Limit transitions in plane homogenization problems for two-phase dielectric composites with extreme material properties of one phase

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Abstract. The plane problem of calculating the effective dielectric constant of two-phase composite consisting of main material with one circular inclusion is considered. To solve the homogenization problem, the method of effective moduli with the support of the energy balance between the composite and the homogeneous comparison medium was used. In the solution obtained, the passage to the limit was made for the case of an inclusion with zero dielectric constant and for the case of a conductive inclusion. The limiting solutions are compared with the solutions of homogenization problems for a medium with a hollow and for a medium with a conducting inclusion boundary.

1. Introduction
Composite materials are widely used to improve the performance characteristics of dielectric materials. Borderline types of two-phase (two-component) dielectric composites are porous dielectric material and dielectric material with conductive inclusions. In the first variant, it can be assumed that the inclusion has a negligible dielectric constant, and in the second variant, it can be assumed that the inclusion has a very large dielectric constant.

Piezoceramics are often used as the main dielectric material of the composite matrix. Then the border properties of the second phase are possessed by porous piezoceramics and piezoceramics with metal inclusions. These active piezocomposites are promising for many applications and have been studied in a large enough of works. For example, the effective properties of porous piezoceramic materials were studied by the methods of the theory of composites and by numerical methods in [1–8] and others. It was noted that porous piezoelectric ceramics, in comparison with dense ones, have a lower acoustic impedance and high piezoelectric sensitivity, which makes it effective for hydroacoustic applications, for use in medical ultrasound devices and in “green energy” piezoelectric generators. Dielectric/metal composites have been investigated in [9–15] etc. In such piezocomposites, the addition of metallic inclusions makes it possible to increase the strength properties of the composite and significantly increase its permittivities.

In this paper, we consider the classical two-dimensional problem of determining the effective permittivity of a dielectric composite in a circular region with a circular central inclusion. This statement corresponds to the three-dimensional problem of determining the effective transverse permeability for a unidirectional fiber composite. We apply the effective moduli method and use the energy criterion to find the effective permeability. The solution of this problem is well known, but we
focus on the analysis of limit transitions in energy relations and in inclusion moduli for the cases of a porous composite and a composite with a conductive inclusion. We compare these solutions with the corresponding solutions of homogenization problems, when only the main material of the composite is considered, and the extreme properties of inclusions are modeled only by boundary conditions.

2. Homogenization of a two-phase dielectric composite by the effective modulus method
Let $\Omega$ be a circular area of radius $b$. We introduce the Cartesian and cylindrical coordinate systems $Oxy$ and $Or\varphi$, where the point $O$ is the center of the circle $\Omega$. The vector of spatial coordinates $x$ in these coordinate systems can be represented as $x = xe_x + ye_y = re_r + \psi e_\varphi$, where $e_x$, $e_y$ are the unit vectors of the Cartesian coordinate system, and $e_r$, $e_\varphi$ are the unit vectors of the cylindrical coordinate system. We connect the coordinates and unit vectors of these systems by the formulas

$$
\cos \psi r = e_r, \quad \sin \psi r = e_\varphi,
$$

$$
\cos \psi r \varphi + \sin \psi r \varphi = e_r, \quad \sin \psi r \varphi - \cos \psi r \varphi = e_\varphi,
$$

moreover, using (1) one can also obtain the following relations

$$
e_r = \cos \psi e_x + \sin \psi e_y, \quad e_\varphi = -\sin \psi e_x + \cos \psi e_y.
$$

Figure 1. Composite medium (a) and homogeneous equivalent medium (b) with boundary condition.

Let as consider the plane homogenization problem for a two-phase dielectric composite in $\Omega$. We will assume that the composite consists of the main material filling the annular area $\Omega_m$, $a \leq r \leq b$, and of circular inclusion $\Omega_i$, $r \leq a$ (figure 1, a). We denote through $\varphi = \varphi(r, \psi)$ the function of the electric potential, and through $\varepsilon_i$ the dielectric constant. Generally speaking, the dielectric constant in each material can be different, but constant for each phase: $\varepsilon_i = \varepsilon_{m}, \ v \in \Omega_m; \varepsilon_i = \varepsilon_i, \ v \in \Omega_i$. We define the vector of the electric field intensity $E$ and the vector of electric induction (electric displacement) $D$ through the function of the electric potential by the formulas

$$
E = -\nabla \varphi = -\frac{\partial \varphi}{\partial r} e_r - \frac{1}{r} \frac{\partial \varphi}{\partial \varphi} e_\varphi, \quad D = \varepsilon_i E.
$$

The electric induction field in the dielectric must satisfy the quasielectrostatics equation $\nabla \cdot D = 0$, which in a cylindrical coordinate system can be written in the following form

$$
\frac{1}{r} \frac{\partial}{\partial r} (r D_r) + \frac{1}{r} \frac{\partial D_\varphi}{\partial \varphi} = 0.
$$
The essence of the homogenization problem is to determine the dielectric constant $\varepsilon$ of a homogeneous medium, which in a sense is equivalent to the original composite. In the method of effective moduli in the region $\Omega$ with piecewise homogeneous values of the dielectric constant, we solve the problem (4), (3) with the following boundary condition

$$\varphi = -{\bf x} \cdot {\bf e}_x E_0, \quad r = b,$$

and with the conditions of full contact on the interface between phases with different material properties

$$[\varphi] = 0, \quad [{\bf n} \cdot {\bf D}] = 0, \quad r = a,$$

where $[\bullet]$ is the jump of the corresponding value across the phase boundary and $n$ is the unit normal vector to the boundary.

Note that the direction of supply of the potential difference of the external electric field along the axis $x = x \cdot {\bf e}_x$ not fundamentally due to the isotropy of the medium, and it can be replaced by any other direction. Having found a solution of the problem (3)–(6), we can calculate the electric energy $U = (E \cdot D) / 2$, and also its average value

$$\langle U \rangle = \langle U \rangle_0 + \langle U \rangle_1,$$

where also highlighted the average values by area $\Omega_0$, which is filled by the matrix of the composite, and by area $\Omega_1$, which is filled by material of inclusions

$$\langle (\bullet) \rangle = \frac{1}{|\Omega|} \int_{\Omega} (\bullet) d\Omega, \quad \langle (\bullet) \rangle_0 = \frac{1}{|\Omega_0|} \int_{\Omega_0} (\bullet) d\Omega, \quad \langle (\bullet) \rangle_1 = \frac{1}{|\Omega_1|} \int_{\Omega_1} (\bullet) d\Omega. \quad (8)$$

We accept, that in a homogeneous comparison medium in the area $\Omega$ with the constant dielectric permittivity $\varepsilon_c = \varepsilon$ (figure 1, b), the Equations (3)–(5) are satisfied. The values related to this homogeneous medium will be marked with a subscript “0”. The solution of the problem (3)–(5) in homogeneous medium easily found and given by formulas: $\varphi_0 = -{\bf x} \cdot {\bf e}_x E_0 = -r \cos \psi E_0$, $E_0 = -\nabla \varphi_0 = E_0 e_x$, $D_0 = \varepsilon E_0 = \varepsilon E_0 e_x$, $U_0 = (E_0 \cdot D_0) / 2 = \varepsilon E_0^2 / 2$.

Then we can find the effective dielectric constant $\varepsilon$ from the equality of energies in the composite and in the homogeneous reference medium

$$\langle U \rangle = \langle U \rangle_0 = U_0. \quad (9)$$

Note that the following important equalities hold for problem (3)–(6)

$$\langle E \rangle = E_0, \quad \langle U \rangle = \langle E \cdot D \rangle / 2 = \langle E \rangle \cdot \langle D \rangle / 2 = E_0 \cdot D_0 / 2 = U_0, \quad (10)$$

moreover, these equalities are generalized to homogenization problems for composites of a more complex nature, for example, for piezoelectric composites [7, 16]. Using (10), one can obtain a simpler equation for determining the effective dielectric constant

$$\langle D \rangle = D_0 = \varepsilon E_0 e_x, \quad (11)$$

which leads to the same formula for $\varepsilon$ as (9).

So, the method of effective moduli consists in solving in the area $\Omega$ the boundary value problem (3)–(6) with inhomogeneous material properties and in the use of the energy relation (9) or (11) to determine the dielectric constant of an equivalent homogeneous medium. This method can be applied to areas $\Omega$ of various structures using numerical methods, for example, the finite element method, to solve the homogenization problem for complex representative volumes [3–7, 16, 17].
considered here circular area $\Omega$ with a circular inclusion, the solution is easily found analytically. Namely, the general solution of Equations (3), (4) can be present in the form

$$
\varphi = \begin{cases}
A_r \cos \psi, & r \leq a, \\
B_r + B_z \frac{1}{r} \sin \psi, & a \leq r \leq b.
\end{cases}
$$

(12)

Then, using (3), (12), we obtain an expression for the electric induction vector $\mathbf{D}$ in basis $\mathbf{e}_r$, $\mathbf{e}_\psi$:

$$
\mathbf{D} = \begin{cases}
-\varepsilon_0 A_r (\cos \psi \mathbf{e}_r - \sin \psi \mathbf{e}_\psi), & r \leq a \\
-\varepsilon_m \left( B_r - B_z \frac{1}{r^2} \cos \psi \right) \mathbf{e}_r - \left( B_z + B_z \frac{1}{r^2} \sin \psi \right) \mathbf{e}_\psi, & a \leq r \leq b.
\end{cases}
$$

(13)

Taking into account (2), the vector $\mathbf{D}$ from (13) can also be represented in the basis $\mathbf{e}_x$, $\mathbf{e}_y$ of the Cartesian coordinate system:

$$
\mathbf{D} = \begin{cases}
-\varepsilon_0 A_r \mathbf{e}_x, & r \leq a \\
-\varepsilon_m \left( B_r - B_z \frac{1}{r^2} \cos \psi \right) \mathbf{e}_x - \left( B_z + B_z \frac{1}{r^2} \sin \psi \right) \mathbf{e}_y, & a \leq r \leq b.
\end{cases}
$$

(14)

The unknown constants $A_1, B_1, B_2$, are determined from the contact conditions on the phase interface (6) and the boundary condition (5), i.e. from relations

$$
\varphi|_{r=a} = \varphi|_{r=b}, \quad D_r|_{r=a} = D_r|_{r=b}, \quad \varphi|_{r=b} = -E_0 b \cos \psi.
$$

(15)

Solving the system of three equations (15) with three unknown constants, we obtain

$$
A_1 = \frac{2E_0}{\Delta}, \quad B_1 = \frac{(1 + \tilde{\varepsilon}_i)E_0}{\Delta}, \quad B_2 = \frac{(1 + \tilde{\varepsilon}_i)pb^2E_0}{\Delta}, \quad \Delta = 1 + p + \tilde{\varepsilon}_i(1 - p),
$$

(16)

where $\tilde{\varepsilon}_i = \varepsilon_i / \varepsilon_m$ is the relative value of the inclusion dielectric constant, and $p = a^2 / b^2$ is the spatial fraction of inclusion in the area $\Omega$ of the composite.

Using the obtained solution (12), (16) and (13), (14), we find from (3), (7), (8), (10) the average values of the electric field intensity, electric induction and energy for individual phases and for the whole composite

$$
\langle \mathbf{E} \rangle = \frac{2pE_0}{\Delta} \mathbf{e}_x, \quad \langle \mathbf{E} \rangle_m = \frac{(1 + \tilde{\varepsilon}_i)(1 - p)E_0}{\Delta} \mathbf{e}_x, \quad \langle \mathbf{E} \rangle = E_0 \mathbf{e}_x
$$

(17)

$$
\langle \mathbf{D} \rangle_i = \frac{2\tilde{\varepsilon}_i p e_m E_0}{\Delta^2} \mathbf{e}_x, \quad \langle \mathbf{D} \rangle_m = \frac{(1 + \tilde{\varepsilon}_i)(1 - p)e_m E_0}{\Delta^2} \mathbf{e}_x, \quad \langle \mathbf{D} \rangle = \frac{1 - p + \tilde{\varepsilon}_i(1 + p)}{1 + p + \tilde{\varepsilon}_i(1 - p)} e_m E_0 \mathbf{e}_x,
$$

(18)

$$
\langle \mathbf{U} \rangle_i = \frac{2\tilde{\varepsilon}_i p e_m E_0^2}{\Delta^2}, \quad \langle \mathbf{U} \rangle_m = \frac{[(1 + \tilde{\varepsilon}_i)^2 + p(1 - \tilde{\varepsilon}_i)^2](1 - p)e_m E_0^2}{2\Delta^2}, \quad \langle \mathbf{U} \rangle = \frac{1 - p + \tilde{\varepsilon}_i(1 + p)}{2[1 + p + \tilde{\varepsilon}_i(1 - p)]} e_m E_0^2.
$$

(19)

Now the last formula (18) and (11) or (19) and (9) allow us to obtain the final value of the relative effective dielectric constant $\tilde{\varepsilon} = \varepsilon / \varepsilon_m$:

$$
\tilde{\varepsilon} = \frac{1 - p + \tilde{\varepsilon}_i(1 + p)}{1 + p + \tilde{\varepsilon}_i(1 - p)}.
$$

(20)
The resulting formula (20) for the effective dielectric constant of the fiber composite is well known and coincides with the formula obtained by the self-consistent method [18] and other methods [19]. Our investigation differs only in the segregation of the averaged electric fields and energies for individual phases and in the analysis of the solutions obtained below for the extreme properties of the inclusion.

3. Limit cases for inclusions

3.1. Pore

If the inclusion is the pore, then such a composite can be modeled as a two-phase composite with a negligible value of the dielectric constant of the second phase. The solution of the problem and other characteristics of the solution for a porous composite can be obtained from formulas (16)–(20) with the passage to the limit \( \varepsilon_i \to 0 \).

\[
A_i = \frac{2E_0}{1 + p}, \quad B_i = -\frac{E_0}{1 + p}, \quad B_z = -\frac{p b^2 E_0}{1 + p}, \quad (21)
\]

\[
\langle E \rangle_i = \frac{2pE_0}{1 + p} e_x, \quad \langle E \rangle_m = \frac{(1-p)E_0}{1 + p} e_x, \quad \langle E \rangle = E_0 e_x, \quad (22)
\]

\[
\langle D \rangle_i = 0, \quad \langle D \rangle_m = \frac{(1-p)e_m E_0^2}{1 + p} e_x, \quad \langle D \rangle = \frac{(1-p)e_m E_0^2}{1 + p} e_x, \quad (23)
\]

\[
\langle U \rangle_i = 0, \quad \langle U \rangle_m = \frac{(1-p)e_m E_0^2}{2(1 + p)}, \quad \langle U \rangle = \frac{(1-p)e_m E_0^2}{2(1 + p)}, \quad (24)
\]

\[
\bar{\varepsilon} = \frac{1-p}{1 + p}. \quad (25)
\]

As can be seen from (22), (23), despite the absence of material in the pore, the electric field there is nonzero \( \langle E \rangle_i \neq 0 \), but the electric induction is zero: \( \langle D \rangle_i = 0 \). Excluding the electric field in the pore \( \langle E \rangle_i \), the average electric field in the base material \( \langle E \rangle_m \) not equal \( E_0 \), that is, the first of equalities (10) is not satisfied. However, given the contribution \( \langle E \rangle_i \) into the full mean field \( \langle E \rangle \) this equality holds.

Because \( \langle D \rangle_i = 0, \langle U \rangle_i = 0 \), the effective modulus \( \varepsilon \) is determined only by the electric fields in the base material. In addition, since for a porous composite \( \langle D \rangle_m = \langle D \rangle, \langle U \rangle_m = \langle U \rangle \), then in formulas (9) and (11) to determine the effective permittivity, one can replace the averaged values \( \langle U \rangle \) and \( \langle D \rangle \) across the entire area by integral characteristics \( \langle U \rangle_m \) и \( \langle D \rangle_m \) across the area of the main material. Note also that, as follows from (13), (21), on the pore boundary \( r = a \), the normal component of the electric induction vector \( D \) vanishes.

3.2. Conductive inclusion

If the inclusion is conductive, for example, metallic, then such a composite can be modeled as a two-phase composite with a very large value of dielectric constant for the second phase. The solution of the problem and other characteristics of the solution for a composite with an absolutely conducting inclusion can be obtained from formulas (16)–(20) with the passage to the limit \( \bar{\varepsilon} \to \infty \). This passage to the limit gives the following formulas
\[ A_i = 0, \quad B_1 = -\frac{E_0}{1 - p}, \quad B_2 = \frac{p b^2 E_0}{1 - p}, \]

\[ \langle E \rangle_i = 0, \quad \langle E \rangle_m = E_0 e_z, \quad \langle E \rangle = E_0 e_z, \]

\[ \langle D \rangle_i = \frac{2 p e_m E_0}{(1 - p)} e_z, \quad \langle D \rangle_m = e_m E_0 e_z, \quad \langle D \rangle = \frac{(1 + p)}{(1 - p)} e_m E_0 e_z, \]

\[ \langle U \rangle_i = 0, \quad \langle U \rangle_m = \frac{(1 + p)}{2(1 - p)} e_m E_0^2, \quad \langle U \rangle = \frac{(1 + p)}{2(1 - p)} e_m E_0^2, \]

\[ \tilde{\varepsilon} = \frac{1 + p}{1 - p}. \]

From (26)–(29) it can be seen that although in the conductive inclusion the electric potential, the electric field intensity and energy are negligible, but the electric induction in \( \Omega_i \) turns out to be not at all a small value. Now, \( \langle E \rangle_m = \langle E \rangle = E_0 e_z \), \( \langle U \rangle_m = \langle U \rangle \), but \( \langle D \rangle_m \neq \langle D \rangle \). Therefore, in formula (9) to determine the effective dielectric constant, it is possible to replace the averaged value \( \langle U \rangle \) over the entire area with the integral energy value \( \langle U \rangle_m \) over the area of the base material. However, in formula (11), when determining the effective dielectric constant, it is impossible to replace the average electric induction \( \langle D \rangle \) over the entire area with an integral value \( \langle D \rangle_m \) only in the area of the dielectric matrix.

Note also that in this case, as follows from (12), (26), on the pore boundary \( r = a \) the electric potential \( \varphi \) vanishes, as in the entire area \( \Omega_i \). Meanwhile, the normal component of the electric induction vector \( D_n \) does not vanish, but the integral value \( D_n \) along the interface is equal to zero

\[ D_n = \frac{2 e_m E_0}{1 - p} \cos \varphi, \quad \int_0^{2\pi} D_n d\varphi = 0, \quad r = a. \]

The most interesting phenomenon for this case of the composite is the nonzero value of the electrical induction \( D \) inside the conductive inclusion \( \Omega_i \) with a zero vector of the electric field \( E \), as well as the fact that the total value \( \langle D \rangle_i \) is important for the correct calculation of the effective dielectric constant.

4. Homogenization problems without considering the extreme material moduli of inclusions

In this section, we will consider the same homogenization problems in a circular domain \( \Omega \) as in the previous section, but without equations inside the inclusion. Thus, we will consider mathematical models only in the area \( \Omega_m \) with the material of the composite matrix, and the inclusion properties will be taken into account only on the phase boundary \( \Gamma_i = \partial \Omega_i = \{ r = a, \ 0 \leq \varphi \leq 2\pi \} \).

4.1. Dielectric material with a pore

Consider a porous material with a dielectric constant \( \varepsilon_r = \varepsilon_m \), in which in the annular area \( \Omega_m \), \( a \leq r \leq b \), Eqs. (3), (4) are satisfied. We assume the boundary conditions on the outer border \( r = b \) in the form (5), and on the pore boundary, we set the condition of a free non-electroded boundary

\[ \mathbf{n} \cdot \mathbf{D} = D_i = 0, \quad r = a. \]
Since now the composite occupying the area $\Omega_m$ is compared with the homogeneous medium occupying the area $\Omega$, then to determine the effective dielectric constant $\varepsilon$, instead of (9) it is necessary to use the condition

$$\langle U_m \rangle = U_0, \quad U_0 = \varepsilon E_0^2/2.$$  \hspace{1cm} (33)

We find the solution of the problem (3)–(5), (32), (33) in the area $\Omega_m$ in the form (12), naturally, only for $a \leq r \leq b$. It is easy to check that the result of solving this problem will be formulas (21)–(25), relating only to the area $\Omega_m$

$$\varphi = \left( B_1 r + B_2 \frac{1}{r} \right) \sin \psi, \quad B_1 = -\frac{E_0}{1 + p}, \quad B_2 = -\frac{p b^2 E_0}{1 + p}, \hspace{1cm} (34)$$

$$\langle E \rangle_m = \left( 1 - p \right) \frac{E_0}{1 + p} \mathbf{e}_x, \quad \langle D \rangle_m = \left( 1 - p \right) \varepsilon_m E_0 \mathbf{e}_x, \quad \langle U \rangle_m = \frac{(1 - p) \varepsilon_m E_0^2}{2(1 + p)}, \quad \bar{\varepsilon} = \frac{1 - p}{1 + p}. \hspace{1cm} (35)$$

Thus, the solution of the homogenization problem in $\Omega$ for a poroule dielectric material, obtained as a result of the limit passage when the dielectric constant tends to zero, coincides with the solution of the homogenization problem in $\Omega_m$ for the material of the composite matrix. Although these two problems at first glance have different conditions on the interface, condition (6) gives (32) upon limit passing, which explains the coincidence of the solutions of both problems. We only note that when calculating the integral values for $\langle E \rangle_m, \langle D \rangle_m, \langle U \rangle_m$ in the homogenization problem in $\Omega_m$, it is still necessary to use the second formula (8) with the assignment of these integral values to the area of the entire area $\Omega$, including the pore. This is quite natural, since we compare the properties of the porous material in the matrix $\Omega_m$ with the properties of an effective medium filling the entire area $\Omega$.

To determine the effective dielectric constant, here it was possible to use also the equality

$$\langle D \rangle_m = D_0 = \varepsilon E_0 \mathbf{e}_x,$$

since for a porous composite $\langle D_m \rangle = \langle D \rangle$.

4.2. Dielectric material with conductive inclusion

In the case of an absolutely conducting inclusion, we, generally speaking, cannot use the equations of quasi-electrostatics inside the inclusion material. Therefore, in this case, we use Eqs. (3), (4) only for the annular area $\Omega_a$, $a \leq r \leq b$, $\varepsilon_r = \varepsilon_m$. The boundary conditions on the outer boundary of the area $r = b$ are again taken in the form (5), and on the boundary of a conducting inclusion, we set the condition of a free electroded boundary, which is usually used for metallized surfaces of piezoelectric media [20, 21]

$$\varphi = \Phi, \quad \int_{\Gamma_r} \mathbf{n} \cdot \mathbf{D} \, d\Gamma = 0, \quad r = a,$$

(37)

where $\Phi$ is an arbitrary constant determined from the second condition (33), which expresses the equality to zero of the total electric charge on $\Gamma_r$.

Looking for a solution to the problem (3)–(5), (37) in the domain $\Omega_m$ in the form (12) and determining the effective dielectric constant from the energy Equation (33), we obtain formulas (26)–(30) related to the domain $\Omega_m$

$$\varphi = \left( B_1 r + B_2 \frac{1}{r} \right) \sin \psi, \quad B_1 = -\frac{E_0}{1 + p}, \quad B_2 = -\frac{p b^2 E_0}{1 + p}, \hspace{1cm} (38)$$
\[ \langle E \rangle_m = E_0 e_x, \quad \langle D \rangle_m = \varepsilon_0 E_0 e_x, \quad \langle U \rangle_m = \frac{(1 + p)}{2(1 - p)} \varepsilon_0 E_0^2, \quad \varepsilon = \frac{1 + p}{1 - p}. \]  

Thus, the solution of the homogenization problem (3)–(6), (9) in \( \Omega \) for a composite dielectric material with an inclusion, obtained as a result of the passage to the limit when the dielectric constant of the inclusion tends to infinity, coincides with the solution of the homogenization problem (3)–(5), (37), (33) in \( \Omega_m \) for the material of the composite matrix.

Note that for such a composite we cannot use condition (36) when determining the effective permittivity, since \( \langle D \rangle_m \neq \langle D \rangle \). It is correct and physically justified here to compare the composite and the effective medium in terms of the energy ratio (33).

5. Conclusion

So, our investigation shows that to determine the effective moduli of dielectric composites with extremal properties, the limit transitions can be used in the solutions for conventional two-phase composites. If, when modeling composites with pores or with conducting inclusions, we restrict ourselves only to the regions occupied by the main material, then energetically justified results are given by problems with free pore boundaries and with equipotential boundaries of conducting inclusions with zero total charges. In this case, to calculate the effective moduli, it is necessary to use the energy criterion, whereas the averaging of the flux quantities can lead to incorrect results.

Note that the considered problem of determining the effective dielectric modulus with accuracy for notation coincides with the problem of determining the effective modulus of thermal conductivity by virtue of the analogy between the quasielectrostatic problem and the stationary thermal problem. Therefore, all the results presented here for dielectric composites will also be valid for similar thermal conducting composite materials. These results are also generalized to more complex types of elastic composites with pores or with rigid inclusions, as well as piezoelectric composites with extreme properties of the second phase moduli.

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References

[1] Dunn H and Taya M 1993 Electromechanical properties of porous piezoelectric ceramics J. of the American Ceramic Society 76 1697–706
[2] Dunn H and Taya M 1993 Micromechanics predictions of the effective electroelastic moduli of piezoelectric composites Int. J. of Solids and Structures 30(2) 161–75
[3] Iyer S and Venkatesh T A 2010 Electromechanical response of porous piezoelectric materials: effects of porosity connectivity Applied Physics Letters 97 072904
[4] Iyer S and Venkatesh T A 2011 Electromechanical response of (3–0) porous piezoelectric materials: effects of porosity shape Applied Physics Letters 110 034109
[5] Kudimova A B, Nadolin D K, Nasedkin A V, Nasedkina A A, Oganysan P A and Soloviev A N 2018 Models of porous piezocomposites with 3-3 connectivity type in ACELAN finite element package Materials Physics and Mechanics 37(1) 16–24
[6] Martinez-Ayuso G, Friswell M I, Adhikari S, Khodaparast H H and Berger H 2017 Homogenization of porous piezoelectric materials Int. J. of Solids and Structures 113–114 218–29
[7] Nasedkin A V and Shevtsova M S 2013 Multiscale computer simulation of piezoelectric devices with elements from porous piezocermics Physics and Mechanics of New Materials and their Applications eds. I A Parinov and S-H Chang (New York: Nova Science Publ.) chapter 16 pp 185–202
[8] Rybyanets A N 2011 Porous piezocermics: theory, technology, and properties *IEEE Transactions on Ultrasonics, Ferroelectrics, and Frequency Control* **58** 1492–507

[9] Bottero C J and Idiart M I 2016 Influence of second-phase inclusions on the electro-deformation of ferroelectric ceramics *Int. J. of Solids and Structures* **80** 381–92

[10] Du H, Lin X, Zheng H, Qu B, Huang Y and Chu D 2016 Colossal permittivity in percolative ceramic/metal dielectric composites *J. of Alloys and Compounds* **663** 848–61

[11] Duan N, Ten Elshof J E, Verweij H, Greuel G and Dannapple O 2000 Enhancement of dielectric and ferroelectric properties by addition of Pt particles to a lead zirconate titanate matrix *Applied Physics Letters* **77**(20) 3263

[12] Li J-F, Takagi K, Terakubo N and Watanabe R 2001 Electrical and mechanical properties of piezoelectric ceramic/metal composites in the Pb(Zr,Ti)O$_3$/Pt system *Applied Physics Letters* **79**(15) 2441

[13] Roscow J I, Bowen C R and Almond D P 2017 Breakdown in the case for materials with giant permittivity? *ACS Energy Letters* **2** 2264–9

[14] Takagi K, Li J-F, Yokoyama S and Watanabe R 2003 Fabrication and evaluation of PZT/Pt piezoelectric composites and functionally graded actuators *J. of the European Ceramic Society* **23** 1577–83

[15] Wang J and Li W 2019 A new piezoelectric hollow cylindrical transducer with multiple concentric annular metal fillers *Materials Research Express* **6** 055701

[16] Nasedkin A and Nassar M E 2020 Effective properties of a porous inhomogeneously polarized by direction piezoceramic material with full metalized pore boundaries: finite element analysis *J. of Advanced Dielectrics* **10**(5) 2050018

[17] Kudimova A B and Nasedkin A V 2019 Analysis of porosity influence on the effective moduli of ceramic matrix PZT composite using the simplified finite element model *J. of Advanced Dielectrics* **9**(6) 1950043

[18] Zarubin V S, Kuvyrkin G N, Savel’eva I Yu 2013 Thermal conductivity of composite reinforced with fibers. *Izv. Vyssh. Uchebn. Zaved., Mashinostr.* [Proc. Univ., Mech. Eng.] **5** 75–81

[19] Milton G W 2002 *The Theory of Composites* (Cambridge: Cambridge University Press)

[20] Nasedkin A V and Nasedkina A A 2015 *Finite Element Modeling of Coupled Problems* (Rostov-on-Don: Southern Federal University Publ.)

[21] Parton V Z and Kudryavtsev B A 1988 *Electromagnetoelasticity: Piezoelectrics and electrically conductive solids* (New York: Gordon and Breach Science Publ.)