Robust Matrix Completion with Mixed Data Types

Daqian Sun, Martin T. Wells*

{*ds653, mtw1}@cornell.edu

Department of Statistics and Data Science
Cornell University

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Abstract

We consider the matrix completion problem of recovering a structured low rank matrix with partially observed entries with mixed data types. Vast majority of the solutions have proposed computationally feasible estimators with strong statistical guarantees for the case where the underlying distribution of data in the matrix is continuous. A few recent approaches have extended using similar ideas these estimators to the case where the underlying distributions belongs to the exponential family. Most of these approaches assume that there is only one underlying distribution and the low rank constraint is regularized by the matrix Schatten Norm. We propose a computationally feasible statistical approach with strong recovery guarantees along with an algorithmic framework suited for parallelization to recover a low rank matrix with partially observed entries for mixed data types in one step. We also provide extensive simulation evidence that corroborate our theoretical results.

1 Introduction

The matrix completion problem is related to recovering a low-rank matrix from an observed subset of its entries and was initially shown to be solvable with strong theoretical guarantees Candes and Tao (2010), subsequently many algorithmic frameworks have been proposed for a variety of data settings Cai and Zhou (2016), Klopp (2014), Lafond (2015), and Udell et al. (2016). However, few of these extensions address the matrix completion problem when the underlying data are mixed data types. On the other hand, mixed typed data matrices are quite common in real world applications. For example, the data matrix could have count and binary data as well as continuous entries. For instance, in recommended systems, the numerical ratings and like/dislike are two different data types but it is quite likely that both entries will be stored together. In this paper we propose a novel scalable algorithmic framework that solves the matrix completion problem for mixed data and provides provable recovery guarantees.

The original problem formulation of matrix completion with a rank constraint is computationally challenging and was in fact shown to be NP-hard Vandenberghhe and Boyd (1996). On the other hand, a convex relaxation version of this problem which uses nuclear norm as a surrogate for rank function gained attention because nuclear norm was shown to be a convex envelop of the rank function Maryam (2002). A series of strong recovery guarantees were given by Candes and Tao (2010) and Candes and Recht (2008). Subsequent articles crystallized the canonical concepts and gave proofs of the main results that were simplified along with sharpened guarantee bounds Recht (2011). From that point on, an expanding literature proposed faster algorithms. The primary bottleneck in the traditional convex algorithm lies in the use of an eigenvalue decomposition or singular value decomposition (SVD) in every iteration. Because of this constraint, non-convex algorithm have also been intensely studied. It has been shown that with proper initialization (usually the SVD of the observed matrix), one can obtain good recovery results with high probability using an alternating minimization type algorithm which does not require eigenvalue decomposition Hardt (2014) and Jain, Netrapalli, and Sanghavi (2013).

While fast computational methods abound, most of them only have provable theoretical recovery guarantees in the continuous data setting. Roughly speaking, for these non-convex methods, one implicitly assumes that

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the underlying distribution of the data is Gaussian. Whether the theoretical recovery guarantees of the current fast non-convex methods can be extended to the more general case where the distribution of the matrix is not necessarily continuous is still an open question. On the other hand, the problem of matrix completion in the more general setting has been partially solved using a convex optimization perspective. Davenport et al. (2014) showed that using the maximum likelihood principle, partially observed binary low rank matrix could be recovered by optimizing a convex objective using spectral gradient methods. More general results follow for more a more general family of distributions Gunasekar, Ravikumar, and Ghosh (2014) and Lafond (2015), it was shown that instead of binary data, one could recover with strong theoretical guarantee a low rank matrix whose data follows a distribution in the exponential family. Klopp et al. (2015) considered the multinomial distribution and it was also shown to have a theoretical recovery bound. Cao and Xie (2015) showed that one could derive, using a different approach, a similar recovery bound to that of Gunasekar, Ravikumar, and Ghosh (2014) and Lafond (2015) in the Poisson distribution setting.

It is worth noting that most of the convex algorithms are based on nuclear norm relaxation. Recent work by Fang et al. (2018) and Cai and Zhou (2016) have given empirical evidence that the matrix max norm works better than nuclear norm when the sampling scheme is not uniform. Several extensions of this result also appeared subsequently. T. Cai and Zhou (2013) presented a novel approach to recover, with theoretical guarantee, a binary matrix using max-norm relaxation. Fang et al. (2018) showed that it is possible to use a hybrid of max-norm and nuclear norm to recover a continuous valued matrix and in the same paper, it was shown that an Alternating Direction Method of Multipliers (ADMM) algorithm is a viable approach to solve max-norm related matrix completion problems with reasonably large input sizes.

The problem of matrix completion when the observed matrix has mixed data types has been essentially overlooked. As mentioned previously, Udell et al. (2016) tried to use an alternating minimization approach to solve this problem. But under such framework the only recovery guarantee that were known is the Gaussian case. Gunasekar et al. (2015) showed that such problem, while ill-formed in general, could be solved when some extra conditions are imposed. Recently, Alaya and Klopp (2019) studied the case of mixed data types with convex optimization using nuclear norm regularization. These authors considered the mixed distributions first to be a mixture of exponential family distributions and showed that it could relaxed to any distributions that satisfies a certain Lipchitz condition.

The problem we address in this article is the following. Suppose we are given some matrix \( M \in \mathbb{R}^{n_1 \times n_2} \) and some observed entries \( (M_{ij})_{(i,j) \in \Omega} \), where \( \Omega \) is the observed sets with \(|\Omega| \ll n_1 n_2\). Also the entries of \( M \) has different data types by the columns (this condition could be relaxed to different deterministic index groups) \( M_{C_i} \sim T_i \), where \( C_i \) is the collection of columns of \( M \) and \( T_i \) represents a data type, which is often chosen from the continuous, binary, or count types. Also we assume that

1. \( M \) is approximately low rank.
2. The missing values could follow other schemes than missing at random, that is, the sampling scheme is can be non-uniform.

Our goal in this article is to develop an analytic framework to study the recovery \( M \) as accurately possible. To formulate it precisely, we solve the following optimization program:

\[
\begin{align*}
\text{minimize} & \quad \| X - M \|_F \\
\text{subject to} & \quad X \text{ is low rank and } X_{C_i} \sim T_i.
\end{align*}
\]

Furthermore, we would like to be able to control the rank of the recovered matrix.

A brief overview our main results here and give an a formal presentation in Section 4.

**Theorem 1.** If we choose \( \lambda \) and \( \lambda_{\text{max}} \), where \( \lambda \) and \( \lambda_{\text{max}} \) are regularization parameters as:

\[
\lambda_* = 2c (U_y \vee K) (\sqrt{n_1} + N_2 + (\log(n_1 \vee N_2))^2) \frac{n_1 N_2}{n_1 n_2}
\]

and \( \frac{\lambda_{\text{max}}}{\lambda_*} \leq \kappa \), where \( \kappa > 0 \) is some constant. Then the following recovery guarantees hold for our proposed algorithm is

\[
\frac{1}{n_1 n_2} \| \hat{\Theta} - \Theta \|_F^2 \leq \frac{\text{rank}(\Theta)(n_1 \vee N_2)}{p^2 n_1 n_2}.
\]
In view of Theorem 1, in order to get a small estimation error, \( p \) should be larger (up to multiplicative constant) than \( \text{rank}(\Theta)/(n_1 \wedge N_2) \) and the expected number of observations \( n \) should follow \( n \geq C \text{rank}(\Theta)(n_1 \wedge N_2) \), where \( C \) is some large constant. The inequality in (4) means up to some term that is \( o(1) \).

Before stating and proving the main result, some pertinent topics will be reviewed. A brief review of the exponential family of distributions is in Section 2.1. One key property used implicitly frequently in our result is the mean parametrization, which was detailed in Section 2.1.1. A brief review of definitions of max-norm and nuclear norm is given in Section 2.2. The specification, assumptions and estimation procedure of our model is detailed in Section 3. In Section 4 we state our main result whose proof will be fully presented in Section 7. Detailed description of our proposed algorithm along with its mathematical properties are in the appendix. The results of our extensive numerical experiments on simulated data are presented in Section 6. Finally, auxiliary lemmas and additional details are listed in the appendix.

2 Preliminaries

In general, the two major approaches to handling data types in statistics are parametric and non-parametric. In the parametric approach, we assume the data follows a distribution that is specified up to a finite dimensional parameter, which in most cases could be represented uniquely by its induced probability measure. Whereas nonparametric models are often indexed by a infinite dimensional family. In this paper, we adopt the parametric approach to represent different data types. Specifically, we restrict the distribution of the underlying data to be nonparametric models are often indexed by an infinite dimensional family. In this paper, we adopt the parametric approach to represent different data types. Specifically, we restrict the distribution of the underlying data to be in the exponential family, of which we will provide a brief review. Much of the review material in this section is adapted from Wainwright and Jordan (2007), we refer interested readers to the original paper for more details.

2.1 Exponential Family

Given a random element \( X = (X_1, \ldots, X_n) \in \otimes_{i=1}^n \mathcal{X} \), where \( \mathcal{X} \) is some arbitrary space with induce probability measure \( \mathbb{P} \). Let \( \phi = (\phi_\alpha, \alpha \in \mathcal{I}) \) be a collection of functions of \( \phi_\alpha : \mathcal{X}^m \rightarrow \mathbb{R} \), sometimes known as sufficient statistics or potential functions. Here \( \mathcal{I} \) is an index set with \( |\mathcal{I}| = d \) such that \( \phi(X) : \otimes_{i=1}^n \mathcal{X} \rightarrow \mathbb{R}^d \) is a vector valued mapping. For a given choice of \( \phi(X) \), we further associate with it another set of vector \( \theta \in \mathbb{R}^d \), which is often called canonical parameters. With this definition, we can then have a relatively generalized definition of exponential family.

Definition 1. The exponential family induced by \( \phi \) is a family of probability density functions (Radon Nikodym derivatives taken with respect to \( d\nu \), a base measure) of the form

\[
\frac{d\mathbb{P}}{d\nu} = p_\theta(x_1, x_2, \ldots, x_n) = \exp \left(\langle \theta, \phi(x) \rangle - A(\theta) \right).
\]

The quantity \( A(\theta) \), known as the log-partition function or cumulant function, is defined by the following integral:

\[
A(\theta) = \log \int_{\mathcal{X}^n} \exp \left(\langle \theta, \phi(x) \rangle \right) \nu(dx).
\]

This integral, if finite, acts as a normalizing factor for the density function \( p_\theta \). Holding the set of \( \phi \) fixed, each parameter vector \( \theta \) corresponds to a particular member \( p_\theta \) of the family. Since \( A(\theta) \) is not always finite, the set of parameter \( \theta \) of interests is the one corresponding to a finite log-partition function, i.e. belong to the set

\[
\Omega := \{ \theta \in \mathbb{R}^d | A(\theta) < +\infty \}.
\]

From now on, unless it’s otherwise defined, we use \( \mathcal{E} \) to denote a exponential family distribution.

While \( \Omega \) is always well-defined because it can be viewed as a pull back of a measurable function from a Borel set, it does not always possess nice topological properties. One example is the case where \( \Omega \) is a closed set. This is not ideal because when \( \theta \) is on the boundary, its \( \epsilon \)-ball is not properly contained in the space, thus rendering the limiting behaviors irregular. Although bad cases such as a closed \( \Omega \) do exists, they are mostly for pedological purposes. In turns out when \( \Omega \) is an open set, it behaves nicer analytically. Thus, we often say an exponential family for which the domain \( \Omega \) is an open set is a regular exponential family. Almost all of the common distributions that we encounter in the exponential family is regular.

3
Given an exponential family with a vector of sufficient statistics \( \phi \), if there does not exist a non-zero vector \( a \in \mathbb{R}^d \) such that the linear combination

\[
\langle a, \phi(x) \rangle = \sum_{\alpha \in \mathcal{I}} a_{\alpha} \phi_\alpha(x)
\tag{7}
\]
is equal to a constant, then we say this exponential has a minimal representation. The notion of minimal representation addresses the problem of identifiability. In other words, with a minimal representation, the canonical parameter \( \theta \) associated with each distribution is unique.

The notion of over-complete representation is the analog to minimal representation. With an over-complete representation, there is a non-zero vector \( a \in \mathbb{R}^d \) such that

\[
\langle a, \phi(x) \rangle = c
\]
for some constant \( c \in \mathbb{R} \).

As one might expect, this might cause problems in identifiability. Indeed, for a member of the exponential family with over-complete representation, the canonical parameter \( \theta \) associated to it is no longer unique, instead, there is an entire affine set of \( \theta \) for it.

### 2.1.1 Mean parameterization and the log partition function

In turns out that many important parametric statistical inference problems are related to the relationship between the canonical parameters and mean parameters of distributions in the exponential family. In the context of the problem at hand, the connection could be formulated as follows: suppose we have observed a low rank matrix \( M \) with missing entries with mixed exponential distributions. What is the most likely recovery, \( \hat{M} \)? Assuming the underlying distribution does not change much, then a natural way to recover the matrix is to resample it for many times and take its mean. However, it is not feasible to sample the such matrix because we are not aware of the exact parameter of the underlying distribution. Therefore, it is natural then to ask for the most likely parameter value given the current observation, which is a classical maximum likelihood estimation problem. For more examples, see Wainwright and Jordan (2007).

It is then natural to explore the relationship between the canonical parameter of an exponential distribution and its corresponding mean. To start with, we state the following result.

**Proposition 1.** The cumulant function in (5) associated with any regular exponential family has the following properties.

1. It has derivatives of all orders on its domain \( \Omega \) and

\[
\frac{\partial A}{\partial \theta_\alpha}(\theta) = \mathbb{E}_\theta[\phi_\alpha(X)] = \int \phi_\alpha(x)p_\theta(x)d\nu,
\tag{8}
\]

\[
\frac{\partial^2 A}{\partial \theta_\alpha \partial \theta_\beta}(\theta) = \mathbb{E}_\theta[\phi_\alpha(X)\phi_\beta(X)] - \mathbb{E}_\theta[\phi_\alpha(X)]\mathbb{E}_\theta[\phi_\beta(X)].
\tag{9}
\]

2. \( A \) is a convex function of \( \theta \) on its domain \( \Omega \) and strictly so if the representation is minimal.

The proof of **Proposition 1** is standard and uses the dominated convergence theorem. This proposition builds a forward mapping from the canonical parameter space to the mean parameter space, which is the gradient map of \( A \). In fact, the following result shows the mapping is surjective with some mild regularity conditions.

**Theorem 2.** Given an exponential family distribution, \( \mathcal{E} \), with a sufficient statistic \( \phi \), then

1. the gradient mapping \( \nabla A : \Omega \to \mathcal{M} \) is injective if and only \( \mathcal{E} \) is minimal, and

2. the gradient mapping \( \nabla A : \Omega \to \mathcal{M} \) is surjective for \( \mu \in \text{Int}(\mathcal{M}) \) if \( \mathcal{E} \) is minimal.

Another important connection between the mean parameterization and the log partition follows using duality theory. The conjugate dual function to \( A \), which we denote by \( A^* \), is defined as follows:

\[
A^*(\mu) = \sup_{\theta \in \Omega} \{ \langle \theta, \mu \rangle - A(\theta) \},
\tag{10}
\]

where \( \mu \in \mathbb{R}^d \) is a fixed vector of so-called dual variables of the same dimension as \( \theta \). These dual variables turn out to have a natural interpretation as mean parameters. The theorems below connect the conjugate dual function of \( A \) to the Shannon Entropy of \( p_\theta(\mu) \), \( H(p_\theta(\mu)) \) and the mean parametrization.
Theorem 3. For any \( \mu \in \text{Int}(\mathcal{M}) \) let \( \theta(\mu) \) denote the unique canonical parameter satisfying the dual matching condition. The conjugate dual function \( A^* \) takes the form
\[
A^*(\mu) = \begin{cases} 
-H(p_{\theta(\mu)}) & \text{if } \mu \in \text{Int}(\mathcal{M}) \\
+\infty & \text{if } \mu \notin \mathcal{M}.
\end{cases}
\] (11)

Theorem 4. The log-partition function has the following variational representation
\[
A(\theta) = \sup_{\mu \in \mathcal{M}} \{ \langle \theta, \mu \rangle - A^*(\mu) \}.
\] (12)

Moreover, for all \( \theta \in \Omega \), the supremum is attained uniquely at \( \text{Int}(\mathcal{M}) \) and is specified by the moment matching conditions
\[
\mu = \int_X \phi(x)p_{\theta}(x)dx = E_{\theta}[\phi(X)].
\] (13)

The final connection with the log partition function connects Bregman and Kullback-Leibler divergences. We introduce the notion of Bregman Divergence since its connection with the Kullback-Leibler divergence of exponential family which will be used frequently in the proof of the recovery upper bound.

**Definition 2.** Let \( S \) be a closed convex subset of \( \mathbb{R}^m \) and \( \Phi : S \subset \text{dom}(\Phi) \rightarrow \mathbb{R} \) a continuously differentiable and strictly convex function. The Bregman divergence of \( \Phi \), denoted as \( d_\Phi : S \times S \rightarrow [0, \infty) \) is defined as
\[
d_\Phi(x, y) = \Phi(x) - \Phi(y) - \langle x - y, \nabla \Phi(y) \rangle.
\] (14)

The next proposition from Wainwright and Jordan (2007) gives the connection of Kullback-Leibler divergence of distributions in exponential family and Bregman divergence.

**Proposition 2.** For exponential family distributions, the Bregman divergence corresponds to the Kullback-Leibler divergence with \( \Phi = A \).

### 2.1.2 Common Examples of Members of the Exponential family

**Example 1 (Gaussian).** The Gaussian distribution is widely used in modeling continuous data. A Gaussian random variable with mean \( \mu \) and variance \( \sigma \) has the following form of density:
\[
f_X(x|\mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left\{ -\frac{(x - \mu)^2}{2\sigma^2} \right\} \quad (\mu, \sigma) \in \mathbb{R} \times \mathbb{R}^+
\] (15)
which in its exponential family form, could be written as
\[
f_X(x|\mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left\{ -\log \sigma - \frac{x^2}{2\sigma^2} + \frac{\mu x}{\sigma^2} - \frac{\mu^2}{2\sigma^2} \right\}
\] (16)
and
\[
f_X(x|\theta) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left\{ \theta(x) - A(\theta) \right\},
\] (17)
where \( \theta = (\mu/\sigma^2, -1/(2\sigma^2)) \) and \( \phi(x) = (x, x^2) \). The canonical parameterization of the Gaussian distributions is the mean parametrization.

**Example 2 (Gamma).** The Gamma distribution is a two parameter continuous probability distribution. It is often used to model the size of insurance claims and rainfalls, see Hewitt and Lefkowitz (1979), Husak, Michaelson, and Funk (2007). In wireless communication, the Gamma distribution is used to model the multipath fading of signal power. It also has wide application in the field of neuroscience, genomics, and oncology. A random variable \( X \) is said to follow Gamma distribution with parameter \( \alpha \) and scale \( \theta \) if it has probability density function
\[
f_X(x|\alpha, \theta) = \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-\frac{x}{\theta}} \text{ for } (\alpha, \theta) \in \mathbb{R}^+ \times \mathbb{R}^+
\] (18)
In its exponential family canonical form, we have
\[
f_X(x|\alpha, \theta) = \exp \left\{ -\log \Gamma(\alpha) - \alpha \log \theta + (\alpha - 1) \log x - \frac{x^2}{\theta} \right\}.
\] (19)

Note that \( E[X] = \theta \alpha \). Hence, the mean parameterization for Gamma distribution can be written as
\[
f_X(x|\mu, \alpha) = \exp \left( x \left( -\frac{\alpha}{\mu} \right) + (\alpha - 1) \log x - \log \Gamma(\alpha) - \alpha \log \mu + \alpha \log \alpha \right).
\] (20)
**Example 3** (Bernoulli). The Bernoulli distribution is the most used distribution to model binary data. In its most common form, the probability mass function (p.m.f) for a Bernoulli random variable as follows

\[
P(X = x|p) = p^x(1 - p)^{1-x}, \quad \text{for } (x, p) \in \{0, 1\} \times [0, 1]
\]

which in its exponential family form, could be rewritten as

\[
P(X = x|p) = \exp \left\{ x \log \frac{p}{1 - p} + \log(1 - p) \right\}.
\]

Since \(\mathbb{E}[X] = p\), (22) is also the mean parameterization with \(p = \mu\).

**Example 4** (Poisson). The Poisson distribution is frequently used in fitting count data. A Poisson random variable is often defined by its being equipped with the following p.m.f

\[
P(X = x|\lambda) = \frac{e^{-\lambda} \lambda^x}{x!} \quad \text{for } (x, \lambda) \in \mathbb{N} \times \mathbb{R}^+.
\]

In its exponential family canonical form,

\[
P(X = x|\lambda) = \frac{1}{x!} \exp \{x \log \lambda - \lambda\}.
\]

Note that \(\mathbb{E}[X] = \lambda\), (24) is also the mean parameterization with \(\lambda = \mu\).

**Example 5** (Negative Binomial). The negative binomial distribution is often used to model the number of failures before \(r\) success in a stream of independent Bernoulli trials. It is also used as a alternative to the Poisson distribution for count data that accounts for possible over-dispersion via the representation as a Poisson-Gamma mixture. A most common way to parameterize negative binomial distribution is by the Poisson for count data that accounts for possible over-dispersion via the representation as a failures before \(r\) success in a stream of independent Bernoulli trials. It is well-known that extended definition (26) could be interpreted as a Poisson-Gamma mixture,

\[
\mathbb{P}(X = k) = \binom{k + r - 1}{k} p^r(1 - p)^k, \quad \text{for } p \in [0, 1], k \in \mathbb{N}.
\]

We note that such definition could be extended to \(k \in \mathbb{R}\) with a slight extension of the definition:

\[
\mathbb{P}(X = k) = \frac{\Gamma(k + r)}{k!\Gamma(r)} p^r(1 - p)^k \quad \text{for } p \in [0, 1], k \in \mathbb{R}.
\]

It’s easy to check that when \(k \in \mathbb{N}\), the extended version falls reduces to the original negative binomial distribution. It is well-known that extended definition (26) could be interpreted as a Poisson-Gamma mixture, which is sometimes useful in some model fitting problems. Its mean parameterization of the following form (cf. Lemma 18):

\[
\mathbb{P}(X = k|\mu, r) = \frac{\Gamma(k + r)}{\Gamma(r)\mu!} \left( \frac{r}{\mu + r} \right)^r \left( \frac{\mu}{\mu + r} \right)^k, \quad \text{for } \mu \in \mathbb{R}^+, r \in \mathbb{N}.
\]

In the previous examples we have seen how the definition of exponential family manifests in simple scalar random variables. To further illustrate the usage of concept of exponential family in the setting of our problem interest, we now showcase the random exponential family matrices.

**Example 6** (Independent Exponential Family Random Matrix). Let \(X \in \mathbb{R}^{m \times n}\) be a random matrix, where each entry \(X_{ij}\) is drawn independently from the same exponential family characterized by the density \(\mathbb{P}(X_{ij}|\Theta_{ij}) = h(X_{ij}) \exp(X_{ij}\Theta_{ij} - G(\Theta_{ij}))\), then it follows from factorization theorem,

\[
\mathbb{P}(X|\Theta) = \prod_{i,j} h(X_{ij}) \exp(X_{ij}\Theta_{ij} - G(\Theta_{ij})) = h(X) \exp ((X, \Theta) - G(\Theta)),
\]

where by slightly abuse of notation we denote \(G : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}\) as \(G(\Theta) = \sum_{ij} G(\Theta_{ij})\) and \(h(X) = \prod_{ij} h(X_{ij})\).

The next example is important as it serves as the basis of our model formulation.
Example 7 (Structurally Heterogeneous Exponential Family Random Matrix). Let \( X = [X^{(1)} \, X^{(2)} \ldots \, X^{(k)}] \) be \( \mathbb{R}^{m \times n} \), where \( X^{(i)} \in \mathbb{R}^{m \times n_i} \) such that \( \sum_{i=1}^{k} n_i = n \) be a random matrix consisting of column-wise disjoint sub-matrices \( X^{(i)} \), and for each \( X^{(s)} \) is an independent exponential family random matrix as in Example 6 with density \( P(X^{(s)}|\Theta^{(s)}) = h^{(s)}(X) \exp\left(\langle X^{(s)}, \Theta^{(s)} \rangle - G^{(s)}(\Theta^{(s)})\right) \). Again, by independence and factorization theorem, we have that the density for \( X \) as

\[
P(X|\Theta) = \prod_{s=1}^{k} h^{(s)}(X^{(s)}) \exp\left(\langle X^{(s)}, \Theta^{(s)} \rangle - G^{(s)}(\Theta^{(s)})\right).
\] (28)

2.2 Matrix Norms

Both the nuclear norm and max norm will serve as important tools in the derivation of a tractable formulation of the matrix completion problem. Before we delve deeper, a few definitions are needed.

Definition 3. Let \( A \in \mathbb{R}^{n_1 \times n_2} \) and \( A = U \Sigma V^T = \sum_{i=1}^{n_1 \wedge n_2} \sigma_i u_i v_i^T \) be its singular value decomposition, then the nuclear norm of \( A \) is defined as

\[
\|A\|_n = \sum_{i=1}^{n_1 \wedge n_2} \sigma_i.
\] (29)

Since the rank function could be defined as the "\( \ell_0 \) norm" of the vector of singular values of a matrix, the nuclear norm, which can be seen as the \( \ell_1 \) counterpart of the same concept, intuitively should be a good approximation of the rank function. Formally, Maryam (2002) showed that the convex envelop of rank(\( X \)) for \( X \in \{X \in \mathbb{R}^{m \times n} : \|X\| \leq 1\} \) is the nuclear norm.

Definition 4. Let \( A \in \mathbb{R}^{n_1 \times n_2} \). The max norm of \( A \) is defined as

\[
\|A\|_{\max} = \min_{U,V \text{ s.t. } A=UVT} \|U\|_{\ell_2 \rightarrow \ell_\infty} \|V\|_{\ell_2 \rightarrow \ell_\infty},
\] (30)

where \( \|\cdot\|_{\ell_2 \rightarrow \ell_\infty} \) is the operator norm from \( \ell_2 \) to \( \ell_\infty \) defined by

\[
\|A\|_{\ell_2 \rightarrow \ell_\infty} = \sup_{\|x\|_2 \leq 1} \|Ax\|_\infty.
\] (31)

The direct connection between max norm and rank is a bit technical. So we avoid it here. Instead, we observe this connection by taking a look at the connection between the max norm and the nuclear norm. It is well known that nuclear norm has the following alternative representation:

\[
\|A\|_n = \min_{\|u\|_2=\|v\|_2=1} \sum_j |\sigma_j|.
\] (32)

On the other hand, Jameson (1987) showed that

\[
\|A\|_{\max} \simeq \min_{\|u\|_\infty=\|v\|_\infty=1} \sum_j |\sigma_j|,
\] (33)

where the factor of equivalence is the Grothendieck’s constant \( K \in (1.67, 1.79) \). Roughly, the similarity in representations suggests that max norm may be a good approximation of rank function. For a more precise characterization of the relationship between rank and the max norm, see Srebro and Shraibman (2005).

Both the max and nuclear norms have their respective semi-definite program representations. This makes the numerical computation easier, especially in the case of max norm, where computing from the original definition is NP-hard. Let \( A \in \mathbb{R}^{m \times n} \) be an arbitrary matrix. Then the nuclear norm of \( A \) was be represented by Maryam (2002) as the solution to the following semi-definite programs:

\[
\max \quad A \cdot X
\]
subject to

\[
\begin{bmatrix}
I_m & Y \\
Y^T & I_n
\end{bmatrix} \succeq 0
\] (34)
(35)
and its dual
\[
\begin{align*}
\min & \quad \text{tr}(W_1) + \text{tr}(W_2) \\
\text{subject to} & \quad \begin{bmatrix} W_1 & -\frac{1}{2}A \\ -\frac{1}{2}A^T & W_2 \end{bmatrix} \succeq 0.
\end{align*}
\] (36)

Furthermore, \(\|A\|_* \leq t\) if and only if there exists \(Y \in \mathcal{S}^m\) and \(Z \in \mathcal{S}^n\) such that
\[
\text{tr}(Y) + \text{tr}(Z) \leq 2t, \quad \begin{bmatrix} Y & X \\ X^T & Z \end{bmatrix} \succeq 0.
\] (37)

On the other hand, Srebro, Rennie, and Jaakkola (2004) showed that \(\|A\|_{\text{max}}\) can be represented as the solution to the following semi-definite program:
\[
\begin{align*}
\min & \quad R \\
\text{subject to} & \quad \begin{bmatrix} W_1 & A \\ A^T & W_2 \end{bmatrix} \succeq 0, \\
& \quad \|\text{diag}(W_1)\|_\infty \leq R, \|\text{diag}(W_2)\|_\infty \leq R.
\end{align*}
\] (38)

We will see later that this representation will facilitate the reformulation of our objective function.

## 3 Model Specification

### 3.1 Set up and assumptions

Let \(\Theta = [\Theta_{ij} \Theta_{i\tau} \ldots \Theta_{ij\tau}] \in \mathbb{R}^{n_1 \times n_2}\) be the full matrix (unknown truth), where \([\Theta_{ij\tau}]_{\tau \in T} \in \mathbb{R}^{n_1 \times n_2}\) represent sub-matrices whose entries follow different distributions in the exponential family, that is,
\[
\frac{d\mathbb{P}_{\Theta_{ij\tau}}}{d\lambda} = h^\tau(\Theta_{ij\tau}) \exp(X_{ij\tau}^\tau - G^\tau(\eta_{ij\tau}))
\] (42)

and \(N_2 := \sum_{\tau \in T} n_{ij\tau}^2\). Let observed matrix be \(Y \in \mathbb{R}^{n_1 \times n_2}\). Additionally, we assume the entries of \(\Theta\) are uniformly bounded, that is, \(\|\Theta\|_{\infty} \leq \gamma\) for some \(\gamma \in \mathbb{R}\). For ease of notation, let \(C(\gamma) := \{X \in \mathbb{R}^{n_1 \times n_2} : \|X\|_{\infty} \leq \gamma\}\) be the \(\ell_{\infty}\) ball with radius \(\gamma\).

Before we delve into estimation, we first address the procedure with which the observed incomplete matrix, \(Y\), is determined. Formally, \(Y\) is generated by associating each full matrix \(X_{ij\tau}\) with a Bernoulli random variable \(\delta_{ij\tau} \sim \text{Bin}(\pi_{ij\tau})\) and let \(Y_{ij\tau} = \delta_{ij\tau} X_{ij\tau}.\) Here, \(\pi_{ij\tau}\) can be thought of as the sampling rate. In the easiest case, uniform sampling scheme, we could consider \(\pi_{ij\tau} = \alpha \in (0, 1)\), where \(\alpha\) is some constant. For an intuitive understanding for this scheme, imagine we are scanning through \(X\) entry by entry in a row-major manner, for each entry, we stop and toss a coin which has a probability of \(\alpha\) landing on a head, and probability of \(1 - \alpha\) on a tail. If landed on a head, we keep \(X_{ij\tau}\) the same; otherwise we let \(X_{ij\tau} = 0\). Notice that in this example, we used the same coin throughout the double loop. In the non-uniform sampling scheme, the same coin analog still holds with one simple modification, we can potentially use a different coin with different head probability for every entry at which we stop.

We now introduce two mild but necessary assumptions for our model. These two assumptions are common in previous literature, cf. Alaya and Klopp (2019), Gunasekar, Ravikumar, and Ghosh (2014), Lafond (2015), and Klopp (2014).

**Assumption 1.** Each entry has a positive probability of being observed, that is,
\[
\min_{\tau} \min_{i,j \in [n_1] \times [n_2]} \pi_{ij\tau}^\tau \geq p
\] (43)

for \(p \in (0, 1)\).

**Assumption 2.** The curvature of \(A^\tau(x)\) is bounded, that is
\[
\begin{align*}
\sup_{\eta \in [-\gamma^{-\frac{1}{2}} \gamma + \frac{1}{2}]} \nabla^2 G^\tau(\eta) & \leq U_\gamma^2 \\
\inf_{\eta \in [-\gamma^{-\frac{1}{2}} \gamma + \frac{1}{2}]} \nabla^2 G^\tau(\eta) & \leq L_\gamma^2
\end{align*}
\] (44) (45)
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
Model & $L_\gamma$ & $U_\gamma$ \\
\hline
Normal & $\sigma^2$ & $\sigma^2$ \\
Binomial & $\frac{N e^{-(\gamma + \frac{1}{K})}}{(1 + e^{-(\gamma + \frac{1}{K})})^2}$ & $\frac{N}{4}$ \\
Gamma & $\frac{\alpha}{\gamma + \frac{1}{K}}$ & $\frac{\alpha}{\gamma + \frac{1}{K}}$ \\
Negative binomial & $\frac{1 - e^{-(\gamma + \frac{1}{K})}}{(1 - e^{-(\gamma + \frac{1}{K})})^2}$ & $\frac{1 - e^{-(\gamma + \frac{1}{K})}}{e^{-(\gamma + \frac{1}{K})}}$ \\
\hline
\end{tabular}
\caption{Examples of $L_\gamma$ and $U_\gamma$ functions in Assumptions 1 and 2 for various member of the exponential family.}
\end{table}

Note that Assumption 1 is natural in the sense that if there are some entries with 0 probability of being sampled, then the problem could become completely intractable in the sense that if we let a whole row to be unobserved then it would be possible that the matrix is full rank and thus non-recoverable. In addition, Assumption 2 is an sufficient condition for $\Theta_{ij}$ to have uniformly bounded variance and sub-exponential tails, which serve as a license that enables us to invoke concentration inequalities in our proof. Alaya and Klopp (2019) shows that a wide range of distributions satisfy Assumptions 1 and 2, some of these are reproduced in Table (3.1).

### 3.2 Estimation Procedure

Since $\{\Theta_{ij}\}$ are independent, by construction, we can write out the (normalized) negative log-likelihood function, $\ell(\Theta)$, as

$$
-\frac{1}{n_1 N_2} \sum_{r \in [T]} \sum_{(i,j) \in [n_1] \times [n_2]} \delta_{ij} (Y_{ij} - G^r(\Theta_{ij})).
$$

(46)

Using maximum likelihood principle, the straightforward approach is to let our estimator $\hat{\Theta}$ be the solution to the following program:

$$
\min_{\Theta \in \mathbb{R}^{n_1 \times N_2}} \ell(\Theta) \quad \text{subject to} \quad \text{rank}(\Theta) \text{ is low}, \|\Theta\|_\infty \leq \gamma.
$$

(47)

(48)

Since this program is non-convex, due to the nature of the rank function, we consider a convex relaxation of the original problem by nuclear norm:

$$
\min_{\Theta \in \mathbb{R}^{n_1 \times N_2}} \ell(\Theta) \quad \text{subject to} \quad \|\Theta\|_* \leq \gamma_1, \|\Theta\|_\infty \leq \gamma.
$$

(49)

(50)

Recent works have shown that nuclear norm alone doesn’t perform well in non-uniform sampling schemes. A max-norm regularization approach is often used to address this issue Fang et al. (2018), T. Cai and Zhou (2013), and Cai and Zhou (2016). However, using max-norm alone could lead to suboptimal recovery result; therefore, we propose using a hybrid norm which combines the max norm and nuclear, in our convex relaxation set up:

$$
\min_{\Theta \in \mathbb{R}^{n_1 \times N_2}} \ell(\Theta) \quad \text{subject to} \quad \|\Theta\|_* \leq \gamma_1, \|\Theta\|_{\max} \leq \gamma_2, \|\Theta\|_\infty \leq \gamma.
$$

(51)

(52)

By convexity and strong duality, the admissible solution $\hat{\Theta}$ of (51) can also be obtained by the following unconstrained program

$$
\hat{\Theta} = \arg\min_{\Theta \in C(\gamma)} \ell(\Theta) + \lambda_* \|\Theta\|_* + \lambda_{\max} \|\Theta\|_{\max}.
$$

(53)
4 Theoretical Properties

We now state the main result regarding the recovery of $\Theta$. Due to spacing limitation, we state an imprecise version of the theorem (ignoring multiplicative constant) and defer the precise versions to Appendix A.

**Theorem 5.** Under Assumption 1 and Assumption 2, if we set $\lambda_* \geq 2\|\nabla\ell(\Theta|Y)\|$ and $\lambda_{\text{max}} \geq 0$, then with probability $1 - \frac{4}{n_1 + N_2}$, we have the following upper bounds,

\[
\frac{1}{n_1 N_2} \|\hat{\Theta} - \Theta\|_{H,F} \leq C \max \left\{ n_1 N_2 \text{rank}(\Theta) \left( \frac{\lambda_*^2 + \lambda_* \lambda_{\text{max}} L_2^2 + L_4^2 \lambda_{\text{max}}^2}{L_5^4} + \left( 1 + \frac{\lambda_{\text{max}}}{\lambda_*} + \frac{\lambda_{\text{max}}^2}{\lambda_*^2} \right) \gamma^2 (E[\|\Sigma_R\|])^2 \right) \right\}
\]

and

\[
\frac{1}{n_1 N_2} \|\hat{\Theta} - \Theta\|_{2}^2 \leq C \max \left\{ n_1 N_2 \text{rank}(\Theta) \left( \frac{\lambda_*^2 + \lambda_* \lambda_{\text{max}} L_2^2 + L_4^2 \lambda_{\text{max}}^2}{L_5^4} + \left( 1 + \frac{\lambda_{\text{max}}}{\lambda_*} + \frac{\lambda_{\text{max}}^2}{\lambda_*^2} \right) \gamma^2 (E[\|\Sigma_R\|])^2 \right) \right\} \gamma^2 \log(n_1 + N_2)
\]

**Theorem 6.** Under Assumption 1 and Assumption 2, if we let choose $\lambda$ and $\lambda_{\text{max}}$ in the following ways:

\[
\lambda_* = \frac{2\epsilon(U_\gamma \vee K)\sqrt{n_1 + N_2} + (\log(n_1 + N_2))^\frac{3}{2}}{n_1 N_2} \quad \text{and} \quad \frac{\lambda_{\text{max}}}{\lambda_*} \leq \kappa,
\]

where $\kappa > 0$ is some constant, then the following recovery guarantees hold:

\[
\frac{1}{n_1 N_2} \|\hat{\Theta} - \Theta\|_{H,F}^2 \leq \frac{\text{Cr ank}(\Theta))(n_1 + N_2)}{pm_1 N_2} \left( 1 + \frac{\log^3(n_1 + N_2)}{n_1 \lor N_2} \right) \left( U_\gamma \lor K \right) \frac{\kappa L_2^2 + \kappa^2 L_4^2}{L_5^4} + \gamma^2 (1 + \kappa + \kappa^2)
\]

and

\[
\frac{1}{n_1 N_2} \|\hat{\Theta} - \Theta\|_{2}^2 \leq \frac{\text{Cr ank}(\Theta))(n_1 + N_2)}{p^2 n_1 N_2} \left( 1 + \frac{\log^3(n_1 + N_2)}{n_1 \lor N_2} \right) \left( U_\gamma \lor K \right) \frac{\kappa L_2^2 + \kappa^2 L_4^2}{L_5^4} + \gamma^2 (1 + \kappa + \kappa^2)
\]

5 Algorithm Framework

In general, there are two dominant approaches on how to solve (53), namely, proximal gradient method and ADMM. We use the latter one mainly because the gradient of the max norm is quite hard to calculate. We propose our solution to Algorithm 1, which is based on the previous work by Fang et al. (2018). We present all of the details of this algorithm in the next section. We note that compared to traditional gradient based method, ADMM has the advantage of easy parallelization, which is powerful in solving large scale inputs.

Recall that our estimator is defined as

\[
\hat{\Theta} := \underset{\Theta \in \mathbb{R}^{n_1 \times N_2}}{\text{argmin}} \sum_{\tau \in \mathcal{T}} \sum_{i,j \in \Omega_{\tau}} (Y_{ij}^\tau \Theta_{ij}^\tau - G^\tau(\Theta_{ij}^\tau)) + \lambda_{\text{max}} \|\Theta\|_{\text{max}} + \lambda_* \|\Theta\|_* \text{, subject to } \|\Theta\|_{\infty} \leq K.
\]

Using definitions of the max-norm and nuclear norm in terms of semi-definite programs, one can get the following equivalent representation.

**Lemma 1.** $\hat{\Theta}$ is has the following equivalent representation.

\[
\tilde{M} = \underset{Z \in \mathbb{R}^{d \times d}}{\text{argmin}} \sum_{\tau \in \mathcal{T}} \sum_{i,j \in \Omega_{\tau}} (Y_{ij}^\tau Z_{ij,\tau}^{12} - G^\tau(Z_{ij,\tau}^{12})) + \lambda \|\text{diag}(Z)\|_{\infty} + \mu (I, Z) \text{ subject to } \|Z^{12}\|_{\infty} \leq \alpha, Z \succeq 0,
\]

where $d = n_1 + N_2$. 

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Algorithm 1 ADMM mixed data matrix completion

Input: $X^0$, $Z^0$, $W^0$, $Y_Q$, $\lambda$, $\mu$, $\alpha$, $\rho$, $\tau$, $t = 0$

while Stopping criterion is not satisfied do

$X^{t+1} \leftarrow \text{proj}_{S^d} \left\{ Z^t - \rho^{-1}(W^t + \mu I) \right\}$

$Z^{t+1} \leftarrow \text{Z}(X^{t+1} + \rho^{-1}W^t)$ by Proposition 3.

$W^{t+1} \leftarrow W^t + \gamma \rho (X^{t+1} - Z^{t+1})$.

$t \leftarrow t + 1$

end while

Output: $\hat{Z} = Z^t$, $\hat{\Theta} = \hat{Z}^{12}$.

ADMM formulation. Now we formulate the objective function described in (62) in a way such that ADMM, a popular algorithm with strong convergence guarantees, could be applied. Note that we can transform the objective function in the following way:

$$
\min_{Z \in \mathbb{R}^{d \times d}} \sum_{\tau \in T, i,j \in \Omega^r} (Y_{ij}^T Z_{ij,\tau}^{12} - G^T(Z_{ij,\tau}^{12})) + \lambda \|\text{diag}(Z)\|_\infty + \mu \langle I, Z \rangle
$$

$$
\iff 
\min_{X,Z \in \mathbb{R}^{d \times d}} \sum_{\tau \in T, i,j \in \Omega^r} (Y_{ij}^T Z_{ij,\tau}^{12} - G^T(Z_{ij,\tau}^{12})) + \lambda \|\text{diag}(Z)\|_\infty + \mu \langle I, X \rangle
s.t. X, Z \succeq 0, \|Z^{12}\|_\infty \leq \alpha, X - Z = 0.
$$

$$
\iff 
\min_{X,Z \in \mathbb{R}^{d \times d}} \mathcal{L}(Z) + \mu \langle I, X \rangle.
$$

s.t. $X, Z \succeq 0, \|Z^{12}\|_\infty \leq \alpha, X - Z = 0$.

Now we can write the augmented Lagrangian function as

$$
L(X,Z;W) = \mathcal{L}(Z) + \mu \langle I, X \rangle + \langle W, X - Z \rangle + \frac{\rho}{2} \|X - Z\|_F^2, \ X \in S^d, Z \in \mathcal{P} := \{Z \in S^d : \|Z^{12}\|_\infty \leq \alpha\}.
$$

Hence, the $t + 1$th update step of the algorithm is

$$
X^{t+1} = \arg\min_{X \in S^d} L(X, Z^t; W^t) = \text{proj}_{S^d} \left\{ Z^t - \rho^{-1}(W^t + \mu I) \right\}, \quad (63)
$$

$$
Z^{t+1} = \arg\min_{Z \in \mathcal{P}} L(X^{t+1}, Z; W^t) = \arg\min_{Z \in \mathcal{P}} \mathcal{L}(Z) + \frac{\rho}{2} \|Z - X^{t-1} - \frac{1}{\rho} W^t\|_F^2, \quad (64)
$$

$$
W^{t+1} = W^t + \tau \rho (X^{t+1} - Z^{t+1}), \quad (65)
$$

where $\tau \in (0,(1 + \sqrt{5})/2)$ is a step length operator. Empirical evidence suggests $\tau = 1.618$ (the golden ratio) works best.

Remark 1. The rate of convergence of ADMM algorithm in the worse case has been established to be $O(t^{-1})$, see Fang et al. (2015).

Details for (63) Note that

$$
L(X, Z^t; W^t) = \sum_{i=1}^n (Y_{i,j_i} - Z_{i_{j_i}}^{12})^2 + \lambda \|\text{diag}(Z)\|_\infty + \mu \langle I, X \rangle + \langle W, X - Z^t \rangle + \frac{\rho}{2} \|X - Z\|_F^2,
$$

where $X \in S^d$ and $Z \in \mathcal{P}$. Differentiating with respect to $X$,

$$
\nabla_X L(X, Z^t; W^t) = \nabla_X \left[ \mu \langle I, X \rangle + \langle W, X \rangle + \frac{\rho}{2} \|X - Z\|_F^2 \right]
$$

$$
= \nabla_X [\mu \langle I, X \rangle] + \nabla_Y [\langle W, X \rangle] + \nabla_X \left[ \frac{\rho}{2} \|X - Z\|_F^2 \right]
$$

$$
= \mu I + W^t + \rho (X - Z).
$$

The critical point is then found by setting gradient to zero:

$$
\nabla_X L(X, Z^t; W^t) = 0 \iff \mu I + W^t + \rho X - \rho Z = 0 \iff Z = \rho^{-1}(W^t + \mu I).
To ensure feasibility of $X$, we need to project the critical point onto the positive semi-definite cone (Boyd et al. (2011)). Hence, combined we get

$$X^{t+1} = \text{proj}_+ (Z - \rho^{-1} (W - \mu I)).$$

**Details for (64)** Note that

$$L(X^{t+1}, Z; W^t) = \mathcal{L}(Z) + \mu \langle I, X^{t+1} \rangle + \langle W^t, X^{t+1} - Z \rangle + \frac{\rho}{2} \|X^{t+1} - Z\|_F^2. \tag{66}$$

First, we show that $\arg\min_{Z \in \mathcal{P}} L(X^{t+1}, Z; W^t) = \arg\min_{Z \in \mathcal{P}} \mathcal{L}(Z) + \frac{\rho}{2} \|Z - X^{t+1} - \frac{1}{\rho} W^t\|_F^2$. We note that

$$\mathcal{L}(Z) + \frac{\rho}{2} \|Z - X^{t+1} - \frac{1}{\rho} W^t\|_F^2 \tag{67}$$

$$= \mathcal{L}(Z) + \frac{\rho}{2} \left\langle Z - X^{t+1} - \frac{1}{\rho} W^t, Z - X^{t+1} - \frac{1}{\rho} W^t \right\rangle \tag{68}$$

$$= \mathcal{L}(Z) + \frac{\rho}{2} \left( Z - X^{t+1} - \frac{1}{\rho} W^t, Z - X^{t+1} \right) - \left( Z - X^{t+1} - \frac{1}{\rho} W^t, \frac{1}{\rho} W^t \right) \tag{69}$$

$$= \mathcal{L}(Z) + \frac{\rho}{2} \left( Z - X^{t+1}, Z - X^{t+1} \right) - \frac{1}{\rho} \langle W^t, Z \rangle + \frac{1}{\rho} \langle W^t, X^{t+1} \rangle - \frac{1}{\rho} \langle W^t, X^{t+1} \rangle + \frac{1}{\rho^2} \langle W^t, W^t \rangle \tag{70}$$

$$= \mathcal{L}(Z) + \frac{1}{\rho^2} \|W^t\|_F^2 - \langle W^t, Z \rangle + \langle W^t, X^{t+1} \rangle + \frac{1}{\rho} \|W^t\|_F^2 \tag{71}$$

Since $\frac{1}{\rho^2} \|W^t\|_F^2$ in the last equation is not related to $Z$, it follows that

$$\arg\min L(X^{t+1}, Z; W^t) = \arg\min \mathcal{L}(Z) + \mu \langle I, X^{t+1} \rangle + \langle W^t, X^{t+1} - Z \rangle + \frac{\rho}{2} \|X^{t+1} - Z\|_F^2 \tag{72}$$

$$= \arg\min \mathcal{L}(Z) + \langle W^t, X^{t+1} - Z \rangle + \frac{\rho}{2} \|X^{t+1} - Z\|_F^2 \tag{73}$$

$$= \arg\min \mathcal{L}(Z) + \langle W^t, X^{t+1} - Z \rangle + \frac{\rho}{2} \|X^{t+1} - Z\|_F^2 + \frac{1}{\rho^2} \|W^t\|_F^2 \tag{74}$$

$$= \arg\min \mathcal{L}(Z) + \frac{\rho}{2} \|Z - X^{t+1} - \frac{1}{\rho} W^t\|_F^2 \tag{75}$$

where the second to last equality is justified by our previous calculation.

Next, we introduce a result that will help us get a closed form of the $Z$-step update.

**Proposition 3.** Let $\Omega = \{(i,j)\}_{i=1}^D$ be the index set of observed entries and let

$$f(Z) = \sum_{\tau \in T} \sum_{i,j \in \Omega} (Y_{ij}^\tau Z_{ij,\tau}^{ij} - G^\tau(Z_{ij,\tau}^{ij})) + \lambda \|\text{diag}(Z)\|_\infty + \frac{\rho}{2} \|Z - C\|_F^2. \tag{76}$$

Then it follows that $\arg\min_{Z \in \mathcal{P}} f(Z) = \mathcal{Z}(C)$, where

$$\mathcal{Z}(C) = \begin{bmatrix} Z^{11}(C) & Z^{12}(C) \\ Z^{21}(C) & Z^{22}(C) \end{bmatrix}, \tag{77}$$

$$Z_{kl}^{12}(C) = \begin{cases} \text{proj}_{-\alpha_{kl}} \arg\min (Y_{ij}^\tau Z_{ij,\tau}^{ij} - G^\tau(Z_{ij,\tau}^{ij})) + \rho(Z_{ij,\tau}^{ij} - C_{ij,\tau})^2, & \text{if } (k, \ell) \in \Omega, \\ \text{proj}_{-\alpha_{kl}} C_{kl}^{12}, & \text{otherwise}, \end{cases} \tag{78}$$

$$Z_{kl}^{11}(C) = C_{kl}^{11} \text{ if } k \neq \ell, \tag{79}$$

$$Z_{kl}^{22}(C) = C_{kl}^{22} \text{ if } k \neq \ell, \tag{80}$$

$$\text{diag}(\mathcal{Z}(C)) = \arg\min_{z \in \mathbb{R}^4} \lambda \|z\|_\infty + \frac{\rho}{2} \|\text{diag}(C) - z\|_2^2. \tag{81}$$

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Proof. The idea for the proof is to decompose (76) into separate disjoint parts based on the blocks of \(Z\). Recall that \(Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12} & Z_{22} \end{bmatrix}\). First, we set the notation \(X_{\text{ND}}\) to be the \(X\) but with its diagonal terms forced to be zero, that is, \(X_{\text{ND}} = X - \text{diag}(X) \cdot I\). Then we note that we can write

\[
\|Z - C\|_F^2 = \sum_{i,j} \left| z_{ij} - c_{ij} \right|^2 = \sum_{(i,j) \in Z_{11}, i \neq j} |z_{ij} - c_{ij}|^2 + \sum_{(i,j) \in Z_{22}, i \neq j} |z_{ij} - c_{ij}|^2 + 2 \sum_{(i,j) \in Z_{12}, i = j} |z_{ij} - c_{ij}|^2
\]

(82)

\[
\|Z_{11}\|_\text{ND} - C_{11}\|_{\text{ND}}^2_F + \|Z_{22}\|_\text{ND} - C_{22}\|_{\text{ND}}^2_F + \|\text{diag}(Z - C)\|_2^2 + 2 \|Z_{12} - C_{12}\|_F^2.
\]

(84)

Hence, it follows that

\[
\argmin_{Z \in S_4} f(Z) = \argmin_{Z_{11}\|_{\text{ND}} \in S_{\text{a}_1}} f_{11}(Z_{11}\|_{\text{ND}}) + f_{22}(Z_{22}\|_{\text{ND}}) + f_{12}(Z_{12}) + f_{\text{diag}}(\text{diag}(Z_{11}), \text{diag}(Z_{22})),
\]

\[
\argmin_{Z_{11}\|_{\text{ND}} \in S_{\text{a}_2}} f_{22}(Z_{22}\|_{\text{ND}}) + \argmin_{Z_{12}\|_{\text{ND}} \in S_{\text{a}_2}} f_{12}(Z_{12}) + \argmin_{Z_{11}\|_{\text{ND}} \in S_{\text{a}_1}} f_{11}(Z_{11}\|_{\text{ND}}).
\]

(85)

where

\[
f_{11}(Z_{11}\|_{\text{ND}}) = \frac{\rho}{2} \left\| Z_{11}\|_{\text{ND}} - C_{11}\|_{\text{ND}} \right\|_F^2 = \frac{\rho}{2} \sum_{(i,j) \in Z_{11}, i \neq j} |z_{ij} - c_{ij}|^2,
\]

(86)

\[
f_{22}(Z_{22}\|_{\text{ND}}) = \frac{\rho}{2} \left\| Z_{22}\|_{\text{ND}} - C_{22}\|_{\text{ND}} \right\|_F^2 = \frac{\rho}{2} \sum_{(i,j) \in Z_{22}, i \neq j} |z_{ij} - c_{ij}|^2,
\]

(87)

\[
f_{12}(Z_{12}) = \sum_{\tau \in T} \sum_{i,j \in \Omega^\tau} (Y_{ij}^\tau Z_{ij,\tau}^2 - G^\tau(Z_{ij,\tau}^2)) + \rho \sum_{\tau \in T} \sum_{(i,j) \in \Omega^\tau} \left( Z_{ij,\tau}^2 - C_{ij,\tau}^2 \right)^2,
\]

(88)

\[
f_{\text{diag}}(Z_{11}, Z_{22}) = \lambda \left\| \text{diag}(Z) \right\|_{\infty} + \frac{\rho}{2} \left\| \text{diag}(Z - C) \right\|_2^2.
\]

(89)

Optimality of \(f_{11}\) and \(f_{22}\) Then note that it is obvious that \(f_{11}(Z_{11}\|_{\text{ND}}) \geq 0\) for any possible candidate of \(Z_{11}\|_{\text{ND}}\) and takes equality sign when \(Z_{11}\|_{\text{ND}} = C_{11}\|_{\text{ND}}\). The same argument can be made for \(f_{22}(Z_{22}\|_{\text{ND}})\). Then, it follows that

\[
\argmin f_{11}(Z_{11}\|_{\text{ND}}) = C_{11}\|_{\text{ND}}, \quad \text{and} \quad \argmin f_{22}(Z_{22}\|_{\text{ND}}) = C_{22}\|_{\text{ND}}.
\]

Optimality of \(f_{12}\) First, we rewrite

\[
f_{12}(Z_{12}) = \sum_{\tau \in T} \sum_{i,j \in \Omega^\tau} (Y_{ij}^\tau Z_{ij,\tau}^2 - G^\tau(Z_{ij,\tau}^2)) + \rho \left\| Z_{12} - C_{12} \right\|_F^2
\]

(90)

\[
= \sum_{\tau \in T} \sum_{i,j \in \Omega^\tau} (Y_{ij}^\tau Z_{ij,\tau}^2 - G^\tau(Z_{ij,\tau}^2)) + \rho \sum_{\tau \in T} \sum_{(i,j) \in \Omega^\tau} \left( Z_{ij,\tau}^2 - C_{ij,\tau}^2 \right)^2 + \rho \sum_{\tau \in T} \sum_{(i,j) \in \Omega^\tau} \left( Z_{ij,\tau}^2 - C_{ij,\tau}^2 \right)^2
\]

(91)

\[
= \sum_{\tau \in T} \sum_{i,j \in \Omega^\tau} \left( (Y_{ij}^\tau Z_{ij,\tau}^2 - G^\tau(Z_{ij,\tau}^2)) + \rho(Z_{ij,\tau}^2 - C_{ij,\tau}^2) \right) + \rho \sum_{\tau \in T} \sum_{(i,j) \in \Omega^\tau} \left( Z_{ij,\tau}^2 - C_{ij,\tau}^2 \right)^2.
\]

(92)

Note that

\[
\frac{\partial f_{12}}{\partial Z_{(i,j) \notin \Omega^\tau}} = 2 \rho (Z_{ij,\tau}^2 - C_{ij,\tau}^2) = 0 \implies Z_{(i,j) \notin \Omega^\tau} = C_{ij,\tau}^2.
\]

Since \(Z_{12}\) has constraint \(Z_{12} \in \mathcal{B}_{\|_{\infty}(\alpha)}\), we need to project it to the constrained space:

\[
Z_{ij,\tau}^2 = \begin{cases} \text{proj}_{[-\alpha,\alpha]} \argmin (Y_{ij}^\tau Z_{ij,\tau}^2 - G^\tau(Z_{ij,\tau}^2)) + \rho(Z_{ij,\tau}^2 - C_{ij,\tau}^2) & \text{if } (i,j) \in \Omega \\ \text{proj}_{[-\alpha,\alpha]} C_{ij} & \text{otherwise} \end{cases}.
\]

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Optimality of \( f_{\text{diag}} \)  
Note that \( \arg\min f_{\text{diag}} \) can be caset into the following program:

\[
\min_{z \in \mathbb{R}^d} \beta \|z\|_{\infty} + \frac{1}{2} \|c - z\|_2^2,
\]

where \( c = (c_1, \ldots, c_d)^T = \text{diag}(C) \) and \( \beta = \frac{1}{\rho} \). A closed form solution could be formed by laying out the KKT condition, see Lemma 16.

Duality on \( X \)  
Assume that \( X^{t+1} \) reaches that optimality, then we have

\[
0 \in \partial \delta_{\mathbb{R}_+^d}(X^{t+1}) + \mu I + W^t,
\]

where we can rewrite the RHS as

\[
(\text{RHS}) = \partial \delta_{\mathbb{R}_+^d}(X^{t+1}) + \mu I + W^t + \rho(X^{t+1} - Z)
\]

\[
= \partial \delta_{\mathbb{R}_+^d}(X^{t+1}) + \rho(\rho^{-1}(\mu I + W^t) + (X^{t+1} - Z)).
\]

Therefore, we can rewrite (94) as

\[
\rho(Z^t - X^{t+1}) - W^t \in \partial \delta_{\mathbb{R}_+^d}(X^{t+1}) + \mu I
\]

\[
\iff \rho(Z^t - X^{t+1}) - W^t \in \partial \delta_{\mathbb{R}_+^d}(X^{t+1}) + \mu I
\]

\[
\iff \rho(Z^t - X^{t+1}) - W^t + W^{t+1} \in \partial \delta_{\mathbb{R}_+^d}(X^{t+1}) + \mu I + W^{t+1}
\]

\[
\iff \rho(Z^t - Z^{t+1}) - W^t + W^{t+1} + \rho Z^{t+1} - \rho X^{t+1} \in \partial \delta_{\mathbb{R}_+^d}(X^{t+1}) + \mu I + W^{t+1}
\]

\[
\iff \rho(Z^t - Z^{t+1}) + W^{t+1} - (W^t + \rho(X^{t+1} - Z^{t+1})) \in \partial \delta_{\mathbb{R}_+^d}(X^{t+1}) + \mu I + W^{t+1}.
\]

Let \( \tilde{W}^{t+1} = W^t + \rho(X^{t+1} - Z^{t+1}) \), then (101) could be written as

\[
\rho(Z^t - Z^{t+1}) + W^{t+1} - \tilde{W}^{t+1} \in \partial \delta_{\mathbb{R}_+^d}(X^{t+1}) + \mu I + W^{t+1}.
\]

Duality on \( Z \)  
Note that we originally have the optimality condition as

\[
0 \in \partial \delta_{\mathbb{R}^d}(Z) + \nabla L(Z) - W.
\]

At iteration \( t + 1 \), if \( Z^{t+1} \) satisfies reaches the optimality condition, we would have \( Z^{t+1} = X^{t+1} \) since we have updated in the first step. Then we have (103) is equivalent to the following

\[
0 \in \partial \delta_{\mathbb{R}^d}(Z^{t+1}) + \nabla L(Z^{t+1}) - W^t + \rho(Z^{t+1} - X^{t+1})
\]

\[
\iff \nabla L(Z^{t+1}) - W^t + \rho(Z^{t+1} - Z^{t+1}) \in \partial \delta_{\mathbb{R}^d}(Z^{t+1}) + \nabla L(Z^{t+1})
\]

\[
\iff \tilde{W}^{t+1} - W^{t+1} \in \partial \delta_{\mathbb{R}^d}(Z^{t+1}) + \nabla L(Z^{t+1}) - W^{t+1}.
\]

Remark 2. The purpose of the rewriting above is to create a get condition for early stopping. Namely, once we have updated all of \( X, Z, W \) in the \((t+1)\)th iteration and hypothetically we have reached the optimality condition

\[
\begin{cases}
0 \in \partial \delta_{\mathbb{R}_+^d}(X^{t+1}) + \mu I + W^{t+1} \\
0 \in \partial \delta_{\mathbb{R}^d}(Z^{t+1}) + \nabla L(Z^{t+1}) - W^{t+1}
\end{cases}
\]

then by the equivalent formulation above, the pair

\[
\left\{ \tilde{W}^{t+1} - W^{t+1}, \rho(Z^t - Z^{t+1}) + W^{t+1} - \tilde{W}^{t+1} \right\}
\]

should be close to 0, i.e. the value \( R_D \) defined as

\[
\max \left\{ \left\| \tilde{W}^{t+1} - W^{t+1} \right\|, \left\| \rho(Z^t - Z^{t+1}) + W^{t+1} - \tilde{W}^{t+1} \right\| \right\}
\]

should be small. We also note any choice should norm should work for \( R_D \) due to the equivalence of norms in finite dimensional vector spaces; however, difference norm might induce a difference convergence rate and as a result impact the effectiveness of the early stopping predicate. Empirically, Frobenous norm works quite well in most cases.

Remark 3. If \( X^{t+1}, Z^{t+1}, W^{t+1} \) produces the optimal solution, aside from satisfying the condition in the previous remark, \( X^{t+1} \) and \( Z^{t+1} \) should also satisfy the primal feasibility condition, i.e. \( X^{t+1} = Z^{t+1} \). Numerically, this means that the value \( R_P := \| X^{t+1} - Z^{t+1} \| \) should be small.
Algorithm 2 Early Stopping Predicate

function EARYSTOPPREDICATE(X, Z, W, tol)
\[ R_P \leftarrow \|X^{t+1} - Z^{t+1}\|_F \]
\[ R_D \leftarrow \max \left\{ \|\tilde{W}^{t+1} - W^{t+1}\|_F, \|\rho(Z^t - Z^{t+1}) + W^{t+1} - \tilde{W}^{t+1}\|_F \right\} \]
if \( \max(R_P, R_D) < \text{tol} \)
    return true
else
    return false
end function

Algorithm 3 Balance Gap

function BALANCEGAP(\( \rho \))
if \( \|R_P^{t+1}\| < 0.5 \|R_D^{t+1}\| \)
    \( \rho \leftarrow 0.7 \rho \)
if \( \|R_D^{t+1}\| < 0.5 \|R_P^{t+1}\| \)
    \( \rho \leftarrow 1.3 \rho \)
end function

Early stopping Based on Remark 2 and Remark 3, we propose the following early stopping predicate to speed up our main algorithm.

Adjust \( \rho \) dynamically According to Fang et al. (2018), dynamically adjusting \( \rho \) according to helps speed up the convergence of the ADMM algorithm. We remark that in the mixed data setting this speed-up procedure still works.

Due the the fact that eigen-decomposition is performed in every iteration of ADMM, we left a few flags in the implemented package for users to choose the eigen-decomposition procedure. For a reasonably large matrix of size 5000 \( \times \) 5000 full eigen decomposition is costly and as we will show in simulation result that the non-dominate eigen values/vector pairs have negligible effects on the final output, a sparse eigen routine is often enough to get the desired recovery.

6 Numerical Experiments
In this section, we present several numerical simulation on random generated low rank matrix data to verify the validity of our proposed model. In additional to tracking recovery rates, we will also focus on

Due to the fact that our computational package is still in development and stability needs further improvement (some of the large scale simulation could not be 100% reproduced), we present a small scale numerical result for the purpose of verifying the correctness of our proposed algorithm.

Small Scale Pure Data 1 In this experiment, we randomly generate 500 \( \times \) 500 matrix of one single distribution (Normal, Gamma, Poisson, Bernoulli and Negative Negative Binomial) and keep its rank fixed while measure the recovery result under different sample rate. The results are shown in Figures 1 to 10.
In this experiment, we randomly generate $500 \times 500$ matrix of five mixed distributions (Normal, Gamma, Poisson, Bernoulli and Negative Binomial) and keep its rank fixed while measure the recovery result under different sample rate. The results are shown in Figures 11 to 19, where each colored line represents the relative error compared to the truth matrix for its corresponding distributions types. The X-axis represents the sampling rate. An averaged relative error over all distributions is shown in figure

**Small Scale Mixed Data 1**

In this experiment, we randomly generate $500 \times 500$ matrix of five mixed distributions (Normal, Gamma, Poisson, Bernoulli and Negative Binomial) and keep its rank fixed while measure the recovery result under different sample rate. The results are shown in Figures 11 to 19, where each colored line represents the relative error compared to the truth matrix for its corresponding distributions types. The X-axis represents the sampling rate. An averaged relative error over all distributions is shown in figure.
**Small Scale Mixed Data**

In this experiment, we test the performance of our algorithm the sampling rate is fixed at 80% while changing the input rank of the input matrix. The resulting figure is in Figure 21.

**Medium Scale Mixed Data**

In this experiment, we reproduce the same previous evaluation procedures on medium scaled input. We generate $2000 \times 2000$ matrix of 5 mixed types (Gaussian, Bernoulli, Poisson, NegBin and Gamma), each of which could be view as a $2000 \times 400$ submatrix. We then measure the performance when holding rank fixed and varying sample rate and vice versa. The results are in Figure 22 and Figure 23.
different Eigen-solvers  This experiment is designed to test the difference in performance when different eigen-solvers were used: full eigen-decomposition or truncated-eigen-decomposition. The result is in Figure 24. The input matrix is a $500 \times 500$ mixed typed matrix with each data type occupying a $500 \times 100$ sub-matrix. We can see that when the rank is low, i.e. less than 20% of the corresponding sub-matrices, the difference between using full and partial eigen decomposition is small.

Observations  The simulation results help verify our theoretical results in that we can see from the plots that

- when the rank is low and fixed, the recovery success is proportional to the sampling rate;
- when the sampling rate is fixed, the recovery success is inversely proportional to the rank of the data matrix;
- the recovery success when recovering mixed distributed low rank matrices is on par with recovering singly-distributed low rank matrices.

Additionally, we note that although in theory the full eigen-decomposition should be used in order to find out all the positive eigen value/vector pairs, in practice when the matrix is sufficiently low rank, e.g. 10% of
min n, m where n, m respectively refer to row count and column count, using truncated eigen-solver therefore only taking not the full positive spectrum but only the dominate ones actually performs on par with taking the full spectrum. However, we should also note that as rank increases, the truncated eigen-version of the algorithm under performs significantly.

7 Concluding Remarks

From a theoretical point of view we have only obtained an upper bound on the recovery rate. However, many of the previous works have developed a lower bound using information theoretic techniques. It would be interesting to see if a similar result could be proved in this general case. Although we have shown that a hybrid of max norm and Schatten norm in the loss function can lead to recovery of the matrix with statistical guarantee, the inequalities between max norm and Schatten norm actually provides a significant bridge in facilitating the final proof. We could not produce a similar result using the same technique without the existence of nuclear norm in the loss function. Hence, an open question is whether we can prove a similar result for max-norm-only loss functions.

While our paper mainly discusses theoretical results, the numerical implementation counterparts are also worth some brief discussion. The algorithms developed and analyzed in this article has been implemented in a Julia package, MatrixCompletion.jl. To the best of our knowledge, this is the first dedicated package in Julia that address the problem of matrix completion of reasonably large input size that uses convex optimization methods. In addition, MatrixCompletion.jl also provides several features that we deem useful for interested readers who want to get hands on experience with our algorithm.

Automatic Data Type Detection  In reality it is often unknown that what are the exact distributions of the underlying data. To address this issue, we provided an API that allows the algorithm to automatically detect the best fitting distributed within the supported range and after doing so, also acquire the MLEs of the corresponding parameters. Traditional goodness-and-fit often has less power when the input data size are large. To address this problem, we adopted a different approach combining a simple trivial decision tree and comparing the empirical distribution to its exponential family candidates in terms of moment generating functions.

Automatic Differentiation and Extensible Loss Function Design  We acknowledge that besides the loss functions we proposed, there are many other possible candidates within or outside the exponential families could be deemed useful in solving the matrix completion problem. MatrixCompletion.jl’s implementation has taken these factors into consideration. Custom loss functions are possible. Furthermore, we also have bundled automatic differentiation support to help facilitate the implementation of custom loss function by removing the need to manually implement another gradient.

More Classical Algorithms  With the help of Github and researchers around the world, we are aiming to make MatrixCompletion.jl a comprehensive library on matrix completion. Currently, we are adding more classical algorithms such as singular value thresholding, manifold optimization based methods. Because of Julia’s multiple dispatch system and its good module system, all these algorithms can be implemented under one polymorphic method call, which is straight forward as well as user-friendly.

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Appendix A: Theoretical Results

Precise statement of upper bounds

Let the collection of matrices \( E_{1,1}^\tau, \ldots, E_{n_1 n_2}^\tau \) be the canonical basis in the space of matrices of size \( n_1 \times n_2 \). Let \( (\varepsilon_{ij}^\tau) \) be an i.i.d. Rademacher sequence. We defined

\[
\Sigma_R = (\Sigma_R^1, \ldots, \Sigma_R^{|T|}),
\]

where

\[
\Sigma_R^\tau = \frac{1}{n_1 n_2} \sum_{i,j \in [n_1] \times [n_2]} \varepsilon_{ij}^\tau E_{ij}^\tau.
\]

The following lemma provides a bound on the operator norm of \( \Sigma_R \).

**Lemma 2** (Lemma 1 in Alaya and Klopp (2019)). There exists an absolute constant \( c \) such that

\[
\mathbb{E}[\|\Sigma_R\|] \leq c \left( \sqrt{p} + \sqrt{\log(n_1 \land n_2)} \right).
\]

Additionally, we let \( \|,\|_{\Pi,F} \) be the weighted Frobenous norm defined by

\[
\|A\|_{\Pi,F} = \sum_{\tau \in T} \sum_{i,j} \pi_{ij}^\tau (A_{ij}^\tau)^2.
\]

**Proof of Theorem 5**

Since by assumption \( \Theta \in \mathcal{B}_{n_1 \times n_2}^\gamma(\gamma) \), it follows that \( \mathcal{L}(\hat{\Theta}|Y) \leq \mathcal{L}(\Theta|Y) \), which expand to

\[
- \frac{1}{n_1 n_2} \sum_{\tau \in T} \sum_{(i,j) \in \Omega_\tau} \delta_{ij}^\tau (Y_{ij}^\tau \Theta_{ij} - A^\tau(\Theta_{ij})) + \|\Theta\|_{\Lambda_{\max},\Lambda_{\max}}^\bullet \geq - \frac{1}{n_1 n_2} \sum_{\tau \in T} \sum_{(i,j) \in \Omega_\tau} \delta_{ij}^\tau (Y_{ij}^\tau \hat{\Theta}_{ij} - A^\tau(\hat{\Theta}_{ij})) + \|\hat{\Theta}\|_{\Lambda_{\max},\Lambda_{\max}}^\bullet,
\]

which, by rearranging, is equivalent to

\[
- \frac{1}{n_1 n_2} \sum_{\tau \in T} \sum_{(i,j) \in \Omega_\tau} \delta_{ij}^\tau (A^\tau(\hat{\Theta}_{ij}) - Y_{ij}^\tau \hat{\Theta}_{ij}) + \|\Theta\|_{\Lambda_{\max},\Lambda_{\max}}^\bullet \leq - \frac{1}{n_1 n_2} \sum_{\tau \in T} \sum_{(i,j) \in \Omega_\tau} \delta_{ij}^\tau (A^\tau(\Theta_{ij}) - Y_{ij}^\tau \Theta_{ij}) + \|\Theta\|_{\Lambda_{\max},\Lambda_{\max}}^\bullet.
\]

Now we massage (112) into a form that’s easier to work with:

\[
\frac{1}{n_1 n_2} \sum_{\tau \in T} \sum_{i,j \in \Omega_\tau} \delta_{ij}^\tau (Y_{ij}^\tau \Theta_{ij} - \Theta_{ij}) - (A^\tau(\hat{\Theta}_{ij}) - A^\tau(\Theta_{ij})) \leq \|\Theta\|_{\Lambda_{\max},\Lambda_{\max}}^\bullet - \|\hat{\Theta}\|_{\Lambda_{\max},\Lambda_{\max}}^\bullet.
\]

Unpacking the norms we get

\[
\frac{1}{n_1 n_2} \sum_{\tau \in T} \sum_{i,j \in \Omega_\tau} \delta_{ij}^\tau \left( (A^\tau(\hat{\Theta}_{ij}) - A^\tau(\Theta_{ij})) - (Y_{ij}^\tau \hat{\Theta}_{ij} - \Theta_{ij}) \right) \leq \lambda_\bullet (\|\Theta\|_{\Lambda_{\max}}^\bullet - \|\hat{\Theta}\|_{\Lambda_{\max}}^\bullet).
\]

Since using the bijection between Bregman divergence and exponential family we can write

\[
\mathcal{KL}(\hat{\Theta}_{ij}, \Theta_{ij}) = A^\tau(\hat{\Theta}_{ij}) - A^\tau(\Theta_{ij}) - (\hat{\Theta}_{ij} - \Theta_{ij})^\tau \nabla A^\tau(\Theta_{ij}),
\]

it follows that

\[
A^\tau(\hat{\Theta}_{ij}) - A^\tau(\Theta_{ij}) = \mathcal{KL}(\hat{\Theta}_{ij}, \Theta_{ij}).
\]

Substitute this back into (114), we get that

\[
\frac{1}{n_1 n_2} \sum_{\tau \in T} \sum_{i,j \in \Omega_\tau} \delta_{ij}^\tau \left( (A^\tau(\hat{\Theta}_{ij}) - A^\tau(\Theta_{ij})) - (Y_{ij}^\tau \hat{\Theta}_{ij} - \Theta_{ij}) \right) \leq \lambda_\bullet (\|\Theta\|_{\Lambda_{\max}}^\bullet - \|\hat{\Theta}\|_{\Lambda_{\max}}^\bullet).
\]

(117)

\[
= \frac{1}{n_1 n_2} \sum_{\tau \in T} \sum_{i,j \in \Omega_\tau} \delta_{ij}^\tau \mathcal{KL}(\hat{\Theta}_{ij}, \Theta_{ij}) + (\hat{\Theta}_{ij} - \Theta_{ij})^\tau \nabla A^\tau(\Theta_{ij}) - (Y_{ij}^\tau \hat{\Theta}_{ij} - \Theta_{ij}) \tau
\]

\[
\leq \lambda_\bullet (\|\Theta\|_{\Lambda_{\max}}^\bullet - \|\hat{\Theta}\|_{\Lambda_{\max}}^\bullet).
\]

(119)
Rearranging the terms, we get

\[
\frac{1}{n_1 N_2} \sum_{\tau \in T} \sum_{i,j \in [n_1] \times [N_2]} \delta_{ij} \text{KL}(\hat{\Theta}_{ij}, \Theta_{ij}) \leq \lambda_*(\|\Theta\|_* - \|\hat{\Theta}\|_*) + \lambda_{\text{max}}(\|\Theta\|_{\text{max}} - \|\hat{\Theta}\|_{\text{max}}) + \frac{1}{n_1 N_2} \sum_{\tau \in T} \sum_{(i,j) \in [n_1] \times [N_2]} \delta_{ij} (Y_{ij} - \nabla A(\Theta_{ij}))(\hat{\Theta}_{ij} - \Theta_{ij})
\]

(120)

\[
\leq \lambda_*(\|\Theta\|_* - \|\hat{\Theta}\|_*) + \lambda_{\text{max}}(\|\Theta\|_{\text{max}} - \|\hat{\Theta}\|_{\text{max}}) + \left\langle \nabla_{\Theta} \mathcal{L}(\Theta; Y), \hat{\Theta} - \Theta \right\rangle
\]

(121)

\[
\leq \lambda_*(\|\Theta\|_* - \|\hat{\Theta}\|_*) + \lambda_{\text{max}}(\|\Theta\|_{\text{max}} - \|\hat{\Theta}\|_{\text{max}}) + \|\nabla_{\Theta} \mathcal{L}(\Theta; Y)\| \|\hat{\Theta} - \Theta\|_*
\]

(122)

\[
\leq \lambda_*(\|\Theta\|_* - \|\hat{\Theta}\|_*) + \lambda_{\text{max}}(\|\Theta\|_{\text{max}} - \|\hat{\Theta}\|_{\text{max}}) + \frac{\lambda_2}{2} \|\hat{\Theta} - \Theta\|_*
\]

(123)

\[
\leq \lambda_{\text{max}}(\|\Theta\|_{\text{max}} - \|\hat{\Theta}\|_{\text{max}}) + \lambda_{\text{max}}(\|\Theta\|_{\text{max}} - \|\hat{\Theta}\|_{\text{max}}) + \frac{\lambda_2}{2} \|\hat{\Theta} - \Theta\|_*
\]

(124)

\[
\leq \lambda_*(\|\Theta\|_* - \|\hat{\Theta}\|_*) + \lambda_{\text{max}}(\|\Theta\|_{\text{max}} - \|\hat{\Theta}\|_{\text{max}}) + \lambda_{\text{max}}(\|\Theta\|_{\text{max}} - \|\hat{\Theta}\|_{\text{max}})
\]

(125)

\[
\leq \frac{3}{2} \lambda_*(\|\Theta\|_* - \|\hat{\Theta}\|_*) - \lambda_{\text{max}}(\|\Theta\|_{\text{max}} - \|\hat{\Theta}\|_{\text{max}})
\]

(126)

\[
\leq \frac{3}{2} \lambda_*(\|\Theta\|_* - \|\hat{\Theta}\|_*) + \lambda_{\text{max}}(\|\Theta\|_{\text{max}} - \|\hat{\Theta}\|_{\text{max}})
\]

(127)

\[
\leq \frac{3}{2} \lambda_*(\|\Theta\|_* - \|\hat{\Theta}\|_*) + \lambda_{\text{max}}(\|\Theta\|_{\text{max}} - \|\hat{\Theta}\|_{\text{max}})
\]

(128)

\[
\leq \frac{3}{2} \lambda_{\text{max}}(\|\Theta\|_{\text{max}} - \|\hat{\Theta}\|_{\text{max}}) + \lambda_{\text{max}}(\|\Theta\|_{\text{max}} - \|\hat{\Theta}\|_{\text{max}})
\]

(129)

\[
= \left( \frac{3}{2} \lambda_{\text{max}}(\|\Theta\|_{\text{max}} - \|\hat{\Theta}\|_{\text{max}}) + \lambda_{\text{max}} \right) \|\Theta - \hat{\Theta}\|_F.
\]

(130)

where we note

- (122) is because of the fact that

\[
\nabla_{\Theta} \mathcal{L}(\Theta; Y) = \nabla_{\Theta} \left[ \sum_{\tau \in T} \sum_{(i,j) \in [n_1] \times [N_2]} \frac{1}{n_1 N_2} \delta_{ij} (Y_{ij} \Theta_{ij} - A(\Theta_{ij})) \right] = \sum_{\tau \in T} \sum_{(i,j) \in [n_1] \times [N_2]} \left[ \delta_{ij} (Y_{ij} - \nabla A(\Theta_{ij})) \right] e_{ij}^\tau
\]

(131)

where \(\{e_{ij}^\tau\}\) is the standard basis in \(\mathbb{R}^{|T| \times n_1 \times N_2}\)

- (123) is due to Cauchy inequality for operator norms

- (124) is due to the assumption that \(\lambda_* \geq 2\|\mathcal{L}_{\Theta}(\Theta; Y)\|\)

Then it follows that

\[
\frac{1}{n_1 N_2} \sum_{\tau \in T} \sum_{i,j \in [n_1] \times [N_2]} \delta_{ij} \text{KL}(\hat{\Theta}_{ij}, \Theta_{ij}) \geq \frac{L_2^2}{2} \frac{1}{n_1 N_2} \sum_{\tau \in T} \sum_{(i,j) \in [n_1] \times [N_2]} \delta_{ij} (\hat{\Theta}_{ij} - \Theta)^2 := \frac{L_2^2}{2} \Delta^2(\hat{\Theta} - \Theta).
\]

(132)

So it follows that

\[
\Delta^2(\hat{\Theta} - \Theta) \leq \frac{2}{L_2^2 \frac{1}{n_1 N_2}} \sum_{\tau \in T} \sum_{i,j \in [n_1] \times [N_2]} \delta_{ij} \text{KL}(\hat{\Theta}_{ij}, \Theta_{ij}) \leq \left( \frac{3}{L_2^2} \lambda_{\text{max}}(\|\Theta\|_{\text{max}} - \|\hat{\Theta}\|_{\text{max}}) \right) \|\Theta - \hat{\Theta}\|_F.
\]

(133)

Now we define the threshold \(\beta = \frac{946 \gamma^2 \log(n_1 + N_2)}{n n_1 D}\) and distinguish the two following cases:

**Case 1.** \(\frac{1}{n_1 N_2} \|\Theta - \hat{\Theta}\|_{\|F} < \beta\) In this case, the theorem is true.
Case 2. \( \frac{1}{n_1 N_2} \| \hat{\Theta} - \Theta \|_{\Pi,F} \geq \beta \) In this case, by Lemma 11, it follows that
\[
\| \hat{\Theta} - \Theta \|_\star \leq 2 \left( \sqrt{8 \text{rank}(\Theta) + \frac{\lambda_{\text{max}}}{\lambda_*}} \right) \| \hat{\Theta} - \Theta \|_F. \tag{134}
\]
Then it follows that \( \hat{\Theta} \in \mathcal{K}(\beta,4(\sqrt{8 \text{rank}(\Theta) + \frac{\lambda_{\text{max}}}{\lambda_*}})^2) \), where
\[
\mathcal{K}(\beta,r) := \left\{ \Xi \in B_\infty(\gamma) : \| \Xi - \Theta \|_\star \leq \sqrt{r} \| \Theta - \Xi \|_F \text{ and } \frac{1}{n_1 N_2} \| \Xi - \Theta \|_{\Pi,F} \geq \beta \right\}. \tag{135}
\]
Then by Lemma 11, it follows that
\[
\left| \Delta^2(\hat{\Theta},\Theta) - \frac{1}{n_1 N_2} \| \hat{\Theta} - \Theta \|^2_{\Pi,F} \right| \leq \frac{\| \hat{\Theta} - \Theta \|^2_{\Pi,F}}{2n_1 N_2} + 1392 \cdot 4 \left( \sqrt{8 \text{rank}(\Theta) + \frac{\lambda_{\text{max}}}{\lambda_*}} \right)^2 r^2(\| \Sigma_R \|)^2 + \frac{5567 \gamma^2}{n_1 N_2}, \tag{136}
\]
which after rearrangement becomes
\[
\Delta^2(\hat{\Theta},\Theta) \geq \frac{\| \hat{\Theta} - \Theta \|^2_{\Pi,F}}{2n_1 N_2} - 5568 \left( \sqrt{8 \text{rank}(\Theta) + \frac{\lambda_{\text{max}}}{\lambda_*}} \right)^2 r^2(\| \Sigma_R \|)^2 + \frac{5567 \gamma^2}{n_1 N_2}. \tag{137}
\]
Then combining (133) and (137), it follows that
\[
\left| \frac{\| \hat{\Theta} - \Theta \|^2_{\Pi,F}}{2n_1 N_2} - 5568 \left( \sqrt{8 \text{rank}(\Theta) + \frac{\lambda_{\text{max}}}{\lambda_*}} \right)^2 r^2(\| \Sigma_R \|)^2 - \frac{5567 \gamma^2}{n_1 N_2} \right| \leq \frac{3 \lambda_* \sqrt{2 \text{rank}(\Theta) + \lambda_{\text{max}}}}{2} \| \Theta - \hat{\Theta} \|_F. \tag{138}
\]
which after rearranging terms becomes
\[
\frac{\| \hat{\Theta} - \Theta \|^2_{\Pi,F}}{2n_1 N_2} \leq \frac{3 \lambda_* \sqrt{2 \text{rank}(\Theta) + \lambda_{\text{max}}}}{2} \| \Theta - \hat{\Theta} \|_F + \frac{5568 \left( \sqrt{8 \text{rank}(\Theta) + \frac{\lambda_{\text{max}}}{\lambda_*}} \right)^2 r^2(\| \Sigma_R \|)^2 + \frac{5567 \gamma^2}{n_1 N_2}}{n_1 N_2}. \tag{139}
\]

Lemma 3. The following identity holds:
\[
(I) \leq \frac{n_1 N_2}{p} \left( \frac{18 \lambda_*^2}{L_g^2} + \frac{12 \lambda_* \lambda_{\text{max}}}{L_g^2} \right) \text{rank}(\Theta) + \lambda_{\text{max}}^2 + \frac{1}{4n_1 N_2} \| \Theta - \hat{\Theta} \|_{\Pi,F}^2. \tag{140}
\]

proof of Lemma 3. Let \( H := \left( \frac{18 \lambda_*^2}{L_g^2} + \sqrt{2 \text{rank}(\Theta) + \lambda_{\text{max}}} \right) \), then it follows we can rewrite the term as
\[
H \| \Theta - \hat{\Theta} \|_F = \left( \frac{\sqrt{2n_1 N_2}}{p} H \right) \left( \frac{\sqrt{p}}{\sqrt{2n_1 N_2}} \| \Theta - \hat{\Theta} \|_F \right) \tag{141}
\]
\[
\leq \frac{1}{2} \left( \frac{2n_1 N_2}{p} H^2 \right) + \frac{1}{2} \left( \frac{p}{2n_1 N_2} \| \Theta - \hat{\Theta} \|_F \right) \tag{142}
\]
\[
\leq \frac{n_1 N_2}{p} \left( \frac{3 \lambda_* \sqrt{2 \text{rank}(\Theta) + \lambda_{\text{max}}}}{L_g^4} \right)^2 + \frac{1}{4n_1 N_2} \| \Theta - \hat{\Theta} \|_{\Pi,F}^2 \tag{143}
\]
\[
= \frac{n_1 N_2}{p} \left( \frac{18 \lambda_*^2 \text{rank}(\Theta)}{L_g^4} + \frac{6 \lambda_* \lambda_{\text{max}}}{L_g^4} \sqrt{2 \text{rank}(\Theta) + \lambda_{\text{max}}} \right) + \frac{1}{4n_1 N_2} \| \Theta - \hat{\Theta} \|_{\Pi,F}^2 \tag{144}
\]
\[
\leq \frac{n_1 N_2}{p} \left( \frac{18 \lambda_*^2}{L_g^4} + \frac{12 \lambda_* \lambda_{\text{max}}}{L_g^4} \right) \text{rank}(\Theta) + \lambda_{\text{max}}^2 \tag{145}
\]

Lemma 4. The following identity holds:
\[
(II) \leq \frac{n_1 N_2}{p} \left[ 44544 \text{rank}(\Theta) + 89088 \right] \frac{\lambda_{\text{max}}}{\lambda_*} + \frac{\lambda_{\text{max}}^2}{\lambda_*^2} \right) \gamma^2(\| \Sigma_R \|)^2 \tag{146}
\]
Proof of Lemma 4. Note that

\[
5568 \left( \sqrt{\text{rank}(\Theta)} + \frac{\lambda_{\text{max}}}{\lambda_*} \right)^2 \gamma^2 \mathbb{E}[\|\Sigma_R\|]^2 = \left[ 5568 \left( 8 \text{rank}(\Theta) + 2 \sqrt{\text{rank}(\Theta)} \frac{\lambda_{\text{max}}}{\lambda_*} + \frac{\lambda_{\text{max}}^2}{\lambda_*^2} \right) \right] \gamma^2 \mathbb{E}[\|\Sigma_R\|]^2 \leq \frac{n_1 N_2}{p} \left[ 44544 \text{rank}(\Theta) + 89088 \text{rank}(\Theta) \frac{\lambda_{\text{max}}}{\lambda_*} + \frac{\lambda_{\text{max}}^2}{\lambda_*^2} \right] \gamma^2 \mathbb{E}[\|\Sigma_R\|]^2. \]  

(147)

Now using Lemma 3, Lemma 4, we have

\[
\frac{1}{4n_1 N_2} \left\| \Theta - \hat{\Theta} \right\|_{\Pi, F}^2 \leq \frac{n_1 N_2}{p} \left( \left( \frac{18 \lambda_*^2}{L_y^2} + \frac{12 \lambda_* \lambda_{\text{max}}}{L_y^2} \right) \text{rank}(\Theta) + \frac{\lambda_{\text{max}}^2}{\lambda_*^2} \right) \gamma^2 \mathbb{E}[\|\Sigma_R\|]^2 + \frac{5567 \gamma^2}{n_1 N_2 p}. \]  

(149)

\[
\leq \frac{n_1 N_2}{p} \left( \left( \frac{18 \lambda_*^2}{L_y^2} + \frac{12 \lambda_* \lambda_{\text{max}}}{L_y^2} \right) \text{rank}(\Theta) + \frac{\lambda_{\text{max}}^2}{\lambda_*^2} \right) \gamma^2 \mathbb{E}[\|\Sigma_R\|]^2 \leq \frac{n_1 N_2}{p} \left( \left( \frac{18 \lambda_*^2}{L_y^2} + \frac{12 \lambda_* \lambda_{\text{max}}}{L_y^2} \right) \text{rank}(\Theta) + \frac{\lambda_{\text{max}}^2}{\lambda_*^2} \right) \gamma^2 \mathbb{E}[\|\Sigma_R\|]^2 + \frac{5567 \gamma^2}{n_1 N_2 p} \]  

(150)

\[
\leq \frac{n_1 N_2}{p} \left( \left( \frac{18 \lambda_*^2}{L_y^2} + \frac{12 \lambda_* \lambda_{\text{max}}}{L_y^2} \right) \text{rank}(\Theta) + \frac{\lambda_{\text{max}}^2}{\lambda_*^2} \right) \gamma^2 \mathbb{E}[\|\Sigma_R\|]^2 \leq \frac{n_1 N_2}{p} \left( \left( \frac{18 \lambda_*^2}{L_y^2} + \frac{12 \lambda_* \lambda_{\text{max}}}{L_y^2} \right) \text{rank}(\Theta) + \frac{\lambda_{\text{max}}^2}{\lambda_*^2} \right) \gamma^2 \mathbb{E}[\|\Sigma_R\|]^2 + \frac{5567 \gamma^2}{n_1 N_2 p} \]  

(151)

\[
\leq \frac{n_1 N_2}{p} \left( \left( \frac{18 \lambda_*^2}{L_y^2} + \frac{12 \lambda_* \lambda_{\text{max}}}{L_y^2} \right) \text{rank}(\Theta) + \frac{\lambda_{\text{max}}^2}{\lambda_*^2} \right) \gamma^2 \mathbb{E}[\|\Sigma_R\|]^2 + 5567 \gamma^2 \left( \frac{n_1 N_2}{p} \right). \]  

(152)

\[
\leq \frac{n_1 N_2}{p} \left( \left( \frac{18 \lambda_*^2}{L_y^2} + \frac{12 \lambda_* \lambda_{\text{max}}}{L_y^2} \right) \text{rank}(\Theta) + \frac{\lambda_{\text{max}}^2}{\lambda_*^2} \right) \gamma^2 \mathbb{E}[\|\Sigma_R\|]^2 + \frac{5567 \gamma^2}{n_1 N_2 p} \]  

(153)

\[
\leq \frac{n_1 N_2}{p} \left( \left( \frac{18 \lambda_*^2}{L_y^2} + \frac{12 \lambda_* \lambda_{\text{max}}}{L_y^2} \right) \text{rank}(\Theta) + \frac{\lambda_{\text{max}}^2}{\lambda_*^2} \right) \gamma^2 \mathbb{E}[\|\Sigma_R\|]^2 + \frac{5567 \gamma^2}{n_1 N_2 p} \]  

(154)

\[
\leq \frac{n_1 N_2}{p} \left( \lambda_*^2 + \lambda_{\text{max}} L_y^2 \right) \gamma^2 \mathbb{E}[\|\Sigma_R\|]^2 + \frac{5567 \gamma^2}{n_1 N_2 p} \]  

(155)

\[
\leq \frac{n_1 N_2}{p} \left( \lambda_*^2 + \lambda_{\text{max}} L_y^2 \right) \gamma^2 \mathbb{E}[\|\Sigma_R\|]^2 + \frac{5567 \gamma^2}{n_1 N_2 p} \]  

(156)

\[
\leq \frac{n_1 N_2}{p} \left( \lambda_*^2 + \lambda_{\text{max}} L_y^2 \right) \gamma^2 \mathbb{E}[\|\Sigma_R\|]^2 + \frac{5567 \gamma^2}{n_1 N_2 p} \]  

(157)
Therefore, the inequality \( a + b \leq 2(a \lor b) \) for \( a, b \in \mathbb{R} \) yields
\[
\frac{1}{n_1 N_2} \| \Theta - \hat{\Theta} \|_{1,F}^2 
\leq 2C_p \max \left\{ 2 \max \left\{ n_1 N_2 \text{rank}(\Theta) \left( \frac{1}{L^4 \gamma} \left( \lambda_*^2 + \lambda_* \lambda_{\max} L^2 \gamma \right) + \left( 1 + \frac{\lambda_{\max}}{\lambda_*} + \frac{\lambda^2_{\max}}{\lambda^2_*} \right) \gamma^2 (\mathbb{E}[\| \Sigma_{\mathcal{R}} \|])^2 \right) \right\} 
+ n_1 N_2 \text{rank}(\Theta) \gamma^2 \frac{\log(n_1 + N_2)}{n_1 N_2}
\right\}
\leq 4C_p \max \left\{ \max \left\{ n_1 N_2 \text{rank}(\Theta) \left( \frac{1}{L^4 \gamma} \left( \lambda_*^2 + \lambda_* \lambda_{\max} L^2 \gamma \right) + \left( 1 + \frac{\lambda_{\max}}{\lambda_*} + \frac{\lambda^2_{\max}}{\lambda^2_*} \right) \gamma^2 (\mathbb{E}[\| \Sigma_{\mathcal{R}} \|])^2 \right) \right\} 
+ n_1 N_2 \text{rank}(\Theta) \gamma^2 \frac{\log(n_1 + N_2)}{n_1 N_2}
\right\}
\leq C_* \max \left\{ n_1 N_2 \text{rank}(\Theta) \left( \frac{1}{L^4 \gamma} \left( \lambda_*^2 + \lambda_* \lambda_{\max} L^2 \gamma \right) + \left( 1 + \frac{\lambda_{\max}}{\lambda_*} + \frac{\lambda^2_{\max}}{\lambda^2_*} \right) \gamma^2 (\mathbb{E}[\| \Sigma_{\mathcal{R}} \|])^2 \right) + \lambda^2_{\max} \gamma^2 \frac{\log(n_1 + N_2)}{n_1 N_2}
\right\},
\]
where the second inequality follows from \( \max(a, b) \leq \max(a, \eta \cdot b) \) for \( \eta > 1 \) and the third inequality follow commutativity of the max function. This completes the proof of Theorem 5.

\[ \Box \]

**Proof of Theorem 6**

For ease of notation, we let
\[
\mathcal{H} = \left( \frac{1}{L^4 \gamma} \left( \lambda_*^2 + \lambda_* \lambda_{\max} L^2 \gamma + L^4 \lambda^2_{\max} \right) + \left( 1 + \frac{\lambda_{\max}}{\lambda_*} + \frac{\lambda^2_{\max}}{\lambda^2_*} \right) \gamma^2 (\mathbb{E}[\| \Sigma_{\mathcal{R}} \|])^2 \right)
\]

Since we let
\[
\lambda_* = 2c \left( \frac{(U_\gamma \cup K)(\sqrt{n_1 \lor N_2} + (\log(n_1 \lor N_2))^{3/2})}{n_1 N_2} \right) \quad \text{and} \quad \lambda_{\max} \leq \kappa \lambda_*,
\]

it follows that
\[
\frac{1}{L^4 \gamma} \left( \lambda_*^2 + \lambda_* \lambda_{\max} L^2 \gamma + L^4 \lambda^2_{\max} \right) \leq \frac{1}{L^4 \gamma} \left( 4c^2 (U_\gamma \cup K)^2 (\sqrt{n_1 \lor N_2} + (\log(n_1 \lor N_2))^{3/2})^2 (1 + \kappa L^2 \gamma + \kappa^2 L^4 \gamma) \right)
\leq c_1 (1 + \kappa L^2 \gamma + \kappa^2 L^4 \gamma) \left( (U_\gamma \cup K)^2 (\sqrt{n_1 \lor N_2} + (\log(n_1 \lor N_2))^{3/2})^2 \right). \quad (168)
\]

And that
\[
\left( 1 + \frac{\lambda_{\max}}{\lambda_*} \right) \gamma^2 (\mathbb{E}[\| \Sigma_{\mathcal{R}} \|])^2 \leq \left[ 1 + \kappa \right] \gamma^2 \cdot c_\Sigma \left( \frac{\sqrt{n_1 \lor N_2} + \sqrt{\log(n_1 \lor N_2)}}{n_1 N_2} \right)^2 \leq \left( \frac{\sqrt{n_1 \lor N_2} + (\log(n_1 \lor N_2))^{3/2}}{n_1 N_2} \right)^2 \cdot c_\Sigma (\kappa + 1) \gamma^2. \quad (170)
\]

\[ \Box \]
And that
\[ \frac{\lambda_{\text{max}}^2}{\lambda^2} \gamma^2 (E[|\Sigma_R|])^2 \leq \kappa^2 \cdot c_{\Sigma}^2 \left( \frac{\sqrt{n_1 \vee N_2} + \log(n_1 \vee N_2)}{n_1 N_2} \right)^2 \gamma^2 \] (172)
\[ \leq c_{\Sigma}^2 \kappa^2 \gamma^2 \left( \frac{\sqrt{n_1 \vee N_2} + \log(n_1 \vee N_2))^{3/2}}{(n_1 N_2)^2} \right)^2. \] (173)

Therefore, it follows that
\[ \mathcal{H} \leq \left( \frac{\sqrt{n_1 \vee N_2} + \log(n_1 \vee N_2))^{3/2}}{(n_1 N_2)^2} \right)^2 \left( \frac{c_1 (1 + \kappa L^2 + \kappa^2 L^4)}{L^4} (U_\gamma \vee K)^2 + c_\Sigma (\kappa + 1) \gamma^2 + c_{\Sigma}^2 \kappa^2 \gamma^2 + 4 \right) \] (174)
\[ \leq \left( \frac{\sqrt{n_1 \vee N_2} + \log(n_1 \vee N_2))^{3/2}}{(n_1 N_2)^2} \right)^2 \left( (U_\gamma \vee K)^2 \frac{c_1 (1 + \kappa L^2 + \kappa^2 L^4)}{L^4} + \gamma^2 (1 + c_\Sigma \kappa + c_{\Sigma}^2 \kappa^2) \right) \] (175)
\[ \leq C \left( \frac{\sqrt{n_1 \vee N_2} + \log(n_1 \vee N_2))^{3/2}}{(n_1 N_2)^2} \right)^2 \left( (U_\gamma \vee K)^2 \frac{(1 + \kappa L^2 + \kappa^2 L^4)}{L^4} + \gamma^2 (1 + \kappa + \kappa^2) \right) \] (176)
\[ \leq C \left( \frac{n_1 \vee N_2}{(n_1 N_2)^2} + \frac{\log^3(n_1 \vee N_2)}{(n_1 N_2)^2} \right)^2 \left( (U_\gamma \vee K)^2 \frac{(1 + \kappa L^2 + \kappa^2 L^4)}{L^4} + \gamma^2 (1 + \kappa + \kappa^2) \right). \] (177)

Hence it follows that
\[ \frac{1}{n_1 N_2} \| \Theta - \hat{\Theta} \|_{\Pi, F} \leq C \left\{ \frac{\log \lambda_{\text{max}}}{n_1 N_2} + c_\Sigma \log(n_1 + N_2) \right\} \] (180)
\[ \leq \frac{C}{p} \max \left\{ \frac{n_1 N_2 \left[ \text{rank}(\Theta) \mathcal{H} + \lambda_{\text{max}}^2 \right]}{n_1 N_2} \right\} \] (181)
\[ \leq \frac{\check{C}}{p} \max \left\{ \frac{\log^3(n_1 \vee N_2)}{(n_1 N_2)^2} \right\} \left( (U_\gamma \vee K)^2 \frac{(1 + \kappa L^2 + \kappa^2 L^4)}{L^4} + \gamma^2 (1 + \kappa + \kappa^2) \right) \] (182)
\[ = \frac{\check{C} \text{rank}(\Theta)}{p} \left( \frac{n_1 \vee N_2}{n_1 N_2} + \frac{\log^3(n_1 \vee N_2)}{n_1 N_2} \right) \left( (U_\gamma \vee K)^2 \frac{(1 + \kappa L^2 + \kappa^2 L^4)}{L^4} + \gamma^2 (1 + \kappa + \kappa^2) \right). \] (183)

Also, using the fact the \( p \|A\|_F \leq \|A\|_{\Pi, F} \), for any matrix \( A \), it follows that
\[ \frac{1}{n_1 N_2} \| \Theta - \hat{\Theta} \|_F \leq \frac{\check{C} \text{rank}(\Theta)}{p} \left( \frac{n_1 \vee N_2}{n_1 N_2} + \frac{\log^3(n_1 \vee N_2)}{n_1 N_2} \right) \left( (U_\gamma \vee K)^2 \frac{(1 + \kappa L^2 + \kappa^2 L^4)}{L^4} + \gamma^2 (1 + \kappa + \kappa^2) \right) \] (184)
\[ = \frac{\check{C} \text{rank}(\Theta)(n_1 \vee N_2)}{p^2 n_1 N_2} \left( 1 + \frac{\log^3(n_1 \vee N_2)}{n_1 \vee N_2} \right) \left( (U_\gamma \vee K)^2 \frac{(1 + \kappa L^2 + \kappa^2 L^4)}{L^4} + \gamma^2 (1 + \kappa + \kappa^2) \right). \] (185)

This completes the proof of Theorem 6.
Appendix B: Technical Lemmas

Lemma 5. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{P}$ be an orthogonal operator. Then $\|\mathcal{P}(f)\| \leq \|f\|$.

Proof. Note that by Cauchy Schwartz inequality, we have

$$\|\mathcal{P}(f)\|^2 = \langle \mathcal{P}(f), \mathcal{P}(f) \rangle = \langle \mathcal{P}(f), f \rangle \leq \|\mathcal{P}(f)\| \|f\|. \quad (186)$$

The result follows by dividing both size by $\|\mathcal{P}(f)\|$.

Lemma 6. For $1 \leq p < q$, the following inequality holds

$$\|x\|_q \leq \|x\|_p \leq n^{\frac{1}{p} - \frac{1}{q}} \|x\|_q \quad (187)$$

for $x \in \mathbb{R}^n$.

Proof. We first show that $\|x\|_q \leq \|x\|_p$. Without loss of generality, it suffices to assume that $\|x\|_p = 1$ since $\|x\|_q \leq \|x\|_p$ if and only if $\left\|\frac{x}{\|x\|_p}\right\|_q \leq \left\|\frac{x}{\|x\|_p}\right\|_p = 1$. For ease of notation, let $z = x/\|x\|_p$. Note that $\|z\|_q \leq 1 \implies z_i \leq 1$ for all $i = 1, \ldots, n$. Now since $x^q \leq x^p$ for all $x \in (0, 1)$, it follows that

$$\|z\|_q = \left(\sum_{i=1}^n |z_i|^q\right)^{\frac{1}{q}} \leq \left(\sum_{i=1}^n |z_i|^p\right)^{\frac{1}{q}} = \|z\|_p^q = 1. \quad (188)$$

The result follows by multiplying both sides by $\|x\|_p$.

Next, we show that $\|x\|_p \leq n^{1/p - 1/q} \|x\|_q$. This follows from Holder’s inequality which states that for $r > 1$,

$$\sum_{i=1}^n \frac{|a_i|}{b_i} \leq \left(\sum_{i=1}^n |a_i|^r\right)^{\frac{1}{r}} \left(\sum_{i=1}^n |b_i|^{-\frac{1}{r}}\right)^{\frac{1}{r}}. \quad (189)$$

Apply (189) to $a_i = |x_i|^p$, $b_i = 1$ and $r = \frac{q}{p} > 1$ and we get

$$\sum_{i=1}^n |x_i|^p \leq \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{q}{p}} \left(\sum_{i=1}^n 1^{1 - \frac{q}{p}}\right)^{\frac{1}{1 - \frac{q}{p}}} = \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{q}{p}} n^{1 - \frac{q}{p}}. \quad (190)$$

Taking the $p$-th root on both sides yields

$$\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} \leq \left[\left(\sum_{i=1}^n |x_i|^q\right)^{\frac{q}{p}} n^{1 - \frac{q}{p}}\right]^{\frac{1}{p}} = \left(\sum_{i=1}^n |x_i|^q\right)^{\frac{1}{q}} \left(n^{1 - \frac{q}{p}}\right)^{\frac{1}{p}} = \|x\|_q n^{\frac{1}{q} - \frac{1}{p}}. \quad (191)$$

Lemma 7. Let $A \in \text{Mat}_{\mathbb{R}}(m \times n)$, then the following inequality holds:

$$\|A\|_* \leq \sqrt{\text{rank}(A)} \|A\|_F. \quad (192)$$

Proof. Let $USV^* = A$ be the singular value decomposition of $A$. Note that $\|A\|_* = \sum_{i=1}^n \sigma_i(A) = \sum_{i=1}^r \Sigma_i, i = \|\text{diag}(\Sigma)\|_1$. On the other hand, note that

$$\|A\|_2 = \text{tr}(A^T A) = \text{tr}(V \Sigma U^* U^* V^*) = \text{tr}(V \Sigma^2 V^*) = \text{tr}(\Sigma^2) = \sum_{i=1}^r \Sigma_i^2 = \|\text{diag}(\Sigma)\|_2. \quad (193)$$

Then, the result follows from an application of Lemma 6 to $\text{diag}(\Sigma)$.

\[\square\]
Lemma 8. Let $A, B$ be compatible matrices and

$$U\Sigma V^* = \begin{bmatrix} U & \tilde{U} \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V \\ \tilde{V} \end{bmatrix} = A = \sum_{i=1}^{rank(A)} \sigma_i(A)u_iv_i^*$$

(194)

be the fat-version of singular value decomposition of $A$. Let

$$T_A = \{u_ka^*, yv_j^* | k = 1, \ldots, r, x \in \mathbb{R}^{m_2}, y \in \mathbb{R}^{n_1}\},$$

(195)

be the generating set of rank $r$ matrices spanned by $A$’s singular vectors. Let $P_T(\cdot)$ be the orthogonal projection onto $T$. Then the following (in)equalities hold:

1. $P_{T_A}(B) = P_U B + BP_V - P_U BP_V = UU^* B + BVV^* - UU^* BVV^*,$
2. $P_{T_A}(B) = (I - P_T)(B) = (I_{n_1} - P_U)X(I_{n_2} - P_V),$
3. $\text{rank}(P_{T_A}(B)) \leq 2\text{rank}(A),$
4. $\|P_{T_A}(B)\|_* \leq \sqrt{2\text{rank}(A)} \|B\|_F,$ for compatible real matrices $A$ and $B$, and
5. $\|A\|_* - \|B\|_* \leq \|P_{T_A}(A - B)\|_* - \|P_{T_A}^-(A - B)\|_*.$

Proof. 1. Suppose $x \in T$, then

\[
x = \sum_{i \in |\{i\}| < \infty} \alpha_i u_i x_i^* + \sum_{j \in |\{j\}| < \infty} \beta_j y v_j^* = \sum_{i=1}^r \tilde{\alpha}_i u_i \tilde{x}_i^* + \sum_{j=1}^r \tilde{\beta}_j y v_j^* = \sum_{i=1}^r \tilde{\alpha}_i u_i \left( \sum_{j=1}^{n_2} \theta_{1,i,j} v_j^* \right) + \sum_{j=1}^r \tilde{\beta}_j \left( \sum_{i=1}^{n_1} \theta_{2,i,j} u_i \right) v_j^* \text{ (for some } v_j \in \{v_1, \ldots, v_k\}, u_i \in \{u_1, \ldots, u_k\}\})
\]

(196)

\[
x = \sum_{i=1}^r \sum_{j=1}^{n_2} \tilde{\alpha}_i \tilde{\beta}_{1,i,j} u_i v_j^* + \sum_{j=1}^r \sum_{i=1}^{n_1} \tilde{\beta}_j \theta_{2,i,j} u_i v_j^* = \sum_{i=1}^r \sum_{j=1}^{n_2} \gamma_{1,i,j} u_i v_j^* + \sum_{j=1}^r \sum_{i=1}^{n_1} \gamma_{2,i,j} u_i v_j^*. \text{ (197)}
\]

Then it follows that

\[
x \in \langle u_i v_j^* | 1 \leq i \leq r, 1 \leq j \leq n_2 \rangle \text{ or } 1 \leq i \leq n_1, 1 \leq j \leq r \rangle,
\]

(198)

the other direction follows using the same argument. Therefore,

\[
T_A = \langle u_i v_j^* | 1 \leq i \leq r, 1 \leq j \leq n_2 \rangle \text{ or } 1 \leq i \leq n_1, 1 \leq j \leq r \rangle.
\]

(199)

Now we calculate the projection onto $T$, $P_T$ : using the projection formula we have that for any matrix $B \in \text{Mat}_R(n_1, n_2),$

\[
P_{T_A}(B) = \sum_{1 \leq i \leq r, 1 \leq j \leq n_2 \text{ or } 1 \leq i \leq n_1, 1 \leq j \leq r} \langle B, u_i v_j^* \rangle u_i v_j^*
\]

(200)

\[
= \sum_{i=1}^r \sum_{j=1}^r \langle B, u_i v_j^* \rangle u_i v_j^* + \sum_{i=1}^r \sum_{j=r+1}^{n_2} \langle B, u_i v_j^* \rangle u_i v_j^* + \sum_{i=r+1}^{n_1} \sum_{j=1}^r \langle B, u_i v_j^* \rangle u_i v_j^*. \text{ (201)}
\]

We analyze it term by term:
\[ \mathcal{P}_{T_A}(B) = P_U B P_V + P_U B P_{V^\perp} + P_U^* B P_V = P_U B + B P_V - P_U X P_V = UU^* B + B V V^* - UU^* B V V^* \] as desired.

2. This is because of by orthogonal decomposition, we have \( \text{Mat}_\mathbb{R}(n_1, n_2) = T \oplus T^\perp \), we have

\[ \mathcal{P}_{T_A}(B) = (I - \mathcal{P}_F(B)) = B - P_U B - B P_V + P_U B P_V = (I - P_U) B (I - P_V). \]

3. Note that

\[ \text{rank}(\mathcal{P}_{T_A}(B)) = \text{rank}(UU^* B + B V V^* - UU^* B V V^*) = \text{rank}(UU^* B + (I - UU^*) B V V^*) \leq \text{rank}(U) + \text{rank}(V) \leq 2 \text{rank}(A), \]

where the second to last inequality follows by keeping applying the basic inequality \( \text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\} \).

4. Note that by Lemma 7 and part-(3)

\[ \|\mathcal{P}_{T_A}(B)\|_* \leq \sqrt{\text{rank}(\mathcal{P}_{T_A}(B))} \|\mathcal{P}_{T_A}(B)\|_F \leq \sqrt{2 \text{rank}(A)} \|\mathcal{P}_{T_A}(B)\|_F = \sqrt{2 \text{rank}(A)} \|\mathcal{P}_{T_A}(B), \mathcal{P}_{T_A}(B)\|^{1/2} \]

\[ = \sqrt{2 \text{rank}(A)} \|\mathcal{P}_{T_A}^* \mathcal{P}_{T_A}(B), B\|^{1/2} = \sqrt{2 \text{rank}(A)} \|\mathcal{P}_{T_A}(B), B\|^{1/2} \leq \sqrt{2 \text{rank}(A)} \|B\|, \]

where the last inequality follows from Lemma 5.

5. Note that

\[ \|B\|_* = \|A + B - A\|_* = \|A + \mathcal{P}_{T_A}^+ (B - A) + \mathcal{P}_{T_A}(B - A)\|_* \geq \|A + \mathcal{P}_{T_A}^+ (B - A)\|_* - \|\mathcal{P}_{T_A}(B - A)\|_* . \]

**Claim 1.** \( \|A + \mathcal{P}_{T_A}^+ (B - A)\|_* \geq \|A\|_* + \|\mathcal{P}_{T_A}^+ (B - A)\|_* . \)

**Proof.** Let \( \tilde{U} \tilde{\Sigma} \tilde{V}^* = (B - A) \) be the singular value decomposition of \( B - A \). Then note that

\[ A + \mathcal{P}_{T_A}^+ (B - A) = U \Sigma V^* + \left[ \tilde{U} \tilde{U}^* U_{B - A} \right] \Sigma_{B - A} \left[ V_{B - A}^* \tilde{V} \tilde{V}^* \right]. \]

Let \( Q_1 R_1 = \tilde{U}^* U_{B - A} \) and \( Q_2 R_2 = \tilde{V}^* V_{B - A} \) be two thin QR decompositions, then it follows that

\[ \left[ \tilde{U} \tilde{U}^* U_{B - A} \right] \Sigma_{B - A} \left[ V_{B - A}^* \tilde{V} \tilde{V}^* \right] = \tilde{U} Q_1 \Sigma_{B - A} R_2 Q_2^* = (\tilde{U} Q_1) \tilde{\Sigma}_{B - A} (\tilde{V} Q_2)^*, \]
where the last equality follows the fact that the product of an upper triangular and lower triangular matrix is a diagonal matrix. We substitute this equation back to (213) we get

\[ A + P_{T_A}(B - A) = [U \quad \tilde{U}Q_1] \left[ \begin{array}{cc} \Sigma & 0 \\ 0 & \tilde{\Sigma}_{B-A} \end{array} \right] [V \quad \tilde{V}Q_2]^*. \]  

(215)

since \((\tilde{U}Q_1)^*(\tilde{U}Q_1) = I\), and \((\tilde{V}Q_2)^*(\tilde{V}Q_2) = I\) and the columns in \(U, \tilde{U}Q_1\) and in \(V, \tilde{V}Q_2\) are orthogonal to each other by construction, it follows that

\[ \|A + P_{T_A}(B - A)\|_* = \|\text{diag}(\Sigma)\|_{\ell_1} + \|\text{diag}(\tilde{\Sigma}_{B-A})\|_{\ell_1} = \|A\|_* + \|P_{T_A}(B - A)\|_* \]  

(216)

as desired.

\[ \text{Claim 2.} \text{ An application of Claim 1 yields that} \]

\[ \|B\|_* \geq \|A\|_* + \|P_{T_A}(B - A)\|_* - \|P_{T_A}(B - A)\|_1, \]  

(217)

which after rearrangement gives the desired inequality.

The following result could be found in plenty of standard Banach space textbooks, see for example, Brézis and Brézis (2011).

**Lemma 9.** Let \(f\) be a function with continuous second partial derivative defined on an open convex set \(U \subset \mathbb{R}^n\). Then for any \(x\) and \(x_0 \in U\), the following identity holds.

\[ f(x) = f(x_0) + \langle \nabla f(x_0), (x - x_0) \rangle + \frac{1}{2}(x - x_0)^T \nabla^2 f(x_0) + c(x - x_0)(x - x_0) \]  

(218)

for \(c \in (0,1)\).

**Lemma 10.** Under Assumption 2, it follows that

\[ L_\gamma^2(x - y) \leq 2d_\gamma^2(x, y) \leq U_\gamma^2(x - y)^2. \]  

(219)

**Proof.** By definition

\[ d_\gamma^2(x, y) = A(x) - A(y) - \langle \nabla A(y), x - y \rangle - \frac{1}{2}(x - y)^2 \nabla^2 A(\xi) \]  

(220)

for some \(\xi \in (x, y)\). Then by Assumption 2, we see that the result holds.

**Proposition 4** (Corollary 3.3 in Bandeira and Handel (2016)). Let \(W\) be the \(n \times m\) rectangular matrix whose entries \(W_{i,j}\) are independent centered bounded random variables. Then there exists a universal constant \(\epsilon\) such that

\[ \mathbb{E}[\|W\|] \leq c(\kappa_1 \vee \kappa_2 + \kappa_3 \sqrt{\log(n \wedge m)}), \]  

(221)

where

\[ \kappa_1 = \max_{i \in [n]} \sqrt{\sum_{j \in [m]} \mathbb{E}[W_{i,j}^2]}, \quad \kappa_2 = \max_{j \in [m]} \sqrt{\sum_{i \in [n]} \mathbb{E}[W_{i,j}^2]}, \quad \kappa_3 = \max_{(i,j) \in [n] \times [m]} |W_{i,j}|. \]  

(222)

**Lemma 11.** Let \(A, B \in B_{\|\cdot\|_\infty}(\gamma)\). If

\[ \ell(A) = \lambda_* \|A\|_* + \lambda_{\text{max}} \|A\|_{\text{max}} \leq \ell(B) = \lambda_* \|B\|_* + \lambda_{\text{max}} \|B\|_{\text{max}}, \]  

(223)

then the following inequalities hold:

1. \(\|P_{B}(A - B)\|_* \leq 3\|P_{B}(A - B)\| + 2\frac{\lambda_{\text{max}}}{\lambda_*} \|A - B\|_F.\)

2. \(\|A - B\|_* = 2 \left( \sqrt{\text{rank}(B)} + \frac{\lambda_{\text{max}}}{\lambda_*} \right) \|A - B\|_F.\)
Proof. 1. First, note that
\[
\lambda_{\max}(\|A\|_{\infty} - \|B\|_{\infty}) = -\lambda_{\max}(\|B\|_{\infty} - \|A\|_{\infty}) \leq -\lambda_{\max}(\|B - A\|_{\max})
\] (224)

Note that
\[
\ell(B|Y) - \ell(A|Y) \geq \lambda_{\ast} (\|A\|_{\infty} - \|B\|_{\infty}) + \lambda_{\max}(\|A\|_{\max} - \|B\|_{\max})
\] (225)
\[
\geq \lambda_{\ast} (\|P_{B}(A - B)\|_{\ast} - \|P_{B}(A - B)\|_{\ast}) + \lambda_{\max}(\|A\|_{\max} - \|B\|_{\max})
\] (226)
\[
\geq \lambda_{\ast} (\|P_{B}(A - B)\|_{\ast} - \|P_{B}(A - B)\|_{\ast}) - \lambda_{\max} \|A - B\|_{\infty}
\] (227)
\[
\geq \lambda_{\ast} (\|P_{B}(A - B)\|_{\ast} - \|P_{B}(A - B)\|_{\ast}) - \lambda_{\max} \|A - B\|_{\ast}.
\] (228)

where (227) follows from the fact that \(\lambda_{\max} \geq 0\) and the fact that \(x \geq -|x|\) for any \(x \in \mathbb{R}\) and (228) follows by Lemma 12. On the other hand, it follows from convexity that
\[
\ell(A|Y) \geq \ell(B|Y) + (\nabla \ell(B|Y), B - A) \implies \ell(B|Y) - \ell(A|Y) \leq \langle \nabla \ell(B|A), B - A \rangle.
\] (229)

An application of operator norm Cauchy Schwartz inequality yield and
\[
\ell(B|Y) - \ell(A|Y) \leq \|\nabla \ell(B|A)\| \|B - A\|_{\ast} \leq \frac{\lambda_{\ast}}{2} \|B - A\|_{\ast},
\] (230)
where the second inequality follows from the assumption stated in the theorem. Combining the (228) and (230), we get that
\[
\lambda_{\ast} (\|P_{B}(A - B)\|_{\ast} - \|P_{B}(A - B)\|_{\ast}) - \lambda_{\max} \|A - B\|_{\ast} \leq \frac{\lambda_{\ast}}{2} \|B - A\|_{\ast},
\] (231)
which by rearranging, becomes
\[
\frac{1}{2} \|P_{B}(A - B)\|_{\ast} \leq \frac{3}{2} \|P_{B}(A - B)\|_{\ast} + \frac{\lambda_{\max}}{\lambda_{\ast}} \|A - B\|_{F},
\] (232)
which is equivalent to
\[
\|P_{B}(A - B)\|_{\ast} \leq 3 \|P_{B}(A - B)\|_{\ast} + \frac{2 \lambda_{\max}}{\lambda_{\ast}} \|A - B\|_{F},
\] (233)

2. A direct application of part (1) yields that
\[
\|A - B\|_{\ast} = \|P_{B}(A - B)\|_{\ast} + \|P_{B}(A - B)\|_{\ast}
\] (234)
\[
\leq 3 \|P_{B}(A - B)\|_{\ast} + \frac{\lambda_{\max}}{\lambda_{\ast}} \|A - B\|_{F} + \|P_{B}(A - B)\|_{\ast}
\] (235)
\[
\leq 4 \|P_{B}(A - B)\|_{\ast} + \frac{\lambda_{\max}}{\lambda_{\ast}} \|A - B\|_{F}
\] (236)
\[
\leq 4 \sqrt{2 \text{rank}(B)} \|A - B\|_{\ast} + \frac{2 \lambda_{\max}}{\lambda_{\ast}} \|A - B\|_{F}
\] (237)
\[
\leq 4 \sqrt{2 \text{rank}(B)} \|A - B\|_{F} + \frac{2 \lambda_{\max}}{\lambda_{\ast}} \|A - B\|_{F}
\] (238)
\[
= 2 \left( \sqrt{2 \text{rank}(B)} + \frac{\lambda_{\max}}{\lambda_{\ast}} \right) \|A - B\|_{F}
\] (239)

Lemma 12. We have \(|\|A\|_{\max} - \|B\|_{\max}| \leq \|A - B\|_{F} \).

Proof. Note that
\[
|\|A\|_{\max} - \|B\|_{\max}| \leq \|A - B\|_{\max} \leq \|A - B\|_{F}.
\] (240)

Another way to see it is that \(\|A - B\|_{\max}\) is the maximum of the \(L^2\) row norms, where as \(\|A - B\|_{F} \) is the sum of all rows’ \(L^2\) norms.  \qed

32
Lemma 13 (Appendix A.1 in Alaya and Klopp (2019)). Let \( \beta = \frac{4965^2 \log(n_1+N_2)}{pn_1N_2} \). Then for all \( \Xi \in \mathcal{K}(\beta,r) \), it follows that

\[
\left| \Delta^2(\Xi, \Theta) - \frac{1}{n_1N_2} \|\Xi - \Theta\|^2_{\Pi,F} \right| \leq \frac{\|\Xi - \Theta\|^2_{\Pi,F}}{2n_1N_2} + 1392r\gamma^2 (E[\|\Sigma_i\|^2])^2 + \frac{5567\gamma^2}{n_1N_2}. \tag{241}
\]

Lemma 14 (Lemma 2 in Alaya and Klopp (2019)). Let Assumption 2 holds. Then there exists an absolute constant \( c \) such that with probability \( 1 - 4/(n_1+N_2) \), we have that

\[
\|\ell(\Theta|Y)\| \leq c \left( \frac{(U_{\Xi} \cup K)(\sqrt{n_1} \vee N_2 + (\log(n_1 \vee N_2))^{3/2}}{n_1N_2} \right). \tag{242}
\]

Proposition 5. Let \( A \in \text{Mat}_R(n_1, n_2) \), then \( \|A\|_{2\to\infty} = \max_{i=1}^{n_1} \|A^*e_i\|_{\ell_2} \), i.e. it is the maximum of the row \( \ell_2 \) norm of \( A \).

Proof. First, note that the equality clearly holds when \( A = 0 \). So without loss of generality, we can assume that \( A \neq 0 \). Note that

\[
\|A\|_{2,\infty} = \sup_{\|x\|_2} \|Ax\|_\infty = \sup_{\|x\|_2} \max_{1 \leq i \leq n_1} \langle Ax, e_i \rangle = \sup_{\|x\|_2} \max_{1 \leq i \leq n_1} \langle x, A^*e_i \rangle \tag{243}
\]

\[
\leq \sup_{\|x\|_2} \max_{1 \leq i \leq n_1} \|x\|_2 \|A^*e_i\|_2 = \max_{1 \leq i \leq n_1} \|A^*e_i\|_{\ell_2}. \tag{244}
\]

On the other hand, let \( \dagger \) be row number of \( A \) that has the largest row \( \ell_2 \) norm (in case of duplicate, pick the first one). In other words, \( \dagger = \arg\max_{1 \leq i \leq n_1} \|A^*e_i\|_{\ell_2} \). Note

\[
\|A\|_{2,\infty} = \max_{1 \leq i \leq n_1} \langle Ax, e_i \rangle \geq \left( A \frac{A^*e_\dagger}{\|A^*e_\dagger\|} , e_\dagger \right) = \frac{1}{\|A^*e_\dagger\|} \langle A^*e_\dagger, A^*e_\dagger \rangle = \|A^*e_\dagger\|_{\ell_2} = \max_{1 \leq i \leq n_1} \|A^*e_i\|_{\ell_2}. \tag{245}
\]

And the proof is completed. \( \square \)

Lemma 15. Let \( M \in \mathbb{R}^{n \times m} \), then it follows that

\[
\|M\|_{\max} \leq \|M\|_{2,\infty} \leq \|M\|_F. \tag{246}
\]

Proof. Since \( \|M\|_{2,\infty} \) is the maximum of the row \( \ell_2 \) norms of \( M \), and \( \|M\|_F \) is the sum of all row \( \ell_2 \) norms, the inequality clearly holds and it suffices to establish the first part of the inequality. Recall from that by definition

\[
\|M\|_{\max} = \min_{U,V \text{ s.t. } M=UV^T} \|U\|_{2,\infty} \|V\|_{2,\infty}. \tag{247}
\]

Note that \( M \) has a trivial decomposition \( M = M \cdot I \), where \( I \in \mathbb{R}^{n \times n} \); it follows that

\[
\min_{U,V \text{ s.t. } M=UV^T} \|U\|_{2,\infty} \|V\|_{2,\infty} \leq \|M\|_{2,\infty} \|I\|_{2,\infty} = \|M\|_{2,\infty}, \tag{248}
\]

and the result follows as desired. \( \square \)

Lemma 16 (Lemma 3.3 in Fang et al. (2018)). Consider the optimization problem

\[
\min_{z \in \mathbb{R}^d} \beta \|z\|_{\infty} + \frac{1}{2} \|c-z\|_2^2. \tag{249}
\]

Assume that \( c_1 \geq c_2 \geq \ldots \geq c_d \geq 0 \). The solution to the problem has the following closed form:

\[
z^* = (t^*, \ldots, t^*, c_{k^*+1}, \ldots, c_d)^T, \tag{250}
\]

where \( t^* = \frac{1}{k^*} \sum_{i=1}^{k^*} (c_i - \beta) \) and \( k^* \) is the index such that \( c_{k^*+1} < \frac{1}{k^*} \sum_{i=1}^{k^*} (c_i - \beta) \leq c_{k^*} \). If no such \( k^* \) exists, then \( z^* = (t^*, \ldots, t^*)^T \), where \( t^* = \frac{1}{d} \sum_{i=1}^{d} (c_i - \beta) \).

Lemma 17 (Negative Binomial Moments). Let \( X \) be a random variable such that \( X \sim \text{NB}(r,p) \). Then
Proof. There are many ways to prove this fact. Here we use the standard factorial moment trick. Note that

\[
E[X] = \sum_{k=0}^{\infty} k \frac{\Gamma(k+r)}{k! \Gamma(r)} p^r (1-p)^k
\]

(251)

\[
= (1-p) \sum_{k=1}^{\infty} \frac{\Gamma(k+r)}{(k-1)! \Gamma(r)} p^r (1-p)^{k-1}
\]

(252)

\[
= (1-p) \sum_{j=0}^{\infty} \frac{\Gamma(j+1+r)}{j! \Gamma(r)} p^r (1-p)^j
\]

(253)

\[
= (1-p) \sum_{j=0}^{\infty} \frac{\Gamma(j+r)(j+r)}{j! \Gamma(r)} p^r (1-p)^j
\]

(254)

\[
= (1-p) \left[ j \left( \sum_{j=0}^{\infty} \frac{\Gamma(j+r)}{j! \Gamma(r)} p^r (1-p)^j \right) + r \left( \sum_{j=0}^{\infty} \frac{\Gamma(j+r)}{j! \Gamma(r)} p^r (1-p)^j \right) \right]
\]

(255)

\[
= (1-p) (E[X] + r),
\]

(256)

Solving yields \( E[X] = \frac{r(1-p)}{p} \). To calculate \( \text{Var}(X) \), we first compute

\[
E[X^2] = \sum_{k=0}^{\infty} k^2 \frac{\Gamma(k+r)}{k! \Gamma(r)} p^r (1-p)^k
\]

(257)

\[
= \sum_{k=0}^{\infty} [k(k-1) + k] \frac{\Gamma(k+r)}{k! \Gamma(r)} p^r (1-p)^k
\]

(258)

\[
= \sum_{k=0}^{\infty} k(k-1) \frac{\Gamma(k+r)}{k! \Gamma(r)} p^r (1-p)^k + \sum_{k=0}^{\infty} k \frac{\Gamma(k+r)}{k! \Gamma(r)} p^r (1-p)^k
\]

(259)

\[
= (1-p)^2 \sum_{j=0}^{\infty} \frac{\Gamma(j+2+r)}{j! \Gamma(r)} p^r (1-p)^j + E[X]
\]

(260)

\[
= (1-p)^2 \sum_{j=0}^{\infty} \frac{\Gamma(j+r)(j+1+r)(j+r)}{j! \Gamma(r)} p^r (1-p)^j + E[X]
\]

(261)

\[
= (1-p)^2 \left[ \sum_{j=0}^{\infty} \left( j^2 + j r + 2jr + j + r \right) \frac{\Gamma(j+r)}{j! \Gamma(r)} p^r (1-p)^j \right] + E[X]
\]

(262)

\[
= (1-p)^2 (E[X^2] + r^2 + 2rE[X] + E[X] + r) + E[X]
\]

(263)

\[
= (1-p)^2 E[X^2] + [2r(1-p)^2 + (1-p)^2 + 1]E[X] + (1-p)^2 (r^2 - r)
\]

(264)

which after arrangement and some bit of algebra yields that \( E[X^2] = \frac{r(p^2 r - 2pr - p + r + 1)}{p^2} \). As a result, we have that

\[
\text{Var}(X) = E[X^2] - (E[X])^2 = \frac{r(p^2 r - 2pr - p + r + 1)}{p^2} - \frac{r^2(1-p)^2}{p^2} = \frac{r(p-1)}{p^2}.
\]

(265)

\[\square\]

Lemma 18 (Negative Binomial mean parametrization). Alternatively, we can parametrize by its mean in the following way: a random variable \( X \) is a negative binomial random variable with mean \( \mu \) and number of success \( r \) if and only if it has the following p.m.f

\[
P(X = k) = \frac{\Gamma(k+r)}{\Gamma(r)k!} \left( \frac{r}{\mu + r} \right)^r \left( \frac{\mu}{\mu + r} \right)^k.
\]

(266)
Proof. This could be directly verified using Lemma 17. Alternatively, one could rewrite (26) in its exponential family canonical form and invoke using the gradient forward map properties. □