Theta operators on Siegel modular forms and Galois representations

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We describe the effect of the differential operators defined by Böcherer-Nagaoka [1], Flander-Ghitza [4] and Yamauchi [12] on the Galois representations (conjecturally) attached to Siegel modular eigenforms.

1 Introduction

Relations between modular forms and Galois representations play a central role in modern arithmetic geometry. The story starts in the 1970s with Deligne’s proof of the Ramanujan conjecture and Serre and Swinnerton-Dyer’s pioneering work on congruences, continues in the 1990s with Wiles’s proof of the Shimura-Taniyama conjecture (and hence of Fermat’s Last Theorem), and leads to the current flurry of activity around the proof of Serre’s conjecture by Khare-Wintenberger and numerous other modularity results due to Taylor and his collaborators.

While this paper is devoted to groups of higher rank, it is modelled on the situation for the group $GL_2$. Let $f$ be a Hecke eigenform (mod $p$). Deligne proved that there exists a continuous group representation

$$\rho_f: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow GL_2(\mathbb{F}_p)$$

that remembers the arithmetic content of $f$ (namely, the Hecke eigenvalues). The algebra of modular forms (mod $p$) is endowed with a differential operator $\vartheta$ that almost commutes with the Hecke operators: $T_\ell \circ \vartheta = \vartheta \circ T_\ell$. A simple consequence is that

$$\rho_{\vartheta f} \cong \chi \otimes \rho_f,$$

where $\chi: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \mathbb{F}_p^\times$ is the cyclotomic character (mod $p$).

Our interest is in Siegel modular forms (mod $p$), which arise from the group $GSp_{2g}$ (the group $GL_2$ being the special case $g = 1$). Such forms conjecturally produce group representations

$$\rho_f: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow GSpin_{2g+1}(\mathbb{F}_p),$$

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where $GSp_{2g+1}$ is the Langlands dual of $GSp_{2g}$. The question is how the operator $\vartheta$ generalises, and how these generalisations relate to the representations $\rho_f$.

Our main result (Theorem 17) takes as input an operator $\vartheta$ satisfying a certain type of commutation relation with Hecke operators and describes its effect on the Galois representations attached to eigenforms. Operators to which Theorem 17 applies include those defined by Flander-Ghitza [4] for arbitrary $g$ and by Yamauchi [12] for $g = 2$. We also analyse the operator $\vartheta_{BN}$ on scalar-valued forms defined by Böcherer-Nagaoka [1]. This requires us to determine in Section 7 the commutation relation between $\vartheta_{BN}$ and Hecke operators, which is a result of independent interest.

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2 The group of symplectic similitudes

We work with the algebraic group $G = GSp_{2g}$ of symplectic similitudes. For any commutative ring $R$, this is defined by

$$GSp_{2g}(R) = \{ M \in GL_{2g}(R) \mid MJM^t = \eta(M)J \text{ for some } \eta(M) \in R^\times \},$$

where $J = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$.

The symplectic group $Sp_{2g}(R)$ is the kernel of $\eta$.

Let $T \subset G$ denote the maximal torus formed by diagonal matrices

$$t = t(u_1, \ldots, u_{g+1}) = \text{diag}(u_1, \ldots, u_g; u_{g+1}u_1^{-1}, \ldots, u_{g+1}u_g^{-1}) \text{ with } u_1, \ldots, u_{g+1} \in \mathbb{G}_m.$$ 

The character lattice $X$ of $G$ is

$$X = \text{Hom}(T, \mathbb{G}_m) = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_{g+1},$$

where

$$e_j(t) = u_j \quad \text{for } j = 1, \ldots, g + 1.$$ 

The cocharacter lattice $X^\vee$ of $G$ is

$$X^\vee = \text{Hom}(\mathbb{G}_m, T) = \mathbb{Z}f_1 \oplus \cdots \oplus \mathbb{Z}f_{g+1},$$

where

$$f_j(u) = t(1, \ldots, 1, u, 1, \ldots, 1) \quad \text{with } u \text{ in the } j\text{-th spot, for } j = 1, \ldots, g + 1$$

$$= \begin{cases} \text{diag}(1, \ldots, 1, u_1, \ldots, 1; 1, \ldots, 1, u_1^{-1}, 1, \ldots, 1) & \text{if } 1 \leq j \leq g \\ \text{diag}(1, \ldots, 1; u, \ldots, u) & \text{if } j = g + 1. \end{cases}$$

Under the natural pairing

$$\langle \cdot, \cdot \rangle : X \times X^\vee \to \text{Hom}(\mathbb{G}_m, \mathbb{G}_m) = \mathbb{Z}$$

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given by composition, we have \( \langle e_i, f_j \rangle = \delta_{ij} \).

We choose simple roots lying in the root basis \( \Delta = \{ \alpha_1, \ldots, \alpha_g \} \), with corresponding coroots \( \Delta^\vee = \{ \alpha_1^\vee, \ldots, \alpha_g^\vee \} \), where

\[
\begin{align*}
\alpha_1 &= e_1 - e_2 \\
\alpha_2 &= e_2 - e_3 \\
\vdots \\
\alpha_{g-1} &= e_{g-1} - e_g \\
\alpha_g &= 2e_g - e_{g+1}
\end{align*}
\]

\[
\begin{align*}
\alpha_1^\vee &= f_1 - f_2 \\
\alpha_2^\vee &= f_2 - f_3 \\
\vdots \\
\alpha_{g-1}^\vee &= f_{g-1} - f_g \\
\alpha_g^\vee &= f_g.
\end{align*}
\]

The positive Weyl chamber is given by

\[
P^+ = \{ \lambda = a_1 f_1 + \cdots + a_{g+1} f_{g+1} \in X^\vee \mid \langle \alpha_j, \lambda \rangle \geq 0 \text{ for } j = 1, \ldots, g \}
= \{ \lambda = a_1 f_1 + \cdots + a_{g+1} f_{g+1} \in X^\vee \mid 2a_1 \geq 2a_2 \geq \cdots \geq 2a_{g-1} \geq a_{g+1} \}.
\]

There is a partial order on \( P^+ \) given by \( \mu \leq \lambda \) if

\[
\mu \leq \lambda \quad \text{if} \quad \lambda - \mu = \sum_{j=1}^{g} n_j \alpha_j^\vee, \quad \text{where } n_j \in \mathbb{Z}_{\geq 0}.
\]

**Proposition 1.** If \( \beta^\vee \) is a coroot of \( \text{GSp}_{2g} \), then \( \eta(\beta^\vee(\ell)) = 1 \).

**Proof.** One observes this to be true for the elements \( \alpha_i^\vee \) above, and these form a (multiplicative) basis for the coroots. \( \square \)

The following consequence will play an important role in the proof of the main result (Theorem 17).

**Corollary 2.** If \( \lambda, \mu \in X^\vee \) are coweights of \( \text{GSp}_{2g} \) such that \( \lambda \geq \mu \), then \( \eta(\lambda(\ell)) = \eta(\mu(\ell)) \).

**Proof.** This follows since the definition of the ordering is that \( \lambda \) and \( \mu \) differ by coroots. \( \square \)

The root datum of \( G \) is the quadruple \( (X, X^\vee, \Delta, \Delta^\vee) \). Let \( \hat{G} \) denote the dual group of \( G \), i.e. the algebraic group whose root datum is \( (X^\vee, X, \Delta^\vee, \Delta) \). It is known that \( \hat{G} \) is isomorphic to \( \text{GSpin}_{2g+1} \), which we describe briefly in the following section.

### 3 The spin group

Fix a commutative ring \( R \) in which 2 is invertible. Let \( C(R) \) be the \( R \)-algebra with generators \( c_1, c_2, \ldots, c_{2g+1} \) subject to the relations

\[
\begin{align*}
e_i^2 &= 1 \quad \text{for all } i, \\
c_i c_j &= -c_j c_i \quad \text{for } i \neq j.
\end{align*}
\]

Then \( C(R) \) is a Clifford algebra of dimension \( 2^g \). It has a \( \mathbb{Z}/2\mathbb{Z} \)-grading with even part \( C_0(R) \) (resp. odd part \( C_1(R) \)) spanned by monomials consisting of even (resp. odd) numbers of generators \( c_i \). As a superalgebra, it is isomorphic to the simple superalgebra

\[
Q(2^g) = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} : a, b \in M_{2^g}(R) \right\}.
\]
The parity automorphism of $C(R)$ is the $R$-linear map $\gamma: C(R) \to C(R)$ defined by

$$\gamma(x) = \begin{cases} x & \text{if } x \text{ is even} \\ -x & \text{if } x \text{ is odd.} \end{cases}$$

Let $M \subset C_1(R)$ be the $R$-span of $\{c_1, \ldots, c_{2g+1}\}$. The special Clifford group is

$$\text{GSpin}_{2g+1}(R) = \{x \in C_0(R)^\times \mid \gamma(x)mx^{-1} \in M \text{ for all } m \in M\}.$$ 

4 The Satake isomorphism

The Satake isomorphism relates the local Hecke algebra of a reductive group $G$ to the representation ring of the dual group $\hat{G}$. We follow the description of this relation given in [6]. Our current interest is of course in the pair $(G = \text{GSp}_{2g}, \hat{G} = \text{GSpin}_{2g+1})$, but the setup applies to any pair of dual reductive groups.

There is a bijective correspondence

$$\{\lambda \in P^+\} \longleftrightarrow \{\text{irreducible representations } V_\lambda \text{ of } \hat{G}\}.$$ 

Fix a prime $\ell \neq p$ and consider the local Hecke algebra

$$\mathcal{H}_\ell = \mathcal{H}(G(\mathbb{Q}_\ell), G(\mathbb{Z}_\ell)) = \left\{f: G(\mathbb{Q}_\ell) \longrightarrow \mathbb{Z} \mid f \text{ locally constant, compactly supported, } G(\mathbb{Z}_\ell)\text{-bi-invariant} \right\}$$

with the convolution product

$$(fg)(x) = \int_{G(\mathbb{Q}_\ell)} f(t)g(t^{-1}x) \, dt,$$

where the Haar measure $dt$ on $G(\mathbb{Q}_\ell)$ is normalised so that the maximal compact subgroup $G(\mathbb{Z}_\ell)$ has volume 1.

We work with the basis of $\mathcal{H}_\ell$ consisting of the characteristic functions

$$c_\lambda = \text{char } (G(\mathbb{Z}_\ell)\lambda(\ell)G(\mathbb{Z}_\ell)) \quad \text{for } \lambda \in P^+.$$ 

The Satake transform is a ring isomorphism

$$S_{\mathbb{Z}, \ell}: \mathcal{H}_\ell \otimes \mathbb{Z}[\ell^{\pm 1/2}] \longrightarrow R(\hat{G}) \otimes \mathbb{Z}[\ell^{\pm 1/2}].$$

We tensor this with $\overline{\mathbb{F}}_p$:

$$S_\ell = S_{\mathbb{F}_p, \ell}: \mathcal{H}_\ell \otimes \overline{\mathbb{F}}_p \longrightarrow R(\hat{G}) \otimes \overline{\mathbb{F}}_p.$$ 

If $\lambda \in P^+$ then the image of $c_\lambda$ can be written

$$S_\ell(c_\lambda) = \sum_{\mu \leq \lambda} b_\lambda(\mu)\ell^{(\mu, \lambda)}\chi_\mu,$$

where $\mu$ runs over the elements in $P^+$ such that $\mu \leq \lambda$, $b_\lambda(\mu) \in \overline{\mathbb{F}}_p$ and $b_\lambda(\lambda) = 1$.

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1The special Clifford group is sometimes denoted $\text{CSpin}_{2g+1}$ or $\text{SF}$.
2The element $\rho$ is the half-sum of the positive roots of $G$, but we will not need to know this, only that it is the same in all the identities related to the Satake isomorphism $S$. 

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There is a similar identity expressing $\chi_\lambda$ in terms of images of characteristic functions:

$$\chi_\lambda = \ell^{-(\rho, \lambda)} \sum_{\mu \leq \lambda} d_\lambda(\mu) S_\ell(c_\mu),$$

where $d_\lambda(\mu) \in \mathbb{F}_p$ and $d_\lambda(\lambda) = 1$.

For future use, let us record a consequence of this identity:

(1) $$S_\ell^{-1}(\chi_\lambda) = \ell^{-(\rho, \lambda)} \sum_{\mu \leq \lambda} d_\lambda(\mu) c_\mu.$$  

5 Siegel modular forms

Let $N \geq 3$ and consider the moduli space $\mathfrak{A}_{g,N}$ parametrising principally polarised $g$-dimensional abelian schemes with principal level $N$ structure. This is a scheme over $\mathbb{Z}/N\mathbb{Z}$ endowed with a rank $g$ locally free sheaf $E$ (the Hodge bundle). Let $\kappa = (k_1 \geq \cdots \geq k_g) \in \mathbb{Z}^g$ be the highest weight of a rational representation of $GL_g$, and let $E^\kappa$ denote the locally free sheaf obtained by applying the representation to the transition functions of $E$. Given a ring $R$ where $N$ is invertible, the space of Siegel modular forms of degree $g$, weight $\kappa$ and level $N$ with coefficients in $R$ is by definition

$$M_\kappa(N; R) = H^0(A_{g,N} \otimes R, E^\kappa).$$

In particular, given a prime $p \nmid N$, the space of Siegel modular forms (mod $p$) is $M_\kappa(N; \mathbb{F}_p)$.

Let

$$\mathcal{F}(g) = \left\{ \mathbf{n} = (n_{ij}) \in \text{Mat}_{g \times g}(\frac{1}{2} \mathbb{Z}) : n_{ii} \in \mathbb{Z}, \mathbf{n} \text{ symmetric semipositive definite} \right\}.$$  

Any $f \in M_\kappa(N; R)$ has a Fourier expansion of the form

$$f(q) = \sum_{\mathbf{n} \in \mathcal{F}(g)} a(\mathbf{n}) q^\mathbf{n} \quad \text{with } a(\mathbf{n}) \in R,$$

where, for $\mathbf{n} = (n_{ij})$, we have

$$q^\mathbf{n} = \left( \prod_i q^{n_{ii}} \prod_{i<j} q^{2n_{ij}} \right)^{1/N}.$$  

If $R = \mathbb{C}$, the space $M_\kappa(N; \mathbb{C})$ consists of holomorphic functions $f : \mathfrak{S}_g \to V_\kappa$ such that

$$f \left( (aq + b)(cz + d)^{-1} \right) = \kappa(cz + d) f(z) \quad \text{for all } \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma(N) \text{ and all } z \in \mathfrak{S}_g,$$

where the principal congruence subgroup of level $N$, resp. the Siegel upper half space, are given by

$$\Gamma(N) = \ker (\text{Sp}_{2g}(\mathbb{Z}) \to \text{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z}))$$

$$= \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{Sp}_{2g}(\mathbb{Z}) : \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \equiv \left( \begin{array}{cc} I_g & 0 \\ 0 & I_g \end{array} \right) \pmod{N} \right\}$$

$$\mathfrak{S}_g = \left\{ z \in \text{Sym}_g(\mathbb{C}) \mid \text{Im}(z) \text{ positive definite} \right\}. \quad \text{An additional condition is required in the case } g = 1, \text{ namely that the sections extend to the compactification of } \mathfrak{A}_{1,N}. $$
The Fourier expansion of \( f \in M_\kappa(N; \mathbb{C}) \) is
\[
f(q) = \sum_{n \in F(g)} a(n)q_N^n, \quad \text{where } q_N^n = e^{\frac{2\pi i}{N} \text{Tr}(nz)}.
\]

Given a power series in \( q_N \) as above and a matrix \( M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{GSp}_2(\mathbb{Q}) \) with positive determinant, we define the weight \( \kappa \) slash operator
\[
M|_\kappa f = \eta(M) \sum_{\kappa_j - g(g+1)/2} \kappa(Cz + D)^{-1} f((Az + B)(Cz + D)^{-1}).
\]

Note that if \( f \) has coefficients \( a(n) \) lying in a ring \( R \), then the coefficients of \( M|_\kappa f \) are in the ring extension \( R[\zeta] \) generated by a root of unity of order \( N \) times the determinant of \( M \).

Let \( \ell \) be a prime not dividing \( N \) and let \( K = \text{GSp}_2(\mathbb{Z}_\ell) \). Given a double coset \( KMK \) and its decomposition into right cosets
\[
KMK = \prod_{i=1}^{h} KM_i,
\]
we define an operator on \( M_\kappa(N; \mathbb{C}) \) by
\[
(KMK)(f) = \sum_{i=1}^{h} M_i|_\kappa f.
\]

For arbitrary characteristic zero rings \( R \), the same approach gives the action of \( KMK \) on \( M_\kappa(N; R) \). For other rings (e.g. of positive characteristic not dividing \( N \)), the slash operator must be replaced by certain isogenies on the moduli space of abelian varieties (see [3, Section VII.3] for details). Note that the operator \( KMK \) does preserve the ring of definition of the coefficients of \( f \).

The various Hecke operators are special cases of (linear combinations of) \( KMK \) for specific types of matrices \( M \):

| Operator | Double coset description |
|----------|-------------------------|
| \( T(\ell) \) | \( K \begin{pmatrix} I_g \\ pI_g \end{pmatrix} K \) |
| \( T_i(\ell^2) \), \( 0 \leq i \leq g \) | \( K \begin{pmatrix} I_{g-i} \\ pI_i \\ p^2 I_{g-i} \end{pmatrix} K \) |
| \( T(\ell^2) \) | \( \sum_{i=0}^{g} T_i(\ell^2) \) |

6 Galois representations attached to Siegel modular forms

A Hecke eigenform \( f \) defines a Hecke \( \ell \)-eigensystem \( \Psi_{f,\ell} \), which we think of as a ring homomorphism
\[
\Psi_{f,\ell} : \mathcal{H}_\ell \otimes \overline{\mathbb{F}}_p \rightarrow \overline{\mathbb{F}}_p
\]
taking an operator $T$ to its eigenvalue: $T(f) = \Psi_{f,\ell}(T)f$.

Using the Satake isomorphism $S_\ell$, we can define a character $\omega_f: R(\hat{G}) \otimes \mathbb{F}_p \to \mathbb{F}_p$ by
$$\omega_f(\chi_\lambda) = \Psi_{f,\ell}(S_\ell^{-1}(\chi_\lambda)).$$

But the characters of the representation ring $R(\hat{G}) \otimes \mathbb{F}_p$ are indexed by the semi-simple conjugacy classes in $\hat{G}(\mathbb{F}_p)$. Given such a class $s$, the corresponding character $\omega_s$ is given by
$$\omega_s(\chi_\lambda) = \chi_\lambda(s).$$

In particular, the character $\omega_f$ given above is indexed by some $s_{f,\ell} \in \hat{G}(\mathbb{F}_p)$, called the $\ell$-Satake parameter of $f$.

**Conjecture 3.** Let $f$ be a degree $g$ Siegel modular form (mod $p$) of level $N$ and suppose $f$ is a Hecke eigenform. There exists a semisimple continuous group representation
$$\rho_f: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \hat{G}(\mathbb{F}_p)$$
that is unramified outside $pN$. If $\ell$ is a prime not dividing $pN$, then $\rho_f(\text{Frob}_\ell) = s_{f,\ell}$, the $\ell$-Satake parameter of $f$.

It is expected that $\rho_f$ is odd (for the meaning of this condition in the general case see [7]).

The case $g = 1$ of Conjecture 3 follows from a well-known result of Deligne ($k \geq 2$) and Deligne-Serre ($k = 1$):

**Theorem 4** (Deligne, see [5, Proposition 11.1]). Let $f = \sum a_nq^n \in M_k(\Gamma_1(N); \mathbb{F}_p)$ be a normalised Hecke eigenform (mod $p$). Then there is a continuous, semisimple Galois representation
$$\rho_f: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \text{GL}_2(\mathbb{F}_p),$$
that is unramified at all primes $\ell \nmid Np$ and such that
$$\text{charpoly}\rho_f(\text{Frob}_\ell) = X^2 - a_\ell X + \varepsilon(\ell)\ell^{k-1}$$
for all primes $\ell \nmid Np$.

**Remark 5.** A priori it is not clear how to reconcile the appearance of level $\Gamma(N)$ in Conjecture 3 with the level $\Gamma_1(N)$ appearing in Deligne’s theorem. However, there is an injective group homomorphism $\Gamma_1(N^2) \to \Gamma(N)$ given by
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & Nb \\ \frac{1}{N}c & d \end{pmatrix}$$
that induces a Hecke-equivariant embedding $\iota: M_k(\Gamma(N); R) \to M_k(\Gamma_1(N^2); R)$ whose effect on Fourier expansions is
$$\sum_n a_nq^n \mapsto \sum_n a_nq^{nN}.$$
(Over $\mathbb{C}$, the map is given by $(\iota f)(z) = f(Nz)$.)

The case $g = 2$ of Conjecture 3 follows from work of Laumon, Taylor and Weissauer.

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4Laumon, Taylor and Weissauer construct $p$-adic Galois representations attached to Siegel modular eigenforms in characteristic zero. The mod $p$ version we are interested in here follows by first lifting a mod $p$ eigenform to characteristic zero and then reducing the resulting representation modulo $p$. 

7
7 The theta operator of Böcherer-Nagaoka

We reinterpret an operator defined by Böcherer and Nagaoka in [1] Section 4, and determine its commutation relation with the Hecke operators.

Let $R$ and $S$ be $g \times g$ symmetric matrix variables and let $x$ be an indeterminate. The following defines polynomials $P_0(R,S), \ldots, P_g(R,S)$:

$$\det(R + xS) = \sum_{j=0}^{g} P_j(R,S)x^j.$$  

Let $k_1,k_2$ be integers such that $2k_i \geq g$. We define

$$Q^{(g)}_{k_1,k_2}(R,S) = \sum_{j=0}^{g} (-1)^j j!(2k_2 - j)\left(\frac{2k_1 - g + j}{g - j}\right)P_j(R,S).$$

We define a matrix differential operator $\partial_i$ in the $g(g + 1)/2$ variables $q_{ij}$ of the Fourier expansion of a Siegel modular form:

$$\partial_i = \left(\frac{1}{2} (1 + \delta_{ij}) \partial_{ij}\right) = \left(\begin{array}{cccc}
\partial_{11} & \frac{1}{2} \partial_{12} & \cdots \\
\frac{1}{2} \partial_{12} & \partial_{22} & \cdots \\
\vdots & \vdots & \ddots \\
\partial_{gg} & & & \partial_{gg}
\end{array}\right),$$

where we write $\partial_{ij} = q_{ij} \frac{\partial}{\partial q_{ij}}$.

We can use this to define a differential operator of order $g$

$$D^{(g)}_{k_1,k_2} = Q^{(g)}_{k_1,k_2}(\partial_{q_1}, \partial_{q_2})$$

and a bilinear operator

$$[F,G] = D^{(g)}_{k_1,k_2}(F(q_1)G(q_2))\bigg|_{q_1 = q_2}.$$  

**Theorem 6** (Eholzer-Ibukiyama). Let $\Gamma \subset \operatorname{Sp}_{2g}(\mathbb{R})$ be a discrete subgroup of finite covolume. If $F \in M_{k_1}(\Gamma; \mathbb{C})$ and $G \in M_{k_2}(\Gamma; \mathbb{C})$ with $2k_i \geq g$, then $[F,G] \in M_{k_1+k_2+2}(\Gamma; \mathbb{C})$.

**Proof.** By the calculations of [2] Section 5.1, the polynomial $Q^{(g)}_{k_1,k_2}$ is associated with an invariant pluriharmonic polynomial of the correct type. Therefore, by [2] Theorem 2.3 (itself a special case of [3] Theorem 2), $[F,G]$ is modular of weight $k_1 + k_2 + 2$. \hfill $\square$

A crucial remark is that the polynomials $P_j$ and $Q^{(g)}_{k_1,k_2}$ have coefficients in $\mathbb{Z}$, so that if $F$ and $G$ have Fourier coefficients in a ring $A$, then so does $[F,G]$.

For the rest of this section, fix $g > 1$ and a prime $p = g(g + 1)/2$. Let $\mathbb{Z}_{(p)}$ denote the subring of $\mathbb{Q}$ consisting of fractions with denominators not divisible by $p$. By [1] Theorem 1, there exists a form $H \in M_{p-1}(1; \mathbb{Z}_{(p)})$ such that its reduction modulo $p$ has Fourier expansion $\overline{H}(\mathbf{q}) = 1$. We define an operator $^3$

$$\vartheta^0_{\text{BN}}: M_k(N; \mathbb{Z}_{(p)}) \to M_{k+p+1}(N; \mathbb{Z}_{(p)})$$

by setting $\vartheta^0_{\text{BN}}(F) = [F,H]$.

By reduction modulo $p$, we obtain the following slight reworking of [1] Theorem 4:

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$^3$Note that, since $p \nmid N$, the local ring $\mathbb{Z}_{(p)}$ is a $\mathbb{Z}[1/N]$-module.
\textbf{Theorem 7.} Suppose $g > 1$, $p > g(g+1)/2$ and $k > g+1$. There is a linear map $\vartheta_{\text{BN}} : M_k(N; \mathbb{F}_p) \to M_{k+p+1}(N; \mathbb{F}_p)$ whose effect on $q$-expansions is given by:

$$\text{if } f(q) = \sum_{n \in F(g)} a(n)q^n \text{ then } (\vartheta_{\text{BN}}f)(q) = \frac{1}{N^g} \sum_{n \in F(g)} \det(n)a(n)q^n.$$

\textit{Proof.} Let $f \in M_k(N; \mathbb{F}_p)$. Since $k > g+1$, \cite{15} Theorem 1.3] says that $f$ can be lifted to a form $F \in M_k(N; \mathbb{Z}_{(p)})$ such that $\tilde{F} = f$. Let $\vartheta_{\text{BN}}(f) \in M_{k+p+1}(N; \mathbb{F}_p)$ be defined by

$$\vartheta_{\text{BN}}(f) = \text{the reduction mod } p \text{ of } \frac{(-1)^g}{(g+1)!} \vartheta_{\text{BN}}^0(F).$$

(Note that we are only dividing by factors strictly smaller than $p$.)

It remains to check the statement about the effect on $q$-expansions. First note that

$$\partial q_2(F(q_1)H(q_2)) = F(q_1) \partial q_2(H(q_2))$$

reduces to zero modulo $p$, since $H$ has constant $q$-expansion modulo $p$. Note also that if $j \geq 1$ then $P_j(R, S)$ is divisible by $S$, so these parts of the operator $D_{k_1, k_2}^{(q)}$ do not contribute anything modulo $p$.

Therefore the only contribution comes from $P_0(R, S) = \det(R)$, so modulo $p$ we obtain

$$(\vartheta_{\text{BN}}f)(q) = \frac{(-1)^g}{(g+1)!} \left(\frac{2p-2}{g}\right)^{\frac{1}{2}} \det(\partial q_1)F(q) = \det(\partial q)f(q),$$

which is easily seen to equal the expression in the statement. \hfill \Box

We record for future use part of the argument in the above proof:

\textbf{Corollary 8.} As an operator on formal power series in characteristic zero, $\vartheta_{\text{BN}}^0$ can be written as

$$\vartheta_{\text{BN}}^0 = \vartheta_{\text{BN}}^1 + p\vartheta_{\text{BN}}^2,$$

where $\vartheta_{\text{BN}}^{0,1,2} : \mathbb{Z}_{(p)}[[q_N]] \to \mathbb{Z}_{(p)}[[q_N]]$ and

$$\vartheta_{\text{BN}}^1f = \frac{1}{N^g} \sum_n \det(n)a(n)q^n \text{ if } f = \sum_n a(n)q^n.$$

Moreover,

$$\vartheta_{\text{BN}}(f) = \text{the reduction mod } p \text{ of } \frac{(-1)^g}{(g+1)!} \vartheta_{\text{BN}}^1(F).$$

\textbf{Remark 9.} Eigenforms typically require going up to field extensions, so they are in $M_k(N; \mathbb{F}_p)$ rather than in $M_k(N; \mathbb{F}_p)$. Hence, we will often work with $\vartheta_{\text{BN}} \otimes_{\mathbb{F}_p} \mathbb{F}_p$ (and abuse notation by simply referring to it as $\vartheta_{\text{BN}}$).

In order to study the interaction between $\vartheta_{\text{BN}}$ and the Hecke operators, we will use the following right coset decomposition proved by Ryan and Shemanske:

\textit{In level one, Böcherer and Nagaoka do not impose any restrictions on the weight $k$, and only need to ensure that $p \geq g+3$. This comes at the expense of getting an operator that is only defined on the subspace of mod $p$ Siegel modular forms of level 1 that are reductions of forms in characteristic zero.}
Proposition 10 (Local version of [10, Proposition 2.10]). Let \( r \geq 0 \) and let \( M \in \text{GSp}_{2g}(\mathbb{Q}_\ell) \) satisfy \( \eta(M) = \ell^r \). There exists a \( g \)-tuple \( b = (b_1, \ldots, b_g) \in \mathbb{Z}^g \) with
\[
 r \geq b_1 \geq \cdots \geq b_g \geq 0
\]
such that
\[
 KMK = K \begin{pmatrix} \ell^{r-b} & 0 \\ 0 & \ell^b \end{pmatrix} K, \quad \text{where } K = \text{GSp}_{2g}(\mathbb{Z}_\ell).
\]
Moreover, \( KMK \) can be decomposed into right cosets of the form
\[
 K \begin{pmatrix} \ell^r (D^T)^{-1} & B \\ 0 & D \end{pmatrix}.
\]

Lemma 11. If \( M = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \text{GSp}_{2g}(\mathbb{Q}_\ell) \), then the following diagram commutes:

\[
\begin{array}{c}
\mathbb{Z}_p[[q^N]] & \xrightarrow{\partial_1^{(1)}} & \mathbb{Z}_p[[q^N]] \\
M|_k & \downarrow & M|_{k+p+1} \\
\mathbb{Z}_p[[\zeta]][[q^N]] & \xrightarrow{\det(M)\partial_1^{(1)}} & \mathbb{Z}_p[[\zeta]][[q^N]]
\end{array}
\]

Proof. Since \( M \) is a symplectic similitude matrix, we have
\[
 A = \eta(M) (D^T)^{-1}.
\]
Suppose
\[
 F = \sum_n a(n) q^n, \quad \text{where } q^n = e^{2\pi i \text{Tr}(nz)}.
\]
Using \( \text{Tr}(XY) = \text{Tr}(YX) \), we see that
\[
 (M|_k F)(z) = \eta(M)^{k-g+g/2} \det(D)^{-k} F ((Az + B)D^{-1})
\]
\[
 = \eta(M)^{k-g+g/2} \det(D)^{-k} \sum_n a(n)c(n) e^{2\pi i \text{Tr}(\eta(M)(D^T)^{-1} z D^{-1})/N}
\]
\[
 = \eta(M)^{k-g+g/2} \det(D)^{-k} \sum_n a(n)c(n) q^n',
\]
where \( n' = \eta(M)D^{-1}n (D^T)^{-1} \) and \( c(n) = e^{2\pi i \text{Tr}(nBD^{-1})/N} \).

Note that
\[
 \det(n') = \eta(M)^g \det(D)^{-2} \det(n).
\]
Therefore

\[
(M|_{k+p+1}(\vartheta^1_{BN}F))(q) = \frac{1}{N^g} \eta(M)^{g-g+g/2} \det(D)^{-(k+p+1)} \times \\
\sum_n \det(n) a(n) c(n) q_n^* \cdot
\]

\[
= (\eta(M)^g \det(D)^{-1})^{p-1} \eta(M)^g \times \\
\frac{1}{N^g} \eta(M)^{g+g+g/2} \det(D)^{-2} \times \\
\sum_n \det(n) a(n) c(n) q_n^*,
\]

\[
(\vartheta^1_{BN}(M|_{k}F))(q) = \frac{1}{N^g} \eta(M)^{g+g+g/2} \det(D)^{-2} \times \\
\sum_n \det(n) a(n) c(n) q_n^*.
\]

\[\square\]

**Theorem 12.** If \( M \in \text{GSp}_{2g}(\mathbb{Q}_{\ell}) \), then the following diagram commutes:

\[
\begin{array}{ccc}
M_k(N; \mathbb{F}_p) & \xrightarrow{\vartheta_{BN}} & M_{k+p+1}(N; \mathbb{F}_p) \\
\downarrow KMK & & \downarrow KMK \\
M_k(N; \mathbb{F}_p) & \xrightarrow{\det(M)\vartheta_{BN}} & M_{k+p+1}(N; \mathbb{F}_p)
\end{array}
\]

**Proof.** By Proposition 10 we can decompose

\[KMK = \prod_i KMi,\]

where \( Mi \) is block upper triangular and \( \det(M_i) = \det(M) \). Lemma 11 then tells us that

\[(KMK) \circ \vartheta_{BN} = \det(M) \vartheta^1_{BN} \circ (KMK),\]

from which the assertion follows by reduction modulo \( p \) and Corollary 8. \( \square \)

**Corollary 13.** If \( f \) is a Hecke eigenform and \( \vartheta_{BN}f \neq 0 \), then \( \vartheta_{BN}f \) is a Hecke eigenform. For any prime \( \ell \nmid pN \), the \( \ell \)-eigensystem of \( \vartheta_{BN}f \) satisfies

\[\Psi_{\vartheta_{BN}, \ell}(c_\lambda) = \det(\lambda(\ell)) \Psi_{f, \ell}(c_\lambda) \quad \text{for all } \lambda \in \mathbb{P}^+.\]

**Proof.** This follows from Theorem 12 with \( M = \lambda(\ell) \). \( \square \)

We can easily describe the explicit form of Theorem 12 for the usual Hecke operators:

**Corollary 14.** As operators on the algebra of scalar-valued Siegel modular forms of degree \( g \) and level \( N \), we have for all primes \( \ell \nmid pN \):

\[T(\ell) \circ \vartheta_{BN} = \ell^g \vartheta_{BN} \circ T(\ell),\]

\[T_i(\ell^2) \circ \vartheta_{BN} = \ell^g \vartheta_{BN} \circ T_i(\ell^2),\]

\[T(\ell^2) \circ \vartheta_{BN} = \ell^g \vartheta_{BN} \circ T(\ell^2).\]
8 The theta operators of Flander-Ghitza and Yamauchi

In [4], Flander and Ghitza generalise Katz’s method in [9] and define a differential operator $\vartheta_{FG}$ using geometric techniques. The construction makes use of the geometric definition of Siegel modular forms as objects arising from a moduli space of abelian varieties. The spaces of Siegel modular forms are related to de Rham cohomology sheaves, on which the Gauss-Manin connection provides a way of taking derivatives.

We refer to [4] for details, and note the effect of $\vartheta_{FG}$ on the Hecke eigensystems.

**Proposition 15.** If $f$ is a Hecke eigenform and $\vartheta_{FG} f \neq 0$, then $\vartheta_{FG} f$ is a Hecke eigenform. For any prime $\ell \nmid pN$, the $\ell$-eigensystem of $\vartheta_{FG} f$ satisfies

$$
\Psi_{\vartheta_{FG} f, \ell}(c\lambda) = \eta(\lambda(\ell)) \Psi_{f, \ell}(c\lambda) \quad \text{for all } \lambda \in \mathbb{P}^+.
$$

In [12], Yamauchi studies a class of theta operators for Siegel modular forms attached to $GSp_4$, i.e. the case $g = 2$. This is similar to the technique used in [4], though more can be said because of the restriction on $g$. For instance, the image of the theta operator does not $a$ priori land in a space of Siegel modular forms with irreducible weight. In this particular setting, one can decompose the space into irreducible pieces explicitly and then project the image onto each component. This gives a set of maps $\theta_i$ that can be studied independently.

Further, in the scalar-valued case, applying the map once does indeed land in an irreducible space. However, if one applies this map twice, we instead have a direct sum that can be decomposed. One of these components corresponds to the space of scalar-valued forms, and projecting onto this gives a degree 2 differential map of scalar-valued Siegel modular forms, denoted $\Theta$. This turns out to be precisely the Böcherer-Nagaoka map $\vartheta_{BN}$ of Section 7.

We note the effect of these maps on Hecke eigensystems:

**Proposition 16** (Yamauchi [12, Proposition 3.9]). Let $f$ be a Hecke eigenform.

- **[general case]** If $\theta_i f \neq 0$, then $\theta_i f$ is a Hecke eigenform. For any prime $\ell \nmid pN$, the $\ell$-eigensystem of $\theta_i f$ satisfies

  $$
  \Psi_{\theta_i f, \ell}(c\lambda) = \eta(\lambda(\ell)) \Psi_{f, \ell}(c\lambda) \quad \text{for all } \lambda \in \mathbb{P}^+.
  $$

- **[scalar-valued case]** If $\Theta f \neq 0$, then $\Theta f$ is a Hecke eigenform. For any prime $\ell \nmid pN$, the $\ell$-eigensystem of $\Theta f$ satisfies

  $$
  \Psi_{\Theta f, \ell}(c\lambda) = \det(\lambda(\ell)) \Psi_{f, \ell}(c\lambda) \quad \text{for all } \lambda \in \mathbb{P}^+.
  $$

9 Effect on Galois representations

We are ready to tackle the main result of the paper, inspired by the interactions between Hecke eigensystems and theta operators detailed in Corollary 13 and Propositions 15 and 16.

**Theorem 17.** Let $\eta^\vee$ be the cocharacter of $GSpin_{2g+1}$ corresponding to $\eta$ by duality. Let $f$ be a level $N$ Hecke eigenform $\pmod{p}$. Let $\vartheta$ be a map of modular forms such that $\vartheta f$ is a Hecke eigenform whose $\ell$-eigensystem satisfies

$$
\Psi_{\vartheta f, \ell}(c\lambda) = \eta^m(\lambda(\ell)) \Psi_{f, \ell}(c\lambda) \quad \text{for all } \lambda \in \mathbb{P}^+ \text{ and all primes } \ell \nmid pN,
$$

\footnote{The form $f$ can be vector-valued or scalar-valued depending on which operator $\vartheta$ we are considering.}
where \( m \in \mathbb{Z}_{\geq 0} \) is fixed. Let \( \chi \) be the cyclotomic character (mod \( p \)). Then

\[
\rho_{\theta} \cong (\eta^\vee \circ \chi^m) \otimes \rho_f
\]

as Galois representations \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GSpin}_{2g+1}(\mathbb{F}_p) \).

**Proof.** Both \( \rho_f \) and \( \rho_{\theta} \) are unramified at primes \( \ell \nmid pN \). Thanks to the density in \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) of the Frobenius classes \( \text{Frob}_\ell \) for \( \ell \nmid pN \), it suffices to prove that Equation (3) holds when both sides are applied to such \( \text{Frob}_\ell \). So we fix \( \ell \nmid pN \) and observe that

\[
\rho_{\theta}(\text{Frob}_\ell) = \lambda_{\theta,\ell}((\eta^\vee \circ \chi^m) \otimes \rho_f)(\text{Frob}_\ell) = \eta^\vee(\ell^m)s_{f,\ell}.
\]

We have to prove that the above right hand sides are equal. Referring back to the definition of the Satake parameters in Section 6, we note that this is equivalent to proving that

\[
\chi_\lambda(s_{\theta,\ell}) = \chi_\lambda(\eta^\vee(\ell^m)s_{f,\ell}) \quad \text{for all } \lambda \in P^+.
\]

Let \( \lambda \in P^+ \). We use Equations (1) and (2) to get

\[
\chi_\lambda(s_{\theta,\ell}) = \omega_{\theta,\ell}(\chi_\lambda) = \Psi_{\theta,\ell}
\]

\[
= \Psi_{\theta,\ell}
\]

\[
= \ell(-\rho,\lambda) \sum_{\mu \leq \lambda} d_\lambda(\mu)c_\mu
\]

\[
\chi_{\lambda}(\eta^\vee(\ell^m)s_{f,\ell}) = \chi_{\lambda}(\eta^\vee(\ell^m))\omega_{f,\ell}(\chi_\lambda)
\]

\[
= \chi_{\lambda}(\eta^\vee(\ell^m))\Psi_{f,\ell}(S_{\ell}^{-1}(\chi_\lambda))
\]

\[
= \chi_{\lambda}(\eta^\vee(\ell^m))\Psi_{f,\ell}
\]

\[
= \ell(-\rho,\lambda) \sum_{\mu \leq \lambda} d_\lambda(\mu)c_\mu
\]

We make two observations:

(a) By Corollary 2 if \( \mu \leq \lambda \) then \( \eta(\mu(\ell)) = \eta(\lambda(\ell)) \).

(b) The character \( \chi_\lambda \) of \( \text{GSpin}_{2g+1} \) is precisely the dual of the cocharacter \( \lambda \) of \( \text{GSp}_{2g} \), so by duality we have

\[
\chi_\lambda \circ \eta^\vee = \lambda^\vee \circ \eta^\vee = \eta \circ \lambda,
\]

in particular \( \chi_\lambda(\eta^\vee(\ell^m)) = \eta(\lambda(\ell^m)) \).

We conclude that Equation (4) holds, and therefore so does Equation (3). \( \square \)

We summarise the results for the various operators in the following table:
| Source                  | Galois representation relation                                      | Conditions                                      |
|------------------------|---------------------------------------------------------------------|-------------------------------------------------|
| Böcherer-Nagaoka [1]   | $\rho_{\text{BN}f} \cong (\eta^V \circ \chi^g) \otimes \rho_f$      | $k > g + 1, p > g(g + 1)/2, \text{scalar-valued}$ |
| Flander-Ghitza [4]     | $\rho_{\text{FG}f} \cong (\eta^V \circ \chi) \otimes \rho_f$         |                                                 |
| Yamauchi [12]          | $\rho_{\Theta f} \cong (\eta^V \circ \chi^2) \otimes \rho_f$        | $g = 2, \text{scalar-valued}$                   |

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