WHEN GAME COMPARISON BECOMES PLAY: ABSOLUTELY CATEGORICAL GAME THEORY

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Abstract. Absolute Universes of combinatorial games, as defined in a recent paper by the same authors, include many standard short normal-misère- and scoring-play monoids. In this note we show that the class is categorical, by extending Joyal’s construction of arrows in normal-play games. Given $G$ and $H$ in an Absolute Universe $U$, we study instead the Left Provisional Game $[G, H]$, which is a normal-play game, independently of the particular Absolute Universe, and find that $G \twoheadrightarrow H$ (implying $G \succcurlyeq H$) corresponds to the set of winning strategies for Left playing second in $[G, H]$. By this we define the category $\text{LNP}(U)$.

1. Introduction

This is the study of game comparison in Combinatorial Game Theory (CGT), specifically, Combinatorial Game Spaces, and their subspaces (universes of games). The concept of a Combinatorial Game Space allows for a general framework, which includes many standard classes of terminating games. One of the most elegant discoveries of normal-play CGT, [1], is that Left wins playing second in the game $G$ if and only if $G \geq 0$. Since normal-play games constitute a group structure, this leads to a constructive (subordinate) general game comparison, $G \geq H$ if and only if Left wins the game $G - H$ playing second. Joyal [3] proved that games, under the normal-play convention, form a category where $H \twoheadrightarrow G$ if Left wins playing second in $G - H$. That is, Left has good replies against any Right moves $G^R - H$ and $G - H^L$ and so forth.

More generally, for any winning convention in CGT, game comparison is axiomatized by: Left prefers $G$ to $H$ if, for all games $X$, Left does at least as well in $G + X$ as in $H + X$. Each different winning convention, possibly coupled with other constraints gives a different partial order.

The authors recently demonstrated [4] that there is a set of properties that define Absolute Universes and together these properties reduce game comparisons to considering only a certain Proviso, and a Common Normal Part (corresponding to Theorem [23] in this paper). Except for normal-play, typically Absolute Universes only have a monoid structure (group structure is not common in scoring-play and non-existent in misère-play), so we cannot use the ‘inverse’ of any game freely. It is generally believed that game comparison in normal-play is a special case, which does not apply to other monoids of combinatorial games.

Here, we construct a normal-play game, called the Left Provisional Game, $[G, H]$ which is essentially playing $G - H$ (as if $H$ were invertible) but where Left’s options

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are restricted by the Proviso, and where the games $G$ and $H$ belong to any Absolute Universe. The previous work [4] implies that in any Absolute Universe, the games $G$ and $H$ satisfy $G \succeq H$ if and only if Left wins the normal-play game $[G, H]$ whenever Right starts (Theorem 2.5). This allows for a construction of arrows, similar to Joyal’s, which shows that Absolute Universes are categorical.

We give the relevant background on Absolute Combinatorial Game Theory [4] in Appendix A at the end of this paper. Appendix B contains code for CGsuite 0.7, which ‘compares’ misère dicot games by, instead, analyzing the Left Provisional Game.

2. Absolute game comparison and the Left Provisional Game

First we recall the Proviso for a pair of games in a given Absolute Universe [4], and we remind the reader that relevant background on outcomes, left-atomic games and so on, is also given in Appendix A.

Definition 2.1 (Proviso). Consider an Absolute Universe $U$, and let $G, H \in U$. The ordered pair of games $[G, H] \in \text{Proviso}(U)$ if

\[
o_L(G + X) \geq o_L(H + X) \text{ for all left-atomic games } X \in U;
\]
\[
o_R(G + X) \geq o_R(H + X) \text{ for all right-atomic games } X \in U.
\]

From now onwards, pairs of games in an Absolute Universe will combine to another (normal-play) game.

Definition 2.2 (Left Provisional Game). Let $U$ be an Absolute Universe. The Left Provisional Game (LPG) is defined on $U \times U$ as follows.

1. The positions are ordered pairs of games $[G, H] \in U \times U$;
2. The Left options of $[G, H]$ are of the form:
   a. $[G^L, H] \in \text{Proviso}(U)$;
   b. $[G, H^R] \in \text{Proviso}(U)$.
3. The Right options are games of the form $[G^R, H]$ or $[G, H^L]$;
4. A player who cannot move loses.

That is Right cannot move and loses playing first if both $G^R$ and $H^L$ are empty. For Left the situation is more intricate. If for all $G^L$, $[G^L, H] \notin \text{Proviso}(U)$ and for all $H^R$, $[G, H^R] \notin \text{Proviso}(U)$, then Left cannot move and loses. Thus, Left Provisional Game $[G, H]$ is in fact a normal-play game, regardless of $U$. Using the standard notation of normal-play, thus $[G, H] = \{[G^L, H]^L \mid [G, H]^R \} = \{[G^L, H] \in \text{Proviso}(U), [G, H^R] \in \text{Proviso}(U) \mid [G^R, H], [G, H^L]\}$, where $G^L$ ranges over all $G$’s Left options, $H^R$ ranges over $H$’s all Right options, etc.

Definition 2.3 (Left’s Maintenance). Consider an Absolute Universe $U$, and let $G, H \in U$. The Left Provisional Game $[G, H] \in \text{Maintain}(U) \subset U \times U$ if, for all Right options $[G, H]^R \in [G, H]^R$, there is a Left option $[G, H]^{RL}$, such that $[G, H]^{RL} \in \text{Maintain}(U)$.

Let us recall the main theorem for comparing games in an Absolute Universe, now stated as an equivalence involving Left Provisional Games (see also Appendix A).

Theorem 2.4 (Basic order of CGT, [4]). Consider an Absolute Universe $U$ and let $G, H \in U$. Then $G \succ H$ if and only if Left Provisional Game $[G, H] \in \text{Proviso}(U) \cap \text{Maintain}(U)$. 
Analogously:

**Theorem 2.5.** Let $G, H$ be games in an Absolute Universe $U$. Then $G \succeq H$ if and only if $[G, H] \in \text{Proviso}(U)$ and the Left Provisional Game $[G, H] \geq 0$.

**Proof.** By Theorem 2.4 it suffices to prove that $[G, H] \geq 0$ is equivalent with $[G, H] \in \text{Maintain}(U)$. This follows precisely because the inequality means Left wins playing second in normal-play, which is Definition 2.2 (3) combined with the definition of Maintain($U$).

To the authors’ knowledge, in each studied Absolute Universe, Proviso($U$) is constructive, in the sense that the condition in Definition 2.1 can be simplified to compare only (variations of) the outcome of the actual games $G$ and $H$, omitting the potentially infinite class of atomic distinguishing games $X$. For example, in the universe of dicot misère-play games, Proviso($U$) = $\{ [G, H] : o(G) \geq o(H) \}$.

**Example 2.6.** The Proviso simplifies to $o(G) \geq o(H)$ in dicot misère play, since the only atomic games are the purely atomic ones. Take $U$ as the dicot misère universe and let $G = _{\wedge} = (0, * | *)$ ("mup", that means "misère up", the simplest dicotic game strictly larger than zero) and $H = 0$. In the Left Provisional Game $[\wedge, 0]$, Left cannot move to $[* , 0]$, because $P = o(*) \not\in U$, $o(0) = N$ gives that the Proviso is not satisfied. The game tree of the Left Provisional Game position $[\wedge, 0]$ is given in Figure 1. This shows that $[G, H] = \uparrow > 0$ and thus $G \succ H$. Still in dicot misère, the Left Provisional Game $[\uparrow , 0] \npreceq 0$, because $R = o(\uparrow) \not\in o(0) = N$, by the Proviso.

### 3. Categories

Joyal’s construction for a category of normal-play games $G$ and $H$ uses that $G \succeq H$ if and only if $G - H \succeq 0$ if and only if Left has a winning strategy playing second in the game $G - H$ (Left’s set of winning strategies is “the arrow”). In our terminology, this corresponds to the Left Maintenance for the free space of normal-play games. This follows since, for normal-play, the Proviso is implied by

\footnote{Where $o(X) = (o_L(X), o_R(X)) \in \{(−1, −1) = R, (−1, +1) = P, (+1, −1) = N, (+1, +1) = L\}, X \in U$, inducing a partial order of outcomes.}
the Maintenance part, which is the condition $G \geq H$ in normal-play. We show that each Absolute Universe is categorical by extending Joyal’s construction to the Left Provisional Game.

In a category, the $\text{Hom}(H, G)$ is a collection of morphisms that link the object $H$ to the object $G$ in a, for the given structure, specific and meaningful way. The morphisms can be functions but it is not a requirement, as we saw with for example Joyal’s winning strategies. The arrows preserve some important property of the given structure, such as “winning” in Joyal’s example. We write $H \rightarrow G$ if $\text{Hom}(H, G)$ is not empty (and $H \leftarrow G$ if we want to particularize an element $f \in \text{Hom}(H, G)$).

To have a categorical structure, three properties must hold:

1. Identity: $G \rightarrow G$ for every object $G$;
2. Composition: given $f \in \text{Hom}(H, J)$ and $g \in \text{Hom}(J, G)$ there is a natural composition $g \circ f \in \text{Hom}(H, G)$;
3. Associativity: the defined composition is associative.

We will give a categorical construction based on a calculus of defined Left Maintenance Strategies of the LPG. Joyal’s and Conway’s “winning” is merely a consequence of being able to maintain an advantage, specifically, being able to move when it is your turn. By using the LPG rather than the actual games, our “arrows” will contain all information of how Left maintains the ability to move, in particular while facing the additional burden of the Proviso part.

**Definition 3.1.** A play in a Left Provisional Game $X_0 = [G, H]$ is a chain of positions $X_0 \rightsquigarrow X_1 \rightsquigarrow \cdots \rightsquigarrow X_n$ where the ‘moves’ correspond to alternating Left and Right (or Right and Left) moves, and where $n \geq 0$.

Thus, we allow for a play to be perhaps the empty sequence of moves. Of course a play can be defined for any combinatorial game, but we only use it in the context of Left Provisional Games.

**Definition 3.2.** A Left Maintenance Strategy in a given LPG is a play with the following property: consider any stage of the play, where Right is to move; if Right has a move, then Left has a response to this move. We write $L_H(G, H)$ for the set of all Left Maintenance Strategies in the game $[G, H]$, assuming that Right starts, and $L_L(H, G)$ for all Left Maintenance Strategies, assuming that Left starts.

Note 1: The reason that we reverse the order of the games in the sets of maintenance strategies is that these will correspond to the homorphisms of the categories, and the order of categorical objects related to “arrows” is reversed as compared with the conventions in game theory.

Note 2: Since the LPG is a normal-play game, if you have a maintenance strategy you will eventually win. The particular winning convention of the component games inside the LPG is irrelevant as long as the universe is absolute.

Choosing a strategy $f \in L_H(G, H)$ is equivalent to choosing a strategy $f_{G^R} \in L_L(H, G^R)$ for each position $G^R \in G^R$ and a strategy $f_{H^L} \in L_L(H^L, G)$ for each position $H^L \in H^L$. Therefore

$$L_R(G, H) \cong \bigcup_{G^R} L_L(H, G^R) \cup \bigcup_{H^L} L_L(H^L, G).$$

---

\(^2\)For normal-play games we use the standard notation for inequality $\geq$, whereas in any other (general) universe we write $\succ$. 
The concept of a residual strategy is crucial in obtaining the composition of morphisms. The fundamental idea is the swivel-chair strategy, using the terminology of \[\text{(or strategy stealing)}, \] see also \[.\]

\textbf{Definition 3.3} (Left’s Residual Strategy). Given two maintenance strategies \(g \in \mathcal{L}_R(J, G)\) and \(f \in \mathcal{L}_R(H, J)\), we construct Left’s residual strategy \(g \ast f\) as follows.

Consider a Right move from \([G, H]\) to \([G^R, H]\). We will find a Left’s maintenance response, given the candidate morphisms \(f\) and \(g\).

Set up the two games \([G, J]\) and \([J, H]\), corresponding to the maintenance strategies \(f\) and \(g\) respectively; see the columns of Figure 2.

If Left’s maintenance response in \([G^R, J]\) is to \([G^{RL}, J]\), then adapt this maintenance strategy for the game \([G^R, H]\), as \([G^{RL}, H]\).

If Left’s maintenance response in \([G^R, J]\) is to \([G^R, J^R]\) then Left considers instead her maintenance response to the Right move in the game \([J^R, H]\). If this is \([J^R, H^R]\), then her response in \([G^R, H^R]\) is to \([G^R, H^{RL}]\). If, instead, the response is to some \([J^{RL}, H]\), she swivels back to the game \([G^R, J^{RL}]\), and finds a response to this Right move, and so on.

In case \([G^R, J^R]\) is a terminal position, then, because \([J^R, H]\) is a Right’s move in a Left Maintenance Strategy, there must exist a Right move in \(H^R\), and so the play will terminate in \([G^R, H^R]\), with a Left win (recall the Left Maintenance Game is normal-play).

In either case, by continuing this idea, because \(J\) is finite and because \(f\) and \(g\) are maintenance strategies, eventually Left’s response shifts to either of the forms \([G^{RL}, J^\alpha]\), with \(\alpha = RL\ldots L\) or \([J^\alpha, H^R]\) with \(\alpha = RL\ldots R\) (i.e. \(\alpha\) is a finite sequence of alternating moves). In the first case, the response in \([G^R, H]\) will be to \([G^{RL}, H]\) and in the second case it will be to \([G^R, H^R]\). Unless this is a terminal position, we may iterate the argument.

The construction of \(g \ast f\) in the case of the Right move \([G, H^L]\) is analogous.

By the definition of \(f\) and \(g\) it is clear that the residual strategy \(g \ast f\) is well defined. As an immediate consequence we get

\textbf{Lemma 3.4}. Consider \(G, J, H \in \mathcal{U}\) and suppose that \(f \in \mathcal{L}_R(H, J)\) and \(g \in \mathcal{L}_R(J, G)\) are Left Maintenance Strategies in the games \([J, H]\) and \([G, J]\) respectively. Then the residual strategy \(f \ast g \in \mathcal{L}_R(H, G)\) is a Left Maintenance Strategy in the game \([G, H]\).

\textit{Proof.} By the swivel-chair construction in the definition of a residual strategy for the game \([G, H]\), Left has a response to any Right move at each stage of play. Thus \(f \ast g \in \mathcal{L}_R(H, G)\). \(\square\)

\textbf{Lemma 3.5}. The operator \(\ast\) is associative.

\textit{Proof.} Given \(f \in \mathcal{L}_R(H, J)\), \(h \in \mathcal{L}_R(J, W)\), and \(g \in \mathcal{L}_R(W, G)\), we construct the composite residual strategy \(g \ast h \ast f \in \mathcal{L}_R(G, H)\) in analogy with the swivel chair construction in Definition 3.3. Against, say, a Right move from \((G, H)\) to \((G^R, H)\), Left executes the stealing procedure over the strategies \(f, h,\) and \(g,\) getting, after a finite number of steps, an option \(G^{RL}\) or \(H^R\). That \(g \ast h \ast f = g \ast (h \ast f) = (g \ast h) \ast f\) is then trivial. \(\square\)

\textbf{Definition 3.6} (Mimic strategy). Consider the Left Provisional Game position \([G, G]\). We define the mimic strategy \(m \in \mathcal{L}_R(G, G)\) (or copy-cat) as the strategy
where Left replies to \([G^R, G]\) and \([G, G^L]\) with \([G^R, G^R]\) and \([G^L, G^L]\) respectively, and repeats this mimic process during the play.

**Lemma 3.7.** The mimic strategy is a Left Maintenance Strategy.

**Proof.** In any game of the form \([X, X]\), the proviso is trivially satisfied, so Left has the same options as Right, and, as a required response, can thus imitate each Right move. □

By using maintenance strategies in the Left Provisonal Game as the morphisms, we generalize Joyal’s results on categories for normal-play, to any Absolute Universe of combinatorial games.

**Theorem 3.8.** Consider an Absolute Universe \(U\) and \(G, H \in U\). If \(G \succ H\), then, the structure \((U, f, \circ)\), where \(f \in \mathcal{L}_R(H, G) = \text{Hom}(H, G)\), and \(g \circ f = g \circ f\), is categorical.

**Proof.** By Theorem 3.4, \(G \succ H\) implies \([G, H] \in \text{Maintain}(U) \cap \text{Proviso}(U)\), which in particular implies that, in the LPG, the set of Left’s maintenance strategies \(\mathcal{L}_R(H, G)\) is nonempty. Moreover, we have seen that the operator is consistent with the residual strategy as composition and the mimic strategy as identity. Indeed, that the following diagram commutes was explained in Lemma 3.4.
That the defined composition (residual strategy) is associative was explained in Lemma 3.5.

For any Absolute Universe \( U \), call this category \( \text{LNP}(U) \), Left Normal Play over \( U \). We finish off by continuing Example 2.6, the dicot misère-play application.

**Example 3.9.** We compare the games of rank 2 in the dicot misère-play universe. The Proviso is \( o(G) \geq o(H) \). The order is given in Figure 3, the value of the LPG \([G, H]\) where \( G \) covers \( H \) in the partial order is written by the appropriate edge. In the picture, the dicot misère-play game values (literal forms) are \( \uparrow = \langle 0|* \rangle \), \( \downarrow = \langle *|0 \rangle \), \( \breve{\lambda} = \langle 0,*|* \rangle \), \( \gamma = \langle *|*,0 \rangle \) (“mown”), \( \breve{\lambda} * = \langle 0,*|0 \rangle \), and \( \gamma = \langle 0|*,0 \rangle \).

**Acknowledgement.** We thank Darien DeWolf for suggesting our category’s name.
Definition 3.11. Consider a totally ordered group $G$. This is known as the disjunctive sum. Here, and elsewhere, an expression of the progress. A player must choose exactly one of these sub-positions and play in it.

Definition 3.12. A combinatorial game space is the structure 

\[ \Omega = ((\Omega, \mathcal{A}), \mathcal{S}, \nu_L, \nu_R, +) \]
where ‘+’ is the disjunctive sum in the free space \((\Omega, A)\), \(S\) is a totally ordered set of game results, and \(\nu_L : A \to S\) and \(\nu_R : A \to S\) are order preserving maps. Moreover, if \(|A| > 1\) then require \(\nu(a) = \nu_L(a) = \nu_R(a)\), for all \(a \in A\).

Suppose \(a, b \in S\) with \(a > b\), the standard convention is that Left prefers \(a\) and Right prefers \(b\). The three winning conventions usually considered in the literature:

- **normal-play** corresponds to: (i) the trivial group \(A = \{0\}\) and the set \(S = \{-1, +1\}\); (ii) the maps \(\nu_L(0) = -1, \nu_R(0) = +1\),
- **misère-play** corresponds to: (i) the trivial group \(A = \{0\}\) and the set \(S = \{-1, +1\}\); (ii) the maps \(\nu_L(0) = +1, \nu_R(0) = -1\),
- **scoring-play** usually corresponds to the adorns being the group of real numbers, with its natural order and addition, and moreover \(S = A = \mathbb{R}\), and where \(\nu\) is the identity map.

The *conjugate* denotes the position where Left and Right have ‘switched roles’.

**Definition 3.13.** The conjugate of \(G \in \Omega\) is

\[
\hat{\hat{G}} = \begin{cases} 
\langle b \mid a \rangle, & \text{if } G = \langle a \mid b \rangle, \quad a, b \in A \\
\langle G^R \mid a \rangle, & \text{if } G = \langle a \mid G^R \rangle \\
\langle a \mid G^C \rangle, & \text{if } G = \langle G^C \mid a \rangle \\
\langle G^C \mid G^R \rangle, & \text{otherwise,}
\end{cases}
\]

where \(G^R\) denotes the list of games \(\hat{\hat{X}}\), for \(X \in G^R\), and similarly for \(G^C\).

By the recursive definition of the free space \((\Omega, A)\), each combinatorial game space is closed under conjugation. In normal-play, the games form an ordered group and each game \(G\) has an additive inverse, appropriately called \(-G\) and \(-G = \hat{\hat{G}}\). However, there are other spaces of games, for example scoring and misère games, where \(\hat{\hat{G}}\) is not necessarily \(-G\) (e.g. [5]).

**Definition 3.14.** A *universe* of games, \(U \subseteq \Omega\), is a subspace of a given combinatorial game space \(\Omega = ((\Omega, A), S, \nu_L, \nu_R, +)\), with:

1. \(A = \langle a \mid a \rangle \in U\) for all \(a \in A\);
2. **options closure**: if \(A \in U\) and \(B\) is an option of \(A\) then \(B \in U\);
3. **disjunctive sum closure**: if \(A, B \in U\) then \(A + B \in U\);
4. **conjugate closure**: if \(A \in U\) then \(\hat{\hat{A}} \in U\).

The mapping of adorns in \(A\) to elements of \(S\) is extended to positions in general via two recursively defined (optimal play) outcome functions.

**Definition 3.15.** Let \(G \in U \subseteq \Omega\) and consider given maps \(\nu_L : A \to S\) and \(\nu_R : A \to S\), where \(S\) is a totally ordered set. The left- and right-outcome functions are \(o_L : \Omega \to S, o_R : \Omega \to S\), where

\[
o_L(G) = \begin{cases} 
\nu_L(\ell) & \text{if } G = \langle \ell \mid G^R \rangle, \\
\max_L \{o_R(G^L)\} & \text{otherwise},
\end{cases}
\]

\[
o_R(G) = \begin{cases} 
\nu_R(r) & \text{if } G = \langle G^C \mid r \rangle, \\
\min_R \{o_L(G^R)\} & \text{otherwise,}
\end{cases}
\]
where the $\max_L$ (min$_R$) ranges over all Left (Right) options.

From this we conclude that each universe is a partially ordered commutative monoid with 0 as the additive identity.

Let $G \in U$. From Definition 3.15 we have that $o_L(G) = \nu_L(\ell)$ and $o_R(G) = \nu_R(r)$ for some $\ell, r \in A$. Therefore we may always assume that the set of (left- and right-) outcomes is $S = \{\nu_L(a) : a \in A\} \cup \{\nu_R(a) : a \in A\}$.

**Definition 3.16.** A universe $U$ of combinatorial games is parental if, for each pair of finite non-empty lists, $A, B \subseteq U$, then $\langle A | B \rangle \in U$.

**Definition 3.17.** A universe $U$ of combinatorial games is dense if, for all $G \in U$, for any $x, y \in S$, there is a $H \in U$ such that $o_L(G + H) = x$ and $o_R(G + H) = y$.

**Definition 3.18.** A universe $U$ of combinatorial games is an Absolute Universe if it is both parental and dense.

A partial order is defined on any universe of additive combinatorial games.

**Definition 3.19.** Let $U$ be any universe of combinatorial games. For $G, H \in U$, $G \succeq H$ modulo $U$ if and only if $o_L(G + X) \succeq o_L(H + X)$ and $o_R(G + X) \succeq o_R(H + X)$, for all games $X \in U$.

The main results for Absolute Combinatorial Game Theory [4] are the following improvements of general game comparison. (The “Common Normal Part” corresponds to the Maintenance part in this paper.)

**Theorem 3.20** (Basic order of games [4]). Consider games $G, H \in U$, an Absolute Universe. Then $G \succeq H$ if and only if the following two conditions hold.

**Proviso:**

- $o_L(G + X) \succeq o_L(H + X)$ for all left-atomic $X \in U$;
- $o_R(G + X) \succeq o_R(H + X)$ for all right-atomic $X \in U$;

**Common Normal Part:**

For all $G^R$, there is $H^R$ such that $G^R \succeq H^R$ or there is $G^RL$ such that $G^RL \succeq H$;

For all $H^L$, there is $G^L$ such that $G^L \succeq H^L$ or there is $L^LR$ such that $G \succeq H^L \succeq L^LR$.

**Corollary 3.21** (Subordinate game comparison [4]). Let $G, H \in U$, an Absolute Universe. Then $G \succeq_U H$ if the Common Normal Part holds and if $U$ is the

- normal-play universe;
- dicot misère-play universe, and $o(G) \succeq o(H)$;
- free misère-play space, and $H^L = 0^0 \Rightarrow G^L = 0^0$ and $G^R = 0^0 \Rightarrow H^R = 0^0$;
- dicot scoring-play universe, and $o(G) \succeq o(H)$;
- guaranteed scoring-play universe, and $\underline{o}_L(G) \succeq \underline{o}_L(H)$ and $\underline{o}_R(G) \succeq \underline{o}_R(H)$, where $\underline{o}$ and $\underline{7}$ denotes Right’s and Left’s pass allowed left- and right-outcomes respectively [4].

**APPENDIX B**

One of the benefits of the Left Provisional Game is that it allows for game comparison in any Absolute Universe in CG-suit. We attach code for version CG-suit 0.7 (coded by C. Santos). The procedure CompareDM requires input Left Provisional Game as a pair of literal form (dicot misère-play) games. We begin by illustrating how to run the below code.
EXAMPLE:

\[ G = \text{literally}\left(\{0,*\}^*\right) \]
\[ H = \text{literally}\left(\{0,*\}{0|0,*}\right) \]
\[ \text{CompareM}([G,H]) \]
\[ M\text{outcome} := \text{proc (G)} \]
\[ \text{local a,b,c,j,w,k,l,r,i; option remember; } \]
\[ l := \text{LeftOptions(G);} \]
\[ r := \text{RightOptions(G);} \]
\[ b := \text{Length(l);} \]
\[ c := \text{Length(r);} \]
\[ \text{if (G==0) then k:=11;} \]
\[ \text{fi;} \]
\[ \text{if (G!=0) then j:=0;} \]
\[ \text{for i from 1 to b do} \]
\[ \text{if (M\text{outcome(l[i]])==0 or M\text{outcome(l[i]])==1) then j:=1;} fi;} \]
\[ \text{od;} \]
\[ w:=0; \]
\[ \text{for i from 1 to c do} \]
\[ \text{if (M\text{outcome(r[i]])==0 or M\text{outcome(r[i]])==1) then w:=1;} fi;} \]
\[ \text{od;} \]
\[ \text{if (j==0 and w==0) then k:=0;} \]
\[ \text{fi;} \]
\[ \text{if (j==0 and w==1) then k:=-1;} \]
\[ \text{fi;} \]
\[ \text{if (j==1 and w==0) then k:=1;} \]
\[ \text{fi;} \]
\[ \text{if (j==1 and w==1) then k:=11;} \]
\[ \text{fi;} \]
\[ \text{return k;} \]
\[ \text{end;} \]
\[ \text{Dual := proc (pos)} \]
\[ \text{local l,r,l1,r1,l2,r2,l11,l12,rr1,rr2,i,aux; option remember;} \]
\[ l := []; \]
\[ r := []; \]
\[ l1 := \text{LeftOptions(pos[1])}; \]
\[ r1 := \text{RightOptions(pos[1])}; \]
\[ l2 := \text{LeftOptions(pos[2])}; \]
r2 := RightOptions(pos[2]);

ll1:=Length(l1);
rr1:=Length(r1);
ll2:=Length(l2);
rr2:=Length(r2);

for i from 1 to rr1 do
aux:=[r1[i],pos[2]];
Add(r,Dual(aux));
od;

for i from 1 to ll2 do
aux:=[pos[1],l2[i]];
Add(r,Dual(aux));
od;

for i from 1 to ll1 do
if (Moutcome(l1[i])==1) then
aux:=[l1[i],pos[2]];
Add(l,Dual(aux));
fi;
if (Moutcome(l1[i])==11 and (Moutcome(pos[2])==11 or Moutcome(pos[2])==-1)) then
aux:=[l1[i],pos[2]];
Add(l,Dual(aux));
fi;
if (Moutcome(l1[i])==0 and (Moutcome(pos[2])==0 or Moutcome(pos[2])==-1)) then
aux:=[l1[i],pos[2]];
Add(l,Dual(aux));
fi;
if (Moutcome(l1[i])==-1 and Moutcome(pos[2])==-1) then
aux:=[l1[i],pos[2]];
Add(l,Dual(aux));
fi;
od;

for i from 1 to rr2 do
if (Moutcome(pos[1])==1) then
aux:=[pos[1],r2[i]];
Add(l,Dual(aux));
fi;
if (Moutcome(pos[1])==11 and (Moutcome(r2[i])==11 or Moutcome(r2[i])==-1)) then
aux:=[pos[1],r2[i]];
Add(l,Dual(aux));
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```
fi;
if (Moutcome(pos[1])==0 and (Moutcome(r2[i])==0 or Moutcome(r2[i])==-1))
then
  aux:=[pos[1],r2[i]];  
  Add(l,Dual(aux));
fi;
if (Moutcome(pos[1])==-1 and Moutcome(r2[i])==-1)
then
  aux:=[pos[1],r2[i]];  
  Add(l,Dual(aux));
fi;
odo;

return {l | r};
end;

CompareDM := proc (pos)
local l,r,l1,r1,l2,r2,ll1,ll2,rr1,rr2,i,a,b,s;
option remember;

l := [];
r := [];

l1 := LeftOptions(pos[1]);
r1 := RightOptions(pos[1]);

l2 := LeftOptions(pos[2]);
r2 := RightOptions(pos[2]);

ll1:=Length(l1);
rr1:=Length(r1);
ll2:=Length(l2);
rr2:=Length(r2);

a:=0; b:=0;

if ((Moutcome(pos[1])==1) or (Moutcome(pos[1])==11 and
(Moutcome(pos[2])==11 or Moutcome(pos[2])==-1)) or
(Moutcome(pos[1])==0 and (Moutcome(pos[2])==0 or
Moutcome(pos[2])==-1)) or (Moutcome(pos[1])==-1 and
Moutcome(pos[2])==-1)) and (Dual(pos)>=0) then
  a:=1;
fi;
if ((Moutcome(pos[2])==1) or (Moutcome(pos[2])==11 and
(Moutcome(pos[1])==11 or Moutcome(pos[1])==-1)) or
(Moutcome(pos[2])==0 and (Moutcome(pos[1])==0 or
Moutcome(pos[1])==-1)) or (Moutcome(pos[2])==-1 and
Moutcome(pos[2])==-1)) and (Dual(pos)>=0) then
  b:=1;
fi;
```
if (Moutcome(pos[1])==-1) or (Moutcome(pos[2])==-1 and
Moutcome(pos[1])==-1)) and (Dual([pos[2],pos[1]])>=0) then
b:=1;
fi;

if (a==1) and (b==1) then s:="G=H"; fi;
if (a==1) and (b==0) then s:="G>H"; fi;
if (a==0) and (b==1) then s:="G<H"; fi;
if (a==0) and (b==0) then s:="G<>H"; fi;

return s;
end;

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