Some examples of non-massive Frobenius manifolds in Singularity Theory

Ignacio de Gregorio

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Abstract

Let $f, g : \mathbb{C}^2 \to \mathbb{C}$ be two quasi-homogeneous polynomials. We compute the $V$-filtration of the restriction of $f$ to any plane curve $C_t = g^{-1}(t)$ and show that the Gorenstein generator $dx \wedge dy/dg$ is a primitive form. Using results of A. Douai and C. Sabbah, we conclude that base space of the miniversal unfolding of $f_t := f|_{C_t}$ is a Frobenius manifold. At the singular fibre $C_0$ we obtain a non-massive Frobenius manifold.

1 Introduction

The axiomatic of Frobenius manifolds as originally defined by B. Dubrovin as for example in [6], represents the geometrisation of the celebrated WDVV or associativity equations in topological quantum field theories (c.f. [4]). This geometrisation made it plain evident that Frobenius manifold, and hence solutions to WDVV equations, already existed in a very different branch of mathematics, namely singularity theory and more particularly deformations of hypersurface singularities. This work had been carried out by K. Saito and M. Saito nearly ten years before (see [14], [15] and [16]).

The other main source of Frobenius manifolds is quantum cohomology, where the solutions are a priori, just formal series and can only be geometrised after some effort if at all. A version of the mirror phenomenon is interpreted in this framework as an isomorphism of two Frobenius manifolds, each coming from one of this two seemingly unrelated sources. In this direction, we have the result of S. Barannikov ([1]) establishing an isomorphism between the quantum cohomology of projective spaces and the Frobenius manifold obtained by unfolding the function $x_0 + \cdots + x_n$ on the affine variety $x_0 \cdots x_n = 1$.

As the mirror of $\mathbb{P}^n$ indicates, in order to find potential mirrors of algebraic varieties, it is not enough to look at Frobenius manifolds produced by unfolding of germs of isolated singularities. Global functions on affine varieties are needed. A. Douai and C. Sabbah in [5] have adapted the results of M. Saito to this global affine situation and, under some mild hypothesis, reduced the existence of Frobenius-Saito structures on the base space of the miniversal unfolding to the existence of a primitive form for the Gauss-Manin system. They used their results to exhibit Frobenius structures for unfoldings of non-degenerate and convenient Laurent polynomials.

In this article, we construct Frobenius manifolds for unfoldings of quasi-homogeneous functions on quasi-homogeneous plane curves. Let $f, g : \mathbb{C}^2 \to \mathbb{C}$ be quasi-homogeneous polynomials with respect to the same weights. We regard $g$ as a family of plane curves $C_t = g^{-1}(t)$, and consider the restriction $f_t := f|_{C_t}$. We show that the $V$-filtration of the Gauss-Manin system, and hence the spectral pairs, of $f_t$ can be computed from $f_0$. In particular, the Gorenstein generator $\alpha := dx \wedge dy/dg$ yields a primitive form with associated spectral number 0. It follows from [5] that the base space of the miniversal deformation of $f_t$ can be endowed with a Frobenius manifold structure. At $t = 0$, the curve $C_0$ has an isolated singularity. We can use the dualising module $\omega_{C_0}$ to define the Gauss-Manin system of $f_0$ and the Grothendieck residue pairing to construct a non-massive Frobenius manifold.

The motivation behind the construction is the following remark: the unfolding of $f = x^a + y^b$ on $C_t : xy = t, t \neq 0$, is the mirror partner of the weighted projective line $\mathbb{P}(a, b)$ (for $a$ and $b$ coprimes, see [11], [17]. At $t = 0$, the multiplication and metric in our construction at the origin coincides with the orbifold cohomology of $\mathbb{P}(a, b)$.
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2 Preliminaries

Let us recall briefly how to obtain a Frobenius manifold from a meromorphic connection. We closely follow C. Sabbah (cf. [13]).

Let \( G \to B \) be a vector bundle on manifold \( B \) and let the rank of \( G \) be equal to the dimension of \( B \), say \( m \). Let \( F \) denote the pull-back of \( G \) via the projection \( \mathbb{P}^1 \times B \to B \). We further assume that \( F \) is equipped with a flat meromorphic connection \( \nabla \) with logarithmic poles along \( \{0\} \times B \) and poles of type 1 along \( \{\infty\} \times B \). From this initial data we obtain the following objects:

(i) the residual connection \( \nabla \) on \( G \to B \): if \( \tau \) denotes the coordinate on the affine chart \( \mathbb{P}^1 \setminus \{\infty\} \) the connection matrix for \( \hat{\nabla} \) is locally written as

\[
\Omega^{\hat{\nabla}} = \frac{d\tau}{\tau} + \sum_{i=1}^{m} \Omega_i du_i
\]

where \( \Omega_\tau \) and \( \Omega_i \) are matrices with holomorphic entries. Here \((u_1, \ldots, u_n)\) denotes a coordinate system on a neighbourhood in \( B \). The residual connection \( \nabla \) on \( B \) is given by

\[
\Omega^{\nabla} = \sum_{i=1}^{m} \Omega_i(0, u_1, \ldots, u_n) du_i.
\]

and the integrability of \( \hat{\nabla} \) implies that of \( \nabla \);

(ii) the residue endomorphism of \( \hat{\nabla} \), that is, an endomorphism \( R_0 \) of \( F|_B \) given in local coordinates by \( \Omega_\tau(0, u) \). The integrability of \( \hat{\nabla} \) implies that \( R_0 \) is covariantly constant with respect to \( \nabla \), i.e., \( \nabla R_0 = 0 \);

(iii) an endomorphism \( R_\infty \) of \( F|_B \), defined (up to constant) by the choice of a coordinate \( \theta \) in \( \mathbb{P}^1 \setminus \{0\} \). Indeed, the connection at infinity has a pole of type 1. If we use \( \theta = \tau^{-1} \) as a coordinate in \( \mathbb{P}^1 \setminus \{0\} \) we see from (1) that \( \hat{\nabla} \) is written near \( \infty \) as

\[
\frac{1}{\theta}(\theta_\theta d\theta + \sum_{i=1}^{m} \Omega_i' du_i)
\]

where \( \theta_{\theta} = -\theta \Omega_\tau \) and \( \theta^2 \Omega_i' = \Omega_i \) have holomorphic entries. The matrix \( \Omega_\theta(0, u_1, \ldots, u_n) \) defines the endomorphism \( R_\infty \) of \( F|_B \). The coordinate \( \theta \) (and hence \( \tau \)) will be kept fixed throughout this article.

(iv) The Higgs field \( \Phi \), defined as follows. We decompose the connection \( \hat{\nabla} = \hat{\nabla}' + \hat{\nabla}'' \) according to the decomposition of 1-forms \( \pi^*_{\mathbb{P}^1 \setminus \{0\}} \Omega^1_{\mathbb{P}^1 \setminus \{0\}} \oplus \pi^*_B \Omega^1_B \). We write \( \hat{\nabla}'' = d_B + \Omega'' \) and set \( \Phi = (\theta \Omega'')|_{\theta=0} \).

It also depends on the choice of the coordinate \( \theta \) (up to constant).

The integrability of \( \hat{\nabla} \) implies the following relations between all of the above objects:

\[
\begin{align*}
\nabla^2 & = 0, \quad \nabla R_0 = 0 \\
\Phi \wedge \Phi & = 0, \quad [R_\infty, \Phi] = 0 \\
\nabla \Phi & = 0, \quad \nabla R_\infty + \Phi = [\Phi, R_0]
\end{align*}
\]

Let \( F[\{0\} \times B] \) denote the module of sections of \( F \) with poles along \( \{0\} \times B \) and let \( F \) denote locally free \( \mathcal{O}_B[\theta] \)-module \((\pi_B)_* F[\{0\} \times B] \). We further assume that \( F \) is equipped with with a non-degenerate \( \mathbb{C} \)-linear pairing

\[
S : F \otimes F \to \theta \mathcal{O}_B[\theta]
\]
satisfying
\[ S(\theta m, m') = \theta S(m, m') = S(m, -\theta m') \]
\[ \text{Lie}_{\partial_\theta} S(m, m') = S(\nabla_{\partial_\theta} m, m') + S(m, -\nabla_{\partial_\theta} m') \]
\[ \text{Lie}_{\partial_t} S(m, m') = S(\nabla_{\partial_t} m, m') + S(m, \nabla_{\partial_t} m') \]

Expanding \( S \) as a series in \( \theta = 0 \) we get

\[ S(m, m') = \theta s_1(m, m') + \theta^2 s_2(m, m') + \ldots. \]

It can be checked that \( s_1 \) is a non-degenerate, symmetric pairing on \( F/\theta F \) which is metric with respect to the connection \( \nabla \). For a \( \nabla \)-horizontal section \( \omega \) of \( G \). We define its associate period mapping \( \varphi_\omega : TB \to G \) by

\[ \varphi_\omega(\xi) := - \Phi(\xi)(\omega) \]

We say that \( \omega \) as above is primitive if

(i) \( \omega \) is an eigenvector of \( R_0 \) and

(ii) \( \varphi_\omega \) is an isomorphism.

If \( \omega \) is a primitive form, we can define a \( \mathcal{O}_B \)-algebra structure on \( \Theta_B \) by setting

\[ \varphi_\omega(\xi \star \eta) := - \varphi_\omega^{-1}(\xi) \varphi_\omega(\eta) \]

and we obtain:

**Theorem 2.1.** \((\text{[13]})\) If \( \omega \) is a primitive form, the triple \( (B, \star, s_\infty) \) is a Frobenius manifold.

**Remark 2.2.** We finish this section with a remark that simplifies enormously the construction of Frobenius manifolds from families of meromorphic connections. Namely, if \( B \) is simply connected, it is enough to check the existence of the primitive form at one single value of the parameter space \( B \). This result is proved in a detailed manner in \([13]\), but it goes back to the work of B. Dubrovin on isomonodromic deformations.

## 3 Functions on curves

Let us recall the definition of the Milnor number of a function \( f_0 \) on a curve-germ given by D. Mond and D. van Straten in \([12]\).

**Definition 3.1.** Let \((C, 0) \hookrightarrow (\mathbb{C}^n, 0)\) be a reduced curve-germ and let \( f_0 : (C, 0) \to (\mathbb{C}, 0) \) be a function non-constant on any branch. The Milnor number \( \mu \) of \( f_0 \) is defined as

\[ \mu := \dim_{\mathbb{C}} \frac{\omega_{C,0}}{\mathcal{O}_{C,0} df_0} \]

where \( \omega_{C,0} = \text{Ext}^{n-1}_{\mathcal{O}_{C,0}}(\mathcal{O}_{C,0}, \Omega^n_{\mathbb{C}^n,0}) \) denotes the dualising module of \( \mathcal{O}_{C,0} \).

**Remark 3.2.** The authors in \([12]\) show that if the curve is unobstructed (i.e. the second cotangent cohomology group \( T^2_{C,0} \) vanishes) then the local Milnor numbers are preserved under flat deformation of \((C, 0)\) and arbitrary deformation of \( f_0 \).

In the case of complete intersection curves the Milnor number is relatively easy to compute. If \((C, 0)\) is a complete intersection curve defined by \( g_1, \ldots, g_n \), the dualising module \( \omega_{C,0} \) can be identified with the module of meromorphic 1-forms \( \omega \) on \((C, 0)\) such that \( \omega \wedge dg_1 \wedge \cdots \wedge dg_n \in \mathcal{O}_{C,0} \otimes \Omega^{n+1}_{\mathbb{C}^{n+1},0} \). It is therefore customary to write \( \omega_{C,0} = \mathcal{O}_{C,0} \alpha \) where

\[ \alpha = \frac{dx_1 \wedge \cdots \wedge dx_{n+1}}{dg_1 \wedge \cdots \wedge dg_n} \]
Given now \( f_0 : (C, 0) \to (C, 0) \), let \( f \) be a representative of \( f_0 \) in \( \mathcal{O}_{C^n, 0} \). We can write \( df_0 = J \alpha \) where \( J \) is the Jacobian determinant of the map \( \varphi = (f, g_1, \ldots, g_n) : (C^{n+1}, 0) \to (C^n, 0) \). Hence \( \mu = \mathcal{O}_{C, 0}/(J) \) and using the Lé-Greuel formula we see that

\[
\mu = \mu_1 + \mu_2 \tag{7}
\]

where \( \mu_1 \) denotes de Milnor number of \((C, 0)\) and \( \mu_2 \) that of the 0-dimensional complete intersection defined by \( \varphi \).

An unfolding of \( f \) over \((B, 0) = (C^n, 0)\) is a function \( F : (C^n \times B, 0) \to (C, 0) \) together with fibration \((g, \mathcal{I}) : (C^n \times B) \to (C^{n-1} \times B, 0)\) such that \( F|_{C_0} = f_0 \). We say that \( F \) is a **miniversal unfolding** (resp. versal) if the Kodaira-Spencer map

\[
\Theta_{B, 0} \ni \frac{\partial}{\partial u_i} \mapsto \frac{\partial F}{\partial u_i} \in \mathcal{O}_{C_0 \times B, 0}/(J) \tag{8}
\]

is an isomorphism (resp. surjection) of \( \mathcal{O}_{B, 0}\)-modules. Here \((u_1, \ldots, u_m)\) denote coordinates on \((B, 0)\). Notice that if \( C_t \) is a Milnor fibre of an appropriate representative of \( g \), conservation of the Milnor number implies that the map

\[
\Theta_{B, 0} \ni \frac{\partial}{\partial u_i} \mapsto \frac{\partial F}{\partial u_i} \in \mathcal{O}_{C_t \times (B, 0)}/(J) \tag{9}
\]

is also an isomorphism (resp. surjection). Hence the restriction of \( F \) to \( C_t \times (B, 0) \) is a miniversal deformation of \( F|_{C_t} \) in the usual left-equivalence sense for multi-gersms.

Notice also that the isomorphisms \( 8 \) and \( 9 \) induce structures of \( \mathcal{O}_B \)-algebras on the tangent sheaf \( \Theta_B \). It is proved in \( 3 \) that these multiplicative structures satisfy certain integrable condition turning them into \( F \)-manifolds (see \( 7 \) and \( 8 \)).

### 3.1 The quasi-homogeneous case

As noted in Remark 3.2, the Milnor number is locally preserved under deformations. Here we wish to show that in the quasi-homogeneous case it is actually globally preserved. Later, this will justify the use of algebraic forms to study the Gauss-Manin system.

Most of the calculations that follow can be carried out for the case of complete intersections curve singularities and we do so. However, our techniques can only be used to construct Frobenius manifolds for functions on plane curves as it is in this case that we are able to extract information about the spectrum of the restriction of the miniversal unfolding of \( f_0 \) to the Milnor fibre of the singularity.

Let us begin by introducing some notation that will be kept for the remainder of this article. Let \( \mathcal{O} \) denote the polynomial ring \( \mathbb{C}[x_1, \ldots, x_{n+1}] \). We make \( \mathcal{O} \) into a graded ring by assigning the positive rational weight \( p_i \) to the variable \( x_i \). Homogeneity will always mean homogeneity with respect to this grading. Let us be given

(i) a polynomial map \( g : \mathbb{C}^{n+1} \to \mathbb{C}^n \) where \( g_i \) is homogeneous of degree \( e_i \), we denote the fibre over \( t \in \mathbb{C}^n \) by \( C_t \) and suppose that the 0-fibre \( C_0 \) is not smooth (see Remark 3.3 below);

(ii) a homogeneous polynomial \( f \in \mathcal{O} \) of degree 1, we write \( f_t \) for the restriction of \( f \) to the fibre \( C_t \) and assume that \( f_0 \) is not constant on any branch of \( C_0 \).

**Remark 3.3.** The smooth case is exceptional as it is the only case for which \( f \) belongs to its Jacobian algebra. On the other hand, the smooth case corresponds to the deformation of the \( A_r \)-singularity in one variable, and it is well-known that the base space of its miniversal deformation does have a Frobenius structure.

Let \( \alpha = dx_1 \wedge \cdots \wedge dx_{n+1}/dg_1 \wedge \cdots \wedge dg_n \) and let \( \omega_J \) be the relative dualising module. As before, let \( J \) be the Jacobian determinant of \((f, g_1, \ldots, g_n)\) so that \( df = J \alpha \). As \( J \) is also homogeneous, the only critical point of \( f_0 \) is the origin and \( \mu = \dim \mathcal{O}/(g_1, \ldots, g_n, J) \). The following proposition shows that this is also the number of critical points of \( f_t \). Let \((t_1, \ldots, t_n)\) be coordinates on the target space of \( g \).

**Proposition 3.4.** The \( \mathbb{C}[t_1, \ldots, t_n] \)-module \( \mathcal{O}/(J) \) is free of rank \( \mu \).
Proof. The module $\mathcal{O}/(J)$ can be seen as a graded module over the graded ring $\mathbb{C}[t_1, \ldots, t_n] := \mathbb{C}[g_1, \ldots, g_n]$. As $\omega_{C_0}/\mathcal{O}_{C_0}df = \mathcal{O}/(g_1, \ldots, g_n, J)$ is a finite dimensional vector space it follows from the graded Nakayama lemma that $\mathcal{O}/(J)$ is finitely generated (we recall that the graded version of Nakayama lemma does not require that the module $\mathcal{O}/(J)$ be finitely generated). As $(g_1, \ldots, g_n, J)$ is a regular sequence, the graded version of the Auslander–Buchsbaum formula tells us that $\mathcal{O}/(J)$ is free as $\mathbb{C}[t_1, \ldots, t_n]$-module.

\section{The Gauss-Manin system}

We keep the notation and hypothesis introduced in the previous section. We define the (algebraic) Gauss-Manin system of $f$ relative to $g$ as the module

$$G := \frac{\omega_g[\tau, \tau^{-1}]}{(d - \tau df \wedge \mathcal{O}[\tau, \tau^{-1}]}$$

where $d$ denotes the relative differential with respect to $g$. It is a $\mathbb{C}[t_1, \ldots, t_n, \tau, \tau^{-1}]$-module endowed with a partial integrable connection with respect to $\partial_\tau$ defined as:

$$\nabla_{\partial_\tau}[\omega] = [-f \omega]$$

(10)

We also consider the (relative) Brieskorn lattice $G$, that is, the image of the canonical map $\omega_g[\tau^{-1}] \to G$. It is a lattice of $G$ as the following proposition shows:

\textbf{Proposition 4.1.} $G$ is a free $\mathbb{C}[t_1, \ldots, t_n, \tau^{-1}]$-module of rank $\mu$.

Proof. According to Prop. 3.4 let $h_1, \ldots, h_\mu$ be a basis of the free $\mathbb{C}[t_1, \ldots, t_n]$-module $\mathcal{O}/J$ consisting of homogeneous elements. Let $\omega_1 = h_1 \alpha$ and let $\omega = a_0 \alpha \in \omega_1$. Then there exist unique $c_1, \ldots, c_\mu \in \mathbb{C}[t_1, \ldots, t_n]$ such that $a_0 = c_1 h_1 + \cdots + c_\mu h_\mu + a_0 J$, which implies that $\omega = c_1 \omega_1 + \cdots + c_\mu \omega_\mu + \tau^{-1} da_0$. Writing $da_0 = a_1 \alpha$ we see that $\deg a_0 > \deg a_1$. The proposition follows by iteration.

We begin by studying the action of $\partial_\tau$ on the holomorphic (algebraic) forms $\Omega_g$. Recall that we are excluding the case in which $C_0$ is smooth.

\textbf{Lemma 4.2.} Let $I = (g_1, \ldots, g_n)$ and let $J_g$ be the ideal generated by all the maximal minors of the Jacobian matrix of $g$. The sequence

$$0 \to J_g + I \to I + (J) \xrightarrow{f} \mathcal{O} \to \mathcal{O} / (f + I + (J)) \to 0$$

is exact.

Proof. Let $\mu_1$ be the Milnor number of $C_0$ and $\mu_2$ that of the 0-dimensional complete intersection defined by $\varphi := (f, g_1, \ldots, g_n)$. We know that $\mu = \mu_1 + \mu_2$. If $E = \sum_{i=1}^n w_i x_i \frac{\partial}{\partial x_i}$ denotes the Euler vector field of $\mathcal{O}$, the homogeneity of $f$ and $g_i$ gives us $t \varphi(E) = f \frac{\partial}{\partial s} + \sum_{i=1}^n e_i g_i \frac{\partial}{\partial x_i}$. Applying Cramer’s rule we obtain

$$w_i x_i J = (-1)^{i+1} f M_i \mod I$$

(12)

where $M_i$ is the minor of the Jacobian matrix of $g$ obtained by deleting the $i$-th column. From here we see that $fJ_g \subset I + (J)$. But we have an exact sequence

$$0 \to (f + I + (J)) \to \mathcal{O} \to \mathcal{O} / (f + I + (J)) \to 0$$

(13)

where the middle term has dimension $\mu_2 + 1$ (cf. [9], Prop. 5.12). So that can also use (12) to conclude that

$$\dim \mathcal{O} / (f + I + (J)) = \mu_2$$

(14)
On the other hand, together with Nakayama lemma tells us that the first term of has dimension 1 so that follows. Back to the original sequence, we conclude that the kernel of has dimension . That is also the dimension of the first term of follows from one more exact sequence:

$$0 \to J_g + I \to \mathcal{O} \to \mathcal{O} \to 0.$$ 

The middle and last term of the above sequence have dimension and respectively. Therefore the first term has dimension and the lemma follows.

The following notation will be useful to describe the action of on .

**Notation.** We set and for a homogeneous element , we define

$$\nu(h) := \deg h + p - e.$$ 

We will also write \(\nu(\omega) := \nu(h)\) where \(\omega = h\alpha\).

**Remark 4.3.** Notice that for \(\omega = h\alpha\) with \(h\) homogeneous we have \(\text{Lie}_E(\omega) = \nu(h)\omega\), where \(\widetilde{E}\) denote the Euler vector field on \(\mathcal{O}\). Also, if \(h \in J_g\) and as before we denote by \(M_t\) the minor of the Jacobian matrix of \(g\) obtained by deleting the \(i\)-th columns, then

$$\deg(h) \geq \min \{\deg M_t : i = 1, \ldots, n + 1\} = e - p + \max \{p_i : i = 1, \ldots, n + 1\} + 1.$$ 

It follows that \(\nu(\omega) > 0\).

**Lemma 4.4.** Let \(\omega \in \Omega_g\) be a homogeneous 1-form. Then, in \(G\) we have

$$\tau \partial_\tau[\omega] = -\nu(\omega)[\omega] + \sum_{j=1}^n t_j \omega_j + \tau \sum_{j=1}^n t_j \omega'_j$$

with \(\nu(\omega_i) \leq \nu(\omega) - e_j\) and \(\nu(\omega'_j) \leq \nu(\omega) + 1 - e_j\).

**Proof.** By linearity, we can assume that \(\omega = hdx_{n+1}\) with \(h\) homogeneous. As \(dx_{n+1} = M_{n+1}\alpha\), we see that \(\nu(\omega) = \deg h + p_{n+1}\). Let us introduce some helpful notation to carry out the calculation: \(i_{\partial_i}\) denotes the contraction with respect to the vector field \(\partial_{x_i}\), \(i_{\partial_{n+1}^n} = i_{\partial_n} \circ \cdots \circ i_{\partial_1}\) and \(i_{n+1,j} = i_{\partial_n} \circ \cdots \circ i_{\partial_j}\). Writing \(dx_{n+1} = i_{n+1} V\) we have

$$-\tau \delta_\tau[\omega] = \tau \partial_\tau[h_{n+1} V] = \tau[h_{n+1}(fV)] = \tau[h_{n+1}(df \wedge i_E V)]$$

$$= \tau \sum_{j=1}^n (-1)^{j+1} [h(i_{\partial_j} df) \wedge i_{n+1,j} i_E V] + (-1)^n \tau [hd f \wedge i_{n+1} i_E V]$$

$$= \tau \sum_{j=1}^n (-1)^{j+1} [h(i_{\partial_j} df) \wedge i_{n+1,j} i_E V] + \tau [hd f \wedge i_E i_{n+1} V]$$

$$= [di_E \omega] + \tau \sum_{j=1}^n (-1)^{j+1} [h(i_{\partial_j} df) \wedge i_{n+1,j} i_E V]$$

$$= [\text{Lie}_E(\omega) - i_E df \omega] + \tau \sum_{j=1}^n (-1)^{j+1} [h(i_{\partial_j} df) \wedge i_{n+1,j} i_E V]$$

$$= \nu(\omega)[\omega] - i_E df \omega + \tau \sum_{j=1}^n (-1)^{j+1} [h(i_{\partial_j} df) \wedge i_{n+1,j} i_E V]$$

Multiplying the second summand by \(dq_1 \wedge \cdots \wedge dq_n\) and applying the determinant theorem we see that \(i_E df \omega = \sum_{j=1}^n t_j \omega_j\) where \(\nu(\omega_j) = \nu(\omega) - e_j\), and analogously for the other summand. This can also be seen by noticing that the expression is homogeneous where \(\deg \tau = -1\).

**Corollary 4.5.** For \(\omega \in \Omega_{G_0}\), we have \(\tau \partial_\tau[\omega] = -\nu(\omega)[\omega]\) in \(G_0 = G/\mathfrak{m}^n_{0,0}\).
4.1 V-filtration and spectral numbers

For a fixed point \( t \in \mathbb{C}^n \), set \( G_t = G/\mathfrak{m}_{\mathbb{C}^n \cdot t} \) and analogously \( G_t = G/\mathfrak{m}_{\mathbb{C}^n \cdot t} \). Let us recall the definition of the Malgrange-Kashiwara \( V_* \)-filtration for \( G_t \). It is the unique filtration \( V_*(G_t) \) indexed by \( \mathbb{Q} \) such that

1. \( V_{\lambda}(G_t) \) is \( \mathbb{C}[\tau] \)-free and \( \mathbb{C}[\tau, \tau^{-1}] \otimes \mathbb{C}[\tau] V_{\lambda}(G_t) = G_t \) for all \( \lambda \in \mathbb{Q} \);
2. \( \tau V_{\lambda}(G_t) \subset V_{\lambda-1}, \partial_\tau V_{\lambda}(G_t) \subset V_{\lambda+1} \)
3. the action of \( \tau \partial_\tau + \lambda \) is nilpotent on the quotient \( \text{gr}_V^\lambda(G_t) := V_{\lambda}(G_t)/V_{\lambda}(G_t) \).

Such a filtration exists and is unique (e.g. [2], pg. 113). Moreover, there exists a finite subset \( A \subset [0, 1) \) such that \( \text{gr}_V^\lambda(G_t) = 0 \) for all \( \lambda \notin A + \mathbb{Z} \).

The filtration \( V_*(G_t) \) induces a filtration on \( G_t/\tau G_t \). The corresponding graded part is given by

\[
\text{gr}_V^\lambda(G_t/	au G_t) := \frac{V_{\lambda}(G_t) \cap G_t}{V_{\lambda-1}(G_t) \cap G_t + V_{\lambda}(G_t) \cap G_t}.
\]

Let \( d(\lambda) \) denote the dimension as a complex vector space of \( \text{gr}_V^\lambda(G_t/\tau G_t) \). The set of pairs \( (\lambda, d(\lambda)) \) for which \( d(\lambda) \neq 0 \) is called the spectrum of \( (G_t, G_t) \).

We can use lemmas 4.2 and 4.4 to compute the \( V_* \)-filtration of the Gauss-Manin system of the function \( f_0 \) and for the case of plane curves, for any \( f_t \). The linear map \((-f_0)^* : \omega_{C_0}/\mathcal{O}_{C_0} df_0 \to \omega_{C_0}/\mathcal{O}_{C_0} df_0 \) is nilpotent and homogeneous. Hence its Jordan basis induces a homogeneous basis of \( G \) of the following form:

\[
[\omega^1_i] = [(-f)^\omega \omega^0_i], \ i = 0, \ldots, N_1 \\
[\omega^2_i] = [(-f)^\omega \omega^0_i], \ i = 0, \ldots, N_2 \\
\ldots \\
[\omega^M_i] = [(-f)^\omega \omega^0_i], \ i = 0, \ldots, N_M \\
[\omega^M+1_i], \ldots, [\omega^{\mu_2}]
\]

It is helpful to set \( \nu^i_j = \nu(\omega^i_j) \). Consider now the following change of basis of \( G \):

\[
\bar{\omega}^j_i = \begin{cases} 
\omega^j_i + (\nu^i_j - 1)\tau^{-1} \omega^{j-1}_i & \text{if } \nu^i_j > 1 \\
\omega^j_i & \text{if } \nu^i_j \leq 1 
\end{cases}
\]

Notice that, a priori it could happen that \( \nu^0_j > 1 \) and the above definition would not be correct. But this does not happen as the following lemma shows:

**Lemma 4.6.** We have \( \nu^0_i \leq 1 \) for all \( i = 1, \ldots, \mu_2 \).

**Proof.** The socle of the 0-dimensional complete intersection defined by \( (f) + I \) has degree \( 1 + e - p \). Hence all the elements of degree greater than \( 1 + e - p \) are contained in the image of the multiplication by \( f \) and the lemma follows. \( \square \)

For \( \omega \in \omega_j \) let us set

\[
\lambda(\omega) := \begin{cases} 
1 & \text{if } \nu(\omega) > 1 \\
\nu & \text{if } 0 \leq \nu(\omega) \leq 1 \\
0 & \text{if } \nu(\omega) < 0 
\end{cases}
\]

and \( \lambda(\omega^j_i) := \lambda^j_i \). In the next result we compute the spectral numbers of \( (G_t, G_t) \) for \( t = 0 \).

**Theorem 4.7.** For any \( \lambda \in \left\{ \lambda^j_i : 1 \leq i \leq \mu_2, 0 \leq j \leq N_i \right\} \) the classes of \( \bar{\omega}^j_i \) for which \( \lambda(\bar{\omega}^j_i) = \lambda \) induce a basis of the vector space \( \text{gr}_V^\lambda(G_0/\tau G_0) \). Hence the numbers \( \lambda^j_i \) together with its multiplicities form the spectrum of \( (G_0, G_0) \).
Theorem 4.9. If \( C, K \sim \) are similarly adapted. Notice that for the elements
\[ \theta = \omega_i N_i, \]
the multiplication is generically semisimple.

Proof. We only need to check that \( \tau \partial_\tau + \lambda \) is nilpotent on \( \mathfrak{g} \) for \( \lambda \in [0,1] \). Notice first that by definition we have \( f \omega_i N_i \in I + (J) \). It follows from Lemma 4.7 and that \( \omega_i N_i \in \Omega C_0 \). An straightforward calculation together with Cor. 4.8 shows that
\[
\begin{align*}
\text{if } j < N_i \text{ then } \tau \partial_\tau \omega_i^j &= \begin{cases} 
\tau \omega_i^{j+1} & \text{if } \nu_i^j \leq 0 \\
-\nu_i^j \omega_i^j + \tau \omega_i^{j+1} & \text{if } 0 < \nu_i^j \leq 1 \\
\omega_i^j + \tau \omega_i^{j+1} & \text{if } \nu_i^j > 1,
\end{cases} \\
\text{and if } j = N_i \text{, } \tau \partial_\tau \omega_i^j &= -\lambda N_i \omega_i^{N_i}.
\end{align*}
\]

We show the nilpotency of \( \tau \partial_\tau + \lambda \) with some detail for the first case in (19) as the others are analogous. As \( \nu_i^j \leq 0 \) we have \( j < N_i \) (see Remark 4.8). If \( \nu_i^j < 0 \), then \( \nu_i^{j+1} < 1 \) so that \( \tau \partial_\tau \omega_i^j \in V_{<0}(G) \). If \( \nu_i^j = 0 \) then \( \nu_i^{j+1} = 1 \) and we get
\[
(\tau \partial_\tau)^2 \omega_i^j = \tau (\tau \partial_\tau + 1) \omega_i^{j+1} = \begin{cases} 
\tau^2 \omega_i^{j+2} & \text{if } j + 1 < N_i \\
\tau [\omega_i^{j+1}] + \tau \omega_i^{j+1} & \text{if } j + 1 = N_i
\end{cases}
\]

In both cases we have \( (\tau \partial_\tau)^2 \omega_i^j \in V_{<0}(G) \).

Corollary 4.8. In the basis of \( G_0 \) induced by \( \omega_i^j \), the matrix of the action of \( \partial_\tau \) takes the form
\[
(A_0 + A_\infty \tau^{-1}) d\tau
\]
where \( A_0 \) and \( A_\infty \) are constant matrices, and \( A_\infty \) diagonal. In particular, \( G_0 \) extends to a bundle on \( \mathbb{P}^1 \) with logarithmic connection on \( \tau = 0 \).

In the case of plane curves, it turns out that the spectrum of the restriction \( f_t \) of \( f \) to the fibre \( C_t \) coincides with that of \( C_0 \). More precisely:

Theorem 4.9. If \( n = 1 \) then the classes of \( \omega_i^j \) induce a basis of \( gr^\chi \mathfrak{g}(C_t/\tau^{-1}G_1) \) for any \( t \in \mathbb{C} \).

Proof. The particularity of family of plane curves is that \( (g) = I \subset J_g \). It follows as in the proof of the previous theorem that \( \omega_i N_i \in \Omega C_1 \). The proof of the theorem now follows almost verbatim, with the difference that we now have to use the full equation \( (20) \) in Lemma 4.8. For example, equation \( (20) \) now becomes
\[
(\tau \partial_\tau)^2 \omega_i^j = \tau (\tau \partial_\tau + 1) \omega_i^{j+1} = \begin{cases} 
\tau^2 \omega_i^{j+2} & \text{if } j + 1 < N_i \\
\tau [\omega_i^{j+1}] + \tau \omega_i^{j+1} & \text{if } j + 1 = N_i
\end{cases}
\]
which again in both cases belong to \( V_{<0}(G_t) \) as \( \nu(\omega_i^{j+1}) \leq \nu_i^{N_i} - e_1 = 1 - e_1 \). The rest of cases are similarly adapted. Notice that for the elements \( \omega_i^j \), we might need to use Lemma 4.8 say \( K \) times, being \( K = \min \{ k \geq 1 : \nu_i^{N_i} - ke_1 < 1 \} \) to ensure that \( (\tau \partial_\tau + \lambda N_i)^K \omega_i^{N_i} \in V_{<\lambda N_i}(G_t) \).

We can then use the results in [5] to construct Frobenius manifolds on the base space of the miniversal deformation of \( f_t \) for \( t \neq 0 \).

Corollary 4.10. If \( n = 1 \), the class of \( \alpha \) in \( G_t \) is a primitive form for any \( t \). Hence for any \( t \neq 0 \), the base space of the miniversal deformation of \( f_t \) has the structure of a massive Frobenius manifold.

Proof. Let \( \omega_i^j = h_i^{(j)} \alpha \) be the basis of \( G_0 \) defined in (17). Then the unfolding \( F = f + \sum_{i=1}^{\mu_2} \sum_{j=0}^{N_i} u_i^{(j)} h_i^{(j)} \) is miniversal. The connection with respect to the deformation parameters is given by
\[
\hat{\nabla}_{\partial u_i^{(j)}} [\omega] = \left[ \frac{\partial \omega}{\partial u_i^{(j)}} \right] - \tau \left[ \frac{\partial F}{\partial u_i^{(j)}} \omega \right].
\]
As \( \alpha = \omega_i^0 \) we have
\[
\hat{\nabla}_{\partial u_i^{(j)}} [\alpha] = -\tau [\omega_i^j], \quad \hat{\nabla}_{\partial u_i^{(j)}} [\omega_i^j] = [\omega_i^j] - \sum_{i=1}^{\mu_2} \sum_{j=0}^{N_i} u_i^{(j)} [\omega_i^j].
\]
It follows that \( \alpha \) is a primitive form. The existence of the metric follows from microlocal Poincaré duality (cf. loc. cit.). Finally, for a generic value of \( u \), all the critical points of \( F \) on \( C_t \times \{ u \} \) are Morse, hence the multiplication is generically semisimple.
It is known that the metric is given by the sum of the residues at the critical points. More precisely, if \((u_1, \ldots, u_\mu)\) are parameters of the base space of the miniversal deformation \(F : (\mathbb{C}^2 \times B, 0) \to (\mathbb{C}, 0)\) and \(dF = J_F \alpha\) denotes the relative differential, then

\[
\left\langle \frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j} \right\rangle_t = \int_{\partial C_t} \left( \frac{dF}{\partial u_i} \frac{dF}{\partial u_j} \right)_{\partial C_t} \tag{25}
\]

where \(\partial C_t\) is the boundary of an appropriate representative of the Milnor fibre of \(g\).

**Corollary 4.11.** The formula \(\text{(25)}\) for \(t = 0\) together with the multiplication defined by \(\text{(8)}\) defines the structure of non-massive Frobenius manifold on \(B\).

**Proof.** We have \(\left\langle \frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j} \right\rangle_t \to \left\langle \frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j} \right\rangle_0\) when \(t \to 0\). The flatness of \(\langle -, - \rangle_t\) implies that of \(\langle -, - \rangle_0\) as it can be seen, for example, writing out explicit formulas for the curvature in terms of the Christoffel symbols. The existence of a potential can be translated into the flatness of the first structure connection (e.g. [10], Th. 1.5). More precisely, for each \(t\), let \(\nabla\) the Levi-Civita connection of \(\langle -, - \rangle_t\). The first structure connection is defined as

\[
\mathfrak{t} \nabla_{z, \partial u_i} \partial u_j := \mathfrak{t} \nabla_{\partial u_i} \partial u_j + z \partial u_i \ast_t \partial u_j
\]

It is of course closely related to the the Gauss-Manin connection \(\nabla\). Notice that \(\partial u_i \ast_t \partial u_j \to \partial u_i \ast_0 \partial u_j\) when \(t \to 0\), and hence \(\mathfrak{t} \nabla \to \mathfrak{t} \nabla\). The result follows.

**5 An example: linear functions on the \(A_k\)-singularity**

Let us illustrate our construction with a worked-out example. We consider the curve \(C_0\) defined by \(g(x, y) = x^k + y^2 = 0\), \(k \geq 2\), and the function \(f_0\) given by the restriction of \(f(x, y) = x\) to \(C_0\).

**Miniversal deformation.** The classes of \(1, \ldots, x^{k-1}\) form a \(\mathbb{C}\)-basis of the Jacobian algebra \(\mathcal{O}_{C_0}/(2y)\) and hence a miniversal unfolding is given by \(F + u_1x^{k-1} + \cdots + u_{k-1}x + u_k\).

**Spectrum.** For a homogeneous polynomial \(h\) we have

\[
\nu(h) = \deg(h) - \frac{k - 2}{2}
\]

According to Theorem 4.7, the spectrum of \(f_t = f|_{C_t}\) is

\[
\begin{cases}
\left\{ \left(0, \frac{k}{2}\right), \left(1, \frac{k}{2}\right) \right\} & \text{if } k \text{ is even and,} \\
\left\{ \left(0, \frac{k-1}{2}\right), \left(\frac{1}{2}, 1\right), \left(1, \frac{k-1}{2}\right) \right\} & \text{if } k \text{ is odd.}
\end{cases}
\tag{27}
\]

**Nilpotent Frobenius structure.** If we set \(F' = \frac{dF}{dx}\), the multiplication table on \(\Theta_{B,0}\) is given by the isomorphism

\[
\partial_{u_i}, \mathfrak{t} \nabla_{\partial u_i} x^{k-i} \in \pi_* \left( \mathcal{O} \left( \frac{O}{(x^k + y^2, 2yF')} \right) \right) \tag{28}
\]

where \(\mathcal{O}\) denotes the sheaf of holomorphic functions on the variables \(x, y, u_1, \ldots, u_k\) and \(\pi : C_0 \times B\) a scheme with two components: \(W_1 := \{0\} \times B\) and the (reduced) variety \(W_2\) defined by \(F' = 0\). As \(W_1\) already has multiplicity \(k = \mu\), the \(F\)-manifold structure extends to \(B \setminus \pi(W_2)\) (notice that \(0 \notin \pi(W_2)\)). We see that this \(F\)-manifold structure is purely nilpotent, in the sense that if \(i \neq k\) (i.e., if \(\partial_{u_i}\) is not the identity), we have \(\partial_{u_i} \ast \cdots \ast \partial_{u_i} = 0\) where the product occurs at most \(k\) times. We remark that this is always the case if the function \(f_0\) is the restriction of a linear function as all the critical points are provided by the singular curve.
The metric is also easy to describe, at least on $T_0B$ (and hence on flat coordinates). The generator of the socle of $\mathbb{C}[x,y]/(x^k+y^2,2y)$ is $x^{k-1}$, so that if we choose a residue form with $\text{Res}(x^{k-1}) = 1$, the metric in the basis $\partial_{u|0}$ is simply given by the matrix with all entries equal 1 in the anti-diagonal, and 0 everywhere else.

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