Sparse hypergraphs: new bounds and constructions

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Abstract

More than forty years ago, Brown, Erdős and Sós introduced the function \( f_r(n,v,e) \) to denote the maximum number of edges in an \( r \)-uniform hypergraph on \( n \) vertices which does not contain \( e \) edges spanned by \( v \) vertices. In other words, in such a hypergraph the union of arbitrary \( e \) edges contains at least \( v + 1 \) vertices. In the literature, the following conjecture is well-known.

**Conjecture:** \( n^{k-o(1)} < f_r(n,e(r-k)+k+1,e) = o(n^k) \) holds for all integers \( r > k \geq 2, e \geq 3 \).

Note that for \( r = 3, e = 3, k = 2 \), this conjecture was solved by the famous Ruzsa-Szemeredi’s (6,3)-theorem. In this paper, we add more evidence for the validity of this conjecture. On one hand, we use the hypergraph removal lemma to prove that the right hand side of the conjecture is true for all fixed integers \( r \geq k+1 \geq e \geq 3 \). Our result implies all known upper bounds which match the conjectured magnitude. On the other hand, we present several constructive results showing that the left hand side of the conjecture is true for \( r \geq 3, k = 2 \) and \( e = 4, 5, 7, 8 \). All previous constructive results meeting the conjectured lower bound satisfy either \( r = 3 \) or \( e = 3 \). Our constructions are the first ones which break this barrier.

**Keywords:** hypergraph Turán problem, sparse hypergraphs, hypergraph removal lemma, hypergraph rainbow cycles, sum-free set

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1 Introduction

In this paper, we will investigate a hypergraph Turán problem introduced by Brown, Erdős and Sós in the early 1970’s. Let us begin with some necessary terminologies. When speaking about a hypergraph, we mean a pair \( \mathcal{H} = (V(\mathcal{H}), E(\mathcal{H})) \),

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\end{itemize}
where the vertex set \( V(H) \) can be regarded as a finite set \( X \) and the edge set \( E(H) \) can be regarded as a collection of subsets of \( X \). We usually denote \( |V(H)| = v(H) \) and \( |E(H)| = e(H) \). For the sake of simplicity, we write \( H \) to represent the edge set \( E(H) \), and hence \( |H| \) stands for \( |E(H)| \). A hypergraph \( H \) is said to be linear if for all distinct \( A, B \in H \) it holds that \( |A \cap B| \leq 1 \). Furthermore, we say \( H \) is \( r \)-uniform if \( |A| = r \) for all \( A \in H \).

Given a set \( \mathcal{H} \) of \( r \)-uniform hypergraphs, an \( \mathcal{H} \)-free \( r \)-uniform hypergraph is a hypergraph containing none of the members of \( \mathcal{H} \). The Turán number \( ex_r(n, \mathcal{H}) \) denotes the maximum number of edges in an \( \mathcal{H} \)-free \( r \)-uniform hypergraph on \( n \) vertices. The Turán-type problems play a central role in the field of extremal combinatorics, see the surveys [9], [13], [17], [29], [31] for the history and the recent developments on this topic.

Let \( \mathcal{G}_r(v, e) \) denote the set of \( r \)-uniform hypergraphs which have \( e \) edges and \( v \) vertices (note that here by “\( v \)” vertices we actually mean “at most \( v \)” vertices, since if a hypergraph has \( e \) edges and \( v' \) (< \( v \)) vertices we can always add \( v - v' \) irrelevant vertices to make it contain exactly \( v \) vertices). Traditionally, we say that these hypergraphs have \( e \) edges which are spanned by \( v \) vertices.

We will focus on a hypergraph Turán-type problem introduced by Brown, Erdős and Sós [6], [7]. They introduced the function \( f_r(n, v, e) \) to denote the maximum number of edges in an \( r \)-uniform hypergraph on \( n \) vertices which does not contain \( e \) edges spanned by \( v \) vertices. This is equivalent to saying that for arbitrary distinct \( e \) edges \( A_1, \ldots, A_e \) of the hypergraph, it holds that \( |A_1 \cup \cdots \cup A_e| \geq v + 1 \). It can be seen directly from the definition that \( f_r(n, v, e) = \text{ex}_r(n, \mathcal{G}_r(v, e)) \). Sometimes \( \mathcal{G}_r(v, e) \)-free graphs are also called sparse hypergraphs [14], due to the sparsity of its edges. In this paper we are interested in the cases when \( r, v, e \) are fixed integers and \( n \) approaches to infinity. It was known that for every \( r > k \geq 2 \) and \( e \geq 3 \), it holds that \( f_r(n, e(r-k)+k, e) = \Theta(n^k) \), where the upper bound follows from a simple counting argument and the lower bound was proved by a standard probabilistic method. However, it is much more difficult to estimate the asymptotic behaviour of \( f_r(n, e(r-k)+k+1, e) \) for \( r > k \geq 2 \) and \( e \geq 3 \). In the literature, there is a famous conjecture (see, for example [2]) concerning the behaviour of the function \( f_r(n, e(r-k)+k+1, e) \).

**Conjecture 1.1.** \( n^{k-o(1)} < f_r(n, e(r-k)+k+1, e) = o(n^k) \) holds for all integers \( r > k \geq 2, e \geq 3 \).

The smallest case, \( r = 3, k = 2, e = 3 \), known as the (6,3)-problem, was not solved until Ruzsa and Szemerédi [25] proved the famous (6,3)-theorem, which pointed out that

\[
n^{2-o(1)} < f_3(n, 6, 3) = o(n^2). \tag{1}
\]

This result was extended by Erdős, Frankl and Rödl [11] to

\[
n^{2-o(1)} < f_r(n, 3(r - 2) + 2 + 1, 3) = o(n^2) \tag{2}
\]

for arbitrary \( r \geq 3 \), and was further extended by Alon and Shapira [2] to

\[
n^{k-o(1)} < f_r(n, 3(r-k) + k + 1, 3) = o(n^k) \tag{3}
\]
for arbitrary $r > k \geq 2$. In [2] the authors also showed that $n^{2-o(1)} < f_3(n, 7, 4)$ and $n^{2-o(1)} < f_3(n, 8, 5)$. These two sporadic cases together with the left hand side of (3) (note that (1) and (2) are implied by (3)) are all known constructions which meet the lower bound of Conjecture 1.1. Sárközy and Selkow [26], [27] also considered the upper bound part of the conjecture. They showed that

$$f_r(n, 4(r - k) + k + 1, 4) = o(n^k)$$

holds for $r > k \geq 3$, and

$$f_r(n, e(r - k) + k + \lfloor \log_2 e \rfloor, e) = o(n^k)$$

holds for all $r > k \geq 2$ and $e \geq 3$. It is easy to see that (5) implies the right hand sides of (1), (2) and (3) since $\lfloor \log_2 e \rfloor = 1$ for $e = 3$. But there is still a gap from the conjectured value for $e \geq 4$. A recent paper of Solymosi and Solymosi [30] proved that $f_3(n, 14, 10) = o(n^2)$, which improves the result of (5) for the special case $r = 3$, $k = 2$ and $e = 10$. This is the first improvement of the asymptotic bound (3) after the silence over ten years.

In this paper, we add more evidence for the validity of Conjecture 1.1. We will summarize our main results in the remaining part of this section.

1.1 Upper bounds from the hypergraph removal lemma

Nowadays removal lemmas are important and powerful tools when studying problems in extremal combinatorics. The reader is referred to [8] for an excellent survey which collects various versions of removal lemmas. One important application of removal lemmas is that they can be used to prove upper bounds for sparse hypergraphs. It is well-known that the bound $f_3(n, 6, 3) = o(n^2)$ (also known as the Ruzsa-Szemerédi theorem) [25] was proved by the first version of the removal lemma which is known as the triangle removal lemma. Ten years later, Erdős, Frankl and Rödl [11] developed the second version of removal lemma where the triangle is generalized to the complete graph $K_r$ (note that a triangle is equivalent to a $K_3$). They used the new lemma to extend Ruzsa-Szemerédi theorem to arbitrary $r \geq 3$ (see the upper bound part of (2)). After that, in 19 years no other tight upper bound of Conjecture 1.1 was known until the year 2005 Sárközy and Selkow [27] proved (4). Their proof relies heavily on a version of hypergraph removal lemma of Frankl and Rödl [12], where the graph is generalized to the hypergraph $K_3^3$, i.e., the complete 3-uniform hypergraph on 4 vertices. After the efforts of many combinatorists (see for example, [15], [16], [19], [20], [22], [23]), finally we have the following modern version of the hypergraph removal lemma.

**Lemma 1.2** (Hypergraph removal lemma, see for example, [8]). For any $r$-uniform hypergraph $G$ and any $\epsilon > 0$, there exists $\delta > 0$ such that any $r$-uniform hypergraph on $n$ vertices which contains at most $\delta n^{v(G)}$ copies of $G$ may be made $G$-free by removing at most $en^r$ edges.

With the help of the hypergraph removal lemma described as above, we are able to prove the following theorem:

**Theorem 1.3.** $f_r(n, e(r - k) + k + 1, e) = o(n^k)$ holds for all integers $r, k, e$ satisfying $r \geq k + 1 \geq e \geq 3$. 

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This theorem implies that the right hand side of Conjecture 1.1 is true for all fixed integers \( r \geq k + 1 \geq e \geq 3 \). We think that it settles all “simple” cases for the upper bound part of the conjecture, since it includes the right hand sides of (1), (2), (3), (4) as special cases, and the first unsettled case is the determination of the magnitude of \( f_3(n, 7, 4) \), which satisfies \( k = 2, e = 4 \) and is widely known as the (7,4)-problem. In the final section of this paper, the authors will discuss why the removal lemmas are not sufficient to prove the conjecture for the case \( k + 1 < e \).

Our proof of Theorem 1.3 actually implies a new asymptotic upper bound for Conjecture 1.1.

**Theorem 1.4.** For integers \( r, k, e, i \) satisfying \( i \geq 0, r \geq k + i + 1 \geq 2, (k+i+1) \geq e \geq 3, \) it holds that \( f_r(n, e(r-k)+k+i+1, e) = o(n^k) \).

One may compare Theorem 1.4 with (5). It is not hard to find that our result is superior to that of (5) when \( r, e, k, i \) satisfy \( r \geq k + i + 1 \geq 2, (k+i+1) \geq e \) and \( \left\lceil \log_2 e \right\rceil \geq i+2 \). For example, if we take \( i = 1 \), then we have \( f_r(n, 10(r-3)+5, e) = o(n^3) \) for \( r \geq 5 \) (5 only implies \( f_r(n, 10(r-3)+6, e) = o(n^3) \)), which is very close to the conjectured formula \( f_r(n, 10(r-3)+4, e) = o(n^3) \).

The proofs of Theorems 1.3 and 1.4 will be postponed to Section 2.

### 1.2 Lower bounds from rainbow-cycle-free hypergraphs and additive number theory

In the literature, constructions of sparse hypergraphs which meet the lower bound of Conjecture 1.1 are rare. Moreover, all of the previously known constructions satisfy either \( r = 3 \) or \( e = 3 \) [2]. In the range \( r \geq 4 \) and \( e \geq 4 \), there are no known constructions satisfying the conjectured lower bound. In this paper we present several new constructions which break this barrier. We prove that the lower bound part of Conjecture 1.1 is true for \( r \geq 3, k = 2 \) and \( e = 4, 5, 7, 8 \). A novel idea of our proof is that we find that rainbow-cycle-free linear hypergraphs are good candidates for sparse hypergraphs. Rainbow cycles are defined on \( r \)-uniform \( r \)-partite linear hypergraphs. They are special types of the Berge cycles [4, 5] with an additional property, that the turning vertices of the cycle must be located in distinct vertex parts of the hypergraph. For the sake of saving space, the definition of the rainbow cycle is postponed to Section 3. We can prove the following theorem:

**Theorem 1.5.** Let \( r \geq 3 \) and \( \mathcal{H} \) be an \( r \)-uniform \( r \)-partite linear hypergraph. Assume that \( \mathcal{H} \) contains no rainbow cycles of lengths three or four. Then it holds that \( \mathcal{H} \) is \( \mathcal{G}_r(3r-3,3) \)-free, \( \mathcal{G}_r(4r-5,4) \)-free, \( \mathcal{G}_r(5r-7,5) \)-free, \( \mathcal{G}_r(7r-11,7) \)-free and \( \mathcal{G}_r(8r-13,8) \)-free.

The proofs for \( e = 3, 4, 5 \) are relatively simple. However, the proofs for \( e = 7, 8 \) are a bit lengthy. The key ingredient is that we find that under the assumptions of Theorem 1.5 if \( \mathcal{H} \) is not \( \mathcal{G}_r(6r-9,6) \)-free, then any six edges \( A_1, \ldots, A_6 \) of \( \mathcal{H} \) satisfying \( |A_1 \cup \cdots \cup A_6| \leq 6r-9 \) must satisfy \( |A_1 \cup \cdots \cup A_6| = 6r-9 \) and up to isomorphism they have only one possible configuration (see Theorem 5.6 below).
By Theorem 1.5 in order to construct large sparse hypergraphs achieving the lower bound of the conjecture, it suffices to construct sufficiently large rainbow-cycle-free hypergraphs. Additive number theory is a useful tool for constructing hypergraphs with forbidden subgraphs, see for example, [1], [14], [23], [28]. The basic strategy is to characterize the structures (or subgraphs) that are not allowed to appear in the desired hypergraph by a couple of equations. Then the existence of an appropriately defined sum-free set (a sum-free set is a set which contains no nontrivial solution to certain equations) will guarantee the existence of a desired hypergraph with some forbidden subgraphs. We find that rainbow cycles can be naturally characterized by several carefully designed equations and fortunately, we are able to construct large rainbow-cycle-free hypergraphs by constructing the corresponding large sum-free sets.

**Theorem 1.6.** For every \( r \geq 3 \) and sufficiently large \( v(H) := n \), there exists an \( r \)-uniform \( r \)-partite linear hypergraph \( H \) with \( |H| > n^{2 - o(1)} \) containing no rainbow cycles of lengths three or four.

The following result is a direct consequence of Theorems 1.5 and 1.6.

**Theorem 1.7.** For sufficiently large \( n \), it holds that \( f_{r}(n, e(r - 2) + 3, e) > n^{2 - o(1)} \) for \( e = 4, 5, 7, 8 \).

Theorem 1.6 will be proved in Section 4 and Theorem 1.5 will be proved in Section 5.

### 1.3 Some remarks on 3-uniform hypergraphs

Theorem 1.7 does not contain the case \( e = 6 \). Although we can not provide an asymptotic tight lower bound for \( G_{3}(6r - 9, 6) \)-free hypergraphs, for \( r = 3 \) we can present a construction which slightly improves the previously known lower bound for \( f_{3}(n, 9, 6) \). In [14], Füredi and Ruszinkó considered the lower bound of the Turán function \( ex_{3}(n, \{I_{\geq 2}, T_{3}, G_{3x3}\}) \) (see (6) of Theorem 1.6 in [14]), which denotes the maximal number of edges of a 3-uniform linear hypergraph \( H \) on \( n \) vertices which does not contain a triangle or a 3 \( \times \) 3 grid. If we further assume that \( H \) is 3-partite, then one can argue that a triangle is a rainbow 3-cycle (see Table 1) and a 3 \( \times \) 3 grid (see Table 8) is a configuration which contains nine vertices and six edges. Note that the construction presented in [14] is 3-partite, then Theorem 5.6 indicates that the hypergraph constructed in Theorem 1.6 of [14] is indeed \( G_{3}(9, 6) \)-free. So the result of [14] implies that \( f_{3}(n, 9, 6) > n^{2 - o(1)} \). To the best of our knowledge, it was the best known lower bound for \( f_{3}(n, 9, 6) \). In this paper we are able to present a construction attaining the bound \( f_{3}(n, 9, 6) = \Omega(n^{\frac{5}{3}}) \), which takes out the \(-o(1)\) term in the exponent of the previous bound. Indeed, we have the following more general result:

**Theorem 1.8.** Let \( \mathbb{F}_{q} \) be a finite field which contains a root of the equation \( x^2 - x + 1 = 0 \). For a fixed positive integer \( k \), let us consider the \( \mathbb{F}_{q} \)-linear space \( \mathbb{F}_{q}^{k} \). Use \( r(\mathbb{F}_{q}^{k}) \) to denote the size of the maximum subset of \( \mathbb{F}_{q}^{k} \) which contains no three points on a same line. Then it holds that \( f_{3}(q^{k}, 9, 6) \geq r(\mathbb{F}_{q}^{k})q^{k} \).

The bound \( f_{3}(n, 9, 6) = \Omega(n^{\frac{5}{3}}) \) follows directly from the theorem above and a result of Lin and Wolf [18] showing that \( r(\mathbb{F}_{q}^{6}) \geq q^{4} \).
It is easy to see that our construction can be improved if one can find a better lower bound for \(r(F_q^k)\). However, this method has an unavoidable bottleneck that we can never use it to find a construction as large as \(f_3(n, 9, 6) > n^{2-o(1)}\). The reason is that the recent breakthrough work of Ellenberg and Gijswijt \([10]\) (see also the blog post of Tao \([32]\)) shows that \(r(F_q^4) < c^k\) for some \(c\) strictly smaller than \(q\). In other words, the maximum size of a subset of \(F_q^k\) containing no three points on a same line is exponentially small (i.e., it can never attain \(q^{k-o(1)}\)).

Theorem 1.8 will be proved in Section 6. Moreover, for 3-uniform linear hypergraphs, we will also classify the possible configurations of \(G_3(6, 3)\)-free hypergraphs which are not \(G_3(12, 9)\)-free.

### 1.4 A general upper bound for \(f_r(n, v, e)\)

Previous papers only discuss the behaviour of \(f_r(n, v, e)\) for \(v = e(r - k) + k\) and \(v = e(r - k) + k + 1\). Certainly, the general behaviour of \(f_r(n, v, e)\) is also of interest. We obtain a general upper bound for \(f_r(n, v, e)\) which can be stated as follows:

**Theorem 1.9.** Assume that \(er = v + p(e - 1) + q\), where \(1 \leq q \leq e - 1\). Then it holds that

\[
f_r(n, v, e) \leq q \left(\frac{n}{p+1}\right) / \left(\frac{r}{p+1}\right) + (e - 1 - q) \left(\frac{n}{p}\right) / \left(\frac{r}{p}\right).
\]

Or more frankly, it holds that

\[
f_r(n, v, e) \leq q \left(\frac{n}{\lceil er-v/e-1 \rceil}\right) / \left(\frac{r}{\lceil er-v/e-1 \rceil}\right) + (e - 1 - q) \left(\frac{n}{\lfloor er-v/e-1 \rfloor}\right) / \left(\frac{r}{\lfloor er-v/e-1 \rfloor}\right).
\]

Our results lead to an upper bound for the general function \(f_r(n, v, e)\). To the best of our knowledge, this is the first bound of this type in the literature. Unfortunately, this upper bound contributes nothing to the conjecture \(f_r(n, e(r - k) + k + 1, e) = o(n^k)\). To see this, just take \(v = e(r - k) + k + 1\), then we can deduce from the theorem above that \(f_r(n, e(r - k) + k + 1, e) \leq (e - 2) \binom{n}{k} / \binom{r}{k} + \binom{n}{e-k-1} / \binom{r}{e-k-1} \). However, the general upper bound has been proved useful for some other related combinatorial objects, see for example, combinatorial batch codes \([21]\) (which are related to \(f_r(n, e-1, e)\)) and perfect hash families \([28]\) (which are related to \(f_r(n, e-r, e)\)).

Theorem 1.9 will be proved in Section 7.

### 1.5 Outline of the paper

The rest of this paper is organized as follows. In Section 2 we will prove the new upper bounds for sparse hypergraphs. The next three sections will be devoted to the constructions for sparse hypergraphs. In Section 3, we will introduce rainbow cycles and \(R_3\)-sum-free sets, and their applications in constructing sparse hypergraphs. In Section 4, we will explain the basic idea about using sum-free sets to construct rainbow-cycle-free hypergraphs. The details of constructions for \(f_r(n, e(r - k) + k + 1, e)\) with \(r \geq 3, k = 2\) and \(e = 4, 5, 7, 8\) will be presented in Section 5. In Section 6, we will discuss some properties of 3-uniform hypergraphs. In Section 7 we present a general upper bound for \(f_r(n, v, e)\). Section 8 contains some concluding remarks.
2 Sparse hypergraphs and the hypergraph removal lemma

Throughout this paper, we always assume that \( r \geq 3 \) and all logarithms are of base 2. We also assume that \( n \) is always a sufficiently large integer. The main task of this section is to prove that the right hand side of Conjecture 1 holds for all integers \( r, k, e \) satisfying \( r \geq k + 1 \geq e \geq 3 \). Using the hypergraph removal lemma described in Lemma 1.2, it is straightforward to deduce the following fact: For any given constant \( \epsilon > 0 \), there exists some \( \delta(\epsilon) > 0 \) such that if one must delete at least \( \epsilon n^e \) edges to make an \( r \)-uniform hypergraph \( \mathcal{H} \) on \( n \) vertices \( G \)-free, then \( \mathcal{H} \) must contain at least \( \delta(\epsilon)n^me \) copies of \( G \).

**Proof of Theorem 1.3** Let \( n \) be a sufficiently large positive integer. For given \( r, k, e \) satisfying the assumption of the theorem, let \( \mathcal{H} \) be an \( r \)-uniform hypergraph which is \( G_r(e(r-k)+k+1, e) \)-free. Assume, to the contrary, that \( |\mathcal{H}| \geq \epsilon n^k \) holds for some constant \( \epsilon > 0 \).

First, we claim that for any \( A_0 \in \mathcal{H} \), there exist at most \( e-2 \) edges in \( \mathcal{H} \setminus \{A_0\} \) such that each of which intersects \( A_0 \) in at least \( k \) vertices. Suppose otherwise, that there exist \( e-1 \) distinct edges \( A_1, \ldots, A_{e-1} \) such that \( |A_0 \cap A_i| \geq k \) for every \( 1 \leq i \leq e-1 \). Then we have \( |\bigcup_{i=0}^{e-1} A_i| \leq e(r-1)k < e(r-k)+k+1 \), which contradicts the fact that \( \mathcal{H} \) is \( G_r(e(r-k)+k+1, e) \)-free. Therefore, for any \( A_0 \in \mathcal{H} \), by deleting at most \( e-2 \) edges which intersect \( A_0 \) in at least \( k \) vertices, we can conclude that there exists a hypergraph \( \mathcal{H}' \subseteq \mathcal{H} \), \( |\mathcal{H}'| \geq \frac{\epsilon}{e-1} n^k \) such that for any distinct \( A, B \in \mathcal{H}' \), it holds that \( |A \cap B| \leq k-1 \).

Next, we would like to construct an auxiliary \( k \)-uniform hypergraph \( \mathcal{H}^* \) which helps to prove the theorem. The vertex set \( V(\mathcal{H}^*) \) satisfies \( V(\mathcal{H}^*) \subseteq V(\mathcal{H}') \subseteq V(\mathcal{H}) \). The edge set \( E(\mathcal{H}^*) \) is formed in the following way: For each \( r \)-uniform edge \( A \in \mathcal{H} \), we construct a hypergraph \( K^k(A) \), which is the complete \( k \)-uniform hypergraph on the vertex set of \( A \). In other words, \( K^k(A) \) is formed by taking all the \( k \)-element subsets of \( A \). It is easy to see that there is a one-to-one correspondence between each \( A \in \mathcal{H}' \) and \( K^k(A) \subseteq \mathcal{H}^* \). \( E(\mathcal{H}^*) \) is formed by taking together the edges of every \( K^k(A) \), that is, \( E(\mathcal{H}^*) = \bigcup_{A \in \mathcal{H}'} K^k(A) \). An important observation is that for every pair of distinct edges \( A, B \in \mathcal{H}' \), the \( k \)-uniform hypergraphs \( K^k(A) \) and \( K^k(B) \) are edge disjoint. This observation follows from the simple fact that \( |A \cap B| \leq k-1 \) for arbitrary distinct \( A, B \in \mathcal{H}' \).

To sum up, we have constructed a \( k \)-uniform hypergraph \( \mathcal{H}^* \) which contains at least \( |\mathcal{H}'| \geq \frac{\epsilon}{e-1} n^k \) edge disjoint copies of \( K^k \). Thus one needs to delete at least \( \frac{\epsilon}{e-1} n^k \) \((k\text{-uniform})\) edges of \( \mathcal{H}^* \) to make it \( K^k \)-free. It follows from the hypergraph removal lemma that \( \mathcal{H}^* \) contains at least \( \delta(\epsilon)n^r \) copies of \( K^k \). Now let us estimate the number of \( K^k \) which contains two edges arising from a same \( A \in \mathcal{H}' \). First of all, notice that two edges of \( K^k \) determine at least \( k+1 \) of its vertices, and there are \( O(n^r) \) choices for such \( A \in \mathcal{H}' \). Moreover, it is obvious that the remaining \( r-k-1 \) vertices of the undetermined \( K^k \) can have at most \( n^{r-k-1} \) choices. Therefore, the number of such \( K^k \)'s is at most \( O(n^k) \cdot n^{r-k-1} = O(n^{r-1}) \), which will be strictly less than \( \delta(\epsilon)n^r \) when \( n \) is sufficiently large (here \( \delta(\epsilon) \) is the constant guaranteed by the hypergraph removal lemma).

According to the discussions above, we can conclude that there always exists a \( K^k \subseteq \mathcal{H}^* \), whose \( \binom{r}{k} \) \( k \)-uniform edges come from \( \binom{r}{k} \) distinct \( r \)-uniform edges
of $\mathcal{H}'$. Especially, for $r \geq k + 1 \geq e$, we can choose a $K^k_{k+1} \subseteq K^k_r$ and $e$ edges from such a $K^k_{k+1}$, denoted by $B_1, \ldots, B_e$. Consider the corresponding $e$ edges $A_1, \ldots, A_e$ of $\mathcal{H}'$, which satisfy $B_i \subseteq A_i$ for $1 \leq i \leq e$. The existence of these $e$ edges is guaranteed by our choice of $K^k_r$. Observe that $B_1, \ldots, B_e$ are $e$ edges spanned by only $k + 1$ vertices. Thus one can deduce that $|A_1 \cup \cdots \cup A_e| \leq re - (ek - (k + 1)) = e(r - k) + k + 1$, contradicting the assumption that $\mathcal{H}'$ is $G_r(e(r - k) + k + 1, e)$-free. Therefore, the theorem is proved by contradiction.

**Proof of Theorem 1.4** Let $\mathcal{H}$ be an $r$-uniform hypergraph which is $G_r(e(r - k) + k + 1, e)$-free. Assume, to the contrary, that $|\mathcal{H}| \geq e\eta^k$ for some constant $\eta > 0$. We follow the line of the proof of Theorem 1.3 As in the last step of the proof, one can choose a $K^k_{k+1} \subseteq K^k_r$ and $e$ edges from such a $K^k_{k+1}$, denoted by $B_1, \ldots, B_e$. Consider the corresponding $e$ edges $A_1, \ldots, A_e$ of $\mathcal{H}'$, which satisfy $B_i \subseteq A_i$ for $1 \leq i \leq e$. Then $B_1, \ldots, B_e$ are $e$ edges spanned by $k + i + 1$ vertices, thus one can deduce that $|A_1 \cup \cdots \cup A_e| \leq re - (ek - (k + i + 1)) = e(r - k) + k + 1$, contradicting the assumption that $\mathcal{H}'$ is $G_r(e(r - k) + k + 1, e)$-free.

**3 Rainbow cycles and sum-free sets**

**3.1 Rainbow cycles**

An $r$-uniform hypergraph $\mathcal{H}$ is $r$-partite if its vertex set $V(\mathcal{H})$ can be colored in $r$ colors in such a way that no edge of $\mathcal{H}$ contains two vertices of the same color. In such a coloring, the color classes of $V(\mathcal{H})$, i.e., the sets of all vertices of the same color, are called vertex parts of $\mathcal{H}$. We use $V_1, \ldots, V_r$ to denote the $r$ color classes of $V(\mathcal{H})$. Then $V(\mathcal{H})$ is a disjoint union of the $V_i$'s and for every $A \in \mathcal{H}$, $|A \cap V_i| = 1$ holds for each $1 \leq i \leq r$. We will always use a table with $r$ rows to represent an $r$-partite hypergraph, where the rows represent the vertex parts and the columns represent the edges. For a column indexed by $A \in \mathcal{H}$ and integers $1 \leq i \leq r$, the symbol in row $i$ and column $A$ is just $A \cap V_i$.

We will use the definition of hypergraph cycles introduced by Berge [4], [5]. For $k \geq 2$, a cycle (of length $k$) in a hypergraph $\mathcal{H}$ is an alternating sequence of vertices and edges of the form $v_1, A_1, v_2, A_2, \ldots, v_k, A_k, v_1$ such that

(a) $v_1, v_2, \ldots, v_k$ are distinct vertices of $\mathcal{H}$,

(b) $A_1, A_2, \ldots, A_k$ are distinct edges of $\mathcal{H}$,

(c) $v_i, v_{i+1} \in A_i$ for $1 \leq i \leq k - 1$ and $v_k, v_1 \in E_k$.

One can check that $v_i \in A_{i-1} \cap A_i$ for $2 \leq i \leq k$ and $v_1 \in A_k \cap A_1$.

Next we will introduce the notion of rainbow cycles. It is introduced by the authors of this paper when studying perfect hash families [28] (perfect hash families have strong connections with sparse hypergraphs, see Section 7 of [28]). Let $\mathcal{H}$ be an $r$-uniform $r$-partite linear hypergraph. A $k$-cycle $v_1, A_1, v_2, A_2, \ldots, v_k, A_k, v_1$ is called a rainbow $k$-cycle of $\mathcal{H}$ if $v_1, \ldots, v_k$ are located in $k$ different parts of $V(\mathcal{H})$. For $r$-partite hypergraphs, a rainbow $k$-cycle exists only if $k \leq r$. A novel idea of this paper is that we find that for certain parameters rainbow-cycle-free hypergraphs are also sparse hypergraphs.
We are most interested in rainbow cycles of lengths three and four. A rainbow 3-cycle is of the form \( v_1, A_1, v_2, A_2, v_3, A_3, v_1 \). Geometrically, one can regard a rainbow 3-cycle as a triangle whose three vertices are located in three distinct vertex parts.

**Theorem 3.1.** Let \( H \) be a linear \( r \)-partite \( r \)-uniform hypergraph. Then \( H \) is \( G_r(3r − 3, 3) \)-free if and only if it contains no rainbow 3-cycles.

**Proof.** The “only if” part is obvious. To prove the “if” part, it suffices to show that if three distinct edges \( A_1, A_2, A_3 \in H \) satisfy \(|A_1 \cup A_2 \cup A_3| ≤ 3r − 3\), then they must also form a rainbow 3-cycle. Since \( H \) is a linear hypergraph, then by the inclusion-exclusion principle it holds that

\[
|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| − |A_1 \cap A_2| − |A_2 \cap A_3| − |A_3 \cap A_1| + |A_1 \cap A_2 \cap A_3| ≥ 3r − 3 + |A_1 \cap A_2 \cap A_3|.
\]

It follows that \(|A_1 \cup A_2 \cup A_3| = 3r − 3\), \(|A_1 \cap A_2| = |A_2 \cap A_3| = |A_3 \cap A_1| = 1\) and \(A_1 \cap A_2 \cap A_3 = \emptyset\). Without loss of generality, assume that \(A_1 \cap A_2 = \{a\}\), \(A_2 \cap A_3 = \{b\}\) and \(A_3 \cap A_1 = \{c\}\). Since \(a, b, c\) are distinct, it is easy to verify that they must belong to three distinct vertex parts. Assume that \(a \in V_1, b \in V_2\) and \(c \in V_3\), then we will have a rainbow 3-cycle \(c, A_1, a, A_2, b, A_3, c\), which can be depicted as the following Table 1.

|      | \(A_1\) | \(A_2\) | \(A_3\) |
|------|--------|--------|--------|
| \(V_1\) | \(a\)   | \(a\)   |        |
| \(V_2\) |        | \(b\)   | \(b\)   |
| \(V_3\) | \(c\)   |        | \(c\)   |

Table 1: Rainbow 3-cycle

Assume that a hypergraph \( G = \{A_1, A_2, A_3, A_4\} \) forms a rainbow 4-cycle \(d, A_1, a, A_2, b, A_3, c, A_4, d\), then as in the lemma above, we can always represent \( G \) by the following Table 2.

|      | \(A_1\) | \(A_2\) | \(A_3\) | \(A_4\) |
|------|--------|--------|--------|--------|
| \(V_1\) | \(a\)   | \(a\)   |        |        |
| \(V_2\) |        | \(b\)   | \(b\)   |        |
| \(V_3\) | \(c\)   |        | \(c\)   |        |
| \(V_4\) |        |        |        | \(d\)   |

Table 2: Rainbow 4-cycle

### 3.2 Sum-free sets

For a positive integer \(n\), we denote \([n] := \{1, \ldots, n\}\). We call a linear equation \(\sum_{i=1}^s a_i m_i = 0\) with integer coefficients \(a_1, \ldots, a_s\) in the unknowns \(m_i\) homogeneous if \(\sum_{i=1}^s a_i = 0\). We say that \(M \subseteq [n]\) has no nontrivial solution to the equation above, if whenever \(m_i \in M\) and \(\sum_{i=1}^s a_i m_i = 0\), it follows that all the \(m_i\)’s are
equal. This definition of the nontrivial solution is a simplification of the original one of Ruzsa [24]. Given a set \( R = \{b_1, \ldots, b_r\} \) of \( r \) distinct nonnegative integers. Given an integer \( 3 \leq L \leq r \), a set \( M \) is said to be \( R_L \)-sum-free if for any integer \( l \) satisfying \( 3 \leq l \leq L \leq r \) and any \( l \)-element subset \( S = \{b_{j_1}, b_{j_2}, \ldots, b_{j_l}\} \subseteq R \), the equation

\[
(b_{j_2} - b_{j_1})m_1 + (b_{j_3} - b_{j_2})m_2 + \cdots + (b_{j_l} - b_{j_{l-1}})m_{l-1} + (b_{j_l} - b_{j_l})m_l = 0
\]

has no solution in \( M \) except for the trivial one \( m_1 = m_2 = \cdots = m_l \).

**Remark 3.2.** The notion of \( R_L \)-sum-free set is a generalization of the traditional sum-free set, which has been studied extensively (see [24] for a detailed introduction). Such generalization was proposed in [28] for \( r = 4 \) and \( L = 4 \).

**Lemma 3.3.** Let \( 0 < a < 1 \) be a fixed constant. Then \( 2^{O(\log^a n)} = o(n^\epsilon) \) for arbitrary small constant \( \epsilon > 0 \) and sufficiently large \( n > n(\epsilon) \).

**Proof.** This lemma follows from the following direct computations.

\[
\lim_{n \to \infty} \frac{2^{O(\log^a n)}}{n^\epsilon} = \lim_{n \to \infty} \frac{2^{O(\log^a n)}}{2^{\epsilon \log n}} = \lim_{n \to \infty} 2^{O(\log^a n) - \epsilon \log n} = 2^{-\infty} = 0.
\]

**Lemma 3.4.** Let \( l \geq 2 \) be a fixed integer. Let \( a_1, \ldots, a_l \in [n^{o(1)}] \) be \( l \) positive integers (not fixed, which could be a function of \( n \)). Then there exists a set \( M \subseteq [n], |M| \geq 2^{O(\sqrt{\log n \log \sum_{i=1}^{l} a_i})} \) with no nontrivial solution to the equation

\[
a_1m_1 + \cdots + a_lm_l = (a_1 + \cdots + a_l)m_{l+1}.
\]

**Proof.** The proof is a standard application of Behrend’s construction [3]. Let \( d \) be an undetermined integer and \( k = \llbracket \frac{\log n}{\log d} \rrbracket \). Then it holds that

\[
\frac{n}{d} = d^{\frac{\log n}{\log d} - 1} < d^k \leq d^{\frac{\log n}{\log d} = n}.
\]

Denote \( D = \sum_{i=1}^{l} a_i \). The desired set \( M \subseteq [n] \) is constructed as follows

\[
M = \{\sum_{i=1}^{k} x_i d^{i-1} | 0 \leq x_i < \frac{d}{D} \text{ and } \sum_{i=1}^{k} x_i^2 = R\},
\]

where \( R \) is an integer in \( \{0, \ldots, k(d/D)^2\} \) which is chosen to maximize the size of \( M \). One can compute that

\[
|M| \geq \frac{n}{d} \cdot \frac{1}{D^k} \cdot \frac{1}{k(d/D)^2} = \frac{n}{kd^3 D^{k-2}}.
\]

Set \( d = 2^{\sqrt{\log n \log D}} \), then it holds that

\[
|M| \geq \frac{n}{\log n \log d} \cdot 2^{3\sqrt{\log n \log D}} \cdot D^{\frac{\log n}{\log d}} \geq \frac{n}{2^{O(\sqrt{\log n \log D})}}.
\]
Now it suffices to show $M$ contains no nontrivial solution to the equation above. Let $\{m_i \in M | 1 \leq j \leq l + 1\}$ be a solution of the equation. By the definition of $M$ we can write $m_j = \sum_{i=1}^{k} x_{j,i}d^i$, where $0 \leq x_{j,i} < \frac{d}{D}$. Thus we have $\sum_{j=1}^{l} a_j (\sum_{i=1}^{k} x_{j,i}d^i) = D \sum_{i=1}^{k} x_{i+1,i}d^i$, which implies that $\sum_{i=1}^{k} (\sum_{j=1}^{l} a_j x_{j,i})d^i = \sum_{i=1}^{k} Dx_{i+1,i}d^i$. Since $\sum_{j=1}^{l} a_j x_{j,i} < d$ holds for every $1 \leq i \leq k$, then for every $1 \leq i \leq k$, it holds that $\sum_{j=1}^{l} a_j x_{j,i} = Dx_{i+1,i}$. Using Cauchy–Schwarz inequality one can argue that for every $1 \leq i \leq k$, it holds that

$$D (\sum_{j=1}^{l} a_j x_{j,i}^2) = (\sum_{j=1}^{l} a_j x_{j,i}) (\sum_{j=1}^{l} a_j) \geq (\sum_{j=1}^{l} a_j x_{j,i})^2 = D^2 x_{i+1,i}^2,$$

thus we have $\sum_{j=1}^{l} a_j x_{j,i}^2 \geq D x_{i+1,i}^2$, and the inequality holds when $x_{1,i} = \cdots = x_{l,i}$.

On the other hand, note that $\sum_{i=1}^{k} x_{j,i}^2 = R$ holds for every $1 \leq j \leq l + 1$, thus one can infer that

$$\sum_{i=1}^{k} (\sum_{j=1}^{l} a_j x_{j,i}^2) = \sum_{j=1}^{l} a_j (\sum_{i=1}^{k} x_{j,i}^2) = \sum_{j=1}^{l} a_j R = DR = \sum_{i=1}^{k} (D x_{i+1,i}^2),$$

implying $x_{1,i} = \cdots = x_{l,i}$ must hold for each $1 \leq i \leq k$. Therefore, it is easy to see that we must have $m_1 = \cdots = m_{l+1}$, which implies that $M$ contains no nontrivial solution to the equation. \qed

**Lemma 3.5.** Let $a_1, a_2, a_3, a_4 \in [n^\Theta(1)]$ be four distinct positive integers (not fixed, which could be a function of $n$) satisfying

1. $a_1 < a_2 < a_3 < a_4$,
2. $a_1 + a_4 = a_2 + a_3$,
3. for arbitrary small constant $\epsilon > 0$, it holds that $a_1 = o(a_3^\epsilon)$, $a_2 = o(a_3^\epsilon)$, $a_4 - a_3 = o(a_3^\epsilon)$ and $a_4 = o(n^\epsilon)$,
4. there exist two constants $0 < a, b < 1$ such that $\log a_2 = \Theta((\log a_3)^a)$ and $\log a_3 = \Theta((\log n)^b)$.

Then there exists a set $M \subseteq [n]$, $|M| \geq \frac{n^{1-\Theta(1)} \log^{\frac{3}{2}}}{2^{\Theta((\log n)^{1+b(\frac{a+1}{a^2})})}} > n^{1-\Theta(1)}$ with no nontrivial solution to the equation $a_1 x + a_4 y = a_2 u + a_3 v$.

**Remark 3.6.** Notice that $1 + \frac{b(a-1)}{2} < 1$ since $a < 1$ and $b < 1$. Thus by Lemma \ref{lemma:existence} we have $2^{\Theta((\log n)^{1+b(\frac{a+1}{a^2})})} = n^{o(1)}$ for sufficiently large $n$.

**Proof.** Let $B \subseteq [0, \frac{a+1}{a_2})$ be a set of integers which has no nontrivial solution to the auxiliary equation

$$a_1 x + (a_4 - a_3 - 1)y + v = a_2 u. \quad (6)$$

Note that by the second condition of the lemma, we have $a_4 + a_4 - a_3 - 1 + 1 = a_2$. Then there exists a $B$ satisfying

$$|B| > \frac{a_3}{2^{\Theta(\sqrt{\log a_3} \log a_2)}} = \frac{a_3}{2 \log a_2 + \Theta(\sqrt{\log a_3} \log a_2)} > \frac{a_3}{2 \log a_2 + \Theta(\sqrt{\log a_3} \log a_2)} > a_3^{1-\Theta(1)},$$

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where the first, second and third inequalities follow from Lemma 3.4, $\log a_2 = \Theta((\log a_3)^n)$ and Lemma 3.3 respectively. Let $M$ be the collection of all integers whose representations in base $a_3 + 1$ contain only digits belonging to $B$. One can compute that

$$|M| \geq |B|^{|\log(a_3+1)n|} = \Omega(|B|^{|\log a_3|}) = \Omega(n^{\frac{\log |B|}{\log (a_3+1)}}) > n^{1-o(1)}.$$ 

We claim that the equation $a_1x + a_3y = a_2u + a_3v$ has no nontrivial solution in $M$. Assume, to the contrary, that there exists some $x, y, u, v$ form a nontrivial solution to the equation above. Let us represent them in base $a_3 + 1$ such that $x = \sum_i x_i(a_3 + 1)^i$, $y = \sum_i y_i(a_3 + 1)^i$, $u = \sum_i u_i(a_3 + 1)^i$ and $v = \sum_i v_i(a_3 + 1)^i$. Since $x, y, u, v$ form a nontrivial solution, there must exist some integer $i$ such that $x_i, y_i, u_i, v_i$ are not all equal. Let $j$ be the least $i$ satisfying such a condition. Then it holds that

$$a_1x_j(a_3 + 1)^j + a_3y_j(a_3 + 1)^j \equiv a_2u_j(a_3 + 1)^j + a_3v_j(a_3 + 1)^j \pmod{(a_3 + 1)^{j+1}}.$$ 

This implies that

$$a_1x_j + (a_4 - a_3 - 1)y_j \equiv a_2u_j - v_j \pmod{a_3 + 1},$$

that is,

$$a_1x_j + (a_4 - a_3 - 1)y_j + v_j \equiv a_2u_j \pmod{a_3 + 1}.$$ 

Note that by our choice of $B$ we have $x_j, y_j, u_j, v_j < \frac{a_3+1}{a_2}$, thus from the congruence relation above we can deduce the following equality

$$a_1x_j + (a_4 - a_3 - 1)y_j + v_j = a_2u_j.$$ 

This is a contradiction since $x_j, y_j, u_j, v_j$ are not all equal while $B$ has no nontrivial solution to equation (6).

Now the lemma follows from the following careful calculation

$$n^{|\log |B||} \geq n\left(\frac{\log a_3}{\log(a_3+1)}\right)^{\Theta((\log a_3)^n)} > n^{1-o(1)} = \frac{n}{\frac{2^{O((\log a_3)^n)}}{\log n}}},$$

under the assumption $\log a_3 = \Theta((\log n)^n)$.

**Lemma 3.7.** Let $\sum_{i=1}^{s} a_i m_i = 0$ be a homogeneous linear equation with the unknowns $m_i$. If $M \subseteq [n]$ has no nontrivial solution to this equation, then the same holds for any shift $(M + b) \cap [n]$ with $b \in \mathbb{Z}$, where $M + b := \{m + b : m \in M\}$.

**Proof.** Assume, to the contrary, that there exists some $b \in \mathbb{Z}$ such that $(M + b) \cap [n]$ contains a nontrivial solution to the equation. Denote this nontrivial solution by $\{b_1, \ldots, b_s\}$ where $b_i = m_i + b$, $m_i \in M$ for each $1 \leq i \leq s$. Then $m_1, \ldots, m_s$ are not all equal by the definition of the nontrivial solution. It follows that

$$0 = \sum_{i=1}^{s} a_i b_i = \sum_{i=1}^{s} a_i (m_i + b) = \sum_{i=1}^{s} a_i m_i + b \sum_{i=1}^{s} a_i = \sum_{i=1}^{s} a_i m_i.$$ 

Therefore, $\{m_1, \ldots, m_s\} \subseteq M$ also forms a nontrivial solution to the equation, which contradicts the assumption of the lemma. \qed
Lemma 3.8. Let $0 < a < 1$ be a fixed constant and $t$ be a fixed positive integer. Let $\sum_{i=1}^{s} a_{ij}m_i = 0$, $1 \leq j \leq t$, be $t$ homogeneous linear equations with the unknowns $m_i$. If for every integer $1 \leq j \leq t$, there exists a set $M_j \subseteq [n]$, $|M_j| \geq \frac{n}{2^t(\log^a n)}$ with no nontrivial solution to the equation $\sum_{i=1}^{s} a_{ij}m_i = 0$. Then there exists a set $M \subseteq [n]$, $|M| \geq \frac{n}{2^t(\log^a n)}$ with no nontrivial solution to any of the equations $\sum_{i=1}^{s} a_{ij}m_i = 0$, $1 \leq j \leq t$.

Proof. Take $t - 1$ integers $\mu_2, \ldots, \mu_t \in \{-n, \ldots, n\}$ randomly, uniformly and independently. Then by Lemma 3.7, it holds that $M = M_1 \cap (M_2 + \mu_2) \cap \cdots \cap (M_t + \mu_t)$ has no nontrivial solution to any of the $t$ equations. Now let us compute the probability that an arbitrary element $m \in M_1$ lies in this intersection. For every $2 \leq j \leq t$, we have $-n \leq m - m_j \leq n$ for each $m_j \in M_j$. Thus one can infer that

$$Pr[m \in (M_i + \mu_i)] = Pr[\exists m_i \in M_i, \text{ s.t. } \mu_i = m - m_i] = \frac{|M_i|}{2n + 1} = \Omega(2^{-O(\log^a n)}).$$

Therefore, it holds that

$$Pr[m \in (M_2 + \mu_2) \cap \cdots \cap (M_t + \mu_t)] = \Omega(2^{-O(\log^a n)} t) = \Omega(2^{-O(\log^a n)}),$$

which implies that the expectation of $|M|$ is at least

$$E[|M|] \geq |M_1| \Omega(2^{-O(\log^a n)}) \geq \frac{n}{2^{O(\log^a n)}}.$$

\[\square\]

The following theorem is the main result of this subsection.

Theorem 3.9. For every integer $r \geq 4$, there exists an $r$-element set $R \subseteq [n]$ and an $R_4$-sum-free set $M \subseteq [n]$ with $|M| > n^{1-o(1)}$.

Proof. To construct the desired sum-free set, we take $R = \{b_1, \ldots, b_r\}$, where $b_1 = 2^{(\log n)^{\frac{t}{4}}}$ and $b_{i+1} = 2^{(\log b_i)^2}$ for $1 \leq i \leq r - 1$. Then $\log b_{i+1} = (\log b_i)^2$ and one can compute that $\log b_i = (\log b_1)^{2^{i-1}}$ and $b_i = 2^{(\log b_1)2^{i-1}} = 2^{(\log n)^{\frac{t}{4} \cdot 2^{i-1}}}$ for $1 \leq i \leq r$. Since $b_i = 2^{\sqrt{\log b_{i+1}}}$, by Lemma 3.8, one can infer that $b_i = o(b_{i+1})$ for $1 \leq i \leq r - 1$ and arbitrary small constant $\epsilon > 0$. Moreover, $b_r = 2^{\sqrt{\log n}} = o(n^\epsilon)$.

For $3 \leq l \leq 4$ and any $l$-element subset $S = \{b_{j_1}, b_{j_2}, \ldots, b_{j_l}\} \subseteq R$, let us classify the nonequivalent equations having the form

$$(b_{j_2} - b_{j_1})m_1 + (b_{j_3} - b_{j_2})m_2 + \cdots + (b_{j_l} - b_{j_{l-1}})m_{l-1} + (b_{j_1} - b_{j_l})m_l = 0. \quad (7)$$

Case 1. For $l = 3$, consider the equation

$$(b_{j_2} - b_{j_1})m_1 + (b_{j_3} - b_{j_2})m_2 + (b_{j_1} - b_{j_3})m_3 = 0.$$ 

By symmetry, we can always assume that $b_{j_1} < b_{j_2} < b_{j_3}$. Then the equation above can be translated to the following equivalent version

$$(b_{j_2} - b_{j_1})m_1 + (b_{j_3} - b_{j_2})m_2 = (b_{j_3} - b_{j_1})m_3. \quad (8)$$

We call this equation Type 1. One can argue that when $l = 3$, all equations have Type 1. The number of Type 1 equations is $\binom{\frac{1}{2}}{3}$. 

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Case 2. For \( l = 4 \), let us choose arbitrary four elements \( b_{j_1}, b_{j_2}, b_{j_3}, b_{j_4} \in R \). Assume that \( b_{j_1} < b_{j_2} < b_{j_3} < b_{j_4} \). Then by simply enumerating all possible combinations, it is not hard to verify that every equation is equivalent to one of the following three versions

\[
(b_{j_2} - b_{j_1})m_1 + (b_{j_3} - b_{j_2})m_2 + (b_{j_4} - b_{j_3})m_3 = (b_{j_4} - b_{j_1})m_4, \quad (9)
\]

\[
(b_{j_2} - b_{j_1})m_1 + (b_{j_4} - b_{j_2})m_2 = (b_{j_4} - b_{j_3})m_3 + (b_{j_3} - b_{j_1})m_4, \quad (10)
\]

\[
(b_{j_4} - b_{j_1})m_1 + (b_{j_3} - b_{j_2})m_2 = (b_{j_3} - b_{j_1})m_3 + (b_{j_4} - b_{j_2})m_4. \quad (11)
\]

We call these equations Type 2, Type 3 and Type 4, respectively. Each type consists of \( \binom{l}{4} \) different equations.

Now we can conclude that all equations having form (11) contain at most \( t := \binom{l}{4} + 3\binom{l}{4} \) different versions, which are denoted by Eq\(_1, \ldots, \) Eq\(_t\), respectively. Thus a set \( M \subseteq \{1, \ldots, n\} \) is \( R_l \)-sum-free with \( R = \{b_1, \ldots, b_r\} \) if and only if it has no nontrivial solution to any of the equations Eq\(_1, \ldots, \) Eq\(_t\). The remaining part of the proof can be summarized as follows: We will first use Lemmas 3.4 and 3.5 to construct \( t \) sufficiently large sets \( M_1, \ldots, M_t \subseteq \{1, \ldots, n\} \) such that \( M_i \) has no nontrivial solution to each Eq\(_i\) for \( 1 \leq i \leq t \). Then Lemma 3.8 will guarantee the existence of a desired large enough set \( M \) with no nontrivial solution to any of the equations Eq\(_1, \ldots, \) Eq\(_t\). The details are presented as follows.

By Lemma 3.4, for every equation of Types 1 or 2, there exists a set \( M \subseteq \{1, \ldots, n\} \),

\[
|M| \geq \frac{n}{2^{O((\log n)^{\frac{1}{2}})}} = \frac{n}{2^{O((\log n)^{\frac{1}{2}})}},
\]

with nontrivial solution to it. It remains to consider equations of Types 3 or 4. Compare equation (10) with Lemma 3.5. We can take \( a_1 := b_{j_2} - b_{j_1}, a_2 := b_{j_3} - b_{j_1}, a_3 := b_{j_4} - b_{j_3}, \) and \( a_4 := b_{j_4} - b_{j_2} \). By our construction of \( R \), it is easy to check that \( a_1, a_2, a_3, a_4 \) satisfy the four constraints of Lemma 3.5. Recall the definition of \( a \) and \( b \) in Lemma 3.5, it is not hard to verify that \( a \leq \frac{1}{2} \) and \( b \geq \frac{1}{2^{r-1}} \), where the inequalities hold when \( j_4 = j_3 + 1 \) and \( j_4 = 4 \), respectively. Therefore, it holds that for every equation of Type 3 or Type 4, there exists a set \( M \subseteq \{1, \ldots, n\} \),

\[
|M| \geq \frac{n}{2^{O((\log n)^{1+\frac{(n-1)}{2^{r-1}}})}} \geq \frac{n}{2^{O((\log n)^{1+\frac{1}{2r-1}})}} = \frac{n}{2^{O((\log n)^{1+\frac{1}{2r-1}})}},
\]

with nontrivial solution to it.

To sum up, if we denote \( c = \max\{\frac{3}{2}, 1 - \frac{1}{2^{r-1}}\} \), then by the discussions above, it holds that for each \( 1 \leq i \leq t \), there exists a set \( M_i \subseteq \{1, \ldots, n\} \), \( |M_i| \geq \frac{n}{2^{O((\log n)^{c})}} \), with no nontrivial solution to Eq\(_i\). By Lemma 3.8 it holds that there exists a set \( M \subseteq \{1, \ldots, n\} \), \( |M| \geq \frac{n}{2^{O((\log n)^{c})}} \) with no nontrivial solution to all the \( t \) equations Eq\(_1, \ldots, \) Eq\(_t\). That is equivalent to saying \( M \subseteq \{1, \ldots, n\} \) is an \( R_4 \)-sum-free set with \( R = \{b_1, \ldots, b_r\} \). Then the cardinality \( |M| \geq n^{1-o(1)} \) is guaranteed by Lemma 3.3.
The proof of Theorem 3.9 implies the following weaker result.

**Theorem 3.10.** For every integer \( r \geq 3 \), there exists an \( r \)-element set \( R \subseteq [n] \) and an \( R_3 \)-sum-free set \( M \subseteq [n] \) with \( |M| > n^{1-o(1)} \).

## 4 Using sum-free sets to construct rainbow-cycle-free hypergraphs

It is well-known that tools from additive number theory can be used to construct hypergraphs satisfying some Turán-type properties, see for example, [1], [14], [25], [28]. Given a positive integer \( r \) and an appropriate sum-free set \( M \subseteq [n] \), we can construct an \( r \)-uniform \( r \)-partite hypergraph \( H_M \) as follows. The vertex set is formed by \( V(H_M) = V_1 \times \cdots \times V_r \), where \( V_1, \ldots, V_r \) are undetermined subsets of positive integers such that \( |V_i| = O(n^{1+o(1)}) \) for every \( 1 \leq i \leq r \). The edge set is formed by

\[
H_M = \{A(y, m) : A(y, m) = (y + b_1 m, y + b_2 m, \ldots, y + b_r m), \ y \in [n], m \in M\} \\
\subseteq V_1 \times V_2 \times \cdots \times V_r.
\]

Here \( A(y, m) \) is an ordered \( r \)-tuple such that \( y + b_j m \in V_i \) for each \( 1 \leq j \leq r \). \( B := \{b_1, b_2, \ldots, b_r\} \subseteq [n^{o(1)}] \) is a fixed \( r \)-element set and it stays the same for every edge of \( H_M \). We call \( B \) the tangent set of the hypergraph \( H_M \).

**Remark 4.1.** Note that the constraint \( A(y, m) \in V_1 \times \cdots \times V_r \) is well-defined. Since we have \( b_j = n^{o(1)} \) for each \( 1 \leq j \leq r \), then for sufficiently large \( n \) it holds that \( y + b_j m \leq n + n^{1+o(1)} \leq n^{1+o(1)} \). Thus we can always choose large enough \( V_j \) to make \( y + b_j m \in V_j \).

**Remark 4.2.** It is easy to see that \( |H_M| = n|M| \), and \( |H_M| > n^{2-o(1)} \) provided that we can construct a sufficiently large sum-free set \( M \) with \( |M| > n^{1-o(1)} \). Note that \( |V(H_M)| = \sum_{i=1}^{r} |V_i| = r \cdot O(n^{1+o(1)}) = O(n^{1+o(1)}) \), then \( |H_M| > n^{2-o(1)} \) also implies \( |H_M| > |V(H_M)|^{2-o(1)} \).

We immediately have the following lemma.

**Lemma 4.3.** The hypergraphs constructed above are always linear.

*Proof.* Assume that \( |A(y, m) \cap A(y', m')| \geq 2 \). Then there exist \( 0 \leq i, j \leq r - 1 \) and \( i \neq j \) such that

\[
\begin{cases}
 y + b_i m = y' + b_i m' \\
 y + b_j m = y' + b_j m'. 
\end{cases}
\]

One can deduce that \( y - y' = b_i (m' - m) = b_j (m' - m) \), implying \( m = m' \) and \( y = y' \) (since \( b_i - b_j \neq 0 \)), which is a contradiction. \( \square \)

**Theorem 4.4.** If \( M \) is an \( R_L \)-sum-free set with \( 3 \leq L \leq r \), then the hypergraph \( H_M \) constructed above is an \( r \)-uniform \( r \)-partite linear hypergraph containing no rainbow cycles of length less than \( L + 1 \).

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Proof. First of all, by our construction $H_M$ is of course an $r$-uniform $r$-partite hypergraph, and its linearity is guaranteed by Lemma 4.3. Recall that by our definition, the length of a hypergraph cycle is at least three. Then it suffices to show that $H_M$ contains no rainbow $l$-cycles with $3 \leq l \leq L$. Assume, to the contrary, that it contains a rainbow $l$-cycle with $3 \leq l \leq L$, which can be denoted by $v_1, A(y_1, m_1), v_2, A(y_2, m_2), \ldots, v_l, A(y_l, m_l), v_1$, where $v_i \in V_j$ for $1 \leq i \leq l$ and $1 \leq j \leq r$. We also have $j_i \neq j_1$ for $i \neq 1$ by the definition of a rainbow cycle. Due to the linearity and the $r$-partite property of the hypergraph, it is easy to check that $A(y_1, m_1) \cap A(y_2, m_2) \cap V_j = \{v_2\}, \ldots, A(y_{l-1}, m_{l-1}) \cap A(y_l, m_l) \cap V_j = \{v_l\}, A(y_l, m_l) \cap A(y_1, m_1) \cap V_j = \{v_1\}$. Moreover, for $1 \leq j \leq r$, by our construction it holds that $A(y, m) \cap V_j = \{y + bj \cdot m\}$, then the following $l$ equations hold simultaneously

$$
\begin{align*}
y_1 + bj_1 \cdot m_1 &= y_2 + bj_2 \cdot m_2, \\
y_2 + bj_3 \cdot m_2 &= y_3 + bj_3 \cdot m_3, \\
\vdots \\
y_{l-1} + bj_j \cdot m_{l-1} &= y_l + bj_j \cdot m_l, \\
y_l + bj_1 \cdot m_l &= y_1 + bj_1 \cdot m_1.
\end{align*}
$$

By a simple elimination, one can infer that

$$(bj_2 - bj_1) \cdot m_1 + (bj_3 - bj_2) \cdot m_2 + \cdots + (bj_l - bj_{l-1}) \cdot m_{l-1} + (bj_1 - bj_l) \cdot m_l = 0,$$

which implies that $m_1 = \cdots = m_l$, taking into account the fact that $M$ is $R_L$-sum-free. Thus $y_1 = \cdots = y_l$ and hence $A(y_1, m_1) = \cdots A(y_l, m_l)$, which is obviously a contradiction. Therefore, we can conclude that $H_M$ contains no rainbow cycles with length less than $L + 1$. □

Proof of Theorem 1.6. This theorem is a direct consequence of Theorems 3.9, 3.10 and 4.4 and Remark 4.2. □

5 Using rainbow-cycle-free hypergraphs to construct sparse hypergraphs

We will use the results presented in the previous sections to construct the desired sparse hypergraphs. We will first prove that $G_r(4r - 5, 4)$-free and $G_r(5r - 7, 5)$-free hypergraphs can both be induced by $G_r(3r - 3, 3)$-free hypergraphs. However, $e = 6$ is a special case, that the $G_r(6r - 9, 6)$-free property can not be implied by the $G_r(3r - 3, 3)$-free property. We are not able to construct $G_r(6r - 9, 6)$-free hypergraphs which match the lower bound of Conjecture 1.1 for $r \geq 3$, $e = 6$ and $k = 2$. Surprisingly, when $e = 7$ and $e = 8$, we can construct sufficiently large $G_r(7r - 11, 7)$-free and $G_r(8r - 13, 8)$-free hypergraphs which match the lower bounds of Conjecture 1.1 for $r \geq 3$, $k = 2$ and $e = 7, 8$. We will begin with several lemmas which are very useful to our proof.

Lemma 5.1. Let $e \geq 4$ be a positive integer and $H$ be an $r$-uniform $r$-partite linear hypergraph. Assume that $H$ is $G_r(3r - 3, 3)$-free and $G_r((e - 1)(r - 2) + 3, e - 1)$-free but not $G_r(e(r - 2) + 3, e)$-free. Then for any $e$ distinct edges $A_1, \ldots, A_e \in H$
violating the $G_r(e(r-2)+3, e)$-free property (i.e., $|\bigcup_{i=1}^e A_i| \leq e(r-2)+3$) and any $A_i \in \{A_1, \ldots, A_e\}$, there exist three distinct edges $A_{i_1}, A_{i_2}, A_{i_3} \in \{A_1, \ldots, A_e\} \setminus \{A_i\}$ such that

1. $A_i$ intersects each of $A_{i_1}, A_{i_2}$ and $A_{i_3}$ in a different vertex, i.e., $|A_i \cap (A_{i_1} \cup A_{i_2} \cup A_{i_3})| = 3$,
2. $A_{i_1}, A_{i_2}$ and $A_{i_3}$ are pairwise disjoint,
3. $|A_1 \cup A_2 \cup A_3 \cup A_4| = 4r - 3$.

**Proof.** Let $A_1, \ldots, A_e \in \mathcal{H}$ be $r$ distinct edges such that $|\bigcup_{i=1}^e A_i| \leq e(r-2)+3$. By the $G_r((e-1)(r-2)+3, e)$-free property of $\mathcal{H}$, we have $|\bigcup_{i=1}^r A_i| \geq (e-1)(r-2)+4$. Denote $X = \bigcup_{i=1}^3 A_i$, then it holds that

$$e(r-2)+3 \geq |X \cup A_e| = |X| + |A_e| - |X \cap A_e| \geq (e-1)(r-2)+4 + r - |X \cap A_e|.$$  

By a simple elimination, one can infer that $|A_e \cap (\bigcup_{i=1}^r A_i)| \geq 3$. Since $\mathcal{H}$ is a linear hypergraph, this intersection restriction implies that there exist three distinct edges $A_{i_1}, A_{i_2}, A_{i_3} \in \{A_1, \ldots, A_{e-1}\}$ such that $A_e$ intersects each of them in a different vertex. We can always assume that $i_1 = 1, i_2 = 2, i_3 = 3$ and $A_1 \cap A_e = \{a\}, A_2 \cap A_e = \{b\}, A_3 \cap A_e = \{c\}$. Note that $\mathcal{H}$ is $r$-partite, then $a, b, c$ must be located in different vertex parts of $\mathcal{H}$. We put $a \in V_1$, $b \in V_2$ and $c \in V_3$, respectively. The intersection relationship of $A_1, A_2, A_3$ and $A_e$ can be represented by Table 3.7

|     | $A_1$ | $A_2$ | $A_3$ | $A_e$ |
|-----|-------|-------|-------|-------|
| $V_1$ | $a$   |       |       |       |
| $V_2$ |       | $b$   |       | $a$   |
| $V_3$ |       |       | $c$   | $c$   |

Table 3: An illustration of Lemma 5.1

We claim that $A_1, A_2$ and $A_3$ are pairwise disjoint. Suppose that $A_1 \cap A_2 = \{d\} \neq \emptyset$. By the linearity of $\mathcal{H}$, it is easy to see that $d \notin \{a, b, c\}$. Observe that $A_1 \cap A_e = \{a\}, A_2 \cap A_e = \{b\}$ and $A_1 \cap A_2 = \{d\}$. Then we have $|A_1 \cup A_2 \cup A_e| \leq 3r - 3$, contradicting the assumption that $\mathcal{H}$ is $G_r(3r-3, 3)$-free. Therefore, our claim is established. It remains to show that $|A_1 \cup A_2 \cup A_3 \cup A_e| = 4r - 3$. This statement follows from the fact that $|A_1 \cup A_2 \cup A_3| = 3r$ and $|A_e \cap (A_1 \cup A_2 \cup A_3)| = 3$. Then the lemma follows by choosing $i_1 = 1, i_2 = 2, i_3 = 3$ and $i = e$. □

**Lemma 5.2.** Assume that $\mathcal{H}$ is a $G_r(3r-3, 3)$-free r-uniform linear hypergraph. Let $A$ and $B$ be two edges of $\mathcal{H}$ satisfying $A \cap B \neq \emptyset$. If some other edge $C \in \mathcal{H} \setminus \{A, B\}$ has nonempty intersection with both $A$ and $B$, then we must have $C \cap A = C \cap B = A \cap B$, i.e., $A, B$ and $C$ contain a common vertex.

**Proof.** The lemma is a direct consequence of the $G_r(3r-3, 3)$-free property and the linearity of $\mathcal{H}$. □

**Lemma 5.3.** Let $e \geq 4$ be a positive integer and $\mathcal{H}$ be an $r$-uniform $r$-partite linear hypergraph formed by exactly $e$ edges. Assume that $\mathcal{H}$ is $G_r(3r-3, 3)$-free and $G_r((e-1)(r-2)+3, e-1)$-free but not $G_r(e(r-2)+3, e)$-free, i.e., $|V(\mathcal{H})| \leq e(r-2)+3$. Then for any vertex $a \in V(\mathcal{H})$ we have $\deg(a) \leq \lfloor \frac{e}{r} \rfloor$. 17
Proof. Without loss of generality, we can assume that \( \max \{ \deg(v) : v \in V(\mathcal{H}) \} = l \). Choose a vertex \( a \in V(\mathcal{H}) \) so that \( \deg(a) = l \). Set \( a \in V_1 \) and let \( A_1, \ldots, A_l \) be the \( l \) edges containing \( a \). Due to the linearity of \( \mathcal{H} \), it is easy to see that \( A_1 \setminus \{a\}, \ldots, A_l \setminus \{a\} \) are pairwise disjoint. For every \( 1 \leq i \leq l \) apply Lemma \ref{5.1} to each \( A_i \), then one can infer that for arbitrary \( A_i \) there exist three edges \( B_{i1}, B_{i2} \) and \( B_{i3} \) satisfying the three conditions of Lemma \ref{5.1}. Notice that for each \( i \) at most one of the edges in \( \{ A_1, \ldots, A_l \} \setminus \{ A_i \} \) may be served as one of the edges in \( \{ B_{i1}, B_{i2}, B_{i3} \} \). So for any \( A_i \), there exist at least two distinct edges, say, \( B_{i1}, B_{i2} \) from \( \mathcal{H} \setminus \{ A_1, \ldots, A_l \} \) such that \( \emptyset \neq B_{i1} \cap A_i \neq B_{i2} \cap A_i \neq \{a\} \). Indeed these \( 2l \) edges \( \{ B_{ij} : 1 \leq i \leq l, 1 \leq j \leq 2 \} \) are all distinct. Assume the opposite. Then there exist \( 1 \leq i \neq i' \leq l \) and some \( B \in \{ B_{ij} : 1 \leq i \leq l, 1 \leq j \leq 2 \} \) such that \( B \cap A_i \neq \emptyset \) and \( B \cap A_{i'} \neq \emptyset \). Thus by Lemma \ref{5.2} the only possible situation is that \( B \cap A_i \cap A_{i'} = \{a\} \), a contradiction.

Now the \( A_i \)'s and the \( B_{ij} \)'s have brought us at least \( 3l \) distinct edges, implying \( 3l \leq e \) and hence \( l \leq \lceil \frac{e}{3} \rceil \) since \( l \) must be an integer. \( \square \)

5.1 \( \mathcal{G}_r(4r - 5, 4) \)-free and \( \mathcal{G}_r(5r - 7, 5) \)-free hypergraphs

The main task of this subsection is to show that if an \( r \)-uniform \( r \)-partite linear hypergraph is \( \mathcal{G}_r(3r - 3, 3) \)-free, then it is also \( \mathcal{G}_r(4r - 5, 4) \)-free and \( \mathcal{G}_r(5r - 7, 5) \)-free.

**Theorem 5.4.** Let \( \mathcal{H} \) be an \( r \)-uniform \( r \)-partite linear hypergraph. If \( \mathcal{H} \) is \( \mathcal{G}_r(3r - 3, 3) \)-free, then it is also \( \mathcal{G}_r(4r - 5, 4) \)-free.

**Proof.** Assume, to the contrary, that \( \mathcal{H} \) is not \( \mathcal{G}_r(4r - 5, 4) \)-free. Then there exist four distinct edges \( A_1, A_2, A_3, A_4 \in \mathcal{H} \) such that \( |A_1 \cup A_2 \cup A_3 \cup A_4| \leq 4r - 5 \). By Lemma \ref{5.1}, \( A_1, A_2, A_3, A_4 \) also satisfy \( |A_1 \cup A_2 \cup A_3 \cup A_4| = 4r - 3 \), a contradiction. \( \square \)

**Theorem 5.5.** Let \( \mathcal{H} \) be an \( r \)-uniform \( r \)-partite linear hypergraph. If \( \mathcal{H} \) is \( \mathcal{G}_r(3r - 3, 3) \)-free, then it is also \( \mathcal{G}_r(5r - 7, 5) \)-free.

**Proof.** Assume, to the contrary, that \( \mathcal{H} \) is not \( \mathcal{G}_r(5r - 7, 5) \)-free. Then there exist five distinct edges \( A_1, A_2, A_3, A_4, A_5 \in \mathcal{H} \) such that \( |A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5| \leq 5r - 7 \). By Theorem \ref{5.4}, \( \mathcal{H} \) is \( \mathcal{G}_r(4r - 5, 4) \)-free. Apply Lemma \ref{5.1} to these five edges. We can infer that they must contain at least three vertices of degree two (for example, vertices \( a, b, c \) appearing in the proof Lemma \ref{5.1}), which is impossible since Lemma \ref{5.3} guarantees that the maximal degree of the vertices contained in \( A_1, A_2, A_3, A_4, A_5 \) can not exceed \( \left\lfloor \frac{5r}{7} \right\rfloor = 1 \). Therefore, we can conclude that \( \mathcal{H} \) is \( \mathcal{G}_r(5r - 7, 5) \)-free. \( \square \)

5.2 Classification of hypergraphs which are not \( \mathcal{G}_r(6r - 9, 6) \)-free

The results in the preceding subsection suggest that all linear \( r \)-uniform \( r \)-partite \( \mathcal{G}_r(3r - 3, 3) \)-free hypergraphs are also \( \mathcal{G}_r(4r - 5, 4) \)-free and \( \mathcal{G}_r(5r - 7, 5) \)-free. However, such a property fails when \( e = 6 \). For example, Table \ref{4} describes a 3-uniform 3-partite hypergraph which is (6,3)-free but not (9,6)-free.
Table 4: A hypergraph of six edges which is (6,3)-free but not (9,6)-free

| $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ | $A_6$ |
|-------|-------|-------|-------|-------|-------|
| $V_1$ | $a_1$ | $b_1$ | $c_1$ | $a_1$ | $b_1$ |
| $V_2$ | $a_2$ | $b_2$ | $c_2$ | $b_2$ | $c_2$ |
| $V_3$ | $a_3$ | $b_3$ | $c_3$ | $a_3$ | $b_3$ |

One can see that such a hypergraph has six edges and it is a 3-uniform 3-partite linear hypergraph. It is also easy to check that this hypergraph is $G_3(6,3)$-free, $G_3(7,4)$-free and $G_2(8,5)$-free. However, $|\bigcup_{i=1}^6 A_i| = 9$ and hence it is not $G_3(9,6)$-free. Surprisingly, if we add some additional restrictions to a hypergraph which is $G_r(3r-3,3)$-free but not $G_r(6r-9,6)$-free, we can prove that if there exist six edges spanned by at most $6r-9$ vertices, then they have only one possible configuration (up to isomorphism).

**Theorem 5.6.** Let $r \geq 3$ and $\mathcal{H}$ be an $r$-uniform $r$-partite linear hypergraph. Assume that $\mathcal{H}$ contains no rainbow cycles of lengths three or four. If there exist six edges $A_1, \ldots, A_6$ of $\mathcal{H}$ such that $|A_1 \cup \cdots \cup A_6| \leq 6r-9$, then $|A_1 \cup \cdots \cup A_6| = 6r-9$ and $A_1, \ldots, A_6$ have only one possible configuration (up to isomorphism).

**Remark 5.7.** Theorem 5.6 indicates that $\mathcal{H}$ contains no rainbow 3-cycles if and only if it is $G_r(3r-3,3)$-free.

**Remark 5.8.** Denote $W = A_1 \cup \cdots \cup A_6$ and $\mathcal{H}_W = \{A_1, \ldots, A_6\}$. By Lemma 5.3 one can infer that the maximal degree of a vertex $x \in W$ is at most two. Assume $W$ contains $\lambda$ degree two vertices and $\mu$ degree one vertices. Then we have $\lambda + \mu \leq 6r - 9$. Moreover, it naturally holds that

$$6r = \sum_{x \in W} \deg(x) = 2\lambda + \mu \leq 2\lambda + (6r - 9 - \lambda) = 6r - 9 + \lambda.$$

One can infer that $\lambda \geq 9$. We are going to show that we actually have $|W| = 6r - 9$ and $W$ contains exactly nine degree two vertices and $6r - 18$ degree one vertices. Moreover, the nine degree two vertices of $\mathcal{H}$ have only one possible configuration (generally speaking, we do not care about the degree one vertices since they are not involved in the intersections), which is equivalent to the hypergraph described by Table 7.

**Proof.** To establish this theorem, the basic strategy is to prove by contradiction. We will achieve our goal after proving several carefully designed claims.

Assume that $\mathcal{H}$ is not $G_r(6r-9,6)$-free. Then there exist six edges $A_1, \ldots, A_6$ of $\mathcal{H}$ such that $|A_1 \cup \cdots \cup A_6| \leq 6r - 9$. Since $\mathcal{H}$ contains no rainbow 3-cycles, it is $G_r(3r-3,3)$-free and hence $G_r(4r-5,4)$-free and $G_r(5r-7,5)$-free. Without loss of generality, we can take $A_1, A_2, A_3$ and $A_6$ to be the four edges which satisfy the conditions of Lemma 5.1. Let $A_1, A_2$ and $A_3$ be the three edges which are pairwise disjoint. Assume that $A_1 \cap A_6 = \{a\} \in V_1$, $A_2 \cap A_6 = \{b\} \in V_2$ and $A_3 \cap A_6 = \{c\} \in V_3$. Denote $X = A_1 \cup A_2 \cup A_3 \cup A_6$ and $Y = A_4 \cup A_5$. The obtained hypergraph is depicted as the following table.
Claim 1. \(|X \cap Y| = 6\) and the six distinct intersections are of the form \(A_i \cap A_j\) with \(i \in \{1, 2, 3\}\) and \(j \in \{4, 5\}\).

Proof. Observe that \(|X \cap Y| \leq 6\) is obvious since \(|X \cap Y| \leq |X \cap A_4| + |X \cap A_5| \leq 3 + 3 = 6\). Thus it suffices to show \(|X \cap Y| \geq 6\). Apply Lemma 5.1 separately to \(A_1, A_2\) and \(A_3\) (take the “\(A_i\)” of Lemma 5.1 to be \(A_1, A_2\) and \(A_3\) separately). Then there exist \(A_{11}, A_{12}, A_{13}, A_{21}, A_{22}, A_{23}, A_{31}, A_{32}, A_{33}\) (not necessarily distinct) such that

\[
|A_1 \cap (A_{11} \cup A_{12} \cup A_{13})| = 3, \quad |A_2 \cap (A_{21} \cup A_{22} \cup A_{23})| = 3, \quad |A_3 \cap (A_{31} \cup A_{32} \cup A_{33})| = 3.
\]

Since \(A_1, A_2\) and \(A_3\) are pairwise disjoint, it is not hard to observe that three formulas above hold if and only if

\[
|A_1 \cap (A_4 \cup A_5 \cup A_6)| = 3, \quad |A_2 \cap (A_4 \cup A_5 \cup A_6)| = 3, \quad |A_3 \cap (A_4 \cup A_5 \cup A_6)| = 3.
\]

The disjointness of \(A_1, A_2\) and \(A_3\) also implies that all the nine vertices involved in the intersections above are distinct. Thus the claim follows immediately.

Claim 2. \(A_4, A_5\) and \(A_6\) are pairwise disjoint.

Proof. Assume the opposite. For example, set \(A_4 \cap A_5 \neq \emptyset\). Pick an arbitrary \(i \in \{1, 2, 3\}\). Since \(A_i \cap A_4 \neq \emptyset\) and \(A_i \cap A_5 \neq \emptyset\), by Lemma 5.2 one can infer that \(A_i, A_4\) and \(A_5\) must contain a common vertex, which is impossible according to Claim 1.

Recall that we have assumed that \(A_1 \cap A_6, A_2 \cap A_6\) and \(A_3 \cap A_6\) are located in \(V_1, V_2\) and \(V_3\), respectively. The next claim is of particular importance: We show that if \(H\) contains no rainbow 4-cycles then the other six vertices involved in \((A_1 \cup A_2 \cup A_3) \cap (A_4 \cup A_5)\) must be also located in \(V_1 \cup V_2 \cup V_3\).

Claim 3. The six vertices involved in the intersections \(A_4 \cap (A_1 \cup A_2 \cup A_3)\) and \(A_5 \cap (A_1 \cup A_2 \cup A_3)\) are all located in vertex parts \(V_1, V_2, V_3\), where we have assumed that \(A_1 \cap A_6 = \{a\} \in V_1, A_2 \cap A_6 = \{b\} \in V_2\) and \(A_3 \cap A_6 = \{c\} \in V_3\).

Proof. We will verify Claim 3 for \(A_4\) and the proof for \(A_5\) is similar. It suffices to show \(A_4 \cap A_1 \in V_1 \cup V_2 \cup V_3\), since the proof for \(A_4 \cap A_2\) or \(A_4 \cap A_3\) is also similar. Note that the claim holds automatically for 3-partite hypergraphs, i.e., for \(r = 3\). For \(r \geq 4\), suppose that there exists some \(d\) such that \(A_4 \cap A_1 = \{d\} \notin V_1 \cup V_2 \cup V_3\). Let \(d \in V_4\) for some vertex part \(V_4\).

We will first show under this circumstance \(A_4 \cap A_2\) and \(A_4 \cap A_3\) must be located in \(V_1 \cup V_2 \cup V_3\). Obviously, neither of these two intersections
can be located in $V_4$, since both of them must be different from $d$ and $H$ is linear and $r$-partite. Thus the statement holds automatically for 4-partite hypergraphs, i.e., for $r = 4$. For $r \geq 5$, suppose $A_4 \cap A_2 = \{e\} \in V_5$ for some $V_5$ not equal to any one of $\{V_1, V_2, V_3, V_4\}$. The intersection relation of these edges is characterized in Table 5. It is not hard to find that $d, A_1, A_6, b, A_2, e, A_4, d$ form a rainbow 4-cycle, which contradicts our assumption of the theorem. By a similar argument, one can show that $A_4 \cap A_3 \in V_1 \cap V_2 \cap V_3$.

| $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ | $A_6$ |
|-------|-------|-------|-------|-------|-------|
| $V_1$ | $a$   |       | $a$   |       |       |
| $V_2$ | $b$   |       |       | $b$   |       |
| $V_3$ |       | $c$   |       | $c$   |       |
| $V_4$ | $d$   |       |       | $d$   |       |
| $V_5$ | $e$   |       |       |       | $e$   |

**Table 5:** $A_4 \cap A_1 \in V_4$ and $A_4 \cap A_2 \in V_5$, the bolder edges form a rainbow 4-cycle

Recall that $A_4 \cap \{a, b, c\} = \emptyset$ and $A_2 \cap V_2 = b$. Then $A_2 \cap A_4$ is located in either $V_1$ or $V_3$. On one hand, if $A_2 \cap A_4 = \{e\} \in V_3$, then Table 6 below indicates that $d, A_1, a, A_6, b, A_2, e, A_4, d$ must form a rainbow 4-cycle, a contradiction.

| $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ | $A_6$ |
|-------|-------|-------|-------|-------|-------|
| $V_1$ | $a$   |       |       |       | $a$   |
| $V_2$ | $b$   |       |       | $b$   |       |
| $V_3$ |       | $e$   | $e$   |       | $e$   |
| $V_4$ | $d$   |       |       | $d$   |       |

**Table 6:** $A_4 \cap A_1 \in V_4$ and $A_4 \cap A_2 \in V_3$, the bolder edges form a rainbow 4-cycle

On the other hand, if $A_2 \cap A_4 = \{e\} \in V_1$, then $A_3 \cap A_4$ must be located in $V_2$ (it can not be located in $V_1$ since $A_2 \cap A_4$ is already in $V_1$, and it can not be located in $V_3$ since $A_3 \cap V_3 = \{e\}$ and $e \not\in A_4$). Let $A_3 \cap A_4 = \{f\} \in V_2$. Then Table 7 below indicates that $d, A_1, a, A_6, c, A_3, f, A_4, d$ again form a rainbow 4-cycle, which is a contradiction, too.

| $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ | $A_6$ |
|-------|-------|-------|-------|-------|-------|
| $V_1$ | $a$   |       |       |       | $a$   |
| $V_2$ | $b$   | $f$   |       | $f$   |       |
| $V_3$ |       | $c$   |       |       | $c$   |
| $V_4$ | $d$   |       |       |       | $d$   |

**Table 7:** $A_4 \cap A_1 \in V_4$, $A_4 \cap A_2 \in V_1$, $A_3 \cap A_4 \in V_2$, the bolder edges form a rainbow 4-cycle
Finally, we can conclude that all six vertices appearing in the intersections $A_4 \cap (A_1 \cup A_2 \cup A_3)$ and $A_5 \cap (A_1 \cup A_2 \cup A_3)$ are located in $V_1$, $V_2$ and $V_3$. The claim is established.

Now we are able to prove the major part of the theorem. Claim 3 suggests that all intersections between arbitrary two members of $\{A_1, \ldots, A_6\}$ must appear in $V_1 \cup V_2 \cup V_3$. Thus $A_1 \setminus (V_1 \cup V_2 \cup V_3), \ldots, A_6 \setminus (V_1 \cup V_2 \cup V_3)$ are pairwise disjoint. So for the sake of simplicity it is reasonable to ignore them. Since $A_1$, $A_2$ and $A_3$ are pairwise disjoint, we can assume that the restrictions of $A_1$, $A_2$, $A_3$ to $V_1$, $V_2$, $V_3$ are as follows.

|   | $A_1$ | $A_2$ | $A_3$ |
|---|-------|-------|-------|
| $V_1$ | $a$   | $f$   | $h$   |
| $V_2$ | $d$   | $b$   | $i$   |
| $V_3$ | $e$   | $g$   | $c$   |

Note that we have assumed that $A_6 \cap A_1 \cap V_1 = \{a\}$, $A_6 \cap A_2 \cap V_2 = \{b\}$, $A_6 \cap A_3 \cap V_3 = \{c\}$. Due to the linearity of $H$ and Claims 1 and 2, it is easy (just by enumerating all the possibilities) to check that there are only two possibilities for the choices of $A_4$ and $A_5$ when considering their restrictions to $V_1 \cup V_2 \cup V_3$, which are

|   | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ | $A_6$ |
|---|-------|-------|-------|-------|-------|-------|
| $V_1$ | $a$   | $f$   | $h$   | $f$   | $h$   | $a$   |
| $V_2$ | $d$   | $b$   | $i$   | $i$   | $d$   | $b$   |
| $V_3$ | $e$   | $g$   | $c$   | $e$   | $g$   | $c$   |

or

|   | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ | $A_6$ |
|---|-------|-------|-------|-------|-------|-------|
| $V_1$ | $a$   | $f$   | $h$   | $h$   | $f$   | $a$   |
| $V_2$ | $d$   | $b$   | $i$   | $d$   | $i$   | $b$   |
| $V_3$ | $e$   | $g$   | $c$   | $g$   | $e$   | $c$   |

Observe that these two configurations are actually equivalent. It is also obvious that $|A_1 \cup \cdots \cup A_6| = 6r - 9$. Now our theorem is established.

Observe that the only configuration guaranteed by Theorem 5.6 can also be described as a $3 \times 3$ grid, which is depicted as the following Table 8. We call this configuration a $G_{3 \times 3}$ for simplicity.

|   | $A_1$ | $A_2$ | $A_3$ |
|---|-------|-------|-------|
| $A_4$ | $a$   | $f$   | $h$   |
| $A_5$ | $d$   | $b$   | $i$   |
| $A_6$ | $e$   | $g$   | $c$   |

Table 8: An illustration for $G_{3 \times 3}$

**Remark 5.9.** One can observe that the only configuration (up to isomorphism) appearing in Theorem 5.6 satisfies three important properties.
(1) For every \( i \in \{1, 2, 3\} \) and \( j \in \{4, 5, 6\} \), \( A_i \cap A_j \neq \emptyset \). Moreover, the nine vertices involved in the intersections are pairwise distinct.

(2) For \( \{i, j\} \subseteq \{1, 2, 3\} \) or \( \{i, j\} \subseteq \{4, 5, 6\} \), \( A_i \cap A_j = \emptyset \).

(3) \( A_1, A_2, A_3 \) and \( A_4, A_5, A_6 \) share the same nine vertices when restricted to \( V_1 \cup V_2 \cup V_3 \).

### 5.3 \( \mathcal{G}_r(7r - 11, 7) \)-free hypergraphs

In this subsection, we will study \( \mathcal{G}_r(7r - 11, 7) \)-free hypergraphs. Our goal is to construct a sufficiently large \( \mathcal{G}_r(7r - 11, 7) \)-free hypergraph that matches the lower bound of Conjecture 1.1 for \( r \geq 3 \), \( k = 2 \) and \( e = 7 \).

**Lemma 5.10.** Let \( r \geq 3 \) be a positive integer and \( \mathcal{H} \) be an \( r \)-uniform \( r \)-partite linear hypergraph. Assume that \( \mathcal{H} \) contains no rainbow cycles of lengths three or four. If there exist six edges \( A_1, \ldots, A_6 \) of \( \mathcal{H} \) such that \( |A_1 \cup \cdots \cup A_6| \leq 6r - 9 \), then for any other edge \( A_7 \in \mathcal{H} \setminus \{A_1, \ldots, A_6\} \), it holds that \( |A_7 \cap (A_1 \cup \cdots \cup A_6)| \leq 1 \).

**Proof.** Recall that our assumption implies that \( \mathcal{H} \) is \( \mathcal{G}_r(3r - 3, 3) \)-free, \( \mathcal{G}_r(4r - 5, 4) \)-free and \( \mathcal{G}_r(5r - 7, 5) \)-free. According to the discussions after Theorem 4.10, the only possible configuration of \( \{A_1, \ldots, A_6\} \) is equivalent to a \( G_{3 \times 3} \).

Denote \( X = A_1 \cup \cdots \cup A_6 \). If there exists some \( A_7 \in \mathcal{H} \setminus \{A_1, \ldots, A_6\} \) such that \( |A_7 \cap X| \geq 2 \). By the linearity of \( \mathcal{H} \), there must be \( i, j \in \{1, \ldots, 6\} \) and \( i \neq j \) such that \( A_i \cap A_j \neq \emptyset \) and \( A_i \cap A_j \neq A_i \cap A_j \). One can infer \( A_i \cap A_j = \emptyset \), since otherwise \( A_i, A_j \) and \( A_k \) will violate the \( \mathcal{G}_r(3r - 3, 3) \)-free property of \( \mathcal{H} \). By (1) and (2) of Remark 5.9 we have \( \{i, j\} \subseteq \{1, 2, 3\} \) or \( \{i, j\} \subseteq \{4, 5, 6\} \). Without loss of generality, assume that \( \{i, j\} \subseteq \{1, 2, 3\} \). Denote \( y_i = A_i \cap A_i \) and \( y_j = A_7 \cap A_j \). Our proof can be divided into three cases, according to the inclusion relations of \( y_i, y_j \) and \( V_1 \cup V_2 \cup V_3 \).

**Case 1.** \( y_i \in V_1 \cup V_2 \cup V_3 \) and \( y_j \in V_1 \cup V_2 \cup V_3 \).

We have assumed that \( \{i, j\} \subseteq \{1, 2, 3\} \). If \( y_j \in V_1 \cup V_2 \cup V_3 \), by (3) of Remark 5.9 we can find a \( j' \in \{4, 5, 6\} \) with \( y_j \in A_{j'} \). Then we have \( A_i \cap A_i = \{y_i\} \), \( A_i \cap A_j = \{y_j\} \), \( y_i \neq y_j \) and \( A_i \cap A_{j'} \neq \emptyset \) (by (1) of Remark 5.9), implying \( |A_7 \cap (A_i \cup A_j)| = 3r - 3 \), which violates the \( \mathcal{G}_r(3r - 3, 3) \)-free property. See Table 9 below for an illustration of our proof.

|   | \( A_i \) | \( A_j \) | \( A_{j'} \) | \( A_7 \) |
|---|---|---|---|---|
| \( V_1 \) | \( y_i \) |   |   |   |
| \( V_2 \) |   | \( y_j \) |   | \( y_j \) |
| \( V_3 \) | \( A_i \cap A_{j'} \) | \( A_i \cap A_{j'} \) |   |   |

Table 9: Case 1 of Lemma 5.10 \( i, j \in \{1, 2, 3\} \) and \( j' \in \{4, 5, 6\} \), the bolder edges form a rainbow 3-cycle
Case 2. \(y_j \in V_1 \cup V_2 \cup V_3\) and \(y_i \not\in V_1 \cup V_2 \cup V_3\).

Observe that this case holds automatically for \(r = 3\). For \(r \geq 4\), again, by (3) of Remark 5.9, there exists a \(j' \in \{4, 5, 6\}\) with \(y_j \in A_{j'}\). Then we have \(A_7 \cap A_j \neq \emptyset\) (by (1) of Remark 5.9), implying \(|A_7 \cup A_j| = 3r - 3\), which again violates the \(G_r(3r - 3, 3)\)-free property. See Table 10 below for an illustration of our proof.

| \(A_i\) | \(A_j\) | \(A_{j'}\) | \(A_7\) |
|---|---|---|---|
| \(V_1\) | \(y_j\) | \(y_j\) | \(y_j\) |
| \(V_2\) | \(y_j\) | \(y_j\) | \(y_j\) |
| \(V_3\) | \(A_i \cap A_{j'}\) | \(A_i \cap A_{j'}\) | \(A_i \cap A_{j'}\) |
| \(V_i\) | \(y_i\) | \(y_i\) | \(y_i\) |

Table 10: Case 2 of Lemma 5.10 \(i, j \in \{1, 2, 3\}\) and \(j' \in \{4, 5, 6\}\), the bolder edges form a rainbow 3-cycle

Case 3. \(y_i \not\in V_1 \cup V_2 \cup V_3\) and \(y_j \not\in V_1 \cup V_2 \cup V_3\).

Under this condition, \(y_i\) and \(y_j\) can not be located in the same vertex part, since \(y_i \neq y_j\) and \(H\) is \(r\)-partite and linear. Assume that \(y_i \in V_i\) and \(y_j \in V_j\), where \(\{i, j\} \cap \{1, 2, 3\} = \emptyset\) and \(i \neq j\). Now \(H\) contains at least five vertex parts. Thus this case holds automatically for \(3 \leq r \leq 4\). For \(r \geq 5\), take an arbitrary \(k \in \{4, 5, 6\}\) such that \(A_k \cap A_i \neq \emptyset\), and \(A_k \cap A_j \neq \emptyset\). Then using the results of Remark 5.9 it is not hard to find that \(x_i, A_i, y_i, A_7, y_j, A_j, x_j, A_k, x_i\) form a rainbow 4-cycle, contradicting the assumption of the theorem. See Table 11 below for an illustration of our proof.

| \(A_i\) | \(A_j\) | \(A_k\) | \(A_7\) |
|---|---|---|---|
| \(V_1\) | \(x_i\) | \(x_i\) | \(x_i\) |
| \(V_2\) | \(x_j\) | \(x_j\) | \(x_j\) |
| \(V_3\) | \(y_i\) | \(y_i\) | \(y_i\) |
| \(V_i\) | \(y_j\) | \(y_j\) | \(y_j\) |

Table 11: Case 3 of Lemma 5.10 \(i, j \in \{1, 2, 3\}\) and \(k \in \{4, 5, 6\}\), the bolder edges form a rainbow 4-cycle

For \(\{i, j\} \subseteq \{4, 5, 6\}\), the proof is similar. Therefore, the lemma is established. \(\square\)

One more lemma is needed before presenting our main result.

Lemma 5.11. Let \(r \geq 4\) be a positive integer and \(H\) be an \(r\)-uniform \(r\)-partite linear hypergraph. Assume that \(H\) contains no rainbow cycles of lengths three or four. Moreover, assume that \(H\) contains no vertex with degree larger than two. Let \(A_1, A_2, A_3, A_4\) be four pairwise disjoint edges of \(H\). Then there exists at most one edge \(B \in H \setminus \{A_1, A_2, A_3, A_4\}\) such that \(|B \cap (A_1 \cup A_2 \cup A_3 \cup A_4)| = 4\).
Proof. Assume, to the contrary, that there exist two distinct edges $B, C \in \mathcal{H} \setminus \{A_1, A_2, A_3, A_4\}$ such that $|B \cap (A_1 \cup A_2 \cup A_3 \cup A_4)| = 4$ and $|C \cap (A_1 \cup A_2 \cup A_3 \cup A_4)| = 4$. The eight vertices involved in the intersections must be all distinct since $\mathcal{H}$ contains vertices with degree larger than two. Then by the $G_r(3r - 3, 3)$-free property of $\mathcal{H}$, it is not hard to verify that $B \cap C = \emptyset$. Our goal is to show that there must exist a rainbow 4-cycle induced by four edges of $\{A_1, A_2, A_3, A_4, B, C\}$. Let $V_1, \ldots, V_8$ be the $r$ vertex parts of $\mathcal{H}$. Without loss of generality, for $1 \leq i \leq 4$ we set $B \cap A_i = \{b_i\} \subseteq V_i$ and $C \cap A_i = \{c_i\}$. Observe that for distinct $1 \leq i_1 \neq i_2 \leq 4$, we have $\{c_{i_1}, c_{i_2}\} \cap (V_{i_1} \cup V_{i_2}) \neq \emptyset$, since otherwise $\{b_{i_1}, b_{i_2}, c_{i_1}, c_{i_2}\}$ are located in four distinct vertex parts and hence $b_{i_1}, A_{i_1}, c_{i_1}, c_{i_2}, A_{i_2}, b_{i_2}, B, b_{i_1}$ form a rainbow 4-cycle. On the other hand, since $a_{i_1} \in V_{i_1}, a_{i_2} \in V_{i_2}$ and $B \cap C = \emptyset$ it is easy to see that $c_{i_1} \notin V_{i_1}$ and $c_{i_2} \notin V_{i_2}$. Thus we have either $c_{i_1} \in V_{i_2}$ or $c_{i_2} \in V_{i_1}$. For each $1 \leq i \leq 4$, let $1 \leq x_i \leq 4$ be four integers such that $c_i \in V_{x_i}$. Then the discussion above implies that for each $\{i_1, i_2\} \subseteq \{1, 2, 3, 4\}$ we have either $x_{i_1} = i_2$ or $x_{i_2} = i_1$. Let $\{i_1, i_2\}$ go through all 2-element subsets of $\{1, 2, 3, 4\}$. One can infer that all the following six equations must hold simultaneously.

\[
\begin{align*}
(x_1 - 2)(x_2 - 1) &= 0, \quad i_1 = 1, \ i_2 = 2, \\
(x_1 - 3)(x_3 - 1) &= 0, \quad i_1 = 1, \ i_2 = 3, \\
(x_1 - 4)(x_4 - 1) &= 0, \quad i_1 = 1, \ i_2 = 4, \\
(x_2 - 3)(x_3 - 2) &= 0, \quad i_1 = 2, \ i_2 = 3, \\
(x_2 - 4)(x_4 - 2) &= 0, \quad i_1 = 2, \ i_2 = 4, \\
(x_3 - 4)(x_4 - 3) &= 0, \quad i_1 = 3, \ i_2 = 4.
\end{align*}
\]

It is not hard to check that this is impossible. Therefore, $\{A_1, A_2, A_3, A_4, B, C\}$ must induce a rainbow 4-cycle, which is a contradiction. Our lemma is then established. 

Theorem 5.12. Let $r \geq 3$ be a positive integer and $\mathcal{H}$ be an $r$-uniform $r$-partite linear hypergraph. Assume that $\mathcal{H}$ contains no rainbow cycles of lengths three or four. Then $\mathcal{H}$ is $G_r(7r - 11, 7)$-free.

Proof. Assume that $\mathcal{H}$ is not $G_r(7r - 11, 7)$-free. Then there exist seven edges $A_1, \ldots, A_7$ of $\mathcal{H}$ such that $|A_1 \cup \cdots \cup A_7| \leq 7r - 11$. Since $\mathcal{H}$ contains no rainbow 3-cycles, it is $G_r(3r - 3, 3)$-free and hence $G_r(4r - 5, 4)$-free and $G_r(5r - 7, 5)$-free. Denote by $\mathcal{H}'$ the subgraph formed by $\{A_1, \ldots, A_7\}$. The proof is divided into two parts, according to whether $\mathcal{H}'$ is $G_r(6r - 9, 6)$-free.

If $\mathcal{H}'$ is not $G_r(6r - 9, 6)$-free, then let $A_1, \ldots, A_6$ be the six edges such that $|A_1 \cup \cdots \cup A_6| \leq 6r - 9$. Denote $X = A_1 \cup \cdots \cup A_6$. By Theorem 5.6, it holds that $|X| = 6r - 9$ and these six edges must form a $G_{3 \times 3}$. Thus we have

\[7r - 11 \geq |X \cup A_7| = |X| + |A_7| - |X \cap A_7| = 6r - 9 + r - |X \cap A_7|,
\]

implying $|X \cap A_7| \geq 2$, which contradicts the result of Lemma 5.10.

Therefore, to prove this theorem, it remains to consider the case when $\mathcal{H}'$ is $G_r(6r - 9, 6)$-free. Lemma 5.3 indicates that $\mathcal{H}'$ contains no vertex of degree three. Assume $\mathcal{H}'$ contains $\lambda$ degree two vertices and $\mu$ degree one vertices. Then we have $\lambda + \mu \leq 7r - 11$. Moreover, it naturally holds that

\[7r = \sum_{x \in W} \deg(x) = 2\lambda + \mu \leq 2\lambda + (7r - 11 - \lambda) = 7r - 11 + \lambda.
\]

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Thus one can infer that \( \lambda \geq 11 \). Let us count the number of pairs \( N := \{(v, A) : v \in A, A \in \mathcal{H}, v \in V(\mathcal{H}), \deg(v) = 2\} \). Observe that \( N = 2\lambda \geq 22 \) and we only have seven edges. Then there exists at least one edge of \( \mathcal{H} \) containing at least four degree two vertices (note that these four vertices must be located in four distinct vertex parts, so the theorem obviously holds for \( r = 3 \)). Without loss of generality, let \( A_7 \) be such an edge and set \( A_1, A_2, A_3, A_4 \) to be the four edges each of which contains a common degree two vertex with \( A_7 \). For \( 1 \leq i \leq 4 \), let \( A_7 \cap A_i = \{a_i\} \). Now we can draw an auxiliary Table 12:

| \( A_1 \) | \( A_2 \) | \( A_3 \) | \( A_4 \) | \( A_5 \) | \( A_6 \) | \( A_7 \) |
|---|---|---|---|---|---|---|
| \( V_1 \) | \( a_1 \) | | | | | |
| \( V_2 \) | | \( a_2 \) | | | | |
| \( V_3 \) | | | \( a_3 \) | | | |
| \( V_4 \) | | | | \( a_4 \) | | |

Table 12: We have four degree two vertices contained in a single edge

By a proof similar to the one in (2) of Lemma 5.1, one can show that \( A_1, A_2, A_3, A_4 \) are pairwise disjoint. Apply Lemma 5.1 repeatedly for \( i = 1, 2, 3, 4 \). Each time for \( i \in \{1, 2, 3, 4\} \) there exist three disjoint edges \( A_{i_1}, A_{i_2}, A_{i_3} \in \{A_1, \ldots, A_7\} \setminus \{A_i\} \) such that \( |A_{i_l} \cap A_i| = 1 \) for every \( 1 \leq l \leq 3 \). Thanks to the disjointness of \( A_1, A_2, A_3, A_4 \), the only possible solution is that \( \{A_{i_1}, A_{i_2}, A_{i_3}\} = \{A_5, A_6, A_7\} \) for all \( 1 \leq i \leq 4 \). Again, due to the disjointness of \( A_1, A_2, A_3, A_4, A_5 \), the twelve vertices involved in \( A_5 \cap (A_1 \cup A_2 \cup A_3 \cup A_4), A_6 \cap (A_1 \cup A_2 \cup A_3 \cup A_4), A_7 \cap (A_1 \cup A_2 \cup A_3 \cup A_4) \) are all distinct, implying \( |A_5 \cap (A_1 \cup A_2 \cup A_3 \cup A_4)| = 4, |A_6 \cap (A_1 \cup A_2 \cup A_3 \cup A_4)| = 4 \) and \( |A_7 \cap (A_1 \cup A_2 \cup A_3 \cup A_4)| = 4 \). Therefore, by Lemma 5.11 \( \mathcal{H}' \) must contain a rainbow 4-cycle, a contradiction.

\[ \square \]

### 5.4 \( G_r(8r - 13, 8) \)-free hypergraphs

In this subsection, we will study \( G_r(8r - 13, 8) \)-free hypergraphs. Our goal is to construct a sufficiently large \( G_r(8r - 13, 8) \)-free hypergraph that matches the lower bound of Conjecture 1.11 for \( r \geq 3, k = 2 \) and \( e = 8 \).

**Theorem 5.13.** Let \( r \geq 3 \) be a positive integer and \( \mathcal{H} \) be an \( r \)-uniform \( r \)-partite linear hypergraph. Assume that \( \mathcal{H} \) contains no rainbow cycles of lengths three or four. Then \( \mathcal{H} \) is \( G_r(8r - 13, 8) \)-free.

**Proof.** Theorem 5.12 implies that \( \mathcal{H} \) is \( G_r(7r - 11, 7) \)-free. If it is not \( G_r(8r - 13, 8) \)-free, then let \( \mathcal{H}' = \{A_1, \ldots, A_8\} \) be a subgraph of eight edges such that \( |A_1 \cup \cdots \cup A_8| \leq 8r - 13 \). Lemma 5.3 indicates that \( \mathcal{H}' \) contains no vertex of degree three. Assume \( \mathcal{H}' \) contains \( \lambda \) degree two vertices and \( \mu \) degree one vertices. Then we have \( \lambda + \mu \leq 8r - 13 \). Moreover, it naturally holds that

\[
8r = \sum_{x \in W} \deg(x) = 2\lambda + \mu \leq 2\lambda + (8r - 13 - \lambda) = 8r - 13 + \lambda.
\]

Thus one can infer that \( \lambda \geq 13 \). Let us count the number of pairs \( N := \{(v, A) : v \in A, A \in \mathcal{H}, v \in V(\mathcal{H}), \deg(v) = 2\} \). Observe that \( N = 2\lambda \geq 26 \) and we only
have eight edges. Then there exists at least one edge of \( \mathcal{H} \) containing at least four degree two vertices (note that these four vertices must be located in four distinct vertex parts, so the theorem obviously holds for \( r = 3 \)). Without loss of generality, let \( A_8 \) be such an edge and set \( A_1, A_2, A_3, A_4 \) to be the four edges each of which contains a common degree two vertex with \( A_8 \). For \( 1 \leq i \leq 4 \), let \( A_8 \cap A_i = \{ a_i \} \).

We can draw an auxiliary Table 13.

| \( A_1 \) | \( A_2 \) | \( A_3 \) | \( A_4 \) | \( A_5 \) | \( A_6 \) | \( A_7 \) | \( A_8 \) |
|---|---|---|---|---|---|---|---|
| \( V_1 \) | \( a_1 \) | | | | | | \( a_1 \) |
| \( V_2 \) | \( a_2 \) | | | | | | \( a_2 \) |
| \( V_3 \) | \( a_3 \) | | | | | | \( a_3 \) |
| \( V_4 \) | \( a_4 \) | | | | | | \( a_4 \) |

Table 13: We have four degree two vertices contained in a single edge

It is easy to verify that \( A_1, A_2, A_3, A_4 \) are pairwise disjoint. We claim that \( A_5, A_6 \) and \( A_7 \) are also pairwise disjoint. Assume the opposite. Without loss of generality, suppose \( A_6 \cap A_7 \neq \emptyset \). Apply Lemma 5.1 repeatedly for \( i = 1, 2, 3, 4 \). Each time for \( i \in \{ 1, 2, 3, 4 \} \) there exist three disjoint edges \( A_{i_1}, A_{i_2}, A_{i_3} \in \{ A_1, \ldots, A_8 \} \setminus \{ A_i \} \) such that \( |A_i \cap A_l| = 1 \) for every \( 1 \leq l \leq 3 \). Due to the disjointness of \( A_1, A_2, A_3, A_4 \), for each \( i \in \{ 1, 2, 3, 4 \} \) the possible candidates for \( A_{i_1}, A_{i_2}, A_{i_3} \) can only be chosen from \( \{ A_5, A_6, A_7, A_8 \} \). Since \( A_i \cap A_k \neq \emptyset \), at least two edges of \( \{ A_5, A_6, A_7 \} \) must have nonempty intersection with \( A_i \). Call such two edges an intersecting pair of \( A_i \) (if three edges \( A_5, A_6, A_7 \) all have nonempty intersection with \( A_i \), we just pick arbitrary two edges to form the intersecting pair). Note that the two edges contained in an intersecting pair are disjoint. Thus under the assumption \( A_6 \cap A_7 \neq \emptyset \), for each \( i \in \{ 1, 2, 3, 4 \} \) the intersecting pair of \( A_i \) must be either \( (A_5, A_6) \) or \( (A_5, A_7) \). Observe that both candidates contain \( A_5 \), which implies that \( A_5 \cap A_i \neq \emptyset \) for each \( 1 \leq i \leq 4 \). Therefore, we have \( |A_5 \cap (A_1 \cup A_2 \cup A_3 \cup A_4)| = 4 \) and \( |A_8 \cap (A_1 \cup A_2 \cup A_3 \cup A_4)| = 4 \), which is a contradiction according to Lemma 5.11. Thus our claim is established and we can conclude that \( A_5, A_6, A_7 \) are pairwise disjoint.

For each \( j \in \{ 5, 6, 7 \} \), by applying Lemma 5.1 to \( A_j \) one can infer that there exists at least \( i \in \{ 1, 2, 3, 4 \} \) satisfying \( A_i \cap A_j \neq \emptyset \). Note that we also have \( A_i \cap A_8 \neq \emptyset \). Thus we have \( A_i \cap A_8 = \emptyset \) since otherwise we will obtain either a vertex of degree at least three or three edges with union at most \( 3r - 3 \). Actually we have proved that \( A_5, A_6, A_7, A_8 \) are pairwise disjoint. On one hand, let us apply Lemma 5.1 repeatedly to \( A_5, A_6 \) and \( A_7 \). Thus for \( 1 \leq j \leq 4 \), each \( A_j \) has at least three distinct intersections with \( A_1 \cup A_2 \cup A_3 \cup A_4 \). On the other hand, Lemma 5.11 implies that the size of intersection is at most three. So indeed we have \( |A_j \cap (A_1 \cup A_2 \cup A_3 \cup A_4)| = 3 \) for each \( j \in \{ 5, 6, 7 \} \). Moreover, all the nine vertices involved in the intersections are distinct. In the following we will prove that \( A_1 \cup A_2 \cup A_3 \cup A_4 \) and \( A_5 \cup A_6 \cup A_7 \cup A_8 \) have intersected in too many vertices (as many as \( 3 \times 3 + 4 = 13 \)) that we can not avoid the appearance of a rainbow 4-cycle. For brevity, in what follows we denote \( X = A_1 \cup A_2 \cup A_3 \cup A_4 \) and \( Y = A_5 \cup A_6 \cup A_7 \).

Since \( |X \cap Y| = 9 \) and \( |A_i \cap Y| \geq 2 \) for each \( 1 \leq i \leq 4 \), there exists exactly one \( i_0 \in \{ 1, 2, 3, 4 \} \) such that \( |A_{i_0} \cap Y| = 3 \) and \( |A_k \cap Y| = 2 \) for \( k \in \{ 1, 2, 3, 4 \} \setminus \{ i_0 \} \).
Let us set this special edge to be $A_4$ (i.e., set $i_0 = 4$). Now notice that each edge in $\{A_1, A_2, A_3\}$ intersects exactly two edges in $\{A_5, A_6, A_7\}$ and conversely, each edge in $\{A_5, A_6, A_7\}$ intersects exactly two edges in $\{A_1, A_2, A_3\}$. Since the maximal degree of vertices of $\mathcal{H}$ is at most two, one can simply show that up to isomorphism there is only one possible intersection relation between $\{A_1, A_2, A_3\}$ and $\{A_5, A_6, A_7\}$. Without loss of generality, assume that

$$A_5 \cap A_1 \neq \emptyset, \quad A_5 \cap A_2 \neq \emptyset, \quad A_6 \cap A_1 \neq \emptyset, \quad A_6 \cap A_3 \neq \emptyset, \quad A_7 \cap A_2 \neq \emptyset, \quad A_7 \cap A_3 \neq \emptyset.$$ 

Furthermore, let $1 \leq x_1, x_2, x_4, y_1, y_4, z_2, z_3, z_4 \leq r$ be nine integers such that

$$A_5 \cap A_1 \in V_{x_1}, \quad A_5 \cap A_2 \in V_{x_2}, \quad A_5 \cap A_4 \in V_{x_4},$$
$$A_6 \cap A_1 \in V_{y_1}, \quad A_6 \cap A_3 \in V_{y_3}, \quad A_6 \cap A_4 \in V_{y_4},$$
$$A_7 \cap A_2 \in V_{z_2}, \quad A_7 \cap A_3 \in V_{z_3}, \quad A_7 \cap A_4 \in V_{z_4}.$$ 

Recall that we have assumed $A_i \cap A_8 \in V_i$ for each $1 \leq i \leq 4$. Thus as in the proof of Lemma 5.11 in order to avoid rainbow 4-cycles, the following nine equations must hold simultaneously.

$$(x_1 - 2)(x_2 - 1) = 0, \quad (x_1 - 4)(x_4 - 1) = 0, \quad (x_2 - 4)(x_4 - 2) = 0,$$
$$(y_1 - 3)(y_3 - 1) = 0, \quad (y_1 - 4)(y_4 - 1) = 0, \quad (y_3 - 4)(y_4 - 3) = 0,$$
$$(z_2 - 3)(z_3 - 2) = 0, \quad (z_2 - 4)(z_4 - 2) = 0, \quad (z_3 - 4)(z_4 - 3) = 0.$$ 

Another important observation is that for each $1 \leq i \leq 4$, no pair of $x_i, y_i, z_i$ (if it does exist) could have equal value, since the hypergraph is $r$-partite. In this sense one can verify that the value of all nine unknowns can be fixed as long as we set a value to arbitrary one unknown. For example, $(x_1 - 2)(x_2 - 1) = 0$ means either $x_1 = 2$ or $x_2 = 1$. Set $x_1 = 2$. Thus in order to guarantee the validity of the remaining two equations in the first row we must set $x_4 = 1$ and $x_2 = 4$. Let us look at the second equation in the second row. Since $x_4 = 1$ and $y_4 \neq x_4$, we must set $y_1 = 4$, implying $y_3 = 1$ and $y_4 = 3$. Now we have set $x_1 = 2, y_1 = 4$ and $x_4 = 1, y_4 = 3$, which implies that $A_5 \cap A_1 \in V_1, A_6 \cap A_1 \in V_4, A_5 \cap A_4 \in V_1, A_6 \in A_4 \in V_4$. It is easy to see that these four intersections are located in four distinct vertex parts and four edges $A_5, A_1, A_6, A_4$ must form a rainbow 4-cycle, a contradiction. Therefore, we have proved that $|A_1 \cup \cdots \cup A_8| > 8r - 13$ and $\mathcal{H}$ must be $\mathcal{G}_r(8r - 13, 8)$-free. 

**Proof of Theorem 6.5.** Theorem 1.5 is a direct consequence of Theorems 5.11 5.14 5.15 5.12 5.13.

### 6 3-uniform hypergraphs

#### 6.1 An improved construction for $\mathcal{G}_3(9, 6)$-free hypergraphs

We will use the method introduced in Section 4 to construct the desired hypergraph $\mathcal{H}$. Let $B = \{\alpha, \beta, \gamma\}$ be an undetermined 3-element set which will be the tangent set for $\mathcal{H}$. Notice that one can first assume that the elements of $B$
are chosen from integers (just as in Section 4), but keep in mind that (which will be mentioned later) here these elements will indeed be chosen from some finite field $\mathbb{F}_q$. Let us assume first that $\mathcal{H}$ contains no rainbow cycles of length three. If $\mathcal{H}$ is not $\mathcal{G}_3(9,6)$-free and assume that there exist $A_1, \ldots, A_6 \in \mathcal{H}$ satisfying $|A_1 \cup \cdots \cup A_6| \leq 6r - 9$, then by Theorem 5.6 we must have $|A_1 \cup \cdots \cup A_6| = 6r - 9$ and all the vertices of $A_1, \ldots, A_6$ must have the following configuration,

|   | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ | $A_6$ |
|---|---|---|---|---|---|---|
| $V_\alpha$ | $y_1 + \alpha m_1$ | $y_2 + \alpha m_2$ | $y_3 + \alpha m_3$ | $y_4 + \alpha m_4$ | $y_5 + \alpha m_5$ | $y_6 + \alpha m_6$ |
| $V_\beta$ | $y_1 + \beta m_1$ | $y_2 + \beta m_2$ | $y_3 + \beta m_3$ | $y_4 + \beta m_4$ | $y_5 + \beta m_5$ | $y_6 + \beta m_6$ |
| $V_\gamma$ | $y_1 + \gamma m_1$ | $y_2 + \gamma m_2$ | $y_3 + \gamma m_3$ | $y_4 + \gamma m_4$ | $y_5 + \gamma m_5$ | $y_6 + \gamma m_6$ |

where $V_\alpha, V_\beta$ and $V_\gamma$ denote the three vertex parts of $\mathcal{H}$. Without loss of generality, we can assume that these vertices are placed in the following form.

|   | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ | $A_6$ |
|---|---|---|---|---|---|---|
| $V_\alpha$ | $a_\alpha$ | $b_\alpha$ | $c_\alpha$ | $a_\alpha$ | $b_\alpha$ | $c_\alpha$ |
| $V_\beta$ | $a_\beta$ | $b_\beta$ | $c_\beta$ | $b_\beta$ | $c_\beta$ | $a_\beta$ |
| $V_\gamma$ | $a_\gamma$ | $b_\gamma$ | $c_\gamma$ | $c_\gamma$ | $a_\gamma$ | $b_\gamma$ |

Therefore, the following nine equations must hold simultaneously.

\[
\begin{align*}
y_4 + \alpha m_4 &= y_1 + \alpha m_1 \\
y_5 + \alpha m_5 &= y_2 + \alpha m_2 \\
y_6 + \alpha m_6 &= y_3 + \alpha m_3 \\
y_4 + \beta m_4 &= y_2 + \beta m_2 \\
y_5 + \beta m_5 &= y_3 + \beta m_3 \\
y_6 + \beta m_6 &= y_1 + \beta m_1 \\
y_4 + \gamma m_4 &= y_3 + \gamma m_3 \\
y_5 + \gamma m_5 &= y_1 + \gamma m_1 \\
y_6 + \gamma m_6 &= y_2 + \gamma m_2
\end{align*}
\]

(12)

If we take $\alpha = 0$, then the first three equations imply $y_4 = y_1, y_5 = y_2$ and $y_6 = y_3$. We can rearrange the remaining six equations to obtain the following identities.

\[
\begin{align*}
y_1 + \beta m_4 &= y_2 + \beta m_2 \\
y_2 + \beta m_5 &= y_3 + \beta m_3 \\
y_3 + \beta m_6 &= y_1 + \beta m_1 \\
y_1 + \gamma m_4 &= y_3 + \gamma m_3 \\
y_2 + \gamma m_5 &= y_1 + \gamma m_1 \\
y_3 + \gamma m_6 &= y_2 + \gamma m_2
\end{align*}
\]

(13)

Let us consider the equations listed above. If we separately add both sides of the first and the fifth, the second and sixth, and the third and the fourth equations, we can obtain the following identities.

\[
\begin{align*}
\beta m_4 + \gamma m_5 &= \beta m_2 + \gamma m_1 \\
\beta m_5 + \gamma m_6 &= \beta m_3 + \gamma m_2 \\
\beta m_6 + \gamma m_4 &= \beta m_1 + \gamma m_3
\end{align*}
\]

(14)
From these identities it is not hard to verify that the following identity must hold
\[(\beta \gamma)m_1 + (\gamma^2 - \beta \gamma)m_3 = (\beta^2 - \beta \gamma + \gamma^2)m_4 + (\beta \gamma - \beta^2)m_2.\]  \hspace{1cm} (15)

Take \(\beta = 1\). Then (15) can be transformed into
\[\gamma m_1 + (\gamma^2 - \gamma)m_3 = (1 - \gamma + \gamma^2)m_4 + (\gamma - 1)m_2.\]  \hspace{1cm} (16)

Note that Equation (16) is indeed a special case of the following more general one
\[sm_1 + tm_3 = (t + 1)m_4 + (s - 1)m_2.\]  \hspace{1cm} (17)

Equations of type (17) have been studied by Ruzsa in [24]. Let \(r_{s,t}(n)\) denote the maximal size of a subset of \([n]\) which contains no solution to (17) except the trivial one \(m_1 = m_2 = m_3 = m_4\). When \(s = t + 1\), it is known that \(r_{t+1,t}(n) = \Theta(\sqrt{n})\). However, the exact order of \(r_{s,t}(n)\) is not known for general \(s, t\). For example, for the smallest case \(r_{2,2}(n)\), Ruzsa [24] showed that \(r_{2,2}(n) = \Omega(\sqrt{n})\), remarked that \(r_{2,2}(n) = o(n)\) and asked whether \(r_{2,2}(n) > n^{1-o(1)}\). The authors of [14] commented that this problem “seems very difficult”. Note that the lower bound of \(r_{2,2}(n)\) will provide a lower bound for (nontrivial) solution-free sets for (16) with \(\gamma = 2\).

If we consider (16) over some finite field \(\mathbb{F}_q\) rather than \(\mathbb{Z}\), then we can benefit from the finite field structure with a carefully chosen \(\gamma \in \mathbb{F}_q\).

**Proof of Theorem 1.8** Let \(\mathbb{F}_{q^k}\) be the field extension of \(\mathbb{F}_q\). It is commonly known that \(\mathbb{F}_{q^k}\) is equivalent to \(\mathbb{F}_q\). Let \(\phi\) be an arbitrary isomorphism from \(\mathbb{F}_{q^k}\) to \(\mathbb{F}_q\). It is obvious that \(\phi\) is \(\mathbb{F}_q\)-linear. Let \(M \subseteq \mathbb{F}_q\) be a subset which contains no three collinear points. Denote by \(\phi^{-1}(M)\) the preimage set of \(M\). By the assumption of the theorem, there exists some \(\gamma \in \mathbb{F}_q\) such that \(\gamma^2 - \gamma + 1 = 0\).

The desired 3-uniform hypergraph \(H_M\) is constructed as follows:
\[H_M = \{A(y, m) : A(y, m) = (y, y + m, y + \gamma m), y \in \mathbb{F}_{q^k}, m \in \phi^{-1}(M)\},\]
where we take the tangent set \(B\) to be \(B = \{0, 1, \gamma\} \subseteq \mathbb{F}_q\).

First it is not hard to verify that \(H_M\) is a linear hypergraph. If it is not \(G_3(6, 3)\)-free, then according to the discussions in the proof of Theorem 3.9 for \(l = 3\), one can verify that there must exist \(m_1, m_2, m_3 \in \phi^{-1}(M)\) satisfying
\[(1 - 0)m_1 + (\gamma - 1)m_2 + (0 - \gamma)m_3 = 0,
\]
implying
\[m_1 + (\gamma - 1)m_2 = m_3,
\]
which means that \(\phi(m_1), \phi(m_2), \phi(m_3) \in M\) are on the same line, a contradiction.

Thus we have proved that \(H_M\) is \(G_3(6, 3)\)-free. Therefore, Theorem 5.6 and the discussions before this theorem indicate that if \(H_M\) is not \(G_3(9, 6)\)-free then there must exist \(m_1, m_2, m_3, m_4 \in \phi^{-1}(M)\) satisfying (16). Note that we have set \(\gamma\) to be a root of \(x^2 - x + 1 = 0\), then (16) implies
\[\gamma m_1 + (\gamma^2 - \gamma)m_3 = (\gamma - 1)m_2,\]  \hspace{1cm} (18)

and hence
\[\gamma m_1 = (\gamma - 1)m_2 + m_3,\]  \hspace{1cm} (19)
where one can compute \(\gamma^2 - \gamma = -1\), which again implies that \(\phi(m_1), \phi(m_2), \phi(m_3) \in M\) are on the same line, a contradiction. Finally our theorem is established. \(\square\)
A result of Lin and Wolf [IS] guarantees the existence of a large $M \subseteq \mathbb{F}_q^k$ meeting our requirements.

**Theorem 6.1** ([IS]). Let $l$ be a positive integer. Let $\mathbb{F}_q$ be the finite field of $q$ elements such that $q \geq l$. Then there is a subset of $\mathbb{F}_q^{2l}$ of size $q^{2(l-1)} + q^{l-1} - 1$ that contains no $l$ points on a line.

**Theorem 6.2.** For sufficiently large $n$, it holds that $f_3(n, 9, 6) = cn^\frac{5}{8}$ for some constant $c > 0$.

**Proof.** Let $q = 6t + 1$ be a prime power and $\gamma$ be a primitive 6th root of unity in $\mathbb{F}_q$. Then it is easy to verify that $\gamma$ is a root of the equation $x^2 - x + 1 = 0$ in $\mathbb{F}_q$. Theorem 6.1 guarantees the existence of a set $M$ of size at least $q^l$ that contains no three points on a same line over $\mathbb{F}_q^6$. By applying Theorem 1.8 with $k = 6$ one can conclude that $f_3(q^6, 9, 6) \geq r(\mathbb{F}_q^6)q^k > q^{10}$. Thus the theorem follows from the distribution of prime powers. \qed

### 6.2 Classification of hypergraphs which are not $G_3(12,9)$-free

The goal of this subsection is to classify the possible configurations of $G_3(6,3)$-free hypergraphs which are not $G_3(12,9)$-free. The following Table 14 is a simple example.

|    | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ | $A_6$ | $A_7$ | $A_8$ | $A_9$ |
|----|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $V_1$ | $a$   | $a$   | $a$   | $f_1$ | $f_2$ | $f_3$ | $f_2$ | $f_3$ | $f_1$ |
| $V_2$ | $d_1$ | $d_2$ | $d_3$ | $b$   | $b$   | $d_1$ | $d_2$ | $d_3$ |
| $V_3$ | $e_1$ | $e_2$ | $e_3$ | $e_1$ | $e_2$ | $e_3$ | $c$   | $c$   |

Table 14: A $G_3(6,3)$-free but not $G_3(12,9)$-free 3-hypergraph

**Theorem 6.3.** Let $\mathcal{H}$ be a 3-uniform 3-partite linear hypergraph. Assume that $\mathcal{H}$ contains no rainbow cycles of length three. If there exist nine edges $A_1, \ldots, A_9$ of $\mathcal{H}$ such that $|A_1 \cup \cdots \cup A_9| \leq 12$, then $|A_1 \cup \cdots \cup A_9| = 12$ and $A_1, \ldots, A_9$ have only one possible configuration (up to isomorphism).

**Proof.** Let $A_1, \ldots, A_9$ be nine edges of $\mathcal{H}$ such that $|A_1 \cup \cdots \cup A_9| \leq 12$. The first step is to prove that the subgraph $\mathcal{A}$ formed by $A_1, \ldots, A_9$ is $G_3(9,6)$-free. Assume the opposite. Without loss of generality, let $A_1, \ldots, A_6$ be the six edges whose union contains at most nine vertices. Denote $X := A_1 \cup \cdots \cup A_6$ and $Y := A_7 \cup A_8 \cup A_9$. Note that $\mathcal{A}$ is of course $G_3(6,3)$-free. By Theorems 5.6 and 5.10 it holds that $|X| = 9$ and $|A_j \cap X| \leq 1$ for each $j \in \{7,8,9\}$. Thus we have $|X \cap Y| \leq 3$. By the inclusion-exclusion principle one can infer

$$12 = |X \cup Y| = |X| + |Y| - |X \cap Y|,$$

implying $|Y| = 12 - |X| + |X \cap Y| = 3 + |X \cap Y| \leq 6$, contradicting the fact that $\mathcal{A}$ is $G_3(6,3)$-free. Thus by the theorems in Section 5 we can conclude that $\mathcal{A}$ is $G_3(e + 3, e)$-free for all $e \in \{3,4,5,6,7,8\}$. Now Lemma 5.3 implies that the maximal degree of $\mathcal{A}$ is at most $|9/3| = 3$. 

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Claim 1. \( A \) contains no degree one vertex.

\textit{Proof.} Assume the opposite. Without loss of generality, let \( a_1 \in A_1 \) be a degree one vertex. Thus \( a \not\in A_2 \cup \cdots \cup A_9 \) and hence \( |A_2 \cup \cdots \cup A_9| \leq 11 \), contradicting the \( G_3(11, 8) \)-free property of \( A \).

Claim 2. The vertex set of \( A \) contains exactly three vertices of degree three and nine vertices of degree two.

\textit{Proof.} It is easy to see that \( V(A) \) contains exactly twelve vertices. Note that each vertex is of degree two or three. Then Claim 2 follows directly from the following obvious fact

\[ \sum_{v \in V(A)} \deg(v) = 3 \times 9 = 27. \]

Claim 3. Each edge of \( A \) contains at most one degree three vertex.

\textit{Proof.} Assume the opposite. Without loss of generality, let \( a \) and \( b \) be two vertices of \( A_1 \) with degree three. Let \( a \in V_1 \) and \( b \in V_2 \). Then we can draw the following Table 15. By the linearity and the \( G_3(6, 3) \)-free property of \( A \), it is easy to verify that \( A_1 \cap V_3, A_2 \cap V_3, A_3 \cap V_3, A_4 \cap V_3 \) and \( A_5 \cap V_3 \) must be all distinct. This is impossible since these five vertices all have degree at least two, which implies that \( A \) should contain at least ten edges, a contradiction.

|     | \( A_1 \) | \( A_2 \) | \( A_3 \) | \( A_4 \) | \( A_5 \) | \( A_6 \) | \( A_7 \) | \( A_8 \) | \( A_9 \) |
|-----|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| \( V_1 \) | a       | a       | a       | A_4     | A_5     | A_6     | A_7     | A_8     | A_9     |
| \( V_2 \) | b       |         | b       | b       |         |         |         |         |         |
| \( V_3 \) |         |         |         |         |         |         |         |         |         |

Table 15: An edge contains two vertices of degree three

Claim 4. Each vertex part contains at most one degree three vertex.

\textit{Proof.} Assume the opposite. Let \( V_1 \) be the vertex part which contains two degree three vertices, say, \( a \) and \( b \). We can draw the following Table 16.

|     | \( A_1 \) | \( A_2 \) | \( A_3 \) | \( A_4 \) | \( A_5 \) | \( A_6 \) | \( A_7 \) | \( A_8 \) | \( A_9 \) |
|-----|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| \( V_1 \) | a       | a       | a       | b       | b       |         |         |         |         |
| \( V_2 \) |         |         |         |         |         |         |         |         |         |
| \( V_3 \) |         |         |         |         |         |         |         |         |         |

Table 16: A vertex part contains two vertices of degree three

Observe that the three vertices \( A_7 \cap V_1, A_8 \cap V_1 \) and \( A_9 \cap V_1 \) should have degree at least two. The only possible situation is \( A_7 \cap V_1 = A_8 \cap V_1 = A_9 \cap V_1 \). Then all three degree three vertices of \( A \) are located in \( V_1 \).
However, for each $i \in \{1, \ldots, 9\}$, $A_i \cap V_2$ has degree at least two. It is easy to check that some vertex $A_i \cap V_2$ must have degree at least three, a contradiction.

By the claims above we can conclude that each vertex part contains exactly one degree three vertex of $A$. Moreover, each edge of $A$ contains at most one degree vertex. These two facts can be described by the following Table 17.

| $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ | $A_6$ | $A_7$ | $A_8$ | $A_9$ |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $V_1$ | a     | a     | a     |       |       |       |       |       |
| $V_2$ |       | b     | b     | b     |       |       |       |       |
| $V_3$ |       |       | c     | c     | c     |       |       |       |

Table 17: Distribution of the degree three vertices

For $i \in \{1, 2, 3\}$, assume $A_i \cap V_2 = \{d_i\}$, $A_i \cap V_3 = e_i$. It is routine to fill in the blanks in $V_2$ and $V_3$ of Table 17. It remains to fill in $V_1$. Let $A_4 \in V_1 = \{f_1\}$, $A_5 \cap V_1 = \{f_2\}$ and $A_6 \cap V_1 = \{f_3\}$. It is obvious that $A_7 \cap V_1, A_8 \cap V_1, A_9 \cap V_1 = \{f_1, f_2, f_3\}$. Due to the $G_3(6, 3)$-free property of $A$, one can verify that $A$ has only two possible configurations, which are

| $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ | $A_6$ | $A_7$ | $A_8$ | $A_9$ |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $V_1$ | a     | a     | a     | $f_1$ | $f_2$ | $f_3$ | $f_2$ | $f_3$ | $f_1$ |
| $V_2$ | $d_1$ | $d_2$ | $d_3$ | $b$   | $b$   | $d_4$ | $d_2$ | $d_3$ |
| $V_3$ | $e_1$ | $e_2$ | $e_3$ | $e_1$ | $e_2$ | $e_3$ | $c$   | $c$   | $c$   |

and

| $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ | $A_6$ | $A_7$ | $A_8$ | $A_9$ |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $V_1$ | a     | a     | a     | $f_1$ | $f_2$ | $f_3$ | $f_1$ | $f_2$ |
| $V_2$ | $d_1$ | $d_2$ | $d_3$ | $b$   | $b$   | $d_4$ | $d_2$ | $d_3$ |
| $V_3$ | $e_1$ | $e_2$ | $e_3$ | $e_1$ | $e_2$ | $e_3$ | $c$   | $c$   | $c$   |

7 A general upper bound for $f_r(n, v, e)$

We will give a recursive inequality which concerns the general behaviour of $f_r(n, v, e)$. Let $R = er - v$ and denote $T_r(n, R, e) := f_r(n, v, e)$. We find that it is more convenient to deal with $T_r(n, R, e)$ rather than $f_r(n, v, e)$. Also, it can be seen that the magnitude of $f_r(n, v, e)$ is more sensitive to $R$ rather than $v$.

Theorem 7.1 (Additive law). For any positive integer $l$ satisfying $1 \leq l \leq r$, it holds that

$$T_r(n, R, e) \leq T_r(n, R - l, e - 1) + \binom{n}{l}/\binom{r}{l}.$$  

Proof. For an $r$-uniform hypergraph $H$ on $n$ vertices, we claim that there exists a family $F_H \subseteq H$ with cardinality at most $\binom{n}{l}/\binom{r}{l}$ such that for any edge $A \in H \setminus F_H$ there exists an edge $B \in F_H$ satisfying $|B \cap A| \geq l$. To prove the claim, it suffices to notice that the maximum cardinality of a subset $F$ of $H$ satisfying the property that no two edges of $F$ can share more than $l - 1$ vertices is $\binom{n}{l}/\binom{r}{l}$. This is
because that $n$ vertices contain $\binom{n}{r}$ $l$-subsets and every $r$-uniform edge contains $\binom{l}{r}$ $l$-subsets, and every pair of distinct edges of $F$ do not share a common $l$-subset. We can get the desired family of the claim if we take $F_H := F$, since if there is some $A \in H \setminus F$ satisfying $|B \cap A| \leq l - 1$ for all $B \in F$, then $A$ can be added into $F$, contradicting the maximality of $F$.

Let $H$ be a $G_r(v, e)$-free hypergraph, where $v = er - R$. Denote $H' = H \setminus F_H$. By the claim above, it is obvious that $|H| = |H'| + |F_H| \leq |H'| + \binom{n}{r}/\binom{l}{r}$. Therefore, to establish the theorem, it suffices to show that $H'$ is $G_r(v', e - 1)$-free with $v' = (e - 1)r - R + l$, since $(e - 1)r - v' = R - l$. Assume, to the contrary, that there exist $A_1, \ldots, A_{e - 1} \in H'$ satisfying $|A_1 \cup \cdots \cup A_{e - 1}| \leq (e - 1)r - R + l$. By the definition of $F_H$, we can always find some $B \in F_H$ such that $|B \cap A_1| \geq l$. Thus we can conclude that

$$|B \cup (\cup_{i=1}^{e-1} A_i)| = |B| + |\cup_{i=1}^{e-1} A_i| - |B \cap (\cup_{i=1}^{e-1} A_i)|$$

$$\leq r + (e - 1)r - R + l = er - R = v,$$

which contradicts the fact that $H$ is $G_r(v, e)$-free. □

Theorem 7.1 has several simple consequences, in the following we list three of them.

**Corollary 7.2.** For any positive integer $l$ satisfying $1 \leq l \leq r$, it holds that

$$f_r(n, v, e) \leq f_r(n, v - r + l, e - 1) + \binom{n}{l}/\binom{r}{l}.$$

**Proof.** Note that $T_r(n, R - l, e - 1) = f_r(n, v - r + l, e - 1)$. □

**Corollary 7.3.** Assume that $R = p(e - 1) + q$, where $1 \leq q \leq e - 1$. Then it holds that

$$T_r(n, R, e) \leq q \left(\frac{n}{p + 1}\right)\left(\frac{r}{p + 1}\right) + (e - 1 - q)\left(\frac{n}{p}\right)\left(\frac{r}{p}\right).$$

**Proof.** One can verify that $R = q(p + 1) + (e - 1 - q)p$. We can apply Theorem 7.1 repeatedly for $e - 1$ times, in which $l$ is chosen to be $p + 1$ for $q$ times and to be $p$ for $e - 1 - q$ times. The theorem then follows from a simple fact that $R_r(n, 0, \{1\}) = 0$. □

**Proof of Theorem 7.9.** This result follows from Corollary 7.2. The proof is similar to that of Corollary 7.3. □

8 Concluding remarks

In this paper we investigate the famous conjecture of Brown, Erdős and Sós on sparse hypergraphs.

For the upper bound part of the conjecture, using hypergraph removal lemma, we prove $f_3(n, e(r-k)+k+1, e) = o(n^k)$ holds for all positive integers $r \geq k+1 \geq e$. The first uncovered case is $r = 3$, $k = 2$ and $e = 4$. In the literature, the determination of the order of $f_3(n, 7, 4)$ is usually termed the $(7,4)$-problem. We call $G \in G_3(e + 3, e)$ an $(e + 3, e)$-configuration. One can argue that proving
$f_3(n, 7, 4) = o(n^2)$ is equivalent to proving that every 3-uniform linear hypergraph with $\Omega(n^2)$ edges must contain two $(6,3)$-configurations with two common edges. However, using the removal lemma one can only guarantee the existence of two $(6,3)$-configurations with only one common edge. Therefore, we suspect that we need more powerful tools to attack the $(7,4)$-problem.

For the lower bound part of the conjecture, on one hand we show that hypergraphs with no rainbow cycles of lengths three or four are good candidates for sparse hypergraphs. On the other hand, using the tools from additive number theory, we develop a general method to construct hypergraphs with no rainbow cycles. Note that we only prove that our constructions meet the conjectured lower bound for $r \geq 3$, $k = 2$ and $e = 3, 4, 5, 7, 8$. We strongly suspect that our construction (with some modifications) can attain the conjectured lower bound for more general parameters. It may be interesting to continue the research on this line.

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