Existence and concentration of positive solutions for a logarithmic Schrödinger equation via penalization method

Claudianor O. Alves1 · Chao Ji2

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Abstract
In this article we are concerned with the following logarithmic Schrödinger equation
\[
\begin{cases}
-\epsilon^2 \Delta u + V(x)u = u \log u^2, & \text{in } \mathbb{R}^N, \\
u \in H^1(\mathbb{R}^N),
\end{cases}
\]
where \( \epsilon > 0 \), \( N \geq 1 \) and \( V : \mathbb{R}^N \to \mathbb{R} \) is a continuous potential. Under a local assumption on the potential \( V \), we use the variational methods to prove the existence and concentration of positive solutions for the above problem.

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1 Introduction
In the past few decades, the nonlinear elliptic equation
\[
-\epsilon^2 \Delta u + V(x)u = f(u), \quad x \in \mathbb{R}^N,
\]
where \( N \geq 1 \), \( \epsilon > 0 \) is a positive parameter, \( V, f \) are continuous functions verifying some assumptions, has been studied by many researchers. A basic motivation for the study of problem \((S_\epsilon)\) is to seek for the standing waves of the following nonlinear Schrödinger equation.
equation
\[ i\epsilon \frac{\partial \Psi}{\partial t} = -\epsilon^2 \Delta \Psi + (V(x) + E)\Psi - f(\Psi), \quad \text{for } x \in \mathbb{R}^N, \]  
\hfill (\text{NLS})

namely, solution of the form \( \Psi(x, t) = \exp(-iEt/\epsilon)u(x) \) with \( u(x) \) is a real value function. There is a broad literature on the existence and concentration of positive solutions for general semilinear elliptic equations \((S_\epsilon)\) for the case \( N \geq 1 \), see for example, Floer and Weinstein [14], Oh [18,19], Rabinowitz [20], Wang [27], Cingolani and Lazzo [8], Ambrosetti, Badiale and Cingolani [6], Gui [15], del Pino and Felmer [12] and their references.

In [20], by a variant of a mountain pass argument, Rabinowitz proved the existence of positive solutions of problem \((S_\epsilon)\) for \( \epsilon > 0 \) small, whenever
\[ V_\infty = \liminf_{|x| \to \infty} V(x) > \inf_{x \in \mathbb{R}^N} V(x) = V_0 > 0. \]

Later, Wang [27] used variational methods to show that these solutions concentrate at global minimum points of \( V \) as \( \epsilon \to 0 \). In [12], del Pino and Felmer found solutions which concentrate around local minimum of \( V \) by introducing a penalization method. More precisely, they assumed that there is an open and bounded set \( \Lambda \subset \mathbb{R}^N \) such that
\[ 0 < V_0 = \inf_{z \in \Lambda} V(z) < \min_{z \in \partial \Lambda} V(z). \]

In the above-mentioned papers, the authors assumed that the nonlinearity \( f \) satisfies superlinear, subcritical growth conditions and the well-known Ambrosetti–Rabinowitz condition, this allow us to employ the variational methods for the class of \( C^1 \) functional to attach these problems.

Recently, the logarithmic Schrödinger equation given by
\[ i\epsilon \partial_t \Psi = -\epsilon^2 \Delta \Psi + (W(x) + w)\Psi - \Psi \log |\Psi|^2, \quad \Psi : [0, \infty) \times \mathbb{R}^N \to \mathbb{C}, \quad N \geq 1, \]
has also received considerable attention. This class of equation has some important physical applications, such as quantum mechanics, quantum optics, nuclear physics, transport and diffusion phenomena, open quantum systems, effective quantum gravity, theory of superfluidity and Bose–Einstein condensation (see [31] and the references therein). In its turn, standing waves solution, \( \Psi \), for this logarithmic Schrödinger equation is related to the solutions of the equation
\[ -\epsilon^2 \Delta u + V(x)u = u \log u^2, \quad \text{in } \mathbb{R}^N. \]  
\hfill (P_\epsilon)

Besides the importance in applications, the equation \((P_\epsilon)\) also raises many difficult mathematical problems. The natural candidate for the associated energy functional would formally be the functional
\[ \hat{I}_\epsilon(u) = \frac{1}{2} \int_{\mathbb{R}^N} (\epsilon^2 |\nabla u|^2 + (V(x) + 1)|u|^2)dx - \frac{1}{2} \int_{\mathbb{R}^N} u^2 \log u^2 dx. \]  
\hfill (1.1)

It is easy to see that each critical point of \( \hat{I}_\epsilon \) is a solution of (1.1). However, this functional is not well defined in \( H^1(\mathbb{R}^N) \) because there is \( u \in H^1(\mathbb{R}^N) \) such that \( \int_{\mathbb{R}^N} u^2 \log u^2 dx = -\infty \). In order to overcome this technical difficulty some authors have used different techniques to study the existence, multiplicity and concentration of the solutions under some assumptions on the potential \( V(x) \), which can be seen in [1–4,10,11,13,16,22,23,25,28] and the references therein. In [11], different from the previous contribution on the this subject, the authors directly faced the loss of the compactness and studied the existence of multiple solutions by
using non-smooth critical point theory, which was also used in [21] to establish the existence and concentration of the solutions for the quasi-linear elliptic equations. We also notice that the soliton dynamics behaviour for the logarithmic Schrödinger equations were studied by some mathematicians, see for example [7]. This class of problems are not fully solved as it depends on the regularity property of the functional.

In a recent paper [1], Alves and de Morais Filho established the existence and concentration of positive solutions to problem \((P_\epsilon)\), for \(\epsilon > 0\), by requiring that \(V\) verifies the global assumption introduced by Rabinowitz [20]

\[
V_\infty := \lim_{|x| \to \infty} V(x) > \inf_{x \in \mathbb{R}^N} V(x) = V_0 > -1. \tag{1.2}
\]

Later, Alves and Ji [3] considered the multiple positive solutions to problem \((P_\epsilon)\) under the same assumption \((1.2)\). More precisely, it was proved that the “shape” of the graph of the function \(V\) affects the number of nontrivial solutions.

It is quite natural to consider the existence and concentration results of the solutions for problem \((P_\epsilon)\) when the potential \(V\) satisfies a local assumption. Inspired by [1,12,24], the main purpose of this paper is to investigate the existence and concentration of positive solutions of problem \((P_\epsilon)\) by combining a local assumption on \(V\) and adapting the penalization method found in del Pino and Felmer [12].

Throughout the paper, we make the following assumptions on the potential \(V\):

\[(V1) \quad V \in C(\mathbb{R}^N, \mathbb{R}) \text{ and } \inf_{x \in \mathbb{R}^N} V(x) = V_0 > -1;\]

\[(V2) \quad \text{There exists an open and bounded set } \Lambda \subset \mathbb{R}^N \text{ satisfying }\]

\[-1 < V_0 = \inf_{x \in \Lambda} V(x) < \min_{x \in \partial \Lambda} V(x).\]

By a change of variable, we know that problem \((P_\epsilon)\) is equivalent to the problem

\[
\begin{cases}
-\Delta u + V(\epsilon x)u = u \log u^2, & \text{in } \mathbb{R}^N, \\
u \in H^1(\mathbb{R}^N). \tag{1.3}
\end{cases}
\]

**Definition 1.1** For us, a positive solution of (1.3) means a positive function \(u \in H_\epsilon\) such that \(u^2 \log u^2 \in L^1(\mathbb{R}^N)\)(i.e., \(J_\epsilon(u) < \infty\)) and

\[
\int_{\mathbb{R}^N} (\nabla u \nabla v + V(\epsilon x)uv)dx = \int_{\mathbb{R}^N} uv \log u^2 dx, \quad \text{for all } v \in C_0^\infty(\mathbb{R}^N). \tag{1.4}
\]

We shall use the variational method found in Szulkin [24] to prove the existence of nontrivial solutions for problem (1.3). Here, we will show that any critical point of the associated energy functional

\[
J_\epsilon(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + (V(\epsilon x) + 1)|u|^2)dx - \frac{1}{2} \int_{\mathbb{R}^N} u^2 \log u^2 dx,
\]

in the sub-differential sense, is a weak solution of (1.3) in \(H^1(\mathbb{R}^N)\). Aiming this approach, let us define the Banach space

\[
H_\epsilon := \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(\epsilon x)|u|^2 dx < \infty \right\}
\]

endowed with the norm

\[
\|u\|_\epsilon = \left( \int_{\mathbb{R}^N} (|\nabla u|^2 + (V(\epsilon x) + 1)|u|^2)dx \right)^{1/2}.
\]

The main result of this paper is the following:
Theorem 1.1 Suppose that $V$ satisfies (V1) – (V2). Then, there exists $\epsilon_0 > 0$ such that, for any $\epsilon \in (0, \epsilon_0)$, the problem $(P_{\epsilon})$ has a positive solution $v_{\epsilon}$. Moreover, if $\eta_0 \in \mathbb{R}^N$ is a global maximum point of $v_{\epsilon}$, we have

$$\lim_{\epsilon \to 0} V(\eta_0) = V_0.$$  

The proof of Theorem 1.1 is inspired from [1,12,24], however we are working with the logarithmic Schrödinger equation, whose the energy functional associated is not continuous, for this reason, some estimates for this problem are also very delicate and different from those used in the Schrödinger equation ($S_\epsilon$). Also for this reason, we shall modify the nonlinearity in a special way to work with a modified problem. Making some estimates we prove that the solutions obtained for the modified problem are solutions of the original problem when $\epsilon > 0$ is sufficient small. On the other hand, a equality of the type $J_{\epsilon}(u) - \frac{1}{2}J_{\epsilon}'(u)u = \frac{1}{2}\int_{\mathbb{R}^N} |u|^2\,dx$ is very important for the study of the logarithmic Schrödinger equations, for example, in [16,22], the authors used it and the logarithmic Sobolev inequality to verify the boundedness of $(PS)$ sequence. But, the functional associated with the modified problem doesn’t satisfy the equality above, so the proof of the boundedness of $(PS)$ sequence is a great challenge, and here we developed a new way to prove this boundedness, see Lemmas 3.2 and 3.3 for more details. Moreover, since the functional associated with the modified problem also lost some other good properties, it is difficult to verify the mountain pass geometry, see Lemma 3.1. The reader is invited to see that the way how we attach these problems in Section 3 is different of that explored in [1,16,22]. After our paper was completed, we learned of some related work due to the approach explored in [1,16,22], due to the lack of smoothness of $J_{\epsilon}$, let us decompose it into a sum of a $C^1$ functional plus a convex lower semicontinuous functional, respectively. For $\delta > 0$, let us define the following functions:

$$F_1(s) = \begin{cases} 
0, & s = 0 \\
-\frac{s^2}{2} \log s^2 & 0 < |s| < \delta \\
-\frac{s^2}{2} (\log \delta^2 + 3) + 2\delta |s| - \frac{1}{2} \delta^2, & |s| \geq \delta
\end{cases}$$

2 Preliminaries

Let us go back to the functional $J_{\epsilon}$. Following the approach explored in [1,16,22], we shall modify the nonlinearity in a special way to work with a modified problem. Making some estimates we prove that the solutions obtained for the modified problem are solutions of the original problem when $\epsilon > 0$ is sufficient small. On the other hand, a equality of the type $J_{\epsilon}(u) - \frac{1}{2}J_{\epsilon}'(u)u = \frac{1}{2}\int_{\mathbb{R}^N} |u|^2\,dx$ is very important for the study of the logarithmic Schrödinger equations, for example, in [16,22], the authors used it and the logarithmic Sobolev inequality to verify the boundedness of $(PS)$ sequence. But, the functional associated with the modified problem doesn’t satisfy the equality above, so the proof of the boundedness of $(PS)$ sequence is a great challenge, and here we developed a new way to prove this boundedness, see Lemmas 3.2 and 3.3 for more details. Moreover, since the functional associated with the modified problem also lost some other good properties, it is difficult to verify the mountain pass geometry, see Lemma 3.1. The reader is invited to see that the way how we attach these problems in Section 3 is different of that explored in [1,16,22]. After our paper was completed, we learned of some related work due to the approach explored in [1,16,22], due to the lack of smoothness of $J_{\epsilon}$, let us decompose it into a sum of a $C^1$ functional plus a convex lower semicontinuous functional, respectively. For $\delta > 0$, let us define the following functions:

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and
\[ F_2(s) = \frac{1}{2} s^2 \log s^2 + F_1(s), \quad s \in \mathbb{R}. \] (2.1)

It was proved in [16,22] that \( F_1 \) and \( F_2 \) verify the following properties:
\[ F_1, F_2 \in C^1(\mathbb{R}, \mathbb{R}). \] (2.2)

If \( \delta > 0 \) is small enough, \( F_1 \) is convex, even, \( F_1(s) \geq 0 \) for all \( s \in \mathbb{R} \) and
\[ F_1'(s)s \geq 0, \quad s \in \mathbb{R}. \] (2.3)

For each fixed \( p \in (2, 2^*) \), there is \( C > 0 \) such that
\[ |F_2'(s)| \leq C|s|^{p-1}, \quad \forall s \in \mathbb{R}. \] (2.4)

Let us define
\[ \Phi_\epsilon(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + (V(\epsilon x) + 1)|u|^2) \, dx - \int_{\mathbb{R}^N} F_2(u) \, dx, \] (2.5)
and
\[ \Psi(u) = \int_{\mathbb{R}^N} F_1(u) \, dx, \] (2.6)
then
\[ J_\epsilon(u) = \Phi_\epsilon(u) + \Psi(u), \quad u \in H_\epsilon. \] (2.7)

Using the above information, it follows that \( \Phi_\epsilon \in C^1(H^1(\mathbb{R}^N), \mathbb{R}) \), \( \Psi \) is convex and lower semicontinuous, but \( \Psi \) is not a \( C^1 \) functional, since we are working on \( \mathbb{R}^N \). Due to this fact, we will look for a critical point in the sub-differential. Here we state some definitions that can be found in [24].

**Definition 2.1** Let \( E \) be a Banach space, \( E' \) be the dual space of \( E \) and \( \langle \cdot, \cdot \rangle \) be the duality paring between \( E' \) and \( E \). Let \( J : E \to \mathbb{R} \) be a functional of the form \( J(u) = \Phi(u) + \Psi(u) \), where \( \Phi \in C^1(E, \mathbb{R}) \) and \( \Psi \) is convex and lower semicontinuous. Let us list some definitions:

(i) The sub-differential \( \partial J(u) \) of the functional \( J \) at a point \( u \in E \) is the following set
\[ \{w \in E' : \langle \Phi'(u), v - u \rangle + \Psi(v) - \Psi(u) \geq \langle w, v - u \rangle, \quad \forall v \in E\}. \] (2.8)

(ii) A critical point of \( J \) is a point \( u \in E \) such that \( J(u) < +\infty \) and \( 0 \in \partial J(u) \), i.e.
\[ \langle \Phi'(u), v - u \rangle + \Psi(v) - \Psi(u) \geq 0, \quad \forall v \in E. \] (2.9)

(iii) A Palais–Smale sequence at level \( d \) for \( J \) is a sequence \( (u_n) \subset E \) such that \( J(u_n) \to d \) and there is a numerical sequence \( \tau_n \to 0^+ \) with
\[ \langle \Phi'(u_n), v - u_n \rangle + \Psi(v) - \Psi(u_n) \geq -\tau_n ||v - u_n||, \quad \forall v \in E. \] (2.10)

(iv) The functional \( J \) satisfies the Palais–Smale condition at level \( d \) ((PS) condition, for short) if all Palais–Smale sequences at level \( d \) has a convergent subsequence.

(v) The effective domain of \( J \) is the set \( D(J) = \{u \in E : J(u) < +\infty\} \).

To proceed further we gather and state below some useful results that leads to a better understanding of the problem and of its particularities. In what follows, for each \( u \in D(J_\epsilon) \), we set the functional \( J_\epsilon'(u) : H^1_\epsilon(\mathbb{R}^N) \to \mathbb{R} \) given by
\[ \langle J_\epsilon'(u), z \rangle = \langle \Phi_\epsilon'(u), z \rangle - \int F_1'(u)z \, dx, \quad \forall z \in H^1_\epsilon(\mathbb{R}^N). \]
and define
\[ \| J'_e(u) \| = \sup \left\{ (J'_e(u), z) : z \in H^1_{\epsilon}(\mathbb{R}^N) \text{ and } \|z\|_\epsilon \leq 1 \right\}. \]

If \( \| J'_e(u) \| \) is finite, then \( J'_e(u) \) may be extended to a bounded operator in \( H_\epsilon \), and so, it can be seen as an element of \( H'_\epsilon \).

**Lemma 2.1** Let \( J_\epsilon \) satisfy (2.7), then:

(i) If \( u \in D(J_\epsilon) \) is a critical point of \( J_\epsilon \), then
\[
\langle \Phi'_e(u), v - u \rangle + \Psi(v) - \Psi(u) \geq 0, \quad \forall v \in H_\epsilon,
\]
or equivalently
\[
\int \nabla u \nabla (v - u) \, dx + \int (V(\epsilon x) + 1)u(v - u) \, dx + \int F_1(v) \, dx - \int F_1(u) \, dx \\
\geq \int F'_2(u)(v - u) \, dx, \quad \forall v \in H_\epsilon.
\]

(ii) For each \( u \in D(J_\epsilon) \) such that \( \| J'_e(u) \| < +\infty \), we have \( \partial J_\epsilon(u) \neq \emptyset \), that is, there is \( w \in H'_\epsilon \), which is denoted by \( w = J'_e(u) \), such that
\[
\langle \Phi'_e(u), v - u \rangle + \int F_1(v) \, dx - \int F_1(u) \, dx \geq \langle w, v - u \rangle, \quad \forall v \in H_\epsilon, \ (\text{see} \ [23, 25])
\]

(iii) If a function \( u \in D(J_\epsilon) \) is a critical point of \( J_\epsilon \), then \( u \) is a solution of (1.3) [(i) in Lemma 2.4, [16]].

(iv) If \( (u_n) \subset H_\epsilon \) is a Palais–Smale sequence, then
\[
\langle J'_e(u_n), z \rangle = o_n(1)\|z\|_\epsilon, \quad \forall z \in H^1_{\epsilon}(\mathbb{R}^N). \tag{2.11}
\]

[see (ii) in Lemma 2.4, [16]].

(v) If \( \Omega \) is a bounded domain with regular boundary, then \( \Psi \) (and hence \( J_\epsilon \)) is of class \( C^1 \) in \( H^1(\Omega) \) (Lemma 2.2 in [22]). More precisely, the functional
\[
\Psi(u) = \int_\Omega F_1(u) \, dx, \quad \forall u \in H^1(\Omega)
\]

belongs to \( C^1(H^1(\Omega), \mathbb{R}) \).

As a consequence of the above proprieties, we have the following results whose the proofs can be found in [1].

**Lemma 2.2** If \( u \in D(J_\epsilon) \) and \( \| J'_e(u) \| < +\infty \), then \( F_1'(u)u \in L^1(\mathbb{R}^N) \).

An immediate consequence of the last lemma is the following.

**Corollary 2.1** For each \( u \in D(J_\epsilon) \setminus \{0\} \) with \( \| J'_e(u) \| < +\infty \), we have that
\[
J'_e(u)u = \int (|\nabla u|^2 + V(\epsilon x)|u|^2) \, dx - \int u^2 \log u^2 \, dx
\]
and
\[
J_\epsilon(u) - \frac{1}{2} J'_e(u)u = \frac{1}{2} \int |u|^2 \, dx.
\]
Corollary 2.2 If \((u_n) \subset H_\varepsilon\) is a \((PS)\) sequence for \(J_\varepsilon\), then \(J'_\varepsilon(u_n)u_n = o_n(1)\|u_n\|_\varepsilon\). If \((u_n)\) is bounded, we have
\[
J_\varepsilon(u_n) = J_\varepsilon(u_n) - \frac{1}{2} J'_\varepsilon(u_n)u_n + o_n(1)\|u_n\|_\varepsilon = \frac{1}{2} \int |u_n|^2 \, dx + o_n(1)\|u_n\|_\varepsilon, \quad \forall n \in \mathbb{N}.
\]

Corollary 2.3 If \(u \in H_\varepsilon\) is a critical point of \(J_\varepsilon\) and \(v \in H_\varepsilon\) verifies \(F'_1(u)v \in L^1(\mathbb{R}^N)\), then \(J'_\varepsilon(u)v = 0\).

3 The modified problem

In order to prove our main theorem, we modify problem (1.3) and then consider the existence of solutions to the modified problem. For our problem, it is direct to consider \(u \log u^2 + u\) as \(f\) appears in [12], but it is easy to verify that it does not satisfy the basic assumptions of \(f\) that were assumed in [12], for example, \(t \log t^2 + t \neq o(t)\) as \(t \to 0\). Thus, we cannot apply directly del Pino and Felmer’s method. By a simple observation, it is easy to see that
\[
\frac{F'_2(s)}{s} \text{ is nondecreasing for } s > 0 \text{ and } \frac{F'_2(s)}{s} \text{ is strictly increasing for } s > \delta,
\]
and
\[
\lim_{s \to +\infty} \frac{F'_2(s)}{s} = +\infty,
\]
and
\[
F'_2(s) \geq 0 \text{ for } s > 0 \text{ and } F'_2(s) > 0 \text{ for } s > \delta.
\]

In what follows we need to fix some notations. Let \(l > 0\) small such that \(V_0 + 1 \geq 2l, a_0 > 0\) such that \(\frac{F'_2(a_0)}{a_0} = l\), it is clear that \(a_0 > \delta\). We define
\[
\tilde{F}'_2(t) = \begin{cases} 
F'_2(s) & 0 \leq s \leq a_0 \\
l s, & s \geq a_0.
\end{cases}
\]

If \(\chi_\Lambda\) denotes the characteristic function of the set \(\Lambda\), we introduce the penalized nonlinearity \(G'_2: \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}\) by setting
\[
G'_2(x, t) = \chi_\Lambda F'_2(t) + (1 - \chi_\Lambda) \tilde{F}'_2(t).
\]

Since our attention is to find the positive solutions of problem, we shall consider the following modified problem
\[
-\Delta u + (V(\varepsilon x) + 1)u = G'_2(\varepsilon x, u^+) - F'_1(u^+), \quad \text{in } \mathbb{R}^N. \quad (P_\varepsilon)^*
\]

We notice that, if \(u_\varepsilon\) is a positive solution of problem \((P_\varepsilon)^*\) with \(0 < u_\varepsilon(x) \leq a_0\) for all \(x \in \mathbb{R}^N \setminus \Lambda_\varepsilon\), then \(G'_2(\varepsilon x, u_\varepsilon) = F'_2(u_\varepsilon)\) and therefore, \(v_\varepsilon = u_\varepsilon(\frac{x}{\varepsilon})\) is also a solution of \((P_\varepsilon)\), where
\[
\Lambda_\varepsilon := \{x \in \mathbb{R}^N : \varepsilon x \in \Lambda\}.
\]

In what follows, we will look for nontrivial critical points for the functional
\[
I_\varepsilon(u) = \frac{1}{2} \int (|\nabla u|^2 + (V(\varepsilon x) + 1)|u|^2) \, dx + \int F_1(u^+) \, dx - \int G_2(\varepsilon x, u^+) \, dx,
\]

in the sub-differential sense, where
\[
u^+ = \max\{u(x), 0\} \quad \text{and} \quad G_2(x, t) = \int_0^t G'_2(x, s) \, ds \quad \forall (x, t) \in \mathbb{R}^N \times \mathbb{R}.
\]
Let \( H_\epsilon^+ \) be the open subset of \( H_\epsilon \) given by
\[
H_\epsilon^+ = \{ u \in H_\epsilon : |\text{supp}(u^+) \cap \Lambda_\epsilon | > 0 \}.
\]
The functional \( I_\epsilon \) satisfies the mountain pass geometry \([29]\).

**Lemma 3.1** For all \( \epsilon > 0 \), the functional \( I_\epsilon \) satisfies the following conditions:

(i) \( I_\epsilon (0) = 0 \);
(ii) there exist \( \alpha, \rho > 0 \) such that \( I_\epsilon (u) \geq \alpha \) for any \( u \in H_\epsilon \) with \( \|u\|_\epsilon = \rho \);
(iii) there exists \( e \in H_\epsilon \) with \( \|e\|_\epsilon > \rho \) such that \( I_\epsilon (e) < 0 \).

**Proof**

(i) It is clear.

(ii) Note that
\[
I_\epsilon (u) \geq \frac{1}{4} \|u\|_\epsilon^2 - \int_{\Lambda_\epsilon} F_2(u^+) dx.
\]
Hence, from (2.4), fixed \( p \in (2, 2^*) \), it follows that
\[
I_\epsilon (u) \geq \frac{1}{4} \|u\|_\epsilon^2 - C \|u\|_\epsilon^p \geq C_1 > 0,
\]
for some \( C_1 > 0 \) and \( \|u\|_\epsilon > 0 \) small enough. Here the constant \( C_1 \) does not depend on \( \epsilon \).

(iii) For each \( u \in H_\epsilon^+ \) and \( t > 0 \). By recalling that
\[
\mathbb{R}^N = (\Lambda_\epsilon \cup [tu^+ \leq a_0]) \cup (\Lambda_\epsilon \cap [tu^+ > a_0]),
\]
the definition of \( G_2 \) gives
\[
\int F_1(tu^+) dx - \int G_2(\epsilon x, tu^+) dx \leq -\frac{1}{2} \int_{\Lambda_\epsilon \cup [tu^+ \leq a_0]} |tu^+|^2 \log(|tu^+|^2) dx
\]
\[
+ \int_{\Lambda_\epsilon \cap [tu^+ > a_0]} F_1(tu^+) dx.
\]
Since \( u \in H_\epsilon \), we know that
\[
\int_{[tu^+ \geq a_0]} |u^+|^2 dx \leq \int_{\mathbb{R}^N} |u^+|^2 dx = D,
\]
and so,
\[
|[tu^+ \geq a_0]| \leq \frac{D}{a_0^2} t^2 = D_1 t^2.
\]
By the definition of \( F_1 \),
\[
F_1(t) \leq a_1 t^2 + b_1, \quad \forall t \geq 0,
\]
then
\[
\int_{\Lambda_\epsilon \cap [tu^+ > a_0]} F_1(tu^+) dx \leq \int_{[tu^+ > a_0]} F_1(tu^+) dx \leq A t^2.
\]
Hence,
\[
I_\epsilon (tu) \leq \frac{t^2}{2} \|u\|_\epsilon^2 - \frac{1}{2} \int_{\Lambda_\epsilon \cup [tu^+ \leq a_0]} |tu^+|^2 \log(|tu^+|^2) dx + A t^2.
\]
or equivalently,
\[ I_\epsilon(tu) \leq \frac{t^2}{2} \|u\|_\epsilon^2 - t^2 \int_{\Lambda_\epsilon \cup \{tu^+ \leq a_0\}} \left( |u^+|^2 \log(t) + |u^+|^2 \log(u^+) \right) dx + At^2. \]

From this,
\[ I_\epsilon(tu) \leq t^2 \left( \frac{1}{2} \|u\|_\epsilon^2 - \log(t) \int_{\Lambda_\epsilon \cup \{tu^+ \leq a_0\}} |u^+|^2 \right) \]
\[ - \frac{1}{2} \int_{\Lambda_\epsilon \cup \{tu^+ \leq a_0\}} |u^+|^2 \log(|u^+|^2) dx + A \].

Since,
\[ \int_{\Lambda_\epsilon \cup \{tu^+ \leq a_0\}} |u^+|^2 dx \geq \int_{\Lambda_\epsilon} |u^+|^2 dx > 0 \]
we derive that
\[ I_\epsilon(tu) \leq t^2 \left( \frac{1}{2} \|u\|_\epsilon^2 - \log(t) \int_{\Lambda_\epsilon \cup \{tu^+ \leq a_0\}} |u^+|^2 \right) \]
\[ - \frac{1}{2} \int_{\Lambda_\epsilon \cup \{tu^+ \leq a_0\}} |u^+|^2 \log(|u^+|^2) dx + A \], \quad \forall t \geq 1.

On the other hand, using the fact that \( I_\epsilon(u) < +\infty \), it follows that \( |u^+|^2 \log |u^+|^2 \in L^1(\mathbb{R}^N) \). Hence,
\[ \left| \int_{\Lambda_\epsilon \cup \{tu^+ \leq a_0\}} |u^+|^2 \log(|u^+|^2) dx \right| \leq \int_{\mathbb{R}^N} |u^+|^2 \log(|u^+|^2) dx < +\infty, \quad \forall t \geq 1. \]

Thereby, setting
\[ C = \sup_{t \geq 1} \left( - \int_{\Lambda_\epsilon \cup \{tu^+ \leq a_0\}} |u^+|^2 \log(|u^+|^2) dx \right) < +\infty, \]
we obtain
\[ I_\epsilon(tu) \leq t^2 \left( \frac{1}{2} \|u\|_\epsilon^2 - \log(t) \int_{\Omega_1} |u^+|^2 + C + A \right), \quad \forall t \geq 1, \]
from where it follows that
\[ I_\epsilon(tu) \rightarrow -\infty \quad \text{as} \quad t \rightarrow +\infty. \]

\[ \square \]

From Lemma 3.1 can define the minimax level
\[ c_\epsilon = \inf_{\gamma \in \Gamma_\epsilon} \max_{t \in [0,1]} I_\epsilon(\gamma(t)), \quad \text{where} \quad \Gamma_\epsilon = \{ \gamma \in C([0,1], H_\epsilon) : \gamma(0) = 0, I_\epsilon(\gamma(1)) < 0 \}. \]

(3.1)

Using a version of the mountain pass theorem without \((PS)\) condition (see [1]), there is a Palais–Smale sequence \((u_n)\) at the level \(c_\epsilon\), that is, \(I_\epsilon(u_n) \rightarrow c_\epsilon\) and
\[
\int \left( \nabla u_n \nabla (v - u_n) + (V(\epsilon x) + 1)u_n(v - u_n) \right) dx - \int G_2'(\epsilon x, u_n^+)(v - u_n) dx \\
+ \int F_1(v^+) dx - \int F_1(u_n^+) dx \geq -\tau_n \|v - u_n\|_\epsilon, \quad \forall v \in H_\epsilon.
\]
In order to show the boundedness of \((PS)\) sequence of \(I_\epsilon\), we will use the following logarithmic inequality, whose the proof can be found in del Pino and Dolbeault [9, pg 153].

**Lemma 3.2** (A new logarithmic inequality) There are constants \(A, B > 0\) such that
\[
\int |u|^2 \log(|u|^2) \, dx \leq A + B \log(\|u\|), \quad \forall u \in H^1(\mathbb{R}^N) \setminus \{0\}.
\]

As an immediate consequence we have the corollary

**Corollary 3.1** There are \(C, R > 0\) such that if \(u \in H^1(\mathbb{R}^N)\) and \(\|u\| \geq R\), then
\[
\int \log(|u|^2)|u|^2 \, dx \leq C(1 + \|u\|).
\]

By the definition of \(G_2\), it is easy to see that
\[G_2(x, s) \leq F_2(s), \quad s \geq 0.\]

Consequently
\[
I_\epsilon(u) \geq J_\epsilon(u) = \frac{1}{2} \|u\|^2_\epsilon + \int F_1(u^+) \, dx - \int F_2(u^+) \, dx, \quad \forall u \in H_\epsilon.
\] (3.2)

**Lemma 3.3** Let \((v_n) \subset H_\epsilon\) be a sequence such that \((I_\epsilon(v_n))\) is bounded in \(\mathbb{R}\). Then, \((v_n)\) is a bounded sequence in \(H_\epsilon\).

**Proof** By the assumption, there is \(M > 0\) such that
\[M \geq I_\epsilon(v_n), \quad \forall n \in \mathbb{N}.\]

Thus,
\[M \geq J_\epsilon(v_n) = \frac{1}{2} \|v_n\|^2_\epsilon + \int F_1(v_n^+) \, dx - \int F_2(v_n^+) \, dx,
\]
that is
\[M \geq \frac{1}{2} \|v_n\|^2_\epsilon - \frac{1}{2} \int |v_n^+|^2 \log(|v_n^+|^2) \, dx,
\]
from where it follows that
\[\|v_n\|^2_\epsilon \leq 2M + \int |v_n^+|^2 \log(|v_n^+|^2) \, dx, \quad \forall n \in \mathbb{N}.
\] (3.3)

Without lost of generality we will assume that \(v_n^+ \neq 0\), because otherwise, we have that inequality
\[\|v_n\|^2_\epsilon \leq 2M.
\]

From this, assume that there is \(n \in \mathbb{N}\) such that \(\|v_n^+\|_\epsilon \geq R\). By Corollary 3.1,
\[\|v_n\|^2_\epsilon \leq 2M + C(1 + \|v_n^+\|_\epsilon) \leq 2M + C(1 + \|v_n\|_\epsilon).
\]

If \(0 < \|v_n^+\|_\epsilon \leq R\), Lemma 3.2 combine with (3.3) to give
\[\|v_n\|^2_\epsilon \leq 2M + A + B \log(R).
\]

The above analysis ensures that \((v_n)\) is bounded. □
As a byproduct of the last lemma we have the boundedness of \((PS)_{c_{\epsilon}}\) sequences for \(I_{\epsilon}\).

**Corollary 3.2** If \((v_n)\) is a \((PS)\) sequence for \(I_{\epsilon}\), then \((v_n)\) is bounded in \(H_{\epsilon}\).

**Lemma 3.4** For any fixed \(\epsilon > 0\), let \((v_n) \subset H_{\epsilon}\) be a \((PS)_{d}\) sequence for \(I_{\epsilon}\). Then, for each \(\zeta > 0\), there is a number \(R = R(\zeta) > 0\) such that

\[
\limsup_{n \to \infty} \int_{\mathbb{R}^N \setminus B_R(0)} (|\nabla v_n|^2 + (V(\epsilon x) + 1)|v_n|^2) dx \leq \zeta.
\]

**Proof** Let \(\phi_R \in C^\infty(\mathbb{R}^N, \mathbb{R})\) be a cut-off function such that

\[
\phi_R = 0 \quad x \in B_{R/2}(0), \quad \phi_R = 1 \quad x \in B_R^c(0), \quad 0 \leq \phi_R \leq 1, \quad \text{and} \quad |\nabla \phi_R| \leq C/R,
\]

where \(C > 0\) is a constant independent of \(R\). Since the sequence \((\phi_R v_n)\) is bounded in \(H_{\epsilon}\), we derive that

\[
I'_\epsilon(u_n)(\phi_R v_n) = o_n(1),
\]

that is

\[
\int \left(|\nabla v_n|^2 + (V(\epsilon x) + 1)|v_n|^2\right) \phi_R dx = \int_{\Lambda_\epsilon} F'_1(v_n^+) \phi_R v_n dx + \int_{\mathbb{R}^N \setminus \Lambda_\epsilon} F'_1(v_n^+) \phi_R v_n dx
\]

\[
- \int v_n \nabla v_n \nabla \phi_R dx - \int F'_1(v_n^+) \phi_R v_n dx + o_n(1).
\]

Choosing \(R > 0\) such that \(\Lambda_\epsilon \subset B_{R/2}(0)\), the Hölder inequality together with the boundedness of the sequence \((v_n)\) in \(H_{\epsilon}\) leads to

\[
\int \left(|\nabla v_n|^2 + (V(\epsilon x) + 1)|v_n|^2\right) \phi_R dx \leq I \int |v_n|^2 \phi_R dx + C \|v_n\|^2 + o_n(1).
\]

So, fixing \(\zeta > 0\) and passing to the limit in the last inequality, it follows that

\[
\limsup_{n \to \infty} \int_{\mathbb{R}^N \setminus B_R(0)} (|\nabla v_n|^2 + (V(\epsilon x) + 1)|v_n|^2) dx \leq \frac{C}{R} < \zeta,
\]

for some \(R\) sufficiently large. \(\square\)

Our next lemma shows that \(I_{\epsilon}\) verifies the \((PS)\) condition.

**Lemma 3.5** Let \((v_n)\) be a \((PS)_{d}\) sequence for \(I_{\epsilon}\) with \(v_n \rightharpoonup v\) in \(H_{\epsilon}\). Then, \(v_n \to v\) in \(H_{\epsilon}\). Moreover,

\[
F_1(v_n^+) \to F_1(v^+) \quad \text{and} \quad F_1'(v_n^+) v_n^+ \to F_1'(v^+) v^+ \quad \text{in} \quad L^1(\mathbb{R}^N).
\]

As a consequence, \(v\) is a critical point of \(I_{\epsilon}\) at level \(d\), that is, \(0 \in \partial I_{\epsilon}(v)\) and \(I_{\epsilon}(v) = d\).

**Proof** Let \((v_n) \subset H_{\epsilon}\) be a \((PS)_{d}\) sequence for \(I_{\epsilon}\). By Corollary 1.2, the sequence \((v_n)\) is bounded in \(H_{\epsilon}\), then without lost of generality we can assume that

\[
v_n \rightharpoonup v \quad \text{in} \quad H_{\epsilon}.
\]

By the last lemma, for any given \(\zeta > 0\), there is \(R > 0\) such that

\[
\limsup_{n \to +\infty} \int_{|x| > R} (|\nabla v_n|^2 + (V(\epsilon x) + 1)|v_n|^2) dx < \zeta.
\]
Since $G'_2$ has a subcritical growth, the above estimate ensures that
\[
\int G'_2(\xi, v^+_n) w \, dx \to \int G'_2(\xi, v^+) w \, dx, \quad \forall w \in C^\infty_0(\mathbb{R}^N),
\]
\[
\int G'_2(\xi, v^+_n) v^+_n \, dx \to \int G'_2(\xi, v^+) v^+ \, dx,
\]
and
\[
\int G'_2(\xi, v^+_n) \, dx \to \int G'_2(\xi, v^+) \, dx.
\]
Now, recalling that $I'_\xi(v_n)w = o_n(1)\|u_n\|_\epsilon$ for all $w \in C^\infty_0(\mathbb{R}^N)$, we deduce that $I'_\xi(v)w = 0$ for all $w \in C^\infty_0(\mathbb{R}^N)$, and so, $I'_\xi(v)v = 0$, that is,
\[
\int (|\nabla v|^2 + (V(\xi x) + 1)|v|^2) \, dx + \int F'_1(v^+) v^+ \, dx = \int G'_2(\xi, v^+) v^+ \, dx.
\]
Moreover, the limit $I'_\xi(v_n)v_n = o_n(1)\|u_n\|_\epsilon$, that is,
\[
\int (|\nabla v_n|^2 + (V(\xi x) + 1)|v_n|^2) \, dx + \int F'_1(v^+_n) v^+_n \, dx = \int G'_2(\xi, v^+_n) v^+_n \, dx + o_n(1).
\]
Gathering the above information, we deduce that
\[
\int (|\nabla v_n|^2 + (V(\xi x) + 1)|v_n|^2) \, dx + \int F'_1(v^+_n) v^+_n \, dx = \int (|\nabla v|^2 + (V(\xi x) + 1)|v|^2) \, dx
\]
\[
+ \int F'_1(v^+) v^+ \, dx + o_n(1),
\]
from where it follows that, for some subsequence,
\[
v_n \to v \quad \text{in} \quad H_\epsilon
\]
and
\[
F'_1(v^+_n) v^+_n \to F'_1(v^+) v^+ \quad \text{in} \quad L^1(\mathbb{R}^N).
\]
Since $F_1$ is convex, even and $F(0) = 0$, we know that $F'_1(t) \geq F_1(t) \geq 0$ for all $t \in \mathbb{R}$. Thus, the last limit together with Lebesgue’s theorem yields
\[
F'_1(v^+_n) \to F'_1(v^+) \quad \text{in} \quad L^1(\mathbb{R}^N).
\]
The above limits permit to conclude that $0 \in \partial I_\epsilon(v)$ and $I_\epsilon(v) = d$. \qed

**Theorem 3.1** The functional $I_\epsilon$ has a positive critical point $u_\epsilon \in H_\epsilon$ such that $I_\epsilon(u_\epsilon) = c_\epsilon$, where $c_\epsilon$ denotes the mountain pass level associated with $I_\epsilon$.

**Proof** The existence of the critical point $u_\epsilon$ is an immediate result of Lemma 3.1, Corollary 3.2 and Lemma 3.5. The function $u_\epsilon$ is nonnegative, because
\[
I'_\epsilon(u_\epsilon)(u^-_\epsilon) = 0 \Rightarrow u^-_\epsilon = 0,
\]
where $u^-_\epsilon = \min\{u_\epsilon, 0\}$. By a slight variant of the argument in [11, Section 3.1] it follows from the maximum principle (see [26, Theorem 1]) that $u_\epsilon(x) > 0$ for a.e. $x \in \mathbb{R}^N$. \qed
In the sequel, we denote by $\mathcal{N}_e$ the set
\[ \mathcal{N}_e = \{ u \in D(I_e) \setminus [0] : I'_e(u)u = 0 \}. \]

The following lemma is important for the proof of Lemma 3.7.

**Lemma 3.6** Assume that hypotheses $(V1)$–$(V2)$ are satisfied. For each $u \in H^+_e$, let $g_u : \mathbb{R}^+ \to \mathbb{R}$ be given by $g_u(t) = I_e(tu)$. Then there exists a unique $t_u > 0$ such that $g_u'(t) > 0$ in $(0, t_u)$ and $g_u'(t) < 0$ in $(t_u, \infty)$.

**Proof** As in the proof of Lemma 3.1, we have $g_u(0) = 0$, $g_u(t) > 0$ for $t > 0$ small and $g_u(t) < 0$ for $t > 0$ large. Therefore, $\max_{t \geq 0} g_u(t)$ is achieved at a global maximum point $t = t_u > 0$ verifying $g'_u(t_u) = 0$ and $tu \in \mathcal{N}_e$. Now we claim that $t_u > 0$ is unique. Indeed, suppose that there exist $t_2 > t_1 > 0$ such that $g_u'(t_1) = g_u'(t_2) = 0$. Then, for $i = 1, 2$,
\[
t_i \int (|\nabla u|^2 + (V(x) + 1)|u|^2)dx - \int_{\Lambda_e} F'_2(t_iu^+)u^+dx - \int_{\mathbb{R}^n \setminus \Lambda_e} \tilde{F}'_2(t_iu^+)u^+dx \\
+ \int F'_1(t_iu^+)u^+dx = 0.
\]
Hence,
\[
\int (|\nabla u|^2 + (V(x) + 1)|u|^2)dx = \int_{\Lambda_e} \frac{F'_2(t_2u^+)u^+}{t_2}dx + \int_{\mathbb{R}^n \setminus \Lambda_e} \frac{\tilde{F}'_2(t_2u^+)u^+}{t_2}dx \\
- \int \frac{F'_1(t_1u^+)u^+}{t_1}dx,
\]
which implies that
\[
\int_{\Lambda_e} \left( \frac{F'_2(t_2u^+)u^+}{t_2} - \frac{F'_2(t_1u^+)u^+}{t_1} \right)dx + \int_{\mathbb{R}^n \setminus \Lambda_e} \left( \frac{\tilde{F}'_2(t_2u^+)u^+}{t_2} - \frac{\tilde{F}'_2(t_1u^+)u^+}{t_1} \right)dx
\]
\[
= \int \left( \frac{F'_1(t_2u^+)u^+}{t_2} - \frac{F'_1(t_1u^+)u^+}{t_1} \right)dx.
\]
Since $u \in H^+_e$, the left side of above equality is positive. For the right side of above equality, we have
\[
\int \left( \frac{F'_1(t_2u^+)u^+}{t_2} - \frac{F'_1(t_1u^+)u^+}{t_1} \right)dx = \int_{u^+ < \frac{\delta}{12}} \left( \frac{F'_1(t_2u^+)u^+}{t_2} - \frac{F'_1(t_1u^+)u^+}{t_1} \right)dx
\]
\[+ \int_{\frac{\delta}{12} < u^+ < \frac{\delta}{11}} \left( \frac{F'_1(t_2u^+)u^+}{t_2} - \frac{F'_1(t_1u^+)u^+}{t_1} \right)dx
\]
\[+ \int_{u^+ > \frac{\delta}{11}} \left( \frac{F'_1(t_2u^+)u^+}{t_2} - \frac{F'_1(t_1u^+)u^+}{t_1} \right)dx
\]
\[= \int_{u^+ < \frac{\delta}{12}} (u^+)^2 \log \left( \frac{t_1}{t_2} \right)^2 dx + \int_{u^+ > \frac{\delta}{11}} \left( \frac{1}{t_2} - \frac{1}{t_1} \right)2\delta u^+dx
\]
\[+ \int_{\frac{\delta}{12} < u^+ < \frac{\delta}{11}} \left( (u^+)^2 \log \frac{t_1^2}{t_2} + 2u^+ \left( \frac{\delta}{t_2} - u^+ \right) \right)dx.
\]
A direct computation shows that the right side of the last last equality is negative, which is a contradiction and $t_u > 0$ is unique. \(\square\)
Remark 3.1 By Lemma 3.6, for each $u \in H^+_c$, there is a unique $m_c(u) \in N_c$. On the other hand, if $u \in N_\epsilon$, then $u \in H^+_c$. Otherwise, we have $|\text{supp}(u^+) \cap \Lambda_\epsilon| = 0$ and

$$
\int (|\nabla u|^2 + (V(0) + 1)|u|^2)\,dx \leq \|u\|^2_c = \int_{\mathbb{R}^N \setminus \Lambda_\epsilon} \tilde{F}_2(u^+)u^+\,dx - \int F'_1(u^+)u^+\,dx \leq 1 \int |u|^2\,dx,
$$

which is impossible since $V(0) + 1 \geq 2t > 0$ and $u \neq 0$.

Related to $\epsilon = 0$, for simplicity, we shall assume that $0 \in \Lambda$, $V(0) = V_0 > -1$ and consider the problem

$$
\begin{align*}
&\{ -\Delta u + V_0u = u \log u^2, \quad \text{in } \mathbb{R}^N, \\
&u \in H^1(\mathbb{R}^N) \}
\end{align*}
$$

The corresponding energy functional associated to (3.4) will be denoted by $J_0 : H^1(\mathbb{R}^N) \to (-\infty, +\infty]$ and defined as

$$
J_0(u) = \frac{1}{2} \int (|\nabla u|^2 + (V_0 + 1)|u|^2)\,dx - \frac{1}{2} \int u^2 \log u^2\,dx.
$$

In [22] is proved that problem (3.4) has a positive ground state solution given by

$$
c_0 := \inf_{u \in N_0} J_0(u) = \inf_{u \in D(J_0) \setminus \{0\}} \left\{ \max_{t \geq 0} J_0(tu) \right\},
$$

where

$$
N_0 = \left\{ u \in D(J_0) \setminus \{0\}; J_0(u) = \frac{1}{2} \int |u|^2\,dx \right\}
$$

and

$$
D(J_0) = \left\{ u \in H^1(\mathbb{R}^N) : J_0(u) < +\infty \right\}.
$$

The next lemma shows that the mountain pass level $c_\epsilon$ in (3.1) is the ground state energy for the functional $I_\epsilon$, it also establishes an important relation between $c_\epsilon$ and $c_0$.

Lemma 3.7 (a) $c_\epsilon > 0$, for $\epsilon > 0$;
(b) $c_\epsilon = \inf_{u \in N_\epsilon} I_\epsilon(u)$, for $\epsilon \geq 0$;
(c) $\lim_{\epsilon \to 0} c_\epsilon = c_0$.

Proof (a) Follows from Lemma 3.1 (ii).
(b) Let $u \in N_\epsilon$ and let us consider $I_\epsilon(t^*u) < 0$, for some $t^* > 0$. If $\gamma_\epsilon : [0, 1] \to H_\epsilon$ is the continuous path $\gamma_\epsilon(t) = t \cdot t^*u$, then

$$
c_\epsilon \leq \max_{t \in [0,1]} I_\epsilon(\gamma_\epsilon(t)) \leq \max_{t \geq 0} I_\epsilon(tu) = I_\epsilon(u)
$$

and consequently $c_\epsilon \leq \inf_{u \in N_\epsilon} I_\epsilon(u)$.

Now we prove the reverse inequality. By Theorem 3.1, there exits $u_\epsilon \in H_\epsilon$ with $u_\epsilon(x) > 0$ for all $x \in \mathbb{R}^N$ such that

$$
I_\epsilon(u_\epsilon) = c_\epsilon \quad \text{and} \quad 0 \in \partial I_\epsilon(u_\epsilon).
$$

Then $u_\epsilon \in N_\epsilon$, and so,

$$
\inf_{u \in N_\epsilon} I_\epsilon(u) \leq I_\epsilon(u_\epsilon) = c_\epsilon.
$$
(c) By [22, Theorem 1.2], the infimum in (3.5) is such that $c_0 = J_0(u_0)$, for some positive function $u_0 \in \mathcal{N}_0$. Note that, if $\varphi \in C_0^\infty(\mathbb{R}_N^N)$, $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ in $B_1(0)$ and $\varphi \equiv 0$ in $B_2(0)^c$, defining $\varphi_R := \varphi(\cdot/R)$ and $u_R(x) = \varphi_R(x)u_0(x)$, we have that

$$u_R \to u_0 \text{ in } H^1(\mathbb{R}_N) \text{ as } R \to +\infty.$$ 

Fixing $R > 0$ and arguing as in the proof of (3.6), for a fixed $\epsilon > 0$ we find

$$c_\epsilon \leq \max_{t \geq 0} I_\epsilon(tu_R) = I_\epsilon(t_\epsilon u_R),$$

and

$$\int \left( |\nabla u_R|^2 + (V(\epsilon x) + 1)u_R^2 \right) dx = \int_{\Lambda_\epsilon} F_2'(t_\epsilon u_R)u_R dx + \int_{\mathbb{R}_N \setminus \Lambda_\epsilon} F_2'(t_\epsilon u_R)u_R dx - \int F_2'(t_\epsilon u_R)u_R dx.$$ 

Since $V(\epsilon x) \to V_0$ as $\epsilon \to 0$, by the Lebesgue Dominated Convergence theorem, we have from the left side of the above equality that

$$\lim_{\epsilon \to 0} \int \left( |\nabla u_R|^2 + (V(\epsilon x) + 1)u_R^2 \right) dx = \int \left( |\nabla u_R|^2 + (V_0 + 1)u_R^2 \right) dx.$$ 

Assuming $t_\epsilon \to +\infty$ as $\epsilon \to 0$, since $\Lambda_\epsilon \to \mathbb{R}_N$ as $\epsilon \to 0$, it is easy to verify that the right side of the above equality goes to $+\infty$ as $\epsilon \to 0$, which is a contradiction. Thus, $(t_\epsilon)$ is bounded in $\mathbb{R}$ for $\epsilon$ small enough. Moreover, since

$$I_\epsilon(t_\epsilon u_R) = J_0(t_\epsilon u_R) + \frac{t_\epsilon^2}{2} \int (V(\epsilon x) - V_0)u_R^2 dx + \int_{\Lambda_\epsilon} F_2(t_\epsilon u_R) dx + \int_{\mathbb{R}_N \setminus \Lambda_\epsilon} \tilde{F}_2(t_\epsilon u_R) dx - \int F_2(t_\epsilon u_R) dx \leq J_0(t_R u_R) + \frac{t_\epsilon^2}{2} \int (V(\epsilon x) - V_0)u_R^2 dx,$$

where $t_R > 0$ satisfies

$$J_0(t_R u_R) = \max_{t \geq 0} J_0(tu_R).$$

Using $\sup_{x \in B(0)} |V(\epsilon x) - V_0| \to 0$ as $\epsilon \to 0$, we get

$$\limsup_{\epsilon \to 0} c_\epsilon \leq \limsup_{\epsilon \to 0} I_\epsilon(t_\epsilon u_R) \leq J_0(t_R u_R). \quad (3.7)$$

Now, we use the fact that $(t_R)$ is also bounded for $R$ large enough, $u_R \leq u_0$ and $F_1$ is increasing for $t \geq 0$ to deduce that

$$F_1(t_R u_R) \leq F_1(k u_0),$$

for some $k > 0$. Since $u_0 \in \mathcal{N}_0$, we can ensure that $F_1(k u_0) \in L^1(\mathbb{R}_N)$ for all $k \geq 0$. Thus, if $R_n \to +\infty$ and $t_{R_n} \to t_*$, the Lebesgue Dominated Convergence theorem yields

$$F_1(t_{R_n} u_{R_n}) \to F_1(t_* u_0) \text{ in } L^1(\mathbb{R}_N).$$
and
\[ F'_1(tR_nu_{R_n})t_{R_n}u_{R_n} \to F'_1(t*_nu_0)t_*u_0 \text{ in } L^1(\mathbb{R}^N). \]

As an immediate consequence, \( t_R \to 1 \) as \( R \to +\infty \) and
\[ J_0(t_Ru_R) \to J_0(u_0) \text{ as } R \to +\infty. \]

This combined with (3.7) gives
\[ \limsup_{\epsilon \to 0} c_\epsilon \leq J_0(u_0) = c_0. \]

Inasmuch as \( I_\epsilon(u) \geq J_\epsilon(u) \geq J_0(u) \), \( \forall \epsilon > 0, u \in D(I_\epsilon) \), and by part (b) with \( \epsilon = 0 \), the reverse inequality holds:
\[ \liminf_{\epsilon \to 0} c_\epsilon \geq c_0. \]

Therefore,
\[ \lim_{\epsilon \to 0} c_\epsilon = c_0. \]

\[ \Box \]

**Lemma 3.8** Let \( (\omega_n) \subset N_0 \) be a sequence satisfying \( J_0(\omega_n) \to c_0 \) such that \( \omega_n \to \omega \) with \( \omega \neq 0 \) and \( J'_0(\omega)\omega \leq 0 \). Then, \( \omega_n \to \omega \) in \( H^1(\mathbb{R}^N) \).

**Proof** Since \( J'_0(\omega)\omega \leq 0 \), there is \( t \in (0, 1] \) such that \( t\omega \in N_0 \), and so,
\[ c_0 \leq J_0(t\omega) = \frac{t^2}{2} \int |\omega|^2 \, dx \leq \liminf_{n \to \infty} \frac{1}{2} \int |\omega_n|^2 \, dx \leq \limsup_{n \to \infty} \frac{1}{2} \int |\omega_n|^2 \, dx = \lim_{n \to \infty} J_0(\omega_n) = c_0. \]

The above argument yields \( t = 1 \) and \( \omega_n \to \omega \) in \( L^2(\mathbb{R}^N) \). For the case \( N \geq 3 \), since \( (\omega_n) \) is bounded in \( L^{2^*(\mathbb{R}^N)} \), by interpolation on the Lebesgue spaces, it follows that
\[ \omega_n \to \omega \text{ in } L^p(\mathbb{R}^N), \text{ for all } 2 \leq p < 2^*, \]
and therefore \( \int F'_2(\omega_n)\omega_n \, dx \to \int F'_2(\omega)\omega \, dx \). For the case \( N = 1, 2 \), for any \( q > 2 \), since \( (\omega_n) \) is bounded in \( L^q(\mathbb{R}^N) \), using interpolation on the Lebesgue spaces again, we have that
\[ \omega_n \to \omega \text{ in } L^p(\mathbb{R}^N), \text{ for all } 2 \leq p < q. \]

Since \( q > 2 \) is arbitrary, thus
\[ \omega_n \to \omega \text{ in } L^p(\mathbb{R}^N), \text{ for all } 2 \leq p < \infty. \]

and \( \int F'_2(\omega_n)\omega_n \, dx \to \int F'_2(\omega)\omega \, dx \).

Finally, using the equalities \( J'_0(\omega)\omega = 0 \) and
\[ \int (|\nabla \omega_n|^2 + (V_0 + 1)|\omega_n|^2) \, dx + \int F'_1(\omega_n)\omega_n \, dx = \int F'_2(\omega_n)\omega_n \, dx, \]
we get
\[ \int (|\nabla \omega_n|^2 + (V_0 + 1)|\omega_n|^2) \, dx \to \int (|\nabla \omega|^2 + (V_0 + 1)|\omega|^2) \, dx \text{ in } H^1(\mathbb{R}^N), \]
from where it follows the desired result. \( \Box \)
Lemma 3.9 Let $\epsilon_n \to 0$ and $u_n \in H_{\epsilon_n}$ such that $I_{\epsilon_n}(u_n) = c_{\epsilon_n}$ and $I'_{\epsilon_n}(u_n) = 0$. Then there exists the sequence $(y_n) \subset \mathbb{R}^N$ such that $\psi_n(x) = u_n(x + y_n)$ has a convergent subsequence in $H^1(\mathbb{R}^N)$. Moreover, there is $y_0 \in \Lambda$ such that

$$
\lim_{n \to \infty} \epsilon_n y_n = y_0 \quad \text{and} \quad V(y_0) = V_0.
$$

Proof Taking into account $u_n \in H_{\epsilon_n}$ such that $I_{\epsilon_n}(u_n) = c_{\epsilon_n}$ and Lemma 3.3, it is easy to see that $(u_n)$ is bounded in $H_{\epsilon_n}$. Moreover, $(u_n)$ is also bounded in $H^1(\mathbb{R}^N)$. Using [17], there exist $r, \gamma > 0$ and a sequence $(y_n) \subset \mathbb{R}^N$ such that

$$
\limsup_{n \to \infty} \int_{B_r(y_n)} |u_n(x)|^2 \, dx \geq \gamma.
$$

(3.8)

Otherwise, we can conclude that $u_n \to 0$ in $L^p(\mathbb{R}^N)$, for all $2 < p < 2^*$. and $\int F'(u_n)u_n \, dx \to 0$, it yields that $I_{\epsilon_n}(u_n) = c_{\epsilon_n} \to 0$ as $\to \infty$, which is a contradiction, since $c_{\epsilon_n} \to c_0 > 0$. Setting $\psi_n(x) = u_n(x + y_n)$, then there is $\psi \in H^1(\mathbb{R}^N) \setminus \{0\}$ such that

$$
\psi_n \to \psi \quad \text{in} \quad H^1(\mathbb{R}^N)
$$

(3.9)

and

$$
\int_{B_r(0)} |\psi(x)|^2 \, dx \geq \gamma.
$$

(3.10)

In the sequel we will prove that the sequence $(\epsilon_n y_n)$ is bounded. To this end, it is enough to show the following claim.

Claim 3.1 $\lim_{n \to \infty} \text{dist}(\epsilon_n y_n, \Lambda) = 0$.

Indeed, if the claim does not hold, there exist $\delta > 0$ and a subsequence of $(\epsilon_n y_n)$, still denoted by itself, such that,

$$
\text{dist}(\epsilon_n y_n, \Lambda) \geq \delta, \quad \forall n \in \mathbb{N}.
$$

Consequently, there is $r > 0$ such that

$$
B_r(\epsilon_n y_n) \subset \Lambda^c, \quad \forall n \in \mathbb{N}.
$$

Using the fact that $\psi$ is a nonnegative function, there is a sequence of nonnegative functions $(\omega_j) \subset H^1(\mathbb{R}^N)$ such that $\omega_j$ has a compact support in $\mathbb{R}^N$ and $\omega_j \to \psi$ in $H^1(\mathbb{R}^N)$ as $j \to \infty$. Now, fixing $j > 0$ and using $w_j$ as a test function, we have

$$
\int \left( \nabla \psi_n \nabla \omega_j + (V(\epsilon_n x + \epsilon_n y_n) + 1)\psi_n \omega_j \right) \, dx = \int \nabla^2 \psi_n(\epsilon_n x + \epsilon_n y_n, \psi_n) \omega_j \, dx - \int F'_{\psi}(\psi_n) \omega_j \, dx.
$$

(3.11)

Note that

$$
\int G'_2(\epsilon_n x + \epsilon_n y_n, \psi_n) \omega_j \, dx = \int_{B_{\epsilon_n}(0)} G'_2(\epsilon_n x + \epsilon_n y_n, \psi_n) \omega_j \, dx
$$

$$
+ \int_{\mathbb{R}^N \setminus B_{\epsilon_n}(0)} G'_2(\epsilon_n x + \epsilon_n y_n, \psi_n) \omega_j \, dx
$$

(3.12)
and so,
\[ \int G'_2(\epsilon_n x + \epsilon_n y_n, \psi_n) \omega_j dx \leq l \int_{B_{\epsilon_n}(0)} \psi_n \omega_j dx + \int_{\mathbb{R}^N \setminus B_{\epsilon_n}(0)} F'_2(\psi_n) \omega_j dx. \]

Therefore,
\[ \int \left( \nabla \psi_n \nabla \omega_j + (V(\epsilon_n x + \epsilon_n y_n) + 1) \psi_n \omega_j \right) dx \leq l \int_{B_{\epsilon_n}(0)} \psi_n \omega_j dx + \int_{\mathbb{R}^N \setminus B_{\epsilon_n}(0)} F'_2(\psi_n) \omega_j dx - \int F'_1(\psi_n) \omega_j dx, \]

implying that
\[ \int_{\mathbb{R}^N} \left( \nabla \psi_n \nabla \omega_j + A \psi \omega_j \right) dx \leq \int_{\mathbb{R}^N \setminus B_{\epsilon_n}(0)} F'_2(\psi_n) \omega_j dx, \]

where \( A = V_0 + 1 - l > 0 \). As \( \omega_j \) has a compact support in \( \mathbb{R}^N \) and \( \epsilon_n \to 0 \), the boundedness of \( (\psi_n) \) imply that
\[ \int_{\mathbb{R}^N \setminus B_{\epsilon_n}(0)} F'_2(\psi_n) \omega_j dx \to 0 \text{ as } n \to \infty, \]

and
\[ \int \left( \nabla \psi_n \nabla \omega_j + A \psi \omega_j \right) dx \to \int \left( \nabla \psi \nabla \omega_j + A \psi \omega_j \right) dx, \text{ as } n \to \infty. \]

Hence,
\[ \int \left( \nabla \psi \nabla \omega_j + A \psi \omega_j \right) dx \leq 0. \]

Since \( j \) is arbitrary, taking the limit of \( j \to +\infty \), we obtain
\[ \int \left( |\nabla \psi|^2 + A |\psi|^2 \right) dx = 0, \]

which contradicts (3.10). This proves Claim 3.1.

From Claim 3.1, there is a subsequence of \( (\epsilon_n y_n) \) and \( y_0 \in \Lambda \) such that
\[ \lim_{n \to \infty} \epsilon_n y_n = y_0. \]

Claim 3.2 \( y_0 \in \Lambda \).

Indeed, by using the definition of \( G'_2 \) and (3.11), we have that
\[ \int \left( \nabla \psi_n \nabla \omega_j + (V(\epsilon_n x + \epsilon_n y_n) + 1) \psi_n \omega_j \right) dx + \int F'_1(\psi_n) \omega_j dx \leq \int F'_2(\psi_n) \omega_j dx. \]

By using (3.9) and the fact that \( \omega_j \) has a compact support, letting \( n \to \infty \), we have
\[ \int \left( \nabla \psi \nabla \omega_j + (V(y_0) + 1) \psi \omega_j \right) dx + \int F'_1(\psi) \omega_j dx \leq \int F'_2(\psi) \omega_j dx. \]

Now, taking the limit of \( j \to +\infty \), it yields that
\[ \int \left( |\nabla \psi|^2 + (V(y_0) + 1) |\psi|^2 \right) dx + \int F'_1(\psi) \psi dx \leq \int F'_2(\psi) \psi dx. \]
Hence, there is $s_1 \in (0, 1]$ such that

$$s_1 \psi \in N_{V(y_0)} = \{ u \in H^1(\mathbb{R}^N) \setminus \{0\} : J'_{V(y_0)}(u)u = 0 \},$$

where $J_{V(y_0)} : H^1(\mathbb{R}^N) \to \mathbb{R}$ is given by

$$J_{V(y_0)}(u) = \frac{1}{2} \int (|\nabla u|^2 + V(y_0)|u|^2)dx - \frac{1}{2} \int u^2 \log u^2 dx.$$  

If $c_{V(y_0)}$ denotes the mountain pass level associated with $J_{V(y_0)}$, we have

$$c_{V(y_0)} \leq J_{V(y_0)}(s_1 \psi) \leq \liminf_{n \to +\infty} I_{\epsilon_n}(u_n) = \liminf_{n \to +\infty} c_{\epsilon_n} = c_0 = c_{V(0)}.$$  

Thus,

$$c_{V(y_0)} \leq c_{V(0)},$$

from where it follows that

$$V(y_0) \leq V(0) \equiv V_0.$$  

As $V_0 = \inf_{x \in \Lambda} V(x)$, the above inequality implies that

$$V(y_0) = V_0.$$  

Moreover, by $(V2)$, $y_0 \notin \partial \Lambda$. Then, $y_0 \in \Lambda$ and the proof of Claim 3.2 is complete.

Now, we are going to prove that $\psi_n \to \psi$ in $H^1(\mathbb{R}^N)$. Fixing $s_n > 0$ such that $\bar{\psi}_n = s_n \psi_n \in N_0$. By (3.2), we can see that

$$c_0 \leq J_0(\bar{\psi}_n) \leq \max_{t \geq 0} I_{\epsilon_n}(t \psi_n) = I_{\epsilon_n}(u_n),$$

which together with Lemma 3.7 implies that $J_0(\bar{\psi}_n) \to c_0$. Since $(\psi_n)$ and $(\bar{\psi}_n)$ are bounded in $H^1(\mathbb{R}^N)$ and $\bar{\psi}_n \to 0$ in $H^1(\mathbb{R}^N)$, we deduce that for some subsequence, still denote by itself, $s_n \to s^* > 0$. Moreover, using that $u_n$ is a solution, we also have that $J'_0(\bar{\psi})\bar{\psi} \leq 0$. Since $\bar{\psi} \neq 0$, by Lemma 3.8,

$$\bar{\psi}_n \to \bar{\psi} \quad \text{in} \quad H^1(\mathbb{R}^N)$$

or equivalently

$$\psi_n \to \psi \quad \text{in} \quad H^1(\mathbb{R}^N),$$

which finishes the proof. □

**Lemma 3.10** Let $(\psi_n)$ the sequence given in Lemma 3.9. Then, $(\psi_n) \subset L^\infty(\mathbb{R}^N)$ and there is $K > 0$ such that

$$|\psi_n|_{\infty} \leq K, \quad \forall n \in \mathbb{N}.$$  

Moreover,

$$\psi_n(x) \to 0 \quad \text{as} \quad |x| \to +\infty,$$

uniformly in $n \in \mathbb{N}$.  

\begin{flushright}
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Proof For any \( R > 0, 0 < r < \frac{R}{2} \), let \( \eta \in C^\infty(\mathbb{R}^N) \), \( 0 \leq \eta \leq 1 \) with \( \eta(x) = 1 \) if \( |x| \geq R \) and \( \eta(x) = 0 \) if \( |x| \leq R - r \) and \( |\nabla \eta| \leq \frac{2}{r} \). For \( L > 0 \), let
\[
\psi_{L,n} := \begin{cases} 
\psi_n, & \text{if } \eta_n \leq L \\
L, & \text{if } \eta_n \geq L.
\end{cases}
\]

We first deal with the case \( N > 2 \). To this end, let \( z_{L,n} = \eta^2 \psi_{L,n}^{2(\beta - 1)} \psi_n \) and \( \omega_{L,n} = \eta \psi_n \psi_{L,n}^{\beta - 1} \) with \( \beta > 1 \) to be determined later. Since \( 0 \leq \eta \leq 1 \) and \( \psi_{L,n} \leq L \), it yields that \( \eta^2 \psi_{L,n}^{2(\beta - 1)} \psi_n \leq L^{2(\beta - 1)} \psi_n \) and \( F_1'(\psi_n) \eta^2 \psi_{L,n}^{2(\beta - 1)} \psi_n \in L^1(\mathbb{R}^N) \). Taking \( z_{L,n} \) as a test function, we have
\[
\int \eta^2 \psi_{L,n}^{2(\beta - 1)} |\nabla \psi_n|^2 dx + \int (V(\epsilon_n x + \epsilon_n y_n) + 1) \eta^2 \psi_{L,n}^{2(\beta - 1)} |\psi_n|^2 dx \\
+ \int F_1'(\psi_n) \eta^2 \psi_{L,n}^{2(\beta - 1)} \psi_n dx \\
= -2 \int \eta \psi_{L,n}^{2(\beta - 1)} \psi_n \nabla \psi_n \nabla \eta dx - 2(\beta - 1) \int \eta^2 \psi_{L,n}^{2(\beta - 3)} \psi_n \nabla \psi_n \nabla \psi_{L,n} dx \\
+ \int G_2'(\epsilon_n x + \epsilon_n y_n, \psi_n) \eta^2 \psi_{L,n}^{2(\beta - 1)} \psi_n dx.
\]

(3.12)

From the definition of \( G_2 \), we have that
\[
G_2'(x, t) \leq \epsilon t^{2^* - 1}, \quad \forall (x, t) \in \mathbb{R}^N \times \mathbb{R}^+.
\]

(3.13)

Using (2.3), (3.12) and (3.13), we can obtain that
\[
\int \eta^2 \psi_{L,n}^{2(\beta - 1)} |\nabla \psi_n|^2 dx \leq 2 \int \eta \psi_{L,n}^{2(\beta - 1)} |\nabla \psi_n|^2 |\nabla \eta| dx \\
+ C \int \eta^2 \psi_{L,n}^{2(\beta - 1)} |\psi_n|^{2^*} dx.
\]

(3.14)

For each \( \delta > 0 \), using Young’s inequality, we have from (3.14) that
\[
\int \eta^2 \psi_{L,n}^{2(\beta - 1)} |\nabla \psi_n|^2 dx \leq 2\delta \int \eta^2 \psi_{L,n}^{2(\beta - 1)} |\nabla \psi_n|^2 dx + 2C\delta \int \psi_{L,n}^{2(\beta - 1)} |\psi_n|^2 |\nabla \eta|^2 dx \\
+ C \int \eta^2 \psi_{L,n}^{2(\beta - 1)} |\psi_n|^{2^*} dx.
\]

Choosing \( \delta \in (0, \frac{1}{4}) \), it yields
\[
\int \eta^2 \psi_{L,n}^{2(\beta - 1)} |\nabla \psi_n|^2 dx \leq C \int \psi_{L,n}^{2(\beta - 1)} |\psi_n|^2 |\nabla \eta|^2 dx \\
+ C \int \eta^2 \psi_{L,n}^{2(\beta - 1)} |\psi_n|^{2^*} dx.
\]

(3.15)

On the other hand, by the Sobolev and Hölder inequalities, we have
\[
|\omega_{L,n}|^{2^*_s} \leq C \int |\nabla \omega_{L,n}|^2 dx = C \int |\nabla (\eta \psi_n \psi_{L,n}^{\beta - 1})|^2 dx \\
\leq C \beta^2 \left( \int \psi_{L,n}^{2(\beta - 1)} |\psi_n|^2 |\nabla \eta|^2 dx + \int \eta^2 \psi_{L,n}^{2(\beta - 1)} |\nabla \psi_n|^2 dx \right).
\]

(3.16)
Combining (3.15) and (3.16), we have

\[ |\omega_{L,n}|^{2^*}_{2} \leq C \beta^2 \left( \int |\psi_{L,n}^{2(\beta-1)}|^2 |\nabla \eta|^2 \ dx + \int \frac{\eta^2 \psi_{L,n}^{2(\beta-1)} |\psi_n|^2}{x} \ dx \right). \quad (3.17) \]

Let \( \beta = \frac{2^*}{2} \), by the definition of \( \omega_{L,n} \) and (3.17), we rewrite the last inequality as

\[
\left( \int (\eta \psi_{L,n}^{(2^*-2)/2})^{2^*} \right)^{2/2^*} \leq C(N, 2) \left\{ \left( \int (\eta \psi_{L,n}^{(2^*-2)/2})^{2^*} \right)^{2/2^*} \right. \\
+ \left. \left( \int \eta \psi_{L,n}^{(2^*-2)/2} |\nabla \eta|^2 \ dx \right) \right\} \leq C(N, 2) \left\{ \left( \int (\eta \psi_{L,n}^{(2^*-2)/2})^{2^*} \right)^{2/2^*} \right. \\
+ \left. \left( \int \psi_{L,n}^{2^*} |\nabla \eta|^2 \ dx \right) \right\}.
\]

In view of \( \psi_n \to \psi \) in \( H^1(\mathbb{R}^N) \), for \( R \) large enough, we conclude that

\[ |\psi_n|^{2^*-2}_{2^*(|x| \geq R/2)} \leq \frac{1}{2C(N, 2)} \] uniformly in \( n \in \mathbb{N} \).

Hence we obtain

\[
\left( \int \psi_{L,n}^{(2^*-2)/2} \right)^{2/2^*} \leq C(N, 2) \int \psi_{L,n}^{2^*} |\nabla \eta|^2 \ dx \leq \frac{C}{r^2} \int |\psi_n|^{2^*} \ dx.
\]

Using the Fatou’s lemma in the variable \( L \), we have

\[ \psi_n \in L^{2^*/2}(|x| \geq R) \quad \text{for } R \quad \text{large enough.} \tag{3.18} \]

Next, we note that if \( \beta = 2^*(t-1)/2t \) with \( t = 2^* / 2(2^* - 2) \), then \( \beta > 1 \) and \( 2t/(t-1) < 2^* \).

Now suppose that \( \psi_n \in L^{2^*/(t-1)}(|x| \geq R - r) \) for some \( \beta \geq 1 \). Using the Hölder inequality with exponent \( t/(t-1) \) and \( t \), then (3.18) gives that

\[
|\omega_{L,n}|^{2^*}_{2^*} \leq C \beta^2 \left\{ \left( \int_{|x| \geq R-r} (\eta \psi_{2^*}^{2\beta})^{1/(t-1)} \ dx \right)^{t-1/t} \left( \int_{|x| \geq R-r} |\nabla \eta|^{2\beta} \ dx \right)^{1/t} \right. \\
+ \left. \frac{(R^N - (R-r)^N)^{1/t}}{r^2} \left( \int_{|x| \geq R-r} |\psi_n|^{2\beta (t-1)} \ dx \right)^{1-1/t} \right\} \leq C \beta^2 \left( 1 + \frac{R^{N/t}}{r^2} \right) \left( \int_{|x| \geq R-r} |\psi_n|^{2\beta (t-1)} \ dx \right)^{1-1/t}. \tag{3.19} \]

Letting \( L \to +\infty \) in (3.19), we obtain

\[ |\psi_n|^{2^*}_{2^* \beta (|x| \geq R)} \leq C \beta^2 \left( 1 + \frac{R^{N/t}}{r^2} \right) |\psi_n|^{2^*}_{2^* \beta (|x| \geq R-r)}. \]

If we set \( \chi := 2^*(t-1)/(2t) \), \( s := 2t/(t-1) \), then

\[ |\psi_n|_{\beta \chi (|x| \geq R)} \leq C^{1/\beta} \beta^{1/\beta} \left( 1 + \frac{R^{N/t}}{r^2} \right)^{1/(2\beta)} |\psi_n|_{\beta s (|x| \geq R-r)}. \tag{3.20} \]
Let \( \beta = \chi^m (m = 1, 2, \ldots) \), we obtain
\[
|\psi_n|_{X^{m+1,s}(|x| \geq R)} \leq C X^{-m} \chi \chi^{-m} \left( 1 + \frac{R^{N/t}}{r^2} \right)^{1/(2\beta)} |\psi_n|_{X^{m,s}(|x| \geq R-r)}.
\]

It is clear that \( 2 > N/t \). So if we take \( r_m = 2^{-(m+1)} R \), then (3.31) implies
\[
|\psi_n|_{X^{m+1,s}(|x| \geq R)} \leq C \psi_n |_{X^{m+1,s}(|x| \geq R-r_m+1)}
\]
\[
\leq C \sum_{i=1}^{m} \chi^{-i} \chi \sum_{i=1}^{m} \chi^{-i} \exp \left( \sum_{i=1}^{m} \frac{\ln(1 + 2^{2(i+1)})}{2\chi^i} \right) |\psi_n|_{X^{s}(|x| \geq R-r)}
\]
\[
\leq C |\psi_n|_{2^s(|x| \geq R/2)}.
\]

Letting \( m \to \infty \) in the last inequality, we get
\[
|\psi_n|_{L^{\infty}(|x| \geq R)} \leq C |\psi_n|_{2^s(|x| \geq R/2)}.
\] (3.21)

Using \( \psi_n \to \psi \) in \( H^1(\mathbb{R}^N) \) again, for any fixed \( a > 0 \), there exists \( R > 0 \) such that \( |\psi_n|_{L^{\infty}(|x| \geq R)} \leq a \) for all \( n \in \mathbb{N} \). Therefore, \( \lim_{|x| \to \infty} \psi_n(x) = 0 \) uniformly in \( n \).

To show that \( |\psi_n|_{L^{\infty}(\mathbb{R}^N)} < +\infty \), we need only show that for any \( x_0 \in \mathbb{N} \), there is a ball \( B_R(x_0) = \{ x \in \mathbb{R}^N : |x - x_0| \leq R \} \) such that \( |\psi_n|_{L^{\infty}(B_R(x_0))} < +\infty \). We can use the same arguments and take \( \eta \in C_c^{\infty}(\mathbb{R}^N) \), \( 0 \leq \eta \leq 1 \) with \( \eta(x) = 1 \) if \( |x - x_0| \leq \rho' \) and \( \eta(x) = 0 \) if \( |x - x_0| > \rho' \) and \( |\nabla \eta| \leq \frac{2}{\rho} \), to prove that
\[
|\psi_n|_{L^{\infty}(|x-x_0| \leq \rho')} \leq C |\psi_n|_{2(|x| \geq 2\rho')}.
\] (3.22)

From (3.21) and (3.22), using a standard covering argument it follows that
\[
|\psi_n|_{L^{\infty}(\mathbb{R}^N)} \leq C
\]
for some positive constant \( C \).

For the case \( N = 2 \), similar with the proof for the case \( N \geq 3 \), we also let \( z_{L,n} = \eta^2 \psi_{L,n} \) and \( \omega_{L,n} = \eta \psi_{L,n} \beta^{-1} \) with \( \beta > 1 \) to be determined later. Since \( 0 \leq \eta \leq 1 \) and \( \psi_{L,n} \leq L \), it yields that \( \eta^2 \psi_{L,n}^{2(\beta-1)} \leq L^2 (\beta-1) \psi_{L,n} \) and \( F_1' (\psi_n) \eta^2 \psi_{L,n}^{2(\beta-1)} \psi_n \in L^1(\mathbb{R}^N) \). Taking \( z_{L,n} \) as a test function, we have
\[
\int \eta^2 \psi_{L,n}^{2(\beta-1)} |\nabla \psi_n|^2 \ dx + \int (V(\epsilon_n x + \epsilon_n y_n) + 1) \eta^2 \psi_{L,n}^{2(\beta-1)} |\psi_n|^2 \ dx
\]
\[
+ \int F_1' (\psi_n) \eta^2 \psi_{L,n}^{2(\beta-1)} \psi_n \ dx
\]
\[
= -2 \int \eta \psi_{L,n}^{2(\beta-1)} \psi_n \nabla \psi_n \nabla \eta \ dx - 2(\beta - 1) \int \eta^2 \psi_{L,n}^{2(\beta-1)} \psi_n \nabla \psi_n \nabla \psi_{L,n} \ dx
\]
\[
+ \int G_2' (\epsilon_n x + \epsilon_n y_n, \eta) \eta^2 \psi_{L,n}^{2(\beta-1)} \psi_n \ dx.
\] (3.23)

From the definition of \( G_2 \), for any \( t \geq 0 \) and \( x \in \mathbb{R}^N \), we have that
\[
G_2' (x, t) \leq t + C t^{q-1},
\] (3.24)
where \( 2 < q < \infty \).
By (2.3), (3.23) and (3.24), we obtain that

\[
\int \eta^2 \psi_{L,n}^{2(\beta-1)} |\nabla \psi_n|^2 \, dx \leq 2 \int \eta \psi_{L,n}^{2(\beta-1)} \psi_n |\nabla \psi_n| |\nabla \eta| \, dx + C \int \eta^2 \psi_{L,n}^{2(\beta-1)} |\psi_n|^q \, dx.
\]  

(3.25)

For any \( \delta > 0 \), using Young’s inequality, we have from (3.25) that

\[
\int \eta^2 \psi_{L,n}^{2(\beta-1)} |\nabla \psi_n|^2 \, dx \leq 2\delta \int \eta^2 \psi_{L,n}^{2(\beta-1)} |\nabla \psi_n|^2 \, dx + 2C\delta \int \psi_{L,n}^{2(\beta-1)} |\psi_n|^2 |\nabla \eta|^2 \, dx \\
+ C \int_{\mathbb{R}^N} \eta^2 \psi_{L,n}^{2(\beta-1)} |\psi_n|^q \, dx.
\]

Choosing \( \delta \in (0, 1/4) \), it yields

\[
\int \eta^2 \psi_{L,n}^{2(\beta-1)} |\nabla \psi_n|^2 \, dx \leq C \int \psi_{L,n}^{2(\beta-1)} |\psi_n|^2 |\nabla \eta|^2 \, dx \\
+ C \int \eta^2 \psi_{L,n}^{2(\beta-1)} |\psi_n|^q \, dx.
\]  

(3.26)

On the other hand, by the Sobolev embedding,

\[
|\omega_{L,n}|_q^2 \leq C \beta^2 \left( \int \psi_{L,n}^{2(\beta-1)} |\psi_n|^2 |\nabla \eta|^2 \, dx + \int \eta^2 \psi_{L,n}^{2(\beta-1)} |\nabla \psi_n|^2 \, dx \right).
\]  

(3.27)

Using (3.26) and (3.27), we have

\[
|\omega_{L,n}|_q^2 \leq C \beta^2 \left( \int \psi_{L,n}^{2(\beta-1)} |\psi_n|^2 |\nabla \eta|^2 \, dx + \int \eta^2 \psi_{L,n}^{2(\beta-1)} |\psi_n|^q \, dx \right).
\]  

(3.28)

Let \( \beta = \frac{q}{2} \), by the definition of \( \omega_{L,n} \) and (3.17), we rewrite the last inequality as

\[
\left( \int (\eta \psi_n \psi_{L,n}^{(q-2)/2})^q \, dx \right)^{2/q} \leq C(2, 2) \left\{ \left( \int (\eta \psi_n \psi_{L,n}^{(q-2)/2})^q \, dx \right)^{2/q} \left( \int_{|x| \geq R-r} |\psi_n|^q \, dx \right)^{(q-2)/q} \right. \\
+ \left. \int \psi_{L,n}^{q-2} |\psi_n|^2 |\nabla \eta|^2 \, dx \right\} \\
\leq C(2, 2) \left\{ \left( \int (\eta \psi_n \psi_{L,n}^{(q-2)/2})^q \, dx \right)^{2/q} \left|\psi_n\right|_{L^q(\mathbb{R}^N \setminus B_R)}^{q-2} \right. \\
+ \left. \int \psi_{L,n}^{q-2} |\psi_n|^2 |\nabla \eta|^2 \, dx \right\}.
\]

In view of \( \psi_n \to \psi \) in \( H^1(\mathbb{R}^N) \), we have \( \psi_n \to \psi \) in \( L^q(\mathbb{R}^N) \). Thus, for \( R \) large enough, we conclude that

\[
\left|\psi_n\right|_{L^q(\mathbb{R}^N \setminus B_{R/2})}^{q-2} \leq \frac{1}{2C(2, 2)} \text{ uniformly in } n \in \mathbb{N}.
\]

Hence we obtain

\[
\left( \int_{|x| \geq R} (\psi_n \psi_{L,n}^{(q-2)/2})^q \, dx \right)^{2/q} \leq 2C(2, 2) \left( \int \psi_{L,n}^{q-2} |\psi_n|^2 |\nabla \eta|^2 \, dx + \int \eta^2 \psi_{L,n}^{2(\beta-1)} |\psi_n|^q \, dx \right) \\
\leq \frac{C}{r^2} \int |\psi_n|^q \, dx.
\]
Using the Fatou’s lemma in the variable $L$, we have
\[ \psi_n \in L^{2/(|x| \geq R)} \] for $R$ large enough. \hfill (3.29)

Next, we note that if $\beta = q(t - 1)/2t$ with $t = q^2/2(q - 2)$, then $\beta > 1$ and $2t/(t - 1) < q$. Now suppose that $\psi_n \in L^{2\beta/(t - 1)}(|x| \geq R - r)$ for some $\beta \geq 1$. Using the Hölder inequality with exponent $t/(t - 1)$ and $t$, then (3.28) gives that
\[
|\omega_{L,n}|^2 \leq C \beta^2 \left\{ \left( \int_{|x| \geq R - r} (|\psi_n|^2)^{t/(t - 1)} dx \right)^{1/t} \left( \int_{|x| \geq R - r} |\psi_n|^{(q - 2)t} dx \right)^{1/t} \right. \\
+ \left. \frac{(R^2 - (R - r)^2)^{1/t}}{r^2} \left( \int_{|x| \geq R - r} |\psi_n|^{2\beta t/(t - 1)} dx \right)^{1/t} \right\}
\leq C \beta^2 \left( 1 + \frac{R^2}{r^2} \right)^{1/(2\beta)} \left( \int_{|x| \geq R - r} |\psi_n|^{\beta t/(t - 1)} dx \right)^{1/t}.
\] \hfill (3.30)

Letting $L \to +\infty$ in (3.30), we obtain
\[
|\psi_n|^{2\beta}_{q(|x| \geq R)} \leq C \beta^2 \left( 1 + \frac{R^2}{r^2} \right) |\psi_n|^{2\beta}_{\beta t/(t - 1)(|x| \geq R - r)}.
\]

If we set $\chi := (q(t - 1)/(2t), s := 2t/(t - 1)$, then
\[
|\psi_n|_{\partial \chi \cdot s (|x| \geq R)} \leq C \beta^2 \beta^1 \beta^{1/\beta} \left( 1 + \frac{R^2}{r^2} \right)^{1/(2\beta)} |\psi_n|_{\beta t s (|x| \geq R - r)}.
\] \hfill (3.31)

Let $\beta = \chi^m (m = 1, 2, \ldots)$, we obtain
\[
|\psi_n|_{\chi^{m+1} s (|x| \geq R)} \leq C \chi^{-m} \chi^m \chi^{-m} \left( 1 + \frac{R^2}{r^2} \right)^{1/(2\beta)} |\psi_n|_{\chi^m s (|x| \geq R - r)}.
\]

It is clear that $2 > 2/t$. So if we take $r_m = 2^{-(m+1)} R$, then (3.31) implies
\[
|\psi_n|_{\chi^{m+1} s (|x| \geq R)} \leq |\psi_n|_{\chi^{m+1} s (|x| \geq R - r_{m+1})} \leq C \sum_{i=1}^m \sum_{i=1}^m \chi^{-i} \chi^{-m} \exp \left( \sum_{i=1}^m \frac{\ln(1 + 2^{2(i + 1)})}{2 \chi^i} \right) |\psi_n|_{\chi^i s (|x| \geq R - r_1)} \leq C |\psi_n|_{q(|x| \geq R/2)}.
\]

Letting $m \to \infty$ in the last inequality, we get
\[
|\psi_n|_{L^\infty (|x| \geq R)} \leq C |\psi_n|_{q(|x| \geq R/2)}.
\] \hfill (3.32)

Using $\psi_n \to \psi$ in $H^1(\mathbb{R}^2)$ again, for any fixed $a > 0$, there exists $R > 0$ such that $|\psi_n|_{L^\infty (|x| \geq R)} \leq a$ for all $n \in \mathbb{N}$. Therefore, $\lim_{|x| \to \infty} \psi_n(x) = 0$ uniformly in $n$.

Similarly, in order to show that $|\psi_n|_{L^\infty (\mathbb{R}^2)} \to +\infty$, we need only show that for any $x_0 \in \mathbb{R}^2$, there is a ball $B_R(x_0) = \{x \in \mathbb{R}^2 : |x - x_0| \leq R\}$ such that $|\psi_n|_{L^\infty (B_R(x_0))} \to +\infty$. We can use the same arguments and take $\eta \in C^\infty (\mathbb{R}^2)$, $0 \leq \eta \leq 1$ with $\eta(x) = 1$ if $|x - x_0| \leq \rho'$ and $\eta(x) = 0$ if $|x - x_0| > 2\rho'$ and $|\nabla \eta| \leq \frac{2}{\rho}$, to prove that
\[
|\psi_n|_{L^\infty (|x - x_0| \leq \rho')} \leq C |\psi_n|_{L^2(|x| \geq 2\rho')}.
\] \hfill (3.33)

From (3.32) and (3.33), using a standard covering argument it follows that
\[
|\psi_n|_{L^\infty (\mathbb{R}^2)} \leq C
\]
for some positive constant $C$.  

\[ \text{Springer} \]
4 Proof of Theorem 1.1

By Theorem 3.1, we know that problem (1.3) has a positive solution $u_\epsilon$ for all $\epsilon > 0$. On the other hand, by Lemma 3.9, there exists a sequence $(y_n) \subset \mathbb{R}^N$ with $\lim_{n \to \infty} \epsilon_n y_n = y_0$ and $V(y_0) = V_0$. Now, we can find $r > 0$, such that $B_r(y_n) \subset \Lambda$ for all $n \in N$. Therefore $B_r/\epsilon_n(y_n) \subset \Lambda_{\epsilon_n}$, $n \in N$. As a consequence

$$\mathbb{R}^N \setminus \Lambda_{\epsilon_n} \subset \mathbb{R}^N \setminus B_{r/\epsilon_n}(y_n) \quad \text{for any } n \in N.$$ 

By using Lemma 3.10, there exists $R > 0$ such that

$$\psi_n(x) < a_0 \quad \text{for } |x| > R, \ n \in N,$$

where $\psi_n(x) = u_{\epsilon_n}(x + y_n)$. Hence $u_{\epsilon_n} < a_0$ for any $x \in \mathbb{R}^N \setminus B_R(y_n)$ and $n \in N$. Then there exists $n_0 \in N$ such that for any $n \geq n_0$ and $r/\epsilon_n > R$ it holds

$$\mathbb{R}^N \setminus \Lambda_{\epsilon_n} \subset \mathbb{R}^N \setminus B_{r/\epsilon_n}(y_n) \subset \mathbb{R}^N \setminus B_R(y_n),$$

which gives $u_{\epsilon_n} < a_0$ for any $x \in \mathbb{R}^N \setminus \Lambda_{\epsilon_n}$ and $n \geq n_0$.

This means that there exists $\epsilon_0 > 0$, problem (1.3) has a positive solution $u_\epsilon$ for all $\epsilon \in (0, \epsilon_0)$. Taking $v_\epsilon(x) = u_\epsilon(x/\epsilon)$, we can infer that $v_\epsilon$ is a solution to problem $(P_\epsilon)$.

Finally, we study the behavior of the maximum points of $v_\epsilon(x)$. Take $\epsilon_n \to 0$ and $(u_{\epsilon_n})$ a sequence of solutions to problem (1.3). By the definition of $G_2$, there exists $\gamma \in (0, a_0)$ such that

$$G'_2(\epsilon x, t) t \leq l \epsilon^2, \quad \text{for all } x \in \mathbb{R}^N, \ 0 \leq t \leq \gamma.$$ 

Using a similar argument above, we can take $R > 0$ such that

$$\|u_{\epsilon_n}\|_{L^\infty(B_R(y_n))} < \gamma. \quad (4.1)$$

Up to a subsequence, we may also assume that

$$\|u_{\epsilon_n}\|_{L^\infty(B_R(y_n))} \geq \gamma. \quad (4.2)$$

Indeed, if $4.2$ does not hold, we have $\|u_{\epsilon_n}\|_{L^\infty(\mathbb{R}^N)} < \gamma$, and it follows from $I'_{\epsilon_n}(u_{\epsilon_n}) = 0$ that

$$\int (|\nabla u_{\epsilon_n}|^2 + (V_0 + 1)|u_{\epsilon_n}|^2)dx \leq \|u_{\epsilon_n}\|^2_{L^\infty} \leq \int G'_2(\epsilon_n x, u_{\epsilon_n}) u_{\epsilon_n} dx \leq l \int |u_{\epsilon_n}|^2 dx.$$ 

This fact shows $u_{\epsilon_n} \equiv 0$ which is a contradiction. Hence $4.2$ holds.

Taking into account $(4.1)$ and $(4.2)$, we can infer that the global maximum points $p_n$ of $u_{\epsilon_n}$ belongs to $B_R(y_n)$, that is $p_n = q_n + y_n$ for some $q_n \in B_R$. Recalling that the associated solution of problem $(P_\epsilon)$ is of form $v_n(x) = u_{\epsilon_n}(x/\epsilon)$, we can see that a maximum point $\eta_{\epsilon_n}$ of $\hat{\nu}_n$ is $\eta_{\epsilon_n} = \epsilon_n y_n + \epsilon_n q_n$. Since $q_n \in B_R$, $\epsilon_n y_n \to y_0$ and $V(y_0) = V_0$, from the continuity of $V$, we can conclude that

$$\lim_{n \to \infty} V(\eta_{\epsilon_n}) = V_0,$$

which concludes the proof of the theorem.
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