ASYMPTOTIC NORMALITY OF ASSOCIATED LAH NUMBERS

WEN ZHANG
Shandong Yutai No.1 Middle School
Yutai 272300, China

LILY LI LIU
School of Mathematical Sciences
Qufu Normal University
Qufu 273165, China

(Communicated by Ana Maria Acu)

Abstract. Based on the results given by Ahuja and Enneking, we show that the generating function of the associated Lah numbers having only real zeros, and further obtain the asymptotic normality of the associated Lah numbers. As application, we get the asymptotic normality of the signless Lah numbers.

1. Introduction.
Let \( a(n, k) \) be a double-indexed sequence of nonnegative numbers and let
\[
p(n, k) = \frac{a(n, k)}{\sum_{j=0}^{n} a(n, j)}
\]
denote the normalized probabilities. Following Bender [6], we say that the sequence \( a(n, k) \) is asymptotically normal by a central limit theorem, if
\[
\lim_{n \to \infty} \sup_{x \in \mathbb{R}} \left| \sum_{k \leq \mu_{n} + x\sigma_{n}} p(n, k) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt \right| = 0,
\]
where \( \mu_{n} \) and \( \sigma_{n}^2 \) are the mean and variance of (1), respectively. We say \( a(n, k) \) is asymptotically normal by a local limit theorem on \( \mathbb{R} \) if
\[
\lim_{n \to \infty} \sup_{x \in \mathbb{R}} \left| \sigma_{n} p(n, [\mu_{n} + x\sigma_{n}]) - \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \right| = 0.
\]
In this case,
\[
a(n, k) \sim \frac{e^{-x^2/2} \sum_{j=0}^{n} a(n, j)}{\sigma_{n} \sqrt{2\pi}}, \quad \text{as } n \to \infty,
\]
where \( k = \mu_{n} + x\sigma_{n} \) and \( x = O(1) \). Clearly, the validity of (2) implies that of (3). The asymptotic normality of many combinatorial statistics has been proved. For
example, Harper [13] proved that the Stirling numbers of the second kind are asymptotically normal distributed. Godsil [11] showed that if the order of a graph is large compared to the maximum degree, then the matching numbers are asymptotically normal distributed. Recently, Wang et al. [24] obtained that the Laplacian coefficients are approximately normal distributed when the size of a graph is large compared to the maximum degree. More results related to asymptotic normality in combinatorics can be referred to [3, 6, 7, 9, 10, 14, 15, 20, 22].

For the positive integer \( s \), the associated Lah numbers \( L(n,k,s) \) defined by

\[
(sx)^{[n]} = \sum_{k=1}^{\infty} L(n,k,s)(x)_k,
\]

where \( (sx)^{[n]} = sx(sx+1)...(sx+n-1) \) and \( (x)_k = x(x-1)...(x-k+1) \). The associated Lah numbers have the following expression

\[
L(n,k,s) = \binom{n}{k!} \sum_{r=0}^{k} (-1)^{k-r} \binom{k}{r} \left( \frac{n+rs-1}{n} \right),
\]

where \( L(n,k,s) = 0 \) for \( k > n \). The exponential generating function of \( L(n,k,s) \) is

\[
\sum_{n=k}^{\infty} L(n,k,s) \frac{x^n}{n!} = \frac{(1-x)^{-s}-1}{k!},
\]

(see [1] for example). It is easy to obtain that \( L(n,k,s) \) is related to the Stirling numbers

\[
\lim_{s \to 0} \frac{L(n,k,s)}{s^k} = |s(n,k)|
\]

and

\[
\lim_{s \to \infty} \frac{L(n,k,s)}{s^n} = S(n,k),
\]

where \( |s(n,k)| \) and \( S(n,k) \) are the signless Stirling number of the first kind and the Stirling number of the second kind respectively. Ahuja and Enneking [2] obtained that \( L(n,k,s) \) satisfying the following three term recurrence

\[
L(n+1,k,s) = sL(n,k-1,s) + (sk+n)L(n,k,s)
\]

and having the strongly log-concave property, i.e,

\[
[L(n,k,s)]^2 > L(n,k+1,s)L(n,k-1,s).
\]

Especially, \( L(n,k,1) = L(n,k) \), the signless Lah numbers, which introduced by Ivan Lah in 1952. The signless Lah numbers \( L(n,k) \) enumerates such distributions with the added proviso that the balls in each urn are to be linearly ordered [11, 18]. The reader is referred to [2, 4, 8, 9, 12, 16, 21] for more information on signless Lah numbers.

In this paper, we present the asymptotic normality of the associated Lah numbers. As applications, we obtain the asymptotic normality of the signless Lah numbers.

2. Main results.

In this section, we show the polynomial \( L_n(x) = \sum_{k=0}^{n} L(n,k,s) x^k \) has only real roots, and further demonstrate the asymptotic normality of the associated Lah numbers \( L(n,k,s) \).
Theorem 2.1. Suppose that $A_n(x) = \sum_{k=0}^n a(n,k) x^k$ has only real roots, and $A_n(x) = \prod_{i=1}^n (x + r_i)$, where all the $a(n,k)$ and $r_i$ are nonnegative. Let
\[
\mu_n = \sum_{i=1}^n \frac{1}{1 + r_i}
\]
and
\[
\sigma_n^2 = \sum_{i=1}^n \frac{r_i}{(1 + r_i)^2}.
\]
Then if $\sigma_n \to +\infty$, the numbers $a(n,k)$ are asymptotically normal with the mean $\mu_n$ and variance $\sigma_n^2$.

Remark 1. ([7]) When a polynomial $A_n(x) = \sum_{n \geq 0} a(n,k) x^k$ has real and non-negative roots, we have a probabilistic interpretation of its coefficients. A bit of algebra shows that the mean and variance of random variables are given by the following “rootfree” expressions
\[
\mu_n = \frac{A'_n(1)}{A_n(1)}, \quad \sigma_n^2 = \frac{A'_n(1)}{A_n(1)} + \frac{A''_n(1)}{A_n(1)} - \left(\frac{A'_n(1)}{A_n(1)}\right)^2.
\] 

We obtain the following result.

Theorem 2.2. The polynomial sequence $L_n(x)$ satisfy the three-term recurrence:
\[
L_n(x) = (sx + n - 1)L_{n-1}(x) + sxL'_n(x), \quad L_0(x) = 1.
\] 

Proof. Multiplying $x^k$ on both side of the recurrence relation (9) of the associated Lah numbers and sum over the values of $k$, we have
\[
L_n(x) = \sum_{k=0}^n sL(n-1, k-1, s)x^k + \sum_{k=0}^n (n + sk - 1)L(n-1, k, s)x^k
\]
\[
= (sx + n - 1)L_{n-1}(x) + sxL'_n(x).
\]

Let $\text{RZ}$ denote the set of real polynomials with only real zeros. Suppose that $f$ and $g$ be two real polynomials with only real zeros and with positive leading coefficients. Denote their zeros by $r_1 \geq r_2 \geq \cdots \geq r_n$ and $s_1 \geq s_2 \geq \cdots \geq s_m$ respectively. For convenience we set that $r_i = +\infty$ for $i < 1$ and $r_i = -\infty$ for $i > n$. We say that $f(x)$ interlaces $g(x)$, denoted by $f \preceq g$, if $n \leq m \leq n + 1$ and $s_i \geq r_i \geq s_{i+1}$ for all $i$. Obviously, if $f$ has only real zeros then $f' \preceq f$.

Theorem 2.3 ([17]). Let $f$ and $g$ be a real polynomials whose leading coefficients have the same sign. Suppose that $f, g \in \text{RZ}$ and $g \preceq f$, if $ad \leq bc$, then $(ax + b)f(x) + (cx + d)g(x) \in \text{RZ}$.

The above Theorem provide the inductive basis for the reality of zeros of polynomials sequence $P_n(x)$ satisfying certain recurrence relations $P_n(x) = a_n(x)P_{n-1}(x) + b_n(x)P'_{n-1}(x)$. So we can obtain the following Theorem immediately.

Theorem 2.4. The polynomial $L_n(x)$ has only real zeros for each $n \geq 1$.

Then, we give the main result of this paper.

Theorem 2.5. The sequence $L(n, k, s)$ are asymptotically normal.
Proof. We let $\sum_{k=0}^{n} L(n, k, s) = L_{n, s}$. Applying (11) to $L_n(x) = \sum_{k=0}^{n} L(n, k, s)x^k$, we obtain

$$ \mu_n = \frac{\sum_{k=0}^{n} kL(n, k, s)}{L_{n, s}}, $$

(13)

$$ \sigma_n^2 = \frac{\sum_{k=0}^{n} k^2 L(n, k, s)}{L_{n, s}} - (\mu_n)^2. $$

(14)

According to the recurrence relation of $L(n, k, s)$, we can obtain

$$ \sum_{k=0}^{n} kL(n, k, s) = \frac{L_{n+1, s} - (n + s)L_{n, s}}{s}, $$

(15)

and

$$ \sum_{k=0}^{n} k^2 L(n, k, s) = \frac{L_{n+2, s} - (2n + 2s + 1)L_{n+1, s} + (n^2 + 2ns)L_{n, s}}{s^2}. $$

(16)

Thus, the (13) and (14) can be expressed

$$ \mu_n = \frac{L_{n+1, s}}{sL_{n, s}} \frac{(n + s)}{s}, $$

(17)

$$ \sigma_n^2 = \frac{L_{n+2, s}}{s^2L_{n, s}} \frac{L_{n+1, s}}{s^2L_{n, s}} - \left( \frac{L_{n+1, s}}{sL_{n, s}} \right)^2 - 1. $$

(18)

Since $\sum_{k=0}^{\infty} L(n, k, s) \frac{x^n}{n!} = \frac{\Gamma(1-x)}{\Gamma(1-s)}$, $k \in N$ and $\sum_{k=0}^{n} L(n, k, s) = L_{n, s}$, so we can get the exponential generating function of the $L_{n, s}$ by

$$ \sum_{n=0}^{\infty} L_{n, s} \frac{x^n}{n!} = \exp \left( (1 - x)^{-s} - 1 \right). $$

(19)

Then, we calculate the asymptotic formula of the $L_{n, s}$ by Cauchy’s formula applied to (19) gives

$$ L_{n, s} = \frac{n!}{2\pi i} \int_{-\infty}^{\infty} \exp \left( (1 - x)^{-s} - 1 \right) \frac{dx}{x^{n+1}}. $$

(20)

We let $x = R e^{i\theta}$, where $R$ will be determined later. Then (20) becomes

$$ L_{n, s} = \frac{n!}{2\pi R^n} \int_{-\pi}^{\pi} \exp \left( (1 - R e^{i\theta})^{-s} - 1 \right) \frac{e^{-i\theta} d\theta}{R^n} $$

$$ = \frac{n!}{2\pi R^n} \int_{-\pi}^{\pi} \exp \left( (1 - R \cos \theta - Ri \sin \theta)^{-s} - 1 - in\theta \right) d\theta. $$

We decompose this last integral into three parts

$$ \left( \int_{-\pi}^{-\varepsilon} + \int_{-\varepsilon}^{\varepsilon} + \int_{\varepsilon}^{\pi} \right) \exp (F(\theta)) d\theta, $$

with $F(\theta) = (1 - R \cos \theta - Ri \sin \theta)^{-s} - 1 - in\theta$ and $\varepsilon = n^{-\frac{1}{4}}$, we prove that the integrals $\int_{-\pi}^{-\varepsilon}$ and $\int_{\varepsilon}^{\pi}$ are negligible, and then the greatest contribution come from the origin. First of all, by differentiating $\theta$, we can get

$$ F'(\theta) = -s(1 - R \cos \theta - Ri \sin \theta)^{-s-1}(R \sin \theta - Ri \cos \theta) - in, $$

and

$$ F''(\theta) = -s((-s - 1)(1 - R \cos \theta - Ri \sin \theta)^{-s-2}(R \sin \theta - Ri \cos \theta)^2 $$

$$ + (1 - R \cos \theta - Ri \sin \theta)^{-s-1}(R \cos \theta + Ri \sin \theta)). $$
Then we get
\[ F(0) = (1 - R)^{-s} - 1, \]
and
\[ F''(0) = -(sR(1 - R)^{-s-1} + s(s + 1)R^2(1 - R)^{-s-2}). \]
We choose R as the only solution of \( F'(0) = 0 \) that is greater than zero and less than one: \( sR(1 - R)^{-s-1} = n \), thus \( F''(0) = -(n + s(s + 1)R^2(1 - R)^{-s-2}) \). Then expanding the integrand in a Taylor series about \( \theta = 0 \), we obtain
\[
\left| \int_\varepsilon^\pi \exp (F(\theta)) \, d\theta \right| \leq \int_\varepsilon^\pi M(\theta) \, d\theta \leq \exp (\left(1 - R\right)^{-s} - 1) \int_\varepsilon^\pi N(\theta) \, d\theta,
\]
where
\[
M(\theta) = \exp \left( (1 - R)^{-s} - 1 - \frac{\theta^2}{2} \left( n + s(s + 1)R^2(1 - R)^{-s-2} \right) + o(\theta^2) \right)
\]
and
\[
N(\theta) = \exp \left( -\frac{\theta^2}{2} \left( n + s(s + 1)R^2(1 - R)^{-s-2} \right) + o(\theta^2) \right).
\]
The integral in the last expression is
\[
\int_\varepsilon^\pi \exp \left( -\frac{\theta^2}{2} \left( n + s(s + 1)R^2(1 - R)^{-s-2} \right) + o(\theta^2) \right) \, d\theta \to 0, \quad n \to \infty.
\]
The same calculation is valid for \( \int_{-\pi}^\varepsilon \). Finally, we obtain
\[
L_{n,s} \sim \frac{n!}{2\pi R^n} \exp \left( (1 - R)^{-s} - 1 \right) \int_{-\pi}^\pi \exp \left( -\frac{\psi^2}{2} + o(\theta^2) \right) \, d\theta, \quad (21)
\]
where \( \psi = \sqrt{n + s(s + 1)R^2(1 - R)^{-s-2}} \). And observing that for \( n \) large enough, we can integrate on the real axis. We obtain
\[
L_{n,s} \sim \frac{n!}{2\pi R^n \sqrt{n + s(s + 1)R^2(1 - R)^{-s-2}}} \int_{-\infty}^{+\infty} \exp \left( -\frac{\psi^2}{2} \right) \, d\psi, \quad (22)
\]
\[
L_{n,s} \sim \frac{n!}{R^n \sqrt{2\pi \left[ n + s(s + 1)R^2(1 - R)^{-s-2} \right]}} \exp \left( 1 - (R)^{-s} - 1 \right), \quad (23)
\]
combining (18) and (23), we can obtain
\[
\sigma_n^2 = \frac{L_{n+2,s}}{s^2 L_{n,s}} - \frac{L_{n+1,s}}{s^2 L_{n,s}} - \left( \frac{L_{n,s}}{s L_{n,s}} \right)^2 - 1
\]
\[
= \frac{(n + 2)(n + 1)}{s^2 R^2} \sqrt{1 - \frac{2}{n + s(s + 1)R^2(1 - R)^{-s-2} + 2}} - \frac{(n + 1)(n + 1)}{s^2 R^2} \left( 1 - \frac{1}{n + s(s + 1)R^2(1 - R)^{-s-2} + 1} \right)
\]
\[
- \frac{n + 1}{s^2 R} \sqrt{1 - \frac{1}{n + s(s + 1)R^2(1 - R)^{-s-2} + 1}} - 1
\]
\[
= \frac{(n + 1)(1 - R)}{s^2 R^2} - 1, \quad n \to \infty.
\]
Thus, $\sigma_n^2 \to \infty$ as $n \to \infty$. By Theorem 2.1, this finally proves the main theorem.

Finally, we intend to prove the asymptotic normality of signless Lah number $L(n, k)$ as application, which has the following recurrence relation [23]:

$$L(n+1, k) = L(n, k-1) + (n+k)L(n, k).$$

We define

$$l_n(x) = \sum_{k=0}^{n} L(n, k)x^k.$$ Then we have the following recurrence relation

$$l_n(x) = (x + n - 1)l_{n-1}(x) + l_n(x), \quad l_0(x) = 1.$$ We know that the exponential generating function of $l_n(x)$ is

$$\sum_{n \geq 0} l_n(x) \frac{x^n}{n!} = \exp\left(\frac{x}{1-x}\right).$$ Thus, we can obtain the following result by the Theorem 2.5.

**Theorem 2.6.** The signless Lah numbers $L(n, k)$ are asymptotically normal.

**Proof.** Applying (17) and (18) to $l_n(x) = \sum_{k=0}^{n} L(n, k)x^k$, let $s = 1$, we obtain

$$\mu_n = \frac{L_{n+1,s}}{L_{n,s}} - (n+1)$$

$$\sigma_n^2 = \frac{L_{n+2,s}}{L_{n,s}} - \frac{L_{n+1,s}}{L_{n,s}} - \left(\frac{L_{n+1,s}}{L_{n,s}}\right)^2 - 1.$$ Then we apply (23) to $L_{n,s}$, let $s = 1$, we have

$$\sigma_n^2 = \frac{(1 - R)(n+1)}{R^2} - 1, \quad n \to \infty.$$ Thus, $\sigma_n^2 \to \infty$ as $n \to \infty$, the Theorem follows. \qed

3. Remarks.

The ordered Lah numbers defined by

$$L^{ord}(n, k) = k!L(n, k) = \begin{cases} \delta_{k0} & n = 0 \\ \frac{n!}{k!} \frac{n-1}{k-1} & n \geq 1,\end{cases}$$

which count the distributions of $n$ labelled balls among $k$ labelled urns, with no urn left, and with the balls in each urn linearly ordered. The recurrence of the ordered Lah numbers is

$$L^{ord}(n+1, k) = kL^{ord}(n, k-1) + (n+k)L^{ord}(n, k).$$

The $s$-associated Lah numbers, which express the number of partitions of $Z_n$ into $k$ order list contains at least $s$, denoted by $\binom{n}{k}^s$. The $s$-associated Lah numbers obey to the following recurrence relation, for $n \geq sk$,

$$\binom{n}{k}^s = \binom{n-1}{k-1}^s \binom{n-s}{k-1}^s + (n+k-1) \binom{n-1}{k}^s.$$ We can easily get the asymptotic normality of the ordered Lah numbers and the 2-associated Lah numbers with the same method as Theorem 2.5, and maybe we can extend this result to the $s$-associated Lah numbers.
Acknowledgments. This work was supported partially by the National Natural Science Foundation of China (No. 11871304) and the Natural Science Foundation of Shandong Province of China (No. ZR2017MA025).

REFERENCES

[1] J. C. Ahuja, Distributions of the Sum of Independent Decapitated Negative Binomial Variables, Ann. Math. Statist., 42 (1971), 383–384.
[2] J. C. Ahuja and E. A. Enneking, Concavity property and a recurrence relation for associated Lah numbers, Fibonacci Quart., 17 (1979), 158–161.
[3] P. Baldi and Y. Rinott, Asymptotic normality of some graph-related statistics, J. Appl. Probab., 26 (1989), 171–175.
[4] P. Barry, Some observations on the Lah and Laguerre transforms of integer sequences, J. Integer Seq., 10 (2007), 18pp.
[5] H. Belbachir and I. E. Bousbaa, Associated Lah numbers and r-Stirling numbers, Mathematics, (2014).
[6] E. A. Bender, Central and local limit theorems applied to asymptotic enumeration, J. Combin. Theory Ser. A, 15 (1973), 91–111.
[7] E. R. Canfield, Asymptotic normality in enumeration, Handbook of enumerative combinatorics, Discrete Math. Appl. (Boca Raton), CRC Press, Boca Raton, FL, (2015), 255–280.
[8] C. A. Charalambides, Enumerative Combinatorics, CRC Press Series on Discrete Mathematics and its Applications, Chapman and Hall/CRC, Boca Raton, FL, 2002.
[9] L. Comtet, Advanced Combinatorics, D. Reidel Publishing Co., Dordrecht, 1974.
[10] D. Galvin, Asymptotic normality of some graph sequences, Graphs Combin., 32 (2016), 639–647.
[11] C. D. Godsil, Matching behavior is asymptotically normal, Combinatorica, 1 (1981), 369–376.
[12] R. L. Graham, D. E. Knuth and O. Patashnik, Concrete Mathematics-A Foundation for Computer Science, 2nd edition, Addison-Wesley Publishing Company, Reading, MA, 1994.
[13] L. H. Harper, Stirling behavior is asymptotically normal, Ann. Math. Statist., 38 (1967), 410–414.
[14] J. Kahn, A normal law for matchings, Combinatorica, 20 (2000), 339–391.
[15] J. L. Lebowitz, B. Pittel, D. Ruelle and E. R. Speer, Central limit theorems, Lee-Yang zeros, and graph-counting polynomials, J. Combin. Theory Ser. A, 141 (2016), 147–183.
[16] J. Lindsay, T. Mansour and M. Shattuck, A new combinatorial interpretation of a q-analogue of the Lah numbers, J. Comb., 2 (2011), 245–264.
[17] L. L. Liu and Y. Wang, A unified approach to polynomial sequences with only real zeros, Adv. in Apple. Math., 38 (2007), 542–560.
[18] T. S. Motzkin, Sorting numbers for cylinders other classification numbers, Proc. Symp. Pure Math, 19 (1971), 167–176.
[19] S. B. Nandi and S. K. Dutta, On associated and generalized Lah numbers and applications to discrete distributions, Fibonacci Quart., 25 (1987), 128–136.
[20] K. Nowick, Asymptotic normality of graph statistics, J. Statist. Plann. Inference, 21 (1989), 209–222.
[21] M. Petkovšek and T. Pisanski, Combinatorial interpretation of unsigned Stirling and Lah numbers, preprint, Univ. of Ljubljana (Available on the Internet), 49 (2002).
[22] A. Ruciński, The behaviour of \( \binom{n}{k} \) is asymptotically normal, Discrete Math., 49 (1984), 287–290.
[23] C. Wagner, Generalized Stirling and Lah numbers, Discrete Math., 160 (1996), 199–218.
[24] Y. Wang, H.-X. Zhang and B.-X. Zhu, Asymptotic normality of Laplacian coefficients of graphs, J. Math. Anal. Appl., 455 (2017), 2030–2037.

Received September 2019; revised June 2021; early access August 2021.

E-mail address: 1411074373@qq.com (Wen Zhang)
E-mail address: liulily@qfnu.edu.cn (Lily Li Liu)