RIGOROUS DERIVATION OF A BINARY-TERNARY BOLTZMANN EQUATION FOR A DENSE GAS OF HARD SPHERES

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ABSTRACT. This paper provides the first rigorous derivation of a binary-ternary Boltzmann equation describing the kinetic properties of a dense hard-spheres gas, where particles undergo either binary or ternary instantaneous interactions, while preserving momentum and energy. An important challenge we overcome in deriving this equation is related to providing a mathematical framework that allows us to detect both binary and ternary interactions. Furthermore, this paper introduces new algebraic and geometric techniques in order to eventually decouple binary and ternary interactions and understand the way they could succeed one another in time.

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1. Introduction

The Boltzmann equation, introduced by L. Boltzmann [10] and J.C. Maxwell [32] describes the time evolution of the probability density of a rarefied, monoatomic gas in thermal non-equilibrium in \( \mathbb{R}^d \), for \( d \geq 2 \). The Boltzmann equation accurately describes very dilute gases since only binary interactions between particles are taken into account. However, when the gas is dense enough, higher order interactions are much more likely to happen, therefore they significantly affect time evolution of the gas. A relevant example is a colloid, which is a homogeneous non-crystalline substance consisting of either large molecules or ultramicroscopic particles of one substance dispersed through a second substance. In [34], authors pointed out importance of including higher order interactions among particles in a colloidal gas. In particular, they show that in addition to binary interactions, interactions among three particles significantly contribute to the grand potential of the colloid. A surprising result of [34], but of invaluable computational importance in numerical simulations, is that interactions among three particles are actually characterized by the sum of the distances between particles, as opposed to depending on different geometric configurations among interacting particles. The results of [34] have been further verified experimentally e.g. [18] and numerically e.g. [28].
Motivated by the observations of [31], in [4] we suggested a model which goes beyond binary interactions incorporating sums of higher order interaction terms. In particular, we introduced the generalized interaction equation
\begin{equation}
\begin{cases}
\partial_t f + v \cdot \nabla_x f = \sum_{k=2}^{m} Q_k(f,f,\ldots,f), & (t,x,v) \in (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d, \\
f(0,x,v) = f_0(x,v), & (x,v) \in \mathbb{R}^d \times \mathbb{R}^d,
\end{cases}
\tag{1.1}
\end{equation}
where, for \( k = 1,\ldots,m \), the expression \( Q_k(f,f,\ldots,f) \) is the \( k \)-th order collisional operator and \( m \in \mathbb{N} \) is the number.

The first challenge we face in deriving (1.1) is to provide a mathematical framework allowing us to detect both binary and ternary interactions among particles. We achieve that by assuming the following:

- Binary interactions are modeled as elastic collisions of hard spheres of diameter \( \epsilon \) i.e. two particles interact when the distance of their centers defined as
  \[ d_2(x_i,x_j) := |x_i - x_j| \]
  becomes equal to the diameter \( \epsilon \).

- Ternary interactions are of interaction zone type as in [4], by which we mean that the particle \( i \) interacts with the particles \( j \) and \( k \) when the non-symmetric ternary distance
  \[ d_3(x_i;x_j,x_k) := \sqrt{|x_i - x_j|^2 + |x_i - x_k|^2} \]
  becomes \( \sqrt{3}\epsilon \).

Simultaneous consideration of both binary and ternary interactions brings us closer to understanding our first obstacle. In particular, in the works on the derivation of the binary Boltzmann equation for hard spheres, pioneered by Lanford [31] and recently completed by Gallagher, Saint-Raymond, Texier [22], the relevant scaling is the Boltzmann-Grad scaling [23, 24]
\begin{equation}
N\epsilon^{d-1} \simeq 1,
\end{equation}
\begin{equation}
(1.2)
\end{equation}
as the number of particles \( N \to \infty \) and their diameter \( \epsilon \to 0^+ \).

On the other hand, the scaling used in [4] to control ternary interactions is a different scaling:
\begin{equation}
N\epsilon^{d-1/2} \simeq 1,
\end{equation}
\begin{equation}
(1.3)
\end{equation}
A crucial, conceptual obstacle is the apparent incompatibility of the Boltzmann-Grad scaling (1.2) dictated by binary interactions and the scaling (1.3) of ternary interactions, if both of them are of order \( \epsilon \). This incompatibility creates major difficulties even at the formal level. We overcome this scaling obstacle by assuming that, at the \( N \)-particle level, hard spheres are of diameter \( \epsilon_2 \) and that particles interact as triplets via an interaction zone \( \epsilon_3 \). Imposing scalings (1.2) with \( \epsilon := \epsilon_2 \) and (1.3) with \( \epsilon := \epsilon_3 \), we obtain the common scaling
\begin{equation}
N\epsilon_2^{d-1} \simeq N\epsilon_3^{d-1/2} \simeq 1,
\end{equation}
\begin{equation}
(1.4)
\end{equation}
as \( N \to \infty \) and \( \epsilon_2, \epsilon_3 \to 0^+ \). Notice that the scaling (1.4) implies that for sufficiently large \( N \), we have
\begin{equation}
\epsilon_2 << \epsilon_3,
\end{equation}
\begin{equation}
(1.5)
\end{equation}
which will have a prominent role in this paper.
The next challenge we address is the need to decouple binary and ternary interactions for a system of finitely many particles. More precisely, our framework a-priori allows i.e. that particles \(i\) and \(j\) interact as hard spheres:

\[
d_2(x_i, x_j) = \epsilon_2,
\]

while at the same time there is another particle \(k\) such that the particle \(i\) interacts with the particles \(j\) and \(k\):

\[
d_3(x_i; x_j, x_k) = \sqrt{2}\epsilon_3.
\]

Such a configuration is illustrated in Figure 1.1. Pathological configurations, including the one we just described, are going to be shown to be negligible. This is far from trivial and for more details on the microscopic dynamics, see Subsection 1.2 and Section 3. In particular, we shall show that as long as \(0 < \epsilon_2 < \epsilon_3 < 1\), only the following two interaction scenarios are possible with non-trivial probability under time evolution:

(i) Two particles interact as hard-spheres while all other particles are not involved in any binary or ternary interactions at the same time. This type of configurations generates the binary collisional operator. It is illustrated in Figure 1.2.

(ii) Three particles interact via an interaction zone, while none of them is involved in a binary interaction with either of the other two particles of the interaction zone at the same time. The rest of the particles are not involved in any binary or ternary interactions. This type of configurations is responsible for generating the ternary collisional operator. It is illustrated in Figure 1.3.

Finally, since we will eventually let the number of particles \(N \to \infty\), the main challenge we need to address is the stability of a good configuration\(^1\) under the adjunction of one or two collisional particles. Assume, for a moment, that we have a good configuration of \(m\)-particles and we add \(\sigma\) particles to the system, where \(\sigma \in \{1, 2\}\), such that a binary or ternary interaction is formed among one of the existing particles and the \(\sigma\) new particles. In general, under backwards time evolution, the system could run into another binary or ternary interaction, see e.g. Figure 1.4 which illustrates the mathematically most

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\(^1\)by which we mean a configuration which does not run into any kind of interactions under backwards time evolution.
difficult case where the newly formed \((m+2)\)-configuration runs into a binary interaction. To the best of our knowledge, this is the first time there was the need to address the possibility of a newly formed interacting configuration running into an interaction of a different type (binary to ternary or ternary to binary) backwards in time. However, in Section 8 and Section 9, we develop novel algebraic and geometric tools which help us eliminate pathological scenarios, including the one described in Figure 1.4 by showing that outside of a small measure set, negligible in the limit, the newly formed configuration does not run into any additional interactions backwards in time. For more details on the technical difficulties faced, see Subsection 1.5.

In the next subsection, we investigate more precisely what happens when a binary or a ternary interactions occurs and describe the time evolution of such a system.

1.2. Dynamics of finitely many particles. Let us describe the evolution in \(\mathbb{R}^d\), \(d \geq 2\), of a system of \(N\) hard spheres of diameter \(\epsilon_2\) and interaction zone \(\epsilon_3\), where \(0 < \epsilon_2 < \epsilon_3 < 1\). The assumption \(\epsilon_2 < \epsilon_3\) is necessary for ternary interactions to be of non trivial probability, see Remark 3.1 for more details.

1.2.1. Interactions considered. We first define the interactions considered in this paper.

Definition 1.1. Let \(N \in \mathbb{N}\), with \(N \geq 3\), and \(0 < \epsilon_2 < \epsilon_3 < 1\). We define binary and ternary interactions, also referred to as collisions, as follows:

- Consider two particles \(i, j \in \{1, \ldots, N\}\) with positions \(x_i, x_j \in \mathbb{R}^d\). We say that the particles \(i, j\) are in an \((i, j)\) binary interaction, if the following geometric condition holds:

  \[
  d_2(x_i, x_j) := |x_i - x_j| = \epsilon_2. \tag{1.6}
  \]

- Consider three particles \(i, j, k \in \{1, \ldots, N\}\), with positions \(x_i, x_j, x_k \in \mathbb{R}^d\). We say that the particles \(i, j, k\) are in an \((i, j, k)\) interaction\(^{2}\) if the following geometric condition holds:

  \[
  d_3(x_i; x_j, x_k) := \sqrt{|x_i - x_j|^2 + |x_i - x_k|^2} = \sqrt{2}\epsilon_3. \tag{1.7}
  \]

When an \((i, j)\) interaction occurs, the velocities \(v_i, v_j\) of the \(i\)-th and \(j\)-th particles instantaneously transform according to the binary collisional law:

\[
\begin{align*}
  v'_i &= v_i + (\omega_1, v_j - v_i)\omega_1, \\
  v'_j &= v_j - (\omega_1, v_j - v_i)\omega_1, \\
  \omega_1 &:= \frac{x_j - x_i}{\epsilon_2}, \tag{1.8}
  \end{align*}
\]

where

\[

\frac{x_j - x_i}{\epsilon_2}.
\]

Thanks to (1.6), we have \(\omega_1 \in S^{d-1}\). The vector \(\omega_1\) is called binary impact direction and it represents the scaled relative position of the colliding particles. Moreover, one can see that the binary momentum-energy system:

\[
\begin{align*}
  v' + v'_1 &= v + v_1, \\
  |v'|^2 + |v'_1|^2 &= |v|^2 + |v_1|^2, \tag{1.10}
  \end{align*}
\]

is satisfied.

\(^{2}\)we use the notation \((i; j, k)\) because the interaction condition is not symmetric. The particle \(i\) is the central particle of the interaction i.e. the one interacting with the particles \(j\) and \(k\) respectively.
When an \((i; j, k)\) interaction happens, the velocities \(v_i, v_j, v_k\) of the \(i\)-th, \(j\)-th and \(k\)-th particles instantaneously transform according to the ternary collisional law derived in [4]

\[
\begin{align*}
v_i^* &= v_i + \frac{\langle \omega_1, v_j - v_i \rangle + \langle \omega_2, v_k - v_i \rangle}{1 + \langle \omega_1, \omega_2 \rangle} (\omega_1 + \omega_2), \\
v_j^* &= v_j - \frac{\langle \omega_1, v_j - v_i \rangle + \langle \omega_2, v_k - v_i \rangle}{1 + \langle \omega_1, \omega_2 \rangle} \omega_1, \\
v_k^* &= v_k - \frac{\langle \omega_1, v_j - v_i \rangle + \langle \omega_2, v_k - v_i \rangle}{1 + \langle \omega_1, \omega_2 \rangle} \omega_2,
\end{align*}
\]

(1.11)

where

\[
(\omega_1, \omega_2) := \left( \frac{x_j - x_i}{\sqrt{2\epsilon_3}}, \frac{x_k - x_i}{\sqrt{2\epsilon_3}} \right).
\]

(1.12)

Thanks to (1.7), we have \((\omega_1, \omega_2) \in S^{2d-1}_1\). The vectors \((\omega_1, \omega_2)\) are called ternary impact directions and they represent the scaled relative positions of the interacting particles. Moreover, it has been shown in [4] that the ternary momentum-energy system:

\[
|v^*|^2 + |v_1^*|^2 + |v_2^*|^2 = |v|^2 + |v_1|^2 + |v_2|^2,
\]

(1.13)

is satisfied.

1.2.2. Phase space and description of the flow. Let \(N \in \mathbb{N}\), with \(N \geq 3\), and \(0 < \epsilon_2 < \epsilon_3 < 1\). The natural phase space \(\mathcal{D}\) to capture both binary and ternary interactions is:

\[
\mathcal{D}_{N,\epsilon_2,\epsilon_3} = \{ Z_N = (X_N, V_N) \in \mathbb{R}^{2dN} : d_2(x_i, x_j) \geq \epsilon_2, \forall (i, j) \in \mathcal{I}_N^2, \text{ and } d_3(x_i, x_j, x_k) \geq \sqrt{2\epsilon_3}, \forall (i, j, k) \in \mathcal{I}_N^3 \},
\]

(1.14)

where \(X_N = (x_1, x_2, ..., x_N), V_N = (v_1, v_2, ..., v_N)\), represent the positions and velocities of the \(N\)-particles, and the index sets \(\mathcal{I}_N^2, \mathcal{I}_N^3\) are given by

\[
\mathcal{I}_N^2 = \{(i, j) \in \{1, ..., N\}^2 : i < j\}, \quad \mathcal{I}_N^3 = \{(i, j) \in \{1, ..., N\}^3 : i < j < k\}.
\]

Let us describe the evolution in time of such a system. Consider an initial configuration \(Z_N \in \mathcal{D}_{N,\epsilon_2,\epsilon_3}\). The motion is described as follows:

(I) Particles are assumed to perform rectilinear motion as long as there is no interaction

\[
\dot{x}_i = v_i, \quad \dot{v}_i = 0, \quad \forall i \in \{1, ..., N\}.
\]

(II) Assume now that an initial configuration \(Z_N = (X_N, V_N)\) has evolved until time \(t > 0\), reaching \(Z_N(t) = (X_N(t), V_N(t))\), and that there is an interaction at time \(t\). We have the following cases:

- The interaction is binary: Assuming there is an \((i, j)\) interaction the velocities of the interacting particles instantaneously transform velocities according to the binary collisional law \((v_i(t), v_j(t)) \rightarrow (v_i^*(t), v_j^*(t))\) given in (1.8).
- The interaction is ternary: Assuming there is an \((i; j, k)\) interaction, the velocities of the interacting particles instantaneously transform velocities according to the ternary collisional law \((v_i(t), v_j(t), v_k(t)) \rightarrow (v_i^*(t), v_j^*(t), v_k^*(t))\) given in (1.11).

Let us note that (I)-(II) are not sufficient to generate a global in time flow for the particle system, since the velocity transformations are not smooth. In general pathologies might arise as time evolves, meaning more than one type of interactions happening at the same time, grazing interaction, or infinitely many interactions in finite time. Although, well-defined dynamics was shown to exist in [1] for hard spheres and in [3] for the purely ternary case, those results do not imply well-posedness of the flow for the mixed case, where both binary and ternary interactions are taken into account. The reason for that is that a binary interaction can be succeeded by a ternary interaction and vice versa, a situation which was not addressed in [1] nor [3]. However we are showing that a non-grazing interaction cannot be succeeded by the same interaction. In other words, when two particles \((i, j)\) interact, the next interaction could be anything, binary or ternary, except a binary recollision of the particles \((i, j)\). Similarly, when three particles there

\[\text{upon symmetrization, one could define the phase space without ordering the particles and obtain a symmetrized version of ternary operator (see [2] for more details). For simplicity, we opt to work upon ordering the particles.}\]
is an \((i; j, k)\) interaction, the next interaction can be anything except a ternary \((i, j, k)\) interaction. This observation allows us to define the flow locally a.e. and then run some combinatorial covering arguments to geometrically exclude a zero Lebesgue measure set such that the flow is globally in time defined on the complement.

Let us informally state this result. For a detailed statement, see Theorem 3.23.

**Existence of a global flow:** Let \(N \in \mathbb{N}\) and \(0 < \varepsilon_2 < \varepsilon_3 < 1\). There is a global in time measure-preserving flow \((\Psi_{\varepsilon_2, \varepsilon_3}^t)_{t \in \mathbb{R}} : \mathcal{D}_{N, \varepsilon_2, \varepsilon_3} \to \mathcal{D}_{N, \varepsilon_2, \varepsilon_3}\) described a.e. by (I)-(II) which preserves kinetic energy. This flow is called the \((\varepsilon_2, \varepsilon_3)\)-interaction flow.

The global measure-preserving interaction flow yields the Liouville equation for the evolution \(f_N\) of an initial \(N\)-particle probability density \(f_{N,0}\):

\[
\partial_t f_N + \sum_{i=1}^N v_i \nabla_{x_i} f_N = 0, \quad (t, Z_N) \in (0, \infty) \times \mathcal{D}_{N, \varepsilon_2, \varepsilon_3},
\]

\[
f_N(t, Z_N) = f(t, Z_N), \quad t \in [0, \infty), \quad Z_N \text{ is a simple binary interaction},
\]

\[
f_N(t, Z_N) = f(t, Z_N), \quad t \in [0, \infty), \quad Z_N \text{ is a simple ternary interaction},
\]

\[
f_N(0, Z_N) = f_{N,0}(Z_N), \quad Z_N \in \mathcal{D}_{N, \varepsilon_2, \varepsilon_3}.
\]

The Liouville equation provides a complete deterministic description of the system of \(N\)-particles. Although Liouville’s equation is a linear transport equation, efficiently solving it is almost impossible in case where the particle number \(N\) is very large. This is why an accurate kinetic description is welcome, and to obtain it one wants to understand the limiting behavior of it as \(N \to \infty\) and \(\varepsilon_2, \varepsilon_3 \to 0^+\), with the hope that qualitative properties will be revealed for a large but finite \(N\).

### 1.3. The binary-ternary Boltzmann equation

To obtain such a kinetic description, we let the number of particles \(N \to \infty\) and the diameter and interaction zone of the particles \(\varepsilon_2, \varepsilon_3 \to 0^+\) in the **common scaling** (1.4):

\[
N\varepsilon_2^{-\frac{d-1}{2}} \approx N\varepsilon_3^{-\frac{d-2}{2}} \approx 1,
\]

which will lead the binary-ternary Boltzmann equation

\[
\partial_t f + v \cdot \nabla_x f = Q_2(f, f) + Q_3(f, f, f), \quad (t, x, v) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d,
\]

\[(1.16)\]

where \(Q_2(f, f)\) is the classical hard sphere binary collisional operator given by

\[
Q_2(f, f) = \int_{S^d_{1} \times \mathbb{R}^d} \langle \omega_1, v_1 - v \rangle+ (f' f_1 - f f_1) \, d\omega_1 \, dv_1,
\]

\[(1.17)\]

The operator \(Q_3(f, f, f)\), introduced for the first time in [3], is the ternary hard interaction zone operator given by

\[
Q_3(f, f, f) = \int_{S^d_{2} \times \mathbb{R}^d} b_+ (f^* f_1 f_2 - f f_1 f_2) \, d\omega_2 \, dv_2 \, dv_3,
\]

\[(1.18)\]

where

\[
b = b(\omega_1, \omega_2, v_1 - v, v_2 - v) := \langle \omega_1, v_1 - v \rangle + \langle \omega_2, v_2 - v \rangle, \quad b_+ = \max\{b, 0\},
\]

\[
f^* = f(t, x, v') = f(x, t, v), \quad f'_i = f'_i(t, x, v'_i), \quad f_i = f(t, x, v_i), \quad \text{for } i \in \{1, 2\}.
\]

\[(1.19)\]

We should mention that in [3], global well-posedness near vacuum has been shown for (1.16) for potentials ranging from moderately soft to hard in spaces of functions bounded by Maxwellian. In fact in

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4Any other permutation of the particle \(i, j, k\) cannot form an interaction since \(i < j < k\). In case one does not order the particles, a subsequent \((j, i, k)\) interaction, for instance, could possibly happen.

5In case \(N = 2\), the ternary boundary condition is not present in (1.18), while if \(N = 1\), equation (1.15) is just the transport equation.

6By simple binary interaction, we mean the only interaction happening is an \((i, j)\) interaction. In this case, we write \(Z_N' = (X_N, V_N')\), where \(V_N' = (v_1, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots, v_{j-1}, v_j, v_{j+1}, \ldots, v_N)\).

7By simple ternary interaction, we mean the only interaction happening is an \((i, j, k)\) interaction. In this case, we write \(Z_N' = (X_N, V_N')\), where \(V_N' = (v_1, \ldots, v_{i-1}, v'_i, v_{i+1}, \ldots, v_{j-1}, v'_j, v_{j+1}, \ldots, v_{k-1}, v'_k, v_{k+1}, \ldots, v_N)\).
It is seen that the ternary collisional operator allows consideration of softer potentials that the binary operator. In other words the ternary correction to the Boltzmann equation does not behave worse than the classical Boltzmann equation.

It is important to point out that, upon symmetrization of the ternary collisional operator (see [2]), the corresponding binary-ternary Boltzmann equation enjoys similar statistical and entropy production properties and conservation laws as the classical Boltzmann equation. Therefore, such a model could serve as a correction of the classical Boltzmann equation to denser gases. This follows after combining the properties of the classical binary operator, see e.g. [12], with the properties of the symmetrized ternary collisional operator investigated for the first time in [3]. It is important to point out that, upon symmetrization of the ternary collisional operator (see [2]), the common scaling (1.4). The most challenging task is to make the argument rigorous, though, is the first difficulty is the apparent incompatibility of scalings (1.2)-(1.3), which we overcome by introducing the first difficulty is the apparent incompatibility of scalings (1.2)-(1.3), which we overcome by introducing the new binary-ternary Boltzmann equation. In [4], we extended this program to include ternary interactions, which led to the rigorous derivation of a purely ternary kinetic equation for particles with hard interaction zone in the scaling (1.3). However, rigorous development of novel geometric and algebraic tools, which we discuss in details in Subsection 1.5. For now, we continue to discuss the process of derivation.

1.4. Strategy of the derivation and statement of the main result. In order to pass from the N-particle system dynamics to the kinetic equation (1.10), we implement the program of constructing linear finite and infinite hierarchies of equations, pioneered by Lanford [31] and refined by Gallagher, Saint-Raymond, Texier [22], and connecting them to the new binary-ternary Boltzmann equation. In [1], we extended this program to include ternary interactions, which led to the rigorous derivation of a purely ternary kinetic equation for particles with hard interaction zone in the scaling (1.3). Therefore, such a model could serve as a correction of the classical Boltzmann equation to denser gases. This follows after combining the properties and conservation laws as the classical Boltzmann equation. Therefore, such a model could serve as a correction of the classical Boltzmann equation to denser gases. This follows after combining the properties and conservation laws as the classical Boltzmann equation.

For the precise form of the operators \( C_{s+1} \), \( C_{s+2} \), see (4.15)-(4.16). Duhamel’s Formula yields that the BBGKY hierarchy can be written in mild form as follows:

\[
f^{(s)}_N(t, Z_s) = T_s^t f_N,0(Z_s) + \int_0^t T_{s-t}^s (C_{s+1} f^{(s+1)}_N + C_{s+2} f^{(s+2)}_N)(\tau, Z_s) d\tau, \quad s \in \mathbb{N},
\]

where for any continuous function \( g_s : D_{s, \epsilon_2, \epsilon_3} \to \mathbb{R} \), we write \( T_s^t g_s(Z_s) := g_s(\Psi^t_s Z_s) \), and \( \Psi^t_s \) is the \((\epsilon_2, \epsilon_3)\)-interaction zone flow of \( s \)-particles.

We then formally let \( N \to \infty \) and \( \epsilon_2, \epsilon_3 \to 0^+ \) in the scaling (1.4) to obtain an infinite, linear, coupled hierarchy of equations, which we call the Boltzmann hierarchy. This hierarchy is given by

\[
\partial_t f^{(s)} + \sum_{i=1}^{N} v_i \cdot \nabla_x f^{(s)} = C_{s+1} \infty f^{(s+1)} + C_{s+2} \infty f^{(s+2)}, \quad s \in \mathbb{N}.
\]

For the precise form of the operators \( C_{s+1, \infty} \), \( C_{s+2, \infty} \), see (4.28), (4.32) respectively. Duhamel’s Formula yields that the Boltzmann hierarchy can be written in mild form as follows:

\[
f^{(s)}(t, Z_s) = S^t_s f_0(Z_s) + \int_0^t S^{t-\tau}_{s} (C_{s+1, \infty} f^{(s+1)} + C_{s+2, \infty} f^{(s+2)})(\tau, Z_s) d\tau, \quad s \in \mathbb{N},
\]

\[\text{by recollisions we mean the possible divergence of the backwards (} \epsilon_2, \epsilon_3 \text{)-interaction flow from the backwards free flow.}
\]

\[\text{Bogoliubov, Born, Green, Kirkwood, Yvon.}\]
where for any continuous function \( g_s : \mathbb{R}^{2d_s} \to \mathbb{R} \), we write \( S^*_t g_s(Z_s) := g_s(\Phi^{-1}_t Z_s) \), and \( \Phi_t^s \) is the \( s \)-particle free flow of \( s \)-particles defined by \( S^*_t Z_s = S^*_t(X_s, V_s) = (X_s - tV_s, V_s) \).

It can be observed that for factorized initial data and assuming that the solution remains factorized in time\(^{10}\), the Boltzmann hierarchy reduces to the binary-ternary Boltzmann equation (1.16). This observation connects the Boltzmann hierarchy with the binary-ternary Boltzmann equation (1.16).

To make this argument rigorous, we first show that the BBGKY and Boltzmann hierarchy are well-posed in the scaling (1.4), at least for short times, and then that the convergence of the BBGKY hierarchy initial data to the Boltzmann hierarchy initial data propagates in the time interval of existence of the solutions. Showing convergence is a very challenging task, and is the heart of our contribution. We describe details in Subsection 1.5.

Now, we informally state our main result. For a rigorous statement of the result see Theorem 6.5.

**Statement of the main result:** Let \( F_0 \) be initial data for the Boltzmann hierarchy (1.23), and \( F_{N,0} \) be some BBGKY hierarchy (1.23) initial data which “approximate” \( F_0 \) as \( N \to \infty \), \( \varepsilon \to 0^+ \) under the scaling (1.4). Let \( F_N \) be the mild solution to the BBGKY hierarchy (1.21) with initial data \( F_{N,0} \), and \( F \) the mild solution to the Boltzmann hierarchy (1.23), with initial data \( F_0 \), up to short time \( T > 0 \). Then \( F_N \) converges in observables\(^{12}\) to \( F \) in \([0, T]\) as \( N \to \infty \), \( \varepsilon \to 0^+ \), under the scaling (1.4).

The convergence obtained implies that the solution of the finite hierarchy indeed approximates the solution of the infinite hierarchy in \([0, T]\), as \( N \to \infty \), \( \varepsilon_2, \varepsilon_3 \to 0^+ \) in the scaling (1.4). For factorized initial data (initial chaotic assumption) the Boltzmann hierarchy reduces to equation (1.16).

1.5. **Difficulties faced in the proof of the main result.** The main idea to obtain convergence (Theorem 6.5) is to inductively use mild forms (1.22), (1.24) of the BBGKY hierarchy and Boltzmann hierarchy respectively, to formally obtain series expansions with respect to the initial data:

\[
j_N^{(s)}(t, Z_s) = T^s_s j_{N,0}^{(s)}(Z_s) + \sum_{k=1}^{\infty} \sum_{s \in S_k} \int_0^t \int_0^{t_1} ... \int_0^{t_{k-1}} T^{t-t_1} ... T^{t_{k-1}-t_2} ... T^{t_1-0} f_{N,0}^{(s)}(Z_s) dt_k ... dt_1,
\]

\[
f^{(s)}(t, Z_s) = S^*_s f_0^{(s)}(Z_s) + \sum_{k=1}^{\infty} \sum_{s \in S_k} \int_0^t \int_0^{t_1} ... \int_0^{t_{k-1}} S^*_{t-t_1} ... S^*_{t_{k-1}-t_2} ... S^*_{t_1-0} f_0^{(s)}(Z_s) dt_k ... dt_1,
\]

where \( S_k \) is defined in (1.24) and given \( s \in S_k, \ell = 1, ..., k \), we write \( \tilde{S}_\ell := \sum_{i=1}^\ell \sigma_i \). We note that the summation over \( S_k \) in (1.25)-(1.26) allows us to keep track of the possible interaction sequences occurring by “adding” one or two particles in each time step. For more details, see Section 7.

Comparing expressions (1.25)-(1.26), we expect to obtain the required convergence under the scaling (1.4) as long as \( f_N^{(s)} \) “approximates” \( f^{(s)} \) under the same scaling. However, it is not possible to directly compare (1.25)-(1.26) because of the possible divergence of the backwards interaction flow from the free flow, which we call recollisions. Although recollisions were also faced in [22] and [4], the mixed case, where both binary and ternary interactions are considered, requires different conceptual treatment in many instances, and is not implied by the results of these works. The reason for that is that a binary interaction can be succeeded by a ternary interaction and vice versa, a situation which was not addressed in [22] [4]. The key to overcome these difficulties is that the diameter of the particles is much smaller than the interaction zone, as implied by the common scaling (1.4). This fact allows us to develop certain delicate algebraic and geometric arguments to extract a small measure set of pathological initial data which lead to recollisions. On the complement of this set, expansions (1.25)-(1.26) are comparable and the required convergence is obtained.

The main idea for eliminating recollisions is an inductive application in each time step of Proposition 9.2 and Proposition 9.3 which treat the binary adjunction, or Proposition 9.6 and Proposition 9.7 which treat the ternary adjunction. More precisely we face the following different cases:

10 this is typically called propagation of chaos assumption
11 see Subsection 6.2 for details
12 for a precise definition of convergence in observables, see Subsection 6.2
(I) **Binary adjunction**: One particle is added forming a binary interaction with one of the existing particles. The pathological situations that might arise under backwards time evolution are the following:

- The newly formed binary collisional configuration runs to a binary interaction under time evolution. This pathological situation is eliminated using arguments inspired by [22]. This is actually the only case which is similar to the cases covered in [22].
- The newly formed binary collisional configuration runs to a ternary interaction under time evolution. This pathological situation did not appear in any of the previous works since merely binary or ternary interactions were studied. However, due to the fact that $\epsilon_2 \ll \epsilon_3$, which comes from the scaling (1.4), this pathological situation can be treated using techniques inspired by [4] and adapting them to the binary case.

Proposition 9.2 and Proposition 9.4 are the relevant results controlling recollisions after a binary adjunction.

(II) **Ternary adjunction**: Two particles are added forming a ternary interaction with one of the existing particles. The pathological situations that might arise under backwards time evolution are the following:

- The newly formed ternary collisional configuration runs to a ternary interaction under time evolution. This case was studied in depth in [4]. We eliminate this pathological situation using Proposition 9.5. For its proof, we refer to [4].
- The newly formed ternary collisional configuration runs to a binary interaction under time evolution. This is the most challenging case to treat and is the heart of the technical contribution, because the scaling (1.4) does not directly help as in the case of the binary adjunction where one of the collisional particles enters an interaction zone. To treat this case, we need to use new algebraic tools (see Proposition 9.6) to exclude sets of initial data which lead to these pathological trajectories and develop elaborate geometric estimates to control its measure. The geometric estimates needed are thoroughly presented in Section 8. In particular, Subsection 8.2 is devoted to developing novel tools which rely on an appropriate representation of $(2d-1)$-spheres (see (8.1)). More specifically, in 8.2.1 we perform some initial truncations to the impact directions, while in 8.2.2 we establish certain spherical cap and conic region estimates needed to control the precollisional case, while 8.2.3 focuses on developing the necessary annuli estimates enabling us to control the postcollisional case using precollisional arguments. After establishing the necessary geometric tools, we employ them in Proposition 9.7 to show that the corresponding set constructed in Proposition 9.6 is negligible.

1.6. **Notation.** For convenience, we introduce some basic notation which will be frequently used throughout the manuscript:

- $d \in \mathbb{N}$ will be a fixed dimension with $d \geq 2$.
- Given $x, y \in \mathbb{R}$, we write
  \begin{align*}
  x \lesssim y &\iff \exists C_d > 0 : x \leq C_d y, \\
  x \simeq y &\iff \exists C_d > 0 : x = C_d y, \\
  x \approx y &\iff \exists C_1, d, C_2, d > 0 : C_1, d y \leq x \leq C_2, d y.
  \end{align*}
- Given $n \in \mathbb{N}$, $\rho > 0$ and $w \in \mathbb{R}^n$, we write $B^\rho_n(w)$ for the $n$-closed ball of radius $\rho > 0$, centered at $w \in \mathbb{R}^n$. In particular, we write $B^\rho_n := B^\rho_n(0)$, for the $\rho$-ball centered at the origin.
- Given $n \in \mathbb{N}$ and $\rho > 0$, we write $S^{n-1}_\rho$ for the $(n-1)$-sphere of radius $\rho > 0$.
- When we write $x << y$, we mean that there is a small enough constant $0 < c < 1$, independent of $x, y$, such that $x < cy$. This constant $c$ is appropriately chosen for the calculations to make sense.

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2. Collisional transformations

In this section we define the collisional transformations of two and three interacting particles respectively. In the two particle case, particles will interact as regular hard spheres, while in the three particle case, particles will interact as triplets of particles with an interaction zone.

2.1. Binary interaction. Here, we define the binary collisional transformation of two interacting hard spheres, induced by an impact direction \( \omega_1 \in S_1^{d-1} \). This will be the law under which the velocities \((v_1, v_2)\) of two interacting hard spheres, with impact direction \( \omega_1 \in S_1^{d-1} \), instananeously transform. The impact direction will represent the scaled relative position of the colliding hard spheres.

Definition 2.1. Consider a binary impact direction \( \omega_1 \in S_1^{d-1} \). We define the binary collisional transformation induced by \( \omega_1 \in S_1^{d-1} \) as the map \( T_{\omega_1} : (v_1, v_2) \in \mathbb{R}^{2d} \rightarrow (v'_1, v'_2) \in \mathbb{R}^{2d} \), where

\[
\begin{align*}
v'_1 &= v_1 + \langle \omega_1, v_2 - v_1 \rangle \omega_1, \\
v'_2 &= v_2 - \langle \omega_1, v_2 - v_1 \rangle \omega_1.
\end{align*}
\]

(2.1)

Let us introduce some notation we will be constantly using. We define the binary cross-section

\[
b_2(\omega_1, \nu_1) := \langle \omega_1, \nu_1 \rangle, \quad (\omega_1, \nu_1) \in S_1^{d-1} \times \mathbb{R}^d.
\]

(2.2)

Under this notation (2.1) can be written as :

\[
\begin{align*}
v'_1 &= v_1 + b_2(\omega_1, v_2 - v_1) \omega_1, \\
v'_2 &= v_2 - b_2(\omega_1, v_2 - v_1) \omega_1.
\end{align*}
\]

(2.3)

One can verify that (2.3) provide the general solution, parametrized by \( \omega_1 \in S_1^{d-1} \), of the binary momentum-energy conservation system:

\[
|v'_1|^2 + |v'_2|^2 = |v_1|^2 + |v_2|^2.
\]

(2.4)

Given a binary impact direction \( \omega_1 \in S_1^{d-1} \), the binary collisional transformation \( T_{\omega_1} \) satisfies the following properties (see, e.g. [12]).

Proposition 2.2. Consider a binary impact direction \( \omega_1 \in S_1^{d-1} \). The induced binary collisional transformation \( T_{\omega_1} \) has the following properties:

(i) Conservation of momentum

\[
v'_1 + v'_2 = v_1 + v_2.
\]

(2.5)

(ii) Conservation of energy

\[
|v'_1|^2 + |v'_2|^2 = |v_1|^2 + |v_2|^2.
\]

(2.6)

(iii) Conservation of relative velocities magnitude

\[
|v'_1 - v'_2| = |v_1 - v_2|.
\]

(2.7)

(iv) Micro-reversibility of the binary cross-section

\[
b_2(\omega_1, v'_2 - v'_1) = -b_2(\omega_1, v_2 - v_1).
\]

(2.8)

(v) \( T_{\omega_1} \) is a linear involution i.e. \( T_{\omega_1} \) is linear and \( T^{-1}_{\omega_1} = T_{\omega_1} \). In particular, \( |\det T_{\omega_1}| = 1 \), so \( T_{\omega_1} \) is measure-preserving.

2.2. Ternary interaction. Now we define the ternary collisional transformation, induced by a given pair of impact directions, and investigate its properties. The interaction considered will be an instantaneous interaction of three particles with an interaction zone (for more details see [11]). This will be the law under which the velocities \((v_1, v_2, v_3)\) of three interacting particles, with impact directions \((\omega_1, \omega_2) \in S_1^{2d-1}\), instananeously transform. The impact directions will represent the scaled relative positions of the three particles in the interaction zone setting.
Definition 2.3. Consider a pair of impact directions \((\omega_1, \omega_2) \in S_1^{2d-1}\). We define the ternary collisional transformation induced by \((\omega_1, \omega_2)\) as the map \(T_{\omega_1, \omega_2} : (v_1, v_2, v_3) \in \mathbb{R}^d \rightarrow (v_1^*, v_2^*, v_3^*) \in \mathbb{R}^d\), where
\begin{align*}
v_1^* &= v_1 + c_{\omega_1, \omega_2, v_1, v_2, v_3}(\omega_1 + \omega_2), \\
v_2^* &= v_2 - c_{\omega_1, \omega_2, v_1, v_2, v_3}\omega_1, \\
v_3^* &= v_3 - c_{\omega_1, \omega_2, v_1, v_2, v_3}\omega_2,
\end{align*}

\begin{equation}
\label{eq:2.9}
c_{\omega_1, \omega_2, v_1, v_2, v_3} = \frac{\langle \omega_1, v_2 - v_3 \rangle + \langle \omega_2, v_3 - v_1 \rangle}{1 + \langle \omega_1, \omega_2 \rangle}.
\end{equation}

We also define the ternary cross-section
\begin{equation}
\label{eq:2.10}
b_3(\omega_1, \omega_2, v_1, v_2) := \langle \omega_1, v_1 \rangle + \langle \omega_2, v_2 \rangle, \quad (\omega_1, \omega_2) \in S_1^{2d-1}, \quad (v_1, v_2) \in \mathbb{R}^d.
\end{equation}

Notice that, given \((\omega_1, \omega_2, v_1, v_2, v_3) \in S_1^{2d-1} \times \mathbb{R}^d\), we clearly have
\begin{equation}
\label{eq:2.11}
b_3(\omega_1, \omega_2, v_2 - v_1, v_3 - v_1) = (1 + \langle \omega_1, \omega_2 \rangle)c_{\omega_1, \omega_2, v_2, v_3, v_3}.
\end{equation}

Remark 2.4. Cauchy-Schwartz inequality and the fact that
\begin{equation}
\label{eq:2.12}
since
\end{equation}

hence for all \((\omega_1, \omega_2, v_1, v_2, v_3) \in S_1^{2d-1} \times \mathbb{R}^d\), relation (2.12) implies
\begin{equation}
\label{eq:2.13}
\frac{2}{3}b_3(\omega_1, \omega_2, v_2 - v_1, v_3 - v_1) \leq c_{\omega_1, \omega_2, v_2, v_3, v_3} \leq 2b_3(\omega_1, \omega_2, v_2 - v_1, v_3 - v_1).
\end{equation}

It has been seen in \([4, 2]\) that (2.13) provide the general solution, parametrized by \((\omega_1, \omega_2) \in S_1^{2d-1}\), of the ternary momentum-energy conservation system:
\begin{equation}
|v_1|^2 + |v_2|^2 + |v_3|^2 = |v_1^*|^2 + |v_2^*|^2 + |v_3^*|^2.
\end{equation}

The main properties of the ternary collisional transformation are summarized in the following Proposition. For the proof, see Proposition 2.3. from \([4]\).

Proposition 2.5. Consider a pair of impact directions \((\omega_1, \omega_2) \in S_1^{2d-1}\). The induced collisional transformation \(T_{\omega_1, \omega_2}\) has the following properties:
(i) Conservation of momentum
\begin{equation}
|v_1|^2 + |v_2|^2 + |v_3|^2 = |v_1| + |v_2| + |v_3|.
\end{equation}

(ii) Conservation of energy
\begin{equation}
|v_1| + |v_2| + |v_3| = |v_1|^2 + |v_2|^2 + |v_3|^2.
\end{equation}

(iii) Conservation of relative velocities magnitude
\begin{equation}
|v_1 - v_2|^2 + |v_2 - v_3|^2 + |v_3 - v_1|^2 = |v_1 - v_2|^2 + |v_1 - v_3|^2 + |v_2 - v_3|^2.
\end{equation}

(iv) Micro-reversibility of the ternary cross-section
\begin{equation}
b_3(\omega_1, \omega_2, v_2 - v_1, v_3 - v_1) = -b_3(\omega_1, \omega_2, v_2 - v_1, v_3 - v_1).
\end{equation}

(v) \(T_{\omega_1, \omega_2}\) is a linear involution i.e. \(T_{\omega_1, \omega_2}\) is linear and \(T_{\omega_1, \omega_2}^{-1} = T_{\omega_1, \omega_2}\). In particular, \(|\det T_{\omega_1, \omega_2}| = 1\), so \(T_{\omega_1, \omega_2}\) is measure-preserving.

3. Dynamics of \(m\)-particles

In this section we rigorously define the dynamics of \(m\) hard spheres of diameter \(\sigma_2\) and interaction zone \(\sigma_3\), where \(0 < \sigma_2 < \sigma_3 < 1\). Heuristically speaking, particles perform rectilinear motion as long as there is no interaction (binary or ternary) and they interact through the binary or ternary collision law when a binary or ternary interaction occurs respectively. However, it is far from obvious that a global dynamics can be defined, since the system might run into pathological configurations e.g. more than one interactions at a time, infinitely many interactions in finite time or interactions which graze under time evolution. The goal of this section is to extract a set of measure zero such that on the complement a global in time, measure preserving flow can be defined.

Throughout this section we consider \(m \in \mathbb{N}\) and \(0 < \sigma_2 < \sigma_3 < 1\).
3.1. Phase space definitions. For convenience we define the following index sets:

For $m \geq 2$: $\mathcal{I}_m^2 = \{(i, j) \in \{1, \ldots, m\}^2 : i < j\}$. \hfill (3.1)

For $m \geq 3$: $\mathcal{I}_m^3 = \{(i, j, k) \in \{1, \ldots, m\}^3 : i < j < k\}$. \hfill (3.2)

Given positions $(x_1, x_2) \in \mathbb{R}^{2d}$, we define the binary distance:

$$d_2(x_1, x_2) := |x_1 - x_2|,$$

and given positions $(x_1, x_2, x_3) \in \mathbb{R}^{3d}$, we define the ternary distance:

$$d_3(x_1; x_2, x_3) = \sqrt{|x_1 - x_2|^2 + |x_1 - x_3|^2}.$$ \hfill (3.4)

For $m \geq 3$, we define the phase space of $m$-particles of diameter $\sigma_2 > 0$ and interaction zone $\sigma_3 > 0$, with $\sigma_2 < \sigma_3 < 1$ as:

$$\mathcal{D}_{m, \sigma_2, \sigma_3} = \{Z_m = (X_m, V_m) \in \mathbb{R}^{2dm} : d_2(x_i, x_j) \geq \sigma_2, \ \forall (i, j) \in \mathcal{I}_m^2, \text{ and } d_3(x_i; x_j, x_k) \geq \sqrt{\sigma_3}, \ \forall (i, j, k) \in \mathcal{I}_m^3\},$$

where $X_m = (x_1, \ldots, x_m) \in \mathbb{R}^{dm}$ represents the positions of the $m$-particles, while $V_m = (v_1, \ldots, v_m) \in \mathbb{R}^{dm}$ represents the velocities of the $m$-particles. For convenience we also define

$$\mathcal{D}_{2, \sigma_2, \sigma_3} = \{Z_2 = (X_2, V_2) \in \mathbb{R}^{2d} : |x_1 - x_2| \geq \sigma_2\}, \quad \mathcal{D}_{1, \sigma_2, \sigma_3} = \mathbb{R}^{3d}.$$ \hfill (3.6)

For $m \geq 3$, the phase space $\mathcal{D}_{m, \sigma_2, \sigma_3}$ decomposes as: $\mathcal{D}_{m, \sigma_2, \sigma_3} = \hat{\mathcal{D}}_{m, \sigma_2, \sigma_3} \cup \partial \mathcal{D}_{m, \sigma_2, \sigma_3}$, where the interior is given by:

$$\hat{\mathcal{D}}_{m, \sigma_2, \sigma_3} = \{Z_m = (X_m, V_m) \in \mathbb{R}^{2dm} : d_2(x_i, x_j) > \sigma_2, \ \forall (i, j) \in \mathcal{I}_m^2, \text{ and } d_3(x_i; x_j, x_k) > \sqrt{\sigma_3}, \ \forall (i, j, k) \in \mathcal{I}_m^3\},$$

and the boundary is given by:

$$\partial \mathcal{D}_{m, \sigma_2, \sigma_3} = \partial_2 \mathcal{D}_{m, \sigma_2, \sigma_3} \cup \partial_3 \mathcal{D}_{m, \sigma_2, \sigma_3};$$ \hfill (3.8)

where $\partial_2 \mathcal{D}_{m, \sigma_2, \sigma_3}$ is the binary boundary:

$$\partial_2 \mathcal{D}_{m, \sigma_2, \sigma_3} = \{Z_m = (X_m, V_m) \in \mathcal{D}_{m, \sigma_2, \sigma_3} : \exists (i, j) \in \mathcal{I}_m^2 \text{ with } d_2(x_i, x_j) = \sigma_2\},$$ \hfill (3.9)

and $\partial_3 \mathcal{D}_{m, \sigma_2, \sigma_3}$ is the ternary boundary:

$$\partial_3 \mathcal{D}_{m, \sigma_2, \sigma_3} = \{Z_m = (X_m, V_m) \in \mathcal{D}_{m, \sigma_2, \sigma_3} : \exists (i, j, k) \in \mathcal{I}_m^3 \text{ with } d_3(x_i; x_j, x_k) = \sqrt{\sigma_3}\}.$$ \hfill (3.10)

Elements of $\mathcal{D}_{m, \sigma_2, \sigma_3}$ are called configurations, elements of $\hat{\mathcal{D}}_{m, \sigma_2, \sigma_3}$ are called noncollisional configurations, and elements of $\partial \mathcal{D}_{m, \sigma_2, \sigma_3}$ are called collisional configurations, or just collisions. Elements of $\partial_2 \mathcal{D}_{m, \sigma_2, \sigma_3}$ are called binary collisions, while elements of $\partial_3 \mathcal{D}_{m, \sigma_2, \sigma_3}$ are called ternary collisions. When we refer to a collision, it will be either binary or ternary.

Clearly the binary boundary can be written as: $\partial_2 \mathcal{D}_{m, \sigma_2, \sigma_3} = \bigcup_{(i,j)\in\mathcal{I}_m^2} \Sigma_{ij}^2$, where $\Sigma_{ij}^2$ are the binary collisional surfaces given by

$$\Sigma_{ij}^2 := \{Z_m \in \mathcal{D}_{m, \sigma_2, \sigma_3} : d_2(x_i, x_j) = \sigma_2\}.$$ \hfill (3.11)

In the same spirit the ternary boundary can be written as: $\partial_3 \mathcal{D}_{m, \sigma_2, \sigma_3} = \bigcup_{(i,j,k)\in\mathcal{I}_m^3} \Sigma_{ijk}^3$, where $\Sigma_{ijk}^3$ are the ternary collisional surfaces given by

$$\Sigma_{ijk}^3 := \{Z_m \in \mathcal{D}_{m, \sigma_2, \sigma_3} : d_3(x_i; x_j, x_k) = \sqrt{\sigma_3}\}.$$ \hfill (3.12)

We now further decompose collisions to simple binary collisions, simple ternary collisions and multiple collisions. In particular we define simple binary collisions as:

$$\partial_{2,sc} \mathcal{D}_{m, \sigma_2, \sigma_3} := \{Z_m = (X_m, V_m) \in \mathcal{D}_{m, \sigma_2, \sigma_3} : \exists (i, j) \in \mathcal{I}_m^2 \text{ with } Z_m \in \Sigma_{ij}^2, \ Z_m \notin \Sigma_{i'j'}^2, \ \forall (i', j') \in \mathcal{I}_m^2 \setminus \{(i, j)\}\}.$$ \hfill (3.13)

We also define simple ternary collisions as:

$$\partial_{3,sc} \mathcal{D}_{m, \sigma_2, \sigma_3} := \{Z_m = (X_m, V_m) \in \mathcal{D}_{m, \sigma_2, \sigma_3} : \exists (i, j, k) \in \mathcal{I}_m^3 \text{ with } Z_m \in \Sigma_{ijk}^3, \ Z_m \notin \Sigma_{i'j'k'}^3, \ \forall (i', j', k') \in \mathcal{I}_m^3 \setminus \{(i, j, k)\}\}.$$ \hfill (3.14)
Remark 3.1. The assumption $\sigma_2 < \sigma_3$ made at the beginning of the section is necessary for $\partial_{3,sc}D_{m,\sigma_2,\sigma_3}$ to be non-empty. Indeed, let $\sigma_2 \geq \sigma_3$ and assume that $\partial_{3,sc}D_{m,\sigma_2,\sigma_3} \neq \emptyset$. Consider $Z_m \in \partial_{3,sc}D_{m,\sigma_2,\sigma_3}$. Then, by (3.14), there is $(i, j, k) \in I_m^3$ such that

$$|x_i - x_j|^2 + |x_i - x_j|^2 = 2\sigma_3^2,$$  \hspace{1cm} (3.15)

and

$$|x_i - x_j| > \epsilon_2, \quad |x_i - x_k| > \epsilon_2.$$  \hspace{1cm} (3.16)

By (3.15), at least one of $|x_i - x_j|$ or $|x_i - x_k|$ has to be smaller than or equal to $\epsilon_3$. Assume, without loss of generality, that $|x_i - x_j| \leq \epsilon_3$. Since $\epsilon_2 \geq \epsilon_3$, we obtain $|x_i - x_j| \leq \epsilon_2$, which contradicts (3.16). Therefore, if $\sigma_2 \geq \sigma_3$, we have $\partial_{3,sc}D_{m,\sigma_2,\sigma_3} = \emptyset$.

A simple collision will be a binary or ternary simple collision i.e.

$$\partial_{sc}D_{m,\sigma_2,\sigma_3} := \partial_{2,sc}D_{m,\sigma_2,\sigma_3} \cup \partial_{3,sc}D_{m,\sigma_2,\sigma_3}.$$  \hspace{1cm} (3.17)

Multiple collisions are configurations which are not simple i.e.

$$\partial_{mc}D_{m,\sigma_2,\sigma_3} := \partial_{2,mc}D_{m,\sigma_2,\sigma_3} \setminus \partial_{sc}D_{m,\sigma_2,\sigma_3}.$$  \hspace{1cm} (3.18)

Remark 3.2. For $m = 2$, there is only binary boundary.

For the binary case, we give the following definitions:

Definition 3.3. Let $m \geq 2$ and $Z_m \in \partial_{2,sc}D_{m,\sigma_2,\sigma_3}$. Then there is a unique $(i, j) \in I_m^2$ such that $Z_m \in \Sigma_{ij}^2$ and $Z_m \notin \Sigma_{i'j'k'}^2$, for all $(i', j', k') \in I_m^3$. In this case we will say $Z_m$ is an $(i, j)$ collision and we will write

$$\Sigma_{ij}^2 := \{Z_m \in D_{m,\sigma_2,\sigma_3} : Z_m \text{ is an } (i, j) \text{ collision}\}.$$  \hspace{1cm} (3.19)

Clearly $\Sigma_{ij}^2 \cap \Sigma_{i'j'}^2 = \emptyset$, for all $(i, j) \neq (i', j') \in I_m^2$ and $\partial_{2,sc}D_{m,\sigma_2,\sigma_3}$ decomposes to:

$$\partial_{2,sc}D_{m,\sigma_2,\sigma_3} = \bigcup_{(i, j) \in I_m^2} \Sigma_{ij}^2.$$  \hspace{1cm} (3.20)

Remark 3.4. Let $m \geq 2$, $(i, j) \in I_m^2$ and $Z_m \in \Sigma_{ij}^2$. Then

$$\omega_1 := \frac{x_i - x_j}{\sigma_2} \in S_1^{d-1}.$$  \hspace{1cm} (3.21)

Therefore, each $(i, j)$ collision naturally induces a binary impact direction $\omega_1 \in S_1^{d-1}$ and consequently a binary collisional transformation $T_{\omega_1}$.

Definition 3.5. Let $m \geq 2$, $(i, j) \in I_m^2$ and $Z_m = (X_m, V_m) \in \Sigma_{ij}^2$. We write $Z_m' = (X_m, V_m')$, where

$$V_m' = (v_1, ..., v_{i-1}, v_i', v_{i+1}, ..., v_{j-1}, v_j', v_{j+1}, ..., v_m),$$

and $(v_i', v_j') = T_{\omega_1}(v_i, v_j)$, $\omega_1 \in S_1^{d-1}$ is given by (3.21).

In the same spirit, for the ternary case, we give the following definitions:

Definition 3.6. Let $m \geq 3$ and $Z_m \in \partial_{3,sc}D_{m,\sigma_2,\sigma_3}$. Then there is a unique $(i, j, k) \in I_m^3$ such that $Z_m \in \Sigma_{ijk}^3$ and $Z_m \notin \Sigma_{i'j'k'}^3$, for all $(i', j', k') \in I_m^3$. In this case we will say $Z_m$ is an $(i, j, k)$ collision and we will write

$$\Sigma_{ijk}^3 := \{Z_m \in D_{m,\sigma_2,\sigma_3} : Z_m \text{ is an } (i, j, k) \text{ collision}\}.$$  \hspace{1cm} (3.22)

Clearly $\Sigma_{ijk}^3 \cap \Sigma_{i'j'k'}^3 = \emptyset$, for all $(i, j, k) \neq (i', j', k') \in I_m^3$ and $\partial_{3,sc}D_{m,\sigma_2,\sigma_3}$ decomposes to:

$$\partial_{3,sc}D_{m,\sigma_2,\sigma_3} = \bigcup_{(i, j, k) \in I_m^3} \Sigma_{ijk}^3.$$  \hspace{1cm} (3.23)

Remark 3.7. Let $m \geq 3$, $(i, j, k) \in I_m^3$ and $Z_m \in \Sigma_{ijk}^3$. Then

$$\omega_1 := \frac{1}{\sqrt{2}\sigma_3} (x_j - x_i, x_k - x_i) \in S_1^{2d-1}.$$  \hspace{1cm} (3.24)

Therefore, each $(i, j, k)$ collision naturally induces ternary impact directions $(\omega_1, \omega_2) \in S_1^{2d-1}$ and consequently a collisional transformation $T_{\omega_1,\omega_2}$. 
Remark 3.12. Let \( m \geq 3 \), \((i, j, k) \in T_m^3\) and \( Z_m = (X_m, V_m) \in \Sigma_{ij}^{k,s} \). We write \( Z_m^* = (X_m, V_m^*) \), where
\[
V_m^{\pm} = (v_1, ..., v_{i-1}, v_i^{\pm}, v_{i+1}, ..., v_{j-1}, v_j^{\pm}, v_{j+1}, ..., v_{k-1}, v_k^{\pm}, v_{k+1}, ..., v_m),
\]
and \( (v_i^+, v_j^+, v_k^+) = T_{\omega_1, \omega_2}(v_1, v_j, v_k) \). \( (\omega_1, \omega_2) \in S_{1}^{2d-1} \) are given by \((3.24)\).

3.2. Classification of simple collisions. We will now classify simple collisions in order to eliminate collisions which graze in time. For this purpose, we come across the following definitions for the binary and the ternary case respectively.

For the binary case:

Definition 3.9. Let \( m \geq 2 \), \((i, j) \in T_m^2\) and \( Z_m \in \Sigma_{ij}^{s} \). The configuration \( Z_m \) is called:

- binary precollisional when \( b_2(\omega_1, v_j - v_i) < 0 \),
- binary postcollisional when \( b_2(\omega_1, v_j - v_i) > 0 \),
- binary grazing when \( b_2(\omega_1, v_j - v_i) = 0 \),

where \( \omega_1 \in S_{1}^{d-1} \) is given by \((3.21)\) and \( b_2 \) is given by \((2.2)\).

Remark 3.10. Let \( m \geq 2 \), \((i, j) \in T_m^2\) and \( Z_m \in \Sigma_{ij}^{s} \). Using \((2.8)\), we obtain the following:

(i) \( Z_m \) is binary precollisional iff \( Z_m^* \) is binary postcollisional.
(ii) \( Z_m \) is binary postcollisional iff \( Z_m^* \) is binary precollisional.
(iii) \( Z_m = Z_m^* \) iff \( Z_m \) is binary grazing.

For the ternary case:

Definition 3.11. Let \( m \geq 3 \), \((i, j, k) \in T_m^3\) and \( Z_m \in \Sigma_{ij}^{k,s} \). The configuration \( Z_m \) is called:

- ternary precollisional when \( b_3(\omega_1, \omega_2, v_j - v_i, v_k - v_i) < 0 \),
- ternary postcollisional when \( b_3(\omega_1, \omega_2, v_j - v_i, v_k - v_i) > 0 \),
- ternary grazing when \( b_3(\omega_1, \omega_2, v_j - v_i, v_k - v_i) = 0 \),

where \( (\omega_1, \omega_2) \in S_{1}^{2d-1} \) is given by \((3.21)\) and \( b_3 \) is given by \((2.11)\).

Remark 3.12. Let \( m \geq 3 \), \((i, j, k) \in T_m^3\) and \( Z_m \in \Sigma_{ij}^{k,s} \). Using \((2.19)\), we obtain the following:

(i) \( Z_m \) is ternary precollisional iff \( Z_m^* \) is ternary postcollisional.
(ii) \( Z_m \) is ternary postcollisional iff \( Z_m^* \) is ternary precollisional.
(iii) \( Z_m = Z_m^* \) iff \( Z_m \) is ternary grazing.

We will just say precollisional, postcollisional or grazing configuration when it is implied whether a simple collision is binary or ternary.

For \( m \geq 2 \), we refine the phase space defining
\[
D_{m,\sigma_2,\sigma_3}^* := D_{m,\sigma_2,\sigma_3} \cup \partial_{sc,ng}D_{m,\sigma_2,\sigma_3}, \tag{3.25}
\]
where \( \partial_{sc,ng}D_{m,\sigma_2,\sigma_3} \) denotes the part of \( \partial D_{m,\sigma_2,\sigma_3} \) consisting of simple, non-grazing collisions i.e. defined as
\[
\partial_{sc,ng}D_{m,\sigma_2,\sigma_3} := \{ Z_m \in \partial D_{m,\sigma_2,\sigma_3} : Z_m \text{ is non-grazing} \}. \tag{3.26}
\]
It is immediate that \( D_{m,\sigma_2,\sigma_3}^* \) is a full measure subset of \( D_{m,\sigma_2,\sigma_3} \) and \( \partial_{sc,ng}D_{m,\sigma_2,\sigma_3} \) is a full surface measure subset of \( \partial D_{m,\sigma_2,\sigma_3} \), since its complement constitutes of lower dimension submanifolds of \( \partial D_{m,\sigma_2,\sigma_3} \) which have zero surface measure.

3.3. Construction of the local flow. Next Lemma shows that the flow can be locally defined for any initial configuration \( Z_m \in D_{m,\sigma_2,\sigma_3} \) up to the time of the first collision.

Lemma 3.13. Let \( m \geq 3 \) and \( Z_m \in D_{m,\sigma_2,\sigma_3} \). Then there is a time \( \tau_{Z_m} \in (0, \infty) \) such that defining \( Z_m(\cdot) : [0, \tau_{Z_m}^1] \to \mathbb{R}^{2dm} \) by:
\[
Z_m(t) = \begin{cases} 
(X_m + tV_m, V_m) & \text{if } Z_m \text{ is noncollisional or postcollisional,} \\
(X_m + tV'_m, V'_m) & \text{if } Z_m \text{ is binary precollisional,} \\
(X_m + tV''_m, V''_m) & \text{if } Z_m \text{ is ternary precollisional,} 
\end{cases}
\]
the following hold:
(i) \( Z_m(t) \in \mathcal{D}_{m,\sigma_2,\sigma_3}, \forall t \in (0, \tau^1_{Z_m}) \).
(ii) if \( \tau^1_{Z_m} < \infty \), then \( Z_m(\tau^1_{Z_m}) \in \partial \mathcal{D}_{m,\sigma_2,\sigma_3} \).
(iii) If \( Z_m \in \Sigma^1_{ijk} \) for some \((i, j) \in \mathcal{T}_m\), then \( Z_m(\tau^1_{Z_m}) \notin \Sigma^1_{ij} \).
(iv) If \( Z_m \in \Sigma^3_{ijk} \) for some \((i, j, k) \in \mathcal{T}_m\), then \( Z_m(\tau^1_{Z_m}) \notin \Sigma^3_{ijk} \).

An analogous statement holds in the case \( m = 2 \), where we just neglect the ternary terms.

Proof. Let us make the convention \( \inf \emptyset = +\infty \). We define
\[
\tau^1_{Z_m} = \begin{cases}
\inf \{ t > 0 : X_m + tV_m \in \partial \mathcal{D}_{m,\sigma_2,\sigma_3} \}, & \text{if } Z_m \text{ is noncollisional or postcollisional}, \\
\inf \{ t > 0 : X_m + tV_m \in \partial \mathcal{D}_{m,\sigma_2,\sigma_3} \}, & \text{if } Z_m \text{ is binary precollisional}, \\
\inf \{ t > 0 : X_m + tV^*_m \in \partial \mathcal{D}_{m,\sigma_2,\sigma_3} \}, & \text{if } Z_m \text{ is ternary precollisional}.
\end{cases}
\]

Since \( \mathcal{D}_{m,\sigma_2,\sigma_3} \) is open, we get \( \tau^1_{Z_m} > 0 \), \( \forall Z_m \in \mathcal{D}_{m,\sigma_2,\sigma_3} \) and claims (i)-(ii) follow immediately for \( Z_m \in \mathcal{D}_{m,\sigma_2,\sigma_3} \).

Assume \( Z_m \in \partial_{\sigma_2,\sigma_3} \mathcal{D}_{m,\sigma_2,\sigma_3} \) which yields that \( Z_m \) is non-grazing. Therefore we may distinguish the following cases:

- \( Z_m \) is an \((i, j)\) binary postcollisional configuration: For any \( t > 0 \), we have
  \[
  \| x_i - x_j + (v_i - v_j)t \|^2 = \| x_i - x_j \|^2 + t^2\| v_i - v_j \|^2 + 2t\langle x_i - x_j, v_i - v_j \rangle \\
  \geq \sigma_2^2 + 2t\sigma_3(x_i - x_j, v_i - v_j) \\
  > \sigma_2^2,
  \]
  since \( b_3(\omega_1, v_j - v_i) > 0 \). This inequality and the fact that \( Z_m \) is a simple binary collision imply that \( \tau^1_{Z_m} > 0 \) and claims (i), (ii), (iii) as well.

- \( Z_m \) is \((i, j)\) binary precollisional configuration: We use the same argument for \( Z'_m \) which is \((i, j)\) binary postcollisional.

- \( Z_m \) is an \((i; j, k)\) ternary postcollisional configuration: For any \( t > 0 \), we have
  \[
  \| x_i - x_j + (v_i - v_j)t \|^2 + \| x_i - x_k + (v_i - v_k)t \|^2 \\
  = \| x_i - x_j \|^2 + \| x_i - x_k \|^2 + t^2\left(\| v_i - v_j \|^2 + \| v_i - v_k \|^2 + 2t(\langle x_i - x_j, v_i - v_j \rangle + \langle x_i - x_k, v_i - v_k \rangle)\right) \\
  \geq 2\sigma_2^2 + 2t\sigma_3(x_i - x_j, v_i - v_j + v_i - v_k) \\
  > 2\sigma_2^2,
  \]
  since \( b_3(\omega_1, \omega_2, v_j - v_i, v_k - v_i) > 0 \). This inequality and the fact that \( Z_m \) is a simple ternary collision imply that \( \tau^1_{Z_m} > 0 \) and claims (i), (ii), (iv) as well.

- \( Z_m \) is an \((i; j, k)\) ternary precollisional configuration: We use the same argument for \( Z'_m \) which is \((i; j, k)\) ternary postcollisional.

\[ \square \]

Let us make an elementary but crucial remark.

**Remark 3.14.** Clearly for configurations with \( \tau^1_{Z_m} = \infty \) the flow is globally defined as the free flow. In the case where \( \tau^1_{Z_m} < \infty \) and \( Z_m(\tau^1_{Z_m}) \) is a non-grazing \((i, j)\) collision or non-grazing \((i; j, k)\) collision, we may apply Lemma 3.13 once more and get a corresponding time \( \tau^2_{Z_m} \) with the property that \( Z_m(\tau^2_{Z_m}) \notin \Sigma^1_{ij} \) or \( Z_m(\tau^2_{Z_m}) \notin \Sigma^3_{ijk} \), respectively, if \( \tau^1_{Z_m} < \infty \). Therefore, in this case the flow can be defined up to time \( \tau^2_{Z_m} \).

**Remark 3.15.** Note that Lemma 3.13 implies that given a non-grazing \((i, j)\) collision, the next collision (if it happens) will not be \((i, j)\). Similarly, given a non-grazing \((i; j, k)\) collision, the next collision (if it happens) will not be \((i; j, k)\). However, Lemma 3.13 it does not imply that the same particles are not involved in a collision of a different type. For instance, one could have the sequence of collisions \((i, j)\) and \((i; j, k)\), or \((i; j, k)\) and \((i, j)\) etc. All these cases will be taken into account when establishing a global flow in Subsection 3.4.
Remark 3.16. Similar results hold for the case $m = 2$ where there are no ternary interactions.

3.4. Extension to a global flow. Now, we extract a zero measure set from $D^*_m,\sigma_2,\sigma_3$ such that the flow is globally defined on the complement. For this purpose, we will first truncate positions and velocities using two parameters $1 < R < \rho$ and then perform time truncation with a small parameter $\delta$ in the scaling:

$$0 < \delta R << \sigma_2 < \sigma_3 < 1 < R << \rho.$$  \hfill  (3.27)

Throughout this subsection, we consider parameters satisfying the scaling (3.27).

Recall that given $r > 0$ we denote the $dm$-ball of radius $r > 0$, centered at the origin as $B^*_r$. We first assume initial positions are in $B^*_\rho$ and initial velocities in $B^*_R$.

For $m \geq 2$, we decompose $D^*_m,\sigma_2,\sigma_3 \cap (B^*_\rho \times B^*_R)$ in the following subsets:

$$I^I_{\text{free}} = \{ Z_m = (X_m, V_m) \in D^*_m,\sigma_2,\sigma_3 \cap (B^*_\rho \times B^*_R) : \tau^1_{Z_m} > \delta \},$$

$$I^1_{\text{sc,ng}} = \{ Z_m = (X_m, V_m) \in D^*_m,\sigma_2,\sigma_3 \cap (B^*_\rho \times B^*_R) : \tau^2_{Z_m} \leq \delta, \ Z_m(\tau^1_{Z_m}) \in \partial_{\text{sc,ng}} D_m,\sigma_2,\sigma_3 \text{ and } \tau^2_{Z_m} > \delta \},$$

$$I^1_{\text{sc,ng}} = \{ Z_m = (X_m, V_m) \in D^*_m,\sigma_2,\sigma_3 \cap (B^*_\rho \times B^*_R) : \tau^1_{Z_m} \leq \delta, \ Z_m(\tau^1_{Z_m}) \in \partial_{\text{sc,ng}} D_m,\sigma_2,\sigma_3 \},$$

$$I^1_{\text{ng}} = \{ Z_m = (X_m, V_m) \in D^*_m,\sigma_2,\sigma_3 \cap (B^*_\rho \times B^*_R) : \tau^2_{Z_m} \leq \delta, \ Z_m(\tau^2_{Z_m}) \in \partial_{\text{ng}} D_m,\sigma_2,\sigma_3 \},$$

$$I^2_{\text{sc,ng}} = \{ Z_m = (X_m, V_m) \in D^*_m,\sigma_2,\sigma_3 \cap (B^*_\rho \times B^*_R) \text{ is grazing} \},$$

$$I^2_{\text{sc,ng}} = \{ Z_m = (X_m, V_m) \in D^*_m,\sigma_2,\sigma_3 \cap (B^*_\rho \times B^*_R) : \tau^1_{Z_m} \leq \delta, \ Z_m(\tau^1_{Z_m}) \in \partial_{\text{sc,ng}} D_m,\sigma_2,\sigma_3 \},$$

$$I^2_{\text{ng}} = \{ Z_m = (X_m, V_m) \in D^*_m,\sigma_2,\sigma_3 \cap (B^*_\rho \times B^*_R) \text{ is grazing} \}.$$

We remark that there is a well-defined flow up to time $\delta$ for $Z_m \in I^I_{\text{free}} \cup I^I_{\text{sc,ng}}$, since in such cases one has at most one simple non-grazing collision in $[0, \delta]$. We aim to estimate the measure of the pathological set $I^3_{\text{sc,ng}} \cup I^3_{\text{ng}} \cup I^2_{\text{sc,ng}}$, with respect to the truncation parameters.

Lemma 3.17. Assume $m \geq 2$. Then $I^3_{\text{sc,ng}}$ is of zero Lebesgue measure.

Proof. Assume first $m \geq 3$. Clearly $I^3_{\text{sc,ng}} \subseteq \bigcup_{(i,j) \in I^2_m} M^1_{ij} \cup \bigcup_{(i,j,k) \in I^3_m} M^3_{ijk}$, where

$$M^1_{ij} = \{ Z_m \in D^*_m,\sigma_2,\sigma_3 \cap (B^*_\rho \times B^*_R) : Z_m(\tau^1_{Z_m}) \text{ is an } (i, j) \text{ grazing collision} \}$$

and

$$M^3_{ijk} = \{ Z_m \in D^*_m,\sigma_2,\sigma_3 \cap (B^*_\rho \times B^*_R) : Z_m(\tau^1_{Z_m}) \text{ is an } (i, j, k) \text{ grazing collision} \}.$$

The above covering consists of lower dimension submanifolds of the space, so it has zero measure. For $m = 2$, we use a similar argument. \hfill \Box

Before proceeding to the next result, let us note that conservation of energy (3.20), (3.21) imply the following elementary but useful remark:

Remark 3.18. The following hold:

- For $m \geq 2$: $Z_m \in \partial_{\text{sc,ng}} D_m,\sigma_2,\sigma_3 \cap (R^d \times B^*_R) \Leftrightarrow Z_m \in \partial_{\text{sc,ng}} D_m,\sigma_2,\sigma_3 \cap (R^d \times B^*_R)$.

- For $m \geq 3$: $Z_m \in \partial_{\text{sc,ng}} D_m,\sigma_2,\sigma_3 \cap (R^d \times B^*_R) \Leftrightarrow Z_m \in \partial_{\text{sc,ng}} D_m,\sigma_2,\sigma_3 \cap (R^d \times B^*_R)$.

Lemma 3.19. For $m \geq 3$, the following inclusion holds:

$$I^3_{\text{ng}} \cup I^3_{\text{sc,ng}} \subseteq U_{22} \cup U_{23} \cup U_{32} \cup U_{33},$$  \hfill  (3.28)

where

$$U_{22} := \bigcup_{(i,j) \neq (i', j')} U^2_{ij} \cup U^2_{ij'},$$  \hfill  (3.29)

$$U_{23} := \bigcup_{(i,j) \in I^2_m, (i', j') \in I^2_m} U^3_{ij} \cup U^3_{ij'},$$  \hfill  (3.30)

$$U_{32} := \bigcup_{(i,j,k) \in I^3_m, (i', j', k') \in I^3_m} U^3_{ijk} \cup U^3_{ij'},$$  \hfill  (3.31)

$$U_{33} := \bigcup_{(i,j,k) \neq (i', j', k')} U^3_{ijk} \cup U^3_{ij'k'},$$  \hfill  (3.32)
and given $(i, j) \in I^2_m$, $(i', j, k) \in I^3_m$, we denote

\[
U^2_{ij} := \left\{ Z_m = (X_m, V_m) \in B^1_{0*} \times B^1_{0*} : \sigma_2 \leq d_2(x_i, x_j) \leq \sigma_2 + 2\delta R \right\}.
\]

(3.33)

\[
U^3_{ijk} := \left\{ Z_m = (X_m, V_m) \in B^1_{0*} \times B^1_{0*} : 2\sigma^2 \leq d^2_2(x_i; x_j, x_k) \leq (\sqrt{2}\sigma_3 + 4\delta R)^2 \right\}.
\]

(3.34)

For $m = 2$, we have $I^2_m = I^2_{sc,nq} = \emptyset$.

**Proof.** For $m = 2$, we have that $\partial_{mu} D_{2, a_2, a_3} = \emptyset$, hence $I^2_{mu} = \emptyset$. Also, since $m = 2$, we trivially obtain $I_3 = \{(1, 2)\}$, hence Remark 3.14 implies that $\tau^2_{2m} = \infty$ i.e. $I^2_{sc,nq} = \emptyset$.

Assume now that $m \geq 3$. We first assume that either $Z_m \in D_{m, a_2, a_3}$ or $Z_m$ is postcollisional. Therefore, up to time $\tau^1_{2m}$, we have free flow i.e. $Z_m(t) = (X_m + tV_m, V_m)$, for all $t \in [0, \tau^1_{2m}]$.

**Inclusion for $I^1_{mu}$:**

We have $\tau^1_{2m} \leq \delta$ and $Z_m(\tau^1_{2m}) \in \partial_{mu} D_{m, a_2, a_3}$. We claim the following which clearly imply inclusion (3.28) for $I^1_{mu}$:

(I) $Z_m(\tau^1_{2m}) \in \Sigma^2_{ij} \cap \Sigma^2_{i'j'} \Rightarrow Z_m \in U^2_{ij} \cap U^2_{i'j'}$, \quad $\forall (i, j), (i', j') \in I^2_m$.

(II) $Z_m(\tau^1_{2m}) \in \Sigma^2_{ij} \cap \Sigma^2_{i'j'} \Rightarrow Z_m \in U^2_{ij} \cap U^2_{i'j'}$, \quad $\forall (i, j) \in I^2_m, \quad (i', j', k') \in I^3_m$.

(III) $Z_m(\tau^1_{2m}) \in \Sigma^3_{ijk} \cap \Sigma^3_{i'j'k'} \Rightarrow Z_m \in U^3_{ijk} \cap U^3_{i'j'k'}$, \quad $\forall (i, j, k) \in I^3_m, \quad (i', j', k', l') \in I^4_m$.

(IV) $Z_m(\tau^1_{2m}) \in \Sigma^3_{ijk} \cap \Sigma^3_{i'j'k'} \Rightarrow Z_m \in U^3_{ijk} \cap U^3_{i'j'k'}$, \quad $\forall (i, j, k, l) \in I^3_m, \quad (i', j', k', l') \in I^4_m$.

Without loss of generality, we prove claim (III). We have $Z_m(\tau^1_{2m}) \in \Sigma^3_{ijk} \cap \Sigma^3_{i'j'k'}$, therefore

\[
d^2_2(x_i; x_j, x_k) \leq \delta R, \quad d_2(x_i; x_j, x_k) = \sigma_2.
\]

(3.35)

Since there is free motion up to $\tau^1_{2m}$, triangle inequality implies

\[
|x_i - x_j| \leq |x_i; (\tau^1_{2m}) - x_j(\tau^1_{2m})| + \delta |\nu_i - \nu_j| \leq |x_i(\tau^1_{2m}) - x_j(\tau^1_{2m})| + 2\delta R.
\]

(3.36)

Since there is an $(i; j, k)$ ternary collision at $\tau^1_{2m}$, we have

\[
|x_i(\tau^1_{2m}) - x_j(\tau^1_{2m})|^2 + |x_k(\tau^1_{2m}) - x_j(\tau^1_{2m})|^2 = 2\sigma^2 \Rightarrow |x_i(\tau^1_{2m}) - x_j(\tau^1_{2m})| \leq \sqrt{2}\sigma_3.
\]

(3.37)

Combining (3.36)-(3.37), we obtain

\[
|x_i - x_j|^2 \leq |x_i(\tau^1_{2m}) - x_j(\tau^1_{2m})|^2 + 4\sqrt{2}\sigma_3\delta R + 4\delta^2 R^2.
\]

(3.38)

Using the same argument for the pair $(i, k)$, adding and recalling the fact that there is $(i; j, k)$ collision at $\tau^1_{2m}$, we obtain

\[
2\sigma^2 \leq d^2_2(x_i; x_j, x_k) \leq 2\sigma^2 + 8\sqrt{2}\sigma_3\delta R + 8\delta^2 R^2 \leq 2\sigma^2 + 8\sqrt{2}\sigma_3\delta R + 16\delta^2 R^2 = (\sqrt{2}\sigma_3 + 4\delta R)^2 \Rightarrow Z_m \in U^3_{ijk}.
\]

(3.39)

where the lower inequality holds trivially since $Z_m \in D_{m, a_2, a_3}$.

For the pair $(i', j')$, (3.39) and triangle inequality yield

\[
\sigma_2 \leq |x_i - x_j| = |x_{i'}(\tau^1_{2m}) - x_{j'}(\tau^1_{2m}) - \tau^1_{2m}(\nu_{i'} - \nu_{j'})| \leq \sigma_2 + 2\delta R \Rightarrow Z_m \in U^3_{i'j'},
\]

(3.40)

where the lower inequality trivially holds because of the phase space.

Combining (3.39)-(3.40), we obtain $Z_m \in U^3_{ijk} \cap U^3_{i'j'}$, and claim (III) is proved. The rest of the claims are proved by similar arguments and we obtain the inclusion

\[
I^1_{mu} \subseteq U_{22} \cup U_{23} \cup U_{32} \cup U_{33}.
\]

(3.41)

**Inclusion for $I^1_{sc,nq}$:** Remark 3.14 guarantees that

\[
\begin{cases}
Z_m(\tau^1_{2m}) \in \Sigma^2_{ij} \Rightarrow Z_m(\tau^1_{2m}) \notin \Sigma^2_{ij}, \\
Z_m(\tau^1_{2m}) \in \Sigma^3_{ijk} \Rightarrow Z_m(\tau^1_{2m}) \notin \Sigma^3_{ijk}.
\end{cases}
\]

(3.42)

We claim the following:

(I) $Z_m(\tau^1_{2m}) \in \Sigma^2_{ij}, \quad Z_m(\tau^1_{2m}) \in \Sigma^2_{i'j'} \Rightarrow Z_m \in U^2_{ij} \cap U^2_{i'j'}, \quad \forall (i, j), (i', j') \in I^2_m$.

(II) $Z_m(\tau^1_{2m}) \in \Sigma^2_{ij}, \quad Z_m(\tau^1_{2m}) \in \Sigma^2_{i'j'} \Rightarrow Z_m \in U^2_{ij} \cap U^2_{i'j'}, \quad \forall (i, j) \in I^2_m, \quad (i', j', k') \in I^3_m$.

(III) $Z_m(\tau^1_{2m}) \in \Sigma^3_{ijk}, \quad Z_m(\tau^1_{2m}) \in \Sigma^3_{i'j'k'} \Rightarrow Z_m \in U^3_{ijk} \cap U^3_{i'j'k'}, \quad \forall (i, j, k) \in I^3_m, \quad (i', j', k', l') \in I^4_m$.

(IV) $Z_m(\tau^1_{2m}) \in \Sigma^3_{ijk}, \quad Z_m(\tau^1_{2m}) \in \Sigma^3_{i'j'k'} \Rightarrow Z_m \in U^3_{ijk} \cap U^3_{i'j'k'}, \quad \forall (i, j, k, l) \in I^3_m, \quad (i', j', k', l') \in I^4_m$.
By (3.42), proving claims (I)-(IV) implies inclusion (3.28) for \( I_{sc,ng} \).

Without loss of generality, we prove claim (III). Clearly all particles perform free motion until \( \tau_{2m}^1 \), so the same argument we used to obtain (3.39) yields

\[
2\sigma_2^2 \leq d_2^2(x_i; x_j, x_k) \leq (\sqrt{2}\sigma_3 + 4\delta R)^2 \Rightarrow Z_m \in U_{ij}^3.
\]  

(3.43)

Moreover, particles keep performing free motion up to time \( \tau_{2m}^2 \), except particles \( i, j, k \) whose velocities instantaneously transform because of the collision at \( \tau_{2m}^1 \).

We wish to prove as well

\[
\sigma_2 \leq d_2(x_{i'}, x_{j'}) \leq \sigma_2 + 2\delta R.
\]  

(3.44)

The first inequality trivially holds because of the phase space. To prove the second inequality, we distinguish the following cases:

(i) \( i', j' \notin \{i, j, k\} \): Since particles \( (i', j') \) perform free motion up to \( \tau_{2m}^2 \), a similar argument to the one we used to obtain (3.40) yields \( Z_m \in U_{ij}^2 \). The only difference is that we apply the argument up to time \( \tau_{2m}^2 \), not \( \tau_{2m}^1 \), hence claim (3.44) is proved.

(ii) There is at least one recollision i.e. at least one of \( i', j' \) belongs to \( \{i, j, k\} \): The argument is similar to (i), the only difference being that velocities of the recolliding particles transform at \( \tau_{2m}^2 \).

Since the argument is similar for all cases, let us provide a detailed proof only for one recollisional case, for instance \( (i', j') = (i, k) \). We have

\[
x_i(\tau_{2m}^2) = x_i(\tau_{2m}^2) + (\tau_{2m}^2 - \tau_{2m}^1)v^*_i = x_i + \tau_{2m}^1 v_i + (\tau_{2m}^2 - \tau_{2m}^1)v^*_i,
\]

(3.43)

\[
x_k(\tau_{2m}^2) = x_k(\tau_{2m}^2) + (\tau_{2m}^2 - \tau_{2m}^1)v^*_k = x_k + \tau_{2m}^1 v_k + (\tau_{2m}^2 - \tau_{2m}^1)v^*_k,
\]

(3.44)

so

\[
x_i - x_k = x_i(\tau_{2m}^2) - x_k(\tau_{2m}^2) - \tau_{2m}^1(v_i - v_k) - (\tau_{2m}^2 - \tau_{2m}^1)(v^*_i - v^*_k).
\]

Therefore, triangle inequality implies

\[
|x_i - x_k| \leq |x_i(\tau_{2m}^2) - x_k(\tau_{2m}^2)| + \tau_{2m}^1|v_i - v_k| + (\tau_{2m}^2 - \tau_{2m}^1)|v^*_i - v^*_k|
\]

\[
\leq |x_i(\tau_{2m}^2) - x_k(\tau_{2m}^2)| + 2\tau_{2m}^1 R + 2(\tau_{2m}^2 - \tau_{2m}^1) R
\]

\[
= |x_i(\tau_{2m}^2) - x_k(\tau_{2m}^2)| + 2\tau_{2m}^2 R
\]

\[
\leq |x_i(\tau_{2m}^2) - x_k(\tau_{2m}^2)| + 2\delta R,
\]

(3.45)

(3.46)

to obtain (3.45), we use triangle inequality and Remark 3.18, and to obtain (3.46), we use the assumption \( \tau_{2m}^2 \leq \delta \). Therefore (3.44) is proved.

Combining (3.43), (3.44), we obtain \( Z_m \in U_{ij}^1 \cap U_{ij}^2 \), and claim (III) follows.

The remaining claims are proved in a similar way. We obtain

\[
I_{sc,ng}^2 \subseteq U_{22} \cup U_{23} \cup U_{32} \cup U_{33}.
\]  

(3.47)

Inclusions (3.41), (3.44) imply inclusion (3.28).

Assume now that \( Z_m \) is precollisional. Therefore, we obtain

\[
Z_m(t) = \begin{cases} (x_m + tv'_m, v'_m), & \forall t \in [0, \tau_{2m}^1], \text{ if } Z_m \in D_{m,2m,2m,3} \\ (x_m + tv_m, v_m), & \forall t \in [0, \tau_{2m}^1], \text{ if } Z_m \in D_{m,2m,2m,3}, \end{cases}
\]

where the collisional transformation is taken with respect to the initial collisional particles. The proof follows the same lines, using Remark 3.18 for the initial collisional particles whenever needed. \(\square\)

Now we wish to estimate the measure of \( I_{sc,g}^1 \cup I_{m}^1 \cup I_{sc,ng}^2 \) in order to show that outside of a small measure set we have a well defined flow. Let us first introduce some notation.

For \( m \geq 2, (i, j) \in I_m^2 \), a permutation \( \pi: \{i, j\} \to \{i, j\} \) and \( x_{x_j} \in \mathbb{R}^d \), we define the set

\[
S_{\pi}(x_{x_j}) = \{ x_{x_i} \in \mathbb{R}^d : (x_i, x_j) \in U_{ij}^3 \}.
\]  

(3.48)

For \( m \geq 3, (i, j, k) \in I_m^3 \), a permutation \( \pi: \{i, j, k\} \to \{i, j, k\} \) and \( (x_{x_j}, x_{x_k}) \in \mathbb{R}^{2d} \), we define the set

\[
S_{\pi}(x_{x_j}, x_{x_k}) = \{ x_{x_i} \in \mathbb{R}^d : (x_i, x_j, x_k) \in U_{ijk}^3 \}.
\]  

(3.49)
Lemma 3.20. The following hold

(i) Let \( m \geq 2, (i, j, k) \in T_m^0 \), a permutation \( \pi : \{i, j\} \to \{i, j\} \) and \( x_{\pi} \in \mathbb{R}^d \). Then

\[
|S_{\pi}(x_{\pi})|_d \leq C_{d,R,d} \delta. \tag{3.50}
\]

(ii) Let \( m \geq 3, (i, j, k) \in T_m^0 \), a permutation \( \pi : \{i, j, k\} \to \{i, j, k\} \) and \( (x_{\pi}, x_{\pi}^2) \in \mathbb{R}^{2d} \). Then

\[
|S_{\pi}(x_{\pi}, x_{\pi}^2)|_d \leq C_{d,R,d} \delta. \tag{3.51}
\]

Proof. For proof of estimate (3.51), we refer to Lemma 3.10, in [4].

Let us prove (3.50). Consider \( (i, j) \in T_m^0 \), and assume without loss of generality that \( \pi(i, j) = (i, j) \). Let \( x_j \in \mathbb{R}^d \). Recalling (3.48), we obtain

\[
S_i(x_j) = \{x_i \in \mathbb{R}^d : \sigma_2 \leq |x_i - x_j| \leq \sigma_2 + 2 \delta R \},
\]

thus \( S_i(x_j) \) is a spherical shell in \( \mathbb{R}^d \) of inner radius \( \sigma_2 \) and outer radius \( \sigma_2 + 2 \delta R \). Therefore, by scaling (3.27), we obtain

\[
|S_i(x_j)|_d \simeq (\sigma_2 + 2 \delta R)^d - \sigma_2^d = 2 \delta R \sum_{i=0}^{d-1} (\sigma_2 + 2 \delta R)^{d-1-i} \sigma_2^i \leq C_{d,R,d} \delta.
\]

\[\square\]

Remark 3.21. Estimates of Lemma 3.20 are not sufficient to generate a global flow because \( \delta \) represents the length of an elementary time step, therefore iterating, we cannot eliminate pathological sets. We will derive a better estimate of order \( \delta^2 \) to achieve this elimination.

Lemma 3.22. Let \( m \geq 2, 1 < R < \rho \) and \( 0 < \delta R < \sigma_2 < \sigma_3 < 1 \). Then the following estimate holds:

\[
|I_{\text{sc,g}}^{1} \cup I_{\text{mu}}^{1} \cup I_{\text{sc,nug}}^{2}|_{2dm} \leq C_{m,d,R} \rho^{d(m-2)} \delta^2. \tag{3.52}
\]

Proof. For \( m = 2 \), the result comes trivially from Lemma 3.17 and Lemma 3.19.

For \( m \geq 3 \), we recall from Lemma 3.17 that \( I_{\text{sc}}^{1} \) is of measure zero and that by Lemma 3.19, we have

\[
I_{\text{mu}}^{1} \cup I_{\text{sc,nug}}^{2} = U_{22} \cup U_{23} \cup U_{33} \cup U_{33},
\]

where \( U_{22}, U_{23}, U_{33}, U_{33} \) are given by (3.29)-(3.32). Therefore it suffices to estimate the measure of \( U_{22}, U_{23}, U_{33}, U_{33} \).

We will strongly rely on Lemma 3.20.

\[\bullet\text{ Estimate of } U_{22}: \] By (3.29), we have

\[
U_{22} = \bigcup_{(i,j) \neq (i',j') \in T_m^0} (U_{ij}^2 \cap U_{i'j'}^2).
\]

Consider \( (i, j) \neq (i', j') \in T_m^0 \). We distinguish the following possible cases:

(I) \( i', j' \neq (i, j) \): By (3.33), followed by Fubini’s Theorem and part (i) of Lemma 3.20, we have

\[
|U_{ij}^2 \cap U_{i'j'}^2|_{2dm} \leq R_{dm}^d \rho^{d(m-4)} \int_{B_{\rho}^d} 1_{S_{ij}^2(x_j) \cap S_{i'j'}^2(x_{i'})} dx_i dx_{i'} dx_j dx_{j'}
\]

\[
\leq R_{dm}^d \rho^{d(m-4)} \left( \int_{B_{\rho}^d} \int_{\mathbb{R}^d} 1_{S_{ij}^2(x_j)} dx_i dx_j \right) \left( \int_{B_{\rho}^d} \int_{\mathbb{R}^d} 1_{S_{i'j'}^2(x_{i'})} dx_{i'} dx_{j'} \right)
\]

\[
\leq C_{d,R} \rho^{d(m-2)} \delta^2.
\]

(II) Exactly one of \( i', j' \) belongs to \( \{i, j\} \): Without loss of generality we consider the case \( (i', j') = (j, j') \), for some \( j' > j \) and all other cases follow similarly. Fubini’s Theorem and part (i) of Lemma 3.20 imply

\[
|U_{ij}^2 \cap U_{jj'}^2|_{2dm} \leq R_{dm}^d \rho^{d(m-3)} \int_{B_{\rho}^d} 1_{S_{ij}^2(x_j) \cap S_{jj'}^2(x_{j'})} dx_j dx_{j'} dx_i
\]

\[
\leq R_{dm}^d \rho^{d(m-3)} \left( \int_{B_{\rho}^d} \int_{\mathbb{R}^d} 1_{S_{ij}^2(x_j)} dx_i dx_j \right) \left( \int_{B_{\rho}^d} \int_{\mathbb{R}^d} 1_{S_{jj'}^2(x_{j'})} dx_{j'} dx_j \right)
\]

\[
\leq C_{d,R} \rho^{d(m-2)} \delta^2.
\]
Combining cases (I)-(II), we obtain

\[ |U_{22}|_{2d} \leq C_{m,d,R} \rho^{d(m-2)} \delta^2. \tag{3.53} \]

- **Estimate of \( U_{23} \):** By (3.20), we have

\[ U_{23} = \bigcup_{(i,j) \in \mathbb{T}^m_n, (i',j',k') \in \mathbb{T}^m_n} (U_{ij}^2 \cap U_{i'j'k'}^3). \]

Consider \((i,j) \in \mathbb{T}^m_n, (i',j',k') \in \mathbb{T}^m_n\). We distinguish the following possible cases:

(I) \( i', j', k' \notin \{i, j\} \): Without loss of generality we consider the case \((i', j', k') := (i', i, j')\), for some \( i' < i < k' \) and all other cases follow similarly. Using Fubini’s Theorem and parts (i)-(ii) of Lemma 3.20, we obtain

\[
|U_{ij}^2 \cap U_{i'j'k'}^3|_{2d} \leq R^{d(m-4)} \int_{B^2_{d}} 1_{S_j^2(x_i) \cap S_{i'}^3(x_{i'}, x_{i'})} \, dx_i \, dx_j \, dx_{i'} \, dx_{j'} \, dx_{k'}
\]

\[
\leq R^{d(m-4)} \rho \delta^2 \).
\]

(II) Exactly one of \( i', j', k' \) belongs in \( \{i, j\} \): Without loss of generality we consider the case \((i', j', k') := (i', i, j)\), for some \( i' < i < k' \) and all other cases follow similarly. Using Fubini’s Theorem and parts (i)-(ii) of Lemma 3.20, we obtain

\[
|U_{ij}^2 \cap U_{i'j'k'}^3|_{2d} \leq R^{d(m-3)} \int_{B^2_{d}} 1_{S_j^2(x_i) \cap S_{i'}^3(x_{i'}, x_{i'})} \, dx_i \, dx_j \, dx_{i'} \, dx_{j'}
\]

\[
\leq R^{d(m-3)} \rho \delta^2 \).
\]

(III) Exactly two of \( i', j', k' \) belong in \( \{i, j\} \): Without loss of generality we consider the case \((i', j', k') := (i', i, j)\), for some \( i' < i < k' \) and all other cases follow similarly. Using Fubini’s Theorem and parts (i)-(ii) of Lemma 3.20, we obtain

\[
|U_{ij}^2 \cap U_{i'j'k'}^3|_{2d} \leq R^{d(m-2)} \int_{B^2_{d}} 1_{S_j^2(x_i) \cap S_{i'}^3(x_{i'}, x_{i'})} \, dx_i \, dx_j \, dx_{i'}
\]

\[
\leq R^{d(m-2)} \rho \delta^2 \).
\]

Combining cases (I)-(III), we obtain

\[ |U_{23}|_{2d} \leq C_{m,d,R} \rho^{d(m-2)} \delta^2. \tag{3.54} \]

- **Estimate of \( U_{32} \):** We use a similar argument to the estimate for \( U_{23} \), to obtain

\[ |U_{32}|_{2d} \leq C_{m,d,R} \rho^{d(m-2)} \delta^2. \tag{3.55} \]

- **Estimate of \( U_{33} \):** We refer to Lemma 3.11. from [4] for a detailed proof. We obtain

\[ |U_{33}|_{2d} \leq C_{m,d,R} \rho^{d(m-2)} \delta^2. \tag{3.56} \]

Combining (3.53), (3.54), (3.55), and (3.56), we obtain (3.52) and the proof is complete.

We inductively use Lemma 3.22 to define a global flow which preserves energy for almost all configuration. For this purpose, given \( Z_m = (X_m, V_m) \in \mathbb{R}^{2dm} \), we define its kinetic energy as:

\[ E_m(Z_m) := \frac{1}{2} \sum_{i=1}^{m} |v_i|^2 \] (3.57)
For convenience, let us define the $m$-particle free flow:

**Definition 3.23.** Let $m \in \mathbb{N}$. We define the $m$-particle free flow as the family of measure-preserving maps $(\Phi_m^t)_{t \in \mathbb{R}} : \mathbb{R}^{2dm} \to \mathbb{R}^{2dm}$, given by

$$
\Phi_m^t Z_m = \Phi_m^t (X_m, V_m) = (X_m + tV_m, V_m).
$$

(3.58)

We are now in the position to state the Existence Theorem of the $m$-particle $(\sigma_2, \sigma_3)$-flow.

**Theorem 3.24.** Let $m \in \mathbb{N}$ and $0 < \sigma_2 < \sigma_3 < 1$. There exists a family of measure-preserving maps $(\Psi_m^s)_{s \in \mathbb{R}} : D_{m, \sigma_2, \sigma_3} \to D_{m, \sigma_2, \sigma_3}$ such that

$$
\Psi_m^s Z_m = (\Psi_m^s \circ \Psi_m^t)(Z_m) = (\Psi_m^s \circ \Psi_m^t)(Z_m), \quad \text{a.e. in } D_{m, \sigma_2, \sigma_3}, \quad \forall t, s \in \mathbb{R},
$$

(3.59)

$$
E_m (\Psi_m^t Z_m) = E_m (Z_m), \quad \text{a.e. in } D_{m, \sigma_2, \sigma_3}, \quad \forall t \in \mathbb{R}, \text{ where } E_m \text{ is given by } (3.57).
$$

(3.60)

Moreover, for $m \geq 3$, we have

$$
\Psi_m^t Z_m' = \Psi_m^t Z_m, \quad \sigma - \text{a.e. on } \partial_{sc,n} D_{m, \sigma_2, \sigma_3} \cap \partial_{2, sc} D_{m, \sigma_2, \sigma_3}, \quad \forall t \in \mathbb{R},
$$

(3.61)

$$
\Psi_m^t Z_m'' = \Psi_m^t Z_m, \quad \sigma - \text{a.e. on } \partial_{sc,n} D_{m, \sigma_2, \sigma_3} \cap \partial_{3, sc} D_{m, \sigma_2, \sigma_3}, \quad \forall t \in \mathbb{R},
$$

(3.62)

while for $m = 2$, we have

$$
\Psi_2^t Z_2 = \Psi_2^t Z_2, \quad \sigma - \text{a.e. on } \partial_{sc,n} D_{2, \sigma_2, \sigma_3} \cap \partial_{2, sc} D_{2, \sigma_2, \sigma_3}, \quad \forall t \in \mathbb{R},
$$

(3.63)

where $\sigma$ is the surface measure induced on $\partial D_m$ by the Lebesgue measure. This family of maps is called the $m$-particle $(\sigma_2, \sigma_3)$-flow.

For $m = 1$, we define $\Psi_1^t := \Phi_1^t \quad \forall t \in \mathbb{R}$.

**Proof.** The proof follows exactly the same steps as the proof of Theorem 3.14. from [4] (for additional details see Theorem 4.9.1 from [2]).

**Remark 3.25.** We have seen that the flow can be defined only a.e. in $D_{m, \sigma_2, \sigma_3}$. However to simplify the notation, without loss of generality, we may assume that the flow is well defined on the whole phase space $D_{m, \sigma_2, \sigma_3}$.

3.5. **The Liouville equation.** Here, we formally derive the Liouville equation for $m$ hard spheres of diameter $\sigma_2$ and interaction zone $\sigma_3$, where $0 < \sigma_2 < \sigma_3 < 1$. Without loss of generality, we derive the equation for $m \geq 3$, and for $m = 2$ we follow a similar argument neglecting the ternary terms. For $m = 1$, the Liouville equation will be trivial since the flow coincides with the free flow. We then introduce the $m$-particle $(\sigma_2, \sigma_3)$ interaction flow operator and the $m$-particle free flow operator.

Let $m \geq 3$ and consider an initial absolutely continuous Borel probability measure $P_0$ on $\mathbb{R}^{2dm}$, with a probability density $f_{m,0}$ satisfying the following properties:

- $f_{m,0}$ is supported in $D_{m, \sigma_2, \sigma_3}$ i.e.

$$
\text{supp } f_{m,0} := \{Z_m \in \mathbb{R}^{2dm} : f_{m,0} (Z_m) \neq 0 \} \subseteq D_{m, \sigma_2, \sigma_3}.
$$

(3.64)

- $f_{m,0}$ is symmetric i.e. for any permutation $p_m$ of the $m$-particles, there holds:

$$
f_{m,0} (p_m (Z_m)) = f_{m,0} (Z_m), \quad \forall Z_m \in \mathbb{R}^{2dm}.
$$

(3.65)

The probability measure $P_0$ expresses the initial distribution in space and velocities of the $m$-particles. We are interested in the evolution of this measure under the flow. For this purpose, given $t \geq 0$ we define $P_t$ to be the push-forward of $P_0$ under the flow i.e.

$$
P_t (A) = P_0 (\Psi_m^{-t} (A)), \quad A \subseteq \mathbb{R}^{2dm} \text{ Borel measurable}.
$$

Conservation of measure under the flow implies that $P_t$ is absolutely continuous with probability density given by

$$
f_m (t, Z_m) = \begin{cases} 
   f_{m,0} \circ \Psi_m^t, & \text{a.e. in } D_{m, \sigma_2, \sigma_3}, \\
   0, & \text{a.e. in } \mathbb{R}^{2dm} \setminus D_{m, \sigma_2, \sigma_3}.
\end{cases}
$$

(3.66)

Clearly $f_m (t, Z_m)$ is symmetric and supported in $D_{m, \sigma_2, \sigma_3}$, for all $t \geq 0$. Moreover, we have

$$
f_m (0, Z_m) = f_{m,0} \circ \Psi_m^0 (Z_m) = f_{m,0} (Z_m), \quad Z_m \in D_{m, \sigma_2, \sigma_3}.
$$

(3.67)
Since \( m > 2 \), \((3.61)\) implies
\[
 f_m(t, Z'_m) = f_{m,0} \circ \Psi^{-1}_m(Z'_m) = f_{m,0} \circ \Psi^{-1}_m(Z_m) = f_m(t, Z_m), \quad \text{a.e. on } \partial_{2,sc} D_m, \forall t \geq 0. \tag{3.68}
\]
Additionally, since \( m \geq 3 \), \((3.62)\) implies
\[
 f_m(t, Z'_m) = f_{m,0} \circ \Psi^{-1}_m(Z'_m) = f_{m,0} \circ \Psi^{-1}_m(Z_m) = f_m(t, Z_m), \quad \text{a.e. on } \partial_{3,sc} D_m, \forall t \geq 0. \tag{3.69}
\]
Finally, recall from \((3.66)\) that
\[
 f_m(t, Z_m) = f_{m,0} \circ \Psi^{-1}_m(Z_m), \quad \text{a.e. in } D_m, \forall t \geq 0. \tag{3.70}
\]
Combining \((3.67)-(3.70)\), and formally assuming that \( f_m \) is smooth in time, by the chain rule, we obtain that \( f_m \) formally satisfies the \( m \)-particle Liouville equation in \( D_m \):
\[
\begin{aligned}
 & \partial_t f_m + m \sum_{i=1}^{m} v_i \cdot \nabla x_i f_m = 0, \quad (t, Z_m) \in (0, \infty) \times D_m, \\
 & f_m(t, Z'_m) = f_m(t, Z_m), \quad (t, Z_m) \in [0, \infty) \times \partial_{2,sc} D_m, \\
 & f_m(t, Z'_m) = f_m(t, Z_m), \quad (t, Z_m) \in [0, \infty) \times \partial_{3,sc} D_m, \\
 & f_m(0, Z_m) = f_{m,0}(Z_m), \quad Z_m \in D_m.
\end{aligned}
\tag{3.71}
\]

With similar arguments, we conclude that, in the case \( m = 2 \), \( f_2 \) formally satisfies the 2-particle Liouville equation \( D_{2,sc} \):
\[
\begin{aligned}
 & \partial_t f_2 + v_1 \cdot \nabla x_1 f_2 + v_2 \cdot \nabla x_2 f_2 = 0, \quad (t, Z_2) \in (0, \infty) \times D_{2,sc}, \\
 & f_2(t, Z'_2) = f_2(t, Z_2), \quad (t, Z_2) \in [0, \infty) \times \partial_{2,sc} D_{2,sc}, \\
 & f_2(0, Z_2) = f_{2,0}(Z_2), \quad Z_2 \in D_{2,sc}.
\end{aligned}
\tag{3.72}
\]

In the case \( m = 1 \), we trivially have \( f(t, x, v) = f_0(\Phi^{-1}_1(x, v)) = f_0(x_1 - tv, v_1) \).

Now, we introduce some notation defining the \( m \)-particle free flow operator and the \( m \)-particle \( (\sigma_2, \sigma_3) \)-flow operator. For convenience, let us denote
\[
 C^0(D_{m,2,sc}) := \{ g_m \in C^0(\mathbb{R}^{2dm}) : \text{supp } g_m \subseteq D_{m,2,sc} \}. \tag{3.73}
\]

**Definition 3.26.** For \( t \in \mathbb{R} \) and \( 0 < \sigma_2 < \sigma_3 < 1 \), we define the \( m \)-particle \( (\sigma_2, \sigma_3) \)-flow operator \( T^t_m : C^0(D_{m,2,sc}) \to C^0(D_{m,2,sc}) \) as:
\[
 T^t_m g_m(Z_m) = \begin{cases}
 g_m(\Psi^{-1}_m(Z_m)), & \text{if } Z_m \in D_{m,2,sc}, \\
 0, & \text{if } Z_m \notin D_{m,2,sc},
\end{cases} \tag{3.74}
\]
where \( \Psi_m \) is the \( m \)-particle \( (\sigma_2, \sigma_3) \)-flow defined in Theorem \((3.34)\).

**Remark 3.27.** Given an initial probability density \( f_{m,0} \), satisfying \((3.64)-(3.65)\), the function \( f_m(t, Z_m) = T^t_m f_{m,0}(Z_m) \) is formally the unique solution to the Liouville equation \((3.74)\) with initial data \( f_{m,0} \).

We also define the free flow and the \( m \)-particle free flow operator.

**Definition 3.28.** For \( t \in \mathbb{R} \) and \( m \in \mathbb{N} \), we define the \( m \)-particle free flow operator \( S^t_m : C^0(\mathbb{R}^{2dm}) \to C^0(\mathbb{R}^{2dm}) \) as:
\[
 S^t_m g_m(Z_m) = g_m(\Psi^{-1}_m(Z_m)) = g_m(X_m - tv_m, V_m). \tag{3.75}
\]

4. BBGKY Hierarchy, Boltzmann Hierarchy and the Binary-Ternary Boltzmann Equation

4.1. The BBGKY hierarchy. Consider \( N \)-particles of diameter \( 0 < \epsilon_2 < 1 \) and interaction zone \( 0 < \epsilon_3 < 1 \), where \( N \geq 3 \) and \( \epsilon_2 < \epsilon_3 \). For \( s \in \mathbb{N} \), we define the \( s \)-marginal of a symmetric probability density \( f_N \), supported in \( D_{N,\epsilon_2,\epsilon_3} \), as
\[
 f^{(s)}_N(Z_s) = \begin{cases}
 \int_{\mathbb{R}^{2(N-s)}} f_N(Z_N) \, dx_{s+1} \ldots dx_N \, dv_{s+1} \ldots dv_N, & 1 \leq s < N, \\
 f_N, & s = N, \\
 0, & s > N.
\end{cases} \tag{4.1}
\]
where for \( Z_s = (X_s, V_s) \in \mathbb{R}^{2d_s} \), we write \( Z_N = (X_s, x_{s+1}, \ldots, x_N, V_s, v_{s+1}, \ldots, v_N) \). One can see, for all \( 1 \leq s \leq N \), the marginals \( f^{(s)}_N \) are symmetric probability densities, supported in \( \mathcal{D}_{s,e_2,e_3} \) and

\[
f^{(s)}_N(Z_s) = \int_{\mathbb{R}^{2d_s}} f^{(s+1)}_N(X_s, V_N) \, dx_{s+1} \, dv_{s+1}, \quad \forall 1 \leq s \leq N - 1.
\]

Assume now that \( f_N \) is formally the solution to the \( N \)-particle Liouville equation (3.71) with initial data \( f_{N,0} \). We seek to formally find a hierarchy of equations satisfied by the marginals of \( f_N \). For \( s \geq N \), by definition, we have

\[
f^{(N)}_N = f_N, \quad \text{and } f^{(s)}_N = 0, \quad \text{for } s > N, \quad (4.2)
\]

We observe that \( \partial \mathcal{D}_{s,e_2,e_3} \) is equivalent up to surface measure zero to \( \Sigma^X \times \mathbb{R}^{dN} \) where

\[
\Sigma^X := \bigcup_{(i,j) \in \mathcal{I}_N^3} \Sigma^{2,ac,X}_{ij} \cup \bigcup_{(i,j,k) \in \mathcal{I}_N^3} \Sigma^{3,ac,X}_{ijk},
\]

\[
\Sigma^{2,ac,X}_{ij} := \{ X_N \in \mathbb{R}^{dN} : (4.5) \}
\]

\[
\Sigma^{3,ac,X}_{ijk} := \{ X_N \in \mathbb{R}^{dN} : (4.6) \}
\]

Notice that (4.3) is a pairwise disjoint union.

**Remark 4.1.** The assumption \( \epsilon_3 < \epsilon_3 \) made at the beginning of the section is necessary for the ternary contribution to be visible. Indeed, if \( \epsilon_2 \geq \epsilon_3 \), Remark 4.1 and (3.22) would imply that \( \Sigma^{3,ac,X}_{ijk} = \emptyset \) for all \( (i,j,k) \in \mathcal{I}_N^3 \), therefore there would not be a ternary collisional term.

The hierarchy for \( s < N \) will come after integrating by parts the Liouville equation (3.71). Consider \( 1 \leq s \leq N - 1 \). The boundary and initial conditions can be easily recovered integrating Liouville’s equation boundary and initial conditions respectively i.e.

\[
\begin{align*}
\frac{df^{(s)}_N}{dt}(t, Z_s) &= f^{(s)}_N(t, Z_s), \quad (t, Z_s) \in [0, \infty) \times \partial_{2,ac}\mathcal{D}_{s,e_2,e_3}, \quad s \geq 2, \\
\frac{df^{(s)}_N}{dt}(t, Z_s) &= f^{(s)}_N(t, Z_s), \quad (t, Z_s) \in [0, \infty) \times \partial_{3,ac}\mathcal{D}_{s,e_2,e_3}, \quad s \geq 3, \\
\frac{df^{(s)}_N}{dt}(0, Z_s) &= f^{(s)}_N(0, Z_s), \quad Z_s \in \mathcal{D}_{s,e_2,e_3}.
\end{align*}
\]

Notice that for \( s = 2 \) there is no ternary boundary condition, while for \( s = 1 \) there is no boundary condition at all.

Consider now a smooth test function \( \phi_s \) compactly supported in \( (0, \infty) \times \mathcal{D}_{s,e_2,e_3} \) such that the following hold:

- For any \( (i,j) \in \mathcal{I}_N^3 \) with \( j \leq s \), we have

\[
\phi_s(t, p_s Z_N) = \phi_s(t, p_s Z_N) = \phi_s(t, Z_N), \quad \forall (t, Z_N) \in (0, \infty) \times \Sigma^{ac,2}_{i,j},
\]

- For any \( (i,j,k) \in \mathcal{I}_N^3 \) with \( j \leq s \), we have

\[
\phi_s(t, p_s Z_N) = \phi_s(t, p_s Z_N) = \phi_s(t, Z_N), \quad \forall (t, Z_N) \in (0, \infty) \times \Sigma^{ac,3}_{i,j,k},
\]

where \( p_s : \mathbb{R}^{2dN} \rightarrow \mathbb{R}^{2d_s} \) denotes the natural projection in space and velocities, given by \( p_s(Z_N) = Z_s \).

Multiplying the Liouville equation by \( \phi_s \) and integrating, we obtain its weak form

\[
\int_{(0,\infty) \times \mathcal{D}_{s,e_2,e_3}} \left( \partial_t f_N(t, Z_N) + \sum_{i=1}^{N} v_i \nabla_x f_N(t, Z_N) \right) \phi_s(t, Z_s) \, dX_N \, dV_N \, dt = 0. \quad (4.7)
\]

For the time derivative in (4.7), we use Fubini’s Theorem, integration by parts in time, the fact that \( f_N \) is supported in \( (0, \infty) \times \mathcal{D}_{s,e_2,e_3} \) and the fact that \( \phi_s \) is compactly supported in \( (0, \infty) \times \mathcal{D}_{s,e_2,e_3} \), to obtain

\[
\int_{(0,\infty) \times \mathcal{D}_{s,e_2,e_3}} \partial_t f_N(t, Z_N) \phi_s(t, Z_s) \, dX_N \, dV_N \, dt = \int_{(0,\infty) \times \mathcal{D}_{s,e_2,e_3}} \partial_t f^{(s)}_N(t, Z_s) \phi_s(t, Z_s) \, dX_s \, dV_s \, dt. \quad (4.8)
\]
For the material derivative term in (4.17), the Divergence Theorem implies that
\[
\int_{\mathcal{D}_{N,s^2,s^3}} \sum_{i=1}^{N} v_i \nabla x_i f_N (t, Z_N) \phi_s(t, Z_s) \, dX_N \, dV_N = \int_{\mathcal{D}_{N,s^2,s^3}} \text{div}_{X_N} [f_N (t, Z_N) V_N] \phi_s(t, Z_s) \, dX_N \, dV_N
\]
\[
= - \int_{\mathcal{D}_{N,s^2,s^3}} V_N \cdot \nabla X_N \phi_s(t, Z_s) f_N(t, Z_N) \, dX_N \, dV_N + \int_{\Sigma_X \times \mathbb{R}^d} \hat{n}(X_N) \cdot V_N f_N(t, Z_N) \phi_s(t, Z_s) \, dV_N \, d\sigma,
\]
where $\Sigma_X$ is given by (4.39), $\hat{n}(X_N)$ is the outwards normal vector on $\Sigma_X$ at $X_N \in \Sigma_X$ and $d\sigma$ is the surface measure on $\Sigma_X$. Using the fact that $f_N$ is supported in $\mathcal{D}_{N,s^2,s^3}$, Divergence Theorem and the fact that $\phi_s$ is compactly supported in $(0, \infty) \times \mathcal{D}_{s^2,s^3}$, we obtain
\[
\int_{\mathcal{D}_{N,s^2,s^3}} V_N \cdot \nabla X_N \phi_s(t, Z_s) f_N(t, Z_N) \, dX_N \, dV_N = - \int_{\mathcal{D}_{N,s^2,s^3}} \sum_{i=1}^{s} v_i \nabla x_i f_N^{(i)}(t, Z_s) \phi_s(t, Z_s) \, dX_s \, dV_s,
\]
Combining (4.17)-(4.19), and recalling the space boundary decomposition (4.39), we obtain
\[
\int_{(0,\infty) \times \mathcal{D}_{s^2,s^3}} \left( \partial_t f_N^{(s)}(t, Z_s) + \sum_{i=1}^{s} v_i \nabla x_i f_N^{(i)}(t, Z_s) \right) \phi_s(t, Z_s) \, dX_s \, dV_s \, dt
\]
\[
= - \int_{(0,\infty) \times \Sigma_X \times \mathbb{R}^d} \hat{n}(X_N) \cdot V_N f_N(t, Z_N) \phi_s(t, Z_s) \, dV_N \, d\sigma \, dt,
\]
\[
= \int_{0}^{\infty} \sum_{(i,j) \in \mathcal{I}_N^3} C_{ij}^2(t) + \sum_{(i,j,k) \in \mathcal{I}_N^3} C_{ijk}^2(t) \, dt,
\]
where for $(i,j) \in \mathcal{I}_N^3$, $t > 0$, we denote
\[
C_{ij}^2(t) = - \int_{\Sigma_{i,j}^2 \times \mathbb{R}^d} \hat{n}^{ij}(X_N) \cdot V_N f_N(t, Z_N) \phi_s(t, Z_s) \, dV_N \, d\sigma_{ij}^3,
\]
for $(i,j,k) \in \mathcal{I}_N^3$, $t > 0$, we denote
\[
C_{ijk}^2(t) = - \int_{\Sigma_{i,j,k}^3 \times \mathbb{R}^d} \hat{n}^{ijk}(X_N) \cdot V_N f_N(t, Z_N) \phi_s(t, Z_s) \, dV_N \, d\sigma_{ijk}^3,
\]
and $\hat{n}^{ij}(X_N)$ is the outwards normal vector on $\Sigma_{ij}^2 \times \mathbb{R}^d$ at $X_N \in \Sigma_{ij}^2 \times \mathbb{R}^d$, $d\sigma_{ij}^3$ is the surface measure on $\Sigma_{ij}^2 \times \mathbb{R}^d$, while $\hat{n}^{ijk}(X_N)$ is the outwards normal vector on $\Sigma_{ijk}^3 \times \mathbb{R}^d$ at $X_N \in \Sigma_{ijk}^3 \times \mathbb{R}^d$ and $d\sigma_{ijk}^3$ is the surface measure on $\Sigma_{ijk}^3 \times \mathbb{R}^d$.

Following similar calculations to [22] which treats the binary case, and [4] which treats the ternary case, we formally obtain the BBGKY hierarchy:
\[
\begin{align*}
\partial_t f_N^{(s)} + \sum_{i=1}^{s} v_i \nabla x_i f_N^{(s)} = C_{s,s+1}^N f_N^{(s+1)} + C_{s,s+2}^N f_N^{(s+2)}, & \quad (t, Z_s) \in (0, \infty) \times \mathcal{D}_{s^2,s^3}, \\
C_{s,s+1}^N = C_{s,s+1}^N - C_{s,s+1}^{-N}, & \quad (s = 1, 2, \ldots, N-1),
\end{align*}
\]
and we use the following notation:
- **Binary notation:** For $1 \leq s \leq N-1$ we denote
\[
C_{s,s+1}^N f_N^{(s+1)}(t, Z_s) = A_{s^2,s^2}^N \sum_{i=1}^{s} \int_{\mathcal{B}_{s^2,s^2} \times \mathbb{R}^d} b_i^+(v_s, v_s+1 - v_i) f_N^{(s+1)}(t, Z_{s^2,s^2}, v_s+1) \, dv_s \, dv_i,
\]
\[
C_{s,s+2}^N f_N^{(s+2)}(t, Z_s) = A_{s^2,s^2}^N \sum_{i=1}^{s} \int_{\mathcal{B}_{s^2,s^2} \times \mathbb{R}^d} b_i^+(v_s, v_s+1 - v_i) f_N^{(s+1)}(t, Z_{s^2,s^2}, v_s+1) \, dv_s \, dv_i,
\]
where
\[ b_2(\omega_1, v_{s+1} - v_i) = (\omega_1, v_{s+1} - v_i), \]
\[ b_3^2 = \max(b_2, 0), \]
\[ A_{s,N,s+2} = (N-s)\epsilon_2^{d-1}, \]
\[ Z_{s+1,s+1,i} = (x_1, ..., x_i, ..., x_s, x_i - \epsilon_2 \omega_1, v_1, ..., v_i - 1, v_i, v_{s+1}, ..., v_s, v_{s+1}), \]
\[ Z_{s+1,s+1,i} = (x_1, ..., x_i, ..., x_s, x_i + \epsilon_2 \omega_1, v_1, ..., v_i - 1, v_i', v_{s+1}, ..., v_s, v_{s+1}', 1). \]

For \( s \geq N \) we trivially define \( C_{s,s+1}^N \equiv 0 \).

- **Ternary notation:** For \( 1 \leq s \leq N-2 \) we denote
\[
C_{s,s+2}^N f_N^{(s+2)}(t, Z_s) = A_{N,s,s+2}^{3} \sum_{i=1}^{N} \int_{\mathbb{R}^{2d-1} \times \mathbb{R}^d} \frac{b_i^3(\omega_1, \omega_2, v_{s+1} - v_i, v_{s+2} - v_i)}{\sqrt{1+\langle\omega_1, \omega_2\rangle}} \times f_N^{(s+2)}(t, Z_{s+2, \epsilon_3, i}) \, d\omega_1 \, dv_{s+1} \, dv_{s+2},
\]
where
\[
A_{N,s,s+2}^{3} = 2^{d-2}(N-s)(N-s-1)\epsilon_3^{d-1},
\]
\[ b_3(\omega_1, \omega_2, v_{s+1} - v_i, v_{s+2} - v_i) = \langle\omega_1, v_{s+1} - v_i\rangle + \langle\omega_2, v_{s+2} - v_i\rangle,
\]
\[ b_3^2 = \max(b_3, 0), \]
\[ Z_{s+2, \epsilon_3, i} = (x_1, ..., x_i, ..., x_s, x_i - \sqrt{2}\epsilon_3 \omega_1, x_i - \sqrt{2}\epsilon_3 \omega_2, v_1, ..., v_i - 1, v_i, v_{s+1}, ..., v_s, v_{s+1}), \]
\[ Z_{s+2, \epsilon_3, i} = (x_1, ..., x_i, ..., x_s, x_i + \sqrt{2}\epsilon_3 \omega_1, x_i + \sqrt{2}\epsilon_3 \omega_2, v_1, ..., v_i - 1, v_i', v_{s+1}, ..., v_s, v_{s+1}', 1). \]

For \( s \geq N-1 \) we trivially define \( C_{s,s+2}^N \equiv 0 \).

Duhamel’s formula implies that the BBGKY hierarchy can be written in mild form as follows
\[
f_N^{(s)}(t, Z_s) = T_s^{d-\sigma} f_N^{(s)}(0, Z_s) + \int_0^t T_s^{d-\sigma} \left( C_{s,s+2}^N f_N^{(s+1)} + C_{s,s+2}^N f_N^{(s+2)} \right)(\tau, Z_s) \, d\tau, \quad s \in \mathbb{N}, \tag{4.23}
\]
where \( T_s^\sigma \) is the \( s \)-particle \((\epsilon_2, \epsilon_3)\)-flow operator given in \((4.74)\).

### 4.2. The Boltzmann hierarchy.

We will now derive the Boltzmann hierarchy as the formal limit of the BBGKY hierarchy as \( N \to \infty \) and \( \epsilon_2, \epsilon_3 \to 0^+ \) under the scaling
\[
N\epsilon_2^{d-1} \simeq N\epsilon_3^{d-1/2} \simeq 1. \tag{4.24}
\]
This scaling implies that \( \epsilon_2, \epsilon_3 \) satisfy
\[
\epsilon_2^{d-1} \simeq \epsilon_3^{d-1/2}. \tag{4.25}
\]

**Remark 4.2.** Using the scaling \((4.24)\), we obtain
\[
\epsilon_2 \simeq N^{-\frac{1}{d-1}} N^{-\infty} 0, \quad \epsilon_3 \simeq N^{-\frac{2}{d-1}} N^{-\infty} 0, \tag{4.26}
\]
thus
\[
\frac{\epsilon_2}{\epsilon_3} \simeq N^{-\frac{1}{d-1}} N^{-\infty} 0. \tag{4.27}
\]

Therefore, for \( N \) large enough, we have \( \epsilon_2 << \epsilon_3 \).

**Remark 4.3.** The scaling \((4.24)\) guarantees that for a fixed \( s \in \mathbb{N} \), we have
\[
A_{N,s,s+2} = (N-s)\epsilon_2^{d-1} \to 1, \quad \text{as} \ N \to \infty,
\]
\[
A_{N,s,s+2} = 2^{d-2}(N-s)(N-s-1)\epsilon_3^{d-1} \to 1, \quad \text{as} \ N \to \infty.
\]
Formally taking the limit under the scaling imposed we may define the following collisional operators:

- **Binary Boltzmann operator:**

\[
C_{s,s+1}^\infty = C_{s,s+1}^{\infty,+} - C_{s,s+1}^{\infty,-},
\]

where

\[
C_{s,s+1}^{\infty,+} f^{(s+1)}(t, Z_s) = \sum_{i=1}^{s} \int \frac{b_2^s (\omega_1, v_{s+2} - v_i) f^{(s+1)} (t, Z_{s,i}') \times d\omega_1 d v_{s+1}}{1 + (\omega_1, \omega_2)}
\]

\[
C_{s,s+1}^{\infty,-} f^{(s+1)}(t, Z_s) = \sum_{i=1}^{s} \int \frac{b_2^s (\omega_1, v_{s+2} - v_i) \times f^{(s+1)} (t, Z_{s,i}') \times d\omega_1 d v_{s+1}}{1 + (\omega_1, \omega_2)}
\]

- **Ternary Boltzmann operator:**

\[
C_{s,s+2}^\infty = C_{s,s+2}^{\infty,+} - C_{s,s+2}^{\infty,-},
\]

where

\[
C_{s,s+2}^{\infty,+} f^{(s+2)}(t, Z_s) = \sum_{i=1}^{s} \int \frac{b_3^s (\omega_1, v_{s+1} - v_i, v_{s+2} - v_i) f^{(s+2)} (t, Z_{s,i}'') \times d\omega_1 d\omega_2 d v_{s+1} d v_{s+2}}{1 + (\omega_1, \omega_2, \omega_3)}
\]

\[
C_{s,s+2}^{\infty,-} f^{(s+2)}(t, Z_s) = \sum_{i=1}^{s} \int \frac{b_3^s (\omega_1, v_{s+1} - v_i, v_{s+2} - v_i) \times f^{(s+2)} (t, Z_{s,i}'') \times d\omega_1 d\omega_2 d v_{s+1} d v_{s+2}}{1 + (\omega_1, \omega_2, \omega_3)}
\]

Now we are ready to introduce the Boltzmann hierarchy. More precisely, given an initial probability density \(f_0\), the Boltzmann hierarchy for \(s \in \mathbb{N}\) is given by:

\[
\begin{align*}
\partial_t f^{(s)} + \sum_{i=1}^{s} v_i \nabla f^{(s)} &= C_{s,s+1}^{\infty} f^{(s+1)}(t, Z_s) + C_{s,s+2}^{\infty} f^{(s+2)}(t, Z_s), \\
&\quad (t, Z_s) \in (0, \infty) \times \mathbb{R}^{2ds}, \\
f^{(s)}(0, Z_s) &= f_0^{(s)}(Z_s), \quad \forall Z_s \in \mathbb{R}^{2ds}.
\end{align*}
\]

Duhamel's formula implies that the Boltzmann hierarchy can be written in mild form as follows

\[
f^{(s)}(t, Z_s) = S_{s}^t f_0^{(s)}(Z_s) + \int_0^t S_{s}^{t-\tau} \left( C_{s,s+1}^{\infty} f^{(s+1)}(\tau, Z_s) + C_{s,s+2}^{\infty} f^{(s+2)}(\tau, Z_s) \right) d\tau, \quad s \in \mathbb{N},
\]

where \(S_t^s\) denotes the \(s\)-particle free flow operator given in (3.35).

### 4.3. The binary-ternary Boltzmann equation.

#### 4.3.1. The binary-ternary Boltzmann equation.

In most applications, particles are initially independently distributed. This translates to tensorized Boltzmann hierarchy initial data i.e.

\[
f_0^{(s)}(Z_s) = f_0^{\otimes s}(Z_s) = \prod_{i=1}^{s} f_0(x_i, v_i), \quad s \in \mathbb{N},
\]

where \(f_0 : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}\) is a given function. One can easily verify that the ansatz:

\[
f^{(s)}(t, Z_s) = f_0^{\otimes s}(t, Z_s) = \prod_{i=1}^{s} f(t, x_i, v_i), \quad s \in \mathbb{N},
\]

(4.39)
solves the Boltzmann hierarchy with initial data given by (4.38), if \( f : [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) satisfies the following nonlinear integro-differential equation:

\[
\begin{aligned}
\partial_t f + v \cdot \nabla_x f &= Q_2(f, f) + Q_3(f, f, f), \\
&\text{ where } (t, x, v) \in (0, \infty) \times \mathbb{R}^{2d},
\end{aligned}
\]

(4.40)

which we call the binary-ternary Boltzmann equation. The binary collisional operator \( Q_2 \) is given by

\[
Q_2(f, f)(t, x, v) = \int_{\mathbb{R}^{2d-1} \times \mathbb{R}^d} b_2^+(\omega_1, v_1 - v) \left( f' f'_1 - f f_1 \right) d\omega_1 dv_1, 
\]

(4.41)

where

\[
\begin{aligned}
b_2(\omega_1, v_1 - v) &= \langle \omega_1, v_1 - v \rangle, \\
b_2^+ &= \max\{0, b_2\}, \\
f' &= f(t, x, v'), \quad f = f(t, x, v), \\
f'_1 &= f(t, x, v'_1), \quad f_1 = f(t, x, v_1).
\end{aligned}
\]

The ternary collisional operator \( Q_3 \) is given by

\[
Q_3(f, f, f)(t, x, v) = \int_{\mathbb{R}_1^{2d-1} \times \mathbb{R}^d} b_3^+(\omega_1, \omega_2, v_1 - v, v_2 - v) \left( f^* f'_1 f'_2 - f f_1 f_2 \right) d\omega_1 d\omega_2 dv_1 dv_2,
\]

(4.43)

where

\[
\begin{aligned}
b_3(\omega_1, \omega_2, v_{i+1} - v_i, v_{i+2} - v_i) &= \langle \omega_1, v_1 - v \rangle + \langle \omega_2, v_2 - v \rangle, \\
b_3^+ &= \max\{0, b_3\}, \\
f^* &= f(t, x, v^*), \quad f = f(t, x, v), \\
f'_1 &= f(t, x, v'_1), \quad f_1 = f(t, x, v_1), \\
f'_2 &= f(t, x, v'_2), \quad f_2 = f(t, x, v_2).
\end{aligned}
\]

Duhamel’s formula implies the binary-ternary Boltzmann equation can be written in mild form as

\[
f(t, x, v) = S^f_1 f_0(x, v) + \int_0^t S^{f_{\tau}}_1 Q(f, f, f)(\tau, x, v) d\tau,
\]

(4.45)

where

\[
S^f_1 g(x, v) = g(x - tv, v), \quad \forall (t, x, v) \in [0, \infty) \times \mathbb{R}^d, \quad g : \mathbb{R}^d \to \mathbb{R}.
\]

\textbf{Remark 4.4.} We will see in Section 5 that both the Boltzmann hierarchy and the binary-ternary Boltzmann equation are well-posed in appropriate functional spaces. It is not hard to see that if \( f \) is formally a solution to the binary-ternary Boltzmann equation with initial data \( f_0 \), then the tensorized product \( F := (f^0)^s \in \mathbb{N} \) is a solution to the Boltzmann hierarchy with initial data \( F_0 := (f_0^0)^s \in \mathbb{N} \). Therefore, the tensorized product of the unique solution to the binary-ternary Boltzmann equation with initial data \( f_0 \) will give the unique mild solution to the Boltzmann hierarchy with initial data \( F_0 \).

\textbf{Remark 4.5.} It is important to point out that in [2], the ternary operator \( Q_3 \) was symmetrized to an operator \( \tilde{Q}_3 \) which shares similar statistical and entropy production properties with the classical binary Boltzmann operator \( Q_2 \) (see [12]). In particular, it has a weak formulation which yields an \( \mathcal{H} \)-Theorem and local conservation laws. Hence, the operator \( Q_2 + Q_3 \) satisfies these statistical properties as well. This observation illustrates that the binary-ternary equation we are studying could serve as an extension term of the classical Boltzmann equation in modeling denser gases.

5. Local well-posedness

In this section, we show that the BBGKY hierarchy, the Boltzmann hierarchy and the binary-ternary Boltzmann equation are well-posed for short times in Maxwellian weighted \( L^\infty \)-spaces. To obtain these results, we combine the continuity estimates on the binary and ternary collisional operators, obtained in [22] and [3] respectively.
5.1. LWP for the BBGKY hierarchy. Consider \((N, \epsilon_2, \epsilon_3)\) in the scaling \((4.24)\), with \(N \geq 3\). For \(s \in \{1, \ldots, N\}\), recall from \((3.73)\) the space of functions

\[
C^0(\mathcal{D}_{s, \epsilon_2, \epsilon_3}) := \{g_m \in C^0(\mathbb{R}^{2d_s}) : \text{supp } g_m \subseteq \mathcal{D}_{s, \epsilon_2, \epsilon_3}\}.
\]

For \(\beta > 0\) we define the Banach space

\[
X_{N, \beta, s} := \left\{ g_{N, s} \in C^0(\mathcal{D}_{m, \epsilon_2, \epsilon_3}) \text{ and } |g_{N, s}|_{N, \beta, s} < \infty \right\},
\]

with norm \(|g_{N, s}|_{N, \beta, s} = \sup_{z \in \mathbb{R}^{2d_s}} |g_{N, s}(Z_s)|^{\beta}E_s(Z_s)\), where \(E_s(Z_s)\) is the kinetic energy of the \(s\)-particles given by \((3.57)\). For \(s > N\) we trivially define \(X_{N, \beta, s} := \{0\}\).

Remark 5.1. Given \(t \in \mathbb{R}\) and \(s \in \mathbb{N}\), conservation of energy under the flow \((3.60)\) implies that the \(s\)-particle of \((\epsilon_2, \epsilon_3)\)-flow operator \(T^s_t : X_{N, \beta, s} \rightarrow X_{N, \beta, s}\), given in \((3.74)\) is an isometry i.e.

\[
|T^s_t g_{N, s}|_{N, \beta, s} = |g_{N, s}|_{N, \beta, s}, \quad \forall g_{N, s} \in X_{N, \beta, s}.
\]

Proof. Let \(g_{N, s} \in X_{N, \beta, s}\) and \(Z_s \in \mathbb{R}^{2d_s}\). If \(Z_s \notin \mathcal{D}_{s, \epsilon_2, \epsilon_3}\), the result is trivial since \(g_{N, s}\) is supported in \(\mathcal{D}_{s, \epsilon_2, \epsilon_3}\). Assume \(Z_s \in \mathcal{D}_{s, \epsilon_2, \epsilon_3}\). Then Theorem \((3.24)\) yields

\[
e^{\beta E_s(Z_s)}|T^s_t g_{N, s}| = e^{\beta E_s(Z_s)}|(g_{N, s} \circ \Psi^s_t)(Z_s)| = e^{\beta E_s(\Psi^s_t Z_s)}|g_{N, s}(\Psi^s_t Z_s)| \leq |g_{N, s}|_{N, \beta, s},
\]

hence \(|T^s_t g_{N, s}|_{N, \beta, s} \leq |g_{N, s}|_{N, \beta, s}\). The other side of the inequality comes similarly using the fact that \(Z_s = \Psi^s_t(\Psi^s_t Z_s)\).

Consider as well \(\mu \in \mathbb{R}\). We define the Banach space

\[
X_{N, \beta, \mu} := \{G_N = (g_{N, s})_{s \in \mathbb{N}} : \|G_N\|_{N, \beta, \mu} < \infty\},
\]

with norm \(\|G_N\|_{N, \beta, \mu} = \sup_{s \in \mathbb{N}} e^{\beta \mu}|g_{N, s}|_{N, \beta, s} = \max_{s \in \{1, \ldots, N\}} e^{\beta \mu}|g_{N, s}|_{N, \beta, s}.
\]

Remark 5.2. Given \(t \in \mathbb{R}\), Remark \((5.1)\) implies that the map \(T^t : X_{N, \beta, \mu} \rightarrow X_{N, \beta, \mu}\) given by

\[
T^t G_N := (T^s_t g_{N, s})_{s \in \mathbb{N}},
\]

is an isometry i.e. \(\|T^t G_N\|_{N, \beta, \mu} = \|G_N\|_{N, \beta, \mu}\), for any \(G_N \in X_{N, \beta, \mu}\).

Finally, given \(T > 0\), \(\beta_0 > 0\), \(\mu_0 \in \mathbb{R}\) and \(\beta, \mu : [0, T] \rightarrow \mathbb{R}\) decreasing functions of time with \(\beta(0) = \beta_0\), \(\beta(T) > 0\), \(\mu(0) = \mu_0\), we define the Banach space

\[
X_{N, \beta, \mu} := L^\infty([0, T] \times X_{N, \beta_0, \mu_0}(t)),
\]

with norm \(\|G_N\|_{N, \beta, \mu} = \sup_{t \in [0, T]} \|G_N(t)\|_{N, \beta(t), \mu(t)}\). Similarly as in Proposition \((6.2)\) from \([2]\), one can obtain the following bounds:

**Proposition 5.3.** Let \(T > 0\), \(\beta_0 > 0\), \(\mu_0 \in \mathbb{R}\) and \(\beta, \mu : [0, T] \rightarrow \mathbb{R}\) decreasing functions with \(\beta_0 = \beta(0)\), \(\beta(T) > 0\), \(\mu(0) = \mu_0\). Then for any \(G_N = (g_{N, s})_{s \in \mathbb{N}} \in X_{N, \beta_0, \mu_0}\), the following estimates hold:

(i) \(\|G_N\|_{N, \beta, \mu} \leq \|G_N\|_{N, \beta_0, \mu_0}\).

(ii) \(\int_0^T T^t G_N \, dt \leq \|G_N\|_{N, \beta_0, \mu_0}\).

From Proposition \((5.3.1)\) in \([22]\) and Lemma \((5.1)\) in \([4]\), we have the following continuity estimates for the binary and ternary collisional operators respectively:

**Lemma 5.4.** Let \(m \in \mathbb{N}\), \(\beta > 0\). For any \(Z_m \in \mathcal{D}_{m, \epsilon_2, \epsilon_3}\) and \(k \in \{1, 2\}\), the following estimate holds:

\[
\|e^{N, m+k} g_{N, m+k}(Z_m)\| \leq \beta^{kd/2} \left( m\beta^{-1/2} + \sum_{l=1}^m |v_l| \right) e^{-\beta E_m(Z_m)} |g_{N, m+k}|_{N, \beta, m+k}, \quad \forall g_{N, m+k} \in X_{N, \beta, m+k}.
\]

Let us now define mild solutions to the BBGKY hierarchy:
Definition 5.5. Consider $T > 0$, $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$ and the decreasing functions $\beta, \mu : [0, T] \rightarrow \mathbb{R}$ with $\beta(0) = \beta_0$, $\beta(T) > 0$, $\mu(0) = \mu_0$. Consider also initial data $G_{N,0} = (g_{N,s,0}) \in X_{N,0,\beta_0,\mu_0}$. A map $G_N = (g_{N,s})_{s \in \mathbb{N}} \in X_{N,\beta,\mu}$ is a mild solution of the BBGKY hierarchy in $[0, T]$, with initial data $G_{N,0}$, if it satisfies:

$$G_N(t) = T^t G_{N,0} + \int_0^t T^{t-\tau} C_N G_N(\tau) \, d\tau,$$

where, given $\beta > 0$, $\mu \in \mathbb{R}$ and $G_N = (g_{N,s})_{s \in \mathbb{N}} \in X_{N,\beta,\mu}$, we write

$$C_N G_N := (C^2_N + C^3_N) G_N, \quad C^2_N G_N := \left(c_{s+1}^N g_{N,s+1}^N\right)_{s \in \mathbb{N}}, \quad C^3_N G_N := \left(c_{s+2}^N g_{N,s+2}^N\right)_{s \in \mathbb{N}},$$

and $T^t$ is given by (5.1).

Using Lemma 5.4 we obtain the following a-priori bounds:

Lemma 5.6. Let $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$, $T > 0$ and $\lambda \in (0, \beta_0/T)$. Consider the functions $\beta_\lambda, \mu_\lambda : [0, T] \rightarrow \mathbb{R}$ given by

$$\beta_\lambda(t) = \beta_0 - \lambda t, \quad \mu_\lambda(t) = \mu_0 - \lambda t. \quad (5.2)$$

Then for any $F(t) \subseteq [0, t]$ measurable, $G_N = (g_{N,s})_{s \in \mathbb{N}} \in X_{N,\beta,\mu}$ and $k \in \{1, 2\}$ the following bounds hold:

$$\left\| \int_{F(t)} T^{t-\tau} C_{k+1}^N G_N(\tau) \, d\tau \right\|_{N,\beta,\mu} \leq C_{k+1} \|G_N\|_{N,\beta,\mu}, \quad (5.3)$$

$$\lambda \lambda^{k+1} = C_{k+1}(d, \beta_0, \mu_0, T, \lambda) = C_d \lambda^{k+1} e^{-\lambda \mu_\lambda(T)} \lambda^{k+1/2} \left(1 + \lambda^{1/2}(T)\right). \quad (5.4)$$

Proof. For the proof of (5.3) for $k = 1$, see Lemma 5.3.1. from [2] and for the proof for $k = 2$ see Lemma 6.4. from [2].

Choosing $\lambda = \beta_0/2T$, Lemma 5.6 implies well-posedness of the BBGKY hierarchy up to short time. The proof follows similar steps to the proof of Theorem 6 from [2] and Theorem 6.4.1 from [2].

Theorem 5.7. Let $\beta_0 > 0$ and $\mu_0 \in \mathbb{R}$. Then there is $T = T(d, \beta_0, \mu_0) > 0$ such that for any initial datum $F_{N,0} = (f_{N,0,s})_{s \in \mathbb{N}} \in X_{N,\beta_0,\mu_0}$ there is unique mild solution $F_N = (f_{N,s})_{s \in \mathbb{N}} \in X_{N,\beta,\mu}$ to the BBGKY hierarchy in $[0, T]$ for the functions $\beta, \mu : [0, T] \rightarrow \mathbb{R}$ given by

$$\beta(t) = \beta_0 - \frac{\beta_0}{2T}, \quad \mu(t) = \mu_0 - \frac{\beta_0}{2T}. \quad (5.5)$$

The solution $F_N$ satisfies the bound:

$$\|F_N\|_{N,\beta,\mu} \leq 2 \|F_{N,0}\|_{N,\beta_0,\mu_0}. \quad (5.6)$$

Moreover, for any $F(t) \subseteq [0, t]$ measurable and $k \in \{1, 2\}$, the following bound holds:

$$\left\| \int_{F(t)} T^{t-\tau} C_{k+1}^N G_N(\tau) \, d\tau \right\|_{N,\beta,\mu} \leq \frac{1}{16} \|G_N\|_{N,\beta,\mu}, \quad \forall G_N \in X_{N,\beta,\mu}. \quad (5.7)$$

The time $T$ is explicitly given by:

$$T \simeq \beta_0 \left(e^{-\mu_0} \frac{\beta_0}{2} \right)^{-d/2} + e^{-2\mu_0 - \beta_0} \left(\frac{\beta_0}{2} \right)^{-d-1} \left(1 + \left(\frac{\beta_0}{2}\right)^{-1/2}\right)^{-1}. \quad (5.9)$$

5.2. LWP for the Boltzmann hierarchy. Similarly to Subsection 5.1, here we establish a-priori bounds and local well-posedness for the Boltzmann hierarchy. Without loss of generality, we will omit the proofs since they are identical to the BBGKY hierarchy case. Given $s \in \mathbb{N}$ and $\beta > 0$, we define the Banach space

$$X_{\infty,\beta,s} := \left\{ g_s \in C^0(\mathbb{R}^{2ds}) : g_s|_{\infty,\beta,s} < \infty \right\},$$

with norm $\|g_s|_{\infty,\beta,s} = \sup_{Z_s \in \mathbb{R}^{2ds}} |g_s(Z_s)| e^{\beta E_s(Z_s)}$, where $E_s(Z_s)$ is the kinetic energy of the $s$-particles given by (5.77).
Remark 5.8. Given $t \in \mathbb{R}$ and $s \in \mathbb{N}$, conservation of energy under the free flow implies that the $s$-particle free flow operator $S_s^t : X_{\infty, \beta, s} \rightarrow X_{\infty, \beta, s}$, given in \eqref{5.72}, is an isometry i.e.
\[
|S_s^t g_s|_{\infty, \beta, s} = |g_s|_{\infty, \beta, s}, \quad \forall g_s \in X_{\infty, \beta, s}.
\]

Consider as well $\mu \in \mathbb{R}$. We define the Banach space
\[
X_{\infty, \beta, \mu} := \{ G = (g_s)_{s \in \mathbb{N}} : \|G\|_{\infty, \beta, \mu} < \infty \},
\]
with norm $\|G\|_{\infty, \beta, \mu} = \sup_{t \in \mathbb{R}} e^{\beta t} |g_s|_{\infty, \beta, s}$.

Remark 5.9. Given $t \in \mathbb{R}$, Remark 5.8 implies that the map $S^t : X_{\infty, \beta, \mu} \rightarrow X_{\infty, \beta, \mu}$ given by
\[
S^t G := (S^t g_s)_{s \in \mathbb{N}},
\]
is an isometry i.e. $\|S^t G\|_{\infty, \beta, \mu} = \|G\|_{\infty, \beta, \mu}$, for any $G \in X_{\infty, \beta, \mu}$.

Finally, given $T > 0$, $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$ and $\beta, \mu : [0, T] \rightarrow \mathbb{R}$ decreasing functions of time with $\beta(0) = \beta_0$, $\beta(T) > 0$, $\mu(0) = \mu_0$, we define the Banach space
\[
X_{\infty, \beta, \mu} := L^\infty ([0, T], X_{\infty, \beta(t), \mu(t)}),
\]
with norm $\|G\|_{\infty, \beta, \mu} = \sup_{t \in [0, T]} \|G(t)\|_{\infty, \beta(t), \mu(t)}$.

Proposition 5.10. Let $T > 0$, $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$ and $\beta, \mu : [0, T] \rightarrow \mathbb{R}$ decreasing functions with $\beta_0 = \beta(0)$, $\beta(T) > 0$ $\mu(0) = \mu(0)$. Then for any $G = (g_s)_{s \in \mathbb{N}} \in X_{\infty, \beta_0, \mu_0}$, the following estimates hold:
\[
\begin{align*}
(i) \quad & \|G\|_{\infty, \beta, \mu} \leq \|G\|_{\infty, \beta_0, \mu_0}. \\
(ii) \quad & \left| \int_0^t S^t G \, dt \right|_{\infty, \beta, \mu} \leq T \|G\|_{\infty, \beta_0, \mu_0}.
\end{align*}
\]

Similarly to Lemma 5.4, we obtain:

Lemma 5.11. Let $m \in \mathbb{N}$ and $\beta > 0$. For any $Z_m \in \mathbb{R}^{2dm}$ and $k \in \{1, 2\}$, the following continuity estimate holds:
\[
|C^s_{\infty, \beta, m+k} g_{m+k}(Z_m)| \leq \beta^{-1/2} \left( m \beta^{-1/2} + \sum_{i=1}^m |v_i| \right) e^{-\beta E_m(Z_m)} |g_m|_{\infty, \beta, m+k}, \quad \forall g_{m+k} \in X_{\infty, \beta, m+k}. \tag{5.11}
\]

Let us now define mild solutions to the Boltzmann hierarchy:

Definition 5.12. Consider $T > 0$, $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$ and the decreasing functions $\beta, \mu : [0, T] \rightarrow \mathbb{R}$ with $\beta_0 = \beta(0)$, $\beta(T) > 0$ $\mu(0) = \mu_0$. Consider also initial data $G_0 = (g_s)_{s \in \mathbb{N}} \in X_{\infty, \beta_0, \mu_0}$. A map $G = (g_s)_{s \in \mathbb{N}} \in X_{\infty, \beta, \mu}$ is a mild solution of the Boltzmann hierarchy in $[0, T]$, with initial data $G_0$, if it satisfies:
\[
G(t) = S^t G_0 + \int_0^t S^t \mathcal{C} \mathcal{G} G(t) \, dt,
\]
where, given $\beta > 0$, $\mu \in \mathbb{R}$ and $\tilde{G} = (\tilde{g}_s)_{s \in \mathbb{N}} \in X_{\infty, \beta, \mu}$, we write
\[
\mathcal{C} G := (C_{\infty}^s, C_{\infty}^2) G, \quad C^2 G := (C_{\infty, \beta, s+1}^2)_{s \in \mathbb{N}}, \quad C^3 G := (C_{\infty, \beta, s+2}^3)_{s \in \mathbb{N}},
\]
and $S^t$ is given by \eqref{5.10}.

Using Lemma 5.11, we obtain the following a-priori bounds:

Lemma 5.13. Let $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$, $T > 0$ and $\lambda \in (0, \beta_0/T)$. Consider the functions $\beta_\lambda, \mu_\lambda : [0, T] \rightarrow \mathbb{R}$ given by \eqref{5.2}. Then for any $F(t) \subseteq [0, t]$ measurable, $G = (g_s)_{s \in \mathbb{N}} \in X_{\infty, \beta, \mu}$ and $k \in \{1, 2\}$, the following bound holds:
\[
\left| \int_{F(t)} S^t \mathcal{C}_\infty^{k+1} G \, dt \right|_{\infty, \beta_\lambda, \mu_\lambda} \leq C_{k+1} \|G\|_{\infty, \beta_\lambda, \mu_\lambda}, \tag{5.12}
\]
where the constant $C_{k+1} = C_{k+1}(d, \beta_0, \mu_0, T, \lambda)$ is given by \eqref{5.4}.

Choosing $\lambda = \beta_0/2T$, Lemma 5.13 directly implies well-posedness of the Boltzmann hierarchy up to short time.
Theorem 5.14. Let $\beta_0 > 0$ and $\mu_0 \in \mathbb{R}$. Then there is $T = T(\beta_0, \mu_0) > 0$ such that for any initial datum $F_0 = (f^{(s)})_{s \in \mathbb{N}} \in X_{\infty, \beta_0, \mu_0}$ there is unique mild solution $F = (f^{(s)})_{s \in \mathbb{N}} \in X_{\infty, \beta, \mu}$ to the Boltzmann hierarchy in $[0, T]$ for the functions $\beta, \mu : [0, T] \to \mathbb{R}$ given by \([5.13]\). The solution $F$ satisfies the bound:

$$\|F\|_{\infty, \beta, \mu} \leq 2\|F_0\|_{\infty, \beta_0, \mu_0}. \tag{5.13}$$

Moreover, for any $F(t) \subseteq [0, t]$ measurable and $k \in \{1, 2\}$, the following bound holds:

$$\left|\int_{F(t)} S^{1-t} C^{k+1}_\infty G(\tau) d\tau\right| \leq \frac{1}{16} \|G\|_{\infty, \beta, \mu}, \quad \forall G \in X_{\infty, \beta, \mu}, \tag{5.14}$$

and the time $T$ is explicitly given by \([5.9]\).

5.3. LWP for the binary-ternary Boltzmann equation and propagation of chaos. Now, we show local well-posedness for the binary-ternary Boltzmann equation and that, for chaotic initial data, their tensorized product produces the unique mild solution of the Boltzmann hierarchy. Therefore uniqueness implies that the mild solution to the Boltzmann hierarchy remains factorized under time evolution, hence chaos is propagated in time.

For $\beta > 0$ let us define the Banach space

$$X_{\beta, \mu} := \left\{ g \in C^0(\mathbb{R}^{2d}) : |g|_{\beta, \mu} < \infty \right\},$$

with norm $|g|_{\beta, \mu} = \sup_{(x,v) \in \mathbb{R}^{2d}} |g(x,v)| e^{\mu_1 |v|^2}$. Notice that for any $t \in [0, T]$, the map $S^t : X_{\beta, \mu} \to X_{\beta, \mu}$ is an isometry.

Consider $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$, $T > 0$ and $\beta, \mu : [0, T] \to \mathbb{R}$ decreasing functions of time with $\beta(0) = \beta_0$, $\beta(T) > 0$ and $\mu(0) = \mu_0$. We define the Banach space

$$X_{\beta, \mu} := L^\infty([0, T], X_{\beta(t), \mu(t)}),$$

with norm $\|g\|_{\beta, \mu} = \sup_{t \in [0, T]} |g(t)|_{\beta(t), \mu(t)}$. One can see that the following estimate holds:

Remark 5.15. Let $T > 0$, $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$ and $\beta, \mu : [0, T] \to \mathbb{R}$ decreasing functions with $\beta_0 = \beta(0)$, $\beta(T) > 0$ $\mu_0 = \mu(0)$. Then for any $g \in X_{\beta_0, \mu_0}$, the following estimate holds:

$$\|g\|_{\beta, \mu} \leq |g|_{\beta_0, \mu_0}.$$
Lemma 5.18. Let $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$, $T > 0$ and $\lambda \in (0, \beta_0/T)$. Consider the functions $\beta_\lambda, \mu_\lambda : [0, T] \to \mathbb{R}$ given by (5.2). Then for any $g, h \in X_{\beta_\lambda, \mu_\lambda}$ the following bounds hold:

$$\left\| \int_0^t S_1^{t-T} [Q_2(g - h, g - h) + Q_3(g - h, g - h)](\tau) d\tau \right\|_{\beta_\lambda, \mu_\lambda} \leq C (\|g\|_{\beta_\lambda, \mu_\lambda} + \|h\|_{\beta_\lambda, \mu_\lambda}) + \|g\|_{\beta_\lambda, \mu_\lambda} + \|h\|_{\beta_\lambda, \mu_\lambda}) \|g - h\|_{\beta_\lambda, \mu_\lambda},$$

where $C = C(d, \beta_0, \mu_0, T, \lambda) = C_2 + C_3$ and $C_2, C_3$ are given by (5.4) for $k = 1, 2$ respectively.

Choosing $\lambda = \beta_0/2T$, this estimate implies local well-posedness of the binary-ternary Boltzmann equation up to short times. Let us write $B_{X_{\beta, \mu}}$ for the unit ball of $X_{\beta, \mu}$.

Theorem 5.19 (LWP for the binary-ternary Boltzmann equation). Let $\beta_0 > 0$ and $\mu_0 \in \mathbb{R}$. Then there is $T = T(d, \beta_0, \mu_0) > 0$ such that for any initial data $f_0 \in X_{\beta_0, \mu_0}$, with $|f_0|_{\beta_0, \mu_0} \leq 1/2$, there is a unique mild solution $f \in B_{X_{\beta, \mu}}$ to the binary-ternary Boltzmann equation in $[0, T]$ with initial data $f_0$, where $\beta, \mu : [0, T] \to \mathbb{R}$ are the functions given by (5.5). The solution $f$ satisfies the bound:

$$\|f\|_{\beta, \mu} \leq 4|f_0|_{\beta_0, \mu_0}.$$  

(5.16)

Moreover, for any $g, h \in X_{\beta, \mu}$, the following estimates hold:

$$\left\| \int_0^t S_1^{t-T} [Q_2(g - h, g - h) + Q_3(g - h, g - h)](\tau) d\tau \right\|_{\beta, \mu} \leq \frac{1}{8}(1 + \|g\|_{\beta, \mu}) \|g\|_{\beta, \mu} \leq \frac{1}{4}\|g\|_{\beta, \mu}.$$  

(5.17)

The time $T$ is explicitly given by (5.5).

Proof. Choosing $T$ as in (5.7), we obtain $C(d, \beta_0, \mu_0, T, \beta_0/2T) = 1/8$. Thus, Lemma 5.18 implies estimate (5.17). Therefore, for any $g \in B_{X_{\beta, \mu}}$, using (5.17) for $h = 0$, we obtain

$$\left\| \int_0^t S_1^{t-T} [Q_2(g, g) + Q_3(g, g, g)](\tau) d\tau \right\|_{\beta, \mu} \leq \frac{1}{8}(1 + \|g\|_{\beta, \mu}) \|g\|_{\beta, \mu} \leq \frac{1}{4}\|g\|_{\beta, \mu}. $$

(5.18)

Let us define the nonlinear operator $\mathcal{L} : X_{\beta, \mu} \to X_{\beta, \mu}$ by

$$\mathcal{L}g(t) = S_1^t f_0 + \int_0^t S_1^{t-T} Q(g, g, g)(\tau) d\tau.$$

By triangle inequality, the fact that the free flow is isometric, Remark 5.16 bound (5.18) and the assumption $|f_0|_{\beta_0, \mu_0} \leq 1/2$, for any $g \in B_{X_{\beta, \mu}}$ and $t \in [0, T]$, we have

$$|\mathcal{L}g|_{\beta(t), \mu(t)} \leq |S_1^t f_0|_{\beta(t), \mu(t)} + \frac{1}{4} \|g\|_{\beta, \mu} = |f_0|_{\beta(t), \mu(t)} + \frac{1}{4} \|g\|_{\beta, \mu} \leq |f_0|_{\beta_0, \mu_0} + \frac{1}{4} \|g\|_{\beta, \mu} \leq \frac{1}{2} \|g\|_{\beta, \mu} \leq \frac{1}{4} \|g\|_{\beta, \mu} \leq \frac{3}{4} \|g\|_{\beta, \mu},$$

(5.19)

Thus, the operator $\mathcal{L} : B_{X_{\beta, \mu}} \to B_{X_{\beta, \mu}}$ is a contraction, so it has a unique fixed point $f \in B_{X_{\beta, \mu}}$ which is clearly the unique mild solution of the binary-ternary Boltzmann equation in $[0, T]$ with initial data $f_0$. To prove (5.10), we use the fact that $f = \mathcal{L}f$. Then for any $t \in [0, T]$, triangle inequality, definition of $\mathcal{L}$, estimate (5.19) for $g = f$ and $g = 0$, free flow being isometric, and Remark 5.17 yield

$$|f|_{\beta(t), \mu(t)} = |\mathcal{L}f|_{\beta(t), \mu(t)} \leq |\mathcal{L}0|_{\beta(t), \mu(t)} + |\mathcal{L}f - \mathcal{L}0|_{\beta(t), \mu(t)} \leq |S_1^t f_0|_{\beta(t), \mu(t)} + \frac{3}{4} \|f\|_{\beta, \mu} = |f_0|_{\beta(t), \mu(t)} + \frac{3}{4} \|f\|_{\beta, \mu} \leq |f_0|_{\beta_0, \mu_0} + \frac{3}{4} \|f\|_{\beta, \mu},$$

(5.10)

thus $\|f\|_{\beta, \mu} \leq |f_0|_{\beta_0, \mu_0} + \frac{3}{4} \|f\|_{\beta, \mu}$, and (5.16) follows. We can now prove that chaos is propagated by the Boltzmann hierarchy.
6. Convergence Statement

In this section we define an appropriate notion of convergence, namely convergence in observables, and we state the main result of this paper.

6.1. Approximation of Boltzmann hierarchy initial data. Here, we approximate Boltzmann hierarchy initial data by BBGKY hierarchy initial data. Let us first introduce some notation we are using from now on.

Given \( \theta > 0 \), we introduce the set of well-separated spatial configurations as follows: For \( m \) in \( \mathbb{N} \), we define

\[
\Delta^X_m(\theta) := \left\{ \vec{x}_m \in \mathbb{R}^d : |\vec{x}_i - \vec{x}_j| > \theta, \quad \forall 1 \leq i < j \leq m \right\}, \quad m \geq 2, \quad \Delta^X_1(\theta) := \mathbb{R}^d.
\]  

(6.1)

For \( m \in \mathbb{N} \), we also define the set of well-separated configurations as:

\[
\Delta(m) := \Delta^X_m(\theta) \times \mathbb{R}^{dm} = \left\{ (\vec{x}_m, \vec{v}_m) \in \mathbb{R}^{2dm} : |\vec{x}_i - \vec{x}_j| > \theta, \quad \forall 1 \leq i < j \leq m \right\}.
\]  

(6.2)

Recall we consider \((N, \varepsilon_2, \varepsilon_3)\) in the scaling

\[
N\varepsilon_2^{d-1} \approx N\varepsilon_3^{d-\frac{1}{2}} \approx 1.
\]  

(6.3)

Let us write \( \varepsilon_{2,N} \), \( \varepsilon_{3,N} \) for the \( \varepsilon_2, \varepsilon_3 \) associated to \( N \) under (6.3). By Remark 4.3, for \( N \) large enough, we have \( 0 < \varepsilon_{2,N} \ll \varepsilon_{3,N} \overset{N \to \infty}{\longrightarrow} 0 \).

We define the following approximating sequence:

**Definition 6.1.** Let \( s \in \mathbb{N} \), \( \beta > 0 \), \( \mu \in \mathbb{R} \) and \( G = (g_s)_{s \in \mathbb{N}} \in X_{\infty, \beta, \mu} \). We define

\[
G_N = (g_{N,s})_{s \in \mathbb{N}}, \quad \text{where} \quad g_{N,s} = \mathbb{1}_{\Delta_s} g_s.
\]  

(6.4)

The sequence \((G_N)_{N \in \mathbb{N}}\) is called approximating BBGKY hierarchy sequence of \( G \).

Similarly to Proposition 7.2. from [2], one obtains the following approximation property:

**Proposition 6.2.** Let \( s \in \mathbb{N} \), \( \beta > 0 \), \( \mu \in \mathbb{R} \), \( G = (g_s)_{s \in \mathbb{N}} \in X_{\infty, \beta, \mu} \) and \((G_N)_{N \in \mathbb{N}}\) the approximating BBGKY hierarchy sequence of \( G \). Then the following hold:

(i) \( G_N \in X_{N, \beta, \mu} \) for all \( N \in \mathbb{N} \). In particular,

\[
\sup_{N \in \mathbb{N}} \| G_N \|_{N, \beta, \mu} \leq \| G \|_{\infty, \beta, \mu}.
\]  

(6.5)

(ii) For any \( s \in \mathbb{N} \) and \( \theta > 0 \), we have

\[
\lim_{N \to \infty} \| g_{N,s} - g_s \|_{L^\infty(\Delta_s(\theta))} = 0.
\]  

(6.6)
6.2. Convergence in observables. Here, we define the convergence in observables. Let us first introduce some notation. Given \( s \in \mathbb{N} \), we define the space of test functions

\[
C_c(\mathbb{R}^d) = \left\{ \phi_s : \mathbb{R}^d \to \mathbb{R} : \phi_s \text{ is continuous and compactly supported} \right\}.
\]

\textbf{Definition 6.3.} Consider \( T > 0 \), \( s \in \mathbb{N} \) and \( g_s \in L^\infty ([0,T], L^\infty (\mathbb{R}^{2ds})) \). Given a test function \( \phi_s \in C_c(\mathbb{R}^d) \), we define the \( s \)-observable functional as

\[
I_{\phi_s} g_s(t)(X_s) = \int_{\mathbb{R}^d} \phi_s(V_s) g_s(t,X_s,V_s) \, dV.
\]

Recalling the set of initially good spatial configurations \( \Delta^X_\theta \) from (6.1), we give the definition of the convergence in observables:

\textbf{Definition 6.4.} Let \( T > 0 \). For each \( N \in \mathbb{N} \), consider \( G_N = (g_{N,s})_{s \in \mathbb{N}} \in \prod_{s=1}^\infty L^\infty ([0,T], L^\infty (\mathbb{R}^{2ds})) \) and \( G = (g_s)_{s \in \mathbb{N}} \in \prod_{s=1}^\infty L^\infty ([0,T], L^\infty (\mathbb{R}^{2ds})) \). We say that the sequence \( (G_N)_{N \in \mathbb{N}} \) converges in observables to \( G \) if for any \( s \in \mathbb{N}, \theta > 0 \) and \( \phi_s \in C_c(\mathbb{R}^d) \), we have

\[
\lim_{N \to \infty} ||I_{\phi_s} g_{N,s}(t) - I_{\phi_s} g_s(t)||_{L^\infty(\Delta^X_\theta)} = 0, \quad \text{uniformly in } [0,T].
\]

6.3. Statement of the main result. We are now in the position to state our main result. The rest of the paper will be devoted to its proof.

\textbf{Theorem 6.5 (Convergence).} Let \( \beta_0 > 0, \mu_0 \in \mathbb{R} \) and \( T = T(d, \beta_0, \mu_0) > 0 \) given by (5.9). Consider some initial Boltzmann hierarchy data \( F_0 = (f_0(s))_{s \in \mathbb{N}} \in X_{\infty, \beta_0, \mu_0} \) with approximating BBGKY hierarchy sequence \( (F_{N,0})_{N \in \mathbb{N}} \). Assume that

- for each \( N \), \( F_N \in X_{N, \beta, \mu} \) is the mild solution (given by Theorem 5.7) of the BBGKY hierarchy in \([0,T]\) with initial data \( F_{N,0} \).
- \( F \in X_{\infty, \beta, \mu} \) is the mild solution (given by Theorem 5.13) of the Boltzmann hierarchy in \([0,T]\) with initial data \( F_0 \).
- \( F_0 \) satisfies the following uniform continuity growth condition: There is a constant \( C > 0 \) such that, for any \( \zeta > 0 \), there is \( q = q(\zeta) > 0 \) such that for all \( s \in \mathbb{N} \), and for all \( Z_s, Z'_s \in \mathbb{R}^{2ds} \) with \( |Z_s - Z'_s| < q \), we have

\[
|f_0(s)(Z_s) - f_0(s)(Z'_s)| \leq C^{s-1} \zeta.
\]

Then, \( F_N \) converges in observables to \( F \).

\textbf{Remark 6.6.} Using the definition of convergence, proving Theorem 6.5 is equivalent to proving that for any \( s \in \mathbb{N}, \phi_s \in C_c(\mathbb{R}^d) \) and \( \theta > 0 \) we have

\[
\lim_{N \to \infty} ||I_{\phi_s}^N(t) - I_{\phi_s}^\infty(t)||_{L^\infty(\Delta^X_\theta)} = 0, \quad \text{uniformly in } [0,T],
\]

where

\[
I_{\phi_s}^N(t)(X_s) := I_{\phi_s} f_0^N(t)(X_s) = \int_{\mathbb{R}^{2ds}} \phi_s(V_s) f_0^N(t,X_s,V_s) \, dV,
\]

\[
I_{\phi_s}^\infty(t)(X_s) := I_{\phi_s} f(t)(X_s) = \int_{\mathbb{R}^{2ds}} \phi_s(V_s) f(t,X_s,V_s) \, dV.
\]

We also obtain the following Corollary 13 of Theorem 6.5.

\textbf{Corollary 6.7.} Let \( \beta_0 > 0, \mu_0 \in \mathbb{R} \) and \( f_0 \in X_{\beta_0, \mu_0} \), with \( |f_0|_{\beta_0, \mu_0} \leq 1/2 \). Assume as well that \( f_0 \) is uniformly continuous. Then for any \( s \in \mathbb{N}, \phi_s \in C_c(\mathbb{R}^d) \) and \( \theta > 0 \), the following convergence holds

\[
\lim_{N \to \infty} ||I_{\phi_s} f^\infty \mathbf{1}_{\Delta_s(\epsilon_3, N)} - I_{\phi_s} f^\infty ||_{L^\infty(\Delta^X_\theta)} = 0,
\]

where \( f \) is the mild solution to the binary-ternary Boltzmann equation in \([0,T]\), with initial data \( f_0 \), given by Theorem 5.19 and \( T \) is given by (5.9).

In order to prove Theorem 6.5, we will first use the local estimates developed in Section 5 to reduce the proof to finitely many observables of bounded energy, which are also well separated in time. Then, we will develop some geometric estimates which will enable us to eliminate recollisions of the backwards \((\epsilon_2, \epsilon_3)\)-flow.

\textsuperscript{13} which can be proved in a similar way as in Corollary 7.5. from \cite{2}
7. Reduction to term by term convergence

In this section we reduce the proof of Theorem 6.5 to term by term convergence after truncating the observables. After introducing the necessary combinatorial notation to take care of all the possible collision sequences occurring, the idea of the truncation is essentially the same as in \cite{22} [2], and it relies on the local estimates developed in Section 5. For this reason, we illustrate the similarities by providing the proof of the first estimate and omit the proofs of the rest of the estimates.

Throughout this section, we consider $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$, the functions $\beta, \mu : [0, T] \to \mathbb{R}$ defined by (5.5), (N, $\varepsilon_2, \varepsilon_3$) in the scaling (1.24) and initial data $F_{N,0} \in X_{N,\beta_0,\mu_0}, F_0 \in X_{\infty,\beta_0,\mu_0}$. Let $F_N = (f_N^{(s)})_{s \in \mathbb{N}} \in X_{N,\beta,\mu}$, $F = (f_s^{(s)})_{s \in \mathbb{N}} \in X_{\infty,\beta,\mu}$ be the mild solutions of the corresponding BBGKY and Boltzmann hierarchies, respectively, in $[0, T]$, given by Theorems 5.7 and Theorem 5.14. Let us note that by (5.5), we obtain

$$\beta(T) = \frac{\beta_0}{2}, \quad \mu(T) = \mu_0 - \frac{\beta_0}{2},$$

(7.1)

thus $\beta(T), \mu(T)$ do not depend on $T$.

For convenience, we introduce the following notation. Given $k \in \mathbb{N}$ and $t \geq 0$, we denote

$$T_k(t) := \left\{ (t_1, ..., t_k) \in \mathbb{R}^k : 0 \leq t_k < ... \leq t_1 \leq t \right\}.$$  

(7.2)

Since the collisions happening can be either binary or ternary we will introduce some additional notation to keep track of the collision sequences. In particular, given $k \geq 1$, we denote

$$S_k := \{ \sigma = (\sigma_1, ..., \sigma_k) : \sigma_i \in \{1, 2\}, \ \forall i = 1, ..., k \}.$$  

(7.3)

Notice that the cardinality of $S_k$ is given by:

$$|S_k| = 2^k, \quad \forall k \geq 1.$$  

(7.4)

Given $k \in \mathbb{N}$ and $\sigma \in S_k$, for any $1 \leq \ell \leq k$ we write

$$\tilde{\sigma}_\ell = \sum_{i=1}^\ell \sigma_i.$$  

(7.5)

We also write $\tilde{\sigma}_0 := 0$. Notice that

$$k \leq \tilde{\sigma}_k \leq 2k, \quad \forall k \in \mathbb{N}.$$  

(7.6)

7.1. Series expansion. Now, we make a series expansion for the mild solution $F_N = (f_N^{(s)})_{s \in \mathbb{N}}$ of the BBGKY hierarchy with respect to the initial data $F_{N,0}$. By Definition 5.5 for any $s \in \mathbb{N}$, we have Duhamel's formula:

$$f_N^{(s)}(t) = T_1 f_N^{(s)}(t) + \int_0^t T_s^{t-t_1} \left[ \mathcal{S}_{s, t_1}^{N+1} f_N^{(s+1)} + \mathcal{C}_{s+2}^{N} f_N^{(s+2)} \right] (t_1) \, dt_1.$$  

(7.9)

Let $n \in \mathbb{N}$. Iterating $n$-times Duhamel's formula, we obtain

$$f_N^{(s)}(t) = \sum_{k=0}^n f_N^{(s,k)}(t) + R_N^{(s,n+1)}(t),$$  

(7.7)

where we use the notation:

$$f_N^{(s,k)}(t) := \sum_{\sigma \in S_k} f_N^{(s,k,\sigma)}(t), \text{ for } 1 \leq k \leq n, \quad f_N^{(s,0)}(t) := T_1 f_N^{(s)}(t).$$  

(7.8)

$$f_N^{(s,k,\sigma)}(t) = \int_{T_k(t)} \mathcal{I}_{\sigma_1}^{N+1} C_{\sigma_2+1}^{N+1} C_{\sigma_3+2}^{N+2} C_{\sigma_4+3}^{N+3} ... C_{\sigma_{k-1}+k}^{N+k} f_N^{(s,\tilde{\sigma}_k)}(t_1) \, dt_1 ... dt_k,$$  

(7.9)

$$R_N^{(s,n+1)}(t) := \sum_{\sigma \in S_{n+1}} R_N^{(s,n+1,\sigma)}(t),$$  

(7.10)

$$R_N^{(s,n+1,\sigma)}(t) := \int_{T_{n+1}(t)} \mathcal{I}_{\sigma_1}^{N} C_{\sigma_2+1}^{N} C_{\sigma_3+2}^{N} C_{\sigma_4+3}^{N} ... C_{\sigma_k+2}^{N+k} f_N^{(s,\tilde{\sigma}_k)}(t_1) \, dt_1 ... dt_k,$$  

(7.11)

$$T_{n+1}^{(s,\tilde{\sigma}_k)}(t) := \int_{T_{n+1}(t)} \mathcal{I}_{\sigma_1}^{N} C_{\sigma_2+1}^{N} C_{\sigma_3+2}^{N} C_{\sigma_4+3}^{N} ... C_{\sigma_{n+1}+2}^{N+n+1} f_N^{(s,\tilde{\sigma}_k)}(t_{n+1}) \, dt_{n+1} ... dt_1.$$
One can make a similar series expansion for the Boltzmann hierarchy. By Definition 5.5 for any \( s \in \mathbb{N} \), we have Duhamel’s formula:

\[
 f^{(s)}(t) = S_t^s f^{(s)}_0 + \int_0^t S_{t-s}^s \left[ c_{s+1}^{(s+1)} f^{(s+1)} + c_{s+2}^{(s+2)} f^{(s+2)} \right] (t) \, dt.
\]

Iterating \( n \)-times Duhamel’s formula, we obtain

\[
 f^{(s)}(t) = \sum_{k=0}^n f^{(s,k)}(t) + R^{(s,n+1)}(t),
\]

where we use the notation:

\[
 f^{(s,k)}(t) := \sum_{\sigma \in S_k} f^{(s,k,\sigma)}(t), \quad \text{for } 1 \leq k \leq n,
\]

\[
 f^{(s,0)}(t) := S_t^s f^{(s)}_0.
\]

Given \( \phi_s \in C_c(\mathbb{R}^{d_\sigma}) \) and \( k \in \mathbb{N} \), let us denote

\[
 I^{N,s}_k(t)(X_s) := \int_{\mathbb{R}^{d_s}} \phi_s(V_s) f^{(s,k)}(t, X_s, V_s) \, dV_s,
\]

\[
 I^{\infty,s}_k(t)(X_s) := \int_{\mathbb{R}^{d_s}} \phi_s(V_s) f^{(s,k)}(t, X_s, V_s) \, dV_s.
\]

We obtain the following estimates:

**Lemma 7.1.** For any \( s, n \in \mathbb{N} \) and \( t \in [0, T] \), the following estimates hold:

\[
 \| I_s^{N}(t) - \sum_{k=0}^n I_{s,k}^{N}(t) \|_{L^\infty} \leq C_s, \beta_0, \mu_0 \cdot \| \phi_s \|_{L^\infty} 4^{-n} \cdot \| F_{N,0} \|_{N, \beta_0, \mu_0},
\]

\[
 \| I_s^{\infty}(t) - \sum_{k=0}^n I_{s,k}^{\infty}(t) \|_{L^\infty} \leq C_s, \beta_0, \mu_0 \cdot \| \phi_s \|_{L^\infty} 4^{-n} \cdot \| F_0 \|_{\infty, \beta_0, \mu_0},
\]

where the observables \( I_s^{N}, I_s^{\infty} \) defined in (5.5) - (5.11).

**Proof.** Fix \( Z_s = (X_s, V_s) \in \mathbb{R}^{2d_s}, t \in [0, T] \) and \( \sigma \in S_{n+1} \). We repeatedly use estimate (5.7) of Theorem 5.7 for \( k = 1 \) if \( \sigma_1 = 1 \) or for \( k = 2 \) if \( \sigma_1 = 2 \), to obtain

\[
 e^{\beta(T) E_s(Z_s) + s \mu(T)} | R^{(s,n+1,\sigma)}(t, X_s, V_s) | \leq 8^{-(n+1)} \cdot \| F_N \|_{N, \beta, \mu},
\]

so adding for all \( \sigma \in S_{n+1} \), using (7.14), (5.6) and the definition of the norms, we take

\[
 | \phi_s(V_s) f^{(s,n+1)}(t, X_s, V_s) | \leq 4^{-(n+1)} e^{-s \mu(T)} \| \phi_s \|_{L^\infty} \| F_N \|_{N, \beta, \mu} e^{-\beta(T) E_s(Z_s)}
\]

\[
 \leq 4^{-n} e^{-s \mu(T)} \| \phi_s \|_{L^\infty} \| F_{N,0} \|_{N, \beta, \mu} e^{-\beta(T) E_s(Z_s)}.
\]

Thus, integrating with respect to velocities and recalling (7.17), (7.18), we obtain

\[
 \| I^{N}_s(t)(X_s) - \sum_{k=0}^n I^{N}_{s,k}(t)(X_s) \| \leq C_s, \beta_0, \mu_0 \cdot \| \phi_s \|_{L^\infty} 4^{-n} \cdot \| F_{N,0} \|_{N, \beta_0, \mu_0} \int_{\mathbb{R}^{d_s}} e^{-\beta(T) E_s(Z_s)} \, dV_s
\]

\[
 \leq C_s, \beta_0, \mu_0 \cdot \| \phi_s \|_{L^\infty} 4^{-n} \cdot \| F_{N,0} \|_{N, \beta_0, \mu_0}.
\]

For the Boltzmann hierarchy, we follow a similar argument using estimates (5.13) and (5.14) instead. 

\[\square\]
7.2. High energy truncation. We will now truncate energies, so that we can focus on bounded energy domains. Let us fix \( s, n \in \mathbb{N} \) and \( R > 1 \). As usual we denote \( B_R^d \) to be the \( 2d \)-ball of radius \( R \) centered at the origin.

We first define the truncated BBGKY hierarchy and Boltzmann hierarchy collisional operators. For \( \ell \in \mathbb{N} \) we define
\[
C_{\ell+1}^N(g_{\ell+1}^1) := C_{\ell+1}^N(g_{\ell+1}^1 \mathbb{1}_{|E_{\ell+1} \leq R^2}), \quad C_{\ell+2}^N(g_{\ell+2}^1) := C_{\ell+2}^N(g_{\ell+2}^1 \mathbb{1}_{|E_{\ell+2} \leq R^2}).
\]

For the BBGKY hierarchy we define
\[
f^{(s,k)}(t, Z_s) := \sum_{N, R} f^{(s,k,\sigma)}(t, Z_s), \quad 1 \leq k \leq n, \quad f^{(s,0)}(t, Z_s) := T^s_t \left(f_{N,R} \mathbb{1}_{|E_s \leq R^2}\right)(Z_s),
\]
where given \( k \geq 1 \) and \( \sigma \in S_k \), we denote
\[
f^{(s,k,\sigma)}(t, Z_s) := \int T^{s-t_1} C_{s_1}^N(t, X_{s_1}, Z_{s_1}) T^{s-t_2} C_{s_2}^N(t, X_{s_2}, Z_{s_2}) \cdots T^{s-t_k} C_{s_k}^N(t, X_{s_k}, Z_{s_k}) \left(f^{(s,0)}\right)(Z_s) dt_1 \cdots dt_k.
\]

For the Boltzmann hierarchy we define
\[
f^{(s,k)}(t, Z_s) := \sum_{N, R} f^{(s,k,\sigma)}(t, Z_s), \quad 1 \leq k \leq n, \quad f^{(s,0)}(t, Z_s) := S^t_t \left(f_{N,R} \mathbb{1}_{|E_s \leq R^2}\right)(Z_s),
\]
where given \( k \geq 1 \) and \( \sigma \in S_k \), we denote
\[
f^{(s,k,\sigma)}(t, Z_s) := \int T^{s-t_1} S_{s_1}^\infty(t, X_{s_1}, Z_{s_1}) T^{s-t_2} C_{s_2}^N(t, X_{s_2}, Z_{s_2}) \cdots T^{s-t_k} S_{s_k}^\infty(t, X_{s_k}, Z_{s_k}) \left(f^{(s,0)}\right)(Z_s) dt_1 \cdots dt_k.
\]

Given \( \phi_s \in C_c(\mathbb{R}^d) \) and \( k \in \mathbb{N} \), let us denote
\[
I_{s,k,R}(t)(X_s) := \int R_{d-s} \phi_s(V_s^t) f^{(s,k)}(t, X_s, V_s) dV_s = \int R_{d-s} \phi_s(V_s^t) f^{(s,k)}(t, X_s, V_s) dV_s.
\]

Recalling the observables \( I_{s,k}^N, I_{s,k}^\infty \), defined in (7.22), we obtain the following estimates:

**Lemma 7.2.** For any \( s, n \in \mathbb{N} \), \( R > 1 \) and \( t \in [0, T] \), the following estimates hold:
\[
\sum_{k=0}^n \| I_{s,k,R}(t) - I_{s,k}^N(t) \|_{L_{\infty}^N} \leq C_{s,\beta_0, \mu_0, T} \| \phi_s \|_{L_{\infty}^N} e^{-\frac{\beta_0 R^2}{2}} \| F_{N,0} \|_{L_{\infty}^N},
\]
\[
\sum_{k=0}^n \| I_{s,k,R}(t) - I_{s,k}^\infty(t) \|_{L_{\infty}^N} \leq C_{s,\beta_0, \mu_0, T} \| \phi_s \|_{L_{\infty}^N} e^{-\frac{\beta_0 R^2}{2}} \| F_0 \|_{L_{\infty}^N}.
\]

**Proof.** For the proof, we use the same ideas as in Lemma 8.4. from [2], and we also use (7.4) to sum over all possible collision sequences. \( \square \)

7.3. Separation of collision times. We will now separate the time intervals we are integrating at, so that collisions occurring are separated in time. For this purpose consider a small time parameter \( \delta > 0 \).

For convenience, given \( t \geq 0 \) and \( k \in \mathbb{N} \), we define
\[
T_{k,\delta}(t) := \{(t_1, \ldots, t_k) \in T_k(t) : 0 \leq t_{i+1} - t_i \leq \delta, \quad \forall i \in [0, k]\},
\]
where we denote \( t_{k+1} = 0, t_0 = t \).

For the BBGKY hierarchy, we define
\[
f^{(s,k)}_{N,R,\delta}(t, Z_s) := \sum_{\sigma \in S_k} f^{(s,k,\sigma)}_{N,R,\delta}(t, Z_s), \quad 1 \leq k \leq n, \quad f^{(s,0)}_{N,R,\delta}(t, Z_s) := T^s_t \left(f_{N,R} \mathbb{1}_{|E_s \leq R^2}\right)(Z_s),
\]
where, given \( k \geq 1 \) and \( \sigma \in S_k \), we denote
\[
f^{(s,k,\sigma)}_{N,R,\delta}(t, Z_s) := \int T_{k,\delta}(t) T^{s-t_1} C_{s_1}^N(t, X_{s_1}, Z_{s_1}) T^{s-t_2} C_{s_2}^N(t, X_{s_2}, Z_{s_2}) \cdots T^{s-t_k} C_{s_k}^N(t, X_{s_k}, Z_{s_k}) \left(f^{(s,0)}\right)(Z_s) dt_1 \cdots dt_k.
\]
In the same spirit, for the Boltzmann hierarchy we define
\[ f_{N,R,δ}^{(s,k)}(t,Z_s) := \sum_{σ ∈ S_k} f_{N,R,δ}^{(s,k,σ)}(t,Z_s), \quad \text{for} \quad 1 ≤ k ≤ n, \quad f_{N,δ}^{(s,0)}(t,Z_s) := \mathcal{S}_t(f_0 1_{|E_s| ≤ R^2})(Z_s), \]
where, given \( k ≥ 1 \) and \( σ ∈ S_k \), we denote
\[ f_{R,δ}^{(s,k,σ)}(t,Z_s) := \int_{T_k(δ)} S^{−1}C^{∞,R}_{s+δ_1} S^{t_1−t_2}C^{∞,R}_{s+δ_2} \cdots C^{∞,R}_{s+δ_k} S^{t_m}_{s+δ_k} f_{0}^{(s+δ_k)}(Z_s) \ dt_k, \cdots dt_1. \]
Given \( φ_s ∈ C_c(ℝ^d) \) and \( k ∈ ℕ \), we define
\[ I_{s,k,R,δ}^{N}(t)(X_s) := \int_{ℝ^d} φ_s(V_s)f_{N,R,δ}^{(s,k)}(t,X_s,V_s) dV_s = \int_{B_{R}^d} φ_s(V_s)f_{N,R,δ}^{(s,k)}(t,X_s,V_s) dV_s, \quad \text{(7.23)} \]
\[ I_{s,k,R,δ}^{∞}(t)(X_s) := \int_{ℝ^d} φ_s(V_s)f_{R,δ}^{(s,k)}(t,X_s,V_s) dV_s = \int_{B_{R}^d} φ_s(V_s)f_{R,δ}^{(s,k)}(t,X_s,V_s) dV_s. \quad \text{(7.24)} \]
\[ \text{Remark 7.3. For} \quad 0 ≤ t ≤ δ, \quad \text{we trivially obtain} \quad T_{k,δ}(t) = 0. \quad \text{In this case the functionals} \quad I_{s,k,R,δ}^{N}(t), I_{s,k,R,δ}^{∞}(t) \quad \text{are identically zero.} \]

Recalling the observables \( I_{s,k,R,δ}^{N}(t), I_{s,k,R,δ}^{∞}(t) \) defined in (7.20)−(7.21), we obtain the following estimates:

**Lemma 7.4.** For any \( s,n ∈ ℕ, R > 0, δ > 0 \) and \( t ∈ [0,T] \), the following estimates hold:
\[ \sum_{k=0}^{n} \| I_{s,k,R,δ}^{N}(t) − I_{s,k,R}(t) \|_{L^∞_{X_s}} ≤ δ∥φ_s∥_{L^∞_{V_s}} C^n_{d,s,β_0,µ_0,T} ∥F_{N,0}∥_{N,β_0,µ_0}, \]
\[ \sum_{k=0}^{n} \| I_{s,k,R,δ}^{∞}(t) − I_{s,k,R}(t) \|_{L^∞_{X_s}} ≤ δ∥φ_s∥_{L^∞_{V_s}} C^n_{d,s,β_0,µ_0,T} ∥F_{0}∥_{β_0,µ_0}. \]

**Proof.** For the proof, we follow similar ideas as in Lemma 8.7. from [2], and we also use bound (7.6) to control the combinatorics occurring.

Combining Lemma 7.1, Lemma 7.2 and Lemma 7.4, we obtain

**Proposition 7.5.** For any \( s,n ∈ ℕ, R > 1, δ > 0 \) and \( t ∈ [0,T] \), the following estimates hold:
\[ \| I_{s}^{N}(t) − \sum_{k=1}^{n} I_{s,k,R,δ}(t) \|_{L^∞_{X_s}} ≤ C_{s,β_0,µ_0,T} ∥φ_s∥_{L^∞_{V_s}} \left( 2^{−n} + e^{−cR^2} + δ C^n_{d,s,β_0,µ_0,T} ∥F_{N,0}∥_{N,β_0,µ_0} \right) \]
\[ \| I_{s}^{∞}(t) − \sum_{k=1}^{n} I_{s,k,R,δ}(t) \|_{L^∞_{X_s}} ≤ C_{s,β_0,µ_0,T} ∥φ_s∥_{L^∞_{V_s}} \left( 2^{−n} + e^{−cR^2} + δ C^n_{d,s,β_0,µ_0,T} ∥F_{0}∥_{β_0,µ_0} \right). \]

Proposition 7.5 implies that, given \( 0 ≤ k ≤ n, R > 1, δ > 0 \), the convergence proof reduces to controlling the differences \( I_{s,k,R,δ}(t) − I_{s,k,R,δ}(t) \), where the observables \( I_{s,k,R,δ}^{N}(t), I_{s,k,R,δ}^{∞}(t) \) are given by (7.23)−(7.24). However this is not immediate since the backwards \((e_2,e_3)\)-flow and the backwards free flow do not coincide in general. The goal is to eliminate some small measure set of initial data, negligible in the limit, such that the backwards \((e_2,e_3)\)-flow and the backwards free flow are comparable.

8. GEOMETRIC ESTIMATES

In this section we present some geometric results which will be essential for estimating the measure of the pathological sets leading to recollisions of the backwards \((e_2,e_3)\) flow (see Section 9). First, we review some of the results we used in [4] which are useful here as well. We then present certain novel results, namely Lemma 8.3, Lemma 8.4, Lemma 8.7 and most importantly Lemma 8.8 which crucially rely on the following symmetric representation of the \((2d − 1)\) sphere of radius \( r > 0 \):
\[ S_{r}^{2d−1} = \{ (ω_1,ω_2) ∈ B^d_s × B^d_s : ω_2 ∈ S^{d−1}r_0 = \sqrt{r^2−|ω_1|^2} \} = \{ (ω_1,ω_2) ∈ B^d_s × B^d_s : ω_1 ∈ S^{d−1}r_0 = \sqrt{r^2−|ω_2|^2} \} \quad \text{(8.1)} \]
Representation (8.1) is very useful when one wants to estimate the intersection of \( S_{r}^{2d−1} \) with sets of the form \( S × ℝ^d \) or \( ℝ^d × S \), where \( S ⊆ ℝ^d \) is of small measure.
8.1. Cylinder-Sphere estimates. Here, we present certain estimates based on the intersection of a sphere with a given solid cylinder. These estimates were used in [4] as well. Similar estimates can be found in [10, 22].

**Lemma 8.1.** Let \( \rho, r > 0 \) and \( K^d_\rho \subseteq \mathbb{R}^d \) be a solid cylinder. Then the following estimate holds for the \((d - 1)\)-spherical measure:

\[
\int_{S^{d-1}} 1_{K^d_\rho} \, d\omega \lesssim r^{d-1} \min \left\{ 1, \left( \frac{\rho}{r} \right)^{\frac{d-1}{2}} \right\}.
\]

**Proof.** After re-scaling we may clearly assume that \( r = 1 \). Then, we refer to the work of R. Denlinger [16], p.30, for the rest of the proof. \( \square \)

Applying Lemma 8.1, we obtain the following geometric estimate, which will be crucially used in Section 9.

**Corollary 8.2.** Given \( 0 < \rho \leq 1 \leq R \), the following estimate holds:

\[
|B^d_R \cap K^d_\rho| \lesssim R^d \rho^{\frac{d-1}{2}}.
\]

**Proof.** The co-area formula and Lemma 8.1 imply

\[
|B^d_R \cap K^d_\rho| = \int_0^R \int_{S^{d-1}} 1_{K^d_\rho} \, d\omega \, dr \\
\lesssim \int_0^R r^{d-1} \min \left\{ 1, \left( \frac{\rho}{r} \right)^{\frac{d-1}{2}} \right\} \, dr \\
\leq \int_0^\rho r^{d-1} \, dr + \rho^{\frac{d-1}{2}} \int_0^\rho r^{\frac{d-1}{2}} \, dr \\
\approx \rho^d + \rho^{\frac{d-1}{2}} R^{\frac{d+1}{2}}, \quad \text{since } d \geq 2 \\
\lesssim R^d \rho^{\frac{d-1}{2}}, \quad \text{since } 0 < \rho \leq 1 \leq R.
\]

\( \square \)

8.2. Estimates relying on the \((2d - 1)\)-sphere representation. Here we present certain geometric estimates relying on the representation (8.1). In particular, up to our knowledge, Lemma 8.3, Lemma 8.6, Lemma 8.7, and most importantly Lemma 8.8 are novel results. Lemma 8.4 is a special case of a result proved in [4].

8.2.1. Truncation of impact directions. We first estimate the intersection of \( S^{2d-1}_1 \) with sets of the form \( B^d_\rho \times \mathbb{R}^d \) or \( \mathbb{R}^d \times B^d_\rho \).

**Lemma 8.3.** Consider \( \rho > 0 \). We define the sets

\[
M_1(\rho) = B^d_\rho \times \mathbb{R}^d = \left\{ (\omega_1, \omega_2) \in \mathbb{R}^{2d} : |\omega_1| \leq \rho \right\},
\]

\[
M_2(\rho) = \mathbb{R}^d \times B^d_\rho = \left\{ (\omega_1, \omega_2) \in \mathbb{R}^{2d} : |\omega_2| \leq \rho \right\}
\]

Then, the following holds

\[
\int_{S^{2d-1}} 1_{M_1(\rho)} \, d\omega_1 \, d\omega_2 = \int_{S^{2d-1}} 1_{M_2(\rho)} \, d\omega_1 \, d\omega_2 \lesssim \min \{1, \rho^d\}.
\]

**Proof.** By symmetry it suffices to estimate the first term. Using (8.3) and representation (8.1), we obtain

\[
\int_{S^{2d-1}} 1_{M_1(\rho)} \, d\omega_1 \, d\omega_2 = \int_{S^{2d-1}} 1_{B^d_\rho \times \mathbb{R}_1} \, d\omega_1 \, d\omega_2 \lesssim \int_{B^d_\rho \cap B^d_1} \int_{S^{d-1}} \frac{d\omega_1 \, d\omega_2}{\sqrt{1 - |\omega_1|^2}} \lesssim \min \{1, \rho^d\}.
\]

\( \square \)

The following result is a special case of Lemma 8.4. from [4]. For the proof, see Lemma 9.5. in [2].
Lemma 8.4. Consider \( \rho > 0 \). Let us define the strip
\[ W^{2d}_\rho = \{ (\omega_1, \omega_2) \in \mathbb{R}^{2d} : |\omega_1 - \omega_2| \leq \rho \}. \tag{8.5} \]
Then, the following estimate holds:
\[ \int_{S^{d-1}_1} 1_{W^{2d}_\rho} \, d\omega_1 \, d\omega_2 \lesssim \min \left\{ 1, \rho \frac{2^{d-1}}{2^{d-1} - 1} \right\}. \]

Proof. For the proof, see Lemma 9.5. in [2]. The main idea is to first use representation (8.1) and then apply Lemma 8.2. □

8.2.2. Conic estimates. Now we establish estimates related to conic regions. We first present a well-known spherical cap estimate.

Lemma 8.5. Consider \( 0 \leq \alpha \leq 1 \) and \( \nu \in \mathbb{R}^d \setminus \{0\} \). Let us define
\[ S(\alpha, \nu) = \{ \omega \in \mathbb{R}^d : |\omega| \geq \alpha |\nu| \}. \tag{8.6} \]
Then, for \( \rho > 0 \), the following estimate holds:
\[ \int_{S^{d-1}_1} 1_{S(\alpha, \nu)} \, d\omega = r^{d-1} |S^{d-2}| \int_0^{2 \arccos \alpha} \sin^{d-2} \theta \, d\theta \lesssim r^{d-1} \arccos \alpha. \]

Proof. After re-scaling, it suffices to prove the result for \( r = 1 \). Notice that \( S^{d-1}_1 \cap S(\alpha, \nu) \) is a spherical cap of angle \( 2 \arccos \alpha \) and direction \( \nu \neq 0 \) on the unit sphere. Therefore, integrating in spherical coordinates, we obtain
\[ \int_{S^{d-1}_1} 1_{S(\alpha, \nu)} \, d\omega = |S^{d-2}| \int_0^{2 \arccos \alpha} \sin^{d-2} \theta \, d\theta \lesssim \arccos \alpha. \]
□

We apply Lemma 8.5 to obtain the following result:

Lemma 8.6. Consider \( 0 \leq \alpha \leq 1 \) and \( \nu \in \mathbb{R}^d \setminus \{0\} \). Let us define
\[ N(\alpha, \nu) = \{ (\omega_1, \omega_2) \in \mathbb{R}^{2d} : \langle \omega_1 - \omega_2, \nu \rangle \geq \alpha |\omega_1 - \omega_2||\nu| \}. \tag{8.7} \]
Then, we have the estimate:
\[ \int_{S^{2d-1}_1} 1_{N(\alpha, \nu)} \, d\omega_1 \, d\omega_2 \lesssim \arccos \alpha. \]

Proof. Recalling (8.6) - (8.7), we have
\[ N(\alpha, \nu) = \{ (\omega_1, \omega_2) \in \mathbb{R}^{2d} : \omega_1 - \omega_2 \in S(\alpha, \nu) \}. \tag{8.8} \]
Let us define the linear map \( T : \mathbb{R}^{2d} \to \mathbb{R}^{2d} \) by
\[ (u_1, u_2) = T(\omega_1, \omega_2) := (\omega_1 + \omega_2, \omega_1 - \omega_2). \]
Clearly
\[ |u_1|^2 + |u_2|^2 = |\omega_1 + \omega_2|^2 + |\omega_1 - \omega_2|^2 = 2|\omega_1|^2 + 2|\omega_2|^2 = 2, \quad \forall (\omega_1, \omega_2) \in S^{2d-1}_1, \]
hence \( T : S^{2d-1}_1 \to S^{2d-1}_1 \). Therefore, using (8.7) and changing variables under \( T \), we have
\[ \int_{S^{2d-1}_1} 1_{N(\alpha, \nu)}(\omega_1, \omega_2) \, d\omega_1 \, d\omega_2 \approx \int_{S^{2d-1}_1} 1_{S(\alpha, \nu)}(\omega_1 - \omega_2) \, d\omega_1 \, d\omega_2 \]
\[ \approx \int_{S^{2d-1}_1} 1_{S(\alpha, \nu)}(u_2) \, du_1 \, du_2 \]
\[ \lesssim \sqrt{2} \int_{S^{2d-1}_1} 1_{S(\alpha, \nu)}(u_2) \, du_1 \, du_2 \]
\[ \lesssim \arccos \alpha, \tag{8.9} \]
where to obtain (8.9) we use the representation of the sphere (8.1), and to obtain (8.10) we use Lemma 8.5. □
8.2.3. Annuli estimates. We present estimates based on the intersection of the unit sphere some appropriate annuli.

**Lemma 8.7.** Let \(0 < \beta < 1/2\), and consider the sets
\[
I_1 = \left\{(\omega_1, \omega_2) \in \mathbb{R}^{2d} : |1 - 2|\omega_1|^2| \leq 2\beta \right\},
\]
\[
I_2 = \left\{(\omega_1, \omega_2) \in \mathbb{R}^{2d} : |1 - 2|\omega_2|^2| \leq 2\beta \right\}.
\]
There hold the estimates:
\[
\int_{S^{d-1}} \mathbf{1}_{I_1} d\omega_1 d\omega_2 = \int_{S^{d-1}} \mathbf{1}_{I_2} d\omega_1 d\omega_2 \lesssim \beta.
\]

**Proof.** By symmetry, it suffices to prove the estimate for \(I_1\). Since \(0 < \beta < 1/2\), we may write
\[
I_1 = \left\{(\omega_1, \omega_2) \in \mathbb{S}^{2d-1} : \sqrt{1/2 - \beta} \leq |\omega_1| \leq \sqrt{1/2 + \beta} \right\}.
\]
Using the representation \((8.1)\) of the \((2d - 1)\)-unit sphere, we obtain
\[
\int_{S^{d-1}} \mathbf{1}_{I_1} d\omega_1 d\omega_2 \leq \int_{\sqrt{1/2 - \beta} \leq |\omega_1| \leq \sqrt{1/2 + \beta}} \int_{\sqrt{1/2 - |\omega_1|^2}}^{d\omega_2} d\omega_1 d\omega_2 \leq \left(\frac{1}{2} + \beta\right)^{d/2} - \left(\frac{1}{2} - \beta\right)^{d/2}
\]
\[
= \frac{2\beta}{\sqrt{1/2 + \beta} + \sqrt{1/2 - \beta}} \sum_{j=0}^{d-1} \left(\frac{1}{2} + \beta\right)^{j/2} \left(\frac{1}{2} - \beta\right)^{d-1-j} \lesssim \beta,
\]
since \(0 < \beta < 1/2\). The proof is complete. \(\square\)

**Lemma 8.8.** Consider \(0 < \beta < 1/4\). Let us define the hemispheres
\[
S_{1,2} = \{(\omega_1, \omega_2) \in \mathbb{S}^{2d-1} : |\omega_1| < |\omega_2|\},
\]
\[
S_{2,1} = \{(\omega_1, \omega_2) \in \mathbb{S}^{2d-1} : |\omega_2| < |\omega_1|\},
\]
and the annuli
\[
I_{1,2} = \{(\omega_1, \omega_2) \in \mathbb{R}^{2d} : |\omega_1|^2 + 2|\omega_1|\omega_2| \leq \beta\},
\]
\[
I_{2,1} = \{(\omega_1, \omega_2) \in \mathbb{R}^{2d} : |\omega_2|^2 + 2|\omega_1|\omega_2| \leq \beta\}.
\]
Then, there holds
\[
\int_{S_{1,2}} \mathbf{1}_{I_{1,2}} d\omega_1 d\omega_2 = \int_{S_{2,1}} \mathbf{1}_{I_{2,1}} d\omega_1 d\omega_2 \lesssim \beta.
\]

**Proof.** By symmetry, it suffices to prove
\[
\int_{S_{2,1}} \mathbf{1}_{I_{2,1}} d\omega_1 d\omega_2 \leq \beta.
\]
Recalling notation from \((8.3)-(8.4)\), let us define
\[
U_\beta = M_1(2\sqrt{\beta}) \cap M_2^c(2\sqrt{\beta}) = \{(\omega_1, \omega_2) \in \mathbb{R}^{2d} : |\omega_1| > 2\sqrt{\beta} \text{ and } |\omega_2| > 2\sqrt{\beta}\}.
\]
Clearly \(U_\beta = M_1(2\sqrt{\beta}) \cup M_2(2\sqrt{\beta})\). Writing \(A := I_{2,1} \cap U_\beta\), we have
\[
\int_{S_{2,1}} \mathbf{1}_{I_{2,1}} d\omega_1 d\omega_2 \leq \int_{S_{2,1}} \mathbf{1}_{U_\beta} d\omega_1 d\omega_2 + \int_{S_{2,1}} \mathbf{1}_A d\omega_1 d\omega_2 \lesssim \beta^{d/2} + \int_{S_{2,1}} \mathbf{1}_A d\omega_1 d\omega_2,
\]
\[(8.18)\]
Recalling (8.21), condition (8.26) implies that (8.24) follows from (8.21). By (8.18), the representation of the sphere (8.1) and (8.19), we have

\[ \int_{S_{2,1}} 1_{I_{2,1}} \omega_1 \, d\omega_2 \leq \beta^{d/2} + \int_{2 \sqrt{\beta} < |\omega_1| \leq 1} \int_{S_{2,1,\omega_1}} 1_{A_{\omega_1}} (\omega_2) \, d\omega_2 \, d\omega_1, \]  

(8.20)

where given \( 2 \sqrt{\beta} < |\omega_1| \leq 1 \), we denote

\[ S_{2,1,\omega_1} = \{ \omega_2 \in S_{d-1}^{d-1} : |\omega_2| < |\omega_1| \}, \]  

(8.21)

\[ A_{\omega_1} = \{ \omega_2 \in \mathbb{R}^d : (\omega_1, \omega_2) \in A \} = \{ \omega_2 \in \mathbb{R}^d : |\omega_2| > 2 \sqrt{\beta} \text{ and } \sqrt{|\omega_1|^2 - \beta} \leq |\omega_1 + \omega_2| \leq \sqrt{|\omega_1|^2 + \beta} \}. \]  

(8.22)

Since \( \beta < 1/4 \), it suffices to control the term:

\[ I' = \int_{2 \sqrt{\beta} < |\omega_1|} \int_{S_{2,1,\omega_1}} 1_{A_{\omega_1}} (\omega_2) \, d\omega_2 \, d\omega_1. \]  

(8.23)

Now we shall prove that, in fact

\[ I' = \int_{2 \sqrt{\beta} \sqrt{1 - |\omega_1|^2}} \int_{S_{2,1,\omega_1}} 1_{A_{\omega_1}} (\omega_2) \, d\omega_2 \, d\omega_1. \]  

(8.24)

Indeed, assume \( \omega_1 \) does not satisfy

\[ 2 \sqrt{\beta} < \sqrt{1 - |\omega_1|^2} < |\omega_1|. \]  

(8.25)

Since we are integrating in the region \( 2 \sqrt{\beta} < |\omega_1| \leq 1 \), exactly one of the following holds:

\[ |\omega_1| \leq \sqrt{1 - |\omega_1|^2}, \]  

(8.26)

\[ \sqrt{1 - |\omega_1|^2} \leq 2 \sqrt{\beta}. \]  

(8.27)

Recalling (8.21), condition (8.26) implies that \( S_{2,1,\omega_1} = \emptyset \), while recalling (8.22), condition (8.27) implies \( S_{2,1,\omega_1} \cap A_{\omega_1} = \emptyset \). Therefore

\[ I' = \int_{2 \sqrt{\beta} \sqrt{1 - |\omega_1|^2}} \int_{S_{2,1,\omega_1}} 1_{A_{\omega_1}} (\omega_2) \, d\omega_2 \, d\omega_1, \]  

and (8.24) follows from (8.21).

Fix any \( \omega_1 \) satisfying (8.25). We first estimate the inner integral:

\[ \int_{S_{d-1}^{d-1}} 1_{\omega_1} (\omega_2) \, d\omega_2. \]  

(8.28)

Notice that (8.25) also yields

\[ |\omega_1| - \sqrt{|\omega_1|^2 - \beta} = \frac{\beta}{|\omega_1| + \sqrt{|\omega_1|^2 - \beta}} < \frac{\beta}{|\omega_1|} \leq \frac{1}{2} \sqrt{\beta} \leq \frac{1}{4} \sqrt{1 - |\omega_1|^2}. \]  

(8.29)

Condition (8.25) guarantees that the vector \( -\omega_1 \) lays outside of the sphere \( S_{d-1}^{d-1} \sqrt{1 - |\omega_1|^2} \), while condition (8.29) guarantees that the sphere is not contained in the annulus \( A_{\omega_1} \). Therefore, the projection of \( S_{d-1}^{d-1} \sqrt{1 - |\omega_1|^2} \cap A_{\omega_1} \) on any plane containing the origin and the vector \( -\omega_1 \) can be visualized as follows:

\[ \text{understood as a point in } \mathbb{R}^d \]
\((OA) = (OB) = \sqrt{1 - |\omega_1|^2}, \quad \overrightarrow{OC} = -\omega_1,\)

\[(AC) = \sqrt{|\omega_1|^2 + \beta}, \quad (CD) = \sqrt{|\omega_1|^2 - \beta}.\]

We conclude that
\[\mathbb{S}^{d-1}_{1 - |\omega_1|^2} \cap A_{\omega_1} = \mathbb{S}^{d-1}_{1 - |\omega_1|^2} \cap (S(\cos \theta_1, -\omega_1) \setminus S(\cos \theta_2, -\omega_1)), \quad (8.30)\]

where recalling the notation introduced in (8.30),
\[\mathbb{S}^{d-1}_{1 - |\omega_1|^2} \cap S(\cos \theta_1, -\omega_1), \quad \mathbb{S}^{d-1}_{1 - |\omega_1|^2} \cap S(\cos \theta_2, -\omega_1),\]

are the spherical shells on \(\mathbb{S}^{d-1}_{1 - |\omega_1|^2}\) of direction \(-\omega_1\) and angles \(2\theta_1, 2\theta_2\) respectively where
\[\theta_1 = \widehat{AOC}, \quad \theta_2 = \widehat{BOC}.\]

Therefore, by (8.30), we have
\[
\int_{\mathbb{S}^{d-1}_{1 - |\omega_1|^2}} 1_{A_{\omega_1}}(\omega_2) d\omega_2 = \int_{\mathbb{S}^{d-1}_{1 - |\omega_1|^2}} 1_{S(\cos \theta_1, -\omega_1) \setminus S(\cos \theta_2, -\omega_1)}(\omega_2) d\omega_2
\]
\[
= (1 - |\omega_1|^2) \frac{2\pi^{d-2}}{\Gamma(d-1)} \int_{2\theta_2}^{2\theta_1} \sin^{d-2} \theta d\theta
\]
\[
\lesssim \theta_1 - \theta_2. \quad (8.31)
\]

where to obtain (8.31), we use Lemma 8.5 and to obtain (8.32) we use the fact that \(d \geq 2\).

Let us calculate \(\alpha_1 = \cos \theta_1, \alpha_2 = \cos \theta_2\). By the cosine law on the triangle AOC, we obtain
\[\alpha_1 = \cos \theta_1 = \frac{(OA)^2 + (OC)^2 - (AC)^2}{2(OA)(OC)} = \frac{1 - |\omega_1|^2 - \beta}{2|\omega_1| \sqrt{1 - |\omega_1|^2}}, \quad (8.33)\]

and by the cosine law on the triangle BOC, we obtain
\[\alpha_2 = \cos \theta_2 = \frac{(OB)^2 + (OC)^2 - (CB)^2}{2(OB)(OC)} = \frac{1 - |\omega_1|^2 + \beta}{2|\omega_1| \sqrt{1 - |\omega_1|^2}}. \quad (8.34)\]

Then, expression (8.33) implies
\[|\alpha_1| \leq \frac{\sqrt{1 - |\omega_1|^2}}{2|\omega_1|} + \frac{\beta}{2|\omega_1| \sqrt{1 - |\omega_1|^2}} < \frac{5}{8}, \quad (8.35)\]

since by (8.35) we have \(|\omega_1| > \sqrt{1 - |\omega_1|^2} > 2\sqrt{\beta}\). In the same spirit, expression (8.34) yields
\[|\alpha_2| < \frac{5}{8}. \quad (8.36)\]
The inverse cosine is smooth in $(-1,1)$, so it is Lipschitz in $[-\frac{\pi}{2}, \frac{\pi}{2}]$, thus by (8.35) and (8.36) and (8.25), we have
\[|\arccos \alpha_1 - \arccos \alpha_2| \leq |\alpha_1 - \alpha_2| = \frac{\beta}{|\omega_1|\sqrt{1 - |\omega_1|^2}}.\]
Therefore (8.32) implies
\[\int_{S^{d-1}_{\omega_1}} 1_{A_{\omega_1}}(\omega_2) \, d\omega_2 \leq |1 - \theta_2| = \arccos \alpha_1 - \arccos \alpha_2 \leq \frac{\beta}{|\omega_1|\sqrt{1 - |\omega_1|^2}}. \tag{8.37}\]
Using (8.37), and recalling (8.21), we have
\[t' = \int_{z<\sqrt{1-|\omega_1|^2}} 1_{A_{\omega_1}}(\omega_2) \, d\omega_2 \, d\omega_1 \leq \beta \int_{B^d_t} \frac{1}{|\omega_1|\sqrt{1 - |\omega_1|^2}} \, d\omega_1 \approx \beta \int_0^1 \frac{r^{d-2}}{\sqrt{1 - r^2}} \, dr \leq \beta \int_0^1 \frac{1}{\sqrt{1 - r^2}} \, dr = \frac{\pi}{2} \beta, \tag{8.38}\]
where to obtain (8.38) we use integration in polar coordinates, and to obtain (8.39) we use the fact that $d \geq 2$. Using (8.20) and (8.40), we obtain
\[\int_{S^d_{t,1}} 1_{I_{t,1}} \, d\omega_1 \, d\omega_2 \leq \beta^{d/2} + \beta \leq \beta,\]
since $\beta < 1/4$. The proof is complete. \hfill \Box

9. Good configurations and stability

9.1. Ad Junction of new particles. In this section, we investigate stability of good configurations under adunctions of collisional particles. Subsection 9.2 investigates binary adjunctions, while Subsection 9.3 investigates ternary adjunctions. To perform the measure estimates needed, we will strongly rely on the results of Section 8.

We start with some definitions on the configurations we are using. Consider $m \in \mathbb{N}$ and $\theta > 0$, and recall from (6.1) the set of well-separated configurations
\[\Delta_m(\theta) = \{ \tilde{Z}_m = (\tilde{X}_m, \tilde{V}_m) \in \mathbb{R}^{2dm} : |\tilde{X}_i - \tilde{X}_j| > \theta, \quad \forall 1 \leq i < j \leq m \}, \quad m \geq 2, \quad \Delta_1(\theta) = \mathbb{R}^d. \tag{9.1}\]
Roughly speaking, a good configuration is a configuration which remains well-separated up backwards time evolution. More precisely, given $\theta > 0$, $t_0 > 0$, we define the set of good configurations as:
\[G_m(\theta, t_0) = \left\{ Z_m = (X_m, V_m) \in \mathbb{R}^{2dm} : Z_m(t) \in \Delta_m(\theta), \quad \forall t \geq t_0 \right\}, \tag{9.2}\]
where $Z_m(t)$ denotes the backwards in time free flow of $Z_m = (X_m, V_m)$, given by:
\[Z_m(t) = ((X_m(t), V_m(t)) := (X_m - tv_m, V_m), \quad t \geq 0. \tag{9.3}\]
Notice that $Z_m$ is the initial point of the trajectory i.e. $Z_m(0) = Z_m$. In other words for $m \geq 2$, we have
\[G_m(\theta, t_0) = \left\{ Z_m = (X_m, V_m) \in \mathbb{R}^{2dm} : |x_i(t) - x_j(t)| > \theta, \quad \forall t \geq t_0, \quad \forall 1 < j \in \{1, ..., m\} \right\}. \tag{9.4}\]

From now on, we consider parameters $R >> 1$ and $0 < \delta, \eta, \epsilon_0, \alpha << 1$ satisfying:
\[\alpha << \epsilon_0 << \eta \delta, \quad R \alpha << \eta \epsilon_0. \tag{9.5}\]
For convenience we choose the parameters in (9.5) in the very end of the paper, see (11.21) and (11.25). Throughout this section, we will write $K^d$ for a cylinder of radius $\eta$ in $\mathbb{R}^d$.

The following Lemma is useful for the adjunction of particles to a given configuration. For the proof, see Lemma 12.2.1 from [22] or Lemma 10.2. from [2].
Lemma 9.1. Consider parameters $\alpha, \epsilon_0, R, \eta, \delta$ as in (9.5) and $\epsilon_2 << \alpha$. Let $\bar{y}_1, \bar{y}_2 \in \mathbb{R}^d$, with $|\bar{y}_1 - \bar{y}_2| > \epsilon_0$ and $v_1 \in B^d_R$. Then there is a d-cylinder $K^d_\eta \subseteq \mathbb{R}^d$ such that for any $y_1 \in B^d_\alpha(\bar{y}_1)$, $y_2 \in B^d_\alpha(\bar{y}_2)$ and $v_2 \in B^d_R \setminus K^d_\eta$, we have

- $(i)$ $(y_1, y_2, v_1, v_2) \in G_2(\sqrt{2}\epsilon_3, 0)$,
- $(ii)$ $(y_1, y_2, v_1, v_2) \in G_2(\epsilon_0, \delta)$.

9.2. Stability under binary adjunction. The main results of this subsection are stated in Proposition 9.2 which will be the inductive step of adding a colliding particle, and Proposition 9.4 which presents the measure estimate of the bad set that appears in this process. The proofs of the Propositions presented below are in part inspired by arguments in [22] and [4] with a caveat that the new scenario needs to be addressed, in the case when the binary collisional configuration formed runs to a ternary interaction under time evolution.

9.2.1. Binary adjunction. For convenience, given $v \in \mathbb{R}^d$, let us denote

$$\left(S^{d-1}_1 \times B^d_R \right)^+(v) = \left\{ (\omega_1, v_1) \in S^{d-1}_1 \times B^d_R : b_2(\omega_1, v_1 - v) > 0 \right\},$$

(9.6)

where $b_2(\omega)1, v_1 - v = (\omega, v_1 - v)$. Recall from (9.3) that given $m \in \mathbb{N}$ and $Z_m = (X_m, V_m) \in \mathbb{R}^{2dm}$, we denote the backwards in time free flow as $Z_m(t) = (X_m - tV_m, V_m)$, $t \geq 0$. Recall also the notation from (9.7)

$$D_{m+1,\epsilon_2,\epsilon_3} = \left\{ Z_{m+1} = (X_{m+1}, V_{m+1}) \in \mathbb{R}^{2d(m+1)} : d_2(x_i, x_j) > \epsilon_2, \, \forall (i, j) \in \mathcal{I}_{m+1}^2, \right.$$  

$$\text{and } d_3(x_i, x_j, x_k) > \sqrt{2}\epsilon_3, \, \forall (i, j, k) \in \mathcal{I}_{m+1}^3, \right\},$$

where $\mathcal{I}_{m+1}^2, \mathcal{I}_{m+1}^3$ are given by (9.1)-(9.2) respectively.

Proposition 9.2. Consider parameters $\alpha, \epsilon_0, R, \eta, \delta$ as in (9.5) and $\epsilon_2 << \epsilon_3 << \alpha$. Let $m \in \mathbb{N}$, $Z_m = (\bar{X}_m, \bar{V}_m) \in \mathcal{G}_m(\epsilon_0, 0)$, $\ell \in \{1, ..., m\}$ and $X_m \in B^d_{\epsilon_0/2}(\bar{X}_m)$. Then there is a subset $B^d_\ell(Z_m) \subseteq (S^{d-1}_1 \times B^d_R)^+(\bar{v}_\ell)$ such that:

- $(i)$ For any $(\omega_1, v_{m+1}) \in (S^{d-1}_1 \times B^d_R)^+(\bar{v}_\ell) \setminus B^d_\ell(\bar{Z}_m)$, one has:

$$Z_{m+1}(t) \in D_{m+1,\epsilon_2,\epsilon_3}, \, \forall t \geq 0,$$

(9.7)

$$Z_{m+1} \in G_{m+1}(\epsilon_0/2, \delta),$$

(9.8)

$$\bar{Z}_{m+1} \in G_{m+1}(\epsilon_0, \delta),$$

(9.9)

where

$$Z_{m+1} = (x_1, ..., x_\ell, ..., x_m, x_{m+1}, \bar{v}_1, ..., \bar{v}_\ell, ..., \bar{v}_m, v_{m+1}),$$

$$x_{m+1} = x_\ell - \epsilon_2 \omega_1,$$

(9.10)

$$\bar{Z}_{m+1} = (\bar{x}_1, ..., \bar{x}_\ell, ..., \bar{x}_m, \bar{x}_{m+1}, \bar{v}_1, ..., \bar{v}_\ell, ..., \bar{v}_m, v_{m+1}),$$

$(ii)$ For any $(\omega_1, v_{m+1}) \in (S^{d-1}_1 \times B^d_R)^+(\bar{v}_\ell) \setminus B^d_\ell(\bar{Z}_m)$, one has:

$$Z'_{m+1}(t) \in D_{m+1,\epsilon_2,\epsilon_3}, \, \forall t \geq 0,$$

(9.11)

$$Z'_{m+1} \in G_{m+1}(\epsilon_0/2, \delta),$$

(9.12)

$$\bar{Z}'_{m+1} \in G_{m+1}(\epsilon_0, \delta),$$

(9.13)

where

$$Z'_{m+1} = (x_1, ..., x_\ell, ..., x_m, x_{m+1}, \bar{v}_1, ..., \bar{v}_\ell, ..., \bar{v}_m, v'_{m+1}),$$

$$x_{m+1} = x_\ell + \epsilon_2 \omega_1,$$

(9.14)

$$\bar{Z}'_{m+1} = (\bar{x}_1, ..., \bar{x}_\ell, ..., \bar{x}_m, \bar{x}_{m+1}, \bar{v}_1, ..., \bar{v}_\ell, ..., \bar{v}_m, v'_{m+1}),$$

$$(\bar{v}_\ell, v'_{m+1}) = T_{\epsilon_2}(\bar{v}_\ell, v_{m+1}).$$

Proof. By symmetry, we may assume that $\ell = m$. For convenience, let us define the set

$$\mathcal{F}_{m+1} = \{(i, j) \in \{1, ..., m+1\} \times \{1, ..., m+1\} : i < \min\{j, m\}\}.$$

Proof of $(i)$: Here we use notation from (9.10). We start by formulating the following claim, which will imply (9.7).
Lemma 9.3. Under the assumptions of Proposition 9.2, there is a subset $\mathcal{B}_{m}^{2,0,-}(\bar{Z}_{m}) \subseteq \mathbb{S}_{1}^{d-1} \times B_{R}^{d}$ such that for any $(\omega_{1}, v_{m+1}) \in (\mathbb{S}_{1}^{d-1} \times B_{R}^{d})^{+}(\bar{v}_{m}) \setminus \mathcal{B}_{m}^{2,0,-}(\bar{Z}_{m})$, there holds:

\[ d_{2}(x_{i}(t), x_{j}(t)) > \sqrt{2} \varepsilon_{3}, \quad \forall t \geq 0, \quad \forall (i, j) \in \mathcal{F}_{m+1}, \tag{9.15} \]

\[ d_{2}(x_{m}(t), x_{m+1}(t)) > \varepsilon_{2}, \quad \forall t \geq 0. \tag{9.16} \]

Notice that $\ref{9.16}$ trivially imply $\ref{9.11}$, since $\varepsilon_{2} < \varepsilon_{3}$.

**Proof of Lemma 9.3.**

**Step 1: The proof of (9.15):** We distinguish the following cases:

- **$j \leq m$:** Since $\bar{Z}_{m} \in G_{m}(\varepsilon_{0}, 0)$ and $j \leq m$, we have $|x_{i}(t) - x_{j}(t)| > \varepsilon_{0}$, for all $t \geq 0$. Therefore, triangle inequality implies that

\[ |x_{i}(t) - x_{j}(t)| = |x_{i} - x_{j} - t(\bar{v}_{i} - \bar{v}_{j})| \geq |x_{i} - x_{j} - t(\bar{v}_{i} - \bar{v}_{j})| - \alpha > \varepsilon_{0} - \alpha > \frac{\varepsilon_{0}}{2} > \sqrt{2} \varepsilon_{3}, \tag{9.17} \]

since $\varepsilon_{3} < \alpha < \varepsilon_{0}$.

- **$j = m + 1$:** Since $(i, m + 1) \in \mathcal{F}_{m+1}$, we have $i \leq m - 1$. Since $\bar{Z}_{m} \in G_{m}(\varepsilon_{0}, 0)$ and $X_{m} \in B_{\alpha/(\delta)}(\bar{X}_{m})$, we conclude

\[ |\bar{x}_{i} - \bar{x}_{m}| > \varepsilon_{0}, \quad |x_{i} - \bar{x}_{m}| \leq \frac{\alpha}{2}, \quad \varepsilon_{2}|\bar{x}_{m} - \bar{v}_{i}| \leq \alpha - \varepsilon_{0} \quad \varepsilon_{2} < \alpha, \quad \varepsilon_{2} < \alpha. \]

Applying part (i) of Lemma 9.1 for $\bar{y}_{i} = \bar{x}_{i}, \bar{y}_{2} = \bar{x}_{m}, y_{1} = x_{i}, y_{2} = x_{m+1}$, we may find a cylinder $K_{d}^{\delta,i}$ such that for any $v_{m+1} \in B_{R}^{d} \setminus K_{d}^{\delta,i}$, we have $|x(t) - x_{m+1}(t)| > \sqrt{2} \varepsilon_{3}$, for all $t \geq 0$. Hence the inequality in (9.15) holds for any $(\omega_{1}, v_{m+1}) \in (\mathbb{S}_{1}^{d-1} \times B_{R}^{d})^{+}(\bar{v}_{m}) \setminus V_{m+1}$, where

\[ V_{m+1} = \mathbb{S}_{1}^{d-1} \times K_{d}^{\delta,i}. \tag{9.18} \]

We conclude that (9.15) holds for any $(\omega_{1}, v_{m+1}) \in (\mathbb{S}_{1}^{d-1} \times B_{R}^{d}) \setminus \bigcup_{i=1}^{m-1} V_{m+1}$.

**Step 2: The proof of (9.10):** We recall notation from (9.10). Considering $t \geq 0$ and $(\omega_{1}, v_{m+1}) \in (\mathbb{S}_{1}^{d-1} \times B_{R}^{d})^{+}(\bar{v}_{m})$. Using the fact that $(\omega_{1}, v_{m+1}) \in (\mathbb{S}_{1}^{d-1} \times B_{R}^{d})^{+}(\bar{v}_{m})$, we obtain

\[ |x_{m}(t) - x_{m+1}(t)|^{2} = |\varepsilon_{2} \omega_{1} - t(\bar{v}_{m} - \bar{v}_{m+1})|^{2} \geq \varepsilon_{2}^{2} |\omega_{1}|^{2} + 2 \varepsilon_{2} t \omega_{1} \omega_{1} > \varepsilon_{2}^{2}. \tag{9.19} \]

Therefore, (9.10) holds for any $(\omega_{1}, v_{m+1}) \in (\mathbb{S}_{1}^{d-1} \times B_{R}^{d})^{+}(\bar{v}_{m})$.

Defining

\[ \mathcal{B}_{m}^{2,0,-}(\bar{Z}_{m}) = \bigcup_{i=1}^{m-1} V_{m+1}^{i}, \tag{9.20} \]

the claim of Lemma 9.3 follows.

Now we go back to the proof of part (i) of Proposition 9.2. We will find a set $\mathcal{B}_{m}^{\delta,-}(\bar{Z}_{m}) \subseteq \mathbb{S}_{1}^{d-1} \times B_{R}^{d}$ such that (9.8) holds for any $(\omega_{1}, v_{m+1}) \in (\mathbb{S}_{1}^{d-1} \times B_{R}^{d}) \setminus \mathcal{B}_{m}^{\delta,-}(\bar{Z}_{m})$.

Let us fix $i, j \in \{1, \ldots, m + 1\}$ with $i < j$. We distinguish the following cases:

- **$j \leq m$:** We use the same argument as in (9.17), to obtain $|x(t) - x_{j}(t)| > \frac{\varepsilon_{0}}{2}$, for all $t \geq 0$.

- **$i, j \in \mathcal{F}_{m+1}, j = m + 1$:** Since $(i, m + 1) \in \mathcal{F}_{m+1}$, we have $i \leq m - 1$. Applying a similar argument to the corresponding case in the proof of (9.10), using part (ii) of Lemma 9.1 instead, we obtain that the inequality $|x_{i}(t) - x_{m+1}(t)| > \varepsilon_{0}$, for all $t \geq \delta$, holds for any $(\omega_{1}, v_{m+1}) \in (\mathbb{S}_{1}^{d-1} \times B_{R}^{d}) \setminus V_{m+1}$, where $V_{m+1}$ is given by (9.18). Notice that the lower bound is in fact $\varepsilon_{0}$.

- **$i = m, j = m + 1$:** Triangle inequality and the fact that $\varepsilon_{2} \varepsilon_{0} < \eta \delta$ imply that for any $t \geq \delta$ and $(\omega_{1}, v_{m+1}) \in \mathbb{S}_{1}^{d-1} \times B_{R}^{d}$ with $|v_{m+1} - \bar{v}_{m}| > \eta$, we have

\[ |x_{m}(t) - x_{m+1}(t)| = |\varepsilon_{2} \omega_{1} - t(\bar{v}_{m} - \bar{v}_{m+1})| \geq |\bar{v}_{m} - \bar{v}_{m+1}| > \eta \delta > \varepsilon_{2} > \varepsilon_{0}. \]

Therefore, the inequality $|x_{m}(t) - x_{m+1}(t)| > \varepsilon_{0}$, for all $t \geq \delta$, holds for any $(\omega_{1}, v_{m+1}) \in (\mathbb{S}_{1}^{d-1} \times B_{R}^{d}) \setminus V_{m+1}$, where

\[ V_{m+1} = \mathbb{S}_{1}^{d-1} \times B_{\eta}^{d}(\bar{v}_{m}). \tag{9.21} \]

Notice that the lower bound is $\varepsilon_{0}$ again.

Defining

\[ \mathcal{B}_{m}^{\delta,-}(\bar{Z}_{m}) = \mathcal{B}_{m}^{2,0,-}(\bar{Z}_{m}) \cup V_{m+1}, \tag{9.22} \]

we conclude that (9.8) holds for any $(\omega_{1}, v_{m+1}) \in (\mathbb{S}_{1}^{d-1} \times B_{R}^{d}) \setminus \mathcal{B}_{m}^{\delta,-}(\bar{Z}_{m})$. 


Let us note that the only case which prevents us from having $Z_{m+1} \in G_{m+1}(\epsilon_0, \delta)$ is the case $1 \leq i < j \leq m$, where we obtain a lower bound of $\epsilon_0/2$. In all other cases we can obtain lower bound $\epsilon_0$.

More precisely, for $(\omega_1, v_{m+1}) \in (\mathcal{S}_1^{d-1} \times B_R^d) \setminus \mathcal{B}_{m}^{\delta,-}(Z_m)$, the inequality $|\bar{x}_i(t) - \bar{x}_j(t)| > \epsilon_0$, for all $t \geq \delta$, holds for all $1 \leq i < j \leq m + 1$ except the case $1 \leq i < j \leq m$. However in this case, for any $1 \leq i < j \leq m$, we have $|\bar{x}_i(t) - \bar{x}_j(t)| > \epsilon_0$, for all $t > 0$, since $\bar{Z}_m \in G_m(\epsilon_0, 0)$. Therefore, (9.9) holds for $(\omega_1, v_{m+1}) \in (\mathcal{S}_1^{d-1} \times B_R^d) \setminus \mathcal{B}_{m}^{\delta,-}(Z_m)$.

We conclude that the set
\[
\mathcal{B}_{m}^{\delta,-}(Z_m) = (\mathcal{S}_1^{d-1} \times B_R^d)^+ (\bar{v}_m) \cap \left( V_{m,m+1} \cup \bigcup_{i=1}^{m-1} V_{m,m+1}^i \right),
\]
(9.23)
is the set we need for the precollisional case.

**Proof of (ii):** Here we use the notation from (9.14). The proof follows the steps of the precollisional case, but we replace the velocities $(\bar{v}_m, v_{m+1})$ by the transformed velocities $(\bar{v}_m, v'_{m+1})$ and then pull-back. It is worth mentioning that the $m$-th particle needs special treatment since its velocity is transformed to $\bar{v}'_m$. Following similar arguments to the precollisional case, we conclude that the appropriate set for the postcollisional case is given by
\[
\mathcal{B}^{2,+}_m(\bar{Z}_m) := (\mathcal{S}_1^{d-1} \times B_R^d)^+ (\bar{v}_m) \cap \left( V_{m,m+1} \cup \bigcup_{i=1}^{m-1} V_{m,m+1}^i \right),
\]
(9.24)
where
\[
V_{m,m+1}^i = \left\{ (\omega_1, v_{m+1}) \in \mathcal{S}_1^{d-1} \times B_R^d : \bar{v}_m \in K_{\eta,1} \right\},
\]
(9.25)
\[
V_{m,m+1} = \mathcal{S}_1^{d-1} \times B_R^d (\bar{v}_m).
\]
(9.27)
The set
\[
\mathcal{B}^{2}_m(\bar{Z}_m) = \mathcal{B}^{2,-}_m(\bar{Z}_m) \cup \mathcal{B}^{2,+}_m(\bar{Z}_m),
\]
(9.28)
is the one we need to conclude the proof.

9.2.2. Measure estimate for binary adjunction. We now estimate the measure of the pathological set $\mathcal{B}_m^2(\bar{Z}_m)$ appearing in Proposition 9.2. To control postcollisional configurations, we will strongly rely on the binary transition map introduced in the Appendix (see Proposition 12.2).

**Proposition 9.4.** Consider parameters $\alpha, \epsilon_0, R, \eta, \delta$ as in (9.9) and $\epsilon_2 \ll \epsilon_0 < \epsilon_3 < \alpha$. Let $m \in \mathbb{N}$, $\bar{Z}_m \in G_m(\epsilon_0, 0)$, $\ell \in \{1, \ldots, m\}$ and $\mathcal{B}_m^2(\bar{Z}_m)$ the set given in the statement of Proposition 9.2. Then the following measure estimate holds:
\[
|\mathcal{B}_m^2(\bar{Z}_m)| \lesssim m R d^{\frac{d-1}{d-2}},
\]
where $|\cdot|$ denotes the product measure on $\mathcal{S}_1^{d-1} \times B_R^d$.

**Proof.** Without loss of generality, we may assume that $\ell = m$. By (9.28) it suffices to estimate the measure of $\mathcal{B}^{2,-}_m(\bar{Z}_m)$ and $\mathcal{B}^{2,+}_m(\bar{Z}_m)$.

**Estimate of $\mathcal{B}^{2,-}_m(\bar{Z}_m)$:** Recalling (9.20), (9.23), (9.22), (9.20), we have
\[
\mathcal{B}^{2,-}_m(\bar{Z}_m) = (\mathcal{S}_1^{d-1} \times B_R^d)^+ (\bar{v}_m) \cap \left( V_{m,m+1} \cup \bigcup_{i=1}^{m-1} V_{m,m+1}^i \right),
\]
(9.29)
where $V_{m,m+1}$ is given by (9.21) and $V_{m+1}$ are given by (9.18). By sub-additivity, it suffices to estimate the measure of each term in (9.29).

- **Estimate of the term corresponding to $V_{m,m+1}$:** By (9.21), we have $V_{m,m+1} = \mathcal{S}_1^{d-1} \times B_R^d (\bar{v}_m)$, therefore
\[
|\mathcal{S}_1^{d-1} \times B_R^d (\bar{v}_m) \cap V_{m,m+1}| \leq |\mathcal{S}_1^{d-1} \times (B_R^d \cap K_{\eta,1}^i)(\bar{v}_m)| \leq |\mathcal{S}_1^{d-1}| \cdot |B_R^d(\bar{v}_m)| \lesssim R d^{\frac{d-1}{d-2}}.
\]
(9.30)
- **Estimate of the term corresponding to $V_{m+1}$:** By (9.18), we have $V_{m+1} = \mathcal{S}_1^{d-1} \times K_{\eta,1}^i$, therefore by Corollary 9.2, we obtain
\[
|\mathcal{S}_1^{d-1} \times B_R^d (\bar{v}_m) \times V_{m+1}^i| \leq |\mathcal{S}_1^{d-1} \times (B_R^d \cap K_{\eta,1}^i)(\bar{v}_m)| \lesssim |\mathcal{S}_1^{d-1}| \cdot |B_R^d(\bar{v}_m)| \lesssim R d^{\frac{d-1}{d-2}}.
\]
(9.31)
Using (9.29)-(9.31), subadditivity, and the fact that $\eta < \epsilon$, $m \geq 1$, we obtain
\[
|B_{m}^{2,-}(Z_{m})| \lesssim mR^{d} \eta^{d/2}. \tag{9.32}
\]

**Estimate of $B_{m}^{2,+}(\bar{Z}_{m})$:** Recalling (9.24), we have
\[
B_{m}^{2,+}(\bar{Z}_{m}) = (S_{1}^{d-1} \times B_{R}^{d})^{+}(\bar{v}_{m}) \cap \left[ V_{m,m+1} \cup \bigcup_{i=1}^{m-1} \left( V_{m}' \cup V_{m+1}' \right) \right], \tag{9.33}
\]
where $V_{m,m+1}$ is given by (9.21) and $V_{m}'$, $V_{m+1}'$ are given by (9.25)-(9.26). By subadditivity, it suffices to estimate the measure of each term in (9.33). The term corresponding to $V_{m,m+1}$ has already been estimated in (9.30). We have
\[
|\left( S_{1}^{d-1} \times B_{R}^{d} \right)^{+}(\bar{v}_{m}) \cap V_{m,m+1}| \lesssim \eta^{d}. \tag{9.34}
\]
To estimate the measure of the remaining terms, we will strongly rely on the properties of the binary transition map defined in Proposition 12.2. We first introduce some notation. Given $0 < r \leq 2R$, let us define the $r$-sphere, centered at $\bar{v}_{m}$:
\[
S_{r}^{d-1}(\bar{v}_{m}) = \left\{ \bar{v}_{m+1} \in \mathbb{R}^{d} : |\bar{v}_{m} - \bar{v}_{m+1}| = r \right\}. \tag{9.35}
\]
Also, given $v_{m+1} \in \mathbb{R}^{d}$, we define the set
\[
S_{1}^{d-1}(v_{m+1}) = \left\{ \omega_{1} \in S_{1}^{d-1} : b_{2}(\omega_{1}, v_{m+1} - \bar{v}_{m}) > 0 \right\} = \left\{ \omega_{1} \in S_{1}^{d-1} : (\omega_{1}, v_{m+1}) \in (S_{1}^{d-1} \times B_{R}^{d})^{+}(\bar{v}_{m}) \right\}. \tag{9.36}
\]
Since $\bar{v}_{m} \in B_{R}^{d}$, triangle inequality implies $B_{R}^{d} \subseteq B_{R}(\bar{v}_{m})$. Under this notation, Fubini’s Theorem, the co-area formula, and relations (9.33)-(9.34) yield
\[
|B_{m}^{2,+}(\bar{Z}_{m})| = \int_{S_{1}^{d-1}(\bar{v}_{m})} \int_{\mathbb{R}^{d}} 1_{B_{R}^{d}}(\bar{v}_{m}) \, d\omega_{1} \, dv_{m+1}
\]
\[
= \int_{B_{R}^{d}} \int_{S_{m}^{d-1}(\bar{v}_{m})} \int_{S_{1}^{d-1}(v_{m+1})} 1_{U_{i=1}^{m-1}(V_{m}' \cup V_{m+1}')} (\omega_{1}) \, d\omega_{1} \, dv_{m+1} \, dr \tag{9.37}
\]
\[
\lesssim \eta^{d} + \int_{0}^{2R} \int_{S_{m}^{d-1}(\bar{v}_{m})} \int_{S_{1}^{d-1}(v_{m+1})} 1_{U_{i=1}^{m-1}(V_{m}' \cup V_{m+1}')} (\omega_{1}) \, d\omega_{1} \, dv_{m+1} \, dr.
\]
Let us estimate the integral:
\[
\int_{S_{m}^{d-1}(\bar{v}_{m})} \int_{S_{1}^{d-1}(v_{m+1})} 1_{U_{i=1}^{m-1}(V_{m}' \cup V_{m+1}')} (\omega_{1}) \, d\omega_{1},
\]
for fixed $0 < r \leq 2R$ and $v_{m+1} \in S_{m}^{d-1}(\bar{v}_{m})$. We introduce a parameter $0 < \beta < \eta$, which will be chosen later in terms of $\eta$, and decompose $S_{1}^{d-1}(v_{m+1})$ as follows:
\[
S_{1}^{d-1}(v_{m+1}) = S_{1}^{d-1}(v_{m+1}) \cup S_{2}^{d-1}(v_{m+1}), \tag{9.38}
\]
where
\[
S_{1}^{d-1}(v_{m+1}) = \left\{ \omega_{1} \in S_{1}^{d-1}(v_{m+1}) : b_{2}(\omega_{1}, v_{m+1} - \bar{v}_{m}) > \beta |v_{m+1} - \bar{v}_{m}| \right\}, \tag{9.39}
\]
and
\[
S_{2}^{d-1}(v_{m+1}) = \left\{ \omega_{1} \in S_{1}^{d-1}(v_{m+1}) : b_{2}(\omega_{1}, v_{m+1} - \bar{v}_{m}) \leq \beta |v_{m+1} - \bar{v}_{m}| \right\}. \tag{9.40}
\]
Notice that $S_{2}^{d-1}(v_{m+1})$ is the union of two unit $(d-1)$-spherical caps of angle $\pi/2 - \arccos \beta$. Thus, integrating in spherical coordinates, we may estimate its measure as follows:
\[
\int_{\mathbb{R}^{d-1}} \int_{S_{2}^{d-1}(v_{m+1})} (\omega_{1}) \, d\omega_{1} \lesssim \int_{\mathbb{R}^{d-1}} \sin^{d-2}(\theta) \, d\theta \leq \frac{\pi}{2} - \arccos \beta = \arcsin \beta.
\]
Thus
\[
\int_{S_{2}^{d-1}(v_{m+1})} 1_{U_{i=1}^{m-1}(V_{m}' \cup V_{m+1}')} (\omega_{1}) \, d\omega_{1} \lesssim \arcsin \beta. \tag{9.41}
\]
We now wish to estimate
\[
\int_{S_{1}^{d-1}(v_{m+1})} 1_{U_{i=1}^{m-1}(V_{m}' \cup V_{m+1}')} (\omega_{1}) \, d\omega_{1},
\]
We will use the binary transition map \( \mathcal{J}_{v_m,v_m+1}: S_{v_m,v_m+1}^+ \to S_{d-1}^1 \), which is given by

\[
\nu_1 := \mathcal{J}_{v_m,v_m+1}(\omega_1) = r^{-1}(v_m' - v_{m+1}'),
\]

(9.42)

to change variables in the above integral. For details on the transition map, see Proposition 12.2 in the Appendix. By Proposition 12.2 for \( \omega \in S_{v_m,v_m+1}^+ \), the Jacobian matrix of the transition map is

\[
\text{Jac}(\mathcal{J}_{v_m,v_m+1})(\omega_1) \simeq r^{-d} b_2(\omega_1, v_{m+1} - v_m) > 0.
\]

Therefore, for \( \omega_1 \in S_{v_m,v_m+1}^+ \), we have

\[
\text{Jac}^{-1}(\mathcal{J}_{v_m,v_m+1})(\omega_1) \simeq r^d b_2^{-d}(\omega_1, v_{m+1} - v_m) \leq r^d \beta^{-d}|v_{m+1} - v_m|^{-d} \lesssim \beta^{-d},
\]

(9.43)
since \( |v_{m+1} - v_m| = r \).

For convenience, we express \( v_m', v_{m+1}' \) in terms of the precollisional velocities \( \bar{v}_m, v_{m+1} \) and \( \nu_1 \) given by (9.42). As a consequence of (2.3), we obtain

\[
v_m' = \frac{v_m + v_{m+1}}{2} + \frac{r}{2} \nu_1,
\]

(9.44)
\[
v_{m+1}' = \frac{v_m + v_{m+1}}{2} - \frac{r}{2} \nu_1
\]

(9.45)

We are now in the position to estimate the integral in (9.41). We first estimate for the term corresponding to \( V_m' \): Recalling (9.25), we have \( V_m' = \{ (\omega_1, v_{m+1}) \in \mathbb{S}_{d-1} \times \mathbb{R}^{d-1} : \bar{v}_m \in \bar{K}_{d-1} \} \). By (9.44),

\[
\bar{v}_m' \in K_{d-1} \rightarrow \nu_1 = \mathcal{J}_{v_m,v_m+1}(\omega_1) \in \bar{K}_{d-1}^{-1}(\nu),
\]

(9.46)

where \( \bar{K}_{d-1}^{-1} \) is a cylinder of radius \( 2\eta/r \). Therefore, we obtain

\[
\int_{S_{v_m,v_m+1}^+} 1_{V_m'}(\omega_1) \, d\omega_1 = \int_{S_{v_m,v_m+1}^+} 1_{v_m' \in \bar{K}_{d-1}^{-1}(\nu)}(\omega_1) \, d\omega_1
\]

(9.47)
\[
\lesssim \beta^{-d} \int_{S_{d-1}^1} 1_{\bar{K}_{d-1}^{-1}(\nu)}(\nu) \, d\nu
\]

(9.48)
\[
\lesssim \beta^{-d} \min \left\{ 1, \left( \frac{\eta}{r} \right)^{d-1} \right\},
\]

(9.49)

where to obtain (9.47) we use (9.46), to obtain (9.48) we use part (iv) of Proposition 12.2 and estimate (9.43), and to obtain (9.49) we use Lemma 8.1.

Hence, for fixed \( v_{m+1} \in S_{v_m}^{d-1}(\bar{v}_m) \), we have

\[
\int_{S_{v_m,v_m+1}^+} 1_{V_m'}(\omega_1) \, d\omega_1 \lesssim \beta^{-d} \min \left\{ 1, \left( \frac{\eta}{r} \right)^{d-1} \right\}.
\]

(9.50)

Recalling also \( V_{m+1}' \) from (9.26), we obtain in an analogous way the estimate:

\[
\int_{S_{v_m,v_m+1}^+} 1_{V_{m+1}'}(\omega_1) \, d\omega_1 \lesssim \beta^{-d} \min \left\{ 1, \left( \frac{\eta}{r} \right)^{d-1} \right\}.
\]

(9.51)

Combining (9.50)–(9.51) and adding for \( i = 1, \ldots, m-1 \), we obtain

\[
\int_{S_{v_m,v_m+1}^+} 1_{\bigcup_{i=1}^{m-1} (V_i' \cup V_{i+1}' \cup V_{m+1}' \cup V_m')} (\omega_1) \, d\omega_1 \lesssim m \beta^{-d} \min \left\{ 1, \left( \frac{\eta}{r} \right)^{d-1} \right\}
\]

(9.52)

Therefore, recalling (9.37) and using estimates (9.40), (9.52), we obtain the estimate:

\[
\int_{S_{v_m,v_m+1}^+} 1_{\bigcup_{i=1}^{m-1} (V_i' \cup V_{i+1}' \cup V_{m+1}' \cup V_m')} (\omega_1) \, d\omega_1 \lesssim \arcsin \beta + m \beta^{-d} \min \left\{ 1, \left( \frac{\eta}{r} \right)^{d-1} \right\},
\]

(9.53)
Hence, (9.30) yields
\[
| \mathcal{B}_m^d(\tilde{Z}_m) | \lesssim \eta^d + \int_0^{2R} \int_{\mathfrak{e}^{d-1}(\tilde{m})} \arcsin \beta + m^d \beta^d \min \left\{ 1, \left( \frac{\eta}{r} \right)^{d-1} \right\} \, dv \, dr
\]
\[
\lesssim \eta^d + \int_0^{2R} r^{-1} \left[ \arcsin \beta + m^d \beta^d \min \left\{ 1, \left( \frac{\eta}{r} \right)^{d-1} \right\} \right] \, dr
\]
\[
\lesssim \eta^d + mR^d \left( \arcsin \beta + \beta^d \eta \frac{d-1}{d^2} \right)
\]
\[
\lesssim mR^d \left( \beta + \beta^d \eta \frac{d-1}{d^2} \right),
\]
(9.54)

after using an estimate similar to (8.2) and the fact that \( \eta << 1, m \geq 1, \beta << 1 \). Choosing \( \beta = \eta \frac{d-1}{d^2} \), we obtain
\[
| \mathcal{B}_m^d(\tilde{Z}_m) | \lesssim mR^d \eta \frac{d-1}{d^2}.
\]
(9.55)

Combining (9.28), (9.32), (9.55), and the fact \( \eta << 1, \) we obtain the required estimate. \( \square \)

9.3. Stability under ternary adjunction. Now, we prove Proposition 9.6 and Proposition 9.7 which will be the inductive step and the corresponding measure estimate of our proof for ternary adjunction of particles. To derive Proposition 9.6 and Proposition 9.7 in addition to results from [3], we develop new algebraic and geometric techniques, thanks to which we can treat the newly formed ternary collisional configuration runs to a binary collision under time evolution.

9.3.1. Ternary adjunction. For convenience, given \( v \in \mathbb{R}^d \), let us denote
\[
\left( S_1^{d-1} \times B_R^d \right)^{+}(v) = \{ (\omega_1, \omega_2, v_1, v_2) \in S_1^{d-1} \times B_R^d : b_3(\omega_1, \omega_2, v_1 - v, v_2 - v) > 0 \},
\]
(9.56)

where \( b_3 \) is the ternary cross-section given in (2.11).

Recall from (9.3) that given \( m \in \mathbb{N} \) and \( Z_m = (X_m, V_m) \in \mathbb{R}^{2dm} \), we denote the backwards in time free flow as \( Z_m(t) = (X_m - tV_m, V_m) \), \( t \geq 0 \).

Proposition 9.5. Consider parameters \( \alpha, \epsilon_0, R, \eta, \delta \) as in (9.5) and \( \epsilon_3 << \alpha \). Let \( m \in \mathbb{N} \), \( \bar{Z}_m = (\bar{X}_m, \bar{V}_m) \in G_m(\epsilon_0, 0), \ell \in \{1, ..., m\} \), and \( X_m \in B_{\epsilon_0/2}(X_m) \). Let us denote
\[
\mathcal{F}_m^{+} = \{(i, j) \in \{1, ..., m+2\} \times \{1, ..., m+2\} : i \neq \ell, i \leq \min \{j, m\} \}.
\]

Then there is a subset \( \mathcal{B}_m^{+}(\bar{Z}_m) \subseteq (S_1^{d-1} \times B_R^d)^{+}(\bar{v}_m) \) such that:

(i) For any \( (\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (S_1^{d-1} \times B_R^d)^{+}(\bar{v}_m) \setminus \mathcal{B}_m^{+}(\bar{Z}_m) \), one has:
\[
d_2(x_i(t), x_j(t)) > \sqrt{2\epsilon_3}, \quad \forall (i, j) \in \mathcal{F}_m^{+}, \quad \forall t \geq 0,
\]
\[
d_3(x_i(t); x_{m+1}(t), x_{m+2}(t)) > \sqrt{2\epsilon_3}, \quad \forall t \geq 0,
\]
\[
Z_{m+2} \in G_{m+2}(\epsilon_0/2, \delta),
\]
\[
\bar{Z}_{m+2} \in G_{m+2}(\epsilon_0, \delta).
\]
(9.57)

where
\[
Z_{m+2} = (x_1, ..., x_\ell, ..., x_m, x_{m+1}, x_{m+2}, v_1, ..., v_\ell, ..., v_m, v_{m+1}, v_{m+2}),
\]
\[
x_{m+i} = x_i + \sqrt{2\epsilon_3} \omega_i, \quad \forall i \in \{1, 2\},
\]
\[
\bar{Z}_{m+2} = (\bar{x}_1, ..., \bar{x}_\ell, ..., \bar{x}_m, \bar{x}_{m+1}, \bar{x}_{m+2}, v_1, ..., v_\ell, ..., v_m, v_{m+1}, v_{m+2}),
\]

(ii) For any \( (\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (S_1^{d-1} \times B_R^d)^{+}(\bar{v}_m) \setminus \mathcal{B}_m^{+}(\bar{Z}_m) \), one has:
\[
d_2(x_i(t), x_j(t)) > \sqrt{2\epsilon_3}, \quad \forall (i, j) \in \mathcal{F}_m^{+}, \quad \forall t \geq 0,
\]
\[
d_3(x_i(t); x_{m+1}(t), x_{m+2}(t)) > \sqrt{2\epsilon_3}, \quad \forall t \geq 0,
\]
\[
Z_{m+2} \in G_{m+2}(\epsilon_0/2, \delta),
\]
\[
\bar{Z}_{m+2} \in G_{m+2}(\epsilon_0, \delta).
\]
(9.58)
where
\[
Z^*_{m+2} = (x_1, \ldots, x_{\ell}, \ldots, x_m, x_{m+1}, x_{m+2}, \tilde{v}_1, \ldots, \tilde{v}_\ell, \ldots, \tilde{v}_m, v^*_{m+1}, v^*_{m+2}),
\]
\[
x_{m+i} = x_i + \sqrt{2} \epsilon_3 \omega_i, \quad \forall i \in \{1, 2\},
\]
\[
Z^*_{m+2} = (\bar{x}_1, \ldots, \bar{x}_\ell, \ldots, \bar{x}_m, \bar{x}_{m+1}, \bar{v}_1, \ldots, \bar{v}_\ell, \ldots, \bar{v}_m, v^*_{m+1}, v^*_{m+2}),
\]
\[
(\bar{v}_\ell, \bar{v}^*_{\ell+1}, v^*_{m+2}) = T_{\omega_1, \omega_2}(\bar{v}_\ell, v_{m+1}, v_{m+2}).
\]

There also holds the measure estimate:
\[
|B_t^1(\bar{Z}_m)| \leq m R^{2d-1} \eta^{\frac{2d-1}{d}}, \quad (9.59)
\]
where \(| \cdot |\) denotes the product measure on \(S^{2d-1}_1 \times B_R^d\).

**Proof.** This Proposition follows from the statement and the proof of Proposition 9.2, and the statement of Proposition 9.4. from [4].

We rely on Proposition 9.5 to derive Proposition 9.6 and Proposition 9.7. Recall the notation from (3.7)
\[
\mathcal{D}_{m+2, \epsilon_2, \epsilon_3} = \{Z_{m+2} = (X_{m+2}, V_{m+2}) \in \mathbb{R}^{2d(m+2)} : d_2(x_i, x_j) > \epsilon_2, \quad \forall (i, j) \in Z_{m+2}^2,
\]
and \(d_3(x_i, x_j, x_k) > \sqrt{2} \epsilon_3, \quad \forall (i, j, k) \in Z_{m+2}^3\},
\]
where \(Z_{m+2}^2, Z_{m+2}^3\) are given by (3.1)-(3.2) respectively.

**Proposition 9.6.** Consider parameters \(\alpha, \epsilon_0, R, \eta, \delta\) as in (9.5) and \(\epsilon_2 < \eta \epsilon_3 < \epsilon_3 < \epsilon_2\). Let \(m \in \mathbb{N}\), \(Z_m = (\bar{X}_m, \bar{V}_m) \in G_m(\epsilon_0, 0), \ell \in \{1, \ldots, m\}\) and \(X_m = B_{R/2}^d(\bar{X}_m)\). Then there is a subset \(B_t^1(\bar{Z}_m) \subseteq (S^{2d-1}_1 \times B_R^d) \ast (\bar{v}_1)\) such that:

(i) For any \((\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (S^{2d-1}_1 \times B_R^d) \ast (\bar{v}_1) \setminus B_t^1(\bar{Z}_m)\), one has:
\[
Z_{m+2}(t) \in \mathcal{D}_{m+2, \epsilon_2, \epsilon_3} \setminus \mathcal{D}_{m+2, \epsilon_2, \epsilon_3}, \quad \forall t \geq 0,
\]
\[
Z_{m+2} \in G_{m+2}(\epsilon_0, 2, \delta),
\]
\[
\bar{Z}_{m+2} \in G_{m+2}(\epsilon_0, \delta),
\]
(ii) For any \((\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (S^{2d-1}_1 \times B_R^d) \ast (\bar{v}_1) \setminus B_t^1(\bar{Z}_m)\), one has:
\[
Z^*_{m+2}(t) \in \mathcal{D}_{m+2, \epsilon_2, \epsilon_3} \setminus \mathcal{D}_{m+2, \epsilon_2, \epsilon_3}, \quad \forall t \geq 0,
\]
\[
Z^*_{m+2} \in G_{m+2}(\epsilon_0, 2, \delta),
\]
\[
\bar{Z}^*_{m+2} \in G_{m+2}(\epsilon_0, \delta).
\]

**Proof.** By symmetry we may assume that \(\ell = m\). Recall the set \(B_{m+2}^1(Z_m)\) from Proposition 9.5 satisfying (9.57)-(9.58).

We will construct a set \(A_m(Z_m) \subseteq (S^{2d-1}_1 \times B_R^d) \ast (\bar{v}_1)\), such that for any \((\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (S^{2d-1}_1 \times B_R^d) \ast (\bar{v}_1) \setminus A_m(Z_m)\):

- Using notation from (9.63) for the precollisional case, we have
\[
|x_i(t) - x_j(t)| > \epsilon_2, \quad \forall i, j \in \{m, m+1, m+2\} \text{ with } i < j.
\]
- Using notation from (9.67) for the postcollisional case, we have
\[
|x_i(t) - x_j(t)| > \epsilon_2, \quad \forall i, j \in \{m, m+1, m+2\} \text{ with } i < j.
\]
Then thanks to Proposition 9.5 and (9.68)-(9.69), the set
\[ B_m^*(Z_m) := \tilde{B}_m^*(Z_m) \cup A_m(Z_m), \]
will satisfy (9.60)-(9.62), (9.64)-(9.66). Let us introduce the following notation:
\[ \gamma := \frac{\varepsilon_2}{\varepsilon_3} << \eta^2, \quad \text{since } \varepsilon_2 << \eta^2 \varepsilon_3, \text{ by assumption}, \]
and
\[ \gamma' = \left(1 - \frac{\gamma}{2}\right)^{1/2} < 1. \]

**Construction of the set satisfying (9.68):** Here we use notation from (9.63). We distinguish the following cases:

- **Case (i,j) = (m,m+1):** Consider \( t \geq 0 \). We have
  \[
  |x_i(t) - x_j(t)|^2 = |x_m(t) - x_{m+1}(t)|^2
  = |\sqrt{2}\varepsilon_3\omega_1 + (v_{m+1} - \bar{v}_m)t|^2
  = 2\varepsilon_3^2|\omega_1|^2 + 2\sqrt{2}\varepsilon_3(\omega_1, v_{m+1} - \bar{v}_m)t + |v_{m+1} - \bar{v}_m|^2t^2.
  \]

We define the sets
\[
\Omega_1 = \{ (\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in S_1^{d-1} \times B_R^d : |\omega_1| \leq \sqrt{7} \},
\]
\[
A_{m+1} = \{ (\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in S_1^{d-1} \times B_R^d : |\omega_1| \geq \gamma|\omega_1||v_{m+1} - \bar{v}_m| \}.
\]
Consider the second degree polynomial in \( t \):
\[
P(t) = (2 - \gamma)|\omega_1|^2 + 2\sqrt{2}\varepsilon_3(\omega_1, v_{m+1} - \bar{v}_m)t + |v_{m+1} - \bar{v}_m|^2t^2.
\]

Let \( (\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (S_1^{d-1} \times B_R^d) \setminus (\Omega_1 \cup A_{m+1}) \). The polynomial \( P(t) \) has discriminant
\[
\Delta = 8\varepsilon_3^2(\omega_1, v_{m+1} - \bar{v}_m)^2 - 4(2 - \gamma)|\omega_1|^2|v_{m+1} - \bar{v}_m|^2
\]
\[
= 8\varepsilon_3^2(\omega_1, v_{m+1} - \bar{v}_m)^2 - 8\varepsilon_3^2|\omega_1|^2|v_{m+1} - \bar{v}_m|^2
\]
\[
= 8\varepsilon_3^2((\omega_1, v_{m+1} - \bar{v}_m)^2 - \gamma^2|\omega_1|^2|v_{m+1} - \bar{v}_m|^2)
\]
\[
< 0.
\]

Since \( (\omega_1, \omega_2, v_{m+1}, v_{m+2}) \notin A_{m+1} \). Since \( \gamma << 1 \), we obtain \( P(t) > 0 \), for all \( t \geq 0 \), or in other words
\[
2\varepsilon_3^2|\omega_1|^2 + 2\sqrt{2}\varepsilon_3(\omega_1, v_{m+1} - \bar{v}_m)t + |v_{m+1} - \bar{v}_m|^2t^2 > \gamma\varepsilon_3^2|\omega_1|^2.
\]

Since \( (\omega_1, \omega_2, v_{m+1}, v_{m+2}) \notin \Omega_1 \), expressions (9.72), (9.76) yield
\[
|x_m(t) - x_{m+1}(t)|^2 > \gamma\varepsilon_3^2|\omega_1|^2 > \gamma\varepsilon_3^2 = \varepsilon_2.
\]

Therefore for any \( (\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (S_1^{d-1} \times B_R^d) \setminus (\Omega_1 \cup A_{m+1}) \), we have
\[
|x_m(t) - x_{m+1}(t)| > \varepsilon_2, \quad \forall t \geq 0.
\]

- **Case (i,j) = (m,m+2):** We follow a similar argument using the sets
  \[
  \Omega_2 = \{ (\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in S_1^{d-1} \times B_R^d : |\omega_2| \leq \sqrt{7} \},
  \]
  \[
  A_{m+2} = \{ (\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in S_1^{d-1} \times B_R^d : |\omega_2| \geq \gamma|\omega_2||v_{m+2} - \bar{v}_m| \},
  \]
  to conclude that for all \( (\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (S_1^{d-1} \times B_R^d) \setminus (\Omega_2 \cup A_{m+2}) \), we have
  \[
  |x_m(t) - x_{m+2}(t)| > \varepsilon_2, \quad \forall t \geq 0.
  \]

- **Case (i,j) = (m+1,m+2):** We follow a similar argument using the sets
  \[
  \Omega_{1,2} = \{ (\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in S_1^{d-1} \times B_R^d : |\omega_1 - \omega_2| \leq \sqrt{7} \},
  \]
  \[
  B_{m+1,m+2} = \{ (\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in S_1^{d-1} \times B_R^d : |\omega_1 - \omega_2| \geq |\omega_1 - \omega_2||v_{m+1} - v_{m+2}| \},
  \]
  to conclude that for all \( (\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (S_1^{d-1} \times B_R^d) \setminus (\Omega_{1,2} \cup B_{m+1,m+2}) \), we have
  \[
  |x_{m+1}(t) - x_{m+2}(t)| > \varepsilon_2, \quad \forall t \geq 0.
  \]
Defining
\[ A_m(\tilde{z}_m) = \Omega_1 \cup \Omega_2 \cup \Omega_{1,2} \cup A_{m,m+1} \cup A_{m,m+2} \cup B_{m+1,m+2}, \]
we obtain that (9.68) holds for \((\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (S_1^{2d-1} \times B_R^{2d}) \setminus A_m(\tilde{z}_m)\).

Construction of the set satisfying (9.69): Here we use notation from (9.64). We distinguish the following cases:
- Case \((i, j) = (m, m+1)\): We follow a similar argument to the precollisional case, using the set \(\Omega_1\), defined in (9.73), and the set
\[ A_{m,m+1}^{*} = \{(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in S_1^{2d-1} \times B_R^{2d} : |\omega_1| \geq \gamma |v_{m+1} - \tilde{v}_m^*|, \}
\]to conclude that for all \((\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (S_1^{2d-1} \times B_R^{2d}) \setminus (\Omega_2 \cup A_{m,m+1}^{*})\), we have
\[ |x_{m+1}(t) - x_m(t)| > \epsilon_2, \quad \forall t \geq 0. \]
- Case \((i, j) = (m, m+2)\): We follow a similar argument to the precollisional case, using the set \(\Omega_{1,2}\), defined in (9.78), and the set
\[ A_{m,m+2}^{*} = \{(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in S_1^{2d-1} \times B_R^{2d} : |\omega_2| \geq \gamma |v_{m+2} - \tilde{v}_m^*|, \}
\]to conclude that for all \((\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (S_1^{2d-1} \times B_R^{2d}) \setminus (\Omega_2 \cup A_{m,m+2}^{*})\), we have
\[ |x_{m+2}(t) - x_m(t)| > \epsilon_2, \quad \forall t \geq 0. \]
- Case \((i, j) = (m+1, m+2)\): We follow a similar argument to the precollisional case, using the set \(\Omega_{1,2}\), defined in (9.80), and the set
\[ B_{m+1,m+2}^{*} = \{(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in S_1^{2d-1} \times B_R^{2d} : |\omega_1 - \omega_2| \geq \gamma |v_{m+1} - v_{m+2}|, \}
\]to conclude that for all \((\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (S_1^{2d-1} \times B_R^{2d}) \setminus (\Omega_2 \cup B_{m+1,m+2}^{*})\), we have
\[ |x_{m+1}(t) - x_{m+2}(t)| > \epsilon_2, \quad \forall t \geq 0. \]

Defining
\[ A_m^{*}(\tilde{z}_m) = \Omega_1 \cup \Omega_2 \cup \Omega_{1,2} \cup A_{m,m+1}^{*} \cup A_{m,m+2}^{*} \cup B_{m+1,m+2}^{*}, \]
we obtain that (9.69) holds for \((\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (S_1^{2d-1} \times B_R^{2d}) \setminus A_m^{*}(\tilde{z}_m)\).

Defining
\[ A_m(\tilde{z}_m) = \Omega_1 \cup \Omega_2 \cup \Omega_{1,2} \cup A_{m,m+1} \cup A_{m,m+2} \cup B_{m+1,m+2}, \]
(9.68) holds for any \((\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (S_1^{2d-1} \times B_R^{2d}) \setminus (A_m(\tilde{z}_m) \cup A_m^{*}(\tilde{z}_m))\).

The set \(B_{m}^{0}(\tilde{z}_m) = \overline{B_{m}^{0}(\tilde{z}_m)} \cup A_m(\tilde{z}_m)\), (9.89) satisfies (9.60)-(9.62), (9.64)-(9.66), thus it is the set we need to conclude the proof.

\[ \square \]

9.3.2. Measure estimate for ternary adjunction. We now provide the corresponding measure estimate for the set \(B_3^{i}(\tilde{z}_m)\) appearing in Proposition 9.6. To estimate the measure of this set, we will strongly rely on the results of Section 8

Proposition 9.7. Consider parameters \(\alpha, \epsilon_0, R, \eta, \delta\) as in (9.5) and \(\epsilon_2 << \eta^2 \epsilon_3 << \alpha\). Let \(m \in \mathbb{N}, Z_m \in G_m(\epsilon_0, 0), \ell \in \{1, \ldots, m\}\) and \(B_{m}^{0}(\tilde{z}_m)\) be the set appearing in the statement of Proposition 9.6. Then the following measure estimate holds:
\[ |B_{m}^{0}(\tilde{z}_m)| \lesssim m R^{2d} \eta \frac{d-1}{d}, \]
where \(|\cdot|\) denotes the product measure on \(S_1^{2d-1} \times B_R^{2d}\).

Proof. By symmetry, we may assume \(\ell = m\). Recall that
\[ B_{m}^{0}(\tilde{z}_m) = \overline{B_{m}^{0}(\tilde{z}_m)} \cup A_m(\tilde{z}_m), \]
where \(B_{m}^{0}(\tilde{z}_m)\) is given by Proposition 9.6 and \(A_m(\tilde{z}_m)\) is given by (9.88). Estimate (9.60) yields
\[ |B_{m}^{0}(\tilde{z}_m)| \lesssim m R^{2d} \eta \frac{d-1}{d}, \]
so it suffices to estimate the measure of $A_m(\bar{Z}_m)$. By (9.88), it suffices to estimate the measure of $A_m(\bar{Z}_m)$ and $A^+_m(\bar{Z}_m)$ which are given by (9.82), (9.87) respectively.

Let us recall the notation from (9.80), (9.81):

$$\gamma = \frac{\epsilon_2}{\epsilon_3} << \eta^2, \quad \gamma' = \sqrt{1 - \frac{\gamma}{2}}.$$  

**Estimate of $A_m(\bar{Z}_m)$:** Recall from (9.82) that

$$A_m(\bar{Z}_m) = \Omega_1 \cup \Omega_2 \cup \Omega_{1,2} \cup A_{m,m+1} \cup A_{m,m+2} \cup B_{m+1,m+2},$$

where $\Omega_1, A_{m,m+1}$ are given by (9.73)-(9.74), $\Omega_2, A_{m,m+2}$ by (9.78)-(9.79) and $\Omega_{1,2}, B_{m+1,m+2}$ are given by (9.80)-(9.81).

- Estimate for $\Omega_1, \Omega_2$: Without loss of generality, it suffices to estimate the measure of $\Omega_1$. Recalling notation from (8.3), Fubini's Theorem and Lemma 8.3 yield

$$(9.93) \quad |\Omega_1| = \int_{B_R^d} \int_{S^d_{1-1}} 1_{M_1(\sqrt{\gamma})} d\omega_1 d\omega_2 dv_{m+1} dv_{m+2} \lesssim R^{2d} \gamma^{d/2},$$

A symmetric argument yields

$$(9.94) \quad |\Omega_2| \lesssim R^{2d} \gamma^{d/2}.$$  

- Estimate for $\Omega_{1,2}$: Recalling notation from (8.3), (9.80) yields

$$(9.95) \quad \Omega_{1,2} = (S^d_{1-1} \cap W_{2d}^d) \times B^d_R,$$

Therefore, Fubini's Theorem and Lemma 8.4 imply

$$(9.96) \quad |\Omega_{1,2}| = \int_{B_R^d} \int_{S^d_{1-1}} 1_{W_{2d}^d} d\omega_1 d\omega_2 dv_{m+1} dv_{m+2} \lesssim R^{2d} \gamma^{d/2}.$$  

- Estimate for $A_{m,m+1}$: Recalling notation from (8.6), the set $A_{m,m+1}$, which was defined in (9.74), can be written as

$$A_{m,m+1} = \left\{ (\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in S_{1-1}^d \times B^d_R : \omega_1 \in S(\gamma', v_{m+1} - \bar{v}_m) \right\}.$$  

Therefore, the representation of the $(2d-1)$-unit sphere (8.1) and Lemma 8.5 yield

$$(9.97) \quad |A_{m,m+1}| \lesssim R^{2d} \arccos \sqrt{1 - \frac{\gamma}{2}}.$$  

- Estimate for $A_{m,m+2}$: We follow a similar argument as in the previous case to obtain

$$(9.98) \quad |A_{m,m+2}| \lesssim R^{2d} \arccos \sqrt{1 - \frac{\gamma}{2}}.$$  

- Estimate for $B_{m+1,m+2}$: Recalling notation from (8.7), (9.81) yields

$$B_{m+1,m+2} = \{ (\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in S_{1-1}^d \times B^d_R : (\omega_1, \omega_2) \in N(\gamma', v_{m+1} - v_{m+2}) \}.$$  

Therefore, using Lemma 8.6 we obtain

$$(9.99) \quad |B_{m+1,m+2}| \leq \int_{B_R^d} \int_{S_{1-1}^d} 1_{N(\gamma', v_{m+1} - v_{m+2})} (\omega_1, \omega_2) d\omega_1 d\omega_2 dv_{m+1} dv_{m+2} \lesssim \int_{B_R^d} \int_{S_{1-1}^d} 1_{N(\gamma', v_{m+1} - v_{m+2})} (\omega_1, \omega_2) d\omega_1 d\omega_2 dv_{m+1} dv_{m+2} \lesssim R^{2d} \arccos \sqrt{1 - \frac{\gamma}{2}}.$$  

Using (9.92) and estimates (9.93)-(9.98), we obtain

$$|A_m(\bar{Z}_m)| \lesssim R^{2d} \left( \gamma^{d/2} + \frac{d+1}{4} \arccos \sqrt{1 - \frac{\gamma}{2}} \right).$$
Estimate of $A_m^*(Z_m)$: Recall from (9.87) that
\[ A_m^*(Z_m) = \Omega_1 \cup \Omega_2 \cup \Omega_{1.2} \cup A_{m,m+1}^* \cup A_{m,m+2}^* \cup B_{m+1,m+2}^*, \] (9.100)
where $\Omega_1, \Omega_2, \Omega_{1.2}, A_{m,m+1}, A_{m,m+2}, B_{m+1,m+2}$ are given by (9.73), (9.75), (9.81), (9.83)-(9.84) respectively. We already have estimates for $\Omega_1, \Omega_2, \Omega_{1.2}$ from (9.93)-(9.95), hence it suffices to derive estimates for $A_{m,m+1}^*, A_{m,m+2}^*, B_{m+1,m+2}$.

For the rest of the proof we consider a parameter $0 < \beta << 1$ which will be chosen later in terms of $\eta$, see (9.150).

- Estimate for $A_{m,m+1}^*$: Recall from (9.83) the set
\[ A_{m,m+1}^* = \{ (\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in S_1^{d-1} \times B_R^d : |(\omega_1, v_{m+1} - \bar{v}_m)| \geq \gamma |\omega_1||v_{m+1} - \bar{v}_m| \} \].
(9.101)
But for any $(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in S_1^{d-1} \times B_R^d$, the ternary collisional law (9.110) implies
\[ v_{m+1}^* - \bar{v}_m = v_{m+1} - \bar{v}_m - 2c_{\omega_1, \omega_2, v_m, v_{m+1}, v_{m+2}} \omega_2 - c_{\omega_1, \omega_2, v_m, v_{m+1}, v_{m+2}} \omega_1, \]
where
\[ c_{\omega_1, \omega_2, v_m, v_{m+1}, v_{m+2}} = \frac{\langle \omega_1, v_{m+1} - \bar{v}_m \rangle + \langle \omega_2, v_{m+2} - \bar{v}_m \rangle}{1 + \langle \omega_1, \omega_2 \rangle}. \]
(9.102)
For convenience, we denote
\[ c := c_{\omega_1, \omega_2, v_m, v_{m+1}, v_{m+2}}. \]
Therefore, by (9.101), we may write
\[ A_{m,m+1}^* = \{ (\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in S_1^{d-1} \times B_R^d : |(\omega_1, v_{m+1} - \bar{v}_m - 2c_{\omega_1, \omega_2, v_m, v_{m+1}, v_{m+2}} \omega_2)| \geq \gamma |\omega_1||v_{m+1} - \bar{v}_m - 2c_{\omega_1, \omega_2} \omega_2| \}. \]

By Fubini’s Theorem we have
\[ |A_{m,m+1}^*| \leq \int_{S_1^{d-1} \times B_R^d} \int_{B_R^d} 1_{v_{m+1}^*_m = v_{m+1} - \bar{v}_m} (v_{m+1}) \, dv_{m+1} \, d\omega_1 \, d\omega_2 \, dv_{m+2} \] (9.103)
where given $(\omega_1, \omega_2, v_{m+2}) \in S_1^{d-1} \times B_R^d$ we write
\[ v_{m+1}^*_m = \{ v_{m+1} \in B_R^d : (\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in A_{m,m+1}^* \} \].
(9.104)
Recall from (8.11) the set
\[ I_1 = \{ (\omega_1, \omega_2) \in \mathbb{R}^d : |1 - 2|\omega_1|^2| \leq 2\beta \} \].
(9.105)
Using (9.103), we obtain
\[ |A_{m,m+1}^*| = |\bar{I}_1 + \bar{I}_1| \] (9.106)
where
\[ |\bar{I}_1| = \int_{(S_1^{d-1} \setminus I_1) \times B_R^d} \int_{B_R^d} 1_{v_{m+1}^*_m = v_{m+1} - \bar{v}_m} (v_{m+1}) \, dv_{m+1} \, d\omega_1 \, d\omega_2 \, dv_{m+2}, \]
(9.107)
\[ |\bar{I}_1| = \int_{(S_1^{d-1} \setminus I_1) \times B_R^d} \int_{B_R^d} 1_{v_{m+1}^*_m = v_{m+1} - \bar{v}_m} (v_{m+1}) \, dv_{m+1} \, d\omega_1 \, d\omega_2 \, dv_{m+2}. \]
(9.108)
We treat each of the terms in (9.106) separately.

Estimate for $\bar{I}_1$: By (9.107), Fubini’s Theorem and Lemma (8.7) we obtain
\[ |\bar{I}_1| \lesssim R^{2d} \int_{S_1^{d-1}} \mathbb{1}_{I_1} \, d\omega_1 \, d\omega_2 \lesssim R^{2d} \beta. \]
(9.109)

Estimate for $\bar{I}_1$: Let us fix $(\omega_1, \omega_2, v_{m+2}) \in (S_1^{d-1} \setminus I_1) \times B_R^d$. We define the smooth map $F_{\omega_1, \omega_2, v_{m+2}}^1 : B_R^d \to \mathbb{R}^d$, by:
\[ F_{\omega_1, \omega_2, v_{m+2}}^1 (v_{m+1}) := v_{m+1}^* - \bar{v}_m = v_{m+1} - \bar{v}_m - 2c_{\omega_1, \omega_2} \omega_2, \]
(9.110)
where $c$ is given by (9.102).

We are showing that we may change variables under $F_{\omega_1, \omega_2, v_{m+2}}^1$, as long as $(\omega_1, \omega_2, v_{m+1}) \in (S_1^{d-1} \setminus I_1) \times B_R^d$ i.e. we are showing that $F_{\omega_1, \omega_2, v_{m+2}}^1$ has non-zero Jacobian and is injective. In particular we will see that the Jacobian is bounded from below by $\beta$.
We first show the Jacobian has a lower bound $\beta$. Differentiating with respect to $v_{m+1}$, we obtain

$$\frac{\partial F_{\omega_1,\omega_2,v_{m+2}}^1}{\partial v_{m+1}} = I_d + (-2\omega_1 - \omega_2)\nabla^T_{v_{m+1}}c.$$ 

Recalling (9.102), we have

$$\nabla^T_{v_{m+1}}c = \frac{1}{1 + \langle \omega_1, \omega_2 \rangle} \omega^T_1.$$ 

Using Lemma 2.3 from the Appendix, we get

$$\text{Jac } F_{\omega_1,\omega_2,v_{m+2}}^1 (v_{m+1}) = \det \left( I_d + \frac{1}{1 + \langle \omega_1, \omega_2 \rangle} (-2\omega_1 - \omega_2)\omega_1^T \right)$$

$$= 1 + \frac{-2|\omega_1|^2 - (\omega_1, \omega_2)}{1 + \langle \omega_1, \omega_2 \rangle}$$

$$= 1 - 2|\omega_1|^2 \frac{2}{1 + \langle \omega_1, \omega_2 \rangle}.$$ 

Since $(\omega_1, \omega_2) \notin I_1$, we have $|1 - 2|\omega_1|^2| > 2\beta$, hence

$$|\text{Jac } F_{\omega_1,\omega_2,v_{m+2}}^1 (v_{m+1})| = \frac{|1 - 2|\omega_1|^2|}{1 + \langle \omega_1, \omega_2 \rangle} \geq \frac{2\beta}{1 + \langle \omega_1, \omega_2 \rangle} \geq \frac{4\beta}{3} > \beta,$$ (9.111)

since $\frac{1}{2} \leq 1 + \langle \omega_1, \omega_2 \rangle \leq \frac{3}{2}$ by (2.13). Thus

$$|\text{Jac } F_{\omega_1,\omega_2,v_{m+2}}^1 (v_{m+1})|^{-1} < \beta^{-1}, \quad \forall v_{m+1} \in B_R^d.$$ (9.112)

We now show that $F_{\omega_1,\omega_2,v_{m+2}}^1$ is injective. For this purpose consider $v_{m+1}, \xi_{m+1} \in B_R^d$ such that

$$F_{\omega_1,\omega_2,v_{m+2}}^1 (v_{m+1}) = F_{\omega_1,\omega_2,v_{m+2}}^1 (\xi_{m+1})$$

$$\Leftrightarrow v_{m+1} - \xi_{m+1} = \frac{\langle \xi_{m+1} - \xi_{m+1}, \omega_1 \rangle}{1 + \langle \omega_1, \omega_2 \rangle} (2\omega_1 + \omega_2),$$ (9.113)

thanks to (9.102). Therefore, there is $\lambda \in \mathbb{R}$ such that

$$v_{m+1} - \xi_{m+1} = \lambda (2\omega_1 + \omega_2),$$ (9.114)

so replacing $v_{m+1} - \xi_{m+1}$ in (9.113) with the right hand side of (9.114), we obtain

$$\lambda(1 - 2|\omega_1|^2) = 0,$$

which yields $\lambda = 0$, since we have assumed $(\omega_1, \omega_2) \notin I_1$. Therefore $v_{m+1} = \xi_{m+1}$, thus $F_{\omega_1,\omega_2,v_{m+2}}^1$ is injective.

Since $(\omega_1, \omega_2, v_{m+2}) \in \mathbb{S}^{2d-1} \times B_R^d$ and $v_m \in B_R^d$, Cauchy-Schwartz inequality yields that, for any $v_{m+1} \in B_R^d$, we have

$$|F_{\omega_1,\omega_2,v_{m+2}}^1 (v_{m+1})| \leq |v_{m+1}| + |\tilde{v}_m| + \frac{\omega_1|\langle \tilde{v}_m, \omega_1 \rangle| + |\tilde{v}_m| + |\omega_2|(|\tilde{v}_m| + |v_{m+2}|)}{1 + \langle \omega_1, \omega_2 \rangle} \leq 26R,$$

since $\frac{1}{2} \leq 1 + \langle \omega_1, \omega_2 \rangle \leq \frac{3}{2}$, by (2.14), and $(\omega_1, \omega_2, v_{m+2}) \in \mathbb{S}^{2d-1} \times B_R^d$. Therefore

$$F_{\omega_1,\omega_2,v_{m+2}}^1 (B_R^d) \subseteq B_{26R}^d.$$ (9.115)

Additionally, recalling (9.104), (9.101) and (9.110), we have

$$V_{\omega_1,\omega_2,v_{m+2}}^m = \{ v_{m+1} \in B_R^d : (\omega_1, F_{\omega_1,\omega_2,v_{m+2}}^1(v_{m+1})) \geq \beta|\omega_1||F_{\omega_1,\omega_2,v_{m+2}}^1(v_{m+1})| \},$$

thus

$$v_{m+1} \in V_{\omega_1,\omega_2,v_{m+2}}^m \Leftrightarrow F_{\omega_1,\omega_2,v_{m+2}}^1 (v_{m+1}) \in U_{\omega_1},$$ (9.116)

where

$$U_{\omega_1} = \{ \nu \in \mathbb{R}^d : \langle \omega_1, \nu \rangle \geq \gamma |\omega_1||\nu| \}.$$ (9.117)

Hence

$$\mathbb{1}_{V_{\omega_1,\omega_2,v_{m+2}}^m} (v_{m+1}) = \mathbb{1}_{U_{\omega_1}} (F_{\omega_1,\omega_2,v_{m+2}}^1(v_{m+1})), \quad \forall v_{m+1} \in B_R^d.$$ (9.118)
Therefore, performing the substitution \( \nu := F_{1, \omega_2, v_{m+2}}^1(v_{m+1}) \), and using (9.112), we obtain
\[
\int_{B^d_R} 1_{B^{m+1}_1}(v_{m+1}) dv_{m+1} = \int_{B^d_R} 1_{U_{\omega_1}}(F_{1, \omega_2, v_{m+2}}^1(v_{m+1})) dv_{m+1} \leq \beta^{-1} \int_{B^d_{2R}} 1_{U_{\omega_1}(\nu)} dv.
\]
Recalling notation from (8.14) and (9.117), we have
\[
1_{U_{\omega_1}(\nu)} = 1_{S(\gamma', \nu)(\omega_1)}, \quad \forall \omega_1 \in B^d_{\bar{R}}, \quad \forall \nu \in B^d_{2R}.
\]
Therefore, using (9.108), (9.119), Fubini’s Theorem and (9.119), we obtain
\[
I_i^* \leq \beta^{-1} \int_{S^d_{d-1}(1) \times B^d_R} 1_{U_{\omega_1}(\nu)} dv_{\omega_2} d\omega_1 dv_{m+2}
\]
\[
\leq \beta^{-1} \int_{B^d_{2R} \times B^d_R} \int_{B^d_R} 1_{S(\gamma', \nu)(\omega_1)} d\omega_1 dv_{\omega_2} dv_{m+2} \leq R^{2d} \beta^{-1} \arccos \gamma'.
\]
\[
= R^{2d} \beta^{-1} \arccos \sqrt{1 - \frac{\gamma^2}{2}}.
\]
where to obtain (9.120) we use Lemma 8.5. Combining (9.106), (9.109), (9.121), we obtain
\[
|A^*_{m, m+1}| \leq R^{2d} \left( \beta + \beta^{-1} \arccos \sqrt{1 - \frac{\gamma^2}{2}} \right).
\]

- Estimate for \( A^*_{m, m+2} \): The argument is entirely symmetric, using the set
\[
V_{m+2} = \{ v_{m+2} \in B^d_{\bar{R}} : (\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in A^*_{m, m+2} \},
\]
for fixed \((\omega_1, \omega_2, v_{m+1}) \in S^d_{d-1} \times B^d_R\) and the map
\[
F_{\omega_1, \omega_2, v_{m+1}}^2(v_{m+2}) = v_{m+2} - \tilde{v}_m - c \omega_1 - 2c \omega_2.
\]
We obtain the estimate
\[
|A^*_{m, m+2}| \leq R^{2d} \left( \beta + \beta^{-1} \arccos \sqrt{1 - \frac{\gamma^2}{2}} \right).
\]

- Estimate for \( B^*_{m+1, m+2} \): The estimate for \( B^*_{m+1, m+2} \) is in the same spirit as the previous estimates, however we will need to distinguish cases depending on the size of the impact directions. The reason for that is that we rely on Lemma 8.6 from Section 8 which provides estimates on hemispheres of the \((2d - 1)\)-unit sphere.

Recall from (9.13) the set
\[
B^*_{m+1, m+2} = \{ (\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in S^d_{d-1} \times B^d_R : |(\omega_1 - \omega_2, v_{m+1} - v_{m+2})| \geq \gamma |\omega_1 - \omega_2| |v_{m+1} - v_{m+2}| \}.
\]
The ternary collision law (2.3) yields \( v_{m+1} - v_{m+2} = v_{m+1} - v_{m+2} - c(\omega_1 - \omega_2) \), where \( c \) is given by (9.102).
Thus we may write
\[
B^*_{m+1, m+2} = \{ (\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in S^d_{d-1} \times B^d_R : |(\omega_2 - \omega_1, v_{m+2} - v_{m+1} - c(\omega_2 - \omega_1))| \geq \gamma |\omega_2 - \omega_1| |v_{m+2} - v_{m+1} - c(\omega_2 - \omega_1)| \}.
\]
Recall from (8.13)-(8.14), the sets
\[
S_{1, 2} = \{ (\omega_1, \omega_2) \in S^d_{d-1} : |\omega_1| < |\omega_2| \}, \quad S_{2, 1} = \{ (\omega_1, \omega_2) \in S^d_{d-1} : |\omega_2| < |\omega_1| \},
\]
We also recall from (8.15)-(8.16) the sets
\[
I_{1, 2} = \{ (\omega_1, \omega_2) \in R^{2d} : |\omega_1|^2 + 2|\omega_1, \omega_2| \leq \beta \}, \quad I_{2, 1} = \{ (\omega_1, \omega_2) \in R^{2d} : |\omega_2|^2 + 2|\omega_1, \omega_2| \leq \beta \}.
\]
We clearly have
\[
|B^*_{m_1,m_2}| = \int_{S_{1,2} \times B_R^d} \mathbb{1}_{B^*_{m_1,m_2}} \, d\omega_1 \, d\omega_2 \, dv_{m_1} \, dv_{m_2} + \int_{S_{1,2} \times B_R^d} \mathbb{1}_{B^*_{m_1,m_2}} \, d\omega_1 \, d\omega_2 \, dv_{m_1} \, dv_{m_2}
\]
\[
= \tilde{I}_{1,2} + \tilde{I}'_{1,2} + \tilde{I}_{2,1} + \tilde{I}'_{2,1},
\]
where
\[
\tilde{I}_{1,2} = \int_{S_{1,2} \times B_R^d} \mathbb{1}_{B^*_{m_1,m_2}} \, d\omega_1 \, d\omega_2 \, dv_{m_1} \, dv_{m_2},
\]
\[
\tilde{I}'_{1,2} = \int_{S_{1,2} \times B_R^d} \mathbb{1}_{B^*_{m_1,m_2}} \, d\omega_1 \, d\omega_2 \, dv_{m_1} \, dv_{m_2},
\]
\[
\tilde{I}_{2,1} = \int_{S_{1,2} \times B_R^d} \mathbb{1}_{B^*_{m_1,m_2}} \, d\omega_1 \, d\omega_2 \, dv_{m_1} \, dv_{m_2},
\]
\[
\tilde{I}'_{2,1} = \int_{S_{1,2} \times B_R^d} \mathbb{1}_{B^*_{m_1,m_2}} \, d\omega_1 \, d\omega_2 \, dv_{m_1} \, dv_{m_2}.
\]

We treat each of the terms in (9.125) separately.

Estimate for \( \tilde{I}_{1,2} \): By (9.126), Fubini’s Theorem and Lemma 8.8 we obtain
\[
\tilde{I}_{1,2} \lesssim R^{2d} \int_{S_{1,2}} \mathbb{1}_{I_{1,2}} \, d\omega_1 \, d\omega_2 \lesssim R^{2d} \beta.
\]

Estimate for \( \tilde{I}_{2,1} \): Similarly, we obtain
\[
\tilde{I}_{2,1} \lesssim R^{2d} \beta.
\]

Estimate for \( \tilde{I}'_{2,1} \): From (9.127), we obtain
\[
\tilde{I}'_{2,1} \leq \int_{S_{1,2} \times I_{1,2}} \int_{B_R^d} \int_{B_R^d} \mathbb{1}_{V_{m_1,m_2}} \, (v_{m_2}) \, dv_{m_2} \, dv_{m_1} \, d\omega_1 \, d\omega_2,
\]
where given \((\omega_1, \omega_2, v_{m+1}) \in (S_{1,2} \setminus I_{1,2}) \times B_R^d\), we denote
\[
V_{m+1,m+2}^{m_1,m_2} = \{v_{m+2} \in B_R^d : (\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in B^*_{m_1,m_2}\}.
\]
Let us fix \((\omega_1, \omega_2, v_{m+1}) \in (S_{1,2} \setminus I_{1,2}) \times B_R^d\). We define the map \(F^*_{1,2,1,2,v_{m+1}} : B_R^d \rightarrow \mathbb{R}^d\) by
\[
F^*_{1,2,1,2,v_{m+1}}(v_{m+2}) = v_{m+2} - 2m_{m+1}v_{m+2} - \omega_2 - \omega_1,
\]
where \(c\) is given by (9.102). In a similar way as in the estimate of \(|A^*_{m,m+1}|\), for any \((\omega_1, \omega_2) \notin I_{1,2}\), we have
\[
|\text{Jac} F^*_{1,2,1,2,v_{m+1}}(v_{m+2})| = \frac{|\omega_1|^2 + 2|\omega_1\omega_2|}{1 + 2|\omega_1\omega_2|} > \beta \frac{\beta}{1 + 2|\omega_1\omega_2|} \geq \frac{2\beta}{3},
\]
Thus
\[
|\text{Jac} F^*_{1,2,1,2,v_{m+1}}(v_{m+2})|^{-1} \leq \frac{3\beta^{-1}}{2}, \quad \forall v_{m+2} \in B_R^d.
\]
Similarly to the estimate for \(|A^*_{m,m+1}|\), we show also that \(F^*_{1,2,1,2,v_{m+1}}\) is injective.

Since \((\omega_1, \omega_2, v_{m+1}) \in S_{1,2}^{1-d} \times B_R^d \) and \(v_{m+1} \in B_R^d\), Cauchy-Schwartz inequality yields that, for any \(v_{m+2} \in B_R^d\), we have
\[
|F^*_{1,2,1,2,v_{m+1}}(v_{m+2})| \leq |v_{m+2}| + |v_{m+1}| + \frac{|\omega_1|(|v_{m+1}| + |v_{m+2}|) + |\omega_2|(|v_{m+2}| + |v_{m+1}|)}{1 + |\omega_1\omega_2|}(|\omega_2| + |v_1|) \leq 18R,
\]
since \(\frac{1}{2} \leq 1 + |\omega_1\omega_2| \leq \frac{3}{2}\). Therefore
\[
F^*_{1,2,1,2,v_{m+1}}(B_R^d) \subseteq B_{18R}^d.
\]
Additionally
\[
v_{m+2} \in V_{m+1,m+2}^{m_1,m_2} \Leftrightarrow F^*_{1,2,1,2,v_{m+1}}(v_{m+2}) \in U_{\omega_1,\omega_2},
\]
where
\[
U_{\omega_1,\omega_2} = \left\{v \in \mathbb{R}^d : |\omega_2 - \omega_1, v| \geq \gamma |\omega_2 - \omega_1| |v| \right\}.
\]
Hence
\[ I_{m+1,m+2}^{1,2}(v_{m+2}) = I_{U_{1},w_{2}}^{1,2}((F_{1,2}^{1,2}(v_{m+2})), \quad \forall v_{m+2} \in B_{R}^{d}. \tag{9.138} \]

Therefore, performing the substitution \( \nu : F_{1,2}^{1,2}(v_{m+2}) \), and using \( \tag{9.135} \), we obtain
\[ \int_{B_{R}^{d}} I_{m+1,m+2}^{1,2}(v_{m+2}) \, dv_{m+2} = \int_{B_{R}^{d}} I_{U_{1},w_{2}}^{1,2}(F_{1,2}^{1,2}(v_{m+2})) \, dv_{m+2} \leq \beta^{-1} \int_{B_{18R}^{d}} I_{U_{1},w_{2}}^{1,2}(\nu) \, d\nu. \tag{9.139} \]

Recalling the set \( N(\gamma', \nu) = \{(\omega_1, \omega_2) \in R^{2d} : (\omega_1 - \omega_2, \nu) \geq \gamma'|\omega_1 - \omega_2|\nu| \} \), from \( \tag{8.7} \) and \( \tag{9.137} \), we have
\[ I_{U_{1},w_{2}}^{1,2}(\nu) = I_{N(\gamma', \nu)}(\omega_1, \omega_2), \quad (\omega_1, \omega_2) \in S_{1}^{d-1}, \quad \forall \nu \in B_{18R}^{d}. \tag{9.140} \]

Therefore, using \( \tag{9.132} \), \( \tag{9.130} \), Fubini’s Theorem and \( \tag{9.140} \), we obtain
\[ I_{1,2}^{1} \leq \beta^{-1} \int_{(S_{1,2} \setminus I_{1,2}) \times B_{R}^{d}} \int_{B_{18R}^{d}} I_{U_{1},w_{2}}^{1,2}(\nu) \, d\omega_1 \, d\omega_2 \, dv_{m+1} \]
\[ \leq \beta^{-1} \int_{B_{R}^{d} \times B_{18R}^{d}} \int_{S_{1,2}^{d-1}} I_{N(\gamma', \nu)}(\omega_1, \omega_2) \, d\omega_1 \, d\omega_2 \, dv_{m+1} \]
\[ \leq R^{2d} \beta^{-1} \arccos \gamma' \]
\[ = R^{2d} \beta^{-1} \arccos \sqrt{1 - \frac{\gamma'}{2}}. \tag{9.141} \]

where to obtain \( \tag{9.141} \), we use Lemma 8.6
Therefore,
\[ I_{1,2}^{1} \leq R^{2d} \beta^{-1} \arccos \sqrt{1 - \frac{\gamma'}{2}}. \tag{9.142} \]

**Estimate for \( I_{1,1}^{1} \):** The argument is entirely symmetric, using the set
\[ V_{m+1,m+2}^{1,1} = \left\{ v_{m+1} \in B_{R}^{d} : (\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in B_{18R}^{d} \right\}, \]
for given \( (\omega_1, \omega_2, v_{m+2}) \in (S_{2,1} \setminus I_{2,1}) \times B_{R}^{d} \) and the map \( F_{1,2}^{1,1}(v_{m+1}) = v_{m+1} - v_{m+2} - c(\omega_1 - \omega_2) \). We obtain
\[ I_{1,1}^{1} \leq R^{2d} \beta^{-1} \arccos \sqrt{1 - \frac{\gamma'}{2}}. \tag{9.143} \]

Recalling \( \tag{9.125} \) and using \( \tag{9.130} \)-\( \tag{9.131} \), \( \tag{9.142} \)-\( \tag{9.143} \), we obtain
\[ |B_{m+1,m+2}^{d}| \leq R^{2d} \left( \beta + \beta^{-1} \arccos \sqrt{1 - \frac{\gamma'}{2}} \right). \tag{9.144} \]

Recalling \( \tag{9.100} \) and using \( \tag{9.93} \)-\( \tag{9.95} \), \( \tag{9.122} \), \( \tag{9.123} \), \( \tag{9.144} \), we obtain
\[ |A_{m}^{d}(Z_{m})| \leq R^{2d} \left( \gamma^{d/2} + \gamma^{\frac{d-1}{2}} + \beta + \beta^{-1} \arccos \sqrt{1 - \frac{\gamma'}{2}} \right). \tag{9.145} \]

Recalling \( \tag{9.88} \), using \( \tag{9.93} \), \( \tag{9.145} \) and using the fact that \( \gamma < 1 \), we obtain
\[ |A_{m}(Z_{m})| \leq R^{2d} \left( \gamma^{\frac{d-1}{2}} + \beta + \beta^{-1} \arccos \sqrt{1 - \frac{\gamma'}{2}} \right). \tag{9.146} \]

**Choice of \( \beta \):** Let us now choose \( \beta \) in terms of \( \eta \). Recalling that \( \epsilon_2 < \eta^{2} \epsilon_3 \) and \( \tag{9.74} \), we have
\[ \gamma^{\frac{d-1}{2}} < \eta^{\frac{d-1}{2}}. \tag{9.147} \]

Moreover, since \( \eta < 1 \), we may assume
\[ \frac{\eta}{\sqrt{2}} \leq \sin \eta \leq \eta. \tag{9.148} \]

Since \( \gamma < \eta^{2} \), \( \tag{9.148} \) implies
\[ \gamma < 2 \sin^{2} \eta \Rightarrow \arccos \sqrt{1 - \frac{\gamma'}{2}} < \eta. \tag{9.149} \]

Choosing
\[ \beta = \eta^{1/2} < 1, \tag{9.150} \]
estimates \( \tag{9.146} \)-\( \tag{9.147} \), \( \tag{9.149} \) imply
\[ |A_{m}(Z_{m})| \leq R^{2d} \left( \eta^{\frac{d-1}{2}} + \eta^{1/2} \right) \leq R^{2d} \eta^{\frac{d-1}{2}}, \tag{9.151} \]
since \( \eta << 1 \) and \( d \geq 2 \). The claim comes from (9.90)-(9.91) and (9.151).

\[ \square \]

10. Elimination of recollisions

In this section we reduce the convergence proof to comparing truncated elementary observables. We first restrict to good configurations and provide the corresponding measure estimate. This is happening in Proposition 10.2. We then inductively apply Proposition 9.2 and Proposition 9.3 or Proposition 9.6 and Proposition 9.7 (depending on whether the adjunction is binary or ternary) to reduce the convergence proof to truncated elementary observables. The convergence proof, completed in Section 11, will then follow naturally, since the backwards proof to truncated elementary observables. The convergence proof, completed in Section 11, will then follow naturally, since the backwards proof to truncated elementary observables.

10.1. Restriction to good configurations. Inductively using Lemma 9.1 we are able to reduce the convergence proof to good configurations, up to a small measure set. The measure of the complement will be negligible in the limit.

For convenience, given \( m \in \mathbb{N} \), let us define the set

\[ G_m(\epsilon_3, \epsilon_0, \delta) := G_m(\epsilon_3, 0) \cap G_m(\epsilon_0, \delta). \]  

(10.1)

For \( s \in \mathbb{N} \), we also recall from (6.1) the set \( \Delta^s_N(\epsilon_0) \) of well-separated spatial configurations.

**Lemma 10.1.** Let \( s \in \mathbb{N} \). Let \( s \in \mathbb{N} \), \( \alpha, \epsilon_0, R, \eta, \delta \) be parameters as in (9.90) and \( \epsilon_2 << \epsilon_3 << \alpha \). Then for any \( X_s \in \Delta^s_N(\epsilon_0) \), there is a subset of velocities \( M_s(X_s) \subseteq B^d_{\eta, \delta} \) of measure

\[ |M_s(X_s)|_{ds} \leq C_{d,s,R,d} \eta \frac{d}{d^2}, \]  

(10.2)

such that

\[ Z_s \in G_s(\epsilon_3, \epsilon_0, \delta), \quad \forall V_s \in B^d_{\eta, \delta} \setminus M_s(X_s). \]  

(10.3)

**Proof.** We use Proposition 11.2. from [2] for \( \epsilon = \epsilon_3 \). \( \square \)

For \( s \in \mathbb{N} \) and \( X_s \in \Delta^s_N(\epsilon_0) \), let us denote \( M^s_s(X_s) = B^d_{\eta, \delta} \setminus M_s(X_s) \). Consider \( 1 \leq k \leq n \) and let us recall the observables \( I^s_{k,R,\delta}, I^\infty_{k,R,\delta} \) defined in (7.18)-(7.24). We restrict the domain of integration to velocities giving good configurations.

In particular, we define

\[ \tilde{I}^s_{k,R,\delta}(t)(X_s) = \int_{M^s_s(X_s)} \phi_s(V_s) f^{(s,k)}_{N,R,\delta}(X_s, V_s) dV_s, \]  

(10.4)

\[ \tilde{I}^\infty_{k,R,\delta}(t)(X_s) = \int_{M^s_s(X_s)} \phi_s(V_s) f^{(s,k)}_{R,\delta}(X_s, V_s) dV_s. \]  

(10.5)

Let us apply Proposition 10.1 to restrict to initially good configurations. To keep track of all the possible adjunctions we recall the notation from (7.13)-(7.15): given \( k \in \mathbb{N} \), we write

\[ S_k = \{ \sigma = (\sigma_1, ..., \sigma_k) : \sigma_i \in \{1,2\} \}. \]

and given \( \sigma \in S_k \), we write

\[ \tilde{\sigma}_\ell = \sum_{i=1}^\ell \sigma_i, \quad 1 \leq \ell \leq k, \quad \tilde{\sigma}_0 = 0. \]

**Proposition 10.2.** Let \( s, n \in \mathbb{N} \), \( \alpha, \epsilon_0, R, \eta, \delta \) be parameters as in (9.90), \( (N, \epsilon_2, \epsilon_3) \) in the scaling (4.4) with \( \epsilon_2 << \epsilon_3 << \alpha \), and \( t \in [0, T] \). Then, the following estimates hold:

\[ \sum_{k=1}^n \| I^s_{k,R,\delta}(t) - \tilde{I}^s_{k,R,\delta}(t) \|_{L^\infty(\Delta^s_N(\epsilon_0))} \leq C_{d,s,R,d} \eta \frac{d}{d^2} \| F_{N,0} \|_{N, \beta_0, \mu_0}, \]  

\[ \sum_{k=1}^n \| I^\infty_{k,R,\delta}(t) - \tilde{I}^\infty_{k,R,\delta}(t) \|_{L^\infty(\Delta^s_N(\epsilon_0))} \leq C_{d,s,R,d} \eta \frac{d}{d^2} \| F_{0} \|_{\infty, \beta_0, \mu_0}. \]
Proof. We present the proof for the BBGKY hierarchy case only. The proof for the Boltzmann hierarchy case is similar. Let us fix \( X_s \in \Delta^X_c(\epsilon_0) \).

We first assume that \( k \in \{1, \ldots, n\} \). Triangle inequality, an inductive application of estimate (5.7), estimate (5.6) and part (ii) of Proposition 5.3 yield

\[
|I_{s,k,R,\delta}^N(t)(X_s) - \tilde{I}_{s,k,R,\delta}^N(t)(X_s)| \leq \sum_{\sigma \in S_k} \int_{M_s(X_s)} |\phi_s(V_{\sigma})f_{s,k,R,\delta}^{s,k,\sigma}(t, X_s, V_{\sigma})|dV_s
\]

where to obtain (10.6), we use (7.4).

For \( k = 0 \), part (i) of Proposition 5.3 and Remark 5.1 similarly yield

\[
|I_{s,0,R,\delta}^N(t)(X_s) - \tilde{I}_{s,0,R,\delta}^N(t)(X_s)| \leq \|\phi_s\|L_{\nu_s}e^{s\mu(T)}\|M_s(X_s)\|dV_s \|F_{N,0}\|_{N,\beta_0,\mu_0},
\]

where we obtain (10.6), we use (7.3).

The claim comes after using (10.7)-(10.8), adding over \( k = 0, \ldots, n \), and using the measure estimate of Proposition 10.1.

Remark 10.3. Given \( s \in \mathbb{N} \) and \( X_s \in \Delta^X_c(\epsilon_0) \), the definition of \( M_s(X_s) \) implies that

\[
\tilde{I}_{s,0,R,\delta}^N(t)(X_s) = \tilde{I}_{s,0,R,\delta}^\infty(t)(X_s).
\]

Therefore, by Proposition 10.2, convergence reduces to controlling the differences \( \tilde{I}_{s,k,R,\delta}^N(t) - \tilde{I}_{s,k,R,\delta}^\infty(t) \), for \( k = 1, \ldots, n \), in the scaled limit.

10.2. Reduction to elementary observables. Here, given \( s \in \mathbb{N} \) and \( 1 \leq k \leq n \), we express the observables \( \tilde{I}_{s,k,R,\delta}(t) \), \( \tilde{I}_{s,k,R,\delta}^\infty(t) \), defined in (10.3)-(10.3), as a superposition of elementary observables.

For this purpose, given \( \ell \in \mathbb{N} \), and recalling (7.19), (11.15), we decompose the BBGKY hierarchy binary truncated collisional operator as:

\[
C_{\ell,\ell+1}^{N,R} = \sum_{i=1}^{\ell} C_{\ell,\ell+1}^{N,R,+i} - \sum_{i=1}^{\ell} C_{\ell,\ell+1}^{N,R,-i},
\]

where

\[
C_{\ell,\ell+1}^{N,R,+i} g_{\ell+1}(Z_{\ell}) = A_{N,s,2,\ell} \int_{B_1^\ell \times B_2^\ell} b_2^i(\omega_1, v_{\ell+1} - v_1) g_{\ell+1}(Z_{\ell+1,s,2})d\omega_1 d\nu_{\ell+1},
\]

\[
C_{\ell,\ell+1}^{N,R,-i} g_{\ell+1}(Z_{\ell}) = A_{N,s,2,\ell} \int_{B_1^\ell \times B_2^\ell} b_2^i(\omega_1, v_{\ell+1} - v_1) g_{\ell+1}(Z_{\ell+1,s,2})d\omega_1 d\nu_{\ell+1},
\]

and the ternary truncated collisional operator as:

\[
C_{\ell,\ell+2}^{N,R} = \sum_{i=1}^{\ell} C_{\ell,\ell+2}^{N,R,+i} - \sum_{i=1}^{\ell} C_{\ell,\ell+2}^{N,R,-i},
\]

where

\[
C_{\ell,\ell+2}^{N,R,+i} g_{\ell+2}(Z_{\ell}) = A_{N,s,3,\ell} \int_{B_1^{2\ell} \times B_2^{2\ell}} \frac{b_2^i(\omega_1, \omega_2, v_{\ell+1} - v_1, v_{\ell+2} - v_2)}{\sqrt{1 + \langle \omega_1, \omega_2 \rangle}} g_{\ell+2}(Z_{\ell+2,s,3})d\omega_1 d\omega_2 d\nu_{\ell+1} d\nu_{\ell+2},
\]

\[
C_{\ell,\ell+2}^{N,R,-i} g_{\ell+2}(Z_{\ell}) = A_{N,s,3,\ell} \int_{B_1^{2\ell} \times B_2^{2\ell}} \frac{b_2^i(\omega_1, \omega_2, v_{\ell+1} - v_1, v_{\ell+2} - v_2)}{\sqrt{1 + \langle \omega_1, \omega_2 \rangle}} g_{\ell+2}(Z_{\ell+2,s,3})d\omega_1 d\omega_2 d\nu_{\ell+1} d\nu_{\ell+2}.
\]

In order to expand the observable \( \tilde{I}_{s,k,R,\delta}(t) \) to elementary observables, we need to take into account all the possible particle adjuctions occurring by adding one or two particles to the system in each step. More precisely, given \( \sigma \in S_k \), and \( i \in \{1, \ldots, k\} \), we are adding \( \sigma_1 \in \{1, 2\} \) particle(s) to the existing \( s + \sigma_{i-1} \) particles.
in either precollisional or postcollisional way. In order to keep track of this process, given $1 \leq k \leq n$, $\sigma \in S_k$, we introduce the notation

$$M_{s,k,\sigma} = \left\{ M = (m_1, ..., m_k) \in \mathbb{N}^k : m_i \in \{1, ..., s + \tilde{\sigma}_{i-1}\}, \quad \forall i \in \{1, ..., k\} \right\},$$

$$J_{s,k,\sigma} = \left\{ J = (j_1, ..., j_k) \in \mathbb{N}^k : j_i \in \{-1, 1\}, \quad \forall i \in \{1, ..., k\} \right\}.$$

$$U_{s,k,\sigma} = J_{s,k,\sigma} \times M_{s,k,\sigma}.$$

Under this notation, the BBGKY hierarchy observable functional $\tilde{T}_{s,k,R,\delta}(t)$ can be expressed, for $1 \leq k \leq n$, as a superposition of elementary observables

$$\tilde{T}_{s,k,R,\delta}(t)(X_s) = \sum_{\sigma \in S_k(J,M)} \sum_{(i)} \left( \prod_{i=1}^{k} j_i \right) \tilde{T}_{s,k,R,\delta,\sigma}^N(t, J, M)(X_s),$$

where the elementary observables are defined by

$$\tilde{T}_{s,k,R,\delta,\sigma}^N(t, J, M)(X_s) = \int_{M_{s,k,\sigma}} \phi_s(V_s) \int_{J_{s,k,\sigma}} \left( \tilde{T}_s^{t-t_1} \mathcal{C}^{N,R,j_1,m_1} T_s^{t_1-t_2} \cdots \right) \left( \tilde{T}_s^{t_{k-1}} \mathcal{C}^{N,R,j_k,m_k} T_s^{t_k-t_1} \right) (Z_s) \, dt_1 ... \, dt_k \, dV_s.$$

Similarly, given $\ell \in \mathbb{N}$, and recalling (4.31), (4.35), we decompose the Boltzmann hierarchy binary and ternary collisional operators as:

$$\mathcal{C}_{\ell,\ell+1}^{\infty,R} = \sum_{i=1}^{\ell} \mathcal{C}_{\ell,\ell+1}^{\infty,R,+i} - \sum_{i=1}^{\ell} \mathcal{C}_{\ell,\ell+1}^{\infty,R,-i},$$

where

$$\mathcal{C}_{\ell,\ell+1}^{\infty,R,+i} g_{\ell+1}(Z_{\ell}) = \int_{S_2^d \times B^d_R} b^i_{\ell+1} (\omega_1, v_{\ell+1} - v_1) g_{\ell+1}(Z_{\ell+1}) \, d\omega_1 \, dv_{\ell+1},$$

$$\mathcal{C}_{\ell,\ell+1}^{\infty,R,-i} g_{\ell+1}(Z_{\ell}) = \int_{S_2^d \times B^d_R} b^i_{\ell+1} (\omega_1, v_{\ell+1} - v_1) g_{\ell+1}(Z_{\ell+1}) \, d\omega_1 \, dv_{\ell+1},$$

$$\mathcal{C}_{\ell,\ell+2}^{\infty,R} = \sum_{i=1}^{\ell} \mathcal{C}_{\ell,\ell+2}^{\infty,R,+i} - \sum_{i=1}^{\ell} \mathcal{C}_{\ell,\ell+2}^{\infty,R,-i},$$

where

$$\mathcal{C}_{\ell,\ell+2}^{\infty,R,+i} g_{\ell+2}(Z_{\ell}) = \int_{S_1^d \times B^d_R} \frac{b^i_{\ell+1} (\omega_1, \omega_2, v_{\ell+1} - v_1, v_{\ell+2} - v_1)}{1 + \langle \omega_1, \omega_2 \rangle} g_{\ell+2}(Z_{\ell+2}) \, d\omega_1 \, dv_{\ell+1} \, dv_{\ell+2},$$

$$\mathcal{C}_{\ell,\ell+2}^{\infty,R,-i} g_{\ell+2}(Z_{\ell}) = \int_{S_1^d \times B^d_R} \frac{b^i_{\ell+1} (\omega_1, \omega_2, v_{\ell+1} - v_1, v_{\ell+2} - v_1)}{1 + \langle \omega_1, \omega_2 \rangle} g_{\ell+2}(Z_{\ell+2}) \, d\omega_1 \, dv_{\ell+1} \, dv_{\ell+2}.$$

Under this notation, the Boltzmann hierarchy observable functional $\tilde{T}_{s,k,R,\delta}^\infty(t)$ can be expressed, for $1 \leq k \leq n$, as a superposition of elementary observables

$$\tilde{T}_{s,k,R,\delta}(t)(X_s) = \sum_{\sigma \in S_k(J,M)} \sum_{(i)} \left( \prod_{i=1}^{k} j_i \right) \tilde{T}_{s,k,R,\delta,\sigma}^\infty(t, J, M)(X_s),$$

where the elementary observables are defined by

$$\tilde{T}_{s,k,R,\delta,\sigma}^\infty(t, J, M)(X_s) = \int_{M_{s,k,\sigma}} \phi_s(V_s) \int_{J_{s,k,\sigma}} \left( \tilde{T}_s^{t-t_1} \mathcal{C}_{\ell,t+1}^{\infty,R,j_1,m_1} \mathcal{C}_{\ell,t+2}^{\infty,R,j_2,m_2} \cdots \right) \left( \tilde{T}_s^{t_{k-1}} \mathcal{C}_{\ell,t+1}^{\infty,R,j_k,m_k} \mathcal{C}_{\ell,t+2}^{\infty,R,j_{k+1},m_{k+1}} \cdots \right) (Z_s) \, dt_1 ... \, dt_k \, dV_s.$$
10.3. Boltzmann hierarchy pseudo-trajectories. We introduce the following notation which we will be constantly using from now on. Let \( s \in \mathbb{N}, Z_s = (X_s, V_s) \in \mathbb{R}^{2ds}, 1 \leq k \leq n, \sigma \in \mathbb{S}_k \) and \( t \in [0, T] \). Let us recall from (7.2) the set

\[
\mathcal{T}_k(t) = \left\{ (t_1, ..., t_k) \in \mathbb{R}^k : 0 = t_{k+1} < t_k < ... < t_1 < t_0 = t \right\}, \quad t_0 = t, \ t_{k+1} = 0.
\]

Consider \( (t_1, ..., t_k) \in \mathcal{T}_k(t) \), \( J = (j_1, ..., j_k), M = (m_1, ..., m_k), (J, M) \in \mathbb{U}_{s, k, \sigma} \). For each \( i = 1, ..., k \), we distinguish two possible situations:

\[
\begin{align*}
\text{If } \sigma_i &= 1, \text{ we consider } (\omega_{s+\vec{\sigma}_i}, v_{s+\vec{\sigma}_i}) \in \mathbb{S}^{-1}_1 \times \mathbb{B}^1_R. \\
\text{If } \sigma_i &= 2, \text{ we consider } (\omega_{s+\vec{\sigma}_{i-1}}, \omega_{s+\vec{\sigma}_i}, v_{s+\vec{\sigma}_{i-1}}, v_{s+\vec{\sigma}_i}) \in \mathbb{S}^{-2}_1 \times \mathbb{B}^2_R.
\end{align*}
\]

(10.16) For convenience, for each \( i = 1, ..., k \), we will write \((\omega_{s, i}, v_{s, i}) \in \mathbb{S}^{d_{s, i}}_1 \times \mathbb{B}^{d_{s, i}}_R \) where \((\omega_{s, i}, v_{s, i})\) is of the form \([\text{10.16}]\) if \( \sigma_i = 1 \) and of the form \([\text{10.17}]\) if \( \sigma_i = 2 \).

We inductively define the Boltzmann hierarchy pseudo-trajectory of \( Z_s \). Roughly speaking, the Boltzmann hierarchy pseudo-trajectory forms the configurations on which particles are adjusted during backwards in time evolution.

Intuitively, assume we are given a configuration \( Z_s = (X_s, V_s) \in \mathbb{R}^{2ds} \) at time \( t_0 = t \). \( Z_s \) evolves under backwards free flow until the time \( t_1 \) when the configuration \((\omega_{s, 1}, v_{s, 1})\) is added, neglecting positions, to the \( m_1 \)-particle, the adjunction being precollisional if \( j_1 = -1 \) and postcollisional if \( j_1 = 1 \). Then we form an \((s + \vec{\sigma}_1)\)-configuration and continue this process inductively until time \( t_{k+1} = 0 \). More precisely, we inductively construct the Boltzmann hierarchy pseudo-trajectory of \( Z_s = (X_s, V_s) \in \mathbb{R}^{2ds} \) as follows:

**Time** \( t_0 = t \): We initially define \( Z_{s, \infty}^\infty(t_0) = (x_{s, \infty}^\infty(t_0), ..., x_{s, \infty}^\infty(t_0), v_{s, \infty}^\infty(t_0), ..., v_{s, \infty}^\infty(t_0)) \) := \( Z_s \).

**Time** \( t_1, i \in \{1, ..., k\} \): Consider \( i \in \{1, ..., k\} \) and assume we know

\[
Z_{s+\vec{\sigma}_{i-1}}^\infty(t_{i-1}) = (x_{s+\vec{\sigma}_{i-1}}^\infty(t_{i-1}), ..., x_{s+\vec{\sigma}_{i-1}}^\infty(t_{i-1}), v_{s+\vec{\sigma}_{i-1}}^\infty(t_{i-1}), ..., v_{s+\vec{\sigma}_{i-1}}^\infty(t_{i-1})).
\]

We define \( Z_{s+\vec{\sigma}_{i-1}}^\infty(t_i) = (x_{s+\vec{\sigma}_{i-1}}^\infty(t_i), ..., x_{s+\vec{\sigma}_{i-1}}^\infty(t_i), v_{s+\vec{\sigma}_{i-1}}^\infty(t_i), ..., v_{s+\vec{\sigma}_{i-1}}^\infty(t_i)) \) as:

\[
Z_{s+\vec{\sigma}_{i-1}}^\infty(t_i) := (X_{s+\vec{\sigma}_{i-1}}(t_i) - (t_i - t_i)V_{s+\vec{\sigma}_{i-1}}(t_i), V_{s+\vec{\sigma}_{i-1}}^\infty(t_i), ..., V_{s+\vec{\sigma}_{i-1}}^\infty(t_i)).
\]

We also define \( Z_{s+\vec{\sigma}_i}^\infty(t_i) = (x_{s+\vec{\sigma}_i}^\infty(t_i), ..., x_{s+\vec{\sigma}_i}^\infty(t_i), v_{s+\vec{\sigma}_i}^\infty(t_i), ..., v_{s+\vec{\sigma}_i}^\infty(t_i)) \) as:

\[
(x_{s+\vec{\sigma}_i}^\infty(t_i), v_{s+\vec{\sigma}_i}^\infty(t_i)) := (x_{s+\vec{\sigma}_i}^\infty(t_i), v_{s+\vec{\sigma}_i}^\infty(t_i)), \quad \forall j \in \{1, ..., s + \vec{\sigma}_{i-1}\} \setminus \{m_i\},
\]

For the rest of the particles, we distinguish the following cases, depending on \( \sigma_i \):

- **\( \sigma_i = 1 \):** If \( j_1 = -1 \):

\[
(x_{s+\vec{\sigma}_i}^\infty(t_i), v_{s+\vec{\sigma}_i}^\infty(t_i)) := (x_{s+\vec{\sigma}_i}^\infty(t_i), v_{s+\vec{\sigma}_i}^\infty(t_i)).
\]

while if \( j_1 = 1 \):

\[
(x_{s+\vec{\sigma}_i}^\infty(t_i), v_{s+\vec{\sigma}_i}^\infty(t_i)) := (x_{s+\vec{\sigma}_i}(t_i), v_{s+\vec{\sigma}_i}(t_i)).
\]

- **\( \sigma_i = 2 \):** If \( j_1 = -1 \):

\[
(x_{s+\vec{\sigma}_i}^\infty(t_i), v_{s+\vec{\sigma}_i}^\infty(t_i)) := (x_{s+\vec{\sigma}_i}^\infty(t_i), v_{s+\vec{\sigma}_i}^\infty(t_i)).
\]

while if \( j_1 = 1 \):

\[
(x_{s+\vec{\sigma}_i}^\infty(t_i), v_{s+\vec{\sigma}_i}^\infty(t_i)) := (x_{s+\vec{\sigma}_i}(t_i), v_{s+\vec{\sigma}_i}(t_i), v_{s+\vec{\sigma}_i}^\infty(t_i)),
\]

where \((v_{s+\vec{\sigma}_i}^\infty(t_i), v_{s+\vec{\sigma}_i}^\infty(t_i)) = T_{\sigma_i+\vec{\sigma}_i}(v_{s+\vec{\sigma}_i}^\infty(t_i), v_{s+\vec{\sigma}_i}).
\]
Time $t_{k+1} = 0$: We finally obtain

$$Z_{s+\bar{\sigma}}^\infty(0^+)^t = Z_{s+\bar{\sigma}}^\infty(t_{k+1}^+) = (X_{s+\bar{\sigma}}^\infty(t_k) - t_k V_{s+\bar{\sigma}}^\infty(t_k^+), V_{s+\bar{\sigma}}^\infty(t_k^+))$$.

The process is illustrated in the following diagram:

\[ \begin{array}{ccccccc}
Z_{s+\bar{\sigma}}^\infty(t_{k+1}^+) & \rightarrow & Z_{s+\bar{\sigma}}^\infty(t_1^+) & \rightarrow & \cdots & \rightarrow & Z_{s+\bar{\sigma}}^\infty(t_0^+)
\end{array} \]

We give the following definition:

**Definition 10.4.** Let $s \in \mathbb{N}$, $Z_s = (X_s, V_s) \in \mathbb{R}^{2d_s}$, $(t_1, ..., t_k) \in \mathcal{T}_k(t)$, $J = (j_1, ..., j_k)$, $M = (m_1, ..., m_k)$, $(J, M) \in U_{s+k}$ and for each $i = 1, ..., k$, $\sigma \in S_{k_i}$, we consider $(\omega_{\sigma, i}, v_{\sigma, i}) \in S^{d_{\sigma, i} - 1}_1 \times B^R_{d_{\sigma, i}}$. The sequence $\{Z_{s+\bar{\sigma}}^\infty(t_i^+)^t\}_{i=0,...,k+1}$ constructed above is called the Boltzmann hierarchy pseudo-trajectory of $Z_s$.

10.4. Reduction to truncated elementary observables. We will now use the Boltzmann hierarchy pseudo-trajectory to define the BBGKY hierarchy and Boltzmann hierarchy truncated observables. The convergence proof will then be reduced to the convergence of the corresponding truncated observables.

Given $\ell \in \mathbb{N}$, recall the notation from (10.1):

$$G_{\ell}(\epsilon_3, \epsilon_0, \delta) = G_{\ell}(\epsilon_3, 0) \cap G_{\ell}(\epsilon_0, \delta)$$.

Given $t \in [0, T]$, we also recall from (7.22) the set $\mathcal{T}_k(t)$ of separated collision times:

$$\mathcal{T}_k(t) := \{(t_1, ..., t_k) \in \mathcal{T}_k(t) \mid 0 \leq t_{i+1} - t_i \leq \delta, \forall i \in [0, k], t_{k+1} = 0, t_0 = t \}$$.

Consider $t \in [0, T]$, $X_s \in \Delta^\epsilon_s(\epsilon_0)$, $1 \leq k \leq n$, $\sigma \in S_k$, and $(J, M) \in U_{s+k}$ and $(t_1, ..., t_k) \in \mathcal{T}_k(t)$. By Proposition 10.4 for any $V_s \in M_s^0(X_s)$, we have $Z_s = (X_s, V_s) \in G_s(\epsilon_3, \epsilon_0, \delta)$ which in turn implies $Z_{s+\bar{\sigma}}^\infty(t_1^+)^t \in G_s(\epsilon_0, 0)$ since $t_0 - t_1 > \delta$. Now we observe that either (9.9), (9.13) from Proposition 9.2 (if the adjunction is binary), or (9.62), (9.66) from Proposition 9.6 (if the adjunction is ternary), yield that there is a set $B_{m_1}(Z_{s+\bar{\sigma}}^\infty(t_1^+)^t) \subseteq S^{d_{\sigma, i} - 1}_1 \times B^R_{d_{\sigma, i}}$ such that:

$$Z_{s+\bar{\sigma}}^\infty(t_1^+)^t \subseteq G_{s+\bar{\sigma}}(\epsilon_0, 0), \forall (\omega_{\sigma, i}, v_{\sigma, i}) \in B_{m_1}(Z_{s+\bar{\sigma}}^\infty(t_1^+)^t)$$,

$$B_{m_1}(Z_{s+\bar{\sigma}}^\infty(t_1^+)^t) := (S^{d_{\sigma, i} - 1}_1 \times B^R_{d_{\sigma, i}})^+ (v_{\sigma, i}) \backslash B_{m_1}(Z_{s+\bar{\sigma}}^\infty(t_1^+)^t)$$.

Clearly this process can be iterated. In particular, given $i \in \{2, ..., k\}$, we have

$$Z_{s+\bar{\sigma}}^\infty(0^+)^t \in G_{s+\bar{\sigma}}(\epsilon_0, 0),$$

so there exists a set $B_{m_1}(Z_{s+\bar{\sigma}}^\infty(t_{i+1}^+)^t) \subseteq S^{d_{\sigma, i} - 1}_1 \times B^R_{d_{\sigma, i}}$ such that:

$$Z_{s+\bar{\sigma}}^\infty(t_{i+1}^+)^t \subseteq G_{s+\bar{\sigma}}(\epsilon_0, 0), \forall (\omega_{\sigma, i}, v_{\sigma, i}) \in B_{m_1}(Z_{s+\bar{\sigma}}^\infty(t_{i+1}^+)^t)$$,

where

$$B_{m_1}(Z_{s+\bar{\sigma}}^\infty(t_{i+1}^+)^t) := (S^{d_{\sigma, i} - 1}_1 \times B^R_{d_{\sigma, i}})^+ (v_{\sigma, i}) \backslash B_{m_1}(Z_{s+\bar{\sigma}}^\infty(t_{i+1}^+)^t)$$.

We finally obtain $Z_{s+\bar{\sigma}}(0^+)^t \subseteq G_{s+\bar{\sigma}}(\epsilon_0, 0)$.

Let us now define the truncated elementary observables. Heuristically we will truncate the domains of adjusted particles in the definition of the observables $\bar{T}_{s+\bar{\sigma}}^{N}_{s+k, R, \delta}$, $\bar{T}_{s+k, R, \delta}$, defined in (10.4)-(10.5).

More precisely, consider $1 \leq k \leq n$, $\sigma \in S_k$, $(J, M) \in U_{s+k}$ and $t \in [0, T]$. For $X_s \in \Delta^\epsilon_s(\epsilon_0)$, Proposition 10.4 implies there is a set of velocities $M_s(X_s) \subseteq B^R_{d_{\sigma, i}}$ such that $Z_s = (X_s, V_s) \in G_s(\epsilon_3, \epsilon_0, \delta)$, $\forall V \in M_s^0(X_s)$. Following the reasoning above, we define the BBGKY hierarchy truncated observables as:

$$J_{s+k, R, \delta}(s, (J, M))(X_s) = \int_{M_{s+k}(X_s)} \phi_s(V_s) \int_{\mathcal{T}_k(t)} T_{s+\bar{\sigma}}^{t_2-t_1} \bar{T}_{s+\bar{\sigma}}^{N_{s+k,R,j,k,m_1}} T_{s+\bar{\sigma}}^{t_1-t_2} ... \bar{T}_{s+\bar{\sigma}}^{N_{s+k,R,j,k,m,k}} T_{s+\bar{\sigma}}^{t_m-t_0} f_{s+\bar{\sigma}}(Z_s) \, dt_{k-1} ... dt_1 dV_s$$.
where for each $i = 1, ..., k$, we denote
\[
\tilde{C}^{N,R,j_m}_{s+\delta_{i-1},s+\delta_i} g_{N,s+\delta_i} = C^{N,R,j_m}_{s+\delta_{i-1},s+\delta_i} \left[ g_{N,s+\delta_i} \mathbb{I}_{(\omega_{s,i}, v_{s,i}) \in B_{m_i} (Z^\infty_{s+\delta_{i-1}} (t^*_i))} \right].
\]

In the same spirit, for $X_s \in X_0 \in \Delta^N (\epsilon_0)$, we define the Boltzmann hierarchy truncated elementary observables as:
\[
J^{N,\infty}_{s,k,R,s,\sigma, \delta, \sigma, \sigma, \sigma} (t, J, M) (X_s) = \int_M \phi_s (V_s) \int_{T_{s+\delta_i} (t)} X_{k-1} \tilde{C}^{N,R,j_m}_{s+\delta_{i-1},s+\delta_i} ^{s+\delta_{i-1}}... \tilde{C}^{N,R,j_m}_{s+\delta_{k-1},s+\delta_i} ^{s+\delta_{k-2}} (Z_s) dt_k, ..., dt_1 dV_s,
\]
where for each $i = 1, ..., k$, we denote
\[
\tilde{C}^{N,R,j_m}_{s+\delta_{i-1},s+\delta_i} g_{s+\delta_i} = C^{N,R,j_m}_{s+\delta_{i-1},s+\delta_i} \left[ g_{s+\delta_i} \mathbb{I}_{(\omega_{s,i}, v_{s,i}) \in B_{m_i} (Z^\infty_{s+\delta_{i-1}} (t^*_i))} \right].
\]

Recalling the observables $\tilde{T}^{N,\infty}_{s,k,R,s,\sigma, \delta, \sigma, \sigma, \sigma}$ from (10.13), (10.15) and using Proposition 9.4 or Proposition 9.7 we obtain:

**Proposition 10.5.** Let $s, n \in \mathbb{N}$, $\alpha, \epsilon_0, R, \eta, \delta$ be parameters as in (9.3), $(N, \epsilon_2, \epsilon_3)$ in the scaling (4.24) with $\epsilon_2 << \epsilon_3 << \alpha$ and $t \in [0, T]$. Then the following estimates hold:
\[
\sum_{k=1}^{n} \sum_{\sigma \in S_k} \sum_{(J,M) \in U_{k,\sigma}} \| \tilde{T}^{N,\infty}_{s,k,R,s,\sigma, \delta, \sigma, \sigma, \sigma} (t, J, M) - J^{N,\infty}_{s,k,R,s,\sigma, \delta, \sigma, \sigma, \sigma} (t, J, M) \|_{L^\infty (\Delta^N (\epsilon_0))} \leq
\]
\[
\leq C^a_{s,\sigma, \delta, \epsilon_0} R^d \| \tilde{\phi}_s \|_{L^\infty (\Delta^N (\epsilon_0))} R^d \| F_{0,\sigma,\delta, \epsilon_0} \|_{L^\infty (\Delta^N (\epsilon_0))}.
\]

**Proof.** As usual, it suffices to prove the estimate for the BBGKY hierarchy case and the Boltzmann hierarchy case follows similarly. Fix $k \in \{1, ..., n\}$, $\sigma \in S_k$ and $(J,M) \in U_{k,\sigma}$. We first estimate the difference:
\[
\tilde{T}^{N,\infty}_{s,k,R,s,\sigma, \delta, \sigma, \sigma, \sigma} (t, J, M) (X_s) - J^{N,\infty}_{s,k,R,s,\sigma, \delta, \sigma, \sigma, \sigma} (t, J, M) (X_s).
\]

- Cauchy-Schwartz inequality and triangle inequality imply
  \[
  |(\omega_1, v_1 - v_2)| \leq 2R, \quad \forall (\omega_1, v_1) \in S^{d-1}_{1}, \quad \forall (\omega_1, v_1) \in B_{R}^d.
  \]
  \[
  |b_2 (\omega_1, \omega_2, v_1 - v_2)| \leq 4R, \quad \forall (\omega_1, \omega_2) \in S^{d-1}_{1}, \quad \forall (\omega_1, \omega_2, v_1, v_2) \in B_{R}^d.
  \]

- so
  \[
  \int_{S^{d-1}_{1} \times B^d_{R}} |(\omega_1, v_1 - v_2)| d\omega_1 d v_1 \leq C_d R^{d+1} \leq C_d R^d, \quad \forall v \in B^d_{R},
  \]
  \[
  \int_{S^{d-1}_{1} \times B^d_{R}} |b_2 (\omega_1, \omega_2, v_1 - v_2)| d\omega_1 d v_1 d v_2 \leq C_d R^{d+1} \leq C_d R^d, \quad \forall v \in B^d_{R}.
  \]

- since $R >> 1$. But in order to estimate the difference (10.21), we integrate at least once over $B_{m_i} (Z^\infty_{s+\delta_{i-1}} (t^*_i))$ for some $i \in \{1, ..., k\}$. For convenience, given $v \in \mathbb{R}^d$, let us write
  \[
  b_{\sigma_i} (\omega_{s,i}, v_{s,i}) := \begin{cases} b_{2} (\omega_{s,i}, v_{s,i}, v_{s,i} - v), & \text{if } \sigma_i = 1, \\
  b_{3} (\omega_{s,i}, v_{s,i} - v_{s,i}, v_{s,i} - v), & \text{if } \sigma_i = 2.
  \end{cases}
  \]

- Under this notation, (10.22) - (10.23) together with Proposition 9.4 or Proposition 9.7, depending on whether the adunction is binary or ternary, yield the estimate
  \[
  \int_{B_{m_i} (Z^\infty_{s+\delta_{i-1}} (t^*_i))} |b_2 (\omega_{s,i}, v_{s,i})| d\omega_{s,i} v_{s,i} \leq C_d (s+\delta_{i-1}) R^{d+1} \eta \frac{d-1}{d+2}, \quad \forall v \in B^d_{R}.
  \]
since $R >> 1$ and $\eta << 1$.

Moreover, we have the elementary inequalities:

$$
\| J_{N,0}^{(s+1)} \|_{L^\infty} \leq e^{-(s+1)\mu_0} \| F_{N,0} \|_{N,\beta_0,\mu_0} \leq e^{-(s+1)\mu_0} \| F_{N,0} \|_{N,\beta_0,\mu_0},
$$

(10.28)

$$
\int_{T_{k,\delta}(t)} dt_{1} \cdots dt_{k} \leq \int_{0}^{t} \int_{0}^{t_1} \cdots \int_{0}^{t_{k-1}} dt_{1} \cdots dt_{k} = \frac{t^k}{k!} \leq \frac{T^k}{k!}.
$$

(10.29)

Therefore, (10.24)-(10.29) imply

$$
[\bar{F}_{k,R,\delta,\sigma}^N(t, J, M)(X_s) - \bar{J}_{k,R,\delta,\sigma}^N(t, J, M)(X_s)]
$$

$$
\leq \| \phi_s \|_{L^\infty} e^{-(s+1)\mu_0} \| F_{N,0} \|_{N,\beta_0,\mu_0} C_d R^d C_d^{k-1} R^{3d(k-1)} (s + 2k) C_d R^{3d} \eta \frac{d-1}{k!} \frac{T^k}{k!}.
$$

Adding for all $(J, M) \in U_{s,k}$ we have $2^k (s + \sigma_1) \cdots (s + \sigma_{k-1}) \leq 2^k (s + 2k)^k$ contributions, thus

$$
\sum_{(J, M) \in U_{s,k}} \| \bar{F}_{k,R,\delta,\sigma}^N(t, J, M) - \bar{J}_{k,R,\delta,\sigma}^N(t, J, M) \|_{L^\infty}(\Delta \mathcal{N}(\epsilon))
$$

$$
\leq C_{d,k,\mu_0,T} \| \phi_s \|_{L^\infty} R^{d(3s+3k)} \eta \frac{d+1}{k!} \frac{T^k}{k!} \| F_{N,0} \|_{N,\beta_0,\mu_0},
$$

since

$$
\frac{(s + 2k)^k}{k!} \leq \frac{2^{k+1} (s + k)^{s+k}}{k!} \leq 2^{k+1} (s + k)^{s+k} \leq C_s,
$$

Summing over $\sigma \in S_k$, $k = 1, \ldots, n$, we get the required estimate. \hfill \square

In the next section, in order to conclude the convergence proof, we will estimate the differences of the corresponding BBGKY hierarchy and Boltzmann hierarchy truncated elementary observables in the scaled limit.

11. Convergence Proof

Recall from Subsection 10.3 that given $s \in \mathbb{N}$, $t \in [0, T]$, and parameters satisfying (9.5), we have reduced the convergence proof to controlling the differences:

$$
\bar{J}_{s,k,R,\delta}(t, J, M) - \bar{J}_{s,k,R,\delta}^\infty(t, J, M)
$$

for given $1 \leq k \leq n$ and $(J, M) \in U_{s,k}$, where $\bar{J}_{s,k,R,\delta}(t, J, M)$, $\bar{J}_{s,k,R,\delta}^\infty(t, J, M)$ are given by (10.29), (10.30). This will be the aim of this section.

Throughout this section $s \in \mathbb{N}$, $\phi_s \in C_c(\mathbb{R}^{d_s})$ will be fixed, $(N, \epsilon_2, \epsilon_3)$ are in the scaling (4.24), $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$, $T > 0$ are given by the statements of Theorem 5.7 and Theorem 5.14 and the parameters $n, \delta, R, \eta, \epsilon_0, \alpha$ satisfy (4.5).

11.1. BBGKY hierarchy pseudo-trajectories and proximity to the Boltzmann hierarchy pseudo-trajectories. In the same spirit as in Subsection 10.3 we may define the BBGKY hierarchy pseudo-trajectory. Consider $s \in \mathbb{N}$, $(N, \epsilon_2, \epsilon_3)$ in the scaling (4.24), $k \in \mathbb{N}$ and $t \in [0, T]$. Let us recall from (7.22) the set

$$
T_{k}(t) = \left\{ (t_1, \ldots, t_k) \in \mathbb{R}^k : 0 = t_{k+1} < t_k < \cdots < t_1 < t_0 = t \right\},
$$

where we use the convention $t_0 = t$ and $t_{k+1} = 0$. Consider $(t_1, \ldots, t_k) \in T_{k}(t)$, $\sigma \in S_k$, $J = (j_1, \ldots, j_k)$, $M = (m_1, \ldots, m_k)$, $(J, M) \in U_{s,k,\sigma}$ and for each $i = 1, \ldots, k$, we consider $(\omega_{i,s}, \upsilon_{i,s}, \lambda) \in S_{d_s,-1} \times B_{d_s}^{\sigma}.$

The process followed is similar to the construction of the Boltzmann hierarchy pseudo-trajectory. The only difference is that we take into account the diameter $\epsilon_2$ or the interaction zone $\epsilon_3$ of the adjusted particles in each step.

More precisely, we inductively construct the BBGKY hierarchy pseudo-trajectory of $Z_s = (X_s, V_s) \in \mathbb{R}^{2d_s}$ as follows:

**Time** $t_0 = t$: We initially define $Z_s^N(t_0) = (x_s^N(t_0), \ldots, x_s^N(t_0), v_s^N(t_0), \ldots, v_s^N(t_0)) := Z_s.$
Time $t_i$, $i \in \{1, ..., k\}$: Consider $i \in \{1, ..., k\}$ and assume we know
\[
Z^N_{i+\sigma_i-1}(t_{i-1}) = \left( x^N_i(t_{i-1}), ..., x^N_{i+\sigma_i-1}(t_{i-1}), v^N_i(t_{i-1}), ..., v^N_{i+\sigma_i-1}(t_{i-1}) \right).
\]
We define $Z^N_{i+\sigma_i-1}(t_i) = \left( x^N_i(t_i), ..., x^N_{i+\sigma_i-1}(t_i), v^N_i(t_i), ..., v^N_{i+\sigma_i-1}(t_i) \right)$ as:
\[
Z^N_{i+\sigma_i-1}(t_i) := \left( X^N_{i+\sigma_i-1}(t_{i-1}) - (t_{i-1} - t_i) V^N_{i+\sigma_i-1}(t_{i-1}), V^N_{i+\sigma_i-1}(t_{i-1}) \right).
\]
We also define $Z^N_{i+\sigma_i-1}(t_i) = \left( x^N_i(t_i), ..., x^N_{i+\sigma_i-1}(t_i), v^N_i(t_i), ..., v^N_{i+\sigma_i-1}(t_i) \right)$ as:
\[
\left( x^N_i(t_i), v^N_i(t_i) \right) := \left( x^N_j(t_j), v^N_j(t_j) \right) \quad \forall j \in \{1, ..., s+\sigma_i-1\} \setminus \{m_i\},
\]
For the rest of the particles, we distinguish the following cases, depending on $\sigma_i$:
- $\sigma_i = 1$: If $j_i = -1$:
  \[
  \left( x^N_{m_i}(t_i), v^N_{m_i}(t_i) \right) := \left( x^N_{m_i}(t_i), v^N_{m_i}(t_i) \right),
  \]
  while if $j_i = 1$:
  \[
  \left( x^N_{m_i}(t_i), v^N_{m_i}(t_i) \right) := \left( x^N_{m_i}(t_i) - \epsilon 2\omega_{i+\sigma_i, i}, v^N_{i+\sigma_i} \right),
  \]
  where $(v^N_{m_i}(t_i), v^N_{i+\sigma_i}) = T_{\omega_{i+\sigma_i}} \left( v^N_{m_i}(t_i), v^N_{i+\sigma_i} \right)$.
- $\sigma_i = 2$: If $j_i = -1$:
  \[
  \left( x^N_{m_i}(t_i), v^N_{m_i}(t_i) \right) := \left( x^N_{m_i}(t_i) + \epsilon 2\omega_{i+\sigma_i, i}, v^N_{i+\sigma_i} \right),
  \]
  while if $j_i = 1$:
  \[
  \left( x^N_{m_i}(t_i), v^N_{m_i}(t_i) \right) := \left( x^N_{m_i}(t_i), v^N_{m_i}(t_i) \right),
  \]
  \[
  \left( x^N_{i+\sigma_i-1}(t_i), v^N_{m_i}(t_i) \right) := \left( x^N_{m_i}(t_i) - \epsilon 2\omega_{i+\sigma_i, i}, v^N_{i+\sigma_i-1} \right),
  \]
  \[
  \left( x^N_{i+\sigma_i-1}(t_i), v^N_{i+\sigma_i-1}(t_i) \right) := \left( x^N_{i+\sigma_i}(t_i) + \epsilon 2\omega_{i+\sigma_i, i}, v^N_{i+\sigma_i-1} \right),
  \]
  where $(v^N_{m_i}(t_i), v^N_{i+\sigma_i-1}, v^N_{i+\sigma_i}) = T_{\omega_{i+\sigma_i-1}} \left( v^N_{m_i}(t_i), v^N_{i+\sigma_i-1}, v^N_{i+\sigma_i} \right)$.

Time $t_{k+1}$: We finally obtain
\[
Z^N_{i+\sigma_i-1}(t_{k+1}) = Z^N_{i+\sigma_i-1}(t_k) = \left( X^N_{i+\sigma_i}(t_k) - t_k V^N_{i+\sigma_i}(t_k), V^N_{i+\sigma_i}(t_k) \right).
\]
The process is illustrated in the following diagram:

We give the following definition:

**Definition 11.1.** Let $s \in \mathbb{N}$, $Z_s = \left( x_s, v_s \right) \in \mathbb{R}^{2d_s}$, $(t_1, ..., t_k) \in T_b(t), J = (j_1, ..., j_k)$, $M = (m_1, ..., m_k)$, $(J, M) \in U_{d_s}$ and for each $i = 1, ..., k$, $\sigma \in S_k$, we consider $(\omega_{\sigma_i, i}, v_{\sigma_i, i}) \in S_{d_s}^{1, -1} \times B_{d_s}^r$. The sequence \( \{Z^N_{i+\sigma_i-1}(t_i)\}_{i=0, ..., k+1} \) constructed above is called the BBGKY hierarchy pseudo-trajectory of $Z_s$.

We now state the following elementary proximity result of the corresponding BBGKY hierarchy and Boltzmann hierarchy pseudo-trajectories.
Lemma 11.2. Let $s \in \mathbb{N}$, $Z_s = (X_s, V_s) \in \mathbb{R}^{2ds}$, $1 \leq k \leq n$, $\sigma \in S_k$, $(M, N) \in U_{k, \sigma}$, $t \in [0, T]$ and $(t_1, \ldots, t_k) \in \mathcal{T}_k(t)$. For each $i = 1, \ldots, k$, consider $(\omega_{\sigma, i}, v_{\sigma, i}) \in u^{d_{\sigma, i} - 1} \times \mathbb{R}^{d_{\sigma, i}}$. Then for all $i = 1, \ldots, k$ and $\ell = 1, \ldots, s + \bar{s} - 1$, we have

$$|x^N_i(t^\ell_1) - x^\infty_i(t^\ell_1)| \leq \sqrt{2} \varepsilon_3(i - 1), \quad v^N_i(t^\ell_1) = v^\infty_i(t^\ell_1).$$

Moreover, if $s < n$, then for each $i \in \{1, \ldots, k\}$, there holds:

$$|X^N_{s+\bar{s}-1}(t^\ell) - X^\infty_{s+\bar{s}-1}(t^\ell)| \leq n^{3/2} \varepsilon_3.$$

Proof. We first prove (11.1) by induction on $i \in \{1, \ldots, k\}$. For $i = 1$ the result is trivial since the pseudotrajectories initially coincide by construction. Assume the conclusion holds for $i \in \{1, \ldots, k - 1\}$ i.e. for all $\ell \in \{1, \ldots, s + \bar{s} - 1\}$, there holds:

$$|x^N_i(t^\ell) - x^\infty_i(t^\ell)| \leq \sqrt{2} \varepsilon_3(i - 1) \quad \text{and} \quad v^N_i(t^\ell) = v^\infty_i(t^\ell).$$

We prove the conclusion holds for $(i + 1) \in \{2, \ldots, k\}$. We need to take different cases for $j_i \in \{-1, 1\}$ and $\sigma_i \in \{1, 2\}$.

- $\sigma_i = 1, j_i = -1$: For the Boltzmann pseudo-trajectory we get

$$x^N_i(t^\ell_{i+1}) = x^\infty_i(t^\ell_{i+1}) = x^N_i(t^\ell_{i+1}) - (t_i - t_{i+1})v^N_i(t^\ell_{i+1}), \quad v^N_i(t^\ell_{i+1}) = v^\infty_i(t^\ell_{i+1}), \quad \forall \ell \in \{1, \ldots, s + \bar{s} - 1\} \setminus \{m_i\},$$

$$x^\infty_{m_i}(t^\ell_{i+1}) = x^\infty_{m_i}(t^\ell_{i+1}) - (t_i - t_{i+1})v^\infty_{m_i}(t^\ell_{i+1}), \quad v^\infty_{m_i}(t^\ell_{i+1}) = v^\infty_{m_i}(t^\ell_{i+1}).$$

while for the BBGKY hierarchy pseudo-trajectory we get

$$x^N_i(t^\ell_{i+1}) = x^N_i(t^\ell_{i+1}) - (t_i - t_{i+1})v^N_i(t^\ell_{i+1}), \quad v^N_i(t^\ell_{i+1}) = v^N_i(t^\ell_{i+1}), \quad \forall \ell \in \{1, \ldots, s + \bar{s} - 1\} \setminus \{m_i\},$$

$$x^\infty_{m_i}(t^\ell_{i+1}) = x^\infty_{m_i}(t^\ell_{i+1}) - (t_i - t_{i+1})v^\infty_{m_i}(t^\ell_{i+1}), \quad v^\infty_{m_i}(t^\ell_{i+1}) = v^\infty_{m_i}(t^\ell_{i+1}).$$

So, for any $\ell \in \{1, \ldots, s + \bar{s} - 1\}$, the induction assumption (11.3) implies

$$v^N_i(t^\ell_{i+1}) = v^\infty_i(t^\ell_{i+1}) = v^N_i(t^\ell_{i+1}),$$

$$|x^N_i(t^\ell_{i+1}) - x^\infty_i(t^\ell_{i+1})| = |x^N_i(t^\ell_{i+1}) - x^\infty_i(t^\ell_{i+1})| \leq \sqrt{2} \varepsilon_3(i - 1).$$

Moreover, since $\varepsilon_2 < \varepsilon_3$, for $\ell = s + \bar{s}$, we get

$$v^\infty_{s+\bar{s}}(t^\ell_{i+1}) = v^\infty_{s+\bar{s}}(t^\ell_{i+1}),$$

$$|x^\infty_{s+\bar{s}}(t^\ell_{i+1}) - x^\infty_{s+\bar{s}}(t^\ell_{i+1})| \leq |x^\infty_{m_i}(t^\ell_{i+1}) - x^\infty_{m_i}(t^\ell_{i+1})| + \varepsilon_2|v^\infty_{s+\bar{s}}| \leq \sqrt{2} \varepsilon_3(i - 1) + \varepsilon_2 < \sqrt{2} \varepsilon_3 i.$$

- $\sigma_i = 1, j_i = 1$: For the Boltzmann hierarchy pseudo-trajectory we get

$$x^\infty_i(t^\ell_{i+1}) = x^\infty_i(t^\ell_{i+1}) - (t_i - t_{i+1})v^\infty_i(t^\ell_{i+1}), \quad v^\infty_i(t^\ell_{i+1}) = v^\infty_i(t^\ell_{i+1}), \quad \forall \ell \in \{1, \ldots, s + \bar{s} - 1\} \setminus \{m_i\},$$

$$x^\infty_{m_i}(t^\ell_{i+1}) = x^\infty_{m_i}(t^\ell_{i+1}) - (t_i - t_{i+1})v^\infty_{m_i}(t^\ell_{i+1}), \quad v^\infty_{m_i}(t^\ell_{i+1}) = v^\infty_{m_i}(t^\ell_{i+1}).$$

and for the BBGKY hierarchy pseudo-trajectory we obtain

$$x^N_i(t^\ell_{i+1}) = x^N_i(t^\ell_{i+1}) - (t_i - t_{i+1})v^N_i(t^\ell_{i+1}), \quad v^N_i(t^\ell_{i+1}) = v^N_i(t^\ell_{i+1}), \quad \forall \ell \in \{1, \ldots, s + \bar{s} - 1\} \setminus \{m_i\},$$

$$x^\infty_{m_i}(t^\ell_{i+1}) = x^\infty_{m_i}(t^\ell_{i+1}) - (t_i - t_{i+1})v^\infty_{m_i}(t^\ell_{i+1}), \quad v^\infty_{m_i}(t^\ell_{i+1}) = v^\infty_{m_i}(t^\ell_{i+1}).$$

For $\ell \in \{1, \ldots, s + \bar{s} - 1\} \setminus \{m_i\}$, the induction assumption (11.3) yields

$$v^N_i(t^\ell_{i+1}) = v^\infty_i(t^\ell_{i+1}) = v^N_i(t^\ell_{i+1}),$$

$$|x^N_i(t^\ell_{i+1}) - x^\infty_i(t^\ell_{i+1})| = |x^N_i(t^\ell_{i+1}) - x^\infty_i(t^\ell_{i+1})| \leq \sqrt{2} \varepsilon_3(i - 1).$$

and for $\ell = m_i$, it yields

$$v^N_{m_i}(t^\ell_{i+1}) = v^\infty_{m_i}(t^\ell_{i+1}) = v^\infty_{m_i}(t^\ell_{i+1}),$$

$$|x^N_{m_i}(t^\ell_{i+1}) - x^\infty_{m_i}(t^\ell_{i+1})| = |x^N_{m_i}(t^\ell_{i+1}) - x^\infty_{m_i}(t^\ell_{i+1})| \leq \sqrt{2} \varepsilon_3(i - 1).$$
Moreover, since $\epsilon_2 \ll \epsilon_3$, for $\ell = s + \bar{\sigma}_i$, we obtain

$$v^N_{\ell+\bar{\sigma}_i}(t_{\ell+1}^+) = v^\infty_{\ell+\bar{\sigma}_i}(t_{\ell+1}^-),$$

$$|x^N_{\ell+\bar{\sigma}_i}(t_{\ell+1}^+) - x^\infty_{\ell+\bar{\sigma}_i}(t_{\ell+1}^-)| \leq |x^N_{\ell}(t_{\ell}^+) - x^\infty_{\ell}(t_{\ell}^-)| + \epsilon_2 |\omega_{\ell+\bar{\sigma}_i}| \leq \sqrt{2}\epsilon_3(i-1) + \epsilon_2 < \sqrt{2}\epsilon_3i.$$

- $\sigma_i = 2, j_i = -1$: For the Boltzmann hierarchy pseudo-trajectory we get

$$x^\infty_{\ell}(t_{\ell+1}^-) = x^\infty_{\ell}(t_{\ell}^-) - (t_{\ell} - t_{\ell+1})v^\infty_{\ell}(t_{\ell}^-), \quad v^\infty_{\ell}(t_{\ell+1}^-) = v^\infty_{\ell}(t_{\ell}^-), \quad \forall \ell \in \{1, \ldots, s + \bar{\sigma}_i - 1\} \setminus \{m_i\},$$

$$x^\infty_{\ell}(t_{\ell+1}^-) = x^\infty_{\ell+1}(t_{\ell+1}^-), \quad v^\infty_{\ell}(t_{\ell+1}^-) = v^\infty_{\ell+1}(t_{\ell+1}^-), \quad \forall \ell \in \{1, \ldots, s + \bar{\sigma}_i - 1\} \setminus \{m_i\},$$

while for the BBGKY hierarchy pseudo-trajectory we get

$$x^N_{\ell+\bar{\sigma}_i-1}(t_{\ell+1}^-) = x^N_{\ell+\bar{\sigma}_i}(t_{\ell+1}^-) - (t_{\ell} - t_{\ell+1})v^N_{\ell+\bar{\sigma}_i-1} - \sqrt{2}\epsilon_3|\omega_{\ell+\bar{\sigma}_i-1}|, \quad v^N_{\ell+\bar{\sigma}_i-1}(t_{\ell+1}^-) = v^N_{\ell+\bar{\sigma}_i}(t_{\ell+1}^-),$$

$$x^N_{\ell+\bar{\sigma}_i}(t_{\ell+1}^-) = x^N_{\ell+\bar{\sigma}_i-1}(t_{\ell+1}^-) - (t_{\ell} - t_{\ell+1})v^N_{\ell+\bar{\sigma}_i}, \quad v^N_{\ell+\bar{\sigma}_i-1}(t_{\ell+1}^-) = v^N_{\ell+\bar{\sigma}_i}(t_{\ell+1}^-).$$

So, for any $\ell \in \{1, \ldots, s + \bar{\sigma}_i - 1\}$, the induction assumption \ref{eq:ind-assumph} implies

$$v^\infty(t_{\ell+1}^-) = v^\infty(t_{\ell}^-) = v^\infty(t_{\ell+1}^-),$$

$$|x^N(t_{\ell+1}^-) - x^\infty(t_{\ell+1}^-)| = |x^N(t_{\ell}^-) - x^\infty(t_{\ell}^-)| \leq \sqrt{2}\epsilon_3(i-1).$$

Moreover, for $\ell = s + \bar{\sigma}_i - 1$ we get

$$v^N_{s+\bar{\sigma}_i-1}(t_{\ell+1}^-) = v^N_{s+\bar{\sigma}_i-1}(t_{\ell+1}^-),$$

$$|x^N_{s+\bar{\sigma}_i-1}(t_{\ell+1}^-) - x^\infty_{s+\bar{\sigma}_i-1}(t_{\ell+1}^-)| \leq |x^N_{s+\bar{\sigma}_i}(t_{\ell}^-) - x^\infty_{s+\bar{\sigma}_i}(t_{\ell}^-)| + \sqrt{2}\epsilon_3|\omega_{s+\bar{\sigma}_i-1}| \leq \sqrt{2}\epsilon_3(i-1) + \sqrt{2}\epsilon_3 = \sqrt{2}\epsilon_3i.$$

and for $\ell = s + \bar{\sigma}_i$ we get

$$v^N_{s+\bar{\sigma}_i}(t_{\ell+1}^-) = v^N_{s+\bar{\sigma}_i}(t_{\ell+1}^-),$$

$$|x^N_{s+\bar{\sigma}_i}(t_{\ell+1}^-) - x^\infty_{s+\bar{\sigma}_i}(t_{\ell+1}^-)| \leq |x^N_{s+\bar{\sigma}_i}(t_{\ell}^-) - x^\infty_{s+\bar{\sigma}_i}(t_{\ell}^-)| + \sqrt{2}\epsilon_3|\omega_{s+\bar{\sigma}_i}| \leq \sqrt{2}\epsilon_3(i-1) + \sqrt{2}\epsilon_3 = \sqrt{2}\epsilon_3i.$$

- $\sigma_i = 2, j_i = 1$: For the Boltzmann hierarchy pseudo-trajectory we get

$$x^\infty(t_{\ell+1}^-) = x^\infty(t_{\ell}^-) - (t_{\ell} - t_{\ell+1})v^\infty(t_{\ell}^-), \quad v^\infty(t_{\ell+1}^-) = v^\infty(t_{\ell}^-), \quad \forall \ell \in \{1, \ldots, s + \bar{\sigma}_i - 1\} \setminus \{m_i\},$$

$$x^\infty(t_{\ell+1}^-) = x^\infty(t_{\ell}^-), \quad v^\infty(t_{\ell+1}^-) = v^\infty(t_{\ell}^-), \quad \forall \ell \in \{1, \ldots, s + \bar{\sigma}_i - 1\} \setminus \{m_i\},$$

while for the BBGKY hierarchy pseudo-trajectory we obtain

$$x^N_{s+\bar{\sigma}_i-1}(t_{\ell+1}^-) = x^N_{s+\bar{\sigma}_i}(t_{\ell+1}^-) - (t_{\ell} - t_{\ell+1})v^N_{s+\bar{\sigma}_i-1} + \sqrt{2}\epsilon_3|\omega_{s+\bar{\sigma}_i-1}|, \quad v^N_{s+\bar{\sigma}_i-1}(t_{\ell+1}^-) = v^N_{s+\bar{\sigma}_i}(t_{\ell+1}^-),$$

$$x^N_{s+\bar{\sigma}_i}(t_{\ell+1}^-) = x^N_{s+\bar{\sigma}_i-1}(t_{\ell+1}^-) - (t_{\ell} - t_{\ell+1})v^N_{s+\bar{\sigma}_i}, \quad v^N_{s+\bar{\sigma}_i-1}(t_{\ell+1}^-) = v^N_{s+\bar{\sigma}_i}(t_{\ell+1}^-).$$

and for the BBGKY hierarchy pseudo-trajectory we obtain

$$x^N(t_{\ell+1}^-) = x^N(t_{\ell}^-) - (t_{\ell} - t_{\ell+1})v^N(t_{\ell}^-), \quad v^N(t_{\ell+1}^-) = v^N(t_{\ell}^-), \quad \forall \ell \in \{1, \ldots, s + \bar{\sigma}_i - 1\} \setminus \{m_i\},$$

$$x^N(t_{\ell+1}^-) = x^N(t_{\ell}^-), \quad v^N(t_{\ell+1}^-) = v^N(t_{\ell}^-), \quad \forall \ell \in \{1, \ldots, s + \bar{\sigma}_i - 1\} \setminus \{m_i\},$$

$$x^N_{s+\bar{\sigma}_i-1}(t_{\ell+1}^-) = x^N_{s+\bar{\sigma}_i}(t_{\ell+1}^-) - (t_{\ell} - t_{\ell+1})v^N_{s+\bar{\sigma}_i-1} + \sqrt{2}\epsilon_3|\omega_{s+\bar{\sigma}_i-1}|, \quad v^N_{s+\bar{\sigma}_i-1}(t_{\ell+1}^-) = v^N_{s+\bar{\sigma}_i}(t_{\ell+1}^-),$$

$$x^N_{s+\bar{\sigma}_i}(t_{\ell+1}^-) = x^N_{s+\bar{\sigma}_i-1}(t_{\ell+1}^-) - (t_{\ell} - t_{\ell+1})v^N_{s+\bar{\sigma}_i}, \quad v^N_{s+\bar{\sigma}_i-1}(t_{\ell+1}^-) = v^N_{s+\bar{\sigma}_i}(t_{\ell+1}^-).$$

For any $\ell \in \{1, \ldots, \bar{\sigma}_i - 1\} \setminus \{m_i\}$, the induction assumption \ref{eq:ind-assumph} yields

$$v^\infty(t_{\ell+1}^-) = v^\infty(t_{\ell}^-) = v^\infty(t_{\ell+1}^-),$$

$$|x^N(t_{\ell+1}^-) - x^\infty(t_{\ell+1}^-)| = |x^N(t_{\ell}^-) - x^\infty(t_{\ell}^-)| \leq \sqrt{2}\epsilon_3(i-1).$$
Thus, for $\ell = m$  
$$v_{m}^{N}(t_{i+1}) = v_{m}^{N\ast}(t_{i}^{+}) = v_{m}^{\infty\ast}(t_{i}^{+}) = v_{m}^{\infty}(t_{i}^{+}) ,$$
$|x_{m}^{N}(t_{i+1}) - x_{m}^{\infty}(t_{i+1})| = |x_{m}^{N}(t_{i}^{+}) - x_{m}^{\infty}(t_{i}^{+})| \leq \sqrt{2}e_{3}(i - 1),$

for $\ell = s + \sigma_{i} - 1$  
$$v_{s+\sigma_{i}-1}^{N}(t_{i+1}) = v_{s+\sigma_{i}-1}^{\ast}(t_{i+1}) = v_{s+\sigma_{i}-1}^{\infty}(t_{i+1}),$$
$|x_{s+\sigma_{i}-1}^{N}(t_{i+1}) - x_{s+\sigma_{i}-1}^{\infty}(t_{i+1})| \leq |x_{s}(t_{i}^{+}) - x_{s}(t_{i}^{+})| + \sqrt{2}e_{3}|\omega_{s+\sigma_{i}-1}| \leq \sqrt{2}e_{3}(i - 1) + \sqrt{2}e_{3} = \sqrt{2}e_{3},$

and for $\ell = s + \sigma_{i}$  
$$v_{s+\sigma_{i}}^{N}(t_{i+1}) = v_{s+\sigma_{i}}^{\ast}(t_{i+1}) = v_{s+\sigma_{i}}^{\infty}(t_{i+1}),$$
$|x_{s+\sigma_{i}}^{N}(t_{i+1}) - x_{s+\sigma_{i}}^{\infty}(t_{i+1})| \leq |x_{s}(t_{i}^{+}) - x_{s+\sigma_{i}}(t_{i}^{+})| + \sqrt{2}e_{3}|\omega_{s+\sigma_{i}}| \leq \sqrt{2}e_{3}(i - 1) + \sqrt{2}e_{3} = \sqrt{2}e_{3}i.$

Combining all cases, (11.2) is proved by induction.

To prove (11.2), it suffices to add for $\ell = 1, \ldots, s + \sigma_{i} - 1$, and use the facts $1 \leq i \leq k - 1$, $\sigma_{i} - 1 < \sigma_{i} \leq \sigma_{k-1} < 2k < 2n$, from (17.6), and the assumption $s < n$. \(\square\)

11.2. Reformulation in terms of pseudo-trajectories. We will now re-write the BBGKY hierarchy and Boltzmann hierarchy truncated elementary observables in terms of pseudo-trajectories.

Let $s \in \mathbb{N}$ and assume $s < n$. For the Boltzmann hierarchy case, there is always free flow between the collision times. Therefore, recalling (10.29) and (10.26), for $x_{s} \in \Delta_{s}(\epsilon_{0})$, $1 \leq s \leq n$, $\sigma \in S_{k}$, $(J, M) \in \mathcal{U}_{s,k,\sigma}$, $t \in [0, T]$ and $(t_{1}, \ldots, t_{k}) \in \mathcal{T}_{s,\sigma}(t)$, the Boltzmann hierarchy truncated elementary observable can be equivalently written as:

$$J_{s}^{\infty}(t, J, M)(X_{s}) = \int_{\mathcal{M}_{s}(X_{s})} \phi_{s}(V_{s}) \int_{\mathcal{T}_{s,\sigma}(t)} \frac{g_{m_{1}}(Z_{s,\sigma}(t_{1}^{+})) \ldots \frac{1}{\epsilon_{3}} \frac{d\omega_{s,1} \ldots d\omega_{s,s}}{dV_{s}} dt_{1} \ldots dt_{k}}{\epsilon_{3}}.$$

Now we shall see that due to Lemma 11.2 it is possible to expand the BBGKY hierarchy truncated elementary observables as well.

More precisely, fix $x_{s} \in \Delta_{s}(\epsilon_{0})$, $1 \leq s \leq n$, $\sigma \in S_{k}$, $(J, M) \in \mathcal{U}_{s,k,\sigma}$, $t \in [0, T]$ and $(t_{1}, \ldots, t_{k}) \in \mathcal{T}_{s,\sigma}(t)$. Consider $(N, \epsilon_{2}, \epsilon_{3})$ in the scaling (4.24) such that $\epsilon_{2} \ll \eta_{3} \epsilon_{3}$ and $n^{3/2} \epsilon_{3} \ll \alpha$. By Lemma 10.1 given $V_{s} \in \mathcal{M}_{s}(X_{s})$, we have $Z_{s} \in G_{s}(\epsilon_{3}, \epsilon_{0}, \delta)$. By the definition of the set $G_{s}(\epsilon_{3}, \epsilon_{0}, \delta)$, see (10.1), and the fact that $\epsilon_{2} \ll \epsilon_{3}$, we have

$$Z_{s} \in G_{s}(\epsilon_{3}, \epsilon_{0}, \delta) \Rightarrow Z_{s}(\tau) \in \hat{D}_{s,\epsilon_{3},\epsilon_{0}}, \quad \forall \tau \geq 0,$$

thus

$$\Psi_{s}^{-\epsilon_{0}}Z_{s}^{N}(t_{0}^{+}) = \Phi_{s}^{-\epsilon_{0}}Z_{s}^{N}(t_{0}^{+}), \quad \forall \tau \in [t_{1}, t_{0}],$$

where $\Psi_{s}$, given in (3.74), denotes the s-particle $(\epsilon_{2}, \epsilon_{3})$-interaction zone flow and $\Phi_{s}$, given in (5.75), denotes the s-particle free flow respectively. We also have

$$Z_{s} = (X_{s}, V_{s}) \in G_{s}(\epsilon_{3}, \epsilon_{0}, \delta) \Rightarrow Z_{s}^{\infty}(t_{i}^{+}) \in G_{s}(\epsilon_{0}, 0).$$

For all $i \in \{1, \ldots, k\}$ inductive application of Proposition 9.2 or Proposition 9.6 depending on whether the adjunction is binary or ternary, implies that

$$Z_{s+\sigma_{i}}^{\infty}(t_{i+1}^{+}) \in G_{s+\sigma_{i}}(\epsilon_{0}, 0), \quad \forall \omega_{s_{i+1}} \in \mathcal{F}_{s_{i+1}}(Z_{s+\sigma_{i}}^{\infty}(t_{i+1}^{+})).$$

Since we have assumed $n^{3/2} \epsilon_{3} \ll \alpha$ and $s < n$, (11.12) from Lemma 11.2 implies

$$|X_{s+\sigma_{i}}^{\infty}(t_{i+1}^{+}) - X_{s+\sigma_{i}}^{\infty}(t_{i}^{+})| \leq \frac{\alpha}{2}, \quad \forall i = 1, \ldots, k.$$

Then, (9.7), (9.11) from Proposition 9.2 or (9.60), (9.61) from Proposition 9.6 depending on whether the adjunction is binary or ternary, yield that for any $i = 1, \ldots, k$, we have

$$\Psi_{s+\sigma_{i}}^{-\epsilon_{0}}Z_{s+\sigma_{i}}^{N}(t_{i}^{+}) = \Phi_{s+\sigma_{i}}^{-\epsilon_{0}}Z_{s+\sigma_{i}}^{N}(t_{i}^{+}), \quad \forall \tau \in [t_{i+1}, t_{i}].$$
where \( \Psi_{s+\bar{\sigma}} \) and \( \Phi_{s+\bar{\sigma}} \) denote the \((s+\bar{\sigma})\)-particle \((\epsilon_2, \epsilon_3)\)-flow and the \((s+\bar{\sigma})\)-particle free flow, given in (3.34) and (3.74) respectively. In other words the backwards \((\epsilon_2, \epsilon_3)\)-flow coincides with the free flow in \([t_{i+1}, t_i]\). Finally, Lemma 11.2 also implies that
\[
\psi^N_{m_i}(t_i^+)^{\infty} = \psi_{m_i}(t_i^+), \quad \forall i = 1, \ldots, k.
\]
Therefore, for \( X_s \in \Delta^N_{s+\bar{\sigma}}(\epsilon_0) \), and \((N, \epsilon_2, \epsilon_3)\) in the scaling (4.24) with \( n\epsilon^3 \ll \alpha \) and \( \epsilon_2 \ll \eta^2 \epsilon_3 \), the BBGKY hierarchy truncated elementary observable can be equivalently written as:
\[
J^N_{s,k,R,\delta,\sigma}(t, J, M)(X_s) = A^N_{s,k,\delta,\sigma} \int_{M^N_s(X_s)} \phi_s(V_s) \int_{T_{k,\delta}(t)} \int_{\mathcal{E}_m^N} (z_s^N(t_i^+)) \cdots \int_{\mathcal{E}_{m_k}^N} (z_s^N(t_k^+))
\]
\[
\times \prod_{i=1}^k b_{\delta_i}^N (\omega_{s+i, i}, v_{s+i, i}, v_{m_i}(t_i^+)) f_{N,s+k}^{(s+\bar{\sigma})} (Z_s^N(0^+)) \prod_{i=1}^k (d\omega_{s+i, i} d\nu_{s+i, i}) dt_k \cdots dt_1 d\nu_s,
\]
where, recalling (4.19), (4.22), we denote
\[
A^N_{s,k,\delta,\sigma} = \prod_{i \in \{1, \ldots, k\}} A^N_{s,k,\delta,\sigma}^1 \prod_{i \in \{1, \ldots, k\} | \sigma_i = 2} A^N_{s,k,\bar{\sigma},\sigma}^1.
\]

**Remark 11.3.** Notice that for fixed \( s \in \mathbb{N} \) and \( k \geq 1 \) and \( \sigma \in S_k \), the scaling (4.24) implies
\[
A^N_{s,k,\delta,\sigma} \to 1, \quad \text{as } N \to \infty.
\]

Let us approximate the BBGKY hierarchy truncated elementary observables by Boltzmann hierarchy truncated elementary observables defining some auxiliary functionals. Let \( s \in \mathbb{N} \) and \( X_s \in \Delta^N_{s+\bar{\sigma}}(\epsilon_0) \). For \( 1 \leq k \leq n \), \( \sigma \in S_k \) and \((J, M) \in \mathcal{U}_{s,k,\sigma}\), we define
\[
J^N_{s,k,R,\delta,\sigma}(t, J, M)(X_s) = \int_{M^N_s(X_s)} \phi_s(V_s) \int_{T_{k,\delta}(t)} \int_{\mathcal{E}_m^N} (z_s^N(t_i^+)) \cdots \int_{\mathcal{E}_{m_k}^N} (z_s^N(t_k^+))
\]
\[
\times \prod_{i=1}^k b_{\delta_i}^N (\omega_{s+i, i}, v_{s+i, i}, v_{m_i}(t_i^+)) f_{N,s+k}^{(s+\bar{\sigma})} (Z_s^N(0^+)) \prod_{i=1}^k (d\omega_{s+i, i} d\nu_{s+i, i}) dt_k \cdots dt_1 d\nu_s,
\]
red write what it is. We conclude that the auxiliary functionals approximate the BBGKY hierarchy truncated elementary observables \( J^N_{s,k,R,\delta,\sigma} \) in (11.8).

**Proposition 11.4.** Let \( s, n \in \mathbb{N} \), with \( s < n \), \( \alpha, \epsilon_0, R, \eta, \delta \) be parameters as in (3.5), and \( t \in [0, T] \). Then for any \( \zeta > 0 \), there is \( N_1 = N_1(\zeta, n, \alpha, \eta, \epsilon_0) \in \mathbb{N} \), such that for all \((N, \epsilon_2, \epsilon_3)\) in the scaling (4.24) with \( N > N_1 \), there holds:
\[
\sum_{k=1}^n \sum_{\sigma \in S_k} \| J^N_{s,k,R,\delta,\sigma}(t, J, M) - J^N_{s,k,R,\delta,\sigma}(t, J, M) \|_{\mathcal{L}^\infty(\Delta^N_{s+\bar{\sigma}}(\epsilon_0))} \leq C^2_{\alpha, \epsilon_0, \eta, \delta} R^{\lfloor \alpha + 3n \rfloor} \zeta^2.
\]

**Proof.** Fix \( 1 \leq k \leq n \), \( \sigma \in S_k \) and \((J, M) \in \mathcal{U}_{s,k,\sigma}\). Consider \((N, \epsilon_2, \epsilon_3)\) in the scaling (4.24). Remark 11.2 guarantees that we can consider \( N \) large enough such that \( \epsilon_2 \ll \eta^2 \epsilon_3 \) and \( n^{3/2} \epsilon_3 \ll \alpha \). Triangle inequality and the inclusion \( \Delta^N_{s+\bar{\sigma}}(\epsilon_0) \subseteq \Delta^N_{s+\bar{\sigma}}(\epsilon_0/2) \) yield
\[
\| J^N_{s,k,R,\delta,\sigma}(t, J, M) - J^N_{s,k,R,\delta,\sigma}(t, J, M) \|_{\mathcal{L}^\infty(\Delta^N_{s+\bar{\sigma}}(\epsilon_0))}
\]
\[
\leq \| J^N_{s,k,R,\delta,\sigma}(t, J, M) - A^N_{s,k,\delta,\sigma} J^N_{s,k,R,\delta,\sigma}(t, J, M) \|_{\mathcal{L}^\infty(\Delta^N_{s+\bar{\sigma}}(\epsilon_0/2))}
\]
\[
+ \| A^N_{s,k,\delta,\sigma} - 1 \| J^N_{s,k,R,\delta,\sigma}(t, J, M) \|_{\mathcal{L}^\infty(\Delta^N_{s+\bar{\sigma}}(\epsilon_0))}.
\]
We estimate each of the terms (11.11)-(11.12) separately.

**Term (11.11):** Let us fix \((t_1, \ldots, t_k) \in T_{k,\delta}(t)\). Applying (10.15) for \( i = k - 1 \), we obtain
\[
Z^\infty_{s+\bar{\sigma}_{k-1}}(t_k^+) \in G_{s+\bar{\sigma}_{k-1}}(\epsilon_0, 0).
\]
Since \( s < n \) and \( n^{3/2} \epsilon_3 \ll \alpha \), (11.2), applied for \( i = k \), implies
\[
| X^N_{s+\bar{\sigma}_{k-1}}(t_k^+) - X^\infty_{s+\bar{\sigma}_{k-1}}(t_k^+) | \leq \frac{\alpha}{2}.
\]
Thus the continuity assumption (6.8) on $F$ that there exists over $U$ imply
\[ Z_{s+\tilde{\sigma}}^N(0^+) \in G_{s+\tilde{\sigma}}(\epsilon_0/2,0) \subseteq \Delta_{s+\tilde{\sigma}}(\epsilon_0/2). \]
(11.13)

Thus (10.24)-(10.25), (10.29), (11.8)-(11.10) and crucially (11.13) imply that for $N$ large enough, we have
\[ \|J_{s,k,R,\sigma}^N(t,J,M) - J_{s,k,R,\sigma}(t,J,M)\|_{L^\infty(\Delta X(\epsilon_0/2))} \leq \frac{C_k}{\mu_{s,k,R,\sigma}} \|\phi_s\|_{L_{0,s}^{\infty}} R^{d(s+3k)} \|J_{s,k,R,\sigma}^N(t,J,M) - J_{s,k,R,\sigma}(t,J,M)\|_{L^\infty(\Delta X(\epsilon_0/2))}. \]
(11.14)

**Term (11.5):** By (10.28), we have
\[ J_{s,k,R,\sigma}^N(t,J,M) = J_{s,k,R,\sigma}(t,J,M) + J_{s,k,R,\sigma}^N(t,J,M) = J_{s,k,R,\sigma}(t,J,M) + J_{s,k,R,\sigma}^N(t,J,M). \]
(11.20)

Adding over all $(J,M) \in U(\epsilon_0)$, we have $\|J_{s,k,R,\sigma}^N(t,J,M) - J_{s,k,R,\sigma}(t,J,M)\|_{L^\infty(\Delta X(\epsilon_0/2))} \leq C_k \|\phi_s\|_{L_{0,s}^{\infty}} R^{d(s+3k)} \|J_{s,k,R,\sigma}^N(t,J,M) - J_{s,k,R,\sigma}(t,J,M)\|_{L^\infty(\Delta X(\epsilon_0/2))}$. Therefore, using (10.24)-(10.25) and (10.29), we obtain
\[ \|J_{s,k,R,\sigma}^N(t,J,M)\|_{L^\infty(\Delta X(\epsilon_0))} \leq C_k \|\phi_s\|_{L_{0,s}^{\infty}} R^{d(s+3k)} \|J_{s,k,R,\sigma}^N(t,J,M)\|_{L^\infty(\Delta X(\epsilon_0/2))}. \]
(11.15)

But since $n \in \mathbb{N}$, $\epsilon_0 > 0$ are fixed, (6.6) implies
\[ \lim_{N \to \infty} \sup_{k \in \{1, \ldots, n\}} \sup_{\sigma \in S_k} \|J_{N,0}^N - f_0^{s+\tilde{\sigma}}k\|_{L^\infty(\Delta X(\epsilon_0/2))} = 0. \]
Moreover, Remark (11.2) yields
\[ \lim_{N \to \infty} \sup_{k \in \{1, \ldots, n\}} \sup_{\sigma \in S_k} |A_{s,k,R,\sigma}^N - 1| = 0, \]
and the result follows.

By the uniform continuity assumption, we also obtain the following estimate:

**Proposition 11.5.** Let $s, n \in \mathbb{N}$ with $s < n$, $\alpha, \epsilon_0, R, \eta, \delta$ be parameters as in (9.5) and $t \in [0,T]$. Then for any $\zeta > 0$, there is $N_2 = N_2(\zeta, n) \in \mathbb{N}$, such that for all $(N, e_t, e_3)$ in the scaling (4.24) with $N > N_2$, there holds
\[ \sum_{k=1}^n \sum_{\sigma \in S_k} \sum_{(J,M) \in U_{s,k,\sigma}} \|J_{s,k,R,\sigma}^N(t,J,M) - J_{s,k,R,\sigma}(t,J,M)\|_{L^\infty(\Delta X(\epsilon_0))} \leq C_k \|\phi_s\|_{L_{0,s}^{\infty}} R^{d(s+3n)} \zeta^2. \]

**Proof.** Let $\zeta > 0$. Fix $1 \leq k \leq n$, $\sigma \in S_k$ and $(J,M) \in U_{s,k,\sigma}$. Since $s < n$, Lemma (11.2) yields
\[ |Z_{s+\tilde{\sigma}}^N(0^+) - Z_{s+\tilde{\sigma}}^N(0^+)| \leq \sqrt{6n}^{3/2} \epsilon_3, \ \forall Z_{s,k} \in \mathbb{R}^{2ds}. \]
(11.16)

Thus the continuity assumption (6.8) on $F_0$, (11.16), the scaling (4.24), and (4.26) from Remark (4.2) imply that there exists $N_2 = N_2(\zeta, n) \in \mathbb{N}$, such that for all $N > N_2$, we have
\[ |f_0^{(s+\tilde{\sigma})}(Z_{s+\tilde{\sigma}}^N(0^+)) - f_0^{(s+\tilde{\sigma})}(Z_{s+\tilde{\sigma}}^N(0^+))| \leq C_k \epsilon_3^{-1/2} \leq C_{s+2k}^{-1} \zeta^2, \ \forall Z_{s,k} \in \mathbb{R}^{2ds}. \]
(11.17)

In the same spirit as in the proof of Proposition (11.4), using (11.17), (10.24)-(10.25), and summing over $(J,M) \in U_{s,k,\sigma}$, $\sigma \in S_k$, $k = 1, \ldots, n$, we obtain the result.
\[ \square \]
11.3. Proof of Theorem 6.5. We are now in the position to prove Theorem 6.5. Fix \( s \in \mathbb{N}, \phi_s \in C_c(\mathbb{R}^d) \) and \( t \in [0, T] \). Consider \( n \in \mathbb{N} \) with \( s < n \), and assume there exist parameters \( \alpha, \epsilon_0, R, \eta, \delta \) satisfying (9.5). Let \( \zeta > 0 \) small enough. Triangle inequality, Propositions 7.5, 10.2, 10.3, 10.5, 11.4, 11.5, Remark 10.3 and part (i) of Proposition 6.2 yield that there is \( N_0(\zeta, n, \alpha, \eta, \epsilon_0) \in \mathbb{N} \) such that for all \( N > N_0 \), we have

\[
\| I_s^n(t) - I_s^\infty(t) \|_{L^\infty(\Delta^\chi(\epsilon_0))} \leq C \left( 2^{-n} + e^{-\frac{\eta}{2} R^2} + \delta C^n \right) + C^n R^{d_{eff}} \eta^{-\frac{d-1}{d+1}} + C^n R^{d_{eff}} \zeta^2,
\]

where

\[
C := C_{d, \alpha, \beta_0, \theta, t} \phi_s \max \{ 1, \| F_0 \|_{\infty, \beta_0, \mu_0} \} > 1,
\]

is an appropriate constant.

Let us fix \( \theta > 0 \). Recall that we have also fixed \( s \in \mathbb{N} \) and \( \phi_s \in C_c(\mathbb{R}^d) \). We will now choose parameters satisfying (9.5), depending only on \( \zeta \), such that the right hand side of (11.18) becomes less than \( \zeta \).

Choice of parameters: We choose \( n \in \mathbb{N} \) and the parameters \( \delta, \eta, \epsilon_0, \alpha \) in the following order:

- \( \max \{ s, \log_2(C \zeta^{-1}) \} << n \), (this implies \( s < n, C2^{-n} << \zeta \)),
- \( \delta << \zeta^{-n+1} \), (this implies \( C^{n+1}\delta << \zeta \)),
- \( \eta << \zeta^{-n+1} \), \( R << \zeta^{-1/4d_{eff}} C^{-1/4d} \), (those imply \( C^n R^{d_{eff}} \eta^{-\frac{d-1}{d+1}} << \zeta \) and \( C^n R^{d_{eff}} \zeta^{-2} << \zeta \)),
- \( \max \left\{ 1, \sqrt{3} \beta_0^{-1/2} \ln^{1/2}(C \zeta^{-1}) \right\} << R \), (this implies \( Ce^{-\frac{\eta R^2}{2}} << \zeta \)),
- \( \epsilon_0 << \eta \delta \), \( \epsilon_0 < \theta \),
- \( \alpha << \epsilon_0 \min \{ 1, R^{-1} \eta \} \).

Clearly (11.24), (11.25) imply the parameters chosen satisfy (9.5) and depend only on \( \zeta \). Then, (11.18) and the choice of parameters imply that we may find \( N_0(\zeta) \in \mathbb{N} \), such that for all \( N > N_0 \), there holds:

\[
\| I_s^n(t) - I_s^\infty(t) \|_{L^\infty(\Delta^\chi(\epsilon_0))} < \zeta.
\]

But by (11.24), we have \( \epsilon_0 < \theta \), therefore we obtain

\[
\| I_s^n(t) - I_s^\infty(t) \|_{L^\infty(\Delta^\chi(\epsilon_0))} \leq \| I_s^n(t) - I_s^\infty(t) \|_{L^\infty(\Delta^\chi(\epsilon_0))} < \zeta,
\]

and Theorem 6.5 is proved.

12. Appendix

In this appendix, we present some auxiliary results which are used throughout the paper.

12.1. Calculation of Jacobians. We first present an elementary Linear Algebra result, which will be useful throughout the manuscript for the calculation of Jacobians. For a proof see Lemma A.1. from [2].

**Lemma 12.1.** Let \( n \in \mathbb{N}, \lambda \neq 0 \) and \( w, u \in \mathbb{R}^n \). Then

\[
\det(\lambda I_n + wu^T) = \lambda^n (1 + \lambda^{-1} \langle w, u \rangle),
\]

where \( I_n \) is the \( n \times n \) identity matrix.

12.2. The binary transition map. Here, we introduce the binary transition map, which will enable us to control binary postcollisional configurations. Recall from [22] the binary cross-section:

\[
b_2(\omega_1, v_1) = \langle \omega, v_1 \rangle, \quad (\omega, v_1) \in S_1^{d-1} \times \mathbb{R}^d.
\]

Given \( v_1, v_2 \in \mathbb{R}^d \), we define the domain \( \Omega := \{ \omega_2 \in \mathbb{R}^d : |\omega_2| \leq 2, \text{ and } b_2(\omega_1, v_2 - v_1) > 0 \} \), and the set \( S_1^{d-1} \times S_1^{d-1} \). We also define the smooth map \( \Psi : \mathbb{R}^d \to \mathbb{R} \) by \( \Psi(\omega_1) := |\omega_1|^2 \). Notice that the unit \((d-1)\)-sphere is given by level sets of \( \Psi \) i.e. \( S_1^{d-1} = \{ \Psi = 1 \} \).

\[\text{[15] we trivially extend the binary cross-section for any } \omega \in \mathbb{R}^d.\]
Proposition 12.2. Consider $v_1, v_2 \in \mathbb{R}^d$ and $r > 0$ such that $|v_1 - v_2| = r$. We define the binary transition map $J_{v_1,v_2} : \Omega \to \mathbb{R}^d$ as follows:

$$J_{v_1,v_2}(\omega) := r^{-1}(v'_1 - v'_2), \quad \omega \in \Omega. \quad (12.1)$$

The map $J_{v_1,v_2}$ has the following properties:

(i) $J_{v_1,v_2}$ is smooth in $\Omega$ with bounded derivative uniformly in $r$ i.e.

$$\|D J_{v_1,v_2}(\omega)\|_{\infty} \leq C_d, \quad \forall \omega_1 \in \Omega, \quad (12.2)$$

where $\| \cdot \|_{\infty}$ denotes the maximum element matrix norm of $D J_{v_1,v_2}(\omega)$.

(ii) The Jacobian of $J_{v_1,v_2}$ is given by:

$$\text{Jac}(J_{v_1,v_2})(\omega_1) \simeq r^{-d} b_2^d(\omega_1, v_2 - v_1) > 0, \quad \forall \omega_1 \in \Omega. \quad (12.3)$$

(iii) The map $J_{v_1,v_2} : S^d_{+1} \to S_{d-1} \setminus \{ r^{-1}(v_1 - v_2) \}$ is bijective. Moreover, there holds

$$S^d_{+1} = \{ \varphi \circ J_{v_1,v_2} = 1 \}. \quad (12.4)$$

(iv) For any measurable $g : \mathbb{R}^d \to [0, +\infty]$, there holds the change of variables estimate:

$$\int_{S^d_{+1}} (g \circ J_{v_1,v_2}(\omega_1)) \text{Jac} J_{v_1,v_2}(\omega_1) \, d\omega_1 \leq \int_{S_{d-1}} g(v_1) \, dv_1. \quad (12.5)$$

Proof. The proof is the binary analogue of the proof of Proposition 8.5. in [4]. □

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\[16\] we trivially extend the binary collisional operator for any $\omega \in \Omega$.\]
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