Thin-shell wormholes in Brans–Dicke gravity

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Abstract

Spherically symmetric thin-shell wormholes are constructed within the framework of Brans–Dicke gravity. It is shown that, for appropriate values of the Brans–Dicke constant, these wormholes can be supported by matter satisfying the energy conditions.

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1 Introduction

After the leading work by Morris and Thorne [1], traversable Lorentzian wormholes [2] have received considerable attention. Such geometries connect two regions of the same universe—or of two universes—by a traversable throat. A central objection against the existence of wormholes is that, within the framework of general relativity, the flare-out condition [3] to be satisfied at the throat requires the presence of exotic matter, that is, matter which violates the energy conditions [1–4]. However, it was shown in Ref. [5] that the amount of exotic matter necessary for the existence of a wormhole can be made infinitesimally small by suitably choosing the geometry, though it may be at the expense of large stresses at the throat [6, 7]. On the other hand, it was demonstrated that in some alternative theories of gravity the requirement of exotic matter can be avoided [8, 9]; in particular, some years ago, Anchordoqui et al. [10] showed that, in Brans–Dicke gravity, Lorentzian wormholes of the Morris–Thorne type are compatible with matter which, apart from the Brans–Dicke scalar field, satisfies the energy conditions. Other related aspects of wormholes in Brans-Dicke or in scalar-tensor theories were also discussed in Refs. [11, 12].

Thin-shell wormholes [13] are mathematically constructed by cutting and pasting two manifolds to yield another one with a throat placed at the joining surface. The mechanical stability and the matter content of spherically symmetric wormholes of this kind have been studied by several authors, both within general relativity [14] and in alternative theories of gravity [6, 8, 15]. Other geometries were also recently explored [16]. In the present work we apply the Darmois–Israel formalism [17,18] generalized to Brans–Dicke gravity [19] for the construction of spherically symmetric thin-shell wormholes. We calculate the energy density and the pressure on the shell. We find that for certain values of the Brans–Dicke constant $\omega$, the wormhole radius can be chosen so that the

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matter on the shell satisfies the energy conditions, being the existence of the throat made possible by the presence of the Brans-Dicke field. Throughout the paper units such that $c = G = 1$ are used.

2 Thin-shells in Brans–Dicke gravity

In the framework of present unified theories, a scalar field should exist besides the metric of the spacetime. Scalar-tensor theories of gravitation would be important when studying the early Universe, where it is supposed that the coupling of the matter to the scalar field could be nonnegligible. In Brans–Dicke theory, matter and non gravitational fields generate a long-range scalar field $\phi$ which, together with them, acts as a source of gravitational field. The metric equations generalizing those of general relativity are

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi}{\phi} T_{\mu\nu} + \frac{\omega}{\phi^2} \phi_{,\mu} \phi_{,\nu} - \frac{\omega}{2\phi^2} g_{\mu\nu} \phi_{,\alpha} \phi^{,\alpha} + \frac{1}{\phi} \phi_{,\mu\nu} - \frac{1}{\phi} g_{\mu\nu} \phi_{,\alpha} \phi^{,\alpha},
\]

where $R_{\mu\nu}$ is the Ricci tensor, $T_{\mu\nu}$ is the energy-momentum tensor of matter and fields –not including the Brans–Dicke field– and $\omega$ is a dimensionless constant. The field $\phi$ is a solution of the equation

\[
\phi_{,\mu} \phi^{,\mu} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \left( \sqrt{-g} g^{\mu\nu} \frac{\partial \phi}{\partial x^\nu} \right) = \frac{8\pi T}{3 + 2\omega},
\]

where $T$ is the trace of $T_{\mu\nu}$. In the limit $\omega \to \infty$ the Einstein equations are recovered, provided that $\phi(\omega \to \infty) = 1/G = 1$. In Brans–Dicke theory, the junction conditions across a smooth timelike hypersurface $\Sigma$ in the four dimensional manifold, can be obtained by projecting on $\Sigma$ the equations above. The extrinsic curvature associated with the two sides of $\Sigma$ in terms of the unit normals $n_\gamma^+ (n_\gamma^+ n_\gamma = 1)$ is given by

\[
K_{ij}^\pm = -n_\gamma^+ \left( \frac{\partial^2 X^\gamma}{\partial \xi^i \partial \xi^j} + \Gamma_{\mu\nu}^{\gamma} \frac{\partial X^\mu}{\partial \xi^i} \frac{\partial X^\nu}{\partial \xi^j} \right) \bigg|_\Sigma,
\]

where $X^\gamma$ are the coordinates of the four dimensional manifold, $\xi^i$ are the coordinates on the hypersurface, and $\Gamma_{\mu\nu}^{\gamma}$ are the components of the connection associated with the metric $g_{\mu\nu}$. Then, with this definition, the junction conditions in Brans–Dicke theory (generalized Darmois–Israel conditions) have the form [19]

\[
-\langle K_j^i \rangle + \langle K \rangle \delta_j^i = \frac{8\pi}{\phi} \left( S_j^i - \frac{S}{3 + 2\omega} \delta_j^i \right),
\]

\[
\langle \phi, N \rangle = \frac{8\pi S}{3 + 2\omega},
\]

where the notation $\langle \cdot \rangle$ stands for the jump of a given quantity across the hypersurface $\Sigma$, $N$ labels the coordinate normal to this surface and $S_j^i$ is the energy-momentum tensor of matter and fields (except the field $\phi$) on the shell located at $\Sigma$. The quantities $K$ and $S$ are the traces of $K_j^i$ and $S_j^i$ respectively. The components of the metric and the Brans–Dicke field are continuous across the shell ($\langle g_{\mu\nu} \rangle = 0$, $\langle \phi \rangle = 0$). Note that in the general relativity limit $\omega \to \infty$ the Lanczos equations [17, 18] are recovered.
3 Spherically symmetric wormholes

Now let us apply the formalism introduced above to the construction of spherically symmetric thin-shell wormholes. We start from the metric

$$ds^2 = -f(r)dt^2 + g(r)dr^2 + h(r)(d\theta^2 + \sin^2 \theta d\varphi^2),$$

so that $X = (t, r, \theta, \varphi)$. From this geometry we choose a radius $a$ and take two copies $\mathcal{M}^+$ and $\mathcal{M}^-$ of the region $r \geq a$, and paste them at the hypersurface $\Sigma$ defined by $r = a$, obtaining a new manifold $\mathcal{M} = \mathcal{M}^+ \cup \mathcal{M}^-$. The radius $a$ is chosen so that there are no horizons and singularities in $\mathcal{M}$. If $h'(a)$ is positive the flare-out condition is satisfied, and the resulting geometry describes a wormhole having a throat of radius $a$ connecting the two regions $\mathcal{M}^+$ and $\mathcal{M}^-$. Introducing the coordinates $\xi = (\tau, \theta, \varphi)$ on $\Sigma$ (with $\tau$ the proper time on the shell), the jump of the components of the extrinsic curvature is given by

$$\langle K^\eta_\tau \rangle = \frac{f'(a)}{f(a) \sqrt{g(a)}},$$

$$\langle K^\varphi_\varphi \rangle = \frac{h'(a)}{h(a) \sqrt{g(a)}}.$$

Then the energy density $\sigma = -S^\tau_\tau$ and the pressures $p = S^\theta_\theta = S^\varphi_\varphi$ of the matter and fields in the shell (apart from the field $\phi$) are given by

$$\sigma = -\frac{\phi(a)}{8\pi \sqrt{g(a)}} \left[ \frac{2h'(a)}{h(a)} + \frac{1}{\omega} \left( \frac{f'(a)}{f(a)} + \frac{2h'(a)}{h(a)} \right) \right],$$

$$p = \frac{\phi(a)}{8\pi \sqrt{g(a)}} \left[ \frac{f'(a)}{f(a)} + \frac{h'(a)}{h(a)} + \frac{1}{\omega} \left( \frac{f'(a)}{f(a)} + \frac{2h'(a)}{h(a)} \right) \right],$$

where $\phi(a)$ is the value of the Brans–Dicke field on the surface $\Sigma$. The constraint given by Eq. (5), which in this case takes the form $\langle \phi, N \rangle = 8\pi(2p - \sigma)/(3+2\omega)$, should be also satisfied. Consequently, we have

$$\sigma + p = \frac{\phi(a)}{8\pi \sqrt{g(a)}} \left[ \frac{f'(a)}{f(a)} - \frac{h'(a)}{h(a)} \right].$$

Non exotic matter should satisfy the weak energy condition, i.e. $\sigma \geq 0$ and $\sigma + p \geq 0$; this condition would be violated if the flare-out condition is to be fulfilled within pure general relativity, which is easy to see from Eq. (7) taking the limit $\omega \to \infty$.

As a particular case, we consider the spherically symmetric vacuum solution of Brans–Dicke equations (see [11] and references therein), in which the functions $f$, $g$ and $h$ are given by

$$f(r) = \left( 1 - \frac{2\eta}{r} \right)^A,$$

$$g(r) = \left( 1 - \frac{2\eta}{r} \right)^B,$$

$$h(r) = \left( 1 - \frac{2\eta}{r} \right)^{1+B} r^2,$$

\footnote{The weak energy condition (WEC) states that for any timelike vector $T_{\mu\nu} V^\mu V^\nu \geq 0$, which means that the local energy density as measured by any timelike observer is positive. In terms of the principal pressures it takes the form $\rho \geq 0$, $\rho + p_j \geq 0 \forall j$. The WEC implies the null energy condition (NEC) $T_{\mu\nu} k^\mu k^\nu \geq 0$ for any null vector, which in terms of the principal pressures takes the form $\rho + p_j \geq 0 \forall j$.}
and the field $\phi$ takes the form

$$\phi(r) = \phi_0 \left(1 - \frac{2\eta}{r}\right)^{-\frac{(A+B)}{2}}, \quad (15)$$

where

$$\phi_0 = \frac{4 + 2\omega}{3 + 2\omega}, \quad A = \frac{1}{\lambda}, \quad B = -\frac{\zeta + 1}{\lambda}, \quad \lambda = \sqrt{(\zeta + 1)^2 - \zeta \left(1 - \frac{\omega\zeta}{2}\right)},$$

with $\eta > 0$ and $\zeta$ constants. If the field $\phi$ is not constant, the geometry presents a naked singularity with radius $r_s = 2\eta$. To construct the wormhole, we take $\omega < -2$ or $\omega > -3/2$ so that $\phi_0 > 0$, and a throat radius larger than $r_s$. As $\lambda$ should be real and non-zero, it follows that $(-1 - \sqrt{-3 - 2\omega})/(2 + \omega) < \zeta < (-1 + \sqrt{-3 - 2\omega})/(2 + \omega)$ if $\omega < -2$ and that $\zeta$ can take any value if $\omega > -3/2$. In the case $B + 1 \geq 0$ we have that $h'(a) > 0 \ \forall a > 2\eta$ and the flare out condition is satisfied; it happens when $2/\omega < \zeta < 0$, for $\omega < -2$, and when $\zeta < 0$ or $\zeta > 2/\omega$, for $\omega > -3/2$. Replacing the explicit form of the metric and field, we have

$$\sigma = -\frac{\phi_0}{4\pi a^2} \left(\frac{a}{a - 2\eta}\right)^{1+B+A/2} \left[2\eta(B - 1) + 2a + \frac{1}{\omega} (\eta A + 2\eta(B - 1) + 2a) \right], \quad (16)$$

$$p = -\frac{\phi_0}{4\pi a^2} \left(\frac{a}{a - 2\eta}\right)^{1+B+A/2} \left[\eta A + \eta(B - 1) + a + \frac{1}{\omega} (\eta A + 2\eta(B - 1) + 2a) \right], \quad (17)$$
with the constraint obtained from Eq. (5):

\[ 2a + \eta \left[ A + 2B - 2 + \omega(A + B) \right] = 0. \]  

(18)

Then we obtain

\[ \sigma + p = \frac{\phi_0}{4\pi a^2} \left( \frac{a}{a - 2\eta} \right)^{1+B+A/2} \left[ \eta A + \eta(1 - B) - a \right]. \]  

(19)

The inequality \( \sigma \geq 0 \) is fulfilled when

\[ \frac{a}{\omega}(\omega + 1) + \frac{\eta}{\omega} \left[ (B - 1)(\omega + 1) + \frac{A}{2} \right] \leq 0, \]  

(20)

and the inequality \( \sigma + p \geq 0 \) is satisfied if

\[ a \leq \eta(A + 1 - B); \]  

(21)

in both cases subject to the aditional condition given by Eq. (18). If both inequalities above are satisfied the weak energy condition is not violated, and then the matter on the shell is not exotic. The throat radii such that conditions (20) and (21) are fulfilled are respectively displayed in figures 1 and 2 for some relevant values of the Brans–Dicke constant \( \omega \) and the parameters \( \zeta \) and \( \eta \). The figure 3 shows where both conditions are simultaneously satisfied, so that configurations defined by
such values contain non exotic matter in the shell placed at the wormhole throat. We see that this happens for \( \omega < -2 \), and the radius \( a \) slightly greater than \( r_s = 2\eta \). The range of the parameters for which exotic matter is not required grows as \( \omega \) approaches \(-2\).

4 Discussion

We have applied the generalized Darmois–Israel formalism to the construction of spherically symmetric thin-shell wormholes in the framework of Brans–Dicke gravity. We have obtained the energy density and the pressure of matter and fields other than \( \phi \), in the shell placed at the throat of the wormhole. We have shown that, for certain negative values of the Brans–Dicke constant, this matter satisfies the energy conditions if the throat radius is suitably chosen. Such values of the constant \( \omega \) seem to be unphysical in the present Universe, but they could make sense in another scenario (i.e. far in the past). If the right hand side of Eq. (4) is understood as an effective source, the junction conditions present the same form as in general relativity:

\[
-\langle K^i_j \rangle + \langle K \rangle \delta^i_j = \frac{8\pi}{\phi} \tilde{S}^i_j,\]

Figure 3: The dashed line corresponds to values of the Brans–Dicke constant \( \omega \) and parameters \( \zeta \) and \( \eta \) for which wormholes with throat radius \( a \) exist. In the grey zone the weak energy condition \((\sigma \geq 0 \text{ and } \sigma + p \geq 0)\) is satisfied by the matter on the shell.
with

$$\tilde{S}^i_j = S^i_j - \frac{S}{3 + 2\omega} \delta^i_j$$

(23)

the effective energy-momentum tensor on the surface. With this definition, we would have that

$$\tilde{\sigma} = -\tilde{S}^\tau_\tau = -\frac{\phi(a)}{4\pi \sqrt{g(a)}} \frac{h'(a)}{h(a)}$$

(24)

is negative because of the flare-out condition. So, in this picture, the energy conditions are not satisfied by this effective surface tensor. The violation of these conditions comes from the Brans–Dicke field, even in the presence of non exotic matter and other fields. This result is analogous to what was obtained by Anchordoqui et al. [10] for wormholes of the Morris–Thorne type.

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**References**

[1] M.S. Morris and K.S. Thorne, Am. J. Phys. 56, 395 (1988).
[2] M. Visser, Lorentzian Wormholes (AIP Press, New York, 1996).
[3] D. Hochberg and M. Visser, Phys. Rev. D 56, 4745 (1997).
[4] D. Hochberg and M. Visser, Phys. Rev. Lett. 81, 746 (1998); D. Hochberg and M. Visser, Phys. Rev D 58, 044021 (1998).
[5] M. Visser, S. Kar and N. Dadhich, Phys. Rev. Lett. 90, 201102 (2003).
[6] E. F. Eiroa and C. Simeone, Phys. Rev. D 71, 127501 (2005).
[7] O. B. Zaslavskii, Phys. Rev. D 76, 044017 (2007).
[8] E. Gravanis and S. Willison, Phys. Rev. D 75, 084025 (2007); M.G. Richarte and C. Simeone, Phys. Rev. D 76, 087502 (2007); ibid. 77, 089903(E) (2008); C. Garraffo, G. Giribet, E. Gravanis and S. Willison, J. Math. Phys. 49, 042503 (2008); C. Garraffo and G. Giribet, Mod. Phys. Lett. A 23, 1801 (2008).
[9] G. Dotti, J. Oliva and R. Troncoso, Phys. Rev. D 75, 024002 (2007); H. Maeda and M. Nozawa, Phys. Rev. D 78, 024005 (2008).
[10] L. A. Anchordoqui, S. Perez Bergliaffa and D. F. Torres, Phys. Rev. D 55, 5226 (1997).
[11] A. G. Agnese and M. La Camera, Phys. Rev. D 51, 2011 (1995).
[12] K. K. Nandi, A. Islam and J. Evans, Phys. Rev. D 55, 2497 (1997); F. He and S.-W. Kim, Phys. Rev. D 65, 084022 (2002); A. Bhadra and K. Sarkar, Mod. Phys. Lett. A 20, 1831 (2005); K. A. Bronnikov and A. A. Starobinsky, JETP Letters 85, 1 (2007).
[13] M. Visser, Phys. Rev. D 39, 3182 (1989); M. Visser, Nucl. Phys. B328, 203 (1989).

[14] E. Poisson and M. Visser, Phys. Rev. D 52, 7318 (1995); C. Barceló and M. Visser, Nucl. Phys. B584, 415 (2000); M. Ishak and K. Lake, Phys. Rev. D 65, 044011 (2002); E.F. Eiroa and G.E. Romero, Gen. Relativ. Gravit. 36, 651 (2004); F.S.N. Lobo and P. Crawford, Class. Quantum Grav. 21, 391 (2004); F.S.N. Lobo and P. Crawford, Class. Quantum Grav. 22, 4869 (2005); E.F. Eiroa and C. Simeone, Phys. Rev. D 76, 024021 (2007); E.F. Eiroa, Phys. Rev. D 78, 024018 (2008); M.G. Richarte and C. Simeone, arXiv:0711.2297 [gr-qc] (2008).

[15] L. A. Anchordoqui, Nuovo Cim. B113, 1497 (1998); M. Thibeault, C. Simeone and E. F. Eiroa, Gen. Relativ. Gravit. 38, 1593 (2006); F. Rahaman, M. Kalam and S. Chakraborty, Gen. Relativ. Gravit. 38, 1687 (2006).

[16] E. F. Eiroa and C. Simeone, Phys. Rev. D 70, 044008 (2004); C. Bejarano, E. F. Eiroa and C. Simeone, Phys. Rev. D 75, 027501 (2007); J. P. S. Lemos and F. S. N. Lobo, Phys. Rev. D 78, 044030 (2008).

[17] N. Sen, Ann. Phys. (Leipzig) 73, 365 (1924); K. Lanczos, ibid. 74, 518 (1924); G. Dar- mois, Mémorial des Sciences Mathématiques, Fascicule XXV, Chap. V (Gauthier-Villars, Paris, 1927); W. Israel, Nuovo Cimento 44B, 1 (1966); ibid. 48B, 463(E) (1967).

[18] P. Musgrave and K. Lake, Class. Quantum Grav. 13, 1885 (1996).

[19] K.G. Suffern, J. Phys. A: Math. Gen. 15, 1599 (1982); C Barrabès and G.F. Bressange, Class. Quantum Grav. 14, 805 (1997); F. Dahia and C. Romero, Phys. Rev. D 60, 104019 (1999).

[20] It can be obtained freely at the address http://grtensor.org