Differential Equations in Special Kähler Geometry

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Abstract

The structure of differential equations as they appear in special Kähler geometry of $N = 2$ supergravity and $(2, 2)$ vacua of the heterotic string is summarized. Their use for computing couplings in the low energy effective Lagrangians of string compactifications is outlined.

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1. Introduction

In order to test string theory as a theory of unifying all known interactions one needs to extract its low-energy limit and compare it with the Standard Model. A first step in this program is the derivation of a low energy effective Lagrangian which only depends on the massless string modes. The process of “integrating out” the heavy string modes depends on the string vacuum chosen and thus leads to a different effective Lagrangian for each vacuum. Phenomenological prejudice focuses our attention on vacua displaying \( N = 1 \) supersymmetry in four space–time dimensions. This subclass of string vacua can be characterized by an \( N = 2, c = 9 \) worldsheet super conformal field theory (SCFT) in the left–moving sector \([1]\). Furthermore, the couplings of the corresponding low energy effective Lagrangian are directly related to correlation functions in the SCFT. Unfortunately, for most string vacua we are currently not able to calculate the relevant correlation functions. However, recently for a particular family of string vacua (a compactification on a specific Calabi–Yau threefold), some of the couplings in the low energy effective Lagrangian have been computed exactly (at the string tree-level, but to all orders in the \( \sigma \)–model coupling) by using techniques of algebraic geometry and without ever relying on the underlying SCFT \([2]\). It was shown that the couplings could be obtained from the solution of a certain 4th-order linear holomorphic differential equation. Subsequently, it was realized that this differential equation is a particular case of the so-called “Picard–Fuchs equations” obeyed by the periods of the holomorphic three-form \( \Omega \) that exists on any Calabi–Yau threefold \([3, 4]\).\(*\) A further step in uncovering the general structure behind the differential equation was undertaken in refs. \([3, 4]\). It was shown that the Picard–Fuchs equations for a Calabi–Yau threefold are just another way of expressing a geometrical structure called “special Kähler geometry” \([8, 13]\).

Special Kähler geometry first arose in the study of coupling vector multiplets to \( N = 2 \) supergravity in four dimensions \([8]\). The manifold spanned by the scalars of the vector multiplets turned out to be a Kähler manifold with an additional constraint dictated by \( N = 2 \) supergravity.\(†\) Consequently, the same structure also appears in

\(*\) Picard–Fuchs equations can be derived for general “Calabi–Yau” \( d \)–folds \([4, 3]\), but we consider only \( d = 3 \) in the following.

\(†\) A coordinate–free characterization of special geometry was given in \([14]\) in the context of \( N = 2 \) supergravity and in \([13, 15]\) for a Calabi-Yau moduli space.
appropriate compactifications of type II string theories \cite{16,19}. However, it can also arise in $N = 1$ vacua of the heterotic string if they display an additional right-moving $N = 2$ worldsheet supersymmetry (so-called \((2,2)\) vacua) \cite{16,18,11}. In this case the couplings of the $N = 1$ effective Lagrangian obey the constraints of special geometry; the Kähler potential $K$ (which encodes the kinetic terms of the scalar fields) is related to the (holomorphic) Yukawa couplings $W_{\alpha\beta\gamma}$. Both quantities can be expressed in terms of so-called holomorphic prepotentials $X^A, F_A$. Therefore, in \((2,2)\) vacua it is sufficient to compute the prepotentials in order to determine the couplings of the effective Lagrangian and the differential equation of ref. \cite{2} is precisely an equation which determines $X^A$ and $F_A$.

The content of ref. \cite{2} is reviewed by P. Candelas in these proceedings. In this talk we discuss the Picard–Fuchs equations from the point of view of special Kähler geometry following refs. \cite{6,7}. This analysis clarifies the exact relation of the Picard–Fuchs equations to the couplings of an effective Lagrangian for \((2,2)\) heterotic vacua. The organisation is as follows. In section 2, we briefly recall the basic definitions and properties of special geometry as they arise in $N = 2$ supergravity and \((2,2)\) vacua of the heterotic string. In section 3 we give an entirely equivalent formulation of this geometrical structure which features a purely holomorphic differential identity. The rest of the talk then evolves around this holomorphic identity. In order to make contact with ref. \cite{2} the special case of a one-dimensional Kähler manifold is treated in section 4. Here, we also observe further properties of the holomorphic identity derived in section 3. The moduli space of Calabi–Yau threefolds is a subclass of special Kähler manifolds \cite{10,13,15} and so we briefly relate the discussion of section 3 to this subclass in section 5. The holomorphic differential identity of section 3 corresponds to the Picard–Fuchs equations obeyed by the periods on the Calabi–Yau manifold. The important point is that the coefficient functions of the differential identity can be computed from the defining polynomial of the Calabi–Yau manifold or equivalently from the Landau-Ginzburg superpotential which characterizes the string vacuum. Once the coefficients are known, the identity turns into a non-trivial linear differential equation whose solutions are the prepotentials $X^A$ and $F_A$ which also determine $W_{\alpha\beta\gamma}$ and $K$. This indicates that it not always necessary to calculate the correlation functions of the SCFT. Instead, the low energy couplings are already encoded in the Landau-Ginzburg superpotential. In section 6 we outline how the coefficients of the differential identity can be computed from the Landau-Ginzburg superpotential.
Special geometry also made its appearance in the context of topological conformal field theories (TCFT) \cite{17,18} and differential equations similar to the Picard–Fuchs equations govern the space of topological deformations. In fact, these observations were one motivation for the investigation performed in refs. \cite{3,4}. In section 7 we conclude by relating special geometry as it arises in TCFT to the analysis of the previous sections.

This talk is based on refs. \cite{6} and \cite{7} where one also finds a lot of the technical details omitted here.

2. Special Kähler Geometry

Let us first briefly summarize the definition and some of the basic properties of special Kähler geometry \cite{8,14}.

The metric of an $n$–dimensional Kähler manifold $\mathcal{M}$ is given by

$$g_{\alpha \overline{\beta}}(z, \overline{z}) = \partial_{\alpha} \partial_{\overline{\beta}} K(z, \overline{z}) , \quad \alpha = 1, \ldots, n ,$$

where the Kähler potential $K(z, \overline{z})$ is a real function of the complex coordinates $z^\alpha$ and $\overline{z}^{\overline{\alpha}}$.

Special Kähler geometry is defined by an additional constraint on the Kähler potential which reads

$$K(z, \overline{z}) = - \ln i \left( X^A(z) F_A(\overline{z}) - \overline{X}^A(\overline{z}) F_A(z) \right) , \quad A = 0, \ldots, n ,$$

where $X^A(z)$ is holomorphic and $F(X^A)$ is a holomorphic functional homogeneous of degree 2 in $X^A$:

$$\partial_{\overline{\alpha}} X^A = 0 , \quad 2F = X^A F_A(X) , \quad F_A \equiv \frac{\partial F}{\partial X^A} .$$

Let us note that $K$ is expressed in terms of the purely holomorphic objects $X^A(z)$ and $F(X^A)$ and their complex conjugates. The metric $g_{\alpha \overline{\beta}}$ as defined in eq. (2.1) is invariant under the Kähler transformations $K(z, \overline{z}) \rightarrow K(z, \overline{z}) + f(z) + \overline{f}(\overline{z})$. For the Kähler potential (2.2) this translates into the transformation properties

$$X^A \rightarrow X^A e^{-f} , \quad F_A \rightarrow F_A e^{-f} .$$

$$-3-$$
It also proves useful to introduce the $(2n + 2)$ dimensional row vector\footnote{We take the expression $(X^A, F_A)$ always as an abbreviation for $(X^0, X^\alpha, F_\alpha, -F_0)$.}

\[
V = (X^A, F_A) \equiv (X^0, X^\alpha, F_\alpha, -F_0) .
\]  \hspace{1cm} (2.5)

In terms of $V$ we find

\[
K = -\ln \left( V(-iQ)V^\dagger \right) ,
\]  \hspace{1cm} (2.6)

where $Q$ is a symplectic metric which satisfies $Q^2 = -1, Q = -Q^T$ and reads

\[
Q = \begin{pmatrix}
  1 & -1 & 0 & \cdots & 0 \\
  1 & 1 & -1 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  1 & 1 & 1 & \cdots & -1 \\
 -1 & 1 & 1 & \cdots & 1 \\
\end{pmatrix} .
\]  \hspace{1cm} (2.7)

However, $V$ is not uniquely defined. Under a global $Sp(2n + 2)$ rotation one finds

\[
(\tilde{X}^A, \tilde{F}_A(\tilde{X}^A)) = (X^A, F_A(X^A)) \cdot M , \quad M \in Sp(2n + 2) ,
\]  \hspace{1cm} (2.8)

where $\tilde{F}_A = (\partial \tilde{F}/\partial \tilde{X}^A)$ and $\tilde{F}$ is again a homogeneous function of $\tilde{X}^A$ of degree 2. From eq. (2.6) we see that $K$ is manifestly invariant under such reparametrizations.

As a consequence of eqs. (2.2) and (2.3) $V$ satisfies the following set of covariant identities

\[
D_\alpha V = U_\alpha , \\
D_\alpha U_\beta = -iC_{\alpha\beta\gamma}g^\gamma_\beta U_\gamma , \\
D_\alpha U_\beta = g^{\alpha\beta}V , \\
D_\alpha V = 0 .
\]  \hspace{1cm} (2.9)

The first equation is the definition of $U_\alpha$ and the Kähler covariant derivative $D_\alpha$ has been defined as follows

\[
D_\alpha V = (\partial_\alpha + \partial_\alpha K)V , \quad D_\alpha V = \partial_\alpha V , \\
D_\alpha U_\beta = (\partial_\alpha + \partial_\alpha K)U_\beta - \Gamma^\gamma_{\alpha\beta}U_\gamma , \quad D_\alpha U_\beta = \partial_\alpha U_\beta .
\]  \hspace{1cm} (2.10)

Here $\Gamma^\gamma_{\alpha\beta} = g^{\alpha\delta} \partial_\alpha g_{\delta\beta}$ denotes the usual Christoffel connection of the Kähler manifold and $\partial_\alpha K, \partial_\alpha K$ act as connections for Kähler transformations (2.4). ($\partial_\alpha K$ is
an Abelian connection of a holomorphic line bundle \( L \) whose first Chern class is the Kähler class.) Finally, we abbreviate

\[
C_{\alpha\beta\gamma} = e^K W_{\alpha\beta\gamma}, \quad W_{\alpha\beta\gamma} = \partial_{\alpha} X^A \partial_{\beta} X^B \partial_{\gamma} X^C F_{ABC},
\]

(2.11)

where \( W_{\alpha\beta\gamma} \) is holomorphic: \( \partial_{\overline{\alpha}} W_{\alpha\beta\gamma} \). As for eq. (2.9) one derives a set of equations including the anti-holomorphic derivative \( D_{\overline{\alpha}} \).

As an integrability condition of the second equation in (2.9) one finds

\[
R_{\overline{\alpha}\beta\delta}^{\gamma} = g_{\overline{\alpha}\beta} \delta_{\delta}^{\gamma} + g_{\overline{\alpha}\delta} \delta_{\beta}^{\gamma} - C_{\beta\delta\mu} g^{\mu\nu} C_{\nu \overline{\alpha}} g^{|\overline{\gamma}}\gamma,
\]

(2.12)

where \( R_{\overline{\alpha}\beta\delta}^{\gamma} \) (\( = \partial_{\overline{\alpha}} \Gamma_{\beta\delta}^{\gamma} \)) denotes the Riemann tensor of the Christoffel connection. The Bianchi identities then imply

\[
\overline{D}_{\overline{\epsilon}} C_{\alpha \beta \gamma} = 0, \quad D_{\epsilon} C_{\alpha \beta \gamma} - D_{\alpha} C_{\epsilon \beta \gamma} = 0.
\]

(2.13)

In (2,2) vacua of the heterotic string eqs. (2.12) and (2.13) were shown to arise from Ward identities of the right-moving \( N = 2 \) world-sheet supersymmetry \([11]\). In such theories the scalar fields \( z^\alpha \) correspond to the so-called moduli fields of the string spectrum and \( C_{\alpha \beta \gamma} \) are the (moduli dependent) Yukawa couplings of the \( 27, (\overline{27}) \) matter multiplets.

3. Holomorphic Differential Identities

So far we recapitulated the basic definitions of special geometry. In this section we show that there is another way of expressing the same constraints. Starting from eqs. (2.9) we will see that it is possible to derive an equivalent but completely holomorphic set of identities which manifestly display the fact that special geometry is determined entirely in terms of the holomorphic sections \( X^A, F_A \). It is these holomorphic identities which can be used to compute the couplings of the effective Lagrangian from the Landau-Ginzburg superpotential. Furthermore, they allow us to make contact with the Picard–Fuchs equations of the Calabi–Yau manifold and the corresponding equations of TCFT.
Let us first observe that eq. (2.9) can be written very compactly as a \((2n + 2) \times (2n + 2)\) matrix equation
\[(\mathbb{1} \partial_\alpha - A_\alpha) U = 0 , \quad (3.1)\]
where \(U = (V, U_\alpha, \overline{U_\alpha^n}, \overline{V})^T\) and
\[
A_\alpha = \begin{pmatrix}
-\partial_\alpha K & \delta^\beta_\alpha & 0 & 0 \\
0 & -\delta^\beta_\gamma \partial_\alpha K + \Gamma_\gamma^\beta & -i C_{\alpha \beta \gamma} g^{\gamma \overline{\gamma}} & 0 \\
0 & 0 & 0 & g_{\alpha \overline{\beta}} \\
0 & 0 & 0 & 0
\end{pmatrix}.
\quad (3.2)
\]
From eq. (2.9) one also infers that in addition \(U\) satisfies
\[(\mathbb{1} \partial_\alpha - A_\overline{\alpha}) U = 0 , \quad (3.3)\]
where
\[
A_\overline{\alpha} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
g_{\alpha \overline{\beta}} & 0 & 0 & 0 \\
i C_{\alpha \overline{\beta} \gamma} g^{\gamma \overline{\gamma}} & 0 & -\delta^\beta_\gamma \partial_{\overline{\alpha}} K + \Gamma_\gamma^\beta & 0 \\
0 & 0 & 0 & -\partial_{\overline{\alpha}} K
\end{pmatrix}.
\quad (3.4)
\]
Strominger observed that as a consequence of (2.12) and (2.11) the connections \(A_\alpha\) and \(A_\overline{\alpha}\) have vanishing curvature \([13]\): \(F_{\alpha \beta} = F_{\overline{\alpha} \overline{\beta}} = 0, \quad F_{\alpha \beta} = \partial_{[\alpha} A_{\beta]} - [A_{\alpha}, A_\beta].\) \quad (3.5)
This zero curvature condition is yet another characterization of special Kähler geometry.

From the definition of the covariant derivatives in eq. (2.10) we see that the set of identities (2.9) (or equivalently (3.1)) is covariant with respect to Kähler and coordinate transformations. However, eqs. (3.1) and (3.3) are also covariant under the more general gauge transformations
\[
U' = S^{-1} \cdot U, \quad A'_\alpha = S^{-1} A_\alpha S - S^{-1} \partial_\alpha S, \quad A'_\overline{\alpha} = S^{-1} A_\overline{\alpha} S - S^{-1} \partial_{\overline{\alpha}} S. \quad (3.6)
\]
The important point is that via a non-holomorphic transformation of the form
\[
S = \begin{pmatrix}
*_{1 \times 1} & 0 & 0 & 0 \\
0 & *_{n \times n} & 0 & 0 \\
0 & 0 & *_{n \times n} & 0 \\
0 & 0 & 0 & *_{1 \times 1}
\end{pmatrix} \in B,
\quad (3.7)
\]
(B denotes the Borel subgroup of $SL(2n + 2, \mathbb{C})$) one can gauge away $A_{\alpha\tau}$ completely and simultaneously make $A_{\alpha}$ purely holomorphic:

$$A'_{\alpha} = 0, \quad \partial_\alpha A_{\alpha} = 0, \quad V = S U, \quad \partial_\alpha V = 0, \quad (3.8)$$

(where we have denoted the holomorphic quantities by $A_{\alpha}$ and $V$ respectively). In this holomorphic gauge eqs. (3.1), (3.3) are replaced by

$$(\mathbb{I} \partial_\alpha - A_{\alpha}) V = 0, \quad (3.9)$$

which is an entirely equivalent characterization of special geometry. It is this holomorphic form which will be the focus of interest in the rest of this talk. Here we note that the condition (3.8) does not completely fix the gauge freedom (3.6). Eq. (3.9) still displays a residual gauge symmetry of purely holomorphic $S$-transformations.

Let us assemble a few more properties of $A_{\alpha}$. It is not an arbitrary $(2n+2) \times (2n+2)$ matrix but can be brought to the form (by using holomorphic $S$-transformations)

$$A_{\alpha} = \Gamma_{\alpha} + \mathcal{C}_{\alpha}, \quad (3.10)$$

where

$$\Gamma_{\alpha} = \begin{pmatrix} -\partial_\alpha \hat{K} & 0 & 0 & 0 \\ 0 & (\hat{\Gamma}_\alpha - \partial_\alpha \hat{K} \mathbb{I})_\beta^\gamma & 0 & 0 \\ 0 & 0 & (\partial_\alpha \hat{K} \mathbb{I} - \hat{\Gamma}_\alpha)_\beta^\gamma & 0 \\ 0 & 0 & 0 & \partial_\alpha \hat{K} \end{pmatrix}, \quad (3.11)$$

and

$$\mathcal{C}_{\alpha} = \begin{pmatrix} 0 & \delta_\alpha^\gamma & 0 & 0 \\ 0 & 0 & (W_\alpha)_\gamma^\beta & 0 \\ 0 & 0 & 0 & \delta_\alpha^\beta \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (3.12)$$

In order to define the “hatted” connection in eq. (3.11) let us first introduce

$$t^a(z) = \frac{X^a(z)}{X^0(z)}, \quad a = 1, \ldots, n. \quad (3.13)$$

In terms of $t^a$ and $X^0$ one finds that the Kähler connection $\partial_\alpha K_\alpha$, as well as the Christoffel connection $\Gamma_\alpha^\gamma_\beta$, split into a holomorphic piece ($\hat{K}_\alpha(z)$ and $\hat{\Gamma}_\alpha^\gamma_\beta(z)$) which
still transforms as a connection, and a non-holomorphic piece with tensorial transfor-
mation properties:
\[
\partial_\alpha K(z, \bar{z}) = \hat{K}_\alpha(z) + K_\alpha(z, \bar{z}), \\
\Gamma^\gamma_{\alpha\beta}(z, \bar{z}) = \hat{\Gamma}^\gamma_{\alpha\beta}(z) + T^\gamma_{\alpha\beta}(z, \bar{z}).
\]
where
\[
\hat{K}_\alpha(z) = -\partial_\alpha \ln X_0(z), \\
\hat{\Gamma}^\gamma_{\alpha\beta}(z) = (\partial_\beta e^a_\alpha)e^{-1\gamma}_a, \\
e^a_\alpha(z) = \partial_\alpha t^a(z).
\]
(3.14)

(The expressions for \( K_\alpha \) and \( T^\gamma_{\alpha\beta} \) can be found in ref. \[6\], they are not essential in the following.) The important point is that the holomorphic objects \( \hat{K}_\alpha \) and \( \hat{\Gamma}^\alpha_{\beta\gamma} \) transform as connections under Kähler and holomorphic reparametrizations respectively; moreover \( T^\gamma_{\beta\gamma} \) is a tensor under holomorphic diffeomorphisms and \( K_\alpha \) is Kähler invariant. As a consequence one can define holomorphic covariant derivatives in analogy with eq. (2.10) where all connections are replaced by their hatted analogue. From eq. (3.11) we see that these holomorphic covariant derivatives exactly appear in (3.9).

It is easy to check that \( \hat{\Gamma}^\gamma_{\beta\gamma} \) is also flat:
\[
\hat{R}^\gamma_{\delta\alpha\beta} \equiv \partial_\delta \hat{\Gamma}^\gamma_{\alpha\beta} - \partial_\alpha \hat{\Gamma}^\gamma_{\delta\beta} + \hat{\Gamma}^\mu_{\alpha\beta} \hat{\Gamma}_{\mu\delta} - \hat{\Gamma}^\mu_{\delta\beta} \hat{\Gamma}_{\mu\alpha} = 0.
\]
(3.16)

The flat coordinates are the so-called “special coordinates” \( t^a = z^\alpha \). In these coordinates we find from (3.15)
\[
e^a_\alpha = \delta^a_\alpha, \quad \hat{\Gamma}^\delta_{\alpha\beta} = 0.
\]
(3.17)

The Kähler gauge choice \( X^0 = 1 \) implies \( \hat{K}_\alpha = 0 \). From eq. (3.11) we learn that in these coordinates also \( \Gamma_\alpha = 0 \) holds whereas eq. (2.11) shows
\[
W_{\alpha\beta\gamma} = \partial_\alpha \partial_\beta \partial_\gamma F.
\]
(3.18)

Let us recapitulate. The Christoffel connection of an \( n \)-dimensional special Kähler manifold splits into a flat holomorphic piece transforming as a connection and a non-
holomorphic term with tensorial transformation properties. In addition, one can de-
fine another holomorphic connection \( A_\alpha \) which acts instead on a \((2n+2)\)-dimensional symplectic bundle. This connection is also flat and the special coordinates of special geometry are the flat coordinates for both connections. It is important to distinguish clearly between these two flat holomorphic connections.
To close this section let us display a few more properties of $A_\alpha$ and eq. (3.9). Firstly, one easily verifies that $A_\alpha$ is valued in $sp(2n+2)$: $QA = (QA)^T$, where $Q$ is the symplectic metric given in (2.7). This is directly related to the symplectic action (2.8) on $V$ or similarly to the fact that $V$ does not contain $(2n+2)$ independent functions. Instead, $X^A$ and $F_A$ are related by eq. (2.3). Furthermore, from its definition (3.12) we learn that $\mathcal{C}_\alpha$ satisfies

$$\mathcal{C}_\alpha \mathcal{C}_\beta \mathcal{C}_\gamma \mathcal{C}_\delta = 0, \quad [\mathcal{C}_\alpha, \mathcal{C}_\beta] = 0, \quad \partial [\alpha, \mathcal{C}_\beta] - A_{[\alpha, \mathcal{C}_\beta]} = 0. \quad (3.19)$$

Thus, $\mathcal{C}_\alpha$ generates an Abelian, $n$-dimensional subalgebra of $sp(2n+2)$ that is nilpotent of order three. $\mathcal{F}_\alpha$ as defined in eq. (3.10) also has zero curvature.

Finally, let us observe that eq. (3.9) can be turned into a set of coupled partial differential equations for $V$. Using eq. (3.10)–(3.12) one finds

$$\hat{D}_\alpha \hat{D}_\beta (W^{-1}) \tilde{\gamma}_{\rho\sigma} \hat{D}_\gamma \hat{D}_\sigma V = 0, \quad (3.20)$$

where $\tilde{\gamma}$ is not summed over. We see that eq. (3.20) is holomorphic and covariant under Kähler and coordinate transformations. It is this ‘solved’ version of (3.9) which in one dimension turns into the 4th-order linear differential equation of ref. [2]. Let us turn to this special case in the following section.

4. Ordinary differential equations and $W$–generators on a one-dimensional special Kähler manifold

In this section we briefly discuss the case of special geometry in one complex dimension. The reason is that the specific example discussed in ref. [4] corresponds to a one-dimensional moduli space of a Calabi–Yau–threefold and so we will be able to make contact with this example rather easily. Also, in one complex dimension some further properties of eq. (3.9) will appear.

In one dimension eq. (3.20) reads

$$\hat{D}_\alpha \hat{D}_\beta W^{-1} \hat{D}_\gamma \hat{D}_\sigma V = 0, \quad (4.1)$$

$$\quad = 9 =$$
where $W$ is the one-dimensional Yukawa coupling. One finds that the coefficients $a_n$ are not arbitrary but related in the following way

$$a_3 = 2\partial a_4, \quad a_4 = W^{-1}, \quad a_1 = \partial a_2 - \frac{1}{2} \partial^2 a_3, \quad (4.2)$$

whereas the coefficients $a_2$ and $a_0$ are complicated functions of $W$ and the connections. (Note that in special coordinates eq. (4.1) becomes very simple and reads $\partial^2 W^{-1} \partial^2 V = 0$).

The coefficients $a_n$ have to obey well-defined transformation laws in order to render eq. (4.1) covariant under coordinate changes ($z \to \tilde{z}(z)$, $\partial \to \xi^{-1} \partial$, $\xi \equiv \partial \tilde{z}/\partial z$) as well as Kähler transformations. It proves convenient to rewrite (4.1) slightly. First, one can scale out $a_4$, and furthermore drop the coefficient proportional to $a_3$ by means of the redefinition $V \to V e^{-1/4 \int a_3(u) du}$. This puts the differential equation into the form

$$(\partial^4 + c_2 \partial^2 + c_1 \partial + c_0) V = 0, \quad (4.3)$$

where the new coefficients $c_n$ are combinations of the $a_n$ and their derivatives. In this basis $V$ transforms as a $-3/2$ differential, but the transformation properties of the $c_n$ are not very illuminating. However, one can find combinations of the $c_n$’s and their derivatives which transform like tensors:

$$w_2 = c_2,$$
$$w_3 = c_1 - c_2',$$
$$w_4 = c_0 - \frac{1}{2} c_1' + \frac{1}{5} c_2'' - \frac{9}{100} c_2^2. \quad (4.4)$$

A straightforward computation shows

$$\tilde{w}_2 = \xi^{-2}[w_2 - 5\{\tilde{z}; z\}],$$
$$\tilde{w}_3 = \xi^{-3} w_3,$$
$$\tilde{w}_4 = \xi^{-4} w_4, \quad (4.5)$$

where $\{\tilde{z}; z\} = (\partial^2 \xi - \frac{3}{2} (\partial \xi)^2)$ is the Schwarzian derivative. ($w_2, w_3, w_4$ form a classical $W_4$-algebra [19], but this will play no role in the following.) However, eq. (4.1) is not the most general 4th-order linear differential equation. Its coefficients satisfy (4.2) and as a consequence one finds $w_3 = 0$ or equivalently

$$[\partial^4 + w_2 \partial^2 + w_2' \partial + \frac{3}{10} w_2'' + \frac{9}{100} w_2^2 + w_4] V = 0. \quad (4.6)$$
Thus, all special geometries in one dimension lead to a 4th-order linear differential equation that is characterized by \( w_3 = 0 \). This is due to the fact that the solution vector \( V \) does not consist of four completely independent elements, but rather is restricted by eqs. (2.5) and (2.3).

From eq. (4.5) we learn that there is always a coordinate system in which \( w_2 = 0 \) holds. On the other hand, \( w_3 \) and \( w_4 \) do characterize a 4th-order differential equation in any coordinate frame. Thus one can discuss the properties of (4.6) when in addition \( w_4 = 0 \) holds. One finds that this corresponds to

\[
F = \frac{1}{6} \left( \frac{X^1(z)}{X^0(z)} \right)^3 + c_{AB} X^A X^B, \quad \hat{D} W = 0, \quad (4.7)
\]

where \( c_{AB} \) are arbitrary constants. For \( c_{AB} = 0 \), (4.7) is the \( F \)-function corresponding to the homogeneous space \( SU(1,1)/U(1) \) (which satisfies the stronger constraint \( D W = 0 \)) [8]. Thus, for covariantly constant Yukawa couplings the differential equation is essentially reduced to the differential equation of a torus. This is similar to the situation for the \( K_3 \) surface where the only non–trivial \( W \)-generator is \( w_2 \) [4]. The possibility of having non–trivial Yukawa couplings, or \( w_4 \neq 0 \), is the new ingredient in special geometry. It reflects the possibility of having instanton corrections to \( W \) or in other words \( w_4 \) measures the deviation from a constant \( W \), which is the large–radius limit of the Calabi–Yau moduli space.

The significance of the \( w \)-generators can also be understood in terms of the first order equation (3.9). Any linear 4th-order differential equation can be cast into the form (3.9) with

\[
A = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-\frac{3}{10} w_2 & 0 & 1 & 0 \\
-\frac{1}{2} w_3 & -\frac{4}{10} w_2 & 0 & 1 \\
-w_4 & -\frac{1}{2} w_3 & -\frac{3}{10} w_2 & 0
\end{pmatrix} \in \text{sl}(4, \mathbb{R}). \quad (4.8)
\]

To understand this form, recall the well-known relationship* between \( W \)-algebras and a special, “principally embedded” \( SL(2) \) subgroup \( \mathcal{K} \) [20] of \( G = SL(N) \) (in fact, \( G \) can be any simple Lie group). The generators of \( \mathcal{K} \) are

\[
J_- = \sum_{\text{simple roots } \alpha} b_{\alpha} E_{\alpha}, \quad J_+ = \sum_{\text{simple roots } \alpha} c_{\alpha}(b_{\alpha}) E_{-\alpha}, \quad J_0 = \rho_G \cdot H, \quad (4.9)
\]

* We thank R.Stora for discussions on this point.
where $b_\alpha$ are arbitrary non-zero constants, $c_\alpha$ depend on the $b_\alpha$ in a certain way and $\rho_G$ is the Weyl vector. An intriguing property \[20\] of $K$ is that the adjoint of any group $G$ decomposes under $K$ in a very specific manner:
\[
\text{adj}(G) \rightarrow \bigoplus r_j ,
\]
(4.10)
where $r_j$ are representations of $SL(2)$ labelled by spin $j$, and the values of $j$ that appear on the r.h.s. are equal to the exponents of $G$. The exponents are just the degrees of the independent Casimirs of $G$ minus one (for $SL(N)$, they are equal to $1, 2, \ldots, N - 1$).

Recalling that the Casimirs are one-to-one to the $W$ generators associated with $G$, one easily sees that the decomposition (4.10) corresponds to writing the connection (4.8) in terms of $W$-generators; more precisely, for an $N$th-order equation related to $G = SL(N)$, the connection (4.8) can be written as \[21,19\]:
\[
A = J_+ - \sum_{m=1}^{N-1} w_{m+1} (J_+)^m ,
\]
(4.11)
where $J_\pm$ are the $SL(2)$ step generators (4.9) (up to irrelevant normalization of the $w_n$).

In our case\[ with $N = 4$, the decomposition (4.10) of the adjoint of $SL(4)$ is given by $j = 1, 2, 3$, which corresponds to $w_2, w_3$ and $w_4$. We noticed above that $w_3 \equiv 0$ for special geometry and this means that $A$ belongs to a Lie algebra that decomposes as $j = 1, 2$ under $K$. It follows that this Lie algebra is $sp(4)$. Indeed, remembering that the algebra $sp(n)$ is spanned by matrices $A$ that satisfy $AQ + QA^T = 0$, we can immediately see from (4.8) that
\[
A \in sp(4) \quad \iff \quad w_3 \equiv 0 .
\]
(4.12)
Above, the symplectic metric $Q$ is taken as in (2.7).

Similarly, if in addition $w_4 = 0$ (which corresponds to a covariantly constant Yukawa coupling), $A$ further reduces to an $SL(2)$ connection. This $SL(2)$ is identical to the principal $SL(2)$ subgroup, $K$, since according to (4.11) the entries labelled by $w_2$ and $1$ in (4.8) are directly given by the $K$ generators $J_+$ and $J_-$.\[ The choice (4.8) for $A$ corresponds to an embedding (4.9) with $b_1 = b_2 = b_3 = 1$, and $c_1 = c_3 = 3/10, c_2 = 4/10$.\]
5. Relation with the moduli space of Calabi–Yau threefolds

It is well known that the moduli space of Calabi–Yau threefolds $\mathcal{M}$ is a special Kähler manifold \([10,13,15]\) and thus our considerations of the previous sections immediately apply. In particular, $U = (V, U_\alpha, U_\beta, V)$ can be identified with the basis elements of the third (real) cohomology of $\mathcal{M}$, $H^3 = H^{(3,0)} \oplus H^{(2,1)} \oplus H^{(1,2)} \oplus H^{(0,3)}$ \([13,15]\). Furthermore, $V = (X^A, F_A)$ are just the periods of the holomorphic three–form $\Omega$ \([22,23,15]\):

\[
X^A = \int_{\gamma^A} \Omega, \quad F_B = \int_{\gamma^B} \Omega, \quad A, B = 0, \ldots, n.
\]  

(5.1)

Here, $\gamma^A, \gamma^B$ are the usual basis cycles of $H_3$. Consequently, $U$ corresponds to the period matrix of $\mathcal{M}$.

The period matrix is defined only up to local gauge transformations, which are precisely of the form (3.7) \([22]\). Thus, from the considerations of section 3 it immediately follows that the period matrix can also be presented in the holomorphic gauge (3.8).* Eq. (3.9) exactly corresponds to the Picard–Fuchs equations obeyed by the periods \([3–5]\). An explicit form of $\Omega$ can be obtained in terms of the defining polynomial of the Calabi–Yau manifold \([23,24]\).

Ref. \([2]\) considered a particular Calabi–Yau manifold (a quintic in $CP_4$) whose moduli space is one-dimensional.‡ $X^A$ and $F_A$ were obtained by explicitly evaluating the period integrals (5.1). It was then noted that the periods do satisfy a 4th-order holomorphic differential equation. This differential equation is exactly eq. (4.1) with specific coefficients $a_n$ which indeed satisfy (4.2). However, for moduli spaces of arbitrary dimension it might be easier to solve the differential equation rather than performing the integrals (5.1). Therefore, let us now turn to an alternative method of computing $V$.

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* In addition, the period matrix is equivalent under conjugation by an integral matrix, $\Lambda$: $U \sim \Lambda U$. These transformations $\Lambda \in Sp(2n + 2, \mathbb{Z})$, which correspond to changes of integral homology bases, preserve the symplectic bilinear intersection form $Q$ of $H_3(\mathcal{M}, \mathbb{Z})$, that is: $\Lambda Q \Lambda^T = Q$. The subset of these transformations that leave $F$ invariant up to redefinitions constitute the “duality group”.

‡ Subsequently, this computation was generalized for a few other examples of Calabi–Yau manifolds with one-dimensional moduli spaces in refs. \([3,25]\).
6. Computation of $K$ and $W_{\alpha\beta\gamma}$

So far we worked within the context of special geometry which is the framework for the effective Lagrangian of $(2,2)$ vacua. In this section we indicate how to explicitly compute the Kähler potential and the Yukawa couplings for such a string vacuum. Various different strategies have been employed. In SCFTs $C_{\alpha\beta\gamma}$ is computed via a (moduli dependent) three-point function of the $27(\overline{27})$ matter fields $[20]$. This can often only be done perturbatively in the moduli fields around a particular point in moduli space $[27]$. Once $C_{\alpha\beta\gamma}$ as a function of the moduli is known, eq. (2.12) can be used as a differential equation which determines $K$. This strategy has been proposed in ref. $[11]$ and in sufficiently simple examples $K$ can indeed be calculated. (In TCFT the same strategy has also been used to determine the metric of “topological–antitopological fusion” $[18]$. We will come back to this point in the next section.)

As we already mentioned in the last section it is sometimes possible to explicitly evaluate eqs. (5.1). In this section we outline an alternative procedure for computing $V$ which was advocated in refs. $[3,4]$. The idea is to calculate $A_\alpha$ of eq. (3.9) and then solve the resulting differential equation for $V$ which determines $K$ and $C_{\alpha\beta\gamma}$ via eqs. (2.2) and (2.11). This computation is particularly simple for the class of $N=2$ string vacua which can be represented by a Landau-Ginzburg superpotential $W$. Therefore, let us recall a few basic properties of $W$. (For a more exhaustive review see for example $[28]$.) The unperturbed superpotential $W_0$ is a quasi-homogeneous function of the chiral superfields $x^i, i = 1,\ldots,N$. The chiral primary operators of the SCFT are represented by all monomials of $x^i$ modulo the equation of motion $\partial_{x^i}W_0 = 0$. These operators form a so called chiral ring where the ring multiplication is identified with polynomial multiplication. Maintaining conformal invariance $W_0$ can be perturbed by exactly marginal operators $p_\alpha(x_i)$. The perturbed superpotential $W$ then reads

$$W(x_i, z^\alpha) = W_0(x_i) - \sum z^\alpha p_\alpha(x_i) , \quad \alpha = 1,\ldots,n , \quad (6.1)$$

where $z^\alpha$ are the (dimensionless) moduli parameters. In a more general situation one can add a perturbation of a relevant operator which induces a RG flow to another SCFT. Here we only focus on marginal perturbations since they correspond to massless fields in the low energy effective Lagrangian. The marginal operators among
themselves generate a chiral \((2n + 2)\) dimensional (sub)-ring \(\{1, p_\alpha, p^\beta, \rho\}\) where \(\rho\) is the unique top element of the chiral ring and \(p^\beta\) is defined by \(p_\alpha p^\beta = \delta^\beta_\alpha \rho\). In this perturbed ring one defines polynomial multiplication via

\[
p_\alpha p^\beta = W^{(p)}_{\alpha\beta\gamma}(z) \rho^\gamma \mod \partial x_i \mathcal{W}.
\]

(6.2)

The non-trivial point is that \(V\) (defined in eq. (2.5)) can be expressed in terms of \(\mathcal{W}\) as follows \cite{23,3,4,24}:

\[
V = \int_{\gamma_A, \gamma_B} \frac{1}{W^\ell(x_i, z^\alpha)} \omega, \quad \omega = \sum_i (-1)^i x^i dx^1 \ldots \hat{dx}^i \ldots dx^N,
\]

(6.3)

where \(\ell = (N - 3)/2\). \footnote{The precise definition of the integral is given in \cite{23,4}, it is not important in the following.} Then the \((2n + 2) \times (2n + 2)\) dimensional matrix \(V\) (defined in eq. (3.8)) can be represented as

\[
V = \begin{pmatrix}
\int \frac{1}{W^{\ell+1} \omega} \\
\int \frac{p_i}{W^{\ell+2} \omega} \\
\int \frac{p_i}{W^{\ell+3} \omega} \\
\int \frac{\rho}{W^{\ell+4} \omega}
\end{pmatrix}.
\]

(6.4)

Using eqs. (6.2) and (6.4) one easily verifies that \(V\) indeed satisfies eq. (3.9). For a given \(\mathcal{W}\) we can use the representation (6.4) and explicitly compute \(A_\alpha\) as a function of \(z^\alpha\) by taking derivatives of \(V\) and rewriting it as \(A_\alpha \cdot V\). Then eq. (3.9) turns into a non-trivial differential equation for \(V\) which (at least in principle) can be solved. For the quintic of ref. \cite{2} \(A_\alpha\) has been computed in refs. \cite{3,4} and the solution of eq. (4.1) for this example is discussed in \cite{2,24} and so we refrain here from repeating this analysis. \footnote{The important point we want to stress is that \(A_\alpha\) is determined from \(\mathcal{W}\) alone. Thus it is not always necessary to solve the SCFT in order to determine the tree-level couplings of the effective Lagrangian. Rather, the necessary information about the couplings is already encoded in the Landau-Ginzburg superpotential \(\mathcal{W}\).}

In general \(A_\alpha\) will not come out in the form (3.10). However, as we noted in the last section eq. (3.9) still displays a gauge covariance with a holomorphic matrix \(S\) of the form (3.7). This gauge freedom can be used to put \(A_\alpha\) into the form \(A_\alpha = \Gamma_\alpha + \mathcal{C}_\alpha\)

\footnote{I like to thank P. Candelas and W. Lerche for discussions on this point.}

\footnote{Further examples are discussed in refs. \cite{5,25}.}
where $\Gamma_\alpha$ and $\xi_a$ are given by (3.11) and (3.12). The matrices $\xi_\alpha$ (which contain $W_{\alpha\beta\gamma}$) can be viewed as the structure constants of the $(2n+2)$-dimensional ring generated by $\{1, p_\alpha, p^\beta, \rho\}$. By going to the gauge (3.12) one finds a relationship between the Yukawa couplings $W_{\alpha\beta\gamma}$ and $W^{(p)}_{\alpha\beta\gamma}$ of eq. (6.2). However, this relation is still not unique. It is clear from eq. (2.4) that the Yukawa couplings are defined only up to a Kähler transformation $W_{\alpha\beta\gamma} \to W_{\alpha\beta\gamma} e^{-2f}$ and we already remarked that eq. (3.9) is covariant under Kähler transformations. Of course, any physical coupling is (Kähler-) gauge independent. Thus, one has to determine $W_{\alpha\beta\gamma}$ and $K$ in the same Kähler gauge and that is exactly what eq. (3.9) does. The solution of (3.9) determines $K$ in the same gauge we have chosen for $W_{\alpha\beta\gamma}$.

A slightly different strategy is to solve $\Gamma_\alpha = 0$, which determines the flat coordinates, instead of solving the differential equation for $V$. As was shown in detail in \cite{4}, imposing this condition gives a non-linear differential equation that determines explicitly the dependence of the Landau-Ginzburg couplings $z^\alpha$ on the $t^\alpha$. (This is closely related to the mirror map of ref. \cite{2}.) Once we have $W_{\alpha\beta\gamma}$ in flat coordinates, $F$ is determined via eq. (3.18) up to three integration constants. These correspond to the initial conditions of eq. (3.9). Unfortunately, they contribute to the physical Yukawa couplings via $e^K$. In ref. \cite{2} they were determined by using the mirror hypothesis \cite{29} and the knowledge of $F$ in the large radius limit. In this limit they can be interpreted as perturbative $\sigma$–model loop corrections in Calabi–Yau compactifications \cite{2}.

Finally, we should mention a disadvantage of the present approach. Not all moduli of a given SCFT can be represented in such a simple fashion as in eq. (6.1). Some of them appear in so-called twisted sectors, and one has to use mirror symmetry to get further information about these twisted moduli. However, the method outlined here allows for the determination of $K$ and $C_{\alpha\beta\gamma}$ of a larger class of string vacua and a larger part of the moduli space than previously known.

7. Conclusions

Special Kähler geometry also made its appearance in topological conformal field theories (TCFT) \cite{17,18,40} and in this final section we briefly comment on this aspect.

Every $N = 2$ SCFT can be ‘twisted’ into a TCFT \cite{31} which leads to a projection onto the chiral primary fields $\phi_i$ as the only physical operators of the TCFT. A family
of TCFT is defined by the action \( S = S_0 + \sum_i t^i \int \phi_i \), where \( t^i \) are the corresponding (complex) coupling parameters. All correlation functions in the TCFT can be expressed in terms of \( \eta_{ij} \equiv \langle \phi_i \phi_j \rangle \) and \( W_{ijk}^{\text{top}} \equiv \langle \phi_i \phi_j \phi_k \rangle \). By using the Ward identities of the TCFT one finds \[ 7.1 \]

\[ \partial_k \eta_{ij} = 0, \quad W_{ijk}^{\text{top}} = \partial_i \partial_j \partial_k F^{\text{top}}, \]

which are precisely the properties of special geometry in flat coordinates (eqs. (3.17), (3.18)). However, the action has been perturbed by all chiral primaries including the relevant operators of the theory. The moduli space of \((2,2)\) SCFT which we discussed so far corresponds to the subspace of the marginal deformations of the TCFT and the (holomorphic) Yukawa couplings coincide (up to the Kähler gauge freedom discussed above) with the topological correlators \( W_{ijk}^{\text{top}} \) \[ 32,7 \].

For the minimal models \( W_{ijk}^{\text{top}} \) has been determined using a Landau-Ginzburg representation exactly analogous to eq. (6.1), where \( \alpha \) now runs over all topological deformations. The flat coordinates \( t^\alpha \) arise as the solution of a Lax equation of the generalized KdV hierarchy where the Landau-Ginzburg superpotential \( W \) is identified with the Lax operator and \( F \) plays the role of the \( \tau \)-function. This corresponds to \( \Gamma_\alpha = 0 \) where \( \Gamma_\alpha \) in this context is the Gauss–Manin connection \[ 4,30 \]. It would be interesting to see what the analogous statement for a TCFT corresponding to a string vacuum is, for example for the quintic of ref. \[ 2 \]. A step in this direction has been reported here at this workshop by B. Dubrovin \[ 33 \] where the integrability of special geometry is shown.

A slightly different perspective was pursued in ref. \[ 18 \]. It was shown that the analogue of the Kähler metric \( g_{ij} \) arises in the “fusion” of a TCFT with its antitopological “partner”. This metric \( g_{ij} \) also satisfies eq. (2.12) and (2.13) and thus is a metric on a (generalized) special manifold. Exactly as before the structure of special geometry is being extended to the much bigger space of all topological deformations. However, when relevant perturbations are present the metric \( g_{ij} \) can no longer be expressed in terms of holomorphic objects as is the case on the subspace of marginal perturbations.

The discovery of special geometry in TCFT were partly the motivation for the investigation of refs. \[ 3,4 \]. We hope to have clarified the structure of the subspace of
marginal perturbations which is the subspace relevant for the effective Lagrangian of (2, 2) string vacua. We should also mention that not only the Picard–Fuchs equations arise from these topological considerations, but it seems that there are further properties of the low energy effective Lagrangian encoded in some appropriate topological field theory [34]. Clearly, this deserves further study.

Finally, we did not touch upon the quantum duality symmetry which imposes a strong constraint on the couplings in the effective Lagrangian. Clearly, the duality group is closely related to the monodromy group of the differential equation [4], which in turn depends on the zeros and poles of the Yukawa couplings. It would be worthwhile to make the relation between the duality group and the monodromy group more precise.

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