Research article

Analysis of the dynamics of new models of nonlinear systems with state variable damping and elastic coefficients

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A B S T R A C T

This paper, is an analysis of the dynamics of new models of nonlinear systems in which the state damping variables with elastic coefficients, given by functions $c \cos(p,x)$, $c \sin(p,x)$, $c \cos(p,x)$ and $c \sin(p,x)$ are investigated in their autonomous and excited states. They exhibit periodic regions of stability and instability in their autonomous states and a rich dynamic behavior. The analysis of limit cycles shows the presence of isolated curves around the origin (0,0), which explains the presence of periodic solutions (limit cycles). The dynamics obtained allows to describe qualitatively the cardiac activity (artificial pacemaker). A chaos analysis shows the appearance of regular and chaotic behaviors. These studies allowed us to show the effect of the damping of the state variable and the elastic coefficients on the dynamics of these models. The presence of analog functions makes the experimental study complex. An implementation based on microcontroller simulation technology has been proposed. The microcontroller results are consistent with the numerical results.

1. Introduction

Owing to their broad importance in many fields and their numerous engineering applications, nonlinear oscillators have recently attracted the attention of a large number of researchers due to their extremely rich dynamics (Joshi, 2021; Dashkovskiy and Pavlichkov, 2020; Ahmed et al., 2017; Cheng and Zhan, 2020; Kudryashov, 2018; Tang et al., 2020; Dashkovskiy and Pavlichkov, 2020; Fonkou et al., 2022; Han et al., 2019; Ramirez et al., 2020; FitzHugh, 1961; Rahman et al., 2021a,b,c,d). Their fields of application are between others: seismology, communication and neurophysiology (Rahman et al., 2021a,b,c,d; Lucero and Schoentgen, 2013; Rowat and Silverston, 1993; Balachandran and Kandiban, 2009). These oscillators exhibit rich dynamics among which limit cycle oscillations of sinusoidal and relaxation nature, since one of their important characteristic is their capacity to present limit cycle behaviors which is an important criterion in the characterization of the artificial pacemaker (Steeb, 1977; Hochstadt and Stephan, 1967; D’Heedene, 1996; Steeb and Kunick, 1987; Steeb et al., 1983). When subjected to an external periodic excitation, numerical studies and singular point analysis have revealed chaotic behaviors, allowing the analysis of phenomena such as control and cardiac activity with numerous technological applications. (Steeb and Kunick, 1982; Steeb and Kunick, 1983; Forger, 1999; Enrique et al., 2020; Rahman et al., 2019; Kai and Tomita, 1979; Rahman et al., 2021a,b,c,d; Van der Pol and Van der Mark, 1926; Van der Pol and Van der Mark, 1928; Alhasnawi et al., 2021).

The Van der Pol oscillator is one of these systems, it is nonlinear and its differential equation is the second order. It is a self-sustained oscillator because it maintains its oscillations by itself. Under certain conditions, it also exhibits the very rich dynamical behaviors (Van der Pol and Van der Mark, 1928; Alhasnawi et al., 2021; Chedjou et al., 1997; Makouo and Woafo, 2017; Han et al., 2018; Simo and Woafo, 2012; Bao et al., 2018; Jasim et al., 2020; Han and Bi, 2012; Ma et al., 2021; Grudzinski and Zebrowski, 2004; Magnitskii and Sidorov, 2004). The refs. (Van der Pol and Van der Mark, 1926; Van der Pol and Van der Mark, 1928) have shown that the qualitative characteristics of the heart action potential are closely related to the dynamic response of the classical Van der Pol oscillator (VdP), because it is the starting point for modeling heartbeats.

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In the work of Refs. (Steeb and Kunick, 1987; Steeb et al., 1983), the authors study the dynamic behavior of anharmonic systems with limit cycles in which the potentials are only dependent variables. The models used are modifications of the Van der Pol oscillator, the appearance of chaotic dynamics and limit cycle dynamics is due solely to the effect of potentials and the damping coefficient.

In the present work, an analysis of the dynamics of four new models of nonlinear systems under external periodic disturbance is proposed: analytical study and experimental simulation by microcontroller. These models have state variable damping and elastic coefficients. They are described by equations (1) and (2).

However, periodic sine or cosine function of system variable may introduce some coexisting attractors based on offset boosting and therefore self-reproducing chaotic system may be devised.

Faced with this, the following question was asked: how can sinusoidal functions show monotonic solutions of limit cycles and the onset of chaotic dynamics under external excitation? To answer this question, the present work has been organized as follows: Section 2 presents a description of the models used. Section 3 presents an analysis of their dynamics in their autonomous states. Section 4 is devoted to the study of the systems in the excited state. Section 5 shows the results through the practical implementation using an Arduino Uno board. Section 6 ends with the conclusion.

2. Description of the models

Four new nonlinear oscillator models in which the variables have been substituted by the terms with state variable damping and elastic coefficients given by functions $c \cos(px)$, $c \sin(px)$, $c \cos(px)$ and $c \sin(px)$ are used. Eq. (1) and Eq. (2) present their dynamics

$$\dot{x} + (a - f(x))x + bx = 0$$

$$\dot{x} + ax + (b - f(x))x = 0$$

With

$$f(x) = \begin{bmatrix} c \cos(px) \\ c \sin(px) \end{bmatrix} \quad \text{and} \quad \dot{f}(x) = \begin{bmatrix} c \cos(px) \\ c \sin(px) \end{bmatrix}$$

With $a, b, c$, and $p$, the characteristic parameters of systems. To maintain the self-oscillating character of the systems, these coefficients must be positive. For an expansion of the functions $f(x)$ and $\dot{f}(x)$ up to the order two, the so-called Van der Pol equation is obtained, which is important for triode oscillators necessary for the characterization of the heart (Van der Pol and Van der Mark, 1928). Quantitatively, the quantity $\frac{c}{a}$ represents the friction, if this one is very $(c >> a)$, one obtains an autonomous oscillatory character and the fixed point unstable tends with the complex eigenvalues of the Jacobean matrix towards a repulsive focus. On the other hand, if it is very strong $(a >> c)$ the oscillatory character is not autonomous and the fixed point stable tends towards an attractive focus with the complex eigenvalues of the Jacobean matrix. Thus $\frac{c}{a}$ influence on the more or less sinusoidal character of these systems.

Quantitatively, the frequency character of the system is linked to the quantity $b$. Thus, when $b$ increases, the frequency of the oscillations increases and their amplitude decreases. On the other hand, the decrease in $b$ leads to a decrease in the frequency of the oscillations but an increase in their amplitude. Being the natural frequency of the systems, $b$ is chosen so that the systems evolve with a frequency of $f_0 = 1$ Hz which is the mean frequency of a healthy heart (Van der Pol and Van der Mark, 1926). Where $p$ is a constant. Using these very rich dynamics, the artificial pacemaker can be a very important application of these models.

3. Analysis of dynamics in their autonomous state

3.1. Analysis of stability

Equation (1) can be rewritten as follows:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -(a - f(x))y - bx \end{cases}$$

Obtaining the equilibrium points be done by solving the system below:

$$\begin{cases} \dot{x} = 0 \\ \dot{y} = 0 \end{cases}$$

For $f(x) = c \cos(px)$ the resulting fixed point is $A_1(0,0)$. The Jacobean matrix obtained from this point $J_1 = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix}$. By solving $\text{Det}(J_1 - \lambda I) = 0$, the resulting characteristic equation is $\lambda^2 - (ca - a)\lambda + b = 0$ and the discriminant is $\Delta_1 = (ca - a)^2 - 4b$.

However, for $f(x) = c \sin(px)$, the fixed point is given by $A_2(0,0)$. Its Jacobean matrix is $J_2 = \begin{bmatrix} 0 & 1 \\ -c - b & -a \end{bmatrix}$. The characteristic equation obtained by solving $\text{Det}(J_2 - \lambda I) = 0$, is $\lambda^2 + a\lambda + b = 0$ and its discriminant $\Delta_2 = a^2 + 4b$.

It can be seen that:

- For $\Delta_1 > 0$ and $\Delta_2 > 0$, $A_1$ unstable and $A_2$ is stable.
- For $\Delta_1 < 0$ and $\Delta_2 < 0$, if $c \in [a; 2\sqrt{b}]$, $A_1$ is unstable. However, if $c \in [-a; a]$, $A_1$ is stable. For $a \in [0; 2\sqrt{b}]$, $A_2$ is stable however if $a \in [-a; a]$, $A_2$ is unstable.
- For $\Delta_1 = 0$ and $\Delta_2 > 0$, if $a < 0$, $A_1$ is unstable, if $a > 0$, $A_1$ is stable. However, $A_2$ is stable.

By considering the Eq. (2), it can still be written by:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -ay - (b - f(y))x \end{cases}$$

From equation (4), we obtain for $f(x) = c \cos(px)$, the fixed point $A_1(0,0)$. Its Jacobean matrix is $J_3 = \begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix}$. By solving $\text{Det}(J_3 - \lambda I) = 0$, its characteristic equation gives $\lambda^2 + a\lambda + b = 0$ and its discriminant $\Delta_3 = a^2 + 4b - c$.

However, for $f(x) = c \sin(px)$, the fixed point is $A_2(0,0)$. The Jacobean matrix is $J_4 = \begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix}$. The characteristic equation obtained by solving $\text{Det}(J_4 - \lambda I) = 0$ is given by $\lambda^2 + a\lambda + b = 0$ and the discriminant by $\Delta_4 = a^2 + 4b - c$.

Thus we have:

- For $\Delta_3 > 0$ and $\Delta_4 > 0$, $A_1$ and $A_2$ are stable.
- For $\Delta_3 < 0$ and $\Delta_4 < 0$, if $a \in [0; 2\sqrt{b}]$, $A_1$ is stable. However, if $a \in [-a; a]$, $A_1$ is unstable. For $a \in [0; 2\sqrt{b}]$, $A_2$ is stable and unstable if $a \in [-a; a]$.

Be it Eq. (1) where Eq. (2) we find that when, $(c < a)$ the equilibrium point is stable and the models converge. On the other hand when $(a < c)$, the equilibrium point is unstable and the models diverge. The only critical point being the origin (zero coordinates), a solution of period one exists. In the $(x, y)$-plane, this solution can describe limit cycle dynamics

3.2. Analysis of the dynamics of limit cycles

Recent studies on the nature of integral curves have shown the presence of singular points in systems of differential equations (Chaté, 1994; Bendixon, 1901). Generally, they are observed in conservative systems. However, for some non-conservative systems, closed trajectories
When the polar coordinates given by Eq. (10) are introduced in Eq. (9), and apart from \( r = 0 \), the expressions of the radii \( R_{\text{min}} \) and \( R_{\text{max}} \) are presented in Eq. (11):

\[
\begin{align*}
  x &= r \cos \theta \\
  y &= r \sin \theta \\
  R_{\text{min}} &= \sqrt{\frac{d(b^2-1)+idg+e}{c^p}} \\
  R_{\text{max}} &= \sqrt{\frac{d(b^2-1)+idg+e}{c^p}}
\end{align*}
\]

Consequently from expressions given in (11), (12), (13), and (14), a ring domain centered at the origin exists and contains all possible limit cycles; these boundaries are the radii \( R_{\text{min}} \) and \( R_{\text{max}} \). They represent the boundaries of the innermost and outermost circles, respectively, that touch the contact curve. These expressions show that if a limit cycle occurs around the origin, it must necessarily be in a ring domain whose center is at the singular point.

In Figs. 1, 2, 3, 4, and 5, the temporal traces and phase portraits have been plotted for the initial conditions \( x(t=0) \) and \( y(t=0) \). This allows us to observe that these systems are nonlinear and self-excited and show limit cycle behaviors of sinusoidal and relaxation nature. The limit cycles are almost isolated curves so the trajectories converge around the origin.

The limit cycles obtained from Eq. (1) (case of \( f(x) = c \sin(px) \)) are presented by figure.
For $c = 2$, the temporal trace and phase portrait are given in Fig. 1. On the other hand, for $c = 8$, we have Fig. 2.

Still by considering Eq. (1) (case of $f(x) = c \sin(px)$), the time serie and the phase portrait are given in Fig. 3.

By considering the Eq. (2) (case of $f(x) = c \cos(px)$), the limit cycles are given in Fig. 4.

When we consider Eq. (2) (case of $f(x) = c \sin(px)$), the time plot and phase portrait are given in Fig. 5.

It can be seen in Figs. 1, 2, 3, 4, and 5 that the models used are nonlinear and self-excited and exhibit limit cycle dynamics of sinusoidal and relaxation nature qualitatively comparable with those exhibited by artificial pacemakers (Van der Pol and Van der Mark, 1928). In addition, the sinusoidal oscillations evolve with a frequency of 1 Hz for $b = \omega_0^2$ where $\omega_0 = 2\pi f_0$ and $f_0 = 1$ Hz.

4. Analysis of dynamics in their excited state

4.1. Chaos analysis

In the anharmonic systems with limit cycles, it has been shown that the variation of their parameters as well as those of the external force can lead to chaotic behaviors. The Refs. (Steer and Kunick, 1987; Steer et al., 1983) have demonstrated the influence of potentials and damped coefficients on the appearance of chaotic dynamics in nonlinear systems. In this study, our objective is to see the effect of state variable damping and elastic coefficients on the appearance of chaotic dynamics in nonlinear systems under the effect of an external excitation of sinusoidal nature $E_0 \sin(\omega t)$. This is done by plotting the bifurcation curves each associated with its maximum Lyapunov exponent with the amplitude of the external excitation $E_0$ as parameter. These curves (bifurcation and maximum Lyapunov exponent) are presented in Figs. 6, 7, 8, and 9. On these figures, we observed the presence of chaotic and multiperiodic dynamics. However, for small values of the parameter $E_0$, the Lyapunov exponent tends towards 0. This behavior reflects the presence of quasiperiodic dynamics. A few are shown in Fig. 6. In Fig. 11 it only about the presentation of some phase portraits illustrating the chaotic states.

When Eq. (1) is excited (case of $f(x) = c \cos(px)$), the bifurcation trees and maximal Lyapunov exponents are given in Figs. 6 and 7.

When $c = 2$, the information given by Fig. 6 are: a period-2T orbit $E_0 \leq 0.2V$ is obtained, this behavior disappears and gives way to the quasiperiodic behavior for $0.2V \leq E_0 < 3.75V$. A second orbit of period-13T appears for $3.75V \leq E_0 < 4.0V$, then disappears to give a quasiperiodic orbit for $4.0V \leq E_0 < 6.0V$. For $6.0V \leq E_0 < 6.2V$, a new orbit of period-11T is observed, the latter disappears and gives a quasiperiodic
behavior for $6.2V \leq E_0 < 9.0V$. When $E_0 \leq 9.0V$, the orbit of period-1T settles.

On the other hand, when $\varepsilon = 8$, on the Fig. 7, an orbit of period-1T appears for $0.0V \leq E_0 \leq 0.25V$ then disappears and gives way to the quasiperiodic orbit for $0.25V \leq E_0 \leq 1.475V$. For $E_0 \in [1.475; 1.625)V$, the first orbit of period-34T appears. When the value of $E_0$ goes from 1.625V to 2.5V, a second quasiperiodic orbit appears and disappears, giving way to the period-32T orbit for $2.5V \leq E_0 \leq 2.625V$. For $2.625V \leq E_0 \leq 3.0V$, quasiperiodic orbits appear, a second orbit of period-30T appears for $3.0V \leq E_0 \leq 3.5V$. This orbit disappears and the first chaotic behavior appears for $3.5V \leq E_0 \leq 3.776V$. This first chaotic behavior is replaced by a second orbit of period-27T when the value of $E_0$ increases from $3.776V$ to $4.385V$. Then a second chaotic orbit appears for $4.385V \leq E_0 \leq 4.559V$. Subsequently, the alternations between periodic and chaotic orbits are observed. For $E_0 \in [4.559; 5.172]V$ an orbit of period-25T is observed. When $5.172V \leq E_0 \leq 5.267V$ a third chaotic orbit is obtained. Then a new orbit of period-23T for $E_0 \in [5.267; 5.91]V$. For $5.91V \leq E_0 \leq 5.975V$ a fourth chaotic orbit appears. Between $5.975V$ and $6.642V$, an orbit of period-21T is also observed. A fifth chaotic orbit between $6.642V$ and $6.696V$. For $6.696V \leq E_0 \leq 7.319V$, a new orbit of period-19T appears. When $E_0 \in [7.319; 7.396]V$ a sixth chaotic orbit appears. Between $7.396V$ and $8.07V$ an orbit of period-17T appears. Between $8.07V$ and $8.14V$ a seventh chaotic orbit is observed. Between $8.14V$ and $8.769V$ an orbit of period-15Tis obtained. For $8.769V \leq E_0 \leq 8.865V$ an eighth chaotic orbit appears. For $8.865V \leq E_0 \leq 9.46V$, an orbit of period-14T appears. Between $9.46V$ and $9.53V$, a ninth chaotic behavior is obtained. Between $9.53V$ and $10.18V$, the orbit of period-13T appears. For $10.18V \leq E_0 \leq 10.27V$, a tenth chaotic orbit is obtained. Between $10.27V$ and $10.89V$, an orbit of period-11T appears and between $10.89V$ and $11.02V$ an eleven chaotic behavior is obtained. For $11.02V \leq E_0 \leq 11.61V$, the orbit of period-9T is observed and for $11.61V \leq E_0 \leq 11.88V$, a twelfth chaotic behavior. When $E_0$ varies between $11.88V$ and $12.4V$, there is the appearance of a period-7T
orbit. However, between $12.4V$ and $13.35V$, the quasiperiodic orbit reappears. From $E_0 = 13.35V$, only the orbit of period 1T evolves. When we excite Eq. (1) (case of $f(x) = c \sin(px)$), we have in Fig. 8 the bifurcation tree and its associated maximal Lyapunov exponent. For this case, a chaotic orbit is obtained for $0.00V < E_0 < 10.00V$, it then disappears and gives way to the period-3T orbit for $10.00V < E_0 < 28.5V$. For $E_0 \in [28.5,36.5]V$ a new chaotic orbit appears. When the value of $E_0$ goes from $36.5V$ to $50V$, a second orbit of period-2T appears, then disappears, giving way to a new chaotic orbit for $50.0V < E_0 < 53.5V$. For $53.5V < E_0 < 62.0V$, new appearance of the period-3T orbit. A third chaotic orbit appears for $62.0V < E_0 < 64.5V$. This orbit disappears and the first quasiperiodic orbit appears for $64.5V < E_0 < 72.5V$. Then, we observe a succession between chaotic and quasiperiodic orbits for $72.5V < E_0 < 79.0V$. Then reappearence of the period-2T orbit for $79.0V < E_0 < 83.0V$. When $83.0V < E_0 < 100.0V$, a new succession between chaotic and quasiperiodic orbits appears.

The bifurcation curves and maximal Lyapunov exponent are given respectively in Fig. 9 for $f(x) = c \cos(px)$ and Fig. 10 for $f(x) = c \sin(px)$. For these cases (Fig. 9 and Fig. 10), an alternation between chaotic orbits and periodic orbits is observed.

To illustrate the chaotic behaviors presented by the bifurcation curves and Lyapunov exponents of Figs. 7, 8, 9, and 10, some phase portraits are given in Fig. 11.

5. Simulation based on microcontroller technology

Microcontroller simulation is one of the simplest because it is performed using a computer in which suitable simulation software is installed. Recent work has shown that it can be used to do electronics using just a programming language (Fonkou et al., 2021a,b,c). Experimentally, it is less cumbersome and requires significant period savings for component wiring, a higher degree of integration, and lower power consumption.

A microcontroller is an integrated circuit that gathers the essential elements of a computer. Recent work has shown that microcontrollers can be used to simulate nonlinear differential equations (Fonkou et al., 2021a,b,c) by producing simple and complex electrical signals. Compared to conventional electronic systems based on separate components, they can reduce the size of the components, as well as the cost of the products. In this work, we will use an Arduino microcontroller from the ATMEGA family (Arduino Card). Its structure is shown in Fig. 12.

To produce the electrical signal described by Eq. (1) and Eq. (2), these equations are discretized. Subsequently, a simple digital code using appropriate software (here the Arduino 1.8.9) is written and loaded into the microcontroller. The signal obtained from Proteus software with its own oscilloscope is shown in Fig. 12.

To produce this signal experimentally from a real oscilloscope, the practical configuration presented in Fig. 13 is used. It consists of a computer for inserting the digital code into the microcontroller and the oscilloscope for viewing the signal delivered by the latter. The real electrical signals obtained are given in Fig. 14, Fig. 15, and Fig. 16.

For $f(x) = c \cos(px)$, we have Fig. 14. For $f(x) = c \sin(px)$, we obtain in figure.

These curves (Fig. 14 and Fig. 15) are in agreement with those obtained respectively in the Fig. 2 and Fig. 3.

On the other hand, when we use the Eq. (2), the dynamics are given by figure. The curves presented in the figures above (Fig. 16a and Fig. 16b) show a good agreement with those obtained numerically (Fig. 4 and Fig. 5).
The used Arduino board to excite the oscillator is shown in Fig. 14a and the complete flowchart used to program this signal are given in Table 1.

The second card is used for programming the self-excited oscillator. The Arduino code and the complete flowchart used to program this oscillator are presented in Table 2. To obtain the result, the $E_0 \sin(\omega t)$ signal produced by the first Arduino board is inserted as an input port of the second Arduino board to excite the self-excited oscillator (Eq. (1) or Eq. (2)) programmed in this second card, using an appropriate line of code.

The device cabled on Proteus is shown in Fig. 17 and its practical device is presented in Fig. 18. As shown in Figs. 19 and 20, only chaotic states have been presented.

The chaotic dynamics obtained for $f(x) = c \cos(p x)$ and $f(x) = c \sin(p x)$ are presented in Fig. 19. These curves show a good agreement between the experimental results and the results obtained numerically. For $f(x) = c \cos(p x)$ and $f(x) = c \sin(p x)$ the chaotic dynamics obtained experimentally are presented in Fig. 20, and a good agreement between the experimental results and those obtained numerically is observed.

### 6. Conclusion

This work was devoted to the study of four new models of nonlinear systems with state variables damping and elastic coefficients defined by functions $c \cos(p x)$, $c \sin(p x)$, $c \cos(p x)$, and $c \sin(p x)$. The study of limit cycles has shown that these systems, to some degree of approximation, can describe the electrical activity of the heart. The study of their dynamics in their autonomous and non-autonomous states has shown that the values of the coefficients influence the nature of the stability/unstability of these systems. The production of real electrical signals by discretizing differential equations and writing digital codes in the microcontroller show that the latter can be used for the design of nonlinear oscillator models. In doing so, it appeared that when these nonlinear
systems are subjected to external and periodic sinusoidal excitation, they presented periodic, quasiperiodic, and chaotic behaviors. Since these models are essentially autonomous oscillators, they continue to deliver a signal if the action of $E(t)$ ceases.

**Declarations**

**Author contribution statement**

Rodrigue Fatou Fonkou, Dr: Conceived and designed the experiments; Performed the experiments; Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data; Wrote the paper. Patrick Louodop, Associate Professor: Conceived and designed the experiments; Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data; Wrote the paper. Pierre Kisito Talla, Professor; Paul Woafo, Professor: Conceived and designed the experiments; Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data.

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**Data availability statement**

No data was used for the research described in the article.

**Declaration of interests statement**

The authors declare no conflict of interest.

**Additional information**

No additional information is available for this paper.

**References**

Ahmed, S.H. Javed, S. Abbas, S.K. Hussain, S., Modified van der pol (Vdp) oscillator based cardiac pacemakers, IEEE Xplore: 23 February (2017).

Allahmawi, Bilal Naji, Jasim, Basil H., Rahman, Zain-Aldeen S.A., Siano, Pierluigi, 2021. A novel robust smart energy management and demand reduction for smart homes based on Internet of energy. Sensors 21 (14), 4756.

Balachandran, V., Kandanb, G., 2009. Experimental and numerical realization of higher order autonomous Van der Pol-Duffing oscillator. Indian J. Pure Appl. Phys. 47, 823–827.

Bao, B.C., Wu, P.Y., Bao, H., Xu, Q., Chen, M., 2018. Numerical and experimental confirmations of quasi-periodic behavior and chaotic bursting in third-order autonomous memristive oscillator. Chaos Solitons Fractals 106, 161–170.

Bendixson, I., 1901. Sur les courbes définies par des équations différentielles. Acta Math. 24, 1–88.

Chaté, H., 1994. Spatiotemporal intermittency regimes of the one-dimensional complex Ginzburg-Landau equation. Nonlinearity 7, 185.

Chedjou, J.C., Potin, H.B., Woafo, P., 1997. Behavior of the Van der Pol oscillator with two external periodic forces. Phys. Scr. 55, 390.

Cheng, J., Zhan, Y., 2020. Nonstationary $i_t = i_e$ filtering for Markov switching repeated scalar nonlinear systems with randomly occurring nonlinearities. Appl. Math. Comput. 365, 124714.

Dushkovskiy, S., Pavlichkov, S., 2020. Stability conditions for infinite networks of nonlinear systems and their application for stabilization. Automatika 112, 108643.

D’Heedene, R.N., 1996. For all real $\mu + \sin x + x = 0$, has an infinite number of limit cycles. Differ. Equ. 5, 564.

Enrique, L., Gonzalez, R., Bermudez, R.Q., Torres, R.Q., 2020. Circuits Syst. Signal Proces. 39, 4775–4791.

FitzHugh, R., 1961. Impulses and physiological states in theoretical models of nerve membrane. Biophys. J. 1 (6), 445–466.

Fonkou, R.F., Louodop, Patrick, Talla, P.K., 2021a. Nonlinear oscillators with state variable damping and elastic coefficients. Pramana J. Phys. 95, 210.

Fonkou, R.F., Louodop, Patrick, Talla, P.K., Woafo, P., 2021b. Van der Pol equation with sine nonlinearity: dynamical behavior and real time control to a target trajectory. Phys. Scr. 96, 125003.

Fonkou, R.F., Louodop, Patrick, Talla, P.K., Woafo, P., 2021c. Dynamic behavior of pacemaker models subjected to a arterial pressure excitation simulator: theoretical and experimental study by microcontroller. Braz. J. Phys. 51, 1448–1458.

Fonkou, R.F., Louodop, Patrick, Talla, P.K., 2022. Dynamical behavior of cardiac conductivity system under external disturbances: simulation based on microcontroller technology. Phys. Scr. 97, 025001.

Forger, D.B., 1999. A simpler model of the human circadian pacemaker. J. Biol. Rhythms 14 (6), 533–537.

Grudzinski, K., Zebrowski, J.J., 2004. Modeling cardiac pacemakers with relaxation oscillators. Physica A 336 (1–2), 153–162.

Han, X., Zhang, Y., Bi, Q., Kurths, J., 2018. Two novel bursting patterns in the Duffing system with multiple frequency slow parametric excitations. Chaos 28, 043111.

Han, X.J., Bi, Q.S., 2012. Complex bursting patterns in Van der Pol system with two slowly changing external forcings. Sci. China, Technol. Sci. 55, 702–708.

Han, Y., Xiang, S., Zhang, L., 2019. Cluster synchronization in mutually-coupled semiconductor laser networks with different topologies. Opt. Commun. 445, 262–267.

Hayashi, C., 1964. Nonlinear Oscillations in Physical Systems. McGraw-Hill, New York, NY.

Hochstadt, H., Stepha, B.H., 1967. On the limit cycle of $\mu + \sin x + x = 0$. Arch. Ration. Mech. Anal.

Jasim, B.H., Rashid, M.T., Omran, K.M., 2020. Synchronization and tracking control of a novel 3 dimensional chaotic system. Iraqi J. Electr. Electron Eng. 3.

Joshi, S.K., 2021. Synchronization of coupled oscillators in presence of disturbance and heterogeneity. Int. J. Dyn. Control 9, 602–618.
Fig. 17. Microcontroller device representing our exciter signal coupled contains Eq. (1) or Eq. (2).

Fig. 18. Experimental device allowing the visualization of the signal.
### Parameters

| Parameters | Numerical results | Experimental results |
|------------|-------------------|-----------------------|
| $f(x) = c \cos(p.x)$ and $E_0 = 4.37V$ | ![Image](image1.png) | ![Image](image2.png) |
| $f(x) = c \sin(p.x)$ and $E_0 = 0.23V$ | ![Image](image3.png) | ![Image](image4.png) |

**Fig. 19.** Chaotic dynamics obtained numerically and by microcontroller, with $x(0) = 0.01$ and $dx/dt(0) = 0.04$.

| Parameters | Numerical results | Experimental results |
|------------|-------------------|-----------------------|
| $f(x) = c \cos(p.x)$ and $E_0 = 34.2V$ | ![Image](image5.png) | ![Image](image6.png) |
| $f(x) = c \sin(p.x)$ and $E_0 = 29.6V$ | ![Image](image7.png) | ![Image](image8.png) |

**Fig. 20.** Chaotic dynamics obtained numerically and by microcontroller, with $x(0) = 0.01$ and $dx/dt(0) = 0.04$.

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**References:**

- Rahman, Zain-Aldeen S.A., Jasim, Basil H., Al-Yasir, Yasir I.A., Abd-Alhameed, Raed A., 2021a. High-security image encryption based on a novel simple fractional-order memristive chaotic system with a single unstable equilibrium point. Electronics 10, 3130.
- Rahman, Zain-Aldeen S.A., Jasim, Basil H., Al-Yasir, Yasir I.A., Abd-Alhameed, Raed A., Alhasnawi, Bilal Naji, 2021c. A new no equilibrium fractional order chaotic system, dynamical investigation, synchronization, and its digital implementation. Inventions 6 (3), 49.
- Rahman, Zain-Aldeen S.A., Jasim, Basil H., Al-Yasir, Yasir I.A., Hu, Yim-Fun, Abd-Alhameed, Raed A., Alhasnawi, Bilal Naji, 2021d. A new fractional-order chaotic system with its analysis, synchronization, and circuit realization for secure communication applications. Mathematics 9 (20), 2593.
- Ramirez, J.P., Garcia, E., Alvarez, J., 2020. Master-slave synchronization via dynamic control. Commun. Nonlinear Sci. Numer. Simul. 80, 104977.
- Rowat, P.F., Selverston, A.I., 1993. Modeling the gastric mill central pattern generator of the lobster with a relaxation-oscillator network. J. Neurophysiol. 70 (3), 1030–1053.
- Simo, H., Woofu, P., 2012. Bifurcation structure of a Van der Pol oscillator subjected to nonsinusoidal periodic excitation. Int. J. Bifurc. Chaos 22 (1), 1250003.
- Steeb, W.H., 1977. Nonlinear autonomous dynamic systems, limit cycles, and one-parameter groups of transformations. Lett. Math. Phys. 2, 171–174.
- Steeb, W.H., Kunick, A., 1982. Limit cycle oscillations and a perturbed harmonic oscillator. Int. J. Non-Linear Mech. 17 (1), 41–45.
- Steeb, W.H., Kunick, A., 1983. Painleve property of anharmonic systems with an external periodic field. Phys. Lett. 95 (6), 269–272.
- Steeb, W.H., Kunick, A., 1987. Chaos in limit cycle systems with external periodic excitations. Int. J. Non-Linear Mech. 22 (5), 349–361.
- Steeb, W.H., Erg, W., Kunick, A., 1983. Chaotic behaviour and limit cycle behaviour of anharmonic systems with periodic external perturbations. Phys. Lett. A 93, 267–270.
Table 2. The Arduino code and the complete flowchart for programming the self-excited oscillator.

| The Arduino code                                                                 | The complete flowchart |
|---------------------------------------------------------------------------------|------------------------|
| //constants                                                                     | Declaration of constants |
| float h=(float)0.1;                                                             | Declaration of initial conditions |
| float a=(float)0.5;                                                             | Declaration of Variables |
| float b=(float)0.25;                                                            | Initialization of the Arduino Card |
| float c=(float)8.;                                                              | Start of the Runge Kutta loop of order 4 |
| float p=(float)5.0;                                                             | Main Program |
| //initial conditions                                                            | End of the Loop |
| float x1=(float)0.01;                                                           |                         |
| float y1=(float)0.04;                                                           |                         |
| float t1=(float)0.0;                                                            |                         |
| // variables                                                                   |                         |
| double atan(double t1);                                                         |                         |
| double sin(double t1);                                                          |                         |
| double cos(double t1);                                                          |                         |
| float x,x2,y2,t1,x;                                                             |                         |
| float y1,r1,t1,x1;                                                              |                         |
| float l1,l2,l3,l4;                                                              |                         |
| float M1,M2,M3,M4;                                                              |                         |
| void setup() {                                                                  |                         |
| pinMode (0, OUTPUT);                                                            |                         |
| pinMode (1, OUTPUT);                                                            |                         |
| pinMode (2, OUTPUT);                                                            |                         |
| pinMode (3, OUTPUT);                                                            |                         |
| pinMode (4, OUTPUT);                                                            |                         |
| pinMode (5, OUTPUT);                                                            |                         |
| pinMode (6, OUTPUT);                                                            |                         |
| pinMode (7, OUTPUT);                                                            |                         |
| pinMode(A0, INPUT);                                                             |                         |
| void loop() {                                                                   |                         |
| L1=y1;                                                                         |                         |
| M1=-(a+c*cos(p*x1))*y1-b*x1;                                                    |                         |
| z=x1+(h/2)*L1;                                                                 |                         |
| r=y1+(h/2)*M1;                                                                  |                         |
| t=t1+(h/2);                                                                    |                         |
| L2=r;                                                                          |                         |
| M2=-(a+c*cos(p*z))*r-b*z;                                                       |                         |
| z=x1+(h/2)*L2;                                                                  |                         |
| r=y1+(h/2)*M2;                                                                  |                         |
| t=t1+(h/2);                                                                    |                         |
| L3=r;                                                                          |                         |
| M3=-(a+c*cos(p*z))*r-b*z;                                                       |                         |
| z=x1+h*L3;                                                                      |                         |
| r=y1+h*M3;                                                                     |                         |
| t=t1+h;                                                                        |                         |
| L4=r;                                                                          |                         |
| M4=-(a+c*cos(p*z))*r-b*z;                                                       |                         |
| x1=x1+(h/6)*(L1+2*L2+2*L3+L4);                                                  |                         |
| y1=y1+(h/6)*(M1+2*M2+2*M3+M4);                                                 |                         |
| t1=t1+h;                                                                       |                         |
| PORTD=(x1*20)+55.5;                                                             |                         |
| x1 = x1;                                                                        |                         |
| y1 = y1;                                                                        |                         |
| t1 = t1;                                                                        |                         |

Tang, Y., Wu, X., Shi, P., Qian, F., 2020. Input-to-state stability for nonlinear systems with stochastic impulses. Automatica 113, 108766.
Van der Pol, B., Van der Mark, J., 1926. On “relaxation-oscillations”. Philos. Mag. J. Sci. Ser. 7 2 (11), 978–992.
Van der Pol, B., Van der Mark, J., 1928. The heartbeat considered as a relaxation oscillation, and an electrical model of the heart. Philos. Mag. J. Sci. Ser. 6, 763–775.