On the true meaning of spatial densities of hadrons

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We clarify the meaning of spatial densities of hadrons. A physical density is given by the expectation value of a local operator for a physical state, and depends on both internal structure and the hadron’s wave packet. In some particular cases, the physical density can be written as a convolution between a density function that depends on internal structure but not wave packet, and a smearing function that depends on wave packet but not internal structure. In these cases, the former constitutes a true internal density of the hadron. We show that the light front densities often encountered in the literature qualify as true densities in this respect, but that instant form densities obtained through wave packet localization do not. We additionally clarify that Breit frame densities constitute apparent internal densities of hadrons prepared in broad wave packets, but do not provide a complete description of hadron spatial structure.

I. INTRODUCTION

There has lately been renewed interest and debate about the proper manner for describing the internal structure of hadrons in terms of spatial coordinates. This debate is especially pertinent now with the increasing focus on the energy momentum tensor and its associated densities in the literature (see e.g. Refs. [1–3]). A major focus of the upcoming Electron Ion Collider [4–6] is to provide a spatial picture of the distributions of partons in hadrons, making it vital that an accurate method for obtaining this spatial picture is used.

For a long time, authors have primarily calculated relativistic densities through three-dimensional Fourier transforms of form factors, giving a result referred to as the Breit frame density [7]. For almost as long, this approach has been criticized [8–10] for failing to follow from the elementary definition of a density in relativistic field theory (i.e., the expectation value of a local operator). While the majority of authors continue to use the Breit frame density uncritically, multiple solutions to addressing the issue of relativistic spatial densities have been advanced.

There are currently three camps on the question of relativistic spatial densities. The first camp uses the formalism of Wigner phase space distributions to justify the Breit frame densities as a slice of a six-dimensional phase space distribution when the hadron momentum is zero [2]. This is a formally consistent approach, but does not provide a physical density.

The second camp obtains densities using expectation values of local currents in light front coordinates, integrating out the coordinate $x^-$ to produce a two-dimensional density in the transverse plane [3, 11, 12]. Within this formalism, one often localizes the hadron’s wave packet [3, 9] or works with Dirac delta functions [10, 11, 13] to remove dependence on the wave packet. Although the densities obtained in this way are true densities, they are often ignored for being two-dimensional.

The last camp attempts to define three-dimensional densities in instant form coordinates in a similar manner to the second camp, namely by taking the expectation value of a local current for a physical state and localizing the wave packet. This idea was first proposed by Fleming in 1974 [8], but the idea has recently been revived [14–16]. Ultimately, the idea behind the Fleming procedure is fallacious. The rationale behind localizing the wave packet to define an internal density is that, in the familiar case of bound states in non-relativistic quantum mechanics, localizing the wave packet does reproduce the internal density of the system. However, it does not necessarily follow that an internal density will result from localizing the wave packet for relativistic bound states. In fact, the results of Refs. [8, 14–16] all depend on the wave packet being spherically symmetric, which already indicates that the result retains wave packet dependence in some form.

The purpose of this work is to address the localization fallacy and clarify what exactly is meant by an internal density of a relativistic system, and under what circumstances an internal density can even be obtained. We also address under which circumstances wave packet localization reproduces the true internal density, and when it does not. Ultimately, we find that localization does reproduce the correct results for a handful of light front densities, but does not produce a meaningful internal density in instant form coordinates, invalidating the Fleming approach.

This work is organized as follows. In Sec. II, we clarify what constitute true, internal densities of hadrons, and how these are related to the physical densities obtained from expectation values of local operators. In Sec. III, we obtain...
light front densities within the constraints laid out in the previous section. These include the electromagnetic current and spatial distributions associated with the energy-momentum tensor, reproducing previous findings where they exist but without requiring wave packet localization. In Sec. IV, we examine three-dimensional densities in instant form coordinates, showing that internal densities do not exist in general. In the special case of diffuse wave packets an infinite tower of internal densities appear—with the Breit frame densities appearing as the leading term—but no internal densities appear for localized wave packets. We summarize and conclude in Sec. V.

II. INTERNAL AND PHYSICAL DENSITIES

A physical density is defined as the expectation value of a local operator for a physical state $|\Psi\rangle$ in Hilbert space:

$$\rho_{\text{phys}}(x) = \langle \Psi | \hat{O}(x) | \Psi \rangle.$$  \hspace{1cm} (1)

We assume $|\Psi\rangle$ is a single-hadron state. The physical density will contain influences both from the internal structure of the hadron, and from the wave packet that it’s prepared in. The goal of hadron structure research is to isolate internal properties of hadrons that are independent of artifacts of state preparation. It is thus vital to determine under what circumstances a purely internal hadron density can be defined, and we therefore refer to such internal densities (when they exist) as true densities.

It is moreover prudent to understand how the true densities contribute to the physical densities defined in Eq. (1). The simplest scenario that can occur is that the physical density corresponds to an internal density being smeared over space at any particular fixed time $\tau$ (which could be time in any of the forms of relativistic dynamics [17]):

$$\rho_{\text{phys}}(x, \tau) = \int d^n x S_{\Psi}(R, \tau) \rho_{\text{true}}(x - R).$$  \hspace{1cm} (2)

Here $S_{\Psi}(R, \tau)$ is a smearing function that depends on the wave packet but not on internal structure, whereas $\rho_{\text{true}}(b)$ contains no wave packet dependence. We have noted $n$ spatial dimensions for generality; we will have $n = 3$ for non-relativistic densities and $n = 2$ for relativistic light front densities. We shall refer to the density in a scenario such as Eq. (2) as a simple density.

Eq. (2) has a clear physical interpretation: $\rho_{\text{true}}(b)$ describes the distribution of some quantity (such as charge or energy) a displacement $b$ from the barycentric position $R$ of the hadron, and $S_{\Psi}(R, \tau)$ gives a distribution for the hadron’s barycenter itself. The convolution of both then gives the physical density at a position $x = R + b$. We shall show in Sec. II A how such a formula naturally arises in the context of non-relativistic quantum mechanics.

In an old work that has found recent attention, Fleming [8] noticed that the non-relativistic charge density took the form of Eq. (2), with the smearing function being the probability density $S_{\Psi}(R) = |\Psi^r(R)|^2 |\Psi(R)|$. If the wave packet is localized in this case, the physical density approaches the true, internal density. In fact, this will always happen when the physical density is a simple density and the smearing function is the barycentric probability density. However, Fleming extrapolated beyond this simple case and postulated that localizing the wave function would always produce an internal density. This assumption is however incorrect.

In fact, the localization procedure fails to produce a true density whenever the physical density does not obey Eq. (2). There will be cases that the physical density is actually a sum of different internal structures, each weighted by a different smearing function:

$$\rho_{\text{phys}}(x, \tau) = \int d^n x S_{\Psi}^{(1)}(R, \tau) \rho_{\text{true}}^{(1)}(x - R, \tau) + \int d^n x S_{\Psi}^{(2)}(R, \tau) \rho_{\text{true}}^{(2)}(x - R, \tau) + \ldots$$  \hspace{1cm} (3)

Here, $\rho_{\text{true}}^{(1)}(x - R, \tau)$ and $\rho_{\text{true}}^{(2)}(x - R, \tau)$ correspond to different internal spatial distributions, which are each independent of the wave packet $\Psi$. We shall refer to such a density as a compound density. In the light front formalism (Sec. III), we shall see that the energy density and stress tensor are examples of compound densities. We shall also see—assuming convergence of the series—that instant form densities are always compound densities with infinitely many contributions.

A. Example from non-relativistic quantum mechanics

Let us consider a simple non-relativistic example of where Eq. (2) naturally arises. We consider two particles with masses $m_1$ and $m_2$ bound to a system with mass $M = m_1 + m_2 - \varepsilon$. The wave function for the barycentric position...
is $\Psi(R, t)$ and the wave function for the relative position is $\psi(r)e^{i\varepsilon t}$. If the positions of the particles are $r_1$ and $r_2$, the barycentric and relative coordinates are defined:

$$R = \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2}$$

$$r = r_2 - r_1.$$  \hfill (4a)

We define a density for some quantity $Q$, which may be electric charge or mass for instance, and for which particles 1 and 2 carry an amount $q_1$ and $q_2$ respectively. The density is:

$$\rho_Q(x, t) = \int d^3R \int d^3r |\Psi(R, t)|^2 |\psi(r)|^2 \left\{ q_1 \delta^3(x - r_1) + q_2 \delta^3(x - r_2) \right\}.$$  \hfill (5)

To integrate out the delta functions, we must use the inversion of Eq. (4), which gives:

$$\rho_Q(x, t) = \int d^3R |\Psi(R, t)|^2 \left\{ q_1 \left| \psi \left( \frac{m_1 + m_2}{m_2} (R - x) \right) \right|^2 + q_2 \left| \psi \left( \frac{m_1 + m_2}{m_1} (x - R) \right) \right|^2 \right\}.$$  \hfill (6)

This has the form of Eq. (2), with:

$$S_\Psi(R, t) = |\Psi(R, t)|^2$$ \hfill (7a)

$$\rho_{\text{true}}(b) = q_1 \left| \psi \left( - \frac{m_1 + m_2}{m_2} b \right) \right|^2 + q_2 \left| \psi \left( \frac{m_1 + m_2}{m_1} b \right) \right|^2.$$ \hfill (7b)

If, at some moment of time $t_0$ we have $|\Psi(R, t_0)|^2 = \delta^3(R)$, then we obtain $\rho_Q(x, t_0) = \rho_{\text{true}}(x)$. This is a generic feature of simple densities where the smearing function is equal to the probability distribution.

### III. INTERNAL DENSITIES IN LIGHT FRONT COORDINATES

In this section, we demonstrate that the light front densities encountered in the literature [2, 3, 9, 11, 18–23] correspond to true internal densities. We also determine how they relate to their associated physical densities. Although much of the previous literature tended to use a localization procedure with a specific wave packet form (or simply used delta function wave packets) to obtain these results, we will show that the results are in fact independent of the wave packet.

For simplicity, we will focus exclusively on spin-zero targets. This work is meant to provide conceptual clarity and rigor to the research program of hadron densities rather than to be an exhaustive catalogue. Higher-spin targets with general polarization will contain complications from helicity-flip contributions [19, 20, 22, 24], so densities that are simple for spin-zero targets will often be compound for higher-spin targets.

#### A. General expression for physical densities

We will derive general expression for the physical light front densities of spin-zero targets, without making any specific assumptions about the wave packet.

The Lorentz-invariant completeness relation for states in the Hilbert subspace of single-particle spin-zero states is:

$$\int \frac{dp^+d^2p_\perp}{2p^+(2\pi)^3} |p^+, p_\perp\rangle\langle p^+, p_\perp| = 1.$$ \hfill (8)

We define the momentum space wave function as:

$$\Psi(p^+, p_\perp) = \frac{1}{\sqrt{2p^+}} \langle p^+, p_\perp | \Psi \rangle,$$ \hfill (9a)

so that it obeys the normalization condition:

$$\int \frac{dp^+d^2p_\perp}{(2\pi)^3} |\Psi(p^+, p_\perp)|^2 = 1.$$ \hfill (9b)
We define the position-space wave function through a Fourier transform:

\[ \Psi(z^-, z_\perp) = \int \frac{dp^+ dp_\perp}{(2\pi)^3} \Psi(p^+, p_\perp) e^{-ip^+ z^-} e^{i p_\perp \cdot z_\perp}, \] (10a)

so that it obeys the normalization condition:

\[ \int dz^- \int d^2 z_\perp |\Psi(z^-, z_\perp)|^2 = 1. \] (10b)

Using the completeness relation twice with momenta \( p \) and \( p' \), integrating out \( x^- \), and defining a change of variables:

\[ P_\perp = \frac{1}{2}(p_\perp + p'_\perp) \] (11a)

\[ \Delta_\perp = p'_\perp - p_\perp \] (11b)

\[ P^+ = p^+ = p'^+ \] (11c)

we obtain:

\[ \rho_{LF}(x_\perp) = \int \frac{dP^+ d^2 P_\perp}{2P^+ (2\pi)^3} \int \frac{d^2 \Delta_\perp}{(2\pi)^2} \langle P^+, p'_\perp | \hat{O}(0) | P^+, p_\perp \rangle \frac{\langle P^+, p'_\perp | \hat{O}(0) | P^+, p_\perp \rangle}{2P^+} \langle P^+, p_\perp | \Psi \rangle e^{-i \Delta_\perp \cdot x_\perp}. \] (12)

Rewriting in terms of the position-space wave functions gives:

\[ \rho_{LF}(x_\perp) = \int dR^- \int d^2 R'_\perp \int dR^- \int d^2 R_\perp \int \frac{dP^+ d^2 P_\perp}{(2\pi)^3} \int \frac{d^2 \Delta_\perp}{(2\pi)^2} e^{-i P^+ (R^- - R^')} e^{i P_\perp \cdot (R'_\perp - R_\perp)} \]

\[ \langle P^+, p'_\perp | \hat{O}(0) | P^+, p_\perp \rangle \frac{\langle P^+, p'_\perp | \hat{O}(0) | P^+, p_\perp \rangle}{2P^+} \langle P^+, p_\perp | \Psi \rangle e^{-i \Delta_\perp \cdot x_\perp}. \] (13)

To proceed, the dependence on \( P^+ \) and \( P_\perp \) must be removed from the matrix element. This can be accomplished by noting (for instance):

\[ \frac{i}{2} \left( \nabla_i^{(R)} - \nabla_i^{(R')} \right) e^{i P_\perp \cdot (R'_\perp - R_\perp)} = P_i^i e^{i P_\perp \cdot (R'_\perp - R_\perp)} \]

\[ \frac{i}{2} \left( \nabla_i^{(R)} - \nabla_i^{(R')} \right) e^{-i \Delta_\perp \cdot (x_\perp - \frac{R'_\perp + R_\perp}{2})} = 0. \]

Thus any factors of \( P_i^i \) that appear in the matrix element can be transformed into differences of gradients, and integration by parts can be used to move the gradients to act on the barycentric wave functions. A similar trick can be applied to \( P^+ \), with the end result being that the following substitutions can be made:

\[ P_\perp \rightarrow - \frac{i}{2} \nabla_i \] (14a)

\[ P^+ \rightarrow + \frac{i}{2} \partial^- \] (14b)

with the two-sided derivatives to be placed between the barycenter wave function and its conjugate. With these substitutions made, it is possible to do the \( P^+ \) and \( P_\perp \) integrals, which produce delta functions, and these delta functions can be eliminated by doing the \( R^- \) and \( R'_\perp \) integrals. Performing these steps gives:

\[ \rho_{LF}(x_\perp) = \int dR^- \int d^2 R_\perp \int \frac{d^2 \Delta_\perp}{(2\pi)^2} e^{-i \Delta_\perp \cdot (x_\perp - R_\perp)} \Psi^*(R^- , R_\perp) \frac{\langle P^+, p'_\perp | \hat{O}(0) | P^+, p_\perp \rangle}{2P^+} \Psi(R^- , R_\perp), \] (15)

where any \( P^+ \) or \( P_\perp \) appearing in this formula should be understood in terms on the substitutions of Eq. (14). To proceed any further, specific local operators must be considered.
B. Electromagnetic current

Let us first consider electromagnetic current of a spin-zero hadron. The matrix element of the electromagnetic four-current \( j^\mu(0) \) between momentum kets is:

\[
\langle P^+, p'_\perp | j^\mu(0) | P^+, p_\perp \rangle = 2P^\mu F(t),
\]

(16)

where \( F(t) \) is the electromagnetic form factor, and:

\[
t = (p' - p)^2 = -\Delta^2_\perp.
\]

(17)

For the plus component of this current—the light front charge density—the physical density takes a simple form:

\[
j_{\perp\text{LF}}(x_\perp) = \int dR^- \int d^2R_\perp |\Psi(R^-, R_\perp)|^2 \int \frac{d^2\Delta_\perp}{(2\pi)^2} F(-\Delta^2_\perp) e^{-i\Delta_\perp \cdot (x_\perp - R_\perp)}.
\]

(18)

This constitutes a simple density in the form of Eq. (2), with an internal density:

\[
j_{\perp\text{true}}(b_\perp) = \int d^2\Delta_\perp \frac{2}{(2\pi)^2} F(-\Delta^2_\perp) e^{-i\Delta_\perp \cdot b_\perp},
\]

(19)

which is the standard result [12], and a smearing function:

\[
P(R_\perp) = \int dR^- |\Psi(R^-, R_\perp)|^2,
\]

(20)

which is the transverse probability density for the barycentric position. We have thus obtained the standard result for the light front charge density of a spin-zero target, but without making any assumptions about the hadron wave packet nor localizing it. Note, however, that if \( P(R_\perp) = \delta^{(2)}(R_\perp) \), then \( j_{\perp\text{LF}}(x_\perp) = j_{\perp\text{true}}(x_\perp) \), which is why previous treatments with localized states obtained the correct internal charge density.

It is possible to also obtain the transverse current. One need only use \( \mu = 1, 2 \) instead of \( \mu = + \). It will be necessary to use the rules of Eq. (14):

\[
\frac{\langle P^+, p'_\perp | j_i(0) | P^+, p_\perp \rangle}{2P^+} = \frac{P_i^+}{P^+} F(t) \rightarrow -\frac{\nabla}{\partial^-} F(t).
\]

(21)

Since \( R^- \) is integrated out in defining smearing functions, it is possible to turn the \( \frac{\nabla}{\partial^-} \) into an expectation value:

\[
-\frac{\nabla}{\partial^-} F(t) \rightarrow -i\nabla F(t).
\]

(22)

We find thus that:

\[
j_\perp(x_\perp) = \frac{1}{P^+} \int d^2R_\perp V_\perp(R_\perp) j_{\perp\text{true}}(x_\perp - R_\perp)
\]

(23)

\[
V_\perp(R_\perp) = \int dR^- \Psi^*(R^-, R_\perp) \frac{-i}{2} \nabla \Psi(R^-, R_\perp).
\]

(24)

The transverse electromagnetic current is thus also a simple density, and in fact involves the same internal density as the electric charge density. The only difference is how the smearing function distributes the true density.

C. Momentum densities and energy density

We next consider momentum and energy densities, which are given by matrix elements of the energy-momentum tensor (EMT). Matrix elements between momentum kets are [1]:

\[
\langle P^+, p'_\perp | T^\mu\nu(0) | P^+, p_\perp \rangle = 2P^\mu P^\nu A(t) + \frac{\Delta^\mu \Delta^\nu - g^{\mu\nu} \Delta^2}{2} D(t),
\]

(25)
where $A(t)$ and $D(t)$ are called gravitational form factors [1, 25, 26], and $t$ is given in Eq. (17). The momentum and energy densities are given by considering $\mu = +$ specifically, for which:

$$\langle P^+, p^+_\perp | T^{+\nu}(0) | P^+, p^+_\perp \rangle = \frac{P^{+\nu}}{2P^+} A(t) + g^{+\nu} \frac{\Delta_\perp^2}{4P^+} D(t).$$  \hspace{1cm} (26)$$

The momenta densities are those with $\nu = +, 1, 2$, while the energy density is given by $\nu = -$ and is the only density to which $D(t)$ contributes.

The $P^+$ density is the most straightforward, since the $P^+$ multiplying $A(t)$ just becomes an expectation value. The physical density is given by:

$$T^{++}_\perp(x_\perp) = \langle P^+ \rangle \int d^2\mathbf{R}_\perp \mathcal{P}(\mathbf{R}_\perp) T^{++}_\perp(\mathbf{x}_\perp - \mathbf{R}_\perp),$$  \hspace{1cm} (27)$$

where the smearing function is defined in Eq. (20) and the internal density is given by:

$$T^{++}_\perp(\mathbf{b}_\perp) = \int d^2\Delta_\perp \frac{A(-\Delta_\perp^2)}{(2\pi)^2} e^{-i\Delta_\perp \cdot \mathbf{b}_\perp},$$  \hspace{1cm} (28)$$

which is the standard result [2, 3]. Although it was unnecessary to consider a localized wave packet to obtain this result, if we consider a localized state $\mathcal{P}(\mathbf{R}_\perp) = \delta^{(2)}(\mathbf{R}_\perp)$, then $T^{++}_\perp(x_\perp) = \langle P^+ \rangle T^{++}_\perp(x_\perp)$, which is why Ref. [3] obtained the correct result.

The $P_\perp$ density is similarly straightforward. The substitution rule of Eq. (14a) must be used, which ends up giving:

$$T^{--}_\perp(x_\perp) = \int d^2\mathbf{R}_\perp \mathbf{V}^i_\perp(\mathbf{R}_\perp) T^{++}_\perp(\mathbf{x}_\perp - \mathbf{R}_\perp),$$  \hspace{1cm} (29)$$

where the smearing function was defined in Eq. (24). So far, the situation is similar to what we saw for the electromagnetic current: the intrinsic $P^+$ density can be convolved with a different smearing function to obtain either the physical $P^+$ density or the physical $P_\perp$ density. The former is in fact proportional to the barycentric probability density.

Let us look at the energy density (i.e., the $P^-$ density) next. The matrix element contains a factor $P^-$, which is given by:

$$P^- \equiv \frac{p^- + \mathbf{p}^-}{2} = \frac{M^2 + \mathbf{P}^2_\perp + \frac{1}{4}\Delta_\perp^2}{2P^+}.$$  \hspace{1cm} (30)$$

The appropriate matrix element thus takes the form:

$$\langle P^+, p^+_\perp | T^{-\nu}(0) | P^+, p^+_\perp \rangle = \frac{M^2 + \mathbf{P}^2_\perp + \frac{1}{4}\Delta_\perp^2}{2P^+} A(t) + \frac{\Delta_\perp^2}{4P^+} D(t)$$

$$\rightarrow \int \left\langle \frac{1}{2P^+} \right\rangle \left( \left( M^2 - \frac{1}{4}\nabla_\perp^2 + \frac{1}{4}\Delta_\perp^2 \right) A(t) + \frac{\Delta_\perp^2}{2} D(t) \right),$$  \hspace{1cm} (31)$$

where we used the substitution rule in Eq. (14a). To proceed, we need to define both a new smearing function:

$$W^-(\mathbf{R}_\perp) = \int d\mathbf{R}^- \Psi^*(\mathbf{R}^-, \mathbf{R}_\perp) - \frac{\nabla_\perp^2}{4} \Psi(\mathbf{R}^-, \mathbf{R}_\perp),$$  \hspace{1cm} (32)$$

and a new internal density:

$$T^{+-}_\perp(\mathbf{b}_\perp) = \int d^2\Delta_\perp \frac{A(-\Delta_\perp^2)}{(2\pi)^2} e^{-i\Delta_\perp \cdot \mathbf{b}_\perp}. $$  \hspace{1cm} (33)$$

With this, we find the physical energy density to be a compound density:

$$T^{++}_\perp(x_\perp) = \left\langle \frac{1}{2P^+} \right\rangle \left( \int d^2\mathbf{R}_\perp W^-(\mathbf{R}_\perp) T^{++}_\perp(x_\perp - \mathbf{R}_\perp) + \int d^2\mathbf{R}_\perp \mathcal{P}(\mathbf{R}_\perp) T^{++}_\perp(x_\perp - \mathbf{R}_\perp) \right).$$  \hspace{1cm} (34)$$

This compound density exhibits a curious structure. The first term can be interpreted as a kinetic term and the second as a dynamical term. The smearing function $W^-(\mathbf{R}_\perp)$ in the first term arises from $\mathbf{P}^2_\perp$ in the momentum-space matrix element, and describes the kinetic motion of the barycenter. The convolution with the internal $P^+$ density.
then distributes the barycentric kinetic motion over the constituents. Because of this first term, if \( P(R_\perp) = \delta^{(2)}(R_\perp) \), then \( T^+_\perp(R_\perp) \) diverges. This is in effect a manifestation of the uncertainty principle [10].

The second term in Eq. (34) encodes every contribution to the hadron energy aside from barycentric kinetic motion. This includes masses of constituents, kinetic energy of the constituents relative to the center-of-momentum, and potential energy. This can be seen by considering the general form of the light-front Hamiltonian for an \( N \)-body system:

\[
H = \sum_{i=1}^{N} \frac{m_i^2 + p_{i\perp}^2}{2p_i^+} + V .
\] (35)

Here, \( V \) is the potential energy, and the absolute momenta are related to relative momenta and light front momentum fractions by:

\[
p_i^+ = x_i P^+ ,
\] (36a)

\[
p_i \perp = x_i P_\perp + k_{i\perp} ,
\] (36b)

and these obey sum rules:

\[
\sum_{i=1}^{N} x_i = 1 \] (36c)

\[
\sum_{i=1}^{N} k_{i\perp} = 0 . \] (36d)

Using these definitions and the sum rules, it follows:

\[
H = \frac{P_\perp^2}{2P^+} + \frac{1}{2P^+} \sum_{i=1}^{N} \frac{m_i^2 + k_{i\perp}^2}{x_i} + V .
\] (37)

The \( P_\perp^2/(2P^+) \) term of course corresponds to the first term in Eq. (34), and thus the remaining terms in Eq. (37)—which are exactly the aspects of the internal energy we have listed—must correspond to the remaining second term of Eq. (34). It is for this reason that we identify \( T_{\text{true}}(b_\perp) \) as the true energy density. In fact, the presence of \( D(t) \) in this density, which has a well-established connection to internal stresses [1–3, 27], helps indicate the dynamical nature of this term.

It is worth mentioning that the true internal energy density is equivalent to the light front energy density in the Drell-Yan frame in the formalism of Wigner phase space distributions, which is discussed in Ref. [2].

D. Stress tensor

The light front stress tensor is given by \( T^{ij}_\text{LF}(x_\perp) \) with \( i, j = 1, 2 \). As remarked previously in Ref. [3], the stress tensor contains contributions from hadron flow and a “comoving” piece. Accordingly, the stress tensor constitutes a compound density in the manner of Eq. (3). We will now derive the exact form of this compound density.

The relevant matrix elements for the stress tensor are:

\[
\langle P^+, P'_\perp | T^{ij}(0) | P^+, P_\perp \rangle = \frac{P_i^i P_j^j}{P^+} A(t) + \frac{\Delta_i^i \Delta_j^j - \delta^{ij} \Delta_\perp^2}{4P^+} D(t) ,
\] (38)

for which the substitution rules of Eq. (14) must be used. The presence of two factors of \( P_\perp \) in front of \( A(t) \) requires defining a new smearing function:

\[
\mathcal{W}^{ij}(R_\perp) = \int dR^- \Psi^*(R^-, R_\perp) \left( -\frac{i}{2} \nabla_i^\perp \right) \left( -\frac{i}{2} \nabla_j^\perp \right) \Psi(R^-, R_\perp) ,
\] (39)

which is related to the smearing function in Eq. (32) by:

\[
\mathcal{W}^{-}(R_\perp) = \delta_{ij} \mathcal{W}^{ij}(R_\perp) .
\] (40)
If we define the internal stress tensor as:

$$T^{ij}_{\text{true}}(b_\perp) = \frac{1}{4} \int \frac{d^2 \Delta_\perp}{(2\pi)^2} \left( \Delta^i_\perp \Delta^j_\perp - \delta^{ij} \Delta^2_\perp \right) D(-\Delta^2_\perp) e^{-i \Delta_\perp \cdot b_\perp} , \tag{41}$$

then the physical stress tensor can be written as a compound density:

$$T^{ij}_{\text{LF}}(x_\perp) = \left( \frac{1}{P^+} \right) \left( \int d^2 R_\perp \mathcal{W}^{ij}(R_\perp) T^{ij}_{\text{true}}(x_\perp - R_\perp) + \int d^2 R_\perp \mathcal{P}(R_\perp) T^{ij}_{\text{true}}(x_\perp - R_\perp) \right) . \tag{42}$$

The quantity $T^{ij}_{\text{true}}(b_\perp)$ is (aside from a factor $P^+$) identical to the pure stress tensor of Ref. [3]. The first and second terms in Eq. (42) can be identified as a flow tensor and comoving stress tensor respectively [3, 28], so that the physical stress tensor essentially takes the classical continuum form [28]. It is interesting to note that while diagonal elements of $T^{ij}_{\text{LF}}(x_\perp)$ correspond to internal or static pressures as seen by observers comoving with the motion of the hadron, diagonal elements of $T^{ij}_{\text{true}}(x_\perp)$ instead correspond to dynamic pressures, which includes impulses that would be imparted by the hadron’s motion and which is akin to radiation pressure (see discussion in Refs. [3, 23, 28, 29]).

### E. The localization fallacy

Deriving Eq. (34) for the energy density required foregoing localization of the wave packet. For a localized wave packet, the energy density actually diverges, which can be understood in terms of the uncertainty principle [10].

Let us consider what might happen if we attempt to localize the wave packet by a procedure similar to that in Refs. [3, 8, 14, 15], but as in Ref. [30] in particular, we absorb the divergence into a normalization constant. The wave function is localized by a scaling transformation:

$$\Psi(R^-, R_\perp) \rightarrow \frac{1}{\sigma} \Psi \left( R^- , \frac{R_\perp}{\sigma} \right) , \tag{43}$$

which transforms the smearing functions as:

$$\mathcal{P}(R_\perp) \rightarrow \frac{1}{\sigma^2} \mathcal{P} \left( \frac{R_\perp}{\sigma} \right) \tag{44a}$$

$$\mathcal{W}^-(R_\perp) \rightarrow \frac{1}{\sigma^4} \mathcal{W}^- \left( \frac{R_\perp}{\sigma} \right) . \tag{44b}$$

Using the variable change $Y_\perp = \sigma R_\perp$, the energy density can be written:

$$T^-_{\text{LF}}(x_\perp) = \left( \frac{1}{2 P^+} \right) \left( \int d^2 Y_\perp \mathcal{P}(Y_\perp) T^{-\text{true}}_{\text{true}}(x_\perp - \sigma Y_\perp) + \frac{1}{\sigma^2} \int d^2 Y_\perp \mathcal{W}^-(Y_\perp) T^{++}_{\text{true}}(x_\perp - \sigma Y_\perp) \right) . \tag{45}$$

Defining the normalization:

$$N_\infty = \lim_{\sigma \rightarrow 0} \frac{1}{\sigma^2} \left\langle \frac{1}{2 P^+} \right\rangle \int d^2 Y_\perp \mathcal{W}^-(Y_\perp) , \tag{46}$$

the energy density in the limit $\sigma \rightarrow 0$ becomes:

$$T^+_{\text{LF}}(x_\perp) \xrightarrow{\sigma \rightarrow 0} N_\infty T^+_{\text{true}}(x_\perp) . \tag{47}$$

The dynamical term was dropped because it remains finite in the $\sigma \rightarrow 0$ limit, and therefore is dominated by the kinetic term. The problem should be immediately clear: by localizing the compound density and keeping only the dominating term, we lose important information about the internal structure of the hadron. In fact, the internal density corresponding to dynamics is precisely what was lost, and the remaining kinetic term is actually trivial and uninteresting in this context.

This example should prove a cautionary tale about the localization procedure. In several special cases—namely charge and $P^+$ density—the correct internal light front densities were obtained from localization because these are simple densities. As we shall see in Sec. IV, however, none of the physical instant form densities associated with the electromagnetic current or energy-momentum tensor are simple, and accordingly wave packet localization does not reproduce meaningful internal densities for instant form coordinates.
where the actual transverse distance from the barycenter to the transverse coordinate—i.e., for the impact parameter. The given by the sum of the single-particle distributions weighted by their charges:

\[ Q \]

There is an up quark with charge \( Q_+ \) and a down antiquark with charge \(-Q_-\). At large \( Q^2 \), or small transverse separations, the system is approximately described by the light front Hamiltonian:

\[ H_{\text{internal}} = \frac{k^2}{x} + \frac{k^2}{1-x} + \kappa^4 x(1-x)y^2_\perp - 2\kappa^2, \tag{48} \]

and the pion ground state is described by the internal wave function:

\[ \psi(x, y_\perp) = \frac{\kappa}{\sqrt{\pi}} \sqrt{x(1-x)} e^{-\frac{1}{2}\kappa^2 x(1-x)y^2_\perp}, \tag{49} \]

where \( \kappa = 0.375 \) GeV. We have followed Soper \[32\] in denoting the transverse separation as \( y_\perp \), and reserve \( b_\perp \), for the actual transverse distance from the barycenter to the transverse coordinate—i.e., for the impact parameter. The transverse separation is related to the actual transverse positions via:

\[ y_\perp = \left( r_{1\perp} - r_{1\perp} \right) = \frac{b_1 \perp}{1-x} = - \left( r_{2\perp} - r_{1\perp} \right) = - \frac{b_2 \perp}{x}. \tag{50} \]

The charge density is the most straightforward case to consider. We assume a \( \pi^+ \) to avoid a trivially zero result. There is an up quark with charge \( Q_+ \) and a down antiquark with charge \(-Q_-\), and the physical charge density is just given by the sum of the single-particle distributions weighted by their charges:

\[ j_{++}^+(x_\perp) = \int dR^- \int d^2R_\perp |\Psi(R^-, R_\perp)|^2 \int_0^1 dx \int d^2y_\perp |\psi(x, y_\perp)|^2 \left\{ Q_u \delta^{(2)}(x_\perp - r_{1\perp}) - Q_d \delta^{(2)}(x_\perp - r_{2\perp}) \right\}. \tag{51} \]

where \( Q_u = 2/3 \) and \( Q_d = -1/3 \). Comparing to Eq. (18) and Eq. (19), the true internal charge density is:

\[ j_{\text{true}}^+(b_\perp) = \int_0^1 dx \left\{ \frac{Q_u}{(1-x)^2} \left| \psi \left( x, \frac{b_\perp}{1-x} \right) \right|^2 - \frac{Q_d}{x^2} \left| \psi \left( x, \frac{b_\perp}{x} \right) \right|^2 \right\}. \tag{52} \]

The integral can be done analytically for the wave function in Eq. (49), and the result is:

\[ j_{\text{true}}^+(b_\perp) = \frac{\kappa^2}{\pi} e^{\kappa^2b_\perp^2} \left( E_1(\kappa^2b_\perp^2) - E_2(\kappa^2b_\perp^2) \right), \tag{53} \]

where

\[ E_n(z) = \int_1^\infty dt e^{-zt} t^n \tag{54} \]

is the generalized exponential integral function \[33\]. The result is plotted in the left panel of Fig. 1.

By a similar token, the internal \( P^+ \) density is given by:

\[ T_{\text{true}}^{++}(b_\perp) = \int_0^1 dx \left\{ \frac{x}{(1-x)^2} \left| \psi \left( x, \frac{b_\perp}{1-x} \right) \right|^2 + \frac{1-x}{x^2} \left| \psi \left( x, \frac{b_\perp}{x} \right) \right|^2 \right\}, \tag{55} \]

which can be evaluated using the wave function in Eq. (49) as:

\[ T_{\text{true}}^{++}(b_\perp) = \frac{\kappa^2}{\pi} e^{\kappa^2b_\perp^2} \left( 2E_1(\kappa^2b_\perp^2) - 4E_2(\kappa^2b_\perp^2) + 2E_3(\kappa^2b_\perp^2) \right). \tag{56} \]

This result is also plotted in the left panel of Fig. 1.
The last internal density we consider is the internal energy density. Now, it is straightforward to find the part of the true energy density associated with quark kinetic energy, using just the substitution rule $k_\perp \rightarrow -\frac{i}{2} \nabla_{(y)}$, since this energy is spatially attached to the quark in question:

$$T_{\text{kin}}^+(b_\perp) = -\int_0^1 dx \left\{ \frac{1}{x(1-x)} \psi^*(x, \frac{b_\perp}{1-x}) \nabla^2 \psi(x, \frac{b_\perp}{1-x}) + \frac{1}{(1-x)x^2} \psi^*(x, \frac{b_\perp}{x}) \nabla^2 \psi(x, \frac{b_\perp}{x}) \right\}. \quad (57)$$

For the wave function in Eq. (49), we note:

$$-\frac{1}{4} \psi^*(x, y_\perp) \nabla_i \nabla_j \psi(x, y_\perp) = \frac{1}{2} \kappa^2 x (1-x) \delta_{ij} \psi^*(x, y_\perp) \psi(x, y_\perp), \quad (58)$$

and the kinetic energy density can be explicitly found to be:

$$T_{\text{kin}}^+(b_\perp) = \frac{2 \kappa^4}{\pi} e^{\kappa^2 b_\perp^2} \left( E_2(\kappa^2 b_\perp^2) - E_3(\kappa^2 b_\perp^2) \right). \quad (59)$$

The potential energy cannot be assigned to either quark, however, so some additional hypothesis about its spatial distribution is needed to proceed. The $y^2_\perp$ term has the form of an elastic potential, so we hypothesize that the corresponding energy is distributed along a string or flux tube between the quarks. The energy density operator so defined is thus:

$$\hat{T}_{\text{string}}^+(s_\perp(\tau)) = \lambda \int_0^1 d\tau (s_\perp(\tau))^2 \delta^{(2)}(x_\perp - s_\perp(\tau)), \quad (60)$$

where $s_\perp(\tau)$ parametrizes the location of the string in the transverse plane and $\lambda$ is the string energy density. For a straight line,

$$s_\perp(\tau) = r_{2\perp} + \tau y_\perp. \quad (61)$$

To reproduce the $y^2_\perp$ term, an energy density $\lambda = \kappa^4 x (1-x)$ is sufficient. For lack of a better hypothesis for the $-2\kappa^2$ term, we distribute this energy evenly over the same string. This makes the necessary string energy density:

$$\lambda(x, b_\perp) = \kappa^4 x (1-x) - \frac{\kappa^2}{y^2_\perp}. \quad (62)$$

The internal energy density associated with this string is then:

$$T_{\text{pot}}^+(b_\perp) = \int_0^1 dx \int_0^1 d\tau \frac{b_\perp^2}{(\tau - x)^4} \left| \psi \left( x, \frac{b_\perp}{\tau - x} \right) \right|^2 \lambda \left( x, \frac{b_\perp}{\tau - x} \right). \quad (63)$$

It is more difficult to obtain an exact analytic result for this integral than for the other integrals we’ve considered. An easy form to work with numerically for the result is:

$$T_{\text{pot}}^+(b_\perp) = \frac{\kappa^4}{\pi} \left\{ e^{\kappa^2 b_\perp^2} \left( E_2(\kappa^2 b_\perp^2) - E_3(\kappa^2 b_\perp^2) \right) - \frac{3\sqrt{\pi}}{\kappa b_\perp} \int_{\pi/2}^{\pi/2} d\theta \sin^2 \theta \cos^2 \theta \text{erfc}(\kappa b_\perp \tan \theta) \right\}, \quad (64)$$

where erfc is the complementary error function $[33]$:

$$\text{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty dz e^{-z^2}. \quad (65)$$

The results for the internal densities in this model are plotted in Fig. 1. The left panel includes both the charge and momentum densities, both of which are normalized to 1. The charge density is broader than the momentum density. The right panel includes the internal energy density, as well as its breakdown into potential and kinetic energy contributions. An interesting feature of the energy density is that it can be negative, in contrast to the momentum density which is positive-definite.
2. Phenomenological multipole model

Although illuminating, a light front Hamiltonian model is not necessary to obtain the light front densities. The form factors \( A(t) \) and \( D(t) \) can be obtained from phenomenology. Multipole forms are reasonable for the form factors, based on analyticity rules for the scattering amplitude. For the pion in particular:

\[
A(t) = \frac{1}{1 - t/m_{f_2}^2}
\]

\[
D(t) = \frac{1}{(1 - t/m_{f_2}^2)} - \frac{1}{(1 - t/m_{K^0}^2)(1 - t/m_{\pi}^2)}.
\]

The monopole form for \( A(t) \) is supported by large-\( N_c \) phenomenology [34], and the presence of an \( f_2(1270) \) pole is suggested by spin-two meson dominance as a gravitational analogy to vector meson dominance in electromagnetic form factors. Moreover, \( m_{f_2} = 1270 \text{ MeV} \) reproduces the pion radius reported by Kumano et al.’s analysis of Belle data [35], as well as the proton radius suggested by Kharzeev’s analysis of GlueX data [36] (using a dipole form for the proton’s \( A(t) \) form factor). The presence of an additional \( \sigma \) pole in \( D(t) \) comes from dressing of the graviton-quark vertex [37], and the value \( D(0) = -1 \) comes from a low-energy pion theorem [1, 38, 39]. Using \( m_{\sigma} = 630 \text{ MeV} \) reproduces the \( D(t) \) slope reported in Ref. [35].

From these multipole forms of the form factors, we obtain an internal momentum density:

\[
T_{\text{true}}^{+-}(b_\perp) = \frac{m_{f_2}^2}{2\pi} K_0(m_{f_2}b_\perp),
\]

where \( K_0(x) \) is a modified Bessel function of the second kind [33]. The internal energy density is given by:

\[
T_{\text{true}}^{++}(b_\perp) = \left(m_{\pi}^2 - \frac{m_{f_2}^2}{4}\right) \frac{m_{f_2}^2}{2\pi} K_0(m_{f_2}b_\perp) + \frac{m_{f_2}^2}{4} \delta^{(2)}(b_\perp) + \frac{1}{4\pi m_{f_2}^2 - m_{\sigma}^2} \left(m_{\sigma}^2 K_0(m_{\sigma}b_\perp) - m_{f_2}^2 K_0(m_{f_2}b_\perp)\right).
\]

In the left panel of Fig. 2, we compare the pion energy density obtained with this phenomenological multipole parametrization to the result found using the soft wall holographic model. The energy densities look radically different, suggesting a need for additional measurements of hard exclusive reactions to further experimentally constrain the pion’s gravitational form factors. A curious feature that models have in common is that the internal energy density becomes negative at short distances.

In the right panel of Fig. 2, we present the physical energy density using a Gaussian wave packet of the form:

\[
\Psi(R_\perp) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}R_\perp^2},
\]

FIG. 1. Internal pion densities in Brodsky’s soft wall holographic model. (Left panel) charge and momentum densities. (Right panel) energy density, including its breakdown into kinetic and potential energy.
FIG. 2. Pion energy densities associated with the phenomenological multipole model. (Left panel) compares the internal energy density in the multipole model to the soft wall holographic model. (Right panel) presents the physical pion energy density (see Eq. (34)) for a Gaussian wave packet for several wave packet widths and with $2P^+ = m_\pi$.

along with the phenomenological multipole model at several values of the wave packet width $\sigma$ and with $2P^+ = m_\pi$ for concreteness. It can be seen clearly that the physical density does not approach the internal density as the wave packet is localized, and is qualitatively quite different. This is a stark manifestation of how wave packet localization will not always distill the desired aspects of a hadron’s internal structure.

IV. LACK OF INTERNAL DENSITIES IN INSTANT FORM COORDINATES

To investigate physical densities in instant form coordinates in a wave packet independent way, we need an analogous formula to that in Eq. (15). We shall proceed to obtain such a relation.

The Lorentz-invariant completeness relation for states in the Hilbert subspace of single-particle spin-zero states is:

$$\int \frac{d^3p}{2E_p(2\pi)^3} |p\rangle \langle p| = 1.$$  \hspace{1cm} (70)

We define the momentum space wave function as:

$$\Psi(p) = \frac{1}{\sqrt{2E_p}} (p|\Psi),$$  \hspace{1cm} (71a)

so that it obeys the normalization condition:

$$\int \frac{d^3p}{(2\pi)^3} |\Psi(p)|^2 = 1.$$  \hspace{1cm} (71b)

We similarly define the position-space wave function:

$$\Psi(r) = \int \frac{d^3p}{(2\pi)^3} \Psi(p) e^{ip \cdot r},$$  \hspace{1cm} (72a)

so that it obeys the normalization condition:

$$\int d^3r |\Psi(r)|^2 = 1.$$  \hspace{1cm} (72b)

Following an analogous process to that outlined in Sec. III, it can be shown that:

$$\rho_{\text{phys}}(x) = \int d^3R' \int d^3R \int \frac{d^3P}{(2\pi)^3} \int \frac{d^3\Delta}{(2\pi)^3} e^{i\Delta \cdot (x - \frac{R + R'}{2})} \Psi^*(R') \frac{\langle p' | \hat{O}(0) | p \rangle}{2\sqrt{E_p E_{p'}}} \Psi(R).$$  \hspace{1cm} (73)
As with the light cone case, we need a substitution rule for $P$ when it shows up in the matrix element. The appropriate rule can be shown to be:

$$P \rightarrow \frac{i}{2} \nabla \cdot.$$ (74)

Wherever $P^0$ shows up, it is necessary to write it out in terms of $P$:

$$P^0 = \frac{E_p + E_{p'}}{2} = \frac{1}{2} \left( \sqrt{M^2 + \left( P - \frac{\Delta}{2} \right)^2} + \sqrt{M^2 + \left( P + \frac{\Delta}{2} \right)^2} \right),$$ (75)

and then use Eq. (74) on each instance of $P$ here. With such substitutions implicitly applied, the instant form equivalent of Eq. (15) is:

$$\rho_{\text{phys}}(x) = \int d^3R \int \frac{d^3\Delta}{(2\pi)^3} e^{-i\Delta \cdot (x-R)} \Psi^*(R) \frac{\langle p'|\hat{O}(0)|p \rangle}{2\sqrt{E_pE_{p'}}} \Psi(R),$$ (76)

where any $P$ are to be implicitly understood in terms of Eq. (74). This is the furthest we can go without looking at specific local operators. Obtaining a simple or finite compound density will depend on the matrix element factorizing in its $\Delta$ and $P$ dependence, or else breaking down into a finite number of terms that factorize as such. We will presently see that this does not happen.

### A. Electric charge and energy densities

The appropriate matrix element for the electric charge density is:

$$\langle p'|T^{00}(0)|p \rangle = 2P^0 F(t) = \left( \sqrt{M^2 + \left( P - \frac{\Delta}{2} \right)^2} + \sqrt{M^2 + \left( P + \frac{\Delta}{2} \right)^2} \right) F(t).$$ (77)

In instant form coordinates, $t$ is given by:

$$t = (p' - p)^2 = -\Delta^2 + 2 \left( M^2 + P^2 + \frac{\Delta^2}{4} \right) - \sqrt{\left( M^2 + P^2 + \frac{\Delta^2}{4} \right)^2 - \left( P \cdot \Delta \right)^2}. \quad (78)$$

The physical charge density thus takes the form:

$$j_{\text{phys}}^0(x) = \frac{1}{2} \int d^3R \int \frac{d^3\Delta}{(2\pi)^3} e^{-i\Delta \cdot (x-R)} \Psi^*(R) F \left( -\Delta^2 + 2 \left( M^2 + \frac{1}{4} \nabla^2 + \frac{1}{4} \Delta^2 \right) - \sqrt{\left( M^2 + \frac{1}{4} \nabla^2 + \frac{1}{4} \Delta^2 \right)^2 + \frac{1}{2} \left( \nabla \cdot \Delta \right)^2} \right)^{1/4} \Psi(R). \quad (79)$$

The problem is immediately clear: it is impossible to write this as a finite number of terms whose $R$ and $\Delta$ dependence factorizes. At best, we could expand the expression as an infinite formal series in $\nabla \cdot \Delta$ and obtain an infinitely compound density. Convergence of the series will depend on the wave packet taking a specific form, which is precisely what we need to avoid to obtain a truly internal density.

Therefore, there is no single internal charge density in instant form coordinates.

The instant form energy density is similarly unfactorizable. The relevant matrix element is [1]:

$$\langle p'|T^{00}(0)|p \rangle = \frac{1}{2} \left( \sqrt{M^2 + \left( P - \frac{\Delta}{2} \right)^2} + \sqrt{M^2 + \left( P + \frac{\Delta}{2} \right)^2} \right)^2 A(t) + \frac{\Delta^2}{2} D(t), \quad (80)$$
where \( t \) is again defined in Eq. (78). This makes the physical instant form energy density:

\[
T_{\text{phys}}^{00}(x) = \int d^3R_\perp \int \frac{d^3\Delta}{(2\pi)^3} e^{-i\Delta \cdot (x - R)} \psi^*(R) \left( \frac{(E_p + E_p')^2}{4\sqrt{E_p E_p'}} A(t) + \frac{\Delta^2}{4\sqrt{E_p E_p'}} D(t) \right) \psi(R),
\]

where we have neglected to explicitly write the substitution rules in order to keep the formula compact. However, a quick glance at Eq. (79) should convince the reader that this expression is not factorizable into either a simple density in the manner of Eq. (2), nor a finite compound density in the manner of Eq. (3).

In fact, we shall see below, that the instant form densities are at best \emph{infinitely compound densities}, but convergence of this series depends on the form of the wave packet.

**B. Diffuse wave packets and Breit frame densities**

We have shown that there are no true densities in instant form coordinates. Nonetheless, it may be worth clarifying the approximate form of the physical densities when they are expanded as a formal series in the derivative, assuming that this expansion is controlled by a small parameter. This can be the case for diffuse wave packets, for which the derivative of the wave function is smaller than the wave function.

Let us consider the first few terms in the physical charge density when expanded out in powers of the derivative. Odd powers of the derivative cancel in the expansion, so we will consider the up to the second power. The leading terms take the form:

\[
\begin{align*}
\hat{j}_{\text{phys}}^{0}(x) &= \int d^3R \mathcal{P}(R) \hat{j}_{\text{Breit}}^{0}(x - R) + \int d^3R Q_{ij}(R) \hat{j}_{\text{new}}^{ij}(x - R) + \ldots, \\
\end{align*}
\]

where the smearing functions:

\[
\begin{align*}
\mathcal{P}(R) &= \Psi^*(R) \Psi(R), \\
Q_{ij}(R) &= \Psi^*(R) \left( -\frac{i \nabla_i}{2M} - \frac{i \nabla_j}{2M} \right) \Psi(R)
\end{align*}
\]

and internal density functions:

\[
\begin{align*}
\hat{j}_{\text{Breit}}^{0}(b) &= \int \frac{d^3\Delta}{(2\pi)^3} F(-\Delta^2) e^{-i\Delta \cdot b} \\
\hat{j}_{\text{new}}^{ij}(b) &= \int \frac{d^3\Delta}{(2\pi)^3} \frac{\Delta^2 \Delta^i \Delta^j}{1 + \frac{\Delta^2}{M^2}} \left( \frac{1}{8M^2} F(-\Delta^2) + \frac{1}{2} F'(-\Delta^2) \right) e^{-i\Delta \cdot b}
\end{align*}
\]

appear. These of course are only the first two terms in an infinite series, and we reiterate that the convergence of the series depends on the wave packet being diffuse.

A similar expansion can be made for the instant form energy density:

\[
T_{\text{phys}}^{00}(x) = \int d^3R \mathcal{P}(R) T_{\text{Breit}}^{00}(x - R) + \int d^3R Q_{ij}(R) Q_{\text{new}}^{ij}(x - R) + \ldots,
\]

where the standard Breit frame density is given by [1]:

\[
T_{\text{Breit}}^{00}(b) = 2M^2 \int \frac{d^3\Delta}{(2\pi)^3} \frac{1}{\sqrt{4M^2 + \Delta^2}} \left( A(-\Delta^2) + \frac{\Delta^2}{4M^2} \left[ A(-\Delta^2) + D(-\Delta^2) \right] \right) e^{-i\Delta \cdot b},
\]

and the new higher-order internal structure by:

\[
Q_{\text{new}}^{ij}(b) = M^2 \int \frac{d^3\Delta}{(2\pi)^3} \frac{1}{\sqrt{4M^2 + \Delta^2}} \left\{ \delta^{ij} \left( A(-\Delta^2) - \frac{\Delta^2}{4M^2 + \Delta^2} D(-\Delta^2) \right) \\
+ 2\Delta^i \Delta^j \left( A'(-\Delta^2) + \frac{\Delta^2}{4M^2 + \Delta^2} \left[ D(-\Delta^2) + D'(-\Delta^2) \right] \right) \right\} e^{-i\Delta \cdot b}.
\]
For both the charge density and energy density, the leading term for diffuse wave packets is a convolution between the probability density of the barycenter and the standard Breit frame density. This suggests that, after all, the Breit frame densities do actually encode an aspect of internal hadron structure—but that, however, they do not encode the truly internal densities, independent of wave packet. In fact, they are the leading internal structure out of infinitely many when the hadron in question is specifically prepared in a diffuse wave packet.

Through the consideration of diffuse wave packets, we have therefore established a rigorous grounds upon which Breit frame densities relate to the physical densities of hadrons, and by providing the next-to-leading order corrections, have also provided a means to numerically test the breakdown of their applicability. We shall numerically study this breakdown in Sec. IV C.

C. Numerical illustration of Breit frame density breakdown

We will illustrate the domain of applicability of Breit frame densities using a simple wave packet—a Gaussian with average momentum \( P \):

\[
\Psi(R) = \frac{1}{(2\pi \sigma^2)^{3/4}} e^{-\frac{R^2}{4\sigma^2}} e^{iP \cdot R}.
\]  

(86)

For such a wave packet:

\[
-\frac{1}{4} \Psi^*(R) \nabla_i \nabla_j \Psi(R) = \left( P_i P_j + \frac{\delta_{ij}}{4\sigma^2} \right) \Psi^*(R)\Psi(R),
\]  

(87)

meaning that the expansion considered in Sec. IV B is valid when both \( |P| \) is small (a slow wave packet) and \( \sigma \) is large (a diffuse wave packet). The second-order smearing function for this wave packet is given by:

\[
Q_{ij}(R) = \left( \frac{P_i P_j}{M^2} + \frac{\delta_{ij}}{4\sigma^2 M^2} \right) P(R).
\]  

(88)

We shall consider a case with proton-like kinematics, using \( M = 940 \text{ MeV} \) and simple multipole models for the form factors:

\[
F(-\Delta^2) = \frac{1}{(1 + \Delta^2/m_{\rho}^2)^2}
\]  

(89a)

\[
A(-\Delta^2) = \frac{1}{(1 + \Delta^2/m_{f_2}^2)^2}
\]  

(89b)

\[
D(-\Delta^2) = \frac{-1}{(1 + \Delta^2/m_{f_2}^2)^3},
\]  

(89c)

with \( m_{\rho} = 776 \text{ MeV} \) and \( m_{f_2} = 1270 \text{ MeV} \). Although the proton is a spin-half particle, this calculation is meant only for illustrative purposes.

For a numerical example, we consider wave packets with zero average momentum. The leading approximations for the physical instant form charge density are presented in Fig. 3 for two wave packet widths. The more diffuse packet has a width equal to the Compton wave length:

\[
\lambda_c = \frac{2\pi}{M} \approx 1.3 \text{ fm},
\]  

(90)

and the more localized packet has a width equal to the reduced Compton wavelength, smaller by a factor \( 2\pi \). For the more diffuse packet, corrections to the charge density from structures beyond the Breit frame density are negligible. For the more localized packet, however, there are significant corrections to the physical density from the first higher-order term. This indicates that while the Breit frame charge density may encode one aspect of internal hadron structure that dominates for diffuse wave packets, it by no means encodes the internal charge density—especially not in a state-independent way.

We next repeat this exercise for the physical instant form energy density in Fig. 4. Just as with the charge density, the Breit frame density dominates other aspects of internal hadron structure for sufficiently diffuse wave packets, and a width as large as the Compton wavelength is sufficient. For smaller wave packets, however, the Breit frame density is no longer sufficient to reproduce the physical density, indicating that the Breit frame density is not the true internal energy density.
There is one significant difference between Fig. 3 and Fig. 4: the higher-order terms in Fig. 4 changes the normalization. In fact, for the smaller wave packet, it should: a more localized wave packet means a greater uncertainty in momentum, and accordingly, a greater expectation value for the total energy. The integral of the physical energy density should give the expected value of the energy, but integrating the leading term in Eq. (83) only gives the mass.

It may be tempting to interpret the leading term of Eq. (83) as a mass density and the corrections as kinetic energy densities, but this would be erroneous. There is no “kinetic charge” in the charge density, yet there were higher-order corrections to the instant form charge density. The higher-order corrections to the energy density certainly contain barycentric kinetic energy, but it is impossible to separate these from higher-order corrections to the mass density. Isolating a barycentric kinetic energy for the light front energy density (see Sec. III) was possible only because of the Galilean subgroup, which does not manifest in instant form coordinates.

D. Localized wave packets and Fleming-like densities

The Fleming procedure [8] (which was rediscovered recently in Refs. [14, 15]) obtains a density by localizing a spherically symmetric wave packet. Fleming claims that this produces an internal density, but we will see that this claim is fallacious.
Let us consider what the charge and energy densities look like if the derivative is treated as a large parameter. We first need to look at the behavior of the invariant momentum transfer at large $P$. Expanding Eq. (78) gives:

$$t \approx \left(-\Delta^2 + \frac{(\Delta \cdot P)^2}{P^2}\right) - \frac{1}{4P^2} \left(-\frac{(\Delta \cdot P)^2}{P^2}(4M^2 + \Delta^2) + \left(\frac{(\Delta \cdot P)^2}{P^2}\right)^2\right) + \ldots$$

(91)

Accordingly, the leading term in the expansion of the charge density is:

$$j^0_{\text{phys}}(x) = \int d^3R \int \frac{d^3\Delta}{(2\pi)^3} e^{-i\Delta \cdot (x-R)} \Psi^*(R) F \left(-\Delta^2 + \frac{(\Delta \cdot \nabla)^2}{\nabla^2}\right) \Psi(R) + \ldots$$

(92)

This leading term reproduces the results of Refs. [8, 14, 15]. The presence of a derivative inside the form factor impedes factorizing the expression, making the scenario worse than in the diffuse wave packet limit. For diffuse wave packets it was at least possible to write the physical charge density as an infinitely compound density and keep the leading terms; for the localized packet, not even this is possible. Refs. [8, 14, 15] proceed from here by making the specific assumption that the wave packet is spherically symmetric, in which case it can be shown that:

$$j^0_{\text{phys}}(x) = \int d^3R \Psi^*(R)\Psi(R) \int \frac{d^3\Delta}{(2\pi)^3} \int_0^1 d\alpha F((\alpha^2 - 1)\Delta^2) e^{-i\Delta \cdot (x-R)} + \ldots$$

(93)

We remind the reader that this expression was found only after very specific assumptions had been made about the wave packet. The quantity:

$$j^0_{\text{lem}}(b) = \int \frac{d^3\Delta}{(2\pi)^3} \int_0^1 d\alpha F((\alpha^2 - 1)\Delta^2) e^{-i\Delta \cdot b}$$

(94)

does not constitute an internal density—not even as a term in a compound density in the sense of Eq. (3)—since the integral over $\alpha$ has absorbed aspects of the wave packet dependence. This actually makes the Fleming density a less desirable candidate for describing hadron structure than the Breit frame density, since the former at least can be interpreted as one aspect of internal structure that dominates when wave packets are diffuse. The Fleming density instead corresponds to a physical density which mixes internal structure and wave packet dependence, and then only for highly localized, spherically symmetric wave packets.

The fact that the Fleming densities fail to encode true internal structures can be seen more viscerally with the energy density. The Fleming-like density found through this method, which has been reported in Ref. [30], is:

$$T^{00}_{\text{Pant}}(b) = \int \frac{d^3\Delta}{(2\pi)^3} \int_0^1 d\alpha A((\alpha^2 - 1)\Delta^2) e^{-i\Delta \cdot b}$$

(95)

which is related to the physical density through:

$$T^{00}_{\text{phys}}(x) = \int d^3R \Psi^*(R)\sqrt{-\frac{\nabla^2}{4}}\Psi(R) T^{00}_{\text{Pant}}(x-R) + \ldots.$$  

(96)

Just as with the charge density, we reiterate that this should not be misread as a term in a compound density, since aspects of the wave packet dependence have actually been absorbed into $T^{00}_{\text{Pant}}(b)$.

An especially telling sign here is that the form factor $D(t)$ is missing. As we saw in Eq. (34), $D(t)$ should appear along with $A(t)$ in the true energy density, since it encodes aspects of stresses acting within a hadron, which contribute to the energy. The Fleming procedure washes away this aspect of hadron dynamics. In fact, as we saw in Sec. III E, the localization procedure causes the barycentric kinetic energy to dominate over all other contributions to energy—including masses and interaction energy.

The form factor $D(t)$ does appear when the next-to-leading terms in the physical energy density are included:

$$T^{00}_{\text{phys}}(x) = \int d^3R \Psi^*(R)\Psi(R) T^{00}_{\text{Pant}}(x-R)$$

$$+ \int d^3R \int \frac{d^3\Delta}{(2\pi)^3} e^{-i\Delta \cdot (x-R)} \Psi^*(R) \left\{ \frac{4}{4} \frac{4M^2 + \Delta^2}{\nabla^2} A_{\text{LO}} + \frac{1}{2} \frac{\Delta^2}{\nabla^2} D_{\text{LO}} + \sqrt{-\frac{\nabla^2}{4}} A_{\text{NLO}} \right\} \Psi(R) + \ldots$$  

(97a)
where, to condense the formulas, we use the notation:

\[ F_{\text{LO}} = F \left( -\Delta^2 + \left( \frac{\Delta \cdot \vec{\nabla}}{\vec{\nabla}_2} \right)^2 \right) \]  

(97b)

\[ F_{\text{NLO}} = \frac{1}{\vec{\nabla}_2^2} \left( -\frac{(\Delta \cdot \vec{\nabla})^2}{\vec{\nabla}_2^2} (4M^2 + \Delta^2) + \left( \frac{(\Delta \cdot \vec{\nabla})^2}{\vec{\nabla}_2^2} \right)^2 \right) F' \left( -\Delta^2 + \left( \frac{\Delta \cdot \vec{\nabla}}{\vec{\nabla}_2} \right)^2 \right) \]  

(97c)



for \( F = A, D \). The form factor \( D(t) \) depends on the internal dynamics of the system [1, 37, 40, 41], and is not an artifact of wave packet dependence. However, as we can see, the amount that \( D(t) \) contributes to the physical instant form density is an artifact of wave packet dependence. Assuming spherical symmetry and localizing the wave packet does not remove this dependence, since the disappearance of \( D(t) \) in this procedure is in fact an artifact of choosing a localized wave function. The claim of Refs. [14, 15] that localization removes information about the wave packet is thus false.

Since localizing wave packets in instant form coordinates does not produce either simple or compound local densities, the localization procedure should not be taken to return any kind of internal hadron structure. Moreover, we reiterate that the densities obtained through this procedure wash out important aspects of internal hadron dynamics that are of great interest. Contrasting to Sec. IV B, we find that even the Breit frame densities have a better claim to encoding internal hadron structure than the localized, Fleming-like densities. However, light front densities alone constitute simple or finitely compound densities that fully exhaust the true internal structure of hadrons.

\section{V. SUMMARY AND CONCLUSIONS}

In this work, we proposed and developed a formalism for identifying internal densities of hadrons in a fully wave packet independent manner. Physical densities are identified as matrix elements of local operators, which necessarily mix internal structure with wave packet dependence. When the dependence on internal structure and wave packet can be cleanly separated within a convolution relation—or the physical density can be written as a sum of such convolutions—an internal density can be defined.

When using light front coordinates with \( x^- \) integrated out, internal densities can always be identified, owing to invariance of the remaining coordinates under the Galilean subgroup. The scenario is thus analogous to the situation in non-relativistic quantum mechanics. For several simple cases such as the charge density and \( P^+ \) density, the physical density approaches the internal density when the wave packet is localized, explaining why previous derivations of these quantities through localized wave packets obtained the correct results.

By contrast, when using instant form coordinates, it is not possible in general to separate wave packet and internal structure dependence. A special case occurs when the hadron is prepared in a diffuse state (i.e., broad in coordinate space), in which case the physical density can be expanded as an infinite series of convolutions between internal densities and wave packet dependent smearing functions. For sufficiently broad wave packets, on the order of the Compton wavelength or wider, the physical density is dominated by a convolution between the barycentric probability density and the conventional Breit frame density. Thus, the Breit frame density can be identified as one of infinitely many internal densities that describe hadron structure in instant form coordinates, but one which dominates for diffuse wave packets. Breit frame densities therefore have a legitimate claim to describe an aspect of hadron structure, but they do not provide a complete description.

In contrast to the light front and non-relativistic cases, an internal density is not obtained in instant form coordinates when the wave packet is localized, as the density thus obtained retains wave packet dependence upon localization.

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