Abstract—We model the buildup of nonlinear interference noise (NLIN) in wavelength division multiplexed systems by considering the pulse collision dynamics in the time domain. The fundamental interactions can be classified as two-pulse, three-pulse, or four-pulse collisions and they can be either complete, or incomplete. Each type of collision is shown to have its unique signature and the overall nature of NLIN is determined by the relative importance of the various classes of pulse collisions in a given WDM system. The pulse-collision picture provides qualitative and quantitative insight into the character of NLIN, offering a simple and intuitive explanation to all of the reported and previously unexplained phenomena.

Index Terms—fiber nonlinearity, NLIN, inter channel interference, XPM, pulse collision, dependence on modulation format, phase noise.

I. INTRODUCTION

The nonlinearity of optical fibers has been long recognized as one of the most important factors limiting the growth of data-rates transmitted in wavelength division multiplexed (WDM) systems [1]. The nonlinear distortions are usually classified as either intra-channel [2], or inter-channel [3]. While intra-channel distortions are generated by each of the WDM channels individually, inter-channel distortions are caused by interference between different WDM channels. In this work we focus on the physical mechanism responsible for inter-channel nonlinear effects as it comes into play in modern, dispersion uncompensated fiber links with coherent detection. Since joint processing is currently considered to be prohibitively complex in commercial systems, inter-channel interference is customarily treated as noise [1], and hence we refer to it in what follows as nonlinear interference noise, or NLIN [4]–[9].

We follow up on the theory published in [4]–[6], [10] and model the build-up of NLIN in the time domain. In this approach, the NLIN is attributed to multiple pulse collisions whose dynamics we examine in detail. As we demonstrate in what follows, the pulse-collision picture provides deep qualitative and quantitative insight into the buildup of NLIN, and explains the various properties of NLIN that have previously only been observed empirically in simulations. For example, issues like the importance of the phase-noise component of the NLIN versus the legitimacy of describing NLIN as complex circular noise, the dependence of the NLIN power on modulation format, the effect of pre-dispersion, the dependence on system-length and type of amplification, etc. [4]–[6], [11]–[13], are all clarified once the essence of pulse collision dynamics is understood.

While the practice of analyzing nonlinear interference by means of pulse collisions has been known for a while (initially in the context of soliton [14] and dispersion-managed-soliton [15], [16] transmission, and subsequently in the more general context of arbitrary pulse interactions [17]), the pulse collision dynamics encountered in modern coherent fiber communication systems, has a number of fundamentally different characteristics. Most prominently, in modern systems, which avoid the use of inline dispersion compensation, simultaneous multi-pulse collisions play a unique and very significant role. In order to illustrate this, consider the case where the channel of interest is perturbed by a nonlinear interaction with a single interfering WDM channel. In the old generation of dispersion managed systems [15]–[17], interference was generated by collisions between pairs of pulses, one from the channel of interest and one from the interfering channel. In the new generation of dispersion uncompensated systems, pulse spreading generates significant temporal overlap within each of the channels, allowing for example, situations in which two different pulses in the interfering channel interact nonlinearly with a third pulse, belonging to the channel of interest, so as to generate interference that affects a fourth pulse, also belonging to the channel of interest.

In what follows we demonstrate that substantial understanding of a broad range of observed nonlinear interference phenomena can be achieved by means of proper classification of the type of collisions taking place in the fiber. Most relevantly, there is a qualitative and a quantitative difference between the interference caused by two-pulse, three-pulse and four-pulse interactions. Additionally, the effect of complete collisions (where interacting pulses that belong to different WDM channels pass by each other completely within the fiber span) may differ considerably from the effect of incomplete collisions. In the case of two-pulse collisions, nonlinear interference always manifests itself as phase-noise, and it is strongly dependent on modulation format. Among the three-pulse collisions some distinctly contribute to phase noise, and are independent of modulation format, whereas others produce complex circular noise, whose variance is modulation format dependent. Finally, all four-pulse collisions produce complex circular noise whose power is independent of modulation format. While the NLIN produced by two-pulse collisions grows monotonically during the collision process and is maximized when the collision is complete, the behavior of three and four-pulse collisions is distinctly different. The NLIN generated by these collisions is built up constructively in the first half of the collision process and then most of it cancels through destructive interference.
when the collision is completed. Hence the effect of three and four-pulse collision ends up being much more significant when the collision is incomplete. The relative importance of complete versus incomplete collisions and therefore the relative importance of two, three and four-pulse collisions, in a given system is determined by the link parameters, predominantly by the length and number of amplified spans, and by the type of amplification that is used. For example, with lumped amplification complete collisions are most significant in short, few-span systems, whereas in the case of distributed amplification (which is the limiting case of Raman amplification) the contribution of complete collisions is dominant regardless of the link length. In links dominated by complete collisions, two-pulse collisions are more significant and the NLIN has a distinct phase-noise-like nature and it is characterized by a strong dependence on modulation format. As the significance of incomplete collisions increases, four-pulse collisions become more significant so that the NLIN progressively evolves into complex circular noise, and its dependence on modulation format reduces.

Our analysis in what follows focuses only on nonlinear interference between two WDM channels, one of which is referred to as the channel of interest, and the other is referred to as the interfering channel. In the traditional jargon of WDM systems, this would qualify as studying only the effect of cross-phase-modulation (XPM), and omitting inter-channel four wave mixing (FWM) processes in which three or four different WDM channels are involved. As is evident from simulations, the contribution of FWM to NLIN in most relevant fiber types is of negligible importance. Under these circumstances, the NLIN that is generated by multiple WDM channels is simply the sum of the NLIN contributions produced by the individual interferers.

The paper is organized as follows. In Sec. II we briefly review the essentials of the time domain model, initially introduced in [4], [6], [10] and establish the background for the subsequent analysis. In Secs. III [V] and V we classify the nonlinear interference processes as two, three and four-pulse collisions, respectively. In each case we describe their distinctive features depending on whether the collision is complete, or incomplete. Several numerical examples illustrating the main ideas are presented in Sec. VI whereas the implications to fiber-communications systems are discussed in Sec. VII. In Sec. VIII we explain the role of chromatic dispersion and relate the time domain theory to the Gaussian noise (GN) model [20]–[24]. Section IX discusses the generalization of the presented principles to the case where polarization multiplexing is explicitly taken into account, and finally, Sec. X is devoted to conclusions.

II. TIME-DOMAIN THEORY

In the case of two WDM channels, we may express the electric field at the fiber input as

$$\sum_{n} a_n g(0, t - nT) + \exp(-i\Omega t) \sum_{n} b_n g(0, t - nT),$$  

where $T$ is the symbol duration and $a_n$ and $b_n$ are the complex valued data symbols transmitted over the channel of interest and over the interfering channel, respectively. The central frequency of the channel of interest is arbitrarily set to 0, whereas the central frequency of the interfering channel is denoted by $\Omega$. The injected fundamental symbol waveforms in both channels are identical and denoted by $g(0, t)$, whereas $g(z, t) = \exp(-i\beta''z \frac{d}{dt^2})g(0, t)$ is the dispersed waveform of the individual pulse when reaching point $z$ along the fiber with $\beta''$ being the fiber dispersion coefficient. The fundamental waveforms are further assumed to be orthogonal with respect to time shifts by an integer number of symbol durations and their energy is assumed to be normalized to 1

$$\int g^*(z, t - nT)g(z, t - n'T)dt = \delta_{n,n'}. \quad (2)$$

After coherent detection, the channel of interest is matched-filtered with the filter’s impulse response being proportional to $g'(L, t)$, where $L$ is the link length. The extracted $n$-th data symbol can be then expressed as $a_n + \Delta a_n$, with $\Delta a_n$ accounting for the presence of NLIN. In [4] we have shown that $\Delta a_n$ due to inter-channel NLIN is given by

$$\Delta a_n = 2i\gamma \sum_{h,k,m} a_h b'_k b_m X_{h,k,m}. \quad (3)$$

where $\gamma$ is the nonlinear coefficient and where we have arbitrarily (and without loss of generality) set the index of the received symbol to 0. The coefficients $X_{h,k,m}$ have been introduced in Ref. [4], where they have been shown to be

$$X_{h,k,m} = \int_{0}^{L} dz \int_{0}^{\infty} dt' g^*(z, t - hT) \times g'(z, t' - kT - \beta''(z)\frac{d}{dt})g(z, t' - mT - \beta''(z)). \quad (4)$$

The function $f(z)$ accounts for the loss/gain profile along the link [4], [10]. For example, $f(z) = 1$ in the case of perfectly distributed amplification and $f(z) = \exp(-\alpha \mod(z, L_s))$ in the case of lumped amplification where $\alpha$ is the loss coefficient and $L_s$ is the span length.

The coefficient $X_{h,k,m}$ is associated with the NLIN that is observed in the measurement of the zeroth data symbol in the channel of interest following the nonlinear interaction between the $h$-th pulse in the channel of interest and the $k$-th and $m$-th pulses in the interfering channel, as illustrated in Fig. 1. The triple product $\xi(z, t) = g(z, t - hT)g(z, t' - kT - \beta''(z))g(z, t' - mT - \beta''(z))$, appearing in the integrand of Eq. (4), describes a classic nonlinear Kerr interaction taking place at time $t$ and position $z$ along the link (where $\beta''(z)$ is the difference between the reciprocal group velocities). This product then propagates towards the receiver, while accumulating the dispersion present in the remaining fiber-length, which is equal to $L - z$. Since the impulse response of the matched filter is proportional to $g'(L, t) = \exp(-i\beta''(L - z) \frac{d}{dt})g^*(z, t)$ (which appears as the ‘fourth’ pulse in the integrand of Eq. (4)), this accumulated dispersion is removed. The sampled perturbation at the receiver, as the inner integral of Eq. (4) suggests, is therefore $\int g^*(z, t)\xi(z, t)dt$, which can be interpreted as if the

1As we are relying on a first order perturbation analysis – the waveform $g(z, t)$ includes only the effect of dispersion and not the effect of nonlinearity. In [4] we used $g^{(0)}(z, t)$ for the same quantity.
The integral over \( z \) in Eq. (4) gathers the nonlinear products produced along the entire link.

We now proceed to classifying the various contributions to the NLIN in terms of the type of collisions that produce them. In the most general case, these contributions are produced by nonlinear interactions between four pulses – the “dummy” zeroth pulse from the channel of interest (accounting for the fact that the signal is matched filtered at the receiver), the \( h \)-th pulse from the channel of interest, and the \( k \)-th and \( m \)-th pulses from the interfering channel. When some of the indices coincide, as we shall see in the following sections, a two-pulse, or three-pulse collision is formed.

### III. Two-Pulse Collisions

A two-pulse collision occurs when one of the pulses is the pulse of interest, which we arbitrarily select as the zeroth-index pulse, whereas the other pulse belongs to the interfering channel, so that \( h = 0 \) and \( k = m \). The strength of this interaction is governed by the coefficient

\[
X_{0,m,m} = \int_{-\infty}^{L} \int_{-\infty}^{\infty} \left| g(z,t) \right|^2 \left| g(z,t-mT-\beta''\Omega z) \right|^2 dz
dt
\]

and, according to Eq. (3), it produces a perturbation equal to \( a_0 (2\gamma X_{0,m,m} |b_m|^2) \). As \( X_{0,m,m} \) is a real-valued quantity the perturbation is at a complex angle of \( \pi/2 \) from the transmitted symbol \( a_0 \). Namely, the perturbation generated by two-pulse interactions has the character of phase-noise.

An interesting perspective into this reality can be obtained by considering the ‘old class’ of two-pulse collisions initially studied in the context of solitons [14]–[16]. There, it has become common knowledge that when a pulse in the channel of interest undergoes a complete collision with an interfering pulse belonging to an adjacent WDM channel, the only notable consequence observed on the pulse of interest is a time-independent phase-shift. However, it is also well known that when the collision between the two pulses is not complete [16], [17], the phase of the interacting pulses is distorted via XPM leading to the formation of a time-dependent phase distortion (nonlinear chirp), which subsequently produces an amplitude (or timing-jitter) perturbation after being affected by the dispersion of the remaining fiber. Note that in the scheme that we consider here, no amplitude perturbation is formed in two-pulse collisions, regardless of whether the collisions are complete, or incomplete. This difference with respect to the old generation of systems can be attributed to coherent detection and the use of a matched filter, which is equivalent to measuring the perturbation at the point along the fiber where the collision takes place, as explained earlier. Namely, the effect of dispersion, which translates the time-dependent phase distortion into amplitude variations is reversed by the matched filter.

Since in addition to being real-valued, the integrand in Eq. (3) is also non-negative, the nonlinear phase-shift is proportional to the “completeness” of the collision, namely the phase-shift is zero when the waveforms do not overlap at all during propagation and it is maximized when the collision is complete. As we demonstrate in the appendix, two-pulse collision coefficients \( X_{0,m,m} \) scale as \( \Omega^{-1} \). In addition, the fact that two-pulse contributions to NLIN are proportional to \( a_0 |b_m|^2 \) implies that the variance of NLIN that is associated with these contributions is proportional to \( \langle |a_0|^2 \rangle (\langle |b_m|^4 \rangle - \langle |b_m|^2 \rangle^2) \), with the angled brackets denoting statistical averaging. While \( \langle |a_0|^2 \rangle \) is simply proportional to the average signal power \( \langle |b_m|^4 \rangle - \langle |b_m|^2 \rangle^2 \), is clearly dependent on modulation format. For example, when the interfering channel is modulated only in phase (e.g. QPSK) the NLIN variance caused by two-pulse collisions is 0.

### IV. Three-Pulse Collisions

Three-pulse interactions are represented by coefficients of the form \( X_{0,k,m} \) (where \( k \neq m \)) and \( X_{h,m,m} \) (where \( h \neq 0 \)). The first form represents an interaction between the pulse

\[
\begin{align*}
\text{Fig. 1. The generic four-pulse interference. The } h\text{-th pulse in the channel of interest interacts with the } k\text{-th and } m\text{-th pulses of the interfering channel so as to create NLIN that affects the zeroth symbol in the channel of interest. Since optimal detection involves filtering that is matched to the waveform of the fundamental pulse centered at the zeroth symbol, as discussed in Sec. II one can refer to this as a four-pulse interaction. The zeroth pulse is plotted only to keep track of the matched-filter waveform, and it is plotted by a dashed line in order to stress that it is not a pulse that is physically participating in the nonlinear interaction during propagation. The left panel shows the situation in the beginning of the link, before the pulses overlap, whereas the right panel illustrates the situation once an overlap between the pulses is formed. When two of the indices overlap (} h = 0 \text{, or } k = m \text{) one observes a three-pulse interaction and when both index pairs overlap (} h = 0 \text{ and } k = m \text{) one obtains a two-pulse interaction.}
\end{align*}
\]
Fig. 2. The evolution of the NLIN coefficients along the fiber axis $z$, where $z_c$ is the position of the collision center. The solid (blue) curves show the contribution from a single fiber increment $dz$ at position $z$ along the fiber to the NLIN component $i\gamma X_{h,k,m}$, this is proportional to the inner integral in Eq. (3), whereas the dashed (green) curves show the NLIN component $i\gamma X_{h,k,m}$ itself, under the assumption that the link is ended at point $z$. The two-pulse component $i\gamma X_{0,20,20}$ is imaginary and hence produces pure phase-noise. It grows monotonically with distance, reaching a maximum after the collision has taken place. The three and four-pulse components $i\gamma X_{1,20,20}$ and $i\gamma X_{1,21,20}$ are complex and their value is largest during the collision itself and reduces significantly when the collision is complete. The figure is plotted for the parameters of a standard single-mode fiber ($\beta = 2 \text{ ps}^2/\text{km}$, $\gamma = 1.3 \text{ W}^{-1} \text{ km}^{-1}$), square-root raised cosine pulses with a roll-off factor of 0.2, a pulse bandwidth of $B = 32 \text{ GHz}$, and a channel separation of $\Omega = 100 \text{ GHz}$ (similar behavior was observed with $\Omega = 50 \text{ GHz}$). Distributed amplification is assumed.

of interest (the zeroth pulse from the channel of interest) and the $k$-th and $m$-th pulses from the interfering channel. The perturbation produced by this interaction is equal to $i\alpha_0(2\gamma X_{0,k,m}b_k^*b_m + 2\gamma X_{0,m,k}b_m^*b_k)$ and since it can be easily verified that $X_{0,k,m} = X_{0,k,m}$, the perturbation is equal to $i\alpha_0\Re\{4\gamma X_{0,k,m}b_k^*b_m\}$ which is at a complex angle of $\pi/2$ from the transmitted symbol $a_0$ and therefore it too has the character of phase noise. Yet unlike in the case of two-pulse collisions, since $k \neq m$ the variance of this phase noise contribution depends only on the average signal power and not on the modulation format. The second form of three-pulse interactions $X_{h,m,m}$ involves an interaction between two pulses in the channel of interest (the zeroth and $h$-th pulses) and a single pulse from the interfering channel (the $m$-th pulse). The perturbation produced by this interaction is equal to $\Delta a_0 = i\alpha_h(2\gamma X_{h,m,m}b_m^2|b_m|^2)$. This contribution has no fixed phase relation with $a_0$ and hence it can be viewed as complex circular noise. Yet, similarly to the case of two-pulse collisions, the dependence on $|b_m|^2$ implies dependence of the NLIN variance due to this contribution on the data modulation format.

We show in Sec. VI and in the appendix that as in the case of three-pulse collisions the interference formed within the initial part of a three-pulse collision, but then cancels through destructive interference once the collision is complete, so that the residual magnitude of the perturbation scales as $\Omega^{-2}$. In the regime where the relative number of complete collisions is significant, the overall contribution of three-pulse collisions practically vanishes relative to that of two-pulse collisions (in spite of the fact that the number of three-pulse collisions is larger). The scaling of incomplete three-pulse collisions with frequency is not monotonous, and depends on the point in which the collision is discontinued.

V. FOUR-PULSE COLLISIONS

The case of four-pulse collisions involves two pulses from the channel of interest (the zeroth and $h$-th pulses, with $h \neq 0$) and two pulses from the interfering channel (the $k$-th and $m$-th pulses with $k \neq m$). Each of these interactions generates a perturbation that is equal to $2i\alpha_h(X_{h,k,m}b_k^2b_m + X_{h,m,k}b_m^2b_k)$ and since there is no fixed phase relation between the various symbols, these contribution produce complex circular noise. Since $h \neq 0$ and $k \neq m$, and the transmitted data on different symbols is assumed to be statistically independent, the average value of NLIN due to four-pulse contributions is 0, and the variance is proportional to $\langle|a_0|^2\rangle\langle|b_0|^2\rangle^2$, which is determined by the average power per symbol independently of the modulation format.

We show in Sec. VI and in the appendix that as in the case of three-pulse collisions the interference formed in the first half of a four-pulse collision process cancels through destructive interference in the second half. The magnitude of the residual interference left after a complete four-pulse collisions scales as $\Omega^{-3}$, which implies that the overall contribution of complete four-pulse collisions is negligible in spite of their much larger number (see appendix for details). With respect to incomplete collisions, four-pulse collisions produce a significant contribution because their number is much larger than that of incomplete two, or three pulse collisions, and because all types of incomplete collisions scale similarly with the frequency separation $\Omega$.

VI. A FEW NUMERICAL EXAMPLES

In order to better illustrate the distinctive features of the various types of collisions we examine the dependence of three particular NLIN coefficients on space and frequency
The range of corresponding blue curves from the link input up to point the NLIN coefficient, which is obtained by integrating the dashed (green) curves show the accumulated contribution to the range of positions in which the collision takes place. The non-negative and hence, the corresponding phase-perturbation the imaginary part of the incremental contributions is strictly argued earlier, the incremental contribution is imaginary (i.e. fiber increment of length \( \Omega = 100 \) GHz, channel separation of \( 85 \) GHz the group velocity differences between the channels are large enough to guarantee that the collision is completed before the fiber’s end, and hence the perturbation in the case of two, three, and four-pulse collisions scales as \( \Omega^{-1} \), \( \Omega^{-2} \), and \( \Omega^{-3} \), respectively. When the frequency separation reduces, the collision is no longer complete, as it is discontinued abruptly at the end of the fiber. In this regime the magnitude of the perturbation depends on the point in which the collision is disrupted and hence, consistently with the discussion of Fig. 2 the dependence on \( \Omega \) is only monotonous in the two-pulse collision case.

VII. SYSTEM IMPLICATIONS OF COLLISION CLASSIFICATION

The nature of NLIN in systems changes depending on whether it is dominated by complete or incomplete collisions. A collision can be safely characterized as complete when the evolution of the individual pulses during the collisions is negligible. This condition can be translated into two requirements. The first is that the dispersive broadening of the pulses during the collision is negligible relative to the pulse width prior to the collisions. This requirement is automatically satisfied when the frequency spacing between the interacting WDM channels is sufficiently larger than the bandwidth of the channels themselves. To see that, we define the collision length, \( L_c \), as the length of the section of fiber in which the collision takes place over the spectral width of the individual pulses, the temporal broadening that is caused by dispersion during the process of a collision can be approximated by \( \beta'' B L_c = \Delta t \frac{B}{\Omega^2} \), and it is certain to become much smaller than \( \Delta t \) when \( \Omega/B \) is large.

The second requirement is that the effects of attenuation or gain are negligible during a collision. In systems using lumped amplification this condition is not satisfied when the collision overlaps with an amplifier site, or when the collision length \( L_c \) is so long that attenuation during the collision cannot be neglected. In modern uncompensated fiber links, where \( \Delta t \) increases due to the accumulated dispersion, \( L_c \) is bound to reach values over which fiber loss becomes significant and the collision can no longer be considered complete. For this reason, truly complete collisions may exist in distributed

![Fig. 3. The magnitude of the nonlinear perturbation \(|\gamma X_{0,20,20}|\), \(|\gamma X_{1,20,20}|\), and \(|\gamma X_{2,20,20}|\) as a function of frequency separation for the case of \( L = 100 \) km and distributed amplification. It can be clearly observed that once the collision becomes complete (as can be deduced from the appendix, this happens when \( \Omega > 85 \) GHz), the two-pulse NLIN coefficient drops as \( \Omega^{-1} \), the three-pulse coefficient drops as \( \Omega^{-2} \) and the four-pulse coefficient drops as \( \Omega^{-3} \) (dashed lines show the scaling with \( \Omega^{-j} \) for \( j = 1, 2 \) and 3).](image)
amplification systems (as can be approximately achieved with Raman technology), or in the first few spans of systems using lumped amplification.

As can be concluded from the combinations of indices, the number of four-pulse collisions is the largest, whereas the smallest number of collisions are those involving only two pulses. Nevertheless, in a regime where complete collisions dominate, the nature of NLIN is dictated primarily by two-pulse collision processes, whose scaling with frequency separation is the most favorable (proportional to $\Omega^{-1}$). In this regime the NLIN is expected to have a strong phase-noise-like character and its variance should be strongly dependent on the modulation format. In the opposite case, where incomplete collisions dominate, the larger number of four-pulse collisions emphasizes their significance and makes the overall NLIN more similar to complex circular noise. These principles are illustrated in Fig. 4 where we show separately the relative aggregate contributions of two-pulse, three-pulse, and four-pulse collisions in the cases of distributed (Fig. 4a)) and lumped (Fig. 4b) amplification. The dominance of two-pulse collisions is clearly evident in the distributed amplification case, whereas in the case of lumped amplification two-pulse collisions are only dominant in few span systems (where a large fraction of the collisions are complete), whereas four-pulse collisions acquire a central role.

In order to illustrate these ideas further we express the received symbols as

$$a_n + \Delta a_n = a_n e^{i\phi_n} + \nu_n$$

(6)

where $\phi_n$ represents the phase-noise component of the NLIN and where $\nu_n$ represents circular nonlinear noise. In the relevant limit of signal to noise ratio being much larger than unity $\Delta a_n \simeq i a_n \phi_n + \nu_n$ and the NLIN variance can be expressed as

$$\sigma^2_{\text{NLIN}} = PT\sigma^2_\phi + \sigma^2_\nu$$

(7)

where $\sigma^2_\phi = \langle |\phi_n|^2 \rangle$, $\sigma^2_\nu = \langle |\nu_n|^2 \rangle$, $P$ is the average signal power and $T$ is the symbol duration (so that $\langle |a_n|^2 \rangle = PT$). The first term on the right-hand-side of Eq. (7) is the contribution of phase-noise, whereas the second term represents the circularly symmetric contributions to the total NLIN variance. In Fig. 5 we plot the ratio between the contributions of phase-noise and of circular noise to the NLIN variance (namely, $PT\sigma^2_\phi/\sigma^2_\nu$), as a function of the number of spans. Considering the origins of NLIN, it is clear that both terms on the right-hand side of Eq. (7) are proportional to $P^3$ and hence their ratio is power independent. In the distributed amplification case (5a), where two-pulse collisions determine the nature of the interaction, phase noise is dominant in the cases of 16 QAM and Gaussian modulation, as expected. Yet, consistently with our discussion in Sec. III, in the case of QPSK modulation the presence of phase noise is negligible. In the case of lumped amplification (5b), phase-noise is only dominant after the first amplified span, but gradually loses its significance subsequently.

We further address the dependence of the phase and circular noise components on modulation format. In order to do that we introduce the concept of Fourth-order modulation factor

$$M = \frac{\langle |b|^4 \rangle}{\langle |b|^2 \rangle^2}$$

(8)

where the random variable $b$ is a symbol in the interfering channel. The fourth-order modulation factor is equal to unity in the case of pure phase-modulation and grows as the power variations of the modulated signal increase. For example in the case of QAM modulation, $M = 1$ for pure phase modulation (like QPSK) and increases asymptotically towards 1.4 with the QAM order (specific values are $M = 1.32$ for 16-QAM, and $M = 1.38$ for 64-QAM). In the case of complex Gaussian modulation $M = 2$. Excluding cases of extremely tight channel-spacing, which require the correction term discussed in [12], all aspects of the modulation-format dependence of the NLIN power have been shown to be captured by the fourth-order modulation factor [3, 5]. In Fig. 6 we plot the phase noise contribution to the NLIN power $PT\sigma^2_\phi$ as well as the contribution of circular noise $\sigma^2_\nu$ as a function of $M$ in the cases of distributed (Fig. 6a) and lumped (Fig. 6b) amplification. Since the NLIN power scales with $P^3$, the curves are displayed for the case in which $P = 1$ mW, and therefore to obtain the results for other input powers,
the vertical axes simply needs to be shifted upward by three times the average input power expressed in dBm units. Evidently, the phase noise component of NLIN strongly depends on modulation format in the cases of both distributed and lumped amplification, and independently of the link length. This can be attributed to the fact that two-pulse collisions are dominant over three-pulse collisions in both regimes of complete and incomplete collisions. It is further evident that while in the distributed amplification case the phase noise becomes dominant almost as soon as some level of variation is introduced into the energies of the transmitted symbols, with lumped amplification phase-noise is only dominant in the case of a single-span, whereas in the case of a 5-span system it exceeds the contribution of the circular noise only when \( M \) is larger than \( \sim 1.7 \). In addition, with distributed amplification or in the case of a single amplified span, complete collisions are dominant and the contribution of three-pulse collisions represented by \( X_{h,m} \) is stronger than the contribution of four-pulse collisions. It is therefore evident that in these cases the circular noise component of NLIN is clearly modulation format dependent. In the case of lumped amplification with more than one fiber span, incomplete four-pulse collisions dominate the generation of circularly symmetric noise and therefore the dependence on modulation format in Fig. 6 is notably weaker.

VIII. RELATION TO THE GAUSSIAN NOISE MODEL AND THE ROLE OF CHROMATIC DISPERSION

The GN model has been derived under the assumption that the electric fields of the interfering channels are statistically independent Gaussian random processes [20]–[24]. Hence its predictions regarding the NLIN power are rigorously accurate (within the obvious limits of the first-order perturbation analysis) only when the interfering channels undergo Gaussian modulation. With other modulation formats, the GN model is known to have certain inaccuracies, which are particularly conspicuous in the case of distributed amplification systems [4], or few-span systems (less than 1000 km) using lumped amplification [5], [6], [21]. The improvement in the GN model’s accuracy with the number of spans is often attributed to chromatic dispersion [11], [21], [25], with the main argument being that with large chromatic dispersion the electric fields of the propagating channels approach Gaussian statistics and hence the assumption of Gaussianity, on which the GN model relies, is better satisfied. The difficulty with
this argument is that it does not explain the fact that in systems using distributed amplification the inaccuracy of the GN model with non-Gaussian modulation does not seem to reduce with the length of the system [4], regardless of the amount of chromatic dispersion that the signals accumulate. The role of chromatic dispersion and its relation to the GN model’s accuracy can be explained in terms of the pulse collision picture introduced in this paper, as we elaborate in what follows.

The GN model is accurate in the regime where modulation-format independent three and four-pulse collisions dominate the formation of NLIN. This does not occur in systems with distributed amplification, where the completeness of collisions implies that the effects of three and four-pulse collisions practically vanishes and the NLIN is dominated by two-pulse collision processes, independently of the link’s length, as can be seen in Fig. 4(a). In the regime of lumped optical amplification incomplete collisions become significant to the formation of NLIN only after the first few spans (see Fig. 4(b)), which is where the GN model’s accuracy begins to improve.

Another interesting aspect of the role of chromatic dispersion has been introduced in [11] and followed in [5]. It has been shown that in the presence of large pre-dispersion the predictions of the GN model become accurate already in the first span, where in the absence of pre-dispersion the GN model is known to be highly inaccurate. It has been suggested [11] that the better accuracy of the GN model in this case results from the fact that pre-dispersion transforms the distribution of the electric field of the interfering channels into Gaussian, independently of their modulation format. While tempting in its simplicity, this argument is inconsistent with an observation that was reported in [5], where it was shown that the inaccuracy of the predictions of the GN model in pre-dispersed systems is smaller than without pre-dispersion only in short links whose accumulated dispersion is much smaller than the pre-dispersion that was applied. For example, when the interfering channels are pre-dispersed by an amount equivalent to 500 km of transmission fiber, the GN model’s prediction improved in the first few hundred kilometers, but in longer links, of the order of 1000 km, it deteriorated to the same level of inaccuracy characterizing systems without pre-dispersion, as illustrated in Fig. 5 of [5]. Clearly, if the GN model’s accuracy improved because of better Gaussianity that followed from pre-dispersion, there is no reason for it to deteriorate in longer systems, where even more dispersion is accumulated. The theory presented in this paper suggests a different explanation, that was briefly illuded to in [5]. Large pre-dispersion creates overlaps between all pulses at the link input so that the collisions taking place in the first few spans of the systems are incomplete. The incompleteness of the collisions emphasizes the significance of modulation format independent three and four-pulse collisions whose effect tends to agree with the predictions of the GN model. As the system becomes longer, the impact of pre-dispersion reduces and the inaccuracy of the GN model resumes its regular value.

IX. A WORD ON POLARIZATION

Throughout this paper we have assumed for simplicity the case of a signal transmitted in a single polarization. While the consideration of polarization multiplexed transmission is considerably more cumbersome, the principles remain unchanged. Equation (3) is generalized to

\[ \Delta \omega_0 = i\gamma \sum_{h,k,m} X_{h,k,m} \left( b_i^h b_i^k I + b_i^h b_i^k \right) \omega_h, \]

where the underline denotes a two-element column vector (i.e. its elements represent the two polarization component of the optical field) and I is the 2 × 2 identity matrix. The coefficient \( X_{h,k,m} \) given in Eq. (4) remains unchanged, and hence its scaling with the frequency separation \( \Omega \) is also unaltered. The main difference is that the specification of indices \( h,k, \) and \( m \) does not uniquely determine the number of colliding pulses. In order to illustrate this, consider for example the NLIN contribution to \( e_{t_0}^{(y)} \) (the y component of \( \omega_0 \)) which is proportional to \( X_{h,k,m} \)

\[ i\gamma X_{h,k,m} \left( 2a_h^{(y)} b_k^{(y)} b_k^{(x)} b_k^{(x)} + a_h^{(y)} b_k^{(x)} b_k^{(x)} + a_h^{(x)} b_k^{(x)} b_k^{(x)} \right). \]  

The first term in Eq. (10) falls into the category of single polarization interactions that we discussed earlier. The second term, in spite of resulting from the interference between the y-polarized symbol in the channel of interest with two x-polarized symbols from the interfering channel, also follows the exact same principles discussed in Secs. III-VI as it describes a classical XPM interaction. The only difference is in the third term in Eq. (10) which results from the interaction of an x-polarized pulse in the channel of interest with two pulses in the interfering channel, one from each state of polarization. Hence, regardless of the choice of indices \( h,k, \) and \( m \) the third term represents a four-pulse collision whose contribution is modulation format independent and manifests itself as complex circular noise. This implies that the dependence on modulation format and the significance of phase-noise will be smaller in the two polarization case than they are in the case of single-polarization transmission [6].

X. CONCLUSIONS

We have shown that the various contributions to NLIN can be classified as pulse collisions involving two, three, or four colliding pulses. Furthermore, we have argued that pulse collisions may either be complete (if they occur on a length-scale where loss/gain and dispersive broadening are negligible), or incomplete. When the majority of collisions are complete (as occurs with distributed gain systems [4] or in few-span systems using lumped amplification [5]) the NLIN is dominated by two-pulse collisions which give it a phase-noise-like character with strong dependence on modulation format. When the majority of collisions are incomplete (high span-count systems with lumped amplification), four-pulse interactions provide the dominant contribution to NLIN, giving it the nature of circularly symmetric noise, as assumed in [20–24]. In this case the dependence of NLIN on the modulation format is much weaker.
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XII. APPENDIX: THE DEPENDENCE ON $\Omega$ IN COMPLETE VERSUS INCOMPLETE COLLISIONS

We consider a generic setup of the kind plotted in Fig. 1. Two pulses, with indices 0 and $h$, belonging to the channel of interest collide with two pulses of the interfering channel, whose indices are $k$ and $m$. Recall that the pulse with the index 0 is not a real pulse that propagates along the fiber, but rather (as we described earlier) it is included in the analysis so as to account for the effect of matched filtering.

We assume the regime of large accumulated dispersion, which is relevant in the case of dispersion uncompensated links and take advantage of the far-field approximation [4, 10].

$$g(z,t) \simeq \sqrt{\frac{i}{2\pi|\beta'|z}} \exp \left(-\frac{i t^2}{2\beta'z}\right) \tilde{g} \left(\frac{t}{\beta'z}\right), \tag{11}$$

where $\tilde{g}(\omega) = \int g(t) \exp(i\omega t) dt$ is the Fourier transform of the fundamental waveform $g(t)$. By substituting Eq. (11) into Eq. (4) and using the change of variables $\omega = t/\beta'z$, we get

$$X_{h,k,m} \simeq \frac{e^{i\Omega T}}{2\pi|\beta'|} \int_{z_0}^{z_1} \frac{f(z)}{z} \tilde{\psi} \left(\Omega + \frac{mT}{\beta'}\right) e^{i\frac{mT}{\beta'}hT} dz, \tag{12}$$

where $\tilde{\psi}$ stands for $(m+k-h)/2$ and

$$\tilde{\psi}(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{g}^*(\omega + u + \frac{hT}{2\beta'z}) \tilde{g} \left(\omega + u - \frac{hT}{2\beta'z}\right) \times \tilde{g}^* \left(\omega + (m-k)T - \frac{hT}{2\beta'z}\right) \tilde{g} \left(\omega - (m-k)T + \frac{hT}{2\beta'z}\right) \times e^{i\omega(h-k+m)^T} d\omega. \tag{13}$$

Note that the function $\tilde{\psi}(u)$ reduces to zero when $z$ satisfies $\frac{|hT|}{2\beta'z} > \frac{B}{2}$ or $\frac{|m-k|T}{2\beta'z} > \frac{B}{2}$, where $B$ is the bandwidth of the fundamental pulse $g(t)$. In the limit of large dispersion, we can hence assume that $\frac{|m-k|T}{2\beta'z} > \frac{B}{2}$ so that the product $\tilde{g}^* \left(\omega + (m-k)T - \frac{hT}{2\beta'z}\right) \tilde{g} \left(\omega - (m-k)T + \frac{hT}{2\beta'z}\right)$ is approximately equal to $|\tilde{g}(\omega)|^2$. In the same manner, by assuming $\frac{|hT|}{2\beta'z} > \frac{B}{2}$ we get

$$\tilde{\psi}(u) \simeq \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{g}^*(\omega + u)^2 |\tilde{g}(\omega)|^2 e^{i\omega(h-k+m)^T} d\omega. \tag{14}$$

Since $\tilde{\psi}(u) \simeq 0$ when $|u| \leq B$ we can conclude that the lower integration limit $z_0$ in Eq. (12) is the largest of the three quantities $\frac{|hT|}{|\beta'|B}$, $\frac{|k-m|T}{|\beta'|B}$, and $\frac{mT}{|\beta'|(|\Omega|+B)}$, whereas the upper limit $z_1$ is the smaller between the link length $L$ and $\frac{mT}{|\beta'|(|\Omega|+B)}$. These integration limits $z_0$ and $z_1$ can be intuitively understood when considering the nature of the nonlinear interaction, as illustrated in Fig. 1. In the general case, two pulses from the channel of interest interact with two pulses of the interfering channel, and the nonlinear interaction takes place while all four pulses overlap with each other temporally. The dispersive broadening of the individual pulses after propagating to point $z$ along the fiber is approximately $|\beta'|Bz$. Within the far field approximation, significant overlap between the zeroth and the $h$-th pulses in the channel of interest is achieved when $|\beta'|Bz \gg |hT|$ and similarly the $k$-th and $m$-th pulses in the interfering channel overlap when $|\beta'|Bz \gg |m-k|T$. In this case, the zeroth and $h$-th pulses form a single block of width $\sim \beta'|Bz$ centered at $hT/2$, whereas the $k$-th and $m$-th pulses form a block of the same width, centered at $(m+k)T/2 - \beta'z$. The overlap between the two blocks starts when $(m+k)T/2 - \beta'z > hT/2 + \beta'|Bz/2$ and ends when $(m+k)T/2 - \beta'z < hT/2 - \beta'|Bz/2$.

The combination of these conditions translates into the integration limits indicated above, implying that when $h$, $k$ and $m$ satisfy $\tilde{m} > \max\{|h|,|k-m|x\}$, a collision is formed if the condition $\tilde{m} < (\Omega + B)|\beta'|L/T$ is being satisfied. The collision is completed before the end of the link if $h$, $k$ and $m$ further satisfy $\tilde{m} < (\Omega - B)|\beta'|L/T$.

Finally, defining $u = \Omega + \frac{mT}{\beta'}$, Eq. (12) can be rewritten as

$$X_{h,k,m} = \frac{1}{2\pi|\beta'|} \int_{u_0}^{u_1} \int_{-\infty}^{\infty} f(u) e^{-iu\eta T} \tilde{\psi}(u) \eta d\eta. \tag{15}$$

where $u_0 = \Omega + \frac{mT}{\beta'}$, and $u_1 = \Omega + \frac{mT}{\beta'2}$. This form will be useful in what follows.

A. Complete collisions

In the case of a complete collision, the integrand in Eq. (15) vanishes at the integration boundaries, which can therefore be replaced by $\pm \infty$. In addition, since in the regime of complete collisions, attenuation and gain during the collision must be assumed negligible, we may replace $f \left(\frac{mT}{|\beta'|(|\Omega|+B)}\right)$ with $f \left(\frac{mT}{|\beta'|(|\Omega|)}\right)$ and move it outside of the integral. Having done so, we obtain

$$X_{h,k,m} = \frac{1}{2\pi|\beta'|} \int_{-\infty}^{\infty} f(\eta) \int_{-\infty}^{\infty} \tilde{\psi}(\eta) e^{-iu\eta T} \eta d\eta. \tag{16}$$

Defining $\psi(v) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\psi}(u) e^{-iu\eta T} \eta d\eta$, as the inverse Fourier transform of $\tilde{\psi}(u)$, Eq. (16) can be rewritten as

$$X_{h,k,m} = \frac{1}{2\pi|\beta'|} \int_{-\infty}^{\infty} f(\eta) \int_{-\infty}^{\infty} \tilde{\psi}(\eta) \eta d\eta. \tag{17}$$

where $\psi^{(n)}(hT)$ is the $n$-th derivative of $\psi(v)$, evaluated at $v = hT$. From Eq. (14) it is evident that

$$\psi(v) = R^*(v)R(v - (h - k + m)T), \tag{18}$$

where $R(v) = \int g^*(t)g(v - t) dt$ is the temporal autocorrelation function of the fundamental pulse $g(t)$. The orthogonality condition of Eq. (4) implies that $R(v)$ has a maximum when $v = 0$ ($R(0) = 1$) and that it vanishes when $v/T$ is any integer other than 0.

It is evident from Eq. (18) that the only contribution that scales with $\Omega^{-1}$ is the one proportional to $\psi^{(0)}(hT) = \psi(hT)$. 


Substitution of \( v = hT \) in Eq. (13) reveals that \( \psi(hT) = R^*(h^2)R((k-m)T) \) is non-zero only when \( h = 0 \) and \( k = m \). This is exactly the case of two-pulse collisions discussed earlier. The next contribution to NLIN, which scales as \( \Omega^{-2} \), follows from \( \psi(hT) = R^*(h^2)R((k-m)T) + R(hT)R^*((k-m)T) \) and it is non-zero only in the case of three-pulse collisions with \( h = 0 \) and \( k \neq m \) (one pulse in the channel of interest and two pulses in the interfering channel), or with \( h \neq 0 \) and \( k = m \) (two pulses in the channel of interest and a single pulse in the interfering channel). Only the second term in the summation in Eq. (17), which scales as \( \Omega^{-3} \) is non-zero in the case of complete four-pulse collisions (with \( h \neq 0 \) and \( k \neq m \)). Hence we may summarize that two-pulse collisions scale as \( \Omega^{-1} \), complete three-pulse collisions scale as \( \Omega^{-2} \) and complete four-pulse collisions scale as \( \Omega^{-3} \). As can be deduced from the integration limits of Eq. (15), the number of complete two, three and four-pulse collisions is proportional to \( \Omega \), \( \Omega^2 \) and \( \Omega^3 \), respectively, and therefore their overall contribution to NLIN (which is proportional to \(|X_{h,k,m}|^2 \)) is dominated by two-pulse collisions (see Sec. VI).

B. Incomplete collisions

The analysis of incomplete collisions is somewhat more complicated and difficult to extend beyond what is given by expressions (12), or (15). Some insight can be gained from considering the case of distributed amplification where \( f(z) = 1 \) and can be taken out of the integral. In this case Eq. (16) still holds, except that the actual integration limits \( u_0 \) to \( u_1 \) need to be applied.

Using the formulation of the previous subsection \( \psi(v) \) is no longer given by Eq. (18), but rather it is the convolution of the expression on the right-hand-side of Eq. (18) with a sinc function corresponding to the inverse Fourier transform of a unit frequency window extending between \( u_0 \) and \( u_1 \). In this situation \( \psi(hT) \) may differ from zero for all combinations of \( h, k, \) and \( m \), implying that the contributions of all types of incomplete pulse collisions scale as \( \Omega^{-1} \).

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