On the existence of certain total recursive functions in nontrivial axiom systems, I.*

N. C. A. da Costa
F. A. Doria

Research Group on Logic and Foundations,
Institute for Advanced Studies, University of São Paulo.
Av. Prof. Luciano Gualberto, trav. J, 374.
05655–010 São Paulo SP Brazil.
NCACOSTA@USP.BR

Research Center on Mathematical Theories of Communication
and Project Griffo,
School of Communications,
Federal University at Rio de Janeiro.
Av. Pasteur, 250. 22295–900 Rio RJ Brazil.
DORIA@OMEGA.LNCC.BR

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Abstract
We investigate the existence of a class of ZFC-provably total recursive unary functions, given certain constraints, and apply some of those results to show that, for Σ₁-sound set theory, ZFC ⊬ P < NP.

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1 Introduction

It is a classical fact that there are total unary recursive functions in the standard model for arithmetic which cannot be proved to be so within reasonably strong axiomatic systems (see [3], p. 257 and references therein). Such a phenomenon can especially be seen to happen within PA (Peano arithmetic) or within ZFC (Zermelo–Fraenkel set theory with the axiom of choice) if they satisfy a soundness condition, $\Sigma_1$–soundness (see Definition 2.1).

We consider in this paper a more particular situation: what can be said about a given partial recursive function when we add several specific restrictions to the objects considered within those $\Sigma_1$–sound systems? We obtain a partial unary recursive function which, if it is true that such a function is total, then that fact cannot be proved within those axiom systems, supposed consistent.

The original motivation for this paper stems out of the authors’ search for incompleteness results with respect to nontrivial “rich” axiom systems, such as ZFC or a sizable fragment of it, which arise out of expressions for mathematical data, that is, which depend on the way we linguistically encode the mathematical objects. Some rather striking undecidable statements may arise in that way, and they settle noted questions as the integrability problem in classical mechanics, the decision problem for chaotic systems [3], or Arnol’d’s problems on the nature of stability in dynamical systems [4, 5]. Those results not only settle open questions, but do so in a linguistic way, that is, they depend on the explicit form of expressions for mathematical concepts that may arise within a formal system. Moreover they are technically simple like the results presented here, and independent of the actual precise formal environment where they are placed.

We develop here the first part of a different (but also linguistically conceived) technique for the construction of independence results. The present technique comes down from a standard diagonal argument (see [10], p. 54, Ex. 5), and deals with provably recursive, denumerable objects which are kept so throughout the procedure. That technique is again rather insensitive to the actual formal setting where its proofs are supposed to happen, as we only require constructs from formal arithmetic in it.

We ask for the existence of a certain set of total recursive functions whose domain is informally understood as a set of expressions that describe Turing machines. We show here that it is consistent (with ZFC, taken as the formal background and supposed $\Sigma_1$–sound) to assume that those functions do not exist within the required axiom system.
As an application we discuss in the last Section the characterization problem for nondeterministic polynomial computations. We show how that question relates to the present work, as well as to previous work in the area. We then show that, if ZFC is $\Sigma_1$-sound, then $\text{ZFC} \not\vdash P < NP$. Related and complementary results will be considered in Part II of this paper.

Conventions, notations, Turing machines

Remark 1.1 The formal setting is ZFC (Zermelo–Fraenkel set theory with the axiom of choice); informal definitions and proofs happen as usual within ordinary mathematics. Here we use Kleene’s $T$ predicate $\mathbb{S}, \mathbb{T}$; $\omega$ is the set of natural numbers.

Restricted bounded arithmetic variables are in general used without making explicit their domains, since they are always clear by context. □

Remark 1.2 We require Turing machines in our argument; their behavior is specified below to avoid ambiguities:

1. Our Turing machines are defined over the set $A_2^*$ of finite words on the binary alphabet $A_2 = \{0, 1\}$.
2. Each machine has $n+1$ states $s_0, s_1, \ldots, s_n$, where $s_0$ is the final state. (The machine stops when it moves to $s_0$.)
3. The machine’s head roams over a two–sided infinite tape.
4. Machines input a single binary word and either never stop or stop when the tape has a finite, and possibly empty set of binary words on it.
5. The machine’s output word will be the one over which the head rests when $s_0$ is reached. (If the head lies on a blank square, then we take the output word to be the empty word.)

It is easy to formalize the concept of Turing machine within ZFC, as a Turing machine is a mathematical structure. For examples of such formalizations see □. In what follows we suppose that we are given the canonical enumeration of binary strings $\emptyset, 0, 1, 00, 01, 10, 11, \ldots$, and that each binary string is coded by the corresponding numeral in the sequence $0, 1, 2, 3, \ldots$ □

2 Preliminary developments

Definition 2.1 A recursively axiomatizable theory $R$ which is sufficient to develop elementary number theory is $\Sigma_1$-sound if and only if whenever $P(x)$ is a primitive recursive predicate such that $R \vdash \exists x P(x)$, there is a natural number $n$ such that $P(n)$ holds. □
Remark 2.2 We will make explicit when we suppose that either PA or ZFC are $\Sigma_1$-sound. □

Definition 2.3

1. A ZFC unary function $f$ is ZFC-provably total recursive if for some Gödel number $e_f$ for $f$, $\text{ZFC} \vdash \forall x \in \omega \exists z \in \omega (T(e_f, x, z) \land \forall y \in \omega f(y) = \{e_f\}(y))$. 

2. A ZFC- unary predicate $Q(x)$ is ZFC-provably total recursive if its characteristic function is ZFC-provably total recursive. Extension of that definition to a $n$-place predicate $Q(x_0, x_1, \ldots, x_n)$ is immediate. □

We need a set $\mathcal{G}$ of ZFC-provably total recursive unary (t.r.u.) functions:

Definition 2.4 The functions $g_n \in \mathcal{G}$, where $n \in \omega$, satisfy, for any $x$ and any $m, n$:

1. $g_n$ is ZFC-provably t.r.u.
2. $g_0(x) > 2$.
3. $g_n(x + 1) > g_n(x)$.
4. $g_n(x) > g_m(x), n > m$. □

It is easy to see that infinitely many such sets $\mathcal{G}$ exist.

Let $M_0, M_1, M_2, \ldots$, be the single-tape Turing machines described above, and ordered by their Gödel numbers; we suppose that every $n \in \omega$ is a Gödel number for some machine.

Let $[g_n(|x|)], n \in \omega, |x|$ the length of $x$, an input to $M_i$, be a clock that stops $M_i(x)$ over input $x$ after $g_n(|x|) - 1$ cycles. We suppose that the clock acts as follows: when the bound in the number of processing steps is reached, the clock stops the machine to which it is coupled and then moves its state to $s_0$. The machine’s output is the word on which the head rests when the clock stops it. Let $\langle \ldots, \ldots \rangle$ denote the usual recursive 1–1 pairing function $\langle \ldots, \ldots \rangle : \omega \times \omega \to \omega$. We define:

Definition 2.5 For $p = \langle i, n \rangle$, $P_p = \langle M_i, [g_n] \rangle$. $\mathcal{P}_0 = \{P_0, P_1, \ldots\};$ we order $\mathcal{P}_0$ by the code sequence $p = 0, 1, 2, \ldots$. □

Remark 2.6 $\mathcal{P}_0$ contains all possible ordered pairs as in the preceding definitions; the number $p$ can intuitively be seen as coding an expression for a Turing machine. From here onwards we will refer to those pairs when we talk about “expressions for Turing machines” of the kind considered in the present paper. □
Definition 2.7

1. If there is a fixed \( n \in \omega \) such that for every \( x \in \omega \), \( M_i(x) \) stops before, or at most at \( g_n(|x|) \) machine cycles, we say that \( M_i \) is \( \mathcal{G} \)-bounded.

2. A total recursive unary function \( f(x) \) is \( \mathcal{G} \)-bounded if there is a \( \mathcal{G} \)-bounded Turing machine that computes it. \( \square \)

Proposition 2.8 If \( M_i \) is a \( \mathcal{G} \)-bounded machine, then there will be infinitely many \( P_p \in \mathcal{P}^0 \) which compute the same function as \( M_i \). \( \square \)

Remark 2.9 We can of course allow index \( n \) in \( g_n \) to range over an initial segment of the ordinals, given a notation system that satisfies the conditions spelled out in Definition 2.4. However we won’t require this fact in the present paper. \( \square \)

The machine \( V \)

Definition 2.10 \( V((x,s)) \) is a fixed but otherwise arbitrary \( \mathcal{G} \)-bounded Turing machine such that:

1. \( V \) is an algorithm for a function \( f_V : \omega \times \omega \to \{0, 1\} \).

2. For every \( x \in \omega \) there is an \( s \in \omega \) such that \( V((x,s)) = 1 \). Given the machine \( P_n \), whenever \( P_n(x) = s \) and \( V((x,s)) = 1 \), we say that \( x \) is acceptable for \( n \).

3. For every \( x \in \omega \) there is an \( s \in \omega \) such that \( V((x,s)) = 0 \).

4. For the empty word \( \emptyset \) and for any nonempty strings \( x, s \):
   
   - (a) \( V((\emptyset, s)) = 1 \).
   - (b) \( V((x, \emptyset)) = 0 \).
   - (c) \( V((\emptyset, \emptyset)) = 0 \). \( \square \)

Proposition 2.11 Given a constant \( x_0 \in \omega \), there are infinitely many \( P_m \) such that \( x_0 \) is acceptable for \( P_m \).

Proof: Let \( s \) be such that \( V((x_0,s)) = 1 \). Let \( C_s \) be one of the constant Turing machines that input anything and output \( s \). Let \( e_s(|x_0|) \) be its operation time when input \( x_0 \) is fed to the machine. Then, given the conditions in Definition 2.4, it is easy to see that \( x_0 \) is acceptable for the \( \mathcal{G} \)-bounded machine
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\langle C_s, g_{c_s(|x_0|)} \rangle$. And consequently, it is acceptable for all similar pairs, with clocks $[g_{c_s(|x_0|)+k}]$, for any natural $k$.

Moreover, since we have agreed that for the empty word $\emptyset$, we always have $V(\langle x, \emptyset \rangle) = 0$, any $x$, one immediately sees that there are infinitely many $G$-bounded Turing machines that, for a fixed $x_0$, will only accept that $x_0$, and no other input. \[ \square \]

**The predicate $P^\phi(m, x)$ and the function $f_P^\phi(m)$**

In what follows we use the coding of binary sequences by natural numbers. We will now slightly strengthen our notational conventions:

**Definition 2.12** A representation for $P^0$ is any map $\phi : P^0 \to P^0$, such that:

1. $\forall p \in \omega \mapsto \phi(p) \in \omega$ is a recursive permutation.
2. $P^0 = \{P^0_0, P^0_1, P^0_2, \ldots \}$, where $P^0_0 = P_0, P^0_1 = P_1, \ldots$
3. $P^\phi_{\phi(p)} = \text{Def } \phi(P^0_p)$.
4. $\phi P^0 = P^\phi = \{P^\phi_0, P^\phi_1, \ldots \}$. \[ \Box \]

**Definition 2.13** For $P^\phi_m \in P^\phi$, $P^\phi(m, x) \iff V(\langle x, P^\phi_m(x) \rangle) = 0$. \[ \Box \]

**Corollary 2.14** $P^\phi(m, x)$ is a ZFC-provably recursive predicate. \[ \Box \]

**Definition 2.15** $f_P^\phi(m) = \text{Def } \mu x P^\phi(m, x)$. \[ \Box \]

**Corollary 2.16** $f_P^\phi$ is partial recursive. \[ \Box \]

**Remark 2.17** If ZFC is $\Sigma_1$-sound, then $f_P^\phi$ is also ZFC-provably partial recursive, as out of the ZFC-provably recursive characteristic function for $P^\phi$ we can compute a Gödel number for a machine that computes $f_P^\phi$. \[ \Box \]

One final result:

**Corollary 2.18** ZFC $\vdash \forall m \in \omega \exists y \in \omega P^\phi(m, y) \iff f_P^\phi$ is total. \[ \Box \]
Equivalence of representations

It doesn’t matter which representation $\phi$ we choose:

**Proposition 2.19** $\text{ZFC} \vdash \forall m \exists x P^\phi(m, x) \iff \forall n \exists x P^{\phi^{-1}}(n, x)$.

**Proof:** $P^\phi(m, x) \iff \text{Def } V(x, P^\phi_m(x)) = 0.$ (We omit the $\langle \ldots \rangle$ for simplicity.)

But $P^\phi_m(x) = P^{\phi^{-1}}_0(m)(x).$ (See Definition 2.12.) Then:

$$V(x, P^\phi_m(x)) = V(x, P^{\phi^{-1}}_0(m)(x)) = 0.$$ Thus

$$\text{ZFC} \vdash P^\phi(m, x) \iff P^{\phi^{-1}}_0(m, x),$$

and $\text{ZFC} \vdash \exists x P^\phi(m, x) \iff \exists x P^{\phi^{-1}}_0(m, x).$

To deal with $\forall$ recall that $\phi$ is 1–1 and onto. Then $\text{ZFC} \vdash \forall m \exists x P^\phi(m, x) \iff \forall m \exists x P^{\phi^{-1}}_0(m, x)$, and

$$\text{ZFC} \vdash \forall m \exists x P^\phi(m, x) \iff \forall n \exists x P^{\phi}(n, x)$$

(with $n = \phi^{-1}(m)$), whereas the conclusion, since the argument holds for arbitrary $\phi$. □

**Corollary 2.20** $\text{ZFC} \vdash \forall \phi \left[ \phi \text{ is a representation } \rightarrow \left( \forall m \exists y P^\phi(m, y) \iff f^\phi_0 \text{ is total} \right) \right].$ □

### 3 Nonexistence of a ZFC–provably total $f^0_0$

In what follows we will frequently refer to “machines” for the sake of a more intuitive argument.

**Proposition 3.1** There is a representation $\phi$ such that, if ZFC is $\Sigma_1$–sound, for every ZFC–provably total recursive unary (t.r.u.) function $F_i$, $\text{ZFC} \vdash f^\phi_0 \neq F_i$.

**Remark 3.2** Therefore either $f^\phi_0$ is partial or it is total but not ZFC–provably total.

The idea of the proof is the following: we will recursively reorder the sequence $P^0$ to obtain $P^{\phi}$ which is such that every ZFC–provably t.r.u. function will differ somewhere from $f^\phi_0$ (see Definition 2.13). More precisely:

- Recall that $P^\phi(m, x) \iff V(\langle x, P^\phi_m(x) \rangle) = 0$ and that $f^\phi_0(m) = \mu_x P^\phi(m, x)$, where $m$ is the machine code in $P^\phi$ and $x$ is the machine’s input.
- If $y = \mu_x P^\phi(m, x) = f^\phi_0(m)$, then $V(\langle y, P^\phi_m(y) \rangle) = 0$, or equivalently by definition, $P^\phi(m, y)$ holds.
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- Then if \( y' = F_i(m) \), and if \( \neg P^\phi(m, y') \) is provable, then that last statement is equivalent to \( V((y', P^\phi_m(y'))) = 1 \).
- Therefore \( y' \neq y = f^\phi_P(m) = \mu_x P^\phi(m, x) \).
- Therefore \( y' = F_i(m) \neq f^\phi_P(m) \).

We must arrange \( P^\phi \) so that for any \( i \) and for at least one \( m, y' \neq y \).

**Proof**: We will consider two sequences of functions and machines: the first one, kept fixed throughout the proof, is the sequence \( F_0, F_1, \ldots \), of ZFC–provably t.r.u. functions. The second sequence is the sequence \( P^0_0, P^0_1, \ldots \) of \( G \)–machines, which will eventually be moved to and fro during the proof.

Start theorem–proving in ZFC. Pick up all theorems of the form

\[
\text{ZFC} \vdash \forall x \in \omega \exists z \in \omega T(e, x, z) \land \forall y \in \omega (f(y) = \{e\}(y)).
\]

(1) We thus generate a r.e. sequence of Gödel numbers

\[
e_{f_0} = e_0, e_{f_1} = e_1, e_{f_2} = e_2, \ldots.
\]

Repetitions are allowed—just pick up the Gödel numbers as they come up in the sequence. Out of them obtain the sequence of Turing machines

\[\mathcal{M}_{e_0}, \mathcal{M}_{e_1}, \ldots\]

which compute the ZFC–provably recursive total functions

\[\{e_0\}(x), \{e_1\}(x), \ldots\]

We use the notation: \( F_j \) is computed by \( \mathcal{M}_{e_j} \). We then reconstruct the sequence \( F_j \) (out of the indices \( e_j \)) within ZFC, where the rest of the argument happens.

Within ZFC, let \( P^0 \) be given, ordered by the indices \( p \) of the \( P^0 \). (See Definitions 2.3 and 2.12.)

- **Step 0**: We obtain a \( f^\phi_P \) such that \( P_0 \neq f^\phi_P \). The idea is to reshuffle \( P^0 \) to suit our purposes. First, generate the list of theorems of ZFC up to

\[
\text{ZFC} \vdash \forall x \in \omega \exists z \in \omega T(e_0, x, z) \land \forall y \in \omega (f(y) = \{e_0\}(y)).
\]

Within ZFC, \( F_0 \) is computed by \( \mathcal{M}_{e_0} \). Get \( \mathcal{M}_{e_0}(0) = y'_0 \). Start from \( P^0 \). Call \( f^\phi_P(0) \) (computed out of \( P^0 \)) the “provisional” \( f^\phi_P(0) \). Now for the provisional value, either \( f^\phi_P(0) \) doesn’t exist, or \( f^\phi_P(0) = y_0 \).

- If the provisional \( f^\phi_P(0) \) doesn’t exist, then \( f^\phi_P \) is a partial function, and the proof stops here. Then the provisional value is the definitive one.
– If, for the provisional value, \( f_P^\phi(0) = y_0 \), we will ensure that \( y_0 \neq y'_0 \).

If required, we will change things (see below) so that for the definitive value \( y_0 \) we ensure that \( y_0 \neq y'_0 \).

There are again two possibilities here:

– If \( V((y'_0, P^\phi_0(y'_0))) = 1 \), proceed to the next step, and put \( P_0^\phi = P_0^\phi \),

the rest being left as in \( P^\phi \) until the next step is reached.

In that case go to an \( m = m_1 = k_0 + 1 > 0 \) (see below) and execute the next step.

– If \( V((y'_0, P^\phi_0(y'_0))) = 0 \), let \( P^\phi_{k_0} \) be the first \( G \)-bounded machine in \( P^\phi \)

such that \( V((y'_0, P^\phi_{k_0}(y'_0))) = 1 \).

(In order to make that test compute \( V((y'_0, P^\phi_j(y'_0))) \), \( j = 0, 1, 2, \ldots \),

up to the first index \( j = k_0 \) such that \( V \ldots = 1 \). Recall that we deal with total machines and that there are infinitely many machines so that \( y'_0 \) is acceptable; pick a machine like the one last described in Proposition 2.11.)

If that machine accepts all instances, then the function is partial and the proof stops here. If not, it will reject some instance \( y \), and since \( y' \) is accepted, \( y' \neq y \).

Then change places between that machine \( P^\phi_{k_0} \), and \( P_0^\phi \), so that we obtain an initial segment of \( P^\phi \), \( P^\phi_0 \) given by \( P^\phi_{k_0}, P^\phi_1, P^\phi_2, \ldots, P^\phi_{k_0-1}, P_0^\phi \).

(That sequence is relabeled:

\[ P^\phi_0 = P^\phi_{k_0}, P^\phi_1 = P_1, P^\phi_2 = P_2, \ldots, P^\phi_{k_0-1}, P^\phi_{k_0} = P_0^\phi \]

and gives the initial segment \( P^\phi_0 \) of \( P^\phi \).) Clearly here \( F_0(0) \neq f_P^\phi(0) \),

since \( f_P^\phi \) either diverges at 0 or differs there from \( F_0 \). The definitive value of \( f_P^\phi(0) \) is also determined here out of the rearranged sequence of the Ps.)

Leave step 0.

• **Step 1:** Out of the listing of theorems of ZFC obtain

\[ ZFC \vdash \forall x \in \omega \exists z \in \omega T(e_1, x, z) \land \forall y \in \omega (f(y) = \{e_1\}(y)). \]

We now obtain \( F_1 \neq f_P^\phi \). Situation is the same as in the first step, with 1 substituted throughout for 0 in the lower indices. For the nontrivial case in Step 1 we get: \( P^\phi_0 \cup \{P^\phi_{k_1}, P^\phi_{k_0+2}, \ldots, P^\phi_{k_0-1}, P^\phi_{k_0+1}\} \) (with a slight abuse of notation), which is the extension \( P^\phi_1 \) of \( P^\phi_0 \) achieved at the present step.

• ...
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• Step $n$: Obtain
  \[ \text{ZFC} \vdash \forall x \in \omega \exists z \in \omega T(e_n, x, z) \land \forall y \in \omega (f(y) = \{e_n\}(y)). \]

  We get $F_n \neq f_P^\phi$. .... Same procedure. \[ \square \]

**Claim 3.3** $P^\phi$, the reordered sequence of $G$–bounded machines that results from the extension of all initial segments $P_m^\phi$, is a recursive sequence of $G$–bounded machines.

**Proof:** The map $\phi : P^0 \to P^0$ is clearly 1–1 and total: let us be given $P_n \in P$. To compute $\phi(P_n)$, obtain an ordered initial segment $P_{\phi}^j$ as above of cardinality $> n + 1$: $\phi(P_n)$ is the machine at position $n$ in that segment $P_{\phi}^m$.

The converse operation is immediate, given $P^\phi$, out of the definition of $P^0$. \[ \square \]

**Claim 3.4** For each $i$, ZFC $\vdash f_P^\phi \neq F_i$. \[ \square \]

**Claim 3.5** For each $i$, ZFC $\vdash \exists x \left[ \neg P^\phi(x, F_i(x)) \right]$. \[ \square \]

**Corollary 3.6** ZFC is $\Sigma_1$–sound (see Definition 2.1) if and only if ZFC $\not\vdash f_P^\phi$ is total.

**Proof:** We have shown that a proof of "$f_P^\phi$ is total" cannot appear in the listing of all proofs of ZFC. \[ \square \]

From the preceding claims and from Proposition 2.19 and Corollaries 2.20 and 3.6.

**Corollary 3.7** If ZFC is $\Sigma_1$–sound then ZFC plus [The partial recursive function $f_P^0$ isn’t total] is consistent. \[ \square \]

### 4 Application: results about the $NP$ class

We consider here the $NP$ class of problems.

**The satisfiability problem**

**Remark 4.1** The motivation here comes from the satisfiability problem for Boolean expressions in conjunctive normal form (cnf).

Suppose that we write a predicate $A^\phi(m, x)$ which means “polynomial machine $P_m^\phi$ accepts $x$,” that is, $P_m^\phi$ inputs a binary string $x$ which polynomially codes a Boolean expression in cnf and outputs another binary string $s$ which satisfies that same instance $x$:
• We can in fact polynomially encode each Boolean expression in cnf as a binary string; that coding may be constructed as an onto map Sat → ω, where Sat is the set of all such Boolean expressions (tautologies and totally false expressions excluded).

• If \( x \in \text{Sat} \), adequately encoded, and if \( P_n \) is the \( n \)-th polynomial machine in the listing given above, let us write \( s = P_n(x) \).

• \( V \) is the “verifying machine” that polynomially checks whether \( s \) satisfies \( x \) or not. That is the meaning of \( V(x, s) = 1 \) (\( s \) satisfies \( x \)) or \( V(x, s) = 0 \) (\( s \) doesn’t satisfy \( x \)).

The conditions we impose on \( V \) (see Definition 2.10) arise out of that motivation.

Polynomial machines

Remark 4.2 Let the set \( G \) (Definition 2.4) be the set of polynomials:

1. \( p_0(x) = x^3 + 3 \).
2. \( p_n(x) = x^{n+3} + (n + 3) \).

Proposition 4.3 The set of \( G \)-bounded Turing machines whose bounds are given by Remark 4.2 is (essentially) the set of all polynomial machines.

Proof: Every polynomial machine can be represented as a pair \( \langle M_n, [p_k] \rangle \). And every Turing machine coupled to a polynomial clock such as \( \langle M_n, [p_k] \rangle \) is polynomial. (See [1].) Throughout this Section, \( P^0 \) is a set of expressions that represent polynomial Turing machines. Let \( V(x, s) \) be the (fixed) polynomial machine that checks whether \( s \) solves instance \( x \) (Definition 2.10). If \( V(x, s) = 1 \), we say that \( s \) satisfies \( x \).

The NP class of problems

Next definition characterizes the NP class [1, 12]:

Definition 4.4 Let \( x, y \) be binary words (identified to the corresponding numbers), let \( p \) be a polynomial, and let \( R(x, y) \) be a 2-place recursive polynomial predicate. Then:

1. \( x \in A_{p,R} \iff \exists y [ |y| \leq p(|x|) \land R(x, y)] \).

2. \( NP = \{ A_{p,R} : p \) is a polynomial and \( R \) is a 2-place recursive polynomial predicate} \).
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Then, after Definition 4.4:

**Definition 4.5** $x \in \text{SAT} \iff \exists s \left[ |s| \leq p_*(|x|) \land P(x) = s \land V((x, s)) = 1 \right]$, where $p_*$ is a fixed polynomial, and $P$ is a polynomial machine. □

**Remark 4.6** The condition $p_*(|x|) \geq |s|$ directly follows from the satisfiability problem. □

**Definition 4.7** For $P^\phi_m \in P^\phi$, $A^\phi(m, x) \iff \text{Def}\left[V(\langle x, P^\phi_m(x) \rangle) = 1 \right] \iff \exists s [P^\phi_m(x) = s \land V(\langle x, s \rangle) = 1]$, where $p^\phi_*$ is a fixed polynomial, and $P$ is a polynomial machine.

**Definition 4.8** (Formalization of $P < NP$ for SAT.) The $P < NP$ conjecture (for SAT) in ZFC is:

$$\forall m \in \omega \exists x \in \omega \neg A^0(m, x).$$

**Proposition 4.9** If $f^0_A(m) = \mu_x \neg A^0(m, x)$, then

$$ZFC \vdash (\forall m \in \omega \exists x \in \omega \neg A^0(m, x)) \iff [f^0_A \text{ is total}],$$

If ZFC is $\Sigma_1$-unsound then ZFC $\vdash P < NP$.

**Lemma 4.10** Let $R$ be a recursively axiomatizable simply consistent extension of PA which is not $\Sigma_1$-sound. Then every partial recursive function is equal to a function which is provably total from $R$. That is, for each $e$, there is an $a$ such that $\varphi_e = \varphi_a$, and $R \vdash [\varphi_a \text{ is total}].$

**Proof**: To construct $a$, fix a primitive recursive predicate $Q(x)$ such that $R \vdash \exists x Q(x)$, but such that $Q(n)$ is false for each natural number $n$. (Possible, since $R$ isn’t $\Sigma_1$-sound.) In Kleene Normal Form, we have $\varphi_e(x) = U(\mu_y T(e, x, y))$, where $U, T$ are as in the reference; recall that they are primitive recursive. Then define $\varphi_a(x) = U[\mu_y (T(e, x, y) \lor Q(y))]$. $\varphi_a = \varphi_e$, since $Q(y)$ is always false, but $R \vdash \forall x \exists y [T(e, x, y) \lor Q(y)]$, so $R \vdash [\varphi_a \text{ is total}].$ □

**Remark 4.11** In order to obtain one such $R$, for example pick up PA and if $P$ is primitive recursive and $\forall x \neg Q(x)$ is true while independent of PA, take $R = \text{PA} + \exists x Q(x).$ □

**Corollary 4.12** If ZFC isn’t $\Sigma_1$-sound, then ZFC $\vdash P < NP$.

**Proof**: From Proposition 4.4 and Lemma 4.10. □
If $\text{ZFC}$ is $\Sigma_1$–sound then $\text{ZFC} \nvdash P < NP$

**Proposition 4.13** If $\text{ZFC}$ is $\Sigma_1$–sound, then:

1. $f_0^A$ isn’t $\text{ZFC}$–provably total.
2. $\text{ZFC} \nvdash P < NP$.
3. $\text{ZFC}$ is consistent if and only if $\text{ZFC} + (P = NP)$ is consistent.

*Proof:* From Proposition 4.9 and Corollary 3.7.

**Corollary 4.14** If $\text{ZFC}$ is consistent, then $\text{ZFC}$ is $\Sigma_1$–sound if and only if $\text{ZFC} \nvdash P < NP$.

**Remark 4.15** We’ve just added Corollary 4.14 as a special assertion in order to emphasize the depth of the $P?NP$ question, which leads to such a beautiful interplay between formal systems and the surrounding metamathematics. The previous results are certainly valid for any $\Sigma_1$–sound fragment of $\text{ZFC}$ which contains $\text{PA}$.

**On the relativized result** $\text{ZFC} \vdash P^A < NP^A$

**Remark 4.16** We comment here on the well–known results in [1]. Some preliminary remarks must be made before we get into details:

- We first notice that the Turing machines considered in the present paper paper are of a more restricted kind than those in the reference, as (in our case) the only special state is $s_0$ (see Remark 3.2). Actually our special machine $V$ stands for their “accepting states,” with several restrictions added (see Definition 2.10).

  The restrictions we impose on $V$ are of course modelled on the way one mechanically verifies the validity of a choice of truth values for Boolean expressions in $\text{cnf}$, and on their possible choices.

- We also recall that, for oracle machines where the oracle is realized as an extra tape in a multitape Turing machine with the oracle set written on it, if $P = NP$ then $P^A = NP^A$, for any oracle $A$ ([1], Remark in p. 437).

- Yet in [1] one supposes that the oracle is consulted, and gives its answer, in a single step.

  But the point is, in which way (if any) does our construction in Proposition 3.1 conflict with, say, the one in the same Theorem 3?
Total recursive function, I

First of all, in their proof of Theorem 3 they use the oracle in an essential way, that is, all machines must nontrivially refer to the oracle during their computations. That is to say, we exclude situations like the machine with the oracle tape which is never accessed by its program, or the machine that proceeds in the same direction never caring whether it does get a “yes” or “no” from the oracle.

So, from start we notice that the freedom we enjoyed in moving machines to and fro in the proof of Proposition 3.1 may not be available in the (very restrictive) relativized setting, as we proceed with the constructions in the reference.

But let us consider the proof of Theorem 3 in the reference [1]: in Theorem 3 one aims at the construction of a language $L(B)$ which will be rejected by every polynomial oracle machine $P_i^B$ which nontrivially consults with oracle $B$. That is to say, at least one instance $x \in L(B)$ isn’t accepted by each $P_i^B$.

If $B$ is the oracle, then $L(B) = \{y : |y| = |x| \land x \in B\}$.

The proof in the reference proceeds by a stepwise construction of $B$: if step $i$ precedes step $j$, then $B_i \subseteq B_j$, and $B = \bigcup_i B_i$. Moreover, $B$ can be made recursive.

Once one has $L(B)$, we can obtain a provably total recursive function $h_B(i) = \min(y : |y| = |x_i| \land x_i \in B)$.

It is clear that $h_B(i) \neq f_{3_A}^B(i)$.

The essential point is: it is not at all clear whether we would be able to make the permutations which are essential to the proof of Proposition 3.1 in the present restricted, relativized setting.

On the diagonalization procedure used here

Remark 4.17 We must also add a few comments on the diagonalization used here. Diagonal constructions have been used out of a ‘lower’ class $X$ to obtain objects in a ‘higher’ class $Y \supset X$. More precisely, out of some listing of classes of $NP$–languages which are known to be in $P$, we try to obtain via a diagonal construction a language which isn’t in $P$.

Our procedure in this paper elaborates on a well–known construction (see [10], p. 54, Ex. 5). We temporarily forget about $P$ and $NP$ and just consider the relation between $f_0^P$ and the listing of all $ZFC$–provably total unary recursive functions. We also try to impose some properties on that $f_0^P$, such that it can be proved to be Turing–computable, and one can prove that there is a Gödel number for the corresponding machine.
The diagonal construction is then used to show that such a function differs from each one of the ZFC–provable t.r.u. functions at least once. To put it in another way, instead of using a single sequence of functions to diagonalize out of it, we couple two sequences, the fundamental one—the sequence of the $F_i$—and the subordinate sequence, the $P_j$. □

Remark 4.18 M. Benda has pointed out to the authors that their original construction leads to a whole hierarchy of problems of the form $G?NG$, which is why we have decided to frame our results in that wider setting from start \[2\]. □

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