Black Hole Evaporation and Large Extra Dimensions

Roberto Casadio\textsuperscript{a} and Benjamin Harms\textsuperscript{b}

\textsuperscript{a}Dipartimento di Fisica, Università di Bologna, and I.N.F.N., Sezione di Bologna, via Irnerio 46, I-40126 Bologna, Italy

\textsuperscript{b}Department of Physics and Astronomy, The University of Alabama, Box 870324, Tuscaloosa, AL 35487-0324, USA

Abstract

We study the evaporation of black holes in space-times with extra dimensions of size $L$. We first obtain a potential which describes the expected behaviors of very large and very small black holes and then show that a (first order) phase transition, possibly signaled by an outburst of energy, occurs in the system when the horizon shrinks below $L$ from a larger value. This is related to both a change in the topology of the horizon and the restoring of translational symmetry along the extra dimensions.

Key words: black holes, extra dimensions, Hawking effect
PACS: 04.50.+h, 04.70.Dy, 04.70.-s, 11.25.Mj

Since Hawking’s semiclassical computation \cite{1}, one of the most elusive riddles of contemporary theoretical physics has been to understand black hole evaporation in a fully dynamical framework which accounts for the backreaction of the emitted particles on the space-time geometry. Intrinsically related to this (technically and conceptually) difficult issue is the role played by quantum gravity, since, by extrapolating from the semiclassical picture, one expects that black holes are capable of emitting particles up to any physical mass scale, including the Planck mass $m_p = \ell_p^{-1}$. Therefore, black holes appear as the most natural window to look through for any theory involving quantum gravity.

In a series of papers \cite{2,3} the point of view of statistical mechanics \cite{4} was taken and the analysis based on the well known fact that the canonical en-
semble cannot be consistently defined for a black hole and its Hawking radiation (conventionally viewed as point-like). Instead, the well posed micro-canonical description led to the conclusion that black holes are (excitations of) extended objects ($p$-branes), a gas of which satisfies the bootstrap condition. This yielded the picture in which a black hole and the emitted particles are of the same nature and an improved law of black hole decay which is consistent with unitarity (energy conservation). The weakness of the statistical mechanical analysis is that it does not convey the geometry of the space-time where the evaporation takes place: although the luminosity of the black hole can be computed as a function of its mass, the backreaction on the metric remains an intractable problem [3].

This scenario has received support from investigations in fundamental string theory, where it is now accepted that extended $Dp$-branes are a basic ingredient [5]. States of such objects were constructed which represent black holes [6] and corroborate [7] the old idea that the area of the horizon is a measure of the quantum degeneracy of the black hole [4]. However, this latter approach works mostly for very tiny black holes and suffers from the same shortcoming that the determination of the space-time geometry during the evaporation is missing.

It appears natural, in the framework of string theory, to consider the case when the black hole is embedded in a space-time of higher dimensionality. Indeed, the interest in models with extra spatial dimensions has been recently revived since they deliver a possible solution to the hierarchy problem without appealing to supersymmetry (see [8] and references therein). One qualitatively views the four dimensional space-time as a $D3$-brane embedded in a bulk space-time of dimension $4 + d$. Since matter is described by open strings with endpoints on the $Dp$-branes, one expects that matter fields are confined to live on the $D3$-brane in the low energy limit (e.g., for energy smaller than the electroweak scale $\Lambda_{EW}$), while gravity, being mediated by closed strings, can propagate also in the bulk. The $d$ extra spatial dimensions can be either compact [8] or infinitely extended [9]. Black holes in the former scenario were considered in [10] and some light on the non-compact case was shed in [11] (see also [12] and References therein).

For compact extra dimensions of typical size $L$, states of the gravitational field living in the bulk have $d$ momentum components quantized in units of $2\pi/L$. For an observer living on the $D3$-brane, they then appear as particles with mass

$$m^{(\vec{n})} \sim m_p \frac{\ell_p}{L} \sum_{i=1}^{d} n_i ,$$  

where all $n_i = 0$ for the (ground state) massless excitations corresponding to four dimensional gravitons, and states with some $n_i > 0$ are known as Kaluza-
Klein (KK) modes which induce short-range deviations from Newton’s law. The masses $m^{(n)}$ can be relatively small and the energy of matter confined on the D3-brane is prevented from leaking into the extra dimensions by the small coupling ($\sim m_p^{-1}$) between KK states of gravity and the matter energy-momentum tensor. This implies that processes involving particles within the standard model should result in energy loss into the extra dimensions and other phenomena potentially observable only above the TeV scale [8]. However, this protection is not effective with Hawking radiation, since a black hole can evaporate into all existing particles whose masses are lower than its temperature, thus providing an independent way of testing the existence of extra dimensions.

The easiest way to estimate deviations from Newton’s law is to evaluate the potential generated by a point-like source of (bare) mass $M$ by means of Gauss’ law [8]. Let us denote by $r_b$ the usual area coordinate on the four dimensional D3-brane. For distances $r_b \gg L$ one then recovers the standard form

$$V_{(4)} = -G_N \frac{M}{r_b}, \quad (2)$$

where $G_N = m_p^{-2}$ is Newton’s constant in four dimensions. For $r_b < L$ one has

$$V_{(4+d)} = -G_{(4+d)} \frac{M}{r_b^{1+d}}, \quad (3)$$

with $G_{(4+d)} = M_{(4+d)}^{-2-d} = L^d G_N$. This implies that the huge Planck mass $m_p^2 = M_{(4+d)}^{2+d} L^d$ and, for sufficiently large $L$ and $d$, the bulk mass scale $M_{(4+d)}$ (eventually identified with the fundamental string scale) can be as small as 1 TeV. Since

$$L \sim \left[ 1 \text{ TeV}/M_{(4+d)} \right]^{1+\frac{d}{d-2}} 10^{\frac{d}{2}-16} \text{ mm}, \quad (4)$$

requiring that Newton’s law not be violated for distances larger than 1 mm restricts $d \geq 2$ [8].

A form for the potential which yields the expected behaviour at both small and large distance and ensures that a test particle on the D3-brane does not experience any discontinuity in the force when crossing $r_b \sim L$ \[\square\] is given by

\[\text{An important effect to be tested by the next generation of table-top experiments, see, e.g., [13].}\]
\[ V = -G_N \frac{M}{r_b} \left[ 1 + \sum_{n=1}^{d} C_n \left( \frac{L}{r_b} \right)^n \right], \tag{5} \]

where \( C_n \) are numerical coefficients (possibly functions of \( M \), see below). The second (Yukawa) contribution in (5) can be related to the exchange of (massive) KK modes and, hence, encodes tidal effects from the bulk due to the presence of the mass \( M \) on the D3-brane. We further note that the case \( d = 2 \) can be used to make estimates for one non-compact extra dimension [12], provided one introduces an effective size \( L^2 \sim -M_5^3/\Lambda \), with \( \Lambda \) the (negative) cosmological constant in the bulk AdS$_5$ [9].

The behavior of both very large (\( R_H \gg L \)) and very small (\( R_H \ll L \)) black holes is by now relatively well understood. In the former case, one can unwrap the compact extra dimensions and regard the real singularity as spread along a (black) \( d \)-brane [14] of uniform density \( M/L^d \), thus obtaining the Schwarzschild metric on the orthogonal D3-brane, in agreement with the weak field limit \( V(4) \), and an approximate “cylindrical” horizon topology \( S^2 \times \mathbb{R}^d \).

In the latter case a solution is known [11], for one infinite extra dimension [9], which still has the form of a black string extending all the way through the bulk AdS$_5$. However, this solution is unstable [15] and believed to further collapse into one point-like singularity [11]. This can be also argued from the observation that the Euclidean action of a black hole is proportional to its horizon area and is thus minimized by the spherical topology \( S^{2+d} \). Hence, small black holes are expected to correspond to a generalization of the Schwarzschild metric to \( 4 + d \) dimensions [16] and should be colder and (possibly much) longer lived [10].

However, there is still a point to be clarified for small black holes, namely one should find an explicit matching between the spherical metric (for \( r_b < L \)) and the cylindrical metric (for \( r_b > L \)). This is not a trivial detail, since the ADM mass of a spherical \( 4 + d \) dimensional black hole is zero as seen from the D3-brane because there is no \( 1/r_b \) term in the large \( r_b \) expansion of the time-time component of the metric tensor [16]. Thus, one concludes that the \( 4 \) dimensional ADM mass of a small black hole can be determined as a function of the \( 4 + d \) dimensional mass parameter only after such a matching is provided explicitly. Even less is known about black holes of size \( R_H \sim L \), and a complete description is likely to be achieved only by solving the entire set of field equations for an evaporating black hole in \( 4 + d \) dimensions.

Instead of tackling this intractable backreaction problem, we extrapolate from the weak field limit on the D3-brane given in (5) the (time and radial components of the) metric in \( 4 + d \) dimensions as

\[ g_{tt} \sim -1 - 2V(r) \]
\[ 1 - 2 G_N \frac{M}{r} \left[ 1 + \sum_{n=1}^{d} C_n \left( \frac{L}{r} \right)^n \right] \tag{6} \]

\[ g_{rr} \simeq -g_{tt}^{-1}, \]

where \( r \) now stands for the area coordinate in \( 4 + d \) dimensions. This yields \( M \) as the ADM mass of the black hole (see [17] for a similar solution in the scenario of [9]) and the radius of the horizon is determined by

\[ g_{tt} = 0 \Rightarrow R_H = 2 G_N M \left[ 1 + \sum_{n=1}^{d} C_n \left( \frac{L}{R_H} \right)^n \right]. \tag{7} \]

The above ansatz does not provide an exact solution of vacuum Einstein equations, since some of the components of the corresponding Einstein tensor in \( 4 + d \) dimensions \( G_{ij} = 8 \pi G_{(4+d)} T_{ij} \neq 0 \). However, it is possible to choose the coefficients \( C_n \) (as functions of \( M \)) in such a way that the “effective matter contribution” \( T_{ij} \) from the region outside the black hole horizon is small. In particular, one can require that the contribution to the ADM mass be negligible,

\[ m \equiv \int_{R_H}^{\infty} d^{4+d}x T^t_t = \frac{1}{8 \pi G_{(4+d)}} \int_{R_H}^{\infty} d^{4+d}x G^t_t \{C_n\} \ll M. \tag{8} \]

In this sense one can render the above metric a good approximation to a true black hole in \( 4 + d \) dimensions. One should also recall that the black hole must be a classical object, to wit its Compton wavelength \( \ell_M \sim \ell_p (m_p/M) \ll R_H \). Further, once Hawking radiation is included, its backreaction on the metric at small \( r \) is likely to be significant for \( R_H \ll L \), so that a true vacuum solution would not be practically much more useful.

When \( R_H \gg L \) one has \( r \simeq r_b \), therefore, the metric (6) is approximately cylindrically symmetric (along the extra dimensions). Further, on setting all \( C_n = 0 \) yields \( m = 0 \), and Eq. (7) coincides with the usual four dimensional Schwarzschild radius \( R_H \simeq 2 \ell_p (M/m_p) \). Correspondingly one has the inverse Hawking temperature \( (\beta_H = T_H^{-1}) \) and Euclidean action \([1,4]\]

\[ \beta_H^2 \simeq 8 \pi \ell_p (M/m_p) \tag{9} \]

\[ S_E^\sim \simeq 4 \pi (M/m_p)^2 \simeq A_{(4)}/4 \ell_p^2, \tag{10} \]

where \( A_{(D)} \) is the area of the horizon in \( D \) space-time dimensions and the condition \( R_H \gg L \) translates into
\[ M \gg m_p (L/\ell_p) \equiv M_c , \]  
\[(11)\]

(e.g., \( M_c \sim 10^{27} \text{ g} \) for \( L \sim 1 \text{ mm} \)). The fact that the extra dimensions do not play any significant role at this stage is further confirmed by \( T_H \ll 2\pi / L \equiv m^{(1)} \) (the mass of the lightest KK mode), therefore no KK particles can be produced.

For \( R_H \ll L \), one can again take advantage of the coefficients \( C_n \) to lower \( m \) as much as possible. For instance, for \( d = 2 \) and \( M = 10^{15} \text{ g} \) (\( \ll M_c \)) the values \( C_2 = -C_1 = 1 \) yield \( m \sim 10^{-3} M \) (more details will be given in [18]). Eq. (7) then leads to

\[ R_H \simeq \left( 2 C_d L^d G_N M \right)^{1/d} , \]
\[(12)\]

and the consistency conditions \( \ell_M \ll R_H \ll L \) hold for

\[ m_p (\ell_p / L)^{d/2 + 1} \ll M \ll M_c . \]
\[(13)\]

Since we have assumed that the spherical symmetry extends to \( 4 + d \) dimensions, one obtains [10]

\[ \beta_H \equiv L (M/M_c)^{1/d} \]
\[(14)\]

\[ S_E \equiv (L/\ell_p)^2 (M/M_c)^{2d/2 + d} \sim A(4 + d)/\ell_p^2 L^d , \]
\[(15)\]

which reduce back to (9) and (10) if one pushes down \( L \rightarrow \ell_p \) (\( M_c \rightarrow m_p \)). For \( M \ll M_c \), the temperature \( T_H \) is sufficient to excite KK modes, although it is lower than that of a four dimensional black hole of equal mass. Correspondingly, the Euclidean action \( S_E(M) \geq S_E(M) \), yielding a smaller probability [1,2]

\[ P \sim \exp (-S_E) \]
\[(16)\]

for the Hawking particles “to come into existence” in the \( 4 + d \) dimensional scenario.

The luminosity of a black hole (provided microcanonical corrections are negligible [3,18]) can be approximated by employing the canonical expression [1]

\[ F(D) \simeq A(D) \sum_s \int_0^\infty \frac{\Gamma_s d\omega^D}{e^{\beta_H \omega} + 1} = A(D) N(D) T_H^D , \]
\[(17)\]
with $\Gamma$ the grey-body factor and $N$ a coefficient which depends upon the number of available particle species $s$ with mass smaller than $T_H$. For small black holes $D = 4 + d$ and $T_H = T_H^c$, hence $F_{(4+d)} \sim (M/M_c)^{1/(1+d)}$ is far less intense than $F_{(4)} \sim (M/M_c)^{-2}$ obtained from $T_H^c$ [10]. It also follows from (17) that $F_{(D)}$ as a function of $R_H$ (not $M$) depends on $D$ only through $N_{(D)}$. Therefore, the energy emitted into KK modes must be a small fraction of the total luminosity,

$$\frac{F_{KK}}{F_{(4+d)}} \sim \frac{N_{KK}}{N_{(4)} + N_{KK}} \ll 1,$$

(18)

because the number of degrees of freedom of KK gravitons ($\sim N_{KK}$) is much smaller than the number of particles in the standard model ($\sim N_{(4)}$), and energy conservation in $4 + d$ dimensions requires that $F_{(4+d)} = F_{(4)} + F_{KK}$ (see also [12] for similar arguments). However, one might question the efficiency of the confining mechanism for matter on the D3-brane at very high energy (say greater than $\Lambda_{EW}$). One should then include bulk standard model fields among KK modes and consider $N_{KK}$ as a growing function of the temperature. If this is the case, the luminosity will greatly increase and the ratio $F_{KK}/F_{(4+d)}$ will eventually approach unity when $T_H^c > \Lambda_{EW}$.

An inspection of the Klein-Gordon equation in $4 + d$ dimensions supports the conclusion that the luminosity of small black holes is relatively fainter. In fact, there is a potential barrier in the equation governing the radial propagation outside the horizon of the form [18]

$$W \sim \frac{d}{2} \left(\frac{d}{2} + 1\right) \frac{1}{r^2} \left[1 - \left(\frac{R_H}{r}\right)^{1+d}\right]^2,$$

(19)

which all Hawking particles have to tunnel through in order to escape. One can roughly reproduce $W$ by assuming that all particles are emitted with an effective angular momentum $\sim l + d/3$, which substantially lowers the grey-body factor [19].

Let us now consider the case when $R_H \sim L$ (i.e., $M \sim M_c$) and try to understand what happens as the horizon shrinks below the size of the extra dimensions. It is easy to solve (7) for the “toy model” $d = 1$ and estimate the relation between the ADM mass and the radius of the horizon at all scales

$$R_H = G_N M \left(1 + \sqrt{1 + (2L/G_N M)}\right).$$

(20)

There is indeed evidence that only the zero modes of standard model fields can be confined, thus allowing matter to leak in the bulk (see, e.g., [20]).
This leaves only the topology of the horizon to be specified for \(2 R_H \leq L\). According to the area law [4] \(S_E \sim A_{(4+d)}/16 \pi G_{(4+d)}\), one has for the “cylinder” \(S^2 \times \mathbb{R}\) and for the three sphere \(S^3\) respectively

\[
S_E^c \sim \frac{R_H^2}{8 \ell_p^2}, \quad S_E^s \sim \frac{R_H^3}{16 \ell_p^2 L},
\]

and the ratio \(S_E^c/S_E^s \sim 4\) for \(2 R_H = L\), so that the spherical topology is favored once the horizon has become small enough to “close up” along the extra dimension. The inverse temperature in the two cases is given by

\[
\beta_H^c \sim \frac{R_H}{4 \ell_p^2} \frac{\partial R_H}{\partial M}, \quad \beta_H^s \sim \frac{3 R_H^2}{16 \ell_p^2 L} \frac{\partial R_H}{\partial M},
\]

and is discontinuous as well, since \(\beta_H^c/\beta_H^s \sim 2/3\) for \(2 R_H = L\). The physically interesting case \(d = 2\) is algebraically more involved but the qualitative picture is the same as for \(d = 1\) if the topology of the extra dimensions is \(\mathbb{R}^2\) [18].

We can now get some further physical insight by appealing to statistical mechanics. Although the use of the canonical ensemble is known to be incorrect, we utilize it in the analysis below in order to conform to the standard description and introduce a partition function for the black hole and its Hawking radiation as the Laplace transform of the microcanonical density of states [21,2,3],

\[
Z_\beta \sim \int dM \ e^{-\beta M} e^{S_E(M)},
\]

where we have estimated the internal degeneracy of a black hole as the inverse of the probability (16), in agreement with the area law [4,7] and the bootstrap relation [2,3]. Since the entropy \(S = \beta^2 (\partial F/\partial \beta)\), where \(\beta F = -\ln Z\) is the (Helmholtz) free energy, it follows from (21) that there is a discontinuity in the first derivative of \(F\) at

\[
\beta_c \sim \beta_H^s(M_c) \sim L \sim 1/m^{(1)}.
\]

This behavior is characteristic of a first order phase transition [21]: In the cold phase, \(T_H < \beta_c^{-1}\), the system appears “condensed” into the low energy standard model fields living on the four dimensional D3-brane and translational invariance in the \(d\) extra directions is therefore broken by the D3-brane itself. For \(T_H > \beta_c^{-1}\) translational invariance begins to restore, and the system starts spreading over all bulk space, with D3-brane vibrations acting as Nambu-Goldstone bosons which give mass to the KK modes [8].

One might argue that it is the cylindrical topology which is translationally sym-
Fig. 1. Sketch of the equation of state $T = T(M)$ and corresponding topology of the horizon. Mass increases to the right (time lapses towards the left).

Of course, in the statistical mechanical analysis the time is missing, and one can more realistically think that, during the evaporation of an initially large black hole, the horizon starts to bend in the extra dimensions when $T^c_H$ approaches $T^c_H(M_c)$ (from below). If the temperature remains constant (and approximately equal to the lower value $T^c_H(M_c) \equiv T_c$) during the transition, then a qualitative picture of the effect is given in Fig. 1. The change in the topology of the horizon could then be accompanied by a burst of energy to get rid quickly of the excess mass (equal to the width of the plateau in Fig. 1).

In the above approximation the specific heats in the cold and hot phases are respectively given by

$$C_V^> \sim -(M/m_p)^2, \quad C_V^< \sim -L M (M/M_c)^{(1+\alpha)/(1+\delta)},$$

(25)

Since $C_V^<$ is negative, one is eventually forced to employ the microcanonical description when the temperature becomes too hot\(^6\), although the fact that $|C_V^<| < |C_V^>|$ suggests that the extra dimensions make the inconsistency of the canonical description milder [18].

This work was supported in part by the U.S. Department of Energy under Grant no. DE-FG02-96ER40967 and by NATO grant no. CRG.973052.

metric along the extra dimensions, however for $T_H < T_c$ there are no physical particles in the bulk, and such a symmetry remains a purely geometrical concept (as well as the bulk itself) with no dynamical counterpart.

\(^6\) In order to make sense of the partition function, one should introduce an ultraviolet cut off $\Lambda_{UV}$ to render the (divergent) integral (23) finite. Then, the canonical description is a good approximation only for $T \ll \Lambda_{UV}$ [18]
References

[1] S.W. Hawking, Nature 248 (1974) 30; Comm. Math. Phys. 43 (1975) 199.

[2] B. Harms and Y. Leblanc, Phys. Rev. D 46 (1992) 2334; Phys. Rev. D 47 (1993) 2438; Ann. Phys. 244 (1995) 262; Ann. Phys. 244 (1995) 272; Europhys. Lett. 27 (1994) 557; Ann. Phys. 242 (1995) 265; P.H. Cox, B. Harms and Y. Leblanc, Europhys. Lett. 26 (1994) 321.

[3] R. Casadio, B. Harms and Y. Leblanc, Phys. Rev. D 57 (1998) 1309; R. Casadio and B. Harms, Phys. Rev. D 58 (1998) 044014; Mod. Phys. Lett. A17 (1999) 1089.

[4] J.D. Bekenstein, Phys. Rev. D 7 (1973) 2333; Phys. Rev. D 9 (1974) 3292. J.M. Bardeen, B. Carter, S.W. Hawking, Commun. Math. Phys. 31 (1973) 161; G.W. Gibbons and S.W. Hawking, Phys. Rev. D 15 (1977) 2752.

[5] J. Polchinski, TASI Lectures on D-branes, hep-th/9702136.

[6] A. Strominger and C. Vafa, Phys. Lett. B379 (1996) 99.

[7] J. Maldacena, Nucl. Phys. Proc. Suppl. 61A (1998) 111.

[8] N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. B429 (1998) 263; Phys. Rev. D59 (1999) 086004; I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. B436 (1998) 257.

[9] L. Randall and R. Sundrum, Phys. Rev. Lett. 83 (1999) 4690.

[10] P.C. Argyres, S. Dimopoulos and J. March-Russell, Phys. Lett. B 441 (1998) 96.

[11] A. Chamblin, S. Hawking and H.S. Reall, Phys. Rev. D 61 (2000) 0605007.

[12] R. Emparan, G.T. Horowitz and R.C. Myers, hep-th/0003118.

[13] J.C. Long, H.W. Chan and J.C. Price, Nucl. Phys. B539, 23 (1999).

[14] G.T. Horowitz and A. Strominger, Nucl. Phys. B360 (1991) 197.

[15] R. Gregory and R. Laflamme, Phys. Rev. Lett. 70 (1993) 2837.

[16] R.C. Myers and M.J. Perry, Ann. Phys. 172 (1986) 304.

[17] N. Dahdhich, R. Maartens, P. Papadopoulos and V. Rezania, hep-th/0003061.

[18] R. Casadio and B. Harms, in preparation.

[19] D. Page, Phys. Rev. D 13 (1976) 198; Phys. Rev. D 16 (1977) 2401.

[20] S.L. Dubovsky, V.A. Rubakov and P.G. Tinyakov, hep-th/0006056; R. Casadio, A. Gruppuso and G. Venturi, in preparation.

[21] K. Huang, Statistical mechanics (John Wiley and Sons, New York, 1987).