An epiperimetric inequality for the lower dimensional obstacle problem

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Abstract

In this paper we give a proof of an epiperimetric inequality in the setting of the lower dimensional obstacle problem. The inequality was introduced by Weiss (Invent. Math., 138 (1999), no. 1, 23-50) for the classical obstacle problem and has striking consequences concerning the regularity of the free-boundary. Our proof follows the approach of Focardi and Spadaro (Adv. Differential Equations 21 (2015), no 1-2, 153-200.) which uses an homogeneity approach and a $\Gamma$-convergence analysis.

Introduction

The obstacle problem consists in finding the minimizer of a suitable energy among all functions, with fixed boundary data, constrained to lie above a given obstacle. The obstacle can live in the whole domain or on a surface of codimension one, these cases are denoted by the classical obstacle and the lower dimensional obstacle (or the thin obstacle) respectively. In this paper we analyse a particular case of a lower dimensional obstacle where the obstacle is laid in a hyperplane of the domain and the energies are the weighted versions of Dirichlet energy. The motivation for studying lower dimensional obstacle problems has roots in many applications. There are examples in physics, mechanics, biology and financial mathematics and many prime examples can be found in [4, 12, 15, 16, 18, 27, 40, 41, 45, 46]. In this paper we consider the energy

$$E(v) := \int_{B_1^+} |\nabla v|^2 x_n^a \, dx,$$

and minimize $E$ among all functions in the class of admissible functions

$$\mathfrak{A}_g := \{ v \in H^1(B_1^+, \mu_a) : v \geq 0 \text{ on } B_1^+, \, v = g \text{ on } (\partial B_1)^+ \},$$

where $H^1(A, \mu_a)$ is the weighted Sobolev Space and $\mu_a$ is the measure $\mu_a := |x_n|^a A^{\alpha \cdot B_1}$ with $a \in (-1,1)$. In what follows we will extend automatically every functions $\mathfrak{A}_g$ by even symmetry with respect to $\{x_n = 0\}$ and for convenience we will indicate any points $x \in \mathbb{R}^n$ as $x = (\tilde{x}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$.

By the direct method of calculus of variations it easy to prove the existence and uniqueness of the minimum of (0.1) on $\mathfrak{A}_g$. Let $u := \min_{\mathfrak{A}_g} E$, we note that $u$ satisfies the following Euler–Lagrange equations:

$$\begin{cases} 
 u(\tilde{x}, 0) \geq 0 & \tilde{x} \in B_1 \\
 u(\tilde{x}, x_n) = u(\tilde{x}, -x_n) & \\
 \text{div}(|x_n|^a \nabla u(\tilde{x}, x_n)) = 0 & x \in B_1 \setminus \{(\tilde{x}, 0) : u(\tilde{x}, 0) = 0\} \\
 \text{div}(|x_n|^a \nabla u(\tilde{x}, x_n)) \leq 0 & x \in B_1 \text{ in distributional sense.}
\end{cases} \quad (0.3)$$

We denote by $\Gamma(u) := \partial \{(\tilde{x}, 0) \in B_1^+ : u(\tilde{x}, 0) = 0\} \cap B_1'$ the free-boundary of $u$. In order to establish the regularity of the solution $u$ and its free-boundary a fundamental tool is the Almgren frequency type function (see [3] for $a = 0$). For all points $x_0 \in \Gamma(u)$:

$$N^x_{\alpha}(r, u) := \frac{r \int_{B_r(x_0)} |\nabla u|^2 \, d\mu_a}{\int_{\partial B_r(x_0)} u^2 |x_n|^a \, d\mathcal{H}^{n-1}}.$$

Caffarelli and Silvestre [14] proved the monotonicity of function $r \mapsto N^x_{\alpha}(r, u)$ and some of its properties such as, the property of being constant over all homogeneous functions; the two authors and Salsa [13] established the property of the frequency function of being bigger than $1 + s$, where $s := \frac{1-a}{n}$ is the exponent of the fractional Laplacian of the trace, on $\mathbb{R}^{n-1} \times \{0\}$, of a global solution of (0.3) (see Section 4).
The points of the subset \( \text{Reg}(u) \) are called regular points, they are the points of the free-boundary with least frequency i.e. \( 1 + s \); we will denote \( \text{Reg}(u) \) as \( \Gamma_{1+s}(u) \). Caffarelli, Salsa and Silvestre in \([13]\) proved that \( \Gamma_{1+s}(u) \) is locally a \( C^{1,\alpha} \) \( (n-1) \)-submanifold. In the case \( s = 1/2 \) the regularity of \( \Gamma_{1/2} \) was already proved by Athanasopoulos, Caffarelli and Salsa in \([3]\), while Focardi and Spadaro \([25]\) and Garofalo, Petrosyan and Smit Vega Garcia in \([31]\) gave alternative proofs of regularity using an epiperimetric inequality (see Theorem \([0.1]\)).

The points of the subset \( \text{Sing}(u) \) are called singular points and are the points of the free-boundary with frequency \( 2m \) with \( m \in \mathbb{N} \), equivalently their contact sets have density zero with respect to \( \mathcal{H}^n \). In the case \( s = 1/2 \) Garofalo and Petrosyan \([28]\) prove that \( \text{Sing}(u) \) is contained in a countable union of \( C^2 \) submanifold. Very recently Garofalo and Ros-Oton \([32]\) extended the result in \([28]\) for \( s \in (0, 1) \).

The subset \( \text{Other}(u) \) is the complement of \( \text{Reg}(u) \cup \text{Sing}(u) \) in \( \Gamma(u) \). Recently Focardi and Spadaro \([29]\) gave a complete description of the subset \( \text{Sing}(u) \cup \text{Other}(u) \) up to a set of \( \mathcal{H}^{n-2} \) measure zero. This result is new also in the framework of the Signorini problem, i.e. in the case \( s = 1/2 \), and it is obtained by a combination of analytical and geometric measure theory arguments.

The goal of this paper is give an alternative proof of the regularity of \( \Gamma_{1+s}(u) \) given by Caffarelli, Salsa and Silvestre \([13]\). Our proof use an epiperimetric inequality and its consequences. We extend the result proved by Focardi and Spadaro \([29]\) in the case \( s \in (0, 1) \). The two authors outline the presence in their proof of two competing variational principles that contribute to the achievement of proof.

In order to enunciate the epiperimetric inequality we introduce a sequence of rescaled functions

\[
 u_{x_0,r} := \frac{u(x_0+rx)}{r^{1+s}}
\]

and an auxiliary energy “à la Weiss”

\[
 W_{1+s}^{2s}(r, u) := \frac{1}{r^{n+s}} \int_{B_r(x_0)} |\nabla u_r|^2 \, d\mu_u - \frac{1 + s}{r^{n+s}} \int_{\partial B_r(x_0)} |u_r|^2 |x_n|^a \, d\mathcal{H}^{n-1},
\]

which is the sum of a volume energy and a boundary energy. We note that \( 1 + s \), the frequency of points of the free-boundary examined, is the exponent of the scaling factor of sequence \( u_{x_0,r} \) (see equation \([3.1]\)) and the coefficient of boundary energy. The existence of blow-ups is a consequence of a gradient estimate of rescaled function in \( L^2(B_1, \mu_u) \); reasoning by contradiction, thanks to properties of the frequency and the optimal regularity of the solution we prove the \((1 + s)\)-homogeneity of blow-ups. So, according to a result of classification by Caffarelli, Salsa and Silvestre \([13]\) we state the result of the classification of \((1 + s)\)-homogeneous global solutions of the fractional obstacle, which constitute the following closed cone

\[
 \delta_{1+s} := \{ \lambda h_{c} : e \in \mathbb{S}^{n-2}, \lambda \in [0, +\infty) \} \subset H^1_{\text{loc}}(\mathbb{R}^n, \mu_u),
\]

with

\[
 h_{c}(x) := \left( s^{-1} \sqrt{e - \sqrt{(\sqrt{e} \cdot e)^2 + x_n^2}} \left( \sqrt{(\sqrt{e} \cdot e)^2 + x_n^2} + \sqrt{e} \cdot e \right) \right)^s.
\]

The key result presented in this paper is an alternative proof (a first proof, with an extra hypothesis, was given by Garofalo, Petrosyan, Smit and Vega Garcia in \([30]\)) of a Weiss’ epiperimetric inequality for the fractional obstacle problem (cf. \([48]\) Theorem 1)).

**Theorem 0.1** (Epiperimetric inequality). Let \( 0 \in \Gamma_{1+s}(u) \). There exists a dimensional constant \( \kappa \in (0, 1) \) such that if \( c \in H^1(B_1, \mu_u) \) is a function \((1 + s)\)-homogeneous for which \( c \geq 0 \) on \( B_1^+ \) then

\[
 \inf_{v \in \mathcal{A}_c} W_{1+s}^{2s}(1, v) \leq (1 - \kappa) W_{1+s}^{2s}(1, c).
\]

Taking the epiperimetric inequality into account, Weiss proved this result in \([48]\) in the classical obstacle case. Recently Garofalo, Petrosyan, Pop and Smit Vega Garcia \([30]\) proved a similar epiperimetric inequality, with an extra hypothesis, for the fractional obstacle problem with drift in the case of \( s \in (1/2, 1) \).
In the case of obstacle 0 and without drift our inequality is stronger. Indeed Garofalo et. al. in [30] require an extra hypothesis of closeness between the function c and a fixed blow-up limit. We do not need such an assumption. On the other hand, due to homogeneity we can reduce to functions c close to cone of global solutions $\mathcal{B}_{1+s}$.

By contradicting the closeness assumption we obtain a quasi-minimality condition for a sequence of auxiliary functionals. Using a $\Gamma$-convergence argument we inspect the $\Gamma$-limits of the sequence of auxiliary energies and analyse their minimizer that represents the directions along which the epiperimetric inequality may fail. Using a variational method we obtain that such minimizers show in the same time contradictory relationship with the cone $\mathcal{B}_{1+s}$.

The epiperimetric inequality is a key ingredient to deduce the following estimate of the decay of energy:

$$W^{x_0}_{1+s}(r, u) \leq C r^\gamma,$$

where $C$ and $\gamma$ are positive constants. Thanks to the decay estimate (0.6) we prove a property of nondegeneration of solutions, from which we deduce that the blow-ups are nonzero. Proceeding as in [25] we can prove the uniqueness of blow-ups and the regularity of $\Gamma_{1+s}(u)$: we state this results in Proposition 5.6 and Theorem 6.1 respectively and do not prove them because they follow by the epiperimetric inequality and its consequences as in [25] Proposition 4.8 and Proposition 4.10.

What follows is a summary of the structure of this paper: in section 2 we introduce the frequency and its properties and define $\Gamma_{1+s}(u)$ the subset of free-boundary with low frequency. In section 3 we prove the existence and $(1+s)$-homogeneity of blow-ups in the points in $\Gamma_{1+s}(u)$ and in section 4 thanks to a result by [13], we characterize the $(1+s)$-homogeneous global solution of the fractional obstacle problem. Section 5 is devoted to establish the epiperimetric inequality and its consequences in the framework of the regularity of the free-boundary, a decay estimate of an auxiliary energy, the nondegeneracy of the solution and the uniqueness of the blow-ups. In section 6 we state the regularity of $\Gamma_{1+s}(u)$.

1 Preliminary results

Let $u \in \min_{\partial B} \mathcal{E}$; we denote by $\Lambda(u)$ its coincidence set, $\Lambda(u) := \{ \tilde{x} \in B'_1 : u(\tilde{x}, 0) = 0 \}$, and by $\Gamma(u)$ its free-boundary $\partial \Lambda(u)$ in $B'_1$ topology.

Caffarelli and Silvestre in [14] showed that the Euler-Lagrange equations of $u$ (0.3) are equivalent to the following equations:

$$\begin{cases}
u(x, 0) \geq 0 & \tilde{x} \in (B'_1)^+ \\
\text{div}(x_n^a \nabla u(\tilde{x}, x_n)) = 0 & x_n > 0 \\
\lim_{x_n \to 0^+} x_n^a \partial_n u(\tilde{x}, x_n) = 0 & u(\tilde{x}, 0) > 0 \\
\lim_{x_n \to 0^+} x_n^a \partial_n u(\tilde{x}, x_n) \leq 0 & \tilde{x} \in (B'_1)^+, \end{cases}$$

which are related to the study of the classical obstacle problem in $\mathbb{R}^{n-1}$ for fractional Laplacian $(\Delta)^s$ with $s \in (0, 1)$, where $a = 1 - 2s$. In particular, for all $v$ solution of $\text{div}(x_n^a \nabla (u(\tilde{x}, x_n))) = 0$ on $B'_1$, with an appropriate extension to the whole $\mathbb{R}^n$, there exists the limit $\lim_{x_n \to 0^+} x_n^a \partial_n v(\tilde{x}, x_n)$ and $\lim_{x_n \to 0^+} x_n^a \partial_n v(\tilde{x}, x_n) = C(-\Delta)^s f(\tilde{x})$ with $f$ the trace of $v$ on $\mathbb{R}^{n-1} \times \{0\}$ and $C$ a constant depending on $n$ and $s$ (cf. [14]).

For $x_n > 0$, $u(\tilde{x}, x_n)$ is smooth so the second condition in (1.1) holds in the classical sense, while the third and fourth condition in (1.1) hold in the weak sense. By Silvestre [17] $u(\tilde{x}, 0) \in C^{\alpha, \alpha}$ with $\alpha < s$, in particular if $\alpha < \alpha < s$ the limit $\lim_{x_n \to 0^+} x_n^a \partial_n u(\tilde{x}, x_n)$ can be considered in the classical sense. By [17] Proposition 3.10 we also know that $\partial_{n} u \geq 0$ for all $e \in \mathbb{S}^{n-2} \subset \mathbb{R}^{n-1} \times \{0\}$, or rather $u$ is semiconvex in the variable $\tilde{x}$; moreover if the obstacle $\varphi \in C^{1,1}$ then $\partial_{n} u \geq - \sup |D^2 \varphi|$.

The function $u$, can be extended by symmetry $u(\tilde{x}, x_n) = u(\tilde{x}, -x_n)$. So, as shown in [14] we can rewrite the problem (1.1) as (0.3).

In order to simplify the notation, we introduce the following symbol:

$$R_n(\psi) := \lim_{\varepsilon \to 0^+} \varepsilon^a \partial_n \psi(\tilde{x}, \varepsilon)$$

for all functions $\psi$ which are solutions for

$$\begin{cases}
\psi(\tilde{x}, x_n) = \psi(\tilde{x}, -x_n) \\
L_n(\psi) := \text{div}(|x_n|^a \nabla \psi(\tilde{x}, x_n)) = 0 \quad \{x_n \neq 0\}.
\end{cases}$$

In what follows, we shall state a uniform estimate on the solution $u$, so we report a quantitative result stated in [25] Theorem 2.1.
Theorem 1.1. For every boundary datum \( g \in H^1(B_1, \mu_a) \) that respects the condition of compatibility with the problem, i.e. \( g(\hat{x}, x_n) = g(\hat{x}, -x_n) \) and \( g(\hat{x}, 0) \geq 0 \), there exists a unique solution \( u \) to the fractional obstacle problem \( \mathcal{O}_\alpha \). Moreover, \( \partial_x u \in C^1(B_1/2) \) for \( i = 1, \ldots, n-1 \) and \( |x_n|^\alpha \partial_x u \in C^\alpha(B_1/2) \) for all \( 0 < \alpha < 1 - s \), and

\[
\|u\|_{C^\alpha(B_1/2)} := \|u\|_{C^0(B_1/2)} + \|
abla u\|_{C^0(B_1)} + \|\partial_x u\|_{C^\alpha(B_1/2)} \leq C\|u\|_{L^2(B_1^+, \mu_a)},
\]

with \( \mathcal{O}_\alpha \). Then there exists a subsequence \( v \in H^1(B_1) \) such that \( \partial_x u \in C^1(B_1/2) \) and \( |x_n|^\alpha \partial_x v \in C^\alpha(B_1/2) \).

Next, we state a version of the Divergence Theorem that will be used frequently in the paper.

Theorem 2.1. (Divergence Theorem). Let \( \varphi \in H^1(B_1, \mu_a) \) and \( \psi \) be a solution of \( \mathcal{O}_\alpha \), then

\[
\int_{B_1} \nabla \varphi \cdot \nabla \psi \, d\mu_a = \int_{\partial B_1} \varphi \nabla \psi \cdot \nu \, d\mu_a - 2\int_{B_1} \varphi R_u(\psi) \, d\mu_a
\]

We conclude the paragraph stating some results related to weighted Sobolev spaces. We rewrite these results for our aims, but these also hold in more general conditions.

We state the analogous of the Banach-Alaoglu-Bourbaki Theorem (see [5] Theorem III.15) for which every bounded and closed set in \( H^1(B_1, \mu_a) \) is relatively compact in the weak topology.

Theorem 1.3 (Banach-Alaoglu-Bourbaki Theorem [39 Theorem 3.1]). Let \( v_j \) be a bounded sequence in \( H^1(B_1, \mu_a) \). Then there exists a subsequence \( v_j \), and a function \( v \in H^1(B_1, \mu_a) \) such that \( v_j \rightharpoonup v \) in \( L^2(B_1, \mu_a) \) and \( \nabla v_j \to \nabla v \) in \( L^2(B_1, \mu_a) \).

Moreover in view of [38 Theorem 8.1], where Heinonen and Keskelä obtained an analogous of the Rellich Theorem on Sobolev metric spaces, we can deduce that every bounded and closed set in \( H^1(B_1, \mu_a) \) is relatively compact in \( L^2(B_1, \mu_a) \).

Theorem 1.4 (Rellich Theorem [38 Theorem 8.1]). Let \( v_j \) be a bounded sequence in \( H^1(B_1, \mu_a) \). Then there exists a subsequence \( v_j \), and a function \( v \in H^1(B_1, \mu_a) \) such that \( v_j \rightharpoonup v \) in \( L^2(B_1, \mu_a) \).

Furthermore, we indicate two Theorems of compact Trace embedding. We are interested in the trace of functions in \( H^1(B_1, \mu_a) \) on \( L^2(B_1) \) and \( L^2(\partial B_1, |x_n|^\alpha \mathcal{H}^{n-1}) \).

Theorem 1.5 (Trace Theorem [20 Theorem 3.4]). For all \( a \in (-1, 1) \) there exists a compact operator \( \mathcal{O}_\alpha : H^1(B_1^+, \mu_a) \to L^2(B_1) \) such that \( \mathcal{O}_\alpha = u \) for every \( u \in C^\infty(B_1) \).

The Trace of Embedding on \( L^2(\partial B_1, |x_n|^\alpha \mathcal{H}^{n-1}) \) is similar to the Theorem of Trace embedding in the classical Sobolev spaces, for its proof we refer to [37 Section 3.7].

Theorem 1.6 (Trace Theorem). For all \( a \in (-1, 1) \) there exists a compact operator \( \mathcal{O}_\alpha : H^1(B_1, \mu_a) \to L^2(\partial B_1, |x_n|^\alpha \mathcal{H}^{n-1}) \) such that \( \mathcal{O}_\alpha = u \) for every \( u \in C^\infty(B_1) \).

2 Frequency formula

Let \( x_0 \in \Gamma(u) \) and \( r \in (0, 1 - |x_0|) \); let \( N^{x_0}(r, u) \) be the frequency function defined by

\[
N^{x_0}(r, u) := \frac{r \int_{B_r(x_0)} |\nabla u|^2 \, d\mu_a}{\int_{B_r(x_0)} a^2 |x_n|^\alpha \, d\mathcal{H}^{n-1}}
\]

if \( u|_{\partial B_r(x_0)} \neq 0 \). We recall the monotonicity result due to Caffarelli and Silvestre [13].

Theorem 2.1. (i) The frequency function \( N^{x_0}(r, u) \) is monotone nondecreasing in the variable \( r \) for all \( r \in (0, 1 - |x_0|) \).

(ii) For all points \( x_0 \in \Gamma(u) \) the function \( N^{x_0}(r, u) = \lambda \) for all \( r \in (0, 1 - |x_0|) \) if and only if \( u(x_0 + \cdot) \) is \( \lambda \)-homogeneous.

(iii) If \( u(x_0 + \cdot) \) is \( \lambda \)-homogeneous then \( \lambda \geq 1 + s \).

(iv) \( N^{x_0}(r, u) \geq 1 + s \) for all \( x_0 \in \Gamma(u) \) and \( r \in (0, 1 - |x_0|) \).

Proof. As far as the proof of (i), (ii) and (iii) is concerned, we refer to [13] Theorem 6.1 and [13] Proposition 5.1. As regards the proof of (iv), see Remark 5.5. \( \square \)
Thanks to Theorem 2.1(i) it is possible to define the limit $N_{x_0}(0^+, u) := \lim_{r \to 0^+} N_{a}^{x_0}(r, u)$. We denote by $\Gamma_{1+s}(u)$ the subset of points of free-boundary with frequency $1+s$:

$$\Gamma_{1+s}(u) := \{ x_0 \in \Gamma_u : N_{a}^{x_0}(0^+, u) = 1 + s \}. \tag{2.2}$$

Note that from the monotonicity of the frequency and by the upper semicontinuity of the function $x \mapsto N_{a}^{x_0}(0^+, u)$ the set $\Gamma_{1+s} \subset \Gamma_u$ is open in the relative topology.

We introduce the notation:

$$D_{\alpha}^{x_0}(r) = \int_{B_r(x_0)} |\nabla u|^2 \, d\mu_a \quad H_{\alpha}^{x_0}(r) = \int_{\partial B_r(x_0)} u^2 |x_n|^a \, d\mathcal{H}^{n-1}$$

and we can omit to write the point $x_0$ if $x_0 = 0$.

All functions $H_{\alpha}^{x_0}(:, )$, $D_{\alpha}^{x_0}(:, )$ and $N_{\alpha}^{x_0}(\cdot)$ are absolutely continuous functions of the radius, so they are differentiable a.e.

We prove two properties of $H_{\alpha}^{x_0}(r)$ (see [1] Lemma 2, [25, A.2.Lemma] for the case $a = 0$).

**Lemma 2.2.**

(i) The function

$$0, 1 - |x_0| \ni r \mapsto \frac{H_{\alpha}^{x_0}(r)}{r^{n+2}} \tag{2.3}$$

is nondecreasing and in particular

$$H_{\alpha}^{x_0}(r) \leq \frac{H_{\alpha}^{x_0}(1 - |x_0|)}{(1 - |x_0|)^{n+2}} r^{n+2} \text{ for all } 0 < r < 1 - |x_0|. \tag{2.4}$$

(ii) Let $x_0 \in \Gamma_{1+s}$. For all $\varepsilon > 0$ there exists an $r_0(\varepsilon)$ such that

$$H_{\alpha}^{x_0}(r) \geq \frac{H_{\alpha}^{x_0}(r_0)}{r_0^{n+2+\varepsilon}} r^{n+2+\varepsilon} \text{ for all } 0 < r < r_0. \tag{2.5}$$

**Proof.** (i) We proceed along a two-step argument. Let $x_0 \in \Gamma_{1+s}(u)$ we recall that $x_0 = (\hat{x}_0, 0)$. Thanks to the Divergence Theorem and the third condition of (1.1) for which $u R_{a}(u) = 0$ in $B'_1$ we can compute the derivative of $\frac{H_{\alpha}^{x_0}(r)}{r^{n+2}}$:

$$\frac{d}{dr} \left( \frac{1}{r^{n+2}} H_{\alpha}^{x_0}(r) \right) = \frac{2}{r^{n+2}} \int_{B_r(x_0)} |\nabla u(x)|^2 \, d\mu_a. \tag{2.6}$$

Next, through the equation (2.6), we compute the derivative of $\frac{H_{\alpha}^{x_0}(r)}{r^{n+2}}$:

$$\frac{d}{dr} \left( \frac{H_{\alpha}^{x_0}(r)}{r^{n+2}} \right) = 2 r^{n-3} \left( r \int_{B_r(x_0)} |\nabla u(x)|^2 \, d\mu_a - (1 + s) \int_{\partial B_r(x_0)} u^2 |x_n|^a \, d\mathcal{H}^{n-1} \right). \tag{2.7}$$

Then, according to item (i) in Theorem 2.1 and recalling that $x_0 \in \Gamma_{1+s}(u)$ we can deduce that $r^{-(n+2)} H_{\alpha}^{x_0}(r)$ is nondecreasing.

(ii) Let $r_0 = r_0(\varepsilon)$ be a radius such that for all $r < r_0$ it holds $N_{a}^{x_0}(u) \leq (1 + s) + \varepsilon/2$. Then, thanks to (2.6), we obtain

$$N_{a}^{x_0}(r, u) = \frac{r}{2} \frac{d}{dr} \log \left( \frac{H_{\alpha}^{x_0}(r)}{r^{n+2}} \right) \leq (1 + s) + \varepsilon/2.$$  

So, dividing to $\frac{r}{2}$ and integrating on $(r, r_0)$ we have

$$H_{\alpha}^{x_0}(r) \geq H_{\alpha}^{x_0}(r_0) \left( \frac{r}{r_0} \right)^{n+2+\varepsilon}. \quad \Box$$

We now prove a version of the Rellich formula for weighted Sobolev spaces:

**Proposition 2.3** (Rellich formula). Let $v$ be a solution of (1.1). Then it holds that:

$$\int_{\partial B_r} |\nabla v|^2 |x_n|^a \, d\mathcal{H}^{n-1} = \frac{n-2+a}{r} \int_{B_r} |\nabla u|^2 \, d\mu_a + 2 \int_{\partial B_r} \left( \frac{\langle \nabla u, x \rangle}{r} \right)^2 |x_n|^a \, d\mathcal{H}^{n-1}. \tag{2.8}$$

**Proof.** We apply the Divergence Theorem and the third condition of (1.1) for which $u R_{a}(u) = 0$ in $B'_1$ and develop

$$\text{div} \left( |\nabla v|^2 \frac{x}{r} |x_n|^a - 2 \frac{\langle \nabla v, x \rangle}{r} \nabla v |x_n|^a \right). \quad \Box$$

\footnote{The function $N_{a}(0^+, u)$ is the infimum on $r$ of continuous functions $N_{a}^+ (r, u)$.}
In view of section [7] we compute the derivative of the volume and boundary energies.

**Lemma 2.4.** The following formulae hold:

(i) \((H^0_a)'(r) = \frac{n-2a}{r} H^0_a(r) + 2 \int_{\partial B_r(x_0)} u \nabla u \cdot \nu |x_n|^a dH^{n-1}\);

(ii) \((D^0_a)'(r) = \frac{n-2a}{r} D^0_a(r) + 2 \int_{\partial B_r(x_0)} (\nabla u \cdot \nu)^2 |x_n|^a dH^{n-1}\);

(iii) \(D^0_a(r) = \int_{\partial B_r(x_0)} |\nabla u|^2 \mu_a + 2 |x|^2 \nu |x_n|^a dH^{n-1}\).

**Proof.** (i) We can obtain the thesis observing that \(\frac{d}{dr} u^2(x_0+ry) = 2u(x_0+ry)\nabla u(x_0+ry) \cdot y\).

(ii) From Coarea and Rellich Formulae we obtain

\[
(D^0_a)'(r) = \int_{\partial B_r(x_0)} |\nabla u|^2 d\mu_a + \frac{n-2a}{r} D^0_a(r) + 2 \int_{\partial B_r(x_0)} (\nabla u \cdot \nu)^2 |x_n|^a dH^{n-1}.
\]

(iii) In order to prove the formula, it is enough to apply the Divergence Theorem and the third condition of (1.1) for which \(uR_\alpha(u) = 0\) in \(B'_1\).

3 The blow-up method: existence and \((1+s)\)-homogeneity of blow-ups

In order to study the properties of the free-boundary, we investigate the properties of the blow-up limits. We shall consider a suitable sequence of rescaled functions of the solution \(u\). Let \(x_0 \in \Gamma_{1+s}(u)\), we set

\[
u_{x_0, r}(x) := \frac{u(x_0 + rx)}{r^{1+s}}, \quad (3.1)
\]

if \(x_0 = 0\) we denote \(u_r(x)\) in the place of \(u_{x_0, r}(x)\). Note that in the choice of the rescaling factor in (5.1) we follow the same approach as in [25] and [30], which is different with respect to the previous approach used in [3].

The first step in the analysis of blow-ups is to prove the existence of the limits of the sequence \((u_{x_0, r})_r\), for all \(x_0 \in \Gamma_{1+s}(u)\). In order to prove their existence, we state the equiboundedness of \((u_{x_0, r})_r\), with respect to the \(H^1(B_1, \mu_a)\)-norm.

**Proposition 3.1** (Existence of blow-ups). Let \(u \in H^1(B_1, \mu_a)\) be the solution of (1.1) and let \(x_0 \in \Gamma_{1+s}(u)\). Then for every sequence \(r_k \downarrow 0\) there exists a subsequence \((r_{kj})_j \subset (r_k)_k\) such that the rescaled functions \((u_{x_0, r_{kj}})_j\) converge in \(L^2(B_1-|x_0|, \mu_a)\).

**Proof.** Since \(x_0 \in \Gamma_{1+s}(u)\),

\[
\|\nabla u_{x_0, r_k}\|_{L^2(\partial B_1, \mu_a)}^2 = \frac{D^0_{x_0}(r_k)}{r_k^{n+2}} \leq \frac{H^0_{x_0}(1-|x_0|)}{(1+s)(1-|x_0|)^{n+2}} \leq \frac{H^0_{x_0}(1-|x_0|)}{(1-|x_0|)^{n+2}} \leq \frac{H^0_{x_0}(1-|x_0|)}{(1-|x_0|)^{n+2}} \leq \frac{H^0_{x_0}(1-|x_0|)}{(1-|x_0|)^{n+2}} \leq \frac{H^0_{x_0}(1-|x_0|)}{(1-|x_0|)^{n+2}} \leq \frac{H^0_{x_0}(1-|x_0|)}{(1-|x_0|)^{n+2}}.
\]

So, according to the Poincaré inequality we have

\[
\sup_k \|u_{x_0, r_k}\|_{L^2(B_1, \mu_a)} \leq \frac{\sup_k \|u_{x_0, r_k}\|_{L^2(\partial B_1, \mu_a)}}{\sup_k \|\nabla u_{x_0, r_k}\|_{L^2(\partial B_1, \mu_a)}} < \infty. \quad (3.3)
\]

Therefore, thanks to Theorem 1.3 for every sequence of radii \(r_k \downarrow 0\), there exists an extracted subsequence \(r_{kj} \downarrow r)\) such that \(u_{x_0, r_{kj}} \rightarrow u_0\) in \(L^2(B_1-|x_0|, \mu_a)\) as \(j \rightarrow +\infty\). \(\Box\)

**Remark 3.2.** So, according to the quantitative estimate (1.3) and inequality (3.3),

\[
\sup_k \|u_{x_0, r_k}\|_{X_{x_0, \alpha}(\partial B_1)} \leq \sup_k \|u_{x_0, r_k}\|_{L^2(B_1, \mu_a)} < \infty. \quad (3.4)
\]

In particular, in view of (3.4) we can easily deduce that

\[
\|u\|_{L^\infty(B_1(x_0))} \leq C r^{1+s}, \quad \|\nabla u\|_{L^\infty(B_1(x_0), \mathbb{R}^n)} \leq C r^s \quad \text{and} \quad \|x_n|^a \partial_{x_n} u\|_{L^\infty(B_1(x_0), \mathbb{R}^n)} \leq C r^{1-s}. \quad (3.5)
\]
Similarly to [43], we consider an energy “à la Weiss” used in [25] and [28] for fractional Laplacian (see [30] for a version in the fractional Laplacian problem with drift and [23,31] for a version in the classical obstacle problem with quadratic energies with variable coefficients):

$$W_{1+\alpha}^{x_0}(r,u) = \frac{1}{r^{n+1}} \int_{B_r(x_0)} |\nabla u|^2 |x_n|\,dx - \frac{1+s}{r^{n+2}} \int_{\partial B_r(x_0)} u^2 |x_n|\,d\mathcal{H}^{n-1}. \quad (3.6)$$

We note that

$$W_{1+\alpha}^{x_0}(r,u) = \frac{H_{\alpha}^{x_0}(r)}{r^{n+2}} (N_{\alpha}^{x_0}(r,u) - (1+s)),$$

thus if $x_0 \in \Gamma_{1+\alpha}(u)$ by (2.2) and Lemma 2.2 (which guarantees the boundedness of $H_{\alpha}^{x_0}(r)$) we have

$$\lim_{r \rightarrow 0} W_{1+\alpha}^{x_0}(r,u) = 0$$

and due to Theorem 2.1 we obtain

$$W_{1+\alpha}^{x_0}(r,u) \geq 0.$$

Moreover, the function $W_{1+\alpha}^{x_0}(\cdot,u)$ satisfies a monotonicity formula in the same essence as Weiss’ monotonicity formula proved in [43]. For a similar proof see [30, Theorem 3.5].

Proposition 3.3 (Weiss’ monotonicity formula). Let $x_0 \in \Gamma_{1+\alpha}(x_0)$ and $u$ be a solution of Problem (1.1); then the function $r \mapsto W_{1+\alpha}^{x_0}(r,u)$ is nondecreasing. In particular, the following formula holds:

$$\frac{d}{dr} W_{1+\alpha}^{x_0}(r,u) = -\frac{1}{r} \int_{\partial B_1} (\nabla u_r \cdot (1+s)u_r) |x_n|\,d\mathcal{H}^{n-1}$$

Next, we prove the homogeneity property of blow-ups. We prove the result through properties of the frequency function and the optimal regularity of the solution. Proceeding as in [23, Proposition 4.2] and thanks to Proposition 3.3, it is possible to obtain the same result.

Proposition 3.4 ((1 + s)-homogeneity of blow-ups). Let $u \in H^1(B_1,\mu_0)$ be a solution of Problem (1.3). Let $x_0 \in \Gamma_{1+\alpha}(u)$ and $(u_{x_0,r})_r$ be a sequence of rescaled functions. Then, for every sequence $(r_j)_j \downarrow 0$ there exists a subsequence $(r_{j_k})_k \subset (r_j)_j$ such that the sequence $(u_{x_0,r_{j_k}})_k$ converges in $C^{1+\alpha}(\mathbb{R}^n)$ (see (1.4)) for all $\alpha < s$ to $u_{x_0}$ a $(1+s)$-homogeneous function.

Proof. In view of (3.3), thanks to the Ascoli-Arzelà Theorem there exists a subsequence (that we do not relabel) $u_{x_0,r_k}$ and $u_{x_0} \in X_{S,\alpha}(B_{1/2})$ such that $\|u_{x_0,r_k} - u_{x_0}\|_{X_{S,\alpha}(B_{1/2})}$ converge to 0 for all $\beta < s$. It is easy to prove that $u_{x_0}$ is a solution of Problem (1.3). In order to conclude the proof, we show that $u_{x_0}$ is $(1+s)$-homogeneous.

We note that for every $\delta > 0$ we can fix $\rho > 0$ such that $N_{\alpha}^{x_0}(\rho,u) \leq (1+s) + \delta$. So for $k >> 1$, for every $t \in (0,1)$ (such that $t r_k < \rho$)

$$N_{\alpha}(t,u_{x_0,r_k}) = N_{\alpha}(t,u_{x_0}) = N_{\alpha}(t r_k,u) = N_{\alpha}(\rho,u) + N_{\alpha}(\rho,u) \leq (1+s) + \delta,$$

where we resort to Theorem 2.1. Now, from the convergence of $u_{x_0,r_k}$ to $u_{x_0}$ and thanks to the arbitrariness of $\delta$, we obtain $N_{\alpha}(t,u_{x_0}) \equiv 1 + s$; then, by Theorem 2.1(ii), $u_{x_0}$ is $(1+s)$-homogeneous.

Remark 3.5. By proceeding in the same way, we can prove Theorem 2.1(iv) as well.

Proof of Theorem 2.1(iv). Let $x_0 \in \Gamma(u)$ and $\lambda = N_{\alpha}^{x_0}(0^+,u)$. Then, if $r_k \searrow 0$ is a suitable sequence of radii, for all $\delta > 0$ we can fix $\rho > 0$ such that $N_{\alpha}^{x_0}(\rho,u) \leq \lambda + \delta$. So, proceeding in much the same way as in (3.7), we deduce

$$\lambda \leq N_{\alpha}(t,u_{x_0,r_k}) \leq \lambda + \delta,$$

thus, by the strong convergence of $u_{x_0,r_k}$, to its blow-up $w_0$ and by the arbitrariness of $\delta$, we have $N_{\alpha}(t,u_0) \equiv \lambda$. So, by the second item of Theorem 2.1, $w_0$ is $\lambda$-homogeneous and by Theorem 2.1(iii), $\lambda \geq 1 + s$.  


4 Classification of the \((1 + s)\)-homogeneous global solutions

Let \(h_c\) be the function defined by

\[
h_c(x) := \left( s^{-1} \hat{x} \cdot e - \sqrt{(\hat{x} \cdot e)^2 + x_n^2} \right) \left( \sqrt{(\hat{x} \cdot e)^2 + x_n^2} + \hat{x} \cdot e \right)^s. \tag{4.1}
\]

From a simple calculation it is possible to prove the following properties:

(i) \(h_c(\hat{x}, x_n) = h_c(\hat{x}, -x_n)\);

(ii) \(h_c(x) \geq 0\) on \(\{x_n = 0\}\) and \(h_c = 0\) on \(\{x_n = 0, \hat{x} \cdot e \leq 0\}\);

(iii) \(\partial_t h_c(x) = \frac{s-2}{s} \left( \sqrt{(\hat{x} \cdot e)^2 + x_n^2} + \hat{x} \cdot e \right)^{s-1}\);

(iv) \(\partial_n h_c(x) = -(1 + s)x_n \left( \sqrt{(\hat{x} \cdot e)^2 + x_n^2} + \hat{x} \cdot e \right)^{s-1}\);

(v) \(h_c\) is solution of \((1.3)\);

(vi)

\[
R_n h_c(\hat{x}) = \begin{cases} 
0 & \hat{x} \cdot e \geq 0 \\
-(1 + s)(2|\hat{x} \cdot e|)^{1-s} & \hat{x} \cdot e < 0.
\end{cases} \tag{4.2}
\]

In particular, we obtain a complementarity property

\[
h_c(\hat{x}, x_n) R_n h_c(\hat{x}) = 0 \quad \text{on} \quad \{x_n = 0\} \tag{4.3}
\]

In view of properties above, \(h_c\) is a solution of problem \((1.1)\), so by \((17)\) \(\partial_{\tau} h_c \geq 0\) for any vector \(\tau \in S^n \subset \mathbb{R}^{n-1} \times \{0\}\). So, thanks to its \((1 + s)\)-homogeneity, \(h_c\) is a solution of \((1.4)\).

We consider the closed convex cone of \((1 + s)\)-homogeneous global solutions:

\[
\mathcal{J}_{1+s} := \{\lambda h_c : e \in S^{n-2}, \lambda \in [0, +\infty)\} \subset H^1_{\text{loc}}(\mathbb{R}^n, \mu_a). \tag{4.5}
\]

Caffarelli, Salsa and Silvestre \([13]\) proved that \(\mathcal{J}_{1+s} \setminus \{0\}\) is the set of blow-ups in the regular points of the free-boundary with lower frequency.

We note that \(\mathcal{J}_{1+s}\) is a closed cone in \(H^1_{\text{loc}}(\mathbb{R}^n, \mu_a)\). The restriction

\[
\mathcal{J}_{1+s}|_{B_1} := \{v|_{B_1} : v \in \mathcal{J}_{1+s}\} \subset H^1(B_1, \mu_a)
\]

is a closed set, and \(\mathcal{J}_{1+s} \setminus \{0\}\) is parameterized by a \((n - 1)\)-manifold by the map

\[
\mathbb{R}^{n-2} \times (0, \infty) \xrightarrow{\Phi} \mathcal{J}_{1+s} \setminus \{0\} \\
(e, \lambda) \quad \mapsto \quad \lambda h_c.
\]

Next we can introduce the tangent plane to space \(\mathcal{J}_{1+s}\) in every point \(\lambda h_c\) as

\[
T_{\lambda h_c} \mathcal{J}_{1+s} := \{d_{(e, \lambda)} \Phi(\xi, \alpha) : \xi \cdot e_n = \xi \cdot e = 0, \alpha \in \mathbb{R}\}. \tag{4.6}
\]

We compute the derivative of the map \(\Phi\) in a point of \(S^{n-2} \times (0, \infty)\):

\[
d_{(e, \lambda)} \Phi(\xi, \alpha) = \left. \frac{d}{dt} h_{\sigma(t)} \right|_{t=0}
\]

with \(\sigma(t) = \frac{e + t \xi}{|e + t \xi|}\), a curve on \(S^{n-2}\) such that \(\sigma(0) = e\) and \(\sigma'(0) = \xi\). By \((1.1)\) and \((1.7)\) we obtain

\[
\left. \frac{d}{dt} h_{\sigma(t)} \right|_{t=0} = \left( s^{-1} - s \right) \hat{x} \cdot \xi \left( \sqrt{(\hat{x} \cdot e)^2 + x_n^2} + \hat{x} \cdot e \right)^s.
\]
Then, we can rewrite (4.6) as
\[ T_{\lambda B_n}(\Omega_{1+s}) := \{ \alpha h_c + v_{c,\xi} : \xi \cdot e_n = \xi \cdot e = 0, \alpha \in \mathbb{R} \} \]
where the function \( v_{c,\xi} \) is defined as follows:
\[ v_{c,\xi} = \hat{x} \cdot \xi \left( \sqrt{(\hat{x} \cdot e)^2 + x_n^2 + \hat{x} \cdot e} \right)^s. \]

We highlight some properties of function \( \psi \in \Phi_{1+s} \). For all \( \varphi \in H^1(B_1, \mu_0) \), integrating by parts, according to Theorem 1.2 and Euler’s homogeneous function Theorem we obtain
\[
\int_{B_1} \nabla \psi \cdot \nabla \varphi d\mu_0 = \int_{\partial B_1} \varphi \nabla \psi \cdot x |x_n|^a dH^{n-1} - 2 \int_{B_1} \varphi R_{a}(\psi) d\mu_0
\]
\[ = (1 + s) \int_{\partial B_1} \varphi \psi |x_n|^a dH^{n-1} - 2 \int_{B_1} \varphi R_a(\psi) d\mu_0. \]

**Remark 4.1.** The first variation of functional \( W_{1+s}^\alpha(1, \cdot) \) in a point \( \psi \in \Phi_{1+s} \) along a direction \( \varphi \in H^1(B_1, \mu_0) \) is:
\[
\delta W_{1+s}^\alpha(1, \psi)[\varphi] = 2 \int_{B_1} \nabla \psi \cdot \nabla \varphi d\mu_0 - 2(1 + s) \int_{\partial B_1} \psi \varphi |x_n|^a dH^{n-1}.
\]

Then, by (4.3)
\[
\delta W_{1+s}^\alpha(1, \psi)[\varphi] = -4 \int_{B_1} \varphi R_{a}(\psi)(\hat{x}) dH^{n-1},
\]
by (4.3)
\[
\delta W_{1+s}^\alpha(1, \psi)[\psi] = 0,
\]
so we can infer that
\[
W_{1+s}^\alpha(1, \psi) = \frac{1}{2} \delta W_{1+s}^\alpha(1, \psi)[\psi] = 0 \quad \forall \psi \in \Phi_{1+s}.
\]

## 5 The epiperimetric inequality and its consequences

In this section we prove an epiperimetric inequality for the points in \( \Gamma_{1+s}(u) \), and its main consequences in the framework of the regularity of the free-boundary. In Paragraph 5.1 we prove the epiperimetric inequality. In Paragraph 5.2 we establish a decay estimate for adjusted boundary energy. In Paragraphs 5.3 and 5.4 we state the nondegeneracy of the solution and the uniqueness of the blow-ups in \( \Gamma_{1+s}(u) \) respectively.

### 5.1 Epiperimetric inequality

We now state the main result of this paper: the epiperimetric inequality “à la Weiss” in our setting. This result is a key ingredient in our approach to the decay of the boundary adjusted energy and to the uniqueness of blow-ups (see \[25\] for the classical case of Laplacian \( s = 1/2 \)).

In this paragraph we state and prove the epiperimetric inequality. For the convenience of readers, the proof will be split into several steps.

**Theorem 5.1** (Epiperimetric inequality). There exists a dimensional constant \( \kappa \in (0, 1) \) such that if \( c \in H^1(B_1, \mu_0) \) is a \((1 + s)\)-homogeneous function with \( c \geq 0 \) on \( B_1 \) and \( c(\hat{x}, x_n) = c(\hat{x}, -x_n) \) then
\[
\inf_{\psi \in \mathcal{A}_e} W_{1+s}^\alpha(\psi) \leq (1 - \kappa) W_{1+s}^\alpha(c).
\]

**Proof.** Without loss of generality it is possible to suppose that the function \( c \) satisfies the following condition
\[
\text{dist}_{H^1(B_1, \mu_0)}(c, \Phi_{1+s}) < \delta.
\]
In fact, according to the \((1 + s)\)-homogeneity of \( c \) and recalling that \( \Phi_{1+s} \) is a cone, for all \( \delta > 0 \) there exists a constant \( \gamma > 0 \) such that
\[
\text{dist}_{H^1(B_1, \mu_0)}(\gamma c, \Phi_{1+s}) < \delta.
\]

\[ \text{The first variation is defined as } \delta W_{1+s}^\alpha(1, \psi)[\varphi] := \lim_{t \to 0} \left( W_{1+s}^\alpha(1, \psi + t \varphi) - W_{1+s}^\alpha(1, \psi) \right). \]
We can observe that if \( v \in A_{\gamma c} \) then \( \gamma^{-1}v \in A_c \). So, if we prove inequality (5.1) for the function \( \gamma_{c} \), or rather
\[
\inf_{w \in A_{\gamma c}} W_{1+s}^{\mu}(1, v) \leq (1 - \kappa)W_{1+s}^{\mu}(1, \gamma_{c}),
\]
then, thanks to \( W_{1+s}^{\mu}(1, \gamma_{c}) = \gamma^2 W_{1+s}^{\mu}(1, c) \) we infer
\[
\inf_{w \in A_{c}} W_{1+s}^{\mu}(1, w) \leq (1 - \kappa)W_{1+s}^{\mu}(1, c).
\]
To simplify the notation we denote the functional \( W_{1+s}^{\mu}(1, \cdot) \) by \( G(\cdot) \).

We argue by contradiction. Let us suppose the existence of sequences of positive numbers \( \kappa_j, \delta_j \downarrow 0 \) and a sequence of \((1 + s)\)-homogeneous functions \( c_j \in H^1(B_1, \mu_a) \) with \( c_j \geq 0 \) on \( B'_1 \) such that
\[
dist_{H^1(B_1, \mu_a)}(c_j, \delta_{1+s}) = \delta_j, \quad (5.3)
\]
In particular, fixing \( h := h_{\xi_j} \), up to change of coordinate depending on \( j \), we assume that there exists \( \lambda_j \geq 0 \) for which \( \psi_j := \lambda_j h \) is the point satisfying the minimum distance between \( c_j \) and \( \delta_{1+s} \), or rather
\[
\|\psi_j - c_j\|_{H^1(B_1, \mu_a)} = \text{dist}_{H^1(B_1, \mu_a)}(c_j, \delta_{1+s}) = \delta_j, \quad \forall j \in \mathbb{N}. \quad (5.5)
\]
We split the proof into some intermediate steps.

**Step 1: Auxiliary functionals.** We can rewrite (5.4) and interpret this inequality as a condition of quasi-minimality for a sequence of new functionals. Setting \( j \in \mathbb{N} \), let \( v \in A_{c_j} \), we use (4.9) (applied twice to \( \psi_j \) with test functions \( c_j - \psi_j \) and \( v - \psi_j \)) and (4.11); we can rewrite (5.4):
\[
(1 - \kappa_j) \left( G(c_j) - G(\psi_j) - \delta G(\psi_j)[c_j - \psi_j] - 4 \int_{B'_1} (c_j - \psi_j) R_a(\psi_j) dH^{n-1} \right)
\leq G(v) - G(\psi_j) - \delta G(\psi_j)[v - \psi_j] - 4 \int_{B'_1} (v - \psi_j) R_a(\psi_j) dH^{n-1}. \quad (5.6)
\]
We can observe that \( G(v_1) - G(v_2) - \delta G(v_2)[v_1 - v_2] = G(v_1 - v_2) \), then for all \( v \in A_{c_j} \) (5.6) can be rewritten as
\[
(1 - \kappa_j) \left( G(c_j - \psi_j) - 4 \int_{B'_1} (c_j - \psi_j) R_a(\psi_j) dH^{n-1} \right)
\leq G(\psi_j - v) - 4 \int_{B'_1} (\psi_j - v) R_a(\psi_j) dH^{n-1}. \quad (5.7)
\]
Next we define new sequences of functions
\[
z_j := \frac{c_j - \psi_j}{\delta_j} \quad (5.8)
\]
(recalling that \( \psi_j = \lambda_j h \)), positive numbers \( \theta_j := \frac{\lambda_j}{\delta_j} \) and sets \( B_j := \set{z \in z_j + H^1_\beta(B_1, \mu_a) : (z + \theta_j h)_{|B_1'} = 0} \). Now we introduce a sequence of auxiliary functionals \( G_j : L^2(B_1, \mu_a) \to (-\infty, +\infty] \)
\[
G_j(z) := \begin{cases} 
\int_{B_1} |\nabla z|^2 d\mu_a - (1 + s) \int_{\partial B_1} z_j^2 |x_n|^a dH^{n-1} - 4\theta_j \int_{B'_1} z R_a(h) dH^{n-1} & \text{if } z \in B_j' \\
+\infty & \text{otherwise.} 
\end{cases} \quad (5.9)
\]
We can observe that the second term in the formula above does not depend on \( z \) but only on its boundary datum \( z_{|\partial B_1} = z_j_{|\partial B_1} \).

We can rewrite (5.7) with the new notation and obtain
\[
(1 - \kappa_j) \left( G(\delta_j z_j) - 4\delta_j \int_{B_1'} z_j R_a(\lambda_j h) dH^{n-1} \right) \leq G(\delta_j z) - 4\delta_j \int_{B_1'} z R_a(\lambda_j h) dH^{n-1}
\]
and dividing by \( \delta_j^3 \) we obtain the condition of quasi-minimality for \( z_j \) with respect to \( G_j \):
\[
(1 - \kappa_j) G_j(z_j) \leq G_j(z) \quad \forall z \in L^2(B_1, \mu_a). \quad (5.10)
\]
Therefore we note that by the very definitions of $z_j$ and $\delta_j$ we have

$$\|z_j\|_{H^1(B_1, \mu_a)} = 1. \quad (5.11)$$

So, by the compactness of Sobolev embedding from $H^1(B_1, \mu_a)$ into the space $L^2(B_1, \mu_a)$ Theorem 1.3 the trace operator from $H^1(B_1, \mu_a)$ into the space $L^2(B_1')$ Theorem 1.5 and the trace operator from $H^1(B_1, \mu_a)$ into $L^2(\partial B_1, |x_n|^{n-1})$ Theorem 1.6 we may extract a subsequence (which we do not relabel) such that

(a) $(z_j)_{j \in \mathbb{N}}$ converges weakly in $H^1(B_1, \mu_a)$ to some $z_\infty$;

(b) the sequences of traces $z_j|_{B_1}$ and $z_j|_{\partial B_1}$ converge respectively in $L^2(B_1')$ and $L^2(\partial B_1, |x_n|^{n-1})$;

(c) $\theta_j$ has a limit $\theta \in [0, \infty]$.

**Step 2: First property of $(G_j)_{j \in \mathbb{N}}$.** In this step we establish the equi-coercivity and some other properties of the family $(G_j)_{j \in \mathbb{N}}$.

We observe that for all $w \in B_j$, since $w|_{\partial B_1} = z_j|_{\partial B_1}$ and $hR_a(h)(\tilde{x}) = 0$, it holds that

$$- \int_{B_1'} wR_a(h)(\tilde{x}) \, d\mathcal{H}^{n-1} = - \int_{B_1'} (w + \theta_j h)R_a(h)(\tilde{x}) \, d\mathcal{H}^{n-1} + \theta_j \int_{B_1'} hR_a(h)(\tilde{x}) \, d\mathcal{H}^{n-1} \geq 0 \quad (5.12)$$

where we used (4.2) for which $R_a(h)(\tilde{x}) \leq 0$ and the condition $w \in B_j$ for which $(w + \theta_j h)|_{B_1'} \geq 0$. Then from the definition of $G_j$ we have

$$\int_{B_1} |\nabla w|^2 \, d\mu_a - (1 + s) \int_{\partial B_1} z_j^2 |x_n|^a \, d\mathcal{H}^{n-1} \leq G_j(w). \quad (5.13)$$

This establishes the equi-coercivity of the sequence $G_j$, in fact from (5.11), thanks to strong convergence of traces, we obtain

$$\liminf_{j \to \infty} G_j(z_j) \geq - (1 + s) \int_{\partial B_1} z_\infty^2 |x_n|^a \, d\mathcal{H}^{n-1} - 4\theta \int_{B_1'} z_\infty R_a(h) \, d\mathcal{H}^{n-1};$$

while if $\theta = +\infty$ from (5.11) and (5.13) we conclude that

$$\liminf_{j \to \infty} G_j(z_j) \geq - (1 + s) \int_{\partial B_1} z_\infty^2 |x_n|^a \, d\mathcal{H}^{n-1}.$$

Note that it is not restrictive (up to subsequence) to assume that $G_j(z_j)$ has a limit in $(-\infty, +\infty]$. Finally we can observe that

$$\lim_{j \to \infty} G_j(z_j) = +\infty \quad \iff \quad \lim_{j \to \infty} \theta_j \int_{B_1'} z_j R_a(h) \, d\mathcal{H}^{n-1} = -\infty. \quad (5.14)$$

**Step 3: Asymptotic analysis of $(G_j)_{j \in \mathbb{N}}$.** In this step we prove a result of $\Gamma$-convergence for the family of functionals $(G_j)_{j \in \mathbb{N}}$.

We can distinguish three cases:

1. If $\theta \in [0, +\infty)$, then $(z_\infty + \theta h)|_{B_1'} \geq 0$ and $\Gamma(L^2(B_1, \mu_a))$-lim $G_j = G^{(1)}_\infty$ with

$$G^{(1)}_\infty(z) := \begin{cases} 
\int_{B_1'} |\nabla z|^2 \, d\mu_a - (1 + s) \int_{\partial B_1} z_\infty^2 |x_n|^a \, d\mathcal{H}^{n-1} + 4\theta \int_{B_1'} z_\infty R_a(h) \, d\mathcal{H}^{n-1} & \text{if } z \in B^{(1)}_\infty \\
+\infty & \text{otherwise,}
\end{cases}$$

where $B^{(1)}_\infty := \{ z \in z_\infty + H^1_0(B_1, \mu_a) : (z + \theta h)|_{B_1'} \geq 0 \}$.

2. If $\theta = +\infty$ and $\lim_{j} G_j(z_j) < \infty$, then $z_\infty|_{B_1'} = 0$ (where $B_1' = B'_1 \cap \{ x_{n-1} \leq 0 \}$) and $\Gamma(L^2(B_1, \mu_a))$-lim $G_j = G^{(2)}_\infty$ with

$$G^{(2)}_\infty(z) := \begin{cases} 
\int_{B_1} |\nabla z|^2 \, d\mu_a - (1 + s) \int_{\partial B_1} z_\infty^2 |x_n|^a \, d\mathcal{H}^{n-1} & \text{if } z \in B^{(2)}_\infty \\
+\infty & \text{otherwise,}
\end{cases}$$

where $B^{(2)}_\infty := \{ z \in z_\infty + H^1_0(B_1, \mu_a) : z|_{B_1'} = 0 \}$. We note that the third addendum of $G_j$ is zero in $B^{(2)}_\infty$, while if $z \in B_j \setminus B^{(2)}_\infty$ the sequence $G_j(z)$ diverges; this heuristically justifies the choice of $G^{(2)}_\infty(z)$.
and $\mathcal{B}_{2}^{(2)}$.

(3) If $\theta = +\infty$ and $\lim j_{\theta} G_{j}(z_{j}) = +\infty$, then $\Gamma(L^{2}(B_{1}, \mu_{a}))$-limit $G_{j}^{(3)}$ with

$$G_{j}^{(3)}(z) = +\infty \quad \text{on } L^{2}(B_{1}, \mu_{a}).$$

For the reader’s convenience we recall the Definition of $\Gamma$-limit (see [17]): the equality $\Gamma(L^{2}(B_{1}, \mu_{a}))$-limit $G_{j} = G_{\infty}^{(i)}$ with $i = 1, 2, 3$ is satisfied if the two following conditions hold:

(a) for all sequences $(w_{j})_{j} \subset L^{2}(B_{1}, \mu_{a})$ and $w \in L^{2}(B_{1}, \mu_{a})$ such that $w_{j} \to w$ in $L^{2}(B_{1}, \mu_{a})$ it holds

$$\liminf_{j} G_{j}(w_{j}) \geq G_{j}^{(i)}(w) \quad \text{(5.15)}$$

(b) for all $w \in L^{2}(B_{1}, \mu_{a})$ there exists a sequence $(w_{j})_{j} \subset L^{2}(B_{1}, \mu_{a})$ such that $w_{j} \to w$ in $L^{2}(B_{1}, \mu_{a})$ and

$$\limsup_{j} G_{j}(w_{j}) \leq G_{j}^{(i)}(w). \quad \text{(5.16)}$$

Proof of the $\Gamma$-convergence: case (1).

(a) Without loss of generality we may suppose that $\liminf_{j} G_{j}(w_{j}) = \lim j G_{j}(w_{j}) < +\infty$, then $w_{j} \in B_{j}$ for all $j \in \mathbb{N}$. Taking (5.15) into account, we deduce

$$\int_{B_{1}} |\nabla w_{j}|^{2} \, d\mu_{a} \leq G_{j}(w_{j}) + (1 + s) \int_{\partial B_{1}} w_{j}^{2} |x_{\alpha}| \, dH^{n-1} < +\infty,$$

then, since $w_{j} \to w$ in $L^{2}(B_{1}, \mu_{a})$ we have $sup_{j} \|w_{j}\|_{H^{1}(B_{1}, \mu_{a})} < +\infty$, so from Theorem 1.3 $\nabla w_{j} \to \nabla w$ in $L^{2}(B_{1}, \mu_{a})$. Then the respective traces converge in $L^{2}(\partial B_{1}, \mu_{a})$ Theorem 1.6 and $L^{2}(B_{1}^{c})$ Theorem 1.5. Hence, we obtain $(w + \theta h)|_{B_{1}} \geq 0$ and, in particular, since $w_{j}|_{B_{1}} = z_{j}|_{B_{1}}$ then $w|_{B_{1}} = z_{j}|_{B_{1}}$ and so $z_{j} \in \mathcal{B}_{2}^{(1)}$. At this point thanks to the convergence of traces of $w_{j}$ and weak semicontinuity of the norm in $L^{2}(B_{1}, \mu_{a})$ we have (5.15).

(b) We observe that it is sufficient to prove the inequality for $w \in \mathcal{B}_{2}^{(1)}$ with

$$\text{supp}(w - z_{\infty}) \subset B_{\rho} \quad \text{for some } \rho \in (0, 1). \quad \text{(5.17)}$$

If we want to deal with the general case, we consider the function

$$w_{t}(x) = t^{1+s} \left( w \left( \frac{x}{t} \right) - z_{\infty} \left( \frac{x}{t} \right) \right)$$

with $t < 1$.

It is easy to prove that $w_{t} \in H^{1}(B_{1}, \mu_{a})$ and $\text{supp}(w_{t} - z_{\infty}) \subset B_{\rho}$; moreover, $w_{t} \to w$ in $H^{1}(B_{1}, \mu_{a})$ (for a similar procedure see [85], Proposition 2.4.1, Chapter 2). If (5.16) holds for all $w_{t}$, resorting to a diagonalization argument we obtain (5.18) for $w$. Therefore for a Uryshon’s type property it is sufficient to prove the following property: fixing $w$ as in (5.17), for all sub sequences $j_{k} \uparrow +\infty$ there exists an extract subsequence $j_{k_{i}} \uparrow +\infty$ and there exists $w_{i} \to w$ in $L^{2}(B_{1}, \mu_{a})$ such that

$$\limsup_{i} G_{j_{k_{i}}}(w_{i}) \leq G_{j}^{(1)}(w).$$

Setting $r \in (\rho, 1)$ let $R := \frac{1 + r}{2}$ and let $\varphi \in C_{c}^{\infty}(B_{1})$ be a cut-off function such that

$$\varphi|_{B_{r}} \equiv 1, \quad \varphi|_{B_{1} \setminus B_{R}} \equiv 0, \quad \|\nabla \varphi\|_{L^{\infty}} \leq \frac{4}{1 - r}.$$

We define

$$w_{k}^{+} := \varphi(w + (\theta - \theta_{j_{k}})h) + (1 - \varphi)z_{j_{k}}$$

leading to a contradiction.
and we verify that \( w_k^r \in B_{j_k} \). In fact \( w \in \mathcal{B}^{(1)}_\infty \), \( z_{j_k} \in B_{j_k} \) and

\[
 w_k^r + \theta_{j_k} h = \varphi(w + \theta h) + (1 - \varphi)(z_{j_k} + \theta_{j_k} h) \geq 0.
\]

Therefore, since \( \theta_{j_k} \to \theta \in [0, +\infty) \) we have \( w_k^r \to \varphi w + (1 - \varphi) z_\infty \) in \( L^2(B_1, \mu_a) \). Thanks to the convergence of traces of \( z_{j_k} \) in \( L^2(B_1^r) \) it is enough to prove the upper bound inequality for the first addendum of \( G_j \) and \( G_j^{(1)} \) respectively. From (5.18), we can infer

\[
 \int_{B_1} |\nabla w_k^r|^2 \, d\mu_a \leq \int_{B_r} |\nabla w + (\theta - \theta_{j_k}) \nabla h|^2 \, d\mu_a + \left( \int_{B_r \setminus B_r^c} |\nabla w_k^r|^2 \, d\mu_a + \int_{B_1 \setminus B_r} |\nabla z_{j_k}|^2 \, d\mu_a. \right) \tag{5.19}
\]

Since \( r > \rho \), from assumption (5.17), we estimate the term \( I_k \) as follows

\[
 I_k \leq 3 \int_{B_r \setminus B_r^c} \varphi^2 |\nabla w + (\theta - \theta_{j_k}) \nabla h|^2 \, d\mu_a
 + 3 \int_{B_r \setminus B_r^c} (1 - \varphi)^2 |\nabla z_{j_k}|^2 \, d\mu_a + 3 \int_{B_r \setminus B_r^c} |\nabla \varphi|^2 |z_\infty - z_{j_k} + (\theta - \theta_{j_k}) \nabla h|^2 \, d\mu_a
\]

So

\[
 \limsup_k \int_{B_1} |\nabla w_k^r|^2 \, d\mu_a \leq \int_{B_r} |\nabla w|^2 \, d\mu_a + 3 \int_{B_r \setminus B_r^c} |\nabla w|^2 \, d\mu_a + 4 \limsup_k \int_{B_1 \setminus B_r} |\nabla z_{j_k}|^2 \, d\mu_a \tag{5.20}
\]

By the \((1 + s)\)-homogeneity of \( z_{j_k} \), we deduce

\[
 \int_{B_1 \setminus B_r} |\nabla z_{j_k}|^2 \, d\mu_a = \int_r^1 \int_{\partial B_1} |\nabla z_{j_k}|^2 |x_n|^a \, d\mathcal{H}^{n-1} \, dt
 = \int_r^1 t^n \int_{\partial B_1} |\nabla z_{j_k}|^2 |x_n|^a \, d\mathcal{H}^{n-1} \, dt = \frac{1 - r^{n+1}}{n+1} \int_{\partial B_1} |\nabla z_{j_k}|^2 |x_n|^a \, d\mathcal{H}^{n-1}
\]

which leads us to

\[
 \int_{\partial B_1} |\nabla z_{j_k}|^2 |x_n|^a \, d\mathcal{H}^{n-1} = \frac{n + 1}{1 - (1/2)^{n+1}} \int_{B_1 \setminus B_r} |\nabla z_{j_k}|^2 \, d\mu_a \leq 2(n + 1)
\]

in turn implying

\[
 \int_{B_1 \setminus B_r} |\nabla z_{j_k}|^2 \, d\mu_a \leq 2 (1 - r) (n + 1). \tag{5.21}
\]

We apply this construction to a subsequence \( r_i \uparrow 1 \) and \( R_i := \frac{1 + r_i}{2} \) and with a diagonal argument we obtain a subsequence \( w_i \to w \) in \( L^2(B_1, \mu_a) \). Thanks to (5.20) and (5.21)

\[
 \limsup_i \int_{B_1} |\nabla w_i|^2 \, d\mu_a \leq \int_{B_1} |\nabla w|^2 \, d\mu_a + 3 \limsup_i \int_{B_{R_i} \setminus B_{R_i}^c} |\nabla w|^2 \, d\mu_a + 4 \limsup_i \int_{B_1 \setminus B_{R_i}} |\nabla z_{j_i}|^2 \, d\mu_a
 \leq \int_{B_1} |\nabla w|^2 \, d\mu_a + 3 \limsup_i \int_{B_{R_i}} |\nabla w|^2 \, d\mu_a
 + 8 \limsup_i \int_{B_1} |\nabla w|^2 \, d\mu_a
\]

and this provides the conclusion.

Proof of the \( \Gamma \)-convergence: case (2).

(a) Without loss of generality we assume that

\[
 \liminf_j G_j(w_j) = \lim G_j(w_j) < +\infty. \tag{5.22}
\]

Let \( w_j \to w \) in \( L^2(B_1, \mu_a) \), since \( w_j \in \mathcal{B}_j \) and (5.22), then \( w \geq 0 \) on \( B_1^r \). From (5.12), we obtain

\[
 0 \leq -\theta_j \int_{B_1} w_j R_a(h) \, d\mathcal{H}^{n-1} \leq G_j(w_j) + (1 + s) \int_{\partial B_1} z_j^2 |x_n|^a \, d\mathcal{H}^{n-1}
 \leq \sup_j \left( G_j(w_j) + (1 + s) \int_{\partial B_1} z_j^2 |x_n|^a \, d\mathcal{H}^{n-1} \right) < +\infty.
\]

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Then dividing by \( \theta_j \), the convergence of traces leads us to

\[
\int_{B_1} w R_a(h) \, dH^{n-1} = \lim_{j} \int_{B_1} w_j R_a(h) \, dH^{n-1} = 0
\]

From (4.12) we deduce that \( \|w\|_{B_1^{\prime,-}} = 0 \), or rather \( w \in B_1^{\prime,2} \). In particular also \( z_\infty \in B_1^{\prime,2} \) because \( \sup \mathcal{G}_j(z_j) < +\infty \). Then, according to the semicontinuity of the norm \( H^1(B_1, \mu_a) \) with respect to weak convergence of gradient, the convergence of \( w_j \) in \( L^2(B_1, \mu_a) \) and the convergence of traces in \( L^2(\partial B_1, |x_n|^a H^{n-1}) \) we obtain the \( \Gamma \)-liminf inequality (5.15).

(b) Now we prove the inequality (5.10). With the same argument used in case (1) we can consider the case of \( \Gamma \). Then dividing by \( w \), we introduce the following auxiliary tools.

Then, according to the semicontinuity of the norm \( H^1(B_1, \mu_a) \) we obtain the \( \Gamma \)-liminf inequality (5.15).

We introduce the positive Radon measures

\[
\nu_k := |\nabla z_{\rho_k}|^2 \|x_n|^a L \setminus B_1 - 4 \theta_{\rho_k}(z_{\rho_k} + \theta_{\rho_k} h) R_a(h) H^{n-1} \setminus B_1^{\prime,-}
\]

Assuming that \( k \gg 1 \), we obtain

\[
\nu_k(B_1) = \mathcal{G}_{\rho_k}(z_{\rho_k}) + (1 + s) \int_{\partial B_1} \frac{z_{\rho_k}^2}{|\nabla z_{\rho_k}|} |x_n|^a H^{n-1} \leq \sup_j \mathcal{G}_j(z_j) + C \sup_j \|z_j\|_{H^1(B_1, \mu_a)} < \infty,
\]

which leads us to

\[
\sup_k \nu_k(B_1) = \lambda_0 < +\infty.
\]

In order to prove \( \nu_k(B_1) = \rho^{n+1} \nu(B_1) \) we observe that setting \( \rho \in (0,1) \) by \( (1 + s) \)-homogeneity of \( z_{\rho_k} \) we obtain

\[
\int_{B_1} |\nabla z_{\rho_k}|^2 \, d\mu_a = \int_0^\rho dt \int_{\partial B_1} |\nabla z_{\rho_k}|^2 |x_n|^a H^{n-1} \quad \overset{\epsilon \to 0}{=} \int_0^{\rho} t^{n-1} \int_{\partial B_1} |\nabla z_{\rho_k}(ty)|^2 |y_n|^a H^{n-1}(y) \, dt
\]

\[
= \rho^{n+1} \int_0^1 t \int_{\partial B_1} |\nabla z_{\rho_k}(y)|^2 |y_n|^a H^{n-1}(y) \, dt = \rho^{n+1} \int_0^1 |\nabla z_{\rho_k}(x)|^2 |x_n|^a H^{n-1}(y) \, dt = \rho^{n+1} \int_{B_1} |\nabla z_{\rho_k}|^2 \, d\mu_a,
\]

and

\[
\int_{B_1} |z_{\rho_k} R_a(h)(\tilde{x})| \, dH^{n-1} = \int_0^{\rho} \int_{\partial B_1} z_{\rho_k} R_a(h)(\tilde{x}) \, dH^{n-2}
\]

\[
= \rho^{n+1} \int_0^{\rho} t \int_{\partial B_1} z_{\rho_k}(t \tilde{y}, 0) \lim_{\epsilon \to 0} \partial \frac{\partial h}{\partial x_n}(t \tilde{y}, t \epsilon) \, dH^{n-2}(\tilde{y})
\]

\[
= \rho^{n+1} \int_{\partial B_1} z_{\rho_k}(\tilde{y}, 0) R_a(h)(\tilde{y}) \, dH^{n-2}(\tilde{y}) = \rho^{n+1} \int_{B_1} z_{\rho_k} R_a(h)(\tilde{x}) \, dH^{n-1}
\]

where in the last equality we did the previous calculus again in reverse order. Since \( \nu_k(B_1) < \infty \) then \( \nu_k(\partial B_1) = 0 \) with \( \rho \in (0,1) \setminus I \) where \( I \) is a set at the most countable. Thus

\[
\nu_k(\partial B_1 \setminus B_2) \leq \lambda_0(\rho_1^{n+1} - \rho_2^{n+1}) \leq c \rho_1 \rho_2 \lambda_0,
\]

for all \( 0 < \rho_1 \leq \rho_2 < 1 \) such that \( \rho_1, \rho_2 \in (0,1) \setminus I \). Repeating the argument in (5.17) we prove the \( \Gamma \)-limsup inequality for function \( w \in B_1^{\prime,2} \) for which there exists some \( \rho \in (0,1) \) such that \( \{w \neq z_\infty\} \subset B_1^{\prime,2} \). We extend \( w \) on \( \mathbb{R}^n \) as \( z_\infty \) in \( B_1^{\prime,2} \) and we indicate the extension by \( w \) again. We fix \( \epsilon > 0 \) and introduce the following auxiliary tools.

Due to the definition of \( H^1(B_1, \mu_a) \) as \( \mathcal{C}^\infty(B_1) \) (cf. Section 1.9 and Lemma 1.15) there exists a function \( u_\delta \in \mathcal{C}^\infty(B_1) \) such that

\[
\|u_\delta - w\|_{H^1(B_1, \mu_a)} < \delta(\epsilon) \quad \text{with} \quad \delta(\epsilon) = o(\epsilon).
\]
Let \( w^\varepsilon(x) := w(x - 3\varepsilon e_{n-1}) \) be the translated function along the direction \( e_{n-1} \). Since \( w \in B^{(2)}_{\infty} \), we observe that
\[
w^\varepsilon(x) = 0 \iff x - 3\varepsilon e_{n-1} \in \{ (\hat{x},0) : x_{n-1} \leq 0 \} \iff x \in \{ (\hat{x},0) : x_{n-1} \leq 3\varepsilon \}.
\]

Let \( I_\sigma \) be the set defined as
\[
I_\sigma = \{ x \in B_1 : \text{dist}(x, B_1^\varepsilon) < \sigma \}
\]

Let \( \phi_\varepsilon \) and \( \chi_\varepsilon \) be two cut-off functions such that
\[
\phi_\varepsilon \in C^\infty(I_{3\varepsilon}), \quad \phi_\varepsilon|_{I_{2\varepsilon}} \equiv 1, \quad \| \nabla \phi_\varepsilon \|_{L^\infty(B_1)} \leq \frac{C}{\varepsilon},
\]
\[
\chi_\varepsilon \in C^\infty(B_{1-\varepsilon}), \quad \chi_\varepsilon|_{B_{1-2\varepsilon}} \equiv 1, \quad \| \nabla \chi_\varepsilon \|_{L^\infty(B_1)} \leq \frac{C}{\varepsilon}.
\]

For all \( 0 < \varepsilon << 1 \) we build the sequence of functions
\[
w^{(\varepsilon)}_k := \chi_\varepsilon(\phi_\varepsilon w^\varepsilon + (1 - \phi_\varepsilon)v^\delta) + (1 - \chi_\varepsilon)(z_{jk}).
\]

Then we can at once infer
\[
w^{(\varepsilon)}_k \in z_{jk} + W^{1,2}_0(B_1)
\]
and since we can write
\[
w^{(\varepsilon)}_k + \theta_{jk} h := \chi_\varepsilon(\phi_\varepsilon(v_{3\varepsilon} + \theta_{jk} h) + (1 - \phi_\varepsilon)(w^\varepsilon + \theta_{jk} h)) + (1 - \chi_\varepsilon)(z_{jk} + \theta_{jk} h),
\]
we prove that \( w^{(\varepsilon)}_k \in B_{3\varepsilon} \). We prove \( w^{(\varepsilon)}_k \) is a convex combination of functions \( v_{3\varepsilon}, w^\varepsilon \) and \( z_{jk} \) with boundary data as \( z_{jk} \) and every addendum is bigger than \( -\theta_{jk} h \) restricted to \( B_1^\varepsilon \). In fact
(i) by definition \( z_{jk} + \theta_{jk} h \geq 0 \) in \( B_1^\varepsilon \);
(ii) if \( x \in \text{supp}(\phi_\varepsilon) \cap B_1^\varepsilon \) then \( x_{n-1} < 3\varepsilon \). Thus \( w^\varepsilon(x) = 0 \) then \( \phi_\varepsilon(x)(w^\varepsilon(x) + \theta_{jk} h(x)) = \phi_\varepsilon(x) \theta_{jk} h(x) \geq 0 \);
(iii) if \( x \in \text{supp}(1 - \phi_\varepsilon) \cap B_1^\varepsilon \) then \( x_{n-1} \geq 2\varepsilon \), so \( h(\hat{x},0) > 0 \) and as \( \theta_{jk} \to +\infty \) \( v^\delta(x) + \theta_{jk} h(x) \geq -\|v_{3\varepsilon}\|_{L^\infty(B_1)} + \theta_{jk} h(x) \geq 0 \) for \( k > k_3 \).

So \( w^{(\varepsilon)}_k \in B_{3\varepsilon} \) for \( k > k_3 \).

Next, consider,
\[
J_k^\varepsilon := -4\theta_{jk} \int_{B_1^\varepsilon} w^{(\varepsilon)}_k R_a(h) \, d\mathcal{H}^{n-1}
\]
\[
I_k^\varepsilon := \int_{B_1} |\nabla w^{(\varepsilon)}_k|^2 \, d\mu_a,
\]
respectively the trace term and the volume term of the energy of \( w^{(\varepsilon)}_k \). By definition we have
\[
J_k^\varepsilon \leq -4\theta_{jk} \int_{B_{1-\varepsilon}} (\phi_\varepsilon w^\varepsilon + (1 - \phi_\varepsilon)v^\delta) R_a(h) \, d\mathcal{H}^{n-1} - 4\theta_{jk} \int_{B_{1-\varepsilon} \setminus B_{1-2\varepsilon}} z_{jk} R_a(h) \, d\mathcal{H}^{n-1} = J_k^{(1)} + J_k^{(2)}.
\]

According to (i), (2.33) and (5.24) we deduce
\[
0 \leq \sup_k J_k^{(2)} \leq \sup_k \nu_k(B_1 \setminus B_{1-2\varepsilon}) \leq C \varepsilon 2\varepsilon.
\]

Instead, due to (ii), the function \( w^\varepsilon|_{B_1^\varepsilon \cap G_{3\varepsilon}} = 0 \) and from definitions of \( I_{2\varepsilon} \) and \( h \) we have \( R_a(h)|_{B_{1-\varepsilon} \setminus I_{2\varepsilon}} = 0 \). From this we infer
\[
0 \leq J_k^{(1)} \leq -4\theta_{jk} \left( \int_{B_{1-\varepsilon} \setminus I_{2\varepsilon}} w^\varepsilon R_a(h) \, d\mathcal{H}^{n-1} + \int_{B_{1-\varepsilon} \setminus I_{2\varepsilon}} v^\delta R_a(h) \, d\mathcal{H}^{n-1} \right) = 0.
\]

Putting (5.28) and (5.29) together yields
\[
\limsup_{k \to \infty} J_k^\varepsilon \leq C \varepsilon.
\]
In order to estimate the functional \( I_k^c \) we observe that
\[
I_k^c \leq \int_{B_{1-2\varepsilon}} |\nabla (\phi_c w^\varepsilon + (1 - \phi_c) v^\varepsilon)|^2 \, d\mu_a + c \int_{B_{1-\varepsilon} \setminus B_{1-2\varepsilon}} |\nabla (\phi_c w^\varepsilon + (1 - \phi_c) v^\varepsilon)|^2 \, d\mu_a
\]
\[
+ \frac{c}{\varepsilon^2} \int_{B_{1-\varepsilon} \setminus B_{1-2\varepsilon}} (|\phi_c w^\varepsilon + (1 - \phi_c) v^\varepsilon - \mu|)^2 \, d\mu_a = I_k^{(1)} + I_k^{(2)} + I_k^{(3)} + I_k^{(4)}.
\]
We estimate the addenda separately. From condition (5.24), we can infer
\[
\sup_k I_k^{(3)} \leq \sup_k \mu_k (B_1 \setminus B_{1-2\varepsilon}) C \varepsilon.
\] (5.31)

We now estimate the first term; recall that \( \phi | \mathcal{Z}_{\delta} = 0 \)
\[
I_k^{(1)} = \int_{B_{1-2\varepsilon} \setminus \mathcal{I}_{3\varepsilon}} |\nabla v^\varepsilon|^2 \, d\mu_a + \int_{B_{1-2\varepsilon} \cap \mathcal{I}_{3\varepsilon}} |\nabla (\phi_c (w^\varepsilon - v^\varepsilon)) \nabla v^\varepsilon|^2 \, d\mu_a
\]
\[
\leq \int_{B_{1-2\varepsilon} \setminus \mathcal{I}_{3\varepsilon}} |\nabla v^\varepsilon|^2 \, d\mu_a + c \int_{B_{1-2\varepsilon} \cap \mathcal{I}_{3\varepsilon}} |\nabla v^\varepsilon|^2 \, d\mu_a
\]
\[
+ \frac{c}{\varepsilon^2} \int_{B_{1-2\varepsilon} \cap \mathcal{I}_{3\varepsilon}} |v^\varepsilon - w^\varepsilon|^2 \, d\mu_a
\]
\[
\leq \int_{B_{1-2\varepsilon} \setminus \mathcal{I}_{3\varepsilon}} |\nabla v^\varepsilon|^2 \, d\mu_a + c \int_{B_{1-2\varepsilon} \cap \mathcal{I}_{3\varepsilon}} |\nabla (v^\varepsilon - w^\varepsilon)|^2 \, d\mu_a
\]
\[
+ \frac{c}{\varepsilon^2} \int_{B_{1-2\varepsilon} \cap \mathcal{I}_{3\varepsilon}} (|v^\varepsilon - w^\varepsilon|^2 + |w - w^\varepsilon|^2) \, d\mu_a.
\] (5.32)

Taking the last addendum above into account, we notice that for all \( \varphi \) smooth functions and \( \tau > 0 \)
\[
|\varphi(x - \tau \varepsilon n_{-1}) - \varphi(x)| \leq \tau \int_0^1 |\nabla \varphi|(x - \tau \varepsilon n_{-1}) \, dt.
\]

Then, by a simple application of Fubini’s theorem we deduce
\[
\frac{c}{\varepsilon^2} \int_{B_{1-2\varepsilon} \cap \mathcal{I}_{3\varepsilon}} |\varphi(x - \tau \varepsilon n_{-1}) - \varphi(x)|^2 \, d\mu_a \leq \frac{c^2}{\varepsilon^2} \int_{(B_{1-2\varepsilon} \cap \mathcal{I}_{3\varepsilon}) + [0, \tau] \varepsilon n_{-1}} |\nabla \varphi|^2 \, d\mu_a
\]
where \((B_{1-2\varepsilon} \cap \mathcal{I}_{3\varepsilon}) + [0, \tau] \varepsilon n_{-1}\) denotes the Minkowski sum between sets. So, thanks to a density argument and for \( \tau = 3\varepsilon \) we infer
\[
\frac{c}{\varepsilon^2} \int_{G_{2\varepsilon} \cap G_{3\varepsilon}} |w - w^\varepsilon|^2 \, d\mu_a \leq c \int_{(B_{1-2\varepsilon} \cap \mathcal{I}_{3\varepsilon}) + [0, \tau] \varepsilon n_{-1}} |\nabla w|^2 \, d\mu_a.
\]

So, from (5.32), according to (5.24) the continuity of translation in \( L^2 \) and the absolute continuity of the integral, and observing that \( \mathcal{L}^d((B_{1-2\varepsilon} \cap \mathcal{I}_{3\varepsilon}) + [0, \tau] \varepsilon n_{-1}) = O(\varepsilon) \) we obtain
\[
I_k^{(1)} \leq \int_{B_{1-2\varepsilon} \setminus \mathcal{I}_{3\varepsilon}} |\nabla v^\varepsilon|^2 \, d\mu_a + O(\varepsilon).
\] (5.33)

Reasoning in the same way as in the estimate of \( I_k^{(1)} \) we obtain
\[
I_k^{(2)} \leq O(\varepsilon)
\] (5.34)

Since \( \sup \rho \in C_{2\varepsilon} \) and recalling that by condition (5.17), if we choose \( \varepsilon \) sufficiently small such that \( \rho < 1 - 5\varepsilon \), \( \sup \rho (w^\varepsilon - z_{3\varepsilon}) \subset B_{1-2\varepsilon} \), we obtain
\[
I_k^{(4)} \leq \frac{c}{\varepsilon^2} \int_{B_{1-\varepsilon} \setminus B_{1-2\varepsilon}} |\phi_c (w^\varepsilon - v^\varepsilon)|^2 \, d\mu_a + \frac{c}{\varepsilon^2} \int_{B_{1-\varepsilon} \setminus B_{1-2\varepsilon}} |v^\varepsilon - z_{jk}|^2 \, d\mu_a
\]
\[
\leq \frac{c}{\varepsilon^2} \int_{B_{1-\varepsilon} \setminus B_{1-2\varepsilon}} (|w^\varepsilon - w|^2 + |w - v^\varepsilon|^2 + |w - z_{jk}|^2) \, d\mu_a.
\]

So, proceeding as in estimate of \( I_k^{(1)} \) and recalling that \( \sup \rho (w - z_{3\varepsilon}) \subset B_\rho \) for \( \varepsilon \) sufficiently small we deduce
\[
\limsup_{k \to \infty} I_k^{(4)} \leq \limsup_{k \to \infty} \frac{c}{\varepsilon^2} \int_{B_{1-\varepsilon} \setminus B_{1-2\varepsilon}} |z_{3\varepsilon} - z_{jk}|^2 \, d\mu_a + O(\varepsilon) \leq O(\varepsilon).
\] (5.35)
Then putting together estimates in (5.31), (5.33), (5.34) and (5.35) leads to
\[
\limsup_{k \to \infty} I^*_k \leq \int_{B_{1-z}\setminus B_{1}} |\nabla v_i|^2 \, d\mu + O(\varepsilon).
\]
So, since
\[
w^{(c)}_k \xrightarrow{k \to \infty} \chi_{\varepsilon}(\phi_xv_0 + (\phi_x)w^\varepsilon) + (1 - \chi_{\varepsilon})z_{\infty} =: w^{(c)} \quad \text{ in } L^2(B_1, \mu_1)
\]
and
\[
w^{(c)} \xrightarrow{\varepsilon \to 0} w \quad \text{ in } L^2(B_1, \mu_1),
\]
we conclude by the lower semicontinuity of the $\Gamma$-lim sup
\[
\Gamma - \limsup_{k \to \infty} G_{j_k}(w) \leq \liminf_{\varepsilon \to 0} \left( \Gamma - \limsup_{k \to \infty} G_{j_k}(w^{(c)}) \right)
\]
\[
\leq \limsup_{\varepsilon \to 0} \left( \limsup_{k \to \infty} (I^*_k + J^*_k) \right) \leq \int_{B_1} |\nabla w|^2 \, d\mu,
\]
that provides the thesis.

**Proof of the $\Gamma$-convergence: case (3).**

(a) From (5.10), we immediately have
\[
\liminf_j G_j(w_j) \geq \liminf_j (1 - \kappa_j)G_j(z_j) = +\infty = G^{(3)}_\infty.
\]
(b) This is trivial, in fact $\liminf_j G_j(w_j) \leq +\infty = G^{(3)}_\infty$.

**Step 4: Improving the convergence of** $(z_j)_j \in \mathbb{N}$ **if** $\lim_j G_j(z_j) < +\infty$. Using a standard result of $\Gamma$-convergence we show that $z_j \to z_{\infty}$ in $H^1(B_1, \mu_1)$. For equi-coercivity of $G_j$ seen in (5.13), [13, Lemma 2.10] (a version of Poincaré inequality for weighted Sobolev spaces) and $\|z_j\|_{H^1(B_1, \mu_1)} = 1$ we have
\[
\|w\|_{H^1(B_1, \mu_1)} \leq C \sqrt{\hat{G}_j(w)} + 1,
\]
so every minimizing sequence converges weakly in $H^1(B_1, \mu_1)$ and thanks to Theorem 1.4 converges strongly in $L^2(B_1, \mu_1)$. Since $G_j$ is semicontinuous with respect to weak topology of $H^1(B_1, \mu_1)$ there exists $\zeta_j$ minimizer of $G_j$. Taking into account [17] Theorem 7.8, with $i = 1, 2$ there exists $\zeta_{\infty} \in H^1(B_1, \mu_1)$ such that
\[
\zeta_j \to \zeta_{\infty}, \quad \text{ in } L^2(B_1, \mu_1) \quad (5.36)
\]
\[
G_j(\zeta_j) \to G_\infty^{(i)}(\zeta_{\infty}), \quad \text{ (5.37)}
\]
\[
\zeta_{\infty} \text{ is the unique minimizer of } G_\infty^{(i)}, \quad (5.38)
\]
where due to (5.38) we have used the strict convexity of $G_\infty^{(i)}$. Therefore using the strong convergence of traces in $L^2(\partial B_1, [x_1]^n \mathcal{H}^{n-1})$ and $L^2(B_1)$, then from the estimates
\[
G_j(\zeta_j) \leq G_j(z_j) \leq \sup_j G_j(z_j) < \infty, \quad (5.39)
\]
and (5.38) we obtain
\[
\int_{B_1} |\nabla \zeta_j|^2 \, d\mu \to \int_{B_1} |\nabla \zeta_{\infty}|^2 \, d\mu,
\]
which implies $\zeta_j \to \zeta_{\infty}$ in $H^1(B_1, \mu_1)$. According to 5.10 and 5.39, $z_j$ is an almost minimizer of $G_j$ in the following sense
\[
0 \leq G_j(z_j) - G_j(\zeta_j) \leq \kappa_j G_j(z_j) \leq \kappa_j \sup_j G_j(z_j).
\]
Since $\kappa_j \downarrow 0$ and $z_j \to z_{\infty}$ in $H^1(B_1, \mu_1)$, (5.37) and Step 3 yield that
\[
G_\infty^{(i)}(z_{\infty}) \leq \liminf_j G_j(z_j) = \lim_j G_j(z_j) = G_\infty^{(i)}(\zeta_{\infty}), \quad (5.40)
\]
with $i = 1, 2$. From (5.10), we infer

$$G_j(z_j) \leq \frac{1}{1 - k_j} G_j(\xi_j);$$

from this, by (5.40) and by strong convergence of traces we obtain

$$\liminf_j \int_{B_1} |\nabla z_j|^2 \, d\mu_a = \int_{B_1} |\nabla z_\infty|^2 \, d\mu_a,$$

that with the weak convergence of in $H^1(B_1, \mu_a)$ proves the convergence

$$z_j \to z_\infty \quad \text{in } H^1(B_1, \mu_a).$$

In particular

$$\|z_\infty\|_{H^1(B_1, \mu_a)} = 1. \quad (5.41)$$

**Step 5: Case (1) cannot occur.** We recall properties of $z_\infty$:

(i) $\|z_\infty\|_{H^1(B_1, \mu_a)} = 1$;

(ii) $z_\infty$ is $(1 + s)$-homogeneous and even with respect to $\{x_n = 0\}$;

(iii) $z_\infty$ is the unique minimizer of $G_\infty^{(1)}$ with respect to its boundary data;

(iv) $z_\infty \in B_\infty^{(1)} = \{ z \in z_\infty + H^1_0(B_1, \mu_a) : (z + \theta h)|_{B_1'} \geq 0 \}$.

These properties imply that

$$w_\infty := z_\infty + \theta h$$

is the minimizer of $\int_{B_1} |\nabla \cdot|^2 \, d\mu_a$ among all functions $w \in w_\infty + H^1_0(B_1, \mu_a)$ and $w|_{B_1'} \geq 0$ in the sense of the trace. So, $w_\infty$ is the solution of the fractional obstacle problem. To prove this claim, for all $z \in B_\infty^{(1)}$ we consider $w := z + \theta h$ and, recalling (4.1), we have

$$G_\infty^{(1)}(z) = \int_{B_1} |\nabla w|^2 \, d\mu_a - \theta^2 \int_{B_1} |\nabla h|^2 \, d\mu_a - (1 + s) \int_{B_1} z_\infty^2 |x_n|^a \, dH^{n-1}$$

$$- 2\theta \int_{B_1} \nabla w \cdot \nabla h \, d\mu_a - 4\theta \int_{B_1} z \lim_{\epsilon \to 0} \left( z^a \frac{\partial h}{\partial x_n}(\tilde{x}, \epsilon) \right) \, dH^{n-1}$$

$$- \theta^2 \int_{B_1} |\nabla h|^2 \, d\mu_a - (1 + s) \int_{B_1} z_\infty^2 |x_n|^a \, dH^{n-1} - 2(1 + s) \int_{\partial B_1} z_\infty h |x_n|^a \, dH^{n-1}.$$  

Since $G_\infty^{(1)}(z) \leq G_\infty^{(1)}(z)$ for all $z \in B_\infty^{(1)}$ then

$$\int_{B_1} |\nabla w|^2 \, d\mu_a \leq \int_{B_1} |\nabla w|^2 \, d\mu_a \quad \forall w \in w_\infty + H^1_0(B_1, \mu_a).$$

Using the $(1 + s)$-homogeneity and [13 Proposition 5.5], the result of classification of global solutions, we deduce that $w_\infty = \lambda_\infty h_{\nu_\infty} \in S_{1+s}$ for some $\lambda_\infty \geq 0$ and $\nu_\infty \in S^n$.

Thanks to (5.5) we have the contradiction: from $z_j \to z_\infty$ in $H^1(B_1, \mu_a)$ and (5.8) we have

$$\frac{c_j}{\delta_j} = \theta_j h + z_j \to \theta h + z_\infty \in S_{1+s} \quad \text{in } H^1(B_1, \mu_a),$$

so for $j >> 1$

$$\text{dist}_{H^1(B_1, \mu_a)}(c_j, S_{1+s}) \leq \|c_j - \delta_j \lambda_\infty h_{\nu_\infty}\|_{H^1(B_1, \mu_a)} \left( \frac{c_j}{\delta_j} \right) < \delta_j = \text{dist}_{H^1(B_1, \mu_a)}(c_j, S_{1+s})$$

where we have used that $\delta_j \lambda_\infty h_{\nu_\infty} \in S_{1+s}$.

**Step 6: Case (3) cannot occur.** To prove that case (3) cannot occur, we conveniently scale the energies so as to get a nontrivial $\Gamma$-limit for the rescaled functionals ultimately leading to a contradiction.

By means (5.14), since $\lim_{j} G_j(z_j) = +\infty$, we have

$$\gamma_j := -4\theta_j \int_{B_1'} z_j R_a(h) \, dH^{n-1} \uparrow +\infty.$$  

(5.43)
Moreover \( z_j \to z_\infty \) in \( L^2(B'_1) \) and (5.12) give us
\[
\lim_j \frac{\gamma_j}{\theta_j} = -4 \lim_j \int_{B'_1} z_j R_\alpha(h) \, dH^{n-1} = -4 \int_{B'_1} z_\infty R_\alpha(h) \, dH^{n-1} \in [0, \infty)
\]
so
\[
\theta_j \gamma_j^{-1/2} \uparrow +\infty.
\] (5.44)
Then we rescale the functional \( \mathcal{G}_j \) dividing by \( \gamma_j \). For all \( z \in B_j \) we consider \( \gamma_j^{-1} \mathcal{G}_j(z) \) and we note that
\[
\gamma_j^{-1} \mathcal{G}_j(z) = \tilde{\mathcal{G}}_j(\gamma_j^{-1/2} z)
\] (5.45)
with
\[
\tilde{\mathcal{G}}_j(w) = \begin{cases} \int_{B_1} |\nabla w|^2 \, d\mu_a - (1 + s) \int_{\partial B_1} w^2 |x_n|^a \, dH^{n-1} - 4 \frac{\theta_j}{\gamma_j} \int_{B'_1} w R_\alpha(h) \, dH^{n-1} & w \in \tilde{B}_j \\ +\infty & \text{otherwise,} \end{cases}
\]
where \( \tilde{B}_j := \{ w \in \gamma_j^{-1/2} z_j + H'_0(B_1, \mu_a) : (w + \theta_j \gamma_j^{-1/2} h)|_{B'_1} \geq 0 \} \).
Setting \( \tilde{z}_j := \gamma_j^{-1/2} z_j \), due to (5.11) and \( \gamma_j \uparrow +\infty \), we have \( \tilde{z}_j \to 0 \) in \( H^1(B_1, \mu_a) \). Moreover the condition (5.45) and the definition of \( \gamma_j \) (5.33) yield
\[
\gamma_j \mathcal{G}_j(\tilde{z}_j) = \frac{\int_{B_1} |\nabla \tilde{z}|^2 \, d\mu_a - (1 + s) \int_{\partial B_1} \tilde{z}^2 |x_n|^a \, dH^{n-1}}{\gamma_j} + 1 = 1 + O(\gamma_j^{-1}).
\] (5.46)
Thanks to (5.46) we can rewrite the inequalities (5.10) as
\[
(1 - \kappa_j) \mathcal{G}_j(\tilde{z}) \leq \mathcal{G}_j(\tilde{z}) \quad \forall \tilde{z} \in \tilde{B}_j.
\]
In particular, by taking into consideration (5.44), \( \tilde{z}_j \to 0 \) in \( H^1(B_1, \mu_a) \), and (5.46) (in other words \( \lim_j \mathcal{G}_j(\tilde{z}_j) < \infty \)) we proceed as in case (2) of Step 3 establishing that
\[
\Gamma(L^2(B_1, \mu_a)) \cap \mathcal{G}_j(\tilde{z}) = \tilde{G}_\infty,
\]
where
\[
\tilde{G}_\infty = \{ \tilde{z} \in H'_0(B_1, \mu_a) : \tilde{z}|_{B'_1} = 0 \}.
\]
From Step 4 and the convergence \( \tilde{z}_j \to 0 \) in \( H^1(B_1, \mu_a) \), the zero function turns out to be the unique minimizer of \( \tilde{G}_\infty \) and \( \lim_j \mathcal{G}_j(\tilde{z}_j) \to \tilde{G}_\infty(0) = 0 \); this is in contradiction with (5.46).

To prove the theorem we have only to exclude case (2) of Step 3. In what follows, we suppose the hypothesis of case (2) of Step 3: \( \theta = +\infty \) and \( \lim_j \mathcal{G}_j(z_j) < +\infty \). In the following steps we exhibit further properties of the limit \( z_\infty \).

**Step 7: An orthogonality condition.** By evaluating that \( \psi_j \) is a point of minimal distance between \( c_j \) and \( B_1^{1+s} \), we prove that \( z_\infty \) is orthogonal to the tangent space \( T_{h}B_{1+s} \):

From the hypothesis \( \theta = +\infty \) we deduce that \( \lambda_j > 0 \) for \( j > 1 \). Therefore, by the condition of minimal distance (5.46), we deduce that for all \( \nu \in \mathbb{S}^{n-2} \) and \( \lambda \geq 0 \),
\[
\|c_j - \psi_j\|_{H^1(B_1, \mu_a)} \leq \|c_j - \lambda h_\nu\|_{H^1(B_1, \mu_a)},
\]
and thanks to definition of \( z_j \) in (5.8) it holds
\[
\delta_j \|z_j\|_{H^1(B_1, \mu_a)} \leq \|\psi_j - \lambda h_\nu + \delta z_j\|_{H^1(B_1, \mu_a)}
\]
or in the same way
\[
-\|\psi_j - \lambda h_\nu\|_{H^1(B_1, \mu_a)}^2 \leq 2\delta_j (\|z_j\|_{H^1(B_1, \mu_a)} - \|\psi_j - \lambda h_\nu\|_{H^1(B_1, \mu_a)})
\] (5.47)
Now we suppose \( (\lambda, \nu) \neq (\lambda_j, c_{n-1}) \) and renormalizing (5.47) we obtain
\[
-\|\psi_j - \lambda h_\nu\|_{H^1(B_1, \mu_a)} \leq 2\delta_j (\|z_j\|_{H^1(B_1, \mu_a)} - \|\psi_j - \lambda h_\nu\|_{H^1(B_1, \mu_a)})
\]
and by passing to the limit \((\lambda, \nu) \to (\lambda_j, \epsilon_{n-1})\), reminding the definition of tangent space \(T\hat{\gamma}_{1+s}\) in (4.4), we deduce

\[
\langle z_j, \zeta \rangle \geq 0 \quad \zeta \in T_{\hat{\gamma}_j}\hat{\gamma}_{1+s} = T_0\hat{\gamma}_{1+s},
\]

where we used \(\lambda_j > 0\) in the computation of the tangent vector. By choosing the sequence \((\lambda, \nu) \to (\lambda_j, \epsilon_{n-1})\) such that \(\lim_{\|y_j - \nu \|_{T(\beta(B_1, \nu_1))}} = -\zeta\) we obtain \(\langle z_j, \zeta \rangle \leq 0\) thus

\[
\langle z_j, \zeta \rangle = 0 \quad \zeta \in T_0\hat{\gamma}_{1+s}.
\]

So, taking the limit \(j \to +\infty\) we conclude

\[
\langle z_\infty, \zeta \rangle = 0 \quad \zeta \in T_0\hat{\gamma}_{1+s}.
\]

**Step 8: Identification of \(z_\infty\) in case (2).** There exist real constants \(a_0, \ldots, a_{n-2}\) such that

\[
z_\infty = a_0h + \left(\sum_{i=1}^{n-2} a_i x_i\right) \left(\frac{2}{n-1} \frac{x_i^2 + x_n^2 + x_n}{x_n}\right)
\]

or rather \(z_\infty \in T_0\hat{\gamma}_{1+s}\).

In view of homogeneity and regularity of \(z_\infty\) (that is solution of a partial differential equation), fixed \(x_{n-1}\) and \(x_n\), we can write the first order Taylor polynomial of \(z_\infty(\cdot, x_{n-1}, x_n)\) in \((\hat{\gamma}_j, x_{n-1}, x_n)\).

To a bidimensional argument we achieve the structure of \(z_\infty\) stated in (A.2). For its proof we refer to [30, Lemma A.3] (for the reader’s convenience we report a proof in Appendix).

**Step 9: Case (2) cannot occur.** We use results of Step 4, 7 and 8 to deduce the contradiction. From (A.2) we deduce that \(z_\infty \in T_0\hat{\gamma}_{1+s}\), by using it as a test function in (5.50), the condition of orthogonality of Step 7 implies

\[
\langle z_\infty, \zeta \rangle = 0 \quad \zeta \in T_0\hat{\gamma}_{1+s}.
\]

Then we have \(z_\infty = 0\) but this is in contradiction with (5.41).

In this way we exclude the occurrence of case (2) of Step 3, thus providing the conclusion of the proof of the theorem.

In what follows we show some important consequences of epiperimetric inequality.

### 5.2 Decay of the boundary adjusted energy

The following proposition establishes a decay estimate for the boundary adjusted energy. In this connection the epiperimetric inequality allows us to estimate from below, up to a constant, the difference between the energy \(W_{1+s}(1, \cdot)\) evaluated respectively in the \((1 + s)\)-homogeneous extension of \(u_{|\partial B_1}\) and in \(u_\ast\) with \(W^\ast_{1+s}(1, u_\ast)\); in this way we obtain a differential inequality from which we deduce the decay estimate.

**Proposition 5.2** (Decay of the boundary adjusted energy). Let \(x_0 \in \Gamma_{1+s}(u)\). There exists a constant \(\gamma > 0\) for which the following property holds:

for every compact set \(K \subset B_1^r\) there exists a positive constant \(C > 0\) such that

\[
W_{1+s}^x(r, u) \leq C r^\gamma,
\]

for all radii \(0 < r < \text{dist}(K, \partial B_1)\) and for all \(x_0 \in \Gamma_{1+s}(u) \cap K\).

**Proof.** Let us assume \(x_0 = 0 \in \Gamma_{1+s}(u)\). Thanks to Lemma 2.4 we calculate the derivative of the boundary adjusted energy \(W_{1+s}^x(\cdot, u)\)

\[
\frac{d}{dr}W_{1+s}^x(r, u) = \frac{n+1}{r^{n+2}} D_a(r) + 1 \frac{\partial}{\partial n} D_a(r) - \frac{1}{r^{n+2}} H_a(r) + \frac{1 + s}{r^{n+3}} H_a(r)
\]

\[
= \frac{n+1}{r^{n+2}} D_a(r) + 1 \frac{\partial}{\partial n} D_a(r) - \frac{(1 + s)(n - 2s)}{r^{n+3}} H_a(r) + \frac{2(1 + s)}{r^{n+3}} H_a(r)
\]

\[
= \frac{n+1}{r} W_{1+s}(r, u) - \frac{(1 + s)(n + 1)}{r^{n+3}} H_a(r) + \frac{1}{r^{n+3}} D_a(r) - \frac{2(1 + s)}{r^{n+2}} D_a(r) + \frac{2(1 + s)^2}{r^{n+3}} H_a(r)
\]

\[
= \frac{n+1}{r} W_{1+s}(r, u) - \frac{(1 + s)(n + 1)}{r^{n+3}} H_a(r) + I.
\]
According to Lemma 2.1 and to the definition of rescaled functions (3.1), we can write
\[
I = \frac{1}{r^{n+2}} \int_{\partial B_r} |\nabla u|^2 |x_n|^a d\mathcal{H}^{n-1} + \frac{2(1+s)^2}{r^{n+3}} \int_{\partial B_r} u^2 |x_n|^a d\mathcal{H}^{n-1} - \frac{2(1+s)}{r^{n+2}} \int_{\partial B_r} u \nabla u \cdot \frac{x}{r} |x_n|^a d\mathcal{H}^{n-1}
\]
\[
= \frac{1}{r} \int_{\partial B_1} (|\nabla u_r|^2 + 2(1+s)^2 u_r^2 - 2(1+s)u_r \nabla u_r \cdot y) |y_n|^a d\mathcal{H}^{n-1} \]
\[
= \frac{1}{r} \int_{\partial B_1} \left( (\nabla u_r \cdot \nu - (1+s)u_r)^2 + |\nabla \theta u_r|^2 + (1+s)^2 u_r^2 \right) |y_n|^a d\mathcal{H}^{n-1}
\]
(5.54)

where by \(\nabla \theta u\) we denote the differential of \(u\) in the tangent direction to \(\partial B_1\). Let \(c_r\) be the \((1+s)\)-homogeneous extension of \(u_r|_{\partial B_1}\)
\[
c_r(x) := |x|^{1+s} \min u \left( \frac{x}{|x|} \right).
\]

Thus, according to \((1+s)\)-homogeneity and by Euler’s homogeneous function Theorem and recalling that \(H_a(\cdot) = r^{n+1}H_a(\cdot)\) and \(W^a_{\Gamma_1+s}(1,u_r) = W^a_{\Gamma_1+s}(r,u)\), by putting together the equations (5.53) and (5.54) we deduce
\[
d\frac{dr}{dr} W^a_{\Gamma_1+s}(r,u) = - \frac{n+1}{r} W^a_{\Gamma_1+s}(r,u) - \frac{n(1+s)}{r} \int_{\partial B_1} u_r^2 |x_n|^a d\mathcal{H}^{n-1}
\]
\[
+ \frac{1}{r} \int_{\partial B_1} (\nabla u_r \cdot \nu - (1+s)u_r)^2 |x_n|^a d\mathcal{H}^{n-1} + \frac{1}{r} \int_{\partial B_1} (|\nabla \theta u_r|^2 + (1+s)^2 u_r^2) |x_n|^a d\mathcal{H}^{n-1}
\]
\[
= - \frac{n+1}{r} W^a_{\Gamma_1+s}(r,u) + \frac{1}{r} \int_{\partial B_1} (\nabla u_r \cdot \nu - (1+s)u_r)^2 |x_n|^a d\mathcal{H}^{n-1}
\]
\[
+ \frac{1}{r} \int_{\partial B_1} (|\nabla \theta c_r|^2 - (1+s)(n-s)c_r^2) |x_n|^a d\mathcal{H}^{n-1}
\]
\[
= - \frac{n+1}{r} W^a_{\Gamma_1+s}(r,u) + \frac{1}{r} \int_{\partial B_1} (\nabla u_r \cdot \nu - (1+s)u_r)^2 |x_n|^a d\mathcal{H}^{n-1}
\]
\[
+ \frac{1}{r} \int_{\partial B_1} (|\nabla c_r|^2 - (1+s)(n+1)c_r^2) |x_n|^a d\mathcal{H}^{n-1}
\]
\[
= \frac{n+1}{r} W^a_{\Gamma_1+s}(1,c_r) - \frac{n+1}{r} W^a_{\Gamma_1+s}(1,u_r) + \frac{1}{r} \int_{\partial B_1} (\nabla u_r \cdot \nu - (1+s)u_r)^2 |x_n|^a d\mathcal{H}^{n-1}.
\]

So, by Proposition 6.3 we have
\[
d\frac{dr}{dr} W^a_{\Gamma_1+s}(r,u) = 2 \frac{n+1}{r} \left( W^a_{\Gamma_1+s}(1,c_r) - W^a_{\Gamma_1+s}(1,u_r) \right).
\]

Then, according to the epiperimetric inequality proved in Theorem 5.1 and recalling that \(u_r\) minimize \(W^a_{\Gamma_1+s}(1,\cdot)\) we obtain
\[
d\frac{dr}{dr} W^a_{\Gamma_1+s}(r,u) \geq 2 \frac{(n+1)\kappa}{r(1-\kappa)} W^a_{\Gamma_1+s}(1,u_r) = 2 \frac{(n+1)\kappa}{r(1-\kappa)} W^a_{\Gamma_1+s}(r,u),
\]
and integrating this inequality in \((0,r_0)\) we have
\[
W^a_{\Gamma_1+s}(r,u) \leq W^a_{\Gamma_1+s}(1,u) r^\gamma,
\]
with \(\gamma := \frac{2(n+1)\kappa}{r(1-\kappa)}\).

**Remark 5.3.** In order to prove the Proposition 5.2 the Weiss’ monotonicity formula is not necessary.

### 5.3 Nondegeneracy of the solution

In order to deduce the nondegeneracy property of the solution we note that the inequality (2.5) is not enough. We state an improved version of (2.5); this is a consequence of epiperimetric inequality and decay estimate of energy above.

**Proposition 5.4 (Nondegeneracy).** Let \(u \in H^1(B_1,\mu_a)\) be a solution of the Problem (0.3). Let us assume that \(0 \in \Gamma_{1+s}(u)\). Then there exists a constant \(H_0 > 0\) for which
\[
H_a(r) \geq H_0 r^{n+2} \qquad \forall 0 < r < 1.
\]
(5.55)
Proof. For the proof of this result we refer to [25 Proposition 4.6].

By means of the nondegeneracy condition (5.5), for all $x_0 \in \Gamma_{1+s}(u)$, we deduce

$$\int_{\partial B_1} u^2_{x_0,r} |x_n|^a \, dx \geq H_0,$$

and if $(u_{x_0,r^k})_{k \in \mathbb{N}}$ is a sequence that converges to $u_0$ in $L^2(B_{1},\mu_a)$, a blow-up function in $x_0$, due to estimate (3.2) and the convergence of the traces in Theorem 1.6 we obtain the convergence of the traces of $u_{x_0,r^k}$ on $\partial B_1$; thus

$$\int_{\partial B_1} u_0 |x_n|^a \, dx \geq H_0 > 0.$$

So we infer $u_0 \neq 0$ for all $u_0$ blow-up functions in a point of $\Gamma_{1+s}(u)$.

So, in view of Propositions 5.2, 5.4 and [13 Proposition 5.5] we can deduce the following result of the classification of blow-ups.

**Proposition 5.5** (Classification of blow-ups). Let $u$ be a solution of the Problem (0.3). Let $u_0$ be a blow-up of $u$ in point $x_0 \in \Gamma_{1+s}(u)$. Then there exist a constant $\lambda > 0$ and a vector $e \in \mathbb{S}^{n-2}$ such that $u_0 = \lambda h_e$.

### 5.4 The blow-up method: Uniqueness of blow-ups

By summarizing what we have been showing so far, due to estimate (3.2) and to Theorem 1.3 for all $x_0 \in \Gamma_{1+s}(u)$ and for all sequences $r_k \to 0$ there exists at least a subsequence (that we do not relabel in what follows) such that $u_{x_0,r_k} \to u_{x_0}$ in $H^1(B_{1},\mu_a)$ for some nontrivial functions $u_{x_0} \in H^1(B_{1},\mu_a)$. It is easy to prove that $u_{x_0}$ is a solution of Problem (0.3). Furthermore $u_{x_0}$ is $(1+s)$-homogeneous. According to Proposition 5.5 the result of the classification of blow-ups, we obtain $u_{x_0} \in B_{1+s}$.

With the next Proposition we prove that the blow-up is unique, i.e. for all $x_0 \in \Gamma_{1+s}(u)$ there exists a function $u_{x_0}$ such that for all $r_k \to 0$ the sequence $(u_{x_0,r_k})_{k \in \mathbb{N}}$ converges to $u_{x_0}$ in $L^2(B_{1},\mu_a)$. This is again a consequence of epiperimetric inequality. In particular, the epiperimetric inequality provides an explicit rate of convergence of the rescaled function $u_{x_0,r}$.

**Proposition 5.6** ([25 Proposition 4.8]). Let $u$ be a solution of the Problem (0.3) and let $K \subset \subset B_{1}'$. Then there exists a positive constant $C > 0$ such that for all $x_0 \in \Gamma_{1+s}(u) \cap K$ the following inequality holds:

$$\int_{\partial B_{1}} |u_{x_0,r} - u_{x_0}| |x_n|^a \, d\mathcal{H}^{n-1} \leq C r^\gamma,$$

where $\gamma > 0$ is the constant defined in Proposition 5.2. In particular the blow-up is unique.

### 6 The regularity of the free-boundary

Thanks to the uniqueness of blow-ups following the proof of [25 Proposition 4.10] it is possible to give a proof of the $C^{1,\alpha}$ regularity of $\Gamma_{1+s}(u)$ the subset of the free-boundary with lower frequency.

**Theorem 6.1.** Let $u \in H^1(B_{1},\mu_a)$ be a solution of the Problem (0.3). Then, there exists a constant $\alpha > 0$ such that for all $x_0 \in \Gamma_{1+s}(u)$ there exists a radius $r = r(x_0)$ for which $\Gamma_{1+s}(u) \cap B_{r'(x_0)}$ is a $C^{1,\alpha}$ regular $(n-2)$-submanifold in $B_{r}'$.

### Appendix

In this Appendix we report a result of structure of a $(1+s)$-homogeneous solution of [A.1] due to Garofalo, Petrosyan, Pop and Smit Vega Garcia [30, Lemma A.3]. Recently Focardi and Spadaro in [26] Proposition A.3] extended this result analysing the structure of $\lambda$-homogeneous solutions of [A.1] (also with different contact set) with $\lambda \in \{m, m + s, m + 2s : m \in \mathbb{N}^+, \lambda \geq 1 + s\}$.

**Lemma A.1.** Let is $z_{\infty}$ a $(1+s)$-homogeneous solution of

$$\begin{cases}
L_\alpha z_{\infty} = 0 & B_1 \setminus B_{r'}' \\
z_{\infty} = 0 & B_{r'}',
\end{cases}
$$

(A.1)
even symmetric w.r.to \( \{x_n = 0\} \). Then there exist real constants \( a_0, \ldots, a_{n-2} \) such that
\[
z_\infty = a_0 h + \left( \sum_{i=1}^{n-2} a_i x_i \right) \left( \sqrt{\frac{x_{n-1}^2 + x_n^2}{x_{n-1}^2 + x_n^2}} \right)^s ,
\]
(A.2)
or rather \( z_\infty \in T_h S_{1+s} \).

**Proof.** For all multi-indices \( \alpha \in \mathbb{N}^{n-2} \) the derivative \( \partial_\alpha z_\infty \) is the solution of
\[
\begin{cases}
L_\alpha \partial_\alpha z_\infty = 0 & B_1 \setminus B_1^{-}\cr
\partial_\alpha z_\infty = 0 & B_1^{-},
\end{cases}
\]
(A.3)
According to [19] Lemma 2.4.1 and [13] Proposition 2.3 the derivative \( \partial_\alpha z_\infty \) are bounded in \( B_{1/2} \), thanks to [19] Theorems 2.3.12 and 2.4.6 they are also continuous in \( B_{1/2} \setminus \{ x_{n-1} = x_n = 0 \} \). We consider the second derivative \( \partial_{ij} z_\infty \) with \( i, j = 1, \ldots, n-2 \); since \( z_\infty \) is \((1 + s)\)-homogeneous, the function \( \partial_{ij} z_\infty \) is \((s - 1)\)-homogeneous; as \( 0 < s < 1 \) from the boundedness of the derivative we deduce
\[
\partial_{ij} z_\infty = 0 \quad \text{in } B_1 \quad \forall i, j = 1, \ldots, n-2.
\]
(A.4)
The solution \( z_\infty \) is a smooth function in \( B_{1/2}^+ \) and \( B_{1/2}^- \) because the coefficients of the strictly elliptic operator \( L_\alpha \) are smooth in these domains. Thus, fixed \( x_{n-1} \) and \( x_n \), we can write the first order Taylor polynomial of \( z_\infty (\cdot, x_{n-1}, x_n) \) in \((\mathbf{0}, x_{n-1}, x_n)\)
\[
z_\infty (x', x_{n-1}, x_n) = c_0(x_{n-1}, x_n) + \sum_{i=1}^{n-2} c_i(x_{n-1}, x_n) x_i ,
\]
with \( c_0(x_{n-1}, x_n) = z_\infty (\mathbf{0}', x_{n-1}, x_n) \) and \( c_i(x_{n-1}, x_n) = \partial_i z_\infty (\mathbf{0}', x_{n-1}, x_n) \). By definition the function \( c_0(x_{n-1}, x_n) \) is \((1 + s)\)-homogeneous and the functions \( c_i(x_{n-1}, x_n) \) are \( s\)-homogeneous. Since \( z_\infty \) and \( \partial_i z_\infty \) are continuous in \( B_{1/2} \setminus \{ x_{n-1} = x_n = 0 \} \) the function \( c_0(x_{n-1}, x_n) \) and \( c_i(x_{n-1}, x_n) \) are continuous in \( B_{1/2} \setminus \{ x_{n-1} = 0 \} \) with \( B_{1/2} \) := \( \{ (x_{n-1}, x_n) \in \mathbb{R}^2 : x_{n-1}^2 + x_n^2 < 1/4 \} \). Thanks to homogeneity with positive degree \( c_0(x_{n-1}, x_n) \) and \( c_i(x_{n-1}, x_n) \) are continuous in \( B_{1/2} \).

Taking into account (A.4), for all \( i = 1, \ldots, n-2 \) we obtain
\[
c_i(x_{n-1}, x_n) = \partial_i z_\infty (x', x_{n-1}, x_n),
c_0(x_{n-1}, x_n) = z_\infty (x', x_{n-1}, x_n),
\]
thus \( c_i, c_0 \in H^1(B_{1/2}^+, |x_n|^a L^2) \) and are solutions of (A.3) on \( B_{1/2}^\pm \). Since \( c_i(x_{n-1}, x_n) \) is \( s\)-homogeneous there exist some constants \( (\tilde{a}_i)_i=1,\ldots,n-2 \) such that \( c_i(x_{n-1}, 0) = \tilde{a}_i x_{n-1}^s \) when \( x_{n-1} > 0 \) and similarly since \( c_0(x_{n-1}, x_n) \) is \((1 + s)\)-homogeneous, there exists a constant \( a_0 \) such that \( c_0(x_{n-1}, 0) = a_0 x_{n-1}^s \) when \( x_{n-1} > 0 \).

We show that
\[
c_i(x_{n-1}, x_n) = \frac{\tilde{a}_i}{2^\alpha} \left( x_{n+1} + \sqrt{x_{n-1}^2 + x_n^2} \right)^s .
\]
(A.5)
Passing to polar coordinates we can write \( c_i(x_{n-1}, x_n) = d_i(r, \theta) = r^s \varphi_i(\theta) \). From \( L_\alpha c_i = 0 \) we deduce that the function \( \varphi_i \) is the solution of the following second order ordinary differential equation
\[
\begin{cases}
\sin \theta \varphi_{\theta \theta} + a \cos \theta \varphi_{\theta} + (a(1 + s)x + (1 + s)^2) \sin \theta \varphi = 0 & \text{in } (0, \pi) \cr
\varphi(0) = \frac{\tilde{a}_i}{2^\alpha} \cr
\varphi(\pi) = 0,
\end{cases}
\]
and so it has a unique solution. Resorting to a direct calculation, we can verify that the function
\[
\varphi_i(\theta) = \frac{\tilde{a}_i}{2^\alpha} (\cos \theta + 1)^s
\]
is solution for all \( \theta \in [0, \pi] \). So the function \( c_i(x_{n-1}, x_n) \) satisfies (A.5).

By proceeding in the same way we prove that the function \( c_0(x_{n-1}, x_n) \) can be written as
\[
c_0(x_{n-1}, x_n) = \frac{\tilde{a}_0}{2^\alpha (s-1)} \left( x_{n+1} + \sqrt{x_{n-1}^2 + x_n^2} \right)^s \left( x_{n+1} - \sqrt{x_{n-1}^2 + x_n^2} \right),
\]
and this provides the conclusion to the proof of the step.
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