DIFFERENTIAL EQUATION AND RECURRENCE RELATIONS OF
THE SHEFFER-APPPELL POLYNOMIAL SEQUENCE: A MATRIX
APPROACH

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Abstract. Motivated by the effective impact of the Pascal functional and the Wron-
skian matrices, we investigate several identities and differential equation for the Sheffer-
Appell polynomial sequence by using matrix algebra. The matrix approach, which we
have used in this article, is convenient to derive the generating functions of the Sheffer-
Appell polynomial sequence. By means of examples, we apply and also illustrate our
results to an extended class of polynomial sequences.

1. INTRODUCTION

Sequences of polynomials play an important role in many problems of pure and ap-
plied mathematical sciences such as those occurring in approximation theory, statistics,
combinatorics and analysis (see, for example, [10, 11, 12, 13]). The class of Sheffer se-
quences is one of the most important classes of polynomial sequences. A polynomial
sequence \( \{s_n(x)\}_{n=0}^{\infty} \) is called a Sheffer polynomial sequence [3, 4, 12, 15] if and only if
its generating function has the following form:

\[
A(y)e^{xH(y)} = \sum_{n=0}^{\infty} s_n(x) \frac{y^n}{n!},
\]

where

\[
A(y) = A_0 + A_1y + \cdots,
\]

and

\[
H(y) = H_1y + H_2y^2 + \cdots,
\]

with \( A_0 \neq 0 \) and \( H_1 \neq 0 \).

Let us recall an alternate definition of the Sheffer sequences in terms of a pair of gen-
erating functions \( (l(y), h(y)) \) (see, for example, [8]):
Let $h(y)$ be a delta series and let $l(y)$ be an invertible series, defined as follows:

$$h(y) = \sum_{n=0}^{\infty} h_n \frac{y^n}{n!} \quad (h(0) = 0; \ h(1) \neq 0) \quad (1.2)$$

and

$$l(y) = \sum_{n=0}^{\infty} l_n \frac{y^n}{n!} \quad (l(0) \neq 0). \quad (1.3)$$

Then there exists a unique sequence of Sheffer polynomials $s_n(x)$ satisfying the orthogonality conditions:

$$\langle l(y)h(y)^k|s_n(x) \rangle = n!\delta_{n,k} \quad (\forall \ n, k \geq 0), \quad (1.4)$$

where $\delta_{n,k}$ is the Kronecker delta.

Roman [12, p. 18, Theorem 2.3.4] introduced the exponential generating function of $s_n(x)$ as follows:

$$\frac{1}{h^{-1}(y)} e^{xy} = \sum_{n=0}^{\infty} s_n(x) \frac{y^n}{n!}. \quad (1.5)$$

The Sheffer sequence for the pair $(l(y), y)$ is called an Appell sequence for $l(y)$. In fact, Roman [12] characterized Appell sequences in several ways:

$$\{A_n(x)\}_{n \in \mathbb{N}} \text{ is an Appell set if either}$$

$$\frac{d}{dx}(\alpha_n(x)) = n\alpha_n(x) \quad n \in \mathbb{N}$$

or if there exists an exponential generating function of the form (see also the recent works [9] [16]):

$$A(y)e^{xy} = \sum_{n=0}^{\infty} \alpha_n(x) \frac{y^n}{n!}, \quad (1.6)$$

where $\mathbb{N}$ denotes the set of positive integers and

$$A(y) = \frac{1}{l(y)}. \quad (1.6)$$

We also note that, for $H(y) = y$, the generating function (1.1) of the Sheffer polynomials $s_n(x)$ reduces to the generating function (1.6) of the Appell polynomials $\alpha_n(x)$.

The polynomials defined as the discrete convolution of known polynomials are used to investigate new families of special functions. For example, the polynomial $h_n^{(A)}(x)$ given by

$$h_n^{(A)}(x) = \sum_{r=0}^{\infty} A_k h_{n-k}(x), \quad (1.7)$$

is known as a discrete Appell convolution by setting $h_n(x) = x^n$ in the above equation.
In the year 2015, Subuhi et al. [7] introduced the determinantal definition and other properties of the Sheffer-Appell polynomials. The Sheffer-Appell polynomial sequences are combination of the families of the Sheffer and the Appell polynomials sequences.

Now, in order to recall the definition of the generalized Pascal functional matrix of an analytic function (see [18]), let

\[ \mathcal{F} = \left\{ h(y) = \sum_{r=0}^{\infty} \frac{\alpha_r}{r!} y^r \mid \alpha_r \in \mathbb{C} \right\} \]

be the set of power series possessing the \( \mathbb{C} \)-algebra. Then the generalized Pascal functional matrix \( [P_n(h(y))] \), which is a lower triangular matrix of order \((n+1) \times (n+1)\) for \( h(y) \in \mathcal{F} \), is defined by

\[
P_n[h(y)]_{ij} = \begin{cases} 
\binom{i}{j} h^{(i-j)}(y) & (i \geq j) \\
0 & \text{(otherwise)},
\end{cases} \tag{1.8}
\]

for all \( i, j = 0, 1, 2, \cdots, n \). Here \( h^{(i)}(y) \) is the \( i \)th order derivative of \( h(y) \).

We next recall the \( n \)th order Wronskian matrix of several analytic functions \( h_1(y), h_2(y), \cdots, h_m(y) \) of order \((n+1) \times m\) as follows:

\[
W_n[h_1(y), h_2(y), \cdots, h_m(y)] = \begin{bmatrix}
    h_1(y) & h_2(y) & h_3(y) & \cdots & h_m(y) \\
    h_1'(y) & h_2'(y) & h_3'(y) & \cdots & h_m'(y) \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    h_1^{(n)}(y) & h_2^{(n)}(y) & h_3^{(n)}(y) & \cdots & h_m^{(n)}(y)
\end{bmatrix}. \tag{1.9}
\]

We also record here some properties and relationships between the Wronskian matrices and the generalized Pascal functional matrices as they are the main tool of our work (see, for example, [19, 20]).

**Property I.** For \( h(y), l(y) \in \mathbb{C} \), \( P_n[h(y)] \) and \( W_n[h(y)] \) are linear, that is,

\[
P_n[uh(y) + vl(y)] = uP_n[h(y)] + vP_n[l(y)]
\]

and

\[
W_n[uh(y) + vl(y)] = uW_n[h(y)] + vW_n[l(y)], \tag{1.10}
\]

where \( u, v \in \mathbb{C} \).

**Property II.** For \( h(y), l(y) \in \mathbb{C} \),

\[
P_n[h(y)l(y)] = P_n[h(y)]P_n[l(y)] = P_n[l(y)]P_n[h(y)]. \tag{1.11}
\]

**Property III.** For \( h(y), l(y) \in \mathbb{C} \),

\[
W_n[h(y)l(y)] = P_n[h(y)]W_n[l(y)] = P_n[l(y)]W_n[h(y)]. \tag{1.12}
\]
Property IV. For \( h(y), l(y) \in \mathbb{C} \), with \( h(0) = 0 \) and \( h'(0) \neq 0 \),

\[
W_n[l(h(y))]_{y=0} = W_n \left[ 1, h(y), h^2(y), \ldots, h^n(y) \right]_{y=0} \Omega_n^{-1} W_n[l(y)]_{y=0},
\]

where \( 0!, 1!, \ldots, n! \) are the diagonal entries in the diagonal matrix given by

\[
\Omega_n = \text{diag}[0!, 1!, 2!, \ldots, n!].
\]

2. The Sheffer-Appell polynomial sequence and its differential equation

He and Ricci ([6]; see also [14]) derived some recurrence relations and differential equation for the Appell polynomial sequence. Further, Youn and Yang ([20]; see also [1]) obtained some identities and differential equation for the Sheffer polynomial sequence by using matrix algebra. Here, in this paper, we study some recursive formulas and differential equation for the Sheffer-Appell polynomial sequence by using matrix algebra.

The Sheffer-Appell polynomial sequence, which is denoted by \( sA_n(x) \), is defined as the discrete Appell convolution of the Sheffer polynomials \( s_n(x) \).

The generating function of the Sheffer-Appell polynomials \( sA_n(x) \) is given by

\[
\frac{1}{l(h^{-1}(y))l(y)} e^{xh^{-1}(y)} = \sum_{n=0}^{\infty} sA_n(x) \frac{y^n}{n!},
\]

where \( h^{-1}(y) \) is the compositional inverse of \( h(y) \), that is, we have (see [19])

\[
h^{-1}(h(y)) = (h^{-1}(y)) = y.
\]

Thus, if the following generating function in (2.1):

\[
\frac{1}{l(h^{-1}(y))h(y)} e^{xh^{-1}(y)}
\]

is analytic, then (by using Taylor’s expansion theorem), we obtain

\[
sA_k(x) = \frac{d^k}{dy^k} \left( \frac{1}{l(h^{-1}(y))l(y)} e^{xh^{-1}(y)} \right) \bigg|_{y=0} \quad (k \geq 0).
\]

The Sheffer-Appell polynomial sequence \( sA_n(x) \) in vector form for the pair \((l(y), h(y))\) is denoted by \( s\vec{A}_n(x) \) and it is defined by

\[
s\vec{A}_n(x) = [sA_0(x), sA_1(x), \ldots, sA_n(x)]^T,
\]

which can also be expressed as follows:

\[
s\vec{A}_n(x) = [sA_0(x), sA_1(x), \ldots, sA_n(x)]^T = W_n \left[ \frac{1}{l(h^{-1}(y))l(y)} e^{xh^{-1}(y)} \right]_{y=0}.
\]
Let us begin with the equation (2.4), that is,

\[ W_n[sA_0(x), sA_1(x), \ldots, sA_n(x)]^T \Omega_n^{-1} = W_n[1, (h^{-1}(y))_{(1)}, (h^{-1}(y))_{(2)}, \ldots, (h^{-1}(y))_{(n)}] \bigg|_{y=0} \]

\[ \cdot \Omega_n^{-1} P_n \left[ \frac{1}{l(1)} \right] \bigg|_{y=0} P_n \left[ \frac{1}{l(h(y))} \right] \bigg|_{y=0} P_n \left[ e^{xy} \right] \bigg|_{y=0}. \]

(2.5)

Proof. Let us begin with the equation (2.4), that is,

\[ sA_n(x) = W_n \left[ \frac{1}{l(h^{-1}(y))l(1)} \right] e^{xy} \bigg|_{y=0}. \]

(2.6)

Applying Property IV in the equation (2.6), we get

\[ sA_n(x) = W_n[1, (h^{-1}(y))_{(1)}, (h^{-1}(y))_{(2)}, \ldots, (h^{-1}(y))_{(n)}] \bigg|_{y=0} \Omega_n^{-1} W_n \left[ \frac{1}{l(1)l(h(y))} \right] e^{xy} \bigg|_{y=0}. \]

(2.7)

In view of the following result:

\[ W_n[e^{xy}] \bigg|_{y=0} = [1, x, x^2, \ldots, x^n]^T, \]

the (2.7) becomes

\[ sA_n(x) = W_n[1, (h^{-1}(y))_{(1)}, (h^{-1}(y))_{(2)}, \ldots, (h^{-1}(y))_{(n)}] \bigg|_{y=0} \]

\[ \cdot \Omega_n^{-1} P_n \left[ \frac{1}{l(1)} \right] \bigg|_{y=0} P_n \left[ \frac{1}{l(h(y))} \right] \bigg|_{y=0} [1, x, x^2, \ldots, x^n]^T. \]

(2.9)

Now, by taking the kth order derivative of both sides of the equation (2.9) with respect to x and dividing the resulting equation by k!, we obtain

\[ \frac{1}{k!} \left[ sA_0^{(k)}(x), sA_1^{(k)}(x), \ldots, sA_n^{(k)}(x) \right]^T \]

\[ = W_n[1, (h^{-1}(y))_{(1)}, (h^{-1}(y))_{(2)}, \ldots, (h^{-1}(y))_{(n)}] \bigg|_{y=0} \]

\[ \cdot \Omega_n^{-1} P_n \left[ \frac{1}{l(1)} \right] \bigg|_{y=0} P_n \left[ \frac{1}{l(h(y))} \right] \bigg|_{y=0} \]

\[ \cdot [0, \ldots, 0, 1, \binom{k+1}{k} x, \binom{k+1}{k} x^2, \ldots, \binom{n}{k} x^{n-k}]^T. \]

(2.10)
Hence, clearly, the right-hand side and left-hand side of the equation (2.10) are the $k$th columns of

$$W_n[1, (h^{-1}(y)), (h^{-1}(y))^2, \ldots, (h^{-1}(y))^n\bigg|_{y=0}^\Omega^{-1},$$

and

$$P_n\left[\frac{1}{l(y)}\bigg|_{y=0}^\Omega^{-1}, P_n\left[\frac{1}{l(h(y))}\bigg|_{y=0}^\Omega^{-1}, P_n[e^{xy}]\bigg|_{y=0}^\Omega^{-1},$$

respectively. Our proof of the Lemma is thus completed.

We now state and prove Theorem 1 below.

**Theorem 1.** The Sheffer-Appell polynomial sequence $A_n(x) \sim (l(y), h(y))$ satisfies the following differential equation:

$$\sum_{k=0}^{n} (xa_k + b_k + c_k)\frac{A_n^{(k)}(x)}{k!} - n_s A_n(x) = 0,$$  \tag{2.11}

where

$$a_k = \left(\frac{h(y)}{h'(y)}\right)^{(k)}\bigg|_{y=0} (k \geq 0),$$

$$b_k = \left(-\frac{h(y)l'(h(y))}{l(h(y))}\right)^{(k)}\bigg|_{y=0} (k \geq 0)$$

and

$$c_k = \left(-\frac{h(y)l'(y)}{h'(y)l(y)}\right)^{(k)}\bigg|_{y=0} (k \geq 0).$$

**Proof.** Let us begin with the following result:

$$W_n\left[\frac{d}{dy}\left(e^{xh^{-1}(y)}\right)\frac{e^{xh^{-1}(y)}}{l(h^{-1}(y))l(y)}\bigg|_{y=0} \right].$$  \tag{2.12}
Also, on the other hand, we can rewrite the equation (2.12) as follows:

\[
W_n \left[ \frac{d}{dy} \left( \frac{e^{xh^{-1}(y)}}{l(h^{-1}(y))l(y)} \right) \right]_{y=0} = P_n[y] W_n \left[ \frac{d}{dy} \left( \frac{e^{xh^{-1}(y)}}{l(h^{-1}(y))l(y)} \right) \right]_{y=0}
\]

\[
= \left[ \begin{array}{cccccc}
0 & 0 & 0 & \cdots & 0 & 0 \\
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 2 & 0 & \cdots & 0 & 0 \\
0 & 0 & 3 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & n-1 & 0 \\
0 & 0 & 0 & \cdots & 0 & n \\
\end{array} \right] \cdot \left[ \begin{array}{c}
sA_1(x) \\
sA_2(x) \\
sA_3(x) \\
\vdots \\
sA_n(x) \\
sA_{n+1}(x) \\
\end{array} \right]. \quad (2.13)
\]

Also, on the other hand, we can rewrite the equation (2.12) as follows:

\[
W_n \left[ \frac{d}{dy} \left( \frac{e^{xh^{-1}(y)}}{l(h^{-1}(y))l(y)} \right) \right]_{y=0} = W_n \left[ \left( \frac{x h(h^{-1}(y))}{h'(h^{-1}(y)) l(h^{-1}(y))} - \frac{h(h^{-1}(y)) l'(h^{-1}(y))}{h'(h^{-1}(y)) l(h^{-1}(y))} \right) e^{x h^{-1}(y)} \right] \frac{l(h^{-1}(y))}{l(h^{-1}(y)) l(y)} \left|_{y=0} \right.
\]

\[
\left. \cdot \frac{e^{x h^{-1}(y)}}{l(h^{-1}(y)) l(y)} \right|_{y=0}. \quad (2.14)
\]

Thus, by using Property IV in the equation (2.14), we have

\[
W_n \left[ \frac{d}{dy} \left( \frac{e^{xh^{-1}(y)}}{l(h^{-1}(y))l(y)} \right) \right]_{y=0} = W_n[1, (h^{-1}(y)), (h^{-1}(y))^2, \cdots, (h^{-1}(y))^n] \left|_{y=0} \right. \Omega_n^{-1}
\]

\[
\cdot W_n \left[ \left( \frac{x h(y)}{h'(y)} - \frac{h(y) l'(h(y))}{l(h(y))} - \frac{h(y) l'(h(y))}{h'(h(y)) l(y)} \right) \frac{e^{xy}}{l(y) l(h(y))} \right] \left|_{y=0} \right. \cdot. \quad (2.15)
\]
Next, by using Property III in the equation (2.15), we get
\[
W_n\left[\frac{d}{dy} \left( \frac{e^{xh^{-1}(y)}}{l(h^{-1}(y))l(y)} \right) \right]_{y=0} = W_n[1, (h^{-1}(y)), (h^{-1}(y))^2, \ldots, (h^{-1}(y))^n] \cdot \Omega_n^{-1}P_n[\frac{e^y}{y}]_{y=0} \cdot P_n\left[\frac{1}{l(y)}\right]_{y=0}
\]

which, by applying the above Lemma, yields
\[
W_n\left[\frac{d}{dy} \left( \frac{e^{xh^{-1}(y)}}{l(h^{-1}(y))l(y)} \right) \right]_{y=0} = W_n[sA_0(x), sA_1(x), \ldots, sA_n(x)]^T \cdot \Omega_n^{-1}W_n\left[\left( \begin{array}{c} x \frac{h(y)}{h'(y)} - \frac{h(y)l'(h(y))}{l(h(y))} - \frac{h(y) l'(y)}{h'(y) l(y)} \end{array} \right) \right]_{y=0}
\]
or, equivalently,
\[
W_n\left[\frac{d}{dy} \left( \frac{e^{xh^{-1}(y)}}{l(h^{-1}(y))l(y)} \right) \right]_{y=0} = \begin{bmatrix} sA_0(x) & 0 & 0 & \cdots & 0 \\ sA_1(x) & \frac{sA_1'(x)}{l} & 0 & \cdots & 0 \\ sA_2(x) & \frac{sA_2'(x)}{l} & \frac{sA_2''(x)}{2l} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ sA_n(x) & \frac{sA_n'(x)}{l} & \frac{sA_n''(x)}{2l} & \cdots & \frac{sA_n^{(n)}(x)}{n!} \end{bmatrix} \begin{bmatrix} xa_0 + b_0 + c_0 \\ xa_1 + b_1 + c_1 \\ xa_2 + b_2 + c_2 \\ \vdots \\ xa_n + b_n + c_n \end{bmatrix}.
\]

Finally, upon equating the nth rows of (2.13) and (2.17), we obtain the desired result (2.11) asserted by Theorem II. \qed

By setting \( l(y) = 1 \) and \( b_k = c_k = 0 \) (\( \forall \ k \geq 1 \)) in Theorem II we get the following corollary.

**Corollary 1.** Let \( qA_n(x) \sim (1, h(y)) \) be the associated polynomial sequence. Then
\[
x \sum_{k=0}^{n} a_k \frac{qA_n^{(k)}(x)}{k!} - n_qA_n(x) = 0.
\]

In its special case when \( a_k = (1)_k, b_k = (\lambda+1)((2)_k-(1)_k) \) and \( c_k = (\lambda+1)[(3)_k-(4)_k] \), Theorem II would apply to the Laguerre polynomials as follows.

**Corollary 2.** Let
\[
L_n^{(\lambda)}(x) \sim \left( (1-y)^{-\lambda-1}, \frac{y}{y-1} \right)
\]
be the generalized Laguerre polynomial of degree \( n \) in \( x \) and with the index (or order) \( \lambda \). Then
\[
\sum_{k=1}^{n} \binom{n}{k} k! \left( x - \frac{k(k-1)(k+4)(\lambda+1)}{6} \right) L_{n-k}(x) = n L_n(x).
\] (2.19)

**Example 1.** By applying Theorem 1 to the Miller-Lee type Appell polynomials \( G^{(m)}(x) \) given by
\[
G^{(m)}(x) \sim \left((1-y)^{m+1}, y\right),
\]
we have
\[
a_k = \begin{cases} 
1 & (k = 1) \\
0 & (k > 1), 
\end{cases}
\]
\[
b_k = \begin{cases} 
0 & (k = 0) \\
(m+1)(1)_k & (k > 0)
\end{cases}
\]
and
\[
c_k = \begin{cases} 
0 & (k = 0) \\
-(m+1)(1)_k & (k > 0)
\end{cases}
\]
Hence we get the following recurrence relation for the Miller-Lee type Appell polynomials \( G^{(m)}(x) \):
\[
n G_A_n(x) - n x G_A_{n-1}(x) = \sum_{k=1}^{n} \binom{n}{k} G_A_{n-k}(x)(b_k + c_k).
\] (2.20)

**3. Recurrence relations for the Sheffer-Appell polynomials**

Here, in this section, we first state and prove Theorem 2 below.

**Theorem 2.** Let \( s A_n(x) \sim (l(y), h(y)) \) be the Sheffer-Appell polynomial sequence. Then the following recursive formula holds true for \( s A_n(x) \):
\[
s A_{n+1}(x) = \sum_{k=0}^{n} (x a_k + b_k + c_k) \frac{s A^{(k)}_n(x)}{k!},
\] (3.1)
where
\[
a_k = \left( \frac{1}{h'(y)} \right)^{(k)} \bigg|_{y=0} \quad (k \geq 0),
\]
\[
b_k = \left( -\frac{l'(h(y))}{l(h(y))} \right)^{(k)} \bigg|_{y=0} \quad (k \geq 0)
\]
and
\[
c_k = \left( -\frac{l'(y)}{h'(y)l(y)} \right)^{(k)} \bigg|_{y=0} \quad (k \geq 0).
\]
Proof. Let us consider

\[ W_n \left[ \frac{d}{dy} \left( \frac{e^{xh^{-1}(y)}}{l(h^{-1}(y))} \right) \right]_{y=0}, \quad (3.2) \]

which, on the one hand, can be written as follows:

\[ W_n \left[ \frac{d}{dy} \left( \frac{e^{xh^{-1}(y)}}{l(h^{-1}(y))} \right) \right]_{y=0} = [sA_1(x), sA_2(x), \ldots, sA_{n+1}(x)]^T. \quad (3.3) \]

Also, on the other hand, we can write the equation (3.2) in the following form:

\[ W_n \left[ \left( x \cdot \frac{1}{h'(h^{-1}(y))} - \frac{l'(y)}{l(y)} - 1 \cdot \frac{l'(h^{-1}(y))}{l(h^{-1}(y))} \right) \frac{e^{xh^{-1}(y)}}{l(h^{-1}(y))} \right]_{y=0}, \quad (3.4) \]

which, on using Property IV, yields

\[ W_n \left[ \left( x \cdot \frac{1}{h'(h^{-1}(y))} - \frac{l'(y)}{l(y)} - 1 \cdot \frac{l'(h^{-1}(y))}{l(h^{-1}(y))} \right) \frac{e^{xh^{-1}(y)}}{l(h^{-1}(y))} \right]_{y=0} = W_n \left[ 1, (h^{-1}(y)), (h^{-1}(y))^2, \ldots, (h^{-1}(y))^n \right]_{y=0} \Omega_n^{-1} \times W_n \left[ x \cdot \frac{1}{h'(y)} - \frac{l'(y)}{l(h(y))} - 1 \cdot \frac{l'(y)}{l(h(y))} \right]_{y=0}. \quad (3.5) \]

Now, if we make use of Property III, we find from (3.5) that

\[ W_n \left[ \left( x \cdot \frac{1}{h'(h^{-1}(y))} - \frac{l'(y)}{l(y)} - 1 \cdot \frac{l'(h^{-1}(y))}{l(h^{-1}(y))} \right) \frac{e^{xh^{-1}(y)}}{l(h^{-1}(y))} \right]_{y=0} = W_n \left[ 1, (h^{-1}(y)), (h^{-1}(y))^2, \ldots, (h^{-1}(y))^n \right]_{y=0} \Omega_n^{-1} \times P_n \left[ e^{xy} \right]_{y=0} \times P_n \left[ \frac{1}{l(y)} \right]_{y=0} \times P_n \left[ \frac{1}{l(h(y))} \right]_{y=0} \times W_n \left[ \left( x \cdot \frac{1}{h'(y)} - \frac{l'(y)}{l(h(y))} - 1 \cdot \frac{l'(y)}{l(h(y))} \right) \right]_{y=0}. \quad (3.6) \]
Finally, by applying the above Lemma, we get

\[
W_n \left[ \left( x \frac{1}{h'(h^{-1}(y))} - \frac{l'(y)}{l(y)} - \frac{1}{h'(h^{-1}(y))} \frac{l'(h^{-1}(y))}{l(h^{-1}(y))} \right) e^{xh^{-1}(y)} \right]_{y=0} \\
= W_n [sA_0(x), sA_1(x), \ldots, sA_n(x)]^T \Omega^{-1} \\
\cdot W_n \left[ \left( x \frac{1}{h'(y)} - \frac{l'(h(y))}{l(h(y))} - \frac{1}{h'(y)} \frac{l'(y)}{l(y)} \right) e^{xh^{-1}(y)} \right]_{y=0}
\]

or, equivalently,

\[
W_n \left[ \left( x \frac{1}{h'(h^{-1}(y))} - \frac{l'(y)}{l(y)} - \frac{1}{h'(h^{-1}(y))} \frac{l'(h^{-1}(y))}{l(h^{-1}(y))} \right) e^{xh^{-1}(y)} \right]_{y=0} \\
= \begin{bmatrix} sA_0(x) & 0 & 0 & \cdots & 0 \\ sA_1(x) & \frac{sA'_1(x)}{1!} & 0 & \cdots & 0 \\ sA_2(x) & \frac{sA'_2(x)}{2!} & \frac{sA''_2(x)}{2!} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ sA_n(x) & \frac{sA'_n(x)}{n!} & \frac{sA''_n(x)}{2!} & \cdots & \frac{sA^{(n)}_n(x)}{n!} \end{bmatrix} \begin{bmatrix} xa_0 + b_0 + c_0 \\ xa_1 + b_1 + c_1 \\ xa_2 + b_2 + c_2 \\ \vdots \\ xa_n + b_n + c_n \end{bmatrix}.
\tag{3.7}
\]

Equating the \(n^{th}\) rows of (3.3) and (3.7), we arrive at the desired result (3.1) asserted by Theorem 2.

\[\square\]

Corollary 3 below follows from Theorem 2 in its special case when \(l(y) = 1\) and \(b_k = c_k = 0\) (\(\forall\ k \geq 1\)) in the recursive formula (3.1).

**Corollary 3.** Let \(qA_n(x) \sim (1, h(y))\) be the associated polynomial sequence. Then

\[
q^{A_{n+1}}(x) = x \sum_{k=0}^{n} a_k \frac{q^{A_n^{(k)}(x)}}{k!},
\tag{3.8}
\]

where

\[
a_k = \left( \frac{1}{h'(y)} \right)^{(k)} \bigg|_{y=0} \quad (k \geq 0).
\]

**Example 2.** Here, in this example, we apply Theorem 2 to the generalized Laguerre polynomials \(L_n^{(\lambda)}(x)\) given by

\[
L_n^{(\lambda)}(x) \sim \left( (1 - y)^{-\lambda - 1}, \frac{y}{y - 1} \right)
\]
with

\[
a_k = \begin{cases} 
-1 & (k = 0) \\
2 & (k = 1) \\
-2 & (k = 2) \\
0 & (k > 2), 
\end{cases}
\]

and

\[
b_k = -\lambda(1)_k.
\]

Hence we get the following recurrence relation for the Laguerre-Appell polynomials:

\[
L_{A_{n+1}}(x) + (x + 2\lambda + 2)L_{A_n}(x) = 2x\lambda L_{A_{n-1}}(x) - 2(x + \lambda + 1)\binom{n}{2}L_{A_{n-2}}(x) + (\lambda + 1)\sum_{k=3}^{n}\binom{n}{k}L_{A_{n-k}}(x)k!.
\] (3.9)

If we apply Theorem 2 to the Miller-Lee type Appell polynomials \(G_{n}^{(m)}(x)\) given by

\[
G_{n}^{(m)}(x) \sim \left((1 - y)^{m+1}, y\right),
\]

we have

\[
a_k = \begin{cases} 
1 & (k = 0) \\
0 & (k > 0), 
\end{cases}
\]

and

\[
b_k = (m + 1)(1)_k.
\]

Hence we get the following recurrence relation for the Miller-Lee type Appell polynomials:

\[
G_{A_{n+1}}(x) - xG_{A_n}(x) = \sum_{k=0}^{n}\binom{n}{k}G_{A_{n-k}}(x)(b_k + c_k).
\] (3.10)

**Theorem 3.** Let \(\{s_{A_n}(x)\} \sim (l(y), h(y))\) be the Sheffer-Appell polynomial sequence. Then the following recursive formula holds true for \(s_{A_n}(x)\):

\[
s_{A_{n+1}}(x)a_0 = x_{s_{A_n}(x)} + \sum_{k=0}^{n}\binom{n}{k}s_{A_{n-k}}(x)(b_k + c_k) - \sum_{k=1}^{n}\binom{n}{k}s_{A_{n+1-k}}(x)a_k, \quad (3.11)
\]
where
\[ a_k = \left( h'(h^{-1}(y)) \right)^{(k)} \bigg|_{y=0}, \]
\[ b_k = \left( -\frac{h'(h^{-1}(y)) l'(y)}{l(y)} \right)^{(k)} \bigg|_{y=0}, \]
and
\[ c_k = \left( -\frac{l'(h^{-1}(y))}{l(h^{-1}(y))} \right)^{(k)} \bigg|_{y=0}. \]

**Proof.** Let us begin with
\[
W_n \left[ h'(h^{-1}(y)) \frac{d}{dy} \left( \frac{e^{xh^{-1}(y)}}{l(h^{-1}(y))l(y)} \right) \right] \bigg|_{y=0},
\]
which, by applying Property III, yields
\[
W_n \left[ h'(h^{-1}(y)) \frac{d}{dy} \left( \frac{e^{xh^{-1}(y)}}{l(h^{-1}(y))l(y)} \right) \right] \bigg|_{y=0} = P_n \left[ \frac{d}{dy} \left( \frac{e^{xh^{-1}(y)}}{l(h^{-1}(y))l(y)} \right) \right] \bigg|_{y=0} W_n \left[ h'(h^{-1}(y)) \right] \bigg|_{y=0}
\]
or, equivalently,
\[
W_n \left[ h'(h^{-1}(y)) \frac{d}{dy} \left( \frac{e^{xh^{-1}(y)}}{l(h^{-1}(y))l(y)} \right) \right] \bigg|_{y=0} = \begin{bmatrix}
  sA_1(x) & 0 & 0 & \cdots & 0 \\
  sA_2(x) & sA_1(x) & 0 & \cdots & 0 \\
  sA_3(x) & \binom{2}{1}sA_2(x) & sA_1(x) & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  sA_{n+1}(x) & \binom{n}{1}sA_n(x) & \binom{n}{2}sA_{n-1}(x) & \cdots & sA_1(x)
\end{bmatrix} \begin{bmatrix}
  a_0 \\
  a_1 \\
  a_2 \\
  \vdots \\
  a_n
\end{bmatrix}. \quad (3.13)
\]

On the other hand, we can write the equation (3.12) as follows:
\[
W_n \left[ h'(h^{-1}(y)) \frac{d}{dy} \left( \frac{e^{xh^{-1}(y)}}{l(h^{-1}(y))l(y)} \right) \right] \bigg|_{y=0} = W_n \left[ \left( x - h'(h^{-1}(y)) l'(y) \right) \frac{e^{xh^{-1}(y)}}{l(h^{-1}(y)) l(y)} \right] \bigg|_{y=0}. \quad (3.14)
\]
Now, if we apply Property I, we get
\[
W_n \left[ h'(h^{-1}(y)) \frac{d}{dy} \left( \frac{e^{xh^{-1}(y)}}{l(h^{-1}(y))l(y)} \right) \right]_{y=0} = xW_n \left[ \frac{e^{xh^{-1}(y)}}{l(h^{-1}(y))l(y)} \right]_{y=0} - W_n \left[ \frac{h'(h^{-1}(y))l'(y)}{l(y)} \frac{e^{xh^{-1}(y)}}{l(h^{-1}(y))l(y)} \right]_{y=0} - W_n \left[ \frac{l'(h^{-1}(y))}{l(h^{-1}(y))l(y)} \frac{e^{xh^{-1}(y)}}{l(h^{-1}(y))l(y)} \right]_{y=0},
\]
which, by using Property III, yields
\[
W_n \left[ h'(h^{-1}(y)) \frac{d}{dy} \left( \frac{e^{xh^{-1}(y)}}{l(h^{-1}(y))l(y)} \right) \right]_{y=0} = xW_n \left[ \frac{e^{xh^{-1}(y)}}{l(h^{-1}(y))l(y)} \right]_{y=0} + P_n \left[ \frac{e^{xh^{-1}(y)}}{l(h^{-1}(y))l(y)} \right]_{y=0} W_n \left[ -\frac{h'(h^{-1}(y))l'(y)}{l(y)} \right]_{y=0},
\]
or, equivalently,
\[
W_n \left[ h'(h^{-1}(y)) \frac{d}{dy} \left( \frac{e^{xh^{-1}(y)}}{l(h^{-1}(y))l(y)} \right) \right]_{y=0} = x \begin{bmatrix} sA_0(x) \\ sA_1(x) \\ sA_2(x) \\ \vdots \\ sA_{n-1}(x) \\ sA_n(x) \end{bmatrix} + \begin{bmatrix} sA_0(x) & 0 & 0 & \cdots & 0 \\ sA_1(x) & sA_0(x) & 0 & \cdots & 0 \\ sA_2(x) & (\frac{2}{1})sA_1(x) & sA_0(x) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ sA_n(x) & (\frac{n}{1})sA_{n-1}(x) & \cdots & sA_0(x) \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} + \begin{bmatrix} sA_0(x) \\ sA_1(x) \\ sA_2(x) \\ \vdots \\ sA_n(x) \end{bmatrix} = x \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.
\]
Equating nth rows of (3.13) and (3.16), we arrive at the desired result (3.11) asserted by Theorem 3.

Upon setting \( l(y) = 1 \) and \( b_k = c_k = 0 \) (\( \forall k \geq 1 \)), in the recursive formula (3.11) asserted by Theorem 2, we can deduce the following corollary.
Corollary 4. Let \( qA_n(x) \sim (1, h(y)) \) be the associated polynomial sequence. Then

\[
qA_{n+1}(x) = x \sum_{k=0}^{n} a_k \frac{qA_n^{(k)}(x)}{k!},
\]

(3.17)

where

\[
a_k = \left( \frac{1}{h'(y)} \right)^{(k)} \bigg|_{y=0}.
\]

Example 3. Applying Theorem 3 to the Miller-Lee type Appell polynomials \( G_n^{(m)}(x) \) given by

\[
G_n^{(m)}(x) \sim \left( (1 - y)^{m+1}, y \right),
\]

we have

\[
a_k = \begin{cases} 
1 & (k = 0) \\
0 & (k > 0) 
\end{cases}
\]

and

\[
b_k = c_k = -(\lambda + 1)(1)_k.
\]

Hence we get the following recurrence relation for Miller-Lee type Appell polynomials:

\[
G A_{n+1}(x) = xG A_n(x) - 2(\lambda + 1) \sum_{k=0}^{n} \binom{n}{k} G A_{n-k}(x)k!.
\]

(3.18)

Theorem 4. Let \( sA_n(x) \sim (l(y), h(y)) \) be the Sheffer-Appell polynomial sequence. Then the following recursive formula holds true for \( sA_n(x) \):

\[
sA_{n+1}(x) = \sum_{k=0}^{n} \binom{n}{k} (xa_k + b_k + c_k) sA_{n-k}(x),
\]

(3.19)

where

\[
a_k = \left( \frac{1}{h'(h^{-1}(y))} \right)^{(k)} \bigg|_{y=0},
\]

\[
b_k = \left( -\frac{l'(y)}{l(y)} \right)^{(k)} \bigg|_{y=0},
\]

and

\[
c_k = \left( -\frac{l'(h^{-1}(y))}{h'(h^{-1}(y))l(h^{-1}(y))} \right)^{(k)} \bigg|_{y=0}.
\]

Proof. Our demonstration of Theorem 4 begins with

\[
W_n \left[ \frac{d}{dy} \left( \frac{e^{yh^{-1}(y)}}{l(h^{-1}(y))l(y)} \right) \right] \bigg|_{y=0},
\]

(3.20)
which, on the one hand, can be rewritten as follows:

\[
W_n \left[ \frac{d}{dy} \left( \frac{e^{xh^{-1}(y)}}{l(h^{-1}(y))l(y)} \right) \right]_{y=0} = [sA_1(x), sA_2(x), \ldots, sA_{n+1}(x)]^T. \tag{3.21}
\]

On the other hand, we can write (3.20) in the following form:

\[
W_n \left[ \frac{d}{dy} \left( \frac{e^{xh^{-1}(y)}}{l(h^{-1}(y))l(y)} \right) \right]_{y=0} = W_n \left[ \left( \frac{1}{h'(h^{-1}(y))} - \frac{l'(y)}{l(y)} \right) \frac{e^{xh^{-1}(y)}}{l'(h^{-1}(y))l'(h^{-1}(y))} \right]_{y=0}, \tag{3.22}
\]

which, by using Property III, yields

\[
W_n \left[ \frac{d}{dy} \left( \frac{e^{xh^{-1}(y)}}{l(h^{-1}(y))l(y)} \right) \right]_{y=0} = P_n \left[ x - \frac{l'(y)}{l(y)} - \frac{l'(h^{-1}(y))}{l'(h^{-1}(y))} \right]_{y=0} W_n \left[ \frac{e^{xh^{-1}(y)}}{l'(h^{-1}(y))l'(y)} \right]_{y=0}
\]

or, equivalently,

\[
W_n \left[ \frac{d}{dy} \left( \frac{e^{xh^{-1}(y)}}{l(h^{-1}(y))l(y)} \right) \right]_{y=0} = \begin{bmatrix}
xa_0 + b_0 + c_0 & 0 & 0 & \cdots & 0 \\
xa_1 + b_1 + c_1 & xa_0 + b_0 + c_0 & 0 & \cdots & 0 \\
xa_2 + b_2 + c_2 & (2)_1 ((xa_1 + b_1 + c_1) & xa_0 + b_0 + c_0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
x_a + b_n + c_n & (n) (xa_n-1 + b_{n-1} + c_{n-1}) & \cdots & \cdots & xa_0 + b_0 + c_0
\end{bmatrix}
\]

Equating the nth rows of (3.21) and (3.23), we arrive at desired result (3.19) asserted by Theorem 4. \hfill \Box

In its special case when \(l(y) = 1\) and \(b_k = c_k = 0\ (\forall k \geq 1)\), the recursive formula (3.19) asserted by Theorem 4, we obtain Corollary 5 below.
Corollary 5. Let \( q A_n(x) \sim (1, h(y)) \) be the associated polynomial sequence. Then

\[
q A_{n+1}(x) = x \sum_{k=0}^{n} \binom{n}{k} a_k q A_{n-k}(x),
\]

where

\[
a_k = \left( \frac{1}{h'(h^{-1}(y))} \right)^{(k)} \bigg|_{y=0}.
\]

4. Concluding remarks and observations

In the preceding sections, we have developed a differential equation and recurrence relations for the Sheffer-Appell polynomials by using the Pascal functional and Wronskian matrices. In order to derive these recursive formulas for the Sheffer-Appell polynomials, we find several interesting recurrence relations for such related polynomials as (for example) the generalized Laguerre polynomials \( L_n^{(\lambda)}(x) \) and the Miller-Lee type Appell polynomials \( G_n^{(m)}(x) \). The results presented in this article are potentially useful in deducing further interesting formulas for other specific classes of orthogonal polynomials.

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