THE ADLER-SHIOTA-VAN MOERBEKE FORMULA
FOR THE BKP HIERARCHY

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Abstract

We study the BKP hierarchy and prove the existence of an Adler–Shiota–van Moerbeke formula. This formula relates the action of the $BW_{1+\infty}$–algebra on tau–functions to the action of the “additional symmetries” on wave functions.

1. Introduction and main result

1.1. Adler, Shiota and van Moerbeke [ASV1-2] obtained for the KP and Toda lattice hierarchies a formula which translates the action of the vertex operator on tau–functions to an action of a vertex operator of pseudo-differential operators on wave functions. This relates the additional symmetries of the KP and Toda lattice hierarchy to the $W_{1+\infty}$, respectively $W_{1+\infty} \times W_{1+\infty}$–algebra symmetries. In this paper we investigate the existence of such an Adler–Shiota–van Moerbeke formula for the BKP hierarchy.

1.2. The BKP hierarchy is the set of deformation equations

$$\frac{\partial L}{\partial t_k} = [(L^n)_+, L], \quad k = 3, 5, \ldots$$

for the first order pseudo-differential operator

$$L \equiv L(x, t) = \partial + u_1(x, t)\partial^{-1} + u_2(x, t)\partial^{-2} + \cdots,$$

here $\partial = \frac{\partial}{\partial x}$ and $t = (t_3, t_5, \ldots)$. It is well–known that $L$ dresses as $L = P\partial P^{-1}$ with

$$P \equiv P(x, t) = 1 + a_1(x, t)\partial^{-1} + a_2(x, t)\partial^{-2} + \cdots$$

$$= \frac{\tau(x - 2\partial^{-1}, t - 2[\partial^{-1}])}{\tau(x, t)},$$

where $\tau$ is the famous $\tau$–function, introduced by the Kyoto group [DJKM1-3] and $[z] = (\frac{z^3}{3}, \frac{z^5}{5}, \ldots)$.

The wave or Baker–Akhiezer function

$$w \equiv w(\cdot, x, t, z) = W(x, t, \partial)e^{xz},$$

where

$$W \equiv W(x, t, z) = P(x, t)e^{\xi(x, t, z)},$$

with $\xi(x, t, z) = \sum_{k=1}^{\infty} t_{2k+1}\partial^{2k+1}$

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is an eigenfunction of $L$, viz.,

$$Lw = zw$$

and

$$\frac{\partial w}{\partial t_k} = (L^k)_+ w.$$

Now introduce, following Orlov and Schulman [OS], the pseudo-differential operator $M \equiv M(x,t) = W x W^{-1}$ which action on $w$ amounts to

$$Mw = \frac{\partial w}{\partial z},$$

then $[L, M] = 1$ and

$$\frac{\partial M}{\partial t_k} = [(L^n)_+, M], \quad k = 3, 5, \ldots$$

Let

$$Y(y, w) = \sum_{\ell=0}^{\infty} \frac{(y - w)^\ell}{\ell!} \sum_{k \in \mathbb{Z}} w^{-k-\ell-1}(M^\ell L^{k+\ell} - (-)^{k+\ell} L^{k+\ell-1} M^\ell L), \quad (1.1)$$

then one has the following main result

**Theorem 1.1.**

$$2(w - y)Y(y, w)w(x, t, z) = (w + y)(e^{-\eta(x,t,z)} - 1) \left( \frac{X(y, w)\tau(x,t)}{\tau(x,t)} \right) w(x, t, z), \quad (1.2)$$

where $X(y, w)$ is the following vertex operator

$$X(y, w) = w^{-1} \exp(x(y - w) + \sum_{j>2, odd} t_j (y^j - w^j) \exp(-2\frac{\partial}{\partial x}(y^{-1} - w^{-1}) - 2 \sum_{j>2, odd} \frac{\partial}{\partial t_j} \frac{y^{-j} - w^{-j}}{j}). \quad (1.3)$$

Formula (1.2) is the Adler–Shiota–van Moerbeke formula for the BKP hierarchy, we will give a proof of this formula in section 6. This formula relates the “additional symmetries” of the BKP hierarchy, generated by $Y(y, w)$, to the $BW_{1+\infty}$-algebra, generated by $X(y, w)$. This $BW_{1+\infty}$-algebra is a subalgebra of $W_{1+\infty}$, which is defined as the $-1$-eigenspace of an anti-involution on $W_{1+\infty}$. 
2. The Lie algebras $o_{\infty}$, $B_{\infty}$ and $BW_{1+\infty}$

2.1. Let $\overline{gl_{\infty}}$ be the Lie algebra of complex infinite dimensional matrices such that all nonzero entries are within a finite distance from the main diagonal, i.e.,

$$\overline{gl_{\infty}} = \{(a_{ij})_{i,j \in \mathbb{Z}} | a_{ij} = 0 \text{ if } |i - j| > 0\}.$$ 

The elements $E_{ij}$, the matrix with the $(i, j)$-th entry 1 and 0 elsewhere, for $i, j \in \mathbb{Z}$ form a basis of a subalgebra $gl_{\infty} \subset \overline{gl_{\infty}}$. The Lie algebra $\overline{gl_{\infty}}$ has a universal central extension $A_{\infty} = \overline{gl_{\infty}} \oplus \mathbb{C} c_A$ with the Lie bracket defined by

$$[a + \alpha c_A, b + \beta c_A] = ab - ba + \mu(a, b)c_A,$$  

for $a, b \in \overline{gl_{\infty}}$ and $\alpha, \beta \in \mathbb{C}$; here $\mu$ is the following 2-cocycle:

$$\mu(E_{ij}, E_{kl}) = \delta_{il}\delta_{jk}(\theta(i) - \theta(j)),$$  

where the function $\theta : \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$\theta(i) = \begin{cases} 0 & \text{if } i > 0, \\ 1 & \text{if } i \leq 0. \end{cases}$$  

The Lie algebra $gl_{\infty}$ and $\overline{gl_{\infty}}$ both have a natural action on the space of column vectors, viz., let $C^\infty = \bigoplus_{k \in \mathbb{Z}} e_k$, then $E_{ij}e_k = \delta_{jk}e_i$. By identifying $e_k$ with $t^{-k}$, we can embed the algebra $D$ of differential operators on the circle, with basis $-t^{i+j}(\frac{\partial}{\partial t})^k$ $(j \in \mathbb{Z}, k \in \mathbb{Z}^+)$, in $\overline{gl_{\infty}}$:

$$\rho : D \rightarrow \overline{gl_{\infty}},$$  

$$\rho(-t^{i+k}(\frac{\partial}{\partial t})^k) = \sum_{m \in \mathbb{Z}} -m(m-1)\cdots(m-k+1)E_{-m-j,-m}.$$  

It is straightforward to check that the 2-cocycle $\mu$ on $\overline{gl_{\infty}}$ induces the following 2-cocycle on $D$:

$$\mu(-t^{i+j}(\frac{\partial}{\partial t})^j, -t^{k+\ell}(\frac{\partial}{\partial t})^\ell) = \delta_{i,-k}(-)^j j!((-1)^j + \frac{j}{j + \ell + 1} c_A).$$  

This cocycle was discovered by Kac and Peterson in [KP] (see also [R], [KR]). In this way we have defined a central extension of $D$, which we denote by $W_{1+\infty} = D \oplus \mathbb{C} c_A$, the Lie bracket on $W_{1+\infty}$ is given by

$$[-t^{i+j}(\frac{\partial}{\partial t})^j + \alpha c_A, -t^{k+\ell}(\frac{\partial}{\partial t})^\ell + \beta c_A] =$$

$$\sum_{m=0}^{\max(j,\ell)} m!(\frac{i+j}{m})(\frac{\ell}{m}) - (\frac{k+\ell}{m})(\frac{j}{m}) (-t^{i+j+k+m}(\frac{\partial}{\partial t})^{i+j+k+m}) + \delta_{i,-k}(-)^j j!(\frac{i+j}{j + \ell + 1}) c_A.$$  

Let $D = t \frac{\partial}{\partial t}$, then we can rewrite the elements $-t^{i+j}(\frac{\partial}{\partial t})^j$, viz.,

$$-t^{i+j}(\frac{\partial}{\partial t})^j = -t^{i}D(D-1)(D-2)\cdots(D-j+1).$$  

Then

$$\rho(t^{k}f(D)) = \sum_{j \in \mathbb{Z}} f(-j)E_{j-k,j},$$  

and the 2-cocycle is as follows [KR]:

$$\mu(t^{k}f(D), t^{\ell}g(D)) = \begin{cases} \sum_{-k \leq j \leq -1} f(j)g(j+k) & \text{if } k = -\ell \geq 0, \\ 0 & \text{otherwise}, \end{cases}$$  

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hence the bracket is
\[ [t^k f(D), t^\ell g(D)] = t^{k+\ell} (f(D+\ell) g(D) - g(D+k) f(D)) + \mu(t^k f(D), t^\ell g(D)). \] (2.10)

2.2. Define on \( gl_\infty \) the following linear anti–involution:
\[ \iota(E_{jk}) = (-)^{j+k}E_{-k,-j}. \] (2.11)

Using this anti–involution we define the Lie algebra \( \overline{o}_\infty \) as a subalgebra of \( gl_\infty \):
\[ \overline{o}_\infty = \{ a \in gl_\infty | \iota(a) = -a \}. \] (2.12)

The elements \( F_{jk} = E_{-j,k} - (-)^{j+k}E_{-k,j} \) with \( j < k \) form a basis of \( o_\infty = \overline{o}_\infty \cap gl_\infty \). The 2–cocycle \( \mu \) on \( gl_\infty \) induces a 2–cocycle on \( \overline{o}_\infty \), and hence we can define a central extension \( B_\infty = \overline{o}_\infty \oplus \mathbb{C}c_B \) of \( \overline{o}_\infty \), with Lie bracket
\[ [a + \alpha c_B, b + \beta c_B] = ab - ba + \frac{1}{2} \mu(a, b) c_B, \] (2.13)
for \( a, b \in \overline{o}_\infty \) and \( \alpha, \beta \in \mathbb{C} \). It is then straightforward to check that the anti–involution \( \iota \) induces
\[ \iota(t) = -t, \quad \iota(D) = -D. \] (2.14)

Hence, it induces the following anti–involution on \( D \):
\[ \iota(t^k f(D)) = f(-D)(-t)^k. \] (2.15)

Define \( D^B = D \cap \overline{o}_\infty = \{ w \in D | \iota(w) = -w \} \), it is spanned by the elements
\[ W_k(f) := t^k f(D) - f(-D)(-t)^k = t^k(f(D) - (-)^k f(-D - k)). \] (2.16)

It is straightforward to check that
\[ \rho(W_k(f)) = \sum_{j \in \mathbb{Z}} f(-j) F_{k-j,j}. \]

The restriction of the 2–cocycle \( \mu \) on \( D \), given by (2.5) or (2.9), induces a 2–cocycle on \( D^B \), which we shall not calculate explicitly here. It defines a central extension \( BW_{1+\infty} = D^B \oplus \mathbb{C}c_B \) of \( D^B \), with Lie bracket
\[ [a + \alpha c_B, b + \beta c_B] = ab - ba + \frac{1}{2} \mu(a, b) c_B, \]
for \( a, b \in D^B \) and \( \alpha, \beta \in \mathbb{C} \).
3. The spin module

3.1. We now want to consider highest weight representations of \( o_\infty \), \( B_\infty \) and \( BW_{1+\infty} \). For this purpose we introduce the Clifford algebra \( B_{\text{Cl}} \) as the associative algebra on the generators \( \phi_j, j \in \mathbb{Z} \), called neutral free fermions, with defining relations

\[ \phi_i \phi_j + \phi_j \phi_i = (-)^j \delta_{i,-j}. \]  

We define the spin module \( V \) over \( B_{\text{Cl}} \) as the irreducible module with highest weight vector the vacuum vector \( |0\rangle \) satisfying

\[ \phi_j |0\rangle = 0 \quad \text{for} \quad j > 0. \]  

The elements \( \phi_{j_1} \phi_{j_2} \cdots \phi_{j_p} |0\rangle \) with \( j_1 < j_2 < \cdots < j_p \leq 0 \) form a basis of \( V \). Then

\[ \pi(F_{jk}) = \frac{(-)^j}{2} (\phi_j \phi_k - \phi_k \phi_j), \]
\[ \hat{\pi}(F_{jk}) = (-)^j : \phi_j \phi_k :, \]
\[ \hat{\pi}(c_B) = I, \]

where the normal ordered product \( : : \) is defined as follows

\[ : \phi_j \phi_k := \begin{cases} \phi_j \phi_k & \text{if} \ k > j, \\ \frac{1}{2}(\phi_j \phi_k - \phi_k \phi_j) & \text{if} \ j = k, \\ -\phi_k \phi_j & \text{if} \ k < j, \end{cases} \]

define representations of \( o_\infty \), respectively \( B_\infty \).

When restricted to \( o_\infty \) and \( B_\infty \), the spin module \( V \) breaks into the direct sum of two irreducible modules. To describe this decomposition we define a \( \mathbb{Z}_2 \)–gradation on \( V \) by introducing a chirality operator \( \chi \) satisfying \( \chi |0\rangle = |0\rangle \), \( \chi \phi_j + \phi_j \chi = 0 \) for all \( j \in \mathbb{Z} \), then

\[ V = \bigoplus_{\alpha \in \mathbb{Z}_2} V_{\alpha} \quad \text{where} \quad V_{\alpha} = \{ v \in V | \chi v = (-)^{\alpha} v \}. \]

Each module \( V_{\alpha} \) is an irreducible highest weight module with highest weight vector \( |0\rangle, |1\rangle = \sqrt{2} \phi_0 |0\rangle \) for \( V_0, V_1 \), respectively, in the sense that \( \hat{\pi}(c_B) = 1 \) and

\[ \pi(F_{-i,j})|\alpha\rangle = \pi(F_{-i,j})|\alpha\rangle = 0 \quad \text{for} \ i < j, \]
\[ \pi(F_{-i,i})|\alpha\rangle = -(-)^i|\alpha\rangle \quad \text{for} \ i > 0, \]
\[ \hat{\pi}(F_{-i,i})|\alpha\rangle = 0. \]

Clearly \( V_{\alpha} \) is also a highest weight module for \( BW_{1+\infty} \), viz

\[ \hat{\pi} \cdot \rho(W_k(f)) = \sum_{j \in \mathbb{Z}} (-)^{k+j} f(-j) : \phi_{k-j} \phi_j :, \]

and

\[ \hat{\pi} \cdot \rho(W_k(f))|\alpha\rangle = 0 \quad \text{for} \ k \geq 0. \]

From now on we will omit \( o_\infty, \hat{\pi} \) and \( \hat{\pi} \cdot \rho \), whenever no confusion can arise.
4. Vertex operators

4.1. Using the boson fermion correspondence (see e.g. [DJKM 3], [K], [tKL] and [Y]), we can express the fermions in terms of differential operators, i.e. there exists an isomorphism $\sigma : V \rightarrow \mathbb{C}[\theta, t_1, t_3, \cdots]$, where $\theta^2 = 0, t_j t_j = t_j t_{-j}$, $\theta t_j = t_j \theta$ and $V_\alpha = \theta^\alpha \mathbb{C}[t_1, t_3, \cdots]$, such that $\sigma(0) = 1$. Define the following two generating series (fermionic fields):
\[
\phi^\pm(z) = \sum_{j \in \mathbb{Z}} \phi^\pm_j z^{-j} = \sum_{j \in \mathbb{Z}} (\pm)^j \phi_j z^{-j},
\]
then one has the following vertex operator for these fields:
\[
\sigma \phi^+(z) \sigma^{-1} = \frac{\theta + \frac{\partial}{\partial t}}{\sqrt{2}} \exp(\pm \sum_{j > 0, \text{odd}} t_j z^j) \exp(\pm 2 \sum_{j > 0, \text{odd}} \frac{\partial}{\partial t_j} z^{-j}).
\]

4.2. Define
\[
W(y, w) = \sum_{\ell = 0}^{\infty} \frac{(y - w)^\ell}{\ell!} W^{(\ell + 1)}(w)
\]
\[
= \sum_{\ell = 0}^{\infty} \frac{(y - w)^\ell}{\ell!} \sum_{k \in \mathbb{Z}} W_k^{(\ell + 1)} w^{-k - \ell - 1}
\]
\[
: \phi^+(y) \phi^-(w) : = \frac{\partial^\ell \phi^+(z) \phi^-(z)}{\partial z^\ell} |_{z = w},
\]
then
\[
W^{(\ell + 1)}(z) = : \frac{\partial^\ell \phi^+(z) \phi^-(z)}{\partial z^\ell} : (4.4)
\]
and
\[
W_k^{(\ell + 1)} = W_k(-\ell \left( \begin{array}{c} D \\ \ell \end{array} \right))
\]
\[
= -t^\ell D(D - 1) \cdots (D - \ell + 1) + (-D)(-D-1) \cdots (-D-\ell+1)(-t)^k
\]
\[
= -t^\ell \partial^\ell + (-1)^{k+\ell} t^k \partial^{k+\ell - 1}.
\]

Using (4.2), we find that for $|w| < |y|
\[
W(y, w) = \frac{1}{2} \frac{y + w}{y - w} (X(y, w) - w^{-1}),
\]
where $X(y, w)$ is the vertex operator defined in (1.3). Hence,
\[
W^{(\ell)}(z) = \frac{w \partial^{\ell} X(y, z)}{\partial z^\ell} |_{y = z} + \frac{1}{2} \frac{\partial^{\ell - 1} X(y, z) - z^{-1}}{\partial z^{\ell - 1}} |_{y = z}
\]
Define
\[
\alpha_j(z) = \begin{cases} 
\frac{1}{2} x & \text{if } j = -1, \\
\frac{1}{2} t_{-j} & \text{if } j < 2 \text{ odd}, \\
\frac{x}{\partial x} & \text{if } j = 1, \\
\frac{t_j}{\partial t_j} & \text{if } j > 2 \text{ odd,}
\end{cases}
\]
(4.8)
and their generating series by
\[
\alpha(z) = \sum_{j \in \mathbb{Z}} \alpha_j z^{-j - 1},
\]
(4.9)
then $[\alpha_j, \alpha_k] = \frac{i}{2} \delta_{j, -k}$. Since $X(z, z) = z^{-1}$, one finds the following expression for $W^{(\ell)}(z)$:
\[
W^{(\ell)}(z) = \frac{2}{\ell} : (2 \alpha(z) + \frac{\partial}{\partial z})^{\ell - 1} \alpha(z) : + \frac{1}{z} : (2 \alpha(z) + \frac{\partial}{\partial z})^{\ell - 2} \alpha(z) :.
\]
(4.10)
For \( \ell = 1, 2, 3 \) one finds respectively
\[
W^{(1)}(z) = 2\alpha(z),
\]
\[
W^{(2)}(z) = 2 : \alpha(z)^2 : + \frac{\partial \alpha(z)}{\partial z} + \frac{\alpha(z)}{z},
\]
\[
W^{(3)}(z) = \frac{8}{3} : \alpha(z)^3 : + \frac{8}{3} \alpha(z) \frac{\partial \alpha(z)}{\partial z} : + \frac{2}{z} : \alpha(z)^2 : + \frac{2}{3} \frac{\partial^2 \alpha(z)}{\partial z^2}.
\]

5. The BKP hierarchy

5.1. The BKP hierarchy is the following equation for \( \tau = \tau(t_1, t_3, \ldots) \) (see e.g [DJM3], [K], [L2], [Y]):
\[
\text{Res}_{z=0} \frac{dz}{z} \phi^+(z) \tau \otimes \phi^-(z) \tau = \frac{1}{2} \theta \tau \otimes \theta \tau. \tag{5.1}
\]

Here \( \text{Res}_{z=0} dz \sum_j f_j z^j = f_{-1} \). We assume that \( \tau \) is any solution of (5.1), so we no longer assume that \( \tau \) is a polynomial in \( t_1, t_3, \ldots \).

We proceed now to rewrite (5.1) in terms of formal pseudo–differential operators. We start by multiplying (5.1) from the left with \( \frac{\partial}{\partial t} \otimes \frac{\partial}{\partial t} \) and divide both the first and the last component of the tensor product by \( \tau(t) \). Let \( x = t_1 \) and \( \partial = \frac{\partial}{\partial x} \), then (5.1) is equivalent to the following bilinear identity:
\[
\text{Res}_{z=0} \frac{dz}{z} w(x, t, z) w(x', t', -z) = 1, \tag{5.2}
\]

where
\[
w(x, t, \pm z) = W(x, t, \pm z) e^{xz} = W(x, t, \partial) e^{xz} \quad \text{with}
\]
\[
W(x, t, z) = P(x, t, z) e^{\xi(t, z)}, \quad \xi(t, z) = \sum_{i=3}^1 t_i z^i \quad \text{and}
\]
\[
P(x, t, z) = \frac{e^{-\eta(x, t, z) \tau(x, t)}}{\tau(x, t)} = \frac{\tau(x - \frac{2}{z}, t_3 - \frac{2}{z^2}, t_5 - \frac{2}{z^3}, \ldots)}{\tau(x, t)} \equiv \tau(x, t, z), \tag{5.4}
\]

where \( \eta(x, t, z) = 2(\frac{\partial}{\partial t} z^{-1} + \sum_{j>2} \frac{\partial}{\partial j} z^{-j}) \), for convenience we also define \( \xi(x, t, z) = \xi(t, z) e^{xz} \).

5.2. As usual one denotes the differential part of a pseudo–differential operator \( P = \sum_j P_j \partial^{-j} \) by \( P_+ = \sum_{j \geq 0} P_j \partial^{-j} \) and writes \( P_- = P - P_+ \). The anti–involution * is defined as follows \( (\sum_j P_j \partial^{-j})^* = \sum_j (-\partial)^{-j} P_j \).

One has the following fundamental lemma.

**Lemma 5.1.** Let \( P(x, t, \partial) \) and \( Q(x, t, \partial) \) be two formal pseudo–differential operators, then
\[
(P(x, t, \partial) Q(x, t', \partial))_- = \pm \sum_{i>0} R_i(x, t, t') \partial^{-i}
\]

if and only if
\[
\text{Res}_{z=0} dz P(x, t, \partial) e^{+xz} Q(x', t', \partial) e^{-xz} = \sum_{i>0} R_i(x, t, t') \frac{(x - x')^i-1}{(i-1)!}.
\]

The proof of this lemma is analogous to the proof of Lemma 4.1 of [L1] (see also [KL]).

5.3. Now differentiate (5.2) to \( t_k \), where we assume that \( x = t_1 \), then we obtain
\[
\text{Res}_{z=0} \frac{dz}{z} \left( \frac{\partial P(x, t, z)}{\partial t_k} + P(x, t, z) z^k \right) e^{\xi(x, t, z)} P(x', t', -z) e^{-\xi(x', t', z)} = 0. \tag{5.5}
\]
Now using lemma 5.1 we deduce that
\[
((\frac{\partial P}{\partial t})^k + P \frac{\partial}{\partial k}) P^* = 0.
\]
From the case \( k = 1 \) we then deduce that \( P^* = \partial P^{-1} \partial^{-1} \), if \( k \neq 1 \), one thus obtains
\[
\frac{\partial P}{\partial x_k} = -(P \partial k P^{-1} \partial^{-1}) \partial P. \tag{5.6}
\]
Since \( k \) is odd, \( \partial^{-1}(P \partial k P^{-1} \partial^{-1}) \partial = -P \partial k P^{-1} \) and hence \( (P \partial k P^{-1} \partial^{-1}) \partial = (P \partial k P^{-1}) \). So \( (5.6) \) turns into Sato’s equation:
\[
\frac{\partial P}{\partial t_k} = -(P \partial k P^{-1}) \partial P. \tag{5.7}
\]

5.4. Define the operators
\[
L = W \partial W^{-1} = P \partial P^{-1}, \quad \Gamma = x + \sum_{j>2} j t_j \partial^{j-1},
\]
\[
M = W x W^{-1} = PT P^{-1} \quad \text{and} \quad N = ML. \tag{5.8}
\]
Then \([L, M] = 1 \) and \([L, N] = L \). Let \( B_k = (L^k)_+ \), using \( (5.7) \) one deduces the following Lax equations:
\[
\frac{\partial L}{\partial t_k} = [B_k, L], \quad \frac{\partial M}{\partial t_k} = [B_k, M] \quad \text{and} \quad \frac{\partial N}{\partial t_k} = [B_k, N]. \tag{5.9}
\]
The first equation of \( (5.9) \) is equivalent to the following Zakharov Shabat equation:
\[
\frac{\partial B_j}{\partial t_k} - \frac{\partial B_k}{\partial t_j} = [B_k, B_j], \tag{5.10}
\]
which are the compatibility conditions of the following linear problem for \( w = w(x, t, z) \):
\[
Lw = zw, \quad Mw = \frac{\partial w}{\partial z} \quad \text{and} \quad \frac{\partial w}{\partial t_k} = B_k w. \tag{5.11}
\]

5.5. The formal adjoint of the wave function \( w \) is (see [DJKM]):
\[
w^* = w^*(x, z) = P^{*-1} e^{-\xi(x, t, z)} = P \partial^{-1} e^{-\xi(x, t, z)}. \tag{5.12}
\]
Now \( L^* = -\partial L \partial^{-1} = -\partial P \partial P^{-1} \partial^{-1} \) and \( M^* = \partial P \partial^{-1} \Gamma \partial P^{-1} \partial^{-1} \), so \([L^*, M^*] = -1 \) and
\[
L^* w^* = zw^*, \quad M^* w^* = -\frac{\partial w^*}{\partial z} \quad \text{and} \quad \frac{\partial w^*}{\partial x_k} = -(L^* k)_+ w^* = -B_k w^*. \tag{5.13}
\]
Finally, notice that by differentiating the bilinear identity \( (5.2) \) to \( x'_1 \) we obtain
\[
\text{Res}_{z=0} dw(x, t, z) w^*(x', t', z) = 0. \tag{5.14}
\]
6. Proof of Theorem 1

6.1. In this section we prove Theorem 1.1. We start from the bilinear identity (5.14) and multiply it by \( \tau(x,t) \), which gives

\[
\text{Res}_{z=0} dz e^{-y(x,t,z)} \tau(x,t) e^{\xi(x,t,z)} \frac{\partial}{\partial x'} \left( \frac{e^{y(x',t',z)} \tau(x',t') e^{-\xi(x',t',z)}}{\tau(x',t')} \right) = 0. \tag{6.1}
\]

Now let \((1-w/y)^{-1}(1+w/y)X(y,w)\) act on this identity, then one obtains

\[
\text{Res}_{z=0} \frac{dz}{wz} 1 + \frac{w}{w} - \frac{1}{y} + \frac{w}{z} - \frac{1}{z} e^{-y(x,t,z)} - \frac{1}{y} + \frac{w}{z} \frac{e^{-y(x,t,z)} - \eta(x,t,y) + \eta(x,t,w) \tau(x,t) e^{\xi(x,t,z) + \xi(x,t,y) - \xi(x,t,w)} \times \\
\frac{\partial}{\partial x'} \left( \frac{e^{y(x',t',z)} \tau(x',t') e^{-\xi(x',t',z)}}{\tau(x',t')} \right) = 0.
\] \tag{6.2}

Next use the fact that \((1-u)^{-1}(1+u) = 2\delta(u,1) - (1-u)^{-1}(1+u^{-1})\), where \(\delta(u,v) = \sum_{j \in \mathbb{Z}} u^{-j} v^{j-1}\), then (6.2) is equivalent to

\[
- \text{Res}_{z=0} \frac{dz}{wz} 1 + \frac{w}{w} - \frac{1}{y} + \frac{w}{z} - \frac{1}{z} e^{-y(x,t,z)} - \frac{1}{y} + \frac{w}{z} \frac{e^{-y(x,t,z)} - \eta(x,t,y) + \eta(x,t,w) \tau(x,t) e^{\xi(x,t,z) + \xi(x,t,y) - \xi(x,t,w)} \times \\
\frac{\partial}{\partial x'} \left( \frac{e^{y(x',t',z)} \tau(x',t') e^{-\xi(x',t',z)}}{\tau(x',t')} \right) = 0.
\] \tag{6.3}

Divide this formula by \( \tau(x,t) \), then it turns into

\[
- \text{Res}_{z=0} \frac{dz}{z} e^{-y(x,t,z)} \left( \frac{1 + w/y X(y,w) \tau(x,t)}{1 - w/y} \right) e^{-y(x,t,z)} \frac{\tau(x,t)}{\tau(x,t)} e^{\xi(x,t,z)} \frac{\partial}{\partial x'} \left( \frac{e^{y(x',t',z)} \tau(x',t') e^{-\xi(x',t',z)}}{\tau(x',t')} \right)
\]

\[
= 2 \text{Res}_{z=0} \frac{dz}{z} \delta(w,z) \left( \frac{e^{-y(x,t,z)} \tau(x,t)}{\tau(x,t)} e^{\xi(x,t,y)} \frac{\partial}{\partial x'} \left( \frac{e^{y(x',t',z)} \tau(x',t') e^{-\xi(x',t',z)}}{\tau(x',t')} \right) \right)
\]

\[
= 2 \text{Res}_{z=0} \frac{dz}{wz} \sum_{\ell=0}^\infty \frac{(y-w) \ell!}{\ell!} \sum_{k \in \mathbb{Z}} \left( \frac{z}{w} \right)^{k+\ell-1} \left( \frac{\partial}{\partial z} \right)^\ell (W(x,t,z) e^{xz}) \frac{\partial}{\partial x'} (W(x',t',z) e^{-xz})
\]

\[
- W(x,t,-z) e^{-xz} \left( \frac{\partial}{\partial z} \right)^\ell (W(x',t',z) e^{xz})
\]

\[
= 2 \text{Res}_{z=0} \frac{dz}{wz} \sum_{\ell=0}^\infty \frac{(y-w) \ell!}{\ell!} \sum_{k \in \mathbb{Z}} w^{-k+\ell-1} (W(x,t,z) e^{x} \partial^{k+\ell-1} e^{-xz} \frac{\partial}{\partial x'} (W(x',t',z) e^{xz}))
\]

\[
- W(x,t,-z) e^{-xz} \frac{\partial}{\partial x'} (W(x',t',z) x^{k+\ell-1} e^{xz})
\].

(6.4)

Now define

\[
\sum_{j=0}^\infty c_j (x,t,y,w) z^{-j} = e^{-y(x,t,z)} \left( \frac{1 + w/y X(y,w) \tau(x,t)}{1 - w/y} \right),
\] \tag{6.5}

then the first line of (6.4) is equal to

\[
- \text{Res}_{z=0} dz \sum_{j=0}^\infty c_j (x,t,y,w) L^{-j-1} (W(x,t,z) e^{xz}) \frac{\partial}{\partial x'} (W(x',t',z) e^{-xz})
\]

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Now using Lemma 5.1 with \( t = t' \), one deduces that
\[
\frac{1}{2} \sum_{j=1}^{\infty} c_j(x, t, y, w) L^{-j} = \\
- \sum_{\ell=0}^{\infty} \frac{(y - w)^{\ell}}{\ell!} \sum_{k \in \mathbb{Z}} w^{-k - \ell - 1} (W(x, t, z)x^\ell \partial^{k+\ell}W(x, t, z)^{-1} - (-)^{k+\ell} W(x, t, z)\partial^{k+\ell-1}x^\ell \partial W(x, t, z)^{-1}).
\]

So finally one has
\[
\frac{1}{2} (e^{-\eta(x,t,z)} - 1) \left( \frac{1 + w/y}{1 - w/y} \frac{X(y, w) \tau(x, t)}{\tau(x, t)} \right) w(x, t, z) = \\
- \sum_{\ell=0}^{\infty} \frac{(y - w)^{\ell}}{\ell!} \sum_{k \in \mathbb{Z}} w^{-k - \ell - 1} (M^{k+\ell} L^{k+\ell} - (-)^{k+\ell} L^{k+\ell-1} M^{k+\ell}) \tau(x, t)^{-1} \partial w(x, t, z),
\]
which is equal to the Adler–Shiota–van Moerbeke formula (1.2) for the BKP case:
\[
(w + y)(e^{-\eta(x,t,z)} - 1) \left( \frac{X(y, w) \tau(x, t)}{\tau(x, t)} \right) w(x, t, z) = 2(w - y)Y(y, w) - w(x, t, z).
\]

6.2. Since the left–hand–side of (1.2) is also equal to
\[
(w + y)(e^{-\eta(x,t,z)} - 1) \left( \frac{X(y, w) \tau(x, t)}{\tau(x, t)} \right) w(x, t, z),
\]
we have the following corollary of Theorem 1.1:

**Corollary 6.1.** For \( k \in \mathbb{Z} \) and \( f \) some polynomial one has
\[
\frac{\sigma \cdot \hat{\pi} \cdot \rho(W_k(f)) \tau(x, t)}{\tau(x, t)} = \frac{(f(N)L^k - (-)^k f(-N)) \tau(x, t)}{w(x, t, z)}. \tag{6.7}
\]

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