Informational Properties of Transmuted Distributions

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Abstract. In this paper, we establish some informational properties of transmuted distributions. Specifically, we derive Shannon entropy, Gini’s mean difference, and Fisher and Bayes Fisher information measures of a transmuted distribution. Two extensions of Shannon entropy and Gini’s mean difference information measures are also provided. Finally, the distances between transmuted distribution and its components based on the Kullback-Leibler, chi-square and energy distance divergences are all derived.

1. The first section

In information theory, there are different criteria to measure the uncertainty of a probabilistic model. Moreover, various divergences measures have been developed in the literature for measuring similarity (closeness) between two probability distributions. Shannon entropy, Fisher information and Gini’s mean difference are three most important information measures that have been used in many different fields. For more details, see Shannon (1948), Fisher (1929) and Gini (1912). More recently, these information measures have been generalized based on Jensen inequality, which have come to be known as Jensen-Shannon, Jensen-Fisher and Jensen-Gini information measures, respectively. For pertinent details, one may see Lin (1991), Sánchez-Moreno et al. (2016) and Mehrali et al. (2018).

Recently, considerable work in distribution theory has focused on the family of transmuted distributions derived through a quadratic rank transmutation map. Although the transmuted distributions are of particular importance in modeling various data sources, these models have not been studied from information theory viewpoint so far.

With this in mind, our main interest here is to establish some informational properties, including Shannon entropy, Fisher information, Bayes Fisher information and Gini’s mean difference for transmuted distributions. These models get formulated by transforming a parent distribution into its generalized counterpart. Transmuted distribution was introduced by Shaw and Buckley (2009) in an unpublished report and applied to uniform, exponential and normal distributions. During the past decade, considerable efforts have been directed at developing more flexible distributions with the use of this construct. Many standard probability distributions have been generalized and developed into some flexible models in this manner; see, for example, Sarabia et al. (2020) and the references therein. Kozubowski and Podgórski

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CDF $F$ be a random sample of size two from the absolutely continuous new probability CDF $F$. Further, the corresponding (1992). Hence, we can rewrite the are given by CDFs $(1992)$. Model (1) is referred to as a transmuted distribution. For a stochastic representation of (1), let $X$ have proposed analogous transmuted distributions based on the theory of records and examined their hazard properties.

In the framework of transmutation map, from a given base cumulative distribution function (CDF) $F$, a new probability CDF $F_{X_I}$ is defined as

$$F_{X_I}(x) = (1 + \lambda)F(x) - \lambda F^2(x), \ x \in \mathcal{R}, |\lambda| \leq 1.$$  \hfill (1)

Model (1) is referred to as a transmuted distribution. For a stochastic representation of (1), let $X_1$ and $X_2$ be a random sample of size two from the absolutely continuous CDF $F$ with probability density function (PDF) $f$. The density functions associated with extreme random variables $\text{min}(X_1, X_2)$ and $\text{max}(X_1, X_2)$ are known to be $f_{\text{min}}(x) = 2f(x)(1 - F(x))$ and $f_{\text{max}}(x) = 2f(x)F(x)$, respectively. Moreover, the corresponding CDFs are given by $F_{\text{min}}(x) = 1 - (1 - F(x))^2$ and $F_{\text{max}}(x) = F^2(x)$, respectively; see, for example, Arnold et al. (1992). Hence, we can rewrite the CDF in (1) based on the components $F$ and $F_{\text{max}}$ as follows:

$$F_{X_I}(x) = (1 + \lambda)F(x) - \lambda F^2(x) = (1 + \lambda)F(x) - \lambda F_{\text{max}}(x).$$  \hfill (2)

Further, the corresponding PDF is given by

$$f_{X_I}(x) = (1 + \lambda)f(x) - 2\lambda f(x)F(x) = (1 + \lambda)f(x) - \lambda f_{\text{max}}(x).$$  \hfill (3)

It is worthwhile to note that (2) and (3) can also be stated based on the variable $\text{min}(X_1, X_2)$ as

$$F_{X_I}(x) = (1 - \lambda)F(x) + \lambda F_{\text{min}}(x)$$

and

$$f_{X_I}(x) = (1 - \lambda)f(x) + \lambda f_{\text{min}}(x),$$

respectively. In order to avoid repetitive results within the present work, we only consider the transmuted distribution in (2).

If the baseline distribution in (3) is considered as Uniform on $(0, 1)$, then the density function of transmuted uniform variable $U_T$ is given by

$$f_{U_T}(u) = (1 + \lambda) - 2\lambda u, \quad u \in (0, 1).$$  \hfill (4)

We now briefly introduce some informational measures that will be used in the sequel. Let $X$ be an absolutely continuous random variable with CDF $F$ and density function $f$. Then, the Shannon entropy of $X$ (or density $f$) is defined as

$$H(X) = H(f) = - \int f(x) \log f(x)dx,$$

where “log” stands for the natural logarithm. For more details, see the pioneering paper of Shannon (1948). To simplify notation, we suppress $X$ for integration with respect to $dx$ throughout the paper, unless a distinction becomes necessary.

Kullback-Leibler (KL) discrimination information is introduced next. Let $X$ and $Y$ be two continuous random variables with absolutely continuous density functions $f$ and $g$, respectively. Then, the Kullback-Leibler distance between $X$ and $Y$ (or $f$ and $g$) is defined as

$$KL(X||Y) = KL(f, g) = \int f(x) \log \frac{f(x)}{g(x)}dx.$$
The Kullback-Leibler discrimination between $Y$ and $X$ can be defined analogously. For more details, see Kullback and Leibler (1951).

Another important diversity measure between two density functions $f$ and $g$ is the chi-square divergence, defined as

$$\chi^2(f, g) = \int \frac{(f(x) - g(x))^2}{g(x)} dx.$$  

In the same way, we can define $\chi^2(g, f)$. For more details, see Nielsen and Nock (2013).

The Fisher information of a random variable $X$, or its PDF $f(x; \theta)$, about the parameter $\theta$ is defined as

$$I(\theta) = \int \left[ \frac{\partial \log f(x; \theta)}{\partial \theta} \right]^2 f(x; \theta) dx.$$  

It is assumed that $\theta$ lies in an open interval in the real line and that $f(x; \theta) > 0$ for all values of $\theta$ in the parameter space and is differentiatable with respect to $\theta$. In Bayesian statistics, it is assumed that the parameter $\theta$ is endowed with a prior $\pi(\theta)$. Then, the expected Fisher information $I(\theta) = E_\pi[I(\theta)]$ is called Bayes Fisher information. For more details, see Asadi et al. (2018) and the references therein.

Gini’s mean difference (GMD) and energy distance are two other prominent information measures that will be considered in this work. Let $X$ be a continuous random variable with distribution function $F$. Then, the GMD associated with $X$ is defined as

$$\text{GMD}(F) = 2 \int F(x) \bar{F}(x) dx,$$

where $\bar{F}(x) = 1 - F(x)$ denotes the survival function of $X$. Yitzhaki (2003) stated that the GMD, as a measure of variability, shares many properties of the variance of $X$, and is more suitable for distributions that are far from normality. Moreover, the energy distance ($L^2$) between two CDFs $F_1$ and $F_2$ is given by

$$D(F_1, F_2) = \int (F_1(x) - F_2(x))^2 dx.$$  

For more details, see Mehrali et al. (2018).

In this paper, we establish some important informational properties of transmuted distributions. In Section 2, we study the Shannon entropy of transmuted density functions and also consider the Kullback-Leibler divergence between transmuted distribution and the densities of each of its components. We then propose transmuted Shannon entropy, which is an extension of Shannon entropy of transmuted distribution, as well as Jensen-Shannon entropy. It is shown that the KL divergence between a general transmuted model and its components is free of the parent distribution. The proposed transmuted Shannon entropy is further expressed based on the transmuted structure of two KL divergences of the general model and its components. In Section 3, by considering the transmuted distribution, we obtain Gini’s mean difference and also discuss the energy divergence between transmuted distribution and its components. The transmuted Gini’s mean difference is also proposed and it is shown that this information measure can be expressed in terms of a quadratic rank structure of two energy distances of the transmuted distribution and its components. Section 4 discusses the chi-square divergence between the general transmuted density function and the density functions of each of its components. In Section 5, the Fisher information for parameter $\lambda$ is derived. It is shown that the Fisher information of the transmuted model about parameter $\lambda$ is connected to chi-square divergence. A detailed discussion about Bayes Fisher information for the parameter $\lambda$ of transmuted distribution under different prior distributions is also discussed in this section. It is specifically shown that the Bayes Fisher information measures for transmuted model under uniform prior are the same as Jeffreys’ divergence. Another result shows that, under triangular and Beta priors, the measures for these two models are different Kullback-Leibler based measures. Finally, Section 6 presents some concluding remarks.
2. Entropic measures

In this section, we first derive the Shannon entropy of the transmuted random variable $X_T$ with PDF as in (3), and then examine the Kullback-Leibler divergence between the transmuted distribution and each of its components. We further propose an extension of the Shannon entropy that is suitable for transmuted distributions.

2.1. Shannon entropy

Suppose random variables $X_1$ and $X_2$ are independent and identically distributed as $X$ with common PDF $f$ and CDF $F$. We now derive the Shannon entropy of the transmuted random variable $X_T$.

**Theorem 2.1.** For the transmuted random variable $X_T$ with PDF as in (3), the Shannon entropy is given by

$$H(X_T) = (1 + \lambda)H(X) - \lambda H(\max(X_1, X_2)) + \lambda H(V) + H(U_T),$$

where the random variable $V$ has Beta$(2, 1)$ distribution with PDF

$$f_v(v) = 2v, \quad v \in (0, 1),$$

and the random variable $U_T$ is the transmuted uniform random variable on $(0, 1)$ with its PDF as in (4).

**Proof:** By the definition of Shannon entropy, we have

$$H(X_T) = - \int_{-\infty}^{\infty} f(x) \log f(x) \, dx$$

$$= - \int_{-\infty}^{\infty} (1 + \lambda) f(x) \log(1 + \lambda) f(x) \, dx$$

$$- \int_{-\infty}^{\infty} 2\lambda f(x) \log(2\lambda f(x)) \, dx$$

$$= (1 + \lambda)H(X) - \lambda H(\max(X_1, X_2)) - \lambda \int_0^1 2v \log(2v) \, dv$$

$$- \int_0^1 \{1 + \lambda\} - 2\lambda u \log(1 + \lambda) - 2\lambda u \, du$$

$$= (1 + \lambda)H(X) - \lambda H(\max(X_1, X_2)) + \lambda H(V) + H(U_T),$$

as required.

If we now define $\phi(\lambda) = \lambda H(V) + H(U_T)$, then by performing some algebraic manipulations, we can obtain

$$\phi(\lambda) = -\lambda \left( \log 2 - \frac{1}{2} \right) + \frac{2\lambda + (\lambda - 1)^2 \log(1 - \lambda) - (\lambda + 1)^2 \log(1 + \lambda)}{4\lambda}.$$  

(6)

In order to examine the shape of the function $\phi(\lambda)$, we get its first derivative to be

$$\frac{\partial \phi(\lambda)}{\partial \lambda} = \frac{1 - \lambda^2}{4\lambda} \log \left( \frac{1 + \lambda}{1 - \lambda} \right) - \frac{0.5 + 0.1931\lambda}{\lambda} = \frac{1 - \lambda^2}{2\lambda} \tanh^{-1}(\lambda) - \frac{0.5 + 0.1931\lambda}{\lambda}.$$  

(7)

The root of (7) is found to be $\lambda = -0.543$. Further, the second derivative of $\phi(\lambda)$ is also obtained to be

$$\frac{\partial^2 \phi(\lambda)}{\partial \lambda^2} = \frac{1}{\lambda^2} \left( 1 - \frac{\tanh^{-1}(\lambda)}{\lambda} \right).$$

Because $\tanh^{-1}(\lambda) \geq \lambda$ for $\lambda \in (0, 1)$ and $\tanh^{-1}(\lambda) \leq -\lambda$ for $\lambda \in (-1, 0)$, we always have $\frac{\partial^2 \phi(\lambda)}{\partial \lambda^2} \leq 0$. A plot of the function $\phi(\lambda)$, for $\lambda$ in the interval $[-1, 1]$, is presented in Figure 1. From Figure 1, it easy to see that for $\lambda \in [-1, 0]$, the function $\phi(\lambda)$ is positive, while for $\lambda \in (0, 1)$, $\phi(\lambda)$ is negative.
Example 2.2. Suppose $X_1$ and $X_2$ are independent and identically distributed as an exponential variable $X$, with PDF $f(x) = \alpha e^{-\alpha x}, \ x > 0$. Then, the Shannon entropy of the corresponding transmuted random variable $X_T$ is given by
\[
H(X_T) = (1 + \lambda)H(X) - \lambda H(\max(X_1, X_2)) + \lambda H(U_T) = (1 + \lambda)(1 - \log \alpha) + \lambda \left\{ \log(2\alpha) - 2 \right\} + \phi(\lambda),
\]
where $\phi(\lambda)$ is as given in (6).

2.2. Kullback-Leibler divergence

In this subsection, we present some results concerning Kullback-Leibler divergence between the transmuted density $f_{X_T}$ and the density functions of each of its components, $f$ and $f_{\max}$. For this purpose, we first derive in the following theorem the Kullback-Leibler divergence between the transmuted uniform random variable and the uniform random variable in the interval $(0, 1)$.

Theorem 2.3. Let variable $X$ have its PDF as $f(x)$ and the corresponding transmuted random variable $X_T$ have its PDF as in (3). Then, the Kullback-Leibler divergences between $f$ and $f_T$ are given, respectively, by
\[
(i) \ KL(f_{X_T}, f) = KL(f_{U_T}, f_U),
(ii) \ KL(f, f_{X_T}) = KL(f_{U_T}, f_{U_T}),
\]
where the random variables $U$ and $U_T$ are uniform and transmuted uniform random variables on $(0, 1)$ with density functions $f_U$ and $f_{U_T}$, respectively.

Proof: By using the definition of Kullback-Leibler divergence and setting $u = F(x)$, we have
\[
KL(f_{X_T}, f) = \int_{-\infty}^{\infty} f_{X_T}(x) \log \left( \frac{f_{X_T}(x)}{f(x)} \right) dx
= \int_{-\infty}^{\infty} f_{X_T}(x) \log \left( \frac{f(x)(1 + \lambda - 2\lambda F(x))}{f(x)} \right) dx
= \int_{0}^{1} (1 + \lambda - 2\lambda u) \log(1 + \lambda - 2\lambda u) du
= KL(f_{U_T}, f_{U_T}).
\]
as required for Part (i). Part (ii) can be proved similarly.

**Theorem 2.4.** Let $X_1$ and $X_2$ be independent and identically distributed as $X$ with PDF $f$. Then, the Kullback-Leibler divergences between $f_{\text{max}}$ and $f_{X_i}$ are given by

(i) $\text{KL}(f_{X_i}, f_{\text{max}(X_1,X_2)}) = \text{KL}(f_{U_1} , f_{X_i})$,

(ii) $\text{KL}(f_{\text{max}(X_1,X_2)}, f_{X_i}) = \text{KL}(f_{V} , f_{U_1})$,

where the random variables $V$ and $U_1$ are as defined earlier in Theorem 2.1.

**Proof:** By using the definition of Kullback-Leibler divergence and using the transformation $u = F(x)$, we get

$$\text{KL}(f_{X_i}, f_{\text{max}}) = \int_{-\infty}^{\infty} f_{X_i}(x) \log \left( \frac{f_{X_i}(x)}{f_{\text{max}}(x)} \right) dx$$

$$= \int_{-\infty}^{\infty} f_{X_i}(x) \log \left( \frac{f(x)(1 + \lambda - 2\lambda F(x))}{2f(x)F(x)} \right) dx$$

$$= \int_{0}^{1} (1 + \lambda - 2\lambda u) \log \frac{1 + \lambda - 2\lambda u}{2u} du$$

$$= \text{KL}(f_{U_1} , f_{X_i}),$$

as in Part (i). Part (ii) follows similarly.

**Remark 2.5.** From Theorems 2.3 and 2.4, it is clear that the KL distances between $f_{X_i}$ and each of its components $f$ and $f_{\text{max}}$ are free of the underlying distribution $F$.

**Remark 2.6.** From Theorems 2.3 and 2.4, Jeffreys’ divergences $J(f, f_{X_i})$ and $J(f_{\text{max}}, f_{X_i})$ can be readily obtained as

$$J(f, f_{X_i}) = \text{KL}(f_{U_1} , f_{X_i}) + \text{KL}(f_{U_1} , f_{X_i}) = J(f_{U_1} , f_{U_1})$$

and

$$J(f_{\text{max}}, f_{X_i}) = \text{KL}(f_{V} , f_{U_1}) + \text{KL}(f_{U_1} , f_{X_i}) = J(f_{V} , f_{U_1}),$$

respectively. Thus, Jeffreys’ divergences are also free of distribution $F$.

With regard to the results in Theorems 2 and 3, it should be noted that these results can also be presented in terms of the invariant properties of Kullback-Leibler divergence under invertible transformations. For more details, see Qiao and Minematsu (2010). Making use of this property of KL divergence and considering the transformations $X \overset{d}{=} F^{-1}(U_1), X_T \overset{d}{=} F^{-1}(U_1)$ and $\text{max}(X_1, X_2) \overset{d}{=} F^{-1}(V)$, the results of Theorems 2 and 3 can be obtained.

**Remark 2.7.** Let the variables $U, V$ and $U_T$ be distributed as uniform, Beta(2, 1) and transmuted uniform on the interval $(0,1)$, respectively. Then:

(i) $\text{KL}(f_{U_1} , f_{U_1}) = \frac{2\lambda((1-\lambda)\log(1-\lambda)-(\lambda+1)\log(\lambda+1))}{2\lambda}$

(ii) $\text{KL}(f_{U_1} , f_{U_1}) = -\frac{2\lambda((1-\lambda)^2\log(1-\lambda)+\lambda^2\log(\lambda+1))}{4\lambda}$

(iii) $\text{KL}(f_{V} , f_{U_1}) = \log 2 - \frac{1}{2} - \frac{(2\lambda^2-2\lambda-1)\log(1-\lambda)-2\lambda(2\lambda+1)+(\lambda+1)^2\log(\lambda+1)}{4\lambda}$

(iv) $\text{KL}(f_{U_1} , f_{V}) = -\frac{2\lambda((1-\lambda)^2\log(1-\lambda)+\lambda^2\log(\lambda+1))}{4\lambda} + \frac{1}{2} + 1 - \log 2$.

**Proof:** By the definition of Kullback-Leibler divergence, we have

$$\text{KL}(f_{U_1} , f_{U_1}) = -\int_{0}^{1} \log(1 + \lambda - 2\lambda u) du$$

$$= \frac{2\lambda + (1 - \lambda) \log(1 - \lambda) - (\lambda + 1) \log(\lambda + 1)}{2\lambda},$$
which yields Part (i). Parts (ii), (iii) and (iv) can be derived in a similar manner. 

The divergence measures $KL(f, f_{X_T}), KL(f_{X_T}, f)$ and $J(f_{X_T}, f)$ shall demonstrate it just for $(2\lambda)$ for which yields Part (i). Parts (ii), (iii) and (iv) can be derived in a similar manner.

From Figure 2, we observe that, for $\lambda$ it is possible to prove formally the above inequalities. As the arguments are quite similar for all cases, we shall demonstrate it just for $KL(f_{X_T}, f) \leq KL(f, f_{X_T})$ in the case when $\lambda \in [0,1]$. Then, from the corresponding expressions from Lemma 2.7, it is equivalent to showing that 

$$6\lambda + (\lambda^2 - 4\lambda + 3) \log(1 - \lambda) - (\lambda^2 + 4\lambda + 3) \log(1 + \lambda) \geq 0$$

for $\lambda \in [0,1]$. Clearly, the left-hand side is 0 when $\lambda = 0$. Moreover, its derivative with respect to $\lambda$, namely, $(2\lambda - 4) \log(1 - \lambda) - (2\lambda + 4) \log(1 + \lambda)$, is clearly positive, meaning it is an increasing function of $\lambda$, thus proving its non-negativity.

Next, let us consider a general mixture distribution with PDF $f_m(x) = pf(x) + (1-p)f(x)F(x)$, $p \in [0,1]$, and the transmuted density $f_{X_T}$ in (3). We then derive the KL divergence between these two densities, generalizing Theorems 2.3 and 2.4.

**Example 2.8.** The KL divergence between the mixture PDF $f_m$ and the transmuted PDF $f_{X_T}$ in (3) is

$$KL(f_m, f_{X_T}) = \int (p f(x) + (1-p)f(x)F(x)) \log \left( \frac{p f(x) + (1-p)f(x)F(x)}{(1 + \lambda)f(x) - 2\lambda f(x)F(x)} \right) dx$$

$$= \int_0^{\lambda} (p + (1-p)u) \log \left( \frac{p + 2(1-p)u}{(1 + \lambda) - 2\lambda u} \right) du$$

$$= p \left( \frac{2 - 2p + (p - 2) \log(2 - p) + p \log(p)}{p - 1} \right)$$

$$+ 2(1-p) \frac{-4p^2 + p^2 \log p + 6p + (p - 2)(3p - 2) \log(2 - p) - 2}{8(p - 1)^2}$$

$$- p \frac{2\lambda}{2\lambda}$$

$$- 2(1-p) \frac{(3\lambda^2 - 2\lambda - 1) \log(1 - \lambda) - 2\lambda(2\lambda + 1) + (\lambda + 1)^2 \log(\lambda + 1)}{8\lambda^2}.$$
Remark 2.9. The above result shows that the KL distance between \( f_m \) and \( f_{X_T} \) is also free of the underlying distribution \( F \).

A 3D-plot of \( KL(f_m, f_{X_T}) \) is presented in Figure 3, from which we observe that the KL divergence gets minimized when \( p = 1 \) (i.e., for the parent density \( f \)) and gets maximized when \( p = 0 \) (i.e., for the density \( f_{max} \)).

![3D-plot of the KL divergence between the mixture PDF \( f_m \) and the transmuted PDF \( f_{X_T} \).](image)

**Fig 3.** 3D-plot of the KL divergence between the mixture PDF \( f_m \) and the transmuted PDF \( f_{X_T} \).

### 2.3. Transmuted Shannon entropy

We now present an extension of Shannon entropy of transmuted distribution as well as Jensen-Shannon divergence, and we refer to it as transmuted Shannon (TS) entropy.

**Definition 2.10.** Let \( X \) be a continuous random variable with CDF \( F \), and the corresponding transmuted random variable \( X_T \) have its PDF as in (3). Then, the transmuted Shannon entropy between components \( f \) and \( f_{max} \) is defined as

\[
TS(f, f_{max}; \lambda) = H(f_{X_T}) - [(1 + \lambda)H(f) - \lambda H(f_{max})], \quad \lambda \in [-1, 1].
\]

**Remark 2.11.** The TS entropy is given by the formula

\[
TS(f, f_{max}; \lambda) = (1 + \lambda)KL(f, f_{X_T}) - \lambda KL(f_{max}, f_{X_T}).
\]

**Proof:** Upon using Theorem 2.1, it is sufficient to show that

\[
(1 + \lambda)KL(f, f_{X_T}) - \lambda KL(f_{max}, f_{X_T}) = \lambda H(V) + H(U_T).
\]

From Theorems 2.3 and 2.4, we have

\[
KL(f, f_{X_T}) = KL(f_U, f_{U_T}).
\]
\[ KL(f_{\text{max}}, f_{X_t}) = KL(f_V, f_{U_T}), \]

where \( U, V \) and \( U_T \) are as defined in Lemma 2.7. Thus, we have

\[
(1 + \lambda)KL(f, f_{X_t}) - \lambda KL(f_{\text{max}}, f_{X_t}) = (1 + \lambda)KL(f_U, f_{U_T}) - \lambda KL(f_V, f_{U_T}),
\]

from which we obtain

\[
(1 + \lambda)KL(f_U, f_{U_T}) - \lambda KL(f_V, f_{U_T}) = -\lambda H(V) + H(U_T),
\]

as required.

**Remark 2.12.** It is easily seen that Jensen-Shannon entropy between \( f \) and \( f_{\text{max}} \) is a special case of TS entropy when \( \lambda \in [-1, 0] \) and using the re-parametrization \( p = 1 + \lambda \), that is,

\[
TS(f, f_{\text{max}}; \lambda) = JS(f, f_{\text{max}}; p),
\]

where \( JS(f, f_{\text{max}}; p) = H(p f + (1 - p) f_{\text{max}}) - [p H(f) + (1 - p) H(f_{\text{max}})] \).

**Remark 2.13.** From Theorems 2.3 and 2.4 and Lemma 2.7, we see that the TS entropy is free of the underlying distribution \( F \) as well, and the exact value of TS is given by

\[
TS(f, f_{\text{max}}; \lambda) = (1 + \lambda)KL(f_U, f_{U_T}) - \lambda KL(f_V, f_{U_T})\]

\[
= (1 + \lambda) - 2\lambda u \left( \ln(1 + \lambda) - \frac{2u}{1 + 2\lambda u} \right) du
\]

\[
= -\lambda \int_0^1 2v \ln(2v) dv
\]

\[
- \int_0^1 \left( (1 + \lambda) - 2\lambda u \right) \ln \left( (1 + \lambda) - 2\lambda u \right) du
\]

\[
= \lambda H(V) + H(U_T),
\]

as required.

**Remark 2.14.** The TS entropy is concave with respect to parameter \( \lambda \).

**Proof:** We need to show that \( \frac{\partial^2 TS(f, f_{\text{max}}; \lambda)}{\partial \lambda^2} \leq 0 \). The second derivative of \( TS(f, f_{\text{max}}; \lambda) \) is obtained from (8) to be

\[
\frac{\partial^2 TS(f, f_{\text{max}}; \lambda)}{\partial \lambda^2} = \frac{2\lambda + \ln(1 - \lambda) - \ln(1 + \lambda)}{2\lambda^3}.
\]

Now, upon using the fact that \( \tanh^{-1}(\lambda) = \frac{1}{2} \ln \left( \frac{1 + \lambda}{1 - \lambda} \right) \) in the above equation, we find
\[
\frac{\partial^2 TS(f, f_{\text{max}}; \lambda)}{\partial \lambda^2} = \frac{1}{\lambda^2} - \frac{\tanh^{-1}(\lambda)}{\lambda^3} = \frac{1}{\lambda^2} \left( 1 - \frac{\tanh^{-1}(\lambda)}{\lambda} \right).
\]

Because \( \tanh^{-1}(\lambda) \geq \lambda \) for \( \lambda \) in \((0, 1)\) and \( \tanh^{-1}(\lambda) \leq \lambda \) for \( \lambda \) in \((-1, 0)\), we obtain

\[
\frac{\partial^2 TS(f, f_{\text{max}}; \lambda)}{\partial \lambda^2} = \frac{1}{\lambda^2} \left( 1 - \frac{\tanh^{-1}(\lambda)}{\lambda} \right) \leq 0,
\]
as required.

Figure 4 plots the \( TS \) entropy as a function of the mixture parameter \( \lambda \), and from it we observe that the \( TS \) entropy is indeed a concave function, supporting Lemma 2.14.

3. Connection with Gini information

Let \( X \) be a continuous random variable with CDF \( F \). Then, Gini’s mean difference (GMD) associated with \( X \) is defined as

\[
GMD(F) = 2 \int F(x)\overline{F}(x)dx,
\]

as required.

Fig 4. Plot of the \( TS \) entropy as a function of \( \lambda \).
where \( F(x) = 1 - F(x) \) denotes the survival function of \( X \). It has been argued by Yitzhaki (2003) that GMD, as a measure of variability, shares many properties of the variance of \( X \), and is suitable for distributions that are non-normal.

If \( X \) and \( Y \) are independent random variables on \( \mathbb{R} \) with CDFs \( F \) and \( G \), respectively, then the energy distance between them is defined as

\[
D(F, G) = \int (F(x) - G(x))^2 \, dx.
\]

(10)

### 3.1. Gini’s mean difference of transmuted distribution

In this subsection, we derive an expression for GMD of transmuted distribution function \( F_{X_T} \).

**Theorem 3.1.** Suppose \( F_{X_T} \) is the transmuted distribution given in (2) based on the components \( F \) and \( F_{\text{max}} \). Then, the GMD associated with \( F_{X_T} \) is given by

\[
\text{GMD}(F_{X_T}) = (1 + \lambda)^2 \text{GMD}(F) + \lambda^2 \text{GMD}(F_{\text{max}}) - 2\lambda(1 + \lambda) \int_0^1 \frac{u(1-u)(1+2u)}{f(F^{-1}(u))} \, du.
\]

(11)

**Proof:** From the definition of GMD in (9), upon setting \( u = F(x) \), we get

\[
\frac{1}{2} \text{GMD}(F_{X_T}) = \int F_{X_T}(x) F_{X_T}(x) \, dx
\]

\[
= \int \left( 1 + \lambda \right) F(x) - \lambda \left( 1 - F^2(x) \right) \left[ 1 - \left( 1 + \lambda \right) F(x) - \lambda \left( 1 - F^2(x) \right) \right] \, dx
\]

\[
= (1 + \lambda) \int F(x) \, dx - \lambda \int (1 - F^2(x)) \, dx - (1 + \lambda)^2 \int F^2(x) \, dx
\]

\[
- 2\lambda(1 + \lambda) \int F(x)(1 - F^2(x)) \, dx - \lambda^2 \int (1 - F^2(x))^2 \, dx
\]

\[
= (1 + \lambda)^2 \int F(x) \, dx + \lambda^2 \int F^2(x)(1 - F^2(x)) \, dx
\]

\[
- \lambda(1 + \lambda) \int_0^1 \frac{u(1-u)(1+2u)}{f(F^{-1}(u))} \, du.
\]

A rearrangement of the last equation yields the expression in (11).

**Example 3.2.** Let \( X \) have a Beta(\( \alpha \), 1) distribution with CDF \( F(x) = x^\alpha \), \( 0 < x < 1 \), \( \alpha > 0 \). Then, the GMD of the transmuted distribution based on \( X \) is

\[
\text{GMD}(F_{X_T}) = (1 + \lambda)^2 \frac{\alpha}{2\alpha^2 + 3\alpha + 1} + \lambda^2 \frac{2\alpha}{8\alpha^2 + 6\alpha + 1} - 2\lambda(1 + \lambda) \left( \frac{1}{2\alpha + 1} - \frac{2}{3\alpha + 1} + \frac{1}{\alpha + 1} \right).
\]

The following theorem gives lower and upper bounds for \( \text{GMD}(F_{X_T}) \) based on the GMD of the component distributions.

**Theorem 3.3.** Suppose \( F \), \( F_{\text{max}} \) and \( F_{X_T} \) have their GMD measures as \( \text{GMD}(F) \), \( \text{GMD}(F_{\text{max}}) \) and \( \text{GMD}(F_{X_T}) \), respectively. Then:

(i) If \( \lambda \in [-1, 0] \), then

\[
\text{GMD}(F_{X_T}) \geq (1 + \lambda)^2 \text{GMD}(F) + \lambda^2 \text{GMD}(F_{\text{max}});
\]
(i) If $\lambda \in [0, 1]$, then
$$\text{GMD}(F_X) \leq (1 + \lambda)^2 \text{GMD}(F) + \lambda^2 \text{GMD}(F_{\text{max}}).$$

Proof: From Theorem 3.1, the above inequalities are obtained readily.

Example 3.4. For the distribution considered in Example 3.2, we find from Theorem 3.3 the following:

(i) If $\lambda \in [-1, 0]$,
$$\text{GMD}(F_X) \geq (1 + \lambda)^2 \frac{\alpha}{2\alpha^2 + 3\alpha + 1} + \lambda^2 \frac{2\alpha}{8\alpha^2 + 6\alpha + 1};$$

(ii) If $\lambda \in [0, 1]$,
$$\text{GMD}(F_X) \leq (1 + \lambda)^2 \frac{\alpha}{2\alpha^2 + 3\alpha + 1} + \lambda^2 \frac{2\alpha}{8\alpha^2 + 6\alpha + 1}.$$

Figure 5 plots the GMD measure and its corresponding bound based on Examples 3.2 and 3.4, for $\alpha = 2$ and $\lambda$ in $[-1, 1]$.

3.2. Transmuted Gini’s mean difference and its connection with energy distance

In this subsection, we introduce an extension of Gini’s mean difference of transmuted distribution as well as Jensen-Gini (JG) measure of divergence, and we refer to it as transmuted Gini (TG) measure of divergence.

Definition 3.5. Let $X$ be a continuous random variable with CDF $F$, and the corresponding transmuted random variable $X_T$ have its PDF as in (3). Then, the transmuted Gini divergence between components $F$ and $F_{\text{max}}$ is defined as
$$\text{TG}(F, F_{\text{max}}, \lambda) = \text{GMD}(F_X) - [(1 + \lambda)\text{GMD}(F) - \lambda \text{GMD}(F_{\text{max}})], \lambda \in [-1, 1].$$

The following theorem provides a connection between TG divergence and energy distance defined in (10).
Theorem 3.6. The TG divergence can be expressed in terms of energy distance as

$$TG(F, F_{max}; \lambda) = (1 + \lambda)D(F, F_{X_1}) - \lambda D(F_{max}, F_{X_1}),$$

where $D(H, F_{X_1})$ is the energy distance between distribution functions $H$ and $F_{X_1}$, with $H$ being either $F$ or $F_{max}$.

Proof: From the definition of TG divergence in (12), we have

$$TG(F, F_{max}; \lambda) = GMD(F_{X_1}) - [(1 + \lambda)GMD(F) - \lambda GMD(F_{max})]$$

$$= \int \bar{F}_{X_1}(x)F_{X_1}(x)dx - \left[(1 + \lambda) \int \bar{F}(x)F(x)dx - \lambda \int \bar{F}_{max}(x)F_{max}(x)dx \right]$$

$$= (1 + \lambda) \int \bar{F}^2(x)dx - \lambda \int \left[1 - F^2(x) \right]^2 dx$$

By denoting $K = (1 + \lambda)D(F, F_{X_1}) - \lambda D(F_{max}, F_{X_1})$, we have

$$K = (1 + \lambda) \int \left[\bar{F}(x) - \bar{F}_{X_1}(x)\right]^2 dx - \lambda \int \left[\bar{F}_{max}(x) - \bar{F}_{X_1}(x)\right]^2 dx$$

which is exactly the same expression derived above for $TG(F, F_{max}; \lambda)$. Thus,

$$TG(F, F_{max}; \lambda) = GMD(F_{X_1}) - [(1 + \lambda)GMD(F) - \lambda GMD(F_{max})]$$

$$= (1 + \lambda)D(F, F_{X_1}) - \lambda D(F_{max}, F_{X_1})$$

$$= (1 + \lambda) \int \bar{F}^2(x)dx - \lambda \int \left[1 - F^2(x) \right]^2 dx$$

proving the theorem.

4. Chi-square divergence between PDF $f_{X_r}$ and its components $f$ and $f_{max}$

Let $f_1$ and $f_2$ be PDFs of random variables $X_1$ and $X_2$, respectively. Then, the chi-square divergence between $f_1$ and $f_2$ is defined as

$$\chi^2(f_1, f_2) = \int \frac{[f_1(x) - f_2(x)]^2}{f_2(x)}dx.$$ (13)
**Theorem 4.1.** Let \(X_1\) and \(X_2\) be two independent random variables identically distributed as \(X\) with PDF \(f(x)\). Then, the chi-square divergence between \(f_{\max}\) and \(f_{X_1}\) (and \(f\) and \(f_{X_1}\)) are given by

(i) \(\chi^2(f_{\max}, f_{X_1}) = \chi^2(f_{V}, f_{U_1}) = (1 + \lambda)^2 \frac{\tanh^{-1}(\lambda) - \lambda}{\lambda^3}\),

(ii) \(\chi^2(f_{X_1}, f) = \chi^2(f_{U_1}, f_U) = \frac{\lambda^2}{3}\),

(iii) \(\chi^2(f, f_{X_1}) = \chi^2(f_{U_1}, f_U) = \frac{\tanh^{-1}(\lambda)}{\lambda} - 1\),

where the random variables \(U, V\) and \(U_1\) are as defined in Lemma 2.7.

**Proof:** From the definition of \(\chi^2\) divergence in (13), upon setting \(u = F(x)\), we get

\[
\chi^2(f_{\max}, f_{X_1}) = \int_{-\infty}^{\infty} \frac{(f_{\max}(x) - f_{X_1}(x))^2}{f_{X_1}(x)} dx
\]

\[
= \int_{-\infty}^{\infty} \frac{2f(x)F(x) - (1 + \lambda)f(x) + 2\lambda f(x)F(x)}{(1 + \lambda)f(x) - 2\lambda f(x)F(x)} dx
\]

\[
= \chi^2(f_{V}, f_{U_1})
\]

\[
= (1 + \lambda)^2 \frac{\tanh^{-1}(\lambda) - \lambda}{\lambda^3},
\]

which proves Part (i). Parts (ii) and (iii) can be proved in a similar manner.

**Remark 4.2.** From Theorem 4.1, we see easily that the chi-square divergence between transmuted PDF \(f_{X_1}\) and its component \(f_{\max}\) is free of the underlying distribution \(F\).

5. Fisher information and Bayes Fisher information for \(\lambda\)

In this section, we derive some results about Fisher information and Bayes Fisher information for the parameter \(\lambda\) of the transmuted distribution.

5.1. Fisher information for parameter \(\lambda\) of transmuted distribution

**Theorem 5.1.** The Fisher information for \(\lambda\) of the transmuted PDF in (3) is given by

\[
I(\lambda) = \int \frac{(f(x) - f_{\max}(x))^2}{f_{X_1}(x)} dx = \frac{1}{(1 + \lambda)^2} \chi^2(f_{\max}, f_{X_1}).
\]

**Proof:** From the definition of Fisher information, we have

\[
I(\lambda) = E \left[ \frac{\partial}{\partial \lambda} \log f_{X_1}(X) \right]^2 = E \left[ \frac{(f(X) - 2f(X)F(X))}{f_{X_1}(X)} \right]^2
\]

\[
= \int \left( \frac{(f(x) - 2f(x)F(x))}{f_{X_1}(x)} \right)^2 f_{X_1}(x) dx
\]

\[
= \frac{1}{(1 + \lambda)^2} \chi^2(f_{\max}, f_{X_1}),
\]

as required.

**Remark 5.2.** From Theorem 5.1, we readily obtain

\[
I(\lambda) = \frac{1}{(1 + \lambda)^2} \chi^2(f_{V}, f_{U_1}) = \frac{\tanh^{-1}(\lambda) - \lambda}{\lambda^3},
\]
From this expression, it is clear that $I(\lambda)$ is symmetric and is free of the underlying distribution $F$.

**Remark 5.3.** The Fisher Information $I(\lambda)$ in (14) can be expressed based on the TS entropy as follows:

$$I(\lambda) = -\frac{\partial^2 TS(f, f_{\text{max}}; \lambda)}{\partial \lambda^2}.$$ \hspace{1cm} (15)

**Proof:** From Lemma 2.14, we have

$$I(\lambda) = -\frac{\partial^2 TS(f, f_{\text{max}}; \lambda)}{\partial \lambda^2} = -\frac{1}{\lambda^2} \left(1 - \tanh^{-1}(\lambda)\right),$$

as required.

Figure 6 plots the Fisher information for parameter $\lambda$, from which we observe that $I(\lambda)$ is minimized when $\lambda = 0$ with $I(0) = 0.5$ and gets maximized when $\lambda$ tends to $\pm 1$.

![Fig 6. Fisher information for $\lambda$ of the transmuted $f_{X_t}$ with underlying density $f$.](image-url)

### 5.2. Bayes Fisher information for $\lambda$ of transmuted distribution

**Theorem 5.4.** The Bayes Fisher information for parameter $\lambda$ of the transmuted PDF in (3), under the uniform prior on $[-1, 1]$, is given by

$$\tilde{I}(\lambda) = \frac{1}{2} KL(f_{\text{max}}, f_{\text{min}}) = \frac{1}{2}.$$ \hspace{1cm} (15)

**Proof:** Upon setting $u = F(x)$, we get

$$\tilde{I}(\lambda) = E[I(\lambda)] = \frac{1}{2} \int_{-1}^{1} I(\lambda)d\lambda = \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{[f(x) - f_{\text{max}}(x)]^2}{f_{X_t}(x)} dx d\lambda$$

$$= \frac{1}{2} \int_{-1}^{1} \log\left(1 - \frac{u}{u}\right)du + \int_{-\infty}^{\infty} f(x)F(x) \log\left(\frac{2 f(x)F(x)}{2 f(x)F(x)}\right)dx$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} f_{\text{max}}(x) \log\left(\frac{f_{\text{max}}(x)}{f_{\text{min}}(x)}\right)dx$$

$$= \frac{1}{2} KL(f_{\text{max}}, f_{\text{min}})$$

$$= \frac{1}{2}.$$
where the fourth equality holds due to the fact that \( \int_0^1 \log \left( \frac{1-u}{u} \right) du = 0 \).

Suppose we consider the following general triangular prior for the parameter \( \lambda \):

\[
\pi_\alpha(\lambda) = \begin{cases} 
\frac{2\lambda}{(1-\alpha)\alpha}, & 0 < \lambda \leq \alpha, \\
\frac{2(1-\lambda)}{(1-\alpha)\alpha}, & \alpha \leq \lambda < 1.
\end{cases}
\]  

\[\text{(16)}\]

**Theorem 5.5.** The Bayes Fisher information for parameter \( \lambda \) in the transmuted distribution with PDF in (3), under the general triangular prior with density \( \pi_\alpha(\lambda) \) as in (16), we have

\[
\tilde{I}(\lambda) = \frac{2}{\alpha(1-\alpha)} \left[ \alpha KL(f_{\min}, f_\alpha) + (1-\alpha)KL(f, f_\alpha) \right],
\]

where \( f_\alpha \) is a transmuted density with PDF

\[
f_\alpha(x) = (1+\alpha)f(x) - \alpha f_{\max}(x).
\]

**Proof:** From (14) and (16), we have

\[
\tilde{I}(\lambda) = E[I(\Lambda)] = \int_0^\alpha I(\lambda)\pi_\alpha d\lambda + \int_\alpha^1 I(\lambda)\pi_\alpha d\lambda
\]

\[
\begin{align*}
&= \frac{2}{\alpha} \int_{-\infty}^{\infty} (f(x) - f_{\max}(x)) \left[ \int_0^\alpha \frac{\lambda(f(x) - f_{\max}(x))}{f_{\chi_1}(x)} d\lambda \right] dx \\
&\quad + \frac{2}{1-\alpha} \int_{-\infty}^{\infty} (f(x) - f_{\max}(x)) \left[ \int_\alpha^1 \frac{(1-\lambda)(f(x) - f_{\max}(x))}{f_{\chi_1}(x)} d\lambda \right] dx \\
&= \frac{2}{\alpha} \int_{-\infty}^{\infty} (f(x) - f_{\max}(x)) \left[ \alpha - \frac{f(x)}{f(x) - f_{\max}(x)} \log \left( \frac{f_\alpha(x)}{f(x)} \right) \right] dx \\
&\quad + \frac{2}{1-\alpha} \int_{-\infty}^{\infty} (f(x) - f_{\max}(x)) \left[ \alpha - 1 - \frac{2f(x)\bar{f}(x)}{f(x) - f_{\max}(x)} \log \left( \frac{f_\alpha(x)}{2f(x)\bar{f}(x)} \right) \right] dx \\
&= \frac{2}{\alpha} \int_{-\infty}^{\infty} f(x) \log \left( \frac{f(x)}{f_\alpha(x)} \right) dx + \frac{2}{1-\alpha} \int_{-\infty}^{\infty} f_{\min}(x) \log \left( \frac{f_{\min}(x)}{f_\alpha(x)} \right) dx \\
&= \frac{2}{\alpha(1-\alpha)} \left[ \alpha KL(f_{\min}, f_\alpha) + (1-\alpha)KL(f, f_\alpha) \right],
\end{align*}
\]

as required.

**Corollary 5.6.** The Bayes Fisher information for \( \lambda \) under the triangular prior distribution with PDF \( \pi(\lambda) = 1 - |\lambda|, |\lambda| \in [0,1] \), is

\[
\tilde{I}(\lambda) = KL(f_{\max}, f_{\min}) - KL(f, f_{\max}) - KL(f, f_{\min}).
\]

**Proof:** From Theorem 5.5, this is readily obtained.

6. Conclusions

In this paper, some informational properties of the transmuted distributions, such as Shannon entropy, Fisher information, Bayes Fisher information and Gini’s mean difference, have been derived. Two new information measures: transmuted Shannon entropy and transmuted Gini’s mean difference, have also been proposed. It has been shown that the transmuted Shannon entropy is based on the transmuted structure of two KL divergences between the general model and its components. A similar result has been provided for transmuted Gini’s mean difference and it has been shown that it can be given in terms of a quadratic rank structure of two energy distances between transmuted distribution and its components. Some interesting results about Fisher information and Bayes Fisher information for parameter \( \lambda \) have been presented under different prior distributions including uniform, beta and triangular. Finally, the divergence between the
general transmuted distribution and the densities of its components have been studied with the use of Kullback-Leibler, chi-square and energy distances. It is of interest to mention that Granzotto et al. (2017) recently introduced a more general family of transmuted distributions, called cubic rank transmuted distributions, and we plan to extend the results here to this family. We hope to report these findings in a future paper.

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