REVISITING THE DERIVATION OF THE FRACTIONAL DIFFUSION EQUATION

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Abstract
The fractional diffusion equation is derived from the master equation of continuous
time random walks (CTRWs) via a straightforward application of the Gnedenko-
Kolmogorov limit theorem. The Cauchy problem for the fractional diffusion equation
is solved in various important and general cases. The meaning of the proper
diffusion limit for CTRWs is discussed.

1. INTRODUCTION
This paper provides a short, but self-contained, introduction to fractional diffusion. The
readers will find the basic ideas behind the derivation of the fractional diffusion equation
starting from continuous-time random walks. We have included formulae for the solution
of the Cauchy problem which can be numerically implemented and used for applications.
Special care has been used to avoid unessential mathematical technicalities. Even if far
Derivation of the fractional diffusion equation from exhaustive, the bibliography should give a sufficient number of entry points for further reading. The following sections are based on a series of papers about the application of fractional calculus to finance. It is our hope that theoretical and experimental condensed matter physicists will find this work useful.

The paper is divided as follows. In Section 2, we outline the theory leading to the time-fractional master equation. In Section 3, the transition to the space-time fractional diffusion equation is discussed. Section 4 is devoted to the solutions of the Cauchy problem for the fractional diffusion equation. The main results are briefly summarized and discussed in Section 5. In Appendix A and Appendix B, we introduce the definitions of fractional derivatives in time and space, respectively, entering the fractional diffusion equation.

2. STEP ONE: TRANSITION TO THE TIME-FRACTIONAL MASTER EQUATION

Let $x$ be the position of a diffusing particle in one dimension. Let us assume that both jumps $\xi_i = x(t_i) - x(t_{i-1})$ and waiting times between two consecutive jumps $\tau_i = t_i - t_{i-1}$ are i.i.d. random variables described by two probability density functions: $w(\xi)$ and $\psi(\tau)$.

According to the model of continuous-time random-walk (CTRW), introduced by Montroll and Weiss, the evolution equation for $p(x,t)$, the probability of finding the random walker at position $x$ at time instant $t$, can be written as follows, assuming the initial condition $p(x,0) = \delta(x)$ (i.e. the walker is initially at the origin $x=0$),

$$p(x,t) = \delta(x) \Psi(t) + \int_0^t \psi(t-t') \left[ \int_{-\infty}^{+\infty} w(x-x') p(x',t') \, dx' \right] \, dt', \quad (2.1)$$

where

$$\Psi(t) = \int_t^\infty \psi(t') \, dt' = 1 - \int_0^t \psi(t') \, dt', \quad \psi(t) = -\frac{d}{dt} \psi(t). \quad (2.2)$$

The master equation of the CTRW can be also derived in the Fourier-Laplace domain.

In general, a CTRW is a non-Markovian process. A CTRW becomes Markovian if (and only if) the above memory function is proportional to a delta function so that $\Psi(t)$ and $\psi(t)$ differ only by a multiplying positive constant. By an appropriate choice of the unit of time, we can write $\Phi(t) = \delta(t)$, $t \geq 0$. In this case, Eq. (2.3) becomes:
$$\frac{\partial}{\partial t} p(x,t) = -p(x,t) + \int_{-\infty}^{+\infty} w(x-x') p(x',t) dx', \quad p(x,0) = \delta(x). \quad (2.4)$$

Up to a change of the unit of time, this is the most general master equation for a Markovian CTRW; Saichev & Zaslavsky call it the Kolmogorov-Feller equation.

Eq. (2.3) allows a natural characterization of a peculiar class of non-Markovian processes, where the memory function, $\Phi(t)$, has power-law time decay. Within this class, an interesting choice is the following:

$$\Phi(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)}, \quad t \geq 0, \quad 0 < \beta < 1. \quad (2.5)$$

In this case, $\Phi(t)$ is a weakly singular function that, in the limit $\beta \rightarrow 1$, reduces to $\Phi(t) = \delta(t)$, according to the formal representation of the Dirac generalized function, $\delta(t) = t^{-1}/\Gamma(0), \quad t \geq 0$. As a consequence of the choice (2.5), Eq. (2.3) can be written as:

$$\frac{\partial^\beta}{\partial t^\beta} p(x,t) = -p(x,t) + \int_{-\infty}^{+\infty} w(x-x') p(x',t) dx', \quad p(x,0) = \delta(x), \quad (2.6)$$

where $\partial^\beta/\partial t^\beta$ is the pseudo-differential operator explicitly defined in the Appendix A, that we usually call the Caputo fractional derivative of order $\beta$. Eq. (2.6) is a time-fractional generalization of Eq. (2.4) and can be called time-fractional Kolmogorov-Feller equation.

Our choice for $\Phi(t)$ implies peculiar forms for the functions $\Psi(t)$ and $\psi(t)$ generalizing the exponential behaviour of the waiting time density in the Markovian case. In fact, we have for $t \geq 0$:

$$\Psi(t) = E_\beta(-t^\beta), \quad \psi(t) = -\frac{d}{dt} E_\beta(-t^\beta), \quad 0 < \beta < 1, \quad (2.7)$$

where $E_\beta$ denotes an entire transcendental function, known as the Mittag-Leffler function of order $\beta$, defined in the complex plane by the power series

$$E_\beta(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta n + 1)}, \quad \beta > 0, \quad z \in \mathbb{C}. \quad (2.8)$$

Detailed information on the Mittag-Leffler-type functions is available in the literature.

From the properties of the Mittag-Leffler function, it can be shown that the corresponding survival probability and waiting-time pdf interpolate between a stretched exponential, for small waiting times, and a power-law decay, for large waiting times. Such a behaviour has been observed in Mainardi et al. and in Raberto et al., where the function $\Psi(\tau)$ has been estimated from empirical financial data.

As a final remark, it is important to notice that different choices of the kernel $\Phi(t)$ in eq. (2.3) are possible, leading to different properties of the waiting-time pdf and different generalized Kolmogorov-Feller evolution equations for $p(x,t)$. 
3. **STEP TWO: THE DIFFUSION LIMIT**

In the physical literature, many authors have discussed the connection between continuous–time random walks and diffusion equations of fractional order, with different degrees of detail. A sound proof of the equivalence between fractional diffusion and CTRW has been given by Hilfer & Anton. However, in order to perform the transition to the diffusion limit, we shall use a different approach. We shall start from Eqs. (2.4) and (2.6), and pass through their Fourier-Laplace counterparts. The stochastic process whose probability density evolves according to those equations is a random walk originating from a sequence of jumps, each jump being a sample of a real random variable $Y$. During the time interval $t_n \leq t < t_{n+1}$, the particle position is $Y_1 + Y_2 + \ldots + Y_n$. The $Y_k$ are i.i.d. random variables all described, as $Y$, by the pdf $w(x)$. Let us denote by $\hat{w}(\kappa)$ the characteristic function corresponding to the probability density $w(x)$.

Let us specify some conditions on the pdf $w(x)$. The requirement is that, if $\alpha = 2$:

$$\sigma^2 = \int_{-\infty}^{+\infty} x^2 w(x) \, dx < \infty, \quad (3.1)$$

whereas, if $0 < \alpha < 2$:

$$w(x) = (b + \epsilon(|x|)) |x|^{-(\alpha+1)}, \quad b > 0, \quad \epsilon(|x|) \to 0 \text{ as } |x| \to \infty. \quad (3.2)$$

In Eq. (3.2), $b > 0$ and $\epsilon(|x|)$ is bounded and $O(|x|^{-\eta})$ with $\eta > 0$ as $|x| \to \infty$. Let us furthermore recall the necessary requirements $w(x) \geq 0$, and the normalization condition $\int_{-\infty}^{+\infty} w(x) \, dx = 1$.

Let us now consider a sequence of random process pdf’s $p_h(x,t)$ describing scaled jumps of size $hY_k$ instead of $Y_k$, with a speed increase of the process by a factor (the scaling factor) $\mu^{1/\beta} h^{-\alpha/\beta}$, where $\mu$ must satisfy some conditions which will be specified later. The pdf of the jump size is $w_h(x) = w(x/h)/h$, so that its characteristic function is $\hat{w}_h(\kappa) = \hat{w}(\kappa h)$. For $0 < \alpha \leq 2$ and $0 < \beta \leq 1$, Eq. (2.6) (including (2.4) in the special case $\beta = 1$) is replaced by the sequence of equations

$$\mu h^\alpha \frac{\partial^\beta}{\partial t^\beta} p_h(x,t) = -p_h(x,t) + \int_{-\infty}^{+\infty} w_h(x-x') p_h(x',t) \, dx'. \quad (3.3)$$

By Fourier-Laplace transforming and by recalling the Laplace transform of the Caputo time-fractional derivative, defined by Eq. (A.2), we have

$$\mu h^\alpha \left\{ \frac{s^\beta}{\Gamma(\beta)} \hat{p}_h(\kappa, s) - s^{\beta-1} \right\} = [\hat{w}_h(\kappa) - 1] \hat{p}_h(\kappa, s). \quad (3.4)$$

We shall now present arguments based on the classical central limit theorem or on the Gnedenko limit theorem, (see the book by Gnedenko & Kolmogorov) both expressed in terms of the characteristic functions. The Gnedenko limit theorem is a suitable generalization of the classical central limit theorem for space pdf’s with infinite variance, decaying according to condition (3.2).

The transition to the diffusion limit is based on the following Lemma introduced by Gorenflo.
With the scaling parameter

$$
\mu = \begin{cases} 
\sigma^2/2, & \text{if } \alpha = 2, \\
\frac{b\pi}{\Gamma(\alpha + 1) \sin(\alpha\pi/2)}, & \text{if } 0 < \alpha < 2,
\end{cases}
$$

we have the relation

$$
\lim_{h \to 0} \frac{\hat{w}(\kappa h) - 1}{\mu h^\alpha} = -|\kappa|^\alpha, \quad 0 < \alpha \leq 2, \quad \kappa \in \mathbb{R}.
$$

Now, it is possible to set

$$
\rho_h(\kappa) = \frac{\hat{w}(\kappa h) - 1}{\mu h^\alpha},
$$

and the sequence of equations (3.4) reads

$$
s^\beta \hat{p}_h(\kappa, s) - s^{\beta - 1} = \rho_h(\kappa) \hat{p}_h(\kappa, s).
$$

Then, passing to the limit $h \to 0$, thanks to (3.6), we get:

$$
s^\beta \hat{p}_0(\kappa, s) - s^{\beta - 1} = -|\kappa|^\alpha \hat{p}_0(\kappa, s), \quad 0 < \alpha \leq 2, \quad 0 < \beta \leq 1.
$$

By inversion and using the Fourier transform of the Riesz space-fractional derivative, defined in Eq. (B.2), we finally obtain the equation:

$$
\frac{\partial^\beta}{\partial \|x\|^\alpha} p_0(x, t) = \frac{\partial^\alpha}{\partial |x|^\alpha} p_0(x, t), \quad p_0(x, 0) = \delta(x),
$$

which is a space-time fractional diffusion equation. In the limiting cases $\beta = 1$ and $\alpha = 2$, Eq. (3.10) reduces to the standard diffusion equation.

We have presented a formally correct transition to the diffusion limit starting from the general master equation of the CTRW, namely Eq. (2.1) or Eq. (2.3). By invoking the continuity theorem of probability theory, see e.g. the book by Lukacs[26], we can see that the random variable whose density is $p_h(x, t)$ converges in distribution ("weakly" or "in law") to the random variable with density $p_0(x, t)$.

Solving (3.8) for $\hat{p}_h(\kappa, s)$, and (3.9) for $\hat{p}_0(\kappa, s)$, gives:

$$
\hat{p}_h(\kappa, s) = \frac{s^{\beta - 1}}{s^\beta - \rho_h(\kappa)}, \quad \hat{p}_0(\kappa, s) = \frac{s^{\beta - 1}}{s^\beta + |\kappa|^\alpha},
$$

which yields:

$$
\hat{p}_h(\kappa, t) = E_\beta \left( \rho_h(\kappa) t^\beta \right), \quad \hat{p}_0(\kappa, t) = E_\beta \left( -|\kappa|^\alpha t^\beta \right).
$$

By (3.6) $\rho_h(\kappa) \to -|\kappa|^\alpha$ as $h \to 0$, hence

$$
p_h(x, t) \to p_0(x, t), \quad \text{for } t > 0, \quad h \to 0.
$$
4. SOLUTIONS AND THEIR SCALING PROPERTIES

For the determination of the fundamental solutions of Eq. (3.12) in the general case \( \{0 < \alpha \leq 2, \ 0 < \beta \leq 1\} \) the reader can consult Gorenflo et al.\(^27\) and Mainardi et al.\(^29\). We also refer to the above references for the particular cases \( \{0 < \alpha \leq 2, \ \beta = 1\} \) and \( \{\alpha = 2, \ 0 < \beta \leq 1\} \), already dealt with in the literature.

For parameters in the interval \( 0 < \alpha \leq 2, \ 0 < \beta \leq 1 \), the Cauchy problem in Eq. (3.12) can be solved by means of the Fourier-Laplace transform method.

The solution (Green function) turns out to be:

\[
p_0(x,t) = \frac{1}{t^{\beta/\alpha}} W_{\alpha,\beta} \left( \frac{x}{t^{\beta/\alpha}} \right). \tag{4.1}\]

The function \( W_{\alpha,\beta}(u) \) is the Fourier transform of a Mittag-Leffler function:

\[
W_{\alpha,\beta}(u) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-iu} E_{\beta}(-|q|^\alpha) dq. \tag{4.2}\]

Indeed, \( E_{\beta} \) is the Mittag-Leffler function of order \( \beta \) and argument \( z = -|q|^\alpha \).

In the limiting case \( 0 < \alpha < 2 \) and \( \beta = 1 \), the solution is:

\[
p_0(x,t) = \frac{1}{t^{1/\alpha}} L_\alpha \left( \frac{x}{t^{1/\alpha}} \right), \tag{4.3}\]

where \( L_\alpha(u) \) is the Lévy standardized probability density function:

\[
L_\alpha(u) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-iu-|q|^\alpha} dq, \tag{4.4}\]

whereas, in the case \( \alpha = 2, \ 0 < \beta < 1 \), we get

\[
p_0(x,t) = \frac{1}{2} t^{-\beta/2} M_{\beta/2} \left( \frac{x}{t^{\beta/2}} \right), \tag{4.5}\]

where \( M_{\beta/2} \) denotes the \( M \) function of Wright type of order \( \beta/2 \).

Remarkably, a composition rule holds true, and it can be shown that the Green function for the space-time fractional diffusion equation of order \( \alpha \) and \( \beta \) can be written in terms of the Green function for the space-fractional diffusion equation of order \( \alpha \) and the Green function for the time-fractional diffusion equation of order \( 2\beta \),

\[
p_0(x,t) = t^{-\beta} \int_0^\infty r^{-1/\alpha} L_\alpha \left( \frac{x}{r^{1/\alpha}} \right) M_\beta \left( \frac{r}{t^{\beta}} \right) dr. \tag{4.6}\]

Finally, as written before, in the case \( \alpha = 2, \ \beta = 1 \), Eq. (3.12) reduces to the standard diffusion equation, and the Cauchy problem is solved by:

\[
p_0(x,t) = t^{-1/2} \frac{1}{2\sqrt{\pi}} \exp(-x^2/(4t)) = t^{-1/2} G \left( \frac{x}{t^{1/2}} \right), \tag{4.7}\]

where \( G(x) \) denotes the Gaussian pdf

\[
G(x) = \frac{1}{2\sqrt{\pi}} \exp(-x^2/4). \tag{4.8}\]
5. SUMMARY AND DISCUSSION

Applications of fractional diffusion equations have been recently reviewed by Uchaikin & Zolotarev and by Metzler & Klafter. After that, other contributions appeared on this issue, among which we quote the paper of Zaslavsky in the book edited by Hilfer, the papers by Meerschaert et al. and by Paradisi et al. and the letter by West and Nonnenmacher.

In this paper, a scaling method has been discussed to get the transition to the diffusion limit in a correct way, starting from the CTRW master equation describing the time evolution of a stochastic process. Moreover, the solutions of the Cauchy problem for the fractional diffusion equation have been listed for the various relevant values of the fractional derivative orders $\alpha$ and $\beta$.

Various formulae which can be useful for applications have been presented. In principle, given a diffusing quantity, the waiting–time density, the jump density, and the probability of finding the random walker in position $x$ at time $t$ are all quantities which can be empirically determined. Therefore, many relationships presented above can be corroborated or falsified in specific contexts.

As a further remark, it may be useful to add some comments on the meaning of the diffusion limit taken in section 3.

The factor $\mu h^\alpha$ can be viewed as causing the jump process to run faster and faster (the waiting times becoming shorter and shorter) as $h$ becomes smaller and smaller. Replacing the density $w(x)$ by the density $w_h(x) = w(x/h)/h$, and, accordingly, the jumps $Y$ by $hY$, means that the jump size becomes smaller and smaller as the scaling length $h$ tends to zero.

An alternative interpretation is that we look at the same process with a discrete number of jumps occurring after finite times, from far away and after long time, so that spatial distances and time intervals of normal size appear very small, being $x$ replaced by $x/h$, $t$ replaced by $t/(\mu^{1/\beta} h^{\alpha/\beta})$. 
APPENDIX A: THE CAPUTO TIME-FRACTIONAL DERIVATIVE

For readers’ convenience, here, we present an introduction to the Caputo fractional derivative starting from its representation in the Laplace domain and pointing out its difference from the standard Riemann-Liouville fractional derivative. In so doing we avoid the subtleties lying in the inversion of fractional integrals.

If \( f(t) \) is a (sufficiently well-behaved) function with Laplace transform \( \mathcal{L} \{ f(t); s \} = \tilde{f}(s) = \int_0^\infty e^{-st} f(t) \, dt \), we have

\[
\mathcal{L} \left\{ \frac{d^\beta}{dt^\beta} f(t); s \right\} = s^\beta \tilde{f}(s) - s^{\beta-1} f(0^+) , \quad 0 < \beta < 1 , \tag{A.1}
\]

if we define

\[
\frac{d^\beta}{dt^\beta} f(t) := \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{df(\tau)}{d\tau} \frac{d\tau}{(t-\tau)^\beta} . \tag{A.2}
\]

We can also write

\[
\frac{d^\beta}{dt^\beta} f(t) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \left\{ \int_0^t [f(\tau) - f(0^+)] \frac{d\tau}{(t-\tau)^\beta} \right\} , \tag{A.3}
\]

\[
\frac{d^\beta}{dt^\beta} f(t) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \left\{ \int_0^t \frac{f(\tau)}{(t-\tau)^\beta} \, d\tau \right\} - \frac{t^{-\beta}}{\Gamma(1-\beta)} f(0^+) . \tag{A.4}
\]

Eqs. (A.1-4) can be extended to any non integer \( \beta > 1 \), (see e.g. the survey by Gorenflo & Mainardi\(^{10}\)). We refer to the fractional derivative defined by (A.2) as the Caputo fractional derivative, as it was used by Caputo for modelling dissipation effects in linear viscoelasticity in the late sixties\(^{35,36,37}\).

This definition differs from the usual one named after Riemann and Liouville, given by the first term in the R.H.S. of (A.4), and defined e.g. in the treatise on Fractional Calculus by Samko, Kilbas & Marichev\(^{38}\).

Gorenflo & Mainardi\(^{10}\) and Podlubny\(^{39}\) have pointed out the usefulness of the Caputo fractional derivative in the treatment of differential equations of fractional order for physical applications. In fact, in physical problems, the initial conditions are usually expressed in terms of a given number of boundary values assumed by the field variable and its derivatives of integer order, despite the fact that the governing evolution equation may be a generic integro-differential equation and therefore, in particular, a fractional differential equation.
APPENDIX B: THE RIEZ SPACE-FRACTIONAL DERIVATIVE

If \( f(x) \) is a (sufficiently well-behaved) function with Fourier transform
\[
\mathcal{F}\{f(x); \kappa\} = \hat{f}(\kappa) = \int_{-\infty}^{+\infty} e^{i\kappa x} f(x) \, dx, \quad \kappa \in \mathbb{R},
\]
we have
\[
\mathcal{F}\left\{ \frac{d^\alpha}{d|x|^\alpha} f(x); \kappa \right\} = -|\kappa|^\alpha \hat{f}(\kappa), \quad 0 < \alpha < 2,
\]
if we define
\[
\frac{d^\alpha}{d|x|^\alpha} f(x) = \Gamma(1 + \alpha) \frac{\sin(\alpha \pi/2)}{\pi} \int_0^\infty \frac{f(x + \xi) - 2f(x) + f(x - \xi)}{\xi^{1+\alpha}} \, d\xi. \tag{B.1}
\]
The fractional derivative defined by (B.2) can be called Riesz fractional derivative, as it is obtained from the inversion of the fractional integral originally introduced by Marcel Riesz, known as the Riesz potential \(38\). The representation (B.2) \(2\), is more explicit and convenient than others found in the literature \(7\). It is based on a suitable regularization of a hyper-singular integral.

For \( \alpha = 2 \), the Riesz derivative reduces to the standard derivative of order 2, as \(-|\kappa|^2 = -\kappa^2\).

For \( \alpha = 1 \), the Riesz derivative is related to the Hilbert transform, resulting in the formula
\[
\frac{d}{d|x|^\alpha} f(x) = -\frac{1}{\pi} \frac{d}{dx} \int_{-\infty}^{+\infty} \frac{f(\xi)}{x - \xi} \, d\xi. \tag{B.3}
\]

We note, by writing \(-|\kappa|^\alpha = -(\kappa^2)^{\alpha/2}\), that the Riesz derivative of order \(\alpha\) can be interpreted as the opposite of the \(\alpha/2\) power of the (positive definite) operator \(-D^2 = -\frac{d^2}{dx^2}\), namely
\[
\frac{d^\alpha}{d|x|^\alpha} = -\left(\frac{d^2}{dx^2}\right)^{\alpha/2}. \tag{B.4}
\]
The notation used above is due to Saichev & Zaslavsky \(7\). A different notation which takes into account asymmetries was used by Gorenflo & Mainardi \(1\).
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6. REFERENCES

[1] E. Scalas, R. Gorenflo and F. Mainardi, *Physica A* **284** (2000) 376.
[2] F. Mainardi, M. Raberto, R. Gorenflo and E. Scalas, *Physica A* **287** (2000) 468.
[3] M. Raberto, E. Scalas, R. Gorenflo and F. Mainardi, *Journal of Quantitative Finance*, submitted. [Pre-print in http://xxx.lanl.gov/abs/cond-mat/0001253](http://xxx.lanl.gov/abs/cond-mat/0001253]
[4] R. Gorenflo, F. Mainardi, E. Scalas and M. Raberto, in *Proc. Workshop on Mathematical Finance*, University of Konstanz, Germany, October 5-7, 2000, eds. M. Kohlmann and S. Tang (Birkhäuser, Boston 2001), to appear.
[5] E.W. Montroll and G.H. Weiss, J. Math. Phys. **6** (1965) 167.
[6] M.F. Shlesinger, in *Encyclopedia of Applied Physics*, ed. G.L. Trigg (VCH Publishers, Inc., New York, 1996), p. 45.
[7] A.I. Saichev and G.M. Zaslavsky, *Chaos* **7** (1997) 753.
[8] I.M. Gel’fand and G.E. Shilov, *Generalized Functions*, Vol. 1, (Academic Press, New York, 1964).
[9] A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, *Higher Transcendental Functions*, Bateman Project, Vol. 3, (McGraw-Hill, New York, 1955). [Ch. 18, pp. 206-227.]
[10] R. Gorenflo and F. Mainardi, in: *Fractals and Fractional Calculus in Continuum Mechanics*, eds. A. Carpinteri and F. Mainardi (Springer Verlag, Wien and New York, 1997), p. 223.
[11] F. Mainardi and R. Gorenflo, *J. Comput. & Appl. Mathematics* **118** (2000) 283.
[12] H.E. Roman and P.A. Alemany, *J. Phys. A: Math. Gen.* **27** (1994) 3407.
[13] H.C. Fogedby, *Phys. Rev. E* **50** (1994) 1657.
[14] B.D. Hughes, *Random Walks and Random Environments, Vol. 1: Random Walks*, (Oxford Science Publ., Clarendon Press, Oxford, 1995) p. 631.
[15] M. Kotulski, *J. Stat. Phys.* **81** (1995) 777.
[16] R. Hilfer and L. Anton, *Phys. Rev. E* **51** (1995) R848.
[17] A. Compte, *Phys. Rev. E* **53** (1996) 4191.
[18] B.J. West, P. Grigolini, R. Metzler and T.F. Nonnenmacher, *Phys. Rev. E* **55** (1997) 9.
[19] R. Metzler and T.F. Nonnenmacher, *Phys. Rev. E* **57** (1998) 6409.
[20] R. Hilfer, in *Applications of Fractional Calculus in Physics*, ed. R. Hilfer, (World Scientific, Singapore, 2000) p. 87.
[21] E. Barkai, R. Metzler and J.Klafter, *Phys. Rev. E* **61** (2000) 132.
[22] R. Metzler and J. Klafter, *Physics Reports* **339** (2000) 1.
[23] V.V. Uchaikin *Int. J. Theor. Phys.* **39** (2000) 2087.
[24] B.V. Gnedenko and A.N. Kolmogorov, *Limit Distributions for Sums of Independent Random Variables*, (Addison-Wesley, Cambridge, Mass., 1954).
[25] R. Gorenflo and F. Mainardi, in *Problems in Mathematical Physics*, eds. J. Elschner, I. Gohberg and B. Silbermann (Birkhäuser Verlag, Basel, 2001), p. 120.
[26] E. Lukacs, *Characteristic Functions*, (Griffin, London, 1960).
[27] R. Gorenflo, A. Iskenderov and Yu. Luchko, *Fractional Calculus and Applied Analysis* **3** (2000) 75.
[28] F. Mainardi, *Chaos, Solitons & Fractals* **7** (1996) 1461.
[29] F. Mainardi, Yu. Luchko and G. Pagnini, *Fractional Calculus and Applied Analysis* **4** (2001), in press.
[30] V.V. Uchaikin and V.M. Zolotarev, *Change and Stability. Stable Distributions and their Applications*, (VSP, Utrecht, 1999).
[31] G.M. Zaslavsky, in *Applications of Fractional Calculus in Physics*, ed. R. Hilfer, (World Scientific, Singapore, 2000), p. 377.
[32] M.M. Meerschaert, D.A. Benson and B. Baeumer, *Phys. Rev. E* **63** (2001), in press.
[33] P. Paradisi, R. Cesari, F. Mainardi and F. Tampieri : *Physica A* **293** (2001), in press.
[34] B.J. West and T. Nonnenmacher, *Phys. Lett. A* **278** (2001) 253.
[35] M. Caputo, Geophys. J. R. Astr. Soc. 13 (1967) 529.
[36] M. Caputo, Elasticità e Dissipazione, (Zanichelli, Bologna, 1969).
[37] M. Caputo and F. Mainardi, Riv. Nuovo Cimento (Ser. II) 1 (1971) 161.
[38] S.G. Samko, A.A. Kilbas and O.I. Marichev, Fractional Integrals and Derivatives, Theory and Applications, (Gordon and Breach, Amsterdam, 1993).
[39] I. Podlubny, Fractional Differential Equations (Academic Press, San Diego, 1999).
[40] R. Gorenflo and F. Mainardi, Fractional Calculus and Applied Analysis 1 (1998) 167.
[41] R. Gorenflo and F. Mainardi, J. Analysis and its Applications (ZAA) 18 (1999) 231.