Correlation of local densities of states on mesoscopic energy scales in random band matrices

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Abstract
We are interested in the phase transition of the correlation function of local densities of states at mesoscopic scales of random band matrices of width $W$ in dimension 2. As a result, we show that the local densities of states are alternately positively and negatively correlated in the diffusive regime $O(\log(L/W))$ times, $L$ being the size of the system.

Keywords Correlation function · Random band matrices · Phase transition

Mathematics Subject Classification 15B52 · 82B44 · 33E99

1 Introduction
Impurities in a disordered quantum system can be modelised by a random band Hermitian matrix $H$ corresponding to the Hamiltonian of the system. Pure materials have an underlying lattice structure which can be represented by a finite lattice in $\mathbb{Z}^d$, say the torus $\mathbb{T} = [-L/2, L/2)^d \cap \mathbb{Z}^d$. Given $x, y \in \mathbb{T}$, corresponding to the lattice sites, the matrix $H_{xy}$ represents the quantum system in a $d$-dimensional discrete box of length $L$. Following the model introduced by Erdős and Knowles in [2, 3], we assume that the matrix $H$ is an Hermitian matrix whose upper triangular entries are independent random variables with zero mean. Let $S_{xy} := \mathbb{E}|H_{xy}|^2$ be a deterministic matrix given

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by an arbitrary profile function \( f \) on the scale \( W \), that is,

\[
S_{xy} = \frac{1}{M - 1} f \left( \frac{[x - y]_L}{W} \right), \quad M := \sum_{x \in \mathbb{Z}^d} f \left( \frac{x}{W} \right)
\]

where \([x]_L \) denotes the canonical representative of \( x \in \mathbb{Z}^d \) in the torus \( \mathbb{T} \) and \( f : \mathbb{R}^d \to \mathbb{R} \) is an even, bounded, nonnegative, piecewise \( C^1 \) function such that \( f \) and \(|\nabla f|\) are integrable and \( \int_{\mathbb{R}^d} dx f(x)|x|^{4+c_1} < \infty \) for some constant \( c_1 > 0 \). Thus, for all \( x \in \mathbb{T} \),

\[
\sum_{y \in \mathbb{T}} S_{xy} = \frac{M}{M - 1} =: I.
\]

We assume that the law of \( H_{xy} \) is symmetric, i.e. \( H_{xy} \) and \(-H_{xy} \) have the same law and that \( A_{xy} := (S_{xy})^{-1/2} H_{xy} \) have uniform subexponential decay, that is,

\[
P(|A_{xy}| > \xi) \leq c_2 e^{-\xi^{c_3}}
\]

for some constants \( c_2, c_3 > 0 \) and for any \( \xi > 0 \). Let \( \phi : \mathbb{R} \to \mathbb{R} \) be a test function defined as a smooth function verifying the following conditions (C):

\[
\int_{\mathbb{R}} \phi(E) dE = 2\pi
\]

and such that for every \( q > 0 \), there exists a constant \( C_q \) satisfying

\[
|\phi(E)| \leq \frac{C_q}{1 + |E|^q}.
\]

We are interested in the correlation of the number of eigenvalues around two energies \( E_1 < E_2 \) such that \( \omega = E_2 - E_1 \) is much larger than the energy window \( \eta \). More precisely, we are concerned with the following correlation

\[
\frac{\langle Y_{\phi_1}^\eta(E_1); Y_{\phi_2}^\eta(E_2) \rangle}{\langle Y_{\phi_1}^\eta(E_1) \rangle \langle Y_{\phi_2}^\eta(E_2) \rangle} \quad \text{where} \quad \langle X \rangle := \mathbb{E} X, \quad \langle X; Y \rangle := \mathbb{E}(XY) - \mathbb{E}X \mathbb{E}Y
\]

and \( Y_{\phi_i}^\eta(E) \) is the smoothed local density of states around energy \( E \) on the scale \( \eta \), defined by

\[
Y_{\phi_i}^\eta(E) := \frac{1}{L^d} \text{Tr} \phi_i^\eta(H/2 - E), \quad i = 1, 2,
\]

\( \phi_i^\eta \) being the rescaled test function \( \phi_i^\eta(E) := \eta^{-1} \phi_i(\eta^{-1} E) \) and \( \phi_i \) a test function satisfying conditions (C). On the mesoscopic energy scale which corresponds to energy scales much larger than the eigenvalue spacing and much smaller than the total macroscopic energy scale of the system, we observe a phase transition at the critical energy, the Thouless energy, given by \( \eta_c = W^2/L^2 \).
Throughout the paper, we make the following assumptions
\[
\omega \gg \eta, \quad W \ll L, \quad L \leq W^C, \quad \eta \ll 1, \quad \eta \gg M^{-1/3},
\]
\[
E_1, E_2 \in [-1 + \kappa, 1 - \kappa], \quad \omega \leq c_*
\]
for some constant \( C, \kappa \) a fixed positive constant, \( c_* \) a small enough positive constant depending on \( \kappa \). We introduce the covariance matrix of \( S_{x0} \), its fourth moment and the covariance matrix of \( f \)
\[
D_{ij} := \frac{1}{2} \sum_{x \in \mathcal{T}} x_i x_j S_{x0}, \quad Q := \frac{1}{32} \sum_{x \in \mathcal{T}} S_{x0} \left| D^{-1/2} \frac{x}{W} \right|^4,
\]
\[
(D_0)_{ij} := \frac{1}{2} \int_{\mathbb{R}^d} dx \, x_i x_j f(x)
\]
and assume that \( c_4 < D_0 < c_5 \) in the sense of quadratic forms for some positive constants \( c_4, c_5 \). Here, we choose the matrix \( D \) as a positive multiple of the \( d \times d \) identity matrix \( I_d \), i.e. \( D = D I_d, D > 0 \). We also introduce the parameters
\[
\alpha := e^{i(\arcsin(E_1+i\eta) - \arcsin(E_2-i\eta))}, \quad u := |1 - \alpha|, \quad \zeta \in S^1 \text{ such that } 1 - \alpha =: u \zeta
\]
\[
b := \frac{1}{D} \left( \frac{\sqrt{u} L}{2 \pi W} \right)^2, \quad R := \frac{\epsilon L}{2 \pi W}, \text{ where } \epsilon > 0 \text{ will be defined later.}
\]
Let \( v \equiv v(E) = 2 \sqrt{1 - E^2}/\pi \) and \( E := (E_1 + E_2)/2 \). Then, \( u \) and \( \zeta \) expand as
\[
u = 2 \sqrt{1 - E^2}/\pi \quad \text{and} \quad E := (E_1 + E_2)/2.
\]
Then, \( u \) and \( \zeta \) expand as
\[
\omega \gg \eta \gg \eta_c \quad \text{which corresponds to large samples, i.e.}
\]
\[
L \quad \text{large with respect to the diffusion length } W/\sqrt{\eta}. \quad \text{In terms of our parameter } b \quad \text{behaving as } \omega L^2/W^2 \quad \text{using (2), it means that } b \gg 1 \quad \text{from assumption (1) } \omega \gg \eta.
\]
On the contrary, the mean-field regime is defined by \( \eta \ll \eta_c \quad \text{corresponding to small samples.}
\]

The diffusive regime is defined by \( \eta \gg \eta_c \quad \text{which corresponds to large samples, i.e.}
\]
\[
L \quad \text{large with respect to the diffusion length } W/\sqrt{\eta}. \quad \text{In terms of our parameter } b \quad \text{behaving as } \omega L^2/W^2 \quad \text{using (2), it means that } b \gg 1 \quad \text{from assumption (1) } \omega \gg \eta.
\]
On the contrary, the mean-field regime is defined by \( \eta \ll \eta_c \quad \text{corresponding to small samples.}
\]

The main contribution of the present paper is an exact asymptotic expression for the local density–density correlation function in dimension 1 valid in both regimes, while in dimension 2 we provide a more precise expression of it with respect to the asymptotics derived in [3] in the diffusive regime improving [3, Proposition 3.5]. In their paper, the authors estimate a series appearing in the main term using a Riemann sum estimate, while the key point here consists in no longer making this approximation. A curious fact is that we observe that for large \( b \) but less than \( (\log(L/W))^2 \) the correlation function has an interesting oscillatory behaviour. The following theorem states our results.

**Theorem 1.1** There exists a constant \( c_0 > 0 \) such that the local density–density correlation satisfies:
(i) In dimension 1 for all $b > 0$,

$$\frac{\langle Y_{\phi_1}^\eta(E_1); Y_{\phi_2}^\eta(E_2) \rangle}{\langle Y_{\phi_1}^\eta(E_1) \rangle \langle Y_{\phi_2}^\eta(E_2) \rangle} = -\frac{1}{16(\pi v)^{5/2} \sqrt{D} \omega^{3/2} LW} \times \left( \frac{\sinh(\pi \sqrt{2b}) + \sin(\pi \sqrt{2b}) + \pi \sqrt{2b} \sinh(\pi \sqrt{2b}) \sin(\pi \sqrt{2b})}{\sinh^2(\pi \sqrt{b/2}) + \sin^2(\pi \sqrt{b/2})} \right) \times O \left( \frac{\omega^{-1/2}}{LW} \left( 1 + \frac{\eta}{\omega^2} \right) + \frac{1}{\omega L^2} \left( 1 + \frac{\eta}{\omega^2} \right) \right) + \frac{1}{\omega W^3} e^{-\pi \sqrt{2\omega L/W}} + \frac{W^{-c_0-1}}{L(\omega + \eta)^{1/2}} + \frac{W^{-c_0}}{L^2(\omega + \eta)} \right).$$

(ii) In dimension 2 for $b \gg 1$,

$$\frac{\langle Y_{\phi_1}^\eta(E_1); Y_{\phi_2}^\eta(E_2) \rangle}{\langle Y_{\phi_1}^\eta(E_1) \rangle \langle Y_{\phi_2}^\eta(E_2) \rangle} = -\frac{1}{2\pi^5 D v^4 L^2 W^2} \left[ \frac{L^2}{\pi DW^2} e^{-\pi \sqrt{2b}} b^{-3/4} \sin \left( \pi \sqrt{2b} - \frac{\pi}{8} \right) - (Q - 1) |\log \omega| \right) + O \left( 1 + \frac{\eta^2}{\omega^3} + \omega |\log \omega| + e^{-\pi \sqrt{2b}} b^{1/4} \left( 1 + \frac{1}{\omega W^2} + \frac{L^2}{W^2 b^{-3}} \right) \right) + \frac{1}{W^{c_0}} |\log(\omega + \eta)| + \frac{W^{2-c_0}}{L^2(\omega + \eta)} \right].$$

We emphasise that for the two-dimensional case, the second term has been calculated in [3], while the first term is new and is dominant for $b \ll (\log(L/W))^2$, showing that the correlation function oscillates around zero $O(\log(L/W))$ times. Then, for $b \gtrsim (\log(L/W))^2$ it is dominated by the logarithmic term. This will be shown in Proposition 2.5 and Corollary 2.6.

An interesting question would be to see whether this oscillatory behaviour is as well observed in other systems and its physical interpretation.

2 Calculation of the correlation function

In [3, Theorem 6.1], it has been shown that, under the assumptions (1), there exists a constant $c_0 > 0$ such that for any $E_1, E_2 \in [-1 + \kappa, 1 - \kappa]$ for small enough $c_* > 0$,
the local density–density correlation satisfies

\[
\frac{\langle Y^n_{\phi_1} (E_1); Y^n_{\phi_2} (E_2) \rangle}{\langle Y^n_{\phi_1} (E_1) \rangle \langle Y^n_{\phi_2} (E_2) \rangle} = \frac{1}{(LW)^d} \left( \Theta^n_{\phi_1, \phi_2} (E_1, E_2) + M^{-c_0} O \left( R(\omega + \eta) + \frac{M}{N(\omega + \eta)} \right) \right) \tag{3}
\]

where \( R(s) := 1 + 1(d = 1)s^{-1/2} + 1(d = 2)|\log s| \) and the leading term \( \Theta^n_{\phi_1, \phi_2} \) is given by

\[
\Theta^n_{\phi_1, \phi_2} (E_1, E_2) = \frac{2W^d}{\pi^4 \nu^4 L^d} \text{Re} \text{Tr} \frac{S}{(1 - \alpha S)^2} (1 + O(\omega)) + O(1). \tag{4}
\]

The proof of the above relation is given in the appendix. For \( q \in [-\pi W, \pi W]^d \), let \( \hat{S}_W (q) \) denote \( \hat{S}(q/W) \) where \( \hat{S}(p) := \sum_{x \in T} e^{-ip \cdot x} S_x \) is defined for all \( p \in [-\pi, \pi]^d \). Also define the following quantity

\[
\mathcal{Q}(q) := \frac{1}{4!} \sum_{x \in T} (x \cdot q/W)^4 S_x 0.
\]

Recall from [3, Lemma B.1] that for any \( \epsilon > 0 \), there is a \( \delta_\epsilon > 0 \) such that \( \hat{S}_W (q) \) is bounded, \( |\hat{S}_W (q)| \leq 1 - \delta_\epsilon \) if \( |q| \geq \epsilon \) for large enough \( W \) and has the following expansion

\[
\hat{S}_W (q) = \mathcal{I} - \mathcal{D}|q|^2 + \mathcal{Q}(q) + O(|q|^{4+c_1})
\]

which comes from a fourth-order Taylor’s expansion of

\[
\mathcal{I} - \hat{S}_W (q) = \frac{1}{M - 1} \sum_{v \in W^{-1}T} (1 - \cos (q \cdot v)) f(v).
\]

Let \( \epsilon > 0 \) be such that \( \hat{S}_W (q) = \mathcal{I} - a(q) \) where \( a \) is a function satisfying \( c_4 |q|^2 \leq a(q) \leq c_5 |q|^2 \leq 1 \) for \( |q| \leq \epsilon \). Equation [3, B.15] states that

\[
\text{Tr} \frac{S}{(1 - \alpha S)^2} = \sum_{q \in W^*} \frac{\hat{S}_W (q)}{(1 - \alpha \hat{S}_W (q))^2} 1(|q| \leq \epsilon) + O \left( \frac{L^d}{\delta_\epsilon W^d} \right) \tag{5}
\]

where \( T^* = \frac{2\pi}{L} T \).

In Subsection 2.1, we study the above trace in the one-dimensional case, while the two-dimensional case is treated in Subsection 2.2.
2.1 Dimension 1

Proposition 2.1  In dimension 1, the following relation holds

\[
\text{Tr} \left\{ \frac{S}{(1 - \alpha S)^2} \right\} = \frac{1}{32\pi^3 D^2} \frac{L^4}{W^4} \frac{1}{(\zeta b)^{3/2}} \left( \coth \left( \pi \sqrt{\zeta b} \right) + \pi \sqrt{\zeta b} \sinh^{-2} \left( \pi \sqrt{\zeta b} \right) \right) + O \left( \frac{L}{W} \frac{1}{\omega^{-1/2}} + \frac{L}{W^2} \frac{1}{\omega^{-5/2}} + \frac{W^2}{\omega^3 L^3} + \frac{L^2}{W^3 \omega^2} e^{-\pi \sqrt{2\omega L/W}} \right).
\]

Proof  With the notations introduced above, equation [3, B.16] reads

\[
\sum_{q \in W_T^*} \frac{\hat{S}_W(q)}{(1 - \alpha \hat{S}_W(q))^2} \mathbf{1}(|q| \leq \epsilon) = I \sum_{q \in W_T^*} \frac{\mathbf{1}(|q| \leq \epsilon)}{(1 - \alpha \hat{S}_W(q))^2} + O \left( \frac{L}{W} R(u) \right)
\]

and equation [3, B.17]

\[
\sum_{q \in W_T^*} \frac{\mathbf{1}(|q| \leq \epsilon)}{(1 - \alpha \hat{S}_W(q))^2} = \sum_{q \in W_T^*} \frac{\mathbf{1}(|q| \leq \epsilon)}{(1 - \alpha + D|q|^2)^2} + \frac{L}{W} O \left( R(u) + \frac{u^{-5/2}}{W} \right)
\]

which together with equation (5) lead to the following expression for the trace appearing in the \( \Theta_{\phi_1, \phi_2}^q (E_1, E_2) \) term in (4)

\[
\text{Tr} \left\{ \frac{S}{(1 - \alpha S)^2} \right\} = I \sum_{q \in W_T^*} \frac{\mathbf{1}(|q| \leq \epsilon)}{(1 - \alpha + D|q|^2)^2} + O \left( \frac{L}{W} u^{-1/2} + \frac{L}{W^2} u^{-5/2} \right). \tag{6}
\]

Let

\[
S_1(b, R) := \sum_{n \in \mathbb{Z}} \frac{\mathbf{1}(|n| \leq R)}{(n^2 + \zeta b)^2},
\]

then

\[
\sum_{q \in W_T^*} \frac{\mathbf{1}(|q| \leq \epsilon)}{(1 - \alpha + D|q|^2)^2} = \frac{1}{D^2} \left( \frac{L}{2\pi W} \right)^4 S_1(b, R). \tag{7}
\]

In dimension 1,

\[
S_1(b, R) = S_1(b) + O \left( R^{-3} \right), \quad \text{with} \; S_1(b) := \sum_{n \in \mathbb{Z}} \frac{1}{(n^2 + \zeta b)^2}.
\]

Using Poisson summation formula,

\[
\sum_{q \in \mathbb{Z}} \frac{1}{z^2 + q^2} = \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} dq \; \frac{e^{-2\pi i n q}}{z^2 + q^2} = \sum_{n \in \mathbb{Z}} \pi \frac{e^{-2\pi |n| z}}{z} = \frac{\pi}{z} \coth(\pi z). \tag{8}
\]
Thus,

\[ \sum_{q \in \mathbb{Z}} \frac{1}{z + q^2} = \frac{\pi}{\sqrt{z}} \coth(\pi \sqrt{z}). \]

By differentiating the above relation with respect to \( z \), we obtain

\[ \sum_{q \in \mathbb{Z}} \frac{1}{(z + q^2)^2} = \frac{\pi}{2z^{3/2}} \coth(\pi \sqrt{z}) + \frac{\pi^2}{2z^3} \frac{1}{\sinh^2(\pi \sqrt{z})}. \]

Hence,

\[ S_1(b) = \frac{\pi}{2(zb)^{3/2}} \coth(\pi \sqrt{zb}) + \frac{\pi^2}{2zb} \frac{1}{\sinh^2(\pi \sqrt{zb})}. \]

The result then follows from (7) and (6) and the fact that \( I = 1 + O(W^{-1}). \)

**Proof of Theorem 1.1(i)** We have

\[
\coth(\pi \sqrt{ib}) = \frac{1}{2} \frac{\sinh(\pi \sqrt{2b}) - i \sin(\pi \sqrt{2b})}{\sinh^2(\pi \sqrt{b/2}) + \sin^2(\pi \sqrt{b/2})}
\]

\[
\frac{1}{\sinh^2(\pi \sqrt{ib})} = \frac{\sin^2(\pi \sqrt{b/2}) \cos(\pi \sqrt{2b}) - \sin^2(\pi \sqrt{b/2}) - i \sin(\pi \sqrt{2b}) \sin(\pi \sqrt{2b})/2}{(\sin^2(\pi \sqrt{b/2}) + \sin^2(\pi \sqrt{b/2}))^2}.
\]

Hence,

\[
\text{Re } S_1(b) = -\frac{\pi}{4\sqrt{2}} \frac{1}{b^{3/2}} \frac{\sin(\pi \sqrt{2b}) + \sin(\pi \sqrt{2b})}{\sinh^2(\pi \sqrt{b/2}) + \sin^2(\pi \sqrt{b/2})} \]

\[
-\frac{\pi^2}{4b} \frac{\sinh(\pi \sqrt{2b}) \sin(\pi \sqrt{2b})}{(\sin^2(\pi \sqrt{b/2}) + \sin^2(\pi \sqrt{b/2}))^2}.
\]

The result follows using Proposition 2.1.

A plot of the following function at the transition \( b \simeq 1 \) is given in Fig. 1.

\[
f(b) = -\frac{\sin(\pi \sqrt{2b}) + \sin(\pi \sqrt{2b})}{\sin^2(\pi \sqrt{b/2}) + \sin^2(\pi \sqrt{b/2})} - \pi \sqrt{2b} \frac{\sin(\pi \sqrt{2b}) \sin(\pi \sqrt{2b})}{(\sin^2(\pi \sqrt{b/2}) + \sin^2(\pi \sqrt{b/2}))^2}.
\]

**Corollary 2.2** In the mean-field regime \( b \ll 1 \),

\[
\frac{\langle Y^\eta_{\phi_1}(E_1); Y^\eta_{\phi_2}(E_2) \rangle}{\langle Y^\eta_{\phi_1}(E_1) \rangle \langle Y^\eta_{\phi_2}(E_2) \rangle} = -\frac{1}{LW} \left[ \frac{1}{2\pi^2 \nu^2 \omega^2} \frac{W}{L} - \frac{1}{360D^2 \pi^4 \nu^4} \left( \frac{L}{W} \right)^3 \right] + O\left( \frac{1}{\sqrt{\omega}} + \frac{1}{\omega^{5/2}W} + \frac{1}{\omega^2 L} + \frac{\eta^2 \nu}{\omega^4 L} + \frac{W^{c_0}}{(\omega + \eta)^{1/2}} + \frac{W}{L(\omega + \eta)} \right)
\]
Fig. 1 The correlation function multiplied by $\omega^{3/2}$ at the phase transition as a function of $\omega(L/W)^2$

which recovers [2, Theorem 2.9 (ii)] up to a multiplicative constant in the leading term and gives the next term in the asymptotic expansion. In the diffusive regime $1 \ll b \ll \min\left((\log(L/W))^2, (\eta/\omega + \omega)^{-2}\right)$,

$$\frac{\langle Y_{\phi_1}^\eta(E_1); Y_{\phi_2}^\eta(E_2) \rangle}{\langle Y_{\phi_1}^\eta(E_1) \rangle \langle Y_{\phi_2}^\eta(E_2) \rangle} = -\frac{1}{16\sqrt{2}\pi^7\nu^4D^2} \frac{L^2}{W^4} b^{-3/2} \left[ 1 + 4\sqrt{2}e^{-\pi\sqrt{2b}} \left[ \sqrt{b} \sin(\pi\sqrt{2b}) + O\left(1 + \frac{1}{\sqrt{\omega}} \frac{L^2}{W^2} \right) \right] \right]$$

which recovers [3, Theorem 2.12 (i)] and gives the next term in the asymptotic expansion.

**Proof** In the mean-field regime $b \ll 1$, $f(b) = -\frac{4\sqrt{2}}{\pi \sqrt{b}} + \frac{4\sqrt{2}}{45} \pi^3 b \sqrt{b} + O(b^{7/2})$ so that the following asymptotic formula holds

$$\text{Tr} \left( 1 - \alpha S \right)^2 = \frac{1}{D^2} \left( \frac{L}{2\pi W} \right)^4 \left( \frac{1}{(\xi b)^2} + \frac{\pi^4}{45} + O \left( b + \frac{1}{b^2 W} + \frac{W^3}{L^3} u^{-1/2} + \frac{W^2}{L^3} u^{-5/2} \right) \right).$$

In the diffusive regime $1 \ll b \ll \min\left((\log(L/W))^2, (\eta/\omega + \omega)^{-2}\right)$, $f(b) = -2 - 8\sqrt{2}\pi e^{-\pi\sqrt{2b}} \sqrt{b} \sin(\pi\sqrt{2b}) + O(e^{-\pi\sqrt{2b}})$, so that the following asymptotic formula holds

$$\text{Tr} \left( 1 - \alpha S \right)^2 = \frac{1}{32\pi^3 D^2 W^4} \frac{L^4}{(\xi b)^{3/2}} \times \left( 1 + 4\pi e^{-2\pi\sqrt{\xi b}} \left[ \sqrt{\xi b} + O \left( 1 + \frac{\sqrt{\omega}}{W^2} + e^{2\pi\sqrt{\xi b}} (u + (uW)^{-1}) \right) \right] \right).$$

The results follow from (4) and (3).
2.2 Dimension 2

From equation (5), it follows

\[
\text{Tr} \frac{S}{(1 - \alpha S)^2} = \sum_{q \in W^*} \frac{\hat{S}_W(q)}{(1 - \alpha \hat{S}_W(q))^2} 1(|q| \leq \epsilon) + O \left( \frac{L^2}{\delta \epsilon W^2} \right)
\]

\[
= \sum_{q \in \frac{2\pi W}{L} \mathbb{Z}^2} \frac{I - D|q|^2}{(1 - \alpha \hat{S}_W(q))^2} 1(|q| \leq \epsilon) + O \left( \frac{L^2}{W^2} \right)
\]

\[
= \sum_{q \in \frac{2\pi W}{L} \mathbb{Z}^2} \frac{I - D|q|^2}{(1 - \alpha + D|q|^2 - Q(q))^2} 1(|q| \leq \epsilon) + O \left( \frac{L^2}{W^2} \right)
\]

where in the second equality we used the fact that

\[
\sum_{q \in W^*} \frac{Q(q)}{(1 - \alpha \hat{S}_W(q))^2} 1(|q| \leq \epsilon) = O \left( \sum_{q \in W^*} \frac{|q|^4}{(1 - \alpha \hat{S}_W(q))^2} 1(|q| \leq \epsilon) \right) = O \left( \frac{L^2}{W^2} \right).
\]

Expanding the denominator yields to

\[
\sum_{q \in \frac{2\pi W}{L} \mathbb{Z}^2} 1(|q| \leq \epsilon) \left( \frac{1}{1 - \alpha + D|q|^2 - Q(q)} \right)^2 = \sum_{q \in \frac{2\pi W}{L} \mathbb{Z}^2} 1(|q| \leq \epsilon) \left( \frac{Q(q)}{(D|q|^2 + u\zeta)^2} \left( 1 + 2 \frac{Q(q)}{D|q|^2 + u\zeta} + O \left( \frac{|q|^4}{|q|^2 + u\zeta} \right) \right) \right).
\]

In [3], the authors estimated this trace by using a Riemann sum approximation for each summation. The real part of the first term then vanishes, while the next terms give a logarithmic contribution. Here, we no longer make the Riemann sum approximation and give a precise estimation of this term. More precisely, rewriting the first summation over the torus \((2\pi W/L)\mathbb{Z}^2\) as a summation over \(\mathbb{Z}^2\), it can be expressed in terms of an inhomogeneous Epstein zeta function

\[
\mathcal{J} := \sum_{q \in \frac{2\pi W}{L} \mathbb{Z}^2} 1(|q| \leq \epsilon) \left( \frac{1}{(D|q|^2 + u\zeta)^2} \right) = \frac{1}{D^2} \left( \frac{L}{2\pi W} \right)^4 S_2(b, R)
\]

where

\[
S_2(b, R) := \sum_{n \in \mathbb{Z}^2} 1(|n| \leq R) \left( \frac{1}{|n|^2 + \zeta b^2} \right).
\]
To express the leading term in terms of $S_2(b)$ defined by

$$S_2(b) := \sum_{n \in \mathbb{Z}^2} \frac{1}{(|n|^2 + ib)^2},$$

we expand $\zeta$ using (2) implying that

$$\mathcal{J} = \frac{1}{D^2} \left( \frac{L}{2\pi W} \right)^4 \left( S_2(b) + O \left( \sum_{n \in \mathbb{Z}^2} \frac{1}{(|n|^2 + ib)^3} \right) \right).$$

We have

$$\sum_{n \in \mathbb{Z}^2} \frac{1}{(|n|^2 + ib)^3} = \frac{1}{b^3} \sum_{m \in b^{-1/2}\mathbb{Z}^2} \frac{1}{(m^2 + i)^3} = \frac{2\pi}{b^2} \int_0^\infty \frac{dr}{(r^2 + i)^3} + O(b^{-5/2}) = O \left( \frac{1}{b^2} \right).$$

Thus,

$$\mathcal{J} = \frac{1}{D^2} \left( \frac{L}{2\pi W} \right)^4 S_2(b) + O \left( \frac{L^2}{W^2} \left( 1 + \frac{\eta^2}{\omega^3} \right) \right)$$

and

$$\text{Tr} \frac{S}{(1 - \alpha S)^2} = \frac{1}{D^2} \left( \frac{L}{2\pi W} \right)^4 S_2(b) + O \left( \frac{L^2}{W^2} \left( \log |u| + \frac{\eta^2}{\omega^3} \right) \right), \quad b \gg 1. \tag{9}$$

**Proposition 2.3** The real part of $S_2(b)$ is given by

$$\text{Re} \ S_2(b) = -\frac{1}{b^2} + 2 \sum_{n=0}^\infty (-1)^n \left( 1 - \left( \frac{\pi b/(2n + 1)}{\sinh(\pi b/(2n + 1))} \right)^2 \right).$$

**Proof** Let $\theta$ be the third Jacobi theta function defined by $\theta(t) = \sum_{n \in \mathbb{Z}} e^{-n^2 t}$. Then, $S_2(b)$ is the Laplace–Mellin transform of the squared Jacobi theta function

$$S_2(b) = \int_0^\infty dt \sum_{n \in \mathbb{Z}^2} e^{-(n_1^2 + n_2^2 + ib)t} = \int_0^\infty dt \ \theta(t)^2 e^{-ibt}.$$
which can be rewritten in a more convenient way using a Jacobi identity (see, for example, [4])

\[ \theta(t)^2 = 1 + 4 \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} (-1)^n e^{-m(2n+1)t}. \]

Hence,

\[ S_2(b) = -\frac{1}{b^2} + 4 \sum_{n=0}^{\infty} (-1)^n \sum_{m=1}^{\infty} \frac{1}{(2n+1)^2 (m + ib/(2n+1))^2}. \]

To calculate the second sum, we rewrite equation (8) as

\[ \frac{1}{z} + i \sum_{m=1}^{\infty} \left( \frac{1}{m + iz} - \frac{1}{m - iz} \right) = \pi \coth(\pi z) \]

which gives by differentiation with respect to \( z \)

\[ -\frac{1}{z^2} + \sum_{m=1}^{\infty} \left( \frac{1}{(m + iz)^2} + \frac{1}{(m - iz)^2} \right) = -\frac{\pi^2}{\sinh^2(\pi z)}, \]

or similarly, for real \( z \),

\[ \text{Re} \sum_{m=1}^{\infty} \frac{1}{(m + iz)^2} = \frac{1}{2z^2} - \frac{\pi^2}{2 \sinh^2(\pi z)}. \] \( \text{(10)} \)

The result follows by replacing \( z \) by \( b/(2n+1) \) in (10).

**Proposition 2.4** For \( b \gg 1 \), the real part of \( S_2(b) \) has the following asymptotic behaviour

\[ \text{Re} S_2(b) = -4\pi^2 e^{-\pi \sqrt{2b}} b^{-3/4} \sin \left( \pi \sqrt{2b} - \frac{\pi}{8} \right) + O(e^{-\pi \sqrt{2b} b^{-11/4}}). \]

**Proof** Given an analytic function \( g \) in \( \{ z \in \mathbb{C} | \text{Re}(z) \geq 0 \} \) such that

(i) \( \lim_{y \to \infty} |g(x \pm iy)|e^{-2\pi y} = 0 \)

uniformly in \( x \) on every finite interval in \([0, \infty)\) and such that

(ii) \( \int_0^{\infty} dy |g(x + iy) - g(x - iy)|e^{-2\pi y} \)
exists for all \( x \geq 0 \) and tends to 0 as \( x \to \infty \), then Abel–Plana summation formula gives an integral representation of an alternating series through the following relation (see, for example, \[ 1\])

\[
\sum_{n=0}^{\infty} (-1)^n g(n) = \frac{1}{2} g(0) + i \int_0^{\infty} dy \frac{g(iy) - g(-iy)}{2 \sinh(y)}.
\]

Define \( g_b(n) := 1 - \left( \frac{\pi b/(2n+1)}{\sinh(\pi b/(2n+1))} \right)^2 \) which is analytic in \( \{ z \in \mathbb{C} \mid \text{Re}(z) \geq 0 \} \). We have

\[
|g_b(x \pm iy)|e^{-2\pi y} \leq \left( 1 + \frac{(\pi b)^2}{|2x + 1 \pm 2iy|^2} \frac{1}{\sinh^2(\pi b/(2x + 1 \pm 2iy))} \right)e^{-2\pi y}
\]

\[
\leq \left( 1 + \frac{(\pi b)^2}{(2x + 1)^2 + 4y^2} \frac{1}{\sinh^2(\pi b(2x + 1) / ((2x + 1)^2 + 4y^2))} \right)e^{-2\pi y}
\]

\[
\leq \left( 2 + \frac{4y^2}{(2x + 1)^2} \right)e^{-2\pi y} \to 0 \text{ as } y \to \infty \text{ uniformly in } x
\]

verifying condition (i). Let \( \varphi = \frac{\pi b(2x+1)}{(2x+1)^2+4y^2} \) and \( \psi = \frac{2\pi by}{(2x+1)^2+4y^2} \). Also,

\[
|g_b(x + iy) - g_b(x - iy)|
= (\pi b)^2 \left| \frac{1}{(2x + 1 + 2iy)^2 \sinh^2(\pi b/(2x + 1 + 2iy))} - \frac{1}{(2x + 1 - 2iy)^2 \sinh^2(\pi b/(2x + 1 - 2iy))} \right|
\]

\[
= \frac{4(\pi b)^2}{((2x + 1)^2 + 4y^2)^2 (\sinh^2(\varphi) + \sin^2(\psi)^2)} |(2x + 1) \cosh \varphi \sin \psi - 2y \sinh \varphi \cos \psi|
\]

\[
\times |(2x + 1) \sinh \varphi \cos \psi + 2y \cosh \varphi \sin \psi|.
\]

Using that \( \sinh \varphi \geq \varphi \), \( \sin \psi \leq \psi \) and \( \cosh \varphi / \sinh^4 \varphi \leq 16 \sinh^4(3\varphi/4) \), for all \( x, y \geq 0 \), we have

\[
|g_b(x + iy) - g_b(x - iy)| \leq \frac{1}{b^2} \left( \gamma_1 \frac{y}{2x + 1} + \gamma_2 \frac{y^3}{(2x + 1)^3} \right)
\]

\[
\times \left( \gamma_3 + \gamma_4 \frac{y^2}{(2x + 1)^2} + \gamma_5 \frac{y^5}{(2x + 1)^5} \right)
\]
for some positive constants $\gamma_i$, $i = 1, \ldots, 5$, which shows that condition (ii) is satisfied. Hence, Abel–Plana summation formula implies that

$$
\sum_{n=0}^{\infty} (-1)^n \left( 1 - \left( \frac{\pi b/(2n + 1)}{\sinh(\pi b/(2n + 1))} \right)^2 \right) = \frac{1}{2} - \frac{(\pi b)^2}{2 \sinh^2(\pi b)} + i \int_0^\infty dy \frac{g_b(iy) - g_b(-iy)}{2 \sinh(\pi y)}.
$$

(11)

We have

$$
i \int_0^\infty dy \frac{g_b(iy) - g_b(-iy)}{2 \sinh(\pi y)} = -2 \text{Im} \int_0^\infty dy \frac{g_b(iy)}{2 \sinh(\pi y)} = 8 \text{Im} \int_0^\infty dy \left( \frac{\pi b}{2iy + 1} \right)^2 \frac{1}{1 + e^{-4\pi b/(2iy+1)} - 2 e^{-2\pi b/(2iy+1)}}
$$

$$
= 8 \text{Im} I(b).
$$

Let $h(y) := -\pi(2/(2iy + 1) + y/b)$ and

$$
j(y) := \left( \frac{\pi b}{2iy + 1} \right)^2 \frac{1}{1 + e^{-4\pi b/(2iy+1)} - 2 e^{-2\pi b/(2iy+1)}} \frac{1}{1 - e^{-2\pi y}}.
$$

The function $\text{Re} h(y)$ attains his maximum at $y_0 = \sqrt{b} e^{-i\pi/4} + i/2$ which satisfies $h(y_0) = -\pi \sqrt{2/b} + i (\pi \sqrt{2/b} - \pi/(2b))$ and $h''(y_0) = 2\pi b^{-3/2} e^{-3i\pi/4}$. From the saddle point method, it follows that

$$
I(b) = \sqrt{\frac{2\pi}{-h''(y_0)}} \frac{1}{\sqrt{b}} e^{h(y_0)} \left( j(y_0) + O(b^{-1}) \right)
$$

$$
= \frac{\pi^2}{4} e^{-\pi \sqrt{2b}} e^{i(\pi \sqrt{2b} - \pi/8)} \left( b^{5/4} + O(b^{-3/4}) \right).
$$

(12)

Putting Proposition 2.3, (11) and (12) together, it comes

$$
\text{Re} S_2(b) = -4\pi^2 e^{-\pi \sqrt{2b}} b^{-3/4} \sin \left( \pi \sqrt{2b} - \frac{\pi}{8} \right) + O(e^{-\pi \sqrt{2b} b^{-11/4}}), \ b \gg 1
$$

showing that the function $\text{Re} S_2(b)$ changes sign infinitely many times. \square

**Proof of Theorem 1.1(ii)** Putting (3), (4), (9), Proposition 2.4 and [3, Proposition 3.3(ii)] together proves Theorem 1.1 (ii). \square

Figure 2 is a plot of $\text{Re} S_2(b)$ and Fig. 3 is a plot of $(4\pi^2)^{-1} b^{3/4} e^{\pi \sqrt{2b}} \text{Re} S_2(b)$ compared to its asymptotic expression.
Proposition 2.5 The two leading terms of the local density–density correlation function are of the same order for $b \sim (\log(L/W))^2$. More precisely,

$$\exists \gamma \in \left[\frac{1}{2\pi^2}, \frac{2}{\pi^2}\right] \text{ such that } \frac{L^2}{W^2} b^{-3/2} e^{-\pi \sqrt{2b}} = |\log \omega| \text{ where } b = \gamma (\log(L/W))^2.$$

Proof Let $L = W^{1+a}$ and $\omega = W^{-c}$ for some positive constants $a$ and $c$. Using that $b \sim \omega L^2/W^2 > 1$, it implies that $2a > c$. Write $b = \gamma_0 (\log W)^2$ for some $\gamma_0$, then the solution to the equation below

$$\frac{L^2}{W^2} b^{-3/2} e^{-\pi \sqrt{2b}} = |\log \omega|$$

corresponds to the root of the following function

$$f(\gamma_0) := W^{2a-\pi \sqrt{2\gamma_0}} - c\gamma_0^{3/2} (\log W)^4$$
which is decreasing in $\gamma_0$ and is such that $f(a^2/(2\pi^2)) > 0$. Indeed

$$f \left( \frac{a^2}{2\pi^2} \right) = W^a - \frac{ca^3}{2^{3/2}\pi^3} (\log W)^4 > g(a)$$

where

$$g(a) := e^{a\beta_1} - \beta_2 a^4, \quad \beta_1 := \log W, \quad \beta_2 := \frac{(\log W)^4}{\sqrt{2}\pi^3}.$$ 

Thus, $g^v(a) = \beta_1^5 e^{a\beta_1} > 0$ implying that $g^v$ is increasing, and $g^v(0) = \beta_1^4 - 24\beta_2 = (\log W)^4 (1 - 24/(\sqrt{2}\pi^3)) > 0$, so that $g^v(a) > 0$ for all $a > 0$. We deduce that $g(a) > 0$ for all $a > 0$. Also, we have

$$f \left( \frac{2a^2}{\pi^2} \right) = 1 - \frac{2^{3/2}ca^3}{\pi^3} (\log W)^4$$

$$= 1 - \frac{2^{3/2}}{\pi^3} \log W^c (\log W^a)^3$$

$$= 1 - \frac{2^{3/2}}{\pi^3} |\log \omega| \left( \log \frac{L}{W} \right)^3 < 0.$$ 

Thus, there is a constant $\gamma_0 \in \left[ \frac{1}{2\pi^2}, \frac{2}{\pi^2} \right]$ such that $f(\gamma_0 a^2) = 0$ showing the proposition. \qed

As a consequence of the above proposition, we deduce the following corollary.

**Corollary 2.6** In dimension 2, the densities of states are alternately positively and negatively correlated $O(\log(L/W))$ times.

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Appendix

For the sake of comprehensiveness, we present the main steps of the proof of relation (4) following [3]. For more details, we refer to Section 3 of Erdős and Knowles’ paper. Let introduce the following parameters

\[ \rho \in (0, \frac{1}{3}), \mu \text{ such that } \rho < \mu < \frac{1}{3} \text{ and } \delta > 0 \text{ satisfying } 2\delta < \mu - \rho < 3\delta \text{ and write } \eta := M^{-\rho}. \]

We also introduce the following notations from [3]:

\[ a_n(t) := \sum_{k \geq 0} \frac{\alpha_{n+2k}(t)}{(M - 1)^k}, \quad \alpha_k(t) := 2(-i)^k \frac{k + 1}{t} J_{k+1}(t), \]

\[ J_{\nu} \text{ denoting the } \nu-\text{th Bessel function of the first kind. For } n \in \mathbb{N}, E \in \mathbb{R} \text{ and } \phi \text{ a test function, let } \gamma_n(E) \text{ and } \tilde{\gamma}_n(E, \phi) \text{ be defined by} \]

\[ \gamma_n(E) := \int_0^\infty dt \ e^{iEt} a_n(t), \quad \tilde{\gamma}_n(E, \phi) := \int_0^{M^\mu} dt \ e^{iEt} \widehat{\phi}(\eta t) a_n(t) \]

\[ \widehat{\phi} \text{ being the Fourier transform of } \phi \]

\[ \widehat{\phi}(t) = \frac{1}{2\pi} \int_{\mathbb{R}} dE e^{iEt} \phi(E). \]

Then, from [2, Lemma 3.2]

\[ \gamma_n(E) = \frac{2(-i)^n e^{i(n+1) \arcsin E}}{1 + (M - 1)^{-1} e^{2i \arcsin E}}. \]

The leading term \( \Theta_{\phi_1, \phi_2}^\eta(E_1, E_2) \) is defined by [3, (4.60)]

\[ \Theta_{\phi_1, \phi_2}^\eta(E_1, E_2) := \frac{W_d}{L^d} \frac{V_{\text{main}}}{\langle Y_{\phi_1}^\eta(E_1) \rangle \langle Y_{\phi_2}^\eta(E_2) \rangle} \]

where the main term \( V_{\text{main}} \) is given in [3, (3.23)]

\[ V_{\text{main}} = \sum_{b_1, b_2 \geq 0} \sum_{(b_3, b_4) \in \mathcal{A}} 1 \left( \sum_{i=1}^4 b_i \leq M^\mu / 2 \right) \]

\[ \times 2 \text{Re} \left( \gamma_{2b_1+b_3+b_4}^\eta(E_1, \phi_1) \right) \text{Re} \left( \gamma_{2b_2+b_3+b_4}^\eta(E_2, \phi_2) \right) T^{b_1+b_2} \text{Tr} S^{b_3+b_4} \]

where \( \mathcal{A} \) is the set \( \mathcal{A} := (\{1, 2, \ldots\} \times (0, 1, \ldots)) \setminus \{(2, 0), (1, 1)\} \). Let \( \psi_i(E) := \phi_i(-E), i = 1, 2, \) where \( \phi_i, i = 1, 2, \) are test functions satisfying conditions (C).
is shown that the above expands as \([3, (3.64)]\)

\[
V_{\text{main}} = \sum_{b_1, b_2 = 0}^{[M^\mu] - 1} \sum_{(b_3, b_4) \in A_\mu} 2 \text{Re} (\gamma_{2b_1 + b_3 + b_4} \ast \psi_{1, < ; \eta}) (E_1) 2 \text{Re} (\gamma_{2b_2 + b_3 + b_4} \ast \psi_{2, < ; \eta}) (E_2) \mathcal{I}^{b_1 + b_2} \text{Tr} S^{b_3 + b_4} + O_q (L^d M^{-q})
\]

\( (14) \)

where \(\psi_{i, < ; \eta} (E) := \eta^{-1} \psi_i (\eta^{-1} E) \chi (M^{-\tau/2} E), i = 1, 2\), \(\chi\) being a smooth nonnegative symmetric function bounded by 1 satisfying \(\chi (E) = 1\) for \(|E| \leq 1\) and \(\chi (E) = 0\) for \(|E| \geq 2\) and \(\tau\) is a positive constant such that \(\eta \leq M^{-\tau} \omega\). The set \(A_\mu\) is the subset \(A_\mu := ([1, 2, \ldots, [M^\mu]] \times \{0, 1, \ldots, [M^\mu] - 1\}) \setminus (2, 0, (1, 1))\), and the convolution of two functions \(\phi\) and \(\psi\) is defined by

\[
(\phi \ast \psi) (E) := \frac{1}{2\pi} \int dE' \phi (E - E') \psi (E').
\]

Using relation \((2 \text{Re} x_1) (2 \text{Re} x_2) = 2 \text{Re} (x_1 \overline{x}_2 + x_1 x_2)\) in \((14)\), the authors in \([3]\) then split \(V_{\text{main}}\) into two parts as \(V_{\text{main}} = 2 \text{Re} (V'_{\text{main}} + V''_{\text{main}}) + O_q (L^d M^{-q})\) where \(V'_{\text{main}}\) identifies with the \(x_1 \overline{x}_2\) part and \(V''_{\text{main}}\) with the \(x_1 x_2\) part, with \(x_1 = \gamma_{2b_1 + b_3 + b_4} \ast \psi_{1, < ; \eta}\) and \(x_2 = \gamma_{2b_2 + b_3 + b_4} \ast \psi_{2, < ; \eta}\). On the one hand, they show in \([3, (3.76)]\) that

\[
|V''_{\text{main}}| \leq C L^d \frac{d}{M}
\]

and on the other hand they split again \(V'_{\text{main}}\) into two terms \(V'_{\text{main}} = V'_{\text{main}, 0} - V'_{\text{main}, 1}\) where \([3, (3.69)]\)

\[
V'_{\text{main}, 1} = O \left( \frac{L^d}{M} \right).
\]

Then, \(V'_{\text{main}, 0}\) is shown to be equal to \([3, (3.73)]\)

\[
V'_{\text{main}, 0} = \left[ T (E_1) \overline{T (E_2)} \frac{e^{i A_1}}{1 + e^{2i A_1} \mathcal{I}} \frac{e^{-i A_2}}{1 + e^{-2i A_2} \mathcal{I}} \text{Tr} e^{i (A_1 - A_2) S} \left( 1 - e^{i (A_1 - A_2) S} \right)^2 \right] \ast \psi_{1, < ; \eta} (E_1) \ast \psi_{2, < ; \eta} (E_2) + O \left( \frac{L^d}{M} \right)
\]

where \(A_i := \arcsin E_i\) and

\[
T (z) := \frac{2}{1 + (M - 1)^{-1} e^{2i \arcsin z}}.
\]
Using the estimate
\[
e^{\pm iA_i} A_1 \frac{1}{1 + e^{\pm 2iA_i}} = \frac{1}{\pi v} \left( 1 + O(M^{-1} + \omega) \right), \quad i = 1, 2,
\]
where \( v = \frac{2}{\pi} \sqrt{1 - E^2} \).

and by definition of the convolution product, we have
\[
Y'_{\text{main}, 0} = \frac{1}{\pi^3 v^2} \int_{\mathbb{R}^2} \psi_1^{\leq, \eta}(v_1) \psi_2^{\leq, \eta}(v_2) \text{Tr} \frac{e^{i(\arcsin(E_1 - v_1) - \arcsin(E_2 - v_2)) S}}{(1 - e^{i(\arcsin(E_1 - v_1) - \arcsin(E_2 - v_2)) S})^2}
\]
\[
\times \left( 1 + O(M^{-1} + \omega) \right) + O \left( \frac{L^d}{M} \right).
\]

On the domain of integration, we have
\[
\arcsin(E_1 - v_1) = \arcsin E - \left( \frac{\omega}{2} + v_1 \right) \frac{1}{\sqrt{1 - E^2}} + O(\omega(\omega + M^{-\tau/2}))
\]
\[
\arcsin(E_2 - v_2) = \arcsin E + \left( \frac{\omega}{2} - v_2 \right) \frac{1}{\sqrt{1 - E^2}} + O(\omega(\omega + M^{-\tau/2}))
\]

which implies
\[
e^{i(\arcsin(E_1 - v_1) - \arcsin(E_2 - v_2))} = 1 - \frac{2i}{\pi v} (\omega + v_1 - v_2) + O(\omega(\omega + M^{-\tau/2})).
\]

Moreover by definition of \( \psi^{\leq, \eta} \) and the test functions, we have for \( i = 1, 2 \)
\[
M^{\tau/2} \int_{-1}^{1} dv \psi_i^{\eta}(M^{\tau/2} v) = \int_{\mathbb{R}} dv \psi(v) + O_q(M^{-(\tau/2+q)})
\]
\[
= 2\pi + O_q(M^{-(\tau/2+q)}), \quad \forall q > 0
\]

and
\[
M^{\tau/2} \int_{1}^{2} dv \psi_i^{\eta}(M^{\tau/2} v) \chi(v) \leq \int_{M^{\tau/2+\rho}}^{2M^{\tau/2+\rho}} dv \psi(v) = O_q(M^{-(\tau/2+q)}), \quad \forall q > 0
\]

implying that
\[
\int_{\mathbb{R}} dv \psi_i^{\leq, \eta}(v) = M^{\tau/2} \int_{-1}^{1} dv \psi_i^{\eta}(M^{\tau/2} v) + 2M^{\tau/2} \int_{1}^{2} dv \psi_i^{\eta}(M^{\tau/2} v) \chi(v)
\]
\[
= 2\pi + O_q(M^{-(\tau/2+q)}), \quad \forall q > 0.
\]

Also,
\[
\int_{\mathbb{R}^2} dv_1 dv_2 \psi_1^{\leq, \eta}(v_1) \psi_2^{\leq, \eta}(v_2)(v_1 - v_2) = 0.
\]
Thus, $\mathcal{V}_{\text{main},0}'$ becomes
\[
\mathcal{V}_{\text{main},0}' = \frac{4}{\pi^2 v^2} \alpha \text{Tr} \frac{S}{(1 - \alpha S)^2} + O \left( \frac{L^d}{W^d} \right)
\]
and
\[
\mathcal{V}_{\text{main}} = \frac{8}{\pi^2 v^2} \text{Re} \text{Tr} \frac{S}{(1 - \alpha S)^2} (1 + O(\omega)) + O \left( \frac{L^d}{W^d} \right)
\]
(15)
since $\alpha = 1 + O(\omega)$. Recall from [3, Lemma 4.17] that the expectation value of the local density of states around energy $E \in [-1 + \kappa, 1 - \kappa]$ is given by
\[
\langle Y^\eta_{\phi_i}(E) \rangle = 2\pi v + O(\eta).
\]
(16)
Putting equations (13), (15) and (16) together ends the proof of (4).

We emphasise that in the above proof, which is derived in [3], is valid in all regimes, i.e. while the authors are considering the diffusive regime in their paper, the hypothesis $\eta \gg \eta_c$ is not used at any stage of it and is thus valid in our case.

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