Lithium abundance is a gravitational model dependent quantity

Aneta Wojnar*1

1Laboratory of Theoretical Physics, Institute of Physics, University of Tartu, W. Ostwaldi 1, 50411 Tartu, Estonia

The dependence of lithium abundance on modified gravity in low-mass stellar objects is demonstrated. This may introduce an additional uncertainty to age determination techniques of young stars and globular clusters if they rely on the light element depletion method.

Many alternatives to General Relativity (GR) have been proposed in order to shed light on the dark energy and dark matter problems [1–4], the existence of space-time singularities [7], and the unification with the high energy physics [8, 9], between others. Issues related to astrophysical objects also contribute to the above list of current shortcomings of GR, among them for example the observations of neutron stars with masses of two solar ones [10, 12], of a compact object with mass 2.6M⊙ [13] sneaking out the mass bounds given by theoretical models for the heaviest neutron stars and the lightest black holes, and very recently, of a binary black hole merger with a total mass of 150M⊙ [14, 15].

Pre-main sequence low-mass stars \( (M \lesssim 0.5M_\odot) \) turn out to be interesting objects to study in the context of modified gravity - it was shown that minimum main sequence mass (MMSM) [16–19] as well as an upper mass’ limit of fully convective stars on the Main Sequence and Hayashi tracks [20] can be used to constrain theories of gravity. Moreover, during the Hayashi contraction phase those young stars fuse lithium \(^7\)Li which depletes before they reach the Main Sequence - that is, the temperature required for lithium burning is lower than MMSM. Therefore, the lithium line is not present in spectroscopically observed red dwarfs with masses \( M \lesssim 0.5M_\odot \) in contrast to brown dwarfs whose core temperatures do not reach \(~ 2.5 \times 10^6\)K required for lithium burning. This fact, called lithium test [21, 22], although not ideal, is used in order to distinguish brown dwarfs from Main Sequence stars in the case when one deals with very low-mass and cool stellar (and substellar) objects occupying overlapping regions of effective temperature and luminosity [23].

The lithium abundance at the photosphere in the pre-main sequence stars is an age-dependent quantity [24–27]. It allows to determine clusters’ age in the age range 20 – 200 Myr, being one of the most reliable method for young globular clusters’ age determination. It also means that the lithium depletion boundary method is usually applied to stars’ groups of similar age; however, it may provide limits on the ages of LMS individuals. More importantly, the procedure is employed to calibrate other techniques used for age estimation since the method is built on solid physical ground, with very few assumptions. Furthermore, the theoretical ages obtained from the lithium depletion method depend weakly on stellar compositions, which is why they do not provide observational uncertainties [28] related to, for instant, star’s metallicity. Together with keeping the effective temperature as a free parameter the technique allows to avoid further uncertainties related to atmosphere and convection models [29, 24].

Another prominent feature of the pre-main sequence low-mass stars, which we are going to use in this letter, is their theoretical description. Being fully convective, they can be modelled as a well-mixed polytrope with \( n = 3/2 \) even during the last stages of contraction, when electron degeneracy starts being important [29]. Due to the simplified relations, low-mass stars can be modelled by the non-relativistic hydrostatic equilibrium equation, which turn out to be altered by modified gravity (see [30] and references therein). That fact does only provide tests for modified gravity as already mentioned, but, what we would like to demonstrate in the following discussion, introduces a new uncertainty to the ages of young stars and globular clusters obtained from the lithium depletion boundary method.

In such stars, the lithium-to-hydrogen ratio \( f \) changes due to the effective convection which mixes lithium-poor and lithium-rich regions throughout a star such that the mixing timescale is much shorter than contraction and lithium destruction times (that is, the star is well mixed). Apart from this process, the proton-capture reactions also contribute to the rate of change of \( f \). Thus, the depletion rate can be written in the following way

\[
\frac{df}{dt} = -\frac{X_f}{m_H} \int_0^M \rho(\sigma v) dM,
\]

where the non-resonant reaction rate for the temperature range \( T < 6 \times 10^6 \)K is given by

\[
N_A(\sigma v) = S f_{sr} T_{c6}^{-2/3} \exp \left[ -a T_{c6}^{-4} \right] \frac{cm^3}{s g},
\]

where \( T_{c6} \equiv T_c/10^6 \)K and \( f_{sr} \) is the screening correction factor while \( S \) and \( a \) are dimensionless parameters in the fit to the reaction rate. For our range of temperatures, the proton-capture rate parameters for the reaction \(^7\)Li\((p,\alpha)\)\(^4\)He are \( S = 7.2 \times 10^{10} \) and \( a = 84.72 \) [27, 31, 32]. Since we are dealing with polytropic stars with the polytropic index \( n = 3/2 \), the temperature is \( T = T_c \theta(\xi) \) while density is expressed as \( \rho = \rho_c \theta^{3/2}(\xi) \). The function \( \theta(\xi) \) is the solution of the (modified) Lane-Emden equation with respect to the radial coordinate.
\[ \xi = r \rho_c \sqrt{\frac{8 \pi G}{2 \rho_c}} \] which crosses zero at \( \xi_R \). For the quadratic model in Palatini \( f(R) \) gravity the equation is written as

\[
\frac{1}{\xi} \frac{d^2}{d\xi^2} \left[ \sqrt{\Phi} \left( \theta - \frac{4 \kappa^2 \rho_c \alpha}{5} \theta^\frac{3}{2} \right) \right] = - \frac{(\Phi + \frac{1}{2} \xi \Phi')^2}{\sqrt{\Phi}} \theta^\frac{3}{2},
\]

with \( \Phi = 1 + 2 \alpha \theta^\frac{3}{2} \) and \( \alpha \) defined as \( \alpha = \kappa^2 \beta \rho_c \) with \( \kappa = -\frac{2}{\sqrt{6}} \). The parameters \( \rho_c \) and \( \rho_c \) stands for central density and central pressure, respectively.

The central temperature and density for that model are modified (via \( \delta, \xi_R \) and \( \theta' \)) and given by

\[
T_c = 1.15 \times 10^6 \left( \frac{\mu_{\text{eff}}}{0.6} \right) \left( \frac{M}{0.1M_\odot} \right) \left( \frac{R_\odot}{R} \right) \frac{\delta^\frac{3}{2}}{\xi_R^2(-\theta'(-\theta_R))^\frac{3}{2}} K \tag{4}
\]

\[
\rho_c = 0.141 \left( \frac{M}{0.1M_\odot} \right) \left( \frac{R}{R_\odot} \right)^3 \frac{\delta^\frac{5}{2}}{\text{cm}^3} \tag{5}
\]

while the radius, with taking into account an arbitrary degeneracy degree \( \eta \) and mean molecular weight \( \mu_{\text{eff}} \), is

\[
R \approx 7.1 \times 10^{-2} \gamma \left( \frac{\mu_{\text{eff}}}{0.1M_\odot} \right)^{\frac{1}{4}} \left( \frac{M}{M_\odot} \right)^{\frac{1}{4}} \tag{6}
\]

where \( F_\eta(\eta) \) is the \( \eta \)th order Fermi-Dirac function. Inserting the Lane-Emden temperature, energy density and radius to \( \xi \) and changing the variables to the spatial ones we will have

\[
\frac{d}{dt} \ln f = -\frac{4 \pi X \rho_c^2 R^3}{\xi_R^4 M N\Lambda M_H} \left( \frac{u}{a} \right)^2 \times \int_{\xi}^{\xi_n} f_{\text{sc}} \xi^2 \theta^\frac{3}{2} \exp(-u \theta^{-1/3}) d\xi \frac{1}{s}, \tag{7}
\]

where \( u \equiv a T_c^{-1/3} \). Approximately, the burning process is restricted to the central region of the star, thus we may apply the near center solution of the modified Lane-Emden equation \( \xi \) to the depletion rate \( \theta \)

\[
\theta(\xi \approx 0) \approx 1 - \frac{\xi^2}{6} \approx \exp \left( -\frac{\xi^2}{6} \right), \tag{8}
\]

which after applying the numerical constants yields

\[
\frac{d}{dt} \ln f \approx -6.54 \left( \frac{X}{0.7} \right) \left( \frac{0.6}{\mu_{\text{eff}}} \right)^3 \left( \frac{0.1M_\odot}{M} \right)^2 \times S f_{\text{sc}} a u^{-2} e^{-u} \left( 1 + \frac{7}{u} \right)^{\frac{3}{2}} \xi_R^2(-\theta'(-\theta_R)). \tag{9}
\]

The integration of the above equation requires the knowledge of the dependence of the central temperature parameter \( u \) on time. In order to find it, let us consider Stefan-Boltzman equation together with the virial theorem after the transformation to the Jordan frame

\[
L = 4\pi R^2 T^4_{\text{eff}} = -\frac{3}{2} \frac{GM^2}{R^2} \frac{dR}{dt}, \tag{10}
\]

where the factor

\[
\Omega = \left( \frac{\Phi^\frac{3}{2}}{1 + \frac{1}{2} \xi_R \Phi'} \right)^{-\frac{1}{4}} \tag{11}
\]

appears due to the frame transformation \( \xi \). Therefore, it is straightforward to get the radius and luminosity as functions of time during the contraction phase

\[
\frac{R}{R_\odot} = 0.85 \Omega \left( \frac{M}{0.1M_\odot} \right)^{\frac{1}{4}} \left( \frac{3000K}{T_{\text{eff}}} \right)^{\frac{1}{4}} \left( \frac{\text{Myr}}{t} \right)^{\frac{1}{4}} \tag{12}
\]

\[
\frac{L}{L_\odot} = 5.25 \times 10^{-2} \Omega \left( \frac{M}{0.1M_\odot} \right)^{\frac{1}{4}} \left( \frac{T_{\text{eff}}}{3000K} \right)^{\frac{1}{4}} \left( \frac{\text{Myr}}{t} \right)^{\frac{1}{4}}, \tag{13}
\]

while the contraction time is given by

\[
t_{\text{cont}} \equiv -\frac{R}{dR/dt} \approx 841.91 \left( \frac{3000K}{T_{\text{eff}}} \right)^{\frac{4}{4}} \left( \frac{0.1M_\odot}{M} \right) \times \left( \frac{0.6}{\mu_{\text{eff}}} \right)^3 \left( \frac{T_{\text{cont}}}{3 \times 10^6 K} \right)^{\frac{3}{2}} \left( e^\frac{R_\odot}{\xi_R^3(-\theta'(-\theta_R))} \Omega \right)^{\frac{1}{3}} \tag{14}
\]

From the relation \( [9] \) and \( [12] \) we may also write down the degeneracy parameter as a function of time

\[
\mu_{\text{eff}} F_{1/2}(\eta) \approx 8.36 \times 10^{-2} \gamma \left( \frac{0.1M_\odot}{M} \right) \left( \frac{T_{3\text{eff}}^4}{\mu_{\text{eff}}^2} \right)^{1/3} \tag{15}
\]

where \( T_{3\text{eff}} \equiv T_{\text{eff}}/3000K \) and \( t_0 \equiv t/10^6 \). Then, using \( \Omega \) together with \( [12] \) and \( [15] \) we find

\[
\frac{u}{a} = 1.15 \left( \frac{M}{0.1M_\odot} \right)^{2/9} \left( \frac{\mu_{\text{eff}} F_{1/2}(\eta)}{t_0 T_{3\text{eff}}^4} \right)^{2/9} \times \left( \frac{\xi_R^3(-\theta'(-\theta_R))}{\gamma^{2/3}} \right)^{1/3}, \tag{16}
\]

which relates the central temperature \( T_c \) with the time during the contraction phase.

Let us consider the case \( M \gtrsim 0.2M_\odot \), that is, when the degeneracy effects are not important and \( \mu_{\text{eff}} \) can be neglected when compared to \( \tilde{R} \). Then, since \( u = a T_c^{-1/3} \) and using \( [4] \) will provide \( du/dR = u/(3R) \), such that

\[
\frac{d}{dt} \ln f \approx \frac{dlnf}{du} \frac{du}{dR} \tilde{R} = \frac{dlnf}{du} \frac{3R}{3R} \tag{17}
\]

which allows to write the depletion rate as

\[
\frac{dlnf}{du} = 1.15 \times 10^{13} T_{3\text{eff}}^{-4} \left( \frac{X}{0.7} \right) \left( \frac{0.6}{\mu_{\text{eff}}} \right) \left( \frac{M_\odot}{M} \right)^3 \times S f_{\text{sc}} a^{16} u^{-2} e^{-u} \left( 1 - \frac{21}{2u} \right) \xi_R^2(-\theta'(-\theta_R))^2 \Omega \tag{18}
\]
Integrating the above equation from \( u_0 = \infty \) to \( u \) and using the properties of the incomplete gamma function gives

\[
F \equiv \ln \frac{f_0}{f} = 1.15 \times 10^{13} T_{\text{eff}}^{-4} \left( \frac{X}{0.7} \right) \left( \frac{0.6}{\mu_{\text{eff}}} \right)^6 \left( \frac{M}{M_\odot} \right)^3 \times Sf_{\text{scr}} a^{16} g(u) \frac{\xi_R^4 (-\theta'(|\xi_R|)^2 \Omega)}{\delta^2},
\]

(19)

where \( g(u) = u^{-37/2} e^{-u} - 29\Gamma(-37/2, u) \) with the function \( \Gamma(-37/2, u) \) being an upper incomplete gamma function. For a given depletion \( F \), the central temperature \( T_c \) is obtained from \( u(F) \) while the star’s age, radius, and luminosity are specified by the equations [13, 12] and [18] respectively. Each of those, as demonstrated, is altered by \( \Omega, \gamma, \delta, \xi_R, \theta(\xi_R), \) and \( \theta'(\xi_R) \) whose values depend on a model of gravity.

Using the similar approach, one may write down the depletion equation [19] for resonant rates (see e.g. [22]), where \( j = 2/3 \) stands for the non-resonant reactions,

\[
F \equiv \ln \frac{f_0}{f} = 1.15 \times 10^{13} T_{\text{eff}}^{-4} \left( \frac{X}{0.7} \right) \left( \frac{0.6}{\mu_{\text{eff}}} \right)^6 \left( \frac{M}{M_\odot} \right)^3 \times Sf_{\text{scr}} a^{18-3j} g(u) \frac{\xi_R^4 (-\theta'(|\xi_R|)^2 \Omega)}{\delta^2},
\]

(20)

with \( g(u) = u^{-41/2-3j} e^{-u} - \frac{68-15j}{2}\Gamma(-41/2, u) \).

The equations [19] and [20] can be solved numerically or be fitted to the observational data; however, we may also find an approximate formula for the central temperature at the time of depletion. Thus, we will compare the local nuclear destruction time at the center of the star (\( X = 0.7 \) being the hydrogen mass fraction while \( m_p = 1.67 \times 10^{-24} \text{g} \) is the proton mass)

\[
t_{\text{dest}} = \frac{m_p}{X p(\sigma v)} = 4.92 \times 10^{-7} \left( \frac{M}{0.1M_\odot} \right) \times \left( \frac{\mu_{\text{eff}}}{0.6} \right)^3 T_{\text{eff}}^{-2} \times \frac{\mu_{\text{eff}}}{Sf_{\text{scr}}} \frac{\varepsilon_{\beta\rho}^{\beta\rho}}{\xi_R} \frac{\delta}{\xi_R} \frac{\theta'(|\xi_R|)}{\theta'|\xi_R|} \text{yr}
\]

(21)

to the contraction time [14].

The approximation \( t_{\text{cont}} = t_{\text{dest}} \) works well so long as the star can be described by the polytropic equation of state with \( n = 3/2 \) (the degeneracy is not important):

\[
\frac{\alpha}{T_{c6}^{1/3}} = 31.78 + \ln(Sf_{\text{scr}}) + \ln \left( \frac{\xi_R^4 \theta'(|\xi_R|)^2 \Omega}{\delta^3} \right) - 6 \ln \left( \frac{\mu_{\text{eff}}}{0.6} \right) - 3 \ln \left( \frac{M}{0.1M_\odot} \right) - 4 \ln \left( \frac{T_{\text{eff}}}{3000 \text{K}} \right) + \frac{16}{3} \ln T_{c6}.
\]

(22)

Let us now consider a star with \( T_{\text{eff}} = 3500 \text{K} \), the mass \( M = 0.5M_\odot \) with \( f_{\text{scr}} = 1 \) (evaluated at the center of the star) and \( \mu_{\text{eff}} = 0.6 \). For the GR values (\( \alpha = 0 \)) of the polytropic solutions \( ^7\text{Li} \) depletes when the central temperature is \( T_c \approx 2.98 \times 10^9 \text{K} \). A few other values of the parameter \( \alpha = \kappa c^2 \beta \rho_c \) and their corresponding central temperatures, ages, radii, and luminosities are given in the table I.

The obtained results clearly demonstrate that modified gravity (here Palatini quadratic model) significantly changes the ages and luminosities of lithium depleted main sequence stars with respect to the GR model. Despite the fact that the values from the table I are given by the approximated expression [22], the deviations from the GR model with \( \alpha = 0 \) in [21] are also expected to occur.

Although it seems to be worrying that the lithium depletion based techniques for the age estimation depend on a model of gravity, it also gives a room for gravitational theories whose modifications shorten any phase of the stellar evolution, as provided by the considered Palatini quadratic model. The discovery of a 0.2M⊙ white dwarf in the binary system KIC 8145411 [36] which according to the commonly accepted model would have to be older then Universe [27], is a clear example of a need of different evolutionary scenarios. It was shown [29] that white dwarfs are also found in young clusters whose progenitor stars’ masses depend crucially on the assumed age of the cluster, which is another argument for being aware of the discussed dependence when the lithium based method is used.

Moreover, staying shorter (longer) in any evolutionary phase has a noticeable effect on the total stars’ luminosity which contributes to the galaxy brightness [35] since the galaxy can have more (less) generations of stars with different luminosities than the ones predicted by the GR model.

In addition, the results discussed in this letter may also provide a test for gravitational theories: prolonging prominently low-mass stars’ lifetimes in comparison to the current widely accepted model would raise doubts on a theory which introduces such effects.

| \( \alpha \) | \( T_{c}/10^9 \text{K} \) | \( \tau_{\text{eff}}/\text{Myr} \) | \( R/R_\odot \) | \( L/L_\odot \) |
|---|---|---|---|---|
| -0.4 | 3.48 | 3.21 | 1.85 | 25.3×10^{-2} |
| -0.1 | 3.18 | 7.48 | 1.28 | 14.4×10^{-2} |
| -0.001 | 3.129 | 7.76 | 1.19 | 14.1×10^{-2} |
| 0 (GR) | 2.98 | 12.42 | 1.03 | 10.3×10^{-2} |
| 0.001 | 3.128 | 7.78 | 1.19 | 14×10^{-2} |
| 0.1 | 3.098 | 7.25 | 1.13 | 14.7×10^{-2} |
| 0.4 | 3.093 | 3.57 | 1.06 | 23.6×10^{-2} |

Table I. Numerical values of central temperatures (in 10^9K), age (in Myr), radius (in R_\odot), and luminosity (in L_\odot) of fully convective low-mass stars with respect to \( \alpha = \kappa c^2 \beta \rho_c \) at the time of \(^7\text{Li} \) depletion. The star’s mass, effective temperature, hydrogen mass fraction, and mean molecular weight are \( M = 0.5M_\odot \), \( T_{\text{eff}} = 3500 \text{K}, X = 0.7, \) and \( \mu_{\text{eff}} = 0.6 \), respectively.

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