DISPLACEMENT ENERGY OF COMPACT LAGRANGIAN SUBMANIFOLD FROM OPEN SUBSET

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Abstract. We prove that for any compact Lagrangian submanifold $L$ the Hofer displacement energy for disjointing $L$ from an open subset $U$ in tame symplectic manifold $(M, \omega)$ is positive, provided $L \cap U \neq \emptyset$. We also give an explicit lower bound in terms of an $\epsilon$-regularity type invariant for pseudo-holomorphic curves relative to $L$ and $U$.

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1. INTRODUCTION

For a compactly supported Hamiltonian diffeomorphism $\phi$, Hofer’s norm $[H]$ is defined to be

$$\|\phi\| = \inf_{H \mapsto \phi} \|H\|$$

(1.1)

where $H \mapsto \phi$ means that $H : M \times [0, 1] \to \mathbb{R}$ is a Hamiltonian such that $\phi = \phi_H^{\cdot}$, and

$$\|H\| = \int_0^1 (\max H_t - \min H_t) \, dt.$$  

(1.2)

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Here $\phi_H^1$ denotes the time-one map of the flow of the Hamilton’s equation $\dot{z} = X_H(z)$.

**Definition 1.1** (Displacement energy). Let $A, B \subset M$ be two closed subsets. The displacement energy $e(A, B)$ is defined to be

$$e(A, B) = \inf_H \{ ||H|| \mid A \cap \phi^1_H(B) = \emptyset \}.$$  

Clearly if $A \cap B = \emptyset$, then $e(A, B) = 0$. But even when $A \cap B \neq \emptyset$, $e(A, B) = 0$ may still happen. For example, if $A, B$ are submanifolds of $\dim A + \dim B < \dim M$, then their displacement energy is zero. When $A$ is a compact submanifold of $M$ and $2 \dim A = \dim M$, Laudenbach-Sikorav [LS] proved that if $A$ is non-Lagrangian, $e(A, A) = 0$, provided its normal bundle has a section without zeros, i.e., as long as there is no topological obstruction to the disjoining.

When both $A$ and $B$ are Lagrangian submanifolds and a certain form of Floer homology $HF(A, B)$ is defined and is non-zero, then $e(A, B) = \infty$. We also refer to the works by Biran [Bi1, Bi2] and Biran-Cornea [BiC] for a different kind of such an obstruction to displacement of a symplectic ball $B(\lambda)$ in $M$ from a Lagrangian submanifold $L$.

However when there is no such obstruction to the disjunction for the pair $(A, B)$ and $e(A, B) < \infty$, this question of measuring the displacement energy, in particular, proving the positivity $e(A, B) > 0$, is a hard problem in general. This is the question we are pursuing in the present paper: More specifically, we study such a measurement when $A = U$ with $U$ an open subset, and $B = L$ is a compact Lagrangian submanifold intersecting $U$, when $\dim M \geq 4$. (The same holds for $\dim M = 2$ which is however easy to see unlike the higher dimensional case.)

Incidentally we would like to recall readers that these two cases, those of Lagrangian submanifolds and of open subsets, are the only general classes of subsets in a symplectic manifold that are known to admit such lower bounds for displacing the subset from itself. We refer readers to [H, LM, O5, U1] for such a measurement of the open subsets in terms of various kind of capacities: Hofer-Zehnder capacity [H, U1], Gromov capacity [LM] and spectral capacity [O5, U1] etc. For the Lagrangian submanifolds, we refer to [P2, Ch, O4] where such a measurement is made with respect to the $\epsilon$-regularity type invariant in the spirit of the present paper.

We recall the definition of tame symplectic manifolds: A symplectic manifold $(M, \omega)$ is called *tame* if it allows an almost complex structure $J_0$ such that the bilinear form $\omega(\cdot, J_0 \cdot)$ defines a Riemannian metric on $M$ with bounded curvature and with injectivity radius bounded away from zero. In this case, we also call *tame* the triple $(M, \omega, J_0)$ or the almost complex structure $J_0$. As usual, when we do our estimates which are implicit mostly in this paper, we will use various norms always in terms of a fixed such metric.

The following theorem provides an answer to the case of mixture of the two.

**Main Theorem** Let $(M, \omega)$ be a tame symplectic manifold and $L \subset M$ be a compact Lagrangian submanifold. Suppose that $U$ is an open subset such that $L \cap U \neq \emptyset$. Then $e(U, L) > 0$.

In fact, we can estimate the displacement energy $e(U, L)$ in terms of an $\epsilon$-regularity type invariant whose description is briefly in order. We refer readers to Theorem 2.14 for the precise statement of the lower bound for $e(U, L)$. 
We start with recalling the (absolute) $\epsilon$-regularity type invariants used in [O2], [Ch], [O4]. For each tame $J_0$, we define

$$A(J_0; M, \omega) = \inf \{ \omega([v]) \mid v: S^2 \to M, \text{ non-constant and } \bar{\partial}_{J_0} v = 0 \}$$

$$A(J_0, L; M, \omega) = \inf \{ \omega([w]) \mid w: (D^2, \partial D^2) \to (M, L), \text{ non-constant and } \bar{\partial}_{J_0} w = 0 \}.$$

It is not difficult to show $A(J_0; M, \omega), A(J_0, L; M, \omega) > 0$ from the $\epsilon$-regularity theorem and tameness of $(M, \omega, J_0)$. (See [O1, Corollary 3.5] for its proof.) We then define

$$A(L; M, \omega) = \sup \min_{J_0} \{ A(J_0; M, \omega), A(J_0, L; M, \omega) \}. \quad (1.4)$$

$A(L; M, \omega)$ could be infinity for general compact $L$ and its finiteness is equivalent to existence of certain pseudo-holomorphic sphere or a disc attached to $L$. However it was proved in [Ch], [O4] that $e(L, L) \geq A(L; M, \omega)$. In particular, if $L$ is displaceable, equivalently, if $e(L, L) < \infty$, then $A(L; M, \omega) < \infty$ also holds.

For the purpose of the present paper, we will also need to construct another more refined invariant which is an analog of the invariant $A(L; M, \omega)$ above by restricting the choice of $J_0$ to those that have some local reflectional symmetry on a symplectic ball. Unlike the case of $A(L; M, \omega)$, defining the $\epsilon$-regularity type invariant relevant to the purpose of the present paper requires some preparation. This kind of invariant in general was introduced and systematically used by Biran-Cornea in [BiC] in their study of mixed symplectic packing number. (See [BiC, Definition 1.1.1] and also [BaC]). We only consider a relative counterpart of the definition of $A(L; M, \omega)$ for the Lagrangian boundary condition, which is defined by using the relative version of isoperimetric inequality for the holomorphic curves with real boundary condition. We will denote the resulting invariant by $\epsilon(U, L; M, \omega)$. We postpone the details of its construction till the next section.

The main geometro-analytic framework of our proof of Main Theorem is an adaptation of the one from [O4] which gave a simple proof of Chekanov’s positivity theorem [Ch] of the displacement energy of general compact Lagrangian submanifold in tame symplectic manifold. In this article, we use the same cut-off version of this Floer’s perturbed Cauchy-Riemann equation as that of [O4] and adapt the scheme used therein to the current context of our interest in the following way:

(1) We identify non-emptiness of $U \cap L$ and emptiness of intersections $U \cap \phi_{J}^{H}(L)$ as an existence criterion of certain solution of Hamiltonian perturbed Cauchy-Riemann equations.

(2) We then combine some basic energy estimate from [O4] with the ideas from [O5], [BiC] to relate the displacement energy to the $\epsilon$-regularity-type invariant relative to $L$.

In addition to this, the scheme of relating the displacement energy with the $\epsilon$-regularity type invariants resembles that of the proof of nondegeneracy of spectral norm of $\text{Ham}(M, \omega)$ given in [O5].

We would like to thank M. Kawasaki for informing us of some positivity result he obtained via a study of Lagrangian spectral invariants for monotone Lagrangian submanifolds whose Floer cohomology is non-trivial. This was the starting point of our investigation of the question of displacing general Lagrangian submanifold from...
an open subset. We also thank him for interesting discussions on the problem in
the early stage of current research while he was a member of IBS-CGP. Discussion
with him much helped the author crystalizing the main scheme of the proof.

After the paper was posted in arXiv, Jun Zhang attracted our attention to
Usher’s previous work \[\text{[U2 Corollary 4.10]}\] in which the same positivity statement
is proved with a slightly different kind of lower bound again by using the framework
of \[\text{[O4]}\] as in the present paper. We thank Zhang for alerting us for \[\text{[U2]}\] and are
sorry for our omission of that article from our attention.

2. Relative \(\epsilon\)-regularity type invariants

In this section, we explain a direct analog of the invariant \(A(L; M, \omega)\) mentioned
in the introduction. Various kinds of \(\epsilon\)-regularity type invariants were introduced
in \[\text{[O5 Section 4]}\] and related to the displacement of symplectic balls. We closely
follow similar scheme therefrom which we adapt to the present relative context of
the pair \((U, L)\).

We start with some general discussion on Darboux-Weinstein chart. Let \(L \subset M\)
be a compact Lagrangian submanifold. Consider the Darboux-Weinstein chart \(\Phi : U \rightarrow V\)
where \(U\) is a neighborhood of \(L\) in \(M\) and \(V\) is a neighborhood of the
zero section \(o_L \subset T^*L\). Then by definition, we have
\[
\omega = \Phi^*\omega_0, \quad \omega_0 = -d\theta
\]
for the Liouville one-form \(\theta\) on \(T^*L\) and \(\Phi|L = id_L\) under the identification of \(L\)
with \(o_L\).

Fix any Riemannian metric \(g\) on \(L\). For \(x \in U\), we define
\[
\|x\|_{g, \Phi} = \|\Phi(x)\|_{g(\pi(\Phi(x)))}
\]
where \(\Phi(x) \in T_{\pi(\Phi(x))}^* L\) and \(\pi : T^*L \rightarrow L\) is the canonical projection, and \(\|\cdot\|_{g(q)}\)
is the norm on \(T_q^*L\) induced by the inner product \(g(q)\).

**Definition 2.1.** Let \(L \subset M\) be a compact Lagrangian submanifold equipped with
a metric \(g\). Consider the Darboux-Weinstein chart \(\Phi : U \rightarrow V\). Define
\[
\mathbf{w}_{\text{DW}}(\Phi; g) := \inf_{\Phi \in L} \left( \sup_{x \in \pi^{-1}(q) \cap U} \|x\|_{g, \Phi} \right)
\]
and
\[
\mathbf{w}_{\text{DW}}(L; M) = \sup_{\Phi} \mathbf{w}_{\text{DW}}(\Phi; g)
\]
over all Darboux-Weinstein chart of \(L\). We call \(\mathbf{w}_{\text{DW}}(L; M)\) the *Weinstein width* of
\(L\) (relative to the metric \(g\)). We will fix this metric \(g\) on \(L\) throughout the paper.

Obviously \(\mathbf{w}_{\text{DW}}(L; M) > 0\) since \(\mathbf{w}_{\text{DW}}(\Phi; g) > 0\) for any Darboux-Weinstein
chart \(\Phi\) for compact Lagrangian submanifold \(L\).

Next following Biran-Cornea \[\text{[BiC]}\], we introduce

**Definition 2.2.** Let \(e : (B^{2n}(r), \omega_0) \rightarrow (M, \omega)\) be a symplectic embedding of the
closed standard ball of \(B^{2n}(r) \subset \mathbb{C}^n\) of radius \(r\). We say \(e\) is *adapted to \(L\)* or simply
\(L\)-adapted if
\[
e^{-1}(L) = B^{2n}(r) \cap \mathbb{R}^n.
\]

We prove the following existence result on such an \(L\)-adapted embedding.
**Proposition 2.3.** Let $L$ be a compact Lagrangian submanifold of $(M, \omega)$ and let $p \in L$. Then there exists an $L$-adapted embedding $e : B^{2n}(r) \to M$ centered at $p$ for some $r > 0$ whose size depends only on the pair $(M, L)$.

**Proof.** We first choose a Darboux-Weinstein neighborhood $U$ of $L$ in $M$. Then we take a canonical coordinate $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ in a neighborhood $V$ of $p$ in $U$. Using this coordinates, we identify $V$ as an open subset of $\mathbb{C}^n$ by identifying the standard complex coordinates $(z_1, \ldots, z_n)$ to be $z_j = q_j + \sqrt{-1}p_j$. Then we can obviously find a symplectic embedding $e : B^{2n}(r) \to V$ centered at $p$ that also satisfies

$$e(B^{2n}(r)) \cap L = e(B^{2n}(r) \cap \mathbb{R}^n), \quad \mathbb{R}^n \subset \mathbb{C}^n$$

if we choose a sufficiently small $r > 0$. In other words, the resulting $e$ is $(U, L)$-adapted.

We remark that the choice of such a radius $r > 0$ depends only on the width $\omega_{DW}(L; M)$ which in turn depends only on the pair $(M, L)$. This finishes the proof. \qed

With this definition, we recall the following Biran-Cornea’s relative version of Gromov area of the pair $(M, L)$ from [BiC].

We denote by $j$ the standard complex structure of $\mathbb{C}^n$ (or on any 2-dimensional Riemann surface in general).

**Definition 2.4.** Let $L \subset (M, \omega)$ be a Lagrangian submanifold and a symplectic embedding $e : B^{2n}(r) \to M$ relative to $L$ be given. We say a compatible almost complex structure $J_0$ adapted to $e$ if $J_0 = e_* j$ on $e(B^{2n}(r)) \subset M$. We denote by $J_{\omega; e}$ the set of $J_0$ adapted to $e$.

**Definition 2.5.** Let $L \subset M$ be given. Consider pairs $(e, J_0)$ with $J_0$ adapted to a symplectic embedding $e$ adapted to $L$. Call it an adapted pair $L$ or simply an $L$-adapted pair.

Consider a compact surface $\Sigma$ with the decomposition

$$\partial \Sigma = \partial_- \Sigma \cup \partial_+ \Sigma$$

so that $\partial_- \Sigma \cap \partial_+ \Sigma$ consists of a finite number of points. Call a subset $C \subset M$ a $J_0$-holomorphic curve if we can represent $C$ as the image of a somewhere injective $J_0$-holomorphic map $w : \Sigma \to M$. We denote $\partial C = w(\partial \Sigma)$ and $\partial_+ C = w(\partial \Sigma_+)$.

When we are given an open subset $U$ intersecting $L$, we consider $e$ that also satisfies $e(B^{2n}(r)) \subset U$.

**Definition 2.6.** Let $(e, J_0)$ be an $L$-adapted pair.

1. We say a $J_0$-holomorphic curve $C \subset M$ is properly $(L, e)$-adapted if $\partial_- C \subset L$ and $\partial_+ C \cap e(\partial B^{2n}(r)) = \emptyset$.

2. Let $U$ be a given open subset and let symplectic embedding $e$ satisfy $e(B^{2n}(r)) \subset U$. In this case we say $(e, J_0)$ is $(U, L)$-adapted, and the curve $C$ given as above is $(U, L; e, J_0)$-adapted.

The following is the relative analog to the monotonicity formula, which is a consequence of the usual monotonicity formula combined with a doubling argument via the reflection principle for $J_0$-holomorphic discs but this time considering only those $J_0$ coming from $J_{\omega; e}$.
Lemma 2.7. Let \((e, J_0)\) be an \((U, L)\)-adapted pair. Then for any properly \((U, L; e, J_0)\)-adapted curve \(C\), there exists \(r_0 > 0\) depending only on \((U, L)\) such that

\[
\int_{C \cap \delta(B^{2n(r)})} \omega \geq \frac{\pi r^2}{2}
\]  

(2.3)

for all \(0 < r \leq r_0\).

\textit{Proof.} Represent \(C\) as the image of \(J_0\)-holomorphic map \(w : \Sigma \to M\). Since \((e, J_0)\) is \((U, L)\)-adapted, \(e^{-1} \circ w\) which is a holomorphic map into \(\mathbb{C}^n\) with respect to standard complex structure \(j\) with real boundary condition.

Therefore we can apply the standard reflection principle in complex one-variable theory to \(e^{-1} \circ w\), and double it to a surface that is reflection-symmetric. Applying \(e\) back to it, we double \(C\) to a proper \(j\)-holomorphic curve \(S = \overline{\# C}\). Then \(e^{-1}(S)\) defines a proper holomorphic curve in \(B^{2n}(r)\) containing \(0 \in B^{2n}(r) \cap \mathbb{R}^n\). Applying the isoperimetric inequality for holomorphic curves \(e^{-1}(S)\) in \(B^{2n}(r) \subset \mathbb{C}^n\) and the symplectic property of the embedding \(e\), we have

\[
\text{Area}(S) \geq \pi r^2.
\]

Since \(S = \overline{\# C}\) and \(\text{Area}(C) = \text{Area}(\overline{C})\),

\[
\text{Area}(S) = 2 \text{Area}(C) \leq 2 \int w^* \omega
\]

Combining these two inequalities, we have finished the proof of \(2.3\). \(\square\)

Based on this relative monotonicity formula, we proceed the process of defining the analog to the invariant \(A_L(M, \omega)\) relative to an open subset \(U\).

For each given properly \((U, L)\)-adapted pair \((e, J_0)\), we define

\[
A(U, L; e, J_0) = \inf C \left\{ \int_{C \cap \delta(B^{2n(r)})} \omega \bigm| C \text{ is } (U, L; e, J_0)\text{-adapted}, e(0) \in C \right\}.
\]

(2.4)

We put \(A(U, L; e, J_0) = \infty\) as usual if there exists no \(L\)-adapted pair \((e, J_0)\) that admits an \((U, L; e, J_0)\)-adapted curve satisfying \(2.4\). Next we define

\[
A(U, L; M, \omega) = \sup_{(e, J_0)} \{ A(U, L; e, J_0) \bigm| (e, J_0) \text{ is properly } (U, L)\text{-adapted} \}.  \quad (2.5)
\]

Next we introduce a restricted version of \(A(J_0; M, \omega)\) and \(A(J_0, L; M, \omega)\) given in \(1.3\). We define \(A^U(J_0; M, \omega)\) (resp. \(A^U(J_0; M, \omega)\)) in the same way as that of \(A(J_0; L; M, \omega)\) (resp. of \(A(J_0; M, \omega)\)) given in the introduction, but restricting \(J_0\) to those contained in \(J_{e, \omega}\) for some \((U, L)\)-adapted embedding \(e\). Then define

\[
A^U(L; M, \omega) = \sup_{(e, J_0)} \min \{ A^U(J_0; M, \omega), A^U(J_0, L; M, \omega) \} \quad (2.6)
\]

where we take the supremum over all \((U, L)\)-adapted pair \((e, J_0)\).

Finally we are arrived at the definition of the invariant we have been seeking for.

\textbf{Definition 2.8.} We denote

\[
\epsilon(U, L; M, \omega) = \min \{ A^U(L; M, \omega), A(U, L; M, \omega) \}.
\]

A priori the possibility of \(\epsilon(U, L; M, \omega) = \infty\) is not ruled out. The following theorem will guarantee that this will not happen under the circumstance of Main Theorem.
Theorem 2.9. Let \((M, \omega)\) is a tame symplectic manifold and \(L \subset M\) be a compact Lagrangian submanifold. Let \(U\) be an open subset such that \(L \cap U \neq \emptyset\) and \(\overline{\cap} \omega (L) = \emptyset\) for some Hamiltonian diffeomorphism \(\phi\). Then \(0 < \epsilon(U, L; M, \omega) < \infty\).

We will give its proof in the course of proving Main Theorem. The main task is to establish an existence result of a \((U, L)\)-adapted pair \((e, J_0)\) that admits a properly \((L, e)\)-adapted \(J_0\)-holomorphic curve \(C\).

One way of producing such a curve \(C\) appearing above is as follows. Consider a map \(v : \mathbb{R} \times [0, 1] \to M\) that satisfies

\[
v(0, 0) \in U \cap L
\]

and the genuine Cauchy-Riemann equation

\[
\begin{cases}
\frac{\partial v}{\partial \tau} + J_0 \frac{\partial v}{\partial t} = 0 \\
v(\tau, 0) \in L, v(\tau, 1) \in \phi^1_H(L).
\end{cases}
\tag{2.7}
\]

We recall that any stationary i.e., any \(\tau\)-independent finite energy solution of (2.7) is a constant solution valued in \(L \cap \phi^1_H(L)\). The following lemma is a key lemma that enters in our construction of an \((U, L; e, J_0)\)-adapted curve which plays an important role in the proof of Main Theorem.

Lemma 2.10. Suppose \(L \cap U \neq \emptyset\) and \(U \cap \phi^1_H(L) = \emptyset\). Let \((e, J_0)\) be a \((U, L)\)-adapted pair. Then for any finite energy solution \(v\) of (2.7), there exists some \(R_0 > 0\) such that

\[
\text{Image } v|_{[\mathbb{R}] \setminus [-R_0, R_0]} \times [0, 1] \subset M \setminus U.
\tag{2.8}
\]

Proof. We prove it by contradiction. Suppose to the contrary that there exists a sequence \(R_j \to \infty\) such that

\[
\text{Image } v|_{[\mathbb{R}] \setminus [-R_j, R_j]} \times [0, 1] \cap U \neq \emptyset.
\tag{2.9}
\]

Pick a point \(q_j\) from \(\text{Image } v|_{[\mathbb{R}] \setminus [-R_j, R_j]} \times [0, 1] \cap U\) for each \(j\). By choosing a subsequence, if necessary, we can express \(q_j = v(R_j^1, t_j)\) or \(q_j = v(-R_j^1, t_j)\) for \(R_j^1 \geq R_j\) for \(j = 1, 2, \ldots\). Without loss of any generality, we may assume \(q_j = v(R_j^1, t_j)\) since the other case can be treated the same. Again by choosing a subsequence, we may assume \(q_j \to q\) for some point \(q \in U\). (Recall we assume that \(M\) is tame and \(L\) is compact. It is easy to derive from the monotonicity formula that the image \(\text{Image } v\) is bounded and so the set (2.9) is pre-compact.)

Then we consider the path \(z_j : [0, 1] \to M\) defined by

\[z_j(t) = v(R_j^1, t)\]

On the other hand, by the finite energy condition of \(v\), we have

\[
\lim_{j \to \infty} E_{J_0} (v|_{[\mathbb{R}] \setminus [-R_j, R_j]} \times [0, 1]) = 0.
\]

Using the standard \(\epsilon\)-regularity theorem (see [O1] Proposition 3.3 for example) applied to \(v\) on the domains of the uniform size

\[|R_j^1 - 1, R_j + 1| \times [0, 1] \cong [-1, 1] \times [0, 1]\]

we obtain a convergence \(\|z_j\|_{C^0} \to 0\) of the \(C^1\)-norm of \(z_j\) as \(j \to \infty\). Therefore since \(q_j = z_j(t_j)\) and \(q_j \to q\), this implies \(z_j\) uniformly converges to a constant path \(z\) valued at \(q\), i.e., \(z(t) = q\) for all \(t \in [0, 1]\). Furthermore the boundary condition

\[v(\tau, 0) \subset L, \ v(\tau, 1) \subset \phi^1_H(L)\]
of \( v \) also implies \( z_j(0) \in L \) and \( z_j(1) \in \phi^1_H(L) \). This implies \( q \in L \cap \phi^1_H(L) \).

Combining the above, we conclude that \( q \in L \cap \phi_H(L) \cap \overline{U} \) which contradicts to the hypothesis \( \overline{U} \cap \phi^1_H(L) = \emptyset \). This finishes the proof. \( \square \)

**Remark 2.11.** If we know that \( v \) uniformly converges as \( \tau \to \pm \infty \) as in the case of transversal intersection \( L \cap \phi^1_H(L) \), we can simply write as \( v(\pm \infty) \in M \setminus U \) instead of (2.8) in the statement of Lemma 2.10. Since we do not impose this transversal intersection property, the statement of this lemma is the only thing we can achieve for the general case. This will be enough for our purpose.

Then the curve \( C = \text{Image } v \) is one that can be used in Theorem 2.9, which will then prove finiteness of \( \epsilon(U, L; M, \omega) \).

Furthermore we also have the following lower bound of the symplectic area of the curve \( v \).

**Proposition 2.12.** Under the same hypotheses as in Lemma 2.10 any finite energy solution \( v \) of (2.7) with \( v(0, 0) = p \) satisfies

\[
\int v^* \omega \geq \frac{\pi r^2}{2}. \tag{2.10}
\]

**Proof.** By the hypothesis \( \phi^1_H(L) \cap \overline{U} = \emptyset \),

\[
v(\tau, 1) \in M \setminus U
\]

for all \( \tau \in \mathbb{R} \) since we have \( v(\tau, 1) \in \phi^1_H(L) \) by the boundary condition at \( t = 1 \). By Lemma 2.10, \( v \) can not be a constant map since \( v(0, 0) = p \in L \).

Furthermore by \((U, L)\)-adaptedness of the embedding \( e \) and since \( v(0, 0) = p \), we have

\[
e(B^{2n}(r) \cap \mathbb{R}^n) = e(B^{2n}(r)) \cap L \subset U \cap L.
\]

Then, \( v \), restricted to the connected component of

\[
v^{-1}(\text{Int } e(B^{2n}(r)) \cap \text{Im } v) \subset \mathbb{R} \times [0, 1]
\]

containing the point \((0, 0)\), defines a surface \( C \) that is \((U, L; e, J_0)\)-adapted. Now Lemma 2.7 finishes the proof. \( \square \)

Therefore we would like to produce a \( J_0 \)-holomorphic map \( v \) used in Proposition 2.12. For this purpose, we exploit the correspondence between the dynamical version and the geometric version for the Lagrangian intersection Floer equations in the spirit of [O5] where this correspondence was extensively used for the applications of spectral invariants to the geometry of Hamiltonian diffeomorphism group \( \text{Ham}(M, \omega) \).

Let \( H = H(t, x) \) be any given compactly supported Hamiltonian and denote \( \phi = \phi^1_H \). We require \( J \) to satisfy the condition

\[
J(t, x) = (\phi^1_H)^* J_0 \tag{2.11}
\]

and consider the associated perturbed Cauchy-Riemann equation

\[
\begin{align*}
\frac{\partial u}{\partial \tau} + J_t(\frac{\partial u}{\partial t} - X_H(u)) &= 0 \\
u(\tau, 0) &\in L, \; u(\tau, 1) \in L.
\end{align*} \tag{2.12}
\]

Let \( u \) be any such solution of (2.12) and \( v \) be the map defined by

\[
v(\tau, t) = \phi^1_H(u(\tau, t)). \tag{2.13}
\]
The associated energy is given by

\[ E_{(J,H)}(u) = \frac{1}{2} \int \int \left( \frac{\partial u}{\partial \tau} \right)^2 + \left( \frac{\partial u}{\partial t} - X_H(u) \right)^2 \, dt \, dr. \]

The following lemma is standard, which follows from direct calculation.

**Lemma 2.13.** Let \( J_t = (\phi_t^H)^* J_0 \). For a given finite energy solution \( u \) of (2.12), consider the map \( v : [0,1] \times \mathbb{R} \to M \) the map as defined in (2.13). Then \( v \) satisfies

\[ E_{(J,H)}(u) = E_{J_0}(v) = \int v^* \omega. \quad (2.14) \]

An immediate corollary of the above discussion is the following

**Corollary 2.14.** Let \( U \cap \phi_H^1(L) = \emptyset \) and \( p \in U \cap L \). Suppose there exists a solution \( u \) of (2.12) satisfying \( u(0,0) = p \). Then

\[ \epsilon(U, L; M, \omega) > 0. \quad (2.15) \]

**Proof.** By Proposition 2.3 there exists a \((U, L)\)-adapted embedding \( e : B^{2n}(r) \to M \), i.e., one satisfying

\[ e(B^{2n}(r)) \subset U, \quad e(B^{2n} \cap \mathbb{R}^n) \subset U \cap L \]

for some \( r > 0 \). Then we consider the map \( v \) defined as in (2.13). This map \( v \) satisfies the conditions given in Proposition 2.12. In particular existence of such a map \( v \) proves \( \epsilon(U, L; M, \omega) > 0 \). \( \square \)

With this definition of \( \epsilon(U, L; M, \omega) \), here is the precise version of Main Theorem.

**Theorem 2.15.** Let \((M, \omega)\) be a tame symplectic manifold and \( L \subset M \) be a compact Lagrangian submanifold. Suppose \( U \cap L \neq \emptyset \). Then

\[ \epsilon(U, L) \geq \epsilon(U, L; M, \omega). \]

In the rest of the paper, we give the proof of Theorem 2.15. Along the way, we will also prove Theorem 2.4.

**Remark 2.16.** In practice, the usage of this theorem is two-fold as in [O5]: one is for the lower bound for the displacement energy between \( L \) and \( U \) and the other is for the upper bound for the areas of relevant pseudoholomorphic curves. The latter measures the maximal possible size of the open subset \( U \) displaceable from \( L \) through the chain of inequalities

\[ \epsilon(U, L) \geq \epsilon(U, L; M, \omega) \geq \frac{\pi r^2}{2} \]

for \((U, L)\)-adapted symplectic embedding \( e : B^{2n}(r) \to M \) displaceable from \( L \).

3. CUT-OFF PERTURBED CAUCHY-RIEMANN EQUATIONS

In this section, we largely borrow verbatim the basic framework that was used by the author in [O4] for the study of displacement energy of compact Lagrangian submanifold from itself.

We first recall the well-known correspondence between the Lagrangian intersections \( \phi_H^1(L) \cap L \) and the set of Hamiltonian chords of \( L \). Let \( \phi \) be a Hamiltonian diffeomorphism of \((M, \omega)\). Let \( L \) be a compact Lagrangian submanifold. We have
one-one correspondence between \( L \cap \phi(L) \) with the set of solutions \( z : [0, 1] \to M \) of
\[
\dot{z} = X_H(t, z), \quad z(0), z(1) \in L. \tag{3.1}
\]
Here is the precise correspondence:
\[
p \in L \cap \phi(L) \longleftrightarrow z = z^H_p \text{ with } z^H_p(t) := \phi^t_H((\phi^1_H)^{-1}(p)). \tag{3.2}
\]
Following [O4], for each \( K \in \mathbb{R}_+ = [0, \infty) \), we define a function \( \rho_K : \mathbb{R} \to [0, 1] \)
as follows: for \( K \geq 1 \), we define
\[
\rho_K(\tau) = \begin{cases} 
0 & \text{for } |\tau| \geq K + 1 \\
1 & \text{for } |\tau| \leq K
\end{cases}
\]
with
\[
\rho'_K < 0 \quad \text{for } K < \tau < K + 1 \\
> 0 \quad \text{for } -K - 1 < \tau < -K
\]
and for \( 0 \leq K \leq 1 \),
\[
\rho_K = K \cdot \rho_1.
\]
In particular, \( \rho_0 \equiv 0 \).

Let \( H : [0, 1] \times M \to \mathbb{R} \) be a Hamiltonian such that
\[
U \cap \phi^1_H(L) = \emptyset, \tag{3.3}
\]
i.e., such that the equation
\[
\begin{cases}
\dot{z} = X_H(z) \\
z(0), z(1) \in L
\end{cases}
\]
has no solution satisfying
\[
z(0) \in L, \quad z(1) \in U.
\]
Then we consider a three-parameter family \( J = \{J(K, \tau, t)\}_{(K, \tau, t) \in \mathbb{R} \times \mathbb{R} \times [0, 1]} \) of tamed almost complex structures such that
\[
J_{(K, \tau, t)} = \begin{cases}
J_0 & \text{for } |\tau| \text{ sufficiently large or for } t = 0, 1 \\
(\phi^t_H)^*J_0 & \text{for } -K \leq \tau \leq K
\end{cases} \tag{3.4}
\]
where \( J_0 \) is a fixed (genuine) almost complex structure on \( M \) that is tamed to \( \omega \).

We would like to remark that it is necessary to vary almost complex structures in terms of \( t \) to get appropriate transversality result for the Floer complex (see [FHS], [O3] for detailed account of the transversality proof).

Throughout this paper, we will exclusively denote by \( J_0 \) any (genuine) almost complex structure and by \( J \) a (domain dependent) two-parameter version of them. We denote a one-parameter family of them by
\[
\overline{J} = \{J_K\}_{K \in (0, +\infty)}
\]
such that
\[
J_K = J_\infty = J_\infty(\tau, t) \quad \text{for sufficiently large } K. \tag{3.5}
\]
For each such pair \( (\overline{J}, H) \), we consider one parameter family of perturbed Cauchy-Riemann equations for the map \( u : \mathbb{R} \times [0, 1] \to M \),
\[
\begin{cases}
\frac{\partial u}{\partial \tau} + J_K(\tau, t, u)\left(\frac{\partial u}{\partial t} - \rho_K(\tau)X_H(u)\right) = 0 \\
u(\tau, 0), u(\tau, 1) \in L
\end{cases} \tag{3.6}
\]
for each $K \in \mathbb{R}_+$.  

**Remark 3.1.** This equation should be regarded as the one used for the chain isotopy in the Floer homology theory connecting the Hamiltonian 0 to $H$ and then to 0 back.

The relevant energy of general smooth map $u : \mathbb{R} \times [0, 1] \to M$ for the equation (3.6) is given by

$$E_{(J_K, H)}(u) = \frac{1}{2} \int \int |\frac{\partial u}{\partial \tau}|_{J_K}^2 + |\frac{\partial u}{\partial t} - \rho_K(\tau)X_H(u)|_{J_K}^2 \, dt \, d\tau.$$  

We will be interested in the solutions of (3.6) with finite energy. We note that the energy is reduced to

$$E_{(J_K, H)}(u) := \int_{-\infty}^{\infty} \int_0^1 |\frac{\partial u}{\partial \tau}|_{J_K}^2 \, dt \, d\tau < \infty$$  

(3.7)

for a solution $u$ of (3.6). We denote by $M_{K}(J, H)$ the set of finite energy solutions thereof.

Noting that $\mathbb{R} \times [0, 1]$ is conformally isomorphic to $D^2 \{ -1, 1 \}$, it follows from the choice of the cut-off function $\rho_K$ that (3.6) and (3.7) imply that the map $u \circ \varphi : (D^2 \{ -1, 1 \}, \partial D^2 \{ -1, 1 \}) \to (M, L)$, with a conformal diffeomorphism $\varphi$, has finite (harmonic) energy and $J_0$-holomorphic near $\{ -1, 1 \}$. Then the removable singularity theorem [O1] enables us to extend this to the whole disc, which we denote by

$$\tilde{u} : (D^2, \partial D^2) \to (M, L)$$

is smooth. We denote by $[u] \in \pi_2(M, L)$ the homotopy class defined by $\tilde{u}$.

Now for each $K \in \mathbb{R}_+$ and for $A \in \pi_2(M, L)$, we study the following moduli space

$$M_K(\mathcal{J}, H; A) = \{ u : \mathbb{R} \times [0, 1] \to M \mid u \text{ satisfies (3.6), } E_{(J_K, H)}(u) < \infty \text{ and } [u] = A \text{ in } \pi_2(M, L) \}.  \quad (3.8)$$

Since (3.6) is a compact perturbation of the standard pseudo-holomorphic equation of discs with Lagrangian boundary condition, the standard index formula from [G] implies

$$\dim M_K(\mathcal{J}, H; A) = \mu_L(A) + n$$

for generic $\mathcal{J}, H$, provided it is non-empty. Here $n$ is the dimension of the Lagrangian submanifold $L$ and $\mu(A)$ is the Maslov index of the map $u : (D^2, \partial D^2) \to (M, L)$ in class $[u] = A$. Then we have the decomposition

$$M_K(\mathcal{J}, H) = \bigcup_{A \in \pi_2(M, L)} M_K(\mathcal{J}, H; A).$$

**Lemma 3.2.** $M_K(\mathcal{J}, H; A)$ for $K = 0$, $A = 0$ in $\pi_2(M, L)$ consists of constant solutions and is Fredholm regular for any almost complex structure $J_0$. In particular $M_0(\mathcal{J}, H; 0)$ is diffeomorphic to $L$. 

Proof. Let \( u \in \mathcal{M}_0(J, H; 0) \). Recall that for \( K = 0 \), \( 3.6 \) becomes
\[
\frac{\partial u}{\partial \tau} + J_0 \frac{\partial u}{\partial t} = 0, \quad u(\tau, 0), u(\tau, 1) \subset L.
\]
Since \( [u] = 0 \), the associated disc \( \tilde{u} \) above must be constant. The Fredholm regularity of constant solutions is not difficult to check and is well-known. The last statement follows by considering the evaluation map \( ev : \mathcal{M}_0(J, H; 0) \rightarrow L \) given by \( ev(u) = u(0, 0) \). □

We next state a simple but a fundamental a priori energy bound for any element \( u : \mathbb{R} \times [0, 1] \rightarrow M \) of the moduli space (3.8), whose proof is given in the proof of Lemma 2.2 [O4]. (See also Remark 2.3 therein.) We omit its proof here referring readers to [O4] or to [O6, Lemma 11.2.6] for the details of the proof.

Lemma 3.3. For all \( K \geq 0 \) and \( A \in \pi_2(M, L) \), we have
\[
E_{(J_K, H)}(u) \leq \omega(A) + \|H\| \tag{3.9}
\]
if \( [u] = A \in \pi_2(M, L) \). In particular, when \( A = 0 \), we have
\[
E_{(J_K, H)}(u) \leq \|H\|. \tag{3.10}
\]
Note that Lemma 3.2 and (3.9) hold for any \( J \) and \( H \) that satisfy (3.6) and (3.7) respectively. Therefore we can do the standard Fredholm theory and the genericity arguments with such pairs \((J, H)\). (See also Remark 5.2 (1) below.) We will always carry out this standard genericity argument without further mentioning details, whenever necessary.

4. Creating a Hamiltonian chord

Let \( H : [0, 1] \times M \rightarrow \mathbb{R} \) be a given (generic) Hamiltonian. For generic \( J = \{J_K\}_{K \in [0, +\infty)} \) satisfying (2.3), we form the parameterized moduli space
\[
\mathcal{M}_{\text{para}}(J, H; A) := \bigcup_{K \in \mathbb{R}_+} \{K\} \times \mathcal{M}_K(J, H; A)
\]
for each \( A \in \pi_2(M, L) \). It becomes a smooth manifold of dimension \( \mu(A) + n + 1 \) with boundary by the parameterized version of the index theorem (2.7). We consider the evaluation map
\[
Ev_A : \mathcal{M}_{\text{para}}(J, H; A) \times \mathbb{R} \rightarrow L \times \mathbb{R}_+ \times \mathbb{R}; \quad (u, K, \tau) \mapsto (u(\tau, 0), K, \tau). \tag{4.1}
\]
We also consider the evaluation map
\[
ev_{A,K} : \mathcal{M}_K(J, H; A) \rightarrow L; \quad ev_{A,K}(u) = u(0, i), \quad i = 0, 1. \tag{4.2}
\]
The following is the main ingredient in our proof. This proposition is the counterpart of Lemma 2.2 [O4].

Proposition 4.1. Let \( K_\alpha \rightarrow \infty \) and let \( u_\alpha \in \mathcal{M}_{K_\alpha}(J, H) \) be a sequence satisfying \( u_\alpha(0, 0) = p \) for a given \( p \in L \) and the energy bound
\[
E_{(J_{K_\alpha}, H)}(u_\alpha) < C \tag{4.3}
\]
for all \( \alpha \) for a constant \( C > 0 \). Then
1. either the given point \( p \) is contained in \( u_0(\mathbb{R} \times \{0\}) \) for some non-stationary solution \( u_0 \) of (2.12),
2. or there exists a non-constant \( J_0 \)-holomorphic disc \( w : (D^2, \partial D^2) \rightarrow (M, L) \) with \( w(\partial D^2) \subset L \) with \( p \in w(\partial D^2) \).

Proof. Using the a priori bound (3.10), we study a local limit of \( u_\alpha \). More specifically, we consider a limit of the sequence

\[
u_\alpha |[-K_\alpha, K_\alpha] \times [0, 1]\]

on every compact subset of \( \mathbb{R} \times [0, 1] \) as \( \alpha \to \infty \), by taking a subsequence if necessary. By the energy bound (3.8) and Dominated Convergence Theorem,

\[ E(J_{K_\alpha}, H) \left( u|_{\mathbb{R} \setminus [-K_\alpha, K_\alpha] \times [0, 1]} \right) \to 0 \]

as \( \alpha \to \infty \). We recall the readers from (3.10) that \( J_K = J_\infty \) for all sufficiently large \( K \). Therefore by the choice of \( \rho_K \) and \( J_K \) that \( u_\alpha \) satisfies the equation

\[
\begin{align*}
\frac{\partial u}{\partial \tau} + J_K \left( \frac{\partial u}{\partial t} - X_H(u) \right) &= 0 \\
u_\alpha(\tau, 0), \ u_\alpha(\tau, 1) &\in L
\end{align*}
\]

on \( [-K_\alpha, K_\alpha] \times [0, 1] \).

Then via the standard diagonal subsequence argument, the energy bound (3.10) and Gromov-Floer compactness theorem (or rather the way how it is proved) applied to \( u_\alpha |[-K_\alpha, K_\alpha] \times [0, 1] \) produce a limit \( u_\infty \) that has the decomposition

\[ u_\infty = u_0 + \sum_i v_i + \sum_j w_j \]

for a collection of \( J_\infty, (\tau, t, \cdot) \)-holomorphic spheres \( v = \{v_i\}_{i=1}^{N_1} \) with some \( (\tau_i, t_i) \in \mathbb{R} \times [0, 1] \), and a collection \( w = \{w_j\}_{j=1}^{N_2} \) of \( J_0 \)-holomorphic discs \( w_j : (D^2, \partial D^2) \to (M, L) \) respectively. And \( u_0 : \mathbb{R} \times [0, 1] \to M \) is a uniform limit of \( u_\alpha \) on compact subsets of \( \mathbb{R} \times [0, 1] \) modulo bubbling and satisfies the equation

\[
\begin{align*}
\frac{\partial u}{\partial \tau} + J_\infty(t, u) \left( \frac{\partial u}{\partial t} - X_H(u) \right) &= 0 \\
u(\tau, 0), \ u(\tau, 1) &\in L.
\end{align*}
\]

We also have the energy bound

\[
E(J_\infty, H)(u_0) + \sum_j \omega([w_j]) + \sum_i \omega([v_i]) \leq C \tag{4.4}
\]

where \( \omega([w_j]), \omega([v_i]) \) are the symplectic areas. There are two alternatives:

1. Either the given point \( p \) is contained in \( u_0(\mathbb{R} \times \{0\}) \),
2. or the point is contained in one of the disc bubbles \( w_j \).

(We recall that \( p \) is contained in boundary of \( \mathbb{R} \times [0, 1] \).) In Case (1), \( u \) cannot be a stationary solution \( u(\tau, t) \equiv z(t) \) where \( z \) is a Hamiltonian chord of \( L \). For otherwise, the map \( v \) defined by (2.13) would be also stationary which means \( v(\tau, t) \equiv \text{const.} \), and hence \( v(0, 1) = v(0, 0) \). Therefore we would have the chain of identities

\[ p = u(0, 0) = v(0, 0) = v(0, 1) = \phi_H^1(u(0, 1)) \]

which implies \( p \in \phi_H^1(L) \cap \overline{U} \). This would contradict to \( \phi_H^1(L) \cap \overline{U} = \emptyset \). This proves that \( u_0 \) cannot be stationary.

In Case (2), it follows from the way how the convergence modulo the bubbles is derived that there is a \( J_0 \)-holomorphic disc \( w : (D^2, \partial D^2) \to (M, L) \) satisfying \( w(\partial D^2) \subset L \) and \( p \in w(\partial D^2) \). This finishes the proof. \( \square \)

An immediate consequence of (4.4) is the following

\[
\]
Corollary 4.2. Suppose \( \overline{U} \cap \phi_H^p(L) = \emptyset \) and \( p \in U \cap L \). If there is a sequence \( K_\alpha \to \infty \) such that \( \mathcal{M}_{K_\alpha}(\overline{J}, H) \cap ev_{0,K_\alpha}(p) \neq \emptyset \), then
\[
\|H\| \geq \epsilon(U, L; M, \omega).
\]

Proof. We start with (4.4) with \( C = \|H\| \) recalling the energy estimate (3.10), and the above two alternatives.

We consider the case (1) first, say \( u_0(\tau_0, 0) = p \) for some \( \tau_0 \in \mathbb{R} \). Consider the map \( v_0 : \mathbb{R} \times [0, 1] \to M \) defined by
\[
v_0(\tau, t) = \phi_H^t(u_0(\tau, t)).
\]
Then \( v_0 \) satisfies \( \overline{J}_\alpha v_0 = 0 \) by the choice (2.11). Furthermore
\[
v_0(\tau_0, 0) = p \in L \cap U, \quad v_0(\tau, 1) = \phi_H^1(v_0(\tau, 1)) \in \phi_H^1(L).
\]
Therefore we have created a \( J_0 \)-holomorphic curve \( C \) which is properly \((U, L)\)-adapted. By definition of \( A(U, L; M, \omega) \), we have obtained
\[
\|H\| \geq E_{(\overline{J}_\alpha, H)}(u_0) = \int v_0^* \omega \geq \int_C \omega = A(U, L; M, \omega).
\] (4.5)

Now consider the case (2). Then \( p \) is in the image of \( w_j \) for some \( j \). Note that if \( w_j \) contains \( p \), then it must also hold that \( w_j(z_0) = p \) for some \( z_0 \in \partial D^2 \) and \( w_j(\partial D^2) \subset L \). Therefore we can take \( C \) to be the connected component of \( \text{Im } w_j \) containing \( p \). This time we have
\[
\|H\| \geq \int w_j^* \omega \geq A^U(L; M, \omega).
\] (4.6)
Combining (4.5) and (4.6), we have finished the proof by the definition \( \epsilon(U, L; M, \omega) \) in Definition 2.8

Another corollary of the existence result stated in Proposition 4.1 is the following positivity result whose proof has been postponed in Theorem 2.9 until now.

Corollary 4.3. Under the hypotheses of Theorem 2.9, we have
\[
0 < \epsilon(U, L; M, \omega) < \infty.
\]

Proof. Take \( C = \|H\| \) in Proposition 4.1. For the case (1), we have \( A(U, L; e, J_0) < \infty \) and for the case of (2) \( A(L; J_0, \omega) < \infty \). This proves \( \epsilon(U, L; M, \omega) < \infty \). On the other hand, \( 0 < \epsilon(U, L; M, \omega) \) follows from Corollary 2.14.

5. PROOF OF THE MAIN THEOREM

We go back to the situation of the Main Theorem, where \( L \) is displaceable from \( U \) and \( L \cap U \neq \emptyset \) so that there exists a Hamiltonian \( H \) such that \( \phi_H^p(L) \cap \overline{U} = \emptyset \). Let \( p \in L \cap U \), fix a symplectic embedding \( e : B^{2n}(r) \to M \) with \( p = e(0) \) adapted to \((U, L)\). Then we consider the set \( \mathcal{J}_{\alpha,e} \) of almost complex structures adapted to \( e \) and form the time-dependent family \( J = \{ J_t \} \) and then \( \overline{J} = \{ J_K \} \) as in Section 3.

The following is the basic structure theorem of \( \mathcal{M}_K(J, H; A) \) whose proof is standard and so is omitted.

Proposition 5.1. (1) For each fixed \( K > 0 \), there exists a generic choice of \( (\overline{J}, H) \) such that \( \mathcal{M}_K(\overline{J}, H; A) \) becomes a smooth manifold of dim \( n + \mu_L(A) \) if non-empty. In particular, if \( A = 0 \), \( \dim \mathcal{M}_K(\overline{J}, H; A) = n \) if non-empty.
For the case $A = 0, K = 0$, all solutions are constant and Fredholm regular and hence $\mathcal{M}_K(\mathcal{J}, H; A) \cong L$. Furthermore the evaluation map $\text{ev}_{0,0} : \mathcal{M}_0(\mathcal{J}, H; 0) \to L : u \mapsto u(0,0)$ is a diffeomorphism.

Let $K_0 > 0$ and assume $\mathcal{M}_{K_0}(\mathcal{J}, H; A)$ is regular. Then the parameterized moduli space $\mathcal{M}^\text{para}_{[0,K_0]}(\mathcal{J}, H; A) := \bigcup_{K \in [0,K_0]} \{K\} \times \mathcal{M}_K(\mathcal{J}, H; A) \to [0,K_0]$ is a smooth manifold with boundary, not necessarily compact, given by $\left(\{0\} \times \mathcal{M}_0(\mathcal{J}, H; A)\right) \coprod \left(\{K_0\} \times \mathcal{M}_{K_0}(\mathcal{J}, H; A)\right)$ and the evaluation map $\text{Ev}_A : \mathcal{M}^\text{para}_{[0,K_0]}(\mathcal{J}, H; A) \times \mathbb{R} \to L \times \mathbb{R}^+ \times \mathbb{R} : ((K, u), \tau) \mapsto (K, u(\tau,0), \tau)$ is smooth.

Remark 5.2. (1) We note that the $J_0$ we are using comes from $J_\omega e$ not from the set of all compatible almost complex structures. Therefore we need to make sure the standard transversality proof such as [Fl, FHS, O3] can be applied for this restricted class. But this can be seen from the fact that there is no non-constant solution of $(3.6)$ or $(2.12)$ whose image is entirely contained in $e(B^{2n}(r))$. (See p.323-324 [O2], especially the top of p.324 for the explanation in a similar context.)

(2) We also mention that the way how we present our proof, especially Proposition 4.1, is deliberately devised so that no transversality result for the bubbles either of the spheres or of the discs, nor the intersection theory between the principal components with bubbles enter in the proof. Only the Gromov-Floer compactness and the transversality result mentioned in Lemma 3.2 and the one in (1) of this remark are used. Both transversality results are easy and standard. This enables us to dispose any virtual cycle technique and any kind of positivity hypothesis of Lagrangian submanifolds or of symplectic manifolds both in the statement of and in the proof of our main theorem.

With this discussion above in mind, we use a priori bound (3.10) to apply the following Gromov-Floer compactness theorem [G], [Y], [Fl], [FO].

Proposition 5.3. Let $K_\alpha$ with $\alpha = 1, \cdots$ converging to $K' \in \mathbb{R}^+$ and $u_\alpha$ be a sequence of solutions of $(3.6)$ for $K = K_\alpha$ with uniform bound $E_{J_\omega, H}(u_\alpha) < C$ for $C$ independent of $\alpha$.

Then there exist a subsequence again enumerated by $u_\alpha$ and a cusp-trajectory $(u_0, v, w)$ such that

1. $u_0$ is a solution of $(3.6)$ with $K = K'$.
2. $v = \{v_i\}_{i=1}^{N_1}$ where each $v_i$ is a $J_{(\tau, t_i)}$-holomorphic sphere and $w = \{w_j\}_{j=1}^{N_2}$ each $w_j$ is a $J_0$-holomorphic disc with boundary lying on $L$.
3. We have the convergence $\lim_{\alpha \to \infty} E_{J_{(K_\alpha, H)}}(u_\alpha) = E_{J_{K'}}(u_0) + \sum_i \omega([v_i]) + \sum_j \omega([w_j])$. 


(4) And $u_\alpha$ converges to $(u_0, v, w)$ in Hausdorff topology and converges in compact $C^\infty$ topology away from the nodes.

Wrap-up of the proof of Theorem 2.15. For the simplicity of notations, we will just denote $ev_K = ev^{0}_{0,K}$ defined in (4.2).

We consider two cases separately. The first case is when the hypothesis of Proposition 4.1 holds so that there exists a sequence $K_\alpha \to \infty$ for which

$$M_{K_\alpha}(\mathcal{J}, H) \cap E^{-1}_{(H_{K_\alpha}, H)}([0, C]) \cap ev_{K_\alpha}^{-1}(p) \neq \emptyset$$

with the choice of

$$C = \|H\|.$$  

Then Proposition 2.12, (2.15) and Corollary 4.2 already prove Theorem 2.15.

Therefore it remains to consider the case where there is a constant $K_0$ for which

$$M_{K}(\mathcal{J}, H) \cap E^{-1}_{(H_{K_\alpha}, H)}([0, C]) \cap ev_{K}^{-1}(p) = \emptyset$$  

for all $K \geq K_0$. We fix one such $K_0 > 0$ in the rest of the proof. In particular, we have

$$M_{K}(\mathcal{J}, H; 0) \cap ev_{K}^{-1}(p) = \emptyset$$

by the energy estimate (3.10).

Now consider an embedded small loop $\gamma : [0, 1] \to L$ such that

$$p = \gamma(0) = \gamma(1), \quad \text{Image} \gamma \subset L \cap U.$$  

(5.2)

We may choose $\gamma$ so that Image $\gamma$ is as close to $p$ as we want.

Then we consider a smooth embedded path $\Gamma : [0, 1] \to L \times \mathbb{R}_+ \times \mathbb{R}$ with

$$\Gamma(s) = (\gamma(s), K(s), \tau(s))$$

such that

$$K(0) = 0, \quad \text{and} \quad K_0 \leq K(1) \leq 2K_0.$$  

(5.3)

Recall that $N_\Gamma$ is regular at $s = 0, 1$: This is because

$$M_{K(1)}(\mathcal{J}, H; 0) \cap ev^{-1}_{K(1)}(p) = \emptyset$$  

(5.4)

by the choice of $K_0$ in (5.3) for which the regularity statement is vacuous. On the other hand for $s = 0$, since $K(0) = 0$

$$M_{K(0)}(\mathcal{J}, H; 0) \cap ev^{-1}_{K(0)}(p) = M_{0}(\mathcal{J}, H; 0) \cap ev^{-1}_{0}(p).$$

But the latter intersection consists of a single element which is a constant map. This constant map is regular by Lemma 3.2.

Applying Proposition 5.1 and the transversality extension theorem, for a generic choice of $\Gamma$, we can make the map (4.1) transversal to the path $\Gamma$ so that $N_{\Gamma} := Ev^{-1}_{0}(\Gamma)$ becomes a one dimensional manifold with its boundary consisting of

$$M_{K(0)}(\mathcal{J}, H; 0) \cap ev^{-1}_{0}(p) \bigoplus M_{K(1)}(\mathcal{J}, H; 0) \cap ev^{-1}_{K(1)}(p).$$

However the second summand is empty by (5.4) and so $\partial N_{\Gamma}$ consists of a single point that is regular. Therefore the one-dimensional cobordism $N_{\Gamma}$ cannot be compact by the classification theorem of compact one-manifolds.

Applying the paramterized version of Proposition 5.3 under the given energy bound, we conclude that there exists a sequence $\{(s_\alpha, u_\alpha)\}$ with $s_\alpha \to s_0$ and
\[0 < s_0 < 1\] and a non-empty set \(v \cup \mathbf{w}\) of bubbles \(v = \{v_i\}_{i=1}^{N_1}, \mathbf{w} = \{w_j\}_{j=1}^{N_2}\) such that \(u_\alpha \in \mathcal{M}_{K(s_0)}(J, H; 0)\) weakly converges to the cusp curve
\[
u = u_0 + \sum_{i=1}^{N_1} v_i + \sum_{j=1}^{N_2} w_j. \tag{5.5}\]

Here \(u_0 \in \mathcal{M}_{K(s_0)}(J, H)\), and \(w_k\)'s and \(v_l\)'s are non-constant \(J_0\)-holomorphic discs and \(J_{(K(s_0), \tau_l, t_\ell)}\)-holomorphic spheres for some \((\tau_l, t_\ell)\) respectively. It follows from a dimension counting that \(s_0\) cannot be either 0 or 1, because the corresponding moduli spaces restricted thereto are Fredholm regular. We also have the energy bound
\[
E_{(J_{K(s_0)}, H)}(u_0) + \sum_i \omega([v_i]) + \sum_j \omega([w_j]) \leq \|H\|. \tag{5.6}
\]

Since the bubble set \(v \cup \mathbf{w}\) is not empty, the energy bound (5.6) implies
\[
A^U(L; M, \omega) \leq \|H\|
\]
by the definition (2.6) of \(A^U(L; M, \omega)\).

Combining Corollary 4.2 and the above analysis of failure of compactness, we have proved
\[
e(U, L; M, \omega) \leq \|H\|
\]
for any Hamiltonian \(H\) such that \(\overline{U} \cap \phi_1^H(L) = \emptyset\). By taking the infimum over all such \(H\), we have obtained
\[
e(\overline{U}, L) \geq e(U, L; M, \omega) > 0.
\]
This finishes the proof of Theorem 2.15. \(\square\)

**Remark 5.4.** In this remark, we would like to compare the arguments used above with that of [O4]. The main difference in the geometric circumstances between [O4] and the present case is as follows: In the former case, \(L\) was displaceable, i.e., there was a Hamiltonian \(H\) such that \(\phi_1^H(L) \cap L = \emptyset\), while in the present case \(L\) is displaceable from an open subset \(U\), i.e.,
\[
L \cap U \neq \emptyset, \quad \phi_1^H(L) \cap \overline{U} = \emptyset
\]
for open subset \(U \subset M\).

Even though a choice of an embedded path \(\gamma : [0, 1] \to L\) and consideration of the one-dimensional cobordism \(N_1\), defined as above, was also made in the proof of the main theorem [O4] (see p.902 therein), such a choice was unnecessary for the purpose of [O4] because it is enough to know non-compactness of the full \((n + 1)\)-dimensional cobordism to prove the inequality \(e(L, L) \geq A(L; M, \omega)\) in the scheme used therein. (As a matter of fact, the author made such a consideration at that time having application to the study of Maslov class obstruction in mind. He did not pursue this further realizing that such a consideration did not gain much in that it does not give rise to anything significant for the study of Maslov class obstruction beyond that of Polterovich [P1].) However consideration here of this one-dimensional cobordism through a point \(p \in L \cap U\), accompanied by the construction of the invariant \(A(U, L; M, \omega)\) relative to the open subset \(U\), is a crucial ingredient needed for the proof of Main Theorem.
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