Evolution of inspiralling neutron star binaries: effects of tidal interactions and orbital eccentricities

Jie-Shuang Wang
Tsung-Dao Lee Institute, Shanghai Jiao Tong University, Shanghai 200240, China

Dong Lai
Cornell Center for Astrophysics and Planetary Science,
Department of Astronomy, Cornell University, Ithaca, NY 14853, USA and
Tsung-Dao Lee Institute, Shanghai Jiao Tong University, Shanghai 200240, China

Neutron star (NS) binaries formed dynamically may have significant eccentricities while emitting gravitational waves (GWs) in the LIGO/VIRGO band. We study tidal effects in such eccentric inspiralling NS binaries using a set of hybrid equations. The NS is modelled as a compressible ellipsoid, which can deform nonlinearly due to tidal forces, while the orbit evolution is treated with the post-Newtonian (PN) theory up to 2.5-PN order. We find that in general, the tidal interaction can accelerate the inspiral, and cause orbital frequency and phase shifts. For circular inspirals, our calculations reproduce previous linear result at large binary separations, but incorporate the dynamical response of the NS at small separations. For eccentric inspirals, the frequency and phase shifts oscillate considerably near pericenter passages, and the oscillating amplitudes increase with eccentricities. As a result, the GW phase is also significantly influenced by the tidal effect. At merger, the cumulative GW phase shift can reach more than 10 radians (for typical NS mass $1.4M_{\odot}$ and radius 11.6 km), much larger than the circular inspiral case. Although the event rate of eccentric NS mergers is likely low, the detection of such mergers could provide a useful constraint on the NS equation of state.

I. INTRODUCTION

Binary neutron star (BNS) mergers are one of the primary sources for multi-messenger astrophysics. They produce gravitational waves (GWs) and copious electromagnetic signals. The recent LIGO detection of the first binary neutron star merger event GW170817, together with its joint detection in multiple electromagnetic bands, heralded the beginning of multi-messenger astronomy [1].

There are two broad classes of formation channels for merging double compact object (CO) binary systems. The first is the standard isolated binary evolution channel, where the binary components are brought closer by dynamical frictions in mass-transferring or common envelope phases [e.g., see 2–8, and references therein]. CO binaries formed in this channel are expected to have circular orbits while emitting high-frequency GWs in the LIGO/Virgo band. The second class of formation channels involve gravitational interactions between multiple stars and COs. For instance, black hole (BH) binaries in dense star clusters can become bound and shrink in separation due to three-body encounters (e.g. an exchange interaction between a binary and a CO) and/or secular interactions [e.g. 9–10]. Dynamical formation of merging compact binaries can also occur in the galactic field or near supermassive BHs, where CO mergers are induced in hierarchical triple or quadruple systems [17–21]. Eccentric inspirals of CO binaries in the LIGO/Virgo band may be produced in these dynamical channels, although the formation rate is uncertain.

GWs from the final inspiral of BNS systems carry important information of the matter properties under extreme conditions. Tidal effects on the gravitational waveform in circular neutron star (NS) binary inspirals have been studied extensively [e.g., see 22–30, and reference therein], including quasi-equilibrium tides [31, 32] and resonant tides [30, 33–41]. Observations of the NS merger events GW170817 [42] and GW190425 [43] provide direct constraints on the dimensionless tidal deformability parameter, which are then used to constrain the equation of state of the NS.

Recently, a number of studies have examined the tidal effects on eccentric NS binary inspirals [11, 14, 17]. These works treat the tidal deformation or forced NS oscillation in the linear approximation. This is expected to break down at small binary separations, where the tidal effects on the gravitational waveform are strongest. In this paper, we study the tidal effects on the orbital evolution and gravitational waveform of eccentric NS binaries, going beyond the linear approximation. We treat the NS as a deformable ellipsoid, and directly incorporate the tidal force in the evolution of the orbit and the NS. The advantage of adopting this ellipsoid model is that it includes the tidal effects in both the linear and nonlinear regimes. In Section [11], we introduce the dynamical equations used to model the NS binary evolution. We present sample numerical results in Section [14] and discuss the tidal effects on the GW signal in Section [15]. The conclusion are presented in Section [17].

* jiesh.wang@gmail.com
II. DYNAMICAL EQUATIONS OF INSPIRALING NS BINARY

We consider an inspiraling NS binary as an example to explore the tidal effects on the gravitational wave. We employ a set of hybrid dynamical equations: The NS is modelled with the Newtonian dynamical equations (Eqs. 1 below) in the compressible ellipsoidal model, where the buoyancy effect is neglected; and we use the hybrid post-Newtonian equations of motion (up to the 2.5-PN order) to evolve the orbital evolution (Eqs. 6 and 7 below).

The NS is modelled as a compressible Riemann-S ellipsoid with mass \( M \) and a polytropic equation of state \( P = K \rho^{\alpha+1} \). The ellipsoid is characterised by three principal axes \( (a_1, a_2, a_3) \), the angular frequency of the ellipsoidal figure \( (\Omega) \), and the angular frequency of the internal circulation \( (\Lambda) \). \( \omega_1 \) is its initial value (with no tidal deformation and rotation). The mean density is \( \rho = 3M/(4\pi a_1 a_2 a_3) \).

The various post-Newtonian correction terms are

\[
A_H = -1 + \frac{1 - M_i/r}{(1 + M_i/r)^3} - \left( \frac{2 - M_i/r}{(1 - (M_i/r)^2) M_i/\rho^2 + v^2} \right) - \eta \left( \frac{2 M_i}{r^3 - 3v^2 + \frac{3}{2} r^2 \eta^2} + \eta \left( \frac{87}{4} \left( \frac{M_i}{r} \right)^2 + (3 - 4\eta) v^4 - \frac{3}{2} (3 - 4\eta) v^2 \right) \right) + \frac{1}{2} (13 - 4\eta) M_i/\rho^2 - (25 + 2\eta) M_i/r^2 \right) \right) \right] \right],
\]

\[
A_{\lambda/2} = - \frac{8}{5} \frac{M_i}{r} \left( \frac{6v^2 - 2M_i}{3r} - 15\eta^2 \right),
\]

where \( \nu^2 = \dot{r}^2 + r^2 \dot{\theta}^2 \), \( \eta = \mu/M_i \), \( \mu = MM'/M_i \) and \( I_{ii} = \kappa_n Ma_i^2/5 \) for \( i = 1, 2, 3 \). Note that the PN terms in the orbital equations (6) are of the “hybrid” form derived in Ref. 51: These hybrid equations augment the Schwarzschild geodesic equations of motion with the finite-mass terms of the PN2.5 equations of motion; they properly account for the transition from orbital inspiral at large separations to plunge at small separations. This hybrid PN model is in essence similar to the effective one-body (EOB) model 52. We find it’s convenient to incorporate the ellipsoid model in the hybrid PN formalism. A detailed comparison of our results with those using EOB models (e.g. Ref. 53) is beyond the scope of this paper. As in previous semi-analytical works on eccentric mergers, our dynamical equations (1) include the dominant gravitational radiation associated with the orbital motion, but not the enhanced radiation associated with the tidally deformed NS. We estimate that this effect will contribute to about \( \lesssim 15\% \) to the phase error due to tidal effects (see the discussion below Eq. 13).

III. NUMERICAL RESULTS

We consider a NS binary system with \( M = M' = 1.4 M_\odot \), \( R_0 = 11.6 \text{ km} \). The NS is modelled as a poly-

\[
\begin{align*}
\dot{a}_1 &= a_1 (\dot{\Omega}^2 + \Lambda^2) - 2a_2 \dot{\Omega} \Lambda - \frac{2\pi}{\eta a_1 A_1} \rho \\
\dot{a}_2 &= a_2 (\ddot{\Omega}^2 + \Lambda^2) - 2a_3 \dot{\Omega} \Lambda - \frac{2\pi}{\eta a_2 A_2} \rho \\
\dot{a}_3 &= \frac{2\pi}{\eta a_3 A_3} \rho \\
\dot{\Omega} &= \left[ 2 \left( \frac{\dot{\Omega}}{\dot{a}_2} + \frac{\Lambda}{\dot{a}_1} \right) \dot{a}_1 - 2 \left( \frac{\dot{\Omega}}{\dot{a}_1} + \frac{\Lambda}{\dot{a}_2} \right) \dot{a}_2 \right] - 3M' \left( \frac{a_1}{a_2} + \frac{a_2}{a_1} \right) \sin 2\alpha \left( \frac{a_2}{a_1} - \frac{a_1}{a_2} \right) \left( \frac{a_2}{a_1} - \frac{a_1}{a_2} \right) \left( \frac{a_2}{a_1} - \frac{a_1}{a_2} \right) \left( \frac{a_2}{a_1} - \frac{a_1}{a_2} \right)^{-1} \\
\dot{\Lambda} &= \left[ 2 \left( \frac{\dot{\Lambda}}{\dot{a}_2} + \frac{\Lambda}{\dot{a}_1} \right) \dot{a}_1 - 2 \left( \frac{\dot{\Lambda}}{\dot{a}_1} + \frac{\Lambda}{\dot{a}_2} \right) \dot{a}_2 \right] - 3M' \sin 2\alpha \left( \frac{a_2}{a_1} - \frac{a_1}{a_2} \right) \left( \frac{a_2}{a_1} - \frac{a_1}{a_2} \right) \left( \frac{a_2}{a_1} - \frac{a_1}{a_2} \right) \left( \frac{a_2}{a_1} - \frac{a_1}{a_2} \right)^{-1} \\
\dot{r} &= 3\kappa_n \frac{M_i}{10} r^4 \left[ a_1^2 (3 \cos^2 \alpha - 1) + a_2^2 (3 \sin^2 \alpha - 1) - a_3^2 \right] + \frac{M_i}{r^2} (1 + A_H + B_H \tilde{r} + A_{\lambda/2} + B_{\lambda/2} \tilde{r}), \\
\dot{\theta} &= \frac{M_i}{r^2} (B_H + B_{\lambda/2}) \tilde{\theta} - \frac{3\kappa_n M_i}{10} (a_1^2 - a_2^2) \sin 2\alpha - \frac{2\tilde{r} \tilde{\theta}}{r}.
\end{align*}
\]
trope with $n = 0.5$, as it represents a reasonable approximation to a large class of NS equations of state, especially stiff ones. The initial condition is obtained by setting $\tilde{a}_1 = \tilde{a}_2 = \Omega = \Lambda = \tilde{r} = \tilde{r} = \tilde{\theta} = 0$ and $\alpha = \theta = \phi = 0$ (so that $a_1$ and $a_2$ are along $x$ and $y$ axes, respectively). The binary is initially at the apocenter with separation $r_0 = r_{p,0}(1 + e_0)/(1 - e_0)$, where $r_{p,0}$ is the initial pericenter distance. The initial eccentricity $e_0$ is defined in terms of the initial orbital angular frequency $\Omega_{\text{orb},0}$ and $\Omega_{\text{circ,orb},0}$, the angular frequency for circular orbit at $r_0$ (in the absence of tidal effects and PN effects): $\sqrt{1 - e_0} = \Omega_{\text{orb},0}/\Omega_{\text{circ,orb},0}$. We set the initial $\Omega_0 = \Omega_{\text{orb},0}$, and $\Lambda_0 = 2a_1a_2\Omega_0/(a_1 + a_2)$, corresponding to an irrotational NS [49]. We integrate Eqs. (1)-(7) forward in time and stop the integration at $t = t_{\text{merge}}$, when the binary separation decays to $r = 2.5R_0$.

The evolution equations (1)-(7) can be stiff because of the small quantity $(a_2/a_1 - a_1/a_2)$ that appears in the denominators of Eqs. (4) and (5). To check the accuracy of our integration, we first turn off all PN terms and consider a pure Newtonian binary system by setting $A_H = B_H = A_{5/2} = B_{5/2} = 0$. Fig. 2 shows the evolution of the separation and various components of the angular momentum and energy of the system for a binary with $e_0 = 0.2$. The angular momentum of the NS ($J_{\text{NS}}$), the orbital angular momentum ($J_{\text{orb}}$), the energy of NS ($E_{\text{NS}}$) and the orbital energy ($E_{\text{orb}}$), as well as the total angular momentum and energy of the system ($J_{\text{tot}} = J_{\text{NS}} + J_{\text{orb}}$, $E_{\text{tot}} = E_{\text{NS}} + E_{\text{orb}}$) are expressed as Eqs. (2.4)-(2.9), (2.12) and (2.15) in Ref. 24. We see that our integration conserves energy and angular momentum to high precision (better than $10^{-6}$).

We then solve the complete hybrid equations (1)-(7), including the tidal effects and PN terms, for NS-BH inspirals with different initial eccentricities (all with the same initial $r_{p,0} = 7R_0$). Fig. 3 shows the evolution of the ellipsoid radii, and the angular frequency of the ellipsoidal figure ($\Omega$), internal circulation ($\Lambda$), and the angular momentum of the NS. The tidal effects result in the elongation of $a_1$, but the contraction of $a_2$ and $a_3$. It also leads to significant oscillating behaviour of $\Omega$ and $\Lambda$, and a slight increase of the NS spin angular momentum as the orbit decays. The changes of these parameters grow as the orbit decays. For better visualisation, we focus on the last 100 ms of the evolution in the figures, where these parameters change significantly. For zero initial eccentricity, these parameters change gradually, while for larger initial eccentricities, their changes oscillate significantly near each pericenter passage, and strongly influenced by the $f$-modes.

To quantitatively examine the tidal effect on the orbital evolution, we compare the time evolution of the orbital phase ($\theta$), angular velocity ($\Omega_{\text{orb}}$) and binary separation ($r$) between the NS-BH inspiral and BH-BH inspiral (with no tidal effect) in the left panels of Fig. 4. It can be seen that the tidal effects accelerate the merger, and influence the evolution of the orbital parameters prior to the merger. We define the shifts of these parameters between the BH-BH and NS-BH cases as $\Delta x = x_{\text{BH-BH}} - x_{\text{NS-BH}}$ for $x = \theta$, $\Omega_{\text{orb}}$, $r$, $t$ to quan-
tify these effects. We show the evolution of the shifts for 
$e_0 = 0.0$, 0.2, 0.8 in the right panels of Fig. 4. For the
circular case, the shifts change gradually. For eccentric
orbits, the shifts oscillate, and the amplitudes of oscilla-
tion increase with the initial eccentricity. For the orbital
phase, we find that the tidal effects will always lead to a
positive shift of orbital phase ($\Delta \theta > 0$).

Fig. 5 shows the net shift at the merger (defined as
$\Delta x_{\text{merge}} = x_{\text{BH--BH}} - x_{\text{NS--BH}}$) as a function of the
initial eccentricity. We see that the tidal effects lead to
faster mergers with $\Delta t_{\text{merge}} \in (22, 58)$ ms, and higher
final orbital frequencies with $\Delta \Omega_{\text{orb,merge}} \in (-86, 0)$
rad/s. The cumulative orbital phase shifts at the merger
are $\Delta \theta_{\text{merge}} \in (3, 7)$. Especially in some cases with
$e_0 \sim 0.5 - 0.8$, the phase shift is $> 2\pi$, meaning that
the NS-BH case takes more than one orbit less than the
BH-BH case to merge.

To contrast with previous works, we first consider the
case of a circular orbit ($e_0 = 0$). The cumulative orbital
phase shift in the linear tidal regime is given by (see Eq.
(66) of Ref. [51]),

$$
\Delta \theta_{\text{Linear}} = \frac{1}{16} k_2 \left(\frac{39}{4} + M_i/M_f\right) \frac{R_0^5}{M^2 M_i^{1/2}} (r_f^{-5/2} - r_i^{-5/2}),
$$

where $k_2 = 3\kappa_n q_n/2$ is the tidal Love number, and $r_i$
and $r_f$ are initial and final orbit separation. This phase
shift is the same as the widely used expression given by
Ref. [22] after translating the binary separation $r$ into the
orbital frequency (see Eqs. (25) and (28) in Ref. [17]).
However, since our dynamical equations (1)-(7) do not
include GW emission from the tidally deformed NS, the
corresponding linear orbital phase shift is

$$
\Delta \theta_{\text{Linear}} = \frac{39}{64} k_2 \frac{R_0^5}{M^2 M_i^{1/2}} (r_f^{-5/2} - r_i^{-5/2}).
$$

(13)

Comparing Eq. (12) with Eq. (13), we see that the enhanced GW emission due to tidally deformed NS in-
troduces only 9% (for $M' > M$) or 17% (for $M' = M$)
unlike the tidal-induced orbital phase shift. The
derivation of the above two equations assumes that the
tidal forcing frequency ($2\Omega_{\text{orb}}$) is much less than the
intrinsic f-mode frequency ($\omega_f$) of the NS. Taking into ac-
count the dynamical response of the f-mode, the above
equation becomes [29],

$$
\Delta \theta_{\text{Linear}}^{\text{dyn}} = \Delta \theta_{\text{Linear}} \frac{1}{1 - 4\Omega_{\text{orb}}^2/\omega_f^2},
$$

(14)

where $\omega_f = 4\pi G \rho/(15 q_n)$ (see Eq. (3.30) in Ref. [50]).
We compare our numerical results with these analytical
results for circular orbits in Fig. 6. We study two cases:
FIG. 4. Left panels: The last 100 ms of the dynamical evolution of the orbital phase ($\theta$), angular velocity ($\Omega_{\text{orb}}$) and separation ($r$) for the BH-BH inspiral (with no tidal effect) and NS-BH inspiral with $e_0 = 0.8$. Right panels: The shifts of orbital parameters between the BH-BH and NS-BH cases for $e_0 = 0.0$, $0.2$, $0.8$, where $\Delta \theta = \theta_{\text{BH-BH}} - \theta_{\text{NS-BH}}$, $\Delta \Omega_{\text{orb}} = \Omega_{\text{orb,BH-BH}} - \Omega_{\text{orb,NS-BH}}$ (scaled by $\Omega_c = \sqrt{\pi G \rho_0}$) and $\Delta r = r_{\text{BH-BH}} - r_{\text{NS-BH}}$. In all cases, the binary has initial pericenter distance $r_{p,0} = 7R_0$, and starts at the apocenter with separation $r_0 = r_{p,0}(1 + e_0)/(1 - e_0)$.

one with full PN terms as in Eqs. (14), and the other with PN2.5 terms only (i.e. setting $A_H = B_H = 0$, while keeping $A_{5/2}$ and $B_{5/2}$). The latter case is considered because of the fact that the analytical results (Eqs. [12] [14] are obtained without the non-dissipative first and second-order PN corrections. We see that our numerical “PN2.5 terms only” result significantly deviates from $\Delta \theta_{\text{linear}}$ at small radii, but is consistent with $\Delta \theta_{\text{dyn}}$ except at $r \lesssim 3R_0$. The “PN2.5 terms only” result also deviates from the “Full PN terms” result at $r \lesssim 7R_0$, with a deviation amount $\Delta \theta_{\text{PN2.5}} \lesssim 1$. The nonlinear effect becomes important only very close to merger ($r \lesssim 3R_0$). Overall, Fig. 6 indicates that the tide-induced phase shift is strongly influenced by the dynamical effect and by the non-dissipative PN effect on the orbit.

Looking back at Fig. 5 we see that $\Delta \theta_{\text{merge}}$ depends on $e_0$ in a non-monotonic way. The reason is that at the merger ($r = 2.5R_0$), the orbit can still be eccentric, and the pericenter distance of the final orbit can be rather different between the NS-BH and BH-BH cases. As a result, the phase shift and orbit frequency shift would depend on the phase at the merger, i.e. whether the binary reaches $r = 2.5R_0$ near the pericenter or apocenter. The behavior of $\Delta \theta_{\text{merge}} - e_0$ for the $r_{p,0} = 6R_0$ case 54 is largely similar to the result obtained in Ref. 47 using the linear tidal theory, where $\Delta \theta_{\text{merge}} < 0$ is observed for some eccentricities. In our example depicted in Fig. 5, $\Delta \theta_{\text{merge}} > 0$ is found for all values of $e_0$. This can be attributed to the fact that we consider the NS with $n = 0.5$ in this paper, whereas Ref. 47 adopts $n = 1$. The difference in the results between the case of $r_{p,0} = 7R_0$ and that of $r_{p,0} = 6R_0$ arises from the fact that the more compact initial orbits can retain a higher eccentricity near merger, giving rise to a stronger dependence of $\Delta \theta$ on the orbital phase at merger.

IV. EFFECTS ON GRAVITATIONAL WAVES

The leading-order gravitational wave forms of the two polarisation modes are given by [e.g. 47]

$$h_+ = \frac{(1 + \cos^2 \Theta) \mu}{d} \left( r^2 \cos 2\theta + r \dot{r} \cos 2\theta - 4r \dot{r} \Omega_{\text{orb}} \sin 2\theta - 2r^2 \Omega_{\text{orb}}^2 \cos 2\theta - r^2 \dot{\theta} \sin 2\theta \right),$$

(15)

$$h_\times = \frac{2 \cos \Theta \mu}{d} \left( r^2 \sin 2\theta + r \dot{r} \sin 2\theta - 4r \dot{r} \Omega_{\text{orb}} \cos 2\theta - 2r^2 \Omega_{\text{orb}}^2 \sin 2\theta + r^2 \dot{\theta} \cos 2\theta \right),$$

(16)

where $d$ is the distance to the source, $\Theta$ is the angle between the line of sight and the orbit normal direction (the $z$-axis), and we have neglected the contribution from the quadrupole moment of the NS. For simplicity, we consider
\( \Theta = 0 \). The GW amplitude \( (A) \) and phase \( (\Psi) \) are given by

\[
A e^{-i\Psi(t)} = h_+ - ih_\times. \tag{17}
\]

Fig. 7 shows the time evolution of the amplitudes and phases of the BH-BH and NS-BH inspirals in the left panels; in the right panels, we show the difference between the two cases, i.e. \( \Delta x = x_{BH-BH} - x_{NS-BH} \) for \( x = A, \Psi \). We see that the tidal effects can significantly modify both the amplitude and the phase evolution. For the case of \( e_0 = 0.8 \), the amplitude difference oscillates considerably with \( |\Delta A|/A \) reaching as large as 40%. The cumulative phase shift also oscillates significantly just like the orbital phase (see Fig. 4 the top right panel), and the phase shift at the merger reaches \( \Delta \Psi \approx 12 \).

Fig. 8 shows the cumulative phase shift \( (\Delta \Psi_{\text{merge}}) \) at the merger. We see that the tidal effects in the NS-BH binary lead to a significant phase shift \( \Delta \Psi_{\text{merge}} \in (7.1, 13.4) \). Note for the circular case, the GW phase shift \( \Delta \Psi_{\text{merge}} = 2\Delta \theta_{\text{merge}} \). The oscillating behaviour in the \( \Delta \Psi_{\text{merge}} - e_0 \) plot is similar to that in the \( \Delta \theta_{\text{merge}} - e_0 \) plot (see Fig. 5), and is caused by the strong dependence of \( \Delta \theta \) on the orbital phase at the merger if the binary is not fully circularised at \( r = 2.5R_0 \).

V. CONCLUSION

In this paper, we have studied the tidal effects in eccentric inspiralling NS binaries, where the NS is modelled as a Newtonian compressible ellipsoid (which can deform non-linearly in response to tidal forcing) and the PN terms (up to the 2.5-PN order) are incorporated for the orbital evolution. Our treatment in this paper complements the linear mode approach in our recent work [12] and other related works (see Section 1). We find that the tidal effects can accelerate the inspiral process, and induce orbital frequency and phase shifts. It can also lead to the elongation of \( a_1 \), but the contraction of \( a_2 \) and \( a_3 \). The change of ellipsoid radii at the merger can reach...
The full numerical PN result is obtained by integrating Eqs. (6)-(7) including all all PN terms. The cumulative orbital phase shift reaches $\Delta \Phi_{\text{merge}} (\in (7.1, 13.4))$ at the merger, as shown in Fig. 8. For most eccentric inspirals, the phase shifts are much larger than that of circular inspirals. Overall, these results are consistent with the previous calculations based on linear models (e.g. [17]).

As noted in Section 1, the event rate of eccentric NS binary mergers is highly uncertain. The results presented in this paper show that if such systems (with sufficiently large eccentricities) are detected, they would provide useful information on the NS equation of state through the enhanced tide-induced phase shift. But this would require more accurate waveform templates for eccentric NS mergers, due to the non-monotonic dependence of $\Delta \Phi_{\text{merge}}$ on $\epsilon_0$.

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FIG. 7. The GW waveforms for inspirals with $e_0 = 0.8$ and the initial $r_{p,0} = 7 R_0$. Left panels: The evolution of GW amplitude and phase for NS-BH and BH-BH inspirals. Right panels: The evolution of the amplitude difference $\Delta A$ and cumulative phase shift $\Delta \Psi$ between the BH-BH and NS-BH cases.

FIG. 8. The cumulative phase shift $\Delta \Psi_{\text{merge}}$ between the BH-BH and NS-BH inspirals as a function of the initial eccentricity for binaries with initial $r_{p,0} = 7 R_0$.

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Note that for $r_{p,0} = 6R_0$, we only show the result with $e_0 < 0.6$. The reason is that for large eccentricities with $r_{p,0} = 6R_0$, the initial condition is not sufficiently accurate (as the initial condition is set by the equilibrium state), which makes the integration inaccurate.