MULTIPARAMETRIC DISSIPATIVE LINEAR STATIONARY DYNAMICAL SCATTERING SYSTEMS: DISCRETE CASE, II: EXISTENCE OF CONSERVATIVE DILATIONS*

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In the present paper we introduce the notion of dilation of a multiparametric linear stationary dynamical system (systems of this type, in particular dissipative, and conservative scattering ones were first introduced in [6]). We establish the criterion for existence of a conservative dilation of a multiparametric dissipative scattering system. This allows to distinguish the class of so-called $N$-dissipative systems preserving the most important properties of one-parametric dissipative scattering systems.

0 Introduction

This paper continues the investigation of multiparametric linear stationary dynamical systems (LSDSs), in particular dissipative, and conservative scattering systems, started in [6]. Such systems represent a generalization of LSDSs with discrete time $t \in \mathbb{Z}$, in particular dissipative (contractive), and conservative (unitary) scattering systems (see [4] and survey [5]) to the case $t \in \mathbb{Z}^N$. In Section 1 we recall the necessary definitions and facts from [6]. In Section 2 we recall the definition of dilation of a one-parametric LSDS and prove the lemma in which a useful equivalent reformulation of this definition is obtained. In Section 3 the notion of dilation of a multiparametric LSDS is introduced, and some of its properties are discussed. In Section 4 the criterion for existence of a conservative dilation of a multiparametric dissipative scattering LSDS is established. This criterion allows to distinguish the subclass of multiparametric dissipative scattering LSDSs that possess conservative dilations (we call them $N$-dissipative scattering systems) and preserve other important properties of one-parametric dissipative scattering LSDSs. In Section 5 we characterize the class of transfer functions of all $N$-dissipative scattering LSDSs with given input and output spaces as the subclass of the generalized Schur class of operator-valued functions on the open unit polydisc $D^N$ (the definition of this class is recalled in Section 1) distinguished by the condition of vanishing at $z = 0$. We prove the existence of minimal $N$-dissipative realizations for operator-valued functions from this subclass. We establish that in the cases $N = 1$ and $N = 2$ the class of $N$-dissipative scattering LSDSs coincides with the class of all dissipative scattering LSDSs. Note that for $N = 1$ it is a well-known result [4] appearing as a system

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analogue of the classical theorem of B. Sz.-Nagy on the existence of a unitary dilation for an arbitrary contractive linear operator on a Hilbert space (see [9]). In the case $N > 2$ the class of $N$-dissipative scattering LSDSs turns out to be a proper subclass of the class of all dissipative scattering LSDSs.

1 Preliminaries

In this section we shall recall some definitions and results from [6] that will be needed in the sequel.

For $t \in \mathbb{Z}^N$ set $|t| := \sum_{k=1}^{N} t_k$, for each $k \in \{1, \ldots, N\}$ let $e_k$ be the $N$-tuple with unit on the $k$-th place and zeroes on the rest, and let $[\mathcal{H}_1, \mathcal{H}_2]$ denote the Banach space of all linear bounded operators mapping a separable Hilbert space $\mathcal{H}_1$ into a separable Hilbert space $\mathcal{H}_2$. Then a multiparametric LSDS is, by definition, the following system of equalities:

\begin{align}
\alpha : \{ \begin{array}{l}
x(t) = \sum_{k=1}^{N} (A_k x(t - e_k) + B_k \phi^-(t - e_k)), \\
\phi^+(t) = \sum_{k=1}^{N} (C_k x(t - e_k) + D_k \phi^-(t - e_k)),
\end{array} \right. \\
(|t| > 0)
\end{align}

where for all $k \in \{1, \ldots, N\}$ $A_k \in [\mathcal{X}, \mathcal{X}]$, $B_k \in [\mathcal{X}^-, \mathcal{X}]$, $C_k \in [\mathcal{X}, \mathcal{N}^+]$, $D_k \in [\mathcal{N}^-, \mathcal{N}^+]$, together with the initial condition

\begin{align}
x(t) = x_0(t)
\end{align}

where $x_0 : \{t \in \mathbb{Z}^N : |t| = 0\} \to \mathcal{X}$ is a prescribed function. We call $\mathcal{X}$, $\mathcal{X}^-$, $\mathcal{N}^+$ respectively the state space, the input space and the output space of $\alpha$. If one denotes the $N$-tuple of operators $T_k$ ($k = 1, \ldots, N$) by $T := (T_1, \ldots, T_N)$ then for such a system one may use the short notation $\alpha = (N; A, B, C, D; \mathcal{X}, \mathcal{X}^-, \mathcal{N}^+)$. Note that in the case $N = 1$ a system of equalities in (1.1) differs from the standard one (see [4] or [5]) by shift in an output signal $\phi^+$, that brings, as it is shown in [6], to unessential changes in the theory of one-parametric LSDSs. The notion of dilation for this case, that will be used in the sequel, doesn’t differ from the standard one (see Section 2).

Set $zT := \sum_{k=1}^{N} z_k T_k$ for $N$-tuples of complex numbers $z = (z_1, \ldots, z_N)$ and operators $T = (T_1, \ldots, T_N)$. Then a $[\mathcal{N}^-, \mathcal{N}^+]$-valued function

\begin{align}
\theta_\alpha(z) = zD + zC(IX - zA)^{-1} zB,
\end{align}

which has to be considered on some neighbourhood of $z = 0$ in $\mathbb{C}^N$, is called the transfer function of a system $\alpha$ of the form (1.1)-(1.2). The system $\alpha = (N; A, B, C, D; \mathcal{X}, \mathcal{N}^-, \mathcal{N}^+)$ is called a dissipative (resp. conservative) scattering LSDS if for each $\zeta \in T^N$ ($N$-dimensional torus)

\begin{align}
\zeta G := \left( \begin{array}{cc}
\zeta A & \zeta B \\
\zeta C & \zeta D
\end{array} \right) \in [\mathcal{X} \oplus \mathcal{N}^-, \mathcal{X} \oplus \mathcal{N}^+]
\end{align}

is a contractive (resp. unitary) operator.

**Theorem 1.1** The transfer function $\theta_\alpha$ of an arbitrary dissipative scattering LSDS $\alpha = (N; A, B, C, D; \mathcal{X}, \mathcal{N}^-, \mathcal{N}^+)$ belongs to the class $B^q_\mathcal{X}(\mathcal{N}^-, \mathcal{N}^+)$ consisting of all functions holomorphic on the open unit polydisc $\mathcal{D}^N$ with contractive values from $[\mathcal{N}^-, \mathcal{N}^+]$ and vanishing at $z = 0$. 
Recall (see [1]) that the generalized Schur class $S_N(N^-, N^+)$ is the class of functions

$$\theta(z) = \sum_{t \in \mathbb{Z}_N^+} \widehat{\theta}_t z^t$$

holomorphic on $D^N$ with values in $[N^-, N^+]$ (here $\mathbb{Z}_N^+ := \{t \in \mathbb{Z}^N : t_k \geq 0, k = 1, \ldots, N\}$ is the discrete positive octant, $z^t := \prod_{k=1}^N z_k^t$ is a usual multipower for $t \in \mathbb{Z}_N^+$), such that for any separable Hilbert space $\mathcal{Y}$, any $N$-tuple $T = (T_1, \ldots, T_N)$ of commuting contractions on $\mathcal{Y}$ and for any positive $r < 1$ one has

$$\|\theta(rT)\| \leq 1$$

where

$$\theta(rT) = \theta(rT_1, \ldots, rT_N) := \sum_{t \in \mathbb{Z}_N^+} \widehat{\theta}_t \otimes (rT)^t \in [N^- \otimes \mathcal{Y}, N^+ \otimes \mathcal{Y}]$$

(the convergence of this series is understood in the sense of norm in the Banach space $[N^- \otimes \mathcal{Y}, N^+ \otimes \mathcal{Y}]$). If $N = 1$ then due to the von Neumann inequality (see [11]) we have $S_N(N^-, N^+) = S(N^-, N^+)$ i.e. the Schur class consisting of all functions holomorphic on the open unit disc $D$ with contractive values from $[N^-, N^+]$.

Denote by $S_{Nv}^0(N^-, N^+)$ the subclass of those functions from $S_N(N^-, N^+)$ that vanish at $z = 0$.

**Theorem 1.2** The class of transfer functions of $N$-parametric conservative scattering LSDSs with the input space $N^-$ and the output space $N^+$ coincides with $S_{Nv}^0(N^-, N^+)$.\[1.3]

In conclusion of this section let us remark that it is not difficult to verify the following inclusion:

$$S_{Nv}^0(N^-, N^+) \subseteq B_{Nv}^0(N^-, N^+)$$

It is known that for $N = 1$ (see [11]) and for $N = 2$ (see [3]) we have in fact the sign "=" in (1.3) for any $N^-$ and $N^+$, i.e. the classes $S_{Nv}^0(N^-, N^+)$ and $B_{Nv}^0(N^-, N^+)$ coincide. For $N > 2$, as it follows from [10], these classes do not coincide, i.e. we have the strict inclusion in (1.3) for any $N^-$ and $N^+$ different from $\{0\}$.

### 2 Lemma on dilations of one-parametric LSDSs

In this section the question is the case $N = 1$, i.e. one-parametric systems of the form

$$\alpha : \begin{cases} x(t) &= Ax(t-1) + B\phi^-(t-1), \\ \phi^+(t) &= Cx(t-1) + D\phi^-(t-1), \end{cases} \quad (t = 1, 2, \ldots)$$

where $A \in [\mathcal{X}, \mathcal{X}], B \in [N^-, \mathcal{X}], C \in [\mathcal{X}, N^+], D \in [N^-, N^+]$, and initial condition will be unessential for our consideration; we shall write $\alpha = (A, B, C, D; \mathcal{X}, N^-, N^+):= (1; A, B, C, D; \mathcal{X}, N^-, N^+)$. Recall (see e.g. [4] or [5]) that the LSDS $\tilde{\alpha} = (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}; \tilde{\mathcal{X}}, N^-, N^+)$ is said to be a dilation of the LSDS $\alpha = (A, B, C, D; \mathcal{X}, N^-, N^+)$ if there are such subspaces $\mathcal{D}$ and $\mathcal{D}_p$ in $\tilde{\mathcal{X}}$ that

$$\tilde{\mathcal{X}} = \mathcal{D} \oplus \mathcal{X} \oplus \mathcal{D}_p,$$

(2.1)
(2.2) \( \tilde{A} \mathcal{D} \subseteq \mathcal{D}, \quad \tilde{C} \mathcal{D} = \{0\}, \quad \tilde{A}^* \mathcal{D}_* \subseteq \mathcal{D}_*, \quad \tilde{B}^* \mathcal{D}_* = \{0\}, \)

(2.3) \( A = P_X \tilde{A}|\mathcal{X}, \quad B = P_X \tilde{B}, \quad C = \tilde{C}|\mathcal{X} \)

(here \( P_X \) denotes the orthogonal projector onto \( \mathcal{X} \) in \( \tilde{\mathcal{X}} \)).

**Lemma 2.1** The LSDS \( \tilde{\alpha} = (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}; \tilde{\mathcal{X}}, \mathcal{N}^-, \mathcal{N}^+) \) is a dilation of the LSDS \( \alpha = (A, B, C, D; \mathcal{X}, \mathcal{N}^-, \mathcal{N}^+) \) if and only if \( \mathcal{X} \subseteq \tilde{\mathcal{X}} \) and for all \( n \in \mathbb{Z}_+ \) the following equalities hold:

\[
(2.4) \quad A^n = P_X \tilde{A}^n|\mathcal{X}, \quad A^nB = P_X \tilde{A}^n\tilde{B}, \quad CA^n = \tilde{C}\tilde{A}^n|\mathcal{X}, \quad CA^nB = \tilde{C}\tilde{A}^n\tilde{B}.
\]

**Proof.** Suppose that \( \tilde{\alpha} \) is a dilation of \( \alpha \). Then \( \mathcal{X} \subseteq \tilde{\mathcal{X}} \), and by (2.3) for \( n = 1 \) the first equality in (2.4) holds (note that for \( n = 0 \) it holds trivially), and for \( n = 0 \) the second and the third equalities in (2.4) hold. Thus we have the base of induction for the proof of the first three relations in (2.4). Let for \( n = k \in \mathbb{Z}_+ \) these relations are true. We will show that then for \( n = k + 1 \) they are also true. We have

\[
A^{k+1} = A \cdot A^k = P_X \tilde{A}P_X \tilde{A}^k|\mathcal{X} = P_X \tilde{A}(I_{\mathcal{X}} - P_{\mathcal{D}} - P_{\mathcal{D}_*})\tilde{A}^k|\mathcal{X} = P_X \tilde{A}^{k+1}|\mathcal{X}
\]

since, according to (2.1) and (2.2), \( P_X \tilde{A}P_{\mathcal{D}_*} = 0 \) and \( P_{\mathcal{D}_*}\tilde{A}^k|\mathcal{X} = 0 \);

\[
A^{k+1}B = A \cdot A^kB = P_X \tilde{A}P_X \tilde{A}^kB = P_X \tilde{A}(I_{\mathcal{X}} - P_{\mathcal{D}} - P_{\mathcal{D}_*})\tilde{A}^kB = P_X \tilde{A}^{k+1}\tilde{B}
\]

since, according to (2.1) and (2.2), \( P_X \tilde{A}P_{\mathcal{D}_*} = 0 \) and \( P_{\mathcal{D}_*}\tilde{A}^kB = 0 \);

\[
CA^{k+1} = CA^k \cdot A = \tilde{C}\tilde{A}^kP_X \tilde{A}^k|\mathcal{X} = \tilde{C}\tilde{A}^k(I_{\mathcal{X}} - P_{\mathcal{D}} - P_{\mathcal{D}_*})\tilde{A}|\mathcal{X} = \tilde{C}\tilde{A}^{k+1}|\mathcal{X}
\]

since, according to (2.1) and (2.2), \( \tilde{C}\tilde{A}^kP_{\mathcal{D}_*} = 0 \) and \( P_{\mathcal{D}_*}\tilde{A}\tilde{A}|\mathcal{X} = 0 \). Thus we established by induction on \( n \) that the first three relations in (2.4) hold. We get from here for an arbitrary \( n \in \mathbb{Z}_+ \)

\[
CA^nB = (\tilde{C}\tilde{A}^n|\mathcal{X}) \cdot (P_X \tilde{B}) = \tilde{C}\tilde{A}^nP_X \tilde{B} = \tilde{C}\tilde{A}^n(I_{\mathcal{X}} - P_{\mathcal{D}} - P_{\mathcal{D}_*})\tilde{B}
\]

since, according to (2.1) and (2.2), \( \tilde{C}\tilde{A}^nP_{\mathcal{D}_*} = 0 \) and \( P_{\mathcal{D}_*}\tilde{B} = 0 \). Thus the fourth relation in (2.4) is also true for any \( n \in \mathbb{Z}_+ \).

Conversely, let \( \mathcal{X} \subseteq \tilde{\mathcal{X}} \) and for all \( n \in \mathbb{Z}_+ \) the equalities in (2.4) hold. Then set

\[
\mathcal{D} := \bigvee_{n=0}^{\infty} \tilde{A}^n((\tilde{A} - A)\mathcal{X} + (\tilde{B} - B)\mathcal{N}^-)
\]

where the symbol \( \bigvee \) denotes the closure of a linear span of some lineals, \( \mathcal{U} + \mathcal{V} := \{u + v : u \in \mathcal{U}, v \in \mathcal{V}\} \) is the sum of lineals \( \mathcal{U} \) and \( \mathcal{V} \) in some space, \( (\tilde{A} - A)\mathcal{X} := \{\tilde{x} \in \tilde{\mathcal{X}} : \tilde{x} = \tilde{A}\mathcal{X} + \mathcal{N}^-\} \)
\[ A_x - Ax, x \in \mathcal{X} \}, (B - B)N^- := \{ \tilde{z} \in \mathcal{X} : \tilde{z} = \tilde{B}\phi^- - B\phi^-, \phi^- \in N^- \}. \] Then \( D \perp \mathcal{X} \).

Indeed, for arbitrary \( x \in \mathcal{X}, \phi^- \in N^- \) and \( n \in \mathbb{Z}_+ \) we have

\[
P_x A^n((\tilde{A} - A)x + (\tilde{B} - B)\phi^-) = P_x A^{n+1}x - (P_x A^n|\mathcal{X}) \cdot (Ax) + P_x \tilde{A}^n \tilde{B}\phi^- - (P_x \tilde{A}^n|\mathcal{X}) \cdot (B\phi^-) = A^{n+1}x - Ax + A^n B\phi^- - A^n \cdot B\phi^- = 0.
\]

Hence \( P_x D = \{0\} \), and \( D \perp \mathcal{X} \). Set

\[ D_* := \tilde{\mathcal{X}} \ominus (\mathcal{X} \oplus D). \]

Then (2.1) is valid. From the definition of \( D \) we obtain that \( \tilde{A}D \subset D \). Further, for arbitrary \( x \in \mathcal{X}, \phi^- \in N^- \) and \( n \in \mathbb{Z}_+ \) we have

\[
C\tilde{A}^n((\tilde{A} - A)x + (\tilde{B} - B)\phi^-) = C\tilde{A}^{n+1}x - (C\tilde{A}^n|\mathcal{X}) \cdot (Ax) + C\tilde{A}^n \tilde{B}\phi^- - (C\tilde{A}^n|\mathcal{X}) \cdot (B\phi^-) = C A^{n+1}x - Ax + C A^n B\phi^- - C A^n \cdot B\phi^- = 0.
\]

From here we obtain that \( \tilde{C}D = \{0\} \). For an arbitrary \( x \in \mathcal{X} \) we have

\[
\tilde{A}x = (\tilde{A} - Ax) + Ax \in \bar{\mathcal{X}} \oplus \mathcal{X} \subset D \oplus \mathcal{X}
\]

(here \( \bar{U} \) denotes the closure of \( U \)). It was shown above that \( \tilde{A}D \subset D \). Hence \( \tilde{A}(D \oplus \mathcal{X}) \subset D \oplus \mathcal{X} \). From here we get \( \tilde{A}^* D_* = \tilde{A}^* (D \oplus \mathcal{X}) \perp \subset (D \oplus \mathcal{X}) \perp = D_* \). For an arbitrary \( \phi^- \in N^- \) we have

\[
\tilde{B}\phi^- = (\tilde{B}\phi^- - B\phi^-) + B\phi^- \in (\tilde{B} - B)N^- \oplus \mathcal{X} \subset D \oplus \mathcal{X} = (D_*) \perp.
\]

From here we get \( \tilde{B}^* D_* = \{0\} \). Thus relations in (2.2) are true. The equalities in (2.3) are the special cases of the equalities in (2.4). Finally, we have obtained that \( \tilde{\alpha} = (\tilde{A}, \tilde{B}, \tilde{C}, D; \tilde{\mathcal{X}}, N^-, N^+) \) is a dilation of \( \alpha = (A, B, C, D; \mathcal{X}, N^-, N^+) \). \( \square \)

**Remark 2.2** From Lemma 2.1, in particular, the well-known result (see e.g. [4]) follows: if \( \tilde{\alpha} = (\tilde{A}, \tilde{B}, \tilde{C}, D; \tilde{\mathcal{X}}, N^-, N^+) \) is a dilation of \( \alpha = (A, B, C, D; \mathcal{X}, N^-, N^+) \) then

\[
\forall n \in \mathbb{Z}_+ \quad A^n = P_X \tilde{A}^n|\mathcal{X},
\]

i.e. the main operator \( \tilde{A} \) of the system \( \tilde{\alpha} \) is a dilation of the main operator \( A \) of the system \( \alpha \).

### 3 The notion of dilation of a multiparametric LSDS

**Definition 3.1** We shall call the LSDS \( \tilde{\alpha} = (N; \tilde{A}, \tilde{B}, \tilde{C}, D; \tilde{\mathcal{X}}, N^-, N^+) \) a dilation of the multiparametric LSDS \( \alpha = (N; A, B, C, D; \mathcal{X}, N^-, N^+) \) if for each \( \zeta \in T^N \) the one-parametric LSDS \( \tilde{\alpha}_\zeta := (\zeta A, \zeta B, \zeta C, \zeta D; \tilde{\mathcal{X}}, N^-, N^+) \) is a dilation of the one-parametric
LSDS $\alpha_\zeta := (\zeta \mathbf{A}, \zeta \mathbf{B}, \zeta \mathbf{C}, \zeta \mathbf{D}; \mathcal{X}, \mathcal{N}^-, \mathcal{N}^+)$, i.e. for each $\zeta \in \mathbb{T}^N$ there are such subspaces $\mathcal{D}_\zeta$ and $\mathcal{D}_{\ast\zeta}$ in $\mathcal{X}$ that
\begin{equation}
\tilde{\mathcal{X}} = \mathcal{D}_\zeta \oplus \mathcal{X} \oplus \mathcal{D}_{\ast\zeta},
\end{equation}
\begin{equation}
\zeta \mathbf{AD}_\zeta \subset \mathcal{D}_\zeta, \quad \zeta \mathbf{CD}_\zeta = \{0\}, \quad (\zeta \mathbf{A})^\ast \mathcal{D}_{\ast\zeta} \subset \mathcal{D}_{\ast\zeta}, \quad (\zeta \mathbf{B})^\ast \mathcal{D}_{\ast\zeta} = \{0\},
\end{equation}
\begin{equation}
\zeta \mathbf{A} = P_\mathcal{X}(\zeta \mathbf{A})|\mathcal{X}, \quad \zeta \mathbf{B} = P_\mathcal{X}(\zeta \mathbf{B}), \quad \zeta \mathbf{C} = (\zeta \mathbf{C})|\mathcal{X}.
\end{equation}

From Lemma 2.1 we obtain the following equivalent reformulation of Definition 3.1.

**Proposition 3.2** The LSDS $\tilde{\alpha} = (N; \tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}}, \tilde{\mathbf{D}}; \tilde{\mathcal{X}}, \mathcal{N}^-, \mathcal{N}^+)$ is a dilation of the LSDS $\alpha = (N; \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}; \mathcal{X}, \mathcal{N}^-, \mathcal{N}^+)$ if and only if $\mathcal{X} \subset \tilde{\mathcal{X}}$ and for all $\zeta \in \mathbb{T}^N$ and $n \in \mathbb{Z}_+$ the following equalities hold:
\begin{align}
(\zeta \mathbf{A})^n &= P_\mathcal{X}(\zeta \mathbf{A})^n|\mathcal{X}, \quad (\zeta \mathbf{B})^n \mathbf{B} = P_\mathcal{X}(\zeta \mathbf{A})^n(\zeta \mathbf{B}),
(\zeta \mathbf{C})^n \mathbf{C} = (\zeta \mathbf{C})^n|\mathcal{X}, \quad (\zeta \mathbf{C} \mathbf{A})^n \mathbf{B} = (\zeta \mathbf{C})^n(\zeta \mathbf{A})^n(\zeta \mathbf{B}).
\end{align}

Equating coefficients of trigonometric polynomials in $N$ variables in both sides of equalities (3.4) we will obtain another equivalent reformulation of Definition 3.1, that is a multiparametric analogue of Lemma 2.1. For convenience of writing of corresponding relations let us recall the notations from [6] for the so-called *symmetrized multipowers* of the $N$-tuple $\mathbf{A}$ and the *symmetrized multipowers* of the $N$-tuple $\mathbf{A}$ bordered from one side (from two sides) by the $N$-tuples $\mathbf{B}$ and $\mathbf{C}$. If
\begin{equation}
c_s := \frac{|s|!}{s_1! \cdots s_N!} \quad (s \in \mathbb{Z}_+^N)
\end{equation}
denotes the *numbers of permutations* of $|s|$ elements of $N$ different types with repetitions (the *polynomial coefficients*) where an element of the $j$-th type repeats itself $s_j$ times, $[k] \in \{1, \ldots, N\}$ denotes the type of an element $k$, and $\sigma$ runs through the set of all such permutations with repetitions, then we set
\begin{align}
\mathbf{A}^s &:= c_s^{-1} \sum_\sigma A_{\sigma([1])} \cdots A_{\sigma([s])}, \quad (s \in \mathbb{Z}_+^N) \\
(\mathbf{A} \mathbf{B})^s &:= c_s^{-1} \sum_\sigma A_{\sigma([1])} \cdots A_{\sigma([s-1])} B_{\sigma([s])}, \quad (s \in \mathbb{Z}_+^N \setminus \{0\}) \\
(\mathbf{C} \mathbf{A})^s &:= c_s^{-1} \sum_\sigma C_{\sigma([1])} A_{\sigma([2])} \cdots A_{\sigma([s])}, \quad (s \in \mathbb{Z}_+^N \setminus \{0\}) \\
(\mathbf{C} \mathbf{A} \mathbf{B})^s &:= c_s^{-1} \sum_\sigma C_{\sigma([1])} A_{\sigma([2])} \cdots A_{\sigma([s-1])} B_{\sigma([s])}, \quad (s \in \mathbb{Z}_+^N \setminus \{0, e_1, \ldots, e_N\})
\end{align}

**Remark 3.3** In case of the commutative $N$-tuple $\mathbf{A}$ we have
\begin{equation}
\mathbf{A}^s = \prod_{k=1}^N A_k^{s_k}
\end{equation}
i.e. a usual multipower.
Proposition 3.4 The LSDS \(\tilde{\alpha} = (N; \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}; \tilde{\mathcal{X}}, N^-, N^+)\) is a dilation of the LSDS \(\alpha = (N; A, B, C, D; \mathcal{X}, N^-, N^+)\) if and only if \(\mathcal{X} \subset \tilde{\mathcal{X}}\) and the following equalities hold:

\[
\begin{align*}
\forall s \in \mathbb{Z}_N^+ & \quad A^s = P_X \tilde{A}^s|\mathcal{X}, \\
\forall s \in \mathbb{Z}_N^+ \setminus \{0\} & \quad (A^s B)^s = P_X (\tilde{A}^s \tilde{B})^s, \\
\forall s \in \mathbb{Z}_N^+ \setminus \{0\} & \quad (C^s A)^s = (\tilde{C}^s \tilde{A})^s|\mathcal{X}, \\
\forall s \in \mathbb{Z}_N^+ \setminus \{0, e_1, \ldots, e_N\} & \quad (C^s B^s)^s = (\tilde{C}^s \tilde{B}^s)^s.
\end{align*}
\tag{3.9}
\]

Remark 3.5 The equalities in the first line of (3.9) mean that the \(N\)-tuple \(\tilde{\mathbf{A}}\) of main operators of the system \(\tilde{\alpha}\) is, by definition, a dilation of the \(N\)-tuple \(\mathbf{A}\) of main operators of the system \(\alpha\) (cf. Remark 2.2). In case of the commutative \(N\)-tuples \(\mathbf{A}\) and \(\tilde{\mathbf{A}}\) this coincides with the definition of dilation for \(N\)-tuples of operators by [9] (see Remark 3.3).

Remark 3.6 It follows from Proposition 3.4 that one can replace \(\zeta \in T^N\) by \(z \in \mathbb{C}^N\) in (3.4) from Proposition 3.2, and in (3.1)–(3.3) from Definition 3.1, thus the LSDS \(\tilde{\alpha}\) is a dilation of the LSDS \(\alpha\) if and only if for each \(z \in \mathbb{C}^N\) there are such subspaces \(D_z\) and \(D_{s,z}\) in \(\tilde{\mathcal{X}}\) that

\[
\tilde{\mathcal{X}} = D_z \oplus \mathcal{X} \oplus D_{s,z},
\]

(3.10)

\[
\begin{align*}
\forall z \in \mathbb{C}^N & \quad z \tilde{\mathcal{A}} D_z \subset D_z, \quad z \tilde{\mathcal{C}} D_z = \{0\}, \quad (z \tilde{\mathcal{A}})^s D_{s,z} \subset D_{s,z}, \quad (z \tilde{\mathcal{B}})^s D_{s,z} = \{0\}, \\
&(z \mathbf{A}) = P_X (z \tilde{\mathbf{A}})|\mathcal{X}, \quad z \mathbf{B} = P_X (z \tilde{\mathbf{B}}), \quad z \mathcal{C} = (z \tilde{\mathcal{C}})|\mathcal{X}.
\end{align*}
\tag{3.11}
\tag{3.12}
\]

Remark 3.7 Symmetrized multipowers that were defined in (3.5)–(3.8) take part in expressions for states \(x(t)\) and output signals \(\phi^+(t)\) of a multiparametric LSDS \(\alpha\) through states \(x_0(\tau)\) from (1.2) and input signals \(\phi^-(\tau)\) at preceding to \(t\) moments \(\tau \neq t\) of “multidimensional time” (we set \(\tau \leq t\) if \(t - \tau \in \mathbb{Z}_N^+\)), that are deduced from the recurrent relations from (1.1) and the initial condition (1.2) (see [6]). Thus the algebraic definition of dilation from Proposition 3.4 is connected with consideration of system in “multidimensional time” domain, whereas the initial geometric Definition 3.1 is connected with considerations in “multidimensional frequency” domain or with so-called Z-transform \(\tilde{\alpha}\) of a system \(\alpha\) (see Remark 3.6 and [6]).

Proposition 3.8 The transfer functions of the system \(\alpha = (N; A, B, C, D; \mathcal{X}, N^-, N^+)\) and of its dilation \(\tilde{\alpha} = (N; \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}; \tilde{\mathcal{X}}, N^-, N^+)\) coincide.

Proof. The transfer functions of \(\alpha\) and \(\tilde{\alpha}\)

\[
\begin{align*}
\theta_{\alpha}(z) &= z \mathcal{D} + z \mathcal{C}(I_{\mathcal{X}} - z \mathbf{A})^{-1} z \mathbf{B} = z \mathcal{D} + \sum_{n=0}^{\infty} z \mathcal{C}(z \mathbf{A})^n z \mathbf{B}, \\
\theta_{\tilde{\alpha}}(z) &= z \tilde{\mathcal{D}} + z \tilde{\mathcal{C}}(I_{\tilde{\mathcal{X}}} - z \tilde{\mathbf{A}})^{-1} z \tilde{\mathbf{B}} = z \tilde{\mathcal{D}} + \sum_{n=0}^{\infty} z \tilde{\mathcal{C}}(z \tilde{\mathbf{A}})^n z \tilde{\mathbf{B}}.
\end{align*}
\tag{3.13}
\tag{3.14}
\]

are defined and holomorphic on some neighbourhood of \(z = 0\) in \(\mathbb{C}^N\). In particular, the series in (3.14) converges to \(\theta_{\tilde{\alpha}}(z)\) in operator norm uniformly and absolutely on compact subsets of the domain \(\Upsilon := \{z \in \mathbb{C}^N : \|z \tilde{\mathbf{A}}\| < 1\}\). In this domain \(\|z \mathbf{A}\| = \|P_X(z \tilde{\mathbf{A}})|\mathcal{X}\| < \|z \tilde{\mathbf{A}}\| < 1\).
Therefore the series in (3.13) converges to $\theta_\alpha(z)$ in operator norm uniformly and absolutely on compact subsets of $\mathbb{T}$. Besides, it follows from Proposition 3.2 and Remark 3.6 that for all $z \in \mathbb{C}^N$ and $n \in \mathbb{Z}_+$, $\bar{z} C(z) A^n \bar{z} B = \bar{z} C(z) A^n \bar{z} B$, and hence for all $z \in \mathbb{T}$, according to (3.13) and (3.14), we have $\theta_\alpha(z) = \theta_\beta(z)$. \(\square\)

**Definition 3.9** We shall call a multiparametric LSDS *minimal* if it is not a dilation of any system other than itself.

**Proposition 3.10** For an arbitrary LSDS $\alpha = (N; A, B, C, D; X, N^- , N^+)$ there exists a minimal LSDS $\alpha_{\text{min}} = (N; A_{\text{min}}, B_{\text{min}}, C_{\text{min}}, D; X_{\text{min}}, N^- , N^+)$ such that $\alpha$ is a dilation of $\alpha_{\text{min}}$.

**Proof.** We will use the Zorn lemma (see e.g. [8]). Consider the set $\Sigma_\alpha$ of all systems $\alpha_\gamma, \gamma \in \Gamma$ (here $\Gamma$ is some set of indices), for which $\alpha$ is a dilation. Then $\Sigma_\alpha$ is a partially ordered set with respect to the relation $\succ$; we shall write $\alpha_\gamma \succ \alpha_\eta$ if $\alpha_\gamma$ is a dilation of $\alpha_\eta$. For the existence in $\Sigma_\alpha$ of a minimal element (which is a minimal system with the dilation $\alpha$) it is sufficient to prove that any chain $\mathcal{C}_\alpha$ in $\Sigma_\alpha$ has a lower bound. Without loss of generality one can suppose that $\mathcal{C}_\alpha$ contains the element $\alpha_0 = \alpha$:

$$\mathcal{C}_\alpha : \alpha = \alpha_0 \succ \ldots \succ \alpha_\gamma \ldots$$

(the directed set of indices for this chain will be denoted by $\Gamma_0$). If $\Gamma_0$ is finite then $\mathcal{C}_\alpha$ has the minimal element $\alpha_{\gamma_0}$ which is a desired lower bound for $\mathcal{C}_\alpha$ in this case. Now let the directed set $\Gamma_0$ be infinite. Evidently, the corresponding state spaces for systems from $\mathcal{C}_\alpha$ are completely ordered by inclusion $\subseteq$, i.e. we obtain the chain

$$\mathcal{C}_X : X_0 \supseteq \ldots \supseteq X_\gamma \supseteq \ldots$$

Set $X_* := \bigcap_{\gamma \in \Gamma_0} X_\gamma$. Then (see e.g. [2])

$$P_{X_*} = \lim_{\gamma \in \Gamma_0} P_{X_\gamma}.$$ 

Set $\alpha_* := (N; A_*, B_*, C_*, D; X_*, N^- , N^+)$ where the $N$-tuples $A_*, B_*, C_*$ of operators are defined by formulas:

$$A_{*,k} = P_{X_*} A_{k|X_*}, \ B_{*,k} = P_{X_*} B_{k}, \ C_{*,k} = C_{k|X_*} \ (k = 1, \ldots, N)$$

Then $\alpha$ is a dilation of $\alpha_*$, i.e. $\alpha_* \in \Sigma_\alpha$. To show this we shall verify the equalities in (3.4) for these two systems and imply Proposition 3.2. For arbitrary $\zeta \in \mathbb{T}^N$ and $n \in \mathbb{Z}_+$ we have

$$(\zeta A_*)^n = (P_{X_*} (\zeta A))^n|X_* = \lim_{\gamma \in \Gamma_0} (P_{X_\gamma} (\zeta A))^n|X_*$$

$$(\zeta A_*)^n \zeta B_* = (P_{X_*} (\zeta A))^n P_{X_*} (\zeta B) = \lim_{\gamma \in \Gamma_0} (P_{X_\gamma} (\zeta A))^n P_{X_\gamma} (\zeta B)$$
and other equalities in (3.4) are verified analogously.

Now let us show that for each \( \gamma \in \Gamma_0 \) \( \alpha_\gamma \) is a dilation of \( \alpha_* \). Indeed, for any \( \gamma \in \Gamma_0 \) and \( n \in \mathbb{Z}_+ \) we have

\[
(\zeta A_*)^n = P_{\gamma_*}((\zeta A)^n|X_\gamma) = P_{\gamma_*}(P_{\gamma_*}((\zeta A)^n|X_\gamma)|X_\gamma) = P_{\gamma_*}(\zeta A_\gamma)^n|X_\gamma, \\
(\zeta A_*)^n B_* = P_{\gamma_*}((\zeta A)^n|B_\gamma) = P_{\gamma_*}((\zeta A)^n|B_\gamma) = P_{\gamma_*}(\zeta B_\gamma), \\
(\zeta C_\gamma((\zeta A)^n|X_\gamma) = (\zeta C_\gamma((\zeta A)^n|X_\gamma) = (\zeta C_\gamma((\zeta A)^n|X_\gamma) = (\zeta C_\gamma((\zeta A)^n|X_\gamma).
\]

Thus \( \alpha_* \) is a lower bound for \( \mathcal{E}_\alpha \), and the proof is complete. □

4 Criterion for existence of a conservative dilation of a multiparametric dissipative scattering LSDS

Definition 4.1 We shall say that \( \tilde{\alpha} = (\tilde{N}; \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}; \tilde{X}, \mathcal{N}^-, \mathcal{N}^+) \) is a conservative dilation of the dissipative scattering LSDS \( \alpha = (N; A, B, C, D; X, \mathcal{N}^-, \mathcal{N}^+) \) if \( \tilde{\alpha} \) is a dilation of \( \alpha \), and \( \tilde{\alpha} \) is a conservative scattering LSDS.

Theorem 4.2 The dissipative scattering LSDS \( \alpha = (N; A, B, C, D; X, \mathcal{N}^-, \mathcal{N}^+) \) allows a conservative dilation if and only if the corresponding linear operator-valued function

\[
L_G(z) := zG = \begin{pmatrix} zA & zB \\ zC & zD \end{pmatrix} \in \mathbb{D}^N
\]

belongs to the class \( S_0^0(X + \mathcal{N}^-, X + \mathcal{N}^+) \) (the definition of this class can be found in Section 1).

Proof. Let the dissipative scattering LSDS \( \alpha = (N; A, B, C, D; X, \mathcal{N}^-, \mathcal{N}^+) \) possess the conservative dilation \( \tilde{\alpha} = (\tilde{N}; \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}; \tilde{X}, \tilde{N}^-, \tilde{N}^+) \). Then for each \( \zeta \in T^N \)

\[
\zeta \tilde{G} := \begin{pmatrix} \zeta \tilde{A} \\ \zeta \tilde{B} \\ \zeta \tilde{C} \\ \zeta \tilde{D} \end{pmatrix} \in [\tilde{X} + \mathcal{N}^-, \tilde{X} + \mathcal{N}^+]
\]

is a unitary operator. This operator allows another block partition:

\[
\zeta \tilde{G} = \begin{pmatrix} \zeta T & \zeta F \\ \zeta H & \zeta S \end{pmatrix} \in [(\tilde{X} \oplus X) \oplus (X + \mathcal{N}^-), (\tilde{X} \oplus X) \oplus (X + \mathcal{N}^+)]
\]

where

\[
\zeta T = P_{\tilde{X} \oplus X}(\zeta \tilde{G})|\tilde{X} \oplus X, \quad \zeta F = P_{\tilde{X} \oplus X}(\zeta \tilde{G})|X + \mathcal{N}^-, \\
\zeta H = P_{X + \mathcal{N}^-}(\zeta \tilde{G})|\tilde{X} \oplus X, \quad \zeta S = P_{X + \mathcal{N}^+}(\zeta \tilde{G})|X + \mathcal{N}^+ = \zeta G.
\]

It is clear that one can correspond to this partition of \( \zeta \tilde{G} \) the conservative scattering LSDS \( \beta = (N; T, F, H, S; \tilde{X} \oplus X, X + \mathcal{N}^-, X + \mathcal{N}^+) \) where \( S = G \). Its transfer function

\[
\theta_\beta(z) = zS + zH(I_{\tilde{X} \oplus X} - zT)^{-1}zF
\]
by Theorem 1.2 belongs to the class $S^p_N(\mathcal{X} \oplus \mathcal{N}^-, \mathcal{X} \oplus \mathcal{N}^+)$. Let us show that $\theta_B(z) = zS = zG = L_G(z)$. This will mean that $L_G \in S^p_N(\mathcal{X} \oplus \mathcal{N}^-, \mathcal{X} \oplus \mathcal{N}^+)$ i.e. the necessary condition of the present theorem. Evidently, it is sufficient to show that for any $z \in \mathbb{C}^N$ and $n \in \mathbb{Z}_+$

$$zH(zT)^nzF = 0.$$ 

According to (3.10) we have

$$zH(zT)^nzF = (P_\mathcal{X} \oplus \mathcal{N}^+)(z\tilde{G})(P_\mathcal{X} \oplus \mathcal{X})(z\tilde{G})(P_\mathcal{X} \oplus \mathcal{N}^-)P_\mathcal{X} \oplus \mathcal{X}(z\tilde{G}) = (P_\mathcal{X} \oplus \mathcal{X})(z\tilde{A})(P_\mathcal{X} \oplus \mathcal{X})(z\tilde{B})$$

$$= \left[ \frac{P_\mathcal{X}(z\tilde{A})}{zC} \right] (P_{D_x} + P_{D_{x^*}})(z\tilde{A}(P_{D_x} + P_{D_{x^*}}))^n[ (z\tilde{A}) \mathcal{X} \quad z\tilde{B} ]$$

$$= \left[ \frac{P_\mathcal{X}(z\tilde{A})}{zC} \right] P_{D_{x^*}}(z\tilde{A}(P_{D_x} + P_{D_{x^*}}))^nP_{D_x}[ (z\tilde{A}) \mathcal{X} \quad z\tilde{B} ]$$

since by (3.11) $P_\mathcal{X}(z\tilde{A})P_{D_x} = 0$, $(z\tilde{C})P_{D_x} = 0$, $P_{D_{x^*}}(z\tilde{A})\mathcal{X} = 0$ and $P_{D_{x^*}}(z\tilde{B}) = 0$. Further, by (3.11) $z\tilde{A}D_z \subset D_z \perp D_{x^*}$, hence

$$P_{D_{x^*}}(z\tilde{A}(P_{D_x} + P_{D_{x^*}}))^nP_{D_x} = P_{D_{x^*}}(z\tilde{A})^nP_{D_x} = 0,$$

and therefore $zH(zT)^nzF = 0$, i.e. the necessary condition of this theorem is fulfilled.

Conversely, let $L_G$ belong to the class $S^p_N(\mathcal{X} \oplus \mathcal{N}^-, \mathcal{X} \oplus \mathcal{N}^+)$. Then by Theorem 1.2 there exists such a conservative LSDS $\beta = (N; T, F, H, S; \mathcal{X} \oplus \mathcal{N}^-, \mathcal{X} \oplus \mathcal{N}^+)$ that for all $z \in \mathbb{D}^N$

$$\theta_B(z) = zS + zH(\mathcal{I}_Y - zT)^{-1}zF = zG = L_G(z).$$

Then $S = G$, and for all $n \in \mathbb{Z}_+$ and $z \in \mathbb{D}^N$ (and hence for all $z \in \mathbb{C}^N$)

$$(4.2) \quad zH(zT)^nzF = 0.$$ 

Conservativity of $\beta$ means that for each $\zeta \in \mathbb{T}^N$

$$\zeta \tilde{G} = \left( \begin{array}{c} \zeta T \\ \zeta H \\ \zeta S \end{array} \right) \in [\mathcal{Y} \oplus (\mathcal{X} \oplus \mathcal{N}^-), \mathcal{Y} \oplus (\mathcal{X} \oplus \mathcal{N}^+)]$$

is a unitary operator. This operator allows another block partition:

$$\zeta \tilde{G} := \left( \begin{array}{c} \zeta \tilde{A} \\ \zeta \tilde{B} \\ \zeta \tilde{C} \\ \zeta \tilde{D} \end{array} \right) \in \left[ (\mathcal{Y} \oplus \mathcal{X}) \oplus \mathcal{N}^-, (\mathcal{Y} \oplus \mathcal{X}) \oplus \mathcal{N}^+ \right]$$

where

$$\zeta \tilde{A} = \left[ \begin{array}{c} \zeta T \\ P_\mathcal{X}(\zeta H) \end{array} \right], \quad \zeta \tilde{C} = \left[ \begin{array}{c} P_\mathcal{X}(\zeta H) \zeta C \end{array} \right], \quad \zeta \tilde{D} = \zeta D.$$

(4.3)

It is clear that one can correspond to this partition of $\zeta \tilde{G}$ the conservative scattering LSDS $\tilde{\alpha} = (N; \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}; \mathcal{Y} \oplus \mathcal{X}, \mathcal{N}^-, \mathcal{N}^+)$. Let us show that $\tilde{\alpha}$ is a dilation of $\alpha$. For this purpose,
according to Proposition 3.2, it is sufficient to verify the equalities in (3.4) for all \( \zeta \in \mathbb{T}^N \).

According to (4.3) \( \zeta A = P_X(\zeta \tilde{A})|\mathcal{X} \), i.e. for \( n = 1 \) the first equality in (3.4) holds (for \( n = 0 \) it holds trivially). Let us apply induction on \( n \). Suppose that \( (\zeta A)^n = P_X(\zeta \tilde{A})^n|\mathcal{X} \) for \( n = k \in \mathbb{Z}_+ \setminus \{0\} \). Then by (4.3) and (4.2) we have

\[
(\zeta A)^{k+1} = (\zeta A)(\zeta A)^k = P_X(\zeta \tilde{A})P_Y(\zeta \tilde{A})^k|\mathcal{X} \\
= P_X(\zeta \tilde{A})(I_{\mathbb{B} \mathcal{X}} - P_Y)(\zeta \tilde{A})^k|\mathcal{X} \\
= P_X(\zeta \tilde{A})^{k+1}|\mathcal{X} - P_X(\zeta \mathcal{H})P_Y(\zeta \tilde{A})^k|\mathcal{X} \\
= P_X(\zeta \tilde{A})^{k+1}|\mathcal{X} - P_X(\zeta \mathcal{H})(\zeta T)(\zeta F)|\mathcal{X} |(\zeta \tilde{A})^{k-1}|\mathcal{X} \\
= P_X(\zeta \tilde{A})^{k+1}|\mathcal{X} - P_X(\zeta \mathcal{H})(\zeta T)^{k-1}P_Y(\zeta \tilde{A})|\mathcal{X} \\
= P_X(\zeta \tilde{A})^{k+1}|\mathcal{X} - P_X(\zeta \mathcal{H})(\zeta T)^{k-1}(\zeta F)|\mathcal{X} = P_X(\zeta \tilde{A})^{k+1}|\mathcal{X}.
\]

Thus the first equality in (3.4) is valid for all \( n \in \mathbb{Z}_+ \). The second and the third equalities in (3.4) are proved analogously. Finally, for an arbitrary \( n \in \mathbb{Z}_+ \) by (4.3), (4.2) and the second equality in (3.4) we have

\[
\zeta C(\zeta A)^n \zeta B = (\zeta \tilde{C})(I_{\mathbb{B} \mathcal{X}} - P_Y)(\zeta \tilde{A})^n \zeta B \\
= \zeta \tilde{C}((\zeta \tilde{A})^n \zeta B - P_{N^+}(\zeta \mathcal{H})P_Y(\zeta \tilde{A})^n \zeta B \\
= \zeta \tilde{C}((\zeta \tilde{A})^n \zeta B - P_{N^+}(\zeta \mathcal{H})(\zeta T)(\zeta F)|\mathcal{X} |(\zeta \tilde{A})^{n-1} \zeta B \\
= \zeta \tilde{C}((\zeta \tilde{A})^n \zeta B - P_{N^+}(\zeta \mathcal{H})(\zeta T)^{n-1}P_Y(\zeta \tilde{A})|\mathcal{X} \\
= \zeta \tilde{C}((\zeta \tilde{A})^n \zeta B - P_{N^+}(\zeta \mathcal{H})(\zeta T)^{n-1}(\zeta F)|\mathcal{X} = \zeta \tilde{C}(\zeta \tilde{A})^n \zeta \tilde{B}.
\]

Note that for \( n = 0 \) this calculation is obviously simplified and does not contain terms like \( (\zeta A)^k \) with \( k < n \). The proof is complete. \( \square \)

In the particular case when \( N^- = N^+ = \{0\} \) we obtain the following result.

**Corollary 4.3** The linear pencil of contractions \( LA(\zeta) := \zeta A \in [\mathcal{X}, \mathcal{X}] (\zeta \in \mathbb{T}^N) \) allows a unitary dilation, i.e. there is a linear pencil of unitary operators \( LX(\zeta) = \zeta \tilde{A} \in [\mathcal{X}, \mathcal{X}] (\zeta \in \mathbb{T}^N) \), for which \( \mathcal{X} \subset \tilde{\mathcal{X}} \) and

\[
(4.4) \quad \forall \zeta \in \mathbb{T}^N, \forall n \in \mathbb{Z}_+ \quad (\zeta A)^n = P_X(\zeta \tilde{A})^n |\mathcal{X},
\]

if and only if \( LA \in S^0_N(\mathcal{X}, \mathcal{X}) \).

**Remark 4.4** Condition (4.4) is equivalent to the family of equalities in the first line of condition (3.9) in Proposition 3.4.

### 5 \( N \)-dissipative scattering LSDSs

It is obvious (see Section 1) that the multiparametric LSDS \( \alpha = (N; A, B, C, D; \mathcal{X}, N^-, N^+) \) is a dissipative scattering LSDS if and only if the corresponding linear function \( LG \) in (4.1) belongs to the class \( B_N^0(\mathcal{X} \oplus N^-, \mathcal{X} \oplus N^+) \).
Definition 5.1 We shall call the system $\alpha = (N; A, B, C, D; X, N^-, N^+)\) a $N$-dissipative scattering LSDS if $L_\alpha \in S_N^0(X \oplus N^-, X \oplus N^+)\). It is clear that by virtue of (1.3) the class of $N$-dissipative scattering LSDSs is a subclass of the class of all dissipative scattering LSDSs. According to Theorem 4.2 it consists of those and only those dissipative systems which allow conservative dilations. It follows from Theorem 1.1 that the class of transfer functions of $N$-parametric dissipative scattering LSDSs with the input space $N^-$ and the output space $N^+$ is a subclass of $B_N^0(N^-, N^+)$, however we have no complete description of this subclass. For $N$-dissipative systems, from Theorem 1.2, Theorem 4.2 and Proposition 3.8 we obtain the complete description of the class of transfer functions.

Theorem 5.2 The class of transfer functions of $N$-dissipative scattering LSDSs with the input space $N^-$ and the output space $N^+$ coincides with $S_N^0(N^-, N^+)\). Let us note that, by Theorem 4.2, if the $N$-dissipative scattering LSDS $\alpha$ is a dilation of some system $\alpha_0$ then $\alpha_0$ is also $N$-dissipative. By virtue of Theorem 1.2, for each operator-valued function $\theta \in S_N^0(N^-, N^+)$ there exists a conservative realization i.e. such a conservative scattering LSDS $\alpha$ that $\theta = \theta_\alpha$. According to Proposition 3.10, for $\alpha$ there exists a minimal system $\alpha_{\min}$ such that $\alpha$ is a dilation of $\alpha_{\min}$, moreover by Proposition 3.8 $\theta = \theta_{\alpha} = \theta_{\alpha_{\min}}$. Thus we obtain the theorem on a minimal $N$-dissipative realization for operator-valued functions from $S_N^0(N^-, N^+)\).

Theorem 5.3 For an arbitrary $\theta \in S_N^0(N^-, N^+)$ there exists a minimal $N$-dissipative scattering LSDS $\alpha_{\min}$ such that $\theta = \theta_{\alpha_{\min}}\). As we remarked in the end of Section 1, for $N = 1$ and $N = 2$ we have equality in (1.3). It follows from here (see Definition 5.1) that for these cases the notions of $N$-dissipative scattering LSDS and dissipative scattering LSDS coincide, and the corresponding classes of systems also coincide. In the case $N > 2$, as we will show, these classes do not coincide.

The following result was obtained in [7].

Theorem 5.4 There exist such triples $T = (T_1, T_2, T_3)$ of commuting contractions on some finite-dimensional Hilbert space $\mathcal{H}$ and $M = (M_1, M_2, M_3)$ of linear operators on $\mathbb{C}^n$, with some integer $n \geq 1$, that the linear homogeneous operator-valued function $L_M(z_1, z_2, z_3) = M_1 z_1 + M_2 z_2 + M_3 z_3$ satisfies

(5.1) $\|L_M(T)\| > \max_{z \in \mathbb{D}^n} \|L_M(z)\|$ (here $L_M(T) := M_1 \otimes T_1 + M_2 \otimes T_2 + M_3 \otimes T_3 \in [\mathbb{C}^n \otimes \mathcal{H}, \mathbb{C}^n \otimes \mathcal{H}]\). Under conditions of this theorem, set

(5.2) $G_k := (\max_{z \in \mathbb{D}^n} \|L_M(z)\|)^{-1} M_k$, \hspace{1cm} (k = 1, 2, 3)

(5.3) $X := \mathbb{C}^{n-1}$, $N^- = N^+ := \mathbb{C}$, so that $\mathbb{C}^n = X \oplus N^- = X \oplus N^+$,

(5.4) $A_k := P_X G_k |X$, $B_k := P_X G_k |N^-$, $C_k := P_{N^+} G_k |X$, $D_k := P_{N^+} G_k |N^-$. \hspace{1cm} (k = 1, 2, 3)
Then the linear operator-valued function $L_G(z) := G_1 z_1 + G_2 z_2 + G_3 z_3 = zG \ (z \in \mathbb{D}^3)$, by virtue of (5.2), belongs to the class $\mathcal{B}_3^0(\mathbb{C}^n, \mathbb{C}^n)$. However by (5.1) and (5.2)

$$
\|L_G(T)\| > \max_{z \in \mathbb{D}^3} \|L_G(z)\| = 1,
$$

and hence there is a positive $r < 1$ for which

$$
\|L_G(rT)\| > 1.
$$

The latter means (see Section 1) that $L_G$ does not belong to the class $\mathcal{S}_3^0(\mathbb{C}^n, \mathbb{C}^n)$. Thus the LSDS $\alpha = (3;A,B,C,D;X,N^-,N^+)$, that is defined in (5.2)-(5.4), is dissipative but not 3-dissipative. For the case $N > 3$ an analogous example of dissipative but not $N$-dissipative system can be easily constructed by supplement of arbitrary operators $M_4, \ldots, M_N$ on $\mathbb{C}^n$ with sufficiently small norms to the triple $\overline{M} = (M_1, M_2, M_3)$ from Theorem 5.4 and setting $T_4 = \ldots = T_N = 0$, so that the inequality analogous to (5.1) holds for the $N$-tuples $\overline{M} := (M_1, M_2, M_3, M_4, \ldots, M_N)$ and $\overline{T} := (T_1, T_2, T_3, 0, \ldots, 0)$, and then defining such a $N$-parametric system in the same way as in (5.2)-(5.4). Thus we have proved the following.

**Theorem 5.5** The class of $N$-dissipative scattering LSDSs for the cases $N = 1$ and $N = 2$ coincides with the class of all $N$-parametric dissipative scattering LSDSs, and for the case $N > 2$ is a proper subclass of the latter.

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