UNIQUENESS FOR THE SIGNATURE OF A PATH OF BOUNDED VARIATION AND THE REDUCED PATH GROUP

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ABSTRACT. We introduce the notions of tree-like path and tree-like equivalence between paths and prove that the latter is an equivalence relation for paths of finite length. We show that the equivalence classes form a group with some similarity to a free group, and that in each class there is one special tree reduced path. The set of these paths is the Reduced Path Group. It is a continuous analogue to the group of reduced words. The signature of the path is a power series whose coefficients are definite iterated integrals of the path. We identify the paths with trivial signature as the tree-like paths, and prove that two paths are in tree-like equivalence if and only if they have the same signature. In this way, we extend Chen’s theorems on the uniqueness of the sequence of iterated integrals associated with a piecewise regular path to finite length paths and identify the appropriate extended meaning for reparameterisation in the general setting. It is suggestive to think of this result as a non-commutative analogue of the result that integrable functions on the circle are determined, up to Lebesgue null sets, by their Fourier coefficients. As a second theme we give quantitative versions of Chen’s theorem in the case of lattice paths and paths with continuous derivative, and as a corollary derive results on the triviality of exponential products in the tensor algebra.

1. Introduction

1.1. Paths with finite length. Paths, that is to say (right) continuous functions \( \gamma \) mapping a non-empty interval \( J \subset \mathbb{R} \) into a topological space \( V \), are fundamental objects in many areas of mathematics, and capture the concept of an ordered evolution of events.

If \((V, d_V)\) is a metric space, then one of \( \gamma \)'s most basic properties is its length \( |\gamma|_J \). This can be defined as

\[
|\gamma|_J := \sup_{D \subset J} \sum_{t_i \in D, i \neq 0} d_V(\gamma_{t_{i-1}}, \gamma_{t_i})
\]

where the supremum is taken over all finite partitions \( D = \{t_0 < t_1 < \cdots < t_r\} \) of the interval \( J \). It is clear that \( |\gamma| \) is positive (although possibly infinite) and independent of the parameterisation for \( \gamma \). Letting \( \tau(t) = |\gamma|_{[0,t]} \) and setting \( \eta(\tau(t)) = \gamma(t) \) one sees that any continuous path of finite length can always be parameterised to
have unit speed. Paths of finite length are often said to be those of bounded or finite variation.

**Definition 1.1.** We denote the set of paths of bounded variation by $BV$, $BV$-paths with values in $V$ by $BV(V)$, and those defined on $J$ by $BV(J,V)$.

If $V$ is a vector space, then for any $\gamma \in BV([0,t],V)$ and $\tau \in BV([0,s],V)$ we can form the concatenation $\gamma * \tau \in BV([0,s+t],V)$

$$\gamma * \tau (u) = \gamma (u), \quad u \in [0,s]$$

$$\gamma * \tau (u) = \tau (u-s) + \gamma (s) - \tau (0), \quad u \in [s,s+t].$$

The operation $*$ is associative, and if $V$ is a normed space, then $|\gamma| + |\tau| = |\gamma * \tau|$.

**1.2. Differential Equations.** One reason for looking at $BV(V)$ is that one can do calculus with these paths, while at the same time the set of paths with $|\gamma|_J \leq l$ is closed under the topology of pointwise convergence (uniform convergence, ...). Differential equations allow one to express relationship between paths in $BV$. If $f_t$ are Lipschitz vector fields on a space $W$ and $\gamma_t = (\gamma_1(t), \ldots, \gamma_d(t)) \in BV(\mathbb{R}^d)$ then the differential equation

$$\frac{dy}{dt} = \sum_i f_i \frac{d\gamma_i}{dt} = f(y) \cdot d\gamma$$

has a unique solution for each $\gamma$. $BV$ is a natural class here for, unless the vector fields commute, there is no meaningful way to make sense of this equation if the path $\gamma$ is only assumed to be continuous.

If $(y, \gamma)$ solves the differential equation and $(\tilde{y}, \tilde{\gamma})$ are simultaneous reparametrisations, then they also solve the equation and so it is customary to drop the $dt$ and write

$$dy = \sum_i f_i \tilde{\gamma}_i dt = \sum_i f_i d\gamma_i = f(y) \cdot d\gamma.$$  

We can regard the location of $y_s$ as a variable and consider the diffeomorphism $\pi_{st}$ defined by $\pi_{st}(y_s) := y_t$. Then $\pi_{st}$ is a function of $\gamma|_{[s,t]}$. One observes that the map $\gamma|_{[s,t]} \rightarrow \pi_{st}$ is a homomorphism from $(BV(V),*)$ to the group of diffeomorphisms of the space $W$.

**1.3. Iterated integrals and the signature of a path.** One could ask which are the key features of $\gamma|_{[s,t]}$, which, with $y_s$, accurately predict the value $y_t$ in equation (1.1). The answer to this question can be found in a map from $BV$ into the free tensor algebra!

**Definition 1.2.** Let $\gamma$ be a path of bounded variation on $[S,T]$ with values in a vector space $V$. Then its signature is the sequence of definite iterated integrals

$$X_{S,T} = (1 + X_{S,T}^1 + \ldots + X_{S,T}^k + \ldots)$$

$$= \left(1 + \int_{S<u<T} d\gamma_u + \ldots + \int_{S<u_1<\ldots<u_k<T} d\gamma_{u_1} \otimes \ldots \otimes d\gamma_{u_k} + \ldots\right)$$

regarded as an element of an appropriate closure of the tensor algebra $T(V) = \bigoplus_{n=0}^\infty V^\otimes n$. 

The signature is the definite integral over the fixed interval where \( \gamma \) is defined; re-parameterising \( \gamma \) does not change its signature. The first term \( X_1^{[S,T]} \) produces the path \( \gamma \) (up to an additive constant). For convenience of notation, when we have many paths, we will sometimes use a symbol such as \( Y_t \) (instead of \( \gamma_t \)) for our path, \( Y_{S,T}^i \) for the \( i \)-th coordinate of the signature of \( Y_t \), and \( Y_{S,T} \) for the signature of the path. In some circumstances we will drop the time interval and just write \( Y \) for the path and \( Y \) for its signature. We call this map the signature map and sometimes denote it by \( S : X \rightarrow S(X) \) when this helps our presentation.

The signature of \( X \) is a natural object to study. The map \( X \rightarrow X \) is a homomorphism (c.f. Chen’s identity [7]) from the monoid of paths with concatenation to (a group embedded in) the algebra \( T(V) \). The signature \( X_0^{[S,T]} = X_{0,T} \) can be computed by solving the differential equation

\[
dX_0 \cdot u = X_0 \cdot u \otimes dX_u
\]

and, in particular, paths with different signatures will have different effects for some choice of differential equation.

There is a converse, although this is a consequence of our main theorem. If \( X \) controls a system through a differential equation

\[
dY_u = f(Y_u) dX_u,
\]

and \( f \) is Lipschitz, then the state \( Y_T \) of the system after the application of \( X|_{[0,T]} \) is completely determined by the signature \( X_{0,T} \) and \( Y_0 \). In other words the signature \( X_{S,T} \) is a truly fundamental representation for the bounded variation path defined on \([S,T]\) that captures its effect on any non-linear system.

This paper explores the relationship between a path and its signature. We determine a precise geometric relation \( \sim \) on bounded variation paths, we prove that two paths of finite length are \( \sim \)-equivalent if and only if they have the same signature:

\[
X|_J \sim Y|_K \iff X_J = Y_K
\]

and hence prove that \( \sim \) is an equivalence relation and identify the sense in which the signature of a path determines the path.

The first detailed studies of the iterated integrals of paths are due to K. T. Chen. In fact Chen [2] proves the following theorems which are clear precursors to our own results:

**Chen Theorem 1:** Let \( d\gamma_1, \cdots, d\gamma_d \) be the canonical 1-forms on \( \mathbb{R}^d \). If \( \alpha, \beta \in [a, b] \rightarrow \mathbb{R}^d \) are irreducible piecewise regular continuous paths, then the iterated integrals of the vector valued paths \( \int_{\alpha(0)}^{\alpha(t)} d\gamma \) and \( \int_{\beta(0)}^{\beta(t)} d\gamma \) agree if and only if there exists a translation \( T \) of \( \mathbb{R}^d \), and a continuous increasing change of parameter \( \lambda : [a, b] \rightarrow [a, b] \) such that \( \alpha = T\beta \lambda \).

**Chen Theorem 2:** Let \( G \) be a Lie group of dimension \( d \), and let \( \omega_1 \cdots \omega_d \) be a basis for the left invariant 1-forms on \( G \). If \( \alpha, \beta \in [a, b] \rightarrow G \) are irreducible piecewise regular continuous paths, then the iterated integrals of the vector valued paths \( \int_{\alpha(0)}^{\alpha(t)} d\omega \) and \( \int_{\beta(0)}^{\beta(t)} d\omega \) agree if and only if there exists a translation \( T \) of \( G \), and a continuous increasing change of parameter \( \lambda : [a, b] \rightarrow [a, b] \) such that \( \alpha = T\beta \lambda \).
In particular, Chen characterised piecewise regular paths in terms of their signatures.

1.4. The main results. There are two essentially independent goals in this paper.

(1) To provide quantitative versions of some of Chen’s results. If \( \gamma \) is continuous, of bounded variation and parameterised at unit speed, then we will obtain lower bounds on the coefficients in the signature in terms of the modulus of continuity of \( \dot{\gamma} \) and the length of the path. For example Theorem 5 shows how one can recover the length of a path \( \gamma \) using the asymptotic magnitudes of these coefficients (c.f. Tauberian theorems in Fourier Analysis). A detailed discussion is to be found in Sections 2 and 3.

(2) To prove a uniqueness theorem characterising paths of bounded variation in terms of their signatures (c.f. the characterisation of integrable functions in terms of their Fourier series) extending Chen’s theorem to the bounded variation setting.

For this second goal we need a notion of tree-like path, our definition codes \( R \)-trees by positive continuous functions on the line, as developed, for instance, in [5].

Definition 1.3. \( X_t, \ t \in [0,T] \) is a tree-like path in \( V \) if there exists a positive real valued continuous function \( h \) defined on \( [0,T] \) such that \( h(0) = h(T) = 0 \) and such that

\[
\| X_t - X_s \|_V \leq h(s) + h(t) - 2 \inf_{u \in [s,t]} h(u).
\]

The function \( h \) will be called a height function for \( X \). We say \( X \) is a Lipschitz tree-like path if \( h \) can be chosen to be of bounded variation.

Definition 1.4. Let \( X, Y \in BV(V) \). We say \( X \sim Y \) if the concatenation of \( X \) and \( Y \) ‘run backwards’ is a Lipschitz tree-like path.

We now focus on \( \mathbb{R}^d \) and state our main results.

Theorem 1. Let \( X \in BV(\mathbb{R}^d) \). The path \( X \) is tree-like if and only if the signature of \( X \) is \( 0 = (1,0,0,\ldots) \).

As the map \( X \to X \) is a homomorphism, and running a path backwards gives the inverse for the signature in \( T(V) \), an immediate consequence of Theorem 1 is

Corollary 1.5. If \( X, Y \in BV(\mathbb{R}^d) \), then \( X = Y \) if and only if the concatenation of \( X \) and \( Y \) run backwards’ is a Lipschitz tree-like path.

Corollary 1.6. For \( X, Y \in BV(\mathbb{R}^d) \) the relation \( X \sim Y \) is an equivalence relation. Concatenation respects \( \sim \) and the equivalence classes \( \Sigma \) form a group under this operation.

There is an analogy between the space of paths of finite length in \( \mathbb{R}^d \) and the space of words \( a^{\pm 1}b^{\pm 1} \cdots c^{\pm 1} \) where the letters \( a, b, \ldots, c \) are drawn from a \( d \)-letter alphabet \( A \). Every such word has a unique reduced form. This reduction respects the concatenation operation and projects the space of words onto the free group. We extend this result from paths on the integer lattice (words) to the bounded variation case.

Corollary 1.7. For any \( X \in BV(\mathbb{R}^d) \) there exists a unique path of minimal length, \( \bar{X} \), called the reduced path, with the same signature \( X = \bar{X} \).
Taking these results together we see that the reduced paths form a group. The multiplication operation is to concatenate the paths and then reduce the result. One should note that this reduction process is not unique (although we have proved that the reduced word one ultimately gets is). This group is at the same time a natural and concrete (a collection of paths of finite length), but also very different to the usual finite dimensional Lie groups. It admits more than one natural topology, and multiplication is not continuous for the topology of bounded variation.

We can restate these results in different language. The space $BV$ with $\ast$, the operation of concatenation, is a monoid. Let $T$ be the set of tree-like paths in $BV$. Then $T$ is also closed under concatenation. If $\gamma \in BV$ and we use the notation $\gamma^{-1}$ for $\gamma$ run backwards. It is clear from the definition that $\gamma^{-1}T \gamma \subset T$ for all $\gamma \in BV$.

As we have proved that tree-like equivalence is an equivalence relation $BV/T$ is well defined, closed under multiplication, and has inverses; it is a group.

We have the following picture

$$0 \rightarrow T \rightarrow BV \rightarrow \Sigma \rightarrow 0$$

where one can regard $\Sigma$ as the $\sim$-equivalence classes of paths or as the subgroup of the tensor algebra. The map $\rightarrow$ takes the class to the reduce path which is an element of $BV$. As $T$ has no natural $BV$-normal sub-monoids, one should expect that any continuous homomorphism of $BV$ into a group will factor through $\Sigma$ if it is trivial on the tree-like elements. It is clear that the set

$$\hat{T} = \{ (\gamma, h) : \gamma \in T, h \text{ a height function for } \gamma \}$$

is contractable. An interesting question is whether $T$ itself is contractable.

We prove in Lemma 6.3 that any $\gamma \in T$ is the limit of weakly piecewise linear tree-like paths and hence $T$ is the smallest multiplicatively closed and topologically closed set containing the trivial path. This universality suggests that $\Sigma$ has similarities to the Free group. One characterising property of the free group is that every function from the alphabet $A$ into a group can be extended to a map from words made from $A$ into paths in the group. The equivalent map for bounded variation paths is Cartan development. Let $\theta$ be a linear map of $\mathbb{R}^d$ to the Lie algebra $\mathfrak{g}$ of a Lie group $G$ and let $X_{|t| \leq T}$ be a bounded variation path. Then Cartan development provides a canonical projection of $\theta(X)$ to a path $Y$ starting at the origin in $G$ and we can define $\hat{\theta} : X \rightarrow Y_T$. This map $\hat{\theta}$ is a homomorphism from $\Sigma$ to $G$.

It is an exercise to prove that this map $\hat{\theta}$ takes all tree-like paths to the identity element in the group $G$. As a consequence, $\hat{\theta}$ is a map from paths of finite variation to $G$ which is constant on each $\sim$ equivalence class and so defines a map from $\Sigma$ to $G$.

Let $X_{|t| \leq T}$ be a path of bounded variation in $\mathbb{R}^d$ and suppose that for every linear map $\theta$ into a Lie algebra $\mathfrak{g}$, that $\hat{\theta}(X)$ is trivial. As the computation of the first $n$ terms in the signature is itself a development (into the free $n$-step nilpotent group) we conclude that $X_{0,T} = (1,0,0,\ldots)$ and so $X$ is tree-like. In this way we have a

**Corollary 1.8.** A path of bounded variation is tree-like if and only if its development into every finite dimensional Lie group is trivial.

The observation that any linear map of $\mathbb{R}^d$ to the Lie algebra $\mathfrak{g}$ defines a map from $\Sigma$ to the Lie group is a universal property of a kind giving further evidence that $\Sigma$ is some sort of continuous analogue of the free group. However, $\Sigma$ is not a
Lie group although it has a Lie algebra and it is not characterised by this property. (Chen’s piecewise regular paths provide another example since they are paths of bounded variation and are dense in the unit speed paths of finite length).

1.5. Questions and Remarks. How important to these results is the condition that the paths have finite length? Does anything survive if one only insists that the paths are continuous?

The space of continuous paths with the uniform topology is another natural generalisation of words - certainly concatenation makes them a monoid. However, despite their popularity in homotopy theory, there seems little hope that a natural closed equivalence relation could be found on this space that transforms it into a continuous ‘free group’ in the sense we mapped out above. The notion of tree-like makes good sense (one simply drops the assumption that the height function \( h \) is Lipschitz). With this relaxation,

**Problem 1.9.** Does \( \sim \) define an equivalence relation on continuous paths?

Homotopy is the correct deformation of paths if one wants to preserve the line integral of a path against a closed one-form. On the other hand tree-like equivalence is the correct deformation of paths if one wants to preserve the line integral of a path against any one form. As we mention elsewhere in this paper, integration of continuous functions against general one forms makes little sense. This is perhaps evidence to suggest the answer to the problem is in the negative. The problem is in the transitivity of the relation.

**Problem 1.10.** Is there a unique tree reduced path associated to any continuous path?

For smooth paths \( \gamma = (\gamma_1, \gamma_2) \) in \( \mathbb{R}^2 \) Cartan development into the Heisenberg group is the map \( (\gamma_1, \gamma_2) \to (\gamma_1, \gamma_2, \int \gamma_1 d\gamma_2) \). One knows [9, Proposition 1.29] that there is no continuous bilinear map extending this definition to any Banach space of paths which carries the Wiener measure. We also know from Levy, that there are many “almost sure” constructions for this integral made in similar ways to “Levy area”. All are highly discontinuous and can give different answers for the same Brownian path in \( \mathbb{R}^2 \). This wide choice for the case of Brownian paths (which have finite \( p \)-variation for every \( p > 2 \)) makes it clear there cannot be a canonical development for all continuous paths.

The paper [7] sets out a close relationship between differential equations, the signature, and the notion of a geometric rough path. These “paths” also form a monoid under concatenation and any linear map from \( \mathbb{R}^d \) into the \((p + \varepsilon)\)-Lipschitz vector fields on a manifold \( M \) induces a canonical homomorphism of the \( p \)-rough paths with concatenation into the group of diffeomorphisms of \( M \) so they certainly have the analogy to the Cartan development property. Similarly, every rough path has a signature, and the map is a homomorphism.

**Problem 1.11.** Given a path \( \gamma \) of finite \( p \)-variation for some \( p > 1 \), is the triviality of the signature of \( \gamma \) equivalent to the path being tree-like?

Our theorem establishes this in the context of \( p = 1 \) or bounded variation paths but our proof uses the one dimensionality of the image of the path in an essential way. An extension to \( p \)-rough paths with \( p > 1 \) would require new ideas to account for the fact that these rougher paths are of higher “dimension”.

There seem to be many other natural questions.

By Corollary 1.7 among paths of finite length with the same signature there is a unique shortest one - the reduced path. Successful resolution of the following question could have wide ramifications in numerical analysis and beyond. The question is interesting even for lattice paths.

**Problem 1.12.** How does one effectively reconstruct the reduced path from its signature?

A related question is to:

**Problem 1.13.** Identify those elements of the tensor algebra that are signatures of paths and relate properties of the paths (for example their smoothness) to the behaviour of the coefficients in the signature.

Some interesting progress in this direction can be found in [3].

We conclude with some wider comments.

1. There is an obvious link between these reduced paths and geometry since each connection defines a closed subgroup of the group of reduced paths (the paths whose developments are loops).

2. It also seems reasonable to ask about the extent to which the intrinsic structure of the space of reduced paths (with finite length) in $d \geq 2$ dimensions changes as $d$ varies.

1.6. **Outline.** We begin in Section 2 by discussing the lattice case. In this setting we can obtain our first quantitative result on the signature. We do not have best possible estimates, but we can prove that a word in the free group of length $L$ in two generators is completely reducible if the first $\lfloor e \log (1 + \sqrt{2}) L \rfloor$ terms in the signature of the path in the lattice corresponding to the word are zero. The case of words in $d$ generators is also treated and if the first $c(d)L$ terms in the signature are zero, the word is reducible, where the constant $c(d)$ grows logarithmically in $d$.

In Section 3 we extend these quantitative estimates to finite length paths. In order to do this we need to discuss the development of a path into a suitable version of hyperbolic space - a technique that has more recently proved useful in [8]. Using this idea we obtain a quantitative estimate on the difference between the length of the developed path and its chord in terms of the modulus of continuity of the derivative of the path. This allows us to obtain, in the case where the derivative is continuous, some estimates on the coefficients in the signature and also shows how to recover the length of the path from the signature.

We can also prove for example that any path with bounded local curvature and the first $N$ terms in the signature zero must be rather long or trivial - a sort of rigidity theorem. We can obtain explicit bounds depending only on the curvature bounds and $N$. However they are far from sharp as we can see from the figure of 8, a path with curvature at most $4\pi$ and length one. It is clear that the first two terms in its signature zero, but our results indicate that it cannot have all of the first 115 terms in the signature zero!

After this we return to the proof of our uniqueness result, the extension of Chen’s theorem. Our proof relies on various analytic tools (the Lebesgue differentiation theorem, the area theorem), and particularly we introduce a mollification of paths that retain certain deeply non-linear properties of these paths to reduce the problem to the case where $\gamma$ is piecewise linear. Piecewise linear paths are irreducible.
piecewise regular paths in the sense of Chen and thus the result follows from Chen’s Theorem. The quantitative estimates we obtained give an independent proof for this piecewise linear result.

In Section 4 we establish the key properties for tree-like paths that we need. In Section 5 we prove that any path $X_t|_{t \in [0,T]} \in BV$ and with trivial signature can, after re-parameterisation, be uniformly approximated by (weakly) piecewise linear paths with trivial signature. This is an essentially non-linear result as the constraint of trivial signature corresponds to an infinite sequence of polynomial constraints of increasing complexity. In Section 6 we show that, by our quantitative version of Chen’s theorem, such piecewise linear paths must be reducible and so tree-like in our language.

This certainly gives us enough to show, in Section 7, that any weakly piecewise linear path with trivial signature is tree-like. It is clear from the definitions that uniform limits of tree-like paths with uniformly bounded length are themselves tree-like. Applying the results of section 5 the argument is complete. We draw together all the parts to give the proofs of our main Theorem and Corollaries in Section 8.

2. Paths on the integer lattice

2.1. A discrete case of Chen’s theorem. Consider an alphabet $A$ and new letters $A^{-1} = \{ a^{-1}, a \in A \}$. Let $\Omega$ be the set of words in $A \cup A^{-1}$. Then $\Omega$ has a natural multiplication (concatenation) and an equivalence relation that respects this multiplication.

**Definition 2.1.** A word $w \in \Omega$ is said to cancel to the empty word if, by applying successive applications of the rule

\[ a \ldots bc^{-1}d\ldots e \rightarrow a \ldots bd\ldots e, \quad a,b,c,d,e,\ldots \in A \cup A^{-1} \]

one can reduce $w$ to the empty word. We will say that $(a \ldots b)$ is equivalent to $(e \ldots f)$

\[ (a \ldots b) \sim (e \ldots f) \]

if $(a \ldots bf^{-1} \ldots e^{-1})$ cancels to the empty word.

An easy induction argument shows that $\sim$ is an equivalence relation. It is well known that the free group $F_A$ can be identified as $\Omega/\sim$. There is an obvious bijection between words in $\Omega$, and lattice paths, that is to say the piecewise linear paths $x_u$ which satisfy $x_0 = 0$ and $\|x_k - x_{k+1}\| = 1$, are linear on each interval $u \in [k,k+1]$, and have $x_k \in \mathbb{Z}^{|A|}$ for each $k$. The length of the path is an integer equal to the number of letters in the word. The equivalence relation between words can be re-articulated in the language of lattice paths: Consider two lattice paths $x$ and $y$, and let $z$ be the concatenation of $x$ with $y$ traversed backwards. Clearly, if $x$ and $y$ are equivalent then, keeping its endpoints fixed, $z$ can be “retracted” step by step to a point while keeping the deformations inside what remains of the graph of $z$. The converse is also true: if $U$ is the universal cover of the lattice, and we identify based path segments in the lattice with points in $U$ then the words equivalent to the empty word correspond with paths $x_i$ in the lattice that lift to loops in $U$. They are the paths that can be factored into the composition of a loop in a tree with a projection of that tree into the lattice. A loop in a tree is a tree-like path, as one can use the distance from the basepoint of the loop as a height function.
Chen’s theorem tells us that any path that is not retractable to a point in the sense of the previous paragraph has a non-trivial signature. Our quantitative approach allows us to prove an algebraic version of this result. Let \( \gamma_w \) be the lattice path associated to the word \( w = a_1^{\sigma_1} \ldots a_L^{\sigma_L} \) (where \( \sigma = (\sigma_1, \ldots, \sigma_L) \in \{\pm 1\}^L \) gives the signs associated to each letter). As the signature is a homomorphism, we have \( S(\gamma_w) = S\left(\gamma_{a_1^{\sigma_1}}\right) \ldots S\left(\gamma_{a_L^{\sigma_L}}\right) \). Since \( \gamma_{a_i} \) is a path that moves \( a_i \) units in a straight line in the \( a_i \) direction, its signature is the exponential and \( S(\gamma_w) = e^{\sigma_1 \alpha_1} \ldots e^{\sigma_L \alpha_L} \).

Our quantitative approach will show in Theorem 2.2 that for a word of length \( L \) in a two letter alphabet, if

\[
e^{\sigma_1 \alpha_1} \ldots e^{\sigma_L \alpha_L} = \left(1, 0, 0, \ldots, 0, X^{N(L)+1}, X^{N(L)+2} \ldots \right), \quad \sigma \in \{\pm 1\}^L
\]

where \( N(L) = \lfloor e \log (1 + \sqrt{2}) \rfloor \), then there is an \( i \) for which \( a_i = -a_{i+1} \) and by induction the reduced word is trivial.

The proof is based on regarding \( \mathbb{R}^d \) as the tangent space to a point in \( d \)-dimensional hyperbolic space \( \mathbb{H} \), scaling the path \( \gamma \) and developing it into hyperbolic space. There are two ways to view this development of the path, one of which yields analytic information out of the iterated integrals, the other geometric information. Together they quickly give the result. We work in two dimensional hyperbolic space and, at the end, show that the general case can be reduced to this one.

2.2. The universal cover as a subset of \( \mathbb{H} \). Let \( X \) be a lattice path in \( \mathbb{R}^2 \), \( \theta \geq 0 \), and \( X^\theta = \theta X \) be the re-scaled lattice path. The development \( Y^\theta \) of \( X^\theta \) into \( \mathbb{H} \) moves along successive geodesic segments of length \( \theta \) in \( \mathbb{H} \), each time \( X^\theta \) turns a corner, so does \( Y^\theta \) and angles are preserved.

For a fixed choice of \( \theta \) we can trace out in \( \mathbb{H} \) the four geodesic segments from the origin, the three segments out from each of these, and the three from each of these, and so on. It is clear that if the scale \( \theta \) is large enough, the negative curvature forces the image to be tree. This will happen exactly when the path that starts by going along the real axis and then always turns anti-clockwise never hits its reflection in the line \( x = y \).

The successive moves can be expressed as iterations of a Mobius transform,

\[
m(x) := \frac{-ir + x}{-i - rx}, \quad x_n = m^{n}(0),
\]

and if \( r = 1/\sqrt{2} \), then the trajectory eventually ends at \((1 + i)/\sqrt{2}\). Hyperbolic convexity ensures that all these trajectories are (after the first linear step) always in the region contained by the geodesic from \((1 + i)/\sqrt{2}\) to \((1 - i)/\sqrt{2}\). In particular they never intersect the trajectories whose first move is from zero to \( i \), to \(-i \), or to \(-1 \). Now, there is nothing special about zero in this discussion, and using conformal invariance it is easy to see that

**Lemma 2.2.** If \( \theta \) is at least equal to the hyperbolic distance from 0 to 1/\( \sqrt{2} \) in \( \mathbb{H} \), then the path \( Y^\theta \) takes its values in a tree. This value 1/\( \sqrt{2} \) is sharp.

We have developed \( X^\theta \) into a tree in \( \mathbb{H} \); we have already observed that a loop in a tree is tree-like. If we can prove that \( Y^\theta = Y^\theta \), the \( Y^\theta \) will be tree-like and hence
so will $X^\theta$ and $X$. To achieve this we must use the assumption that the path has finite length and that all its iterated integrals are zero from a different perspective.

2.3. **Cartan development as a linear differential equation.** If $G$ is a closed subgroup of the matrices, and $X_t|_{t \leq T}$ is a path in its Lie algebra $\mathfrak{g}$, then the equation for the Cartan development $M_T \in G$ of $X_t|_{t \leq T} \in \mathfrak{g}$ is given by the differential equation

$$M_{t+\delta t} \approx M_t \exp(\delta X_t) \quad \text{or equivalently} \quad dM_t = M_t dX_t.$$

The development of a smooth path in the tangent space to 0 in $H = \{ z \in \mathbb{C} \ | \ ||z|| < 1 \}$, is also expressible as a differential equation. However, it is easier to express this development in terms of Cartan development in the group of isometries regarded as matrices in $GL(2, \mathbb{C})$ rather than on the points of $H$. We identify $\mathbb{R}^2$ with the Lie subspace

$$\left( \begin{array}{cc} 0 & x + iy \\ x - iy & 0 \end{array} \right).$$

In this representation, the equation for $M_t$ is linear and so we have an expansion for $M$:

$$M_T = M_0 \left( I + \int_{0 < u < T} dX_u + \int_{0 < u_1 < u_2 < T} dX_{u_1} dX_{u_2} + \ldots \right) = M_0 \times \left( \begin{array}{cc} a & b \\ \bar{b} & \bar{a} \end{array} \right),$$

where

$$a = 1 + \sum_k \int_{0 < u_1 < u_2 < \ldots < u_{2k} < T} dX_{u_1} d\bar{X}_{u_2} \ldots dX_{u_{2k-1}} d\bar{X}_{u_{2k}}$$

$$b = \sum_k \int_{0 < u_1 < u_2 < \ldots < u_{2k-1} < T} dX_{u_1} d\bar{X}_{u_2} \ldots dX_{u_{2k-1}}$$

and $\int_{0 < u_1 < \ldots < u_{2k} < T} dX_{u_1} d\bar{X}_{u_2} \ldots dX_{u_{2k-1}} d\bar{X}_{u_{2k}}$ is now, with an abuse of notation, a complex number. We have an a priori bound:

**Lemma 2.3.** If $X$ is a path of length exactly $\theta L$, then

$$\left| \int_{0 < u_1 < u_2 < \ldots < u_{2k} < T} dX_{u_1} d\bar{X}_{u_2} \ldots dX_{u_{2k-1}} d\bar{X}_{u_{2k}} \right| < \frac{(\theta L)^{2k}}{(2k)!}.$$

To use this lemma we need to be able to estimate the tail of an exponential series. The following lemma (based on Stirling’s formula) articulates a convenient inequality.

**Lemma 2.4.** Let $x \geq 1/e$. (1) $\frac{x^m}{m!} \leq \frac{e^x}{m^{1/2}}$ holds for all $m \geq ex$.

(2) If for any $k$ one has $m \geq ex + k$, then

$$\sum_{r \geq m} \frac{x^r}{r!} \leq \frac{e^x}{\sqrt{2\pi(e-1)}} e^{-kx^{1/2}} \approx 0.38 e^{-kx^{1/2}}.$$
Proof. By Stirling’s formula \( \lim_{y \to \infty} \frac{e^{-y \frac{1}{2} + y}}{y!} = \frac{1}{\sqrt{2\pi}} \) and is approached monotonely from below. It is an upper bound and also a good global approximation to \( \frac{e^{-y \frac{1}{2} + y}}{y!} \) valid for all \( y \geq 1 \). Putting \( y = ex \) gives

\[
\frac{e^{-ex} (ex)^{\frac{1}{2}+ex}}{(ex)!} < \frac{1}{\sqrt{2\pi}},
\]

\[
\frac{x^{ex}}{(ex)!} < e^{-\frac{1}{2}x} \frac{1}{\sqrt{2\pi}}.
\]

Moreover the recurrence relation for the \( ! \) function implies, for every \( k \in \mathbb{Z} \) with \( ex + k > 0 \), that

\[
\frac{x^{ex+k}}{(ex+k)!} < e^{-\frac{1}{2}x} \frac{1}{\sqrt{2\pi}}.
\]

establishing the first claim. Now summing this bound we have

\[
\sum_{k \geq 0} \frac{x^{ex+k}}{(ex+k)!} \leq e^{-\frac{1}{2}x} \frac{1}{\sqrt{2\pi}} \sum_{k \geq 0} e^{-k} = \frac{e^{\frac{1}{2}}}{e - 1} \frac{1}{\sqrt{2\pi}} x^{-\frac{1}{2}}.
\]

Since for \( ex > 0 \) the function \( k \to \frac{x^{ex+k}}{(ex+k)!} \) is monotone decreasing on \( \mathbb{R}^{+} \) we see that

\[
\sum_{m \geq ex} \frac{x^{m}}{m!} \leq \frac{e^{\frac{1}{2}}}{e - 1} \frac{1}{\sqrt{2\pi}} x^{-\frac{1}{2}},
\]

completing the proof of the lemma. Finally we note the approximate value of the constant:

\[
\frac{e^{\frac{1}{2}}}{\sqrt{2\pi}(e - 1)} \simeq 0.38.
\]

\[\Box\]

2.4. The signature of a word of length \( L \). We deduce the following totally algebraic corollary for paths \( X \) that have traversed at most \( L \) vertices.

**Theorem 2.** If a path of length \( L \) in the two dimensional integer lattice (corresponding to a word with \( L \) letters drawn from a two letter alphabet and its inverse), has the first \( |e \log(1 + \sqrt{2})L| \) \( GL(2, \mathbb{C}) \)-iterated integrals \(^2\) zero, then all iterated integrals (in the tensor algebra) are zero, the path is tree-like, and the corresponding reduced word is trivial.

\(^2\)\(GL(2, \mathbb{C})\)-iterated integrals: since our path is in a vector subspace of the algebra \( GL(2, \mathbb{C}) \) we may compute the iterated integrals in the algebra \( GL(2, \mathbb{C}) \) or in the tensor algebra over the vector subspace. There is a natural algebra homomorphism of the tensor algebra onto \( GL(2, \mathbb{C}) \). The \( GL(2, \mathbb{C}) \)-iterated integrals are the images of those in the tensor algebra under this projection and \( \text{a priori} \) contain less information.
Proof. Any Mobius transformation preserving the disk can be expressed as
\[ M = \begin{pmatrix} z & 0 & 0 \\ 0 & \bar{z} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{1-r^2}} & \frac{r}{\sqrt{1-r^2}} \\ \frac{r}{\sqrt{1-r^2}} & \frac{1}{\sqrt{1-r^2}} \end{pmatrix} \begin{pmatrix} \omega & 0 & 0 \\ 0 & \bar{\omega} & 0 \end{pmatrix}, \]
where \(|z| = |\omega| = 1\) and \(r\) is the Euclidean distance from 0 to \(M_0\).

Now \(Tr[AB] = \sum \sum a_{ij} \bar{b}_{ji}\)
\[ = Tr[B^TA] \]
and
\[ \left( \begin{array}{cc} \omega & 0 \\ 0 & \bar{\omega} \end{array} \right) \left( \begin{array}{cc} \omega & 0 \\ 0 & \bar{\omega} \end{array} \right)^T = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \]
hence
\[ Tr[M^T] = Tr\left[ \left( \begin{array}{cc} \frac{1}{\sqrt{1-r^2}} & \frac{r}{\sqrt{1-r^2}} \\ \frac{r}{\sqrt{1-r^2}} & \frac{1}{\sqrt{1-r^2}} \end{array} \right)^2 \right] \]
\[ = \frac{2(1+r^2)}{(1-r^2)}. \]
Letting \(r = 1/\sqrt{2}\) we see that if
\[ Tr[M^T] < 6, \]
then the image of 0 under the Mobius transformation must lie in the circle of radius \(1/\sqrt{2}\). On the other hand
\[ Tr\left[ \left( \begin{array}{cc} a & b \\ \bar{b} & \bar{a} \end{array} \right) \right) \left( \begin{array}{cc} \bar{a} & b \\ \bar{b} & a \end{array} \right) \right) = 2(|a|^2 + |b|^2) \]
and in our context, where the first \(N\) iterated integrals are zero, this gives the inequality
\[ \left| 1 + \sum_{k>N} \int_{0<u_1<\cdots<u_{2k}<T} dX_{u_1} d\tilde{X}_{u_2} \cdots dX_{u_{2k_1}} d\tilde{X}_{u_{2k}} \right|^2 + \]
\[ \left| \sum_{k>N} \int_{0<u_1<\cdots<u_{2k_1}<T} dX_{u_1} d\tilde{X}_{u_2} \cdots dX_{u_{2k_1}} \right|^2 < 6. \]
Using our a priori estimate from Lemma 2.3 we have that the inequality will hold if
\[ \left( 1 + \sum_{k>N} \frac{(\theta L)^{2k-1}}{(2k-1)!} \right)^2 + \left( \sum_{k>N} \frac{(\theta L)^{2k}}{(2k)!} \right)^2 < 6. \]
Observing that, as we will choose \(N > \theta L\), the terms in the sums are decreasing, we have
\[ \sum_{k>N} \frac{(\theta L)^{2k}}{(2k)!} < s \]
and then
\[ \sum_{k>N} \frac{(\theta L)^{2k}}{(2k)!} < s. \]
We see that (2.1) will always be satisfied if we choose $s$ such that
\[ 2 (s + s^2) < 5 \quad \text{and} \quad \sum_{k>N} (\theta L)^{2k-1} (2k-1)! < s \]
Hence, if
\[ \sum_{k>N} (\theta L)^{2k-1} (2k-1)! < \frac{\sqrt{11} - 1}{2}, \]
then $Y_T = 0$. By Lemma 2.4 (2) with $x = \log(1 + \sqrt{2})L$, we have if $N \geq e \log(1 + \sqrt{2})L$,
\[
\sum_{m \geq N} \frac{(\log(1 + \sqrt{2})L)^m}{m!} \leq 0.38 \cdot (\log(1 + \sqrt{2})L)^{-1/2} < \frac{\sqrt{11} - 1}{2}
\]
for all $L \geq 1$.

Observe that if $\theta \geq \log \left[ 1 + \sqrt{2} \right]$ then $Y_t = M_0 \theta$ lies in a tree, and the development of $Y$ is such that every vertex of the tree is at least a distance $\theta$ from the origin except the origin itself. By our hypotheses and the above argument $d \{Y_T, 0\} < \theta$ and hence $Y_T = 0$. Therefore $Y$ is tree-like and the reduced word is trivial. \(\square\)

Finally we note that the case of the free group with two generators is enough to obtain a general result as the free group on $d$ generators can be embedded in it.

**Lemma 2.5.** Suppose that $\Gamma_d$ is the free group on $d$ letters $e_i$, and that $\Gamma$ is the free group on the letters $a, b$. Then we can identify $f_i \in \Gamma$ so that the homomorphism induced by $e_i \rightarrow f_i$ from $\Gamma_d$ to $\Gamma$ is an isomorphism and so that the length of the reduced words $f_i$ are at most $|f_i| \leq 2 \lceil \log_3\frac{d}{2} \rceil + 3$.

**Proof.** It is enough to show that we can embed $\Gamma_{2^{d+1}}$ into $\Gamma$ so that each $f_i$ has length $l$. Consider the collection of all reduced words of length $l$ in $\Gamma$. There are $43^{l-1}$ of them if $l > 0$. Partition them into pairs, so that the left most letter of each of the words in a pair is the same up to inverses. Order them lexicographically. Now consider the space which is the ball in the Cayley graph of $\Gamma$ comprising reduced words with length at most $l$. It is obvious that this is a contractable space. Now adjoin new edges connecting the ends of our pairs. Associate with each of the new edges the alternate letter and orient the edge to point from the lower to the higher word in the lexicographic order. Then this new space $\Delta$ is contractable to $23^{l-1}$ loops and so has the free group $\Gamma_{2^{d+1}}$ as its fundamental group. On the other hand, we can obviously lift any path in $\Delta$ to the Cayley graph of $\Gamma$; the map from loops in $\Delta$ to $\Gamma$ is a homomorphism. The homomorphism is injective. So we see that the image is a copy of the free group $\Gamma_{2^{d+1}}$. The generators of the classes in $\Delta$ clearly lift to paths of length $2l + 1$ in $\Gamma$ and we take the end points of these paths to be the $f_i$. \(\square\)
Theorem 3. If $X$ is a path of length $L$ in the $d$-dimensional integer lattice and the projections into $GL(2, \mathbb{C})$ of the first $\left\lfloor (2 \lceil \log_3 \frac{d}{2} \rceil + 3) c \log(1 + \sqrt{2}) L \right\rfloor$ iterated integrals are zero, then the path is tree-like.

In this section, our arguments depend on the tree-like nature of the development of the path in the lattice and little else - this is a property of the development into any rank one symmetric space but is still plausible, if less obvious for general homogeneous spaces. Each space will give rise to a different class of iterated integrals that are sufficient to determine the tree-like nature of a path in a ‘jungle gym’. One should note that computing the iterated integrals is not the most efficient way to determine if a word is reducible if the word, as opposed to its signature, is presented.

3. Quantitative versions of Chen’s Theorem

We work in the hyperboloid model for $H$ (which embeds the space $\mathbb{H}$ into a $d+1$-dimensional Lorentz space) because the isometries of $H$ extend to linear maps.

Consider the quadratic form on $\mathbb{R}^{d+1}$ defined by

$$I_d(x, y) = \sum_{i=1}^{d} x_i y_i - x_{d+1} y_{d+1}$$

and the surface

$$H = \{ x, I_d(x, x) = -1 \}.$$

Then $H$ is hyperbolic space with the metric obtained by restricting $I_d$ to the tangent spaces to $H$. (If $x \in H$ then $\{ y | I_d(y, x) = 0 \}$ is the tangent space to $H$ in $\mathbb{R}^{d+1}$ and moreover $I_d(x, z)$ is positive definite on $z \in \{ y | I_d(y, x) = 0 \}$ and so this inner product is a Riemannian structure on $H$). In fact, (see [1], p83) distances in $H$ can be calculated using

$$- \cosh d(x, y) = I_d(x, y)$$

(3.1)

If $SO(I_d)$ denotes the group of matrices with positive determinant preserving the quadratic form $I_d$ then one can prove this is exactly the group of orientation preserving isometries of $H$. The Lie algebra of $SO(I_d)$ is easily recognised as the $d+1$ dimensional matrices that are antisymmetric in the top left $d \times d$ block and symmetric in the last column and bottom row and zero in the bottom right corner. Then the development of a path $\gamma \in \mathbb{R}^d$ to $SO(I_d)$ and $H$ (chosen to commute with the action of multiplication on the right in $SO(I_d))$ is given by solving the following differential equation

$$d\Gamma_t = \begin{pmatrix}
0 & \cdots & 0 & d\gamma^1_t \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & d\gamma^d_t \\
d\gamma^1_t & \cdots & d\gamma^d_t & 0
\end{pmatrix} \Gamma_t.$$

(3.2)

We define $X$ to be the development of the path $\gamma$ to the path in $\mathbb{H}$ starting at $o = (0, \cdots, 0, 1)^t$ and given by

$$X_t = \Gamma_t o.$$

3Precisely, $M \in SO(I_n)$ if $I_d \left( (My^t)^t, (Mx^t)^t \right) \equiv I_d(y, x)$
Now we can write \( d\Gamma_t = F(d\gamma_t)\Gamma_t \) where

\[
F : x \rightarrow \begin{pmatrix}
0 & \cdots & 0 & x_1 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & x_d \\
x_1 & \cdots & x_d & 0
\end{pmatrix}
\]

is a map from \( \mathbb{R}^d \) to \( \text{Hom}(\mathbb{R}^{d+1}, \mathbb{R}^{d+1}) \), where for precision we choose the Euclidean norm on \( \mathbb{R}^d \) and \( \mathbb{R}^{d+1} \) and the operator norm on \( \text{Hom}(\mathbb{R}^{d+1}, \mathbb{R}^{d+1}) \).

**Lemma 3.1.** In fact \( \|F\|_{\text{Hom}(\mathbb{R}^{d+1}, \mathbb{R}^{d+1})} = 1 \).

**Proof.** Let \( e \in \mathbb{R}^d \) and \( f \in \mathbb{R} \). Then for \( x \in \mathbb{R}^d \)

\[
F(x) \begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} fx \\ e.x \end{pmatrix}
\]

and computing norms

\[
\left\| \begin{pmatrix} fx \\ e.x \end{pmatrix} \right\|^2 \leq f^2 \|x\|^2 + \|e\|^2 \|x\|^2 = \left\| \begin{pmatrix} e \\ f \end{pmatrix} \right\|^2 \|x\|^2
\]

and hence \( \|F\|_{\text{Hom}(\mathbb{R}^{d+1}, \mathbb{R}^{d+1})} = 1 \). \( \Box \)

### 3.1. Paths close to a geodesic.

We are interested in developing paths \( \gamma \) of fixed length \( l \) into paths \( \Gamma \) in \( SO(I_d) \) and in the function

\[
\varrho(\gamma) := d(o, \Gamma o)
\]

giving the length of the chord connecting the beginning and end of the development of \( \gamma \) into Hyperbolic space. Amongst these paths \( \gamma \) of fixed length, straight lines maximise \( \varrho \) as the developments are geodesics. The function \( \varrho \) is a smooth function on path space \([6]\). Therefore one would expect that for some constant \( K \)

\[
\varrho(\gamma) \geq l - K\varepsilon^2
\]

whenever \( \gamma \) is in the \( \varepsilon \)-neighbourhood (for the appropriate norm) of a straight line. We will make this precise using Taylor’s theorem.

Suppose our straight line is in the direction of a unit vector \( v \). If our path \( \gamma \) is parameterised at unit speed we can represent it by

\[
d\gamma_t = \Theta_t v dt,
\]

where \( \Theta_t \) is a path in the isometries of \( \mathbb{R}^d \). In this discussion we assume that \( \Theta_t \) is continuous and has modulus of continuity \( \delta \). Of course, \( \gamma \) is close to \( t \to tv \) if \( \Theta \) is uniformly close to the identity. Consider the development \( \Gamma_t o = (\hat{x}_t, x_t) \) of \( \gamma \) into the hyperboloid model of \( \mathbb{H} \) defined by

\[
\begin{align*}
x_t &\in \mathbb{R} \\
\hat{x}_t &\in \mathbb{R}^d \\
dx_t &= \hat{x}_t, \Theta_t v dt \quad x_0 = 1 \\
\hat{d} x_t &= x_t, \Theta_t v dt \quad \hat{x}_0 = 0
\end{align*}
\]

We know that \( \|\hat{x}_t\|^2 + 1 = |x_t|^2 \) and that

\[
cosh d(\Gamma_t o, o) = -I_d \left( \begin{pmatrix} \hat{x}_t \\ x_t \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = x_t
\]
in other words \( \cosh \theta (\gamma_{[0,t]}) = x_t \).

**Proposition 3.2.** Suppose one can express \( \Theta_t \) in the form \( e^{A_t} \) where \( A_t \) is a continuously varying anti-symmetric matrix and that \( \|A\|_\infty \leq \eta < 1 \). Then

\[
|\cosh T - x_T| \leq 4T \frac{\|A\|_\infty^2}{2}
\]

**Proof.** Suppose \( \varepsilon \in [-1,1] \). We can introduce a family of paths \( \gamma_{\varepsilon}^t \) with \( \gamma_1^t \equiv \gamma_t \) and with \( \gamma_0^t \) the straight line \( tv \) by setting

\[
d\gamma_{\varepsilon}^t = e^{\varepsilon A_t} v dt,
\]

\[
\gamma_0^t = 0.
\]

We can then consider the real valued function \( f \) on \([-1,1]\) comparing the length of the development of \( \gamma_{\varepsilon}^t \) and the straight line

\[
f(\varepsilon) := \cosh (\gamma_{\varepsilon}^t |_{t \in [0,T]}) - \cosh T.
\]

Of course \( f(0) = 0 \) and \( f \leq 0 \). Now [6, Theorem 2.2] proves that development of a path \( \gamma \) is Frechet differentiable as a map from paths to paths in all \( p \)-variation norms with \( p \in [1,2) \).

It is elementary that

\[
d \left( \gamma_{\varepsilon}^t - \gamma_{\varepsilon + h}^t \right) = e^{\varepsilon A_t} (1 - e^{h A_t}) v dt = h A_t e^{\varepsilon A_t} v dt + \frac{1}{2} h^2 A_t^2 e^{\varepsilon A_t} v dt,
\]

where \( \tilde{h}_t \in [0,h] \). Working towards the 1-variation derivative

\[
\int_{\varepsilon \in [0,T]} |d \left( \gamma_{\varepsilon}^t - \gamma_{\varepsilon + h}^t \right) - h A_t e^{\varepsilon A_t} v dt| \leq \int_{\varepsilon \in [0,T]} \left| \frac{1}{2} h^2 A_t^2 e^{\varepsilon A_t} v dt \right|
\]

\[
\leq \frac{h^2}{2} \int_{\varepsilon \in [0,T]} |A_t^2| dt
\]

and \( \varepsilon \to \gamma_{\varepsilon}^t \) is differentiable with derivative

\[
d^{(1)} \gamma_{\varepsilon} = A_t e^{\varepsilon A_t} v dt,
\]

providing \( \int_{\varepsilon \in [0,T]} |A_t^2| dt < \infty \). A similar estimate shows that the derivative of \( \gamma^{(1)}_{\varepsilon} \) exists and is

\[
d^{(2)} \gamma_{\varepsilon} := A_t^2 e^{\varepsilon A_t} v dt,
\]

providing \( \int_{\varepsilon \in [0,T]} |A_t^3| dt < \infty \). From [6, Theorem 2.2] we know that the development map is certainly twice differentiable in the 1-variation norm and applying the chain rule it follows that \( f \) is a twice differentiable function on \([-1,1] \). On the other hand \( f(0) = 0 \) and \( f(\varepsilon) \leq 0 \) for \( \varepsilon \in [-1,1] \) so that \( f'(0) = 0 \) and applying Taylor’s theorem

\[
0 \geq f(1) \geq \inf_{\varepsilon \in [0,1]} \frac{\varepsilon^2}{2} f''(\varepsilon).
\]

In fact the derivatives in \( \varepsilon \) form a simple system of differential equations. If

\[
\begin{pmatrix}
\hat{x}_{\varepsilon}^{t + h} \\
x_{\varepsilon}^{t + h}
\end{pmatrix} =
\begin{pmatrix}
\hat{x}_{\varepsilon}^t \\
x_{\varepsilon}^t
\end{pmatrix} + h
\begin{pmatrix}
\hat{y}_{\varepsilon}^t \\
y_{\varepsilon}^t
\end{pmatrix} + \frac{h^2}{2}
\begin{pmatrix}
\hat{z}_{\varepsilon}^t \\
z_{\varepsilon}^t
\end{pmatrix} + o(h^2),
\]
Lemma 3.3. If the distance $c$ from $A$ to $B$ is at least $\log \left(\frac{\cos |\theta_A| + 1}{1 - \cos |\theta_A|}\right)$, then $|\theta_B| \leq |\theta_A|$. 

Proof. Fix $c$ and the angle $\theta_A$, the angle $\theta_B$ is zero if $b = 0$ and monotone increasing as $b \to \infty$. Suppose that $|\theta_B| > |\theta_A|$. We may reduce $b$ so that $|\theta_B| = |\theta_A|$, now the triangle has two equal edges and applying the cosine rule to compute the base length:

\[ c = \log \left(\frac{(\cos |\theta_A|) e^{2a} + e^{2a} - \cos |\theta_A| + 1}{-e^{2a} + (\cos |\theta_A|) e^{2a} - \cos |\theta_A| - 1}\right) \]

\[ < \lim_{a \to \infty} \log \left(\frac{(\cos |\theta_A|) e^{2a} + e^{2a} - \cos |\theta_A| + 1}{-e^{2a} + (\cos |\theta_A|) e^{2a} - \cos |\theta_A| - 1}\right) \]

\[ = \log \left(\frac{\cos |\theta_A| + 1}{1 - \cos |\theta_A|}\right). \]

Lemma 3.4. We have $a \geq b + c - \log \frac{2}{1 - \cos \theta_A}$, and thus if $\max(b, c) \geq \log \frac{2}{1 - \cos \theta_A}$, then $a > \min(b, c)$. 
Proof. Suppose consider triangles with fixed angle $\theta_A$ and with side lengths $\lambda b$, $\lambda c$ and resulting length $a(\lambda)$ for the opposite. Then

$$\lambda b + \lambda c - a(\lambda)$$

is monotone increasing in $\lambda$ with a finite limit. Now

$$\frac{\sinh(\lambda b) \sinh(\lambda c) \cos(\theta_A)}{\sinh(\lambda b) \sinh(\lambda c) - \cos(\theta_A)} = \frac{\cosh(\lambda b) \cosh(\lambda c) - \cosh(a(\lambda))}{\sinh(\lambda b) \sinh(\lambda c)}$$

$$\lim_{\lambda \to \infty} \log \frac{\cosh(a(\lambda))}{\sinh(\lambda b) \sinh(\lambda c)} = \lim_{\lambda \to \infty} \left( (a(\lambda) - \lambda b - \lambda b) + \log 2 \right)$$

$$\lambda b + \lambda c - a(\lambda) \leq \lim_{\lambda \to \infty} \frac{2}{1 - \cos \theta_A}.$$

Thus

$$a \geq b + c - \log \frac{2}{1 - \cos \theta_A}.$$

Also, providing $\max(b, c) \geq \log \frac{2}{1 - \cos \theta_A}$, one has $a \geq \min(b, c)$. □

**Corollary 3.5.** If the distance $c$ from $A$ to $B$ is at least $\log \left( \frac{2}{1 - \cos |\theta_A|} \right)$, then

$$|\theta_B| \leq |\theta_A|,$$

and $a \geq b$.

The above lemma is useful in the case where the angles of interest are acute. But in some contexts we are interested in one angle is very obtuse in which case the following lemma gives much better information.

**Lemma 3.6.** Suppose that $\theta_A > \pi/2$ and that the distance $c$ from $A$ to $B$ is at least $\log \left( \frac{2}{1 - \cos \theta_A} \right)$ then $\theta_B < (\pi - \theta_A)/2$.

Proof. The second hyperbolic cosine rule states that

$$\sin(\theta_B) \sin(\theta_A) \cosh(c) = \cos(\theta_C) + \cos(\theta_B) \cos(\theta_A)$$

$$\cosh(c) = \frac{\cos(\theta_C) + \cos(\theta_B) \cos(\theta_A)}{\sin(\theta_B) \sin(\theta_A)}$$

Fix $\theta_A > \pi/2$. By our assumptions $\cosh(c) \geq \sqrt{2}$, and so

$$\frac{\cos(\theta_C) + \cos(\theta_B) \cos(\theta_A)}{\sin(\theta_B) \sin(\theta_A)} \geq \sqrt{2}.$$

Since the sum of interior angles in a Hyperbolic triangle is less than $\pi$ one can conclude that $\theta_B = \alpha (\pi - \theta_A)$ where $0 < \alpha < 1$ and that $\theta_B$ and $\theta_C$ are in $[0, \pi/2)$. To prove this lemma we need to show further, that $\alpha \leq \frac{1}{2}$. It is enough to demonstrate that, in the case $\theta_A > \pi/2$, and $\frac{1}{2} < \alpha < 1$, we have

$$\frac{\cos(\theta_C) + \cos(\theta_B) \cos(\theta_A)}{\sin(\theta_B) \sin(\theta_A)} < \sqrt{2}.$$

It is enough to prove that

$$\frac{1 + \cos(\theta_B) \cos(\theta_A)}{\sin(\theta_B) \sin(\theta_A)} < \sqrt{2}.$$
Replacing \((\pi - \theta_A)\) by \(\tau\) and rewriting
\[
f(\alpha, \tau) := \frac{1 - \cos(\alpha \tau) \cos(\tau)}{\sin(\alpha \tau) \sin(\tau)}
\]

it is enough to prove that \(f(\alpha, \tau) < \sqrt{2}\) if \(\tau < \pi/2\) and \(\frac{1}{2} < \alpha < 1\). The derivative in \(\alpha\) of \(f(\alpha, \tau)\) is
\[
\frac{\tau (\cos(\tau) - \cos(\alpha \tau))}{\sin(\tau) \sin(\alpha \tau)^2}
\]
and so \(f\) is strictly decreasing in \(\alpha\) in our domain. Hence, if \(\alpha > 1/2\) then
\[
f(\alpha, \tau) < f\left(\frac{1}{2}, \tau\right)
\]
The derivative of \(f\left(\frac{1}{2}, \tau\right)\) is readily computed as
\[
\frac{1}{8} \frac{(1 + 2 \cos(\tau/2)) \tan(\tau/4)}{\cos(\tau/2)^4 \cos(\tau/2)^2}
\]
and this is seen to be positive so that
\[
f(\alpha, \tau) < f\left(\frac{1}{2}, \tau\right) < f\left(\frac{1}{2}, \pi/2\right) = \sqrt{2}
\]
which completes the argument. \(\square\)

**Lemma 3.7.** Let \(0 = T_0 < \ldots < T_i < \ldots T_n = T\) be a partition of \([0, T]\). Let \((X_t)_{t \in [0, T]}\) be a continuous path, geodesic on the intervals \([T_i, T_{i+1}]|_{i=0,\ldots,n-1}\) in hyperbolic space with \(n \geq 1\) where, at each \(T_i\), the angle between the two geodesic segments: \(\angle X_{i-1}X_T, X_{i+1}\), is in \([\theta, \pi]\). Suppose that each geodesic segment has length at least \(K(\theta) = \log \left(\frac{2}{1 - \cos(\theta)}\right)\).

(1) \(d(X_0, X_{T_i})\) is increasing in \(i\) and for each \(i \leq n\)

\[
d(X_0, X_{T_i}) \geq d(X_0, X_{T_{i-1}}) + d(X_{T_{i-1}}, X_{T_i}) - K(\theta)
\]

and the angle between \(\overrightarrow{X_{T_{i-1}}X_{T_i}}\) and \(\overrightarrow{X_0X_{T_i}}\) is at most \(\theta\).

(2) We also have
\[
0 \leq \sum_{i=1}^{n} d(X_{T_{i-1}}, X_{T_i}) - d(X_0, X_{T_n}) \leq (n-1)K(\theta).
\]

**Proof.** We proceed by induction. Suppose \(d(X_0, X_{T_i}) \geq K(\theta)\) and the angle \(\angle X_{T_{i-1}}X_T, X_{T_{i+1}}\) is at most \(\theta\). Now the angle \(\angle X_{T_{i-1}}X_T, X_{T_{i+1}}\) is at least \(2\theta\) so that the angle \(\angle X_{T_{i}}X_T, X_{T_{i+1}}\) is at least \(\theta\). As \(d(X_0, X_{T_i}) \geq K(\theta)\) and our supposition \(d(X_{T_{i}}, X_{T_{i+1}}) \geq K(\theta)\), Lemma 3.3 and Corollary 3.5 imply
\[
d(X_0, X_{T_{i+1}}) \geq d(X_0, X_{T_i}) + d(X_{T_i}, X_{T_{i+1}}) - K(\theta),
\]
and that \(\angle X_0X_{T_{i+1}}X_{T_i} \leq \theta\) proving the main inequality. Using the induction one also has the second part of the inequality
\[
d(X_0, X_{T_{i+1}}) \geq K(\theta).
\]
The second claim is obtained by iterating (3.3).

\[ d (X_0, X_{T_{i+1}}) \geq d (X_0, X_{T_i}) + \sum_{j=1}^{i} d (X_{T_j}, X_{T_{j+1}}) - iK (\theta). \]

Now rearrange to get the result. \( \square \)

3.3. The main quantitative estimate. Let \( \gamma \) in \( \mathbb{R}^d \) be a continuous path of finite length \( l \), and parameterised at unit speed. With this parameterisation \( \gamma \) can be regarded as a path on the unit sphere in \( \mathbb{R}^d \). We consider the case where \( u \to \gamma (u) \) is continuous with modulus of continuity \( \delta_\gamma \). If \( \alpha \in \mathbb{R} \), then the path \( \gamma \) is also parameterised at unit speed, its length is \( \alpha l \) and its derivative has modulus of continuity \( \delta_\gamma (\alpha h) = \delta_\gamma (h) \). Its development from the identity matrix (defined in (3.2)) into \( SO(I_d) \) is denoted by \( \Gamma_\alpha \).

The goal of this section is to provide a quantitative understanding for \( \Gamma_\alpha \) as we let \( \alpha \to \infty \). Our estimates will only depend on \( \delta_\gamma \) and the length of the path. We let \( R_0 = \log(1 + \sqrt{2}) \).

**Proposition 3.8.** Let \( \gamma \) in \( \mathbb{R}^d \) be a continuous path of length \( l \). For each \( C < 1 \) and \( 1 \leq M \in \mathbb{N} \) then for any \( \alpha \) chosen large enough that \( \alpha l \geq MR_0 \) and \( \delta (\frac{M+1}{M}R_0) \) is denoted by \( \Gamma_\alpha \), one has

\[ |d (o, \Gamma_\alpha o) - \alpha l| \leq \left( \frac{4^M R_0}{2C} + \frac{16 \log 2}{\pi^2} \right) \frac{\alpha l}{R_0} \delta_\gamma \left( \frac{M+1}{M} \right)^2 \]

and if \( \alpha = 1 \), \( l \geq MR_0 \), providing \( \delta (\frac{M+1}{M}R_0) < \sqrt{2} (\sqrt{1} + C^2) \), we have for paths of any length

\[ |d (o, \Gamma o) - l| \leq \left( \frac{4 M+1}{2C} + \frac{16 \log 2}{\pi^2} \right) \frac{l}{R_0} \delta_\gamma \left( \frac{M+1}{M} \right)^2. \]

We set \( D_1 (C, M) = \left( \frac{4 M+1}{2C} + \frac{16 \log 2}{\pi^2} \right) \frac{l}{R_0} \), and \( D_2 (M) = \frac{M+1}{M} R_0 \) so that the inequality becomes

\[ |d (o, \Gamma o) - \alpha l| \leq D_1 \delta_\gamma (D_2/\alpha) \alpha l. \]

We note that \( R_0 \approx 0.881374 \), \( 4 R_0 \approx 3.34393 \leq 4^{(M+1)/R_0/M} \leq 4^{2 R_0} \approx 11.5154, \frac{16 \log 2}{\pi^2} \approx 1.12369 \). Fixing \( M = 1 \), one immediately sees that the distance \( d (o, \Gamma o) \) grows linearly with the scaling and the chordal distance \( d (o, \Gamma o) \) behaves like the length of the path \( \gamma \) as \( \alpha \to \infty \). We also note that the shape of this result is reminiscent of the elegant result of Fawcett [3] Lemma 68: that among \( C^2 \)-curves \( \gamma \) with modulus of continuity \( \delta_\gamma (h) \leq \kappa h \) one has sharp estimates on the minimal value of \( d (o, \Gamma o) \) given by

\[ \inf_{\gamma} \cosh (d (o, \Gamma o)) = \frac{\cosh (\alpha l \sqrt{1 - \kappa^2}) - \kappa^2}{1 - \kappa^2}. \]

A natural question to ask is whether our estimate (which is non-infinitesimal and only needs information about \( \delta_\gamma (2R_0) \)) can be improved to this shape and even to this sharp form.
Proof. The path \( \gamma_\alpha := t \rightarrow \alpha \gamma(t/\alpha) \) is of length \( \alpha l \) and parameterised at unit speed; its derivative has modulus of continuity \( \delta_\alpha : t \rightarrow \delta_\alpha (t/\alpha) \). Because \( \alpha l \geq M R_0 \) we can fix \( R = \alpha l / N \), where \( R \in \left[ R_0, \frac{M+1}{M} R_0 \right] \) and \( N \) is a positive integer depending on \( \alpha \). Let \( t_i = i R \) where \( i \in [0,N] \). Let \( G_i \in SO(I_d) \) be the development of the path segment \( \gamma_\alpha |_{[t_{i-1},t_i]} \) into \( SO(I_d) \) and \( \Gamma_{\alpha,t} \) be the development of the path segment \( \gamma_\alpha |_{[0,t]} \). We define \( X_0 := o \in \mathbb{H} \) and \( X_j := G_i X_{j-1} \in \mathbb{H} \). Then \( X_j \) are the points \( \Gamma_{\alpha,t} \) on the path \( \Gamma_{\alpha,t} \).

As the length of the path is greater than any chord
\[
|\alpha l - d(o, \Gamma_{\alpha,t})| = \alpha l - \sum_{i=1}^{N} d(X_{i-1}, X_i)
\]
and
\[
\alpha l - \sum_{i=1}^{N} d(X_{i-1}, X_i) \geq 0 \quad \sum_{i=1}^{N} d(X_{i-1}, X_i) - d(X_0, X_N) \geq 0.
\]
We now estimate each of these terms from above.

For the first term we use our result on paths close to a geodesic. By Proposition \ref{prop:convexity} we have
\[
cosh d(X_{i-1}, X_i) \geq \cosh R - \frac{\delta_\alpha(R)^2}{2} 4^R
\]
Thus, using the convexity of \( \cosh \) and hyperbolic trig identities,
\[
\frac{\delta_\alpha(R)^2}{2} 4^R \geq \cosh R - \cosh d(X_{i-1}, X_i)
\]
\[
\geq (R - d(X_{i-1}, X_i)) \sinh d(X_{i-1}, X_i)
\]
\[
= (R - d(X_{i-1}, X_i)) \sqrt{\cosh d(X_{i-1}, X_i)^2 - 1}
\]
\[
\geq (R - d(X_{i-1}, X_i)) \sqrt{\left( \cosh R - \frac{\delta_\alpha(R)^2}{2} 4^R \right)^2 - 1}
\]
\[
\geq (R - d(X_{i-1}, X_i)) \sqrt{\left( \sqrt{2} - \frac{\delta_\alpha(R)^2}{2} 4^R \right)^2 - 1}.
\]
Now, for \( C < 1 \), providing
\[
\left( \sqrt{2} - \frac{\delta_\alpha(R)^2}{2} 4^R \right)^2 \geq 1 + C^2,
\]
we have
\[
\frac{\delta_\alpha(R)^2}{2C} 4^R \geq (R - d(X_{i-1}, X_i)).
\]
This will follow if we choose \( \alpha \) large enough such that our condition
\[
\delta_\alpha \left( \frac{M+1}{M} R_0 \frac{1}{\alpha} \right)^2 \leq 2 \left( \sqrt{2} - \sqrt{1+C^2} \right) 4^{-\frac{M+1}{M} R_0},
\]
holds.

Hence, summing over all the pieces, we have
\[
\left( \frac{\alpha l}{R} \right) R \delta_\gamma \left( \frac{R}{\alpha} \right) \frac{2 R^2}{2C} \geq \left( \alpha l - \sum_{i=1}^{N} d(X_{i-1}, X_i) \right).
\]

We now use our bounds on \( R \) to obtain
\[
(3.6) \quad \left( \frac{\alpha l}{R_0} \right) R_0 \delta_\gamma \left( \frac{M+1}{\alpha} \frac{R_0}{\alpha} \right) \frac{2 R_0^2}{2C} \geq \left( \alpha l - \sum_{i=1}^{N} d(X_{i-1}, X_i) \right).
\]

Applying Lemma 3.6 and Lemma 3.7 we see that as \( R \geq R_0 = \log(1 + \sqrt{2}) \), then the angle \( \angle X_0 X_n X_{n+1} \) is at least \( \pi - 2 \delta_\gamma \left( \frac{R}{\alpha} \right) \) for each \( 0 < n < N - 1 \).

Thus
\[
\sum_{i=1}^{N} d(X_{i-1}, X_i) - d(X_0, X_{T_n}) \leq (N - 1)K(\pi - 2 \delta_\gamma \left( \frac{R}{\alpha} \right)) \]
\[
= \left( \frac{\alpha l}{R} - 1 \right) \log \left( \frac{2}{1 - \cos \left( \pi - 2 \delta_\gamma \left( \frac{R}{\alpha} \right) \right)} \right).
\]

Since \( \log \left( \frac{2}{1 - \cos \left( \pi - 2 \delta_\gamma \left( \frac{R}{\alpha} \right) \right)} \right) \) is increasing in \( u \) for \( u \leq \pi \), we have
\[
\log \left( \frac{2}{1 - \cos \left( \pi - 2 \delta_\gamma \left( \frac{R}{\alpha} \right) \right)} \right) \leq 4 \delta_\gamma \left( \frac{R}{\alpha} \right) \frac{2 \log 2}{\pi^2} \] since
\[
\delta_\gamma \left( \frac{R}{\alpha} \right) \leq \sqrt{2 \left( \sqrt{2} - \sqrt{1 + C^2} \right) 4^{-\frac{M+1}{M}} R_0}
\]

which is less than \( \pi/4 \) for all \( C \) and \( M \).

Observing that \( \delta_\gamma \left( \frac{R}{\alpha} \right) \) is increasing and that \( \frac{M+1}{M} R_0 \geq R \) gives the second part of our estimate
\[
(3.7) \sum_{i=1}^{n} d(X_{i-1}, X_i) - d(X_0, X_{T_n}) \leq \frac{16 \log 2}{\pi^2} \left( \frac{\alpha l}{R} - 1 \right) \delta_\gamma \left( \frac{R}{\alpha} \right)^2.
\]

(3.8)
\[
\leq \frac{16 \log 2}{\pi^2} \left( \frac{\alpha l}{R_0} - 1 \right) \delta_\gamma \left( \frac{M+1}{M} \frac{R_0}{\alpha} \right)^2.
\]

Combining the estimates (3.6) and (3.8) completes proof. \( \square \)

3.4. Recovering the length of the path from its signature. From the last section we know that if \( \alpha \) is large enough then
\[
(3.9) \quad |d(o, \Gamma_\alpha o) - l\alpha| \leq D_1 \delta_\gamma \left( D_2/\alpha \right)^2 \alpha l
\]
and in particular will go to zero as \( \alpha \to \infty \) if \( \delta_\gamma (\varepsilon) = o(\varepsilon^{1/2}) \).

The lower bound on \( d(o, \Gamma_\alpha o) \) implicit in (3.9) leads to a lower bound on the norm of \( \Gamma_\alpha \) as a matrix. We will compare it with the upper bound that comes from expressing the matrix \( \Gamma_\alpha \) as a series whose coefficients are iterated integrals. We have an upper bound for each coefficient in the series, and taken together these provide a bound for the sum. This bound is so close to the lower bound that it
allows us to conclude a lower bound for each coefficient and relate the decay rate for the norms of the iterated integrals directly to the length of $\gamma$.

It is an open question as to whether signatures with given decay rate correspond to paths of finite length.

**Proposition 3.9.** Let $G \in SO(I_d)$. Then $\|G\| \geq e^{d(o,Go)}$ where $\|G\|$ is the operator norm for $G \in \text{Hom}(\mathbb{R}^{d+1}, \mathbb{R}^{d+1})$ where $\mathbb{R}^{d+1}$ has the Euclidean norm.

**Proof.** If

$$F_\rho := \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 1 & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & 0 & 0 \\ 0 & \cdots & \cdots & 0 & \cosh \rho & \sinh \rho \\ 0 & \cdots & \cdots & 0 & \sinh \rho & \cosh \rho \end{pmatrix},$$

then $F_\rho F_\gamma = F_{\rho+\gamma}$ and the set of such elements forms a (maximal) abelian subgroup of $SO(I_d)$. Any element $G$ of $SO(I_d)$ can be factored into a Cartan Decomposition $KF_\rho \tilde{K}$ where $K$ and $\tilde{K}$ are built out of rotations $\Theta$ of $\mathbb{R}^d$

$$
\left( \begin{array}{c} \Theta \\ 0 \\ \Theta^t \end{array} \right)
$$

and $\rho \in \mathbb{R}_+$. As an operator on Euclidean space, $G$ has norm $\|G\| = \|KF_\rho \tilde{K}\| = \|F_\rho\|$ since $K, \tilde{K}$ are isometries. In addition, the matrix $F_\rho$ is symmetric and hence has a basis comprising eigenfunctions; its norm is at least as large as its largest eigenvalue. Computation shows that the eigenvalues of $F_\rho$ are $\{e^{\rho}, e^{-\rho}, 1, \cdots, 1\}$ so that, given $\rho > 0$, one has

$$\|G\| \geq e^\rho.$$

On the other hand

$$-\cosh d(o, Go) = I_d \left( \begin{pmatrix} x_1 \\ \vdots \\ x_d \\ \cosh \rho \end{pmatrix}, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right) = -\cosh \rho$$

and so $\|G\| \geq e^{d(o, Go)}$. \hfill \Box

If $\gamma$ is a path of finite length then the development (3.2) into hyperbolic space $\mathbb{H}$ is defined by

$$d\Gamma_t = F(d\gamma_t) \Gamma_t,$$

where by Lemma 3.1 $F : \mathbb{R}^d \to \text{hom}(\mathbb{R}^{d+1}, \mathbb{R}^{d+1})$ has norm one as a map from Euclidean space to the operators on Euclidean space. As a result the development of a path $X$ is given by

$$G = I + \int_{0<u<} F(dX_u) + \cdots + \int_{0<u_1<\cdots<u_k<T} F(dX_{u_1}) \otimes \ldots \otimes F(dX_{u_k}) + \cdots$$
and, as in Lemma 2.3, if $X$ is a path of length $\theta$, then we have an a priori bound

$$\left\| \int_{0<u_1<\ldots<u_k<T} dX_{u_1} \otimes \ldots \otimes dX_{u_k} \right\| \leq \frac{\theta^n}{n!}.$$  

Applying this to $\alpha \gamma$, we conclude that

$$e^{d(o, \Gamma_\alpha o)} \leq \left\| \frac{\Gamma_\alpha}{\alpha} \right\| \leq \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} b_k \leq 1,$$

where $l$ is the length of $\gamma$. Letting

$$b_n = n! \left\| \int_{0<u_1<\ldots<u_k<1} d\gamma_{u_1} \otimes \ldots \otimes d\gamma_{u_k} \right\|,$$

one has for all $\alpha$

$$e^{d(o, \Gamma_\alpha o) - \alpha l} \leq e^{-\alpha l} \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} b_k \leq 1,$$

$$0 \leq b_k \leq l^k.$$

Thus the expectation of $b_n$ with respect to a Poisson measure with mean $\alpha l$ is close to one while at the same time the $b_n$ are all bounded above by one and positive. In particular

$$\sum_{k=0}^{\infty} \frac{\alpha^k}{k!} |l^k - b_k| \leq e^{\alpha l} - e^{d(o, \Gamma_\alpha o)} \leq e^{\alpha l} \left( 1 - e^{-D_1 \delta (D_2/\alpha)^2} \right),$$

and so

$$|l^k - b_k| \leq \inf_{\alpha>1} k! \alpha^{-k} e^{\alpha l} \left( 1 - e^{-D_1 \delta (D_2/\alpha)^2} \right)$$

applying Stirling’s formulae that $k! = e^{k \log k - k + \frac{1}{2} \log k + C_k}$ where $C_k = o(1)$ and setting $\alpha = k/l$ gives

$$|l^k - b_k| \leq e^{C_k k \sqrt{k}} \left( 1 - e^{-D_1 \delta (D_2/k)^2} \right) \leq l^k \tilde{C} \delta (lD_2/k)^2 k \sqrt{k},$$

where $\tilde{C} = D_1 e^{C_k}$ and so we see that, if $\delta_\gamma (lD_2/k)^2 k^{3/2} \to 0$ as $k \to \infty$, then $b_k/l^k \to 1$. Thus we have shown the following

**Theorem 4.** For any path of finite length with $\delta_\gamma (\varepsilon) = o(\varepsilon^{3/4})$,

$$l^{-k} k! \left\| \int_{0<u_1<\ldots<u_k<1} d\gamma_{u_1} \otimes \ldots \otimes d\gamma_{u_k} \right\| \to 1,$$

as $k \to \infty$.

This is of course quite a strong result obtained by making strong assumptions. One could ask less and so we give a weaker but more widely applicable result.
Theorem 5. Let $\gamma$ be a path of finite length $l$, and suppose its derivative, when parameterised at unit speed, is continuous. Then the Poisson averages $C_\alpha$ of the $b_k$ defined by

$$C_\alpha = e^{-\alpha} \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} b_k$$

satisfy

$$\lim_{\alpha \to \infty} \frac{1}{\alpha} \log C_\alpha = l - 1$$

Note that the $C_\alpha$ are averages of the $b_k$ against Poisson measures; it is standard that these are close to Gaussian with mean $\alpha$ and variance $\alpha$.

Proof. Note that

$$e^{d(o, \Gamma_\alpha o) - \alpha l} \leq e^{-\alpha} \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} b_k \leq 1$$

and so

$$(3.10) \quad \frac{d(o, \Gamma_\alpha o)}{\alpha} - l \leq \frac{1}{\alpha} \log \left( e^{-\alpha} \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} b_k \right) + 1 - l \leq 0$$

and using (3.9) we have

$$\left| \frac{d(o, \Gamma_\alpha o)}{\alpha} - l \right| \leq D_1 \delta_\gamma \left( D_2 / \alpha \right)^2 l$$

and so the left hand side in (3.10) goes to zero. $\square$

In particular we see that the high order coefficients of the signature already determine the length of the path and in fact one can obtain quantitative estimates in terms of the modulus of continuity for the derivative of $\gamma$.

Conjecture 3.10. The length of $\tilde{\gamma}$ can be recovered from the asymptotic behaviour of averages of the $b_k$.

It might be that $\lim_{\alpha \to \infty} 1 + \frac{1}{\alpha} \log C_\alpha$ gives the length of $\tilde{\gamma}$ directly although the Poisson averages may have to be replaced in some way.

We conclude with an analogous result to that proved for the lattice case in Proposition 2.

Theorem 6. Let $\gamma$ be a path of length $l$ parameterised at unit speed, and let $\delta_\gamma$ be the modulus of continuity for $\gamma$. Fix $C < 1$ and $1 \leq M \in \mathbb{N}$. Suppose that $\delta_\gamma(0) < \frac{1}{\sqrt{D_1(C,M)}}$, then there is an integer $N(l, \delta) = N(l, \delta)$ such that at least one of the first $N$ terms in the signature must be non-zero.

Proof. In the case where the first $eal$ coefficients in the signature of the path $\gamma$ are zero, by Lemma 2.4 we have some explicit constant $C_1$ such that

$$\|\Gamma_\alpha\| \leq 1 + \sum_{m > eal} \frac{(\alpha l)^m}{m!} \leq 1 + C_1 (\alpha l)^{-1/2}.$$
By Proposition 3.9 and letting \( \alpha \) be sufficiently large so that we can apply (3.9),

\[
\| \Gamma_\alpha \| \geq e^{d(0, \Gamma_\alpha)} \\
\geq e^{l \alpha - D_1 \delta_\gamma (D_2/\alpha)^2 \alpha \lambda} \\
\geq 1 + l \alpha - D_1 \delta_\gamma (D_2/\alpha)^2 \alpha \lambda.
\]

These two statements lead to a contradiction if for large \( \alpha \), we have

\[
l \alpha - D_1 \delta_\gamma (D_2/\alpha)^2 \alpha \lambda > C_1 (\alpha l)^{-1/2} \quad \text{or} \quad \alpha^{3/2} \left(1 - D_1 \delta_\gamma (D_2/\alpha)^2\right) > C_1 l^{-3/2}.
\]

Thus providing \( 1 > D_1 \delta_\gamma (D_2/\alpha)^2 \) for some large \( \alpha \) (continuity of the derivative is enough) then the left hand side goes to infinity as \( \alpha \to \infty \). This always gives a contradiction and shows that existence of \( N(l, \delta) \).

An explicit estimate is

\[
N(l, \delta) = \left\lceil e^{l} \right\rceil,
\]

where one chooses the smallest \( \alpha \geq MR_0 \), large enough so that

\[
\delta_\gamma \left(\frac{M + 1 R_0}{M} \alpha\right) < \sqrt{2 \left(\sqrt{2} - \sqrt{1 + C^2}\right) 4^{-\frac{M + 1}{2} R_0}}
\]

and so that \( \alpha^{3/2} \left(1 - D_1 \delta_\gamma (D_2/\alpha)^2\right) > C_1 l^{-3/2} \). To give an idea of the numerical size of \( N \), the number of iterated integrals required for this result, we note that if the path has \( \delta(h) \leq h \), then the optimal value of \( \alpha \) is around 15.2 at \( C = 0.8875 \) (with \( M = 1 \)) and the number \( N \) is the integer greater than 41.3875 (for large \( l \)).

With a more careful optimization of the constants (varying \( M \)) our estimate can be reduced to the integer greater than 13.2838 (for large \( l \)).

Remark 3.11. We note that this proof did not require that \( \delta_\gamma (0) = 0 \) or that \( \dot{\gamma} \) is continuous.

Remark 3.12. An easy way to produce a path with each of the first \( N \) iterated integrals zero is take two paths with the same signature up to the level of the \( N \)’th iterated integral and to take the first path concatenated with the second with time run backwards. Since these paths will, except at the point of joining, have the same smoothness as they did before, all focus goes to the point where they join. One could hope that a development of these ideas would prove that the two paths must be nearly tangential. If this were exactly true, then it would give a reconstruction theorem.

We have obtained quantitative lower bounds on the signature of \( \gamma \) when \( \gamma \) is parameterised at unit speed and \( \dot{\gamma} \) is close to continuously differentiable. In fact one could obtain estimates whenever the \( \dot{\gamma} \) is piecewise continuous and the jumps are less than \( \pi \). However the main extra idea is already visible in the case where \( \dot{\gamma} \) is piecewise constant. We give an explicit estimate in Theorem 9 in Section 6.

4. Tree-Like paths

We now turn to our proof of the extension of Chen’s theorem to the case of finite length paths. In this section we suppose that \( X \in [0, T] \) is a path in a Banach or metric space \( E \) and we recall our definition (3.3) of tree-like paths in this more general setting.
Theorem 7. If $X$ is a tree-like path with height function $h$ and, if $X$ is of bounded variation, then there exists a new height function $\tilde{h}$ having bounded variation and hence $X$ is a Lipschitz tree-like path; moreover, the variation of $\tilde{h}$ is bounded by the variation of $X$.

Proof. The function $h$ allows one to introduce a partial order and tree structure on $[0,T]$. Let $t \in [0,T]$. Define the continuous and monotone function $g_t(\cdot)$ by

$$g_t(v) = \inf_{v \leq u \leq t} h(u), \quad v \in [0,t].$$

The intermediate value theorem ensures that $g_t$ maps $[0,t]$ onto $[0,h(t)]$. Let $\tau_t$ be a maximal inverse of $h$ in that

$$\tau_t(x) = \sup \{u \in [0,t] | g_t(u) = x\}, \quad x \in [0,h(t)].$$

As $g_t$ is monotone and continuous

$$\tau_t(x) = \inf \{u \in [0,t] | g_t(u) > x\}$$

for $x < h(t)$.

Now say $s \leq t$ if and only if $s$ is in the range of $\tau_t$; that is to say if there is an $x \in [0,h(t)]$ so that $s = \tau_t(x)$. Since $\tau_t(h(t)) = \sup \{u \in [0,t] | g_t(u) = h(t)\}$, it follows that $\tau_t(h(t)) = t$ and so $t \leq s$. Since $h(\tau_t(x)) = x$ for $x \in [0,h(t)]$ we see there is an inequality-preserving bijection between the $\{s|s \leq t\}$ and $[0,h(t)]$.

Suppose $t_1 \leq t_0$ and that they are distinct; then $h(t_1) < h(t_0)$. We may choose $x_1 \in [0,h(t_0))$ so that $t_1 = \tau_{t_0}(x_1)$, it follows that

$$t_1 = \tau_{t_0}(x_1) = \inf \{u \in [0,t_0] | g_{t_0}(u) > x_1\},$$

and that

$$h(t_1) = x_1 < h(u), \quad u \in (t_1,t_0).$$

Of course

$$g_{t_0}(t_1) = \inf_{t_1 \leq u \leq t_0} h(u) = h(t_1) = g_t(t_1),$$

and hence $g_{t_0}(u) = g_t(u)$ for all $u \in [0,t_1]$. Hence, $\tau_{t_0}(x) = \tau_t(x)$ for any $x < g_{t_1}(t_1) = h(t_1) = x_1$; we have already seen that $\tau_{t_1}(h(t_1)) = t_1 = \tau_{t_0}(x_1)$. It follows that the range $\tau_{t_1}([0,h(t_1)])$ is contained in the range of $\tau_{t_0}$. In particular, we deduce that if $t_2 \leq t_1$ and $t_1 \leq t_0$ then $t_2 \leq t_0$.

We have shown that $\leq$ is a partial order, and that $\{t|t \leq t_0\}$ is totally ordered under $\leq$, and in one to one correspondence with $[0,h(t_0)]$.

Now, consider two generic times $s < t$. Let $x_0 = \inf_{s \leq u \leq t} h(u)$ and $I = \{v \in [s,t] | h(v) = x_0\}$. Since $h$ is continuous and $[s,t]$ is compact the set $I$ is non-empty and compact. By the construction of the function $g_t$ it is obvious that $g_t \leq g_s$ on $[0,s]$ and that if $g_t(u) = g_s(u)$, then $g_t(v) = g_s(v)$ for $v \in [0,u]$. Thus, there will be a unique $r \in [0,s]$ so that $g_s = g_t$ on $[0,r]$ and $g_t < g_s$ on $(r,s]$. Observe that $g_t(r) = x_0$ and that $\tau_t(x_0) = \sup I$ and, essentially as above $\tau_s = \tau_t$ on $[0,h(r)]$. Observe also that if $\tilde{t} \in [s,t]$ then $g_s = g_{\tilde{t}}$ on $[0,\tilde{t}]$ so that $\tau_s = \tau_{\tilde{t}}$ on $[0,h(\tilde{t})]$. 

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Having understood \( h \) and \( \tau \) to the necessary level of detail, we return to the path \( X \). For \( x, y \in [0, h(t)] \) one has, for \( x < y \),
\[
\| X_{\tau_i(x)} - X_{\tau_i(y)} \| \leq h(\tau_i(x)) + h(\tau_i(y)) - 2 \inf_{u \in [\tau(x), \tau(y)]} h(u) \\
\leq x + y - 2 \inf_{z \in [x, y]} h(\tau_i(z)) \\
= y - x
\]
so we see that \( X_{\tau_i(\cdot)} \) is continuous and of bounded variation.

The intuition is that \( X_{\tau_i(\cdot)} \) is the branch of a tree corresponding to the time \( t \). Consider two generic times \( s < t \), then \( X_{\tau_i(s)} \) and \( X_{\tau_i(t)} \) agree on the initial segment \([0, h(r)]\) but thereafter \( \tau_s(\cdot) \in [r, s] \) while \( \tau_t(\cdot) \in [\sup I, t] \). The restriction of \( X_{\tau_i(\cdot)} \) to the initial segment \([0, h(r)]\) is the path \( X_{\tau_i(h(r))} \). As \( h(r) = \inf [h(u) | u \in [s, t]] \) they have independent trajectories after \( h(r) \).

Let \( \tilde{h}(t) \) be the total 1-variation of the path \( X_{\tau_i(\cdot)} \). The claim is that \( \tilde{h} \) has total 1-variation bounded by that of \( X \) and is also a height function for \( X \).

As the paths \( X_{\tau_i(\cdot)} \) and \( X_{\tau_i(\cdot)} \) share the common segment \( X_{\tau_i(\cdot)} \) we have
\[
\| X_s - X_t \| \leq \tilde{h}(t) - \tilde{h}(r) + \tilde{h}(s) - \tilde{h}(r),
\]
and in particular
\[
\| X_s - X_t \| \leq \tilde{h}(s) + \tilde{h}(t) - 2 \tilde{h}(r).
\]
On the other hand \( \tilde{h}(r) = \tilde{h}(\sup I) = \inf_{s \leq u \leq t} (\tilde{h}(u)) \) and so
\[
\| X(s) - X(t) \| \leq \tilde{h}(s) + \tilde{h}(t) - 2 \inf_{s \leq u \leq t} (\tilde{h}(u)).
\]
and \( \tilde{h} \) is a height function for \( X \).

Finally we control the total variation of \( \tilde{h} \) by \( \omega_X \), the total variation of the path. In fact,
\[
\left| \tilde{h}(s) - \tilde{h}(t) \right| \leq \tilde{h}(s) + \tilde{h}(t) - 2 \inf_{s \leq u \leq t} (\tilde{h}(u)) \\
\leq \omega_X(s, t),
\]
where \( \omega_X(s, t) = \sup_{D \in D} \sum_{D} \| X_{t_{i+1}} - X_{t_i} \| \), with \( D \) denoting the set of all partitions of \([s, t]\) and for \( D \in D \), then \( D = \{s < t_1 < t_2 < \cdots < t \leq t\} \). The first of these inequalities is trivial, the second needs explanation. As before, notice that the paths \( X_{\tau_i(\cdot)} \) and \( X_{\tau_i(\cdot)} \) share the common segment \( X_{\tau_i(\cdot)} \) and that
\[
\inf_{s \leq u \leq t} (\tilde{h}(u)) = \tilde{h}(r).
\]
So \( \tilde{h}(s) + \tilde{h}(t) - 2 \inf_{s \leq u \leq t} (\tilde{h}(u)) \) is the total length of the two segments \( X_{\tau_i(h(r))} \) and \( X_{\tau_i(h(s))} \). Now the total variation of \( X_{\tau_i(h(r), h(s))} \) is obviously bounded by \( \omega_X(\sup I, t) \), as the path \( X_{\tau_i(h(r), h(s))} \) is a time change of \( X_{\sup I, t} \).

It is enough to show that the total length of \( X_{\tau_i(h(r), h(s))} \) is controlled by \( \omega_X(s, \inf I) \) to conclude that
\[
\tilde{h}(s) + \tilde{h}(t) - 2 \inf_{s \leq u \leq t} (\tilde{h}(u)) \leq \omega_X(s, t).
\]
In order to do this we work backwards in time. Let
\[
f_s(u) = \inf_{s \leq v \leq u} h(v) \\
\rho_s(x) = \inf \{u \in [s, T] | f_s(u) = x\} \]
then, because $X$ is tree-like

$$X_{\tau_{r}(\cdot)}|_{[0, h(s)]} = X_{\rho_{r}(\cdot)}|_{[0, h(s)]},$$

and in particular, the path segment $X_{\rho_{r}(\cdot)}|_{[h(r), h(s)]}$ is a time change (but backwards) of $X|_{[s, \inf t]}$. \hfill \Box

The property of being tree-like is re-parameterisation invariant. We see informally that a tree-like path $X$ is the composition of a contraction on the $R$-tree defined by $h$ and the based loop in this tree obtained by taking $t \in [0, T]$ to its equivalence class under the metric induced by $h$ (for definitions and a proof see the Appendix).

Any path that can be factored through a based loop of finite length in an $R$-tree and a contraction of that tree to the space $E$ is a Lipschitz tree-like path. If $0$ is the root of the tree and $\phi$ is the based loop defined on $[0, T]$, then define $h(t) = d(0, \phi(t))$. This makes $\phi$ a tree-like path. Any Lipschitz image of a tree-like path is obviously a Lipschitz tree-like path.

We have the following trivial lemma.

**Lemma 4.1.** A Lipschitz tree-like path $X$ always has bounded variation less than that of any height function $h$ for $X$.

**Proof.** Let $D = \{ t_0 < \ldots < t_n \}$ be a partition of $[0, T]$. Choose $u_i \in [t_{i-1}, t_i]$ maximising $h(t_i) + h(t_{i-1}) - 2h(u_i)$ and let $\tilde{D} = \{ t_0 \leq u_1 \ldots \leq t_{n-1} \leq u_n \leq u_n \}$. Relabel the points of $\tilde{D} = \{ v_0 \leq v_1 \ldots \leq v_m \}$ Then

$$\sum_D \| X_{t_i} - X_{t_{i-1}} \| \leq \sum_D \| h(v_i) - h(v_{i-1}) \|.$$

\hfill \Box

We now prove a compactness result.

**Lemma 4.2.** Suppose that \{ $h_n$ \} are a sequence of height functions on $[0, T]$ for a sequence of tree-like paths \{ $X_n$ \}. Suppose further that the $h_n$ are parameterised at speeds of at most one and that the $X_n$ take their values in a common compact set within $E$. Then we may find a subsequence \{ $(X_{n(k)}, h_{n(k)})$ \} converging uniformly to a Lipschitz tree-like path $(Y, h)$. The speed of traversing $h$ is at most one.

**Proof.** The $h_n$ are equi-continuous, and in view of \[26\] the $X_n$ are as well. Our hypotheses are sufficient for us to apply the Arzela-Ascoli theorem to obtain a subsequence \{ $(X_{n(k)}, h_{n(k)})$ \} converging uniformly to some $(Y, h)$. In view of the fact that the Lip norm is lower semi-continuous in the uniform topology, we see that $h$ is a bounded variation function parameterised at speed at most one and that $Y$ is of bounded variation; of course $h$ takes the value 0 at both ends of the interval $[0, T]$.

Now $h_{n(k)}$ converge uniformly to $h$ and hence $\inf_{u \in [s, t]} h_{n(k)}(u) \to \inf_{u \in [s, t]} h(u)$; meanwhile the $h_n$ are height functions for the tree-like paths $X_n$ and hence we can take limits through the definition to show that $h$ is a height function for $Y$. \hfill \Box

**Corollary 4.3.** Every Lipschitz tree-like path $X$ has a height function $h$ of minimal total variation and its total variation measure is boundedly absolutely continuous with respect to the total variation measure of any other height function.
Proof. We see that this is an immediate corollary of Proposition 7 and Lemma 4.1 and □

There can be more than one minimiser \( h \) for a given \( X \).

5. Approximation of the path

5.1. Representing the path as a line integral against a rank one form. Let \( \gamma \) be a path of finite variation in a finite dimensional Euclidean space \( V \) with total length \( T \) and parameterised at unit speed. Its parameter set is \([0, T]\). We note that the signature of \( \gamma \) is unaffected by this choice of parameterisation.

Definition 5.1. Let \( \gamma ([0, T]) \) denote the range of \( \gamma \) in \( V \) and let the occupation measure \( \mu \) on \((V, B(V))\) be denoted

\[
\mu (A) = |\{s < T| \gamma (s) \in A\}|, \ A \subset V.
\]

Let \( n(x) \) be the number of points on \([0, T]\) corresponding under \( \gamma \) to \( x \in E \). By the area formulae \([11]\) p125-126, one has the total variation, or length, of the path \( \gamma \) is given by

\[
(5.1) \quad \text{Var} (\gamma) = \int n(x) \Lambda_1 (dx),
\]

where \( \Lambda_1 \) is one dimensional Hausdorff measure. Moreover, for any continuous function \( f \)

\[
\int f (\gamma (t)) \ dt = \int f (x) n(x) \Lambda_1 (dx).
\]

Note that \( \mu = n(x) \Lambda_1 \) and that \( n \) is integrable.

Lemma 5.2. The image under \( \gamma \) of a Lebesgue null set is null for \( \mu \). That is to say \( \mu (\gamma (N)) = |\gamma^{-1}\gamma (N)| = 0 \) if \( |N| = 0 \).

Definition 5.3. We will say that \( N \subset [0, T] \) is \( \gamma \)-stable if \( \gamma^{-1}\gamma (N) = N \).

As a result of Lemma 5.2 we see that any null set can always be enlarged to a \( \gamma \)-stable null set.

The Lebesgue differentiation theorem tells us that \( \gamma \) is differentiable at almost every \( u \) in the classical sense, and with this parameterisation the derivative will be absolutely continuous and of modulus one.

Corollary 5.4. There is a set \( G \) of full \( \mu \) measure in \( V \) so that \( \gamma \) is differentiable with \( |\gamma' (t)| = 1 \) whenever \( \gamma (t) \in G \). We set \( M = \gamma^{-1}G \). \( M \) is \( \gamma \)-stable.

Now it may well happen that the path visits the same point \( x \in G \) more than once. A priori, there is no reason why the direct ions of the derivative on \( \{t \in M| \gamma (t) = m\} \) should not vary. However this can only occur at a countable number of points.

Lemma 5.5. The set of pairs \((s, t)\) of distinct times in \( M \times M \) for which

\[
\gamma (s) = \gamma (t), \quad \gamma' (s) \neq \pm \gamma' (t)
\]

is countable.
Proof: If $\gamma(s_*) = \gamma(t_*)$ but $\gamma'(s_*) \neq \pm \gamma'(t_*)$ then, by a routine transversality argument, there is an open neighbourhood of $(s_*, t_*)$ in which there are no solutions of $\gamma(s) = \gamma(t)$ except $s = s_*, t = t_*$. 

Up to sign and with countably many exceptions, the derivative of $\gamma$ does not depend on the occasion of the visit to a point, only the location. Sometimes we will only be concerned with the unsigned or projective direction of $\gamma$ and identify $v \in S$ with $-v$.

**Definition 5.6.** For clarity we introduce $\gamma'_\pm$ as the equivalence relation that identifies $v$ and $-v$ and let $[\gamma'_t]_-^\pm \in S/\sim^\pm$ denote the unsigned direction of $\gamma$.

$\gamma'_\pm$ is defined on the full measure subset of $[0, T]$ where $\gamma'$ is defined and in $S$.

**Corollary 5.7.** There is a function $\phi$ defined on $G$ with values in the projective sphere $S/\sim^\pm$ so that $\phi(\gamma(t)) = [\gamma'_t]_-^\pm$.

As a result we may define a useful vector valued 1-form $\mu$-almost everywhere on $G$. If $\xi$ is a vector in $S$, then $\langle \xi, u \rangle \xi$ is the linear projection of $u$ onto the subspace spanned by $\xi$. As $\langle \xi, u \rangle \xi = -\langle \xi, u \rangle (-\xi)$ it defines a function from $S/\sim^\pm$ to $\text{Hom}(V, V)$.

**Definition 5.8.** We define the tangential projection 1-form $\omega$. Let $\xi$ be a unit strength vector field on $G$ with $[\xi]_-^\pm = \phi$. Then

$$
\omega(g, u) = \langle \xi(g), u \rangle \xi(g), \forall g \in G, \forall u
$$

defines a vector 1-form. The 1-form depends on $\phi$, but is otherwise independent of the choice of $\xi$.

The 1-form $\omega$ is the projection of $u$ onto the line determined by $\phi(g)$.

**Theorem 8.**

**Proposition 5.9.** The tangential projection $\omega$, defined $\mu$ a.e. on $G$, is a linear map from $V \to V$ with rank one. For almost every $t$ one has

$$
\gamma'(t) = \omega(\gamma(t), \gamma'(t))
$$

and as a result, using the fundamental theorem of calculus for Lipschitz functions,

$$
\gamma(t) = \int_{0 < u < t} d\gamma_u + \gamma(0) = \int_{0 < u < t} \omega \circ d\gamma_u + \gamma(0),
$$

for every $t \leq T$.

By approximating $\omega$ by other rank one 1-forms we will be able to approximate $\gamma$ by (weakly) piecewise linear paths that also have trivial signature. It will be easy to see that such paths are tree-like. The set of tree-like paths is closed. This will complete the argument.
5.2. Iterated integrals of iterated integrals. We now prove that if \( \gamma \) has a trivial signature \((1,0,0,\ldots)\), then it can always be approximated arbitrarily well by weakly piecewise linear paths with shorter length and trivial signature. Our approximations will all be line integrals of 1-forms against our basic path \( \gamma \). Two key points we will need are that the integrals are continuous against varying the 1-form, and that a line integral of a path with trivial signature also has trivial signature. The Stone-Weierstrass theorem will allow us to reduce this second problem to one concerning line integrals against polynomial 1-forms, and in turn this will reduce to the study of certain iterated integrals. The application of the Stone-Weierstrass theorem requires a commutative algebra structure and this is provided by the coordinate iterated integrals and the shuffle product. For completeness we set this out below.

Suppose that we define

\[
Z_u := \int \cdots \int_{0<u_1<\ldots<u_r<u} d\gamma_{u_1} \cdots d\gamma_{u_r} \in V^\otimes r
\]

and

\[
\tilde{Z}_u := \int \cdots \int_{0<u_1<\ldots<u_r<u} d\gamma_{u_1} \cdots d\gamma_{u_r} \in V^\otimes r,
\]

then it is interesting as a general point, and necessary here, to consider iterated integrals of \( Z \) and \( \tilde{Z} \)

\[
\int \cdots \int_{0<u_1<u_2<T} d\tilde{Z}_{u_1} dZ_{u_2} \in V^\otimes \tilde{r} \otimes V^\otimes r.
\]

It will be technically important to us to observe that such integrals can also be expressed as linear combinations of iterated integrals of \( \gamma \) so we do this with some care. Some of the results stated below follow from the well known shuffle product and its relationship with multiplication of coordinate iterated integrals.

Definition 5.10. The truncated or \( n \)-signature \( X_{s,t}^{(n)} = (1, X_{s,t}^{1}, X_{s,t}^{2}, \ldots, X_{s,t}^{n}) \) is the projection of the signature \( X_{s,t} \) to the algebra \( T^{(n)}(V) := \bigoplus_{r=0}^{n} V^\otimes r \) of tensors with degree at most \( n \).

Definition 5.11. If \( e \) is an element of the dual space \( V^* \) to \( V \), then \( \gamma^e_u = \langle e, \gamma_u \rangle \) is a scalar path and \( d\gamma^e_u = \langle e, d\gamma_u \rangle \). If \( e = (e_1, \ldots, e_r) \) is a list of elements of the dual space to \( V \), then we define the coordinate iterated integral

\[
X^e_{s,t} := \int \cdots \int_{s<u_1<\ldots<u_r<t} d\gamma^{e_1}_{u_1} \cdots d\gamma^{e_r}_{u_r} = \langle e, X^r_{s,t} \rangle.
\]

Lemma 5.12. The map \( e \to X^e_{s,t} \) defined above extends uniquely as a linear map from \( T^{(n)}(V^*) \) to the space of real valued functions on paths of bounded variation.

Proof. Let \( e \in T^{(n)}(V^*) \). Since \( T^{(n)}(V^*) \) is dual to \( T^{(n)}(V) \) the pairing \( e \to \langle e, X^r_{s,t}^{(n)} \rangle \) defines a real number of each path \( \gamma \). If \( e = (e_1, \ldots, e_r) \) then this coincides with \( X^e_{s,t} \); since such vectors span \( T^{(n)}(V^*) \) the result is immediate. \( \square \)

We therefore extend definition 5.11.
Definition 5.13. For any \( n, e \in T^{(n)}(V^*) \) we call \( X_{s,t}^e \) the e-coordinate iterated integral of \( \gamma \) over the interval \([s,t]\).

These functions on path space are important because they form an algebra under pointwise multiplication and because they are like polynomials and so it is easy to define a differentiation operator on this space. Given two tensors \( e, f \) there is a natural product \( e \cup f \), called the shuffle product, derived from the above. For basic tensors

\[
e = e_1 \otimes \ldots \otimes e_r \in V^\otimes r
\]
\[
f = f_1 \otimes \ldots \otimes f_s \in V^\otimes s
\]

and a shuffle \((\pi_1, \pi_2)\) (a pair of increasing injective functions from \((1, \ldots, r), (1, \ldots, s)\) to \((1, \ldots, r+s)\) with disjoint range) one can define a tensor of degree \( r+s \):

\[
\omega_{(\pi_1, \pi_2)} = \omega_1 \otimes \ldots \otimes \omega_{r+s},
\]

where \( \omega_{\pi_1(j)} = e_j \) for \( j = 1, \ldots r \) and \( \omega_{\pi_2(j)} = f_j \) for \( j = 1, \ldots s \). Since the ranges of \( \pi_1 \) and \( \pi_2 \) are disjoint a counting argument shows that the union of the ranges is \( 1, \ldots, r+s \), and that \( \omega_k \) is well defined for all \( k \) in \( 1, \ldots r+s \) and hence \( \omega_{(\pi_1, \pi_2)} \) is defined. By summing over all shuffles

\[
e \cup f = \sum_{(\pi_1, \pi_2)} \omega_{(\pi_1, \pi_2)}
\]

one defines a multilinear map of \( V^\otimes r \times V^\otimes s \to V^\otimes (r+s) \).

Definition 5.14. The unique extension of \( \cup \) to a map from \( T(V) \times T(V) \to T(V) \) is called the shuffle product.

The following is standard.

Lemma 5.15. The class of coordinate iterated integrals is closed under pointwise multiplication. For each \( \gamma \) the point-wise product of the e-coordinate iterated integral and the f-coordinate iterated integral is the \((e \cup f)\)-coordinate iterated integral:

\[
X_{s,t}^e X_{s,t}^f = X_{s,t}^{e \cup f}.
\]

Corollary 5.16. Any polynomial in coordinate iterated integrals is a coordinate iterated integral.

Remark 5.17. It is at first sight surprising that any polynomial in the linear functionals on \( T(V) \) coincides with a unique linear functional on \( T(V) \) when restricted to signatures of paths and reflects the fact that the signature of a path is far from being a generic element of the tensor algebra.

A slightly more demanding remark relates to iterated integrals of coordinate iterated integrals.

Proposition 5.18. The iterated integral

\[
\int \cdots \int_{s < u_1 < \ldots < u_r < t} dX_{s,u_1}^{e_1} \ldots dX_{s,u_r}^{e_r}
\]

is itself a coordinate iterated integral.
Proof. A simple induction ensures that it suffices to consider the case
\[
\int \int_{s < u_1 < u_2 < t} dX_{s, u_1}^e dX_{s, u_2}^f,
\]
where
\[
e = e_1 \otimes \ldots \otimes e_r \in (V^*)^r
\]
\[
f = f_1 \otimes \ldots \otimes f_s \in (V^*)^s
\]
and in this case
\[
\int \int_{s < u_1 < u_2 < t} dX_{s, u_1}^e dX_{s, u_2}^f = \int \ldots \int d\gamma_{v_1}^{e_1} \ldots d\gamma_{v_r}^{e_r} d\gamma_{w_1}^{f_1} \ldots d\gamma_{w_s}^{f_s}.
\]
Expressing the integral as a sum of integrals over the regions where the relative ordering of the \(v_i\) and \(w_j\) are preserved (i.e. all shuffles for which the last card comes from the right hand pack) we have
\[
\int \int_{s < u_1 < u_2 < t} dX_{s, u_1}^e dX_{s, u_2}^f = \chi_{s,t}^{(e \otimes f)} \otimes f_s
\]
\[
\tilde{f} = f_1 \otimes \ldots \otimes f_{s-1}.
\]
\[\square\]

From this it is, of course, clear that

Lemma 5.19. If a path has trivial signature, then all iterated integrals of its iterated integrals are zero.

5.3. Bounded, measurable, and integrable forms. Recall that \(\gamma\) is a path of finite length in \(V\), and that it is parameterized at unit speed. The occupation measure is \(\mu\) and has total mass equal to the length \(T\) of the path \(\gamma\). Let \((W, \|\|)\) be a normed space with a countable base (usually \(V\) itself). If \(\omega\) is a \(\mu\)-integrable 1-form with values in \(W\) then we write
\[
\|\omega\|_{L^1(V, B(V), \mu)} = \int_V \|\omega(y)\|_{\text{Hom}(V, W)} \mu(dy) = \int_0^T \|\omega(\gamma_t)\|_{\text{Hom}(V, W)} dt.
\]

Proposition 5.20. Let \(\omega \in L^1(V, B(V), \mu)\) be a \(\mu\)-integrable 1-form with values in \(W\). Then the indefinite line integral \(y_t := \int_0^t \omega(d\gamma_t)\) is well defined, linear in \(\omega\), and a path in \(W\) with 1-variation at most \(\|\omega\|_{L^1(V, B(V), \mu)}\).

Proof. Since \(\omega\) is a 1-form defined \(\mu\)-almost surely, \(\omega(\gamma_t) \in \text{Hom}(V, W)\) (where \(\text{Hom}(V, W)\) is equipped with the operator norm \(\|\|\)) is defined \(dt\) almost everywhere. Since \(\omega\) is integrable, it is measurable, and hence \(\omega(\gamma_t)\) is measurable on \([0, T]\). Since \(\gamma\) has finite variation and is parameterized at unit speed, it is differentiable almost everywhere and its derivative is measurable with unit length \(dt\) almost surely. Hence \(\omega(\gamma_t)(\gamma_t')\) is measurable and dominated by \(\|\omega(\gamma_t)\|\), which
is an integrable function, and hence \( \omega(\gamma)(\dot{\gamma}) \) is integrable. Thus the line integral can be defined to be

\[
y_t = \int_0^t \omega(\gamma_u)(\dot{\gamma}_u) \, du
\]

and so has 1-variation bounded by \( \|\omega\|_{L^1(V,B(V))} \).

\[\square\]

**Proposition 5.21.** Let \( \omega_n \in L^1(V,B(V),\mu) \) be a uniformly bounded sequence of integrable 1-forms with values in a vector space \( W \). Suppose that they converge in \( L^1(V,B(V),\mu) \) to \( \omega \), then the signatures of the line integrals \( \int \omega_n(d\gamma) \) converge to the signature of \( \int \omega(d\gamma) \).

**Proof.** The \( r \)’th term in the iterated integral of the line integral \( \int \omega_n(d\gamma) \) can be expressed as

\[
\int_{0<u_1<\ldots<u_r<T} \cdots \int \omega_n(\gamma_{u_1}) \otimes \ldots \otimes \omega_n(\gamma_{u_r})(\dot{\gamma}_{u_1}) \ldots (\dot{\gamma}_{u_r}) \, du_1 \ldots du_r
\]

and since \( \omega_n \) converge in \( L^1(V,B(V),\mu) \), it follows that from the definition of \( \mu \) that \( \omega_n(\gamma_u) \) converge to \( \omega(\gamma_u) \) in \( L^1([0,T],B(\mathbb{R}),du) \) almost everywhere. Thus \( \omega_n(\gamma_{u_1}) \otimes \ldots \otimes \omega_n(\gamma_{u_r})(\dot{\gamma}_{u_1}) \ldots (\dot{\gamma}_{u_r}) \) converges in \( L^1([0,T]^r, B(\mathbb{R}),du_1\ldots du_r) \) to

\[
\omega(\gamma_{u_1}) \otimes \ldots \otimes \omega(\gamma_{u_r})(\dot{\gamma}_{u_1}) \ldots (\dot{\gamma}_{u_r}).
\]

Thus, integrating over \( 0<u_1<\ldots<u_r<T \), the proposition follows. \[\square\]

**Corollary 5.22.** Let \( \omega \in L^1(V,B(V),\mu) \). If \( \gamma \) has trivial signature, then so does \( \int \omega(d\gamma) \). That is to say, for each \( r \),

\[
\int_{0<u_1,\ldots,u_r<T} \cdots \int \omega(d\gamma_{u_1}) \ldots \omega(d\gamma_{u_r}) = 0 \in W^\otimes r.
\]

**Proof.** It is a consequence of Proposition 5.21 that the set of \( L^1(V,B(V),\mu) \) forms producing line integrals having trivial signature is closed. By Lusin’s theorem, one may approximate, in the \( L^1(V,B(V),\mu) \) norm, any integrable form by bounded continuous forms. If the initial form is uniformly bounded then the approximations can be chosen to satisfy the same uniform bound.

The support of \( \mu \) is compact, so by the Stone Weierstrass theorem, we can uniformly approximate these continuous forms by polynomial forms \( \omega = \sum_i p_i e_i \), where the \( p_i \) are polynomials and \( e_i \) are a basis for \( V^* \). Using the fact that

\[
\frac{(\gamma^e_1)^r}{r!} = \int_{0<u_1,\ldots,u_r<T} \cdots \int d\gamma_{e_1}^u \ldots d\gamma_{e_1}^u,
\]
with Corollary 5.16 and Proposition 5.18 we have that the line integrals against these polynomial forms and their iterated integrals can be expressed as linear combinations of coordinate iterated integrals. If \( \gamma \) has trivial signature, then by Lemma 5.19 these integrals will all be zero. It follows from the \( L_1 \) \((V, B(V), \mu)\) continuity of the truncated signature, that the signature of the path formed by taking the line integral against any form \( \omega \) in \( L_1 \) \((V, B(V), \mu)\) will always be trivial. \( \Box \)

5.4. Approximating rank one 1-forms.

**Definition 5.23.** A vector valued 1-form \( \omega \) is (at each point of \( V \)) a linear map between vector spaces. We say the 1-form \( \omega \) is of rank \( k \in \mathbb{N} \) on the support of \( \mu \) if \( \dim (\omega(V)) \leq k \) at \( \mu \)-almost every point in \( V \).

A linear multiple of a form has the same rank as the original form, but in general the sum of two forms has any rank less than or equal to the sum of the ranks of the individual components. However, we will now explain how one can approximate any rank one 1-form by piecewise constant rank one 1-forms \( \omega \). Additionally we will choose the approximations so that, for some \( \varepsilon > 0 \), if \( \omega(x) \neq \omega(y) \) and \( |x - y| \leq \varepsilon \), then either \( \omega(x) \) or \( \omega(y) \) is zero.

In other words \( \omega \) is rank one and constant on patches which are separated by thin barrier regions on which it is zero. The patches can be chosen to be compact and so that the \( \mu \)-measure of the compliment is arbitrarily small.

We will use the following easy consequence of Lusin’s theorem for one forms defined on a \( \mu \)-measurable set \( K \): 

**Lemma 5.24.** Let \( \omega \) be a measurable 1-form \( \omega \) in \( L^1(V, B(V), \mu) \). For each \( \varepsilon > 0 \) there is a compact subset \( L \) of \( \gamma [0, T] \) so that \( \omega \) restricted to \( L \) is continuous, while \( \int_{K \setminus L} \| \omega \|_{Hom(V,W)} \mu(dx) < \varepsilon \).

**Lemma 5.25.** If \( \omega \) is a measurable 1-form in \( L^1(V, B(V), \mu) \), then for each \( \varepsilon > 0 \) there are finitely many disjoint compact subsets \( K_i \) of \( K \) and a 1-form \( \tilde{\omega} \), that is zero off \( \bigcup_i K_i \) and constant on each \( K_i \), such that \( \int_K \| \omega - \tilde{\omega} \|_{Hom(V,W)} \mu(dx) \leq 4 \varepsilon \) and with the property that \( \tilde{\omega} \) is rank one if \( \omega \) is.

**Proof.** Let \( L \) be the compact subset introduced in Lemma 5.24. Now \( \omega(L) \) is compact. Fix \( \varepsilon > 0 \) and choose \( I_1, \ldots, I_n \) so that 

\[ \omega(L) \subset \bigcup_{i=1}^n B_i \left( \omega(I_i), \frac{\varepsilon}{\mu(L)} \right) \]

and put 

\[ F_j = \omega^{-1} \left( \bigcup_{i=1}^j B_i \left( \omega(I_i), \frac{\varepsilon}{\mu(L)} \right) \right) . \]

Now choose a compact set \( K_j \subset F_j \setminus F_{j-1} \) so that 

\[ \mu((F_j \setminus F_{j-1}) \setminus K_j) \leq \frac{\varepsilon 2^{-j}}{\| \omega \|_{L^\infty(L, B(L), \mu)}}. \]
Then the $K_j$ are disjoint and of diameter $\frac{2\varepsilon}{\mu(L)}$. Moreover

$$L = P_n \varepsilon$$

$$\mu (L \cup \bigcup_{i=1}^n K_j) \leq \|\omega\|_{L^\infty(L,\mathcal{B}(L),\mu)}$$

and

$$\int_{L \setminus \bigcup_{i=1}^n K_j} \|\omega\|_{Hom(V,W)} \mu (dx) < \varepsilon.$$

For each non-empty $K_j$ choose $k_j \in K_j$. Define $\tilde{\omega}$ as follows.

$$\tilde{\omega} (k) = \omega (k_j), \quad k \in K_j$$

$$\tilde{\omega} (k) = 0, \quad k \in K \setminus \bigcup_{i=1}^n K_j.$$ Then

$$\int_{L \cup \bigcup_{i=1}^n K_j} \|\omega - \tilde{\omega}\|_{Hom(V,W)} \mu (dx) < \varepsilon$$

$$\int_{\bigcup_{i=1}^n K_j} \|\omega - \tilde{\omega}\|_{Hom(V,W)} \mu (dx) < \frac{2\varepsilon}{\mu(L)} \mu(L),$$

using Lemma 5.24 one has

$$\int_{K \setminus L} \|\omega - \tilde{\omega}\|_{Hom(V,W)} \mu (dx) < \varepsilon$$

and finally

$$\int_{K} \|\omega - \tilde{\omega}\|_{Hom(V,W)} \mu (dx) \leq 4\varepsilon.$$

If $\omega$ had rank 1 at almost every point of $K$, then it will have rank 1 everywhere on $L$ since $\omega$ is continuous. As either $\tilde{\omega} (k) = \omega (k_j)$ for some $k_j \in L$ or is zero, the form $\tilde{\omega}$ has rank one also. □

**Proposition 5.26.** Consider the set $P$ of one forms on a set $K$ for which there exists finitely many disjoint compact subsets $K_i$ of $K$ so that the 1-form is zero off the $K_i$ and constant on each $K_i$. The set of rank one 1-forms in $P$ is a dense subset in the $L^1 (V,\mathcal{B}(V),\mu)$ topology of the set of rank one 1-forms in $L^1 (V,\mathcal{B}(V),\mu)$.

**6. Piecewise linear paths with no repeated edges**

We call a path $\gamma$ piecewise linear if it is continuous, and if there is a finite partition

$$0 = t_0 < t_1 < t_2 < \ldots < t_r = T$$

such that $\gamma$ is linear (or more generally, geodesic) on each segment $[t_i, t_{i+1}]$.

**Definition 6.1.** We say the path is non-degenerate if we can choose the partition so that $[\gamma_{t_{i-1}}, \gamma_{t_i}]$ and $[\gamma_{t_i}, \gamma_{t_{i+1}}]$ are not collinear for any $0 < i < r$ and if the $[\gamma_{t_{i-1}}, \gamma_{t_i}]$ are non-zero for every $0 < i \leq r$.

The positive length condition is automatic if the path is parameterised at unit speed and $0 < T$. If $\theta_i$ is the angle $\angle \gamma_{t_{i-1}} \gamma_{t_i} \gamma_{t_{i+1}}$, then $\gamma$ is non-degenerate if we can find a partition so that for each $0 < i < r$ one has

$$|\theta_i| \neq 0 \mod \pi.$$
This partition is unique, and we refer to the \( \gamma_{t_{i-1}} \) as the \( i \)-th linear segment in \( \gamma \). We see, from the quantitative estimate in part 1 of Lemma 3.7, that if we choose \( \theta = \frac{1}{2} \min |\theta| \) and scale it so that the length of the minimal segment is at least \( K(\theta) = \log \left( \frac{2}{1 - \cos |\theta|} \right) \), then its development into hyperbolic space is non-trivial and so its signature is not zero. That is to say, Lemma 3.7 contains all the information you need to give a quantitative form of Chen’s uniqueness result in the context of piecewise linear paths:

**Theorem 9.** If \( \gamma \) is a non-degenerate piecewise linear path, \( 2\theta \) is the smallest angle between adjacent edges, and \( D > 0 \) is the length of the shortest edge then there is at least one \( n \) for which

\[
\left( \frac{2}{1 - \cos |\theta|} \right)^{1/2} \leq n! \left\| \int \cdots \int d\gamma_{n} \right\|
\]

and in particular \( \gamma \) has non-trivial signature.

**Proof.** Choose \( \alpha = K(\theta)/D \). Isometrically embed \( V \) into \( SO(I_d) \) and let \( \Gamma_{\alpha} \) be the development of \( \alpha \gamma \). Then \( \Gamma_{\alpha,o} \) is a piecewise geodesic path in hyperbolic space satisfying the hypotheses in Lemma 3.7. Thus we can deduce that the distance \( d(o, \Gamma_{\alpha,o}) \) is at least \( K(\theta) > 0 \). As in the discussion before Theorem 3.4 in Section 3.4 we have

\[
e^{K(\theta)} \leq \left\| \Gamma_{\alpha} \right\| = \sum_{n=0}^{\infty} \alpha^n \left\| \int \cdots \int d\gamma_{1} \cdots d\gamma_{n} \right\| = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{K(\theta)}{D} \right)^n n! \left\| \int \cdots \int d\gamma_{1} \cdots d\gamma_{n} \right\|.
\]

Now multiplying both sides by \( e^{-K(\theta)/D} \) we have

\[
e^{-K(\theta)/D} e^{K(\theta)} \leq e^{-K(\theta)/D} \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{K(\theta)}{D} \right)^n n! \left\| \int \cdots \int d\gamma_{1} \cdots d\gamma_{n} \right\|.
\]

Since any integrable function has at least one point where its value equals or exceeds its average and since

\[
1 = e^{-K(\theta)/2} \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{K(\theta)}{D} \right)^n
\]

we can conclude an absolute lower bound on the \( L^1 \) norm of the signature, against the Poisson measure and conclude that there is an \( n \) for which

\[
e^{K(\theta)} \left( 1 - \frac{1}{D} \right) \leq n! \left\| \int \cdots \int d\gamma_{1} \cdots d\gamma_{n} \right\|.
\]

Recalling the form of \( K(\theta) \) we have the result. \( \Box \)

**Corollary 6.2.** Any piecewise linear path \( \gamma \) that has trivial signature is tree-like with a height function \( h \) having the same total variation as \( \gamma \).
Proof. We will proceed by induction on the number $r$ of edges in the minimal partition
\[ 0 = t_0 < t_1 < t_2 < \ldots < t_r = T \]
of $\gamma$. We assume that $\gamma$ is linear on each segment $[t_i, t_{i+1}]$ and that $\gamma$ is always parameterised at unit speed.

We assume that $\gamma$ has trivial signature. Our goal is to find a continuous real valued function $h$ with $h \geq 0$, $h(0) = h(T) = 0$, and so that for every $s, t \in [0, T]$ one has
\[
|h(s) - h(t)| \leq |t - s| \\
|\gamma_s - \gamma_t| \leq h(s) + h(t) - 2 \inf_{u \in [s,t]} h(u).
\]

If $r = 0$ the result is obvious; in this case $T = 0$ and the function $h = 0$ does the job.

Now suppose that the minimal partition into linear pieces has $r > 0$ pieces. By Corollary 9, it must be a degenerate partition. In other words one of the $\theta_i = \angle \gamma_{t_{i-1}} \gamma_{t_i} \gamma_{t_{i+1}}$ must have $|\theta_i| = 0 \mod \pi$.

If $\theta_i = \pi$ the point $t_i$ could be dropped from the partition and the path would still be linear. As we have chosen the partition to be minimal this case cannot occur and we conclude that $\theta_i = 0$ and the path retraces its trajectory for an interval of length
\[ s = \min(|t_i - t_{i-1}|, |t_{i+1} - t_i|) > 0. \]

Now $\gamma(t_i - u) = \gamma(t_i + u)$ for $u \in [0, s]$ and either $t_i - s = t_{i-1}$ or $t_i + s = t_{i+1}$. Suppose that the former holds. Consider the path segments obtained by restricting the path to the disjoint intervals
\[
\gamma_- = \gamma|_{0,t_{i-1}} \\
\gamma_+ = \gamma|_{t_i+s,T} \\
\tau = \gamma|_{[t_i-s,t_i+s]},
\]
then $\gamma = \gamma_- \ast \tau \ast \gamma_+$ where $\ast$ denotes concatenation.

Because the signature map $\gamma \rightarrow S(\gamma)$ is a homomorphism one sees that the product of the signatures associated to the segments is the signature of the concatenation of the paths and hence is trivial,
\[
S(\gamma_-) \otimes S(\tau) \otimes S(\gamma_+) = S(\gamma) = 1 \oplus 0 \oplus 0 \oplus \ldots \in T(V).
\]

On the other hand the path $\tau$ is a linear trajectory followed by its reverse and as reversal produces the inverse signature
\[
S(\tau) = 1 \oplus 0 \oplus 0 \oplus \ldots \in T(V).
\]

Thus
\[
S(\gamma_-) \otimes S(\gamma_+) = 1 \oplus 0 \oplus 0 \oplus \ldots \in T(V)
\]
and so the concatenation of $\gamma_-$ and $\gamma_+$ ($\gamma$ with $\tau$ excised) also has a trivial signature. As it is piecewise linear with at least one less edge we may apply the induction
hypothesis to conclude that this reduced path is tree-like. Let \( \tilde{h} \) be the height function for the reduced path. Then define

\[
\begin{align*}
    h(u) &= \tilde{h}(u), \ u \in [0, t_{i-1}] \\
    h(u) &= \tilde{h}(u - 2s), \ u \in [t_i + s, T] \\
    h(u) &= s - |t_i - u| + \tilde{h}(t_{i-1}), \ u \in [t_i - s, t_i + s].
\end{align*}
\]

It is easy to check that \( h \) is a height function for \( \gamma \) with the required properties. □

The reader should note that the main result of the paper Theorem 1 linking the signature to tree-like equivalence relies on Chen’s result only through the above Corollary and hence only requires a version for piecewise linear paths with no repeated edges. Our quantitative Theorem 9 provides an independent proof of this result but, in this context, is stronger than is necessary; Chen’s non-quantitative result could equally well have been used.

We end this section with two straightforward results which will establish half of our main theorem.

**Lemma 6.3.** If \( \gamma \) is a Lipschitz tree-like path with height function \( h \), then one can find piecewise linear Lipschitz tree-like paths converging in total variation to a re-parameterisation of \( \gamma \).

**Proof.** Without loss of generality we may re-parametrise time to be the arc length of \( h \). Since \( h \) is of bounded variation, using the area formula (5.1), we can find finitely many points \( u_n \) within \( \delta \) of one another and increasing in \([0, T]\) so that \( h \) takes the value \( h(u_n) \) only finitely many times and only at the times \( u_n \). Consider the path \( \gamma_n \) that is linear on the intervals \((u_n, u_{n+1})\) and agrees with \( \gamma \) at the times \( u_n \). Define \( h_n \) similarly. Then \( h_n \) is a height function for \( \gamma_n \) and so \( \gamma_n \) is a tree. The paths \( \gamma_n \) converge to \( \gamma \) uniformly, and in \( p \)-variation for all \( p > 1 \). However, as we have parameterised \( h \) by arc length, it follows that the total variation of \( \gamma \) is absolutely continuous with respect to arc length. As \( \gamma_n \) is a martingale with respect to the filtration determined by the successive time partitions, applying the martingale convergence theorem, it follows that \( \gamma_n \) converges to \( \gamma \) in \( L_1 \). □

**Corollary 6.4.** Any Lipschitz tree-like path has all iterated integrals equal to zero.

**Proof.** For piecewise linear tree-like paths it is obvious by induction on the number of segments that all the iterated integrals are 0. Since the process of taking iterated integrals is continuous in \( p \)-variation norm for \( p < 2 \), and Lemma 6.3 proves that any Lipschitz tree-like path can be approximated by piecewise linear tree-like paths in \( 1 \)-variation the result follows.

In the next section we introduce the concept of a weakly piecewise linear path. After reading the definition, the reader should satisfy themselves that the arguments of this section apply equally to weakly piecewise linear paths.

### 7. Weakly piecewise linear paths

Paths that lie in lines are special.

**Definition 7.1.** A continuous path \( \gamma_t \) is weakly linear (geodesic) on \([0, T]\) if there is a line \( l \) (or geodesic \( l \)) so that \( \gamma_t \in l \) for all \( t \in [0, T] \).

Suppose that \( \gamma \) is smooth enough that one can form its iterated integrals.
Lemma 7.2. If \( \gamma \) is weakly linear, then the \( n \)-signature of the path \( \gamma(t)_{t \in [0,T]} \) is
\[
\sum_{n=0}^{\infty} \frac{(\gamma_T - \gamma_0)^{\otimes n}}{n!}.
\]
In particular the signature of a weakly linear path is trivial if and only if the path has \( \gamma_T = \gamma_0 \) or, equivalently, that it is a loop.

Lemma 7.3. A weakly geodesic, and in particular a weakly linear, path with \( \gamma_0 = \gamma_T \) is always tree-like.

Proof. By definition, \( \gamma \) lies in a single geodesic. Define \( h(t) = d(\gamma_0, \gamma_t) \).
Clearly
\[
h(0) = h(T) = 0
\]
\[
h(t) \geq 0.
\]
If \( h(u) = 0 \) at some point \( u \in (s,t) \) then
\[
d(\gamma_s, \gamma_t) \leq d(\gamma_0, \gamma_s) + d(\gamma_0, \gamma_t) = h(s) + h(t) - 2 \inf_{u \in [s,t]} h(u)
\]
while if \( h(u) > 0 \) at all points \( u \in (s,t) \) then \( \gamma_s \) and \( \gamma_t \) are both on the same side of \( \gamma_0 \) in the geodesic. Assume that \( d(\gamma_0, \gamma_s) \geq d(\gamma_0, \gamma_t) \), then
\[
d(\gamma_s, \gamma_t) = d(\gamma_0, \gamma_s) - d(\gamma_0, \gamma_t) \leq h(s) + h(t) - 2 \inf_{u \in [s,t]} h(u).
\]
as required. \( \square \)

There are two key operations, splicing and excising, which preserve the triviality of the signature and (because we will prove it is the same thing) the tree-like property. However, the fact that excision of tree-like pieces preserves the tree-like property will be a consequence of our work.

Definition 7.4. If \( \gamma \in V \) is a path taking \([0,T]\) to the vector space \( V \), \( t \in [0,T] \) and \( \tau \) is a second path in \( V \), then the insertion of \( \tau \) into \( \gamma \) at the time point \( t \) is the concatenation of paths
\[
\gamma|_{[0,t]} * \tau * \gamma|_{[t,T]}.
\]

Definition 7.5. If \( \gamma \in V \) is a path on \([0,T]\) with values in a vector space \( V \), and \( [s,t] \subset [0,T] \), then \( \gamma \) with the segment \([s,t]\) excised is
\[
\gamma|_{[0,s]} * \gamma|_{[t,T]}.
\]

Remark 7.6. Note that these definitions make sense for paths in manifolds as well as in the linear case, but in this case concatenation requires the first path to finish where the second starts. We will use these operations for paths on manifolds, but it will always be a requirement for insertion that \( \tau \) is a loop based at \( \gamma_t \), for excision we require that \( \gamma|_{[s,t]} \) is a loop.

We have the following two easy lemmas:

Lemma 7.7. Suppose that \( \gamma \in M \) is a tree-like path in a manifold \( M \), and that \( \tau \) is a tree-like path in \( M \) that starts at \( \gamma_t \), then the insertion of \( \tau \) into \( \gamma \) at the point \( t \) is also tree-like. Moreover, the insertion at the time point \( t \) of any height function for \( \tau \) into any height function coding \( \gamma \) is a height function for \( \gamma|_{[0,t]} * \tau * \gamma|_{[t,T]} \).
Proof. Assume $\gamma \in M$ is a tree-like path on a domain $[0, T]$, by definition there is a positive and continuous function $h$ so that for every $s, \tilde{s}$ in the domain $[0, T]$

$$d(\gamma_s, \gamma_{\tilde{s}}) \leq h(s) + h(\tilde{s}) - 2 \inf_{u \in [s, \tilde{s}]} h(u),$$

$$h(0) = h(T) = 0.$$  

In a similar way, let the domain of $\tau$ be $[0, R]$ and let $g$ be the height function for $\tau$

$$d(\tau_s, \tau_{\tilde{s}}) \leq g(s) + g(\tilde{s}) - 2 \inf_{u \in [s, \tilde{s}]} g(u),$$

$$g(0) = g(R) = 0.$$  

Now insert $g$ in $h$ at $t$ and $\tau$ in $\gamma$ at $t$. Let $\tilde{h}, \tilde{\gamma}$ be the resulting functions defined on $[0, T + R]$. Then

$$\tilde{\gamma}(s) = \gamma(s), \quad 0 \leq s \leq t$$

$$\tilde{\gamma}(s) = \tau(s - t), \quad t \leq s \leq t + R$$

$$\tilde{\gamma}(s) = \gamma(s - R), \quad t + R \leq s \leq T + R,$$

and

$$\tilde{h}(s) = h(s), \quad 0 \leq s \leq t$$

$$\tilde{h}(s) = g(s - t), \quad t \leq s \leq t + R$$

$$\tilde{h}(s) = h(s - R), \quad t + R \leq s \leq T + R,$$

where the definition of these functions for $s \in [t + R, T + R]$ uses the fact that $\tau$ and $g$ are both loops.

Now it is quite obvious that if $s, \tilde{s} \in [0, T + R] \setminus [t, t + R]$, then

$$d(\tilde{\gamma}_{s}, \tilde{\gamma}_{\tilde{s}}) \leq \tilde{h}(s) + \tilde{h}(\tilde{s}) - 2 \inf_{u \in [s, \tilde{s}] | [t, t + R]} \tilde{h}(u)$$

$$\leq \tilde{h}(s) + \tilde{h}(\tilde{s}) - 2 \inf_{u \in [s, \tilde{s}]} \tilde{h}(u)$$

$$\tilde{h}(0) = \tilde{h}(T + R) = 0$$

and that for $s, \tilde{s} \in [t, t + R],$

$$d(\tilde{\gamma}_{s}, \tilde{\gamma}_{\tilde{s}}) = d(\tau_{s-t}, \tau_{\tilde{s}-t})$$

$$\leq g(s - t) + g(\tilde{s} - t) - 2 \inf_{u \in [s-t, \tilde{s}-t]} g(u)$$

$$= \tilde{h}(s) + \tilde{h}(\tilde{s}) - 2 \inf_{u \in [s, \tilde{s}]} \tilde{h}(u).$$

To finish the proof we must consider the case where $0 \leq s \leq t \leq \tilde{s} \leq t + R$ and the case where $0 \leq t \leq s \leq t + R \leq \tilde{s} \leq T + R$. As both cases are essentially identical we only deal with the first. In this case

$$d(\tilde{\gamma}_{s}, \tilde{\gamma}_{\tilde{s}}) = d(\gamma_s, \gamma_{\tilde{s}-t})$$

$$\leq d(\gamma_s, \gamma_t) + d(\tau_0, \tau_{\tilde{s}-t})$$

$$\leq h(s) + h(t) - 2 \inf_{u \in [s, t]} h(u) + g(\tilde{s} - t) - g(0)$$

$$= \tilde{h}(s) + \tilde{h}(\tilde{s}) - 2 \inf_{u \in [s, t]} \tilde{h}(u)$$

$$\leq \tilde{h}(s) + \tilde{h}(\tilde{s}) - 2 \inf_{u \in [s, \tilde{s}]} \tilde{h}(u).$$

$\Box$
Remark 7.8. The argument above is straightforward and could have been left to the reader. However, we draw attention to the converse result, which also seems very reasonable: that a tree-like path with a tree-like piece excised is still tree-like. This result seems very much more difficult to prove. The point is that the height function one has initially, as a consequence of \( \gamma \) being tree-like, may well not certify that \( \tau \) is tree-like even though there is a second height function defined on \([s, t]\) that certifies that it is. A direct proof that there is a new height function simultaneously attesting to the tree-like nature of \( \gamma \) and \( \tau \) seems difficult. Using the full power of the results in the paper, we can do this for paths of bounded variation.

**Lemma 7.9.** Let be \( \gamma \) a path defined on \([0, T]\) with values in \( V \) and suppose that \( \gamma|_{[s, t]} \) has trivial signature where \([s, t] \subset [0, T]\). Then \( \gamma \) has trivial signature if and only if \( \gamma \) with the segment \([s, t]\) excised has trivial signature.

*Proof.* This is also easy. Since the signature map is a homomorphism we see that

\[
\gamma = \gamma|_{[0, s]} \ast \gamma|_{[s, t]} \ast \gamma|_{[t, T]}
\]

\[
S(\gamma) = S(\gamma|_{[0, s]}) \otimes S(\gamma|_{[s, t]}) \otimes S(\gamma|_{[t, T]})
\]

and by hypothesis \( S(\gamma|_{[s, t]}) \) is the identity in the tensor algebra. Therefore

\[
S(\gamma) = S(\gamma|_{[0, s]}) \otimes S(\gamma|_{[t, T]})
\]

\[
= S(\gamma|_{[0, s]} \ast \gamma|_{[t, T]}).
\]

□

**Definition 7.10.** A continuous path \( \gamma \), defined on \([0, T]\) is weakly piecewise linear (or more generally, weakly geodesic) if there are finitely many times

\[
0 = t_0 < t_1 < t_2 < \ldots < t_r = T
\]

such that for each \( 0 < i \leq r \) the path segment \( \gamma|_{[t_{i-1}, t_i]} \) is weakly linear (geodesic)\(^4\).

Our goal in this section is to prove, through an induction, that a weakly linear path with trivial signature is tree-like and construct the height function. As before, every such path admits a unique partition so that

**Lemma 7.11.** If \( \gamma \) is a weakly piecewise linear path, then there exists a unique partition \( 0 = t_0 < t_1 < t_2 < \ldots < t_r = T \) so that the linear segments associated to \( \gamma|_{[t_{i-1}, t_i]} \) and \( \gamma|_{[t_i, t_{i+1}]} \) are not collinear for any \( 0 < i < r \).

We will henceforth only use this partition and refer to \( r \) as the number of segments in \( \gamma \).

**Definition 7.12.** We say \( \gamma \) is fully non-degenerate if, in addition, \( \gamma|_{[t_{i-1}, t_i]} \neq \gamma|_{[t_i, t_{i+1}]} \) for every \( 0 < i \leq r \).

**Lemma 7.13.** If \( \gamma \) is a weakly linear path with trivial signature and at least one segment, then there exist \( 0 < i \leq r \) so that \( \gamma|_{[t_{i-1}, t_i]} = \gamma|_{[t_i, t_{i+1}]} \).

*Proof.* The arguments in the previous section on piecewise linear paths apply equally to weakly piecewise linear and weakly piecewise geodesic paths. In particular Corollary 3.7 only refers to the location of \( \gamma \) at the times \( t_i \) at which the path changes direction (by an angle different from \( \pi \)). □

\(^4\)The geodesic will always be unique since the path has unit speed and \( t_i < t_{i+1} \) so contains at least two distinct points.
Proposition 7.14. Any weakly piecewise linear path $\gamma$ with trivial signature is tree-like with a height function whose total variation is the same as that of $\gamma$.

Proof. The argument is a simple induction using the lemmas above. If it has no segments we are clearly finished with $h \equiv 0$. We now assume that any weakly piecewise linear path $\gamma^{(r-1)}$, consisting of at most $r - 1$ segments, with trivial signature is tree-like with a height function whose total variation is the same as that of $\gamma^{(r-1)}$. Suppose that $\gamma^{(r)}$ is chosen so that it is a weakly piecewise linear path of $r$ segments with trivial signature but there was no height function coding it as a tree-like path with total variation controlled by that of $\gamma^{(r)}$. Then, by Lemma 7.13, in the standard partition there must be $0 < i \leq r$ so that $\gamma^{(r)}_{t_i-1} = \gamma^{(r)}_{t_i}$, and by assumption $t_{i-1} < t_i$. In other words, the segment $\gamma^{(r)}|_{[t_{i-1}, t_i]}$ is a weakly linear segment and a loop. It therefore has trivial signature, is tree-like and the height function we constructed for it in the proof of Lemma 7.3 was indeed controlled by the variation of the loop.

Let $\hat{\gamma}$ be the result of excising the segment $\gamma|_{[t_{i-1}, t_i]}$ from $\gamma^{(r)}$. As $\gamma^{(r)}|_{[t_{i-1}, t_i]}$ has trivial signature, by Lemma 7.13 $\hat{\gamma}$ also has trivial signature. On the other hand, $\hat{\gamma}$ is weakly piecewise linear with fewer edges than $\gamma$ (it is possible that $\gamma$ restricted to $[t_{i-2}, t_{i-1}]$ and $[t_i, t_{i+1}]$ are collinear and so the number of edges drops by more than one in the canonical partition - but it will always drop!). So by induction, $\hat{\gamma}$ is tree-like and is controlled by some height function $\hat{h}$ that has total variation controlled by the variation of $\hat{\gamma}$.

Now insert the tree-like path $\gamma^{(r)}|_{[t_{i-1}, t_i]}$ into $\hat{\gamma}$. By Lemma 7.11 this will be tree-like and the height function is simply the insertion of the height function for $\gamma^{(r)}|_{[t_{i-1}, t_i]}$ into that for $\hat{\gamma}$ and by construction is indeed controlled by the variation of $\gamma^{(r)}$ as required. Thus we have completed our induction.

8. Proof of the main theorem

We can now combine the results of the last sections to conclude the proof of our main theorem and its corollaries.

Proof of Theorem 4. Corollary 6.4 establishes that tree-like paths have trivial signature.

Thus we only need to establish that if the path of bounded variation has trivial signature, then it is tree-like. By Lemma 5.9 we can write the path as an integral against a rank one 1-form. By Corollary 5.20 we can approximate any rank one 1-form by a sequence of rank one 1-forms with the property that each 1-form is piecewise constant on finitely many disjoint compact sets and 0 elsewhere. By integrating $\gamma$ against the sequence of 1-forms we can construct a sequence of weakly piecewise linear paths approximating $\gamma$ in bounded variation. By Corollary 5.22 these approximations have trivial signature. By Proposition 7.14 this means that these weakly piecewise linear paths must be tree-like. Hence we have a sequence of tree-like paths which approximate $\gamma$. By re-parametrizing the paths at unit speed and using Lemma 4.12 $\gamma$ must be tree-like, completing the proof.

Proof of Corollary 1.4. Recall that we defined $X \sim Y$, by the relation that $X$ then $Y$ run backwards is tree-like. The transitivity is the part that is not obvious. However, we can now say $X \sim Y$ if and only if the signature of $XY^{-1}$ is trivial. As multiplication in the tensor algebra is associative, it is now simple to check the
conditions for an equivalence relation. Denoting the signature of \( X \) by \( \mathbf{X} \) etc. one sees that
1. The path run backward has signature \( \mathbf{YX}^{-1} = -\mathbf{XY}^{-1} = 0 \).
2. \( \mathbf{XX}^{-1} = 0 \) by definition.
3. If \( X \sim Y \) and \( Y \sim Z \), then \( \mathbf{XY}^{-1} = 0 \) and \( \mathbf{YZ}^{-1} = 0 \). Thus
   \[
   0 = (\mathbf{XY}^{-1})(\mathbf{YZ}^{-1}) = \mathbf{X}(\mathbf{Y}^{-1}\mathbf{Y})Z^{-1} = \mathbf{X}(\mathbf{0})Z^{-1} = \mathbf{XZ}^{-1}
   \]
and hence \( X \sim Z \) as required.

It is straightforward to see that the equivalence classes form a group. □

Proof of Corollary 1.7. In order to deduce the existence and uniqueness of minimisers for the length within each equivalence class we observe that;
1. We can re-parameterise the paths to have unit speed and thereafter to be constant. Then by the compactness of the equivalence classes of paths with the same signature and bounded length, any sequence of paths will have a subsequential uniform limit with the same signature. As length is lower-semicontinuous in the uniform topology, the limit of a sequence of paths with length decreasing to the minimum will have length less than or equal to the minimum. We have seen, through a subsubsequence where the height functions also converge, that it will also be in the same equivalence class as far as the signature is concerned, so it is a minimiser.
2. Within the class of paths with given signature and finite length there will always be at least one minimal element. Let \( X \) and \( Y \) be two minimisers parameterised at unit speed, and let \( h \) be a height function for \( \mathbf{XY}^{-1} \). Let the time interval on which \( h \) is defined be \([0, T]\) and let \( \tau \) denote the time at which the switch from \( X \) to \( Y \) occurs. The function \( h \) is monotone on \([0, \tau]\) and on \([\tau, T]\) for otherwise there would be an interval \([s, t] \subset [0, \tau]\) with \( h(s) = h(t) \). Then the function \( u \to h(u) - h(s) \) is a height function confirming that the restriction of \( X \) to \([s, t]\) is tree-like. Now we know from the associativity of the product in the tensor algebra that the signature is not changed by excision of a tree-like piece. Therefore, \( X \) with the interval \([s, t]\) excised is in the same equivalence class as \( X \) but has strictly shorter length. Thus \( X \) could not have been a minimiser - as it is, we deduce the function \( h \) is strictly monotone. A similar argument works on \([\tau, T]\).

Let \( \sigma : [0, \tau] \to [\tau, T] \) be the unique function with
\[
h(t) = h(\sigma(t)).
\]
Then \( \sigma \) is continuous decreasing and \( \sigma(0) = T \) and \( \sigma(\tau) = \tau \). Moreover, \( X_u = Y_{T-\sigma(u)} \) and so we see that (up to reparameterisations), the two paths are the same.

Hence we have a unique minimal element! □

Appendix A.

A.1. Trees and paths - background information. We have shown in this paper that trees have an important role as the negligible sets of control theory, quite analogous to the null sets of Lebesgue integration. The trees we need to consider are analytic objects in flavour, and not the finite combinatorial objects of undergraduate courses. In this appendix we collect together a few related ways of looking at them, and prove a basic characterisation generalising the concept of height function.

We first recall that
1. Graphs \((E, V)\) that are acyclic and connected are generally called trees. If such a tree is non-empty and has a distinguished vertex \( v \) it is called a rooted tree.
(2). A rooted tree induces and is characterised by a partial order on $V$ with least element $v$. The partial order is defined as follows

$$a \preceq b$$

if the circuit free path from the root $v \to b$ goes through $a$.

This order has the property that for each fixed $b$ the set $\{a \preceq b\}$ is totally ordered by $\preceq$.

Conversely any partial order on a finite set $V$ with a least element $v$ and the property that for each $b$ the set $\{a \preceq b\}$ is totally ordered defines a unique rooted tree on $V$.

(3). Alternatively, let $(E, V)$ be a graph extended into a continuum by assigning a length to each edge. Let $d(a, b)$ be the infimum of the lengths of paths between the two vertices $a, b$ in the graph. Then $g$ is a geodesic metric on $V$. Trees are exactly the graphs that give rise to $0$-hyperbolic metrics in the sense of Gromov (see for example [4]).

(4). There are many ways to enumerate the edges and nodes of a finite rooted tree. One way is to think of a family tree recording the descendants of a single individual (the root). Start with the root. At the root, if all children have been visited stop, at any other node, if all the children have been visited, move up to the parent. If there are children who have not been visited, then visit the oldest unvisited child. At each time $n$ the enumeration either moves up an edge or down an edge - each edge is visited exactly twice. Let $h(n)$ denote the distance from the top of the family tree after $n$ steps in this enumeration with the convention that $h(0) = 0$, then $h$ is similar to the path of a random walk, moving up or down one unit at each step, except that it is positive and returns to zero exactly as many times as there are edges coming from the root. Hence $h(2|E|) = 0$.

*The function $h$ completely describes the rooted tree.* The function $h$ directly yields the nearest neighbour metric on the tree. If $h$ is a function such that $h(0) = 0$, it moves up or down one unit at each step, is positive and $h(2|E|) = 0$, then $d$ defined by

$$d(m, n) = h(m) + h(n) - 2 \inf_{u \in [m, n]} h(u),$$

is a pseudo-metric on $[0, 2|V|]$. If we identify points in $[0, 2|V|]$ that are zero distance apart and join by edges the equivalence classes of points that are distance one apart, then one recovers an equivalent rooted tree.

Put less pedantically, let the enumeration be $a$ at step $n$ and $b$ at step $m$ and define

$$d(a, b) = h(m) + h(n) - 2 \inf_{u \in [m, n]} h(u),$$

then it is simple to check that $d$ is well defined and is a metric on vertices making the set of vertices a tree.

Thus excursions of simple (random) walks are a convenient (and well studied) way to describe abstract graphical trees. This particular choice for *coding a tree with a positive function on the interval* can be extended to describe continuous trees. This approach was used by Le Gall [5] in his development of the Brownian snake associated to the measure valued Dawson-Watanabe process.

---

5The sum of the lengths of the edges
A.2. **R-trees are coded by continuous functions.** One of the early examples of a continuous tree is the evolution of a continuous time stochastic process, where, as is customary in probability, one identifies the evolution of two trajectories until the first time they separate. (This idea dates back at least to Kolmogorov and his introduction of filtrations). Another popular and equivalent approach to continuous trees is through $\mathbf{R}$-trees ([10] p425 and the references there).

Interestingly, analysts and probabilists have generally rejected the abstract tree as too wild an object, and usually add extra structure, essentially a second topology or Borel structure on the tree that comes from thinking of the tree as a family of paths in a space which also has some topology. This approach is critical to the arguments used here, as we prove our tree-like paths are tree-like by approximating them with simpler tree-like paths. (They would never converge in the ‘hyperbolic’ metric). In contrast, group theorists and low dimensional topologists have made a great deal of progress by studying specific symmetry groups of these trees and do not seem to find their hugeness too problematic.

Our goal in this subsection of the appendix is to prove the simple representation: that the general $\mathbf{R}$-tree arises from identifying the contours of a continuous function on a locally connected and connected space. The height functions we considered on $[0, T]$ are a special case.

**Definition A.1.** An $\mathbf{R}$-tree is a uniquely arcwise connected metric space, in which the arc between two points is isometric to an interval.

Such a space is locally connected, for let $B_x$ be the set of points a distance at most $1/n$ from $x$. If $z \in B_x$, then the arc connecting $x$ with $z$ is isometrically embedded, and hence is contained in $B_x$. Hence $B_x$ is the union of connected sets with non-empty common intersection (they contain $x$) and is connected. The sets $B_x$ form a basis for the topology induced by the metric. Observe that if two arcs meet at two points, then the uniqueness assertion ensures that they coincide on the interval in between.

Fix some point $v$ as the ‘root’ and let $x$ and $y$ be two points in the $\mathbf{R}$-tree. The arcs from $x$ and $y$ to $v$ have a maximal interval in common starting at $v$ and terminating at some $v_1$, after that time they never meet again. One arc between them is the join of the arcs from $x$ to $v_1$ to $y$ (and hence it is the arc and a geodesic between them). Hence

$$d(x, y) = d(x, v) + d(y, v) - 2d(v, v_1).$$

**Example A.2.** Consider the space $\Omega$ of continuous paths $X_t \in E$ where each path is defined on an interval $[0, \xi(\omega))$ and has a left limit at $[0, \xi(\omega))$. Suppose that if $X \in \Omega$ is defined on $[0, s)$, then $X|_{[0, s)} \in \Omega$ for every $s$ less than $\xi$. Define

$$d(\omega, \omega') = \xi(\omega) + \xi(\omega') - 2 \sup \{t < \min(\xi(\omega), \xi(\omega')) | \omega(s) = \omega'(s) \ \forall s \leq t\}.$$  

Then $(\Omega, d)$ is an $R$-tree.

We now give a way of constructing $\mathbf{R}$-trees. The basic idea for this is quite easy, but the core of the argument lies in the detail so we proceed carefully in stages.

Let $I$ be a connected and locally connected topological space, and $h : I \rightarrow \mathbb{R}$ be a positive continuous function that attains its lower bound at a point $v \in I$.

**Definition A.3.** For each $x \in I$ and $\lambda \leq h(x)$ define $C_{x, \lambda}$ to be the maximal connected subset of $\{y \mid h(y) \geq \lambda\}$ containing $x$. 
Lemma A.4. The sets $C_{x,\lambda}$ exist, and are closed. Moreover, if $C_{x,\lambda} \cap C_{x',\lambda'} \neq \emptyset$ and $\lambda \leq \lambda'$, then

$$C_{x',\lambda'} \subset C_{x,\lambda}.$$  

Proof. An arbitrary union of connected sets with non-empty intersection is connected, taking the union of all connected subsets of $\{y \mid h(y) \geq \lambda\}$ containing $x$ constructs the unique maximal connected subset. Since $h$ is continuous the closure $D_{x,\lambda}$ of $C_{x,\lambda}$ is also a subset of $\{y \mid h(y) \geq \lambda\}$. The closure of a connected set is always connected hence $D_{x,\lambda}$ is also connected. It follows from the fact that $C_{x,\lambda}$ is maximal that $C_{x,\lambda} = D_{x,\lambda}$ and so is a closed set.

If $C_{x,\lambda} \cap C_{x',\lambda'} \neq \emptyset$ and $\lambda \leq \lambda'$, then

$$x \in C_{x,\lambda} \cup C_{x',\lambda'} \subset \{y \mid h(y) \geq \lambda\},$$

and since $C_{x,\lambda} \cap C_{x',\lambda'} \neq \emptyset$, the set $C_{x,\lambda} \cup C_{x',\lambda'}$ is connected. Hence maximality ensures $C_{x,\lambda} = C_{x,\lambda} \cup C_{x',\lambda'}$ and hence $C_{x',\lambda'} \subset C_{x,\lambda}$. □

Corollary A.5. Either $C_{x,\lambda}$ equals $C_{x',\lambda}$ or it is disjoint from it.

Proof. If they are not disjoint, then the previous Lemma can be applied twice to prove that $C_{x',\lambda} \subset C_{x,\lambda}$ and $C_{x,\lambda} \subset C_{x',\lambda}$. □

Corollary A.6. If $C_{x,\lambda} = C_{x',\lambda}$, then $C_{x,\lambda''} = C_{x',\lambda''}$ for all $\lambda'' < \lambda$.

Proof. The set $C_{x,\lambda}$, $C_{x',\lambda}$ are nonempty and have nontrivial intersection. $C_{x,\lambda} \subset C_{x,\lambda''}$ and $C_{x',\lambda} \subset C_{x',\lambda''}$ hence $C_{x,\lambda''}$ and $C_{x',\lambda''}$ have nontrivial intersection. Hence they are equal. □

Corollary A.7. $y \in C_{x,\lambda}$ if and only if $C_{y,h(y)} \subset C_{x,\lambda}$.

Proof. Suppose that $y \in C_{x,\lambda}$, then $C_{y,h(y)}$ and $C_{x,\lambda}$ are not disjoint. It follows from the definition of $C_{x,\lambda}$ and $y \in C_{x,\lambda}$ that $h(y) \geq \lambda$. By Lemma A.4 $C_{y,h(y)} \subset C_{x,\lambda}$.

Suppose that $C_{y,h(y)} \subset C_{x,\lambda}$, since $y \in C_{y,h(y)}$ it is obvious that $y \in C_{x,\lambda}$. □

Definition A.8. The set $C_x := C_{x,h(x)}$ is commonly referred to as the contour of $h$ through $x$.

The map $x \to C_x$ induces a partial order on $I$ with $x \preceq y$ if $C_x \supseteq C_y$. If $h$ attains its lower bound at $x$, then $C_x = I$ since $\{y \mid h(y) \geq h(x)\} = I$ and $I$ is connected by hypothesis. Hence the root $v \preceq y$ for all $y \in I$.

Lemma A.9. Suppose that $\lambda \in [h(y), h(x)]$, then there is a $y$ in $C_{x,\lambda}$ such that $h(y) = \lambda$ and, in particular, there is always a contour $(C_{x,\lambda})$ at height $\lambda$ through $y$ that contains $x$.

Proof. By the definition of $C_{x,\lambda}$ it is the maximal connected subset of $h \geq \lambda$ containing $x$; assume the hypothesis that there is no $y$ in $C_{x,\lambda}$ with $h(y) = \lambda$ so that it is contained in $h > \lambda$, hence $C_{x,\lambda}$ is a maximal connected subset of $h > \lambda$. Now $h > \lambda$ is open and locally connected, hence its maximal connected subsets of $h > \lambda$ are open and $C_{x,\lambda}$ is open. However it is also closed, which contradicts the connectedness of the $I$. Thus we have established the existence of the point $y$. □

The contour is obviously unique, although $y$ is in general not. If we consider the equivalence classes $x \sim y$ if $x \preceq y$ and $y \preceq x$, then we see that the equivalence classes $[y]$ of $y \preceq x$ are totally ordered and in one to one correspondence with points in the interval $[h(y), h(x)]$. 


Lemma A.10. If \( z \in C_{y,\lambda} \) and \( h(z) > \lambda \), then \( z \) is in the interior of \( C_{y,\lambda} \). If \( C_{x',\lambda'} \subset C_{z,\lambda} \) with \( \lambda' > \lambda \), then \( C_{z,\lambda} \) is a neighbourhood of \( C_{x',\lambda'} \).

Proof. \( I \) is locally connected, and \( h \) is continuous, hence there is a connected neighbourhood \( U \) of \( z \) such that \( h(z) \geq \lambda \). By maximality \( U \subset C_{z,\lambda} \). Since \( C_{z,\lambda} \cap C_{y,\lambda} \neq \emptyset \) we have \( C_{z,\lambda} = C_{y,\lambda} \) and thus \( U \subset C_{y,\lambda} \). Hence \( C_{y,\lambda} \) is a neighbourhood of \( z \). The last part follows trivially once by noting that for all \( z \in C_{x',\lambda'} \) we have \( h(z) \geq \lambda' > \lambda \) and hence \( C_{y,\lambda} \) is a neighbourhood of \( z \).

We now define a pseudo-metric on \( I \). Lemma A.10 (the only place we will use local connectedness) is critical to showing that the map from \( I \) to the resulting quotient space is continuous.

Definition A.11. If \( y \) and \( z \) are points in \( I \), define \( \lambda(y, z) = \min(h(y), h(z)) \) such that \( C_{y,\lambda} = C_{z,\lambda} \)

\[
\lambda(y, z) = \sup\{ \lambda \mid C_{y,\lambda} = C_{z,\lambda}, \lambda \leq h(y), \lambda \leq h(z) \}.
\]

The set

\[
\{ \lambda \mid C_{y,\lambda} = C_{z,\lambda}, \lambda \leq h(y), \lambda \leq h(z) \}
\]

is a non-empty interval \([h(v), \lambda(y, z)]\) or \([h(v), \lambda(y, z)]\) where \( \lambda(y, z) \) satisfies

\[
h(v) \leq \lambda(y, z) \leq \min(h(y), h(z)).
\]

Clearly \( \lambda(x, x) = h(x) \).

Lemma A.12. The function \( \lambda \) is lower semi-continuous

\[
\liminf_{z \to z_0} \lambda(y, z) \geq \lambda(y, z_0).
\]

Proof. Fix \( y, z_0 \) and choose some \( \lambda' < \lambda(y, z_0) \). By the definition of \( \lambda(y, z_0) \) we have that \( C_{y,\lambda'} = C_{z_0,\lambda'} \). Since \( h(z_0) \geq \lambda' \) there is a neighbourhood \( U \) of \( z_0 \) so that \( U \subset C_{z_0,\lambda'} \). For any \( z \in U \) one has \( z \in C_{z,\lambda'} \cap C_{z_0,\lambda'} \). Hence \( C_{z_0,\lambda'} = C_{z,\lambda'} \) and \( C_{y,\lambda'} = C_{z,\lambda'} \). Thus \( \lambda(y, z) \geq \lambda' \) for \( z \in U \) and hence

\[
\liminf_{z \to z_0} \lambda(y, z) \geq \lambda'.
\]

Since \( \lambda' < \lambda(y, z_0) \) was arbitrary

\[
\liminf_{z \to z_0} \lambda(y, z) \geq \lambda(y, z_0)
\]

and the result is proved.

Lemma A.13. The following inequality holds

\[
\min\{ \lambda(x, z), \lambda(y, z) \} \leq \lambda(x, y).
\]

Proof. If \( \min\{ \lambda(x, z), \lambda(y, z) \} = h(v) \), then there is nothing to prove. Recall that

\[
\{ \lambda \mid C_{y,\lambda} = C_{z,\lambda}, \lambda \leq h(y), \lambda \leq h(x) \}
\]

is connected and contains \( h(v) \). Suppose \( h(v) \leq \lambda < \min\{ \lambda(x, z), \lambda(y, z) \} \), then it follows that the identity \( C_{x,\lambda} = C_{z,\lambda} \) holds for \( \lambda \). Similarly \( C_{y,\lambda} = C_{z,\lambda} \). As a result \( C_{x,\lambda} = C_{y,\lambda} \) and \( \lambda(x, y) \geq \lambda \).

Definition A.14. Define \( d \) on \( I \times I \) by

\[
d(x, y) = h(x) + h(y) - 2\lambda(x, y).
\]
Lemma A.15. The function $d$ is a pseudo-metric on $I$. If $\left(I, d\right)$ is the resulting quotient metric space, then the projection $I \rightarrow \tilde{I}$ from the topological space $I$ to the metric space is continuous.

Proof. Clearly $d$ is positive, symmetric and we have remarked that for all $x$, $\lambda(x, x) = h(x)$ hence it is zero on the diagonal. To see the triangle inequality, assume
$$\lambda(x, z) = \min \{\lambda(x, z), \lambda(y, z)\}$$
and then observe
$$d(x, y) = h(x) + h(y) - 2\lambda(x, y)$$
$$\leq h(x) + h(y) - 2\lambda(x, z)$$
$$= h(x) + h(z) - 2\lambda(x, z) + h(y) - h(z)$$
$$\leq d(x, z) + |h(y) - h(z)|$$
but $\lambda(y, z) \leq \min (h(y), h(z))$ and hence
$$|h(y) - h(z)| = h(y) + h(z) - 2 \min (h(y), h(z))$$
$$\leq h(y) + h(z) - 2\lambda(y, z)$$
$$= d(y, z)$$
hence
$$d(x, y) \leq d(x, z) + d(y, z)$$
as required.

We can now introduce the equivalence relation $x^\sim y$ if $d(x, y) = 0$ and the quotient space $I/\sim$. We write $I/\sim = \tilde{I}$ and $i : I \rightarrow \tilde{I}$ for the canonical projection. The function $d$ projects onto $\tilde{I} \times \tilde{I}$ and is a metric there.

It is tempting to think that $x^\sim y$ if and only if $C_x = C_y$ and this is true if $I$ is compact Hausdorff. However the definitions imply a slightly different criteria: $x^\sim y$ iff
$$h(x) = h(y) = \lambda$$
and $C_{x, \lambda''} = C_{y, \lambda''}$ for all $\lambda'' < \lambda$.

The stronger statement $x^\sim y$ if and only if $C_x = C_y$ is not true for all continuous functions $h$ on $\mathbb{R}^2$ as it is easy to find a decreasing family of closed connected sets there whose limit is a closed set that is not connected.

Consider again the new metric space $\tilde{I}$ that has as its points the equivalence classes of points indistinguishable under $d$. We now prove that the projection $i$ taking $I$ to $\tilde{I}$ is continuous. Fix $y \in I$ and $\varepsilon > 0$. Since $\lambda(y, .)$ is lower semi-continuous and $h$ is (upper semi)continuous there is a neighbourhood $U$ of $y$ so that for $z \in U$ one has $\lambda(y, z) = \lambda(y, y) - \varepsilon/4$ and $h(z) < h(y) + \varepsilon/2$. Thus $d(y, z) < \varepsilon$ for $z \in U$. Hence $d(i(y), i(z)) < \varepsilon$ if $z \in U$. The function $i$ is continuous and as continuous images of compact sets are compact we have the following.

Corollary A.16. If $I$ is compact, then $\tilde{I}$ is a compact metric space.

To complete this section we will show $\tilde{I}$ is a uniquely arcwise connected metric space, in which the arc between two points is isometric to an interval and give a characterisation of compact trees.
Proposition A.17. If $I$ is a connected and locally connected topological space, and $h : I \to \mathbb{R}$ is a positive continuous function that attains its lower bound, then its “contour tree” the metric space $(\tilde{I}, \tilde{d})$ is an $\mathbb{R}$-tree. Every $\mathbb{R}$-tree can be constructed in this way.

Proof. It is enough to prove that the metric space $\tilde{I}$ we have constructed is really an $\mathbb{R}$-tree and that every $\mathbb{R}$-tree can be constructed in this way. Let $\tilde{x}$ any point in $\tilde{I}$ and $x \in I$ satisfy $i(x) = \tilde{x}$. Then $h(x)$ does not depend on the choice of $x$. Fix $h(v) < \lambda < h(x)$. We have seen that there is a $y$ such that $h(y) = \lambda$ and $y < x$ moreover any two choices have the same contour through them and hence the same $\tilde{y}(\lambda)$. In this way we see that there is a map from $[h(v), h(x)]$ into $\tilde{I}$ that is injective. Moreover, it is immediate from the definition of $\tilde{d}$ that it is an isometry and that $\tilde{I}$ is uniquely arc connected.

Remark A.18. 1. In the case where $I$ is compact, obviously $\tilde{I}$ is both complete and totally bounded as it is compact.

2. An $\mathbb{R}$-tree is a metric space; it is therefore possible to complete it. Indeed the completion consists of those paths, all of whose initial segments are in the tree we have not identified a simple sufficient condition on the continuous function and topological space $\Omega$ to ensure this. An $\mathbb{R}$-tree is totally bounded if it is bounded and for each $\varepsilon > 0$ there is an $N$ so that for each $t$ the paths that extend a distance $t$ from the root have at most $N$ ancestral paths between them at time $t - \varepsilon$. In this way we see that the $\mathbb{R}$-tree that comes out of studying the historical process for the Fleming-Viot or the Dawson-Watanabe measure-valued processes is, with probability one, a compact $\mathbb{R}$-tree for each finite time.

Lemma A.19. Given a compact $\mathbb{R}$-tree, there is always a height function on a closed interval that yields the same tree as its quotient.

Proof. As the tree is compact, path connected and locally path connected, there is always as based loop mapping $[0, 1]$ onto the tree. Let $h$ denote the distance from the root. Its pullback onto the interval $[0, 1]$ is a height function and the natural quotient is the original tree. In this way we see that there is always a version of Le Gall’s snake traversing a compact tree.

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