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To cite this version:
Boris Gralak, Arismendi Genaro, Sébastien Guenneau, Avril Benoît, André Diatta. Analysis in temporal regime of dispersive invisible structures designed from transformation optics. Physical Review B: Condensed Matter and Materials Physics (1998-2015), 2016, 93, pp.121114(R). 10.1103/PhysRevB.93.121114 . hal-01310953

HAL Id: hal-01310953
https://hal.science/hal-01310953
Submitted on 7 Sep 2023

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Analysis in temporal regime of dispersive invisible structures designed from transformation optics

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(Dated: June 13, 2021)

A simple invisible structure made of two anisotropic homogeneous layers is analyzed theoretically in temporal regime. The frequency dispersion is introduced and analytic expression of the transient part of the field is derived for large times when the structure is illuminated by a causal excitation. This expression shows that the limiting amplitude principle applies with transient fields decaying as the power $-3/4$ of the time. The quality of the cloak is then reduced at short times and remains preserved at large times. The one-dimensional theoretical analysis is supplemented with full-wave numerical simulations in two-dimensional situations which confirm the effect of dispersion.

PACS numbers: 78.20.Bh, 41.20.Jb, 78.67.Pt, 42.25.Bs

In 2006, Pendry et al. [1] and Leonhardt [2] designed an invisibility cloak for electromagnetic radiation by blowing up a hole in optical space and hiding an object inside it. These proposals have been validated by microwave experiments [3]. However, these metamaterials are subject to an inherent frequency dispersion which may affect the quality of the optical function designed in time harmonic regime. Hence, there is a renewed interest in the propagation in dispersive media, originally investigated by Brillouin [4]. The effect of dispersion has been addressed in the cases of the flat lens [5–9] and cylindrical layered cloak, where the presence of additional modes is confirmed in the transient regime.

We start with the definition of a system of invisible layers. Let $x = (x_1, x_2, x_3)$ be a Cartesian coordinate system in the space $\mathbb{R}^3$. At the oscillating frequency $\omega$, the electric field amplitude $E(x)$ is governed in free space by the Helmholtz equation

$$ -\nabla \times \nabla \times E(x) + \omega^2 \mu_0 \varepsilon_0 E(x) = 0, $$

where $\varepsilon_0$ and $\mu_0$ are the vacuum permittivity and permeability. The invisible layered structure is then deduced using the coordinate transform $x \rightarrow x'$ (see Fig. 1):

$$ x_1' = \frac{a}{\alpha} x_1 \quad 0 \leq x_1 \leq \alpha, $$

$$ x_1' = a + \frac{b-a}{\alpha} (x_1 - \alpha) \quad \alpha \leq x_1 \leq b, $$

$$ x_1' = x_1 \quad x_1 \leq 0, \quad b < x_1, $$

where $0 < a < \alpha < b$, $x_2' = x_2$ and $x_3' = x_3$ being invariant. The effect of this geometric transform is to map the layer $0 \leq x_1 \leq \alpha$ onto the layer $0 \leq x_1' \leq a$.

The originality of our approach is to consider a simple invisibility system made of two layers allowing analytic calculations. Indeed, the invisible nature of the system leads to a simple expression of the transmitted field, since there is no reflexion at the interfaces. Also, the absence of branch cut in the integral expression of the time dependent field in multilayered structures is exploited. The method is presented in detail and the derivation of the transient regime shows that the electromagnetic field includes contributions generated by the singular values of the permittivity and permeability (zeros and infinities). An explicit expression of the transient fields is obtained for long times, which is similar to the one obtained by Brillouin [4] for wavefronts (forerunners). Next, the limiting amplitude principle is considered to show that cloaking can be addressed in temporal regime after the transient regime. These results are supplemented with numerical simulations in the case of a two-dimensional cylindrical layered cloak, where the presence of additional modes is confirmed in the transient regime.

FIG. 1. Coordinate transform for invisible layers. Left: change of coordinate $x_1 \rightarrow x_1'$. Center: free space before coordinate transform. Right: invisible set of homogeneous anisotropic layers after coordinate transform.
where equal to the tensor \( \epsilon \) since \( x \cdot x^\prime = 1 \) in vacuum, i.e., when \( \nu^\parallel(x) = \nu^\perp(x) = 1 \). This implies that the transfer matrix \( T_b T_a = \exp[\imath M_0 b] \), associated with layers A and B, is exactly the same as the one of a vacuum layer of thickness \( b \). Hence the system of layers A and B is invisible to any incident field.

Nevertheless, as pointed out by V. Veselago when he introduced negative index materials \([16]\), causality principle and passivity require that permittivity and permeability be frequency dispersive when they take relative value below unity \([17, 18]\). According to this requirement, frequency dispersion is introduced in the components of \( \nu_a \) and \( \nu_b \) with value below unity, assuming the simple Drude-Lorentz model \([15]\):

\[
\nu_a^\perp(\omega) = 1 - \frac{\Omega_a^2}{\omega^2 - \omega_0^2}, \quad \Omega_a^2 = \frac{\alpha - a}{a} (\omega_0^2 - \omega_a^2),
\]

\[
\nu_b^\perp(\omega) = 1 - \frac{\Omega_b^2}{\omega^2 - \omega_0^2}, \quad \Omega_b^2 = \frac{\alpha - b}{b - a} (\omega_0^2 - \omega_b^2).
\]

Under this assumption, the functions \( \nu_a^\perp(\omega) \) and \( \nu_b^\perp(\omega) \) take the appropriate values for the invisibility at \( \omega = \omega_0 \). Notice that the resonance frequencies \( \omega_a \) and \( \omega_b \) must be smaller than the operating frequency \( \omega_0 \) in order to ensure that the oscillator strengths \( \Omega_a^2 \) and \( \Omega_b^2 \) are positive. For frequencies different from \( \omega_0 \), the system has no reason to be invisible.

The effect of dispersion is analyzed using illumination with sinusoidal time-dependence oscillating at \( \omega_0 \) and switched on at an initial time. Such a “causal” incident field, originally used by L. Brillouin \([1]\) and more recently in \([15, 17]\), is assumed to be in normal incidence for simplicity. Hence the following current source is considered:

\[
S(x, t) = S_0 \delta(x - x_0) \theta(t) \sin[\omega_0 t],
\]

where \( \delta \) is the Dirac “function”, \( \theta(t) \) the step function (equal to 0 if \( t < 0 \) and 1 otherwise), and \( S_0 \) the constant component of the source parallel to the field component \( U(x) \). In the domain of complex frequencies \( \omega = \omega^\prime + i\eta \), the electric field radiated in vacuum by this source is

\[
U_0(x, z) = \frac{S_0 \mu_0 c}{2} \frac{\omega_0}{z^2 - \omega_0^2} \exp[iz|x - x_0|/c].
\]

The positive imaginary part \( \eta \) has been added to the frequency \( \omega \) to ensure a correct definition of the Fourier transform with respect to time of the source \([12]\). The time dependent incident field radiated in vacuum is, with
The resulting contribution \( E_T^{(b)} \) in the transmitted field is
\[
E_T^{(b)}(x, t) \approx -2S_0 \mu_0 \pi c \frac{\omega_0 \omega_b}{\omega_0^2 - \omega_b^2} \theta(\tau) \frac{1}{\sqrt{\tau/\beta}} \times J_1(2\omega_b \sqrt{\tau/\beta} \cos(\omega_b(\tau + \beta/2))),
\]
(20)
where \( J_1 \) is the Bessel function (see the supplemental material). It is stressed that a similar behavior, given by the Bessel function \( J_1 \) with argument proportional to \( \sqrt{\tau} \), has been highlighted by Brillouin [4] but for short relative time \( \tau \) (forerunners). In both cases, \( J_1 \) is a consequence of the dispersion given by the Drude-Lorentz model (11), but for different frequency ranges: near the resonance frequencies \( \pm \omega_b \) in the present case, and for the high frequencies in the case considered by Brillouin (forerunners). Forerunners at \( \tau \to 0 \) can also be characterized here.

The asymptotic form \( J_1(u) \approx \sqrt{2/(\pi u)} \cos[u - 3\pi/4] \) provides an explicit expression for long time \( \tau \gg \beta \). The contribution in the transmitted field becomes
\[
E_T^{(b)}(x, t) \approx -2S_0 \mu_0 c \frac{\omega_0 \omega_b}{\omega_0^2 - \omega_b^2} \frac{\sqrt{\pi}}{\omega_b \beta} \theta(\tau) \times (\tau/\beta)^{-3/4} \cos[2\omega_b \beta \sqrt{\tau/\beta} - 3\pi/4] \times \cos[\omega_b \beta(\tau + \beta/2)],
\]
(21)
This expression shows that this second contribution has a first factor oscillating at the frequency \( \omega_b \) and a second factor with more complex oscillating behavior with
The left panel of Fig. 3 shows that the cylindrical cloak works almost perfectly in time harmonic regime oscillating at the frequency $\omega_0 = 2\pi c/\lambda_0$, where $\lambda_0 = R_0/2$. Note that a purely dielectric structure is used for this 2D cloak, and thus interfaces between different concentric layers are subject to reflections producing effective dispersion. Hence, it is expected to observe an effect of dispersion even if all the dielectric layers are non dispersive [15]. The right panel of Fig. 3 shows the longitudinal magnetic field amplitude when the cloak is illuminated by the causal incident field given by Eq. (12) and Fig. 2 (right).

The cloaking effect appears to be of similar quality in both panels of Fig. 3. We now analyze the magnetic field at short times. In Fig. 4, cylindrical modes are excited in the multi-layers when the incident front wave reaches the cloak (left), what produces a superluminal concentric wave (see [21] for a design without supraluminal component). These modes can propagate in the cloak faster than the front wave in vacuum since the frequency dispersion is not introduced in the dielectrics, especially those with index values below unity. The cylindrical modes excited in the multi-layers then radiate cylindrical waves outside the cloak, as evidenced by the right panel in Fig. 4, which explains the tiny perturbation of the field observed on right panel of Fig. 3 (the field perturbation is smoothed down at long times, in agreement with the analytical part). In addition, Fig. 4 shows a picture of the transient part of the field produced by the causal source.

Here, we take benefit of the supra-luminal propagation of

- **TABLE I.** Relative permittivity values of the layered cloak from inside (layer 1) to outside (layer 20).

| layer | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-------|---|---|---|---|---|---|---|---|---|----|
| $\varepsilon/\varepsilon_0$ | 0.0012 | 8.0 | 0.02 | 8.0 | 0.07 | 8.0 | 0.12 | 8.0 | 0.18 | 8.0 |
| layer | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| $\varepsilon/\varepsilon_0$ | 0.24 | 8.0 | 0.3 | 8.0 | 0.38 | 8.0 | 0.44 | 8.0 | 0.5 | 8.0 |
the modes in the cloak to observe that the radiated transient part is almost isotropic. We deduce that the radial dependence of this transient part does not correspond to the function $J_1$ found by A. Sommerfeld and L. Brillouin [4], and exhibited in the present Eq. (20). There is no contradiction since the $J_1$ dependence is clearly related to the Drude-Lorentz model of the dispersion, while the transient field around the 2D cloak is related to the effective dispersion produced by the cylindrical multilayered geometry. Nonetheless, one can conclude that both situations considered in this letter attest that the quality of cloaking deteriorates at short times under illumination by a causal incident field.

In summary, a new method to analyze propagation of electromagnetic waves in dispersive media has been proposed. The major ideas are to consider a layered structure to eliminate branch cuts, and an invisible structure (with $\epsilon = \mu$) to eliminate reflections in normal incidence. In this situation, the transient regime can be highlighted and, especially, an explicit expression is obtained in the long time limit. As a result the amplitude of the transient part decreases like $(t - x/c)^{-3/4}$. Hence the technique proposed in this letter brings new elements to the transient part decreases like $(t - x/c)^{-3/4}$ and exponential can be expanded around $\xi$ and $\epsilon$.

The proposed method opens new possibilities for investigating transient regime of dispersive systems, notably structures designed from transformation optics like cloaks, carpets, concentrators and rotators. This method can be also applied to optical systems moving like cloaks, carpets, concentrators and rotators. This notably structures designed from transformation optics investigating transient regime of dispersive systems, and, especially, an explicit expression is obtained in the long time limit. As a result the amplitude of the transient part decreases like $(t - x/c)^{-3/4}$. Hence the technique proposed in this letter brings new elements to the transient part decreases like $(t - x/c)^{-3/4}$ and exponential can be expanded around $\xi$ and $\epsilon$.

B. Avril, A. Diatta and S. Guenneau acknowledge ERC funding (ANAMORPHISM). G. Arismendi and B. Gralak acknowledge S. Enoch for his support.

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Supplemental material: calculation of the transient field

The contribution $E^{(b)}_R(x, t)$ of the two isolated singularities at $z = \pm \omega_b$ in the integral expression (15) is estimated for large values of the relative time $\tau$ [given by (19)] after the front wave. These two singularities are present in the transmission coefficient $T(z)$ given by (18). Decomposing the ratio $z/(z^2 - \omega_b^2)$ in simple poles, the whole function under the integral in (15) can be formulated as

$$f(z) = f_\pm(z) \exp \left[-i \frac{(b-a)\Omega_b^2/(2c)}{z - (\pm \omega_b)} \right],$$  

(24)

where $f_\pm(z)$ are analytic around $\pm \omega_b$. Let $\xi = z - (\pm \omega_b)$, then the functions $f_\pm$ and exponential can be expanded in power series around $\xi = 0$:

$$f(z) = \sum_{q \in \mathbb{N}} \frac{f_\pm^{(q)}(\pm \omega_b)}{q!} \xi^q \sum_{p \in \mathbb{N}} \frac{[(b-a)\Omega_b^2/(2ic)]^p}{p!} \xi^{-p},$$

(25)
where \( f^{(q)}(\pm \omega_b) \) is the derivative of order \( q \) of \( f_{\pm}(z) \) evaluated at \( \pm \omega_b \). Thanks to the convergence of the series, the terms of this product can be arranged in order to obtain the coefficients of the poles \( \xi^{-1} \), i.e., the residues \( \text{Res}(\pm \omega_b) \) of the function \( f(z) \) at \( z = \pm \omega_b \):

\[
\text{Res}(\pm \omega_b) = \sum_{p \in \mathbb{N}\setminus\{0\}} \frac{f^{(p-1)}(\pm \omega_b) \cdot (b-a)\Omega_b^2/(2ic)^p}{(p-1)! p!}.
\]

(26)

Notice that it can be checked that the series above converges as well as the series expansion of the exponential function. Hence the residues \( \text{Res}(\pm \omega_b) \) are well-defined.

Using that the complex conjugated of \( f(z) \) is \( f(-z) = f(-\bar{z}) \), the contribution of the singularities at \( \pm \omega_b \) in the time dependent transmitted field is

\[
E^{(b)}_T(x,t) = \theta(t - (x-x_0 - \alpha - a)/c) \text{Imag} \left\{ 4\pi \text{Res}(\omega_b) \right\}.
\]

(27)

The exact calculation of this second contribution, corresponding to the transient regime, cannot be performed in general. However, the \((x,t)\) dependence can be analyzed from the one of \( f_{\pm}(z) \) which can be expressed as

\[
f_{\pm}(z) = g_{\pm}(z) \exp[-iz\tau], \quad \tau = t - (x-x_0 + \alpha - a)/c.
\]

(28)

where the functions \( g_{\pm}(z) \) are \((x,t)\) independent, and the time quantity \( \tau \) defines the arrival of the signal (from \( \tau = 0 \)). Denoting \( \beta = (b-a)\Omega_b^2/(2ic) \) and recalling that \( \xi = z - (\pm \omega_b) \), the function \( f(z) \) becomes

\[
f(\xi \pm \omega_b) = g_{\pm}(\xi \pm \omega_b) \exp[\mp i\omega_b \tau] \exp[-i(\tau \xi + \omega_b^2 \beta/\xi)].
\]

(29)

Then the residues can be expressed as

\[
\text{Res}(\pm \omega_b) = \frac{1}{2i\pi} \int_{\xi=\pm \omega_b} d\xi f(\xi \pm \omega_b)
\]

(30)

as soon as the functions \( g_{\pm}(z) \) are analytic in the disks of radius \( \pm \omega_b \) and centered at \( \pm \omega_b \). In particular, this expression can be estimated for \( \tau \) tending to infinity. Let the radius of the disks set to \( d = \omega_b \sqrt{\beta/\tau} \) and the complex number \( \xi = \omega_b \sqrt{\beta/\tau} \exp[i\phi] \). For \( \tau/\beta \to \infty \), the functions \( g_{\pm}(\xi \pm \omega_b) \approx g_{\pm}(\pm \omega_b) \) and the residues can be approached by

\[
\text{Res}(\pm \omega_b) \approx \frac{1}{2i\pi} g_{\pm}(\pm \omega_b) \exp[-i(\pm \omega_b)\tau] i\omega_b \sqrt{\beta/\tau} \times \int_{[0,2\pi]} d\phi \exp[i\phi - i2\omega_b \sqrt{\beta/\tau} \cos \phi].
\]

(31)

Using the integral representation of the Bessel function

\[
J_1(u) = -\frac{1}{2i\pi} \int_{[0,2\pi]} d\phi \exp[i\phi - iu \cos \phi],
\]

(32)

it is deduced that, for \( \tau/\beta \to \infty \),

\[
\text{Res}(\pm \omega_b) \approx -i \frac{S_0\mu_0 c}{2} \frac{\omega_b \omega_b \sqrt{\beta/\tau}}{\omega_b^2 - \omega_0^2} \times \exp[i\omega_b (\tau + \beta/2)] J_1(2\omega_b \sqrt{\beta/\tau}).
\]

(33)

Replacing this estimate of the residues in (27) provides the expression (20) of the time dependent transmitted field in the letter.