Genus Two Siegel Quasi-Modular Forms and Gromov–Witten Theory of Toric Calabi–Yau Threefolds

Yongbin Ruan\textsuperscript{1}, Yingchun Zhang\textsuperscript{2}, Jie Zhou\textsuperscript{3}

\textsuperscript{1} Institute of Advanced Study for Mathematics, Zhejiang University, Hangzhou, People’s Republic of China. E-mail: yongbin.ruan@yahoo.com

\textsuperscript{2} Department of Mathematics, University of Michigan, 2074 East Hall, 530 Church Street, Ann Arbor, MI 48109, USA. E-mail: zyingchu@umich.edu

\textsuperscript{3} Yau Mathematical Sciences Center, Tsinghua University, Beijing 100084, People’s Republic of China. E-mail: jzhou2018@mail.tsinghua.edu.cn

Received: 27 June 2020 / Accepted: 17 September 2022
Published online: 12 November 2022 – © The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2022

Abstract: We first formulate theories of differential rings of Siegel quasi-modular and Siegel quasi-Jacobi forms for genus two. Then we apply them to Eynard–Orantin topological recursion of the toric Calabi–Yau threefold $\mathbb{C}^3/\mathbb{Z}_6$ equipped with brane whose mirror curve is a genus-two hyperelliptic curve. By the proof of the Remodeling Conjecture, we prove that the corresponding open- and closed- Gromov–Witten potentials are essentially Siegel quasi-Jacobi and Siegel quasi-modular forms for genus two, respectively.

Contents

1. Introduction ........................................ 758
   1.1 Differential ring of Siegel quasi-modular forms for genus two ............ 759
   1.2 Modularity in Eynard–Orantin topological recursion and in open-closed GW theory ........................................ 762

2. Review on Genus Two Curves and Their Jacobians ................ 763
   2.1 Hyperellipticity of genus two algebraic curves and their Jacobians .... 763
   2.2 Weierstrass $\sigma$-function and functional field of hyperelliptic Jacobian . 765
   2.3 Families of genus two curves ................................ 767
      2.3.1 Scalar-valued Siegel modular forms. .......................... 767
      2.3.2 Binary invariants, Igusa absolute invariants, and Torelli map. 768
      2.3.3 Other models: quintic and Rosenhain normal form. ........ 769

3. Differential Rings of Quasi-Modular and Quasi-Jacobi Forms for Genus Two 770
   3.1 Quasi-modular forms .................................... 771
      3.1.1 Quasi-periods: archetype. ................................. 771
      3.1.2 Sheaf-theoretic and analytic definitions of quasi-modular forms. 773
      3.1.3 Ring structure inherited from representations. ............. 775
      3.1.4 Almost-meromorphic modular forms. ........................ 776
      3.1.5 Differential structure from Picard–Vessiot extension. ....... 777
1. Introduction

One of the most exciting aspects of Gromov–Witten (GW) theory is its interaction with the theory of modular forms. Now it is well accepted from the B-model of mirror symmetry that the generating functions of GW theory carry certain non-holomorphic signature in the sense that they should be viewed as certain holomorphic limits of non-holomorphic generating functions. It is amazing that it coincides with the non-holomorphic signature in the theory of the so-called quasi-modular forms. We should mention that there is similar and yet different kind of non-holomorphic signature in mock modular forms which made its appearance recently in quantum K-theory.

There is a great deal of literature discussing this connection between GW theory and quasi-modular forms since the works [Dij95, KZ95, OP06] discussing the elliptic curve case. Despite of many surprising progresses that have been made, so far most of the results on quasi-modularity are established only for the cases where the dimension of the moduli space is one. The present work is an attempt to push the modularity phenomenon to higher-dimensional moduli spaces. An attractive place to start our program is toric Calabi–Yau (CY) 3-folds whose B-models are described by algebraic curves. Therefore, it is natural to expect that there should be a marriage between GW theory of toric Calabi–Yau 3-folds and Siegel modular forms. However, a moment of thought suggests that this expectation may be premature. The natural domain of B-model GW generating functions is the moduli space $M_g$ of complex structures for mirror curves of genus $g$. On the other hand, the natural domain of Siegel modular forms is the moduli space $A_g$ of dimension-$g$ Abelian varieties which has a higher dimension than that of $M_g$ except for $g = 1, 2$ cases. In addition to the classical case $g = 1$, it leaves the only other possibility at $g = 2$. In this article, we try to solve the case $g = 2$. We focus on a concrete example $\mathbb{C}^3/\mathbb{Z}_6$ which is the most frequently studied one in the literature. However, the general set-up for modular and quasi-modular forms developed in this article should apply to any toric Calabi–Yau 3-fold whose mirror curve family is a 3-dimensional family of genus 2 curves equipped with hyperelliptic structures. It would be a very interesting problem to classify these examples. In a future work, we plan to continue our search

---

1 We refer the interested readers to the introduction part of [SZ17] for an incomplete list of them.

2 Throughout this work, by modular forms and quasi-modular forms we mean Siegel modular forms and Siegel quasi-modular forms for genus two respectively, unless stated otherwise. Similar nomenclature is used for other variants of modular forms and for variants of Jacobi forms. See Definition 3.4, Definition 3.5, Definition 3.6, Definition 3.7, and Definition 3.13 for their precise definitions.
of modularity in the GW theory of quotients of K3 surfaces where there is even greater room for progress.

Technically, one of the main difficulties in generalizing the quasi-modularity investigations to higher dimensional moduli spaces appearing as Hermitian symmetric spaces is that it is not clear what the appropriate notion of "quasi" is (see an early work [KPSWR16]). In the first half of this paper, we deal with the case of the moduli space of genus two Riemann surfaces by formulating a theory of differential ring of Siegel quasi-modular forms. One of the main results is that the differential ring is finitely generated by explicit generators. Our formulation is a combination of various perspectives of periods and quasi-periods such as those in complex differential geometry, differential equations, representation theory and Hodge theory, and is based on earlier results from [Igu60, Igu62, Igu67, BZ00, vdG08, Urb14]. The second half of the paper is devoted to the application of quasi-modular forms to the GW theories of toric Calabi–Yau 3-folds whose B-model mirror theories are governed by Eynard–Orantin topological recursion [EO07] on genus two mirror curves, thereby extending the earlier results in [FRZZ19] from toric CYs with genus one mirror curves to those with genus two ones.

1.1. Differential ring of Siegel quasi-modular forms for genus two. Our main objects of study are various geometric objects defined by Eynard–Orantin topological recursion. The bulk of our work is to bridge the gap between these geometric objects and number-theoretic modular forms. We do so by developing an intermediate Hodge-theoretic theory of modular forms which we believe should be useful in other setting such as K3 period domains.

To state our results, we first need to recall some notations. Let

\[ H = \{ \tau \in M_2(\mathbb{C}) \mid \text{Im} \tau > 0, \tau^t = \tau \} \]  

be the Siegel upper-half space of genus two. Let \( \Gamma \) be a congruence subgroup of the full Siegel modular group \( \Gamma(1) = \text{Sp}_4(\mathbb{Z}) \) that acts on \( H \) by

\[ \tau \mapsto \gamma \tau = (a \tau + b)(c \tau + d)^{-1}, \quad \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma. \]  

(1.2)

Here to avoid technicality we assume that it has no torsion, later this assumption will be lifted. Let \( S = \Gamma \backslash H \) be the corresponding Siegel modular variety which by Baily-Borel [BB66] admits the structure of a quasi-projective algebraic variety, and \( \pi : Z \to S \) be the universal family of Abelian varieties with level structure determined by \( \Gamma \). Define the following locally free sheaves (or vector bundles) on \( S \)

\[ \omega = R^0 \pi_* \Omega^1_{Z|S}, \quad \mathcal{V}_S = R^1 \pi_* \mathcal{C}_Z \otimes \mathcal{O}_S, \]  

(1.3)

where \( \Omega^1_{Z|S} \) is the sheaf of relative differentials.

Generalizing the discussions in [vdG08, Urb14], we arrive at a sheaf-theoretic definition of quasi-modular forms.

**Definition 1.1 (Definition 3.1).** Let the notations be as above. A weakly holomorphic quasi-modular form of weight \( k \) and order \( m \geq 0 \) for the modular group \( \Gamma \) is a holomorphic section of \( \text{Sym}^{\otimes (k-m)} \omega \otimes \text{Sym}^{\otimes m} \mathcal{V}_S \).

---

3 Here we are following the terminology used for the genus one case (see e.g. [Zag08]) in which "weak holomorphicity" stands for holomorphicity away from \( \bar{S} - S \) where \( \bar{S} \) is a certain compactification of \( S \).
By working with a particular frame adapted to the Hodge filtration on the pull-back sheaves along \( H \to S \), global sections of \( \text{Sym}^\otimes(k-m) \omega \otimes \text{Sym}^m \mathcal{V}_S \) can be described in terms of the coordinates of the corresponding pull-back sections. As explained in [Urb14], this particular frame is naturally motivated from the monodromy weight filtration around a singular fiber in the family \( \pi : \mathcal{Z} \to S \). This then yields an equivalent analytic definition of quasi-modular forms in the spirit of [KZ95].

**Definition 1.2 (Definition 3.2).** Let the notations be as above. A weakly holomorphic quasi-modular form of weight \( k \) and order \( m \) for the modular group \( \Gamma \) is a collection of holomorphic functions \( \{ f_i(\tau) \}_{i=0,1,...,m} \) on \( H \) which satisfy the transformation law

\[
f_i(\tau) = \sum_{\ell=0}^m \left( \begin{array}{c} \ell \\ i \end{array} \right) f_\ell(\gamma \tau) ((c \tau + d)^{-1})^{\otimes(k-\ell)}(-c')^{\otimes(\ell-i)}((c \tau + d)^{d})^{\otimes i}, \quad \forall \gamma \in \Gamma.
\]

In particular, weakly holomorphic quasi-modular forms with \( m = 0 \) are weakly holomorphic, vector-valued modular forms corresponding to global sections of the sheaves \( \text{Sym}^{\geq 0} \omega := \{ \text{Sym}^k \omega \}_{k \geq 0} \).

Using the further structure of \( \mathcal{H} = \text{Sp}_4(\mathbb{R})/\mathbb{U}_2 \) as a Hermitian symmetric domain, the bundles above can be regarded as homogeneous vector bundles defined by representations of \( \mathbb{U}_2 \). The bundle \( \omega \) corresponds to the fundamental representation, while \( \mathcal{V}_S \) the trivial representation. Moreover, one has the following elementary fact

\[
\omega^{\otimes 2} = \text{Sym}^{\otimes 2} \omega \oplus \wedge^2 \omega, \quad \wedge^2 \omega = \text{det} \omega.
\] (1.4)

This then tells that meromorphic scalar-valued modular forms correspond to global sections of the sheaves \( \text{det}^\bullet \omega \), which are studied thoroughly by Igusa in [Igu60, Igu62, Igu64, Igu67].

By the Weyl character formula, any holomorphic representation of \( \mathbb{U}_2 \) decomposes into a direct sum whose summands lie in \( \text{Sym}^\bullet \omega \), \( \text{det}^\bullet \omega \). Therefore, the set of representations \( \{ \text{det}^\bullet \omega \otimes \text{Sym}^\bullet \omega \otimes \oplus, \otimes \} \) carries a module structure over \( \{ \text{det}^\bullet \omega, \oplus, \otimes \} \). See [vdG08] for related discussions. Hence in addition to a direct examination by using the analytic definition, the representation-theoretic description leads to another way of showing the ring structure of meromorphic vector-valued modular forms graded by the representations \( \text{Sym}^\bullet \omega \), \( \text{det}^\bullet \omega \). Furthermore, since \( \mathcal{V}_S \) corresponds to the trivial representation, similar module structure holds for \( \{ \text{det}^\bullet \omega \otimes \text{Sym}^\bullet \omega \otimes \text{Sym}^{\geq 0} \mathcal{V}_S \} \).

For practical reasons, in this paper we shall work analytically with a collection of subspaces of \( H^0(S, \text{det}^\bullet \omega \otimes \text{Sym}^\bullet \omega \otimes \text{Sym}^{\geq 0} \mathcal{V}_S) \) given by

\[
H^0(S, \text{det}^\bullet \omega \otimes \text{Sym}^{\geq 0} H^0(S, \omega) \otimes \text{Sym}^{\geq 0} H^0(S, \mathcal{V}_S)).
\] (1.5)

Note that the subspace we consider here is finitely generated, while the full collection \( H^0(S, \text{det}^\bullet \omega \otimes \text{Sym}^\bullet \omega \otimes \text{Sym}^{\geq 0} \mathcal{V}_S) \) is not by Grundh [vdG08].

**Definition 1.3 (Definition 3.4, Definition 3.5).** Let \( R(\Gamma) \) be the fractional ring of the graded ring of scalar-valued modular forms for \( \Gamma \). Define a subring of vector-valued modular forms to be the \( R(\Gamma) \)-module given by

\[
\mathcal{R}(\Gamma) = R(\Gamma)[\text{Sym}^{\geq 0} H^0(S, \omega)]
\] (1.6)

Define a subring of quasi-modular forms to be the \( \mathcal{R}(\Gamma) \)-module given by

\[
\mathcal{R}(\Gamma) = R(\Gamma)[\text{Sym}^{\geq 0} H^0(S, \omega), \text{Sym}^{\geq 0} H^0(S, \mathcal{V}_S)].
\] (1.7)

\(^4\) Here \( H^0(X, -) \) denotes the global section functor.
Convention 1.4. Hereafter unless stated otherwise, by modular forms we mean elements in $R(\Gamma)$ and by quasi-modular forms elements in $\tilde{R}(\Gamma)$.

The sheaf-theoretic origin of Definition 1.3 above not only gives the ring structure by construction, but also allows to describe the differential structure in terms of very concrete geometric terms. We now briefly explain this. Denote the rational functional field of the quasi-projective algebraic variety $S$ by $k(S)$. Up to algebraic extension, $k(S)$ is the same as the rational functional field of $\Gamma(1)\backslash \mathcal{H}$ which admits very explicit presentation via scalar-valued Siegel modular forms [Igu62, Igu64], as shall be reviewed in Sect. 2.3 below. We take a set of three algebraically independent generators $\{t_1, t_2, t_3\}$ from $k(S)$ and denote them collectively by $t$. Hence up to algebraic extension $k(S)$ coincides with $k(t_1, t_2, t_3)$. Denote the differentials $\partial_{\tau_{ij}}, i, j = 1, 2$ collectively by $\partial_{\tau}$ and similarly for $\partial_{t}$. Taking a weak Torelli marking $\{A, B\}$ and a frame $\omega, \eta$ adapted to the Hodge filtration of the universal family $\pi : Z \rightarrow S$ of Abelian varieties. Let $\Pi_A(\omega), \Pi_A(\eta)$ be the $A$-cycle periods and quasi-periods.

**Theorem 1.5** (Theorem 3.8 [BZ00], Theorem 3.9 [BZ00], Corollary 3.11, Corollary 3.14). Let the notations be as above. Let $\mathcal{D}$ be the differential ring obtained by adjoining to $k(S)$ all of the $\partial_{\tau_1}^n, n \geq 0$ derivatives of rational functions in $k(S)$.

1. The differential ring $\mathcal{D}$ is also stable under $\partial_t$. The fractional field of $\mathcal{D}$ has transcendental degree 10 over $\mathbb{C}$.
2. Up to algebraic extension, the differential ring $\mathcal{D}$ is generated over $k(S)$ by the entries of

$$\partial_{\tau} t, \quad \partial_{t}^2 t.$$  (1.8)

Elements in $k(S)[\partial_{\tau} t]$ are modular forms for $\Gamma$ while $\partial_{t}^2 t$ quasi-modular.

3. Up to algebraic extension, the differential ring $\mathcal{D}$ is identical to the ring of quasi-modular forms:

$$\mathcal{D} = \tilde{R}(\Gamma), \quad \text{up to algebraic extension.}$$  (1.9)

Let $\Pi_A(\omega)$ and $\Pi_A(\eta)$ be the periods and quasi-periods associated to the family $\pi : Z \rightarrow S$, respectively. The generators of $\tilde{R}(\Gamma)$ over $k(S)$ can be taken to be the entries of

$$\Pi_A(\omega), \quad \Pi_A(\eta),$$  (1.10)

subject to the only relation given by the 1st Riemann–Hodge bilinear relation

$$\Pi_A(\omega)\Pi_A(\eta) = \Pi_A(\eta)\Pi_A(\omega).$$  (1.11)

Elements in $k(S)[\Pi_A(\omega)]$ are modular forms for $\Gamma$, while $\Pi_A(\eta)$ is quasi-modular with the obstruction of modularity given by the 2nd Riemann–Hodge bilinear relation/Legendre period relation.

4. Up to algebraic extension, the differential ring $\mathcal{D}$ is generated over $k(S)$ by

$$\partial_{\tau} \log f, \quad [z^0] \partial_{z}^2 \log \vartheta_{\delta},$$  (1.12)

where $f$ is any nonzero element in $k(S)$, $\vartheta_{\delta}$ is any Jacobi theta function with an odd characteristic $\delta$ defined on dimension two principally polarized Abelian varieties, and $[z^0] \partial_{z}^2 \log \vartheta_{\delta}$ stands for the degree zero term in the Laurent expansion of $\partial_{z}^2 \log \vartheta_{\delta}$. Elements in $k(S)[\partial_{\tau} \log f]$ are modular forms for $\Gamma$, while $[z^0] \partial_{z}^2 \log \vartheta_{\delta}$ is quasi-modular.
We should mention that an attempt in developing a theory of quasi-modular forms for higher genus cases was made in [KPSWR16] using differential operators on vector-valued modular forms. In fact, they defined a ring of almost-holomorphic Siegel modular forms which is graded by certain types of representations. As far as we understand, there is no finite generation result for a theory of ring of quasi-modular forms in their setting. In addition to our finite generation theorem, our results differ from [KPSWR16] and other previous works in that we provide various realizations of the differential ring from various perspectives such as the connection to quasi-periods. These aspects are crucial to understand points such as the reason behind the failure of modularity. More importantly, the ring of quasi-modular forms that we build is universal in a certain sense (detailed in Sect. 3.1.5) and can be generalized to other cases such as the moduli space of principally polarized Abelian varieties of higher dimensions.

1.2. Modularity in Eynard–Orantin topological recursion and in open-closed GW theory. The second half of the paper is a continuation of the paper [FRZZ19]. When the mirror curve of the toric CY is of genus one, a detailed analysis of Eynard–Orantin topological recursion in terms of quasi-modular forms and their Jacobian cousins is performed therein.

Using the theory of quasi-modular forms developed earlier and the results of [Gra85, Gra90] for the differential field of rational functions of genus two hyperelliptic Jacobians, we can then define a ring of meromorphic quasi-Jacobi forms (see Definition 3.13). Then we generalize the previous analysis in [FRZZ19] to Eynard–Orantin topological recursion for the cases where the curve family is a 3-dimensional family of genus-two curves equipped with hyperelliptic structures. Specializing to a particular genus-two curve family, namely the one arising from the mirror of the resolution $X$ of $\mathbb{C}^3/\mathbb{Z}_6$, these results on quasi-modularity in Eynard–Orantin topological recursion immediately apply to the open-closed GW theory of $X$ thanks to the proof of the Remodeling Conjecture [BKMnP09, FLZ20]. That is, the open and closed GW potentials produced from Eynard–Orantin topological recursion are pull-backs of meromorphic quasi-Jacobi forms along the Abel–Jacobi map and quasi-modular forms, respectively. Part of the main results is phrased as follows.

**Theorem 1.6 (Theorem 4.6).** Let $\mathcal{X}$ be the resolution of the toric CY 3-fold $\mathbb{C}^3/\mathbb{Z}_6$. Let $\pi : \mathcal{C} \rightarrow S$ be its compactified mirror curve family of genus two with generic fiber $C$. The open GW potentials $\omega_{g,n}, 2g - 2 + n > 0, n > 0$ are pull-backs of meromorphic quasi-Jacobi forms along the Abel–Jacobi map $\phi : C \rightarrow \text{Pic}^0(C)$ based at a Weierstrass point on $C$. The closed GW potentials $F_g = \omega_{g,0}, g \geq 2$ are quasi-modular forms.

Our results not only offers an effective way to package the open- and closed- GW invariants in terms of nice quasi-Jacobi forms and quasi-modular forms, but also provides a promising tool in analyzing the global behavior (such as the singularities and the domain of convergence) of the open-closed GW potentials. Furthermore, it offers some hints about the global property of Kähler moduli space and of the open-moduli space in open GW theory whose rigorous definition has so far resisted various attempts.

**Organization of the paper.** In Sect. 2 we review the basics on genus two algebraic curves and the function theory of their Jacobian varieties, as well as some properties of different families of genus two algebraic curves that will be used in later sections. Readers who are familiar with these can skip this section.
In Sect. 3 we formulate the Hodge-theoretic theory of quasi-modular forms for genus two. We also define a differential ring of meromorphic quasi-Jacobi forms, which includes the Bergman kernel as a generator and is suited for the purpose of studying Eynard–Orantin topological recursion.

In Sect. 4 we extend the analysis in [FRZZ19] to prove the quasi-modularity in Eynard–Orantin topological recursion for genus two algebraic curves. We apply the discussions to the mirror curve of a specific toric CY 3-fold and obtain the quasi-modularity of its open-closed GW theory using the Remodeling Conjecture.

Due to the computational nature of Eynard–Orantin topological recursion, we have strived to make the paper self-contained by collecting some useful facts and computations in the body of the paper.

2. Review on Genus Two Curves and Their Jacobians

We review some basics about genus two curves and their Jacobians, following [Mum83, Mum84, Gra85, Gra90, Mum99, BL13, GH14].

2.1. Hyperellipticity of genus two algebraic curves and their Jacobians. Let \( C \) be a smooth complex curve of genus \( g \). Let \( C^{(n)} \) be the \( n \)th symmetric product of \( C \). Consider the map

\[
\rho_n : C^{(n)} \to \text{Pic}^n(C), \quad (p_1, p_2, \ldots, p_n) \mapsto \mathcal{O}(p_1 + p_2 + \cdots + p_n). \tag{2.1}
\]

When \( n = g \), \( \rho_n \) gives a birational map. When \( n = g - 1 \), the image of \( \rho_n \) is a divisor.

Fix a degree \( n \) divisor \( D_n = nc \) on the curve \( C \) for some point \( c \) on \( C \). Consider the Abel–Jacobi map

\[
\alpha_n : C^{(n)} \to J(C) := \text{Pic}^0(C), \quad (p_1, p_2, \ldots, p_n) \mapsto \mathcal{O}(p_1 + p_2 + \cdots + p_n) \otimes \mathcal{O}(-D_n). \tag{2.2}
\]

When \( n = 1 \), one can show that \( \alpha_1 \) is an embedding. When \( n = g - 1 \), Riemann’s theorem says that one has

\[
\rho_{g-1}(C^{(g-1)}) = \kappa + \Theta \tag{2.3}
\]

for some theta divisor \( \Theta \) on \( J(C) \) defining a principal polarization on \( J(C) \) and a line bundle \( \kappa \in \text{Pic}^{g-1}(C) \). If \( \Theta \) is symmetric, that is \((-1)^g \Theta = \Theta\) where \(-1\) is the involution on \( J(C) \), then \( \kappa \) is a theta characteristic satisfying \( \kappa^2 = K_C \). See the textbooks [BL13, GH14] for more details on this.

We now restrict ourselves to the genus two case which is the main interest of this work. It is a classical result that any genus two curve \( C \) is hyperelliptic. In fact, the curve can be realized as

\[
Y^2 = F(X, Z) := \sum_{k=0}^{6} a_k X^{6-k} Z^k \tag{2.4}
\]

for some degree 6 polynomial \( F \) in \( X, Z \) with \( a_0 \neq 0 \). The polynomial \( F \) can be factored to be

\[
F(X, Z) = a_0 \prod_{k=0}^{5} (X - r_k Z), \tag{2.5}
\]
where \( r_k, k = 0, \ldots, 5 \) are the roots. In the above coordinates, the hyperelliptic cover \( C \to \mathbb{P}^1 \) is given by \([X, Y, Z] \mapsto [X, Z]\). The ramification points \([X, Y, Z] = [r_k, 0, 1]\) for the hyperelliptic cover have intrinsic meaning: they are exactly the Weierstrass points on the hyperelliptic curve \( C \).

By applying a \( \text{PGL}_2(\mathbb{C}) \)-action on the base \( \mathbb{P}^1 \) of the above hyperelliptic cover, we can assume that \( r_0 = \infty \). Then the affine curve \( C \setminus \{r_0\} \) can be represented by an equation

\[
y^2 = f(x; b) := x^5 + b_1 x^4 + b_2 x^3 + b_3 x^2 + b_4 x + b_5 = \prod_{k=1}^{5} (x - e_k). \tag{2.6}
\]

The hyperelliptic involution, denoted by \( \sigma \) below, is given by

\[
\sigma : C \to C, \quad (x, y) \mapsto (x, -y). \tag{2.7}
\]

The model (2.6) remains in its form under the change of variables

\[
x \mapsto s^2 x + r, \quad y \mapsto s^5 y, \tag{2.8}
\]

where \( s \neq 0, r \) are complex numbers. The curve \( C \) is the normalization of the projective closure of \( C \setminus \{r_0\} \) in \( \mathbb{P}^2 \). Its projective algebraic structure is the one obtained by gluing another patch with \( C \setminus \{r_0\} \) in the usual way, see e.g. [Gra90, Corollary 2.15]. Alternatively, the curve can be embedded into the weighted projective space \( \mathbb{W} \mathbb{P}^2[1, 3, 1] \). In the corresponding weighted homogeneous coordinates the point \( r_0 \), which we shall also regard as a root of the quintic \( f \) in (2.6) and denote by \( e_0 \), is given by \([1, 0, 0]\).

In the current genus two case, the map \( \rho_2 : C^{(2)} \to \text{Pic}^2(C) \) reviewed in (2.1) is the map that blows down the locus of unordered pairs

\[
\{(p, \sigma(p)) | p \in C\} \tag{2.9}
\]
to \( K_C \). The curve itself is embedded into \( \text{Pic}^2(C) \) via the map \( \rho_1 \).

We now make the Abel–Jacobi map in (2.2) more explicit for the quintic model (2.6), following [Gra90]. We take the following basis for the space \( H^0(C, K_C) \) of the 1st kind Abelian differentials

\[
\omega_1 = \frac{dx}{2y}, \quad \omega_2 = \frac{xdx}{2y}. \tag{2.10}
\]

There is a standard and canonical way to pick a symplectic basis \( \{A_1, A_2, B_1, B_2\} \) for \( H_1(C, \mathbb{Z}) \), with the dual basis \( \{\alpha_1, \alpha_2, \beta_1, \beta_2\} \) for \( H^1(C, \mathbb{Z}) \). We shall denote these bases by \( \omega = (\omega_1, \omega_2)^t, (A, B) \) for simplicity. Using these data one can form the period matrix

\[
\begin{pmatrix}
\int_{A_1} \omega_1 & \int_{A_2} \omega_1 \\
\int_{A_1} \omega_2 & \int_{A_2} \omega_2 \\
\int_{B_1} \omega_1 & \int_{B_2} \omega_1 \\
\int_{B_1} \omega_2 & \int_{B_2} \omega_2
\end{pmatrix} = \left( \Pi_A(\omega), \Pi_B(\omega) \right). \tag{2.11}
\]

It is a standard fact that \( \Pi_A(\omega) \) is nondegenerate for the smooth curve \( C \). We shall then use the familiar notation for the normalized matrix

\[
\tau = \Pi_A(\omega)^{-1} \Pi_B(\omega). \tag{2.12}
\]

It lies on the upper-half space \( \mathcal{H} \) consisting of symmetric \( 2 \times 2 \) complex matrices with positive definite imaginary parts.

Let \( \Lambda_\tau = \mathbb{Z}^2 \oplus \tau \mathbb{Z}^2 \) be the lattice in \( \mathbb{C}^2 \). The Jacobian \( J(C) \) is then identified with the Albanese variety \( \mathbb{C}^2/\Lambda_\tau \) via integration with respect to the basis \( \omega_1, \omega_2 \). Choosing
the reference point $c$ in the definition of the Abel–Jacobi map in (2.2) to be $e_0 = \infty$, then $O_C(2e_0) = K_C$. That is, $e_0$ gives a theta-characteristic. Then we have a morphism from $C^{(2)}$ to the Jacobian $\mathbb{C}^2/\Lambda_\tau$ via

$$\Phi : (p_1, p_2) \mapsto u = \left( \begin{array}{c} u_1 \\ u_2 \end{array} \right) = \left( \begin{array}{c} \int_{\infty}^{p_1} + \int_{\infty}^{p_2} \omega_1 \\ \int_{\infty}^{p_1} + \int_{\infty}^{p_2} \omega_2 \end{array} \right).$$

(2.13)

The curve $C$ is embedded into $J(C)$ by

$$\phi : p \mapsto \Phi(p, \infty).$$

(2.14)

By construction, the image $\Theta$ gives a principal polarization on $J(C)$. The line bundle $O_{J(C)}(\Theta)$ is ample and the linear system $|m\Theta|$, $m \gg 0$ gives an embedding for the principally polarized Abelian variety $(J(C), \Theta)$. In fact, the latter is embedded into a projective space when $m \geq 3$, see [Mum83].

The holomorphic section of $O_{J(C)}(\Theta)$, unique up to multiplication by a constant, is given by a theta function with certain characteristic. Theta functions with characteristics $(a, b)$ are defined on the Jacobian $\mathbb{C}^2/\Lambda_\tau$ in the usual way, see [Mum83],

$$\vartheta(a, b)(v, \tau) = \sum_{n \in \mathbb{Z}^2} e^{\pi i (n+a)^t \tau (n+a) + 2\pi i (n+a)^t (v+b)}, \quad a, b \in \mathbb{Q}^2.$$

(2.15)

They are Jacobi forms with multiplier systems [EZ84] which enjoy nice transformation properties. We refer the interested readers to our earlier work [FRZZ19] for a short exposition to its definition and properties which should be enough for the purpose of this work.

It turns out that with the above particularly chosen frame (2.10) and carefully selected marking $(A, B)$, one has

$$\Theta = (\vartheta_\delta), \quad \delta = \left( \begin{array}{c} 1/2 \\ 1/2 \end{array} \right).$$

(2.16)

A different choice of the base point $c$ from the set of Weierstrass points corresponds to a different theta function with an odd characteristic from a total number of 6. The above carefully chosen one is convenient in performing many computations.

For any such choice of $c$, the ramification points on $C$ are mapped to 2-torsion points on $J(C)$ by construction. In particular, the point $e_0 := \infty$ is sent to the origin in $J(C)$. These are the only 2-torsion points lying on $\phi(C)$. In fact, the images generate the whole group of 2-torsion points on $J(C)$. See [Gra85, Gra90] for detailed computations on the corresponding theta-characteristics.

2.2. Weierstrass $\sigma$-function and functional field of hyperelliptic Jacobian. For later use, we also review some basics of hyperelliptic $\sigma$-functions for the genus two curve in (2.6) following [Gra90].

We take the following 2nd kind Abelian differentials on $C$ due to Baker [Bak07]

$$\eta_1 = \frac{(3x^3 + 2b_1x^2 + b_2x)dx}{2y}, \quad \eta_2 = \frac{x^2dx}{2y}.$$  (2.17)
These differentials are holomorphic except at $\infty$. The differentials $\eta = (\eta_1, \eta_2)^t$ are dual to the ones $\omega$ in (2.10) under the Poincaré residue pairing
\[ \langle \omega, \eta \rangle = 2\pi i 1. \] (2.18)

We organize the integrals over the $A$-cycles by
\[ \Pi_A(\eta) = \left( \int_{A_1} \eta_1 \int_{A_2} \eta_1 \int_{A_1} \eta_2 \int_{A_2} \eta_2 \right). \] (2.19)
The entries are called quasi-periods.

The 2-dimensional Weierstrass $\sigma$-function is defined by
\[ \sigma(u, \Pi_A(\omega), \Pi_B(\omega)) = e^{-\frac{1}{2}u^t \Pi_A(\eta) \Pi_A(\omega)^{-1} u} \partial_\delta (\Pi_A(\omega)^{-1} u, \Pi_A(\omega)^{-1} \Pi_B(\omega)). \] (2.20)

Similar to the genus one curve, we define the 2-dimensional Weierstrass $\wp$-functions by
\[ \wp_{ij}(u) = -\partial_{u_i} \partial_{u_j} \log \sigma(u, \Pi_A(\omega), \Pi_B(\omega)). \] (2.21)

One can consider furthermore higher derivatives such as $\wp_{ijk} = \partial_{u_k} \wp_{ij}$.

Let $k = \mathbb{C}$ and denote $\partial_u := \{ \partial_{u_1}, \partial_{u_2} \}$ and $k_{\partial_u} \langle \wp_{ij} \rangle$ to be the differential closure of $k(\wp_{ij})$ obtained by adjoining all derivatives of $\wp_{ij}$ under $\partial_u$. One then has the following result.

**Theorem 2.1** [Gra85,Gra90,BEL97,Ôni98]. The differential field $k_{\partial_u} \langle \wp_{ij} \rangle$ is finitely generated by $\wp_{ij}$, $\wp_{ijk}$. Moreover, derivatives of $\wp_{ijk}$ are polynomials in the generators $\wp_{ij}$, $\wp_{ijk}$.

In fact, derivatives of $\wp_{ij}$ give rise to bases of the Riemann-Roch spaces $\{ \mathcal{L}(m\Theta) \}_m$. For example, one has
\[ \mathcal{L}(\Theta) = \mathbb{C}, \]
\[ \mathcal{L}(2\Theta)/\mathcal{L}(\Theta) = \mathbb{C}\wp_{11} \oplus \mathbb{C}\wp_{12} \oplus \mathbb{C}\wp_{22}. \]
\[ \mathcal{L}(3\Theta)/\mathcal{L}(2\Theta) = \mathbb{C}\wp_{111} \oplus \mathbb{C}\wp_{112} \oplus \mathbb{C}\wp_{122} \oplus \mathbb{C}\wp_{222} \oplus \mathbb{C}(\wp_{11}\wp_{22} - \wp_{12}^2). \] (2.22)

The complete linear system $|3\Theta|$ defines a projective embedding of $J(C)$ into $\mathbb{P}^8$. The set of defining equations are worked out in [Gra90]. As a consequence, the differential field $k_{\partial_u} (J(C))$ of the functional field $k(J(C))$ is finitely generated by the generators in $\mathcal{L}(3\Theta)$. See [Gra88,Gra90,BEL97,Ôni98,BEL12] for some examples where the 4th derivatives are expressed in terms of polynomials of the generators in $\mathcal{L}(3\Theta)$.

**Remark 2.2.** An alternative explanation of the above statement is as follows. From the $\partial_\delta$ or $\sigma$-function above, one can construct the prime form and hence the bifundamental in the usual way. Particularly for hyperelliptic curves, there is another algebraic way of constructing this bifundamental (called the Klein bifundamental). The result then follows by comparing the Laurent expansions of the bifundamental constructed in these two different ways. See [Bak98,Buc97,BEL97,Ôni98,Mat01,Fay06,BEL12] for related discussions. The hyperellipticity will be essential in our work in studying Eynard–Orantin topological recursion, as we shall explain below in Remark 4.2.

---

5 Strictly speaking, the above expressions only describe the corresponding cohomology classes they represent in $H^1(C, \mathbb{C})$ in an affine patch of the curve: the full description can be obtained by tracing the de-Rham Čech resolution for the constant sheaf via the machinery of the algebraic de-Rham cohomology.
Since \( C^{(2)} \) is birational to \( J(C) \), the pullback of rational functions in \( k(J(C)) \) are rational functions on \( C \times C \) that are invariant under permutation. It is shown in [Gra85] that
\[
k(J(C)) = \mathbb{C}(x_1 + x_2, x_1 x_2, y_1 + y_2).
\]
(2.23)

Explicit formulae for \( \wp_{ij}, \wp_{ijk} \in k(J(C)) \subseteq k_{\phi}(J(C)) \) in terms of the generators \( x_1 + x_2, x_1 x_2, y_1 + y_2 \) can be found in [Gra85,Gra90]. For example, one has
\[
\wp_{22} = (x_1 + x_2), \quad \wp_{12} = -x_1 x_2, \quad \wp_{11} = \frac{G(x_1, x_2) - 2y_1 y_2}{(x_1 - x_2)^2},
\]
(2.24)

where
\[
G(x_1, x_2) = (x_1 + x_2)(x_1 x_2)^2 + 2b_1(x_1 x_2)^2 + b_2(x_1 + x_2)(x_1 x_2) \\
+ 2b_3(x_1 x_2) + b_4(x_1 + x_2) + 2b_5.
\]
(2.25)

The pull-back of \( \wp_{ij} \) to the curve \( C \) along \( \phi \), which is essentially equivalent to the restriction to \( \phi(C) \), are discussed in [Gra88,Gra90,Gra91]. This amounts to replacing the variables \( (x_2, y_2) \) by the coordinates for \( \infty \) which then reduces the map \( \Phi \) in (2.13) to the map \( \phi \) in (2.14). In particular, it is found that
\[
\wp_{22}|_{\phi(C)} = \frac{\sigma_2}{\sigma_1}|_{\phi(C)} = -\frac{1}{x}, \quad y = -\frac{\wp_{11}\wp_{22} - \wp_{12}^2}{2\wp_{22}}|_{\phi(C)}.
\]
(2.26)

The Laurent expansions of \( \sigma \) in terms of the algebraic local uniformizers made from \( x, y \), and the analytic coordinates from \( u_1, u_2 \) on \( \phi(C) \cong C \) can be found in [Gra90,Ôni98,Ôni02]. In fact, it is found that around a point \( p \) a local uniformizer can be taken to be \( x - x(p) \) if \( p \neq e_k; y \) if \( p = e_k, k \neq 0; 1/x^{\frac{1}{2}} \) if \( p = e_0 \). Also away from \( e_0 \), the analytic coordinate \( u_1 - u_1(p) \) serves as a local uniformizer, and around \( e_0 \) the analytic coordinate \( u_2 \) does so.

The dependence of all of the above constructions on the choice of marking \( (A, B) \) and the choice of frame \( \omega \) is discussed in [Gra85].

2.3. Families of genus two curves.

2.3.1. Scalar-valued Siegel modular forms. It is known since Igusa [Igu62,Igu64] that the ring of scalar-valued Siegel modular forms for \( \Gamma(1) = \text{Sp}_4(\mathbb{Z}) \) is generated by the Eisenstein series
\[
E_{2k}(\tau) = \sum_{(c,d)=1} \det(c\tau + d)^{-2k}, \quad k \geq 2,
\]
(2.27)

which are defined as the summation over all coprime symmetric pairs. They can be regarded as holomorphic sections of \( K_{A}^{\otimes k} \), where \( K_{A} \) is the canonical bundle of the Siegel modular variety \( A := \Gamma(1)\backslash H \).

The following normalized cusp forms (that is, those holomorphic ones vanishing at the boundary of \( A \)) \( \chi_{10}, \chi_{12} \) are convenient
\[
\chi_{10} = -43867 \cdot 2^{-12}3^{-5}5^{-2}7^{-1}53^{-1}(E_4E_6 - E_{10}), \\
\chi_{12} = 131 \cdot 593 \cdot 2^{-13}3^{-7}5^{-3}7^{-2}337^{-1}(3^27^2E_4^3 + 2 \cdot 5^2E_6^2 - 691E_{12}).
\]
(2.28)
In the above we have omitted the argument $\tau$ in the Siegel modular forms. It is shown in [Igu62,Igu64] that the ring $M(\Gamma(1))$ of scalar-valued Siegel modular forms of genus two for the Siegel modular group $\Gamma(1)$ is given by the ring with five generators subject to one relation

$$M(\Gamma(1)) = \mathbb{C}[E_4, E_6, \chi_{10}, \chi_{12}, \chi_{35}]/\chi_{35}^2 \chi_{10} \cdot P(E_4, E_6, \chi_{10}, \chi_{12}),$$

(2.29)

where all of the generators and the polynomial $P$ are explicit in terms of Eisenstein series given in [Igu62,Igu64]. The relations of the above Eisenstein series to theta constants are also worked out in [Igu67]. The rings of modular forms for a class of congruence subgroups of $\Gamma(1)$ are shown to be generated by theta constants [Igu64].

Limits of these Siegel modular forms around singular points on the Siegel modular variety $\Gamma(1) \backslash \mathcal{H}$ are also discussed [Igu62]. This will potentially be useful for the later purpose of studying the enumerative content of the Gromov–Witten potentials near singular points in the moduli space.

2.3.2 Binary invariants, Igusa absolute invariants, and Torelli map. Let $\mathcal{M}$ be the moduli space of smooth sextics. By assigning the (canonical) principal polarized Jacobian $J(C)$ to a smooth genus two curve $C$ defined by $Y^2 = F(X; a)$ in (2.4), we obtain the Torelli map from the moduli space $\mathcal{M}$ of smooth genus two curves to the Siegel modular variety $\mathcal{A} = \Gamma(1) \backslash \mathcal{H}$. Recall that the Siegel modular variety is the moduli space of principally polarized Abelian varieties of dimension 2. The Torelli theorem states that the Torelli map is an injective, birational map. In particular rational functions on the moduli space $\mathcal{M}$ are identified with modular functions for the Siegel modular group $\Gamma(1)$.

Consider a family of genus two curves given by (2.4) which in inhomogenized coordinates of $\mathbb{WP}^2[1, 3, 1]$ is given by

$$Y^2 = F(X; a) = \sum_{k=0}^{6} a_k X^{6-k} = a_0 \prod_{k=0}^{5} (X - r_k), \quad a_0 \neq 0. \quad (2.30)$$

The binary invariants $A, B, C, D$ for the sextic $F(X; a)$ are defined in terms of the roots of $F(x; a)$, see [Igu62] (also [Igu67]). For example, $D$ is the discriminant $a_0^6 \prod_{0 \leq i < j \leq 5} (r_i - r_j)^2$. They can be expressed in terms of the coefficients $a_k, k = 0, 1, \ldots, 6$ of the sextic $F(x; a)$. For example, one has

$$A = 6a_3^2 - 16a_2a_4 + 40a_1a_5 - 240a_0a_6, \quad \text{Resultant}(F, \partial_X F) = -a_0 D, \quad (2.31)$$

where Resultant represents the resultant. Here we omit the detailed expressions and refer the interested readers to [KPSWR16] for a collection of the relevant formulae. The Igusa absolute invariants are given in terms of the binary invariants by

$$j_1 = \frac{2^4 3^2 B}{A^2}, \quad j_2 = \frac{2^6 3^3 (3C - AB)}{A^3}, \quad j_3 = 2 \cdot 3^5 \frac{D}{A^5}. \quad (2.32)$$

The Torelli theorem concerns the period map which assigns to a smooth genus 2 curve with its normalized period $\tau$. It turns out that the absolute invariants (2.32) are identified [Igu62] with the following modular functions $j_1, j_2, j_3$ for $\Gamma(1)$, which are generators of the rational function field of the Siegel modular variety $\Gamma(1) \backslash \mathcal{H}$,

$$j_1 = \frac{E_4 \chi_{10}^2}{\chi_{12}^2}, \quad j_2 = \frac{E_6 \chi_{10}^3}{\chi_{12}^3}, \quad j_3 = \frac{\chi_{10}^6}{\chi_{12}^5}. \quad (2.33)$$
Explicit relations between the binary invariants and the Eisenstein series for the modular group $\text{Sp}_4(\mathbb{Z})$, that involve the Jacobian from the $\tau$-variables to the absolute invariants, are worked out in [Igu62]. In particular, the relations between the absolute invariants and Eisenstein series are laid out there.

We now look at the singular loci of the space of sextics in the model (2.30). As explained in [Igu60], the binary invariants $B, C, D$ (as symmetric polynomials in the roots) vanish simultaneously at sextics with triple roots, and all of such genus two curves are mapped to the same point in $\mathcal{M}$. Blowing up this point then recovers the space parametrizing genus two curves with $A \neq 0$ and their degenerations. From the perspective of $A$, a Jacobian variety corresponding to a point on $H_1 := \{ \tau_{12} = 0 \}$, called the Humbert surface of degree 1, would satisfy $\chi_{10} = 0, \chi_{12} \neq 0$ and $[A, B, C, D] = [1, 0, 0, 0]$. It is also known that points on the locus $H_4 = \{ \tau_{11} = \tau_{22} \}$ in $\mathcal{A}$, called the Humbert surface of degree 4, correspond to sextic curves which have extra automorphism groups. Hence this Humbert surface serves as the orbifold loci.

2.3.3. Other models: quintic and Rosenhain normal form. Assume that the discriminant $D$ of the curve in (2.30) is non-vanishing and that $r_0 \neq 0$. The coordinate transformation

$$X = \frac{x + c_1}{x + c_2}r_0, \quad Y = (a_0r_0(c_1 - c_2)) \prod_{k=1}^{5}(r_0 - r_k)^{\frac{1}{2}}b_0^{-\frac{1}{2}} \cdot \frac{y}{(x + c_2)^\frac{5}{2}}, b_0 \neq 0,$$  \hspace{1cm} (2.34)

where $c_1, c_2$ are undetermined coefficients, changes (2.30) to the form in (2.6)

$$y^2 = b_0 \prod_{k=1}^{5}(x + \frac{c_1r_0 - c_2r_k}{r_0 - r_k}) := b_0x^5 + \sum_{k=1}^{5} b_kx^{5-k} = f(x; b).$$ \hspace{1cm} (2.35)

The binary invariants in terms of the coefficients for the general quintic case are studied in [Igu60, KSV05] (see also [Gra94] for the special case $b_0 = 1$). Relations between theta constants and branch points are given by Rosenhain’s and Thomae’s formula, see for example [Mum83, Gra85, Gra88, BEL97, ER07, Eil18].

Now we apply the results in [Gra85, Gra90] to the curve in (2.35), which are reviewed in Sect. 2.2. This yields explicit expressions for the rational functions $x, y$ on the curve (2.35) and hence those on the curve (2.30) in terms of pull-back of rational functions in $\wp_{ij}, \wp_{ijk}$ which are Jacobi forms.

As reviewed earlier, the model in (2.35) remains its form under the transformation (2.8). A transformation in (2.8) amounts to changing the values for $c_1, c_2$ in the coordinate transformation (2.34). Insisting that the equation takes the following Rosenhain normal form

$$y^2 = x(x - 1)(x - \lambda_1)(x - \lambda_2)(x - \lambda_3)$$ \hspace{1cm} (2.36)

fixes the transformation (2.8) up to permutations of the roots. One such choice is such that the overall transformation in (2.34) is given by

$$X = r_0 \frac{r_4 - r_2}{r_4 - r_0} x - \frac{r_2}{r_0}, \quad \frac{r_4 - r_2}{r_4 - r_0} x - 1.$$ \hspace{1cm} (2.37)

That is,

$$c_1 = -\frac{r_2}{r_0} \cdot \frac{r_4 - r_0}{r_4 - r_2}, \quad c_2 = -1 \cdot \frac{r_4 - r_0}{r_4 - r_2}.$$ \hspace{1cm} (2.38)
The roots $r_k, k = 2, 4, 0$ in the $X$-coordinate get changed to 0, 1, $\infty$ in the $x$-coordinate, while the rest of the roots $r_1, r_3, r_5$ become

$$\lambda_k = \frac{r_4 - r_0}{r_4 - r_2} \cdot \frac{r_2k - r_2}{r_2k - r_0}, \quad k = 1, 2, 3.$$  (2.39)

Explicit relations between the binary invariants and the parameters $\lambda_k, k = 1, 2, 3$ can be found in [Igu60]; see also [KSV05] for a nice summary on the explicit formulae. The works [MS17b, MS17a] also collect formulae about these relations.

It turns out that the Rosenhain normal form gives the universal family of smooth genus two curves with level two structures. To be more precise, for any ordered triple $(\lambda_1, \lambda_2, \lambda_3)$, where $\lambda_1, \lambda_2, \lambda_3$ are all distinct and are distinct from 0, 1, $\infty$, the curve (2.36) determines a point in the coarse moduli space $M(2)$ of genus two curves with level two structures. The period matrix $\tau$ gives a point in the Siegel modular variety $A(2) := \Gamma(2) \backslash H$, here

$$\Gamma(2) := \{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) = Sp_4(\mathbb{Z}) \mid \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod 2 \}.$$  (2.40)

Conversely, for any point $\tau \in A(2)$, there is a genus two curve in the Rosenhain normal form whose normalized period is $\tau$. In particular, the parameters $\lambda_k, k = 1, 2, 3$ are modular functions for the modular subgroup $\Gamma(2)$ whose field of rational functions consists of rational functions in the level 2 theta functions with characteristics $[Mum83]$.

It is a standard fact that the modular subgroup $\Gamma(2)$ fixes the theta characteristics and leaves $\lambda_k$’s invariant, and that the group $\Gamma(2) \backslash \Gamma(1)$ is the Galois group for the field extension $\mathbb{C}(\lambda_1, \lambda_2, \lambda_3)/\mathbb{C}(j_1, j_2, j_3)$. The forgetful map $M(2) \to M$ is a Galois cover of degree $|G_6| = 720$, where $G_6$ acts on the roots of a sextic by permutations and hence induces action on the ordered triple $(\lambda_1, \lambda_2, \lambda_3)$. Therefore up to this action, the Rosenhain family (2.36) gives the universal family over $M(2)$.

The relations between the parameters $\lambda_k$ and rational functions in the level two theta constants are determined up to the Galois action. One such relation is given by Rosenhain and can be found in [Igu62],

$$\lambda_1 = \frac{\theta_{1100}\theta_{1000}}{\theta_{1000}\theta_{1000}}, \quad \lambda_2 = \frac{\theta_{1001}\theta_{1100}}{\theta_{0001}\theta_{1000}}, \quad \lambda_3 = \frac{\theta_{1001}\theta_{1000}}{\theta_{0001}\theta_{0000}}.$$  (2.41)

Here we have followed the convention in [Igu62] for the theta constants, which is related to the one $\vartheta_{(a,b)}$ in (2.15) by

$$\vartheta_{xyzw} = \vartheta_{(a,b)}, \quad a = \frac{1}{2}(x, y)^t, \quad b = \frac{1}{2}(z, w)^t.$$  (2.42)

The work [MS17a] provides a nice summary on the relations among the parameters $\lambda_k$ in the Rosenhain normal form, Eisenstein series, and theta constants.

3. Differential Rings of Quasi-Modular and Quasi-Jacobi Forms for Genus Two

Following the sheaf language in [Urb14], we shall first define the ring $\tilde{R}$ of quasi-modular forms. It is obtained by adjoining extra generators to the ring $R$ of modular forms valued in certain representations. These extra generators are not modular, but quasi-modular in the sense generalizing the one in [KZ95]. Geometrically they are related to the periods of the 2nd kind Abelian differentials called quasi-periods. The ring of quasi-modular
Genus Two Siegel Quasi-Modular Forms and Gromov–Witten Theory

forms turns out to be a differential ring due to the connection to periods and quasi-periods whose differential closure can be studied via differential Galois theory [BZ00]. Our results reveal various facets of quasi-modular forms from different angles such as representation theory, Hodge theory and complex differential geometry.

We then construct a ring \( J \) of meromorphic Jacobi forms consisting of higher dimensional analogues of Weierstrass \( \wp \)-function and their derivatives. Due to hyperellipticity this ring exhibits a very simple differential ring structure [Gra90, Ôni98]. Finally we define a ring \( \tilde{J} \) of meromorphic quasi-Jacobi forms to be the \( \tilde{\mathbb{R}} \)-module \( \tilde{\mathbb{R}} \otimes J \).

3.1. Quasi-modular forms. The theory of quasi-modular forms will be formulated for universal families of smooth polarized Abelian varieties \( \pi : \mathcal{Z} \to S \) which do not necessarily arise as families of Jacobian varieties attached to families of genus two curves.

We shall only work with universal families in which \( S \) is a Siegel modular variety (more precisely, stack) \( \Gamma \backslash \mathcal{H} \), where \( \Gamma \) is a congruence subgroup of \( \text{Sp}_4(\mathbb{Z}) \). The resulting variety \( S \) is algebraic and in fact quasi-projective, with the embedding given by theta constants. The frequently studied cases in the literature are the full modular group \( \text{Sp}_4(\mathbb{Z}) \) with the embedding given by the Igusa absolute invariants, and the Igusa modular groups \( \Gamma(2), \Gamma_{2,4}, \Gamma_{4,8} \) with the embeddings given by theta constants of level 4, 2 and 1 respectively. See [vdG08,CM19] for a collection of the definitions of these groups and the corresponding theta constants.

The subtlety of compactification of the modular variety \( \Gamma \backslash \mathcal{H} \) is in fact not important for our purpose, we can simply use Baily-Borel if needed. The reason is that we shall only need to use the functional field of the variety \( \Gamma \backslash \mathcal{H} \). However, interested readers are referred to [vdG08,Urb14,Liu19] for details on how the various constructions can be extended to the compactification.

We shall also ignore the subtlety on multiplier system in the definition of modular forms: we may pass to a smaller congruence subgroup or work freely with algebraic extensions if needed (cf. Lemma 4.1 below). Most of the constructions in this section will be independent of the precise choice of \( \Gamma \). For this reason we shall often omit the symbol \( \Gamma \) in the notations, if no confusion should arise.

3.1.1. Quasi-periods: archetype. For the given family \( \pi : \mathcal{Z} \to S = \Gamma \backslash \mathcal{H} \), the function field \( k(S) \) has transcendental degree 3 over \( k = \mathbb{C} \). We take three modular functions \( t_1, t_2, t_3 \) for \( \Gamma \) which generate \( k(S) \) up to algebraic extension. We denote these generators collectively by \( t \) and similarly the partial derivatives collectively by \( \partial_t \).

The relevant locally free sheaves/bundles that enter the story are the cotangent bundle \( \Omega^1_S, \omega := R^0\pi_*\Omega^1_{\mathcal{Z}/S} \), and the Hodge bundle \( \mathcal{V}_S := R^1\pi_*\mathcal{C}_Z \otimes \mathcal{O}_S \). For a universal family, the Kodaira-Spencer map gives

\[
\Omega^1_S \cong \text{Sym}^2 \omega.
\]

(3.1)

The Hodge filtration is the relative version of the following sequence for an Abelian variety \( Z \)

\[
0 \to H^0(Z, \Omega^2_Z) \to H^1_{\text{dR}}(Z) \to H^1(Z, \mathcal{O}_Z) \to 0.
\]

(3.2)

We fix a locally constant frame \( \{ A, B \} \) for the local system \( R^1\pi_*\mathcal{C}_Z \) with dual frame \( \{ \alpha, \beta \} \). We also choose a Hodge frame \( \{ \omega, \eta \} \) adapted for the Hodge filtration for the
family $\pi : Z \to S$: $\omega$ is a frame for $\omega$, and $\{\omega, \eta\}$ a frame for $V_S$ satisfying a relation similar to (2.18)

$$\langle \omega, \eta \rangle = 2\pi i \mathbb{1}.$$  

(3.3)

The pairings between elements in the frame $\{\omega, \eta\}$ with those in $\{A, B\}$ are given by the corresponding period integrals. In the case where the family $\pi : Z \to S$ does arise as the family of Jacobian varieties of a genus two curve family $C \to S$, these period integrals coincide with the period integrals for the curves. This is due to the fact that the Jacobian $J(C)$ of a curve $C$, defined earlier as Pic$^0(C)$ or the Albanese variety, is isomorphic to the intermediate Jacobian of $C$. See [BL13] for details. For this reason, we shall use the same notations for the matrices of periods and quasi-periods for the family $\pi : C \to S$.

Since the Kodaira-Spencer map (3.1) is an isomorphism for the universal family $\pi : Z \to S$, we can in fact take

$$\eta = \nabla_{v_0} \omega$$

for some $v_0 \in T_\Sigma$. The convention for the period matrix and the quasi-period matrix that we adapt in studying modularity is

$$\Pi := \begin{pmatrix} \Pi_B(\omega) & \Pi_B(\eta) \\ \Pi_A(\omega) & \Pi_A(\eta) \end{pmatrix} = \begin{pmatrix} \int_{B_1} \omega_1 & \int_{B_1} \omega_2 & \int_{B_1} \eta_1 & \int_{B_1} \eta_2 \\ \int_{B_2} \omega_1 & \int_{B_2} \omega_2 & \int_{B_2} \eta_1 & \int_{B_2} \eta_2 \\ \int_{A_1} \omega_1 & \int_{A_1} \omega_2 & \int_{A_1} \eta_1 & \int_{A_1} \eta_2 \\ \int_{A_2} \omega_1 & \int_{A_2} \omega_2 & \int_{A_2} \eta_1 & \int_{A_2} \eta_2 \end{pmatrix}. $$

(3.5)

This convention seems to be more convenient than the one given in Sect. 2. For example, under this convention the action on $\tau = \Pi_B(\omega)\Pi_A(\omega)^{-1}$ is given by (1.2), while the action on (2.12) would be less appealing. Straightforward computation by using the Riemann–Hodge bilinear relation (2.18) tells that under this convention the transformation law of the normalized quasi-period matrix $\Pi_A(\omega)^{-1}\Pi_A(\eta)$ under $\gamma = (a, b; c, d) \in \Gamma$ is

$$\Pi_A(\omega)^{-1}\Pi_A(\eta) \mapsto \Pi_A(\omega)^{-1}\Pi_A(\eta) + 2\pi i \Pi_A(\omega)^{-1}(c\tau + d)^{-1}c\Pi_A(\omega)^{-1}. $$

(3.6)

Henceforward we shall adapt this convention. Explicit modular transformation rules will be computed in this one, and then transferred to the one in Sect. 2 if needed.

The normalized quasi-period matrix $\Pi_A(\omega)^{-1}\Pi_A(\eta)$ is the archetype of the so-called quasi-modular forms that we shall define below. Furthermore, later in Sect. 3.2 we shall see that it is nothing but the degree zero term in the Laurent expansion of the Bergman kernel defined in a way similar to (2.21).

We now briefly review some explicit formulae for the periods and quasi-periods of hyperelliptic curves in terms of theta constants. Consider the sextic model in (2.4) or (2.30), with the marking chosen as before and the frame chosen in a way similar to the ones in (2.10) and (2.17) [Bak07]. The quasi-period matrix $\Pi_A^{(6)}(\omega)^{-1}\Pi_A^{(6)}(\eta)$ for the case $a_0 = 1$ can be computed explicitly in terms of the coefficients $a_k$ and theta constants [Eil16] to be

$$\Pi_A^{(6)}(\omega)^{-1}\Pi_A^{(6)}(\eta)$$

$$= -\frac{1}{10} \begin{pmatrix} 4a_4 & a_3 & 4a_2 \\ a_3 & 4a_4 & 4a_2 \\ 4a_2 & a_3 & 4a_4 \end{pmatrix} + \frac{1}{10} \sum_{1 \leq i < j \leq 5} \frac{\partial^2 \vartheta_{\gamma_{ij}u_1} \partial^2 \vartheta_{\gamma_{ij}u_2}}{\partial u_1 \partial u_2} \bigg|_{u=0} \vartheta_{(a,b)} \left( \Pi_A^{(6)}(\omega)^{-1} \cdot u \right).$$

(3.7)

\footnote{The convention in [Eil16] is different from the one in [Gra90] that we utilize in this work, and we have fixed the constants arising from this difference.}
where \((a, b)\) is the theta-characteristic corresponding to \(r_i + r_j - 2r_0\). Here and in what follows the superscript \((6)\) stands for constructions for the sextic model and similarly \((5)\) for the quintic model when confusion might arise. The above result for the case with \(a_0 = 1\) can be used to recover those for general \(a_0\) easily. That the above quasi-period matrix transforms according to \((3.6)\) can be proved directly by working out the transformation of the derivatives of theta functions, as shown in [Gra85]. The expressions for the periods themselves can be obtained by inverting Thomae’s formula which relates the branch points to certain theta constants.

Consider also the quintic model \((2.6)\) or \((2.35)\) with \(b_0 = 1\). Based on Thomae formula it is proved in [Gra88] that

\[
\prod_{1 \leq k < \ell \leq 5} (e^k - e^\ell)^2 = \pm \left( \frac{\det \Pi_A^{(5)}(\omega)}{\pi^2} \right)^{-10} \prod_{v \text{ even}} \vartheta_v^2(0, \tau), \tag{3.8}
\]

where \(\Pi_A^{(5)}(\omega)\) is the matrix of \(A\)-cycle integrals of the 1st kind Abelian differentials as explained earlier in Sect. 2.1, and the product is over the 10 even theta-characteristics.

Further relations between theta constants and branch points are also given by Rosenhain’s and Thomae’s formulae, see for example [Mum83, Gra85, Gra88, BEL97, ER07, Eil18]. The expressions for the periods and quasi-periods can be found in [ER07, Eil18] for this quintic model. To be more precise, let \(\epsilon^k\) (where \(\epsilon^1\) follows the superscript \((5)\)) be the theta-characteristic corresponding to \(r_i + r_j - 2r_0\). Here and in what follows the superscript \((5)\) stands for constructions for the sextic model and similarly \((4)\) for the quintic model when confusion might arise. The above result for the case with \(a_0 = 1\) can be used to recover those for general \(a_0\) easily. That the above quasi-period matrix transforms according to \((3.6)\) can be proved directly by working out the transformation of the derivatives of theta functions, as shown in [Gra85]. The expressions for the periods themselves can be obtained by inverting Thomae’s formula which relates the branch points to certain theta constants.

Consider also the quintic model \((2.6)\) or \((2.35)\) with \(b_0 = 1\). Based on Thomae formula it is proved in [Gra88] that

\[
\Pi_A^{(5)}(\omega) = \frac{\theta_{pq}^{\epsilon^p \epsilon^q}}{\theta_{\epsilon^\ell}(e^k - e^\ell)^2} \left( \frac{\partial_{v^1}|_{v=0} \theta_{\epsilon^\ell}}{\partial_{v^2}|_{v=0} \theta_{\epsilon^k}} \right) \cdot \left( \frac{1}{\vartheta_v} \begin{pmatrix} 0 & 1 & -e^\ell \\ 0 & 1 & 0 \\ 1 & 0 & e^k \end{pmatrix} \right). \tag{3.9}
\]

Choosing \(k = 2, \ell = 4\) and normalizing the curve such that \(e_2 = 0, e_4 = 1\), this leads to the Rosenhain normal form \((2.36)\) with a modular representation for the period matrix. In particular, one has

\[
\left( \frac{\det \Pi_A^{(5)}(\omega)}{\pi^2} \right)^{-1} = -\frac{1}{\pi^2} \theta_{1100} \theta_{0010} \theta_{0101} \theta_{0000} \theta_{0001} \theta_{1111} \theta_{0100} \tag{3.10}
\]

\[
\left( \frac{\theta_{1100} \theta_{0000} \theta_{1001}}{\theta_{0110} \theta_{0100} \theta_{0011}} \right)^4 = \lambda_1 \lambda_2 \lambda_3. \tag{3.11}
\]

Here we have followed the convention for \(\theta_{xyzw}\) as in \((2.42)\). Note that the relation displayed above differs from the one obtained in \((2.41)\) by an action in \(\Gamma(2) \backslash \Gamma(1)\).

### 3.1.2. Sheaf-theoretic and analytic definitions of quasi-modular forms

In this subsection we shall define quasi-modular forms, following the analytic and sheaf-theoretic formulation in [KZ95, Urb14, Liu19].

For this purpose we need some elementary representation-theoretic aspects of the vector bundles involved which we now briefly review, see [vdG08, PSS15, S19, KPSWR16].

\footnote{Note that in [Gra88], it is assumed that \(b_0 = 1\) and the basis \(\omega\) therein is 2 times that in this work.}

\footnote{Note that in [ER07], it is assumed that \(b_0 = 4\) and the basis \(\omega\) therein is 2 times that in this work. Also the convention for the period matrix is different from ours.}
for related studies. Vector bundles on \( S = \Gamma \setminus \mathcal{H} \) in consideration descend from homogeneous vector bundles on \( \mathcal{H} = \text{Sp}_4(\mathbb{R})/\mathbb{U}_2 \). Sections of the former bundles are equivalent to \( \Gamma \)-invariant sections of the homogeneous vector bundles. The basic building block for the ring of modular forms is the bundle \( \omega \). Its pull-back from \( S = \Gamma \setminus \mathcal{H} \) to \( \mathcal{H} \) is the homogeneous vector bundle associated to the fundamental representation of \( \mathbb{U}_2 \). One can then construct various bundles by taking tensor product \( \otimes^k \omega \), \( k \geq 0 \), symmetric tensor product \( \text{Sym}^k \omega \), \( k \geq 0 \), etc on the corresponding representations. The \( \Gamma \)-invariant sections of these bundles are modular forms valued in these representations.

To motivate the sheaf-theoretic definition of quasi-modular forms [Urb14], we consider sections of \( \text{Sym}^{(k-m)} \omega \otimes \text{Sym}^m \mathcal{V}_S \) with \( m \geq 0 \). Although the local system \( R^1 \pi_* \mathbb{C}_Z \) does not extend across cusps on the Satake compactification of \( S \), the sheaf \( \mathcal{V}_S \) does by Schmid’s nilpotent orbit theorem [Sch73] or Deligne’s canonical extension. We focus on a cusp singularity where the local monodromy is given by the Dehn twist. Denote \( p : \mathcal{H} \to S = \Gamma \setminus \mathcal{H} \). Then the natural frame for the lift along \( p \) of the extended sheaf \( \mathcal{V}_S \) is given by \( \{ e = \beta \tau + \alpha, \beta \} \) by Deligne’s canonical extension. This frame is the one used in the study of mixed Hodge structure and is induced by the monodromy weight filtration. We shall therefore call this frame the monodromy weight frame. This frame satisfies \( \Pi_A (e) = 1 \) and has the following transformation under the action \( \gamma \) on the marking

\[
(e, \beta) \mapsto (e, \beta) \begin{pmatrix} (c \tau + d)^{-1} & -c\ell' \\ 0 & (c \tau + d)^{\ell} \end{pmatrix}.
\]  

(3.12)

Consider the transformation of the coordinate of the lifted section \( p^* s \) for the section \( s \) of \( \text{Sym}^{(k-m)} \omega \otimes \text{Sym}^m \mathcal{V}_S \) in terms of this frame. The lift \( p^* s \) is given by

\[
p^* s = \sum_{a=0}^{m} f_a(\tau) e^{\otimes(k-a)} \otimes \beta^{\otimes a}.
\]  

(3.13)

The transformation law for \( \{ f_a \}_{a=0, \ldots, m} \) is hence

\[
f_a(\tau) = \sum_{\ell=a}^{m} \binom{\ell}{a} f_\ell(\gamma \tau) ((c \tau + d)^{-1})^{\otimes(k-\ell)} (-c\ell')^{\otimes(\ell-a)} ((c \tau + d)^{\ell})^{\otimes a}.
\]  

(3.14)

The above discussions yield the following definition of weakly holomorphic “quasi-modular” forms, borrowing the terminology from the genus one case [KZ95].

**Definition 3.1 (Quasi-modular forms by sheaf).** A weakly holomorphic quasi-modular form of weight \( k \) and order \( m \geq 0 \) is a holomorphic section of \( \text{Sym}^{(k-m)} \omega \otimes \text{Sym}^m \mathcal{V}_S \).

By working with the monodromy weight frame \( \{ e, \beta \} \) of \( \mathcal{V}_S \) adapted to the Hodge filtration which transforms according to (3.12), the above definition is equivalent to the following analytic definition of weakly holomorphic quasi-modular forms in the spirit of [KZ95].

**Definition 3.2 (Quasi-modular forms by transformation).** A weakly holomorphic quasi-modular form of weight \( k \) and order \( m \) is a collection of holomorphic functions \( \{ f_a(\tau) \}_{a=0,1,\ldots, m} \) which satisfy the transformation law

\[
f_a(\tau) = \sum_{\ell=a}^{m} \binom{\ell}{a} f_\ell(\gamma \tau) ((c \tau + d)^{-1})^{\otimes(k-\ell)} (-c\ell')^{\otimes(\ell-a)} ((c \tau + d)^{\ell})^{\otimes a}.
\]  

(3.15)
We remark that sections of the lifts to $\mathcal{H}$ of the above vector bundles on $S$ are by definition well behaved under transformations by the modular group. It is the local frame, naturally singled out from the monodromy weight filtration, which ruins the coordinates presentations of the $\Gamma$-invariant sections of the vector bundles.

The above algebraic definition given in Definition 3.1 and the analytic one in Definition 3.2 provide different perspectives on the weakly holomorphic quasi-modular forms, and we shall work with both throughout this paper.

Example 3.3. A weakly holomorphic quasi-modular form with $m = 0$ is a weakly holomorphic vector-valued modular form. For example, a section of the $\omega$ is a vector-valued modular form with $k = 1, m = 0$. The quantity $\eta = \omega \Pi_A(\eta) + \beta \Pi_B(\eta)$ is a section of $\text{Sym}^2(\omega) \otimes \text{Sym}^{m} \nu_S$ with $k = m = 1$ and is a quasi-modular form.

To see this analytically, recall that the dual frame of $\{ e, \beta \}$ is given by $\{ A, B - \tau A \}$.

That is, the coordinate of $\omega$ in the frame $\{ e \}$ is $\frac{1}{\Pi_A(\omega)}$ which is easily checked to be a modular form. For the section $\eta$, one has

$$\eta = e \int_A \eta + \beta \int_{B-\tau A} \eta = e \cdot \Pi_A(\eta) + \beta \cdot (\Pi_B(\eta) - \tau \Pi_A(\eta)).$$

(3.17)

The coordinates are weakly holomorphic quasi-modular forms whose transformation follows straightforwardly from (3.6).

3.1.3. Ring structure inherited from representations. Let $M(\Gamma)$ be the graded ring of holomorphic scalar-valued modular forms for $\Gamma$, generated over $k$ by $\det^{\geq 0} \omega$. The ring $R(\Gamma)$ of meromorphic scalar-valued modular forms is defined to be the fractional field of the ring $M(\Gamma)$. It includes the functional field of the projective closure of the variety $\Gamma \backslash \mathcal{H}$ since rational functions on $S$ which constitute the field $k(S)$ correspond to modular functions $k(\Gamma)$ (weight zero scalar-valued modular forms for $\Gamma$).

For the case in study where $\omega$ has rank two corresponding to the fundamental representation of $U_2$, we have $\wedge^2 \omega = \det \omega$ whose sections are scalar-valued modular forms of weight 1. Note that

$$\omega^{\otimes 2} = \text{Sym}^{\otimes 2} \omega \oplus \wedge^2 \omega.$$  

(3.18)

With the space of scalar-valued modular forms well understood thanks to the work of [Igu62], working with $\text{Sym}^{\otimes 2} \omega$ is essentially equivalent to working with $\omega^{\otimes 2}$. In fact, by the Weyl character formula, any holomorphic representation decomposes into a direct sum of $\text{Sym}^k \omega$. Therefore, the set of representations $\{ \det^\bullet \omega \otimes \text{Sym}^{\geq 0} \omega, \oplus, \otimes \}$ carries a module structure over $\{ \det^\bullet \omega, \oplus, \otimes \}$. See [vdG08] for related discussions.

For the purpose of this work, we shall focus on a subclass of weakly holomorphic modular forms and quasi-modular forms.

Definition 3.4 (Ring of modular forms by sheaf). Let $R(\Gamma)$ be the ring of scalar-valued meromorphic modular forms for $\Gamma$. Define a subring of modular forms to be the $R(\Gamma)$-module given by

$$\mathcal{R}(\Gamma) = R(\Gamma)[\text{Sym}^{\bullet}_{\geq 0} H^0(S, \omega)].$$  

(3.19)
From either the sheaf-theoretic or the analytic definition of quasi-modular forms, we can see that the space of quasi-modular forms (including modular forms as a special case with \( m = 0 \)) is a \( R(\Gamma) \)-module and carries a graded ring structure, with the grading given by representations.

Recall that \( V_S \) corresponds to the trivial representation of \( U_2 \) and hence the decomposition of its tensor product is also clear. We can then define the following ring of quasi-modular forms.

**Definition 3.5 (Ring of quasi-modular forms by sheaf).** Let \( R(\Gamma) \) be the ring of scalar-valued meromorphic modular forms for \( \Gamma \). Define the ring of meromorphic quasi-modular forms to be the \( R(\Gamma) \)-module given by

\[
\tilde{\mathcal{R}}(\Gamma) = R(\Gamma)[\text{Sym}^{\geq 0} H^0(S, \omega), \text{Sym}^{\geq 0} H^0(S, V_S)].
\] (3.20)

The corresponding analytic definition of quasi-modular forms is as follows.

**Definition 3.6 (Ring of quasi-modular forms, analytic).** Define the ring of meromorphic quasi-modular forms to be

\[
\tilde{\mathcal{R}}(\Gamma) = R(\Gamma)[\Pi_A(\omega), \Pi_A(\eta)].
\] (3.21)

3.1.4. Almost-meromorphic modular forms. By working with the real category and choosing the frame \( \{ e, \bar{e} \} \) for \( V_S \), the notion of real-analytic modular forms can be defined. See [Kat76] for details. From its transformation law, it is easy to see that the quantity \( (\tau - \bar{\tau})^{-1} \) is the coordinate description of a real-analytic modular form lying in \( \text{Sym}(\omega) \otimes \text{Sym}(\bar{\omega}) \otimes C^\infty \) in the sheaf description. In this work we however shall not need this notion but only a more restricted one called almost-meromorphic modular forms [KZ95] or nearly holomorphic modular forms [Shi86, Shi87, Urb14].

Straightforward calculation shows that one can make \( \Pi_A(\omega) - \Pi_A(\eta) \) modular by adding a non-holomorphic term to it

\[
\Pi_A(\omega)^{-1} \Pi_A(\eta) := \Pi_A(\omega)^{-1} \Pi_A(\eta) + 2\pi i \Pi_A(\omega)^{-1}(\tau - \bar{\tau})^{-1} \Pi_A(\omega)^{-1}.
\] (3.22)

**Definition 3.7 (Ring of almost-meromorphic modular forms, analytic).** Let the notations be as above. Define the ring of almost-meromorphic modular forms to be

\[
\hat{\mathcal{R}}(\Gamma) = R(\Gamma)[\Pi_A(\omega), \Pi_A(\eta)].
\] (3.23)

The splitting of the Hodge filtration sequence defined by using Hodge decomposition induces the projection

\[
\text{Sym}^{\otimes(k-m)} \omega \otimes \text{Sym}^{\otimes m} V_S \to \text{Sym}^{\otimes k} \omega \otimes C^\infty_S.
\] (3.24)
One can show that it induces an isomorphism between the ring $\hat{R}$ and the ring $\hat{R}$. Therefore, the notions of quasi-modular forms and almost-meromorphic modular forms are equivalent through the Hodge decomposition. This is the generalization of the map from quasi-modular forms to almost-meromorphic modular forms for the dimension one case. The constant term map and modular completion for the genus one case [KZ95] can be defined either sheaf-theoretically or analytically in the same way. Interested readers are referred to [KZ95,Urb14] for details.

### 3.1.5. Differential structure from Picard–Vessiot extension.

As mentioned in the previous section, Definition 3.1 and Definition 3.2 of quasi-modular forms offer both algebraic and analytic descriptions of quasi-modular forms. Two immediate remarks are in order.

1. Firstly, it is easy to check that the derivatives of a modular form of nonzero weight yield quasi-modular forms. Ideally a ring of quasi-modular forms should be stable under differentiation $\partial_{t_{ij}}, i, j = 1, 2$, similar to the genus one case [KZ95]. It is not clear from the definitions whether the ring $\hat{R}(\Gamma)$ in Definition 3.6 is a differential ring.

2. Secondly, it is not clear if the ring $\hat{R}(\Gamma)$ is “minimal” in the sense that it contains no nontrivial sub-differential ring that includes the ring $k(\Gamma)$ of modular functions.

In this section we shall address these points.

First we shall show that the ring of quasi-modular forms as defined in Definition 3.6 is closed under the differential $\partial_{t_{ij}},$ where $\partial_{t_{ij}}$ denotes collectively the partial derivatives $\partial_{t_{ij}}, i, j \in \{1, 2\}$. To be precise, we shall prove the following statements.

**Theorem 3.8** [BZ00]. Let $R = R(\Gamma), \hat{R} = \hat{R}(\Gamma)$ be the ring of scalar-valued meromorphic modular forms, the ring of meromorphic quasi-modular forms for $\Gamma$, respectively. Let $R(\Pi_A(\omega), \Pi_A(\eta))$ be the fractional field of $\hat{R}$. Then the following statements hold.

1. The fractional field $R(\Pi_A(\omega), \Pi_A(\eta))$ of $\hat{R}$ is stable under $\partial_{t_{ij}}$.
2. The fractional field $R(\Pi_A(\omega), \Pi_A(\eta))$ of $\hat{R}$ has transcendental degree 10 over $\mathbb{C}$.
3. The presentation of the fractional field $R(\Pi_A(\omega), \Pi_A(\eta))$ is the following. As a field extension of $R$, the generators are the entries of $\Pi_A(\omega), \Pi_A(\eta)$, subject to the only relation given by the Riemann–Hodge bilinear relation

   $$\left(\Pi_A(\omega)^{-1}\Pi_A(\eta)\right)^t = \Pi_A(\omega)^{-1}\Pi_A(\eta). \tag{3.25}$$

The proof of Theorem 3.8 is essentially given in [BZ00] (see also [Zud00]) which was used to prove the following statement. Let $(K, \partial)$ be a differential field such that the field of constant under $\partial$ is $k = \mathbb{C}$. Let $L$ be another field extension of $k$. Denote by $K_\partial(L)$ the differential field obtained by adjoining to $K$ all $\partial$-differentials of elements from $L$.

As before, we take a set of three algebraically independent generators $\{t_1, t_2, t_3\}$ from $k(S) = k(\Gamma)$ and denote them collectively by $t$. Hence up to algebraic extension $k(S)$ coincides with $k(t_1, t_2, t_3)$. Denote the differentials $\{\partial_{t_{ij}}, i, j = 1, 2\}$ collectively by $\partial_t$ and similarly for $\partial_{\tau}$.

**Theorem 3.9** [BZ00]. Consider the differential field

$$D := k_\partial \{ k(S) \}, \quad k = \mathbb{C}, \tag{3.26}$$

which up to algebraic extension coincides with $F := k_\partial \{ t \}$ defined by adjoining all $\partial_{t}$-differentials of the generators $t = \{ t_1, t_2, t_3 \}$ of $k(t)$ to the field $k$. The differential field
\( F \), and hence \( \mathcal{D} = k_{\mathcal{D}}(k(S)) \), is a finite extension of the field generated over \( k(t) \) by the \( \tau \)-derivatives of \( t \) of order \( \leq 2 \). It has transcendental degree 10 over \( k \).

The idea of the proofs is to translate the problem into a problem on period integrals, and then utilize differential Galois theory. Our proof of Theorem 3.8 is a little more Hodge-theoretic reformulation of the proof of Theorem 3.9 given in [BZ00].

**Proof of Theorem 3.8.** We prove this theorem in several steps.

**Step 1. Picard–Vessiot extension.** Let

\[
\mathcal{E}' := k(t)\partial_t \langle \Pi_A(\omega), \Pi_B(\omega), \Pi_A(\eta), \Pi_B(\eta) \rangle
\]

be the Picard–Vessiot extension corresponding to the Picard-Fuchs system for the family \( \pi : Z \to S = \Gamma \backslash \mathcal{H} \). Here again the notation \( \Pi_A(\omega) \) denotes collectively its entries and similar notations are used for others. The notation \( k(t)\partial_t \langle \Pi_A(\omega), \Pi_B(\omega), \Pi_A(\eta), \Pi_B(\eta) \rangle \) denotes the differential closure of \( k(t)\langle \Pi_A(\omega), \Pi_B(\omega), \Pi_A(\eta), \Pi_B(\eta) \rangle \). By differential Galois theory, the field \( \mathcal{E}' \) has transcendental degree 10 over \( k(t) \) and hence transcendental degree 10 + 3 over \( k \).

Let

\[
\mathcal{E} := k(t)\partial_t \langle \Pi_A(\omega), \Pi_A(\eta) \rangle.
\]

Due to the universality of the family in (3.4), it follows that

\[
\mathcal{E} = k(t)\partial_t \langle \Pi_A(\omega) \rangle \subseteq \mathcal{E}' = k(t)\partial_t \langle \Pi_A(\omega), \Pi_B(\omega) \rangle.
\]

By the fact that the Picard-Fuchs system is of order two, we have

\[
\mathcal{E} = k(t)(\Pi_A(\omega), \partial_t \Pi_A(\omega)) \subseteq \mathcal{E}' = k(t)(\Pi_A(\omega), \Pi_B(\omega), \partial_t \Pi_A(\omega), \partial_t \Pi_B(\omega)).
\]

Denote for simplicity \( \pi = \Pi_A(\omega) \) which is nondegenerate with \( \Pi_B(\omega) = \tau \pi \). Then

\[
\mathcal{E} = k(t)\partial_t \langle \pi \rangle = k(t)(\pi, \partial_t \pi),
\]

\[
\mathcal{E}' = k(t)\partial_t \langle \pi, \tau \rangle = k(t)(\pi, \tau, \partial_t \pi, \partial_t \tau) = k(t)(\pi, \tau, \partial_t \pi, \partial_t \tau).
\]

**Step 2. Properties of \( \partial_t \) and \( \pi \).** The quantity \( \pi \) is a modular form valued in the fundamental representation of \( \Gamma_2 \). This implies that \( \det \pi \) lies in the graded ring \( M = M(\Gamma) \) of holomorphic scalar-valued modular forms

\[
\det \pi \in M.
\]

Therefore, \( k(t)(\pi) \) contains a weight-one scalar-valued modular form and hence the fractional ring \( R \) of the ring \( M \). That is

\[
M \subseteq R \subseteq k(t)(\det \pi) \subseteq k(t)(\pi).
\]

Furthermore, it is easy to check that for any modular function \( f \in k(t), d_\tau f \in \text{Sym}^{\geq 2}\omega \) and \( \pi^{-1} \cdot (\partial_{\tau_{ij}}) f \cdot \pi^{-t} \) is a (matrix-valued) modular function. This tells that

\[
\partial_\tau f \in k(t)(\pi), \quad \forall f \in k(t).
\]

In particular one can take \( f \) to be one of the generators \( t_1, t_2, t_3 \). This leads to

\[
\partial_\tau t \in k(t)(\pi).
\]
Similar to (3.33), we have

\[ M \subseteq R \subseteq (k(t)(\det \partial f))^\text{alg} \subseteq (k(t)(\partial \tau f))^\text{alg}, \quad \forall f \in k(t). \quad (3.36) \]

Here for a field \( K \), \( K^\text{alg} \) stands for its algebraic closure.

**Step 3. Eliminating \( \partial \tau t \).** According to (3.35), we know that \( \partial \tau t \in E \subseteq E' \). It follows from (3.31) that

\[ E = k(t)\partial_t \langle \pi \rangle = k(t)\partial_t \langle \pi \rangle, \quad E' = k(t)\partial_t \langle \pi, \tau \rangle = k(t)\partial_t \langle \pi, \tau \rangle. \quad (3.37) \]

We can also get rid of the generators \( \partial \tau t \) to obtain

\[ E' = k(t)(\pi, \partial_t \pi) = k(t)(\pi, \tau, \partial \tau \pi). \quad (3.38) \]

By definition we have \( \partial_t \Pi_A(\omega) = \Pi_A(\nabla_{\partial_t} \omega) \), with \( \nabla_{\partial_t} \omega \) being a \( O_S \)-linear combination of \( \omega, \eta \). Since \( O_S \subseteq R \), from (3.31), (3.38) we have

\[ E = k(t)(\pi, \partial \tau \pi) = k(t)(\Pi_A(\omega), \Pi_A(\eta)), \quad E' = k(t)(\Pi_A(\omega), \tau, \Pi_A(\eta)). \quad (3.39) \]

**Step 4. Computing transcendental degrees.** By the Riemann–Hodge bilinear relation (3.25), the number of algebraically independent entries in \( \Pi_A(\omega), \tau, \Pi_A(\eta) \) is at most \( 4+3+3 = 10 \). According to the presentation in (3.39), that \( E' \) has transcendental degree 13 over \( k \) immediately tells that these generators are in fact algebraically independent.

The differential field \( E \) is embedded into the fractional field of the ring of convergent series in \( e^{2\pi i \tau t} \), \( i, j \in \{1, 2\} \) which is linearly disjoint from \( k(\tau) \). Hence \( E' \) has nonzero transcendental degree 3 over \( E \). Therefore \( E \) has transcendental degree \( 13 - 3 = 10 \) over \( k \). Clearly this has generators being \( t_1, t_2, t_3 \) and the entries of \( \Pi_A(\omega), \Pi_A(\eta) \), subject to the only relation given by the Riemann–Hodge bilinear relation (3.25).

**Step 5. Differential structure of the fractional field.** Clearly

\[ k(t) \subseteq R \subseteq \tilde{R} \subseteq R(\Pi_A(\omega), \Pi_A(\eta)). \]

From the relation \( R \subseteq k(t)(\Pi_A(\omega)) \) in (3.33) we have

\[ R(\Pi_A(\omega), \Pi_A(\eta)) \subseteq k(t)(\Pi_A(\omega), \Pi_A(\eta)). \]

Combing the above two we obtain

\[ R(\Pi_A(\omega), \Pi_A(\eta)) = k(t)(\Pi_A(\omega), \Pi_A(\eta)). \]

From (3.39), the right hand side of the above is

\[ k(t)(\Pi_A(\omega), \Pi_A(\eta)) = E = k(t)(\Pi_A(\omega), \partial_t \Pi_A(\omega)). \]

Therefore, the fractional field \( R(\Pi_A(\omega), \Pi_A(\eta)) \) of the ring \( \tilde{R} \) is isomorphic to \( E \) and hence has transcendental degree 10 over \( k \). This finishes the proof. \( \square \)
In fact, due to the universality of the family $\pi : \mathcal{Z} \to S$, the entries $\partial_t \tau$ form a basis for modular forms valued in $\Omega^1_S \cong \text{Sym}^2 \omega$. That is, all binomials in the entries of $\pi$ lie in the field $R(\partial_t \tau)$. This says that $\pi$ is algebraic over $R(\partial_t \tau)$. That is, $\pi \in (R(\partial_t \tau))^{\text{alg}}$. (3.40)

where $(R(\partial_t \tau))^{\text{alg}}$ denotes the algebraic closure of $R(\partial_t \tau)$. Combining (3.36) and (3.40), we obtain

$$\pi \in (R(\partial_t \tau))^{\text{alg}} \subseteq (k(t)(\partial_t \tau))^{\text{alg}}.$$ (3.41)

This implies that $\partial_t \pi \in (k(\tau)(\partial_t \tau))^{\text{alg}} \subseteq (k(\tau))(\partial_t \tau)^{\text{alg}}$. (3.42)

The two relations (3.40), (3.42) tell that the $\partial_t$-stable field $\mathcal{E}$ satisfies

$$\mathcal{F} = k(\partial_t \langle \tau \rangle) \subseteq \mathcal{E} = k(t)_{\partial_t} \langle \pi \rangle = k(t)_{\partial_t} (\langle \pi \rangle) \subseteq \mathcal{F}^{\text{alg}}.$$ (3.43)

This proves that $\mathcal{F}$ has the same transcendental degree with $\mathcal{E}$ and is in fact a finite extension of the field generated over $k(t)$ by the $t$-derivatives of $t$ of order $\leq 2$. Since $k(S)$ is a finite extension of $k(t)$, by the proof for Theorem 3.8 both of $\mathcal{F}$ and $k(\partial_t) (k(S))$ have transcendental degree 10. This proves Theorem 3.9.

**Remark 3.10.** In the proof of Theorem 3.8 we proved $\Pi_A(\omega)$ is modular by looking at its transformation property. Another argument provided in [BZ00, Zud00] is to firstly pick a particular family (with modular group $\Gamma_{4,8}$) and explicit basis $\omega$ constructed by theta functions, and then prove the statement by direct checking. Again an algebraic extension does not affect any of the discussions used in the proof. The relation in (3.40) is shown similarly. Interesting computations on the presentation of the differential field for the particular subgroup $\Gamma = \Gamma_{4,8}$ can be found in [BZ00].

**Corollary 3.11.** The following statements hold for the rings

$$\tilde{\mathcal{R}} = R[\Pi_A(\omega), \Pi_A(\eta)], \quad \tilde{\mathcal{R}}^\circ := R[\Pi_A(\omega), \Pi_A(\omega)^{-1} \Pi_A(\eta)].$$ (3.44)

1. They are stable under $\partial_t$ and $\partial_t$.
2. Their fractional fields have transcendental degree 10 over $\mathbb{C}$.
3. The presentation of the former ring $\tilde{\mathcal{R}}$ is the following. Over the ring $R$, the generators are the entries of $\Pi_A(\omega)$, $\Pi_A(\eta)$, subject to the only relation given by the Riemann–Hodge bilinear relation

$$\Pi_A(\omega) \Pi_A(\eta)^t = \Pi_A(\eta) \Pi_A(\omega)^t.$$ (3.45)

The presentation of the latter ring $\tilde{\mathcal{R}}^\circ$ is the following. Over the ring $R$, the generators are the entries of $\Pi_A(\omega)$, $\Pi_A(\omega)^{-1} \Pi_A(\eta)$, subject to the only relation given by the Riemann–Hodge bilinear relation

$$\Pi_A(\omega)^{-1} \Pi_A(\eta) = \left( \Pi_A(\omega)^{-1} \Pi_A(\eta) \right)^t.$$ (3.46)

The reason we have chosen the generators $\Pi_A(\omega)$, $\Pi_A(\omega)^{-1} \Pi_A(\eta)$ in constructing the ring $\tilde{\mathcal{R}}^\circ$ is that in later applications it is the quantity $\Pi_A(\omega)^{-1} \Pi_A(\eta)$ that naturally enters the stage through the Bergman kernel.
Proof. Recall that the Gauss-Manin connection is an $O_S$-derivation. This tells that $\partial_t^2 \pi$ is a $k(t)$-linear combination of $\pi, \partial_t \pi$. Therefore, $k(t)[\pi, \partial_t \pi]$ is closed under $\partial_t$.

The derivation of (3.35) in the proof of Theorem 3.8 in fact shows
\[ \partial_t t \in k(t)[\pi]. \] (4.47)
By the chain rule, this tells that $k(t)[\pi, \partial_t \pi]$ is also $\partial_t$-stable. Since $R \subseteq k(t)[\pi]$, the ring $R[\pi, \partial_t \pi]$ and hence $R[\Pi_A(\omega), \Pi_A(\eta)]$ is also $\partial_t$-stable. Applying (3.4), we have proved the first statement. The rest follows from Theorem 3.8. $\square$

Corollary 3.11 says that the ring $\tilde{R}$ of quasi-modular forms is a differential ring. It is not only a $k(t)_{\partial_t}$-module, but also up to algebraic extension it coincides with $k(t)_{\partial_t}$. That is, any ring including $k(t)$ and closed under differentiation $\partial_t$ must be $\tilde{R}$, up to an algebraic extension. Therefore, $\tilde{R}$ is the “minimal” differential ring including $k(t)$. On the other hand, the presentation in Corollary 3.11 gives an extremal simple dimension count [BZ00]
\[ 3 + 4 + 4 - 1 \] (4.48)
for the transcendental degree of its factional field: the 3 occurs as that for the field $k(S)$, the first 4 from the $A$-cycle periods forming a vector-valued modular form, the second 4 from the $A$-cycle quasi-periods, and the $-1$ from the 2nd Riemann–Hodge bilinear relation. This structure is universal in the sense that it is completely determined by differential Galois theory for the universal family and does not rely on the details on the congruence subgroup $\Gamma$.

From the presentation of the ring $\tilde{R}_0^o$ we see that all of the “quasi”-ness is completely encoded in the quasi-period matrix $\Pi_A(\omega)^{-1} \Pi_A(\eta)$. For the case of a family of Jacobian varieties of genus two curves, its explicit expression is given in (3.7) in terms of special values of the derivatives of theta functions. The structure of the ring $\tilde{R}_0^o$ from the differential field perspective gives another way of presenting the quasi-modular generator up to algebraic extension: $\tilde{R}_0^o$ can be generated over $\tilde{R}$ from the derivative of any scalar-valued modular form of nonzero weight. The reason is that according to the transformation law, such a derivative can not be modular and therefore must be quasi-modular. For example, one can take $\partial_t \log \chi_{10}$ or $\partial_t \log \chi_{12}$.

Example 3.12. Consider the quintic model (2.6) or (2.35) with $b_0 = 1, b_6 = 0$. The period matrix $\Pi_A(\omega)$ are given in (3.9). The quasi-period matrix $\Pi_A(\eta)$ can be derived from (3.7). One has the following Rauch’s variational formula for the periods and quasi-periods, see [ER07] and references therein,\footnote{In [ER07], it is assumed that $b_0 = 4$ and the basis $\omega$ therein is 2 times that in this work. Also the convention for the period matrix is different from ours. We have rewritten the results therein in our convention.}
\[ \frac{\partial}{\partial e_k} \begin{pmatrix} \Pi_A(\omega) & -\Pi_A(\eta) \\ \Pi_B(\omega) & -\Pi_B(\eta) \end{pmatrix} = \begin{pmatrix} \Pi_A(\omega) & -\Pi_A(\eta) \\ \Pi_B(\omega) & -\Pi_B(\eta) \end{pmatrix} \begin{pmatrix} \alpha_k^t & \gamma_k^t \\ \beta_k^t & -\alpha_k \end{pmatrix}, \quad k = 1, 2, \ldots, 5. \] (3.49)
The matrices $\alpha_k, \beta_k, \gamma_k$ are given as follows. Let
\[ U(x) = \begin{pmatrix} 1 \\ x \end{pmatrix} := \begin{pmatrix} U_1(x) \\ U_2(x) \end{pmatrix}, \quad V(x) = \begin{pmatrix} \sum_{\ell=1}^4 \ell b_{3+\ell} x^\ell \\ 4 \sum_{\ell=2}^3 (\ell-1) b_{2+\ell} x^\ell \end{pmatrix} := \begin{pmatrix} V_1(x) \\ V_2(x) \end{pmatrix}. \]
Then
\[ \alpha_k = -\frac{1}{2} \left( \begin{pmatrix} 1 \\ f'(e_k) \end{pmatrix} U(e_k) V(e_k)^t - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right), \]
\[ \begin{align*}
\beta_k &= -2 \left( \frac{1}{f'(e_k)} U(e_k) U(e_k)' \right), \\
\gamma_k &= \frac{1}{8} \left( \frac{1}{f'(e_k)} V(e_k) V(e_k)' - e_k \left( \begin{array}{ll}
V_1(e_k) & e_k \\
V_1(e_k) & e_k
\end{array} \right) - \left( \begin{array}{ll}
V_1(e_k) & 0 \\
0 & V_2(e_k)
\end{array} \right) \right). \end{align*} \]

From this one can immediately derive the Picard-Fuchs equations for the periods and quasi-periods. One also has

\[ \frac{\partial}{\partial e_k} \tau = 2\pi i \Pi_A(\omega)^{-1} \beta_k^t (\Pi_A(\omega)^t)^{-1}. \]

These relations exhibit the differential ring structure of \( \tilde{R} = R[\Pi_A(\omega), \Pi_A(\eta)] \).

3.2. Quasi-Jacobi forms

3.2.1. Differential function field of Jacobian variety. We have mentioned in Theorem 2.1 in Sect. 2.2 that the functional field of the Jacobian \( J(C) \) is a differential field stable under \( \partial_u \).

We now consider the relative version of the ring \( k_{\partial_u}(\wp_{ij}) \). We again work with the family \( \mathcal{J} \to S = \Gamma \backslash \mathcal{H} \). By Theorem 2.1 we then have

\[ k(S)_{\partial_u}(\wp_{ij}) = k(S)(\wp_{ij}, \wp_{ijk}). \] (3.50)

It is easy to check that elements in the above ring are meromorphic Jacobi forms for \( \Gamma \), see [Gra85]. This then motivates the following definition.

**Definition 3.13.** Let \( R, \tilde{R}, \hat{R} \) be the ring of modular, quasi-modular, almost-meromorphic modular forms, respectively. Define

\[ \mathcal{J} := R \otimes k(S)_{\partial_u}(\wp_{ij}) = R \otimes k(S)(\wp_{ij}, \wp_{ijk}), \]

\[ \tilde{\mathcal{J}} := \tilde{R}^o \otimes k(S)_{\partial_u}(\wp_{ij}) = \tilde{R}^o \otimes k(S)(\wp_{ij}, \wp_{ijk}), \]

\[ \hat{\mathcal{J}} := \hat{R}^o \otimes k(S)_{\partial_u}(\wp_{ij}) = \hat{R}^o \otimes k(S)(\wp_{ij}, \wp_{ijk}). \]

Here the ring \( \tilde{R}^o \) is defined in (3.44), while the ring \( \hat{R}^o \) is defined by replacing \( \Pi_A(\eta) \) in \( \tilde{R}^o \) by \( \Pi_A(\eta) \). Elements in the ring \( \mathcal{J} \) are meromorphic Jacobi forms for \( \Gamma \). We shall call elements in \( \tilde{\mathcal{J}}, \hat{\mathcal{J}} \) quasi-Jacobi and almost-meromorphic Jacobi forms, respectively.

Note that we should have used a covariant derivative (raising operator in the context of Jacobi forms [EZ84]) on sections in \( H^0(J(C), O_J(C)(m\Theta)) \). However, for rational functions (Jacobi forms of index \( m = 0 \)), it turns out that the connection matrix vanishes. This fact can be checked explicitly. Namely, the \( u \)-derivatives of index zero Jacobi forms are Jacobi forms. Hence as long as we restrict to index zero objects, the derivatives \( \partial_u \) respect the Jacobi form property.
3.2.2. The Bergman kernel. The Bergman kernel is characterized by some reproducing property and normalization condition depending on the marking. See for example [Tyu78, Tak01] for details. It can be constructed from the prime form and turns out to be [Bak98, Buc97, BEL97, Oni98, Mat01, Fay06, BEL12]

\[ B(p, q) = H \log \vartheta_\delta \left( (\Pi_A(\omega))^{-1} (\phi(p) - \phi(q)) \right) \cdot d\phi(p) \boxtimes d\phi(q), \]  

(3.51)

where \( H \) is the Hessian operator \((\partial_{u_iu_j}^2)\) and \( \phi(p) = u(p + \infty) \). The above is regarded as a matrix-valued two-form lying in the space whose basis is given by the components of \( du \otimes du' \).

According to (2.20) adapted to our convention (3.5) for the matrix \( \Pi \), we see that it is related to the Weierstrass \( \wp \)-function by

\[ B(p, q) = \left( \wp(u(p) - u(q)) - \Pi_A(\omega)^{-1}\Pi_A(\eta) \right) \cdot d\phi(p) \boxtimes d\phi(q). \]  

(3.52)

From (2.22), we see that the quasi-period matrix \( \Pi_A(\omega)^{-1}\Pi_A(\eta) \) is nothing but the degree zero term in the Laurent expansion of \( B \).

A different choice of odd characteristic corresponds to a different choice of the marking that is related to the previous one by an action in \( \Gamma(1) = S_4(\mathbb{Z}) \), as mentioned before in Sect. 2.3.3. The embedding \( \phi : C \to J(C) \) and the theta-characteristic \( \delta \) in (2.16) is fixed once and for all so that we can apply results in [Gra88, Gra90, Gra91] such as (2.26). Therefore we restrict ourselves to the modular subgroup \( \Gamma(2) \) that fixes \( \delta \), and pass to the modular group \( \Gamma \cap \Gamma(2) \) in the constructions of the fractional rings \( \mathcal{R}, \bar{\mathcal{R}} \) of modular and quasi-modular forms in Sect. 3.1, and the rings of quasi-Jacobi forms and almost-meromorphic Jacobi forms in Definition 3.13. Under the action of \( \gamma = (a, b; c, d) \in \Gamma \cap \Gamma(2) \) on the marking, \( B \) transforms according to

\[ B \mapsto B + 2\pi i(c\tau + d)^{-1}c \cdot d\phi(p) \boxtimes d\phi(q), \quad \forall \gamma = (a, b; c, d) \in \Gamma \cap \Gamma(2). \]  

(3.53)

This agrees with the transformation property implied by the characterization of the Bergman kernel [Tyu78]. Combining (3.6), it follows then that \( \wp(u(p) - u(q))d\phi(p) \boxtimes d\phi(q) \) is modular invariant for the subgroup preserving the characteristic \( \delta \), and that \( B \in \hat{\mathcal{F}} \) is a quasi-Jacobi form. If we replace the term \( \Pi_A(\omega)^{-1}\Pi_A(\eta) \) in \( B \) by \( \Pi_A(\omega)^{-1}\Pi_A(\eta) \), then the Bergman kernel \( B \) gets changed to the so-called Schiffer kernel \( S \) which is an almost-meromorphic Jacobi form according to our definition, see [Tyu78, Tak01] for details.

More intrinsically, from the fact that \( \delta \) is odd, it is easy to see that \( B(p, q) \) is an element of \( H^0(C \times C, K_C \boxtimes K_C(2\Delta))_{\mathbb{Z}_2} \), where \( \Delta \) is the diagonal in \( C \times C \). In fact, consider the following sequence of sheaves associated to the nonreduced divisor \( 2\Delta \) in \( C \times C \)

\[ 0 \to K_C \boxtimes K_C \to K_C \boxtimes K_C(2\Delta) \to K_C \boxtimes K_C(2\Delta)|_{2\Delta} \to 0. \]  

(3.54)

The third map above is the polar part of an element in the middle sheaf. It is also called the biresidue map or symbol map depending on the context. Consider further the sequence of sheaves

\[ 0 \to (K_C \boxtimes K_C) \otimes O_{\Delta}(-\Delta) \to K_C \boxtimes K_C(2\Delta)|_{2\Delta} \to K_C \boxtimes K_C(2\Delta)_{|\Delta} \to 0. \]  

(3.55)

By the adjunction formula, one has \( K_C \boxtimes K_C(2\Delta)_{|\Delta} \cong O_{\Delta} \). Also one has \( K_C \otimes K_C \otimes O_{\Delta}(-\Delta) \cong K_C \). It can be shown that the restriction \( K_C \boxtimes K_C(2\Delta)|_{2\Delta} \) has a
Corollary 3.14. The Bergman kernel $\mathcal{B}$ is a lift of its principal part in $KC \otimes KC(2\Delta)|_{2\Delta}$ that in turn is the unique symmetric element lifting the canonical section of $KC \otimes KC(2\Delta)|_{\Delta}$. The degree zero term of $\mathcal{B}$ is a projective connection that is an extension of the above unique symmetric element in $KC \otimes KC(2\Delta)|_{2\Delta}$ to $KC \otimes KC(2\Delta)|_{3\Delta}$. The ring $\mathcal{R}_{0}$ of quasi-modular forms is generated by the Bergman projective connection over the ring $\mathcal{R}$ of modular forms. Concretely, one has

$$\mathcal{B}(p, q) = H \log \vartheta_{\delta} \left( \Pi_{A}(\omega)^{-1}(\phi(p) - \phi(q)) \right) \cdot d\phi(p) \otimes d\phi(q),$$

(3.56)

where $H$ is the Hessian operator $(\partial_{u_{i}, u_{j}})$ and $\delta$ can be any odd theta-characteristic.

As explained in Remark 2.2, for the quintic model in (2.6) or (2.35) with $b_{0} = 1$

$$y^{2} = f(x; b) = x^{5} + b_{1}x^{4} + b_{2}x^{3} + b_{3}x^{2} + b_{4}x + b_{5},$$

(3.57)

the Bergman kernel (3.52) admits an algebraic expression using (2.24)

$$\mathcal{B}(p, q) = \frac{G(x_{1}, x_{2}) + 2\gamma_{12} \frac{dx_{1} dx_{2}}{(x_{1} - x_{2})^{2}} - \Pi_{A}(\omega)^{-1} \Pi_{A}(\eta) d\phi(p) \otimes d\phi(q),}$$

(3.58)

with $p = (x_{1}, y_{1})$, $q = (x_{2}, y_{2})$ and $G$ is given in (2.25). For the sextic model given in (2.4)

$$y^{2} = F(X; a) = \sum_{k=0}^{6} a_{k} X^{6-k},$$

the algebraic form for the Bergman kernel is given by [BEL12, Page24]. To be precise, let the affine coordinates of $p, q$ be $(X_{1}, Y_{1}, 1)$ and $(X_{2}, Y_{2}, 1)$ respectively, then

$$\mathcal{B}(p, q) = \frac{G^{(6)}(X_{1}, X_{2}) + 2Y_{1}Y_{2} \frac{dX_{1} dX_{2}}{Y_{1} Y_{2}} - \Pi_{A}(\omega)^{-1} \Pi_{A}(\eta) d\phi(p) \otimes d\phi(q),}$$

(3.59)

where now

$$G^{(6)}(X_{1}, X_{2}) = 2a_{0}X_{1}^{3}X_{2}^{3} + \sum_{i=0}^{2} X_{1}^{i} X_{2}^{3} (2a_{6-2i} + a_{6-2i-1})(X_{1} + X_{2}).$$

(3.60)

As we shall see below, in Eynard–Orantin topological recursion, all of the quasi-modularity in the open GW potentials enter through $\mathcal{B}$. Therefore, keeping track of the failure of modularity (in the built-in combinatorial structure in Eynard–Orantin topological recursion) in these potentials is equivalent to keeping track of the non-holomorphic dependence in the Schiffer kernel.
Remark 3.15. For the genus one case, we can take $\omega = \frac{dx}{y}, \eta = \frac{dx}{y}$ in the Weierstrass normal form. By uniformization, the lift of this frame is $\tilde{\omega} = dz, \tilde{\eta} = \wp dz$. We normalize the period $\Pi_A(\tilde{\omega})$. Then $\Pi_A(\tilde{\eta}) = \int_A \wp dz = -\eta_1$, where $\eta_1 = \zeta(z + 1) - \zeta(z)$ is a quasi-period. It turns out that

$$B(u(p) - u(q)) = \partial_p \partial_q \ln \theta_{(\frac{1}{2}, \frac{1}{2})}(u(p) - u(q))du(p) \otimes du(q) = (\wp + \eta_1)du(p) \otimes du(q),$$

where $\eta_1 = \frac{\pi^2}{3}E_2$. It is well known that $\eta_1$ is an elliptic quasi-modular form satisfying

$$\eta_1(\alpha \tau + \beta c \tau + \delta) = (c \tau + d)^2 \eta_1(\tau) - 2\pi ic(c \tau + d), \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}).$$

On the other hand, we have

$$\frac{1}{\text{Im}(\alpha \tau + b c \tau + d)} = (c \tau + d)^2 \frac{1}{\text{Im}\tau} - 2ic(c \tau + d).$$

Then $\hat{\eta}_1 = \eta_1 - 2\pi i/\tau - \bar{\tau})$ is modular and is called an almost-holomorphic modular form [KZ95].

4. Eynard–Orantin Topological Recursion for Genus Two Mirror Curve Families and Applications

According to the proof of the Remodeling Conjecture [BKMnP09,FLZ20], the open GW potentials of a toric CY 3-fold $X$ equal the differentials $\{\omega_{g,n}\}_{g,n}$ constructed from Eynard–Orantin topological recursion [EO07] for the mirror curve, under the so-called open and closed mirror maps. In what follows we shall first study the geometry of the mirror curve and discuss the construction of the mirror maps, focusing on a particular example. Then we apply results in Sect. 3 to prove modularity of Eynard–Orantin topological recursion. Finally we prove the modularity of open and closed GW potentials based on the proof of the Remodeling Conjecture.

4.1. Geometry of genus two curves constructed from mirror symmetry. Our approach in establishing modularity applies to a large class of genus two mirror curves with hyperelliptic structure constructed from the brane structure. In this paper we shall illustrate our strategy by analyzing in detail the resolution of the orbifold $\mathbb{C}^3/\mathbb{Z}_6$ which is one of the simplest toric Calabi–Yau threefolds with genus two mirror curves. This example has been studied extensively in the context of mirror symmetry, see e.g., [KKV97,CKYZ99] and in particular [KPSWR16] with which our present work in this part is closely related.

4.1.1. Construction from mirror symmetry. The local toric Calabi–Yau threefold $X$ in the A-model is the resolution of the orbifold $\mathbb{C}^3/\mathbb{Z}_6$, with

$$h^2(X) = 3. \quad (4.1)$$

The height one slice of its fan is depicted in Fig. 1 below.

The mirror curve $C_0 \subseteq (\mathbb{C}^*)^2$ is then given by

$$1 + X + Y + q_1X^2 + q_2X^3 + q_3X^6Y^{-1} = 0. \quad (4.2)$$
We denote the compactified mirror curve by $C$ and the resulting families by $\pi^o: C^o \to S$, $\pi: C \to S$, respectively.

As elaborated in [FRZZ19], we shall choose a suitable Lagrangian with framing $(\mathcal{L}, \mathcal{f})$ in studying the open GW theory of $X$. In order to simplify the B-model computation we choose $\mathcal{f} = 0$ and $X: C^o \to \mathbb{C}$ gives a hyperelliptic structure of $C$. Let

$$R = \{ r \in C^o \mid dX|_r = 0 \}$$

be the set of ramification points, then $R$ consists of all 6 Weierstrass points of $C$.

### 4.1.2. Closed and open moduli.

After a simple change of coordinate $Y \mapsto Y + h(X)$, the affine part of the mirror curve in (4.2) takes the hyperelliptic form. To be more precise, let

$$h(X) = \frac{1}{2}(1 + X + q_1X^2 + q_2X^3).$$

Let $\tilde{Y} = Y + h(X)$, and then (4.2) is reduced to

$$\tilde{Y}^2 = (Y + h(X))^2 = -q_3X^6 + h^2(X) = \left(\frac{1}{4}q_2^2 - q_3\right)X^6 + \frac{1}{2}q_1q_2X^5 + \left(\frac{1}{4}q_1^2 + \frac{1}{2}q_2\right)X^4 + \left(\frac{1}{2}q_1 + \frac{1}{2}q_2\right)X^3 + \left(\frac{1}{4} + \frac{1}{2}q_1\right)X^2 + \frac{1}{2}X + \frac{1}{4}.$$ 

We again denote the sextic on the right hand side of the above equation (4.5) by

$$F(X; a) = \sum_{k=0}^{6} a_k X^{6-k} = a_0 \prod_{k=0}^{5} (X - r_k),$$

where now the coefficients $a_k, k = 0, 1 \ldots, 6$ are polynomials in $q_1, q_2, q_3$. The following result concerns the modularity of the parameters $q_1, q_2, q_3$ in the mirror curve family (4.2).

**Lemma 4.1.** The parameters $q_1, q_2, q_3$ in the mirror curve family (4.2) are algebraically independent modular functions with respect to a certain congruence subgroup $\Gamma < \Gamma(1) = \text{Sp}_4(\mathbb{Z})$.

**Proof.** Clearly the Igusa absolute invariants $j_1, j_2, j_3$ in (2.32) are rational functions in $q_1, q_2, q_3$. It is routine, for example by using a computer program, to check that $\mathbb{C}(q_1, q_2, q_3)$ is an algebraic extension of $\mathbb{C}(j_1, j_2, j_3)$ using the Jacobian criterion.
Since the field \( \mathbb{C}(j_1, j_2, j_3) \) is identified with the field \( \mathbb{C}(j_1, j_2, j_3) \) of modular functions for the modular group \( \Gamma(1) \), \( \mathbb{C}(q_1, q_2, q_3) \) is an algebraical extension of \( \mathbb{C}(j_1, j_2, j_3) \). By looking at the transcendental degrees over \( \mathbb{C} \) of these fields, one sees that in particular the generators \( q_1, q_2, q_3 \) are algebraically independent over \( \mathbb{C} \). Consider the group homomorphism
\[
\Psi : \Gamma(1) \to \text{Gal}(\mathbb{C}(q_1, q_2, q_3)/\mathbb{C}(j_1, j_2, j_3)).
\] (4.7)

Its kernel \( \Gamma = \text{Ker} \Psi \) is of finite index in \( \Gamma(1) \) since the Galois group is finite due to the algebraic extension property. By construction, elements in the field extension \( \mathbb{C}(q_1, q_2, q_3) \) are invariant under the subgroup \( \Gamma \). From the classical fact that finite index subgroups of \( \text{Sp}_4(\mathbb{Z}) \) are congruence subgroups, one concludes that the generators \( q_1, q_2, q_3 \) of \( \mathbb{C}(q_1, q_2, q_3) \) are modular functions for the congruence subgroup \( \Gamma \).

The same argument in Lemma 4.1 above also shows that the roots \( r_k, k = 1, \ldots, 6 \) are modular functions, and one has the tower of fields attached to the sextic in (4.6) as depicted in Fig. 2.

In mirror symmetry, one often needs to translate results between the A- and B- sides using the mirror map. In our case, these maps are explicitly given as follows. Let
\[
s_1 = q_1, \quad s_2 = q_1^{-2} q_2, \quad s_3 = q_2^{-2} q_3.
\] (4.8)

Define
\[
A_1 = 0,
\]
\[
A_2 = \sum_{d_1 \geq 2d_2, d_2 \geq 2d_3, d_3 \geq 0, d_1 > 0} \frac{(-1)^{d_2-1}(2d_1 - d_2 - 1)!s_1^{d_1}s_2^{d_2}s_3^{d_3}}{d_1!(d_1 - 2d_2)!(d_2 - 2d_3)!(d_3!)^2},
\]
\[
A_3 = \sum_{0 \leq 2d_1 \leq 2d_2, d_2 > 0, d_2 \geq 2d_3 \geq 0} \frac{(-1)^{d_1-1}(2d_2 - d_1 - 1)!s_1^{d_1}s_2^{d_2}s_3^{d_3}}{d_1!(d_2 - 2d_1)!(d_2 - 2d_3)!(d_3!)^2}.
\]
\[ A_4 = - \sum_{d_3 > 0} \frac{(2d_3 - 1)!s_3^{d_3}}{(d_3!)^2}. \]

Then the closed mirror map \( Q = (Q_1, Q_2, Q_3) \), constructed by solving Picard-Fuchs system [CKYZ99, FLZ20] for the non-compact Calabi-Yau that is mirror to \( \mathcal{X} \), is given by

\[
\begin{align*}
Q_1 &= s_1 \exp(-2A_2 + A_3), \\
Q_2 &= s_2 \exp(A_2 - 2A_3 + A_4), \\
Q_3 &= s_3 \exp(-2A_4).
\end{align*}
\]

The open mirror map \( \mathcal{X} = (\mathcal{X}_1, \ldots, \mathcal{X}_n) \) are linear functions in the rational functions \( X_1, X_2, \ldots, X_n \) on \( C \) whose coefficients are polynomials in the \( A_k \)’s above. These quantities are understood as expansions near the point \( s_1 = s_2 = s_3 = 0 \) and the so-called open GW point \( s_0 = (0, -1) \). For this reason, we shall call \( Q, \mathcal{X} \) the closed- and open-moduli respectively. See [FRZZ19] for detailed discussions on the explicit expressions and modularity for them in the genus one case.

Lemma 4.1 above implies that the closed-moduli (4.9) are Siegel modular functions, modulo the complicated exponential factors \( Q_k s_k^{-1}, k = 1, 2, 3 \). In this work, we shall rarely need to carry out the expansions in terms of \( Q, \mathcal{X} \), and hence shall loosely call \( q = (q_1, q_2, q_3) \) and \( X = (X_1, \ldots, X_n) \) the closed- and open-moduli. This makes it more convenient to phrase statements about modularity.

The same argument in Sect. 2.3.3 applying to (4.6) enables us to obtain explicit expressions (see (2.26)) for the rational functions \( X, \tilde{Y} \) on the mirror curve in terms of pull-backs of rational functions in \( \varphi_{ij}, \varphi_{ijk} \) which are Jacobi forms.

### 4.2. Modularity in Eynard–Orantin topological recursion

Recall that Eynard–Orantin topological recursion [EO07] defines differentials \( \omega_{g,n} \) \((2g - 2 + n > 0)\) recursively as follows. One starts with an affine curve \( C^o \rightarrow \mathbb{A}^1 \) equipped with a simply ramified cover structure whose ramification locus is \( R^o \). Then one defines the following basic ingredients on \( C^o \) and on its compactification \( C \)

\[ \lambda = \ln Y dX X, \quad \omega_{0,1} = 0, \quad \omega_{0,2} = B. \]

Let \( * : p \rightarrow p^* \) be the local involution near a simple ramification point \( p_0 \). In the present hyperelliptic cover case, \( C \rightarrow \mathbb{P}^1 \) is a 2 : 1 cover and the * action is the hyperelliptic involution \( \sigma \) given in (2.7). In general, one defines

\[
\begin{align*}
\omega_{g,n}(p_1, \ldots, p_n) &= \sum_{p_0 \in R^o} \text{Res}_{p \rightarrow p_0} \frac{f_{\xi = p}^* B(p_n, \xi)}{2(\lambda(p) - \lambda(p^*))} \left( \omega_{g-1,n+1}(p, p^*, p_1, \ldots, p_{n-1}) \right. \\
&\left. + \sum_{g_1 + g_2 = g} \sum_{J \cup K = \{1, \ldots, n-1\}} \omega_{g_1, |J|+1}(p, p_J) \omega_{g_2, |K|+1}(p^*, p_K) \right), \\
\omega_{g,0} &= \sum_{p_0 \in R^o} \text{Res}_{p \rightarrow p_0} d\lambda^{-1}(p) \omega_{g,1}(p).
\end{align*}
\]

\[(4.10)\]

\[(4.11)\]
Thanks to the proof of the Remodeling Conjecture [BKMnP09, FLZ20], studying the open GW theory for the toric CY 3-fold with the given Lagrangian and framing data is reduced to studying Eynard–Orantin topological recursion on the curve $C$. Following the same strategy in [FRZZ19] in establishing the modularity and structure theorems for the open GW potentials, we shall first work with the Schiffer kernel $S$ instead of the Bergman kernel $B$ for Eynard–Orantin topological recursion to obtain globally defined differentials $\hat{\omega}_{g,n}$. In the final step we then take the holomorphic limit of the resulting differentials $\hat{\omega}_{g,n}$ to produce the open GW potentials $\omega_{g,n}$.

Due to the residue calculus nature in Eynard–Orantin topological recursion [EO07], the algorithm is independent of the choice of coordinates which arises from biregular morphism on the algebraic curve $C$. This can be further checked by induction on $g, n$ in the recursive construction of $\hat{\omega}_{g,n}$. Hence we can freely choose coordinates to simplify computations.

4.2.1. Expansions of basic ingredients. We now study the Laurent expansions of the basic ingredients such as the differential $\lambda$ and the Schiffer kernel $S$ near the ramification points.

The differential $\lambda$ used in the proof of the Remodeling Conjecture is given by

$$\lambda = \log Y \frac{dX}{X} = \log(\tilde{Y} - h(X)) \frac{dX}{X}. \quad (4.12)$$

As reviewed in Sect. 2.1, the image of the ramification points are 2-torsion points on $J(C)$. Rational functions of derivatives of $\wp_{ij}$, if not identically zero or infinity on $C$, will be valued in Siegel modular forms at torsion points. This implies that the Laurent expansion in coordinate $\tilde{Y}$ of the quantity $\lambda(p) - \lambda(p^*)$, that is of central importance in Eynard–Orantin topological recursion, has Siegel modular forms as coefficients.

Remark 4.2. In the original set-up of Eynard–Orantin topological recursion [EO07], it is assumed that a cover $C \to \mathbb{P}^1$ is a ramified cover with ramification index 2 or 1. The local involution $\ast$ is the action which switches the two branches at a ramification point with ramification index 2 and fixes the rest of the branches. For computational purpose, one needs to realize the action of the local involution $\ast$ on the functional field of the algebraic curve $C$. This is easiest to do if the cover is Galois. The simplest case is the hyperelliptic cover. Without the hyperellipticity, similar statements about the ring structure and holomorphic anomaly in Eynard–Orantin topological recursion still hold, but now one needs a case by case analysis to figure out the action on the functional field based on the details of the equation for the curve.

To expand the Schiffer kernel $S$, we again use the algebraic local uniformizer. From the algebraic formula (3.58) or (3.59) for the principal part of the Bergman kernel, it is easy to obtain the expansion of $S(p, q) - S(p, \sigma(q))$ around $q \in R$. The Laurent coefficients are rational functions in the coordinates of the points $p, q$. This can also be checked by first showing that the $X, \tilde{Y}$ coordinates of ramification points are scalar-valued meromorphic modular forms using Lemma 4.1 and then using (3.59).

4.2.2. Modularity of initial values. The initial values for Eynard–Orantin topological recursion are the differentials $\hat{\omega}_{g,n}$ satisfying $2g - 2 + n \leq 0$. These are the disk potential $\partial_x W$, the annuals potential $\hat{\omega}_{0,2}$ and the genus one potential $\hat{F}_1$. The open GW potentials will be the holomorphic limits of them.
The disk potential is given by \( \partial_x W = \log Y/X \) and the annulus potential is nothing but the Bergman kernel \( B \). The modularity (strictly speaking, quasi-Jacobiness) of both are evident, as shown above. Specializing the above computations to the mirror curve family in (4.5), the formula for \( \hat{F}_1 \) in [EO07] reads

\[
\hat{F}_1 = -\frac{1}{2} \log \tau_B - \frac{1}{12} \log \prod_{r \in R^o} \frac{d(Y + h(X))}{d(X - r_k)^{\frac{1}{2}}} + \frac{1}{2} \ln \det(\text{Im} \tau)^{-1},
\]

where \( R^o \) is the set of finite ramification points which in the current case is the set of all Weierstrass points.

In general, for the sextic model in (2.4) or (2.30), the Bergman tau-function is given by\(^{10} \) [KK04]

\[
\tau_B = 2^2 \det \Pi^{(6)}_A(\omega) \prod_{0 \leq i < j \leq 5} (r_i - r_j)^{\frac{1}{2}}.
\]

Here again the superscript (6) means the period matrix for the sextic model and similarly (5) for the quintic model. Direct calculation shows that

\[
\log \prod_{r \in R^o} \frac{dY}{d(X - X_r)^{\frac{1}{2}}} = \log \left( a_0^3 \prod_{0 \leq i < j \leq 5} (r_i - r_j) \right).
\]

This yields

\[
\hat{F}_1 = -\frac{1}{2} \log \det \Pi^{(6)}_A(\omega) - \frac{5}{24} \log \prod_{0 \leq i < j \leq 5} (r_i - r_j) - \frac{1}{4} \log(16a_0) + \frac{1}{2} \ln \det(\text{Im} \tau)^{-1}.
\]

We now rewrite the above quantities in the quintic model obtained from the coordinate transformation (2.34) which changes the sextic model to the quintic model in (2.6) or (2.35) with \( b_0 = 1 \). Straightforward calculation tells that

\[
\det \Pi^{(6)}_A(\omega) = \det \Pi^{(5)}_A(\omega) \cdot \frac{(c_1 - c_2)^2 r_0^2}{\prod_{k=1}^5 (r_0 - r_k)},
\]

\[
\prod_{0 \leq i < j \leq 5} (r_i - r_j) = r_0^{15} (c_2 - c_1)^{15} \prod_{k=1}^5 (c_2 + e_k)^5 \prod_{1 \leq i < j \leq 5} (e_j - e_i).
\]

Note that

\[
c_2 + e_k = \frac{r_0}{r_0 - r_k} (c_2 - c_1).
\]

Simplifying (4.16) we obtain

\[
\hat{F}_1 = -\frac{1}{2} \log \det \Pi^{(5)}_A(\omega) - \frac{5}{24} \log \prod_{1 \leq i < j \leq 5} (e_j - e_i) + \frac{1}{2} \ln \det(\text{Im} \tau)^{-1}
\]

\[
-\frac{1}{4} \log(16a_0) + \frac{13}{24} \frac{r_0^2 (c_1 - c_2)^2}{\prod_{k=1}^5 (r_0 - r_k)}.
\]

\(^{10} \) The formula is valid for the sextic model with \( a_0 = 1 \).
For definiteness, we take the values for $c_1, c_2$ as in (2.38). The period matrix $\Pi_A^{(5)}(\omega)$ is given in (3.9) and its determinant in (3.10).

According to Lemma 4.1, the last term in (4.20) is the logarithm of a modular function. From (3.8) proved in [Gra88], one has

$$D = \prod_{i<j} (e_i - e_j)^2 = \pm (\det(\frac{\Pi_A^{(5)}(\omega)}{\pi^2}))^{-10} \prod_{\nu \text{ even}} \theta_\nu^2(0, \tau).$$  (4.21)

The product term in the above expression is nothing but essentially the cusp form $\chi_{10}$ [Igu67] according to

$$-214 \chi_{10} = \prod_{\nu \text{ even}} \theta_\nu^2(0, \tau).$$  (4.22)

Combining these facts, we obtain that $\tilde{F}_1$ is the logarithm of a modular form, modulo a constant and the $\det \Im \tau$ term which is real-analytic modular

$$\tilde{F}_1 = \frac{13}{24} \log \frac{(\theta_{0110}\theta_{1000}\theta_{0011})^3}{\theta_{1100}\theta_{0010}\theta_{1001}\theta_{0000}\theta_{0001}\theta_{1111}\theta_{0100}}$$

$$- \frac{5}{48} \log \chi_{10} + \frac{1}{2} \ln \det(\Im \tau)^{-1}$$

$$- \frac{1}{4} \log a_0 + \frac{13}{24} \log \frac{(r_0 - r_2)^2(r_0 - r_4)^2}{(r_2 - r_4)^2} \prod_{k=1}^{5} (r_0 - r_k).$$  (4.23)

We now summarize the above discussions in the following.

**Proposition 4.3.** For the mirror curve family of the resolution of $\mathbb{C}^3/\mathbb{Z}_6$, we obtain the following modularity for the initial values for Eynard–Orantin topological recursion.

1. The disk potential $\partial_x W$ is the logarithm of the pull-back of a meromorphic Jacobi form.
2. The annulus amplitude $\omega_{0,2} = B$ is the pull-back of a weight 2, index 0, meromorphic quasi-Jacobi form. It is symmetric in its arguments. The recursion kernel $K = d^{-1} \omega_{0,2}/(\lambda - \lambda^*)$ is the pull-back of a formal almost-meromorphic Jacobi form of formal weight 0.
3. Up to addition by a constant, the genus one potential $F_1$ is the logarithm of a modular form.

Note that the first and last statements in Proposition 4.3 above match the results in [KPSWR16] obtained by other means.

4.2.3. Modularity in Eynard–Orantin topological recursion. Exactly the same procedure as the genus one case in [FRZZ19] yields the modularity and polynomial structure of the differentials produced by Eynard–Orantin topological recursion. For example, the structure on the upper bound of the order of pole in $\tilde{\omega}_{g,n}$ is a consequence of that for $\lambda, \tilde{\omega}_{0,2}$ and the combinatorial pattern in the recursion algorithm, and is independent of details such as the specific family one chooses. Below we phrase the statements whose proofs are the same for the genus one case [FRZZ19] and are therefore omitted.

**Theorem 4.4.** The following statements hold for $\tilde{\omega}_{g,n}$ with $2g - 2 + n > 0$. 

1. The differential $\tilde{\omega}_{g,n}(\tilde{Y}_1, \ldots, \tilde{Y}_n)$, $n \neq 0$ is symmetric in its arguments. In each argument, it only has poles at the ramification points in $R^g$. At any of the ramification points, the order of pole in any argument is at most $6g + 2n - 4$. Furthermore, the sum of orders of poles over all arguments in each term in $\tilde{\omega}_{g,n}(\tilde{Y}_1, \ldots, \tilde{Y}_n)$ is at most $6g + 4n - 6$.

2. The differential $\tilde{\omega}_{g,n}(\tilde{Y}_1, \ldots, \tilde{Y}_n)$, $n \neq 0$ is a differential polynomial in $S(\tilde{Y}_k - \tilde{Y}_r)$, $k = 1, 2, \ldots, n, r \in R$; the coefficients are elements in the ring of almost-meromorphic modular forms. In particular, $\tilde{\omega}_{g,n}(\tilde{Y}_1, \ldots, \tilde{Y}_n)$, $n \neq 0$ is the pull-back of an almost-meromorphic Jacobi form.

3. The quantities $\tilde{F}_g$, $g \geq 2$ are almost-meromorphic modular forms.

Example 4.5. The following gives the explicit expression for $\tilde{\omega}_{0,3}$. Let the affine coordinates of $p_i$ on the mirror curve (4.5) be $(X_i, \tilde{Y}_i, 1)$, $i = 1, 2, 3$. Then we have

$$
\tilde{\omega}_{0,3}(p_1, p_2, p_3) = \sum_{r_k \in R} \frac{r_k h(r_k)}{4} \prod_{i=1}^{3} \left( \frac{G^{(6)}(X_i, r_k)}{(X_i - r_k)^2} - \frac{1}{X_i} \cdot \Pi_A(\omega)^{-1} \Pi_A(\eta) \cdot \frac{1}{r_k} \right) dX_i / 2\tilde{Y}_i,
$$

(4.24)

where $G^{(6)}(X_1, X_2)$ is as displayed in (3.60).

4.3. Applications to open-closed Gromov–Witten theory. The statements on modularity and structure theorems for the differentials $\omega_{g,n}$ can be translated into those for open and closed GW potentials by the proof of the Remodeling Conjecture. The Yagamuchi-Yau functional equation also follows from keeping track of the non-holomorphic dependence in $\tau$ of the Schiffer kernel $S$, in the same way described in [FRZZ19].

We again only list the main results for completeness. Interested readers are referred to [FRZZ19] for more details.

Theorem 4.6. The open GW potentials $\omega_{g,n}$, $2g - 2 + n > 0$, $n > 0$ are pull-backs of quasi-meromorphic Jacobi forms. The structure as meromorphic quasi-Jacobi forms is as exhibited in Theorem 4.4, with the Schiffer kernel $S$ replaced by the Bergman kernel $B$. The closed GW potentials $F_g = \omega_{g,0}$, $g \geq 2$ are quasi-modular forms.

Theorem 4.7. Let $T_a = \ln Q_a$, $a = 1, 2, 3$ be the coordinates on the base $S$ of the mirror curve family $\pi : \mathcal{C} \to S$ of the resolved manifold $\mathcal{X}$ of $\mathbb{C}^3 / \mathbb{Z}_6$, where $Q_a$, $a = 1, 2, 3$ are given in (4.9). Let $F_0$ be the genus zero potential. Define the quantity $P^{cd}$ to be a solution to

$$
\bar{\partial}_b^{cd} P^{cd} = C^{cd}_{b} := (2\text{Im}\tau)^{-1,c} \left( \frac{\bar{\partial}^3 F_0}{\partial T^b \partial T^c \partial T^d} \right) (2\text{Im}\tau)^{-1,d\bar{\alpha}},
$$

(4.25)

which can alternatively be computed from the Weil-Petersson geometry of the moduli space of complex structures of the mirror CY 3-fold $\hat{X}$. Let

$$
\eta = \Pi_A(\omega)^{-1} \Pi_A(\eta), \quad \hat{\eta} = \eta + Y, \quad Y = 2\pi i \Pi_A(\omega)^{-1} (\tau - \bar{\tau})^{-1} \Pi_A(\omega)^{-t}.
$$

(4.26)
Then the open GW potentials \( \omega_{g,n}, 2g - 2 + n > 0, n > 0 \) satisfy the holomorphic anomaly equations

\[
\left( \frac{\partial}{\partial \eta^{cd}} + \sum_{k,r} \frac{\partial}{\partial B_{kr}^{cd}} \right) \omega_{g,I+1} = \frac{\partial Y}{\partial \eta^{cd}} \cdot \frac{1}{2} \left( \partial_{T^a} \partial_{T^b} \omega_{g-1,I+1} + \sum_{g_1+g_2=g, I=J\cup K, (g_1, J) \neq (0,\emptyset), (g, I)} \partial_{T^a} \omega_{g_1,1} \cdot \partial_{T^b} \omega_{g_2,K} \right),
\]

(4.27)

where \( B_{kr} := B(\tilde{Y}_k - \tilde{Y}_r) \) and the indices \( a, b, c, d \) label the components of the corresponding quantities. The closed GW potentials \( F_g, g \geq 2 \) satisfy

\[
\frac{\partial}{\partial \eta^{cd}} F_g = \frac{\partial Y}{\partial \eta^{cd}} \cdot \frac{1}{2} \left( \partial_{T^a} \partial_{T^b} F_{g-1} + \sum_{g_1+g_2=g, g_1 \neq 0, g} \partial_{T^a} F_{g_1} \cdot \partial_{T^b} F_{g_2} \right).
\]

(4.28)

In summary, we proved the modularity of the differentials constructed from Eynard–Orantin topological recursion, for a special family of genus two hyperelliptic curves. Using the proof of the Remodeling Conjecture this gets translated to modularity of the open-closed GW potentials under the mirror maps.

Modularity (quasi-Jacobiness, strictly speaking) in the open moduli is more or less automatic, using results on the functional field of hyperelliptic Jacobians. The modularity for the differentials relies on results on the differential structure of the functional field. Modularity in the closed moduli, as exhibited in Lemma 4.1, requires that the base of the family is a cover of the Siegel modular variety \( \Gamma(1) \backslash H \). In particular, our approach does not carry over to a two-parameter family of genus two mirror curves in a straightforward manner.

Besides the hyperellipticity mentioned in Remark 4.2, the genus two condition is also heavily replied upon in this work. At genus two, the ring of meromorphic modular forms is graded by the representations \( \text{Sym}^k \omega, \det^k \omega \) of \( \mathbb{U}_2 \) according to the Weyl character formula. For higher genus cases, the irreducible representation decomposition is more involved, making the ring structure of quasi-modular forms less trackable. Furthermore, for the genus two case the moduli space of smooth curves is essentially the moduli space of principally polarized Abelian varieties of dimension two. Hence constructions on the latter yield automatically those on the former. For higher genus cases, we need to pull back the theory of quasi-modular forms from the moduli space of principally polarized Abelian varieties to the Jacobian locus. The description of this locus (the Schottky problem) is still an open problem. The lack of understanding of this problem makes the generalization of our approach to higher genus curves difficult.

Acknowledgements. We would like to thank Bohan Fang for collaboration and discussions at the early stage of this work, and Chiu-Chu Melissa Liu for helpful communications. The bulk of this work was done during Y. R.’s tenure at University of Michigan and he is forever grateful for the support and wonderful environment in the department. Y. R. would like to thank Albrecht Klemm for many stimulating discussion on the topics over the years. He is partially supported by NSF grant DMS 1807079 and NSF FRG grant DMS 1564457. J. Z. would like to thank Kathrin Bringmann and Michael Mertens for discussions on modular forms. J. Z. is partially
supported by a start-up grant from Tsinghua University, the young overseas high-level talents introduction plan of China, and national key research and development program of China (NO. 2020YFA0713000). Part of J. Z.’s work was done while he was a postdoc at the Mathematical Institute of University of Cologne and was supported by German Research Foundation Grant CRC/TRR 191.

Publisher’s Note  Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.

References

[BB66] Baily, W., Borel, A.: Compactification of arithmetic quotients of bounded symmetric domains. Ann. Math. 84, 442–528 (1966)

[Bak98] Baker, H.: On the hyperelliptic sigma functions. Math. Ann. 50(2–3), 462–472 (1898)

[Bak07] Baker, H.: An Introduction to the Theory of Multiply Periodic Functions. Cambridge University Press, Cambridge (1907)

[BEL97] Buchstaber, V., Enolski, V., Leykin, D.: Kleinian functions, hyperelliptic Jacobians and applications. In: Novikov, S.P., Krichever, I.M. (eds.) Reviews in Mathematics and Mathematical Physics (London), vol. 10, no. 2, pp. 1–125. Gordon and Breach, London (1997)

[BEL12] Buchstaber, V., Enolski, V., Leykin, D.: Multi-dimensional sigma-functions (2012). arXiv:1208.0990

[BKMNp09] Bouchard, V., Klemm, A., Mariño, M., Pasquetti, S.: Remodeling the B-model. Commun. Math. Phys. 287(1), 117–178 (2009)

[BL13] Birkenhake, C., Lange, H.: Complex Abelian Varieties, vol. 302. Springer, Berlin (2013)

[BR96] Biswas, I., Raina, A.K.: Projective structures on a Riemann surface. Int. Math. Res. Not. 1996, 753–753 (1996)

[Buc97] Buchstaber, V. M.; Enolskii, V. Z.; Leikin, D. V. Hyperelliptic Kleinian functions and applications. Solitons, geometry, and topology: on the crossroad, 1–33, Amer. Math. Soc. Transl. Ser. 2, 179, Adv. Math. Sci., 33, Amer. Math. Soc., Providence, RI (1997)

[BZ00] Bertrand, D., Zudilin, W.: On the Transcendence Degree of the Differential Field Generated by Siegel Modular Forms, vol. 248 Prépubl. de l’Institut de Math. de Jussieu (2000)

[BZB03] Ben-Zvi, D., Biswas, I.: Theta functions and Szegö kernels. Int. Math. Res. Not. 2003(24), 1305–1340 (2003)

[BZB04] Ben-Zvi, D., Biswas, I.: Opers and theta functions. Adv. Math. 181(2), 368–395 (2004)

[CKYZ99] Chiang, T.-M., Klemm, A., Yau, S.-T., Zaslow, E.: Local mirror symmetry: calculations and interpretations. Adv. Theor. Math. Phys. 3, 495–565 (1999)

[CM19] Clingher, A., Malmendier, A.: Normal forms for Kummer surfaces. Lond. Math. Soc. Lect. Note Ser. 459(2), 107–162 (2019)

[Dij95] Dijkgraaf, R.: Mirror symmetry and elliptic curves. In: The Moduli Space of Curves (Texel Island, 1994), Progress in Mathematics, vol. 129, pp. 149–163. Birkhäuser Boston, Boston (1995)

[Eil16] Eilers, K.: Modular form representation for periods of hyperelliptic integrals. Symmetry Integr. Geom. Methods Appl.: SIGMA 12, 060 (2016)

[Eil18] Eilers, K.: Rosenhain–Thomae formulae for higher genera hyperelliptic curves. J. Nonlinear Math. Phys. 25(1), 86–105 (2018)

[EO07] Eynard, B., Orantin, N.: Invariants of algebraic curves and topological expansion. Commun. Number Theory Phys. 1(2), 347–452 (2007)

[ER07] Enolski, V., Richter, P.: Periods of hyperelliptic integrals expressed in terms of constants by means of Thomae formulae. Philos. Trans. R. Soc. A Math. Phys. Eng. Sci. 366(1867), 1005–1024 (2007)

[EZ84] Eichler, M., Zagier, D.: The theory of Jacobi forms. Progress in Mathematics, vol. 55. Birkhäuser Boston, Inc., Boston, MA (1985)

[Fay06] Fay, J.: Theta Functions on Riemann Surfaces. Lecture Notes in Mathematics, vol. 352. Springer, Berlin (2006)

[FLZ20] Fang, B., Liu, C.-C., Zong, Z.: On the remodeling conjecture for toric Calabi–Yau 3-orbifolds. J. Am. Math. Soc. 33, 135–222 (2020)
Genus Two Siegel Quasi-Modular Forms and Gromov–Witten Theory

[FRZZ19] Fang, B., Ruan, Y., Zhang, Y., Zhou, J.: Open Gromov–Witten theory of \( K_{g,2}, K_{F_1 \times P^1}, K_{W[1,1,2]}, K_{F_1} \) and Jacobi forms. Commun. Math. Phys. 369, 675–719 (2019)

[GH14] Griffiths, P., Harris, J.: Principles of Algebraic Geometry. Wiley, Hoboken (2014)

[Gra85] Grant, D.: Theta functions and division points on abelian varieties of dimension two, Ph.D. thesis, Massachusetts Institute of Technology (1985)

[Gra88] Grant, D.: A generalization of Jacobi’s derivative formula to dimension two. J. Reine Angew. Math. 392, 125–136 (1988)

[Gra90] Grant, D.: Formal groups in genus two. J. Reine Angew. Math. 411, 96–121 (1990)

[Gra91] Grant, D.: A generalization of a formula of Eisenstein. Proc. Lond. Math. Soc. 3(1), 121–132 (1991)

[Gra94] Grant, D.: Units from 3- and 4-torsion on Jacobians of curves of genus 2. Compos. Math. 94(3), 311–320 (1994)

[Igu60] Igusa, J.: Arithmetic variety of moduli for genus two. Ann. Math. 72, 612–649 (1960)

[Igu62] Igusa, J.: On Siegel modular forms of genus two. Am. J. Math. 84(1), 175–200 (1962)

[Igu64] Igusa, J.: On Siegel modular forms of genus two (ii). Am. J. Math. 86(2), 392–412 (1964)

[Igu67] Igusa, J.: Modular forms and projective invariants. Am. J. Math. 89(3), 817–855 (1967)

[Kat76] Katz, N.: p-adic interpolation of real analytic Eisenstein series. Ann. Math. 104, 459–571 (1976)

[KK04] Kokotov, A., Korotkin, D.: Tau-functions on Hurwitz spaces. Math. Phys. Anal. Geom. 7(1), 47–96 (2004)

[KKV97] Katz, S., Klemm, A., Vafa, C.: Geometric engineering of quantum field theories. Nucl. Phys. B 497, 173–195 (1997)

[KPSWR16] Klemm, A., Poretschkin, M., Schimanze, T., Westerholt-Raum, M.: On direct integration for mirror curves of genus two and an almost meromorphic Siegel modular form. Commun. Number Theory Phys. 10(4), 587–701 (2016)

[KSV05] Krishnamoorthy, V., Shaska, T., Völklein, H.: Invariants of Binary Forms, Progress in Galois Theory, pp. 101–122. Springer, Berlin (2005)

[KZ95] Kaneko, M., Zagier, D., A generalized Jacobi theta function and quasi-modular forms. In: The Moduli Space of Curves (Texel Island, 1994): Progress in Mathematics, vol. 129. Birkhäuser Boston, Boston, pp. 156–172 (1995)

[Liu19] Liu, Z.: Nearly overconvergent Siegel modular forms. Annales de l’Institut Fourier 69, 2439–2506 (2019)

[Mats83] Matsutani, S.: Hyperelliptic solutions of KdV and KP equations: re-evaluation of baker’s study on hyperelliptic sigma functions. J. Phys. A Math. Gen. 34(22), 4721 (2001)

[MSS17a] Malmendier, A., Shaska, T.: The Satake sextic in F-theory. J. Geom. Phys. 120, 290–305 (2017)

[MSS17b] Malmendier, A., Shaska, T.: A universal genus-two curve from Siegel modular forms. Symmetry Integr. Geom. Methods Appl. 13, 089 (2017)

[Mum83] Mumford, D.: Tata lectures on theta. I. With the assistance of C. Musili, M. Nori, E. Previato and M. Stillman. Progress in Mathematics, 28. Birkhäuser Boston, Inc., Boston, MA (1983)

[Mum84] Mumford, D.: Tata lectures on theta. II. Jacobian theta functions and differential equations. With the collaboration of C. Musili, M. Nori, E. Previato, M. Stillman and H. Umemura. Progress in Mathematics, 43. Birkhäuser Boston, Inc., Boston, MA (1984)

[Mum99] Mumford, D., Appendix: Curves and Their Jacobians. In: The Red Book of Varieties and Schemes. Lecture Notes in Mathematics, vol 1358. Springer, Berlin, Heidelberg (1999)

[Ôni98] Ônishi, Y.: Complex multiplication formulae for hyperelliptic curves of genus three. Tokyo J. Math. 21(2), 381–431 (1998)

[Ôni02] Ônishi, Y.: Determinant expressions for abelian functions in genus two. Glasg. Math. J. 44(3), 353–364 (2002)

[OP06] Okounkov, A., Pandharipande, R.: Gromov–Witten theory, Hurwitz theory, and completed cycles. Ann. Math. (2) 163(2), 517–560 (2006)

[PSS15] Pitale, A., Saha, A., Schmidt, R.: Representations of \( SL_2(\mathbb{R}) \) and nearly holomorphic modular forms. arXiv:1501.00525 [math.AG]

[PSS19] Pitale, A., Saha, A., Schmidt, R.: Lowest weight modules of \( Sp_4(\mathbb{R}) \) and nearly holomorphic Siegel modular forms. Kyoto J. Math. 61(4), 745–814 (2021)

[Sch73] Schmid, W.: Variation of Hodge structure: the singularities of the period mapping. Invent. Math. 22(3), 211–319 (1973)

[Shi86] Shimura, G.: On a class of nearly holomorphic automorphic forms. Ann. Math. 123(2), 347–406 (1986)

[Shi87] Shimura, G.: Nearly holomorphic functions on Hermitian symmetric spaces. Math. Ann. 278(1–4), 1–28 (1987)

[SZ17] Shen, Y., Zhou, J.: Ramanujan identities and quasi-modularity in Gromov–Witten theory. Commun. Number Theory Phys. 11(2), 405–452 (2017)
[Tak01] Takhtajan, L.: Free bosons and tau-functions for compact Riemann surfaces and closed smooth Jordan curves. Current correlation functions. Lett. Math. Phys. 56(3), 181–228 (2001)
[Tyu78] Tyurin, A.: On periods of quadratic differentials. Russ. Math. Surv. 33(6), 169–221 (1978)
[Urb14] Urban, E.: Nearly Overconvergent Modular Forms, Iwasawa Theory, 2012, pp. 401–441. Springer, Berlin (2014)
[vdG08] van der Geer, G.: Siegel Modular Forms and Their Applications, pp. 181–245. Springer, Berlin (2008)
[Zag08] Zagier, D.: Elliptic modular forms and their applications. In: The 1-2-3 of Modular Forms, Universitext, pp. 1–103. Springer, Berlin (2008)
[Zud00] Zudilin, W.: Thetanulls and differential equations. Sb. Math. 191(12), 1827–1871 (2000)