Semiclassical Properties and Chaos Degree for the Quantum Baker’s Map

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Abstract

We study the chaotic behaviour and the quantum-classical correspondence for the baker’s map. Correspondence between quantum and classical expectation values is investigated and it is numerically shown that it is lost at the logarithmic timescale. The quantum chaos degree is computed and it is demonstrated that it describes the chaotic features of the model. The correspondence between classical and quantum chaos degrees is considered.

1 Introduction

The study of chaotic behaviour in classical dynamical systems is dating back to Lobachevsky and Hadamard who have been studied the exponential instability property of geodesics on manifolds of negative curvature and to Poincare, who initiated the inquiry into the stability of the solar system. One believes now that the main features of chaotic behaviour in the classical dynamical systems are rather well understood, see for example [1, 2]. However the status of “quantum chaos” is much less clear although the significant progress has been made on this front.
Sometimes one says that an approach to quantum chaos, which attempts to generalize the classical notion of sensitivity to initial conditions, fails for two reasons: first there is no quantum analogue of the classical phase space trajectories and, second, the unitarity of linear Schrödinger equation precludes sensitivity to initial conditions in the quantum dynamics of state vector. Let us remind, however, that in fact there exists a quantum analogue of the classical phase space trajectories. It is quantum evolution of expectation values of appropriate observables in suitable states. Also let us remind that the dynamics of a classical system can be described either by the Hamilton equations or by the linear Liouville equations. In quantum theory the linear Schrödinger equation is the counterpart of the Liouville equation while the quantum counterpart of the classical Hamilton’s equation is the Heisenberg equation. Therefore the study of quantum expectation values should reveal the chaotic behaviour of quantum systems. In this paper we demonstrate this fact for the quantum baker’s map.

If one has the classical Hamilton’s equations

\[ \frac{dq}{dt} = p, \quad \frac{dp}{dt} = -V'(q), \]

then the corresponding quantum Heisenberg equations have the same form

\[ \frac{dq_h}{dt} = p_h, \quad \frac{dp_h}{dt} = -V'(q_h), \]

where \( q_h \) and \( p_h \) are quantum canonical operators of position and momentum. For the expectation values one gets the Ehrenfest equations

\[ d\langle q_h \rangle /dt = \langle p_h \rangle, \quad d\langle p_h \rangle /dt = -\langle V'(q_h) \rangle. \]

Note that the Ehrenfest equations are classical equations but for nonlinear \( V'(q_h) \) they are neither Hamiltonian’s equations nor even differential equations because one can not write \( \langle V'(q_h) \rangle \) as a function of \( \langle q_h \rangle \) and \( \langle p_h \rangle \). However these equations are very convenient for the consideration of the semiclassical properties of quantum system. The expectation values \( \langle q_h \rangle \) and \( \langle p_h \rangle \) are functions of time and initial data. They also depend on the quantum states. One of important problems is to study the dependence of expectation values from the initial data. In this paper we will study this problem for the quantum baker’s map.

The main objective of “quantum chaos” is to study the correspondence between classical chaotic systems and their quantum counterparts in the semiclassical limit \[3, 4\]. The quantum-classical correspondence for dynamical systems has been studied for many years, see for example \[5, 6, 7, 8, 9, 10\].
and reference therein. A significant progress in understanding of this correspondence has been achieved in the WKB approach when one considers the Planck constant $h$ as a small variable parameter. Then it is well known that in the limit $h \to 0$ quantum theory is reduced to the classical one [11]. However in physics the Planck constant is a fixed constant although it is very small. Therefore it is important to study the relation between classical and quantum evolutions when the Planck constant is fixed. There is a conjecture [12, 13, 14, 8] that a characteristic timescale $\tau$ appears in the quantal evolution of chaotic dynamical systems. For time less than $\tau$ there is a correspondence between quantum and classical expectation values, while for times greater than $\tau$ the predictions of the classical and quantum dynamics no longer coincide. The important problem is to estimate the dependence $\tau$ on the Planck constant $h$. Probably a universal formula expressing $\tau$ in terms of $h$ does not exist and every model should be studied case by case. It is expected that certain quantum and classical expectation values diverge on a timescale inversely proportional to some power of $h$ [13]. Other authors suggest that a breakdown may be anticipated on a much smaller logarithmic timescale [16, 17, 18, 19, 20, 21, 22, 23]. The characteristic time $\tau$ associated with the hyperbolic fixed points of the classical motion is expected to be of the logarithmic form $\tau = \frac{1}{\lambda} \ln \frac{C}{h}$ where $\lambda$ is the Lyapunov exponent and $C$ is a constant which can be taken to be the classical action. Such the logarithmic timescale has been found in the numerical simulations of some dynamical models. [7]. It was shown also that the discrepancy between quantum and classical evolutions is decreased by even a small coupling with the environment, which in the quantum case leads to decoherence [7].

The chaotic behaviour of the classical dynamical systems is often investigated by computing the Lyapunov exponents. An alternative quantity measuring chaos in dynamical systems which is called the chaos degree has been suggested in [24] in the general framework of information dynamics [25]. The chaos degree was applied to various models in [26]. An advantage of the chaos degree is that it can be applied not only to classical systems but also to quantum systems as well.

In this work we study the chaotic behaviour and the quantum-classical correspondence for the baker’s map [15, 27]. The quantum baker’s map is a simple model invented for the theoretical study of quantum chaos. Its mathematical properties have been studied in numerical works. In particular its semiclassical properties have been considered [16, 17, 18, 19, 20, 21, 22, 23], quantum computing and optical realizations have been proposed [28, 29, 30], various quantization procedures have been discussed [31, 32, 18, 33], a symbolic dynamics representation has been given [34].

It is well known that for the consideration of the semiclassical limit in
quantum mechanics it is very useful to use coherent states. We define an
analogue of the coherent states for the quantum baker’s map. We study the
quantum baker’s map by using the correlation functions of the special form
which corresponds to the expectation values of Weyl operators, translated in
time by the unitary evolution operator and taken in the coherent states.

To explain our formalism we first discuss the classical limit for correlation
functions in ordinary quantum mechanics. Correspondence between quantum
and classical expectation values for the baker’s map is investigated and it
is numerically shown that it is lost at the logarithmic timescale. The chaos
degree for the quantum baker’s map is computed and it is demonstrated that
it describes the chaotic features of the model. The dependence of the chaos
degree on the Planck constant is studied and the correspondence between
classical and quantum chaos degrees is established.

2 Quantum vs. Classical Dynamics

In this section we discuss an approach to the semiclassical limit in quantum
mechanics by using the coherent states, see [6]. Then in the next section an
extension of this approach to the quantum baker’s map will be given.
Consider the canonical system with the Hamiltonian function

\[ H = \frac{p^2}{2} + V(x) \]  

in the plane \((p, x) \in \mathbb{R}^2\). We assume that the canonical equations

\[ \dot{x}(t) = p(t), \quad \dot{p}(t) = -V'(x(t)) \]  

have a unique solution \((x(t), p(t))\) for times \(|t| < T\) with the initial data

\[ x(0) = x_0, \quad p(0) = v_0 \]  

This is equivalent to the solution of the Newton equation

\[ \ddot{x}(t) = -V'(x(t)) \]  

with the initial data

\[ x(0) = x_0, \quad \dot{x}(0) = v_0 \]  

We denote

\[ \alpha = \frac{1}{\sqrt{2}}(x_0 + iv_0) \]
The quantum Hamiltonian operator has the form

\[ H_h = \frac{p_h^2}{2} + V(q_h) \]

where \( p_h \) and \( q_h \) satisfy the commutation relations

\[ [p_h, q_h] = -i \hbar \]

The Heisenberg evolution of the canonical variables is defined as

\[ p_h(t) = U(t) p_h U(t)^*, \quad q_h(t) = U(t) q_h U(t)^* \]

where

\[ U(t) = \exp(-itH_h/\hbar) \]

For the consideration of the classical limit we take the following representation

\[ p_h = -i\hbar^{1/2}\partial/\partial x, \quad q_h = \hbar^{1/2}x \]

acting to functions of the variable \( x \in \mathbb{R} \). We also set

\[ a = \frac{1}{\sqrt{2\hbar^{1/2}}} (q_h + ip_h) = \frac{1}{\sqrt{2}} \left( x + \frac{\partial}{\partial x} \right), \quad a^* = \frac{1}{\sqrt{2\hbar^{1/2}}} (q_h - ip_h) = \frac{1}{\sqrt{2}} \left( x - \frac{\partial}{\partial x} \right), \]

then

\[ [a, a^*] = 1. \]

The coherent state \( |\alpha\rangle \) is defined as

\[ |\alpha\rangle = W(\alpha) |0\rangle \] (7)

where \( \alpha \) is a complex number, \( W(\alpha) = \exp(\alpha a^* - a\alpha^*) \) and \( |0\rangle \) is the vacuum vector, \( a |0\rangle = 0 \). The vacuum vector is the solution of the equation

\[ (q_h + ip_h) |0\rangle = 0 \] (8)

In the \( x \) - representation one has

\[ |0\rangle = \exp(-x^2/2) / \sqrt{2\pi}. \] (9)

The operator \( W(\alpha) \) one can write also in the form
\[ W(\alpha) = Ce^{i q_0/v_0} e^{-i p_0 x_0} \]  

where \( C = \exp(-v_0 x_0/2h) \).

The mean value of the position operator with respect to the coherent vectors is the real valued function

\[ q(t, \alpha, h) = \langle h^{-1/2} \alpha | q(t) | h^{-1/2} \alpha \rangle \]  

Now one can present the following basic formula describing the semiclassical limit

\[ \lim_{h \to 0} q(t, \alpha, h) = x(t, \alpha) \]  

Here \( x(t, \alpha) \) is the solution of (4) with the initial data (5) and \( \alpha \) is given by (6).

Let us notice that for time \( t = 0 \) the quantum expectation value \( q(t, \alpha, h) \) is equal to the classical one:

\[ q(0, \alpha, h) = x(0, \alpha) = x_0 \]  

for any \( h \). We are going to compare the time dependence of two real functions \( q(t, \alpha, h) \) and \( x(t, \alpha) \) these functions are approximately equal. The important problem is to estimate for which \( t \) the large difference between them will appear. It is expected that certain quantum and classical expectation values diverge on a timescale inversely proportional to some power of \( h \) \[6\]. Other authors suggest that a breakdown may be anticipated on a much smaller logarithmic timescale \[16, 17, 18, 19, 20, 21, 22, 23\]. One of very interesting examples \[5\] of classical systems with chaotic behaviour is described by the hamiltonian function

\[ H = \frac{p_1^2}{2} + \frac{p_2^2}{2} + \lambda x_1^2 x_2^2 \]

The consideration of this classical and quantum model within the described framework will be presented in another publication.

### 3 Coherent States for the Quantum Baker’s Map

The classical baker’s transformation maps the unit square \( 0 \leq q, p \leq 1 \) onto itself according to
\((q,p) \rightarrow \begin{cases} 
(2q,p/2), & \text{if } 0 \leq q \leq 1/2 \\
(2q - 1, (p + 1)/2), & \text{if } 1/2 < q \leq 1 
\end{cases}\)

This corresponds to compressing the unit square in the \(p\) direction and stretching it in the \(q\) direction, while preserving the area, then cutting it vertically and stacking the right part on top of the left part.

The classical baker’s map has a simple description in terms of its symbolic dynamics \[34\]. Each point \((q,p)\) is represented by a symbolic string

\[
\xi = \cdots \xi_2 \xi_1 \xi_0 \xi_1 \xi_2 \cdots, \quad \text{(14)}
\]

where \(\xi_k \in \{0, 1\}\), and

\[
q = \sum_{k=1}^{\infty} \xi_k 2^{-k}, \quad p = \sum_{k=0}^{\infty} \xi_{-k} 2^{-k-1}
\]

The action of the baker’s map on a symbolic string \(s\) is given by the shift map (Bernoulli shift) \(U\) defined by \(U\xi = \xi'\), where \(\xi'_k = \xi_{k+1}\). This means that, at each time step, the dot is shifted one place to the right while entire string remains fixed. After \(n\) steps the \(q\) coordinate becomes

\[
q_n = \sum_{k=1}^{\infty} \xi_{n+k} 2^{-k}
\]

This relation defines the classical trajectory with the initial data

\[
q = q_0 = \sum_{k=1}^{\infty} \xi_k 2^{-k}
\]

Quantum baker’s maps are defined on the \(D\)-dimensional Hilbert space of the quantized unit square. To quantize the unit square one defines the Weyl unitary displacement operators \(\hat{U}\) and \(\hat{V}\) in \(D\)-dimensional Hilbert space, which produces displacements in the momentum and position directions, respectively, and the following commutation relation is obeyed

\[
\hat{U}\hat{V} = \epsilon \hat{V}\hat{U},
\]

where \(\epsilon = \exp(2\pi i/D)\). We choose \(D = 2^N\), so that our Hilbert space will be the \(N\) qubit space \(\mathbb{C}^{\otimes N}\). The constant \(\hbar = 1/D = 2^{-N}\) can be regarded as the Plank constant. The space \(\mathbb{C}^2\) has a basis

\[
|0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]
The basis in $C^\otimes N$ is

$$|\xi_1\rangle \otimes |\xi_2\rangle \otimes \cdots \otimes |\xi_N\rangle, \quad \xi_k = 0, 1$$

We write

$$\xi = \sum_{k=1}^{N} \xi_k 2^{N-k}$$

then $\xi = 0, 1, \ldots, 2^N - 1$ and denote

$$|\xi\rangle = |\xi_1\xi_2\cdots\xi_N\rangle = |\xi_1\rangle \otimes |\xi_2\rangle \otimes \cdots \otimes |\xi_N\rangle$$

We will use for this basis also notations $\{|\eta\rangle = |\eta_1\eta_2\cdots\eta_N\rangle, \quad \eta_k = 0, 1\}$ and $\{|j\rangle = |j_1j_2\cdots j_N\rangle, j_k = 0, 1\}$.

The operators $\hat{U}$ and $\hat{V}$ can be written as

$$\hat{U} = e^{2\pi i \hat{q}}, \quad \hat{V} = e^{2\pi i \hat{p}}$$

where the position and momentum operators $\hat{q}$ and $\hat{p}$ are operators in $C^\otimes N$ which are defined as follows. The position operator is

$$\hat{q} = \sum_{j=0}^{2^N-1} q_j |j\rangle \langle j| = \sum_{j_1,\ldots,j_N} q_{j_1\ldots j_N} |j_1\ldots j_N\rangle \langle j_1\ldots j_N|$$

where

$$|j\rangle = |j_1j_2\cdots j_N\rangle, j_k = 0, 1$$

is the basis in $C^\otimes N$,

$$j = \sum_{k=1}^{N} j_k 2^{N-k}$$

and

$$q_j = \frac{j + 1/2}{2^N}, \quad j = 0, 1, \ldots, 2^N - 1$$

The momentum operator is defined as

$$\hat{p} = F_N \hat{q} F_N^*$$

where $F_N$ is the quantum Fourier transform acting to the basis vectors as
\[ F_N |j\rangle = \frac{1}{\sqrt{D}} \sum_{\xi=0}^{D-1} e^{2\pi i \xi j / D} |\xi\rangle, \]

here \( D = 2^N \).

The symbolic representation of quantum baker’s map \( T \) was introduced by Schack and Caves \cite{33} and studied in \cite{35, 36}. Let us explain the symbolic representation of quantum baker’s map as a special case \cite{33}: By applying a partial quantum Fourier transform \( G_m = I \otimes \cdots \otimes I \otimes F_{N-m} \) to the position eigenstates, one obtains the following quantum baker’s map \( T \):

\[ T |\xi_1 \cdots \xi_N\rangle \equiv |\xi_1 \cdot \cdot \cdot \xi_N\rangle, \]

where

\[ T = G_{N-1} \circ G_N^{-1} \]

and

\[ |\xi_1 \cdots \xi_{N-m} \cdot \xi_{N-m+1} \cdots \xi_N\rangle \equiv G_m |\xi_{N-m+1} \cdots \xi_N \xi_{N-m} \cdots \xi_1\rangle = |\xi_{N-m+1} \rangle \otimes \cdots \otimes |\xi_N\rangle \otimes F_{N-m} |\xi_{N-m}\rangle \otimes \cdots \otimes |\xi_1\rangle. \]

The quantum baker’s map \( T \) is the unitary operator in \( \mathbb{C}^{\otimes N} \) with the following matrix elements

\[ \langle \xi | T |\eta\rangle = \frac{1 - i}{2} \exp \left( \frac{\pi}{2} i |\xi_1 - \eta_N| \right) \prod_{k=2}^{N} \delta (\xi_k - \eta_{k-1}), \tag{17} \]

where \( |\xi\rangle = |\xi_1 \xi_2 \cdots \xi_N\rangle, \ |\eta\rangle = |\eta_1 \eta_2 \cdots \eta_N\rangle \) and \( \delta(x) \) is the Kronecker symbol, \( \delta(0) = 1; \ \delta(x) = 0, x \neq 0. \)

We define the coherent states by

\[ |\alpha\rangle = C e^{2\pi i \hat{q} v} e^{-2\pi i \hat{p} x} |\psi_0\rangle \tag{18} \]

Here \( \alpha = x + iv, x \) and \( v \) are integers, \( C \) is the normalization constant and \( |\psi_0\rangle \) is the vacuum vector. This definition should be compared with \cite{10}. The vacuum vector can be defined as the solution of the equation

\[ (q_{x} + ip_{v}) |\psi_0\rangle = 0 \]

(compare with \cite{8}). We will use the simpler definition which in the position representation is

\[ \langle q_j |\psi_0\rangle = C \exp \left( -q_j^2 / 2 \right) \]

(compare with \cite{4}). Here \( C \) is a normalization constant.
4 Chaos Degree

Let us review the entropic chaos degree defined in [24]. This entropic chaos degree is given by a probability distribution \( \varphi \) and a dynamics (channel) \( \Lambda^* \) sending a state to a state; \( \varphi = \sum_k p_k \delta_k \), where \( \delta_k \) is the delta measure such as \( \delta_k(j) \equiv \begin{cases} 1 & (k = j) \\ 0 & (k \neq j) \end{cases} \). Then the entropic chaos degree is defined as

\[
D(\varphi; \Lambda^*) = \sum_k p_k S(\Lambda^* \delta_k) \tag{19}
\]

with the von Neumann entropy \( S \), equivalently to the Shannon entropy because the probability distribution \( \varphi \) is a classical object.

A dynamics \( F \) of the orbit produces the above channel \( \Lambda^* \), so that let \( \{x_n\} \) be the orbit and \( F \) be a map from \( x_n \) to \( x_{n+1} \).

Take a finite partition \( \{B_k\} \) of \( I = [a, b]^l (a, b \in \mathbb{R}) \subset \mathbb{R}^l \) such as

\[
I = \bigcup_k B_k \quad (B_i \cap B_j = \emptyset, i \neq j)
\]

for a map \( F \) on \( I \) with \( x_{n+1} = F(x_n) \) (a difference equation). The state \( \varphi(n) \) of the orbit determined by the difference equation is defined by the probability distribution \( (p^{(n)}_i) \), that is, \( \varphi(n) = p^{(n)} = \sum_i p^{(n)}_i \delta_i \), where for an initial value \( x \in I \) and the characteristic function \( 1_A \)

\[
p^{(n)}_i \equiv \frac{1}{m+1} \sum_{k=n}^{m+n} 1_{B_i} (F^k x).
\]

When the initial value \( x \) is distributed due to a measure \( \nu \) on \( I \), the above \( p^{(n)}_i \) is given as

\[
p^{(n)}_i \equiv \frac{1}{m+1} \int_I \sum_{k=n}^{m+n} 1_{B_i} (F^k x) \, d\nu.
\]

In the case that \( F \) is a classical baker’s transformation, if the orbit is not stable and periodic, then it is shown that the \( m \to \infty \) limit of \( p^{(n)}_i \) exists and equals to a natural invariant measure for a fixed \( n \in \mathbb{N} \) [37].

The joint distribution \( (p^{(n,n+1)}_{ij}) \) between the time \( n \) and \( n + 1 \) is defined by

\[
p^{(n,n+1)}_{ij} \equiv \frac{1}{m+1} \sum_{k=n}^{m+n} 1_{B_i} (F^k x) \, 1_{B_j} (F^{k+1} x)
\]

or
\[ p_{ij}^{(n,n+1)} \equiv \frac{1}{m+1} \int \sum_{k=n}^{m+n} 1_{B_i} (F^k x) 1_{B_j} (F^{k+1} x) \, d\nu. \]

Then the channel \( \Lambda_n^* \) at \( n \) is determined by
\[
\Lambda_n^* \equiv \left( \frac{p_{ij}^{(a,n+1)}}{p_i^{(n)}} \right) \implies p^{(n+1)} = \Lambda_n^* p^{(n)},
\]
and the chaos degree is given by
\[
D_c \left( p^{(n)}; \Lambda_n^* \right) = \sup_{\{B_k\}} \left\{ \sum_i p_i^{(n)} S(\Lambda_n^* \delta_i) = \sum_{i,j} p_{ij}^{(n,n+1)} \log \frac{p_i^{(n)}}{p_{ij}^{(n,n+1)}}, \{B_k\} \right\}.
\]

(20)

We can judge whether the dynamics causes a chaos or not by the value of \( D \) as

\[
D > 0 \iff \text{chaotic}, \\
D = 0 \iff \text{stable}.
\]

Therefore it is enough to find a partition \( \{B_k\} \) such that \( D \) is positive when the dynamics produces chaos.

This classical chaos degree was applied to several dynamical maps such as logistic map, Baker’s transformation and Tinkerbel map, and it could explain their chaotic characters\[24, 26\]. Our chaos degree has several merits compared with usual measures such as Lyapunov exponent.

5 Expectation Values and Chaos Degree

In this section, we show a general representation of the mean value of the position operator \( \hat{q} \) for the time evolution, which is constructed by the quantum baker’s map. Then we give the algorithm to compute the chaos degree for the quantum baker’s map.

To study the time evolution and the classical limit \( \hbar \to 0 \) which corresponds to \( N \to \infty \) of the quantum baker’s map \( T \), we introduce the following mean value of the position operator \( \hat{q} \) for time \( n \in \mathbb{N} \) with respect to a single basis|\( \xi \)|:
where \(|\xi\rangle = |\xi_1\xi_2\cdots\xi_N\rangle\).

From (\ref{eq:17}), the following formula of the matrix elements of \(T^n\) for any \(n \in \mathbb{N}\) is easily obtained.

\[
\langle \xi | T^n | \xi \rangle = \begin{cases} 
\left(\frac{1-\mu}{2}\right)^n \left(\prod_{k=1}^{N-n} \delta(\xi_{n+k} - \xi_k)\right) \left(\prod_{l=1}^{n} A_{\xi_l\xi_{N-n+l}}\right) & \text{if } n < N \\
\left(\frac{1-\mu}{2}\right)^n \left(\prod_{k=1}^{n} A_{\xi_k\xi_k}\right) & \text{if } n = N \\
\left(\frac{1-\mu}{2}\right)^n \left(\prod_{k=1}^{p} (A^{m+1})_{\xi_k\xi_{N-p+k}}\right) \left(\prod_{l=1}^{N-p} (A^m)_{\xi_{p+l}\xi_l}\right) & \text{if } n = mN + p \\
\left(\frac{1-\mu}{2}\right)^n \left(\prod_{k=1}^{N} (A^m)_{\xi_k\xi_k}\right) & \text{if } n = mN,
\end{cases}
\]

where \(A\) is the \(2 \times 2\) matrix with the element \(A_{x_1x_2} = \exp\left(\frac{\pi i}{2} |x_1 - x_2|\right)\) for \(x_1, x_2 = 0, 1, p = 1, \cdots, N - 1\) and \(m \in \mathbb{N}\).

Using these formula, the following theorems are obtained and their proofs are given in Appendix.

**THEOREM 5.1**

\[
r_n^{(N)} = \begin{cases} 
\sum_{k=1}^{N-n} \xi_{n+k} 2^{-k} + \frac{2^n}{2^{N-n}} & \text{if } n < N \\
\frac{1}{2^n} \sum_{j=0}^{2^{N-1}} \frac{j+1/2}{2^{N}} \left(\prod_{k=1}^{p} (A^{m+1})_{\xi_k\xi_{N-p+k}}\right) \left(\prod_{l=1}^{N-p} (A^m)_{\xi_{p+l}\xi_l}\right)^2 & \text{if } n = mN + p \\
\frac{1}{2^n} \sum_{j=0}^{2^{N-1}} \frac{j+1/2}{2^{N}} \left(\prod_{k=1}^{N} (A^m)_{\xi_k\xi_k}\right)^2 & \text{if } n = mN,
\end{cases}
\]

where \(A\) is the \(2 \times 2\) matrix with the element \(A_{x_1x_2} = \exp\left(\frac{\pi i}{2} |x_1 - x_2|\right)\) for \(x_1, x_2 = 0, 1, p = 1, \cdots, N - 1\) and \(m \in \mathbb{N}\).

By diagonalizing the matrix \(A\), we obtain the following formula of the absolute square of the matrix elements of \(A^n\) for any \(n \in \mathbb{N}\).

**LEMMA 5.2** For any \(n \in \mathbb{N}\), we have

\[
\left| (A^n)_{kj} \right|^2 = \begin{cases} 
2^n \cos^2 \left(\frac{\pi k}{4}\right) & \text{if } k = j \\
2^n \sin^2 \left(\frac{\pi k}{4}\right) & \text{if } k \neq j.
\end{cases}
\]

Combining the above theorem and lemma, we obtain the following two theorems with respect to the mean value \(r_n^{(N)}\) of the position operator.
THEOREM 5.3 For the case $n = mN + p$, $p = 1, 2, \ldots N - 1$ and $m \in \mathbb{N}$, we have

$$r^{(N)}_n = \begin{cases} 
\sum_{k=1}^{N-p} \xi_{p+k} 2^{-k} + \frac{2^p}{2^{N+1}} & \text{if } m = 0 \pmod{4} \\
\sum_{k=N-p+1}^{N} \eta_k - (N-p) 2^{-k} + \frac{2^N - 2^p + 1}{2^{N+1}} & \text{if } m = 1 \pmod{4} \\
\sum_{k=1}^{N-p} \eta_{p+k} 2^{-k} + \frac{2^p}{2^{N+1}} & \text{if } m = 2 \pmod{4} \\
\sum_{k=N-p+1}^{N} \xi_k - (N-p) 2^{-k} + \frac{2^N - 2^p + 1}{2^{N+1}} & \text{if } m = 3 \pmod{4},
\end{cases}$$

(24)

where $\eta_k = \xi_k + 1 \pmod{2}$, $k = 1, \ldots, N$.

THEOREM 5.4 For the case $n = mN, m \in \mathbb{N}$, we have

$$r^{(n)}_N = \begin{cases} 
\sum_{k=1}^{N} \xi_k 2^{-k} + \frac{1}{2^{N+1}} & \text{if } m = 0 \pmod{4} \\
\sum_{k=1}^{N} \eta_k 2^{-k} + \frac{1}{2^{N+1}} & \text{if } m = 1, 3 \pmod{4} \\
\sum_{k=1}^{N} \eta_k 2^{-k} + \frac{1}{2^{N+1}} & \text{if } m = 2 \pmod{4}.
\end{cases}$$

(25)

Using these formulas (23), (24) and (25), the probability distribution $(p^{(n)}_i)$ of the orbit of mean value $r^{(N)}_n$ of the position operator $\hat{q}$ for the time evolution, which is constructed by the quantum baker’s map, is given by

$$p^{(n)}_i \equiv \frac{1}{m+1} \sum_{k=n}^{m+n} 1_{B_i}(r^{(N)}_k)$$

for an initial value $r^{(N)}_0 \in [0, 1]$ and the characteristic function $1_A$. The joint distribution $(p^{(n,n+1)}_{ij})$ between the time $n$ and $n + 1$ is given by

$$p^{(n,n+1)}_{ij} \equiv \frac{1}{m+1} \sum_{k=n}^{m+n} 1_{B_i}(r^{(N)}_k) 1_{B_j}(r^{(N)}_{k+1}) .$$

Thus the chaos degree for the quantum baker’s map is calculated by

$$D_q (p^{(n)}; \Lambda^n) = \sum_{i,j} p^{(n,n+1)}_{ij} \log \frac{p^{(n)}_i}{p^{(n,n+1)}_{ij}},$$

(26)

whose numerical value is shown in the next section.
6 Numerical Simulation of the Chaos Degree and Classical-Quantum Correspondence

We compare the dynamics of the mean value \( r_n^{(N)} \) of position operator \( \hat{q} \) with that of the classical value \( q_n \) in the \( q \) direction. We take an initial value of the mean value as

\[
r_0^{(N)} = \sum_{i=1}^{N} \xi_i 2^{-i} + 1/2^{N+1} = 0.\xi_1\xi_2 \cdots \xi_N1,
\]

where \( \xi_i \) is a pseudo-random number valued with 0 or 1. At the time zero we assume that the classical value \( q_0 \) in the \( q \) direction takes the same value as the mean value \( r_0^{(N)} \) of position operator \( \hat{q} \). The distribution of \( r_n^{(N)} \) for the case \( N = 500 \) is shown in Fig.1 up to the time \( n = 1000 \). The distribution of the classical value \( q_n \) for the case \( N = 500 \) in the \( q \) direction is shown in Fig.2 up to the time \( n = 1000 \).

![Fig.1. The distribution of \( r_n^{(N)} \) for the case \( N=500 \)](image-url)
Fig. 2. The distribution of the classical value $q^{(n)}$ for the case $N=500$

Fig. 3 presents the change of the chaos degree for the case $N = 100, 300, 500, 700$ up to the time $n = 1000$.

The correspondence between the chaos degree $D_q$ for the quantum baker’s map and the chaos degree $D_c$ for the classical baker’s map for some fixed $N$s (100, 300, 500, 700 here) is shown for the time less than $T = \log_2 \frac{1}{\hbar} = \log_2 2^N = N$, and it is lost at the logarithmic time scale $T$. Here we took a finite partition $\{B_k\}$ of $I = [0, 1]$, such as $B_k = \left[\frac{k}{100}, \frac{k+1}{100}\right]$ ($k = 0, 1, \ldots, 98$) and $B_{99} = \left[\frac{99}{100}, 1\right]$ to compute the chaos degree numerically.

The difference of the chaos degrees between the chaos degree $D_q$ for the quantum baker’s map and the chaos degree $D_c$ for the classical baker’s map for a fixed time $n = 1000$, here is displayed w.r.t. $N$ in Fig. 4.
Thus we conclude that the dynamics of the mean value $q_n$ reduces the classical dynamics $q_n$ in the $q$ direction in the classical limit $N \to \infty (h \to 0)$.

The appearance of the logarithmic timescale have been proved rigorously in our recent paper [38].

7 Acknowledgments

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8 Appendix

Proof of THEOREM 5.1: By a direct calculation, we obtain

$$r_n^{(N)} = \langle \xi | T^n \tilde{q} T^{-n} | \xi \rangle$$

$$= \langle \xi | T^n \left( \sum_{j=0}^{2^{N-1}} \frac{j + 1/2}{2^N} |j\rangle \langle j| \right) T^{-n} | \xi \rangle$$

$$= \sum_{j=0}^{2^{N-1}} \frac{j + 1/2}{2^N} \langle \xi | T^n |j\rangle \langle j| T^{-n} | \xi \rangle$$
\[
\begin{align*}
&= \sum_{j=0}^{2^{N-1}} \frac{j+1/2}{2^N} \langle \xi| T^n |j\rangle \langle j| T^n |\xi\rangle \\
&= \sum_{j=0}^{2^{N-1}} \frac{j+1/2}{2^N} \langle \xi| T^n |j\rangle \overline{\langle \xi| T^n |j\rangle} \\
&= \sum_{j=0}^{2^{N-1}} \frac{j+1/2}{2^N} |\langle \xi| T^n |j\rangle|^2.
\end{align*}
\]

Using (22) the mean value \( r_n^{(N)} \) in the case \( n < N \) can be expressed as

\[
\begin{align*}
\quad \quad \quad \quad \quad r_n^{(N)} &= \sum_{j=0}^{2^{N-1}} \frac{j+1/2}{2^N} |\langle \xi| T^n |j\rangle|^2. \\
&= \sum_{j=0}^{2^{N-1}} \frac{j+1/2}{2^N} \left( \frac{1-i}{2} \right)^n \left( \prod_{k=1}^{N-n} \delta (\xi_{n+k} - j_k) \right) \left( \prod_{l=1}^{n} A_{\xi_ljN-n+l} \right)^2 \\
&= \sum_{j=0}^{2^{N-1}} \frac{j+1/2}{2^N} \left( \frac{1-i}{2} \right)^n \left( \prod_{k=1}^{N-n} \delta (\xi_{n+k} - j_k) \right) \left( \prod_{l=1}^{n} A_{\xi_ljN-n+l} \right)^2 \\
&= \sum_{j=0}^{2^{N-1}} \frac{j+1/2}{2^N} \left( \frac{1+i}{2} \right)^n \left( \prod_{k=1}^{N-n} \delta (\xi_{n+k} - j_k) \right) \left( \prod_{l=1}^{n} A_{\xi_ljN-n+l} \right)^2 \\
&= \sum_{j=0}^{2^{N-1}} \frac{j+1/2}{2^N} \left( \frac{1+i}{2} \right)^n \left( \prod_{k=1}^{N-n} \delta (\xi_{n+k} - j_k) \right) \left( \prod_{l=1}^{n} A_{\xi_ljN-n+l} \right)^2 \\
&= \frac{1}{2^{N+n}} \sum_{j_1, \ldots, j_N} \left\{ \left( \sum_{k=1}^{N} j_k 2^{N-k} \right) + 1/2 \right\} \left( \prod_{k=1}^{N-n} \delta (\xi_{n+k} - j_k) \right) \\
&= \frac{1}{2^{N+n}} \sum_{j_1, \ldots, j_N} \left( \sum_{k=1}^{N} j_k 2^{N-k} \right) \left( \prod_{k=1}^{N-n} \delta (\xi_{n+k} - j_k) \right) \\
&= \frac{1}{2^{N+n}} \sum_{j_1, \ldots, j_N} \left( \prod_{k=1}^{N-n} \delta (\xi_{n+k} - j_k) \right) \\
&\quad + \frac{1}{2^{N+n+1}} \sum_{j_1, \ldots, j_N} \left( \prod_{k=1}^{N-n} \delta (\xi_{n+k} - j_k) \right)
\end{align*}
\]
For the case $n = N$, we similarly obtain

$$
\gamma_n^{(N)} = \sum_{j=0}^{2^{n-1}} \frac{j + 1/2}{2^{2n}} \left| \langle \xi | T^n | j \rangle \right|^2.
$$

For $n = mN + p, p = 1, 2, \cdots, N - 1$, $m \in \mathbb{N}$,
\[
\begin{align*}
\Phi_N &= \sum_{j=0}^{2N-1} \frac{j + 1/2}{2N} |\langle \xi | T^n | j \rangle|^2 \\
&= \sum_{j=0}^{2N-1} \frac{j + 1/2}{2N} \left| \left( \frac{1 - i}{2} \right)^n \left( \prod_{k=1}^{N} (A^{m+1})_{\xi_{k,j_{N-p+k}}} \right) \left( \prod_{l=1}^{N-p} (A^m)_{\xi_{p+l,j_l}} \right) \right|^2 \\
&= \frac{1}{2^n} \sum_{j=0}^{2N-1} \frac{j + 1/2}{2N} \prod_{k=1}^{p} \left| (A^{m+1})_{\xi_{k,j_{N-p+k}}} \right|^2 \prod_{l=1}^{N-p} \left| (A^m)_{\xi_{p+l,j_l}} \right|^2 \\
\end{align*}
\]

and for \( n = mN, m \in \mathbb{N} \),

\[
\begin{align*}
\Phi_N^{(m)} &= \sum_{j=0}^{2N-1} \frac{j + 1/2}{2N} |\langle \xi | T^n | j \rangle|^2 \\
&= \sum_{j=0}^{2N-1} \frac{j + 1/2}{2N} \left( \frac{1 - i}{2} \right)^n \left( \prod_{k=1}^{N} (A^{m+1})_{\xi_{k,j}} \right)^2 \\
&= \frac{1}{2^n} \sum_{j=0}^{2N-1} \frac{j + 1/2}{2N} \prod_{k=1}^{N} \left| (A^{m})_{\xi_{k,j}} \right|^2.
\end{align*}
\]

**Proof of LEMMA 5.2:** By a direct calculation, the matrix \( A \) is diagonalized as follows:

\[
A = FDF^*,
\]

where

\[
F = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & -1 \\
1 & 1
\end{pmatrix},
D = \begin{pmatrix}
1 + i & 0 \\
0 & 1 - i
\end{pmatrix}.
\]

From (27), we have

\[
\begin{align*}
A^n &= FD^nF^* \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & -1 \\
1 & 1
\end{pmatrix} \begin{pmatrix}
(1 + i)^n & 0 \\
0 & (1 - i)^n
\end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 1 \\
-1 & 1
\end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix}
(1 + i)^n + (1 - i)^n & (1 + i)^n - (1 - i)^n \\
(1 + i)^n - (1 - i)^n & (1 + i)^n + (1 - i)^n
\end{pmatrix}.
\end{align*}
\]

Using (28), it follows that for any \( k = j, k = 1, 2 \),
\[ |(A^n)_{kj}|^2 = \frac{1}{2} \{ (1 + i)^n + (1 - i)^n \} \frac{1}{2} \{ (1 + i)^n + (1 - i)^n \} \]
\[ = \frac{1}{4} \{ (1 + i)^n + (1 - i)^n \} \{ (1 - i)^n + (1 + i)^n \} \]
\[ = \frac{1}{4} \{ (1 + i)^n + (1 - i)^n \}^2 \]
\[ = \frac{1}{4} \{ \left( \sqrt{2} \frac{1 + i}{\sqrt{2}} \right)^n + \left( \sqrt{2} \frac{1 - i}{\sqrt{2}} \right)^n \}^2 \]
\[ = \frac{1}{4} \{ \left( \sqrt{2} \right)^n \left( \frac{1 + i}{\sqrt{2}} \right)^n + \left( \sqrt{2} \right)^n \left( \frac{1 - i}{\sqrt{2}} \right)^n \}^2 \]
\[ = 2^n \left\{ \left( \exp \left( \frac{n\pi}{4} i \right) \right)^n + \left( \exp \left( -\frac{n\pi}{4} i \right) \right)^n \right\} \]
\[ = 2^n \left\{ \exp \left( \frac{n\pi}{4} i \right) + \exp \left( -\frac{n\pi}{4} i \right) \right\}^2 \]
\[ = 2^n \left\{ \left( \sqrt{2} \right)^n \left( 1 + i \sqrt{2} \right)^n \right\} \]
\[ = 2^n \cos^2 \left( \frac{n\pi}{4} \right) \]

and for any \( k \neq j, k = 1, 2, \)
\[ \left| (A^n)_{kj} \right|^2 = \frac{1}{2} \{ (1 + i)^n - (1 - i)^n \} \frac{1}{2} \{ (1 + i)^n - (1 - i)^n \} \]
\[ = \frac{1}{4} \{ (1 + i)^n - (1 - i)^n \} \{ (1 - i)^n - (1 + i)^n \} \]
\[ = -\frac{1}{4} \{ (1 + i)^n - (1 - i)^n \}^2 \]
\[ = -\frac{1}{4} \{ \left( \sqrt{2} \frac{1 + i}{\sqrt{2}} \right)^n - \left( \sqrt{2} \frac{1 - i}{\sqrt{2}} \right)^n \}^2 \]
\[ = -\frac{1}{4} \{ \left( \sqrt{2} \right)^n \left( \frac{1 + i}{\sqrt{2}} \right)^n - \left( \sqrt{2} \right)^n \left( \frac{1 - i}{\sqrt{2}} \right)^n \}^2 \]
\[ = -2^n \left\{ \left( \exp \left( \frac{n\pi}{4} i \right) \right)^n - \left( \exp \left( -\frac{n\pi}{4} i \right) \right)^n \right\} \]
\[ = -2^n \left\{ \cos \left( \frac{n\pi}{4} \right) + i \sin \left( \frac{n\pi}{4} \right) \right\} \]
\[ = -2^n \left\{ \left( \sqrt{2} \right)^n \left( 1 + i \sqrt{2} \right)^n \right\} \]
\[ = -\frac{2^n}{4} \left\{ 2i \sin \left( \frac{n\pi}{4} \right) \right\}^2 \]
By a direct calculation, we obtain

\[ r_n^{(N)} = \frac{1}{2^n} \sum_{j=0}^{2^N-1} \frac{j + 1/2}{2^N} \prod_{k=1}^{p} (A^{m+1})^{\xi_{N-p+k}} \prod_{l=1}^{N-p} (A^{m})^{\xi_{p+l}}. \]

Proof of THEOREM 5.3: For the case \( n = mN + p, \ p = 1, \ldots, N - 1 \) and \( m \in \mathbb{N}, \)

\[ r_n^{(N)} = \frac{1}{2^n} \sum_{j=0}^{2^N-1} \frac{j + 1/2}{2^N} \prod_{k=1}^{p} (A^{m+1})^{\xi_{N-p+k}} \prod_{l=1}^{N-p} (A^{m})^{\xi_{p+l}}. \]

By a direct calculation, we obtain

\[
\begin{align*}
    r_n^{(N)} &= \frac{1}{2^n} \sum_{j=0}^{2^N-1} \frac{j + 1/2}{2^N} \prod_{k=1}^{p} (A^{m+1})^{\xi_{N-p+k}} \prod_{l=1}^{N-p} (A^{m})^{\xi_{p+l}} \\
    &= \frac{1}{2^n} \sum_{j=0}^{2^N-1} \frac{j + 1/2}{2^N} \prod_{l=1}^{N-p} (A^{m})^{\xi_{p+l}} \prod_{k=1}^{p} (A^{m+1})^{\xi_{N-p+k}} \\
    &= \frac{1}{2^n} \sum_{j=0}^{2^N-1} \left( \frac{j + 1}{2} \right)^{N-p} \prod_{l=1}^{N-p} (A^{m})^{\xi_{p+l}} \prod_{k=1}^{p} (A^{m+1})^{\xi_{N-p+k}} \\
    &= \frac{1}{2^n} \sum_{j_1, \ldots, j_N} \left( \sum_{k=1}^{N-p} j_k 2^{N-k} \right) + \frac{1}{2} \prod_{l=1}^{N-p} (A^{m})^{\xi_{p+l}} \prod_{k=1}^{p} (A^{m+1})^{\xi_{N-p+k}} \\
    &= \frac{1}{2^{n+N}} \sum_{j_1, \ldots, j_N} \left( \sum_{k=1}^{N-p} j_k 2^{N-k} \right) + \frac{1}{2} \prod_{l=1}^{N-p} (A^{m})^{\xi_{p+l}} \prod_{k=1}^{p} (A^{m+1})^{\xi_{N-p+k}} \\
    &= \prod_{l=1}^{N-p} (A^{m})^{\xi_{p+l}} \prod_{k=1}^{p} (A^{m+1})^{\xi_{N-p+k}} \\
    &= 2^n \sin^2 \left( \frac{n\pi}{4} \right) \] 

By the above lemma, we have

\[ \left| (A^{m})^{\xi_{p+l}} \right|^2 = \begin{cases} 
2^m & \text{if } j_l = \xi_{p+l} \\
0 & \text{if } j_l \neq \xi_{p+l} 
\end{cases}, \quad \left| (A^{m+1})^{\xi_{N-p+j_k}} \right|^2 = 2^m \]
for any \( l = 1, \cdots, N - p \) and \( k = N - p + 1, \cdots, N \). Using this formula the product of absolute squares can be expressed as

\[
\left| \prod_{l=1}^{N-p} (A^m)_{\xi_p+j_l} \right|^2 \prod_{k=N-p+1}^{N} (A^{m+1})_{\xi_k-(N-p)j_k} = \begin{cases} 
(2^m)^{N-p} \ (2^m)^p \text{ if } j_l = \xi_{p+l} \text{ for all } l = 1, \cdots, N - p \\
0 & \text{ otherwise}
\end{cases} = \begin{cases} 
2^{mN} \text{ if } j_l = \xi_{p+l} \text{ for all } l = 1, \cdots, N - p \\
0 & \text{ otherwise}
\end{cases}
\]

\[
(29) \text{ can be rewritten as}
\]

\[
r_n^{(N)} = \frac{1}{2^{(m+1)N+p}} \sum_{j_1, \cdots, j_N} \left\{ \left( \sum_{k=1}^{N-p} j_k 2^{N-k} \right) + \left( \sum_{k=N-p+1}^{N} j_k 2^{N-k} \right) + \frac{1}{2} \right\} \\
\times \prod_{l=1}^{N-p} \left| (A^m)_{\xi_p+j_l} \right|^2 \prod_{k=N-p+1}^{N} \left| (A^{m+1})_{\xi_k-(N-p)j_k} \right|^2
\]

\[
= \frac{2^{mN}}{2^{(m+1)N+p}} \sum_{j_{N-p+1}, \cdots, j_N} \left\{ \left( \sum_{k=1}^{N-p} \xi_{p+k} 2^{N-k} \right) + \left( \sum_{k=N-p+1}^{N} j_k 2^{N-k} \right) + \frac{1}{2} \right\} \tag{30}
\]

\[
= \frac{1}{2^{N+p}} \left( \sum_{k=1}^{N-p} \xi_{p+k} 2^{N-k} \right) \left( \sum_{j_{N-p+1}, \cdots, j_N} 1 \right) + \frac{1}{2^{N+p}} \sum_{j_{N-p+1}, \cdots, j_N} \left( \sum_{k=N-p+1}^{N} j_k 2^{N-k} \right)
\]

\[
+ \frac{1}{2^{N+p} 2} \left( \sum_{j_{N-p+1}, \cdots, j_N} 1 \right)
\]

\[
= \frac{1}{2^N} \sum_{k=1}^{N-p} \xi_{p+k} 2^{N-k} + \frac{1}{2^{N+p}} \sum_{j_{N-p+1}, \cdots, j_N} \left( \sum_{k=N-p+1}^{N} j_k 2^{N-k} \right) + \frac{1}{2^{N+1}}
\]

\[
= \frac{1}{2^N} \sum_{k=1}^{N-p} \xi_{p+k} 2^{N-k} + \frac{1}{2^{N+p}} \sum_{j_{N-p+1}, \cdots, j_N} \left( \sum_{k=1}^{p} j_{N-p+k} 2^{p-k} \right) + \frac{1}{2^{N+1}}
\]

\[
= \frac{1}{2^N} \sum_{k=1}^{N-p} \xi_{p+k} 2^{N-k} + \frac{1}{2^{N+p}} \sum_{k=0}^{2^p-1} k + \frac{1}{2^{N+1}}
\]

\[
= \frac{1}{2^N} \sum_{k=1}^{N-p} \xi_{p+k} 2^{N-k} + \frac{1}{2^{N+p}} \left( 2^p - 1 \right) 2^p + \frac{1}{2^{N+1}}
\]
From the above lemma, we have

\[\prod_{k=1}^{N-p} |(A^m)_{\xi_p+k}|^2 = 2^{m-1}, \quad \prod_{k=N-p+1}^{N} |(A^{m+1})_{\xi_{k-(N-p)}}|^2 = \begin{cases} 2^{m+1} & \text{if } j_k \neq \xi_{k-(N-p)} \\ 0 & \text{if } j_k = \xi_{k-(N-p)} \end{cases}\]

for any \(l = 1, \ldots, N-p\) and \(k = N-p+1, \ldots, N\). Using this formula the product of absolute squares can be expressed as

\[
\prod_{l=1}^{N-p} |(A^m)_{\xi_{p+l}}|^2 \prod_{k=N-p+1}^{N} |(A^{m+1})_{\xi_{k-(N-p)}}|^2 = \begin{cases} (2^{m-1})^{N-p} (2^{m+1})^p & \text{if } j_k \neq \xi_{k-(N-p)} \text{ for all } k = N-p+1, \ldots, N \\ 0 & \text{otherwise} \end{cases}
\]

= \begin{cases} 2^{(m-1)N+2p} & \text{if } j_k \neq \xi_{k-(N-p)} \text{ for all } k = N-p+1, \ldots, N \\ 0 & \text{otherwise} \end{cases}.

Let \(\eta_{k-(N-p)} = \xi_{k-(N-p)} + 1 \pmod 2\), \(k = N-p+1, \ldots, N\). It follows that

\[
r_{n}^{(N)} = \frac{1}{2^{(m+1)N+p}} \sum_{j_1, \ldots, j_N} \left\{ \left( \sum_{k=1}^{N-p} j_k 2^{N-k} \right) + \left( \sum_{k=N-p+1}^{N} j_k 2^{N-k} \right) + \frac{1}{2} \right\}
\times \prod_{l=1}^{N-p} |(A^m)_{\xi_{p+l}}|^2 \prod_{k=N-p+1}^{N} |(A^{m+1})_{\xi_{k-(N-p)}}|^2
\]

= \frac{2^{(m-1)N+2p}}{2^{(m+1)N+p}} \sum_{j_1, \ldots, j_{N-p}} \left\{ \left( \sum_{k=1}^{N-p} j_k 2^{N-k} \right) + \left( \sum_{k=N-p+1}^{N} \eta_{k-(N-p)} 2^{N-k} \right) + \frac{1}{2} \right\} \quad (31)

= \frac{1}{2^{2N-p}} \sum_{j_1, \ldots, j_{N-p}} \left\{ \left( \sum_{k=1}^{N-p} j_k 2^{N-k} \right) + \left( \sum_{k=N-p+1}^{N} \eta_{k-(N-p)} 2^{N-k} \right) + \frac{1}{2} \right\}

= \frac{1}{2^{2N-p}} \sum_{j_1, \ldots, j_{N-p}} \left( \sum_{k=1}^{N-p} j_k 2^{N-k} \right) + \frac{1}{2^{2N-p}} \left( \sum_{k=N-p+1}^{N} \eta_{k-(N-p)} 2^{N-k} \right) \left( \sum_{j_1, \ldots, j_{N-p}} 1 \right)

+ \frac{1}{2^{2N-p}} \left( \sum_{j_1, \ldots, j_{N-p}} 1 \right)
\]
\[
\frac{1}{2^{N-p}} \sum_{j_1, \ldots, j_{N-p}} \left( \sum_{k=1}^{N-p} j_k 2^{N-k} \right) + \frac{2^{N-p}}{2^{N+1}} \left( \sum_{k=N-p+1}^{N} \eta_{k-(N-p)} 2^{N-k} \right) + \frac{2^{N-p}}{2^{N+1}} \frac{1}{2} \\
= \frac{1}{2^{N-p}} \sum_{j_1, \ldots, j_{N-p}} \left( \sum_{k=1}^{N-p} j_k 2^{N-k} \right) + \sum_{k=N-p+1}^{N} \eta_{k-(N-p)} 2^{N-k} + \frac{1}{2^{N+1}} \\
= \frac{2^p}{2^{N+1}} \sum_{j_1, \ldots, j_{N-p}} \left( \sum_{k=1}^{N-p} j_k 2^{N-p-k} \right) + \sum_{k=N-p+1}^{N} \eta_{k-(N-p)} 2^{N-k} + \frac{1}{2^{N+1}} \\
= \frac{2^p}{2^{N+1}} \frac{1}{2} (2^{N-p} - 1) 2^{N-p} + \sum_{k=N-p+1}^{N} \eta_{k-(N-p)} 2^{N-k} + \frac{1}{2^{N+1}} \\
= \sum_{k=N-p+1}^{N} \eta_{k-(N-p)} 2^{N-k} + \frac{2^N - 2^p + 1}{2^{N+1}}.
\]

(iii) \( m = 2 \pmod{4} \)

From the above lemma, we have

\[
\left| (A^m)_{\xi_{p+l} j_l} \right|^2 = \begin{cases} 
2^m & \text{if } j_l \neq \xi_{p+l} \\
0 & \text{if } j_l = \xi_{p+l} 
\end{cases},
\]

\[
\left| (A^{m+1})_{\xi_{k-(N-p)} j_k} \right|^2 = 2^m
\]

for any \( l = 1, \ldots, N - p \) and \( k = N - p + 1, \ldots, N \). Using this formula the product of absolute squares can be expressed as

\[
\prod_{l=1}^{N-p} \left| (A^m)_{\xi_{p+l} j_l} \right|^2 \prod_{k=N-p+1}^{N} \left| (A^{m+1})_{\xi_{k-(N-p)} j_k} \right|
\]

\[
= \begin{cases} 
(2^m)^{N-p} (2^m)^p & \text{if } j_l \neq \xi_{p+l} \text{ for all } l = 1, \ldots, N - p \\
0 & \text{otherwise}
\end{cases}
\]

\[
= \begin{cases} 
2^{mN} & \text{if } j_l \neq \xi_{p+l} \text{ for all } l = 1, \ldots, N - p \\
0 & \text{otherwise}
\end{cases}
\]

Let \( \eta_{p+l} = \xi_{p+l} + 1 \pmod{2}, l = 1, \ldots, N - p \). It follows that

\[
r_n^{(N)} = \frac{1}{2^{(m+1)N+p}} \sum_{j_1, \ldots, j_N} \left\{ \left( \sum_{k=1}^{N-p} j_k 2^{N-k} \right) + \left( \sum_{k=N-p+1}^{N} j_k 2^{N-k} \right) + \frac{1}{2} \right\}
\]
Substituting $\xi$ from the above lemma, we have

$$(iv) \quad m = \frac{N-p}{2(m+1)N+p} \sum_{j_{N-p+1}, \cdots, j_N} \left\{ \left( \sum_{k=1}^{N-p} \eta_{p+k} 2^{N-k} \right) + \left( \sum_{k=N-p+1}^{N} j_k 2^{N-k} \right) + \frac{1}{2} \right\}$$

Substituting $\eta_{p+k}$ for $\xi_{p+k}$ in (31), we get

$$r_n^{(N)} = \sum_{k=1}^{N-p} \eta_{p+k} 2^{-k} + \frac{2^p}{2^{N+1}}$$

(iv) $m = 3 \pmod{4}$

From the above lemma, we have

$$\left| (A^m)_{\xi_{p+lj}} \right|^2 = 2^{m-1}, \quad \left| (A^{m+1})_{\xi_{k-(N-p)jk}} \right|^2 = \begin{cases} 2^{m+1} & \text{if } j_k = \xi_{k-(N-p)} \\ 0 & \text{if } j_k \neq \xi_{k-(N-p)} \end{cases}$$

for any $l = 1, \cdots, N-p$ and $k = N-p+1, \cdots, N$. Using this formula the product of absolute squares can be expressed as

$$\prod_{l=1}^{N-p} \left| (A^m)_{\xi_{p+lj}} \right|^2 \prod_{k=N-p+1}^{N} \left| (A^{m+1})_{\xi_{k-(N-p)jk}} \right|^2 = \begin{cases} (2^{m-1})^{N-p} (2^{m+1})^p & \text{if } j = \xi_{k-(N-p)} \text{ for all } k = N-p+1, \cdots, N \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 2^{(m-1)N+2p} & \text{if } j_k \neq \xi_{k-(N-p)} \text{ for all } k = N-p+1, \cdots, N \\ 0 & \text{otherwise} \end{cases}$$

(29) can be rewritten as

$$r_n^{(N)} = \frac{1}{2(m+1)N+p} \sum_{j_1, \cdots, j_N} \left\{ \left( \sum_{k=1}^{N-p} j_k 2^{N-k} \right) + \left( \sum_{k=N-p+1}^{N} j_k 2^{N-k} \right) + \frac{1}{2} \right\}$$

$$\times \prod_{l=1}^{N-p} \left| (A^m)_{\xi_{p+lj}} \right|^2 \prod_{k=N-p+1}^{N} \left| (A^{m+1})_{\xi_{k-(N-p)jk}} \right|^2$$

$$= \frac{2^{(m-1)N+2p}}{2(m+1)N+p} \sum_{j_{N-p+1}, \cdots, j_N} \left\{ \left( \sum_{k=1}^{N-p} j_k 2^{N-k} \right) + \left( \sum_{k=N-p+1}^{N} \xi_{k-(N-p)} 2^{N-k} \right) + \frac{1}{2} \right\}$$

Substituting $\xi_{k-(N-p)}$ for $\eta_{k-(N-p)}$ in (31), we get
\[ r^{(N)}_n = \sum_{k=N-p-1}^{N-p} \xi_{k-(N-p)} 2^{-k} + \frac{2^N - 2^p + 1}{2^{N+1}}. \]

**Proof of Theorem 5.4:** For any \( n = mN, m \in \mathbb{N} \),

\[ r^{(N)}_n = \frac{1}{2^n} \sum_{j=0}^{2^{N-1}} \left( \frac{j + 1/2}{2N} \prod_{k=1}^{N} |(A^m)_{\xi_{jk}}|^2 \right). \]

By a direct calculation, we obtain

\[ r^{(N)}_n = \frac{1}{2^n} \sum_{j=0}^{2^{N-1}} \left( \frac{j + 1/2}{2N} \prod_{k=1}^{N} |(A^m)_{\xi_{jk}}|^2 \right) \]

\[ = \frac{1}{2^{n+N}} \sum_{j=0}^{2^{N-1}} \left( j + 1/2 \right) \prod_{k=1}^{N} |(A^m)_{\xi_{jk}}|^2 \]

\[ = \frac{1}{2^{(m+1)N}} \sum_{j_1, \cdots, j_N} \left\{ \left( \sum_{k=1}^{N} j_k 2^{N-k} \right) + \frac{1}{2} \right\} \prod_{k=1}^{N} |(A^m)_{\xi_{jk}}|^2 \]

(i) \( m = 0 \) (mod4)

From the above lemma, we have

\[ |(A^m)_{\xi_{jk}}|^2 = \begin{cases} 2^m & \text{if } j_k = \xi_k \\ 0 & \text{if } j_k \neq \xi_k \end{cases} \]

for any \( k = 1, \cdots, N \). Using this formula the product of absolute squares can be expressed as

\[ \prod_{l=1}^{N} |(A^m)_{\xi_{jk}}|^2 = \begin{cases} 2^{mN} & \text{if } j_k = \xi_k \text{ for all } k = 1, \cdots, N \\ 0 & \text{otherwise} \end{cases} \]

Using this formula the mean value \( r^{(N)}_n \) of the position operator can be expressed as

\[ r^{(N)}_n = \frac{1}{2^{(m+1)N}} \sum_{j_1, \cdots, j_N} \left\{ \left( \sum_{k=1}^{N} j_k 2^{N-k} \right) + \frac{1}{2} \right\} \prod_{k=1}^{N} |(A^m)_{\xi_{jk}}|^2 \]

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\[
\begin{align*}
&= \frac{2^{mN}}{2^{(m+1)N}} \left\{ \left( \sum_{k=1}^{N} \xi_k 2^{N-k} \right) + \frac{1}{2} \right\} \\
&= \sum_{k=1}^{N} \xi_k 2^{-k} + \frac{1}{2^{N+1}}.
\end{align*}
\]

(ii) \( m = 1, 3 \pmod{4} \)

From the above lemma, we have

\[
\left| (A^m)_{\xi_k j_k} \right|^2 = 2^{m-1}
\]

for any \( k = 1, \ldots, N \). Note that

\[
\prod_{l=1}^{N} \left| (A^m)_{\xi_k j_k} \right|^2 = 2^{(m-1)N}.
\]

Using this formula the mean value \( r_n^{(N)} \) of the position operator can be expressed as

\[
r_n^{(N)} = \frac{1}{2^{(m+1)N}} \sum_{j_1, \ldots, j_N} \left\{ \left( \sum_{k=1}^{N} j_k 2^{N-k} \right) + \frac{1}{2} \right\} \prod_{k=1}^{N} \left| (A^m)_{\xi_k j_k} \right|^2
\]

\[
= \frac{2^{(m-1)N}}{2^{(m+1)N}} \sum_{j_1, \ldots, j_N} \left\{ \left( \sum_{k=1}^{N} j_k 2^{N-k} \right) + \frac{1}{2} \right\}
\]

\[
= \frac{1}{2^{2N}} \left\{ \sum_{j_1, \ldots, j_N} \left( \sum_{k=1}^{N} j_k 2^{N-k} \right) + \frac{1}{2} \left( \sum_{j_1, \ldots, j_N} 1 \right) \right\}
\]

\[
= \frac{1}{2^{2N}} \sum_{k=0}^{2^{N-1}} k + \frac{1}{2^{N+1}}
\]

\[
= \frac{1}{2^{2N}} 2^{N-1} (2^N - 1) + \frac{1}{2^{N+1}}
\]

\[
= \frac{1}{2}.
\]

(iii) \( m = 2 \pmod{4} \)

From the above lemma, we have

\[
\left| (A^m)_{\xi_k j_k} \right|^2 = \begin{cases} 
2^m & \text{if } j_k \neq \xi_k \\
0 & \text{if } j_k = \xi_k
\end{cases}
\]
for any \( k = 1, \cdots, N \). Using this formula the product of absolute squares can be expressed as

\[
\prod_{l=1}^{N} |(A^m)_{\xi_kj_k}|^2 = \begin{cases} 
2^{mN} & \text{if } j_k \neq \xi_k \text{ for all } k = 1, \cdots, N \vspace{10pt} \\ 
0 & \text{otherwise}
\end{cases}
\]

Let \( \eta_k = \xi_k + 1 \pmod 2 \), \( k = 1, \cdots, N \). It follows that

\[
\begin{align*}
\ell_n^{(N)} &= \frac{1}{2^{(m+1)N}} \sum_{j_1, \cdots, j_N} \left\{ \left( \sum_{k=1}^{N} j_k 2^{N-k} \right) + \frac{1}{2} \right\} \prod_{k=1}^{N} |(A^m)_{\xi_kj_k}|^2 \\
&= \frac{2^{mN}}{2^{(m+1)N}} \left\{ \sum_{k=1}^{N} \eta_k 2^{N-k} \right\} + \frac{1}{2} \\
&= \sum_{k=1}^{N} \eta_k 2^{-k} + \frac{1}{2^{N+1}}
\end{align*}
\]

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