On the frequency of hydrodynamic perturbations. From the early transient through the intermediate term to the asymptotic state

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Abstract. We present recent findings concerning the frequency in the transient evolution of three-dimensional perturbations in sheared flows. Sheared incompressible flows are usually considered non-dispersive media, as a consequence, the frequency evolution in transients has received much less attention than the wave energy density or growth factor. By carrying out a large number of long term transient simulations, we observe frequency jumps which appear quite far along within the time interval needed to reach the asymptotic state. We interpret these jumps as the end of the early transient and the beginning of the intermediate transient, where the process of the establishment of the final wave characteristics takes place. In the presence of a wall, where the no-slip condition applies, regardless of the symmetry of the initial condition, the frequency of non orthogonal waves jumps to values that can be 30–40% higher than those observed in the early transient. In the case of the wake, which is the free flow considered in this paper, the situation is similar, but the jumps are generally lower. These are accompanied by oscillations that begin in the early transient, may last throughout the intermediate term and disappear when the asymptotic state is reached. The early transient and the intermediate term durations are of the same order in the wake and in the channel solely in the case of long waves. Medium-short waves in the channel flow have an intermediate evolution one order of magnitude longer than the early transient. In both cases, the frequency and the phase speed fall to zero when $\phi$, the obliquity angle between the perturbation and basic flow plane, approaches $\pi/2$. As a consequence, orthogonal waves are standing waves, a result that can be explained in terms of the symmetry of the system and agrees with laboratory findings concerning the unstable-turbulent spots observed in wall flows during the transition to turbulence.

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1. Introduction

Although both Kelvin (1887 a,b) and Orr (1907 a,b) recognized that the early transient contains important information, only in recent decades have many contributions been devoted to the study of the transient dynamics of three-dimensional perturbations in shear flows (Schmid and Henningson, 2001; Criminale et al., 2003). For a long time linear modal analysis, developed by Orr (1907 a,b) and Sommerfeld (1908), has been considered a sufficient and efficient tool to analyze hydrodynamic stability. More recently it has been observed that the early stages of the disturbance evolution can deeply affect the stability of the flow (Butler and Farrell, 1992; Bergström, 2005; Gustavsson, 1991; Criminale and Drazin, 1990; Reddy and Henningson, 1993). In fact, early algebraic growth can show exceptionally large amplitudes long before an exponential mode is able to set in. It is believed that this kind of behaviour is able to promote rapid transition to fluid turbulence, a phenomenon known as bypass transition (Biau and Bottaro, 2009; Henningson et al., 1993; Lasseigne et al., 1999; Luchini, 1996). Transient decay of asymptotically unstable waves is also possible, which makes the situation rich and complex at the same time. An example of this possible scenario is represented by pipe flow. Linear modal analysis assures stability for all the Reynolds numbers (Drazin, 2002), but this result is in contrast with the experimental evidence, which shows that the flow becomes turbulent at sufficiently large Reynolds numbers. The disagreement between the linear modal prediction and laboratory results has motivated several recent works (Faisst and Eckhardt, 2003; Hof et al., 2004; Duguet et al., 2010) that focus on transient travelling waves and their link to the transition process. In general, it is now considered possible that inside the transient life of travelling waves, which modal analysis cannot describe, some important events for the stability of the flow can take place.

In this work, we focus on the temporal evolution of the wave frequency in two archetypical shear flows, the plane channel flow and the bluff-body wake. Probably due to
the fact that incompressible shear flows are viewed as non-dispersive media, the frequency transient has been poorly investigated so far. For instance, attention has been mainly devoted to the frequency of vortex shedding for the most unstable spatial scales (Williamson, 1989; Strykowski and Sreenivasan, 1990). The situation is quite different within the context of atmosphere and climate dynamics. Here, the interaction between low-frequency and high-frequency phenomena, which is related to the existence of very different spatial and temporal scales, is believed to be one of the main reasons for planetary-scale instabilities (Swanson, 2002; Nakamura et al., 1997). However, due to the inherent strong nonlinearity, the evolution of single scales cannot be observed in the geophysical systems and thus also these studies usually do not account for the frequency transient evolution of a single wave.

The organization of the paper is as follows. To observe the transient life of a perturbation, an initial-value problem must be formulated, see Section 2. Further details on the mathematical formulation are given in the Appendix. The short-term behaviour of transient lives shows trends which are not easily predictable. High maxima of energy followed by asymptotic damping or very low minima of energy reached before the ultimate amplification occurs are just some relevant examples of the scenarios observed. Though the dynamics we consider are linear, different temporal scales develop in the different stages of the life of the perturbation of a given wavelength. The transition between the early transient and asymptotic state does not occur smoothly: frequency jumps appear at an intermediate stage located in between the beginning of the time evolution and the setting of the asymptotic state. We interpret the appearance of the frequency jumps, which are usually preceded and followed by modulating fluctuations, as the beginning of the dynamic process yielding to the final state. In so doing we introduce the intermediate term, which separates the early and final stage of the evolution of the perturbations. We observe that the length of the early and intermediate stages is in general comparable, except for medium/short waves in the channel flow, where the intermediate term is much longer. These results are described in Section 3.

In Section 4, the asymptotic frequency and phase velocity are considered. We present results concerning longitudinal perturbations that extend data obtained in laboratory experiments which are mainly relevant to long waves only. The study of oblique and orthogonal perturbations confirms the reduction of the phase speed when the obliquity angle is increased. Purely orthogonal waves have zero frequency and phase velocity, and present an intense energy transient growth. Conclusive remarks are in Section 5.

2. Mathematical framework

The transient and longterm behaviour is studied using the initial-value problem formulation. We consider two typical shear flows, the plane Poiseuille flow, the archetype of wall flows, and the bluff-body wake, one of the few free flow archetypes (see Fig. 1b and 1c, respectively). The viscous perturbation equations are combined in terms of the vorticity and velocity (Criminale and Drazin, 1990), and then solved by means of a combined Fourier–Fourier (channel) and Laplace–Fourier (wake) transform in the plane normal to the basic flow. This slightly different formulation is due to the fact that the channel flow is homogeneous in the
Figure 1. (a) Perturbation geometry scheme. Perturbations propagate in the direction of the polar wavenumber, \( k = \sqrt{\alpha^2 + \gamma^2} \). (\( \alpha \) and \( \gamma \) are the streamwise (x) and spanwise (z) wavenumbers, respectively). \( \phi \) is the angle of obliquity with respect to the basic flow. (b)-(c) Base flow velocity profiles, \( U(y) \) (thick curves), and symmetric and antisymmetric initial conditions of the perturbation transversal velocity, the \( \hat{v} \) component along the \( y \) direction, \( \hat{v}(y, t = 0) \) (thin curves).

streamwise (x) and spanwise (z) directions, while the wake flow is homogeneous in the z direction and slightly inhomogeneous in the x direction. For the wake, the domain is defined for \( x \geq 0 \) (\( x = 0 \) is the position of the body which generates the flow).

Unlike traditional methods where travelling wave normal modes are assumed as solutions, we follow Criminale et al. (1997) and use arbitrary initial conditions that can be specified without having to recur to eigenfunction expansions. Within our framework, for any initial small-amplitude three-dimensional disturbance, this method allows the determination of the complete temporal behaviour, including both the early and intermediate transients and the long-time asymptotics. It should be recalled that an arbitrary initial disturbance could be expanded in terms of the complete set of discrete and continuum eigenfunctions, as it was demonstrated in the more general case of open flows by Salwen and Grosch (1981). In bounded flows, in fact, it would be sufficient to expand in terms of discrete eigenfunctions.

In literature, various initial conditions were used to explore transient behaviour at subcritical Reynolds numbers. The important physical issue is however the ability to make, in a simple manner, arbitrary specifications. Since a normal mode decomposition provides a complete set of eigenfunctions, it is true that any arbitrary specification can (theoretically) be written in terms of an eigenfunction expansion. Nevertheless, it should be noted that there is nothing special about the eigenfunctions when it comes to specifying initial conditions. They
simply represent the most convenient means of specifying the long-term solution.

Furthermore, the adoption of non-orthogonal eigenfunctions in the try to build any real arbitrary initial condition introduces unnecessary mathematical complications. Physically, it seems that the natural issues affecting the initial specification are whether the disturbances are, first, symmetric or antisymmetric and, second, local or more distributes across the basic profile of the flow. The cases we used here satisfy both of these needs and use functions that can be employed to represent any arbitrary initial condition. If not otherwise specified, with symmetric and antisymmetric conditions we intend the initial conditions as specified in the Appendix and shown in Fig. 1b-c.

The exploratory analysis is carried out with respect to physical quantities, such as the polar wavenumber, \( k \), the angle of obliquity with respect to the basic flow plane, \( \phi \), the symmetry of the perturbation with respect to \( y \) (which is the coordinate orthogonal to the wavenumber vector), and the flow control parameter, \( Re \). We define the longitudinal wavenumber, \( \alpha = k \cos(\phi) \), and the transversal wavenumber, \( \gamma = k \sin(\phi) \), see Fig. 1a. The perturbation and the flow schemes are presented in Fig. 1(a,b,c). More details on the formulation are provided in the Appendix.

As longitudinal observation points we selected for the wake, which is near parallel, two positions downstream of the body: \( x_0 = 10 \), which is a position inside the intermediate part of the wake spatial development, and \( x_0 = 50 \), which is a location inside the far field. The frequency has been evaluated in these sections at a transversal observation point, which is in the following is called \( y_0 \). For the channel flow, which is homogeneous in the streamwise direction, it is sufficient to specify the transversal position, \( y_0 \).

In Fig. 2(a, b), one can see two examples of snapshots of the perturbation velocity, which are displayed in the physical space, for an oblique perturbation with wavenumber equal to 1.5 for the channel and equal to 0.5 for the wake flow. The wave lengths are normalized over the channel half height and the body diameter, respectively. The time, \( t \), which appears in most figures is the independent temporal variable normalized with respect to the basic flow eddy turn over time.

To measure the growth of the perturbations, we define the kinetic energy density,

\[
e(t; \alpha, \gamma) = 1/2 \int_{-y_f}^{+y_f} (|\hat{u}|^2 + |\hat{v}|^2 + |\hat{w}|^2)dy,
\]

where \(-y_f\) and \( y_f \) are the computational limits of the domain, while \( \hat{u}(y,t;\alpha,\gamma) \), \( \hat{v}(y,t;\alpha,\gamma) \) and \( \hat{w}(y,t;\alpha,\gamma) \) are the transformed velocity components of the perturbed field. For the channel flow, which is bounded, the computational limits coincide with the walls (\( y_f = 1 \)). The wake is an unbounded flow and the value \( y_f \) is defined so that the numerical solutions are insensitive to further extensions of the computational domain size (\( y_f \sim 30 \)). We then introduce the amplification factor, \( G \), as the kinetic energy density normalized with respect to its initial value,

\[
G(t; \alpha, \gamma) = e(t; \alpha, \gamma)/e(t = 0; \alpha, \gamma).
\]
Figure 2. Snapshots of the perturbation velocity field in the physical plane $y,z$ normal to the streamwise direction ($u, v, w$ are obtained by a discrete 2D anti-transform of the solved quantities, $\hat{u}, \hat{v}, \hat{w}$). (a) Channel flow: $Re = 10000$, $t = 20$, angle of obliquity $\phi = 3/8 \pi$, antisymmetric initial condition, $k = 1.5$. (b) Wake flow: $Re = 100$, wake section: 50 body lengths downstream, $t = 45$, angle of obliquity $\phi = 3/8 \pi$, antisymmetric initial condition, $k = 0.7$. The wavelengths are normalized over the flow external spatial scales: the channel height $2h$ and the bluff body diameter $D$ for the wake. The times are normalized over the flow relevant eddy turnover times, $2h/U$ and $D/U$, where $U$ is the reference velocity scale of the base flow.

Assuming that the temporal asymptotic behaviour of the linear perturbations is exponential, the temporal growth rate, $r$, can be defined \cite{Criminale2003} as

$$r(t; \alpha, \gamma) = \log(e)/(2t). \quad (3)$$

The frequency, $\omega$, of the perturbation is defined as the temporal derivative of the unwrapped wave phase, $\theta(y, t; \alpha, \gamma)$, at a specific spatial point along the $y$ direction. The wrapped phase,

$$\theta_w(y, t; \alpha, \gamma) = \arg(\hat{v}(y, t; \alpha, \gamma)), \quad (4)$$

is a discontinuous function of $t$ defined in $[-\pi, +\pi]$, while the unwrapped phase, $\theta$, is a continuous function obtained by introducing a sequence of $2\pi$ shifts on the phase values in correspondence to the periodical discontinuities, see Fig. 4 and 5. In the case of the wake we use as reference transversal observation point $y = y_0 = 1$, and in the case of the channel flow the point $y = y_0 = 0.5$. The frequency \cite{Scarsoglio2009} is thus

$$\omega(t; y_0, \alpha, \gamma) = |d\theta(t; y_0, \alpha, \gamma)|/dt. \quad (5)$$

It should be noted that when $r$ and $\omega$ become constant, the asymptotic state is reached.

The phase velocity is defined as

$$C = (\omega/k)\hat{k}, \quad (6)$$

where $\hat{k} = (\cos(\phi), \sin(\phi))$ is the unitary vector in the $k$ direction, and represents the rate at which the phase of the wave propagates in space.
3. Frequency transient

Transient dynamics offer a great variety of different behaviours and phenomena, which are not easy to predict \textit{a priori}. It is interesting to note that these phenomena develop in the context of the linear dynamics, where interaction among different perturbations (and even self-interaction) is absent.

We start the discussion by presenting an overview for transient dynamics, comparing evolutions of kinetic energy, frequency and growth factors of the perturbation. We then focus on the frequency and phase evolutions.

In Fig. 3 we propose some examples of transient dynamics in terms of the amplification factor, \(G\), the frequency, \(\omega\), and the temporal growth rate, \(r\). The wavenumber is \(k = 15\) for the channel flow and \(k = 0.7\) for the wake flow. Two angles of obliquity (\(\phi = 0, \pi/4\)), with symmetric and antisymmetric initial conditions, are considered. One can see that oblique stable waves present maxima of energy in time before being asymptotically damped (see in particular the case of channel flow in panel a). On the contrary, non-orthogonal perturbations can be significantly damped before an ultimate growth occurs (see the wake flow in panel b). An important observation is that quite far along within the transient, frequency discontinuously jumps to a value close to the asymptotic value, which is in general higher than the average value in the transient. The relative variation between transient and asymptotic values can change from a few percentages (about 5%) in case of the wake flow (see panel d), to values up to \(30 - 40\%\) in the case of the channel flow (see panel c). In Fig. 3d, frequency jumps for the wake are only observable for antisymmetric initial conditions. However, this is not true in general. Indeed, different symmetric inputs (see Fig. 4) can lead to discontinuous frequency transients as well. Therefore, the symmetry of initial conditions does not affect the occurrence of frequency jumps in either the channel or wake flows. Moreover, we observe that even if symmetric and antisymmetric perturbations can have slightly different frequency values along the transient, they always reach the same asymptotic value eventually.

The different values of frequency in the early transient (\(\omega_t = 7.95\)) and in the asymptotic state (\(\omega_a = 10.30\)) displayed in Fig. 5a for the channel flow are caused by an abrupt temporal variation of the phase (see Fig. 5c-d for the wrapped and unwrapped wave phases temporal evolution). Two distinct temporal periods, \(T_t\) and \(T_a\), are shown in Fig. 5b, by highlighting the real (or imaginary) part of the perturbation transversal velocity, \(\hat{v}\), at a fixed spatial point, \(y_0 = 0.5\). The discontinuity separates the time interval with the transient period, \(T_t = 0.79\), from the time intermediate interval where the asymptotic period, \(T_a = 0.61\), is reached.

Frequency discontinuity is found for the wake too, see Fig. 6. The sudden variations of the frequency reported in Fig. 6a are due to the phase changes (see Fig. 6c1-c2)-(d), where the wrapped and unwrapped wave phases are reported, respectively). Here again, the temporal region where the frequency jumps appear separates the early transient, where the period is \(T_t = 15\), (Fig. 6b1) from the intermediate transient at the end of which the asymptotic period, \(T_a = 13.5\), is obtained (see Fig. 6b2). We can also observe a further periodicity, \(T_f\), related to the temporal modulation of the frequency during the early and intermediate terms (see Fig. 3c-d). This period is shorter \((T_f \equiv 1)\) for medium-short waves \((k > 10)\) and longer
Figure 3. Transient lives of the perturbations observed through the amplification factor, $G$ (top), the frequency, $\omega$ (middle), and the temporal growth rate, $r$ (bottom), at a fixed polar wavenumber and varying the obliquity ($\phi = 0$, $\pi/4$) and the symmetry of the initial conditions. Left column: channel flow, $k = 15$, $Re = 10000$. Right column: wake flow, $Re = 100$, $k = 0.7$. Here the wake profile is observed at a distance past the body equal to 50 length scales, $x_0 = 50$. The angular frequency $\omega$ is computed at a distance from the wall equal to 0.5 channel half-width, $h$, ($y_0 = 0.5$, panel c), and at a distance equal to one body length, $D$, from the centre of the wake flow ($y_0 = 1$, panel d). The quantity $T_f$ (see panels c-d) indicates the temporal periodicity related to the frequency fluctuations observed in the early and intermediate dynamics.
Figure 4. Frequency jumps observed for symmetric initial conditions in the wake flow ($Re = 100$, observation points $x_0 = 10$ and $y_0 = 1$ diameters downstream the body, $\phi = 0, \pi/4, k = 2.2$): energy (top panel) and frequency (bottom panel) transients. The inset in the bottom panel shows the specific symmetric initial condition in terms of $\hat{v}(y, t = 0) = \exp(-(y - 5)^2) + \exp(-(y + 5)^2)$ considered here.

($T_f \equiv 10^1 - 10^2$) for long waves ($k < 2$), and is in general different from $T_t$ and $T_a$.

These observations yield an interesting result: the perturbation temporal evolution has a three-part structure, with an early stage, an intermediate stage and an asymptotic stage. This is clearly seen by the fact that events like frequency jumps and associated fluctuations split the transient into two parts, which have extended and comparable length in most cases. We interpret these events as the beginning of the process that leads to the settlement of the asymptotic perturbation characteristics, that is the characteristics predicted by modal theory. The intermediate stage is the stage where this process takes place, while the early transient is the stage where the perturbation is most affected by the influence of the initial conditions. This observation should be framed in the general context of the ‘intermediate asymptotics’ where dynamical systems present solutions valid for times and distances from boundaries, large enough for the influence of the fine details of the initial /or boundary conditions to disappear, but small enough to keep the system far from the equilibrium state (Barenblatt, 1996).

Besides the three temporal scales ($T_t$, $T_f$, and $T_a$, see Figures 3, 4, 6) that can be observed in the frequency transient, the system presents two other temporal scales: the external scale related to the base flow (see caption of Figure 2) and the length of the transient (which can be determined by observing the time instant beyond which the growth rate, $r$, and the angular frequency, $\omega$, are both constant). Therefore, for each wavenumber, it is possible to count up to five different time scales.
Figure 5. Channel flow, $Re = 10000$, $k = 15$, symmetric initial condition, $\phi = \pi/4$, observed at $y_0 = 0.5$, a distance from the wall of 1/2 the channel half-width. (a) Frequency temporal evolution: $\omega_t$ is the value in the early transient while $\omega_a$ is the asymptotic one. (b) Perturbation transversal velocity $\hat{v}$ (real and imaginary parts). Temporal periods ($T_t$: transient value, $T_a$: asymptotic value). (c) Wrapped wave phase, $\theta_w(t)$. (d) Unwrapped wave phase, $\theta(t)$. For a visualization of the wave solution, see the film ChannelFrequency.avi.
Figure 6. Wake flow, $Re = 100$, $k = 0.7$, symmetric initial condition, $\phi = \pi/4$, observed at $x_0 = 50$ and $y_0 = 1$. (a) Temporal frequency evolution: $\omega_t$ is the value in the early transient, $\omega_a$ is the asymptotic value. (b1)-(b2) Perturbation transversal velocity $\hat{v}$ (real and imaginary parts) near the frequency jumps highlighted in the panel (a) by the blue rectangles. (c) Wrapped wave phase, $\theta_w(t)$, in the surroundings of the jumps highlighted in the panel (a) by the blue rectangles. (d) Unwrapped wave phase, $\theta(t)$. Here again, the blue rectangles highlight the temporal ranges where the frequency jumps occur. For a visualization of the wave solution, see the film WakeFrequency.avi.
Discontinuities on the frequency are well observable when transient dynamics are sufficiently extended in time. In general we observe that, for fixed wavenumbers, the transient behaviour for the channel flow lasts longer than for the wake flow. For both flows, short wavelengths lead to short transients, while long waves slowly extinguish their transient (for $k = 1$ transients can last up to $10^3 - 10^4$ base flow time units, for $k = 100$ only up to $10^{-1}$ units). Moreover, for long waves in the wake, antisymmetric perturbations can in general present transients lasting longer than those observed for symmetric perturbations (Scarsoglio et al., 2009). However, as shown in Fig. 4, a different shape of symmetric initial conditions can lengthen transient dynamics. In synthesis, jumps of frequency are clearly seen both for symmetric and antisymmetric perturbations in the channel and wake flows when the asymptotic state is not immediately reached, that is in the case of the longitudinal most amplified or damped waves.

Temporal growth rates, $r$, are reported in panels (e) and (f) of Fig. 3 for the channel and wake flows, respectively. When transients monotonically grow or damp they are quite short and the temporal growth rates, $r$, become constant after few temporal scales (see the channel flow configurations in panel (e) and the symmetric perturbations for the wake flow in panel (f)). On the contrary, some transients can last thousands of time units. Examples of this are reported by the antisymmetric longitudinal perturbations acting on the wake flow (light curves in panel f, $\phi = 0, \pi/4$). The temporal growth rates change their trend at about $t = 1000$ and $t = 1600$ for $\phi = 0$ and $\phi = \pi/4$, respectively. They still consistently increase beyond these points until the asymptotic states are reached ($t \simeq 3500 - 4000$ not reported in the Figure). The sudden variations of the temporal growth rates, $r$, are in correspondence with the instants where the frequency values, $\omega$, become constant and the amplification factors, $G$, change their trends (see panels d and b at about $t = 1000$ and $t = 1600$).

In Fig. 7 we show the frequency transient as observed at different transversal points $y_0$, for the channel flow (panel a) and the wake (panel b). The points, $y_0$, are chosen in the high shear region. We consider $y_0$ ranging from 0.2 to 0.6 in the channel, and from 0.5 to 1.5 in
The wake. By varying $y_0$, the asymptotic values of the frequency remain unaltered, while its transient dynamics can change. This behaviour is more evident for the channel flow case (see panel a). However, for both base flows, the presence of the frequency discontinuity is not affected by the specific choice of $y_0$.

To summarize, the end of the early transient and the subsequent beginning of the intermediate transient is announced by the occurrence of the frequency jumps. Many temporal scales beyond this instant, the frequency temporal variations disappear and a constant value emerges. The system, however, is not yet close to its ultimate state. The intermediate transient can be considered extinguished only when the temporal growth rate, $r$, also becomes constant. The intermediate temporal interval is of the same order or greater than the length of the early transient and, when entering this state, the perturbation suddenly changes its behaviour by varying its phase velocity. A measure of the temporal scales related to the end of the early transient and the reaching of the asymptotic state ($t_e$ and $t_a$, respectively) is reported in Fig. 8 by considering different perturbation wavelengths for the wake and channel flows. The length
Figure 9. Spectrum in the wavenumber space of the asymptotic frequency for a collection of longitudinal waves ($\phi = 0$). The plane channel flow is on the left ($Re = 500$ top panel, $Re = 10000$ bottom panel), the bluff body wake is on the right ($Re = 50$ and $x_0 = 10$ top panel, $Re = 100$ and $x_0 = 50$ bottom panel). The observation transversal points are $y_0 = 1$ and $y_0 = 0.5$ for the wake and channel flows, respectively. Symmetric and asymmetric initial conditions are indicated by circles and triangles, respectively. The data in this figure are compared to the available laboratory and theoretical data where frequencies are associated to specific wavenumber values, see references (Nishioka et al., 1975; Ito, 1974; Asai and Floryan, 2006; Williamson, 1989; Tordella et al., 2006; Scarsoglio et al., 2009).

of the intermediate transient can be obtained by calculating the difference between $t_a$ and $t_e$, and is at least of the order of $t_e$.

4. Asymptotic behaviour of the dispersion relation

In this section we describe the distribution of the frequency and phase velocity of longitudinal and transversal waves in correspondence to the settlement of the asymptotic condition. As mentioned in Section 2, this condition can be considered reached when both the temporal growth rate, $r$, and the angular frequency, $\omega$, approach a constant value. It should be noted that in all the cases we observed, the frequency settles before the growth rate. The frequency determination can be validated through the comparison of the temporal asymptotic behaviour obtained by means of the initial-value analysis with other theoretical and experimental
data in literature. To our knowledge, this data collection, unfortunately, does not contain information on three-dimensional perturbations. Indeed, the normal mode theory is quite restricted to longitudinal perturbations. For the channel we refer to the available results in Nishioka et al. (1975); Ito (1974); Asai and Floryan (2006), for the wake, to the results in Williamson (1989); Tordella et al. (2006); Scarsoglio et al. (2009). The asymptotic frequency dependence on the wavenumber is presented in spectral form in Fig. 9. We observe a good agreement between the different literature data and the present results. It should be noted that this is much so for long waves, the most unstable ones. Indeed, these perturbations are those easily observed in the laboratory, even if, usually in their nonlinear regime. We see that although experimental results are affected by the nonlinear interaction, the agreement between laboratory data and linear IVP analysis is very good. It has been shown (Delbende and Chomaz, 1998) that nonlinear terms limit the amplitude of the wave packet, leaving unaffected its frequency, see also the laboratory and normal mode data comparison in Tordella et al. (2006); Belan and Tordella (2006). This good data agreement validates the use of linear stability analysis to predict the frequency transient and asymptotic behaviour.

Figure 9 presents an extended spectral dependence of the frequency, $\omega$, on the polar wavenumber, $k$. In fact, it contains a three decade range of wavenumbers, which is uncommon in literature. For longitudinal waves ($k = \alpha$), see Fig. 1 a, and large enough wavenumber values ($k > 30$ for the wake, $k > 100$ for the channel flow) we observe that $\omega = kc$, where $c$ is constant with respect to $k$ and depends on the basic flow. This means that $\omega$ is directly proportional to $k$ (see red curves in Fig. 9) and the behaviour is non-dispersive. For smaller wavenumbers, instead, $c$ is a complicated function of $k$ and the behaviour becomes dispersive. In Fig. 10(a)-(b) the frequency, $\omega$, and the module of the phase velocity, $|C|$, are reported as functions of the obliquity angle, $\phi$, for two different polar wavenumbers, $k$. For both the flows, the ordinate axis on the left represents the frequency, $\omega$, while the one on the right, the module of the phase velocity, $|C|$. We verified that for short waves

$$\omega = kc \cos(\phi),$$

where $|C| = c \cos(\phi)$ is the module of the phase velocity. For longer waves ($k < 30$ for the wake, $k < 100$ for the channel flow), we observe that $c(k)$ is highly dependent on $k$ and on the basic flow. Thus,

$$\omega = kc(k) \cos(\phi),$$

and the phase velocity vector, $C$, has components

$$C_x = c(k) \cos^2(\phi), \quad C_z = c(k) \cos(\phi) \sin(\phi),$$

In Fig. 10(c)-(d), the module, $|C|$, is shown both for the channel and wake flows as a function of $k$ for three different angles of obliquity. Non-orthogonal long waves are dispersive as the shape of $c(k)$ strongly depends on $k$ and on the base flow considered. For shorter waves, $|C|$ approaches a constant value which only depends on the angle of obliquity, $\phi$.

Disturbances normal to the mean flow ($\phi = \pi/2$) present zero frequency and phase velocity throughout their lives (see Fig. 10). This is true regardless the wavelength of the
perturbations. A zero phase velocity implies that orthogonal waves are stationary. Indeed, there is no reason for an orthogonal wave to move in either of the two possible directions along the $z$ coordinate, as the base flows we considered do not have a component in the spanwise direction. On the contrary, the phase velocity is maximum for longitudinal waves (see Fig. 10) because these have the same direction of the base flow.

Orthogonal waves, although always asymptotically stable, can experience a quick initial growth of energy. This behaviour is evidenced in Fig. 11 where two examples of normal transient growths are reported for the channel and wake flows. Both configurations are asymptotically stable but, before this state is reached, the perturbative waves have strong amplifications, which can last up to hundreds of time units. On the basis of these findings, the role of orthogonal waves, usually underestimated due to their asymptotic stability, seems to be very important for the understanding of mechanisms such as the nonlinear interaction of waves and the bypass transition. It should be also recalled that the laboratory images that describe
turbulent spots (Cantwell et al., 1978; Gad-El-Hak et al., 1981) show in their upstream part an orthogonal wave which is not moving across the channel. In the case of wall flows, the present results can thus offer a possible interpretation for the morphology of turbulent spots.

5. Conclusions

We show evidence of a discontinuous behaviour in the frequency within the transient life of three-dimensional travelling perturbation waves in two typical sheared flows. In the wall flow case, the presence of the discontinuity is barely influenced by the symmetry of the initial condition or by the obliquity angle of the perturbation wave. In the free flow, the wake, the jumps are more marked for antisymmetric perturbations, either aligned with the basic flow or oblique to it. The discontinuity appears after many basic flow eddy turn over times have elapsed and last from 10% to 50% of the global transient length. We interpret this phenomenon as the signature of both the end of the early transient, the part of the evolution most affected by the initial condition, and the beginning of the intermediate term, where the accomplishment of the final values of the wave characteristics take place in accordance with the modal theory. In general, these sudden variations are preceded by a modulation of the constant frequency value observed in the early transient and followed by higher values with a modulation that progressively extinguishes as the asymptotic state is approached. The only waves which do not show frequency discontinuities are those that asymptotically are the most stable or unstable. These waves have a very compact transient which monotonically grows or decays in time. Since the physical definition of intermediate asymptotics is tightly linked to the concept of self-similarity, we think that these findings could open a way to the determination of eventual self-similar properties for a general transient perturbation.

The investigation of the dispersion relation in the asymptotic regime reveals that longitudinal long waves and all the perturbations not aligned with the base flow present a
dispersive behaviour, while only longitudinal short waves are non-dispersive.

This work also contains a numerical investigation of the frequency dependence on the obliquity angle, which indicates that purely orthogonal waves, always stable in the long-term, are standing waves. Since any of these waves arriving in the system will have a zero phase velocity and since during the early transient orthogonal waves can present intense algebraic growth, the system in this condition faces a situation where, in principle, instability can be incentivated.

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Appendix A. Initial-value problem formulation

The base flow system is perturbed with small three-dimensional disturbances. The perturbed system can be linearized and the continuity and Navier-Stokes equations describing its spatio-temporal evolution can be expressed as:

\[
\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} + \frac{\partial \tilde{w}}{\partial z} = 0,
\]

\[
\frac{\partial \tilde{u}}{\partial t} + U \frac{\partial \tilde{u}}{\partial x} + \tilde{v} \frac{dU}{dy} + \frac{\partial \tilde{p}}{\partial x} = \frac{1}{Re} \nabla^2 \tilde{u},
\]

\[
\frac{\partial \tilde{v}}{\partial t} + U \frac{\partial \tilde{v}}{\partial x} + \frac{\partial \tilde{p}}{\partial y} = \frac{1}{Re} \nabla^2 \tilde{v},
\]

\[
\frac{\partial \tilde{w}}{\partial t} + U \frac{\partial \tilde{w}}{\partial x} + \frac{\partial \tilde{p}}{\partial z} = \frac{1}{Re} \nabla^2 \tilde{w},
\]

where \( (\tilde{u}(x, y, z, t), \tilde{v}(x, y, z, t), \tilde{w}(x, y, z, t)) \) and \( \tilde{p}(x, y, z, t) \) are the perturbation velocity and pressure, respectively. \( U \) and \( dU/dy \) indicate the base flow profile (under the near-parallelism assumption) and its first derivative in the shear direction, respectively. For the channel flow, the independent spatial variable, \( z \), is defined from \(-\infty\) to \(+\infty\), the \( x \) variable from \(-\infty\) to \(+\infty\), and the \( y \) from \(-1\) to \(+1\). For the wake flow, \( z \) is defined from \(-\infty\) to \(+\infty\), \( x \) from \(0\) to \(+\infty\), and \( y \) from \(-\infty\) to \(+\infty\). All the physical quantities are normalized with respect to a typical velocity (the free stream velocity, \( U_f \), and the centerline velocity, \( U_0 \), for the 2D wake and the plane Poiseuille flow, respectively), a characteristic length scale (the body diameter, \( D \), and the channel half-width, \( h \), for the 2D wake and the plane Poiseuille flow, respectively), and the reference density, \( \rho_0 \).

The plane channel flow is homogeneous in the \( x \) direction and is represented by the Poiseuille solution, \( U(y) = 1 - y^2 \). Assuming that the bluff-body wake slowly evolves in the streamwise direction, the base flow is approximated at each longitudinal station past the body, \( x_0 \), by using the first orders (\( n = 0, 1 \)) of the Navier-Stokes expansion...
solutions described in [Tordella and Belan 2003]. Under this approximation, \( U(y; x_0, Re) = 1 - a x_0^{-1/2} \exp \left( -\frac{Re \xi^2}{4 x_0} \right) \), where \( a \) is related to the drag coefficient.

By combining equations (10) to (13) to eliminate the pressure terms, the perturbed system can be expressed in terms of velocity and vorticity [Criminale and Drazin 1990]. A two-dimensional Fourier transform is then performed in the \( x \) and \( z \) directions for perturbations in the channel flow. Two real wavenumbers, \( \alpha \) and \( \gamma \), are introduced along the \( x \) and \( z \) coordinates, respectively. A combined two-dimensional Laplace-Fourier decomposition is instead performed for the wake flow in the \( x \) and \( z \) directions. In this case, a complex wavenumber, \( \alpha = \alpha_r + i \alpha_i \), is introduced along the \( x \) coordinate, as well as a real wavenumber, \( \gamma \), along the \( z \) coordinate. To obtain a finite perturbation kinetic energy, the imaginary part, \( \alpha_i \), of the Laplace transformed complex longitudinal wavenumber can only assume non-negative values and can thus be defined as a spatial damping rate in the streamwise direction. Here, for the sake of simplicity, we have \( \alpha_i = 0 \), therefore \( \alpha = \alpha_r \). The following governing partial differential equations are thus obtained

\[
\frac{\partial^2 \hat{v}}{\partial y^2} - k^2 \hat{v} = \hat{\Gamma},
\]

\[
\frac{\partial \hat{\Gamma}}{\partial t} = -i k \cos(\phi) U \hat{\Gamma} + i k \cos(\phi) \frac{d^2 U}{dy^2} \hat{v} + \frac{1}{Re} \left( \frac{\partial^2 \hat{\Gamma}}{\partial y^2} - k^2 \hat{\Gamma} \right),
\]

\[
\frac{\partial \hat{\omega}_y}{\partial t} = -i k \cos(\phi) U \hat{\omega}_y - i k \sin(\phi) \frac{dU}{dy} \hat{v} + \frac{1}{Re} \left( \frac{\partial^2 \hat{\omega}_y}{\partial y^2} - k^2 \hat{\omega}_y \right),
\]

where the superscript \(^\wedge\) indicates the transformed perturbation quantities. The quantity \( \hat{\Gamma} \) is defined through the kinematic relation \( \hat{\Gamma} = \partial \hat{\omega}_z / \partial x - \partial \hat{\omega}_x / \partial z \) that in the physical plane links the perturbation vorticity components in the \( x \) and \( z \) directions (\( \hat{\omega}_x \) and \( \hat{\omega}_z \)) and the perturbed velocity field \( \hat{v}, \phi = \tan^{-1}(\gamma/\alpha) \) is the perturbation obliquity angle with respect to the \( x-y \) plane, \( k \) is the polar wavenumber, \( \alpha = k \cos(\phi), \gamma = k \sin(\phi) \) are the wavenumber components in the \( x \) and \( z \) directions, respectively, see Fig.1.

Unlike traditional methods where travelling wave normal modes are assumed as solutions, we follow [Criminale et al 1997] and use arbitrary initial specifications without having to resort to eigenfunction expansions, for more details see the Mathematical Framework in Section 2. For any initial small-amplitude three-dimensional disturbance, this approach allows the determination of the full temporal behaviour, including both early-time and intermediate transients and the long-time asymptotics. Among all the possible inputs, we focus on arbitrary symmetric and antisymmetric initial conditions distributed over the whole shear region.

The transversal vorticity \( \hat{\omega}_y(y, t) \) is initially taken equal to zero to highlight the three-dimensionality net contribution on its temporal evolution (see Scarsoglio 2008; Criminale et al 1997), to consider the effects of non-zero initial transversal vorticity. Therefore, initial conditions can be shaped in terms of the transversal velocity (see thin curves in Fig. 1b and 1c), as follow:

Channel flow: \( \hat{v}(y, t = 0) = (1 - y^2)^2, \quad \hat{v}(y, t = 0) = y(1 - y^2)^2, \)
Wake flow: \[ \hat{v}(y, t = 0) = \exp(-y^2) \cos(y), \quad \hat{v}(y, t = 0) = \exp(-y^2) \sin(y). \]

For the channel flow no-slip and impermeability boundary conditions are imposed, while for the wake flow uniformity at infinity and finiteness of the energy are imposed.

Equations (14)-(16) are numerically solved by the method of lines: the equations are first discretized in the spatial domain using a second-order finite difference scheme, and then integrated in time by means of an explicit Runge-Kutta formula.

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