Lower bounds on the absorption probability of beam splitters

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We derive a lower limit to the amount of absorptive loss present in passive linear optical devices such as a beam splitter. We choose a particularly simple beam splitter geometry, a single planar slab surrounded by vacuum, which already reveals the important features of the theory. It is shown that, using general causality requirements and statistical arguments, the lower bound depends on the frequency of the incident light and the transverse resonance frequency of a suitably chosen single-resonance model only. For symmetric beam splitters and reasonable assumptions on the resonance frequency $\omega_r$, the lower absorption bound is $p_{\text{min}} \approx 10^{-6} (\omega/\omega_r)^4$.

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I. INTRODUCTION

Passive linear optical elements such as beam splitters are indispensable tools in photonic interferometry. Most recently, they became even more important with the emergence of ideas to use beam splitters as central units in all-optical quantum information processing [1]. The action of a beam splitter can be understood by assuming that light impinges on a single slab of material with a refractive index $n$. The differences in refractive indices at the faces of the beam splitter cause light to be partially reflected from the surface, while other parts are transmitted.

In a mathematical description of it, the beam splitter is mostly assumed to be lossless, at least in the frequency window associated with the bandwidth of the incident light. In this case, the action of a beam splitter on two amplitude operators of photons impinging on it is given by an element of the unitary group $\text{SU}(2)$ which reflects energy conservation by splitting into transmitted and reflected light [2]. If two identical photons in well-defined single-photon Fock states impinge on both sides of a lossless beam splitter, one would for instance observe the well-known Hong–Ou–Mandel quantum interference effect [3] with perfect visibility.

Realistically, however, perfectly lossless beam splitter do not exist and cannot exist. Causality and the resulting Kramers–Kronig relations for the dielectric permittivity imply that a (real part of the) refractive index $n(\omega)$ different from unity is always accompanied by an imaginary part $\kappa(\omega)$ responsible for describing absorption and dissipation [4]. An additional feature of the so-called superconvergence rule [5] which follows from the Kramers–Kronig relations is that the real part of the refractive index is unity on average over all frequencies, i.e. $\int_0^\infty d\omega [n(\omega) - 1] = 0$.

On the other hand, causality itself does not immediately imply that the absorption coefficient has to be strictly larger than zero for all non-zero frequencies. In fact, the Kramers–Kronig relations in principle allow for a situation in which there is only one $\delta$ function peak in the absorption spectrum at some particular frequency and vanishing absorption elsewhere. Physically, this implies an infinitely sharp resonance and thus an infinitely long lifetime of the corresponding transition which is not realistic either. Here some statistical arguments come to the rescue. For macroscopically large systems a statistical description in terms of response functions has to be used which leads to a particular form of the absorption spectrum. In the linear-response approximation, this prevents the absorption function to become zero at any point except the origin of the frequency spectrum.

A causal response of a macroscopic physical quantity (such as the dielectric polarization) to an external excitation (such as an electric field) results in relations between the correlation function for the responding quantities and the imaginary part of the response function. Such relations are known as fluctuation-dissipation theorems [6]. They have wide applications in all branches of statistical physics. Of particular interest to us is that fluctuation-dissipation theorems have helped to establish limits on physical processes such as frequency stabilization of lasers with cavities [7]. Generalizations of these theorems have been used to provide bounds on the performance of solid-state two-qubit gates [8].

A realistic description of a beam splitter that includes absorption and dispersion has been formulated in [9] and makes use of the quantum-optical input-output relations derived in [10]. It is shown in [11] that the action of a lossy beam splitter on the amplitude operators of the incoming light (plus the amplitude operators of the device excitation) corresponds to an element of the group $\text{SU}(4)$. From there the $\text{SU}(2)$-action of a lossless beam splitter follows as a limiting case in which photons and device would effectively decouple which, by the above arguments, means that there is no beam splitter action present at all.

The consequence of this argument is that any beam splitter, no matter how well it is fabricated, must show some non-vanishing absorption. In this article we will
show how to obtain lower bounds on the amount of absorption. We will use a simple model of a single planar slab for which the input-output relations are well-known \[10\] and we will use a single-resonance Drude–Lorentz model for the dielectric permittivity (Sec. \[11\]). This, together with some lower bounds on the resonance line width, will be sufficient to derive such lower bounds that only depend on the frequency of the impinging light (Sec. \[11\]). We summarize our results in Sec. \[14\].

**II. THE BEAM SPLITTER MODEL**

We will use a particularly simple model for a beam splitter, a single planar slab of thickness \(l\) made of a material with a complex refractive index \(n(\omega) = \eta(\omega) + i\kappa(\omega)\). In this approximation, we consider linearly polarized light travelling in a particular direction in space. Following \[10\], the transmission and reflection coefficients \(T(\omega)\) and \(R(\omega)\), respectively, can be written in the form

\[
T(\omega) = \frac{4n e^{i(\eta-n)\omega l/c}}{(1+n)^2 - (1-n)^2 e^{2i\eta\omega l/c}},
\]

\[
R(\omega) = \frac{n-1}{n+1} e^{-\omega l/c} \left[ 1 - T(\omega)e^{i(\eta-n)\omega l/c} \right],
\]

\[n \equiv n(\omega)\]. Because of absorption, they fulfill \(|T(\omega)|^2 + |R(\omega)|^2 \leq 1\).

In the next step, we need to model the complex refractive index \(n(\omega)\). In the linear approximation in which the macroscopic polarization responds linearly to an external electric field, we need to define a complex susceptibility function \(\chi(\omega)\). In this approximation the quasi-excitations associated with the linear response are harmonic oscillator-like, hence the susceptibility can only be a (possibly infinite) sum of Drude–Lorentz functions, i.e.

\[
\chi(\omega) = \sum_i \frac{\omega_{T,i}^2}{\omega_{T,i}^2 - \omega^2 - i\gamma_i \omega},
\]

where \(\omega_{T,i}\) is the resonance frequencies of the \(i\)th resonance with a decay constant or line width \(\gamma_i\), and strengths \(\omega_{p,i}\) that are related to the static dielectric permittivity. Note that Drude–Lorentz susceptibilities are the only allowed type of response functions. From Eq. \[3\] it is seen that the susceptibility has a strictly positive imaginary part for all positive frequencies. That means that even at frequencies far away from all resonances, a beam splitter will absorb light and we have the strict inequality \(|T(\omega)|^2 + |R(\omega)|^2 < 1\) except for the point \(\omega = 0\). In particular, for frequencies below all resonances, we can write for the real and imaginary parts of the refractive index to their respective lowest orders in \(\omega\)

\[
\eta(\omega) \approx \left(1 + \sum_i \frac{\omega_{p,i}^2}{\omega_{T,i}^2} \right)^{1/2},
\]

\[
\kappa(\omega) \approx \frac{\omega}{2\eta(\omega)} \sum_i \frac{1}{\omega_{T,i}} \left( \frac{\gamma_i}{\omega_{T,i}} \right) \left( \frac{\omega_{p,i}^2}{\omega_{T,i}^2} \right).
\]

This limit is of special importance for our consideration since it is the regime in which absorption is lowest while \(n(\omega)\) retains a strong diffractive part.

**III. MINIMAL ABSORPTION PROBABILITY**

Let us assume that we would like to build a beam splitter with a certain splitting ratio \(x = |T|^2/|R|^2\) out of such a single slab. Then there are several parameters which we can tune: the static permittivity, the frequency of the incoming light, the thickness of the slab, and the line widths of the resonances. If we restrict ourselves to a single-resonance model with a transition frequency \(\omega_T\), line width \(\gamma\), and strength \(\omega_p\), we are left with four parameters: the static refractive index \(\eta = 1 + \omega_p^2/\omega_T^2\), the scaled line width \(\gamma_T = \gamma/\omega_T\), the scaled frequency \(\tilde{\omega} = \omega/\omega_T\), and the scaled thickness \(d = \omega_T l/c\). The product \(\eta d\) is proportional to the optical path length and thus determined by the chosen beam splitting ratio \(x\). Then, given the frequency \(\tilde{\omega}\) and the line width \(\gamma_T\), we minimize the absorption probability \(p = 1 - |T|^2 - |R|^2\) over the beam splitter thickness \(d\). For low frequencies and narrow line widths we can expand Eqs. \[1\] and \[2\] with respect to those small parameters. The result can be cast in the form

\[
p_{\text{min}} = \alpha(x) \tilde{\gamma} \tilde{\omega},
\]

where \(\alpha(x)\) is a numerical coefficient that depends on the beam splitting ratio \(x\). Figure \[4\] shows the result of a numerical calculation of the coefficient \(\alpha(x)\) for different beam splitting ratios \(x = |T|^2/|R|^2\). The maximum is obtained around a symmetric splitting ratio.

![Coefficient α in Eq. (6) for different beam splitting ratios](image_url)

**FIG. 1:** Coefficient \(\alpha\) in Eq. \[6\] for different beam splitting ratios \(x = |T|^2/|R|^2\). The maximum is obtained around a symmetric splitting ratio.

The minimal absorption probability \(\alpha(x)\) still depends on two parameters, the resonance line width \(\gamma_T\) and the frequency of the incoming light, \(\tilde{\omega}\). In order to find the
absolute minimum of Eq. (6) we need to give a lower bound on the scaled line width \( \gamma \). For a non-vanishing susceptibility we need non-zero dipole moment which implies the existence of some (electronic) excited states that can be populated by optical excitation. This is because the dipole moment is proportional to the off-diagonal matrix element of the position operator. These electronic states are coupled to the electromagnetic vacuum and are thus subject to spontaneous decay. Disregarding fabrication errors and impurities, spontaneous decay is the limiting factor when determining the line width. This fundamental process is also responsible for giving lower bounds on factorization times in quantum computers [11].

Because each individual dipole (atom) is surrounded by many other dipoles of the same kind, it will feel not only the vacuum fluctuations but enhanced fluctuations due to the presence of the surroundings. These effects are known as local-field corrections, and several models exist to calculate them. In particular, the so-called real-cavity model has attracted much attention because it consistently places the radiating dipoles in vacuum [12, 13] rather than in front of a smeared dielectric background which is known to cause spurious divergences [14, 15, 16].

Because we are interested in the low-frequency limit in which absorption is very small, we can ignore its influence on spontaneous decay and approximate the effect of the surrounding atoms by the (classical) real-cavity correction factor [12, 13],

\[
\Gamma = \eta(\omega)\Gamma_0 \left( \frac{3\eta^2(\omega)}{2\eta^2(\omega) + 1} \right)^2,
\]

where \( \Gamma_0 \) is the free-space decay rate which is defined as

\[
\Gamma_0 = \frac{\omega^3\hbar^2}{3n\hbar\varepsilon_0 c^3},
\]

where \( d \) is the (matrix element of the) dipole moment of the transition. The dipole moment is related to the static refractive index in the following way. Consider a two-level atom with a ground state \( |g\rangle \) and an excited state \( |e\rangle \) (there is no need to consider more complicated situations because the single-resonance model corresponds directly to the response of an ensemble of two-level atoms). The atomic polarizability in the zero-frequency limit can be written as [17]

\[
\alpha_{ij}(0) = \frac{n_g - n_e}{\hbar \omega_T n V} \left[ d_i d_j^* + d_i^* d_j \right] \approx \frac{d_i d_j^* + d_i^* d_j}{\hbar \omega_T}
\]

where \( n_{g,e} \) are the population numbers of ground-state and excited-state atoms, respectively, and \( n \) the total number density of atoms in the volume \( V \). The matrix elements of the dipole operator are denoted as \( d_i = \langle e|d_i|g\rangle \). The approximation made on the rhs of Eq. (9) is justified because at room temperature and optical frequencies the thermal occupation of the excited state is negligible, and in the linear-response approximation there are no population inversions. Therefore, the zero-frequency susceptibility can be written as

\[
\chi(0) \approx \frac{2d_i^2 n}{3\hbar \omega_T \varepsilon_0}.
\]

In that way, we can relate the dipole moment to the static refractive index \( \eta \) as

\[
d_i^2 = \frac{3\hbar \omega_T \varepsilon_0(\eta^2 - 1)}{2n}.
\]

Inserting Eq. (11) into (3) and subsequently into Eq. (2), we find that

\[
\Gamma = \frac{4\pi^2 \omega^3 \eta(\eta^2 - 1)}{n V_T} \left( \frac{3\eta^2}{2\eta^2 + 1} \right)^2,
\]

where \( V_T \) is the volume of a cube with side length equal to the transition wavelength \( \lambda_T \). The product \( n V_T \) is therefore the number of atoms within this cube.

Finally, we associate Eq. (2) with the lower bound on the line width \( \gamma \) of the response function. Hence, we obtain the following formula for the minimal absorption probability as

\[
p_{\min} = \frac{4\pi^2 \alpha(x) \omega^4 \eta(\eta^2 - 1)}{n V_T} \left( \frac{3\eta^2}{2\eta^2 + 1} \right)^2.
\]

Note that the static refractive index \( \eta \) appearing in Eq. (13) has to obtained from the minimization procedure over the beam splitter thickness \( d \) that led to the numerical coefficient \( \alpha(x) \) in Fig. 1. The dependence of the static permittivity \( \varepsilon_x = 1 + \chi(0) = \eta^2 \) on the beam splitter ratio \( x \) is shown in Fig. 2.

![FIG. 2: Static permittivity \( \varepsilon_x \) leading to the minimal value \( p_{\min} \) [Eq. (13)] for different beam splitter ratios \( x = |T|^2/|R|^2 \).](image-url)

As an example, let us consider the symmetric beam splitter with \( x = 1 \). The numerical minimization leads to \( \alpha \approx 0.9 \) at \( \varepsilon_x \approx 6.2 \). Let us further assume that the resonance frequency \( \omega_T \) lies somewhere in the UV such that we can take \( n V_T \approx 10^9 \). With these numbers, the minimal absorption probability reads

\[
p_{\min} \approx 10^{-6} \omega^4.
\]

If additionally the incident light has a frequency that lies in the IR region, i.e. \( \tilde{\omega} \approx 0.1 \), then we obtain
$p_{\text{min}} \approx 10^{-10}$. Currently, the best beam splitters that have been fabricated show an absorption probability of around 2 ppm \cite{18}. This means that on the one hand that there is still room for improvement by eliminating technical imperfections such as impurities or diffuse scattering from interfaces. On the other hand, the lower bound is large enough to become an obstacle in future generations of beam splitter fabrication.

Equation \ref{13} shows that the minimal absorption probability strongly depends on the static permittivity (or, equivalently, the static refractive index). For large values of \(\eta\), \(p_{\text{min}}\) increases as \(p_{\text{min}} \propto \varepsilon_s^{3/2}\) or \(p_{\text{min}} \propto \eta^3\), respectively. From Fig. \ref{fig:2} it follows that this is the case for small beam splitting ratios, i.e. when the slab is highly reflecting. For example, choosing \(x = 0.05\) and all other parameters as before, we obtain \(p_{\text{min}} \approx 2 \cdot 10^{-5} \omega^4\), an increase by a factor of 20.

IV. CONCLUSIONS

We have shown that one can obtain lower bounds on the absorption probability of a beam splitter by causality arguments. This shows the intimate relation between manipulability of light — described by the real part of the linear response function — and decoherence or absorption which is described by the imaginary part of the response function. Although it is intuitively clear that such relations must exist, we have presented in a particularly simple example how they emerge in a realistic physical situation.

From the Kramers–Kronig relations one infers that a beam splitter material which necessarily has \(\eta(\omega) \neq 1\) shows absorption somewhere in the frequency spectrum. However, it is only the assumption of linear response and the resulting validity of Drude–Lorentz models to describe the dielectric susceptibility that leads to the absorption coefficient being strictly positive for positive frequencies. In the single-resonance Drude–Lorentz model employed here the width of the resonance is limited by the spontaneous decay rate between excited and ground state. This is certainly a lower limit when all line-broadening effects are disregarded.

The final result, Eq. \ref{13}, is seen to depend only on the transverse resonance frequency \(\omega_T\) and the frequency of the incident light \(\omega\). The other parameters (beam splitter thickness \(l\), static susceptibility \(\varepsilon_s\), and line width \(\gamma\)) are fixed by the beam splitter condition \((l)\), the minimization \((\varepsilon_s)\), and the spontaneous decay rate \((\gamma)\), respectively. For reasonable assumptions on the transverse resonance frequency, we obtain that the absorption limit for a symmetric beam splitter is approximately \(p_{\text{min}} \approx 10^{-6} \omega^4\). This result will set a possible limit on experimental performances where ultra-low absorption is necessary, such as in gravitational-wave interferometry. Currently, absorption probabilities of 2 ppm have already been achieved, and with progress being made in eliminating material impurities, surface roughnesses, and other experimental imperfections it is reasonable to assume that the theoretical limits of the kind that have been considered here will impose ultimate performance limitations that have to be taken seriously.

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