Lower bound for the mean-square distance between classical and quantum spin correlations

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Received 9 November 2010, in final form 21 January 2011
Published 14 February 2011
Online at stacks.iop.org/JPhysA/44/095307

Abstract
Bell’s theorem prevents local Kolmogorov simulations of the singlet state of two spin-1/2 particles. We derive a positive lower bound for the $L^2$-distance between the quantum mechanical spin-singlet anticorrelation function $\cos$ and any of its classical approximants $C$ formed by the stationary autocorrelation functions of mean-square-continuous, $2\pi$-periodic, $\pm 1$-valued, stochastic processes. This bound is given by $\|C - \cos\| \geq (1 - \frac{8}{\pi^2})/\sqrt{2} \approx 0.13395$.

PACS number: 03.65.Ud

1. Introduction

Consider a Stern–Gerlach correlation experiment performed on the two distant members of a spin-singlet state of two spin-1/2 particles. The two apparatuses’ magnetic fields point into the directions $a$ and $b$. Quantum mechanics states that the stochastic correlation $-Q(a, b)$ between the two measurements’ results $\pm 1$ is given by $-a \cdot b$, the negative scalar product between the two analyzing directions’ unit vectors $a$ and $b$. To get rid of the minus sign, we employ in the following the anticorrelation function $Q$.

Bell demonstrated that any local Kolmogorov simulation of such experiments is at variance with $Q(a, b) = a \cdot b$ [1]. The basic reason for this being his inequality, which subjects any triple of values $C(a, b), C(a, c), C(b, c)$ of the autocorrelation function $C: S^2 \times S^2 \rightarrow [-1, 1], (a, b) \mapsto (f_a f_b)$ of any local Kolmogorov simulation to

$$|C(a, b) - C(a, c)| \leq 1 - C(b, c).$$

The quantum mechanical function $Q$, however, does not obey Bell’s inequality for all choices of directions $a, b, c$. Thus, $Q = C$ cannot hold.

This raises the question for how close the quantum mechanical function $Q$ can be approximated by a local Kolmogorov autocorrelation function $C$. Clearly, a notion of 'closeness' is needed in order to make the question meaningful. The mean-square deviation between $C$ and $Q$ averaged over a suitable set of pairs of directions is a natural choice, which
we adopt. First, it is experimentally better accessible than, e.g., the uniform norm, as the latter is sensitive to the function’s behavior in arbitrarily small regions of the domain. (Note that the correlation function is confined to take values in the interval $[-1, 1]$.) Second, the mean-square norm derives from a scalar product. Thus, it behaves more like the Euclidean distance than the various $L^p$-norms do, which, for $p \neq 2$, do not have an associated scalar product. And it is exactly this scalar product which leads to our result.

How should one choose the set of directions to be averaged over? Rotation invariance suggests to keep one point $a \in \mathbb{S}^2$ fixed and to average over all $b$ from a great circle or equivalently a half great circle through $a$, because from the $SO(3)$ invariance of $C$ it follows that $C$ is completely determined by the mapping $b \mapsto C(a, b)$ for any fixed $a$ and all directions $b$ lying on a meridian emanating from $a$. After all $C$ is represented by a function of the (unoriented) angle between $a$ and $b$ only, i.e. there exists a function $C : [0, \pi] \to [-1, 1]$ such that $C(a, b) = C(\theta_{a,b})$ for all $a, b \in \mathbb{S}^2$. Here, $\theta_{a,b} \in [0, \pi]$ is uniquely determined by $a \cdot b = \cos \theta_{a,b}$.

Within the class of local Kolmogorov autocorrelation functions which are rotation invariant and continuous, we have found that the mean-square deviation of $C$ from its quantum counterpart $\cos$ obeys the estimate

$$\sqrt{\frac{1}{\pi} \int_0^\pi |C(\theta) - \cos \theta|^2 d\theta} \geq \left(1 - \frac{8}{\pi^2}\right)/\sqrt{2} \approx 0.13395. \tag{2}$$

The search for such ‘integrated’ conditions which constrain the local Kolmogorov approximants $C$ of the quantum anticorrelation function $\cos$ seems to have been initiated by Żukowski, who in [2] derived an upper bound for the modulus of the $L^2$-scalar product between $\cos$ and $C$. See estimate (10) in [2], where it was taken for granted that certain integrability conditions are fulfilled by the stochastic variables of the local Kolmogorov simulation under consideration. In [3], the idea has been generalized further in order to cover more general, entangled many-particle states.

In a first step, we prove a scalar product bound, which in certain cases is equivalent to Żukowski’s one. Our bound only holds for the singlet state but it does not rely on Żukowski’s integrability assumption. The class of Kolmogorov simulations for which it holds is specified precisely in our lemma 5. From this we then derive in proposition 6 a lower bound for the mean-square distance between $C$ and $\cos$. Although fairly straightforward, this latter bound seems to have remained unnoticed till now.

2. Local Kolmogorov simulation

Let $(\chi_+, \chi_-)$ denote an orthonormal basis of $\mathbb{C}^2$. Then, the vector

$$\chi = \frac{1}{\sqrt{2}} (\chi_+ \otimes \chi_- - \chi_- \otimes \chi_+)$$

represents the spin-singlet state of two spin-1/2 particles. A Stern–Gerlach experiment on each of the two constituents of the singlet state, which are assumed to be approximately localized in well separated regions, measures the two observables $A = \sigma (a) \otimes id$ and $B = id \otimes \sigma (b)$, where $a, b \in \mathbb{S}^2 \subset \mathbb{R}^3$ are two arbitrary vectors from the unit sphere $\mathbb{S}^2$. The unit vectors $a$ and $b$ specify the Stern–Gerlach apparatuses’ orientations. For $a = (a^1, a^2, a^3) \in \mathbb{R}^3$ the following holds:

$$\sigma (a) = \begin{pmatrix} a^3 & a^1 - ia^2 \\ a^1 + ia^2 & -a^3 \end{pmatrix}.$$
The possible results of such measurements are the pairs \((\varepsilon, \eta) \in \{1, -1\} \times \{1, -1\}\). The quantum mechanically determined probability of the outcome \((\varepsilon, \eta)\) is given in terms of the Euclidean scalar product \(a \cdot b\) between \(a\) and \(b\) by

\[
p_{a,b}(\varepsilon, \eta) = \frac{1 - \varepsilon \eta \, a \cdot b}{4}.
\]  

(3)

This probability does not depend on the time order of the two Stern–Gerlach experiments since \([A, B] = 0\).

Thus, any choice of two directions \(a\) and \(b\) determines a probability measure on the event space \(\{1, -1\} \times \{1, -1\}\). The situation is analogous to a random experiment in which a pair of widely separated coins is tossed, while each coin of the pair is exposed to a magnetic field of direction \(a\) and \(b\), respectively. And the magnetic fields take an influence on the distribution of the possible results.

The family of the probability functions \(\{p_{a,b} : a, b \in S^2\}\) can be obtained as the distributions of stochastic variables \(\{X_{a,b} : a, b \in S^2\}\):

\[
X_{a,b} = (f_{a,b}, g_{a,b}) : \Omega \to \{1, -1\} \times \{1, -1\}
\]

on a single Kolmogorov probability space \((\Omega, W)\) as follows [4]. Take as the space of events \(\Omega\) the square \([0, 1] \times [0, 1]\) with the uniform distribution \(W\). Then, decompose the square \(\Omega\) into a set of four nonoverlapping rectangles, i.e.

\[
\Omega = \bigcup_{\varepsilon, \eta \in \{1, -1\}} R_{a,b}(\varepsilon, \eta),
\]

such that the area of the rectangle \(R_{a,b}(\varepsilon, \eta)\) has the value \(p_{a,b}(\varepsilon, \eta)\). Finally, assume

\[
X_{a,b}(\omega) = (\varepsilon, \eta) \quad \text{for} \quad \omega \in R_{a,b}(\varepsilon, \eta).
\]

Then, by construction we obviously have

\[
p_{a,b}(\varepsilon, \eta) = W(\{\omega \in \Omega : X_{a,b}(\omega) = (\varepsilon, \eta)\}).
\]

In such a model the outcome of each coin flip is determined by a randomly chosen ‘hidden variable’ \(\omega \in \Omega\) in a way that in general the outcome also depends on both of the magnetic fields \(a\) and \(b\)! How come that a coin gets influenced by faraway circumstances?

Trying to save locality, Bell considered the question whether there exists a probability space \((\Omega, W)\) with a stochastic variable

\[
X_{a,b} = (f_a, g_b) : \Omega \to \{1, -1\} \times \{1, -1\}
\]

for each pair \((a, b) \in S^2 \times S^2\) such that

\[
p_{a,b}(\varepsilon, \eta) = W(\{\omega \in \Omega : f_a(\omega) = \varepsilon \text{ and } g_b(\omega) = \eta\}) \quad \text{for all } a, b \in S^2.
\]  

(4)

Such structure would constitute a local Kolmogorov simulation of the probability distributions generated by composite Stern–Gerlach experiments on a spin-singlet state. The assumption that the stochastic variables \(f_a, g_b\) depend on their respective Stern–Gerlach orientation \(a\) or \(b\) only is made in order to take care of the principle of locality.

Now equations (3) and (4) imply

\[
(f_a g_b) = \sum_{\varepsilon, \eta \in \{1, -1\}} \varepsilon \eta p_{a,b}(\varepsilon, \eta) = -a \cdot b
\]  

(5)

for all \(a, b \in S^2\). Yet Bell’s theorem [1] rules out exactly that \(f_a g_b = -a \cdot b\) for all \(a, b \in S^2\). Therefore, a local Kolmogorov simulation of the singlet state does not exist. For the sake of completeness we spell out Bell’s theorem precisely.
Theorem 1 (Bell). Let \((\Omega, W)\) be a probability space with two stochastic variables \(f_a, g_a : \Omega \to \{1, -1\}\) for every \(a \in S^2\). Then, there exist points \(a, b \in S^2\) such that \(\langle f_a g_b \rangle \neq -a \cdot b\).

Proof. That \(\langle f_a g_b \rangle = -a \cdot b\) cannot hold for all \(a, b \in S^2\) may be proven by contradiction. Choosing \(b = a\), the equation \(\langle f_a g_a \rangle = -1\) implies \(g_a = -f_a\) in the sense of stochastic variables, i.e. almost everywhere on \(\Omega\). Thus, assuming \(\langle f_a g_b \rangle = -a \cdot b\) for all \(a, b \in S^2\) leads to

\[-\langle f_a g_b \rangle = \langle f_a f_b \rangle = a \cdot b\]

for all \(a, b \in S^2\). Bell noticed that, due to \(f_b^2 = 1\), there holds

\[|\langle f_a f_b \rangle - \langle f_a f_c \rangle| = |\langle f_a f_b (1 - f_b f_c) \rangle| \leq 1 - \langle f_b f_c \rangle\]

for all \(a, b, c \in S^2\). Bell’s famous inequality, however, is in contradiction with \(\langle f_a g_b \rangle = a \cdot b\). Choose, e.g., three coplanar vectors \(a, b, c\), with \(a \cdot b = \frac{1}{\sqrt{2}} = b \cdot c\) and \(a \cdot c = 0\). Then,

\[
\frac{1}{\sqrt{2}} = |a \cdot b - a \cdot c| \leq 1 - b \cdot c = \sqrt{2} - \frac{1}{\sqrt{2}}
\]

and thus the falsity \(2 \leq \sqrt{2}\) follows from Bell’s inequality. \(\square\)

3. Quality of a classical singlet model

Bell’s theorem poses the following problem. Determine the infimum of the set of numbers

\[
\frac{1}{(4\pi)^2} \int_{S^2 \times S^2} |\langle f_a g_b \rangle - [-a \cdot b]|^2 \, da \, db,
\]

obtained from all \textit{local} Kolmogorov models of a spin-singlet state. Any such model\(^2\) consists of a probability space \((\Omega, W)\) and two families of \([1, -1]\)-valued stochastic variables \(\{f_a : a \in S^2\}\) and \(\{g_a : a \in S^2\}\) such that \(\langle f_a g_a \rangle = -1\) for all \(a \in S^2\). Thus, again \(g_a = -f_a\) holds for all \(a \in S^2\). Here, \(da\) and \(db\) denote the rotation-invariant area element on the unit sphere normalized to \(4\pi\).

The solution to this problem would quantify and limit the optimal approximation to the quantum mechanical covariance function

\[
S^2 \times S^2 \ni (a, b) \mapsto \langle \chi, \sigma (a) \otimes \sigma (b) \chi \rangle = -a \cdot b
\]

through classical singlet models.

In this paper, we address a somewhat simpler but related problem. We first confine the admissible direction vectors \(a, b\) from \(S^2\) to a great circle \(S^1 \subset S^2\). Then, we restrict to such \(S^1\)-parametrized families of stochastic variables \(f_a = -g_a\) for all \(a \in S^1\), for which there exists a \textit{continuous} function \(\tilde{C} : \mathbb{R} \to \mathbb{R}\) such that

\[
\langle f_a f_b \rangle = \tilde{C} (a \cdot b) \quad \text{holds for all} \quad a, b \in S^1.
\] (7)

The assumption that \(\langle f_a f_b \rangle\) depends on the scalar product \(a \cdot b\) only amounts to postulating \(O(2)\)-invariance for the mapping \((a, b) \mapsto \langle f_a f_b \rangle\), i.e. the relation \(\langle f_R a f_R b \rangle = \langle f_a f_b \rangle\) for each orthogonal mapping \(R\) which stabilizes the great circle \(S^1\). Under these premises, we shall derive a positive lower bound for

\[
\frac{1}{(2\pi)^2} \int_{S^1 \times S^1} |\langle f_a f_b \rangle - a \cdot b|^2 \, da \, db.
\] (8)

Here, \(da\) and \(db\) denote the rotation-invariant line element on the unit circle normalized to \(2\pi\).

\(^2\) We denote such models as classical singlet models.
An equivalent but simpler formulation of our problem is obtained by periodically parametrizing the circle $S^1$ through real numbers $s$ and $t$, e.g., such that $a(s) = (\cos(s), \sin(s), 0)$ and $b(s) = (\cos(t), \sin(t), 0)$. Then, there exists a continuous $2\pi$-periodic function $C : \mathbb{R} \rightarrow \mathbb{R}$ with $C(a \cdot b) = C(t - s)$ and we obtain

$$\frac{1}{(2\pi)^2} \int_{|s|,|b|} |\langle f_s, f_b \rangle - a \cdot b|^2 \, ds \, db$$

$$= \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} |C(t - s) - \cos(t - s)|^2 \, ds \, dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} |C(t) - \cos(t)|^2 \, dt.$$  

Thus, we try to find a positive lower bound for

$$\frac{1}{2\pi} \int_0^{2\pi} |C(t) - \cos(t)|^2 \, dt,$$

where the continuous function $C : \mathbb{R} \rightarrow \mathbb{R}$ is related to a $2\pi$-periodic, $\{1, -1\}$-valued, stochastic process $\{f_s : s \in \mathbb{R}\}$ as $C(t) = \langle f_s, f_{s+t} \rangle$ for all $s, t \in \mathbb{R}$.

Note that, because of

$$C(-t) = \langle f_{s-t}, f_s \rangle = \langle f_{s-t}, f_{s-t} \rangle = \langle f_s, f_{s+t} \rangle = C(t),$$

the function $C$ is even and it also obeys $C(0) = 1$.

If, in addition to $C(t) = \langle f_s, f_{s+t} \rangle$, the mapping $s \mapsto \langle f_s \rangle$ is constant, the process $\{f_s : s \in \mathbb{R}\}$ is called stationary in the wide sense [5, 6]. In case of $\langle f_s \rangle = 0$ for all $s \in \mathbb{R}$, the function $C$ specializes to the autocorrelation function of a wide-sense-stationary process $\{f_s : s \in \mathbb{R}\}$. However, we will not need any assumption on the expectation values $\langle f_s \rangle$.

The following results from the theory of stationary processes make it clear that insistence on the continuity of $C$ is much less a restriction than it appears to be. They show that the condition of continuity of $C$ can be replaced by the seemingly weaker condition that $C$ is continuous at 0. See, e.g., section 8.10 from [5]. For the sake of completeness, we include the proofs.

**Lemma 2.** Let $\{f_s : s \in \mathbb{R}\}$ be a $\{1, -1\}$-valued stochastic process such that there exists a function $C : \mathbb{R} \rightarrow \mathbb{R}$ with $C(t) = \langle f_s, f_{s+t} \rangle$ for all $s, t \in \mathbb{R}$. Then, $C$ is continuous everywhere if and only if it is continuous at 0.

**Proof.** Observe first that

$$C(t + \varepsilon) - C(t) = \langle f_0, f_{s+\varepsilon} \rangle - \langle f_0, f_s \rangle = \langle f_0, f_{s+\varepsilon} - f_s \rangle.$$ 

Thus, we have, due to the Cauchy–Schwarz inequality and due to $f_t^2 = 1$, that

$$|C(t + \varepsilon) - C(t)|^2 = |\langle f_0, (f_{s+\varepsilon} - f_s) \rangle|^2 \leq |f_0|^2 \langle (f_{s+\varepsilon} - f_s)^2 \rangle$$

$$= \langle (f_{s+\varepsilon} - f_s)^2 \rangle = f_{s+\varepsilon}^2 + f_s^2 - 2 f_{s+\varepsilon} f_s$$

$$= 2(1 - C(\varepsilon)) = 2(C(0) - C(\varepsilon)).$$

From this it follows that

$$\lim_{\varepsilon \to 0} |C(t + \varepsilon) - C(t)| \leq \sqrt{2 \lim_{\varepsilon \to 0} |C(0) - C(\varepsilon)|}.$$  

(10)
Thus, \( \lim_{\varepsilon \to 0} |C(t + \varepsilon) - C(t)| = 0 \) if \( \lim_{\varepsilon \to 0} C(\varepsilon) = C(0) \). Clearly, if \( C \) is continuous everywhere, it is in particular continuous at 0.

**Definition 3.** A stochastic process \( \{f_s : s \in \mathbb{R}\} \), with \( \lim_{\varepsilon \to 0} ((f_{s+\varepsilon} - f_s)^2) = 0 \) for all \( t \in \mathbb{R} \) is called mean-square-continuous.

**Lemma 4.** A stochastic process \( \{f_s : s \in \mathbb{R}\} \) with \( C(t) = \langle f_s f_{s+t} \rangle \) for all \( s, t \in \mathbb{R} \) with values in \([1, -1]\) is mean-square-continuous if and only if \( C \) is continuous at 0.

**Proof.** In the case of a \([1, -1]\)-valued process with the stationary correlation function \( C \) the following holds:

\[
\lim_{\varepsilon \to 0} ((f_{s+\varepsilon} - f_s)^2) = \lim_{\varepsilon \to 0} \left[ (f_{s+\varepsilon}^2) + (f_s^2) - 2 \langle f_{s+\varepsilon}, f_s \rangle \right] = 2 \lim_{\varepsilon \to 0} (1 - C(\varepsilon)) = 2 \lim_{\varepsilon \to 0} (C(0) - C(\varepsilon)).
\]

Thus, in the case of a \([1, -1]\)-valued stochastic process \( \{f_s : s \in \mathbb{R}\} \) for which there exists a function \( C : \mathbb{R} \to \mathbb{R} \) such that \( C(t) = \langle f_s f_{s+t} \rangle \) holds for all \( s, t \in \mathbb{R} \), the following three conditions are equivalent.

(i) \( C \) is continuous.

(ii) \( C \) is continuous at 0.

(iii) \( \{f_s : s \in \mathbb{R}\} \) is mean-square-continuous.

**4. Lower bound for the quality**

Our main tool is the following estimate for the Fourier coefficients of the correlation functions \( C \) which appear in the present context.

**Lemma 5.** Let \( \{f_s : s \in \mathbb{R}\} \) be a \( 2\pi \)-periodic, \([1, -1]\)-valued, mean-square-continuous stochastic process such that \( C(t) = \langle f_s f_{s+t} \rangle \) holds for all \( s, t \in \mathbb{R} \). Then, all the Fourier coefficients \( c_k \) of \( C \) exist and for all \( k \in \mathbb{Z} \) the following estimates hold:

\[
0 \leq c_k = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} C(t) \, dt \leq \frac{1}{2\pi} \int_0^{2\pi} \cos(kt) C(t) \, dt \leq \left(\frac{2}{\pi}\right)^2.
\]

**Proof.** Since the function \( C \) is continuous, its Fourier coefficients exist. Since \( C \) is real valued and even, there holds \( c_k = c_{-k} = c_k \). Thus, the mapping \( k \mapsto c_k \) is real valued and even too. In particular, because of \( c_k \in \mathbb{R} \) we have

\[
\int_0^{2\pi} e^{-ikt} C(t) \, dt = \int_{-\pi}^{\pi} e^{-ikt} C(t) \, dt = \int_{-\pi}^{\pi} (\cos(kt) - i \sin(kt)) C(t) \, dt
\]

\[
= \int_{-\pi}^{\pi} \cos(kt) C(t) \, dt = \int_{-\pi}^{\pi} \cos(kt) C(t) \, dt.
\]

Now for the estimate \( c_k \geq 0 \),

\[
c_k = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} \langle f_s f_{s+t} \rangle \, dt = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} \left( \frac{1}{2\pi} \int_0^{2\pi} \langle f_s f_{s+t} \rangle \, ds \right) \, dt
\]

\[
= \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi} \sin(kt) \left( \int_0^{2\pi} e^{-ikt} \langle f_s f_{s+t} \rangle \, dt \right) \, ds
\]

\[
= \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi} \sin(kt) \left( \int_0^{2\pi} e^{-ikt} \langle f_s f_t \rangle \, dt \right) \, ds
\]
The sum in the last line can be bounded from above as follows. First observe that for all \( \tau \in \mathbb{R} \) there holds
\[
M \left( \sum_{n=1}^{N} \frac{\exp(-i\pi \frac{n}{N})}{N} \frac{f_{2\pi} \frac{\pi}{N}}{\tau} \right) = \left| e^{i\tau} \sum_{n=1}^{N} \frac{\exp(-i\pi \frac{n}{N})}{N} f_{2\pi} \frac{\pi}{N} \right|. 
\]
(11)

Now there exists a number \( \tau_{N} \in [0, \frac{2\pi}{N}] \) depending on \( N \), such that
\[
0 \leq e^{i\tau_{N}} \sum_{n=1}^{N} \frac{\exp(-i\pi \frac{n}{N})}{N} f_{2\pi} \frac{\pi}{N} = \left| e^{i\tau_{N}} \sum_{n=1}^{N} \frac{\exp(-i\pi \frac{n}{N})}{N} f_{2\pi} \frac{\pi}{N} \right|
\]
holds. It then follows for such \( \tau_{N} \) that
\[
\sum_{n=1}^{N} \frac{\exp(-i\pi \frac{n}{N})}{N} f_{2\pi} \frac{\pi}{N} = \sum_{n=1}^{N} \frac{\cos \left(2\pi \frac{n}{N} - \tau_{N}\right)}{N} f_{2\pi} \frac{\pi}{N}.
\]

Since \( f_{2\pi} \frac{\pi}{N} \) assumes values from \((-1, 1)\) only, we obtain from the triangle inequality
\[
\sum_{n=1}^{N} \frac{\cos \left(2\pi \frac{n}{N} - \tau_{N}\right)}{N} f_{2\pi} \frac{\pi}{N} \leq \frac{1}{2\pi} \sum_{n=1}^{N} \frac{2\pi}{N} \left| \cos \left(2\pi \frac{n}{N} - \tau_{N}\right) \right| f_{2\pi} \frac{\pi}{N}.
\]
(12)

The points \( Z_{N} = \left\{ 2\pi \frac{n}{N} - \tau_{N} : n = 1, \ldots, N \right\} \) partition the interval \([0, 2\pi] - \tau_{N}\) whose length is \(2\pi\). The mesh \(2\pi/N\) of \( Z_{N}\) tends to zero for \( N \to \infty\). Furthermore, \(2\pi\) is a period of the function \(x \mapsto |\cos(kx)|\). Therefore, the right-hand side of inequality (12) converges toward the Riemannian integral
\[
\frac{1}{2\pi} \int_{0}^{2\pi} |\cos(kt)| \, dt
\]
(13)

for \( N \to \infty\). Thus, for each \( \varepsilon > 0 \), there exists a number \( N_{0} \), such that for all \( N > N_{0} \)
\[
\sum_{n=1}^{N} \frac{\cos \left(2\pi \frac{n}{N} - \tau_{N}\right)}{N} f_{2\pi} \frac{\pi}{N} \leq \frac{1}{2\pi} \int_{0}^{2\pi} |\cos(kt)| \, dt + \varepsilon.
\]

For \( k \in \mathbb{Z} \setminus 0 \), we have
\[
\int_{0}^{2\pi} |\cos(kt)| \, dt = 4k \int_{0}^{\frac{\pi}{2}} \cos(kt) \, dt = 4k \sin(kt)|_{0}^{\pi/2k} = 4.
\]

Thus, for any \( \varepsilon > 0 \) there exists a \( N_{0} \), such that for all \( N > N_{0} \) the following holds:
\[
\sum_{n=1}^{N} \frac{\exp(-i\pi \frac{n}{N})}{N} f_{2\pi} \frac{\pi}{N} \leq \frac{2}{\pi} + \varepsilon.
\]

We therefore have proven for any \( k \in \mathbb{Z} \setminus 0 \) that
\[
c_{k} = \lim_{N \to \infty} \left\| \sum_{n=1}^{N} \frac{\exp(-i\pi \frac{n}{N})}{N} f_{2\pi} \frac{\pi}{N} \right\|^{2} \leq \left( \frac{2}{\pi} \right)^{2}.
\]
(14)
For $k = 0$, the estimate $c_0 \leq \left(\frac{2}{\pi}\right)^2$ follows from

$$
c_0 = \frac{1}{2\pi} \int_0^{2\pi} C(t) \, dt = 0.
$$

Note that in this proof we did not interchange the limiting process of integration with the probabilistic expectation value, which in general also involves a limit process. Such an interchange can be misleading since the realizations $t \mapsto f_t(\omega)$ need not be integrable for almost all $\omega \in \Omega$. If, however, the two limits can be interchanged, the proof gets abbreviated considerably [2, 7].

From the lemma’s estimate for the case $k = 1$, namely $0 \leq c_1 \leq \left(\frac{2}{\pi}\right)^2$, we now obtain our lower bound for the $L^2$-distance between the quantum mechanical spin-singlet correlation function and its classical approximants.

**Proposition 6.** Let $\{f_s : s \in \mathbb{R}\}$ be a $2\pi$-periodic, $\{1, -1\}$-valued, mean-square-continuous stochastic process such that $C(t) = \langle f_s, f_{s+t} \rangle$ holds for all $s, t \in \mathbb{R}$. Then, the mean-square deviation of $C$ from the quantum mechanical correlation function $\cos$ obeys

$$
\|C - \cos\| = \sqrt{\frac{1}{2\pi} \int_0^{2\pi} (C(t) - \cos(t))^2 \, dt} \geq \frac{1 - \frac{8}{\pi^2}}{\sqrt{2}} \approx 0.133 95. \tag{15}
$$

**Proof.** Note that

$$
\|\cos\|^2 = \frac{1}{2\pi} \int_0^{2\pi} \cos^2(t) \, dt = \frac{1}{2}.
$$

We thus can decompose $C$ into a component parallel to $\cos$ and one orthogonal to it according to

$$
C = \sqrt{2} \cos(\sqrt{2} \cos, C) + (C - \sqrt{2} \cos(\sqrt{2} \cos, C)).
$$

Here, the scalar product between two continuous functions $f, g : \mathbb{R} \to \mathbb{C}$ with a period of $2\pi$ is denoted by

$$
\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(t)g(t) \, dt. \tag{16}
$$

From the estimate

$$
c_1 = \frac{1}{2\pi} \int_0^{2\pi} \cos(t)C(t) \, dt \leq \frac{4}{\pi^2}
$$

it follows that

$$
\|C - \cos\|^2 = \frac{1}{2\pi} \int_0^{2\pi} (C(t) - \cos(t))^2 \, dt
$$

$$
= \|C - \sqrt{2} \cos(\sqrt{2} \cos, C)\|^2 + \|\sqrt{2} \cos(\sqrt{2} \cos, C) - \cos\|^2
$$

$$
\geq \|\sqrt{2} \cos(\sqrt{2} \cos, C) - \cos\|^2
$$

$$
= (1 - 2 \langle \cos, C \rangle)^2 \|\cos\|^2
$$

$$
= \frac{1}{2} (1 - 2 \langle \cos, C \rangle)^2.
$$

Because of $0 \leq 2 \langle \cos, C \rangle = 2c_1 \leq \frac{8}{\pi^2} = 0.810 57$ we finally have

$$
\|C - \cos\| \geq \frac{1 - \frac{8}{\pi^2}}{\sqrt{2}} = 0.133 95. \tag{15}
$$

From this proof it is obvious that the estimate $\|C - \cos\| \geq (1 - \frac{8}{\pi^2}) / \sqrt{2}$ is saturated if and only if $C$ is proportional to $\cos$ which, because of $C(0) = 1$, in turn implies $C = \cos$. Thus, the estimate cannot be saturated and stronger estimates might exist.
5. Bell’s example

Bell [1] constructed a local spin-singlet model with the $2\pi$-periodic autocorrelation function given by $C : \mathbb{R} \rightarrow [-1, 1]$: 

$$C(t) = 1 - 2 \frac{|t|}{\pi} \quad \text{for} \quad 0 \leq |t| \leq \pi. \quad (17)$$

$C$ is continuous and even. Bell’s stochastic variables $\{f_a : a \in S^2\}$ are defined on the set $S^2$ endowed with the uniform distribution. They are given by 

$$f_a(\omega) = \begin{cases} 1 & \text{for } \omega \cdot a > 0 \\ -1 & \text{otherwise} \end{cases}, \quad (18)$$

and indeed yield $\langle f_a f_b \rangle = 1 - 2 \frac{\theta}{\pi}$ with $\theta \in [0, \pi]$ such that $a \cdot b = \cos \theta$ holds.

The Fourier coefficient $c_k$ of $C$ is given for $k \in \mathbb{Z} \setminus 0$ by 

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikt} \left(1 - 2 \frac{|t|}{\pi}\right) dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(kt) \left(1 - 2 \frac{|t|}{\pi}\right) dt$$

$$= \frac{1}{\pi} \int_{0}^{\pi} \cos(kt) \left(1 - 2 \frac{t}{\pi}\right) dt = \frac{1}{k\pi} \int_{0}^{\pi} \frac{d}{dt} \left[\sin(kt)\right] \left(1 - 2 \frac{t}{\pi}\right) dt$$

$$= \frac{1}{k\pi} \left[\frac{\sin(kt)}{k} - \frac{2}{\pi} \int_{0}^{\pi} \sin(kt) dt\right] = \frac{1}{(k\pi)^2} \left[1 - (-1)^k\right]$$

$$= \begin{cases} \frac{4}{(k\pi)^2} & \text{for odd } k \\ 0 & \text{for even } k \end{cases}. \quad (19)$$

Obviously, $c_0 = 0$ holds. The mapping $k \mapsto c_k$ indeed is real valued and even. The estimate $0 \leq c_k \leq 4/\pi^2$ is realized and in the case of $c_1$ saturated. Therefore, it holds that 

$$\|C - \cos\|^2 = \|C\|^2 + \|\cos\|^2 - 2 \langle \cos, C \rangle$$

$$= \|C\|^2 + \frac{1}{2} - 2c_1$$

$$= \|C\|^2 + \frac{1}{2} - \frac{8}{\pi^2}.$$ 

The value of $\|C\|^2$ is given by 

$$\|C\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} C^2(t) dt = \frac{1}{\pi} \int_{0}^{\pi} C^2(t) dt$$

$$= \frac{1}{\pi} \int_{0}^{\pi} \left(1 - 2 \frac{t}{\pi}\right)^2 dt = \int_{0}^{1} (1 - 2x)^2 dx$$

$$= \frac{1}{2} \int_{-1}^{1} y^2 dy = \int_{0}^{1} y^2 dy = \frac{1}{3}. \quad (17)$$

Thus, we have 

$$\|C - \cos\|^2 = \frac{1}{3} + \frac{1}{2} - \frac{8}{\pi^2} = \frac{5}{6} - \frac{8}{\pi^2}.$$ 

and in consequence 

$$\|C - \cos\| = \sqrt{\frac{5}{6} - \frac{8}{\pi^2}} = 0.15088. \quad (19)$$
Note that $C$ has the following particularly simple uniformly converging Fourier series representation:

$$C(t) = \sum_{k=1}^{\infty} \left( c_k e^{ikt} + c_{-k} e^{-ikt} \right) = \sum_{k=1}^{\infty} c_k \left( e^{ikt} + e^{-ikt} \right)$$

$$\begin{align*}
&= \sum_{k=1}^{\infty} 2c_k \cos(kt) \quad \text{(by Parseval's theorem)} \\
&= \sum_{k=1}^{\infty} 2c_{2k+1} \cos((2k+1) t) \\
&= \frac{8}{\pi^2} \sum_{k=0}^{\infty} \frac{\cos((2k+1) t)}{(2k+1)^2} \\
&= \frac{8}{\pi^2} \left[ \cos(t) + \frac{\cos(3t)}{3^2} + \frac{\cos(5t)}{5^2} + \cdots \right].
\end{align*}$$

Acknowledgments

We are indebted to Gregor Weihs for advising us of [2]. Critical remarks by Markus Penz and Tobias Griesser on an earlier version of the manuscript have been helpful.

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