On the multiplicative form of the Lagrangian

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Abstract

An alternative class of the Lagrangian called the multiplicative form is successfully derived for a system with one degree of freedom. This new Lagrangian can be considered as a 1-parameter extended class from the standard additive form of the Lagrangian since both yield the same Newtonian equation.

1 Introduction

In classical mechanics, if one considers the trajectory of a particle on the configuration space, the action functional associated with the trajectory is given by

\[ S[q(t)] = \int_{0}^{T} L(\dot{q}(t), q(t)) dt , \quad \text{(1.1)} \]

where \( L(\dot{q}(t), q(t)) \) is the Lagrange function. We also define \( q = (x, y, z) \) and \( \dot{q} = \frac{dq}{dt} \) as a set of generalised coordinates and generalised velocities. According to Hamilton’s principle or the principle of least actions, the particle evolves in time on the trajectory (curve), called classical path, on the configuration space whose action is stationary under the local deformation. Taking arbitrary infinitesimal local deformation \( q(t) \to q(t) + \delta q(t) \) with the conditions \( \delta q(t = 0) = \delta q(t = T) = 0 \), one obtains

\[ \frac{\partial L}{\partial q} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) = 0 . \quad \text{(1.2)} \]

Eq. (1.2) is called the Euler-Lagrange’s equation and it determines the time evolution of the system along the classical path subject to initial conditions.
Normally, the Lagrange function takes the form $L = T - V$, where $T(q) = \frac{1}{2} m \dot{q}^2$ is the kinetic energy and $V = V(q)$ is the potential energy. Eq. (1.2) gives

$$m \frac{d^2 q}{dt^2} + \frac{dV(q)}{dq} = 0,$$

(1.3)

which is the equation of motion in Newtonian mechanics. The solution $q(t)$ gives the trajectory of the particle subject to initial conditions.

On the equivalent level, there is an alternative way to study the system through the function called Hamilton function (Hamiltonian) $H = T + V$. We find that the Hamiltonian is connected with the Lagrangian through Legendre transformation $H(p, q) = p \dot{q} - L(q, \dot{q})$. Inserting this transformation into the action, we obtain

$$S = \int_0^T [p \dot{q} - H(p, q)] dt,$$

(1.4)

which is associated with the trajectory on the phase space with constant $H$. The variation of the action $\delta S = 0$ gives a system of two coupled first order differential equations

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}.$$  

(1.5)

These two equations also yield Eq. (1.3). It is also found that the time derivative of the first one of Eq. (1.3) (with the help with the second Hamilton’s equation in 1.5) gives

$$\frac{\partial^2 H}{\partial q \partial p} \frac{\partial H}{\partial \dot{p}} - \frac{\partial^2 H}{\partial \dot{p}^2} \frac{\partial H}{\partial q} + \frac{1}{m} \frac{\partial V}{\partial q} = 0.$$  

(1.6)

Interestingly, in [1], a special solution of Eq. (1.6), called a multiplicative form for the Hamiltonian, was solved for the system with one degree of freedom, apart from a standard form, namely, additive form. The multiplicative Hamiltonian takes the form of

$$H_c(x, p) = 4mc^2 \cosh \left( \frac{p}{2mc} \right) \sqrt{1 + \frac{V(x)}{2mc^2}}.$$  

(1.7)

In the limit where the speed of light $c$ becomes extremely large, it is found that

$$\lim_{c \to \infty} (H_c(x, p) - 4mc^2) = \frac{p^2}{2m} + V(x) = H_N(x, p).$$  

(1.8)

This is the standard Hamiltonian found in classical mechanics. This seems to suggest that the parameter $c$ plays a role of speed of light and Eq. (1.8) can be viewed as a non-relativistic limit of the Hamiltonian $H_c$.

From the result in [1], it is relatively natural to ask what a multiplicative form of the Lagrangian is; and hence, this is the main question in which this paper tries to address. In section 2, we set out to find the multiplicative Lagrangian. In section 3, we establish the Legendre transformation between multiplicative Hamiltonian and multiplicative Lagrangian. In section 4, we solve directly the multiplicative Hamiltonian and compare the result with the Legendre transformation. In section 5, we generalise a system of one particle to a system of two identical particles in one dimension. Finally in the last section, we will give the summary with some remarks.
2 Multiplicative Lagrangian

In this section we will construct the multiplicative form of the Lagrangian for the case of one particle in one dimension. We start with an Ansatz form of the Lagrangian
\[ L = F(\dot{x})G(x)\]
where \( F \) and \( G \) are yet to be determined, respectively. We now find that the action functional is
\[ S[x] = \int_0^T L(\dot{x}, x) dt = \int_0^T F(\dot{x})G(x) dt . \] (2.1)

According to the variational principle, it is found that
\[ S[x+\delta x] = \int_0^T L(\dot{x} + \delta \dot{x}, x + \delta x) dt = \int_0^T F(\dot{x} + \delta \dot{x}) G(x + \delta x) dt \]
\[ S + \delta S + ... = \int_0^T \left( F(\dot{x})G(x) + \delta \dot{x} \frac{dF(\dot{x})}{dx} G(x) + \delta x F(\dot{x}) \frac{dG(x)}{dx} + ... \right) dt \]
\[ \delta S \simeq \int_0^T \left( \delta \dot{x} \frac{dF(\dot{x})}{dx} G(x) + \delta x F(\dot{x}) \frac{dG(x)}{dx} \right) dt \]
\[ \simeq \int_0^T \left( -\frac{d}{dt} \left( G(x) \frac{dF(\dot{x})}{dx} \right) + F(\dot{x}) \frac{dG(x)}{dx} \right) \delta x dt , \] (2.2)
with the use of \( \delta x(0) = \delta x(T) = 0. \) According to Eq. (2.3), \( \delta S \) vanishes if
\[ F(\dot{x}) \frac{dG(x)}{dx} - \frac{d}{dt} \left( G(x) \frac{dF(\dot{x})}{dx} \right) = 0 , \] (2.3)
which could be treated as a new Euler-Lagrange equation for the multiplicative Lagrangian. Furthermore, Eq. (2.3) can be re-written in the form
\[ \frac{1}{\dot{x}G} \frac{dG}{dx} \left( F - \dot{x} \frac{dF}{d\dot{x}} \right) - \frac{d^2 F}{d\dot{x}^2} = 0 . \] (2.4)

Using the technique of separation of variables, we set
\[ \frac{1}{\dot{x}G} \frac{dG}{dx} = A \Rightarrow \frac{1}{G} \frac{dG}{dx} = A \dot{x} , \] (2.5)

Using the Newtonian equation
\[ \ddot{x} = -\frac{1}{m} \frac{dV(x)}{dx} , \]
Eq. (2.5) becomes
\[ \frac{1}{G} \frac{dG}{dx} = -\frac{A}{m} \frac{dV(x)}{dx} , \] (2.6)
The parameter \( A \) is a constant which will be determined later. The solution of Eq. (2.3) is simply taken the form of
\[ G(x) = \alpha_1 e^{-\frac{A V(x)}{m}} , \] (2.7)
where $\alpha_1$ is constant. Using Eq. (2.5), Eq. (2.4) now reduces to

$$A \left( F - \dot{x} \frac{dF}{dx} \right) - \frac{d^2 F}{dx^2} = 0 ,$$

(2.8)

and the solution for $F$ is

$$F(\dot{x}) = \alpha_2 \dot{x} - \alpha_3 \left( e^{-\frac{A\dot{x}^2}{2}} + \dot{x} \int e^{-\frac{A\dot{x}^2}{2}} d\dot{x} \right),$$

(2.9)

where $\alpha_2$ and $\alpha_3$ are constant.

Using the results of $F$ and $G$, we then obtain the multiplicative form of the Lagrangian which is given by

$$L(\dot{x}, x) = \left[ k_1 \dot{x} - k_2 \left( e^{-\frac{A\dot{x}^2}{2}} + \dot{x} \int e^{-\frac{A\dot{x}^2}{2}} d\dot{x} \right) \right] e^{-\frac{AV(x)}{m}},$$

(2.10)

where $k_1 = \alpha_1 \alpha_2$ and $k_2 = \alpha_1 \alpha_3$ are constant and yet to be determined. In order to identify all the remaining constants, we may need to consider the limit such that

$$L = \left( k_1 \dot{x} - k_2 \left[ 1 - \frac{A\dot{x}^2}{2} + \frac{1}{2!} \left( \frac{A\dot{x}^2}{2} \right)^2 - \ldots \right] - k_2 \dot{x} A \int \left[ 1 - \frac{A\dot{x}^2}{2} + \frac{1}{2!} \left( \frac{A\dot{x}^2}{2} \right)^2 - \ldots \right] d\dot{x} \right) \left[ 1 - \frac{AV(x)}{m} + \frac{1}{2!} \left( \frac{AV(x)}{m} \right)^2 - \ldots \right].$$

(2.11)

It is found that if we take $A$ to be an inverse square of the velocity: $\lambda^{-2}$, $k_1$ to be zero and $k_2$ to be in energy unit: $-m\lambda^2$, the Lagrangian Eq. (2.11) in the limit that $\lambda$ approaches infinity is

$$\lim_{\lambda \to \infty} \left( L - m\lambda^2 \right) = \lim_{\lambda \to \infty} \left( m\lambda^2 \left[ 1 - \frac{\dot{x}^2}{2\lambda^2} + \ldots \right] + m\dot{x} \int \left[ 1 - \frac{\dot{x}^2}{2\lambda^2} + \ldots \right] d\dot{x} \right) \times \left[ 1 - \frac{V(x)}{m\lambda^2} + \ldots \right]$$

$$= \lim_{\lambda \to \infty} m\lambda^2 \left[ 1 + \frac{\dot{x}^2}{2\lambda^2} + \ldots \right] \left[ 1 - \frac{V(x)}{m\lambda^2} + \ldots \right]$$

$$= \lim_{\lambda \to \infty} m\lambda^2 \left[ 1 + \frac{\dot{x}^2}{2\lambda^2} - \frac{V(x)}{m\lambda^2} + \ldots \right]$$

$$\lim_{\lambda \to \infty} (L - m\lambda^2) = \frac{m\dot{x}^2}{2} - V(x) = L_N ,$$

(2.12)

where $L_N$ is the standard Lagrangian in the additive form. Thus the Lagrangian takes the form of

$$L_\lambda(\dot{x}, x) = m\lambda^2 \left[ e^{-\frac{\dot{x}^2}{2\lambda^2}} + \frac{\dot{x}}{\lambda^2} \int e^{-\frac{\dot{x}^2}{2\lambda^2}} d\dot{x} \right] e^{-\frac{V(x)}{m\lambda^2}},$$

(2.13)

which can be treated as the 1-parameter, namely $\lambda$, extended class of the Lagrangian. Indeed, Lagrangian (2.13) includes the standard Lagrangian in the limit case (2.12)
Next, we define the momentum as
\[ p_\lambda = \frac{\partial L_\lambda}{\partial \dot{x}} = m \left[ \int e^{-\frac{\dot{x}^2}{2\lambda^2}} d\dot{x} \right] e^{-\frac{V(x)}{m\lambda^2}}, \] (2.14)
which in the limit that \( \lambda \) is very large we recover the standard momentum: \( \lim_{\lambda \to \infty} p_\lambda = p = m\dot{x} \). The mass is also given by
\[ m_\lambda = \frac{\partial^2 L_\lambda}{\partial \dot{x}^2} = m \left[ e^{-\frac{\dot{x}^2}{2\lambda^2}} \right] e^{-\frac{V(x)}{m\lambda^2}}, \] (2.15)
which becomes the mass in classical physics in the limit \( \lim_{\lambda \to \infty} m_\lambda = m \).

Next, we find that
\[ \frac{\partial L_\lambda}{\partial x} = \left[ e^{-\frac{\dot{x}^2}{2\lambda^2}} + \frac{\dot{x}}{\lambda^2} \int e^{-\frac{\dot{x}^2}{2\lambda^2}} d\dot{x} \right] e^{-\frac{V(x)}{m\lambda^2}} \left[ -\frac{dV(x)}{dx} \right], \] (2.16)
\[ \frac{\partial^2 L_\lambda}{\partial x \partial \dot{x}} = \frac{1}{\lambda^2} \left[ \int e^{-\frac{\dot{x}^2}{2\lambda^2}} d\dot{x} \right] e^{-\frac{V(x)}{m\lambda^2}} \left[ -\frac{dV(x)}{dx} \right]. \] (2.17)
Substituting Eq. (2.16) and Eq. (2.17) into the Euler-Lagrange equation,
\[ \frac{\partial L_\lambda}{\partial x} - \frac{d}{dt} \left[ \frac{\partial L_\lambda}{\partial \dot{x}} \right] = 0, \]
with the help of the relation
\[ \frac{d}{dt} \left[ \frac{\partial L_\lambda}{\partial \dot{x}} \right] = \ddot{x} \frac{\partial^2 L_\lambda}{\partial \dot{x}^2} + \dot{x} \frac{\partial^2 L_\lambda}{\partial x \partial \dot{x}}, \] (2.18)
we obtain
\[ 0 = \left[ e^{-\frac{\dot{x}^2}{2\lambda^2}} + \frac{\dot{x}}{\lambda^2} \int e^{-\frac{\dot{x}^2}{2\lambda^2}} d\dot{x} \right] e^{-\frac{V(x)}{m\lambda^2}} \left[ -\frac{dV(x)}{dx} \right] - m\ddot{x} \left[ e^{-\frac{\dot{x}^2}{2\lambda^2}} \right] e^{-\frac{V(x)}{m\lambda^2}} \]
\[ - \frac{\dot{x}}{\lambda^2} \left[ \int e^{-\frac{\dot{x}^2}{2\lambda^2}} d\dot{x} \right] e^{-\frac{V(x)}{m\lambda^2}} \left[ -\frac{dV(x)}{dx} \right] \]
\[ = e^{-\frac{\dot{x}^2}{2\lambda^2}} \left[ -\frac{dV(x)}{dx} \right] - m\ddot{x}. \] (2.19)
Then we obtain the equation of motion similar to that in the Newtonian mechanics
\[ m\ddot{x} = -\frac{dV(x)}{dx}, \] (2.20)
which completes the quest for searching an alternative class of the Lagrangian.

3 Legendre transformation

In the previous section, the multiplicative form of the Lagrangian was established. In the additive case, the Hamiltonian and the Lagrangian are connected through the Legendre transformation such that
\[ H_N(p, x) = \frac{\partial L_N}{\partial \dot{x}} \dot{x} - L_N(\dot{x}, x). \]
Here it is interesting to see how it works in the case of the multiplicative Lagrangian. According to the Noether’s theorem, we also find that

\[
\frac{dL_\lambda}{dt} = \frac{\partial L_\lambda}{\partial x} \dot{x} + \frac{\partial L_\lambda}{\partial \dot{x}} \ddot{x}
\]

\[
= \frac{d}{dt} \left( \frac{\partial L_\lambda}{\partial x} \right) \dot{x} + \frac{\partial L_\lambda}{\partial \dot{x}} \ddot{x}
\]

\[
\Rightarrow 0 = \frac{d}{dt} \left( \frac{\partial L_\lambda}{\partial \dot{x}} \dot{x} - L_\lambda(\dot{x}, x) \right),
\]

(3.1)

which implies that the terms in the bracket must be invariant in time leading to

\[
H_\lambda(p, x) = \frac{\partial L_\lambda}{\partial \dot{x}} \dot{x} - L_\lambda(\dot{x}, x),
\]

(3.2)

where \( H_\lambda \) is the Hamilton function. Substituting \( \frac{d}{dt} \left( \frac{\partial L_\lambda}{\partial x} \right) \dot{x} \) into Eq. (3.2) and using \( p = m \dot{x} \), we find

\[
H_\lambda(p, x) = m \lambda^2 \left[ \left( \frac{1}{\lambda^2} \int e^{-\frac{p^2}{2m^2\lambda^2}} \frac{dp}{m} \right) \frac{p}{m} - m \lambda^2 \left( e^{-\frac{p^2}{2m^2\lambda^2}} + \frac{p}{m^2\lambda^2} \int e^{-\frac{p^2}{2m^2\lambda^2}} \frac{dp}{m} \right) \right]
\]

\[
\times e^{-\frac{V(x)}{m^2\lambda^2}}
\]

\[
= -m \lambda^2 e^{-\frac{p^2}{2m^2\lambda^2}} e^{-\frac{V(x)}{m^2\lambda^2}},
\]

(3.3)

which can be considered as the 1-parameter, namely \( \lambda \), extended class of the Hamiltonian and is obviously in the multiplicative form.

### 4 Multiplicative Hamiltonian

In the previous section, the multiplicative Hamiltonian from the Legendre transformation was successfully established. In this section, we invented the method to directly construct the multiplicative Hamiltonian.

It is assumed that the Hamiltonian takes an Ansatz form \( H_\lambda = K(p)B(x) \) and it satisfies the Hamilton equation.

\[
\dot{x} = \frac{\partial H_\lambda}{\partial p}, \quad \dot{p} = -\frac{\partial H_\lambda}{\partial x}.
\]

(4.1)

Using Eq. (4.1), the time derivative of the momentum \( p = m \dot{x} \) can be re-written in the form of

\[
- \frac{\partial H_\lambda}{\partial x} = m \frac{d}{dt} \left( \frac{\partial H_\lambda}{\partial p} \right)
\]

\[
\frac{1}{m} \frac{\partial H_\lambda}{\partial x} = \frac{\partial^2 H_\lambda}{\partial p^2} + \frac{p}{m} \frac{\partial^2 H_\lambda}{\partial p \partial x}.
\]

(4.2)

Inserting \( H_\lambda \) into Eq. (4.2) can be expressed in terms of functions \( K \) and \( B \) as the following

\[
\frac{d^2 K}{dp^2} + \frac{1}{mB} \frac{dB}{dx} \left( p \frac{dK}{dp} + K \right) = 0.
\]

(4.3)
To solve the differential equation, we define
\[
\frac{1}{m} \frac{dB}{dx} = W
\]
where \( W \) is the constant to be determined. Using the Newtonian equation \( \dot{p} = -\frac{\partial V(x)}{\partial x} \), we find that the \( B \) takes the form of
\[
B(x) = \beta_1 e^{-mWV(x)},
\]
where \( \beta_1 \) is constant. Substituting Eq. (4.4) into Eq. (4.2), it is easily found that
\[
K(p) = \beta_2 e^{-\frac{Wp^2}{x}},
\]
where \( \beta_2 \) is also a constant. Then the multiplicative Hamiltonian is given by
\[
H_\lambda(p, x) = \kappa e^{-\frac{Wp^2}{2x} - mAV(x)}
\]
where \( \kappa = \beta_1 \beta_2 \) is a new constant. In order to determine the parameters \( W \) and \( k \), we may proceed the same way as we did in the case of the Lagrangian by choosing \( W = \frac{1}{m^2 \lambda^2} \) and \( k = -m \lambda^2 \) and considering the limit such that \( \lambda \) approaches to infinity
\[
\lim_{\lambda \to \infty} H_\lambda = \lim_{\lambda \to \infty} \left[ -m \lambda^2 \left( 1 - \frac{p^2}{2m^2 \lambda^2} + \ldots \right) \left( 1 - \frac{V(x)}{m \lambda^2} + \ldots \right) \right] = \left[ -m \lambda^2 \left( 1 - \frac{p^2}{2m^2 \lambda^2} - \frac{V(x)}{m \lambda^2} + \ldots \right) \right],
\]
\[
\lim_{\lambda \to \infty} (H_\lambda + m \lambda^2) = \frac{p^2}{2m} + V(x) = H_N,
\]
which is the standard Hamiltonian in the additive form. Finally, the multiplicative Hamiltonian
\[
H_\lambda(p, x) = -m \lambda^2 e^{-\frac{Wp^2}{2m^2 \lambda^2} - \frac{V(x)}{m \lambda^2}},
\]
which takes exactly the same form as the one obtained through the Legendre transformation.

The last step is to show that the Hamiltonian in Eq. (4.8) yields the same equation of motion as that obtained in the Newtonian mechanics. In order to do this task, we substitute the Hamiltonian into Eq. (4.2)
\[
- \frac{dV(x)}{dx} = - \frac{p^2}{m^2 \lambda^2} \frac{dV(x)}{dx} + \dot{p} \left( - \frac{p^2}{m^2 \lambda^2} + 1 \right)
\]
\[
- \frac{dV(x)}{dx} \left( - \frac{p^2}{m^2 \lambda^2} + 1 \right) = \dot{p} \left( - \frac{p^2}{m^2 \lambda^2} + 1 \right),
\]
which leads again to the Newtonian equation
\[
\dot{p} = - \frac{dV(x)}{dx}.
\]

Unsurprisingly, the Hamiltonian obtained in this section has different form from the one given in [1]. This is because we start with the definition of the momentum \( p = m \dot{x} \) instead of the Newtonian equation.
5 One dimensional two-particle system

In this section, the idea is extended to the case of a system of two identical particles in one dimension. The Hamiltonian of the system is given by

$$H_N(p_1, p_2, x_1, x_2) = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + V(x_1 - x_2),$$  \hspace{1cm} (5.1)

where $V(x_1 - x_2)$ is the even function and may take the form of

$$V(x_1 - x_2) = \begin{cases} \frac{g^2(x_1 - x_2)^2}{2} : \text{Harmonic interaction}, \\ \frac{g^2}{(x_1 - x_2)^2} : \text{Calogero-Moser interaction} \end{cases}$$  \hspace{1cm} (5.2)

where $g$ is a coupling constant. The equations of motion for each particle reads

$$\ddot{x}_1 = \frac{1}{m} \frac{\partial V(x_1 - x_2)}{\partial x_1},$$  \hspace{1cm} (5.3a)

$$\ddot{x}_2 = -\frac{1}{m} \frac{\partial V(x_1 - x_2)}{\partial x_2}.$$  \hspace{1cm} (5.3b)

The additive Lagrangian of the system is

$$L_N(\dot{x}_1, \dot{x}_2, x_1, x_2) = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2 - V(x_1 - x_2),$$  \hspace{1cm} (5.4)

which can be obtained through the Legendre transformation $L_N = p_1 \dot{x}_1 + p_2 \dot{x}_2 - H_N$. To decouple the variables $x_1$ and $x_2$ in Eq. (5.3a) and Eq. (5.3b), the new set of variables, namely, $X = x_1 + x_2$ and $x = x_1 - x_2$ are used; and hence, Eq. (5.3a) and Eq. (5.3b) can be re-written as

$$\ddot{X} = 0,$$  \hspace{1cm} (5.5a)

$$\ddot{x} = -\frac{2}{m} \frac{dV(x)}{dx}.$$  \hspace{1cm} (5.5b)

Eq. (5.5a) describes the motion of the centre of mass whereas (5.5b) describes the motion of the system in terms of the relative position between two particles.

We are now looking for the multiplicative Lagrangian in terms of the variables $X$ and $x$ corresponding to Eq. (5.5a) and Eq. (5.5b). Employing the result in the case of one particle, the multiplicative Lagrangian for two particles which describes the motion of free particle and the particle in the potential is

$$L_\lambda(X, \dot{x}, x) = \frac{m \lambda^2}{2} \left[ f(X) + f(\dot{x})g(x) \right],$$  \hspace{1cm} (5.6)

where $F$ and $G$ are already defined in Section 2, however, we repeatedly give them here again

$$f(\dot{u}) = e^{-\frac{\dot{u}^2}{2\lambda^2}} + \frac{\dot{u}}{\lambda^2} \int e^{-\frac{\dot{u}^2}{2\lambda^2}} d\dot{u},$$

$$g(u) = e^{-\frac{V(u)}{m\lambda^2}}.$$

To see how the Lagrangian given by Eq. (5.6) leads to the equations of motion given by Eq. (5.5), we first put the Lagrangian into the Euler-Lagrange equation for the
variable $X$

\[
\frac{\partial L_\lambda}{\partial X} - \frac{d}{dt} \left[ \frac{\partial L_\lambda}{\partial \dot{X}} \right] = 0
\]

\[
0 - \frac{d}{dt} \left[ \frac{df(\dot{X})}{dX} \right] = 0
\]

\[
\dot{X} \int m e^{\frac{-\dot{X}^2}{2\lambda^2}} d\dot{X} = 0.
\]

This results in $\ddot{X} = 0$. Next, we substitute the Lagrangian into the Euler-Lagrange equation for the variable $x$

\[
\frac{\partial L_\lambda}{\partial x} - \frac{d}{dt} \left[ \frac{\partial L_\lambda}{\partial \dot{x}} \right] = 0
\]

\[
f(\dot{x}) \frac{dg(x)}{dx} - \ddot{x} g(x) \frac{d^2 f(\dot{x})}{d\dot{x}^2} = 0
\]

\[
[ -2 \frac{dV(x)}{m\lambda^2} dx - \ddot{x} \frac{2V(x)}{m\lambda^2} d\dot{x} ] e^{-\frac{2V(x)}{m\lambda^2}} = 0
\]

\[
[ -2 \frac{dV(x)}{m\lambda^2} dx - \ddot{x} \frac{\dot{V}(x)}{\lambda^2} ] e^{-\frac{\dot{V}(x)}{\lambda^2}} = 0
\]

\[
\Rightarrow \ddot{x} = -\frac{2}{m} \frac{dV(x)}{dx},
\]

which is indeed the equation of motion.

In addition, in the limit where $\lambda$ is very large, the multiplicative Lagrangian becomes

\[
\lim_{\lambda \to \infty} L_\lambda = \lim_{\lambda \to \infty} \left\{ m\lambda^2 \left( 1 + \frac{\dot{X}^2}{4\lambda^2} + \frac{\dot{x}^2}{4\lambda^2} + \frac{V(x)}{m\lambda^2} + \ldots \right) \right\}
\]

\[
= \lim_{\lambda \to \infty} \left\{ m\lambda^2 + \frac{m(\dot{x}_1 + \dot{x}_2)^2}{4} + \frac{m(\dot{x}_1 - \dot{x}_2)^2}{4} - V(x_1 - x_2) + \ldots \right\}
\]

\[
= \lim_{\lambda \to \infty} \left\{ m\lambda^2 + \frac{m\dot{x}_1^2}{2} + \frac{m\dot{x}_2^2}{2} - V(x_1 - x_2) + \ldots \right\}
\]

\[
\lim_{\lambda \to \infty} \{ L_\lambda - m\lambda^2 \} = \frac{m\dot{x}_1^2}{2} + \frac{m\dot{x}_2^2}{2} - V(x_1 - x_2) = L_N
\]

which yields the additive Lagrangian.

The multiplicative form of the Hamiltonian is given by

\[
H_\lambda = \frac{m\lambda^2}{2} \left[ k(P) + k(p)b(x) \right], \quad (5.7)
\]

where

\[
k(P) = e^{-\frac{P^2}{2m^2\lambda^2}}, \quad (5.8)
\]

\[
b(x) = e^{-\frac{V(x)}{m\lambda^2}}, \quad (5.9)
\]

and $p = p_1 - p_2$ and $P = p_1 + p_2$. The $p_1 = m\dot{x}_1$ and $p_2 = m\dot{x}_2$ are momenta for the primary and secondary particles, respectively. It is easy to show that the
Hamiltonian given by Eq. (5.8) gives the equations of motion, i.e., Eq. (5.5). Firstly, we consider the equation of motion for the centre of mass

$$\frac{\partial H_\lambda}{\partial X} = m \frac{d}{dt} \left( \frac{\partial H_\lambda}{\partial P} \right)$$

$$0 = -\frac{m^2 \lambda^2}{2} \frac{d}{dt} \left( \frac{dk(P)}{dP} \right)$$

$$0 = \dot{P} k(P) ,$$

which \(P\) is a constant as we expected. Secondly, we consider the equation for \(x\) variable

$$-\frac{\partial H_\lambda}{\partial x} = \frac{1}{2} \frac{d}{dt} \left( pk(p) b(x) \right)$$

$$= \frac{1}{2} \left[ 2p^2 k(p) b(x) \frac{dV(x)}{dx} + b(x) \dot{p} \left(-\frac{p^2}{m^2 \lambda^2} k(p) + k(p) \right) \right]$$

$$- \frac{dV(x)}{dx} = \frac{1}{2} \left[ \frac{2p^2}{m^2 \lambda^2} \frac{dV(x)}{dx} + \dot{p} \left(-\frac{p^2}{m^2 \lambda^2} + 1 \right) \right]$$

$$-2 \frac{dV(x)}{dx} \left( -\frac{p^2}{m^2 \lambda^2} + 1 \right) = \dot{p} \left(-\frac{p^2}{m^2 \lambda^2} + 1 \right) \Rightarrow \dot{p} = -2 \frac{dV(x)}{dx} ,$$

which is again the equation of motion that we expected.

Finally, we are interested to see how the multiplicative Hamiltonian reduces to in the limit for vary large \(\lambda\)

$$\lim_{\lambda \to \infty} H_\lambda = \frac{m \lambda^2}{2} \lim_{c \to \infty} \left[ 2 - \frac{p^2 + p^2}{2m^2 \lambda^2} - \frac{2V}{m \lambda^2} + ... \right]$$

$$\lim_{\lambda \to \infty} (H_\lambda + m \lambda^2) = \frac{p^2 + p^2}{4m} + V = \frac{p^2}{2m} + \frac{p^2}{2m} + V = H_N ,$$

which is nothing but the standard Hamiltonian in the additive form.

6 Summary

For the system with one degree of freedom, we successfully derive an alternative class of the Lagrangian which \(L_\lambda\) is an extended class of the Lagrangian through the variable \(\lambda\) of the standard Lagrangian \(L_N\) in the case that the potential depends only on the position. The result in this paper again confirms a feature called the nonuniqueness of the Lagrangian apart from multiplying by a constant or adding a total time derivative term. We also consider the system with two identical particles interacting through an even potential function and succeed to construct the multiplicative form of the Lagrangian. For the case of higher number of particles, especially the Calogero-Moser type systems [3, 4, 5], there exists a hierarchy of the Lagrangians which are all in the additive form. Then it is interesting to see whether we could find the corresponding hierarchy of the multiplicative Lagrangians. We shall leave this problem for another publication. One more remark that we would like to make is about the quantisation of the system. It is known that that we can
quantise the system through the Feynman path integrals for the standard additive form of the Lagrangian. One may ask what is the Feynman’s quantisation method for the multiplicative Lagrangian. We will seriously answer this question elsewhere.

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