CONDITIONALLY RISK-AVERSE CONTEXTUAL BANDITS

Mónika Farsang  
Vienna University of Technology  
monika.farsang@tuwien.ac.at

Paul Mineiro  
Microsoft Research  
pmineiro@microsoft.com

Wangda Zhang  
Microsoft Research  
wangdazhang@microsoft.com

ABSTRACT

Contextual bandits with average-case statistical guarantees are inadequate in risk-averse situations because they might trade off degraded worst-case behaviour for better average performance. Designing a risk-averse contextual bandit is challenging because exploration is necessary but risk-aversion is sensitive to the entire distribution of rewards; nonetheless we exhibit the first risk-averse contextual bandit algorithm with an online regret guarantee. We conduct experiments from diverse scenarios where worst-case outcomes should be avoided, from dynamic pricing, inventory management, and self-tuning software; including a production exascale data processing system.

1 Introduction

Contextual bandits [Auer et al., 2002; Langford and Zhang, 2007] are a mature technology with numerous applications; however, adoption has been most aggressive in recommendation scenarios [Bouneffouf and Rish, 2019], where the worst-case outcome is user annoyance. At the other extreme are medical and defense scenarios where worst-case outcomes are literally fatal. In between are scenarios of interest where bad outcomes are tolerable but should be avoided, e.g., logistics; finance; and self-tuning software, where the term tail catastrophe highlights the inadequacy of average case performance guarantees in real-world applications [Marcus et al., 2021]. These scenarios demand risk-aversion, i.e., decisions should sacrifice average performance in order to avoid worst-case outcomes, and incorporating risk-aversion into contextual bandits would facilitate adoption. More generally, risk aversion is essential for making informed decisions that align with the risk preferences of the decision maker by balancing the potential benefits and risks of a particular action.

This paper solves risk-averse decision making for contextual bandits via reduction to regression, resulting in the first risk-averse contextual bandit algorithm with an online regret guarantee. The regret guarantee applies over adversarially chosen context sequences and includes the exploration choices made by the algorithm. The approach utilizes arbitrary (online learnable) function classes and extends to infinite action spaces; introduces no computational overhead relative to the risk-neutral setting; introduces statistical overhead directly related to the desired level of risk-aversion, with no overhead in the risk-neutral limit; and composes with other innovations within the Decision-to-Estimation framework [Foster et al., 2021], e.g., linear representability [Zhu and Mineiro, 2022].

We make the following contributions:

- We explain the problem setting (Section 2) with careful definitions which facilitate the application of theory and reveal the unique status of expectile loss.
- We state the resulting algorithms (Section 3), which arise via application of the Estimation-to-Decisions framework [Foster et al., 2021].
- We discuss the superior utility of expectile loss for algorithm design over more commonly used risk-measures VaR and CVaR (Section 2 and 3).
- We provide experimental support for the technique via diverse scenarios (Section 4). Empirically, tail control is proportionally inexpensive relative to average-case degradation, justifying the criticism of average-case guarantees in the self-tuning software literature.
2 Problem Setting

This section contains tedious exposition, necessary because (i) this work draws heavily on results from mathematical finance that cannot be presumed known by the general machine learning audience; and (ii) careful definitions are key to our contribution. For the impatient reader wanting to skip directly to Section [3] we provide the following summary: use expectile loss. The rest of this section answers the question "why?".

Contextual Bandits We describe the contextual bandit problem, which proceeds over $T$ rounds. At each round $t \in [T]$, the learner receives a context $x_t \in \mathcal{X}$ (the context space), selects an action $a_t \in \mathcal{A}$ (the action space), and then observes a loss $l_t(a_t)$, where $l_t : \mathcal{A} \rightarrow [0, 1]$ is the underlying loss function. We assume that for each round $t$, conditioned on $x_t$, $l_t$ is sampled from a distribution $\mathbb{P}_{l_t} (\cdot | x_t)$. We allow both the contexts $x_1, \ldots, x_T$ and the distributions $\mathbb{P}_{l_1}, \ldots, \mathbb{P}_{l_T}$ to be selected in an arbitrary, potentially adaptive fashion based on the history.

Risk Measures In seminal work [Artzner et al., 1999] presented an axiomatic approach to measuring risk. A risk measure is a function which maps a random variable to $\mathbb{R} \cup \{\infty\}$ and obeys certain axioms such as normalization, translation contravariance, and monotonicity. Risk measures embed previous approaches to measuring risk: we refer the interested readers to [Meyfired, 2004].

Conditional Risk-Aversion When considering extensions of risk-averse bandit algorithms to the contextual setting, two possible choices are apparent: marginal risk-aversion, corresponding to applying a risk measure to the distribution of losses realized over the joint context-action distribution; and conditional risk-aversion, corresponding to computing a risk measure on a per-context basis and then summing over encountered contexts. For now our focus is conditional risk-aversion, but after introducing terminology, we revisit the relationship between these two at the end of this section.

Contextual Bandit Regret Conditional risk-aversion motivates our definition of regret for finite action sets,

$$\text{Reg}_{CB}(T) \doteq \sum_{t=1}^{T} \mathbb{E}_{a_t} \left[ \rho \left( \langle l_t \rangle_{a_t} \right) - \min_{a} \rho \left( \langle l_t \rangle_{a} \right) \right] x_t,$$

where $\rho$ is a risk measure, and the expectation is with respect to (the algorithm’s) action distribution; note $\rho$ is a function of the adversary’s loss random variable and not the realization. For infinite action sets we use a smoothed regret criterion: instead of competing with the best action, we compete with any action distribution $Q$ with limited concentration $\frac{dQ}{dh} \leq h^{-1}$ relative to a reference measure $\mu$,

$$\text{Reg}_{CB}^{(h, \mu)}(T) \doteq \sum_{t=1}^{T} \left( \mathbb{E}_{a_t} \left[ \rho \left( \langle l_t \rangle_{a_t} \right) | x_t \right] - \min_{Q} \mathbb{E}_{a \sim Q} \left[ \rho \left( \langle l_t \rangle_{a} \right) | x_t \right] \right).$$

Note the finite action regret is a special case, corresponding to the uniform reference measure $\mu$ and $h^{-1} = |\mathcal{A}|$. In practice $\mu$ is a hyperparameter while $h$ can be tuned using contextual bandit meta-learning: see experiments for details.

Reduction to Regression We attack the contextual bandit problem via reduction to regression, working with a user-specified class of regression functions $\mathcal{F} \subseteq (\mathcal{X} \times \mathcal{A} \rightarrow [0, 1])$ that aims to estimate a risk measure $\rho$ of the conditional loss distribution. We make the following realizability assumption:

$$\forall a \in \mathcal{A}, t \in [T] : \exists f^* \in \mathcal{F} : f^*(x_t, a) = \rho \left( \langle l_t \rangle_{a} \right),$$

i.e., our function class includes a function which correctly estimates the value of the risk measure arising from any action $a$ in context $x_t$. This constrains the adversary’s choices, as $l_t$ must be consistent with realizability, but there are many random variables that achieve a particular risk value.

Motivation for EVaR We describe additional desirable properties of a risk measure which ultimately determine our choice of risk measure. A law-invariant risk measure is invariant to transformations of the random variable that preserve the distribution of outcomes, i.e., is a function of distribution only [Kusuoka, 2001]. An elicitable risk measure can be defined as the minimum of the expectation of a loss function. Because our algorithm operates via reduction to regression, we require an elicitable risk measure. A coherent risk measure satisfies the additional axiom of convexity: coherence is desirable because it implies risk reduction from diversification. To avoid confusion, note the convexity of a risk measure

[Foster et al., 2020] demonstrate misspecification is tolerable, but we do not complicate the exposition here.
We assume access to an online regression oracle which is an algorithm for sequential prediction under strongly convex losses using \( F \) as a benchmark class. More specifically, the oracle operates in the following protocol: at each round \( t \in [T] \), the algorithm receives a context \( x_t \in X \), makes a prediction \( \hat{f}_t \), where \( f_t(x_t, a) \) is interpreted as the prediction for action \( a \), and then observes an action \( a_t \in \mathcal{A} \) and realized outcome \( l_t(a_t) \in [0, 1] \) and incurs instantaneous expectile loss

\[
g_t(\hat{f}_t) = \left( 1 - q \right) \left( (\hat{v} - \hat{v})_+^2 + q \left( (\hat{v} - v)_+^2 \right) \right) \bigg|_{v=l_t(a_t), \hat{v} = \hat{f}_t(x_t, a_t)}.
\]

We assume \( \text{Alg}_{\text{Reg}} \) guarantees that for any (potentially adaptively chosen) sequence \( (x_t, a_t, l_t)_{t=1}^T \),

\[
\sum_{t=1}^T \left( g_t(\hat{f}_t) - g_t(f^*) \right) \leq \text{Reg}_{\text{EVaR}_q}(T),
\]

for some (non-data-dependent) function \( \text{Reg}_{\text{EVaR}_q}(T) \). Online regression is well-studied with many known algorithms in various cases, e.g., for linear \( F \) on the \( d \)-dimensional hypersphere, online Newton step achieves \( \text{Reg}_{\text{EVaR}_q}(T) = \mathcal{O} \left( \frac{d}{\log(T)} \right) \) [Hazan et al., 2007]. Furthermore, for any finite \( F \) we can achieve \( \text{Reg}_{\text{EVaR}_q}(T) = \mathcal{O} \left( \frac{1}{\min(\rho,1-q)} \log |F| \right) \) using Vovk’s aggregation algorithm [Vovk, 1998]. Section 2.3 of Foster and Rakhlin [2020] has a more complete list of oracles.

**Optimization Oracle** We assume an approximate (possibly randomized) optimization oracle \( \text{Alg}_{\text{Opt}} : F \times \Delta(\mathcal{A}) \times \mathbb{R}^+ \rightarrow \Delta(\mathcal{A}) \) which guarantees

\[
\forall \hat{f} \in F : \mathbb{E}_{\hat{a} \sim \text{Alg}_{\text{Opt}}(\hat{f}, \mu, \delta)} \left[ \mathbb{E}_{\mu} \left[ \max \left( 0, \hat{f}(\hat{a}) - \hat{f}(a) \right) \right] \right] \leq \delta,
\]

i.e., given an (estimated reward) function \( \hat{f} \) the optimization oracle can find an approximate minimizer \( \hat{a} \) w.r.t the reference measure \( \mu \). For finite action sets we can compute \( \text{Alg}_{\text{Opt}} \) in \( \mathcal{O}(|A|) \) for all \( \mu \) with \( \delta = 0 \). For infinite action sets we can compute \( \text{Alg}_{\text{Opt}} \) with high probability via the empirical argmin over \( \mathcal{O} \left( \frac{1}{\delta^2} \right) \) i.i.d. samples from \( \mu \), independent of the cardinality or dimensionality of the action space. Of course specific function classes may admit superior customized strategies.

\[\text{Ziegler [2016]}\] shows the class of elicitable law-invariant coherent risk measures for real-valued random variables is precisely Entropic Value at Risk (EVaR) for \( q \in \left( \frac{1}{2}, 1 \right] \), defined as

\[
\text{EVaR}_q(D) = \arg \min_{v \in [0,1]} \mathbb{E}_{v \sim D} \left[ (1 - q) \left( (v - \hat{v})_+^2 + q \left( (\hat{v} - v)_+^2 \right) \right) \right],
\]

where \((x)_+ = \max(x,0)\). This asymmetrical strongly convex loss encourages overprediction relative to the mean, implying infrequent large losses correspond to increased risk. A minimizer of equation (3) is called an *expectile*. Certain technical qualifications are necessary for the minimum to be achieved (bounded realization suffices). We refer to the elicitation loss function as *expectile loss*.

EVaR is less familiar to the machine learning community than VaR or CVaR, but is a popular risk-measure in financial applications [Bellini and Di Bernardino, 2017], whose proponents champion the superior finite-sample guarantees induced by strong convexity [Rossello, 2022]. Waltrup et al. [2015] reveal connections between EVaR and the risk measures VaR and CVaR; in particular noting that both VaR and CVaR can be computed from EVaR. See Section 3 for additional commentary.

When \( q \in \left( \frac{1}{2}, 1 \right) \), EVaR\(_q\) is risk-seeking. While not our focus, the analysis remains valid therefore we state results in terms of \( \min(q, 1 - q) \).

The relationship involves differences which induces ambiguous curvature and is therefore not viable for incorporating VaR or CVaR into decision-to-estimation.
Algorithm 1 Finite Action Set

1: for $t = 1, 2, \ldots, T$ do
2:     Receive context $x_t$.
3:     $\hat{f}_t \leftarrow \text{Alg}_{\text{Reg}}. \text{predict}(x_t)$.
4:     $\hat{a}_t \leftarrow \text{Alg}_{\text{Opt}}(\hat{f}_t, 0)$.
5:     Sample $a_t \sim \text{AL}(\hat{f}_t, \hat{a}_t)$.
6:     Play $a_t$ and observe loss $l_t$.
7:     Call $\text{Alg}_{\text{Reg}}. \text{update}(x_t, a_t, l_t)$.

Algorithm 2 Infinite Action Set

1: for $t = 1, 2, \ldots, T$ do
2:     Receive context $x_t$.
3:     $\hat{f}_t \leftarrow \text{Alg}_{\text{Reg}}. \text{predict}(x_t)$.
4:     $\hat{a}_t \leftarrow \text{Alg}_{\text{Opt}}(\hat{f}_t, \frac{1}{T})$.
5:     Sample $a_t \sim \text{Cont-AL}(\hat{f}_t, \hat{a}_t)$.
6:     Play $a_t$ and observe loss $l_t$.
7:     Call $\text{Alg}_{\text{Reg}}. \text{update}(x_t, a_t, l_t)$.

Figure 1: (Left) Finite action set with exact optimization oracle. (Right) Infinite action set with approximate optimization oracle. Hyperparameters $\mu$ and $h$ are elided to facilitate comparison.

Marginal vs. conditional, revisited Now consider an oblivious stationary environment where $(x, l)$ is drawn from a fixed joint distribution $D$: further assume a law-invariant risk measure to ease exposition, i.e., assume $\rho$ is a function of distribution only. Marginal risk-aversion regret for a policy $\pi : X \to \mathcal{P}(A)$ over a policy class $\Pi$ is defined as

$$\text{Reg}_{\text{Marg}}(\pi) \equiv \rho(D_{\text{Marg}}(\pi)) - \min_{\pi \in \Pi} \rho(D_{\text{Marg}}(\pi))$$

where $D_{\text{Marg}}(\pi)$ is defined via

$$\mathbb{E}_{z \sim D_{\text{Marg}}(\pi)} [f(z)] \equiv \mathbb{E}_{(x, l) \sim D} [f(l)]$$

Marginal risk-aversion does not correspond to the expectation of a per-context function, because the risk measure is a function of the complete distribution. Thus, if we apply an online-to-batch conversion to a conditional risk-aversion regret guarantee, we end up with a regret guarantee with respect to the expected conditional risk under $D$ rather than the marginal risk. For coherent risk measures, minimizing expected conditional risk minimizes an upper bound on marginal risk, which is sensible. However this is unlike the risk-neutral setting, where an adversarial guarantee provides a tight stochastic guarantee. In financial parlance, an algorithm designed for the stochastic case could benefit from diversification opportunities across context. However, conditional risk-aversion is the appropriate metric for scenarios where re-distributing risk across contexts is not acceptable, e.g., software quality-of-service guarantees where the contexts are customers.

Conditional risk alternative For conditional risk there is a plausible alternative definition. Equation (1) is defined by averaging the per-action risk over the policy action distribution, but another quantity of interest is the risk measure of the complete conditional (on context) action distribution. Due to coherence of the risk measure, the definition in equation (1) upper bounds this alternative,

$$\mathbb{E}_{a_{t}} [\rho(D(l_{t}, a_{t}|x_{t}))] \geq \rho(D(l_{t}, a_{t}|x_{t})),$$

where $D(l_{t}, a_{t}|x_{t})$ is the joint distribution of the action and loss under the algorithm’s conditional action distribution. Fortunately, unlike the marginal vs. conditional case, this is tight because we are competing with the best single action and the bound is tight for degenerate distributions. Thus optimizing our regret also controls the risk measure of the complete conditional action distribution.

3 Algorithms

Proofs are elided to the supplemental. The proof technique has useful generality, e.g. enables the use of an approximate minimizer in the continuous case.

By using the Estimation-to-Decision framework, we derive the resulting algorithm, which is the first risk-averse contextual bandit with an online guarantee. We present two versions for finite and infinite action sets.

3.1 Finite Action Set

Algorithm 1 states the finite action version of our algorithm. It is the SquareCB algorithm [Foster and Rakhlin 2020] instantiated with an expectile loss regression oracle.

Theorem 3.1. Algorithm 1 guarantees $\text{Reg}_{\text{CB}}(T) \leq O\left(\frac{1}{\theta} \sqrt{|A|T \text{Reg}_{\text{EVaR}_q}(T)}\right)$, where $\theta = \min(q, 1 - q)$. 

4
When describing experiments, we will use a reward convention when it is more natural, despite the analysis using loss convention. We will also describe experiments using the natural reward range rather than explicitly transforming to $\theta=0$.

We emphasize this regret is with respect to the risk measure of the best action for each context, and includes the average learning performance and the likelihood of these worst-case outcomes. To prevent these adverse outcomes, one may make a trade-off between maximizing average-case and minimizing worst-case outcomes.

Our experiments emphasize scenarios where average-case guarantees are inadequate, and are intended to exhibit a trade-off between competing with an extreme expectile. The parameter $\theta$ is the strong convexity parameter of the expectile loss. The distribution in line 5 of Algorithm 1 is

$$\text{Cont-AL}(\hat{f}_t, \hat{a}_t) = \begin{cases} 
1 & a = \hat{a}_t \\
\frac{1}{|A|} & a \neq \hat{a}_t 
\end{cases}.$$ 

**Remark 3.1.** VaR and CVaR are alternative popular risk measures that differ from EVaR: VaR lacks coherence, and CVaR is not elicitable (only jointly elicitable) [Fissler and Ziegel, 2016]. Both VaR and CVaR do not have strongly convex elicitation losses and hence are not compatible with the decision-to-estimation framework.

**Remark 3.2.** It is possible to obtain a regret bound which depends upon the loss of the optimal predictor ($L^*$) by eliciting expectile loss via asymmetric KL divergence combined with a FastCB-style reduction [Foster and Krishnamurthy, 2021]. It is difficult to envision a realistic risk-averse scenario in which $L^*$ is expected to be small, i.e., in which the risk measure is expected to obtain small values yet average case guarantees are insufficient, so we have neglected this direction in this paper. However in a risk-seeking scenario small $L^*$ is plausible and of potential interest.

### 3.2 Infinite Action Set

Algorithm 2 states the infinite action version of our algorithm. It is the SmoothCB algorithm [Zhu and Mineiro, 2022], adjusted to allow for approximate minimization and instantiated with expectile loss.

**Theorem 3.2.** Algorithm 2 guarantees $\text{Reg}_{\text{CB}}^{(h, \mu)}(T) \leq O\left(\frac{1}{\theta} \sqrt{\frac{1}{\pi} T \text{Reg}_{\text{EVaR}}^{(q)}(T)}\right)$, where $\theta = \min(q, 1-q)$.

**Proof.** See Appendix A.1.

The distribution in line 5 of Algorithm 2 is

$$\text{AL}(\hat{f}_t, \hat{a}_t) = \begin{cases} 
\frac{1}{|A|} & a = \hat{a}_t \\
1 & a \neq \hat{a}_t 
\end{cases}.$$ 

**Remark 3.3.** The strong convexity of expectile loss admits other infinite action strategies for specialized function classes, e.g., linearly structured action spaces [Zhu and Mineiro, 2022]. Relative to squared loss, expectile loss introduces no computational overhead, and the statistical overhead is $\min(q, 1-q)$ due to the reduction in the strong convexity parameter.

### 4 Experiments

Our experiments emphasize scenarios where average-case guarantees are inadequate, and are intended to exhibit a trade-off between maximizing average-case and minimizing worst-case outcomes.

Table 3 in Appendix B gives an overview of the scenarios and associated datasets. None of the datasets used contain either personally identifying information or offensive content. The selected datasets present various risks, such as overestimating prices in dynamic pricing, incurring unnecessary inventory costs in inventory management, and selecting worse-than-baseline configurations in self-tuning software. These risks lead to undesirable outcomes such as no-sale, financial losses, or software performance issues. To prevent these adverse outcomes, one may make a trade-off between average learning performance and the likelihood of these worst-case outcomes.

Across many domains, we found comparing a risk-averse setting with $q = 0.2$ and the risk-neutral technique with $q = 0.5$ exhibited a clear tradeoff. Note that $q = 0.5$ is the same as using the standard squared loss function.

When describing experiments, we will use a reward convention when it is more natural, despite the analysis using loss convention. We will also describe experiments using the natural reward range rather than explicitly transforming to $[0, 1]$. In our first experiment we assess realized online expectiles directly, but in subsequent experiments we focus on key metrics whose control is a consequence of risk-aversion.
Continuous action experiments are implemented in PyTorch, using Lebesgue reference measure $\mu$; selecting $h$ adaptively via Corral [Agarwal et al., 2017]; and computing $\hat{a}$ via the empirical minimum over $\gamma$ samples from $\mu$. Finite action experiments are implemented in Vowpal Wabbit [Langford et al., 2007]. Hyperparameters are tuned using best of 59 random trials. Confidence intervals are 95% coverage bootstrap intervals of online performance. Code to reproduce all results, along with the “Query Opt” dataset, is available at https://github.com/zwd-ms/risk_averse_cb. All experiments run comfortably on a commodity laptop.

4.1 Dynamic Pricing

Prudential Our first dataset is from the Prudential Life Insurance Assessment Competition, which contains customer features along with an associated discrete integral risk level between 1 and 8 inclusive. We convert this to a dynamic pricing simulation as follows. First, the algorithm is asked to predict a risk level given the customer features. It is assumed that the risk level is associated with a price quote which, when correctly assessed, leads to maximum profit. If the algorithm overpredicts the risk level, the reward is 0; this corresponds to quoting the customer too large of a premium and losing business to a competitor (“no sale”). If the risk level is not overpredicted the reward is a linear function of the difference between the predicted and actual risk level; this corresponds to charging too little for the premium. Denoting the ground truth label as $\gamma$ and the predicted label as $\hat{\gamma}$, we have $\text{Profit}(y,\hat{y};\beta) = (1 - \beta (y - \hat{y})) I_{y \geq \hat{y}}$. We use $\beta = 0.1$ in our experiments.

Housing Datasets Our next two datasets are King County and Perth home prices, both of which contain home features along with a ground truth listing price. We convert these to a dynamic pricing simulation as follows. The algorithm must choose a listing price, and if it is lower than the ground truth listing price, the algorithm receives 0 reward (“no sale”). Denoting the ground truth listing price $y$ and the chosen listing price $\tilde{y}$, we have $\text{Profit}(y,\tilde{y}) = \tilde{y} I_{y \geq \tilde{y}}$. We treat (normalized) prices as continuous actions on $[0, 1]$ and utilize Algorithm 2 with Lebesgue reference measure. For our regressor class, we first predict $\tilde{\gamma} : X \rightarrow [0, 1] \times (0, \infty)$ using a linearized Cauchy kernel machine [Rahimi and Recht, 2007], and then induce a prediction function $\hat{f}$,

$$
\hat{f}(x, a) = a \frac{\text{erf} \left( \frac{1 - \tilde{\gamma}(x)}{\tilde{\gamma}(x)} \right) - \text{erf} \left( \frac{\alpha - \tilde{\gamma}(x)}{\tilde{\gamma}(x)} \right)}{\text{erf} \left( \frac{1 - \tilde{\gamma}(x)}{\tilde{\gamma}(x)} \right) + \text{erf} \left( \frac{\alpha - \tilde{\gamma}(x)}{\tilde{\gamma}(x)} \right)},
$$

(5)

This functional form is inspired by a truncated Gaussian random variable, but does not imply any particular generative model. It is simply a suitable function which is easy to implement in Pytorch.

Online Performance Figure 3 shows multiple realized marginal expectiles on the Prudential dataset when the algorithm is either risk-averse or risk-neutral. This figure deviates from our theoretical analysis in two ways. First, it displays realized marginal expectiles (i.e., expectiles computed from the actual sequence of rewards experienced online) rather than summed conditional expectiles. Second, it extrapolates results to expectiles not optimized by the algorithm. Nonetheless, the result exhibits the desired tail control and the extrapolation is reasonable.

Complete results are in Table 1. All CIs in the table are computed from the online realizations. In particular, $\text{EVaR}_{0.2}$ is the empirical marginal expectile experienced by the algorithm. We see that learning with risk-aversion ($q = 0.2$) trades average performance (profit) for tail control. Furthermore, learning with risk-aversion reduces the frequency of no sale in exchange for a reduction in profit. Fractionally, reduction in profit is less than the reduction in the frequency of no sale.

| Dataset     | Learn $q$ | $\text{EVaR}_{0.2}$ ($) | Profit ($) | No Sale (%) |
|-------------|-----------|-------------------------|------------|-------------|
| King        | 0.2       | [18.2, 18.7]            | [26.3, 26.7]| [8.8, 9.1]  |
|             | 0.5       | [17.2, 17.6]            | [28.0, 28.4]| [17.5, 18.1]| |
| Perth       | 0.2       | [22.2, 22.5]            | [29.6, 29.9]| [9.5, 9.9]  |
|             | 0.5       | [18.0, 18.5]            | [31.0, 31.4]| [23.3, 23.8]| |
| Prudential  | 0.2       | [41.4, 41.7]            | [53.4, 53.8]| [0.05, 0.09]|
|             | 0.5       | [38.7, 39.4]            | [60.6, 61.2]| [16.4, 17.0]| |
Conditionally Risk-Averse Contextual Bandits

| Dataset   | q   | Profit ($)   | Sold Out (%) |
|-----------|-----|--------------|--------------|
| Chicago   | 0.2 | [2.1,2.3]    | [50.0,50.7]  |
|           | 0.5 | [3.7,3.9]    | [20.0,20.4]  |
| DC        | 0.2 | [7.8,8.1]    | [30.7,31.6]  |
|           | 0.5 | [9.8,10.3]   | [18.9,19.6]  |
| London    | 0.2 | [3.5,3.7]    | [56.3,57.8]  |
|           | 0.5 | [4.7,5.0]    | [28.1,29.1]  |

Figure 2: Inventory management results. (Left) Risk-aversion results in lower profits but higher chance of inventory fully selling out. (Right) Risk-aversion conservatively explores into larger allocations from a region of safety.

Approximate vs. exact  \( \hat{a} \) Unimodality of equation (5) allows us to compare an approximate maximizer, computed over \( \gamma \) samples from \( \mu \); with an exact maximizer, computed using Brent’s method. Table 2 compares on the Perth dataset. Statistically results are similar. Computationally, Brent’s method is slower as it is not vectorized.

| Dataset | Learn q | Exact? | EVaR_{0.2} | Profit ($) | No Sale (%) |
|---------|---------|--------|------------|------------|-------------|
| Perth   | 0.2     | Y      | [21.7,22.1]| [29.5,29.8]| [8.5,8.8]   |
|         | N       |        | [22.2,22.5]| [29.6,29.9]| [9.5,9.9]   |
|         | 0.5     | Y      | [18.0,18.6]| [31.3,31.7]| [23.4,24.0]|
|         | N       |        | [18.0,18.5]| [31.0,31.4]| [23.3,23.8]|

4.2 Inventory Management

Chicago, DC, London Our next three datasets are public bicycle demand datasets which contain weather and date information along with a count of the number of bicycles demanded. We convert these to an inventory management simulation in which an inventory manager wants to avoid paying for inventory which is not purchased by customers. First, the algorithm is asked to choose an allocation level given the weather and date information. A fixed cost per allocated bicycle is assumed. Then, the empirical demand level produces a fixed revenue per demanded bicycle. We treat (normalized) bicycle allocations as continuous actions on \([0,1]\) and allow for fractional allocation. Denoting the ground truth demand as \( y \) and the allocation as \( \hat{y} \), we have
\[
\text{Profit}(y, \hat{y}) = \min(y, \hat{y}) - \beta \hat{y}.
\]
For our regressor class, we first predict \( \hat{z} : X \rightarrow [0,1] \times (0,\infty) \) using a linearized Cauchy kernel machine [Rahimi and Recht 2007], and then induce a prediction function \( \hat{f} \),
\[
\hat{f}(x,a) = -\beta a + \int_0^1 \min(a,p) \, dN(p; \hat{z}_0(x), \hat{z}_1(x)) + \int_0^1 \, dN(p; \hat{z}_0(x), \hat{z}_1(x))
\]
which has a (lengthy) closed form when \( N(\cdot; \hat{z}_0(x), \hat{z}_1(x)) \) is a Gaussian with mean \( \hat{z}_0(x) \) and variance \( \hat{z}_1(x) \). Although inspired by a truncated Gaussian random variable, this does not imply any particular generative model. We use \( \beta = 1/3 \).

Online Performance Complete results are in Table 2a. All CIs in the table are computed from the online realizations. Learning with risk-aversion trades average performance (profit) in exchange for a higher percentage that all allocated inventory is demanded (sold out). Figure 2b shows the cumulative sold out percentage as the DC dataset is consumed. Compared to risk-neutral learning, risk-averse learning underestimates demand and then starts to approach more accurate estimates from below.

---

Training with Brent’s method is circa 2x slower on an author’s laptop, but this is problem dependent.
Conditionally Risk-Averse Contextual Bandits

Figure 3: Realized aggregate expectiles on the Prudential dataset when the algorithm is risk-neutral ($q = 0.5$) vs risk-averse ($q = 0.2$). A tradeoff between average-case guarantee and tail control is clearly evident.

Figure 4: Query Optimization results. Varying the learning expectile ($q$) yields different realized lift and regression. The Pareto front is in green. With moderate $q$, reductions in regression are proportionally larger than reductions in lift.

4.3 Self-Tuning Software

Query Optimization  Our final dataset is from the exascale cloud data processing system Scope [Power et al., 2021]. The Scope query optimizer is highly configurable and uses a contextual bandit framework to select optimizer flags on a per-query basis [Zhang et al., 2022]. For this application, there is no single optimal flag configuration working for all input queries, and the best configuration depends on the specific query. While it is valuable to increase the overall average performance of queries, it is important to avoid regressions (queries with worse performance than the default strategy), which lead to user frustration and extra investigation work. In our experience, a non-contextual risk-neutral policy only results in marginal performance lift, with a lot of regressions.

We assembled query information (as context) and assessed the performance of multiple configurations (as actions) per query relative to a default strategy, using fractional change as the reward. The number of actions per example varies depending upon constraints imposed by the optimizer: it ranges from 2 to 22, with a mean of 4.3 and a median of 3. We use this dataset to construct a query optimization simulator as follows. First, the algorithm is presented the query information and the configuration choices. Then the algorithm selects a configuration and receives the reward for that configuration.

Figure 4 summarizes the results, where the x-axis is the average performance regression for the regressed queries, and the y-axis shows the overall average performance lift. Varying the learning expectile ($q$) illuminates the trade-off between lift and regression. As seen in other experiments, there is a moderate $q$ regime where reductions in regression are proportionally larger than reductions in lift. Comparing risk-averse $q = 0.2$ with risk-neutral $q = 0.5$, the regressions drop by over 50% relatively while almost maintaining the same level of lift. For $q < 0.0001$ every point is Pareto-dominated, as anticipated by the theoretical analysis (the regret bound degrades at extreme quantiles).

5 Related Work

Risk-aversion has received extensive attention in the (non-contextual) bandit literature, utilizing various risk measures. Even-Dar et al. [2006], Sani et al. [2012], Yu and Nikolova [2013], Vakili and Zhao [2016], Zhu and Tan [2020] minimize the mean-variance, while Szöregyi et al. [2015], David and Shimkin [2016], Howard and Ramdas [2019], Nikolakakis et al. [2021] use quantiles for optimization. Numerous prior works utilize Conditional Value at Risk (CVaR) [Tamkin et al., 2019, Cardoso and Xu, 2019, Bhat and Prashanth, 2019, Chang et al., 2020, Baudry et al., 2021, Khajonchotpanya et al., 2021]. General risk criteria are studied in Cassel et al. [2018], Torossian et al. [2019]. Axelrod et al. [2016], Aryania et al. [2021] consider expectiles. Galichet [2015] states algorithms for both CVaR and the essential infimum.

Prior work on risk-averse contextual bandits is comparatively limited. Sun et al. [2017] address the adversarial contextual setting by treating total risk as a constraint, but requires an additional risk value observed along with cost. Bouneffouf [2016] presents a contextual UCB algorithm which optimizes for mean reward, but which modulates the level of $\epsilon$-greedy exploration based upon a risk estimate. Concurrent to our work, Saux and Maillard [2023] recently also use the UCB framework solving a convex problem under the assumption of linear bandits, which do not apply...
to any of our non-linear predictors (e.g. Equation 5) in the experiments. Huang et al. [2021] study the finite sample behaviour of off-policy estimation for a broad class of risk measures.

The inadequacy of average-case guarantees is a recurring theme in real-time systems applications. Jalaparti et al. [2013] improve tail latencies of request-response workflows by minimizing variance. Schad et al. [2010] use the same performance measure in cloud computing. CVaR optimization is also present in systems applications. Mena et al. [2014] propose a multi-objective optimization technique with CVaR as risk metric in a sizing and allocation problem of renewable generation, whereas Moreno and Srirac [2015] limit risk exposure to high impact low probability events in distribution substations through this metric. However, only a small number of related bandit studies tackle risk-aware optimization in systems applications. Marcus et al. [2021] present a bandit optimizer to improve the tail latency of queries. Sachidananda and Sivaraman [2021] design an autoscaler using a multi-armed bandit algorithm to optimize median or tail latency for microservice applications.

6 Conclusions and Future Work

This paper studies the application of contextual bandits to scenarios where average-case statistical guarantees are inadequate. We show that the composition of reduction to online regression and expectile loss is analytically tractable, computationally convenient, and empirically effective. Our experiments demonstrate the trade-off between maximizing average-case outcomes and minimizing worst-case performance. These results highlight the usefulness of our method, which can be easily applied to problems that require risk aversion.

Our reduction method exhibits an adversarial conditional risk guarantee but empirically it is also effective at controlling realized marginal risk. However, it is possible an algorithm designed for the stochastic case could explicitly guarantee marginal risk, e.g. via reduction to offline reduction qua Simchi-Levi and Xu [2021].

For many applications, risk-aversion is a desired end goal. However, explicit constraints on key metrics are also of practical interest. Although risk-aversion implicitly controlled key metrics computed from the complete reward distribution in our experiments, it is complementary to approaches for constrained contextual bandits such as Badanidiyuru et al. [2014]. In particular constrained contextual bandits can control key metrics unrelated to the reward distribution, e.g., guaranteeing quality of service while being rewarded on cost of delivery. Combining risk-aversion with constraints is also a promising topic for future work.

References

Chicago divvy bicycle sharing data. https://www.kaggle.com/datasets/yingwurenjian/chicago-divvy-bicycle-sharing-data.

London bike sharing dataset. https://www.kaggle.com/datasets/hmavrodiev/london-bike-sharing-dataset.

Prudential life insurance assessment competition. https://www.kaggle.com/competitions/prudential-life-insurance-assessment.

Alekh Agarwal, Haipeng Luo, Behnam Neyshabur, and Robert E Schapire. Corralling a band of bandit algorithms. In Conference on Learning Theory, pages 12–38. PMLR, 2017.

Philippe Artzner, Freddy Delbaen, Jean-Marc Eber, and David Heath. Coherent measures of risk. Mathematical finance, 9(3):203–228, 1999.

Azra Aryania, Hadi S Aghdasi, Rasoul Heshmati, and Andrea Bonarini. Robust risk-averse multi-armed bandits with application in social engagement behavior of children with autism spectrum disorder while imitating a humanoid robot. Information Sciences, 573:194–221, 2021.

Peter Auer, Nicolo Cesa-Bianchi, Yoav Freund, and Robert E Schapire. The nonstochastic multiarmed bandit problem. SIAM journal on computing, 32(1):48–77, 2002.

Allan Axelrod, Luca Carlone, Girish Chowdhary, and Sertac Karaman. Data-driven prediction of evar with confidence in time-varying datasets. In 2016 IEEE 55th Conference on Decision and Control (CDC), pages 5833–5838, 2016. doi: 10.1109/CDC.2016.7799166.

Ashwinkumar Badanidiyuru, John Langford, and Aleksandrs Slivkins. Resourceful contextual bandits. In Conference on Learning Theory, pages 1109–1134. PMLR, 2014.

Dorian Baudry, Romain Gautron, Emilie Kaufmann, and Odalric Maillard. Optimal thompson sampling strategies for support-aware cvaR bandits. In Marina Meila and Tong Zhang, editors, Proceedings of the 38th International
**Conditionally Risk-Averse Contextual Bandits**

Conference on Machine Learning, volume 139 of Proceedings of Machine Learning Research, pages 716–726. PMLR, 18–24 Jul 2021. URL https://proceedings.mlr.press/v139/baudry21a.html

Fabio Bellini and Elena Di Bernardino. Risk management with expectiles. The European Journal of Finance, 23(6): 487–506, 2017.

Sanjay P. Bhat and L. A. Prashanth. Concentration of Risk Measures: A Wasserstein Distance Approach. Curran Associates Inc., Red Hook, NY, USA, 2019.

Djallel Bouneffouf. Contextual bandit algorithm for risk-aware recommender systems. 2016 IEEE Congress on Evolutionary Computation (CEC), pages 4667–4674, 2016.

Djallel Bouneffouf and Irina Rish. A survey on practical applications of multi-armed and contextual bandits. CoRR, abs/1904.10040, 2019. URL http://arxiv.org/abs/1904.10040.

Adrian Rivera Cardoso and Huan Xu. Risk-averse stochastic convex bandit. In Kamalika Chaudhuri and Masashi Sugiyama, editors, Proceedings of the Twenty-Second International Conference on Artificial Intelligence and Statistics, volume 89 of Proceedings of Machine Learning Research, pages 39–47. PMLR, 16–18 Apr 2019. URL https://proceedings.mlr.press/v89/cardoso19a.html

Asaf Cassel, Shie Mannor, and Assaf Zeevi. A general approach to multi-armed bandits under risk criteria. In Sébastien Bubeck, Vianney Perchet, and Philippe Rigollet, editors, Proceedings of the 31st Conference On Learning Theory, volume 75 of Proceedings of Machine Learning Research, pages 1295–1306. PMLR, 06–09 Jul 2018. URL https://proceedings.mlr.press/v75/cassel18a.html

Joel QL Chang, Qiuyu Zhu, and Vincent YF Tan. Risk-constrained thompson sampling for cvar bandits. arXiv preprint arXiv:2011.08046, 2020.

Yahel David and Nahum Shimkin. Pac lower bounds and efficient algorithms for the max k -armed bandit problem. In Maria Florina Balcan and Kilian Q. Weinberger, editors, Proceedings of The 33rd International Conference on Machine Learning, volume 48 of Proceedings of Machine Learning Research, pages 878–887, New York, New York, USA, 20–22 Jun 2016. PMLR. URL https://proceedings.mlr.press/v48/david16.html

Eyal Even-Dar, Michael Kearns, and Jennifer Wortman. Risk-sensitive online learning. In Proceedings of the 17th International Conference on Algorithmic Learning Theory, ALT’06, page 199–213, Berlin, Heidelberg, 2006. Springer-Verlag. ISBN 3540466495. doi: 10.1007/11894841_18. URL https://doi.org/10.1007/11894841_18

Tobias Fissler and Johanna F Ziegel. Higher order elicitability and osband’s principle. The Annals of Statistics, 44(4): 1680–1707, 2016.

Dylan Foster and Alexander Rakhlin. Beyond ucb: Optimal and efficient contextual bandits with regression oracles. In International Conference on Machine Learning, pages 3199–3210. PMLR, 2020.

Dylan J Foster and Akshay Krishnamurthy. Efficient first-order contextual bandits: Prediction, allocation, and triangular discrimination. Advances in Neural Information Processing Systems, 34, 2021.

Dylan J Foster, Claudio Gentile, Mehrayr Mohri, and Julian Zimmert. Adapting to misspecification in contextual bandits. Advances in Neural Information Processing Systems, 33:11478–11489, 2020.

Dylan J Foster, Sham M Kakade, Jian Qian, and Alexander Rakhlin. The statistical complexity of interactive decision making. arXiv preprint arXiv:2112.13487, 2021.

Nicolas Galichet. Contributions to Multi-Armed Bandits : Risk-Awareness and Sub-Sampling for Linear Contextual Bandits. Theses, Université Paris Sud - Paris XI, September 2015. URL https://tel.archives-ouvertes.fr/tel-01277170

Elad Hazan, Amit Agarwal, and Satyen Kale. Logarithmic regret algorithms for online convex optimization. Machine Learning, 69(2-3):169–192, 2007.

Steven R. Howard and Aaditya Ramdas. Sequential estimation of quantiles with applications to a/b-testing and best-arm identification. arXiv: Statistics Theory, 2019.

Audrey Huang, Liu Leqi, Zachary Lipton, and Kamyar Azizzadenesheli. Off-policy risk assessment in contextual bandits. In M. Ranzato, A. Beygelzimer, Y. Dauphin, P.S. Liang, and J. Wortman Vaughan, editors, Advances in Neural Information Processing Systems, volume 34, pages 23714–23726. Curran Associates, Inc., 2021. URL https://proceedings.neurips.cc/paper/2021/file/c7502c55f8d815b540625b59d9a42638520-Paper.pdf

Virajith Jalaparti, Peter Bodik, Srikanth Kandula, Ishai Menache, Mikhail Rybalkin, and Chenyun Yan. Speeding up distributed request-response workflows. Proceedings of the ACM SIGCOMM 2013 conference on SIGCOMM, 2013.

Najakorn Khajonchotpanya, Yilin Xue, and Napat Rujeerapaiboon. A revised approach for risk-averse multi-armed bandits under cvar criterion. Oper. Res. Lett., 49:465–472, 2021.
Conditionally Risk-Averse Contextual Bandits

Shigeo Kusuoka. On law invariant coherent risk measures. In Advances in mathematical economics, pages 83–95. Springer, 2001.

John Langford and Tong Zhang. The epoch-greedy algorithm for contextual multi-armed bandits. Advances in neural information processing systems, 20(1):96–1, 2007.

John Langford, Lihong Li, and Alex Strehl. Vowpal wabbit online learning project, 2007.

Ryan Marcus, Parimarjan Negi, Hongzi Mao, Nesime Tatbul, Mohammad Alizadeh, and Tim Kraska. Bao: Making learned query optimization practical. In Proceedings of the 2021 International Conference on Management of Data, pages 1275–1288, 2021.

Rodrigo Mena, Martin Hennebel, Yan-Fu Li, Carlos Ruiz, and Enrico Zio. A risk-based simulation and multi-objective optimization framework for the integration of distributed renewable generation and storage. Renewable and Sustainable Energy Reviews, 37:778–793, 2014. ISSN 1364-0321. doi: https://doi.org/10.1016/j.rser.2014.05.046. URL https://www.sciencedirect.com/science/article/pii/S1364032114003712

Jean-Christophe Meyrefdi. History of the risk concept and risk modeling. EDHEC–Risk Publications. Lille: EDHEC Risk and Asset Management Research Center, pages 1–8, 2004.

Rodrigo Moreno and Goran Strbac. Integrating high impact low probability events in smart distribution network security standards through cvar optimisation. 2015.

Konstantinos E. Nikolakakis, Dionysios S. Kalogerias, Or Sheffet, and Anand D. Sarwate. Quantile multi-armed bandits: Optimal best-arm identification and a differentially private scheme. IEEE Journal on Selected Areas in Information Theory, 2:534–548, 2021.

Conor Power, Hiren Patel, Alekh Jindal, Jyoti Leeka, Bob Jenkins, Michael Rys, Ed Triou, Dexin Zhu, Lucky Katahanas, Chakrapani Bhat Talapady, et al. The Cosmos big data platform at Microsoft: over a decade of progress and a decade to look forward. Proceedings of the VLDB Endowment, 14(12):3148–3161, 2021.

Ali Rahimi and Benjamin Recht. Random features for large-scale kernel machines. Advances in neural information processing systems, 20, 2007.

Damiano Rossello. Performance measurement with expectiles. Decisions in Economics and Finance, pages 1–32, 2022.

Vighnesh Sachidananda and Anirudh Sivaraman. Learned autoscaling for cloud microservices with multi-armed bandits. ArXiv, abs/2112.14845, 2021.

Amir Sani, Alessandro Lazaric, and Rémi Munos. Risk-aversion in multi-armed bandits. In NIPS, 2012.

Patrick Saux and Odalric Maillard. Risk-aware linear bandits with convex loss. In International Conference on Artificial Intelligence and Statistics, pages 7723–7754. PMLR, 2023.

Jörg Schad, Jens Dittrich, and Jorge-Arnulfo Quiané-Ruiz. Runtime measurements in the cloud. Proceedings of the VLDB Endowment, 3:460 – 471, 2010.

David Simchi-Levi and Yunzong Xu. Bypassing the monster: A faster and simpler optimal algorithm for contextual bandits under realizability. Mathematics of Operations Research, 2021.

Wen Sun, Debadeepta Dey, and Ashish Kapoor. Safety-aware algorithms for adversarial contextual bandit. In Doina Precup and Yee Whye Teh, editors, Proceedings of the 34th International Conference on Machine Learning, volume 70 of Proceedings of Machine Learning Research, pages 3280–3288. PMLR, 06–11 Aug 2017. URL https://proceedings.mlr.press/v70/sun17a.html

Balázs Szörényi, Róbert Busa-Fekete, Paul Weng, and Eyke Hüllermeier. Qualitative multi-armed bandits: A quantile-based approach. In ICML, 2015.

Alex Tamkin, Ramtin Keramati, and Emma Brunskill. Distributionally-aware exploration for cvar bandits. 2019.

Léonard Torossian, Aurélien Garivier, and Victor Picheny. $\alpha$-armed bandits: Optimizing quantiles, cvar and other risks. In ACM, 2019.

Sattar Vakili and Qing Zhao. Risk-averse multi-armed bandit problems under mean-variance measure. IEEE Journal of Selected Topics in Signal Processing, 10(6):1093–1111, 2016. doi: 10.1109/JSTSP.2016.2592622.

Joaquin Vanschoren, Jan N Van Rijn, Bernd Bischl, and Luís Torgo. Openml: networked science in machine learning. ACM SIGKDD Explorations Newsletter, 15(2):49–60, 2014.

Vladimir Vovk. A game of prediction with expert advice. Journal of Computer and System Sciences, 56(2):153–173, 1998.

Linda Schulze Waltrup, Fabian Sobotka, Thomas Kneib, and Göran Kauermann. Expectile and quantile regression—david and goliath? Statistical Modelling, 15(5):433–456, 2015.
Jia Yuan Yu and Evdokia Nikolova. Sample complexity of risk-averse bandit-arm selection. In *IJCAI*, 2013.

Wangda Zhang, Matteo Interlandi, Paul Mineiro, Shi Qiao, Nasim Ghazanfari, Karlen Lie, Marc Friedman, Rafah Hosn, Hiren Patel, and Alekh Jindal. Deploying a steered query optimizer in production at Microsoft. In *Proceedings of the 2022 International Conference on Management of Data*, 2022.

Qiuyu Zhu and Vincent Tan. Thompson sampling algorithms for mean-variance bandits. In Hal Daumé III and Aarti Singh, editors, *Proceedings of the 37th International Conference on Machine Learning*, volume 119 of *Proceedings of Machine Learning Research*, pages 11599–11608. PMLR, 13–18 Jul 2020. URL https://proceedings.mlr.press/v119/zhu20d.html.

Yinglun Zhu and Paul Mineiro. Contextual bandits with smooth regret: Efficient learning in continuous action spaces. In *International Conference on Machine Learning*, pages 27574–27590. PMLR, 2022.

Johanna F Ziegel. Coherence and elicitability. *Mathematical Finance*, 26(4):901–918, 2016.
A Regret Bound Proofs

Note the SmoothCB bound is more general, and the finite action case a specialization.

A.1 Proof of SmoothCB Bound

**Theorem 3.2.** Algorithm $\text{CB}^{(h, \mu)}$ guarantees $\text{Reg}_{\text{CB}}^{(h, \mu)}(T) \leq O\left(\frac{1}{h} \sqrt{\frac{1}{T} \text{Reg}_{\text{EVaR}}(T)}\right)$, where $\theta = \min(q, 1 - q)$.

**Proof.** We have

$$\text{Reg}_{\text{CB}}^{(h, \mu)}(T) \leq T \left(\frac{3}{4 \min(q, 1 - q) \gamma h} + \gamma \text{Reg}_{\text{EVaR}}(T), \right)$$

where (a) follows from Corollary [B.2] and (b) follows from Lemma [C.1]. Optimizing over $\gamma$ yields $\gamma^*$. Note the bound is more general, and the finite action case a specialization.

A.2 Proof of SquareCB Bound

**Theorem 3.1.** Algorithm $\text{CB}^{(h, \mu)}$ guarantees $\text{Reg}_{\text{CB}}(T) \leq O\left(\frac{\sqrt{|A| T \text{Reg}_{\text{EVaR}}(T)}}{\gamma h}\right)$, where $\theta = \min(q, 1 - q)$.

**Proof.** Analogous to Theorem 3.2 but using Corollary [B.3]. i.e., $h^{-1} = |A|$ and $\mu$ is the uniform distribution.

B Proof of convex conjugate lemma

The following Lemma concerns bounding $\max \left(\frac{1}{h} \sqrt{\frac{1}{T} \text{Reg}_{\text{EVaR}}(T)}\right)$. Optimizing over $\gamma$ yields $\gamma^*$.
Conditionally Risk-Averse Contextual Bandits

**Proof** Consider \( P \) of the form \( P = (1 - M(A)) N + M \) and elide \( x \) dependence.

\[
\begin{align*}
\min_P \max_Q & \max_{f^*} \mathbb{E}_{a \sim P} [f^*(a)] - \mathbb{E}_{a \sim Q} [f^*(a)] - x \mathbb{E}_{a \sim P} \left[ g_t(\hat{f}_t) - g_t(f^*) \right] \\
& \leq \min_P \max_Q \max_{f^*} \mathbb{E}_{a \sim P} [f^*(a)] - \mathbb{E}_{a \sim Q} [f^*(a)] - \gamma \mathbb{E}_{a \sim P} \left[ \phi \left( \hat{f}(a) - f^*(a) \right) \right] \\
& \overset{(a)}{=} \min_P \max_Q \mathbb{E}_{a \sim P} \left[ \hat{f}(a) \right] - \mathbb{E}_{a \sim Q} \left[ \hat{f}(a) \right] \\
& \quad + \max_z \left( \mathbb{E}_{a \sim Q} [z(a)] - \mathbb{E}_{a \sim M} [z(a) + \gamma \phi(z(a))] \right) \\
& \quad - (1 - M(A)) \mathbb{E}_{a \sim N} [z(a) + \gamma \phi(z(a))] \\
& \overset{(b)}{=} \min_P \max_Q \mathbb{E}_{a \sim P} \left[ \hat{f}(a) \right] - \mathbb{E}_{a \sim Q} \left[ \hat{f}(a) \right] + \mathbb{E}_{a \sim M} \left[ \gamma \phi^* \left( \frac{1}{\gamma} \left( \frac{dQ}{d\mu}(a) - 1 \right) \right) \right] \\
& \quad + (1 - M(A)) \gamma \phi^* \left( \frac{1}{\gamma} \right) \\
& = \min_P \max_Q \mathbb{E}_{a \sim P} \left[ \hat{f}(a) \right] + \beta + \mathbb{E}_{a \sim \mu} [\kappa(a)] \\
& \quad + \mathbb{E}_{a \sim \mu} \left[ -\kappa(a) - \frac{dQ}{d\mu}(a) \left( \hat{f}(a) + \beta \right) + \xi \left( \frac{dM}{d\mu}(a), \frac{dQ}{d\mu}(a) \right) \right] \\
& \leq \mathbb{E}_{a \sim P} \left[ \hat{f}(a) \right] + \beta + \mathbb{E}_{a \sim \mu} [\kappa(a)],
\end{align*}
\]

where \((a)\) substitutes \( z(a) \equiv \hat{f}(a) - f^*(a) \); and \((b)\) is because \( x + \gamma \phi(x) \) and \( \gamma \phi^* \left( \frac{1}{\gamma} (x^* - 1) \right) \) are convex conjugates.

**Corollary B.2.** (Continuous Approximate Abe-Long) For \( \text{EVaR}_q \), the distribution satisfying

\[
\hat{P} = \left( 1 - \hat{M}(A) \right) 1_{\hat{a}} + \hat{M},
\]

\[
\frac{d\hat{M}}{d\mu}(a) = \frac{1}{1 + 4 \min(q, 1 - q) \gamma h \max \left( 0, \hat{f}(a) - \hat{f}(\hat{a}) \right)} \leq 1,
\]

where \( \hat{a} \) satisfies

\[
\mathbb{E}_{a \sim \mu} \left[ \max \left( 0, \hat{f}(\hat{a}) - \hat{f}(a) \right) \right] \leq \frac{1}{4 \min(q, 1 - q) \gamma h},
\]

guarantees game value bound

\[
\frac{3}{4 \min(q, 1 - q) \gamma h}.
\]

**Proof** For \( \text{EVaR}_q \), \( \phi(x) = \min(q, 1 - q) x^2 \) lower bounds the expected regret by strong convexity; the convex conjugate is \( \phi^*(x) = \frac{x^2}{\min(q, 1 - q)} \), for which \( \hat{M}(a) \) satisfies equation \((6)\) with

\[
\beta = \frac{1 - 2h}{4 \min(q, 1 - q) \gamma h} - \hat{f}(\hat{a}),
\]

\[
\kappa(a) = \frac{\max \left( 0, \hat{f}(\hat{a}) - \hat{f}(a) \right)}{h} + \frac{1}{4 \min(q, 1 - q) \gamma}.
\]
A bound is therefore
\[ \frac{1}{h} E_{a \sim \mu} \left[ \max \left( 0, \hat{f}(a) - \hat{f}(\hat{a}) \right) \right] + \frac{1}{4 \min(q, 1 - q) \gamma h} E_{a \sim \hat{\nu}} \left[ \hat{f}(a) - \hat{f}(\hat{a}) \right] \leq \frac{1}{2 \min(q, 1 - q) \gamma h} + E_{a \sim \hat{\nu}} \left[ \hat{f}(a) - \hat{f}(\hat{a}) \right] \leq \frac{1}{2 \min(q, 1 - q) \gamma h} + E_{a \sim \hat{\nu}} \left[ \hat{f}(a) - \hat{f}(\hat{a}) \right] \leq \frac{1}{4 \min(q, 1 - q) \gamma h} . \]

**Corollary B.3.** (Discrete Abe-Long) For EVaR\(_q\), given a finite action set \( A \), the distribution satisfying
\[ \hat{P} = \left( 1 - \hat{M}(A) \right) 1_{\hat{a}} + \hat{M} \]
\[ \hat{M}(a) = \frac{1}{|A| + 4 \min(q, 1 - q) \gamma} \left( \hat{f}(a) - \hat{f}(\hat{a}) \right) \leq \frac{1}{|A|} , \]
where \( \hat{a} \) is an exact minimizer of \( \hat{f} \), guarantees game value bound \( \frac{3|\mathcal{A}|}{4 \min(q, 1 - q) \gamma} \) when competing with the best action.

**Proof** Follows from above with \( h^{-1} = |A|, \nu = \hat{f}(\hat{a}) \), and \( \mu \) uniform over \( A \).

### C Proof of expected regret lemma

Our goal is to relate the total realized regret defined as
\[ \sum_{t=1}^{T} \left( g_t(\hat{f}_t) - g_t(f^*) \right) \leq \sum_{t=1}^{T} Z_t \leq \text{Reg}_{\text{EVaR}_q}(T) . \]
to the total expected regret
\[ \overline{\text{Reg}_{\text{EVaR}_q}}(T) = \sum_{t=1}^{T} \mathbb{E}_t [ Z_t ] , \]
where \( \mathbb{E}_t \) denotes expectation conditioned (\( \{(x_s, a_s, l_s)\}_{s \leq t}, x_t, P_{l_t} \), i.e., averaged over the conditional action and loss distribution.

**Lemma C.1.** The total expected regret is bounded by
\[ \overline{\text{Reg}_{\text{EVaR}_q}}(T) \leq 3 + 3 \text{Reg}_{\text{EVaR}_q}(T) + \frac{4}{\min(q, 1 - q)} . \]

**Proof** Note
\[ M_t = \sum_{s=1}^{t} (\mathbb{E}_s [ Z_s ] - Z_s) = \sum_{s=1}^{t} \Delta M_t , \]
is a martingale. Freedman’s inequality says
\[ \Pr ( M_T \geq \epsilon ) \leq \exp \left( -\frac{\epsilon^2}{\sigma^2 + \frac{\epsilon^2}{5}} \right) \]
where a.s. \( \sigma^2 \geq \sum_{t=1}^{T} \mathbb{E}_t \left[ (\Delta M_t)^2 \right] \). Integrating the tail bound,
\[ \exp \left( -\frac{\epsilon^2}{\sigma^2 + \frac{\epsilon^2}{5}} \right) \leq \exp \left( -\frac{\epsilon^2}{2 \max (\sigma^2, \frac{\epsilon^2}{5})} \right) \leq \exp \left( -\frac{\epsilon^2}{2 \sigma^2} \right) + \exp \left( -\frac{3 \epsilon}{2} \right) . \]
\[ \mathbb{E} \left[ |M_T| \mid \{x_t\}_{t=1}^{T} \right] \leq \beta_1 + \beta_2 , \]
\[ \beta_1 = \int_{0}^{\infty} \min \left( 1, \exp \left( -\frac{\epsilon^2}{2 \sigma^2} \right) \right) \, de \leq 2 \sqrt{\sigma^2} , \]
\[ \beta_2 = \int_{0}^{\infty} \min \left( 1, \exp \left( -\frac{3 \epsilon}{2} \right) \right) \, de = \frac{2}{3} + \frac{2}{3 \exp(1)} \leq 1 . \]
Thus
\[
\sum_{t=1}^{T} \mathbb{E}_t [Z_t] \leq \mathbb{E} \left[ \sum_{t=1}^{T} Z_t \right] + 2\sqrt{\sigma^2 + 1}
\]
\[
\leq \text{Reg}_{\text{EVaR}_q}(T) + 2\sqrt{\sigma^2 + 1},
\]
where the second inequality is because the regret guarantee applies pointwise. It suffices to bound \(\sigma\). From below we have
\[
\mathbb{E}_s \left[ Z_t^2 \big| a_t \right] \leq \frac{1}{\min(q, 1 - q)} \mathbb{E}_s [Z_t].
\]
So
\[
\sum_{t=1}^{T} \mathbb{E}_t [Z_t] \leq \text{Reg}_{\text{EVaR}_q}(T) + 2\sqrt{\frac{1}{\min(q, 1 - q)} \sum_{t=1}^{T} \mathbb{E}_t [Z_t]} + 1
\]
\[
\Rightarrow \sum_{t=1}^{T} \mathbb{E}_t [Z_t] \leq 3 + 3\text{Reg}_{\text{EVaR}_q}(T) + \frac{4}{\min(q, 1 - q)}.
\]

C.1 Bound for \(\sigma\)

\[
\mathbb{E}_t [(\Delta M_t)^2] \leq \mathbb{E}_t [Z_t^2] = \mathbb{E}_t \left[ \mathbb{E}_t [Z_t^2 | a_t] \right],
\]
\[
\mathbb{E}_t [Z_t^2 | a_t] = \mathbb{E}_t \left[ \left( g_t(\hat{f}_t) - g_t(f^*) \right)^2 a_t \right].
\]
From convexity and \(|\nabla g_t| \leq \max(q, 1 - q) \leq 1\), we have
\[
|\hat{f}_t - f^*| \leq |g_t(\hat{f}_t) - g_t(f^*)| \leq |\hat{f}_t - f^*|,
\]
thus
\[
\mathbb{E}_t [Z_t^2 | a_t] \leq \mathbb{E}_t \left[ \left( \hat{f}_t - f^* \right)^2 a_t \right]
\]
\[
= \frac{1}{\min(q, 1 - q)} \mathbb{E}_t \left[ \left( \min(q, 1 - q) \left( \hat{f}_t - f^* \right) \right)^2 a_t \right]
\]
\[
\leq \frac{1}{\min(q, 1 - q)} \mathbb{E}_t [Z_t | a_t].
\]

D Datasets

| Scenario               | Name                  | License   | \(T\)  | Actions |
|------------------------|-----------------------|-----------|--------|---------|
| Dynamic Pricing        | King County (42092)   | CC-BY     | 21613  | [0,1]   |
|                        | Perth (43822)         | CC-BY     | 33656  | [0,1]   |
|                        | [Pru] Custom          | Custom    | 59381  | 8       |
| Inventory Management   | Chicago[Chi]          | CC-0      | 34617  | [0,1]   |
|                        | DC (42712)            | CC-BY     | 17379  | [0,1]   |
|                        | London[Lon]           | OGL       | 17414  | [0,1]   |
| Self-Tuning Software   | Query Opt             | CC-BY     | 48681  | Finite Variadic |

[Vanschoren et al., 2014]

\[\text{https://creativecommons.org/licenses/by/2.0/}\
\[\text{https://www.kaggle.com/competitions/prudential-life-insurance-assessment/rules}\
\[\text{https://creativecommons.org/share-your-work/public-domain/cc0}\
\[\text{https://en.wikipedia.org/wiki/Open_Government_Licence}\

16