Ordinary-derivative formulation of conformal
totally symmetric arbitrary spin bosonic fields

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Abstract

Conformal totally symmetric arbitrary spin bosonic fields in flat space-time of even dimension greater than or equal to four are studied. Second-derivative formulation for such fields is developed. We obtain gauge invariant Lagrangian and corresponding gauge transformations. Gauge symmetries are realized by involving the Stueckelberg fields. Realization of global conformal boost symmetries on conformal gauge fields is obtained. On-shell degrees of freedom of the conformal field are analyzed. Interrelations between the second-derivative formulation of the conformal fields and the gauge invariant formulation of massive fields are discussed.
1 Introduction

The present paper is a sequel to our paper [1] where ordinary-derivative formulation of conformal fields was developed. Commonly used Lagrangian formulations of most conformal fields involve higher derivatives (for review, see [2]). In Ref. [1], we developed ordinary-derivative, gauge invariant, and Lagrangian formulation for free low spin conformal fields. This is to say that our Lagrangians for free bosonic fields do not involve higher than second order terms in derivatives, while our Lagrangians for free fermionic fields do not involve higher than first order terms in derivatives. In the present paper, we generalize results in Ref.[1] to the case of arbitrary spin bosonic conformal fields.

A long term motivation for our study of conformal fields comes from the following potentially important applications.

In Ref.[3], it has been conjectured that string theory, theory of massless higher-spin fields in $\text{AdS}$ space, and theory of conformal fields, though different, eventually may turn out to be different phases of one and the same unified field theory with new forces mediated by higher-spin gauge fields. According to this conjecture, in the ultra high-energy domain, dynamics of the unified theory is governed by conformal higher-spin field theory generalizing Weyl gravity, i.e., in the ultra high-energy domain, microscopic degrees of freedom of the unified theory are described by conformal low- and higher-spin gauge fields. The spontaneous conformal symmetry breaking leads to massless higher-spin field theory in $\text{AdS}$ space [4] generalizing $\text{AdS}$ supergravity. One expects that, in the $\text{AdS}$ phase, symmetries of the unified theory are realized as infinite-dimensional gauge symmetries of massless low- and higher-spin fields. Further, $\text{AdS}$ massless higher-spin gauge symmetry breaking leads to the string theory in a flat space. We believe that use of the ordinary-derivative approach might be helpful to study the conjecture in Ref.[3]. Note that actions of string theory free massive fields and free massless higher-spin $\text{AdS}$ fields do not involve higher derivatives. Therefore, the ordinary-derivative approach to conformal field theory seems to be most suitable for investigation of possible interrelations between conformal fields, string theory, and massless higher-spin field theory in $\text{AdS}$.

The second application is to the conjectured duality [6] of large $N$ conformal $\mathcal{N} = 4$ SYM theory and type IIB superstring theory in $\text{AdS}_5 \times S^5$. In the framework of AdS/CFT correspondence, the conformal fields manifest themselves in two related ways at least. Firstly, they appear as boundary values of non-normalizable solution of equations of motion for bulk fields of IIB supergravity in $\text{AdS}_5 \times S^5$ background (see e.g. [7]-[11]). Secondly, conformal fields appearing in spin 2 field supermultiplet of $\mathcal{N} = 4$ superconformal algebra constitute

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1 Recent interesting discussion of conformal symmetries of massless higher-spin $\text{AdS}_4$ field theory may be found in [5].

2 In the earlier literature, discussion of conformal field dualities may be found in [12] [13].
multiplet of $\mathcal{N} = 4$ conformal supergravity. The conformal supergravity multiplet couples with currents constructed out the fields of $\mathcal{N} = 4$ supersymmetric YM theory. It turns out that IIB supergravity expanded over AdS background and evaluated over Dirichlet problem reproduces action of $\mathcal{N} = 4$ conformal supergravity \cite{14}. Note also that $\mathcal{N} = 4$ conformal supergravity has the same global supersymmetries, though realized in different way, as supergravity/superstring theory in $AdS_5 \times S^5$ Ramond-Ramond background. In view of these relations to IIB supergravity/superstring in $AdS_5 \times S^5$ and supersymmetric YM theory we think that ordinary-derivative formulation of conformal fields might be useful to understand string/gauge theory dualities better. This is to say that bulk equation of motions of free IIB supergravity/superstring in $AdS_5 \times S^5$ background do not involve higher derivatives. We believe therefore that the ordinary-derivative approach to conformal field theory should be most suitable for investigation of bulk/boundary dualities.

In this paper, we discuss ordinary-derivative formulation of bosonic free arbitrary spin conformal fields in space-time of even dimension $d \geq 4$. As is well known any higher-derivative theory can be represented in ordinary-derivative form by introducing additional field degrees of freedom. Our purpose is to introduce the additional field degrees of freedom so that to respect the following three requirements: i) ordinary-derivative formulation should be Lagrangian; ii) the additional fields should be supplemented by appropriate additional gauge symmetries so that to retain on-shell D.o.F of the generic higher-derivative theory; iii) realization of global conformal symmetries should be local.

2 Preliminaries

2.1 Notation

Our conventions are as follows. $x^A$ denotes coordinates in $d$-dimensional flat space-time, while $\partial_A$ denotes derivative with respect to $x^A$, $\partial_A \equiv \partial / \partial x^A$. Vector indices of the Lorentz algebra $so(d-1,1)$ take the values $A, B, C, E = 0, 1, \ldots, d - 1$. We use mostly positive flat metric tensor $\eta^{AB}$. To simplify our expressions we drop $\eta_{AB}$ in scalar products, i.e. we use $X^{A} Y^{A} \equiv \eta_{AB} X^{A} Y^{B}$. We adopt the notation

$$\Box = \partial^A \partial_A, \quad \alpha \partial = \alpha^A \partial_A, \quad \bar{\alpha} \partial = \bar{\alpha}^A \partial_A, \quad \alpha^2 = \alpha^A \alpha^A, \quad \bar{\alpha}^2 = \bar{\alpha}^A \bar{\alpha}^A. \quad (2.1)$$

To avoid complicated tensor expressions we use a set of the creation operators $\alpha^A, \zeta, \upsilon^\circ, \upsilon^\circ$, and the respective set of annihilation operators $\bar{\alpha}^A, \bar{\zeta}, \bar{\upsilon}^\circ, \bar{\upsilon}^\circ$,

$$\bar{\alpha}^A |0\rangle = 0, \quad \bar{\zeta} |0\rangle = 0, \quad \bar{\upsilon}^\circ |0\rangle = 0, \quad \bar{\upsilon}^\circ |0\rangle = 0. \quad (2.2)$$

3 Also, conformal symmetries manifest themselves in the tensionless limit of strings \cite{15} (see also \cite{16}).

4 To realize those additional gauge symmetries we adopt the approach of Refs.\cite{17} \cite{18} which turns out to be the most useful for our purposes.
These operators satisfy the commutators
\[
\begin{align*}
[\bar{\alpha}^A, \alpha^B] &= \eta^{AB}, \\
[\bar{\zeta}, \zeta] &= 1,
\end{align*}
\]
and will often be referred to as oscillators in what follows. The oscillators \(\alpha^A, \bar{\alpha}^A\) and \(\zeta, \bar{\zeta}\), \(\upsilon^\oplus, \upsilon^\ominus, \bar{\upsilon}^\oplus, \bar{\upsilon}^\ominus\) transform in the respective vector and scalar representations of the \(so(d - 1, 1)\) Lorentz algebra and satisfy the following hermitian conjugation rules:
\[
\begin{align*}
\alpha^A + &= \bar{\alpha}^A, \\
\zeta^+ &= \bar{\zeta}, \\
\upsilon^{\oplus} + &= \bar{\upsilon}^\oplus, \\
\upsilon^{\ominus} + &= \bar{\upsilon}^\ominus.
\end{align*}
\]
Throughout this paper we use operators constructed out the oscillators,
\[
\begin{align*}
N_\alpha &\equiv \alpha^A \bar{\alpha}^A, \\
N_\zeta &\equiv \zeta \bar{\zeta}, \\
N_\upsilon^\oplus &\equiv \upsilon^{\oplus} \bar{\upsilon}^{\oplus}, \\
N_\upsilon^\ominus &\equiv \upsilon^{\ominus} \bar{\upsilon}^{\ominus}, \\
N_\upsilon &\equiv N_\upsilon^\oplus + N_\upsilon^\ominus, \\
\Delta' &\equiv N_\upsilon^\oplus - N_\upsilon^\ominus.
\end{align*}
\]

### 2.2 Global conformal symmetries

The conformal algebra \(so(d, 2)\) of \(d\) dimensional space-time taken to be in basis of the Lorentz algebra \(so(d - 1, 1)\) consists of translation generators \(P^A\), conformal boost generators \(K^A\), and generators \(J^{AB}\) which span \(so(d - 1, 1)\) Lorentz algebra. We assume the following normalization for commutators of the conformal algebra:
\[
\begin{align*}
[D, P^A] &= -P^A, \\
[D, K^A] &= K^A, \\
[P^A, J^{BC}] &= \eta^{AB} P^C - \eta^{AC} P^B, \\
[K^A, J^{BC}] &= \eta^{AB} K^C - \eta^{AC} K^B, \\
[P^A, K^B] &= \eta^{AB} D - J^{AB}, \\
[J^{AB}, J^{CE}] &= \eta^{BC} J^{AE} + 3 \text{ terms}.
\end{align*}
\]

\(^5\) We use oscillator formulation \([19, 20, 21]\) to handle the many indices appearing for tensor fields. It can also be reformulated as an algebra acting on the symmetric-spinor bundle on the manifold \(M\) \([22]\). Note that the scalar oscillators \(\zeta, \bar{\zeta}\), which appeared in gauge invariant formulation of massive fields, arise naturally by a dimensional reduction \([23, 22]\) from flat space. It is natural to expect that ‘conformal’ oscillators \(\upsilon^{\oplus}, \upsilon^{\ominus}, \bar{\upsilon}^{\oplus}, \bar{\upsilon}^{\ominus}\) also allow certain interpretation via dimensional reduction.
Let $|\phi\rangle$ denotes field propagating in flat space-time of dimension $d \geq 4$. Let Lagrangian for the free field $|\phi\rangle$ be conformal invariant. This implies, that Lagrangian is invariant with respect to transformation (invariance of the Lagrangian is assumed to be by module of total derivatives)

$$\delta_{\hat{G}}|\phi\rangle = \hat{G}|\phi\rangle,$$

where realization of the conformal algebra generators $\hat{G}$ in terms of differential operators takes the form

$$P^A = \partial^A,$$

$$J^{AB} = x^A \partial^B - x^B \partial^A + M^{AB},$$

$$D = x \partial + \Delta,$$

$$K^A = K_{\Delta,M}^A + R^A,$$

and we use the notation

$$K_{\Delta,M}^A \equiv -\frac{1}{2} x^2 \partial^A + x^A D + M^{AB} x^B,$$

$$x \partial \equiv x^A \partial^A, \quad x^2 = x^A x^A.$$

In (2.19)-(2.21), $\Delta$ is operator of conformal dimension, $M^{AB}$ is spin operator of the Lorentz algebra,

$$[M^{AB}, M^{CE}] = \eta^{BC} M^{AE} + 3 \text{ terms},$$

and $R^A$ is operator depending on derivatives with respect to space-time coordinates and not depending on space-time coordinates $x^A$,

$$[P^A, R^B] = 0.$$
3 Field content and Lagrangian of conformal field

To discuss ordinary-derivative and gauge invariant formulation of arbitrary spin-$s$ conformal field in flat space of even dimension $d \geq 4$ we use the following set scalar, vector and tensor fields of the Lorentz algebra $so(d-1,1)$:

$$
\phi^{A_1 \ldots A_{s'}}_{s',k''}, \quad s' = 0, 1, \ldots, s - 1, s;
$$

$$
k'' = -k', -k' + 2, \ldots, k' - 2, k';
$$

$$
k' \equiv s' + \frac{d - 6}{2}.
$$

(3.1)

The subscript $s'$ denotes that the field $\phi^{A_1 \ldots A_{s'}}_{s',k''}$ is rank-$s'$ tensor field of the Lorentz algebra $so(d-1,1)$, while the subscript $k''$ determines the conformal dimension of the field $\phi^{A_1 \ldots A_{s'}}_{s',k''}$,

$$
\Delta(\phi^{A_1 \ldots A_{s'}}_{s',k''}) = \frac{d - 2}{2} + k''.
$$

(3.2)

Alternatively, field content (3.1) can be represented as

$$
\phi^{A_1 \ldots A_{s'}}_{s,k''}, \quad k'' = -k, -k + 2, \ldots, k - 2, k; \quad (3.3)
$$

$$
\phi^{A_1 \ldots A_{s-1}}_{s-1,k''}, \quad k'' = -k + 1, -k + 3, \ldots, k - 3, k - 1; \quad (3.4)
$$

$$
\ldots \ldots \ldots \ldots \ldots \ldots \ldots
$$

$$
\phi^A_{1,k''}, \quad k'' = s - k - 1, s - k + 1, \ldots, k - s - 1, k - s + 1; \quad (3.5)
$$

$$
\phi^0_{0,k''}, \quad k'' = s - k, s - k + 2, \ldots, k - s - 2, k - s; \quad (3.6)
$$

$$
k \equiv s + \frac{d - 6}{2}.
$$

(3.7)

We note that

i) In (3.1), the fields $\phi^0_{0,k''}$ and $\phi^A_{1,k''}$ are the respective scalar and vector fields of the Lorentz algebra, while the fields $\phi^{A_1 \ldots A_{s'}}_{s',k''}, s' > 1$, are rank-$s'$ totally symmetric tensor fields of the Lorentz algebra $so(d-1,1)$. Note that the scalar fields $\phi^0_{0,k''}$ enter in the field content only when $d \geq 6$.

ii) The tensor fields $\phi^{A_1 \ldots A_{s'}}_{s',k''}$ with $s' \geq 4$ satisfy the double-tracelessness constraint

$$
\phi^{AABBA_5 \ldots A_{s'}}_{s',k''} = 0, \quad s' \geq 4.
$$

(3.8)

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6 Lagrangian higher-derivative formulation of totally symmetric conformal fields was developed in [2, 24]. Discussion of equations for mixed symmetry conformal fields may be found in [25].
To illustrate the field content given in (3.1) we note that if
\[ d \geq 6, \quad s - \text{arbitrary}, \quad k = s + \frac{d - 6}{2}, \]
then the field content in (3.1) can be represented as
\[
\begin{align*}
\phi(s, -k) & \quad \phi(s, -k + 2) \quad \ldots \quad \ldots \quad \phi(s, k - 2) \quad \phi(s, k) \\
\phi(s - 1, -k + 1) & \quad \phi(s - 1, -k + 3) \quad \ldots \quad \ldots \quad \phi(s - 1, k - 3) \quad \phi(s - 1, k - 1) \\
\quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
\phi(1, s - k - 1) & \quad \phi(1, s - k + 1) \quad \ldots \quad \ldots \quad \phi(1, k - s - 1) \quad \phi(1, k - s + 1) \\
\phi(0, s - k) & \quad \phi(0, s - k + 2) \quad \ldots \quad \ldots \quad \phi(0, k - s - 2) \quad \phi(0, k - s)
\end{align*}
\]
where \( \phi(s', k'') \) stands for the field \( \phi^{A_1 \ldots A_s}_{s', k''} \).

As we have said, the scalar fields do not enter in the field content when \( d = 4 \). This, if
\[ d = 4, \quad s - \text{arbitrary}, \quad k = s - 1, \]
then the field content in (3.1) can be represented as
\[
\begin{align*}
\phi(s, 1 - s) & \quad \phi(s, 3 - s) \quad \ldots \quad \phi(s, s - 3) \quad \phi(s, s - 1) \\
\phi(s - 1, 2 - s) & \quad \phi(s - 1, 4 - s) \quad \ldots \quad \phi(s - 1, s - 4) \quad \phi(s - 1, s - 2) \\
\quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
\phi(2, -1) & \quad \phi(2, 1) \\
\phi(1, 0)
\end{align*}
\]

We note that \( d = 6 \) is the lowest space-time dimension when the scalar fields appear in the field content. This, if
\[ d = 6, \quad s - \text{arbitrary}, \quad k = s, \]
then the field content in (3.1) can be represented as
\[
\begin{align*}
\phi(s, -s) & \quad \phi(s, 2 - s) \quad \ldots \quad \phi(s, s - 2) \quad \phi(s, s) \\
\phi(s - 1, 1 - s) & \quad \phi(s - 1, 3 - s) \quad \ldots \quad \phi(s - 1, s - 3) \quad \phi(s - 1, s - 1) \\
\quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
\phi(1, -1) & \quad \phi(1, 1) \\
\phi(0, 0)
\end{align*}
\]
In order to obtain the gauge invariant description in an easy–to–use form we use a set of
the creation operators
\[ \alpha^A, \zeta, \upsilon^\oplus, \upsilon^\ominus, \] and the respective set of annihilation operators,
\[ \bar{\alpha}^A, \bar{\zeta}, \bar{\upsilon}^\ominus, \bar{\upsilon}^\oplus. \] Then, fields (5.1) can be collected into a ket-vector \( |\phi\rangle \) defined by
\[
|\phi\rangle \equiv \sum_{s' = 0}^{s} \zeta^{s-s'}|\phi_{s'}\rangle, \tag{3.12}
\]
\[
|\phi_{s'}\rangle \equiv \alpha^{A_1} \ldots \alpha^{A_{s'}} \sum_{k''} (\upsilon^{\oplus})^{k''} (\upsilon^{\ominus})^{k''-k''} \phi_{s',k''}^A |0\rangle, \tag{3.13}
\]
\[
k'' = -k', -k' + 2, \ldots, k' - 2, k';
\]
\[
k' \equiv s' + \frac{d - 6}{2}. \tag{3.13}
\]
From (3.12), (3.13), we see that the ket-vector \( |\phi\rangle \) is degree-\( s \) homogeneous polynomial in the oscillators \( \alpha^A, \zeta \) and degree-\( k \) homogeneous polynomial in the oscillators \( \upsilon^{\oplus}, \upsilon^{\ominus} \). In other words, the ket-vector \( |\phi\rangle \) satisfies the relations
\[
(N_{\alpha} + N_{\zeta} - s)|\phi\rangle = 0, \tag{3.14}
\]
\[
(N_{\zeta} + N_{\upsilon} - k)|\phi\rangle = 0, \quad k \equiv s + \frac{d - 6}{2}. \tag{3.15}
\]
Also, note that the ket-vector \( |\phi_{s'}\rangle \) is degree-\( s' \) homogeneous polynomial in the oscillators \( \alpha^A \) and degree-\( k' \) homogeneous polynomial in the oscillators \( \upsilon^{\oplus}, \upsilon^{\ominus} \), i.e., the ket-vector \( |\phi_{s'}\rangle \) satisfies the relations
\[
N_{\alpha}|\phi_{s'}\rangle = s'|\phi_{s'}\rangle, \tag{3.16}
\]
\[
N_{\upsilon}|\phi_{s'}\rangle = k'|\phi_{s'}\rangle, \quad k' = s' + \frac{d - 6}{2}. \tag{3.17}
\]
We note that ket-vector \( |\phi_0\rangle \), which collects the scalar fields (3.6), enters in the \( |\phi\rangle \) only when \( d \geq 6 \) (i.e. \( k \geq 2 \)).

In terms of the ket-vector \( |\phi\rangle \), double-tracelessness constraint (3.8) takes the form
\[
(\bar{\alpha}^2)^2|\phi\rangle = 0. \tag{3.18}
\]

**Lagrangian.** Lagrangian we found takes the form
\[
\mathcal{L} = \frac{1}{2} \langle \phi | E | \phi \rangle, \tag{3.19}
\]

\[\text{7 In this paper we adopt the Lagrangian formulation in terms of the double traceless gauge fields [26]. Discussion of various Lagrangian formulations in terms of unconstrained gauge fields may be found in [27]–[31].}\]
where operator $E$ is given by

\begin{align}
E &= E_{(2)} + E_{(1)} + E_{(0)}, \\
E_{(2)} &\equiv \Box - \alpha \partial \bar{\alpha} \partial + \frac{1}{2} (\alpha \partial)^2 \alpha^2 + \frac{1}{2} \alpha^2 (\bar{\alpha} \partial)^2 - \frac{1}{2} \alpha^2 \Box \alpha^2 - \frac{1}{4} \alpha^2 \alpha \partial \bar{\alpha} \partial \bar{\alpha}^2, \\
E_{(1)} &\equiv e_1 A - e_1 ^\dagger \bar{A}, \\
E_{(0)} &\equiv m_1 + \alpha^2 \bar{\alpha} \partial m_2 + m_3 \alpha^2 + m_4 \bar{\alpha}^2, \\
A &\equiv \alpha \partial - \alpha^2 \bar{\alpha} \partial + \frac{1}{4} \alpha^2 \alpha \partial \bar{\alpha}^2, \\
\bar{A} &\equiv \bar{\alpha} \partial - \alpha \partial \bar{\alpha} + \frac{1}{4} \alpha^2 \alpha \partial \bar{\alpha}^2, \\
e_1 &\equiv \psi \bar{e} \bar{1} \zeta, \\
e_1 ^\dagger &\equiv \zeta \bar{e} \bar{1} \bar{\psi}, \\
m_1 &\equiv \frac{2s + d - 2 - N_\zeta}{2s + d - 2 - 2N_\zeta} (N_\zeta - 1) \psi \bar{\psi}, \\
m_2 &\equiv \frac{2(2s + d - 2) + (2s + d - 7)N_\zeta - N_\zeta^2}{4(2s + d - 2 - 2N_\zeta)} \psi \bar{\psi}, \\
m_3 &\equiv \frac{1}{2} \psi \bar{e} \bar{1} \zeta \bar{e} \bar{1} \zeta, \\
m_3 ^\dagger &\equiv \frac{1}{2} \zeta \bar{e} \bar{1} \zeta \bar{e} \bar{1} \zeta, \\
\bar{e} \bar{1} &\equiv \left( \frac{2s + d - 4 - N_\zeta}{2s + d - 4 - 2N_\zeta} \right)^{1/2}, \\
\bar{e} \bar{1} ^{(1)} &\equiv \left( \frac{2s + d - 5 - N_\zeta}{2s + d - 6 - 2N_\zeta} \right)^{1/2}.
\end{align}

We note that $E_{(2)}$ \ref{3.21} is standard second-order Fronsdal operator represented in terms of the oscillators.
4 Gauge symmetries of conformal field

We now discuss gauge symmetries of the Lagrangian. To this end we introduce the following set of gauge transformation parameters:

\[ \epsilon^{A_{1} \ldots A_{s-1}}_{s', k''-1}, \quad s' = 0, 1, \ldots, s - 2, s - 1; \]
\[ k'' = -k' - 1, -k' + 1, \ldots, k' - 1, k' + 1; \]
\[ k' \equiv s' + \frac{d - 6}{2}. \]  (4.1)

Alternatively, gauge parameters (4.1) can be represented as

\[ \epsilon^{A_{1} \ldots A_{s-1}}_{s-1, k''-1}, \quad k'' = -k, -k + 2, \ldots, k - 2, k; \]  (4.2)
\[ \epsilon^{A_{1} \ldots A_{s-2}}_{s-2, k''-1}, \quad k'' = -k + 1, -k + 3, \ldots, k - 3, k - 1; \]  (4.3)
\[ \ldots \ldots \ldots \ldots \]  
\[ \ldots \ldots \ldots \ldots \]  
\[ \epsilon^{A_{1}}_{1, k''-1}, \quad k'' = s - k - 2, s - k, \ldots, k - s, k - s + 2; \]  (4.4)
\[ \epsilon^{0}_{0, k''-1}, \quad k'' = s - k - 1, s - k + 1, \ldots, k - s - 1, k - s + 1; \]  (4.5)
\[ k \equiv s + \frac{d - 6}{2}. \]  (4.6)

We note that

i) In (4.1), the gauge parameters \( \epsilon_{0, k''-1} \) and \( \epsilon^{A_{1} \ldots A_{s-1}}_{1, k''-1} \) are the respective scalar and vector fields of the Lorentz algebra, while the gauge parameters \( \epsilon^{A_{1} \ldots A_{s-1}}_{s', k''-1}, \quad s' > 1, \) are rank-\( s' \) totally symmetric tensor fields of the Lorentz algebra \( \text{so}(d - 1, 1). \)

ii) The gauge parameters \( \epsilon^{A_{1} \ldots A_{s'}}_{s', k''-1} \) with \( s' \geq 2 \) satisfy the tracelessness constraint

\[ \epsilon^{AA_{3} \ldots A_{s'}}_{s', k''-1} = 0, \quad s' \geq 2. \]  (4.7)

iii) The gauge parameters \( \epsilon^{A_{1} \ldots A_{s'}}_{s', k''-1} \) have the conformal dimensions

\[ \Delta(\epsilon^{A_{1} \ldots A_{s'}}_{s', k''-1}) = \frac{d - 2}{2} + k'' - 1. \]  (4.8)
Now, as usually, we collect gauge transformation parameters in ket-vector $|\epsilon\rangle$ defined by

$$|\epsilon\rangle \equiv \sum_{s'=0}^{s-1} \zeta^{s-1-s'}|\epsilon_{s'}\rangle,$$

(4.9)

$$|\epsilon_{s'}\rangle \equiv \alpha^{A_1}\ldots\alpha^{A_s'} \sum_{k''} (\nu^\oplus)^{\nu'+k''} (\nu^\otimes)^{\nu'+k''} \epsilon_{A_1\ldots A_s'} |0\rangle,$$

$$k'' = -k' - 1, -k' + 1, \ldots, k' - 1, k' + 1;$$

$$k' = s' + \frac{d - 6}{2}.$$

(4.10)

The ket-vector $|\epsilon\rangle$ satisfies the algebraic constraints

$$(N_\alpha + N_\zeta - s + 1)|\epsilon\rangle = 0,$$

(4.11)

$$(N_\zeta + N_\nu - k)|\epsilon\rangle = 0, \quad k = s + \frac{d - 6}{2},$$

(4.12)

which tell us that $|\epsilon\rangle$ is a degree-$(s - 1)$ homogeneous polynomial in the oscillators $\alpha^A, \zeta$ and degree-$k$ homogeneous polynomial in the oscillators $\zeta, \nu^\oplus, \nu^\otimes$. The ket-vector $|\epsilon_{s'}\rangle$ satisfies the algebraic constraints

$$N_\alpha|\epsilon_{s'}\rangle = s'|\epsilon_{s'}\rangle,$$

(4.13)

$$N_\nu|\epsilon_{s'}\rangle = (k' + 1)|\epsilon_{s'}\rangle, \quad k' = s' + \frac{d - 6}{2},$$

(4.14)

which imply that $|\epsilon_{s'}\rangle$ is degree-$s'$ homogeneous polynomial in the oscillators $\alpha^A$ and degree-$(k' + 1)$ homogeneous polynomial in the oscillators $\nu^\oplus, \nu^\otimes$.

In terms of the ket-vector $|\epsilon\rangle$, tracelessness constraint (4.7) takes the form

$$\bar{\alpha}^2|\epsilon\rangle = 0.$$

(4.15)

Gauge transformations can entirely be written in terms of $|\phi\rangle$ and $|\epsilon\rangle$. This is to say that gauge transformations take the form

$$\delta|\phi\rangle = (\alpha \partial + b_1 + b_2 \alpha^2)|\epsilon\rangle,$$

(4.16)

$$b_1 = \zeta \tilde{e}_1 \nu^\otimes,$$

(4.17)

$$b_2 = -\frac{1}{2s + d - 6 - 2N_\zeta} \nu^\otimes \tilde{e}_1 \tilde{\zeta},$$

(4.18)
where $\tilde{e}_1$ is defined in (3.32).

We now consider structure of gauge transformations (4.16). Making use of simplified notation for derivatives, fields and flat metric tensor

$$
\phi(s', k') \sim \phi^{A_1 \ldots A_s'}, \quad \epsilon(s', k') \sim \epsilon^{A_1 \ldots A_s'}, \quad \partial \sim \partial^A, \quad \eta \sim \eta^{AB},
$$

(4.19)
gauge transformations (4.16) can schematically be represented as

$$
\delta\phi(s', k') \sim \partial\epsilon(s', k') + \epsilon(s', k') + \eta\epsilon(s'-2,k'), \quad s' = 2, 3, \ldots, s - 1, s,
$$

(4.20)

$$
\delta\phi(1, k') \sim \partial\epsilon(0, k') + \epsilon(1, k'),
$$

(4.21)

$$
\delta\phi(0, k') \sim \epsilon(0, k').
$$

(4.22)

From (4.20)-(4.22), we see that all the scalar and vector fields $\phi_{0,k'}, \phi^{A}_{1,k'}$ and some of the tensor fields can be gauged away, i.e. all the scalar and vector fields and some of the tensor fields are nothing but the Stueckelberg fields in the framework of ordinary-derivative approach.

We now find set of fields that are realized as Stueckelberg fields and demonstrate that with the exception of gauge symmetry related to the parameter $\epsilon(s-1,-k-1)$ all the gauge symmetries are realized as Stueckelberg (Goldstone) gauge transformation. To this end we note that the fields $\phi(s', k')$ can be decomposed into traceless tensor fields

$$
\phi(s', k') = \phi^T(s', k') \oplus \phi^{TT}(s'-2,k'), \quad s' = 2, 3, \ldots, s - 1, s,
$$

(4.23)

where $\phi^T$ and $\phi^{TT}$ stand for traceless tensors of the Lorentz algebra $so(d-1,1)$. From (4.20)-(4.22), we see that using the gauge symmetries generated by

$$
\epsilon(s', k'), \quad s' = 0, 1, \ldots, s - 2, s - 1, \quad k'' \neq -k' - 2,
$$

(4.24)

where $k'$ is given in (3.1), we can gauge away the following fields:

$$
\phi^T(s', k'), \quad s' = 2, 3, \ldots, s - 2, s - 1,
$$

(4.25)

$$
\phi^{TT}(s', k'), \quad \phi(0, k'),
$$

(4.26)

with $k''$ given in (3.1). After gauging away fields (4.25),(4.26), we are left with the following set of fields:

$$
\phi(s, k''), \quad k'' = -k, -k + 2, \ldots, k - 2, k,
$$

(4.27)

$$
\phi^{TT}(s', k''), \quad s' = 0, 1, \ldots, s - 4, s - 3,
$$

$$
k'' = -k' - 2, -k', \ldots, k', k' + 2, \quad k' \equiv s' + \frac{d-6}{2},
$$

(4.28)
and surviving gauge symmetries generated by gauge parameters

\[ \epsilon(s', -k'-2), \quad s' = 0, 1, \ldots, s - 1, \quad k' = s' + \frac{d - 6}{2}. \quad (4.29) \]

Gauge symmetries related to gauge parameters (4.29) with \( s' = 0, 1, \ldots, s - 2 \) are realized as Stueckelberg gauge symmetries. Therefore these gauge symmetries can also be used to gauge away some of the fields in (4.27), (4.28). This is to say that using the gauge symmetry generated by the parameter \( \epsilon(s_{-2}, -k) \) we can gauge away the field \( \phi^{TT}_{(s_{-2}, -k)} \) in (4.27), while using the gauge symmetries generated by the parameters

\[ \epsilon(s', -k'-2), \quad s' = 0, 1, \ldots, s - 3, \quad (4.30) \]

we can gauge away the following set of fields in (4.28):

\[ \phi^{TT}_{(s', -k'-2)}, \quad s' = 0, 1, \ldots, s - 3, \quad (4.31) \]

i.e. we are left with the fields

\[
\begin{align*}
\phi^T_{(s, -k)}, & \quad (4.32) \\
\phi_{(s, k'')}, & \quad k'' = -k + 2, -k + 4, \ldots, k - 2, k \quad (4.33) \\
\phi^{TT}_{(s', k''')}, & \quad s' = 0, 1, \ldots, s - 4, s - 3, \quad k''' = -k' + 2, \ldots, k', k' + 2, \quad k' \equiv s' + \frac{d - 6}{2}, \quad (4.34)
\end{align*}
\]

and one surviving gauge symmetry generated by gauge parameter

\[ \epsilon(s_{-1}, -k-1). \quad (4.35) \]

To summarize, we are left with one spin-\( s \) traceless field (4.32), set of spin-\( s \) double-traceless fields (4.33), and set of traceless fields (4.34). The surviving gauge symmetry is generated by gauge parameter (4.35). By using equations of motion, all fields in (4.33), (4.34) can be solved in terms of field (4.32). This leads to the standard higher-derivative formulation \[2\],\[24\] of conformal field theory in terms of single traceless field (4.32).

5 Realization of conformal boost symmetries

To complete ordinary-derivative formulation of spin-\( s \) conformal field we provide realization of the conformal algebra symmetries on the space of the ket-vector \( |\phi\rangle \). All that is required is to fix operators \( M^{AB}, \Delta \) and \( R^A \) for the case of spin-\( s \) conformal field and then use these
operators in (2.18)-(2.21). For the case of arbitrary spin-$s$ conformal field the spin matrix of the Lorentz algebra takes the form

$$M^{AB} = \alpha^A \bar{\alpha}^B - \alpha^B \bar{\alpha}^A. \quad (5.1)$$

Realization of the operator of conformal dimension $\Delta$ on space of $|\phi\rangle$ can be read from (3.2),

$$\Delta = \frac{d - 2}{2} + \Delta', \quad (5.2)$$

$$\Delta' \equiv N_{\nu^\oplus} - N_{\nu^\ominus}. \quad (5.3)$$

Representation of the operator $R^A$ on space of $|\phi\rangle$ is given by

$$R^A = R^A_0 + R^A_1 + R^A_G, \quad (5.4)$$

$$R^A_0 = r_{0,1}(\alpha^A - \alpha^2 \frac{1}{2N_{\alpha} + d - 2} \bar{\alpha}^A) - r_{0,1}^\dagger \bar{\alpha}^A, \quad (5.5)$$

$$R^A_1 = r_{1,1} \partial^A, \quad (5.6)$$

$$R^A_G = G\Pi r^A_G, \quad (5.7)$$

$$\Pi \equiv 1 - \alpha^2 \frac{1}{2(2N_{\alpha} + d)} \bar{\alpha}^2, \quad (5.8)$$

$$r^A_G = r_{g,1} \bar{\alpha}^A + r_{g,2} \alpha^A + r_{g,3} \alpha^A \bar{\alpha}^2 + r_{g,4} \bar{\alpha}^A \bar{\alpha}^2, \quad (5.9)$$

$$r_{0,1} = 2\nu^\oplus \bar{e}_1 \bar{\zeta}, \quad (5.10)$$

$$r_{0,1}^\dagger = 2\bar{\zeta} \bar{e}_1 \nu^\ominus, \quad (5.11)$$

$$r_{1,1} = -2\nu^\ominus \bar{\nu}^\ominus. \quad (5.12)$$

$$r_{g,a} = \nu^\ominus \bar{r}_{g,a} \bar{\nu}^\ominus, \quad a = 1, 3; \quad (5.13)$$

$$r_{g,2} = \nu^\ominus \nu^\ominus \bar{r}_{g,2} \bar{\zeta}^2, \quad (5.14)$$

$$r_{g,4} = \zeta^2 \bar{r}_{g,4} \bar{\nu}^\ominus \bar{\nu}^\ominus, \quad (5.15)$$

$$\bar{r}_{g,a} = \bar{r}_{g,a}(N_{\zeta}, \Delta'), \quad a = 1, 2, 3, 4, \quad (5.16)$$

where $G$ (5.7) stands for operator of gauge transformation (4,16), $G = \alpha \partial + b_1 + b_2 \alpha^2$, and $\bar{e}_1$ is given in (3.32). In (5.16), $\bar{r}_{g,a}, a = 1, 2, 3, 4$, are arbitrary functions of the operators $N_{\zeta}, \Delta'$. 14
Two remarks are in order.

**i)** $R_A^0$ and $R_A^1$ parts of the operator $R_A$ are fixed uniquely, while $R_A^G$ part, in view of arbitrary $\tilde{\tau}_{G,a}$, $a = 1, 2, 3, 4$, is still to be arbitrary. Reason for arbitrariness in $R_A^G$ is obvious. Global transformations of gauge fields are defined by module of gauge transformations. Since $R_A^G$ is proportional to $G$, action of $R_A^G$ on gauge field takes the form of some special gauge transformation.

**ii)** Evaluating commutator $[K^A, K^B]$, we obtain $[K^A, K^B] \sim Gr^{AB}$, where $r^{AB}$ is some differential operator. In other words, $[K^A, K^B]$ is proportional to the operator of gauge transformation, as it should in gauge theory. If we impose requirement $[K^A, K^B] = 0$, which amounts to $r^{AB} = 0$, then we obtain equations for $\tilde{\tau}_{G,a}$. Because explicit form of solution of those equations is not illuminating we do not present its here. Note that the simplest representation for $R_A$ that respects the commutator $[K^A, K^B] = 0$ is achieved by taking $R_A^G = 0$.

To summarize, we note that, having been introduced field content, the Lagrangian and the operator $R_A$ are fixed uniquely by requiring that

- **i)** Lagrangian should not involve higher than second order terms in derivatives;
- **ii)** the operator $R_A$ should not involve higher than first order terms in derivatives;
- **iii)** Lagrangian should be invariant with respect to global conformal algebra symmetries.

## 6 On-shell degrees of freedom of conformal field

Let us now discuss on-shell D.o.F of the conformal theory under consideration. For this purpose it is convenient to use fields transforming in irreps of the $so(d-2)$ algebra. Namely, we decompose on-shell D.o.F into irreps of the $so(d-2)$ algebra. One can prove that on-shell D.o.F are described by the following set of fields:

---

8 In the framework of higher-derivative formulation, uniqueness of interacting spin 2 conformal field theory was discussed in [32].

9 For the case of 4$d$ spin 2 conformal field theory, decomposition of D.o.F into irreps of the $so(2)$ algebra was carried out in [33]. We use light-cone approach to count on-shell D.o.F of conformal theory. Discussion of alternative method for counting of on-shell D.o.F may be found in [33, 34].

---
where vector indices of the $so(d-2)$ take values $I = 1, 2, \ldots, d-2$. Note that scalar on-shell D.o.F (6.4) appear only when $d \geq 6$ (i.e. $k \geq 2$). The tensor fields $\phi_{s',k''}^{I_1\ldots I_s}$ with $s' \geq 2$ satisfy the tracelessness constraint
\[ \phi_{s',k''}^{I_1\ldots I_s} = 0, \quad s' \geq 2, \] (6.6)
i.e. the tensor fields $\phi_{s',k''}^{I_1\ldots I_s}$ transform as irreps of the $so(d-2)$ algebra. Obviously, set of fields (6.1)-(6.4) is related to non-unitary representation of the conformal algebra $so(d,2)^{10}$.

Total number of on-shell D.o.F shown in (6.1)-(6.4) is given by
\[ \nu = \frac{1}{2} (d-3)(2s+d-2)(2s+d-4)\frac{(s+d-4)!}{s!(d-2)!}. \] (6.7)
Relation (6.7) gives generalization of the results found for the particular cases in the earlier literature [2].

\[ \nu \bigg|_{s-\text{arbitrary}; d=4} = s(s+1), \] (6.8)
\[ \nu \bigg|_{s=1; d-\text{arbitrary}} = \frac{1}{2}d(d-3), \] (6.9)
\[ \nu \bigg|_{s=2; d-\text{arbitrary}} = \frac{1}{4}d(d-3)(d+2). \] (6.10)

We note that $\nu$ given in (6.7) is a sum of D.o.F of fields given in (6.1)-(6.4). This is to say that $\nu$ can be represented as

\[ 10 \] By now, representations of (super)conformal algebras that are relevant for elementary particles are well understood (for discussion of conformal algebras see e.g. [35]-[39] and superconformal algebras in [40, 41]). In contrast to this, non-unitary representation deserves to be understood better.

\[ 11 \] For the case of $s = 1, d$-arbitrary, see Ref.[1].
\[ \nu = \sum_{s' = 0}^{s} \nu_{s'}, \quad (6.11) \]

where \( \nu_{s'} \) is a sum of D.o.F of rank-\( s' \) traceless tensor fields,

\[ \nu_{s'} \equiv \sum_{k''} \nu(\phi_{s', k''}), \quad (6.12) \]

\[ k'' = -k', -k' + 2, \ldots, k' - 2, k', \quad k' = s' + \frac{d - 6}{2}, \quad (6.13) \]

and \( \nu(\phi_{s', k''}) \) stands for D.o.F of rank-\( s' \) traceless tensor field \( \phi_{s', k''} \), i.e. \( \nu(\phi_{s', k''}) \) is a dimension of spin-\( s' \) irrep of the \( so(d - 2) \) algebra. Taking into account the well-known relation

\[ \nu(\phi_{s', k''}) = \frac{(s' + d - 5)!}{(d - 4)! s'!} (2s' + d - 4), \quad (6.14) \]

and the fact that there are \( k' + 1 \) rank-\( s' \) traceless tensor fields, we find the relation

\[ \nu_{s'} = \frac{1}{2} \frac{(s' + d - 5)!}{(d - 4)! s'!} (2s' + d - 4)^2, \quad (6.15) \]

which together with (6.11) leads to

\[ \nu = \sum_{s' = 0}^{s} \frac{1}{2} \frac{(s' + a)!}{(d - 4)! s'!} (2s' + d - 4)^2. \quad (6.16) \]

Then, making use of sum rule

\[ \sum_{s' = 0}^{s} \frac{(s' + a)!}{s'!} = \frac{(s + a + 1)!}{(a + 1)s!}, \quad (6.17) \]

gives (6.7).

7 Interrelation between ordinary-derivative description of conformal field and gauge invariant description of massive field

The ordinary-derivative formulation of conformal field involves Stueckelberg fields. As is well known, the gauge invariant description of massive field is also formulated by using Stueckelberg fields. Of course, the sets of the Stueckelberg fields in conformal field theory and the
massive gauge field theory are different. However, it turns out that there are interesting inter-
relations between ordinary-derivative description of the conformal field and gauge invariant
description of the massive field. These interrelations can straightforwardly be illustrated by
using oscillator formulation we exploit to discuss the conformal field. To do this, we need os-
cillator form of the Lagrangian and gauge transformations for the massive field. In terms of the
component tensor fields, Lagrangian and gauge transformations for arbitrary spin massive field
were obtained in [17]. We start with representing results in Ref.[17] in terms of ket-vectors
depending on the oscillators $\alpha^A$ and $\zeta$.

7.1 Gauge invariant description of massive field

Gauge invariant formulation of massive totally symmetric spin-$s$ field in flat space of dimen-
sion $d \geq 4$ involves the following set scalar, vector and tensor fields of the Lorentz algebra
$so(d−1,1)$ (see [17]):

$$\phi^{A_1...A_{s'}}_{s'}, \quad s' = 0, 1, \ldots, s - 1, s . \quad (7.1)$$

We note that

i) In (7.1), the fields $\phi_0$ and $\phi^A_1$ are the respective scalar and vector fields of the Lorentz algebra,
while the fields $\phi^{A_1...A_{s'}}_{s'}, \quad s' > 1$, are rank-$s'$ totally symmetric tensor fields of the Lorentz
algebra $so(d−1,1)$.

ii) The tensor fields $\phi^{A_1...A_{s'}}_{s'}$ with $s' \geq 4$ satisfy the double-tracelessness constraint

$$\phi^{A_1...A_{s'}}_{s'} = 0 , \quad s' \geq 4 . \quad (7.2)$$

In order to obtain the gauge invariant description in an easy–to–use form we use a set of
the creation operators $\alpha^A$, $\zeta$, and the respective set of annihilation operators, $\bar{\alpha}^A$, $\bar{\zeta}$. The fields
(7.3) can then be collected into a ket-vector $|\phi\rangle$ defined by

$$|\tilde{\phi}\rangle \equiv \sum_{s'=0}^{s} \zeta^{s-s'}|\phi_{s'}\rangle , \quad (7.3)$$

$$|\phi_{s'}\rangle \equiv \alpha^{A_1} \ldots \alpha^{A_{s'}} \phi^{A_1...A_{s'}}|0\rangle . \quad (7.4)$$

From (7.3),(7.4), we see that the ket-vector $|\phi\rangle$ is degree-$s$ homogeneous polynomial in the
oscillators $\alpha^A$, $\zeta$. In other words, the ket-vector $|\phi\rangle$ satisfies the relation

$$(N_\alpha + N_\zeta - s)|\phi\rangle = 0 . \quad (7.5)$$

Also, note that the ket-vector $|\phi_{s'}\rangle$ is degree-$s'$ homogeneous polynomial in the oscillators $\alpha^A$, i.e.,
this ket-vector satisfies the relation

$$N_\alpha|\phi_{s'}\rangle = s'|\phi_{s'}\rangle . \quad (7.6)$$
We note that in terms of ket-vector $|\phi\rangle$ (7.3), double-tracelessness constraint (7.2) takes the form

$$(\alpha^2)^2 |\phi\rangle = 0.$$ (7.7)

In terms of ket-vector (7.3), Lagrangian of the massive field takes the form

$$L = \frac{1}{2} \langle \phi | E | \phi \rangle,$$ (7.8)

where operator $E$ is given by

$$E = E_{(2)} + E_{(1)} + E_{(0)},$$ (7.9)

$$E_{(2)} \equiv \Box - \alpha \partial \bar{\alpha} \partial + \frac{1}{2} \left( \alpha^2 \partial^2 - \frac{1}{2} \alpha^2 \partial^2 \right) - \frac{1}{4} \alpha^2 \alpha^2 \partial \bar{\partial} \alpha^2,$$ (7.10)

$$E_{(1)} \equiv e_1 A - e_1^\dagger \bar{A},$$ (7.11)

$$E_{(0)} \equiv m_1 + \alpha^2 \bar{\alpha}^2 m_2 + m_3 \alpha^2 m_1^\dagger \bar{\alpha}^2,$$ (7.12)

$$A \equiv \alpha \partial - \alpha^2 \bar{\alpha} \partial + \frac{1}{4} \alpha^2 \alpha^2 \partial \bar{\partial} \alpha^2,$$ (7.13)

$$\bar{A} \equiv \bar{\alpha} \partial - \alpha^2 \alpha \partial + \frac{1}{4} \alpha^2 \alpha^2 \partial \bar{\partial} \alpha^2,$$ (7.14)

$$e_1 = m \tilde{e}_1 \zeta,$$ (7.15)

$$e_1^\dagger = \zeta \tilde{e}_1^\dagger m,$$ (7.16)

$$m_1 = \frac{2s + d - 2 - N_\zeta}{2s + d - 2 - 2N_\zeta} (N_\zeta - 1) m^2,$$ (7.17)

$$m_2 = \frac{2(2s + d - 2) + (2s + d - 7)N_\zeta - N_\zeta^2}{4s + d - 2 - 2N_\zeta} m^2,$$ (7.18)

$$m_3 = \frac{1}{2} m^2 \tilde{e}_1 \tilde{e}_1^{(1)} \zeta^2,$$ (7.19)

$$m_3^\dagger = \frac{1}{2} m^2 \zeta^2 \tilde{e}_1 \tilde{e}_1^{(1)} ,$$ (7.20)

$$\tilde{e}_1 = \left( \frac{2s + d - 4 - N_\zeta}{2s + d - 4 - 2N_\zeta} \right)^{1/2},$$ (7.21)

$$\tilde{e}_1^{(1)} = \left( \frac{2s + d - 5 - N_\zeta}{2s + d - 6 - 2N_\zeta} \right)^{1/2}.$$ (7.22)
Gauge transformations. We now discuss gauge symmetries of the Lagrangian. To this end we introduce the following set of gauge transformation parameters:

\[ \epsilon_{s'...s'}^{A_1...A_{s'}} , \quad s' = 0, 1, \ldots, s - 2, s - 1. \]  \hspace{1cm} (7.23)

We note that

i) In (7.23), the gauge parameters \( \epsilon_0 \) and \( \epsilon_1^A \) are the respective scalar and vector fields of the Lorentz algebra, while the gauge parameters \( \epsilon_{s'...s'}^{A_1...A_{s'}} \), \( s' > 1 \), are rank-s' totally symmetric tensor fields of the Lorentz algebra \( so(d-1,1) \).

ii) The gauge parameters \( \epsilon_{s'...s'}^{A_1...A_{s'}} \) with \( s' \geq 2 \) satisfy the tracelessness constraint

\[ \epsilon_{s'...s'}^{AAA_1...A_{s'}} = 0 , \quad s' \geq 2. \]  \hspace{1cm} (7.24)

Now, as usually, we collect gauge transformation parameters in ket-vector \( |\epsilon\rangle \) defined by

\[ |\epsilon\rangle \equiv \sum_{s'=0}^{s-1} \zeta^{s-1-s'} |\epsilon_{s'}\rangle , \]  \hspace{1cm} (7.25)

\[ |\epsilon_{s'}\rangle \equiv \alpha^{A_1}...\alpha^{A_{s'}} \epsilon_{s'...s'}^{A_1...A_{s'}} |0\rangle . \]  \hspace{1cm} (7.26)

The ket-vector \( |\epsilon\rangle \) satisfies the algebraic constraint

\[ (N_\alpha + N_\zeta - s + 1)|\epsilon\rangle = 0 , \]  \hspace{1cm} (7.27)

which tells us that \( |\epsilon\rangle \) is a degree-(s - 1) homogeneous polynomial in the oscillators \( \alpha^A, \zeta \). The ket-vector \( |\epsilon_{s'}\rangle \) satisfies the algebraic constraint

\[ N_\alpha |\epsilon_{s'}\rangle = s'|\epsilon_{s'}\rangle , \]  \hspace{1cm} (7.28)

which implies that \( |\epsilon_{s'}\rangle \) is degree-\( s' \) homogeneous polynomial in the oscillators \( \alpha^A \). In terms of the ket-vector \( |\epsilon\rangle \), tracelessness constraint (7.24) takes the form

\[ \bar{\alpha}^2 |\epsilon\rangle = 0 . \]  \hspace{1cm} (7.29)

Gauge transformations can entirely be written in terms of \( |\phi\rangle \) and \( |\epsilon\rangle \). This is to say that gauge transformations take the form

\[ \delta |\phi\rangle = (\alpha \partial + b_1 + b_2 \alpha^2) |\epsilon\rangle , \]  \hspace{1cm} (7.30)

\[ b_1 = m \zeta \bar{e}_1 , \]  \hspace{1cm} (7.31)

\[ b_2 = -\frac{1}{2s + d - 6 - 2N_\zeta} m \bar{e}_1 \bar{\zeta} , \]  \hspace{1cm} (7.32)

where \( \bar{e}_1 \) is defined in (7.21).
7.2 Comparison of ordinary-derivative description of conformal field and gauge invariant description of massive field

We are now ready to make comparison of the ordinary-derivative description of the conformal field and the gauge invariant description of the massive field. First of all we note that field contents of conformal field and massive field are different (see (3.1) and (7.1)). We note however that formulation in terms of the respective ket-vectors (3.12), (7.3) leads to remarkable interrelations between structures of Lagrangian and gauge transformations for the conformal field and the ones for the massive field. This is to say that

i) Dependence of both the operator \( E \), which appears in conformal field Lagrangian (3.20), and operator \( G = \alpha \partial + b_1 + b_2 \alpha^2 \), which appears in gauge transformations (4.16), on the oscillators \( \alpha^A, \zeta \) and the operator \( N_\zeta \) takes the same form as the one of the massive field (see (7.9), (7.30)).

ii) Operator \( E \) (3.20) and operator of gauge transformations of conformal field, \( G = \alpha \partial + b_1 + b_2 \alpha^2 \) (4.16), can be obtained from the ones of massive field (see (7.9), (7.30)) by substitution

\[
\zeta m \rightarrow \zeta \bar{\nu}, \quad \zeta^2 m^2 \rightarrow \zeta^2 \nu \bar{\nu}, \quad m \bar{\zeta} \rightarrow \nu \bar{\zeta}, \quad m^2 \bar{\zeta}^2 \rightarrow \nu \nu \bar{\zeta} \bar{\zeta}, \quad (7.33)
\]

\[
m^2 \rightarrow \nu \nu. \quad (7.34)
\]

We note that these substitutions are realizable unambiguously. For the case of fermionic fields, suitable substitutions are discussed in [42].

By now, in the literature, there are various approaches to gauge invariant formulations of massive fields. Obviously, use of just mentioned interrelations between conformal and massive fields might be helpful for straightforward generalization of those approaches to the case of conformal fields. Note however that it is important to keep in mind the following important difference between conformal and massive fields. Lagrangian and gauge transformations of massive field are uniquely determined by requiring the Lagrangian to be gauge invariant. For the case of conformal field such requirement does not allow to determine Lagrangian uniquely. One needs to impose additional requirement. Namely, one needs to require that the Lagrangian be invariant with respect to global conformal boost symmetries.

8 Conclusions

In this paper we applied ordinary-derivative approach developed in Ref.[1] to study of higher-spin conformal bosonic fields in flat space of arbitrary dimension. Because the approach we
exploit is based on use of oscillator realization of spin degrees of freedom of gauge fields it allows straightforward generalization to the fermionic arbitrary spin conformal fields [42].

The results presented here should have a number of interesting applications and generalizations, some of which are:

ii) extension of our approach to interacting conformal higher-spin field theories. In the frameworks of various approaches, such theories were discussed in Refs. [3], [43], [24]. As we have illustrated, in our approach, use of Stueckelberg fields is very similar to the one in gauge invariant formulation of massive fields. Stueckelberg fields provide interesting possibilities for study of interacting massive gauge fields (see e.g. [44]-[47]). We expect therefore that application of our approach to the interacting (super)conformal fields should lead to new interesting development.

iii) application of BRST methods to study of ordinary-derivative conformal field theories. In recent time, BRST approach was extensively exploited for study of ordinary-derivative Lagrangian theories of massive fields (see e.g. [31], [48], [49]). Because gauge invariant formulations of massive fields and ordinary-derivative formulation of conformal fields have common features, application of previously developed BRST methods to study of ordinary-derivative conformal field theories should be relatively straightforward.

iv) representation of ordinary-derivative approach to conformal theories in terms of unconstrained fields. Discussion of various approaches to higher-spin dynamics based of unconstrained gauge fields may be found in [27]-[31].

v) extension to mixed symmetry higher-spin gauge fields. Mixed symmetry fields [50], [51] have attracted considerable interest in recent time (see e.g. [52]-[57]). In AdS space, massless mixed symmetry fields, in contrast to massless fields in Minkowski space whose physical degrees of freedom transform in irreps of \( o(d-2) \) algebra, reduce to a number of irreps of \( so(d-2) \) algebra. In other words, not every massless field in flat space admits a deformation to \( AdS_d \) with the same number of degrees of freedom [58]. It would be interesting to check this phenomena for the case of mixed symmetry conformal fields.

vi) extension of our approach to light-cone gauge conformal fields and application to analysis of interaction of conformal fields with currents constructed out of fields of supersymmetric Yang-Mills theory [12]. Action of Green-Schwarz superstring in AdS/Ramond-Ramond background simplifies considerably in the light-cone gauge [59], [60]. Therefore we expect that from the stringy perspective of \( AdS/CFT \) correspondence the light-cone approach to conformal field theory might be helpful for study of various aspect of the string/gauge theory duality.

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12 We note that the general method of building interaction vertices of light-cone massive fields was developed in [61]. This method can straightforwardly be generalized to the case of conformal fields.
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