Decentralized Stochastic Gradient Tracking for Empirical Risk Minimization

Jiaqi Zhang
Department of Automation
Tsinghua University
Beijing, China
ZJQ16@MAILS.TSINGHUA.EDU.CN

Keyou You
Department of Automation
Tsinghua University
Beijing, China
YOUKY@TSINGHUA.EDU.CN

Abstract
Recent works have shown superiorities of decentralized SGD to centralized counterparts in large-scale machine learning, but their theoretical gap is still not fully understood. In this paper, we propose a decentralized stochastic gradient tracking (DSGT) algorithm over peer-to-peer networks for empirical risk minimization problems, and explicitly evaluate its convergence rate in terms of key parameters of the problem, e.g., algebraic connectivity of the communication network, mini-batch size, and gradient variance. Importantly, it is the first theoretical result that can exactly recover the rate of the centralized SGD, and has optimal dependence on the algebraic connectivity of the networks when using stochastic gradients. Moreover, we explicitly quantify how the network affects speedup and the rate improvement over existing works. Interestingly, we also point out for the first time that both linear and sublinear speedup can be possible. We empirically validate DSGT on neural networks and logistic regression problems, and show its advantage over the state-of-the-art algorithms.

1. Introduction
1.1 Decentralized empirical risk minimization
Empirical risk minimization (ERM) arise in many machine learning applications [7], which is to solve an optimization problem of the form

$$\min_{x \in \mathbb{R}^m} f(x) \triangleq \sum_{u=1}^{N} l(x; d_u)$$

(1)

where $\mathcal{D} = \{d_1, \cdots, d_N\}$ is the dataset, $N = |\mathcal{D}|$ is the number of samples, and $l(x; d_u)$ is the loss of sample $d_u$ given parameter $x$. The dataset is often large in applications and it may be too slow to train on a single computing node such as CPU, GPU, or server. Moreover, in some applications the dataset is collected by spatially distributed nodes and each local dataset is privately preserved by the associated node. Both cases suggest the necessity of using multiple networked nodes for training.

Two types of architectures (networks) are commonly adopted for training. 1) master-slave architecture [9, 19, 22], where a master node in the network collects and aggregates information (e.g., local gradients) from all slave nodes at each iterate for updating. 2)
peer-to-peer architecture, where each node performs local updates and communicates only with neighbor nodes to share information. Their associated algorithms in this work are referred to as centralized and decentralized algorithms, respectively. Since the centralized one is vulnerable to the damage of master node, has poor privacy protection, and suffers from the communication bottleneck of the master node [20, 5], this paper focuses on the design of decentralized algorithms.

For decentralized training with \( n \) nodes, each node \( i \) collects or is assigned a local dataset \( D_i = \{d_1^{(i)}, \ldots, d_{N_i}^{(i)}\} \) with \( N_i \equiv |D_i| \) samples. We allow local datasets to have different sizes and be sampled from different distribution. Thus, each node \( i \) has a local objective function \( f_i(x) \), and (1) can be rewritten as

\[
\min_{x \in \mathbb{R}^m} f(x) = \sum_{i=1}^n f_i(x), \quad f_i(x) \triangleq \sum_{u=1}^{N_i} l(x; d_u^{(i)}).
\] (2)

The interactions of nodes is modeled by a communication graph \( G = (V, E) \), where \( V = \{1, \ldots, n\} \) is the set of nodes, \( E \) is the set of edges and \((i, j) \in E \) if and only if nodes \( i \) and \( j \) can communicate with each other. The set \( N_i = \{j|(i, j) \in E\} \) is called the neighbors of node \( i \). In decentralized training, each node only performs local computations and only communicates with its neighbors.

1.2 Our contribution

We propose a novel decentralized stochastic gradient tracking (DSGT) algorithm to solve (2) with multiple nodes. DSGT extends the gradient tracking method in [26, 29], and allows nodes to have local datasets with different sample sizes and distributions. In DSGT, each node computes stochastic gradients using mini-batch samples from its local dataset, and communicate with its neighbors at each iterate.

For convex and Lipschitz smooth cost functions, we provide an explicit convergence rate of DSGT in terms of key parameters of the problem, i.e., \( O\left(\frac{\sigma}{\sqrt{K} \sqrt{\eta N} \sqrt{1-\rho}}\right) \), where \( K \) is the number of iterations, \( \eta N \) is the mini-batch size, \( \sigma^2 \) is the gradient variance, and \((1-\rho) \in (0, 1]\) is the well-studied algebraic connectivity of the communication graph (formally defined in Assumption 1) [8, 3]. Note that the optimal dependence of convergence rate of decentralized algorithms on the algebraic connectivity is \( O(1/\sqrt{1-\rho}) \) [31, 32]. To the best of our knowledge, DSGT the first to have this optimal dependence using stochastic gradients.

Moreover, our theoretical result is the first one that exactly recovers the rate of centralized stochastic gradient descent (SGD) algorithm [18, 27, 2]. This also improves the rate of D²2[36], which is, to our best knowledge, the only existing algorithm that can effectively handle decentralized datasets.

Importantly, we show that a linear speedup in training time w.r.t. number of nodes is achievable under the same conditions as existing works [20, 36, 5]. However, we point out, for the first time, that there are scenarios where the speedup is only sublinear for both DSGT and existing algorithms, which suggests the importance of designing good decentralized algorithms.

Finally, we validate our theoretical results on neural networks and logistic regression problems, and show the advantage of DSGT over the state-of-the-art algorithms.
1.3 Related work

The most relevant algorithm is $D^2$ [36], which is, as far as we know, the only existing algorithm with convergence rate independent of the variance among different local datasets. $D^2$ can be seen as an extension of EXTRA [34] to stochastic gradient case, or an extension of D-PSGD [20] to handle decentralized data. The convergence rate of $D^2$ and D-PSGD are respectively $O\left(\frac{L\sigma}{\sqrt{nK}} + \frac{L}{K}\right)$ and $O\left(\sigma\sqrt{nK} + \frac{n^{3/2}K^{2/3}}{3}\right)$, both of which are slower than the $O\left(\frac{\sigma}{\sqrt{nK}}\right)$ rate of DSGT. Moreover, the convergence rate of DSGT has optimal dependence $O\left(1/(1-\rho)^{2}\right)$ on the algebraic connectivity, which is better than $O\left(1/(1-\rho)^{2}\right)$ of D-PSGD and $D^2$. We will compare $D^2$ and D-PSGD with DSGT in more detail later.

DSGT is inspired by DIGing [26, 29], which uses full gradients with a gradient tracking method, and achieves linear convergence rate for strongly convex and Lipschitz smooth objective functions. Our extension is non-trivial since the strongly convex condition is essential in their proof and cannot be easily removed. For example, [28, 42] studies DIGing with stochastic gradients for strongly convex functions, and shows the convergence to a neighborhood of the optimal solution with convergence rate implicitly given by the spectral radius of a complicated matrix. No result is given for speedup and mini-batch size. In contrast, this work removes the strongly convex condition and derives the convergence rate with explicit dependence on key parameters of the problem.

There are many other decentralized algorithms that focus on orthogonal aspects to ours. 1) Algorithms using full local gradients such as DGD[25], DDA[1], EXTRA[34], DIGing[26], MSDA[31] and MSPD[32]. MSDA is shown to have optimal dependence on the algebraic connectivity of the network, but it requires to calculate the gradient of Fenchel conjugate, which makes it difficult to utilize stochastic gradients. MSPD is designed for non-smooth cost functions. 2) Algorithms for directed networks, including DEXTRA [39], SGP [5, 24], SAB [42, 41], etc. In particular, DEXTRA reduces to DGD and SGP reduces to D-PSGD for undirected graphs. 3) Algorithms with asynchronous updates such as AD-PSGD[21], AsySONATA[38], AsySPA [44, 4], APPG [45], and [12, 4]. Algorithms for unreliable network [37], compressed or efficient communication [16, 33, 14, 14], stale gradient information [35, 5], general stochastic approximation [6], information diffusion [43], distributed SGD with momentum [11], non-convex problems [5, 11], etc. It is interesting to extend DSGT to the above situations in the future.

Finally, it is also possible to accelerate DSGT with variance reduction methods such as SAG [30], SVRG [13], and SAGA [10], which can achieve linear convergence rate with stochastic gradients. For example, [23] combines EXTRA and SAG to obtain DSA, which converges linearly. However, its performance for non-strongly convex functions and speedup w.r.t. number of nodes are not clear.

Notations Throughout this paper, $\| \cdot \|$ denotes the $l_2$ norm of vectors or induced $l_2$ norm of matrices. $\| \cdot \|_F$ denotes the Frobenius norm of matrices. $\nabla f(x)$ denotes the gradient of $f$ at $x$. $f^*$ and $x^*$ denote the optimal solution and an optimal point of (1), respectively. $I$ denotes the identity matrix, and $1$ denotes the vector with all ones, the dimension of which depends on the context. A matrix $W$ is doubly-stochastic if $W1 = 1$ and $W^T1 = 1$. 3
2. The Decentralized Stochastic Gradient Tracking (DSGT) algorithm

The Decentralized Stochastic Gradient Tracking (DSGT) is given in Algorithm 1. Each node $i$ keeps local variables $x_{i,k}$, $\tilde{x}_{i,k}$, and $y_{i,k}$, and updates them only using locally computed stochastic gradients and information from neighbors. $[W]_{ij}$ used in (5) and (6) is the $(i,j)$-th element of $W$, which is a doubly-stochastic matrix in $\mathbb{R}^{n \times n}$ and $[W]_{ij} = 0$ if $i$ and $j$ cannot communicate, i.e., $(i,j) \notin \mathcal{E}$.

From a global viewpoint, Algorithm 1 can be written as the following compact form
\begin{align}
X_{k+1} &= W(X_k - \gamma Y_k) \\
Y_{k+1} &= WY_k + \partial F(X_{k+1}; \xi_{k+1}) - \partial F(X_k; \xi_k)
\end{align}
(3)
where $k \geq 1$, $X_k$, $Y_k$ and $\partial F(X_k; \xi_k)$ are $n \times m$ matrices defined as
\begin{align*}
X_k &= [x_{1,k}, x_{2,k}, \ldots, x_{n,k}]^T, \\
Y_k &= [y_{1,k}, y_{2,k}, \ldots, y_{n,k}]^T, \\
\partial F(X_k; \xi_k) &= [\partial f_1(x_{1,k}; \xi_1), \ldots, \partial f_n(x_{n,k}; \xi_n)]^T.
\end{align*}
with $\xi_k = \{\xi_{1,k}, \xi_{2,k}, \ldots, \xi_{n,k}\}$, $\partial f_i(x_{i,k}; \xi_{i,k}) = \sum_{d \in \xi_{i,k}} \partial l(x; d)$, and $\xi_{i,k}$ is a set of $\eta N_i (\eta \in (0, 1))$ data points uniformly randomly sampled from local dataset $D_i$ at iteration $k$. The initial state is $X_1 = WX_0, Y_1 = \partial F(X_1; \xi_1)$.

2.1 Mini-batch size proportional to the size of local dataset

A notable feature of DSGT is that the proportion of mini-batch size to local dataset size should be the same among nodes, i.e., $\eta$ in Algorithm 1, which is different from most existing decentralized algorithms where all nodes use the same batch size regardless of the local dataset’s size. We now provide an interpretation for it. Let $\mathcal{F}_k = \{X_0, \xi_0, Y_0, \cdots, X_{k-1}, \xi_{k-1}, Y_{k-1}, X_k, \xi_k, Y_k, X_{k+1}\}$ be the history sequence of random variables and define
\begin{align*}
\bar{x}_k &\triangleq \frac{1}{n} \sum_{i=1}^n x_{i,k} = \frac{1}{n} X_k^T 1, \\
\bar{y}_k &\triangleq \frac{1}{n} \sum_{i=1}^n y_{i,k} = \frac{1}{n} Y_k^T 1^{13} - \frac{1}{n} (Y_k + \partial F(X_k; \xi_k) - \partial F(X_1; \xi_1))^T 1 = \frac{1}{n} \partial F(X_k; \xi_k)^T 1, \\
\bar{g}_k &\triangleq \frac{1}{n} \sum_{i=1}^n \nabla f_i(x_{i,k}) = \frac{1}{n} \nabla F(X_k)^T 1, \text{ where } \nabla F(X_k) \triangleq [\nabla f_1(x_{1,k}), \ldots, \nabla f_n(x_{n,k})]^T.
\end{align*}
Then, we have $\mathbb{E}[\bar{y}_k | \mathcal{F}_{k-1}] = \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\partial f_i(x_{i,k}; \xi_{i,k})] = \frac{1}{n} \sum_{i=1}^n \frac{\eta N_i}{N_i} \sum_{u=1}^{N_i} \partial l(x_{i,k}; d_u^{(i)}) = \eta \bar{g}_k$. If all $x_{i,k}$ almost achieve consensus, i.e., $x_{i,k} \approx \bar{x}_k, \forall i$, it follows from (3) that the update of $\mathbb{E}[\bar{x}_k]$ is just a standard gradient descent step, i.e.,
\begin{align*}
\mathbb{E}[\bar{x}_{k+1} | \mathcal{F}_{k-1}] &= \bar{x}_k - \gamma \eta \bar{g}_k \approx \bar{x}_k - \gamma \eta \sum_{i=1}^n \nabla f_i(\bar{x}_k) = \bar{x}_k - \gamma \eta \nabla f(\bar{x}_k)
\end{align*}
which promises the convergence of $\bar{x}_k$ to an optimal solution of (1) for appropriate learning rate. In fact, the consensus is guaranteed due to the weighted averaging in (5) and (6).

However, if all nodes use the same mini-batch size, say $M$, then
\begin{align*}
\mathbb{E}[\bar{y}_k | \mathcal{F}_{k-1}] = \frac{1}{n} \sum_{i=1}^n \frac{M}{N_i} \sum_{u=1}^{N_i} \partial l(x_{i,k}; d_u^{(i)}) &\approx \frac{M}{n} \sum_{i=1}^n \frac{1}{N_i} \partial f_i(\bar{x}_k).
\end{align*}
The algorithm is actually solving a problem that is generally different from problem (1). This analysis also holds for many existing works (e.g. [20, 36, 5]).
Algorithm 1 Decentralized Stochastic Gradient Tracking (DSGT) — from the view of node $i$

**Input:** Initial states $x_{i,0}, y_{i,0} = 0, s_{i,0} = 0$, learning rate $\gamma$, weight matrix $W$, batch size proportion $\eta \in (0,1)$, and the maximum number of iterations $K$.

1: for $k = 0, 1, \cdots, K - 1$ do
2: Receive or fetch $x_{j,k}$ and $y_{j,k}$ from neighbors, and update $x_{i,k+1}$ by the weighted average:
\[ x_{i,k+1} = \sum_{j \in N_i} [W]_{ij} (\tilde{x}_{j,k} - \gamma y_{j,k}). \]  (5)
3: Uniformly randomly sample a mini-batch $\xi_{i,k+1}$ of size $\eta N_i a$ from the local dataset $D_i$, compute the stochastic gradient and set $s_{i,k+1} = \partial f_i(x_{i,k+1}; \xi_{i,k+1}) = \sum_{d \in \xi_{i,k+1}} \partial l(d_{i,k+1}; d)$.
4: Update $y_{i,k+1}$ and $x_{i,k+1}$ by
\[ y_{i,k+1} = \sum_{j \in N_i} [W]_{ij} y_{j,k} + s_{i,k+1} - s_{i,k} \]  (6)
5: Send $x_{i,k+1}$ and $y_{i,k+1}$ to all neighbors.
6: end for

**Output:** $\frac{1}{n} \sum_{i=1}^{n} x_{i,K}$.

---

a. We assume $\eta N_i$ is an integer to simplify notation. Any rounding method can be used in practice.

2.2 Comparison with $D^2$ and D-PSGD

DSGT shares some similarities with $D^2$[36] except the batch size. $D^2$ has the following update rule
\[ X_{k+1} = 2WX_k - WX_{k-1} - \gamma W(\partial F(X_k; \xi_k) - \partial F(X_{k-1}; \xi_{k-1})) \]  (7)
and DSGT can be rewritten as the following form by eliminating $Y_k$
\[ X_{k+1} = 2WX_k - W^2X_{k-1} - \gamma W(\partial F(X_k; \xi_k) - \partial F(X_{k-1}; \xi_{k-1})). \]  (8)

It turns out that the only difference is the weight matrix before $X_{k-1}$, which requires a completely different approach to examine the behavior of DSGT and brings the following advantages:

- The convergence of $D^2$ requires $W + \frac{1}{3} I$ to be positive definite while DSGT does not. This implies that DSGT can be adapted to a wider class of communication graphs. In fact, the optimal weight matrix $W$ with the smallest $\rho$ (defined later in Assumption 1) for a given topology often violates $W > -\frac{1}{3} I$ [40] and leads to the divergence of $D^2$. See an example in Appendix A.

- We will show that the convergence rate of DSGT has the optimal dependence $O(1/\sqrt{T-\rho})$ on $\rho$, which is better than $O(1/(1-\rho)^2)$ of $D^2$. 

5
• $y_{i,k}$ in DSGT is to track the aggregated local stochastic gradients of nodes and the expectation of $\bar{y}_k$ is an approximation of the full gradient. This fact offers an insight into the algorithm and brings convenience to extend DSGT. For example, one may consider to treat $y_{i,k}$ as true global stochastic gradients and applying existing accelerated methods such as momentum to $y_{i,k}$, while it lacks such an intuition in $D^2$.

• We also empirically validate that DSGT converges faster than $D^2$ in Section 4.

DSGT also improves D-PSGD [20] (or SGP [5] over static undirected graphs) by removing the effect of variance among local datasets to the convergence rate, which is similar to $D^2$. With some calculations, we can in fact rewrite (3) as

$$X_{k+1} = WX_k - \gamma \partial F(X_k; \xi_k) - \gamma \bigg( \sum_{t=0}^{k-1} W(W-I)^t \partial F(X_{k-t}; \xi_{k-t}) - \partial F(X_k; \xi_k) \bigg)$$

(9)

which can be seen as D-PSGD with a carefully designed momentum term. Note that the momentum in (9) uses history information from neighbors, which is quite different from naive acceleration schemes like Nesterov’s momentum with local history stochastic gradients.

3. Theoretical result

This section establishes the convergence rate result of DSGT. To this end, we make the following standard assumption as in [20, 5, 36].

**Assumption 1** The following assumptions are made throughout this paper.

1. All local objective functions are convex with Lipschitz continuous gradients, i.e., there exist $L_i > 0, i \in V$ such that for all $x, y \in \mathbb{R}^m$ and $i \in V, \|\nabla f_i(x) - \nabla f_i(y)\| \leq L_i \|x - y\|$.
2. The communication graph is connected and the weight matrix $W$ is doubly-stochastic satisfying $\rho \triangleq \|W - \frac{1}{n} \mathbf{1} \mathbf{1}^T\| < 1$.
3. All local stochastic gradients have bounded variance w.r.t. local datasets, i.e., there exist $\sigma_i > 0, i \in V$ such that $\mathbb{E}[\|\partial f_i(x_{i,k}; \xi_{i,k}) - \eta \nabla f_i(x_{i,k})\|^2 | F_{k-1}] \leq \sigma_i^2, \forall i, x, k$.

We define $L \triangleq \max_i L_i$ and $\rho \triangleq \|W - \frac{1}{n} \mathbf{1} \mathbf{1}^T\|$. Note that $1 - \rho$ is the well-studied algebraic connectivity of $G$ [8, 3], which reflects the information mixing over the graph. The weight matrix $W$ satisfying the assumption can be constructed by, e.g., Metropolis method [34, 40]. In particular, [40] discusses how to find an optimal weight matrix maximizing the algebraic connectivity for a given topology. If $W$ is further symmetric, then $\rho$ is equal to the second largest eigenvalue of $W$. Note that $L_i$ depends on the local dataset size $N_i$ and $\sigma_i^2$ is related to $\eta$ in general. We let $\sigma_s^2 \triangleq \mathbb{E}[\|\partial F(X_k; \xi_k) - \eta \nabla F(X_k)\|^2] \leq \sum_{i=1}^n \sigma_i^2$. The following is the main result of this paper.

1. See Appendix B
2. A graph is connected means there exists a path between any two nodes in the graph. This is obviously a necessary condition.
Theorem 1  Under Assumption 1, if the learning rate $\gamma$ satisfies $\gamma < \frac{(1-\rho)^2}{\eta L (1+\rho)^2}$, then all $x_{i,k}, i \in V$ in Algorithm 1 converges to an optimal solution of (1) with the following convergence rate

$$
\frac{1}{N} \left( \mathbb{E}[f(\bar{x}_K)] - f^* \right) \leq \frac{n\|\bar{x}_0 - x^*\|^2}{2KN\gamma\eta} + \frac{\sigma^2\gamma}{\eta nN} + \frac{4\rho\sigma^2\gamma L(1-\rho^K)}{(1-\sqrt{\rho})N} + \frac{(1-\rho^K)LC}{KN(1-\sqrt{\rho})}
$$

where $\bar{x}_K = \frac{1}{K} \sum_{t=2}^{K+1} \bar{x}_t$ and $C = 2(\rho + 1)^2 \mathbb{E}[\|X_0 - 1\bar{x}_0^T\|^2] + 2\gamma \mathbb{E}[\|\nabla F(WX_0)\|^2] + 2\gamma \sigma^2$.

The proof is given in Appendix C. Theorem 1 evaluates the convergence rate of DSGT with respect to key parameters of the problem and network topology explicitly, including $n, \gamma, \rho, \eta, \sigma^2$ and $L$. The constant $C$ shows the effect of initial condition to the convergence rate. By initializing all nodes with the same value and appropriately choosing the learning rate $\gamma$, we have the following corollary.

Corollary 2  Under Assumption 1, let $x_{1,0} = \cdots = x_{n,0} = x_0$, $D = \|x_0 - x^*\|$, $\delta = \max(2\sqrt{nLn}, 1)$, and set learning rate $\gamma = \frac{nD}{\sqrt{2K}nL\sqrt{1-\rho}}$. If $K \geq 8\left(\frac{nD}{\eta nL(1-\rho)^2}\right)^2$, then Algorithm 1 converges at the following rate

$$
\frac{1}{N} \left( \mathbb{E}[f(\bar{x}_K)] - f^* \right) \leq \frac{2D\sigma_s}{\eta n\sqrt{K} \sqrt{1-\rho}} + \frac{2\rho LnD \left( \sum_{i=1}^{n} \|\nabla f_i(x_0)\|^2 + \sigma^2 \right)}{NK \sqrt{K} (1-\sqrt{\rho}) \sigma_s \delta}
$$

Proof  Let $q = \sqrt{1 + \frac{4\rho L n}{1-\sqrt{\rho}}}$. Notice that $\gamma < \frac{(1-\rho)^2}{\eta L (1+\rho)^2}$ and $1 - \rho^K < 1$. It follows from Theorem 1 that

$$
\frac{1}{N} \left( \mathbb{E}[f(\bar{x}_K)] - f^* \right) \leq \frac{\sqrt{2}D\sigma_s q}{\eta n\sqrt{K} \sqrt{1-\rho}} + \frac{\sqrt{2} \rho L n D \left( \sum_{i=1}^{n} \|\nabla f_i(x_0)\|^2 + \sigma^2 \right)}{NK \sqrt{K} (1-\sqrt{\rho}) \sigma_s q}
$$

The desired result follows from $q \leq \delta \sqrt{1 + \rho/(1 - \sqrt{\rho})} < \delta \sqrt{1/(1 - \sqrt{\rho})} = \sqrt{1 + \sqrt{\rho}} \sqrt{1 - \rho} < \sqrt{2} \sqrt{1 - \rho}$ and $\frac{1}{q(1-\sqrt{\rho})} \leq \frac{\sqrt{2}}{\sqrt{1-\rho}}$.

Corollary 2 suggests that the convergence rate of DSGT is $O\left(\frac{\sigma_s}{\eta n\sqrt{K} \sqrt{1-\rho}}\right)$ if $K$ is sufficiently large.

Optimal dependence on $\rho$. Notably, Corollary 2 points out that the dependence of convergence rate on algebraic connectivity is $O(1/\sqrt{1 - \rho})$, which is shown to be optimal in decentralized optimization [31, 32], and improves on existing works [20, 5, 36].

3.1 Recover the rate of centralized SGD

We now show that DSGT exactly recovers the convergence rate of centralized SGD for complete graphs, i.e., each node connects with all other nodes. The weight matrix becomes $W = \frac{1}{n} n1^T$, and thus $\rho = 0$. Note that in this case the state $x_{i,k}$ are identical among nodes at every time $k$. Thus, the update of DSGT is equivalent to the centralized SGD with mini-batch size $M \triangleq \eta n$, and hence the convergence rate should be $O(\sigma_s/\sqrt{KM})$. 
in [18, 27], where \( \sigma \) is given by the following standard assumption (i.e., bounded variance) for centralized SGD to solve (1),

\[
\mathbb{E}_{d \sim \mathcal{D}} \left[ \| \partial l(x; d) - \frac{1}{N} \sum_{u=1}^{N} \partial l(x; d_u) \| \right] \leq \sigma^2.
\]

In view of Assumption 1, \( \sigma^2 s \) can be set as \( \sigma^2 s = \eta N \sigma^2 \) since

\[
\mathbb{E} \left[ \| \partial F(X; \xi) - \eta \nabla F(X) \|_2^2 \right] \leq \eta N \mathbb{E}_{d \sim \mathcal{D}} \left[ \| \partial l(x; d) - \frac{1}{N} \sum_{u=1}^{N} \partial l(x; d_u) \| \right] \leq \eta N \sigma^2.
\]

The proof is given in Appendix D. By setting \( \rho = 0 \) and combining \( \sigma^2 s = \eta N \sigma^2 \), we obtain from Theorem 1 that

\[
\frac{1}{N} \left( \mathbb{E}[f(\bar{x}_K)] - f^* \right) \leq \frac{n \| \bar{x}_0 - x^* \|}{2KM \gamma} + \frac{\sigma^2 \gamma}{n}.
\]

which is exactly the same as centralized SGD with learning rate \( \gamma/(Kn) \) [18].

Plugging the learning rate in Corollary 2 to (11) (or minimizing the right-hand-of (11) w.r.t \( \gamma \)), we obtain the optimal learning rate \( \gamma = \frac{n \| \bar{x}_0 - x^* \|}{\sqrt{2KM \sigma}} \) and the optimal convergence rate of DSGT

\[
\frac{1}{N} \left( \mathbb{E}[f(\bar{x}_K)] - f^* \right) \leq \frac{\sqrt{2} \| \bar{x}_0 - x^* \|}{\sqrt{KM \sigma}}
\]

for sufficiently large \( K \), which is the well-known convergence rate \( O(1/\sqrt{KM}) \) of SGD [18, 2], and the constant factor is also identical. A notable observation is that the optimal learning rate should be proportional to the number of nodes for decentralized learning.

Note that (12) is obtained by directly setting \( \rho = 0 \) in Theorem 1, which shows the tightness of the upper bound in Theorem 1 to some extent. In contrast, previous theoretical analysis cannot reproduce the rate (12) by simply setting \( \rho = 0 \) [20, 5, 36]. In this sense, DSGT is the first decentralized algorithm with theoretical convergence rate exactly consistent with centralized SGD.

**Superiority over D^2 and D-PSGD** If we let the total mini-batch size \( M \) be proportional to the number of nodes \( n \), i.e., each node uses fixed batch size as in many existing works (e.g. [20, 36]), then the convergence rate of DSGT becomes \( O\left(\frac{\sigma}{\sqrt{nK}} + \frac{n^{1/4} \zeta^{1/2}}{K^{1/2}}\right) \) from (12), which improves the rate \( O\left(\frac{L}{\sqrt{nK}} + \frac{L}{K}\right) \) of D^2 by eliminating the dependence of \( L \), and improves the rate \( O\left(\frac{\sigma}{\sqrt{nK}} + \frac{n^{1/4} \zeta^{1/2}}{K^{1/2}}\right) \) of D-PSGD by removing the dependence of \( \zeta \), the variance among different local datasets.

### 3.2 Linear or sublinear speedup

Speedup is defined as \( S = T_1/T_n \), where \( T_n \) is the time cost to solve (1) to achieve a given accuracy using \( n \) nodes with equal computational capabilities. Speedup is important to reflect the scalability of a decentralized algorithm and is apparently up-bounded by \( n \).

We first consider the case that all local datasets have equal sizes. (12) shows that the number of iterates to achieve a certain accuracy of DSGT is exactly the same with that of centralized SGD. Note that in centralized SGD each iterate needs to compute \( \eta N \) samples’
gradients, while in DSGT each node only needs to compute $\eta N_i = \frac{1}{n} \eta N$ samples’ gradients. If the time spent on communication is negligible, then finishing an iterate in DSGT is $n$ times faster than that of centralized SGD, and hence a linear speedup $S = n$ is achievable. However, in some applications communications are the most time-consuming part, in which case the speedup may be largely reduced, and one may consider to combine DSGT with some communication compressed technique such as [16, 33, 14, 14].

We now consider the speedup for a general connected graph where $\rho \neq 0$. The convergence rate becomes $O(\sqrt{1 + \frac{4nLc}{1 - \sqrt{\rho}}})$ from Corollary 2. Hence, the speedup is reduced and becomes $S = cn$, where $c = 1/\sqrt{1 + \frac{4nLc}{1 - \sqrt{\rho}}}$. To examine $c$, we introduce the standard Lipschitz smoothness assumption in centralized SGD to solve (1):

$$\|\nabla f(x) - \nabla f(y)\| \leq L_c \|x - y\|, \forall x, y.$$ 

In many applications, the data in all $D_i$ are randomly sampled from the same distribution, and thus it is reasonable to assume that $L_c$ is proportional to the number of samples in $D$. Therefore, we can choose $L = \frac{1}{n} L_c$ and hence $c = 1/\sqrt{1 + \frac{4nLc}{1 - \sqrt{\rho}}}$, which is constant and not related to $n$ if $\rho$ is fixed. Thus, the speedup is still linear but with a smaller slope. Note that most existing algorithms (e.g. D-PSGD[20], D2[36], SGP[5], [11]) implicitly adopt the assumption that $L = L_c/n$ and $\rho$ is fixed to obtain the linear speedup result.

However, $\rho$ is often related to $n$, and the speedup may be further reduced if $\rho$ is increasing w.r.t $n$. Generally speaking, the graph should have denser edges to keep the $\rho$ unchanged for larger $n$, but a dense graph will introduce heavy communication overhead, which may slow down the convergence rate in wall time. Thus, there is a tradeoff in the design of communication topology. In practice, it is preferable to design a network with given $\rho$ by as few edges as possible to reduce communication overhead, which is addressed in [40]. To our best knowledge, most of existing decentralized methods (e.g. [20, 36, 5]) face such a tradeoff and it is empirically observed in [5].

For applications where $L = L_c/n$ does not hold, we always have $L \leq L_c^3$. Thus, the speedup becomes $S = cn = n/\sqrt{1 + \frac{4nLc}{1 - \sqrt{\rho}}} \approx \sqrt{\frac{(1 - \sqrt{\rho})n}{4nLc}} = O(\sqrt{n})$, where the approximation holds for large $n$ and $\rho$. Therefore, the speedup is sublinear, which may be the case if nodes collect data from different local distributions or local datasets are poorly partitioned from the whole dataset $D$.

We assume all local datasets have equal sizes so far, where nodes use identical minibatch sizes and hence can finish an iterate in the same time. If the sizes of local datasets are different, the node with smaller dataset computes the local stochastic gradients faster, and has to wait for nodes with larger datasets at each iterate, which decreases the speedup. Recall that the number of iterates are the same to achieve a given accuracy no matter what the local datasets’ sizes are. Thus, the speedup becomes $S = N/N_m < n$, where $N_m = \max_i N_i$ is the largest size of local datasets.

Nonetheless, there are scenarios where different local datasets’ sizes are preferable. If nodes have different computational capacities, we can assign the local dataset with size

---

3. $f(x)$ is $L$-smooth $\Rightarrow \frac{1}{2} \|x\|^2 - f(x)$ is convex $\Rightarrow \frac{1}{2} \|x\|^2 - f_i(x)$ is convex $\Rightarrow f_i(x)$ is $L$-smooth
4. Experiment

We numerically evaluate the DSGT on two tasks and compare it with the two state-of-the-art algorithms D-PSGD [20] and D$^2$ [36]. Note that the more recent algorithm SGP [5] reduces to D-PSGD for undirected graphs and hence we omit the comparison with it. The first task is training a logistic regression classifier on CIFAR-10 dataset [15], which is a convex problem. The second task is training LeNet [17] on CIFAR-10, which is a non-convex neural network training problem.

We test all the three algorithms over $n = 1, 6, 12, 18, 24$ nodes$^4$, and each node is assigned a randomly partitioned subset with equal sizes. The total mini-batch sizes is 1024 and hence each node uses batch size $1024/n$, which is for a fair comparison with centralized SGD. We use a constant learning rate tuned for each algorithm and each task from the grid $[0.001n, 0.005n, 0.01n, 0.03n, 0.05n, 0.1n, 0.2n]$. The communication graph is constructed using methods in [40]. In particular, we firstly randomly generate the topology with each node has $2\log_2 n$ neighbors on average, and determine the weight matrix $W$ by applying the FDLA method [40] for DSGT and D-PSGD, and Metropolis method for D$^2$. The weight matrix obtained by FDLA method generally has smaller $\rho$ than Metropolis method, but it always results in the divergence of D$^2$ as explained in Section 2.2. The implementation is based on PyTorch 0.4 and OpenMPI 1.10+.

For logistic regression task, Figs. 1(a) and 1(b) depict the decreasing training loss w.r.t. number of epochs and the wall time over 12 nodes, respectively. We can see DSGT has a proportional to the computation capacity. It ensures that nodes finish an iterate in the same time without waiting for each other, which leads to a linear speedup.

4. Each node is a physical core of Intel Xeon CPU E5-2660 v4 on a server.
faster convergence rate than both D-PSGD and D^2. The performances for other numbers of nodes are almost identical, and hence are omitted for saving space. Fig. 1(c) plots the training loss of DSGT w.r.t. epochs over different number of nodes, which shows that the number of iterates required to achieve a certain accuracy is almost not related to the number of nodes, and validates our analysis in Section 3.2. Fig. 1(d) illustrates the linear speedup of DSGT in training time, which is also consistent with the analysis in Section 3.2. The performance of DSGT on LeNet is similar as displayed in Fig. 2.

We then test the accuracy of the trained models on the test data and the results are presented in Table 1. It shows that the accuracy of any method remains almost unchanged over different number of nodes, which indicates that decentralized algorithms can keep the generalization capability of a model. An interesting observation is that DSGT achieves a bit higher accuracy than D-PSGD and D^2 on LeNet, which empirically shows the superiority of DSGT from a different point of view.

5. Conclusion

We propose DSGT in this paper, which is a distributed optimization algorithm for empirical risk minimization problems. DSGT leverages the stochastic gradient tracking method to achieve variance reduction, and allow nodes to have local datasets with different variances and sizes. We rigorously prove the convergence rate of DSGT, and show the exact consistency between DSGT and vanilla centralized SGD over a complete graph. DSGT achieves a linear speedup w.r.t. the number of nodes under the same assumptions in many existing works. However, we are the first to point out that DSGT and many existing algorithms may only achieve sublinear speedup in some scenarios. Experiments validate the efficiency of DSGT and show its advantages to existing algorithms.

Appendix A. An example where optimal weight matrix violates $W \succ -\frac{1}{3} I$

![Figure 3: A graph.](image-url)
Consider the communication topology given in Fig. 3. From [40], the optimal weight matrix minimizing \( \rho \) (maximizing the algebraic connectivity) is given by

\[
W = \begin{bmatrix}
0.5 & 0.5 & 0 \\
0.5 & 0 & 0.5 \\
0 & 0.5 & 0.5
\end{bmatrix}
\] (13)

which has negative eigenvalues -0.5 and \( \rho = 0.5 \). Thus, it does not satisfy \( W \succ -\frac{1}{\rho} I \). Moreover, \( D^2 \) indeed diverges for some problems with such \( W \). For example, consider a simple least square problem where \( f_i(x) = \frac{1}{2}(x - a_i)^2 \) and the global cost function is \( f(x) = \sum_{i=1}^{3} f_i(x) \). It follows from (7) that \( D^2 \) has the update rule

\[
X_{k+1} = (2 - \gamma)WX_k - (1 - \gamma)WX_{k-1}
\] (14)

where we use the full gradient \( \nabla f_i(x_i) = x_i - a_i \) instead of a stochastic gradient. To study the behavior of (14), we can rewrite it as

\[
\begin{bmatrix}
X_{k+1} - 1(x^*)^T \\
X_k - 1(x^*)^T
\end{bmatrix} = \begin{bmatrix}
(2 - \gamma)W & -(1 - \gamma)W \\
I & 0
\end{bmatrix} \begin{bmatrix}
X_k - 1(x^*)^T \\
X_{k-1} - 1(x^*)^T
\end{bmatrix} \text{H}
\]

where \( x^* \) is the minimum point of \( f(x) \). Thus, \( X_k \) converges to \( 1(x^*)^T \) if and only if \( H \) has spectral radius less than 1. However, it can be readily obtained that the spectral radius of \( H \) with \( W \) given in (13) is larger or equal than 1 for any \( \gamma < 0.5 \), and hence \( D^2 \) diverges for any \( \gamma < 0.5 \). In contrast, DSGT converges for any \( \gamma \leq 1 \).

**Appendix B. Proof of (9)**

It follows from (3) and \( Y_k^T1 = \partial F(X_k; \xi_k)^T1 \) that

\[
Y_{k+1} - \partial F(X_{k+1}; \xi_{k+1}) = WY_k - \partial F(X_k; \xi_k)
= (W - \frac{1}{n}11^T)(Y_k - \partial F(X_k; \xi_k)) + (W - I)\partial F(X_k; \xi_k).
\]

Using the relation \( Y_1 = \partial F(X_1; \xi_1) \), we have

\[
Y_{k+1} - \partial F(X_{k+1}; \xi_{k+1}) = \sum_{t=1}^{k} (W - I)^t \partial F(X_{k-t+1}; \xi_{k-t+1})
\]

which combined with (3) implies

\[
X_{k+1} = WX_k - \gamma WY_k
= WX_k - \gamma \partial F(X_k; \xi_k) - \gamma W(Y_k - \partial F(X_k; \xi_k)) + \gamma (I - W)\partial F(X_k; \xi_k)
= WX_k - \gamma \partial F(X_k; \xi_k) - \gamma \sum_{t=1}^{k-1} W(W - I)^t \partial F(X_{k-t}; \xi_{k-t}) + \gamma (I - W)\partial F(X_k; \xi_k)
= WX_k - \gamma \partial F(X_k; \xi_k) - \gamma \left( \sum_{t=0}^{k-1} W(W - I)^t \partial F(X_{k-t}; \xi_{k-t}) - \partial F(X_k; \xi_k) \right)
\]

where we have used the relation \( Y_{k+1} = \partial F(X_{k+1}; \xi_{k+1}) \).
Appendix C. Proof of Theorem 1

The proof is roughly divided into four steps. We first bound $\mathbb{E}[\|X_k - 1\bar{x}_k^T\|_F]$ and $\mathbb{E}[\|Y_k - 1\bar{y}_k^T\|_F]$ in Step 1 by constructing an linear matrix inequality. We then bound $\mathbb{E}[\|\mathbf{g}_k\|^2]$ defined in (4) in Step 2 by showing $\mathbf{g}_k$ can be seen as an inexact gradient. We next bound $\sum_{t=1}^{k} \mathbb{E}[\|X_t - 1\bar{x}_t\|_F^2]$ and $\sum_{t=1}^{k} \mathbb{E}[\|\bar{y}_t\|_F^2]$ using previous results in Step 3, and finally bound $\mathbb{E}[\|\bar{x}_k - \bar{x}^*\|^2]$ by following similar arguments as the proof of centralized SGD in Step 4 to complete the whole proof.

The following relations will be frequently used in the proof without special mention.

\[ \|W - \frac{1}{n} 11^T\| \leq \rho < 1, \quad \|I - \frac{1}{n} 11^T\| = 1, \quad \|AB\|_F \leq \|A\|\|B\|_F \]

where $A, B$ are any two matrices.

**Step 1: Bound $\mathbb{E}[\|Y_k - 1\bar{y}_k^T\|_F]$ and $\mathbb{E}[\|X_k - 1\bar{x}_k^T\|_F]$**

From (3) we have
\[ \|Y_{k+1} - 1\bar{y}_{k+1}^T\|_F = \|WY_k - 1\bar{y}_k^T + (\partial F(X_{k+1}; \xi_{k+1}) - \partial F(X_k; \xi_k)) - (1\bar{y}_{k+1}^T - 1\bar{y}_k^T)\|_F \]
\[ \leq \|WY_k - 1\bar{y}_k^T\| + \|(\partial F(X_{k+1}; \xi_{k+1}) - \partial F(X_k; \xi_k)) - (1\bar{y}_{k+1}^T - 1\bar{y}_k^T)\|_F. \]  
(15)

Recall from (4) that
\[ WY_k - 1\bar{y}_k^T = (W - \frac{1}{n} 11^T)(Y_k - 1\bar{y}_k^T) \]
\[ (\partial F(X_{k+1}; \xi_{k+1}) - \partial F(X_k; \xi_k)) - (1\bar{y}_{k+1}^T - 1\bar{y}_k^T) = \]
\[ (I - \frac{1}{n} 11^T)(\partial F(X_{k+1}; \xi_{k+1}) - \partial F(X_k; \xi_k)) \]

Using $\|W - \frac{1}{n} 11^T\| \leq \rho$, $\|I - \frac{1}{n} 11^T\| = 1$ and the relation $\|AB\|_F \leq \|A\|\|B\|_F$, we have
\[ \|WY_k - 1\bar{y}_k^T\| \leq \rho \|Y_k - 1\bar{y}_k^T\|_F \]
\[ \|(\partial F(X_{k+1}; \xi_{k+1}) - \partial F(X_k; \xi_k)) - (1\bar{y}_{k+1}^T - 1\bar{y}_k^T)\|_F \leq \|\partial F(X_{k+1}; \xi_{k+1}) - \partial F(X_k; \xi_k)\|_F \]  
(16)

Recall that $\mathbb{E}[\partial F(X_k; \xi_k)|\mathcal{F}_{k-1}] = \eta \nabla F(X_k)$. We have
\[ \mathbb{E}[\|\partial F(X_{k+1}; \xi_{k+1}) - \partial F(X_k; \xi_k)\|_F |\mathcal{F}_{k-1}] \]
\[ = \mathbb{E}[\|\partial F(X_{k+1}; \xi_{k+1}) - \eta \nabla F(X_{k+1}) + (\eta \nabla F(X_k) - \partial F(X_k; \xi_k)) + \eta(\nabla F(X_{k+1}) - \nabla F(X_k))\|_F |\mathcal{F}_{k-1}] \]
\[ \leq \mathbb{E}[\|\partial F(X_{k+1}; \xi_{k+1}) - \eta \nabla F(X_{k+1}) + (\eta \nabla F(X_k) - \partial F(X_k; \xi_k))\|_F |\mathcal{F}_{k-1}] + \eta \mathbb{E}[\|X_{k+1} - X_k\|_F |\mathcal{F}_{k-1}] \]  
(17)

where we used Assumption 1 to obtain the last inequality. Moreover,
\[ \mathbb{E}[\|\partial F(X_{k+1}; \xi_{k+1}) - \eta \nabla F(X_{k+1}) + (\eta \nabla F(X_k) - \partial F(X_k; \xi_k))\|^2] \]
\[ = \mathbb{E}[\|\partial F(X_{k+1}; \xi_{k+1}) - \eta \nabla F(X_{k+1})\|^2] + \mathbb{E}[\|\partial F(X_k; \xi_k) - \eta \nabla F(X_k)\|^2] + 2\mathbb{E}[(\partial F(X_{k+1}; \xi_{k+1}) - \eta \nabla F(X_{k+1}))^T (\partial F(X_k; \xi_k) - \eta \nabla F(X_k))] \]
\[ \leq 2\sigma_x^2 \]
where the inequality follow from Assumption 1, \( \mathbb{E}[\partial F(X_k; \xi_k)] = \eta \nabla F(X_k) \), and the independence of \( \xi_{k+1} \) and \( \xi_k \).

Taking the full expectation on both sides of (16) and combining (17) and (18) yields

\[
\mathbb{E}[\|\partial F(X_{k+1}; \xi_{k+1}) - \partial F(X_k; \xi_k)\|_F] \leq \eta L \mathbb{E}[\|X_{k+1} - X_k\|_F] + \sqrt{2} \sigma_s \quad (19)
\]

By taking the full expectation on both sides of (15) and substituting (16) and (19), we obtain

\[
\mathbb{E}[\|Y_{k+1} - 1_y^T X_{k+1}\|_F] \leq \rho \mathbb{E}[\|Y_k - 1_y^T X_k\|_F] + \eta L \mathbb{E}[\|X_{k+1} - X_k\|_F] + \sqrt{2} \sigma_s. \quad (20)
\]

Moreover,

\[
\mathbb{E}[\|X_{k+1} - X_k\|_F] = \mathbb{E}[\|WX_k - X_k - \gamma WY_k\|_F] = \mathbb{E}[\|(W - I)(X_k - 1_x^T) - \gamma WY_k\|_F] 
\leq \left\| (W - \frac{1}{n} 11^T) + \frac{1}{n} 11^T - I \right\| \mathbb{E}[\|X_k - 1_x^T\|_F] + \gamma \|W\|_2 \mathbb{E}[\|Y_k\|_F] 
\leq (1 + \rho) \mathbb{E}[\|X_k - 1_x^T\|_F] + \gamma \mathbb{E}[\|Y_k - 1_y^T\|_F] + \gamma \sqrt{n} \mathbb{E}[\|\bar{y}_k\|] 
\]

which combined with (20) yields

\[
\mathbb{E}[\|Y_{k+1} - 1_y^T X_{k+1}\|_F] \leq (\rho + \eta L \gamma) \mathbb{E}[\|Y_k - 1_y^T X_k\|_F] + \eta L (1 + \rho) \mathbb{E}[\|X_{k+1} - 1_x^T\|_F] 
+ \eta L \gamma \sqrt{n} \mathbb{E}[\|\bar{y}_k\|] + \sqrt{2} \sigma_s. \quad (21)
\]

Now we bound \( \mathbb{E}[\|X_{k+1} - 1_x^T\|_F] \). It follows from (3) that

\[
\|X_{k+1} - 1_x^T\|_F = \|WX_k - 1_x^T + \gamma 1_y^T X_k - \gamma WY_k\|_F 
= \left\| (W - \frac{1}{n} 11^T)(X_k - 1_x^T) + \gamma (1_y^T - 1_x^T) \right\|_F 
\leq \rho \|X_k - 1_x^T\|_F + \gamma \rho \|Y_k - 1_y^T\|_F
\]

Taking the full expectation on both sides implies

\[
\mathbb{E}[\|X_{k+1} - 1_x^T\|_F] \leq \rho \mathbb{E}[\|X_k - 1_x^T\|_F] + \gamma \rho \mathbb{E}[\|Y_k - 1_y^T\|_F] \quad (22)
\]

Combing (21) and (22) implies the following linear matrix inequality

\[
\begin{bmatrix}
\mathbb{E}[\|X_{k+1} - 1_x^T\|_F] \\
\mathbb{E}[\|Y_{k+1} - 1_y^T\|_F]
\end{bmatrix} \preceq
\begin{bmatrix}
\rho \\
\eta L (1 + \rho) + \gamma \rho \eta L
\end{bmatrix}
\begin{bmatrix}
\mathbb{E}[\|X_k - 1_x^T\|_F] \\
\mathbb{E}[\|Y_k - 1_y^T\|_F]
\end{bmatrix} + \begin{bmatrix}
0 \\
\eta L \sqrt{n} \mathbb{E}[\|\bar{y}_k\|] + \sqrt{2} \sigma_s
\end{bmatrix}
\triangleq z_{k+1}
\]

where \( \preceq \) denotes the element-wise less than or equal sign and \( k \geq 1 \). We have

\[
z_k \preceq P^{k-1} z_1 + \sum_{t=1}^{k-1} P^{t-1} u_{k-t}
\quad (23)
\]

14
Let $\theta_1$ and $\theta_2$ be the two eigenvalues of $M$ and $\theta_2 > \theta_1$, then $P$ can be diagonalized as

$$M = T \Lambda T^{-1}, \quad \Lambda = \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{bmatrix}.$$

With some tedious calculations, we obtain

$$\theta_1 = \rho + \frac{\gamma \eta L - \sqrt{\gamma^2 \eta^2 L^2 + 4 \gamma \eta L \rho + 4 \gamma \eta \rho^2}}{2},$$
$$\theta_2 = \rho + \frac{\gamma \eta L + \sqrt{\gamma^2 \eta^2 L^2 + 4 \gamma \eta L \rho + 4 \gamma \eta \rho^2}}{2},$$
$$T = \begin{bmatrix} -\frac{\eta L + \eta \rho}{\sqrt{\gamma \eta L(4 \rho^2 + 4 \rho \eta L \gamma) + \eta L \gamma}} & \frac{\sqrt{\gamma \eta L(4 \rho^2 + 4 \rho \eta L \gamma) - \eta L \gamma}}{2(\eta L + \eta \rho)} \\ \frac{\eta L + \eta \rho}{\sqrt{\gamma \eta L(4 \rho^2 + 4 \rho \eta L \gamma) + \eta L \gamma}} & \frac{\sqrt{\gamma \eta L(4 \rho^2 + 4 \rho \eta L \gamma) - \eta L \gamma}}{2(\eta L + \eta \rho)} \end{bmatrix},$$
$$T^{-1} = \begin{bmatrix} -\frac{\eta L + \eta \rho}{\sqrt{\gamma \eta L(4 \rho^2 + 4 \rho \eta L \gamma) + \eta L \gamma}} & \frac{1}{3} - \frac{\sqrt{\gamma \eta L}}{2 \sqrt{4 \rho^2 + 4 \rho \eta L \gamma}} \\ \frac{\eta L + \eta \rho}{\sqrt{\gamma \eta L(4 \rho^2 + 4 \rho \eta L \gamma) + \eta L \gamma}} & \frac{1}{3} + \frac{\sqrt{\gamma \eta L}}{2 \sqrt{4 \rho^2 + 4 \rho \eta L \gamma}} \end{bmatrix},$$

and

$$P^k = T \Lambda^k T^{-1} = \begin{bmatrix} \frac{\theta_1^k + \rho^k}{2} + \frac{\sqrt{\gamma \eta L}}{2 \sqrt{4 \rho^2 + 4 \rho \eta L \gamma}} (\theta_1^k - \theta_2^k) & \frac{\rho \sqrt{\gamma \eta L}}{\eta L \sqrt{4 \rho^2 + 4 \rho \eta L \gamma}} (\theta_2^k - \theta_1^k) \\ \frac{\eta L + \eta \rho}{\sqrt{\gamma \eta L(4 \rho^2 + 4 \rho \eta L \gamma) + \eta L \gamma}} (\theta_1^k - \theta_2^k) & \frac{\rho \sqrt{\gamma \eta L}}{\eta L \sqrt{4 \rho^2 + 4 \rho \eta L \gamma}} (\theta_2^k - \theta_1^k) \end{bmatrix}. \quad (24)$$

Note that $\gamma \eta L < \frac{(1-\rho)^2}{(1+\rho)^2} \leq 1$ and hence $(\gamma \eta L)^2 \leq \gamma \eta L \leq \sqrt{\gamma \eta L}$, we define

$$\theta \triangleq \theta_2 \leq \rho + \frac{\gamma \eta L}{2} + \frac{\sqrt{\gamma \eta L(2 \rho + 1)^2}}{2} \leq \rho + (\rho + 1) \sqrt{\gamma \eta L} < \rho + (\rho + 1) \frac{1 - \rho}{1 + \rho} = 1. \quad (25)$$

One can also show that $\theta_1 \geq 0$. Thus, $\theta_2^k - \theta_1^k \leq \theta_2^k - \theta_1^k = \theta^k$, and we obtain from (24) that

$$P^k \mathbf{z}_1 \leqslant \begin{bmatrix} \frac{\eta L + \eta \rho}{\sqrt{\gamma \eta L(4 \rho^2 + 4 \rho \eta L \gamma) + \eta L \gamma}} \mathbb{E}[\|X_1 - 1 \mathbf{x}_T^T \|_F] + \frac{\rho \sqrt{\gamma \eta L}}{\eta L \sqrt{4 \rho^2 + 4 \rho \eta L \gamma}} \mathbb{E}[\|Y_1 - 1 \mathbf{y}_T^T \|_F] \\ \frac{\eta L + \eta \rho}{\sqrt{\gamma \eta L(4 \rho^2 + 4 \rho \eta L \gamma) + \eta L \gamma}} \mathbb{E}[\|X_1 - 1 \mathbf{x}_T^T \|_F] + \frac{\sqrt{\gamma \eta L(4 \rho^2 + 4 \rho \eta L \gamma) - \eta L \gamma}}{2 \sqrt{4 \rho^2 + 4 \rho \eta L \gamma}} \mathbb{E}[\|Y_1 - 1 \mathbf{y}_T^T \|_F] \end{bmatrix} \mathbf{1} \quad (26)$$

$$P_{k-t} \mathbf{z}_{k-t} \leq \theta^{-1} (\eta L \gamma \sqrt{n} \mathbb{E}[\|\mathbf{y}_{k-t}\|] + \sqrt{2} \sigma_s) \begin{bmatrix} \frac{\rho \sqrt{\gamma \eta L}}{\eta L \sqrt{4 \rho^2 + 4 \rho \eta L \gamma}} \mathbb{E}[\|Y_1 - 1 \mathbf{y}_T^T \|_F] \\ \frac{\sqrt{\gamma \eta L(4 \rho^2 + 4 \rho \eta L \gamma) + \eta L \gamma}}{2 \sqrt{4 \rho^2 + 4 \rho \eta L \gamma}} \mathbb{E}[\|Y_1 - 1 \mathbf{y}_T^T \|_F] \end{bmatrix}.$$

It then follows from (23) and (26) that

$$\mathbb{E}[\|X_k - 1 \mathbf{x}_T^T \|_F] \leq C_0 \theta^{k-1} + \sum_{i=1}^{k-1} \theta^{t-1} (C_1 \mathbb{E}[\|\mathbf{y}_{k-t}\|] + C_2 \sigma_s) \quad (27)$$
Moreover, the last equality follows from where the first inequality is from the Lipschitz continuous gradients in Assumption 1 and

\[
C_0 = \mathbb{E}[\|X_1 - 1\mathbf{x}_1^T\|_F] + \frac{\rho \sqrt{\eta L \gamma}}{\eta L \sqrt{\gamma + 4 \rho + 4 \rho^2}} \mathbb{E}[\|Y_1 - 1\mathbf{y}_1^T\|_F]
\]
\[
C_1 = \frac{\gamma \rho \sqrt{n \eta L \gamma}}{\sqrt{4 \rho^2 + 4 \rho + \eta L \gamma}}, \quad C_2 = \frac{\rho \sqrt{2 \eta L \gamma}}{\eta L \sqrt{4 \rho^2 + 4 \rho + \eta L \gamma}}. \tag{28}
\]

**Step 2: Bound** \(\mathbb{E}[\|g_k\|^2]\)

Let \(\tilde{f}_k \triangleq \sum_{i=1}^n \left[ f_i(x_{i,k}) + \nabla f_i(x_{i,k})^T (\mathbf{x}_k - x_{i,k}) \right] \leq f(\mathbf{x}_k)\).

Then, for any \(x \in \mathbb{R}^m\), we have

\[
f(x) = \sum_{i=1}^n f_i(x) \geq \sum_{i=1}^n \left[ f_i(x_{i,k}) + \nabla f_i(x_{i,k})^T (x - x_{i,k}) \right] \]
\[
= \sum_{i=1}^n \left[ f_i(x_{i,k}) + \nabla f_i(x_{i,k})^T (\mathbf{x}_k - x_{i,k}) \right] + \sum_{i=1}^n \nabla f_i(x_{i,k})^T (x - \mathbf{x}_k) \tag{29}
\]
\[
= \tilde{f}_k + ng_k^T (x - \mathbf{x}_k)
\]

Moreover,

\[
f(x) \leq \sum_{i=1}^n \left[ f_i(x_{i,k}) + \nabla f_i(x_{i,k})^T (x - x_{i,k}) + \frac{L_i}{2} \|x - x_{i,k}\|^2 \right] \]
\[
= \sum_{i=1}^n \left[ f_i(x_{i,k}) + \nabla f_i(x_{i,k})^T (\mathbf{x}_k - x_{i,k}) \right] + \sum_{i=1}^n \nabla f_i(x_{i,k})^T (x - \mathbf{x}_k) + \frac{L}{2} \sum_{i=1}^n \|x - x_{i,k}\|^2 \]
\[
= \tilde{f}_k + ng_k^T (x - \mathbf{x}_k) + \frac{L}{2} \sum_{i=1}^n \|x - \mathbf{x}_k\|^2 + \|x_{i,k} - \mathbf{x}_k\|^2 + L \sum_{i=1}^n (x - \mathbf{x}_k)^T (\mathbf{x}_k - x_{i,k}) \]
\[
= \tilde{f}_k + ng_k^T (x - \mathbf{x}_k) + \frac{nL}{2} \|x - \mathbf{x}_k\|^2 + \frac{L}{2} \|X_k - 1\mathbf{x}_k\|^2 + L \sum_{i=1}^n (x - \mathbf{x}_k)^T (\mathbf{x}_k - x_{i,k}) \]
\[
= \tilde{f}_k + ng_k^T (x - \mathbf{x}_k) + \frac{nL}{2} \|x - \mathbf{x}_k\|^2 + \frac{L}{2} \|X_k - 1\mathbf{x}_k\|^2 \tag{30}
\]

where the first inequality is from the Lipschitz continuous gradients in Assumption 1 and the last equality follows from

\[
\sum_{i=1}^n (x - \mathbf{x}_k)^T (\mathbf{x}_k - x_{i,k}) = nx^T \mathbf{x}_k - n\|\mathbf{x}_k\|^2 - \mathbf{x}^T \sum_{i=1}^n x_{i,k} + \mathbf{x}_k^T \sum_{i=1}^n x_{i,k}
\]
\[
= nx^T \mathbf{x}_k - n\|\mathbf{x}_k\|^2 - nx^T \mathbf{x}_k + n\mathbf{x}_k^T \mathbf{x}_k = 0.
\]
Let $x = \bar{x}_{k+1}$ in (30) and take the conditional expectation on both sides. By noticing that $E[X_k|F_{k-1}] = X_k$, $E[g_k|F_{k-1}] = g_k$ and $E[y_k|F_{k-1}] = \eta g_k$, we have

$$E[f(\bar{x}_{k+1})|F_{k-1}]$$

\[= \bar{f}_k + n \eta g_k^T E[(\bar{x}_{k+1} - x_k)|F_{k-1}] + \frac{n L}{2} E[||\bar{x}_{k+1} - x_k||^2|F_{k-1}] + \frac{L}{2} ||X_k - 1 \bar{x}_k||^2_F \]

\[= \bar{f}_k + n \eta g_k^T E[-\gamma \bar{y}_k|F_{k-1}] + \frac{n L}{2} E[||\gamma \bar{y}_k||^2|F_{k-1}] + \frac{L}{2} ||X_k - 1 \bar{x}_k||^2_F \]

\[= \bar{f}_k - n \eta \gamma ||g_k||^2 + \frac{n L \gamma^2}{2} E[||\gamma \bar{y}_k||^2] + \frac{L}{2} ||X_k - 1 \bar{x}_k||^2_F \]

\[= \bar{f}_k - n \eta \gamma (1 - \frac{\gamma L}{2}) ||g_k||^2 + \frac{L}{2} ||X_k - 1 \bar{x}_k||^2_F + \frac{L \gamma^2 \sigma_s^2}{2n} \]  \hspace{1cm} (31)

where the last equality follows from

$$E[||\bar{y}_k - \eta g_k||^2|F_{k-1}]$$

\[= E[||\frac{1}{n} 1^T(\partial F(X_k; \xi_k) - \eta \nabla F(X_k))||^2|F_{k-1}] \]

\[= E[||\frac{1}{n} (\partial f_1(x_{1,k}; \xi_{1,k}) - \eta \nabla f_1(x_{1,k})) + \cdots + \frac{1}{n} (\partial f_n(x_{n,k}; \xi_{n,k}) - \eta \nabla f_n(x_{n,k}))||^2|F_{k-1}] \]

\[= \frac{1}{n^2} \sum_{i=1}^{n} E[||\partial f_i(x_{i,k}; \xi_{i,k}) - \eta \nabla f_i(x_{i,k})||^2|F_{k-1}] \leq \frac{1}{n^2} \sum_{i=1}^{n} \sigma_i^2 = \frac{\sigma_s^2}{n^2} \]  \hspace{1cm} (32)

where we used the fact that $\xi_{i,k}$ and $\xi_{j,k}$ are independent if $i \neq j$ to obtain the last equality. Taking the full expectation on both sides of (31) yields

$$E[f(\bar{x}_{k+1})] \leq E[\bar{f}_k] - n \gamma \eta (1 - \frac{\gamma L}{2}) E[||g_k||^2] + \frac{L}{2} E[||X_k - 1 \bar{x}_k||^2_F] + \frac{L \gamma^2 \sigma_s^2}{2n}. \hspace{1cm} (33)$$

Since $\gamma \eta L < 1$, it follows from (33) that

$$E[||g_k||^2]$$

\[\leq \frac{2}{n \eta \gamma (2 - \gamma \eta L)} \left( E[\bar{f}_k] - E[f(\bar{x}_{k+1})] + \frac{L}{2} ||X_k - 1 \bar{x}_k||^2_F + \frac{L \gamma^2 \sigma_s^2}{2n} \right) \]

\[\leq \frac{2}{n \eta \gamma (2 - \gamma \eta L)} \left( E[f(\bar{x}_k)] - E[f(\bar{x}_{k+1})] + \frac{L}{2} ||X_k - 1 \bar{x}_k||^2_F \right) + \frac{L \gamma \sigma_s^2}{n^2 \eta (2 - \gamma \eta L)}. \hspace{1cm} (34)$$

Combining (32) and (34) implies that

$$\sum_{t=1}^{k} E[||\bar{y}_t||^2] = \sum_{t=1}^{k} \left( E[||\eta g_t||^2] + E[||y_t - \eta g_t||^2] \right)$$

\[\leq \eta \left( 2(E[f(\bar{x}_1)]) - E[f(\bar{x}_{k+1})] + L \sum_{t=1}^{k} E[||X_t - 1 \bar{x}_t||^2_F] \right) + \frac{k \eta L \gamma \sigma_s^2}{n^2 (2 - \gamma \eta L)} + \frac{k \sigma_s^2}{n^2} \]

\[\leq \frac{2 \eta (E[f(\bar{x}_1)]) - E[f(\bar{x}_{k+1})]}{n \gamma (2 - \gamma \eta L)} + \frac{\eta L}{n \gamma (2 - \gamma \eta L)} \sum_{t=1}^{k} E[||X_t - 1 \bar{x}_t||^2_F] + \frac{2k \sigma_s^2}{n^2 (2 - \gamma \eta L)}. \hspace{1cm} (35)$$

17
Step 3: Bound $\sum_{t=1}^{k} \mathbb{E}[\|X_t - 1\bar{x}_t\|_F^2]$ and $\sum_{t=1}^{k} \mathbb{E}[\|\bar{y}_t\|_F^2]$

We first bound $\sum_{t=1}^{k} \mathbb{E}[\|X_t - 1\bar{x}_t\|_F^2]$. For any $k \in \mathbb{N}$, define

$$\mu_k = [C_0, C_1 \mathbb{E}[\|\bar{y}_1\|] + C_2 \sigma_s, \ldots, C_1 \mathbb{E}[\|\bar{y}_{k-1}\|] + C_2 \sigma_s]^T \in \mathbb{R}^k$$

$$\vartheta_t = [\theta^t_1, \theta^t_2, \ldots, \theta, 1, 0, \ldots, 0]^T \in \mathbb{R}^k$$

$$\Theta_k = \sum_{t=1}^{k} \vartheta_t \vartheta_t^T \in \mathbb{R}^{k \times k}$$

where the elements are defined in (25) and (28). Clearly, $\Theta_k$ is nonnegative and positive semi-definite. We have from (27) that

$$\mathbb{E}[\|X_t - 1\bar{x}_t\|_F] \leq \mu_k^T \vartheta_t$$

and hence

$$\sum_{t=1}^{k} \mathbb{E}[\|X_t - 1\bar{x}_t\|_F^2] = \mu_k^T \Theta_k \mu_k \leq \|\Theta_k\| \|\mu_k\|^2. \tag{36}$$

To bound $\|\Theta_k\|$, let $[\Theta_k]_{ij}$ be the element in the $i$-th row and $j$-th column of $\Theta_k$. For any $0 < i \leq j \leq k$, we have

$$[\Theta_k]_{ij} = \sum_{t=i-1}^{k} \theta^{t-i+1} \theta^{t-j+1} = \sum_{t=i-1}^{k} \theta^{2t-i-j} + 2 = \frac{\theta^{2j-2}(1 - \theta^{2(k-j+2)})}{1 - \theta^2}. \tag{37}$$

Since $\Theta_k$ is symmetric, $\|\Theta_k\|$ equals to its spectral radius. By invoking the Gershgorin circle theorem, we have

$$\|\Theta_k\| \leq \max_{j} \sum_{i=1}^{j} |[\Theta_k]_{ij}| = \max_{j} \left[ \sum_{i=1}^{j} |[\Theta_k]_{ij}| + \sum_{i=j+1}^{k} |[\Theta_k]_{ij}| \right] = \max_{j} \left[ \sum_{i=1}^{j} |[\Theta_k]_{ij}| + \sum_{i=j+1}^{k} |[\Theta_k]_{ji}| \right]$$

$$= \max_{j} \left[ \frac{j \theta^{j-i}(1 - \theta^{2(k-j+2)})}{1 - \theta^2} + \frac{\theta^{j-i}(1 - \theta^{2(k+j+2)})}{1 - \theta^2} \right]$$

$$= \max_{j} \left[ \frac{(1 - \theta^j)(1 - \theta^{2(k-j+2)})}{(1 - \theta)(1 - \theta^2)} + \frac{\theta(1 - \theta^{k-j}) + \theta^2(1 - \theta^{2k+2})}{(1 - \theta)(1 - \theta^2)} \right]$$

$$= \frac{(1 + \theta^2)(1 - \theta^k)}{1 - \theta}. \quad \tag{37}$$

It then follows from (36) that

$$\sum_{t=1}^{k} \mathbb{E}[\|X_t - 1\bar{x}_t\|_F^2] \leq \|\Theta_k\| \left[ C_0^2 + \sum_{t=1}^{k-1} \mathbb{E}[\|\bar{y}_t\|_F^2] + 2C_2^2 \sigma_s^2 \right]$$

$$= 2C_1^2 \|\Theta_k\| \sum_{t=1}^{k} \mathbb{E}[\|\bar{y}_t\|_F^2] + \left( 2(k-1)C_2^2 \sigma_s^2 + C_0^2 \right) \|\Theta_k\|. \tag{38}$$
Combining this with (35) implies

\[
\sum_{t=1}^{k} \mathbb{E}[\|\bar{y}_t\|^2] \leq \frac{2\eta}{n\gamma(2 - \gamma L)} (\mathbb{E}[f(\bar{x}_1)] - \mathbb{E}[f(\bar{x}_{k+1})]) + \frac{2\eta LC_1^2 \|\Theta_k\|}{n\gamma(2 - \eta L)} \sum_{t=1}^{k} \mathbb{E}[\|\bar{y}_t\|^2]
\]

\[
+ \frac{\eta L (2(k - 1)C_2^2 \sigma_2^2 + C_0^2) \|\Theta_k\|}{n\gamma(2 - \gamma \eta L)} + \frac{2k\sigma_1^2}{n^2(2 - \gamma \eta L)}.
\]

(39)

An important fact is

\[
\frac{2\eta LC_1^2 \|\Theta_k\|}{n\gamma(2 - \eta L \gamma)} < 1
\]

(40)

where the proof is deferred to Appendix C.1.

It then follows from (39) that

\[
\left( 1 - \frac{2\eta LC_1^2 \|\Theta_k\|}{n\gamma(2 - \eta L \gamma)} \right) \sum_{t=1}^{k} \mathbb{E}[\|\bar{y}_t\|^2]
\leq \frac{2\eta (\mathbb{E}[f(\bar{x}_1)] - \mathbb{E}[f(\bar{x}_{k+1})]) + \eta L (2(k - 1)C_2^2 \sigma_2^2 + C_0^2) \|\Theta_k\|}{n\gamma(2 - \gamma \eta L)} + \frac{2k\sigma_1^2}{n^2(2 - \gamma \eta L)}
\]

Therefore,

\[
\sum_{t=1}^{k} \mathbb{E}[\|\bar{y}_t\|^2] \leq \frac{2\eta (\mathbb{E}[f(\bar{x}_1)] - \mathbb{E}[f(\bar{x}_{k+1})]) + \eta L (2(k - 1)C_2^2 \sigma_2^2 + C_0^2) \|\Theta_k\|}{n\gamma(2 - \gamma \eta L)} + \frac{2k\sigma_1^2}{n^2(2 - \gamma \eta L)}.
\]

(41)

**Step 4: Bound** \(\mathbb{E}[\|\bar{x}_k - x^*\|^2]\)

Let \(x^*\) be an optimal solution of (1) and \(e_k = \|\bar{x}_k - x^*\|^2\). We have

\[
e_{k+1} = \|\bar{x}_k - x^* - \gamma \bar{y}_k\|^2 = e_k - 2\gamma \bar{y}_k^T(\bar{x}_k - x^*) + \gamma^2\|\bar{y}_k\|^2.
\]

Taking the expectation\(^5\) and applying (29) implies that

\[
\mathbb{E}[e_{k+1}] = \mathbb{E}[e_k] - 2\gamma \mathbb{E}[g_k^T(\bar{x}_k - x^*)] + \gamma^2 \mathbb{E}[\|\bar{y}_k\|^2]
\leq \mathbb{E}[e_k] + \frac{2\gamma \eta}{n} (f^* - \mathbb{E}[\tilde{f}_k]) + \gamma^2 \mathbb{E}[\|\bar{y}_k\|^2]
\]

(42)

Recall that by letting \(x = \bar{x}_k\) in (30) and taking the expectation we have

\[
\mathbb{E}[f(\bar{x}_k)] \leq \mathbb{E}[\tilde{f}_k] + \frac{L}{2} \mathbb{E}[\|X_k - 1\bar{x}_k\|^2]
\]

which combined with (42) implies

---

5. Technically, we should first take the conditional expectation on \(F_{k-1}\), and then take the full expectation.
\[
\frac{2\gamma \eta}{n} \sum_{t=1}^{k} (\mathbb{E}[f(\bar{x}_t)] - f^*) \\
\leq \frac{2\gamma \eta}{n} \sum_{t=1}^{k} (\mathbb{E}[\tilde{f}_t] - f^*) + \frac{\eta L \gamma}{n} \sum_{t=1}^{k} \mathbb{E}[\|X_t - 1\bar{x}_t\|^2] \\
\leq \mathbb{E}[e_1] - \mathbb{E}[e_{k+1}] + \gamma^2 \sum_{t=1}^{k} \mathbb{E}[\|\tilde{y}_t\|^2] + \frac{\eta L \gamma}{n} \sum_{t=1}^{k} \mathbb{E}[\|X_t - 1\bar{x}_t\|^2] \\
\leq \mathbb{E}[e_1] + \gamma^2 \left( 1 + \frac{2C_1^2 \eta \|\Theta_k\|}{n \gamma} \right) \sum_{t=1}^{k} \mathbb{E}[\|\tilde{y}_t\|^2] + \frac{\eta L \gamma}{n} \left( 2(k-1)C_2^2 \sigma_s^2 + C_0^2 \right) \|\Theta_k\|
\]

where the last inequality follows from (35) and (38). Thus,

\[
\sum_{t=1}^{k} (\mathbb{E}[f(\bar{x}_t)] - f^*) \\
\leq \mathbb{E}[e_1] + \frac{n \gamma}{2 \gamma \eta} + \frac{\eta L \gamma}{2 \eta} \sum_{t=1}^{k} \mathbb{E}[\|\tilde{y}_t\|^2] + \frac{L}{2} \left( 2(k-1)C_2^2 \sigma_s^2 + C_0^2 \right) \|\Theta_k\| + \frac{2k \sigma_s^2 \gamma}{\eta n} \tag{43}
\]

From (41) we have

\[
\left( \frac{n \gamma}{2 \eta} + C_1^2 \eta \|\Theta_k\| \right) \sum_{t=1}^{k} \mathbb{E}[\|\tilde{y}_t\|^2] \leq \frac{n \gamma + 2 \eta L C_1^2 \|\Theta_k\|}{2n \eta (2 - \eta L \gamma) - 2 \eta L C_1^2 \|\Theta_k\|} \\
\left( 2 \eta (\mathbb{E}[f(\bar{x}_1)] - \mathbb{E}[f(\bar{x}_{k+1})]) + \eta L \left( 2(k-1)C_2^2 \sigma_s^2 + C_0^2 \right) \|\Theta_k\| + \frac{2k \sigma_s^2 \gamma}{\eta n} \right) \tag{44}
\]

Note that

\[
\frac{n \gamma + 2 \eta L C_1^2 \|\Theta_k\|}{n \gamma (2 - \eta L \gamma) - 2 \eta L C_1^2 \|\Theta_k\|} = \frac{(4 \rho^2 + 4 \rho + \eta L \gamma) + 2 \rho^2 \eta^2 L^2 \gamma^2 \|\Theta_k\|}{(2 - \eta L \gamma)(4 \rho^2 + 4 \rho + \eta L \gamma) - 2 \rho^2 \eta^2 L^2 \gamma^2 \|\Theta_k\|} \leq 1 \tag{45}
\]

the proof of which can be found in Appendix C.2. Thus, it follows from (44) and (45) that

\[
\left( \frac{n \gamma}{2 \eta} + C_1^2 \eta \|\Theta_k\| \right) \sum_{t=1}^{k} \mathbb{E}[\|\tilde{y}_t\|^2] \\
\leq \mathbb{E}[f(\bar{x}_1)] - \mathbb{E}[f(\bar{x}_{k+1})] + \frac{L}{2} \left( 2(k-1)C_2^2 \sigma_s^2 + C_0^2 \right) \|\Theta_k\| + \frac{k \sigma_s^2 \gamma}{\eta n} \tag{46}
\]

\[20\]
where we used the following important relation to obtain the last inequality
\[ C \]
the proof of which is provided in Appendix C.3.

We now bound the constant \( C_0^2 \) that appears in the last term of (48). Recall that \( Y_1 = \partial F(X_1; \xi_1) \), and hence it follows from (28) that
\[ C_0^2 \leq 2E\|X_1 - \mathbf{1}x_1^T\|^2 + \frac{2\rho^2 \gamma}{\eta L \gamma + 4 \rho + 4 \rho^2} \left( \frac{E\left[ \|Y_1 - \mathbf{1}y_1^T\|^2 \right]}{\eta L \gamma + 4 \rho + 4 \rho^2} \right) \]
\[ \leq 2E\|X_1 - \mathbf{1}x_1^T\|^2 + \frac{2\rho^2 \gamma}{\eta L \gamma + 4 \rho + 4 \rho^2} \left( \frac{E\left[ \|Y_1\|^2 \right]}{\eta L \gamma + 4 \rho + 4 \rho^2} \right) \]
\[ \leq 2E\left[ \left( W - \frac{1}{n} \mathbf{1}^T X_0 - \mathbf{1}x_0^T \right) \right]^2 + \frac{2\rho^2 \gamma \left( \|\nabla F(X_1) - \partial F(X_1; \xi_1)\|^2 \| \| \nabla F(X_1)\|^2 \| \right)}{\eta L \gamma + 4 \rho + 4 \rho^2} \]
\[ \leq 2E\left[ \left( W - \frac{1}{n} \mathbf{1}^T X_0 - \mathbf{1}x_0^T \right) \right]^2 + \frac{2\rho^2 \gamma \left( \|\nabla F(W X_0)\|^2 \right)}{\eta L \gamma + 4 \rho + 4 \rho^2} \]
Substituting (49) and (48) into (47) implies
\[ \sum_{t=1}^{k} (\mathbb{E}[f(\mathbf{x}_t)] - f^*) \]
\[ \leq \mathbb{E}[f(\mathbf{x}_1)] - \mathbb{E}[f(\mathbf{x}_{k+1})] + \frac{n\mathbb{E}[e_1]}{2\gamma \eta} + \frac{4(k - 1)\rho \sigma^2 \gamma (1 - \rho)^k}{1 - \sqrt{\rho}} + \frac{k \sigma^2 \gamma}{\eta m} + \frac{\rho (1 - \rho)^k L C}{1 - \sqrt{\rho}} \]
or equivalently,
\[ \sum_{t=2}^{k+1} (\mathbb{E}[f(\mathbf{x}_t)] - f^*) \leq \frac{n\mathbb{E}[e_1]}{2\gamma \eta} + \frac{4(k - 1)\rho \sigma^2 \gamma (1 - \rho)^k}{1 - \sqrt{\rho}} + \frac{k \sigma^2 \gamma}{\eta m} + \frac{\rho (1 - \rho)^k L C}{1 - \sqrt{\rho}} \]
where \( C \) is defined in Theorem 1.
Let $\tilde{x}_K = \frac{1}{K} \sum_{t=2}^{K+1} x_t$, it follows from (50) and the convexity of $f(x)$ that

$$
E[f(\tilde{x}_K)] - f^* \leq \frac{1}{K} \sum_{t=2}^{K+1} E[f(x_t)] - f^* \\
\leq \frac{nE[e_1]}{2K\gamma \eta} + \frac{4\rho \sigma^2 \gamma L(1 - \rho^K)}{1 - \sqrt{\rho}} + \frac{\sigma^2 \gamma}{\eta m} + \frac{\rho(1 - \rho^K)LC}{K(1 - \sqrt{\rho})} \\
= \frac{n\|\bar{x}_0 - x^*\|^2}{2K\gamma \eta} + \frac{4\rho \sigma^2 \gamma L(1 - \rho^K)}{1 - \sqrt{\rho}} + \frac{\sigma^2 \gamma}{\eta m} + \frac{\rho(1 - \rho^K)LC}{K(1 - \sqrt{\rho})}.
$$

Divide it by $N$ on both sides implies Theorem 1.

C.1 Proof of (40)

To simplify notation, let $\tilde{\gamma} = \eta L \gamma$. Thus, $\tilde{\gamma} < \frac{(1 - \rho^2)}{(1 + \rho)^2} \leq 1$. It follows from (28) and (37) that

$$
\frac{2\eta LC^2}{n\gamma (2 - \eta L \gamma)} \left\| \Theta_k \right\| \leq \frac{2\tilde{\gamma}^2 \rho^2 (1 - \theta^k)(1 + \theta^2)}{(1 - \theta)(4\rho^2 + 4\rho + \tilde{\gamma})(2 - \tilde{\gamma})} \leq \frac{4\tilde{\gamma} \rho^2}{1 - \theta}
$$

where we used the relation $\tilde{\gamma} \leq 4\rho^2 + 4\rho + \tilde{\gamma}$. Then, it is sufficient to show that $4\tilde{\gamma} \rho^2 < 1 - \theta$, or

$$
4\tilde{\gamma} \rho^2 + \theta < 1.
$$

By (25), we have

$$
4\tilde{\gamma} \rho^2 + \theta < \rho + \frac{\tilde{\gamma}}{2} + \frac{\sqrt{\tilde{\gamma}(2\rho + 1)}^2}{2} + 4\tilde{\gamma} \rho^2.
$$

Clearly, $4\tilde{\gamma} \rho^2 + \theta$ is an increasing function w.r.t $\tilde{\gamma}$. Thus,

$$
4\tilde{\gamma} \rho^2 + \theta < \rho + \frac{(1 - \rho)^2}{2(1 + \rho)^2} + \frac{1 - \rho}{1 + \rho} \frac{2\rho + 1}{2} + \frac{4(1 - \rho)^2 \rho^2}{(1 + \rho)^2}
$$

$$
= \rho + \frac{(1 - \rho)[1 - \rho + 2\rho + 1 + 8(1 - \rho)\rho^2]}{2(1 + \rho)^2} = \frac{8\rho^4 - 14\rho^3 + 11\rho^2 + \rho + 2}{2(1 + \rho)^2}
$$

Since

$$
8\rho^4 - 14\rho^3 + 11\rho^2 + \rho + 2 - 2(1 + \rho)^2 = \rho(\rho - 1)(8\rho^2 - 6\rho + 3) \leq 0, \forall \rho \in [0, 1].
$$

The desired result follows immediately.

C.2 Proof of (45)

Let $\tilde{\gamma} = \eta L \gamma \leq \frac{(1 - \rho^2)}{(1 + \rho)^2} \leq 1$. We first show that

$$
\theta = \rho + \frac{\tilde{\gamma}}{2} + \frac{\sqrt{\tilde{\gamma}^2 + 4\rho(\rho + 1)\tilde{\gamma}}}{2} \leq 1 - \frac{\tilde{\gamma}(1 - \tilde{\gamma})}{2}. \quad (51)
$$

To this end, let

$$
q(\tilde{\gamma}) = \left(1 - \frac{\tilde{\gamma}(1 - \tilde{\gamma})}{2} - \rho - \frac{\tilde{\gamma}}{2}\right)^2 - \left(\frac{\sqrt{\tilde{\gamma}^2 + 4\rho(\rho + 1)\tilde{\gamma}}}{2}\right)^2.
$$

22
Clearly, which implies (51). Since both the numerator and denominator are positive, it is sufficient to show that

\[ q'(\tilde{\gamma}) = 2(1 - \rho) - 2\tilde{\gamma} + \tilde{\gamma}^2(\tilde{\gamma} - 1) - \tilde{\gamma}/2 - \rho(\rho + 1), \]
\[ q''(\tilde{\gamma}) = 2(1 - \rho) - 2\tilde{\gamma} + \tilde{\gamma}^2 + 2(1 - \tilde{\gamma})^2 - 0.5 \]
\[ q'''(\tilde{\gamma}) = 2\tilde{\gamma} - 2 + 4(1 - \tilde{\gamma}) = 2(1 - \tilde{\gamma}) > 0, \ \forall \tilde{\gamma} \in [0, 1). \]

Since \( q''(0) = 2(1 - \rho)^2 + 1.5 > 0 \) and \( q'''(\tilde{\gamma}) \) is increasing, we know that \( q'(\tilde{\gamma}) \) is increasing. Thus,

\[ q(\tilde{\gamma}) > q\left(\frac{(1 - \rho)^2}{(1 + \rho)^2}\right) = \left(1 - \rho - \frac{2(1 - \rho)^2}{(1 + \rho)^4} - \frac{(1 - \rho)^2}{2(1 + \rho)^2}\right)^2 - \frac{(1 - \rho)^4}{4(1 + \rho)^4} - \frac{\rho(1 - \rho)^2}{1 + \rho}, \]

which implies (51).

Now we come back to (45), which can be rewritten as

\[ \frac{n\gamma + 2\eta LC^2_k\|\Theta_k\|}{n\gamma(2 - \eta L\gamma) - 2\eta LC^2_k\|\Theta_k\|} = \frac{(4\rho^2 + 4\rho + \tilde{\gamma})^2\|\Theta_k\|}{(2 - \tilde{\gamma})(4\rho^2 + 4\rho + \tilde{\gamma}) - 2\rho^2\tilde{\gamma}^2\|\Theta_k\|}. \]

Since both the numerator and denominator are positive, it is sufficient to show that

\[ (4\rho^2 + 4\rho + \tilde{\gamma}) + 2\rho^2\tilde{\gamma}^2\|\Theta_k\| - ((2 - \tilde{\gamma})(4\rho^2 + 4\rho + \tilde{\gamma}) - 2\rho^2\tilde{\gamma}^2\|\Theta_k\|) \leq 0 \]

In fact, we have

\[ (4\rho^2 + 4\rho + \tilde{\gamma}) + 2\rho^2\tilde{\gamma}^2\|\Theta_k\| - ((2 - \tilde{\gamma})(4\rho^2 + 4\rho + \tilde{\gamma}) - 2\rho^2\tilde{\gamma}^2\|\Theta_k\|) \]
\[ = (\gamma - 1)(4\rho^2 + 4\rho + \tilde{\gamma}) + 4\rho^2\tilde{\gamma}^2\|\Theta_k\| \]
\[ \leq (4\rho^2 + 4\rho + \tilde{\gamma}) \left(\tilde{\gamma} - 1 + \frac{8\rho^2\tilde{\gamma}^2}{(1 - \theta)(4\rho^2 + 4\rho + \tilde{\gamma})}\right) \]
\[ \leq (4\rho^2 + 4\rho + \tilde{\gamma}) \left(\tilde{\gamma} - 1 + \frac{16\rho^2\tilde{\gamma}}{(1 - \tilde{\gamma})(4\rho^2 + 4\rho + \tilde{\gamma})}\right) \triangleq (4\rho^2 + 4\rho + \tilde{\gamma})\tilde{q}(\tilde{\gamma}) \]

where the first inequality follows from (37), the second inequality is from (51), and

\[ \tilde{q}(\tilde{\gamma}) \triangleq \tilde{\gamma} - 1 + \frac{16\rho^2\tilde{\gamma}}{(1 - \tilde{\gamma})(4\rho^2 + 4\rho + \tilde{\gamma})}. \]

Clearly, \( r(\tilde{\gamma}) \) is an increasing function. Thus

\[ \tilde{q}(\tilde{\gamma}) < \tilde{q}\left(\frac{(1 - \rho)^2}{(1 + \rho)^2}\right) = \frac{-4\rho}{(1 + \rho)^2} + \frac{4\rho(1 - \rho)^2(1 + \rho)^2}{4\rho(1 + \rho)^3 + (1 - \rho)^2} = \frac{4\rho^3(\rho^4 + 2\rho^3 - 5\rho^2 - 16\rho - 14)}{(1 + \rho)^2(4\rho(1 + \rho)^3 + (1 - \rho)^2) \]

It can be easily checked that \( \rho^4 + 2\rho^3 - 5\rho^2 - 16\rho - 14 \) is negative when \( \rho \in [0, 1] \). Thus,

\[ \tilde{q}(\tilde{\gamma}) < 0 \] and the desired result follows from the above arguments.
C.3 Proof of (48)

Let \( \tilde{\gamma} = \eta L \gamma \in (0, (1 - \rho)^2) \). In viewing of (37), it is sufficient to show that

\[
\frac{2\rho (1 - \theta^k)}{(4\rho^2 + 4\rho + \tilde{\gamma})(1 - \theta)} \leq \frac{1 - \rho^k}{1 - \sqrt{\rho}}.
\]

Clearly, \( \theta \geq \rho \), which implies that \( 1 - \theta^k \leq 1 - \rho^k \). Then, we just need to prove \( (4\rho^2 + 4\rho + \tilde{\gamma})(1 - \theta) \geq 2\rho(1 - \sqrt{\rho}) \). Notice that

\[
(4\rho^2 + 4\rho + \tilde{\gamma})(1 - \theta) = (4\rho^2 + 4\rho + \tilde{\gamma})(1 - \rho - \frac{\tilde{\gamma}}{2} - \frac{\sqrt{\tilde{\gamma}^2 + 4\rho(\rho + 1)\tilde{\gamma}}}{2})
\]

\[
> (4\rho^2 + 4\rho + \tilde{\gamma})(1 - \rho - \frac{\tilde{\gamma}}{2} - \frac{1 - \rho}{1 + \rho} \sqrt{\frac{(1 - \rho)^2}{(1 + \rho)^2} + 4\rho(\rho + 1)}) \triangleq r(\tilde{\gamma}).
\]

\( r(\tilde{\gamma}) \) is a quadratic function w.r.t. \( \tilde{\gamma} \) and the derivative of \( r(\tilde{\gamma}) \) is

\[
r'(\tilde{\gamma}) = 1 - \rho - \frac{\tilde{\gamma}}{2} - \frac{1 - \rho}{2(1 + \rho)} \sqrt{(1 - \rho)^2 + 4\rho(\rho + 1) - 2\rho(\rho + 1) - \frac{\tilde{\gamma}}{2} = b - \tilde{\gamma}} \quad (52)
\]

where \( b = 1 - 3\rho - 2\rho^2 - \frac{1 - \rho}{2(1 + \rho)} \sqrt{(1 - \rho)^2 + 4\rho(\rho + 1)} \leq 1 \). It follows from (52) that \( r(\tilde{\gamma}) \) is increasing for \( \tilde{\gamma} \leq b \), and is decreasing for \( \tilde{\gamma} \geq b \). Thus, \( r(\tilde{\gamma}) \geq \min\{r(0), r\left(\frac{(1 - \rho)^2}{(1 + \rho)^2}\right)\} \) for \( \tilde{\gamma} \in (0, (1 - \rho)^2) \). We then show \( r(0) \geq 2\rho(1 - \sqrt{\rho}) \) and \( r\left(\frac{(1 - \rho)^2}{(1 + \rho)^2}\right) \geq 2\rho(1 - \sqrt{\rho}) \).

Firstly, we have

\[
r(0) - 2\rho(1 - \sqrt{\rho}) = 4\rho(\rho + 1) \left(1 - \rho - \frac{1 - \rho}{2(1 + \rho)} \sqrt{\frac{(1 - \rho)^2}{(1 + \rho)^2} + 4\rho(\rho + 1)} - 2\rho(1 - \sqrt{\rho})\right)
\]

\[
\geq 4\rho(1 - \rho^2) \left(1 - \frac{1}{2(1 + \rho)} \sqrt{1 + 4\rho(\rho + 1)}\right) - 2\rho(1 - \sqrt{\rho})
\]

\[
= 2\rho(1 - \rho)(2(1 + \rho) - (2\rho + 1)) - 2\rho(1 - \sqrt{\rho})
\]

\[
= 2\rho(\sqrt{\rho} - \rho) \geq 0.
\]

Secondly,

\[
r\left(\frac{(1 - \rho)^2}{(1 + \rho)^2}\right)
\]

\[
= \left(\frac{(1 - \rho)^2}{(1 + \rho)^2} + 4\rho(\rho + 1)\right) \left(1 - \rho - \frac{(1 - \rho)^2}{2(1 + \rho)^2} - \frac{(1 - \rho)}{2(1 + \rho)} \sqrt{\frac{(1 - \rho)^2}{(1 + \rho)^2} + 4\rho(\rho + 1)}\right)
\]

\[
= \frac{(1 - \rho)^2 + 4\rho(1 + \rho)^3}{2(1 + \rho)^4} \left(2(1 - \rho)(1 + \rho)^2 - (1 - \rho)^2 - (1 - \rho)\sqrt{(1 - \rho)^2 + 4\rho(\rho + 1)^3}\right)
\]

\[
\geq \frac{(1 - \rho)^2 + 4\rho(1 + \rho)^3}{2(1 + \rho)^4} \left(2(1 - \rho)(1 + \rho)^2 - (1 - \rho)^2 - (1 - \rho)(-2\rho^3 + 7\rho^2 + \rho + 1)\right)
\]

\[
= \frac{(1 - \rho)^3 + 4\rho(1 + \rho)^3(1 - \rho)}{2(1 + \rho)^4} \left(2(1 + \rho)^2 - (1 - \rho) - (-2\rho^3 + 7\rho^2 + \rho + 1)\right)
\]

(53)
where the inequality follows from the relation

\[
(1 - \rho)^2 + 4\rho(\rho + 1)^3 - (-2\rho^3 + 7\rho^2 + \rho + 1)^2 = -\rho^2(4\rho^4 - 28\rho^3 + 41\rho^2 - 2\rho + 2) \\
\leq -\rho^2(4\rho^4 + 13\rho^2 + 2(1 - \rho)) \leq 0.
\]

Thus, letting \(\tilde{\rho} = \sqrt{\rho}\), it follows from (53) that

\[
2(1 + \rho)^4 \left( r \left( \frac{(1 - \rho)^2}{(1 + \rho)^3} \right) - 2\rho(1 - \sqrt{\rho}) \right) \\
= \left( (1 - \rho)^3 + 4\rho(1 + \rho)^3(1 - \rho) \right) (2(1 + \rho)^2 + 2\rho^3 - 7\rho^2 - 2)) - 4(1 + \rho)^4 \rho(1 - \sqrt{\rho}) \\
= \tilde{\rho}^2 \left( (1 - \tilde{\rho}^3) + 4\tilde{\rho}^2(1 + \tilde{\rho}^3)(1 - \tilde{\rho}^3) \right) (2\tilde{\rho}^4 - 5\tilde{\rho}^2 + 4) - 4\tilde{\rho}^2(1 + \tilde{\rho}^2)^4(1 - \tilde{\rho}) \\
= -\tilde{\rho}^3(1 - \tilde{\rho})^2 \times \\
\left( 8\tilde{\rho}^{11} + 16\tilde{\rho}^{10} + 20\tilde{\rho}^9 + 24\tilde{\rho}^8 + 6\tilde{\rho}^7 - 16\tilde{\rho}^6 - 29\tilde{\rho}^5 - 58\tilde{\rho}^4 - 14\tilde{\rho}^3 + 6\tilde{\rho}^2 + 9\tilde{\rho} - 4 \right) \\
\geq -\tilde{\rho}^3(1 - \tilde{\rho})^2( -29\tilde{\rho}^4 - 14\tilde{\rho}^3 + 6\tilde{\rho}^2 + 9\tilde{\rho} - 4) \\
= \tilde{\rho}^3(1 - \tilde{\rho})^2( (29\tilde{\rho}^4 - 6\tilde{\rho}^2 + 1) + (14\tilde{\rho}^3 - 9\tilde{\rho} + 3) \right) \\
\geq 0
\]

where the first inequality follows from \(\tilde{\rho}^a < \tilde{\rho}^b\) for all \(\tilde{\rho} \in [0, 1]\) and \(a > b \geq 1\), and the last inequality follows from \(29\tilde{\rho}^4 - 6\tilde{\rho}^2 + 1 > 0\) and \(14\tilde{\rho}^3 - 9\tilde{\rho} + 3 > 0\) for all \(\tilde{\rho} \in (0, 1)\), both of which can be easily checked. Therefore, We obtained \(r(0) \geq 2\rho(1 - \sqrt{\rho})\) and 
\(r(\frac{(1 - \rho)^2}{(1 + \rho)^2}) \geq 2\rho(1 - \sqrt{\rho})\), and the result follows by combining the above arguments.
Appendix D. Proof of (10)

Note that $X_k = 1x_k^T$ since $\rho = 0$. We have

$$E[\|\partial F(X; \xi) - \eta \nabla F(X)\|^2] = \sum_{i=1}^{n} E_{\xi_i} [\|\partial f_i(\bar{x}; \xi_i) - \eta \nabla f_i(\bar{x})\|^2]$$

$$= \sum_{i=1}^{n} E_{\xi_i} [\|\sum_{d \in \xi_i} (\nabla l(\bar{x}; d) - \eta \partial f_i(\bar{x}))\|^2]$$

$$= \eta \sum_{i=1}^{n} N_i E_{d \sim D_i} [\|\nabla l(\bar{x}; d) - \eta \nabla f_i(\bar{x})\|^2]$$

$$= \eta \sum_{i=1}^{n} \sum_{u=1}^{N_i} \|\nabla l(\bar{x}; d_u^{(i)}) - \eta \nabla f_i(\bar{x})\|^2$$

$$\leq \eta \sum_{i=1}^{n} \sum_{u=1}^{N_i} \|\nabla l(\bar{x}; d_u) - \eta \nabla f(\bar{x})\|^2$$

$$= \eta N E_{d \sim D} [\|\nabla l(\bar{x}; d) - \eta \nabla f(\bar{x})\|^2] \leq \eta N \sigma^2$$

where equality (a) is from the definition of $\partial f_i(\bar{x}; \xi_i)$ and $|\xi_i| = \eta N_i$, equality (b) follows from $E_{d \sim D_i} [\nabla l(\bar{x}, d)] = \frac{1}{N_i} \nabla f_i(\bar{x})$ and the independence of samples in $\xi_i$, equality (c) uses the definition of expectation, equality (d) is obtained by expanding the quadratic form and using the following relation

$$\sum_{i=1}^{n} \sum_{u=1}^{N_i} 2(\nabla l(\bar{x}; d_u^{(i)}) - \eta \nabla f(\bar{x}))^T (\frac{1}{N_i} \nabla f(\bar{x}) - \frac{1}{N_i} \nabla f_i(\bar{x})) + \frac{1}{N_i} \nabla f(\bar{x}) - \frac{1}{N_i} \nabla f_i(\bar{x}))\|^2$$

$$= \frac{2}{N} \|\nabla f(\bar{x})\|^2 - 2 \sum_{i=1}^{n} \frac{1}{N_i} \|\nabla f_i(\bar{x})\|^2 + \sum_{i=1}^{n} \frac{1}{N_i} \|\nabla f_i(\bar{x})\|^2 - \frac{1}{N} \|\nabla f(\bar{x})\|^2$$

$$= \frac{1}{N} \|\nabla f(\bar{x})\|^2 - \sum_{i=1}^{n} \frac{1}{N_i} \|\nabla f_i(\bar{x})\|^2$$

and inequality (e) is from the convexity of $h(x, y) = \frac{\|x\|^2}{y}$ for $y > 0$, which implies that

$$\frac{1}{N} \|\nabla f(\bar{x})\|^2 = n h(\frac{1}{n} \sum_{i=1}^{n} \nabla f_i(\bar{x}), \frac{1}{n} \sum_{i=1}^{n} N_i) \leq n \cdot \frac{1}{n} \sum_{i=1}^{n} h(\nabla f_i(\bar{x}), N_i) = \sum_{i=1}^{n} \frac{1}{N_i} \|\nabla f_i(\bar{x})\|^2.$$
References

[1] Alekh Agarwal, Martin J Wainwright, and John C Duchi. Distributed dual averaging in networks. In *Advances in Neural Information Processing Systems*, pages 550–558, 2010.

[2] Zeyuan Allen-Zhu. Katyusha: The first direct acceleration of stochastic gradient methods. *The Journal of Machine Learning Research*, 18(1):8194–8244, 2017.

[3] Rosario Aragues, Guodong Shi, Dimos V Dimarogonas, Carlos Sagüés, Karl Henrik Johansson, and Youcef Mezouar. Distributed algebraic connectivity estimation for undirected graphs with upper and lower bounds. *Automatica*, 50(12):3253–3259, 2014.

[4] Mahmoud Assran and Michael Rabbat. Asynchronous subgradient-push. *arXiv preprint arXiv:1803.08950*, 2018.

[5] Mahmoud Assran, Nicolas Loizou, Nicolas Ballas, and Michael Rabbat. Stochastic gradient push for distributed deep learning. *arXiv preprint arXiv:1811.10792*, 2018.

[6] Pascal Bianchi, Gersende Fort, and Walid Hachem. Performance of a distributed stochastic approximation algorithm. *IEEE Transactions on Information Theory*, 59(11):7405–7418, 2013.

[7] Léon Bottou, Frank E Curtis, and Jorge Nocedal. Optimization methods for large-scale machine learning. *Siam Review*, 60(2):223–311, 2018.

[8] Fan RK Chung and Fan Chung Graham. *Spectral graph theory*. Number 92. American Mathematical Soc., 1997.

[9] Jeffrey Dean, Greg Corrado, Rajat Monga, Kai Chen, Matthieu Devin, Mark Mao, Andrew Senior, Paul Tucker, Ke Yang, Quoc V Le, et al. Large scale distributed deep networks. In *Advances in neural information processing systems*, pages 1223–1231, 2012.

[10] Aaron Defazio, Francis Bach, and Simon Lacoste-Julien. SAGA: A fast incremental gradient method with support for non-strongly convex composite objectives. In *Advances in neural information processing systems*, pages 1646–1654, 2014.

[11] Yu Hao, Jin Rong, and Sen Yang. On the linear speedup analysis of communication efficient momentum SGD for distributed non-convex optimization. *arXiv preprint arXiv:1905.03817*, 2019.

[12] Hadrien Hendrikx, Francis Bach, and Laurent Massoulié. Asynchronous accelerated proximal stochastic gradient for strongly convex distributed finite sums. *arXiv preprint arXiv:1901.09865*, 2019.

[13] Rie Johnson and Tong Zhang. Accelerating stochastic gradient descent using predictive variance reduction. In *Advances in neural information processing systems*, pages 315–323, 2013.
[14] Anastasia Koloskova, Sebastian Stich, and Martin Jaggi. Decentralized stochastic optimization and gossip algorithms with compressed communication. In Proceedings of the 36th International Conference on Machine Learning, pages 3478–3487. PMLR, 2019.

[15] Alex Krizhevsky and Geoffrey Hinton. Learning multiple layers of features from tiny images. Technical report, Citeseer, 2009.

[16] Guanghui Lan, Soomin Lee, and Yi Zhou. Communication-efficient algorithms for decentralized and stochastic optimization. Mathematical Programming, pages 1–48, 2017.

[17] Yann LeCun, Léon Bottou, Yoshua Bengio, Patrick Haffner, et al. Gradient-based learning applied to document recognition. Proceedings of the IEEE, 86(11):2278–2324, 1998.

[18] Mu Li, Tong Zhang, Yuqiang Chen, and Alexander J Smola. Efficient mini-batch training for stochastic optimization. In Proceedings of the 20th ACM SIGKDD international conference on Knowledge discovery and data mining, pages 661–670. ACM, 2014.

[19] Xiangru Lian, Yijun Huang, Yuncheng Li, and Ji Liu. Asynchronous parallel stochastic gradient for nonconvex optimization. In Advances in Neural Information Processing Systems, pages 2737–2745, 2015.

[20] Xiangru Lian, Ce Zhang, Huan Zhang, Cho-Jui Hsieh, Wei Zhang, and Ji Liu. Can decentralized algorithms outperform centralized algorithms? A case study for decentralized parallel stochastic gradient descent. In Advances in Neural Information Processing Systems, pages 5330–5340, 2017.

[21] Xiangru Lian, Wei Zhang, Ce Zhang, and Ji Liu. Asynchronous decentralized parallel stochastic gradient descent. In Proceedings of the 35th International Conference on Machine Learning, pages 3049–3058, 2018.

[22] Brendan McMahan, Eider Moore, Daniel Ramage, Seth Hampson, and Blaise Agüera y Arcas. Communication-efficient learning of deep networks from decentralized data. In Proceedings of the 20th International Conference on Artificial Intelligence and Statistics, volume 54, pages 1273–1282, Fort Lauderdale, FL, USA, 2017. PMLR.

[23] Aryan Mokhtari and Alejandro Ribeiro. DSA: Decentralized double stochastic averaging gradient algorithm. The Journal of Machine Learning Research, 17(1):2165–2199, 2016.

[24] Angelia Nedić and Alex Olshevsky. Stochastic gradient-push for strongly convex functions on time-varying directed graphs. IEEE Transactions on Automatic Control, 61(12):3936–3947, 2016.

[25] Angelia Nedić and Asuman Ozdaglar. Distributed subgradient methods for multi-agent optimization. IEEE Transactions on Automatic Control, 54(1):48–61, 2009.
[26] Angelia Nedić, Alex Olshevsky, and Wei Shi. Achieving geometric convergence for distributed optimization over time-varying graphs. *SIAM Journal on Optimization*, 27(4):2597–2633, 2017.

[27] Arkadi Nemirovski, Anatoli Juditsky, Guanghui Lan, and Alexander Shapiro. Robust stochastic approximation approach to stochastic programming. *SIAM Journal on Optimization*, 19(4):1574–1609, 2009.

[28] Shi Pu and Angelia Nedić. A distributed stochastic gradient tracking method. In 2018 *IEEE Conference on Decision and Control (CDC)*, pages 963–968. IEEE, 2018.

[29] Guannan Qu and Na Li. Harnessing smoothness to accelerate distributed optimization. *IEEE Transactions on Control of Network Systems*, 5(3):1245–1260, 2018.

[30] Nicolas L Roux, Mark Schmidt, and Francis R Bach. A stochastic gradient method with an exponential convergence rate for finite training sets. In *Advances in neural information processing systems*, pages 2663–2671, 2012.

[31] Kevin Scaman, Francis Bach, Sébastien Bubeck, Yin Tat Lee, and Laurent Massoulié. Optimal algorithms for smooth and strongly convex distributed optimization in networks. In *International Conference on Machine Learning*, pages 3027–3036, 2017.

[32] Kevin Scaman, Francis Bach, Sébastien Bubeck, Laurent Massoulié, and Yin Tat Lee. Optimal algorithms for non-smooth distributed optimization in networks. In *Advances in Neural Information Processing Systems*, pages 2745–2754, 2018.

[33] Zebang Shen, Aryan Mokhtari, Tengfei Zhou, Peilin Zhao, and Hui Qian. Towards more efficient stochastic decentralized learning: Faster convergence and sparse communication. In *Proceedings of the 35th International Conference on Machine Learning*, volume 80, pages 4624–4633, Stockholmsmssan, Stockholm Sweden, 2018. PMLR.

[34] Wei Shi, Qing Ling, Gang Wu, and Wotao Yin. EXTRA: An exact first-order algorithm for decentralized consensus optimization. *SIAM Journal on Optimization*, 25(2):944–966, 2015.

[35] Benjamin Sirb and Xiaojing Ye. Decentralized consensus algorithm with delayed and stochastic gradients. *SIAM Journal on Optimization*, 28(2):1232–1254, 2018.

[36] Hanlin Tang, Xiangru Lian, Ming Yan, Ce Zhang, and Ji Liu. D²: Decentralized training over decentralized data. In *Proceedings of the 35th International Conference on Machine Learning*, pages 4848–4856, 2018.

[37] Hanlin Tang, Chen Yu, Cedric Renggli, Simon Kassing, Ankit Singla, Dan Alistarh, Ji Liu, and Ce Zhang. Distributed learning over unreliable networks. *ArXiv preprint arXiv:1810.07766*, 2018.

[38] Ye Tian, Ying Sun, and Gesualdo Scutari. Achieving linear convergence in distributed asynchronous multi-agent optimization. *ArXiv preprint arXiv:1803.10359*, 2018.
[39] Chenguang Xi and Usman A Khan. DEXTRA: A fast algorithm for optimization over directed graphs. *IEEE Transactions on Automatic Control*, 62(10):4980–4993, 2017.

[40] Lin Xiao and Stephen Boyd. Fast linear iterations for distributed averaging. *Systems & Control Letters*, 53(1):65–78, 2004.

[41] R. Xin and U. A. Khan. A linear algorithm for optimization over directed graphs with geometric convergence. *IEEE Control Systems Letters*, 2(3):315–320, July 2018. ISSN 2475-1456.

[42] Ran Xin, Anit Kumar Sahu, Usman A Khan, and Soummya Kar. Distributed stochastic optimization with gradient tracking over strongly-connected networks. *arXiv preprint arXiv:1903.07266*, 2019.

[43] Kun Yuan, Sulaiman A Alghunaim, Bicheng Ying, and Ali H Sayed. On the performance of exact diffusion over adaptive networks. *arXiv preprint arXiv:1903.10956*, 2019.

[44] Jiaqi Zhang and Keyou You. AsySPA: An exact asynchronous algorithm for convex optimization over digraphs. *arXiv preprint arXiv:1808.04118*, 2018.

[45] Jiaqi Zhang and Keyou You. Asynchronous decentralized optimization in directed networks. *arXiv preprint arXiv:1901.08215*, 2019.