On a restriction problem of de Leeuw type for Laguerre multipliers

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Dedicated to Károly Tandori on the occasion of his 70th birthday

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Abstract. In 1965 K. de Leeuw [3] proved among other things in the Fourier transform setting: If a continuous function $m(\xi_1, \ldots, \xi_n)$ on $\mathbb{R}^n$ generates a bounded transformation on $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, then its trace $\tilde{m}(\xi_1, \ldots, \xi_m) = m(\xi_1, \ldots, \xi_m, 0, \ldots, 0)$, $m < n$, generates a bounded transformation on $L^p(\mathbb{R}^m)$. In this paper, the analogous problem is discussed in the setting of Laguerre expansions of different orders.

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1 Introduction

The purpose of this paper is to discuss the question: suppose $\{m_k\}_{k \in \mathbb{N}_0}$ generates a bounded transformation with respect to a Laguerre function expansion of order $\alpha$ on some $L^p$–space, does it also generate a corresponding bounded map with respect to a Laguerre function expansion of order $\beta$? To become more precise let us first introduce some notation. Consider the Lebesgue spaces

$$L^p_{w(\gamma)} = \{ f : \| f \|_{L^p_{w(\gamma)}} = (\int_0^\infty |f(x)e^{-x/2}|^p x^{\gamma} dx)^{1/p} < \infty \}, \quad 1 \leq p < \infty,$$

$$L^\infty_{w(\gamma)} = \{ f : \| f \|_{L^\infty_{w(\gamma)}} = \text{ess sup}_{x>0} |f(x)e^{-x/2}| < \infty \}, \quad p = \infty,$$

where $\gamma > -1$. Let $L_n^\alpha(x)$, $\alpha > -1$, $n \in \mathbb{N}_0$, denote the classical Laguerre polynomials (see Szegő [15, p. 100]) and set

$$P_n^\alpha(x) = L_n^\alpha(x)/L_n^\alpha(0), \quad L_n^\alpha(0) = A_n^\alpha = \binom{n+\alpha}{n} = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}.$$
Associate to $f$ its formal Laguerre series

$$f(x) \sim (\Gamma(\alpha + 1))^{-1} \sum_{k=0}^{\infty} \hat{f}_\alpha(k) L^\alpha_k(x),$$

where the Fourier Laguerre coefficients of $f$ are defined by

$$\hat{f}_\alpha(n) = \int_0^\infty f(x) R^\alpha_n(x) x^\alpha e^{-x} dx$$ (1)

(if the integrals exist). A sequence $m = \{m_k\}_{k \in \mathbb{N}_0}$ is called a (bounded) multiplier from $L^p_{w(\gamma)}$ into $L^q_{w(\delta)}$, notation $m \in M_{p,q}^{\alpha;\gamma;\delta}$, if

$$\| \sum_{k=0}^{\infty} m_k \hat{f}_\alpha(k) L^\alpha_k \|_{L^q_{w(\delta)}} \leq C \| \sum_{k=0}^{\infty} \hat{f}_\alpha(k) L^\alpha_k \|_{L^p_{w(\gamma)}}$$

for all polynomials $f$; the smallest constant $C$ for which this holds is called the multiplier norm $\| m \|_{M_{p,q}^{\alpha;\gamma;\delta}}$. For the sake of simplicity we write $M_{p,q}^{\alpha;\gamma} := M_{p,q}^{\alpha;\gamma;\delta}$ if $\gamma = \delta$ and, if additionally $p = q$, $M_{p,q}^{\alpha} := M_{p,q}^{\alpha;\gamma}$. We are mainly interested in the question: when is $M_{p,q}^{\alpha;\alpha}$ continuously embedded in $M_{p,q}^{\beta;\beta}$:

$$M_{p,q}^{\alpha;\alpha} \subset \subset M_{p,q}^{\beta;\beta}, \quad 1 \leq p \leq q \leq \infty?$$

The Plancherel theory immediately yields

$$l^\infty = M_2^{\alpha;\alpha} = M_2^{\beta;\beta}, \quad \alpha, \beta > -1.$$

A combination of sufficient multiplier conditions with necessary ones indicates which results are to be expected. To this end, define the fractional difference operator $\Delta^\delta$ of order $\delta$ by

$$\Delta^\delta m_k = \sum_{j=0}^{\infty} A_j^{-\delta-1} m_{k+j}$$

(whenever the series converges), the classes $wBV_{q,\delta}$, $1 \leq q \leq \infty$, $\delta > 0$, of weak bounded variation (see [5]) of bounded sequences which have finite norm $\| m \|_{q,\delta}$, where

$$\| m \|_{q,\delta} := \sup_k |m_k| + \sup_{N \in \mathbb{N}_0} \left( \sum_{k=N}^{2N} |(k+1)^{\delta} \Delta^\delta m_k| \frac{1}{k+1} \right)^{1/q}, \quad q < \infty,$$

$$\| m \|_{\infty,\delta} := \sup_k |m_k| + \sup_{N \in \mathbb{N}_0} \left( |(k+1)^{\delta} \Delta^\delta m_k| \right), \quad q = \infty.$$ 

Observing the duality (see [14])

$$M_{p,q}^{\alpha;\gamma} = M_{\alpha;\alpha p'-\gamma p'/p}, \quad -1 < \gamma < p(\alpha + 1) - 1, \quad 1 < p < \infty,$$ (2)
where $1/p + 1/p' = 1$, we may restrict ourselves to the case $1 < p < 2$. The Corollary 1.2 b) in [14] gives the embedding
\[ M^p_{\alpha; \alpha} \subset \subset \mathcal{W}^p_{\alpha', s}, \quad s = (2\alpha + 2/3)(1/p - 1/2), \quad \alpha > -1/3, \]
(3)
when $(2\alpha + 2)(1/p - 1/2) > 1/2$. Theorem 5 in [5] gives the first embedding in
\[ \mathcal{W}^p_{\alpha', s} \subset \subset \mathcal{W}^2_{s}, \quad s = (2\alpha + 2/3)(1/p - 1/2), \quad \alpha > -1/3, \]
whereas the last one follows from Corollaries 1.2 and 4.5 in [14] provided $s > \max\{(2\beta + 2)(1/p - 1/2), 1\}$, $\beta > -1$. Hence, choosing $\gamma = \alpha$ in (2), we obtain
**Proposition 1.1** Let $1 < p < \infty$ and $\alpha$ be such that $(2\alpha + 2/3)(1/p - 1/2) > 1$. Then
\[ M^p_{\alpha; \alpha} \subset \subset M^p_{\beta; \beta}, \quad -1 < \beta < \alpha - 2/3. \]
In the same way we can derive a result for $M^{p,q}$-multipliers. The necessary condition in [6, Cor. 1.3] can easily be extended in the sense of [6, Cor. 2.5 b)] to
\[ \sup \left| (k + 1)^{\sigma} m_k \right| + \sup \left( \sum_{k=n}^{2n} \left| (k + 1)^{\sigma + s} \Delta^s m_k \right| q'/k \right)^{1/q'} \leq C\|m\|_{M^{p,q}_{\alpha; \alpha}}, \]
where $\alpha > -1/3$, $1/q = 1/p - \sigma/(\alpha + 1)$, $1 < p < q < 2$, $(\alpha + 1)(1/q - 1/2) > 1/4$, and $s = (2\alpha + 2/3)(1/q - 1/2) > 0$. Using this and the sufficient condition for $M^{p,q}_{\beta; \beta}$-multipliers given in [4, Cor. 1.2], which is proved only for $\beta \geq 0$, we obtain
\[ M^{p,q}_{\alpha; \alpha} \subset \subset M^{p,q}_{\beta; \beta}, \quad 0 \leq \beta < \alpha - 2/3, \quad (2\alpha + 2/3)(1/q - 1/2) > 1, \quad 1 < p < q < 2. \]
In this context let us mention that the same technique yields for $1 < p, q < 2$
\[ M^p_{\alpha; \alpha} \subset \subset M^q_{\beta; \beta}, \quad (2\alpha + 2/3)(1/p - 1/2) > \max\{(2\beta + 2)(1/q - 1/2), 1\}. \]
(4)
This embedding is in so far interesting as it allows to go from $p, 1 < p < 2$, to $q \neq p, 1 < q < 2$, connected with a loss in the size of $\beta$ if $q < p$ or a gain in $\beta$ if $1 < p < q < 2$; e.g.
\[ M^{4/3}_{10; 10} \subset \subset M^q_{5; 5}, \quad 1.08 \leq q \leq 2, \quad \text{or} \quad M^{8/7}_{2; 2} \subset \subset M^q_{4; 4}, \quad 3/2 \leq q \leq 2. \]
Improvements of [14] can be expected by better necessary conditions and/or better sufficient conditions; but this technique cannot give something like
\[ M^p_{\alpha; \alpha} \subset \subset M^q_{\beta; \beta}, \quad (\alpha + 1)(1/p - 1/2) > (\beta + 1)(1/q - 1/2), \quad 1 < p, q < 2, \]
which is suggested by (4) when choosing “large” $\alpha$ with $p$ near 2 since then the number $4(1/p - 1/2)/3$, which describes the smoothness gap between the necessary conditions and the sufficient conditions in [14, Cor. 1.2], is “negligible”.

Concerning the general problem “When does $M_{p,q}^{\alpha;\gamma_1,\delta_1} \hookrightarrow M_{p,q}^{\alpha;\gamma_2,\delta_2}$ hold?”, we mention results in Stempak and Trebels [14, Cor. 4.3]: For $1 < p < \infty$ there holds

$$M_{p;\beta p/2+\delta}^p = M_{p;\delta}^p$$

if $\left\{ \begin{array}{ll} -1 - \beta p/2 < \delta < p - 1 + \beta p/2, & -1 < \beta < 0, \\ -1 < \delta < p - 1, & 0 \leq \beta, \end{array} \right.$

which for $\delta = 0$ contains Kanjin’s [9] result and for $\delta = p/4 - 1/2$ Thangavelu’s [16].

In particular, there holds for $-1 < \beta < \alpha$, $1 < p < \infty$,

$$M_{p;\beta}^p = M_{p;\beta p/2+\beta p(1/p-1/2)}^p = M_{p;\alpha p/2+\beta p(1/p-1/2)}^p,$$

$$(2\beta + 2)|1/p - 1/2| < 1. \quad (5)$$

These results are based on Kanjin’s [9] transplantation theorem and its weighted version in [14]. The latter gives further insight into our problem in so far as it implies that the restriction $\beta < \alpha - 2/3$ in Proposition 1.1 is not sharp.

To this end we first note that the following extension of Corollary 4.4 in [14] holds

$$wbv_{2,s} \hookrightarrow M_{p;\alpha p/2+\eta(p/2-1)}, \quad 0 \leq \eta \leq 1, \quad 1 < p \leq 2, \quad s > 1/p.$$ (For the proof observe that for $\alpha = 0$ the parameter $\gamma = \eta(p/2 - 1)$, $0 \leq \eta \leq 1$, is admissible in [14, Theorem 1.1] and then follow the argumentation of [14, Cor. 4.4].)

This combined with (3) yields for $s = (2\alpha + 2/3)(1/p - 1/2) > 1/p$

$$M_{p;\beta}^{p,\alpha} \hookrightarrow wbv_{2,s} \hookrightarrow M_{p;\alpha p/2+\eta(p/2-1)}, \quad 1 < p \leq 2, \quad \alpha > (p+1)/(6-3p).$$

Thus, by interpolation with change of measure,

$$M_{p;\alpha}^{p,\alpha} \hookrightarrow M_{p;\alpha p/2+\delta}^p, \quad p/2 - 1 \leq \delta \leq \alpha - \alpha p/2, \quad \alpha > (p+1)/(6-3p).$$

Since (3) gives

$$M_{p;\alpha p/2+\beta p(1/p-1/2)}^p = M_{p;\beta}^p,$$

we arrive at

**Proposition 1.2** Let $1 < p \leq 2$ and $\alpha > (p+1)/(6-3p)$. Then

$$M_{p;\alpha}^{p,\alpha} \hookrightarrow M_{p;\beta}^{p,\beta}, \quad (2\beta + 2)(1/p - 1/2) < 1, \quad -1 < \beta < \alpha.$$ The first restriction on $\beta$ is equivalent to $\beta < (2p - 2)/(2 - p)$. This combined with the restriction on $\alpha$ gives $\alpha - \beta > (7 - 5p)/(6 - 3p)$, the latter being decreasing in $p$ and taking the value $2/3$ at $p = 1$. Hence Proposition 1.2 is an improvement of the previous one for all $1 < p < 2$ provided $(p+1)/(6-3p) < \alpha \leq (2p - 2)/(2 - p).$
For big $\alpha$’s, Proposition 1.1 is certainly better. If in the transplantation theorem in [4] higher exponents could be allowed in the power weight – which is possible in the Jacobi expansion case as shown by Muckenhoupt [12] – the technique just used would give the embedding when $-1 < \beta < \alpha$, $1 < p < 2$, and $\alpha > (p + 1)/(6 - 3p)$. Summarizing, it is reasonable to conjecture

$$M^{p,q}_{\alpha;\alpha} \subset \subset M^{p,q}_{\beta;\beta}, \quad -1 < \beta < \alpha, \quad 1 < p \leq q \leq \infty.$$ 

Apart from the above fragmentary results, so far we can only prove the conjecture in the extreme case when $q = \infty$ and $\beta \geq 0$; the latter restriction arises from the fact that we have to make use of the twisted Laguerre convolution (see [7]) which is proved till now only for Laguerre polynomials $L^n_\alpha(x)$ with $\alpha \geq 0$. Our main result is

**Theorem 1.3** If $1 \leq p \leq \infty$, then

$$M^{p,\infty}_{\alpha;\alpha} \subset \subset M^{p,\infty}_{\beta;\beta}, \quad 0 \leq \beta < \alpha.$$ 

**Remarks.** 1) One could speculate that an interpolation argument applied to

$$M^2_{\alpha;\alpha} = M^2_{\beta;\beta}, \quad M^\infty_{\alpha;\alpha} = M^1_{\alpha;\alpha} \subset M^1_{\beta;\beta} = M^\infty_{\beta;\beta}, \quad \beta < \alpha,$$

could give the open case $M^{p}_{\alpha;\alpha} \subset \subset M^{p}_{\beta;\beta}, \quad 1 < p < 2$. In this respect we mention a result of Zafran [17, p. 1412] for the Fourier transform pointed out to us by A. Seeger:

*Denote by $M^p(R)$ the set of bounded Fourier multipliers on $L^p(R)$ and by $M^\wedge(R)$ the set of Fourier transforms of bounded measures on $R$. Then $M^p(R), \quad 1 < p < 2$, is not an interpolation space with respect to the pair $(M^\wedge(R), L^\infty(R))$.*

Thus de Leeuw’s result mentioned at the beginning cannot be proved by interpolation.

2) It is perhaps amazing to note that the $wbv$–classes do not play only an auxiliary role in dealing with the above formulated general problem. In the framework of one-dimensional Fourier transforms/series this was shown by Muckenhoupt, Wheeden, and Wo-Sang Young [13]. That this phenomenon also occurs in the framework of Laguerre expansions can be seen from the following two theorems.

**Theorem 1.4** If $\alpha > -1, \quad \alpha \neq 0$, then

$$wbv^2_{2,1} \subset \subset M^2_{\alpha;\alpha+1}.$$ 

In the case $-1 < \alpha < 0$ the multiplier operator is defined only on the subspace $\{f \in L^2_{w(\alpha+1)} : \hat{f}_\alpha(0) = 0\}$. 

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Theorem 1.5 If $\alpha > -1$, then
\[ M_{\alpha;\alpha+1}^2 \subset \subset wbv_{2,1}. \]

A combination of these two results leads to
\[ M_{\alpha;\alpha+1}^2 = M_{\beta;\beta+1}^2 = wbv_{2,1}, \quad \alpha, \beta > -1, \quad \alpha, \beta \neq 0, \]  
and a combination with [14, (19)] gives
\[ M_{\alpha;\alpha+1}^2 \subset \subset M_{\alpha;\alpha}^p, \quad \alpha \geq 0, \quad (2\alpha + 2)/(\alpha + 1) < p \leq 2. \]

2 Proof of Theorem 1.3

Theorem 1.3 is an immediate consequence of the combination of the following two theorems.

Theorem 2.1 Let $f \in L_{w(\alpha)}^p$ with $\alpha > -1$ when $1 \leq p < \infty$ and $\alpha \geq 0$ when $p = \infty$. Then there exists a function $g \in L_{w(\beta)}^p$, $-1 < \beta < \alpha$, with
\[ g(x) \sim (\Gamma(\beta + 1))^{-1} \sum_{k=0}^{\infty} \hat{f}_\alpha(k) L_\beta^\alpha(x), \quad \|g\|_{L_{w(\beta)}^p} \leq C \|f\|_{L_{w(\alpha)}^p}. \]

Proof

First let $1 \leq p < \infty$ and, without loss of generality, let $f$ be a polynomial (these are dense in $L_{w(\alpha)}^p$). We recall the projection formula (3.31) in Askey and Fitch [2]
\[ e^{-x} L_\alpha^\beta(x) = \frac{1}{\Gamma(\alpha - \beta)} \int_x^{\infty} (y - x)^{\alpha - \beta - 1} e^{-y} L_\alpha^\beta(y) \, dy, \quad -1 < \beta < \alpha. \]

Then the following computations are justified.
\[ \|g\|_{L_{w(\beta)}^p} = C \left( \int_0^{\infty} \left| \sum_{k=0}^{\infty} \hat{f}_\alpha(k) L_\beta^\alpha(x) e^{-x/2} x^{\beta/2} \right|^p x^{\beta/2} \, dx \right)^{1/p} \]
\[ = C \left( \int_0^{\infty} \left| \int_x^{\infty} (y - x)^{\alpha - \beta - 1} e^{-y} \sum_{k=0}^{\infty} \hat{f}_\alpha(k) L_\alpha^\beta(y) \, dy \right| x^{\beta/2} e^{-x/2} x^{\beta/2} \, dx \right)^{1/p} \]
\[ \leq C \int_1^{\infty} (t - 1)^{\alpha - \beta - 1} \left( \int_0^{\infty} \left| \sum_k \hat{f}_\alpha(k) L_\alpha^\beta(xt) x^{\alpha - \beta + \beta/p} e^{-x(t-1/2)^2} \right|^p \, dx \right)^{1/p} \, dt \]
after a substitution and application of the integral Minkowski inequality. Additional substitutions lead to

\[
\|g\|_{L_w^p} \leq C \int_0^\infty s^{\alpha-\beta-1}(s+1)^{\beta/p'-\alpha-1/p} \times \\
\left( \int_0^\infty | \sum_k \hat{f}_\alpha(k) L_k^\alpha(y) e^{-y/2} y^{(\alpha-\beta)/p'} e^{-ys/(s+1)} |^{p'} dy \right)^{1/p} ds \\
\leq C \int_0^\infty s^{(\alpha-\beta)/p-1}(s+1)^{-(\alpha+1)/p} \left( \int_0^\infty | \sum_k \hat{f}_\alpha(k) L_k^\alpha(y) e^{-y/2} y^{\alpha} dy \right)^{1/p} ds,
\]

where we used the inequality \( y^{(\alpha-\beta)/p'} e^{-ys/(s+1)} \leq C((s+1)/s)^{(\alpha-\beta)/p'} \). Since \(-1 < \beta < \alpha\) it is easily seen that the outer integration only gives a bounded contribution.

If \( f \in L_w^{\infty} \), then \( \| (k+1)^{-1/2} \hat{f}_\alpha(k) \| \leq C \| f \|_{L_w^\infty} \) by \([10, \text{Lemma 1}]\) and, therefore, the Abel-Poisson means of an arbitrary \( f \in L_w^{\infty} \) can be represented by

\[
P_r f(x) = (\Gamma(\alpha + 1))^{-1} \sum_k r^k \hat{f}_\alpha(k) L_k^\alpha(x), \quad 0 \leq r < 1, \quad x \geq 0,
\]

and, by the convolution theorem in Görlich and Markett \([7, \text{p. 169}]\),

\[
\| P_r f \|_{L_w^\infty} \leq C \| f \|_{L_w^\infty}, \quad 0 \leq r < 1, \quad \alpha \geq 0.
\]

A slight modification of the argument in the case \( 1 \leq p < \infty \) shows that

\[
\| g_r \|_{L_w^{\infty}} := \| (\Gamma(\beta + 1))^{-1} \sum_k r^k \hat{f}_\alpha(k) L_k^\beta \|_{L_w^{\infty}} \leq C \| P_r f \|_{L_w^{\infty}} \leq C \| f \|_{L_w^{\infty}}.
\]

By the weak* compactness there exists a function \( g \in L_w^{\infty} \) with \( \hat{g}_\beta(k) = \hat{f}_\alpha(k) \) and \( \| g \|_{L_w^{\infty}} \leq \liminf_{k \to \infty} \| g_{r_k} \|_{L_w^{\infty}} \) for a suitable sequence \( r_k \to 1^- \); hence also the assertion in the case \( p = \infty \).

**Theorem 2.2** For \( \alpha \geq 0 \) there holds

i) \( M_{1; \alpha}^1 = M_{\alpha, \alpha}^p \leq L_w^p, \quad 1 < p \leq \infty \),

ii) \( M_{1; \alpha}^1 = M_{\alpha, \alpha}^\infty = \{ m = \{ m_k \}_{k \in \mathbb{N}_0} : \| P_r(m) \|_{L_w^1} = O(1), \ r \to 1^- \} \),

where \( P_r(m)(x) = (\Gamma(\alpha + 1))^{-1} \sum_k r^k m_k L_k^\alpha(x) \).

**Proof**

The first equalities in i) and ii) are the standard duality statements. Let us briefly indicate the second equalities (which are also more or less standard).

If \( m = \{ m_k \}_{k \in \mathbb{N}_0} \) are the Fourier Laguerre coefficients of an \( L_w^p \) function, \( 1 < p \leq \infty \), or in the case \( p = 1 \) of a bounded measure with respect to the weight \( e^{-x/2} x^\alpha \), then
Young’s inequality in Görlich and Markett [7] (or a slight extension of it to measures in the case \( p = 1 \)) shows that \( m \in M_{p',\infty}. \)

Conversely, associate formally to a sequence \( m = \{m_k\} \) an operator \( T_m \) by

\[
T_{m}f(x) \sim (\Gamma(\alpha + 1))^{-1} \sum_{k=0}^{\infty} m_k \hat{f}_\alpha(k) L_\alpha^0(x). \tag{7}
\]

Then, in essentially the notation of Görlich and Markett [7],

\[
T_{m}(P_rf)(x) = P_r(m) \ast f(x) = \int_{0}^{\infty} T_\alpha^0(P_r(m)(y))f(y)e^{-y/2}dy,
\]

where \( T_\alpha^0 \) is the Laguerre translation operator. If \( \|f\|_{L_{p'}^w(\alpha)} = 1 \) then

\[
\|T_{m}(P_rf)\|_{L_{\infty}^w(\alpha)} \leq \|m\|_{M_{p',\infty}^{\alpha}} \|P_rf\|_{L_{p'}^w(\alpha)} \leq C \|m\|_{M_{p',\infty}^{\alpha}},
\]

and hence, by the converse of Hölder’s inequality,

\[
\sup_{\|f\|_{L_{p'}^w(\alpha)} = 1} \left| \int_{0}^{\infty} T_\alpha^0(P_r(m)(y))e^{-y/2}y^{\alpha+p}\hat{f}(y)e^{-y/2}y^{\alpha/p'}dy \right| \leq \|T_\alpha^0(P_r(m))\|_{L_{p'}^p(\alpha)} \leq C \|m\|_{M_{p',\infty}^{\alpha}}
\]

for \( x \geq 0, 0 \leq r < 1 \). In particular, for \( x = 0 \) we obtain

\[
\|P_r(m)\|_{L_{p'}^p(\alpha)} \leq C \|m\|_{M_{p',\infty}^{\alpha}} , \quad 0 \leq r < 1.
\]

Now weak* compactness gives the desired converse embedding.

### 3 Proof of Theorems 1.4 and 1.5

The proof relies heavily on the Parseval formula

\[
\frac{1}{\Gamma(\alpha + 1)} \sum_{k=0}^{\infty} A_k^\alpha |\hat{f}_\alpha(k)|^2 = \int_{0}^{\infty} |f(x)e^{-x/2}|^2 x^\alpha dx \tag{8}
\]

and its extension

\[
\sum_{k=0}^{\infty} A_k^{\alpha+\lambda} |\Delta^\lambda \hat{f}_\alpha(k)|^2 \approx \int_{0}^{\infty} |f(x)e^{-x/2}|^2 x^{\alpha+\lambda} dx, \quad \lambda \geq 0, \tag{9}
\]

which is a consequence of the formula

\[
\Delta^\lambda \hat{f}_\alpha(k) = C_{\alpha,\lambda} \hat{f}_{\alpha+\lambda}(k) \tag{10}
\]
(see e.g. the proof of Lemma 2.1 in [3]). For the proof of Theorem 1.4 we further need the following discrete analog of the \( p = 2 \) case of a weighted Hardy inequality in Muckenhoupt [11] whose proof can at once be read off from [11] by replacing the integrals there by sums and using the fact that
\[
a \leq 2(a + b)^{1/2}[(a + b)^{1/2} - b^{1/2}]
\]
when \( a, b \geq 0 \); also see the extensions in [4, Sec. 4].

Lemma 3.1. Let \( \{u_k\}_{k \in \mathbb{N}_0}, \{v_k\}_{k \in \mathbb{N}_0} \) be non-negative sequences (if \( v_k = 0 \) we set \( v_k^{-1} = 0 \)). Then
\[
a) \quad \sum_{k=0}^{\infty} \left| \sum_{j=0}^{k} a_j \right|^2 u_k \leq C \sup_{N} \left( \sum_{k=N}^{\infty} u_k \sum_{k=0}^{N} v_k^{-1} \right) \sum_{j=0}^{\infty} \left| a_j \right|^2 v_j.
\]
\[
b) \quad \sum_{k=0}^{\infty} \left| \sum_{j=k}^{\infty} a_j \right|^2 u_k \leq C \sup_{N} \left( \sum_{k=0}^{N} u_k \sum_{k=0}^{\infty} v_k^{-1} \right) \sum_{j=0}^{\infty} \left| a_j \right|^2 v_j.
\]

Proof of Theorem 1.4. Using (9) and the operator \( T_m \) defined in (6), we obtain
\[
\int_0^\infty |T_m f(x) e^{-x/2} x^\alpha dx| \approx \sum_{k=0}^{\infty} A_k^{\alpha+1} |\Delta (m_k \hat{f}_\alpha(k))|^2.
\]
Since
\[
\Delta (m_k \hat{f}_\alpha(k)) = m_k \Delta \hat{f}_\alpha(k) + \hat{f}_\alpha(k+1) \Delta m_k
\]
we first observe that
\[
\sum_{k=0}^{\infty} A_k^{\alpha+1} |m_k|^2 |\Delta \hat{f}_\alpha(k)|^2 \leq \|m\|_\infty \sum_{k=0}^{\infty} A_k^{\alpha+1} |\Delta \hat{f}_\alpha(k)|^2 \leq C \|m\|_\infty^2 \|f\|_{L^2_{\omega^{\alpha+1}}}^2.
\]
To dominate the term containing \( \Delta m_k \) we deduce from (8) that for \( \alpha \geq 0 \) the Fourier Laguerre coefficients tend to zero as \( k \to \infty \). Hence
\[
\sum_{k=0}^{\infty} A_k^{\alpha+1} |\hat{f}_\alpha(k+1) \Delta m_k|^2 = \sum_{k=0}^{\infty} A_k^{\alpha+1} |\Delta m_k|^2 \sum_{j=k+1}^{\infty} |\Delta \hat{f}_\alpha(j)|^2 =: I.
\]
In order to apply Lemma 3.1 b), we choose \( u_k = A_k^{\alpha+1} |\Delta m_k|^2 \) and \( v_k = A_k^{\alpha+1} \), and observe that when \( M \in \mathbb{N}, 2^{M-1} \leq N < 2^M \), we have that
\[
\left( \sum_{k=0}^{N} u_k \sum_{k=N}^{\infty} v_k^{-1} \right) \leq C(N+1)^{-\alpha} \sum_{j=0}^{M} 2^{j+1-2} \sum_{k=2^j-1}^{2^j} (k+1) |\Delta m_k|^2 A_k^{\alpha+1} \frac{A_k^{\alpha+1}}{k+1}
\]
\[
\leq C(N+1)^{-\alpha} \sum_{j=0}^{M} (2^j)^\alpha \|m\|_{2,1}^2 \leq C \|m\|_{2,1}^2.
\]

uniformly in $N$ if $\alpha > 0$. Then Lemma 3.1 b) gives

$$I \leq C\|m\|_{2,1}^2 \sum_{j=0}^{\infty} A_j^{\alpha+1} |\Delta \hat{f}_\alpha(j)|^2 \leq C\|m\|_{2,1}^2 \|f\|_{L^2_{w(\alpha+1)}}^2,$$

by (9). Thus there remains to consider the case $-1 < \alpha < 0$. For the same choice of $u_k$ and $v_k$ one easily obtains

$$\left( \sum_{k=N}^{\infty} u_k v_k^{-1} \right) \leq C\|m\|_{2,1}^2.$$

Now assume that $\hat{f}(0) = 0$. Then we have

$$\sum_{k=0}^{\infty} A_k^{\alpha+1} |\hat{f}_\alpha(k+1)\Delta m_k|^2 = \sum_{k=0}^{\infty} A_k^{\alpha+1} |\Delta m_k|^2 \sum_{j=0}^{k} |\Delta \hat{f}_\alpha(j)|^2 \leq C\|m\|_{2,1}^2 \|f\|_{L^2_{w(\alpha+1)}}^2,$$

where the last estimate follows by Lemma 4.1 a); thus Theorem 1.4 is established.

The proof of Theorem 1.5 is essentially contained in [3]. As in [3], consider a monotone decreasing $C^\infty$-function $\phi(x)$ with

$$\phi(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 2 \\ 0 & \text{if } x \geq 4 \end{cases}, \quad \phi_i(x) = \phi(x/2^i).$$

Then the $\phi_i(k)$ are the Fourier Laguerre coefficients of an $L^2_{\alpha+1}$-function $\Phi^{(i)}$ with norm $\|\Phi^{(i)}\|_{L^2_{\alpha+1}} \leq C (2^i)^{\alpha/2}$ and

$$\sum_{k=2^i}^{2^{i+1}} A_k^{\alpha+1} |\Delta m_k|^2 = \sum_{k=2^i}^{2^{i+1}} A_k^{\alpha+1} |\Delta (m_k \phi_i(k))|^2 \leq \sum_{k=0}^{2^{i+2}} A_k^{\alpha+1} |\Delta (m_k \phi_i(k))|^2 \leq C\|m\|_{M^2_{\alpha+1}} \|\Phi^{(i)}\|_{L^2_{\alpha+1}}^2 \leq C 2^{i\alpha} \|m\|_{M^2_{\alpha+1}}.$$

This immediately leads to

$$\|m\|_{\infty} + \left( \sum_{k=2^i}^{2^{i+1}} |(k+1)\Delta m_k^2| \frac{1}{k+1} \right)^{1/2} \leq C\|m\|_{M^2_{\alpha+1}},$$

uniformly in $i$, since by [3] (10) there holds $\|m\|_{\infty} \leq C\|m\|_{M^2_{\alpha+1}}$; thus Theorem 1.5 is established.

Remark. 3) (Added on Aug. 10, 1994) The characterization (6) can easily be extended to

$$M^2_{\alpha,\alpha+l} = w_b v_{2,l}, \quad \alpha > -1, \quad \alpha \neq 0, \ldots, l - 1, \quad l \in \mathbb{N}.$$ (12)
In the case \( \alpha < l - 1 \) the multiplier operator is defined only on the subspace \( \{ f \in L^2_{w^v} : \hat{f}_\alpha(k) = 0, \ 0 \leq k < (l - 1 - \alpha)/2 \} \).

The necessity part carries over immediately (see also [3]). The sufficiency part will be proved by induction. Thus suppose that (12) is true for \( l = 1, \ldots, n \) and \( \alpha \)'s as indicated. Then, as in the case \( n = 1 \), by (11)

\[
\int_0^\infty |T_m f(x) e^{-x/2}|^2 x^{\alpha + n + 1} dx \approx \sum_{k=0}^\infty A_{k}^{\alpha + n + 1} |\Delta^\alpha \Delta(m_k \hat{f}_\alpha(k))|^2
\]

\[
\leq C \sum_{k=0}^\infty A_{k}^{\alpha + n + 1} |\Delta^\alpha (m_k \Delta \hat{f}_\alpha(k))|^2 + C \sum_{k=0}^\infty A_{k}^{\alpha + n + 1} |\Delta^\alpha (\hat{f}_\alpha(k+1) \Delta m_k)|^2 =: I + II
\]

By the assumption and (11)

\[
I \leq C \|m\|_{w^v, \ldots, n}^2 \sum_{k=0}^\infty A_{k}^{\alpha + n + 1} |\Delta^\alpha \hat{f}_{\alpha+1}(k)|^2 \leq C \|m\|_{w^v, \ldots, n}^2 \int_0^\infty |f(x) e^{-x/2}|^2 x^{\alpha + n + 1} dx
\]

on account of the embedding properties of the \( w^v \)-spaces [3]. Analogously \( II \) can be estimated by

\[
II \leq C \|(k+1) \Delta m_k\|_{w^v, \ldots, n} \sum_{k=0}^\infty A_{k}^{\alpha + n + 1} |\Delta^\alpha (\hat{f}_\alpha(k+1)/k)|^2.
\]

By the Leibniz formula for differences there holds

\[
\Delta^\alpha \left( \frac{\hat{f}_\alpha(k+1)}{k+1} \right) \leq C \sum_{j=0}^n |\Delta^j \hat{f}_\alpha(k+1)| |\Delta^{n-j} \frac{1}{j+k+1}| \leq C \sum_{j=0}^n (j+k+1)^{j-n-1} |\Delta^j \hat{f}_\alpha(k+1)|.
\]

Hence we have to dominate for \( j = 0, \ldots, n \)

\[
II_j := \sum_{k=0}^\infty A_{k}^{\alpha - n - 1 + 2j} |\Delta^j \hat{f}_\alpha(k+1)|^2.
\]

If \( \alpha > n \) then \( c_j := -\alpha - 2j + n + 1 < 1 \) for all \( j = 0, \ldots, n \), \( \Delta^j \hat{f}_\alpha(k+1) = \sum_{i=k+1}^\infty \Delta^j \hat{f}_\alpha(i) \), and we can apply [3, Theorem 346] repeatedly to obtain

\[
II_j \leq C \sum_{k=0}^\infty A_{k}^{\alpha - n - 1 + 2j} |(k+1) \Delta^{j+1} \hat{f}_\alpha(k+1)|^2 \approx \sum_{k=0}^\infty A_{k}^{\alpha - n + 2j + 1} |\Delta^{j+1} \hat{f}_\alpha(k+1)|^2 \leq \ldots \leq C \sum_{k=0}^\infty A_{k}^{\alpha + n + 1} |\Delta^{n+1} \hat{f}_\alpha(k+1)|^2 \leq C \int_0^\infty |f(x) e^{-x/2}|^2 x^{\alpha + n + 1} dx.
\]
Since \( \|\{(k + 1)\Delta m_k\}\|_{w^{b_2,n}} \leq C\|m\|_{w^{b_2,n+1}} \), this gives the assertion for the weight \( x^{n+1} \) in the case \( \alpha > n \).

If \( \alpha < n \), \( \alpha \neq 0, \ldots, n \), then some \( c_j > 1 \). For the application of \([8, \text{Theorem 346}]\) one needs \( c_j \neq 1 \); this is guaranteed by the hypothesis \( \alpha \neq 0, \ldots, n \) (in the case of an additional weight \( x^{n+1} \)). For the \( j \) for which \( c_j > 1 \) we have to use the representation

\[
\Delta^j \hat{f}_\alpha(k + 1) = -\sum_{i=0}^{k} \Delta^{j+1} \hat{f}_\alpha(i), \quad \text{if } \Delta^j \hat{f}_\alpha(0) = 0,
\]
i.e., the first \( (j + 1) \) Fourier-Laguerre coefficients have to vanish to ensure this representation. But \( 0 \leq j \leq j_0 \), where \( j_0 \) is chosen in such a way that \( c_{j_0} > 1 \) and \( c_{j_0+1} < 1 \), hence \( j_0 = \left[ (n - \alpha)/2 \right] \) (with respect to the additional weight \( x^{n+1} \)); here we used the standard notation for \( [a], a \in \mathbb{R} \), to be the greatest integer \( \leq a \). Hence the condition that the first \( \left[ (n - \alpha)/2 \right] + 1 \) Fourier-Laguerre coefficients have to vanish is needed if the additional weight is \( x^{n+1} \). A repeated application of \([8, \text{Theorem 346}]\) with appropriate \( c > 1 \) or \( c < 1 \) now gives the assertion.

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