Euclidean 2D Gravity with Torsion

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Abstract

Closing a gap in the literature on the subject, the local solutions of 2D-gravity with torsion are given for Euclidean signature. For the topology of a cylinder the system is quantized in a Dirac approach.

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The main reason to consider \([1]\)
\[
\mathcal{L} = e \left( -\frac{\gamma}{4} R^2 + \frac{\beta}{2} T^2 - \lambda \right)
\]  
(1)
was to provide a 'kinetic term' for the gravitational sector of 2D string theory. Whereas, however, for Lorentz signature of the world sheet metric \(g_{\mu\nu}\) all the classical and quantum solutions to \([1]\) have been found (cf. \([2]\), \([3]\), \([4]\), \([5]\), \([6]\), \([7]\), \([8]\), and references therein), the corresponding solutions for Euclidean signature have been still missing up to now. It is the incentive of this letter to fill this gap.

The basic quantities in \([1]\) are the orthonormal one–form \(e^a\) and and the SO(2) connection \(\omega\); from them one obtains the Ricci scalar \(R = 2 \ast d\omega\), the 'torsion function' \(T^a = \ast De^a\), and \(e \equiv \det e^a_{\mu a}\). The first order form of \([1]\) is
\[
\mathcal{L}_H = e \left( \frac{\pi_2}{2} R + \pi_a T^a + E \right)
\]  
(2)
with
\[
E \equiv \frac{1}{4\gamma} (\pi_2)^2 - \frac{1}{2\beta} \pi^2 - \lambda,
\]  
(3)
from which one infers the following canonical structure: The phase space is spanned by the canonically conjugates \((e^a_1, \omega_1; \pi_a, \pi_2)\) subject to the first class constraints
\[
G_a = \partial \pi_a + \varepsilon_{ab} E e^b_1 + \varepsilon_{ab} \pi_b \omega_1 \approx 0
\]  
(4a)
\[
G_2 = \partial \pi_2 + \varepsilon_{ab} \pi_a e^b_1 \approx 0.
\]  
(4b)
The Hamiltonian is
\[
\mathcal{H} = -e_0^a G_a - \omega_0 G_2 + \partial (e_0^a \pi_a + \omega_0 \pi_2),
\]  
(5)
and \(e_0^a\) and \(\omega_0\) play the role of Lagrange multipliers.

To have a well-behaved Hamiltonian formulation (ensuring e. g. that the reduced phase space is even dimensional, a necessary condition for a symplectic structure on it) one needs also specific boundary conditions. One possibility is to compactify the \(x^1\)-direction (topology of a cylinder), which is implemented by means of periodic boundary conditions in the \(x^1\)-variable. This option will be chosen when quantizing the model. In the first part of this letter, however, we will be interested only in local classical Euclidean solutions of \([1]\), postponing a more elaborate global analysis to some later work. The boundaries need less careful treatment then and the gauge freedom present locally (i.e. in some surrounding of each point) is usually larger than the one present globally, since in the latter case some specific boundary conditions must not be violated by the gauge transformations.

\[1\] In \([1]\) the second term of \([1]\) is rather \(-e^\beta T^{\mu\nu}T_{\mu\nu} = -e(-)^M \beta T^2\) with \(\det g_{ab} = (-)^M\). We changed the sign of \(\beta\) in the Euclidean case \(M = 0\), since only then the Hamiltonian formulations differ just by the different tangent space metric \(g_{ab} = diag(1, (-)^M)\) and can be compared more easily.
To better understand this last point as well as to get some feeling for the 'physical' (i.e. gauge independent) content of the theory, let us have recourse to the canonical abelianization of the Minkowskian version of the model as found in [7]. Introducing
\[ \pi_{\pm} = \frac{1}{\sqrt{2}} (\pi_0 \pm i \pi_1) \]  
(6)
eq 0, use either of the sets of new canonical variables
\[ \left( \frac{\pm i G_\pm}{\pi_{\pm}}, \mp i \frac{G_2}{\pi_{\pm}}, Q; \pi_2, \pi_{\pm}, P^{(\pm)} \right), \]  
(7)
and
\[ Q = -\beta \exp\left(\frac{\pi^2}{\beta}\right) \left[ E - \frac{\beta}{2\gamma} \pi_2 + \frac{\beta^2}{2\gamma} \right], \quad P^{(\pm)} = -\exp\left(-\frac{\pi_{2}}{\beta}\right) \frac{\varepsilon_1^{\mp}}{\pi_{\pm}}. \]  
(8)
The factors \( i \) in (7) come about since here \( \varepsilon_{+-} = -i \) when \( \varepsilon_{01} = 1 \). Since
\[ \partial Q = \exp\left(\frac{\pi^2}{\beta}\right) [\pi^a G_a - EG_2] \approx 0, \]  
(9)
all of the new generalized coordinates are (strongly commuting) first class constraints except for the constant part \( Q_0 \) of \( Q(x^1) \), which is a Dirac observable. With regard to the gauge independent portion of the corresponding momenta, however, the above mentioned dependence on the boundaries appears: Whereas locally \( \partial Q \) can shift all of \( P^{(\pm)} \) [locally any (non-diverging) function has an integral], periodic boundary conditions obviously make the (otherwise ill-defined) zero mode \( P_0^{(\pm)} \) insensitive to gauge transformations.

Let us dwell some more on the latter case of a compact \( x^1 \)-direction. Due to the absence of surface contributions as well as \( Q(x^1) \approx Q_0 = \text{const} \) one verifies \( P_0^{(+)} \approx P_0^{(-)} \), so that we can also use
\[ P_0 := \frac{1}{2} (P_0^{(+)} + P_0^{(-)}) = -\oint_{S^1} \exp\left(-\frac{\pi_2}{\beta}\right) \frac{\pi^a \varepsilon_1^{+}}{\pi_{\pm}} \, dx^1 \]  
(10)
as the momentum conjugate to \( Q_0 \). \( P_0 \) has the advantage of being already real such as \( Q_0 \). Thus it is these two coordinates spanning the reduced phase space (RPS) of Euclidean 2D gravity with torsion on the cylinder, although only locally (in the RPS) since we operated out points with \( \pi^2 \neq 0 \) to obtain the simple canonical splitting in (7). To interpret \( P_0 \) let us focus on such global solutions which allow for closed curves \( C \) of constant curvature \( \pi_2 \) (which is at least the case when \( \pi^2 \neq 0 \) everywhere on the space–time manifold, as becomes clear from the solutions below): Then one can check that \( P_0 \mid_C = f_C \sqrt{g_{11}} \, dx^1 \). That is to say, the quantity \( P_0 \) is a measure for the metric

\footnote{The normalization of \( Q \) is different to the one in [9], [10], where it was chosen so as to coincide with the constant \( A \) in Katanaev’s work. Here we multiplied it by \( (\beta^3/4\gamma) \) so that it is better behaved under contractions of [11] to \( R^2 \)-Gravity, the Jackiw–Teitelboim model, and other models, as reported elsewhere [9].}
determined circumference of the cylinder; as such it is a gauge independent quantity, but intrinsically global.

We could proceed using some complexification argument to obtain the classical Euclidean solutions from the already known Minkowskian ones \[2\], \[3\]. But this shall be postponed to an appendix for the following reason: Within the conformal gauge \[2\], for which a 'Wick rotation' poses no problem, the Minkowskian solutions have been found only up to a first order differential equation. Moreover, lines with \( \pi^2 = 0 \) could be covered only by cumbersome glueing of the local solutions (valid for \( \pi^2 \neq 0 \)) via geodetic completing \[10\]; the \( C^\infty \)-structure of the resulting space–time manifolds \( \mathcal{M} \) remained unproven in this way. Within the 'light cone gauge' \( e_0^+ = 1, e_0^- = \omega_0 = 0 \) \[3\], on the other hand, a simple algebraic solution was available for \( \pi_+ \neq 0 \), but also in the vicinity of points with \( \pi_a = 0 \) \[7\]. Solutions such as the latter allowed to complete the proof of the \( C^\infty \)-structure of \( \mathcal{M} \) \[11\]. In the case of the light cone gauge, however, a continuation to the Euclidean solutions is more complicated as one might think at first sight (cf. Appendix). Since a direct Hamiltonian calculation of the Euclidean solutions is straightforward and yields also algebraic solutions, we will present this possibility in the following.

To find the classical solutions within the Hamiltonian framework there are basically two (equivalent) procedures: The first more traditional one, which is also closer related to the Lagrangian formulation, fixes the Hamiltonian by means of a specific choice for the Lagrange multipliers. The flow of \( \mathcal{H} \) on the constraint surface yields then the \( x^0 \)-dependence of the basic fields, and the still remaining gauge equivalence of the 'initial data' is factored out in a final step. The alternative procedure, the advantages of which we pointed out already in \[7\], starts from a gauge fixation of the constraints within the phase space; identifying then a gauge fixing parameter with the evolution parameter \( x^0 \) allows [for the case of independent (effective) constraints] to determine the Lagrange multipliers by purely algebraic means. Since the constraints are not completely independent in our case\[1\] there remains one ordinary differential equation in \( x^1 \) for the Lagrange multipliers yielding an integration 'constant' \( f(x^0) \), which has to be gauge fixed as well. As a last step one then has to integrate the flow of \( \mathcal{H} \) for the remaining phase space variables. It is the second method which turns out to be so powerful in our context.

First we have to find good gauge conditions. In a region where the curvature is non-constant and has also no extremal point, slices of constant \( \pi_2 \) provide a good (local) foliation. (Note that according to \( \{2\} \), \( \{3\} \) on–shell \( \pi_2 = -\gamma R, \pi_a = \beta T_a \). Further, by SO(2) rotations of the zweibein frame we can always manage to set \( \pi_0 = 0 \); and for the case that \( \pi_1 \neq 0 \) this can be made a perfect cross section to the frame rotations, when eliminating the Gribov ambiguity \( \pi_1 \rightarrow -\pi_1 \) by the requirement \( \pi_1 > 0 \). The remaining gauge freedom \( x^1 \rightarrow a(x^\mu) \) can be used to obtain \( e_1^1 = 1 \), since the differential equation \( e_1^1(x^0, a(x^\mu))\partial_1 a(x^\mu) = 1 \) is known to always have a solution for \( a \) locally. For the case of a closed \( x^1 \) direction, however, it is at this point where one could obtain only \( \partial_e e_1^1 = 0 \),

\[3\]

Integrating \( \{3\} \) over \( S^1 \) reveals that the constraints are dependent globally; dropping possible but ill-defined boundary contributions when calculating the flow of the Hamiltonian (there appear \( \partial_\delta \)–terms to be integrated by parts) this dependence becomes visible also on a local level.
since the zero mode $\oint e_1^1 dx^1$ is diffeomorphism invariant. So let us start with
\[ \pi_0 = 0, \quad \pi_2 = x^0, \quad e_1^1 = 1. \] (11)
Note that there still remains the gauge freedom $x^1 \to x^1 + F(x^0)$, which we will make use of below when determining the Lagrange multipliers.

From the 'Faddeev-Popov matrix'
\[
\begin{pmatrix}
\{\pi_2, G_0\} & \{e_1^1, G_0\} & \{\pi_0, G_0\} \\
\{\pi_2, G_1\} & \{e_1^1, G_1\} & \{\pi_0, G_1\} \\
\{\pi_2, G_2\} & \{e_1^1, G_2\} & \{\pi_0, G_2\}
\end{pmatrix}
= \begin{pmatrix}
-\pi_1 \delta & 0 & 0 \\
\ast & -\partial \delta & 0 \\
0 & \ast & \pi_1 \delta
\end{pmatrix},
\]
(12)
in which we used the constraints as well as the gauge conditions (11), we learn that our gauge breaks down only when $\pi_1$ becomes zero, i.e. on-shell there is no need for further restrictions on the curvature.\(^4\) Indeed, a (gauge independent) analysis of the constraints and the flow of the Hamiltonian, or, equivalently, the covariant form of the field equations (cf. e.g. [5])
\[ \pi_{a;b} = -\varepsilon_{ab} E, \quad \pi_{2;a} = \varepsilon_{ab} \pi^b, \] (13)
reveal that points with $\partial_\mu \pi_2 = 0$ are always points with vanishing torsion $\pi_a = 0$. Furthermore, all solutions to the field equations (13) with constant curvature are $\pi_a \equiv 0$, $\pi_2 \equiv \pm \sqrt{4 \gamma \lambda}$. These describe a general two dimensional Euclidean (Anti-)de-Sitter space. Being well-known already, we will not treat these solutions here further. The analysis of the solutions around extremal points of $\pi_2$, on the other hand, shall be taken up after exploring the gauge (11).

Inverting $G_0 = 0$ and $G_2 = 0$ in the above gauge (11), one obtains
\[ \omega_1 = -\frac{E}{\pi_1}, \quad e_1^0 = 0, \] (14)
respectively. Due to (9) we can use the first equation of (8) instead of $G_1 = 0$, yielding
\[ \pi_1 = \sqrt{W(x^0)}, \quad W(x^0) \equiv 2Q_0 \exp\left(-\frac{x^0}{\beta}\right) + \frac{\beta}{2\gamma} (x^0)^2 - \frac{\beta^2}{\gamma} (x^0 - \beta) - 2\beta \lambda, \] (15)
in which $Q_0$ plays the role of an integration constant (due to (3) $Q_0$, being a Dirac observable, is also a constant of the motion). The 'Lagrange multipliers' are now obtained by differentiating (11) with respect to $x^0$ and requiring that $\partial_0$ is generated by the Hamiltonian. This yields two algebraic and one ordinary differential equation in $x^1$ for the Lagrange multipliers, the latter giving rise to an undetermined function $f(x^0)$:
\[ e_0^0 = \frac{1}{\pi_1}, \quad e_0^1 = -\left(\frac{E}{(\pi_1)^2} + \frac{1}{\beta}\right)x^1 + f(x^0), \quad \omega_0 = -\frac{e_0^1 E}{\pi_1}. \] (16)

\(^4\)At first sight one could think $\det(12) = 0$ since $Ker \partial \delta \neq 0$ on $C^\infty(R)$ and thus $\partial \delta$ is not invertible. Note, however, that $Im \partial \delta = C^\infty(R)$. This is the relevant condition for the attainability of the gauge. So, calculating the Faddeev-Popov determinant, we should regard $\partial \delta$ as an operator $\partial \delta : C^\infty(R) / Ker \partial \delta \to C^\infty(R)$ rather than as an operator $\partial \delta : C^\infty(R) \to C^\infty(R)$. This subtlety arises, as we did not specify boundary conditions.
It is straightforward to check that $f$ can be made to vanish by means of the residual
gauge freedom $x^1 \rightarrow x^1 + F(x^0)$ when $F$ obeys

$$
\dot{F} - \left( \frac{E}{\beta^2} + \frac{1}{\beta} \right) F(x^0) = f(x^0);
$$

or even simpler by observing that due to (4) a shift in $f$ does not change the Hamiltonian (up to boundary terms). Thus around points on the space-time manifold with nonvanishing torsion the geometrical quantities can be always brought into the form (11, 14 - 16) with $f \equiv 0$.

Let us analyse the solutions in the neighbourhood of points with vanishing torsion, which according to (13) are also extremal points of the curvature. Since it is instructive, we will do this simultaneously for both signatures $(-)^M$. Due to the choice of signs in (4) the field equations (13) are valid also for $M = 1$; beside the different tangent space metric the difference of the signature enters only in the definition of the momenta:

$$
\pi_a = (-)^M \beta T_a, \quad \pi_2 = (-)^{M+1} \gamma R.
$$

Differentiating the second equation of (13) covariantly and then using the first one, one obtains

$$
\pi_{2,a;b} = (-)^M g_{ab} E. \tag{18}
$$

Since at extremal points of $\pi_2$ the covariant derivative at the lefthand side of (18) can be replaced by a normal one, and since at a point of vanishing torsion $E = 0$ only in the (Anti-)de-Sitter case of constant curvature, this equation tells us the following: whereas for Minkowskian signature extremal points of the curvature are always saddle points (cf. also [7]), in the Euclidean case they are true maxima or minima. This allows us to choose a frame and coordinates such that

$$
\pi_0 = x^0, \quad \pi_1 = x^1. \tag{19}
$$

Note that according to (8) lines of constant curvature are also lines of constant $\pi^2 \equiv \pi_0^2 + (-)^M \pi_1^2$, so that (19) captures exactly the different character of the extremal points of $\pi_2$ for each signature. As the third gauge condition we require

$$
\omega_1 = 0. \tag{20}
$$

To see that this gauge is attainable on-shell, one first calculates the corresponding Faddeev-Popov determinant. One can verify then that the determinant does not vanish at the origin since $E(x^\mu = 0) \neq 0$; due to the continuity of the solutions this implies that the determinant does not vanish in a neighborhood of the considered point $\pi_n = 0$. Finally one has to check that the resulting solutions satisfy $e \neq 0$, which is indeed the case here. The remaining steps to find the solutions are the same as before: The inversion of the constraints yields

$$
e^1_1 = 0, \quad e^0_1 = \frac{1}{E}, \tag{21}
$$

and $\pi_2$ is determined implicitly through (8) (with $Q = Q_0 = \text{const}$); note that there can exist several branches of $\pi_2$ (or even none), depending on the value of $Q_0$. The equations
for the Lagrange multipliers give:

\[ e_0^0 = -\frac{\omega_0 x^0}{E}, \quad e_0^1 = \frac{(-)^{M+1} \omega_0 x^1 - 1}{E}, \quad \partial \omega_0 = \frac{1}{2\gamma} e \pi_2. \] (22)

The last first order differential equation cannot be solved by means of elementary functions, since \( \pi_2 \) cannot be inverted analytically from (8). Nevertheless, the existence theorems for such differential equations guarantee us the \( C^\infty \)-structure of the resulting solutions around points of vanishing torsion.

Although we do not treat all the global aspects of the classical solutions to (1) within this letter, the above analysis allows an interesting conclusion about a difference between global Minkowskian and global Euclidean solutions. For Minkowskian signature the only maximally extended solutions with nowhere diverging curvature (and torsion) are the de-Sitter and Anti-de-Sitter space \([10]\). For Euclidean signature of (1), however, there exist additional globally complete solutions with everywhere bounded curvature and torsion. As one learns from (8), depending on \( Q_0 \) there exist 0, 1, 2, or 3 values of \( \pi_2 \) for which \( \pi_2 \) vanishes. Let us choose \( Q_0 \) such that there exist at least two such values \( B_1 \) and \( B_2 \) of \( \pi_2 \). If one starts from a point of the space time manifold with \( B_1 < \pi_2 < B_2 \), one can take the fields in the form of (11, 14 - 16). This solution extends to a region \( \pi_2 = x^0 \in [B_1, B_2] \) since within this region \( \pi_2 \neq 0 \). At \( \pi_2 = B_1 \), however, \( \pi_2 = 0 \) and contrary to the Minkowskian signature this also implies \( \pi_a = 0 \). According to our considerations following equation (18) this implies that \( \pi_2 = B_1 \) is a minimum of the curvature and \( \pi_2 = B_2 \) a maximum. Thus in the Euclidean case there exist global solutions with oscillating curvature and torsion.

This completes the present analysis of the Euclidean solutions to (1). However, let us by means of the acquired knowledge about the solutions of the field equations test our gauge fixing procedure to find all the classical solutions. From (12) we see no reason (as far as \( \pi_1 \neq 0 \)) to not choose e. g.

\[ \pi_2 = \text{const}, \quad \pi_0 = 0, \quad e_1^1 = k(x^\mu) \] (23)

with some function \( k \) fulfilling \( \dot{k} \neq 0 \) as our gauge fixing conditions. From the resulting space-time manifolds, on the other hand, we know that with this gauge choice we either are on a de-Sitter space or we are stuck to a line of constant curvature. The framework passes this test: For constant \( \pi_2 \) the \( G_2 = 0 \) constraint as well as \( \partial_0 \pi_2 = 0 \) yield, respectively

\[ \pi_0 e_1^1 - \pi_1 e_1^0 = 0 \]
\[ \pi_0 e_0^1 - \pi_1 e_0^0 = 0. \]

From this we learn that either \( \pi_a = 0 \), in which case these two equations are empty and a further analysis, equivalent to the use of (13), leads to the above mentioned (Anti-)de-Sitter solution, or

\[ \partial_0 \propto \partial_1. \] (24)
telling us that the flow of the Hamiltonian indeed does not leave the line with the initial values. Since (24) implies $e \equiv 0$ of the resulting solutions one is warned automatically that something is wrong with the ’gauge’ (23).

Let us turn to the Dirac quantization of the Euclidean form of (1). For this purpose we choose a momentum representation for the wave functionals and the operator ordering within the quantum version of the constraints (4) such that all functional derivatives have been put to the right; the constraint algebra has no anomalies then since it still satisfies

$$[G_a, G_2] = \varepsilon_{ab} G_b \delta$$

$$[G_a, G_b] = \varepsilon_{ab} (-\frac{1}{\beta} \pi_c G_c + \frac{1}{2\gamma} \pi_2 G_2) \delta.$$  

To find the physical wave functionals, which are defined as the kernel of the quantum constraints, we can again have recourse to [7], where we obtained the physical wave functionals of the Minkowskian theory. Making use of (6), we only have to substitute $i \bar{\hbar}$ by $\bar{\hbar}$ due to the imaginary $\varepsilon_{\pm \pm}$. Then we know from there that except for distributional functionals located at $\pi_a = 0$, the kernel of the constraints is given by the functionals

$$\Psi = \exp(\pm \frac{1}{\hbar} \oint \ln \pi_\pm d\pi_2) \bar{\Psi}[Q], \quad \partial Q \bar{\Psi}[Q] = 0.$$  

The second equation is a restriction on the support of $\bar{\Psi}[Q]$, due to which the first equation holds for either of the two signs with the same $\bar{\Psi}[Q] —$ as far as one has compactified the $x^1$–direction. In the Dirac quantization it is at this point where the boundary conditions are to be specified consistently. Since $\frac{1}{2} \ln(\frac{\pi_+}{\pi_-}) = i \arctan(\frac{\pi_1}{\pi_0})$ the Euclidean physical wavefunctionals can be rewritten as

$$\Psi = \exp(\frac{i}{\hbar} \int_{S^1} \Theta \partial \pi_2 dx^1) \bar{\Psi}[Q], \quad \partial Q \Psi = 0,$$

when using the polar coordinates $(\pi^2, \Theta)$ instead of $(\pi_0, \pi_1)$. Obviously the set of physical wave functionals emerges as equivalent to the set of ordinary functions $\bar{\Psi}(Q_0)$, as one would expect from (10) or [7]. To find the appropriate inner product, we require the Dirac observables $Q_0$ and $P_0$ to become hermitian. Substituting $e_{1a}$ by $i \hbar(\delta/\delta \pi_a)$ in (10) and applying this operator to (26) one finds that $P_0$ acts as $-i \hbar(d/dQ_0)$ on $\bar{\Psi}(Q_0)$. Therefore the measure within the inner product has to be the Lebesgue measure,

$$\langle \Psi, \Phi \rangle = \int dQ_0 \bar{\Psi}(Q_0)^* \hat{\Phi}(Q_0),$$

so that we end up with an $L^2(R)$ as in the Minkowskian case. Since to treat the ’issue of time’ within a Euclidean theory seems somewhat artificial, we will skip this point here. (The reader interested in this topic — there is no meaningful Schroedinger equation since the Hamiltonian vanishes on (27) — shall be refered to the Minkowskian counterpart in [7].)

As a next step one should extend the present considerations to the case of a string action coupled to (1). For the Minkowskian signature this has been done quite recently by Katanaev [12].
A Appendix

In general, gauge conditions on metric and connection, which can be realized by diffeomorphisms and local Lorentz transformations on a space with Minkowski type metric are not available in a Euclidean space. Complexifying the tangent bundle over the space time manifold, it may, although, be possible to apply such gauges to the Euclidean case.

The method is well established for the gauge condition \( g_{00} = g_{11} = 0 \). Given any metric there will always exist two lightlike independent vector fields in the complexified tangent bundle. We may then find two (in the Euclidean case complex) functions \( z_0, z_1 \) such that each of them is constant in one of the lightlike directions and the exterior derivatives of these functions form a basis in the complexified cotangent bundle. In this basis the metric obviously has the form \( g = f dz_0 \otimes dz_1 \) with a complex coefficient function \( f \). As \( g \) is real we have \( g = \overline{g} = \overline{f} d\bar{z}_0 d\bar{z}_1 \). With the Ansatz \( d\bar{z}_0 = adz_0 + bdz_1, \) \( d\bar{z}_1 = cdz_0 + edz_1 \) on deduces easily that either \( b = c = 0 \) or \( a = e = 0 \). The first case corresponds to the Minkowski signature. In the second case we have \( dz_1 = (1/b)dz_0 \).

From \( d^2 = 0 \) we conclude that \( b \) depends on \( \bar{z}_0 \) only. After calculating \( b \) from the conditions \( g = \bar{g} \) and \( \bar{z}_i = z_i \) we can express \( z_1 \) as a function of \( \bar{z}_0 \) and then return to real coordinates \( x_0 = z_0 + \bar{z}_0 \) and \( x_1 = i(\bar{z}_0 - z_0) \).

The aim of this appendix is, to apply this method to the local solutions of the equations of motion of the Lagrangian (1) which were calculated in [3] in a gauge characterized by the conditions \( e_0^+ = 1, e_0^- = 0, \omega_0 = 0 \) (light cone gauge). It is well known that this gauge is always obtainable for a Minkowski-type metric.

In a Euclidean space, however, we need complex coordinates as well as a complex local rotation to achieve this gauge. With some additional gauge fixing we then find the general solution in the neighborhood of a point, where \( \pi^2 \neq 0 \) to be determined by a real constant \( Q_0 \):

\[
\begin{align*}
\pi_\omega &= x^0, & \pi_+ &= i & \pi_- &= -i/2 W(x^0) \\
\omega_1 &= -e_1^- \pi_- & e_1^+ &= -ie_1^- \pi_- \\
e_1^- &= \exp(x^0/\beta) & \quad (A1)
\end{align*}
\]

The quantity \( W(x^0) \) was defined in (15).

As \( \pi_2 \) is a function on the space time manifold, we have \( \bar{\pi}_2 = \pi_2 \) and thus \( x^0 \) turns out to be real. This also guarantees \( \pi^2 \) to be real. With the ansatz

\[
dz^1 = adx^0 + bdz^1 \quad (A2)
\]

in \( g = \bar{g} \), where the metric \( g \) is given by the expression

\[
g = 2 \exp(x^0/\beta) dx^0 dz^1 - 2i \exp(2x^0/\beta) \pi_- dz^1 dz^1, \quad (A3)
\]

we obtain a set of equations for \( a \) and \( b \):

\[
Aa^2 + a = 0 \quad 2Aab + b - 1 = 0 \quad Ab^2 - A = 0 \quad (A4)
\]
where \( A \equiv -i \exp(x^0/\beta)\pi_- \). They are simultaneously solved by

\[
a = -\frac{1}{A}, \quad b = -1
\]  

(A second solution \( a = 0, b = 1 \) corresponds to the Minkowski space). As a check of consistency one may verify that \( d^2 \bar{z}^1 = 0 \) and \( d\bar{z}^1 = dz^1 \). We may now introduce the real coordinate \( x^1 = i(\bar{z}^1 - z^1) \) and express \( z^1 \) in terms of \( x^0 \) and \( x^1 \):

\[
z^1 = \frac{1}{2}(ix^1 - \int \frac{1}{A} dx^0)
\]  

Note that the solution (A1) does not depend on \( z^1 \). Thus we need not solve the integral in (A6). To calculate the Euclidean solutions it is sufficient to know \( dz^1 \) as a linear combination of the \( dx^i \).

But still the components of the torsion \( \pi_0 = \frac{1}{\sqrt{2}}(\pi_+ + \pi_-) \) and \( \pi_1 = \frac{-i}{\sqrt{2}}(\pi_+ - \pi_-) \) as well as the connection \( \omega \) are not real. This is not surprising, as we had to apply a complex local rotation to achieve the light cone gauge. It is easy to verify that \( \pi_0, \pi_1, \) and \( \omega \) become real by the transformation

\[
\pi_+ \rightarrow \alpha \pi_+ \quad \pi_- \rightarrow \frac{1}{\alpha} \pi_- \quad \omega \rightarrow \omega + id \ln \alpha \quad \alpha = \sqrt{\frac{W}{2}}
\]  

The construction of the Euclidean solution is thus complete:

\[
\begin{align*}
\pi_2 &= x^0 & \pi_0 &= 0 & \pi_1 &= \sqrt{W} \\
\omega_0 &= 0 & \omega_1 &= -\frac{1}{4}W \exp(x^0/\beta)
\end{align*}
\]  

\[
e_0^0 = \sqrt{\frac{1}{W}} \quad e_1^0 = e_0^1 = 0 \quad e_1^1 = \frac{1}{2} \sqrt{W} \exp(x^0/\beta)
\]  

This result is transformed into the conformal gauge \((g_{01} = 0, \ g_{00} = g_{11})\) by the coordinate transformation

\[
\begin{align*}
\bar{x}^0 &= \int \frac{2}{W(x^0)} \exp(x^0/\beta) dx^0 & \bar{x}_1 &= x_1 \\
\Rightarrow \quad g &= \frac{1}{4}W(h(\bar{x}^0)) \exp(2h(\bar{x}^0)/\beta)(d\bar{x}^0 d\bar{x}^0 + d\bar{x}^1 d\bar{x}^1)
\end{align*}
\]  

where \( h(\bar{x}^0) = \pi_2(\bar{x}^0) \) is the inverse function of \( \bar{x}^0(x^0) \) and solves Katanaev’s differential equation

\[
h = \frac{1}{4}W(h) \exp(h/\beta) \quad (A9) \]  

agrees with the result obtained by a Wick rotation of the solution given in [2].
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