Markov constant and quantum instabilities

Edita Pelantová¹, Štěpán Starosta² and Miloslav Znojil³

¹ Faculty of Nuclear Sciences and Physical Engineering, Czech Technical University in Prague, Prague, Czech Republic
² Faculty of Information Technology, Czech Technical University in Prague, Prague, Czech Republic
³ Nuclear Physics Institute ASCR, Řež, Czech Republic

E-mail: stepan.starosta@fit.cvut.cz

Received 8 October 2015, revised 26 January 2016
Accepted for publication 4 February 2016
Published 1 March 2016

Abstract

For a qualitative analysis of spectra of certain two-dimensional rectangular-well quantum systems several rigorous methods of number theory are shown productive and useful. These methods (and, in particular, a generalization of the concept of Markov constant known in Diophantine approximation theory) are shown to provide a new mathematical insight in the phenomenologically relevant occurrence of anomalies in the spectra. Our results may inspire methodical innovations ranging from the description of the stability properties of metamaterials and of certain hiddenly unitary quantum evolution models up to the clarification of the mechanisms of occurrence of ghosts in quantum cosmology.

Keywords: renormalizable quantum theories with ghosts, Pais–Uhlenbeck model, singular spectra, square-well model, number theory analysis, physical applications, metamaterials

1. Introduction

The main mathematical inspiration of our present physics-oriented paper may be traced back to the theory of Diophantine approximations in which an important role is played by certain sets of real numbers possessing an accumulation point called Markov constant [1]. The related ideas and techniques (to be shortly outlined below) are transferred to an entirely different context. Briefly, we show that and how some of the results of number theory may appear applicable in an analysis of realistic quantum dynamics.

The sources of our phenomenological motivation are more diverse. Among them, a distinct place is taken by the problems of quantum stability which are older than the quantum theory itself. Their profound importance already became clear in the context of the Niels Bohr’s model of atom [2]. In this light one of the main achievements of the early quantum
theory may be seen precisely in the explanation of the well verified experimental observation that many quantum systems (like hydrogen atom, etc) are safely stable.

During the subsequent developments of the quantum theory, the rigorous mathematical foundation of the concept of quantum stability found its safe ground in the spectral theory of self-adjoint operators in Hilbert space [3]. Although it may sound like a paradox, a similar interpretation of the loss of quantum stability is much less developed at present. This does not imply that the systematic study of instabilities would be less important. The opposite is true because the majority of existing quantum systems ranging from elementary particles to atomic nuclei and molecules are unstable.

In this direction of study one could only feel discouraged by the fact that the existing theoretical descriptions of quantum instabilities require complicated mathematics, be it in quantum field theory, in statistical quantum physics or, last but not least, in the representations of quantum models using non-selfadjoint operators [4]. For this reason we believe that our present approach combining a sufficiently rigorous level of mathematics with a not too complicated exemplification of quantum systems might offer a fresh and innovative perspective to quantum physics and, in particular, to some of its stability and instability aspects.

It is certainly encouraging for us to notice that a combination of Diophantine analysis with phenomenological physics already appeared relevant in the context of study of certain stable quantum systems controlled by point interactions and living on rectangular lattices [5] or on hexagonal lattices [6], where typically, the band spectra may depend on certain number-theoretical characteristics of the system. In what follows, we intend to turn our attention from complicated quantum graphs to a maximally elementary and exactly solvable model in which the hyperbolic partial differential equation

$$\Box f(x, y) = \lambda f(x, y), \quad \Box = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}, \quad f|_{\partial R} = 0$$

(1)

is studied and in which the instability is immanently present, in a way to be discussed below, via the unboundedness of the spectrum from below.

In our model the eigenfunctions are required to satisfy the most common Dirichlet boundary conditions, i.e., they are expected to vanish along the boundary of the two-dimensional rectangle

$$R = \{(x, y) : 0 \leq x \leq a, \ 0 \leq y \leq b\}.$$  

(2)

In sections 2–5 we describe and prove rigorous results of analysis of such a model. After a systematic presentation of these mathematical observations we return, in sections 6 and 7, to the problem of their various potential connections with physics. We also list there a few not entirely artificial samples of placing the Klein–Gordon-resembling equation (1) into a broader phenomenological context.

2. Spectral problem

2.1. Separation of variables

Our present analysis is fully concentrated upon the properties of spectra of hyperbolic partial differential operators $\Box$ of equations (1) + (2) which act upon twice differentiable functions $f(x, y)$ of two real variables. Setting $f(x, y) = g(x)h(y)$ we find that the eigenvalue problem is easily solvable by separation of variables, i.e., that there exist constants $C$ and $D$ such that
The solution of the corresponding ordinary differential equation for unknown \( g(x) \) (and, \textit{mutatis mutandis}, for \( h(y) \)) yields

\[
g(x) = \alpha \sin(\sqrt{-C}x) + \beta \cos(\sqrt{-C}a)
\]

for \( C < 0 \)

\[
g(x) = \alpha x + \beta
\]

for \( C = 0 \) and

\[
g(x) = \alpha e^{-\sqrt{C}x} + \beta e^{-\sqrt{C}a}
\]

for \( C > 0 \). Under our Dirichlet boundary conditions, a non-zero solution is obtained only for \( C < 0 \). We obtain

\[
a \sqrt{-C} = m\pi
\]

for \( m \in \mathbb{Z} \). Analogously, we obtain

\[
b \sqrt{-D} = k\pi
\]

for \( k \in \mathbb{Z} \). Since \( \lambda = C - D \), we have, finally

\[
\lambda_{k,m} = \frac{k^2 \pi^2}{a^2} - \frac{m^2 \pi^2}{b^2} = \frac{\pi^2 m^2}{a^2} \left( \frac{k^2}{m^2} - \frac{a^2}{b^2} \right) = \frac{\pi^2 m^2}{a^2} \left( \frac{k}{m} - \frac{a}{b} \right) \left( \frac{k}{m} + \frac{a}{b} \right)
\]

for all \( k, m \in \mathbb{Z} \). Thus, the spectrum equals the closure of the set of all \( \lambda_{k,m} \):

\[
\sigma(\Box) = \{ \lambda_{k,m} : k, m \in \mathbb{Z} \}.
\]

2.2. The number theory approach

Up to a multiplicative factor, the singular part of the spectrum \( \sigma(\Box) \) coincides with the set

\[
S(\alpha) = \text{set of all accumulation points of } \left\{ m^2 \left( \frac{k}{m} - \alpha \right) : k, m \in \mathbb{Z} \right\},
\]

where the ratio \( \alpha = a/b \) is a dynamical parameter of the model. The structure of such sets is well understood in the theory of Diophantine approximations. In particular, the smallest accumulation point of the displayed set—the so-called Markov constant of \( \alpha \)—is in the center of interest of many mathematicians.

This observation is in fact a methodical starting point of our present paper. In essence, our analysis of stability/instability issues are mainly inspired by the results of the existing number-theory literature on Markov constant.

3. Simple properties of \( S(\alpha) \)

Assume \( \alpha \in \mathbb{R} \). As the set \( \mathbb{Z}^2 \) is countable, the set \( \left\{ m^2 \left( \frac{k}{m} - \alpha \right) : k, m \in \mathbb{Z} \right\} \) can be viewed as the range of a real sequence. Let us rephrase the definition of \( S(\alpha) \): a number \( x \) belongs to \( S(\alpha) \) if there exist strictly monotone sequences of integers \( (k_n) \) and \( (m_n) \) such that

\[
x = \lim_{n \to \infty} m_n^2 \left( \frac{k_n}{m_n} - \alpha \right).
\]
We list several simple properties of $S(\alpha)$.

(1) Since the set of accumulation points of any real sequence is closed, the set $S(\alpha)$ is a topologically closed subset of $\mathbb{R}$.

(2) $S(\alpha)$ is closed under multiplication by $z^2$ for each $z \in \mathbb{Z}$.

**Proof.** If $x \in S(\alpha)$, i.e., $m_n^2\left(\frac{k_n}{m_n} - \alpha\right) \to x$, then $(m_nz)^2\left(\frac{k_nz}{m_nz} - \alpha\right) \to xz^2$, thus $xz^2 \in S(\alpha)$. □

(3) If $\alpha \in \mathbb{Q}$, then $S(\alpha)$ is empty.

**Proof.** If $\alpha = \frac{r}{s}$ with $r, s \in \mathbb{Z}$, then $m^2\left(\frac{k}{m} - \frac{r}{s}\right) = \frac{t}{s}$ for some $t \in \mathbb{Z}$. It means that 
\[
\{m^2\left(\frac{k}{m} - \alpha\right) : k, m \in \mathbb{Z}\}
\] is a subset of the discrete set $\frac{1}{s}\mathbb{Z}$. □

(4) If $\alpha \notin \mathbb{Q}$, then $S(\alpha)$ has at least one element in the interval $[-1, 1]$.

**Proof.** According to Dirichlet’s theorem, there exist infinitely many rational numbers $\frac{k}{m}$ such that $\left|\frac{k}{m} - \alpha\right| < \frac{1}{m^2}$. □

In order to present another remarkable property of $S(\alpha)$ we exploit simple rational transformations connected with
\[
G = \{g \in \mathbb{Z}^{2 \times 2} : \det(g) \neq 0\} \quad \text{and} \quad \text{SL}_2(\mathbb{Z}) = \{g \in G : \det(g) = 1\}.
\]
Note that $G$ is a monoid, whereas $\text{SL}_2(\mathbb{Z})$ is a group. We define the action of
\[
g = \begin{pmatrix} c & d \\ e & f \end{pmatrix} \in G \quad \text{on the set} \quad \mathbb{R} \quad \text{by} \quad \alpha \mapsto g\alpha = \frac{ca + d}{ea + f}.
\]

**Proposition 1.** Let $\alpha \in \mathbb{R}$ and $g \in G$. We have
\[
\det(g)S(\alpha) \subset S(g\alpha).
\]

In particular, $S(g\alpha) = S(\alpha)$ if $g \in \text{SL}_2(\mathbb{Z})$.

**Proof.** Let $g = \begin{pmatrix} c & d \\ e & f \end{pmatrix} \in G$. Let $x \in S(\alpha)$ and let $(k_n)$ and $(m_n)$ be sequences such that
\[
m_n^2\left(\frac{k_n}{m_n} - \alpha\right) \to x.
\]
We set
\[
k'_n = ck_n + dm_n \quad \text{and} \quad m'_n = ek_n + fm_n.
\]
We obtain
\[
\left(\frac{k'_n}{m'_n} - g\alpha\right)m'^2_n = \left(\frac{ck_n + dm_n}{ek_n + fm_n} - \frac{ca + d}{ea + f}\right)^2(ek_n + fm_n)^2
\]
\[
= \frac{k_n(cf - de) - cm_n(cf - de)}{(ea + f)(ek_n + fm_n)}(ek_n + fm_n)^2
\]
as $\frac{k_n}{m_n} \to \alpha$. It means that $\det(g)x$ belongs to $S(g\alpha)$.

If $\det g = 1$, then $S(\alpha) \subset S(\alpha)$ and $g^{-1} \in \text{SL}_2(\mathbb{Z})$ as well. Therefore, $S(g\alpha) \subset S(g^{-1}\alpha) = S(\alpha)$, too. \hfill \Box

In the sequel, $[x]$ stands for the integer part of $x$, i.e., the largest integer $n$ such that $n \leq x$. Since $\alpha - [\alpha] = g\alpha$, where $g = \left( \begin{array}{c} 1 & \lfloor \alpha \rfloor \\ 0 & 1 \end{array} \right) \in \text{SL}_2(\mathbb{Z})$, the previous proposition immediately implies the following corollary.

**Corollary 2.** For any $\alpha \in \mathbb{R}$ we have $S(\alpha) = S(\alpha - [\alpha])$.

Note that $g\alpha = \alpha$ for any $g = \left( \begin{array}{c} z & 0 \\ 0 & z \end{array} \right)$ with $z \in \mathbb{Z}$. Proposition 1 implies $z^2S(\alpha) \subset S(\alpha)$, as already observed.

### 4. Continued fractions and convergents

The theory of continued fractions plays a crucial role in Diophantine approximation, i.e., in approximation of an irrational number by a rational number. The definition of $S(\alpha)$ indicates that the quality of an approximation of $\alpha$ by fractions $\frac{k}{m}$ governs the behavior of $S(\alpha)$. The continued fraction of an irrational number $x$ is a coding of the orbit of $x$ under a transformation $T$ defined by

$$
T : \mathbb{R} \setminus \mathbb{Q} \to (1, +\infty) \setminus \mathbb{Q} \quad \text{and} \quad T(x) = \frac{1}{x - [x]}.
$$

**Definition 3.** Let $x \in \mathbb{R} \setminus \mathbb{Q}$. The continued fraction of $x$ is the infinite sequence of integers $[a_0, a_1, a_2, a_3, ...]$, where

$$a_i = \lceil T^i(x) \rceil \quad \text{for all} \quad i = 0, 1, 2, 3, ... .$$

Clearly, for all $i \geq 1$ the coefficient $a_i$ is a positive integer. Only the coefficient $a_0$ takes values in the whole range of integers.

If $\alpha$ is an irrational number, then $T(\alpha) = g\alpha$ with $g = \left( \begin{array}{c} 0 & 1 \\ 1 & -\lfloor \alpha \rfloor \end{array} \right) \in G$. As $\det(g) = -1$, proposition 1 implies $S(T(\alpha)) = -S(\alpha)$. A number $x$ is usually identified with its continued fraction and we also write $x = [a_0, a_1, a_2, a_3, ...]$. Using this convention, the previous fact can be generalized as follows.
Corollary 4. Let \([a_0, a_1, a_2, a_3, \ldots]\) be a continued fraction. We have
\[S([a_{n+k}, a_{n+1+k}, a_{n+2+k}, \ldots]) = (-1)^k S([a_n, a_{n+1}, a_{n+2}, \ldots])\] for any \(k, n \in \mathbb{N}\).

The knowledge of the continued fraction of \(x\) allows us to find the best rational approximations, in a certain sense, of the number \(x\). To describe these approximations, we use the following notation: \([a_0, a_1, a_2, a_3, \ldots, a_n]\), where \(a_0 \in \mathbb{Z}\) and \(a_1, \ldots, a_n \in \mathbb{N} \setminus \{0\}\), denotes the fraction
\[
a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_n}}}}.
\]
The number \(a_i\) is said to be the \(i\)th partial quotient of \(x\).

Definition 5. Let \(x\) be an irrational number, \([a_0, a_1, a_2, a_3, \ldots]\) its continued fraction and let \(N \in \mathbb{N}\). Let \(p_N \in \mathbb{Z}\) and \(q_N \in \mathbb{N} \setminus \{0\}\) denote coprime numbers such that \(\frac{p_N}{q_N} = [a_0, a_1, a_2, a_3, \ldots, a_N]\). The fraction \(\frac{p_N}{q_N}\) is called the \(N\)th-convergent of \(x\).

We list the relevant properties of convergents of an irrational number \(\alpha\). They can be found in any textbook of number theory, for example [1].

1. We have \(p_0 = a_0, q_0 = 1, p_1 = a_0a_1 + 1\) and \(q_1 = a_1\). For any \(N \in \mathbb{N}\), we have
\[
p_{N+1} = a_{N+1}p_N + p_{N-1}\quad \text{and} \quad q_{N+1} = a_{N+1}q_N + q_{N-1}.
\]

2. For \(N \in \mathbb{N}\), set \(\alpha_{N+1} = [a_{N+1}, a_{N+2}, a_{N+3}, \ldots]\). We have
\[
\frac{p_N}{q_N} - \alpha = \frac{(-1)^{N+1}}{q_N(\alpha_{N+1}q_N + q_{N-1})}\quad \text{and in particular} \quad \left|\frac{p_N}{q_N} - \alpha\right| < \frac{1}{q_N^2}.
\]

3. For \(N \in \mathbb{N}\) and \(a \in \mathbb{Z}\) satisfying \(1 \leq a \leq a_{N+1} - 1\) we have
\[
\frac{ap_N + p_{N-1}}{aq_N + q_{N-1}} - \alpha = \frac{(-1)^{N+1}(\alpha_{N+1} - a)}{(aq_N + q_{N-1})}\left(\alpha_{N+1}q_N + q_{N-1}\right).
\]

These rational approximations are known as secondary convergents of \(\alpha\).

Corollary 6. Let \(\alpha\) be an irrational number and \(I\) be an interval. There exists \(\beta \in I\) such that \(S(\alpha) = S(\beta)\).

Proof. Without loss of generality, let \(I\) be an open interval and \(\gamma \in I\) be an irrational number. Let \([a_0, a_1, a_2, \ldots]\) and \([c_0, c_1, c_2, \ldots]\) be continued fractions of \(\alpha\) and \(\gamma\), respectively. Find \(\varepsilon > 0\) such that \((\gamma - 2\varepsilon, \gamma + 2\varepsilon) \subseteq I\). In virtue of (4) one can find an integer \(N\) such that the \(N\)th-convergent \(\frac{p_N}{q_N}\) of \(\gamma\) satisfies
\[
\left|\gamma - \frac{p_N}{q_N}\right| < \frac{1}{q_N^2} < \varepsilon.
\]
Define
\[ \beta = [c_0, c_1, \ldots, c_N, a_{N+1}, a_{N+2}, \ldots]. \]

As the \( N \)th-convergents of \( \beta \) and \( \gamma \) coincide and due to (4), we have \( |\gamma - \beta| < \frac{2}{q_Nq_{N+1}^2} < 2\varepsilon \) and thus \( \beta \in I \). Corollary 4 implies that the sets \( S(\alpha) \) and \( S(\beta) \) coincide as well. \( \square \)

**Theorem 7.** Let \( \alpha \) be an irrational number and \( \left( \frac{a_k}{b_k} \right)_{k \in \mathbb{N}} \) be the sequence of its convergents. If \( x \) belongs to \( S(\alpha) \cap \left( -\frac{1}{2}, \frac{1}{2} \right) \), then \( x \) is an accumulation point of the sequence

\[
\left( \frac{p_N}{q_N} - \alpha \right)_{N \in \mathbb{N}}.
\]

**Proof.** The theorem is a direct consequence of Legendre’s theorem (see for instance [1], theorem 5.12): Let \( \alpha \) be an irrational number and \( \frac{p}{q} \in \mathbb{Q} \). If \( \left| \frac{p}{q} - \alpha \right| < \frac{1}{2q^2} \), then \( \frac{p}{q} \) is a convergent of \( \alpha \). \( \square \)

Therefore, we start to investigate the accumulation points of the sequence (6).

**Lemma 8.** Let \( \alpha \) be an irrational number and \( \left( \frac{a_k}{b_k} \right)_{k \in \mathbb{N}} \) be the sequence of its convergents. For any \( N \in \mathbb{N} \) we have

\[ q_N^2 \left( \frac{p_N}{q_N} - \alpha \right) = (-1)^{N+1}(a_{N+1}, a_{N+2}, \ldots) + [0, a_N, a_{N-1}, \ldots, a_1]^{-1}. \]

In particular, for any \( N \in \mathbb{N} \)

\[ \frac{1}{2 + a_{N+1}} < \left| q_N^2 \left( \frac{p_N}{q_N} - \alpha \right) \right| < \frac{1}{a_{N+1}}. \]

**Proof.** Using (4) we obtain

\[ q_N^2 \left( \frac{p_N}{q_N} - \alpha \right) = \frac{(-1)^{N+1}}{a_{N+1} + \frac{q_{N-1}}{q_N}}. \]

By definition \( a_{N+1} = [a_{N+1}, a_{N+2}, \ldots] \). It remains to show that \( \frac{q_{N-1}}{q_N} = [0, a_N, a_{N-1}, \ldots, a_1] \).

We exploit the recurrent relation (3) for \( (q_k) \). We proceed by induction:

If \( N = 1 \), then \( q_0 = 1 \) and \( q_1 = a_1 \). Clearly \( \frac{q_0}{q_1} = \frac{1}{a_1} = [0, a_1] \).

If \( N > 1 \), then

\[ q_{N-1} = a_Nq_{N-1} + q_{N-2} = \frac{q_{N-1}}{q_N} = \frac{1}{a_{N+1} + \frac{q_{N-2}}{q_{N-1}}}. \]

The number \( \beta \in (0, 1) \) has its continued fraction in the form \([0, b_1, b_2, \ldots]\). If \( 1 \leq B \in \mathbb{Z} \), then the algorithm for construction of continued fraction assigns to the number \( \frac{1}{B + \beta} \) the continued fraction \([0, B, b_1, b_2, \ldots]\). We apply this rule and the induction assumption to (7) with \( B = a_N \) and \( \beta = \frac{q_{N-1}}{q_N} = [0, a_{N-1}, a_{N-2}, \ldots, a_1] \). \( \square \)
4.1. Spectra of quadratic numbers

A famous theorem of Lagrange says that an irrational number \( \alpha \) is a root of the quadratic polynomial \( A x^2 + B x + C \) with integer coefficients \( A, B, C \) if and only if the continued fraction of \( \alpha \) is eventually periodic, i.e., \( \alpha = [a_0, a_1, \ldots, a_v, (a_{v+1}, \ldots, a_{v+z})^\infty] \), where \( v^\infty \) denotes the infinite string formed by the repetition of the finite string \( v \).

Theorem 9. Let \( \alpha \) be a quadratic number and \( \left( \frac{p_n}{q_n} \right) \) be the sequence of its convergents. Let \( \ell \) be the smallest period of the repeating part of the continued fraction of \( \alpha \). The sequence \( \left( q_n \left( \frac{p_n}{q_n} - \alpha \right) \right)_{n \in \mathbb{N}} \) has at most \( \ell \) accumulation points if \( \ell \) is even; \( \ell^2 \) accumulation points if \( \ell \) is odd.

Proof. According to corollary 4 we can assume that the continued fraction of \( \alpha \) is purely periodic, i.e., \( \alpha = [a_0, a_1, \ldots, a_{v-1}, a_v, (a_{v+1}, \ldots, a_{v+z})^\infty] \), for some \( v > 0 \) and that the first digit satisfies \( a_v = \max \{a_0, a_1, \ldots, a_{v-1}\} \). Let \( D \) denote the set of the accumulation points of \( \left( q_n \left( \frac{p_n}{q_n} - \alpha \right) \right)_{n \in \mathbb{N}} \).

Suppose \( \ell \) is even. By lemma 8 and since \( \alpha = [a_0, a_1, \ldots, a_{v-1}, a_v, (a_{v+1}, \ldots, a_{v+z})^\infty] \), it follows that all the elements of \( D \) are the limit-points of the sequences \( \left( c_j^{(k)} \right)_{k \in \mathbb{N}} \), where

\[
c_j^{(k)} = (-1)^{j+k-1} \left( [a_v, a_{v+1}, \ldots] + [0, a_{j+k-1}, \ldots, a_1] \right)^{-1}
\]

for each \( j \) with \( 0 \leq j < \ell \). As \( \ell \) is even, the term \( (-1)^{j+k-1} \) equals \( (-1)^{j-1} \) and a limit exists. Thus, \#\( D \leq \ell \).

If \( \ell \) is odd, we define the number \( c_j^{(k)} \) for \( 0 \leq j < 2\ell \) in the same way and the elements of \( D \) are exactly the limit-points of the sequences \( \left( c_j^{(k)} \right)_{k \in \mathbb{N}} \). The term \( (-1)^{j+k-1} \) in the expression of \( c_j^{(k)} \) equals again \( (-1)^{j-1} \) and a limit exists for all \( j \). Thus, \#\( D \leq 2\ell \).

If \( a_0 = 1 \), then \( \alpha = [1^\infty] \), i.e., it is the golden ratio. We have

\[
\lim_{k \to +\infty} c_0^{(k)} = -((1^\infty) + [0, 1^\infty])^{-1} = -\left( \frac{1 + \sqrt{5}}{2} + \frac{2}{1 + \sqrt{5}} \right)^{-1} = -\frac{1}{\sqrt{5}} \geq -\frac{1}{2}.
\]

Thus, in this case, \( D \cap \left( -\frac{1}{2}, \frac{1}{2} \right) \) is not empty.

If \( a_0 = 2 \), then

\[
|c_0^{(k)}| = |[2, a_{k+2}, \ldots] + [0, a_{k+1}, \ldots, a_1]|^{-1}
\]

\[
= |2 + [0, a_{k+2}, \ldots] + [0, a_{k+1}, \ldots, a_1]|^{-1} \leq \frac{1}{2}.
\]

It implies that \( D \cap \left( -\frac{1}{2}, \frac{1}{2} \right) \) is not empty. \( \square \)

We add some remarks on the last theorem. The following observation follows from the last proof: if \( \ell \) is odd, then \( D \) is symmetric around 0.

Let \( \eta_j^{(k)} = [a_j, \ldots, a_{j+\ell-1}^\infty] \). The number \( \eta_j^{(k)} \) is a reduced quadratic surd and its conjugate \( \bar{\eta}_j^{(k)} \) satisfies
Therefore,
\[
\lim_{k \to +\infty} c_{2k}^{(j)} = \frac{(-1)^{j-1}}{\eta^{(j)} - \eta^{(j)}}.
\]
(8)

As follows from the last proof, the bound of theorem 9 is tight. On the other hand, there exist quadratic numbers such that the bound is not attained. It suffices to set \( \alpha = [(1, 2, 1, 1)^{\infty}] \). We have
\[
[(1, 2, 1, 1)^{2}] = \frac{2}{5} \sqrt{6} + \frac{2}{5} \quad \text{and}
\]
\[
[(1, 1, 1, 2)^{2}] = \frac{2}{5} \sqrt{6} + \frac{3}{5}.
\]
Using (8) we obtain \( \#D < \ell = 4 \). In fact, \( D = \left\{-\frac{5}{4} \sqrt{6}, \frac{1}{4} \sqrt{6}, \frac{3}{4} \sqrt{6}\right\} \).

5. Well and badly approximable numbers

The search for the best rational approximation of irrational numbers motivates the notion of Markov constant.

**Definition 10.** Let \( \alpha \) be an irrational number. The number
\[
\mu(\alpha) = \inf \left\{ c > 0 : \left| \alpha - \frac{k}{m} \right| < \frac{c}{m^2} \text{ has infinitely many solutions } k, m \in \mathbb{Z} \right\}
\]
is the *Markov constant of* \( \alpha \).

The number \( \alpha \) is said to be *well approximable* if \( \mu(\alpha) = 0 \) and *badly approximable* otherwise.

We give several comments on the value \( \mu(\alpha) \):

(1) Theorem of Hurwitz implies \( \mu(\alpha) \leq \frac{1}{\sqrt{5}} \) for any irrational real number \( \alpha \).

(2) A pair \((k, m)\) which is a solution of \( \left| \alpha - \frac{k}{m} \right| < \frac{c}{m^2} \) with \( c \leq \frac{1}{\sqrt{5}} \) satisfies \( k = \lfloor m\alpha \rfloor \), where we use the notation \( \lfloor x \rfloor = \min \{|x - n| : n \in \mathbb{Z}\} \). Therefore
\[
\mu(\alpha) = \lim_{m \to +\infty} \inf m\|m\alpha\| \quad \text{and} \quad \mu(\alpha) = \min \mathcal{S}(\alpha)
\]
as the set \( \mathcal{S}(\alpha) \) is topologically closed.

(3) Due to the inclusion \( \det(g)\mathcal{S}(\alpha) \subset \mathcal{S}(g\alpha) \) for \( g \in G \) we can write
\[
|\det(g)| \mu(\alpha) \geq \mu(g\alpha).
\]

(4) The inequality in lemma 8 implies
\[
\mu(\alpha) = 0 \iff (a_N) \text{ is not bounded} \iff 0 \in \mathcal{S}(\alpha).
\]
In other words, an irrational number \( \alpha \) is well approximable if and only if the sequence \((a_N)\) of its partial quotients is bounded.
5.1. Badly approximable numbers

As noted above, quadratic irrational numbers serve as an example of badly approximable
numbers. The spectrum \( \mathcal{S}(\alpha) \) of such a number has only finite number of elements in the interval \((-\frac{1}{2}, \frac{1}{2})\). Theorems 11 and 13 give two examples of spectra of badly approximable
numbers of different kinds.

**Theorem 11.** There exists an irrational number \( \alpha \) such that
\( \mathcal{S}(\alpha) = (-\infty, -\varepsilon] \cup [\varepsilon, +\infty) \), where \( \varepsilon = \frac{\sqrt{2}}{8} \approx 0.18 \).

We first recall that the natural order on \( \mathbb{R} \) is represented by an alternate order in continued
fractions. More precisely, let \( x, y \) be two irrational numbers with the continued fractions
\([x_0, x_1, \ldots] \) and \([y_0, y_1, \ldots] \), respectively. Set \( k = \min \{i \in \mathbb{N} : x_i \neq y_i\} \). We have \( x < y \) if and
only if
\[ (k \text{ is even and } x_k < y_k) \text{ or } (k \text{ is odd and } x_k > y_k). \]
To study the numbers with bounded partial quotients we define the following sets:
\[ F(r) = \{ [t, a_1, a_2, \ldots] : t \in \mathbb{Z}, 1 \leq a_i \leq r \} \]
and
\[ F_0(r) = \{ [0, a_1, a_2, \ldots] : 1 \leq a_i \leq r \}. \]
These sets are ‘sparse’ and they are Cantor sets: perfect sets that are nowhere dense (see for
instance \([7]\)). For example, the Hausdorff dimension of \( F(2) \) satisfies
\[ 0.44 < \dim H(F(2)) < 0.66 \] (see example 10.2 in \([8]\)). Taking into account the alternate
order, the maximum and minimum elements of \( F_0(r) \) can be simply determined. Thus,
\[ \max F_0(r) = [0, 1, r, 1, r, 1, r, 1, \ldots] \text{ and } \min F_0(r) = [0, r, 1, r, 1, r, 1, \ldots]. \] A crucial
result which enables us to prove theorem 11 is due to \([9]\) (see also \([7]\)):
\[ F(4) + F(4) = \mathbb{R}. \] (9)
It is worth mentioning that \( r = 4 \) is the least integer for which \( F(r) + F(r) = \mathbb{R}, \) i.e., in
particular, \( F(3) + F(3) = \mathbb{R} \) (see \([10]\)). Applying Theorem 2.2 and Lemma 4.2 of \([7]\) we
obtain the following modification of (9):
\[ F_0(4) + F_0(4) = [2 \min F_0(4), 2 \max F_0(4)] = [\sqrt{2} - 1, 4(\sqrt{2} - 1)]. \] (10)
We use the last equality to construct the number \( \alpha \) for the proof of theorem 11. The
construction is based on the following observation.

**Lemma 12.** Let \( \mathbf{a} = a_0a_1a_2\ldots \) be an infinite word over the alphabet \( \mathcal{A} = \{1, 2, \ldots, r\} \)
such that any finite string \( w_1w_2\ldots w_k \) over the alphabet \( \mathcal{A} \) occurs in \( \mathbf{a}, \) i.e., there exists index \( n \in \mathbb{N} \)
such that \( a_n\ldots a_{n+k-1} = w_1w_2\ldots w_k. \) Any number \( z \in \mathcal{A} + F_0(r) + F_0(r) \) is an
accumulation point of the sequence \( (S_{2N}) \) and the sequence \( (S_{2N+1}) \) with
\[ S_N = [a_{N+1}, a_{N+2}, \ldots] + [0, a_N, a_{N-1}, \ldots, a_1]. \] (11)

**Proof.** Let \( x = [0, x_1, x_2, x_3, \ldots], y = [0, y_1, y_2, y_3, \ldots] \in F_0(r) \) and \( b \in \mathcal{A}. \) For any string
\( w_1w_2\ldots w_k \) there exist infinitely many finite strings \( u_1u_2\ldots u_{k-1}u_b \) such that \( w_1w_2\ldots w_k \) is a
prefix and a suffix of \( u_1u_2\ldots u_{k-1}u_b. \) According to our assumptions each of them occurs at
least once in \( \mathbf{a}. \) It means that any string \( w_1w_2\ldots w_k \) occurs in \( \mathbf{a} \) infinitely many times on both
odd and even positions. In particular, for any \( n \) there exists infinitely many odd and infinitely many even indices \( N \) such that
\[
a_{N-n+1} \cdots a_N a_{N+1} \cdots a_{N+n} = x_1 y_1 y_2 \cdots y_{n-2} y_{n-1}.
\]
Obviously, the number \( S_N \) given by (11) equals
\[
b + [0, y_1, y_2, \ldots, y_{n-1}, a_{N+n}, a_{N+n+1}, \ldots] + [0, x_1, x_2, \ldots, x_n].
\]
As \( b + y + x \) is the limit of the previous sequence, it is an accumulation point of the sequence (11).

We can complete the proof of theorem 11.

**Proof of theorem 11.** We construct an infinite word \( a \) with letters in \( \{1, 2, 3, 4\} \) satisfying the assumptions of lemma 12. We define a sequence \( (u_n)_{n=0}^{\infty} \) recursively as follows: \( u_0 \) is the empty word and \( u_n = u_{n-1} v_n \), where \( v_n \) is the word which the concatenation of all words over \( \{1, 2, 3, 4\} \) of length \( n \) ordered lexicographically. We have
\[
u_0 = 1 2 3 4 \quad \text{and} \quad u_2 = 1 2 3 4 11 12 13 14 21 22 23 24 31 32 33 34 41 42 43 44.
\]
As \( u_{n-1} \) is a prefix of \( u_n \), we can set \( a \) to be the unique infinite word which has a prefix \( u_n \) for any \( n \in \mathbb{N} \). One can easily see that \( a \) satisfies the assumptions of lemma 12.

Let \( \alpha \) be the number with the continued fraction \([0, a_1, a_2, a_3, \ldots]\), where \( a_1 a_2 a_3 \ldots = a \). Combining lemmas 8 and 12 and the equality (10) we obtain that \( \frac{1}{z} - \frac{1}{z^2} \) belong to \( S(\alpha) \) for any \( z \in [b + \sqrt{2} - 1, b + 4(\sqrt{2} - 1)] \) with \( b \in \{1, 2, 3, 4\} \). Overall, we obtain
\[
[-\frac{1}{\sqrt{2}}, -\frac{1}{4\sqrt{2}}] \cup \left[\frac{1}{4\sqrt{2}}, \frac{1}{\sqrt{2}}\right] \subseteq S(\alpha).
\]
The property that \( S(\alpha) \) is closed under multiplication by \( z^2 \) for each positive integer \( z \), in particular under multiplication by 4, already proves theorem 11.

**Theorem 13.** There exists an irrational number \( \alpha \) such that the Hausdorff dimension of \( S(\alpha) \cap \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \) is positive but less than 1. In particular, \( S(\alpha) \cap \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \) is an uncountable set and its Lebesgue measure is 0.

**Proof.** Let \( a \) be an infinite word with letters in \( \{4, 5\} \) such that it contains any finite string over \( \{4, 5\} \) infinitely many times. A word with such properties can be constructed in the same way as in the proof of theorem 11.

In accordance with the previous notation we set
\[
F_0(\{4, 5\}) = \{[0, a_1, a_2, \ldots] : a_i \in \{4, 5\}\}.
\]
To simplify, we write \( F = F_0(\{4, 5\}) \). Theorem 1.2 in [7] implies that
\[
\dim_H (F + F) \geq 0.263 \ldots
\]
To obtain an upper bound on the Hausdorff dimension of \( F + F \), we first give a construction of \( F \). Let \( I \) denote the interval \( I = [\min F, \max F] \). Clearly, \( F \subseteq I \).
For both letters \( z = 4 \) and \( z = 5 \) we define \( f_z : I \rightarrow I \) as follows:

\[
f_z(x) = \frac{1}{z + x}
\]

for all \( x \in I \).

Using the mean value theorem, one can easily derive that

\[
\left| \frac{f_z(x) - f_z(y)}{|x - y|} \right| \leq \max_{\xi \in I} |f'(\xi)| \leq L = \frac{1}{(\min F + 4)^2}
\]

for all \( x, y \in I, x \neq y \). Thus, the mappings \( f_4 \) and \( f_5 \) are contractive and one can see that \( F \) is the fixed point of the iterated function system generated by these mappings. In other words, we have

\[
F = \lim_{n \to +\infty} Z_n \quad \text{with} \quad Z_n = \bigcup_{a_1, a_2, \ldots, a_n \in \{4, 5\}} f_{a_1} f_{a_2} \cdots f_{a_n}(I).
\]

Let us stress that \( \lim_{n \to +\infty} \) on the previous row is defined via the Hausdorff metric on the space of compact subsets of \( \mathbb{R} \).

Let \( n \in \mathbb{N} \). It follows that there exists a covering of the set \( Z_n \) consisting of \( 2^n \) intervals of length at most \( |I| \cdot L^n \). Similarly, the set \( Z_n + Z_n \) can be covered by \( 4^n \) intervals of length at most \( |I| \cdot L^n \).

Since \( F + F = \lim_{n \to +\infty} Z_n + \lim_{n \to +\infty} Z_n = \lim_{n \to +\infty} (Z_n + Z_n) \) and \( Z_n + 1 \subset Z_n \), we can use this covering to estimate the Hausdorff dimension of \( F + F \) (see \cite{8}, proposition 4.1) as follows:

\[
dim_H (F + F) \leq \lim_{n \to +\infty} \frac{\log 4^n}{-\log(|I| \cdot L^n)} = \frac{\log 4}{\log L} = \frac{\log 2}{\log(4 + \min F)}.
\]

As \( \min F = [0, (5, 4)^\circ] \), we obtain \( \min F = \frac{1}{5 + \frac{1}{|I| \cdot L^n}} \). Thus \( \min F = 2\left(\frac{6}{5} - 1\right) \) and we deduced the upper bound

\[
\dim_H (F + F) \leq \frac{\log 2}{\log 2 + \log \left(\frac{6}{5} + 1\right)} < \frac{1}{2}.
\]

The rest of the proof is analogous to the end of the proof of theorem 1. We use lemma 8 and an analogous modification of lemma 12 for the alphabet \( \mathcal{A} = \{4, 5\} \) to obtain that

\[
\pm \frac{1}{x} \in S(\alpha) \quad \text{for each} \quad x \in \{4, 5\} + F + F.
\]

By theorem 7 we have

\[
\left\{ \frac{1}{x} : |x| \in \{4, 5\} + F + F \right\} \cap \left( -\frac{1}{2}, \frac{1}{2} \right) = S(\alpha) \cap \left( -\frac{1}{2}, \frac{1}{2} \right).
\]

Clearly, the union of the four sets \( 4 + F + F, 5 + F + F, -4 - F - F \) and \( -5 - F - F \) with the same Hausdorff dimension is a set of the same dimension. Moreover, the Hausdorff dimensions of \( f(M) \) and \( M \) coincide for any continuous mapping \( f \), in particular for \( f(x) = \frac{1}{x} \).

It implies that the estimates on the Hausdorff dimension of \( F + F \) are valid also for \( S(\alpha) \cap \left( -\frac{1}{2}, \frac{1}{2} \right) \).

\[\square\]

5.2. Well approximable numbers

We consider \( \alpha = [a_0, a_1, a_2, \ldots] \) with unbounded partial quotients. Using second convergents defined in (5) we can write for any \( N \in \mathbb{N} \) and \( a \in \mathbb{N} \) with \( 1 \leq a < a_{N+1} \)
\begin{equation}
(aq_N + q_{N-1})^2 \frac{ap_N + p_{N-1}}{aq_N + q_{N-1}} = (a + q_{N-1})^2 (a + q_{N-1}) - \alpha = (-1)^{N+1} \left( a + \frac{q_{N-1}}{q_N} \right) \frac{\alpha_{N+1} - a}{\alpha_{N+1} + \frac{2q_N}{q_{N+1}}} \tag{12}
\end{equation}

Recall that \(\alpha_{N+1} = [a_N, a_{N+2}, a_{N+3}, \ldots]\). Let \((q_N)\) be a strictly increasing sequence of integers such that \(\lim_{N \to +\infty} a_1 + b_\alpha = +\infty\). Clearly, \(\lim_{N \to +\infty} \alpha_{1+b_\alpha} = +\infty\). Let us fix \(a \in \mathbb{N}\) and put \(k_N = ap_{q_\alpha} + p_{q_{\alpha-1}}\) and \(m_N = aq_{q_\alpha} + q_{q_{\alpha-1}}\), we have

\[m_N \left| \frac{k_N}{m_N} - \alpha \right| = \alpha + \frac{q_{q_{\alpha-1}}}{q_{q_\alpha}} E_N,
\]

where we set \(E_N = \frac{\alpha_{q_{\alpha-1}} - a}{\alpha_{q_{\alpha-1}} + \frac{2q_{q_{\alpha-1}}}{q_{q_\alpha}}}\). Obviously, \(\lim_{N \to +\infty} E_N = 1\). Since the sequence \((q_N)\) is a strictly increasing sequence of integers, the ratio \(\frac{q_{q_{\alpha-1}}}{q_{q_\alpha}}\) belongs to \((0, 1)\). This implies that the sequence \(m_N^2 \left| \frac{k_N}{m_N} - \alpha \right|\) has at least one accumulation point in the interval \([a, a + 1]\). Therefore we can conclude the next lemma.

**Lemma 14.** Let \(\alpha\) be an irrational well approximable number. For any \(n \in \mathbb{N}\) the interval \([n, n + 1]\) or the interval \([-n - 1, -n]\) has a non-empty intersection with \(S(\alpha)\).

**Example 15.** Unlike the number \(\pi\), the continued fraction of the Euler constant has a regular structure:

\(e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, \ldots]\).

In general, for \(e = [2, a_1, a_2, \ldots]\) we have

\[a_{3n+1} = 1, \quad a_{3n+2} = 2(n + 1) \quad \text{and} \quad a_{3n+3} = 1 \quad \text{for any} \quad n \in \mathbb{N}.
\]

We demonstrate that

\[\left| \frac{q_{3n}^2 \left( p_{3n} - e \right)}{q_{3n}^2 - q_{3n-1}} \right| \to \frac{1}{2}.
\]

By lemma 8 we need to show

\[A_{3n} := [a_{3n+1}, a_{3n+2}, a_{3n+3}, \ldots] + [0, a_{3n}, a_{3n+1}, \ldots, a_1] \to 2.
\]

Using the simple estimate valid for any continued fraction

\[b_0 + \frac{1}{b_1 + \frac{1}{b_2}} < [b_0, b_1, b_2, \ldots] < b_0 + \frac{1}{b_1 + \frac{1}{b_2}},
\]

we obtain the following bounds:

\[1 + \frac{1}{2(N + 1) + 1} + \frac{1}{1 + \frac{1}{2N}} < A_{3n} < 1 + \frac{1}{2(N + 1) + 1} + \frac{1}{1 + \frac{1}{2N + 1}}.
\]

Both bounds have the same limit, namely 2, as we wanted to show. Analogously one can deduce that

\[\left| \frac{q_{3n-1}^2 \left( p_{3n-1} - e \right)}{q_{3n-1}^2 - q_{3n-1}} \right| \to \frac{1}{2} \quad \text{and} \quad \left| \frac{q_{3n+1}^2 \left( p_{3n+1} - e \right)}{q_{3n+1}^2 - q_{3n+1}} \right| \to 0.
\]

Since \((-1)^{3n}\) takes positive and negative signs, the values 0, ±\(\frac{1}{2}\) belong to the spectrum of \(e\) and moreover
As $a_{3N+2} = 2N > 1$, we can use the second convergents as well and for any fixed $a \in \mathbb{N}$ and any $N$ such that $a < a_{3N+2}$ we obtain

$$(aq_{3N+1} + q_{3N})^2 \left| e - \frac{aq_{3N+1} + p_{3N}}{aq_{3N+1} + q_{3N}} \right| = \left( a + \frac{q_{3N}}{q_{3N+1}} \right) E_{3N},$$

where $\lim_{N \to \infty} E_{3N} = 1$, see (12). By the proof of lemma 8, we have $\frac{q_{3N}}{q_{3N+1}} = [0, a_{3N+1}, a_{3N}, a_{3N-1}, \ldots, a_1] \to \frac{1}{2}$.

We conclude for the spectrum of the Euler number satisfies

$$\{0\} \bigcup \{ a + \frac{1}{2} : a \in \mathbb{Z} \} \subset \mathcal{S}(e).$$

Of course, the inclusion cannot be replaced by an equality. The reason is simple; the spectrum is closed under multiplication by the factor 4, and thus

$$\{4a + 2 : a \in \mathbb{Z} \} \subset \mathcal{S}(e)$$

as well.

**Theorem 16.** There exists an irrational number $\alpha$ such that $\mathcal{S}(\alpha) = \mathbb{R}$.

**Proof.** Suppose that $a = a_1 a_2 \ldots$ is an infinite word such that any sequence of the form $w_1 w_2 \ldots w_N w_{k+1} w_{k+2} \ldots w_{2k}$, where the symbols $w_i$ are from the alphabet $\{1, 2, 3, 4\}$ and $N > 1, N \in \mathbb{Z}$ occurs in $a$ infinitely many times. The same reasoning as in the proofs of lemma 12 and theorem 11 together with the equality (10) imply the statement of the theorem. Therefore, it is enough to describe $a$.

Fix $n \in \mathbb{N}$ and consider a word $w = w_1 w_2 \ldots w_n$ of length $n$ over the alphabet $\{1, 2, 3, 4\}$. Copy$(w)$ denotes the concatenation of $n$ words of length $(n + 1)$, each in the form $wh = w_1 w_2 \ldots w_n h$ with $h = 1, 2, \ldots, n$. Thus Copy$(w)$ is a word of length $n(n + 1)$. The word $v_n$ is created by concatenation of Copy$(w)$ for all words $w$ of length $n$ over the alphabet $\{1, 2, 3, 4\}$. In particular, the length of $v_n$ is $4^n(n + 1)$.

The infinite word $a$ is given by its prefixes $(u_n)$ which are constructed recursively: $u_0$ is the empty word and $u_n = u_{n-1} v_n$. \hfill \Box

**Remark 17.** We note that the behavior of $\alpha = [a_1, a_2, \ldots]$ defined in the proof of the previous theorem is typical. In [11], Bosma, Jager and Wiedijk described the distribution of the sequence $q_n |p_n - \alpha q_n|$. A direct consequence of their result is that $\mathcal{S}(\alpha) = \mathbb{R}$ for almost all $\alpha \in [0, 1]$. Thus, it is also true for almost all $\alpha \in \mathbb{R}$.

### 6. Discussion and remarks

In the context of physics it must be emphasized that our choice of the elementary model (1) + (2) is motivated not only by its appealing number-theoretical properties but also by its possible straightforward phenomenological applicability. We feel motivated by the persuasion that the related constructive exemplification of certain spectral anomalies might prove attractive even from the point of view of a physicist who need not necessarily care about the deeper mathematical subtleties.
Using our purely mathematical tools we are able to arrive at a better understanding of certain purely formal connections between various structural aspects of the spectra, with the main emphasis put on its unboundedness from below (which could result into instabilities under small perturbations) in an interplay with the emergence of accumulation points in the point spectrum (in the latter case it makes sense to keep in mind the existing terminological ambiguities [12]).

Needless to add that the phenomenological role of the spectral accumulation points remains strongly model-dependent (see the rest of this section for a few samples). In the most elementary quantized hydrogen atom, for example, such a point represents just an entirely innocent lower bound of the continuous spectrum. A more interesting interpretation of these points is obtained in the case of the so called Efimov three-body bound states [13, 14], etc.

### 6.1. The context of systems with position-dependent mass

Irrespectively of the concrete physical background of quantum stability [15], its study encounters several subtle mathematical challenges [16–19]. In our present hyperbolic-operator square-well model living on a compact domain $R$, a number of interesting spectral properties is deduced and proved by the means and techniques of mathematical number theory, without any recourse to the abstract spectral theory. Still, the standard spectral theory is to be recalled. For example, once we return to the explicit units we may reinterpret our present hyperbolic partial differential operator $\Box$ in equation (1) as a result of a drastic deformation of an elliptic non-equal-mass Laplacean

$$\Delta = \frac{1}{2m_x} \frac{\partial^2}{\partial x^2} + \frac{1}{2m_y} \frac{\partial^2}{\partial y^2}$$

or rather of an even more general kinetic-energy operator

$$T(x, y) = \frac{1}{2m_x(x, y)} \frac{\partial^2}{\partial x^2} + \frac{1}{2m_y(x, y)} \frac{\partial^2}{\partial y^2}$$

containing the position-dependent positive masses. In the ultimate and decisive step one simplifies the coordinate dependence in the masses $m_{x, y}(x, y)$ (say, to piecewise constant functions) and, purely formally, allows one of them to become negative.

In such a setting our present mathematical project is also guided by the specific position-dependent mass physical projects of [20, 21] inspired, in their turn, by the non-Hermitian (a.k.a. $PT$-symmetric [22]) version of quantum Kepler problem. In these papers the mass $m(x)$ is allowed to be complex and, in particular, negative. In [20] the onset of the spectral instability is analyzed as an onset of an undesirable unboundedness of the discrete spectrum from below. A return to a stable system with vacuum is then shown controllable only via an energy-dependent mass $m(x, E)$, i.e., via an ad hoc spectral cut-off (see also [23]).

### 6.2. The context of generalized quantum waveguides

Before one recalls the boundary conditions (2), the majority of physicists would perceive our hyperbolic partial differential equation (1) as the Klein–Gordon equation describing the free relativistic one-dimensional motion of a massive and spinless point particle. Whenever one adds an external (say, attractive Coulomb) field, the model becomes realistic (describing, say, a pionic atom). Now, even if we add the above-mentioned Dirichlet boundary conditions
A certain physical interpretation of the spectrum survives the characterization of, say, the bound states in a ‘relativistic quantum waveguide’.

One of the most interesting consequences of the latter approach may be seen in the possibility of a collapse of the system in a strong field. The most elementary illustrations of such a type of instability may even remain non-relativistic: Landau and Lifshitz [24] described the phenomenon in detail. Another, alternative, type of quantum instabilities connected with the emergence of spectral accumulation points occur also in Horava–Lifshitz gravity with ghosts [25, 26] or in the conformal theories of gravity [27–29] etc.

Our present choice of the elementary illustrative example with compact and rectangular \( R \) changes the physics and becomes more intimately related to the problems of the so called quantum waveguides with impenetrable walls [30]. Most of the mathematical problems solved in the latter context are very close to the present ones. Typically, they concern the possible relationship between the spectra and geometry of the spatial boundaries. In this setting, various transitions to the infinitely thin and/or topologically non-trivial domains \( R \) (one may then speak about quantum graphs) and, possibly, also to the various anomalous point-interaction forms of the interactions are being also studied [6].

Up to now, people only very rarely considered a replacement of the positive-definite kinetic-energy operator (i.e., Laplacian) by its hyperbolic alternative. Thus, in spite of some progress [31], such a ‘relativistic’ generalization of the concept of quantum waveguide and/or of quantum graph still remains to be developed.

6.3. The context of classical optical systems with gain and loss

One of the most characteristic features of modern physics may be seen in the multiplicity of overlaps between its apparently remote areas. *Pars pro toto* let us mention here the unexpected productivity of the transfer of several quantumtheoretical concepts beyond the domain of quantum theory itself [32]. One of the best known recent samples of such a transfer starts in quantum field theory [33] and ends up in classical electrodynamics [34]. A common mathematical background consists in the requirements of the Krein-space self-adjointness (alias parity-times-time-reversal symmetry (\( \mathcal{PT} \)-symmetry)).

It is worth adding that the latter form of a transfer of ideas already proceeded in both directions. The textbook formalism of classical electrodynamics based on Maxwell equations was enriched by the mathematical techniques originating in spectral theory of quantum operators in Hilbert space (see e.g., section 9.3 of the review paper [36] for more details). In parallel, the \( \mathcal{PT} \) – symmetry-related version of quantum theory (see also its older review [22]) took an enormous profit from the emergence and success of its experimental tests using optical metamaterials [37]. People discovered that the time is ripe to think about non-elliptic versions of Maxwell equations reflecting the quick progress in the manufacture of various sophisticated metamaterials which possess non-real elements of the permittivity and/or permeability tensors [38–41].

Naturally, the mutual enrichments of the respective theories would not have been so successful without the progress in experimental techniques, and vice versa. In fact, the availability of the necessary optical metamaterials (which can simulate the \( \mathcal{PT} \)-symmetry of quantum interactions via classical gain-loss symmetry of prefabricated complex refraction indices) was a highly non-trivial consequence of the quick growth of the know-how in nanotechnologies [42, 43]. In opposite direction, the experimental simulations of various quantum loss-of-stability phenomena in optical metamaterials encouraged an intensification of the related growth of interest in the questions of stability of quantum systems with respect to perturbations [44–46].
6.4. The context of unbounded spectra

Our last comment on the possible phenomenological fructification of our present study of the toy model (1) + (2) concerns its possible, albeit purely formal, connection to the traditional Pais–Uhlenbeck (PU) oscillator [47]. The idea itself is inspired by the Smilga’s paper [48] which provides us with a compact review of the appeal of the next-to-elementary PU model in physics.

We imagine, first of all, that the unboundedness of the spectrum of the PU oscillators parallels the same ‘threat of instability’ feature of our rectangular model. At the same time, in the broad physics community, the PU oscillator is much more widely accepted as a standard model throwing a new light on several methodical aspects of the loss of stability, especially in the context of quantum cosmology and quantization of gravity (see also [49–51]). In particular, the PU model contributes to the understanding of the role of renormalizability in higher-order field theories [25, 52], etc.

For these reasons we skip the problems connected with the ambiguity of transition from Lagrangians to Hamiltonians [53] and we restrict our attention just to one of the specific, PU-related quantum Hamiltonian(s), viz., to the operator picked up for analysis, e.g., in [48],

\[ H = (- \partial_x^2 + \Omega_x^2 x^2) - (- \partial_y^2 + \Omega_y^2 y^2). \]  

In a way resembling our present results, the related quantum dynamics looks pathological because even the choice of the incommensurable oscillator frequencies $\Omega_x$ and $\Omega_y$ leads to a quantum system in which the bound-state energy spectrum (i.e., in the language of mathematics, point spectrum—see a comment Nr. 2 in [48]) is real but dense and unbounded

\[ E_{nm} = \left(n + \frac{1}{2}\right) \Omega_x - \left(m + \frac{1}{2}\right) \Omega_y, \quad n, m = 0, 1, 2, \ldots \]  

In the related literature (see e.g., [54–56]) several remedies of the pathologies are proposed ranging from the use of the Wick rotation of $y \rightarrow iy$ [57] up to a suitable modification of the Hamiltonian as performed already before quantization, on classical level [58–60].

This being said, an independent disturbing feature of the PU toy model (15) may be seen in an abrupt occurrence of a set of spectral accumulation points in the equal-frequency limit $\Omega_x = \Omega_y \rightarrow 0$ [61, 62]. The emergent new technical difficulty originates from the fact that the resulting Hamiltonian becomes non-diagonalizable, acquiring a rather peculiar canonical matrix structure of an infinite-dimensional Jordan block.

This is one the most dangerous loss-of-quantum-meaning aspects of the model. Its serious phenomenological consequences are discussed, e.g., in the scalar field cosmology (see the freshmost papers [63, 64] with further references). In a narrower context of specific pure fourth-order conformal gravity, such a spectral discontinuity cannot be circumvented at all [65].

7. Summary

The aim of this paper is to demonstrate the variability of spectra in dependence on the number-theoretical properties of the ratio $\alpha = a/b$ of the sides of the rectangular $R$. In particular, we show that in an arbitrarily short interval $I \subseteq \mathbb{R}$ one can find numbers $\alpha, \beta, \gamma, \delta, \varepsilon$ such that the spectrum of $S(\alpha)$ is empty, the spectrum of $S(\beta)$ forms an infinite discrete set, the spectrum $S(\gamma) = \mathbb{R}$ covers the whole real line, the spectrum $S(\delta) = \mathbb{R} \setminus (-a, a)$ has a ‘hole’ with some positive real $a = a(\delta)$. Finally the spectrum $S(\varepsilon)$ has zero Lebesgue measure, it is uncountable, but it has a positive Hausdorff dimension.
which is less than 1. It means that a small change of the dynamical parameter $\alpha = a/b$
dramatically influences the spectrum.

Although we give just an extremely elementary example for the detailed and rigorous
analysis, we would like to emphasize that our present approach proves productive in spite of
lying far beyond the standard scope and methods of spectral analysis. A non-trivial insight in
the underlying physics is provided purely by the means of number theory.

From the point of view of number theory, various results on the Markov constant, i.e.,
$$\mu_k(a) = \min \left\{ m^2 \left| k \cdot \frac{a}{m} - \alpha \right| : k, m \in \mathbb{Z} \right\},$$
may be found. In the present article, we provide some insight into the behavior of all the accumulation points of the concerned set. Since we restrict ourselves to some special cases, naturally, the next step would be to fully investigate the properties of $S(\alpha)$.

Acknowledgments

EP acknowledges financial support from the Czech Science Foundation grant 13-03538S and
SŠ acknowledges financial support from the Czech Science Foundation grant 13-35273P. MZ
was supported by RVO61389005 and acknowledges financial support from the Czech Science Foundation grant 16-22945S.

References

[1] Burger E B 2000 Exploring the Number Jungle: A Journey Into Diophantine Analysis
(Providence, RI: American Mathematical Society)
[2] Bohr N 1913 Phil. Mag. 26 476
[3] Reed M and Simon B 1980 Methods of Modern Mathematical Physics vol 1–4 (New York: Academic)
[4] Bagarello F, Gazeau J P, Szafraniec F H and Znojil M (ed) 2015 Non-Selfadjoint Operators in Quantum Physics: Mathematical Aspects (New York: Wiley)
[5] Exner P and Gawlista R 1996 Phys. Rev. B 53 7275–86
[6] Exner P and Turek O 2015 Integr. Equat. Oper. Th. 81 535–57
[7] Astels S 1999 Trans. Am. Math. Soc. 352 133–70
[8] Falconer K 2003 Fractal Geometry: Mathematical Foundations and Applications 2nd edn (New York: Wiley)
[9] Hall M Jr 1947 Ann. of Math. 48 966–93
[10] Diviš B 1973 Acta Arith. 22 157–73
[11] Bosma W, Jager H and Wiedijk F 1983 Indagat. Math. 86 281–99
[12] Krejčířík D and Siegl P 2015 Elements of spectral theory without the spectral theorem Non-Selfadjoint Operators in Quantum Physics: Mathematical Aspects 1st edn ed F Bagarello et al (New York: Wiley) pp 241–92
[13] Efimov V 1970 Phys. Lett. B 33 563–4
[14] Tung S K, Jiménez-García K, Johansen J, Parker C V and Chin C 2014 Phys. Rev. Lett. 113 240402
[15] Li J Q, Li Q and Miao Y G 2012 Commun. Theor. Phys. 58 497–503
[16] Behrndt J and Krejčířík D 2014 J. Anal. Math. arXiv:1407.7802 to appear
[17] Kostyrkin V 2012 The div(a,grad) operator without ellipticity self-adjointness and spectrum Geometric Aspects of Spectral Theory, 33/2012 Mathematisches Forschungsinstitut Oberwolfach p 2061 joint work with Amru Hussein, David Krejčířík, Stepan Schmitz
[18] Hussein A 2013 PhD Thesis Johannes Gutenberg-Universität Mainz
[19] Schmitz S 2014 PhD Thesis Johannes Gutenberg-Universität Mainz
[20] Znojil M and Lévai G 2012 Phys. Lett. A 376 3000–5
[21] Lévai G, Siegl P and Znojil M 2009 J. Phys. A: Math. Theor. 42 295201
[22] Bender C M 2007 Rep. Prog. Phys. 70 947
[23] Chen T J, Fasiello M, Lim E A and Tolley A J 2013 J. Cosmol. Astropart. Phys. JCAP(2013)042
