GROUP SPLITTINGS AND ASYMPTOTIC TOPOLOGY

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Abstract. It is a consequence of the theorem of Stallings on groups with many ends that splittings over finite groups are preserved by quasi-isometries. In this paper we use asymptotic topology to show that group splittings are preserved by quasi-isometries in many cases. Roughly speaking we show that splittings are preserved under quasi-isometries when the vertex groups are fundamental groups of aspherical manifolds (or more generally ‘coarse $PD(n)$-groups’) and the edge groups are ‘smaller’ than the vertex groups.

§0. Introduction

The notion of quasi-isometry and the study of the relation of large-scale geometry of groups and algebraic properties has become predominant in group theory after the seminal papers of Gromov [G1,G2].

A classical theorem of Stallings ([St]) implies, that if $G$ splits over a finite group and $H$ is a group quasi-isometric to $G$ then $H$ also splits over a finite group.

Bowditch has shown recently ([Bo]) that splittings of hyperbolic groups over 2-ended groups are preserved by quasi-isometries.

By a theorem of Kapovich and Leeb [K-L] it follows that a group quasi-isometric to a non-geometric Haken 3-manifold splits over a group commensurable to a surface group. In this paper we show that group splittings are preserved by quasi-isometries in many cases. Our approach is based on asymptotic topology (‘coarse topology’) methods. Schwartz’s asymptotic version of the Jordan separation theorem ([Sch,F-S]) is our main tool. In fact we will need a stronger version of Schwartz’s theorem that has been given recently by Kapovich and Kleiner ([K-K]) in the context of their work on coarse $PD(n)$-spaces and groups. We will therefore formulate our results in this more general setting.

We say that a group $G$ is a coarse $PD(n)$ group if it acts discretely co-compactly on a coarse $PD(n)$-space. We say that $G$ is a coarse $PD(n)$ group of dimension $n$ if it is a coarse $PD(n)$-group that has an $n$-dimensional $K(G,1)$.

We note that examples of coarse $PD(n)$-groups are fundamental groups of closed aspherical $n$-manifolds.

Using the theory of coarse $PD(n)$-spaces we show that groups that are quasi-isometric to ‘trees of coarse $PD(n)$-spaces’ split. To pass from geometry to algebra
we use a recent result of Scott and Swarup ([S-S]).
We explain briefly the notation we use for graphs of groups. For more details see [Ba], [Se]. A graph of groups is given by the following data:
a) A finite graph $\Gamma$. Each edge $e \in \Gamma$ is oriented. We denote by $\partial_0 e$ the initial vertex of $e$ and by $\partial_1 e$ the terminal vertex of $e$.
b) To each vertex $v \in \Gamma$ and each edge $e \in \Gamma$ there correspond groups $A_v, A_e$. If $v = \partial_0 e$ or $v = \partial_1 e$ we have monomorphisms (respectively) $i_0 : A_e \to A_v$ and $i_1 : A_e \to A_v$. We denote this collection of groups and morphisms by $A$.
Using these data one defines the fundamental group $\pi_1(\Gamma, A)$.

**Theorem 3.1.** Let $G$ be a finitely generated group admitting a graph of groups decomposition $G = \pi_1(\Gamma, A)$ such that all edge and vertex groups are coarse $PD(n)$ groups of dimension $n$. Suppose further that $(\Gamma, A)$ is not a loop with all edge to vertex maps isomorphisms and that it is not a graph of one edge with both edge to vertex maps having as image an index 2 subgroup of the vertex group. If $H$ is quasi-isometric to $G$ then $H$ splits over a group that is quasi-isometric to an edge group of $(\Gamma, A)$.

We obtain the following corollaries:

**Corollary 3.2 ([F-M1,2]).** Let $G$ be a solvable Baumslag-Solitar group. If $H$ is a group quasi-isometric to $G$ then $H$ is commensurable to a solvable Baumslag-Solitar group.

**Corollary 3.3.** Let $G$ be a finitely generated group admitting a graph of groups decomposition $G = \pi_1(\Gamma, A)$ such that all edge and vertex groups are virtually $\mathbb{Z}^n$. Suppose further that $(\Gamma, A)$ is not a loop with all edge to vertex maps isomorphisms and that it is not a graph of one edge with both edge to vertex maps having as image an index 2 subgroup of the vertex group. If $H$ is quasi-isometric to $G$ then $H = \pi_1(\Delta, B)$ where all vertex and edge groups of $\Delta$ are virtually $\mathbb{Z}^n$.

It turns out that splittings are invariant under quasi-isometries in the case that edge groups are ‘smaller’ than vertex groups:

**Theorem 3.4.** Let $G$ be a finitely generated group admitting a graph of groups decomposition $G = \pi_1(\Gamma, A)$ such that all vertex groups are coarse $PD(n)$-groups and all edge groups are dominated by coarse $PD(n-1)$ spaces. If $H$ is quasi-isometric to $G$ then $H$ splits over some group quasi-isometric to an edge group of $\Gamma$.

We say that a group $G$ is dominated by a coarse $PD(n)$-space $X$ if there is a uniform embedding $f : G \to X$ (for more details see sec. 3). Note that a subgroup of a coarse-$PD(n)$ group is dominated by a coarse $PD(n)$-space. So for example a free group is dominated by a coarse $PD(2)$-space.
The main geometric observation on which our results are based is that the groups in theorems 3.1, 3.4 are ‘trees of spaces’. The simplest example of such groups are products $\mathbb{F}_k \times \mathbb{Z}^n$. To make the exposition easily accessible to readers not familiar with the geometry of graphs of groups and with ‘coarse $PD(n)$-spaces’ we treat this special case first in section 2. All ‘asymptotic topology’ arguments that we need are already present in this case. The link between algebra and geometry is
provided by a result of Scott and Swarup ([S-S]) generalizing the algebraic torus theorem of Dunwoody-Swenson ([D-S]).

In section 3 we explain how to generalize these arguments to graphs of groups in which all edge and vertex groups are ‘coarse PD(n)-groups of dimension n’. For this it suffices to understand the ‘tree-like shape’ of graphs of groups in general. The geometries of such groups have been described in several places (see [S-W], [Ep], [F-M1,2], [Wh]). We use similar arguments to treat the case of graphs of groups with vertex groups coarse PD(n) groups and edge groups, groups that are ‘dominated’ by coarse PD(n - 1) groups.

In section 4 we discuss how the results of this paper (and Stallings’ theorem) could be generalized and we ask some specific questions.

In the course of this work we found out that some of our results had been obtained earlier, independently, by Mosher, Sageev and Whyte. In particular they have shown a stronger version of theorem 3.1, Corollary 3.4 and some cases of theorem 3.4 ([MSW], [MSW1]). The main novelty (apart from the difference in the proofs) of this paper compared to [MSW] is that we improve n - 2 to n - 1 in theorem 3.4. So for example from our results it follows that if G, H are quasi-isometric groups and G is an amalgam of two aspherical 3-manifold groups along a surface group (or free group) then H also splits over a virtual surface (or virtually free) group. The work of [MSW] implies a similar result when G is an amalgam over Z.

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§1. Preliminaries

A (K,L)-quasi-isometry between two metric spaces X, Y is a map f : X → Y such that the following two properties are satisfied:
1) \( \frac{1}{K}d(x, y) - L \leq d(f(x), f(y)) \leq Kd(x, y) + L \) for all \( x, y \in X \).
2) For every \( y \in Y \) there is an \( x \in X \) such that \( d(y, f(x)) \leq K \).

We will usually simply say quasi-isometry instead of (K,L)-quasi-isometry. Two metric spaces X, Y are called quasi-isometric if there is a quasi-isometry \( f : X \to Y \).

A geodesic metric space is a metric space in which any two points \( x, y \) are joined by a path of length \( d(x, y) \). In what follows we will be interested in graphs that we always turn into geodesic metric spaces by giving each edge length 1.

We recall now some notation and results from [K-K]. We refer the reader to this paper for more details.

Let X be a connected locally finite simplicial complex. The 1-skeleton \( X^1 \) is a graph and we turn it into a geodesic metric space as explained above. If \( A \subset X^1 \) the r-neighborhood of A is the set of points in \( X^1 \) at distance less or equal to r from A. More generally if K is a subcomplex of X we define the r-neighborhood of \( K, N_r(K) \), to be the set of simplices intersecting \( N_r(K^1) \).

The diameter, \( \text{diam}(K) \), of K is by definition the diameter of \( K^1 \).

We call a map f : X → Y between metric spaces a uniform embedding (see [G2]) if the following two conditions are satisfied:
1) There are K, L such that for all \( x, y \in X \) we have \( d(f(x), f(y)) < Kd(x, y) + L \).
2) For every \( E > 0 \) there is \( D > 0 \) such that \( \text{diam}(A) < E \Rightarrow \text{diam}(f^{-1}A) < D \).

We say that the distortion of f is bounded by h, where h : \( \mathbb{R}^+ \to \mathbb{R}^+ \), if for all \( A \subset Y \), \( \text{diam}(f^{-1}A) \leq h(\text{diam}(A)) \).

If f satisfies only condition 1 above we say that f is (K,L) Lipschitz maps.
X is called uniformly acyclic if for each $R_1$ there is an $R_2$ such that for each subcomplex $K \subset X$ with $\text{diam}(K) < R_1$ the inclusion $K \to N_{R_2}(K)$ induces zero on reduced homology groups.

If $K \subset X$ is a subcomplex of $X$ and $R > 0$ we say that a component of $X - N_R(K)$ is **deep** if it is not contained in $N_{R_1}(K)$ for any $R_1 > 0$.

We say that $K$ coarsely separates $X$ if there is an $R > 0$ such that $X - N_R(K)$ has at least two deep components.

The appropriate context for the results in this paper seems to be that of ‘coarse $PD(n)$ spaces’. We refer to [K-K] for a definition and an exposition of the theory of ‘coarse $PD(n)$ spaces’. Important examples of ‘coarse $PD(n)$ spaces’ are uniformly acyclic PL-manifolds of bounded geometry.

We say that $X$ is a ‘coarse $n$-dimensional PL-manifold’ if $X$ is quasi-isometric to a uniformly acyclic $n$-dimensional PL-manifold of bounded geometry.

A reader not familiar with [K-K] can read this paper by replacing everywhere ‘coarse $PD(n)$ spaces’ by ‘coarse $n$-dimensional PL-manifold’. The only result that we will use from [K-K] is the coarse Jordan theorem stated below (see theorem 7.7, footnote 11 and corollary 7.8 of [K-K]).

**Proposition 1 (Coarse Jordan theorem).** Let $X, X'$ be coarse $PD(n)$, $PD(n-1)$ spaces respectively, $Z \subset X'$ and let $g : Z \to X$ be a uniform embedding such that $g$ is a $(K, L)$ lipschitz map that is a uniform embedding with distortion bounded by $f$. Then

1. If $Z = X'$ then there is an $R > 0$ such that $X - N_R(g(X'))$ has exactly 2 deep components.
2. There is an $N > 0$ such that every non-deep component of $X - g(Z)$ is contained in the $N$-neighborhood of $g(Z)$.
3. There is an $M > 0$ such that if $X - g(Z)$ has more than one deep component, $X'$ is contained in the $M$-neighborhood of $Z$.
4. If $Z = X'$ then for every $r$, each point of $N_r(g(X'))$ lies within uniform distance from each of the deep components of $X - N_r(g(X'))$.

The constants $M, N, R$ depend only on $K, L, f, X, Y$.

We note that the metric on $Z$ in the above proposition is the metric induced by $X'$.

We say that $A \subset X$ coarsely contains $B$ if for some $R, B \subset N_R(A)$. Let $f : X \to Y$ be a uniform embedding of a coarse $PD(n-1)$ space $X$ to a coarse $PD(n)$ space $Y$. Let $K > 0$ be such that $Y - N_K(f(X))$ has two deep components. We will call a deep component of $Y - N_K(f(X))$ a **half coarse $PD(n)$-space**.

We call a group $G$ a coarse $PD(n)$-**group** if it acts discretely co-compactly simplicially on a coarse $PD(n)$ space. We say that $G$ is a coarse $PD(n)$ group of **dimension** $n$ if it is a coarse $PD(n)$-group that has an $n$-dimensional $K(G, 1)$.

We call $(G; \{F_1, \ldots, F_i\})$ a coarse $PD(n)$-**pair** if:

1. $G$ acts discretely simplicially on a coarse $PD(n)$ space $X$ and there is a connected $G$-invariant subcomplex $K \subset X$ such that the stabilizer of each component of $X - K$ is conjugate to one of the $F_i$’s.
2. The $F_i$’s are coarse $PD(n-1)$ groups.

### §2. The geometry of direct products of abelian and free groups

**Theorem 2.1.** Let $G = \mathbb{F}_h \times \mathbb{Z}^b$, where $\mathbb{F}_h$ is the free group on $h > 1$ generators.
If $H$ is quasi-isometric to $G$ then $H = \pi_1(\Delta, B)$ where all vertex and edge groups of $\Delta$ are virtually $\mathbb{Z}^n$.

**Proof.** Let $\Gamma_G$ be the Cayley graph of $G$ with respect to the standard generators. $\Gamma$ is isometric to $T_{2k} \times \mathbb{R}^n$ where $T_{2k}$ is the homogeneous tree of degree $2k$. Clearly $G$ acts discretely and co-compactly on $T_{2k} \times \mathbb{R}^n$.

Let's denote $T_{2k} \times \mathbb{R}^n$ by $X$ and let $p : X \to T_{2k}$ be the natural projection from $X = T_{2k} \times \mathbb{R}^n$ to $T_{2k}$. With this notation we have:

**Lemma 2.2.** Let $f : X \to X$ be a quasi-isometry. Then for any vertex $v$ of $T_{2k}$ there is a vertex $u \in T_{2k}$ such that $f(v \times \mathbb{R}^n)$ and $u \times \mathbb{R}^n$ are at finite distance from each other.

**Proof.**

We note that if $v$ is a vertex of $T_{2k}$ $v \times \mathbb{R}^n$ separates $T_{2k} \times \mathbb{R}^n$ in more than 2 deep components.

Moreover there are geodesic rays $r_1, r_2, r_3 : [0, \infty) \to X$ such that $r_1, r_2, r_3$ lie in distinct components of $X - p^{-1}(v)$ and $d(r_1(t), p^{-1}(v)) = t$.

Let $K > 0$ be such that $N_K(f(p^{-1}(v)))$ separates $X$ in more than 2 deep components. Clearly for $K$ sufficiently big $f(r_i), i = 1, 2, 3$ are coarsely contained in distinct deep components of $X - N_K(f(p^{-1}(v)))$. We pick $K$ so that this holds. Let's call $C_i$ the deep component coarsely containing $f(r_i)$.

To simplify notation we set $S = N_K(f(p^{-1}(v)))$.

$f(r_i)$ is not necessarily connected. Let $R_i$ be the path obtained by joining $f(r_i(n))$ to $f(r_i(n + 1))$ by a geodesic path for all $n \in \mathbb{N}$. Clearly $R_i$ is coarsely contained in $C_i$. If we parametrize $R_i$ by arclength we have that $d(R_i(t), S)$ is a proper function from $[0, \infty)$ to $[0, \infty)$. Let $l_1 = p(R_1), l_2 = p(R_2)$. We pick now a geodesic $l \subset T_{2k}$ such that $p^{-1}l \cap R_1$ and $p^{-1}l \cap R_2$ are both unbounded. We explain how to find such an $l$: If $l_1, l_2$ are finite then there are closed edges $e_1, e_2$ of $T_{2k}$ such that $p^{-1}e_1 \cap R_1, p^{-1}e_2 \cap R_2$ are both unbounded, so we simply pick $l$ to be any geodesic containing both these edges. If both $l_1, l_2$ are infinite, since they are connected they contain at least one geodesic ray each. We pick therefore $l$ to be a line having an infinite intersection with both geodesic rays. If one of them is infinite and one finite we pick $l$ similarly, requiring that it has an infinite intersection with a given ray and passes from a given edge.

We have therefore that $S$ separates $p^{-1}l$. By part 3 of proposition 1 we conclude that there is a $K_1$ such that $S$ is contained in the $K_1$ neighborhood of $p^{-1}l$. We note that by parts 1, 2 of proposition 1, $p^{-1}l \cap R_3$ is bounded. We can therefore find a geodesic ray $r \in T_{2k}$ intersecting $l$ only at one point $u$, such that $p^{-1}r \cap R_3$ is unbounded. We have then $l = r_1' \cup r_2'$ and $r_1' \cap r_2' = u$ where $r_1', r_2'$ are geodesic rays. As before we have that $S$ separates both $p^{-1}(r \cup r_1')$ and $p^{-1}(r \cup r_2')$ so it lies in a finite neighborhood of both. We conclude that $S$ lies in a finite neighborhood of all three: $p^{-1}(r \cup r_1'), p^{-1}(r \cup r_2')$ and $p^{-1}l$. Therefore $S$ lies in a finite neighborhood of $p^{-1}u$.

We remark that if $f$ is an $(A, B)$-quasi-isometry then the proof above shows that there is a $C$ that depends only on $A, B$ and $X$ such that $f(v \times \mathbb{R}^n)$ and $u \times \mathbb{R}^n$ are in the $C$-neighborhood of each other. This is so because the constants in proposition 1 depend only on $A, B$. We will use this fact in the next lemma.

**Lemma 2.3.** Let $G = \mathbb{F}_k \times \mathbb{Z}^n$ where $\mathbb{F}_k$ is the free group on $k > 1$ generators. If $H$ is quasi-isometric to $G$ then $H$ splits over a group commensurable to $\mathbb{Z}^n$. 
Proof.

Let $\Gamma_H$ be the Cayley graph of $H$. If $v \in T_{2k}$ is a vertex and $f : X \to \Gamma_H$ a quasi-isometry we have that $f(v \times \mathbb{R}^n)$ coarsely separates $\Gamma_H$ to more than 2 deep components.

Let $K$ be such that $\Gamma_H - N_K(f(v \times \mathbb{R}^n))$ has more than 2 deep components. We set $S = N_K(f(v \times \mathbb{R}^n))$. Note that $\Gamma_H - S$ has a finite number of deep components.

We pick $M > 0$ such that the following holds:

For all $h \in H$ if $hS \cap S \neq \emptyset$ then $hS \subset N_M(S)$.

It follows from our remark at the end of the proof of lemma 2.2 that such an $M$ exists.

We will show that there is a subgroup $J$ of $H$ quasi-isometric to $S$ such that $\Gamma_H/J$ has more than two ends.

We fix a vertex $e \in S$. We define an equivalence relation on the set of vertices $x \in S$:

Let $g_x \in H$ such that $g_x x = e$. Let $C_1, ..., C_k$ be the deep components of $\Gamma_H - S$ and let $D_1, ..., D_m$ be the deep components of $\Gamma_H - N_M(S)$.

We note that each $D_i, i = 1, ..., m$ is contained in some $C_j, j = 1, ..., k$.

For each $g_x$ and $C_j$ we have that there are $i_1, ..., i_r$ such that $g_xC_j$ and $D_{i_1} \cup ... \cup D_{i_r}$ are at finite distance from each other. We use this to define a map $f_x : \{C_1, ..., C_k\} \to \mathcal{P}(\{D_1, ..., D_m\})$ where $f_x(C_j) = \{D_{i_1}, ..., D_{i_r}\}$ if and only if $g_xC_j$ and $D_{i_1} \cup ... \cup D_{i_r}$ are at finite distance from each other.

Let’s write now $x \sim y$ for $x, y$ vertices of $S$ if $f_x = f_y$. Clearly $\sim$ is an equivalence relation with finitely many equivalence classes.

Let $R$ be such that $B_R(e) \cap S$ contains all elements of the equivalence classes with finitely many elements and at least one element of each equivalence class with infinitely many elements.

For each vertex $y \in S$ we pick $t \in B_R(e) \cap S$, $t \sim y$ and we consider the group $J$ generated by $\{g_y^{-1}g_t\}$. Clearly $S \subset J(B_R(e) \cap S)$. We claim that $Je$ is contained in a neighborhood of $S$. Indeed for each $g \in J$ we have that $gC_j$ and $C_j$ are at finite distance from each other for all $j$. Let $A > 0$ be such that for any $v \in \Gamma_H$ we have that if $d(v, S) > A$ then $v$ lies in a deep component of $\Gamma_H - S$. One sees easily that such an $A$ exists by prop. 1, part 2.

If $Je$ is not contained in any neighborhood of $S$ then there is a $g \in J$ such that $gS \cap N_A(S) = \emptyset$. This follows again by lemma 2.2 and the remark at the end of the proof of lemma 2.2. Clearly then $S$ intersects a single component of $\Gamma_H - gS$.

Therefore there is some $C_j$ such that all $gC_j$ except one are contained in a single deep component of $\Gamma_H - S$. This however contradicts the fact that $gC_j$ is at finite distance from $C_j$ for all $j$.

We have therefore shown that $J$ is quasi-isometric to $S$. Clearly $\Gamma_H/J$ has more than two ends. By the algebraic torus theorem of Dunwoody-Swenson ([D-S]) we have that $H$ splits over a group commensurable with $\mathbb{Z}^n$. In fact by proposition 3.1 of ([D-S]) we have that $H$ splits over a group commensurable with $J$. To see this note that if $T$ is an essential track corresponding to $H$ as in lemma 2.3 of [D-S] then from lemma 2.2 it follows that no translate of $T$ crosses $T$. 

We return now to the proof of the theorem. We proceed inductively: We assume that we can write $H$ as the fundamental group of a graph of groups $H = \pi_1(\Delta, B)$ where edge groups are commensurable with $J$ and $(\Delta, B)$ can not be further refined. By this we mean that no vertex group $B_v$ of $\Delta$ admits a graph of groups refinements.

We note that each $g \in H$ stabilizes a neighborhood of $gS \cap S$. By our assumption there is a group commensurable with $J$ that stabilizes this neighborhood. But this means that $H$ has a finite decomposition into graphs of groups.

This completes the proof of the theorem.

decomposition with edge groups commensurable to $J$ and such that all edge groups $B_v$ of edges adjacent to $v$ are subgroups of vertex groups of the graph of groups decomposition of $B_v$.

By the accessibility theorem of Bestvina-Feighn ([B-F]) this procedure terminates. Each vertex group of the graph of groups decomposition, say $H$, is a non-trivial $J$-almost invariant subset of $G$ and the intersection number $\langle H, J \rangle$ is a non-trivial $J$-almost invariant subset of $G$, so by applying the algebraic torus theorem once again we see that we can refine $(\Delta, B)$, a contradiction. We conclude therefore that all edge and vertex groups are commensurable to $J$ which proves the theorem. ■

§3. Graphs of groups

To deal with splittings over coarse $PD(n)$ groups in general rather than $\mathbb{Z}^n$ we need a theorem of Scott-Swarup ([S-S]) generalizing proposition 3.1 of [D-S]. We recall their notation and results:

Definitions. Two sets $P, Q$ are almost equal if their symmetric difference is finite. If $G$ acts on the right on a set $Z$ a subset $P$ is almost invariant if $Pg$ is almost equal to $P$ for all $g \in G$.

If $G$ is a finitely generated group and $J$ is a subgroup then a subset $X$ of $G$ is $J$-almost invariant if $gX = X$ for all $g \in J$ and $J \setminus X$ is an almost invariant subset of $J \setminus G$.

$X$ is a non-trivial $J$-almost invariant subset of $G$ if, in addition, $J \setminus X$ and $J \setminus (G - X)$ are both infinite.

If $G$ splits over $J$ there is a $J$-almost invariant subset $X$ of $G$ associated to the splitting in a natural way. If $G = A \ast_J B$ let $X_A, X_B, X_J$ be complexes with $\pi_1(X_A) = A, \pi_1(X_B) = B, \pi_1(X_J) = J$ and let $X_G$ be the complex obtained as usual by gluing $X_J \times I$ to $X_A \cup X_B$, so that $\pi_1(X_G) = G$. Let $\tilde{X}_G$ be the universal covering of $X_G$ and let $p : \tilde{X}_G \to X_G$ be the covering projection. We note that each connected component of $p^{-1}(X_J \times (0, 1))$ separates $\tilde{X}$ in two sets. Each of these two sets gives a $J$-almost invariant subset.

If $X, Y$ are non-trivial $J$-invariant subsets of $G$ we say that $Y$ crosses $X$ if all the four intersections $X \cap Y, X^* \cap Y, X \cap Y^*, X^* \cap Y^*$ (where $X^* = G - X, Y^* = G - Y$) project to infinite sets in $J \setminus G$. If $X$ is a non-trivial $J$-invariant subset we say that the intersection number $i(J \setminus X, J \setminus X)$ is 0 if $gX$ does not cross $X$ for any $g \in G$.

With this notation Scott and Swarup ([S-S]) show the following:

Theorem. Let $G$ be a finitely generated group with a finitely generated subgroup $J$ such that $\Gamma_G/J$ has more than one end. If there is a non-trivial $J$ almost invariant set $X$ of $G$ such that $i(J \setminus X, J \setminus X) = 0$ then $G$ has a splitting over some subgroup $J'$ commensurable with $J$.

We note that from theorem 7.7 of [K-K] it follows that if $f : X \to Y$ is a uniform embedding of a coarse $PD(n)$-space $X$ to a coarse $PD(n)$-space $Y$ then $Y$ is coarsely contained in $f(X)$. This implies that if $h : G \to H$ is a monomorphism between coarse $PD(n)$-groups then $|H : h(G)| < \infty$.

Theorem 3.1. Let $G$ be a finitely generated group admitting a graph of groups decomposition $G \leftarrow \Gamma \rightarrow \Delta$ such that all edge and vertex groups are coarse $PD(n)$-spaces. Then $G$ is $PD(n)$.
groups of dimension \( n \). Suppose further that \((\Gamma, A)\) is not a loop with all edge to vertex maps isomorphisms and that it is not a graph of one edge with both edge to vertex maps having as image an index 2 subgroup of the vertex group. If \( H \) is quasi-isometric to \( G \) then \( H \) splits over a group that is quasi-isometric to an edge group of \((\Gamma, A)\).

**Proof.** We recall from [S-W] the topological point of view on graphs of groups: To each vertex \( v \in \Gamma \) and each edge \( e \in \Gamma \) we associate finite simplicial complexes \( X_v, X_e \) such that \( \pi_1(X_v) = A_v, \pi_1(X_e) = A_e \). Let \( I \) be the unit interval. We construct a complex \( X \) such that \( \pi_1(X) = G \) by gluing the complexes \( X_v \) and \( X_e \times I \) as follows: Let \( v \) be an endpoint of \( e \in \Gamma \), say \( v = \partial_0 e \). Then there is a monomorphism \( i_0 : A_e \rightarrow A_v \). Let \( f : X_e \rightarrow X_v \) be a simplicial map such that \( f_* = i_0 \). We identify then \((t, 0) \in X_e \times I \) to \( f(t) \in X_v \).

Similarly we define an identification between \( X_e \times \{1\} \) and \( X_v \) if \( v = \partial_1 e \).

Doing all these identifications for the vertices \( v \in \Gamma \) and the edges \( e \in \Gamma \) we obtain a complex \( X \) such that \( \pi_1(X) = G \). We metrize the 1-skeleton of \( \tilde{X} \) as usual by giving each edge length 1. We note that with this metric \( \tilde{X} \) is quasi-isometric to the Cayley graph of \( G \).

We can obtain the Bass-Serre tree \( T \) associated to \( \pi_1(\Gamma, A) \) from \( \tilde{X} \), by collapsing each copy of \( \tilde{X}_v \subset X \) to a vertex and each copy of \( \tilde{X}_e \times I \) to \( I \). This collapsing gives a \( G \)-equivariant map \( p : \tilde{X} \rightarrow T \).

We note now that if \( v \in T \) is a vertex then \( p^{-1}v \) separates \( \tilde{X} \) into more than 2 deep components. Moreover if \( l \) is an infinite geodesic in \( T \) \( p^{-1}l \) is a coarse \( PD(n+1) \)-space (see theorem 11.13 of [K-K]). As in lemma 2.2 we have that if \( h : \tilde{X} \rightarrow \tilde{X} \) is a quasi-isometry then for any vertex \( v \in T \) there is a vertex \( u \in T \) such that \( h(p^{-1}(v)) \) and \( p^{-1}(u) \) are at finite distance from each other.

Let \( \Gamma_H \) be the Cayley graph of \( H \). Arguing as in lemma 2.3 we show that there is a subgroup \( J \) of \( H \) quasi-isometric to an edge group of \((\Gamma, A)\) such that \( \Gamma_H / J \) has more than one end. As in lemma 2.3 we show that there is a connected subset \( S \) of \( \Gamma_H \) such that \( \Gamma_H - S \) has more than 1 deep components and \( S / J \) is finite. Without loss of generality we can assume that for any \( v \in S \), \( Jv \subset S \). As in lemma 2.3 we see that we can find \( J \) as above such that ,in addition, \( hC_i \) and \( C_i \) are at finite distance from each other for all deep components \( C_i \) of \( \Gamma_H - S \). We fix now \( v \in \Gamma_H \) and we identify \( H \) with the orbit \( Hv \). The set \( X = J(C_i \cap Hv) \) is clearly a non-trivial \( J \) almost invariant subset of \( H \). It is easy to verify that \( i(J \setminus X, J \setminus X) = 0 \) and theorem 4.1 follows from the result of Scott-Swarup quoted above. \( \blacksquare \)

**Corollary 3.2 ([F-M1,2]).** Let \( G \) be a solvable Baumslag-Solitar group. If \( H \) is a group quasi-isometric to \( G \) then \( H \) is commensurable to a solvable Baumslag-Solitar group.

**Proof.** By theorem 3.1 \( H \) splits over a 2-ended group. Since \( H \) is amenable and is not virtually abelian \( H \) can be written as a graph of groups with a single vertex and a single edge such that the edge group is two ended and exactly one edge to vertex map is an isomorphism. If \( a \) is an element of the edge group generating an infinite normal subgroup of the edge group and \( t \) is the generator corresponding to the edge then \( tat^{-1} = a^k \) for some \( k \in \mathbb{Z} \) and the solvable Baumslag-Solitar subgroup of \( H \) generated by \( < t, a > \) is a subgroup of finite index.

**Corollary 3.3.** Let \( G \) be a finitely generated group admitting a graph of groups decomposition \( G = \pi(\Gamma, A) \) such that all edge and vertex groups are virtually \( \mathbb{Z} \).
Suppose further that $(\Gamma, \mathcal{A})$ is not a loop with all edge to vertex maps isomorphisms and that it is not a graph of one edge with both edge to vertex maps having as image an index 2 subgroup of the vertex group. If $H$ is quasi-isometric to $G$ then $H = \pi_1(\Delta, \mathcal{B})$ where all vertex and edge groups of $\Delta$ are virtually $\mathbb{Z}^n$.

Proof. By theorem 3.1 $H$ splits over a subgroup quasi-isometric to $\mathbb{Z}^n$ and hence virtually $\mathbb{Z}^n$. We apply the same argument as in the proof of theorem 2.1 to conclude that $H = \pi_1(\Delta, \mathcal{B})$ where all vertex and edge groups of $\Delta$ are virtually $\mathbb{Z}^n$. ■

Definition. A metric space $X$ is dominated by a coarse $PD(n)$-space if there is a uniform embedding $f : Y \to X$ where $X$ is a coarse $PD(n)$-space. We say that a finitely generated group $G$ is dominated by a coarse $PD(n)$-space if $G$ equipped with the word metric is dominated by a coarse $PD(n)$ space.

Some examples: A coarse $PD(n-k)$ group is dominated by a coarse $PD(n)$ space where $k = 0, \ldots, n-1$. A free group is dominated by a coarse $PD(2)$ space.

Theorem 3.4. Let $G$ be a finitely generated group admitting a graph of groups decomposition $G = \pi_1(\Gamma, \mathcal{A})$ such that all vertex groups are coarse $PD(n)$-groups and all edge groups are dominated by coarse $PD(n-1)$ spaces. If $H$ is quasi-isometric to $G$ then $H$ splits over some group quasi-isometric to an edge group of $\Gamma$.

Proof. We construct a complex $X$ such that $\pi_1(X) = G$ as in theorem 3.1. We have as in 3.1 that there is a map $p : \tilde{X} \to T$ where $T$ is the Bass-Serre tree of $(\Gamma, \mathcal{A})$. We note that $\tilde{X}$ is contained in a neighborhood of the vertex spaces of $\tilde{X}': p^{-1}(v)$ is a coarse $PD(n)$ space and $\tilde{X}$ is contained in the 1-neighborhood of $\bigcup p^{-1}(v)$ (where $v$ ranges over vertices of $T$). Moreover if $v, w$ are vertices of $T$ adjacent to an edge $e$ then for $t > 1 N_t(p^{-1}(v)) \cap N_t(p^{-1}(w))$ is a path connected subset of $\tilde{X}$ quasi-isometric to $p^{-1}e$ (here we consider this subset equipped with its path metric). In fact $p^{-1}e$ and $N_t(p^{-1}(v)) \cap N_t(p^{-1}(w))$ are contained in a finite neighborhood of each other.

Let $f : \tilde{X} \to \Gamma_H$ be a quasi-isometry from $\tilde{X}$ to the Cayley graph of $H$. If $e$ is an edge of $T$ then $f(p^{-1}e)$ coarsely separates $\Gamma_H$. Let $e$ be an edge of $T$ adjacent to the vertices $v, w$. Let $R_0$ be such that $f(p^{-1}e) \subset N_{R_0}(f(p^{-1}v)) \cap N_{R_0}(f(p^{-1}w))$. Note that we can pick $R_0$ uniformly for all $e \in T$. We distinguish now two cases:

Case 1: $p^{-1}e$ is not quasi-isometric to a coarse $PD(n-1)$-space.

Let $r$ be such that $f(p^{-1}v), f(p^{-1}w)$ are coarsely contained in distinct components of $\Gamma_H - N_r(f(p^{-1}e))$. Let’s call $F_1, F_2$ respectively these 2 components. Again we can pick $r$ uniformly for all $e \in T$. We set $S = N_r(f(p^{-1}v)), C_1 = f(p^{-1}v), C_2 = f(p^{-1}w)$. Let $R'_1 > 2R_0$ be such that the following holds: Let $u$ be a vertex of $T$ \( C = hf(p^{-1}u) \) (where $h \in H$) and let $F$ be the component of $\Gamma_H - S$ coarsely containing $C$. Then if $x \in C - F$ we have $d(x, S) < R'_1$. Note that the existence of an $R'_1$ with this property follows from prop. 1, part 2.

Let $R_1 > R'_1$ be such the following holds: If $x \in S$ then neither $B_x(R_1) \cap C_1$ nor $B_x(R_1) \cap C_2$ is contained in the $R'_1$-neighborhood of $S$.

Again we can pick $R'_1, R_1$ uniformly for all $e, u \in T, h \in H$.

Case 2: $p^{-1}e$ is quasi-isometric to a coarse $PD(n-1)$-space.

We pick in this case too constants $r, R'_1, R_1$ with similar properties:

Let $r$ be such that $f(p^{-1}v), f(p^{-1}w)$ are not contained coarsely in the same component of $\Gamma_H - N_r(f(p^{-1}e))$. We pick $r$ so that $f(p^{-1}v), f(p^{-1}w)$ intersect each $2$ different components of $\Gamma_H - N_r(f(p^{-1}e))$. Again we can pick $R'_1, R_1$ uniformly for all $e, u \in T, h \in H$.
deep components of $\Gamma_{H} - N_{e}(f(p^{-1}e))$ along a half coarse PD($n$) space. We set $S = N_{e}(f(p^{-1}e)), C_{1} = f(p^{-1}v), C_{2} = f(p^{-1}w)$. Let $R_{1} > 2R_{0}$ be such that the following hold: If $u$ is a vertex of $T, C = h f(p^{-1}u)$ (where $h \in H$) and $F_{1}, F_{2}$ are distinct components of $\Gamma_{H} - S$ such that $C \cap F_{1}, C \cap F_{2}$ are coarse half PD($n$)-spaces then if $x \in C - (F_{1} \cup F_{2})$ then $d(x, S) < R_{1}$. We suppose further that the following holds: If $F$ is any deep component of $\Gamma_{H} - S$ that intersects $C_{1}$ (or $C_{2}$) along a half coarse PD($n$)-space then $B_{v}(R_{1}) \cap F \cap C_{1}$ is not contained in the $R_{1}$-neighborhood of $S$ (and similarly for $C_{2}$.) Again we can pick $r, R_{1}, R_{1}$ uniformly for all $e, u \in T, h \in H$.

We note that there is an $R_{2} > R_{1}$ such that the following hold:

a) for any $x \in S$ any two points in $B_{x}(R_{1})$ that lie in the same deep component can be joined by a path lying in $B_{x}(R_{2}) - S$

b) for any $v \in B_{x}(R_{1})$ $d(v, S) = d(v, S \cap B_{x}(R_{2}))$

We fix a vertex $v \in S$. For any vertex $x \in S$ we pick a $g_{x} \in H$ such that $g_{x}x = v$.

We call the set of vertices in $B_{v}(R_{2}) \cap g_{x}S$ the type of $x$. We will show the following:

**Lemma 3.5.** There is an $M > 0$ such that if $x, y$ are of the same type then $g_{x}S$ and $g_{y}S$ lie in the $M$-neighborhood of each other.

**Proof.** We note that two points in $B_{v}(R_{1})$ lie in the same deep component of $\Gamma_{H} - g_{x}S$ if and only if they lie in the same deep component of $\Gamma_{H} - g_{y}S$.

We distinguish two cases:

Case 1: $S$ is not quasi-isometric to a coarse PD($n-1$)-space.

Let $C_{1} = f(p^{-1}v), C_{2} = f(p^{-1}w)$ where $v, w$ are vertices adjacent to $e$. $g_{x}C_{1}, g_{x}C_{2}$ are coarsely contained in distinct components, say $F_{1}, F_{2}$ of $\Gamma_{H} - g_{x}S$. Let $c_{1} \in B_{v}(R_{1}) \cap g_{x}C_{1}$ such that $d(c_{1}, g_{x}S) > R_{1}$. Then $c_{1}$ lies in $g_{x}F_{1}$. Since $x, y$ are of the same type $d(c_{1}, g_{y}S) > R_{1}$ so $c_{1}$ lies in the deep component of $\Gamma_{H} - g_{y}S$ that coarsely contains $g_{x}C_{1}$. We pick similarly $c_{2} \in B_{v}(R_{1}) \cap g_{x}C_{2}$. Since $c_{1}, c_{2}$ are not contained in the same component of $\Gamma_{H} - g_{y}S$ we have that $g_{x}C_{1}, g_{x}C_{2}$ are coarsely contained in distinct deep components of $\Gamma_{H} - g_{y}S$. We claim that $N_{R_{0}}(g_{x}C_{1}) \cap N_{R_{0}}(g_{x}C_{2})$ is contained in the $R_{1} + R_{0}$ neighborhood of $g_{y}S$ so $g_{x}S$ is coarsely contained in $g_{y}S$. Indeed let $a \in N_{R_{0}}(g_{x}C_{1}) \cap N_{R_{0}}(g_{x}C_{2})$. Let $a_{1} \in g_{x}C_{1}, a_{2} \in g_{x}C_{2}$ with $d(a, a_{1}) \leq R_{0}, d(a, a_{2}) \leq R_{0}$. If $a_{1}$ (or $a_{2}$) is not contained in the deep component of $\Gamma_{H} - g_{y}S$ that contains $g_{x}C_{1}$ we have $d(a_{1}, g_{y}S) \leq R_{1}$ so $d(a, g_{y}S) \leq R_{0} + R_{1}$. Otherwise we have that $a_{1}, a_{2}$ lie in distinct components of $\Gamma_{H} - g_{y}S$ and there is a path joining them of length less or equal to $2R_{0}$. This path intersects $g_{y}S$ so in this case $a$ is at distance less than $R_{0}$ from $g_{y}S$. In the same way we see that $g_{y}S$ is coarsely contained in $g_{x}S$. Clearly if $M = R_{1} + R_{0}$ we have that $g_{x}S$ and $g_{y}S$ lie in the $M$-neighborhood of each other. We note in particular that $M$ does not depend on $x, y$.

Case 2: $S$ is quasi-isometric to a coarse PD($n-1$)-space.

Let $C_{1} = f(p^{-1}v), C_{2} = f(p^{-1}w)$ where $v, w$ are vertices adjacent to $e$. $g_{x}C_{1}, g_{x}C_{2}$ are not contained coarsely in the same component of $\Gamma_{H} - g_{x}S$. Say $g_{x}C_{1}, g_{x}C_{2}$ intersect respectively the (distinct) deep components $F_{1}, F_{2}$ of $\Gamma_{H} - g_{x}S$ along half coarse PD($n$) spaces. Let $c_{1} \in g_{x}C_{1} \cap B_{v}(R_{1})$ and $c_{2} \in g_{x}C_{2} \cap B_{v}(R_{1})$ such that $d(c_{1}, g_{x}S) > R_{1}, d(c_{2}, g_{x}S) > R_{1}$. As in case 1 we have that $c_{1}, c_{2}$ lie in distinct components of $\Gamma_{H} - g_{y}S$ and so $g_{x}C_{1} \cap F_{1}, g_{x}C_{2} \cap F_{2}$ are coarsely contained in distinct components of $\Gamma_{H} - g_{y}S$. We have then that $N_{R_{0}}(g_{x}C_{1}) \cap N_{R_{0}}(g_{x}C_{2})$ is contained in the $R_{1} + R_{0}$ neighborhood of $g_{y}S$ so $g_{x}S$ is coarsely contained in $g_{y}S$. In the same way we see that $g_{x}S$ is coarsely contained in $g_{y}S$. So if $M = R_{1} + R_{0}$
we have that that $g_x S$ and $g_y S$ lie in the $M$-neighborhood of each other. ■

We consider now the group $J_1$ generated by all elements \{${g_x^{-1}g_y}$\} where $x, y$ are of the same type. We claim that it follows from lemma 3.5 that there is an $M_1 > 0$ such that for any $h \in J_1$, if $hS \cap S \neq \emptyset$ then $hS$ is contained in the $M_1$-neighborhood of $S$.

Indeed we remark that there are finitely many types of vertices of $S$. Let $x_1, ..., x_n$ be vertices of $S$ representing these types and let $g_{x_1}, ..., g_{x_n}$ be the corresponding group elements. There is a $M_0 > 0$ such that for any $i, j \in \{1, ..., n\}$ if $g_{x_i} S$ and $g_{x_j} S$ are contained in a finite neighborhood of each other then they are contained in the $M_0$-neighborhood of each other. Consider now an $h \in J_1$ such that $hS \cap S \neq \emptyset$.

Since every generator of $J_1$ maps $S$ in a finite neighborhood of itself we have that $hS$ and $S$ are contained in a finite neighborhood of each other.

Similarly we see that for any $K > 0$ there is a $M_K > 0$ such that for any $h \in J_1$ if $hS$ intersects the $K$-neighborhood of $S$ then $hS$ is contained in the $M_K$-neighborhood of $S$.

Let $a \in hS \cap S$. We have that $g_a hS$ and $g_a S$ are contained in a finite neighborhood of each other. Therefore they are contained in an $M_0 + 2M$ neighborhood of each other. Hence we can take $M_1 = M_0 + 2M$.

We fix $R > 0$ such that $S \subset J_1(B_v(R) \cap S)$. We distinguish two cases:

Case 1: There is an $M_2 > 0$ such that for any $h \in J_1$, $hS$ is contained in the $M_2$-neighborhood of $S$. Then clearly $\Gamma_H / J_1$ has more than 1 end. Let $S' = J_1 S$. Then $\Gamma_H - S'$ has finitely many deep components, say $F_1, ..., F_n$ and $S'/J_1$ is finite.

Moreover $J_1$ acts on the set of deep components by permutations. Therefore there is a finite index subgroup of $J_1$, say $J_1'$, such that $F_1$ is a non-trivial $J_1'$-almost invariant subset of $H$. Clearly $hF_1$ does not cross $F_1$ for any $h \in H$. It follows by [S-S] that $H$ splits over a subgroup commensurable with $J_1$.

Case 2: Suppose now that no such $M_2 > 0$ exists. It follows that there is an $h_0 \in J_1$ such that $h_0 S$ does not intersect the $R_1'$-neighborhood of $S$. This implies that $\Gamma_H - (S \cup h_0 S)$ has at least 3 deep components. We can now apply the same argument as in lemma 2.3 to conclude that $H$ splits. We do this in detail here:

Let $A > M$ be such that $h_0 S \subset N_A(S)$. We set $S' = N_A(S)$. $\Gamma_H - S'$ has at least 3 deep components.

For each $x \in S$ we pick $h_x \in J_1$ such that $h_x x \in B_v(R) \cap S$. If $F_1, ..., F_k$ are the deep components of $\Gamma_H - S$ and $D_1, ..., D_m$ the deep components of $\Gamma_H - S'$ we have that for each $j$ there are $D_{j_1}, ..., D_{j_r}$ such that $h_x F_j$ and $D_{j_1} \cup ... \cup D_{j_r}$ are at finite distance from each other.

We use this to define a map $f_x : \{F_1, ..., F_k\} \rightarrow \mathcal{P}\{\{D_1, ..., D_m\}\}$ where $f_x(F_j) = \{D_{i_1}, ..., D_{i_r}\}$ if and only if $h_x F_j$ and $D_{i_1} \cup ... \cup D_{i_r}$ are at finite distance from each other.

Let's write now $x \sim y$ for $x, y$ vertices of $S$ if $f_x = f_y$. Clearly $\sim$ is an equivalence relation with finitely many equivalence classes.

Let $R_3$ be such that $B_{R_3}(v) \cap S$ contains all elements of the equivalence classes with finitely many elements and at least one element of each equivalence class with infinitely many elements.

For each vertex $y \in S$ we pick $t \in B_R(v) \cap S$, $t \sim y$ and we consider the group $J$ generated by $\{h_y^{-1}h_t\}$. Clearly $S \subset J(B_{R_3}(v))$. We note that for any $h \in J h F_i$ and $F_i$ are in a finite neighborhood of each other for all $i$.

We claim that there is an $M_3 > 0$ such that for any $h \in J$, $hS$ is contained in the $M_3$-neighborhood of $S$.
Therefore the set $S$ recall that if $x$ for any $\Gamma$ the same deep component of $\Gamma$ $H$ more than one deep component. Using the criterion of $[S,S]$ as before we conclude that $H$ splits over a group commensurable with $J$. 

We argue similarly when $S$ is a coarse $PD(n-1)$-space. It follows that there is a $M_3 > 0$ such that for any $h \in J$, $hS$ is contained in the $M_3$-neighborhood of $S$. Therefore the set $JS$ is at finite distance from $S$, $JS/J$ is finite and $\Gamma_H - JS$ has more than one deep component. Using the criterion of $[S,S]$ as before we conclude that $H$ splits over a group commensurable with $J$. 

One can get a finer result when all edge groups are virtually $\mathbb{Z}^{n-1}$:

**Theorem 3.6.** Let $G$ be a finitely generated group admitting a graph of groups decomposition $G = \pi_1(\Gamma,A)$ such that all vertex groups are coarse $PD(n)$-groups and all edge groups are virtually $\mathbb{Z}^{n-1}$. If $H$ is quasi-isometric to $G$ then $H = \pi_1(\Delta,B)$ where all edge groups of $\Delta$ are virtually $\mathbb{Z}^{n-1}$ and all vertex groups $H_v$ of $\Delta$ are either coarse $PD(n)$-groups or virtually $\mathbb{Z}^{n-1}$ or the pair $(H_v,\{H_{e_i}\})$ is a coarse $PD(n)$-pair, where $H_{e_i}$ are the edge groups of the edges containing $v$.

**Proof.** We construct a complex $X$ such that $\pi_1(X) = G$ as in theorem 3.4. We have as in 3.4 that there is a map $p : \tilde{X} \to T$ where $T$ is the Bass-Serre tree of $(\Gamma,A)$. By theorem 3.4 $H$ splits over a group $J$ commensurable to $\mathbb{Z}^{n-1}$. Moreover $J$ lies at finite distance from $f(p^{-1}e)$ for some $e \in T$, where $f : \tilde{X} \to \Gamma_H$ is a quasi-isometry. We proceed now inductively: We assume that we can write $H$ as the fundamental group of a graph of groups $H = \pi_1(\Delta,B)$ where edge groups are commensurable with $\mathbb{Z}^{n-1}$ lie at finite distance from $f(p^{-1}e)$ for some $e \in T$ and $(\Delta,B)$ can not be further refined. By this we mean that no vertex group $B_v$ of $\Delta$ admits a graph of groups decomposition with edge groups of the same type and such that all edge groups $B_e$ of edges adjacent to $v$ are subgroups of vertex groups of the graph of groups decomposition of $B_v$.

By the accessibility theorem of Bestvina Freighn ([B-F]) this procedure terminates. We note now that if $H_v$ is a vertex group of $(\Delta,B)$ then $f^{-1}(H_v)$ is coarsely contained in $p^{-1}u$ for some vertex $u \in T$. Indeed if not we could refine $(\Delta,B)$ further as in proposition 3.3.

If $H_v$ is not a coarse $PD(n-1)$-group and it is not quasi-isometric to $p^{-1}u$ we have that $H_v$ acts by quasi-isometries (via $f$) on the coarse $PD(n)$-space $p^{-1}u$ and it easily follows that the pair $(H_v,\{H_{e_i}\})$ is a coarse $PD(n)$-pair, where $H_{e_i}$ are the edge groups of the edges containing $v$. 

§4. Questions

It is reasonable to wonder whether theorems 3.1 and 3.4 can be subsumed under a theorem posing no restriction on edge groups. More precisely we have the following:

**Question 1.** Let $G$ be a finitely generated group admitting a graph of groups decomposition $G = \pi_1(\Gamma,A)$ such that all vertex groups are coarse $PD(n)$-groups. Is it true that $G$ admits a graph of groups decomposition as in theorem 3.6 with all edge groups virtually $\mathbb{Z}^{n-1}$?
true that if $H$ is quasi-isometric to $G$ then $H$ splits over some group quasi-isometric to an edge group of $\Gamma$?

The main motivation of this paper was to generalize Stallings’ theorem on groups with infinitely many ends to splittings over groups that are not necessarily finite. This has been achieved to a large extent for splittings over virtually cyclic groups (see [Bo], [P]). However generalizing Stallings’ theorem for splittings over any group poses serious difficulties. For example even in the virtually cyclic case we have that surface groups do split over $\mathbb{Z}$ but triangle groups that are quasi-isometric to them don't. In general there are examples of groups that are quasi-isometric to groups that split but which do not split themselves and even do not virtually split (that is none of their finite index subgroups splits). Moreover it is a hard problem to determine whether a group virtually splits, it is not known even for 3-manifold hyperbolic groups.

On the other hand looking closer at Stallings’ theorem one realizes that it splits in 3 cases: Groups with 2 ends, virtually free groups and groups with more than 2 ends that are not virtually free groups. Or to rephrase it in asymptotic topology terminology, the first two cases correspond to groups of asymptotic dimension 1 which are coarsely separated by subsets of asymptotic dimension 0 (compact sets) and the third case corresponds to groups of asymptotic dimension $\geq 2$ which are coarsely separated by subsets of asymptotic dimension 0 (compact sets). The first two cases can be considered as ‘exceptional’ as they belong to only 2 quasi-isometry (in fact commensurability) classes. It seems that these two cases are the harder to generalize. On the other hand the third case (the ‘codimension 2’ case) might be easier to deal with. More precisely we have the following:

**Question 2.** Let $G$ be a finitely generated group of asymptotic dimension $\geq n$. Suppose that a uniformly embedded subset $S$ of asymptotic dimension $\leq n-2$ coarsely separates the Cayley graph of $G$. Is it true then that $G$ splits?

See [G2] for a definition of asymptotic dimension.

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