HYDRODYNAMICS OF POROUS MEDIUM MODEL WITH SLOW RESERVOIRS

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ABSTRACT. We analyze the hydrodynamic behavior of the porous medium model (PMM) in a discrete space \( \{0, \ldots, n\} \), where the sites 0 and \( n \) stand for reservoirs. Our strategy relies on the entropy method of Guo, Papanicolaou and Varadhan \cite{12}. However, this method cannot be straightforwardly applied, since there are configurations that do not evolve according to the dynamics (blocked configurations). In order to avoid this problem, we slightly perturbed the dynamics in such a way that the macroscopic behavior of the system keeps following the porous medium equation (PME), but with boundary conditions which depend on the reservoirs strength’s.

1. INTRODUCTION

In recent years, there has been an intensive research activity around the derivation of partial differential equations (PDEs) with boundary conditions from interacting particle systems \cite{15}. This derivation is known as hydrodynamic limit, which consists in proving, rigorously, that the conserved quantities of a random microscopic dynamics are described by the solution of some PDE. Therefore, this PDE coins the name hydrodynamic equation. The aforementioned procedure, consists, probabilistically speaking, in a Law of Large Numbers for the empirical measure associated to the conserved quantities of the system. More recently, there has been quite a lot of attention devoted to the analysis of microscopic systems with local perturbations, and one of the puzzling questions is to see whether these perturbations have an impact at the macroscopic behavior of the system. Usually, these perturbations, being local, do not destroy the nature of the PDE, but instead they bring up additional boundary conditions to the PDE, see for instance \cite{10} and references therein.

In light of these questions, in this paper we present the derivation of the porous medium equation (PME) with boundary conditions from a microscopic dynamics which is placed in contact with reservoirs. Up to our knowledge, this is the first derivation of a nonlinear degenerate PDE with boundary conditions which can be obtained as the hydrodynamic limit of an underlying microscopic random dynamics. More specifically, we obtain three different types of boundary conditions (Dirichlet, Robin, and Neumann) depending on the intensity of the rate at the reservoir’s dynamics. We remark however that the first microscopic derivation of the PME was obtained in \cite{7} and \cite{8}, in which the authors considered a model with continuous occupational variables. The first microscopic derivation considering discrete occupational variables was obtained in \cite{11}. There, the authors considered the porous medium model (PMM) evolving in the discrete d-dimensional torus \( \mathbb{T}_n^d \) without the presence of reservoirs and therefore, the PME did not have any type of boundary conditions. The article \cite{11} motivated us to work with discrete occupational variables in order to derive the PME, that is, to consider as the random microscopic dynamics, an ad-hoc version of the PMM analyzed there. With the aim to derive boundary conditions in the PME, we combined the microscopic dynamics of \cite{11} with the boundary dynamics of \cite{2}. In the latter article, the dynamics at the bulk was given by the simple symmetric exclusion process (SSEP), then the authors obtained the heat equation with different types of boundary conditions, namely Dirichlet, Robin, and Neumann.

Now we describe precisely what is the random dynamics that we analyze in this article: the \textit{PMM with slow reservoirs}. First, we fix the discrete space where the particle will be moving around, that is, the space \( \Sigma_n = \{1, \ldots, n-1\} \), which we call \textit{bulk}. For \( x \in \Sigma_n \), the occupation variable \( \eta(x) \) takes values...
in \{0, 1\} and \(\eta(x) = 0\) (resp. \(\eta(x) = 1\)) stands for empty (resp. occupied) site. The configuration of particles, that we denote by \(\eta\) is, therefore, an element of \(\{0, 1\}^{\mathbb{Z}}\). The PMM is an exclusion process (since only one particle per site is allowed) with dynamical constraints, i.e., a particle at site \(x\) can jump to \(x+1\), if and only if there is at least one particle at sites \(x-1\) or \(x+2\). The jump rate is given by the sum of the number of particles at sites \(x-1\) and \(x+2\). Due to the constraint of the model’s rates, and since one can have configurations in which the distance between two successive particles is larger than two, the model exhibits the so-called blocked configurations, that is, configurations that do not evolve under the dynamics. To avoid them, we superpose the PMM dynamics with the dynamics of the SSEP on the bulk in such a way that the macroscopic hydrodynamic behavior of this perturbed dynamics still evolves according to the PME. This means that when scaling the time diffusively, we tune the SSEP dynamics in such a way that its impact is not seen at the macroscopic level. At this point this is exactly the same dynamics considered in [11] but restricted to the bulk. At the boundary, we used the same dynamics introduced in [2], that is, a Glauber dynamics at sites 1 and \(n-1\), which plays the role of reservoirs. These reservoirs will also be scaled by a parameter which can be taken to infinity, and the highest its value, the slowest is the boundary dynamics. More specifically, the dynamics of the reservoirs can be described as follows. Particles can be inserted into the system at the site 1 (resp. \(n-1\)) with rate \(\frac{m}{\alpha}\) (resp. \(\frac{m}{\beta}\)), and can be removed from the system through the site 1 (resp. \(n-1\)) at rate \(\frac{n}{\theta}\) (resp. \(\frac{n}{\theta}\)). The factor \(n^{-\theta}\) is the one scaling the boundary dynamics and the higher the value of \(\theta\) the slower is the boundary dynamics, see Figure 1 for an illustration. Throughout the text we use the parameters \(\alpha, \beta \in (0, 1)\), \(m > 0\) and \(\theta \geq 0\).

The PMM just described, belongs to the class of kinetically constrained lattice gases, which are interacting particle systems used to model the liquid/glass transition, see, for example, [4, 19] for a review on the subject. This class of models was introduced in the physics literature in [11], and they are usually classified as cooperative or non-cooperative. In this classification, the PMM is a non-cooperative model, since its dynamical constraints are defined in such a way that it is possible to construct a finite group of particles (the mobile cluster), which can be moved to any position of the discrete space where particles evolve, by using strictly positive exchange rates; and any exchange is allowed when the mobile cluster is brought to the vicinity of the jumping particle. The non-cooperativity of the PMM and the fact that we can perturb its dynamics with the SSEP dynamics, are crucial properties of the model that will be extensively used in the proofs of our arguments. More precisely, when proving the hydrodynamic limit, in order to recognize the solution as a weak solution to the PME, we will have to derive some replacement lemmas, which are stated and proved in Section 5. In their proofs we will have to analyze the irreducibility of the model in the sense that we will have to send a particle from a site \(x\) to some site \(y\) at a distance of order \(O(n)\). In spite of having available the SSEP dynamics, one could think that this could be accomplished easily. Nevertheless the problem cannot be overcome just by using the SSEP jumps since they will be scale in a time less than the diffusive one and for this reason, particles cannot travel to sites at a distance of order \(O(n)\). To push the argument further, we could try to use the PMM jumps, but to do that we need the jumping particle to have particles in its vicinity and many times that does not happen. The trick is then to fix a finite size window around the jumping particle, create a mobile cluster in that window and once the mobile cluster is created we can just use the PMM jumps to move the particles. After sending the particle to where we wanted we destroy the mobile cluster and we put the particles back to their initial position. We remark that the jumps that are used to create and destroy the mobile cluster on the finite size window are the SSEP jumps, in all the rest of the path, we use the PMM jumps. The reader can see Figure 2 and the proof of Lemma (5.3) for a complete description of this argument.

As mentioned above, the main contribution of this article is to derive for the first time the hydrodynamic limit for the PMM with slow reservoirs. Then, we finally present the hydrodynamic equation for that model. The solution of the hydrodynamic equation is called hydrodynamic profile. Our hydrodynamic profiles are weak solutions of the PME with different boundary conditions depending on the range of the parameter \(\theta\). For \(0 \leq \theta < 1\), we obtain the PME with Dirichlet boundary conditions, which
is given by,
\[
\begin{align*}
\partial_t \rho_t(u) &= \Delta(\rho_t(u))^2, \quad (t,u) \in (0,T] \times (0,1), \\
\rho_t(0) &= \alpha, \quad \rho_t(1) = \beta, \quad t \in (0,T].
\end{align*}
\] (1.1)

For \( \theta = 1 \), the boundary dynamics is slowed enough so the boundary conditions of Dirichlet type are replaced by a type of Robin boundary conditions,
\[
\begin{align*}
\partial_t \rho_t(u) &= \Delta(\rho_t(u))^2, \quad (t,u) \in (0,T] \times (0,1), \\
\partial_n \rho_t(0)^2 &= \kappa(\rho_t(0) - \alpha), \quad t \in (0,T], \\
\partial_n \rho_t(1)^2 &= \kappa(\beta - \rho_t(1)), \quad t \in (0,T],
\end{align*}
\] (1.2)

where \( \kappa \in [0,\infty) \). Finally, for \( \theta > 1 \), the boundary is sufficiently slowed so that the Robin boundary conditions are replaced by Neumann boundary conditions (taking \( \kappa = 0 \) in (1.2)) which dictate that, macroscopically, there is no flux of particles from the boundary reservoirs.

In order to better understand the hydrodynamic behavior of our model, we start by observing that the PME, \( \partial_t \rho = \Delta(\rho^M) \), \( M > 1 \), is a nonlinear evolution equation of parabolic type. This equation has received a lot of attention in the last decades due to the mathematical difficulties of building a theory for nonlinear versions of the heat equation. One can rewrite the equation in divergence form as
\[
\partial_t \rho = \nabla(D(\rho)\nabla \rho),
\] (1.3)

where \( \rho = \rho(t,u) \) is a scalar function and \( D(\rho) = M\rho^{M-1} \) is the diffusion coefficient. The space variable \( u \) takes values in some bounded or unbounded domain of \( \mathbb{R}^d \) and the variable \( t \) satisfies \( t \geq 0 \). As mentioned above, the PME is also a degenerate parabolic equation, since the diffusion coefficient vanishes when \( \rho \) goes to zero. Because of that, the regularity results for its solutions is weaker than the solutions of classical parabolic equations and the techniques for the study of PME are much more refined. Matters as existence and uniqueness of classical and weak solutions are also affected by the degeneracy of this equation. From the physical point of view, one of the main differences between the PME and the heat equation is the so-called finite speed of propagation, that is, its solutions can be compactly supported at each fixed time. This property implies the appearance of a free boundary that separates the support of the solution from the empty region. Across this boundary, the solution loses regularity. See [20] and references therein for a more detailed study of this equation.

The name PME was motivated by the work [19], in which the equation (with \( M = 2 \)) was used to model the density of a gas flowing through a porous medium. There are a lot of physical applications of the PME with several values of \( M \), most of them being used to describe processes involving diffusion or heat transfer. In [22], the equation was used to study the heat radiation in plasmas, and in [13,14], the authors used the PME to describe migratory diffusion of biological populations.

We consider the one-dimensional boundary-value problem to the PME in a spatial domain \([0,1] \subset \mathbb{R}\) given in (1.3) with \( M = 2 \). The spatial domain \([0,1]\) is the macroscopic space that corresponds to the discretized space \( \Sigma_n \) defined above. We remark that it is possible to extend our results to higher values of \( M \) simply by taking the jump rates of the process in accordance to that. For example, when \( M = 2 \) in order to have a jump we required to have, at least, one particle close to the jumping particle, but if \( M = 3 \) is taken, we then need to require two particles instead of one, see (2.5) for the precise expression of the jump rates in this case. For simplicity of the presentation, all the arguments are given for the case \( M = 2 \) but they extend easily to other values of \( M \).

Now we explain the difficulties that we face when trying to derive the hydrodynamic limit for this model. The proof goes by showing tightness and characterizing uniquely the limit point. We remark that in the characterization of limit points, one important property of this model is that it is a gradient system. This means that the instantaneous current of the system can be written as a discrete gradient of some local function of the dynamics, see (2.2). In our case this function is a two degree function, that is, it is a function given by sums of terms of the form \( \eta(x)\eta(y) \) for \( |y-x| \leq 2 \). Due to this fact, one needs a replacement lemma in the whole bulk which allows to write this function in terms of an
average of particles around a box of size $O(n)$. Since we are in the presence of reservoirs the proof of \cite{11} does not apply in this setting and we have to redo the whole argument. The idea consists in removing the boundary points from the bulk which do not allow these replacements; show that this removal is negligible in the limit and on the remaining points we do a step-by-step replacement in the following fashion: at first step fix one of the variables $\eta(x)$ and do the replacement of $\eta(x+1)$ by the particle average to the right of $x+1$ on a box of size $O(n)$. Then, fix this average and repeat the previous replacement but now for the variable $\eta(x)$ and a box of size $O(n)$ to the left of $x$; this left-right argument is crucial so that the two boxes do not overlap and variables do not correlate. When doing all these replacements one has to use the arguments described above, in which we need to create a mobile cluster capable of making particles move. Due to the reservoir’s action, we also have to control the terms that arise at the boundary and we need to derive a couple of replacements to deal with these extra terms.

For the uniqueness of the limiting point we also had to derive the uniqueness of the weak solution of the PME with the different types of boundary conditions. The Dirichlet case could be easily proved but the Robin case deserved a special attention. Since we did not find in the literature the exact statement of uniqueness we needed, we had to adapt the arguments in \cite{9} to our particular setting and for completeness we presented here the whole proof. Indeed, we obtain uniqueness for a Robin boundary condition for a function $u^2$ (in the place of a function $\beta^{-1}(u)$ in the notation of the article \cite{9}) that is not Lipschitz, which is an important hypothesis for the proof given in \cite{9}.

There are a couple of questions that still have no answer and are left for a future work. We highlight one which is concerned with the hydrostatic limit. In our result on the hydrodynamic limit we need to impose the starting measure to be associated to a profile, see \cite{2.12}. We note that when the boundary rates $\alpha$ and $\beta$ coincide with $\rho$, the Bernoulli product measure with constant parameter $\rho$ is a reversible measure for this model and, in particular, it is invariant. Nevertheless, when $\alpha \neq \beta$, this measure is no longer invariant and we have no information on the invariant measure of the system. The matrix method of Derrida \cite{5} cannot be applied to this model due to the complicated action of the bulk dynamics. One way to prove that the invariant measure of the model is associated with a profile, namely the stationary profile of the respective hydrodynamic equation (see Remark \cite{2.7}) is to prove that its space correlations decay to 0 when $n \to +\infty$. For this model it is not easy to obtain information of the correlations since the equations satisfied by the correlation function are not closed and again this is a consequence of the complicated action of the bulk dynamics. Another interesting problem is to derive the hydrodynamic limit for the PMM without the perturbation with the SSEP jumps. The difficulty we will face is the lack of mobility of the system: the creation of the mobile cluster now is not possible. These are problems that we will attack in the near future.

Here follows an outline of the article. In Section \ref{sec:2} we state our results. In Subsection \ref{sec:2.1} we introduce some notations and we define precisely the PMM. In Subsection \ref{sec:2.2} we present the notion of weak solution of the hydrodynamic equations, and we also present their stationary solutions. In Subsection \ref{sec:2.3} we state our main result. In Section \ref{sec:3}, we prove tightness for the sequence of probability measures of interest. In Section \ref{sec:4} we characterize the limit points. In Section \ref{sec:5} we provide estimates on Dirichlet forms and we present the proofs of all the replacement lemmas that are needed along the proof’s arguments. Section \ref{sec:6} deals with energy estimates, and we finish the paper with Section \ref{sec:7} by presenting a proof of the uniqueness of weak solutions of each hydrodynamic equation.

2. Statement of results

2.1. The model. Let $n \geq 1$ a scaling parameter, and fix the following real numbers: $\theta \geq 0$, $m > 0$, $a \in (1,2)$ and $\alpha, \beta \in (0,1)$. Let $\Sigma_n$ be the discrete space $\{1, \ldots, n-1\}$ which we call bulk. The dynamics of the PMM with a superposed SSEP dynamics and a Glauber dynamics can be described as follows: we associate a Poisson clock at each bond $\{x, x+1\}$, with $x = 1, \ldots, n-2$ and with a parameter depending on the exclusion rule and on the constraints of the process. At the left boundary (resp. right boundary) we artificially add Poisson clocks at the bonds $\{0,1\}$ (resp. $\{n-1,n\}$) and $\{1,0\}$ (resp. $\{n,n-1\}$) with
Above, the exchange rates are given by 

\[ a_{x,y} = \eta(x)(1 - \eta(y)), \quad x \neq y, \]

\[ l_x^b(\eta) = b(1 - \eta(x)) + (1 - b)\eta(x), \]

for \( x, y \in \{1, \ldots, n-2\}, \ z \in \{1, n-1\} \) and \( b \in \{\alpha, \beta\} \). Note that, throughout the text, we use the convention

\[ \eta(0) = \alpha, \quad \eta(n) = \beta, \]

where \( \alpha, \beta \in (0,1) \). Figure 1 below shows the dynamics of the model.

**Figure 1.** The porous medium model with slow reservoirs.

**Remark 2.1.** We stress that (2.1) is related to the diffusion coefficient of (1.3) when \( M = 2 \). Considering general values of \( M \) in (1.3), we have to consider different values in (2.1). For example, when \( M = 3 \), the diffusion coefficient of (1.3) is given by \( D(\rho) = 3\rho^2 \), and the exchange rate in (2.1) is given by

\[ c_{x,x+1}(\eta) = \eta(x-2)\eta(x-1) + \eta(x-1)\eta(x+2) + \eta(x+2)\eta(x+3). \]  

(2.5)
Remark 2.3. Note that the dynamics is degenerate, gradient, and does not conserve the total number of particles. Note also that since the process is superposed with the SSEP dynamics, it is an irreducible Markov process on a finite state space, therefore only one invariant measure exists. In the equilibrium state, that is, when $\alpha = \beta$, the interested reader can verify that the invariant measure of the process is the Bernoulli product measure, with a constant parameter, let us say, $\rho = \alpha = \beta$. For the non-equilibrium state, that is, when $\alpha \neq \beta$, we need to put more effort to obtain the invariant measure of this process. We stress that a possible way to get some information about this measure is to apply the matrix ansatz (also called matrix product state), introduced in [5], and we leave this issue to a future work.

Remark 2.4. From now on let $\{\eta_{tn^2}\}_{t \geq 0}$ denote the Markov process speeded up in the diffusive time scale $tn^2$ and driven by the infinitesimal generator $n L_n$.

2.2. Hydrodynamic equations. We first introduce some notations and definitions to state the hydrodynamic limit. Fix an interval $\mathcal{S} \subset \mathbb{R}$ and $m, n \in \mathbb{Z}$. We denote by:

- $C^{m,n}([0, T] \times \mathcal{S})$, the set of real-valued functions defined on $[0, T] \times \mathcal{S}$ that are $m$ times differentiable on the first variable and $n$ times differentiable on the second variable (with continuous derivatives);
- $C^m_0([0, 1])$, the set of all $m$ continuously differentiable real-valued functions defined on $[0, 1]$ with compact support;
- $C^{m,n}_0([0, T] \times [0, 1])$, the set of real-valued functions $G \in C^{m,n}([0, T] \times [0, 1])$ such that $G_t(0) = G_t(1) = 0$, for all $s \in [0, T]$.

The inner product in $L^2([0, 1])$ is denoted by $\langle \cdot, \cdot \rangle$ with corresponding norm $\| \cdot \|_2$. The semi-inner product $\langle \cdot, \cdot \rangle_1$ is defined on the set $C^{\infty}([0, 1])$ by

$$
\langle G, H \rangle_1 = \int_0^1 (\partial_u G)(u)(\partial_u H)(u) du,
$$

with corresponding semi-norm $\| \cdot \|_1$.

Definition 1. The Sobolev space $\mathcal{H}^1$ on $(0, 1)$ is the Hilbert space defined as the completion of $C^{\infty}((0, 1))$ for the norm $\| \cdot \|_{\mathcal{H}^1} := \| \cdot \|_2^2 + \| \cdot \|_1^2$. The space $L^2(0, T; \mathcal{H}^1)$ is the set of measurable functions $f : [0, T] \to \mathcal{H}^1$ such that $\int_0^T \| f_s \|_{\mathcal{H}^1}^2 ds < \infty$.

After both definitions and notations outlined above, we may move forth to define the notion of weak solution of the hydrodynamic equations that we will use along this article.

Definition 2. Let $\alpha, \beta \in (0, 1)$ and $g : [0, 1] \to [0, 1]$ a measurable function. We say that $\rho : [0, T] \times [0, 1] \to [0, 1]$ is a weak solution of the PME with Dirichlet boundary conditions

$$
\begin{align*}
\partial_t \rho_t(u) &= \Delta (\rho_t(u))^2, \quad (t, u) \in (0, T) \times (0, 1), \\
\rho_t(0) &= \alpha, \quad \rho_t(1) = \beta, \quad t \in (0, T], \\
\rho_0(\cdot) &= g(\cdot),
\end{align*}
$$

if the following conditions hold:

1. $\rho \in L^2(0, T; \mathcal{H}^1)$;
2. $\rho$ satisfies the integral equation:

$$
F_{Di}(G, t, \rho, g) := \langle \rho_t, G_t \rangle - \langle g, G_0 \rangle - \int_0^t \langle \rho_s, (\partial_u G_s + \rho_s \Delta G_s) \rangle ds
$$

$$
+ \int_0^t \{\beta^2 \partial_u G_s(1) - \alpha^2 \partial_u G_s(0)\} ds = 0,
$$

for all $t \in [0, T]$ and any function $G \in C^{1,2}_0([0, T] \times [0, 1])$;

3. $\rho_t(0) = \alpha$ and $\rho_t(1) = \beta$ for all $t \in (0, T]$.
**Definition 3.** Let $\kappa \geq 0$, $\alpha, \beta \in (0, 1)$ and $g : [0, 1] \rightarrow [0, 1]$ a measurable function. We say that $\rho : [0, T] \times [0, 1] \rightarrow [0, 1]$ is a weak solution of the PME with Robin boundary conditions

\[
\begin{aligned}
&\partial_t \rho(t,u) = \Delta (\rho(t,u))^2, \quad (t,u) \in (0, T) \times (0, 1), \\
&\partial_n \rho(t,0) = \kappa (\rho(t,0) - \alpha), \quad t \in (0, T), \\
&\partial_n \rho(t,1) = \kappa (\beta - \rho(t,1)), \quad t \in (0, T), \\
&\rho(0) = g(\cdot),
\end{aligned}
\]

if the following conditions hold:

1. $\rho \in L^2(0, T; \mathcal{H}^1)$;
2. $\rho$ satisfies the integral equation:

\[
F_{\text{Rob}}(G, t, \rho, g) := \langle \rho_t, G_s \rangle - \left( g, G_0 \right) - \int_0^t \langle \rho_s, (\partial_s G_s + \rho_s \Delta G_s) \rangle \, ds \tag{2.9}
\]

\[
+ \int_0^t \left\{ (\rho_s(1))^2 \partial_n G_s(1) - (\rho_s(0))^2 \partial_n G_s(0) \right\} \, ds \tag{2.9}
\]

\[- \kappa \int_0^t \left\{ G_s(0)(\alpha - \rho_s(0)) + G_s(1)(\beta - \rho_s(1)) \right\} \, ds = 0,
\]

for all $t \in [0, T]$ and any function $G \in C^{1,2}([0, T] \times [0, 1])$.

**Remark 2.5.** For $\kappa = 0$ we obtain in (2.8) Neumann boundary conditions.

**Lemma 2.6.** The weak solutions of (2.6) and (2.8) are unique.

The proof of last lemma can be found in Section 7.

**Remark 2.7.** In order to get more information about the invariant measures of the process in the non-equilibrium state, it is good to know the stationary solution of each hydrodynamic equation. Thus, a simple computation shows that the stationary solution of (2.6) is given on $u \in (0, 1)$ by

\[
\bar{\rho}(u) = \sqrt{(\beta^2 - \alpha^2)u + \alpha^2},
\]

and the stationary solution of (2.8) is given on $u \in (0, 1)$ by

\[
\bar{\rho}(u) = \sqrt{au + b}, \tag{2.10}
\]

where

\[
a = \kappa (\sqrt{b} - \alpha) \quad \text{and} \quad b = \left( \frac{\kappa \alpha + (\alpha + \beta)^2}{2(\alpha + \beta) + \kappa} \right)^2. \tag{2.11}
\]

The stationary solution for the Neumann case is simply a constant. But, in fact, we observe that, looking back at the stationary solution that we just computed, when we take $\kappa = 0$, the stationary solution is given on $u \in (0, 1)$ by $\bar{\rho}(u) = \frac{a + \beta}{2}$.

### 2.3. Hydrodynamic Limit.

For any configuration $\eta \in \Omega_n$, we define the empirical measure $\pi^n(\eta, du)$ on $[0, 1]$ by

\[
\pi^n(\eta, du) = \frac{1}{n} \sum_{x \in \mathbb{Z}_n} \eta(x) \delta_x (du),
\]

where $\delta_a$ is a Dirac mass on $a \in [0, 1]$. We also define $\pi^n(\eta, du) := \pi^n(\eta_{\text{tri}}, du)$. For a test function $G : [0, 1] \rightarrow \mathbb{R}$, we denote by $\langle \pi^n_t, G \rangle$ the integral of $G$ with respect to the measure $\pi^n_t$, which is equal to

\[
\langle \pi^n_t, G \rangle = \frac{1}{n} \sum_{x \in \mathbb{Z}_n} G(x) \eta_{\text{tri}}^n(x).
\]

Fix $T > 0$ and $\theta > 0$. Let $\mathcal{M}_+$ be the space of positive measures on $[0, 1]$ with total mass bounded by 1 equipped with the weak topology. Let $\mu_n$ be a measure on $\Omega_n$. We denote by $P_{\mu_n}$ the probability measure in the Skorokhod space $\mathcal{D}([0, T], \Omega_n)$, that is, the space of càdlàg trajectories induced by the accelerated Markov process $\{\eta_{\text{tri}}\}_{t \geq 0}$ and the initial measure $\mu_n$. We denote by $E_{\mu_n}$ the expectation
with respect to $\mathbb{P}_{\mu_n}$. Let $\{Q_n\}_{n \in \mathbb{N}}$ be the sequence of probability measures on $\mathcal{D}([0, T], \mathcal{M}_u)$ induced by the Markov processes $\{\pi^*_n\}_{t \geq 0}$ and by $\mathbb{P}_{\mu_n}$.

Given a measurable function $g : [0, 1] \to [0, 1]$, we say that a sequence of probability measures $\{\mu_n\}_{n \in \mathbb{N}}$ on $\Omega_n$ is associated with $g(\cdot)$, if for any continuous function $G : [0, 1] \to \mathbb{R}$ and any $\delta > 0$

$$\lim_{n \to +\infty} \mu_n \left( \eta \in \Omega_n : \frac{1}{n} \sum_{x \in \Sigma_n} G\left( \frac{x}{n} \right) \eta(x) - (G, \eta) > \delta \right) = 0. \tag{2.12}$$

**Theorem 2.8.** Let $g : [0, 1] \to [0, 1]$ be a measurable function and let $\{\mu_n\}_{n \in \mathbb{N}}$ be a sequence of probability measures on $\Omega_n$ associated with $g(\cdot)$. Then, for any $t \in [0, T]$ and any $\delta > 0$,

$$\lim_{n \to +\infty} \mathbb{P}_{\mu_n} \left( \eta \in \mathcal{D}([0, T], \Omega_n) : \frac{1}{n} \sum_{x \in \Sigma_n} G\left( \frac{x}{n} \right) \eta_t(x) - (G, \rho_t) > \delta \right) = 0,$$

where

- for $\theta < 1$, $\rho_t(\cdot)$ is a weak solution of (2.6);
- for $\theta = 1$, $\rho_t(\cdot)$ is a weak solution of (2.8) with $\kappa = m$;
- for $\theta > 1$, $\rho_t(\cdot)$ is a weak solution of (2.8) with $\kappa = 0$.

To prove Theorem 2.8 we will use the classical entropy method of Guo, Papacipanoul, and Varadhan [12]. In Section 3 we prove that the sequence of probability measures $\{Q_n\}_{n \in \mathbb{N}}$ is tight, i.e., the sequence has limit points $Q$. In Section 4, we prove that the density $\rho_t(u)$ is a weak solution of the corresponding hydrodynamic equation. In Section 5, we present some estimates for the Dirichlet forms that are necessary to prove the replacement lemmas, and we present the proofs of the replacement lemmas. Then, in Section 6, we prove the energy estimates, that is, $\rho \in L^2(0, T; \mathcal{M}^1)$. To conclude, in Section 7 we prove uniqueness of weak solutions for each hydrodynamic equation presented above, and due to this, we guarantee the uniqueness of the limit point $Q$.

### 3. Tightness

In this section we prove that the sequence $\{Q_n\}_{n \in \mathbb{N}}$, defined in Section 2 is tight. Before start proving tightness, let us present some results we shall use within this section.

Fix a function $G \in C^{1,2}([0, T] \times [0, 1])$. We know by Dynkin’s formula, see Lemma A1.5.1 of [15], that

$$M^n_t(G) = \langle \pi^n_t, G_t \rangle - \langle \pi^n_0, G_0 \rangle - \int_0^t \left( \frac{n^2}{\tau} + n^2 L_P + n^a L_S + n^2 L_P \right) \langle \pi^n_s, G_s \rangle \, ds \tag{3.1}$$

is a martingale with respect to the natural filtration $\{\mathcal{F}_t\}_{t \geq 0}$, where $\mathcal{F}_t = \langle \sigma(\eta) : s \leq t \rangle$. Assume, for argument’s sake, that $G$ is time independent. For $\eta \in \Omega_n$ and $x \in \Sigma_n$, we denote by $j_{x,x+1}(\eta)$ the instantaneous current associated to the bond $\{x, x+1\}$, which is given by

$$j_{0,1}(\eta) = \frac{m}{n} (\alpha - \eta(1)), \quad j_{x,x+1}(\eta) = \tau_x h(\eta) - \tau_{x+1} h(\eta), \quad j_{n-1,n}(\eta) = \frac{m}{n} (\eta(n-1) - \beta), \tag{3.2}$$

where

$$\tau_x h(\eta) = \eta(x-1)\eta(x) + \eta(x)\eta(x+1) - \eta(x-1)\eta(x+1) + n^{a-2} \eta(x).$$

Using the computations above, we have that $n^2 L_P/\langle \pi^n_s, G \rangle$ is given by

$$\frac{1}{n} \sum_{x=1}^{n-1} \Delta_n G\left( \frac{x}{n} \right) \tau_x h(\eta_{x+1}) + \nabla_n^+ G(0) \tau_1 h(\eta_{1n}) - \nabla_n^- G(1) \tau_{n-1} h(\eta_{n-1}) + n G\left( \frac{x}{n} \right) \frac{m}{n} (\alpha - \eta(1)) + n G\left( \frac{x+1}{n} \right) \frac{m}{n} (\beta - \eta(x)(n-1)), \tag{3.3}$$

where for $x \in \Sigma_n$, the discrete Laplacian is given by

$$\Delta_n G\left( \frac{x}{n} \right) = n^2 \left( G\left( \frac{x+1}{n} \right) - 2 G\left( \frac{x}{n} \right) + G\left( \frac{x-1}{n} \right) \right).$$
and the discrete derivatives are given by
\[ \nabla_n G \left( \frac{x}{n} \right) = n \left( G \left( \frac{x+1}{n} \right) - G \left( \frac{x}{n} \right) \right) \quad \text{and} \quad \nabla_n^- G \left( \frac{x}{n} \right) = n \left( G \left( \frac{x-1}{n} \right) - G \left( \frac{x}{n} \right) \right). \]

Since the function \( G \) is time independent and using the convention \((2.4)\), the martingale in \((3.1)\) is equal to
\[ \left\langle \pi_t^n, G \right\rangle \right] - \left\langle \pi_0^n, G \right\rangle \right] - \int_0^t \frac{1}{n} \sum_{k=1}^{n-1} \Delta_n G \left( \frac{k}{n} \right) \tau_s h(\eta_{sn^2}) \, ds \]
\[ - \int_0^t \nabla_n^+ G(0) \tau_1 h(\eta_{sn^2}) \, ds + \int_0^t \nabla_n^- G(1) \tau_{n-1} h(\eta_{sn^2}) \, ds \]
\[ - \frac{m}{n^\beta} \int_0^t \left\{ G \left( \frac{1}{n} \right) (\alpha - \eta_{sn^2}(1)) + G \left( \frac{n-1}{n} \right) (\beta - \eta_{sn^2}(n-1)) \right\} \, ds. \]

**Remark 3.1.** By the mean value theorem and since \(|\eta_{sn^2}(x)| \leq 1\), we have that
\[ |\Delta_n G \left( \frac{x}{n} \right)| \leq 2\|G''\|_\infty, \quad |\nabla_n^+ G(0)| \leq 2\|G'\|_\infty, \quad \text{and} \quad |\nabla_n^- G(1)| \leq 2\|G'\|_\infty, \]
for all \( s \geq 0 \) and \( x \in \Sigma_n \).

**Remark 3.2.** Note that when \( n \to +\infty \) the terms that come from the SSEP jumps vanish, so that, throughout the paper we ignore them and we look only at the remaining terms.

**Proposition 3.3.** The sequence of measures \( \{\mathbb{Q}_n\}_{n \in \mathbb{N}} \) is tight with respect to the Skorokhod topology of \( D([0,T], \mathcal{M}_+). \)

**Proof.** From Proposition 4.1.6 of \([15]\), it is enough to show tightness of the real-valued process \( \{\langle \pi_t^n, G \rangle\}_{0 \leq t \leq T} \) for a time independent function \( G \in C([0,1]) \). We claim that for each \( \epsilon > 0 \),
\[ \lim_{N \to 0} \lim_{n \to +\infty} \sup_{t \in [0,T]} \mathbb{P}_{\mu_n} \left( |\langle \pi_t^{n+\sigma}, G \rangle \right] - \left\langle \pi_t^n, G \right\rangle | > \frac{\epsilon}{2} \right) = 0, \]
where \( \mathcal{F}_t \) is the set of stopping times bounded by \( T \). By Proposition 4.1.7 of \([15]\), it is enough to show that \((3.5)\) holds for functions \( G \) in a dense subset of \( C([0,1]) \), with respect to the uniform topology of \( C([0,1]) \). From \((3.1)\), Markov’s and Chebyshev’s inequalities, the probability in \((3.5)\) can be bounded from above by
\[ \mathbb{P}_{\mu_n} \left( \left| M_{t+\sigma}^n (G) - M_t^n (G) \right| > \frac{\epsilon}{2} \right) + \mathbb{P}_{\mu_n} \left( \left| \int_\tau^{t+\sigma} n^2 L_n (\pi_r^n, G) \, dr \right| > \frac{\epsilon}{2} \right) \]
\[ \leq \frac{4}{\epsilon^2} \mathbb{E}_{\mu_n} \left( \left| M_{t+\sigma}^n (G) - M_t^n (G) \right|^2 \right) + \frac{2}{\epsilon} \mathbb{E}_{\mu_n} \left( \left| \int_\tau^{t+\sigma} n^2 L_n (\pi_r^n, G) \, dr \right| \right). \]

So, if we prove that
\[ \lim_{N \to 0} \lim_{n \to +\infty} \sup_{t \in [0,T], \sigma \leq y} \mathbb{E}_{\mu_n} \left( \left| M_{t+\sigma}^n (G) - M_t^n (G) \right|^2 \right) = 0, \]
and
\[ \lim_{N \to 0} \lim_{n \to +\infty} \sup_{t \in [0,T], \sigma \leq y} \mathbb{E}_{\mu_n} \left( \left| \int_\tau^{t+\sigma} n^2 L_n (\pi_r^n, G) \, dr \right| \right) = 0, \]
the claim follows. We have divided the proof of \((3.6)\) and \((3.7)\) into two cases: \( \theta \geq 1 \) and \( \theta \in [0,1) \).

**Case \( \theta \geq 1 \):** We begin by analyzing \((3.6)\). Let \( G \in C^2([0,1]) \), where \( C^2([0,1]) \) is a dense subset of \( C([0,1]) \) with respect to the uniform topology. Define
\[ F_s^n (G) := n^2 \left( L_n (\pi_s^n, G) \right)^2 - 2 \left( \pi_s^n, G \right) L_n (\pi_s^n, G) \].

Note that
\[ \mathbb{E}_{\mu_n} \left( \left( M_{t+\sigma}^n (G) - M_t^n (G) \right)^2 \right) = \mathbb{E}_{\mu_n} \left( \int_\tau^{t+\sigma} F_s^n (G) \, ds \right), \]
since \((M^n_{\tau+\sigma}(G) - M^n_\tau(G))^2 - \int_\tau^{\tau+\sigma} F^n_s(G) ds\) is a mean zero martingale. Note that, \((3.6)\) holds if we show that \(\int_\tau^{\tau+\sigma} F^n_s(G) ds\) converges to zero uniformly in \(t \in [0, T]\), when \(n \to +\infty\). From Remark 3.1 a simple computation shows that \(F^n_t(G)\) is bounded from above by a constant, times

\[
\frac{1}{n} \| (G')^2 \|_\infty + C(\alpha, \beta) \frac{m}{n^\alpha} \| G^2 \|_\infty + n^\alpha \| (G')^2 \|_\infty,
\]

(3.8)

where \(C(\alpha, \beta)\) is a real constant depending on \(\alpha\) and \(\beta\). Taking \(n \to +\infty\) in the previous display, the result follows.

It remains to prove (3.7). Recall (3.3). From Remark 3.1 we can bound the bulk term from above by

\[
\left| \Delta_n G(n) \tau_x h(\eta_{in^2}) \right| \leq 2\| G' \|_\infty,
\]

(3.9)

and the boundary terms by

\[
\nabla^n_\tau G(0) \tau_x h(\eta_{in^2}) + n G(n) \frac{m}{n} (\alpha - \eta_{in^2}(1)) \leq \| G' \|_\infty + n^{1-\theta} m \| G \|_\infty,
\]

\[
-\nabla^n_\tau G(n) \tau_x h(\eta_{in^2}) + n G(n+1) \frac{m}{n} (\beta - \eta_{in^2}(n-1)) \leq \| G' \|_\infty + n^{1-\theta} m \| G \|_\infty.
\]

(3.10)

So, since \(\theta \geq 1\), by (3.9), (3.10), and (3.10), we have that

\[
\lim_{\gamma \to 0} \lim_{n \to +\infty} \sup_{\tau \in \mathcal{T}, \sigma \leq \gamma} \mathbb{E}_{\mu_n} \left( \int_\tau ^{\tau+\sigma} n^2 L_\mathcal{E}(\pi^n_\tau, G) d\tau \right) = 0.
\]

This proves (3.7). Note that the proof of (3.6) works for any \(\theta > 0\), but does not work for \(\theta = 0\) since the second term in (3.6) does not vanish when we take \(n \to +\infty\). We treat this case below.

**Case \(\theta \in [0,1)\):** Note that if we try to apply the strategy used above, we will have problems trying to control the expression \(\int_\tau ^{\tau+\sigma} n^2 L_\mathcal{E}(\pi^n_\tau, G) d\tau\). This happens because for these values of \(\theta\), the terms that come from the boundary go to infinity with \(n\). Due to this fact, since these terms also depend on the value of \(G(\frac{1}{n})\) and \(G(\frac{n-1}{n})\), we can get rid of them by asking the test function \(G\) to have compact support in \((0,1)\). With this assumption, we can show that (3.6) and (3.7) are still valid when \(G \in C^2(0,1)\) only by using the computations done for \(\theta \geq 1\). To finish the proof, we need to show that (3.6) and (3.7) hold for \(G \in C(0,1)\). The idea then is to approximate \(G \in C(0,1)\) in \(L_1\) by functions in \(C^2(0,1)\). We leave the interested reader to look for the proof of this in, for example, Section 4 of [2].

### 4. Characterization of Limit Points

We begin by fixing some notations used along the text. Fix \(x \in \Sigma_n\), \(\ell \in \mathbb{N}\), \(\varepsilon > 0, \delta > 0\) and recall that \(a \in (1,2)\). In what follows \(\varepsilon n\) denotes \([\varepsilon n]\). Let

\[
\Sigma^{\ell}_n = \{1+\varepsilon n, \ldots, n-1-\varepsilon n\},
\]

(4.1)

and

\[
\overleftarrow{\Lambda}^{\ell}_x := \{x-\ell+1, \ldots, x\} \quad (\text{resp. } \overrightarrow{\Lambda}^{\ell}_x := \{x, \ldots, x+\ell-1\}),
\]

be the box of size \(\ell\) to the left (resp. right) of the site \(x\). We denote by

\[
\overleftarrow{\eta}^{\ell}(x) = \frac{1}{\ell} \sum_{y \in \overleftarrow{\Lambda}_x^{\ell}} \eta(y) \quad \text{and} \quad \overrightarrow{\eta}^{\ell}(x) = \frac{1}{\ell} \sum_{y \in \overrightarrow{\Lambda}_x^{\ell}} \eta(y)
\]

(4.2)

the empirical densities in the boxes \(\overleftarrow{\Lambda}^{\ell}_x\) and \(\overrightarrow{\Lambda}^{\ell}_x\), respectively.

From Section 3 we know that limit points \(Q\) of the sequence \(\{Q_n\}_{n \in \mathbb{N}}\) exist. We now observe that, as a consequence of the exclusion rule, they are concentrated on trajectories of measures, that are absolutely continuous with respect to the Lebesgue measure, see [15] for more details. Moreover, we claim that the density \(\pi_t(u)\) is a weak solution of the corresponding hydrodynamic equation. This is proved in the next proposition.

**Proposition 4.1.** Let \(Q\) be a limit point of the sequence \(\{Q_n\}_{n \in \mathbb{N}}\). Then

\[
Q\left( \pi, \in \mathcal{G}(\mathcal{M}_t^\varepsilon), G \right) : F_\theta(G, t, \pi, p, \omega) = 0, \forall t \in [0, T], \forall G \in C_\theta\) = 1.
\]

Above \(F_\theta = F_{Di,\theta}\) and \(C_\theta = C^{1,2}_0([0, T] \times [0, 1])\) for \(\theta < 1\); and \(F_\theta = F_{Rob,\theta}\) and \(C_\theta = C^{1,2}_0([0, T] \times [0, 1])\) for \(\theta \geq 1\).
Proof. The proof ends as long as we show that for any $\delta > 0$ and $G \in C_\theta$

$$Q\left( \pi \in \mathcal{D}([0,T], \mathcal{M}_+): \sup_{0 \leq t \leq T} |F_\theta(G,t,\rho,g)| > \delta \right) = 0, \quad (4.3)$$

for each regime of $\theta$. We start with the case $\theta = 1$. Recall from Definition 2.9 the definition of $F_{\text{Rob}}$. We note that the set inside last probability is not an open set in the Skorokhod space. To avoid this problem, we fix $\varepsilon > 0$ and we consider two approximations of the identity, for fixed $u \in [0,1]$, which are given on $v \in [0,1]$ by

$$\tilde{t}_\varepsilon^u(v) = \frac{1}{\varepsilon} 1_{(u-\varepsilon,u)}(v) \quad \text{and} \quad \tilde{t}_\varepsilon^u(v) = \frac{1}{\varepsilon} 1_{[u,u+\varepsilon]}(v).$$

We use the notation

$$\langle \pi_s, \tilde{t}_\varepsilon^u \rangle = \frac{1}{\varepsilon} \int_{u-\varepsilon}^u \rho_s(v) dv \quad \text{and} \quad \langle \pi_s, \tilde{t}_\varepsilon^u \rangle = \frac{1}{\varepsilon} \int_u^{u+\varepsilon} \rho_s(v) dv.$$

By summing and subtracting proper terms, we bound the probability in (4.3) from above by the sum of

$$Q\left( \sup_{0 \leq t \leq T} \left| \langle \rho_t, G_t \rangle - \langle \rho_0, G_0 \rangle - \int_0^t \langle \rho_s, \tilde{t}_\varepsilon^u \rangle \partial_s G_s ds + \int_0^t \int_0^{1-\varepsilon} \langle \pi_s, \tilde{t}_\varepsilon^u \rangle \Delta G_s(u) ds \right| > \frac{\delta}{5} \right),$$

$$Q\left( \sup_{0 \leq t \leq T} \left| \int_0^t \left( \langle \rho_s^2, \Delta G_s \rangle - \int_0^{1-\varepsilon} \langle \pi_s, \tilde{t}_\varepsilon^u \rangle \Delta G_s(u) ds \right) ds \right| > \frac{\delta}{5} \right), \quad (4.6)$$

$$Q\left( \sup_{0 \leq t \leq T} \left| m \int_0^t G_s(0) \left\{ \langle \pi_s, \tilde{t}_\varepsilon^0 \rangle - \rho_s(0) \right\} ds \right| > \frac{\delta}{5} \right), \quad (4.7)$$

$$Q\left( \sup_{0 \leq t \leq T} \left| \int_0^t \left( \langle \rho_s(0)^2 - \langle \pi_s, \tilde{t}_\varepsilon^0 \rangle \langle \pi_s, \tilde{t}_\varepsilon^u \rangle \rangle \frac{\partial_s G_s(0)}{ds} \right) \right| > \frac{\delta}{5} \right), \quad (4.8)$$

plus two terms similar to the last ones but with respect to the right boundary. The term (4.5) is equal to zero since $Q$ is a limit point of $\{Q_n\}_{n \in \mathbb{N}}$ and $Q_0$ is induced by $\mu_\varepsilon$, which satisfies (2.12). To treat the terms (4.7) and (4.8), we use the fact that $\rho \in L^2(0,T;\mathcal{H}^1)$, which will be proved in Section 5. From this result we have that

$$\left| \rho_s(u) - \langle \pi_s, \tilde{t}_\varepsilon^u \rangle \right| \lesssim \sqrt{\varepsilon \left( \|\partial_s \rho_s\|_2^2 + 1 \right)}, \quad (4.9)$$

and the same bound holds replacing $\langle \pi_s, \tilde{t}_\varepsilon^u \rangle$ by $\langle \pi_s, \tilde{t}_\varepsilon^u \rangle$. The notation $f(x) \lesssim g(x)$ means that there exist a constant $C$ independent of $x$, such that $f(x) \leq C g(x)$ for every $x$. The inequality in (4.9) is important to prove that the terms (4.6), (4.7) and (4.8) converge to 0, as $\varepsilon \to 0$. The term (4.7) goes to zero, with a simple application of (4.9). For (4.8), besides using (4.9), we also need to add and subtract suitable terms, use that $\rho_s(\cdot)$ is bounded from above by 1, $\pi_s \in \mathcal{M}_+$ and the fact that $|\pi_s - \rho_s(\varepsilon)| \lesssim \sqrt{\varepsilon} \|\partial_s \rho_s\|_2$. To treat (4.6), we use (4.9) and we note that $\rho \in L^2(0,T;\mathcal{H}^1)$ which, as mentioned before, is proved in Section 5. Then, we get that

$$\left| \langle \rho_s^2, \Delta G_s \rangle - \int_0^{1-\varepsilon} \langle \pi_s, \tilde{t}_\varepsilon^u \rangle \langle \pi_s, \tilde{t}_\varepsilon^u \rangle \Delta G_s(u) ds \right|$$

$$\lesssim \varepsilon C(\varepsilon) + \int_0^{1-\varepsilon} \left| \Delta G_s(u) \right| \left\{ \left| \rho_s(u) - \langle \pi_s, \tilde{t}_\varepsilon^u \rangle \right| + \left| \rho_s(u) - \langle \pi_s, \tilde{t}_\varepsilon^u \rangle \right| \right\} du$$

$$\lesssim \varepsilon + \sqrt{\varepsilon \left( \|\partial_s \rho_s\|_2^2 + 1 \right)},$$
so that (4.6) vanishes as \( \varepsilon \to 0 \).

Therefore, it remains only to look at (4.4). Note that we still cannot use Portmanteau’s Theorem, since the functions \( \tilde{\tau}_n^\varepsilon \) and \( \tilde{\tau}_n^\varepsilon \) are not continuous. Nevertheless, we can approximate each one of these functions by continuous functions, in such a way that the error vanishes as \( \varepsilon \to 0 \). Then, since the set inside the probability in (4.4) is an open set with respect to the Skorokhod topology, we can use Portmanteau’s Theorem and bound (4.4) from above by

\[
\liminf_{n \to +\infty} Q_n \left( \sup_{0 \leq t \leq T} \left| (\pi_n, G_t) - (\pi_0, G_0) - \int_0^t \langle \pi_n, \tilde{\tau}_n^\varepsilon \rangle \tilde{\tau}_n^\varepsilon \Delta G_s(u) \, du \right| \right)
\]

Summing and subtracting \( \int_0^t n^2 L_n(\pi_n, G_t) \, ds \) to the term inside the supremum in (4.10), and recalling (3.4), we bound the probability in (4.10) from above by the sum of the next two terms

\[
\mathbb{P}_{\mu_0} \left( \sup_{0 \leq t \leq T} \left| M^\varepsilon_n(G) \right| > \frac{\delta}{14} \right),
\]

and

\[
\mathbb{P}_{\mu_0} \left( \sup_{0 \leq t \leq T} \left| \int_0^t n^2 L_n(\pi_n, G_s) \, ds - \int_0^t \langle \pi_n^\varepsilon, \tilde{\tau}_n^\varepsilon \rangle \tilde{\tau}_n^\varepsilon \Delta G_s(u) \, du \right| \right)
\]

From Doob’s inequality and (3.8), the term (4.11) vanishes as \( n \to +\infty \). Finally, for \( \delta > 0 \), we can bound (4.12) from above by the sum of the following terms

\[
\mathbb{P}_{\mu_0} \left( \sup_{0 \leq t \leq T} \left| \frac{1}{n} \sum_{x \in \Sigma_n} \Delta_\varepsilon G_s \left( \pi_s^\varepsilon \right) \sigma_x h(\eta_{sn^2}) - \int_0^t \langle \pi_n^\varepsilon, \tilde{\tau}_n^\varepsilon \rangle \tilde{\tau}_n^\varepsilon \Delta G_s(u) \, du \right| \right) > \frac{\delta}{14},
\]

\[
\mathbb{P}_{\mu_0} \left( \sup_{0 \leq t \leq T} \left| \int_0^t \left\{ \nabla^+ G_s(0) \right\} \sigma_x h(\eta_{sn^2}) - \langle \pi_n^\varepsilon, \tilde{\tau}_n^\varepsilon \rangle \tilde{\tau}_n^\varepsilon \Delta G_s(u) \, du \right| \right) > \frac{\delta}{14},
\]

\[
\mathbb{P}_{\mu_0} \left( \sup_{0 \leq t \leq T} \left| \int_0^t \left\{ G_s(0) \right\} \sigma_x h(\eta_{sn^2}) - \langle \pi_n^\varepsilon, \tilde{\tau}_n^\varepsilon \rangle \tilde{\tau}_n^\varepsilon \Delta G_s(u) \, du \right| \right) > \frac{\delta}{14},
\]

plus two terms which are similar to the last ones, but which involve the right boundary. Now, we show that (4.15) vanishes when \( n \to +\infty \) and then \( \varepsilon \to 0 \). By Taylor expansion on \( G \), the terms which involve \( \alpha \) vanish when \( n \to +\infty \). Recall (4.2). Observing that \( \langle \pi_n^\varepsilon, \tilde{\tau}_n^\varepsilon \rangle = \tilde{\eta}_{sn^2}(1) \), from Lemma (5.15), (4.15) goes to zero as \( n \to +\infty \) and \( \varepsilon \to 0 \). Now, we treat (4.14). Using Taylor expansion, \( \partial_x G_s(0) \) can be replaced by its discrete derivative \( \nabla^+_n G_s(0) \). Since

\[
\langle \pi_n^\varepsilon, \tilde{\tau}_n^\varepsilon \rangle = \tilde{\eta}_{sn^2}(1) \tilde{\eta}_{sn^2}(en + 1) + O \left( \frac{1}{n^2} \right),
\]

and \( \sigma_x h(\eta) = \eta(1)\eta(2) + \alpha(\eta(1) - \eta(2)) \), we can use Theorem (5.10) to replace the product \( \eta(1)\eta(2) \) by \( \tilde{\eta}_{sn^2}(1) \tilde{\eta}_{sn^2}(en + 1) \). Applying Remark (5.6) twice, the term with \( \eta(1) - \eta(2) \) vanishes, since we can replace \( \eta_{sn^2}(1) \) by \( \tilde{\eta}_{sn^2}(1) \), \( \eta_{sn^2}(2) \) by \( \tilde{\eta}_{sn^2}(2) \) and \( \tilde{\eta}_{sn^2}(1) - \tilde{\eta}_{sn^2}(2) \leq \frac{\delta}{14} \). Then, from these observations, (4.14) vanishes, as \( n \to +\infty \) and \( \varepsilon \to 0 \). Finally, we treat (4.13). Recall (4.1). Note that
the sum in \( \Sigma_n \) can be written as a sum over \( \Sigma_n^c \) by paying a price of order \( O(\varepsilon) \). Now, note that the error from changing the integral in the space variable by its Riemann sum is of order \( O(\frac{1}{n}) \), and therefore we can bound (4.13) from above by

\[
P_{\mu_n} \left( \sup_{0 \leq t \leq T} \left| \int_0^t \frac{1}{n} \sum_{x \in \Sigma_n} \left\{ \Delta_n G_s \left( \frac{x}{n} \right) \tau_s h(\eta_{sn^2}) - \left( \pi_{\varepsilon}, \tau_{\varepsilon}^{n/\varepsilon} \right) \Delta G_s \left( \frac{x}{n} \right) \right\} ds \right| > \delta \right). \tag{4.16}
\]

By Taylor expansion on the test function \( G \), we can replace its Laplacian by its discrete Laplacian, by paying a price of order \( O(\frac{1}{n}) \). Since for \( x \in \Sigma_n \), \( \tau_s h(\eta) = \eta(x-1) \eta(x) + \eta(x+1) \eta(x+1) - \eta(x) \eta(x+1) \eta(x+1) + O(\frac{1}{n}) \),

(4.16) can be bounded from above by the sum of the following terms

\[
P_{\mu_n} \left( \sup_{0 \leq t \leq T} \left| \int_0^t \frac{1}{n} \sum_{x \in \Sigma_n^c} \left\{ \Delta_n G_s \left( \frac{x}{n} \right) \eta_{sn^2}(x-1) - \tau_{\varepsilon}^{n/\varepsilon} \eta_{sn^2}(x+1) \right\} ds \right| > \delta \right), \tag{4.17}
\]

\[
P_{\mu_n} \left( \sup_{0 \leq t \leq T} \left| \int_0^t \frac{1}{n} \sum_{x \in \Sigma_n^c} \Delta_n G_s \left( \frac{x}{n} \right) \eta_{sn^2}(x-1) - \eta_{sn^2}(x+1) \right| ds \right| > \delta \right). \tag{4.18}
\]

From Theorem 5.2 and the application of Remark 5.6 twice, (4.17) and (4.18) vanish, respectively, as \( n \to +\infty \) and \( \varepsilon \to 0 \). This ends the proof in the case \( \theta = 1 \). We observe that the case \( \theta > 1 \) is contained in the previous proof.

Finally, we present the proof in the case \( \theta \in [0,1) \). Recall the definition of \( F_{\text{Dir}} \) from Definition 2.7.

Following the same ideas presented in the case \( \theta = 1 \), we can bound (4.3) from above by the sum of

\[
\begin{align*}
\mathcal{Q} \left( \sup_{0 \leq t \leq T} & \left| \langle \pi_{\varepsilon}, G_t \rangle - \langle \pi_0, G_0 \rangle + \int_0^t \int_0^{1-\varepsilon} \langle \pi_{\varepsilon}, \tau_{\varepsilon}^{n/\varepsilon} \rangle \Delta G_s(u) du ds \\
& - \int_0^t \langle \pi_s, \partial_s G_s \rangle ds + \int_0^t \left\{ \beta^2 \partial_s G_s(1) - \alpha^2 \partial_s G_s(0) \right\} ds \right| > \delta \right), \tag{4.19}
\end{align*}
\]

\[
\begin{align*}
\mathcal{Q} \left( |(\rho_0 - g, G_0)| > \frac{\delta}{\varepsilon} \right), \tag{4.20}
\end{align*}
\]

\[
\begin{align*}
\mathcal{Q} \left( \sup_{0 \leq t \leq T} \left| \int_0^t \left\{ \rho_s^2 \Delta G_s - \int_0^{1-\varepsilon} \langle \pi_s, \tau_{\varepsilon}^{n/\varepsilon} \rangle \Delta G_s(u) du \right\} ds \right| > \frac{\delta}{\varepsilon} \right). \tag{4.21}
\end{align*}
\]

Using the same arguments that we used above to treat (4.5) and (4.9), we can see that (4.20) and (4.21) vanish. Therefore, it remains only to bound (4.19). By the same arguments used in case \( \theta = 1 \), (4.19) is bounded from above by

\[
\begin{align*}
\liminf_{n \to +\infty} \mathcal{Q}_n \left( \sup_{0 \leq t \leq T} & \left| \langle \pi_{\varepsilon}, G_t \rangle - \langle \pi_0, G_0 \rangle + \int_0^t \int_0^{1-\varepsilon} \langle \pi_{\varepsilon}, \tau_{\varepsilon}^{n/\varepsilon} \rangle \Delta G_s(u) du ds \\
& - \int_0^t \langle \pi_s, \partial_s G_s \rangle ds + \int_0^t \left\{ \beta^2 \partial_s G_s(1) - \alpha^2 \partial_s G_s(0) \right\} ds \right| > \frac{\delta}{\varepsilon} \right). \tag{4.22}
\end{align*}
\]

Summing and subtracting \( \int_0^t \eta^2 L_\eta \langle \pi_{\varepsilon}, G_t \rangle ds \) to the term inside the supremum in (4.22) and recalling (3.4), we can bound the probability in (4.22) from above by

\[
\begin{align*}
P_{\mu_n} \left( \sup_{0 \leq t \leq T} & \left| \int_0^t \eta^2 L_\eta \langle \pi_{\varepsilon}, G_t \rangle ds + \int_0^t \int_0^{1-\varepsilon} \langle \pi_{\varepsilon}, \tau_{\varepsilon}^{n/\varepsilon} \rangle \Delta G_s(u) du ds \\
& + \int_0^t \left\{ \beta^2 \partial_s G_s(1) - \alpha^2 \partial_s G_s(0) \right\} ds \right| > \frac{\delta}{\varepsilon} \right). \tag{4.23}
\end{align*}
\]

plus \( P_{\mu_n} \left( \sup_{0 \leq t \leq T} |M^n_\eta(G)| > \frac{\delta}{\varepsilon} \right) \), which we showed above that vanishes when \( n \to +\infty \) without using the fact that \( \theta = 1 \).
From (4.23) and following again the steps of the case \( \theta = 1 \), we need to bound the next terms

\[
P_{\mu_n}\left( \sup_{0 \leq t \leq T} \left| \frac{1}{n} \sum_{x \in \Sigma_0} \Delta_x G_\theta \left( \frac{t}{n} \right) \tau_x h(\eta_{n,t}) - \int_0^{1-t} \langle \pi_x^n, \pi_x^n \rangle \Delta G_\theta(u) du \right| ds > \delta \right),
\]

(4.24)

\[
P_{\mu_n}\left( \sup_{0 \leq t \leq T} \left| \frac{1}{n} \sum_{x \in \Sigma_0} \Delta_x G_\theta \left( \frac{t}{n} \right) \tau_x h(\eta_{n,t}) - \int_0^{1-t} \langle \pi_x^n, \pi_x^n \rangle \Delta G_\theta(u) du \right| ds > \delta \right).
\]

(4.25)

\[
P_{\mu_n}\left( \sup_{0 \leq t \leq T} \left| \int_0^t \alpha \nabla G_\theta(0) \left( \eta_{n,t}^2(1) - \eta_{n,t}^2(2) \right) ds \right| > \delta \right).
\]

(4.26)

\[
P_{\mu_n}\left( \sup_{0 \leq t \leq T} \left| \int_0^t \left( \eta_{n,t}^2(1) - \eta_{n,t}^2(2) \right) \Delta G_\theta(0) ds \right| > \delta \right).
\]

(4.27)

\[
P_{\mu_n}\left( \sup_{0 \leq t \leq T} \left| \int_0^t \eta_{n,t}^2(1) \eta_{n,t}^2(2) \left( \nabla G_\theta(0) - \partial_u G_\theta(0) \right) ds \right| > \delta \right).
\]

(4.28)

plus three other terms similar to the last ones which come from the right boundary. Note that from the previous computations done for (4.17), we have that (4.24) vanishes, as \( n \to +\infty \). Not only (4.25) vanishes, (since from Theorem 5.9 we can replace \( \eta_{n,t}^2(1) \) by \( \alpha \) and \( \eta_{n,t}^2(n-1) \) by \( \beta \), but also (4.26) vanishes for the same reasons used in (4.14) to show that the difference \( \eta_{n,t}^2(2) - \eta_{n,t}^2(1) \) vanishes. For (4.27), we also replace \( \eta_{n,t}^2(2) \) by \( \eta_{n,t}^2(1) \), and we apply Theorem 5.9 twice to replace \( \eta_{n,t}^2(1) \) by \( \alpha \).

Finally, since \( G \in C^{1,2}_\nu([0,T] \times [0,1]) \), we have that \( \nabla G_\theta(0) \to \partial_u G_\theta(0) \) uniformly in \( s \), which implies that (4.28) vanishes as \( n \to +\infty \).

\[\square\]

5. Replacement lemmas

This section is divided in four subsections as follows. In Subsection 5.1 we state some estimates that will be used along the proofs of the replacement lemmas. We define Dirichlet forms, the \textit{carré du champ} operator, and the Bernoulli product measure. Thereafter, we compare the Dirichlet forms and the \textit{carré du champ} operator in order to state some of the estimates that will be used in the proofs of the replacement lemmas. In Subsections 5.2 and 5.3, we present the proofs of the several replacement lemmas at the bulk and at the boundary, respectively. Finally, in Subsection 5.4, we prove item 3. in Definition 2.

5.1. Dirichlet forms. Let \( \mu \) be a probability measure on \( \Omega_n \), and \( f : \Omega_n \to \mathbb{R} \) a density with respect to \( \mu \). The Dirichlet form of the process is defined as

\[
\{f, -L_n f\}_\mu = \{f, -L_p f\}_\mu + n^{a-2} \{f, -L_S f\}_\mu + \{f, -L_B f\}_\mu,
\]

where

\[
\langle h, g \rangle_\mu = \sum_{\eta \in \Omega_n} h(\eta) g(\eta) \mu(\eta),
\]

for all functions \( h, g : \Omega_n \to \mathbb{R} \). Moreover, recalling (2.2) and (2.3), we define the \textit{carré du champ} operator, denoted by \( D_n \), acting on functions \( f : \Omega_n \to \mathbb{R} \), with respect to \( \mu \) as

\[
D_n(f, \mu) := D_p(f, \mu) + n^{a-2} D_S(f, \mu) + D_B(f, \mu),
\]

where

\[
D_p(f, \mu) := \sum_{x=1}^{n^2} \int c_{x,x+1}(\eta) \left\{ a_{x,x+1}(\eta) + a_{x+1,x}(\eta) \right\} (\nabla_{x,x+1} f(\eta))^2 d\mu,
\]

\[
D_S(f, \mu) := \sum_{x=1}^{n^2} \int \left\{ a_{x,x+1}(\eta) + a_{x+1,x}(\eta) \right\} (\nabla_{x,x+1} f(\eta))^2 d\mu.
\]

The rates \( c_{x,x+1}(\eta) \) and \( a_{x,x+1}(\eta) \) are given in (2.4) and (2.2), respectively, and

\[
D_B(f, \mu) := \frac{m}{n} \left( F_t^2(f, \mu) + F_{n-1}^2(f, \mu) \right),
\]

where \( F_t^2(f, \mu) \) and \( F_{n-1}^2(f, \mu) \) are given in (2.5) and (2.3), respectively.
where $F_1^\alpha$ and $F_{n-1}^\beta$ are given by
\[ F_x^\ell(f, \mu) = \int I_x^\ell(\eta)(\nabla_x f(\eta))^2 \, d\mu, \] (5.1)
with $I_x^\ell$ given in (2.3) for $e \in \{\alpha, \beta\}$ and $x \in \{1, n-1\}$. For a measurable profile $\rho : [0, 1] \to [0, 1]$, we define the Bernoulli product measure $\nu_{\rho(x)}^n$ on $\Omega_n$ with marginals given by
\[ \nu_{\rho(x)}^n \{ \eta : \eta(x) = 1 \} = \rho(x). \]

Let $f$ be a density with respect to $\nu_{\rho(x)}^n$. The goal of this part of the section is to state the following estimate for the Dirichlet form $\langle L_n \sqrt{f}, \sqrt{f} \rangle_{\nu_{\rho(x)}^n}$ that is necessary in the proofs of the replacement lemmas.

**Lemma 5.1.** Let $\rho : [0, 1] \to [0, 1]$ be a Lipschitz profile such that for all $u \in (0, 1)$,\n\[ \alpha = \rho(0) \leq \rho(u) \leq \beta, \] (5.2)
and which is locally constant at the boundary. Then, the Dirichlet form satisfies\n\[ \langle L_n \sqrt{f}, \sqrt{f} \rangle_{\nu_{\rho(x)}^n} \leq -\frac{1}{4} D_n(\sqrt{f}, \nu_{\rho(x)}^n) + O\left(\frac{1}{n}\right). \] (5.3)

In case $\rho : [0, 1] \to [0, 1]$ is a constant profile, then\n\[ \langle L_n \sqrt{f}, \sqrt{f} \rangle_{\nu_{\rho(x)}^n} \leq -\frac{1}{4} D_n(\sqrt{f}, \nu_{\rho(x)}^n) + O\left(\frac{1}{n}\right). \]

**Proof.** We note that since it is not difficult to prove the result, the interested reader can find the proof of it in Section 5 of [5]. \hfill \Box

Now, we state all the replacement lemmas that were used in Section 4. We divide this part of the section into two subsections: one to prove the replacements lemmas concerning the bulk, and another to prove the replacements lemmas concerning the boundary.

### 5.2. Replacement lemmas at the bulk.

For the bulk, we basically have to prove that we can replace $\eta(x)$ by $\eta^{en}(x)$ and $\eta(x+1)$ by $\eta^{en}(x+1)$, as stated in Theorem 5.2. We remark that the sites $x \in \Sigma_n \setminus \Sigma_n^e$, where $\Sigma_n^e$ is defined in (4.1), are the ones where we do not have space to replace $\eta(x)$ by $\eta^{en}(x)$ (nor $\eta^{en}(x)$), and are those where we do not need to make the replacement.

**Theorem 5.2.** Let $G^n_s : [0, 1] \to \mathbb{R}$ be such that $\|G^n_s\|_\infty \leq M < \infty$, for all $n \in \mathbb{N}$ and $s \in [0, T]$. For any $t \in [0, T]$, we have that\n\[ \lim_{\epsilon \to 0} \lim_{n \to +\infty} \mathbb{E}_{\mu_n} \left( \left| \int_0^t \frac{1}{n} \sum_{x \in \Sigma_n} G^n_s \left( \frac{x}{n} \right) \{ \eta_{in^2}(x) \eta_{in^2}(x+1) - \eta^{en}_{in^2}(x) \eta^{en}_{in^2}(x+1) \} \, ds \right| \right) = 0. \]

Let us explain the idea behind the proof of this theorem. The proof is divided in three steps which are proved in the lemmas below. For $x \in \Sigma_n^e$ and $\delta > 0$

1) replace $\eta(x)\eta(x+1)$ by $\eta^{en}(x)\eta^{en}(x+1)$, for $\ell = n^{a-1-\delta}$; (Lemma 5.3)
2) replace $\eta^{en}(x)\eta^{en}(x+1)$ by $\eta^{en}(x)\eta^{en}(x+1)$; (Lemma 5.7)
3) replace $\eta^{en}(x)\eta^{en}(x+1)$ by $\eta^{en}(x)\eta^{en}(x+1)$, where $L = \ell m$ and $m = \ell^{a-\delta}$. (Lemma 5.8)

**Lemma 5.3.** Let $G^n_s : [0, 1] \to \mathbb{R}$ be such that $\|G^n_s\|_\infty \leq M < \infty$, for all $n \in \mathbb{N}$ and $s \in [0, T]$. For any $t \in [0, T]$, $\epsilon > 0$ and $\ell = n^{a-1-\delta}$ with $\delta > 0$ such that $a-1-\delta > 0$, we have that\n\[ \lim_{n \to +\infty} \mathbb{E}_{\mu_n} \left( \left| \int_0^t \frac{1}{n} \sum_{x \in \Sigma_n^e} G^n_s \left( \frac{x}{n} \right) \{ \eta_{in^2}(x) - \eta^{en}_{in^2}(x) \} \eta_{in^2}(x+1) \, ds \right| \right) = 0. \] (5.4)
Proof. Let $\nu_0^\rho(\cdot)$ be a Bernoulli product measure associated with the profile $\rho(\cdot)$ satisfying Lemma 5.1. Let $H\left(\mu_n|\nu_0^\rho(\cdot)\right)$ be the entropy of $\mu_n$ with respect to $\nu_0^\rho(\cdot)$ and $B > 0$. By entropy's and Jensen's inequalities, the expected value in (5.4) can be bounded from above by

$$\frac{1}{nB} \log \mathbb{E}_{\nu_0^\rho(\cdot)} \left( \exp \left\{ nb \int_0^t \frac{1}{n} \sum_{x \in \Sigma_n} G_n^\mu \left( \frac{x}{n} \right) \left( \eta_{s+t}^2(x) - \eta_{s+t}^2(x) \right) \eta_{s+t}(x+1) \, ds \right\} \right).$$

(5.5)

Since $\rho(\cdot)$ satisfies (5.2), it holds that

$$H\left(\mu_n|\nu_0^\rho(\cdot)\right) \leq \log \left( \frac{1}{(\alpha \land (1-\beta))^n} \right) \sum_{\eta \in \Omega_n} \mu_n(\eta) \leq nC(\alpha, \beta).$$

(5.6)

Thus, we only need to treat the term in (5.5). From Feynman-Kac's formula, (5.5) is bounded from above by

$$\int_0^t \sup_f \left( \left| \left| \int_{\Omega_n} \frac{1}{n} \sum_{x \in \Sigma_n} G_n^\mu \left( \frac{x}{n} \right) \left( \eta(x) - \eta(x+1) \right) \eta(x+1) f(\eta) \, d\nu_0^\rho(\cdot) \right| + \frac{n}{B} \left( L_n \sqrt{\tilde{f}_s} \sqrt{\tilde{f}_t} \right) \nu_0^\rho(\cdot) \right) \, ds,$$

where the supremum is carried over all densities $f$ with respect to $\nu_0^\rho(\cdot)$. To bound the first integral in the last display, we note that $\eta(x) - \eta(x+1) = \frac{1}{2} \sum_{y \in \Lambda_x} \eta(x) - \eta(y)$, and that $\eta(x) - \eta(y) = \sum_{z=x}^{y-1} \eta(z+1) - \eta(z)$. Therefore, by summing and subtracting the term $\frac{1}{2} f(\eta_{z+1})$ and using the hypothesis on $G$, we can bound that integral from above by

$$\frac{M}{2f_n} \sum_{x \in \Sigma_n} \sum_{y \in \Lambda_x} \sum_{z=x}^{y-1} \int_{\Omega_n} \left( \eta(x+1) - \eta(z) \right) \eta(x+1) \left( f(\eta) + f(\eta_{z+1}) \right) \, d\nu_0^\rho(\cdot)$$

$$+ \frac{M}{2f_n} \sum_{x \in \Sigma_n} \sum_{y \in \Lambda_x} \sum_{z=x}^{y-1} \int_{\Omega_n} \left( \eta(x+1) - \eta(z) \right) \eta(x+1) \left( f(\eta) - f(\eta_{z+1}) \right) \, d\nu_0^\rho(\cdot).$$

(5.7)

Let $\tilde{\eta}$ denote the configuration $\eta$ removing its value at the sites $z$ and $z+1$. Thus, we can write the first integral in (5.7) as

$$\left| \sum_{\tilde{\eta} \in \Omega_{n-2}} \left( \tilde{\eta}(x+1) \left( f(\tilde{\eta}, 0, 1) + f(\tilde{\eta}, 1, 0) \right) \left( 1 - \rho \left( \frac{x+1}{n} \right) \right) \right) \rho \left( \frac{x+1}{n} \right)$$

$$- \tilde{\eta}(x+1) \left( f(\tilde{\eta}, 0, 1) + f(\tilde{\eta}, 1, 0) \right) \rho \left( \frac{x}{n} \right) \left( 1 - \rho \left( \frac{x+1}{n} \right) \right) \right| \nu_{\rho(\cdot)}^{n-2}(\tilde{\eta}),$$

(5.8)

where the notation $f(\tilde{\eta}, 1, 0)$ means that we are computing $f(\eta)$ with $\eta(z) = 1$ and $\eta(z+1) = 0$. Using the fact that $\rho(\cdot)$ satisfies (5.2), (5.8) is bounded from above by a constant (depending on $\rho(\cdot)$) times

$$\frac{1}{n} \sum_{\tilde{\eta} \in \Omega_{n-2}} \left( f(\tilde{\eta}, 0, 1) + f(\tilde{\eta}, 1, 0) \right) \nu_{\rho(\cdot)}^{n-2}(\tilde{\eta}).$$

Since last term is bounded from above by

$$\frac{2}{n} \sum_{x \in \{0, 1\}} \sum_{y \in \Omega_0} \rho \left( \frac{x}{n} \right)^{\eta(y)} \left( \prod_{y=z+1} 1 - \rho \left( \frac{z+1}{n} \right) \right)^{-1} \nu_{\rho(\cdot)}^{n}(\eta),$$

and $f$ is a density with respect to $\nu_0^\rho(\cdot)$, (5.8) is of order $O(\frac{1}{n})$. Thus, the first expression in (5.7) is bounded from above by a constant, times $\frac{t}{B}$. It remains to treat the second integral term in (5.7). Note that for two nonnegative numbers $a$ and $b$, $a - b = \left[ \sqrt{a} - \sqrt{b} \right] \left( \sqrt{a} + \sqrt{b} \right)$. So, from Young's inequality
we have that for any $A > 0$ the second integral in \((5.7)\) is bounded from above by

\[
\frac{M}{4nA} \sum_{x \in \Sigma_n^t, y \in \Lambda^t_x} \sum_{z=0}^{x-1} \left| \int_{\Omega_n} \frac{1}{a_s(x+1)(\eta)} \left( \sqrt{f(\eta)} + \sqrt{f(\eta^{s,x+1})} \right)^2 d\nu^\rho \right| + \frac{AM}{4n\ell} \sum_{x \in \Sigma_n^t, y \in \Lambda^t_x} \sum_{z=0}^{x-1} \left| \int_{\Omega_n} a_s(x+1)(\eta) \left( \sqrt{f(\eta)} - \sqrt{f(\eta^{s,x+1})} \right)^2 d\nu^\rho \right|,
\]

where $a_s(x+1)(\eta)$ is defined in \((2.2)\). A simple computation, based on the fact that $f$ is a density, shows that the previous display is bounded from above by

\[
\frac{M\ell}{A} + \frac{MA}{4} D_5(\sqrt{f}, \nu^\rho(\cdot)). \tag{5.9}
\]

Now, recall from \((5.3)\) that

\[
\langle L_n \sqrt{f}, \sqrt{f} \rangle_{\nu} \geq -\frac{n^{a-2}}{4} D_5(\sqrt{f}, \nu^\rho(\cdot)) + O\left(\frac{1}{n} \right).
\]

Taking $A = \frac{n^{a-1}}{4M}$ in \((5.9)\), from last inequality and the previous computations, we have that the expectation in the statement of the lemma is bounded from above by a constant, times

\[
\frac{1}{B} + T\left(\frac{\ell}{n} + \frac{B\ell}{n-1}\right).
\]

Therefore, from our choice of $\ell$, taking $n \to +\infty$ and then $B \to +\infty$, the proof ends. $\square$

**Remark 5.4.** We stress that, in the proof above and the ones below, we present the replacement lemmas using $\nu^\rho(\cdot)$ and asking $\rho(\cdot)$ to satisfy the conditions stated in the first part of Lemma 5.7. Nevertheless, in the case $\theta \geq 1$, it is enough to consider the constant profile $\rho(\cdot)$, due to the bound obtained in the second part of Lemma 5.7.

**Remark 5.5.** We observe that the restriction imposed above Remark 2.1 that the parameters $\alpha, \beta \in (0,1)$ comes from the estimate in \((5.6)\). Since, as mentioned above, in the case $\theta \geq 1$ we can take any constant profile, that restriction on the parameters is only needed in Dirichlet case, that is when $\theta < 1$.

**Remark 5.6.** A simple modification of the proof of Lemma 5.6 also shows that, for all $\varepsilon > 0$

\[
\lim_{n \to +\infty} E_{\mu_n} \left( \left| \int_0^t \frac{1}{n} \sum_{x \in \Sigma_n^t} G_s^\eta \left( \frac{x}{n} \right) \eta_{x+1} - \eta_{x+1} \eta_{x,1}(x+1) \right| ds \right) = 0.
\]

**Lemma 5.7.** Let $G_s^\eta : [0,1] \to \mathbb{R}$ be such that $\|G_s^\eta\|_\infty \leq M < \infty$, for all $n \in \mathbb{N}$ and $s \in [0,T]$. For any $t \in [0,T]$ and $\ell = n^{a-1-\delta}$ with $\delta > 0$ such that $\alpha - 1 - \delta > 0$, we have that

\[
\lim_{\varepsilon \to 0} \lim_{n \to +\infty} E_{\mu_n} \left( \left| \int_0^t \frac{1}{n} \sum_{x \in \Sigma_n^t} G_s^\eta \left( \frac{x}{n} \right) \eta_{x,1}(x) \eta_{x+1} - \eta_{x+1} \eta_{x,1}(x+1) \right| ds \right) = 0. \tag{5.10}
\]

**Proof.** As in the previous lemma, let $\nu^\rho(\cdot)$ be a Bernoulli product measure associated with the profile $\rho(\cdot)$ satisfying Lemma 5.1. Then, for $B > 0$, the expectation in \((5.10)\) is bounded from above by $\frac{C(\alpha, \beta)}{B}$, plus

\[
\int_0^t \sup_f \left( \left| \int \frac{1}{n} \sum_{x \in \Sigma_n^t} G_s^\eta \left( \frac{x}{n} \right) \eta_{x,1}(x) \eta_{x+1} - \eta_{x+1} \eta_{x,1}(x+1) \right| f(\eta) d\nu^\rho \right) + \frac{n}{B} \left( L_n \sqrt{f}, \sqrt{f} \right)_{\nu} \right| ds.
\]

Recall the definition of $\Lambda^t_x$ in \((2.2)\). Denote by $X_1 = \{ \eta \in \Omega_n : \eta_{1}(x) \geq \frac{\alpha}{\lambda} \}$ the set of configurations that have at least two particles in $\Lambda^t_x$. Thus, we can write the first integral inside the supremum above as the sum of the integral over the set $X_1$, plus the integral over its complementary $X_1^c$. By the hypothesis on $G$, the fact that $|\eta(x)| \leq 1$ for $x \in \Sigma_n$, and since $f$ is a density, the integral over $X_1^c$ is bounded from above by a constant, times $\frac{1}{B}$. Taking $n \to +\infty$, and by the hypothesis in $\ell$, the integral over $X_1^c$
vanishes. Moreover, to treat the remaining integral term, we just need to follow the same computations done in Lemma [5.3]. Hence, it is enough to estimate the next two terms

\begin{equation}
\frac{M}{2n^2\epsilon} \sum_{x \in \mathbb{Z}^d} \sum_{y \in \Lambda_{x+1}^n} \left| \int_{X_1} \tilde{\eta}^\ell(x) \left( \eta(x+1) - \eta(y) \right) \left( f(\eta) + f(\eta^{x+1,y}) \right) d\nu^n_{\rho(\cdot)} \right|
\end{equation}

\begin{equation}
+ \frac{M}{2n^2\epsilon} \sum_{x \in \mathbb{Z}^d} \sum_{y \in \Lambda_{x+1}^n} \left| \int_{X_1} \tilde{\eta}^\ell(x) \left( \eta(x+1) - \eta(y) \right) \left( f(\eta) - f(\eta^{x+1,y}) \right) d\nu^n_{\rho(\cdot)} \right|.
\end{equation}

We begin by estimating the first term in the previous display. We use the notation $\tilde{\eta}$ for the configuration $\eta$ removing its value at the sites $x + 1$ and $y$. Since $x + 1$ and $y$ do not intersect $\Lambda_x^\ell$, that term can be written as

\begin{equation}
\left| \sum_{\tilde{\eta} \in \Omega_{n-2}} \int_{X_1} \tilde{\eta}^\ell(x) \left( f(\tilde{\eta}, 0, 1) + f(\tilde{\eta}, 1, 0) \right) \left( 1 - \rho(\frac{\tilde{\eta}}{n}) \right) \frac{\rho(\frac{\tilde{\eta}+1}{n}) - \rho(\frac{\tilde{\eta}}{n}) (1 - \rho(\frac{\tilde{\eta}+1}{n}))}{\rho(\tilde{\eta})} d\nu^n_{\rho(\cdot)}(\tilde{\eta}) \right|.
\end{equation}

Using the fact that $\rho(\cdot)$ satisfies (5.2), last expression is of order $O(\frac{1}{n})$. To bound the second term in (5.11) we need to be more careful. Recall that the idea behind this lemma is to replace a particle at $x + 1$ by the empirical density in the box $\Lambda_{x+1}^n$. To accomplish this we have to construct a path (with allowed jumps from the SSEP and the PMM dynamics), in such a way that we can send a particle from the site $x + 1$ to the site $y$, for any $y \in \Lambda_{x+1}^n$. This is explained in the next paragraph.

Recall that we are integrating over $X_1$, so that we have at least two particles in $\Lambda_x^\ell$. Suppose, without loss of generality, that we have a particle at site $x_1 \in \Lambda_x^\ell$, and another one at site $x_2 \in \Lambda_x^\ell$, with $x_1 < x_2$. Using the SSEP jumps, we can take the particle from the site $x_1$ close to the particle at the site $x_2$, in such a way that the distance between them is less than or equal to 2. Denoting by $\bullet$ an occupied site and by $\circ$ an empty site, this approximation is done by jumps of the SSEP and at the end we get one of the following structures ($\bullet \circ \bullet$ or $\circ \bullet \bullet$). When we reach a structure of the previous form, we say that a mobile cluster has been created. Now, since we have a mobile cluster, there exists a sequence of nearest-neighbor jumps (with the PMM dynamics) which allow us to move the mobile cluster to any position on the box $\Lambda_{x+1}^n$. Note that the SSEP jumps are used to approximate particles inside a box of size $\ell$, with the choice of $\ell$ as in the statement of this lemma. However, the PMM jumps can be used in the presence of the mobile cluster, to take a particle from a site $x + 1$ to a site $y$ at a distance at most $\epsilon n$. After the creation of the mobile cluster with SSEP jumps, we move it to a vicinity of the site $x + 1$ until the distance between them is less than or equal to 2. Then, using the PMM jumps we take a particle to the site $y$ and we bring back the mobile cluster to the same position where it was created. When we reach this step, we use the SSEP jumps again to put the particles back to their initial positions, $x_1$ and $x_2$, respectively. To illustrate all the steps mentioned above the reader can see Figure 2.

Note that, in this path, we use at most $4\ell$ jumps from the SSEP and $6(\ell + \epsilon n)$ jumps from the PMM. From this, it follows that for any configuration $\eta \in X_1$, if $x_1$ and $x_2$ denote the position of the two closest particles to $x + 1$, then there exist $N(x_1) \leq \ell + \epsilon n$ and a sequence of allowed moves $\{x(i)\}_{i=0}^{N(x_1)}$, which takes values in the set of points $\{x_1, \ldots, y\}$, such that $\eta^{(0)} = \eta, \eta^{(i+1)} = (\eta^{(i)})^{x(i),x(i)+1}$ and the final configuration is $\eta^{(N(x_1))} = \eta^{x+1,y}$. Note that the rates for each exchange is strictly positive. With this in mind, we can rewrite the exchange $f(\eta) - f(\eta^{x+1,y})$ as

\begin{equation}
f(\eta) - f(\eta^{x+1,y}) = \sum_{i=1}^{N(x_1)} f(\eta^{(i-1)}) - f(\eta^{(i)}) = \sum_{iexc} f(\eta^{(i-1)}) - f(\eta^{(i)}) + \sum_{ipmm} f(\eta^{(i-1)}) - f(\eta^{(i)}),
\end{equation}

where $iexc$ (resp. $ipmm$) are related to the bonds used with SSEP jumps (resp. PMM jumps) along the path. Take into account the fact that the SSEP jumps are used only to create and to destroy the mobile cluster, while all the rest of the path is done with PMM jumps. Now, substituting (5.12) in the second
Taking the term of (5.11) and using the triangular inequality, we need to estimate the following expressions

\[ \frac{M}{2n^\frac{\alpha}{2}} \sum_{x, y \in \Sigma_n^{\ell}} \sum_{\{i \in \Sigma_n^{\ell} \}} \left| \int_{X_1} \tilde{\eta}^i(x)(\eta(x+1)-\eta(y))(f(\eta((i-1)))-f(\eta(i)))d\nu^\eta_{\rho(x)} \right| \]

\[ + \frac{M}{2n^\frac{\alpha}{2}} \sum_{x, y \in \Sigma_n^{\ell}} \sum_{\{i \in \Sigma_n^{\ell} \}} \left| \int_{X_1} \tilde{\eta}^i(x)(\eta(x+1)-\eta(y))(f(\eta((i-1)))-f(\eta(i)))d\nu^\eta_{\rho(x)} \right| \]  

(5.13)

The way to estimate these terms is the same as the one used to bound the second integral term in (5.7). The difference is that in this case we have to take into account that there is an error which comes from each change of variables \( \eta^{i-1} \) into \( \eta \), since the measure \( \nu^\eta_{\rho(x)} \) is not invariant for this exchange. Since the function \( \rho(\cdot) \) is assumed to be Lipschitz, this error is of order \( O(\frac{1}{n}) \) times the size of the path involved. Therefore, after some computations we have that, for \( A, \tilde{A} > 0 \), the expression (5.13) is bounded from above by a constant, times

\[ \frac{M\ell}{A} + \frac{MA}{4} D_S(\sqrt{f}, \nu^\eta_{\rho(x)}) + \frac{M\tilde{A}}{4} D_P(\sqrt{f}, \nu^\eta_{\rho(x)}). \]

Taking \( A = \frac{M}{B} \) and \( \tilde{A} = \frac{M}{Bn^\frac{\alpha}{2}} \), from the previous computations, the expectation in the statement of the lemma is bounded from above by a constant, times

\[ \frac{1}{B} + T \left( \frac{1}{n} + \frac{\ell B}{n^{\alpha-1}} + \epsilon B \right). \]  

(5.14)

Taking \( n \to +\infty \), the second and the third term of (5.14) vanish by the choice of \( \ell \). Taking \( \epsilon \to 0 \), the fourth term of (5.14) vanishes. To finish, we send \( B \to +\infty \) and the remaining term vanishes. \( \square \)

\[ \text{Figure 2. Path used to send a particle from site } x + 1 \text{ to } y \text{ inside the box of size } \epsilon n. \]

**Lemma 5.8.** Let \( G^\eta : [0, 1] \to \mathbb{R} \) be such that \( \|G^\eta\|_\infty \leq M < \infty \), for all \( n \in \mathbb{N} \) and \( s \in [0, T] \). For any \( t \in [0, T] \) and \( \ell = n^{\alpha-1-\delta} \) with \( \delta > 0 \) such that \( \alpha - 1 - \delta > 0 \), we have that

\[ \lim_{\epsilon \to 0} \lim_{n \to +\infty} \mathbb{E}_{\mu^n} \left( \left| \int_0^t \frac{1}{n} \sum_{x \in \Sigma_n^{\ell}} G^\eta_i \left( \frac{x}{n} \right) \left( \frac{1}{\eta^{\ell} (x)} - \frac{1}{\eta^{\ell} (x+1)} \right) d\eta^{\ell} (x+1) ds \right| \right) = 0. \]  

(5.15)
Proof. The proof follows exactly the argument of the proof of Lemma 5.7. Again, letting \( \nu^\eta_{\rho(\cdot)} \) be a Bernoulli product measure associated with the profile \( \rho(\cdot) \) satisfying Lemma 5.1, the expectation in (5.15) can be bounded from above by \( \frac{C(\rho,\rho_0)}{B} \) plus

\[
\int_0^t \sup_f \left( \left\{ \int_{\Omega_n} \frac{1}{n} \sum_{x \in \Sigma^n} G^m_i \left( \frac{x}{\xi(\eta)} \right) \left\{ \eta^-(x) - \eta^+(x) \right\} \eta^\epsilon_n(x+1)f(\eta)d\nu^\eta_{\rho(\cdot)} \right\} + \frac{\eta}{B} \left( L_n \sqrt{f}, \sqrt{f} \right) \nu^\eta_{\rho(\cdot)} \right) ds,
\]

for any \( B > 0 \). Take \( L = \ell m \) with \( m = \frac{M}{2} \) and note that

\[
\eta^-(x) - \eta^+(x) = \frac{1}{m} \sum_{j=1}^{m-1} \left( \eta^-(x) - \eta^-(x-j\ell) \right).
\]

From last identity, to bound the first integral inside the supremum in (5.16), it is enough to estimate the term

\[
\frac{M}{mn} \sum_{x \in \Sigma^n} \sum_{j=1}^{m-1} \left| \int_{\Omega_n} \left\{ \eta^-(x) - \eta^+(x-j\ell) \right\} \eta^\epsilon_n(x+1)f(\eta)d\nu^\eta_{\rho(\cdot)} \right|.
\]

Let \( X_2 = \{ \eta \in \Omega_n : \eta^-(x) \geq \frac{T}{2} \} \cup \{ \eta \in \Omega_n : \eta^+(x-j\ell) \geq \frac{T}{2} \} \). The integral in the previous display can be written as the integral over \( X_2 \) plus the integral over its complementary \( \widetilde{X}_2 \). We observe that the integral over \( \widetilde{X}_2 \) is of order \( O \left( \frac{1}{L} \right) \), and that we can write the integral over \( X_2 \) as

\[
\frac{M}{mn} \sum_{x \in \Sigma^n} \sum_{j=1}^{m-1} \left| \int_{X_2} \frac{1}{\ell} \sum_{z \in \Lambda_x} \left( \eta(z) - \eta(z-j\ell) \right) \eta^\epsilon_n(x+1)f(\eta)d\nu^\eta_{\rho(\cdot)} \right|.
\]

Basically the idea above is to send a particle \( z \in \Lambda^\ell_x \) to a site inside a box of size \( j\ell \), given that we have at least two particles in \( \Lambda^\ell_x \) or \( \Lambda^\ell_{x-j\ell} \), see Figure 3. We stress that the path argument used here is the same used above to prove Lemma 5.7.

![Figure 3. Sending a particle from site \( z \) to \( z-j\ell \).](image)

Summing and subtracting \( \frac{1}{\ell}f(\eta^zz^{j\ell}) \) in (5.17), we rewrite (5.17) as:

\[
\frac{M}{2mn\ell} \sum_{x \in \Sigma^n} \sum_{j=1}^{m-1} \sum_{z \in \Lambda_x} \left| \int_{X_2} \left( \eta(z) - \eta(z-j\ell) \right) \eta^\epsilon_n(x+1) \left( f(\eta) + f(\eta^zz^{j\ell}) \right) d\nu^\eta_{\rho(\cdot)} \right|
\]

\[
+ \frac{M}{2mn\ell} \sum_{x \in \Sigma^n} \sum_{j=1}^{m-1} \sum_{z \in \Lambda_x} \left| \int_{X_2} \left( \eta(z) - \eta(z-j\ell) \right) \eta^\epsilon_n(x+1) \left( f(\eta) - f(\eta^zz^{j\ell}) \right) d\nu^\eta_{\rho(\cdot)} \right|.
\]

Note that, as in Lemma 5.7 using the fact that \( \rho(\cdot) \) is Lipschitz, the first term of (5.18) is bounded from above by a constant times \( \frac{M}{\ell} \). Since \( m = \frac{M}{2} \), that term is of order \( O(\epsilon) \). It remains to estimate the second term in (5.18). The idea is to exchange a particle from the site \( z \) to the site \( z-j\ell \). This can be done since we are restricted to the set \( X_2 \), so that we know that there are at least two particles either in the box \( \Lambda^\ell_x \) or in the box \( \Lambda^\ell_{x-j\ell} \). With this in mind, we can again construct a path using the SSEP jumps to create a mobile cluster in the box where there are for sure two particles. Now, we use the PMM jumps to move the mobile cluster close to the particle at site \( z \), and to send it to the site \( z-j\ell \). Then, we put the mobile cluster back to its starting point using the PMM jumps, and we then
put the two particles back to their initial position using the SSEP jumps. As in the previous lemma, for $A, \hat{A} > 0$, we can bound the second term in (5.18) from above by a constant, times

$$\frac{\ell}{A} + AD_S(\sqrt{f}, \nu^\mu_{\rho^\cdot}) + \frac{\ell m}{A} + \hat{A}D_P(\sqrt{f}, \nu^\mu_{\rho^\cdot}).$$

By choosing $A = \frac{n^{n-1}}{\mu}$ and $\hat{A} = \frac{\mu}{n}$, we can bound (5.16) from above by a constant, times

$$\frac{1}{B} + T\left(\epsilon + \frac{\ell B}{n^{n-1}} + \frac{\ell mB}{n}\right). \quad (5.19)$$

From the choice of $\ell$ and $m$, (5.19) can be bounded from above by $\frac{1}{B} + TB(n^{-\delta} + \epsilon)$, which vanishes when we take $n \to +\infty$, then $\epsilon \to 0$ and finally $B \to +\infty$. □

5.3. Replacement lemmas at the boundary. In this subsection we prove the replacement lemmas that are necessary for the boundary terms. Throughout this subsection let $\rho(\cdot)$ be a profile satisfying Lemma 5.2.

Lemma 5.9. Fix $\theta < 1$. Let $\varphi : \Omega_n \to \Omega_n$ be a positive and bounded function which does not depend on the value of the configuration $\eta$ at the site $1$. For any $t \in [0, T]$, we have that

$$\lim_{n \to +\infty} \mathbb{E}_{\mu_n}\left(\left|\int_0^t \varphi(\eta_{\mu_n})(\alpha - \eta_{\mu_n}(1))d\eta_{\mu_n}\right|\right) = 0.$$

The same is true replacing $\alpha$ by $\beta$, $1$ by $n^{-1}$ and requiring $\varphi$ not to depend on $\eta$ at the site $n-1$.

Proof. As in the previous replacement lemmas, we have that the expectation in the statement of the theorem is bounded from above by $C(a, \beta)B$, plus

$$T sup_f \left(\int_{\Omega_n} \varphi(\eta)(\alpha - \eta(1))f(\eta)\nu^n_{\rho^\cdot}d\nu^n_{\rho^\cdot}\right) + \frac{n}{B} \int_{\Omega_n} \nu^n_{\rho^\cdot}d\nu^n_{\rho^\cdot}. \quad (5.20)$$

where $B > 0$ and the supremum is carried over all the densities $f$ with respect to $\nu^n_{\rho^\cdot}$. Summing and subtracting $\frac{1}{2}f(\eta^1)$ in the first integral term inside the supremum in (5.20), we can bound this integral term from above by

$$\frac{1}{2} \int_{\Omega_n} \varphi(\eta)(\alpha - \eta(1))f(\eta)\nu^n_{\rho^\cdot}d\nu^n_{\rho^\cdot} + \frac{1}{2} \int_{\Omega_n} \varphi(\eta)(\alpha - \eta(1))f(\eta)\nu^n_{\rho^\cdot}d\nu^n_{\rho^\cdot}. \quad (5.21)$$

Since $\varphi$ is bounded, from Young’s inequality and from similar computations made in Theorem 5.2, the first term in (5.21) is bounded from above by a constant, times

$$\frac{A}{4} + \frac{1}{4A}I^n_1(S, \nu^n_{\rho^\cdot}), \quad (5.22)$$

where $A > 0$ and $I^n_1(S, \nu^n_{\rho^\cdot})$ is defined in (5.1). To bound the remaining term in (5.21) we follow exactly the same idea used to bound the second expression in (5.7). Then, after some computations we have that this term is bounded from above by a constant times $|\alpha - \rho(\frac{1}{n})|$. Now, from (5.2), (5.22), and with the choice $A = Bn^{\theta-1}m^{-1}$, we have that (5.20) is bounded from above by a constant, times

$$\frac{Bn^{\theta-1}}{4m} + \left|\rho(\frac{1}{n}) - \alpha\right|.$$

Taking $n \to +\infty$ and using the fact that $\rho(\cdot)$ is Lipschitz and $\rho(0) = \alpha$, we have that these terms vanish since $\theta < 1$. □

Theorem 5.10. For any $t \in [0, T]$, we have

$$\lim_{t \to \infty} \lim_{n \to +\infty} \mathbb{E}_{\mu_n}\left(\left|\int_0^t \{\eta_{\mu_n}(1)\eta_{\mu_n}(2) - \eta_{\mu_n}(1)\eta_{\mu_n}(\epsilon n + 1)\}d\eta_{\mu_n}\right|\right) = 0. \quad (5.23)$$
and
\[
\lim_{\varepsilon \to 0} \lim_{n \to +\infty} \mathbb{E}_{\mu_n} \left( \int_0^\ell \{ \eta_{n^2}(n-1)\eta_{n^2}(n-2) - \frac{\eta_n}{n^2}(n-1)\frac{\eta_n}{n^2}(n-1-\varepsilon n) \} \, ds \right) = 0. \tag{5.24}
\]

For simplicity of the presentation, we only prove (5.23), that is, the left boundary part. We note that the result concerning the right boundary in (5.24) can be proved with an analogous argument, therefore we leave the details to the reader. We divide the proof of (5.23) in the following steps:

1) replace \( \eta(1)\eta(2) \) by \( \eta(1)(\ell + 1) \); (Lemma 5.11)
2) replace \( \eta(1)\eta(\ell + 1) \) by \( \eta(1)\eta(\ell + 1) \); (Lemma 5.12)
3) replace \( \eta(1)\eta(\ell + 1) \) by \( \eta(1)\eta(\ell + 1) \); (Lemma 5.13)
4) replace \( \eta(1)\eta(\ell + 1) \) by \( \eta(1)\eta(\ell + 1) \). (Lemma 5.14)

Lemma 5.11. For any \( t \in [0,T] \), \( \ell = n^{a-1-\delta} \) with \( \delta > 0 \) such that \( a - 1 - \delta > 0 \), we have
\[
\lim_{n \to +\infty} \mathbb{E}_{\mu_n} \left( \int_0^\ell \eta_{n^2}(1)\eta_{n^2}(2-\eta_{n^2}(\ell + 1)) \, ds \right) = 0.
\]

Proof. Following the same steps of the proof of Lemma 5.3 the expectation in the statement of the lemma is bounded from above by \( \frac{C(a,\beta)}{B} \), plus
\[
T \sup_f \left( \int_{\Omega_n} \{ \eta(1)\eta(\ell + 1) \} f(\eta) \, d\nu_A^\beta \right) + \frac{n}{B} \left( L_n \sqrt{f}, \sqrt{f} \right) \nu_A^\beta.
\]

where \( B > 0 \) and the supremum is carried over all the densities \( f \) with respect to \( \nu_A^\beta \). Write \( \eta(2) - \eta(\ell + 1) = \sum_{y=2}^{\ell} \eta(y) - \eta(y+1) \). Using the same strategy that we used to bound the term in (5.7), for \( A > 0 \), the first term inside the supremum in (5.25) is bounded from above by a constant, times
\[
\frac{\ell}{n} + \frac{\ell B}{2n^{a-1}}.
\]

With the choice \( A = \frac{n^{a-1}}{B} \), from (5.3), (5.25), and (5.26), we have that the expectation in the statement of the lemma is bounded from above by a constant times
\[
\frac{1}{B} + T \left( \frac{\ell}{n} + \frac{\ell B}{n^{a-1}} \right).
\]

Taking \( n \to +\infty \), and from the choice of \( \ell \), we have that the right-hand side of last expression vanishes. By sending \( B \to +\infty \) we finish the proof.

Lemma 5.12. For any \( t \in [0,T] \), \( \ell = n^{a-1-\delta} \) with \( \delta > 0 \) such that \( a - 1 - \delta > 0 \), we have
\[
\lim_{n \to +\infty} \mathbb{E}_{\mu_n} \left( \int_0^\ell \eta_{n^2}(1) - \eta_{n^2}(\ell + 1) \, ds \right) = 0.
\]

Proof. Following the same steps of previous lemmas, we have that the expectation in (5.27) is bounded from above by \( \frac{C(a,\beta)}{B} \), plus
\[
T \sup_f \left( \int_{\Omega_n} \{ \eta(1) - \eta(\ell + 1) \} f(\eta) \, d\nu_A^\beta \right) + \frac{n}{B} \left( L_n \sqrt{f}, \sqrt{f} \right) \nu_A^\beta,
\]

where \( B > 0 \) and the supremum is carried over all densities \( f \) with respect to \( \nu_A^\beta \). Now, following exactly the same computations done in the proof of Lemma 5.3 the expectation in the statement of the lemma is bounded from above by a constant times
\[
\frac{1}{B} + T \left( \frac{\ell}{n} + \frac{\ell B}{n^{a-1}} \right).
\]

Taking \( n \to +\infty \) and then \( B \to +\infty \), the expression above vanishes due to our choice of \( \ell \).
Lemma 5.13. For any \( t \in [0, T] \), \( \delta > 0 \) and \( \ell = n^{a-1-\delta} \) such that \( a-1-\delta > 0 \), we have
\[
\lim_{\epsilon \to 0} \lim_{n \to +\infty} \mathbb{E}_\mu_n \left( \left| \int_0^t \overline{\eta}_n (\ell \nu + 1) - \overline{\eta}^e_n (\epsilon \nu + 1) \, ds \right| \right) = 0.
\] (5.28)

Proof. Following the same steps of the proof of Lemma 5.8 we have that the expectation in (5.28) is bounded from above by \( \frac{C(a, \delta)}{B} \) plus
\[
T \sup_f \left( \left| \int_{\Omega_n} \overline{\eta}^e (1) \left\{ \overline{\eta}(\ell + 1) - \overline{\eta}^e (\epsilon \nu + 1) \right\} f(\eta) \, d\nu_{\rho(\cdot)} \right| + \frac{n}{B} \left\{ L_n, \sqrt{f}, \sqrt{f} \right\}_{\rho(\cdot)},
\]
where \( B > 0 \) and the supremum is carried over all densities \( f \) with respect to \( \nu_{\rho(\cdot)} \). Let \( X_3 = \{ \eta \in \Omega_n : \overline{\eta}^e(1) \geq \frac{2}{n} \} \). Write the first integral inside the supremum as the integral over the set \( X_3 \). Note that
\[
\eta(\ell + 1) - \overline{\eta}^e (\epsilon \nu + 1) = \frac{1}{\epsilon n} \sum_{y=\epsilon \nu + 1}^{2 \epsilon n} \eta(\ell + 1) - \eta(y).
\]

Now, following the same computations done in the proof of Lemma 5.7, we have that the expectation in the statement of the lemma is bounded from above by a constant times
\[
\frac{1}{B} + T \left( \frac{\ell}{n} + \frac{1}{n} + \epsilon + \frac{\ell B}{n^{a-1}} + B \epsilon \right).
\]
Taking \( n \to +\infty \), then \( \epsilon \to 0 \), and finally \( B \to +\infty \), the result follows due to our choice of \( \ell \). \( \square \)

Lemma 5.14. For any \( t \in [0, T] \) and \( \ell = n^{a-1-\delta} \) with \( \delta > 0 \) such that \( a-1-\delta > 0 \), we have
\[
\lim_{\epsilon \to 0} \lim_{n \to +\infty} \mathbb{E}_\mu_n \left( \left| \int_0^t \overline{\eta}_n (\ell \nu + 1) - \overline{\eta}_n^e (\epsilon \nu + 1) \, ds \right| \right) = 0.
\] (5.29)

Proof. Following the same steps of Lemma 5.8 we have that the expectation in (5.29) is bounded from above by \( \frac{C(a, \delta)}{B} \) plus
\[
T \sup_f \left( \left| \int_{\Omega_n} \overline{\eta}^e (1) - \overline{\eta}^e (1) \right\} f(\eta) \, d\nu_{\rho(\cdot)} \right| + \frac{n}{B} \left\{ L_n, \sqrt{f}, \sqrt{f} \right\}_{\rho(\cdot)},
\]
where \( B > 0 \) and the supremum is carried over all densities \( f \) with respect to \( \nu_{\rho(\cdot)} \). Take \( L = \ell m \) with \( m = \frac{T}{\ell} \). As in Lemma 5.8 let \( X_4 = \{ \eta \in \Omega_n : \overline{\eta}^e(1) \geq \frac{2}{n} \} \cup \{ \eta \in \Omega_n : \overline{\eta}^e(1 + \ell) \geq \frac{2}{n} \} \). Now, following exactly the same computations done in the proof of that lemma, we have that the expectation in (5.29) is bounded from above by a constant times
\[
\frac{1}{B} + T \left( \epsilon + \frac{\ell B}{n^{a-1}} + B \epsilon \right).
\]
Taking \( n \to +\infty \), then \( \epsilon \to 0 \), and \( B \to +\infty \), the result follows due to our choice of \( \ell \) and \( m \). \( \square \)

Lemma 5.15. For any \( t \in [0, T] \) we have
\[
\lim_{\epsilon \to 0} \lim_{n \to +\infty} \mathbb{E}_\mu_n \left( \left| \int_0^t \overline{\eta}_n (1) - \overline{\eta}_n^e (1) \, ds \right| \right) = 0
\]
and
\[
\lim_{\epsilon \to 0} \lim_{n \to +\infty} \mathbb{E}_\mu_n \left( \left| \int_0^t \overline{\eta}_n (n-1) - \overline{\eta}_n^e (n-1) \, ds \right| \right) = 0.
\]

Proof. This proof is similar to the proof presented in Lemma 5.3 and it has two steps. The first one is to replace \( \overline{\eta}(1) \) by \( \overline{\eta}_n^e(1) \) and the second one is to replace \( \overline{\eta}_n^e(1) \) by \( \overline{\eta}_n^e(1) \). We leave the details to the reader. \( \square \)
5.4. Fixing the profile at the boundary for the case $\theta < 1$. In this subsection we intend to prove item 3 in Definition $\mathbb{2}$ that is, $\rho_t(0) = \alpha$ and $\rho_t(1) = \beta$ for all $t \in (0, T]$. We note that it is a simple observation to show that these facts are, in fact, a consequence of combining both Lemma $5.9$ with $\varphi \equiv 1$ and Lemma $5.15$. We refer the interested reader to Appendix A.4 of $\mathbb{10}$.

6. Energy estimates

The idea of this section is to prove that any limit point $\mathbb{Q}$ of the sequence $\{\mathbb{Q}_n\}_{n \in \mathbb{N}}$ is concentrated on trajectories $\rho_t(u)du$, in which $\rho_t(u)$ belongs to $L^2(0, T; \mathcal{H}^1)$, see Definition $\mathbb{1}$. For $\pi \in \mathfrak{D}([0, T], \mathcal{M}_+)$, we define the linear functional $\langle \langle \pi, \cdot \rangle \rangle$ on $C^{0,1}_0([0, T] \times (0, 1))$ by

$$
\langle \langle \pi, G \rangle \rangle := \int_0^T \int_0^1 G_s(u) \pi_s(du) \, ds = \int_0^T \langle \pi_s, G_s \rangle \, ds.
$$

If $\pi_t$ has a density $\rho_t(\cdot)$ for all $t \in [0, T]$, we also use the same notation. Note that Proposition $6.1$ shows that $\langle \langle \pi, \cdot \rangle \rangle$ is $\mathbb{Q}$ almost surely continuous, then the linear functional can be extended to $L^2([0, T] \times (0, 1))$. Furthermore, by the Riesz’s Representation Theorem we can find $\xi \in L^2([0, T] \times (0, 1))$ such that

$$
\langle \langle \pi, G \rangle \rangle = -\int_0^T \int_0^1 G_s(u) \xi_s(u) \, duds,
$$

for all $G \in C^{0,1}_0([0, T] \times (0, 1))$, which implies that $\rho \in L^2(0, T; \mathcal{H}^1)$.

**Proposition 6.1.** There exist positive constants $K_0$ and $c$ such that

$$
\mathbb{E}_\mathbb{Q} \left( \sup_{G \in C^{0,2}_c([0, T] \times (0, 1))} \langle \langle \rho, \partial_u G \rangle \rangle - c \|G\|_2^2 \right) \leq K_0 < \infty,
$$

where $\mathbb{Q}$ is a limit point of $\mathbb{Q}_n$, and $\|G\|_2$ is the norm of a function $G \in L^2([0, T] \times (0, 1))$.

**Proof.** By density and by the Monotone Convergence Theorem, it is enough to prove that for a countable dense subset $\{G^m\}_{m \in \mathbb{N}}$ on $C^{0,2}_c([0, T] \times (0, 1))$ it holds that

$$
\mathbb{E} \left( \max_{k \leq m} \langle \langle \rho, \partial_u G^k \rangle \rangle - c \|G^k\|_2^2 \right) \leq K_0,
$$

for any $m$ and for some $K_0$ independent of $m$. Note that the function that associates to a trajectory $\pi_n \in \mathfrak{D}([0, T], \mathcal{M}_+)$ the number

$$
\max_{k \leq m} \langle \langle \pi_n, \partial_u G^k \rangle \rangle - c \|G^k\|_2^2,
$$

is continuous and bounded with respect to the Skorokhod topology of $\mathfrak{D}([0, T], \mathcal{M}_+)$. For that reason, the expectation in (6.1) is equal to

$$
\lim_{n \to +\infty} \mathbb{E}_{\mu_n} \left( \max_{k \leq m} \langle \langle \pi_n, \partial_u G^k \rangle \rangle - c \|G^k\|_2^2 \right).
$$

Recall (5.6). By entropy’s and Jensen’s inequalities, and the fact that $\exp \left( \max_{k \leq m} a_k \right) \leq \sum_{k=1}^m \exp(a_k)$, the previous display is bounded from above by

$$
C(\alpha, \beta) + \frac{1}{n} \log \mathbb{E}_{\mu_n} \left( \sum_{k=1}^m \exp \left( n \langle \langle \pi_n, \partial_u G^k \rangle \rangle - c n \|G^k\|_2^2 \right) \right),
$$

where $C(\alpha, \beta)$ is a constant which depends on $\alpha$ and $\beta$. For a fixed function $G \in C^{0,2}_c([0, T] \times (0, 1))$, to treat the second term in the previous display it is enough to bound the term

$$
\lim_{n \to +\infty} \frac{1}{n} \log \mathbb{E}_{\mu_n} \left( \exp \left( \int_0^T \sum_{x \in \mathbb{V}_n} \partial_u G_s(\frac{x}{n}) \eta_s(x) - cn \|G_s\|_2^2 \, ds \right) \right).
$$

(6.2)
by a constant independent of $G$. This can be done using the following facts: the linearity of the expectation, the property
\[ \lim_{n \to +\infty} n^{-1} \log(a_n + b_n) = \max \left( \lim_{n \to +\infty} n^{-1} \log(a_n), \lim_{n \to +\infty} n^{-1} \log(b_n) \right), \]
the definition of $\langle \langle \cdot, \cdot \rangle \rangle$, and the definition of the empirical measure. Therefore, by the Feynman-Kac’s formula, the expression (6.2) is bounded from above by
\[ \int_0^T \sup_f \left( \frac{1}{n} \int_{\Omega_n} \sum_{x \in \Omega_n} \nabla G(x) \eta(x) f(\eta) d\nu_{\rho(c)} - c\|G\|_2^2 + n(L_n \sqrt{f}, \sqrt{f}, \rho(c)) \right) ds, \quad (6.3) \]
where the supremum is carried over all the densities $f$ with respect to $\nu_{\rho(c)}$. Note that by a Taylor expansion on $G$, it is easy to see that we can replace its space derivative by the discrete gradient $\nabla n G\left( \frac{x-1}{n} \right)$ plus an error of order $O\left( \frac{1}{n^2} \right)$. Then, from a summation by parts, we obtain that the first term above is equal to
\[ \int_{\Omega_n} \sum_{x=1}^{n-2} G\left( \frac{x}{n} \right) \{\eta(x) - \eta(x+1)\} f(\eta) d\nu_{\rho(c)} + O\left( \frac{1}{n^2} \right). \]
By writing the previous term as one half of it plus one half of it, and in one of the halves we swap the occupation variables $\eta(x)$ and $\eta(x+1)$, last display becomes equal to
\[ \frac{1}{2} \int_{\Omega_n} \sum_{x=1}^{n-2} G\left( \frac{x}{n} \right) \{\eta(x) - \eta(x+1)\} f(\eta) - f(\eta^{x,x+1}) d\nu_{\rho(c)} \]
\[ + \frac{1}{2} \int_{\Omega_n} \sum_{x=1}^{n-2} G\left( \frac{x}{n} \right) \{\eta(x) - \eta(x+1)\} f(\eta) + f(\eta^{x,x+1}) d\nu_{\rho(c)}. \quad (6.4) \]
Repeating similar arguments to the ones used in the proof of Theorem 5.2, the first term in (6.4) is bounded from above by
\[ \frac{1}{4n} \sum_{x=1}^{n-2} \int_{\Omega_n} \left( \frac{1}{a_{x,x+1}(\eta)} (G(x) )^2 (\sqrt{f(\eta)} + \sqrt{f(\eta^{x,x+1})})^2 + a_{x,x+1}(\eta) (\sqrt{f(\eta)} - \sqrt{f(\eta^{x,x+1})})^2 \right) d\nu_{\rho(c)} \]
\[ \leq \frac{C}{n} \sum_{x \in \Omega_n} (G(x) )^2 + \frac{1}{4n} D_\delta \left( \sqrt{f}, \nu_{\rho(c)} \right), \]
for some $C > 0$. To treat the second term in (6.4) we use similar computations to those performed in the first integral of (5.7) and we can show that it is of order $O\left( \frac{1}{n^2} \right)$. From (5.5) we get that (6.5) is bounded from above by
\[ C \int_0^T \left( 1 + \frac{1}{n} \sum_{x \in \Omega_n} (G(x) )^2 \right) ds - c\|G\|_2^2, \]
plus an error of order $O\left( \frac{1}{n^2} \right)$. Above $C$ is a positive constant independent of $G$. Since $\frac{1}{n} \sum_{x \in \Omega_n} (G(x) )^2$ converges to $\|G\|_2^2$, as $n \to +\infty$, then it is enough to choose $c > C$ to conclude that
\[ \lim_{n \to +\infty} \left\{ C \int_0^T \left( 1 + \frac{1}{n} \sum_{x \in \Omega_n} (G(x) )^2 \right) ds - c\|G\|_2^2 \right\} \preceq 1. \]
Taking $K_0 = C(\alpha, \beta) + CT$ the result follows.

7. Uniqueness of weak solutions

In this section we prove Lemma 2.6 that is, the uniqueness of weak solutions of the hydrodynamic equations defined in Section 2. We start covering the Dirichlet case, in which we use the Oleinik’s trick, and we finish the section presenting the uniqueness for the Robin case. We remark that both methods presented below, cover the Neumann case. We decided to include a brief description at the end of the proof for the Dirichlet case stating what would be the differences for the Neumann case.
Before presenting the proofs suppose that \( \rho_1(t,u) \) and \( \rho_2(t,u) \) are weak solutions of the PME starting from the same initial condition \( g(\cdot) \) and with suitable boundary conditions for each problem. We stress that throughout this section we will denote \( w_t(u) = \rho_1(t,u) - \rho_2(t,u) \) and \( v_t(u) = \rho_1(t,u) + \rho_2(t,u) \), for \((t,u) \in [0,T] \times [0,1]\).

7.1. The Dirichlet and Neumann cases. Suppose that \( \rho_1(t,u) \) and \( \rho_2(t,u) \) are weak solutions of (2.6) starting from the same initial condition \( g(\cdot) \). Doing an integration by parts in (2.6), we have that

\[
\langle w_T, G_T \rangle + \int_0^T \langle \partial_t \rho_1^2(t, \cdot) - \partial_u \rho_2^2(t, \cdot), \partial_t G_1 \rangle \, dt - \int_0^T \langle w_t, \partial_t G_1 \rangle \, dt = 0, \tag{7.1}
\]

for all \( G \in C_0^{1,2}([0,T] \times [0,1]) \). We consider the function \( \zeta \in C_0^{1,2}([0,T] \times [0,1]) \) given by

\[
\zeta(t,u) = \begin{cases} 
\int_0^T w_t(u) v_t(u) \, ds, & \text{if } 0 < t < T, \\
0, & \text{if } t \geq T,
\end{cases}
\]

where \( T > 0 \). Note that \( \zeta(t,0) = \zeta(t,1) = 0 \) for all \( t \in [0,T] \), comes from the fact that \( \rho_1(t,u) \) and \( \rho_2(t,u) \) satisfy item (3) of Definition 2. From this, and from the fact that \( \mathcal{C}^{1,2}_0 \) is equal to the set of functions \( \mathcal{C}^1 \) vanishing at 0 and 1, we have that for a.e. time \( t \in (0,T) \), \( w_t(\cdot) \in \mathcal{C}^1 \), then \( v(\cdot) \in L^2(0,T; \mathcal{C}^1) \). We also note that by mollifying \( \zeta \) we can approximate it by smooth functions, in such a way that we can consider that it belongs to the space of test functions \( C^{1,2}_0([0,T] \times [0,1]) \) and therefore we can plug it back into (2.7). We leave the details to the reader and we refer to (20) for more details.

Now, observe that

\[
\begin{align*}
\partial_t \zeta(t,u) &= -w_t(u) v_t(u) \in L^2([0,T] \times [0,1]), \\
\partial_u \zeta(t,u) &= \int_0^T \left( \partial_u \rho_1^2(s,u) - \partial_u \rho_2^2(s,u) \right) \, ds \in L^2([0,T] \times [0,1]).
\end{align*}
\tag{7.2}
\]

Replacing \( G \) by \( \zeta \) in (7.1), we have

\[
\int_0^T \langle \partial_u \rho_1^2(t, \cdot) - \partial_u \rho_2^2(t, \cdot), \partial_u \zeta \rangle \, dt - \int_0^T \langle w_t, \partial_u \zeta \rangle \, dt = 0.
\]

Using (7.2) it follows that

\[
\int_0^1 \int_0^T \left( w_t^2(u) v_t(u) + \left( \partial_u \rho_1^2(t,u) - \partial_u \rho_2^2(t,u) \right) \left( \int_t^T \left( \partial_u \rho_1^2(s,u) - \partial_u \rho_2^2(s,u) \right) \, ds \right) \right) \, dt \, du = 0,
\]

that is

\[
\int_0^T \langle w_t, w_t v_t \rangle \, dt + \frac{1}{2} \int_0^1 \int_0^T \left( \partial_u \rho_1^2(t,u) - \partial_u \rho_2^2(t,u) \right) \, dt \, du = 0.
\]

From last identity, we conclude that \( \rho_1(t,u) = \rho_2(t,u) \) a.s. in \([0,T] \times [0,1]\).

Now, we remark that the proof above also shows uniqueness in the Neumann case. The only difference with respect to the proof above is that we do not need to require the profile \( \rho(\cdot) \) to have a fixed value at the boundary. We give now a sketch of the proof in this case. Suppose that \( \rho_1(t,u) \) and \( \rho_2(t,u) \) are now weak solutions of (2.9) with \( \kappa = 0 \), starting from the same initial condition \( g(\cdot) \). Doing an integration by parts in (2.9) with \( \kappa = 0 \) we have that,

\[
\langle w_T, G_T \rangle + \int_0^T \langle \partial_u \rho_1^2(t, \cdot) - \partial_u \rho_2^2(t, \cdot), \partial_u G_1 \rangle \, dt - \int_0^T \langle w_t, \partial_u G_1 \rangle \, dt = 0,
\]

for all \( G \in C^{1,2}_0([0,T] \times [0,1]) \). Note that the last equation is exactly the same as in (7.1). Now, by the same arguments used in the Dirichlet case, we can reach the same conclusion for the Neumann case.
7.2. The Robin case. We adapt Filo’s proof to our model (see [9], Theorem 3), and we present it in details below. Although the proof there holds for any spatial dimension, we consider only the one-dimensional case. Before starting the proof, we need some technical results. The following result is concerning a parabolic value problem with Robin conditions:

**Lemma 7.1.** Suppose that \( a = a(t,u) \) is a positive \( C^{2,2}([0,T] \times [0,1]) \) function, \( b = b(t,u) \) is a positive \( C^{2}([0,T]) \) function, for \( u = 0 \) and \( u = 1 \), \( h = h(u) \in C^{2}_{0}([0,1]) \), and \( \lambda \geq 0 \). Then, for \( t \in (0,T] \), the problem with Robin conditions

\[
\begin{align*}
\frac{\partial \zeta}{\partial t} + a \Delta \zeta &= \lambda \zeta, & (s,u) \in [0,t) \times (0,1), \\
\partial_{u} \zeta(s,0) &= b(s,0) \zeta(s,0), & s \in [0,t), \\
\partial_{u} \zeta(s,1) &= -b(s,1) \zeta(s,1), & s \in [0,t), \\
\zeta(t,u) &= h(u), & u \in (0,1),
\end{align*}
\]

(7.3)

has a unique solution \( \zeta \) in \( C^{1,2}([0,t] \times [0,1]) \). Moreover, if \( 0 \leq h \leq 1 \), then

\[
0 \leq \zeta(s,u) \leq e^{-\lambda(t-s)}, \quad \text{for } (s,u) \in [0,t] \times [0,1].
\]

(7.4)

**Proof.** First, observe that by setting \( \tau = t-s \) and \( \zeta(t,u) = e^{-\lambda(t-s)} \zeta(t,s) \), (7.3) is equivalent to

\[
\begin{align*}
\frac{\partial \zeta}{\partial \tau} - \Delta \zeta &= 0, & (\tau,u) \in (0,t] \times (0,1), \\
\partial_{u} \zeta(\tau,0) &= b(\tau,0) \zeta(\tau,0), & \tau \in (0,t], \\
\partial_{u} \zeta(\tau,1) &= -b(\tau,1) \zeta(\tau,1), & \tau \in (0,t], \\
\zeta(\tau,u) &= e^{-\lambda \tau} h(u), & u \in (0,1),
\end{align*}
\]

(7.5)

which has a unique \( C^{1,2}([0,t] \times [0,1]) \) solution \( \zeta(\tau,u) \) according to [16] (see Theorem 5.3) or [17] (see Theorem 4). Now, we need to show that \( 0 \leq \zeta(0) \leq e^{-\lambda t} \) in \( [0,t] \times [0,1] \), under the assumption that \( 0 \leq h \leq 1 \). Suppose that

\[
\max_{[0,t] \times [0,1]} \zeta(0) > e^{-\lambda t}.
\]

From the maximum principle for parabolic equations,

\[
\max_{[0,1]} \zeta_{0} = \max_{\Sigma_{t} \cap (0 \times [0,1])} \zeta_{0},
\]

where \( \Sigma_{t} = ([0,t] \times \{0\}) \cup ([0,t] \times \{1\}) \). Since \( \zeta_{0}(0,u) = e^{-\lambda t} h(u) \leq e^{-\lambda t}, \) for \( 0 \leq u \leq 1 \), there exists some \( (\tau_{1}, u_{1}) \in \Sigma_{t} \) that realizes the maximum of \( \zeta_{0} \). Suppose, without loss of generality, that \( u_{1} = 0 \). Observe that \( \tau_{1} > 0 \), due to the fact that \( \zeta_{0} \) is continuous in \( [0,t] \times [0,1] \) and \( \zeta_{0}(0,0) = e^{-\lambda t} h(0) = 0 \). Since \( \zeta_{0}(\tau_{1},u_{1}) > e^{-\lambda t} \) and \( b \) is positive, it follows that

\[
\partial_{u} \zeta(\tau_{1},0) = b(\tau_{1},0) \zeta(\tau_{1},0) > 0.
\]

Hence, for \( u > 0 \) sufficiently close to 0, we have

\[
\zeta(0,\tau_{1},u) \leq \zeta(\tau_{1},0),
\]

contradicting the fact that \( (\tau_{1},0) \) is a point of maximum of \( \zeta_{0} \). Therefore, \( \zeta_{0} \leq e^{-\lambda t} \). By an analogous argument, we can prove that \( \zeta_{0} \geq 0 \), concluding that \( 0 \leq \zeta_{0} \leq e^{-\lambda t} \).

Now, let \( \varphi(0,u) = e^{\lambda t} \zeta_{0}(t-s,u) \). As we have already mentioned, since \( \zeta_{0} \) is the solution of (7.3), then \( \varphi_{0}(s,u) \) is the solution of (7.3). Furthermore, since \( 0 \leq \zeta_{0} \leq e^{-\lambda t} \), we have that \( 0 \leq \varphi_{0}(s,u) \leq e^{-\lambda(t-s)} \), which proves the lemma.

**Lemma 7.2.** Let \( \varphi_{0} \) be the solution of the parabolic problem (7.3). There exists a positive constant \( C = C(b,h) \) such that

\[
\int_{0}^{t} \int_{0}^{1} a(s,u)(\Delta \varphi_{0}(s,u))^{2} \, ds \, du \leq C(b,h).
\]
Proof. Multiplying the first line of (7.3) by $\Delta \varphi_0(s, u)$, and integrating it in space and time, we obtain
\[
\int_0^t \int_0^1 \partial_\xi \varphi_0 \Delta \varphi_0 \, duds + \int_0^t \int_0^1 a(\Delta \varphi_0)^2 \, duds - \int_0^t \int_0^1 \lambda \varphi_0 \Delta \varphi_0 \, duds = 0.
\]
Integrating last equation by parts, we have
\[
\int_0^t \partial_t \varphi_0 (s, 1) \partial_t \varphi_0 (s, 1) \, ds - \int_0^t \partial_t \varphi_0 (s, 0) \partial_t \varphi_0 (s, 0) \, ds
\]
\[
- \frac{1}{2} \int_0^t \int_0^1 \partial_\xi |\partial_u \varphi_0|^2 \, duds + \int_0^t \int_0^1 a(\Delta \varphi_0)^2 \, duds
\]
\[- \int_0^t \lambda \varphi_0 (s, 1) \partial_t \varphi_0 (s, 1) \, ds - \int_0^t \lambda \varphi_0 (s, 0) \partial_t \varphi_0 (s, 0) \, ds
\]
\[- \int_0^t \lambda |\partial_u \varphi_0|^2 \, duds = 0.
\]
Integrating the third term in the last equation and using the boundary conditions, it follows that
\[
\int_0^t \int_0^1 (a(\Delta \varphi_0)^2 + \lambda |\partial_u \varphi_0|^2) \, duds + \int_0^t \lambda b(s, 1)(\varphi_0(s, 1))^2 \, ds + \int_0^t \lambda b(s, 0)(\varphi_0(s, 0))^2 \, ds
\]
\[- \int_0^t \partial_t \varphi_0 (s, 1) b(s, 1) \varphi_0 (s, 1) \, ds - \int_0^t \partial_t \varphi_0 (s, 0) b(s, 0) \varphi_0 (s, 0) \, ds
\]
\[- \frac{1}{2} \int_0^t \int_0^1 |\partial_u \varphi_0|^2 (t, u) - |\partial_u \varphi_0|^2 (0, u) \, du = 0.
\]
Now, doing an integration by parts on the fourth and fifth terms in the above display, and using the initial condition, we obtain:
\[
\int_0^t \int_0^1 (a(\Delta \varphi_0)^2 + \lambda |\partial_u \varphi_0|^2) \, duds + \int_0^t \lambda b(s, 1)(\varphi_0(s, 1))^2 \, ds + \int_0^t \lambda b(s, 0)(\varphi_0(s, 0))^2 \, ds
\]
\[- \frac{1}{2} b(t, 1)(\varphi_0(t, 1))^2 + \frac{1}{2} b(0, 1)(\varphi_0(0, 1))^2 + \frac{1}{2} \int_0^t \partial_\xi b(s, 1)(\varphi_0(s, 1))^2 \, ds
\]
\[- \frac{1}{2} b(t, 0)(\varphi_0(t, 0))^2 + \frac{1}{2} b(0, 0)(\varphi_0(0, 0))^2 + \frac{1}{2} \int_0^t \partial_\xi b(s, 0)(\varphi_0(s, 0))^2 \, ds
\]
\[- \frac{1}{2} \int_0^1 |h'(u)|^2 \, du + \frac{1}{2} \int_0^1 |\partial_u \varphi_0|^2 (0, u) \, du = 0.
\]
Therefore,
\[
\int_0^t \int_0^1 a(\Delta \varphi_0)^2 \, duds \leq \frac{1}{2} \int_0^1 |h'(u)|^2 \, du
\]
\[
+ \frac{1}{2} b(t, 1)(\varphi_0(t, 1))^2 - \frac{1}{2} b(0, 1)(\varphi_0(0, 1))^2 - \frac{1}{2} \int_0^t \partial_\xi b(s, 1)(\varphi_0(s, 1))^2 \, ds
\]
\[
+ \frac{1}{2} b(t, 0)(\varphi_0(t, 0))^2 - \frac{1}{2} b(0, 0)(\varphi_0(0, 0))^2 - \frac{1}{2} \int_0^t \partial_\xi b(s, 0)(\varphi_0(s, 0))^2 \, ds.
\]
Since $\varphi_0$ is bounded, according to Lemma 7.1 the right-hand side of last inequality is bounded from above by some constant $C$, that depends only on $h$ and $b$. \qed

Before presenting the uniqueness of weak solutions of the hydrodynamic equation with Robin boundary conditions, we need two more technical results:
Lemma 7.3. Let \( b \) be a nonnegative and bounded measurable function in \([0, T]\) and \( 1 \leq p < +\infty \). There exists a sequence \( \{b_k\}_{k \geq 0} \) of positive functions in \( C^\infty([0, T]) \), such that \( b_k \) converges to \( b \) in \( L^p([0, T]) \) and

\[
\left\| \frac{b}{b_k} - 1 \right\|_{L^p(A)} \to 0,
\]

where \( A = \{ t \in (0, T) : b(t) > 0 \} \).

Proof. Let \( \varepsilon_k = 1/k > 0 \). Consider a sequence of positive numbers \( \{\delta_j\}_{j \geq 0} \), such that \( \delta_j \to 0 \). Since \( b > 0 \) in \( A \), we have

\[
\frac{b(t)}{b(t) + \delta_j} - 1 \to 0 \quad \text{for any } t \in A \text{ as } j \to +\infty, \quad \text{and} \quad \left| \frac{b(t)}{b(t) + \delta_j} - 1 \right| < 2.
\]

From the dominated convergence theorem, \( b/(b + \delta_j) - 1 \) converges to 0 in \( L^p(A) \). Hence, for a large \( j_0 \), we have

\[
\left\| \frac{b}{b + \delta_{j_0}} - 1 \right\|_{L^p(A)} < \frac{\varepsilon_k}{2}. \tag{7.6}
\]

Let \( \{c_m\}_{m \geq 0} \) be a sequence in \( C^\infty([0, T]) \), such that \( c_m \to b + \delta_{j_0} \) in \( L^p([0, T]) \). Since \( b + \delta_{j_0} \geq \delta_{j_0} \), we can assume that \( c_m \geq \delta_{j_0} \). Then

\[
\left\| \frac{b}{c_m} - \frac{b}{b + \delta_{j_0}} \right\|_{L^p(0,T)} = \left\| \frac{b(b + \delta_{j_0} - c_m)}{c_m(b + \delta_{j_0})} \right\|_{L^p(0,T)} \leq \frac{\|b\|_{L^\infty([0,T])}\|b + \delta_{j_0} - c_m\|_{L^p([0,T])}}{\delta_{j_0}^2}.
\]

Hence, using that \( c_m \to b + \delta_{j_0} \) in \( L^p([0, T]) \), for a large \( m_0 \), we have that

\[
\left\| \frac{b}{c_{m_0}} - \frac{b}{b + \delta_{j_0}} \right\|_{L^p(0,T)} < \frac{\varepsilon_k}{2}. \tag{7.7}
\]

Defining \( b_k = c_{m_0} \), (7.6) and (7.7) imply that

\[
\left\| \frac{b}{b_k} - 1 \right\|_{L^p(A)} < \varepsilon_k,
\]

proving the result. \( \square \)

Remark 7.4. Using the same argument above, we can prove the following result that is used in [9]: if \( a \) is a nonnegative bounded measurable function in \([0, T] \times [0, 1]\), then there exists a sequence \( \{a_k\}_{k \geq 0} \) of positive \( C^\infty \) functions in time and space, such that

\[
\frac{1}{k} \leq a_k \leq \|a\|_{L^\infty} \quad \text{and} \quad \frac{|a - a_k|}{\sqrt{a_k}} \to 0 \quad \text{in } L^2([0,T] \times [0,1]).
\]

Proof of Lemma 2.6 for the Robin case ([9]): Although the proof that we will present is true for \( \kappa \geq 0 \), we will only consider the case \( \kappa > 0 \). But the interested reader can check that for \( k = 0 \), the proof also holds. Suppose that \( \rho_1(t,u) \) and \( \rho_2(t,u) \) are weak solutions of (2.8). Since \( \rho_1(t,u) \) and \( \rho_2(t,u) \) satisfy (2.9), we conclude that

\[
\langle w_t, G_t \rangle - \int_0^t \langle w_s, \partial_s G_s \rangle ds - \int_0^t \langle w_s, v_s G_s \rangle ds + \int_0^t w_s(1)v_s(1)\partial_u G_s(1) - w_s(0)v_s(0)\partial_u G_s(0) ds
\]

\[
+ \kappa \int_0^t w_s(1)G_s(0) + w_s(1)G_s(1) ds = 0.
\]

Therefore, this equation can be rewritten as

\[
\langle w_t, G_t \rangle = \int_0^t \left( w_s(1)\partial_u G_s + v_s\Delta G_s \right) ds - \int_0^t w_s(1)(\kappa G_s(1) + v_s(1)\partial_u G_s(1)) ds
\]

\[
+ \int_0^t w_s(0)(v_s(0)\partial_u G_s(0) - \kappa G_s(0)) ds. \tag{7.8}
\]
To estimate the integrals above we need to use a suitable test function, which is the solution of the parabolic equation (7.3). Unfortunately, the function \( v \) above does not have regularity enough. To avoid this difficulty, observe that 0 ≤ \( v(t, u) ≤ 2 \), since 0 ≤ \( \rho_1 \) and \( \rho_2 \) ≤ 1. Then, according to Lemma 7.3, taking \( b \) equal to \( v \), for \( \varepsilon > 0 \) and \( p = 1 \), there exists a positive function \( b_\varepsilon \in C^2([0, T] \times [0, 1]) \) such that

\[
\left\| \frac{v(t, u_i)}{b_\varepsilon(t, u_i)} - 1 \right\|_{L^1(A_i)} < \varepsilon \quad \text{for} \quad i \in \{0, 1\},
\]

where \( u_0 = 0, u_1 = 1 \) and \( A_i = \{ t \in (0, T) : v(t, u_i) > 0 \} \). Moreover, from Remark 7.4 with \( a = v \), there exists a sequence of functions \( \{a_n\}_{n \geq 0} \) in \( C^\infty \) in time and space, such that

\[
\frac{1}{n} \leq a_n \leq 2 + \frac{1}{n} \quad \text{and} \quad \frac{a_n - v}{\sqrt{a_n}} \to 0 \quad \text{in} \quad L^2([0, T] \times [0, 1]) \quad \text{as} \quad n \to +\infty.
\]

For fixed \( \lambda = 0 \) and \( h \in C^0_c([0, 1]) \), consider the parabolic problem (7.3) with \( a \) and \( b \) replaced by \( a_n \) and \( \kappa/b_\varepsilon \), respectively. Observe that \( \kappa/b_\varepsilon \) is a positive \( C^2 \) function. Then, from Lemma 7.1 there exists a unique solution \( \varphi_n(s, u) \) to this problem associated to \( a_n \) and \( \kappa/b_\varepsilon \).

Now, for \( G(s, u) = \varphi_n(s, u) \), we estimate each integral of the right-hand side of (7.8). For the first integral, using the fact that \( \varphi_n \) is a solution of (7.3) (with \( \lambda = 0 \)), and the Cauchy-Schwarz inequality, we obtain

\[
\int_0^t \left\langle w_s, \partial_t \varphi_n(s, \cdot) + v_s \Delta \varphi_n(s, \cdot) \right\rangle \, ds \\
= \int_0^t \left\langle w_s, \partial_t \varphi_n(s, \cdot) + a_n(s, u) \Delta \varphi_n(s, \cdot) \right\rangle \, ds + \int_0^t \left\langle w_s, (v_s - a_n(s, \cdot)) \Delta \varphi_n(s, \cdot) \right\rangle \, ds \\
\leq \int_0^t \left\| w_s \varphi_n \right\|_{L^2([0, 1])} \\left\| \Delta \varphi_n \right\|_{L^2([0, 1])} \, ds.
\]

Hence, from Cauchy-Schwarz inequality, Lemma 7.2 and \( \|w_s\| = |\rho_0 - \rho_1| \leq 2 \), we have

\[
\int_0^t \left\langle w_s, \partial_t \varphi_n(s, \cdot) + v_s \Delta \varphi_n(s, \cdot) \right\rangle \, ds \leq 2 \left\| \frac{v_s - a_n}{\sqrt{a_n}} \right\|_{L^2([0, T] \times [0, 1])} \left\| \varphi_n \right\|_{C^\infty} \frac{\sqrt{C(\kappa/b_\varepsilon, h)}}{\kappa b_\varepsilon(s, 1)}. \tag{7.11}
\]

For the boundary integrals of (7.8) we use the Robin condition satisfied by \( \varphi_n \). For the right-hand side of the boundary (\( u_1 = 1 \)), we have

\[
\partial_{u_1} \varphi_n(s, 1) = -\frac{\kappa}{b_\varepsilon(s, 1)} \varphi_n(s, 1).
\]

Then, for \( G(s, u) = \varphi_n(s, u) \), the second integral on the right-hand side of (7.8) becomes

\[
\int_0^t w_s(1)(\kappa \varphi_n(s, 1) + v_s(1) \partial_{u_1} \varphi_n(s, 1)) \, ds = \int_0^t w_s(1) \left( \kappa \varphi_n(s, 1) - v_s(1) \frac{\kappa}{b_\varepsilon(s, 1)} \varphi_n(s, 1) \right) \, ds.
\]

Note that if \( s_0 \notin A'_1 := \{ s \in [0, t] : v_s(1) > 0 \} \), then \( \rho_1(s_0, 1) = \rho_2(s_0, 1) = 0 \) and, therefore, \( w(s_0, 1) = 0 \). Hence, from the fact that \( |w| \leq 2 \), and (7.4) together with the choice \( \lambda = 0 \), we get

\[
\left| \int_0^t w_s(1)(\kappa \varphi_n(s, 1) + v_s(1) \partial_{u_1} \varphi_n(s, 1)) \, ds \right| = \int_{A'_1} w_s(1) \left( \kappa \varphi_n(s, 1) - v_s(1) \frac{\kappa}{b_\varepsilon(s, 1)} \varphi_n(s, 1) \right) \, ds \\
\leq 2\kappa \left\| 1 - \frac{v_s(1)}{b_\varepsilon(s, 1)} \right\|_{L^1(A'_1)}.
\]

Then, using (7.9) and that \( A'_1 \subset A_1 \), we have

\[
\left| \int_0^t w_s(1)(\kappa \varphi_n(s, 1) + v_s(1) \partial_{u_1} \varphi_n(s, 1)) \, ds \right| \leq 2\kappa \varepsilon. \tag{7.12}
\]
By an analogous argument, we also have
\[\left| \int_0^t w_t(s)(v_t(s)\partial_s\varphi_n(s,0) - \kappa\varphi_n(s,0))ds \right| \leq 2\kappa\varepsilon. \tag{7.13}\]

Therefore, from the fact that \(\varphi_n(t,u) = h(u)\), and from (7.8), (7.11), (7.12), and (7.13), we conclude that
\[\langle w_t , h \rangle \leq 2\left(\frac{\nu-a_n}{\sqrt{a_n}}\right)\|h\|_{L^2([0,T]\times[0,1])} + 4\kappa\varepsilon.\]

Taking \(n \rightarrow +\infty\) and using (7.10), it follows that
\[\langle w_t , h \rangle \leq 4\kappa\varepsilon.\]

Since \(\varepsilon > 0\) is arbitrary,
\[\langle w_t , h \rangle \leq 0,\]
for any \(h \in C^2_0(0,1)\). Now consider a sequence \(h_n \in C^2_0(0,1)\) such that \(h_n(\cdot) \rightarrow 1_{[u\in[0,1]:w_t(u)>0]}(t,\cdot)\) in \(L^2([0,1])\). Then, from the last inequality, we obtain
\[\int_0^1 w^+(t,u)du \leq 0,\]
where \(w^+ = \max\{w,0\}\). Therefore, for any \(t \in [0,T]\), \(\rho_1(t,u) \leq \rho_2(t,u)\) for almost every \(u \in [0,1]\). That is, \(\rho_1 \leq \rho_2\) for almost every \((t,u) \in [0,T] \times [0,1]\). In the same way, \(\rho_2 \leq \rho_1 \) a.e., completing the proof. \(\square\)

**Acknowledgements**

A.N. was supported through a grant “L’ORÉAL - ABC - UNESCO Para Mulheres na Ciência”. R.P thanks FCT/Portugal for support through the project Lisbon Mathematics PhD (LisaMath). This project has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovative programme (grant agreement No 715734).

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