THE MAXIMAL ORDER OF ITERATED MULTIPLICATIVE FUNCTIONS

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Abstract. Following Wigert, various authors, including Ramanujan, Gronwall, Erdős, Ivić, Schwarz, Wirsing, and Shiu, determined the maximal order of several multiplicative functions, generalizing Wigert’s result

$$\max_{n \leq x} \log d(n) = \log \frac{x}{\log \log x} \log 2 + o(1).$$

On the contrary, for many multiplicative functions, the maximal order of iterations of the functions remains widely open. The case of the iterated divisor function was only solved recently, answering a question of Ramanujan from 1915.

Here we determine the maximal order of $$\log f(f(n))$$ for a class of multiplicative functions $$f$$. In particular, this class contains functions counting ideals of given norm in the ring of integers of an arbitrary, fixed quadratic number field. As a consequence, we determine such maximal orders for several multiplicative $$f$$ arising as a normalized function counting representations by certain binary quadratic forms. Incidentally, for the non-multiplicative function $$r_2$$ which counts how often a positive integer is represented as a sum of two squares, this entails the asymptotic formula

$$\max_{n \leq x} \log r_2(r_2(n)) = \sqrt{\log x} \log \frac{c}{\sqrt{2} + o(1)}$$

with some explicitly given constant $$c > 0$$.

1. Introduction

1.1. Maximal orders of multiplicative functions. The study of the maximal order of arithmetic functions (for example of the divisor functions $$d$$ or $$\sigma$$) is an integral part of introductory number theory textbooks. For the divisor functions $$d$$ and $$\sigma$$, satisfactory answers are well known; see, for example, Wigert [36] and Gronwall [9]. Their proofs make use of the fact that $$d$$ and $$\sigma$$ are multiplicative functions. For the maximal order of magnitude of iterated arithmetic functions much less is known. Here are some reasons which show that this is generally a very delicate subject:

1. The iterate of a multiplicative function need not be multiplicative; for instance, for any pairwise distinct primes $$p_1, \ldots, p_r$$,

$$\frac{d(d(p_1)) \cdots d(d(p_r))}{d(d(p_1 \cdots p_r))} = \frac{d(2)^r}{d(2^r)} = \frac{2^r}{r+1} \neq 1.$$
Let $a(n)$ denote the number of abelian groups of order $n$. By results of Erdős and Ivić [6] it is known that
\[
\exp((\log x)^{1/2+o(1)}) \ll \max_{n \leq x} a(a(n)) \ll \exp((\log x)^{7/8+o(1)}),
\]
leaving a large gap between lower and upper bounds. Improving these bounds would seem to require understanding the multiplicative structure of the number $p(n)$ of unrestricted partitions, about which very little is known beyond certain congruences.

Let $\sigma_1(n) = \sigma(n)$ be the sum of divisors function, and $\sigma_k(n) = \sigma_1(\sigma_{k-1}(n))$ its iterates. Schinzel [30] conjectured that
\[
\liminf_{n \to \infty} \frac{\sigma_k(n)}{n} < \infty.
\]
This is only known for $k = 1, 2$ and 3 by results of Mąkowski [22] and Maier [21], and conditionally on Schinzel’s Hypothesis H. In light of studying maximal orders of magnitude: the equivalent question
\[
\limsup_{n \to \infty} \frac{\sigma_k(n)}{n} > 0
\]
is equally open.

For the iterated Euler $\varphi$-function the situation is quite different, compared to the iterated $\sigma$-function. In view of $\varphi(2^n) = 2^{n-1}$ and $\varphi_k(2^n) = 2^{n-k}$, it is evident that, for any fixed $k$, the extremal order of magnitude is $\limsup_{n \to \infty} \frac{\varphi_k(n)}{n} \geq \frac{1}{2^k}$. As Maier [21] points out, the situation changes if one discards such thin sets of prime powers. If one studies large values of $\varphi_k(n)$ that occur for about $\frac{x}{\log x}$ values of $n \leq x$, then the situation is very similar to the situation with the iterated $\sigma$-function, see above.

However, there are other non-trivial results on the iterated $\varphi$-function, e.g. concerning the range of the values of $\varphi$ by Ford [8] and $\varphi_k$ (Luca and Pomerance [20]). Moreover, it is well known that the iterated $\varphi$-function has applications to Pratt-trees, see e.g. [3].

In the case of multiplicative functions, the maximal order of magnitude was initially proved in a number of individual cases: the maximal order of the divisor function $d$ has been determined by Wigert [36] and Ramanujan [29]. They proved that
\[
\limsup_{n \to \infty} \frac{\log d(n) \log \log n}{\log n} = \log 2,
\]
where $\log$ denotes the logarithm with base $e$. (Note that for functions of this magnitude one typically has an asymptotic for $\log(f(n))$ rather than for $f(n)$ itself. From our perspective we will still say that the maximal order has been determined.) This study subsequently influenced (via results of Hardy and Ramanujan, Turán and Erdős and Kac) the development of probabilistic number theory.

Ramanujan studied the multiplicative function $\delta$ that counts the number of representations of its argument as a sum of two squares ignoring sign, i.e.,
\[
\delta(n) = \frac{1}{4} \# \{ (x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x^2 + y^2 = n \}. \tag{1.1}
\]
If $\nu_p$ denotes the $p$-adic valuation, then it is well-known (see, e.g., [10, Theorem 278]) that
\begin{equation}
\delta(n) = \prod_{\text{prime } q|n \atop q \equiv 1 \mod 4} (\nu_q(n) + 1) \times \prod_{\text{prime } p|n \atop p \equiv 3 \mod 4} \frac{1}{2}(1 + (-1)^{\nu_p(n)}).
\end{equation}

(To be precise, Ramanujan called this function $Q_2(n)$, but here we follow the notation used by Hardy and Wright [10, Theorem 278].) We observe that $4\delta(n) = r_2(n)$, where $r_2(n)$ is the sum of two squares function which also takes care of signs. Ramanujan [28] showed that for some positive constant $a$
\[
\max_{n \leq x} \delta(n) = \exp\left(\frac{\log 2}{2} \text{li}(2 \log x) + O((\log x) \exp(-a\sqrt{\log x}))\right),
\]
where the right hand side can be simplified to
\[
\exp\left((\log 2 + o(1)) \frac{\log x}{\log \log x}\right).
\]
This implies the very same logarithmic maximum order:
\[
\limsup_{n \to \infty} \frac{\log r_2(n) \log \log n}{\log n} = \log 2.
\]
Knopfmacher [18] and Nicolas [24], who were unaware\(^1\) of Ramanujan’s work, later also observed this.

Ramanujan (see [28, §§ 55–56]) also achieved the very same result
\[
\max_{n \leq x} \bar{Q}_2(n) = \exp\left(\frac{\log 2}{2} \text{li}(2 \log x) + O((\log x) \exp(-a\sqrt{\log x}))\right),
\]
for the function $\bar{Q}_2(n)$ counting non-negative pairs $(x, y)$ with $n = x^2 + xy + y^2$,
\begin{equation}
\bar{Q}_2(n) = \prod_{\text{prime } q|n \atop q \equiv 1 \mod 3} (\nu_q(n) + 1) \times \prod_{\text{prime } p|n \atop p \equiv 2 \mod 3} \frac{1}{2}(1 + (-1)^{\nu_p(n)}).
\end{equation}

Note that this quadratic form corresponds to the Eisenstein lattice $\mathbb{Z}[e^{2\pi i/3}]$, and non-negative coordinates correspond to a sector of 60 degrees, which explains the factor 1/6 in (2.3); for a more conceptual explanation for the factor 1/6, see the last display formula before Section 4.

Krätzel [19] proved for the number $a(n)$ of non-isomorphic abelian groups of order $n$:
\[
\limsup_{n \to \infty} \frac{\log a(n) \log \log n}{\log n} = \frac{1}{4} \log 5,
\]
and Knopfmacher [17] proved for the number $\beta(n)$ of squareful divisors of $n$:
\[
\limsup_{n \to \infty} \frac{\log \beta(n) \log \log n}{\log n} = \frac{1}{3} \log 3.
\]

\(^1\)At that time Ramanujan’s work was unpublished: quite remarkably, the end of Ramanujan’s paper [29] of 1915 was not intended to be the end. In fact, Ramanujan’s manuscript was considerably longer and due to a shortage of resources during wartime the London Mathematical Society printed only part of the manuscript. The second part has been recovered and published many years later, first in [27], but later with detailed annotations by Nicolas and Robin [28], and also [1].
Note that all of the functions $a$, $\beta$ and $d$ are prime independent, where a multiplicative arithmetic function $f$ is said to be prime independent if $f(p^\nu) = f(2^\nu)$ for every prime power $p^\nu$.

A number of authors independently observed that such limits can be worked out more generally for the class of prime independent multiplicative functions. Of these results we only mention the one by Shiu [32], but there are others—see [2, 5, 11, 12, 18, 23, 25, 26, 35].

Shiu [32] proved: let $f : \mathbb{N} \to \mathbb{R}$ be a multiplicative function satisfying the following conditions:

(1) There exist constants $A$ and $0 < \theta < 1$ such that $f(2^\nu) \leq \exp(A \nu^\theta)$ where $\nu \geq 1$, and

(2) for all primes $p$ and all $a \geq 1$ one has $f(p^\nu) = f(2^\nu) \geq 1$, then the following holds:

$$\limsup_{n \to \infty} \frac{\log f(n) \log \log n}{\log n} = \log \max_{\nu \geq 1} (f(2^\nu))^{1/\nu}.$$  

1.2. On iterates of arithmetic functions. The quest for the maximal order of the iterated divisor function was raised by Ramanujan [29] in his paper on highly composite numbers. At the very end of that paper he gave a construction of integers, namely, $N_k = \prod_{i=1}^k p_i^{i-1}$, where $p_i$ denotes the $i$-th prime, and observed that for these integers $d(d(N_k)) \geq \exp((\sqrt{2} \log 4 + o(1)) \frac{\log N_k}{\log \log N_k})$ holds. Erdős and Kátai [7], Ivić [14] and Smati [33, 34] gave results on the maximal order, but a satisfying answer on the maximal order of the iterated divisor function was only given almost 100 years after Ramanujan’s paper: Buttkewitz, Elsholtz, Ford and Schlage-Puchta [4] proved:

$$\limsup_{n \to \infty} \frac{\log d(d(n)) \log \log n}{\sqrt{\log n}} = c,$$

where

$$c = \left(8 \sum_{l=1}^{\infty} \left( \log \left(1 + \frac{1}{7} \right) \right)^2 \right)^{1/2}.$$  

However, it seems that no similar result is known for either of the functions $\delta$ or $r_2$. Neither of these functions is prime independent in the sense defined above, but nonetheless they are still quite similar to $d$ (compare (1.2)). For the latter reason, results concerning $\delta$ and $r_2$ have often been an intuitive next step following results concerning $d$. In fact, let us recall the development for sums of multiplicative functions, where Landau investigated the number of integers representable as sums of two squares. Subsequently, this was generalized many times, for example to the number of integers consisting of primes in certain residue classes only, and eventually led to the celebrated mean value results of Wirsing and Halász.

Motivated by this development, we study a class of multiplicative functions which includes important functions, such as the divisor function $d$, $\delta$, and—more generally—a number of functions connected with counting ideals in quadratic number fields. In the spirit of Shiu’s theorem, we also investigate which hypotheses on the function $f$ and which growth rates of
of the following expressions $f(p^r)$, depending on $\nu$, allow us to bound the maximum order magnitude of $f(f(n))$. In some cases (including $\delta$, $r_2$ and $Q_2$), we are able to give an asymptotic for the logarithmic size of this maximum.

1.3. Plan of the paper. The rest of the paper is structured as follows: first, we present our results in ascending generality. Results for $\delta$, $r_2$, $Q_2$, and some related functions are presented in Section 2. In Section 3, these results are then cast into a more conceptual light from the point of view of basic algebraic number theory. All of these results follow from general results we describe in Section 4. The rest of the paper deals with supplying all the deferred proofs (which mostly concerns our statements from Section 4).

2. The prototypes: $\delta$, $r_2$, $Q_2$, and relatives

The following result is an immediate consequence of (1.2) and Corollary 4.3 below.

**Theorem 2.1.** Let $\delta$ be given by (1.1). Then

$$\max_{n \leq x} \log \delta(\delta(n)) = \frac{\sqrt{\log x}}{\log_2 x} \left( \frac{c}{\sqrt{2}} + O\left(\frac{\log x}{\log_2 x}\right)\right),$$

where $c$ is given in (1.4).

Incidentally, this implies a result for $r_2$ at no additional effort—even though $r_2$ is not multiplicative:

**Corollary 2.2.** The assertion of Theorem 2.1 remains valid if $\delta$ is replaced with $r_2$, where $r_2(n) = \delta(n) = \#\{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x^2 + y^2 = n\}$.

**Proof.** For $n \in \mathbb{N}$ write $\delta(n) = 2^\nu m(n)$ with some odd integer $m(n)$. Then, using multiplicativity of $\delta$ and $\delta(2^{2+\nu}) = 1 = \delta(2^\nu)$, we have

$$r_2(r_2(n)) = 4\delta(2^{2+\nu}m(n)) = 4\delta(2^{2+\nu})\delta(m(n)) = 4\delta(2^\nu)\delta(m(n)) = 4\delta(2^\nu m(n)) = 4\delta(\delta(n)).$$

Hence, $\log r_2(r_2(n)) = \log \delta(\delta(n)) + 2\log 2$ and the assertion of the corollary follows from Theorem 2.1. □

It turns out that our arguments are not just limited to the binary quadratic form $x^2 + y^2$ appearing in (1.2). A more refined explanation can be found in the next section. Here we content ourselves with stating the next result in a very modest form:

**Theorem 2.3.** Fix $k \in \{2, 3, 5, 11, 17, 41\}$ and let $f(n)$ be defined by either of the following expressions

\begin{align*}
(2.1) & \quad \frac{1}{4}\#\{(x, y) \in \mathbb{Z}^2 \mid x^2 + y^2 = n\}, \\
(2.2) & \quad \frac{1}{2}\#\{(x, y) \in \mathbb{Z}^2 \mid x^2 + 2y^2 = n\}, \\
(2.3) & \quad \frac{1}{6}\#\{(x, y) \in \mathbb{Z}^2 \mid x^2 + xy + y^2 = n\}, \\
& \quad \frac{1}{2}\#\{(x, y) \in \mathbb{Z}^2 \mid x^2 + xy + ky^2 = n\}
\end{align*}

for all positive integers $n$ and put $f(0) = 1$. Then

$$\max_{n \leq x} \log f(f(n)) = \frac{\sqrt{\log x}}{\log_2 x} \left( \frac{c}{\sqrt{2}} + O\left(\frac{\log x}{\log_2 x}\right)\right),$$
where \( c \) is given in (1.4).

Certainly Theorem 2.3 implies Theorem 2.1 (see (2.1)). Moreover, it covers the choice \( f = Q_2 \) with \( Q_2 \) defined in (1.3) (see (2.3)). Corollary 2.2 allows one to drop the factor \( \frac{1}{2} \) in (2.1) and the same trick used for proving Corollary 2.2 can also be used to show that one can drop the factor \( \frac{1}{2} \) in (2.2).

3. Examples from algebraic number theory

Our next objective is to fit Theorem 2.3 into a broader context. We begin with some notation. Let \( K \) be an arbitrary number field and \( \mathcal{O} \) its ring of algebraic integers. The norm of an ideal \( a \subseteq \mathcal{O} \) is denoted by \( N(a) = \#(\mathcal{O}/a\mathcal{O}) \in \mathbb{N} \). Consider

\[
(3.1) \quad f_K(n) = \#\{\text{ideals } a \subseteq \mathcal{O} : N(a) = n\}.
\]

For two ideals \( a, b \subseteq \mathcal{O} \) their product \( ab \) is defined element-wise and we have \( N(ab) = (N(a))(N(b)) \). On combining this with the classical facts that ideals of \( \mathcal{O} \) admit a unique factorisation into prime ideals and that the norm of a prime ideal is always a power of a prime in \( \mathbb{N} \), one easily deduces that \( f_K : \mathbb{N}_0 \to \mathbb{N}_0 \) as defined above is multiplicative.

In order to understand the behaviour of \( f_K \) on prime powers \( p^\nu \), we start by considering any ideal \( a \) with \( N(a) = p^\nu \). By the multiplicativity of the norm, the factorisation of \( a \) consists only of prime ideals whose norm is again a power of \( p \). From algebraic number theory one knows that the number of prime ideals \( p \subseteq \mathcal{O} \) with \( p \) dividing \( N(p) \) is finite. Thus, enumerating the prime ideals with said property by \( p_1, \ldots, p_k \), we may write

\[
a = p_1^{\nu_1} \cdots p_k^{\nu_k}
\]

for some exponents \( \nu_1, \ldots, \nu_k \in \mathbb{N}_0 \). The condition that \( N(a) = n \) then takes the form

\[
(3.2) \quad \sum_{j=1}^k \nu_j \frac{\log N p_j}{\log p} = \nu.
\]

On the other hand, this argument also works in the opposite direction. Hence, the map

\[
(3.3) \quad \{(\nu_1, \ldots, \nu_k) \in \mathbb{N}_0^k : (3.2) \text{ holds}\} \longrightarrow \{\text{ideals } a \subseteq \mathcal{O} \text{ with } N(a) = p^\nu\},
\]

\[
(\nu_1, \ldots, \nu_k) \longrightarrow p_1^{\nu_1} \cdots p_k^{\nu_k}
\]

is bijective. Moreover, one knows that the principal ideal \( (p) = p\mathcal{O} \) factors as \( (p) = p_1^{e_1} \cdots p_k^{e_k} \), so that

\[
(3.4) \quad \sum_{j=1}^k e_j \frac{\log N(p_j)}{\log p} = \frac{N(p)}{\log p} = [K : \mathbb{Q}],
\]

where the right hand side is the degree of the field extension \( K/\mathbb{Q} \).

If \( [K : \mathbb{Q}] > 2 \), then (3.2) allows for too much freedom in the choice of the exponents \( \nu_1, \ldots, \nu_k \). Consequently, the value of \( f_K \) at prime powers \( p^\nu \) may depend too loosely on the prime \( p \) and the quick growth of \( f_K(p^\nu) \) as a function of \( \nu \) poses additional problems. The situation becomes appreciably better if \( K \) is a quadratic extension of \( \mathbb{Q} \) and we shall henceforth restrict ourselves to this case. Then, by (3.4), for any prime \( p \), only one of the following three cases may occur:
(1) \( p \) is ramified, that is, \((p) = p_1^2 \) for some prime ideal \( p_1 \) of \( \mathcal{O} \) with \( \mathcal{N}p = p \);

(2) \( p \) is split, that is, \((p) = p_1p_2 \) with some distinct prime ideals \( p_1, p_2 \subseteq \mathcal{O} \) each having norm \( p \);

(3) \( p \) is inert, that is, \((p) = p_1 \) is a prime ideal of \( \mathcal{O} \) and \( \mathcal{N}p_1 = \mathcal{N}(p) = p^2 \).

From multiplicativity of \( f_K \) in combination with the map in (3.3) being bijective, it follows that

\[
(3.5) \quad f_K(n) = \prod_{\text{prime } p \mid n \text{ ramified}} 1 \times \prod_{\text{prime } q \mid n \text{ split}} (\nu_q(n) + 1) \times \prod_{\text{prime } p \mid n \text{ inert}} \frac{1}{2}(1 + (-1)^{\nu_p(n)}).
\]

From this representation of \( f_K(n) \), we see that our results from Section 4 below can be applied to establish the following:

**Theorem 3.1.** Suppose that \( K \) is some fixed quadratic number field and let \( f_K \) be given by (3.1). Then

\[
\max_{n \leq x} \log f_K(f_K(n)) = \sqrt{\log x} \left( \frac{c}{\log_2 x} + O\left(\frac{\log_3 x}{\log_2 x}\right)\right),
\]

where the implied constant may depend on \( K \), and \( c \) is given in (1.4).

The missing pieces for the proof of the above theorem are given in Section 5 below.

In the remainder of this section we briefly sketch how to obtain Theorem 2.3 from Theorem 3.1. To this end, assume that \( K \) is some imaginary quadratic number field with class number one. Then the value \( f_K(n) \) can be viewed as the number of solutions to some binary quadratic equation. Indeed, the assumption about the class number implies that all ideals of \( \mathcal{O} \) are principal and \( K \) being imaginary quadratic implies that \( \mathcal{O} \) has finitely many units. Therefore,

\[
f_K(n) = \frac{\#\{\xi \in \mathcal{O} : N(\xi) = n\}}{\#\{\text{units in } \mathcal{O}\}}.
\]

The celebrated Baker–Heegner–Stark theorem gives a complete classification (up to isomorphism) of all imaginary quadratic number fields with class number one. Theorem 2.3 then follows immediately by going through that list, rewriting the norm equation \( N(\xi) = n \) with respect to some integral basis and applying Theorem 3.1; here the choice of the integral basis may affect the particular form one gets, but any such form can be readily checked to be equivalent to one of the ones implicit in Theorem 2.3 by using the well-known reduction theory for binary quadratic forms.

### 4. Hypotheses and the General Results

In what follows, we give a description of a class of arithmetic functions for which the subsequent reasoning works. The imposed restrictions could be relaxed somewhat, but the model cases we primarily aim at are given in (1.2), (1.3) and (3.5). The important features here are the following: the arithmetic function \( f \) to be iterated is multiplicative, acts affinely on the exponents of powers of primes \( q \) from a certain subset of primes \( Q \subseteq \mathbb{P} \) (e.g., primes \( \equiv 1 \mod 4 \) in the case \( f = \delta \) as seen in (1.2)), and takes only the
values 0, 1 on powers of primes $p \in \mathbb{P} \setminus Q$ (subject to a rule which—under the assumptions below—turns out to be irrelevant).

In [4], the case $Q = \mathbb{P}$ with the multiplicative arithmetic function $d$ acting as $d(p^\nu) = \nu + 1$ is studied. In our approach, we assume that $f(q^\nu) = g(\nu)$ for powers of primes $q \in Q$ with a function $g$ satisfying suitable axioms as listed below. By elaborating on the method of [4], we obtain upper and lower bounds on the maximal order of first iterates of arithmetic functions $f$ which enjoy similar properties as those observed for $d$ and $\delta$, see Theorem 4.1 and Theorem 4.2.

In detail, we start with a strictly increasing sequence of primes $(q_j)_{j \geq 1}$.

By the prime number theorem, the sequence of all primes $(p_j)_{j \geq 1}$ satisfies $p_j = j(\log j + \log(\log j) + \mathcal{O}(1))$, so it seems reasonable to assume similar asymptotics for $(q_j)_{j \geq 1}$ (see (A.1) below). Set $Q = \{q_j : j \in \mathbb{N}\}$ and let $\langle Q \rangle$ be the monoid (multiplicatively) generated by $Q$. Furthermore, fix a map $g : \mathbb{N}_0 \rightarrow \mathbb{N}$ with $g(0) = 1$ and let

\[ g^1(y) = \inf\{x \in \mathbb{N} : g(x) = y\} \in \mathbb{N} \cup \{+\infty\}. \]  

Finally, assume that

(A.1) $(q_j)_{j \geq 1}$ satisfies the asymptotic expansion

\[ q_j = \kappa j(\log j + \log(\log j) + \mathcal{O}(1)), \]

where $\kappa > 0$ is some constant,

(A.2) $g$ is monotonically increasing,

(A.3) $g(\mathbb{N}) \supseteq \langle Q \rangle$,

(A.4) $g^1(b) + c_s g^1(a) \leq g^1(ab)$ for all $a, b \in \langle Q \rangle$ such that $q_1 \leq a \leq b$, where $c_s > 1/q_1$ is some constant,

(A.5) $g(i)/g(i - 1) = 1 + \mathcal{O}(i^{-1/2 - \epsilon})$ for some $\epsilon > 0$,

(A.6) $g(x) \leq c_f x$ for all $x \in \mathbb{N}$, where $c_f > 0$ is some constant,

(A.7) $g^1(q) = c_1 q + \mathcal{O}(q/\log q)$ as $Q \ni q \rightarrow \infty$, where $c_1 > 0$ is some constant. (Note that $g^1(q)$ is finite due to (A.3).)

Now let $f$ be a multiplicative arithmetic function satisfying

\[ f(p^\nu) = \begin{cases} g(\nu) & \text{if } p \in Q, \\ \in \{0, 1\} & \text{if } p \notin Q \end{cases} \]

for a prime power $p^\nu \geq 1$. Furthermore, let $f(0) = 1$. We write

\[ M(x) = \max_{n \leq x} \log f(f(n)). \]

On writing $\log_k$ for the $k$-fold iterate of the natural logarithm, our main results may now be stated as follows:

**Theorem 4.1.** Let $M$ be as in (4.3). Then,

\[ M(x) \leq \frac{\sqrt{\log x}}{\log_2 x} \left( \frac{C_g}{\sqrt{\kappa c_1}} + \mathcal{O}\left(\frac{\log_3 x}{\log_2 x}\right) \right), \]

The symbol $g^1$ was chosen to allude to a pseudo inverse.
where the implied constant depends on $Q, f$ and

$$C_g = \left(8 \sum_{j=1}^{\infty} \left(\log \frac{g(j)}{g(j-1)}\right)^2\right)^{1/2}. \tag{4.5}$$

Throughout this paper, $C_g$ always denotes the constant defined in (4.5). We also note in passing that throughout all implied constants may depend on the function $f$ and the set $Q$ and an $\epsilon$, where obvious.

**Theorem 4.2.** Letting $g(\nu) = \alpha \nu + 1$ and assuming the above hypotheses, the following holds

$$M(x) \geq \frac{\sqrt{\log x}}{\log_2 x} \left(\frac{C_g}{\sqrt{\kappa/\alpha}} + O\left(\frac{\log_3 x}{\log_2 x}\right)\right). \tag{4.6}$$

Upon combining Theorem 4.1 and Theorem 4.2, we immediately deduce the following corollary:

**Corollary 4.3.** Letting $g(\nu) = \alpha \nu + 1$ for some $\alpha \in \mathbb{N}$, and on the above hypotheses, it holds that

$$M(x) = \frac{\sqrt{\log x}}{\log_2 x} \left(\frac{C_g}{\sqrt{\kappa/\alpha}} + O\left(\frac{\log_3 x}{\log_2 x}\right)\right).$$

We note in passing that, for $Q = \mathbb{P}$ in the setting of Corollary 4.3, the function $f$ in (4.2) arises naturally as number of divisors of monic monomials, i.e., $f(n) = d(n^\alpha)$.

### 5. Proof of Theorem 3.1

We assume the notation of Theorem 3.1 and recall (3.5). In order to use Corollary 4.3 to obtain an asymptotic formula for

$$\max_{n \leq x} \log f_K(f_K(n)),$$

it only remains to verify Assumption (A.1), where $q_j$ therein is taken to be the $j$-th smallest prime which splits in $K$. The next lemma furnishes a prime number theorem for such primes and can be used to verify that Assumption (A.1) holds with $\kappa = \frac{1}{2}$ and thereby finishes the proof of Theorem 3.1.

**Lemma 5.1.** Suppose that $K$ is some fixed quadratic number field. Then

$$\#\{\text{primes } q \leq x \text{ which are split in } K\} = \frac{1}{2} \frac{x}{\log x} (1 + O(1/\log x))$$

as $x \to \infty$.

The above lemma is certainly well-known. Nevertheless, we sketch a proof for the convenience of the reader.
Proof of Lemma 5.1. By [13, Proposition 13.1.3] there is some integer $\Delta$ such that
\[
\#\{\text{primes } q \leq x \text{ which are split in } K\} = \#\{\text{primes } q \leq x \text{ with } \left(\frac{\Delta}{q}\right) = 1\} + O(d(\Delta)) = \frac{1}{2}\#\{\text{primes } p \leq x\} + \frac{1}{2} \sum_{\text{primes } q \leq x} \left(\frac{\Delta}{q}\right) + O(d(\Delta))
\]
where $\left(\frac{\Delta}{q}\right)$ denotes the Kronecker symbol. Consequently, the assertion of the lemma then follows from the prime number theorem and [15, Corollary 5.29] (see also [15, Exercise 6 in §3.5]).

\[\square\]

6. Notation and auxiliary results

6.1. Notation. At this point it is convenient to introduce some additional notation used throughout the rest of the paper. Let
- $\Omega(n) = \sum_{p|n} \nu_p(n)$, $\omega(n) = \sum_{p|n} 1$,
- $\Pi_Q(n) = \max\{m \in \langle Q \rangle : m \mid n\}$,
- $\Omega_Q = \Omega \circ \Pi_Q$,
- $\omega_Q = \omega \circ \Pi_Q$,
- $\pi_Q(x) = \#\{q \in Q : q \leq x\}$.

6.2. Auxiliary results. We would like to give the reader our perspective on the problem at hand. In order to keep the notation simple, let $Q = \mathbb{P}$ for the moment. Then, for any positive integer $n$,
\[
\log f(f(n)) = \sum_{q\in Q \atop q|f(n)} \log g(q^{\nu_q(f(n))}).
\]

Vaguely speaking, in order to give estimates on $M(x)$, one needs to exhibit some control over the prime factors of integers $N$, which appear as values $N = f(n)$ for $n \leq x$. This sort of control is provided by Lemma 6.1.

Additionally, one might like to remove $g$ from the above sum and perhaps also take advantage of the fact that (weighted) sums of $\nu_q(f(N))$ over $q$ are more readily controlled than values of $\nu_q(f(N))$ for some individual $q$. Lemma 6.2 makes this happen and is the source of the main term in Theorem 4.1 and Theorem 4.2.

Finally, Lemma 6.3 is a technical tool used to handle the case when $N = f(n)$ does not have sufficiently many prime factors $q$ with small exponent $\nu_q(N)$.

Lemma 6.1. For an $N \in \langle Q \rangle$, let $m_N$ be the least positive integer $m$ such that $f(m) = N > 1$. Then the following assertions hold:

1. The number $m_N$ factors as $m_N = q_1^{n_1} \cdots q_r^{n_r}$ with some $r \geq 1$, exponents $n_1 \geq \ldots \geq n_r$ and the primes $q_1, q_2, \ldots$, from Section 4.
2. If $N'$ divides $N$, then $m_{N'} \leq m_N$.
3. If $q_j > q_r^{1/s_k}$ for some $j \leq r$, then $\Omega(g(\nu_j)) \leq k$, where $s_k = c_nq_1^{1/s}$. 


Proof. Pick some $p \notin Q$ and let $\nu = \nu_p(m_N)$. Then $1 < N = f(m_N) = f(p^\nu)f(m_N/p^\nu)$, so that $m_N = m_N/p^\nu$. Hence, $\nu = 0$ and $p \mid m_N$. Now, writing $m_N = q_1^{\nu_1} \cdots q_r^{\nu_r}$, note that one can permute the exponents without changing the value under $f$. Therefore, by minimality of $m_N$, we must have $\nu_1 \geq \ldots \geq \nu_r$. This proves (1).

Turning to (2), if we write $m_N = q_1^{\nu_1} \cdots q_r^{\nu_r}$, then, by (6.2),
\begin{equation}
N = f(m_N) = \prod_{j \leq r} g(\nu_j),
\end{equation}
and since $N' \mid N$ there is a partition $\nu_k = \nu_{k,1} + \ldots + \nu_{k,r}$ such that $N_j' = \prod_{k \leq s} g_k^{\nu_{k,j}} | g(\nu_j)$. By Assumption (A.3) on $g$, the value $N_j'$ is attained by $g$. Hence, we may look at $m_* = q_1^{\nu'_1} \cdots q_r^{\nu'_r}$, where $\nu'_j = g(N_j')$. Clearly, $f(m_*) = N'$, and, by monotonicity of $g$, $\nu'_j \leq \nu_j$, so that $m_N' \leq m_* \leq m_N$.

To prove (3), let us assume for the sake of contradiction that $q_j > q_{r+1}^{1/k}$, $\Omega(g(\nu_j)) > k$. Using (6.1) and $N \in (Q)$, we have $\Omega_Q(g(\nu_j)) = \Omega(g(\nu_j)) > k$, so that there is a decomposition $g(\nu_j) = ab$, where $a \geq q_1, b \geq q_1^r$. Recalling the definition of $g^\dagger$ in (4.1) and using that both $a$ and $b$ are contained in $\langle Q \rangle$, we see that both $g^\dagger(a)$ and $g^\dagger(b)$ are finite and we may consider
\[
m_* = q_j^{g^\dagger(b)} q_{r+1}^{g^\dagger(a)} \prod_{i \neq j} q_i^{\nu_i}.
\]
Since (A.4) implies
\[
g^\dagger(b) - \nu_j \leq g^\dagger(b) - g^\dagger(ab) = g^\dagger(b) \left(1 - \frac{g^\dagger(ab)}{g^\dagger(b)}\right) \leq -c_4 g^\dagger(a)b,
\]
which, by assumption, is $\leq -c_4 q_1^k$, we infer
\[
f(m_*) = \frac{m_*}{m_N} = q_j^{g^\dagger(b) - \nu_j} q_{r+1}^{g^\dagger(a)} \leq q_j^{-c_4 g^\dagger(a)b} q_{r+1}^{g^\dagger(a)}.
\]
However, this shows that $m_* < m_N$, which contradicts the definition of $m_N$, for we have
\[
f(m_*) = g(g^\dagger(b))g(g^\dagger(a)) \prod_{i \neq j} g(\nu_i) = \prod_{i \leq r} g(\nu_i) = N.
\]
Hence, we conclude that $\Omega_Q(g(\nu_j)) \leq k$. \hfill \Box

**Lemma 6.2.** Let $\nu_1, \ldots, \nu_t$ be positive integers. Then
\begin{equation}
\sum_{j \leq t} \log g(\nu_j) \leq C_4 \left(\sum_{j \leq t} j\nu_j\right)^{1/2},
\end{equation}
where $C_4$ is given by (4.5). If additionally $\nu_t \geq \nu$, then
\[
\sum_{j \leq t} \log g(\nu_j) \ll \sqrt{\frac{1}{\nu^{2\epsilon}} + \frac{\log g(\nu)}{\nu} \left(\sum_{j \leq t} j\nu_j\right)^{1/2}},
\]
with $\epsilon$ from (A.5).
Proof. (Compare [4, Lemma 3.3].) First note that the right hand side of (6.2) is minimal if the \( \nu_j \)'s are decreasing. Hence, we may subsequently assume that \( \nu_1 \geq \nu_2 \geq \ldots \geq \nu_t \). Let \( y_i = \# \{ j : \nu_j \geq i \} \) and observe that

\[
\sum_{j \leq t} j \nu_j = \sum_{j \leq t} \sum_{i \leq \nu_j} j = \sum_{i=1}^{\infty} \sum_{j \leq y_i} j = \frac{1}{2} \sum_{i=1}^{\infty} y_i (y_i + 1) \geq \frac{1}{2} \sum_{i=1}^{\infty} y_i^2.
\]

By partial summation,

\[
\sum_{j \leq t} \log g(\nu_j) = \sum_{i=1}^{\infty} (y_i - y_{i+1}) \log g(i) = \sum_{i=1}^{\infty} y_i \log \left( \frac{g(i)}{g(i-1)} \right).
\]

The first claim now follows by applying the Cauchy–Schwarz inequality to the right hand side, and taking (6.3) into account.

Moreover, if \( \nu_1 \geq \nu \), then \( y_1 = y_2 = \ldots = y_\nu \) and

\[
\sum_{i \leq A} y_i \log \left( \frac{g(i)}{g(i-1)} \right) = y_1 \log g(\nu).
\]

By splitting up the sum in (6.4) into sums over the ranges \( i \leq \nu \) and \( i > \nu \), and applying the Cauchy–Schwarz inequality, we obtain

\[
\sum_{j \leq t} \log g(\nu_j) \leq \left( \sum_{i=1}^{\infty} y_i^2 \right)^{1/2} \left( \frac{(\log g(\nu))^2}{\nu} + \sum_{i > \nu} \left( \frac{\log g(i)}{g(i-1)} \right)^2 \right)^{1/2}.
\]

By (A.5) and \( \log (1 + 1/i) < 1/i \), the second sum is \( \ll \nu^{-2\varepsilon} \). In view of (6.3), we have established the second claim. \( \square \)

Lemma 6.3. For every \( \varepsilon > 0 \), and \( s := \omega_Q(n) \geq 2 \),

\[
f(n) \ll \left( \frac{(c_f + \varepsilon) \log n}{s \log s} \right)^s.
\]

Proof. See [4, Lemma 3.2] and, recalling that there \( g \) is \( x \mapsto x + 1 \), use (A.6) instead of \( x + 1 \leq 2x \). \( \square \)

7. Proof of Theorem 4.1

Let \( n \) be a positive integer such that \( f(f(n)) > 1 \) and \( N = \Pi_Q(f(n)) \). As before, \( f(f(n)) = f(N) \).

We now write \( N \) as a product of powers of elements in \( Q \) and split these into two groups according to the size of their exponents. More precisely, we write \( N = N'N'' \), where

\[
N' = u_1^{\nu_1} \ldots u_w^{\nu_w}, \quad N'' = v_1^{\nu_1} \ldots v_s^{\nu_s}
\]

and \( u_1 < \ldots < u_w, v_1 < \ldots < v_s \) all belong to \( Q \), are all distinct, and \( a_i \leq (\log_2 n)^K \) and \( b_i > (\log_2 n)^K \), for \( K = \max \{ 6, 2/\epsilon \} \), with \( \epsilon \) from (A.5).

Clearly, \( \log f(N) = \log f(N') + \log f(N'') \), so that it suffices to deal with \( f(N') \) and \( f(N'') \) separately. The main term in (4.4) comes from \( \log f(N'') \) (see (7.3)) and the term \( \log f(N') \) is seen to be somewhat smaller (see (7.1)).
7.1. Bounding \( f(N') \). Write \( m_{N'} = q_1^{\beta_1} \cdots q_h^{\beta_h} \). Due to Lemma 6.1 (2) we have \( m_{N'} \leq m_N \leq n \) and, hence, \( h \ll \log n \). Lemma 6.1 (3) yields \( \Omega(g(\beta_i)) \ll \log_2 h \ll \log_3 n \) for every \( i \). Therefore, there are \( \gg b_j/\log_3 n \) values of \( i \) such that \( u_j \mid g(\beta_i) \). Furthermore, assuming, as we may, that \( n \) is sufficiently large, Lemma 6.2 with \( \nu = \lfloor (\log_2 n)^K \rfloor \) shows that, for \( \epsilon' = K/2 - 2 \),

\[
\log f(N') = \sum_{j \leq w} \log g(b_j) \ll (\log_2 n)^{-\min\{\epsilon K, 2\}} \left( \sum_{j \leq w} j b_j \right)^{1/2}.
\]

Moreover,

\[
\frac{1}{\log_3 n} \sum_{j \leq w} j b_j \leq \sum_{j \leq w} \frac{u_j b_j}{\log_3 n} \ll \sum_{i \leq h} \sum_{p \mid g(\beta_i)} p \leq \sum_{i \leq h} \log_3 n \ll \log m_{N'} \leq \log n.
\]

Hence,

\[
(7.1) \quad \log f(N') \ll \sqrt{\log n} \frac{\log_3 n}{(\log_2 n)^2}.
\]

7.2. Bounding \( f(N'') \). To estimate \( f(N'') \) we may assume that

\[
(7.2) \quad s > \frac{\sqrt{\log n}}{(\log_2 n)^{K/2}},
\]

for otherwise Lemma 6.3 implies that

\[
\log f(N'') \ll \frac{\sqrt{\log n}}{(\log_2 n)^{K/2 - 1}}.
\]

We shall prove the following proposition that is crucial for estimating \( f(N') \); it relates \( m_{N''} \) with upper bounds as in Lemma 6.2.

**Proposition 7.1.** Let \( K = \max\{6, 2/\epsilon\} \), with \( \epsilon \) from (A.5). Suppose \( N'' = v_1^{a_1} \cdots v_s^{a_s} \) where \( u_1 < \ldots < u_w \), \( v_1 < \ldots < v_s \) all belong to \( Q \), are all distinct, and \( a_i \leq (\log_2 n)^K \), and \( s \) satisfies (7.2). Then,

\[
\log m_{N''} \geq \left( 1 + O\left( \frac{\log_3 n}{\log_2 n} \right) \right) c_1 K \left( \frac{\log_2 n}{4} \right)^2 \sum_{j \leq s} j a_j.
\]

Let us suppose for the moment that Proposition 7.1 is proved. We can conclude by Lemma 6.1 (2) that

\[
\log n \geq \log m_{N''} \geq \left( 1 + O\left( \frac{\log_3 n}{\log_2 n} \right) \right) c_1 K \left( \frac{\log_2 n}{4} \right)^2 \sum_{j \leq s} j a_j.
\]

Inequality (6.2) implies that

\[
(7.3) \quad \log f(N'') \leq \sqrt{\log n} \left( \frac{C_g}{\sqrt{\epsilon' \pi}} + O\left( \frac{\log_3 n}{\log_2 n} \right) \right),
\]

which concludes the proof of Theorem 4.1.
Proof of Proposition 7.1. Write \(m_{N''} = q_1^{a_1} \cdots q_r^{a_r}\) for the minimal element of \(f^{-1}(N'')\), as in Lemma 6.1. Our first goal is to establish that \(r\) cannot be too small. By Lemma 6.1, and letting \(s_0 = 1\) for the moment, the last sum in
\[
\Omega(N'') = \sum_{j \leq s} a_j = \sum_{i \leq r} \Omega(g(\alpha_i))
\]
is seen to be
\[
\sum_{k=1}^{\infty} k\left(\pi_Q(q_{r+1}^{1/s_k-1}) - \pi_Q(q_{r+1}^{1/s_k})\right) = r + 1 + \sum_{k=1}^{\infty} \pi_Q(q_{r+1}^{1/s_k}) =: r + E.
\]
To handle \(E\), we split the term for \(k = 1\) from the sum and estimate the rest trivially, thereby obtaining \(E \ll \pi(q_{r+1}^{1/s_1})\). Also, by (7.4), \(\Omega(N'') \geq r\), so that
\[
\Omega(N'') = r + \mathcal{O}(\pi(q_{r+1}^{1/c_s q_1})).
\]
Hence,
\[
r \leq \Omega(N'') \leq r + r^\theta,
\]
where \(\theta \in (1/c_s q_1, 1)\) is some constant (recall that by (A.4) this interval is non-empty). In particular, \(r \gg s\) so that by (7.2), \(r\) must be large if \(n\) is sufficiently large. The next goal is to determine \(g(\alpha_i)\) for all \(i\) in a suitable range. To this end, first note that by Lemma 6.1 (3) we find that \(g(\alpha_i)\) is prime for all \(i > r\). By (7.4), \(\Omega(N'') \geq r\), so that
\[
\Omega(N'') = r + \mathcal{O}(\pi(q_{r+1}^{1/c_s q_1})).
\]
Also, by (7.4),
\[
2r^\theta \leq 2(\Omega(N''))^\theta \leq 2\left(s(\log_2 n)^K\right)^\theta \leq s^{1-\varepsilon} \leq \sum_{s-s^{1-\varepsilon} < j < s} a_j,
\]
for \(n\) sufficiently large. Hence,
\[
\sum_{j \leq s-s^{1-\varepsilon}} a_j \leq \Omega(N'') - 2r^\theta \leq r - r^\theta.
\]
As explained above, \(g(\alpha_i)\) is prime for all \(i > r^\theta\) and from (7.7) we know that this surely is the case for all \(i \geq r - \sum_{k \leq j} a_k\), where \(j \leq s - s^{1-\varepsilon}\). Since, by Lemma 6.1 (1) the values \(g(\alpha_i)\) are decreasing as \(i\) increases, this yields that \(g(\alpha_i) = v_j\) for \(r - \sum_{k \leq j} a_k < i \leq r - \sum_{k < j} a_k\). By (7.5) and (7.6),
\[
r - \sum_{k \leq j} a_k = r - \Omega(N'') + \sum_{k \leq s - j} a_{j+k} \geq s - j - r^\theta \geq 1/2 s^{1-\varepsilon}.
\]
From (A.7) and (A.1) we deduce that
\[
g^\dagger(q_j) \geq c_l q_j + \mathcal{O}(q_j / \log q_j) \geq c_l k_j \log j
\]
for all sufficiently large \(j\). Hence, by (7.9) and (7.8),
\[
\log m_{N''} \geq c_l k \sum_{s^{1-\varepsilon} \leq j \leq s - s^{1-\varepsilon}} j(\log j) a_j (\log s + \mathcal{O}(\log_3 n)).
\]
By (7.2), we find that the right hand side above exceeds
\[
\sum_{s^{1-\varepsilon} \leq j \leq s-s^{1-\varepsilon}} (1 - \varepsilon)(\log s)^2 ja_j \left( 1 + \mathcal{O}\left( \frac{\log_3 n}{\log s} \right) \right) \\
\geq \sum_{s^{1-\varepsilon} \leq j \leq s-s^{1-\varepsilon}} (1 + \mathcal{O}(\varepsilon))(\log_2 n)^2 ja_j.
\]
By the choice of \( \varepsilon \), we get \( s^{\varepsilon} \gg \left( \log \frac{1}{\log x} \right)^{K-1} \). Also, \( \sum_{j \leq s} ja_j \geq \frac{1}{2}s^2 \). Now recalling that \( a_j \leq \left( \log \frac{1}{\log x} \right)^K \) for every \( j \), we infer that
\[
\sum_{s^{1-\varepsilon} \leq j \leq s-s^{1-\varepsilon}} ja_j = \sum_{j \leq s} ja_j + \mathcal{O}(s^{2-\varepsilon}(\log_2 n)^K) \\
= \left( 1 + \mathcal{O}\left( \frac{1}{\log_2 n} \right) \right) \sum_{j \leq s} ja_j,
\]
thus completing the proof. \( \square \)

8. Proof of Theorem 4.2

Recall that the main term in the upper bound in Theorem 4.1 stems from an application of Lemma 6.2. Given some large \( x > 1 \), we wish to find an integer \( n \) smaller than \( x \), such that
\[
\log f(f(n)) = \sum_{q \in Q, q|f(n)} \log g(\nu_q(f(n)))
\]
is large. The idea is to realise equality in Lemma 6.2. Therefore, recalling that the inequality was obtained by applying the Cauchy–Schwarz inequality to (6.4), we would like to have
\[
\#\{q \in Q : \nu_q(f(n)) \geq i\} \approx \text{const} \times \log \frac{g(i)}{g(i-1)} \quad (i \geq 1)
\]
with some constant, independent of \( i \). Furthermore, to have suitable control over \( f(n) \) it seems reasonable to choose \( n \) such that the factorisation of \( f(n) \) is known. With this in mind, let \( \varepsilon = c_0 \frac{\log x}{\log_2 x} \) for \( c_0 \) sufficiently large, where
\[
t = \left\lfloor \left( \frac{8 \log g(1)}{C_g^0} - \varepsilon \right) \sqrt{\log x} \log_2 x \right\rfloor,
\]
and consider
\[
\nu_j := \left[ 1 - \frac{1}{\alpha} + \frac{1}{(\alpha + 1)^{j/t} - 1} \right] \quad (1 \leq j \leq t).
\]
Evidently,
\[
(8.1) \quad \nu_j = \frac{1}{\log(\alpha + 1)^{j/t}} + \mathcal{O}(1)
\]
Letting
\[
n = \prod_{j \leq t} \prod_{i \leq \nu_j} \frac{g^i(q_j)}{q_{\nu_1 + \ldots + \nu_{j-1} + i}},
\]
we find that
\[
f(n) = \prod_{j \leq t} g(g^i(q_j))^{\nu_j} = \prod_{j \leq t} q_j^{\nu_j^i}.
\]
Now it remains to give a good lower bound on \( \log f(f(n)) \) and an upper bound on \( n \). To obtain the upper bound, let

\[
(8.2) \quad y_i = \# \{ j : \nu_j \geq i \} = \left\lfloor \frac{t}{\log(\alpha + 1)} \log \left( 1 + \frac{1}{i - 1 + \alpha^{-1}} \right) \right\rfloor.
\]

Observe that \( \nu_1 + \ldots + \nu_t \ll t \log t \). Using (A.1) we find that

\[
\log q_{\nu_1 + \ldots + \nu_t} \leq \log t + 2 \log_2 t + O(1).
\]

Hence,

\[
\log n \leq \sum_{j \leq t} \nu_j g_j \log q_{\nu_1 + \ldots + \nu_j} \leq \frac{\kappa}{\alpha} \left( (\log t)^2 + 3(\log_2 t) \log t + O(\log t) \right) \sum_j j \nu_j.
\]

Since \( y_i = O(t/i) \) and by (8.1) and (4.5),

\[
\sum_{j \leq t} j \nu_j = \frac{1}{2} \sum_{i \leq \nu_1} y_i (y_i + 1) = \frac{t^2}{2(\log(\alpha + 1))^2} \sum_{i=1}^\infty \left( \log \left( 1 + \frac{1}{i - 1 + \alpha^{-1}} \right) \right)^2 + O(t \log t)
\]

\[
= \frac{t^2 C_g^2}{16(\log(\alpha + 1))^2} + O(t \log t).
\]

By the definition of \( t \), \( \log t = \frac{1}{2} \log_2 x - \log_3 x + O(1) \) and \( \log_2 t = \log_3 x + O(1) \). By choosing \( c_\varepsilon \) sufficiently large, we get

\[
\left( 1 + O\left( \frac{\log_3 x}{\log_2 x} \right) \right) \left( 1 - \frac{C_g c_\varepsilon}{8 \log(\alpha + 1) \log_2 x} \right)^2 \leq 1.
\]

Thus, we infer

\[
\log n \leq \frac{\kappa}{\alpha} \left( 1 + \frac{\log_3 x}{\log_2 x} \right) \left( 1 - \frac{\varepsilon C_g}{8 \log(\alpha + 1)} \right)^2 \log x
\]

so that \( n \leq x^{\kappa/\alpha} \) if \( x \) is sufficiently large. Next, we estimate \( \log f(f(n)) \): Using partial summation and (8.2),

\[
\log f(f(n)) = \sum_{j \leq t} \log g_j = \sum_{i=1}^\infty (y_i - y_{i+1}) \log g(i) = \sum_{i=1}^\infty y_i \log \frac{g(i)}{g(i - 1)}.
\]
Due to the construction of $n$ the last sum simplifies to:
\[
\sum_{i \leq \nu_1} y_i \log \frac{g(i)}{g(i-1)} = \sum_{i \leq \nu_1} \left( \frac{t}{\log(\alpha + 1)} \left( \frac{\log g(i)}{g(i-1)} \right)^2 + O(1/i) \right)
\]
\[
= \frac{C_2^2}{8 \log(\alpha + 1)} t + O(\log t)
\]
\[
= \frac{\sqrt{\log x}}{\log_2 x} \left( C_g + O\left( \frac{\log_3 x}{\log_2 x} \right) \right).
\]
Since $M(x^{n/\alpha}) \geq \log f(f(n))$, we infer (4.6). This concludes the proof.

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