A benchmark generator for boolean quadratic programming

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Abstract—For boolean quadratic programming (BQP), we will show that there is no duality gap between the primal and dual problems under some conditions by using the classical Lagrangian duality. A benchmark generator is given to create random BQP problems which can be solved in polynomial time. Several numerical examples are generated to demonstrate the effectiveness of the proposed method.

Index Terms—Lagrangian duality, Boolean quadratic programming, Polynomial time, Benchmark generator

I. INTRODUCTION

Boolean quadratic programming (BQP) has found many applications (see [1], the references therein), for example, the Boolean networks [2], and the well-known maxcut problem [3], can be formulated as BQP. Various approaches have been used to solve BQP, including semidefinite programming (SDP) relaxation [4], second order cone programming (SOCP) relaxation [5], decomposition method [6] and randomized heuristics [7]. However, these approaches are only able to find inexact solutions to BQP. On the other hand, methods based on branch-and-cut [8] can find exact solutions, but they are usually time-consuming. Therefore, there exists a necessity to evaluate the performance of approaches to solving BQP. To provide testbed for these approaches, a benchmark generator for BQP problems is proposed in this study.

II. THEORETICAL BASICS

Considering the following boolean quadratic programming (BQP) problem

$$\min_{x \in \mathbb{R}^n} \left\{ f(x) = \frac{1}{2} x^T Q x - c^T x \mid x \in \{-1,1\}^n \right\},$$  \hspace{1cm} (1)

where, $Q = Q^T \in \mathbb{R}^{n \times n}$ is a given indefinite matrix, $c \in \mathbb{R}^n$ is a given nonzero vector.

By introducing the Lagrange multiplier $\lambda_i$ associated with each constraint $\frac{1}{2} (x_i^2 - 1) = 0$, the Lagrangian $L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ can be defined as

$$L(x, \lambda) = \frac{1}{2} x^T Q x - c^T x + \sum_{i=1}^{n} \frac{1}{2} \lambda_i (x_i^2 - 1) = \frac{1}{2} x^T Q(\lambda)x - c^T x - \frac{1}{2} \lambda^T e$$ \hspace{1cm} (2)

where, $\lambda = (\lambda_1, \cdots, \lambda_n)^T$, $e \in \mathbb{R}^n$ is a vector with all entries one, and $Q_\lambda$ is

$$Q(\lambda) = Q + \text{diag}(\lambda),$$  \hspace{1cm} (3)

here, diag($\lambda$) is a diagonal matrix with its diagonal entries $\lambda_1, \cdots, \lambda_n$.

The Lagrangian dual function can be obtained by

$$g(\lambda) = \inf_{x \in \{-1,1\}^n} L(x, \lambda).$$ \hspace{1cm} (4)

Let define the following dual feasible space

$$S^+ = \{ \lambda \in \mathbb{R}^n \mid Q(\lambda) \succ 0 \},$$ \hspace{1cm} (5)

then the Lagrangian dual function can be written explicitly as

$$g(\lambda) = -\frac{1}{2} c^T Q^{-1}(\lambda)c - \frac{1}{2} \lambda^T e, \text{ s.t. } \lambda \in S^+,$$ \hspace{1cm} (6)

associated with the Lagrangian equation

$$Q(\lambda)x = c.$$ \hspace{1cm} (7)

Finally, the Lagrangian dual problem can be obtained as

$$\max_{\lambda \in S^+} \left\{ g(\lambda) = -\frac{1}{2} c^T Q^{-1}(\lambda)c - \frac{1}{2} \lambda^T e \right\}. \hspace{1cm} (8)$$

Theorem 1. If $\bar{X}$ is a critical point of the Lagrangian dual function and $\lambda \in S^+$, then the corresponding $\bar{x} = Q^{-1}(\bar{X})c$ is a global solution to the BQP problem.

Proof. The derivative of $g(\lambda)$ gives

$$\frac{\partial g(\lambda)}{\partial \lambda_i} = -\frac{1}{2} \frac{\partial [c^T Q^{-1}(\lambda)c]}{\partial \lambda_i} - \frac{1}{2} \frac{\partial Q(\lambda)}{\partial \lambda_i} Q^{-1}(\lambda)c - \frac{1}{2}$$

$$= -\frac{1}{2} c^T Q^{-1}(\bar{X}) \frac{\partial Q(\bar{X})}{\partial \lambda_i} Q^{-1}(\bar{X})c - \frac{1}{2}$$

$$= \frac{1}{2} \bar{x}_i^2 - \frac{1}{2}$$

where, $\bar{x} = Q^{-1}(\bar{X})c$. Since $\bar{X}$ is a critical point of the Lagrangian dual function, we have $\frac{1}{2} \bar{x}_i^2 - \frac{1}{2} = 0$. 

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To continue, we have
\[
    f(x) - f(\hat{x}) = \frac{1}{2} x^T Q x - c^T x - \frac{1}{2} x^T Q \hat{x} + c^T \hat{x} - \frac{1}{2} \sum_{i=1}^{n} \lambda_i (\hat{x}_i^2 - 1)
\]
\[
= \frac{1}{2} (x - \hat{x})^T Q(\lambda)(x - \hat{x}) - \frac{1}{2} x^T \text{diag}(\lambda) x + x^T Q(\lambda) \hat{x} - \frac{1}{2} \lambda^T e
\]
\[
= \frac{1}{2} (x - \hat{x})^T Q(\lambda)(x - \hat{x}) + (x - \hat{x})^T (Q(\lambda) \hat{x} - c)
\]
\[
= \frac{1}{2} (x - \hat{x})^T Q(\lambda)(x - \hat{x}) > 0, \forall x \in \{-1, 1\}^n
\]
that is to say, \( \bar{x} = Q^{-1}(\lambda)c \) is the global minimizer of the BQP problem. This completes the proof. \( \square \)

**Remark 1.** Under the conditions in Theorem 7 it is easy to verify that
\[
g(\lambda) = -\frac{1}{2} c^T Q^{-1}(\lambda)c - \frac{1}{2} \lambda^T e
\]
\[
= \frac{1}{2} \bar{x}^T Q(\lambda) \bar{x} + (\bar{x}^T Q(\lambda) \bar{x} - c^T \bar{x}) - \frac{1}{2} \lambda^T e
\]
\[
= \frac{1}{2} \bar{x}^T Q(\lambda) \bar{x} - c^T \bar{x} - \frac{1}{2} \lambda^T e = L(\bar{x}, \lambda)
\]
\[
= \frac{1}{2} \bar{x}^T Q \bar{x} - c^T \bar{x} + \frac{n}{2} \sum_{i=1}^{n} \lambda_i (\bar{x}_i^2 - 1)
\]
\[= f(\bar{x}),
\]
which shows that there is no duality gap between the primal and dual problems.

### III. A Benchmark Generator for BQP

As stated above, if there exists a critical point of Lagrangian dual function in \( S^+ \), zero duality gap will be met. In this section, we will construct a benchmark generator for BQP problem such that these conditions are satisfied.

The inverse problem can be simplified as follows
\[
\begin{array}{ll}
\text{find} & Q, x, \lambda, c \\
\text{s.t.} & (Q + \text{diag}(\lambda))x = c \\
& Q + \text{diag}(\lambda) > 0 \\
& x \in \{-1, 1\}^n
\end{array}
\]
\[
(9)
\]
where, \( Q = Q^T \in \mathbb{R}^{n \times n}, c, x, \lambda \in \mathbb{R}^n \).

Suppose \( Q \) is a freely random symmetric matrix, to guarantee \( Q + \text{diag}(\lambda) > 0 \), let \( \lambda \) satisfy
\[
\lambda_i \geq \sum_{j=1}^{n} |Q_{ij}|,
\]
to make sure that \( Q(\lambda) \) is a diagonally dominant matrix.

Suppose \( x \) is a freely random “true” solution, then \( c \) should satisfy
\[
c = (Q + \text{diag}(\lambda))x.
\]

The matlab scripts for generating a benchmark BQP are given in the following

```matlab
function [Q,c,x,lambda] = generate_Qc(n)
base = 10;
Q = round((Q + Q')/2);
Q = base*randn(n);
lambda = zeros(n,1);
Q = base*randn(n);
base = 10;
function 
[x,lambda] = generate_Qc(n)
```

where, \( \text{base} \) is set to control the range of elements in \( Q \).

### IV. Numerical Experiments

The Lagrangian dual problem can be reformulated as the following SDP problem easily via Schur complement [9]
\[
\begin{array}{ll}
\text{(SDP)} : & \min \frac{1}{2} t + \frac{1}{2} \lambda^T e \\
\text{s.t.} & (Q(\lambda) \ c \ t) \succeq 0
\end{array}
\]
\[
12
\]
In the next, several examples are created by the above mentioned generator function `generate_Qc(n)`. All of the experiments are run on MATLAB R2010b on Intel(R) Core(TM) i3-2310M CPU @2.10GHz under Window 7 environment. The SDPT3 [10] is used as a solver embedded in YALMIP [11] for the SDP problem.

#### Example 1.

\[
Q = \begin{pmatrix}
-4 & -3 & 6 & -3 & -6 \\
-3 & 13 & -25 & -5 & 3 \\
6 & -25 & -2 & -5 & -1 \\
-3 & -5 & -5 & -8 & -7 \\
-6 & 3 & -1 & -7 & -5
\end{pmatrix},
\]
\[
c = \begin{pmatrix}
-18 \\
92 \\
-62 \\
-10 \\
0
\end{pmatrix}
\]

By solving the corresponding Lagrangian dual problem in 0.9204 seconds, we can get \( \bar{X} = (21.9996, 48.9999, 39.0000, 27.9996, 21.9998)^T \) and \( \bar{x} = Q^{-1}(\lambda)c = (-1.0000, 1.0000, -1.0000, -1.0000, -1.0000) \).
Example 2.

\[ Q = \begin{pmatrix} -6 & -9 & -7 & -3 & 2 & -2 & -12 & -11 & 8 & -6 \\ -9 & 14 & -4 & 3 & -2 & -4 & 6 & 23 & 8 & 3 \\ -7 & -4 & 1 & 21 & -10 & 2 & 6 & -13 & -9 & 4 \\ -3 & 3 & 21 & -18 & -2 & 7 & -2 & 16 & 1 & 3 \\ 2 & -2 & -10 & -2 & 10 & 8 & -11 & -3 & -2 & -2 \\ -2 & -4 & 2 & 7 & 8 & 0 & -10 & -10 & -2 & -7 \\ -12 & 6 & 6 & -2 & -11 & -10 & -7 & -10 & 3 & 0 \\ -11 & 23 & -13 & 16 & -3 & -10 & -10 & 6 & -8 & 10 \\ 8 & 8 & -9 & 1 & -2 & -2 & 3 & -8 & -10 & -5 \\ -6 & 3 & 4 & 3 & -2 & -7 & 0 & 10 & -5 & -9 \end{pmatrix}, \quad c = \begin{pmatrix} 24 \\ 72 \\ -18 \\ 16 \\ 38 \\ 54 \\ -66 \\ 76.0010 \\ 51.9993 \\ 66.9988 \\ 109.9979 \\ 55.9985 \\ 48.9995 \end{pmatrix}^T \]

By solving the Lagrangian dual problem in 0.9204 seconds, we can get \( \bar{\lambda} = (66.0010, 75.9997, 76.9970, 76.0010, 51.9993, 51.9988, 66.9988, 109.9979, 55.9985, 48.9995)^T \) and \( \bar{\pi} = \mathcal{Q}^{-1}(\bar{\lambda})c = (1.0000, -1.0000, 1.0000, -1.0000, -1.0000, 1.0000, 1.0000, 1.0000, -1.0000, 1.0000)^T \).

Example 3.

\[ Q = \begin{pmatrix} 11 & 2 & -10 & 15 & 21 & -7 & -8 & 6 & -3 & 11 & 2 & -1 & 3 & -2 & -1 \\ 2 & 6 & 7 & -5 & 10 & -14 & -1 & -8 & 3 & 6 & 6 & 0 & -7 & 1 & -2 \\ -10 & 7 & 21 & 12 & 13 & -9 & -1 & 2 & -5 & 9 & 2 & -1 & -2 & 4 & 8 \\ 15 & -5 & 12 & 12 & -7 & 0 & -3 & -17 & -3 & 6 & -1 & -1 & -1 & 6 & -5 \\ 21 & 10 & 13 & -7 & 3 & 6 & 3 & -1 & -10 & 0 & -9 & -1 & -4 & -7 & -2 \\ -7 & -14 & -9 & 0 & 6 & 5 & 1 & 7 & 3 & 2 & -1 & 3 & -4 & 3 & 8 \\ -8 & -1 & -1 & -3 & 3 & 1 & -5 & -5 & 3 & 6 & 17 & -13 & 6 & 14 & -10 \\ -14 & -9 & -9 & 0 & 6 & 5 & 1 & 7 & 3 & 2 & -1 & 3 & -4 & 3 & 8 \\ -8 & -1 & -1 & -3 & 3 & 1 & -5 & -5 & 3 & 6 & 17 & -13 & 6 & 14 & -10 \end{pmatrix}, \quad c = \begin{pmatrix} -140 \\ 86 \\ -118 \\ -114 \\ -134 \\ -72 \\ 92 \\ -100 \\ 120 \\ 98 \\ -80 \\ 70 \\ 90 \\ 120 \end{pmatrix} \]

By solving the Lagrangian dual problem in 0.9204 seconds, we can get \( \bar{\lambda} = (102.9996, 77.9974, 105.9976, 93.9972, 96.9967, 72.9976, 95.9974, 80.9970, 94.9967, 113.9981, 98.9980, 64.9976, 88.9972, 101.9973, 85.9979)^T \) and \( \bar{\pi} = \mathcal{Q}^{-1}(\bar{\lambda})c = (-1.0000, 1.0000, -1.0000, -1.0000, -1.0000, 1.0000, -1.0000, 1.0000, 1.0000, 1.0000)^T \).

**Remark 2.** All of the tested problems are solved in 0.9204 seconds, which can be regarded as an indicator for polynomial time complexity.

Example 4. Other large random BQP problems are created by the same procedure, and the corresponding running time for solving these problems can be found in Fig. 1.

![Fig. 1: Running time for other large random BQP cases](image-url)
V. CONCLUSION

In this short note, a benchmark generator for boolean quadratic programming (BQP) was implemented in Matlab. And these BQP instances can be used for evaluating the performance of various approaches.

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