^2 \Delta \rho^2} - \frac{1}{12\pi^2 R^2} + 0 \left( (\Delta \rho/R)^2 \right), \quad \Delta \rho = R \Delta \chi, \tag{22}
\]

which when compared with Equation (19), yields an imaginary gravitational temperature

\[
T = i \frac{\sqrt{2}}{2\pi R}, \quad k = -1. \tag{23}
\]

Note that in this case, the Casimir effect calculations give only the real part of the effective gravitational temperature, which is zero [8].

For sufficiently slowly expanding universes, where particle creation can be ignored, the gravitational temperature can be generalized simply by replacing \( R \) with its time dependent expression, \( R(t) = R_0 e^{\sigma(t)/2} \). Similarly, granted that the time dependence is sufficiently slow, the effective gravitational temperature for the time dependent flat, \( k = 0 \), FRW models is defined as zero.

\section*{A. Connection with the Unruh Temperature}

To gain a better understanding of what this gravitational temperature means, we apply this method to an accelerated particle detector coupled to a massless conformal scalar field,
where the Green function is given as [15]

\[ G^{(+)}(x,x') = -\frac{1}{16\pi^2\alpha^2} \sinh^2 \left( \frac{\tau - \tau'}{2\alpha} \right), \quad \alpha = \text{const.} \tag{24} \]

The proper time \( \tau \) is related to \( t \) as

\[ t = \alpha \sinh(\tau/\alpha), \tag{25} \]

where the detector moves along a hyperbolic trajectory with the proper acceleration \( \omega = 1/\alpha \).

Using \( G^{(1)} = 2 \text{Re} G^{(+)} \), we can write the equal time expansion as

\[ G^{(1)} = -\frac{1}{2\pi^2\Delta^2} + \frac{\omega^2}{24\pi^2} + O(\Delta^2). \tag{26} \]

Comparing this with the expansion of the thermal Green function in Equation (19) with \( t = \tau \):

\[ M^\beta G^{(1)}(x,x') \simeq -\frac{1}{2\pi^2} \left[ \frac{1}{\Delta \tau^2} - \Delta \rho^2 \right]^2 + 0(\Delta^2), \tag{27} \]

and with \( \Delta \rho = 0 \), and as \( \Delta \tau \to 0 \):

\[ M^\beta G^{(1)}(x,x') \simeq -\frac{1}{2\pi^2} \left[ \frac{1}{\Delta \tau^2} - \frac{\pi^2 T^2}{3} \right] + 0(\Delta^2), \tag{28} \]

we obtain the Unruh temperature [15]

\[ T = \frac{\omega}{2\pi}. \tag{29} \]

Note that for the uniformly accelerated detector [Eq. (24)], using the identity

\[ \csc^2 \pi x = \pi^{-2} \sum_{k=-\infty}^{\infty} (x - k)^{-2} \tag{30} \]

and the relation \( G^{(1)} = 2 \text{Re} G^{(+)} \), we can also write \( G^{(1)}(\Delta \tau) \) as

\[ G^{(1)}(\Delta \tau) = -(2\pi^2)^{-1} \sum_{k=-\infty}^{\infty} (\Delta \tau + 2\pi i\alpha k)^{-2}. \tag{31} \]

Comparing this with Equation (18) and with the substitutions \( t = \tau \) and \( \Delta \rho = 0 \):

\[ M^\beta G^{(1)}(x,x') = -\frac{1}{2\pi^2} \sum_{n=-\infty}^{\infty} \frac{1}{(\Delta \tau + in\beta)^2}, \tag{32} \]

we see that the two Green functions are actually identical to all orders for the Unruh temperature \( T = 1/2\pi\alpha \).
In other words, for the massless conformal scalar field, the vacuum Green function for the uniformly accelerating detector is identical to the thermal Green function of an inertial detector at the Unruh temperature. Since the Green functions for the FRW models, \( G^{(1)}_{(1)}, G^{(1)}_{(0)}, G^{(1)}_{(-1)} \), differ from \( M G^{(1)}_{\beta} \) to second and higher orders in \( \Delta \rho / R \), the gravitational temperature we deduce is only an effective temperature in terms of the Maxwell-Boltzmann statistics. Potential implications of this will be discussed in Section VIII.

V. THE LOCAL EQUATION OF STATE

In Friedmann thermodynamics, the flat space-time equation of state, \( P_0 = P_0(\rho_0) \), is also the local equation of state. That is, the equation of state that one would observe when the size of the thermodynamic system is sufficiently small compared to the current radius of the universe:

\[
r \ll R_0, \quad (33)
\]

where \( r \) is any point within the local system. Now, the FRW line element [Eq. (3)] can be expanded as

\[
ds^2 \simeq dt^2 - e^{g(t)} \left[ \left( 1 + k \frac{r^2}{R_0^2} + \cdots \right) dr^2 + r^2 d^2 \Omega \right], \quad k = \pm 1,
\]

and the field equations become

\[
8\pi \rho \simeq \frac{3}{R^2} \left( 1 - k \frac{r^2}{R_0^2} + \cdots \right) + \frac{3k}{R^2} \left( 1 - \frac{5}{3} \frac{k r^2}{R_0^2} + \cdots \right), \quad (35)
\]

\[
8\pi P \simeq -2 \frac{\dot{R}}{R} - \frac{\dot{R}^2}{R^2} - \frac{k}{R^2} \left( 1 - k \frac{r^2}{R_0^2} + \cdots \right), \quad (36)
\]

\[
8\pi P \simeq -2 \frac{\dot{R}}{R} - \frac{\dot{R}^2}{R^2} - \frac{k}{R^2} \left( 1 - 2k \frac{r^2}{R_0^2} + \cdots \right), \quad (37)
\]

where \( R(t) = R_0 e^{g(t)/2} \). The crucial point in the above equations is that \( R(t) \) is still exact. Granted that the size of the local thermodynamic system is sufficiently small with respect to the current radius, \( r \ll R_0 \), and also the inequalities

\[
\frac{|k|}{R^2} \ll \left( \frac{\dot{R}}{R} \right)^2 \quad \text{and} \quad \frac{|k|}{R^2} \ll \left| 2 \frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} \right|, \quad k = \pm 1, \quad (38)
\]
hold, the global pressure $P$ and the global density $\rho$, approach to their local expressions $P_0$ and $\rho_0$, hence the time dependent local field equations become

$$8\pi P \rightarrow 8\pi P_0 = -\frac{2\dot{R}}{R} - \frac{\dot{R}^2}{R^2},$$  \hspace{1cm} (39)$$

$$8\pi \rho \rightarrow 8\pi \rho_0 = \frac{3\dot{R}^2}{R^2}. $$  \hspace{1cm} (40)$$

Note that within the local thermodynamic system, the line element is

$$ds^2 \simeq dt^2 - R(t)^2 \left[ dr^2 + r^2 d^2 \Omega \right],$$  \hspace{1cm} (41)$$

where $R(t)$ satisfies the inequalities in Equation (38). Using the local equation of state, $P_0 = P_0(\rho_0)$, one can now determine $R(t)$. Of course, after $R(t)$ is found, one has to check the inequalities in Equation (38) for self-consistency.

VI. THE FIELD EQUATIONS AND THE GIBBS ENERGY

In terms of the global pressure, $P(\rho, T)$, and the global density, $\rho(\rho, T)$, the field equations [Eq. (4)] give two equations for the four unknowns, $P, \rho, T, R$, as

$$8\pi P(\rho, T) = -\frac{k}{R^2} - \frac{2\ddot{R}}{R} - \frac{\dot{R}^2}{R^2},$$  \hspace{1cm} (42)$$

$$8\pi \rho(P, T) = \frac{3k}{R^2} + \frac{3\dot{R}^2}{R^2}. $$  \hspace{1cm} (43)$$

Along with the definition of the gravitational temperature

$$T = \frac{\beta}{R}, \quad \beta = \begin{cases} 1/2\pi & , \ k = 1 \\ 0 & , \ k = 0 \\ iv\sqrt{2}/2\pi & , \ k = -1 \end{cases},$$  \hspace{1cm} (44)$$

we can complete this set with a global equation of state, that is, a relation between $P, \rho$ and $T$, and solve for the four unknowns: \{ $P(t), \rho(t), T(t), R(t)$ \}. However, the global equation of state requires a priori knowledge of the geometry of the universe, which is exactly what we are trying to find out, hence this is not a viable option.

Instead, considering that the global variables $P$ and $\rho$ depend implicitly on the local conditions described by $P_0$ and $\rho_0$, and the gravitational temperature $T$, we rewrite the field
equations [Eqs. (42) and (43)] as

\[
8\pi P(P_0, \rho_0, T) = -\frac{k}{R^2} - \frac{2\dot{R}}{R} - \frac{\dot{R}^2}{R^2},
\]

\[
8\pi \rho(P_0, \rho_0, T) = \frac{3k}{R^2} + \frac{3\dot{R}^2}{R^2}.
\]

(45)  

We now have a system of two equations with six unknowns: \(\{P, \rho, P_0, \rho_0, T, R\}\). For a solution, we have to supplement this set with four independent equations, which we take as

(i) The local equation of state \(P_0 = P_0(\rho_0)\), which is locally accessible.

(ii) The definition of gravitational temperature, which comes from QFT [Eq. (44)].

(iii–iv) The two equations that the local pressure, \(P_0\), and the local density, \(\rho_0\), satisfy [Eqs. (39, 40)]:

\[
8\pi P_0 = -\frac{2\dot{R}}{R} - \frac{\dot{R}^2}{R^2},
\]

\[
8\pi \rho_0 = \frac{3\dot{R}^2}{R^2}.
\]

(47)  

(48)  

A. Determining the Geometry of the Universe

Using the definition of the gravitational temperature [Eq. (44)], and the Equations (47) and (48), we can write the field Equations (45) and (46) as

\[
8\pi P = -\frac{kT^2}{\beta^2} + 8\pi P_0(\rho_0),
\]

\[
8\pi \rho = \frac{3kT^2}{\beta^2} + 8\pi \rho_0.
\]

(49)  

(50)  

Eliminating \(\rho_0\) among the above equations gives the Gibbs energy density, \(\rho_{(k)}(P, T)\), \(k = 0, \pm 1\), or the global equation of state, \(P_{(k)}(\rho, T)\), \(k = 0, \pm 1\), of the universe. Now the three geometries are no longer isentropic, hence we can pick the \(k\) value that has the lowest Gibbs energy as the preferred geometry of the universe with respect to the Friedmann thermodynamics. Finally, with \(k\) determined, using the local equation of state, \(P_0 = P_0(\rho_0)\), we can solve Equations (47) and (48) for \(P_0, \rho_0\) and \(R\) as functions of time to complete the solution as

\[
\{P(t), \rho(t), P_0(t), \rho_0(t), T(t), R(t)\}.
\]

(51)
VII. THE GEOMETRY OF THE UNIVERSE

To demonstrate the above protocol for determining the geometry of the universe, we consider the local equation of state $P_0 = \alpha \rho_0$, $\alpha > 0$, which covers a wide range of physically realistic cases like radiation for $\alpha = 1/3$ and dust for very small values of $\alpha$. The corresponding Gibbs energy densities, $\rho^{(k)}(P, T)$, $k = 0, \pm 1$, can now be found as

$$\rho^{(1)}(P, T) = \left( 3 + \frac{1}{\alpha} \right) \frac{\pi T^2}{2} + \frac{P}{\alpha},$$

(52)

$$\rho^{(0)}(P, T) = \frac{P}{\alpha},$$

(53)

$$\rho^{(-1)}(P, T) = -\left( 3 + \frac{1}{\alpha} \right) \frac{\pi |T|^2}{4} + \frac{P}{\alpha},$$

(54)

which for all $T$ and $P$, indicate that the model with $k = -1$ has the lowest Gibbs energy density. Hence with respect to the Friedmann thermodynamics and the local equation of state, $P_0 = \alpha \rho_0$, $\alpha > 0$, the stable geometry of the universe is Lobachevskian. Of course, all these models being locally equal, what determines their stability is their gravitational entropy. To complete the solution, we find $R(t)$ using Equations (47) and (48) as

$$R(t) = R_0 \left[ \frac{3 H_0 (\alpha + 1)}{2} (t - t_0) + 1 \right] \frac{2}{3 (\alpha + 1)},$$

(55)

where $t_0$ is the current time (age) and $R_0$ is the current radius.

Finally, we check the inequalities in Equation (55) for the self-consistency of the model, which indicate that the current radius of the universe has to be much larger than the visible universe:

$$R_0 \gg c/3 \alpha H_0.$$

(56)

Now the global equation of state is determined [Eq. (54)], the consistency of the model can also be confirmed by the first approach by supplementing the system of equations (42-43) with the global equation of state [Eq. (54)]. It should be noted that our arguments about the geometry are local and does not say anything about the global topology of the universe, which is an other issue that the Friedmann thermodynamics at this point can not answer. This prediction is consistent with the WMAP data, which indicates that the universe may deviate from flatness by as much as 1%. Also, recently Liddle and Cortes [11] argued that the observed large-scale asymmetry of the microwave background radiation may
be explained by assuming an open (Lobachevskian) universe slightly curved just beyond the cosmic horizon.

VIII. CONCLUSIONS

The effective gravitational temperature we give is in terms of the Maxwell-Boltzmann statistics and valid for regions large enough for the effects of curvature to be important but yet small enough compared to the current radius/size of the universe. We also showed that this method reproduces the correct expressions for the Unruh temperature for the uniformly accelerated reference frames and also the Hawking temperature for the black holes [15]. However, for the accelerated observers and the black holes the Green function expansions match to all orders, thus indicating that in Friedmann thermodynamics the Maxwell-Boltzmann statistics is only approximately true. In this regard, the gravitational temperature we give is at best an effective temperature.

Using the concept of local thermodynamic equilibrium, we have suggested a generalization of our gravitational temperature to arbitrary space-times [8]. For the Schwarzschild geometry, in the black hole limit, that is, when the surface of the star approaches to its horizon, this temperature reduces precisely to the Hawking temperature. This supports the view that black holes are essentially the equilibrium state of any self gravitating system. In this regard, the underlying statistical mechanics of the Friedmann thermodynamics can not be the Maxwell-Boltzmann statistics. But considering that the black hole thermodynamics obeys Maxwell-Boltzmann statistics, the attractor of the Friedmann thermostatistics should be the Maxwell-Boltzmann statistics.

An alternative to Maxwell-Boltzmann statistics is given by Tsallis et. al. [17] and Tsallis [18], where the entropy is defined as

\[ S_q = k \left( 1 - \sum_{i=1}^{W} p_i^q \right) / (q - 1). \]  

(57)

In the limit as \( q \to 1 \), the Tsallis entropy, \( S_q \), reduces to the Boltzmann entropy

\[ S = -k \sum_{i=1}^{W} p_i \ln p_i. \]  

(58)

With equal probability assumption, \( S_q \) can be written as

\[ S_q = k \frac{(W^{1-q} - 1)}{(1 - q)}. \]  

(59)
When $S_q$ is maximized with the condition that the variance is finite and the total probability is 1, the probability of the $i$th state with the energy $\epsilon_i$ being occupied becomes

$$ p_i \propto \left[ 1 - (1 - q) \frac{\epsilon_i}{kT} \right]^{1/(1-q)}. \quad (60) $$

In the limit as $q \to 1$, this reduces to the Boltzmann weight factor

$$ p_i \propto e^{-\epsilon_i/kT}. \quad (61) $$

An important feature of the Tsallis thermostatistics is that when two independent systems with entropies $S_q(A)$ and $S_q(B)$ are combined, the total entropy is given as

$$ S_q(A + B) = S_q(A) + S_q(B) + (1 - q)S_q(A)S_q(B). \quad (62) $$

This reduces to the sum of the individual entropies only when $q = 1$. The Tsallis thermodynamics is nonextensive, where $q$ is a measure of the nonextensivity. For $q = 1$, the corresponding distribution function is a Gaussian. For general $q < 5/3$, it is not possible to give an analytic expression for the distribution function. However, from the central limit theorem, the final distribution always tends to a Gaussian after many steps. In this regard, Tsallis statistics is a possible candidate for the Friedmann thermodynamics.

In this exploratory but parameter free model, the left hand side of the Einstein’s field equations remain untouched, hence the vacuum field equations are the same. However, even though the energy-momentum tensor is conserved, geometry and matter are coupled through the global equation of state in a rather intricate way, that can not be covered by the conventional approaches. It is this profound connection that we exploit in Friedmann thermodynamics that allows us to predict the geometry of the universe [7-10].

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