On the Time Dependent Gross Pitaevskii- and Hartree Equation

P. Pickl *

August 8, 2008

Abstract

We are interested in solutions $Ψ_t$ of the Schrödinger equation of $N$ interacting bosons under the influence of a time dependent external field, where the range and the coupling constant of the interaction scale with $N$ in such a way, that the interaction energy per particle stays more or less constant. Let $N^{\varphi_0}$ be the particle number operator with respect to some $\varphi_0 \in L^2(\mathbb{R}^3 \to \mathbb{C})$. Assume that the relative particle number of the initial wave function $N^{-1}(\psi_0, N^{\varphi_0} \psi_0)$ converges to one as $N \to \infty$. We shall show that we can find a $\varphi_t \in L^2(\mathbb{R}^3 \to \mathbb{C})$ such that $\lim_{N \to \infty} N^{-1}(\psi_t, N^{\varphi_t} \psi_t) = 1$ and that $\varphi_t$ is — dependent of the scaling of the range of the interaction — solution of the Gross-Pitaevskii or Hartree equation.

We shall also show that under additional decay conditions of $\varphi_t$ the limit can be taken uniform in $t < \infty$ and that convergence of the relative particle number implies convergence of the $k$-particle density matrices of $\psi_t$.

1 Introduction

In this paper we wish to analyze the dynamics of a Bose condensate of $N$ interacting particles when the external trap — described by an external potential $A_t$ — is changed, for example removed.

We are interested in solutions of the $N$-particle Schrödinger equation

$$i \frac{d}{dt} \psi_t = H \psi_t$$

(1)

with some symmetric $\psi_0$ we shall specify below and the Hamiltonian

$$H = -\sum_{j=1}^{N} \Delta_j + \sum_{j \neq k=1}^{n} v_N^{j}\,(x_j - x_k) + \sum_{j=1}^{N} A_t(x_j)$$

(2)

*Mathematisches Institut der Universität München, Theresienstr. 39, 80333 München, E-mail: pickl@mathematik.uni-muenchen.de
acting on the Hilbert space $L^2(\mathbb{R}^{3N} \to \mathbb{C})$, where $\beta \in \mathbb{R}$ stands for the scaling behavior of the interaction. The $v^N_\beta$ we wish to analyze scales with the particle number in such a way, that the total interaction energy scales in the same way as the total kinetic energy of the $N$ particles. For the heuristic arguments we shall give first one should think of an interaction which is given by $v^N_\beta(x) = N^{-1 + 3\beta}v(N^\beta x)$ for a compactly supported, spherically symmetric, positive potential $v \in L^\infty$. The interactions we shall choose below will be of a more general form. The $A_t$ describing the trap potential is a time dependent external potential which we shall choose — in contrast to $v^N_\beta$ — not $N$-dependent. Note, that $H$ conserves symmetry, i.e. for any symmetric function $\Psi_0$ also $H\Psi_0$ and thus $\Psi_t$ is symmetric.

Assume moreover that the initial wave function $\Psi_0$ is a condensate, i.e. that there exists a $L^2$ function $\phi_0$ such that

$$\lim_{N \to \infty} \langle \Psi_0, \hat{n}_\phi \Psi_0 \rangle = 1$$

where $\hat{n}_\phi$ is the particle number operator of particles in $\phi_0$ (see Definition 2.1 (c) and Lemma 2.3 (a)).

Under these and some additional technical assumptions we shall show that also $\Psi_t$ will be a condensate, i.e. that there exist $L^2$ functions $\phi_t$ such that

$$\lim_{N \to \infty} \langle \Psi_0, \hat{n}_\phi \Psi_0 \rangle = 1$$

uniform in $t$ on any compact subset of $\mathbb{R}^+$ and — under additional decay conditions on $\phi_t$ — uniform in $t \in \mathbb{R}^+$.

Even more: We shall show that $\phi_t$ solves the differential equation

$$i \frac{d}{dt} \phi_t = - (\Delta + A_t + V_{\phi_t}) \phi_t$$

with $\phi_0$ as above, where the “mean field” $V_{\phi_t}$ depends on $\phi_t$ itself, so (3) is a non-linear equation. For different regimes of $\beta$ different effective mean field potentials will appear:

| $\beta$ | $V_{\phi_t}$ |
|---|---|
| $\beta < 0$ | $0$ |
| $\beta = 0$ | $v \ast |\phi_t|^2$ |
| $0 < \beta \leq 1$ | $2a|\phi_t|^2$ |
| $\beta > 1$ | $0$ |

We explain the table. For $\beta < 0$ $\lim_{N \to \infty} \left\| \sum_{j=2}^N v^N_\beta(x_1 - x_j) \right\|_\infty = 0$, so it is heuristically clear that the mean field is zero.

$v^N_0 = v/N$ and hence particle (say number one) feels $N^{-1} \sum_{j=2}^N v(x_1 - x_j) \approx \int v(x-y)|\phi_t|^2(y)dy$ assuming that the particles are $|\phi_t|^2$-distributed. In this case (3) is called “Hartree equation”. This limit has already been proven in the literature [8]. A sketch of an alternative proof shall be given below to motivate the technique used in this paper for the cases $0 < \beta \leq 1$ we shall focus on here.

For $0 < \beta$ the interaction becomes $\delta$-like. To be able to “average out” the potential it is important to control the microscopic structure of $\Psi_t$. Assuming that the energy of $\Psi_t$ is small, the microscopic structure is — whenever two
particles approach — roughly given by the zero energy scattering length of the potential $1/2v_0^N$ (the factor $1/2$ comes from the fact that one has to go to relative coordinates of the two particles).

For $\beta = 1$ the scaling of the potential is such that the zero energy scattering state of $f^N(x)$ of the potential $v_0^{N\beta}/2$ scales like $f^N = f_1(Nx)$. It follows that the mean field is given by $2a|\varphi_1|^2$, where $a$ is the scattering length of $v/2$.

For $0 < \beta < 1$ the scaling is “softer” and the microscopic structure disappears as $N \to \infty$. Thus the mean field is given by $V\varphi \equiv \|v\|_1|\varphi|^2$. One can also argue, that for “soft scalings” the scattering length is in good approximation given by the first order Born approximation, i.e. by the $L_1$-norm of the interaction.

For $\beta > 1$ note, that the scattering length of a spherically symmetric potential is always smaller than the radius of its support, thus for $\beta > 1$ $Na_N \to 0$ for $N \to \infty$, implying that the interaction becomes negligible for $\beta > 1$ as $N \to \infty$.

The cases $\beta = 1$ and $0 < \beta < 1$ have been proven recently for the special case $A_1 \equiv 0$ [1, 2, 3, 4]. We shall give an alternative proof including time dependent external potentials and with weaker conditions on $\Psi_0$ and also generalizing to hard core potentials for $\beta = 1$.

2 Definition of the Projectors

Before we consider the different cases of $0 \leq \beta \leq 1$ we define the following operators acting on $L^2(\mathbb{R}^{3N} \to \mathbb{C})$ we shall need in the proofs below

**Definition 2.1** For any $\varphi \in L^2(\mathbb{R}^3 \to \mathbb{C})$ we define

(a) for any $1 \leq j \leq N$ and any $\varphi \in L^2(\mathbb{R}^3 \to \mathbb{C})$ the orthogonal projector $p^\varphi_j$ of the $j^{th}$ particle onto $\varphi$ defined by

$$p^\varphi_j \Psi := \varphi(x_j) \int \varphi(x_j)^* \Psi(x_1, \ldots, x_N) d^3 x_j$$

for any $\Psi \in L^2(\mathbb{R}^{3N} \to \mathbb{C})$. We shall also need $q^\varphi_j = 1 - p^\varphi_j$.

(b) For any $0 \leq k \leq j \leq N$ we define the set

$$A^j_k := \{(a_1, a_2, \ldots, a_j) : a_i \in \{0, 1\}; \sum_{i=1}^j a_i = k\}.$$

For any $0 \leq k \leq j \leq N$ and any $\varphi(x_j) \in L^2(\mathbb{R}^3 \to \mathbb{C})$ we define the orthogonal projector $P^\varphi_{j,k}$ acting on $L^2(\mathbb{R}^{3N} \to \mathbb{C})$ as

$$P^\varphi_{j,k} := \sum_{a \in A^j_k} \prod_{i=1}^j (p^\varphi_{N-j+i})^{1-a_i} (q^\varphi_{N-j+i})^{a_i}$$

and denote the special case $j = N$ by $P^\varphi_k := P^\varphi_{N,k}$. For negative $k$ and $k > N$ we set $P^\varphi_k := 0$.  

3
(c) For any function $f : \{0, 1, \ldots, N\} \to \mathbb{R}^+$ we define the operator $\hat{f} : L^2(\mathbb{R}^{3N} \to \mathbb{C}) \to L^2(\mathbb{R}^{3N} \to \mathbb{C})$ as
\[
\hat{f} := \sum_{j=0}^{N} f(j) P_j^\varphi .
\] (4)

We shall also need translations of the operators $\hat{f}$: Let $f : \{0, 1, \ldots, N\} \to \mathbb{R}^+$ and $d \in \mathbb{Z}$. We define the operator $\hat{f}_d : L^2(\mathbb{R}^{3N} \to \mathbb{C}) \to L^2(\mathbb{R}^{3N} \to \mathbb{C})$ as
\[
\hat{f}_d := \sum_{j=d}^{N+d} f(j-d) P_j^\varphi .
\]

Notation 2.2 Throughout the paper hats $\hat{\cdot}$ shall solemnly be used in the sense of Definition 2.1 (c). In what follows the letter $C$ will be used for various constants that need not be identical even within the same equation.

With Definition 2.1 we arrive directly at the following Lemma based on combinatorics of the $p_j^\varphi$ and $q_j^\varphi$:

Lemma 2.3

(a) For any functions $f, g : \{0, 1, \ldots, N\} \to \mathbb{R}^+$ we have that
\[
\hat{f} \hat{g} = \hat{g} \hat{f} = \hat{g} \hat{f} [\hat{p}_j = \hat{p}_j \hat{f} \quad \hat{f} \hat{p}_j = \hat{p}_j \hat{f} .
\]
(b) Let $n : \{0, 1, \ldots, N\} \to \mathbb{R}^+$ be given by $n(k) := \sqrt{k/N}$. Then the respective $(\hat{n}^\varphi)^2$ (c.f. [4]) equals the relative particle number operator of particles not in the state $\varphi$, i.e.
\[
(\hat{n}^\varphi)^2 = N^{-1} \sum_{j=1}^{N} q_j^\varphi .
\]

(c) For any function $f : \{0, 1, \ldots, N\} \to \mathbb{R}^+$ and any symmetric $\Psi \in L^2(\mathbb{R}^{3N} \to \mathbb{C})$ we have
\[
\left\| \hat{f} \hat{q}_1^\varphi \Psi \right\|^2 = \left\| \hat{f} \hat{n}^\varphi \Psi \right\|^2
\]
\[
\left\| \hat{f} \hat{q}_1^\varphi \hat{q}_2^\varphi \Psi \right\|^2 \leq \frac{N}{N-1} \left\| \hat{f} \hat{n}^\varphi \right\|^2 .
\] (5) (6)

(d) For any function $f : \{0, 1, \ldots, N\} \to \mathbb{R}^+$, any function $v : \mathbb{R}^6 \to \mathbb{R}$ and any $j, k = 0, 1, 2$ we have
\[
\hat{f} \hat{Q}_j^v v(x_1, x_2) Q_k^c = Q_j^c v(x_1, x_2) \hat{f} \hat{Q}_{k-j} Q_k^c ,
\]
where $Q_0^c := q_1^c q_2^c$, $Q_1^c := q_1^c q_2^c$ and $Q_2^c := q_1^c q_2^c$. 
(e) For any \( w \in L^\infty(\mathbb{R}^3 \to \mathbb{C}) \) and any symmetric \( \Psi \in L^2(\mathbb{R}^{3N} \to \mathbb{C}) \)

\[
\langle \Psi, w(x_1) \rangle \Psi - \langle \varphi, w\varphi \rangle| \leq 4\|w\|_\infty \left( N^{-1/4} + \|\hat{\mathbf{r}}\|^{1/2} \|\Psi\|^2 \right) .
\]

\( (7) \)

Proof: (a) follows immediate from definition 2.1 using that \( p_j \) and \( q_j \) are orthogonal projectors.

For (b) note that \( 1 = \sum_{k=1}^N P_k^{\mathbf{r}} \). Using also \( (q_k^{\mathbf{r}})^2 = q_k^{\mathbf{r}} \) and \( q_k^{\mathbf{r}} p_k^{\mathbf{r}} = 0 \) we get

\[
N^{-1} \sum_{k=1}^N q_k^{\mathbf{r}} = N^{-1} \sum_{k=1}^N q_k^{\mathbf{r}} \sum_{j=1}^N P_j^{\mathbf{r}} = N^{-1} \sum_{j=1}^N \sum_{k=1}^N q_k^{\mathbf{r}} P_j^{\mathbf{r}} = N^{-1} \sum_{j=1}^N j P_j^{\mathbf{r}}
\]

and (b) follows.

For (c) we can write using symmetry of \( \Psi \)

\[
\|\hat{f}^{\mathbf{r}} \hat{\mathbf{n}}^{\mathbf{r}} \Psi\|^2 = \langle \Psi, (\hat{f}^{\mathbf{r}})^2 (\hat{\mathbf{n}}^{\mathbf{r}})^2 \Psi \rangle = N^{-1} \sum_{k=1}^N \langle \Psi, (\hat{f}^{\mathbf{r}})^2 q_k^{\mathbf{r}} \Psi \rangle
\]

\[
= \langle \Psi, (\hat{f}^{\mathbf{r}})^2 q_1^{\mathbf{r}} \Psi \rangle + \sum_{k=2}^N \langle \Psi, (\hat{f}^{\mathbf{r}})^2 q_k^{\mathbf{r}} \Psi \rangle = \|\hat{f}^{\mathbf{r}} q_1^{\mathbf{r}} \Psi\|^2 .
\]

Similarly we have for (c)

\[
\|\hat{f}^{\mathbf{r}} (\hat{\mathbf{n}}^{\mathbf{r}})^2 \Psi\|^2 = \langle \Psi, (\hat{f}^{\mathbf{r}})^2 (\hat{\mathbf{n}}^{\mathbf{r}})^2 \Psi \rangle = N^{-2} \sum_{j,k=1}^N \langle \Psi, (\hat{f}^{\mathbf{r}})^2 q_j^{\mathbf{r}} q_k^{\mathbf{r}} \Psi \rangle
\]

\[
= N^{-1} \langle \Psi, (\hat{f}^{\mathbf{r}})^2 q_1^{\mathbf{r}} q_2^{\mathbf{r}} \Psi \rangle + N^{-1} \langle \Psi, (\hat{f}^{\mathbf{r}})^2 q_1^{\mathbf{r}} \Psi \rangle + N^{-1} \|\hat{f}^{\mathbf{r}} q_1^{\mathbf{r}} \Psi\|^2 .
\]

and (c) follows.

Using the definitions above we have for (d)

\[
\hat{f}^{\mathbf{r}} Q_j^{\mathbf{r}} v(x_1, x_2) Q_k^{\mathbf{r}} = \sum_{l=0}^N f(l) P_l^{\mathbf{r}} Q_j^{\mathbf{r}} v(x_1, x_2) Q_k^{\mathbf{r}}
\]

\[
= \sum_{l=0}^{N} f(l) P_{N-2,l-j}^{\mathbf{r}} Q_j^{\mathbf{r}} v(x_1, x_2) Q_k^{\mathbf{r}} = \sum_{l=k-j}^{N+k-j} Q_j^{\mathbf{r}} v(x_1, x_2) f(l+j-k) P_{N-2,l-k}^{\mathbf{r}} Q_k^{\mathbf{r}}
\]

\[
= \sum_{l=k-j}^{N+k-j} Q_j^{\mathbf{r}} v(x_1, x_2) f(l+j-k) P_l^{\mathbf{r}} Q_k^{\mathbf{r}}
\]

For (e) we have

\[
|\langle \Psi, w(x_1) \rangle \Psi - \langle \varphi, w\varphi \rangle| = \langle \hat{p}_1^{\mathbf{r}} \Psi, w(x_1) \hat{p}_1^{\mathbf{r}} \Psi \rangle + \langle \hat{p}_1^{\mathbf{r}} \Psi, w(x_1) \hat{q}_1^{\mathbf{r}} \Psi \rangle + \langle \hat{q}_1^{\mathbf{r}} \Psi, w(x_1) \hat{p}_1^{\mathbf{r}} \Psi \rangle
\]

\[
+ \langle \hat{q}_1^{\mathbf{r}} \Psi, w(x_1) \hat{q}_1^{\mathbf{r}} \Psi \rangle - \langle \varphi, w\varphi \rangle \leq \langle \varphi, w\varphi \rangle \left( 1 - \|\hat{p}_1^{\mathbf{r}} \Psi\|^2 \right) + \|w\|_\infty \|q_1^{\mathbf{r}} \Psi\|^2 + 2\|\hat{q}_1^{\mathbf{r}} \Psi, w(x_1) \hat{p}_1^{\mathbf{r}} \Psi\|
\]

\[
\leq 2\|w\|_\infty \|q_1^{\mathbf{r}} \Psi\|^2 + 2\|\hat{\mathbf{n}}^{\mathbf{r}}\|^{-1/2} q_1^{\mathbf{r}} \Psi \| \|\hat{\mathbf{n}}^{\mathbf{r}}\|^{-1/2} w(x_1) \hat{p}_1^{\mathbf{r}} \Psi\| .
\]
Using that $\sqrt{k+1} < \sqrt{k} + 1$ (thus $n(k+1) < n(k) + N^{-1/2}$) and part (d)\\
\begin{align*}
\| (\hat{n}^2)^{1/2} w(x_1)p_1^2 \Psi \|^2 &= \langle \Psi, p_1^2 w(x_1)\hat{n}^2 q_1^2 w(x_1)p_1^2 \Psi \rangle + \langle \Psi, p_1^2 w(x_1)\hat{n}^2 p_1^2 w(x_1)p_1^2 \Psi \rangle \\
&= ( (\hat{n}^2)^{1/2} \Psi, p_1^2 w(x_1)q_1^2 w(x_1)p_1^2 (\hat{n}^2)^{1/2} \Psi ) \\
&\quad + ( (\hat{n}^{-1})^{1/2} \Psi, p_1^2 w(x_1)p_1^2 w(x_1)p_1^2 (\hat{n}^{-1})^{1/2} \Psi ) \\
&\leq \| w(x_1) \|^2_{\infty} 2( (\hat{n}^2)^{1/2} \Psi \|^2 + N^{-1/2} )
\end{align*}
thus part (c) of the Lemma yields\\
\begin{align*}
|\langle \Psi, w(x_1)\Psi \rangle - \langle \varphi, w\varphi \rangle | &\leq 4\| w \|_{\infty} ( |\| \hat{n}^2 \Psi \|^2 + N^{-1/4} + |(\hat{n}^2)^{1/2} \Psi |^2 )
\end{align*}
With the operator inequality $(\hat{n}^2)^{\lambda} < (\hat{n}^2)^{\gamma}$ for any $\lambda < \gamma$ we get (e).

\begin{proof}
\end{proof}

\section{2.1 Convergence of the Reduced Density Matrix}

\begin{lemma}
Let $j > 0$, $\varphi \in L^2$ and let $\Psi \in L^2(\mathbb{R}^N \to \mathbb{C})$ be symmetric, let $\mu(\Psi)$ be the reduced one particle density matrix of $\Psi$. Then\\
(a)\\
\begin{align*}
\lim_{N \to \infty} |\| \hat{n}^2 \Psi \| | = 0 \iff \lim_{N \to \infty} \left\langle \Psi, (\hat{n}^2)^j \Psi \right\rangle = 0 .
\end{align*}
(b)\\
\begin{align*}
\lim_{N \to \infty} \left\langle \Psi, (\hat{n}^2)^{j} \Psi \right\rangle = 0 \implies \lim_{N \to \infty} \mu(\Psi) = |\varphi \rangle \langle \varphi |
\end{align*}
in weak-$*$ sense.

\begin{proof}
We shall show that\\
\begin{align*}
\lim_{N \to \infty} \left\langle \Psi, (\hat{n}^2)^j \Psi \right\rangle = 0 \implies \lim_{N \to \infty} \left\langle \Psi, (\hat{n}^2)^j \Psi \right\rangle = 0
\end{align*}
for any $j,l > 0$, which is equivalent to (a).

Let $\lim_{N \to \infty} \left\langle \Psi, (\hat{n}^2)^j \Psi \right\rangle = 0$ for some $j > 0$. It follows, that there exists a function $\delta(N)$ with $\lim_{N \to \infty} \delta(N) = 0$ such that\\
\begin{align*}
\sum_{k=0}^{N} \left( \frac{k}{N} \right)^j \| P_k \Psi \| < \delta(N) .
\end{align*}
Let $k(N)$ be the smallest integer such that $\left( \frac{k(N)}{N} \right)^j < \sqrt{\delta(N)}$. It follows that\\
\begin{align*}
\left( \frac{k(N)+1}{N} \right)^j \geq \sqrt{\delta(N)} \text{ and thus } \sum_{k=0}^{N} \| P_k \Psi \| \leq \sqrt{\delta(N)} .
\end{align*}
Hence\\
\begin{align*}
\sum_{k=0}^{N} \left( \frac{k}{N} \right)^j \| P_k \Psi \| &\leq \sum_{k=0}^{k(N)} \left( \frac{k}{N} \right)^j \| P_k \Psi \| + \sum_{k=0}^{N} \| P_k \Psi \| \\
&\leq \left( \frac{k(N)}{N} \right)^j + \sqrt{\delta(N)} \leq \left( \sqrt{\delta(N)} \right)^{1/j} + \sqrt{\delta(N)} .
\end{align*}

\end{proof}
Thus \( \lim_{N \to \infty} \langle \Psi, (\hat{n} \phi)^j \rangle = 0 \) and (a) follows.

With (a) we can choose without loss of generality \( j = 2 \) to prove (b). So let

\[
\lim_{N \to \infty} \langle \Psi, (\hat{n} \phi)^2 \rangle = 0.
\]

With Lemma 2.3 (c) we have using symmetry of \( \Psi \) that \( \lim_{N \to \infty} \| q_1 \phi \| = 0 \) and \( \lim_{N \to \infty} \| p_1 \phi \| = 1 \). Note, that

\[
\mu(\Psi) = \int \Psi(\cdot, x_2, \ldots, x_N) \Psi^*(\cdot, x_2, \ldots, x_N) d^{3N-3}x
\]

\[
= \int p_1^2 \Psi(\cdot, x_2, \ldots, x_N) p_1^2 \Psi^*(\cdot, x_2, \ldots, x_N) d^{3N-3}x
+ \int q_1^2 \Psi(\cdot, x_2, \ldots, x_N) q_1^2 \Psi^*(\cdot, x_2, \ldots, x_N) d^{3N-3}x
+ \int p_1^2 \Psi(\cdot, x_2, \ldots, x_N) q_1^2 \Psi^*(\cdot, x_2, \ldots, x_N) d^{3N-3}x
+ \int q_1^2 \Psi(\cdot, x_2, \ldots, x_N) q_1^2 \Psi^*(\cdot, x_2, \ldots, x_N) d^{3N-3}x
\]

The first summand equals \( \| p_1 \phi \|^2 \| \phi \rangle \langle \phi | \), the other summands have operator norm \( \| q_1 \phi \| \| p_1 \phi \| \) and \( \| q_1 \phi \|^2 \) respectively and the Lemma follows.

\[\Box\]

**Remark 2.5** Similarly one can proof that \( \lim_{N \to \infty} \langle \Psi, (\hat{n} \phi) \gamma \rangle = 0 \) for \( \gamma \in \mathbb{R}^+ \) implies convergence of the reduced \( k \)-particle density matrix for any fixed \( k < \infty \).

### 3 Derivation of the Hartree equation

Let us now consider the different cases for \( \beta \). To motivate the technique we shall use below, we first take a short look at \( \beta = 0 \). In this case we have that the mean field is of the form \( \nu \| \phi \|^2 \) and (3) becomes the Hartree equation.

Let \( \phi_t \) be a solution of the Hartree equation, let \( T < \infty \) be such that \( \| \phi_t \| < \infty \) for all \( t < T \).

Defining

\[
\alpha_t := \| \hat{n} \phi_t \| = \langle \Psi_t, (\hat{n} \phi_t)^2 \Psi_t \rangle
\]

and assuming that \( \alpha_0 \to 0 \) as \( N \to \infty \) we wish to show that \( \alpha_t \to 0 \) uniform in \( t < T \).

Note, that \( \alpha_t \) is \( 1/N \) times the expectation of particles which are not in the state \( \phi_t \), i.e. \( 1 - \alpha_t = \langle \Psi_t, (1 - (\hat{n} \phi_t)^2) \Psi_t \rangle \) is \( 1/N \) times the expectation of particles which are in the state \( \phi_t \).

By (8)

\[
\frac{d}{dt} \alpha_t := -i \langle \Psi_t, [H - H^H, (\hat{n} \phi_t)^2] \Psi_t \rangle
\]
where

\[ H^H := \sum_{j=1}^{N} -\Delta_j + A_t(x_j) + (v \ast |\varphi_t|^2)(x_j) \, . \]

Using symmetry of \( \Psi_t \) and Definition \[2.3\] we have

\[
\alpha'_t = -i N^{-1} \sum_{j=1}^{N} \langle \Psi_t, \sum_{k \neq \ell} v_j^N (x_k - x_\ell) - v \ast |\varphi_t|^2(x_j) \rangle \Psi_t \]

\[
= -i \langle \Psi_t, \sum_{k \neq \ell} v_j^N (x_k - x_\ell) - v \ast |\varphi_t|^2(x_j) \rangle \Psi_t \]

\[
= -i \langle \Psi_t, ((N - 1)v_j^N (x_2 - x_1) - v \ast |\varphi_t|^2(x_1))q_1^{\varphi_t} \Psi_t \rangle -i \langle \Psi_t, (N - 1)v_j^N (x_2 - x_1) - v \ast |\varphi_t|^2(x_1))q_1^{\varphi_t} \Psi_t \rangle +i \langle \Psi_t, q_1^{\varphi_t} (v_2^N (x_2 - x_1) - v \ast |\varphi_t|^2(x_1)) \Psi_t \rangle \]

\[
= -i \langle \Psi_t, q_1^{\varphi_t} ((N - 1)v_j^N (x_2 - x_1) - v \ast |\varphi_t|^2(x_1))q_1^{\varphi_t} \Psi_t \rangle +i \langle \Psi_t, q_1^{\varphi_t} (v_2^N (x_2 - x_1) - v \ast |\varphi_t|^2(x_1)) \Psi_t \rangle \]

Using selfadjointness of the multiplication operators the first and third summand cancel out and we get

\[ |\alpha'_t| \leq 2 \langle \Psi_t, p_1^{\varphi_t} ((N - 1)v_j^N (x_2 - x_1) - v \ast |\varphi_t|^2(x_1))q_1^{\varphi_t} \Psi_t \rangle \]

Using \( \langle \Psi, p_2^{\varphi_t} v_j^N (x_1 - x_2)p_2^{\varphi_t} \Psi \rangle = \langle \Psi, (v \ast |\varphi_t|^2)(x_1)p_2^{\varphi_t} \Psi \rangle \) and Lemma \[2.3\] (d)

\[
|\alpha'_t| \leq 2 |\langle \Psi_t, p_1^{\varphi_t} p_2^{\varphi_t} ((N - 1)v_j^N (x_2 - x_1) - v \ast |\varphi_t|^2(x_1))q_1^{\varphi_t} q_2^{\varphi_t} \Psi_t \rangle |
\]

\[
+ 2 |\langle \Psi_t, p_1^{\varphi_t} q_2^{\varphi_t} ((N - 1)v_j^N (x_2 - x_1) - v \ast |\varphi_t|^2(x_1))q_1^{\varphi_t} p_2^{\varphi_t} \Psi_t \rangle |
\]

\[
+ 2 |\langle \Psi_t, q_1^{\varphi_t} q_2^{\varphi_t} ((N - 1)v_j^N (x_2 - x_1) - v \ast |\varphi_t|^2(x_1))q_1^{\varphi_t} q_2^{\varphi_t} \Psi_t \rangle |
\]

\[
\leq 2 \left( \|p_1^{\varphi_t} q_2^{\varphi_t} \Psi_t\|^2 + \|p_1^{\varphi_t} q_2^{\varphi_t} \Psi_t\|^2 \right) \left( \|v_j^N\|_\infty + \|v \ast |\varphi_t|^2\|_\infty \right) \right) .
\]

Remember that in the case \( \beta = 0 \) the scaling is such that \( v_j^N = N^{-1} v \), thus

\[ \|v_j^N\|_1 = N^{-1} \|v\|_1 \text{ and } \|v_j^N\|_\infty = N^{-1} \|v\|_\infty . \]

Note also that \( \frac{\sqrt{N^2 - 1}}{\sqrt{N}} < \frac{\sqrt{N}}{\sqrt{N}} + \frac{2}{\sqrt{N}} \)
and thus \( \tilde{N}^{\varphi_t} \leq \tilde{N}^{\varphi_t} + \frac{2}{\sqrt{N}} \). It follows that

\[ |\alpha'_t| \leq C \left( \|p_1^{\varphi_t} q_2^{\varphi_t} \Psi_t\|^2 + \|p_1^{\varphi_t} q_2^{\varphi_t} \Psi_t\|^2 \right) \left( \tilde{N}^{\varphi_t} \right) \left( \tilde{N}^{\varphi_t} \right) 
\]

Using Lemma \[2.3\] (c) it follows in view of \[8\] that one can find a \( C < \infty \) such that

\[ |\alpha'_t| \leq C \alpha + CN^{-1/2} , \]

thus by Gronwalls Lemma \( \alpha_t \to 0 \) for \( N \to \infty \) uniform in \( t < T \) (under the assumptions above, in particular \( \alpha_0 \to 0 \) for \( N \to \infty \)).
4 Derivation of the Gross-Pitaevskii equation

Let us now consider the case $0 < \beta \leq 1$. Then (3) becomes the Gross Pitaevskii equation

$$i \frac{d}{dt} \varphi^G_t = (-\Delta + A_t) \varphi^G_t + 2a|\varphi^G_t|^2 \varphi^G_t := h^G \varphi^G_t.$$  

(9)

The respective Gross Pitaevskii energy is given by

$$E^G_t = E^{kin} + E^{pot} := \langle \nabla \varphi^G_t, \nabla \varphi^G_t \rangle + \langle \varphi^G_t, (A_t + a|\varphi^G_t|^2) \varphi^G_t \rangle$$

$$= \langle \varphi^G_t, (h^G - a|\varphi^G_t|^2) \varphi^G_t \rangle.$$  

(10)

To control $\langle \Psi_t, \hat{n} \varphi^G_t \Psi_t \rangle$, the solutions $\varphi^G_t$ of the Gross Pitaevskii equation we shall consider have to satisfy some additional conditions. If we have in addition sufficiently strong decay conditions on $\varphi^G_t$ in $t$ we can even get control on the respective $\alpha_t$ uniform in $t < \infty$. Therefore we shall define next the sets $\mathcal{G}$ and $\mathcal{G}_{dec}$ of solutions of (9) which satisfy these conditions.

**Definition 4.1**

$$\mathcal{G} := \{ \varphi^G_t : i \frac{d}{dt} \varphi^G_t = h^G \varphi^G_t ; \| \varphi^G_t \|_\infty + \| \nabla \varphi^G_t \|_\infty + \| \Delta \varphi^G_t \|_\infty < \infty \ \forall \ t \geq 0 \}$$

and

$$\mathcal{G}_{dec} := \{ \varphi^G_t \in \mathcal{G} : \int_0^\infty \| \varphi^G_t \|_\infty + \| \nabla \varphi^G_t \|_\infty \, dt < \infty \}$$

Furthermore we shall — depending on $\beta$ — need some conditions on the interaction $v^N_\beta$. These conditions shall include the potentials we used in the introduction, i.e. potentials which scale like $v^N_\beta(x) = N^{-1+3\beta} v(N^\beta x)$ for a compactly supported, spherically symmetric, positive potential $v \in L^1 \cap L^\infty$.

**Definition 4.2** For any $0 < \beta \leq 1$ let

$$\mathcal{W}_\beta := \{ v^N_\beta \text{ pos. and spher. symm. } v^N_\beta (x) = 0 \ \forall \ x > RN^{-\beta} \text{ for some } R < \infty \} .$$

For any $0 < \beta < 1$ let

$$\mathcal{V}_\beta := \{ v^N_\beta \in \mathcal{W}_\beta : \lim_{N \to \infty} N^{1-3\beta} \| v^N_\beta \|_\infty < \infty ; \lim_{N \to \infty} N^{-1+\delta} (\| v^N_\beta \|_1 - a/N) < \infty \text{ for some } \delta > 0 \}$$

and let

$$\mathcal{V}_1 := \{ v^N_\beta \in \mathcal{W}_1 : \lim_{N \to \infty} N^{1+\delta} (\text{scat}(v^N_\beta) - a/N) < \infty \text{ for some } \delta > 0 \} ,$$

where $\text{scat}(v)$ is the scattering length of the potential $v$.

With these definitions we arrive at the main Theorem:
Theorem 4.3 Let $0 < \beta \leq 1$, let $v^N_\beta \in V_\beta$ and let $\varphi^GP_t \in G$. Let $T < \infty$ ($T \leq \infty$ if $\varphi^GP_t \in G_{dec}$). Let $A_t$ be such that $\int^T_0 \|A_t\| dt < \infty$. Let $\Psi_0$ be symmetric with $\|\Psi_0\| = 1$,

$$\lim_{N \to \infty} N^\delta \left\langle \Psi_0, \left(\hat{n}^{GP}_t\right)^2 \Psi_0 \right\rangle = 0$$  \hspace{1cm} (11)

and

$$\lim_{N \to \infty} N^\delta (N^{-1} \langle \Psi_0, H \Psi_0 \rangle - E^GP_t) = 0$$ \hspace{1cm} (12)

for some $\delta > 0$. Then

$$\lim_{N \to \infty} \left\langle \Psi_t, \left(\hat{n}^{GP}_t\right)^2 \Psi_t \right\rangle = 0$$ \hspace{1cm} (13)

uniform in $0 < t < T$.

Remark 4.4 (a) Lemma 2.4 implies convergence of the reduced one-particle density matrix.

(b) For $\beta = 1$ the conditions on $v^N_\beta$ include the hard sphere case (and potentials which scale like $v^N_\beta = N^\gamma v(Nx)$ for any $\gamma > 2$) with compactly supported $v$ with support-radius $a$: Such potentials satisfy all conditions one needs to be in $W_1$ and the respective scattering length equals $a/N$ (converges against $a/N$) as $N \to \infty$.

(c) It has been proven for a large class of external potentials that the $N$-particle ground state wave function $\Psi$ satisfies the conditions (11) and (12) \cite{5,6,7}. So the Theorem fits well for describing the physics of a trapped, cooled Bose gas when the trap is removed.

(d) Condition (12) can be understood as smoothness condition on $\Psi_0$. For the case $0 < \beta < 1$ this is clear on a heuristic level: If all particles of $\Psi_0$ are more or less equal to $\varphi^GP_t$ and if $\Psi_0$ is smooth enough, then the energy of $\Psi_0$ is of course close to $N E^GP_t$.

For $\beta = 1$ note, that $L^2$ density arguments can be used, i.e. if (13) holds for some $\Psi_0$, then it also holds for a sequence $\Psi^N_0$ which converges in $L^2$ against $\Psi_0$. Thus we can equip $\Psi_0$ with a microscopic structure which does not change the $L^2$ norm significantly in such a way, that the energy gets close to $N E^GP_t$.

With the technique we shall present in this paper this can be done rigorously.

4.1 Proof of the Theorem

Notation 4.5 In the following all projectors shall be with respect to $\varphi^GP_t$. We shall omit the upper index $\varphi^GP_t$ on $p_j$, $q_j$, $P_j$, $P_{j,k}$ and $\hat{\cdot}$.
Note that due to Lemma 2.4 (a) we have some flexibility in choosing which term we wish to control: To prove the Theorem we can choose to control \( \langle \Psi_t, \hat{n} \rangle \Psi_t \) for arbitrary \( \gamma > 0 \). We shall use \( \gamma = 1 \) since we shall estimate the kinetic energy (see Lemma 5.4 below) in terms of \( \langle \Psi_t, \hat{n} \rangle \Psi_t \).

**Definition 4.6** Using the notation

\[
h_{j,k} := (N - 1)v_j^N (x_j - x_k) - \frac{a}{2} |\varphi_t^{GP}|^2(x_j) - \frac{a}{2} |\varphi_t^{GP}|^2(x_k)
\]

we define the functional \( \alpha : L^2(\mathbb{R}^3N \to \mathbb{C}) \to \mathbb{R}^+ \) by

\[
\alpha(\Psi) := \langle \Psi, \hat{n} \rangle = \| (\hat{n})^{1/2} \Psi \|
\]

and the functionals \( \alpha'_1, \alpha'_2 : L^2(\mathbb{R}^3N \to \mathbb{C}) \to \mathbb{R}^+ \) by

\[
\begin{align*}
\alpha'_1(\Psi) &= N \Im (\langle \Psi, h_{1,2}(\hat{n} - \hat{n}_2)p_1p_2 \rangle) \\
\alpha'_2(\Psi) &= N \Im (\langle \Psi, h_{1,2}(\hat{n} - \hat{n}_1)p_1q_2 \rangle).
\end{align*}
\]

**Lemma 4.7** For any solution of the Schrödinger equation \( \Psi_t \), we have

\[
\frac{d}{dt} \alpha(\Psi_t) = 2\alpha'_1(\Psi_t) + 4\alpha'_2(\Psi_t).
\]

**Proof:** We have for \( 0 < \beta \leq 1 \) for the time derivative

\[
\frac{d}{dt} \alpha(\Psi) = \frac{d}{dt} \langle \Psi, \hat{n} \rangle \\
= -i \langle H^{\beta} \Psi, \hat{n} \rangle + i \langle \Psi, H^{\beta} \hat{n} \Psi \rangle + i \langle \Psi, [H_t^{GP}, \hat{n}] \Psi \rangle \\
= -i \langle \Psi, [H^{\beta} - H_t^{GP}, \hat{n}] \Psi \rangle.
\]

Using symmetry of \( \Psi \) it follows that

\[
\frac{d}{dt} \alpha(\Psi) = -i(N - 1)^{-1} \sum_{j \neq k} \langle \Psi, [h_{j,k}, \hat{n}] \Psi \rangle \\
= -iN \langle \Psi, h_{1,2} \hat{n} \rangle - \langle \Psi, \hat{n} h_{1,2} \Psi \rangle = 2N \Im (\langle \Psi, h_{1,2} \hat{n} \rangle \Psi).
\]
Note that we can write for any \( m : \{1, \ldots, N\} \to \mathbb{R}^+ \)
\[
\hat{m} = \sum_{k=0}^{N} m(k) P_k \tag{17}
\]
\[
= \sum_{k=0}^{N-2} (m(k)p_1 p_2 P_{N-2,k} + m(k)q_1 q_2 P_{N-2,k-1})
+ m(k)q_1 p_2 P_{N-2,k-1} + m(k)(1 - p_1 q_2 - q_1 p_2) P_{N-2,k-2}
\]
\[
= \sum_{k=0}^{N} (m(k)p_1 p_2 P_{N-2,k} + m(k)q_1 p_2 P_{N-2,k-1})
+ \sum_{k=0}^{N} m(k+1)p_1 p_2 P_{N-2,k-1} - m(k+1)q_1 p_2 P_{N-2,k-1}
- \sum_{k=0}^{N} m(k+2)p_1 p_2 P_{N-2,k}
\]
\[
= (\hat{m} - \hat{m}_2)p_1 p_2 + (\hat{m} - \hat{m}_1)q_1 q_2 + (\hat{m} - \hat{m}_1)q_1 p_2 + \sum_{k=0}^{N} m(k)P_{N-2,k-2}
\]
Using symmetry of \( \Psi \) and selfadjointness of \( h_{1,2} P_{N-2,k-2} \) it follows that
\[
\frac{d}{dt} \alpha(\Psi) = \Im \left( \langle \Psi, h_{1,2} (N(\hat{n} - \hat{n}_2)p_1 p_2 + 2(\hat{n} - \hat{n}_1)p_1 q_2) \Psi \rangle \right).
\]

\[\square\]

5 The Gross Pitaevskii equation for \( 0 < \beta < 1/3 \)

In this section we shall control \( \alpha_{1,\Psi} \) and \( \alpha_{2,\Psi} \) under additional conditions on \( \beta \), namely \( \beta < 1/3 \) for \( \alpha_{1,\Psi} \) and \( \beta < 1 \) for \( \alpha_{2,\Psi} \).

**Lemma 5.1** We have under the conditions of Theorem 4.3 that there exists a \( C < \infty \) and a \( \xi > 0 \) such that for any \( \Psi \in L^2(\mathbb{R}^{3N} \to \mathbb{C}) \) with \( \nabla_1 \Psi \in L^2(\mathbb{R}^{3N} \to \mathbb{C}) \) that

(a) for \( 0 < \beta < 1/3 \)
\[
|\alpha'(\Psi)| \leq C(\|\varphi_G^{GP}\|_\infty + \|\nabla \varphi_G^{GP}\|_\infty)(\alpha(\Psi) + N^{-\xi})
\]

(b) for \( 0 < \beta < 1 \)
\[
|\alpha'_{2}(\Psi)| \leq C(\|\varphi_{1,2}^{GP}\|_\infty + \|\nabla \varphi_{1,2}^{GP}\|_\infty)(\alpha(\Psi) + \|\nabla_1 q_1 \Psi\|) + N^{-\xi})
\]

Proof: Using (14) and 1 = p_1p_2 + p_1q_2 + q_1p_2 + q_1q_2

\[ \alpha'_1(\Psi) = \Im (\langle \Psi, p_1p_2h_{1,2}N(\tilde{n} - \tilde{n}_2)p_1p_2\Psi \rangle) + \Im (\langle \Psi, p_1q_2h_{1,2}N(\tilde{n} - \tilde{n}_2)p_1p_2\Psi \rangle) + \Im (\langle \Psi, q_1q_2h_{1,2}N(\tilde{n} - \tilde{n}_2)p_1p_2\Psi \rangle) \]

\[ \alpha'_2(\Psi) = \Im (\langle \Psi, p_1p_2h_{1,2}N(\tilde{n} - \tilde{n}_1)p_1q_2\Psi \rangle) + \Im (\langle \Psi, q_1q_2h_{1,2}N(\tilde{n} - \tilde{n}_1)p_1q_2\Psi \rangle) \]

Using that \( \Im (\langle \Psi, A\Psi \rangle) = -\Im (\langle \Psi, A^{\dagger}\Psi \rangle) \) for any operator \( A \) and that \( \Psi \) is symmetric (note that \( p_1q_2h_{1,2}q_1p_2 \) is invariant under adjunction plus exchange of the variable \( x_1 \) and \( x_2 \)) and Lemma 5.3 (dc) we get

\[ \alpha'_1(\Psi) = 2\Im (\langle \Psi, p_1q_2h_{1,2}N(\tilde{n} - \tilde{n}_2)p_1p_2\Psi \rangle) + \Im (\langle \Psi, q_1q_2h_{1,2}N(\tilde{n} - \tilde{n}_2)p_1p_2\Psi \rangle) \]

\[ \alpha'_2(\Psi) = \Im (\langle \Psi, N(\tilde{n} - \tilde{n}_2)p_1p_2h_{1,2}p_1q_2\Psi \rangle) + \Im (\langle \Psi, q_1q_2h_{1,2}N(\tilde{n} - \tilde{n}_1)p_1q_2\Psi \rangle) \]

Note that

\[ \sqrt{k/N} - \sqrt{(k-2)/N} = (k/N - (k-2)/N) / \left( \sqrt{k/N} + \sqrt{(k-2)/N} \right) \]

\[ \leq \frac{2/N}{\sqrt{k/N}} = 2(Nk)^{-1/2}, \]

so we have that \( 0 \leq (\tilde{n} - \tilde{n}_1), (\tilde{n}_1 - \tilde{n}_2) \leq (\tilde{n} - \tilde{n}_2) \leq 2(N\tilde{n})^{-1} \) and Lemma 5.1 follows from

**Lemma 5.2** Let \( m : \{1, \ldots, N\} \to \mathbb{R}^+ \) with \( m \leq n^{-1} \), \( 0 < \beta < 1 \). Then we have under the conditions of the Theorem that there exists a \( C < \infty \) and a \( \xi > 0 \) such that

(a) for any \( 0 < \beta < 1 \)

\[ |\langle \Psi_t, p_1p_2h_{1,2}\tilde{n}q_1p_2\Psi_t \rangle| \leq C(\| \varphi_{t,\xi}^{GP} \|_\infty + \| \nabla \varphi_{t,\xi}^{GP} \|_\infty)N^{-\xi} \]

\[ |\langle \Psi_t, p_1q_2h_{1,2}\tilde{n}^{1/2}q_1q_2\Psi_t \rangle| \leq C(\| \varphi_{t,\xi}^{GP} \|_\infty + \| \nabla \varphi_{t,\xi}^{GP} \|_\infty)\alpha(\Psi_t) + N^{-\gamma} + \| \nabla_1 q_1 \Psi \|^2 \]

(b) for any \( 0 < \beta < 1/3 \)

\[ |\langle \Psi_t, p_1p_2h_{1,2}\tilde{n}^{1/2}q_1q_2\Psi_t \rangle| \leq C(\| \varphi_{t,\xi}^{GP} \|_\infty + \| \nabla \varphi_{t,\xi}^{GP} \|_\infty)\alpha(\Psi_t) + N^{-\xi} \]

The proof of Lemma 5.2 shall be given in the Appendix for later reference in a more general form.

\[ \square \]

### 5.1 Control of the kinetic energy for \( \beta < 1 \)

To finish the control of \( \alpha(\Psi_t) \) we shall provide a sufficient estimate on the kinetic energy of \( \Psi_t \), in particular \( \| \nabla_1 q_1 \Psi_t \| \). This estimate shall be given in terms of \( \alpha(\Psi_t) \), thus finally our estimate on \( \alpha'(\Psi_t) \) shall depend on \( \alpha(\Psi_t) \) making \( \alpha(\Psi_t) \) controllable by a Gronwall argument. For that we need
Lemma 5.3 Let \( m : \{1, \ldots, N\} \to \mathbb{R}^+ \) with \( m \leq n^{-1} \), \( 0 < \beta < 1 \). Then we have under the conditions of the Theorem that there exists a \( C < \infty \) and a \( \xi > 0 \) such that for any \( 0 < \beta < 1 \)

\[
|\langle \Psi, p_1p_2 ((N-1)v_\beta^N(x_1,x_2) - a|\varphi_t^{GP}|^2(x_1)) p_1p_2\Psi \rangle| \leq C(\|\varphi_t^{GP}\|_\infty + \|\nabla\varphi_t^{GP}\|_\infty)N^{-\xi}
\]

and

\[
|\langle \Psi, p_1p_2v_\beta^N(x_1,x_2)q_1q_2\Psi \rangle| \leq CN^{-1}(\|\varphi_t^{GP}\|_\infty + \|\nabla\varphi_t^{GP}\|_\infty)(\alpha(\Psi) + N^{-\xi}) .
\]

The proof of which shall be given together with the proof of Lemma 5.2 in the Appendix.

Lemma 5.4 Let \( 0 < \beta < 1 \). Then we have under the conditions of Theorem 4.3 that there exists a \( \xi > 0 \) such that uniform in \( 0 < t < T \)

\[
\|\nabla q_1\Psi\|^2 \leq C \left( \sup_{0 \leq s \leq t} \{ \alpha(\Psi_s) \} + N^{-\xi} \right)
\]

Proof: Using symmetry of \( \Psi_t \)

\[
N^{-1}\langle \Psi_t, H\Psi_t \rangle = -\|\nabla_1\Psi_t\|^2 + (N-1)\langle \Psi_t, v_\beta^N(x_1-x_2)\Psi_t \rangle + \langle \Psi_t, A_t(x_1)\Psi_t \rangle ,
\]

Thus

\[
\|\nabla\varphi_t^{GP}\|^2 - \|\nabla_1\Psi_t\|^2 = N^{-1}\langle \Psi_t, H\Psi_t \rangle - E_t^{GP} - \langle \Psi_t, (A_t(x_1) + a|\varphi_t^{GP}|^2(x_1))\Psi_t \rangle + \langle \Psi_t, (N-1)v_\beta^N(x_1-x_2) - a|\varphi_t^{GP}|^2(x_1)\rangle \Psi_t \rangle .
\]

Using symmetry of \( \Psi_t \)

\[
\frac{d}{dt} (N^{-1}\langle \Psi_t, H\Psi_t \rangle - E_t^{GP}) = \langle \Psi_t, A'_t(x_1)\Psi_t \rangle - \langle \varphi_t^{GP}, a \left( \frac{d}{dt}|\varphi_t^{GP}|^2 \right) \varphi_t^{GP} \rangle - \langle \varphi_t^{GP}, A'_t\varphi_t^{GP} \rangle - \langle \varphi_t^{GP}, (h^{GP} - a|\varphi_t^{GP}|^2)h^{GP}\varphi_t^{GP} \rangle = \langle \Psi_t, A'_t(x_1)\Psi_t \rangle - \langle \varphi_t^{GP}, A'_t\varphi_t^{GP} \rangle + \langle \varphi_t^{GP}, [a|\varphi_t^{GP}|^2, h^{GP}]\varphi_t^{GP} \rangle - \langle \varphi_t^{GP}, [a|\varphi_t^{GP}|^2, h^{GP}]\varphi_t^{GP} \rangle \leq 4\|A'_t\|_{\infty} \left( N^{-1/4} + \alpha(\Psi_t) \right) ,
\]

where we used Lemma 2.3 (e) in the last step. It follows using condition \( 12 \) that for \( N \) sufficiently small (i.e. such that \( N^\delta (N^{-1}\langle \Psi_0, H\Psi_0 \rangle - E_0^{GP}) < 1 \))

\[
(N^{-1}\langle \Psi_t, H\Psi_t \rangle - E_t^{GP}) \leq N^{-\delta} + \int_0^t \|A'_s\|_{\infty} \left( \frac{2}{N} \right)^{1/4} + \alpha(\Psi_s) + \sup_{0 \leq s \leq t} \{ \alpha(\Psi_s) \}) ds
\]

\[
\leq C(N^{-\delta} + N^{-1/4} + \sup_{0 \leq s \leq t} \{ \alpha(\Psi_s) \})
\]
uniform in $t < T$. Note that due to Lemma 5.2 (e)

$$\langle \Psi_t, (A_t(x_1) + a|\varphi_t^{GP}|^2(x_1)) \Psi_t \rangle - \langle \varphi_t^{GP}, (A_t + a|\varphi_t^{GP}|^2) \varphi_t^{GP} \rangle \leq 4(\|A_t\|_\infty + a\|\varphi_t^{GP}\|_\infty^2) \left( N^{-1/4} + a(\Psi_t) \right),$$

thus

$$\|\nabla_1 \Psi_t\|^2 - \|\nabla \varphi_t^{GP}\|^2 \leq C(N^{-\delta} + N^{-1/4} + \sup_{0 \leq s \leq t} \alpha(\Psi_s)) \quad (18)$$

We get using symmetry of $\Psi_t$ and self adjointness of the multiplication operators for the last summand in (18)

$$\langle \Psi_t, (N-1)v_N^\beta (x_1 - x_2) - a|\varphi_t^{GP}|^2(x_1) \rangle \Psi_t \rangle$$

$$= \langle p_1 p_2 \Psi_t, (N-1)v_N^\beta (x_1 - x_2) - a|\varphi_t^{GP}|^2(x_1) \rangle p_1 p_2 \Psi_t \rangle$$

$$+ 2R \langle p_1 p_2 \Psi_t, (N-1)v_N^\beta (x_1 - x_2) - a|\varphi_t^{GP}|^2(1 - p_1 p_2) \Psi_t \rangle$$

$$+(N-1) \langle (1 - p_1 p_2) \Psi_t, v_N^\beta (x_1 - x_2)(1 - p_1 p_2) \Psi_t \rangle$$

$$- a \langle (1 - p_1 p_2) \Psi_t, |\varphi_t^{GP}|^2(1 - p_1 p_2) \Psi_t \rangle. \quad (19)$$

Using symmetry of $\Psi_t$, the absolute value of the second term is bounded by

$$4(N-1) \left| \langle p_1 p_2 \Psi_t, (N-1)v_N^\beta (x_1 - x_2) - a|\varphi_t^{GP}|^2 \right| p_1 q_2 \Psi_t \rangle \right|$$

$$+ 2(N-1) \left| \langle p_1 p_2 \Psi_t, v_N^\beta (x_1 - x_2)q_1 q_2 \Psi_t \rangle \right|$$

Using Lemma 5.2 in its more general form as given in the Appendix and using positivity of $v_N^\beta$ (implying positivity of line (19)) we get that

$$\langle \Psi_t, \left( \sum_{j \neq 1} v_j^\beta \right) \Psi_t \rangle$$

$$\geq -C(\alpha(\Psi_t) + N^{-\delta} - a|\varphi_t^{GP}|^2 \|1 - p_1 p_2\|_\Psi_t \|^2.$$

Writing $1 - p_1 p_2 = p_1 q_2 + q_1 p_2 + q_1 q_2$ Lemma 2.3 yields

$$\langle \Psi_t, \left( \sum_{j \neq 1} v_j^\beta \right) \Psi_t \rangle \geq -C(\alpha(\Psi_t) + N^{-\delta} + N^{-1/4} \sum_{0 \leq s \leq t} \alpha(\Psi_s)) \leq -C(\alpha(\Psi_t) + N^{-\delta} + N^{-1/4} + \sum_{0 \leq s \leq t} \alpha(\Psi_s)).$$

(20)

so with (18)

$$\|\nabla \Psi_t\|^2 - \|\nabla \varphi_t^{GP}\|^2 \leq C(N^{-\delta} + N^{-1/4} + \sup_{0 \leq s \leq t} \alpha(\Psi_s)) + N^{-\delta} + N^{-1/4} + \sum_{0 \leq s \leq t} \alpha(\Psi_s).$$

Note also that $\|\nabla \Psi_t\|^2 = \|\nabla p_1 \Psi_t\|^2 + \|\nabla q_1 \Psi_t\|^2$ and

$$\|\nabla p_1 \Psi_t\|^2 - \|\nabla q_1 \Psi_t\|^2 = \|\nabla \varphi_t^{GP}\|^2(\|p_1 \Psi_t\|^2 - 1) = \|\nabla \varphi_t^{GP}\|^2\|q_1 \Psi_t\|^2 \leq \|\nabla \varphi_t^{GP}\|^2 \|\nabla \varphi_t^{GP}\|^2 \alpha(\Psi_t).$$

Choosing $\xi \leq \min\{\delta, 1/4, \beta, 1 - \beta\}$ Lemma 5.4 follows.

\[ \square \]
5.2 Proof of Theorem 4.3 for $\beta < 1/3$

Lemma 4.7 with Lemma 5.1 and Lemma 5.4 gives
\[ |\alpha'(\Psi_t)| \leq C(||\varphi^G_t||_{\infty} + ||\nabla \varphi^G_t||_{\infty})(\sup_{0 \leq s \leq t}\{\alpha(\Psi_s)\} + N^{-\xi}). \] (21)

We shall use a Gronwall argument to control $\alpha(\Psi_t)$: Consider the differential equation
\[ \gamma'_t = C(||\varphi^G_t||_{\infty} + ||\nabla \varphi^G_t||_{\infty})(\sup_{0 \leq s \leq t}\{\gamma_s\} + N^{-\xi}). \] (22)

Since the right hand side of (22) is positive, the solution $\gamma_t$ with $\gamma_0 = \alpha(\Psi_0)$ dominates $\alpha(\Psi_t)$. Moreover $\gamma_t$ increases monotonously, thus $\sup_{0 \leq s \leq t}\{\gamma_s\} = \gamma_t$ and
\[ \gamma'_t = C(||\varphi^G_t||_{\infty} + ||\nabla \varphi^G_t||_{\infty})(\gamma_t + N^{-\xi}). \]

It follows that
\[ \ln(\gamma_t + N^{-\xi}) = C \int_0^t (||\varphi^G_s||_{\infty} + ||\nabla \varphi^G_s||_{\infty}) ds + K_N \]
where the integration constant $K_N$ is such that $\gamma_0 = \alpha(\Psi_0)$, i.e.
\[ \gamma_t = C_N \exp\left(C \int_0^t ||\varphi^G_s||_{\infty} + ||\nabla \varphi^G_s||_{\infty} ds\right) - N^{-\xi} \]
where $C_N = e^{K_N}$. Note that Lemma 2.4 implies with (11) that $\lim_{N \to \infty} \alpha(\Psi_0) = 0$, thus $\lim_{N \to \infty} C_N = 0$ and Theorem 4.3 follows for $0 < \beta < 1$.

6 The Gross Pitaevskii equation for $1/3 \leq \beta \leq 1$

6.1 Microscopic Structure

Definition 6.1 Let $0 < \beta_1 < \beta_2 < 1$, $v^N_{\beta_2} \in V_{\beta_2}$. We define the potential $W^N_{\beta_1, \beta_2}$ via
\[ W^N_{\beta_1, \beta_2}(x) := \begin{cases} aN^{-1+3\beta_1}, & \text{for } RN^{-\beta_2} < x < R^N_{\beta_1, \beta_2}, \\ 0, & \text{else.} \end{cases} \]

Here $RN^{-\beta_2}$ is an upper bound on the radius of the support of $v^N_{\beta_2}$ (see Definition 4.3) and $R^N_{\beta_1, \beta_2}$ is the minimal value which ensures that the scattering length of $v^N_{\beta_2} - W^N_{\beta_1, \beta_2}$ is zero.

The respective zero energy scattering state shall be denoted by $f^N_{\beta_1, \beta_2}$, i.e.
\[ (-\Delta + v^N_{\beta_2} - W^N_{\beta_1, \beta_2}) f^N_{\beta_1, \beta_2} = 0, \]
we shall also need
\[ g^N_{\beta_1, \beta_2} = 1 - f^N_{\beta_1, \beta_2} \]
Lemma 6.2 For any $0 < \beta_1 < \beta_2 \leq 1$, $v_{\beta_2}^N \in \mathcal{V}_{\beta_2}$

(a) \[ \| g_{\beta_1, \beta_2}^N \| \leq \sqrt{8\pi a N^{-1-\beta_1/2}} \quad \| g_{\beta_1, \beta_2}^N \|_1 \leq 16\pi a N^{-1-2\beta_1} \]

(b) \[ W_{\beta_1, \beta_2}^N f_{\beta_1, \beta_2}^N \in \mathcal{V}_{\beta_1} \]

(c) The operator $h := -\Delta + v_{\beta_2}^N - W_{\beta_1, \beta_2}^N$ is positive.

(d) For any $\beta < \gamma < 1$ let $B_{\gamma} := \{ x \in \mathbb{R}^3 : |x| \leq N^{-\gamma} \}$. Then for any $\Psi \in \mathcal{D}(H)$
\[ \| 1_{B_{\gamma}} \nabla \Psi \| + \langle \Psi, (v_{\beta_2}^N - W_{\beta_1, \beta_2}^N) \Psi \rangle \geq 0 \]

Proof: Let $j_{\beta_2}^N$ be the zero energy scattering state of the potential $\frac{1}{2}v_{\beta_2}^N$. Since $v_{\beta_2}^N$ is positive and has compact support of radius $r_N$ it follows, that $1 > j_{\beta_2}^N(x) \geq 1 - a/(Nx)$ for any $x \geq r_N$. Note, that the potential $W_{\beta_1, \beta_2}^N$ is zero inside the Ball around zero of radius $RN^{-\beta_2}$, hence $f_{\beta_1, \beta_2}^N$ is inside this Ball a multiple of $j_{\beta_2}^N$.

Let $R_{\beta_1, \beta_2}^N$ be such that $K_{\beta_1, \beta_2}^N f_{\beta_1, \beta_2}^N(x) = j_{\beta_2}^N(x)$ for any $x < N^{-\beta_1}$. By definition of the potential $W_{\beta_1, \beta_2}^N$ we have that $\partial_r f_{\beta_1, \beta_2}^N(x) \geq 0$: $R_{\beta_1, \beta_2}^N$ was defined to be the minimal value which ensures that the scattering length of $v_{\beta_2}^N - W_{\beta_1, \beta_2}^N$ is zero, thus $\partial_r f_{\beta_1, \beta_2}^N(r) \geq 0$ for $r < R_{\beta_1, \beta_2}^N$. It follows in particular that $f_{\beta_1, \beta_2}^N \leq 1$. Furthermore we have, since $W_{\beta_1, \beta_2}^N$ is positive, that $K_{\beta_1, \beta_2}^N \partial_r f_{\beta_1, \beta_2} \leq \partial_r j_{\beta_2}^N$ and $K_{\beta_1, \beta_2}^N f_{\beta_1, \beta_2} \leq j_{\beta_2}^N$.

Since $f_{\beta_1, \beta_2}^N(x) = 1$ for $x > 2N^{-\beta_1}$ and $\lim_{x \to \infty} j_{\beta_2}^N(x) = 1$ we get that $K_{\beta_1, \beta_2}^N \leq 1$, thus $1 > f_{\beta_1, \beta_2}^N \geq j_{\beta_2}^N$. Since $j_{\beta_2}^N(x) \geq 1 - a/(Nx)$ it follows that
\[ |g_{\beta_1, \beta_2}^N(x)| \leq a/(Nx). \] (23)

Since $g_{\beta_1, \beta_2}^N(x) = 0$ for $x > 2N^{-\beta_1}$ it follows that
\[ \| g_{\beta_1, \beta_2}^N \|^2 \leq a^2 N^{-2} \int_0^{2N^{-\beta_1}} |x|^{-2} d^3 x = 8 N^{-\beta_1} \pi a^2 N^{-2} \]
\[ \| g_{\beta_1, \beta_2}^N \|_1 \leq a N^{-1} \int_0^{2N^{-\beta_1}} |x|^{-1} d^3 x = 16 N^{-2\beta_1} \pi a N^{-1} \]

which is (a).

Next we have to show that $W_{\beta_1, \beta_2}^N f_{\beta_1, \beta_2}^N \in \mathcal{V}_{\beta_1}$.

Since $K_{\beta_1, \beta_2}^N f_{\beta_1, \beta_2}^N > 1 - a N^{-1-\beta_1}$ on the support of $W_{\beta_1, \beta_2}^N$
\[ \| W_{\beta_1, \beta_2}^N f_{\beta_1, \beta_2}^N \|_1 < \| W_{\beta_1, \beta_2}^N f_{\beta_1, \beta_2}^N \|_1 < \| W_{\beta_1, \beta_2}^N \|_1 \]
\[ < (1 - a N^{-1+\beta_1})^{-1} \| W_{\beta_1, \beta_2}^N K_{\beta_1, \beta_2}^N f_{\beta_1, \beta_2}^N \|_1 \]
\[ < (1 - a N^{-1+\beta_1})^{-1} \| W_{\beta_1, \beta_2}^N f_{\beta_1, \beta_2}^N \|_1 . \] (24)
Next we shall control the kinetic energy ∥∇ᵢΨ∥₁ = a/N: Read ρ³ᵢ := Wᵢ²⁺θᵢ β₁,β₂ ᶠ²⁺θᵢ β₁,β₂ as a classical charge distribution which must compensate the charge a/N (recall that Kᵢ β₁,β₂ ᶠ²⁺θᵢ β₁,β₂ (x) = 1 − a/(Nx) for rᵢ < x < N⁻β) to get that the potential \( \varphi_N \equiv Kᵢ β₁,β₂ ᶠ²⁺θᵢ β₁,β₂ \) is zero outside the support of \( Wᵢ β₁,β₂ \). With (24) it follows that
\[
\lim_{N \to \infty} N^{1-β₁}(∥Wᵢ β₁,β₂ ᶠ²⁺θᵢ β₁,β₂∥₁ − a/N) < \infty \tag{25}
\]
and again using (24)
\[
\lim_{N \to \infty} N^{1-β₁}(∥Wᵢ β₁,β₂∥₁ − a/N) < \infty .
\]
It follows that the support of \( Wᵢ β₁,β₂ \) is of order \( N^{3β₁} \). Since \( Wᵢ β₁,β₂ ᶠ²⁺θᵢ β₁,β₂ \) is spherically symmetric, positive and equal to zero for \( x > Rᵢ β₁,β₂ \) it follows that \( Wᵢ β₁,β₂ ᶠ²⁺θᵢ β₁,β₂ \in Wᵢ β₁ \).

With (25) and using that \( Wᵢ β₁,β₂ \) is defined such that \( ∥Wᵢ β₁,β₂ ᶠ²⁺θᵢ β₁,β₂∥₁ < a/N^{1+3β₁} \) it follows that \( Wᵢ β₁,β₂ ᶠ²⁺θᵢ β₁,β₂ \in \mathcal{V} β₁ \).

We show (c) by contradiction. Assume that \( h \) is not positive, thus it has a ground state \( χ \). Since \( ᶠ²⁺θᵢ β₁,β₂ \) is by construction a positive function and so is the ground state \( χ \) it follows that \( ∫ ᶠ²⁺θᵢ β₁,β₂ (x)χ^*(x)d^3x > 0 \). But \( ᶠ²⁺θᵢ β₁,β₂ \) is the generalized eigenfunction of \( h \) with energy 0, so \( ∫ ᶠ²⁺θᵢ β₁,β₂ (x)χ^*(x)d^3x = 0 \) which leads to contradiction and (c) follows.

We shall also prove (d) by contradiction. Assume that there exists a \( χ \in L^2 \) such that \( ∫ (∇χ, 1_B, ∇χ) + ⟨χ, (vᵢ β⁺ - Wᵢ β⁺,β₂)χ⟩ < 0 \). Since our potential is spherically symmetric we can assume without loss of generality that \( χ \) is spherically symmetric. Defining the function \( ϕ(r) := χ(r) \) for \( r ≤ N^{−γ} \) and \( ϕ(r) = χ(N^{−γ}) \) for \( r > N^{−γ} \) it follows that
\[
⟨ϕ, hϕ⟩ = ⟨∇χ, 1_B, ∇χ⟩ + ⟨χ, (vᵢ β⁺ - Wᵢ β⁺,β₂)χ⟩ < 0 .
\]
This contradicts (c) and (d) follows.

□

6.2 Control of the kinetic energy for \( β = 1 \)

Next we shall control the kinetic energy \( ∥∇₁Ψ∥ \) for \( β = 1 \). Note that in this case, a relevant part of the kinetic energy is absorbed to form the microscopic structure. That part of the kinetic energy is concentrated around the scattering centers.

The microscopic structure can — as long as there are no three particle interactions — be controlled using Lemma 6.2 So we shall first cutoff three particle interactions without disturbing \( ∇₁Ψ \), i.e. we define a cutoff function which does not depend on \( x₁ \) and cuts off all parts of the wave function where two particles \( x_j, x_k \) with \( j ≠ k, j, k ≠ 1 \) come to close \( R₁ \) given by Definition 6.3.

After that we shall subtract that part of the kinetic energy which is used to form the microscopic structure. The latter is concentrated around the scattering centers (i.e. on the set \( S_j \) given by Definition 6.3).
Definition 6.3 For any \( j, k = \{1, \ldots, N\} \) let
\[
s_{j,k} := \{ X \in \mathbb{R}^{3N} : |x_j - x_k| < N^{-26/27}\}
\]
(26)
\[
\overline{S}_j := \bigcup_{k \neq j} s_{j,k} \quad S_j := \mathbb{R}^{3N} \setminus \overline{S}_j \quad R_{j,k} := \bigcup_{l \neq j, k} s_{k,l} \quad R_{j,k} := \mathbb{R}^{3N} \setminus R_{j,k}
\]

Proposition 6.4
\[
\|\Psi_t - \mathbb{I}_{R_j} \Psi_t\| < CN^{-7/54}.
\]

Proof: Using Hölder and Sobolev we get
\[
\|\Psi_t - \mathbb{I}_{R_j} \Psi_t\| = \|\Psi_t \mathbb{I}_{\mathbb{R}^{3N} \setminus R_{j,k}}\|^2 \leq \|\mathbb{I}_{\mathbb{R}^{3N} \setminus R_{j,k}}\|_{\infty} \|\Psi_t\|^2 \\
\leq (N-1)\|\nabla_1 \Psi_t\|^2 (NN^{-26/9})^{2/3} \leq N^{-7/27} \|\nabla_1 \Psi_t\|^2.
\]
Since \( \|\nabla_1 \Psi_t\| < C \) the Proposition follows.

\[
(27)
\]

Lemma 6.5 Let under the conditions of the Theorem \( \beta \leq 1 \). Then there exists a \( \gamma > 0 \) such that for any \( t \in \mathbb{R} \)
\[
\|\mathbb{I}_{S_j} \nabla_1 q_1 \Psi_t\| < C\alpha(\Psi_t) + \int_0^t \alpha(\Psi_s) ds + N^{-\gamma}
\]
(27)
\[
\|\mathbb{I}_{\mathbb{R}^{3N} \setminus \overline{S}_j} \sqrt{v_N} (x_1 - x_2) \Psi_t\| < CN^{-1}(\alpha(\Psi_t) + \int_0^t \alpha(\Psi_s) ds + N^{-\gamma})
\]
(28)

Proof: Below we shall use from time to time that for any \( f \in L^2 \), any \( g \in L^1 \) and any normalized \( \Psi, \chi \)
\[
\|f(x_1 - x_2)p_1 \Psi\|^2 = \langle \Psi p_1 f^2 (x_1 - x_2) p_1 \Psi \rangle \leq \|f^2\|_1 \|\varphi^{GP}\|_\infty^2
\]
and
\[
\langle \chi p_1 g(x_1 - x_2) p_1 \Psi \rangle \leq \|g\|_1 \|\varphi^{GP}\|_\infty^2
\]
Thus
\[
\|f(x_1 - x_2)p_1\|_{op} \leq \|f\|_1 \|\varphi^{GP}\|_\infty
\]
(29)
and
\[
\|p_1 g(x_1 - x_2)p_1\|_{op} \leq \|g\|_1 \|\varphi^{GP}\|_\infty^2,
\]
(30)
where \( \| \cdot \|_{op} \) stands for the operator norm
\[
\|A\|_{op} := \inf_{\|\Psi\|=1} \|A \Psi\|.
\]
Let us now prove Lemma 6.5. Recall (18)

\[ C(N^{-\delta} + N^{-1/4} + \sup_{0 \leq s \leq t} \{ \alpha(\Psi_s) \}) \]

\[ \geq \| \nabla_1 \Psi_t \|^2 - \| \nabla \varphi^{GP}_t \|^2 + \langle \Psi_t, ((N - 1)v^N_1(x_1 - x_2) - a|\varphi^{GP}_t|^2(x_1)) \Psi_t \rangle \]

\[ = \langle \nabla_1 \Psi_t, \mathbb{1}_{\mathcal{S}_1} \nabla_1 \Psi_t \rangle + \langle \nabla_1 \Psi_t, \mathbb{1}_{\mathcal{S}_1} \nabla_1 \Psi_t \rangle - E_{\text{kin}}^{GP} \]

\[ + \langle \Psi_t, \sum_{j \neq 1} \mathbb{1}_{\mathcal{R}_{1,j}} (v^N_1(x_1 - x_j) - W^N_{\beta,1}(x_1 - x_j)) \Psi_t \rangle \]

\[ + \langle \Psi_t, \left( \sum_{j \neq 1} \mathbb{1}_{\mathcal{R}_{1,j}} W^N_{\beta,1}(x_1 - x_j) - a|\varphi^{GP}_t|^2(x_1) \right) \Psi_t \rangle \]

\[ + \langle \Psi_t, \sum_{j \neq 1} \mathbb{1}_{\mathcal{R}_{1,j}} v^N_1(x_1 - x_j) \Psi_t \rangle . \]

By definition of the set \( \mathcal{S}_1 \) the support of the potentials \( v^N_1(x_1 - x_j) \) and \( W^N_{\beta,1}(x_1 - x_j) \) are subsets of \( \mathcal{S}_1 := \mathbb{R}^{3N} \setminus \mathcal{S}_1 \). Furthermore we have by definition of the set \( \mathcal{R}_{1,j} \) that the support of the potentials \( \mathbb{1}_{\mathcal{R}_{1,j}} (v^N_1(x_1 - x_j) - W^N_{\beta,1}(x_1 - x_j)) \) are pairwise disjoint for different \( j \). It follows with Lemma 6.2 (d) that

\[ \langle \nabla_1 \Psi_t, \mathbb{1}_{\mathcal{S}_1} \nabla_1 \Psi_t \rangle + \langle \Psi_t, \sum_{j \neq 1} \mathbb{1}_{\mathcal{R}_{1,j}} (v^N_1(x_1 - x_j) - W^N_{\beta,1}(x_1 - x_j)) \Psi_t \rangle \]

is positive and

\[ \langle \nabla_1 \Psi_t, \mathbb{1}_{\mathcal{S}_1} \nabla_1 \Psi_t \rangle - E_{\text{kin}}^{GP} + \langle \Psi_t, \left( \sum_{j \neq 1} \mathbb{1}_{\mathcal{R}_{1,j}} W^N_{\beta,1}(x_1 - x_j) - a|\varphi^{GP}_t|^2(x_1) \right) \Psi_t \rangle \]

\[ + \langle \Psi_t, \sum_{j \neq 1} \mathbb{1}_{\mathcal{R}_{1,j}} v^N_1(x_1 - x_j) \Psi_t \rangle \leq C(N^{-\delta} + N^{-1/4} + \sup_{0 \leq s \leq t} \{ \alpha(\Psi_s) \}) . \] (31)
For the last summand in the first line we have using positivity of $W_{\beta,1}^N$

$$\langle \Psi_t, \sum_{j \neq 1} I_{R_{1,j}} W_{\beta,1}^N(x_1 - x_j) - a|\varphi_t^{GP}(x_1)|^2 \rangle \Psi_t$$

$$= \langle \Psi_t, p_1 p_2 \left( \sum_{j \neq 1} W_{\beta,1}^N(x_1 - x_j) - a|\varphi_t^{GP}(x_1)|^2 \right) p_1 p_2 \Psi_t \rangle$$

$$- \langle \Psi_t, p_1 p_2 \sum_{j \neq 1} I_{R_{1,j}} W_{\beta,1}^N(x_1 - x_j)p_1 p_2 \Psi_t \rangle$$

$$+ 2R \left( \langle (1 - p_1 p_2) \Psi_t, \left( \sum_{j \neq 1} W_{\beta,1}^N(x_1 - x_j) - a|\varphi_t^{GP}(x_1)|^2 \right) p_1 p_2 \Psi_t \rangle \right)$$

$$+ 2R \left( \langle (1 - p_1 p_2) \Psi_t, \sum_{j \neq 1} I_{R_{1,j}} W_{\beta,1}^N(x_1 - x_j)p_1 p_2 \Psi_t \rangle \right)$$

$$+ \langle (1 - p_1 p_2) \Psi_t, \sum_{j \neq 1} I_{R_{1,j}} W_{\beta,1}^N(x_1 - x_j)((1 - p_1 p_2) \Psi_t) \rangle$$

$$- \langle (1 - p_1 p_2) \Psi_t, a|\varphi_t^{GP}(x_1)|^2((1 - p_1 p_2) \Psi_t) \rangle$$

$$= : \sum_{j=1}^{6} S_j;$$

We already got bounds on $S_1$, $S_3$ and $S_6$: All these terms appeared in $[13]$ above and could be estimated by the right hand side of Lemma 5.4 $S_5 > 0$ since $W_{1,\beta}^N$ is positive. For $S_2$ we have

$$\left| \langle \Psi_t, p_1 p_2 \sum_{j \neq 1} I_{R_{1,j}} W_{\beta,1}^N(x_1 - x_j)p_1 p_2 \Psi_t \rangle \right|$$

$$\leq (N - 1) \| p_1 W_{\beta,1}^N(x_1 - x_2)p_1 \|_{op} \| I_{R_{1,j}} \|_{2} \| \Psi_t \|^2$$

$$\leq (N - 1) \| p_1 W_{\beta,1}^N(x_1 - x_2)p_1 \|_{op} \left( \| I_{R_{1,j}} \| \| \Psi_t \| + \| I_{R_{1,j}} \| p_2 \Psi_t \| \right)^2$$

which is in view of Proposition 6.4 bounded by the right hand of (27).

For $S_4$ we have

$$2 \left| R \left( \langle (1 - p_1 p_2) \Psi_t, \sum_{j \neq 1} I_{R_{1,j}} W_{\beta,1}^N(x_1 - x_j)p_1 p_2 \Psi_t \rangle \right) \right|$$

$$\leq 2(N - 1) \| W_{\beta,1}^N(x_1 - x_2)(1 - p_1 p_2) \Psi_t \| \| I_{R_{1,j}} \| \| W_{\beta,1}^N(x_1 - x_2)p_1 p_2 \Psi_t \|$$

\[21\]
Using Hölder and Sobolev we have

\[
\left\| \sqrt{W_{\beta,1}^N}(x_1 - x_2)\Psi_t \right\|^2 \leq \|W_{\beta,1}^N\|_{3/2} \|\Psi_t\|_0^2 \\
\leq \left( \int |W_{\beta,1}^N(x)|^{3/2} dx \right)^{2/3} \|\nabla \Psi\|^2 \leq C(N^{-3/2+3\beta/2})^{2/3} = CN^{-1+\beta}.
\]

Since

\[
\|\mathbb{I}_{\mathbb{R}^3,2} \sqrt{W_{\beta,1}^N(x_1 - x_2)p_1p_2} \| \leq \|p_2\mathbb{I}_{\mathbb{R}^3,2}p_2\|_{op} \|\sqrt{W_{\beta,1}^N(x_1 - x_2)p_1}\|_2 \\
\leq \|\nabla \Psi\|^2 \leq CN^{-1} \leq CN^{-26/9}N^{-1} = CN^{-26/9},
\]

it follows with (32) that $S_4$ is bounded by the right hand side of (27). Hence

\[
\langle \nabla_1 \Psi_t, \mathbb{I}_{\mathbb{R}_1} \nabla_1 \Psi_t \rangle - E_{kin}^P + \langle \Psi_t, \sum_{j \neq 1} \mathbb{I}_{\mathbb{R}_1,j} v_j^N(x_1 - x_j)\Psi_t \rangle \\
\leq C(N^{-\delta} + N^{-1/4} + \sup_{0 \leq s \leq t} \{\alpha(\Psi_s)\}). \quad (33)
\]

For the first summand in (33) we can write

\[
\|\mathbb{I}_{\mathbb{S}_1} \nabla_1 \Psi_t\|^2 \geq \|\mathbb{I}_{\mathbb{S}_1} \nabla_1 p_1 \Psi_t\|^2 + \|\mathbb{I}_{\mathbb{S}_1} \nabla_1 q_1 \Psi_t\|^2 - 2 \langle \nabla_1 q_1 \Psi_t, \nabla_1 p_1 \Psi_t \rangle \\
- 2 \langle \nabla_1 q_1 \Psi_t, \mathbb{I}_{\mathbb{S}_1} \nabla_1 \Psi_t \rangle \\
\geq \|\nabla_1 p_1 \Psi_t\|^2 + \|\mathbb{I}_{\mathbb{S}_1} \nabla_1 q_1 \Psi_t\|^2 - \|\mathbb{I}_{\mathbb{S}_1} \nabla_1 \Psi_t\|^2 \\
+ \langle q_1 \Psi_t, \Delta_1 p_1 \Psi_t \rangle - \|\nabla_1 q_1 \Psi_t\| \|\mathbb{I}_{\mathbb{S}_1}\|_{1/2} \|\nabla_1 \varphi_t^GP\|_\infty \\
\geq \|\nabla_1 \varphi_t^GP\| \|p_1 \Psi_t\| + \|\mathbb{I}_{\mathbb{S}_1} \nabla_1 q_1 \Psi_t\|^2 \\
- \|\mathbb{I}_{\mathbb{S}_1}\|^2 \|\nabla_1 \varphi_t^GP\|_\infty^2 - \langle q_1 \Psi_t, \Delta_1 \varphi_t^GP \rangle \|p_1 \Psi_t\| \\
- \|\mathbb{I}_{\mathbb{S}_1}\|_{1/2} \|\nabla_1 \varphi_t^GP\|_\infty.
\]

Since

\[
\|\mathbb{I}_{\mathbb{S}_1}\|_1 \leq N|s_{1,2}| = 4/3\pi N^{-17/9}
\]

we can find a $\gamma > 0$ such that

\[
\|\mathbb{I}_{\mathbb{S}_1} \nabla_1 q_1 \Psi_t\|^2 + \langle \Psi_t, \mathbb{I}_{\mathbb{S}_1} \sum_{j \neq 1} v_j^N(x_1 - x_j)\Psi_t \rangle \\
\leq E_{kin}(1 - \|p_1 \Psi_t\|^2) + C(\alpha(\Psi_0) + \alpha(\Psi_t)) + C \int_0^t \alpha(\Psi_s)ds + CN^{-\gamma}.
\]

Using that $1 - \|p_1 \Psi_t\|^2 = \|q_1 \Psi_t\|^2 < \alpha(\Psi)$ and that both summands are positive the Lemma follows. □
6.3 Redefinition of $\alpha$ for $1/3 < \beta \leq 1$

As mentioned in the introduction one has to control the microscopic structure of $\Psi$ when $\beta$ increases. On the technical level that means, that for $\beta > 1/3$ the $\alpha_1'$ and for $\beta = 1$ the $\alpha_2'$ can’t be controlled. We have to equip the $\alpha_{1/2}$ with the respective microscopic structure. We shall do that by adding the derivatives $\lambda_{1,2}$ to $\alpha_{1/2}$ and $\lambda_{1,2}'$ to $\alpha_{1/2}'$ in such a way, that $\lambda_{1,2}'(\Psi_t)$ is the time derivative $\lambda_{1,2}(\Psi_t)$ if $\Psi_t$ solves the Schrödinger equation and $\alpha_{1,2} + \lambda_{1,2}$ becomes controllable.

First note that we can replace in the estimate of the second term in Lemma 5.2 (a) $\|\nabla \Psi_1 \|$ by $\|\mathcal{L}_t \nabla \Psi_1 \|$:

Lemma 6.6 Under the conditions of Lemma 5.2 we have for $0 < \beta < 1$

$$|\langle \Psi_t, p_1 q_2 \hat{m}^{1/2} h_{1,2} \hat{m}^{1/2} q_1 q_2 \Psi_t \rangle| \leq C(\|\varphi_t^{GP}\|_{\infty} + \|\nabla \varphi_t^{GP}\|_{\infty}) (\alpha(\Psi) + N^{-\gamma} + \|\mathcal{L}_t \nabla \Psi_1 \|^2)$$

The proof shall be given in the Appendix.

Definition 6.7 Let $v_1^N \in \mathcal{V}_1$.

We define

$$\lambda_2(\Psi) := N(N-1)\Re \left( \langle \Psi, g_{8/9,1}^N(x_1-x_2) (\hat{m} - \hat{n}) p_1 q_2 \Psi \rangle \right)$$

and

$$\lambda_2'(\Psi) := N(N-1)\Re \left( \langle \Psi, \left[ H, g_{8/9,1}^N(x_1-x_2) (\hat{m} - \hat{n}) p_1 q_2 \right] \Psi \rangle \right) - N(N-1)\Re \left( \langle \Psi, g_{8/9,1}^N(x_1-x_2) [H^{GP}, (\hat{m} - \hat{n}) p_1 q_2] \Psi \rangle \right).$$

Lemma 6.8 There exists a $\gamma > 0$ such that

(a) For any solution of the Schrödinger equation $\Psi_t \in L^2(\mathbb{R}^{3N} \rightarrow \mathbb{C})$

$$i \frac{d}{dt} \lambda_2(\Psi_t) = \lambda_2'(\Psi_t)$$

(b) There exist a $C < \infty$ such that for any $\Psi \in L^2(\mathbb{R}^{3N} \rightarrow \mathbb{C})$

$$|\lambda_2'(\Psi) - \alpha_2'(\Psi)| \leq C(\|\varphi_t^{GP}\|_{\infty} + \|\nabla \varphi_t^{GP}\|_{\infty}) \left( N^{-\gamma} + (\ln N)^{1/3} \alpha(\Psi) \right).$$  \hspace{1cm} (35)

(c)  \hspace{1cm} 

$$|\lambda_2(\Psi)| \leq CN^{-\gamma} \|\varphi_t^{GP}\|_{\infty}$$

Proof: (a) follows as above, using that $\frac{d}{dt} \hat{m} = -[H^{GP}, \hat{m}]$.

For (c) we have with Lemma 5.2 (a)

$$|\lambda_2(\Psi)| \leq N^2 \|g_{8/9,1}^N\| \|\varphi_t^{GP}\|_{\infty} \|\hat{m} - \hat{n}\| q_2 \Psi \| \leq CN^{-4/3} \|\varphi_t^{GP}\|_{\infty}$$

23
For (b) we have since
\[
[H, g^N_{8/9,1}(x_1 - x_2)] = [H, f^N_{8/9,1}(x_1 - x_2)] = -[\Delta_1 + \Delta_2, f^N_{8/9,1}(x_1 - x_2)]
\]
\[
= (\Delta_1 + \Delta_2) f^N_{8/9,1}(x_1 - x_2) + (\nabla_1 f^N_{8/9,1}(x_1 - x_2)) \nabla_1 + (\nabla_2 f^N_{8/9,1}(x_1 - x_2)) \nabla_2
\]
\[
= (v^N_\beta - W^N_{8/9,1}) f^N_{8/9,1}(x_1 - x_2) + (\nabla_1 g^N_{8/9,1}(x_1 - x_2)) \nabla_1 + (\nabla_2 g^N_{8/9,1}(x_1 - x_2)) \nabla_2
\]
that
\[
\lambda'_2(\Psi) = N(N-1)\Im \left( \langle \Psi, \left( H - H^{GP}, g^N_{8/9,1}(x_1 - x_2)(\hat{n} - \hat{n}_1)p_1q_2 \right) \right)
\]
\[
+ N(N-1)\Im \left( \langle \Psi, (v^N_\beta - W^N_{8/9,1}) f^N_{8/9,1}(x_1 - x_2) \nabla_1 + (\nabla_1 g^N_{8/9,1}(x_1 - x_2)) \nabla_1 \rangle \right)
\]
\[
+ N(N-1)\Im \left( \langle \Psi, (\nabla_2 g^N_{8/9,1}(x_1 - x_2)) \nabla_2 \rangle \right)
\]
It follows that
\[
\lambda'_2(\Psi) - \alpha'_2(\Psi) = N(N-1)\Im \left( \langle \Psi, \left[ -a_N \sum_{j=1}^N |\varphi^N_{GP,2}(x_j), g^N_{8/9,1}(x_1 - x_2)(\hat{n} - \hat{n}_1)p_1q_2 \right] \right)
\]
\[
+ N(N-1)\Im \left( \langle \Psi, W^N_{8/9,1} f^N_{8/9,1}(x_1 - x_2)(\hat{n} - \hat{n}_1)p_1q_2 \right)
\]
\[
+ N(N-1)\Im \left( \langle \Psi, (\nabla_1 g^N_{8/9,1}(x_1 - x_2)) \nabla_1 \rangle \right)
\]
\[
- N(N-1)\Im \left( \langle \Psi, g^N_{8/9,1}(x_1 - x_2)(\hat{n} - \hat{n}_1)p_1q_2 \sum_{j<k} v^N_\beta(x_j - x_k) \Psi \rangle \right)
\]
\[
+ N(N-1)\Im \left( \langle \Psi, \sum_{j=1}^2 \sum_{k=3}^N v^N_\beta(x_j - x_k) g^N_{8/9,1}(x_1 - x_2)(\hat{n} - \hat{n}_1)p_1q_2 \Psi \rangle \right)
\]
\[
+ N(N-1)\Im \left( \langle \Psi, \sum_{2<j<k}^N v^N_\beta(x_j - x_k) g^N_{8/9,1}(x_1 - x_2)(\hat{n} - \hat{n}_1)p_1q_2 \Psi \rangle \right).
\]
Using symmetry of $\Psi$ and $\nabla_1 g_{\beta_1, \beta_2}^N = -\nabla_2 g_{\beta_1, \beta_2}^N$

$$\lambda_2^*(\Psi) - \alpha_2^*(\Psi)$$

$$= N(N-1)3 \left( \left( \Psi, \left[ -a_N \sum_{j=1}^{N} |\varphi_{1}^{GP}|^2(x_j), g_{6/9,1}^N(x_1-x_2)(\hat{n} - \hat{n}_1)p_1q_2 \right] \Psi \right) \right)$$

$$+ N(N-1)3 \left( \left( \Psi, W_{8/9,1}^N f_{8/9,1}^N(x_1-x_2)(\hat{n} - \hat{n}_1)p_1q_2 \Psi \right) \right)$$

$$- N(N-1)3 \left( \left( \Psi, (\nabla_2 g_{6/9,1}^N(x_1-x_2)) \nabla_1 p_1q_2 (\hat{n} - \hat{n}_1) \Psi \right) \right)$$

$$- N(N-1)3 \left( \left( \Psi, (\nabla_1 g_{6/9,1}^N(x_1-x_2)) \nabla_2 p_1q_2 (\hat{n} - \hat{n}_1) \Psi \right) \right)$$

$$- N(N-1)3 \left( \left( \Psi, g_{6/9,1}^N(x_1-x_2)(\hat{n} - \hat{n}_1)p_1q_2 v_{\beta}^N (x_1-x_2) \Psi \right) \right)$$

$$- \frac{N!}{(N-3)!}3 \left( \left( \Psi, g_{6/9,1}^N(x_1-x_2)(\hat{n} - \hat{n}_1)p_1q_2 v_{\beta}^N (x_2-x_3) \Psi \right) \right)$$

$$- \frac{N!}{(N-4)!}3 \left( \left( \Psi, g_{6/9,1}^N(x_1-x_2)(\hat{n} - \hat{n}_1)p_1q_2 v_{\beta}^N (x_3-x_4) \Psi \right) \right)$$

$$+ \frac{N!}{(N-3)!}3 \left( \left( \Psi, v_{\beta}^N (x_1-x_2) g_{6/9,1}^N(x_1-x_2)(\hat{n} - \hat{n}_1)p_1q_2 \Psi \right) \right)$$

$$+ \frac{N!}{(N-4)!}3 \left( \left( \Psi, v_{\beta}^N (x_2-x_3) g_{6/9,1}^N(x_1-x_2)(\hat{n} - \hat{n}_1)p_1q_2 \Psi \right) \right)$$

$$=: \sum_{j=0}^{10} S_j . \quad (36)$$

For the first summand we have

$$|S_0| \leq 2N^2a ||\varphi_{t}^{GP}||^3 ||g_{6/9,1}^N|| ||(\hat{n} - \hat{n}_1)p_1q_2 \Psi|| .$$

With Lemma 6.5, it follows that $|S_0|$ is bounded by the right hand side of (35).

Using as above (see proof of Lemma 5.4) that $\Im(\langle \Psi, A \Psi \rangle) = -\Im(\langle \Psi, A^\dagger \Psi \rangle)$ for any operator $A$ and that $\Psi$ is symmetric (note that $p_1q_2 v_{\beta}^N (x_1-x_2) q_1 p_2$ is invariant under adjunction plus exchange of the variable $x_1$ and $x_2$) and Lemma 2.3 (d) we get for $S_1$

$$|S_1| \leq N(N-1) \left| \Im \left( \left( \Psi, p_1 p_2 W_{8/9,1}^N f_{8/9,1}^N(x_1-x_2)(\hat{n} - \hat{n}_1)p_1q_2 \Psi \right) \right) \right|$$

$$+ N(N-1) \left| \Im \left( \left( \Psi, q_1 q_2 W_{8/9,1}^N f_{8/9,1}^N(x_1-x_2)(\hat{n} - \hat{n}_1)p_1q_2 \Psi \right) \right) \right|$$

Since $W_{8/9,1}^N f_{8/9,1}^N \in V_{\beta_2}$ (see Lemma 6.2 (b)) it follows with Lemma 6.6 that

$$|S_1| \leq C(||\varphi_{t}^{GP}||_\infty + ||\nabla \varphi_{t}^{GP}||_\infty)(\alpha(\Psi) + \|I_8 \nabla_1 \Psi_t\| + N^{-\xi}) .$$
With Lemma 6.5 it follows that $|S_1|$ is bounded by the right hand side of (36).

For $S_2$ and $S_3$ we get integrating by parts

$$S_2 + S_3 = N(N-1)\Im \left( \left\langle \nabla_2 \Psi, (g_8^{N,1}(x_1-x_2))\nabla_1 p_1 q_2 (\hat{\n} - \hat{n}_1) \Psi \right\rangle \right)$$

$$+ N(N-1)\Im \left( \left\langle (q_1 q_2 + 1 - q_1 q_2) \nabla_1 \Psi, (g_8^{N,1}(x_1-x_2))\nabla_2 p_1 q_2 (\hat{\n} - \hat{n}_1) \Psi \right\rangle \right)$$

Thus

$$|S_2 + S_3| \leq N(N-1)\Im \left\| \nabla_2 \Psi \right\| \left\| g_8^{N,1}(x_1-x_2) \right\| \left\| \nabla_1 \varphi_t^{GP} \right\| \infty \left\| p_1 q_2 (\hat{\n} - \hat{n}_1) \Psi \right\|$$

$$+ N(N-1)\Im \left\| q_1 \nabla_2 q_2 (\hat{n}_2 - \hat{n}_3) \Psi \right\| \left\| g_8^{N,1}(x_1-x_2) \right\| \left\| \varphi_t^{GP} \right\| \infty \left\| \nabla_2 p_1 q_2 (\hat{\n} - \hat{n}_1) \Psi \right\|$$

$$+ 2N(N-1)\Im \left\| q_1 \nabla_2 (\hat{n}_2 - \hat{n}_3) \Psi \right\| \left\| g_8^{N,1}(x_1-x_2) \right\| \left\| \varphi_t^{GP} \right\| \infty \left\| \nabla_2 \nabla_2 q_2 (\hat{\n} - \hat{n}_1) \Psi \right\|$$

With Lemma 6.5 it follows that $|S_2 + S_3|$ is bounded by the right hand side of (36).

For $S_4$ we have

$$|S_4| \leq N(N-1)\Im \left\| g_8^{N,1}(x_1-x_2) \right\| \left\| \varphi_t^{GP} \right\| \infty \left\| \sqrt{\n} \right\| \left\| \sqrt{\n} \right\| + a \| \varphi_t^{GP} \| \infty \left\| \Psi \right\|$$

With Lemma 6.5 it follows that $|S_4|$ is bounded by the right hand side of (36).

For $S_5$ we have using $q_2 = 1 - p_2$

$$|S_5| \leq \frac{N!}{(N-3)!} \Im \left( \left\langle (p_1 + q_1) \sqrt{\n} \Psi, g_8^{N,1}(x_1-x_2)(\hat{n} - \hat{n}_1) p_1 \sqrt{\n} \right\rangle \right)$$

$$+ \frac{N!}{(N-3)!} \Im \left( \left\langle \Psi, (q_1 q_2 + 1 - q_1 q_2) g_8^{N,1}(x_1-x_2)(\hat{n} - \hat{n}_1) p_1 p_2 \right\rangle \right)$$

For the first summand we have

$$\frac{N!}{(N-3)!} \Im \left( \left\langle (p_1 + q_1) \sqrt{\n} \right\rangle \right)$$

$$\leq \frac{N!}{(N-3)!} \left\| \sqrt{\n} \right\| \left\| \sqrt{\n} \right\| + \frac{N!}{(N-3)!} \left\| \sqrt{\n} \right\|$$

which is due to Lemma 6.5 and Lemma 2.3 bounded by the right hand side of
For the second summand we have in view of Lemma 2.3 (d)

\[
\frac{N!}{(N-3)!} \left| \mathbf{3} \left( \left< \Psi, (q_1 q_2 + 1 - q_1 q_2) g_{8/9,1}^N (x_1 - x_2) (\hat{n} - \tilde{n}_1) p_1 p_2 v_\beta^N (x_2 - x_3) \Psi \right> \right) \right| \\
\leq \frac{N!}{(N-3)!} \| (\hat{n} - \tilde{n}_2) q_1 q_2 \Psi \| \| g_{8/9,1}^N \| \| \varphi_1^{GP} \|_\infty \| \sqrt{v_\beta^N} \| (x_1 - x_3) \Psi \| \\
+ \frac{N!}{(N-3)!} \| g_{8/9,1}^N \|_1 \| \varphi_1^{GP} \|_2 \| \hat{n} - \tilde{n}_1 \|_\op \| \varphi_1^{GP} \|_\infty \| \sqrt{v_\beta^N} \| (x_2 - x_3) \Psi \|.
\]

With Lemma 6.5 it follows that \(|S_6|\) is bounded by the right hand side of (35).

Similarly we get for \(S_6\) using Lemma 2.3 (d)

\[
|S_6| = \frac{N!}{(N-3)!} \left| \mathbf{3} \left( \left< \Psi, (q_1 q_2 + 1 - q_1 q_2) g_{8/9,1}^N (x_1 - x_2) (\hat{n} - \tilde{n}_1) p_1 q_2 v_\beta^N (x_1 - x_3) \Psi \right> \right) \right| \\
\leq \frac{N!}{(N-3)!} \| (\hat{n} - \tilde{n}_2) q_1 q_2 \Psi \| \| g_{8/9,1}^N \| \| \varphi_1^{GP} \|_\infty \| \sqrt{v_\beta^N} \| (x_1 - x_3) \Psi \| \\
+ \| (\hat{n} - \tilde{n}_2) q_1 q_2 \Psi \| \| g_{8/9,1}^N \| \| \varphi_1^{GP} \|_\infty \| \sqrt{v_\beta^N} \| (x_2 - x_3) \Psi \|.
\]

With Lemma 6.5 it follows that \(|S_6|\) is bounded by the right hand side of (35).

For \(S_7 + S_{10}\) we use (17) to get

\[
S_7 + S_{10} = - \frac{N!}{(N-4)!} \left| \mathbf{3} \left( \left< \Psi, g_{8/9,1}^N (x_1 - x_2) Q v_\beta^N (x_3 - x_4) \Psi \right> \right) \right| \\
+ \frac{N!}{(N-4)!} \left| \mathbf{3} \left( \left< \Psi, g_{8/9,1}^N (x_1 - x_2) v_\beta^N (x_3 - x_4) Q \Psi \right> \right) \right|
\]

with

\[
Q = (\hat{n} - \tilde{n}_2 - \tilde{n}_3) p_1 q_2 p_3 p_4 + (\hat{n} - 2\tilde{n}_1 + \tilde{n}_2) (p_1 q_2 p_3 q_4 + p_1 q_2 q_3 p_4).
\]

Since \(\sqrt{k} - \sqrt{k - 2} - \sqrt{k - 1} + \sqrt{k - 3} < Ck^{-3/2}\) and \(\sqrt{k} - 2\sqrt{k - 1} + \sqrt{k - 2} < Ck^{-3/2}\) it follows that

\[
Q < N^{-2}\tilde{n}^{-3/2}(p_1 q_2 p_3 p_4 + p_1 q_2 p_3 q_4 + p_1 q_2 q_3 p_4).
\]
It follows using symmetry and Lemma 2.3 that

\[
|S_7 + S_{10}| \
\leq N^2 \left| \Im \left( \Psi, q_1 q_2 \tilde{n}^{-1} g_{8/9,1}^{N} (x_1 - x_2) \tilde{n}^{-1/2} (p_1 q_2 p_3 p_4 + 2 p_1 q_2 p_3 q_4) v_\beta^N (x_3 - x_4) \right) \right| \\
+ N^2 \left| \Im \left( \Psi, (1 - q_1 q_2) g_{8/9,1}^{N} (x_1 - x_2) \tilde{n}^{-3/2} (p_1 q_2 p_3 p_4 + 2 p_1 q_2 p_3 q_4) v_\beta^N (x_3 - x_4) \right) \right| \\
+ N^2 \left| \Im \left( \Psi, q_1 q_2 v_\beta^N (x_3 - x_4) \tilde{n}^{-1} g_{8/9,1}^{N} (x_1 - x_2) \tilde{n}^{-1/2} (p_1 q_2 p_3 p_4 + 2 p_1 q_2 p_3 q_4) \right) \right| \\
+ N^2 \left| \Im \left( \Psi, (1 - q_1 q_2) g_{8/9,1}^{N} (x_1 - x_2) v_\beta^N (x_3 - x_4) \tilde{n}^{-3/2} (p_1 q_2 p_3 p_4 + 2 p_1 q_2 p_3 q_4) \right) \right| \\
\leq 3 N^2 \| \tilde{n}^{-1} q_1 q_2 \Psi \| \| g_{8/9,1}^{N} \| \| \varphi_t^{GP} \|_2 \left\| \varphi_\beta^N \right\|_\infty \left\| v_\beta^N \right\|_\infty \\
+ 3 N^2 \| g_{8/9,1}^{N} \|_1 \| \varphi_t^{GP} \|_3 \left\| \tilde{n}^{-3/2} q_2 \right\| \| v_\beta^N \|_\infty \left\| \varphi_\beta^N \right\|_3 \left\| v_\beta^N \right\|_\infty \\
+ 3 N^2 \| \tilde{n}^{-1} \|_3 \| q_1 q_2 \Psi \| \| g_{8/9,1}^{N} \| \| \varphi_t^{GP} \|_2 \left\| v_\beta^N \right\|_\infty \\
+ 3 N^2 \| \tilde{n}^{-3/2} \|_2 \| q_2 \Psi \| \| g_{8/9,1}^{N} \| \| \varphi_\beta^N \|_3 \left\| v_\beta^N \right\|_\infty \\
\right|
\]

With Lemma 6.5 it follows that \(|S_7 + S_{10}|\) is bounded by the right hand side of (35).

For \(S_8\) and \(S_9\) note first, that

\[
\left| \frac{N!}{(N - 3)!} \Im \left( \Psi, v_\beta^N (x_2 - x_3) g_{8/9,1}^{N} (x_1 - x_2) (\tilde{n} - \tilde{n}_1) p_1 p_2 \Psi \right) \right| \\
\leq C N^2 \left| \Im \left( \Psi, p_1 v_\beta^N (x_2 - x_3) g_{8/9,1}^{N} (x_1 - x_2) \tilde{n}^{-1/2} p_1 p_2 \Psi \right) \right| \\
+ C N^2 \left| \Im \left( \Psi, q_1 v_\beta^N (x_2 - x_3) g_{8/9,1}^{N} (x_1 - x_2) \tilde{n}^{-1/2} p_1 p_2 \Psi \right) \right| \\
\leq C N^2 \left\| \tilde{n}^{-1/2} \|_2 \| v_\beta^N \|_2 \left\| g_{8/9,1}^{N} \right\|_1 N^{1/2} \| \varphi_t^{GP} \|_1 \left\| v_\beta^N \right\|_\infty \left\| \varphi_\beta^N \right\|_3 \left\| v_\beta^N \right\|_\infty \right| \\
+ \sum_{j=1}^{3} C N^2 \left\| \tilde{n}^{-1/2} \|_2 \| v_\beta^N \|_2 \left\| q_1 \Psi \right\| \| \varphi_t^{GP} \| \left\| g_{8/9,1}^{N} \right\| \| \varphi_\beta^N \| \left\| v_\beta^N \right\|_\infty \right| \\
\right|
\]

With Lemma 6.5 it follows that the latter is bounded by the right hand side of (35), thus it suffices to control

\[
\tilde{S}_8 := (N - 1)(N - 2) \Im \left( \Psi, v_\beta^N (x_2 - x_3) g_{8/9,1}^{N} (x_1 - x_2) (\tilde{n} - \tilde{n}_1) p_1 \Psi \right) \\
\text{and}
\tilde{S}_9 := \frac{N!}{(N - 3)!} \Im \left( \Psi, v_\beta^N (x_1 - x_3) g_{8/9,1}^{N} (x_1 - x_2) (\tilde{n} - \tilde{n}_1) p_1 \Psi \right)
\]

instead of \(S_8\) and \(S_9\). For \(\tilde{S}_8\) we have

\[
|\tilde{S}_8| \leq N^2 \left\| \tilde{n}^{-1/2} \|_2 \| v_\beta^N \|_2 \left\| \varphi_t^{GP} \| \left\| g_{8/9,1}^{N} \right\| \left\| v_\beta^N \right\|_\infty \right|
\]
which is again bounded by the right hand side of (35). For $S_9$

$$|S_9| \leq N^2 \|v_N^{1/2} (x_1 - x_2) \Psi\| \|g_{9/9,1}^N (x_1 - x_2) \sqrt{v_N^{1/2}} (x_1 - x_3) p_1 \Psi\| .$$

Note that due to (23) $\sqrt{v_{1,3} g_{9/9,1}^N (x_1 - x_2)} < v_{1,3} a/(N|x_1 - x_2|)$ and $g_{9/9,1}^N (x_1 - x_2) < C$, thus $\sqrt{v_{1,3} g_{9/9,1}^N (x_1 - x_2)} < v_{1,3} \bar{g}(x_2 - x_3)$ with $\bar{g}(x) < C/(N|x| + 1)$ and $g(x) = 0$ for $x > CN^{-8/9}$. It follows that

$$\|g_{9/9,1}^N (x_1 - x_2) \sqrt{v_N^{1/2}} (x_1 - x_3) p_1 \Psi\| \leq C \|\mathcal{V}_{\Gamma}^G\|_\infty \|\sqrt{v_N^{1/2}} (x_1 - x_3)\| \|\bar{g}(x_2 - x_3) \Psi\| .$$

Using Hölder and Sobolev it follows that for sufficiently large $N$

$$\|\bar{g}(x_2 - x_3) \Psi\|^2 \leq \|\bar{g}^2\|^3 \|\Psi^2\|^3 = \|\bar{g}\|_3^2 \|\Psi\|_6^2$$

$$\leq \|\nabla \Psi\|^2 \left(\int \bar{g}^3 d^3 x\right)^{2/3}$$

$$\leq C \|\nabla \Psi\|^2 \left(N^{-3} \int_{N^{-1} < |x| < 1} |x|^{-3} d^3 x + N^{-3}\right)^{2/3}$$

$$\leq CN^{-2} \|\nabla \Psi\|^2 (\ln N)^{2/3}$$

It follows that also $S_9$ is bounded by the right hand side of (35) and (b) follows.

(c)
we get furthermore
\[
\| \lambda_1(\Psi) \| \leq C N^2 \| \varphi_i^{GP} \|_\infty \| g_{2/\beta}^N \|_1 N^{-1/2} + C N^2 \| \varphi_i^{GP} \|_\infty \| g_{2/\beta}^N \|^{-1} N^{-1}.
\]

For |\lambda_1'(\Psi) - \alpha_1'(\Psi)| we can use (36), replacing \(g_{6/9,\beta}^N\) by \(g_{2/\beta}^N\) and \((\hat{n} - \hat{n}_1)p_1q_2\) by \((\hat{n} - \hat{n}_2)p_1p_2\). Using symmetry, \(1 = p_j + q_j\) and (17) and reordering the summands we get
\[
\lambda_1'(\Psi) - \alpha_1'(\Psi) = N(N - 1)3 \left( p_1 \Psi, -a_N \sum_{j=1}^N |\varphi_i^{GP}|^2(x_j), g_{2/\beta}^N(x_1 - x_2)(\hat{n} - \hat{n}_2)p_1p_2 \right) \Psi
\]
\[
+ N(N - 1)3 \left( q_1 \Psi, -a_N \sum_{j=1}^N |\varphi_i^{GP}|^2(x_j), g_{2/\beta}^N(x_1 - x_2)(\hat{n} - \hat{n}_2)p_1p_2 \right) \Psi
\]
\[
+ N(N - 1)3 \left( \Psi, g_{2/\beta}^N(x_1 - x_2)(\hat{n} - \hat{n}_2)p_1p_2 v_\beta^N(x_1 - x_2) \right) \Psi
\]
\[
- 2N(N - 1)3 \left( \Psi, p_1(\nabla_2 g_{2/\beta}^N(x_1 - x_2)) \nabla_1 p_1 p_2 (\hat{n} - \hat{n}_1) \Psi \right)
\]
\[
- 2N(N - 1)3 \left( \Psi, q_1(\nabla_2 g_{2/\beta}^N(x_1 - x_2)) \nabla_1 p_1 p_2 (\hat{n} - \hat{n}_1) \Psi \right)
\]
\[
- N(N - 1)3 \left( \Psi, g_{2/\beta}^N(x_1 - x_2)(\hat{n} - \hat{n}_2) p_1 p_2 v_\beta^N(x_1 - x_2) \right) \Psi
\]
\[
- 2 \frac{N!}{(N - 3)!} 3 \left( \Psi, p_1 g_{2/\beta}^N(x_1 - x_2)(\hat{n} - \hat{n}_2) p_1 p_2 v_\beta^N(x_2 - x_3) \right) \Psi
\]
\[
- 2 \frac{N!}{(N - 3)!} 3 \left( \Psi, g_{2/\beta}^N(x_1 - x_2)(\hat{n} - \hat{n}_2) p_1 p_2 v_\beta^N(x_2 - x_3) \right) \Psi
\]
\[
+ 2 \frac{N!}{(N - 3)!} 3 \left( \Psi, g_{2/\beta}^N(x_2 - x_3) g_{2/\beta}^N(x_1 - x_2)(\hat{n} - \hat{n}_2) p_1 p_2 \right) \Psi
\]
\[
+ 2 \frac{N!}{(N - 3)!} 3 \left( \Psi, g_{2/\beta}^N(x_2 - x_3) g_{2/\beta}^N(x_1 - x_2)(\hat{n} - \hat{n}_2) p_1 p_2 \right) \Psi
\]
\[
+ 2 \frac{N!}{(N - 4)!} 3 \left( \Psi, v_\beta^N(x_3 - x_4) g_{2/\beta}^N(x_1 - x_2)(\hat{n} - \hat{n}_1 - \hat{n}_2 + \hat{n}_3) p_1 p_2 p_3 q_3 \Psi \right)
\]
\[
- 2 \frac{N!}{(N - 4)!} 3 \left( \Psi, g_{2/\beta}^N(x_1 - x_2)(\hat{n} - \hat{n}_1 - \hat{n}_2 + \hat{n}_3) p_1 p_2 p_3 q_3 v_\beta^N(x_3 - x_4) \Psi \right)
\]
\[
+ \frac{N!}{(N - 4)!} 3 \left( \Psi, g_{2/\beta}^N(x_3 - x_4) g_{2/\beta}^N(x_1 - x_2)(\hat{n} - 2\hat{n}_2 + \hat{n}_4) p_1 p_2 p_3 q_4 \Psi \right)
\]
\[
- \frac{N!}{(N - 4)!} 3 \left( \Psi, g_{2/\beta}^N(x_3 - x_4) g_{2/\beta}^N(x_1 - x_2)(\hat{n} - 2\hat{n}_2 + \hat{n}_4) p_1 p_2 p_3 q_4 \Psi \right)
\]
\[
=: \sum_{j=0}^{13} T_j. \quad (37)
\]

For \(T_0\) to \(T_{11}\) one can copy the estimates of \(S_0\) to \(S_{10}\) above and gets, that
\sum_{j=0}^{11} T_j \text{ is bounded by }
\begin{equation}
C(||\varphi^G_P||_\infty + ||\nabla\varphi^G_P||_\infty)N^{-\gamma} + (\ln N)^{1/3}\alpha(\Psi).
\tag{38}
\end{equation}

Instead of controlling \(T_{12}\) and \(T_{13}\) we add another term which pays respect to higher orders of for the microscopic structure, i.e. we define
\begin{align*}
\lambda_3(\Psi) & := \frac{N!}{(N - 4)!} 3 \left( \left\langle \Psi, g^N_{2/\gamma, 3}(x_3 - x_4)g^N_{2/\gamma, 3}(x_1 - x_2)(\widehat{n} - 2\widehat{n}_2 + \widehat{n}_4)p_1p_2p_3p_4\Psi \right\rangle \right), \\
& \quad - \frac{N!}{(N - 4)!} 3 \left( \left\langle \Psi, g^N_{2/\gamma, 3}(x_1 - x_2)(\widehat{n} - 2\widehat{n}_2 + \widehat{n}_4)p_1p_2p_3p_4g^N_{2/\gamma, 3}(x_3 - x_4)\Psi \right\rangle \right)
\end{align*}
and the respective \(\lambda'_3(\Psi)\), again with \(\frac{d}{dt}\lambda_3(\Psi) = \lambda'_3(\Psi)\).

Controlling \(\lambda'_3(\Psi)\) we get similar terms as the \(T_j\) above, the only difference being an additional operator \(N^2g^N_{2/\gamma, 3}(x_j - x_k)p_jp_k\) and a higher order derivative of \(\widehat{n}\) (interpreting \(\widehat{n} - \widehat{n}_1\) as the derivative of \(\widehat{n}\)). We arrive at terms which are bounded by \(38\) and the respective \(T_{12}\) and \(T_{13}\), i.e.
\begin{align*}
\frac{N!}{(N - 6)!} 3 \left( \left\langle \Psi, V_{5,6}g^N_{2/\gamma, 3}(x_1 - x_2)g^N_{2/\gamma, 3}(x_3 - x_4)(\widehat{n} - 3\widehat{n}_2 + 3\widehat{n}_4 - \widehat{n}_6)p_1p_2p_3p_4p_5\Psi \right\rangle \right), \\
\frac{N!}{(N - 6)!} 3 \left( \left\langle \Psi, g^N_{2/\gamma, 3}(x_1 - x_2)g^N_{2/\gamma, 3}(x_3 - x_4)(\widehat{n} - 3\widehat{n}_2 + 3\widehat{n}_4 - \widehat{n}_6)p_1p_2p_3p_4p_5V_{5,6}\Psi \right\rangle \right)
\end{align*}

and
\begin{align*}
\frac{N!}{(N - 6)!} 3 \left( \left\langle \Psi, V_{5,6}g^N_{2/\gamma, 3}(x_1 - x_2)(\widehat{n} - 3\widehat{n}_2 + 3\widehat{n}_4 - \widehat{n}_6)p_1p_2p_3p_4p_5g^N_{2/\gamma, 3}(x_3 - x_4)\Psi \right\rangle \right)
\end{align*}

Iteratively we add higher orders of the microscopic structure for the remaining terms. Each iteration yields another operator \(N^2g^N_{2/\gamma, 3}(x_j - x_k)p_jp_k\) and a “higher order derivative of \(\widehat{n}\)”, thus a factor \(N^{-1/7}\). We stop the iteration as soon as all the remaining terms can be estimated by \(38\). Thus we get

**Lemma 6.9** There exists a \(\gamma > 0\) and functionals \(\lambda_1(\Psi)\) and \(\lambda'_1(\Psi)\) such that
(a) For any solution of the Schrödinger equation \(\Psi_t \in L^2(\mathbb{R}^{3N} \to \mathbb{C})\)
\[\frac{d}{dt}\lambda_1(\Psi_t) = \lambda'_1(\Psi_t)\]
(b) There exist a \(C < \infty\) such that for any \(\Psi \in L^2(\mathbb{R}^{3N} \to \mathbb{C})\)
\[|\lambda'_1(\Psi) - \alpha'_1(\Psi)| \leq C(||\varphi^G_P||_\infty + ||\nabla\varphi^G_P||_\infty)\left(N^{-\gamma} + (\ln N)^{1/3}\alpha(\Psi)\right)\]
(c) \[\|\lambda_1(\Psi)\| \leq CN^{-\gamma}\|\varphi^G_P\|\_\infty.\]
Summarizing (21), Lemma 6.8 and Lemma 6.9 and setting $\lambda(\Psi) := \alpha(\Psi) + \lambda_1(\Psi) + \lambda_2(\Psi)$ we arrive at

**Corollary 6.10** There exists a $\gamma > 0$ and functionals $\lambda(\Psi)$ and $\lambda'(\Psi)$ such that

(a) For any solution of the Schrödinger equation $\Psi_t \in L^2(\mathbb{R}^3 \rightarrow \mathbb{C})$

$$i \frac{d}{dt} \lambda(\Psi_t) = \lambda'(\Psi_t)$$

(b) There exist a $C < \infty$ such that for any $\Psi \in L^2(\mathbb{R}^3 \rightarrow \mathbb{C})$

$$|\lambda'(\Psi)| \leq C(\|\varphi_t^{GP}\|_\infty + \|\nabla \varphi_t^{GP}\|_\infty) \left( N^{-\gamma} + (\ln N)^{1/3} \lambda(\Psi) \right).$$

(c) $\|\lambda(\Psi) - \alpha(\Psi)\| \leq CN^{-\gamma} \|\varphi_t^{GP}\|_\infty$.

6.4 Proof of Theorem 4.3 for $\beta \geq 1/3$

In view of Corollary 6.10 (c) and Lemma 2.4 (b) it suffices to prove that

$$\lim_{N \rightarrow \infty} \lambda(\Psi_t) = 0$$

under the assumption $\lim_{N \rightarrow \infty} N^\gamma \lambda(\Psi_0) = 0$. Therefore we use the estimates we get from Corollary 6.10 (b) on the time derivative of $\lambda(\Psi_t)$ and a Gronwall-like argument.

Using that

$$\frac{d}{dt} \lambda(\Psi_t) \leq C(\|\varphi_t^{GP}\|_\infty + \|\nabla \varphi_t^{GP}\|_\infty) \left( N^{-\gamma} + (\ln N)^{1/3} \lambda(\Psi_t) \right)$$

it follows that $\lambda(\Psi_t)$ is bounded from above by the solution $\mu_t$ of the differential equation

$$\frac{d}{dt} \mu_t = C(\|\varphi_t^{GP}\|_\infty + \|\nabla \varphi_t^{GP}\|_\infty) \left( N^{-\gamma} + (\ln N)^{1/3} \mu_t \right) \quad (39)$$

with $\mu_0 = \lambda(\Psi_0)$.

Defining $\zeta_t := N^{-\gamma} + (\ln N)^{1/3} \mu_t$ we get from (39)

$$(\ln N)^{-1/3} \frac{d}{dt} \zeta_t = C(\|\varphi_t^{GP}\|_\infty + \|\nabla \varphi_t^{GP}\|_\infty) \zeta_t.$$.

Thus

$$\zeta_t := K \exp \left( C(\ln N)^{1/3} \int_0^t (\|\varphi_s^{GP}\|_\infty + \|\nabla \varphi_s^{GP}\|_\infty) ds \right)$$

with

$$K = \zeta_0 = N^{-\gamma} + (\ln N)^{1/3} \mu_0 = N^{-\gamma} + (\ln N)^{1/3} \lambda(\Psi_0) < N^{-\gamma}(1 + (\ln N)^{1/3})$$.
for $N$ large enough.

Note, that under the assumptions of the Theorem $\int_0^T (\|\varphi^{GP}_x\|_\infty + \|\nabla \varphi^{GP}_x\|_\infty) ds$ is bounded. Note also, that $e^{C(ln N)^{1/3}} = e^{C(ln N)(ln N)^{-2/3}} = N^{C(ln N)^{-2/3}}$. Since $\lim_{N \to \infty} (ln N)^{-2/3} = 0$ it follows that $\lim_{N \to \infty} N^{-\gamma} e^{C(ln N)^{1/3}} = 0$ for any $\gamma > 0$.

Thus $\zeta_t$ tends to zero as $N \to \infty$ uniform in $t < T$, so does $\mu_t$ and so does $\lambda(\Psi_t)$. With Corollary 6.10 (c) the Theorem follows.

### Acknowledgments

Helpful discussions with Detlef Dürr, Jakob Yngvason and Jean-Bernard Bru are gratefully acknowledged.

### 7 Appendix

It is left to prove the Lemma 5.2, Lemma 5.3 and Lemma 6.6. Since $\|\varphi^{GP}_t\|_\infty$ is bounded we have that for any $m : \{1, \ldots, N\} \to \mathbb{R}^+$ with $m \leq n^{-1}$

$$\langle \Psi, p_1 p_2 |(N-1)v^N_\beta(x_1, x_2) - a|\varphi^{GP}_t|^2 \rangle \leq C \|q_1 p_2 \Psi\| \|\widehat{m} q_1 q_2 \Psi\| < C \alpha(\Psi).$$

Note also that $p_j f(x_k)q_j = 0$ for any $k \neq j$ and any function $f$. So Lemma 5.2, Lemma 5.3 and Lemma 6.6 follow once we have

**Lemma 7.1** Let $m : \{1, \ldots, N\} \to \mathbb{R}^+$ with $m \leq n^{-1}$, $0 < \beta < 1$. Then we have under the conditions of the Theorem that there exists a $C < \infty$ and a $\xi > 0$ such that for any $m : \{1, \ldots, N\} \to \mathbb{R}^+$ with $m \leq \sqrt{n}$

(a) for $0 < \beta < 1$

$$\langle \Psi, p_1 p_2 ((N-1)v^N_\beta(x_1, x_2) - a|\varphi^{GP}_t|^2 \rangle p_1 p_2 \Psi \rangle \leq C (\|\varphi^{GP}_t\|_\infty + \|\nabla \varphi^{GP}_t\|_\infty) N^{-\xi}$$

(b) for any $0 < \beta < 1/3$

$$\langle \Psi, p_1 p_2 v^N_\beta(x_1, x_2) \widehat{m} q_1 q_2 \Psi \rangle \leq C N^{-1}(\|\varphi^{GP}_t\|_\infty + \|\nabla \varphi^{GP}_t\|_\infty)(\alpha(\Psi) + N^{-\xi})$$
Proof:
The right hand side of (40) is bounded by

$$S_1 := \sup_{x_1 \in \mathbb{R}^3} \left\{ \left( \langle \psi^*_i(x_2), ((N-1)v^N_\delta(x_1, x_2) - a|\psi^*_i v^N_\delta(x_1) \rangle \psi^*_i(x_2) \right) \right\}$$

$$\leq \sup_{x_1 \in \mathbb{R}^3} \left\{ \left( \langle \psi^*_i(x_1), ((N-1)v^N_\delta(x_1 - x_2)\psi^*_i(x_1)) \psi^*_i(x_2) \right. \right.$$  

$$+ (N-1) \sup_{|x_1 - x_2| < C N^{-\beta}} \{ (|\psi^*_i(x_1)|^2 - (\psi^*_i(x_2))^2) \} \|v^N_\delta \|_1.$$  

The first term is equal to $((N-1)\|v^N_\delta \|_1 - a)\|\psi^*_i v^N_\delta(x_1)\|_\infty^2$ and in view of Definition 1.2 bounded by $C\|\psi^*_i v^N_\delta\|_\infty^2 N^{-\delta}$. Using Taylors formula the second term is of order $\|\nabla\psi^*_i v^N_\delta\|_\infty N^{-\beta}$, thus

$$|S_1| \leq C(\|\psi^*_i v^N_\delta\|_\infty + \|\nabla\psi^*_i v^N_\delta\|_\infty(N^{-\beta} + N^{-\delta}).$$

(45)

Since under our assumptions $\|\psi^*_i v^N_\delta\|_\infty + \|\nabla\psi^*_i v^N_\delta\|_\infty < \infty$ (40) follows.

The left hand side of (41) is bounded by

$$S_1 \|p_1 p_2 \Psi\| \|\tilde{m} q_1 q_2 \Psi\| \leq S_1 \|p_1 p_2 \Psi\| \|\tilde{m} q_1 q_2 \Psi\|$$

With (45) and Lemma 2.3 we get (41).

Next we shall prove (44). To estimate this term note, that the operator norm of $p_1 p_2 v^N_\beta(x_1, x_2)\hat{m} q_1 q_2$ restricted to subspace of symmetric functions is much smaller than the operator norm on full $L^2(\mathbb{R}^{3N} \rightarrow \mathbb{C})$. Therefore one has to use symmetry of $\Psi$ to get good control of this term. We define for some $\delta > 0$ we shall specify below the functions $m^{a,b} : \{1, \ldots, N\} \rightarrow \mathbb{R}^+$ by $m^{a}(k) := m(k)$ for $k < N^{1-\delta}$, $m^{a}(k) = 0$ for $k \geq N^{1-\delta}$ and $m^{b} = m - m^{a}$. It follows that (44) is bounded by

$$|\langle \Psi, p_1 p_2 v^N_\beta(x_1, x_2)\hat{m}^a q_1 q_2 \Psi \rangle| + |\langle \Psi, p_1 p_2 v^N_\beta(x_1, x_2)\hat{m}^b q_1 q_2 \Psi \rangle|.$$  

Defining also $g : \{1, \ldots, N\} \rightarrow \mathbb{R}^+$ by $g(k) = 1$ for $k < N^{1-\delta}$, $g(k) = 0$ for $k \geq N^{1-\delta}$ we have that $m^{a} = m^{a}s$ and thus

$$\langle \Psi, \tilde{g}_{-2} p_1 p_2 v^N_\beta(x_1, x_2)q_1 q_2 \hat{m}^a \Psi \rangle$$

$$= (N-1)^{-1} \langle \Psi, \sum_{j=2}^N \tilde{g}_{-2} p_1 p_j v^N_\beta(x_1, x_j)q_1 q_j \hat{m}^a \Psi \rangle$$

$$\leq (N-1)^{-1} \| \sum_{j=2}^N \tilde{g}_{-2} q_j v^N_\beta(x_1, x_j)p_1 p_j \Psi \| \|\hat{m}^a q_1 \Psi\|.$$  

(46)
Choosing \( \delta > 0 \) where we used Lemma 2.3 as well as that under our conditions

\[
\| q_j v^N_j (x_1, x_j) p_1 p_j \| \| \sqrt{\Delta} (\hat{m}^N)^{1/2} q_1 \| \leq (N - 1)^{-1} \sum_{j=2}^{N} q_j v^N_j (x_1, x_j) p_1 p_j \| \sqrt{\Delta} (\hat{m}^N)^{1/2} q_1 \| \leq (N - 1)^{-1} \sum_{j=2}^{N} q_j v^N_j (x_1, x_j) p_1 p_j \| \alpha (\Psi) \). \tag{47}
\]

For any \( h : \{1, \ldots, N\} \rightarrow \mathbb{R}^+ \) we have that

\[
\| \sum_{j=2}^{N} q_j v^N_j (x_1, x_j) p_1 p_j \|^2 = \sum_{j \neq k \neq 1} (\hat{h} \Psi, p_1 q_k \sqrt{v^N_k (x_1, x_k)} p_j \sqrt{v^N_j (x_1, x_j)}) + \sum_{j=2}^{N} (\hat{h} \Psi, p_1 p_j v_N (x_1, x_j) q_j v_N (x_1, x_j) p_1 p_j \hat{h} \Psi)
\]

\[
\leq (N - 1)(N - 2) \| \sqrt{\Delta} (\hat{m}^N)^{1/2} q_3 \|^2 + C N^{1/2} (N - 1) \| \sqrt{v^N_j} \| \| \phi \|_{\infty}^2 \| \hat{h} \|^2 \leq C (N - 1)(N - 2) N^{-2} \| \phi \|_{\infty}^2 \| \hat{h} \|^2 + C (N - 1) N^{1/2} N^{-2+3\beta} \| \phi \|_{\infty}^2 \sup_{1 \leq k \leq N} |h(k)|^2
\]

where we used Lemma 2.3 as well as that under our conditions \( \| v^N_j \|_{\infty} \leq C N^{3\beta} \).

Note that \( \sup_{1 \leq k \leq N} | g(k) | = 1 \) and \( \sup_{1 \leq k \leq N} | m(k) | = N^\delta \). Note also that

\[
C N^{-1} \| \phi \|_{\infty}^2 \leq (N - 1)^{-1} \| \Delta \| \| \phi \|_{\infty}^2 \leq \alpha (\Psi) + 2 N^{-1/2} \text{ and } s(k - 2) n(k) < C N^{-\delta}. \]

Thus

\[
| \langle \Psi, p_1 p_2 v^N_j (x_1, x_2) \hat{m} q_1 q_2 \rangle | \leq C N^{-1} \| \phi \|_{\infty}^2 \left( \alpha (\Psi) + N^{-1+3\beta+2\delta} + N^{-\delta} \right).
\]

Choosing \( 0 < \delta < (-1 + 3\beta)/2 \) and \( \xi < \min \{ -1 + 3\beta + 2\delta, \delta \} \) \tag{44} follows.

\[\text{(44)}\]

for \( 0 \leq \beta < 1/3 \) can be proven in the same way replacing \( \hat{m} \) by 1. For \( 1/3 \leq \beta < 1 \) we define

\[
U_N (x) := \begin{cases} \frac{3}{4\pi} \| v^N \|_1 N^{3/4}, & \text{for } x < N^{-1/4}; \\ 0, & \text{else.} \end{cases}
\]

and

\[
h_N (x) := \int |x - y|^{-1} (v^{N}_j (y) - U_N (y)) d^3 y \tag{48}
\]
By this Definition it follows that \( h_N(x) = 0 \) for \( x > N^{-1/4} \), \( |h_N| < \|v_N^\beta\|_1 |x|^{-1} \), \( |\nabla h_N| < \|v_N^\beta\|_1 |x|^{-2} \), thus
\[
\|h_N\|_\infty < CN^{-1+3\beta} \quad \|h_N\| < CN^{-1-\beta/2}
\] (49)
and
\[-\Delta h_N = v_N^\beta - U_N.\]

So having proven (42) for \( \beta < 1/3 \), (42) follows once we have
\[
|\langle \Psi, p_1 p_2 (\Delta h_N) (x_1 - x_2) q_1 q_2 \Psi \rangle| \leq C(\|\varphi_t^{GP}\|_\infty + \|\nabla \varphi_t^{GP}\|_\infty) N^{-\xi}. (50)
\]

Integration by parts and Lemma 2.3 (d) yield
\[
|\langle \Psi, p_1 p_2 (\Delta h_N) q_1 q_2 \Psi \rangle| \leq |\langle \Psi, p_1 p_2 (\nabla_1 h_N (x_1 - x_2)) \nabla_1 q_1 q_2 \Psi \rangle| + |\langle \nabla_1 p_1 p_2 \Psi, (\nabla_1 h_N (x_1 - x_2)) q_1 q_2 \Psi \rangle| =: S_2 + S_3.
\]

For \( S_2 \) we have similar as above
\[
|\langle \Psi, p_1 p_2 (\nabla_1 h_N (x_1 - x_2)) \nabla_1 q_1 q_2 \Psi \rangle| \leq (N-1)^{-1} \sum_{j=2}^N |\langle \Psi, p_1 p_j (\nabla_1 h_N (x_1 - x_j)) \nabla_1 q_1 q_j \Psi \rangle| (52)
\]
\[
\leq (N-1)^{-1} \|\nabla_1 q_1 \Psi\| \| \sum_{j=2}^N q_j (\nabla_1 h_N (x_1 - x_j)) p_1 p_j \Psi \| (53)
\]

For the last factor we write
\[
\| \sum_{j=2}^N q_j (\nabla_1 h_N (x_1 - x_j)) p_1 p_j \Psi \|
= \sum_{j \neq k \neq 1} |\langle \Psi, p_1 p_k (\nabla_1 h_N (x_1 - x_j)) q_k q_j (\nabla_1 h_N (x_1 - x_k)) p_1 p_j \Psi \rangle|
+ \sum_{j=2}^N |\langle \Psi, p_1 p_j (\nabla_1 h_N (x_1 - x_j))^2 p_1 p_j \Psi \rangle|
=: S_4 + S_5.
\]

Note, that \( \nabla_1 h_N (x_1 - x_2) = -\nabla_2 h_N (x_1 - x_2) \), thus
\[
S_4 = \sum_{j \neq k \neq 1} |\langle \Psi, p_1 p_k q_j (\nabla_j h_N (x_1 - x_j)) (\nabla_k h_N (x_1 - x_k)) p_1 p_j q_k \Psi \rangle|
\]

36
Partial integrations yield

\[ S_4 = \sum_{j \neq k \neq 1} \langle \nabla_j \nabla_k p_1 p_k q_j, h_N(x_1 - x_j)h_N(x_1 - x_k)p_1 p_j q_k \rangle \]
\[ + \sum_{j \neq k \neq 1} \langle \nabla_j p_1 p_k q_j, h_N(x_1 - x_j)h_N(x_1 - x_k)\nabla_j p_1 p_j q_k \rangle \]
\[ + \sum_{j \neq k \neq 1} \langle \nabla_k p_1 p_k q_j, h_N(x_1 - x_j)h_N(x_1 - x_k)p_1 \nabla_j p_j q_k \rangle \]
\[ + \sum_{j \neq k \neq 1} \langle p_1 p_k q_j, h_N(x_1 - x_j)h_N(x_1 - x_k)\nabla_j \nabla_k p_1 p_k q_j \rangle , \]

so as above  \( S_4 \) is bounded by the right hand side of (42). For  \( S_5 \) we estimate

\[ \|(\nabla_1 h_N(x_1 - x_j))^2\|_1 = h_N(x_1 - x_j)\Delta_1 h_N(x_1 - x_j) \]

which is (see below (48)) of order  \( N^{-2+3\beta} \). Thus  \( S_2 \) is bounded by the right hand side of (42).

For  \( S_3 \), note that  \( \nabla_1 h_N(x_1 - x_2) = -\nabla_2 h_N(x_1 - x_2) \). Integration by parts yields

\[ S_3 \leq |\langle \nabla_1 \nabla_2 p_1 p_2 \Psi, h_N(x_1 - x_2)q_1 q_2 \Psi \rangle | \]
\[ + |\langle \nabla_1 p_1 p_2 \Psi, h_N(x_1 - x_2)\nabla_2 q_1 q_2 \Psi \rangle | \]
\[ \leq \|\nabla \varphi^{GP}_{x_1}\|_2 \|h_N^2(x_1 - x_2)\|_1^{1/2} \|q_1 q_2 \Psi \| \]
\[ + \|\nabla \varphi^{GP}_{x_1}\|_\infty \|\varphi^{GP}_{x_1}\|_\infty \|h_N^2(x_1 - x_2)\|_1^{1/2} \|\nabla_2 q_1 q_2 \Psi \| \]

and (50) and thus (42) follows.

Next we shall prove (43). We define

\[ h_N(x) := \int |x - y|^{-1} v_{\beta}^N(y) dy \quad (54) \]

As above this definition implies that  \( |h_N| < \|v_{\beta}^N\|_1 |x|^{-1} \),  \( |\nabla h_N| < \|v_{\beta}^N\|_1 |x|^{-2} \),  \( \|h_N\|_\infty < C N^{-1+3\beta} \),  \( \|h_N\| < C N^{-1-\beta/2} \) and

\[ -\Delta h_N = v_{\beta}^N . \]

\[ |\langle \Psi, q_1 p_2 v_{\beta}^N(x_1 - x_2)\tilde{m} q_1 q_2 \Psi \rangle | = |\langle \Psi, \tilde{m} q_1 p_2 (\Delta h_N)(x_1 - x_2)q_1 q_2 \Psi \rangle | \]
\[ = |\langle \Psi, q_1 p_2 \tilde{m} (\Delta h_N) q_1 q_2 \Psi \rangle | \]
\[ \leq |\langle \Psi, q_1 p_2 \tilde{m} (\nabla_1 h_N(x_1 - x_2)) \mathbf{1}_{S_1} \nabla_1 q_1 q_2 \Psi \rangle | \]
\[ + |\langle \Psi, q_1 p_2 \tilde{m} (\nabla_1 h_N(x_1 - x_2)) \mathbf{1}_{S_1} \nabla_1 q_1 q_2 \Psi \rangle | \]
\[ + |\langle \nabla_1 q_1 p_2 \tilde{m} \nabla_1 h_N(x_1 - x_2)q_1 q_2 \Psi \rangle | \]
\[ =: S_6 + S_7 + S_8 . \]
For $S_6$ we have

\[
S_6 = (N - 1)^{-1} \sum_{j=2}^{N} \langle \Psi, q_1 p_j \hat{m}_2 (\nabla_1 h_N(x_1 - x_j)) \mathbb{1}_{S_1} \nabla_1 q_j \Psi \rangle | \leq (N - 1)^{-1} \| \mathbb{1}_{S_1} \nabla_1 q_1 \Psi \| \| \sum_{j=2}^{N} q_j (\nabla_1 h_N(x_1 - x_j)) \hat{m}_2 q_1 p_j \Psi \|
\]

For the last factor we write

\[
\| \sum_{j=2}^{N} q_j (\nabla_1 h_N(x_1 - x_j)) \hat{m}_2 q_1 p_j \Psi \|^2
\]

\[
= \sum_{j \neq k \neq 1} \langle \Psi, \hat{m}_2 q_1 p_k (\nabla_1 h_N(x_1 - x_j)) q_k q_j (\nabla_1 h_N(x_1 - x_j)) \hat{m}_2 q_1 p_j \Psi \rangle
\]

\[
+ \sum_{j=2}^{N} \langle \Psi, \hat{m}_2 q_1 p_j (\nabla_1 h_N(x_1 - x_j))^2 \hat{m}_2 q_1 p_j \Psi \rangle
\]

\[
=: S_9 + S_{10}.
\]

Note, that $\nabla_1 h_N(x_1 - x_2) = -\nabla_2 h_N(x_1 - x_2)$, thus

\[
S_9 = \sum_{j \neq k \neq 1} \langle \Psi, \hat{m}_2 q_1 p_k q_j (\nabla_j h_N(x_1 - x_j))(\nabla_k h_N(x_1 - x_k)) \hat{m}_2 q_1 p_j q_k \Psi \rangle
\]

Partial integrations yield

\[
S_9 = \sum_{j \neq k \neq 1} \langle \nabla_j \nabla_k \hat{m}_2 q_1 p_k q_j \Psi, h_N(x_1 - x_j) h_N(x_1 - x_k) \hat{m}_2 q_1 p_j q_k \Psi \rangle
\]

\[
+ \sum_{j \neq k \neq 1} \langle \nabla_j \hat{m}_2 q_1 p_k q_j \Psi, h_N(x_1 - x_j) h_N(x_1 - x_k) \nabla_k \hat{m}_2 q_1 p_j q_k \Psi \rangle
\]

\[
+ \sum_{j \neq k \neq 1} \langle \nabla_k \hat{m}_2 q_1 p_k q_j \Psi, h_N(x_1 - x_j) h_N(x_1 - x_k) \hat{m}_2 q_1 \nabla_j q_k \Psi \rangle
\]

\[
+ \sum_{j \neq k \neq 1} \langle \hat{m}_2 q_1 p_k q_j \Psi, h_N(x_1 - x_j) h_N(x_1 - x_k) \nabla_j \nabla_k \hat{m}_2 q_1 p_j q_k \Psi \rangle,
\]

Using symmetry of $\Psi$

\[
|S_9| \leq 2(N - 1)(N - 2) | \langle \mathbb{1}_{S_2} \nabla_2 \nabla_3 \hat{m}_2 q_1 p_3 q_2 \Psi, h_N(x_1 - x_2) h_N(x_1 - x_3) \hat{m}_2 q_1 p_2 q_3 \Psi \rangle |
\]

\[
+ 2(N - 1)(N - 2) | \langle \mathbb{1}_{S_2} \nabla_2 \hat{m}_2 q_1 p_3 q_2 \Psi, h_N(x_1 - x_2) h_N(x_1 - x_3) \mathbb{1}_{S_3} \nabla_3 \hat{m}_2 q_1 p_2 q_3 \Psi \rangle |
\]

\[
+ 2(N - 1)(N - 2) | \langle \mathbb{1}_{S_2} \nabla_3 \hat{m}_2 q_1 p_3 q_2 \Psi, h_N(x_1 - x_2) h_N(x_1 - x_3) \hat{m}_2 q_1 p_2 q_3 \Psi \rangle |
\]

\[
+ 2(N - 1)(N - 2) | \langle \mathbb{1}_{S_2} \nabla_2 \nabla_3 \hat{m}_2 q_1 p_3 q_3 \Psi, h_N(x_1 - x_2) h_N(x_1 - x_3) \mathbb{1}_{S_3} \nabla_3 \hat{m}_2 q_1 p_2 q_3 \Psi \rangle |,
\]

so as above $S_9$ is bounded by the right hand side of (12). For $S_{10}$ we estimate

\[
\| (\nabla_1 h_N(x_1 - x_j))^2 \|_1 = h_N(x_1 - x_j) \Delta_1 h_N(x_1 - x_j)
\]

38
which is (see below (18)) of order $N^{-2+3\beta}$. Using $|a b| < a^2 + b^2$ we get that $S_6$ is bounded by the right hand side of (13).

For $S_7$ we have using symmetry

$$S_7 = \frac{1}{N-1} \left| \sum_{j=2}^{N} \langle \Psi, q_j p_j \hat{m}_2(\nabla_1 h_N(x_1 - x_j)) \mathbb{1}_{\mathcal{F}_1} \nabla_1 q_j \Psi \rangle \right|$$

$$\leq \frac{1}{N-1} \left\| \sum_{j=2}^{N} q_j(\nabla_1 h_N(x_1 - x_j)) q_j p_j \hat{m}_2 \Psi \right\| \left\| \mathbb{1}_{\mathcal{F}_1} \nabla_1 q_j \Psi \right\| .$$

Using again symmetry we have

$$\left\| \sum_{j=2}^{N} q_j(\nabla_1 h_N(x_1 - x_j)) q_j p_j \hat{m}_2 \Psi \right\|^2 \leq \sum_{j=2}^{N} \left\| q_j(\nabla_1 h_N(x_1 - x_j)) q_j p_j \hat{m}_2 \Psi \right\|^2$$

\[+ \sum_{j \neq k \neq 1} \langle \Psi, \hat{m}_2 q_j p_k h_N(x_1 - x_k) \rangle q_k q_j(\nabla_1 h_N(x_1 - x_j)) q_j p_j \hat{m}_2 \Psi \rangle \]

\[= (N-1)\| q_j(\nabla_1 h_N(x_1 - x_j)) q_j p_2 \hat{m}_2 \Psi \|^2 \]

\[+ (N-1)(N-2)\langle \Psi, \hat{m}_2 q_1 p_2(\nabla_1 h_N(x_1 - x_2)) (\nabla_1 h_N(x_1 - x_3)) q_1 q_2 p_3 \hat{m}_2 \Psi \rangle \]

For the first summand we have

\[(N-1)\| q_j(\nabla_1 h_N(x_1 - x_2)) q_1 p_2 \hat{m}_2 \Psi \|^2 \leq (N-1)\| (\nabla_1 h_N(x_1 - x_2)) \|^2 \| \varphi_1^{GP} \|^2_{\infty} \| \hat{m}_2 q_1 p_2 \Psi \|^2 \]

\[\leq C(N-1)\| \varphi_1^{GP} \|_{\infty}^2 N^{-2} N^{\beta} \leq C N^{\beta-1} \]

For the second summand of the right hand side of (58) we get using $\nabla_1 h_N(x_1 - x_2) = -\nabla_2 h_N(x_1 - x_2)$, integrating by parts and using symmetry

\[= 2(N-1)(N-2)\langle \Psi, \hat{m}_2 q_1 q_2 p_2(\nabla_2 h_N(x_1 - x_2)) (\nabla_3 h_N(x_1 - x_3)) q_1 q_2 p_3 \hat{m}_2 \Psi \rangle \]

\[+ 2(N-1)(N-2)\langle \Psi, \hat{m}_2 q_1 q_3 p_2(\nabla_2 h_N(x_1 - x_2)) (\nabla_3 h_N(x_1 - x_3)) q_1 q_2 p_3 \hat{m}_2 \Psi \rangle \]

\[+ (N-1)(N-2)\langle \Psi, \hat{m}_2 q_1 q_2 p_3(\nabla_2 h_N(x_1 - x_2)) (\nabla_3 h_N(x_1 - x_3)) q_1 \mathbb{1}_{\mathcal{F}_2} \nabla_2 q_2 p_3 \hat{m}_2 \Psi \rangle \]

\[+ (N-1)(N-2)\langle \Psi, \hat{m}_2 q_1 q_3 p_3(\nabla_2 h_N(x_1 - x_2)) (\nabla_3 h_N(x_1 - x_3)) q_1 q_2 p_3 \hat{m}_2 \Psi \rangle \]

\[\leq \left(2 C N^{2} \left\| \mathbb{1}_{\mathcal{F}_1} \nabla_3 q_3 \hat{m}_2 \Psi \right\| \left\| \nabla_1 \varphi_1^{GP} \right\|_{\infty} \left\| h_N \right\|^2 \left\| \varphi_1^{GP} \right\|_{\infty} \right) \left\| q_1 q_2 \hat{m}_2 \Psi \right\|^2 \]

\[+ 2 N^2 \left\| \nabla_1 q_3 \hat{m}_2 \Psi \right\| \left\| \nabla_1 \varphi_1^{GP} \right\|_{\infty} \left\| h_N \right\| \left\| \nabla_1 \varphi_1^{GP} \right\|_{\infty} \sqrt{N} \left\| \mathbb{1}_{\mathcal{F}_1} \right\| \left\| \varphi_1^{GP} \right\|_{\infty} \left\| q_1 q_2 \hat{m}_2 \Psi \right\|^2 \]

\[+ N^2 \left\| \mathbb{1}_{\mathcal{F}_2} \nabla_1 q_3 q_2 \hat{m}_2 \Psi \right\| \left\| \nabla_1 \varphi_1^{GP} \right\|_{\infty} ^2 \left\| h_N \right\| \left\| \nabla_1 \varphi_1^{GP} \right\|_{\infty} ^2 \left\| q_1 q_2 \hat{m}_2 \Psi \right\|^2 \]

\[+ 2 N^2 \left\| \mathbb{1}_{\mathcal{F}_1} \right\| \left\| \varphi_1^{GP} \right\|_{\infty} \left\| \nabla_3 q_3 \hat{m}_2 \Psi \right\| \left\| \nabla_1 \varphi_1^{GP} \right\|_{\infty} ^2 \left\| h_N \right\| \left\| \nabla_1 \varphi_1^{GP} \right\|_{\infty} ^2 \left\| q_1 q_2 \hat{m}_2 \Psi \right\|^2 \].
Since with (34)
\[ \|1_{S_3}\| = \|1_{S_3}\|^{1/2} = \frac{3}{4\pi} N^{-17/9} \]
(58) is bounded by \(C(\alpha(\Psi) + N^{-1/18})\). With \((55), (56)\) and \((57)\) and using \(|ab| < a^2 + b^2\) it follows that \(S_7\) is bounded by the right hand side of \((43)\).

For \(S_8\) note, that \(\nabla_1 h_N(x_1 - x_2) = -\nabla_2 h_N(x_1 - x_2)\). Integration by parts yields
\[
S_8 \leq |\langle \nabla_1 \nabla_2 q_1 p_2 \Psi, h_N(x_1 - x_2) \hat{m}_1 q_1 q_2 \Psi \rangle| + |\langle \nabla_1 q_1 p_2 \Psi, h_N(x_1 - x_2) \nabla_2 \hat{m}_1 q_1 q_2 \Psi \rangle| \\
\leq \|\nabla \varphi_{GP}^{(1)}\|_\infty \|h_N\| \|\hat{m}_1 q_1 q_2 \Psi\| \\
+ \|\nabla \varphi_{GP}^{(2)}\|_\infty \|\nabla \varphi_{GP}^{(1)}\|_\infty \|h_N\| \|\hat{m}_1 q_1 \nabla_2 q_2 \Psi\|
\]
which is in view of Lemma 2.3 and (49) of order \(N^{-1-\beta/2}\) and (33) follows. 

\[\square\]

References

[1] Erdös, L.; Schlein, B.; Yau, H.-T.: Derivation of the Gross-Pitaevskii Hierarchy for the Dynamics of Bose-Einstein Condensate. Commun. Pure Appl. Math. 59 (2006), no. 12, 16591741.

[2] Erdös, L.; Schlein, B.; Yau, H.-T.: Derivation of the cubic non-linear Schrodinger equation from quantum dynamics of many-body systems. Invent. Math. 167 (2007), 515614.

[3] Erdös, L.; Schlein, B.; Yau, H.-T.: Derivation of the Gross-Pitaevskii Equation for the Dynamics of Bose-Einstein Condensate. Preprint arXiv:math-ph/0606017 To appear in Ann. Math.

[4] Erdös, L.; Schlein, B.; Yau, H.-T.: Rigorous Derivation of the Gross-Pitaevskii Equation with a Large Interaction Potential arXiv:math-ph/0802.3877v2

[5] Lieb, E.H.; Seiringer, R.: Proof of Bose-Einstein condensation for dilute trapped gases. Phys. Rev. Lett. 88 (2002), 170409-1-4.

[6] Lieb, E.H.; Seiringer, R.; Solovej, J.P.; Yngvason, J.: The mathematics of the Bose gas and its condensation. Oberwolfach Seminars, 34 Birkhauser Verlag, Basel, 2005.

[7] Lieb, E.H.; Seiringer, R.; Yngvason, J.: Bosons in a trap: a rigorous derivation of the Gross-Pitaevskii energy functional. Phys. Rev A 61 (2000), 043602.

[8] Fröhlich, J.; Knowles, A. and Pizzo, A.: Atomism and quantization. J. Phys. A: Math. Theor. 40 (2007) 3033-3045