EXOTIC SYMMETRIC SPACE
OVER A FINITE FIELD, I

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Abstract. Let $V$ be a $2n$-dimensional vector space over an algebraically closed field $k$ with $\text{ch} k \neq 2$. Let $G = \text{GL}(V)$ and $H = \text{Sp}_{2n}$ be the symplectic group obtained as $H = G^\theta$ for an involution $\theta$ on $G$. We also denote by $\theta$ the induced involution on $\mathfrak{g} = \text{Lie} G$. Consider the variety $G/H \times V$ on which $H$ acts naturally. Let $\mathfrak{g}^\theta_{\text{nil}}$ be the set of nilpotent elements in the $-1$ eigenspace of $\theta$ in $\mathfrak{g}$. The role of the unipotent variety for $G$ in our setup is played by $\mathfrak{g}^\theta_{\text{nil}} \times V$, which coincides with Kato's exotic nilpotent cone. Kato established, in the case where $k = \mathbb{C}$, the Springer correspondence between the set of irreducible representations of the Weyl group of type $C_n$ and the set of $H$-orbits in $\mathfrak{g}^\theta_{\text{nil}} \times V$ by applying Ginzburg theory for affine Hecke algebras. In this paper we develop a theory of character sheaves on $G/H \times V$, and give an alternate proof for Kato's result on the Springer correspondence based on the theory of character sheaves.

Introduction

Let $G' = \text{GL}_n$ acting on the $n$-dimensional vector space $V'$ over an algebraically closed field $k$, and $\mathfrak{g}' = \text{Lie} G'$. Let $G'_{\text{uni}}$ (respectively $\mathfrak{g}'_{\text{nil}}$) be the unipotent variety of $G'$ (respectively the nilpotent cone of $\mathfrak{g}'$). We consider the action of $G'$ on the variety $\mathfrak{g}'_{\text{nil}} \times V'$, where $G'$ acts on $\mathfrak{g}'_{\text{nil}}$ by the adjoint action, and on $V'$ by the natural action. By Achar–Henderson [AH] and Travkin [T], $\mathfrak{g}'_{\text{nil}} \times V'$ has finitely many $G'$-orbits parametrized by double partitions of $n$. Following [AH], we call $\mathfrak{g}'_{\text{nil}} \times V'$ the enhanced nilpotent cone. In [AH], they studied the intersection cohomology of the closure of such orbits, and showed that associated Poincaré polynomials give Kostka polynomials labelled by double partitions, introduced in [S2], which is an analogue of the classical result by Lusztig [L1] relating nilpotent orbits in $\mathfrak{g}_{\text{nil}}$ and Kostka polynomials. Passing to the group case, we consider the action of $G'$ on the variety $G' \times V'$, where $G'$ acts on $G'$ by conjugation, and on $V'$ by the natural action. Finkelberg–Ginzburg–Travkin [FGT] constructed a family

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of \(G'\)-equivariant simple perverse sheaves on \(G' \times V'\), and developed an analogue of the theory of character sheaves on \(G'\), where \(G'_{\text{uni}} \times V' \simeq g_{\text{nil}}' \times V'\) plays a role of the unipotent variety of \(G'\). They conjecture that the characteristic functions of such character sheaves on \(G' \times V'\) provide a basis of the space of \(G'(\mathbb{F}_q)\)-invariant functions on \((G' \times V')(\mathbb{F}_q)\).

Assume that \(k\) is an algebraic closure of a finite field \(\mathbb{F}_q\) with \(\text{ch} k \neq 2\), and let \(V\) be a 2\(n\)-dimensional vector space over \(k\). Let \(H = \text{Sp}_{2n}\) be the symplectic group obtained as the fixed point subgroup \(G^\theta\) for an involutive automorphism \(\theta\) on \(G = \text{GL}(V)\), and consider the symmetric space \(G/H\). In [BKS], Bannai–Kawanaka–Song studied the characters of the Hecke algebra \(\mathcal{H} = \mathcal{H}(G(\mathbb{F}_q), H(\mathbb{F}_q))\) associated to the pair \(H(\mathbb{F}_q) \subset G(\mathbb{F}_q)\), and showed that the character table of \(\mathcal{H}\) is basically obtained from the character table of \(\text{GL}_n(\mathbb{F}_q)\) by replacing \(q\) by \(q^2\) in an appropriate sense. On the other hand, in [H1] Henderson tried to reconstruct the result of [BKS] in terms of the geometry of the symmetric space \(G/H\). Let 
\[
g_{\text{nil}}^\theta = \{g \in g_{\text{nil}} \mid \theta(g) = -g\}
\]
for the involution \(\theta\) induced on \(g = \text{Lie} G\). Then \(H\) acts on \(g_{\text{nil}}^\theta\), and \(H\)-orbits are labelled by partitions of \(n\). He showed, in particular, that Poincaré polynomials associated to the intersection cohomology of the closure of those orbits provide Kostka polynomials, replacing the variable \(q\) by \(q^2\), which is a geometric counterpart of the result of [BKS].

In this paper, we consider the variety \(G/H \times V\) as a generalization of the above two cases. \(H\) acts on \(G/H \times V\) as a left multiplication on \(G/H\), and as the natural action on \(V\). In this setup, the role of the unipotent variety for \(G\) is played by the variety \(g_{\text{nil}}^\theta \times V\), which is nothing but the exotic nilpotent cone introduced by Kato [Ka1]. So we shall call \(G/H \times V\) the exotic symmetric space. \(H\) acts on \(g_{\text{nil}}^\theta \times V\). Kato showed that the number of \(H\)-orbits is finite and they are parametrized by double partitions of \(n\) (a reformulation by Achar–Henderson [AH]). An interesting relationship is expected between the intersection cohomology of the closure of those orbits and Kostka polynomials labelled by double partitions. Our aim is to construct a theory of character sheaves on \(G/H \times V\) as an analogue of the theory for \(G'\) and \(G' \times V'\). In fact, in [HT] Henderson–Trapa propose a construction of character sheaves on \(G/H \times V\), as a natural generalization of mirabolic character sheaves due to [FGT]—“exotic character sheaves” in their terminology. In their framework, the character sheaves constructed in this paper cover just the principal series part. However, we expect that any exotic character sheaf can be obtained by our construction.

The main result in this paper is the Springer correspondence between the set of irreducible representations of the Weyl group of type \(C_n\) and the set of \(H\)-orbits of \(g_{\text{nil}}^\theta \times V\) through the intersection cohomology of the closure of \(H\)-orbits. In fact, the Springer correspondence for the exotic nilcone was first established by [Ka1], by using the Ginzburg theory of affine Hecke algebras. In [Ka2], he determined the correspondence explicitly by computing Joseph polynomials associated to \(H\)-orbits. So our result gives an alternate approach to Kato’s result based on the theory of character sheaves, which is quite similar to the original proof of the Springer correspondence due to Borho–MacPherson [BM]. We prove the restriction theorem for Springer representations, which is an analogue of Lusztig’s restriction theorem [L2] with respect to the generalized Springer correspondence, and we