On the number of connected components in complements to arrangements of submanifolds

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Abstract

We consider arrangements of \( n \) connected codimensional one submanifolds in closed \( d \)–dimensional manifold \( M \). Let \( f \) be the number of connected components of the complement in \( M \) to the union of submanifolds. We prove the sharp lower bound for \( f \) via \( n \) and homology group \( H_{d-1}(M) \). The sets of all possible \( f \) – values for given \( n \) are studied for hyperplane arrangements in real projective spaces and for subtori arrangements in \( d \) – dimensional tori.

Introduction

The theory of plane arrangements in affine or projective spaces has been investigated rather thoroughly, see the book of P. Orlic, H. Terao \[A\] and V. A. Vassiliev’s review \[8\]. Inspired by a conjecture of B. Grünbaum \[2\], N. Martinov \[3\] found all possible pairs \((n, f)\) such that there is a real projective plane arrangement of \( n \) pseudolines and \( f \) regions. It turns out, that some facts concerning arrangements of hyperplanes or oriented matroids could be generalized to arrangements of submanifolds, see P. Deshpande dissertation \[1\]. So we are going to study the sets \( F(M, n) \) of connected components numbers of the complements in the closed manifold \( M \) to the unions of \( n \) closed connected codimensional one submanifolds. Sometimes it seems reasonable to restrict the type of submanifolds, for example, author \[7\] found sets \( F(M, n) \) of region numbers in arrangements of \( n \) closed geodesics in the two dimensional torus and the Klein bottle with locally flat metrics.

Homological bound of the number of connected components

Let \( M^n \) be connected \( n \)–dimensional smooth compact manifold without boundary, let \( A_i \subset M^n \) be distinct connected \( (n-1) \)–dimensional closed submanifolds in \( M^n \) for \( 1 \leq i \leq k \). Let us consider the union

\[
A = \bigcup_{i=1}^{k} A_i.
\]

We shall denote by \( f \) the number \( |\pi_0(M^n \setminus A)| \) of connected components of the complement to \( A \) in \( M^n \). Let \( UA \) be regular open neighbourhood of \( A \) in \( M^n \). Let

\[
M^n \setminus UA \cong \bigcup_{j=1}^{f} N_j,
\]

where \( N_j \) are the connected components of the complement to \( UA \) in \( M^n \). If \( M^n \) and all submanifolds \( A_i \) are orientable, then we assume \( G = \mathbb{Z} \). If some \( A_i \) or \( M^n \) is not orientable, then \( G = \mathbb{Z}_2 \).

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Lemma 1. If closed \((n-1)\) – dimensional submanifolds \(A_i \subset M^n\), \(i = 1, \ldots, k\) intersect each other transversally, then
\[
\dim H_{n-1}(UA, G) = \dim H_{n-1}(A, G) \geq k
\]

Proof. The regular neighbourhood of \(UA\) is homotopically equivalent to \(A\) and so all homology groups of \(A\) and \(UA\) are the same. By induction on \(k\) let us prove
\[
\dim H_{n-1}\left(\bigcup_{i=1}^{k} A_i, G\right) \geq k.
\]
It is obvious for \(k = 1\) because for connected closed \((n-1)\) – dimensional manifold \(H_{n-1}(A_1, G) \cong G\).

Suppose the statement is true for \(k - 1\) submanifolds and let us prove it for \(k\) submanifolds. Let
\[
A' = \bigcup_{i=1}^{k-1} A_i.
\]
Then by induction assumption
\[
\dim H_{n-1}(A', G) \geq k - 1.
\]
By Meyer-Vietoris exact sequence for pair \(A', A_k\) we have:
\[
\rightarrow H_{n-1}(A' \cap A_k) \rightarrow H_{n-1}(A') \oplus H_{n-1}(A_k) \rightarrow H_{n-1}(A' \cup A_k) \rightarrow
\]
As submanifolds \(A_i\) and \(A_j\) intersect transversally then \(A' \cap A_k\) is a finite union of at most \((n-2)\) – dimensional submanifolds in \(M^n\). Hence \(H_{n-1}(A' \cap A_k) = 0\) and the map
\[
H_{n-1}(A') \oplus H_{n-1}(A_k) \rightarrow H_{n-1}(A' \cup A_k)
\]
is monomorphic. Therefore,
\[
\dim H_{n-1}(A' \cup A_k) \geq \dim H_{n-1}(A') + \dim H_{n-1}(A_k) \geq k.
\]
\[\square\]

Lemma 2. \[H_n(M^n, UA, G) \cong G^f\]

Proof.
\[
H_n(M^n, UA, G) = \widetilde{H}_n(M^n/UA, G) = \\
= \widetilde{H}_n\left(\sqcup_{j=1}^{f} N_j / \sqcup_{j=1}^{f} \partial N_j, \ G\right) = \widetilde{H}_n\left(\bigvee_{j=1}^{f} N_j / \partial N_j, \ G\right) = \\
= \bigoplus_{j=1}^{f} \widetilde{H}_n(N_j/\partial N_j, G) = G^f,
\]
where \(n \geq 1\), \(\bigvee\) is one point union, \(\widetilde{H}_n\) is the reduced homology group, \(\partial N_j\) is the boundary of \(N_j\).
\[\square\]

Theorem 1. Let \(A_1, \ldots, A_k\) be connected closed codimensional one submanifolds in a connected closed \(n\) – dimensional manifold \(M^n\). Suppose that the submanifolds \(A_i\) intersect each other transversally and \(A = \bigcup_i A_i\). Then
\[
|\pi_0(M^n \setminus A)| \geq k + 1 - \dim H_{n-1}(M^n, G),
\]
where \(G\) is chosen as before.
Proof. Let us write the exact homological pair sequence for inclusion $i : UA \rightarrow M^n$ with coefficients in $G$:

$$H_n(UA) \rightarrow H_n(M^n) \rightarrow H_n(M^n, UA) \rightarrow H_{n-1}(UA) \rightarrow H_{n-1}(M^n) \rightarrow$$

Notice that

$$H_n(UA) = H_n(A) = 0, \quad H_n(M_n) = G.$$

It follows from the exactness of sequence in $H_n(M^n)$, that the map

$$H_n(M^n) \rightarrow H_n(M^n, UA)$$

is monomorphic. By lemma 2

$$H_n(M^n, UA, G) \cong G^f$$

One can see that

$$\dim H_{n-1}(UA) \leq \dim \operatorname{Im} \partial_* + \dim \operatorname{Im} i_*,$$

where the homomorphisms are

$$\partial_* : H_n(M^n, UA) \rightarrow H_{n-1}(UA), \quad i_* : H_{n-1}(UA) \rightarrow H_{n-1}(M^n).$$

Notice that

$$\dim \operatorname{Im} i_* \leq \dim H_{n-1}(M^n), \quad \dim \operatorname{Im} \partial_* \leq f - 1.$$

By lemma \[\square\] \[\dim H_{n-1}(UA) \geq k\] and so $k \leq f - 1 + \dim H_{n-1}(M^n)$. \[\square\]

Remark 1. One can see that the inequality of the theorem is sharp for arrangements of

$$n \geq \dim H_{n-1}(M^n)$$

submanifolds in projective spaces, spheres, $n$ – dimensional tori and Riemann surfaces of genus $g$.

**Toric arrangements**

**Definition 1.** By a flat $d$ – dimensional torus $T^d$ we mean a quotient of affine $d$ – dimensional space by a nondegenerate $d$ – lattice $\mathbb{Z}^d$ (which is not surely integer lattice). A codimensional one subtorus is given by equation

$$\sum_i a_i x_i = c,$$

where $a_i$ are rational, $x_i$ are coordinates of $\mathbb{R}^d$ in some lattice basis, $c$ is real.

A codimensional one subtorus is closed submanifold of $T^d$ homeomorphic to $(d-1)$ – dimensional torus. Let $A$ be the union of $n$ codimensional one subtori in the flat $d$ – dimensional torus $T^d$. Consider the connected components of the complement $T^d \setminus A$; denote the number of connected components by $f = |\pi_0(T^d \setminus A)|$; let $F(T^d, n)$ be the set of all possible numbers $f$.

**Theorem 2.** For $n > d$

$$F(T^d, n) \supseteq \{n - d + 1, \ldots, n\} \cup \{l \in \mathbb{N} \mid l \geq 2(n - d)\}.$$  

For $2 \leq n \leq d$ we have $F(T^d, n) = \mathbb{N}$.  

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Proof. Let $T^d = \mathbb{R}^d / \mathbb{Z}^d$ and $e$ be the basis of $\mathbb{Z}^d$. Let $(x_1, \ldots, x_d)$ be the coordinates of $\mathbb{R}^d$ in the basis $e$. We shall construct examples for $\leq n$ and $\geq 2n - 2d$ regions separately.

Let us consider $n$ hyperplanes in $\mathbb{R}^d$ (an equation corresponds to a hyperplane):

$$ x_i = 0, \quad 1 \leq i \leq k, $$

$$ x_{k+1} = c_{i-k}, \quad k + 1 \leq i \leq n $$

for some integer $k$, $0 \leq k \leq d - 1$ and real $c_{i-k}$ with different fractional parts. By the factorization map $\mathbb{R}^d \to \mathbb{R}^d / \mathbb{Z}^d$ we shall get a set $\{T^{d-1}_i, i = 1, \ldots, n\}$ of $n$ codimensional one subtori. And the complement is homeomorphic to the prime product

$$ T^d \setminus \bigcup_i T^{d-1}_i \approx \mathbb{R}^k \times \left(S^1 \setminus \{p_1, \ldots, p_{n-k}\}\right) \times \left(S^1\right)^{d-k-1}, $$

where $S^1 \setminus \{p_1, \ldots, p_{n-k}\}$ denotes a circle without $n - k$ points. Hence the number of complement regions equals $n - k$, for an integer $k$ such that $0 \leq k \leq d - 1$.

Now let us take integer nonnegative $k$ and construct an arrangement with $2n - 2d + k$ connected components of the complement. We shall determine the subtori by equations:

$$ x_i = 0, \quad \text{for} \quad 2 \leq i \leq d, $$

$$ x_2 = kx_1 + \frac{1}{2}, $$

$$ x_1 = c_j \quad \text{for} \quad j = 1, \ldots, n - d, $$

whereas numbers $kc_j + \frac{1}{2}$ are not integer for any $j$. (This means that the intersection of three subtori

$$ x_2 = kx_1 + \frac{1}{2}, \quad x_1 = c_j, \quad x_2 = 0 $$

is an empty set.) One may see that

$$ T^d \setminus \bigcup_{i=3}^d \{x_i = 0\} \approx T^2 \times \mathbb{R}^{d-2}. $$

In the two-dimensional torus the equations

$$ x_2 = 0, $$

$$ x_2 = kx_1 + \frac{1}{2}, $$

$$ x_1 = c_j \quad \text{for} \quad j = 1, \ldots, n - d $$

produce the arrangement of $n - d + 2$ closed geodesics. The geodesics’ union divides the torus into $2n - 2d + k$ connected components (for more details on arrangements of closed geodesics in the flat torus see author’s paper [7]).

□

Conjecture 1. It seems believable that the inclusion in the theorem is indeed the equality for all $d \geq 2$ and $n \geq d$. Yet the equality is proved for $d = 2$ in [7].

Sets of region’s numbers in hyperplane arrangements

By an arrangement of $n$ hyperplanes in the real projective space $\mathbb{RP}^d$ we mean a set of $n$ hyperplanes, such that there are no point belonging to all the hyperplanes. The arrangement produce the cell decomposition of the $\mathbb{RP}^d$, let $f$ denotes the number of open $d$–cells. Let $F_n^{(d)}$ denotes the set of all possible numbers $f$ arising in arrangements of $n$ hyperplanes in $\mathbb{RP}^d$. Let $m$ be the maximal number of hyperplanes, passing through one point.
Lemma 3. For arrangements of \( n \) hyperplanes in \( \mathbb{R}P^d \) we have

\[
f \geq (m - d + 1) \sum_{j=0}^{\left\lfloor \frac{d}{2} \right\rfloor} C_{m-2j}^d C_{m-2j}^{d-2j}.
\]

Proof. It follows from Zaslavsky formula for number of regions and some inequalities concerning the Möbius function of the arrangement poset. \( \square \)

Lemma 4. For arrangement of \( n \) hyperplanes in the real projective space \( \mathbb{R}P^d \)

\[
f \geq (n - m + 1)(m - d + 2)2^{d-2}.
\]

Proof. Let \( m \) hyperplanes \( A_1, \ldots, A_m \) have nonempty intersection \( Q \) (\( Q \) is a point). The family \( A_1, \ldots, A_m \) is a cone over some arrangement \( B \) of \( m \) planes in \( \mathbb{R}P^{d-1} \). The number \( f(B) \) of regions in arrangement \( B \) could be estimated (see Shannon paper [5], where this result is referred to McMullen) as:

\[
f(B) \geq (m - d + 2)2^{d-2}.
\]

Each of the remaining hyperplane of the former arrangement intersects the family \( A_1, \ldots, A_k \) by an arrangement \( B_i \), projective equivalent to \( B \). Thus

\[
f \geq f(B) + \sum_i f(B_i) = (n - m + 1)f(B).
\]

\( \square \)

Theorem 3. Let \( d \geq 3 \) and \( n \geq 2d + 5 \). Then the first four increasing numbers of \( F_n^{(d)} \) are the following:

\[
(n - d + 1)2^{d-1}, \quad 3(n - d)2^{d-2}, \quad (3n - 3d + 1)2^{d-2}, \quad 7(n - d)2^{d-3}.
\]

Proof. We are going to prove that the four mentioned numbers are the only realizable ones among numbers not greater than \( 7(n - d)2^{d-3} \). After it one may see how to construct examples of arrangements with required numbers \( f \). Let us prove that if \( m \leq d + 1 \), then

\[
f \geq 7(n - d)2^{d-3}.
\]

For \( m = d \) we have an arrangement of hyperplanes in general position and the number of regions is the largest possible. If \( m = d + 1 \), then by lemma 3 we have

\[
f \geq \frac{C_n^{d+1}}{n - d} = \frac{n(n - 1)}{3} \cdot \frac{(n - 2)}{d} \cdot \frac{(n - 3)}{d - 1} \cdot \frac{m}{4} \cdot \frac{m}{n - d + 1} \geq 7 \cdot 2^{d-3}(n - d)
\]

because \( n \geq 2d + 5 \).

Now we prove the theorem for \( d = 3, n \geq 11 \). Let us consider three cases.

1. If \( m = n - 1 \), then \( f = 2\varphi \), where \( \varphi \in F_{n-1}^{(2)} \). The set \( F_{n-1}^{(2)} \) is known due to N. Martinov [3]

\[
\{ f \in F_{n-1}^{(2)} \mid f \leq 4n - 16 \} = \{ 2n - 4, 3n - 9, 3n - 8, 4n - 16 \}.
\]

2. \( m = n - 2 \). The arguments are the same as in the inductive step further (Martinov theorem [3] for the set \( F_n^{(2)} \) is also used).

3. If \( 5 \leq m \leq n - 3 \), then by using lemma 4 we have

\[
f \geq 2(n - m + 1)(m - 1) \geq 8n - 32 \geq 7n - 21
\]

for \( n \geq 11 \).
Now we use induction on $d \geq 3$. Base is the validity of the theorem for $d = 3$. The assumption is the validity of the theorem for all integers $3 \leq d' < d$ and $n' \geq 2d' + 5$. To prove the induction step we shall consider three cases.

1. If $m = n - 1$, then $f = 2\varphi$, where $\varphi \in F_{n-1}^{(d-1)}$. By induction assumption for the set $F_{n-1}^{(d-1)}$ (note that $n - 1 \geq 2(d - 1) + 1$) we get that either $\varphi$ is equal to one of four numbers

\[(n - d + 1)2^{d-2}, \quad 3(n - d)2^{d-3}, \quad (3n - 3d + 1)2^{d-3}, \quad 7(n - d)2^{d-4},\]

or $\varphi > 7(n - d)2^{d-4}$.

2. $m = n - 2$. Consider $n - 2$ hyperplanes $p_1, \ldots, p_{n-2}$, passing through one point. These hyperplanes cut $\mathbb{RP}^d$ into $\varphi$ regions and $\varphi \in F_{n-2}^{(d-1)}$. Let $l$ denote the intersection of the two remaining hyperplanes. By the inductive assumption we have either

$$\varphi = (n - d)2^{d-2} \quad \text{or} \quad \varphi \geq 3(n - d - 1)2^{d-3}$$

(note that assumption may be used as $n - 2 \geq 2(d - 1) + 5$). If

$$l \in \bigcup_{i=1}^{n-2} p_i$$

then $f = 3\varphi$ and the case is over. If

$$l \notin \bigcup_{i=1}^{n-2} p_i$$

then let $B$ be the set of planes $p_i \cap l$ in the $l$, where $l$ is regarded as the ambient $(d - 2)$ - dimensional projective space. One may prove, that $B$ is an arrangement of at least $n - 3$ planes in $l$. Then $f(B) \geq (n - d)2^{d-3}$ by Shannon theorem \[5\]. Since

$$f = 3\varphi + f(B) \geq 7(n - d)2d - 3,$$

the case is over.

3. If $d + 2 \leq m \leq n - 3$ then by lemma \[4\] we have

$$f \geq (n - m + 1)(m - d + 2)2^{d-2} \geq (4n - 4d - 4)2^{d-2} \geq 7(n - d)2^{d-3}$$

for $n \geq d + 8$. \hfill \Box

**Lemma 5.** For arrangement of $n$ hyperplanes in the real projective space $\mathbb{RP}^d$

$$f \geq 2 \frac{n^2 - n}{m - d + 5}.$$

*Proof.* It follows from the similar inequality for arrangement of lines in the projective plane, see details in \[2\]. \hfill \Box

**Theorem 4.** First 36 increasing numbers of the set $F_{n}^{(3)}$ for $n \geq 50$ are the following (i.e. all realizable numbers up to $12n - 60$)

\[
\begin{align*}
4n - 8, & \quad 6n - 18, \quad 6n - 16, \quad 7n - 21, \quad 7n - 20, \quad 8n - 32, \quad 8n - 30, \quad 8n - 28, \\
8n - 26, & \quad 9n - 36, \quad 9n - 33, \quad 9n - 31, \quad 9n - 30, \quad 10n - 50, \quad 10n - 48, \quad 10n - 46, \\
10n - 44, & \quad 10n - 42, \quad 10n - 40, \quad 10n - 39, \quad 10n - 38, \quad 10n - 37, \quad 10n - 36, \quad 10n - 35, \\
11n - 44, & \quad 11n - 43, \quad 11n - 42, \quad 11n - 41, \quad 11n - 40, \quad 12n - 72, \quad 12n - 70, \quad 12n - 68, \\
12n - 66, & \quad 12n - 64, \quad 12n - 62, \quad 12n - 60. 
\end{align*}
\]
Proof. Let \( m \) be the maximal number of hyperplanes, passing through one point. Examples for this numbers could be constructed for arrangements with \( m \geq n - 5 \). Let us prove that there are no other realizable numbers, smaller then \( 12n - 60 \). Consider three cases.

1. If \( m \geq n - 5 \), then by enumeration of possibilities we have that either \( f \) belongs to given set or \( f \geq 12n - 60 \).
2. If \( 8 \leq m \leq n - 6 \), then by lemma 4 we have \( f \geq 7n - 49 \).
3. If \( m \leq 7 \) then by lemma 5

\[
f \geq 2 \frac{n^2 - n}{9} \geq 12n - 60
\]

for \( n \geq 50 \). \( \square \)

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