Random spectrahedra

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Abstract. Spectrahedral cones are linear sections of the cone of positive semidefinite symmetric matrices. We study statistical properties of random spectrahedral cones (intersected with the sphere)

\[ \mathcal{F}_{\ell,n} = \{ (x_0, \ldots, x_\ell) \in S^\ell \mid x_0 1 + x_1 R_1 + \cdots + x_\ell R_\ell \succ 0 \} \]

where \( R_1, \ldots, R_\ell \) are independent GOE\((n)\)-distributed matrices rescaled by \((2n\ell)^{-1/2}\).

We relate the expectation of the volume of \( \mathcal{F}_{\ell,n} \) and the expectation of the volume of its boundary \( \partial \mathcal{F}_{\ell,n} \) with some statistics of the smallest eigenvalue of a GOE\((n)\) matrix, by providing explicit formulas for these quantities. These formulas imply that as \( \ell, n \to \infty \) on average \( \mathcal{F}_{\ell,n} \) keeps a positive fraction of the volume of the sphere \( S^\ell \) (the exact constant is \( \Phi(-1) \approx 0.1587 \), where \( \Phi \) is the cumulative distribution function of a standard gaussian variable). Furthermore, we prove that as \( \ell, n \to \infty \) the volume of the boundary behaves as 

\[ (1 - (2\ell)^{-1} + O(\ell^{-2}) + O(n^{-1/2})) \cdot \text{vol}(S^\ell) \]

For \( \ell = 2 \) spectrahedra are generically smooth, but already when \( \ell = 3 \) singular points on their boundaries appear with positive probability. We relate the average number \( E\sigma_n \) of singular points on the boundary of a three-dimensional spectrahedron \( \mathcal{F}_{3,n} \) to the volume of the set of symmetric matrices whose two smallest eigenvalues coincide. In the case of quartic spectrahedra \((n = 4)\) we show that \( E\sigma_4 = 6 - \frac{\pi}{2} \). Moreover, we prove that the average number \( E\rho_n \) of singular points on the random symmetroid surface

\[ \Sigma_{3,n} = \{ (x_0, x_1, x_2, x_3) \in S^3 \mid \det(x_0 1 + x_1 R_1 + x_2 R_2 + x_3 R_3) = 0 \} \]

equals \( n(n - 1) \). This quantity is related to the volume of the variety of real symmetric matrices with repeated eigenvalues.

1. Introduction

A spectrahedron is an affine-linear section of the cone \( \mathcal{P}_n \subset \text{Sym}(n, \mathbb{R}) \) of positive semidefinite symmetric matrices. On the space \( \text{Sym}(n, \mathbb{R}) \) of \( n \times n \) real symmetric matrices there is a partial order defined by \( A \succ B \), if and only if \( A - B \in \mathcal{P}_n \). Every spectrahedron can then be parametrized as the set of solutions of a linear matrix inequality:

\[ M_0 + x_1 M_1 + \cdots + x_\ell M_\ell \succ 0, \quad x = (x_1, \ldots, x_\ell) \in \mathbb{R}^\ell, \]

for some symmetric matrices \( M_0, \ldots, M_\ell \in \text{Sym}(n, \mathbb{R}) \).

Optimization of a linear function over a spectrahedron is called semidefinite programming [17, 3]. This is a useful generalization of linear programming, i.e. optimization of a linear function over a polyhedron. Such problems as finding the smallest eigenvalue of a symmetric matrix or optimizing a polynomial function on the sphere can be approached using semidefinite programming. The presence of singularities on the boundary of a three-dimensional spectrahedron is relevant for optimization: with a positive probability a linear function constrained on a polyhedron attains its maximum in a vertex of the polyhedron, and, similarly, with a positive probability a linear function constrained on a spectrahedron attains its maximum in a singular point of the boundary of the spectrahedron.
Figure 1.1. On the left is the cubic spectrahedron from the introduction. On the right is a quartic spectrahedron (in [6] it is called the “pillow”). The singularities on the boundaries of both spectrahedra are visible.

For example, consider the cubic spectrahedron shown on Figure 1.1:

$$\mathcal{S} = \{ (x, z, y) \in \mathbb{R}^3 \mid \begin{pmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{pmatrix} \in \mathcal{P}_n \}.$$

The linear function $$\psi_w(x, y, z) = \langle w, (x, y, z) \rangle, w \in S^3$$, constrained on $$\mathcal{S}$$ attains its maximum at a point $$(x, y, z) \in \partial \mathcal{S}$$ on the boundary of $$\mathcal{S}$$ at which the normal cone to $$\partial \mathcal{S}$$ contains $$w$$. At a singular point of the boundary the normal cone has positive dimension and hence the set of $$w \in S^3$$ for which the maximum of $$\psi_w$$ is attained at a singular point of the boundary of the spectrahedron has positive volume in $$S^3$$.

Besides mentioned applications spectrahedra also appear in modern real algebraic geometry: in [10] Helton and Vinnikov gave a beautiful characterization of two-dimensional spectrahedra and in [8] Degtyarev and Itenberg described all generic possibilities for the number of singular points on the boundary of a quartic three-dimensional spectrahedron. The reader can also look at [23] for a survey article.

1.1. Random spectrahedra. In order to perform a probabilistic study, it is more convenient to work instead with spherical spectrahedra, defined by:

$$x_0M_0 + x_1M_1 + \cdots + x_\ell M_\ell \succ 0, \quad x = (x_0, x_1, \ldots, x_\ell) \in S^\ell.$$

A generic spherical spectrahedron has nonempty interior and, after a change of coordinates in the space of symmetric matrices, it can be presented as:

$$\mathcal{S}_{\ell, n} = \{ x = (x_0, \ldots, x_\ell) \in S^\ell \mid x_0 I + x_1 R_1 + \cdots + x_\ell R_\ell \succ 0 \}.$$

From now on, for simplicity, we abuse the terminology and use the term “spectrahedron” to refer to a spherical spectrahedron.

Our choice of random model for spectrahedra is as follows. In the representation (1.1) we sample the matrices $$R_1, \ldots, R_\ell$$ independently and identically distributed from the Gaussian Orthogonal Ensemble GOE(n) [16, 20], rescaled by $$(2n\ell)^{-1/2}$$:

$$R_i = \frac{1}{\sqrt{2n\ell}} Q_i, \quad \text{where} \quad Q_i \sim \text{GOE}(n), \quad i = 1, \ldots, \ell.$$

By $$Q \sim \text{GOE}(n)$$ we mean that the joint probability density of the entries of the symmetric matrix $$Q \in \text{Sym}(n, \mathbb{R})$$ is $$\varphi(Q) = \frac{1}{C_n} \exp(-\frac{1}{2} \text{tr}(Q^2))$$, where $$C_n$$ is the normalization constant with $$\int_{\text{Sym}(n, \mathbb{R})} \varphi(Q) \, dQ = 1$$. That is, the entries of $$Q$$ are centered gaussian random variables, the diagonal entries having variance 1 and the off-diagonal entries having variance 1/2. The scaling factor $$(2n\ell)^{-1/2}$$ serves to balance the order of magnitudes of eigenvalues of the two summands $$x_0 I$$ and $$x_1 R_1 + \cdots + x_\ell R_\ell$$. 2
1.2. Statistical properties of random spectrahedra. In the following, for a semialgebraic subset $X \subset \mathbb{R}^d$ of dimension $d$ by $|X|$ we denote the $d$-dimensional volume of the set of smooth points of $X$ and by $|X|_{\text{rel}} := |X|/|S^d|$ we denote the relative volume of $X$. The statistics we will be interested in are the expected values of the relative volumes $|\mathcal{A}_{n}\text{rel}|, |\partial \mathcal{A}_{n}\text{rel}|$ of the spectrahedron and of its boundary and the expected number of singular points on $\partial \mathcal{A}_{n}\text{rel}$. The boundary $\partial \mathcal{A}_{n}$ of a 3-dimensional spectrahedron $\mathcal{A}_{n}$ is in general singular and, generically, has finitely many singular points that are all nodes (see Proposition 5.2). Let us denote the number of singular points on $\mathcal{A}_{n}$ by $\sigma_n$. The boundary $\partial \mathcal{A}_{n}$ is a semialgebraic subset of the symmetroid surface $\Sigma_{3,n} = \{x \in S^3 \mid \det(x_0\mathbb{1} + x_1R_1 + x_2R_2 + x_3R_3) = 0\}$. Hence $\sigma_n$ is smaller than the number $\rho_n$ of singular points on $\Sigma_{3,n}$. Summarizing, we will be interested in:

$$E|\mathcal{A}_{n}\text{rel}|, E|\partial \mathcal{A}_{n}\text{rel}|, E\sigma_n \text{ and } E\rho_n.$$ 

Our main results on those three quantities follow next.

1.3. Main results. To state our first main result, let $\lambda_{\text{min}}(Q)$ denote the smallest eigenvalue of the matrix $Q$. For the scaled smallest eigenvalue we write

$$\tilde{\lambda}_{\text{min}}(Q) := \frac{\lambda_{\text{min}}(Q)}{\sqrt{2n}}.$$ 

The following is proved in Section 3 below.

**Theorem 1.1** (Expected volume of the spectrahedron). Let $F_\ell$ denote the cumulative distribution function of the student’s $t$-distribution with $\ell$ degrees of freedom [13, Chapter 28] and $\Phi(x)$ denote the cumulative distribution function of the normal distribution [19, 40:14:2]. Then:

1. $E|\mathcal{A}_{n}\text{rel}| = E_{Q \sim \text{GOE}(n)} F_\ell(\tilde{\lambda}_{\text{min}}(Q))$.

Moreover, we have

2. $E|\mathcal{A}_{n}\text{rel}| = F_\ell(-1) + O(n^{-2/3})$ uniformly in $\ell$,
3. $E|\partial \mathcal{A}_{n}\text{rel}| = F_\ell(-1) + O(\ell^{-1}) + O(n^{-2/3})$.

Note that $\Phi(-1) \approx 0.1587$. This means that asymptotically (in both $n$ and $\ell$) the average volume of a spectrahedron is at least 15% of the volume of the sphere. Next we consider the expected volume of the boundary. The following theorem will be proved in Section 4.

**Theorem 1.2** (Expected volume of the boundary of the spectrahedron). Let $\chi_{\ell,n}^2$ denote the chi-square distribution with $\ell - 1$ degrees of freedom [13, Chapter 18] and define the function

$$f_{\ell,n}(x) = \frac{\ell}{\ell + x^2} \mathbb{E}_{w \sim \chi_{\ell,n}^2 \text{tr}} \left[ 1 + \frac{x^2}{\ell} + \frac{w}{2n\ell} \right].$$

Then:

1. $E|\partial \mathcal{A}_{n}\text{rel}| = E_{Q \sim \text{GOE}(n)} f_{\ell,n}(\tilde{\lambda}_{\text{min}}(Q))$.

Moreover, we have

2. $E|\partial \mathcal{A}_{n}\text{rel}| = \sqrt{\frac{\ell}{\ell + 1}} + O(n^{-1/2})$.
3. $E|\partial \mathcal{A}_{n}\text{rel}| = 1 - \frac{1}{\sqrt{\ell}} + O(\ell^{-2}) + O(n^{-1/2})$.

For the average number of singular points $\sigma_n$ on $\partial \mathcal{A}_{n}$ and number of singular points $\rho_n$ on $\Sigma_{3,n}$ the result is more delicate to state. We denote the dimension of $\text{Sym}(n, \mathbb{R})$ by $N := \frac{n(n + 1)}{2}$ and the unit sphere there by $S^{N-1} := \{Q \in \text{Sym}(n, \mathbb{R}) \mid ||Q||^2 = \text{tr}(Q^2) = 1\}$. Let $\Delta \subset S^{N-1}$ be
the set of symmetric matrices of unit norm and with repeated eigenvalues and let \(\Delta_1 \subset \Delta\) be its subset consisting of symmetric matrices whose two smallest eigenvalues coincide:

\[
\Delta := \{ Q \in \text{Sym}(n, \mathbb{R}) \cap S^{n-1} \mid \lambda_i(Q) = \lambda_j(Q) \text{ for some } i \neq j \},
\]

\[
\Delta_1 := \{ Q \in \text{Sym}(n, \mathbb{R}) \cap S^{n-1} \mid \lambda_1(Q) = \lambda_2(Q) \}.
\]

Note that \(\Delta\) and \(\Delta_1\) are both semialgebraic subsets of \(S^{n-1}\) of codimension two; \(\Delta\) is actually algebraic. The following theorem relates \(E\sigma_n\) and \(E\rho_n\) to the volumes of \(\Delta_1\) and \(\Delta\), respectively. We give its proof in Section 5.

**Theorem 1.3** (The average number of singular points). *The average number of singular points on the boundary of a random 3-dimensional spectrahedron \(\mathcal{S}_{3,n} \subset S^3\) equals*

\[
(1) \quad E\sigma_n = 2|\Delta_1|_{\text{rel}}.
\]

*The average number of singular points on the symmetroid \(\Sigma_{3,n} \subset S^3\) equals*

\[
(2) \quad E\rho_n = 2|\Delta|_{\text{rel}}.
\]

In [7, Thm. 1.1] it was proved that \(|\Delta|_{\text{rel}} = \binom{n}{2}\). This immediately yields the following.

**Corollary 1.4.** *\(E\rho_n = n(n-1)\).*

For the expectation of \(\sigma_n\) we are lacking such an explicit formula. However combining Theorem 1.3 (1) with the formula in [7, Remark 6] one obtains

\[
(1.3) \quad \mathbb{E}\sigma_n = \frac{2^n}{\sqrt{\pi} n!} \binom{n}{2} \int_{u \in \mathbb{R}} \mathbb{E} \frac{\mathbb{E}}{n-2} \left[ \det(Q - uI) \right]^2 \mathbb{1}_{[Q-uI^+0]} e^{-u^2} du.
\]

We expect that \(\lim_{n \to \infty} \frac{\mathbb{E}\sigma_n}{\mathbb{E}\rho_n} = 0\), but it is difficult to predict how small is \(\mathbb{E}\sigma_n\) compared to \(\mathbb{E}\rho_n\). The main challenge is to handle the indicator function \(\mathbb{1}_{[Q-uI^+0]}\) in the integral above.

Quartic spectrahedra are a special case of our study, corresponding to \(n = 4\). In this case the symmetroid surface

\[
\Sigma_{3,4} = \{ x \in S^3 \mid \det(x_0I + x_1R_1 + x_2R_2 + x_3R_3) = 0 \}
\]

has degree four, since \(1, R_1, R_2, R_3 \in \text{Sym}(4, \mathbb{R})\). In [8] Degtyarev and Itenberg proved that all possibilities for \(\sigma_4\) and \(\rho_4\) are realized by some generic spectrahedra \(\mathcal{S}_{3,4}\) and their symmetroids \(\Sigma_{3,4}\) under the following constraints:

\[
(1.4) \quad \sigma_4 \text{ is even and } 2 \leq \sigma_4 \leq 10; \quad \rho_4 \text{ is a multiple of } 4 \quad \text{and} \quad 4 \leq \rho_4 \leq 20.
\]

(Degtyarev and Itenberg proved this for the spectrahedron and its symmetroid in projective space, that is why in our condition (1.4) above we have to double their estimates.) An “average picture” of this result is given in the following proposition.

**Proposition 1.5** (The average number of nodes on the boundary of a quartic spectrahedron). *We have

\[
E\sigma_4 = 6 - \frac{4}{\sqrt{3}} \approx 3.69 \quad \text{and} \quad E\rho_4 = 12.
\]

It would be interesting to understand the distribution of the random variables \(\sigma_4, \rho_4\) and compare it with the deterministic picture in (1.4).
1.4. Another possible random model. Another natural model of random spectrahedra is by defining them as linear sections of $P_n \cap S^{N-1}$ by a uniformly distributed $(\ell + 1)$-dimensional plane $V$ in $\text{Sym}(n, \mathbb{R})$:

\begin{equation}
S_{\ell,n}(V) := P_n \cap S^{N-1} \cap V.
\end{equation}

Before proceeding we argue in favor of the model (1.1) over the random linear section model (1.5). The main reason for this is that the expected volume of $S_{\ell,n}(V)$ decays to zero for fixed $\ell$ and $n \to \infty$, which we prove in Proposition 1.6 below. In fact, typically a spherical spectrahedron of the form $S_{\ell,n}(V)$ is empty (this is essentially due to the fact that the volume of the positive semidefinite cone $P_n$ decays exponentially fast as $n \to \infty$), and this model is inaccessible for probabilistic studies. For the model introduced in (1.1) this appears differently: for large $n$ and $\ell$ the spectrahedron $\mathcal{J}_{\ell,n}$ keeps a fraction of about 15% of the volume of the sphere $S^\ell$; cf. Theorem 1.1. In fact, the spectrahedron $\mathcal{J}_{\ell,n}$ is never empty, as it contains an open neighborhood of $(1, 0, \ldots, 0) \in S^\ell$.

**Proposition 1.6** (Decay of the random linear section model). Let $V$ be uniformly distributed in the Grassmannian of $(\ell + 1)$-dimensional subspaces of $\text{Sym}(n, \mathbb{R})$. Then for every $c > 0$ we have $\mathbb{P}[S_{\ell,n}(V) \neq \emptyset] \leq O(n^{-c})$.

**Proof.** Let us denote by $\mu(V)$ the maximum number of positive eigenvalues that a matrix in $V$ has. For $n \geq 4$, we have the simple bound $\mathbb{P}[S_{\ell,n}(V) \neq \emptyset] = \mathbb{P}[\mu(V) = n] \leq \mathbb{P}[\mu(V) \geq \frac{\lambda}{2} + \sqrt{n}]$. By [15, Lemma 4] the last quantity is smaller than $O(n^{-c})$ for every $c > 0$. \hfill $\square$

1.5. **Notation.** Throughout the article some symbols are repeatedly used for the same purposes: $\text{Sym}(n, \mathbb{R})$ stands for the space of $n \times n$ real symmetric matrices. By the symbols $\mathcal{Q} = (Q_1, \ldots, Q_{\ell}) \in (\text{Sym}(n, \mathbb{R}))^\ell$ and $\mathcal{R} = (R_1, \ldots, R_{\ell}) \in (\text{Sym}(n, \mathbb{R}))^\ell$ we denote a collection of $\ell$ symmetric matrices and its rescaled version respectively, i.e. $R_i = \frac{1}{\sqrt{\alpha}} Q_i$. The $k$-dimensional sphere endowed with the standard metric is denoted $S^k$. The symbol $\mathbb{1}$ stands for the unit matrix (of any dimension). For $x = (x_0, x_1, \ldots, x_\ell) \in S^\ell$ we denote the matrices $Q(x) = x_1 Q_1 + \cdots + x_\ell Q_\ell$ and $A(x) = x_0 \mathbb{1} + Q(x)$. By $\mathcal{J}_{\ell,n}$, $\partial \mathcal{J}_{\ell,n}$ and $\Sigma_{\ell,n}$ we denote a (random) spectrahedron, its boundary and a symmetroid hypersurface respectively. Letters $\alpha, \lambda$ and $\mu$ are used to denote eigenvalues and $\tilde{\lambda} = \frac{1}{\sqrt{2n}} \lambda$ stands for the rescaled eigenvalue $\lambda$.

1.6. **Organization of the article.** The organization of the paper is as follows. In the next section we recall some known deviation inequalities for the smallest eigenvalue of a GOE$(n)$-matrix. In Sections 3–5 we prove our main theorems. Finally, Section 6 deals with the case of quartic spectrahedra.

2. **Deviation inequalities for the smallest eigenvalue**

In this section we want to summarize known inequalities for the deviation of $\lambda_{\text{min}}(Q)$ from its expected value in the GOE$(n)$ random matrix model. The results that we present are due to [14]. Note that in that reference, however, the inequalities are given for the largest eigenvalue $\lambda_{\text{max}}(Q)$. Since the GOE$(n)$-distribution is symmetric around 0, we have $\lambda_{\text{max}}(Q) \sim -\lambda_{\text{min}}(Q)$. Using this we translate the deviation inequalities for $\lambda_{\text{max}}(Q)$ from [14] into deviation inequalities for $\lambda_{\text{min}}(Q)$. Furthermore, note that in [14, (1.2)] the variance for the GOE$(n)$-ensemble is defined differently than it is here: eigenvalues of a random matrix in [14] are $\sqrt{2}$ times eigenvalues in our definition.

We express the deviation inequalities in terms of the scaled eigenvalue $\tilde{\lambda}_{\text{min}}(Q)$, cf. (1.2). The following Proposition is [14, Theorem 1]. We will not need this result in the rest of the paper directly, but we decided to recall it here because it gives an idea of the behavior of the smallest eigenvalue of a random GOE$(n)$ matrix, in terms of which our theorem on the volume of random spectrahedra is stated.
Proposition 2.1. For some constant $C > 0$, all $n \geq 1$ and $0 < \epsilon < 1$, we have
\[
\Pr_{Q \in \text{GOE}(n)} \left\{ \tilde{\lambda}_{\min}(Q) \leq -(1 + \epsilon) \right\} \leq C e^{-C^{-1} n^{3/2}}
\]
and
\[
\Pr_{Q \in \text{GOE}(n)} \left\{ \tilde{\lambda}_{\min}(Q) \geq -(1 - \epsilon) \right\} \leq C e^{-C^{-1} n^{3/2}}.
\]

Proposition 2.1 shows that for large $n$ the mass of $\tilde{\lambda}_{\min}(Q)$ concentrates exponentially around $-1$. Thus $\mathbb{E} \tilde{\lambda}_{\min}(Q)$ converges to $-1$ as the following proposition shows.

Proposition 2.2. For some constant $C > 0$ and all $n \geq 1$ we have
\[
|\mathbb{E} \tilde{\lambda}_{\min}(Q) + 1| \leq \mathbb{E} |\tilde{\lambda}_{\min}(Q) + 1| \leq C n^{-2/3}.
\]

Proof. By [14, Equation after Corollary 3] we have
\[
(2.1) \quad \lim_{n \to \infty} \sup \, 2^p n^{2p/3} \mathbb{E} |\tilde{\lambda}_{\min}(Q) + 1|^p < \infty.
\]
The assertion follows from monotonicity of the integral: $|\mathbb{E} \tilde{\lambda}_{\min}(Q) + 1| \leq \mathbb{E} |\tilde{\lambda}_{\min}(Q) + 1|$ and (2.1) with $p = 1$. □

Remark. The distribution of the scaled largest eigenvalue of a GOE($n$) matrix for $n \to \infty$ is known as the Tracy-Widom distribution [21]. Suprisingly, this distribution appears in branches of probability that at first sight seem unrelated. For instance, the length of the longest increasing subsequence in a permutation that is chosen uniformly at random in the limit follows the Tracy-Widom distribution [5]. In the survey article [22] Tracy and Widom give an overview of appearances of the distribution in growth processes, random tilings, statistics, queuing theory and superconductors. The present article adds spectrehedra to that list.

3. Expected volume of the spectrahedron

In this section we prove Theorem 1.1.

3.1. Proof of Theorem 1.1 (1). Note that due to the rotational invariance of the standard Gaussian distribution $N(0,1)$ the volume of a spectrahedron $\mathcal{F}_n$ can be computed as follows:
\[
(3.1) \quad |\mathcal{F}_{n,\ell}|_{\text{rel}} = \frac{|\mathcal{F}_n|}{|S^n|} = \Pr_{\xi \sim N(0,1)} \left\{ \xi_0 \mathbb{I} + \frac{1}{\sqrt{2n}} \sum_{i=1}^{\ell} \xi_i Q_i = 0 \right\}
\]
Using this and the following shorthand notation
\[
(3.2) \quad Q(x) = \sum_{i=1}^{\ell} x_i Q_i, \quad A(x) = x_0 \mathbb{I} + \frac{1}{\sqrt{2n}} Q(x)
\]
we now write the expectation $\mathbb{E} |\mathcal{F}_{n,\ell}|_{\text{rel}}$ of the relative volume of the random spectrahedron as:
\[
\mathbb{E} |\mathcal{F}_{n,\ell}|_{\text{rel}} = \mathbb{E}_{Q \in \text{GOE}(n)^\ell} \Pr_{\xi_0, \ldots, \xi_{\ell} \sim N(0,1)} \left\{ \xi_0 \mathbb{I} + \frac{1}{\sqrt{2n}} \sum_{i=1}^{\ell} \xi_i Q_i = 0 \right\} = \mathbb{E}_{Q} \mathbb{E}_{\xi} 1_{\{A(\xi) > 0\}} =: (*)
\]
where $1_Y$ denotes the characteristic function of the set $Y$. Using Tonelli’s theorem the two integrations can be exchanged:
\[
(3.3) \quad (*) = \mathbb{E}_{\xi} \mathbb{E}_{Q} 1_{\{A(\xi) > 0\}} = \mathbb{E}_{\xi} \Pr_{Q \in \text{GOE}(n)^\ell} \left\{ \xi_0 \mathbb{I} + \frac{1}{\sqrt{2n}} Q(\xi) > 0 \right\}.
\]
For a unit vector \( x = (x_1, \ldots, x_\ell) \in S^{\ell-1} \) by the orthogonal invariance of the GOE-ensemble we have \( Q(x) \sim \text{GOE}(n) \) which leads to

\[
(*) = \mathbb{E}_\xi \left[ \left\{ \frac{\xi_0}{(\xi_1^2 + \cdots + \xi_\ell^2)^{1/2}} \parallel + \frac{1}{\sqrt{2n}} Q > 0 \right\} \right]
\]

\[
= \mathbb{E}_{Q \in \text{GOE}(n)} \left( \mathbb{P}_\xi \left( \left\{ \frac{\xi_0 \sqrt{\ell}}{(\xi_1^2 + \cdots + \xi_\ell^2)^{1/2}} \parallel + \frac{1}{\sqrt{2n}} Q > 0 \right\} \right) \right) ,
\]

where in the second equality we again used Tonelli’s theorem. Let us put \( t_\ell := \frac{\xi_0 \sqrt{\ell}}{(\xi_1^2 + \cdots + \xi_\ell^2)^{1/2}} \). Note that by [13, (28.1)] the random variable \( t_\ell \) follows the Student’s t-distribution with \( \ell \) degrees of freedom. Since this distribution is symmetric around the origin and \( t_\ell \parallel + \frac{1}{\sqrt{2n}} Q > 0 \) is equivalent to \( -t_\ell \leq \frac{1}{\sqrt{2n}} \lambda_{\min}(Q) \), we have

\[
(*) = \mathbb{E}_{Q \in \text{GOE}(n)} \left( \mathbb{P}_\xi \left( t_\ell \leq \frac{1}{\sqrt{2n}} \lambda_{\min}(Q) \right) \right) = \mathbb{E}_{Q \in \text{GOE}(n)} F_\ell(\lambda_{\min}(Q)),
\]

where \( F_\ell \) is the cumulative distribution function of the random variable \( t_\ell \). This proves Theorem 1.1 (1) since \( (*) = \mathbb{E}[\mathcal{F}_{\ell,n}]_{\text{rel}} \). \( \square \)

3.2. Proof of Theorem 1.1 (2). The random variable \( t_\ell \) is absolutely continuous. Moreover, its density \( F'_\ell \) is continuous and bounded uniformly in \( \ell \) [13, (28.2)]. This combined with the following lemma proves Theorem 1.1 (2):

**Lemma 3.1.** Let \( f_\ell : \mathbb{R} \to \mathbb{R} \) be a sequence of smooth functions such that there exists a constant \( c > 0 \) with \( \|f'_\ell\|_\infty \leq c, \ell \geq 1 \). Then \( \mathbb{E}_{Q \in \text{GOE}(n)} f_\ell(\lambda_{\min}(Q)) = f_\ell(-1) + O(n^{-2/3}) \) uniformly in \( \ell \). 

**Proof.** Write \( f_\ell(\lambda_{\min}) \) as \( f_\ell(\lambda_{\min}) = f_\ell(-1) + f'_\ell(x)(\lambda_{\min} + 1) \) for some \( x = x(\lambda_{\min}) \in \mathbb{R} \). Taking expectation we obtain

\[
\mathbb{E} f_\ell(\lambda_{\min}) = f_\ell(-1) + \mathbb{E}(f'_\ell(x)(\lambda_{\min} + 1)) \leq f_\ell(-1) + c \mathbb{E} |\lambda_{\min} + 1| = f_\ell(-1) + O(n^{-2/3}),
\]

where the last inequality follows from Proposition 2.2. \( \square \)

3.3. Proof of Theorem 1.1 (3). In the preceding subsection we have shown the (uniform in \( \ell \)) asymptotic \( \mathbb{E}[\mathcal{F}_{\ell,n}]_{\text{rel}} = F_\ell(-1) + O(n^{-2/3}) \), where \( F_\ell(x) = \mathbb{P}(\{t_\ell \leq x\}) \) and the random variable \( t_\ell \) follows the Student’s t-distribution with \( \ell \) degrees of freedom. By [13, (28.15)] for fixed \( x \) we have \( F_\ell(x) = \Phi(x)(1 + O(\ell^{-1})) \), where \( \Phi \) is the cumulative distribution function of the standard normal distribution. Plugging in \( x = -1 \) settles Theorem 1.1 (3). \( \square \)

4. Expected volume of the boundary of the spectrahedron

In this section we prove Theorem 1.2.

4.1. Proof of Theorem 1.2 (1). We use the Kac-Rice formula for volume of random manifolds [1, Theorem 12.1.1]. Let \( Q(x) = \sum_{i=1}^\ell x_i Q_i \) and \( A(x) = x_0 \parallel + \frac{1}{\sqrt{2n}} Q(x) \) be as in (3.2) and denote by \( \mu_1(x) \leq \ldots \leq \mu_n(x) \) the ordered eigenvalues of \( Q(x) \). We write the eigenvalues of \( A(x) \) as \( \alpha_j(x) = x_0 + \frac{1}{\sqrt{2n}} \mu_j(x) \). Later we will also need \( \lambda_1 \leq \ldots \leq \lambda_n \), the eigenvalues of the first matrix \( Q_1 \).

The set of smooth points of \( \partial \mathcal{F}_{\ell,n} \) is described as follows:

\[
(\partial \mathcal{F}_{\ell,n})_{\text{sm}} = \{ x \in S^{\ell} \mid \alpha_1(x) = 0, \alpha_1(x) \neq \alpha_2(x) \}
\]

In the following we omit ‘\( \varepsilon \in S^{\ell} \)’ in the notation of sets. By continuity of Lebesgue measure, we have \( |\partial \mathcal{F}_{\ell,n}| = \lim_{\varepsilon \to 0} |\{ \alpha_1 = 0 \} \cap \{ |\alpha_1 - \alpha_2| \geq \varepsilon \}| \) and, consequently, taking expectation over \( Q \sim \text{GOE}(n) \) we have

\[
(\partial \mathcal{F}_{\ell,n})_{\text{sm}} = \mathbb{E} \lim_{\varepsilon \to 0} |\{ \alpha_1 = 0 \} \cap \{ |\alpha_1 - \alpha_2| \geq \varepsilon \}| = \lim_{\varepsilon \to 0} \mathbb{E} |\{ \alpha_1 = 0 \} \cap \{ |\alpha_1 - \alpha_2| > \varepsilon \}|
\]
for the last equality we have used monotone convergence to exchange the limit with the expectation. For a fixed $\epsilon$ the function $\alpha_1$ is smooth on the set $\{\alpha_1 - \alpha_2 > \epsilon\}$ and we can use the Kac-Rice formula\footnote{Here we are applying a generalization of [1, Theorem 12.1.1] with the choice $M = S^n$, $f = \alpha_1 : S^k \to \mathbb{R}$, $h = \alpha_1 - \alpha_2 : S^k \to \mathbb{R}$ (two random fields) and $B = (-\infty, \epsilon) \cup (\epsilon, \infty) \subset \mathbb{R}$. The higher generality comes from the fact that $f^{-1}(0)$ has codimension one in $S^n$, in this, and in the more general case when $f : M \to \mathbb{R}$ with $\text{dim}(M) \geq k$ (the codimension-$k$ case), we have to modify the statement of [1, Theorem 12.1.1] as follows. Under the assumption that “0” is a regular value of the random map $f$ on $h^{-1}(B)$ with probability one, the expectation of the geometric (dim$(M) - k$)-dimensional volume of $f^{-1}(0) \cap h^{-1}(B)$ equals:}

\begin{equation}
E |\partial \mathcal{F}_{\ell,n}| = \lim_{\epsilon \to 0} \int_{S^{\ell}} E \{ |\nabla S^{\ell} \alpha_1(x)| \cdot 1_{\{\alpha_1(x) - \alpha_2(x) > \epsilon\}} \mid \alpha_1(x) = 0 \} \cdot p_{\alpha_1}(0) \, dx.
\end{equation}

Here $\nabla S^{\ell}$ denotes the gradient with respect to an orthogonal basis of $T_y S^{\ell}$, also called the spherical gradient.

Since for a unit vector $x \in S^{\ell-1}$ we have $Q(x) \sim \text{GOE}(n)$ statistical properties of $\partial \mathcal{F}_{\ell,n}$ are invariant under rotations preserving the axis through the point $(1,0,\ldots,0) \in S^\ell$. Therefore the integrand in (4.3) only depends on $y = (x_0, \sqrt{1 - x_0^2}, 0, \ldots, 0)$. Moreover, the uniform distribution on $S^\ell$ induces the uniform distribution on the first entry $x_0$. Hence,

\begin{equation}
E |\partial \mathcal{F}_{\ell,n}| = \lim_{\epsilon \to 0} \int_{x_0 \in [-1,1]} E \{ |\nabla S^{\ell} \alpha_1(y)| \cdot 1_{\{\alpha_1(y) - \alpha_2(y) > \epsilon\}} \mid \alpha_1(y) = 0 \} \cdot p_{\alpha_1}(0) \, dx =: (\star)
\end{equation}

Before continuing with the evaluation of (\star) we examine the integrand and the random variables therein. Recall that $\alpha_j(y)$ is the $j$-the eigenvalue of $A(y)$, i.e., $\alpha_1(y) = x_0 + \frac{1}{\sqrt{2n\ell}} \mu_1(y)$. Note

\begin{equation}
\alpha_1(y) = 0 \quad \text{if and only if} \quad \mu_1(y) = -x_0 \sqrt{2n\ell}.
\end{equation}

Let us denote by $u(x)$ the normalized eigenvector of $Q(x)$ associated to the eigenvalue $\mu_1(x)$. The spherical gradient $\nabla S^{\ell} \alpha_1(x)$ is the projection of the (ordinary) gradient $\nabla S^{\ell} \alpha_1(x)$ onto $T_x S^\ell$:

\begin{equation}
\nabla S^{\ell} \alpha_1(x) = \nabla S^{\ell+1} \alpha_1(x) - \langle \nabla S^{\ell+1} \alpha_1(x), x \rangle x.
\end{equation}

Using Hadamard’s first variation [20, Section 1.3] we can write

\begin{equation}
\nabla S^{\ell+1} \alpha_1(x) = \left(1, \frac{1}{\sqrt{2n\ell}} u(x)^T Q_1 u(x), \ldots, \frac{1}{\sqrt{2n\ell}} u(x)^T Q_\ell u(x) \right).
\end{equation}

Since the $Q_1,\ldots,Q_\ell$ are all symmetric, there is an orthogonal change of basis that makes $Q_1$ diagonal. Then, $Q(y) = \sqrt{1 - x_0^2} Q_1$ is also diagonal and we can assume that the eigenvector corresponding to $\mu_1(y)$ is $u(y) = (1,0,\ldots,0)$. Recall that we have denoted the smallest eigenvalue of $Q_1$ by $\lambda_1$ and note that $\sqrt{1 - x_0^2} \lambda_1 = \mu_1(y)$. Consequently, $u(y)^T Q_1 u(y) = \lambda_1$ and therefore we can assume that $\nabla S^{\ell+1} \alpha_1(y)$ has the form

\begin{equation}
\nabla S^{\ell+1} \alpha_1(y) = \left(1, \frac{\lambda_1}{2n\ell}, \frac{\xi_2}{\sqrt{2n\ell}}, \ldots, \frac{\xi_\ell}{\sqrt{2n\ell}} \right)
\end{equation}

where $\xi_k = (Q_k)_{11}$, $k = 2,\ldots,\ell$, are standard independent gaussian variables (recall that in our definition the diagonal entries of a GOE($n$)-matrix are standard normal variables). In particular, by (4.5),

\begin{equation}
|\nabla S^{\ell} \alpha_1(y)|^2 = |\nabla S^{\ell+1} \alpha_1(y)|^2 - \langle \nabla S^{\ell+1} \alpha_1(y), y \rangle^2 = 1 + \frac{\lambda_1^2}{2n\ell} + \frac{\lambda_1^2}{2n\ell} - \left(x_0 + \frac{\mu_1(y)}{\sqrt{2n\ell}} \right)^2,
\end{equation}
where \( \chi_{\ell-1}^2 = \sum_{i=2}^\ell \xi_i^2 \) is a chi-squared distributed random variable with \( \ell - 1 \) degrees of freedom. Observe that the inner expectation in (\( \star \)) is conditioned on the event \( \alpha_1(y) = 0 \). Given this equality we have that \( x_0 + \sqrt{\frac{1}{\ell}} h_1(y) = 0 \). This deletes the last summand in (4.8). Furthermore, we rewrite the right-hand side of (4.4) as

\[
\frac{\lambda_1}{\sqrt{2n}} = -\frac{x_0 \sqrt{\ell}}{\sqrt{1 - x_0^2}}.
\]

Recall that \( \tilde{\lambda} = \lambda / \sqrt{2n} \) denotes the scaled eigenvalue and define now the function

\[
h(\tilde{\lambda}, w) = \sqrt{1 + \frac{\tilde{\lambda}^2}{\ell} + \frac{w}{2n\ell}}.
\]

Denote by \( p_{(\tilde{\lambda}, \tilde{\lambda}_2)} \) the conditional density of \( (\tilde{\lambda}_1, \tilde{\lambda}_2) \) on \( \alpha_1(y) = \alpha_1(x_0, \sqrt{1 - x_0^2}, 0, \ldots, 0) = 0 \) and by \( p_{\tilde{\lambda}_1} \) the marginal density of \( \tilde{\lambda}_1 \). The expectation inside (\( \star \)) is with respect to \( (\alpha_1(y), \alpha_2(y)) \), and we have \( \alpha_1(y) = x_0 + \sqrt{\frac{1-x_0^2}{\ell}} \tilde{\lambda}_1 \). Instead of integrating over \( (\alpha_1, \alpha_2) \), we integrate over \( (\tilde{\lambda}_1, \tilde{\lambda}_2) \) and, using (4.10), the expectation becomes:

\[
E \left\{ |\nabla S| \cdot \alpha_1(y) \right\} \cdot 1_{|\alpha_1(y) - \alpha_2(y)| > \epsilon} \bigg| \alpha_1(y) = 0 \bigg\} = \mathbb{E}_{\tilde{\lambda}, w} \left[ h(\tilde{\lambda}, w) \mathbf{1}_{\hat{A}(\epsilon)} \big| \tilde{\lambda}_1 = \frac{-x_0 \sqrt{\ell}}{\sqrt{1 - x_0^2}} \right]
\]

where \( w \sim \chi_{\ell-1}^2 \) and \( A(\epsilon) = \{ \sqrt{\frac{1-x_0^2}{\ell}} | \tilde{\lambda}_1 - \tilde{\lambda}_2 | > \epsilon \} \).

We now go back to our integral. First observe that the densities of the eigenvalues at zero are related by:

\[
p_{\alpha_1(y)}(0) = \sqrt{\frac{\ell}{1-x_0^2}} p_{\tilde{\lambda}_1} \left( \frac{-x_0 \sqrt{\ell}}{\sqrt{1-x_0^2}} \right).
\]

Using this and (4.11), we get

\[
(\star) = \lim_{\epsilon \to 0} \int_{x_0 \in [-1, 1]} \mathbb{E}_{\tilde{\lambda}, w} \left[ h(\tilde{\lambda}, w) \mathbf{1}_{\hat{A}(\epsilon)} \big| \tilde{\lambda}_1 = \frac{-x_0 \sqrt{\ell}}{\sqrt{1-x_0^2}} \right] \sqrt{\frac{\ell}{1-x_0^2}} p_{\tilde{\lambda}_1} \left( \frac{-x_0 \sqrt{\ell}}{\sqrt{1-x_0^2}} \right) \, dx_0.
\]

Using the monotone convergence theorem we can perform the limit within the expectation. Thereafter, \( \lambda_2 \) does not appear in the variable we take the expected value from. We may omit it to get

\[
(\star) = \int_{x_0 \in [-1, 1]} \mathbb{E}_{\tilde{\lambda}, w} \left[ h(\tilde{\lambda}_1, w) \big| \tilde{\lambda}_1 = \frac{-x_0 \sqrt{\ell}}{\sqrt{1-x_0^2}} \right] \sqrt{\frac{\ell}{1-x_0^2}} p_{\tilde{\lambda}_1} \left( \frac{-x_0 \sqrt{\ell}}{\sqrt{1-x_0^2}} \right) \, dx_0.
\]

where in the last step we have used the independence of \( \tilde{\lambda}_1 \) and \( w \). Let us denote by \( p_{\lambda_{\ell-1}}(w) \) the density of the \( \chi_{\ell-1}^2 \)-random variable. Then we can write

\[
(\star) = \int_{x_0 \in [-1, 1]} \int_{w > 0} \sqrt{\frac{\ell}{1-x_0^2}} h \left( \frac{-x_0 \sqrt{\ell}}{\sqrt{1-x_0^2}}, w \right) p_{\tilde{\lambda}_1} \left( \frac{-x_0 \sqrt{\ell}}{\sqrt{1-x_0^2}} \right) p_{\lambda_{\ell-1}}(w) \, dw \, dx_0.
\]

We use the formula (4.9) to make the change of variables \( x_0 \mapsto \tilde{\lambda}_1 \) (this trick allows to perform the \( x_0 \)-integration over the smallest rescaled eigenvalue of a GOE(\( n \)) matrix). For the differentials
we get \( dx_0 = -\frac{\ell}{(\ell + \lambda_1)^2} \, d\lambda_1 \) and, writing \( \hat{\lambda}_{\text{min}}(Q) \) instead of \( \hat{\lambda}_1 \) we obtain:

\[
(*) = \int \mathbb{E}_{w \sim \chi_{\ell}^2} \left( \frac{\ell}{\ell + \lambda_1} \cdot h(\lambda_1, w) \right) p_{\lambda_1}(\lambda_1) d\lambda_1
\]

\[
= \mathbb{E}_{Q \sim \text{GOE}(n)} \mathbb{E}_{w \sim \chi_{\ell}^2} \left( \frac{\ell}{\ell + \min(Q)}^2 \cdot h(\hat{\lambda}_{\text{min}}(Q), w) \right)
\]

\[
= \mathbb{E}_{Q \sim \text{GOE}(n)} \left[ \frac{\ell}{\ell + \min(Q)}^2 \mathbb{E}_{w \sim \chi_{\ell}^2} \sqrt{1 + \frac{\min(Q)}{\ell} + \frac{w}{2n\ell}} \right] = f_{\ell,n}(\hat{\lambda}_{\text{min}}(Q))
\]

Because \( (*) = \mathbb{E} |\partial f_{\ell,n}| \), the assertion follows.

\[\square\]

4.2. Proof of Theorem 1.2 (2). The overall idea to prove Theorem 1.2 (1) is to use Lemma 3.1 on the function

\[
f_{\ell,n}(x) = \frac{\ell}{\ell + x^2} \cdot \mathbb{E}_{w \sim \chi_{\ell}^2} \sqrt{1 + \frac{x^2}{\ell} + \frac{w}{2n\ell}}.
\]

Because \( \sqrt{a + b} \leq \sqrt{a} + \sqrt{b} \) for non-negative \( a \) and \( b \) we have

\[
\sqrt{1 + \frac{x^2}{\ell} + \frac{w}{2n\ell}} \leq \sqrt{1 + \frac{x^2}{\ell}} + \sqrt{\frac{w}{2n\ell}}.
\]

Taking expectation (the expectation of \( \sqrt{w} \) can be found, e.g., in [?]) we get

\[
\mathbb{E}_{w \sim \chi_{\ell}^2} \sqrt{1 + \frac{x^2}{\ell} + \frac{w}{2n\ell}} \leq \sqrt{1 + \frac{x^2}{\ell}} + \frac{1}{\sqrt{2\pi \ell}} \Gamma \left( \frac{\ell + 1}{2} \right)
\]

See, e.g., [19, 43:9:3] for \( \frac{\Gamma \left( \frac{\ell + 1}{2} \right)}{\sqrt{\ell \Gamma \left( \frac{\ell + 1}{2} \right)}} = \mathcal{O}(1) \), from which we get

\[
(4.13) \quad \mathbb{E}_{w \sim \chi_{\ell}^2} \sqrt{1 + \frac{x^2}{\ell} + \frac{w}{2n\ell}} = \sqrt{1 + \frac{x^2}{\ell}} + \mathcal{O}(n^{-1/2}).
\]

Multiplying by \( \frac{\ell}{\ell + x^2} \) we have

\[
(4.14) \quad f_{\ell,n}(-1) = \sqrt{\frac{\ell}{\ell + 1}} + \mathcal{O}(n^{-1/2}).
\]

Next, we need to compute the derivative of \( f_{\ell,n}(x) \):

\[
(4.15) \quad f'_{\ell,n}(x) = \frac{-2x\ell}{(\ell + x^2)^2} \mathbb{E}_{w \sim \chi_{\ell}^2} \sqrt{1 + \frac{x^2}{\ell} + \frac{w}{2n\ell}} + \frac{\ell}{\ell + x^2} \mathbb{E}_{w \sim \chi_{\ell}^2} \frac{x}{\ell} \sqrt{1 + \frac{x^2}{\ell} + \frac{w}{2n\ell}}
\]

Note that we can bound \( x \leq \sqrt{\ell} \sqrt{1 + \frac{x^2}{\ell} + \frac{w}{2n\ell}} \), so that the right-hand-side summand in (4.15) is bounded by \( \frac{\sqrt{\ell}}{\sqrt{\ell + x^2}} \mathbb{E}_{w \sim \chi_{\ell}^2} 1 \leq 1 \). Furthermore, for bounding the left-hand-side summand in (4.15) we use (4.13) to get

\[
\left| \frac{-2x\ell}{(\ell + x^2)^2} \mathbb{E}_{w \sim \chi_{\ell}^2} \sqrt{1 + \frac{x^2}{\ell} + \frac{w}{2n\ell}} \right| = \frac{2|x|\ell}{(\ell + x^2)^2} \left( \sqrt{1 + \frac{x^2}{\ell} + \mathcal{O}(n^{-1/2})} \right),
\]

which shows that it is bounded. Altogether, we have shown that \( f'_{\ell,n}(x) \) is a bounded function, hence there exists some \( c \in \mathbb{R} \) with \( \|f_{\ell,n}(x)\| \leq c \). We envoke Lemma 3.1 to deduce that

\[
\mathbb{E}_{Q \sim \text{GOE}(n)} f_{\ell,n}(\hat{\lambda}_{\text{min}}(Q)) = f_{\ell,n}(-1) + \mathcal{O}(n^{-2/3}).
\]

By (4.14) the latter is \( \sqrt{\frac{\ell}{\ell + 1}} + \mathcal{O}(n^{-1/2}) \). This shows the assertion. \[\square\]
4.3. Proof of Theorem 1.2 (3). The assertion is implied by Theorem 1.2 (2) and the asymptotic \( \sqrt{\frac{t}{t+1}} = 1 - \frac{1}{2t} + O(t^{-2}) \).

5. The average number of singular points

For the study of the average number of singular points on the boundary of a random spectrahedron and on its symmetroid surface we will rely on the following proposition, which implies that this number is generically finite.

**Proposition 5.1.** Let \( S^{(k)}_{t,n} \) be the set of matrices of corank \( k \) in the spectrahedron \( S^{(k)}_{t,n} \) and \( \Sigma^{(k)}_{t,n} \) the set of matrices of corank \( k \) in the symmetroid hypersurface \( \Sigma^{(k)}_{t,n} \). For a generic choice of \( R = (R_1, \ldots, R_\ell) \in \text{Sym}(n, \mathbb{R})^\ell \) the sets \( S^{(k)}_{t,n}, \Sigma^{(k)}_{t,n} \subset S^\ell \) are semialgebraic of codimension \( \left(\frac{k+1}{2}\right) \).

**Proof.** In the space \( \text{Sym}(n, \mathbb{R}) \) consider the semialgebraic stratification given by the corank \( \text{Sym}(n, \mathbb{R}) = \bigcup_{k=0}^n Z(k) \), where \( Z(k) \) denotes the set matrices of corank \( k \), and the induced stratification on the cone \( P_n \) of positive semidefinite matrices \( P_n = \bigcup_{k=0}^n (Z(k) \cap P_n) \). These are Nash stratifications [2, Proposition 9] and the codimensions of both \( Z(k) \) and \( Z(k) \cap P_n \) are equal to \( \left(\frac{k+1}{2}\right) \).

Consider now the semialgebraic map

\[
F : S^\ell \times (\text{Sym}(n, \mathbb{R}))^\ell \rightarrow \text{Sym}(n, \mathbb{R}), \quad (x, R) \mapsto x_0 I + x_1 R_1 + \cdots + x_\ell R_\ell.
\]

Then \( \Sigma^{(k)}_{t,n} = \{ x \in S^\ell \mid F(R, x) \in Z(k) \} \) and \( S^{(k)}_{t,n} = \{ x \in S^\ell \mid F(R, x) \in Z(k) \cap P_n \} \) and hence they are semialgebraic.

We now prove that \( F \) is transversal to all the strata of these stratifications. Then the parametric transversality theorem [11, Chapter 3, Theorem 2.7] will imply that for a generic choice of \( R \) the set \( S^{(k)}_{t,n} \) is stratified by the \( S^{(k)}_{t,n} \) and the same for the set \( \Sigma^{(k)}_{t,n} \). To see that \( F \) is transversal to all the strata of the stratifications we compute its differential. At points \( (x, R) \) with \( x \neq e_0 = (1, 0, \ldots, 0) \) we have \( D_{(x, R)} F(0, \hat{R}) = x_1 \hat{R}_1 + \cdots + x_\ell \hat{R}_\ell \) and the equation 
\[
D_{(x, R)} F(\hat{x}, \hat{Q}) = P
\]

is solved by taking \( \hat{x} = 0 \) and \( \hat{R} = (0, \ldots, 0, x_i^{-1} P, 0, \ldots, 0) \) where \( x_i^{-1} P \) is in the \( i \)-th entry and \( i \) is such that \( x_i \neq 0 \) (in other words, already variations in \( R \) ensure surjectivity of \( D_{(x, R)} F \)). All points of the form \( (e_0, R) \) are mapped by \( F \) to the identity matrix \( I \) which belongs to the open stratum \( Z^{(0)} \), on which transversality is automatic (because this stratum has full dimension). This concludes the proof. \( \square \)

The following result on the number of singular points on a generic symmetroid surface is well-known among the experts.

**Proposition 5.2.** For generic \( R \in \text{Sym}(n, \mathbb{R})^3 \) the number of singular points \( \rho_n \) on the symmetroid \( \Sigma_{3,n} \) and hence the number of singular points \( \sigma_n \) on \( \partial S^{(k)}_{3,n} \) is finite and satisfies

\[
\sigma_n \leq \rho_n \leq \frac{n(n+1)(n-1)}{3}.
\]

Moreover, for any \( n \geq 1 \) there exists a generic symmetroid \( \Sigma_{3,n} \) with \( \rho_n = \frac{n(n+1)(n-1)}{3} \) singular points on it.

**Proof.** The fact that \( \sigma_n \leq \rho_n \) are generically finite follows from Proposition 5.1 with \( k = 2 \), as remarked before. Observe that \( \rho_n \) is bounded by twice (since \( \Sigma_{3,n} \) is a subset of \( S^3 \)) the number \( \# \text{Sing}(\Sigma_{3,n}) \) of singular points on the complex symmetroid projective surface

\[
\Sigma_{3,n}^C = \{ x \in \mathbb{C}P^3 \mid \det(x_0 I + x_1 R_1 + x_2 R_2 + x_3 R_3) = 0 \}
\]
Since \( \text{Sing}(\Sigma^{C}_{3,n}) \) is obtained as a linear section of the set \( \mathcal{Z}^{(2)}_{C} \) of \( n \times n \) complex symmetric matrices of corank two (using similar transversality arguments as in Proposition 5.1) we have that generically \( \# \text{Sing}(\Sigma^{C}_{3,n}) = \deg(\mathcal{Z}^{(2)}_{C}) \). The latter is equal to \( \frac{n(n+1)(n-1)}{6} \), see [9].

Now comes the proof of the second claim, we are thankful to Bernd Sturmfels and Simone Naldi for helping us with this. For a generic collection of \( n+1 \) linear forms \( L_1, \ldots, L_{n+1} \) in \( \ell+1 \) variables we denote by \( p(x) := L_1(x) \cdots L_{n+1}(x) \) their product and by \( P = \{x \in \mathbb{R}^{\ell+1} | L_i(x) > 0, i = 1, \ldots, n+1 \} \) the polyhedral cone. Let \( e \in \text{int}(P) \) be any interior point of \( P \). Then [18, Thm 1.1] implies that the derivative \( \langle \nabla_p, e \rangle \) of \( p \) along the constant vector field \( e \in \mathbb{R}^{\ell+1} \) is a hyperbolic polynomial in direction \( e \) and that the closure of the connected component of \( \mathbb{R}^{\ell+1} \setminus \{ \langle \nabla_p, e \rangle = 0 \} \) containing \( e \) is a spectrahedral cone. Let’s consider the intersection of this spectrahedral cone with the generic linear 4-space \( V \subset \mathbb{R}^{\ell+1} \) and denote by \( \mathcal{F}_{3,n}^{(2)}, \Sigma_{3,n}^{(2)} \) the corresponding spectrahedron and its symmetroid surface respectively. It is straightforward to check that the triple intersections of the hyperplanes \( L_1, \ldots, L_{n+1} \) when intersected with \( V \) produce \( 2^{(n+1)} = \frac{(n+1)!}{(n-1)!} \) singular points on \( \Sigma_{3,n} \).

We now prove Theorem 1.3.

5.1. Proof of Theorem 1.3. By Proposition 5.1 for a generic choice of \( \mathcal{R} = (R_1, R_2, R_3) \in \text{Sym}(n, \mathbb{R})^3 \) matrices of corank 2 in \( \mathcal{F}_{3,n}^{(2)} = \{ x \in S^{3} : \text{det}(x_0 I + x_1 R_1 + x_2 R_2 + x_3 R_3) = 0 \} \) and in \( \Sigma_{3,n} = \{ x \in S^{3} : x_0 I + x_1 R_1 + x_2 R_2 + x_3 R_3 > 0 \} \) constitute the singular loci of \( \mathcal{F}_{3,n} \) and \( \Sigma_{3,n} \) respectively, i.e.,

\[
\mathcal{F}_{3,n}^{(2)} = \text{Sing}(\partial \mathcal{F}_{3,n}), \quad \Sigma_{3,n}^{(2)} = \text{Sing}(\Sigma_{3,n}),
\]

and the sets \( \mathcal{F}_{3,n}^{(2)} \subset \Sigma_{3,n}^{(2)} \) are finite. When \( \mathcal{R} \) is generic the sets \( R \cap \Delta_1 \subset R \cap \Delta \) are finite as well, where \( R = \text{span}(R_1, R_2, R_3) \subset \text{Sym}(n, \mathbb{R}) \) denotes the 3-dimensional linear space defined by \( \mathcal{R} \), \( \Delta \subset S^{n-1} \) is the algebraic set of real symmetric matrices of unit Frobenius norm with repeated eigenvalues and \( \Delta_1 \subset \Delta \) is its semialgebraic subset consisting of matrices whose two smallest eigenvalues coincide. We first show that the number \( \#(\text{Sing}(\partial \mathcal{F}_{3,n})) \) of singular points on the spectrahedron defined by \( \mathcal{R} = (R_1, R_2, R_3) \) coincides with the number \( \#(R \cap \Delta_1) \) of matrices in \( R = \text{span}(R_1, R_2, R_3) \) whose two smallest eigenvalues \( \lambda_1, \lambda_2 \) coincide. For this observe that

\[
\lambda_i(x_0 I + R(x)) = x_0 + \lambda_i(R(x)), \quad i = 1, \ldots, n,
\]

where we denote \( R(x) = x_1 R_1 + x_2 R_2 + x_3 R_3 \). If \( R(x) \in \Delta_1 \), i.e., \( \lambda_1(R(x)) = \lambda_2(R(x)) \), then, due to (5.2) and (5.3),

\[
\frac{(-\lambda_1(R(x)), x_1, x_2, x_3)}{\sqrt{\lambda_1(R(x))^2 + x_1^2 + x_2^2 + x_3^2}} \in \mathcal{F}_{3,n}^{(2)} = \text{Sing}(\partial \mathcal{F}_{3,n})
\]

is a singular point of \( \partial \mathcal{F}_{3,n} \). Vice versa, if \( (x_0, x_1, x_2, x_3) \in \text{Sing}(\mathcal{F}_{3,n}) \) we have that \( x = (x_1, x_2, x_3) \neq 0 \), \( \lambda_1(R(x)) = \lambda_2(R(x)) \) (by (5.2) and (5.3)) and hence \( R(x) || R(x) || \in \Delta_1 \). Moreover, one can easily see that the described identification is one-to-one.

When \( Q_1, Q_2, Q_3 \in \text{Sym}(n, \mathbb{R}) \) are GOE(\(n\))-matrices and \( R_i = Q_i/\sqrt{6n}, i = 1, 2, 3 \) the space \( R = \text{span}(R_1, R_2, R_3) \) is uniformly distributed in the Grassmanian of 3-planes in \( \text{Sym}(n, \mathbb{R}) \). Applying the integral geometry formula [12, p. 17] to \( \Delta_1 \subset S^{n-1} \) and the random space \( R \subset \text{Sym}(n, \mathbb{R}) \) we write

\[
\mathbb{E} \#(R \cap \Delta_1) = 2 \frac{|\Delta_1|}{|S^{n-1}|} = 2|\Delta_1|_{\text{rel}}
\]

From the above it follows that with probability one \( \#(\Delta_1 \cap R) = \#(\text{Sing}(\partial \mathcal{F}_{3,n})) = \sigma_n \). Thus \( \mathbb{E} \#(\Delta \cap R) = \mathbb{E} \sigma_n \) which combined with (5.5) completes the proof of Theorem 1.3 (1).

The proof of Theorem 1.3 (2) is completely analogous.
6. Quartic Spectrahedra

In this section we prove Proposition 1.5: the identity $\mathbb{E}\rho_4 = 12$ follows immediately from Corollary 1.4 for $n = 4$. For the other identity we apply (1.3) and the formula [7, (3.2)]:

$$\mathbb{E}\sigma_4 = \frac{2^4}{\sqrt{\pi}} \frac{1}{4!} \int_0^\infty \mathbb{E} \left[ \chi_{Q-u1>0} \det(Q-u\mathbb{I})^2 \right] e^{-u^2} du$$

$$= \frac{4}{\sqrt{\pi}} \int \frac{1}{Z_2} \int_{\mathbb{R}^2} (\lambda_1 - u)^2(\lambda_2 - u)^2|\lambda_1 - \lambda_2|e^{-\frac{(\lambda_1 + \lambda_2)}{4}} \chi_{\{\lambda_1 > 0, \lambda_2 > 0\}} d\lambda_1 d\lambda_2 e^{-u^2} du$$

$$= (*),$$

where $Z_2 = 4\sqrt{\pi}$ is the normalization constant for the density of eigenvalues of a GOE(2)-matrix (see [7, (3.1)]).

We now apply the change of variables $\alpha_1 = \lambda_1 - u, \alpha_2 = \lambda_2 - u$ in the innermost integral, obtaining:

$$(*): = \frac{4}{\sqrt{\pi}} \int_0^\infty \frac{1}{Z_2} \int_{\mathbb{R}_{+}^2} (\alpha_1 \alpha_2)^2|\alpha_1 - \alpha_2|e^{-\frac{\alpha_1^2 + \alpha_2^2}{2}} e^{-u^2-u(\alpha_1 + \alpha_2)} d\alpha_1 d\alpha_2 e^{-u^2} du$$

$$= \frac{4}{\sqrt{\pi}} \int_0^\infty \frac{1}{Z_2} \int_{\mathbb{R}_{+}^2} (\alpha_1 \alpha_2)^2|\alpha_1 - \alpha_2|e^{-\frac{\alpha_1^2 + \alpha_2^2}{2}} \left( \int_\mathbb{R} e^{-2u^2-u(\alpha_1 + \alpha_2)} du \right) d\alpha_1 d\alpha_2$$

$$= \frac{1}{\pi} \int_{\mathbb{R}_{+}^2} (\alpha_1 \alpha_2)^2|\alpha_1 - \alpha_2|e^{-\frac{\alpha_1^2 + \alpha_2^2}{2}} \left( \sqrt{\frac{\pi}{2}} e^{\frac{\alpha_1^2 + \alpha_2^2}{2}} \right) d\alpha_1 d\alpha_2$$

$$= \frac{2}{\sqrt{\pi}} \int_{\mathbb{R}_{+}^2 \cap \{\alpha_1 < \alpha_2\}} (\alpha_1 \alpha_2)^2|\alpha_1 - \alpha_2|e^{-\frac{\alpha_1^2 + \alpha_2^2}{2}} + \frac{(\alpha_1 + \alpha_2)^2}{2} d\alpha_1 d\alpha_2.$$

In the last equality we have used the fact that the integrand is invariant under the symmetry $(\alpha_1, \alpha_2) \mapsto (\alpha_2, \alpha_1)$. Consider now the map $F: \mathbb{R}_{+}^2 \cap \{\alpha_1 < \alpha_2\} \rightarrow \mathbb{R}[x] \simeq \mathbb{R}^2$ given by

$$F(\alpha_1, \alpha_2) = (x - \alpha_1)(x - \alpha_2) = x^2 - (\alpha_1 + \alpha_2)x + \alpha_1 \alpha_2.$$

Essentially, $F$ maps the pair $(\alpha_1, \alpha_2)$ to a monic polynomial of degree two whose ordered roots are $(\alpha_1, \alpha_2)$. Observe that $F$ is injective on the region $\mathbb{R}_{+}^2 \cap \{\alpha_1 < \alpha_2\}$ with never-vanishing Jacobian $|JF(\alpha_1, \alpha_2)| = |\alpha_1 - \alpha_2|$. What is the image of $F$ in the space of polynomials $\mathbb{R}[x]$?

Denoting by $a_1, a_2$ the coefficients of a monic polynomial $p(x) = x^2 - a_1 x + a_2 \in \mathbb{R}[x]$, we see first that the conditions $\alpha_1, \alpha_2 > 0$ imply $a_1, a_2 > 0$. Moreover the polynomial $p(x) = F(\alpha_1, \alpha_2)$ has by construction real roots, hence its discriminant $a_1^2 - 4a_2$ must be positive. Viceversa, given $(a_1, a_2)$ such that $a_1, a_2 > 0$ and $a_1^2 - 4a_2 > 0$, the roots of $p(x) = x^2 - a_1 x + a_2$ are real and positive. Hence, $F(\mathbb{R}_{+}^2 \cap \{\alpha_1 < \alpha_2\}) = \{(a_1, a_2) \in \mathbb{R}^2 | a_1, a_2 > 0, a_1^2 - 4a_2 > 0\}$. Thus we can write the above integral as

$$(*) = \frac{2}{\sqrt{\pi}} \int_0^\infty \int_0^\infty \frac{a_1^2}{2} e^{-\frac{a_1^2 + 2a_2}{2}} + \frac{a_1^2}{2} da_2 da_1 = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-\frac{3a_1^2}{2}} \left( \int_0^{\sqrt{\frac{3}{2}}} a_2^2 e^{a_2^2} da_2 \right) da_1$$

and performing elementary integration we obtain $(*) = \mathbb{E}\sigma_4 = 6 - \frac{4}{\sqrt{\pi}}$.

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