Convergence of random sums and statistics constructed from samples with random sizes to the Linnik and Mittag-Leffler distributions and their generalizations

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Abstract: We present some product representations for random variables with the Linnik, Mittag-Leffler and Weibull distributions and establish the relationship between the mixing distributions in these representations. Based on these representations, we prove some limit theorems for a wide class of rather simple statistics constructed from samples with random sizes including, e. g., random sums of independent random variables with finite variances, maximum random sums, extreme order statistics, in which the Linnik and Mittag-Leffler distributions play the role of limit laws. Thus we demonstrate that the scheme of geometric summation is far not the only asymptotic setting (even for sums of independent random variables) in which the Mittag-Leffler and Linnik laws appear as limit distributions. The two-sided Mittag-Leffler and one-sided Linnik distribution are introduced and also proved to be limit laws for some statistics constructed from samples with random sizes.

Key words: Linnik distribution; Mittag-Leffler distribution; exponential distribution; Weibull distribution; Laplace distribution; strictly stable distribution; random sum; central limit theorem; normal scale mixture; half-normal distribution; extreme order statistic; sample with random size

1 Introduction

Usually the Mittag-Leffler and Linnik distributions are mentioned in the literature together as examples of geometric stable distributions. Since these distributions are very often pointed at as weak limits for geometric random sums, there might have emerged a prejudice that the scheme of geometric summation is the only asymptotic setting within which these distributions can be limiting for sums of independent and identically distributed random variables. This prejudice is accompanied by the suspicion that non-trivial (δ < 1, α < 2) Mittag-Leffler and Linnik laws can be limiting only for sums in which the summands have infinite variances.

The aim of this paper is to dispel this prejudice by presenting some examples of limit theorems for a wide class of rather simple statistics constructed from samples with random sizes including, e. g., random sums of independent random variables with finite variances, maximum random sums, extreme order statistics in which the Linnik and Mittag-Leffler distributions play the role of limit laws. We will demonstrate that the scheme of geometric summation is far not the only asymptotic setting (even for sums of independent random variables!) in which the Mittag-Leffler and Linnik laws appear as limit distributions.

The main tools used to prove the limit theorems in this paper are mixture representations for the Linnik, Mittag-Leffler and Weibull distributions also presented here. Some of these representations were known (mixture representations for the Linnik and Mittag-Leffler laws were the objects of investigation in [7, 9, 10, 29, 37, 32, 33]), some of them are new. These representations open the way to establish the close analytic and asymptotic relations between these two laws.

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Another obvious reason for which the Mittag-Leffler and Linnik distributions are often brought
together in the literature is the formal similarity of the Laplace transform of the former and the Fourier–
Stieltjes transform of the latter. We will show that actually the link between these two laws (and some
laws related to them) is much more interesting than the formal coincidence of their transforms. We
develop the known results on mixture representability of the Linnik and Mittag-Leffler distributions and
prove some new results of this kind, thus finding a tight and clear analytical link between the Linnik,
Mittag-Leffler, stable and related distributions. For example, it turns out that the Linnik distribution
with parameter $\alpha$ is a scale mixture of the normal distributions with the mixing distribution being
the Mittag-Leffler law with parameter $\delta = \alpha/2$. Product representations for the random variables
with the Linnik and Mittag-Leffler distributions obtained in the previous works were aimed at the
construction of convenient algorithms for the computer generation of pseudo-random variables with
these distributions. At the same time, mixture representation for the Linnik distribution as a scale
mixture of normals opens the way for the construction of a random-sum central limit theorem with
the Linnik distribution as the limit law. Moreover, in the “if and only if” version of the random-sum
central limit theorem presented in this paper the Mittag-Leffler distribution must be the limit law for
the normalized number of summands.

Strange as it may seem, the results concerning the possibility of representation of the Linnik
distribution as a scale mixture of normals have never been explicitly presented in the literature in
full detail although the property of the Linnik distribution to be a normal scale mixture is something
almost obvious. Perhaps, the paper [29] is the closest to this conclusion and exposes the representability
of the Linnik law as a scale mixture of Laplace distributions with the mixing distribution written out
explicitly.

Other examples of the results obtained here are the representations of the Mittag-Leffler law as a
scale mixture of Weibull or half-normal distributions, based on which we prove theorems establishing
the conditions for the distributions of extreme order statistics in samples with random sizes or maximum
random sums of independent random variables with finite variances to converge to the Mittag-Leffler
law.

The paper is organized as follows. Section 2 presents the definitions and basic properties of the
Linnik and Mittag-Leffler distributions. Section 3 contains basic definitions and auxiliary results. The
proofs of our main results are purposely indirect and essentially rely on some new mixture properties
of the Weibull distribution also presented in Section 3. In Section 4 we prove the representability of the
Linnik distribution as the scale mixture of normal laws with the Mittag-Leffler mixing distribution.
We use this result to describe the asymptotics of the tail behavior of the Linnik distribution. Here
we also obtain the representation of the Linnik distribution as a scale mixture of the Laplace laws
with the mixing distribution explicitly determined as that of the ratio of two independent random
variables with the same strictly stable distribution concentrated on the nonnegative halfline. We use
this representation together with the result of [29] to obtain a by-product corollary which is the explicit
representation of the distribution density of the ratio of two independent positive strictly stable random
variables, thus giving a new proof of a result of [3]. In Section 5 we prove some representations of
the Mittag-Leffler distribution as a mixed exponential or a mixed half-normal law. In Section 6 we
prove and discuss some criteria (that is, necessary and sufficient conditions) for the convergence of
the distributions of rather simple statistics constructed from samples with random sizes including,
e. g., random sums of independent random variables with finite variances, maximum random sums,
and the Linnik and Mittag-Leffler laws. The asymptotic theory of extreme
values in samples with random sizes is well-developed. The basics of this theory were presented, say, in
[3, 2, 39, 15]. A detailed review of this theory can be found in [11]. Dealing with extreme order statistics
in Section 6 we consider a special but rather important case where the sample size is generated by a
doubly stochastic Poisson process and consider the asymptotic behavior of the so-called max-compound
Cox processes introduced and studied in [23]. Here we also present two examples of the construction of
“appropriate” random indices possessing the desired asymptotic properties. In Section 7 the symmetric
two-sided Mittag-Leffler distribution and the one-sided Linnik distribution are introduced. Here we
prove theorems stating that these laws can also be limit distributions for statistics constructed from samples with random sizes such as random sums of independent random variables with finite variances, maximum random sums or extreme order statistics.

2 The Mittag-Leffler and Linnik distributions

2.1 The Mittag-Leffler distributions

The Mittag-Leffler probability distribution is the distribution of a nonnegative random variable $M_\delta$ whose Laplace transform is

$$\psi_\delta(s) \equiv \mathbb{E}e^{-sM_\delta} = \frac{1}{1 + \lambda s^\delta}, \quad s \geq 0,$$

(1)

where $\lambda > 0$, $0 < \delta \leq 1$. For simplicity, in what follows we will consider the standard scale case and assume that $\lambda = 1$.

The origin of the term Mittag-Leffler distribution is due to that the probability density corresponding to Laplace transform (1) has the form

$$f_{M_\delta}(x) = \frac{1}{x^{1-\delta}} \sum_{n=0}^{\infty} (-1)^n x^{\delta n} \frac{x^{\delta n}}{\Gamma(\delta n + 1)}, \quad x \geq 0,$$

(2)

where $E_\delta(z)$ is the Mittag-Leffler function with index $\delta$ that is defined as the power series

$$E_\delta(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\delta n + 1)}, \quad \delta > 0, \quad z \in \mathbb{Z}.$$

Here $\Gamma(s)$ is Euler’s gamma-function,

$$\Gamma(s) = \int_0^{\infty} z^{s-1} e^{-z} dz, \quad s > 0.$$

The distribution function corresponding to density (2) will be denoted $F_{M_\delta}(x)$.

With $\delta = 1$, the Mittag-Leffler distribution turns into the standard exponential distribution, that is, $F_1(x) = [1 - e^{-x}] \mathbf{1}(x \geq 0)$, $x \in \mathbb{R}$ (here and in what follows the symbol $\mathbf{1}(C)$ denotes the indicator function of a set $C$). But with $\delta < 1$ the Mittag-Leffler distribution density has the heavy power-type tail: from the well-known asymptotic properties of the Mittag-Leffler function it can be deduced that if $0 < \delta < 1$, then

$$f_{M_\delta}(x) \sim \frac{\sin(\delta \pi)\Gamma(\delta + 1)}{\pi x^{\delta+1}}$$

(3)

as $x \to \infty$, see, e. g., [17].

It is well-known that the Mittag-Leffler distribution is stable with respect to geometric summation (or geometrically stable). This means that if $X_1, X_2, \ldots$ are independent random variables and $V_p$ is the random variable independent of $X_1, X_2, \ldots$ and having the geometric distribution

$$\mathbb{P}(V_p = n) = p(1 - p)^{n-1}, \quad n = 1, 2, \ldots, \quad p \in (0, 1),$$

(4)

then for each $p \in (0, 1)$ there exists a constant $a_p > 0$ such that $a_p(X_1 + \ldots + X_{V_p}) \Longrightarrow M_\delta$ as $p \to 0$, see, e. g., [10] or [20] (the symbol $\Longrightarrow$ hereinafter denotes convergence in distribution). Moreover, as far ago as in 1965 it was shown by I. Kovalenko [31] that the distributions with Laplace transforms (1) are the only possible limit laws for the distributions of appropriately normalized geometric sums of the form $a_p(X_1 + \ldots + X_{V_p})$ as $p \to 0$, where $X_1, X_2, \ldots$ are independent identically distributed nonnegative random variables and $V_p$ is the random variable with geometric distribution (4) independent of the sequence $X_1, X_2, \ldots$ for each $p \in (0, 1)$. The proofs of this result were reproduced in [13, 14] and [12].
In these books the class of distributions with Laplace transforms (1) was not identified as the class of Mittag-Leffler distributions but was called class $K$ after I. Kovalenko.

Twenty five years later this limit property of the Mittag-Leffler distributions was re-discovered by A. Pillai in [39, 40] who proposed the term Mittag-Leffler distribution for the distribution with Laplace transform (1). Perhaps, since the works [31, 13, 14] were not easily available to probabilists, the term class $K$ distribution did not take roots in the literature whereas the term Mittag-Leffler distribution became conventional.

The Mittag-Leffler distributions are of serious theoretical interest in the problems related to thinned (or rarefied) homogeneous flows of events such as renewal processes or anomalous diffusion or relaxation phenomena, see [43, 16] and the references therein.

2.2 The Linnik distributions

In 1953 Yu. V. Linnik [35] introduced the class of symmetric probability distributions defined by the characteristic functions

$$f_{\alpha}^L(t) = \frac{1}{1 + |t|^\alpha}, \quad t \in \mathbb{R},$$

(5)

where $\alpha \in (0, 2]$. Later the distributions of this class were called Linnik distributions [30] or $\alpha$-Laplace distributions [39]. In this paper we will keep to the first term that has become conventional. With $\alpha = 2$, the Linnik distribution turns into the Laplace distribution corresponding to the density

$$f^\Lambda(x) = \frac{1}{2}e^{-|x|}, \quad x \in \mathbb{R}.$$  

(6)

A random variable with Laplace density (6) and its distribution function will be denoted $\Lambda$ and $F^\Lambda(x)$, respectively.

The Linnik distributions possess many interesting analytic properties such as unimodality [34] and infinite divisibility [7], existence of an infinite peak of the density for $\alpha \leq 1$ [7], etc. In [27, 28] a detailed investigation of analytic and asymptotic properties of the density of the Linnik distribution was carried out. However, perhaps, most often Linnik distributions are recalled as examples of geometric stable distributions.

A random variable with the Linnik distribution with parameter $\alpha$ will be denoted $L_{\alpha}$. Its distribution function and density will be denoted $F_{\alpha}^L$ and $f_{\alpha}^L$, respectively. As this is so, from (5) and (6) it follows that $F_{2}^L(x) \equiv F^\Lambda(x), x \in \mathbb{R}$.

3 Basic notation and auxiliary results

Most results presented below actually concern special mixture representations for probability distributions. However, without any loss of generality, for the sake of visuality and compactness of formulations and proofs we will represent the results in terms of the corresponding random variables assuming that all the random variables mentioned in what follows are defined on the same probability space $(\Omega, \mathcal{A}, P)$.

The random variable with the standard normal distribution function $\Phi(x)$ will be denoted $X$,

$$P(X < x) = \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2} dz, \quad x \in \mathbb{R}.$$

Let $\Psi(x), x \in \mathbb{R},$ be the distribution function of the maximum of the standard Wiener process on the unit interval, $\Psi(x) = 2\Phi(\max\{0, x\}) - 1, x \in \mathbb{R}$. It is easy to see that $\Psi(x) = P(|X| < x)$. Therefore, sometimes $\Psi(x)$ is said to determine the half-normal distribution.

Throughout the paper the symbol $\d equiv$ will denote the coincidence of distributions.

The distribution function and the density of the strictly stable distribution with the characteristic exponent $\alpha$ and shape parameter $\theta$ defined by the characteristic function

$$g_{\alpha, \theta}(t) = \exp\left\{ -|t|^\alpha \exp\left(-\frac{i\pi \theta}{2} \text{sign} t\right)\right\}, \quad t \in \mathbb{R},$$

(7)
with $0 < \alpha \leq 2$, $|\theta| \leq \min\{1, \frac{2}{\alpha} - 1\}$, will be denoted by $G_{\alpha,\theta}(x)$ and $g_{\alpha,\theta}(x)$, respectively (see, e. g., [44]). Any random variable with the distribution function $G_{\alpha,\theta}(x)$ will be denoted $S_{\alpha,\theta}$.

From (7) it follows that the characteristic function of a symmetric ($\theta = 0$) strictly stable distribution has the form

$$g_{0,0}(t) = e^{-|t|^\alpha}, \quad t \in \mathbb{R}. \quad (8)$$

**Lemma 1.** Let $\alpha \in (0, 2]$, $\alpha' \in (0, 1]$. Then

$$S_{\alpha,\alpha',0} \overset{d}{=} S_{0,\alpha,0} S_{1/\alpha,1}^{1/\alpha}$$

where the random variables on the right-hand side are independent.

**Proof.** See, e. g., theorem 3.3.1 in [44].

**Corollary 1.** A symmetric strictly stable distribution with the characteristic exponent $\alpha$ is a scale mixture of normal laws in which the mixing distribution is the one-sided strictly stable law ($\theta = 1$) with the characteristic exponent $\alpha/2$:

$$S_{\alpha,0} \overset{d}{=} X \sqrt{S_{\alpha/2,1}} \quad (9)$$

with the random variables on the right-hand side being independent.

In terms of distribution functions the statement of corollary 1 can be written as

$$G_{\alpha,0}(x) = \int_0^\infty \Phi\left(\frac{x}{\sqrt{z}}\right) dG_{\alpha/2,1}(z), \quad x \in \mathbb{R}.$$  

Let $\gamma > 0$. The distribution of the random variable $W_\gamma$:

$$P(W_\gamma < x) = [1 - e^{-x^\gamma}] I(x \geq \gamma), \quad x \in \mathbb{R},$$

is called the Weibull distribution with shape parameter $\gamma$. It is obvious that $W_1$ is the random variable with the standard exponential distribution; $P(W_1 < x) = [1 - e^{-x}] I(x \geq 0)$. The Weibull distribution with $\gamma = 2$, that is, $P(W_2 < x) = [1 - e^{-x^2}] I(x \geq 0)$ is called the Rayleigh distribution.

It is easy to see that if $\gamma > 0$ and $\gamma' > 0$, then $P(W_{\gamma/\gamma'} \geq x) = P(W_{\gamma'} \geq x^{\gamma'}) = e^{-x^{\gamma'}} = P(W_{\gamma'\gamma'} \geq x), x \geq 0$, that is, for any $\gamma > 0$ and $\gamma' > 0$

$$W_{\gamma'\gamma'} \overset{d}{=} W_{\gamma'}^{1/\gamma'} \quad (10)$$

It can be shown that each Weibull distribution with parameter $\gamma \in (0, 1]$ is a mixed exponential distribution. In order to prove this we first make sure that each Weibull distribution with parameter $\gamma \in (0, 2]$ is a scale mixture of the Rayleigh distributions.

For $\alpha \in (0, 1]$ denote $T_\alpha = S_{\alpha,1}^{-1}$, where $S_{\alpha,1}$ is a random variable with one-sided strictly stable density $g_{\alpha,1}(x)$.

**Lemma 2.** For any $\gamma \in (0, 2]$ we have

$$W_\gamma \overset{d}{=} W_2 \sqrt{T_{\gamma/2}},$$

where the random variables on the right-hand side are independent.

**Proof.** Write relation (9) in terms of characteristic functions with the account of (7):

$$e^{-|t|^\alpha} = \int_0^\infty \exp\left(-\frac{1}{2} t^2 z\right) g_{\alpha/2,1}(z) dz, \quad t \in \mathbb{R}. \quad (11)$$

Formally letting $|t| = x$ in (11), where $x \geq 0$ is an arbitrary nonnegative number, we obtain

$$P(W_\gamma > x) = e^{-x^\gamma} = \int_0^\infty \exp\left(-\frac{1}{2} x^2 z\right) g_{\gamma/2,1}(z) dz. \quad (12)$$
At the same time it is obvious that if $W_2$ and $S_{\gamma/2,1}$ are independent, then
\[
P(W_2 \sqrt{T_{\gamma/2}} > x) = P(W_2 > x \sqrt{\frac{1}{2} S_{\gamma/2,1}}) = \int_0^\infty \exp\{-\frac{1}{2} x^2 z\} g_{\gamma/2,1}(z)dz.
\] (13)

Since the right-hand sides of (12) and (13) coincide identically in $x \geq 0$, the left-hand sides of these relations coincide as well. The lemma is proved.

**Lemma 3.** For any $\gamma \in (0,1]$, the Weibull distribution with parameter $\gamma$ is a mixed exponential distribution:
\[
W_\gamma \overset{d}{=} W_1 T_\gamma,
\] (14)
where the random variables on the right-hand side of (14) are independent.

**Proof.** It is easy to see that $P(W_1^{1/\gamma} \geq x) = P(W_1 \geq x^\gamma) = e^{-x^\gamma} = P(W_\gamma \geq x)$, $x \geq 0$, that is,
\[
W_\gamma \overset{d}{=} W_1^{1/\gamma}
\] (15)

for any $\gamma > 0$. From (15) it follows that $W_2 \overset{d}{=} \sqrt{W_1}$. Therefore, from lemma 2 it follows that for $\gamma \in (0,2]$ we have
\[
W_\gamma \overset{d}{=} W_2 \sqrt{T_{\gamma/2}} = \sqrt{W_1 T_{\gamma/2}}
\]
or, with the account of (15),
\[
W_{\gamma/2} \overset{d}{=} W_2^{1/2} W_1 T_{\gamma/2}.
\]
Re-denoting $\gamma/2 \mapsto \gamma \in (0,1]$, we obtain the desired assertion.

In [2] the following statement was proved. Here its formulation is extended with the account of (10).

**Lemma 4** [7]. For any $\alpha \in (0,2]$, the Linnik distribution with parameter $\alpha$ is a scale mixture of a symmetric stable distribution with the Weibull mixing distribution with parameter $\alpha/2$, that is,
\[
L_\alpha \overset{d}{=} S_{\alpha,0} W_\alpha \overset{d}{=} S_{\alpha,0} \sqrt{W_{\alpha/2}},
\]
where the random variables on the right-hand side are independent.

**Lemma 5.** For any $\delta \in (0,1]$, the Mittag-Leffler distribution with parameter $\delta$ is a scale mixture of a one-sided stable distribution with the Weibull mixing distribution with parameter $\delta/2$, that is,
\[
M_\delta \overset{d}{=} S_{\delta,1} W_\delta \overset{d}{=} S_{\delta,1} \sqrt{W_{\delta/2}},
\]
where the random variables on the right-hand side are independent.

**Proof.** This statement has already become folklore. For the purpose of convenience we give its elementary proof without any claims for priority. Let $S_{\delta,1}$ be a positive strictly stable random variable. As is known, its Laplace transform is $\psi(s) = E e^{-s S_{\delta,1}} = e^{-s^\delta}$, $s \geq 0$. Then with the account of (10) by the Fubini theorem the Laplace transform of the product $S_{\delta,1} W_\delta$ is
\[
E \exp\{-s S_{\delta,1} W_\delta\} = E \exp\{-s S_{\delta,1} W_1^{1/\delta}\} = E E(\exp\{-s S_{\delta,1} W_1^{1/\delta}\}|W_1) = \int_0^\infty e^{-(s z^{1/\delta})} e^{-z}dz = 
\]
\[
= \int_0^\infty e^{-z(s^{1/\delta}+1)}dz = \frac{1}{1+s^\delta} = E e^{-s M_\delta}, \quad s \geq 0.
\]
The lemma is proved.

Let $\rho \in (0,1)$. In [32] it was demonstrated that the function
\[
f^{K}_\rho(x) = \frac{\sin(\pi \rho)}{\pi \rho(x^2 + 2x \cos(\pi \rho) + 1)}, \quad x \in (0,\infty),
\] (16)
is a probability density on $(0, \infty)$. Let $K_\rho$ be a random variable with density (16).

**Lemma 6** [32]. Let $0 < \delta < \delta' \leq 1$ and $\rho = \delta/\delta' < 1$. Then

$$M_\delta \overset{d}{=} M_\rho K_\rho^{1/\delta}$$

where the random variables on the right-hand side are independent.

With $\delta' = 1$ we have

**Corollary 2** [32]. Let $0 < \delta < 1$. Then the Mittag-Leffler distribution with parameter $\delta$ is mixed exponential:

$$M_\delta \overset{d}{=} K_\delta^{1/\delta} W_1$$

where the random variables on the right-hand side are independent.

Let $0 < \alpha < \alpha' \leq 2$. In [29] it was shown that the function

$$f_{\alpha,\alpha'}^Q(x) = \frac{\alpha' \sin(\pi \alpha/\alpha') x^{\alpha-1}}{\pi[1 + x^{2\alpha} + 2x^\alpha \cos(\pi \alpha/\alpha')]} , \quad x > 0, \quad (17)$$

is a probability density on $(0, \infty)$. Let $Q_{\alpha,\alpha'}$ be a random variable whose probability density is $f_{\alpha,\alpha'}^Q(x)$.

**Lemma 7** [29]. Let $0 < \alpha < \alpha' \leq 2$. Then

$$L_\alpha \overset{d}{=} L_{\alpha'} Q_{\alpha,\alpha'},$$

where the random variables on the right-hand side are independent.

With $\alpha' = 2$ we have

**Corollary 3** [29]. Let $0 < \alpha < 2$. Then the Linnik distribution with parameter $\alpha$ is a scale mixture of Laplace distributions corresponding to density (5):

$$L_\alpha \overset{d}{=} \Lambda Q_{\alpha,2}$$

where the random variables on the right-hand side are independent.

For the sake of completeness, we will demonstrate that the Weibull distributions possess the same property as the Linnik and Mittag-Leffler distributions presented in lemmas 6 and 7: any distribution of the corresponding class can be represented as a scale mixture of a distribution from the same class with larger parameter.

Relation (14) implies the following statement generalizing lemmas 2 and 3 and stating that the Weibull distribution with an arbitrary positive shape parameter $\gamma$ is a scale mixture of the Weibull distribution with an arbitrary positive shape parameter $\gamma' > \gamma$.

**Lemma 8.** Let $\gamma' > \gamma > 0$ be arbitrary numbers. Then

$$W_\gamma \overset{d}{=} W_{\gamma'} \cdot T_{\alpha}^{1/\gamma},$$

where $\alpha = \gamma/\gamma' \in (0, 1)$ and the random variables on the right-hand side are independent.

**Proof.** In lemma 3 we showed that a Weibull distribution with parameter $\alpha \in (0, 1]$ is a mixed exponential distribution. Indeed, from (14) it follows that

$$e^{-x^\alpha} = P(W_\alpha > x) = P(W_1 > \frac{1}{2} S_{\alpha,1} x) = \int_0^\infty e^{-\frac{1}{2} x^2} g_{\alpha,1}(z) dz, \quad x \geq 0.$$

Therefore, for any $\gamma' > \gamma > 0$, denoting $\alpha = \gamma/\gamma'$ (as this is so, $\alpha \in (0, 1)$), for any $x \in \mathbb{R}$ we obtain

$$P(W_\gamma > x) = e^{-x^\gamma} = e^{-x^{\gamma'}} = P(W_\alpha > x^{\gamma'}) = P(W_1 > \frac{1}{2} S_{\alpha,1} x^{\gamma'}).$$
The lemma is proved.

It should be noted that if $0 < \gamma < \gamma' < 2$, then the assertion of lemma 8 directly follows from theorem 3.3.1 of [43] due to the formal coincidence of the characteristic function of a strictly stable law and the complementary Weibull distribution function (see the proof of lemma 2).

Corollary 4. Let $\gamma \geq 1$ be arbitrary. Then the exponential distribution is a scale mixture of the Weibull laws with parameter $\gamma$:

$$W_1 \overset{d}{=} W_{\gamma'} \cdot T_{1/\gamma'},$$

where the random variables on the right-hand side are independent.

4 Representation of the Linnik distribution as a scale mixture of normal or Laplace distributions and related results

4.1 The representation of the Linnik distribution as a normal scale mixture

In all the products of random variables mentioned below the multipliers are assumed independent.

Theorem 1. Let $\alpha \in (0, 2]$, $\alpha' \in (0, 1]$. Then

$$L_{\alpha\alpha'} \overset{d}{=} S_{\alpha,0} M_{\alpha'}^{1/\alpha}.$$

Proof. From lemma 4 we have

$$L_{\alpha\alpha'} \overset{d}{=} S_{\alpha,0} S_{\alpha',0}^{1/2} W_{\alpha\alpha'}/2. \quad (18)$$

Continuing (18) with the account of lemma 1, we obtain

$$L_{\alpha\alpha'} \overset{d}{=} S_{\alpha,0} S_{\alpha',1}^{1/\alpha} W_{\alpha\alpha'}/2. \quad (19)$$

From (10) and lemma 5 it follows that

$$S_{\alpha',1}^{1/\alpha} W_{\alpha\alpha'}/2 \overset{d}{=} S_{\alpha',1}^{1/\alpha} W_{\alpha\alpha'} \overset{d}{=} (S_{\alpha',1} W_{\alpha'})^{1/\alpha} \overset{d}{=} M_{\alpha'}^{1/\alpha}. $$

The theorem is proved.

As far as we know, the following result has never been explicitly presented in the literature in full detail although the property of the Linnik distribution to be a normal scale mixture is something almost obvious.

Corollary 5. For each $\alpha \in (0, 2]$, the Linnik distribution with parameter $\alpha$ is the scale mixture of zero-mean normal laws with mixing Mittag-Leffler distribution with twice less parameter $\alpha/2$:

$$L_{\alpha} \overset{d}{=} X \sqrt{M_{\alpha/2}},$$

where the random variables on the right-hand side are independent.
4.2 The tail behavior of the Linnik distribution

From (20) we can easily characterize the tail behavior of the Linnik distribution. For this purpose we will use the following statement proved in [1].

**Lemma 9** [1]. *If a distribution function $F(x)$ has the form

$$ F(x) = \int_0^\infty \Phi\left(\frac{x}{\sqrt{u}}\right)dG(u), \quad x \in \mathbb{R}, $$

where $G(u)$ is a distribution function such that $G(0) = 0$, and $\rho$ and $C$ are positive numbers, then the conditions

$$ \limsup_{x \to \infty} x^\rho [1 - F(x)] = C $$

and

$$ \limsup_{u \to \infty} u^{\rho/2} [1 - G(u)] = 2C $$

are equivalent.*

From lemma 9 with the account of (3) and (20) we obtain the following statement.

**Corollary 6.** *The tail behavior of the Linnik distribution $L_\alpha$ with parameter $\alpha \in (0, 2)$ as $x \to \infty$ is described by the relation

$$ \limsup_{x \to \infty} x^{\alpha/2} [1 - L_\alpha(x)] = \frac{\alpha}{2\pi} \sin\left(\frac{\alpha\pi}{2}\right) \Gamma\left(\frac{\alpha}{2} + 1\right). $$

In other words, the above reasoning gives one more proof that if $0 < \alpha < 2$, then $1 - L_\alpha(x) = O(x^{-\alpha/2})$ as $x \to \infty$, not involving the tail properties of stable distributions.*

4.3 The representation of the Linnik distribution as a scale mixture of Laplace distributions

It should be noted that by lemma 5 representation (20) can be rewritten as

$$ L_\alpha \overset{d}{=} X \sqrt{\frac{S_{\alpha/2,1}W_{\alpha/2}}{S'_{\alpha/2,1}}}. \quad (21) $$

From lemma 3 we have

$$ W_{\alpha/2} \overset{d}{=} \frac{2W_1}{S'_{\alpha/2,1}}. $$

Hence, from (21) it follows that

$$ L_\alpha \overset{d}{=} X \sqrt{\frac{2W_1 S_{\alpha/2,1}}{S'_{\alpha/2,1}}} $$

where the independent random variables $S_{\alpha/2,1}$ and $S'_{\alpha/2,1}$ have one and the same one-sided strictly stable distribution with characteristic exponent $\alpha/2$ and are independent of the exponentially distributed random variable $W_1$. It is well known that

$$ X \sqrt{2W_1} \overset{d}{=} \Lambda \quad (22) $$

(see, e. g., the example on p. 272 of [3] or lemma 10 below). Therefore we obtain one more mixture representation for the Linnik distribution.

**Theorem 2.** *For each $\alpha \in (0, 2]$, the Linnik distribution with parameter $\alpha$ is the scale mixture of the Laplace laws corresponding to density (6) with mixing distribution being that of the ratio of*
two independent random variables having one and the same one-sided strictly stable distribution with characteristic exponent $\alpha/2$:

$$L_\alpha \overset{d}{=} \Lambda \sqrt{\frac{S_{\alpha/2,1}}{S'_{\alpha/2,1}}}$$

where the random variables on the right-hand side are independent.

It is easy to see that scale mixtures of Laplace distribution (6) are identifiable, that is, if

$$\Lambda Y \overset{d}{=} \Lambda Y'$$

where $Y$ and $Y'$ are nonnegative random variables independent of $\Lambda$, then $Y \overset{d}{=} Y'$. Indeed, with the account of (22), the last relation turns into

$$X \sqrt{2W_1 Y^2} \overset{d}{=} X \sqrt{2W_1 (Y')^2},$$

where the random the multipliers on both sides are independent. But, as is known, scale mixtures of zero-mean normals are identifiable (see [42]). Therefore, (23) implies that

$$W_1 Y^2 \overset{d}{=} W_1 (Y')^2.$$  

(24)

The complementary mixed exponential distribution functions of the random variables related by (24) are the Laplace transforms of $Y^2$ and $(Y')^2$, respectively. Relation (24) means that these Laplace transforms identically coincide:

$$\int_0^\infty e^{-sz}dP(Y^2 < z) \equiv \int_0^\infty e^{-sz}dP((Y')^2 < z), \quad s \geq 0.$$

Hence, the distributions of the random variables $Y^2$ and $(Y')^2$ coincide and hence, the distributions of $Y$ and $Y'$ coincide as well since these random variables were originally assumed nonnegative.

4.4 Some properties of the mixing distributions

Comparing the statement of theorem 2 with the assertion of corollary 3 with the account of identifiability of scale mixtures of Laplace distributions (6) we arrive at the relation

$$Q_{\alpha,2} \overset{d}{=} \sqrt{\frac{S_{\alpha/2,1}}{S'_{\alpha/2,1}}}.$$  

(25)

The combination of (17) and (25) gives one more, possibly simpler, proof of the following by-product result concerning the properties of stable distributions obtained in [8]. This result offers an explicit representation for the density of the ratio of two independent stable random variables in terms of elementary functions although with the exception of one case, the Lévy distribution ($\alpha = \frac{1}{2}$), such representations for the densities of nonnegative stable random variables themselves do not exist.

**Corollary 7.** Let $S_{\alpha,1}$ and $S'_{\alpha,1}$ be two independent random variables having one and the same one-sided strictly stable distribution with characteristic exponent $\alpha \in (0, 1)$. Then the probability density $p_\alpha(x)$ of the ratio $S_{\alpha,1}/S'_{\alpha,1}$ has the form

$$p_\alpha(x) = 2xf_{2\alpha,2}^Q(x^2), \quad x > 0,$$

where $f_{2\alpha,2}^Q$ was defined in (17), that is,

$$p_\alpha(x) = \frac{4 \sin(\pi\alpha)x^{4\alpha-1}}{\pi [1 + x^{8\alpha} + 2x^{4\alpha}\cos(\pi\alpha)]}, \quad x > 0.$$  

(26)

By the reasoning similar to that used to prove corollary 6 we can obtain the following relation linking the distributions of the random variables $K_\delta$ and $Q_{\alpha,2}$: for any $\delta \in (0, 1)$

$$K_\delta^{1/\delta} \overset{d}{=} 2 \frac{S_{\delta,1}}{S'_{\delta,1}} \overset{d}{=} 2Q_{2\delta,2}^2.$$  

(27)
Hence, \( K_\delta = 2^d Q_{2^\delta}^{\delta} \).

Using lemmas 1, 4 and 5 it is possible to obtain more product representations for the Mittag-Leffler- and Linnik-distributed random variables and hence, more mixture representations for these distributions.

5 Exponential and half-normal mixture representations for the Mittag-Leffler distribution

5.1 The Mittag-Leffler distribution as a mixed exponential distribution

As concerns the Mittag-Leffler distribution, from lemmas 3 and 5 we obtain the following statement analogous to theorem 2.

**Theorem 3.** For each \( \delta \in (0, 1] \), the Mittag-Leffler distribution with parameter \( \delta \) is the mixed exponential distribution with mixing distribution being that of twice the ratio of two independent random variables having one and the same one-sided strictly stable distribution with characteristic exponent \( \delta \):

\[
M_\delta \stackrel{d}{=} 2 W_{1|X}^{S_{\delta,1}/S_{\delta,1}},
\]

where the random variables on the right-hand side are independent.

From theorem 3 and corollary 7 we obtain the following representation of the Mittag-Leffler distribution function \( F_M(x) \):

\[
F_M(x) = 1 - \frac{4 \sin(\pi \delta)}{\pi} \int_0^\infty \frac{z^{4\delta - 1} e^{-2zx} dz}{1 + z^{5\delta} + 2z^{4\delta} \cos(\pi \delta)}, \quad x > 0,
\]

whence for the Mittag-Leffler density \( f_M(x) \) we obtain the integral representation

\[
f_M(x) = \frac{8 \sin(\pi \delta)}{\pi} \int_0^\infty \frac{z^{4\delta} e^{-2zx} dz}{1 + z^{5\delta} + 2z^{4\delta} \cos(\pi \delta)}, \quad x > 0.
\]

A representation for the Linnik distribution similar to (28) was obtained in [32]. We will use that representation in Section 7.4.

5.2 The Mittag-Leffler distribution as a mixture of half-normal distributions

For a more thorough analysis of properties of the Mittag-Leffler law as the limit distribution for random sums of independent random variables we need the product representation of an exponential random variable presented in what follows.

**Lemma 10.** The exponential distribution is a scale mixture of half-normal laws. Namely, the relation

\[
W_1 \stackrel{d}{=} \sqrt{2} |X|
\]

holds, where the random variables on the right-hand side are independent.

**Proof.** For \( x > 0 \) we have

\[
P(|X|\sqrt{W_1} < x) = E\Psi(x/\sqrt{W_1}) = 2E\Phi(x/\sqrt{W_1}) - 1 = 2 \int_0^\infty \Phi(x/\sqrt{z}) [1 - e^{-z}] - 1 =
\]

\[
= 2 \int_0^\infty \left[ 1 + \frac{1}{\sqrt{2\pi}} \int_0^{x/\sqrt{z}} e^{-u^2/2} du \right] e^{-z} dz - 1 = \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^\infty \int_0^{x/\sqrt{z}} e^{-u^2/2} - z du dz =
\]

\[
= \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^\infty \int_0^{x^2/u^2} e^{-z} e^{-u^2/2} du dz = \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^\infty \left( 1 - \exp\left\{ -\frac{x^2}{u^2} \right\} \right) e^{-u^2/2} du =
\]

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\[1 - \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^\infty \exp\left\{-\frac{u^2}{2} - \frac{x^2}{u^2}\right\} du = 1 - e^{-\sqrt{2}x} = P(W_1 < \sqrt{2}x),\]

see, e. g., [18], formula 3.325. This is nothing else than the exponential distribution with parameter \(\sqrt{2}\). The lemma is proved.

From theorem 3 and lemma 10 we obtain the following representation of the Mittag-Leffler distribution as a scale mixture of half-normal laws.

**Theorem 4.** For \(\delta \in (0,1]\) the Mittag-Leffler distribution with parameter \(\delta\) is a scale mixture of half-normal laws:

\[M_\delta \overset{d}{=} \mid X \mid \sqrt{8W_1 \left(\frac{S_{\delta,1}}{S_{\delta,1}'}\right)^2},\] (30)

In the subsequent sections an important role will be played by the distribution which is mixing in (30). Denote

\[H_\delta(x) = P\left(W_1 \left(\frac{S_{\delta,1}}{S_{\delta,1}'}\right)^2 < \frac{x}{8}\right), \quad x \geq 0,\] (31)

so that the assertion of theorem 4 can be written as

\[F_\delta^M(x) = \int_0^\infty \Psi\left(\frac{x}{\sqrt{u}}\right) dH_\delta(u) = 2 \int_0^\infty \Phi\left(\frac{x}{\sqrt{u}}\right) dH_\delta(u) - 1, \quad x \geq 0.\] (32)

With the account of corollary 7, for \(0 < \delta < 1\) the density corresponding to the distribution function \(H_\delta(x)\) can be written as

\[h_\delta(x) = \frac{d}{dx} H_\delta(x) = \frac{d}{dx} P\left(W_1 < \frac{x}{8} \left(\frac{S_{\delta,1}'}{S_{\delta,1}}\right)^2\right) = \frac{\sin(\pi \delta)}{4\pi} \int_0^\infty \frac{z^{2\delta-1/2} e^{-xz/8} dz}{1 + z^{4\delta} + 2z^{2\delta} \cos(\pi \delta)}, \quad x \geq 0.\]

If \(\delta = 1\), then obviously \(H_1(x) = 1 - e^{-x/8}, x \geq 0.\)

From (32) and (3) it follows that if \(0 < \delta < 1\), then, as \(x \to \infty\),

\[1 - F_\delta^M(x) \sim \frac{\delta \sin(\delta \pi) \Gamma(\delta + 1)}{\pi x^\delta}.\]

Therefore, by lemma 9 from theorem 4 we obtain that

\[\limsup_{x \to \infty} x^{\delta/2} |1 - H_\delta(x)| = \frac{\delta \sin(\delta \pi) \Gamma(\delta + 1)}{\pi},\]

that is, if \(0 < \delta < 1\), then \(1 - H_\delta(x) = O(x^{-\delta/2})\) as \(x \to \infty.\)

### 6 Convergence of the distributions of random sums and statistics constructed from samples with random sizes to the Linnik and Mittag-Leffler distributions

#### 6.1 Convergence of the distributions of random sums to the Linnik distribution

In applied probability it is a convention that a model distribution can be regarded as well-justified or adequate, if it is an asymptotic approximation, that is, if there exists a rather simple limit setting (say, schemes of maximum or summation of random variables) and the corresponding limit theorem in which the model under consideration manifests itself as a limit distribution. The existence of such limit setting can provide a better understanding of real mechanisms that generate observed statistical regularities.

As it has already been noted in the introduction, both the Mittag-Leffler and Linnik laws are geometrically stable and are therefore limit distributions for geometric random sums. In this and
subsequent sections we will demonstrate that the scheme of geometric summation is far not the only asymptotic setting (even for sums of independent random variables!) in which the Mittag-Leffler and Linnik laws appear as limit distributions.

Product representations for the random variables with the Linnik and Mittag-Leffler distributions obtained in the previous works were aimed at the construction of convenient algorithms for the computer generation of pseudo-random variables with these distributions. The mixture representation for the Linnik distribution as a scale mixture of normals obtained in corollary 4 opens the way for the construction in this section of a random-sum central limit theorem with the Linnik distribution as the limit law. Moreover, in this “if and only if” version of the random-sum central limit theorem the Mittag-Leffler distribution must be the limit law for the normalized number of summands.

Recall that the symbol $\implies$ denotes the convergence in distribution.

Consider independent not necessarily identically distributed random variables $X_1, X_2, \ldots$ with $\mathbb{E}X_i = 0$ and $0 < \sigma_i^2 = \text{D}X_i < \infty$, $i \geq 1$. For $n \in \mathbb{N}$ denote

$$S_n^* = X_1 + \ldots + X_n, \quad B_n^2 = \sigma_1^2 + \ldots + \sigma_n^2.$$ 

Assume that the random variables $X_1, X_2, \ldots$ satisfy the Lindeberg condition: for any $\tau > 0$

$$\lim_{n \to \infty} \frac{1}{B_n^2} \sum_{i=1}^{n} \int_{|x| \geq \tau B_n} x^2 d\mathbb{P}(X_i < x) = 0. \quad (33)$$

It is well known that under these assumptions

$$\mathbb{P}(S_n^* < B_n x) \implies \Phi(x)$$

(this is the classical Lindeberg central limit theorem).

Let $N_1, N_2, \ldots$ be a sequence of integer-valued nonnegative random variables defined on the same probability space so that for each $n \in \mathbb{N}$ the random variable $N_n$ is independent of the sequence $X_1, X_2, \ldots$. Denote $S_{N_n}^* = X_1 + \ldots + X_{N_n}$. For definiteness, in what follows we assume that $\sum_{j=1}^{0} = 0$.

Recall that a random sequence $X_1, N_2, \ldots$ is said to infinitely increase in probability ($N_n \xrightarrow{P} \infty$), if $\mathbb{P}(N_n \leq m) \to 0$ as $n \to \infty$ for any $m \in (0, \infty)$.

Let $\{d_n\}_{n \geq 1}$ be an infinitely increasing sequence of positive numbers.

The proof of the main result of this section is based on the following version of the random-sum central limit theorem.

**Lemma 11** [21]. Assume that the random variables $X_1, X_2, \ldots$ and $N_1, N_2, \ldots$ satisfy the conditions specified above. In particular, let Lindeberg condition (33) hold. Moreover, let $N_n \xrightarrow{P} \infty$ as $n \to \infty$. A distribution function $F(x)$ such that

$$\mathbb{P}\left(\frac{S_{N_n}^*}{d_n} < x\right) \implies F(x)$$

as $n \to \infty$ exists if and only if there exists a distribution function $H(x)$ satisfying the conditions

$$H(0) = 0, \quad F(x) = \int_{0}^{\infty} \Phi\left(\frac{x}{\sqrt{y}}\right) dH(y), \quad x \in \mathbb{R},$$

and $\mathbb{P}(B_{N_n}^2 < x d_n^2) \implies H(x)$ ($n \to \infty$).

**Proof.** This statement is a particular case of a result proved in [21], also see theorem 3.3.2 in [12].

The following theorem gives a criterion (that is, necessary and sufficient conditions) of the convergence of the distributions of random sums of independent identically distributed random variables with finite variances to the Linnik distribution.

**Theorem 4.** Let $\alpha \in (0, 2]$. Assume that the random variables $X_1, X_2, \ldots$ and $N_1, N_2, \ldots$ satisfy the conditions specified above. In particular, let Lindeberg condition (33) hold. Moreover, let $N_n \xrightarrow{P} \infty$
as \( n \to \infty \). Then the distributions of the normalized random sums \( S^*_N \) converge to the Linnik law with parameter \( \alpha \), that is,

\[
P \left( \frac{S^*_N}{d_n} < x \right) \to F^L_\alpha (x)
\]

with some \( d_n > 0, d_n \to \infty \) as \( n \to \infty \), if and only if

\[
\frac{B^2_{N_n}}{d_n^2} \to M_{\alpha/2} \quad (n \to \infty).
\]

**Proof.** This statement is a direct consequence of corollary 4 and lemma 11 with \( H(x) = F^M_{\alpha/2} (x) \).

Note that if the random variables \( X_1, X_2, \ldots \) are identically distributed, then \( \sigma_i = \sigma, i \in \mathbb{N} \), and the Lindeberg condition holds automatically. In this case it is reasonable to take \( d_n = \sigma \sqrt{n} \). Hence, from theorem 4 in this case it follows that for the convergence

\[
\frac{S^*_N}{\sigma \sqrt{n}} \to L_\alpha
\]

to hold as \( n \to \infty \) it is necessary and sufficient that

\[
\frac{N_n}{n} \to M_{\alpha/2}.
\]

One more remark is that with \( \alpha = 2 \) Theorem 4 involves the case of convergence to the Laplace distribution.

### 6.2 Convergence of the distributions of statistics constructed from samples with random sizes to the Linnik distribution

In classical problems of mathematical statistics, the size of the available sample, i.e., the number of available observations, is traditionally assumed to be deterministic. In the asymptotic settings it plays the role of infinitely increasing known parameter. At the same time, in practice very often the data to be analyzed is collected or registered during a certain period of time and the flow of informative events each of which brings a next observation forms a random point process. Therefore, the number of available observations is unknown till the end of the process of their registration and also must be treated as a (random) observation. For example, this is so in insurance statistics where during different accounting periods different numbers of insurance events (insurance claims and/or insurance contracts) occur and in high-frequency financial statistics where the number of events in a limit order book during a time unit essentially depends on the intensity of order flows. Moreover, contemporary statistical procedures of insurance and financial mathematics do take this circumstance into consideration as one of possible ways of dealing with heavy tails. However, in other fields such as medical statistics or quality control this approach has not become conventional yet although the number of patients with a certain disease varies from month to month due to seasonal factors or from year to year due to some epidemic reasons and the number of failed items varies from lot to lot. In these cases the number of available observations as well as the observations themselves are unknown beforehand and should be treated as random to avoid underestimation of risks or error probabilities.

Therefore it is quite reasonable to study the asymptotic behavior of general statistics constructed from samples with random sizes for the purpose of construction of suitable and reasonable asymptotic approximations. As this is so, to obtain non-trivial asymptotic distributions in limit theorems of probability theory and mathematical statistics, an appropriate centering and normalization of random variables and vectors under consideration must be used. It should be especially noted that to obtain reasonable approximation to the distribution of the basic statistics, both centering and normalizing values should be non-random. Otherwise the approximate distribution becomes random itself and, for example, the problem of evaluation of quantiles or significance levels becomes senseless.
In asymptotic settings, statistics constructed from samples with random sizes are special cases of random sequences with random indices. The randomness of indices usually leads to that the limit distributions for the corresponding random sequences are heavy-tailed even in the situations where the distributions of non-randomly indexed random sequences are asymptotically normal see, e. g., [3, 4, 12]. For example, if a statistic which is asymptotically normal in the traditional sense, is constructed on the basis of a sample with random size having negative binomial distribution, then instead of the expected law for this statistic.

Consider a problem setting that is traditional for mathematical statistics. Let random variables \( N_1, N_2, \ldots, X_1, X_2, \ldots \) be defined on one and the same probability space \( (\Omega, \mathcal{A}, P) \). Assume that for each \( n \geq 1 \) the random variable \( N_n \) takes only natural values and is independent of the sequence \( X_1, X_2, \ldots \). Let \( T_n = T_n(X_1, \ldots, X_n) \) be a statistic, that is, a measurable function of \( X_1, \ldots, X_n \). For every \( n \geq 1 \) define the random variable \( T_{N_n}(\omega) \) as

\[
T_{N_n}(\omega) = T_{N_n(\omega)}(X_1(\omega), \ldots, X_{N_n(\omega)}(\omega))
\]

for each \( \omega \in \Omega \). As usual, the symbol \( \Rightarrow \) denotes convergence in distribution.

A statistic \( T_n \) is said to be

asymptotically normal,

if there exist \( \delta > 0 \) and \( \theta \in \mathbb{R} \) such that

\[
P\left( \delta \sqrt{n}(T_n - \theta) < x \right) \Rightarrow \Phi(x) \quad (n \to \infty).
\]

(34)

**Lemma 12** [23]. Assume that \( N_n \to \infty \) in probability. Let the statistic \( T_n \) be asymptotically normal in the sense of (34). A distribution function \( F(x) \) such that

\[
P\left( \delta \sqrt{n}(T_{N_n} - \theta) < x \right) \Rightarrow F(x) \quad (n \to \infty),
\]

exists if and only if there exists a distribution function \( H(x) \) satisfying the conditions

\[
H(0) = 0, \quad F(x) = \int_0^\infty \Phi(x \sqrt{y}) dH(y), \quad x \in \mathbb{R}, \quad P(N_n < nx) \Rightarrow H(x) \quad (n \to \infty).
\]

The following theorem gives a criterion (that is, necessary and sufficient conditions) of the convergence of the distributions of statistics, which are suggested to be asymptotically normal in the traditional sense but are constructed from samples with random sizes, to the Linnik distribution.

**Theorem 5.** Let \( \alpha \in (0, 2] \). Assume that the random variables \( X_1, X_2, \ldots \) and \( N_1, N_2, \ldots \) satisfy the conditions specified above and, moreover, let \( N_n \overset{P}{\to} \infty \) as \( n \to \infty \). Let the statistic \( T_n \) be asymptotically normal in the sense of (34). Then the distribution of the statistic \( T_{N_n} \) constructed from samples with random sizes \( N_n \) converges to the Linnik law \( F_\alpha^L(x) \) as \( n \to \infty \), that is,

\[
P\left( \delta \sqrt{n}(T_{N_n} - \theta) < x \right) \Rightarrow F_\alpha^L(x),
\]

if and only if

\[
\frac{N_n}{n} \Rightarrow M_{\alpha/2}^{-1} \quad (n \to \infty).
\]

(35)

**Proof.** This statement is a direct consequence of corollary 5 and lemma 12 with \( H(x) = P(M_{\alpha/2}^{-1} < x) \).

From (28) and the absolute continuity of the Mittag-Leffler distribution it follows that condition (35) can be written as

\[
\sup_{x>0} \left| P(N_n < nx) - \frac{4 \sin(\pi \alpha/2)}{\pi} \int_0^\infty \frac{z^{2\alpha-1} e^{-2z/x} dz}{1 + z^{4\alpha} + 2z^{2\alpha} \cos(\pi \alpha/2)} \right| = 0.
\]

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6.3 Convergence of the distributions of extreme order statistics constructed from samples with random sizes to the Mittag-Leffler distribution

Using lemmas 5 and 6 we can obtain the following representation of the Mittag-Leffler distribution as a mixed Weibull distribution: if $0 < \delta < \delta' \leq 1$, then

$$M_\delta \overset{d}{=} W_{\delta'}S_{\delta,1}K_{\delta/\delta'}^{1/\delta}.$$  \hspace{1cm} (36)

It is well known that the Weibull distribution is a limit law for extreme order statistics under an appropriate linear normalization. This fact together with (36) open the way to prove that the Mittag-Leffler distribution can be limiting for extreme order statistics constructed from samples with random sizes.

In the book [24] it was proposed to model the evolution of non-homogeneous chaotic stochastic processes, in particular, the dynamics of financial markets by compound doubly stochastic Poisson processes (compound Cox processes). This approach got further grounds and development, say, in [4] [26]. According to this approach the flow of informative events, each of which generates the next possesing the properties:

For vividness consider the case where in the model under consideration the parameter $U$ distribution function $F_U$ is described by the stochastic point process $P$ [3, 26]. According to this approach the flow of informative events, each of which generates the next non-decreasing and right-continuous. The process $P(U(t))$, $t \geq 0$, is called a doubly stochastic Poisson process (Cox process) [19].

Within this model, for each $t$ the distribution of the random variable $P(U(t))$ is mixed Poisson. For vividness, consider the case where in the model under consideration the parameter $t$ is discrete: $U(t) = U(n) = U_n$, $n \in \mathbb{N}$, where $\{U_n\}_{n \geq 1}$ is an infinitely increasing sequence of nonnegative random variables such that $U_{n+1}(\omega) \geq U_n(\omega)$ for any $\omega \in \Omega$, $n \geq 1$. Here the asymptotics $n \to \infty$ may be interpreted as that the intensity of the flow of informative events is (infinitely) large.

From the assumptions formulated above it follows that the random variable $U_n$ is independent of the standard Poisson process $P(t)$, $t \geq 0$. For each natural $n$ let $N_n = P(U_n)$, $n \geq 1$. It is obvious that the random variable $N_n$ so defined has the mixed Poisson distribution

$$P(N_n = k) = P(P(U_n) = k) = \int_0^\infty e^{-nz}(nz)^k k!dP(U_n < z) \quad k = 0, 1, \ldots$$

Let $X_1, X_2, \ldots$ be independent identically distributed random variables with the common distribution function $F(x) = P(X_i < x)$, $x \in \mathbb{R}$, $i \geq 1$. Denote $\text{lext}(F) = \inf\{x: F(x) > 0\}$. Assume that for each $k \in \mathbb{N}$ the random variable $N_k$ is independent of the sequence $X_1, X_2, \ldots$ In the book [24] the following statement was proved.

**Lemma 13** [24]. Assume that there exist an infinitely increasing sequence of positive numbers $\{d_k\}_{k \geq 1}$ and a nonnegative random variable $U$ such that

$$\frac{U_k}{d_k} \Rightarrow U \quad (k \to \infty).$$

Also assume that $\text{lext}(F) > -\infty$ and the distribution function $A_F(x) = F(\text{lext}(F) - x^{-1})$ satisfies the condition: there exists a positive number $\delta'$ such that for any $x > 0$

$$\lim_{y \to \infty} \frac{A_F(yx)}{A_F(y)} = x^{-\delta'}. \hspace{1cm} (37)$$

Then there exist sequences of numbers $a_k$ and $b_k$ such that

$$P\left( \min_{1 \leq j \leq N_k} X_j - a_k < b_kx \right) \Rightarrow \left[ 1 - \int_0^\infty e^{-ux^{d'}}dP(U < u) \right] 1(x \geq 0) \quad (k \to \infty).$$

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Moreover, the numbers $a_k$ and $b_k$ can be defined as

$$a_k = \text{lex}(F), \quad b_k = \sup \{ x : F(x) \leq d_k^{-1} \} - \text{lex}(F), \quad k \geq 1. \quad (38)$$

From representation (36) and lemma 13 we obtain the following result.

**Theorem 6.** Let $\delta \in (0, 1)$. For the existence of numbers $a_k \in \mathbb{R}$ and $b_k > 0$ such that

$$\frac{1}{b_k} \left( \min_{1 \leq j \leq N_k} X_j - a_k \right) \implies M_\delta \quad (k \to \infty),$$

it is sufficient that:

(i) there exists a $\delta' \in (\delta, 1]$ such that the distribution function $F$ belongs to the domain of min-attraction of the Weibull distribution with some shape parameter $\delta' \in (0, 1]$, that is, $\text{lex}(F) > -\infty$ and condition (37) holds;

(ii) there exists an infinitely increasing sequence $\{d_k\}_{k \geq 1}$ such that

$$\frac{U_k}{d_k} \implies S^{-\delta'}_\delta K^{\delta'/\delta}_{\delta/\delta'} \quad (k \to \infty). \quad (39)$$

Moreover, the numbers $a_k$ and $b_k$ can be defined in accordance with (38).

**Proof.** The desired result is a direct consequence of lemma 13 and representation (11) with the account of the relation $K^{-1}_{\delta/\delta'} d = K_{\delta/\delta'}'$ implied by (27).

So, the randomness of the sample size can make the tails of the limit distribution considerably more heavy than this is so in the “classical” case. For example, let the distribution of the sample element $X_1$ belong to the domain of min-attraction of the exponential law, that is, condition (37) holds with $\delta' = 1$, but the sample size is random and has the form $N_k = P(U_k)$ and for some $d_k$ condition (39) holds with some $\delta \in (0, \delta')$. Then the actual Mittag-Leffler limit distribution of the minimum order statistic has power-type decreasing tails unlike the “originally assumed” exponentially decreasing tails.

### 6.4 Convergence of the distributions of maximum random sums to the Mittag-Leffler distribution

In this section we will demonstrate that the Mittag-Leffler distribution can be the limit law for maximum random sums. The main role here will be played by representations (30) and (32) of the Mittag-Leffler distribution as a scale mixture of half-normal distributions. We will show that this distribution can be limiting for maximum sums of a random number of independent random variables (maximum random sums), minimum random sums and absolute values of random sums.

As in Section 6.1, consider independent not necessarily identically distributed random variables $X_1, X_2, \ldots$ with $EX_i = 0$ and $0 < \sigma_i^2 = DX_i < \infty$, $i \in \mathbb{N}$. In addition to the notation introduced in Section 6.1, for $n \geq 1$ denote $\overline{S}_n = \max_{1 \leq i \leq n} S_i$, $\underline{S}_n = \min_{1 \leq i \leq n} S_i$. Assume that the random variables $X_1, X_2, \ldots$ satisfy the Lindeberg condition (33).

It is well known that under these assumptions $P(\overline{S}_n < B_n x) \implies \Psi(x)$ and $P(\underline{S}_n < B_n x) \implies 1 - \Psi(-x)$ as $n \to \infty$ (this is one of manifestations of the invariance principle).

Let $N_1, N_2, \ldots$ be a sequence of nonnegative random variables such that for each $n \in \mathbb{N}$ the random variables $N_n, Y_1, Y_2, \ldots$ are independent. For $n \in \mathbb{N}$ let $S^*_n = X_1 + \ldots + X_{N_n}$, $\overline{S}_n = \max_{1 \leq i \leq N_n} S_i$, $\underline{S}_n = \min_{1 \leq i \leq N_n} S_i$ (for definiteness assume that $S_0 = \overline{S}_0 = \underline{S}_0 = 0$). Let $\{d_n\}_{n \geq 1}$ be an infinitely increasing sequence of positive numbers.

**Lemma 14** [21]. Assume that the random variables $X_1, X_2, \ldots$ and $N_1, N_2, \ldots$ satisfy the conditions specified above. In particular, let Lindeberg condition (33) hold. Moreover, let $N_n \overset{P}{\to} \infty$ as $n \to \infty$. 

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Then the distributions of normalized extremal random sums and absolute values of random sums weakly converge to some distributions, that is, there exist random variables $Y, \bar{Y}$ and $Y$ such that

$$\frac{S'_{N_n}}{d_n} \Rightarrow Y, \quad \frac{S'_{N_n}}{d_n} \Rightarrow \bar{Y}, \quad \frac{|S'_{N_n}|}{d_n} \Rightarrow |Y|$$

as $n \to \infty$ if and only if there exists a nonnegative random variable $U$ such that

$$B_{N_k}^2 \Rightarrow U \quad (n \to \infty).$$

Moreover, for $x \in \mathbb{R}$ we have

$$P(\bar{Y} < x) = P(|Y| < x) = E\Psi\left(\frac{x}{\sqrt{U}}\right), \quad P(Y < x) = 1 - E\Psi\left(-\frac{x}{\sqrt{U}}\right).$$

The proof of lemma 14 was given in [21].

Lemma 14 and theorem 4 imply the following statement.

**Theorem 7.** Let $\delta \in (0, 1]$. Let $H_\delta(x)$ be the distribution function defined in (31). Assume that the random variables $X_1, X_2, \ldots$ and $N_1, N_2, \ldots$ satisfy the conditions specified above. In particular, let Lindeberg condition (33) hold. Moreover, let $N_n \xrightarrow{P} \infty$ as $n \to \infty$. Then, as $n \to \infty$, the following statements are equivalent:

$$\frac{S'_{N_n}}{d_n} \Rightarrow M_\delta; \quad \frac{S'_{N_n}}{d_n} \Rightarrow -M_\delta; \quad \frac{|S'_{N_n}|}{d_n} \Rightarrow M_\delta; \quad P(B_{N_n}^2 < d_n^2x) \Rightarrow H_\delta(x).$$

### 6.5 Examples of appropriate random indices

The convergence of the distributions of the normalized indices $N_n/n$ to a special law is a crucial conditions in all the theorems presented above concerning the convergence of random sums or statistics constructed from samples with random sizes to the Linnik and Mittag-Leffler distributions. For example, the convergence of the distributions of the normalized indices $N_n/n$ to the Mittag-Leffler distribution $F_\delta^M$ is the main condition in theorem 4. Here we will give two examples of the situation where this condition can hold. The first example is trivial and is based on the geometric stability of the Mittag-Leffler distribution. The second example relies on a useful general construction of nonnegative integer-valued random variables which, under an appropriate normalization, converge to a given nonnegative (not necessarily discrete) random variable, whatever the latter is. Hence, this construction can be correspondingly modified to give examples of indices possessing the asymptotic properties required in other convergence criteria presented in this paper.

**Example 1.** Let $\delta \in (0, 1)$ be arbitrary. For every $n \in \mathbb{N}$ let $V_{1/n}$ be a random variable having the geometric distribution (4) with $p = \frac{1}{n}$ independent of the sequence $Y_1, Y_2, \ldots$ of independent identically distributed nonnegative random variables such that

$$n^{-1/\delta} \sum_{j=1}^{V_{1/n}} Y_j \Rightarrow M_\delta \quad (40)$$

as $n \to \infty$. To provide (40), the distributions of the random variables $Y_1, Y_2, \ldots$ should belong to the domain of the normal attraction of the one-sided strictly stable law with characteristic exponent $\delta$. As $N_n$ for each $n \in \mathbb{N}$ take

$$N_n = \left[n^{1-1/\delta} \sum_{j=1}^{V_{1/n}} Y_j\right].$$
that is, the random variables $N$.

Definition 1. 7.1 The symmetric two-sided Mittag-Leffler distribution

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construction to the required distributions.

This random variable $N_n$ can be interpreted as the number of events registered up to time $n$ in the Poisson process with the stochastic intensity having the Mittag-Leffler density $f^M_\delta$.

Denote $A_n(z) = P(N_n < nz)$, $z \geq 0$ ($A_n(z) = 0$ for $z < 0$). It is easy to see that $A_n(z) \Rightarrow F^M_\delta(z)$ as $n \to \infty$. Indeed, as is known, if $\Pi(x; \ell)$ is the Poisson distribution function with the parameter $\ell > 0$ and $E(x; c)$ is the distribution function with a single unit jump at the point $c \in \mathbb{R}$, then $\Pi(\ell x; \ell) \Rightarrow E(x; 1)$ as $\ell \to \infty$. Since for $x \in \mathbb{R}$

$$A_n(x) = \int_0^\infty \Pi(nx; nz)dF^M_\delta(z),$$

then by the Lebesgue dominated convergence theorem, as $n \to \infty$, we have

$$A_n(x) \Rightarrow \int_0^\infty E(x/z; 1)dF^M_\delta(z) = \int_0^x dF^M_\delta(z) = F^M_\delta(x),$$

that is, the random variables $N_n$ defined above satisfy the condition of theorem 4. Moreover, $N_n \xrightarrow{P} \infty$ as $n \to \infty$ since $P(M_\delta = 0) = 0$.

As it has already been noted, instead of $M_\delta$ any other positive random variable can be taken to provide the convergence of the distributions of indices constructed according to the presented construction to the required distributions.

7 Two-sided Mittag-Leffler and one-sided Linnik distributions

7.1 The symmetric two-sided Mittag-Leffler distribution

Definition 1. Let $\delta \in (0,1)$. By the symmetric two-sided Mittag-Leffler distribution with parameter $\delta$ we will mean the distribution of the random variable $\tilde{M}_\delta$ defined by the density

$$f^M_\delta(x) = \begin{cases} \frac{1}{2} f^M_\delta(-x), & x < 0, \\ \frac{1}{2} f^M_\delta(x), & x \geq 0, \end{cases}$$

where the Mittag-Leffler density $f^M_\delta(x)$ was defined in (2) for $x \in \mathbb{R}^+$. The distribution function corresponding to the density $f^M_\delta(x)$ has the form

$$F^M_\delta(x) = \begin{cases} \frac{1}{2} [1 - F^M_\delta(-x)], & x < 0, \\ \frac{1}{2} [1 + F^M_\delta(x)], & x \geq 0. \end{cases}$$
The two-sided Mittag-Leffler distribution so defined is obviously symmetric, that is, \( -\tilde{M}_\delta \overset{d}{=} \tilde{M}_\delta \). Furthermore, we can say that \( \tilde{M}_\delta \) is the randomization symmetrization of \( M_\delta \). The randomization symmetrization of a random variable \( Y \) can be formally defined in the following way.

**Definition 2.** Let \( Z \) be a random variable such that \( \mathbb{P}(Z = -1) = \mathbb{P}(Z = 1) = \frac{1}{2} \). Assume that the random variables \( Y \) and \( Z \) are independent. The random variable \( \tilde{Y} = Z \cdot Y \) is called the randomization symmetrization of \( Y \).

It can be easily verified that \( \tilde{M}_\delta \overset{d}{=} Z \cdot M_\delta \).

It is also easy to verify that the random variable \( Y \) is symmetric, then \( \tilde{Y} \overset{d}{=} Z |Y| \) where the random variables on the right-hand side are independent and \( Z \) is the same as in definition 2.

To describe the symmetric two-sided Mittag-Leffler distribution in more detail, we will use the following well-known fact (see, e.g., [25]). Let \( \tilde{Y} \) be the characteristic function of the random variable \( Y \), and \( \tilde{f}(t) \) be the characteristic function of the randomization symmetrization of the random variable \( Y \), \( t \in \mathbb{R} \). Then from definition 2 it follows that

\[
\tilde{f}(t) = \frac{1}{2} \mathbb{E} e^{-itY} + \frac{1}{2} \mathbb{E} e^{itY} = \frac{1}{2} [\mathbb{E} \cos(tY) - i\mathbb{E} \sin(tY)] + \frac{1}{2} [\mathbb{E} \cos(tY) + \mathbb{E} \sin(tY)] = \mathbb{E} \cos(tY) = \text{Re} \tilde{f}(t).
\]

That is, the characteristic function of the randomization symmetrization of any random variable \( Y \) coincides with the real part of the characteristic function of \( Y \).

Hence, to find the characteristic function \( \tilde{f}^M(t) \) of the symmetric two-sided Mittag-Leffler distribution, we should calculate the real part of the characteristic function \( \tilde{f}^M(t) \) of the ordinary (one-sided) Mittag-Leffler distribution.

It is clear that \( \tilde{f}^M(t) = \psi_\delta(-it) \), \( t \in \mathbb{R} \) (see (1)). So, we have

\[
\tilde{f}^M(t) = \psi_\delta(-it) = \frac{1}{1 + (-it)^\delta} = \frac{1}{1 + |t|^\delta e^{-\frac{\pi i}{2} \delta \text{sign } t}} = \frac{1 + |t|^\delta \cos(\frac{\pi}{2} \delta \text{sign } t) + i|t|^\delta \sin(\frac{\pi}{2} \delta \text{sign } t)}{(1 + |t|^\delta \cos(\frac{\pi}{2} \delta \text{sign } t))^2 + |t|^{2\delta}(\sin(\frac{\pi}{2} \delta \text{sign } t))^2} = \frac{1 + |t|^\delta \cos(\frac{\pi}{2} \delta \text{sign } t)}{1 + |t|^{2\delta} + 2|t|^\delta \cos(\frac{\pi}{2} \delta)} + i \frac{|t|^\delta \sin(\frac{\pi}{2} \delta \text{sign } t)}{1 + |t|^{2\delta} + 2|t|^\delta \cos(\frac{\pi}{2} \delta)}.
\]

That is,

\[
\tilde{f}^M(t) = \text{Re} \tilde{f}^M(t) = \frac{1 + |t|^\delta \cos(\frac{\pi}{2} \delta)}{1 + |t|^{2\delta} + 2|t|^\delta \cos(\frac{\pi}{2} \delta)}, \quad t \in \mathbb{R}.
\] (43)

Having compared the right-hand side of (43) with the characteristic function \( \tilde{f}^L_{2\delta}(t) \) of the Linnik distribution (see (5)) we notice that the former differs from the latter by the presence of the addends of the form \( |t|^\delta \cos(\frac{\pi}{2} \delta) \) in both the numerator and denominator. These addends vanish as \( \delta \to 1 \), that is, as both the symmetric two-sided Mittag-Leffler and Linnik laws turn into the Laplace distribution.

In general, the above reasoning may be treated as a proof of the fact that the function on the right-hand side of (43) is a characteristic function (of a probability distribution). The properties of this distribution will be considered in more detail in the next subsection.

### 7.2 A normal scale mixture representation for the symmetric two-sided Mittag-Leffler distribution

**Theorem 8.** The real part of the characteristic function of any mixed exponential distribution is the characteristic function of a normal scale mixture.
Proof. Let a random variable $Y$ be represented as $Y \overset{d}{=} W_1 \cdot U$ where $U$ is a nonnegative random variable independent of the standard exponential random variable $W_1$. This means that the distribution of $Y$ is mixed exponential, the mixing distribution being that of $U^{-1}$. From lemma 10 we obtain the representation

$$Y = |X|U \sqrt{2W_1}$$

with all the random variables on the right-hand side independent. Now, as it has been demonstrated above,

$$\Re \{Y(t)\} = E \exp\{itY\} = E \exp\{itZ|X|U \sqrt{2W_1}\} = E \exp\{itX \sqrt{2W_1U^2}\}, \quad t \in \mathbb{R},$$

where all the involved random variables are independent. But relation (44) means that

$$P(\tilde{Y} < x) = \int_0^\infty \Phi\left(\frac{x}{\sqrt{u}}\right) dP(2W_1U^2 < u), \quad x \in \mathbb{R}.$$ 

The theorem is proved.

Note that the mixing distributions of the original mixed exponential distribution and the corresponding normal scale mixture are related in a simple way: the latter is the mixed exponential distribution with the mixing law being that of the squared original mixing random variable.

It is easy to make sure that $X \overset{d}{=} Z \cdot |X|$, where, as above, $Z$ is a random variable such that $P(Z = -1) = P(Z = 1) = \frac{1}{2}$ and the random variables $X$ and $Z$ are independent. So, from theorem 8, corollary 2 and (27) we obtain the following result.

**Theorem 9.** The symmetric two-sided Mittag-Leffler distribution (42) is a normal scale mixture:

$$\tilde{M}_\delta \overset{d}{=} X \sqrt{2W_1K_{2/\delta}^{2/\delta} = X \sqrt{8W_1 \left(\frac{S_{\delta,1}}{S_{\delta,1}'}\right)^2}}.$$

We see that in theorem 9 the mixing distribution is the same as in theorem 4 and was denoted as $H_\delta(x)$, see (31), so that the assertion of theorem 9 can be written as

$$P(\tilde{M}_\delta < x) = \int_0^\infty \Phi\left(\frac{x}{\sqrt{u}}\right) dH_\delta(u), \quad x \in \mathbb{R}$$

To compare the symmetric two-sided Mittag-Leffler distribution with the Linnik distribution, we will use the representations of these distributions as scale mixtures of the Laplace distribution. For $0 < \alpha < 1$ Denote the distribution function corresponding to density $p_\alpha(x)$ (see (26) by $P_\alpha(x)$, $x \geq 0$. Let $\delta \in (0, 1)$. Then from theorem 2 it follows that

$$P(L_{2\delta} < x) = \int_0^\infty F^{\Lambda}(\frac{x}{\sqrt{u}}) dP_\delta(u), \quad x \in \mathbb{R}.$$

At the same time, from (22) and theorem 9 we obtain the representation

$$P(\tilde{M}_\delta < x) = \int_0^\infty F^{\Lambda}(\frac{x}{\sqrt{u}}) dP_\delta(\sqrt{2u}), \quad x \in \mathbb{R}.$$ 

Representations (47) and (48) differ only by that the power of the argument of the mixing distribution in (47) is twice the power of the argument of the mixing distribution in (48) resulting in that the exponent of the power-type tail asymptotics of the Linnik distribution (33) is twice greater than that of the symmetric two-sided Mittag-Leffler distribution (48).
7.3 Convergence of the distributions of random sums and statistics constructed from samples with random sizes to the symmetric two-sided Mittag-Leffler law

Following the lines of the reasoning used to prove the results of Sect. 6.1 and 6.2, here we will present similar results concerning the convergence of the distributions of random sums and statistics constructed from samples with random sizes to the symmetric two-sided Mittag-Leffler law.

Product representations for the random variables with the symmetric two-sided and Mittag-Leffler distributions obtained in the preceding sections open the way for the construction in this section of a random-sum central limit theorem with the symmetric two-sided and Mittag-Leffler distribution as the limit law. Moreover, as in Sections 6.1 and 6.2, the results of this section have the “if and only if” character.

We again consider a sequence of independent identically distributed random variables with necessary and sufficient conditions) of the convergence of the distributions of random sums of integer-valued nonnegative random variables defined on the same probability space so that for each $n \geq 1$ the random variable $N_n$ is independent of the sequence $X_1, X_2, \ldots$. For definiteness, in what follows we assume that $\sum_{j=1}^{\infty} = 0$.

Using lemma 11 and theorem 9 we obtain the following statement which is a criterion (that is, necessary and sufficient conditions) of the convergence of the distributions of random sums of independent identically distributed random variables with finite variances to the symmetric two-sided and Mittag-Leffler distribution.

Let $\{a_n\}_{n \geq 1}$ be an infinitely increasing sequence of positive numbers.

**Theorem 10.** Let $\delta \in (0, 1]$. Assume that the random variables $X_1, X_2, \ldots$ and $N_1, N_2, \ldots$ satisfy the conditions specified above. In particular, let Lindeberg condition (33) hold. Moreover, let $N_n \xrightarrow{P} \infty$ as $n \to \infty$. Then the distributions of the normalized random sums $S_{n}^{*}$ converge to the symmetric two-sided and Mittag-Leffler distribution with parameter $\delta$, that is,

$$P \left( \frac{S_{n}^{*}}{a_n} < x \right) \Rightarrow F_{\delta}^{M}(x)$$

as $n \to \infty$, if and only if

$$P \left( \frac{B_n^2}{d_n^2} < x \right) \Rightarrow H_{\delta}(x) \quad (n \to \infty).$$

Now let random variables $N_1, N_2, \ldots, X_1, X_2, \ldots$, be defined on one and the same probability space $(\Omega, \mathfrak{A}, P)$. Assume that for each $n \geq 1$ the random variable $N_n$ takes only natural values and is independent of the sequence $X_1, X_2, \ldots$. Let $T_n = T_n(X_1, \ldots, X_n)$ be a statistic, that is, a measurable function of $X_1, \ldots, X_n$. Recall that for every $n \geq 1$ the random variable $T_{N_n}$ is defined as

$$T_{N_n}(\omega) = T_{N_n(\omega)}(X_1(\omega), \ldots, X_{N_n(\omega)}(\omega))$$

for each $\omega \in \Omega$. We will assume that the statistic $T_{N_n}$ is asymptotically normal in the sense of (34). Using lemma 12 and theorem 9 we obtain the following criterion (that is, necessary and sufficient conditions) for the convergence of the distributions of statistics, which are suggested to be asymptotically normal in the traditional sense but are constructed from samples with random sizes, to the symmetric two-sided Mittag-Leffler distribution.

**Theorem 11.** Let $\delta \in (0, 1]$. Assume that the random variables $X_1, X_2, \ldots$ and $N_1, N_2, \ldots$ satisfy the conditions specified above and, moreover, let $N_n \xrightarrow{P} \infty$ as $n \to \infty$. Let the statistic $T_n$ be asymptotically normal in the sense of (34). Then the distribution of the statistic $T_{N_n}$ constructed from samples with random sizes $N_n$ converges to the symmetric two-sided Mittag-Leffler distribution $F_{\delta}^{M}(x)$ as $n \to \infty$, that is,

$$P \left( \delta \sqrt{\hat{n}} (T_{N_n} - \theta) < x \right) \Rightarrow F_{\delta}^{M}(x),$$

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if and only if
\[ P(N_n < nx) \implies 1 - H_\delta\left(\frac{1}{x}\right) \quad (n \to \infty). \quad (49) \]

From the absolute continuity of the distribution function \( H_\delta(x) \) it follows that condition (49) can be written as
\[ \sup_{x>0} \left| P(N_n > nx) - H_\delta\left(\frac{1}{x}\right) \right| = 0. \]

### 7.4 The one-sided Linnik distribution

In Section 7.1 we noted that if a random variable \( Y \) is symmetric, that is, \( Y \overset{d}{=} -Y \), and \( Z \) is a random variable independent of \( Y \) such that \( P(Z = -1) = P(Z = 1) = \frac{1}{2} \), then \( Y \overset{d}{=} Z|Y| \). As this is so, the nonnegative random variable \(|Y|\) can be treated as a “de-symmetrization” of \( Y \).

Following this logic, the distribution of the random variable \(|L_\alpha|\) with \( \alpha \in (0, 2) \) will be called the one-sided Linnik distribution.

It is easy to see that
\[ \hat{F}_\alpha^L(x) \equiv P(|L_\alpha| < x) = 2F_\alpha^L(x) - 1, \quad x \geq 0. \]

In [32] it was shown that for the Linnik distribution density the following integral representation holds:
\[ f_\alpha^L(x) = \frac{\sin(\pi \alpha/2)}{\pi} \int_0^\infty \frac{y^\alpha e^{-y|x|} dy}{1 + y^{2\alpha} + 2y^\alpha \cos(\pi \alpha/2)}, \quad x \in \mathbb{R}. \]

Hence, the density \( \hat{f}_\alpha^L(x) \) of the one-sided Linnik law has the form
\[ \hat{f}_\alpha^L(x) = \frac{2\sin(\pi \alpha/2)}{\pi} \int_0^\infty \frac{y^\alpha e^{-yx} dy}{1 + y^{2\alpha} + 2y^\alpha \cos(\pi \alpha/2)}, \quad x \geq 0. \]

From corollary 5 we obviously obtain the representation
\[ |L_\alpha| = |X|\sqrt{M_{\alpha/2}}. \quad (50) \]

Based on this representation, below we will present the conditions for the one-sided Linnik distribution to be the limit law for statistics constructed from samples with random sizes, namely, for maximum random sums and extreme order statistics.

### 7.5 Convergence of the distributions of maximum random sums to the one-sided Linnik law

In this section we will demonstrate that the one-sided Linnik distribution can be the limit law for maximum or minimum random sums. The main role here will be played by representation (50) of the one-sided Linnik law as a scale mixture of half-normal distributions. The results of this section are complementary to those of Section 6.1.

We will use the same notation as in Section 6.4. As in Sections 6.1 and 6.4, assume that the random variables \( X_1, X_2, \ldots \) satisfy the Lindeberg condition (33)

Let \( N_1, N_2, \ldots \) be a sequence of nonnegative random variables such that for each \( n \in \mathbb{N} \) the random variables \( N_n, X_1, X_2, \ldots \) are independent. Let \( \{d_n\}_{n \geq 1} \) be an infinitely increasing sequence of positive numbers.

Lemmas 11 and 14 together with representation (50) with the account of the identifiability of scale mixtures of half-normal laws imply the following statement.
Theorem 12. Let $\alpha \in (0,2]$. Assume that the random variables $X_1, X_2, \ldots$ and $N_1, N_2, \ldots$ satisfy the conditions specified above. In particular, let Lindeberg condition (33) hold. Moreover, let $N_n \xrightarrow{P} \infty$ as $n \to \infty$. Then, as $n \to \infty$, the following statements are equivalent:

$$\frac{B_n^2}{d_n} \Rightarrow M_{\alpha/2}; \quad \frac{S_{N_n}^*}{d_n} \Rightarrow L_\alpha; \quad \frac{S_{N_n}^*}{d_n} \Rightarrow |L_\alpha|; \quad \frac{S_{N_n}^*}{d_n} \Rightarrow -|L_\alpha|; \quad \frac{|S_{N_n}|}{d_n} \Rightarrow |L_\alpha|.$$

7.6 Convergence of the distributions of extreme order statistics to the one-sided Linnik law

Let $\alpha \in (0,2]$. Since obviously $W_1 \overset{d}{=} |A|$, from corollaries 3 and 4 it follows that for any $\gamma \geq 1$ we have

$$|L_\alpha| \overset{d}{=} W_1 Q_{\alpha,2} = W_2 T_{1/\gamma,\alpha} = \frac{2^{1/\gamma} W_{\gamma}}{S_{1/\gamma,1}^{1/\gamma}} \sqrt{S_{\alpha/2,1}^{\alpha/2}} \quad (51)$$

We will use the same construction as in Section 6.3. Assume that for each $n \in \mathbb{N}$ the random variable $U_n$ is independent of the standard Poisson process $P(t), t \geq 0$, and let $N_n = P(U_n)$. Let $X_1, X_2, \ldots$ be independent identically distributed random variables with the common distribution function $F(x) = P(X_i < x), x \in \mathbb{R}, i \geq 1$. Denote $\text{lext}(F) = \inf \{x : F(x) > 0\}$. Assume that for each $n \in \mathbb{N}$ the random variable $N_n$ is independent of the sequence $X_1, X_2, \ldots$.

From representation (51) and lemma 13 we obtain the following result.

Theorem 13. Let $\alpha \in (0,2]$. For the existence of numbers $a_n \in \mathbb{R}$ and $b_n > 0$ such that

$$\frac{1}{b_n} \left( \min_{1 \leq j \leq N_n} X_j - a_n \right) \Rightarrow |L_\alpha| \quad (n \to \infty),$$

it is sufficient that:

(i) there exists a $\gamma \geq 1$ such that the distribution function $F$ belongs to the domain of min-attraction of the Weibull distribution with shape parameter $\gamma$, that is, $\text{lext}(F) > -\infty$ and condition (37) holds with $\delta' = \gamma$;

(ii) there exists an infinitely increasing sequence $\{d_n\}_{n \geq 1}$ such that

$$\frac{U_n}{d_n} \Rightarrow T_{1/\gamma}^{-1/\gamma} Q_{\alpha,2} = \frac{W_{\gamma}}{S_{1/\gamma,1}^{1/\gamma}} \sqrt{S_{\alpha/2,1}^{\alpha/2}} \quad (k \to \infty),$$

where all the random variables on the right-hand side are independent. Moreover, the numbers $a_n$ and $b_n$ can be defined in accordance with (38).

Proof. The desired result is a direct consequence of lemma 13 and representation (51) with the account of the relation $Q_{\alpha,2}^{-1} \overset{d}{=} Q_{\alpha,2}$ implied by (27).

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