POSITIVE LYAPUNOV EXPONENTS FOR SYMPLECTIC COCYCLES

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Abstract. In the present paper we give a positive answer to a question posed by Viana in [22] on the existence of positive Lyapunov exponents for symplectic cocycles. Actually, we prove that for an open and dense set of Hölder symplectic cocycles over a non-uniformly hyperbolic diffeomorphism there are non-zero Lyapunov exponents with respect to any invariant ergodic measure with the local product structure.

1. Introduction

1.1. Symplectic cocycles and Hamiltonian linear differential systems. Let $M$ denote a $d$-dimensional compact Hausdorff space $M$, $f: M \to M$ a diffeomorphism and $A: M \to sp(2\ell, \mathbb{R})$ a continuous, sometimes smooth, map, where $sp(2\ell, \mathbb{R})$, $\ell \geq 1$, stands for the symplectic Lie group of $2\ell \times 2\ell$ matrices $A$ and with entries over the reals such that $A^TJA = J$, where $J$ is the skew-symmetric matrix defined on (3.2) and $A^T$ is the transpose of $A$. In this paper we will try to understand the dynamics defined by

$$A^n(x) = A(f^{n-1}(x)) \circ \cdots \circ A(f(x)) \circ A(x),$$

for most systems $A$ when $n \to \infty$, namely the grow of its norm. This setup of linear symplectic cocycles can be seen as a toy model aiming to understand the dynamics of symplectomorphisms, where $A$ is defined to be the tangent map to a symplectomorphism $f$ defined in a symplectic $d$-manifold with $d = 2\ell$.

The framework of symplectic cocycles is a good start if one aims to understand the behavior of the dynamical cocycle associated to a given symplectomorphism. However, the cocycle Achilles heel tends to be the independent relation between the base and fiber dynamics which is a natural counterweight to its great generality. In other words we are able to perturb the fiber keeping unchanged the base dynamical system (or vice-versa) but on the other hand we allow a vast number of symplectic actions in the fiber. We should keep in mind that any perturbation in the action of the fiber of a dynamical cocycle should begin with a perturbation in the dynamical system itself which, in general, cause extra difficulties.

1.2. Lyapunov exponents. Given a cocycle $A$ over a map $f$ the Lyapunov exponents detect if there are any exponential asymptotic behavior on the evolution of the norm of (1.1) along orbits (see [4]). Under certain measure preserving assumptions on $f$ and integrability of $A$ the existence of Lyapunov exponents for almost every point is guaranteed by Oseledets’ theorem ([20]). Non-zero Lyapunov exponents assure, in average, exponential rate of divergence or convergence of two neighboring trajectories, whereas zero exponents give us the lack of any kind of average exponential behavior. A dynamical system is said to be non-uniformly hyperbolic if its Lyapunov exponents are all different from zero. The correspondent definitions for the continuous-time case are completely analogous.

A central question in dynamical systems is to determine whether we have non-uniform hyperbolicity for the original dynamics and some or the majority of nearby systems. Such an answer usually depends on the smoothness and richness of the dynamical system, among other aspects.

1.3. Overview. Concerning with discrete-time cocycles over compact Hausdorff spaces, in the works by Bochi and Viana [7, 9], yields the dichotomy: either the Oseledets decomposition along the orbit of almost every point has a weak form of hyperbolicity called dominated splitting or else the spectrum is trivial mean that all the Lyapunov exponents vanish. The main idea behind the proof of these results,
also used by Novikov [19] and by Mañe [18], is to use the absence of dominated splitting to cause a decay of the Lyapunov exponents by perturbing the system rotating Oseledets’ subspaces thus mixing different expansion rates. Let us mention that, in [14], Cong proved that a discrete generic bounded cocycle has simple spectrum and that the Oseledets splitting is dominated. In consequence, uniform hyperbolicity is generic among discrete area-preserving bounded cocycles.

Other approaches were given in [1, 2, 6] where it was proved abundance of trivial spectrum but with respect to $L^p$ topologies. Let us also mention [8] where the authors, instead of perturbing the fiber, perturb the base dynamics and recover again the dichotomy: hyperbolicity versus zero Lyapunov exponents.

In this work we are interested in proving abundance of non-zero Lyapunov exponents. In fact, a major breakthrough in the analysis of the Lyapunov exponents of H"older continuous cocycles over non-uniformly hyperbolic base map was obtained recently by an outstanding paper by Viana [22] and our purpose here is to contribute to the better understanding of the ergodic theory of symplectic cocycles and to answer some of the questions raised in that article, namely Problem 4 in [22, pp. 678]. More precisely, we generalize to the setting of symplectic cocycles the results obtained by Viana [22] for conservative cocycles.

First, it is proven that fiber-bunched cocycles admit center dynamics called holonomies. Then, using a generalization of Ledrappier’s criterium ([17]), Viana proved that zero Lyapunov exponents correspond to a highly non-generic condition on the system $A$; conditional measures associated to invariant measures for the cocycle are preserved under holonomies. Finally, for sl$(d, \mathbb{R})$-cocycles the map $A \mapsto H_{A,x,y}$ is a submersion and this leads to show that the set of cocycles $A \in C^{\nu}(M, sl(d, \mathbb{R}))$ satisfying the later is a closed subset of empty interior. The case of sp$(2\ell, \mathbb{R})$-symplectic cocycles have more subtleties as pointed out by Viana [22 page 678], since the fundamental and elegant lemma asserting that the holonomy maps are submersions as function of $A$ in $C^{\nu}(M, sp(2\ell, \mathbb{K}))$ fails to be true because the symplectic group sp$(2\ell, \mathbb{R})$ has dimension $2\ell(2\ell + 1)$ which is smaller than the necessary dimension $2\ell(2\ell - 1)$, ($\ell \geq 2$). This lead to the question of understanding which groups can be taken to obtain non-trivial spectrum.

In this article we use a symplectic perturbative approach (see Section 4.4) in small neighborhoods of heteroclinic points to show that every cocycle $A$ is $C^{r,\nu}$-approximated by open sets of cocycles so that unstable holonomies remain unchanged while stable holonomies are modified in order not to satisfy a rigid condition.

Our paper is organized as follows. In Section 2 we state our main results. We collect some preliminary results on groups of symplectic matrices, symplectic geometry of Oseledets spaces, in Section 3 while the proofs of the main results are given in Section 4.

2. Statement of the results

2.1. Some definitions. This section is devoted to recall some necessary notions and to state our main results. Our main result answers in an affirmative way to Problem 4 in [22]. We recall some definitions. Given $A \in C^{r,\nu}(M, sp(2\ell, \mathbb{K}))$ we will denote, by a slight abuse of notation, by $cocycle$ the skew-product

$$F_A : M \times \mathbb{K}^{2\ell} \rightarrow M \times \mathbb{K}^{2\ell}$$

$$(x,v) \rightarrow (f(x), A(x)v)$$

where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We denote by $f_A$ the natural cocycle induced by $F_A$ in the projective spaces $PK^{2\ell}$. If $\mu$ is an $f$-invariant probability measure such that $\|A^{\pm 1}\| \in L^1(\mu)$ then it follows from Oseledets theorem ([20]) that for $\mu$-almost every $x$ there exists a decomposition $E_x^d = E_x^{1} \oplus E_x^{2} \oplus \cdots \oplus E_x^{k(x)}$, called, the Oseledets splitting, and for $1 \leq i \leq k(x)$ there are well defined real numbers

$$\lambda_i(A, f, x) = \lim_{n \to \pm \infty} \frac{1}{n} \log \|A^n(x)v_i\|, \quad \forall v_i \in E_x^{i}(\{0\})$$

called the Lyapunov exponents associated to $A$, $f$ and $x$. It is well known that, if $\mu$ is ergodic, then the Lyapunov exponents are almost everywhere constant. Since we are dealing with symplectic cocycles and $sp(2\ell, \mathbb{R}) \subset sl(2\ell, \mathbb{R})$, this implies that $\sum_{i=1}^{k(x)} \lambda_i(A, f, x) = 0$. Notice that the spectrum of a symplectic linear transformation is symmetric with respect to the $x$-axis and to $S^1$. In fact, if $\sigma \in \mathbb{C}$ is an eigenvalue with multiplicity $m$ so is $\sigma^{-1}$, $\sigma$ and $\sigma^{-1}$ keeping the same multiplicity (see e.g. [21].
Proposition 1.5). Therefore, since Lyapunov exponents come in pairs in the symplectic setting, then 
\[ \lambda_i(A, f, x) = -\lambda_{2-i}(A, f, x) \] 
for all \( i \in \{1, \ldots, t\} \). So, not counting the multiplicity and abbreviating \( \Lambda(A, f, x) = \lambda(x) \), we have the increasing set of real numbers,
\[ \lambda_1(x) \geq \lambda_2(x) \geq \ldots \geq \lambda_t(x) \geq 0 \geq -\lambda_t(x) \geq \ldots \geq -\lambda_2(x) \geq -\lambda_1(x), \]
or, equivalently,
\[ \lambda_1(x) \geq \lambda_2(x) \geq \ldots \geq \lambda_t(x) \geq 0 \geq -\lambda_1(x) \geq \ldots \geq -\lambda_2(x) \geq -\lambda_t(x). \]

Associated to the Lyapunov exponents we have the Oseledets splitting
\[ \mathbb{K}^{2\ell} = E^{1}_x \oplus E^{2}_x \oplus \ldots \oplus E^{\ell}_x \oplus E^{\ell+1}_x \oplus \ldots \oplus E^{2\ell-\ell}_x \oplus E^{2\ell}_x. \tag{2.1} \]

We will see in [3.1] that the vector space \( \mathbb{K}^{2\ell} \) can be decomposed into \( \ell \) symplectic 2-dimensional subspaces.

Recall that an \( f \)-invariant probability measure \( \mu \) is hyperbolic if it has only non-zero Lyapunov exponents. In that case, for every regular point \( x \) let \( E^{\pm}_x \) (respectively \( E^0_x \)) denote the sums of all Lyapunov subspaces corresponding to all negative (respectively positive) Lyapunov exponents. It follows from Pesin’s stable manifold theorem (see e.g. [1]) that for \( \mu \)-almost every \( x \) there exists a \( C^1 \)-embedded disk \( W^s_{\text{loc}}(x) \) (local stable manifold at \( x \)) such that \( T_x W^s_{\text{loc}}(x) = E^+_x \), and it is Lipschitz with respect to the metric \( d \) on \( W^s_{\text{loc}}(x) \) given by
\[ d(f^n(y), f^n(z)) \leq K_e e^{-n\tau} d(y, z) \]
for every \( y, z \in W^s_{\text{loc}}(x) \). Local unstable manifolds \( W^u_{\text{loc}}(x) \) are defined analogously using \( E_u^0 \) and \( f^{-1} \).

Moreover, since local invariant manifolds and the constants above vary measurably with the point \( x \) one can consider hyperbolic blocks \( H(K, \tau) \) (sometimes with the notation \( \Lambda_{x,K} \) used also in [16, 22]) with measure arbitrary close to 1 by taking larger \( K \) and smaller \( \tau \) such that \( K_e \leq K, \tau_e \geq \tau \) and both the local invariant manifolds \( W^s_{\text{loc}}(x) \) and \( W^u_{\text{loc}}(x) \) vary continuously with \( x \) in \( H(K, \tau) \). In consequence, if \( x \in H(K, \tau) \) and \( \delta > 0 \) is small enough, then for every \( y, z \in B(x, \delta) \) the intersection \( [y, z] := W^u_{\text{loc}}(y) \cap W^s_{\text{loc}}(z) \neq \emptyset \) consists of a unique point. Set \( N^u_y(\delta) = \{ [y, x] \in W^u_{\text{loc}}(x) : y \in H(K, \tau) \cap B(x, \delta) \} \) to be a \( \nu \)-neighborhood of \( x \) and \( N^s_y(\delta) = \{ [x, y] \in W^s_{\text{loc}}(x) : y \in H(K, \tau) \cap B(x, \delta) \} \) an \( \nu \)-neighborhood of \( x \).

It is not hard to check that the map
\[ \Upsilon_x : N^s_y(\delta) \rightarrow N^s_{\tau^s}(\delta) \times N^s_{\tau^s}(\delta) \]
\[ y \mapsto ([x, y], [y, x]) \]
is a homeomorphism, where \( N^s_y(\delta) := H(K, \tau) \cap B(x, \delta) \) is a neighborhood of \( x \) in \( H(K, \tau) \). Now, we recall the notion of local product structure (cf. [22, Page 646]).

**Definition 2.1.** A hyperbolic measure \( \mu \) has local product structure if for every \( x \in \text{supp}(\mu) \) (supp(\( \mu \)) stands for the support of the measure \( \mu \)) and a small \( \delta > 0 \) the measure \( \mu_{|N^s_x(\delta)} \) is equivalent to the product measure \( \mu^s_x \times \mu^u_x \), where \( \mu^s_x \) denotes the conditional measure of \( \Upsilon_x \circ (\mu |_{N^s_x(\delta)}) \) on \( N^s_y(\delta) \), for \( i \in \{u, s\} \).

At this point we describe the space of cocycles that we shall consider. We say that \( A : M \rightarrow sp(2\ell, \mathbb{K}) \) is a \( C^{r,\nu} \)-cocycle if the map \( A \) is \( r \)-times differentiable and \( D^r A \) is \( \nu \)-Hölder continuous and denote by \( C^{r,\nu}(M, sp(2\ell, \mathbb{K})) \) the vector space of \( C^{r,\nu} \)-cocycles. It is not difficult to prove that the vector space \( C^{r,\nu}(M, sp(2\ell, \mathbb{K})) \) of \( C^{r,\nu} \)-cocycles endowed with the norm \( \| \cdot \|_{r,\nu} \) defined as
\[ \| A \|_{r,\nu} = \sup_{0 < j < r} \| D^j A(x) \| + \sup_{x \neq y} \frac{\| D^r A(x) - D^r A(y) \|}{d(x, y)\nu}, \]
is a Banach space. Let us also mention that for the proofs it is enough to consider the case when \( \nu = 1 \), that is, of Lipschitz matrices. In fact, if \( A \) is \( \nu \)-Hölder continuous with respect to the metric \( d(\cdot, \cdot) \) then it is Lipschitz with respect to the metric \( d(\cdot, \cdot)^\nu \). Hence, up to a change of metric we may assume that \( A \) is Lipschitz and we will do so throughout the paper.
2.2. Statement of Theorem A

We are now in a position to state our main result.

**Theorem A.** Let $M$ be a compact Riemannian manifold. Take $f \in \text{Diff}^{1+\alpha}(M)$ ($\alpha > 0$) and an $f$-invariant, ergodic and hyperbolic probability measure $\mu$ with local product structure. Then, there exists an open and dense set of maps $\mathcal{O}$ in $C^{r}(M, \text{sp}(2\ell, K))$ such that for any $A \in \mathcal{O}$ the cocycle $F_A$ has at least one positive Lyapunov exponent at $\mu$-almost every point. Moreover, the complement is a set with infinite codimension.

It follows from the symplectic geometry of the Oseledets subspace (see [3.2]) that previous result also assures that the cocycle $F_A$ has at least one negative Lyapunov exponent at $\mu$-almost every point.

Let us recall the notion of fiber-bunched (or dominated) cocycles over some uniformly hyperbolic homeomorphisms. Let $X$ be a compact metric space, let $f : X \to X$ be an homeomorphism and $\Lambda \subset X$ be a compact $f$-invariant set. We say that $f|_\Lambda$ is uniformly hyperbolic if there exists a distance $d$ and constants $0 < \lambda < 1$ and $\varepsilon, \delta > 0$ such that:

(i) $d(f^n(y), f^n(z)) \leq \lambda^nd(y, z)$ for all $y, z \in W^s(x)$ and $n \geq 0$;

(ii) $d(f^{-n}(y), f^{-n}(z)) \leq \lambda^nd(y, z)$ for all $y, z \in W^u(x)$ and $n \geq 0$;

(iii) if $x, y \in \Lambda$ and $d(x, y) < \delta$ then the intersection $[x, y] := W^s(x) \cap W^u(y)$ consists of exactly one point and depends continuously on $x$ and $y$.

For simplicity we say that $\Lambda$ is a uniformly hyperbolic set.

**Definition 2.2.** Let $X$ be a compact metric space, $f : X \to X$ be a homeomorphism and $\Lambda \subset X$ is a uniformly hyperbolic set. We say that an $\alpha$-Hölder continuous cocycle $A : \Lambda \to \text{sp}(2\ell, K)$ is fiber-bunched if $|A(x)| A(x)^{-1}| \lambda^n < 1$ for all $x \in \Lambda$.

In view of the recent developments by Avila and Viana [3] we expect that most symplectic cocycles do have simple Lyapunov spectrum. We make these assertions more precisely:

**Conjecture 1:** If $(f, \mu)$ is as above then there exists a residual subset of fiber-bunched maps $\mathcal{R} \subset C^{r}(M, \text{sp}(2\ell, K))$ such that each cocycle $F_A$ has simple spectrum for every $A \in \mathcal{R}$.

We note that this conjecture holds true in the class of fiber-bunched cocycles and the base transformation has an at most countable Markov partition. This is a consequence of the work of [3], using that the twisting and pinching conditions extend to the symplectic setting. Moreover, the following conjecture, which is the symplectic version of Bonatti, Viana [12] remains open:

**Conjecture 2:** Let $\Lambda$ be a hyperbolic set and $\mu$ be an $f$-invariant probability measure with local product structure. Then there exists a residual subset $\mathcal{R} \subset C^{r}(M, \text{sp}(2\ell, K))$ of fiber-bunched maps such that each cocycle $F_A$ has simple spectrum for every $A \in \mathcal{R}$.

More recently, Cambraia [13] has announced an affirmative solution to the previous conjectures.

We end this section by recalling that a time-continuous version of our results for Hamiltonian linear differential systems is given in [5] while the general statement on the typical simplicity of the Lyapunov spectrum still remains an open question.

3. Preliminaries

3.1. The symplectic group of matrices. We collect some necessary preliminary results on symplectic structures. Let $\omega$ be a symplectic form, i.e., a closed and nondegenerate 2-form. A linear automorphism $A : (V, \omega) \to (V, \omega)$ in a symplectic vector space $V$ is called symplectic if $A^{\ast}\omega = \omega$, that is

$$\omega(u, v) = \omega(A(u), A(v))$$

for all $u, v \in V$. Clearly $\dim(V) = 2\ell$ for some $\ell \geq 1$ and the $\ell$-times wedging $\omega \wedge \omega \wedge \cdots \wedge \omega$ is a volume-form (see e.g. [24, Lemma 1.3]). We identify the symplectic linear automorphisms with the set of matrices and denote by $\text{sp}(2\ell, R)$ ($\ell \geq 1$), the non-compact $\ell(2\ell + 1)$-dimensional Lie group of $2\ell \times 2\ell$ matrices $A$ and with real entries satisfying $A^T J A = J$, where

$$J = \begin{pmatrix} 0 & -I_{\ell} \\ I_{\ell} & 0 \end{pmatrix}$$
denotes the skew-symmetric matrix, $1_\ell$ is the $\ell$-dimensional identity matrix and $A^T$ stands for the transpose matrix of $A$.

Given a subspace $S \subset V$, where $\dim(V) = 2\ell$, we denote its $\omega$-orthogonal complement by $S^\perp$ which is defined by those vectors $u \in V$ such that $\omega(u, v) = 0$, for all $v \in S$. Clearly $\dim(S^\perp) = 2\ell - \dim(S)$. When, for a given subspace $S \subset V$, we have that $\omega|_{S \times S}$ is non-degenerate (say $S^\perp \cap S = \{0\}$), then $S$ is said to be a symplectic subspace. On the other hand, when $\omega|_{S \times S} = 0$ (or $S \subset S^\perp$) we say that the subspace $S$ is isotropic. Finally, Lagrangian subspaces $S$ are isotropic subspaces such that $\dim(S) = \ell$ or, in other words, $S^\perp = S$, i.e. these subspaces are maximal subspaces such that the form $\omega$ degenerates when restricted to them. We say that the basis $\{e_1, ..., e_\ell, e_{\ell+1}, ..., e_{2\ell}\}$ is a symplectic base of $\mathbb{R}^{2\ell}$ if $\omega(e_i, e_j) = 0$, for all $j \neq i$ and $\omega(e_i, e_i) = 1$.

3.2. The symplectic geometry of Oseledets’ spaces. We now present the main geometric properties of the subspaces given by the Oseledets theorem. Through this section consider $f \in \text{Diff}^r(M)$ and let $\mu$ be an $f$-invariant probability measure. Moreover, let $F_A$ be the cocycle over $f$ induced by $A \in \mathcal{C}^r(M, sp(2\ell, \mathbb{R}))$.

**Lemma 3.1.** Assume that $x$ is an Oseledets $\mu$-regular point with $2\ell$ distinct Lyapunov exponents and with Oseledets decomposition in 1-dimensional subspaces

$$\mathbb{R}^{2\ell} = E^1_x \oplus E^2_x \oplus ... \oplus E^\ell_x \oplus E^\ell_{x} \oplus ... \oplus E^\ell_x \oplus E^\ell_x.$$ (3.3)

Then, there exists a symplectic basis $\{e_1, ..., e_\ell, e_{\ell+1}, ..., e_{2\ell}\}$ in the fiber over $x$ formed by the invariant directions given by (3.3). Furthermore, each 2-dimensional subspace $E^i_x \oplus E^j_x$ is symplectic.

**Proof.** Let $\{\lambda_i : i = 1, ..., 2\ell\}$ be the set of Lyapunov exponents for the cocycle $F_A$ at the point $x$ and, for each $i$, let $E^i_x$ denote the corresponding 1-dimensional Oseledets invariant direction. We proceed to prove that there exists a symplectic basis formed by vectors in the $2\ell$ invariant 1-dimensional subspaces in (3.3). Fix $\lambda_1$ and $\lambda_2$ and the correspondent invariant directions $E^1$ and $E^2$, which we divide in cases. Firstly, if $\lambda_1 + \lambda_2 < 0$ then for every $\epsilon \in (0, |\lambda_1 + \lambda_2|/2)$, $u_{i} \in E^1$ and $u_{j} \in E^2$, it follows from the theory of non-uniform hyperbolicity that there exists $K_\epsilon > 0$ (depending only on $x$) such that for all $n \in \mathbb{N}$

$$K_\epsilon^{-1}e^{(\lambda_1 - \epsilon)n}\|u_{i}\| \leq \|A^n(x) \cdot u_{i}\| \leq K_\epsilon e^{(\lambda_2 + \epsilon)n}\|u_{j}\|,$$

where $\sigma = i, j$. Using Cauchy-Schwartz inequality and that $A$ is symplectic we obtain that

$$|\omega(u_{i}, u_{j})| = |\omega(A^n(x) \cdot u_{i}, A^n(x) \cdot u_{j})| \leq \|A^n(x) \cdot u_{i}\| \|A^n(x) \cdot u_{j}\| \leq K_\epsilon^2 e^{(\lambda_1 + \lambda_2 + 2\epsilon)n}\|u_{i}\|\|u_{j}\|,$$

which converges to zero as $n \rightarrow +\infty$, proving that $\omega(u_{i}, u_{j}) = 0$. A completely analogous reasoning for the symplectic action $A^{-1}$ shows that if $\lambda_1 + \lambda_2 > 0$, $u_{i} \in E^1$ and $u_{j} \in E^2$ and $\epsilon > 0$ is small there exists $K_\epsilon > 0$ so that

$$|\omega(u_{i}, u_{j})| \leq K_\epsilon^2 e^{(-\lambda_1 - \lambda_2 + 2\epsilon)n}\|u_{i}\|\|u_{j}\|,$$

which also tends to zero as $n \rightarrow +\infty$. This also proves that $\omega(u_{i}, u_{j}) = 0$.

Finally, it remains the case that $\lambda_1 = -\lambda_2$. If the 2-dimensional space $E^1 \oplus E^2$ is isotropic then we change $\lambda_1$ by any of the $2\ell - 1$ remaining Lyapunov exponents and proceed as above. Since $\omega$ is nondegenerated there exists some $j'$ such that $E^1 \oplus E^{j'}$ is a symplectic 2-dimensional subspace. Hence $\omega(u_{i}, u_{j'}) = 0$ for all $u_{i} \in E^1$ and $u_{j'} \in E^{j'}$. Hence, up to normalization and denoting $i = j'$, we get $\omega(e_i, e_i) = 1$. If one proceeds analogously and reorganize the Lyapunov exponents we obtain $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_\ell \geq 0 \geq \lambda_{i} \geq ... \geq \lambda_{2} \geq \lambda_{1}$ and $\ell$ symplectic subspaces $E^i \oplus E^j$. This completes the proof of the lemma. \[\square\]

**Lemma 3.2.** Assume that $x$ is an Oseledets $\mu$-regular point with some zero Lyapunov exponent. Then, the associated invariant Oseledets subspace corresponding to the zero Lyapunov exponent has even dimension and it is symplectic.

**Proof.** Let $E^0$ be the subspace associated to $\lambda = 0$. By symmetry of the Lyapunov spectrum of $A$ we get that $\dim(E^0)$ is even, say equal to $2k$. To obtain that $E^0$ is symplectic we prove that $(E^0)^\perp \cap E^0 = \{0\}$. Given any non-zero vector $e_1 \in E^0$, then there exists $e_2 \in \mathbb{R}^{2\ell}$ such that $\omega(e_1, e_2) = 0$, because otherwise
the form would be degenerate. Clearly, $e_{i} \in E^{0}$. We complete (cf. [21, Lemma 1.2]) the symplectic base of $E^{0}$ obtaining $\{e_{1}, e_{2}, e_{3}, \ldots, e_{k}\}$. Let $u \in (E^{0})^{\perp} \cap E^{0}$, then since $u \in E^{0}$, $u = \sum_{i=1}^{k} \alpha_{i} e_{i} + \beta_{i} e_{i}$ for some $\alpha_{i}, \beta_{i} \in \mathbb{R}$. In one hand we have $\omega(u, e_{i}) = \omega(\sum_{i=1}^{k} \alpha_{i} e_{i} + \beta_{i} e_{i}) = -\beta_{i}$ and one the other hand we also have $\omega(u, e_{i}) = \omega(\sum_{i=1}^{k} \alpha_{i} e_{i} + \beta_{i} e_{i}) = \alpha_{i}$. Moreover, since $u \in (E^{0})^{\perp}$ we must have $\alpha_{i} = \beta_{i} = 0$ and so $u = 0$.

In overall, given $A \in C^{r,v}(M, sp(2\ell, \mathbb{R}))$ over a $\mu$-invariant diffeomorphism $f : M \to M$, $x$ an Oseledets regular point displaying $2k \leq 2\ell$ distinct Lyapunov exponents $\lambda_{1}, \ldots, \lambda_{k}, \lambda^{\perp}_{1}, \ldots, \lambda_{1}$ and with associated invariant 1-dimensional subspaces $E^{1}, \ldots, E^{k}, E^{k}_{\perp}, \ldots, E^{1}_{\perp}$, then

(i) the subspaces $E^{1} \oplus \ldots \oplus E^{k}$ and $E^{k}_{\perp} \oplus \ldots \oplus E^{1}_{\perp}$ are isotropic;
(ii) the dimension of $E^{0}$ is equal to $2(\ell - k)$, and the invariant subspace can be decomposed into $\ell - k$ 2-dimensional symplectic subspaces;
(iii) the space $E^{0}$ is symplectic;
(iv) each 2-dimensional subspaces $E^{i} \oplus E^{i}_{\perp}$ for $i = 1, \ldots, 2k$ are symplectic and
(v) each subspace $E^{+} := E^{1} \oplus \ldots \oplus E^{k}$ and $E^{-} := E^{k}_{\perp} \oplus \ldots \oplus E^{1}_{\perp}$ is Lagrangian if $k = \ell$ (see [9] Lemma 2.4 (1)).

Let us mention that if $A \in sp(2\ell, \mathbb{R})$ is an automorphism, then analogous conclusions of both Lemma 3.1 and Lemma 3.2 can be deduced.

4. Linear symplectic cocycles

4.1. A quick tour on the proof of Theorem A

In this section we prove Theorem A for symplectic cocycles over non-uniformly hyperbolic maps. Let $\mu$ be an $f$-invariant, ergodic and hyperbolic measure. We proceed as described at the end of Section 2.

First we describe the strategy, that goes along some ideas developed in [12, 22] for discrete time cocycles over non-uniformly hyperbolic maps in the more subtle symplectic perturbation theory. Consider a $C^{1+\alpha}$-diffeomorphism $f$ preserving an $f$-invariant, ergodic and hyperbolic measure with local product structure.

By non-uniform hyperbolicity of $(f, \mu)$ there are, not necessarily invariant, compact Pesin sets $H(K, \tau)$ of points with uniform hyperbolicity constants along the stable and unstable Pesin submanifolds. Then, if the cocycle $F_{A}$ associated to $A \in C^{r,v}(M, sp(2\ell, \mathbb{R}))$ has only zero Lyapunov exponents then there exist compact sets (i.e. domination blocks cf. [4, 12]) where the cocycle behaves like a partially hyperbolic map in a sense that the fiber behaves like a central manifold which is dominated by the base dynamics. In consequence, there are well defined stable holonomy transformations

$$H^{s}_{x,y} := H^{s}_{A,x,y} : \{x\} \times \mathbb{R}^{2\ell} \to \{y\} \times \mathbb{R}^{2\ell}$$

$$v \to \lim_{n \to +\infty} A^{-n}(f^{n}(y)) A^{n}(x) \cdot v,$$

for points $y \in W^{s}(x)$, which correspond to central manifolds to the partially hyperbolic dynamics in $F_{A}$ and are linear symplectic transformations. Unstable holonomies $H^{u}_{x,y}$ are defined analogously using unstable manifolds and $f^{-1}$. Let $h^{s}_{x,y}$ and $h^{u}_{x,y}$ denote the projectivization on the fibers of the stable and unstable holonomies respectively (see [4, 12] for full details). Furthermore, given any $f_{A}$-invariant probability measure $m$ such that $\Omega_{A} m = \mu$ (recall $\mu$ has local product structure) there exists a continuous disintegration $(m_{y})_{y \in Y}$ of $m$ such that $(h^{s}_{x,y})_{y} m_{x} = m_{y}$ for all $y \in W^{s}(x)$ and $(h^{u}_{x,y})_{y} m_{x} = m_{y}$ for all $y \in W^{u}(x)$ with $x$ belonging to the holonomy block. Moreover, such property holds for all periodic points $p_{1}, p_{2}, \ldots, p_{k}$ in the domination block that are homoclinically related, meaning that there exists

$$z_{i} \in W^{u}_{loc}(p_{i}) \cap W^{s}_{loc}(p_{i+1}) \text{ for } i = 1 \ldots k - 1.$$
The existence of such periodic points in domination blocks was guaranteed in [22]. Thus, if $A$ has only zero Lyapunov exponents and $(p_i)_{i=1}^k$ are periodic points whose Lyapunov exponents are all distinct then
\[ (h_i^A p_i, z_i) = m_i = (h_i^A p_{i+1}, z_i) = m_{i+1} \quad \text{for all } i = 1 \ldots k-1, \]
and is a finite convex combination of Dirac measures. Hence, we are reduced to show that the condition described in (1.1) is a highly non-generic condition on $A$. This was done for $\text{sl}(d, \mathbb{K})$-cocycles by Viana using an elegant argument to prove that the map $A \mapsto H_{A,x,y}$ is a submersion and, consequently, the set of cocycles $A \in C^{r,\nu}(M, \text{sl}(d, \mathbb{K}))$ satisfying (1.1) is contained in a closed subset of empty interior and has infinite codimension. As pointed out by Viana [22] page 678 this argument fails to extend to groups of matrices with dimension smaller than $2((2\ell-1)$, as the symplectic group $\text{sp}(2\ell, \mathbb{K})$. To overcome these difficulties, in [4.3] we use a symplectic perturbative approach in very small neighborhoods of heteroclinic points to show that every cocycle $A$ is $C^{r,\nu}$-approximated by open sets of cocycles so that unstable holonomies remain unchanged while stable holonomies are modified in order not to satisfy the rigid condition in (1.1). This finishes the sketch of the proof.

### 4.2 Zero Lyapunov exponents lead to rigidity

In this subsection we shall collect some ingredients from [22] and show that cocycles $A$ whose Lyapunov exponents are all zero exhibit a rigid condition for all $f_A$-invariant measures on the fibered projective space. Recall the following definition.

**Definition 4.1.** Let $A \in C^0(M, \text{sp}(2\ell, \mathbb{K}))$ be a continuous cocycle. Given $N \geq 1$ and $\theta > 0$, consider the set $\mathcal{D}_A(N, \theta)$ of points $x \in M$ satisfying
\[
\sum_{j=0}^{k-1} \left| A^N(f^j(x)) \right| \left\| A^N(f^jN(x))^{-1} \right\| \leq e^{kN\theta} \quad \text{for all } k \in \mathbb{N}.
\]

For simplicity we say that $\mathcal{O}$ is a *holonomy block* for $A$ if it is a compact subset of $\mathcal{H}(K, \tau) \cap \mathcal{D}_A(N, \theta)$ for some constants $K, \tau, N, \theta$ satisfying $30 < \tau$. This property means that the cocycle $f_A$ behaves like a partially hyperbolic dynamics with the central direction corresponding to the fibers, leading to strong-stable and strong-unstable foliations. Moreover, since domination is an open condition for the cocycle this enables to obtain strong-stable and strong-unstable foliations for all nearby cocycles. More precisely, we have the following result.

**Proposition 4.2.** For every $x \in \mathcal{O}$ and $y \in W^u_{\text{loc}}(x)$, there exists $C_1 > 0$ and a symplectic linear transformation $H^u_{A,x,y} : \{x\} \times \mathbb{P}K^{2\ell} \to \{y\} \times \mathbb{P}K^{2\ell}$ such that:

1. $H^u_{x,z} = \text{id}$ and $H^u_{y,z} = H^u_{y,z} \circ H^u_{x,y}$;
2. $A(f^{-1}(y)) \circ H^u_{y,z} \circ f^{-1}(y) \circ A(x)^{-1} = H^u_{x,y}$;
3. $\left\| H^u_{x,y} - \text{id} \right\| \leq C_1 \|x,y\|$ and
4. $H^u_{f^j(y), f^j(z)} = A^j(z) \circ H^u_{y,z} \circ A^j(y)^{-1}$ for all $j \in \mathbb{Z}$, for every $x$, $y$, $z$ in the same local unstable manifold.

**Proof.** This is a consequence of [12] Proposition 1.2. In fact, since the cocycle $F_A$ varies Lipschitz continuously on the fibers and satisfies the fiber-bunched property in the domination blocks, then it follows by [11] [12] [22] that the limit
\[
H^u_{x,y} = \lim_{n \to \infty} [A^n(y)]^{-1} A^n(x) = \lim_{n \to \infty} A^n(f^{-n}(y)) A^{-n}(x)
\]
does exist for every $y \in W^u(x)$. Since $\{A^n(f^{-n}(y)) A^{-n}(x)\}_{n \in \mathbb{N}}$ is a Cauchy sequence in the complete metric space $\text{sp}(2\ell, \mathbb{K})$ then it follows that $H^u_{x,y}$ defines a symplectic linear map. Properties (1) and (2) are immediate from the definition while property (3) is a consequence of Proposition 1.2 in [12]. Finally, property (4) follows easily from (4.2). This finishes the proof of the proposition. 

Notice that each cocycle $A$ induces a projectivized cocycle $f_A : M \times \mathbb{P}K^{2\ell} \to M \times \mathbb{P}K^{2\ell}$ with compact fiber $\mathbb{P}K^{2\ell}$. In particular, the set of $f_A$-invariant probability measures is non-empty by the Krylov-Bogoliubov theorem. Let $h^u_{A,x,y}$ (respectively $h^s_{A,x,y}$) be the transformations obtained from $H^u_{A,x,y}$ (respectively $H^s_{A,x,y}$) by projectivization on the fibers and let us refer as *unstable (respectively stable)*
holonomies for the projectivized cocycle \( f_A \). We omit its dependence on \( A \) for notational simplicity when no confusion is possible. Given a holonomy block \( \mathcal{O} \) and \( \delta > 0 \) small enough (depending only on \( K, \tau \)), for all \( x \in \text{supp}(\mu \mid \mathcal{O}) \) let \( \mathcal{N}_x^\mathcal{O}(\delta) \), \( \mathcal{N}_x^{\mathcal{O}}(\mathcal{O}, \delta) \) and \( \mathcal{N}_x^{\mathcal{O}}(\mathcal{O}, \delta) \) be the induced neighborhoods of \( x \) in \( \mathcal{O} \) with local product structure defined similarly as in [22].

We say that an \( f_A \)-invariant probability measure \( m \) admits a continuous disintegration on \( \tilde{M} \subset M \) if \( m(A) = \int m_x(A) \, d\mu(x) \) for all measurable subset \( A \subset M \times P\mathbb{R}^2 \) and \( \tilde{M} \ni x \mapsto m_x \) is continuous in the weak* topology. The following result assert roughly that any \( f_A \)-invariant probability measure \( m \) such that \( \Pi_x m = \mu \) admits such disintegration for all points in the support of holonomy blocks.

**Proposition 4.3.** [22 Proposition 3.5] Let \( \mathcal{O} \) be a positive \( \mu \)-measure holonomy block, consider \( x \in \text{supp}(\mu \mid \mathcal{O}) \) and set the neighborhoods \( \mathcal{N}_x^\mathcal{O}(\delta) \) of \( x \) as above. Then, every \( f_A \)-invariant probability measure \( m \) with \( \Pi_x m = \mu \) admits a continuous disintegration on \( \text{supp}(\mu \mid \mathcal{N}_x^\mathcal{O}(\mathcal{O}, \delta)) \). Moreover, \( m_z = (h_{y,z}^x)_* m_y \) and \( m_z = (h_{w,z}^x)_* m_w \) for all \( y, z, w \in \text{supp}(\mu \mid \mathcal{N}_x^\mathcal{O}(\mathcal{O}, \delta)) \) such that \( y, z \) belong to the same strong-stable local manifold and \( z, w \) belong to the same strong-unstable local manifold.

Moreover, an analogous result yields invariance of the disintegrated measures by unstable holonomies. Therefore, the purpose is to consider periodic dominated points that are homoclinically related, where the characterization of the disintegrated measures \( m_z \) is simple. This is done in the following subsection.

4.3. Obstructions using periodic points. In this section we collect some results on the obstruction to zero Lyapunov exponents obtained using heteroclinic orbits associated to periodic points. Throughout this section let \( \mathcal{O} \) be a positive \( \mu \)-measure holonomy block for a cocycle \( A \) such that \( \lambda^+(A, \mu) = 0 \) and let \( x \in \text{supp}(\mu \mid \mathcal{O}) \) as above. We begin with the following result.

**Lemma 4.4.** If \( m \) is an \( f_A \)-invariant probability measure such that \( \Pi_x m = \mu \) and \( p \in \text{supp}(\mu \mid \mathcal{N}_x^\mathcal{O}(\mathcal{O}, \delta)) \) is a periodic point of period \( \pi \) for \( f \), then \( A^\pi_x m_p = m_p \). Moreover, if \( A^\pi_x \) has all real and distinct eigenvalues, then there exist elements \( \{v_i\}_{i=1}^{\ldots 2\pi} \) in \( P\mathbb{R}^2 \) and a probability vector \( \{\alpha_i\}_{i=1}^{\ldots 2\pi} \) such that \( m_p = \sum_{i=1}^{\ldots 2\pi} \alpha_i \delta_{v_i} \).

**Proof.** Let \( m \) be an arbitrary \( f_A \)-invariant probability measure and \( (m_x)_z \) be a continuous disintegration of \( m \) on \( \text{supp}(\mu \mid \mathcal{N}_x^\mathcal{O}(\mathcal{O}, \delta)) \). Since \( m = \int m_x \, d\mu(x) \) and \( (f_A)_x m = m \) then, for all \( k \), we have \( A^k(x)_* m_x = m_{f^k(x)} \) for \( \mu \)-almost every \( x \). Hence, by continuity in the weak* topology at \( p \in \text{supp}(\mu \mid \mathcal{N}_x^\mathcal{O}(\mathcal{O}, \delta)) \) it follows that \( A^{\pi \mathcal{O}} m_p = m_p \), which proves the first claim. Now, if \( \{v_i\}_{i=1}^{\ldots 2\pi} \) are linearly independent unitary eigenvectors of \( A^\pi_x \) in \( \mathbb{R}^2 \) then every such vectors are the unique fixed points for the Morse-Smale action of \( A^\pi_x \) on the projective space. Thus, every \( A^\pi_x \)-invariant probability is a convex combination of Dirac measures at the points \( \{v_i\}_{i=1}^{\ldots 2\pi} \). This finishes the proof of the lemma. \( \square \)

Following [22], given a periodic point with hyperbolicity constants \( K, \tau \) we say that \( p \) is dominated if there exists \( P \geq 1 \) such that \( p \in \mathcal{P}(P\pi, \theta) \), where \( \pi \) is the period of \( p \) and \( 3\theta < \tau \). In particular it follows that \( H^A_{s,p,z} = \lim_{n \to \infty} A^{\pi n}(z)^{-1} A^{\pi n}(p) \) defines the stable holonomies for all points in the local stable manifold of \( p \), and similarly using \( f^{-1} \) for unstable holonomies. We say that a periodic point is simple if it has all eigenvalues of different norm. The following is a direct consequence of Proposition 4.3.

**Corollary 4.5.** Let \( m \) be an \( f_A \)-invariant probability measure such that \( \Pi_x m = \mu \) and assume that \( p, q \in \text{supp}(\mu \mid \mathcal{N}_x^\mathcal{O}(\mathcal{O}, \delta)) \) are dominated periodic points for \( f \) and \( z \) is the unique point in the heteroclinic intersection \( W^u_{\text{loc}}(q) \cap W^s_{\text{loc}}(p) \). Then \( m_z = (h_{y,z}^x)_* m_q = (h_{w,z}^x)_* m_p \).

The former corollary will be of particular interest in the case that \( p \) and \( q \) are simple periodic points. In fact, if this is the case, while on the one hand \( m_z \) is an atomic measure whose atoms are Dirac measures at the vectors \( h_{y,z}^x(v_i) \) where \( v_i \) are the eigenvectors for \( A^{\pi_1}(p) \) (being \( \pi_1 \) the period of \( p \)), on the other hand it also belongs to the convex hull of the Dirac measures at points \( h_{w,z}^x(v_i) \) where \( v_i \) are the eigenvectors for \( A^{\pi}(q) \) (being \( \pi_2 \) the period of \( q \)). In the next subsection we prove that the set of cocycles satisfying these invariance-like properties is small from the topological viewpoint.
4.4. Perturbation results. In this section we develop perturbative arguments for symplectic cocycles to prove that the holonomy invariance is a rigid condition. As we already said the technique of [22] does not hold due to lack of dimension for \( sp(2\ell, \mathbb{R}) \subset sl(2\ell, \mathbb{R}) \). First, we consider a basic perturbative result (Lemma 4.6) which allows us to obtain simple and real spectrum by making small perturbations once we assume that we have a periodic point with large period. This type of perturbations, to the best of our knowledge, goes back to Moser’s work and were honed in [12, Proposition 9.1] in our context of cocycles. Then, we show that generally simple periodic points on holonomy blocks do exist (Proposition 4.7).

It follows from [21, Proposition 1.5] that the eigenvalues of an element in \( sp(2\ell, \mathbb{R}) \) come in quadruples, say if \( \sigma \) is an eigenvalue, then \( \sigma^{-1}, \bar{\sigma} \) and \( \bar{\sigma}^{-1} \) are also eigenvalues. Moreover, from a simple transversality argument we can and will assume that some generic perturbation is made \textit{a priori} in order to put a certain matrix with all its eigenvalues distinct (although some of them may be complex). In conclusion, we reduce our analysis to invariant subspaces corresponding to eigendirections with dimension at most equal to four which, in the case of the symplectic group, encloses all difficulties. We say that \( A \in sp(4, \mathbb{R}) \) is (see Figure 1):

- (A) a complex saddle if it has a non real eigenvalue with norm greater than 1;
- (B) a saddle center if it has a non real eigenvalue with norm 1 and a real eigenvalue with norm greater than 1;
- (C) a generic center if it has four different non real eigenvalues with norm 1;
- (D) a degenerated center if it has two non real eigenvalues with norm 1 and with multiplicity two.

![Figure 1](image)

Notice that (A) is \textit{symplectically maximal} meaning that there are no 2-dimensional symplectic subspaces. The degenerated center (D) is excluded by our generic assumption. Finally, once we obtain the simplicity of the spectrum it is easy to show that we can always provide a basis formed by eigendirections and the space spanned by the eigendirections associated to a quadruple of eigenvalues is a symplectic subspace (see e.g. [15, Lemma 4.1] where the authors treat with a similar type of symplectic perturbations). In the next table we see the \( sp(4, \mathbb{R}) \) matrices representing all possible cases displaying complex eigenvalues and with respect to the canonical symplectic basis (cf. Figure 1).
The following result is the symplectic version of [12, Proposition 9.1].

**Lemma 4.6.** Given \( p \) a hyperbolic periodic point for \( f \) with an homoclinic point \( z \neq p \), \( \epsilon > 0 \) and \( A \in C^{r,\nu}(M, \text{sp}(2\ell, \mathbb{R})) \), there exist \( B \in C^{r,\nu}(M, \text{sp}(2\ell, \mathbb{R})) \) such that \( \|A - B\|_{r,\nu} < \epsilon \) and a periodic orbit \( q \in M \) arbitrarily close to \( p \) such that all the eigenvalues of \( B^{\pi}(q) \) are real and have different norm.

**Proof.** Since the proof follows closely the one in [12, §9] we present the highlights emphasizing the main differences. We assume without loss of generality that \( p \) is a fixed point for the base dynamics \( f \). Moreover, since it is a generic condition as described before the lemma, up to a small \( C^{r,\nu} \)-perturbation we can also assume that \( A \) has simple spectrum. Thus, we get pairs of complex conjugates of norm one associated to a symplectic 2-dimensional eigenspace (a center as in (B) and (C) above) or else we get quadruples of complex eigenvalues of norm different from one associated to a symplectic 4-dimensional eigenspace (a complex saddle as in (A) above). Let \( n \geq 0 \) be the number of those complex pairs or quadruples and let \( z \) denote an homoclinic point related to \( p \). Consider the horseshoe \( \Lambda \) generated by local stable and unstable manifolds of \( p \) crossing through \( z \), that is, \( \Lambda = \bigcup_{n \in \mathbb{Z}} f^n(U_0 \cup U_1) \) with \( U_0, U_1 \) disjoint neighborhoods of \( p \) and \( z \) respectively. Hence, for each \( n \) there exists a periodic point \( x_n \), of increasing period equal to \( l + n \), such that the first \( n \) iterates of \( x_n \) belong to \( U_0 \) and the following \( l \) iterates belong to \( U_1 \) (see Figure 2). Those \( l \) iterates are precisely the ones equal to the orbit of \( z \) different from \( p \). Defined in this way \( x_n \),

![Figure 2. Periodic points \( x_n \).](image-url)
as \( n \) increases, the point \( x_n \) is as close as desired to \( p \) and the matrix \( A^{1+n}(x_n) \) inherits the dynamical behavior of \( A(p) \) as \( n \) increases. Let us define

\[
K_n := \bigcup_{n \geq m} f^i(x_m) \quad \text{and} \quad K_\infty := \bigcup_{i \in \mathbb{Z}} f^i(z).
\]

By [12] Lemma 9.2, which is an abstract result also holds for our symplectic context, for every large enough \( n \), the cocycle \( A \) admits a dominated decomposition \( E^n_A \cap \cdots \cap E^0_A \) over the invariant set \( K_n \) coinciding with the decomposition into eigenspaces at the point \( p \). Furthermore, due to [10] Theorem 11 we have, in fact, a partial hyperbolic splitting \( E^n_A \cap \cdots \cap E^0_A \). The persistence of dominated splitting under \( C^0 \) (thus \( C^{r,v} \)) perturbations allows us to conclude that any \( B \in C^{r,v}(M, sp(2\ell, \mathbb{K})) \) sufficiently \( C^{r,v} \)-close to \( A \) also has a dominated splitting \( E^1_B \cap \cdots \cap E^k_B \) over the invariant set \( K_n \). Moreover, this dependence is continuous and keeps the same dimension of each fiber. From now on we consider a symplectic basis of the fiber \( K^*_p \) adapted to the splitting \( E^{1}_A \cap \cdots \cap E^{k}_A \) at \( p \), i.e., by Lemmas 3.1 and 3.2 the splitting is decomposed into

\[
\begin{aligned}
E^{u_1}_A & \cap E^{u_2}_A \cap \cdots \cap E^{u_i}_A \cap E^{c_1}_A \cap E^{c_2}_A \cap \cdots \cap E^{c_i}_A \cap E^{c_{i-1}}_A \cap \cdots \cap E^{c_1}_A \cap E^{s_1}_A \cap E^{s_2}_A \cap \cdots \cap E^{s_i}_A, \\
\text{expansive} & \quad \text{central} & \quad \text{contractive}
\end{aligned}
\]

such that:

(I) every vector in the basis is in some of the fibers listed in (4.3);

(II) for any \( k \in \{1, \ldots, i\} \) the fiber \( E^{u_k}_A \) is 1 or 2-dimensional;

(III) if the fiber \( E^{u}_A \) is 1-dimensional, then \( E^{s}_A \) is also 1-dimensional, \( E^{u}_A \cap E^{s}_A \) is symplectic and \( A(p)|_{E^{u}_A \cap E^{s}_A} \) is a rotation relative to this basis;

(IV) if the fiber \( E^{u}_A \) is 2-dimensional, then \( E^{s}_A \) is also 2-dimensional, \( E^{u}_A \cap E^{s}_A \) is symplectic and \( A(p)|_{E^{u}_A \cap E^{s}_A} \) is a complex saddle relative to this basis;

(V) for any \( k \in \{1, \ldots, j\} \) the fiber \( E^{s}_A \) is 1-dimensional and \( E^{c}_A \) is also 1-dimensional, \( E^{s}_A \cap E^{c}_A \) is symplectic and \( A(p)|_{E^{s}_A \cap E^{c}_A} \) is a rotation relative to this basis.

Now, we proceed by induction considering the lowest index such that \( E^{u}_A \) is associated to a complex case as (A), (B) or (C). The main step is to use [12] Lemma 9.3 which allows us to obtain a one-parameter path \( A_t \in C^{r,v}(M, sp(2\ell, \mathbb{K})) \) (\( t \in [0, 1] \)) sufficiently \( C^{r,v} \)-close to \( A \) and such that \( A_0 = A \) and for any \( t \in [0, 1] \) there exists \( n(t) \in \mathbb{N} \) such that \( A^{1+n}(x_n) \) has some real eigenvector along the subspace \( E^{u}_A \).

Using this result we have to consider four cases:

(i) If \( E^{u}_A \) is a 2-dimensional subspace associated to a 4-dimensional symplectic eigenspace \( E^{u}_A \cap E^{u}_A \) (i.e. a complex saddle) and the achieved real eigenvalue is equal to \( \sigma > 1 \), then the other eigenvalue, associated to the symplectic eigenspace \( E^{u}_A \), is \( \sigma^{-1} \) and both have multiplicity two. We perturb once more to obtain simple spectrum in this 4-dimensional symplectic eigenspace \( E^{u}_A \cap E^{u}_A \). The perturbation is simply to consider a saddle center perturbation \( S \in sl(4, \mathbb{R}) \) defined, for \( \eta \approx 0 \), by:

\[
S := \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 + \eta & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & (1 + \eta)^{-1}
\end{pmatrix},
\]

in \( E^{u}_A \cap E^{u}_A \) and the identity on all the remaining fibers in (4.3). Finally, we take \( A' := S \circ A_t \).

(ii) If \( E^{u}_A \) is a 2-dimensional subspace associated to a 4-dimensional symplectic eigenspace \( E^{u}_A \cap E^{u}_A \) (i.e. a complex saddle) and the achieved real eigenvalue is equal to 1. We perturb once more to obtain simple spectrum in this 4-dimensional symplectic eigenspace \( E^{u}_A \cap E^{u}_A \). The perturbation
is simply to consider the hyperbolic perturbation $S \in sl(4, \mathbb{R})$ defined, for $\eta \approx 0$, by:

$$S := \begin{pmatrix} 1 - \eta & 0 & 0 & 0 \\ 0 & 1 + \eta & 0 & 0 \\ 0 & 0 & (1 - \eta)^{-1} & 0 \\ 0 & 0 & 0 & (1 + \eta)^{-1} \end{pmatrix},$$

in $E^m_A \oplus E^n_A$ and the identity on all the remaining fibers in (4.3). Finally, we take $A' := S \circ A_t$.

(iii) If $E^m_A$ is a direction associated to a 2-dimensional symplectic eigenspace $E^m_A \oplus E^n_A$ (i.e. a center) and the achieved real eigenvalue is equal to $\sigma > 1$, then the other eigenvalue is $\sigma^{-1}$ and we are done.

(iv) If $E^m_A$ is a direction associated to a 2-dimensional symplectic eigenspace $E^m_A \oplus E^n_A$ (i.e. a center) and the achieved real eigenvalue is equal to 1. We perturb once more to obtain simple spectrum in this 2-dimensional symplectic eigenspace $E^m_A \oplus E^n_A$. The perturbation is simply to consider the hyperbolic perturbation $S \in sl(2, \mathbb{R})$ defined, for $\eta \approx 0$, by:

$$S := \begin{pmatrix} 1 - \eta & 0 \\ 0 & (1 - \eta)^{-1} \end{pmatrix},$$

in $E^m_A \oplus E^n_A$ and the identity on all the remaining fibers in (4.3). Finally, we take $A' := S \circ A_t$.

In any of these previous situations we obtain that $(A')^{l+n}(x_n)$ has at most $c - 1$ pair or quadruples complex eigenvalues. Now, we let $p_2 = x_n$ and $A'$ instead of $p$ and $A$ and repeat the same argument at most $c - 1$ times until obtain the statement of Lemma 4.6.

In the proof of Lemma 4.6 we used a version of [12] Lemma 9.3 for symplectic cocycles. For the sake of completeness we will give the underlying idea of this simple but beautiful result. The main idea is to define $A_t := R_{\tau \varepsilon} \circ A$, where $t \in [0, 1]$, $R_{\tau \varepsilon}$ is the rotation of angle $\tau \varepsilon$ in its associated 2 or 4-dimensional symplectic eigenspace over $p$ and $\varepsilon > 0$ is very small to assure that $A_t$ is close to $A$. The only novelty is when $E^m_A$ is 2-dimensional. In this case our map $R_{\tau \varepsilon}$ is defined by the complex saddle rotation

$$
\begin{pmatrix}
\cos(\tau \varepsilon) & -\sin(\tau \varepsilon) & 0 & 0 \\
\sin(\tau \varepsilon) & \cos(\tau \varepsilon) & 0 & 0 \\
0 & 0 & \cos(\tau \varepsilon) & -\sin(\tau \varepsilon) \\
0 & 0 & \sin(\tau \varepsilon) & \cos(\tau \varepsilon)
\end{pmatrix},
$$

in $E^m_A \oplus E^n_A$ and the identity all the remaining fibers in (4.3). Now, since for some $n(t)$ sufficiently large most of the iterated of $x_n$ remain very close to $p$, as we said before, the dynamics of $x_n$ (for any $n \geq n(t)$) is mostly given by $A(p)$. Since $A(p)$ has a complex behavior in $E^m_A$ if we input more rotations, then for a certain parameter $t \in [0, 1]$, we reach the real axis and $\mu(A_t)^{l+n}(x_n) \oplus E^n_A$ must have a real eigenvalue $\sigma$ with multiplicity equal to two. Since the initial matrix was a complex saddle and the rotation was also a complex saddle the (non-symplectic) eigendirection $E^n_A$ has a real eigenvalue equal to $\sigma^{-1}$.

Lemma 4.6 will now be used to obtain the following useful result.

**Proposition 4.7.** Given $\varepsilon > 0$ and $k \geq 2$ there exists a holonomy block $O$ for $A$ so that $\mu(O) > 1 - \varepsilon$, distinct dominated periodic points $\{p_i\}_{i=1}^k$ in $O$ and a cocycle $B \in C^{\nu,\omega}(M, sp(2\ell, \mathbb{K}))$ such that the following properties hold:

1. $W^u_{loc}(p_i) \cap W^s_{loc}(p_{i+1}) \neq \emptyset$ consists of one point;
2. $p_i \in \text{supp}(\mu | O \cap \pi^{-1}(O))$, where $\pi_i$ is the period of the periodic point $p_i$, for all $1 \leq i \leq k$;
3. $\|A - B\|_{\nu,\omega} < \varepsilon$ and
4. the Lyapunov spectrum of $B^{\pi_i}(p_i)$ is real and simple.

Finally, the set of cocycles $B$ satisfying (1), (2) and (4) is open in the $C^{\nu,\omega}$-topology.
Proof. The arguments of this proof are borrowed from the arguments in [22] §4 together with Lemma 4.6. For that reason we just sketch the main argument for reader’s convenience focusing on the symplectic perturbative argument. Let \( \mathcal{O} \) be a holonomy block for \( A \) with large measure and \( x \in \text{supp}(\mu \mid \mathcal{O}) \). It follows from [22] Corollary 4.8 that there exists \( \rho > 0 \) and there are \( k \geq 2 \) distinct periodic points \( p_1, \ldots, p_k \in B(x, \rho/2) \) of periods \( \pi_1, \ldots, \pi_k \) such that:

(a) \( \text{dist}(f^n(y), f^n(z)) \leq K e^{-\tau n} \text{dist}(y, z) \) for all \( n \geq 0 \) and \( y, z \in W^u_{\text{loc}}(p_i) \).

(b) \( W^u_{\text{loc}}(p_i) \) has size at least \( \rho \) and intersects every \( W^u_{\text{loc}}(p_j) \) for all \( i, j \) (and analogously for local unstable manifolds).

Moreover, following \textit{ipsis litteris} [22] §4.3 there exists a holonomy block \( \mathcal{O} \supset \mathcal{O} \) such that all periodic points belong to \( \mathcal{O} \) and \( p_i \in \text{supp}(\mu \mid \mathcal{O} \cap f^{-n}(\mathcal{O})) \). This yields properties (1) and (2) above. Thus, we are reduced to prove that there exists a symplectic cocycle \( B \in C^{r,\nu}(M, sp(2\ell, \mathbb{R})) \) such that \( \|A - B\|_{r,\nu} < \varepsilon \) and the Lyapunov spectrum of \( B^{n}(p_i) \) is real and simple for all \( i \).

If the later property holds for the cocycle \( A \) with respect to the periodic points \( p_i \) we are done. Otherwise, up to a small perturbation we may assume that there exists a complex eigenvalue \( \sigma \) of largest norm for \( A^{\pi_i}(p_i) \), and that the only complex eigenvalue of equal norm \( |\sigma| = |\pi| \). Recall that it is guaranteed by the symplectic structure that \( \sigma^{-1} \) and \( \pi^{-1} \) are also eigenvalues (see [21] Proposition 1.5). We will assume that they are the only complex eigenvalues, since otherwise recursive perturbations can be made to reduce the number of complex eigenvalues, and follow the strategy of [12]. It follows from the \( \lambda \)-lemma that there exist homoclinic points \( z_i \) for \( p_i \). Consider \( \Lambda_i \) to be the hyperbolic set obtained as the maximal invariant set in a small neighborhood of \( p_i \) and \( z_i \). Thus using Lemma 4.6 one can find a cocycle \( B \) that is \( C^{r,\nu} \)-close to \( A \) and such that \( B^{\pi_i}(x_{n_i}) \) has simple and real Lyapunov spectrum, i.e., \( 2\ell \) real and distinct eigenvalues of different norm. On the other hand, if the neighborhood of \( p_i \) and \( z_i \) is small enough, then the local stable and unstable manifolds of the periodic points \( p_i = x_{n_i} \) obtained recursively as above are uniformly long in such a way that properties (1) and (2) still hold. Since the last assertion in the proposition is immediate this finishes its proof.

The next crucial lemma asserts that one can perturb, in the \( C^{r,\nu} \)-topology, any cocycle with all its Lyapunov exponents equal to zero to show that open and densely there is no holonomy invariance of the disintegrated measures. More precisely,

**Lemma 4.8.** (Breaking lemma) Let \( W = \{w_i : i = 1 \ldots 2\ell\} \) be any linearly independent set of vectors in the fiber \( \mathbb{R}P^{2\ell} \) over \( z \in W^u_{\text{loc}}(p) \cap W^u_{\text{loc}}(q) \). Given \( \varepsilon > 0 \), \( A \in C^{r,\nu}(M, sp(2\ell, \mathbb{R})) \) and a symplectic base \( \{v_i : i = 1 \ldots 2\ell\} \) in the fiber \( \mathbb{R}P^{2\ell} \) over \( p \), there exists a cocycle \( B \in C^{r,\nu}(M, sp(2\ell, \mathbb{R})) \) such that \( \|A - B\|_{r,\nu} < \varepsilon \), the unstable holonomies coincide \( H_{B;\pi z}^{\pi_i} = H_{A;\pi z}^{\pi_i} \) and \( H_{B;v_i}^{\pi_i} \) does not belong to the 1-dimensional subspace generated by \( w_j \) for all \( j \). Moreover, the later property is open in the \( C^{r,\nu} \)-topology.

**Proof.** Let \( \pi_1 = \pi(p) \) and \( \pi_2 = \pi(q) \) be the periods of the periodic point \( p \) and \( q \), respectively. Since the limits do exist recall that \( H_{A;\pi z}^u = \lim_{n \to \infty} [A^{-\pi n}(z)]^{-1} A^{-\pi n}(q) \) and also, for all \( j \geq 0 \),

\[
H_{A;\pi z}^u = \lim_{n \to \infty} A^{\pi j n}(z)^{-1} A^{\pi j n}(p) = [A^{\pi n}(z)]^{-1} H_{A;\pi j n}^{\pi j n}(z). \tag{4.4}
\]

Our strategy is to perform a small symplectic perturbation on a small neighborhood \( V \) around the point \( f^{\pi}(z) \) such that the unstable holonomy \( H_{B;v_i}^{\pi_j}(z) \) for the perturbed cocycle \( B \) remains equal while

\[
H_{B;\pi z}^u(v_j) = [B^{\pi j}(z)]^{-1} H_{B;\pi j n}(v_j).
\]

is linearly independent with each \( w_j \) for all \( j \), as we now detail.

Since the forward orbit of \( z \) is convergent to the one of \( p \) and the backward orbit of \( z \) is convergent to the periodic orbit of \( q \), then there exists a neighborhood \( V \) of \( f^{\pi}(z) \) such that \( V \cap \{ f^i(z) : i \in \mathbb{Z} \} \) consists of \( f^{\pi}(z) \). We are going to construct the desired local perturbation \( B \in C^{r,\nu}(M, sp(2\ell, \mathbb{R})) \) of the cocycle \( A \) in the neighborhood \( V \). Since all constructions are done in local charts using the \( C^\infty \) Riemannian structure of the manifold \( M \) one may assume without loss of generality that \( V \subset \mathbb{R}^d \), where \( d = \dim(M) \). Now, fix a small \( \delta > 0 \) with \( B(f^{\pi j}(z), \delta) \subset V \) being the support of the perturbation and such that \( B \) is \( C^{r,\nu} \)-close to \( A \) as follows.
Let $K^2_x = E_x^1 \oplus E_x^2 \oplus \cdots \oplus E_x^d \oplus E_x^d \oplus \cdots \oplus E_x^2 \oplus E_x^1$ be the symplectic decomposition of $K^2_x$ over $x = f^{\tau_1}(z)$ cf. (3.3) into 2-dimensional symplectic subspaces $E_x^i \oplus E_x^{d-i}$ induced by the vectors $e_i = H_{A,p, f^{\tau_1}(z)}(v_i)$ for $i = 1 \ldots 2d$. Fix $\varepsilon > 0$ and consider a $C^\infty$ bump function $\varphi : [0, +\infty) \to [0, +\infty)$ such that $\varphi(t) = 0$ if $t > \delta$ and $\varphi(t) = 1$ if $t < \frac{\delta}{2}$. Define $S \in C^{r,\eta}(M, sp(2f, \mathbb{R}))$ so that the map $S(y)$ rotates an angle $\varphi(\|y - f^{\tau_1}(z)\|)\eta$ in each symplectic 2-dimensional subspace $E_y^i \oplus E_y^{d-i} = E_y^i \oplus E_y^{d-i}$, where $\eta > 0$ is a degree of freedom to be considered in the sequel such that $\eta \to 0$ as $\varepsilon \to 0$. More precisely, if $y \in B(f^{\tau_1}(z), \delta)$ we define $S(y) : K^2_x \to K^2_x$ as the symplectic automorphism whose representation in the symplectic base $\{e_i\}$ is given by

$$S(y) e_i = \cos(\eta \varphi(\|y - f^{\tau_1}(z)\|^2)) e_i - \sin(\eta \varphi(\|y - f^{\tau_1}(z)\|^2)) e_i$$

and

$$S(y) e_i = \sin(\eta \varphi(\|y - f^{\tau_1}(z)\|^2)) e_i + \cos(\eta \varphi(\|y - f^{\tau_1}(z)\|^2)) e_i$$

for all $i$.

It is clear by the construction that the cocycle $S$ and the developments in $3.2$ that it is symplectic and coincides with the identity outside $B(f^{\tau_1}(z), \delta)$.

Moreover, $S$ is a $C^{r,\eta}$-small perturbation of the identity cocycle. In fact, if $\Delta(y) = \|y - f^{\tau_1}(z)\|^2$ then it follows from Faà di Bruno’s formula that

$$\frac{\partial^n \varphi(\Delta(y))}{\partial y_k} = \sum_{m_1! \cdots m_n!} \frac{n!}{m_1! \cdots m_n!} \varphi^{m_1+\cdots+m_n}(\Delta(y)) \prod_{j=1}^n \left( \frac{\partial^j \Delta(y)}{\partial y_k} \right)^{m_j},$$

where the sum is over all vectors with nonnegative integers entries $(m_1, \ldots, m_n)$ such that we have $\sum_{j=1}^n j \cdot m_j = n$. It is not hard to check that $\eta \frac{\partial^N \varphi(N(z))}{\partial y_k}$ is close to zero, and, in consequence, the cocycle $B = A \circ S$ satisfies $\|A - B\|_{r,\nu} \leq \varepsilon$, provided that $\eta$ is small enough. More precisely, if $r = 0$ then $|A - B|_{r,\nu}$ reduces to

$$\|A - B\|_\nu = \sup_{x+y} \frac{\|A(x) - (A - B)(y)\|}{d(x, y)^\nu} = \sup_{x+y} \frac{\|A(x)[S(x) - id] - A(y)[S(y) - id]\|}{d(x, y)^\nu} \leq \|A\|_{r,\nu} \|S(x) - id\| + \|A\| \|S(x)\|_1 \operatorname{diam}(M)^{1-\nu} \leq \varepsilon$$

since $\|S(y) - id\|$ and $\|S(y)\|_1 \operatorname{diam}(M)^{1-\nu}$ are arbitrarily small by choice of the constant $\eta$. For $r \in \mathbb{N}$, we have $D^r(A \circ S)(y) = D^rA(S(y)) D^rS(y)$ and we can use the Cauchy-Schwarz inequality to deal with the $C^r$ norm of the cocycles and estimate the Hölder constant of $D^r(A \circ S)$ as above and we leave the details to the reader.

Moreover, $B = A \circ S$ coincides with $A$ outside of $V$. On the other hand, since $H^s_{B,q,z}$ depends only on the values of the cocycle $B$ on the points $\{f^{-j}(z) : j \geq 0\} \cup \{f^{-j}(z) : j \geq 0\}$ which are outside of $V$, then $H^s_{B,q,z} = H^s_{A,q,z}$. Similarly, one has that $H^s_{A,p,f^{\tau_1}(z)} = H^s_{A,p, f^{\tau_1}(z)}$ and, using the equality (4.4), we are reduced to check that $H^s_{B,p,z}(v_i) = [H^s_{B,p,z}(z)]^{-1} H^s_{B,p,f^{\tau_1}(z)}(v_i) = [H^s_{B,p,f^{\tau_1}(z)}(z)]^{-1} (e_i)$ does not belong to any subspace generated by proper subsets of $W$. See Figure 3.

In fact, this holds because

$$B^{2\pi_1}(z) = B(f^{2\pi_1-1}(z)) \circ \cdots \circ B(f^{\pi_1}(z)) \circ B(f^{\pi_1-1}(z)) \circ \cdots \circ B(z)$$

and $S$ induces a rotation of angle $\eta$ in consecutive 2-dimensional symplectic subspaces $E_y^i \oplus E_y^{d-i}$ on each symplectic 2-dimensional subspace $E_i \oplus E_i$, which are disjoint from the previous ones for some small $\eta$. Finally, since the holonomies $B \mapsto H^s_{B,p,z}$ and $B \mapsto H^s_{B,q,z}$ are differentiable in a neighborhood of $A$ in
the $C^{r,\nu}$-topology (cf. Lemma 2.9 in [22]) then it is clear that the previous property is an open condition. This finishes the proof of the lemma. \hfill \Box

Let us mention that the previous result will be of particular interest in §4.5 when the vectors $\{w_i\}_{i=1}^{2\ell}$ are related to eigenvectors of periodic points with simple spectrum and obtained by means of unstable holonomies.

4.5. Finishing the proof of Theorem A. Fix $f \in \text{Diff}^{1+\alpha}(M)$ and an $f$-invariant, ergodic, hyperbolic measure $\mu$ with local product structure and $\ell \geq 1$. We prove that the set of cocycles in $C^{r,\nu}(M, sp(2\ell, \mathbb{K}))$ that have all zero Lyapunov exponents is contained in a compact set with empty interior and infinite codimension.

Let $A \in C^{r,\nu}(M, sp(2\ell, \mathbb{K}))$ be such that $\lambda^+(A, \mu) = 0$, that is, so that $F_A$ has only zero Lyapunov exponents and take an arbitrary $\varepsilon > 0$ and also $k \geq 2$. It follows from Proposition 4.7 that there exists a holonomy block $\mathcal{O}$ and distinct dominated periodic points $\{p_i\}_{i=1}^k$ in $\mathcal{O}$ and a cocycle $B \in C^{r,\nu}(M, sp(2\ell, \mathbb{K}))$ satisfying $|A - B|_{r,\nu} < \varepsilon/2$ and such that $W^{u}_{\text{loc}}(p_i) \cap W^{u}_{\text{loc}}(p_{i+1}) \neq \emptyset$ consists of one point $z_i$ and the Lyapunov spectrum of $B^{\pi_i}(p_i)$ is real and simple, where $\pi_i$ is the period of the periodic point $p_i$ for all $i = 1, \ldots, k$.

Let $\{v_i^j : j = 1 \ldots 2\ell\}$ be a symplectic base of eigenvectors for $A^{\pi_i}(p_i)$ for each $i = 1 \ldots k$. Consider also the symplectic base $\{w_i^j = H_{B,p_i,z_i}^u(k_i^j) : j = 1 \ldots 2\ell\}$ on the fiber of the heteroclinic point $z_i$, for $i = 1 \ldots k - 1$. Since the assertion of the Breaking Lemma (Lemma 4.8) is an open condition one can apply it recursively to each homoclinic point $z_i$ to prove that there exists $B \in C^{r,\nu}(M, sp(2\ell, \mathbb{K}))$ with $|B - \tilde{B}|_{r,\nu} < \varepsilon/2$ and so that the unstable holonomies $H_{B,p_i,z_i}^u = H_{B,p_{i+1},z_i}^u$ coincide and $H_{B,p_i,z_i}^{u}(v_i^j)$ does not belong to the subspace generated by any proper subset of $W$. Again, the later property is open in the $C^{r,\nu}$-topology and clearly implies

$$(h_{B,p_i,z_i}^s)_{*} m_{p_i} \neq (h_{B,p_{i+1},z_i}^s)_{*} m_{p_{i+1}}, \quad \text{for all } i = 1 \ldots k - 1.$$ 

Together with Corollary 4.7 this implies that $B$ has at least one non-zero Lyapunov exponent and proves that the set of cocycles in $C^{r,\nu}(M, sp(2\ell, \mathbb{K}))$ with at least one non-zero Lyapunov exponent is an open and dense set. In addition, we also deduce that cocycles in $C^{r,\nu}(M, sp(2\ell, \mathbb{K}))$ with all zero Lyapunov exponents are contained in codimension $k$ topological submanifolds. Since $k \geq 2$ was chosen arbitrary then it follows the second assertion and completes the proof of the theorem.

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