I. INTRODUCTION

Understanding the nature of the interplay between strong interactions (generally emerging at the stoichiometric electron density due to the possibility of umklapp scattering) and dimensionality is an important open matter relevant to a large class of materials including the cuprate superconductors and quasi one-dimensional (1D) organic conductors. The latter can be viewed as possible realizations of coupled one-dimensional Hubbard chains at half-filling where electrons can hop from chain to chain (although the quarter-filled chains are weakly dimerized allowing half-filled umklapps, there is increasing evidence that quarter-filled umklapp scattering dominates the physics). In particular, the Fabre salt family \((\text{TMTSF})_2X\) \((X = \text{ClO}_4, \text{Br} \text{and PF}_6)\) displays insulating behavior at ambient pressure up to quite high temperatures. This reflects the presence of strong umklapp scattering (interaction, \(U\)) along the chains due to the commensurate filling resulting in a 1D Mott transition; essentially one has virtually uncoupled insulating chains \((\text{confinement})\).

By contrast, substitution of Se for S to create the Bechgaard salt \((\text{TMTSF})_2\text{PF}_6\), increases the hopping between chains \((t_{\perp})\) or alternately decreases the dimerization sufficiently to delocalize particles in the transverse direction (inducing deconfinement). The system thus exhibits a crossover to a regime of metallic planes (as \(t_{\perp}\) scales to strong coupling faster than \(U\)). Experimentally this dimensional crossover takes place around 100K in \((\text{TMTSF})_2\text{PF}_6\); This is manifested by a change from \(T\) (single-chain Luttinger liquid) to \(T^2\) (Fermi liquid) behavior of the dc transport along the chain axis. However, such compounds are located quite close to the confinement transition such that the optical conductivity of \((\text{TMTSF})_2\text{PF}_6\) exhibits many features in common with the single-chain Mott insulator \((\text{TMTTF})_2\text{PF}_6\). Indeed, only 1% of the spectral weight contributes to the dc (Fermi liquid) transport. Some theoretical attempts to reproduce this result (which bears a superficial resemblance to our half-filled chains for small \(N\) (see Fig. 1)), have been performed recently using Dynamical Mean Field Theory (DMFT) and RPA.

Here, we are additionally motivated to tackle the physics of 2D Cu-O planes of cuprate materials at and close to half-filling where Mott physics (“Mottness”) is also ubiquitous. We systematically investigate the behavior of the optical conductivity for half-filled \(N\)-leg Hubbard ladders (=coupled chains) far into the “deconfinement” regime (the bare transverse hopping amplitude is almost equal to the longitudinal hopping amplitude). For weak on-site repulsion, we can make use of our previous band approach for half-filled \(N\)-leg Hubbard ladders. This allows us to properly take into account the renormalization of the different coupling channels (umklapp, Cooper,...) and also to provide a rigorous study of the ground state. The \(N\)-leg Hubbard ladder is equivalent to an \(N\)-band model where at half-filling the Fermi velocities obey \(v_1 = v_N < v_2 = v_{N-1} < ...\).

For small \(N\), the model results in a cascade of energy scales, where band pairs \((i, N + 1 - i)\) successively flow to the D-Mott state which possesses an enlarged SO(8) symmetry; This is reminiscent of the two-leg ladder behavior. This leads to a complex behavior of the optical conductivity: a Drude peak and a high-frequency particle-hole continuum coexist, with sharp exciton peaks appearing below the continuum reflecting the underlying SO(8) symmetry.

By increasing considerably the number of chains, “four-band” couplings (Fig. 2) become relevant on the 2D Fermi surface. At sufficiently low temperatures, a 2D insulating transition with strongly enhanced SDW corre-
lations arise. The low-energy physics already converges to that of the purely 2D Hubbard model. We rigorously establish that charge excitations above the Mott gap consist of bound hole-pairs (preformed pairs) and that the excitonic peaks cannot survive in this limit. This could be relevant to an understanding of optical conductivity measurements on the high-$T_c$ cuprates at and close to half-filling. Indeed, umklapp scattering should not be ignored in the underdoped regime of the cuprates–this is the point we are making.

II. THE MODEL

Our starting point is the N-leg Hubbard ladder model,

$$H_{Kin} = -t \sum_{x,s} d_{i,s}^\dagger(x+a) d_{i,s}(x) + h.c. - t_\perp \sum_{x,i,s} d_{i+1,s}^\dagger(x) d_{i,s}(x) + h.c.;$$

(1)

t and $t_\perp$ denote the hopping matrix elements along and perpendicular to the chains, $a$ is a lattice step, and $d_{i,s}(x)$ annihilates an electron with spin $s$ on chain $i$ at the rung $x$. The interaction term reads

$$H_{Int} = U \sum_{i,x} d_{i,x}^\dagger(x) d_{i,x}^\dagger(x) d_{i,x}(x),$$

(2)

where $U$ is the on-site Hubbard repulsion. Again, here we are investigating the perfect “deconfinement” regime which implies that the perpendicular hopping $t_\perp$ is sufficiently large to deconfine all the electrons in the transverse direction. More precisely, below, we consider the weak-interaction limit $0 < U \ll (t, t_\perp)$ where tractable (and rigorous) calculations are indeed possible. It is convenient to use the band picture where the kinetic part simply takes a diagonal form:

$$H_{Kin} = \sum_{i=1...N,s} \int dk \epsilon_i(k) \Psi_{i,s}^\dagger(k) \Psi_{i,s}(k);$$

(3)

$\Psi_{i,s}^\dagger$ and $\Psi_{i,s}$ are the creation and annihilation operators for the band $i$ and

$$\epsilon_i(k) = -2t \cos(ka) - 2t_\perp \cos(k_\perp a).$$

(4)

The transverse (Fermi) momenta here simply obey $k_\perp = \pm \pi i/(a(N+1))$ and at half-filling the longitudinal momenta are exactly determined by $\epsilon_i(F_i) = 0$. The resulting Fermi velocities $v_i = \frac{2a}{\hbar} \sin(k_i) \hbar$ then take the form

$$v_i = v_i = \frac{2a}{\hbar} \sqrt{t^2 - \{t_\perp \cos[\pi i/(N+1)]\}^2},$$

(5)

where $\tilde{i} = N + 1 - i$. The transformation from chain to band (with open boundary conditions) reads:

$$d_{i,s} = \sum_m \sqrt{\frac{2}{N+1}} \sin \left( \frac{\pi m i}{N+1} \right) \Psi_{m,s}.$$
The low-energy physics emerging from bands 1 and 3 depends on a single coupling $g$ (whose bare value is of the order of $U$). It is straightforward to show that the interacting part of $H_{1,3}$ both opens a charge and spin gap equal to $2\Delta_1 = 4g < \Psi_{1s}^\dagger \Psi_{1s} > \sim 2T_1$. The ground state corresponds to the “D-Mott state”: a Mott insulator having short-range pairing correlations with approximate $d$-wave symmetry. Additionally, fluctuations of the gap around its vacuum value can generate attractive interactions between the fermions $\Psi_{a}$ leading to the formation of excitons, whose mass satisfies $M_1 = V_d a$ (i.e. is smaller than the charge gap of the (localized) preformed-pair continuum). Unlike the single-chain, here excitons already appear for purely on-site repulsion.

Band 2 is described by a single chain at half-filling ($\mu = 0$), which is embodied by gapless spinons (spin-1/2 excitations) and the following charge Hamiltonian:

$$H_2^c = \int dx \left\{ \frac{u_2}{2} \left[ K_\rho (\partial_x \Phi_\rho)^2 + K_\rho (\partial_x \Theta_\rho)^2 \right] - \frac{g_\rho}{(\pi a)^2} \cos(\sqrt{8\pi} \Phi_\rho) - \mu \phi \Phi_\rho \right\}; \quad (9)$$

$\partial_x \Phi_\rho \partial_x \Theta_\rho$ describes charge density fluctuations and $\partial_x \Theta_\rho$ current excitations. $g_\rho$ is of the order of $U$ and $u_2$ is almost equal to $v_2$. It is immediate to check that $H_2^c$ induces a charge gap $2\Delta_2 \sim 2T_2 < 2T_1$. Let us now compute the optical conductivity of the three-leg ladder.

Using formula (9), the electrical current can be written in the chain basis as:

$$J(x) = \frac{e}{2i\sqrt{\pi}} \sum_{is} \frac{1}{\delta t_i} \left( d_{is}^\dagger(x+a) d_{is}(x) - d_{is}(x+a) d_{is}^\dagger(x) \right) = \frac{e}{2i\sqrt{\pi}} \frac{(v_{D1} + v_{D2})}{a} \left( \Psi_{1s}^\dagger(x+a) \Psi_{1s}(x) - \Psi_{1s}(x+a) \Psi_{1s}^\dagger(x) \right)$$

where we have summed up over the contribution coming from each spin $s$ and each leg $i = 1, 2, 3$ of the ladder. The Fermi surface (FS) (drawn for $t_z = t$) exhibits a blatant truncation, meaning that bands 1 and 3 form the D-Mott state opening a gap whereas the band 2 is still metallic (and described by a Luttinger theory). The optical conductivity then exhibits a Drude peak (due to band 2), a high-energy continuum (treated here at the mean field level) above the charge gap $2\Delta_1$ and a sharp exciton peak due to the SO(8) symmetry of the D-Mott state.

FIG. 1: Optical conductivity of the three-band model for $T_2 \ll T < T_1$. The Fermi surface (FS) (drawn for $t_z = t$) exhibits a blatant truncation, meaning that bands 1 and 3 form the D-Mott state opening a gap whereas the band 2 is still metallic (and described by a Luttinger theory). The optical conductivity then exhibits a Drude peak (due to band 2), a high-energy continuum (treated here at the mean field level) above the charge gap $2\Delta_1$ and a sharp exciton peak due to the SO(8) symmetry of the D-Mott state.

In the three-leg ladder, we find the following current:

$$J = \frac{\sqrt{2}}{\pi} e \left( v_{D1} \partial_x \Theta_\rho + \frac{v_{D1} + v_{D2}}{2} \sqrt{2} \sin(k_{F1} a) \partial_x \Theta_\rho \right) = J_{1\text{band}} + J_{2\text{band}}, \quad (13)$$

where one can re-express the 2-band contribution in terms of Majorana fields (real chiral fermions $\psi_{R/L} = \eta_{R/La} - i \eta_{R/Lb}$) as

$$J_{2\text{band}} = -\frac{(v_{D1} + v_{D2}) e}{\sqrt{2\pi a}} \sin(k_{F1} a) \bar{\psi}_1 \tau^z \psi_1 = \frac{(v_{D1} + v_{D2}) e}{\sqrt{2\pi a}} \sin(k_{F1} a) (\eta_{R1} \eta_{R2} - \eta_{L1} \eta_{L2}). \quad (14)$$

Using Eq. 12, the contribution of these terms to the conductivity is,

$$\text{Re } \sigma_{2\text{band}}(\omega) = -\frac{e^2 (v_{D1} + v_{D2})^2}{\pi a^2 \hbar \omega} \sum_{P \sim R, L} \text{Re } \langle T_{\tau} \eta_{P1} \eta_{P2}(x, \tau) \eta_{P1} \eta_{P2}(0, 0) \rangle \int dx$$

As pointed out in Ref. 12, since the two-band model is equivalent to the SO(8) Gross-Neveu model whose excitation spectrum is known, below the D-Mott gap mass $M_1$ bound-states of charge $\pm 2e$ fundamental fermions can form. The latter factor of Eq. 15 bears resemblance to see Ref. 12. The low-energy physics emerging from bands 1 and 3 depends on a single coupling $g$ (whose bare value is of the order of $U$). It is straightforward to show that the interacting part of $H_{1,3}$ both opens a charge and spin gap equal to $2\Delta_1 = 4g < \Psi_{1s}^\dagger \Psi_{1s} > \sim 2T_1$. The ground state corresponds to the “D-Mott state”: a Mott insulator having short-range pairing correlations with approxi-
the ($k = 0$) Green’s function of a localized particle of mass $M_1$, so that

\[- \int dx \int dr e^{i \omega r} e^{i \underx} < T_r \xi_1(x, r)(\xi_2(0, 0)) > \approx - \int dx \int dr e^{i \omega r} A e^{- \frac{|x|}{\bar{M}_1} - \frac{|r|}{\bar{M}_1}}\]

where we have introduced a normalization constant, $A$, the length of the system, $L$, which we can take to be infinite relative to the correlation length, $\xi$ as we have power law correlations in this system. This means that for $\omega < 2\Delta_1$, the 2-band contribution to the optical conductivity at T=0 should be

\[\text{Re} \sigma_{\text{band}}(\omega) = \frac{2e^2(v_{D1} + v_{D2})^2 \xi^2 \sin^2(k_{F1}a)}{a^2\hbar\omega} \delta(\omega - M_1).\]

As in principle there is only one velocity scale in the (massive) SO(8) Gross-Neveu model, $v_1$, one might expect to find $(v_{D1} + v_{D2}) \propto v_1$. However, one might expect the opening of this gap to strongly renormalize $v_1$.

An interesting situation emerges when the Fermi surface is partially truncated, i.e., when $T_3 \ll T \ll T_1$: Band 2 is still metallic whereas bands 1 and 3 already form a D-Mott insulating state. In fact, the umklapp term $-\frac{g_0}{\pi a} \cos(\sqrt{8\pi} \Phi_p)$ is still small, because its renormalization has been (completely) cutoff by thermal effects. Band 2 then behaves as a single-chain Luttinger liquid which produces a prominent Drude peak of height $\sim 2e^2v_{D1}/\hbar \sim 2e^2u_\mu K_\rho/\hbar$ at $\omega = 0$; from this we see that $v_{D1} = v_2$, the Fermi energy of the second band. Moreover, following Ref. [12], the current operator $\Psi_1^\dagger \tau^2 \Psi_1$ clearly excites the mass $\sqrt{8}\Delta_1$ excitons, as well as higher energy continuum scattering states (preformed pairs) with energy above $2\Delta_1$. For clarity’s sake, results have been summarized in Fig. 1. At very high-frequency, we approximately recover the behavior of a single-chain at half-filling, i.e., $\sigma(\omega) \propto \omega^{-1}$ (see Appendix A).

IV. SMALL N: THE EFFECTS OF TEMPERATURE

As mentioned previously, one certainly would expect a non-trivial role to be played by the temperature of the system. While at strictly $T=0$, one would expect to see a $\delta$-function peak arising from the bound-state excitonic contribution to the optical conductivity, this peak should be thermally broadened. At $T \neq 0$, a mean field treatment would predict the spectral peak in the high-energy continuum to have a square-root singularity reflecting the van Hove singularity at the bottom of a band. At $T \neq 0$, the scattering time between the fermions $\Psi_1$ becomes finite and one would expect $\sigma(\omega = 2\Delta_1, T) \propto e^{\Delta_1/T}$ (finite). On the other hand, since the SO(8) Gross-Neveu model is integrable, the optical conductivity for this state can be found exactly in principle, as it consists of the bound-state, 2 particle scattering, and higher particle scattering contributions. Konik and Ludwig have calculated the (zero temperature) optical conductivity of the bound-state and the 2-particle scattering contribution of this model to be ($k=0$),

\[\text{Re} \sigma_{\text{band}}(\omega) = A^2 \left[ \frac{\delta(\omega - \sqrt{3}m)}{\frac{(\omega)^{1/3}}{9m^2}} \right] \sqrt{\frac{3}{\omega}} \exp \left[ -2 \int_0^\infty dx \frac{G_c(x) \sin^2(x/3)}{\sin(x)} \right] \]

\[+ \theta(\omega - 4m^2) \frac{12m^2}{(\omega^2 - 3m^2)^2} \sqrt{\omega^2 - 4m^2} \omega \exp \left[ \int_0^\infty \frac{dx}{x} \right] \]

\[\times \frac{G_c(x)(1 - \cosh(x) \cos(\frac{x}{2}))}{\sin(x)} \]
tice that the square-root singularity predicted by mean field theory does not survive in the exact calculation. Integration of their result, (assuming higher order scattering processes do not dominate) shows that an astounding 69% of the spectral weight of the two-leg ladder (at zero temperature) is expected to lie within the bound-state. As noted, their results are exact up to the onset of the three-particle continuum, provided one stays at $\omega < \omega_c$, where $\omega_c$ is the temperature at which the RG couplings reach their approximate SO(8) coupling ratios. Above this frequency, presumably the chains will begin to behave more and more as individual chains, such that one might expect a crossover from the $\frac{1}{\omega}$ high frequency behavior found by Konig and Ludwig to an approximate $\frac{1}{\omega^2}$ as found by Giamarchi. The effects of such a crossover have been considered in Appendix A.

In the hopes of making a more simple comparison to experimental curves, we here make the simplest assumptions consistent with a low temperature broadening of the bound-state Gross-Neveu peaks following the treatment by Lin et al. There, it was argued that for temperatures $T \lesssim m$, one could essentially solve the classical Boltzmann equation to obtain a particle density due to thermal excitations of $\mu^{\text{class}}$.

Let us now increase considerably the number of chains (we consider the solvable case where the energy difference between band pairs $\sim 1/N$ is larger than the relevant gaps). As long as “4-band” interactions between bands $(i,k,i',k')$ $(kF_i \rightarrow kF_{i'},k_{F,i'} \rightarrow k_{F,i})$ –which stand for 2D-like antiferromagnetic interactions (Fig. 3)– remain small, our system is a bunch of (almost) decoupled bands and is still 1D-like. However, at low-energy, the system clearly behaves as in 2D (See Ref. [13])! Indeed, two-band and single-band interactions will be dominantly

![Graph showing optical conductivity of the N-band model for large (but finite) N and $T < T_{sdw}$. 2D antiferromagnetic (AFM) interactions break the SO(8) symmetry (exciton peaks vanish); The system flows to a 2D SDW state with a uniform charge gap.](image-url)
renormalized by the “four-band” interactions: Certain couplings of the band pairs \((i, i)\) still grow and tend to approach fixed ratios, however these differ from the small \(N\) ratios. Furthermore, for bands \(k\) and \(i\) which are close together on the FS, as \(k \rightarrow i\), the four-band couplings become the same as the corresponding two-band couplings. The one-loop RGEs and theoretical details are given explicitly in Ref.\[12\]. To summarize, all the band pairs now strongly interact on the 2D FS, and this leads to a unique single-particle (Mott) gap \(\Delta\) which can be calculated with the asymptotic ratios. Using Eq. \(7\) for \(t_{\perp} \rightarrow t\), we find

\[
\Delta \sim T_{sdw} = t e^{-\alpha t/(U \ln N)}. \tag{19}
\]

The logarithmic corrections come from the fact that \(v t_{\perp} \approx t a/N\), a precursor of van Hove singularities in 2D. Let us repeat that here \(\ln N < t/U\) for the (strict) validity of our calculations. In this limit, it is yet possible to derive the ground-state properties using bosonization

The pinning of the antisymmetric spin mode between bands \(i\) and \(\bar{i}\), \(\Phi_{\bar{i}i-} \approx 0\), produces spinon confinement leaving as physical particles spin-1 magnons. In the symmetric spin sector, only the differences \(\Theta_{\bar{i}i+} - \Theta_{\bar{k}k+}\) are fixed, such that the total magnon mode(s), given by \(\Theta_{\bar{i}i} = \sqrt{2/N} \sum_{i=1}^{N/2} \Theta_{\bar{i}i+}\) and \(\Phi_{\bar{i}i} = \sqrt{2/N} \sum_{i=1}^{N/2} \Phi_{\bar{i}i+}\), remain gapless. The spin system behaves increasingly as a 2D SDW in agreement with the strong \(U\) limit, and 2D AFM interactions explicitly break the \(SO(8)\) symmetry of individual band pairs as it becomes more favorable to pin the mode \(\Theta_{\bar{i}i+} - \Theta_{\bar{k}k+}\) rather than \(\Phi_{\bar{i}i+}\). Finally, 2D AFM interactions do not much affect the ground-state charge properties of band pairs, i.e., as in the small-\(N\) limit \(\Phi_{\bar{i}i+} \approx 0\) (symmetric charge mode) and \(\Theta_{\bar{i}i-} \approx 0\) (antisymmetric superfluid mode).

Let us now discuss the optical conductivity in this large-\(N\) limit. First, one can easily show that exciton peaks (that were a blatant signature of the underlying \(SO(8)\) symmetry for small-\(N\)) cannot persist. Integrating out the spin sector -which now produces antiferromagnetism and then totally decouples from the gapped charge sector at low energy- the symmetric charge mode \(\Phi_{\bar{i}i+}\) is simply described by a Sine-Gordon model [11] with \(\beta \approx \sqrt{4\pi}\). The spectrum now contains only solitons and anti-solitons (bound hole- (electron) pairs) with dispersion \(\epsilon(p) = \sqrt{p^2 + \Delta^2}\). At \(\omega \approx 0\) and \(T \ll \Delta\), the density of excited carriers evolves like \(n \approx e^{-\Delta/T}\), which implies that the Drude peak has an exponentially vanishing weight, reflecting the simultaneous opening of a Mott gap on the whole 2D FS. Finally, at high-frequency, we still find \(\sigma(\omega) \propto \omega^{-1}\) (we have almost decoupled bands).

VI. CONCLUSIONS

In closing, we are able to provide a microscopic (low energy) theory for half-filled \(N\)-leg Hubbard ladders – even when \(N\) is very large– taking into account both the ubiquity of Mott physics and the emergence of prominent antiferromagnetism. At small \(N\), we have demonstrated that it is at least possible within the Hubbard model at half-filling for a system to exhibit metallic behavior over a finite temperature range while maintaining a large Mott contribution, a situation not unlike that seen in \((\text{TMTSF})_2\beta \text{PF}_6\)-although the \(T^2\) resistivity encountered in that case has not been recovered here, the relative spectral weight of the Drude feature would be \(\mathcal{O}(\frac{1}{t})\) rather than \(1\%), and the relevant energy scales for this system are well outside the range of validity of our method. At high frequencies one should be able to employ a memory function approach to extract a detailed frequency dependence of the optical conductivity even in regions of moderate coupling. For large \(N\) (and \(U < \ell/\ln N, t_{\perp}/\ln N\)), the Mott gap opens simultaneously on the 2D FS and, as a precursor of superconductivity, charge excitations resemble hard-core bosons (preformed pairs), whereas spin excitations are gapless magnons (bosons). The optical conductivity is mostly sensitive to the Mott gap opening on the 2D FS (not to the long-range magnetism) and excitons cannot persist! This seems to be in agreement with results on half-filled cuprates [12]. When doping, we stress the occurrence of a spin gap and the progressive growth of \(d\)-wave phase coherence on the 2D FS. In order to properly discuss the pseudogap phase of cuprates, we must include the effect of a next nearest neighbor hopping \(t'\). It is conceivable that such a term might provide a mechanism for the creation of the “mid-gap” states of the doped cuprates [13,22] (which bear a qualitative resemblance to the two-leg ladder contribution as outlined in Fig. 2), if the effective value of \(U\) were to vary substantially around the Fermi surface.

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APPENDIX A: GENERIC FEATURES FROM ONE TO THREE CHAINS

A serious problem for this theory is the weakness of \(U\) required for the theory to flow to the strong coupling \(SO(8)\) fixed point. In particular, this imposes that our gap sizes and hence all of our temperature scales are outrageously low, as the price to pay for the rigor of our treatment. In particular, to actually reach \(SO(8)\) within the validity of the 1 loop RG, the half-filled two-leg ladder requires \(U \lesssim 2 \times 10^{-6}\), yielding a gap \(\Delta \sim t e^{-500,000}\), which for all intents and purposes experimentally might as well be zero. This can be seen from Fig.4, an explicit plot of the ratios of the couplings for the two-leg ladder as a function of the normalized number of steps. For a definition of the couplings of the two and three leg ladder, the reader is referred to Ref.11 and Ref.22. One sees that the curves described by the 1-loop RG equations are universal, although the region of validity decreases as \(U\) increases. In the inset, this is seen as a function of the maximal coupling, with ratios expected to reach 1 and 0.
if the SO(8) fixed point is reached.

FIG. 4: Validity of the 1-loop RG: (main) Scaling of the RG equations as a function of the number of steps, normalized by the point at which approximate SO(8) symmetry is found \((l_{max})\). For a definition of the couplings of the two and three leg ladder, the reader is referred to Ref. 11 and Ref. 22. \(c_{12}^{\prime}, c_{12}, u_{12}^{\prime}\) and \(u_{12}^{\prime}\) have been plotted for \(U=10^{-8}\) due to their close proximity, while the remaining curves have been plotted for \(U=10^{-6}, 10^{-4}, 10^{-2}, 1\) with \(l_{max}=(5)099340, 508230, 5118.84, 51.7196, 0.5174\) respectively. The vertical black dashed line delineates the point where the 1-loop RG equations are expected to break down for \(U=10^{-2}\), the vertical gray dotted line for \(U=10^{-4}\). (inset) The same curves plotted as a function of the ratio of the coupling \(c_{12}^{\prime}/c_{12}\). The vertical dashed line at 0.2, represents the point beyond which the 1-loop RG should not be valid. In principle, these curves could be used to extract the frequency dependence of the umklapp contributions to the two- and three- leg ladders as sketched in Fig. 7 and Fig. 8.

Even in the absence of SO(8) symmetry, one clearly can say something about the evolution of the couplings (until the higher order contributions in \(U\) become relevant), which means that in principle one could calculate the \(\omega\)-dependence of the optical conductivity at high frequencies for the two and three leg ladders using a memory function approach akin to Giamarchi’s work on the single chain. In this spirit, here we provide a heuristic sketch of such a derivation for the 3-leg ladder, and what one would expect to be the region of its applicability of this perturbative approach. At very high frequencies, one would expect the optical conductivity to break into two components roughly as,

\[
\sigma(\omega) = \frac{2e^2u_{\rho}K_{\rho}}{h(\omega + M_{1\text{band}}(\omega))} + \frac{4e^2u_{\rho+}K_{\rho+}}{h(\omega + M_{2\text{band}}(\omega))},
\]

(20)

where the memory functions, \(M_{1\text{band}}(\omega)\) and \(M_{2\text{band}}(\omega)\) can be computed perturbatively in linear response theory as outlined in Ref. 19 and 24 respectively. In particular one finds that \(M_{\text{band}}\) in the limit \(\omega \gg T\),

\[
M_{\text{band}} = \kappa g_{\rho}^2 T^2 (1 - 2K_{\rho}) \sin(2\pi K_{\rho}) e^{-i\pi(2K_{\rho} - 1)\omega} 4K_{\rho}^{-3},
\]

(21)

where \(\kappa \approx \frac{K_{\rho}}{\pi u_{\rho}} \left(\frac{u_{\rho}}{u_{\rho}}\right) 4K_{\rho}^{-2}\), and

\[
M_{2\text{band}} = G_{\chi} \sum_{\chi=1}^{4} \left( (g_{\chi})^2 T^2 (1 - (K_{\chi})) \sin(\pi(\pi K_{\chi})) \times e^{-i\pi(K_{\chi} - 1)\omega} (2K_{\chi}^{-3}) \right),
\]

(22)

where \(G_{\chi} \approx \frac{K_{\rho+}^2\sin^2(\pi K_{\rho+})}{4\pi a^2} 2K_{\chi}^{-2}\),

\[
g_{\chi} = \begin{cases} 
2u_{1331}^{\sigma} & \chi = 1 \\
u_{1331}^{\sigma} + 4u_{1331}^{\rho} & \chi = 2 \\
u_{1331}^{\sigma} - 4u_{1331}^{\rho} & \chi = 3 \\
16u_{1133}^{\sigma} & \chi = 4,
\end{cases}
\]

and

\[
K_{\chi} = \begin{cases} 
K_{\rho+} + K_{\rho+} & \chi = 1 \\
K_{\rho+} - K_{\rho+} & \chi = 2 \\
K_{\rho+} & \chi = 3 \\
K_{\rho+} & \chi = 4.
\end{cases}
\]

For the three-leg ladder, the Luttinger coefficients can be written as, \(K_{\rho} = \sqrt{\frac{\pi u_{1331}^{\sigma}}{\pi u_{1331}^{\rho} + f_{12}^{\rho}}}, K_{\rho+} = \sqrt{\frac{\pi u_{1331}^{\rho} - f_{12}^{\rho}}{\pi u_{1331}^{\rho} + f_{12}^{\rho}}},\)

\[
K_{\rho-} = \sqrt{\frac{\pi u_{1331}^{\rho} - f_{12}^{\rho}}{\pi u_{1331}^{\rho} + f_{12}^{\rho}}}, K_{\rho+} = \sqrt{\frac{\pi u_{1331}^{\rho} + f_{12}^{\rho}}{\pi u_{1331}^{\rho} + f_{12}^{\rho}}}, \text{ and}
\]

\[
K_{\sigma-} = \sqrt{\frac{\pi u_{1331}^{\rho} + f_{12}^{\rho}}{\pi u_{1331}^{\rho} + f_{12}^{\rho}}}.\]

In the high frequency limit, all couplings, with the exception of \(u_{1331}^{\sigma}\), which vanishes so that one recovers the form of Ref. 24, \(U\) are of order \(Ua/\hbar\) (if one is interested in the exact values, the reader is referred to Table I of Ref. 22), such that the Luttinger coefficients all closely approach 1. Eq. 22 is then the sum of three identical terms. In particular, one finds that \(\Im M(\omega) \approx U^2 \omega^{2K_{\rho+} - 1} \Gamma^2 (-K_{\rho+}) \sin^2 (\pi K_{\rho+})\). In the opposite limit, the strong coupling SO(8) fixed point corresponds to \(0 < g \simeq c_{13}^{\sigma} \simeq 4 f_{13}^{\rho} \simeq 4 f_{13}^{\rho} \simeq 4 u_{1331}^{\rho} \simeq 8 u_{1133}^{\rho} \simeq u_{1331}^{\rho} \simeq e_{11}^{\sigma}\), so that one sees that while \(K_{\rho}\) remains close to 1, spin and charge gaps drive \(K_{\rho+} \rightarrow 1\), \(K_{\sigma+} \rightarrow 0\). If this limit were strictly attainable, then \(g_{3} = 0\), and Eq. 22 again would become the sum of three identical terms, with a naive \(\omega^{-3}\) dependence to \(\Im M(\omega)\), although it is essential to note both that: the coefficient of such a term would also be highly frequency dependent, and for the small frequencies required for the validity of our theory, \(M(\omega)\) would outgrow the perturbative regime required for the memory function’s linear response to be valid (despite the smallness of \(G\)) a little before this point.

Unlike at the strong coupling point, in the high frequency limit, 3-band umklapp processes are also relevant,
with,

$$b_{\nu} = \begin{cases} 2\nu 223 & \nu = 1 \\ u_{123}^2 + 4u_{123}^2 & \nu = 2 \\ u_{123}^2 - 4u_{123}^2 & \nu = 3 \\ u_{123}^2 + 2K_{\rho} & \nu = 4, \end{cases}$$

and

$$K_{\nu} = \begin{cases} \frac{1}{4} (K_{\sigma -} + K_{\rho -} + \frac{1}{K_{\rho +}} + \frac{1}{K_{\sigma +}} + 2K_{\rho} + 2K_{\sigma}) & \nu = 1 \\ \frac{1}{4} (K_{\sigma -} + K_{\rho -} + \frac{1}{K_{\sigma +}} + \frac{1}{K_{\sigma +}} + 2K_{\rho} + 2K_{\sigma}) & \nu = 2 \\ \frac{1}{4} (K_{\sigma +} + K_{\rho +} + \frac{1}{K_{\sigma +}} + \frac{1}{K_{\rho +}} + 2K_{\rho} + 2K_{\sigma}) & \nu = 3 \\ \frac{1}{4} (K_{\sigma -} + K_{\rho -} + \frac{1}{K_{\sigma +}} + \frac{1}{K_{\sigma +}} + 2K_{\rho} + 2K_{\sigma}) & \nu = 4. \end{cases}$$

Here, $K_{\sigma} = \sqrt{\frac{4\nu^2 - \omega^2}{\pi}}$.

The real part of the optical conductivity arises as,

$$\Re \sigma_{2\text{band}}(\omega) = \frac{4e^2 \rho K_{F} (3m(M_{2\text{band}}(\omega)))^{2/3}}{\hbar (\omega + \Re (M_{2\text{band}}(\omega))} ,$$

the condition $0.2 \geq \omega^{-1}[M_{2\text{band}}(\omega)]$ approximates the region where this formula can be applied. On a 3-leg ladder the region about which our linearization is valid is $\sim \frac{1}{\pi}$ of the band-width ($4t$). This gives a high energy cut-off on the frequencies we can describe for the 3-leg ladder: $\omega < \frac{4t}{\pi}$ so that one can describe a large range of frequencies (see Fig. 5). This constraint becomes more severe for N large—for N=16 one is restricted to $\omega < 0.08t$—but is expected to still admit a region where the memory function returns $\sigma(\omega) \approx \frac{1}{\omega}$.

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FIG. 6: The effects of temperature on a half-filled single chain system in weak coupling. Here, \( \tau = 2\pi e^{A/T}/T \), the 1/\( \omega \) term is cut off at high frequencies by \( K_\rho \rightarrow 1 \), where \( K_\rho \) and \( u_\rho \) are the Luttinger parameters of the single chain, and at intermediate temperatures we would expect to see a partial shift of weight from above the gap to form an \( \omega = 0 \) Drude peak. A, B and c are constants, \( g_\rho \) is the strength of the umklapp scattering as defined in Eq.9 and 21, while \( \sigma_0 \) is the weight \( 2 e^2u_\rho K_\rho/\hbar \).
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FIG. 7: The evolution of the two-leg ladder optical conductivity as a function of temperature, were the frequencies available experimentally. (top) The majority of the spectral weight would lie in a sharp peak at $\sqrt{3}\Delta$; the onset of 2-particle scattering would be as $\sqrt{\omega - 2\Delta}$; the decrease up to $3\Delta$ would go as $\omega^{-3}$, and cross over to the $\omega^{-1}$ form of Giamarchi, cut off at high frequencies by the larger of the resistive umklapp scattering terms, when $K_\chi \rightarrow 2$, (where $K_\chi$ is defined in the text), such that the optical conductivity sum rule is satisfied. Note that this need not be a simple cross-over of exponents, one might find that this asymptotic behavior arrives higher than the exact low frequency result, leading to a second bump at higher frequencies as outlined with dashed lines. Less weight would be attributed to the exciton as the temperature approached that of the gap, with increasing weight at $\omega = 0$. 
FIG. 8: The evolution of the three-leg ladder as a function of temperature.