Deriving identities for Wigner \{nj\}-symbols

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Abstract. We show how a simple and elegant graphical notation can be used to derive the Biedenharn-Elliot identity for the 6j-symbol and we demonstrate how the same technique can be applied to obtain new identities for the 6j. We then employ the same method also in the context of 4D spin-foam gravity and propose an analogous identity for the 15j symbol.

1. INTRODUCTION

The framework of spin foam quantum gravity provides a description for the discrete structure of space-time expressed in terms of variables of the representation theory of a certain gauge group (see [1] for introductory material). The 4D theory has undergone substantial development in the last fifteen years: since the seminal Barrett-Crane paper [2], a number of spin foam models have been suggested (for instance [3] [4]). The 3D formulation of the theory, instead, has not change dramatically since the introduction of the very first spin foam model by Ponzano and Regge (PR) in 1968 [5] (see [6] for a thorough review). The model is now well understood and is known to reproduce the expected low energy behaviour [7].

Since the PR model is based on the SU(2) group the amplitudes involve familiar objects of angular momentum theory. In particular, the most recurrent one is the 6j-symbol, which represent the amplitude of the four-valent vertex of the theory. In spin-foam the vertex is the node of a graph dual to a triangulation and represents a “quanta” of spacetime; knowing the properties of the vertex means understanding the fundamental properties of the theory. In fact, the 6j-symbol is known to satisfy some identities (see [8] [9]) that are employed to prove triangulation independence of the PR model (see for instance [10]). The most common of these identities is the one due to Biedenharn-Elliot (BE). The latter is also been used by Freidel et al. as a discretization of the equation that imposes the scalar constraint in 3D spin-foam gravity [11]. In particular, the authors of [11] use the BE identity to impose flatness on a generic tetrahedron spin-network.

Our work uses the paper [11] as a starting point. In section 1.3 we review the results of [11] and we show how the BE identity can be easily envisaged with a diagrammatic notation we propose. In section 2 instead, we employ our diagrammatic formalism to
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derive new identities for the 6j. Finally and most importantly, in section 3 we show that the same graphical method can be used to derive identities for the 15j-symbol, which are relevant in the context of the four-dimensional theory.

1.1. 3D Spin foam gravity and the Biedenharn-Elliot identity

Spin foam models are defined by assigning probability amplitudes to elements of a triangulation. In general, given a triangulated space-time manifold, the spin foam model partition function will look like the following

\[ Z_{SF} = \sum_{j's} \prod_{f} A_f(j \ldots j) \prod_{e} A_e(j \ldots j) \prod_{v} A_v(j \ldots j), \]  

where the labels denote respectively faces, edges and vertices of the graph dual to the triangulation. The \( j \)'s are representation variables that depend on the group the spin-foam model is based on. For instance, in three dimensions, in the Ponzano-Regge model \( A_v \) is the Wigner 6j-symbol from angular momentum theory which is known to satisfy the Biedenharn-Elliot identity.

The aim of this section is not to introduce the reader to spin foam models (for which we refer to [12] [13]), but rather to introduce the graphical notation we will extensively use in the subsequent sections.

In section 1.2 we briefly review the construction of the Ponzano-Regge (PR) model, while in section 1.3 we derive the Biedenharn-Elliot identity following [11] and employing our graphical method.

1.2. The Ponzano Regge Model

Let us consider a three-dimensional manifold \( M \) and principal \( SU(2) \)-bundle over it. Gravity then can be defined in terms of a cotriad \( e^a_i \) and a connection \( A^a_i \), where the indices \( a, i = (1, 2, 3) \). Both \( e \) and \( A \) are 1-forms and the latter takes values in the \( su(2) \) Lie algebra. The action for gravity in three dimension is then

\[ S_{BF}[e, A] = \frac{1}{4\pi G} \int_{M} e^i \wedge F^i(A), \]  

where the 2-form \( F^i(A) = dA^i + \frac{1}{2} \varepsilon^{ijk} A^j \wedge A^k \) is the curvature of the connection \( A \). Finally, if we define the density \( E^a_i = \frac{1}{2} \varepsilon^{abc} \varepsilon_{ijk} e^j_b e^k_c \) the phase space of gravity is spanned by the canonical pair

\[ \{ A^a_i(x), E^b_j(y) \} = \delta_a^b \delta_{ij} \delta^{(2)}(x - y). \]

We can now discretise the theory by triangulating the manifold \( M \). The fundamental “block” of spacetime is then the tetrahedron, and our variable \( E^a_i \) has to be averaged over its edges. In fact, upon quantization, the canonical pair \( (A^a_i, E^b_j) \) is replaced by \( (g, X_e) \in SU(2) \times su(2) \), that are defined on the dual graph to the
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To be precise the group element \(g\) is associated with the dual edge and defines the notion of parallel transport between to contiguous vertices; \(X_e\) instead is the flux variable given by the smearing of the densitised triad \(E^\alpha\)

\[
X_e = \int_e E_i(x) \tau^i dx, \tag{3}
\]

where the \(\tau^i\) are anti-Hermitian matrices\(^\dagger\).

Without delving into details we can state the PR partition function. Discretising the action in (2) we can derive:

\[
Z_{PR} = \sum_j d_j \prod_v A_v(j \ldots j), \quad d_j = 2j + 1, \tag{4}
\]

where the sum is over half-integers\(^\S\) and, as mentioned, the dual vertex amplitude is given by the \(6j\)-symbol known from angular momentum theory:

\[
A_v(j \ldots j) = \{ j_1 j_2 j_3 j_4 j_5 j_6 \}.
\]

**Figure 1.** The dual vertex in 3 dimensions.

### 1.2.1. Spin Networks

Quantum geometries in spin foam are identified with spin-network states. These are simply dual graphs coloured (labelled) with representation variables and group elements. To be more precise, the edges of the triangulation carry a representation variable \(j\), the dual edges \(e\) are labelled by group elements \(g\), finally invariant tensors \(\iota\) sit on the dual vertices \(^\dagger\). Spin networks span the kinematical Hilbert space of the theory and can be seen as elements of \(L^2(SU(2)^e/SU(2)^v)\), i.e. functions over the group elements on the dual edges and invariant under translations acting at the dual vertices. Commonly a spin-networks is written in the from

\(^\dagger\) Here \(\tau_i = i\frac{\sigma_i}{2}\), where \(\sigma^i\) are the Pauli matrices.

\(^\S\) The Ponzano-Regge model is based on \(SU(2)\).
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\[
s^{(j_k)}(g_1 \ldots g_n) = \bigotimes_f D^{(j)}(\text{hol}_f) \bigotimes_v \iota_v; \tag{5}
\]

where \(D^{(j)}(g)\) is the representation matrix of an element of the group \(g \in G\). In this case \(g = \text{hol}\) which stands for “holonomy,” i.e. the (ordered) product of all the group elements around a dual face \(f\):

\[\text{hol}_f = \prod_{g_i \in \partial f} g_i.\]

The simplest non trivial spin network one can envision in 2+1 dimensions is obtained by triangulating a 2-sphere. The result is a tetrahedron which, in 2 dimensions, is in turn dual to a tetrahedron (see figure 2).

![Figure 2. Two dimensional spin network and its dual.](image_url)

Given the expression for the \(\{6j\}\) in (A.6) we can easily write the spin-network state of the tetrahedron in figure 2

\[
s^{\{6j\}}_{\text{Tet}}(g_1, \ldots, g_6) = (-1)^{j_2 - a_2 + j_3 - a_3} (-1)^{j_4 - a_4} (-1)^{j_1 - a_1 + j_5 - a_5 + j_6 - a_6} \prod_i \langle j_i, a_i | g_i | j_i, b_i \rangle \]

\[
\begin{pmatrix}
  j_1 & j_2 & j_3 \\
-a_1 & -a_2 & -a_3
\end{pmatrix}
\begin{pmatrix}
  j_3 & j_4 & j_5 \\
  b_3 & b_4 & b_5
\end{pmatrix}
\begin{pmatrix}
  j_4 & j_2 & j_6 \\
-a_4 & b_2 & b_6
\end{pmatrix}
\begin{pmatrix}
  j_1 & j_5 & j_6 \\
  b_1 & -a_5 & -a_6
\end{pmatrix}, \tag{6}
\]

where the \((3jm)\)-symbols are the invariant tensors \(\iota_v\) sitting on each vertex (see appendix Appendix A) and the \(D^j_{mn}(g) = \langle j, m | g | j, n \rangle\) are representation matrices of the group elements in their respective representations.

The key point is that in 3D, as can be seen at the classical level from the equations of motion of (2), the physical states are flat geometries. This translates, in terms of quantum geometries, to imposing the holonomy \(g_1 \ldots g_n\) around dual faces to be trivial. Therefore, concerning the spin network (i.e. the quantum geometry) in (6), the physical configuration is given by contracting with a series of Dirac deltas on the group. In fact, if we take
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$$\psi_{\text{phys}}(g_1, \ldots, g_6) = \delta(g_4 g_5 g_6) \delta(g_1 g_6 g_2^{-1}) \delta(g_2 g_4 g_3^{-1}).$$  \hspace{1cm} (7)

we can obtain the 6j-symbol by taking the inner product of the spin network function $s_{\text{tet}}(g_1, \ldots, g_6)$:

$$\psi_{\text{phys}}(j_1, \ldots, j_6) = \int \prod_{e=1}^{6} dg_e \ s_{\text{tet}}(j_e) \psi_{\text{phys}}(g_1, \ldots, g_6),$$

$$= \begin{bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{bmatrix}. \hspace{1cm} (8)$$

1.3. The Biedenharn-Elliot identity

The authors of [11] work in the context of (2 + 1) spin-foam quantum gravity. They introduce a quantum Hamiltonian constraint and prove that the BE identity can be seen a consequence of imposing such constraint on a flat tetrahedron.

Following [11], define a quantum constraint $\hat{H}$ that acts on spin-networks $|s\rangle$. The idea is to choose two contiguous fluxes variables $X_i, X_j \in \mathfrak{su}(2)$, and impose that their scalar product is invariant under parallel transport around the dual face:

$$\hat{H}_{ij}|s\rangle = 0, \quad \text{where} \quad \hat{H}_{ij} = X_i \cdot X_j - X_i \cdot \text{Ad}_{\text{hol}} X_j. \hspace{1cm} (9)$$

Here the flux $X_j$ is parallel-transported with the holonomy through the action of $G$ on its Lie algebra: $g \triangleright X = gXg^{-1} = \text{Ad}_{g} X$. We note that the mere action of $\hat{H}_{ij}$ on a node of the spin-network does not impose flatness, it implies only the fact that the holonomy $g$ lives in the Cartan subalgebra spanned by the Lie-algebra element $X_j$. However imposing the constraint (9) at each node does imply that the holonomy is trivial. In other words, if the constraint is applied to each node, this imposes flatness of the holonomies around all the faces (see equation 7).

One could take the opposite point of view. Assuming flatness of the holonomy (again see equation 7), we notice that the constraint in (9) is trivially satisfied. The next step is to show that such condition, when $|s\rangle$ is the tetrahedron spin-network in (6), implies a second order recursion relation on the 6j-symbol [11]. This recursion relation is commonly known in Angular Momentum theory as the Biedenharn-Elliot (BE) identity and it reads

$$A_{+1}(j_1) \begin{bmatrix} j_1 + 1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{bmatrix} + A_0(j_1) \begin{bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{bmatrix} + A_{-1}(j_1) \begin{bmatrix} j_1 - 1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{bmatrix} = 0. \hspace{1cm} (10)$$

The coefficients are given by
$A_0(j_1) = (-1)^{j_2+j_4+j_6} \left\{ \begin{array}{ccc} j_2 & j_2 & 1 \\ j_6 & j_6 & j_4 \end{array} \right\} +$

$+(-1)^{2j_1+j_2+j_3+j_5+j_6}(2j_1 + 1) \left\{ \begin{array}{ccc} j_1 & j_1 & 1 \\ j_2 & j_2 & j_3 \end{array} \right\} \left\{ \begin{array}{ccc} j_1 & j_1 & 1 \\ j_6 & j_6 & j_5 \end{array} \right\},$  \hspace{1cm} (11)

and

$A_{\pm 1}(j_1) = (-1)^{2j_1+j_2+j_3+j_5+j_6+1}(2(j_1 \pm 1) + 1) \left\{ \begin{array}{ccc} j_1 \pm 1 & j_1 & 1 \\ j_2 & j_2 & j_3 \end{array} \right\} \left\{ \begin{array}{ccc} j_1 \pm 1 & j_1 & 1 \\ j_6 & j_6 & j_5 \end{array} \right\}.$  \hspace{1cm} (12)

In the following we show how to prove the BE identity the same way it is proved in [11]. In this section we will also introduce the graphical notation we employed to find our results. We show that with such graphical notation the derivation of the result is greatly simplified.

We begin by choosing the contiguous fluxes to be $X_2$ and $X_6$ (see figure 2). Then we can write the equation for $\hat{H}$ action on $s_{\text{Tot}}^{(j_e)}$ (to avoid cluttering of the notation we dropped the ket around the spin-network state) as

$$(X_2 \cdot X_6) s_{\text{Tot}}^{(j_e)}(g_1, \ldots, g_6) = (X_2 \cdot Ad_{g_2^{-1}g_1g_6^{-1}}X_6) s_{\text{Tot}}^{(j_e)}(g_1, \ldots, g_6)$$  \hspace{1cm} (13)

where the $X_e$'s act by inserting a $\tau_e$ in front of the corresponding ket $|j_e, a_e\rangle$ in (6). The right hand side (rhs) is given by taking the same product after we parallel transported $X_6$ around the dual face as in

$$X_6 \rightarrow Ad_{g_2^{-1}g_1g_6^{-1}}X_6.$$  

The key point of our analysis is that relation (13) (and any other of this kind) can be represented graphically as inserting “grasping” operators in different places of the spin network. In this case (13) is simply given by the figure 3 where the grasping operator is represented by a dashed line. We notice that the grasping operator has been moved along the lines 2 and 6.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{Figure3.png}
\caption{Biedenharn-Elliot identity.}
\end{figure}
In the following section we will give the details of how to prove the BE identity explaining every step diagrammatically. We split the computation in left and right hand sides.

1.3.1. LHS of BE identity proof

To obtain the BE identity we first evaluate the LHS of (13). Graphically, figure 4 shows how we want to proceed. We can already see that this side of the equation is rather trivial in the graphical notation.

What exactly happens is that the “grasping” in fig 4 shows us where the operator $X_2 \cdot X_6$ is acting. Then we can use recoupling theory to extract a $6j$-symbol which is represented by a small tetrahedron (for convenience of the reader we listed some useful recoupling theory relations in the appendix Appendix A).

In formulas the action of the grasping operator is given by the insertion of $-(\tau_2)^n \otimes (\tau_6)_n$ in the spin network.

Following [11] we adopt the relations

$$
(\tau_e)^n \otimes (\tau_{e'})_n = \sum_{n=-1}^{+1} (-1)^{1-n} (L_e)^n \otimes (L_{e'})_{-n},
$$

where we defined $L_0 = -\frac{\sigma_z}{2}$ and $L_{\pm} = \frac{(\pm \sigma_x + i \sigma_y)}{2\sqrt{2}}$.

To avoid further cluttering of the notation, in the following we will omit the subscript $e$ that indicates which edge the operators act on.

The action of the $L_n$ operators on $SU(2)$ kets is easily defined using $\{3jm\}$ symbols:

$$
L_n |j,a\rangle = -N_j \sum_b (-1)^{j-b} \left( \begin{array}{ccc} 1 & j & j \\ n & a & -b \end{array} \right) |j,b\rangle = (-1)^{2j+1}N_j \sum_b (-1)^{1+j-n-a} \left( \begin{array}{ccc} 1 & j & j \\ -n & -a & b \end{array} \right) |j,b\rangle,
$$

|| Since we are using anti-Hermitian generators $X \cdot X = -X^tX$
where the normalization factor is

$$N_j = \sqrt{j(j + 1)d_j}, \quad d_j = 2j + 1.$$  \hspace{1cm} (16)$$

We can now see how, using recoupling theory, we obtain the desired result on the LHS. Writing only relevant parts of the spin-network state $s^{\{j_\text{e}\}}_{\text{Tet}}$, we have:

\[
(X_2 \cdot X_6) \sum_{a,b} (-1)^{j_4 - a_4} \begin{pmatrix} j_4 & j_2 & j_6 \\ -a_4 & b_2 & b_6 \end{pmatrix} |j_2, b_2\rangle \otimes |j_6, b_6\rangle = \\
- \sum_{a,b} (-1)^{j_4 - a_4} \begin{pmatrix} j_4 & j_2 & j_6 \\ -a_4 & b_2 & b_6 \end{pmatrix} \tau_i |j_2, b_2\rangle \otimes \tau^i |j_6, b_6\rangle = \\
- \sum_{n=-1}^{+1} (-1)^{1-n} \sum_{a,b} (-1)^{j_4 - a_4} \begin{pmatrix} j_4 & j_2 & j_6 \\ -a_4 & b_2 & b_6 \end{pmatrix} L^n |j_2, b_2\rangle \otimes L^{-n} |j_6, b_6\rangle = \\
N_{j_2}N_{j_6}(-)^{j_2 + j_4 + j_6} \left\{ \begin{array}{c}
\hat{j}_2 \\
\hat{j}_6 \\
\hat{j}_4 
\end{array} \right\} \sum_{a,b} (-1)^{j_4 - a_4} \begin{pmatrix} j_4 & j_2 & j_6 \\ -a_4 & k_2 & k_6 \end{pmatrix} |j_2, k_2\rangle \otimes |j_6, k_6\rangle.
\]  \hspace{1cm} (17)$$

In the last step we used the relations (A.5) and (A.7). Therefore the overall effect of the insertion of the grasping operator on the left-hand-side of the equation in (13) is simply:

\[
(X_2 \cdot X_6) s^{\{j_\text{e}\}}_{\text{Tet}}(g_1, \ldots, g_6) = \quad N_{j_2}N_{j_6}(-)^{j_2 + j_4 + j_6} \left\{ \begin{array}{c}
\hat{j}_2 \\
\hat{j}_6 \\
\hat{j}_4 
\end{array} \right\} s^{\{j_\text{e}\}}_{\text{Tet}}(g_1, \ldots, g_6)  \hspace{1cm} (18)$$

1.3.2. RHS of BE identity proof

Before proceeding with the evaluation of the right hand side of (13) we define $\tilde{X} = g^{-1}Xg$, so that

\[
(X_2 \cdot Ad_{g_2^{-1}g_1g_6^{-1}}X_6) s^{\{j_\text{e}\}}_{\text{Tet}}(g_1, \ldots, g_6) = \quad (\tilde{X}_2 \cdot Ad_{g_1\tilde{X}_6}) s^{\{j_\text{e}\}}_{\text{Tet}}(g_1, \ldots, g_6).  \hspace{1cm} (19)$$

This way the evaluation of the action of the quantum operator is more straightforward. However we note that the insertion of $g^{-1}Xg$ in front of the respective kets reduces to $X$ effectively acting on the bra:

\[
\langle j, a| g^{-1}Xg|j, b\rangle = \langle j, a| X|j, b\rangle.  \hspace{1cm} (20)$$

The operator $Ad(g)$ acts as multiplicative operator, contributing adding representation matrices in the adjoint (spin 1) representation as factors:

$$Ad(g) \rightarrow D^1(g)_{mn}.$$ 

Analogously to (15) we have
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\[ \langle j_2, a_2 | L_m = (-1)^{2j_2+1} N_{j_2} \sum_{k_2} (-1)^{1-m} (-1)^{j_2-k_2} \begin{pmatrix} 1 & j_2 & j_2 \\ -m & -k_2 & a_2 \end{pmatrix} \langle j_2, k_2 |, \]

\[ \langle j_6, a_6 | (-1)^{1-n} L_{-n} = (-1)^{2j_6+1} N_{j_6} \sum_{k_6} (-1)^{j_6-k_6} \begin{pmatrix} 1 & j_6 & j_6 \\ n & -k_6 & a_6 \end{pmatrix} \langle j_6, k_6 | \]

with the same normalization factors [16].

The operation to carry out can be also easily described with the formalism that we are proposing. The steps are in figure 5.

![Figure 5. Diagrammatic computation of RHS.](image)

Precisely, each operator $\tilde{X}_e^m$ acts by inserting a $\tau_e^m$ in front of the respective bra $\langle j_e, a_e |$, while $Ad_{ge}$ acts as a multiplicative operator in the adjoint representation as $\langle 1, m | g_1 | 1, n \rangle$. We notice that the insertion of the grasping operators, after employing recoupling theory, yields two extra \{6j\}'s:

\[
(\tilde{X}_2 \cdot Ad_{g_2} \cdot \tilde{X}_6) \langle 1, m | g_1 | 1, n \rangle \langle j_1, a_1 | j_1, b_1 \rangle \langle j_2, a_2 | g_2 \otimes j_6, a_6 | g_6 =
\]

\[ (-) \sum \langle 1, m | g_1 | 1, n \rangle \langle j_1, a_1 | g_1 | j_1, b_1 \rangle (-1)^{j_2-a_2+j_3-a_3} \langle -1 \rangle^{j_1-a_1+j_6-a_6} \begin{pmatrix} j_1 & j_2 & j_3 \\ a_1 & -a_2 & -a_3 \end{pmatrix} \]

\[ \begin{pmatrix} j_1 & j_5 & j_6 \\ b_1 & -a_5 & -a_6 \end{pmatrix} \langle 1, m | g_1 | 1, n \rangle \langle j_1, a_1 | g_1 | j_1, b_1 \rangle \langle j_2, a_2 | L_m g_2 \otimes j_6, a_6 | L_{-n} (-1)^{1-n} g_6, \]

which after using the recoupling relation [A.8] becomes:

\[
N_{j_2} N_{j_6} \sum_{J_1 = j_1 - 1}^{j_1 + 1} d_{J_1} \langle -1 \rangle^J \begin{pmatrix} J_1 & J_1 \\ j_3 & j_2 \end{pmatrix} \begin{pmatrix} 1 & J_1 & J_1 \\ J_3 & J_2 & J_2 \end{pmatrix} \times
\]

\[ (-1)^{j_2-k_2+j_3-a_3} \langle -1 \rangle^{J_1-A_1+j_6-k_6} \begin{pmatrix} J_1 & J_1 \\ -A_1 & -k_2 & -a_3 \end{pmatrix} \begin{pmatrix} J_1 & J_5 & J_6 \\ B_1 & -a_5 & -k_6 \end{pmatrix} \]

\[ \langle J_1, A_1 | g_1 | J_1, B_1 \rangle \langle j_2, k_2 | g_2 \otimes j_6, k_6 | g_6 \rangle. \]
Finally we obtain
\[(\tilde{X}_2 \cdot \text{Ad}_{g_1}\tilde{X}_6) s^{(\{j\}_c)}_{\text{Tet}}(g) = \]
\[N_{j_2}N_{j_6} \sum_{J_1 = j_{j_1}-1}^{j_{j_1}+1} d_{J_1}(-1)^{j_1+j_i+j_2+j_3+j_5+j_6+1} \left\{ \begin{array}{ccc} 1 & J_1 & j_1 \\ j_3 & j_2 & j_2 \end{array} \right\} \left\{ \begin{array}{ccc} 1 & J_1 & j_1 \\ j_5 & j_6 & j_6 \end{array} \right\} s^{(\{j_1\}_{...,j_6})}_{\text{Tet}}(g). \]
(23)

1.3.3. The BE identity Putting together the results in (18) and in (23) we have:
\[\hat{H} s^{(\{j\}_c)}_{\text{Tet}} = N_{j_2}N_{j_6} \left[ A_0(j) s^{(\{j\}_c)}_{\text{Tet}} + A_{-1}(j) s^{(\{j_{1-1},j_2-,j_6\})}_{\text{Tet}} + A_{+1}(j) s^{(\{j_1+1,j_2-,j_6\})}_{\text{Tet}} \right] = 0. \] (24)

The coefficients \(A_i(j)\) are exactly the ones in (11).

Now the last step. We evaluated the action of the quantum operator \(\hat{H}\) on a specific spin-network \(s^{(\{j\}_c)}_{\text{Tet}}\), therefore we know its action also on a linear superposition of such states like \(\Psi(g) = \sum \prod d_j \psi(j) s_{\text{Tet}}(g)\):
\[\hat{H} \Psi(g_1 \ldots g_6) = \left[ \prod_{i=1}^{6} d_j \right] \psi(j_1 \ldots j_6) \hat{H} s^{(\{j\}_c)}_{\text{Tet}}(g_1, \ldots, g_6) = \sum_{j_1 \ldots j_6} \left[ \prod_{i=2}^{6} d_j \right] N_{j_2}N_{j_6} [d_{j_1} A_0(j) \psi(j_1 \ldots j_6) + d_{j_1-1} \psi(j_1 + 1 \ldots j_6) A_{-1}(j+1) + d_{j_1+1} \psi(j_1 + 1 \ldots j_6) A_{+1}(j-1)] s^{(\{j\}_c)}_{\text{Tet}} = 0. \] (25)

Finally we note that the following relations hold for the coefficients
\[d_{j+1} A_{\pm}(j) = d_j A_{\mp}(j).\] (26)

Thus we obtain that the operator \(\hat{H}\) imposes
\[A_0(j) \psi(j_1 \ldots j_6) + A_{+1}(j) \psi(j_1 + 1 \ldots j_6) + A_{-1}(j) \psi(j_1 - 1 \ldots j_6) = 0, \] (27)

which is exactly the Biedenharn-Elliot identity in (10).

2. New Identity

In the previous section we showed how we can define an operator \(\hat{H}\) to impose flatness on the dual faces of a spin network. Most importantly we showed how, assuming flatness of the tetrahedron, the equation in (11) implies the BE identity on the 6\(j\)-symbol, which is used in spin foam to prove triangulation independence of PR model and to study the “low energy” limit of the theory.
We saw that the action of the operator \( \hat{H} \) on the spin networks can be seen as the insertion of a grasping operators (represented with a dashed line in figure 3) moved along the edges of the spin-network.

In this section we want find a new recursion identity for the \( \{6j\} \) by adopting the same principles. However we will insert the grasping operator to connect two edges that belong to different dual faces and move the grasping along these edges. For convenience of the reader we state the new identity before delving into the details of the derivation. The new identity for the \( \{6j\} \) derived with our graphical method reads

\[
A_{10}(j_1)\psi(j_1 \ldots j_6) + A_{1+1}(j_1)\psi(j_1 + 1 \ldots j_6) + A_{1-1}(j_1)\psi(j_1 - 1 \ldots j_6) \\
+ A_{40}(j_4)\psi(j_1 \ldots j_6) + A_{4+1}(j_4)\psi(j_4 + 1 \ldots j_6) + A_{4-1}(j_4)\psi(j_4 - 1 \ldots j_6) = 0,
\]

(28)

Where the coefficients are:

\[
A_{10}(j) = (-1)^{j_1 + j_2 + j_3 + 1}d_{j_1} \left\{ \begin{array}{c} 1 \\ j_3 \\ j_2 \\ 1 \\ j_6 \\ j_5 \\ j_5 \end{array} \right\} \left\{ \begin{array}{c} 1 \\ j_1 \\ j_1 \\ j_6 \\ j_5 \\ j_5 \end{array} \right\}
\]

(29)

\[
A_{1\pm1}(j) = (-1)^{j_1 + j_2 + j_3}d_{j_1\pm1} \left\{ \begin{array}{c} 1 \\ j_3 \\ j_2 \\ 1 \\ j_6 \\ j_5 \\ j_5 \end{array} \right\} \left\{ \begin{array}{c} 1 \\ j_1 \pm 1 \\ j_1 \\ j_6 \\ j_5 \\ j_5 \end{array} \right\}
\]

\[
A_{40}(j) = (-1)^{2j_4 + j_2 + j_3 + j_5 + j_6 + 1}d_{j_4} \left\{ \begin{array}{c} 1 \\ j_3 \\ j_5 \\ j_5 \\ 1 \\ j_6 \\ j_2 \\ j_2 \end{array} \right\} \left\{ \begin{array}{c} 1 \\ j_4 \\ j_4 \\ j_6 \\ j_2 \\ j_2 \end{array} \right\}
\]

\[
A_{4\pm1}(j) = (-1)^{2j_4 + j_2 + j_3 + j_5 + j_6 + 1}d_{j_4\pm1} \left\{ \begin{array}{c} 1 \\ j_3 \\ j_5 \\ j_5 \\ 1 \\ j_6 \\ j_2 \\ j_2 \end{array} \right\} \left\{ \begin{array}{c} 1 \\ j_4 \pm 1 \\ j_4 \\ j_6 \\ j_2 \\ j_2 \end{array} \right\}
\]

In the following we give a detailed derivation for the new identity in equation 28 using the method introduced in the previous section. In the graphical method, the new starting equation is represented in figure 6. In this case we will use the fact that imposing flatness on the dual faces implies that also the holonomies around edges are trivial: looking at figure 2, imposing the deltas in (7), then also \( g_1g_2g_4g_5 = 1 \).
Thus we can define a new $\hat{H}_{\text{new}} = (\tilde{X}_2 \cdot Ad_{g_1}\tilde{X}_5) - (X_2 \cdot Ad_{g_4}X_5)$ operator which yields the identity:

$$
(\tilde{X}_2 \cdot Ad_{g_1}\tilde{X}_5) s_{\text{Tet}}^{\{j_e\}}(g_1, \ldots, g_6) = (X_2 \cdot Ad_{g_4}X_5) s_{\text{Tet}}^{\{j_e\}}(g_1, \ldots, g_6).
$$

Let us split the calculation into two parts and evaluate both sides. Graphically the calculation we want to follow for the left hand side of (30) goes as in figure 7.

![Figure 7. Steps for the LHS](image)

Again the dashed lines represent grasping operators. In the second step we used, as before, recoupling theory rule (A.7).

As in section 1.3 Each operator $\tilde{X}_e^m$ acts by inserting a $\tau_e^m$ in front of the respective bra $\langle j_e, a_e |$, while $Ad_{g_e}$ acts as a multiplicative operator in the adjoint representation as $D^1_{mn}(g)$. Explicitly it reads

$$
(\tilde{X}_2 \cdot Ad_{g_1}\tilde{X}_5) s_{\text{Tet}}^{\{j_e\}}(g) = 
$$

$$
N_{j_2} N_{j_5} \sum_{J_1=j_1-1}^{j_1+1} d_{J_1}(-1)^{J_1+j_2+j_3+1} \left\{ \begin{array}{ccc} 1 & J_1 & j_1 \\ j_3 & j_2 & j_2 \end{array} \right\} \left\{ \begin{array}{ccc} 1 & J_1 & j_1 \\ j_6 & j_5 & j_5 \end{array} \right\} s_{\text{Tet}}^{\{J_1, \ldots, j_e\}}(g),
$$

where $N_j$ is the same normalization factor in (16).

On the right hand side we have instead (see figure 8).

![Figure 8. Steps for the RHS](image)
(X_2 \cdot Ad_{g_4} X_5) s^{\{j_e\}}_{\text{Tet}}(g) =
\sum_{J_4 = J_4 - 1}^{J_4 + 1} d_{J_4} (-1)^{2J_4 + j_2 + j_3 + j_6 + 1} \left\{ \begin{array}{ccc}
1 & J_4 & j_4 \\
J_3 & J_5 & j_5
\end{array} \right\} \left\{ \begin{array}{ccc}
1 & J_4 & j_3 \\
J_6 & J_2 & j_2
\end{array} \right\} s^{\{J_4 \ldots j_e\}}_{\text{Tet}}(g).

(32)

All together:

\sum_{J_1 = J_1 - 1}^{J_1 + 1} d_{J_1} (-1)^{J_1 + j_2 + J_3 + j_3 + 1} \left\{ \begin{array}{ccc}
1 & J_1 & j_1 \\
J_3 & J_2 & j_2
\end{array} \right\} \left\{ \begin{array}{ccc}
1 & J_1 & j_3 \\
J_6 & J_5 & j_5
\end{array} \right\} s^{\{J_1 \ldots j_e\}}_{\text{Tet}} = 0

\sum_{J_4 = J_4 - 1}^{J_4 + 1} d_{J_4} (-1)^{2J_4 + j_2 + j_3 + j_5 + j_6} \left\{ \begin{array}{ccc}
1 & J_4 & j_4 \\
J_3 & J_5 & j_5
\end{array} \right\} \left\{ \begin{array}{ccc}
1 & J_4 & j_3 \\
J_6 & J_2 & j_2
\end{array} \right\} s^{\{J_4 \ldots j_e\}}_{\text{Tet}} = 0.

(33)

As in equation (25), we can use the result to evaluate the action of \( \hat{H}_{\text{new}} \) on a superposition of states. We get

\[ \hat{H}_{\text{new}} \Psi(g_1 \ldots g_6) = N_{j_2} N_{j_5} \sum_{j_1 \ldots j_6} \prod_{i = 2,3,5,6} d_{j_i} \times \]

\[ d_{j_4} \left( d_{j_4} A_{10}(j_1) \psi(j_e) + d_{j_4+1} A_{1-1}(j_1 + 1) \psi(j_1 + 1) + d_{j_1-1} A_{1+1}(j_1 - 1) \psi(j_1 - 1) \right) \]

\[ + d_{j_1} \left( d_{j_4} A_{40}(j_4) \psi(j_4) + d_{j_4+1} A_{4-1}(j_4 + 1) \psi(j_4 + 1) + d_{j_4-1} A_{4+1}(j_4 - 1) \psi(j_4 - 1) \right) \]

\[ s^{\{j_e\}}_{\text{Tet}} = 0, \]

(34)

where the coefficients are the one in (29).

Using again the relation (26) we obtain the identity for the 6j-symbol in (28).

We want to stress that the derivation of this new identity is itself a proof of it. In fact, starting from the assumption of flatness of the holonomy \( g_1 g_2 g_4 g_5 \), we can write the equation in (30); in turn, we showed how the latter implies the identity in (28).

Finally we should say that we cannot exclude that the identity in (28) was already known before, however we were not able to find it anywhere in the literature.

3. A new identity for the 15j

After showing how the BE identity can be thought as moving a grasping operator along edges of a tetrahedron, we want to extend the concept to four spacetime dimensions, thus deriving a recursion relation for the 15j-symbol. In order to do that we have to choose a specific 4-simplex (figure 9).

We can write the spin-network state for this 4-simplex explicitly; using the convention \( g_{ij} = g_{ji}^{-1} \in G \).
Now we want to assume flatness on the 10 faces. The flat 4-simplex configuration is given by the analogous of (7):

$$\psi_{\text{flat}}(g_{ij}) = \prod_{f} \delta(hol_{f}). \quad (35)$$

The corresponding \{15j\}-symbol is given by:

$$\psi_{15j}(j_{01} \ldots j_{89}) = \int \prod_{i<j} dg_{ij} s_{4\text{-sim}}^{i<j} \prod_{i<j} g_{ij} \psi_{\text{flat}}(\prod_{i<j} g_{ij}),$$

$$= \{15j\}. \quad (36)$$

As before, to derive an identity we use the fact that, given flatness on a face, the grasping on a vertex is equivalent to the grasping on the other two. We show this identity in figure 10.

We notice that in our case we can apply our condition to two different “kinds” of faces, which are identified by red(dotted) and green(dashed) colouring, as in figure 11.
However, on a closer look, these apparently diverse shapes turn out to be very similar (see fig. 12). In the following we will evaluate the new identity for the particular face coloured in red (dotted).

3.1. The new identity

Using the fact that \( g_{58} g_{83} g_{32} g_{27} g_{75} = 1 \), the identity we obtain is schematically the one in figure 13.

In formulas, we can define a new quantum operator \( \hat{H}_{(4D)} \) as following
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Figure 13. Diagram of the new identity.

\[
\hat{H}_{(4D)} s_4^{\sim} = 0, \quad \text{where} \quad \hat{H}_{(4D)} = X_{58} \cdot X_{57} - X_{58} \cdot Ad_{hol} X_{57},
\]

which yields the identity

\[
X_{58} \cdot X_{57} s_4^{\{je\}} = X_{58} \cdot Ad(g_{58} g_{83} g_{32} g_{27} g_{75}) X_{57} s_4^{\{je\}} = \nonumber \\
\tilde{X}_{58} \cdot Ad(g_{83} g_{32} g_{27}) \tilde{X}_{57} s_4^{\{je\}}.
\]

Let us split the calculation once again in left and right hand sides and proceed step by step.

3.1.1. LHS of new \( \{15j\} \) identity Following an analogous calculation as in section 2 this part of the calculation is trivial. We get

\[
X_{58} \cdot X_{57} s_4^{\{je\}} = (-1)^{(j_{58}+j_{57}+j_{05})} N_{j_{58}j_{57}} N_{j_{58}} \left\{ \begin{array}{ccc} j_{58} & j_{58} & 1 \\ j_{57} & j_{57} & j_{05} \end{array} \right\} s_4^{\{je\}},
\]

For the sake of simplicity we will define the amplitude

\[
A_0(j_{01} \cdots j_{89}) = (-1)^{(j_{58}+j_{57}+j_{05})} \left\{ \begin{array}{ccc} j_{58} & j_{58} & 1 \\ j_{57} & j_{57} & j_{05} \end{array} \right\}.
\]

3.1.2. RHS of new \( \{15j\} \) identity The right hand side is more complicated. We are going to follow the simple scheme in figure 14.

Figure 14. Diagrams for the computation of the RHS.

The \( Ad(g) \) in (39) acts as a multiplicative operator in the adjoint representation:
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\[ \text{Ad}(g_{32} g_{27}) \to D^1(g_{32})_{\alpha \beta} D^1(g_{27})_{\beta n}. \]

First we note that for the representation matrices the following relation holds

\[ D^j_{mn}(g^{-1}) = (D^j_{nm}(g))^* = (-1)^{n-m} D^j_{-n-m}(g). \]

Now, as in \([A.8]\), we can compute the products:

\[
(-1)^{(\beta - \alpha)} \langle 1, -\beta | g_{23} | 1, -\alpha \rangle \langle j_{23}, a_{23} | g_{23} | j_{23}, b_{23} \rangle \\
= (-1)^{2j_{23}} \sum_{j_{23}=j_{23}-1}^{j_{23}+1} d_{j_{23}} \left( \begin{array}{ccc} 1 & j_{23} & J_{23} \\ \alpha & -b_{23} & B_{23} \end{array} \right) (-1)^{J_{23}-A_{23}} \left( \begin{array}{ccc} 1 & j_{23} & J_{23} \\ -\beta & a_{23} & -A_{23} \end{array} \right) \\
(-1)^{1-\beta + j_{23} - b_{23}} \langle J_{23}, A_{23} | g_{32} | J_{23}, B_{23} \rangle, \tag{41}
\]

also

\[
(-1)^{(\alpha - m)} \langle 1, -\alpha | g_{38} | 1, -m \rangle \langle j_{38}, a_{38} | g_{38} | j_{38}, b_{38} \rangle \\
= (-1)^{2j_{38}} \sum_{j_{38}=j_{38}-1}^{j_{38}+1} d_{j_{38}} \left( \begin{array}{ccc} 1 & j_{38} & J_{38} \\ m & -b_{38} & B_{38} \end{array} \right) (-1)^{J_{38}-A_{38}} \left( \begin{array}{ccc} 1 & j_{38} & J_{38} \\ -\alpha & a_{38} & -A_{38} \end{array} \right) \\
(-1)^{1-\alpha + j_{38} - b_{38}} \langle J_{38}, A_{38} | g_{38} | J_{38}, B_{38} \rangle, \tag{42}
\]

and at last

\[
\langle 1, \beta | g_{27} | 1, n \rangle \langle j_{27}, a_{27} | g_{27} | j_{27}, b_{27} \rangle \\
= (-1)^{2j_{27}} \sum_{j_{27}=j_{27}-1}^{j_{27}+1} d_{j_{27}} \left( \begin{array}{ccc} 1 & j_{27} & J_{27} \\ \beta & a_{27} & -A_{27} \end{array} \right) (-1)^{J_{27}-A_{27}} \left( \begin{array}{ccc} 1 & j_{27} & J_{27} \\ -n & -b_{27} & B_{27} \end{array} \right) \\
(-1)^{1-n + j_{27} - b_{27}} \langle J_{27}, A_{27} | g_{27} | J_{27}, B_{27} \rangle. \tag{43}
\]

Now I evaluate the action of the operator \( L_{m, n} \) on the states:

\[
\langle j_{58}, a_{58} | L_m = (-1)^{2j_{58}+1} N_{j_{58}} \sum_{k_{58}} (-1)^{1-m} (-1)^{j_{58}-k_{58}} \left( \begin{array}{ccc} 1 & j_{58} & j_{58} \\ -m & -k_{58} & a_{58} \end{array} \right) \langle j_{58}, k_{58} \rangle, \\
(-1)^{1-n} L_{-n} | j_{57}, b_{57} \rangle = (-1)^{2j_{57}+1} N_{j_{57}} \sum_{k_{57}} (-1)^{j_{57}-b_{57}} \left( \begin{array}{ccc} 1 & j_{57} & j_{57} \\ n & -b_{57} & k_{57} \end{array} \right) | j_{57}, k_{57} \rangle. \tag{44}
\]

The result on the rhs is the following:
\[ \mathcal{X}_{58} \cdot \text{Ad}(g_{83} g_{32} g_{27}) \mathcal{X}_{57} \ P^{\{j_e\}_{4}}_{\text{sim}} \]

\[ = \sum_{j_{23}=j_{23}-1}^{j_{23}+1} d_{j_{23}} \sum_{j_{38}=j_{38}-1}^{j_{38}+1} d_{j_{38}} \sum_{j_{27}=j_{27}-1}^{j_{27}+1} d_{j_{27}} \ N_{j_{58}} N_{j_{57}} (-1)^{(j_{12}+j_{23}+j_{27}+2j_{38}+j_{34}+j_{57}+j_{58}+j_{68}+j_{79}+1)} \]

\[
\begin{cases}
1 & j_{38} \\
1 & j_{34} \\
1 & j_{12}
\end{cases}
\begin{cases}
1 & j_{38} \\
1 & j_{34} \\
1 & j_{12}
\end{cases}
\begin{cases}
j_{23} & j_{27} \\
j_{23} & j_{27} \\
j_{57} & j_{57}
\end{cases}
\begin{cases}
j_{27} & j_{27} \\
j_{79} & j_{57} \\
j_{79} & j_{57}
\end{cases}
\{\{j_{23}, j_{27}, j_{38}, j_{e}\} \} \ .
\]

(45)

3.1.3. The new identity We evaluated both left and right hand sides of \[38\]; now we can state the result which is a new identity for the 15j-symbol. Let us call \(A_{(a,b,c)}\) the amplitude

\[
A_{(a,b,c)}(j_{01} \ldots j_{89}) = d_{j_{23}+a} d_{j_{27}+b} d_{j_{38}+c} (-1)^{(j_{12}+2j_{23}+a+2j_{27}+2j_{38}+j_{34}+j_{57}+j_{58}+j_{68}+j_{79})}
\]

\[
\begin{cases}
1 & j_{38} + c \\
j_{68} & j_{58} \\
j_{68} & j_{58}
\end{cases}
\begin{cases}
1 & j_{38} + c \\
j_{34} & j_{23} \\
j_{12} & j_{27}
\end{cases}
\begin{cases}
1 & j_{23} + a \\
j_{23} & j_{23} + a \\
j_{12} & j_{27}
\end{cases}
\begin{cases}
1 & j_{27} + b \\
j_{79} & j_{57} \\
j_{79} & j_{57}
\end{cases}
\]

(46)

where \((a, b, c) = -1, 0, +1\). Then, from \[39\] and \[45\] we get

\[
\hat{H}_{(AD)} P^{\{j_e\}_{4}}_{\text{sim}} = N_{j_{58}} N_{j_{57}} \left( A_{0}(j_{23}, j_{27}, j_{38} \ldots j_{89}) \ P^{\{j_e\}_{4}}_{\text{sim}} + \sum_{(a,b,c)=(-,-,-)} A_{a,b,c}(j_{23} + a, j_{27} + b, j_{38} + c, \ldots j_{89}) P^{\{j_{23}+a,j_{27}+b,j_{38}+c,j_{e}\}_{4}}_{\text{sim}} \right) = 0.
\]

(47)

Finally, if we define the linear combination of spin-networks

\[
\Psi(g_{01} \ldots g_{89}) = \sum_{all \ j's} \prod_{i<j} d_{j_{ij}} \ \psi(j_{01} \ldots j_{89}) \ P^{\{j_{ij}\}_{4}}_{\text{sim}}(g_{01} \ldots g_{89}),
\]

then the action of the new operator \(\hat{H}_{(AD)} \ \Psi(g_{01} \ldots g_{89}) = 0\) imposes the following identity for the 15j-symbol:

\[
A_{0}(j) \ \psi(j_{23}, j_{27}, j_{38} \ldots j_{89}) + \sum_{(a,b,c)=(-,-,-)} A_{a,b,c}(j) \ \psi(j_{23} + a, j_{27} + b, j_{38} + c, \ldots j_{89}) = 0.
\]

(49)

Here we derived only one possible identity for the 15j-symbol. Obviously we could apply the same method to obtain many more identities just shifting around the grasping operator in different equivalent configurations. For instance we could have chosen to
grasp over two different faces as we did for the tetrahedron in sec. 2. However the scope of this paper is only to illustrate the method, not to list all possible identities for a particular \( nj \)-symbol.

Yet again, we want to stress that the derivation of the new identity can be considered itself a proof of it. Indeed the fact that we imposed the triviality of the holonomy implied the identity in equation (38), which, as we showed, implied the resulting identity in (49).

4. DISCUSSION

In this paper we described a method to derive recursion relations on Wigner \( nj \)-symbols; in particular we derived and proved new identities for the \( 6j \) and \( 15j \). Our method consists in a simple and elegant graphical way to describe the action of a certain \textit{grasping} operator on different nodes of a spin-network. More precisely we assumed flatness on the dual faces of an \( n \)-simplex spin-network and used such condition to “move” around a grasping operator to form equivalent configurations. We then showed that equating these equivalent states implies an identity for the relative \( nj \)-symbol.

The inspiration for this work came from the paper in [11]. In this paper Freidel et al. define a quantum \textit{Hamiltonian} operator in the context of \( 2 + 1 \) spin foam quantum gravity, then prove that a common Angular Momentum identity for the \( 6j \)-symbol called Biedenharn-Elliot identity can be seen a consequence of imposing flatness on the tetrahedron. Such identity plays a key role in three-dimensional Spin-Foam quantum gravity to prove triangulation independence of the theory.

In section 1.3 we showed how the results of the paper [11] can be envisioned in a simple and elegant manner through the adoption of our graphical method. We saw that the BE identity is the result of applying the grasping operator on a node of a flat tetrahedron and moving it alongside the edges of the same face. This lead us to the formulation of a \textit{new} identity, in sec. 2, for the \( 6j \)-symbol. To obtain this new identity we grasped two different nodes and moved the grasping along the edges of the different faces.

Furthermore, in sec. 3 we employed the same graphical method to derive an identity for the \( 15j \)-symbol. Again, imposing flatness on the dual faces, we obtained an identity grasping on a node and sliding the grasping operator along the edges of a face. The result, presented at the end of sec. 3.1.3 is strikingly simple and can be considered the four-dimensional equivalent of the BE identity.

5. ACKNOWLEDGMENTS

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Appendix A. Recoupling theory compendium

We present a short summary of the graphical notation we used for angular momentum theory. For further details we refer to [16].

- The Wigner $3m_j$ – Symbol is represented as a trivalent node. It is assumed that the momenta have to be read in a counter-clockwise manner, unless the node is marked with a minus sign:

  \[
  \begin{array}{c}
  a \\
  |
  \end{array} \quad b
  \quad a \quad b \\
  c
  = (-1)^{a+b+c}, \quad (A.1)
  \]

  \[
  \begin{pmatrix}
  a & c & b \\
  \alpha & \gamma & \beta
  \end{pmatrix}
  = (-1)^{a+b+c} \begin{pmatrix}
  a & b & b \\
  \alpha & \beta & \gamma
  \end{pmatrix}. \quad (A.2)
  \]

- A straight line represents the Kronecker delta

  \[
  \delta_{ab\delta_{\alpha\beta}} = \quad (A.3)
  \]

- The Anti-symmetric tensor is represented as a line marked with an arrow:

  \[
  (-1)^{a+\alpha} \delta_{ab\delta_{\alpha-\beta}} = \quad j \\
  \]

  Flipping the arrow produces a phase

  \[
  \quad j \\
  = (-1)^{2j} \quad j \\
  \quad (A.5)
  \]

- We remind the formula for the 6-j symbol:

  \[
  \begin{pmatrix}
  a_1 & b_1 & c_1 \\
  a_2 & b_2 & c_2
  \end{pmatrix} = \sum_{a_3, b_3} (-1)^{(j_3 + j_2 + j_1 - a_3 - a_2 - a_4)} \begin{pmatrix}
  a_1 & a_2 & a_3 \\
  a_4 & a_5 & a_6
  \end{pmatrix} \begin{pmatrix}
  j_1 & j_2 & j_3 \\
  j_4 & j_5 & j_6
  \end{pmatrix} \begin{pmatrix}
  j_1 & j_2 & j_3 \\
  j_4 & j_5 & j_6
  \end{pmatrix} \begin{pmatrix}
  j_1 & j_2 & j_3 \\
  j_4 & j_5 & j_6
  \end{pmatrix}. \quad (A.6)
  \]

- When the internal lines (momenta) of the graph are combined to “close” a triangle, one can use the following relation to extract a $\{6j\}$ (we follow conventions of [16]):
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Figure A1. Tetrahedron in graphical notation

\[
\sum_{\alpha, \beta, \gamma, \delta, \epsilon, \phi} (-1)^{d + e + f - \delta - \epsilon - \phi} \left( \begin{array}{ccc}
    e & d & a \\
    -\epsilon & \delta & \alpha \\
    \phi & \beta & -\delta
\end{array} \right) \left( \begin{array}{ccc}
    f & b & d \\
    \epsilon & \gamma & -\phi
\end{array} \right) \left( \begin{array}{ccc}
    e & c & f \\
    -\epsilon & \delta & \alpha
\end{array} \right)
\]

Figure A2. Recoupling closed triangle.

\[
\sum_{\alpha, \beta, \gamma, \delta, \epsilon, \phi} (-1)^{d + e + f - \delta - \epsilon - \phi} \left( \begin{array}{ccc}
    e & d & a \\
    -\epsilon & \delta & \alpha \\
    \phi & \beta & -\delta
\end{array} \right) \left( \begin{array}{ccc}
    f & b & d \\
    \epsilon & \gamma & -\phi
\end{array} \right) \left( \begin{array}{ccc}
    e & c & f \\
    -\epsilon & \delta & \alpha
\end{array} \right)
\]

\[
\left\{ \begin{array}{ccc}
    c & b & a \\
    d & e & f
\end{array} \right\} \left( \begin{array}{ccc}
    a & c & b \\
    \alpha & \gamma & \beta
\end{array} \right) (-1)^{a + b + c}
\]

Two parallel lines carrying the same group element \( g \) can be “re-coupled” as in figure A3

\[
\langle j_1, a_1 | g | j_1, b_1 \rangle \langle j_2, a_2 | g | j_2, b_2 \rangle = (-1)^{j_1 + j_2} \sum_{J = |j_1 - j_2|} d_{j_1}(-1)^{J - A} (-1)^{j_2 - b_2 + j_1 - b_1} \times
\]

\[
\left( \begin{array}{ccc}
    j_2 & j_1 & J \\
    a_2 & a_1 & -A_1
\end{array} \right) \left( \begin{array}{ccc}
    j_2 & j_1 & J \\
    -b_2 & -b_1 & B_1
\end{array} \right) \langle J, A | g | J, B \rangle.
\]

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