ON THE DIRICHLET PROBLEM FOR FULLY NONLINEAR ELLIPTIC EQUATIONS ON ANNULI OF METRIC CONES

CHUNHUI QIU AND RIRONG YUAN

School of Mathematical Sciences
Xiamen University
Xiamen 361005, China

(Communicated by Yanyan Li)

Abstract. In this paper, we study a class of fully nonlinear elliptic equations on annuli of metric cones constructed from closed Sasakian manifolds and derive the a priori estimates assuming the existence of subsolutions. Moreover, such a priori estimates can be applied to certain degenerate equations. A condition for the solvability of Dirichlet problem for non-degenerate fully nonlinear elliptic equations is discovered. Furthermore, we also discuss degenerate equations.

1. Introduction. Sasakian manifolds, which can be viewed as the odd dimensional counterparts of Kähler manifolds, are odd dimensional Riemannian manifolds whose metric cone \((C(M), \bar{g}) = (M \times \mathbb{R}^+, r^2g + dr^2)\) admits a Kähler structure. These manifolds can be used to construct new Einstein manifolds in geometry, and play an important role in AdS/CFT correspondence in Mathematical physics. We refer the readers to [18, 20, 42, 43] for relevant works on Sasakian geometry from mathematicians and physicists.

Let \((M, \xi, \eta, \Phi, g)\) be a Sasakian manifold of dimension \(2n - 1\), then the metric cone \((C(M), \bar{g}) = (M \times \mathbb{R}^+, r^2g + dr^2)\) is a Kähler manifold, here \(r\) is the coordinate on \(\mathbb{R}^+\). The Sasakian structure \((\xi, \eta, \Phi, g)\) consists of a Reeb field \(\xi\) of unit length on \(M\), a \((1,1)\)-type tensor field \(\Phi(X) = \nabla_X \xi\) and a contact 1-form \(\eta\) with \(\eta(X) = g(\xi, X)\).

Denote by \(\bar{\omega}\) the Kähler form of \((C(M), \bar{g})\), \(\omega_T\) the transverse Kähler form of the Sasakian manifold \((M, \xi, \eta, \Phi, g)\), \(C^k_B(M) = \{ u \in C^k(M) : \xi u = 0 \}\) and \(C^{k,\alpha}_B(M) = C^k_B(M) \cap C^{k,\alpha}(M)\). Please see Section 2 below for more notations and definitions.

In this paper we consider the following Dirichlet problem for fully nonlinear elliptic equations on \(\bar{M} = M \times [a, b], 0 < a < b < +\infty\),

\[
\begin{aligned}
F(U[u]) &= \psi & \text{in } \text{Int}(\bar{M}) := M \times (a, b), \\
u &= \varphi_a & \text{on } M \times \{a\}, \\
u &= \varphi_b & \text{on } M \times \{b\},
\end{aligned}
\]

2010 Mathematics Subject Classification. 35J15, 35J70, 58J05, 35B45.

Key words and phrases. Sasakian manifold, metric cone, degenerate fully nonlinear elliptic equation, a priori estimates, Dirichlet problem, subsolution, cone condition, concavity.

Research supported in part by NSF in China, No. 11571288, No. 11671330, No. 11571332, No. 11625106 and No. 11131007.
where \( \psi \in C^2_b(\text{Int}(\Omega)) \cap C^{0,1}(\bar{\Omega}) \) is a prescribed function, \( U[u] = \bar{\chi} + \sqrt{-1} \partial\bar{\partial} u - \frac{\partial f}{\partial r} \partial r \), \( \bar{\chi} \) is a real \((1,1)\)-form satisfying \( \nabla_\xi \bar{\chi} = 0 \), and \( \nabla \) is Chern connection with respect to \( \bar{\eta} \). Moreover \( \varphi_a, \varphi_b \in C^2_b(\bar{\Omega}) \) and we denote by \( u = \varphi \) the Dirichlet boundary value for convenience.

Equation (1.1) shall be of the form: There is a smooth and symmetric function \( f \) defined on an open symmetric convex cone \( \Gamma \) with \( \Gamma \) most important one is the complex Monge-Ampère equation (in this case \( f \) boundary value for convenience.

Equation (1.1) shall be of the form: There is a smooth and symmetric function \( f \) defined on an open symmetric convex cone \( \Gamma \) with \( \Gamma_n \subseteq \Gamma \subseteq \Gamma_1 \), such that

\[
F(U) = f(\lambda(U)),
\]

where \( \lambda(\eta) = (\lambda_1, \ldots, \lambda_n) \) are the eigenvalues of the real \((1,1)\)-form \( \eta \) with respect to \( \omega \), \( \Gamma_k = \{ \lambda \in \mathbb{R}^n : S_i(\lambda) > 0 \text{ for } 1 \leq i \leq k \} \) and \( S_k(\lambda) \) is the k-th elementary symmetric polynomial of \( \lambda \in \mathbb{R}^n \).

The study of this type of equations can trace back to Ivochkina [36] where the author treated some special cases, and Caffarelli, Nirenberg and Spruck [5] in which the authors dealt with the Dirichlet problem for Hessian equations on the bounded domains \( \Omega \subset \mathbb{R}^n \). As in [5], the function \( f \) shall satisfy the following fundamental conditions,

\[
f_i = \frac{\partial f}{\partial \lambda_i} > 0 \text{ in } \Gamma, \quad 1 \leq i \leq n,
\]

(1.2)

and \( \psi : \bar{\Omega} \rightarrow (\text{sup}_{\partial \Omega} f, \text{sup}_\Gamma f) \) satisfies

\[
\delta_{\psi,f} = \inf \psi - \sup f > 0,
\]

(1.3)

where \( \text{sup}_{\partial \Omega} f \equiv \sup_{\lambda_0 \in \partial \Omega} \lim sup_{\lambda \rightarrow \lambda_0} f(\lambda) \).

The central issue for solving equation (1.1) is to obtain \( C^2 \)-estimates for the admissible solution \( u \). Conditions (1.2)-(1.4) play fundamental roles in the studies of such equations. More precisely, condition (1.2) guarantees equation (1.1) to be elliptic for solutions \( u \in C^2(\bar{\Omega}) \) satisfying \( \lambda(\bar{\Omega}, u) \subseteq \Gamma \), which as in [5], we call such functions admissible, while condition (1.3) ensures that \( F(\lambda(A)) = f(\lambda(A)) \) is concave for any Hermitian matrixes \( A \) with \( \lambda(A) \subseteq \Gamma \), and (1.4) prevents equation (1.1) from becoming degenerate and guarantees that it is uniformly elliptic once \( a \) priori bounds of (complex) Hessian are established for solutions. Then one can apply the Evans-Krylov theorem [15, 38], together with the method of deriving real Hessian due to Guan and Li [25], to derive the \( C^{2,\alpha} \) bounds.

Furthermore, we assume that

\[
\text{For any } \sigma < \text{sup}_\Gamma f \text{ and } \lambda \in \Gamma \text{ we have } \lim_{t \rightarrow +\infty} f(t \lambda) > \sigma.
\]

(1.5)

Such condition is closely related to (1.12) below and allows one to derive the gradient estimates using the blow-up argument presented in [9, 50]. Moreover, Székesyihidi [50] proved that if \( f \) satisfies (1.2)-(1.5) then there exists a positive constant depending only on \( \sigma \) and \( f \) (may depend on \( \delta_{\psi,f} \)) such that

\[
\sum_{i=1}^{n} f_i \geq \kappa > 0 \text{ in } \left\{ \lambda \in \Gamma : f(\lambda) = \sigma, \inf_M \psi \leq \sigma \leq \text{sup}_M \psi \right\}.
\]

(1.6)

In case of equation (1.1) becoming degenerate \( (\delta_{\psi,f} \geq 0) \), we assume that

The positive constant \( \kappa \) in (1.6) is independent of \( \delta_{\psi,f} \).

(1.7)

For the standard equations on complex manifolds, i.e. \( U[u] = \bar{\chi} + \sqrt{-1} \partial\bar{\partial} u \), the most important one is the complex Monge-Ampère equation (in this case \( f(\lambda) =
Given any \(\sigma_n(\lambda)^{1/n}\) which is related to the representation of Ricci form of the Kähler manifold. A related problem is the well known Calabi conjecture which requires one to solve the smooth non-degenerate complex Monge-Ampère equation, c.f. Calabi [8]. Yau [56] solved it and showed that every closed Kähler manifold of \(C_1(M) \leq 0\) admits a unique Kähler-Einstein metric contained in the given Kähler class. The same result on a Kähler manifold with \(C_1(M) < 0\) was obtained by Aubin [1] independently. Another fundamental work about the complex Monge-Ampère equation is due to Caffarelli, Kohn Nirenberg and Spruck [7] who dealt with the Dirichlet problem for the complex Monge-Ampère equation on a strictly pseudoconvex domain in \(\mathbb{C}^n\). This work was extended to general bounded domains \(\Omega \subset \mathbb{C}^n\) by Guan [22], in which the author used the admissible subsolution satisfying (1.8) below to relax the restrictions to \(\partial \Omega\). This notion of the subsolution plays crucial roles in various works, c.f. [9 23 30 33]. It would be worthwhile to note that the Dirichlet problem is not always solvable without assuming subsolutions.

Observing that equation (1.1) involves the radial derivative term \(\partial_u\), equation (1.1) is different from the standard equations on complex manifolds, c.f. [1 2 7 8 10 13 47 50 53 56]. The equations containing gradient terms arise naturally in complex geometry, complex analysis and mathematical physics, c.f. [17 19 46 32 10 13 47 50 53 56] and the references therein.

By contrast with the real setting, it turns out to be a rather challenging task to derive a priori estimates for second derivatives of solutions of fully nonlinear elliptic equations containing gradient terms in complex setting. The underlying reason is the two different types of complex derivatives. For more information, we refer the reader to [17 26 44 51] for related works.

We state our main results as follows.

**Theorem 1.1.** Given any admissible solution \(u \in C^4(\text{Int}(\bar{M})) \cap C^2(\bar{M})\) of Dirichlet problem (1.1). Suppose, in addition to \(f\) satisfies (1.2)-(1.4) and \(\psi \in C^2_B(\text{Int}(\bar{M})) \cap C^{1,1}_B(\bar{M})\), that the Dirichlet problem admits an admissible subsolution \(\underline{u} \in C^2_B(\bar{M})\) satisfying

\[
\begin{aligned}
&f(\lambda([U[u]])) \geq \psi & \text{in } \text{Int}(\bar{M}), \\
&\underline{u} = \varphi & \text{on } \partial \bar{M}.
\end{aligned}
\]  

(1.8)

Then there exists a uniform bounded positive constant \(C\) depending on \(\varepsilon^{-1}, \beta^{-1}, \|u\|_{C^4(\text{Int}(\bar{M}))}, \|\psi\|_{C^{1,1}(\bar{M})}, \|\psi\|_{C^{1,1}(\bar{M})}, \|\chi\|_{C^{1,1}(\bar{M})}\) and other data under control (but not on \(\|\nabla u\|_{C^{0}(\bar{M})}\)), such that

\[
\sup \frac{\Delta u}{M} \leq C \left( 1 + \sup \frac{\|\nabla u\|_{\bar{M}}^2}{\partial M} + \sup \frac{\Delta u}{\partial M} \right),
\]  

(1.9)

where \(\nu_m = Df(\mu)/|Df(\mu)|, \Delta \) is the Laplacian operator of \(\bar{g}\) with respect to Chern connection, \(\beta = \frac{1}{2} \min_{\gamma} \text{dist}(\nu_m, \partial \Gamma_n)\) and \(\varepsilon\) is the constant in Lemma 2.2 below. Moreover, the constant \(C\) in (1.9) is independent of \((\delta\varphi, f)^{-1}\).

We now turn to the second order boundary estimates

\[
\sup \frac{\Delta u \bar{g}}{M} \leq C.
\]  

(1.10)

In [5], Caffarelli, Nirenberg and Spruck derived (1.10) for the Dirichlet problem in a bounded domain \(\Omega \subset \mathbb{R}^n\)

\[
\begin{aligned}
f(\lambda(\nabla^2 u)) = \psi & \text{ in } \Omega, \\
u = \varphi & \text{ on } \partial \Omega
\end{aligned}
\]  

(1.11)
Theorem 1.2. Let boundary estimates for second derivatives hold and Dirichlet problem (1.5) therein. This result was extended by Li [39] to the general case. Furthermore, condition (1.11) was removed by Trudinger [55]. Since then boundary estimates (1.10) were extended to very general Hessian equations, c.f. [22, 23, 24, 31, 37] and reference therein.

In this paper, identity (1.8) below allows one to derive the following precise boundary estimates for second derivatives.

**Theorem 1.2.** Let \( \psi \in C^2_{\bar{B}}(\text{Int}(\bar{M})) \cap C^{0,1}(\bar{M}) \). Assume that conditions (1.12)-(1.15) hold and Dirichlet problem (1.1) admits an admissible subsolution \( u \in C^2_{\bar{B}}(\bar{M}) \). Suppose that there are smooth functions \( a_i(x, t), b(z, t) : M \times (\sup_{\partial\Omega} f, \sup f) \to \mathbb{R} \), such that the equation can be written as the form

\[
\sum_{k=1}^{n} a_k(z, \psi) U^k \wedge \bar{\omega}^{n-k} = b(z, \psi).
\]

Then for any admissible solution \( u \in C^2(\text{Int}(\bar{M})) \cap C^2(\bar{M}) \) to the Dirichlet problem, we have

\[
\sup_{\partial M} |\nabla^2 u|_g \leq C(1 + \sup_M |\nabla u|_g^2).
\]

where \( C \) is a uniform positive constant depending on \((\delta_{\psi, f})^{-1}, \kappa^{-1} (\kappa \text{ is the constant in (1.6)}, |\psi|_{C^{0,1}(M)}, |\psi|_{C^{2,1}(M)}, |\varphi|_{C^{2,1}(M)} \) and other known data under control (but not on \( \sup_{\partial M} |\nabla u|_g \)).

Suppose, in addition condition (1.17) holds or \( \psi \) is a constant function, that \( a_k(z, \psi) = a_k(z) \) and there is a uniform constant \( C_{II} \) depending not on \((\delta_{\psi, f})^{-1}\) such that

\[
\sup_{\partial M} |b(z, \psi)| \leq C_{II}.
\]

Then the constant \( C \) in (1.14) does not depend on \((\delta_{\psi, f})^{-1}\).

Combining Theorem 1.1 and Theorem 1.14 the quadratic power of gradient term can dominate the Laplacian

\[
\sup_M |\Delta u| \leq C(1 + \sup_M |\nabla u|_g^2).
\]

Therefore, the gradient estimate can be obtained by the blow-up argument presented in [23, 50].

The following condition allows one to construct the desired basic admissible subsolutions of Dirichlet problem (1.1).

Given a smooth real \((1, 1)\)-form \( \xi \), we set

\[
g^T(\cdot, \cdot) = g|_{P^*D^{1,0} \times P^*D^{0,1}} : P^*D^{1,0} \times P^*D^{0,1} \to \mathbb{C},
\]

where \( P^*D^C \) is the pullback contact bundle \( D^C = D^{1,0} \oplus D^{0,1} \) and \( P : C(M) \to M \) is the natural projective map. In particular, \( U[v]^T = \chi^T + \sqrt{-1} \partial_B \bar{\partial} v \) for \( v \in C^2_B(\bar{M}) \).

Denote by \( C_B^k(M) = \{ u \in C^k(M) : \xi u = 0 \} \), \( C_B^{k,\alpha}(M) = C^{k,\alpha}(M) \cap C_B^k(M) \) and

\[
C_\alpha(\bar{\chi}) = \{ v \in C^{k,\alpha}_B(M) : \exists R > 0 \text{ such that } (\lambda'(U[v]^T), R) \in \Gamma \},
\]
where $\alpha \in (0, 1)$, and $\lambda(U[v]^T) = (\lambda'_1, \ldots, \lambda'_{n-1})$ are the eigenvalues of $U[v]^T$ with respect to $r^2 \omega^T$.

For Dirichlet problem (1.1), we assume that there exists a function $v \in C_0(\bar{\chi})$ with the boundary value condition $v|_{\partial\bar{M}} = \varphi$, such that

$$\lim_{R \to +\infty} f(\lambda(U[v] + \sqrt{-1} R\theta^a \wedge \bar{\theta}^a)) > \psi, \quad (1.17)$$

where $\theta^a = dr + \sqrt{-1} \eta$ and $\bar{\theta}^a = dr - \sqrt{-1} \eta$. Then we can construct a basic admissible subsolution of Dirichlet problem (1.1). A typical example of (1.17) is the function $f$ satisfying (1.11). We shall point out that the assumption $(\lambda(U[v]^T), R) \in \Gamma$ and Lemma 6.1 below ensure that the subsolution $v$ constructed in (6.1) is admissible. Please refer to Section 6 for more details.

In Theorem 1.3 below, we show that condition (1.17) is sufficient to solve (non-degenerate) Dirichlet problem (1.1). Moreover, condition (1.17) can be viewed as a cone condition. We also refer the readers to [16, 27, 48, 49, 50] for related cone conditions of other fully nonlinear elliptic equations.

**Theorem 1.3.** Let $\psi \in C^{k+\alpha}_{B}(\bar{M})$ and $\varphi_a, \varphi_b \in C^{k+2,\alpha}_B(M)$, $k \geq 2$, $\forall 0 < \alpha < 1$. Suppose, in addition conditions (1.2), (1.5) and (1.13) hold, that there exists a basic function $v \in C_0(\bar{\chi})$ satisfying (1.17). Then the Dirichlet problem admits a unique basic admissible solution $v \in C^{k+2,\alpha}_B(M)$.

From Theorem 1.3 and Lemma 6.1 below, we know that the solvability of Dirichlet problem (1.1) is almost determined by the action of $g[v]$ on the pullback contact bundle $P^*\mathcal{D}$.

Let us turn our attention to degenerate fully nonlinear elliptic equations. Degenerate elliptic equations are crucial for understanding both certain geometric objects in differential geometry and analysis of PDEs, c.f. [2, 3, 6, 9, 25, 30, 31, 33, 40, 37, 45, 47]. Motivated by Donaldson’s conjecture in Kähler geometry [12, 41, 47, 9], Guan and Zhang [32, 43] studied the geodesic equation in the space of Sasakian metrics $\mathcal{H}$,

$$\begin{cases}
\{\Omega[u]\}^n = 0 & \text{in } \text{Int}(\bar{M}), \\
u|_{r=1} = \varphi_1, \\
u|_{r=\frac{3}{2}} = \varphi_{\frac{3}{2}},
\end{cases} \quad (1.18)$$

where $\varphi_1, \varphi_{\frac{3}{2}} \in \mathcal{H} := \{\varphi \in C^\infty_B(M) : \omega^T + \frac{\sqrt{-1}}{2} \partial_B \bar{\partial}_B \varphi > 0\}$ and $\Omega[u] = \bar{\omega} + \frac{\pi}{2} \sqrt{-1} (\partial \bar{\partial} u - \frac{1}{n} \partial u \partial \bar{\partial}).$ Guan and Zhang proved that the weak solutions of geodesic equations (1.18) are of class $C^{1,\alpha}$, which extended Chen’s work to the Sasakian setting.

In this paper, we also study the following Dirichlet problem of degenerate fully nonlinear elliptic equations

$$\begin{cases}
f(\lambda(\{U_{ij}[u]\})) = \psi & \text{in } \text{Int}(\bar{M}), \\
\lambda(\{U_{ij}[u]\}) \in \bar{\Gamma}, \\
u|_{r=a} = \varphi_a, \\
u|_{r=b} = \varphi_b,
\end{cases} \quad (1.19)$$

where $\varphi_a, \varphi_b \in C^{1,\gamma}_B(M)$ and $\psi \in C^{2,\gamma}_B(\bar{M})$ with $\delta_{\psi,f} \geq 0$ for some $\gamma \in (0, 1)$.

The main theorem for degenerate equations can be stated as follows.
Theorem 1.4. Suppose, in addition conditions \((1.2)-(1.3), (1.5)\) and \((1.7)\) hold, that condition \((1.13)\) holds for \(a_i(z, \psi) = a_i(z), \ |b(z, \psi)| \leq C_{11}, \) where \(C_{11}\) is a uniform positive constant. If there exists a \textbf{basic} function \(v \in C_\gamma(\bar{\gamma})\) such that \((1.17)\) holds for Dirichlet problem \((1.19)\), then Dirichlet problem \((1.19)\) admits a (weakly) \textbf{basic} solution \(u \in C^{1,\alpha}_{\bar{B}}(M), \ \forall 0 < \alpha < 1, \) with \(\lambda(U[u]) \in \bar{\Gamma}\) and \(\Delta u \in L^\infty(M)\).

Clearly, the geodesic equation in \(\mathcal{H}\) satisfies all the above assumptions.

This paper is organized as follows. In Section 2, we briefly outline some knowledge of Sasakian manifolds and some useful lemmas. In Section 3 we establish second order estimates for admissible solutions. In Section 4 we derive \textbf{a priori} boundary estimates precisely if equation \((1.1)\) can be rewritten as the form \((1.13)\). Gradient estimate will be given in Section 5 where we use the blow-up argument. Finally, \textbf{basic} admissible subsolutions are constructed under condition \((1.17)\). Moreover, the existence of the solutions can be proved by the method of continuity.

2. Preliminaries. In this section, we briefly outline some knowledge of Sasakian manifolds and some useful lemmas.

2.1. Sasakian manifold. We provide some background of Sasakian manifolds here. We also refer the reader to [4, 21] for more extensive treatment on the subject.

A Sasakian manifold \((M, \xi, \eta, \Phi, g)\) is a \((2n-1)\)-dimensional Riemannian manifold with the property that the cone manifold \((C(M), \bar{g}) = (M \times \mathbb{R}^+, r^2 g + dr^2)\) is Kähler. An important fact is that \(\Phi\) determines a complex structure on the contact sub-bundle \(D = \ker\{\eta}\). Furthermore, \((D, \Phi|_D, d\eta)\) provides \(M\) with a transverse Kähler structure admitting a Kähler form \(\frac{1}{2} d\eta\) and a metric \(g^T\) defined by \(g^T(\cdot, \cdot) = \frac{1}{2} d\eta(\cdot, \Phi\cdot)\). The complexification \(D^\mathbb{C}\) of the sub-bundle \(D\) can be decomposed into its eigenspaces with respect to \(\Phi|_D\) as \(D^\mathbb{C} = D^{1,0} \oplus D^{0,1}\).

It is easy to see that the exterior differential preserves \textbf{basic} forms. The transverse complex structure follows the splitting of the complexification of the bundles of \textbf{basic} \(p\)-forms \(\wedge^p_B(M)\) on \(M\),

\[
\wedge^p_B(M) \otimes \mathbb{C} = \oplus_{i+j=p} \wedge^{i,j}_B(M),
\]

where \(\wedge^{i,j}_B(M)\) denotes the bundle of \textbf{basic} forms of type \((i, j)\). Set \(d_B = d|_{\wedge^{i,j}_B}\), we then can decompose \(d|_B = \partial_B + \bar{\partial}_B\), where \(\partial_B : \wedge^{i,j}_B \rightarrow \wedge^{i+1,j}_B, \bar{\partial}_B : \wedge^{i,j}_B \rightarrow \wedge^{i,j+1}_B\). Furthermore, \(\partial_B^2 = \bar{\partial}_B^2 = 0, \partial_B \bar{\partial}_B + \bar{\partial}_B \partial_B = 0\).

Definition 2.1 (Basic p-form). A \(p\)-form \(\theta\) on a Sasakian manifold \((M, g)\) is called \textbf{basic} if \(i_\xi \theta = 0\) and \(L_\xi \theta = 0\), where \(i_\xi\) is the contraction with the Reeb field \(\xi\) and \(L_\xi\) is the Lie derivative with respect to \(\xi\).

The Sasakian structure \((\xi, \eta, \Phi, g)\) of \((M, g)\) determines the following almost complex structure on \(C(M)\):

\[
J(Y) = \Phi(Y) - \eta(Y)r \frac{\partial}{\partial r}, \ J(r \frac{\partial}{\partial r}) = \xi,
\]

which turns \((C(M), \bar{g}, J)\) into a Kähler manifold.

From now on, \(P^* \eta\) and \(P^* d\eta\) will be used to denote pull-backs by \(\eta\) and \(d\eta\), respectively, where \(P : C(M) \rightarrow M\) is the projective map.

As in the Kähler setting, the Sasakian metric can be locally generated by a free real function of \(2(n-1)\) variables. More precisely, for any \(p \in M\), there is a local
We know that a basic function $h$ and a local coordinate chart $(z^1, \cdots, z^{n-1}, x) \in \mathbb{C}^{n-1} \times \mathbb{R}$ on a small neighborhood $U$ around $p$ such that

$$
\left\{
\begin{array}{l}
\xi = \frac{\partial}{\partial x}, \ g = \eta \otimes \eta + 2h_{ij} dz^i \otimes d\bar{z}^j, \ \eta = dx - \sqrt{-1}(h_{ij} dz^i - h_{ij} d\bar{z}^j), \\
\Phi = \sum_{i=1}^{n-1} \left\{ \sqrt{-1}(\frac{\partial}{\partial z^i} + \sqrt{-1}h_i \frac{\partial}{\partial x}) \otimes dz^i - \sqrt{-1}(\frac{\partial}{\partial \bar{z}^i} - \sqrt{-1}h_i \frac{\partial}{\partial x}) \otimes d\bar{z}^i \right\},
\end{array}
\right.
$$

(2.2)

and $D \otimes \mathbb{C}$ is spanned by

$$
X_i = \frac{\partial}{\partial z^i} + \sqrt{-1}h_i \frac{\partial}{\partial x}, \bar{X}_i = \frac{\partial}{\partial \bar{z}^i} - \sqrt{-1}h_i \frac{\partial}{\partial x}, 1 \leq i \leq n - 1,
$$

(2.3)

where $2dz^i d\bar{z}^j = dz^i \otimes d\bar{z}^j + d\bar{z}^i \otimes dz^j$, $h_i = \frac{\partial h}{\partial z^i}$, $h_{ij} = \frac{\partial^2 h}{\partial z^i \partial z^j}$. Then

$$
\Phi X_i = \sqrt{-1}X_i, \ \Phi \bar{X}_i = -\sqrt{-1}\bar{X}_i, i < n.
$$

Thus

$$
JX_i = \sqrt{-1}X_i, \ J\bar{X}_i = -\sqrt{-1}\bar{X}_i \text{ for } i = 1, \cdots, n - 1.
$$

(2.4)

Moreover, one can change the local coordinates to normal coordinates such that

$$
h_i(p) = 0, h_{ij}(p) = \delta_{ij} \text{ and } dh_{ij}\big|_p = 0.
$$

(2.5)

We refer the reader to Godliniski, Kopczynski and Nurowski [21] for more details.

Then the transverse Kähler form is given by

$$
\omega^T = \frac{1}{2} d\eta = \sum_{i,j=1}^{n-1} \sqrt{-1} h_{ij}^2 dz^i \wedge d\bar{z}^j.
$$

(2.6)

For the normal local coordinate chart $(z^1, \cdots, z^{n-1}, \bar{z})$ on a Sasakian manifold $(M, \eta, \xi, \Phi, g)$, set

$$
(z^1, \cdots, z^{n-1}, \bar{z}) \text{ on } U \times \mathbb{R}^+ \subset C(M), \text{ where } \bar{z} = r + \sqrt{-1}x,
$$

(2.7)

and

$$
X_n = \frac{1}{2} \left( \frac{\partial}{\partial r} - \frac{1}{r} \frac{\partial}{\partial \bar{z}} \right), \bar{X}_n = \frac{1}{2} \left( \frac{\partial}{\partial r} + \frac{1}{r} \frac{\partial}{\partial \bar{z}} \right).
$$

(2.8)

where $h$ is the Sasakian potential function (which is basic). Then

$$
JX_n = \sqrt{-1}X_n, \ J\bar{X}_n = -\sqrt{-1}\bar{X}_n.
$$

(2.9)

It is easy to verify that

$$
\left\{
\begin{array}{l}
[X_n, \bar{X}_n] = -\frac{1}{2} \sqrt{-1}r^{-2} \xi, \ [X_i, \bar{X}_n] = 0, \ [X_i, \bar{X}_j] = -2\sqrt{-1}h_{ij} \xi, \\
[X_i, X_j] = [X_i, X_n] = [\bar{X}_i, \bar{X}_j] = 0, \\
[\xi, X_i] = [\xi, \bar{X}_i] = [\xi, X_n] = [\xi, \bar{X}_n] = 0 \text{ for } i, j < n.
\end{array}
\right.
$$

(2.10)

Set

$$
\theta^i = dz^i, 1 \leq i \leq n - 1, \ \theta^n = dr + \sqrt{-1}r \eta.
$$

(2.11)

We know that $\{\theta^i, \bar{\theta}^j\}$ is the dual basis of $\{X_i, \bar{X}_j\}$. 

2. Some lemmas. The key ingredients in this article are the following Lemma 2.2 and Lemma 2.3.

**Lemma 2.2 (28).** Let \( f \) be a symmetric function on the open symmetric and convex cone \( \Gamma \). Suppose that \( f \) satisfies \([1.2]-[1.4]\) and \( \Gamma_n \subseteq \Gamma \subseteq \Gamma_1 \). Let \( K \) be a compact subset of \( \Gamma \) and \( \beta > 0 \). There is a constant \( \varepsilon > 0 \) such that, for any \( \mu \in K \) and \( \lambda \in \Gamma \), when \( |\nu_{\mu} - \nu_{\lambda}| \geq \beta \),

\[
\sum f_i(\lambda)(\mu_i - \lambda_i) \geq f(\mu) - f(\lambda) + \varepsilon(1 + \sum f_i(\lambda)),
\]

where \( \nu_{\lambda} = Df(\lambda)/|Df(\lambda)| \) denotes the unit normal vector to the level surface of \( f \) through \( \lambda \).

To overcome the difficulty due to the two different types of complex derivatives, as in Guan and Zhang [33], we establish the following lemma.

**Lemma 2.3.** Let \( u \in C^3(\text{Int}(\bar{M})) \cap C^1(\bar{M}) \) be a admissible solution of Dirichlet problem \([1.1]\). Suppose that \( f \) satisfies \([1.2]-[1.4]\). If the boundary data of \( u \) is **basic** then \( \xi u \equiv 0 \) on \( \bar{M} \).

**Proof.** The proof is the same as that used by Guan and Zhang [33]. Choosing normal coordinates \((\bar{z}^1, \cdots, \bar{z}^{n-1}, x)\) on \( M \) with properties \((2.2)\) and \((2.5)\), we then have local coordinates \((z^1, \cdots, z^{n-1}, \bar{z})\), where \( \bar{z} = r + \sqrt{-1}x \).

We know that \( T^{1,0}M \) is spanned by \( X_i \) for \( i = 1, \cdots, n \), where \( X_i \) are the vectors defined in \((2.3)\) and \((2.8)\). Set \( \bar{\chi} = \sum_{i,j=1}^{n} \sqrt{-1} X_{i,j} \theta^i \wedge \bar{\theta}^j \) and

\[
U[u] = \sum_{i,j=1}^{n} \sqrt{-1} U_{i,j} \theta^i \wedge \bar{\theta}^j,
\]

where \( \theta^i \) is the dual basis of \( X_i \).

Firstly, \( \nabla \bar{\chi} = 0 \) implies that \( \nabla \frac{\partial}{\partial \bar{z}} \bar{\chi}_{i,j} = 0 \). Following Guan and Zhang’s calculations in [33], we have

\[
\frac{\partial}{\partial x}(u_{i,j} - \frac{\partial u}{\partial r} r_{i,j}) = (X_i X_j \frac{\partial u}{\partial x} - \frac{1}{2} [X_i, X_j] \frac{\partial u}{\partial x}) = \frac{1}{2}(X_i X_j \frac{\partial u}{\partial x} + X_j X_i \frac{\partial u}{\partial x}).
\]

Since \( \nabla \) is the Chern connection of the Kähler manifold \((\bar{M}, J, \bar{g})\), we know that

\[
\nabla_{X_i} \bar{X}_j + \nabla_{\bar{X}_j} X_i = [X_i, \bar{X}_j],
\]

and

\[
\nabla_{X_i} \bar{X}_j + \nabla_{\bar{X}_j} X_i = \sqrt{-1} J([X_i, \bar{X}_j]).
\]

Combining \((2.10)\) with \( J(\frac{\partial}{\partial x}) = -r \frac{\partial}{\partial r} \), we know that

\[
\nabla_{X_i} \bar{X}_j + \nabla_{\bar{X}_j} X_i = -2r^{-1} g_{i,j}^* \frac{\partial}{\partial r},
\]

where \( g^*(\cdot, \cdot) = \omega^*(\cdot, J\cdot) \), \( \omega^* = \frac{1}{2} r^2 P^* d\eta + \frac{1}{2} r^2 d\eta \wedge \eta \).

Then we have

\[
\left\{ \begin{array}{ll}
F^{i,j} \left[ 2\partial \bar{\partial} (\xi u)(X_i, \bar{X}_j) - r^{-1} g^*_{i,j} \frac{\partial}{\partial r} (\xi u) \right] = 0 & \text{in } \text{Int}(\bar{M}), \\
\xi u = 0 & \text{on } \partial \bar{M},
\end{array} \right.
\]

where \( F^{i,j} = \frac{\partial F^i_{\partial \xi_j}}{\partial (\xi_j)} \). Thus \( \xi u \equiv 0 \) in \( \bar{M} \) by the maximum principle for linear elliptic equations. \( \square \)
Throughout this paper we shall use the following notations: for a given Hermitian matrix $A = \{a_{ij}\}$, we write

$$F^{ij}(A) = \frac{\partial F}{\partial a_{ij}}(A),$$

$$F^{ij,k\bar{l}}(A) = \frac{\partial^2 F}{\partial a_{ij} \partial a_{k\bar{l}}}(A).$$

The matrix $\{F^{ij}\}$ has eigenvalues $f_1, \ldots, f_n$ and is positive definite by assumption (1.2), while (1.3) implies that $F$ is a concave function of $a_{ij}$. Moreover, when $A$ is diagonal, so is $\{F^{ij}(A)\}$, see e.g. [5]. Furthermore,

$$F^{ij}(A)a_{ij} = \sum f_i \lambda_i, \quad F^{ij}(A)a_{k\bar{l}} = \sum f_i \lambda_i^2.$$

In local complex coordinates $z = (z^1, \ldots, z^n)$, we shall use notations such as

$$v_i = \nabla_{\bar{z}^i}v, \quad v_{ij} = \Delta_{\bar{z}^i, \bar{z}^j}v, \quad v_{i\bar{j}} = \nabla_{\bar{z}^i} \nabla_{\bar{z}^j}v, \ldots.$$

We also write

$$U_{ij} = U_{ij}[u] = \bar{\chi}_{ij} + u_{ij} - \frac{\partial u}{\partial r}r_{ij},$$

$$\bar{U}_{i\bar{j}} = U_{i\bar{j}}[\bar{u}] = \bar{\chi}_{i\bar{j}} + u_{i\bar{j}} - \frac{\partial u}{\partial r}r_{i\bar{j}}.$$

3. Second order estimates. In this section we give the proof of Theorem 1.1 which derives second order estimates.

Firstly, we give $C^0$-estimate and boundary gradient estimate assuming the existence of admissible subsolution $u$. Let $h$ be a $C^2$ solution to

$$\begin{cases}
\frac{1}{2}(\Delta_{\bar{g}}h - \frac{\partial h}{\partial r}\Delta_{\bar{g}}r) + \bar{g}^{ij}\bar{\chi}_{ij} \leq 0 & \text{in } \bar{M}, \\
h = \varphi & \text{on } \partial \bar{M},
\end{cases} \quad (3.1)$$

where $\Delta_{\bar{g}}$ is the standard Laplacian with respect to the Levi-Civita connection. We refer the readers to [52] for the solvability of (3.1).

By the maximum principle

$$\begin{cases}
u \leq u \leq h & \text{in } \bar{M}, \\
u = u = h = \varphi & \text{on } \partial \bar{M}.
\end{cases} \quad (3.2)$$

Hence

$$\sup_{\bar{M}} |u| + \sup_{\partial \bar{M}} |\nabla u| \bar{g} \leq C^*, \quad (3.3)$$

where the constant $C^*$ depends only on $u$ and $h$.

Next we prove Theorem 1.1.

Proof of Theorem 1.1. Firstly, Lemma 2.3 concludes that $u$ is basic. Denote the eigenvalues of the matrix $A = \{A^i\} = \{\bar{g}^{ij}U_{ij}\}$ by $(\lambda_1, \ldots, \lambda_n)$, and $\lambda_1 : \bar{M} \to \mathbb{R}$ is the largest eigenvalue and $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_n$ at each point. We want to apply the maximum principle to $H$,

$$H := \lambda_1 e^\phi,$$

where $\{\bar{g}^{ij}\} = \{\bar{g}_{pq}\}^{-1}$ and $\phi$ is the test function to be chosen later. Suppose $H$ achieves its maximum at an interior point $p_0 = (x_0, r_0), x_0 \in \bar{M}, a < r_0 < b$. 

also achieves its maximum at the origin $p_0$, say $(w^1, \cdots, w^n)$, such that under this coordinate, at $p_0$

$$U_{ij} = \lambda_i \delta_{ij}, F^{ij} = f_i \delta_{ij}, \bar{g}_{ij} = \delta_{ij}, d \bar{g}_{ij} = 0.$$ 

From now on, we denote by $\partial_i = \frac{\partial}{\partial w^i}$, $\partial \bar{i} = \frac{\partial}{\partial \bar{w}^i}$.

In what follows, we use derivatives with respect to the Chern connection of $\bar{g}$ and the computations will be given at the origin $p_0$, and we assume

$$\lambda_1(p_0) \geq 1 + \sup_M |\nabla u|^2_{\bar{g}}.$$

(3.4)

Since the eigenvalues of $A$ do not need to be distinct at the point $p_0$, $H$ may only be continuous. To circumvent this difficulty we use the perturbation argument used by Collins, Jacob and Yau [11], which is a modification of the treatment of Székelyhidi [50]. To do this, we can suppose that $B$ is sufficiently small. Moreover, in our fixed local coordinates, $B$ is a constant diagonal matrix $B^p_q$ with real entries satisfying $B^1_1 = B^n_n = 0$ and $B^a_{n-1} < \cdots < B^2_2 < 0$. Then we define the matrix $\tilde{A} = A + B$ with the eigenvalues $\tilde{\lambda} = (\tilde{\lambda}_1, \cdots, \tilde{\lambda}_n)$. At the origin $p_0$, $\tilde{\lambda}_1 = \lambda_1, \tilde{\lambda}_i = \lambda_i + B^1_i$ if $i \geq 2$ and the eigenvalues of $\tilde{A}$ define $C^2$-functions near the origin. Notice

$$\tilde{H} = \tilde{\lambda}_1 e^\phi$$

also achieves its maximum at the origin $p_0$ (we may assume $\lambda_1(p_0) = \tilde{\lambda}_1(p_0) > 1$). Then we have at $p_0$

$$\begin{cases} 
\frac{\tilde{\lambda}_1}{\lambda_1} + \phi_k = 0, \\
\frac{\tilde{\lambda}_1}{\lambda_1} - |\tilde{\lambda}_1|_2^2 + \phi_{kk} \leq 0.
\end{cases}$$

(3.5)

By straightforward calculations, one obtains

$$\tilde{\lambda}_{1,k} = U_{11k} + (B^1_{1})_{k}.$$ (3.6)

Moreover

$$\tilde{\lambda}_{1,\bar{k}} = U_{11\bar{k}} + \sum_{p>1} \frac{|U_{p1k}|^2 + |U_{1pk}|^2}{\lambda_1 - \lambda_p} - (B^1_{1})_{\bar{k}}$$

$$+ 2 \text{Re} \sum_{p>1} \frac{U_{p1k}(B^p_{1})_{\bar{k}} + U_{1pk}(B^p_{1})_{k}}{\lambda_1 - \lambda_p} + \tilde{\lambda}_{pq,\bar{r}s} (B^p_{q})_{k}(B^r_{s})_{\bar{k}},$$

(3.7)

where

$$\tilde{\lambda}_{pq,\bar{r}s} = (1 - \delta_{1p}) \frac{\delta_{1q} \delta_{1r} \delta_{pq}}{\lambda_1 - \lambda_p} + (1 - \delta_{1q}) \frac{\delta_{1p} \delta_{1r} \delta_{r\bar{q}}}{\lambda_1 - \lambda_r}.$$ c.f. [11] [50].

Evaluating this expression at $p_0$, and using that $B$ is constant, $B^1_1 = 0$, and that we are working in normal coordinates for $\bar{g}$, we have

$$\begin{cases} 
\tilde{\lambda}_{1,k} = U_{11k}, \\
U_{ijk} = \tilde{\lambda}_{ij,k} + u_{ijk} - \frac{\partial u}{\partial r} k r_{ij} - \frac{\partial u}{\partial \bar{r}} \bar{r}_{ij},
\end{cases}$$

(3.8)

and

$$\tilde{\lambda}_{1,\bar{k}} = U_{11\bar{k}} + \sum_{p>1} \frac{|U_{p1k}|^2 + |U_{1pk}|^2}{\lambda_1 - \lambda_p}.$$ (3.9)
We need to estimate \( \hat{\lambda}_1 - \hat{\lambda}_p \) near the origin \( p_0 \) for \( p > 1 \). Since \( \Gamma_n \subseteq \bar{\Gamma} \subset \Gamma_1 \), we may assume \( \sum \lambda_i > 0 \) (otherwise we are done). Then \( |\lambda_i| \leq (n-1)\lambda_1 \) for all \( i \), and so \( (\hat{\lambda}_1 - \hat{\lambda}_p)^{-1} \geq \left( \frac{n\lambda_1}{1} \right)^{-1} \). As in [11, 50], we know that

\[
\hat{\lambda}_{1,kk} \geq U_{1kk} + \frac{1}{2n\lambda_1} \sum_{p>1} (|U_{pikk}|^2 + |U_{pik}|^2) - C.
\]  

(3.10)

Then there exists two uniform constant \( C_0 \) and \( C_1 \) depending only on known data under control such that

\[
\hat{\lambda}_{1,kk} \geq U_{1kk} + \frac{1}{4n\lambda_1} \sum_{p>1} (|u_{pikk}|^2 + |u_{pik}|^2) - \frac{C_1}{2n\lambda_1} \sum_{p>1} \left( 1 + |\frac{\partial u}{\partial r}|^2 + |\frac{\partial u}{\partial r}|^2 \right)^2 - C_0,
\]  

(3.11)

where we use the inequality such that \( |h + g|^2 \geq \frac{1}{2}|h|^2 - |g|^2 \).

We compute

\[
U_{1kk} = U_{k11} + (\bar{X}_{11,kk} - \bar{X}_{kk,11}) + (u_{1kk} - u_{kk11}) - \frac{\partial u}{\partial r} (r_{1kk} - r_{kk11}) + \left( \frac{\partial u}{\partial r} \right)_{11} r_{kk11} - \left( \frac{\partial u}{\partial r} \right)_{kk} r_{11}
\]  

(3.12)

+ \left( \frac{\partial u}{\partial r} \right)_{r_{kk11} - \left( \frac{\partial u}{\partial r} \right)_{k_{r_{11}}} (3.12)

Differentiating equation (1.1) twice (using the covariant derivative), we have at \( p_0 \)

\[
\{ F^{kk} U_{kk} = \psi_k, F^{kk} U_{kkl} = \psi_{kl}, F^{kk} U_{kkl1} + F^{ij,pq} U_{ij1l} U_{pql} = \psi_{11} \}.
\]

In this local holomorphic coordinates \((w^1, \cdots, w^n)\), there is a transform \( T \) of vectors \((2.3) \) and \((2.8) \) such that

\[
(X_1, \cdots, X_n) T = (\partial_1, \cdots, \partial_n),
\]  

(3.13)

here we use \((2.4) \) and \((2.9) \). Near \( p_0 \), the transform \( T \) allows one to present \( \frac{\partial}{\partial r} \) by the linear combinations of \( X_i \) thanks to \((2.3) \) and \((2.8) \), i.e. \( \frac{\partial}{\partial r} = \sum_{k=1}^n T^j_k X_k \).

We may set \( X_i = \sum_{k=1}^n S^i_j k, \) where \( T^j_k S^i_j = \delta_{ji} \). Note that \( u \) is basic,

\[
\frac{\partial u}{\partial r} = 2 X_n u = 2 \bar{X} n u,
\]

then by \((3.4) \) one derives

\[
|\frac{1}{2} (\frac{\partial u}{\partial r})_k| = \frac{1}{2} (\frac{\partial u}{\partial r})_k = \frac{1}{2} (\frac{\partial u}{\partial r})(\frac{\partial u}{\partial r}) \leq C_A \lambda_1,
\]  

(3.14)

where \( C_A \) is a positive uniform constant depending on sup \( |T| \) and other known data.

We know that \([X_n, \bar{X}_\alpha] = [X_n, X_\alpha] = [X_n, \bar{X}_n] = 0,[X_n, \bar{X}_\beta] = -2\sqrt{-1} \frac{\partial^2 h}{\partial \bar{z} \partial z} \xi \) for \( 1 \leq \alpha, \beta \leq n - 1 \); \([X_n, \bar{X}_n] = -\frac{1}{2} r^{-2} \xi \). Thus

\[
[X_k, \bar{X}_i] u = [\bar{X}_k, \bar{X}_i] u = [X_k, X_i] u = 0, \text{since } u \text{ is basic.}
\]

Then by straightforward calculations, one obtains

\[
X_k \bar{X}_i X_n u = X_n X_k \bar{X}_i u,
\]  

(3.15)

here we use \([\xi, X_i] = [\xi, \bar{X}_i] = 0 \).
Note that
\[ \partial_t \partial_i v = T_k^i T_i^j (X_k X_i v - (\nabla X_k X_i)v) \] for \( v \in C^2(\text{Int}(M)) \).

Applying (3.13) and (3.15), one obtains
\[ (\frac{\partial u}{\partial r})_{\bar{\bar{ij}}} = \nabla_{\bar{ii}} u_{\bar{ii}} + O(\lambda_1 + |\nabla u|). \] (3.16)

Commuting derivatives
\[ u_{\bar{ij}k} - u_{\bar{ik}j} = 0, u_{\bar{ij}k} - u_{\bar{ki}j} = -\bar{g}^{im} R_{i\bar{j}k\bar{m}} u_l \] (3.17)
and
\[ u_{\bar{ij}k} - u_{\bar{kij}} = \bar{g}^{qi}(R_{k\bar{i}q\bar{m}} u_{\bar{pj}} - R_{k\bar{ij}q} u_{\bar{pi}}). \] (3.18)

Applying (3.13), (3.15), (3.8) and (3.16)-(3.18), we have
\[ F_{\bar{k}\bar{k}} \bar{\lambda}_{1,kk} \geq -F_{ij,\bar{m}} U_{ij} U_{\bar{m}1} - \lambda_1 \frac{\partial \phi}{\partial r} F_{\bar{k}\bar{k}} - C \lambda_1 \left( 1 + \sum F^{ii} \right). \]

Denote by \( L \) the linearized operator of equation (1.1),
\[ L v = F^{ij} \left( v_{ij} - \frac{\partial u}{\partial r} T_{ij} \right) \] for \( v \in C^2(M) \). (3.19)

Then we have by (3.5) and (3.15)
\[ L(\log \bar{\lambda}_1) \geq - \frac{F_{ij,\bar{m}} U_{ij} U_{\bar{m}1}}{\lambda_1} - F_{\bar{k}\bar{k}} |U_{\bar{1}1\bar{k}}|^2 \lambda_1 - C \left( 1 + \sum F^{ii} \right). \]

By (3.17), (3.4) and (3.14), we obtain
\[ |U_{\bar{1}1\bar{k}}|^2 \leq |U_{k11}|^2 + C_1 (1 + \lambda_1^2) + C_1 (1 + \lambda_1^2) |U_{11\bar{k}}|, \forall \bar{k} \geq 2, \] (3.20)
where \( C_1 \) depends on the constant \( C_A \) in (3.14) and other known data.

We set \( \Psi : [0, +\infty) \to \mathbb{R} \),
\[ \Psi(x) = \frac{A}{(1 + x)^N}, \] (3.21)
where \( A \geq 1, N \in \mathbb{N} \) to be chosen later. Let
\[ \phi = \Phi(|\nabla u|^2) + \Psi(u - u), \]
As in (33), we set
\[ \Phi(t) = -\frac{1}{2} \log(1 - \frac{t}{2K}), \] where \( K = 1 + 2n \sup_M (|\nabla u|^2 + |\nabla u|^2). \)

Then we know that
\[ L \phi = \Phi' L(|\nabla u|^2) + \Psi' L(u - u) + \Phi'' F^{ii}(|\nabla u|^2)|^2 + \Psi'' F^{ii}|(u - u)|^2, \]
and
\[ \Phi'' = 2\Phi'^2, \quad (4K)^{-1} < \Phi' < (2K)^{-1} \text{ for } x \in [0, \sup_M |\nabla u|^2]. \]

Differentiating equation (1.1) one obtains
\[ F^{ii} \left( u_{\bar{ii}k} - \frac{\partial u}{\partial r} T_{\bar{ii}k} \right) = \psi_k - F^{ii} \bar{\chi}_{\bar{ii}, k}. \]

Then by Cauchy-Schwarz inequality
\[ L(|\nabla u|^2) \geq \frac{7}{8} F^{ii} \left( |u_{kk}|^2 + |u_{\bar{k}i}|^2 \right) - C(1 + |\nabla u|^2) \left( 1 + \sum F^{ii} \right). \] (3.22)
Therefore
\[
0 \geq \mathcal{L} \log \lambda_1 + \mathcal{L} \phi \geq \frac{F^{ij,l}U_{ij}U_{l\bar{i}\bar{m}}}{\lambda_1} - \frac{F^{\bar{i}i}|U_{1i}|^2}{\lambda_1^2} + \Psi' \mathcal{L}(u-\bar{u})
\]
\[
+ \frac{3}{4} \Phi' F^{\bar{i}i}(|u_{ki}|^2 + |u_{k\bar{i}}|^2) + \Psi'' F^{\bar{i}i}|(u-\bar{u})_i|^2 + \Phi'' F^{\bar{i}i}(|\nabla u|^2)_i^2
\]
\[
- C \left(1 + \sum F^{\bar{i}i}\right) - CF^{k\bar{k}}|U_{1i}|\frac{|U_{1i}|^2}{\lambda_1}.
\]  
(3.23)

Note that in this computation $\lambda_1$ denotes the largest eigenvalue of the perturbed endomorphism $\tilde{A} = A + B$. At the origin $p_0$ where we compute, $\lambda_1$ coincides the largest eigenvalue of $A$ with respect to the metric $\{\tilde{g}_{ij}\}$, but at nearby points it is a small perturbation. We would take $B \to 0$, and obtain the above differential inequality (3.23) for the largest eigenvalue of $A$ as well, but this would only hold in a viscosity sense because the largest eigenvalue of $A$ may not be $C^2$ at the origin $p_0$, if some eigenvalues coincide.

**Case I.** Assume that $\delta \lambda_1 \geq -\lambda_n (0 < \delta \ll \frac{1}{2})$. Set
\[
I = \left\{i : F^{\bar{i}i} > \delta^{-1} F^{1\bar{1}}\right\} \text{ and } J = \left\{i : F^{\bar{i}i} \leq \delta^{-1} F^{1\bar{1}}\right\}.
\]
Clearly $1 \in J$. The identity (3.5) implies that for fixed $k$ one has
\[
\frac{|\lambda_{1,k}|^2}{\lambda_1^2} \geq -2 |\Psi''| (u-\bar{u})_k^2 - 2 |\nabla u|^2_k |(u-\bar{u})_k^2 - \Phi'' |(\nabla u)|^2_k^2.
\]  
(3.24)

The assumption $\delta \lambda_1 \geq -\lambda_n$ implies that
\[
\frac{1 - \delta}{\lambda_1 - \lambda_k} \geq \frac{1 - 2\delta}{\lambda_1}.
\]
Then
\[
-F^{ij,l\bar{m}}U_{ij}U_{l\bar{i}\bar{m}} \geq \sum_{k \in I} \frac{F^{k\bar{k}} - F^{1\bar{1}}}{\lambda_1 - \lambda_k} |U_{k1i}|^2
\]
\[
\geq \frac{1 - 2\delta}{\lambda_1} \sum_{k \in I} F^{k\bar{k}} |U_{k1i}|^2.
\]  
(3.25)

We use some of the good term $\Psi'' F^{k\bar{k}} |(u-\bar{u})_k|^2$ in (3.23) to control the last term of (3.24). So we choose small $\delta$ such that
\[
4\delta \Psi'' < \frac{1}{2} \Psi''.
\]
For $k \in J$, we know that $F^{k\bar{k}} \leq \delta^{-1} F^{1\bar{1}}$. Then
\[
- \sum_{k \in J} \frac{F^{k\bar{k}} |\lambda_{1,k}|^2}{\lambda_1^2} \geq -2 |\Psi''| \delta^{-1} F^{1\bar{1}} K - \Phi'' \sum_{k \in J} F^{k\bar{k}} |(\nabla u)|^2_k^2.
\]  
(3.26)

By (3.20), (3.23)–(3.26), we have
\[
0 \geq \frac{1}{8K} F^{k\bar{k}} (|u_{ik}|^2 + |u_{k\bar{i}}|^2) + \frac{1}{2} \Psi'' F^{k\bar{k}} |(u-\bar{u})_k|^2 + \Psi' \mathcal{L}(u-\bar{u})
\]
\[
- CF^{k\bar{k}} |U_{1i}| - 2 \Psi'' \delta^{-1} F^{1\bar{1}} K - C \left(1 + \sum F^{\bar{i}i}\right).
\]
From (3.25) one derives
\[ F^{kk\lambda_1k\lambda_1} = F^{kk\lambda_1} = F^{kk\lambda_1} |\Psi'(u - u)_k + \Phi'(|\nabla u|^2)_k | \leq \frac{1}{2K^{1/2}} F^{kk\lambda_1}(|u_{ik}| + |u_{ik}|) + |\Psi'F^{kk\lambda_1}|(u - u)_k. \]

Therefore
\[ 0 \geq \Psi'_C(u - u) + \frac{1}{64K} F^{kk\lambda_1}(|u_{ik}|^2 + |u_{ik}|^2) + F^{11\lambda_1} \left( \frac{\lambda_1^2}{128K} - 2\delta^{-1} \Psi'^2 K \right) - C^2 |\Psi'|^2 \frac{1}{2|\Psi'|} \sum F^{ii} - C_o \left( 1 + \sum F^{ii} \right), \tag{3.27} \]

where we use the elementary inequality \( ax^2 - bx \geq -\frac{b^2}{4a} \) for \( a > 0 \).

Denote by \( \lambda[v] := \lambda(\{U_{ij}[v]\}) \) for convenience. We briefly outline a useful property of the symmetric function \( f \) as follows, c.f. [24].

For \( |\lambda| \geq 1 + \sup_\overline{M} |\lambda[u]| \), we have
\[ |\lambda|^2 \sum f_i \geq \frac{4b_0|\lambda|}{\lambda} > 0, \text{ where } b_0 = f(|\lambda|1) - \sup_{z \in \overline{M}} f(\lambda[u](z)). \tag{3.28} \]

Next, we will give the proof of this property.

By the concavity of \( f \), we have
\[ |\lambda| \sum F^{ii} = |\lambda| \sum f_i \geq f(|\lambda|1) - f(\lambda[u]) + \sum f_i \lambda_i \]
\[ \geq f(|\lambda|1) - f(\lambda[u]) - \frac{1}{4|\lambda|} \sum f_i \lambda_i^2 - |\lambda| \sum f_i. \]

Then
\[ |\lambda|^2 \sum f_i \geq \frac{|\lambda|}{2} \left( f(|\lambda|1) - f(\lambda[u]) \right) - \frac{1}{8} \sum f_i \lambda_i^2. \]

Suppose that \( |\lambda| \geq 1 + \sup_\overline{M} |\lambda[u]| \). Then (3.28) holds.

For the admissible subsolution \( u \), \( \lambda[u] \) falls in a compact subset of \( \Gamma \), and
\[ \beta = \frac{1}{2} \min_{\overline{M}} \text{dist}(\nu_{\lambda[u]}, \partial \Gamma_n) > 0. \]

**Subcase (a).** Suppose that \( |\nu_{\lambda[u]} - \nu_{\lambda[u]}| \geq \beta \). Then by Lemma 2.2 we have
\[ \mathcal{L}(u - u) \geq \varepsilon \left( 1 + \sum F_{ij} \tilde{g}_{ij} \right), \]
where \( \varepsilon \) is the positive constant in Lemma 2.2 determined by \( f \) and \( \beta \).

Choosing \( N \gg 1 \) so that \( C^2 |\Psi'| \frac{1}{|\Psi'|} \geq \frac{\varepsilon}{2} \). Then we choose \( A \gg 1 \) such that
\[ -\Psi' \geq \frac{AN}{(1 + \sup_\overline{M}(u - u) - \inf_\overline{M}(u - u))^{N+1}} \gg C_o + 2. \]

Hence (3.27) follows that
\[ \lambda_1 \leq CK. \tag{3.29} \]

**Subcase (b).** Fix the constants \( A, N \) and \( \delta \) as above. Suppose that \( |\nu_{\lambda[u]} - \nu_{\lambda[u]}| < \beta \) and therefore \( \nu_{\lambda[u]} - \beta \overline{1} \in \Gamma_n \). Then
\[ f_i \geq \frac{\beta}{\sqrt{n}} \sum_{j=1}^{n} f_j, \forall i. \]
Note that
\[ |u_{kk}|^2 = |\lambda_k - \chi_{kk} + \frac{\partial u}{\partial r} r_{kk}|^2 \geq \frac{1}{2} |\lambda_k|^2 - C K. \]

Combining it with (3.27) and (3.28), we derive that
\[ \lambda_1 \leq C K. \]  

**Case II.** We assume that \( \delta \lambda_1 < -\lambda_n \) with the constants \( A, N \) and \( \delta \) fixed as in previous case. In particular, \(|\lambda_1| \leq \frac{1}{3} |\lambda_n|\).

\[-\sum F^{kk} \frac{|U_{11kk}|^2}{\lambda_1^2} = -\sum F^{kk} |\Psi'(u - \bar{u})_k + \Phi'(|\nabla u|^2)_k| \]
\[ \geq -2\Psi'^2 \sum F^{kk} |(u - \bar{u})_k|^2 - 2\Phi'^2 \sum F^{kk} |(\nabla u)^2)_k|^2. \]

Then by (3.23)
\[ 0 \geq -2\Psi'^2 F^{ii} |(u - \bar{u})_i|^2 + \Psi' \mathcal{L}(u - \bar{u}) + \frac{1}{8K} F^{ii} |u_{ki}|^2 + |u_{ki}|^2 \]
\[ + \Psi'' F^{ii} |(u - \bar{u})_i|^2 - C \lambda_1^{-1} F^{ii} |U_{11i}| - C(1 + \sum F^{ii}). \]

Note that \( F^{ii} \geq \frac{1}{n} \sum F^{ii} \) and
\[ |u_{ii}|^2 = |\lambda_i - \chi_{ii} + \frac{\partial u}{\partial r} r_{ii}|^2 \geq \frac{1}{2} |\lambda_i|^2 - C K. \]

and
\[ -F^{ii} \frac{|U_{11i}|}{\lambda_1} = -F^{ii} |\phi_i| = -F^{ii} |\Psi'(u - \bar{u})_i + \Phi'(|\nabla u|^2)_i| \]
\[ \geq -|\Psi'| F^{ii} |(u - \bar{u})_i| - \frac{1}{2} F^{ii} |(u_{ki}| + |u_{ki}|). \]

Then
\[ 0 \geq \Psi' \mathcal{L}(u - \bar{u}) - 2\Psi'^2 K \sum F^{ii} + \frac{\delta^2}{64nK} |\lambda_i|^2 \sum F^{ii} - C \left(1 + \sum F^{ii}\right). \]

Thus by (3.28), one obtains
\[ \lambda_1 \leq C K. \]  

Finally, it follows from (3.29), (3.30) and (3.32) that there is a positive constant \( C \) depending on \( \beta^{-1}, \varepsilon^{-1}, |u|_{C^0(\tilde{M})}, |\phi|_{C^{1,1}(\tilde{M})}, |u|_{C^{1,1}(\tilde{M})}, |\tilde{x}|_{C^{1,1}(\tilde{M})}, \) and other data under control (but not on \( |\nabla u|_{C^0(\tilde{M})} \)), such that
\[ \sup_{\tilde{M}} |\Delta u| \leq C \left(1 + \sup_{\tilde{M}} |\nabla u|^2 + \sup_{\partial \tilde{M}} |\Delta u| \right), \]
where \( \varepsilon \) and \( \beta = \frac{1}{2} \min_{\tilde{M}} \text{dist}(\nu_{\lambda_{\tilde{M}}}, \partial \tilde{M}) \) are the constants in Lemma 2.2 \( \square \)

4. **Boundary estimates for second derivatives.** In this section we study the second order boundary estimates for the admissible solution \( u \) via constructing barrier functions.

Given a point \( p \in \partial \tilde{M} \), let \( \rho(z) = \text{dist}_{\tilde{M}}(z, p) \) be the distance function from \( z \) to \( p \) and
\[ \Omega_\delta = \{ z \in \text{Int}(\tilde{M}) : \rho(z) < \delta \}, 0 \leq \delta \ll 1. \]

Denote by \( \sigma \) the distance function to \( \partial \tilde{M} \) with respect to \( \tilde{g} \).
To construct a barrier function, we will follow Guan [22] and employ a barrier function of the form
\[ v = u - u - \frac{N}{2} \sigma^2 + t \sigma \text{ in } \Omega_{\delta_0} \text{ for } 0 < \delta_0 \ll 1, \]
where \( u \in C^2_{\bar{B}}(M) \) is the admissible subsolution of Dirichlet problem (1.1).

The following lemma was first proved in [22] for domains in \( \mathbb{C}^n \) and holds for general compact Hermitian manifolds with smooth boundaries.

**Lemma 4.1.** Suppose that \( f \) satisfies (1.2)–(1.5). If Dirichlet problem (1.1) admits an admissible subsolution \( \overline{u} \in C^2_{\bar{B}}(M) \), then in \( \Omega_{\delta_0} \) \((0 < \delta_0 < 1)\), \( v \geq 0 \) and

(i). \( \mathcal{L}v \leq -\frac{\kappa}{2} F^{ij} \tilde{g}_{ij} \),

(ii). \( \mathcal{L}v \leq -\frac{\kappa \varepsilon}{2(1 + \kappa)} \left( 1 + F^{ij} \tilde{g}_{ij} \right) \),

where \( \kappa \) is the constant in (1.6), and \( \varepsilon \) is the constant in Lemma 2.2.

**Proof.** By the maximum principle, \( u \geq \overline{u} \). The concavity of equation imply \( \mathcal{L}(u - \overline{u}) \geq 0 \) in \( M \). Recall that there is a constant \( \alpha_0 \) such that for small \( \delta_0 \), we have

\[ |\mathcal{L}\sigma| \leq \alpha_0 F^{ij} \tilde{g}_{ij} \text{ in } \Omega_{\delta_0}. \]

Choosing \( 0 < \delta_0 \ll 1 \) such that \( N \sigma \leq N \delta_0 < t, \sigma \) is smooth and \( \frac{1}{2} \leq |\nabla \sigma|_{\tilde{g}} \leq 1 \) in \( \Omega_{\delta_0} \), then we know that \( v \geq 0 \) in \( \Omega_{\delta_0} \) and \( v = 0 \) on \( \partial M \).

Note that
\[ \mathcal{L}v = \mathcal{L}(u - \overline{u}) - NF^{ij} \sigma_i \sigma_j - (t - N \sigma) \mathcal{L}\sigma \leq \mathcal{L}(u - \overline{u}) - NF^{ij} \sigma_i \sigma_j + t \alpha_0 F^{ij} \tilde{g}_{ij}. \]

Denote by \( \lambda[v] := \lambda\{U_{ij}[v]\} \). For the admissible subsolution \( u, \lambda[u] \) falls into a compact set of \( \Gamma \). Set
\[ \beta = \frac{1}{2} \min_M (\nu_{\lambda[u]} \cap \partial \Gamma_n) > 0. \]

In case I, \( |\nu_{\lambda[u]} - \nu_{\lambda[\overline{u}]}| \geq \beta \), then Lemma 2.2 implies that
\[ \mathcal{L}(u - \overline{u}) \leq -\varepsilon \left( 1 + F^{ij} \tilde{g}_{ij} \right), \]
where \( \varepsilon \) is the positive constant in Lemma 2.2. If \( t \leq \frac{\varepsilon}{2\alpha_0} \), then
\[ \mathcal{L}v \leq (t \alpha_0 - \varepsilon) F^{ij} \tilde{g}_{ij} \leq -\frac{\varepsilon}{2} F^{ij} \tilde{g}_{ij} - \varepsilon \leq -\frac{\varepsilon}{2} \left( 1 + F^{ij} \tilde{g}_{ij} \right). \]

Fix \( t \) small enough so that \( t \leq \frac{\varepsilon}{2\alpha_0} \). In case II, \( |\nu_{\lambda[u]} - \nu_{\lambda[\overline{u}]}| < \beta, \nu_{\lambda[u]} - \beta \Gamma \in \Gamma_n \).

Hence
\[ f_i \geq \frac{\beta}{\sqrt{n}} \sum_k f_k. \]

If we choose \( N \geq \frac{16 \sqrt{n} \varepsilon}{\beta} \), then
\[ \mathcal{L}v \leq \left( t \alpha_0 - \frac{\beta N}{16 \sqrt{n}} \right) F^{ij} \tilde{g}_{ij} \leq \left( \frac{\varepsilon}{2} - \frac{\beta N}{16 \sqrt{n}} \right) F^{ij} \tilde{g}_{ij} \]
\[ \leq -\frac{\beta N}{32 \sqrt{n}} F^{ij} \tilde{g}_{ij} \leq -\frac{\varepsilon}{2} F^{ij} \tilde{g}_{ij} \leq -\frac{\varepsilon \kappa}{2(1 + \kappa)} \left( 1 + F^{ij} \tilde{g}_{ij} \right). \]

\( \square \)
For any given point \( p = (q, a) \in M \times \{a\} \) (or \( p = (q, b) \in M \times \{b\} \)), we may pick a local coordinate chart \((z^1, \cdots, z^{n-1}, x)\) around \( q \) with properties \((2.2), (2.5)\) and
\[
\frac{1}{4} \delta_{ij} \leq h_{ij}(z) \leq \delta_{ij}, \sum_{i<n} |h_i|^2(z) \leq 1, \forall z \in U \subset M.
\]
(4.2)
Then we have local coordinates \((z^1, \cdots, z^{n-1}, \tilde{z})\), where \( \tilde{z} = r + \sqrt{-1}x \). Set
\[
V_\delta = U \times [a, a+\delta]( \text{ or } V_\delta = U \times [b-\delta, b]).
\]
Then \( \partial\tilde{M} \cap V_\delta = \{r = a\} \) (or \( r = b\)).

Let \( X_i \) be the vectors defined in \((2.3)\) and \((2.8)\). Then \( \{X_1, \cdots, X_{n-1}, X_n\} \) is a basis of \( T^{1,0}(M) \) and \( \{\theta^1, \cdots, \theta^n\} \) is the dual basis. The Kähler form \( \tilde{\omega} \) of \( (M, g) \) can be written as
\[
\tilde{\omega} = \sqrt{-1} \left( \sum_{i,j=1}^{n-1} r^2 h_{ij} dz^i \wedge d\tilde{z}^j + \frac{1}{2} \theta^n \wedge \bar{\theta}^n \right).
\]
(4.3)

Let \( D \) be any locally defined constant linear first order operator (with respect to the chosen coordinate chart) near the boundary (e.g., \( D = D_i = \pm \frac{\partial}{\partial x_i}, \pm \frac{\partial}{\partial y_i} \) for \( i = 1, \cdots, n-1 \)).

**Lemma 4.2.** There is a positive constant \( C_0 \) depending on \( |\tilde{\omega}|_{C^{0,1}(\tilde{M})}, |\varphi|_{C^{2,1}(\tilde{M})} \) and other known data such that
\[
\mathcal{L}D(u - \varphi) \leq \sup_{\text{Int}(\tilde{M})} |D\psi|_{\tilde{g}} + C_0 F^{ij} \bar{g}_{ij}.
\]
(4.4)

**Proof.** In \( \text{Int}(\tilde{M}) \), one has
\[
\mathcal{L}D(u - \varphi) = F^{ij} X_i X_j (Du) - \mathcal{L}D\varphi
= F^{ij} DX_i X_j u - \mathcal{L}D\varphi
= F^{ij} D(U_{ij} - \bar{X}_{ij}) - \mathcal{L}D\varphi
\leq \sup_{\text{Int}(\tilde{M})} |D\psi|_{\tilde{g}} + C_0 F^{ij} \bar{g}_{ij},
\]
where we use the fact that \( u \) is **basic**, \( |X_n, D| = 0 \) and \( X_i \) is the vectors defined in \((2.3)\) and \((2.8)\). \( \square \)

**Proposition 1.** Assume that \( \psi \in C^1_b(\text{Int}(\tilde{M})) \cap C^{0,1}(\tilde{M}) \) and conditions \((1.1)-(1.5)\) hold and Dirichlet problem \((1.1)\) admits an admissible subsolution \( \underline{u} \in C^2_b(\tilde{M}) \). Then for any admissible solution \( u \in C^3(\text{Int}(\tilde{M})) \cap C^2(\tilde{M}) \) to the Dirichlet problem
\[
\sup_{\partial\tilde{M}} |D_i D_j u|_{\tilde{g}} \leq C,
\]
(4.5)
\[
|\nabla_{\partial\tilde{g}} Du(p)|_{\tilde{g}} \leq C(1 + \sup_M |\nabla u|_{\tilde{g}}),
\]
(4.6)
where \( C \) is a positive constant depending on \( \|u\|_{C^2(\tilde{M})}, |\varphi|_{C^{2,1}(\tilde{M})}, |\psi|_{C^{0,1}(\tilde{M})} \) and other known data. Suppose in addition that \( f \) satisfies \((1.7)\) or \( \psi \) is a constant function. Then the constants in \((4.6)\) and \((4.5)\) depend not on \( \kappa^{-1}(\kappa \text{ is the constant in } (1.6)) \).
Proof. The second order boundary estimates for pure tangential derivatives is standard. To be precise, there exists a uniform positive constant $C_1'$ depending on
\[
\sup_{\partial M} |\frac{\partial (u-v)}{\partial n}|_\delta \text{ and other known data under control such that}
\]
\[
\sup_{\partial M} |D_i D_j u|_\delta \leq C_1'.
\]
Combining it with (3.3), the gradient on boundary is uniform. We thus have (4.5).

The tangential-normal case will be proved by constructing barrier functions. This type of construction of barrier functions follows from [22, 29, 34].

The barrier function will be given as follows,
\[
\Psi = A_1(1 + \sup_{M} |\nabla u|_\delta) v + A_2(1 + \sup_{M} |\nabla u|_\delta) \rho^2 \pm D(u - u),
\]
where $A_1 \gg A_2 \gg 1$ and $v$ is the function as in (4.1), i.e., $v = u - u - \frac{N}{2}\sigma^2 + t\sigma$.

Choose $0 < \delta \ll 1$ so that $0 \leq \sigma < 1$ in $\Omega_{\delta}$.

On the one hand, by Lemma 4.1 and Lemma 4.2, we know that $\mathcal{L}\Psi \leq 0$ in $\Omega_{\delta}$, if $A_1 \gg A_2 \gg 1$.

On the other hand,
\[
\begin{cases}
D(u - u) = 0 & \text{on } \partial M \cap \Omega_{\delta},
\rho = \delta, |D(u - u)|_\delta \leq a_0(1 + \sup_{M} |\nabla u|_\delta) & \text{on } \partial \Omega_{\delta} \setminus \partial M.
\end{cases}
\]

Therefore $\Psi \geq 0$ on $\partial \Omega_{\delta}$ since (3.2) holds. Hence $\Psi \geq 0$ in $\Omega_{\delta}$. Note that $\Psi(p) = 0$, then the maximum principle implies that
\[
|\nabla_{\partial_{\delta}} Du(p)|_\delta \leq C(1 + \sup_{M} |\nabla u|_\delta),
\]

where $C$ is a positive constant depending on $|u|_{C^{2}(M)}$, $|\varphi|_{C^{2,1}(M)}$, $|\psi|_{C^{0,1}(M)}$ and other known data.

Moreover, if $\psi$ is a constant function, then the term $\sup_{\text{int}(M)} |D\psi|$ in (4.4) must be zero. Therefore, condition (1.7) can be removed when $\psi$ is constant. \qed

Next we will complete the proof of Theorem 1.2

Proof of Theorem 1.2: From the proof of Proposition 1 we only need considering pure tangential derivative case. And we also use the notations and computations there. Given $p = (q, r_p) \in \partial M$, $r_p = a$ or $b$.

Since $u - \underline{u} = 0$ on $\partial M$, we can therefore write $u - \underline{u} = \hat{h}\eta$ in $V_{\delta}$ for some

$0 < \delta \ll 1$, where
\[
\eta(p) = (r - a) \text{ or } (b - r), \text{ where } p = (q, r) \in M, q \in M.
\]

It follows that for $\alpha, \beta \leq n - 1$, we have $X_{\alpha}\bar{X}_{\beta}u = X_{\alpha}\bar{X}_{\beta}\underline{u}$ at $p$, i.e.,
\[
U_{\alpha\beta} = U_{\alpha\beta} \text{ at } p,
\]

where
\[
U = \sqrt{-1}U_{nn}\theta^n \wedge \bar{\theta}^n + \sum_{\alpha + \beta < 2n} \sqrt{-1}U_{\alpha\beta}\theta^n \wedge \bar{\theta}^\beta.
\]

By direct computations
\[
U^k \wedge \bar{\omega}^{n-k} = kU_{n\bar{n}} \cdot U^{k-1} \wedge \sqrt{-1}\theta^n \wedge \bar{\theta}^n \wedge \bar{\omega}^{n-k} + (\ast_1),
\]
where \((*)_1\) does not contain \(\sqrt{-1}U_n \theta \wedge \tilde{\theta}^n\). By assumption \((1.13)\), equation \((1.1)\) can be written as of the form

\[
U_n \left[ \sum_{k=1}^{n} k \alpha_k(z, \psi) U^{k-1} \wedge \sqrt{-1} \theta^n \wedge \tilde{\theta}^n \wedge \tilde{\omega}^{n-k} \right] = b(z, \psi) + (*_2),
\]

where \((*_2)\) does not contain \(\sqrt{-1}U_n \theta^n \wedge \tilde{\theta}^n\).

Next we claim that

\[
\sum_{k=1}^{n} k \alpha_k(z, \psi) U^{k-1} \wedge \sqrt{-1} \theta^n \wedge \tilde{\theta}^n \wedge \tilde{\omega}^{n-k} \neq 0.
\]

If the claim does not hold, i.e., there exists a point \(z\) such that at this point

\[
\sum_{k=1}^{n} k \alpha_k(z, \psi) U^{k-1} \wedge \sqrt{-1} \theta^n \wedge \tilde{\theta}^n \wedge \tilde{\omega}^{n-k} = 0,
\]

then we know that at \(z\),

\[
f(\lambda(U[u] + t\sqrt{-1} \theta^n \wedge \tilde{\theta}^n)) = \psi, \forall t \geq 0.
\]

It is a contradiction to the ellipticity of equation \((1.1)\).

Note that at \(p \in \partial M\), \(U_{\alpha \beta} = U_{\alpha \beta}, (\alpha, \beta < n)\), then

\[
\sum_{k=1}^{n} k \alpha_k(z, \psi) U^{k-1} \wedge \sqrt{-1} \theta^n \wedge \tilde{\theta}^n \wedge \tilde{\omega}^{n-k} = \sum_{k=1}^{n} k \alpha_k(z, \psi) U^{k-1} \wedge \sqrt{-1} \theta^n \wedge \tilde{\theta}^n \wedge \tilde{\omega}^{n-k} \text{ at } p.
\]

Therefore, it follows from \((4.8)\), \((4.5)\) and \((4.6)\) that \(U_{n\bar{n}} \leq C(1 + \sup_{M} |\nabla u|^2)\). Hence \((1.14)\) holds.

\section{Gradient estimate and blow-up argument}

In [50], Székelyhidi proved a Liouville type theorem which extended that of Dineen and Kolodziej [14]. Based on the blow-up argument in [9] [50], we obtain the gradient estimate as follows. Moreover, assumption \((1.5)\) is crucial for the blow-up argument in Székelyhidi [50].

\begin{proposition}
Assume that \(\psi \in C^{2,1}(\text{Int}(\hat{M})) \cap C^{1,1}(\hat{M})\), \((1.2)\) - \((1.5)\) and \((1.13)\) hold. Suppose that Dirichlet problem \((1.1)\) admits an admissible subsolution \(u \in C^{2}(\hat{M})\). Then for any admissible solution \(u \in C^{4}(\text{Int}(\hat{M})) \cap C^{2}(\hat{M})\) of the Dirichlet problem, there is a positive uniform constant \(C\) depending on \(\delta_{\psi, f}^{-1}\), \(\kappa^{-1}\) (\(\kappa\) is the constant in \((1.6)\)), \(|\varphi|_{C^{2,1}(\hat{M})}\), \(|u|_{C^{2}(\hat{M})}\), \(|\psi|_{C^{1,1}(\hat{M})}\) and other known data under control, such that

\[
\sup_{\hat{M}} |\nabla u|^2 \leq C.
\]

Furthermore, suppose, in addition to \(f\) satisfies \((1.7)\) or \(\psi\) is a constant function, that condition \((1.13)\) holds for \(a_k(z, \psi) = a_k(z)\) and \(|b(z, \psi)| \leq C_{11}\) for a uniform constant \(C_{11}\) (but not on \(\delta_{\psi, f}^{-1}\)). Then the constant \(C\) in \((5.1)\) does not depend on \((\delta_{\psi, f})^{-1}\).

\end{proposition}

\begin{proof}
It follows from \((3.3)\), Theorem \((1.1)\) and Lemma \((1)\) that there is a uniform positive constant \(C_5\) depending on \(|\psi|_{C^{1,1}(\hat{M})}\) and other known data under control such that

\[
\sup_{\hat{M}} |\Delta u| \leq C_5(1 + \sup_{\hat{M}} |\nabla u|^2).
\]

\end{proof}
Suppose $N_i := \sup_{\bar{M}} |\nabla u_i| g \to +\infty$. Let $N_i = |\nabla u_i(z_i)| g$. From the bound of $\sup_{\partial \bar{M}} |\nabla u_i| g$, it follows that there is a constant $n_0$, such that for any $i \geq n_0$, $z_i \in Int(M)$. By (5.2), we obtain
$$\sup_{\bar{M}} |\Delta u_i| \leq C N_i^2.$$ We choose a convergent subsequence of $\{z_i\}$, also denote by $\{z_i\}$, such that $z_i \to z$.

Suppose $z \in \partial \bar{M}$. Then by the argument used in [9], we have a contradiction. Therefore, $z$ stays in the interior of $\bar{M}$. In this case, we use the argument of Székelyhidi [50] where assumption (1.5) is important. By combining the blow-up argument with a Liouville type theorem in [50], we obtain that $\sup_M |\nabla u| g \leq C$. Note that in the blow-up argument the only difference from the setup here is the linear term $\frac{\partial u}{\partial r}$ in $U_{ij}$. However, the order of $\frac{\partial u}{\partial r}$ is only one which ensures the terms containing $\frac{\partial u}{\partial r}$ converge to zero uniformly on compact sets under the rescaling procedure of Székelyhidi [50]. Thus the blow-up argument in [50] still works. Therefore we obtain (5.1). We refer the reader to [9, 50] for more details.

6. Existences and basic subsolutions. In this section we solve the equations assuming condition (1.17) holds. We construct basic admissible subsolutions under condition (1.17), and the existences of Dirichlet problems can be derived via the method of continuity and a priori estimates above.

6.1. Basic admissible subsolution. Under cone condition (1.17), we can construct a basic admissible subsolution of the Dirichlet problem. Firstly, let us present the following lemma due to Caffarelli, Nirenberg and Spruck [5].

**Lemma 6.1** ([5]). Consider the $n \times n$ symmetric matrix

$$M = \begin{pmatrix} d_1 & a_1 \\ d_2 & a_2 \\ \vdots & \vdots \\ a_1 & a_2 & \cdots & a_{n-1} & a_{n-1} & a \end{pmatrix}$$

with $d_1, \ldots, d_{n-1}$ fixed, $|a|$ tends to infinity and
$$|a_i| \leq C, i = 1, \ldots, n.$$

Then the eigenvalues $\lambda_1, \ldots, \lambda_n$ behave like
$$\lambda_\alpha = d_\alpha + o(1), 1 \leq \alpha \leq n - 1,$$
$$\lambda_n = a \left(1 + O \left(\frac{1}{a}\right)\right),$$
where the $o(1)$ and $O(1/a)$ are uniform–depending only on $d_1, \ldots, d_{n-1}$ and $C$.

Now, we construct the desired subsolution as follows.

Given a function $v \in C_0(\bar{\chi})$ satisfying condition (1.17), we set
$$u = v + 2A(r - a)(r - b), A \gg 1. \quad (6.1)$$
By a simple computation, we have
$$\sqrt{-1} \left( \partial \tilde{\partial} - \partial \tilde{\partial}r \frac{\partial}{\partial r} \right) ((r - a)(r - b)) = \frac{\sqrt{-1}}{2} \theta^n \wedge \tilde{\theta}^n,$$
where $\theta^n = dr + \sqrt{-1}r \eta$ and $\tilde{\theta}^n = dr - \sqrt{-1}r \eta.$
Applying Lemma 6.1, we know that $u \in C_B^{4,\alpha}(\bar{M})$ and $\lambda(U[u]) \in \Gamma$ for $A \gg 1$ thanks to $v \in C^{2,\alpha}(\chi)$. Moreover, $u$ is a basic admissible subsolution of Dirichlet problem (1.1) for large $A$ if in addition condition (1.17) holds.

6.2. Solving equations. In this subsection $u$ is the basic admissible subsolution constructed in (6.1). Based on a priori estimates as previous, we solve the equations by the method of continuity.

Proof of Theorem 1.3. We consider a family of equations

\[ \begin{aligned}
F(U[u_t]) &= (1-t)F(U[u]) + t\psi & \text{in } \text{Int}(\bar{M}), \\
u_t &= \varphi_a & \text{on } M \times \{a\}, \\
u_t &= \varphi_b & \text{on } M \times \{b\}.
\end{aligned} \]

(6.2)

Set

\[ I = \{ t \in [0,1] : \text{there exists } u_t \in C^{4,\alpha}(\bar{M}) \text{ solving equation } (6.2) \}. \]

Clearly $0 \in I$ by taking $u_t = \bar{u}$. One can verify that

(i) $u$ is the basic admissible subsolution along the whole method of continuity.

(ii) The right hand side of (6.2) is basic.

So we can apply the a priori estimates above. Lemma 2.3 implies that $u_t$ is basic if it exists. By applying (3.3), Theorems 1.1 and Proposition 2, one derives a uniform bound for complex Hessian of basic admissible solution $u_t$, i.e., $u_t$ is basic. Hence, the equations become to be uniform elliptic and concave because of conditions (1.2), (1.3) and (1.4). The argument used by Guan and Li [25] gives us a uniform bound of real Hessian. Thus the theory of Evans-Krylov [15, 38] can be applied to derive $|u_t|_{C^{2,\alpha}(\bar{M})} \leq C$. The higher regularities can be derived by classical Schauder theory.

Therefore our estimates conclude that $I$ is closed. The openness is follows from implicit function theorem and the a priori estimates above, hence then $I = [0,1]$. So there is a basic admissible solution $u \in C^{4,\alpha}(\bar{M})$ of Dirichlet problem (1.1). The uniqueness of the solution $u$ follows from maximum principle.

To solve Dirichlet problem (1.19) of the degenerate fully nonlinear elliptic equation, we consider the following perturbation equation

\[ \begin{aligned}
f(\lambda(\{U[z|u]\})) &= \psi + \epsilon \text{ in } M \times (a,b), \\
\lambda(\{U[z|u]\}) &\in \Gamma, \\
\epsilon|_{r=a} &= \varphi_a, \epsilon|_{r=b} = \varphi_b,
\end{aligned} \]

(6.3)

where $0 < \epsilon \ll 1$, $\delta_{\psi,f} = \inf_{\mathcal{M}} \psi - \sup_{\partial\mathcal{M}} f \geq 0$. Moreover, we have

Proposition 3. Assume that (1.2), (1.3), (1.5) and (1.7) hold. Suppose, in addition to Dirichlet problem (6.3) has a basic admissible subsolution $u \in C_B^{4,\gamma}(\bar{M})$ for some $0 < \gamma < 1$, that equation can be written as the form of (1.13) such that $a_k(z,\psi) = a_k(z)$ and $C_{ij}$ in (1.15) is independent of $(\delta_{\psi,f} + \epsilon)^{-1}$. Then there is a unique admissible basic solution $u_t \in C^{4,\gamma}(\bar{M})$ to equation (6.3). Moreover, there exists a positive constant $C$ which is independent of $(\delta_{\psi,f} + \epsilon)^{-1}$ such that

\[ \sup_{\bar{M}} |u_t| + \sup_{\bar{M}} |\nabla u_t| + \sup_{\bar{M}} |\Delta u_t| \leq C. \]

(6.4)

Moreover, assumption (1.7) can be removed if $\psi$ is a constant function.
Proof. For the perturbation equation (6.3) of degenerate equation (1.19), Lemma 2.3 concludes that the admissible solution $u$ is basic. Then the existence and estimates follow from (3.3), Theorems 1.1 [1] and Propositions 2.

Proof of Theorem 1.4. From Proposition 3, we need only to construct a basic admissible subsolution of the perturbation equation (6.3).

Set
\[ w = u + 2A(r - a)(r - b), \]
where $u \in C^{4,\gamma}_{\tilde{M}}$, which is constructed in (6.1), is the basic admissible subsolution of Dirichlet problem (1.19). Recall that
\[ U[w] = U[u] + A\sqrt{-1} \theta^n \wedge \bar{\theta}^n. \]
For $A > 0$, then
\[ f(\lambda(U[w])) > f(\lambda(U[u])) \geq \psi. \]
Thus for $1 \gg \epsilon > 0$, $f(\lambda(U[w])) \geq \psi + \epsilon$. hence we construct the desired basic admissible subsolution.

Acknowledgments. The authors would like to thank Professor Bo Guan for bringing this topic to their attentions. The second author wants to thank Professor Bo Guan for his constant support and encouragement. The second author would also like to thank Professor Xi Zhang for his hospitality while he was visiting University of Science and Technology of China.

REFERENCES

[1] T. Aubin, Équations du type Monge-Ampère sur les variétés Kähleriennes compactes, (French), Bull. Sci. Math., 102 (1978), 63–95.
[2] E. Bedford and B. Taylor, The Dirichlet problem for a complex Monge-Ampère equation, Invent. Math., 37 (1976), 1–44.
[3] Z. Blocki, On geodesics in the space of Kähler metrics, Advances in geometric analysis, Adv. Lect. Math. (ALM), Int. Press, Somerville, MA, 21 (2012), 3–19.
[4] C. Boyer and K. Galicki, Sasakian Geometry, Oxford: Oxford Mathematical Monographs, Oxford University press, 2008.
[5] L. Caffarelli, L. Nirenberg and J. Spruck, The Dirichlet problem for nonlinear second-order elliptic equations III: Functions of eigenvalues of the Hessians, Acta Math., 155 (1985), 261–301.
[6] L. Caffarelli, L. Nirenberg and J. Spruck, The Dirichlet problem for the degenerate Monge-Ampère equation, Rev. Mat. Iberoamericana, 2 (1986), 19–27.
[7] L. Caffarelli, J. Kohn, L. Nirenberg and J. Spruck, The Dirichlet problem for nonlinear second-order elliptic equations. II. Complex Monge-Ampère, and uniformly elliptic, equations, Comm. Pure Applied Math., 38 (1985), 209–252.
[8] E. Calabi, The Space of Kähler Metrics, Proc. Internat. Congress of Mathematicians, Amsterdam, Holland. 1954.
[9] X.-X. Chen, The space of Kähler metrics, J. Diff. Geom., 56 (2000), 189–234.
[10] X.-X. Chen, A new parabolic flow in Kähler manifolds, Comm. Anal. Geom., 12 (2004), 837–852.
[11] T. Collins, A. Jacob and S.-T. Yau, (1,1) forms with special Lagrangian type: A priori estimates and algebraic obstructions, arXiv:1508.01834.
[12] S.-K. Donaldson, Symmetric spaces, Kähler geometry and Hamiltonian dynamics, Northern California Symplectic Geometry Seminar, American Mathematical Society Translations: Series 2, 196. Providence, RI: American Mathematical Society, 45 (1999), 13–33.
[13] S.-K. Donaldson, Moment maps and diffeomorphisms, Asian J. Math., 3 (1999), 1–16.
[14] S. Dinew and S. Kolodziej, Liouville and Calabi-Yau type theorems for complex Hessian equations, American Journal of Mathematics, 139 (2017), 403–415, arXiv:1203.3995.
[15] L. Evans, Classical solutions of fully nonlinear convex, second order elliptic equations, Comm. Pure Applied Math., 35 (1982), 333–363.
[16] H. Fang, M.-J. Lai and X.-N. Ma, On a class of fully nonlinear flows in Kähler geometry, J. Reine Angew. Math., 653 (2011), 189–220.
[17] J.-X. Fu, On non-Kähler Calabi-Yau threefolds with balanced metrics, *Proc. Internat. Congress of Mathematicians*, Hyderabad, India. New Delhi: Hindustan Book Agency, (2010), 705–716.

[18] A. Futaki, H. Ono and G.-F. Wang, Transverse Kähler geometry of Sasaki manifolds and toric Sasaki-Einstein manifolds, *J. Diff. Geom.*, 83 (2009), 585–635.

[19] P. Gauduchon, La 1-forme de torsion d’une variété hermitienne compacte, *Math. Ann.*, 267 (1984), 495–518.

[20] J. Gauntlett, D. Martelli, J. Sparks and D. Waldram, A new infinite class of Sasaki-Einstein manifolds, *Adv. Theor. Math. Phys.*, 8 (2004), 987–1000.

[21] M. Godlini, W. Kopczynski and P. Nurowski, Locally Sasakian manifolds, *Classical quant. gravity*, 17 (2000), 105–115.

[22] B. Guan, The Dirichlet problem for complex Monge-Ampère equations and regularity of the pluri-complex Green function, *Comm. Anal. Geom.*, 6 (1998), 687–703.

[23] B. Guan, Second order estimates and regularity for fully nonlinear elliptic equations on Riemannian manifolds, *Duke Math. J.*, 163 (2014), 1491–1524.

[24] B. Guan, The Dirichlet problem for fully nonlinear elliptic equations on Riemannian manifolds, preprint.

[25] B. Guan and Q. Li, The Dirichlet problem for a complex Monge-Ampère type equation on Hermitian manifolds, *Adv. Math.*, 246 (2013), 351–367.

[26] B. Guan and X.-L. Nie, Fully nonlinear elliptic equations on Hermitian manifolds, preprint.

[27] B. Guan and W. Sun, On a class of fully nonlinear elliptic equations on Hermitian manifolds, *Calc. Var. PDE.*, 54 (2013), 901–916.

[28] B. Guan, S.-J. Shi and Z.-N. Sui, On estimates for fully nonlinear parabolic equations on Riemannian manifolds, *Anal. PDE.*, 8 (2015), 1145–1164.

[29] B. Guan and J. Spruck, Boundary-value problems on $\mathbb{S}^n$ for surfaces of constant Gauss curvature, *Ann. Math.*, 138 (1993), 601–624.

[30] P.-F. Guan, The extremal function associated to intrinsic norms, *Ann. Math.*, 156 (2002), 197–211.

[31] P.-F. Guan, N. Trudinger and X.-J. Wang, On the Dirichlet problem for degenerate Monge-Ampère equations, *Acta Math.*, 182 (1999), 87–104.

[32] P.-F. Guan and X. Zhang, A geodesic equation in the space of Sasake metrics, *Geometry and Analysis, Adv. Lect. Math.*, Somerville: International Press, 17 (2011), 303–318.

[33] P.-F. Guan and X. Zhang, Regularity of the geodesic equation in the space of Sasake metrics, *Adv. Math.*, 230 (2012), 321–371.

[34] D. Hoffman, H. Rosenberg and J. Spruck, Boundary value problems for surfaces of constant Gauss Curvature, *Comm. Pure Applied Math.*, 45 (1992), 1051–1062.

[35] Z. Hou, X.-N. Ma and D.-M. Wu, A second order estimate for complex Hessian equations on a compact Kähler manifold, *Math. Res. Lett.*, 17 (2010), 547–561.

[36] N. Ivochkina, The integral method of barrier functions and the Dirichlet problem for equations with operators of the Monge-Ampère type, (Russian) *Izv. Akad. Nauk SSSR Ser. Mat.*, 47 (1983), 75–108.

[37] N. Krylov, Boundedly inhomogeneous elliptic and parabolic equations in a domain, (Russian) *Izv. Akad. Nauk SSSR Ser. Mat.*, 47 (1983), 75–108.

[38] Y.-Y. Li, Some existence results of fully nonlinear elliptic equations of Monge-Ampère type, *Comm. Pure Applied Math.*, 43 (1990), 233–271.

[39] Y.-Y. Li, Degenerate conformally invariant fully nonlinear elliptic equations, *Arch. Ration. Mech. Anal.*, 186 (2007), 25–51.

[40] T. Mabuchi, Some symplectic geometry on Kähler manifolds. I, *Osaka J. Math.*, 24 (1987), 227–252.

[41] D. Martelli and J. Sparks, Toric geometry, Sasaki-Einstein manifolds and a new infinite class of AdS/CFT duals, *Comm. Math. Phys.*, 262 (2006), 51–89.

[42] D. Martelli, J. Sparks and S.-T. Yau, Sasaki-Einstein manifolds and volume minimisation, *Comm. Mathe. Phys.*, 280 (2008), 611–673.

[43] D.-H. Phong, S. Picard and X.-W. Zhang, On estimates for the Fu-Yau generalization of a Strominger system, *arXiv:1507.08193*.

[44] D.-H. Phong and J. Sturm, The Dirichlet problem for degenerate complex Monge-Ampère equations, *Comm. Anal. Geom.*, 18 (2010), 145–170.
[46] D. Popovici, Aeppli cohomology classes associated with Gauduchon metrics on compact complex manifolds, *Bulletin de la SMF*, 143 (2015), 763–800, [arXiv:1310.3685](https://arxiv.org/abs/1310.3685).

[47] S. Semmes, Complex Monge-Ampère and symplectic manifolds, *Amer. J. Math.*, 114 (1992), 495–550.

[48] J. Song and B. Weinkove, On the convergence and singularities of the J-Flow with applications to the Mabuchi energy, *Comm. Pure Applied Math.*, 61 (2008), 210–229.

[49] W. Sun, Generalized complex Monge-Ampère type equations on closed Hermitian manifolds, [arXiv:1412.8192](https://arxiv.org/abs/1412.8192).

[50] G. Székelyhidi, Fully non-linear elliptic equations on compact Hermitian manifolds, [arXiv:1501.02762](https://arxiv.org/abs/1501.02762).

[51] G. Székelyhidi, V. Tosatti and B. Weinkove, Gauduchon metrics with prescribed volume form, [arXiv:1503.04491](https://arxiv.org/abs/1503.04491).

[52] M. E. Taylor, *Partial Differential Equations I, Basic Theory*, New York, Berlin, Heidelberg: Applied Mathematical Sciences, 115, Springer-Verlag, 1996.

[53] G. Tian and S.-T. Yau, Complete Kähler manifolds with zero Ricci curvature. I, *J. Amer. Math. Soc.*, 3 (1990), 579–609.

[54] V. Tosatti and B. Weinkove, Hermitian metrics, $(n-1,n-1)$-forms and Monge-Ampère equations, [arXiv:1310.6326](https://arxiv.org/abs/1310.6326).

[55] N. Trudinger, On the Dirichlet problem for Hessian equations, *Acta Math.*, 175 (1995), 151–164.

[56] S.-T. Yau, On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I, *Comm. Pure Applied Math.*, 31 (1978), 339–411.

Received for publication May 2017.

E-mail address: chqiu@xmu.edu.cn

E-mail address: rirongyuan@stu.xmu.edu.cn