Singular limit of an integrodifferential system related to the entropy balance

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Dedicated to Professor Mauro Fabrizio,
nice friend and valuable collaborator,
wishing him the very best “ad multos annos”

Abstract

A thermodynamic model describing phase transitions with thermal memory, in terms of an entropy equation and a momentum balance for the microforces, is addressed. Convergence results and error estimates are proved for the related integrodifferential system of PDE as the sequence of memory kernels converges to a multiple of a Dirac delta, in a suitable sense.

Key words: entropy equation, thermal memory, phase field model, nonlinear partial differential equations, asymptotics on the memory term

AMS (MOS) Subject Classification: 35K55, 35B40, 35Q79, 80A22.

1 Introduction

In this paper, we deal with the following integrodifferential PDE system, describing a phase transition process in the case when thermal memory effects are included. Indeed, here \(\vartheta_\tau\) stands for the absolute temperature, \(\chi_\tau\) for the phase parameter, and \(k_\tau\) for a

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memory kernel

\[
\begin{align*}
\partial_t (\ln \vartheta + \lambda(\chi)) & - \kappa_0 \Delta \vartheta - \Delta (k_\tau * \vartheta) = f \\
\partial_t \chi + \Delta \chi + \xi + \sigma'(\chi) & = \lambda'(\chi) \vartheta, \quad \xi \in \beta(\chi) \\
\vartheta |_{\Gamma} & = \vartheta_0 \quad \text{and} \quad \partial_n \vartheta |_{\Gamma} = 0 \\
\ln \vartheta(0) & = \ln \vartheta_0 \quad \text{and} \quad \chi(0) = \chi_0.
\end{align*}
\]

Each of the partial differential equations (1.1)–(1.2) is meant to hold in a three-dimensional bounded domain Ω, endowed with a smooth boundary Γ, and in some time interval (0, T). In (1.1), the memory kernel \(k_\tau\) may depend on a positive parameter \(\tau\). Moreover, the symbol \(*\) denotes the usual time convolution formally defined by \((a * b)(t) = \int_0^t a(t - s) b(s) \, ds\) for functions that depend just on time, and then extended to functions that also depend on space. Furthermore, \(f\) is some given source term. In (1.2), \(\beta\) is a maximal monotone graph in \(\mathbb{R}^2\), while \(\lambda\) and \(\sigma\) are real functions defined on the whole of \(\mathbb{R}\). The boundary conditions (1.3) must be satisfied in \(\Gamma \times (0, T)\), while the initial conditions (1.4) are written for the functions \(\ln \vartheta\) and \(\chi\): of course, \(\vartheta_\Gamma, \vartheta_0\), and \(\chi_0\) are given boundary and initial data.

Equation (1.1) may be interpreted as an entropy balance equation. Note in particular that the equation is singular with respect to the temperature, mainly for the presence of the logarithm, forcing the temperature to assume only positive values (which is in accordance with physical consistency). Similar systems have been studied in the literature from the point of view of the existence and regularity of solutions (see, among the others, [3, 4, 5, 6, 7, 12]).

The well-posedness of a proper variational formulation of (1.1)–(1.4) has been proved in [5]. Here, our main goal is the following. By assuming that \(k_\tau\) converges to \(\kappa_0' \delta\) at \(\tau \downarrow 0\) in a suitable sense, where \(\delta\) is the Dirac mass at the origine of the real line and \(\kappa_0'\) is a real constant satisfying \(\kappa := \kappa_0 + \kappa_0' > 0\), we prove that the solution \((\vartheta_\tau, \chi_\tau)\) to (1.1)–(1.4) converges in a proper topology to the solution \((\vartheta, \chi)\) of the problem stated below

\[
\begin{align*}
\partial_t (\ln \vartheta + \lambda(\chi)) & - \kappa \Delta \vartheta = f \\
\partial_t \chi + \Delta \chi + \xi + \sigma'(\chi) & = \lambda'(\chi) \vartheta, \quad \xi \in \beta(\chi) \\
\vartheta |_{\Gamma} & = \vartheta_0 \quad \text{and} \quad \partial_n \vartheta |_{\Gamma} = 0 \\
\ln \vartheta(0) & = \ln \vartheta_0 \quad \text{and} \quad \chi(0) = \chi_0.
\end{align*}
\]

This convergence result is obtained by the use of an a priori estimates technique and passage to the limit arguments, based on monotonicity and compactness. Moreover, an error estimate, i.e. an estimate of suitable norms or quantities involving the difference of solutions, is shown.

Our paper is organized as follows. In the next section, we discuss a derivation of the system (1.5)–(1.6) from the basic laws of thermomechanics. Section 3 is devoted to the statement of our assumptions and of our results on the mathematical problem. In Section 4 we present some auxiliary material that is needed for the proof of our convergence Theorem 3.3 mainly. The last section is devoted to the proofs of the above theorem and of the error estimate stated in Theorem 3.4.
2 The model

In this section, we briefly introduce the modeling derivation of the equations (1.1)–(1.2) and discuss the convergence to (1.5)–(1.6), as the parameter $\tau$ (in the memory kernel) tends to 0. Here, the argument is mainly developed from a physical point of view, while we refer to subsequent sections for a more precise setting of analytical assumptions and comments. In particular, we aim to focus on the fact that (1.1) accounts for thermal evolution involving memory effects, on the basis of the memory kernel $k_\tau$.

Materials with thermal memory have been deeply studied in the literature, both from a modeling and analytical point of view. We refer, in particular, to the approach by Gurtin and Pipkin (see [15]) for thermal memory materials. Several authors have investigated phase transitions in special materials with thermal memory, both concerning modeling and analysis. For a fairly complete and detailed presentation of this kind of problems, let us mention the very recent monograph [1]. Now, we combine thermal memory with a new theory for phase transitions models, based on a generalization of the principle of virtual powers (see [11]). The idea is that micro-forces, which are responsible for the phase transition, have to be included in the whole energy balance of the system. Consequently, the phase (evolution) equation is derived as a micro-forces balance equation and it is coupled with an entropy evolution equation. This approach has been recently investigated in the literature by several authors (among the others, we mainly refer to the papers [3] and [7], in which the derivation of the model is detailed in the case when possible thermal memory effects are included, as in equations (1.1)–(1.2)).

Indeed, let us recall that in [3] the theory by Gurtin-Pipkin is considered, allowing the free energy functional to depend on the past history of the temperature gradient. The resulting functional accounts for non-dissipative contributions in the heat flux, which may be combined with additional dissipative instantaneous contributions coming from a pseudo-potential of dissipation. The use of an entropy balance has been recovered, in this approach, from a rescaling (with respect to the absolute temperature) of the energy balance, under the small perturbations assumption (see also [4, 5]). In [7] a fairly general theory is introduced. The model is derived by a dual approach (mainly in the sense of convex analysis) in which the entropy and the history of the entropy flux are chosen as state variables (together with the phase parameter and possibly its gradient). Then, the dissipative functional is written in terms of a dissipative contribution in the entropy flux and for the time derivative of the phase parameter.

Let us point out that the above mentioned approach is not far from the theory proposed by Green-Naghdi [13] and Podio-Guidugli [16], in which some thermal displacement is introduced as state variable (it is a primitive of the temperature) and the equations come from a generalization of the principle of virtual powers, in which thermal forces are included. As a consequence, in this framework, the entropy equation is formally obtained as a momentum balance (i.e., a balance of thermal forces acting in the system). The reader may also examine [9, 10], where some asymptotic analyses are carried out to find the interconnections among peculiar Green and Naghdi types.

We aim to observe that the model we are investigating actually may be obtained by combining the above two theories, i.e. generalizing the principle of virtual powers accounting for microforces as well as thermal stresses. Let us present our position. First,
we specify the expression of the power of internal forces. The power of interior forces is written for any virtual micro-velocity $\gamma$ and thermal velocity $v$, as follows

$$P_i = \int_\Omega B\gamma + \mathbf{H} \cdot \nabla \gamma + \mathbf{Q} \cdot \nabla v,$$

(2.1)

where $B$ and $\mathbf{H}$ are interior forces responsible for the phase transition (as introduced in [11]), and $\mathbf{Q}$ stands for a thermal stress (corresponding to the entropy flux by [16]). Hence, the resulting balance equations are written as momentum balance equations. It is assumed that an external (density of) entropy source $f$ is applied. A thermal momentum is introduced to measure reluctance to the order of the system (in analogy with the mechanical momentum measuring reluctance to quiet). We prescribe that it is given by the entropy $s$. It results that (see (2.1))

$$s_t + \text{div} \mathbf{Q} = f.$$  

(2.2)

As far as the microscopic momentum balance is concerned, we assume that no acceleration and no external force are contributing, so that we have

$$B - \text{div} \mathbf{H} = 0.$$  

(2.3)

Henceforth, (2.2)–(2.3) are combined with suitable boundary conditions. As usual, we assume that the flux through the boundary $\mathbf{H} \cdot \mathbf{n}$ is null, while (mainly for analytical reasons) we prescribe a known temperature on the boundary.

The entropy $s$, the entropy flux $\mathbf{Q}$, and the new interior forces $B$ and $\mathbf{H}$ are recovered by suitable energy and dissipation functionals, that we are going to make precise, in terms of state variables. The state variables are related to the equilibrium of the thermodynamical system: they are the absolute temperature $\vartheta$, the phase parameter $\chi$, the gradient $\nabla \chi$ (actually accounting for local interactions), and the history variable $\nabla \vartheta^t$, which is defined as

$$\nabla \vartheta^t(s) = \int_{t-s}^{t} \nabla \vartheta(r)dr, \quad s > 0.$$  

(2.4)

As in [3], we assume that the free energy of the system (depending on $(\vartheta, \nabla \vartheta^t, \chi, \nabla \chi)$) is split into two contributions: the first is related to present variables at time $t$ ($\Psi_P$), the second accounts for some history in the system ($\Psi_H$), measured through a memory kernel (related to $k_r$ in the equations). In particular, the history contribution of the free energy is given by

$$\Psi_H(\nabla \vartheta^t) = \frac{1}{2} |\nabla \vartheta^t|^2_{S_r}$$  

(2.5)

where $S_r$ is the space of the past histories (as it is introduced in the theory of thermal memory materials by Gurtin and Pipkin), defined by

$$S_r := \{ f : (0, +\infty) \to \mathbb{R}^3 \text{ measurable s.t. } \int_0^{+\infty} h_r(s)|f(s)|^2ds < +\infty \}.$$  

(2.6)

Here, $h_r : (0, +\infty) \to (0, +\infty)$ (possibly depending on a parameter $\tau$) is a continuous, decreasing function such that

$$\int_0^{+\infty} s^2 h_r(s)ds < +\infty.$$  

(2.7)
The space $S_\tau$ is endowed with the natural norm
\[ |f|^2_{S_\tau} = \int_0^{+\infty} h_\tau(s)|f(s)|^2 ds \] (2.8)
and the related scalar product is $(v, u)_{S_\tau} = \int_0^{+\infty} h_\tau(s)v(s) \cdot u(s) ds$. Let us comment that in our system, to derive (1.1), we have introduced a kernel $k_\tau$ such that $-k'_\tau = h_\tau$. More precisely, let $k_\tau : (0, +\infty) \to \mathbb{R}$ and require that $k_\tau \in W^{2,1}(0, +\infty)$, $\lim_{s \to +\infty} k_\tau(s) = 0$. (2.9)

Hence, by virtue of the assumptions on $h_\tau$ we also have
\[ k'_\tau \leq 0 \quad \text{and} \quad k''_\tau \geq 0 \quad \text{a.e. in } (0, +\infty). \] (2.10)

Note that $k'_\tau(t)$ vanishes for $t$ going to $+\infty$ and that $k_\tau$ is a non-increasing function with $k_\tau(0) \geq 0$, and in the case $k_\tau(0) = 0$ one has $k_\tau \equiv 0$. These assumptions on $k_\tau$ actually ensure that the model is thermodynamically consistent, as it is detailed in [3].

Then, the free energy functional $\Psi_P$ (written at the present time $t$) is addressed
\[ \Psi_P(\vartheta, \chi, \nabla \chi) = c_V \vartheta (1 - \ln \vartheta) - \lambda(\chi) \vartheta + \sigma(\chi) + \hat{\beta}(\chi) + \frac{1}{2} |\nabla \chi|^2 \] (2.11)
where $c_V > 0$ (in the sequel let us take $c_V = 1$) is the specific heat, $\sigma$ and $\lambda$ are sufficiently smooth functions (with $\lambda'(\chi)$ denoting the latent heat), $\hat{\beta}$ is a proper convex and lower semicontinuous function, possibly accounting for internal constraints on the phase variable $\chi$. For instance, a fairly classical choice is $\hat{\beta}(\chi) = I_{[0, 1]}(\chi)$, which is equal to 0 if $\chi \in [0, 1]$ and takes value $+\infty$ elsewhere (thus forcing $\chi \in [0, 1]$).

Dissipation is rendered in terms of the time derivative $\chi_t$ and of the dissipative variable $\nabla \vartheta$. It is derived by a pseudo-potential of dissipation (in the sense of Moreau, i.e. a convex, non-negative function assuming its minimum 0 for null dissipation):
\[ \Phi(\chi_t, \nabla \vartheta) = \frac{1}{2} |\chi_t|^2 + \frac{\kappa_0}{2} |\nabla \vartheta|^2. \] (2.12)

Note that, in order to ensure the validity of the second principle of thermodynamics, it is required that $\kappa_0 \geq 0$.

Now, we are in a position to recover our system, after specifying constitutive relations for the involved physical quantities. We have that
\[ s = -\frac{\partial \Psi}{\partial \vartheta} = \ln \vartheta + \lambda(\chi) \] (2.13)
and
\[ B \in \frac{\partial \Psi}{\partial \chi} + \frac{\partial \Phi}{\partial \chi_t} = \beta(\chi) + \sigma'(\chi) - \lambda'(\chi) \vartheta + \chi_t \] (2.14)
$\beta$ being the subdifferential (in the sense of convex analysis) of $\hat{\beta}$, and
\[ H = \frac{\partial \Psi}{\partial (\nabla \chi)} = \nabla \chi. \] (2.15)
Hence, the entropy flux vector $\mathbf{Q}$ is specified by
\[ -\mathbf{Q} = (-\mathbf{Q}^{nd}) + (-\mathbf{Q}^d) \tag{2.16} \]
where $-\mathbf{Q}^{nd}$ results to be defined in $\mathcal{S}_\tau$. It is obtained taking the derivative in $\mathcal{S}_\tau$ of the history functional with respect to the history variable. Integrating by parts in time, using the Fréchet derivative, and exploiting the hypotheses on $k_\tau$ (see [3] for any further detail) lead to
\[ -\mathbf{Q}^{nd}(t) = -\int_{-\infty}^{t} k_\tau(t-s)\nabla \vartheta(s)ds. \tag{2.17} \]
We have now to make precise the dissipative part of the entropy flux
\[ -\mathbf{Q}^d = \frac{\partial \Phi}{\partial \nabla \vartheta} = \kappa_0 \nabla \vartheta. \tag{2.18} \]

Hence, equations (1.1)–(1.2) are obtained by (2.2) and (2.3) exploiting the above introduced constitutive relations. We point out that in (1.1), the past history contribution of (2.17) (actually its divergence), i.e. \( \int_{0}^{\infty} k_\tau(t-s)\nabla \vartheta(s)ds \), is assumed to be known and included in the external entropy source $f$ (we have used the same notation as in (2.2) for the sake of simplicity).

As we have already pointed out in the Introduction, the main aim of this paper is to investigate the asymptotic behavior of system (1.1)–(1.2) as the thermal memory kernel converges to $\kappa'_0 \delta$, $\delta$ being the Dirac mass at the origin of the real line and $\kappa'_0 > 0$, in a suitable sense
\[ k_\tau \to \kappa'_0 \delta. \tag{2.19} \]

More precisely, we are interested in proving that solutions to the system (1.1)–(1.2) converge to solutions to (1.5)–(1.6) (at least in some weak topology). Let us briefly comment that the system (1.5)–(1.6), obtained in our proof as a suitable limit of (1.1)–(1.2), can be actually derived by an analogous procedure as the one we have performed to formally derive (1.1)–(1.2). Indeed, (1.5)–(1.6) follow from (2.2) and (2.3) when exploiting (2.13)–(2.16). Here, the new energy and dissipative functionals are $\Psi = \Psi_P$ (i.e., no history contribution of type (2.5) in the free energy is given) and (cf. (2.12))
\[ \Phi(\chi_t, \nabla \vartheta) = \frac{1}{2}|\chi_t|^2 + \frac{(\kappa_0 + \kappa'_0)}{2}|
abla \vartheta|^2. \tag{2.20} \]

In particular, it results that in (2.16) $\mathbf{Q}^{nd} = 0$, while (due to (2.18)) $-\mathbf{Q}^d = (\kappa_0 + \kappa'_0) \nabla \vartheta$.

## 3 Statement of the mathematical problem

In this section, we make our assumptions precise and state our results. First of all, we assume $\Omega$ to be a bounded connected open set in $\mathbb{R}^3$ (lower-dimensional cases could be considered with minor changes) whose boundary $\Gamma$ is supposed to be smooth. Next, we fix a final time $T \in (0, +\infty)$ and set:
\[ Q := \Omega \times (0, T), \quad \Sigma := \Gamma \times (0, T) \tag{3.1} \]
\[ V := H^1(\Omega), \quad V_0 := H^1_0(\Omega), \quad H := L^2(\Omega) \tag{3.2} \]
\[ W := \{v \in H^2(\Omega) : \partial_n v = 0 \text{ on } \Gamma\}, \tag{3.3} \]
\( \partial \nu \) denoting the normal derivative operator on the boundary. We endow the spaces (3.2)–(3.3) with their standard norms, for which we use a self-explained notation like \( \| \cdot \|_V \). Moreover, for \( p \in [1, +\infty] \), we write \( \| \cdot \|_p \) for the usual norm in \( L^p(\Omega) \); as no confusion can arise, the symbol \( \| \cdot \|_p \) is used for the norm in \( L^p(Q) \) as well. In the sequel, the same symbols are used for powers of the above spaces and the corresponding natural induced norms. It is understood that \( H \subset V^*_0 \) as usual, i.e., any element \( u \in H \) is identified with the functional \( V_0 \ni v \mapsto \int_\Omega uv \) which actually belongs to the dual space \( V^*_0 = H^{-1}(\Omega) \) of \( V_0 \). We observe that \( L^2(0,T; V^*_0) \) coincides with the dual space of \( L^2(0,T; V_0) \) and use the symbol \( \langle \cdot, \cdot \rangle \) for the corresponding duality pairing.

As far as the structure of the system is concerned (see (1.1), (1.5) and (1.2), (1.6)), we are given the three functions \( \widehat{\beta}, \lambda, \sigma \), the constant \( \kappa_0 \) and the memory kernel \( k_\tau \) depending on the parameter \( \tau > 0 \) and we assume that the conditions listed below are satisfied.

\[
\begin{align*}
\widehat{\beta} : \mathbb{R} &\to [0, +\infty] \text{ is convex, proper, lower semicontinuous, and } \widehat{\beta}(0) = 0 \quad (3.4) \\
\lambda, \sigma &\in C^1(\mathbb{R}) \text{ and } \lambda', \sigma' \text{ Lipschitz continuous} \quad (3.5) \\
\kappa_0 &> 0 \quad \text{and} \quad k_\tau \in W^{1,1}(0,T). \quad (3.6)
\end{align*}
\]

We define the graph \( \beta \) in \( \mathbb{R} \times \mathbb{R} \) by

\[
\beta := \partial \widehat{\beta} \quad (3.7)
\]

and note that \( \beta \) is maximal monotone and that \( \beta(0) \ni 0 \). In the sequel, we write \( D(\widehat{\beta}) \) and \( D(\beta) \) for the effective domains of \( \widehat{\beta} \) and \( \beta \), respectively, and we use the same symbol \( \beta \) for the maximal monotone operators induced on \( L^2 \) spaces.

As far as the data of our problem are concerned, we assume that the functions \( f, \vartheta_\Gamma, \vartheta_0, \chi_0 \) and the constants \( \vartheta_\ast \) and \( \vartheta^\ast \) are given such that

\[
\begin{align*}
f &\in L^2(0,T; H) \quad (3.8) \\
\vartheta_\Gamma &\in H^1(0,T; H^{-1/2}(\Gamma)) \cap W^{1,1}(0,T; L^\infty(\Gamma)) \cap C^0([0,T]; H^{1/2}(\Gamma)) \quad (3.9) \\
0 &< \vartheta_\ast \leq \vartheta^\ast < +\infty \quad (3.10) \\
\vartheta_\ast &\leq \vartheta_\Gamma \leq \vartheta^\ast \quad \text{a.e. on } \Gamma \times (0,T) \quad (3.11) \\
\vartheta_0 &\in L^\infty(\Omega), \quad \vartheta_\ast \leq \vartheta_0 \leq \vartheta^\ast \quad \text{a.e. in } \Omega \quad (3.12) \\
\chi_0 &\in V \quad \text{and} \quad \widehat{\beta}(\chi_0) \in L^1(\Omega). \quad (3.13)
\end{align*}
\]

The function \( \vartheta_\Gamma \) is the boundary datum for the temperature and we would like to consider a function \( u := \vartheta - \vartheta_\chi \) vanishing on the boundary as associated unknown function. Hence, a natural choice of \( \vartheta_\chi \) is the harmonic extension of \( \vartheta_\Gamma \), so that \( \Delta u = \Delta \vartheta \). Therefore, we define \( \vartheta_\chi : Q \to \mathbb{R} \) as follows

\[
\vartheta_\chi(t) \in V, \quad \Delta \vartheta_\chi(t) = 0, \quad \text{and} \quad \vartheta_\chi(t)|_{\Gamma} = \vartheta_\Gamma(t) \quad \text{for a.a. } t \in (0,T). \quad (3.14)
\]

For the regularity of \( \vartheta_\chi \) (induced by (3.9)) see the subsequent Proposition 4.1.

Next, we list the a priori regularity conditions we require for any solution \((\vartheta, \chi, \xi)\) of either (1.1)–(1.4) or (1.5)–(1.8). We ask that

\[
\begin{align*}
\vartheta &\in L^2(0,T; V) \quad \text{and} \quad u := \vartheta - \vartheta_\chi \in L^2(0,T; V_0) \quad (3.15) \\
\vartheta &> 0 \quad \text{a.e. in } Q \quad \text{and} \quad \ln \vartheta \in H^1(0,T; V_0^*) \cap L^\infty(0,T; H) \quad (3.16) \\
\chi &\in H^1(0,T; H) \cap L^\infty(0,T; V) \cap L^2(0,T; W) \quad (3.17) \\
\xi &\in L^2(0,T; H). \quad (3.18)
\end{align*}
\]
At this point, we are ready to state the problems we are dealing with in a precise form. For fixed $\tau > 0$, we look for a triplet $(\vartheta, \chi, \xi)$ satisfying (3.15)–(3.18) and the following system

\begin{align}
\partial_t (\ln \vartheta(t) + \lambda(\chi(t))) - \kappa_0 \Delta \vartheta(t) - \Delta (k_\tau * \vartheta)(t) &= f(t) \\
&\text{in } V_0^*, \text{ for a.a. } t \in (0,T) \quad (3.19)
\end{align}

\begin{align}
\partial_t \chi - \Delta \chi + \xi + \sigma'(\chi) &= \lambda'(\chi) \vartheta \\
&\text{and } \xi \in \beta(\chi) \quad \text{a.e. in } Q \quad (3.20)
\end{align}

\begin{align}
(\ln \vartheta)(0) &= \ln \vartheta_0 \quad \text{and} \quad \chi(0) = \chi_0. \quad (3.21)
\end{align}

We note that the boundary conditions (1.3) are contained in (3.15) and (3.17) (see the definitions (3.14) and (3.2)–(3.3)). We also remark that (3.16) implies that $\ln \vartheta$ is a continuous $V_0^*$-valued function (while no continuity of $\vartheta$ is known), so that the Cauchy condition for $\ln \vartheta$ contained in (3.21) makes sense. Similar remarks hold for the limit problem we are going to state (i.e., (1.5)–(1.8) in a precise form).

The following well-posedness result deals with a fixed $\tau > 0$ and essentially follows from [5]. Just the notation is different, indeed.

**Theorem 3.1.** Assume that both (3.4)–(3.7) and (3.8)–(3.13) hold. Then, there exists a unique triplet $(\vartheta, \chi, \xi)$ satisfying (3.15)–(3.18) and solving problem (3.19)–(3.21).

Our aim is to study the limit of the solution $(\vartheta, \chi, \xi)$ as $\tau$ tends to zero, under suitable assumptions on the behavior of the memory kernel $k_\tau$. Namely, we assume that

\[ 1 * k_\tau \to \kappa' \quad \text{strongly in } L^1(0,T) \quad \text{as } \tau \searrow 0 \quad (3.22) \]

for some real constant $\kappa'$ and set

\[ \kappa := \kappa_0 + \kappa'. \quad (3.23) \]

In (3.22) and later on, we use the same symbol for any real constant (like 1 and $\kappa'$) and for the corresponding constant function. We advice the reader that $\kappa > 0$ in the sequel, either by assumption or as a consequence of some condition we require, so that the limit problem we are going to state is parabolic with respect to $\vartheta$. Such a problem consists in looking for a triplet $(\vartheta, \chi, \xi)$ satisfying (3.15)–(3.18) and the following system

\begin{align}
\partial_t (\ln \vartheta(t) + \lambda(\chi(t))) - \kappa \Delta \vartheta(t) &= f(t) \\
&\text{in } V_0^*, \text{ for a.a. } t \in (0,T) \quad (3.24)
\end{align}

\begin{align}
\partial_t \chi - \Delta \chi + \xi + \sigma'(\chi) &= \lambda'(\chi) \vartheta \\
&\text{and } \xi \in \beta(\chi) \quad \text{a.e. in } Q \quad (3.25)
\end{align}

\begin{align}
(\ln \vartheta)(0) &= \ln \vartheta_0 \quad \text{and} \quad \chi(0) = \chi_0. \quad (3.26)
\end{align}

By just taking $k_\tau = 0$ and replacing $\kappa_0$ by $\kappa$ in Theorem 3.1, we obtain

**Corollary 3.2.** Assume that both (3.4)–(3.7) and (3.8)–(3.13) hold and that $\kappa > 0$. Then, there exists a unique triplet $(\vartheta, \chi, \xi)$ satisfying (3.15)–(3.18) and solving problem (3.24)–(3.26).

However, we can prove a convergence result under further assumptions, namely

\[ \int_0^T (\kappa_0 v(t) + (k_\tau * v)(t)) v(t) dt \geq \kappa_* \|v\|_{L^2(0,T)}^2 \quad \text{and} \quad \|k_\tau\|_{L^1(0,T)} \leq \kappa^* \quad (3.27) \]
for some constants $\kappa_*, \kappa^* > 0$ and every $v \in L^2(0,T)$ and $\tau > 0$. By taking $v = 0$ on $(T', T)$, we clearly see that the time $T$ can be replaced by any $T' \in (0, T)$ in the first inequality of (3.27). Moreover, we observe that (3.22) and (3.27) imply that $\kappa \geq \kappa_*$, so that $\kappa > 0$ as a consequence. Here is our first result.

**Theorem 3.3.** Assume $(3.4)$–$(3.7)$, $(3.8)$–$(3.13)$, and let $(\vartheta_\tau, \chi_\tau, \xi_\tau)$ be the unique solution to problem $(3.19)$–$(3.21)$ given by Theorem 3.1. Moreover, assume $(3.22)$–$(3.23)$ and $(3.27)$. Then, $(\vartheta_\tau, \chi_\tau, \xi_\tau)$ converges in a proper topology to the unique solution $(\vartheta, \chi, \xi)$ to problem $(3.24)$–$(3.26)$ satisfying $(3.15)$–$(3.18)$.

The topology mentioned in Theorem 3.3 will be clear from the proof we give in Section 3.3 and is rather strong. Provided that a much weaker topology is considered, an error estimate can be proved. We have indeed

**Theorem 3.4.** Under the assumptions of Theorem 3.3 the following estimate holds true

$$
\begin{align*}
\|1 * (\vartheta_\tau - \vartheta)\|_{L^\infty(0,T;V)}^2 + \|\chi_\tau - \chi\|_{L^\infty(0,T;H) \cap L^2(0,T;V)}^2 \\
+ \int_Q (\ln \vartheta_\tau - \ln \vartheta)(\vartheta_\tau - \vartheta) + \int_Q (\xi_\tau - \xi)(\chi_\tau - \chi)
\leq M\|1 * k_\tau - \kappa_0\|_{L^1(0,T)}
\end{align*}
$$

(3.28)

for $\tau$ small enough, where $M$ depends on the structure and the data, only.

**Remark 3.5.** We observe that a sufficient condition for (3.27) is that $k_\tau$ has the form $k_\tau(t) = \tau^{-1}k(t/\tau)$ with $k \in L^1(0, +\infty)$. Moreover, a sufficient condition for (3.27) to hold is that $k_\tau$ is a positive type kernel, i.e.,

$$
\int_0^T (k_\tau * v)(t) v(t) \, dt \geq 0 \quad \text{for every } v \in L^2(0,T).
$$

(3.29)

In such a case, one can take $\kappa_* = \kappa_0$, indeed. The fact that the kernel is of positive type is actually in accordance with the assumptions required on $k$ to ensure thermodynamical consistency of the model [3]. We remark that a sufficient condition for a kernel $k$ to be of positive type is the following (see, e.g., [2], Prop. 4.1, p. 237] or [14]): $k$ is smooth, nonnegative, decreasing, and convex. So, any positive multiple of $\exp(-t/\tau)$ plays the role of the kernel given by $k_\tau(t) := (k_0/\tau)\exp(-t/\tau)$ is a prototype for both (3.22) and (3.27) since $1 * k_\tau$ converges to $k_0$ strongly in $L^1(0,T)$. More generally, assume that $k_\tau$ can be split as $k_\tau = p_\tau + r_\tau$, where $p_\tau$ is of positive type and $r_\tau$ is a remainder. If

$$
1 * p_\tau \to \kappa'_1 \quad \text{and} \quad 1 * r_\tau \to \kappa'_2 \quad \text{strongly in } L^1(0,T)
$$

$$
\|r_\tau\|_{L^1(0,T)} \leq \eta_0 \quad \text{for every } \tau > 0
$$

for some constants $\kappa'_1, \kappa'_2$, and $\eta_0 < \kappa_0$, then both assumptions (3.22) and (3.27) are still fulfilled. Indeed, we can take $\kappa'_0 = \kappa'_1 + \kappa'_2$, clearly, and $\kappa_* = \kappa_0 - \eta_0$, since

$$
\int_0^T (k_0 v + k_\tau * v) v \, dt \geq \int_0^T (k_0 v + r_\tau * v) v \, dt \geq (\kappa_0 - \eta_0) \int_0^T v^2 \, dt \quad \text{for every } v \in L^2(0,T)
$$

thanks to the Hölder and Young inequalities (see also (3.37)). Note that $\kappa'_1 \geq 0$ since $p_\tau$ is of positive type, while $\kappa'_2$ can be any real constant.
Remark 3.6. Assumption (3.22) is a well-defined reinforcement of the condition roughly mentioned in the Introduction as \( k_\tau \to k'_0 \delta \), where \( \delta \) is the Dirac mass at the origin. Indeed, if we introduce the Heaviside function \( H \) on \((-\infty, T)\), i.e., \( H(t) = 0 \) for \( t < 0 \) and \( H(t) = 1 \) for \( t \in (0, T) \), and the trivial extension \( \tilde{k}_\tau \) of \( k_\tau \), (3.22) reads

\[
H \ast \tilde{k}_\tau \to k'_0 H \quad \text{strongly in } L^1(-\infty, T)
\]

with an obvious new meaning of the convolution. By differentiating and observing that \( (H \ast \tilde{k}_\tau)' = \delta \ast \tilde{k}_\tau = \tilde{k}_\tau \), we deduce that

\[
\tilde{k}_\tau \to k'_0 \delta \quad \text{in the sense of distributions on } (-\infty, T)
\]

where \( \delta \) is the actually well-defined Dirac mass at 0 in the open set \((-\infty, T)\).

Remark 3.7. By checking the proofs in the next sections, the reader will be able to realize that our results can be suitably extended to the case of coefficients \( k_0 \tau \) possibly depending on \( \tau \), with boundedness and convergence properties as \( \tau \to 0 \).

We recall that \( \Omega \) is bounded and smooth. So, throughout the paper, we owe to some well-known embeddings of Sobolev type, namely \( V \subset L^p(\Omega) \) for \( p \in [1, 6] \), together with the related Sobolev inequality

\[
\|v\|_p \leq C \|v\|_V \quad \text{for every } v \in V \text{ and } 1 \leq p \leq 6 \quad (3.30)
\]

and \( W^{1,p}(\Omega) \subset C^0(\Omega) \) for \( p > 3 \), together with

\[
\|v\|_\infty \leq C_p \|v\|_{W^{1,p}(\Omega)} \quad \text{for every } v \in W^{1,p}(\Omega) \text{ and } p > 3. \quad (3.31)
\]

In (3.30), \( C \) depends only on \( \Omega \), while \( C_p \) in (3.31) depends also on \( p \). In particular, the continuous embedding \( W \subset W^{1,6}(\Omega) \subset C^0(\Omega) \) holds. Some of the previous embeddings are in fact compact. This is the case for \( V \subset L^4(\Omega) \) and \( W \subset C^0(\Omega) \). We note that also the embeddings \( V \subset G \), \( V \subset H \), \( V_0 \subset H \), and \( H \subset V_0^* \) are compact. Moreover, we often account for the well-known Poincaré inequalities

\[
\|v\|_V \leq C \|\nabla v\|_H \quad \text{for every } v \in V_0 \quad (3.32)
\]

\[
\|v\|_V \leq C \left( \|\nabla v\|_H + |f_\Omega| \right) \quad \text{for every } v \in V \quad (3.33)
\]

where \( C \) depends only on \( \Omega \). Furthermore, we repeatedly make use of the notation

\[
Q_t := \Omega \times (0, t) \quad \text{for } t \in [0, T] \quad (3.34)
\]

and of well-known inequalities, namely, the Hölder inequality and the elementary Young inequality:

\[
ab \leq \delta a^2 + \frac{1}{4\delta} b^2 \quad \text{for every } a, b \geq 0 \text{ and } \delta > 0. \quad (3.35)
\]

As far as properties of the convolution are concerned, we take advantage of the elementary formulas (which hold whenever they make sense)

\[
a \ast b = a(0)(1 \ast b) + a_t \ast 1 \ast b \quad \text{and} \quad (a \ast b)_t = a(0)b + a_t \ast b \quad (3.36)
\]
and of the well-known Young theorem
\begin{equation}
\|u \ast v\|_{L^r(0,T;X)} \leq \|u\|_{L^p(0,T)} \|v\|_{L^q(0,T;X)}
\end{equation}
(3.37)
where \(X\) is a Banach space, \(1 \leq p, q, r \leq \infty\), and \(1/r = (1/p) + (1/q) - 1\) (cf., e.g., [14]).

Finally, again throughout the paper, we use a small-case italic \(c\) for different constants, that may only depend on \(\Omega\), the final time \(T\), the shape of the nonlinearity \(\lambda, \beta, \sigma\), and the properties of the data involved in the statements at hand; a notation like \(c_\delta\) signals a constant that depends also on the parameter \(\delta\). The reader should keep in mind that the meaning of \(c\) and \(c_\delta\) might change from line to line and even in the same chain of inequalities, whereas those constants we need to refer to are always denoted by capital letters, just like \(C\) in (3.31).

4 Auxiliary material

This section contains a very short summary on the properties of the harmonic extension \(\vartheta_\Omega\) of the boundary datum \(\vartheta_T\) (see (3.14)) and a preliminary result dealing with a generalized version of the limit problem (3.24)–(3.26). The properties listed in the following proposition will be extensively used in the sequel.

Proposition 4.1. Assumptions (3.9)–(3.11) yield
\[
\vartheta_\Omega \in H^1(0,T; H) \cap W^{1,1}(0,T; L^\infty(\Omega)) \cap C^0([0,T]; V).
\] (4.1)

More precisely, owing to the theory of harmonic functions, in particular to the maximum principle, we have that
\[
\begin{align*}
\|\vartheta_\Omega\|_{L^2(0,T; V)} &\leq C\|\vartheta_T\|_{L^2(0,T; H^{1/2}(\Gamma))}, \\
\|\vartheta_\Omega\|_{L^1(0,T; H)} &\leq C\|\vartheta_T\|_{L^1(0,T; H^{-1/2}(\Gamma))}, \\
\|\partial_t \vartheta_\Omega\|_{L^2(0,T; H)} &\leq C\|\partial_t \vartheta_T\|_{L^2(0,T; H^{-1/2}(\Gamma))}, \\
\vartheta_\ast &\leq \vartheta_\Omega \leq \vartheta_\ast \text{ a.e. in } Q, \\
\|\partial_t \vartheta_\Omega\|_{L^1(0,T; L^\infty(\Omega))} &\leq \|\partial_t \vartheta_T\|_{L^1(0,T; L^\infty(\Gamma))}
\end{align*}
\]
where \(C\) is a constant depending on \(\Omega\), only.

Now, in order to help the reader, we sketch the outline of the proof of Theorem 3.3, we are going to develop in the next section. By accounting for a number of a priori estimates and using well-known compactness results, we derive that the family of solutions \((\vartheta_T, \chi_T, \xi_T)\) converges (for a subsequence) to a generalized solution of the problem (3.24)–(3.26), in which \(\ln \vartheta\) is understood in a non standard sense. Next, in order to conclude that such a solution actually is the solution given by Corollary 3.2, we prove a preliminary well-posedness result for generalized solutions (Theorem 4.2). Therefore, we first have to introduce the ingredients that are needed to explain such a notion of solution.

We define a generalized logarithm by following [12] Def. 4.2. However, we confine ourselves to consider the functions \(\vartheta \in L^2(0,T; V)\) such that \(\vartheta = \vartheta_T\) on the boundary, i.e., \(\vartheta \in \vartheta_\Omega + L^2(0,T; V_0)\). First, we introduce the function \(\psi : \mathbb{R} \rightarrow (-\infty, +\infty]\) by setting
\[
\psi(r) = r(\ln r - 1) \text{ if } r > 0, \quad \psi(0) = 0, \quad \text{and} \quad \psi(r) = +\infty \text{ if } r < 0.
\] (4.2)
Then, for $\vartheta \in \vartheta_{2\kappa} + L^2(0; T; V_0)$, we term $\ln \vartheta$ the set of $\zeta \in L^2(0, T; V_0^*)$ satisfying
\[
\langle \zeta, v - \vartheta \rangle + \int_Q \psi(\vartheta) \leq \int_Q \psi(v) \quad \text{for every } v \in \vartheta + L^2(0, T; V_0)
\]  
(4.3)
where $\langle \cdot, \cdot \rangle$ stands for the duality pairing between $L^2(0, T; V_0^*)$ and $L^2(0, T; V_0)$. It can be checked (see [12 Thm. 4.7]) that $\vartheta > 0$ a.e. in $Q$ whenever $\ln \vartheta$ is not empty. Moreover, even though $\ln \vartheta$ might contain elements that are not functions (they are just Radon measure in such a case), its definition actually generalizes the usual logarithm. Indeed, for $\vartheta \in \vartheta_{2\kappa} + L^2(0, T; V_0)$ we have (see [12 Rem. 4.3])
\[
\ln \vartheta \in \ln \vartheta \quad \text{whenever} \quad \vartheta > 0 \text{ a.e. in } Q \text{ and } \ln \vartheta \in L^2(Q).
\]  
(4.4)
Furthermore, the generalized logarithm is related to the theory of subdifferentials as follows. We define the function $\Psi : L^2(0, T; V_0) \rightarrow (-\infty, +\infty]$ by
\[
\Psi(v) := \int_Q \psi(v + \vartheta_{2\kappa}) \quad \text{for } v \in L^2(0, T; V_0)
\]
being understood that the integral is infinite if $\psi(v + \vartheta_{2\kappa}) \notin L^1(Q)$. Then, $\Psi$ turns out to be convex proper and lower semicontinuous on $L^2(0, T; V_0)$, so that its (possibly multivalued) subdifferential $\partial \Psi : L^2(0, T; V_0) \rightarrow L^2(0, T; V_0^*)$ is well-defined. Precisely, we have (see [12 Rem. 4.4] for details)
\[
\ln \vartheta = \partial \Psi(\vartheta - \vartheta_{2\kappa}) \quad \text{for every } \vartheta \in \vartheta_{2\kappa} + L^2(0, T; V_0).
\]  
(4.6)
At this point, we can state the generalized version of problem (3.24)–(3.26) as follows. We look for a quadruplet $(\vartheta, \zeta, \chi, \xi)$ satisfying (3.15), (3.17)–(3.18), and
\[
\begin{align*}
\zeta & \in H^1(0; T; V_0^*) \quad \text{and} \quad \zeta \in \ln \vartheta \quad \text{(4.7)} \\
\partial_t (\zeta(t) + \lambda(\chi(t))) - \kappa \Delta \theta(t) &= f(t) \quad \text{in } V_0^* \text{ for a.a. } t \in (0, T) \quad \text{(4.8)} \\
\partial \chi - \Delta \chi + \xi + \delta' = 0 \quad \text{and} \quad \chi \in \beta(\chi) \quad \text{a.e. in } Q \\
\zeta(0) &= \ln \vartheta_0 \quad \text{and} \quad \chi(0) = \chi_0.
\end{align*}
\]  
(4.9)
(4.10)
The following result holds

**Theorem 4.2.** Assume that both (3.4)–(3.7) and (3.8)–(3.13) hold and that $\kappa > 0$. Then, problem (4.7)–(4.10) has a unique solution $(\vartheta, \zeta, \chi, \xi)$ satisfying (3.15) and (3.17)–(3.18). Moreover, we have that
\[
\ln \vartheta \in L^\infty(0, T; H) \quad \text{and} \quad \zeta = \ln \vartheta.
\]  
(4.11)

**Proof.** We first prove uniqueness. Our proof closely follows [5 Sect. 5]. However, we repeat at least a part of the argument for the reader’s convenience. We first observe that, if $(\vartheta, \zeta, \chi, \xi)$ is a generalized solution, by integrating (4.8) in time we obtain
\[
\zeta + \lambda(\chi) - \kappa \Delta (1 * \vartheta) = \ln \vartheta_0 + \lambda(\chi_0) + 1 * f.
\]  
(4.12)
Now, we pick two solutions $(\vartheta_i, \zeta_i, \chi_i, \xi_i), i = 1, 2$, and set for convenience
\[
\begin{align*}
\vartheta := \vartheta_1 - \vartheta_2, \quad \zeta := \zeta_1 - \zeta_2, \quad \chi := \chi_1 - \chi_2, \quad \text{and} \quad \xi := \xi_1 - \xi_2.
\end{align*}
\]  
(4.13)
Now, we write (4.12) for both solutions, take the difference, and test it by $\vartheta \mathcal{C}_t$ in the duality $L^2(0, T; V_0') - L^2(0, T; V_0)$, where $t \in (0, T)$ is arbitrary and $\mathcal{C}_t$ is the characteristic function of the interval $(0, t)$. We observe that all the terms but one are in fact integrals. Namely, we have

$$\langle \zeta, \vartheta \mathcal{C}_t \rangle + \frac{\kappa}{2} \int_\Omega |\nabla (1 + \vartheta)(t)|^2 = - \int_{Q_t} (\lambda(x_1) - \lambda(x_2)) \vartheta. \tag{4.14}$$

At the same time, we write (4.9) for both solutions, take the difference, multiply the resulting equality by $\chi$, and integrate over $Q_t$. We obtain

$$\frac{1}{2} \int_\Omega |\chi(t)|^2 + \int_{Q_t} |\nabla \chi|^2 + \int_{Q_t} \xi \chi = \int_{Q_t} (\lambda' (x_1) \vartheta_1 - \lambda' (x_2) \vartheta_2) \chi - \int_{Q_t} (\sigma' (x_1) - \sigma' (x_2)) \chi. \tag{4.15}$$

Finally, we add (4.15) to (4.14) and proceed exactly as in [5]. However, let us point out that, in view of Taylor’s expansion and the Hölder and Sobolev inequalities, we have

$$\int_{Q_t} \left\{ - (\lambda(x_1) - \lambda(x_2)) \vartheta + (\lambda'(x_1) \vartheta_1 - \lambda'(x_2) \vartheta_2) \chi \right\}$$

$$= \int_{Q_t} \vartheta_1 (\lambda(x_2) - \lambda(x_1) - \lambda'(x_2)(x_2 - x_1))$$

$$+ \int_{Q_t} \vartheta_2 (\lambda(x_1) - \lambda(x_2) - \lambda'(x_2)(x_2 - x_1))$$

$$\leq c \int_{Q_t} |\vartheta_1 + \vartheta_2| |\chi|^2 \leq c \int_0^t \|\vartheta_1(s) + \vartheta_2(s)\|_4 \|\chi(s)\|_4 \|\chi(s)\|_2 ds$$

$$\leq c \int_0^t \|\vartheta_1(s) + \vartheta_2(s)\|_V (\|\nabla \chi(s)\|_H + \|\chi(s)\|_H) \|\chi(s)\|_H ds$$

$$\leq \frac{1}{4} \int_{Q_t} |\nabla \chi|^2 + c \int_0^t (\|\vartheta_1(s)\|_V^2 + \|\vartheta_2(s)\|_V^2) \|\chi(s)\|_H^2 ds.$$

Therefore, we derive that

$$\langle \zeta, \vartheta \mathcal{C}_t \rangle + \frac{\kappa}{2} \int_\Omega |\nabla (1 + \vartheta)(t)|^2 + \frac{1}{2} \int_\Omega |\chi(t)|^2 + \int_{Q_t} |\nabla \chi|^2 + \int_{Q_t} \xi \chi$$

$$\leq c \int_0^t (1 + \|\vartheta_1(s)\|_V^2 + \|\vartheta_2(s)\|_V^2) \|\chi(s)\|_H^2 ds + \frac{1}{4} \int_{Q_t} |\nabla \chi|^2.$$

As the last term can be easily controlled by the left-hand side, what remains to observe is that all the terms on the left-hand side are non negative. The integral containing $\xi$ is non negative by monotonicity. Let us deal with the duality term. For $i = 1, 2$ we have $\zeta_i \in \text{Ln} \vartheta_i$. Due to [133], this means that

$$\langle \zeta_i, v_i - \vartheta_i \rangle + \int_Q \psi(\vartheta_i) \leq \int_Q \psi(v_i) \quad \text{for every } v_i \in \vartheta_i + L^2(0, T; V_0) \text{ and } i = 1, 2.$$
Now, we choose the admissible functions \( v_1 = \vartheta_1 - \partial_t \varphi_c \) and \( v_2 = \vartheta_2 + \partial_t \varphi_c \), we sum up and split the integrals. We have

\[
\langle \zeta, -\partial_t \varphi_c \rangle + \int_{Q_t} (\psi(\vartheta_1) + \psi(\vartheta_2)) + \int_{Q \setminus Q_t} (\psi(\vartheta_1) + \psi(\vartheta_2)) \leq \int_{Q_t} (\psi(v_1) + \psi(v_2)) + \int_{Q \setminus Q_t} (\psi(v_1) + \psi(v_2)).
\]

As \( v_1 = \vartheta_2 \) in \( Q_t \) and \( v_1 = \vartheta_1 \) in \( Q \setminus Q_t \) and similarly for \( v_2 \), all the integrals are finite and cancel out. We deduce that \( \langle \zeta, -\partial_t \varphi_c \rangle \leq 0 \), i.e., what we wanted to prove. At this point, we can apply the Gronwall lemma (see, e.g., [8, pp.156-157]) and obtain, in particular, that \( \chi = 0 \) and \( \nabla(1 \ast \vartheta) = 0 \) a.e. in \( Q \). As \( 1 \ast \vartheta \) is \( V_0 \)-valued, this implies that \( 1 \ast \vartheta = 0 \) a.e. in \( Q \), whence also \( \vartheta = 0 \) a.e. in \( Q \). All this means that \( \vartheta_1 = \vartheta_2 \) and \( \chi_1 = \chi_2 \). By comparison in (4.12) and (4.9), we conclude that \( \zeta_1 = \zeta_2 \) and \( \xi_1 = \xi_2 \) as well.

Once uniqueness of the generalized solution is proved, we can easily conclude. Indeed, our assumptions allow us to apply Corollary 3.2. Hence, a solution exists in the strong sense, i.e., satisfying the regularity requirements (3.15)–(3.18). On the other hand, such a solution is also a generalized solution due to (4.4). Finally, it satisfies (4.11). \( \square \)

## 5 Proofs of Theorems 3.3 and 3.4

The argument we follow for our first proof uses compactness and monotonicity methods. So, we start estimating. However, we often proceed formally for the sake of simplicity. The correct procedure could be based on performing similar estimates on the solution of some approximating problem. One approximation is constructed in [5] and depends on the parameter \( \varepsilon \): the solution is smoother than the solution to the problem we are dealing with (actually the limit as \( \varepsilon \searrow 0 \) keeps such estimates). Furthermore, in order to simplify the notation, we often avoid the subscript \( \tau \) (on the solutions) during the calculation and restore it just at the end of each estimate.

**First a priori estimate.** We would like testing (3.19) by

\[
u_\tau := \vartheta_\tau - \vartheta_\varphi
\]

in the duality \( V_0^*-V_0 \) and integrate over \((0,t)\), where \( t \in (0,T) \) is arbitrary. However, we proceed formally, as just said. In particular, we behave as if the logarithmic term were smoother. At the same time we multiply (3.20) by \( \partial_t \chi_\tau \) and integrate over \( Q_t \) (see (3.31)). Finally, we sum up and remark that the terms containing \( \partial_t \chi_\tau \) partially cancel. Hence, by avoiding some subscripts in the notation for a while and adding the same integral
\[ \int_{\Omega} |\chi(t)|^2 \] to both sides for convenience, we obtain
\[
\int_{\Omega} \vartheta(t) - \int_{Q_t} \partial_t (\ln \vartheta) \vartheta_{3\ell} + \int_{Q_t} (\kappa_0 |\nabla u|^2 + (k_r \ast \nabla u) \cdot \nabla u)
+ \int_{Q_t} |\partial_\chi\chi|^2 + \frac{1}{2} \int_{\Omega} |\nabla \chi(t)|^2 + \int_{\Omega} \widehat{\beta}(\chi(t)) + \int_{\Omega} |\chi(t)|^2
= \int_{Q_t} \partial_t \lambda(\chi) \vartheta_{3\ell} - \int_{Q_t} \left( \kappa_0 \nabla \vartheta_{3\ell} + (k_r \ast \nabla \vartheta_{3\ell}) \right) \cdot \nabla u + \int_{Q_t} f u
+ \int_{\Omega} (|\chi(t)|^2 - \sigma(\chi(t))) + \int_{\Omega} \vartheta_0 + \frac{1}{2} \int_{\Omega} |\nabla \chi_0|^2 + \int_{\Omega} (\widehat{\beta} + \sigma)(\chi_0).
\] (5.2)

Note that we have \( \vartheta > 0 \) and can make use of the chain rule for subdifferentials. Now, we recall that \( \widehat{\beta} \) is nonnegative (cf. (3.4)) and treat each of the non-trivial terms, separately. We integrate the second integral on the left-hand side by parts with respect to time and get
\[
\int_{Q_t} \partial_t (\ln \vartheta) \vartheta_{3\ell} = \int_{\Omega} \vartheta_{3\ell}(t) \ln \vartheta(t) - \int_{\Omega} \vartheta_{3\ell}(0) \ln \vartheta_0 - \int_{Q_t} \ln \vartheta \partial_t \vartheta_{3\ell}
\leq \int_{\Omega} \vartheta_{3\ell}(t) \ln^+ \vartheta(t) - \int_{\Omega} \vartheta_{3\ell}(t) \ln^- \vartheta(t) + \int_{Q_t} |\ln \vartheta| |\partial_t \vartheta_{3\ell}| + c.
\]

Hence, by recalling Proposition [4.4] and observing that \( r - \vartheta^* \ln^+ r \geq (r/2) - c \) for every \( r > 0 \), we deduce that
\[
\int_{\Omega} \vartheta(t) - \int_{Q_t} \partial_t (\ln \vartheta) \vartheta_{3\ell}
\geq \int_{\Omega} \left( \vartheta(t) - \vartheta^* \ln^+ \vartheta(t) + \vartheta^* \ln^- \vartheta(t) \right) - \int_{Q_t} |\ln \vartheta| |\partial_t \vartheta_{3\ell}| - c
\geq \int_{\Omega} \left( \frac{1}{2} \vartheta(t) + \vartheta^* \ln^- \vartheta(t) \right) - \int_{Q_t} |\ln \vartheta| |\partial_t \vartheta_{3\ell}| - c.
\]

On the other hand, we notice that \( |\ln r| \leq c((r/2) + \vartheta^* \ln^- r) \) for \( r > 0 \), so that
\[
\int_{Q_t} |\ln \vartheta| |\partial_t \vartheta_{3\ell}| \leq c \int_{Q_t} ((\vartheta/2) + \vartheta^* \ln^- \vartheta) |\partial_t \vartheta_{3\ell}|
\]
and the last integral can be treated on the right-hand side via the Gronwall lemma since \( \partial_t \vartheta_{3\ell} \in L^1(0, T; L^\infty(\Omega)) \). Next, thanks to (3.27) and to (3.32), we infer that
\[
\int_{Q_t} \left( \kappa_0 |\nabla u|^2 + (k_r \ast \nabla u) \cdot \nabla u \right) \geq \kappa_* \int_{Q_t} |\nabla u|^2 \geq \frac{\kappa_*}{C} \int_0^t \| u(s) \| V^2 ds
\]
where \( C \) is the constant in (3.32). Now, let us deal with the right-hand side. With the help of (3.27), the Young theorem (3.37), and (3.8), we immediately have
\[
- \int_{Q_t} \left( \kappa_0 \nabla \vartheta_{3\ell} + (k_r \ast \nabla \vartheta_{3\ell}) \right) \cdot \nabla u + \int_{Q_t} f u \leq c + c \int_0^t \| u(s) \| V^2 ds.
\]
Moreover, we observe that
\[
\int_{Q_t} \partial_t \lambda(\chi) \vartheta_{3\ell} = \int_{Q_t} \lambda'(\chi) \partial_t \chi \vartheta_{3\ell} \leq \frac{1}{4} \int_{Q_t} |\partial_t \chi|^2 + c \int_{Q_t} |\chi|^2 + c
\]
since $|\lambda'(r)| \leq c(1+|r|)$ by (3.5) and $0 \leq \partial_\kappa \leq \vartheta^*$ by Proposition 4.1. Finally, we observe that (3.5) also yields $|\sigma(r)| \leq c(1+|r|^2)$ for every $r$ and deduce that
\[
\int_{\Omega} \left(|\chi(t)|^2 - \sigma(\chi(t))\right) \leq c + c \int_{\Omega} |\chi(t)|^2 \leq c + c \int_{0}^{t} 2\chi(s) \partial_t \chi(s) \, ds
\]
\[
\leq c + \frac{1}{4} \int_{Q_t} |\partial_t \chi|^2 + c \int_{Q_t} |\chi|^2.
\]
By combining all the estimates we have derived with (5.2), applying the Gronwall lemma, and owing to the Poincaré inequality (3.33), we conclude that
\[
\hat{\beta}(\chi_r) \leq \frac{c}{\epsilon} \left( \frac{c}{\epsilon^{\frac{q}{2}}(\Omega)} \right) + \frac{c}{\epsilon^{\frac{q}{2}}} \chi_{H^1((0,T);H)} + \frac{c}{\epsilon^{\frac{q}{2}}} \chi_{L^2((0,T);V)} \leq c
\]
besides an estimate for
\[
(\hat{\beta}(\chi_r) \leq \frac{c}{\epsilon} \left( \frac{c}{\epsilon^{\frac{q}{2}}(\Omega)} \right) + \frac{c}{\epsilon^{\frac{q}{2}}} \chi_{H^1((0,T);H)} + \frac{c}{\epsilon^{\frac{q}{2}}} \chi_{L^2((0,T);V)} \leq c
\]

**Second a priori estimate.** We write (3.20) in the form of a nonlinear monotone elliptic equation, namely
\[
-\Delta \chi_r + \xi_r = \lambda'(\chi_r) \partial_t \chi_r - \partial_t \chi_r - \sigma'(\chi_r) \quad \text{and} \quad \xi_r \in \beta(\chi_r)
\]
and notice that each term on the right-hand side of (5.4) is bounded in $L^2(0,T; H)$ by $\lambda(\Omega)$ and (5.3). Concerning the first term, notice that $\chi_r$ and $\partial_t \chi_r$ are bounded in $L^\infty(0,T; L^q(\Omega))$ and $L^2(0,T; L^q(\Omega))$, respectively, due to the Sobolev inequality (3.30). Then, a quite standard argument (formally test (5.4) by either $-\Delta \chi_r$ or $\xi_r$ in order to estimate both of them and then use the regularity theory for elliptic equations) yields
\[
\||\chi||_{L^2(0,T;W)} + ||\xi||_{L^2(0,T;H)} \leq c.
\]

**Third a priori estimate.** We want to estimate $\partial_t \chi_r$ in $L^1(0,T; W^{-1,q}(\Omega))$ for some $q > 1$ satisfying $L^2(\Omega) \subset W^{-1,q}(\Omega)$, and the choice $q = 4/3$ will work. Therefore, by proceeding formally, we take any $v \in W_0^{1,q}(\Omega)$ satisfying $||v||_{W^{1,q}(\Omega)} \leq 1$ and test (3.19) written at almost any time $t$ by $\chi_r(t) v$, which is a good test function belonging to $V_0$, due to (3.30)-(3.31) and (5.3). We obtain (the first integral is intended as a duality pairing)
\[
\int_{\Omega} \partial_t \chi_r(t) \, v = \int_{\Omega} \left\{ f(t) - \lambda'(\chi_r(t)) \partial_t \chi_r(t) \right\} \partial_t \chi(t) \, v
\]
\[
- \int_{\Omega} \kappa_0 \nabla \chi_r(t) \cdot \nabla (\vartheta_r(t) \, v) - \int_{\Omega} (k_r \ast \nabla \partial_t \chi_r)(t) \cdot \nabla (\vartheta_r(t) \, v).
\]
We simplify the notation by dropping the time $t$ (and the subscript $\tau_r$ as usual) for a while and estimate each term on the right-hand side of (5.6), separately. We account for the Lipschitz continuity of $\lambda'$ (see (3.5)) and for the Hölder and Sobolev inequality (3.30). Moreover, we observe that $||v||_\infty \leq c$ thanks to (3.31). We have
\[
\int_{\Omega} \left\{ f - \lambda'(\chi) \partial_\kappa \right\} \partial_t \chi \, v \leq ||f||_2 ||\vartheta||_2 ||v||_\infty + ||\lambda'(\chi)||_4 ||\partial_\kappa \chi||_2 ||\vartheta||_4 ||v||_\infty
\]
\[
\leq c ||f||_2 ||\vartheta||_2 + c(1 + ||\lambda||_4) ||\partial_\kappa \chi||_2 ||\vartheta||_4.
\]
\[
- \int_{\Omega} \kappa_0 \nabla \partial \cdot \nabla (\partial \, v) \leq \int_{\Omega} \kappa_0 \nabla \partial \cdot \nabla (\partial \, v) \leq c ||\nabla \partial||_2 ||v||_\infty + c ||\nabla \partial||_2 ||\vartheta||_4 ||\nabla v||_4
\]
\[
\leq c ||\nabla \partial||_2^2 + c ||\nabla \partial||_2 ||\vartheta||_4.
\]
\[
- \int_{\Omega} (k_r \ast \nabla \partial_t \chi_r) \cdot \nabla (\partial_t \chi \, v) \leq ||k_r \ast \nabla \partial||_2 (||\nabla \partial||_2 ||v||_\infty + ||\vartheta||_4 ||\nabla v||_4)
\]
\[
\leq c ||k_r \ast \nabla \partial||_2 (||\nabla \partial||_2 + ||\vartheta||_4).
\]
We term $C$ the maximum of the values of the above constant $c$’s, for clarity. Then, we first collect the estimates just obtained and \((5.6)\). Finally, we take the supremum with respect to $v \in W_0^{1,4}(\Omega)$ under the constraint $\|v\|_{W^{1,4}(\Omega)} \leq 1$. We conclude that

$$\|\partial_t \varphi(t)\|_{W^{-1,4/3}(\Omega)} \leq C \phi_r(t) \quad \text{for a.a. } t \in (0,T)$$

where $\phi_r : (0,T) \to \mathbb{R}$ is defined by

$$\phi_r(t) := \|f(t)\|_2 \|\varphi(t)\|_2 + (1 + \|\varphi(t)\|_4)\|\varphi(t)\|_4 + \|\nabla \varphi(t)\|_2$$

Thus, the estimate

$$\|\partial_t \varphi\|_{L^1(0,T;W^{-1,4/3}(\Omega))} \leq c \quad \text{(5.7)}$$

follows once we prove that $\phi_r$ is bounded in $L^1(0,T)$. In view of the previous estimates \((5.3)\), \((5.5)\), and of the Sobolev inequality, we see that the only trouble could come from the term containing the convolution. By owing to the Young theorem (see \((3.37)\)) and to \((5.3)\), \((5.5)\), and of the Sobolev inequality, we see that the only trouble could come from the term containing the convolution. By owing to the Young theorem (see \((3.37)\)) and to the Sobolev inequality once more, we have

$$\int_0^T \|k_r \times \nabla \varphi(t)\|_2 \|\nabla \varphi(t)\|_2 + \|\varphi(t)\|_4 \, dt$$

$$\leq c \|k_r \times \nabla \varphi\|_{L^2(0,T;H)} \|\varphi\|_{L^2(0,T;V)} \leq c \|k_r\|_{L^1(0,T)} \|\varphi\|_{L^2(0,T;V)}^2 \cdot \quad \text{(5.8)}$$

Now, we recall \((3.27)\) and \((5.3)\) and conclude that $\phi_r$ is bounded in $L^1(0,T)$. Therefore, \((5.7)\) is established.

**Fourth a priori estimate.** By testing \((3.19)\) by any $v \in L^2(0,T;V_0)$ and integrating over $(0,T)$, we deduce that

$$\left| \int_0^T \langle \partial_t \varphi(t), v(t) \rangle \, dt \right| \leq c M_r \|v\|_{L^2(0,T;V)} \quad \text{for every } v \in L^2(0,T;V_0)$$

where we have set

$$M_r := \kappa_0 \|\nabla \varphi\|_{L^2(0,T;H)} + \|k_r \times \nabla \varphi\|_{L^2(0,T;H)} + \|f\|_{L^2(0,T;H)}$$

$$+ \left(1 + \|\varphi\|_{L^\infty(0,T;L^4(\Omega))}\right) \|\partial_t \varphi\|_{L^2(0,T;H)} \cdot \quad \text{(5.8)}$$

Thus, the estimate

$$\|\partial_t \ln \varphi\|_{L^2(0,T;V_0^*')} \leq c \quad \text{(5.9)}$$

follows whenever we prove that $M_r \leq c$. So, let us examine each term of \((5.8)\) but the third one, of course, by accounting for \((5.3)\). The first and last ones are bounded by \((3.27)\) and \((3.5)\). For the second term, we use the Young inequality \((3.37)\) and \((3.27)\) and obtain

$$\|k_r \times \nabla \varphi\|_{L^2(0,T;H)} \leq \|k_r\|_{L^1(0,T)} \|\nabla \varphi\|_{L^2(0,T;H)} \leq c.$$

Hence, \((5.9)\) is established.
**Convergence and conclusion.** From estimates (5.3) and (5.9) we derive the following convergence

\[ \vartheta_\tau \to \vartheta \quad \text{weakly in } L^2(0, T; V) \]  
\[ u_\tau \to u \quad \text{weakly in } L^2(0, T; V_0) \]  
\[ \chi_\tau \to \chi \quad \text{weakly star in } H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W) \]  
\[ \xi_\tau \to \xi \quad \text{weakly in } L^2(0, T; H) \]  
\[ \ln \vartheta_\tau \to \zeta \quad \text{weakly in } H^1(0, T; V_0^*) \]

for suitable functions \( \vartheta, \chi, \xi, \zeta \), and \( u := \vartheta - \vartheta_\infty \), possibly for a subsequence \( \tau = \tau_n \searrow 0 \). We are going to show that the triplet \( (\vartheta, \chi, \xi) \) is a solution to problem \( (3.24) - (3.26) \). Once this is proved, the whole family \( (\vartheta_\tau, \chi_\tau, \xi_\tau) \) converges (in the above topology) to \( (\vartheta, \chi, \xi) \), since the solution to problem \( (3.24) - (3.26) \) is unique. Now, we recall Theorem 4.2 and observe that it is sufficient to show that the quadruplet \( (\vartheta, \zeta, \chi, \xi) \) is a generalized solution, i.e., it solves \( (4.7) - (4.10) \). So, we just prove this fact. The regularity requirements \( (3.15), (3.17) - (3.18) \), and \( (4.7) \) are already clear. First, we observe that \( (5.12) \) and \( (5.14) \) imply at least weak convergence in \( C^0([0, T]; H) \) and \( C^0([0, T]; V_0^*) \), respectively, whence \( \zeta \) and \( \chi \) satisfy the Cauchy conditions \( \zeta(0) = \ln \vartheta_0 \) and \( \lambda(0) = \chi_0 \). Furthermore, as the embeddings \( W \subset V, V \subset L^4(\Omega) \), and \( V \subset H \) are compact, we can apply, e.g., [17, Sect. 8, Cor. 4] and deduce that \( (5.12), (5.10) \) and \( (5.7) \) imply

\[ \chi_\tau \to \chi \quad \text{strongly in } L^2(0, T; V) \cap C^0([0, T]; L^4(\Omega)) \]  
\[ \vartheta_\tau \to \vartheta \quad \text{strongly in } L^2(0, T; H). \]

Moreover, we can even assume that

\[ \chi_\tau \to \chi \quad \text{and} \quad \vartheta_\tau \to \vartheta \quad \text{a.e. in } Q. \]  

In particular, by noting that \( (3.5) \) implies Lipschitz continuity for \( \lambda' \) and \( \sigma' \) and the estimate (via Taylor’s formula)

\[ |\lambda(\chi_\tau) - \lambda(\chi)| \leq c(1 + |\chi|)|\chi_\tau - \chi| + c|\chi_\tau - \chi|^2 \]

we infer that

\[ \lambda'(\chi_\tau) \to \lambda'(\chi) \quad \text{strongly in } C^0([0, T]; L^4(\Omega)) \text{ and a.e. in } Q \]  
\[ \sigma'(\chi_\tau) \to \sigma'(\chi) \quad \text{strongly in } C^0([0, T]; L^4(\Omega)) \text{ and a.e. in } Q \]  
\[ \lambda(\chi_\tau) \to \lambda(\chi) \quad \text{strongly in } C^0([0, T]; H) \text{ and a.e. in } Q. \]

Next, we observe that \( (5.10) \) implies that \( \Delta \vartheta_\tau \) converges to \( \Delta \vartheta \) weakly in \( L^2(0, T; V_0^*) \). By \( (3.22) - (3.23) \) and the Young theorem (see \( (3.37) \)), we infer that

\[ (\kappa_0 + 1 \ast k_\tau) \ast \Delta \vartheta_\tau \to \kappa \ast \Delta \vartheta \quad \text{weakly in } L^2(0, T; V_0^*) \]

and we can take the limit in the integrated version of \( (3.19) \). Namely, we obtain

\[ \zeta + \lambda(\chi) - \kappa \ast \Delta \vartheta = 1 \ast f + \ln \vartheta_0 + \lambda(\chi_0) \]  

and we conclude that the limit functions we have constructed satisfy \( (4.8) \) and

\[ \partial_t \chi - \Delta \chi + \xi + \sigma'(\chi) = \lambda'(\chi) \vartheta \quad \text{a.e. in } Q. \]
Thus, it just remains to prove that $\xi \in \beta(\chi)$ a.e. in $Q$ and $\zeta \in L^2(0, T)$ in the framework of maximal monotone operators in $L^2(Q)$ by accounting for \((5.13)\) and \((5.15)\). On the contrary, the second claim needs much more work. We use the framework of the maximal monotone graphs in $L^2(0, T; V_0) \times L^2(0, T; V'_0)$ and consider the subdifferential $\partial \Psi$ of the function $\Psi$ given by \((4.5)\), which is related to the multivalued operator $L_n$ by \((4.6)\). So, we have to prove that $\zeta \in \partial \Psi(u)$ by starting from $\ln \vartheta \in \partial \Psi(u_\tau)$ for $\tau > 0$ (see \((4.4)\) and the weak convergence given by \((5.11)\) and \((5.14)\). It is well known (see once more, e.g., [2, Lemma 1.3, p. 42]) that a condition that allows to conclude is the following

$$\limsup_{\tau \searrow 0} \langle \ln \vartheta, u_\tau \rangle \leq \langle \zeta, u \rangle, \quad \text{that is,} \quad \limsup_{\tau \searrow 0} \langle \ln \vartheta - \zeta, \vartheta - \vartheta \rangle \leq 0. \quad (5.22)$$

In order to prove \((5.22)\), we consider the integrated version of \((3.20)\) and equation \((5.21)\). We take the difference and test the equality obtained by $\vartheta - \vartheta$. We have

$$\langle \ln \vartheta - \zeta, \vartheta - \vartheta \rangle = -\kappa_0 \int_Q \nabla (1 * (\vartheta - \vartheta)) \cdot \nabla (\vartheta - \vartheta)$$

$$- \int_Q (1 * k_\tau * \nabla \vartheta - \kappa'_0 * \nabla \vartheta) \cdot \nabla (\vartheta - \vartheta) - \int_Q (\lambda(X_\tau) - \lambda(\chi))(\vartheta - \vartheta)$$

$$= -(\kappa_0 + \kappa'_0) \int_Q \nabla (1 * (\vartheta - \vartheta)) \cdot \nabla (\vartheta - \vartheta)$$

$$- \int_Q ((1 * k_\tau - \kappa'_0) * \nabla \vartheta) \cdot \nabla (\vartheta - \vartheta) - \int_Q (\lambda(X_\tau) - \lambda(\chi))(\vartheta - \vartheta).$$

Now, the first integral of the last chain is nonnegative and $\kappa_0 + \kappa'_0 = \kappa > 0$. Next, the last integral tends to zero by \((5.16)\) and \((5.20)\). Finally, the middle term is estimated by the Hölder inequality, the Young theorem, and \((5.3)\) this way

$$- \int_Q ((1 * k_\tau - \kappa'_0) * \nabla \vartheta) \cdot \nabla (\vartheta - \vartheta)$$

$$\leq \|(1 * k_\tau - \kappa'_0) * \nabla \vartheta\|_{L^2(0,T;H)} \|\nabla (\vartheta - \vartheta)\|_{L^2(0,T;H)}$$

$$\leq \|1 * k_\tau - \kappa'_0\|_{L^1(0,T)} \|\nabla \vartheta\|_{L^2(0,T;H)} \|\nabla (\vartheta - \vartheta)\|_{L^2(0,T;H)}$$

$$\leq c \|1 * k_\tau - \kappa'_0\|_{L^1(0,T)}.$$ \quad (5.23)

Hence, by recalling \((5.22)\), we conclude that \((5.22)\) holds.

**Proof of Theorem 3.4.** By arguing as in the last part of the previous proof, we consider the integrated version of the equations for temperature and test the difference by $\vartheta - \vartheta$. However, in the present situation we already know that $\zeta = \ln \vartheta$ and can integrate over $Q_t$ rather than $Q$. Thus, a quite similar calculation yields

$$\int_{Q_t} (\ln \vartheta - \ln \vartheta)(\vartheta - \vartheta) + \kappa \int_{Q_t} \nabla (1 * (\vartheta - \vartheta)) \cdot \nabla (\vartheta - \vartheta)$$

$$= -\int_{Q_t} ((1 * k_\tau - \kappa'_0) * \nabla \vartheta) \cdot \nabla (\vartheta - \vartheta) - \int_{Q_t} (\lambda(X_\tau) - \lambda(\chi))(\vartheta - \vartheta). \quad (5.24)$$
At the same time, we multiply the difference between (3.20) and (3.25) by $\chi_\tau - \chi$ and integrate over $Q_t$. We obtain

$$\frac{1}{2} \int_Q |(\chi_\tau - \chi)(t)|^2 + \int_{Q_t} |\nabla(\chi_\tau - \chi)|^2 + \int_{Q_t} (\xi_\tau - \xi)(\chi_\tau - \chi)$$

$$= \int_{Q_t} (\lambda'(\chi_\tau) \vartheta_\tau - \lambda'(\chi_\vartheta)(\chi_\tau - \chi) - \int_{Q_t} (\sigma'(\chi_\tau) - \sigma'(\chi))(\chi_\tau - \chi). \quad (5.25)$$

At this point, we sum (5.25) to (5.24), and it is clear that all the terms on the left-hand side are nonnegative. Thus, we estimate each term on the right-hand side. As far as the term containing the convolution kernel is concerned, we can repeat the argument that led to (5.23). Thus, we have

$$- \int_Q \left( (1 * k_\tau - \kappa_0') \nabla \vartheta_\tau \right) \cdot \nabla (\vartheta_\tau - \vartheta) \leq c \|1 * k_\tau - \kappa_0'\|_{L^1(0,T)}.$$

The sum of all the terms involving $\lambda$ and $\lambda'$ can be first transformed and then estimated as follows (we use the Taylor formula and (3.5), besides standard inequalities, as usual)

$$- \int_{Q_t} (\vartheta_\tau \{ \lambda(\chi_\tau) - \lambda(\chi) - \lambda'(\chi_\tau)(\chi_\tau - \chi) \} - \int_{Q_t} \vartheta \{ \lambda(\chi) - \lambda(\chi_\tau) + \lambda'(\chi)(\chi_\tau - \chi) \}$$

$$\leq c \int_{Q_t} (\vartheta_\tau + \vartheta) |\chi_\tau - \chi|^2$$

$$\leq c \int_0^t \|\vartheta_\tau(s) + \vartheta(s)\|_4 \|\chi_\tau(s) - \chi(s)\|_4 \|\chi_\tau(s) - \chi(s)\|_2 ds$$

$$\leq c \int_0^t \|\vartheta_\tau(s) + \vartheta(s)\|_V \left( \|\chi_\tau(s) - \chi(s)\|_H + \|\nabla(\chi_\tau(s) - \chi(s))\|_H \right) \|\chi_\tau(s) - \chi(s)\|_H ds$$

$$\leq \frac{1}{2} \int_{Q_t} |\nabla(\chi_\tau - \chi)|^2 + c \int_0^t \left( 1 + \|\vartheta_\tau(t) + \vartheta(t)\|^2_1 \right) \|\chi_\tau(t) - \chi(t)\|^2_H ds.$$  

As the integral involving $\sigma'$ can be treated in a trivial way due to (3.3), we can apply the Gronwall lemma (see, e.g., [8, p. 156]) and infer that the left-hand side of (3.28) is bounded by

$$c \|1 * k_\tau - \kappa_0'\|_{L^1(0,T)} \exp \left( c \int_0^T \left( 1 + \|\vartheta_\tau(t) + \vartheta(t)\|^2_1 \right) dt \right).$$

As the last integral is bounded by a constant thanks to (3.3), inequality (3.28) follows.

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