An Orthogonal Stabilization of Quadratic and Generalized Quadratic Functional Equations

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Abstract

This study is devoted to the stabilization of following quadratic and modified quadratic functional equations in orthogonal space

\[ h(3x \pm y) = 16h(x) + h(x \pm y), \]

and

\[ h(x + ay) + h(x - ay) = 2a^2 h(y) + 2h(x). \]

**Keywords:** Orthogonal spaces, Quadratic and Modified functional equations.

1. Introduction

In 1975, Gudder et al.\(^1\) first established the orthogonal stability of the Cauchy functional equations \( h(r + s) = h(r) + h(s) \) with \( r \perp s \). This result was further extended and studied to examine the orthogonal stability for the mapping \( h \) by Ger and Sikorska\(^1\) on the steps of Ratz\(^6\). Further, the stability of the functional equation

\[ h(r + s) + h(r - s) = 2h(r) + 2h(s), \text{ with } x \perp y \]

On Hilbert orthogonal space was studied by famous mathematician Vajzovic\(^3\). The results of Vajzovic\(^3\) were generalized by Szabo\(^2\), Driljevic\(^5\), Fochi\(^8\). Furthermore, for more study on orthogonal spaces one may refer to \(^1, 7, 10, 13, 14\).

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This article deals with the orthogonal stabilization of the following functional equations defined as

\[ h(3x \pm y) - 16h(x) - h(x \pm y) = 0 \]
\[ h(x + ay) + h(x - ay) - 2a^2h(y) - 2h(x) = 0 \]

The paper is divided into four sections. Section 2 is introductory in nature. Sections 3 and 4 present the stability of quadratic and modified quadratic functional equations.

2. Preliminaries:

This section contains the following orthogonality result studied by many researchers such as Ratz⁶, James⁴, Birkhoff⁹, etc.

**Definition 1.** Let X be a linear space with dimension greater then equal to two and perpendicular (⊥) is the operator defined on X which satisfies the following conditions:

(A1) \( r \perp 0 \), \( 0 \perp r \), \( \forall r \in X \) (Totality)
(A2) \( r, s \in X - \{0\} \Rightarrow r \perp s \) (Independence)
(A3) if \( r \perp s \), then \( \alpha r \perp \beta s \) for \( \alpha, \beta \in \mathbb{R} \) and for all \( r, s \in X \), (Homogeneity)
(A4) Let Y is a subspace of X, \( r \in Y \) and \( \lambda \) be a positive scalar number, then for \( y_0 \in Y \) and \( r \perp y_0 \) we have \( r + y_0 \perp \lambda r - y_0 \). (Thalesian property)

Then the combination \((X, \perp)\) is known as orthogonality space. It is also known as symmetric if \( r \perp s \) and \( s \perp r \), \( \forall r, s \in X \).

**Definition 2.** Let \((X, \perp)\) be an orthogonal space and Z be a Banach space. Then, the relation \( h : X \rightarrow Z \) is called orthogonal quadratic map if it satisfies the system (1).

3. Orthogonal stability of quadratic equations:

In this section we prove that orthogonal stability of following quadratic functional equations

\[ P(h) = h(3r \pm s) - 16h(r) - h(r \pm s) \]  \hspace{1cm} (1)

**Theorem 1.** Let us consider \( h \) be the quadratic function which satisfies

\[ \|P(h)\|_Z \leq \eta(\|r\|_X^p + \|s\|_X^p) \]  \hspace{1cm} (2)

\( \forall r, s \in X \) with \( x \perp y \) and \( p < 2 \). Then, the mapping \( R : X \rightarrow Z \) satisfying

\[ \|h(r) - R(r)\|_Z \leq \frac{\eta}{2(3^2 - 3^p)} \|r\|_X^p \]  \hspace{1cm} (3)

is unique orthogonality solution.

Proof. Putting \( s = 0 \) in (2), we get

\[ \|2h(3r) - 16h(r) - 2h(r)\|_Z \leq \eta(\|r\|_X^p + \|0\|_X^p) \]
\[ \|2h(3r) - 18h(r)\|_Z \leq \eta(\|r\|_X^p) \]
Changing $r = 3r$ and then dividing throughout by $3^2$ in inequality (4) and also summing the obtained result with (4), we get

$$
\left\| h(r) - \frac{h(3r)}{3^2} \right\|_z \leq \frac{\eta}{2.3^2} \left\| r \right\|_x^p
$$

(4)

Now, to prove that the sequence $< h(3^n r) / 3^{2n} >$ is a Cauchy sequence. Changing $r$ with $3^m r$ and then dividing throughout by $3^{2m}$ in (6) we get for all $n, m > 0$.

$$
\left\| h(3^m r) - \frac{h(3^{n+m} r)}{3^{2n+2m}} \right\|_z \leq \frac{\eta}{2.3^2} \sum_{k=0}^{n-1} \frac{3^{pk}}{3^{2k+2m}} \left\| r \right\|_x^p,
$$

(5)

As $m \to \infty$ for all $r \in X$ and $p < 2$ the sequence $< h(3^n r) / 3^{2n} >$ is converges to a point in $Z$. Further, as the Banach space $Z$ is a complete, thus $< h(3^n r) / 3^{2n} >$ is a Cauchy sequence. Thus, we can say

$$
R(r) = \lim_{n \to \infty} \{ h(3^n r) / 3^{2n} \}, \forall r \in X.
$$

(8)

Putting $3^n r$ and $3^n s$ for $r$ and $s$ in (2) respectively and then dividing by the number $3^{2n}$, we have

$$
\left\| P(h) \right\|_{3^{2n}} \leq \frac{\eta}{3^{2n}} \left( \left\| 3^n r \right\|_x^p + \left\| 3^n s \right\|_x^p \right)
$$

(9)

Letting $n \to \infty$, we obtain

$$
\left\| R(3r \pm s) - 16R(r) - R(r \pm s) \right\|_z \leq 0
$$

$$
R(3r \pm s) = 16R(r) + R(r \pm s), \forall r, s \in X.
$$

Hence $R$ is orthogonally quadratic relation.

Taking $n \to \infty$ in (6) we get

$$
\left\| R(r) - h(s) \right\|_z \leq \frac{\eta}{2(3^2 - 3^p)} \left\| r \right\|_x^p, \forall r \in X.
$$
For uniqueness of $R: X \to Z$, let us consider the relation $R': X \to Z$ which satisfies (2), then we get
\[
\|R'(r) - R(r)\|_Z \leq \frac{1}{3^2} \left\{ \|h(3^n r) - R'(3^n r)\|_Z + \|R(3^n r) - h(3^n r)\|_Z \right\}
\leq \frac{\eta}{(3^2 - 3^p)3^{n(2-p)}} \|r\|_X^n \to 0 \text{ as } n \to \infty
\]
Thus, $R' = R$, that means $R$ is unique.

**Theorem 2.** Let $h$ be the quadratic function which satisfies the inequality (2) for all $r, s \in X$ with $r \perp s$ and $p > 2$. Then, the mapping $R: X \to Z$ satisfying
\[
\|h(r) - R(s)\|_Z \leq \frac{\eta}{2(3^p - 3^2)} \|r\|_X^p
\]
is a unique quadratic orthogonal mapping.

**Proof.** Putting $r/3$ at the place of $r$ and then multiplying by $3^2$ in inequality (4), we get
\[
\left\|3^2 h\left(\frac{r}{3}\right) - h(r)\right\|_Z \leq \frac{\eta}{2} \left\|\frac{r}{3}\right\|_X^p,
\]
\[
\left\|3^2 h\left(\frac{r}{3}\right) - h(r)\right\|_Z \leq \frac{\eta}{2.3^p} \|r\|_X^p
\]
$r \perp 0$ for all $r \in X$. Proceeding in this way $n$-times we get the following inequality
\[
\left\|3^n h\left(\frac{r}{3^n}\right) - h(r)\right\|_Z \leq \frac{\eta}{2.3^p} \left(1 + \frac{3^2}{3^p}\right) \|r\|_X^p
\]
\[
\left\|3^{2n} h\left(\frac{r}{3^n}\right) - h(r)\right\|_Z \leq \frac{\eta}{2.3^p} \sum_{k=0}^{n-1} \frac{3^{2k}}{3^{3k}} \|r\|_X^p
\]
\[
\leq \frac{\eta}{2.3^p} \sum_{k=0}^{n} \frac{3^{2k}}{3^{3k}} \|r\|_X^p
\]
Now, to prove the sequence $< h(3^n r) / 3^{2n} >$ is convergent. Replacing $r$ with $r / 3^m$ and then multiplying by $3^{2m}$ in the inequality (13), we get
\[
\left\|3^{2n+2m} h\left(\frac{r}{3^{n+m}}\right) - 3^{2m} h\left(\frac{r}{3^m}\right)\right\|_Z \leq \frac{\eta}{2.3^m(3^p - 3^2)} \sum_{k=0}^{\infty} \frac{3^{2k}}{3^{3(k-1)}} \|r\|_X^p
\]
Which tends to 0 as $m \to \infty$ for all in the right hand side of (14). Therefore, we prove that the sequence $<3^{2n}h(r/3^n)>$ converges in the Banach space $Y$, hence the $<3^{2n}h(r/3^n)>$ is a Cauchy sequence. Thus, we get the orthogonal quadratic system $R : X \to Z$ such that

$$\lim_{n \to \infty} \{3^{2n}h(r/3^n)\} = R(r) \text{ for all } r \in X.$$  

(15)

Taking $n \to \infty$ in (14) and using (15), we get the required result.

4. **Orthogonal stability for generalized quadratic equation** :

This section deals with the orthogonal stability of the following modified quadratic equation

$$h(x + ay) + h(x-ay) - 2a^2 h(y) - 2h(x) = 0$$  

(16)

**Theorem 3.** Let us consider $X$ be a normed linear space, $Y$ be a Banach space and $\zeta : X \times X \to [0, \infty)$ be a mapping such that

$$\lim_{n \to \infty} \frac{\zeta(a^n x, a^n y)}{a^{2n}} = 0$$  

(17)

for all $x, y \in X$. If the function $h : X \to Y$ with $h(0) = 0$, satisfies

$$\|h(x + ay) + h(x-ay) - a^2 h(y) - 2h(x)\| \leq \zeta(x, y)$$  

(18)

for all $x, y \in X$. Then, the map $R : X \to Y$ is a unique quadratic function satisfying the relation

$$\|R(y) - h(y)\| \leq \frac{1}{2a^2} \sum_{i=0}^{\infty} \frac{1}{a^{2i}} \zeta(0, a^i y)$$  

(19)

The quadratic map $R$ is defined as

$$R(y) = \lim_{n \to \infty} \frac{h(a^n y)}{a^{2n}}.$$  

(20)

Proof: Letting $x = 0$ in the relation (18), we obtain

$$\|2a^2 h(y) - 2h(ay)\| \leq \zeta(0, y)$$  

(21)

that

$$\left\|h(y) - \frac{h(ay)}{a^2}\right\| \leq \frac{1}{2a^2} \zeta(0, y)$$  

(22)

Now, putting $y = ay$ in (22) and dividing throughout with $a^2$ and then adding the final equation with (22), we have

$$\left\|h(y) - \frac{h(a^2 y)}{a^4}\right\| \leq \frac{1}{2a^2} \zeta(0, y) + \frac{1}{2a^4} \zeta(0, a y)$$  

(23)

$$\leq \frac{1}{2a^2} \left[\frac{1}{a^2} \zeta(0, ay) + \zeta(0, y)\right]$$
Proceeding in this way \(n\)-times for a positive integer \(n\), we get

\[
\left\| h(y) - \frac{h(a^n y)}{a^{2n}} \right\| \leq \frac{1}{2a^2} \sum_{i=0}^{n-1} \frac{1}{a^{2i}} \zeta(0, a^i y) \geq \frac{1}{2a^2} \sum_{i=0}^{\infty} \frac{1}{a^{2i}} \zeta(0, a^i y)
\]

(24)

Now, we will prove the convergence of the sequence \(< h(a^n y) / a^{2n} >\), changing \(y\) with \(a^k y\) and then dividing relation (24) by \(a^{2k}\), we obtain for \(n, k > 0\),

\[
\left\| h(a^k y) - \frac{h(a^{n+k} y)}{a^{2(n+k)}} \right\| \leq \frac{1}{a^{2k}} \left\| h(a^k y) - \frac{h(a^{n+k} y)}{a^{2n}} \right\|
\]

\[
\leq \frac{1}{a^{2k}} \sum_{i=0}^{\infty} \frac{\zeta(0, a^{i+k} y)}{a^{2i}}
\]

\[
\leq \frac{1}{a^{2k}} \sum_{i=0}^{\infty} \frac{\zeta(0, a^{i+k} y)}{a^{2(i+k)}}
\]

(25)

As \(k \to \infty\), the sequence \(< h(a^n y) / a^{2n} >\) is a Cauchy sequence. Further, as \(Y\) is a Banach space, the sequence \(< h(a^n y) / a^{2n} >\) approaches to a point \(R(y) \in Y\) and thus \(R\) can be defined as

\[
R(y) = \lim_{n \to \infty} \frac{h(a^n y)}{a^{2n}}.
\]

Now, we replace \(x\) and \(y\) with \(a^n x, a^n y\) in (16) and then dividing throughout with \(a^{2n}\), to show that \(R\) is a solution of (16)

\[
\left\| \frac{h(a^n (x + ay))}{a^{2n}} - 2h(a^n x) + 2a^2 h(a^n y) + \frac{h(a^n (x - ay))}{a^{2n}} \right\| \leq \zeta(a^n x, a^n y).
\]

As \(n \to \infty\), then \(R\) satisfies (16).

Now, Let us consider \(R' : X \to Y\) be the second quadratic mapping which is the solution of (16) and (19). Thus, we get
\[ \| R'(y) - R(y) \| = \frac{1}{a^{2n}} \| R'(ay) - R(ay) \| \]

\[ \leq \frac{1}{a^{2n}} (\| R'(ay) - h(ay) \| + \| R(ay) - h(ay) \|) \]

\[ \leq \frac{1}{a^2} \sum_{i=0}^{\infty} \frac{\zeta(0, a^{i+n} y)}{a^{2(i+n)}} \]  \hspace{1cm} (26)

As \( n \to \infty \), we get \( R(y) = R'(y) \) for all \( y \in X \). This completes the result.

**Corollary 1.** Let us consider \( X \) and \( Y \) are normed linear and Banach spaces, respectively. Let \( h: X \to Y \) with the condition \( h(0) = 0 \) satisfies

\[ \| h(x + ay) + h(x - ay) - 2a^2 h(y) - 2h(x) \| \leq \varepsilon \]

\( \varepsilon \geq 0 \) be a real number.

Then, \( \exists \) a unique quadratic mapping \( R: X \to Y \) defined by

\[ \lim_{n \to \infty} \frac{h(a^n y)}{a^{2n}} = R(y) \]

Satisfying the inequality (20) and the relation

\[ \| R(y) - h(y) \| \leq \frac{\varepsilon}{2(a^2 - 1)} \]

for all \( y \in X \).

Moreover, for each \( y \in X \) the function \( m \to h(my) \) from \( R \) to \( Y \) is continuous function, then we get \( a^2 R(y) = R(ay) \).

**Corollary 2.** Let us consider \( X \) and \( Y \) are normed linear and Banach spaces, respectively. Let \( h: X \to Y \) with the condition \( h(0) = 0 \) satisfies

\[ \| h(x + ay) + h(x - ay) - 2a^2 h(y) - 2h(x) \| \leq \alpha \| x \| + \| y \| \]

where \( \alpha \geq 0, 0 < p < 2 \).

Then, \( \exists \) a unique quadratic mapping \( R: X \to Y \) satisfying the inequality (16) and the relation

\[ \| R(y) - h(y) \| \leq \frac{\varepsilon}{2(n^2 - a^p)} \| y \| \]

Where the function \( R \) is defined as

\[ \lim_{n \to \infty} \frac{h(a^n y)}{a^{2n}} = R(y) \]

Moreover, for each \( y \in X \) the function \( m \to h(my) \) from \( R \) to \( Y \) is continuous function, then we get \( a^2 R(y) = R(ay) \).

**Theorem 2.3.4.** Let us consider \( X \) and \( Y \) are normed and Banach spaces, respectively and \( \zeta: X \times Y \to [0, \infty) \) is a mapping such that

\[ \lim_{n \to \infty} a^n \zeta \left( \frac{x}{a^n}, \frac{y}{a^n} \right) = 0 \]  \hspace{1cm} (27)
If the function \( h : X \to Y \) with \( h(0) = 0 \), satisfies
\[
\| h(x + ay) + h(x-ay) - 2a^2 h(y) - 2h(x) \| \leq \zeta(x, y)
\] (28)

Then, the map \( R : X \to Y \) is a unique quadratic function satisfying the relation
\[
\| R(y) - h(y) \| \leq \frac{1}{2} \sum_{i=0}^{\infty} a^{2i} \zeta \left( 0, \frac{y}{a^{i+1}} \right)
\] (29)

where the quadratic map \( R \) is defined as
\[
\lim_{n \to \infty} a^{2n} h \left( \frac{y}{a^n} \right) = R(y), \quad \text{for all } y \in X.
\] (30)

Proof: Putting \( y = \frac{y}{a} \) in (16) and multiplying throughout by \( a^2 \), then, we have
\[
\left\| a^2 h \left( \frac{y}{a} \right) - h(y) \right\| \leq \frac{1}{2} \zeta \left( 0, \frac{y}{a^2} \right)
\] (31)

Again changing \( y = \frac{y}{a} \) and then multiplying throughout by \( a^2 \) in (31).
\[
\left\| a^4 h \left( \frac{y}{a^2} \right) - h(y) \right\| \leq \frac{a^2}{2} \zeta \left( 0, \frac{y}{a^2} \right) + \frac{1}{2} \zeta \left( 0, \frac{y}{a} \right)
\]

Thus, we obtain
\[
\| h(y) - R(y) \| \leq \frac{1}{2} \sum_{i=0}^{\infty} a^{2i} \zeta \left( 0, \frac{y}{a^{i+1}} \right)
\] (32)

For the convergence of \( \left\{ a^{2n} h \left( \frac{y}{a^n} \right) \right\} \), putting \( y = \frac{y}{a^k} \) and then multiplying throughout by \( a^{2k} \) in (32), we get
\[
\left\| a^{2k} h \left( \frac{y}{a^k} \right) - a^{2n+2k} h \left( \frac{y}{a^{n+k}} \right) \right\| \leq \frac{1}{2} \sum_{i=0}^{\infty} a^{2(i+k)} \zeta \left( 0, \frac{y}{a^{i+k}} \right)
\]

Then, from (32) the sequence \( \left\{ a^{2n} h \left( \frac{y}{a^n} \right) \right\} \), is a Cauchy sequence. But \( Y \) is a Banach space thus the sequence \( \left\{ a^{2n} h \left( \frac{y}{a^n} \right) \right\} \) converges in \( Y \). So, let us define a mapping \( h : X \to Y \) by
\[
\lim_{n \to \infty} a^{2n} h\left(\frac{y}{a^n}\right) = R(y)
\]

Then, using Theorem 3, the map \( R: X \to Y \) is quadratic. Further, the remaining part is similar to the Theorem 3.

**Corollary 3.** Let \( h: X \to Y \) be a mapping and \( h(0) = 0 \) which satisfies the inequality \[
\|h(x + ay) + h(x - ay) - 2a^2h(y) - 2h(x)\| \leq \varepsilon
\]
for all \( x, y \in X \), then, \( \exists \) a mapping \( R: X \to Y \) which satisfies the relation \[
\|R(y) - h(y)\| \leq \frac{q}{2(1 - a^2)}
\]
where the mapping \( R \) is defined as \[
\lim_{n \to \infty} a^{2n} h\left(\frac{y}{a^n}\right) = R(y) , \text{ for all } y \in X.
\]

**Corollary 4.** Let \( h: X \to Y \) be a mapping and \( h(0) = 0 \) which satisfies the inequality \[
\|h(x + ay) + h(x - ay) - 2a^2h(y) - 2h(x)\| \leq \varepsilon \|x\| + \|y\|
\]
for some \( p > 2 \), then, \( \exists \) a mapping \( R: X \to Y \) which satisfies the relation \[
\|R(y) - h(y)\| \leq \frac{1}{2} \frac{\varepsilon}{a^p - a^2} \|y\|
\]
where the mapping \( R \) is defined as \[
R(y) = \lim_{n \to \infty} a^{2n} h\left(\frac{y}{a^n}\right) , \text{ for all } y \in E_1.
\]

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