ON G_2-STRUCTURES, SPECIAL METRICS AND RELATED FLOWS

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ABSTRACT. We review results about G_2-structures in relation to the existence of special metrics, such as Einstein metrics and Ricci solitons, and the evolution under the Laplacian flow on non-compact homogeneous spaces. We also discuss some examples in detail.

1. Introduction

A G_2-structure on a seven-dimensional manifold M is characterized by the existence of a globally defined 3-form \( \varphi \) which can be pointwise written as

\[
\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245},
\]

with respect to a suitable basis \( \{e^1, \ldots, e^7\} \) of the cotangent space. Here, the shorthand \( e^{ijk} \) stands for \( e^i \wedge e^j \wedge e^k \). Such a 3-form \( \varphi \) gives rise to a Riemannian metric \( g_\varphi \) and an orientation \( dV_\varphi \) on \( M \).

The intrinsic torsion of a G_2-structure \( \varphi \) can be identified with the covariant derivative \( \nabla^\varphi \varphi \), \( \nabla^\varphi \) being the Levi Civita connection of \( g_\varphi \). By [24], it vanishes identically if and only if both \( d\varphi = 0 \) and \( d^* \varphi = 0 \), where \( * \varphi \) denotes the Hodge operator of \( g_\varphi \). When this happens, the G_2-structure is said to be torsion-free, its associated Riemannian metric \( g_\varphi \) is Ricci-flat and the corresponding Riemannian holonomy group is a subgroup of the exceptional Lie group \( G_2 \).

G_2-structures can be divided into classes, which are characterized by the expression of the exterior derivatives \( d\varphi \) and \( d^* \varphi \) [11, 24]. A G_2-structure \( \varphi \) is called closed (or calibrated according to [31]) if \( d\varphi = 0 \), while it is called coclosed (or cocalibrated) if \( d^* \varphi = 0 \).

Since the Ricci tensor and the scalar curvature of the metric induced by a G_2-structure can be expressed in terms of the intrinsic torsion [11], it may happen that certain restrictions on the curvature give rise to some constraints on the intrinsic torsion. For instance, a calibrated G_2-structure on a compact manifold induces an Einstein metric if and only if it is also cocalibrated, i.e., if and only if it is torsion-free [11, 16]. A natural problem consists then in investigating whether this happens also in the non-compact case, and whether similar results also hold when the metric is a Ricci soliton. These problems were studied for calibrated G_2-structures on homogeneous spaces in [21, 22], and for the wider class of locally conformal calibrated G_2-structures in [26]. We shall review the results in Section 3.

A useful tool to study geometric structures on manifolds is represented by geometric flows. Let \( M \) be a 7-manifold endowed with a calibrated G_2-structure \( \varphi_0 \). The Laplacian...
flow starting from $\phi_0$ is the initial value problem
\[
\begin{aligned}
\frac{\partial}{\partial t} \phi(t) &= \Delta \phi(t), \\
\frac{d}{dt} \phi(t) &= 0, \\
\phi(0) &= \phi_0,
\end{aligned}
\]
where $\Delta \phi = dd^* \phi + d^* d \phi$ is the Hodge Laplacian of $\phi$ with respect to the metric $g_\phi$. This flow was introduced by Bryant in [11] to study 7-manifolds admitting calibrated $G_2$-structures. Short-time existence and uniqueness of the solution when $M$ is compact were proved in the unpublished paper [13]. Recently, the analytic and geometric properties of the Laplacian flow have been deeply investigated in the series of papers [45, 46, 47]. In particular, the authors obtained a long-time existence result, and they proved that the solution exists for all positive times and it converges to a torsion-free $G_2$-structure modulo diffeomorphism provided that the initial datum $\phi_0$ is sufficiently close to a given torsion-free $G_2$-structure.

The first noncompact examples with long-time existence of the solution were obtained on seven-dimensional nilpotent Lie groups in [22], while further solutions on solvable Lie groups were described in [27, 43, 44, 49]. Moreover, a cohomogeneity one solution converging to a torsion-free $G_2$-structure on the 7-torus was worked out in [35]. In Section 4, we shall discuss the results on nilpotent Lie groups obtained in [22].

2. Preliminaries

Let $M$ be a seven-dimensional manifold endowed with a $G_2$-structure $\phi$. The Riemannian metric $g_\phi$ and the volume form $dV_\phi$ are determined by $\phi$ via the equation
\[
g_\phi(X, Y) dV_\phi = \frac{1}{6} i_X \phi \wedge i_Y \phi \wedge \phi,
\]
for all vector fields $X, Y$ on $M$.

The vanishing of the intrinsic torsion $T_\phi$ of a $G_2$-structure $\phi$ can be stated in the following equivalent ways.

**Theorem 2.1** ([24]). Let $\phi$ be a $G_2$-structure on a seven-dimensional manifold $M$. Then, the following conditions are equivalent:

a) the intrinsic torsion of $\phi$ vanishes identically;

b) $\nabla^e \phi = 0$, where $\nabla^e$ denotes the Levi Civita connection of $g_\phi$;

c) $d \phi = 0$ and $d^* \phi \phi = 0$;

d) $\text{Hol}(g_\phi)$ is isomorphic to a subgroup of $G_2$.

A $G_2$-structure satisfying any of the above conditions is said to be *torsion-free* or *parallel*. By [9], the Riemannian metric induced by a torsion-free $G_2$-structure $\phi$ is Ricci-flat, i.e., $\text{Ric}(g_\phi) = 0$.

More generally, as the intrinsic torsion $T_\phi$ is a section of a vector bundle over $M$ with fibre $\mathbb{R}^7 \otimes g_2^1$, $G_2$-structures can be divided into classes according to the vanishing of the components of $T_\phi$ with respect to the $G_2$-irreducible decomposition
\[
\mathbb{R}^7 \otimes g_2^1 \cong X_1 \oplus X_2 \oplus X_3 \oplus X_4 = \mathbb{R} \oplus g_2 \oplus S_0^2(\mathbb{R}^7) \oplus \mathbb{R}^7,
\]
where $S_0^2(\mathbb{R}^7)$ denotes the space of traceless symmetric 2-tensors and $\mathfrak{g}_2 = \text{Lie}(G_2)$. This gives rise to sixteen classes of $G_2$-structures, which were first described in [24].

By [11], it is also possible to characterize each class in terms of the exterior derivatives $d\varphi$ and $d*\varphi$. In detail, the spaces $\Lambda^k(\mathbb{R}^7^*)$, $k = 2, 3$, admit the following $G_2$-irreducible decompositions (cf. [10])

$$
\Lambda^2(\mathbb{R}^7^*) = \Lambda_0^2(\mathbb{R}^7^*) \oplus \Lambda_1^2(\mathbb{R}^7^*),
$$
$$
\Lambda^3(\mathbb{R}^7^*) = \Lambda_1^3(\mathbb{R}^7^*) \oplus \Lambda_2^3(\mathbb{R}^7^*) \oplus \Lambda_3^3(\mathbb{R}^7^*),
$$

where the subscript in $\Lambda^k(\mathbb{R}^7^*)$ denotes the dimension of the summand as an irreducible $G_2$-module, and $\Lambda_1^2(\mathbb{R}^7^*) \cong \mathfrak{g}_2$, $\Lambda_2^3(\mathbb{R}^7^*) \cong S_0^2(\mathbb{R}^7)$. Consequently, on $M$ there exist a unique function $\tau_0 \in \mathcal{C}^\infty(M)$ and unique differential forms $\tau_1 \in \Omega^1(M)$, $\tau_2 \in \Omega^2(M)$, $\tau_3 \in \Omega^3(M)$ such that

$$
d\varphi = \tau_0 \ast \varphi + 3\tau_1 \wedge \varphi + \ast \varphi \tau_3,
$$
$$
d*\varphi = 4\tau_1 \wedge \ast \varphi + \tau_2 \wedge \varphi.
$$

The differential forms $\tau_0, \tau_1, \tau_2, \tau_3$ are called intrinsic torsion forms of the $G_2$-structure $\varphi$, and they can be identified with the components of the intrinsic torsion $T_\varphi$ belonging to the $G_2$-modules $\mathcal{X}_1, \mathcal{X}_4, \mathcal{X}_2, \mathcal{X}_3$, respectively.

Some classes of $G_2$-structures with the defining conditions are recalled in Table 1.

| class | type | conditions |
|-------|------|------------|
| $\mathcal{X}_1$ | nearly parallel | $\tau_1, \tau_2, \tau_3 = 0$ |
| $\mathcal{X}_2$ | closed, calibrated | $\tau_0, \tau_1, \tau_3 = 0$ |
| $\mathcal{X}_4$ | locally conformal parallel | $\tau_0, \tau_2, \tau_3 = 0$ |
| $\mathcal{X}_1 \oplus \mathcal{X}_3$ | coclosed, cocalibrated | $\tau_1, \tau_2 = 0$ |
| $\mathcal{X}_2 \oplus \mathcal{X}_4$ | locally conformal calibrated | $\tau_0, \tau_3 = 0$ |

Table 1. Some classes of $G_2$-structures

2.1. Link with SU(3)-structures. An SU(3)-structure on a six-dimensional manifold $N$ is the data of an almost Hermitian structure $(g, J)$ with fundamental 2-form $\omega := g(J\cdot, \cdot)$ and a complex volume form $\Psi = \psi + i\hat{\psi} \in \Omega^{3,0}(M)$ of nonzero constant length.

By [34], an SU(3)-structure $(g, J, \Psi)$ is completely determined by the real 2-form $\omega$ and the real 3-form $\psi$.

Since $G_2$ acts transitively on the 6-sphere with isotropy SU(3), every $G_2$-structure on a 7-manifold $M$ induces an SU(3)-structure on each oriented hypersurface. In particular, if $M$ is endowed with a torsion-free $G_2$-structure $\varphi$, and $N \subset M$ is an oriented hypersurface, then $\varphi$ induces an SU(3)-structure $(\omega, \psi)$ on $N$ which is half-flat according to the definition given in [14]. This means that the differential forms $\omega$ and $\psi$ satisfy the conditions

$$
d(\omega \wedge \omega) = 0, \quad d\psi = 0.$$

The inverse problem, i.e., establishing whether a half-flat SU(3)-structure on a 6-manifold is induced by an immersion into a 7-manifold with a torsion-free $G_2$-structure, can be analyzed using the so-called Hitchin flow equations (see [12, 34] for details).

We now recall the definition of some special types of half-flat SU(3)-structures.

**Definition 2.2.** A half-flat SU(3)-structure $(\omega, \psi)$ such that $d\omega = c\psi$ for some real number $c$ is said to be coupled if $c \neq 0$, while it is called symplectic half-flat if $c = 0$, i.e., if the 2-form $\omega$ is symplectic. A coupled SU(3)-structure satisfying the additional condition $d\tilde{\psi} = -\frac{2}{3}c\omega \wedge \omega$ is called nearly Kähler.

If $N$ is a 6-manifold endowed with an SU(3)-structure $(\omega, \psi)$, then the product manifold $N \times \mathbb{R}$ admits a $G_2$-structure defined by the 3-form

$$\varphi := \omega \wedge dt + \psi,$$

where $dt$ is the global 1-form on $\mathbb{R}$. Such $\varphi$ induces the product metric $g_\varphi = g + dt^2$. Moreover, $\varphi$ is calibrated (resp. locally conformal calibrated) if the SU(3)-structure $(\omega, \psi)$ is symplectic half-flat (resp. coupled), while $\varphi$ is locally conformal parallel if $(\omega, \psi)$ is nearly Kähler.

### 3. $G_2$-Structures and Special Metrics

By [11], the Ricci tensor and the scalar curvature of the metric induced by a $G_2$-structure $\varphi$ can be expressed in terms of the intrinsic torsion forms $\tau_i$. In particular, the scalar curvature is given by

$$\text{Scal}(g_\varphi) = 12d^*\tau_1 + \frac{21}{8}\tau_0^2 + 30|\tau_1|^2 - \frac{1}{2}|\tau_2|^2 - \frac{1}{2}|\tau_3|^2,$$

where $|\cdot|$ denotes the pointwise norm induced by $g_\varphi$. Consequently, it has a definite sign for certain classes of $G_2$-structures. For instance, when $\varphi$ is calibrated, then $\text{Scal}(g_\varphi) = -\frac{1}{2}|\tau_2|^2$ is non-positive, while a nearly-parallel $G_2$-structure always induces an Einstein metric with positive scalar curvature $\text{Scal}(g_\varphi) = \frac{21}{8}\tau_0^2$.

A generalization of Einstein metrics is given by Ricci solitons. We recall the definition here.

**Definition 3.1.** A (complete) Riemannian metric $g$ on a smooth manifold $M$ is a *Ricci soliton* if its Ricci tensor satisfies the equation

$$\text{Ric}(g) = \lambda g + \mathcal{L}_X g,$$

for some real constant $\lambda$ and some (complete) vector field $X$, where $\mathcal{L}$ denotes the Lie derivative. If in addition $X$ is the gradient of a smooth function $f \in C^\infty(M)$, i.e., $X = \nabla f$, then $g$ is said to be of *gradient type*.

Equivalently, a Riemannian metric $g$ on $M$ is a Ricci soliton if and only if there exists a positive real valued function $h(t)$ and a family of diffeomorphisms $\eta_t$ such that $g(t) = h(t)\eta_t^*(g)$ is a solution of the Ricci flow starting from $g$ (see e.g. [15, Lemma 2.4]).

Depending on the sign of $\lambda$, a Ricci soliton is called *expanding* ($\lambda < 0$), *steady* ($\lambda = 0$) or *shrinking* ($\lambda > 0$). Moreover, a Ricci soliton is said to be *trivial* if it is either Einstein or the product of a homogeneous Einstein metric with the Euclidean metric. According to
if $M$ is a compact manifold with a Ricci soliton $g$ which is steady or expanding, then $g$ is Einstein.

A special class of Ricci solitons is given by homogeneous ones, which are defined as follows

**Definition 3.2.** A Ricci soliton $g$ on a smooth manifold $M$ is homogeneous if its isometry group acts transitively on $M$.

Properties of non-trivial homogeneous Ricci solitons were given by Lauret in [42]. In particular, he proved the following.

**Proposition 3.3 ([42]).** Let $g$ be a non-trivial homogeneous Ricci soliton on a smooth manifold $M$. Then, $g$ is expanding and it cannot be of gradient type. Moreover, $M$ has to be non-compact.

Currently, all known examples of nontrivial homogeneous Ricci solitons are solvsolitons, that is left-invariant Ricci solitons on simply connected solvable Lie groups.

Since requiring that the metric induced by a $G_2$-structure is Einstein might impose some constraints on the intrinsic torsion, a natural problem is to investigate which types of $G_2$-structures can induce an Einstein (or, more generally, a Ricci soliton) non-Ricci-flat metric, and to see whether there is any difference between the compact and noncompact cases. For instance, if $M$ is a 7-manifold endowed with a locally conformal nearly parallel $G_2$-structure $\varphi$ (torsion class $\lambda_1 \oplus \lambda_4$) with $g_\varphi$ complete and Einstein, then $(M, \varphi)$ is either nearly parallel or conformally equivalent to the standard 7-sphere (17).

In what follows, we consider the cases of calibrated and locally conformal calibrated $G_2$-structures.

**3.1. Calibrated $G_2$-structures.** A calibrated $G_2$-structure $\varphi$ satisfies the equations

$$d\varphi = 0, \quad d^* \varphi = \tau_2 \wedge \varphi,$$

with $\tau_2 \in \Omega^2_{14}(M)$. We collect some known properties of such type of $G_2$-structures in the next results.

**Proposition 3.4 ([11]).** Let $\varphi$ be a calibrated $G_2$-structure on $M$. Then,

1) $\text{Scal}(g_\varphi) \leq 0$ and $\text{Scal}(g_\varphi) = 0$ if and only if $g_\varphi$ is Ricci-flat;

2) $\varphi$ defines an Einstein metric on $M$ if and only if $d^* \varphi = \tau_2 \wedge \varphi$, with $d\tau_2 = \frac{3}{14} |\tau_2|^2 \varphi + \frac{1}{2} \varphi^1 (\tau_2 \wedge \tau_2)$.

**Corollary 3.5 ([11, 16]).** Let $M$ be a compact 7-manifold with a calibrated $G_2$-structure $\varphi$. If the underlying metric $g_\varphi$ is Einstein, then $d^* \varphi = 0$ or, equivalently, the holonomy group of $g_\varphi$ is a subgroup of $G_2$.

The proof of the corollary follows from the identity $d \left( \frac{1}{3} \tau_2^3 \right) = \frac{2}{7} |\tau_2|^4 \varphi^1$ and Stokes’ theorem. In detail, $\tau_2$ must vanish identically since

$$0 = \int_M d \left( \frac{1}{3} \tau_2^3 \right) = \int_M \frac{2}{7} |\tau_2|^4 \varphi^1.$$

In the non-compact case, there is a known non-existence result involving the $^*\text{Ricci tensor}$ and the $^*\text{scalar curvature}$, where

$$\text{Ric}^*(g_\varphi)_{sm} = R_{ijkl} \varphi^i \varphi^j \varphi^k \varphi^l, \quad \text{Scal}^*(g_\varphi) = \text{tr}_{g_\varphi}(\text{Ric}^*(g_\varphi)).$$
Such a result can be stated as follows.

**Theorem 3.6** ([16]). Let \( \varphi \) be a calibrated \( G_2 \)-structure on a 7-manifold \( M \). If \( g_\varphi \) is Einstein and \( \ast \)-Einstein, i.e., \( \text{Ric}^\ast(g_\varphi) = \frac{\text{Scal}^\ast(g_\varphi)}{7} g_\varphi \), then \( g_\varphi \) is Ricci-flat.

In light of the previous results, one might investigate the existence of calibrated \( G_2 \)-structures that are Einstein but non-Ricci-flat on non-compact manifolds. This problem can be viewed as a \( G_2 \)-analogue of the Goldberg conjecture [29], which states that a compact Einstein almost-Kähler manifold has to be Kähler. Recall that a non-compact homogeneous example of Einstein strictly almost Kähler 6-manifold was constructed in [2].

In the homogeneous setting, an answer to the above problem for calibrated \( G_2 \)-structures was given in [21].

All known examples of non-compact homogeneous Einstein manifolds are solvmanifolds, that is, simply connected solvable Lie groups endowed with a left-invariant Einstein metric. The long-standing Alekseevskii conjecture [7, Question 7.5] states that a connected homogeneous Einstein space \( G/K \) of negative scalar curvature must be diffeomorphic to the Euclidean space. Thus, Einstein solvmanifolds might exhaust the class of non-compact homogeneous Einstein manifolds. The conjecture is known to be true in dimensions five and lower by [38, 50], and in dimension seven by [3]. So, seven-dimensional non-compact homogeneous Einstein manifolds are necessarily solvmanifolds.

We now review some general results about Einstein metrics on solvmanifolds of arbitrary dimension.

**Theorem 3.7** ([41]). Every Einstein solvmanifold \( (S, g) \) is standard, i.e., the corresponding solvable metric Lie algebra \( (\mathfrak{s}, \langle \cdot, \cdot \rangle) \) admits the orthogonal decomposition \( \mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a} \), with \( \mathfrak{n} = [\mathfrak{s}, \mathfrak{s}] \) and \( \mathfrak{a} \) abelian.

Recall that the dimension of the abelian summand \( \mathfrak{a} \) in the decomposition \( \mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a} \) is called the rank of the standard solvable metric Lie algebra \( (\mathfrak{s}, \langle \cdot, \cdot \rangle) \).

In contrast to the compact homogeneous case (see e.g. [32, §5] and the references therein), standard Einstein metrics are essentially unique.

**Theorem 3.8** ([32]). A standard Einstein metric is unique up to isometry and scaling among invariant metrics.

**Remark 3.9.**

1. The study of standard Einstein solvmanifolds reduces to those with \( \dim \mathfrak{a} = 1 \) (cf. [32, Thm. 4.18]).

2. The Lie algebra of any standard Einstein solvmanifold resembles an Iwasawa subalgebra of a semisimple Lie algebra, since \( \text{ad}_A \) is symmetric and non-zero for any \( A \neq 0 \in \mathfrak{a} \), and there exists some \( A^0 \in \mathfrak{a} \) such that \( \text{ad}_{A^0}|_{\mathfrak{n}} \) is positive definite (see [32, Thm. 4.10]).

Using the rank of a standard solvable metric Lie algebra, it is possible to get a classification of seven-dimensional Einstein solvmanifolds (see e.g. [21, Thm. 4.4]). Then, using the obstructions to the existence of calibrated \( G_2 \)-structures on Lie algebras given in [18], we have the following result.
Theorem 3.10 ([21]). Let $g_\varphi$ be the metric determined by a left-invariant calibrated $G_2$-structure $\varphi$ on a solvmanifold. Then, $g_\varphi$ is Einstein if and only if $g_\varphi$ is flat.

Remark 3.11. Note that a similar theorem can be proved also for cocalibrated $G_2$-structures [21]. Moreover, Theorem 3.10 shows that left-invariant calibrated $G_2$-structures behave differently from almost Kähler structures [2].

The situation is different if we require that $g_\varphi$ is a non-trivial Ricci soliton. Indeed, non-compact examples of manifolds admitting a calibrated $G_2$-structure inducing a non-trivial Ricci soliton were constructed in [21], and they are all nilsolitons (see Theorem 3.17 and Example 3.19 below).

Definition 3.12. Let $N$ be a simply connected nilpotent Lie group endowed with a left-invariant Riemannian metric $g$, and denote by $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$ the corresponding metric nilpotent Lie algebra. The metric $g$ is called nilsoliton if its Ricci endomorphism $\text{Ric}(g)$ on $\mathfrak{n}$ differs from a derivation $D$ of $\mathfrak{n}$ by a scalar multiple of the identity map $I$, i.e.,

$$\text{Ric}(g) = \lambda I + D,$$

for some real number $\lambda$.

By [40, Prop. 1.1], a left-invariant Riemannian metric on a simply connected nilpotent Lie group is a nilsoliton if and only if it is a Ricci soliton according to Definition 3.1. It is worth recalling here that non-abelian nilpotent Lie groups cannot admit any left-invariant Einstein metric unless it is flat [48].

Remark 3.13. As the existence of a nilsoliton on a simply connected nilpotent Lie group $N$ implies the existence of a non-zero symmetric derivation on the corresponding nilpotent Lie algebra $\mathfrak{n}$, nilsolitons might not exist. This is the case, for instance, of Lie algebras having nilpotent derivation algebra. Such Lie algebras are nilpotent by Engel’s Theorem, and they are known as characteristically nilpotent in literature.

Before reviewing some properties of nilsolitons, we recall the following.

Definition 3.14. Let $(\mathfrak{n}, [\cdot, \cdot]_\mathfrak{n}, \langle \cdot, \cdot \rangle_\mathfrak{n})$ be a metric nilpotent Lie algebra. A metric Lie algebra $(\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}, [\cdot, \cdot], \langle \cdot, \cdot \rangle)$ is a metric solvable extension of $(\mathfrak{n}, [\cdot, \cdot]_\mathfrak{n}, \langle \cdot, \cdot \rangle_\mathfrak{n})$ if the restriction to $\mathfrak{n}$ of the Lie bracket $[\cdot, \cdot]$ of $\mathfrak{s}$ coincides with $[\cdot, \cdot]_\mathfrak{n}$ and $\langle \cdot, \cdot \rangle_{\mathfrak{n} \times \mathfrak{n}} = \langle \cdot, \cdot \rangle_\mathfrak{n}$.

Theorem 3.15 ([40]). Let $N$ be a simply connected nilpotent Lie group with Lie algebra $\mathfrak{n}$. Then,

1) A nilsoliton metric on $N$ is unique up to isometry and scaling;

2) $N$ has a nilsoliton metric $g$ if and only if the corresponding metric Lie algebra $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$ is an Einstein nilradical, i.e., it has a metric solvable extension $\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}$, with $\mathfrak{a}$ abelian, whose corresponding solvmanifold is Einstein.

From now on, we will use the following notation to define a Lie algebra. Suppose that $\mathfrak{g}$ is a seven-dimensional Lie algebra, whose dual space $\mathfrak{g}^*$ is spanned by $\{e^1, \ldots, e^7\}$ satisfying

$$de^i = 0, \quad 1 \leq i \leq 4, \quad de^5 = e^{12}, \quad de^6 = e^{13}, \quad de^7 = 0,$$

where $d$ is the Chevalley-Eilenberg differential of $\mathfrak{g}$. Then, we will write

$$\mathfrak{g} = (0, 0, 0, 0, e^{12}, e^{13}, 0)$$
with the same meaning.

In order to show the existence of nilpotent Lie algebras with a calibrated $G_2$-structure inducing a nilsoliton, we need to recall the classification of the nilpotent Lie algebras admitting a calibrated $G_2$-structure given in [18].

**Theorem 3.16** ([18]). Up to isomorphism, there are exactly twelve nilpotent Lie algebras admitting a calibrated $G_2$-structure. They are:

\[
\begin{align*}
n_1 &= (0, 0, 0, 0, 0, 0), \\
n_2 &= (0, 0, 0, e^{12}, e^{13}, 0), \\
n_3 &= (0, 0, 0, e^{12}, e^{13}, e^{23}, 0), \\
n_4 &= (0, 0, e^{12}, 0, 0, e^{13} + e^{24}, e^{15}), \\
n_5 &= (0, 0, e^{12}, 0, 0, e^{13}, e^{14} + e^{25}), \\
n_6 &= (0, 0, 0, e^{12}, e^{13}, e^{14}, e^{15}), \\
n_7 &= (0, 0, 0, e^{12}, e^{13}, e^{14} + e^{23}, e^{15}), \\
n_8 &= (0, 0, e^{12}, e^{13}, e^{23}, e^{15} + e^{24}, e^{16} + e^{34}), \\
n_9 &= (0, 0, e^{12}, e^{13}, e^{23}, e^{15} + e^{24}, e^{16} + e^{34} + e^{25}), \\
n_{10} &= (0, 0, e^{12}, 0, e^{13} + e^{24}, e^{14}, e^{46} + e^{34} + e^{15} + e^{23}), \\
n_{11} &= (0, 0, e^{12}, 0, e^{13}, e^{24} + e^{23}, e^{25} + e^{34} + e^{15} + e^{16} - 3e^{26}), \\
n_{12} &= (0, 0, 0, e^{12}, e^{23}, -e^{13}, 2e^{26} - 2e^{34} - 2e^{16} + 2e^{25}).
\end{align*}
\]

Comparing the previous classification with the results in [20], it turns out that, up to isomorphism, $n_9$ is the unique nilpotent Lie algebra with a calibrated $G_2$-structure but not admitting any nilsoliton. Moreover, the existence of a nilsoliton on the Lie algebra $n_{10}$ was shown in [19, Example 2], but its explicit expression is not known. Therefore, it is still an open problem to determine whether the Lie algebra $n_{10}$ admits a calibrated $G_2$-structure inducing the nilsoliton. For the remaining Lie algebras, we have the following.

**Theorem 3.17** ([22]). Up to isomorphism, $n_2, n_4, n_6$ and $n_{12}$ are the unique $s$-step nilpotent Lie algebras ($s = 2, 3$) with a nilsoliton inner product determined by a calibrated $G_2$-structure.

**Remark 3.18.** Note that the Lie algebra $n_i$, $i = 3, 5, 7, 8, 11$, has a nilsoliton inner product but no calibrated $G_2$-structure defining the nilsoliton [22].

In the next example, we write the expression of a calibrated $G_2$-structure inducing the nilsoliton inner product on $n_i$ for $i = 2, 4, 6, 12$. Moreover, in each case we also specify the negative real number $\lambda$ and the derivation $D$ of $n_i$ for which $\text{Ric} = \lambda I + D$. 


Example 3.19 ([22]). Consider the nilpotent Lie algebras $n_2, n_4, n_6$ with the structure equations given in Theorem 3.16. Then,

$$n_2: \quad \varphi_2 = e^{147} + e^{267} + e^{357} + e^{123} + e^{156} + e^{245} - e^{346},$$
$$\lambda = -2, \quad D = \text{diag} \left( 1, \frac{3}{2}, \frac{3}{2}, 2, \frac{5}{2}, \frac{5}{2}, 2 \right);$$

$$n_4: \quad \varphi_4 = -e^{124} - e^{456} + e^{347} + e^{135} + e^{167} + e^{257} - e^{236},$$
$$\lambda = -\frac{5}{2}, \quad D = \text{diag} \left( 1, \frac{3}{2}, \frac{5}{2}, 2, 2, \frac{7}{2}, 3 \right);$$

$$n_6: \quad \varphi_6 = e^{123} + e^{145} + e^{167} + e^{257} - e^{246} + e^{347} + e^{356},$$
$$\lambda = -\frac{5}{2}, \quad D = \text{diag} \left( \frac{1}{2}, 2, 2, \frac{5}{2}, \frac{5}{2}, 3, 3 \right).$$

For $n_{12}$, we firstly consider a basis $\{e^1, \ldots, e^7\}$ of its dual space $n_{12}^*$ for which the structure equations are

$$\left( 0, 0, 0, \frac{\sqrt{3}}{6} e^{12}, \frac{\sqrt{3}}{12} e^{13} - \frac{1}{4} e^{23}, -\frac{\sqrt{3}}{12} e^{23} - \frac{1}{4} e^{13}, \frac{\sqrt{3}}{12} e^{16} - \frac{\sqrt{3}}{6} e^{34} + \frac{\sqrt{3}}{12} e^{25} + \frac{1}{4} e^{26} - \frac{1}{4} e^{15} \right).$$

Then, a calibrated $G_2$-structure satisfying the required properties is

$$\varphi_{12} = -e^{124} + e^{167} + e^{257} - e^{246} + e^{347} + e^{356},$$

with $\lambda = -\frac{1}{4}$ and $D = \frac{1}{8} \text{diag}(1, 1, 1, 2, 2, 2, 3)$.

Remark 3.20. The nilsoliton condition is less restrictive for cocalibrated $G_2$-structures. Indeed, on each 2-step nilpotent Lie algebra admitting cocalibrated $G_2$-structures there exists a cocalibrated $G_2$-structure inducing the nilsoliton inner product (see [41]).

3.2. Locally conformal calibrated $G_2$-structures. A $G_2$-structure $\varphi$ is said to be locally conformal calibrated if the intrinsic torsion forms $\tau_0$ and $\tau_3$ vanish identically (cf. Table 1). In this case, equations (2.1) reduce to

$$d\varphi = 3\tau_1 \wedge \varphi, \quad d*\varphi = 4\tau_1 \wedge *\varphi + \tau_2 \wedge \varphi.$$  

Let $\theta := 3\tau_1 = -\frac{1}{4} *\varphi (\ast \varphi, d\varphi \wedge \varphi)$ denote the Lee form of the $G_2$-structure. Taking the exterior derivative of both sides of the equation $d\varphi = \theta \wedge \varphi$, we get $d\theta \wedge \varphi = 0$. This implies $d\theta = 0$. Consequently, each point of the manifold has an open neighborhood $U$ where $\theta = df$ for some $f \in C^\infty(U)$, and the 3-form $e^{-f}\varphi$ defines a calibrated $G_2$-structure on $U$. Hence, locally conformal calibrated $G_2$-structures are locally conformal equivalent to calibrated $G_2$-structures.

Motivated by Corollary 3.5 and Theorem 3.10 it is natural to investigate the existence of locally conformal calibrated $G_2$-structures whose associated metric is Einstein and non-Ricci-flat. In what follows, we refer to a locally conformal calibrated $G_2$-structure $\varphi$ with $g_{\varphi}$ Einstein as an Einstein locally conformal calibrated $G_2$-structure.

On compact manifolds, the following constraint on the scalar curvature holds.

Theorem 3.21 ([24]). An Einstein locally conformal calibrated $G_2$-structure on a compact seven-dimensional manifold has non-positive scalar curvature.
Since homogeneous Einstein manifolds with negative scalar curvature are non-compact (cf. [7, Thm. 7.4]) and since every homogeneous Ricci-flat metric is flat (see [1]), an immediate consequence of the previous result is the following.

**Corollary 3.22** ([26]). A compact homogeneous 7-manifold cannot admit an invariant Einstein locally conformal calibrated $G_2$-structure $\varphi$ unless the underlying metric $g_\varphi$ is flat.

**Remark 3.23.** By [8, Thm. 3.1], the result of Corollary 3.22 is valid more generally on every compact locally homogeneous space.

In the non-compact setting, there is an example of a simply connected solvable Lie group endowed with a left-invariant locally conformal calibrated $G_2$-structure $\varphi$ such that $g_\varphi$ is Einstein non-Ricci-flat. Thus, the result of [21] recalled in Theorem 3.10 is not anymore for locally conformal calibrated $G_2$-structures. Before describing the example, we recall some useful results.

Consider a 6-manifold $N$ endowed with a coupled SU(3)-structure $(\omega, \psi)$, with $d\omega = c\psi$ (cf. Definition 2.2). As we mentioned in §2.1, the 3-form $\varphi := \omega \wedge dt + \psi$ defines a locally conformal calibrated $G_2$-structure on the product manifold $N \times \mathbb{R}$. It is not difficult to check that the corresponding Lee form is $\theta = -cdt$.

In [26], the classification of six-dimensional nilpotent Lie algebras admitting a coupled SU(3)-structure inducing a nilsoliton was achieved (see also [25, §4.1]). We recall it in the next theorem.

**Theorem 3.24** ([26]). A non-abelian six-dimensional nilpotent Lie algebra admitting a coupled SU(3)-structure is isomorphic to one of the following

$$h_1 = (0, 0, 0, e_{12}, e_{14} - e_{23}, e_{15} + e_{34}), \quad h_2 = (0, 0, 0, 0, e_{13} - e_{24}, e_{14} + e_{23}).$$

Moreover, the only one admitting a coupled SU(3)-structure inducing a nilsoliton is $h_2$.

**Remark 3.25.** Notice that $h_2$ is the Lie algebra of the three-dimensional complex Heisenberg group.

We are now ready to describe the example.

**Example 3.26** ([26]). Consider the coupled SU(3)-structure on $h_2$ defined by the pair

$$\omega = e_{12} + e_{34} - e_{56}, \quad \psi = e_{136} - e_{145} - e_{235} - e_{246}.$$ 

It satisfies the equation $d\omega = -\psi$, and it induces the nilsoliton inner product $g = \sum_{k=1}^{6}(e^k)^2$ with Ricci operator

$$\text{Ric} = -3I + 4 \text{diag} \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, 1 \right),$$

where $D = \text{diag} \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, 1 \right)$ is a symmetric derivation of $h_2$. Consequently, the metric rank-one solvable extension $s = h_2 \oplus \langle e_7 \rangle$ of $h_2$ with structure equations

$$\left( \frac{1}{2}e^{17}, \frac{1}{2}e^{27}, \frac{1}{2}e^{37}, \frac{1}{2}e^{47}, e^{13} - e^{24} + e^{57}, e^{14} + e^{23} + e^{67}, 0 \right)$$

is endowed with the Einstein (non-Ricci-flat) inner product $g + (e^7)^2$. This is precisely the inner product $g_\varphi$ induced by the 3-form

$$\varphi = \omega \wedge e^7 + \psi,$$
which defines a locally conformal calibrated $G_2$-structure on $\mathfrak{s}$. A simple computation shows that the non-vanishing intrinsic torsion forms of $\varphi$ are
\[
\tau_1 = -\frac{1}{3} e^7, \quad \tau_2 = -\left(\frac{5}{3} e^{12} + \frac{5}{3} e^{34} + \frac{10}{3} e^{56}\right).
\]

Clearly, left multiplication allows to extend $\varphi$ to a left-invariant Einstein locally conformal calibrated $G_2$-structure on the simply connected nilpotent Lie group corresponding to $\mathfrak{s}$.

We conclude this section recalling a general structure result for compact 7-manifolds endowed with a locally conformal calibrated $G_2$-structure with nowhere vanishing Lee form.

**Theorem 3.27** ([23]). Let $M$ be a compact, connected seven-dimensional manifold endowed with a locally conformal calibrated $G_2$-structure $\varphi$, with nowhere vanishing Lee form $\theta$. Suppose that $\mathcal{L}_X \varphi = 0$, where $X$ is the $g_{\varphi}$-dual vector field of $\theta$. Then,
1) $M$ is the total space of a fibre bundle over $\mathbb{S}^1$, and each fibre is endowed with a coupled $SU(3)$-structure;
2) $M$ has a locally conformal calibrated $G_2$-structure $\hat{\varphi}$ such that $d\hat{\varphi} = \hat{\theta} \wedge \hat{\varphi}$, where $\hat{\theta}$ is a 1-form with integral periods.

The previous theorem implies in particular that $M$ is the mapping torus of a diffeomorphism $\nu$ of a certain 6-manifold $N$, i.e., $M$ is diffeomorphic to the quotient of $N \times \mathbb{R}$ by the infinite cyclic group of diffeomorphisms generated by $(p,t) \mapsto (\nu(p), t + 1)$.

**Remark 3.28.** It is worth recalling here that compact locally conformal parallel $G_2$-manifolds can be characterized as fibre bundles over $\mathbb{S}^1$ with compact nearly Kähler fibre (see [39, 51]).

### 4. The Laplacian flow on Lie groups

Consider a 7-manifold $M$ endowed with a calibrated $G_2$-structure $\varphi_0$. The Laplacian flow starting from $\varphi_0$ is the initial value problem
\begin{align}
\frac{\partial}{\partial t} \varphi(t) &= \Delta_{\varphi(t)} \varphi(t), \\
\frac{d}{dt} \varphi(t) &= 0, \\
\varphi(0) &= \varphi_0,
\end{align}
where $\Delta_{\varphi}$ denotes the Hodge Laplacian of the Riemannian metric $g_{\varphi}$ induced by $\varphi$. This flow was introduced by Bryant in [11] to study seven-dimensional manifolds admitting calibrated $G_2$-structures. Notice that the stationary points of the flow equation in (4.1) are harmonic $G_2$-structures, which coincide with torsion-free $G_2$-structures on compact manifolds.

Short-time existence and uniqueness of the solution of (4.1) when $M$ is compact were proved in [13].

**Theorem 4.1** ([13]). Assume that $M$ is compact. Then, the Laplacian flow (4.1) has a unique solution defined for a short time $t \in [0, \varepsilon)$, with $\varepsilon$ depending on $\varphi_0$.

As a consequence of the condition $\frac{d}{dt} \varphi(t) = 0$, the solution $\varphi(t)$ must belong to the open set
\[
[\varphi_0]_+ := [\varphi_0] \cap \Omega^3_+(M)
\]
in the cohomology class $[\varphi_0]$ as long as it exists.
Remark 4.2. By \([11, 33]\), the evolution equation in (4.1) is the gradient flow of Hitchin’s volume functional

\[
[\varphi_0]_+ \ni \varphi \mapsto \int_M dV_\varphi,
\]

with respect to a suitable \(L^2\)-metric on \([\varphi_0]_+\).

4.1. Solutions to the Laplacian flow on nilpotent Lie groups. Lie groups admitting left-invariant calibrated \(G_2\)-structures constitute a convenient setting where it is possible to investigate the behaviour of the Laplacian flow in the non-compact case. In literature, results in this direction have been obtained on nilpotent and solvable Lie groups in various works \([22, 27, 43, 44, 49]\). In the non-solvable case, the first examples of calibrated \(G_2\)-structures have been exhibited only recently in \([28]\), and the study of the Laplacian flow starting from any of them is still in progress.

The main peculiarity of the known non-compact examples is that the solution of (4.1) exists on an infinite time interval. We recall here a result of \([22]\), while we refer the reader to \([43, 44]\) for further examples.

Example 4.3 (\([22]\)).

1) On the nilpotent Lie algebra \(n_2\), the solution of the Laplacian flow starting from the calibrated \(G_2\)-structure \(\varphi_2\) given in Example 3.19 is

\[
\varphi(t) = e^{147} + e^{267} + e^{357} + \left(\frac{10}{3}t + 1\right)^{3/5} e^{123} + e^{156} + e^{245} - e^{346},
\]

where \(t \in (-\frac{3}{10}, +\infty)\).

2) On the nilpotent Lie algebra \(n_{12}\), the solution of the Laplacian flow starting from the calibrated \(G_2\)-structure \(\varphi_{12}\) given in Example 3.19 is

\[
\varphi(t) = -e^{124} + e^{167} + e^{257} + e^{347} - e^{456} + \left(\frac{1}{3}t + 1\right)^{3/4} (e^{135} - e^{236}),
\]

with \(t \in (-3, +\infty)\).

In the previous example, both the calibrated \(G_2\)-structures considered as initial value for the Laplacian flow induce the nilsoliton inner product on the corresponding nilpotent Lie algebra (cf. Theorem 3.17 and Example 3.19). Furthermore, using suitable analytic techniques it is possible to show that the solution \(\varphi(t)\) of (4.1) with \(\varphi(0) = \varphi_4\) on \(n_4\) and \(\varphi(0) = \varphi_6\) on \(n_6\) exists for \(t \in (T, +\infty)\) with \(T < 0\), see \([22, Thm. 4.7, Thm. 4.8]\). In all cases, it is then possible to analyze the behaviour of the solution \(\varphi(t)\) when \(t \to +\infty\).

Theorem 4.4 (\([22]\)). On the simply-connected nilpotent Lie groups \(N_i\), \(i = 2, 4, 6, 12\), the Laplacian flow starting from the left-invariant calibrated \(G_2\)-structure \(\varphi_i\) has a global solution defined for \(t \in (T, +\infty)\), with \(T < 0\). Moreover, all solutions converge to a flat \(G_2\)-structure when \(t \to +\infty\).

Remark 4.5. The nilpotent Lie algebra \(n_2\) may be seen as a product algebra \(n_2 = n' \oplus \mathbb{R}\) with \(\dim(n') = 6\) and \(\mathbb{R} = \langle e_7 \rangle\), and the calibrated \(G_2\)-structure \(\varphi_2\) on it can be written as

\[
\varphi_2 = \omega \wedge e^7 + \psi,
\]
where \((\omega, \psi)\) is a symplectic half-flat SU(3)-structure on \(n'\) (cf. Definition 2.2). Moreover, the solution of the Laplacian flow starting from \(\varphi_2\) at \(t = 0\) is of the form

\[
\varphi(t) = f(t) \omega(t) \wedge e^7 + \psi(t),
\]

where \((\omega(t), \psi(t))\) is a family of symplectic half-flat SU(3)-structures on \(n'\), and the function \(f : (-\frac{3}{10}, +\infty) \to \mathbb{R}^+\) is given by \(f(t) = (\frac{10}{3}t + 1)^{-1/10}\).

This fact is a consequence of a more general result which holds for a suitable class of symplectic half-flat SU(3)-structures and allows to construct new examples of solutions of (4.1) on solvable Lie groups. For more details, we refer the reader to [27].

**Remark 4.6.** The investigation reviewed in this section can be carried out also for the Laplacian coflow for cocalibrated \(G_2\)-structures [39] and its modified version introduced in [30]. It turns out that the behaviour of these flows on solvable Lie groups is slightly different from the behaviour of the Laplacian flow. We refer the reader to [5, 6] for a detailed treatment.

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