Variational theory of elastic manifolds with correlated disorder and localization of interacting quantum particles

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Abstract

We apply the gaussian variational method (GVM) to study the equilibrium statistical mechanics of the two related systems: (i) classical elastic manifolds, such as flux lattices, in presence of columnar disorder correlated along the \( \tau \) direction (ii) interacting quantum particles in a static random potential. We find localization by disorder, the localized phase being described by a replica symmetry broken solution confined to the mode \( \omega = 0 \). For classical systems we compute the correlation function of relative displacements. In \( d = 2 + 1 \), in the absence of dislocations, the GVM allows to describes the Bose glass phase. Along the columns the displacements saturate at a length \( l_{\perp} \) indicating flux-line localization. Perpendicularly to the columns long range order is destroyed. We find divergent tilt modulus \( c_{44} = \infty \) and a \( x \sim \tau^{1/2} \) scaling. Quantum systems are studied using the analytic continuation from imaginary to real time \( \tau \rightarrow it \). We compute the conductivity and find that it behaves at small frequency as \( \sigma(\omega) \approx \omega^2 \) in all dimensions \( (d < 4) \) for which disorder is relevant. We compute the quantum localization length \( \xi \). In \( d = 1 \), where the model also describes interacting fermions in a static random potential, we find a delocalization transition and obtain analytically both the low and high frequency behavior of the conductivity for any value of the interaction. We show that the marginality condition appears as the condition to obtain the correct physical behavior. Agreement with renormalization group results is found whenever it can be compared.

I. INTRODUCTION

Two related longstanding problems of statistical mechanics have recently received renewed attention. On the one hand the problem of the pinning of the flux lattice in high-Tc superconductors has motivated further studies of the effect of disorder on an elastic manifold. Pinning by correlated disorder, e.g. in the form of columnar defects introduced in the sample
by heavy-ion irradiation, is particularly interesting for technological applications since it leads to a large increase in critical current. On the other hand, the problem of interacting quantum particles in a random potential, as the one of quantum fluctuations of an elastic manifold in a random potential, have been recently studied by a number of authors. The latter problems have many experimental realizations, such as Charge Density Waves (CDW), Wigner crystals, quantum vortex creep in superconductors, superconducting-insulating transition in disordered superconductors. There is a deep analogy between these two classes of problems which rests on the well known identification of the bosons worldlines in the imaginary-time path integral of quantum mechanics with actual flux lines. This relation has been reanalyzed in details and has led for instance, to the prediction of a new physical state of vortex systems with vanishing linear resistivity, where 3d flux lines are localized by columnar disorder, very much like 2d quantum bosons become localized in presence of disorder and form the so-called Bose glass state. Although there is some evidence for a Bose glass, or localized phase, few analytical methods are available to study this problem. One has to resort to series expansions, numerical simulations. Experimental evidence is also controversial. In the case of $d = 1$ (one spatial dimension, one “time” dimension) where interacting bosons and fermions can be related, the problem can be studied using powerful bosonization and Renormalization Group (RG) techniques. These methods show a transition away from the superfluid phase when the repulsion is increased and allow to compute the crossover towards the localized phase. However they do not allow to describe the low energy properties of the resulting (presumably) localized phase, since RG trajectories runaway to strong coupling. Only in the case of non-interacting fermions, or equivalently bosons interacting with hard-core repulsion, is the “Bose glass” phase rigorously known to exist. The conductivity was calculated in that case by Berezinskii and found to behave as $\sigma(\omega) \sim \omega^2 \ln(\omega)^2$.

An interesting method to study problems with an interplay of elasticity and disorder is the Gaussian Variational method (GVM) introduced by Mezard and Parisi to study the problem of elastic manifolds, such as interfaces, in a random uncorrelated potential. In this method one finds the best quadratic Hamiltonian approximation to the original complicated Hamiltonian. This quadratic approximation is performed in replica space and the optimization naturally leads to Replica Symmetry Breaking (RSB) whenever the energy landscape is complicated and contains metastable states. Within the gaussian ansatz, RSB appears as the natural way to encode the distribution of the metastable states induced by disorder. The possible optimal Hamiltonian is parametrized by a function $\sigma(u)$, which leads to a large number of optimization parameters. Thus one can hope that it produces a reasonable approximation of the real problem. Indeed, in the interface problem, the method leads to a reasonable, Flory-like, approximation for the roughness exponent; It also leads to the correct physics of sample to sample fluctuations in response functions, due to rare events. This was recently demonstrated in $d = 1 + 1$ using the mapping onto the Burgers equation. In recent papers, we have applied the GVM to the problem of an elastic lattice in a random uncorrelated potential. It was shown that the periodicity of the lattice leads to logarithmic disorder-induced displacements at large scales in $d \geq 2$. Comparison with two RG calculations suggests that the GVM captures the correct physics, and leads to reasonable quantitative approximations. Although the GVM captures important non-perturbative features such as the existence of many local equilibrium states,
it may miss some details of the nonlinearities and instanton-type configurations (kinks) between metastable states.

In the present paper we apply the Gaussian Variational Method to classical problems with correlated disorder and to quantum problems with disorder. A summary of the results of this work has appeared in Ref. 34. There are some additional subtleties in applying the method, due to translational invariance in one direction (the “imaginary time” direction) and extra care should be paid in treating boundary conditions and taking the large “imaginary time” limit $\beta \to \infty$ properly. We then find that the GVM naturally leads to localization by disorder and we compute the properties in the localized phase. For the classical system we obtain the tilt modulus and find that $c_{44}(q_z) \to \infty$ as $q_z \to 0$ in the localized phase, i.e a signature of the transverse Meissner effect predicted in Ref. 4. For quantum systems we investigate in detail dynamical correlation functions such as the conductivity. We study in detail the case of of $d = 1$ which describes either the classical problem of flux lines in a plane with correlated (columnar) disorder or interacting fermions (or bosons) in a $d = 1$ static random potential. We also give the main features and the essential physics of the solution for general $d$. In $d = 2$ we study simplified models relevant to experimental systems such as the three dimensional vortex lattice with columnar disorder, disordered quantum bosons in $d = 2$, the pinned Wigner crystal and Charge Density waves.

In a celebrated work, Fukuyama and Lee (FL) analyzed the quantum problem of a one-dimensional CDW with disorder ($d = 1$) using a diagrammatic self-consistent method. Physically, their method, and its further applications to interacting fermions in $d = 1$, amounts to choose an unknown classical solution ($h = 0$) of the equations of motion in the random potential and to treat the quantum fluctuations as an harmonic oscillator around this solution. Since it corresponds to an expansion around an unknown classical background, the FL method contains an ad-hoc phenomenological parameter. Although the static correlations are nearly independent of this ad-hoc parameter, completely different physical behaviors can be obtained for frequency-dependent quantities depending on its precise value. To obtain the presumed correct physics the parameter had therefore to be adjusted by hand. Remarkably, the physics of the GVM solution parallels the FL theory and allows to establish it on a firmer footing. As in FL, the physics of the GVM solution involves an elastic expansion around the classical solution, but this comes out naturally as the solution of the variational equations. In addition the GVM allows for a detailed description of the classical solution. In particular, the phenomenological parameter adjusted by hand in FL exactly corresponds to the breakpoint parameter $u_c$ in the RSB solution! It is thus determined by the theory itself. The GVM therefore provides a more general theoretical framework which can easily be extended to higher dimensions and more complicated systems.

We have not attempted to treat the strong pinning case, i.e a Poisson distribution of strong pins, although this extension of the method is possible in principle. For flux line lattices, disorder can be considered as gaussian and weak when $R_c > D$ where $D$ is the distance between columnar defects and $R_c$ the Larkin-Ovchinnikov pinning length. Since columnar defects produced by irradiation are quite strong pinning centers this corresponds to high fields or temperature. Furthermore our theory corresponds to a large pinning length $\xi \gg a$, i.e the elastic limit is assumed, as well as the absence of dislocations.

The plan of the paper is as follows: in Section II we introduce the two classes of models
studied in this paper, quantum problems and classical problems with correlated disorder. In Section II we apply the variational method to these models and derive the corresponding saddle point equations. We discuss general features of the solution and show that, within the GVM, only the zero Matsubara frequency mode can exhibit replica symmetry breaking. In Section IV we give the solution for a classical model in $d = 1$ and $d = 2$ describing respectively lines and vortices in a correlated random potential, as well as the solution for general $d$. In Section V we discuss the properties of the solution specific to quantum problems, such as the analytical continuation to real time and we perform the calculation of the conductivity. The GVM is applied to the problem of interacting quantum particles in $d=1+1$. Conclusions can be found in section VI.

II. MODELS

This section introduces the various classical and quantum models which we study using the variational method.

A. general model

A general model for an elastic manifold of internal dimension $d+1$ in a correlated random potential is:

$$H = \int d^d x d\tau \frac{c}{2} [(\nabla_x \phi)^2 + (\partial_\tau \phi)^2] + \int d^d x d\tau W((\phi(x, \tau), x)$$

(1)

$\phi$ can in general be a $N$-component field (to describe a manifold embedded in $N$ dimensions). The partition function is $Z = \int D\phi(x, \tau) \exp(-\frac{H}{T})$. The disorder potential is Gaussian, correlated in the direction $\tau$ and the correlator is:

$$W(\phi, x)W(\phi', x') = -C(\phi - \phi')\delta^d(x - x')$$

(2)

This model describes a large number of classical models and we give important physical realizations in the next sections. It also describes the equilibrium statistical mechanics of quantum problems with disorder. The Hamiltonian can also be viewed as the action of a quantum problem and $T$ corresponds to $\hbar$. $\tau = it$ is the imaginary time of the quantum problem in the Matsubara representation. $\beta$ is the inverse temperature of the quantum problem, and by imposing periodic boundary conditions $\phi(x, \tau = L) = \phi(x, \tau = 0)$ one describes bosonic systems, with $L = \beta \hbar$. Disorder is then time independent which is the natural situation for a quantum problem. $d = 0$ corresponds to a single quantum particle in a random potential. Applications to quantum problems are considered in more details in Section V. Note that in this paper $\beta$ and $T$ are two unrelated parameters.

In order to average over disorder one uses the replica trick by introducing $k$ replicas of the system $\phi_a$, $a = 1, ..k$ and taking the limit $k \to 0$ at the end. The replicated Hamiltonian averaged over disorder reads:

$$\frac{H_{\text{eff}}}{T} = \int d^d x d\tau \frac{c}{2} \sum_a [(\nabla_x \phi_a)^2 + (\partial_\tau \phi_a)^2] + \frac{1}{2T} \int d^d x d\tau d\tau' \sum_{a,b} C(\phi_a(x, \tau) - \phi_b(x, \tau'))$$

(3)
B. vortex lattice: all harmonics

An important physical realization of the system (1) is the Abrikosov vortex lattice in the presence of columnar disorder. The columns are oriented parallel to the \( \tau \) direction, the magnetic field being aligned with the columns. The \( d = 1 \) version of this problem describes flux lines in a plane pinned by columns of defects while the \( d = 2 \) version describes the flux lattice with columns aligned with the direction of the flux lines.

We denote by \( R_i \) the equilibrium position of the vortex lines labeled by an integer \( i \), which defines a perfect lattice of spacing \( a \), and by \( u(R_i, \tau) \) their displacements transverse to the field. As will be seen later, the disorder defines a pinning length \( \xi \) (in the transverse direction) such that the relative displacement of vortices separated by \( \xi \) (for equal \( \tau \)) is of order \( a \). For weak disorder \( a/\xi \ll 1 \) it is legitimate to assume that \( u(R_i, \tau) \) is slowly varying on the scale of the lattice and to use a continuum elastic energy, in terms of the continuous variable \( u(x, \tau) \). Impurity disorder is modeled by a \( \tau \) independent gaussian random potential \( V(x) \) with correlations: \( \langle V(x) V(x') \rangle = W \delta(x-x') \). The total energy is:

\[
H = \frac{1}{2} \int d^dxd\tau \left[ (\partial_x u)^2 + (\nabla_x u)^2 \right] + \int dxd\tau V(x) \rho(x, \tau)
\]

where the density is \( \rho(x, \tau) = \sum_i \delta(x - R_i - u(R_i, \tau)) \). For simplicity we use an isotropic elastic hamiltonian. Instead of using the displacements \( u \), it is more useful to introduce a smooth “labelling” field \( \eta(x,\tau) = x - u(\eta(x,\tau),\tau) \). The density can be rewritten as

\[
\rho(x, \tau) = \rho_0 \det[\partial_{\alpha} \eta_{\beta}] \sum_K e^{iK \eta(x,\tau)} \approx \rho_0 (1 - \partial_{\alpha} \eta_{\alpha}(\eta(x,\tau),\tau) + \sum_{K \neq 0} e^{iK x} \rho_K(x))
\]

where \( \rho_K(x) = e^{-iK \cdot u(\eta(x,\tau),\tau)} \) is the usual translational order parameter defined in terms of the reciprocal lattice vectors \( K \) and \( \rho_0 \) is the average vortex density.

Using the replica trick on (4) the disorder term gives

\[
- \frac{W}{2T} \sum_{a,b} \int dxd\tau d\tau' \rho^a(x, \tau) \rho^b(x, \tau')
\]

The above decomposition for the density leads to:

\[
H_{\text{eff}} = \frac{1}{2} \int d^dxd\tau \sum_a c[(\partial_x u^a)^2 + (\nabla_x u^a)^2]
\]

\[
- \int d^dxd\tau d\tau' \sum_{a,b} \left[ \frac{\rho^2_0 W}{2T} \partial_x u^a(x, \tau) \partial_x u^b(x, \tau') + \sum_{K \neq 0} \frac{\rho^2_0 W}{2T} \cos(K(u^a(x, \tau) - u^b(x, \tau'))) \right]
\]

The range of validity of (1) is as discussed in Ref. 28, 29: (i) elastic limit \( a/\xi \ll 1 \) (ii) neglect of dislocations, (which is fully justified in \( d = 1 \)). We redefine \( \rho^2_0 W \to W \) in the following.

The slowly varying part of the disorder, i.e the \( q \sim 0 \) Fourier components of \( V(q) \), couples to the long wavelength part of the density fluctuations and gives rise in (2) to the quadratic part. The higher Fourier components of the random potential, specifically those of Fourier components near \( q \sim K \), lead to the cosine terms. The \( q \sim 0 \) part of the disorder can easily be treated, either in the replica form of (2), or before averaging by simply redefining:
\begin{equation}
  u_\alpha(x, \tau) \rightarrow u_\alpha(x, \tau) + f_\alpha(x, \tau) \\
  f_\alpha(q, \omega) = \delta_{\omega,0} \frac{i \rho_0 q_\alpha V_{q \sim 0}}{cq^2}
\end{equation}

where \( V_{q \sim 0} \) is a truncation of the random potential \( V(q) \) keeping only the Fourier components close to zero. Since this redefinition of the \( u \) field is \( \tau \)-independent, it does not affect the cosine terms in (7). The \( q \sim 0 \) part of the disorder decouples completely from the Hamiltonian (4) and we thus drop it in the remainder of this paper. It leads to a simple additional contribution to the correlation functions which can be computed using (8).

The model (7) which describes vortex lattices in correlated disorder is thus a particular case of the general model (1,2).

C. single cosine model

A further simplification can be obtained by keeping only the lowest harmonic \( K_0 \) in (7). This leads to another model, which for simplicity we define for a scalar field \( \phi \). This model is studied in detail in this paper:

\begin{equation}
  \frac{H_{\text{eff}}}{T} = \int d^d x d\tau \frac{c}{2} \sum_a (\nabla_x \phi_a)^2 + (\partial_x \phi_a)^2 - \int d^d x d\tau d\tau' \sum_{a,b} \frac{W}{2T} \cos(2(\phi_a(x, \tau) - \phi_b(x, \tau'))) \tag{9}
\end{equation}

It applies directly to CDW and interacting fermions in \( d = 1 \) as discussed in Section V. It is also relevant for flux lines and correspondence with model (7) can be made by defining \( u = a\phi/\pi \). As we will see, in the limit of weak disorder \( a/\xi \ll 1 \), the contribution of higher harmonics becomes irrelevant at large scales, for separations transverse to the magnetic field. As for point disorder, an intermediate “random manifold” regime exists at intermediate scale \( x < \xi \) (see Ref. 29). For separations along \( \tau \), higher harmonics can be included, as in (7), but do not change the main physics.

D. domain of application of the models

In this paper we consider Gaussian disorder, i.e a large number of very weak pins, rather than the Poissonian disorder which corresponds to few strong pins. In addition, we are in the elastic limit of small relative displacements between nearest neighbors. Finally in \( d = 2 \), dislocations have been neglected.

Because of the elastic forces (interactions) the modelization by Gaussian disorder is often appropriate. Its domain of validity can be estimated at low \( T \) as follows. Let us consider defects of strength \( U_0 \) separated by a distance \( D \), i.e a density \( n \sim 1/D^d \). The disorder potential is \( V(x) = U_0 \sum_i \delta(x - x_i) \) where \( x_i \) are the positions of the defects. Using (5), and keeping the lowest harmonic, the coupling to density is \( \sim \rho_0 \sum_i \cos(K_0.(u - x_i)) \). If density of defects is very small, or \( U_0 \) very strong, the phase of the lattice adjusts such that \( K_0.u = K_0.x_i \) at each defect. In fact one has to balance the cost in elastic energy with the gain in potential energy. Assuming that \( K_0.u \) varies by \( 2\pi \) over distances of order \( \xi \), the elastic (kinetic) energy is \( c\xi^{d-2} \) and the potential energy is, assuming effectively a Gaussian disorder, \( U_0 \sqrt{n\xi^d} \). Optimizing over \( \xi \), one finds \( \xi \sim (1/nU_0^2)^{1/(4-d)} \). The Gaussian-like
regime fails when the distance $\xi$ becomes smaller than the distance $D$ between defects. One then enters the Poissonian regime where $\xi$ then saturates at $\xi = D$. From the expression $\xi/D = (n^{2(2-d)/d}/U_0^2)^{1/(4-d)}$ one finds the domain of validity of Gaussian disorder in any dimensions. In $d = 1$ one has $\xi/D = (n/U_0)^{2/3}$ and the Gaussian regime holds if the density of defects is not too small $n \gg U_0$. In $d = 2$ one has $\xi/D = 1/U_0$ and the Gaussian regime holds at any density of defects provided disorder is small enough $U_0 \ll 1$. In $d = 3$ one has $\xi/D = 1/(n^{2/3}U_0^2)$ and the Gaussian regime holds for all density and disorder provided they are small.

For models described by a single harmonic, such as charge density waves, the above argument always holds. For flux lattices, disorder can vary at scale $\xi_0$ much smaller than $\alpha$ and higher harmonics may be important. There is an additional length scale, the Larkin Ovchinnikov length $R_c$, for which relative displacements are of order $\xi_0$. At weak disorder $R_c$ and $\xi$ (also called $R_a$) are very different. A sufficient criterion for Gaussian disorder to hold is then that $R_c \gg D$.

### III. VARIATIONAL METHOD : GENERAL PROPERTIES

#### A. saddle point equations

We study the statistical mechanics described by the partition function:

$$Z = \int D\phi(x, \tau) \exp\left(-\frac{H}{T}\right)$$

(10)

for a system of finite size $\beta$ in the $\tau$ direction. $H$ is the hamiltonian (1). In view of further applications to quantum problems we impose periodic boundary conditions in the $\tau$ direction. For infinitely thick vortex systems of length $L = \beta \rightarrow \infty$ boundary conditions should not matter. However, imposing definite boundary conditions and working first on a finite size sample is crucial for obtaining well-defined equations in the case of correlated disorder.

Introducing replicas and averaging over disorder the model becomes (3). We now use the variational method as in Ref. 25. This is done by choosing a Gaussian variational Hamiltonian:

$$H_0 = \frac{1}{2} \int \frac{d^dq}{(2\pi)^d} \sum_n G^{-1}_{ab}(q, \omega_n)\phi_a(q, \omega_n)\phi_b(-q, -\omega_n)$$

(11)

where the discrete summation is over $\omega_n = 2\pi n/\beta$ analogous to Matsubara frequencies. Note that $\frac{1}{\Omega} \sum_q \rightarrow \int \frac{d^dq}{(2\pi)^d}$ and $\frac{1}{\beta} \sum_n \rightarrow \int \frac{d\omega}{2\pi}$ in the large volume ($\Omega$, resp. $\beta$) limit.

By the same arguments as in Ref. 25 this method is exact for a choice of correlator of the form $C(\phi - \phi') = NV((\phi - \phi')^2/N)$ in the limit $N \rightarrow \infty$. This corresponds to a manifold in infinite dimensions, either a classical manifold with correlated disorder or a quantum manifold with static disorder. We obtain the variational free energy $F_{\text{var}} = F_0 + \langle H - H_0 \rangle_{H_0}$ as

$$F_{\text{var}} = \frac{T}{2} \sum_a \sum_{q, \omega_n} \left( c(\omega_n^2 + q^2)G(q, \omega_n)_{aa} - [\ln(TG)]_{aa} \right)$$

$$+ \frac{1}{2T} \sum_{ab} \int d^dx \int_0^\beta \int_0^\beta d\tau d\tau' \ V(B_{ab}(x = 0, \tau - \tau'))$$

(12)
where

\[ B_{ab}(x, \tau) = \langle [\phi_a(x, \tau) - \phi_b(0, 0)]^2 \rangle \]

\[ = \frac{T}{\beta} \int \frac{d^d q}{(2\pi)^d} \sum_n (G_{aa}(q, \omega_n) + G_{bb}(q, \omega_n) - 2\cos(q x + \omega_n \tau)G_{ab}(q, \omega_n)) \]

In general the function \( V(B) \) which appears in the variational energy is defined as:

\[ \langle C(\phi) \rangle = V(\langle \phi^2 \rangle) \]

(14)

where \( \langle . \rangle \) denotes an average over any (one-component) gaussian random variable \( \phi \). For the case of model (14) \( V(B) = -W \exp(-2B) \) and \( V(B) = -W \sum K \exp(-K^2 B/2) \) for model (4). Defining the self-energy \( G_{ab}^{-1} = c q^2 \delta_{ab} - \sigma_{ab} \), and \( G_{c}^{-1}(q) = \sum_b G_{ab}^{-1}(q) \) we obtain by minimization of the variational free energy \( F_{\text{var}} \) the saddle point equations:

\[ G_{c}^{-1} = c(q^2 + \omega_n^2) + \frac{2}{T} \int_0^\beta d\tau (1 - \cos(\omega_n \tau))[V'(B_{aa}(0, \tau)) + \sum_{b \neq a} V'(B_{ab}(0, \tau))] \]

\[ \sigma_{a \neq b} = \frac{2}{T} \int_0^\beta d\tau \cos(\omega_n \tau)V'(B_{ab}(0, \tau)) \]

(15)

Taking the limit of the number of replicas \( k \rightarrow 0 \) and considering the most general solution, i.e. a replica symmetry broken solution, we denote \( \tilde{G}(q, \omega_n) = G_{aa}(q, \omega_n) \) and parameterize \( G_{ab}(q, \omega_n) \) by \( G(q, \omega_n, u) \) where \( 0 < u < 1 \), and similarly for \( B_{ab}(x, \tau) \) and \( B(x, \tau, u) \). Equ. (15) and the algebraic rules for hierarchical matrices give:

\[ G_{c}^{-1} = c(q^2 + \omega_n^2) + \frac{2}{T} \int_0^\beta d\tau (1 - \cos(\omega_n \tau))[V'(\tilde{B}(\tau)) - \int_0^1 du V'(B(\tau, u))] \]

\[ \sigma(\omega_n, u) = \frac{2}{T} \int_0^\beta d\tau \cos(\omega_n \tau)V'(B(\tau, u)) \]

(16)

with

\[ B(\tau, u) = \frac{2T}{\beta} \sum_n \int \frac{d^d q}{(2\pi)^d} [\tilde{G}(q, \omega_n) - G(q, \omega_n, u) \cos(\omega_n \tau)] \]

\[ \tilde{B}(\tau) = \frac{2T}{\beta} \sum_n \int_0^\beta \frac{d^d q}{(2\pi)^d} \tilde{G}(q, \omega_n)(1 - \cos(\omega_n \tau)) \]

(17)

where we denote \( \tilde{B}(\tau) = \tilde{B}(x = 0, \tau) \) and \( B(\tau, u) = B(x = 0, \tau, u) \). A replica symmetric solution would correspond quantities constant in \( u \) in (16), and is analyzed in Appendix A. Note that the connected part \( G_{c}^{-1}(q, \omega) \), is renormalized by correlated disorder, while it is not the case for point disorder [2]. This is a crucial difference which leads to localization and stiffening of the tilt modulus.

**B. general features of the solution**

A general and important property of quantum problems with disorder is that off-diagonal quantities like \( B_{a \neq b}(x, \tau) \) are in fact always independent of \( \tau \). The general argument [3] is
that in each realization of the random potential $W$, the disorder does not depend on $\tau$. Therefore before averaging over disorder:

$$G_{ab,W} = \langle \phi_a(x,\tau)\phi_b(0,0) \rangle = \langle \phi_a(x,\tau) \rangle \langle \phi_b(0,0) \rangle = \langle \phi_a(x,0) \rangle \langle \phi_b(0,0) \rangle$$ (18)

It is important to note that such a property crucially depends on the assumption that the Hamiltonian is $\tau$-independent and on the fact that equilibrium has being attained. This is the case considered here. Perturbations which explicitly depend on imaginary or real time destroy this property and may lead to more complex behavior, as discussed in Section VI.

It is easy to check on the saddle point equations (16) that they indeed admit a solution such that off-diagonal quantities are $\tau$-independent. For this solution the saddle point equations simplify greatly:

$$G_c^{-1} = c(q^2 + \omega_n^2) + \frac{2}{T} \int_0^\beta d\tau (1 - \cos(\omega_n\tau))[V'(\tilde{B}(\tau)) - \int_0^1 duV'(B(u))]$$

$$\sigma(\omega_n, u) = \frac{2\beta}{T} V'(B(u)) \delta_{n,0}$$ (19)

with

$$B(u) = \frac{2T}{\beta} \sum_{n \neq 0} \int \frac{d^d q}{(2\pi)^d} G_c(q, \omega_n) + \frac{2T}{\beta} \int \frac{d^d q}{(2\pi)^d} [\tilde{G}(q, \omega_n = 0) - G(q, \omega_n = 0, u)]$$

$$\tilde{B}(\tau) = \frac{2T}{\beta} \sum_n \int_0^\beta \frac{d^d q}{(2\pi)^d} G_c(q, \omega_n)(1 - \cos(\omega_n\tau))$$ (20)

where we denote $B(\tau, u) = B(u)$. The simplifications are the following. Since $B(\tau, u)$ is independent of $\tau$ the self energy $\sigma(\omega_n, u)$ vanishes for all modes excepted $\omega_n = 0$. One also has $G(q, \omega_n, u) = 0$ for $\omega_n \neq 0$, and thus $\tilde{G}(q, \omega_n) = G_c(q, \omega_n)$ for $\omega_n \neq 0$.

An obvious property of the above saddle point is thus that replica symmetry breaking is confined to the mode $\omega_n = 0$. The equation for this mode:

$$\sigma(\omega_n, u) = \delta_{n,0} \sigma(u) = \frac{2\beta}{T} \delta_{n,0} V'(B(u))$$ (21)

is identical to the one for a model with pointlike disorder in $d$ dimensions, studied in. The only (important) difference is that the potential is effectively multiplied by $\beta$, i.e the total length along the $\tau$ direction. This is why a single mode ($\omega_n = 0$) can lead to finite contributions to correlation functions in the limit $\beta \to \infty$. In that limit the corresponding model with point disorder in $d$ dimensions can, in some sense, be considered as effectively at zero temperature $\sim T/\beta$.

Known results can thus be transposed to study the zero mode $\omega_n = 0$. For the potentials with power law correlators $V(x) = gx^{1-\gamma}/(2(1-\gamma))$ there are two generic cases: (i) long range potentials, if $\gamma(1-d/2) < 1$, for which one has full replica symmetry breaking and (ii) short range potentials, if $\gamma(1-d/2) > 1$, for which one has one step replica symmetry breaking (and a transition to a high temperature Replica Symmetric (RS) phase). In the case of the single cosine model (9) there is a one step solution for $d \leq 2$ and a full RSB solution for $d > 2$. Potentials $V(x)$ which are realistic to describe media with several scales, such as the vortex lattice (7), will include several of the above regimes.
The general way to study (21) is as follows. There is a breakpoint \( u_c \) such as \( \sigma(u) \) is constant for \( u > u_c \). For \( u > u_c \) the inversion rules applied to (21) give:

\[
B(u > u_c) = B(u_c) = \frac{2T}{\beta} \int \frac{d^d q}{(2\pi)^d} \sum_{n \neq 0} (G_c(q, \omega_n) + \frac{2T}{\beta} \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 + \Sigma_1}
\]

(22)

where by definition \( \Sigma_1 = [\sigma](u_c) \) and the function \([\sigma](u) = u\sigma(u) - \int_0^u dv \sigma(v)\). One searches for a full RSB solution by applying to (21) the inversion formula for \( 1 \) where by definition \( \Sigma \)

Differentiating (21) with respect to \( u \) for a full RSB solution by applying to (20) the inversion formula for \( 1 \) where by definition \( \Sigma \)

\[
B(u) = B(u_c) + \frac{2T}{\beta} \int_u^{u_c} dv \int \frac{d^d q}{(2\pi)^d} \frac{\sigma'(v)}{G_c(q, \omega_n) = 0 - 1 + [\sigma](v)^2}
\]

(23)

Differentiating (21) with respect to \( u \), using (23) and \( [\sigma]'(u) = u\sigma'(u) \) one can then divide by \( \sigma'(u) \neq 0 \) since full RSB is assumed. The quantities \( B(u) \), \([\sigma](u)\) and \( \sigma(u) \) are then determined by elimination from the resulting system:

\[
1 = -4V''(B(u)) \int \frac{d^d q}{(2\pi)^d} \frac{1}{(cq^2 + [\sigma](u))^2}
\]

\[
\sigma(u) = \frac{2\beta}{T} V'(B(u))
\]

(24)

and the definition of \( [\sigma](u) \). These equations, which govern the mode \( \omega_n = 0 \), are identical to the one obtained previously for pointlike disorder in \( d \) total dimension, up to a rescaling of \( u \) by \( \beta \), i.e. \( u \) is smaller by a factor of \( \beta \) in this problem. \([\sigma] \) and \( B \) do not scale but \( \sigma \) is larger by a factor \( \beta \). The full RSB solution for the power law potential is thus \([\sigma](u) = \Sigma_1(\beta u / u_c)^{2/\theta} \) with \( \theta = (2 + (d - 2)\gamma)/(1 + \gamma) \). Once the equation is solved for \( u < u_c \), provided that a full RSB solution exists, which is the case when \( \theta > 0 \), the unknown constants \( \Sigma_1 \) and \( B(u_c) \) are determined by matching or equivalently by elimination in:

\[
1 = -4V''(B(u_c)) \int \frac{d^d q}{(2\pi)^d} \frac{1}{(cq^2 + \Sigma_1)^2}
\]

\[
B(u_c) = \frac{2T}{\beta} \int \frac{d^d q}{(2\pi)^d} \sum_{n \neq 0} G_c(q, \omega_n) + \frac{2T}{\beta} \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 + \Sigma_1}
\]

(25)

For the single cosine model (3) which we focus on here, the above system (24) simplifies into:

\[
1 = \beta \sigma(u) \int \frac{d^d q}{(2\pi)^d} \frac{1}{(cq^2 + [\sigma](u))^2}
\]

(26)

and the solution of (24) is in \( d > 2 \) \([\sigma](u) = (\beta u / u_0)^{2/\theta} \) and \( u_0 = 8Tc_d e^{-d/2}/(4 - d) \). The exponent \( \theta = d - 2 \) characterize the scaling of the fluctuations of the energy of the metastable states with distance, \( \Delta F \sim x^\theta \). A pair of states characterized by \( u \) in Parisi’s hierarchy are typically \( x \) apart in space and \( \Delta F = T/u \) apart in energy. The large scale behavior is thus controlled by small \( u \).

In \( d \leq 2 \), the single cosine model does not admit a full RSB solution. Instead one finds a one step RSB solution with \( \sigma(u) = 0 \) for \( u < u_c \) and \( \sigma(u) = \sigma(u_c) \) for \( u \geq u_c \). One then has:
\[ [\sigma](u) = 0 \quad u < u_c \]  
\[ [\sigma](u) = \Sigma_1 = \beta u c \frac{2}{T} V'(B) \quad u > u_c \]  

where we have used (21) and the fact that \( \Sigma_1 = [\sigma](u_c) = u_c \sigma(u_c) \) for such a solution. One has also \( B(u) = \infty \) for \( u < u_c \) and we denote \( B(u > u_c) = B \). For the single cosine model \( \Sigma_1 = \beta u c \frac{4\pi}{\beta} e^{-2B} \). To determine \( \Sigma_1, B \) and \( u_c \) in the one step case, one can use only equations (28,23). One equation is still missing. Before we discuss the additional equation needed to determine \( u_c \), let us further simplify the saddle point equations.

Irrespective of whether the solution is one step or full RSB, one can always rewrite the variational equations as:

\[
G_c^{-1}(q, \omega_n) = c(q^2 + \omega_n^2) + \Sigma_1(1 - \delta_{n,0}) + I(\omega_n) 
\]
\[
I(\omega_n) = \frac{2}{T} \int_0^\beta d\tau (1 - \cos(\omega_n \tau))(V'(\tilde{B}(\tau)) - V'(B)) 
\]

with

\[
B = \frac{2T}{\beta} \int \frac{d^dq}{(2\pi)^d} \sum_{n \neq 0} G_c(q, \omega_n) + \frac{2T}{\beta} \int \frac{d^dq}{(2\pi)^d} \frac{1}{q^2 + \Sigma_1} 
\]
\[
\tilde{B}(\tau) = \frac{2T}{\beta} \sum_n \frac{d^dq}{(2\pi)^d} G_c(q, \omega_n)(1 - \cos(\omega_n \tau)) 
\]

where \( B(u > u_c) = B \). To obtain (23) from (16) we have splitted:

\[
\int_0^1 du V'(B(u)) = (1 - u_c)V'(B) + \int_0^{u_c} du V'(B(u)) = V'(B) - \frac{T}{2\beta} \Sigma_1 
\]

and used the equation (21) and the definition of \( [\sigma](u) \). The form (23) is convenient because when \( \beta \) becomes large, \( B(\tau) \) converges at large \( \tau \) towards \( B \) and thus \( I(\omega_n) \) in (30) goes to zero at small \( \omega_n \). One has:

\[
B - \tilde{B}(\tau) = \frac{2T}{\beta} \sum_n \int \frac{d^dq}{(2\pi)^d} \cos(\omega_n \tau) \frac{1}{cq^2 + c\omega_n^2 + \Sigma_1 + I(\omega_n)} 
\]

and thus \( \lim_{\tau \to \infty} \lim_{\beta \to \infty} \tilde{B}(\tau) = B \).

One thus sees on (23) that disorder produces a “mass” term \( \Sigma_1 \) and thus localization of the elastic manifold. The mechanism by which it is generated is subtle, however. Such term cannot exist at \( \omega_n = 0 \) as can be seen from (16). Indeed \( G_c^{-1}(q, \omega = 0) = 0 \) as it must from translational invariance after averaging over disorder. As soon as \( \omega_n = 2\pi n/\beta \neq 0 \) the first term in (29) becomes non zero equal to \( \Sigma_1 \) while the second one is a smooth function of \( \omega_n \) and therefore remains small near \( \omega = 0 \). Thus we find, and observe in a numerical solution of equations (29,31), that in the limit \( \beta \to \infty \), \( G_c^{-1}(q, \omega) \) develops a discontinuity at \( \omega = 0 \). In fact the only way to obtain a finite localization length in the limit \( \beta \to \infty \) is to break replica symmetry. The formula (31) for \( B \) shows that \( \Sigma_1 \) plays the role of an effective mass which makes \( B \) finite in \( d \leq 2 \). There is no such mechanism in the replica
symmetric solution, which cannot describe properly the localized phase and is studied in detail in Appendix A.

Remarkably, Eqs. (29-31) now forms a closed system of equations, depending only on one additional quantity \( \Sigma_1 \), and do not depend on how the replica symmetry is broken or on the detailed structure of the solution \( B(u) \) for \( u < u_c \). We will discuss further this property in Section VI.

IV. ELASTIC MANIFOLDS IN CORRELATED POTENTIALS

It is possible to solve analytically the saddle point equations (29-31) in the limit of low temperature \( T \to 0 \). Such solution, as we will show, contains the essential physics of the localized phase. It is important to retain the finite temperature only for few quantities that we will discuss explicitly.

In the limit when \( T \to 0 \), one can expand the potential in the expression (30) of \( I(\omega_n) \).

One gets a self-consistent equation for \( I(\omega_n) \):

\[
I(\omega_n) = -4V''(0) \int \frac{d^d q}{(2\pi)^d} \left( \frac{1}{cq^2 + \Sigma_1} - \frac{1}{cq^2 + \Sigma_1 + c\omega_n^2 + I(\omega_n)} \right)
\]

(34)

It is important to note that the low-T expansion of the potential can be performed safely only on the form (29) of the saddle point equations, \( \Sigma_1 \) being fixed. It would be incorrect to expand directly (19) since \( B(u) \) diverges at small \( u \) and the low-T expansion breaks down. We now study different cases.

A. Solution for the cosine model in \( d = 2 \)

In \( d = 2 \) the solution for the single cosine model (9) is one step but marginally so. The value of \( u_c \) is determined unambiguously either from the full RSB solution in the limit \( d \to 2^+ \), or by minimizing the free energy \( F_{\text{var}} \) in (12). This is done similarly to the case of point like disorder. For model (9) in the continuous limit:

\[
\beta u_c = \frac{T K_0^2}{4\pi c}
\]

(35)

and thus \( \beta u_c = T/\pi c \) for model (9). Note that there is a transition to a high temperature phase where disorder is irrelevant at \( T = T_c = \pi \beta c \). For \( \beta \) not very small this temperature however is very high and in realistic systems other mechanisms, e.g dislocation induced melting, will cause a transition out of the glass phase before this one occurs. Thus we confine our study to the low temperature glass phase \( T \ll T_c \).

One then obtains from (34)

\[
I(\omega_n) = 4\pi c \Sigma_1 \int \frac{d^d q}{(2\pi)^d} \left[ \frac{1}{cq^2 + \Sigma_1} - \frac{1}{cq^2 + \Sigma_1 + c\omega_n^2 + I(\omega_n)} \right]
\]

(36)

which leads to, in the limit where the cutoff goes to infinity.

12
\[ I(\omega_n) = \Sigma_1 \log(1 + \frac{I(\omega_n) + c\omega_n^2}{\Sigma_1}) \] (37)

The solution of (37) can be written as
\[ I(\omega_n) = \Sigma_1 f(\sqrt{c|\omega_n|}/\sqrt{\Sigma_1}) \] (38)

\( \Sigma_1 \) defines a natural length scale \( \xi = \sqrt{c/\Sigma_1} \). \( \xi \) is called the pinning length and corresponds to the scale above which the long range order is destroyed due to disorder.

For small \( \omega_n \) the solution of (37) is
\[ I(\omega_n) \sim \sqrt{2c\Sigma_1|\omega_n|} \] (39)

and for large \( \omega_n \)
\[ I(\omega_n) \sim \Sigma_1 \log(c\omega_n^2/\Sigma_1) \] (40)

This formula is valid up to the cutoff \( \omega_n \sim \Lambda \). At large \( \omega \), \( I(\omega) \to I(\infty) = \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 + \Sigma_1} \). \( \Sigma_1 \) is determined from (25, 35, 28). At low \( T \) one finds \( \Sigma_1 = \frac{4W}{\pi} \).

The general expression for the correlation functions for a one step solution is:
\[
\tilde{B}(x, \tau) = \langle [\phi(x, \tau) - \phi(0, 0)]^2 \rangle = \frac{2T}{\beta} \sum_n \int \frac{d^d q}{(2\pi)^d} \frac{1}{c\omega_n^2 + cq^2 + \Sigma_1 + I(\omega_n)}(1 - \cos(qx + \omega_n \tau))
+ \frac{2T}{\beta u_c} \int \frac{d^d q}{(2\pi)^d} \frac{\Sigma_1}{cq^2(cq^2 + \Sigma_1)}(1 - \cos(qx))
\] (41)

At \( x = 0 \), \( \tilde{B}(0, \tau) \) goes to a constant when \( \tau \to \infty \). This describes the localization of a flux line due to the correlated disorder. One defines a localization length \( l_\perp \) as:
\[ l_\perp^2 = \frac{a^2}{\pi^2} \lim_{\tau \to \infty} \langle [\phi(x, \tau) - \phi(0, 0)]^2 \rangle \] (42)

remembering that we have defined the displacements as \( u = a\phi/\pi \). One finds from (41):
\[ l_\perp^2 = \frac{a^2}{\pi^2} B = \frac{a^2}{\pi^2} \frac{2T}{\beta} \sum_n \int \frac{d^d q}{(2\pi)^d} \frac{1}{c\omega_n^2 + cq^2 + \Sigma_1 + I(\omega_n)} \] (43)

In \( d = 2 \) one finds:
\[ l_{\perp}^2 = l_T^2 - \frac{a^2}{\pi^2} B = \frac{a^2}{\pi^2} \frac{2T}{\beta} \sum_n \int \frac{d^d q}{(2\pi)^d} \frac{1}{c\omega_n^2 + cq^2 + \Sigma_1 + I(\omega_n)} \] (44)

where \( l_T^2 = 2T a^2/(c\pi^2) \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 + \omega^2} \) is the Lindemman length. This formula can be used to compute the upward shift in the melting temperature due to correlated disorder using a Lindemman criterion.

On the other hand for fixed \( \tau \), the \( \omega_n = 0 \) mode dominates at large \( x \) and \( \tilde{B}(\tau, x) \) grows as:
\[ \tilde{B}(x, \tau) \sim \frac{T}{c\beta u_c \pi} \log(x/\xi) = \log\left(\frac{x}{\xi}\right) \]  

(45)

The prefactor in (45) is universal and lead to a power law decay of the translational order parameter. The additional contribution of the \( q \sim 0 \) Fourier components of the random potential can be computed from (8) and gives a small non universal correction.

One can define the \( \omega \)-dependent (i.e. \( q_z \) dependent, in usual notations) tilt modulus \( c_{44}(\omega) \):

\[ \frac{T}{c_{44}(\omega)} = \lim_{q \to 0} \langle \partial_{\tau} \phi \partial_{\tau} \phi \rangle = \lim_{q \to 0} \omega^2 \langle \phi(q, \omega) \phi(-q, -\omega) \rangle \]  

(46)

Using equation (29) one finds:

\[ \frac{1}{c_{44}(\omega)} = \frac{\omega^2}{c\omega^2 + \Sigma_1 + I(\omega)} \]  

(47)

a formula which holds in any dimension. Thus we find that \( c_{44} = c_{44}(\omega \to 0) \) diverges in the Bose glass phase, a signature of the transverse Meissner effect.

We find from the GVM three important properties of the Bose Glass phase: the finite localization length \( l_{\perp} \), the decay of the translational order in the direction transverse to the columnar defects, and the divergence of the tilt modulus.

**B. Solution for the cosine model in \( d = 1 \)**

In \( d = 1 \) thermal fluctuations are much stronger and in the absence of disorder correlation functions grow logarithmically at large distances:

\[ \tilde{B}(x, \tau) \sim \frac{T}{2\pi e} \ln[(x^2 + \tau^2)/a^2] \]  

(48)

In the presence of disorder one expects a low temperature localized phase were disorder is relevant and a replica symmetric high temperature phase were thermal fluctuations dominate. The transition at \( T_{BG} = 6\pi c/K_0^2 \) (at infinitesimal disorder) can be seen on the RS solution (see appendix A) and is found to coincide with the results of the RG. Eq. (34) leads to:

\[ I(\omega_n) = -\frac{4V''(0)}{2\sqrt{c}} \left[ \Sigma_1^{-1/2} - \left( \omega_n^2 + \Sigma_1 + I(\omega_n) \right)^{-1/2} \right] \]  

(49)

To solve this equation one needs to know \( \Sigma_1 \), and thus determine \( u_c \). Most of the physical properties, such as the relative displacements in the transverse direction, the localization along the columns, or the vanishing of the tilt modulus are independent of the precise value of \( u_c \). However, quantities for which the asymptotic \( \tau \)-dependence is important are crucially dependent on \( u_c \).

The standard determination of \( u_c \) in the GVM in the one step case is by minimizing the free energy \( F_{\text{par}} \) in (12) with respect to \( u_c \). This can be done similarly to the case of point like disorder. The result for the cosine model in \( d < 2 \) is:
\[ \beta u_c = 4T^2 - d \frac{d}{d} j_d \Sigma_1^{(d-2)/2} \]  

(50)

where \( j_d = \int_{-\infty}^{\infty} \frac{d^d q}{(2\pi)^d} (q^2 + 1)^{-1} \). Thus \( \beta u_c = 2T \Sigma_1^{-1/2} c^{1/2} \) for \( d = 1 \) (\( j_1 = 1/2 \)). A similar result for a general short range potential can be obtained and involves \( V'(B) \). For the cosine model if one uses the value of \( u_c \) obtained from minimization one gets:

\[ I(\omega_n) = \Sigma_1^{3/2} [\Sigma_1^{-1/2} - (\omega_n^2 + \Sigma_1 + I(\omega_n))^{-1/2}] \]  

(51)

Expanding the second term in powers of \( I(\omega_n) \) one finds that, unlike what happens in \( d = 2 \), the linear order does not vanish. As a consequence the solution of (51) behaves as \( I(\omega_n) \sim |\omega_n| \) at small \( \omega_n \). This solution would lead to an exponential decay of \( B(\tau) \) with \( \tau \) towards its asymptotic value. For the classical problem this type of behavior cannot be excluded, though it would lead to a surprising difference between \( d = 2 \) and \( d = 1 \). However, for the associated quantum problem there is strong evidence that this solution gives incorrectly the physics, for example it leads to unphysical result for the conductivity as we discuss in Section V. We will thus not explore this solution further.

A look at (34) shows that the general condition for the vanishing of the linear term in the expansion in \( I(\omega_n) \), leading to \( I(\omega_n) \sim \omega_n \) at small \( \omega_n \) reads:

\[ 1 = -4V''(B) \int \frac{d^d q}{(2\pi)^d} \frac{1}{(cq^2 + \Sigma_1)^2} \]  

(52)

One recognize this condition as the condition of marginal stability, i.e the vanishing of the replicon eigenvalue. As discussed in the next section, such a criterion is true at all temperatures. This condition involves the second derivative of the potential and is thus different from the equation obtained by minimization of the free energy over \( u_c \). The marginal stability condition is known to arise naturally in the problem of the Langevin dynamics of models which have a one step RSB solution in the statics (see discussion in Section VI). This is the condition that we choose here, on physical grounds. For \( d = 2 \) this condition coincides with the one obtained by minimizing the free energy studied in section \( \nabla A \).

With this choice of \( u_c \), \( \beta u_c = T \Sigma_1^{-1/2} c^{1/2} \) (51) becomes:

\[ I(\omega_n) = 2\Sigma_1^{3/2} [\Sigma_1^{-1/2} - (\omega_n^2 + \Sigma_1 + I(\omega_n))^{-1/2}] \]  

(53)

The solution of (53) can be written as

\[ I(\omega_n) = \Sigma_1 f(\sqrt{c}|\omega_n|/\sqrt{\Sigma_1}) \]  

(54)

where

\[ f(x) \sim \frac{2}{\sqrt{3}} x - \frac{4}{9} x^2 \quad \text{when} \quad x \sim 0 \]  

(55)

\[ f(x) = 2 - \frac{1}{\sqrt{x}} \quad \text{when} \quad x \to \infty \]  

(56)

Using the above solution and formulae (41) one can compute the correlation functions. For fixed \( \tau \), the \( \omega_n = 0 \) mode dominates at large \( x \) and \( B \) grows as:
\[ \tilde{B}(\tau, x) \sim \frac{T}{\beta u_c} \frac{|x|}{c} = \frac{|x|}{\xi} \]  

(57)

where we have used the value \( \beta u_c = T/\sqrt{c\Sigma_1} \) and \( \xi \) is the pinning length.

In the limit of small disorder \( W \), and away from the transition, \( \Sigma_1 \) and thus \( \xi \) can be computed as a function of the temperature \( T \) (for \( \beta \to \infty \)). Introducing an ultraviolet cutoff \( q_{\text{max}} = \Lambda \), and rescaling the variables using (54) one gets from (31):

\[
B = 2T \int_{\Lambda/\sqrt{\Sigma_1}} dq d\omega \frac{1}{(2\pi)^2 cq^2 + c\omega^2 + 1 + f(\sqrt{c}|\omega|)} \sim \frac{T}{\pi c} \log(\sqrt{c}\Lambda/\sqrt{\Sigma_1})
\]  

(58)

where one has used (55). Inserting the value of \( B \) (58) in (28) and using \( \beta u_c = T/\sqrt{c\Sigma_1} \) one gets:

\[
c/\xi^2 = \Sigma_1 = (4W/\sqrt{c})^{2/3-2/\alpha} \Lambda^{-4T/c\Lambda} \]  

(59)

This formula for the weak disorder behavior is similar to the one that was derived for the pinning of charge density waves by Fukuyama and Lee\(^3\) (for \( T = 0 \)) and extended to take into account the quantum fluctuations (here the thermal fluctuations \( T > 0 \)) by Suzumura and Fukuyama\(^3\). The formula (58) is not identical however to the one of Suzumura and Fukuyama since it involves an integration over \( I(\omega) \) which it determined self-consistently in our theory. In Ref. 36 the corresponding \( I(\omega) \) is zero, which if taken seriously would lead to unphysical results for quantities such as \( c_{44} \) or \( \sigma(\omega) \). In addition, contrarily to their solution, it does not contain any unknown prefactor. The scaling with disorder of (59) coincides with the result from the renormalization group analysis\(^2\). The phase transition at \( T = T_{BG} \) appears on these formula. Additional calculations at finite temperature have been performed in Appendix B.

At \( x = 0 \), \( \tilde{B}(0, \tau) \) goes to a constant when \( \tau \to \infty \). This describes the localization of the lines due to the correlated disorder. One finds the expression of the localization radius:

\[
l^2_\perp = \frac{a^2 T}{\pi^2 \pi c} \ln(\xi/a)
\]  

(60)

A comparison with the result (18) shows that \( \tilde{B}(\tau) \) grows until \( \tau \sim \xi \) when it saturates at \( l^2_\perp \). Note the difference between the two lengths \( l^2_\perp \) and \( \xi \). At the transition one expects that \( \xi \) diverges as \( \xi \sim \exp(b/(T_{BG} - T)^\alpha) \) with \( \alpha = 1/2 \) from the RG\(^2\). Thus relation (60) would predict that:

\[
l_\perp \sim \frac{1}{(T_{BG} - T)^{\alpha/2}}
\]  

(61)

which could be measured in numerical simulations.

For the single cosine model in \( d = 1 \) and for \( \beta \) finite, one does not expect a real phase transition (at \( T > 0 \)) since the system is quasi-one dimensional. The GVM gives however a one step RSB solution at finite large enough \( \beta, \beta \sim T \). This solution is of the kind discussed in Appendix C: it is discontinuous and disappears when disorder is decreased below a finite threshold. It is an artifact of the GVM and should not be interpreted as a genuine phase transition in \( d = 1 \). However, in the limit \( \beta = \infty \) the GVM gives a continuous solution which we believe corresponds to a genuine phase transition.
C. solution for the general case

We now present the results for the general case of an elastic manifold in a correlated potential. Upon rescaling with β, the solution for the zero mode ω_n = 0 is identical to the problem of uncorrelated disorder in d dimension. However, the values of the breakpoint u_c, and the quantities associated to it, [σ(u_c) = Σ_1 and B(u_c) are different. Indeed Σ_1 and B(u_c) are determined by equations (25): as one sees, the difference comes from the second equation and B(u_c) is determined by fluctuations in d + 1 space rather than d. At T → 0 there is no difference and the length ξ is equal to the Larkin-Ovchinnikov pinning length R_c for the problem with point disorder in d dimension. However, Σ_1 is less renormalized by thermal fluctuations and is larger than Σ_{1_d,uncorr} at finite temperature and thus ξ < R_c at finite temperature.

Once Σ_1 is determined, the equations which determine the connected correlation function, (29,31) are closed. The general expression for the correlation function is then:

\[
\tilde{B}(x, \tau) = \langle [\phi(x, \tau) - \phi(0, 0)]^2 \rangle = \tilde{B}_{d+1}(x, \tau) + \tilde{B}_d(x)
\]

where the τ independent part \( \tilde{B}_d(x) \) has formally the same expression as for the d-dim problem with point disorder, with different values of Σ_1 and u_c. It is the sum of two pieces:

\[
\tilde{B}_d(x) = \frac{2T}{\beta u_c} \int \frac{d^d q}{(2\pi)^d} \left( \frac{\Sigma_1}{cq^2(cq^2 + \Sigma_1)} + \int_0^1 \frac{dv}{v^2} \frac{[\sigma](u_c v)}{cq^2(cq^2 + [\sigma](u_c v))(1 - \cos(qx))} \right)
\]

This has a well defined limit when \( \beta \to \infty \) since \( \beta u_c \) goes to a constant. We have dropped the term \( \frac{2T}{\beta} \int \frac{d^d q}{(2\pi)^d} \frac{1}{cq^2(1 - \cos(qx))} \) which corresponds to the contribution of the zero mode in the absence of disorder, since it vanishes as \( \beta \to \infty \). The part \( \tilde{B}_{d+1}(x, \tau) \) is truly d + 1 dimensional and contains the contribution of all the modes. It reads:

\[
\tilde{B}_{d+1}(x, \tau) = \frac{2T}{\beta} \sum_n \int \frac{d^d q}{(2\pi)^d} \frac{1}{cw_n^2 + cq^2 + \Sigma_1 + I(\omega_n)}(1 - \cos(qx + \omega_n \tau))
\]

This contribution has a different scaling than the previous one. The characteristic length scale is \( \xi = \sqrt{c/\Sigma_1} \) in both x and τ, and x ∼ τ^{1/2} (diffusion in the localization tube). \( \tilde{B}_d(x) \) has the same long distance behavior as in the d-dim problem and, if the problem has several length scales, exhibits crossover regimes as discussed in Ref. 30. The quantities I(ω), B(u_c) and Σ_1 are determined by equations (30) and (25).

Expanding (30) to lowest order in T gives:

\[
I(\omega_n) = 2\Sigma_1[1 - (1 + \frac{cw_n^2 + I(\omega_n)}{\Sigma_1})^{(d-2)/2}] \tag{65}
\]

after simplifications using the replicon condition (25) which gives Σ_1 = (-4j_dV''(0))^{2/(4-d)}. (65) is solved for any d as:

\[
I(\omega_n) = \Sigma_1 f(\sqrt{c|\omega_n|}/\sqrt{\Sigma_1}) \tag{66}
\]

where f(z) is determined by inverting the equation:
\[ z^2 = (1 - \frac{(2 - d)}{2}f(z))^{2/(d-2)} - f(z) - 1 \]  
(67)

For \( d \geq 2 \) (67) is valid for \( \omega_n \ll \Lambda \). The small \( z \) expansion of \( f(z) \) is:

\[ f(z) = \frac{2}{\sqrt{4 - d}} z + \frac{2(d - 3)}{3(4 - d)} z^2 + O(z^{3/2}) \]  
(68)

Thus the feature that \( I(\omega) \sim |\omega| \) is always verified (with the choice of the replicon condition) at \( T = 0 \). One can see easily that the above low-T solution will give the correct picture of the localized phase. This is because \( \tilde{B}(\tau) - B \) becomes small at large \( \tau \). To obtain the small \( \omega \) behavior one can expand (30):

\[ I(\omega_n) = V''(B) \frac{2}{T} \int_0^\beta \! d\tau (1 - \cos(\omega_n \tau))(\tilde{B}(\tau) - B) + A(\omega_n) \]  
(69)

where \( A(\omega_n) = \frac{2}{T} \int_0^\beta \! d\tau (1 - \cos(\omega_n \tau))(H(\tilde{B}(\tau)) - H(B)) \) and \( H(x) = V(x) - xV'(x) \). The part \( A(\omega_n) \sim Z\omega_n^2 \) with \( Z \sim T \) because the integral:

\[ Z = \frac{2}{T} \int_0^\infty \! d\tau \frac{\tau^2}{2} (H(\tilde{B}(\tau)) - H(B)) \]  
(70)

is convergent at large \( \tau \) as we check self-consistently. Thus one has:

\[ I(\omega_n) = 2\Sigma_1[1 - (1 + \frac{c\omega_n^2 + I(\omega_n)}{\Sigma_1})(d-2)/2] + A(\omega_n) \]  
(71)

where the exact replicon condition \( \Sigma_1 = (-4j_d V''(B))^{2/(4-d)} \) has been used. Expanding in powers of \( I(\omega_n) \) and using \( A(\omega_n) \sim Z\omega_n^2 \) at small \( \omega_n \), one finds:

\[ I(\omega_n) \sim \sqrt{\frac{4(c + Z)\Sigma_1}{4 - d}} |\omega_n| \]  
(72)

This behavior \( I(\omega) \sim |\omega| \) at small \( \omega \) comes from the decay \( B - \tilde{B}(\tau) \sim 1/\tau^2 \) at large \( \tau \). This property holds for any temperature inside the localized phase, as we confirmed using numerical solution of the saddle point equations.

V. QUANTUM PROBLEMS

A. models

The Hamiltonian (1) introduced in section II is also of great interest for the study of quantum problems. The dimension denoted as \( \tau \) is viewed as the time and the Hamiltonian becomes the action of a quantum problem. Correlated disorder along the “time” direction arises naturally in quantum problems since for these problems disorder is usually time-independent. The quantum statistical mechanics of bosons systems corresponds to considering \( \tau \) as the imaginary (Matsubara) time and taking periodic boundary conditions
along the time direction. The size of the system along the imaginary time direction $\tau$ is, in this representation, the inverse temperature $\beta$ of the quantum problem. The temperature $T$ of the corresponding classical problem is a measure of the quantum effects, since it corresponds to $\hbar$. One of the difficulties in treating quantum problems from the action given in section II comes from the fact that all quantities are computed in imaginary time. The physical dynamical quantities (such as the conductivity) which are of interest for the quantum problem come from retarded correlation functions, obtained from the imaginary time ones via an analytic continuation. Such a step is in general highly non trivial. Fortunately, as we show here, it can be performed directly on the variational formulas, allowing to obtain the physical quantities of interest.

The action (1) corresponds to the general quantum problem of a manifold of internal dimension $d$ ($d = 0$ corresponds to a quantum particle) in a $N$-dimensional random potential. One experimental realization is the quantum behavior of flux lines in high-Tc superconductors ($d = 1$, $N = 2$ for a single line, $d = 3$, $N = 2$ for an elastic vortex lattice), which is important in low temperature experiments. The solution given here for the problem (1) is exact in the limit $N \to \infty$.

Many quantum problems can also be described by the action (7) and (9). It coincides with the phase Hamiltonian used to describe charge density waves and Wigner crystals. The density of the system depends on a phase $\phi$ by

$$\rho(x) = \rho_0 \cos(Qx + \phi(x))$$

(73)

where $Q$ is the modulation vector of the charge density wave. The Hamiltonian describing the energy cost in variation of the phase $\phi$ is

$$\mathcal{H} = \int d^d x \left( \frac{\Pi^2}{2M} + (\nabla \phi)^2 \right)$$

(74)

where $\Pi$ is the conjugate momentum to the phase $\phi$ and $M$ is the mass of the charge density wave (usually a large number for phonon-induced charge density waves). If one introduces a random potential coupling to density and goes to the action representation one obtains exactly the action (3). The current is given by a continuity equation as $j \sim \partial_t \phi$.

In one dimension (9) also describes the problem of interacting fermions in the presence of disorder. The solution presented here describes the Anderson localization of interacting particles. Let us consider for simplicity spinless fermions in $d = 1$ interacting via the Hamiltonian

$$\mathcal{H} = -t \sum_{\langle i,j \rangle} c_i^\dagger c_j + V \sum_i n_i n_{i+1}$$

(75)

where $\langle i,j \rangle$ denotes nearest neighbors. As is well known in one dimension the fermion operators can be represented in terms of boson ones. The bosons represents charge fluctuations of the fermion system. The complete Hamiltonian becomes

$$H = \frac{1}{2\pi} \int dx \left[ (vK)(\pi\Pi)^2 + \left( \frac{v}{K} \right)(\partial_x \phi)^2 \right]$$

(76)
Π and φ are canonically conjugate variables and πΠ = ∂xθ. The constants v (velocity of excitations) and K incorporate all the effects of the interaction V. More generally, the Hamiltonian (76) describes correctly all the low energy properties of any one dimensional spinless fermion system provided one uses the correct v and K parameters. For the Hamiltonian (75) v and K can be computed from the Bethe-Ansatz solution\[44],\[45\], but we will consider them in the following as parameters. K = 1 corresponds to non-interacting fermions, whereas K < 1 (resp. K > 1) corresponds to repulsive (resp. attractive) interaction V. Using (76) one gets for the Lagrangian

\[ \mathcal{L} = \frac{1}{2\pi K} \int dx d\tau \left[ \frac{1}{v} (\partial_{\tau} \phi)^2 + v (\partial_x \phi)^2 \right] \] (77)

In the following calculations we will set v = 1 which can be done by rescaled time and space. Therefore in order to identify (77) with (9) one must have:

\[ \frac{c}{2T} = \frac{1}{2\pi K} \] (78)

The limit K → 0, which is the classical limit of the quantum problem (at zero temperature β = ∞), corresponds to the zero temperature limit T → 0 for the associated classical problem in d+1 dimension. One can now introduce a random potential

\[ \mathcal{L}_{\text{imp}} = \sum_i \int d\tau V(i, \tau) n_i(\tau) \] (79)

After replicating the fermions and averaging over the random potential one gets

\[ \mathcal{L} = -D \int d\tau \sum_{a,b} \sum_i n_i^a(\tau) n_i^b(\tau) \] (80)

where (a, b) are the replica indices. In the continuum limit the density operator will have Fourier components around q = 0 and q = 2k_F, and using the boson representation the density becomes

\[ n(x, \tau) = \frac{1}{\pi} \partial_x \phi(x, \tau) + \frac{1}{2\pi \alpha} (e^{i2k_F x + 2i\phi(x, \tau)} + \text{h.c.}) \] (81)

where \( \alpha \) is a cutoff of the order of the lattice spacing. Physically this means that to describe the low energy properties of the system one can only consider the Fourier components of the disorder around q = 0 and q = 2k_F, and consider them as independent random variables. Such an approximation will be valid provided the disorder is weak enough compared to the Fermi energy, and therefore if the fermion density is not too small (for a fixed value of the disorder). Using the boson representation the disorder term (80) becomes

\[ \mathcal{L} = -D \sum_{a \neq b} \int dx d\tau d\tau' \frac{1}{\pi^2} (\partial_x \phi^a(x, \tau))(\partial_x \phi^b(x, \tau')) + \frac{2}{(2\pi \alpha)^2} \cos(2\phi^a(x, \tau) - 2\phi^b(x, \tau')) \] (82)

As can be shown either on the fermion or the boson representation the q ∼ 0 part of the disorder can be eliminated by a simple redefinition of variables, as in (8), which does not
affect the \( q \sim 2k_F \) part. Since the current is \( j = \frac{1}{4} \partial_t \phi \) such a change has no effect on the conductivity, and its effect on other correlation functions can be simply computed. In the following we therefore keep only the \( q \sim 2k_F \) part, which can lead to the localization of the fermions. The total action we consider in the following is identical to model (9) with \( W/(2T) = 2D/(2\pi\alpha)^2\nu \).

A similar mapping can be made for repulsive bosons in one space dimension. The resulting Lagrangian is identical to (77) with the identifications \( v/(\pi K) = \rho_0/m \), and \( v^2 = 1/(\kappa\rho_0 m) \). \( \kappa \) is the compressibility, \( \rho_0 \) the average density, and \( m \) the mass of the bosons. The excited states in the absence of disorder are sound waves with phase velocity \( v \), which are the phonon modes typical of a Bose superfluid. The existence of such modes is sufficient for true superfluidity to exist.

The point \( K = 1 \) is the best known. It corresponds to non interacting fermions, or equivalently to bosons with a hard core repulsion. There, it is well known that all states are localized by disorder, and that the conductivity behaves as \( \sigma(\omega) \sim \omega^2 \log^2(\omega) \). For small disorder one can use the renormalization group (RG) and one finds a transition between a regime where the disorder is irrelevant, and a regime where the disorder renormalizes to large values (there is no perturbative fixed point accessible in the localized phase). The localized regime is found to happen for \( K < 3/2 \), i.e. \( T < T_{BG} = 3\pi c/2 \) with the identification (78). This coincides with the value predicted by the variational method (see section IV B and Appendix A). The localization length, the crossover towards the localized regime, and the high-frequency part of the conductivity can be also computed using RG. When \( K \rightarrow 0 \) the hamiltonian becomes similar to the one describing the pinning of a charge density wave, for which a self-consistent formula for the pinning length was obtained by Fukuyama and Lee. The conductivity was computed analytically for \( K = 0 \) using a transfer matrix approach and is also found to behave as \( \sigma(\omega) \sim \omega^2 \log^2(\omega) \), as for \( K = 1 \).

Most of the physical properties can be derived from the solution of the GVM in imaginary time obtained in section IV B. Contrarily to the case of the elastic manifold, the length \( \xi \), e.g given by (59) (the length at which the phase \( \phi \) disorders) is interpreted as the localization length for the interacting fermions. This is because it set the frequency scale in the conductivity as can be seen on the GVM result below and on the RG. For \( K = 1 \) it coincides with the usual definition of the localization length at the Fermi energy. At a given point in space formula (60) and (42) show that the phase fluctuations saturate to a constant at large time. Thus, in the localized phase, at each point there is a well defined average value of the phase that can be viewed as constant in time but which adjusts spatially in order to take advantage of the random potential. This agrees with the physical picture of Fukuyama and Lee.

### B. conductivity: generalities

In addition to the results which can be directly transposed from section IV, let us examine here dynamical correlation functions. The most interesting quantity is the conductivity. We will compute it using the Kubo formula (see Ref. 50 Ch. 3 for a derivation, and also Ref. 42):
\[ \sigma(q, \omega) = \frac{i}{\omega} \left( \frac{vK}{\pi} + \chi_{\text{ret}}(q, \omega) \right) \]  

(83)

Here \( \omega \) is a real frequency associated to real time \( t \). It is understood that \( \omega \) always stands for \( \omega + i\delta \), i.e contains an infinitesimal imaginary part. The first term is the diamagnetic contribution and the second term is the retarded current-current correlation function:

\[ \chi_{\text{ret}}(q, \omega) = -i \int dt e^{i\omega t} \theta(t) \langle [j^+(q, t), j(q, 0)] \rangle \]  

(84)

The conductivity is defined as \( \sigma(\omega) = \lim_{q \to 0} \sigma(q, \omega) \). For the actual calculation one must work with imaginary time \( \tau = it \), use the imaginary Matsubara frequencies \( i\omega_n \) and then perform the analytic continuation \( i\omega_n \to \omega + i\delta \). Introducing:

\[ \chi(q, i\omega_n) = -\int_0^\beta d\tau e^{i\omega_n \tau} \langle T^\tau (T^\tau \delta(q, \omega_n)) \rangle \]  

(85)

et done \( \chi_{\text{ret}}(q, \omega) = \chi(q, i\omega_n)|_{\omega_n \to \omega + i\delta} \). Using \( j = \frac{1}{\pi} \partial_t \phi \), the integration by part over time in (83) cancels the diamagnetic term and one finally finds the general formula for the conductivity:

\[ \sigma(\omega) = \frac{1}{\pi^2} \frac{i}{\omega + i\delta} \left( \omega_n^2 \langle T^\tau \phi(q = 0, \omega_n) \phi(q = 0, -\omega_n) \rangle \right)_{\omega_n \to \omega + i\delta} \]  

(86)

One can apply this formula to a system of interacting spinless fermions (or bosons) in the absence of disorder. The correlator is then given by \( \langle T^\tau \phi(q, \omega_n) \phi(-q, \omega_n) \rangle = T/((c\omega_n^2 + cq^2)) \) and thus one finds:

\[ \sigma(\omega) = \lim_{q \to 0} \frac{i}{\pi^2(\omega + i\delta)} \left( \frac{T\omega_n^2}{c\omega_n^2 + cq^2} \right)_{\omega_n \to -\omega + i\delta} = \frac{iT}{c\pi^2(\omega + i\delta)} = D \left( \delta(\omega) + \frac{i}{\pi} P\left( \frac{1}{\omega} \right) \right) \]  

(87)

where \( P \) denotes the principal part. \( D = T/(\pi c) = vK \) is the charge stiffness and the strength of the Drude peak. The charge stiffness \( D \) can be defined quite generally by \( \Re(\sigma(\omega)) = D\delta(\omega) + \sigma_{\text{reg}}(\omega) \). When \( D > 0 \) it means that the system is a perfect conductor.

To explicit the content of the flux line-quantum boson analogy it is useful to indicate the relation between the conductivity of the quantum particles and the tilt modulus of the corresponding classical system. Using (46) and (86) one has:

\[ \frac{T}{c_{44}(\omega_n)} \bigg|_{\omega_n \to -i\omega + i\delta} = -\pi^2 i\omega \sigma(\omega) \]  

(88)

This relation between \( c_{44} \), and a dynamic quantity, \( \sigma(\omega) \), involves a non trivial analytic continuation and one should be careful before drawing conclusions from one system to the other. Taking the limit \( \omega \to 0 \) in the above equation one gets:

\[ \frac{T}{c_{44}} = \lim_{\omega \to 0} \pi^2 Im(\omega \sigma(\omega)) = \pi D \]  

(89)

where the last relation follows from the Kramers-Kronig relations. Thus the tilt modulus \( c_{44} \) is related to the charge stiffness \( D \) of the quantum system. Both \( D \) and \( c_{44} \) correspond to
“dynamical” quantities, i.e the limit \( q \to 0 \) is performed before the \( \omega \to 0 \). Note that both quantities are unrenormalized by interactions for a galilean invariant system. In general, \( 1/c_{44} \sim D \) should be distinguished from “static” quantities such as the superfluid density \( \rho_s \sim \partial^2 F/\partial \Phi^2 \) which can also be obtained from the current-current correlation function, but in the opposite limit \( \omega \to 0 \) first then \( q \to 0 \) (\( F \) is the free energy and \( \Phi \) an external flux). A priori \( D \) and \( \rho_s \) are different quantities\(^5\) (for free fermions \( \rho_s = 0 \) and \( D \neq 0 \)). At \( \beta = \infty \), \( D \sim \partial^2 E_0/\partial \Phi^2 \) where \( E_0 \) is the ground state energy\(^6\). In \( d = 2 \) and \( \beta = \infty \) a relation between \( c_{44} \) and \( \rho_s \) applies\(^7\), i.e in the superfluid phase for the bosons.

In the presence of disorder, in the Bose glass phase, both \( \sigma(\omega = 0) \) and \( c_{44}^{-1} \) are expected to vanish. A point which deserves further investigation is whether the two quantities \( D \) and \( \rho_s \) may generally become proportional in the presence of disorder.

C. conductivity computed from the GVM

In order to compute the conductivity in the most general case one needs the analytic continuation of (29-31). Let us first examine the limit \( K \to 0 \), i.e \( T \to 0 \), where the continuation is straightforward. The analytical continuation to real frequency \( \omega \) can be done directly on the equation (34) and gives:

\[
I(\omega) = -4V''(0) \int \frac{d^d q}{(2\pi)^d} \left( \frac{1}{cq^2 + \Sigma_1} - \frac{1}{cq^2 + \Sigma_1 - c\omega^2 + I(\omega)} \right)
\]  
(90)

In the case of the single cosine model (i.e for interacting fermions) in \( d = 1 \) it gives:

\[
I(\omega) = 2\Sigma_1^{3/2}[\Sigma_1^{-1/2} - (\Sigma_1 - \omega^2 + I(\omega))^{-1/2}]
\]  
(91)

The solution of this equation, and of the general case of an arbitrary \( V(B) \), can be written:

\[
I(\omega) = \Sigma_1 f(\sqrt{ci\omega}/\sqrt{\Sigma_1})
\]  
(92)

where the scaling function \( f(z) \) was computed in Section IV.

The expression (86) for the conductivity can be rewritten as:

\[
\sigma(\omega) = \frac{-T\omega}{\pi^2} \frac{I''(\omega) + i(I'(\omega) - c\omega^2)}{I''(\omega)^2 + (I'(\omega) - c\omega^2)^2}
\]  
(93)

where \( I(i\omega_n) \to I(\omega + i\delta) = I'(\omega) + iI''(\omega) \). It can also be written in the scaling form:

\[
\sigma(\omega) = \frac{-iT\omega}{\pi^2\Sigma_1} s(\sqrt{ci\omega}/\sqrt{\Sigma_1})
\]  
(94)

where \( s(z) \) is an analytic scaling function of \( z \). We are primarily interested in the small frequency behavior. Writing \( f(z) = 1 + bz + O(z^2) \), with \( b = 2/\sqrt{3} \) for \( d = 1 \), one finds the asymptotic behavior at small frequency:

23
\[
Re(\sigma(\omega)) \sim \frac{bT\omega^2}{\pi^2\Sigma_1} \sqrt{\frac{c}{\Sigma_1}}
\]
\[
Im(\sigma(\omega)) \sim -\frac{T\omega}{\pi^2\Sigma_1}
\]

(95)

To compute the conductivity at positive temperature \(T > 0\) is more difficult because the analytic continuation of (29-31) becomes non trivial. The continuation of the mass term \(\Sigma_1(1 - \delta_{n,0})\) give simply \(\Sigma_1\), but the continuation of the self-consistent equation for \(I(i\omega_n)\) is more involved and is performed in appendix [D]. The result reads, for the single cosine model:

\[
P^{ret}(\omega) = \frac{4We^{-2B}}{T} \left[ \int_{0}^{\beta} d\tau \left( e^{\frac{4T}{\pi} \int_{0}^{\infty} du A(u)(e^{-u|\tau|}+2N_u \cosh(u\tau))} - 1 \right) 
+ 2 \int_{0}^{\infty} dte^{i\omega t} \text{Im} \left[ e^{\frac{4T}{\pi} \int_{0}^{\infty} du A(u)(e^{-iu|t|}+2N_u \cos(u|t|))} \right] \right] \]

(96)

where \(N_u = 1/(e^{\beta u} - 1)\) is the Bose factor and \(A(u)\) is the \(q\) integrated spectral function

\[
A(u) = \int \frac{d^4q}{(2\pi)^4} \frac{I''(u)}{(q^2 - u^2 + \Sigma_1 + I'(u))^2 + (I''(u))^2}
\]

(97)

with the property \(A(-u) = -A(u)\). One must check that indeed in the limit of \(K \rightarrow 0\) (equivalently \(T \rightarrow 0\)), when one can simply expand the exponential one recovers the above expression (90). The expansion gives:

\[
I(\omega) = 16We^{-2B} \int_{0}^{\infty} du A(u) \frac{1}{\pi} \left( \frac{2}{u + \frac{2u}{(\omega + i\delta)^2 - u^2}} \right)
\]

(98)

Using again (D5) one obtains back (90).

We do not expect that finite \(T (K > 0)\) will change qualitatively the low-frequency result apart from a change in the coefficient \(b\) when \(T\) varies, presumably an increase of \(b\) with \(T\). The high frequency behaviour however is strongly dependent on \(T\) and is computed in Appendix [D].

D. discussion

We have found a conductivity \(\sigma(\omega) \sim \omega^2\) in any dimension \(d\) using the GVM. This is a direct consequence of the \(i\omega\) term being generated in the self-energy. While this term appears naturally when there is full RSB, in \(d = 1\) for the single cosine model, it appears only as the consequence of the marginality condition in the replica language. The other choice (minimization over \(u_c\)) leads to \(I(\omega) \sim \omega^2\) and to a gap. Indeed generically, simply adding a mass term to a \(\omega^2\) term in the self-energy leads to a gap in the excitation spectrum. If one has \(\langle T_\tau \phi(q, \omega_n)\phi(-q, \omega_n) \rangle = T/(c_q^2 + c'\omega_n^2 + \Sigma_1)\), then using (86) one finds:

\[
\sigma(\omega) \sim \frac{\omega^2}{\Sigma_1 - \omega^2 - i\delta} \sim \delta(\omega - \sqrt{\Sigma_1}) + iP\left( \frac{1}{\omega - \Sigma_1} \right)
\]

(99)
Thus there is a gap and no absorption at small frequency.

Within the framework of the variational method the present result, $\sigma(\omega) \sim \omega^2$, is obtained by taking into account only small oscillations around the equilibrium position (in the following we call “phonons” such excitations, not to be confused with real phonons which are absent here). Our result is in reasonable agreement with previous results in $d = 1$, up to logarithmic corrections. Previous results were: (i) for free fermions at $K = 1$, $\sigma(\omega) \sim \omega^2$ by Mott, and $\sigma(\omega) \sim \omega^2 \log^2 \omega$ through exact calculation by Berezinskii and through a physical argument by Halperin, (ii) in the classical limit $K = 0$, $\sigma(\omega) \sim \omega^2 \log^2 \omega$ using a transfer matrix technique by Feigelman and Vinokur (FV). In Mott’s physical derivation, the conductivity is estimated from computing the energy absorbed by the system of electrons. An external driving frequency $\omega$ will cause electrons within $\omega$ of the Fermi surface to make transitions from one localized state to another. The density of the active electrons is thus $\sim \omega$ and since they absorb a photon of energy $\omega$, one finally finds a conductivity $\sigma(\omega) \sim \omega^2$. Thus in this derivation, it seems that the important excitations are, when cast back in the phase language, the soliton-like excitations of the phase, corresponding to an electron being transferred from one localized state to another. In FV derivation however, only “phonons” modes are considered.

One may thus speculate that both solitons and “phonons”, i.e small oscillations of the phase, give the same contribution $\omega^2 \log^2 (\omega)$ to the conductivity. One must note that since we have considered Gaussian disorder, corresponding to the limit of a large number of very weak pins, it is always possible to create phonon-like excitations down to $\omega = 0$. For Poissonian disorder, on the other hand, phonon-like excitations should be impossible for frequencies lower than the strength of an individual pin $V$. In that case the $d = 1$ wire is cut into segments with fixed boundary conditions, and the conductivity from “phonons” is exponentially activated. On the other hand soliton like excitations are still allowed. One can therefore expect that above $\omega = V$ both phonons and solitons will contribute to the $\omega^2 \log^2 (\omega)$ term in the conductivity, while below $\omega = V$, only the contribution of the soliton part should remain. In the near classical limit, for small $K$, solitons are energetically much less favorable than phonons and the contribution of the soliton part should therefore be small. The conductivity should therefore remain as $\omega^2 \log^2 (\omega)$ down to arbitrary low temperatures but the prefactor dropping dramatically below $\omega = V$. In the other case of large quantum fluctuations, for example free electrons, there should be practically no effect of Poissonian disorder.

VI. CONCLUSIONS

In this paper we have extended the application of the gaussian variational method (GVM) to quantum systems with disorder and, using the analogy between the two problems, to classical elastic systems with correlated disorder. By construction this method is exact in the limit of a large number of components, i.e in large embedding dimension. It allows to describe the equilibrium statistical mechanics of both problems. A glass phase with replica symmetry breaking (RSB) is found at low temperature.

Compared to the case of point disorder treated in Refs., the solution in the case of correlated disorder presents three new non-trivial features. These are not merely technical
complications but originate from the physics being very different in the present case.

First a correct treatment of the boundary conditions is crucial. The disorder being correlated along $\tau$ (imaginary time for the quantum particles) a single mode, the mode $\omega = 0$, controls most (but not all) of the physics in the glass phase. The RSB is confined to this mode and there is a non trivial mechanism to generate an effective mass term for all the other modes, leading to localization by disorder. The associated physics which naturally emerges from the GVM is that of small distortions around a classical background configuration. The method therefore confirms the physical picture developed by Fukuyama and Lee. In addition it allows for a calculation from first principles of all correlation functions. The properties of the classical configuration is fully described by the RSB and is not assumed from the start. For classical elastic manifolds we have obtained the pinning length $\xi$, the localization length $l_\perp$ and the non trivial frequency dependence of the self-energy $I(\omega)$. The GVM predicts a space vs “time” scaling $x \sim \tau^z$ with $z = 2$ within the localized phase and up to the transition. For the single cosine model, this agrees with a recent RG prediction in $d = 4 - \epsilon$ obtained directly from the RG fixed point for the mode $\omega = 0$ previously derived in Ref. [28,29]. These results could in principle be checked experimentally by measuring correlations along the columns using neutron diffraction. We have computed the tilt modulus $c_{44}$ and shown that $c_{44} = \infty$ in the Bose Glass phase, a signature of the transverse Meissner effect predicted in Ref. [4]. By including higher harmonics and realistic elastic energy the GVM can be used to obtain a more detailed description of the Bose Glass phase.

In order to apply the GVM to quantum problem and compute dynamical linear response quantities, such as the conductivity, we had to perform a non trivial analytic continuation to real time, $t = i\tau$. We showed that within the GVM such a continuation can be performed in full generality. We computed the frequency-dependent conductivity $\sigma(\omega)$ for phase Hamiltonians, relevant e.g. for Wigner crystals and Charge Density Waves. These systems are localized by the disorder and we computed the localization length $\xi$. We find that $\sigma(\omega) \sim \omega^2$ at low frequency in any dimension. We have also studied interacting fermions and bosons in $d = 1$. The method then predicts the localization-delocalization transition correctly. It yields an analytic form for $\sigma(\omega)$ which is in agreement with known high frequency results and which at low frequency also agrees, up to logarithmic factors, with the exactly solvable cases (i.e non-interacting fermions and classical limit). Such a behavior of the conductivity results from a new “friction” term $i\omega$ generated by the random potential as predicted by the GVM. Finally, a general relation between the tilt modulus of the classical problem and the conductivity of the quantum problem has been derived.

The third new feature arises when looking at the $d = 1$ quantum problem, modeled by a single cosine Hamiltonian. We found that the natural way to obtain a physically correct dynamical response (conductivity) was to use the so-called marginality condition at the replica saddle point. It is known that the marginality condition arises naturally in the solution of off-equilibrium Langevin Dynamics (LD) for spin glass models. The equilibrium quantum statistical mechanics studied here thus plays the same role with respect to the full problem of quantum dynamics as the equilibrium FDT-regime does in the off-equilibrium Langevin Dynamics (LD) of a classical system (in which the fluctuation dissipation theorem (FDT) and time translational invariance holds, and thus called the FDT-regime). This scenario agrees with the fact that the quantum equilibrium dynamical equations that we obtained
depend on the quantity $\Sigma_1$ only. This quantity is determined at the breakpoint from the RS side (short scales) independently of the details of the RSB at larger scales. This is completely analogous to the role played by the “anomaly” $\mathcal{M}$ with respect to the FDT-regime of the off-equilibrium LD. The LD for the $d$-dimensional classical problem (i.e. the $\omega_n = 0$ mode of the quantum problem) was studied in Refs. 57,58 and the expressions for the anomaly $\mathcal{M}$ and for the large time limit $\tilde{b}_1$ of the displacement in the FDT-regime were derived. One can check explicitly that in the limit $T \to 0$, $\Sigma_1$ and $B$ in the quantum problem are identical to the anomaly and $\tilde{b}_1$ from Refs. 58,57. Thus we speculate that there also exists an off-equilibrium quantum aging regime, at finite waiting time $t_w$. The regime studied here of equilibrium time-translational invariant quantum dynamics being reached only at infinite waiting time $t_w \to \infty$ and fixed $\tau = it$. This aging regime might be relevant for computing some response functions, in particular the nonlinear responses.

As we have discussed, the GVM studied here takes into account only the small gaussian deviations around a non trivial, spatially dependent, equilibrium (i.e static) configuration. It would very interesting to extend the method and take also include kinks and solitons. This would allow to compute the response to a finite transverse field (i.e study the transverse Meissner phase) as well as non linear dynamical quantum responses and variable range hopping phenomenon.

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APPENDIX A: REPLICA SYMMETRIC SOLUTION

Assuming replica symmetry in (16) one finds:

$$G_c^{-1} = c(q^2 + \omega_n^2) + \frac{2}{T} \int_0^\beta d\tau (1 - \cos(\omega_n \tau)) [V'(\tilde{B}(\tau)) - \int_0^1 du V'(B))]$$

$$\sigma(\omega_n) = \frac{2\beta}{T} \delta_{n,0} V'(B)$$

where

$$B = \frac{2T}{\beta} \sum_n \int \frac{d^dq}{(2\pi)^d} G_c(q, \omega_n)$$

$$\tilde{B}(\tau) = \frac{2T}{\beta} \sum_n \int_0^\beta \frac{d^dq}{(2\pi)^d} \tilde{G}(q, \omega_n)(1 - \cos(\omega_n \tau))$$

(A2)

Since the $\omega = 0$ mode is infrared divergent in $q$, $B = \infty$ for $d \leq 2$. The equations simplify in:

$$G_c^{-1} = c(q^2 + \omega_n^2) + \frac{2}{T} \int_0^\beta d\tau (1 - \cos(\omega_n \tau)) V'(\tilde{B}(\tau))$$

$$\tilde{B}(\tau) = \frac{2T}{\beta} \sum_n \int_0^\beta \frac{d^dq}{(2\pi)^d} G_c(q, \omega_n)(1 - \cos(\omega_n \tau))$$

(A3)
Contrarily to the RSB case no good $T = 0$ limit exists. Thus the solution cannot describe a low temperature localized phase with a finite localization length. A localized solution of (A3) can be found numerically (adding a mass term), but has no good limit $\beta \to \infty$ since the integral in the first equation produces a term proportional to $\beta$.

The RS solution describes however the high temperature phase. For power-law models such a phase exists when in the short-range or marginal case, $\gamma \geq 2/(2 - d)$. For the single cosine model it exists in $d = 1$. It also exists in $d = 2$ but since $T_c \propto \beta$ it disappears in the limit $\beta \to \infty$. Starting from the elastic theory of the flux lattice in $d = 2$ correlated disorder always lead to the Bose glass phase. Only the liquid, such as the case $d = 1$ of lines confined to a plane, may show a transition.

In the high temperature phase, the tilt modulus $c_{44}$ is renormalized, and given by (in the limit $\beta = \infty$):

$$c_{44} = c + \frac{1}{T} \int_0^\infty \frac{d\tau}{2\pi} \tau^2 V'(\tilde{B}(\tau))$$ (A4)

The divergence of the integral to first order in perturbation theory signals the transition at $T_{BG}$. For the model which describes the vortex lattice one finds $V(x) = -W \exp(-K_0^2 x/2)$, $T_{BG} = 6\pi c/K_0^2$. Within the self consistent method one has:

$$c_{44} = c + \frac{W K_0^2}{T} \int_0^\beta \frac{d\tau}{2\pi} \tau^2 \exp(-\frac{2TK_0^2}{\sqrt{4\pi c_{44}c}} \log(\tau/\Lambda))$$ (A5)

there is no divergence of $c_{44}$ at the transition, but the solution for $c_{44}$ ceases to exist below $T_{BG}$. One recovers correctly some of features of the RG approach.

**APPENDIX B: STUDY OF THE RSB SOLUTION FOR THE SINGLE COSINE MODEL IN $D = 1$ AT FINITE $T$**

We now study the equations for the single cosine model in $d = 1$ at finite $T$. We will study the limit where $\Sigma_1/\Lambda \ll 1$, i.e $\xi \gg a$. This happens in two cases: (i) at weak disorder $W \to 0$ at fixed $T$ (ii) near the transition $T \to T_{BG}$. We will only study the case $\beta = \infty$. We take $c = 1$.

The closed system of equation (for the displacements, self energy and effective mass) is:

$$\dot{B}(\tau) = T \int_{-\Omega}^{\Omega} \frac{d\omega}{2\pi} \cos(\omega\tau) \sqrt{\omega^2 + \Sigma_1 + I(\omega)}$$

$$I(\omega) = \frac{(\Sigma_1)^{3/2}}{T} \int_0^\infty d\tau (1 - \cos(\omega\tau))(e^{2B(\tau)} - 1)$$ (B1)

where $\Omega = \Lambda$ is the ultraviolet cutoff, and we have denoted by $\dot{B}(\tau) = \tilde{B}(\tau) - B$. All variables can be rescaled when $\Sigma_1 \to 0$. We thus define $\tau = (\Sigma_1)^{-1/2} u$, $\omega = (\Sigma_1)^{1/2} w$, and as before $I(\omega) = \Sigma_1 f(w)$. The cutoff becomes $w_m = (\Sigma_1)^{-1/2} \Omega$ and we are thus interested in the limit $w_m \to \infty$. Eqs. (B1) become:
\[
\hat{B}(u) = T \int_{-w_m}^{w_m} \frac{dw}{2\pi} \frac{\cos(wu)}{\sqrt{w^2 + 1 + f(w)}}
\]
\[
f(w) = \frac{1}{T} \int_0^\infty du (1 - \cos(wu))(e^{2\hat{B}(u)} - 1)
\]
\[
(\Sigma_1)^{3/2} = 4We^{-2\hat{B}(0)} \quad w_m = (\Sigma_1)^{-1/2}\Omega
\]

The first two equations have a well defined limit when \(w_m \to \infty\) as long as \(T < T_{BG} = 3\pi/2\). Thus the functions \(\hat{B}(u)\) for \(u > 0\) and \(f(w)\) for \(w < \infty\) also have a well defined limit. The approach of the BG transition from below is characterized by divergences in the integrals at small rescaled time \(u\). As we show below, when \(T < T_{BG}\) one has \(f(w) \ll w^2\) at large frequency \(w \gg 1\). Thus one has \(\hat{B}(u) \sim -(T/\pi)\ln(u/u^*)\) at small \(u\), and reporting in the second equation one gets:

\[
f(w) \sim \frac{1}{T} \int_0^\infty du \left(\frac{u^*}{u}\right)^{2K} (1 - \cos(wu))
\]

One sees that this integral converges at small \(u\) provided \(T < T_{BG} = 3\pi/2\). There is thus a well defined weak disorder continuum limit in \(d = 1\).

Analyzing further the solution one finds the following. For \(T < \pi/2\) (i.e \(K < 1/2\)) \(\lim_{w \to \infty} f(w) = f(\infty)\) is finite. For \(\pi/2 < T < 3\pi/2\) one has \(f(w) \sim w^\alpha\) with \(\alpha = \frac{2T}{\pi} - 1\) (for \(1 \ll w \ll w_m\). One sees that the point \(K = 1\) (free fermions) corresponds to \(f(w) \sim w\) both at small and large frequency. The transition at \(T = T_{BG} = 3\pi/2\) occurs when \(\alpha = 2\), i.e when the term \(I(\omega)\) becomes of the same order as \(\omega^2\) at large frequencies. Of course the small frequency behavior is always the same \(\sim |w|\) as discussed in the text.

Using the formula (93) for the conductivity and the above result \(I(\omega) \sim \omega^{2K-1}\) at large \(\omega\), one finds:

\[
\sigma(\omega) \sim \frac{1}{\omega^{4-2K}}
\]

This power-law behavior of the conductivity coincides with previously known results. The GVM therefore describes the correct physical behavior for the conductivity for all frequency scales.

**APPENDIX C: REMARKS ON THE VARIATIONAL METHOD**

For finite number of components \(N\) the Hartree or Gaussian Variational Method is only an approximation. It is important to determine whether it is a good approximation. It is in principle possible to answer in a \(1/N\) expansion, but this is a rather difficult task. The GVM seems to work well e.g for the cosine model in the case of point disorder where it can be compared with RG calculations. In \(d = 4 - \epsilon\) the values of the amplitude of logarithmic growth of displacements was found to agree within 10% in the FRG and GVM. In \(d = 2\) it predicts the transition temperature \(T_G\) exactly.

In this Appendix we illustrate some of the artifacts produced by the method. When properly understood these examples may help decide to which extent and in which cases the results of the GVM can be trusted. The following simple model is studied:
\[
H = \int d^d x \frac{c}{2} (\nabla \phi)^2 - g \cos \phi(x)
\]  
(C1)

The variational Hamiltonian is \( H = \int d^d q / (2\pi)^d (1/2) G^{-1}(q) \phi(q) \phi(-q) \) and the variational free energy \( F_{\text{var}} = -T \ln \text{Tr} \exp(-H_0/T) + \langle H - H_0 \rangle H_0 \) is found to be:

\[
\frac{F_{\text{var}}}{V} = -\frac{T}{2} \int \frac{d^d q}{(2\pi)^d} \ln(TG(q)) + \frac{Tc}{2} \int \frac{d^d q}{(2\pi)^d} q^2 G(q) - g \exp(-\frac{T}{2} \int \frac{d^d q}{(2\pi)^d} G(q))
\]  
(C2)

where \( V \) is the volume of the system. The saddle point equation thus gives \( G^{-1}(q) = cq^2 + \sigma \) with:

\[
\sigma = g \exp(-\frac{T}{2} \int \frac{d^d q}{(2\pi)^d} \frac{1}{cq^2 + \sigma})
\]  
(C3)

Let us consider \( d = 1 \), i.e. a string directed along \( z \), in a periodic potential independent of \( z \). In the limit of a large ultraviolet cutoff, \( q_{\text{max}} = \Lambda \to \infty \), one finds, up to a diverging but \( \sigma \)-independent contribution:

\[
\frac{F_{\text{var}}}{V} = \frac{T^2}{16c} (x - \tilde{g} e^{-1/x})
\]  
(C4)

where we have defined \( x = 4\sqrt{c\sigma}/T \) and \( \tilde{g} = 16cg/T^2 \). The total field fluctuation is \( \langle \phi^2 \rangle = 1/2x \). One sees that at small \( g \), \( F_{\text{var}} \) is a monotonic function with a minimum at \( x = 0 \). The first term, the entropy, dominates and the string is delocalized, \( \langle \phi^2 \rangle = \infty \), which is the correct result. However, for \( \tilde{g} > e^2/4 \) a second minimum at finite \( x \) appears (it appears at \( x = 1/2 \) for \( \tilde{g} = e^2/4 \)). There are then three roots to the saddle point equation. The absolute minimum remains in \( x = 0 \) until \( \tilde{g} \) reaches \( \tilde{g} = e \) (when \( x = 1 \)), at which the (unphysical) finite \( x \) solution becomes of lower free energy.

The fact that in \( d = 1 \) the GVM predicts a transition at \( \tilde{g} = e \) to a localized phase is an artifact. The physical reason for this result is very clear, however. The first term in the variational free energy is the entropy cost \( \sim T^2/cR^2 \) of confining the string to a transverse extension \( R \), i.e. the kinetic energy of a bound state in the quantum mechanical analogy. This is a correct estimate. The energy cost of delocalizing the string away from a minimum is, however, strongly overestimated at large \( \tilde{g} \). Indeed it is computed by assuming that the localized packet is gaussian, while in fact the packet will be strongly peaked near the minima of the potential. The true delocalization will thus proceed via kinks and be less costly in energy. At small \( \tilde{g} \) this effect is not a problem, and one expects the method to work well.

In the case \( d = 0 \) a similar artifact occurs, though in a weaker sense. The equation \( \sigma = g \exp(-T/(2\sigma)) \) has always the physically correct solution \( \sigma = 0 \). If \( g/T > e/2 \) two additional solutions appear: \( \sigma = \sigma_+ \) and \( \sigma = \sigma_- \), such that at \( g/T = e/2 \), \( \sigma_- = \sigma_+ = T/2 \). However in the GVM in \( d = 0 \) entropy always wins and the absolute minimum in the variational free energy is at \( \langle \phi^2 \rangle = \infty \), which is the correct physical result. At large enough \( g \), however, a second unphysical minimum appears, which leads to some artifacts in the presence of an extra mass term.

Our conclusion is thus that one may trust the GVM only for weak coupling or disorder. A transition at intermediate couplings, in low dimension, which cannot be tuned to zero, is
usually the sign of an artifact. We believe a similar artifact to occur in the case of \( d = 1 \) with point disorder (see Appendix C in Ref. 29 and Section III above). This artifact may be indicative of tendencies of the system, e.g. to be more glassy, but not taken as the sign of a true transition. In the disordered case, one may also be able to determine when statistical entropy and disorder energy are incorrectly evaluated. One generally expects that the main corrections should be produced by kinks or instanton configurations, while small displacements, e.g. phonons, should be described correctly.

**APPENDIX D: ANALYTIC CONTINUATION**

We illustrate the method of analytic continuation to real time on the single cosine model (9), with \( V(x) = -W \exp(-2x) \), which is relevant for interacting fermions in \( d = 1 \). The extension to arbitrary \( V(x) \) is straightforward.

We want to perform the analytical continuation of:

\[
F(i\omega_n) = \frac{4W}{2\pi} \int_0^\beta d\tau (1 - \cos(\omega_n \tau)) \left( e^{\frac{4T}{\beta}} \int_0^{\frac{2\pi}{\beta}} \sum_p \frac{\cos(\omega_p \tau)}{c(\omega_p^2 + q^2) + \Sigma + \Gamma(i\omega_p)} - 1 \right)
\]  

where \( I(i\omega_n) \) is the self energy with Matsubara (imaginary) frequencies. Physical quantities are expressed in terms of the retarded self-energy \( I(\omega) = I'(\omega) + iI''(\omega) \) obtained by the continuation \( i\omega_n \rightarrow \omega + i\delta \). The term coming from the 1 in the integral (D1) is a simple constant term, independent of frequency and is therefore unchanged under the analytic continuation. Difficulties come from the cosine term. This term is the Fourier transform in Matsubara frequencies of the imaginary time function

\[
I_{\cos}(\tau) = e^{\frac{4T}{\beta}} \int_0^{\frac{2\pi}{\beta}} \sum_p \frac{\cos(\omega_p \tau)}{c(\omega_p^2 + q^2) + \Sigma + \Gamma(i\omega_p)} - 1
\]  

since the \( \omega_n = \frac{2\pi n}{\beta} \) are Matsubara boson frequencies. In general, obtaining \( I_{ret}(t) \) from (D2), is a difficult task. Here, however, \( I \) is computed using the Gaussian variational approximation. Within this approximation \( I_{\cos} \) is itself a Green function. One can therefore use for \( I_{\cos} \) the well known relation between the real time time ordered boson Green function

\[
I_{\cos}(t) = \theta(t)I_1(t) + \theta(-t)I_2(t)
\]  

and the retarded one to get

\[
I_{ret}^\cos(t) = \theta(t)[I_1(t) - I_2(t)]
\]  

to perform the analytical continuation, it is therefore sufficient to know the real time, time ordered, function \( I_{\cos}(t) \).

In order to do so, we use a spectral representation.

\[
G_c[q, i\omega_p] = \frac{1}{c(\omega_p^2 + q^2) + \Sigma + \Gamma(i\omega_p)} = -\frac{1}{\pi} \int_{-\infty}^{\infty} du A(q, u) \frac{1}{i\omega_p - u}
\]  

where the spectral function \( A(q, u) \) is the imaginary part of the retarded function (see Ref. 70 sections 3-2 to 3-5) defined by

\[
A(q, u) = \frac{1}{\pi} \int_{-\infty}^{\infty} dt e^{i(q\cdot\vec{x} - \omega t)} G_c(\vec{x}, i\omega)
\]
\[ A(q, u) = \text{Im}(G_c[q, i\omega_p \rightarrow \omega + i\delta]) = \frac{I''(u)}{(c(q^2 - u^2) + \Sigma_1 + I'(u))^2 + (I''(u))^2} \quad (D6) \]

The above identity uses \( \delta(\omega - u) = \frac{1}{\pi} \text{Im}[1/(\omega + i\delta - u)] \). The real part of (D5) gives \( \Re(G_c[q, i\omega_p \rightarrow \omega + i\delta]) = \frac{1}{\pi} \int du A(q, u) \mathcal{P}(1/(\omega - u)) \), which is the Kramers-Kronig relation. 

The term in the exponential in (D2) then becomes

\[
\frac{-4T}{\beta \pi} \int \frac{d^d q}{(2\pi)^d} \int_{-\infty}^{\infty} du \sum_p A(q, u) \cos(\omega_p \tau) \frac{2u}{\omega_p^2 + u^2} A(q, u) \quad (D7)
\]

Using the fact that \( A(q, -u) = -A(q, u) \) for bosons systems one can rewrite (D7) as

\[
\frac{4T}{\beta \pi} \int \frac{d^d q}{(2\pi)^d} \int_0^\infty du \sum_p \cos(\omega_p \tau) \frac{2u}{\omega_p^2 + u^2} A(q, u) \quad (D8)
\]

One recognize in (D8) the Fourier transform of a simple phonon propagator which gives

\[
\frac{4T}{\pi} \int \frac{d^d q}{(2\pi)^d} \int_0^\infty du [e^{-|\tau|u} + 2N_u \cosh(u\tau)] A(q, u) \quad (D9)
\]

where \( N_u = 1/(e^{\beta u} - 1) \) is the boson occupation factor. The time continuation to real time \( \tau = it \) is now easily performed\(^5\) and gives

\[
\frac{4T}{\pi} \int \frac{d^d q}{(2\pi)^d} \int_0^\infty du [e^{-|t|u} + 2N_u \cos(ut)] A(q, u) \quad (D10)
\]

Using (D3) and (D4) one finally obtains

\[
I^{\text{ret}}(\omega) = 4We^{-2B\beta T} \left[ \int_0^\beta d\tau \left( e^{4T \int_0^\infty du A(u)(e^{-u|\tau|} + 2N_u \cosh(u\tau))} - 1 \right) + 2 \int_0^\infty dt e^{i\omega t} \text{Im} \left[ e^{4T \int_0^\infty du A(u)(e^{-iu|t|} + 2N_u \cos(ut))} \right] \right] \quad (D11)
\]

where \( A(u) = \int \frac{d^d q}{(2\pi)^d} A(q, u) \) is the \( q \) integrated spectral function.
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