Sum of squares bounds for the total ordering principle

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Abstract

In this paper, we analyze the sum of squares hierarchy (SOS) on the total ordering principle on \( n \) elements. We show that degree \( \tilde{O}(\sqrt{n}) \) SOS can prove the total ordering principle so in this setting SOS is considerably more powerful than resolution, polynomial calculus, and the Sherali-Adams hierarchy. We also show superconstant degree SOS lower bounds which we believe can be improved to degree \( \tilde{\Omega}(\sqrt{n}) \).
1 Introduction

The total ordering principle states that if we have elements \(x_1, \ldots, x_n\) which have an ordering and no two elements are equal then some element \(x_i\) must be minimal. The total ordering principle is a very interesting example in proof complexity because it has a small size resolution proof based on induction \([7]\) yet any resolution proof must have \(\Omega(n)\) width. This example can be modified slightly to reduce the width of the initial clauses and show that the width/size lower bounds of Ben-Sasson and Wigderson \([2]\) (which were first shown for polynomial calculus by Impagliazzo, Pudlák, and Sgall \([4]\)) are tight \([1]\).

\(\Omega(n)\) degree lower bounds for the total ordering principle are also known for polynomial calculus \([3]\) and for the Sherali-Adams hierarchy. However, non-trivial degree bounds for the total ordering principle for the sum of squares hierarchy (SOS) were previously unknown. In this paper, we show that degree \(\tilde{O}(\sqrt{n})\) SOS can prove the total ordering principle, so SOS is more powerful than resolution, polynomial calculus, and the Sherali-Adams hierarchy in this setting. We also show a superconstant degree SOS lower bound, i.e. for any constant \(d > 0\) there is an \(n_0 > 0\) such that for all \(n \geq n_0\), degree \(d\) SOS cannot prove the total ordering principle on \(n\) elements. We expect that this lower bound can be improved to \(\tilde{\Omega}(\sqrt{n})\) and we describe why.

2 The total ordering principle

We analyze the following system of equations corresponding to the total ordering principle. We have variables \(x_{ij}\) where we want that \(x_{ij} = 1\) if \(x_i < x_j\) and \(x_{ij} = 0\) if \(x_i > x_j\).

1. \(\forall i \neq j, x_{ij} = x_{ji}\)
2. \(\forall i < j, x_{ij} = 1 - x_{ji}\)
3. For all distinct \(i, j, k\), \(x_{ij} x_{jk} (1 - x_{ik}) = 0\) (transitivity)
4. \(\forall j, \sum_{i \neq j} x_{ij} = 1 + z_j^2\) (totality)

3 Pseudo-expectation values for the total ordering principle

To obtain pseudo-expectation values for the total ordering principle, we consider the uniform distribution \(S_n\) over all true orderings.

**Definition 3.1.** Given a polynomial \(p(x_1, \ldots, x_n)\) of degree at most \(d\), we define \(\tilde{E}[p] = \mathbb{E}_{S_n}[p]\)

**Example 3.2.** \(\forall i \neq j, \tilde{E}[x_{ij}] = \frac{1}{2}\) because there is a \(\frac{1}{2}\) chance that \(i\) comes before \(j\) in a random ordering.

**Example 3.3.** For all distinct \(i, j, k\), \(\tilde{E}[x_{ij} x_{jk}] = \frac{1}{6}\) because there is a \(\frac{1}{6}\) chance that \(i < j < k\) in a random ordering.

So far, these are actual expectation values over a distribution of solutions. However, we have to define \(\tilde{E}[p]\) for monomials involving the \(z\) variables. We can do this as follows.
Proposition 3.4. Any monomial $p$ is equal to $\left( \prod_{j \in A} z_j \right) f(x_1, \ldots, x_n)$ for some polynomial $f$ and $A \subseteq [1, n]$.

Proof. Observe that whenever we have a factor of $z_j^2$, we can replace it by $\sum_{i \neq j} x_{ij} - 1$. \qed

Definition 3.5. We set $\tilde{E}\left[ \left( \prod_{j \in A} z_j \right) f(x_1, \ldots, x_n) \right] = 0$ whenever $A$ is non-empty because each $z_j$ could be either positive or negative. When $A$ is empty we set $\tilde{E}[f(x_1, \ldots, x_n)] = E_{S_n}[f]$. 

To analyze these pseudo-expectation values, it is convenient to create a new variable $w_j$ which is equal to $z_j^2$.

Definition 3.6. Define $w_j = \sum_{i \neq j} x_{ij} - 1$.

Remark 3.7. Viewing everything in terms of the variables $\{x_{ij}\}$ and $\{w_j\}$, we are taking the actual expected values over a distribution of solutions. However, each $w_j$ is supposed to be a square but this is not actually the case for this distribution. This is the one way in which $\tilde{E}$ can fail to give valid pseudo-expectation values.

Lemma 3.8. If $\tilde{E}\left[ \left( \prod_{j \in A} w_j \right) g^2 \right] \geq 0$ whenever $A \subseteq [1, n]$, $g \in \mathbb{R}[x_1, \ldots, x_n]$, and $|A| + 2\deg(g) \leq d$ then $\tilde{E}$ gives pseudo-expectation values.

Proof. Given a polynomial $g$, decompose $g$ as

$$g = \sum_{A \subseteq [1, n]} \left( \prod_{j \in A} z_j \right) g_A(x_1, \ldots, x_n)$$

Now observe that

$$\tilde{E}[g^2] = \tilde{E} \left[ \sum_{A, A' \subseteq [1, n]} \left( \prod_{j \in A} z_j \prod_{j \in A'} z_j \right) g_A(x_1, \ldots, x_n) g_{A'}(x_1, \ldots, x_n) \right]$$

$$= \sum_{A \subseteq [1, n]} \tilde{E} \left[ \left( \prod_{j \in A} w_j \right) g_A^2(x_1, \ldots, x_n) \right]$$ \qed

4 $\tilde{O}(\sqrt{n})$ degree upper bound

As a warmup, we consider single-variable polynomials in $w_1$. We show that these pseudo-expectation values fail at degree $\tilde{O}(\sqrt{n})$. Using this, we deduce a $\tilde{O}(\sqrt{n})$ degree upper bound.

Consider $\tilde{E}[w_1 g^2(w_1)] = E_{S_n}[w_1 g^2(w_1)]$. Observe that over the uniform distribution of orderings, $w_1$ is equally likely to be any integer in $[-1, n - 2]$. To make $\tilde{E}[w_1 g^2(w_1)]$ negative, we want $g(w_1)$ to have high magnitude at $w_1 = -1$ and small magnitude on $[1, n - 2]$. For this, we can use Chebyshev polynomials. From Wikipedia,
Definition 4.1. Recall that the mth Chebyshev polynomial can be expressed as

1. \( T_m(x) = \cos(m \cos^{-1}(x)) \) if \(|x| \leq 1\)

2. \( T_m(x) = \frac{1}{2} \left( (x + \sqrt{x^2 - 1})^m + (x - \sqrt{x^2 - 1})^m \right) \) if \(|x| \geq 1\)

Take \( g(w_1) = T_m(-1 + \frac{2w_1}{n}) \). Observe that near \( x = -1 \),
\[
\sqrt{x^2 - 1} = \sqrt{(-1 + (x + 1))^2 - 1} \approx \sqrt{-2(x + 1)}
\]
so
\[
T_m(-1 - \Delta) \approx \frac{1}{2} \left( (-1 + 2\sqrt{\Delta})^m + (-1 - 2\sqrt{\Delta})^m \right)
\]
Taking \( \Delta = \frac{2}{n} \), we see that if we take \( m = \sqrt{n \log(n)} \), \(|g(-1)| > n\) while \(|g(w_1)| \leq 1\) whenever \( w_1 \in [0, n-2] \). Thus, \( \bar{E}[w_1 g^2(w_1)] < 0\), as needed.

Intuitively, by symmetry \( \bar{E} \) are the correct pseudo-expectation values to take so if they break there should be an upper bound. We now show that this is indeed the case.

The idea behind the proof is to show that for all \( k \), the value of
\[
\sum_j \left( \sum_{i \neq j} x_{ij} \right)^k
\]
is fixed. For \( k = 1 \) we observe that
\[
\sum_j \left( \sum_{i \neq j} x_{ij} \right) = \sum_{i<j} (x_{ij} + x_{ji}) = \sum_{i<j} 1 = \binom{n}{2}
\]
For larger \( k \) we use the following lemma

Lemma 4.2. Given the ordering and transitivity axioms, for all \( k \) and all indices \( i_1, \ldots, i_k \),
\[
1 = \sum_{\sigma \in S_k} \prod_{j=1}^{k-1} x_{i_{\sigma(j)} i_{\sigma(j+1)}}
\]
and there is a degree \( k + 2 \) proof of this fact.

Proof. The idea is that the indices \( i_1, \ldots, i_k \) must have some ordering and we can determine this ordering using insertion sort. In particular, we have the following iterative algorithm. Assume that we the monomial \( \prod_{j=1}^{r-1} x_{i_{\sigma(j)} i_{\sigma(j+1)}} \) for some \( \sigma \in S_r \). We now determine where \( x_{i_{r+1}} \) should be inserted as follows. For \( j \in [1, r] \):

1. If \( j = 1 \), assume that we have the monomial \( \prod_{j=1}^{r-1} x_{i_{\sigma(j)} i_{\sigma(j+1)}} \). If \( j > 1 \), assume that we have the monomial \( \left( \prod_{j=1}^{r-1} x_{i_{\sigma(j)} i_{\sigma(j+1)}} \right) x_{i_{\sigma(j-1)} i_{r+1}} \)
2. Observe that \( 1 = x_{i_{\sigma(j)} i_{r+1}} + x_{i_{r+1} i_{\sigma(j)}} \). This splits us into two cases depending on whether \( x_{i_{r+1}} < x_{i_{\sigma(j)}} \) or \( x_{i_{r+1}} > x_{i_{\sigma(j)}} \).
3. If $x_{ir+1} < x_{i\sigma(j)}$ (i.e. we gained $x_{ir+1}$) then we have determined where $x_{ir+1}$ fits in and we are done except that we may have an extra variable. If $j > 1$ then we have both $x_{i\sigma(j-1)r+1}$ and $x_{ir+1}$ and can use transitivity to eliminate $x_{i\sigma(j-1)r+1}$.

4. If $x_{ir+1} > x_{i\sigma(j)}$ (i.e. we gained $x_{i\sigma(j)r+1}$) then we have both $x_{i\sigma(j-1)r+1}$ and $x_{ir+1}$ and can use transitivity to eliminate $x_{i\sigma(j-1)r+1}$. If $j < r$ then we move on to the next $j$. If $j = r$ then we have determined where $x_{ir+1}$ fits in and we are done.

\[\] \hfill \square

**Corollary 4.3.** Given the ordering and transitivity axioms, for all $k$ and all indices $i_1, \ldots, i_k$,

\[1 = \sum_{j=1}^{k} \left( \prod_{j' \in [1,k] \setminus \{j\}} x_{ij'j} \right)\]

and there is a degree $2k + 1$ proof of this fact.

**Proof sketch.** First use Lemma 4.2 to split into all the possible orderings of $x_{i_1}, \ldots, x_{i_k}$. Then remember which element is largest and use Lemma 4.2 in reverse to forget all other information. \hfill \square

To see how this implies there cannot be any valid pseudo-expectation values, assume that we have pseudo-expectation values $\tilde{E}$. Note that since our problem is symmetric, we can symmetrize $\tilde{E}$ so we can assume without loss of generality that $\tilde{E}$ is symmetric. Now observe that the above corollary implies that $\tilde{E}[w_1g^2(w_1)]$ is the same for all symmetric $\tilde{E}$ which satisfy the problem equations whether or not they are non-negative on squares. Since we found a symmetric $\tilde{E}$ which satisfies the problem equations such that $\tilde{E}[w_1g^2(w_1)]$ is negative, we must have that $\tilde{E}[w_1g^2(w_1)] < 0$, which is a contradiction.

## 5 Lower Bound Overview

Proving the lower bound is surprisingly subtle. We proceed as follows

1. Using symmetry reduction, it is sufficient to show that $\tilde{E}[g^2] \geq 0$ whenever $g$ is symmetric under permutations of all but $d/2$ indices. By symmetry, we can assume without loss of generality that these indices are $[1, d/2]$.

2. Observe that we can split into cases for the ordering of $x_1, \ldots, x_{d_1}$ using the equality

\[1 = \sum_{\sigma \in S_{d_1}} \prod_{j=1}^{d_1-1} x_{i\sigma(j)i\sigma(j+1)}\]

Since

\[\tilde{E}[g^2] = \sum_{\sigma \in S_{d_1}} \tilde{E} \left[ \left( \prod_{j=1}^{d_1-1} x_{i\sigma(j)i\sigma(j+1)} \right) g^2 \right]\]

it is sufficient to consider one $\sigma$ at a time. Without loss of generality, consider the case when $\sigma$ is the identity.
3. We create new variables $u_0, \ldots, u_{d_1}$ where for all $i \in [0, d_1]$, $u_i = |\{j : x_i < x_j < x_{i+1}\}|$ (where $x_0$ is automatically first and $x_{d_1+1}$ is automatically last). Note that the variables $u_0, \ldots, u_{d_1}$ are symmetric to each other. Note that $w_1 = u_0 - 1$ and for all $i \in [2, d_1]$, $w_i$ is a square (as $x_1$ is guaranteed to be first among $x_1, \ldots, x_{d_1}$).

We then note that it is sufficient to show that $\tilde{E}[(u_0 - 1)g^2] \geq 0$ whenever $g(u_0, u_1, \ldots, u_{d_1})$ has degree at most $d$.

4. $\tilde{E}$ is the expected values under the uniform distribution of

$\{u_0, \ldots, u_{d_1} : u_0, \ldots, u_{d_1} \in \mathbb{Z}, \forall i \in [0, d_1], u_i \geq 0, \sum_{i=0}^{d_1} u_i = n'\}$

To make things easier to analyze, we approximate this with a continuous distribution. In particular, instead of considering the uniform distribution over

$\{u_0, \ldots, u_{d_1} : u_0, \ldots, u_{d_1} \in \mathbb{Z}, \forall i \in [0, d_1], u_i \geq 0, \sum_{i=0}^{d_1} u_i = n'\}$

where $n' = n - d_1$, we take the uniform distribution over

$\{u_0, \ldots, u_{d_1} : u_0, \ldots, u_{d_1} \in \mathbb{R}, \forall i \in [0, d_1], u_i \geq 0, \sum_{i=0}^{d_1} u_i = n'\}$

5. We can eliminate the variables $u_1, \ldots, u_{d_1}$ with the following trick. For any polynomial $g(u_0, \ldots, u_{d_1})$,

$E_{u_1, \ldots, u_{d_1}}[g^2]$

is a polynomial in $u_0$. Moreover, this polynomial is non-negative for all $u_0$ so it must be a sum of squares. Thus, it is sufficient to analyze polynomials of the form $(u_0 - 1)g(u_0)^2$

6. We analyze polynomials of the form $(u_0 - 1)g(u_0)^2$ directly, showing that the total contribution from $u_0 \geq 1$ is at least 10 times larger in magnitude than the contribution from $u_0 = 0$.

7. We must show that we did not incur too large of an error by moving to the continuous distribution. We can show this by showing that shifting to the continuous distribution changes the contribution when $u_0 = 0$ and the contribution when $u_0 \geq 1$ by a factor of at most 2.

We make these steps more precise in the next sections.

6 Restricting to particular squares

To limit the squares which we need to check are non-negative (this is steps 1 and 2 in the plan above), we prove the following theorem
Theorem 6.1. If $\tilde{E}$ is a linear map from polynomials to $\mathbb{R}$ which is symmetric and $d_1 \geq \frac{d}{2}$ then if

$$\tilde{E}\left(\left(\prod_{i=1}^{d_1-1} x_{i(i+1)}\right) g^2\right) \geq 0$$

for all $g$ of degree at most $\frac{d}{2}$ such that $g$ is symmetric under permutations of $[1, n] \setminus [1, d_1]$, then $\tilde{E}[g^2] \geq 0$ for all $g$ of degree at most $\frac{d}{2}$.

Proof. We use the following theorem in [5] which is essentially implied by Corollary 2.6 of [6].

Theorem 6.2. If $\tilde{E}$ is a linear map from polynomials to $\mathbb{R}$ which is symmetric with respect to permutations of $[1, n]$ then for any polynomial $g$, we can write

$$\tilde{E}[g^2] = \sum_{I \subseteq [1, n], j:|I| \leq \text{indexdeg}(g)} \tilde{E}[g_{Ij}^2]$$

where for all $I, j$,

1. $g_{Ij}$ is symmetric with respect to permutations of $[1, n] \setminus I$.
2. $\text{indexdeg}(g_{Ij}) \leq \text{indexdeg}(g)$
3. $\forall i \in I, \sum_{\sigma \in S_{[1,n]\setminus(I\setminus{i})}} \sigma(g_{Ij}) = 0$

By Theorem 6.2, to verify that $\tilde{E}[g^2]$ is non-negative for all polynomials $g$ of degree at most $\frac{d}{2}$, it suffices to check polynomials $g$ which are symmetric with respect to permutations of all but $\frac{d}{2}$ indices. Without loss of generality, we can assume that these indices are $[1, \frac{d}{2}]$.

By Lemma 4.2, we have that

$$1 = \sum_{\sigma \in S_{k}} \prod_{i=1}^{k-1} x_{\sigma(i)\sigma(i+1)}$$

Thus, for any $g$, we can reexpress $\tilde{E}[g^2]$ as

$$\tilde{E}[g^2] = \sum_{\sigma \in S_k} \tilde{E}\left[\left(\prod_{i=1}^{k-1} x_{\sigma(i)\sigma(i+1)}\right) g^2\right]$$

By symmetry, if

$$\tilde{E}\left[\left(\prod_{i=1}^{d_1-1} x_{i(i+1)}\right) g^2\right] \geq 0$$

for all $g$ of degree at most $\frac{d}{2}$ such that $g$ is symmetric under permutations of $[1, n] \setminus [1, d_1]$ then $\tilde{E}[g^2] \geq 0$ for all $g$ of degree at most $\frac{d}{2}$ such that $g$ is symmetric under permutations of $[1, n] \setminus [1, d_1]$. By the discussion above, this in turn implies that $\tilde{E}[g^2] \geq 0$ for all $g$ of degree at most $\frac{d}{2}$, as needed.

Remark 6.3. Here the polynomial $g$ may involve the variables $\{z_j\}$. However, we now observe that once we the only interesting case is when we just have $z_1$.
Corollary 6.4. For our specific $\tilde{E}$, if $\tilde{E}[w_1 \left( \prod_{i=1}^{d_1-1} x_i(i+1) \right) g^2] \geq 0$ for all polynomials $g$ on the variables $\{x_{ij}\}$ such that $\deg(g) \leq \frac{d}{2}$ and $g$ is symmetric under permutations of $[1,n] \setminus [1,d_1]$ then our specific $\tilde{E}$ gives valid pseudo-expectation values.

Proof. Recall the decomposition

$$\tilde{E}[g^2] = \sum_{A \subseteq [1,n]} \tilde{E} \left[ \left( \prod_{j \in A} w_j \right) g_A^2(x_1, \ldots, x_n) \right]$$

Since are multiplying by $\left( \prod_{i=1}^{d_1-1} x_i(i+1) \right)$, any $w_j$ except $w_1$ can be reexpressed as $\sum_{i \in [2,n] \setminus \{j\}} x_{ij}^2$ because we are guaranteed that $x_1 < x_j$ (as otherwise $\left( \prod_{i=1}^{d_1-1} x_i(i+1) \right) = 0$ so the contribution is 0 anyways).

7 Changing variables

We now reexpress our polynomials in terms of more convenient variables (step 3 of our plan).

We assume that $x_1 < x_2 < \cdots < x_{d_1}$ which is enforced by multiplying our polynomials by $\prod_{i=1}^{d_1-1} x_i(i+1)$.

Definition 7.1.

1. We define $u_0 = w_1 = \sum_{i=d_1+1}^n x_{i1}$ which is the number of elements less than $x_1$
2. For $j \in [0, d_1]$, we define $u_j = \sum_{i=d_1+1}^n x_{ji} x_i(j+1)$ which is the number of elements between $x_j$ and $x_{j+1}$
3. We define $u_{d_1} = \sum_{i=d_1+1}^n x_{d_1i}$ which is the number of elements bigger than $x_{d_1}$

Lemma 7.2. If $g$ is a polynomial which is symmetric under permutations of $[1,n] \setminus [1,d_1]$ then up to the ordering and transitivity axioms we can express $\left( \prod_{i=1}^{d_1-1} x_i(i+1) \right) g$ as $\left( \prod_{i=1}^{d_1-1} x_i(i+1) \right) g'$ where $g'$ is a polynomial in $u_0, \ldots, u_{d_1}$ and $\deg(g') \leq 2 \deg(g)$

Proof sketch. As a base case, consider the expression

$$\sum_{i_1, \ldots, i_z \text{ are distinct elements of } [d_1+1,n]} x_{j_1} \left( \prod_{a=1}^{z-1} x_{i_{a+1} j_{a+1}} \right) x_{i_z j'}$$

where $j < j'$ are distinct elements in $[1,d_1]$. Let $m = \sum_{i=j}^{j'-1} u_i$. This expression is equivalent modulo the ordering and transitivity axioms to

$$\prod_{a=0}^{z} (m - a)$$
Following similar logic,
\[
\sum_{i_1, \ldots, i_z} \left( \prod_{a=1}^{z-1} x_{i_a i_{a+1}} \right) x_{i z_1} \quad \text{is equivalent modulo the ordering and transitivity axioms to } \prod_{a=0}^{z} (m - a) \text{ where } m = \sum_{i=j=0}^{d_1} u_i
\]
for \(i_1, \ldots, i_z\) are distinct elements of \([d_1+1, n]\). 

Given a sum which consists of all permutations of \([d_1 + 1, n]\) applied to a monomial \(p\), we can use the following steps to reduce this sum to the base case:

1. **Split** \(p\) into one monomial for each possible ordering of the elements involved. For example, if \(p = x_{i_1} x_{i_2}\) then we would split \(p\) into the cases \(x_{i_1} x_{i_2}\), \(x_{i_2} x_{i_1}\), and \(x_{i_1} x_{i_2}\).

2. **Sum** over all permutations of \([d_1 + 1, n]\) gives a product of base case expressions.

\[\mathbb{E}[(u_0 - 1)g^2] \geq 0\]

\section{Shifting to a continuous distribution}

The probability that \(u_0 = 0\) is at most \(\frac{4d_1}{n}\). Thus, to prove that \(\mathbb{E}[(u_0 - 1)g^2] \geq 0\) it is sufficient to show that

\[\mathbb{E}_{u_0=0}[g^2] \leq \frac{n}{4d_1} \mathbb{E}_{u_0\geq1}[(u_0 - 1)g^2]\]

To make this easier to analyze, we shift to a continuous distribution (step 4 of our plan) and show that with this continuous distribution,

\[\mathbb{E}_{u_0=0}[g^2] \leq \frac{n}{16d_1} \mathbb{E}_{u_0\geq1}[(u_0 - 1)g^2]\]

In Section \ref{sec:continuation} we argue that for sufficiently large \(n\) both \(\mathbb{E}_{u_0=0}[g^2]\) and \(\mathbb{E}_{u_0\geq1}[(u_0 - 1)g^2]\) are off by a factor of at most 2 in the continuous distribution so this implies that for the discrete distribution,

\[\mathbb{E}_{u_0=0}[g^2] \leq \frac{n}{4d_1} \mathbb{E}_{u_0\geq1}[(u_0 - 1)g^2]\]
as needed. After shifting to a continuous distribution, we show that it is sufficient to consider polynomials in just $u_0$.

Let $n' = n - d_1$. Our discrete distribution is the uniform distribution on the set

$$\{u_0, \ldots, u_{d_1} : u_0, \ldots, u_{d_1} \in \mathbb{Z}, \forall i \in [0, d_1], u_i \geq 0, \sum_{i=0}^{d_1} u_i = n'\}$$

For $u_0 = 0$ we take the uniform distribution over

$$\{u_1, \ldots, u_{d_1} : u_1, \ldots, u_{d_1} \in \mathbb{R}, \forall i \in [1, d_1], u_i \geq 0, \sum_{i=1}^{d_1} u_i = n'\}$$

For $u_0 \geq 1$ we take the uniform distribution over

$$\{u_0, \ldots, u_{d_1} : u_0, \ldots, u_{d_1} \in \mathbb{R}, u_0 \geq 1, \forall i \in [1, d_1], u_i \geq 0, \sum_{i=0}^{d_1} u_i = n'\}$$

We now use the following lemma to show that with the continuous distributions, we only need to consider polynomials $g$ which only depend on $u_0$.

**Lemma 8.1.** For all $u_0 \in [0, n']$ (where $n' = n - d_1$), for all polynomials $g(u_0, \ldots, u_{d_1})$, we can reexpress $E_{u_1, \ldots, u_{d_1}; u_1, \ldots, u_{d_1} \in \mathbb{R}, \forall i \in [1, d_1], u_i \geq 0, \sum_{i=1}^{d_1} u_i = n' - u_0}[g^2]$ as

$$E_{u_1, \ldots, u_{d_1}; u_1, \ldots, u_{d_1} \in \mathbb{R}, \forall i \in [1, d_1], u_i \geq 0, \sum_{i=1}^{d_1} u_i = n' - u_0}[g^2] = \sum_j E_{u_1, \ldots, u_{d_1}; u_1, \ldots, u_{d_1} \in \mathbb{R}, \forall i \in [1, d_1], u_i \geq 0, \sum_{i=1}^{d_1} u_i = n' - u_0}[g_j^2]$$

where each $g_j$ is a polynomial of degree at most $\deg(g)$ which only depends on $u_0$.

**Proof sketch.** We first observe that $E_{u_1, \ldots, u_{d_1}; u_1, \ldots, u_{d_1} \in \mathbb{R}, \forall i \in [1, d_1], u_i \geq 0, \sum_{i=1}^{d_1} u_i = n' - u_0}[g^2]$ is a polynomial in $u_0$ of degree at most $2\deg(g)$. To see this, note that this expression is equivalent to

$$\frac{\int_{u_1, \ldots, u_{d_1}; u_1, \ldots, u_{d_1} \in \mathbb{R}, \forall i \in [1, d_1], u_i \geq 0, \sum_{i=1}^{d_1} u_i = n' - u_0} g^2}{\int_{u_1, \ldots, u_{d_1}; u_1, \ldots, u_{d_1} \in \mathbb{R}, \forall i \in [1, d_1], u_i \geq 0, \sum_{i=1}^{d_1} u_i = n' - u_0} 1}$$

Using calculus, for any monomial $p$ in the variables $u_1, \ldots, u_{d_1}$,

$$\int_{u_1, \ldots, u_{d_1}; u_1, \ldots, u_{d_1} \in \mathbb{R}, \forall i \in [1, d_1], u_i \geq 0, \sum_{i=1}^{d_1} u_i = n' - u_0} p$$

will be a constant times $(n' - u_0)^{\deg(p)+d_1-1}$.

Since the denominator is a constant times $(n' - u_0)^{d_1-1}$, the final result will be a polynomial in $(n' - u_0)$ of degree at most $2\deg(g)$, as needed.

Finally, we observe that $E_{u_1, \ldots, u_{d_1}; u_1, \ldots, u_{d_1} \in \mathbb{R}, \forall i \in [1, d_1], u_i \geq 0, \sum_{i=1}^{d_1} u_i = n' - u_0}[g^2]$ is non-negative for all $u_0$. Since a polynomial $f(x)$ in one variable is non-negative if and only if it is a sum of squares, the result follows.
9 Analyzing polynomials in $u_0$

We now analyze the case when $g$ only depends on $u_0$ (step 6 of our lower bound strategy).

The probability distribution for $u_0$ is proportional to $(\frac{u_0}{n})^{d_1}$. This implies that the probability distribution for $x = \frac{d_1}{n} u_0$ is

$$
\left(1 - \frac{x}{d_1}\right)^{d_1} \approx e^{-x}
$$

Remark 9.1. We might think that the probability that $x \leq \frac{d_1}{n}$ is very small and can be ignored. If so, than using Chebyshev polynomials would give us a degree upper bound of $\tilde{O}(\sqrt{\frac{n}{d_1}})$ which is much less than $\sqrt{n}$. However, this is not correct. Intuitively, since we are considering polynomials of degree up to $d$, we should consider the point where $x^d e^{-x}$ becomes negligible, which is when $x$ is a sufficiently large constant times $d \log(d)$. Thus, we can only ignore the tail when $u_0$ is a sufficiently large constant times $n' \log(d) \frac{d}{d_1}$. This is why we have both $d$ and $d_1$ as we want $d_1$ to be at least $Cd \log(d)$ for some sufficiently large constant $C$.

With this rescaling, we want to show that for all polynomials $g(x)$ of degree at most $d$,

$$
g^2 \left( -\frac{d_1}{n'} \right) \leq \frac{n'^2}{40d_1^2} \int_{x=0}^{\infty} x e^{-x} g^2(x) dx
$$

In other words, we want to show that for all polynomials $g(x)$ of degree at most $d$,

$$
\int_{x=0}^{\infty} x e^{-x} g^2(x) dx \leq \frac{n'^2}{40d_1^2}
$$

Remark 9.2. The extra factor of $\frac{n'}{d_1}$ comes because we are shifting from $u_0 - 1$ to $x$.

To find the maximum possible value of $\frac{g^2(-\frac{d_1}{n'})}{\int_{x=0}^{\infty} x e^{-x} g^2(x) dx}$, we find an orthonormal basis $\{h_k\}$ for the measure $\mu(x) = xe^{-x}$ with dot product

$$
f \cdot g = \int_{x=\infty}^{\infty} f(x) g(x) \mu(x) dx
$$

Lemma 9.3. If $\{h_k\}$ is the orthonormal basis for the measure $\mu(x) = xe^{-x}$ then

$$
\frac{g^2 \left( -\frac{d_1}{n'} \right)}{\int_{x=0}^{\infty} x e^{-x} g^2(x) dx}
$$

is maximized by the polynomial $g = \sum_{k=0}^{d} h_k \left( -\frac{d_1}{n'} \right) h_k$ and has maximum value $\sum_{k=0}^{d} h_k^2 \left( -\frac{d_1}{n'} \right)$

Proof. Given a polynomial $g$ of degree at most $d$, writing $g = \sum_{k=0}^{d} c_k h_k$ and using Cauchy-Schwarz we have that

$$
\frac{\int_{x=0}^{\infty} x e^{-x} g^2(x) dx}{\sum_{k=0}^{d} c_k^2} \leq \left( \sum_{k=0}^{d} c_k h_k \left( -\frac{d_1}{n'} \right) \right)^2 \sum_{k=0}^{d} c_k^2
$$

with equality if and only if $c_k$ is proportional to $h_k \left( -\frac{d_1}{n'} \right)$

\[\square\]
Lemma 9.4.

\[ h_k(x) = \frac{1}{\sqrt{k!(k+1)!}} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \frac{(k+1)!}{(j+1)!} x^j \]

Proof.

Proposition 9.5. \(x^p \cdot x^q = (p + q + 1)!\)

Computing directly using Gram-Schmidt, the first few polynomials in the orthonormal basis are

1. \(h_0 = 1\)
2. \(h_1 = \frac{1}{\sqrt{2}}(x - 2)\)
3. \(h_2 = \frac{1}{\sqrt{12}}(x^2 - 6x + 6)\)
4. \(h_3 = \frac{1}{\sqrt{144}}(x^3 - 12x^2 + 36x - 24)\)
5. \(h_4 = \frac{1}{\sqrt{2880}}(x^4 - 20x^3 + 120x^2 - 240x + 120)\)

To check the general pattern, we need to check that for all \(i \in [0, k-1]\), \(h_k \cdot x^i = 0\) and \(h_k \cdot h_k = 1\).

To see this, observe that for all \(i \geq 0\),

\[ h_k \cdot x^i = \frac{1}{\sqrt{k!(k+1)!}} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \frac{(k+1)!}{(j+1)!} (i+j+1)! \]

Now observe that for all \(k\) and all functions \(f(j)\),

\[ \sum_{j=0}^{k} (-1)^{k-j} f(j) = (\Delta^k f)(0) \]

where \((\Delta f)(x) = f(x+1) - f(x)\)

Proposition 9.6. If \(f = x^i\) then \(\Delta^k f = 0\) if \(i < k\) and \(\Delta^k f = k!\) if \(i = k\).

Viewing \(\binom{i+j+1}{j+1}\) as a polynomial in \(j\),

\[ \binom{i+j+1}{j+1} = \frac{(i+j+1)! \cdot j^i}{i!(j+1)! \cdot i!} + \text{lower order terms} \]

Putting everything together,

1. \(h_k \cdot x^i = 0\) whenever \(i \leq k\).
2. \(h_k \cdot h_k = \frac{1}{\sqrt{k!(k+1)!}} (h_k \cdot x^k) = \frac{k!(k+1)!}{k!(k+1)!} (\Delta^k \binom{k+j+1}{j+1}(0)) = 1\)
We now estimate $h_k(-\frac{d}{m})$ and use this to upper bound $\sum_{k=0}^{d} h_k^2(-\frac{d}{m})$. Observe that if $x > 0$ is small,

$$h_k(-x) \approx (-1)^k \sqrt{k + 1} (1 + \frac{k}{2!} x + \frac{k^2}{2!3!} x^2 + \ldots)$$

Thus, as long as $kx << 1$, $h_k^2(-x) \leq 4 (k + 1)$. Summing this from $k = 0$ to $k = d$ gives a bound of $O(d^2)$ on $\sum_{k=0}^{d} h_k^2(-\frac{d}{m})$. This will be much less than $\frac{d^2}{40d_1^2}$ as long as $dd_1 << n$. Thus, we expect our $\tilde{E}$ to be valid as long as $d << \sqrt{\log(n)}$.

### 10 Bounding the difference between distributions

We now sketch how to show that we did not introduce too much error by shifting the distribution. Our error has two parts

1. The corresponding continuous distribution on $x = \frac{d_1}{m} u_0$ was proportional to $(1 - \frac{x}{d_1})^{d_1}$ rather than $e^{-x}$ and has domain $[0, d_1]$ rather than $[0, \infty)$

2. Our distribution was discrete rather than continuous.

For the first part, instead of using the exact distribution $e^{-x}$, we upper and lower bound most of the actual continuous distribution by multiples of $e^{-\frac{x}{2}}$ and $e^{-2x}$ respectively. We then argue that the remaining tail where the lower bound fails is negligible.

For the second part, we observe that if $d, d_1$ are fixed then as $n \to \infty$ the discrete distribution gets closer and closer to the continuous distribution. For our continuous distribution, we can in fact show that $\tilde{E}[(u_0 - 1)g^2] > 0$ for any $g$ which is not equivalent to 0. This implies that for any fixed $d$, for sufficiently large $n$, we will have that

$$E_{\text{discrete}}[(u_0 - 1)g^2] > \frac{1}{2} E_{\text{continuous}}[(u_0 - 1)g^2] > 0$$

for any $g$ which is not equivalent to 0

We expect that this lower bound can be improved to $\tilde{\Omega}(\sqrt{n})$ but this will require a more careful argument bounding the difference between the continuous and discrete distributions.

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