Local formulae for the hydrodynamic pressure and applications

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Abstract. We provide local formulae for the pressure of incompressible fluids. The pressure can be expressed in terms of its average and averages of squares of velocity increments in arbitrarily small neighbourhoods. As an application, we give a brief proof of the fact that $C^{\alpha}$ velocities have $C^{2\alpha}$ (or Lipschitz) pressures. We also give some regularity criteria for 3D incompressible Navier–Stokes equations.

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1. Introduction

We provide local formulae for the pressure in incompressible fluids. By this we mean expressions that compute a solution of the equation

$$-\Delta p = \sum_{i,j=1}^{3} \frac{\partial^2}{\partial x_i \partial x_j} (u_i u_j),$$

where $u$ is a divergence-free velocity, at a point $x \in \Omega \subset \mathbb{R}^3$ in terms of the spherical average

$$\bar{p}(x, r) = \frac{1}{4\pi r^2} \int_{|x-y|=r} p(y) \, dS(y).$$

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of the pressure and the integrals of the squared increments \((u_i(y) - u_i(x))(u_j(y) - u_j(x))\) over a ball \(|y - x| \leq r\) with arbitrarily small \(r\). No knowledge of the behaviour of \(u\) outside a small ball is needed. The main ingredient of the proof is a kind of monotonicity equation for a modified object

\[
b(x, r) = \overline{p}(x, r) + \frac{1}{4\pi r^2} \int_{|x-y|=r} \left( \frac{y-x}{|y-x|} \cdot u(y) \right)^2 dS(y).
\]

This enables us to express the pressure in the form

\[
p(x) = \beta(x, r) + \pi(x, r),
\]

where \(\beta\) is just a local average of the pressure,

\[
\beta(x, r) = \frac{1}{r} \int_r^{2r} \overline{p}(x, \rho) d\rho,
\]

and \(\pi(x, r)\) is given by a pair of integrals (39) of the squares of the velocity increments over a ball of radius 2\(r\) and over a spherical annulus of radii \(r\) and 2\(r\). Thus, we write the pressure as a sum of two local terms, one small and the other sufficiently well-behaved. Indeed, \(\beta \in L^\infty(\mathbb{R}^3)\) is bounded in space (for any \(r\)) if \(u \in L^2(\mathbb{R}^3)\) (see (34)), and \(\|\nabla \beta\|_{L^2(\mathbb{R}^3)}\) is bounded in terms of \(\|u\|_{L^4(\mathbb{R}^3)}^2\) (see (47)).

On the other hand, \(\pi\) is of the order \(r^2|\nabla u|^2\) for small \(r\). There are well-known criteria for regularity of the 3D incompressible Navier–Stokes equations in terms of the pressure [1], [7]. If the pressure would have the same bounds as \(\beta\), then regularity of solutions of the 3D Navier–Stokes equations would easily follow. Because \(\pi(x, r) \to 0\) as \(r \to 0\), the assumption that \(p\) has the same bounds as \(\beta\) is not unreasonable. On the other hand, bounds on \(\pi\) require some smoothness of the velocity field. For weak solutions of the 3D Navier–Stokes equations a higher regularity in space for the velocity field was obtained in [4] (see also [9]). These bounds imply that \(\pi(x, r)\) is small for almost all time. For instance, \(\|\pi\|_{L^3(\mathbb{R}^3)} \leq C(t) r^2 t\)-almost everywhere (52), (59). The problem is that in general the time integrability of \(C(t)\) is too poor to conclude regularity \((C(t)^{1/3}\) is time-integrable, whereas time integrability of \(C(t)\) would be sufficient for regularity).

The organization of this paper is as follows. In §2 we present the basic calculations which lead to the formulae for the pressure. In §3 we give ensuing bounds for \(\beta\) and \(\pi\). In §4 we give a quick proof of bounds for higher derivatives of solutions of the 3D Navier–Stokes equations in the whole space. These follow from the classical paper [4] and have been well known for decades, although because [4] deals with spatially periodic solutions, a proof of one of the results for the whole space was given only in 2001 [2] (due originally to Luc Tartar; see the acknowledgment in [4]). The 2012 preprint [8] contains also a proof of this result and more references. In §5 we give two applications. The first is a simple proof that if \(u \in C^\alpha\), then \(p \in C^{2\alpha}\) (for \(2\alpha < 1\); if \(2\alpha > 1\), then \(p\) is Lipschitz). This result was used recently in [5], with a proof based on the Littlewood–Paley decomposition. A different proof (closer to ours) was obtained previously by L. Silvestre, but was not published. The 3D Navier–Stokes equations are regular if \(u \in L^\infty([0, T], L^3(\mathbb{R}^3))\) [3], [6]. As a second application, we present regularity criteria for the 3D Navier–Stokes equations in
terms of $\pi$. These say essentially that if we can find a small $r(t)$ such that in some sense $\pi$ is small, and if some integral of $r(t)^{-1}$ is finite, then we have regularity.

Some elementary calculations needed for deriving the formulae are presented in the Appendix (§6).

2. Spherical averages

Let
\[
\tilde{f}(x, r) = \frac{1}{4\pi r^2} \int_{|x-y|=r} f(y) dS(y) = \int_{|\xi|=1} f(x + r\xi) dS(\xi),
\]
where $\int$ denotes the normalized integral. We consider solutions of the equation
\[
-\Delta p = \nabla \cdot (u \cdot \nabla u)
\]
in $\Omega \subset \mathbb{R}^3$. We assume that $\nabla \cdot u = 0$ and that $u$ is smooth. Let us start by computing
\[
\partial_r \tilde{p}(x, r) = \int_{|\xi|=1} \xi \cdot \nabla_x p(x + r\xi) dS(\xi) = \frac{1}{4\pi r} \int_{|\xi|=1} \xi \cdot \nabla_\xi p(x + r\xi) dS(\xi)
\]
\[
= \frac{1}{4\pi r} \int_{|\xi|<1} \Delta_\xi p(x + r\xi) d\xi = \frac{r}{4\pi} \int_{|\xi|<1} \Delta_x p(x + r\xi) d\xi.
\]
We use the equation (2). Note that, in view of the incompressibility $\nabla \cdot u = 0$, we have
\[
\Delta p = -\partial_i \partial_j ((u_i - v_i)(u_j - v_j))
\]
for any constant vector $v$. (We use the summation convention unless explicitly stated otherwise.) Thus,
\[
\partial_r \tilde{p}(x, r) = -\frac{r}{4\pi} \int_{|\xi|<1} \partial_i \partial_j ((u_i - v_i)(u_j - v_j))(x + r\xi) d\xi
\]
\[
= -\frac{1}{4\pi} \int_{|\xi|<1} \partial_\xi_i \partial_\xi_j ((u_i - v_i)(u_j - v_j))(x + r\xi) d\xi
\]
\[
= -\frac{1}{4\pi r} \int_{|\xi|=1} \xi_i \partial_\xi_j ((u_i - v_i)(u_j - v_j))(x + r\xi) dS(\xi)
\]
\[
= -\frac{1}{4\pi r} \int_{|\xi|=1} \xi_i \partial_\xi_j ((u_i - v_i)(u_j - v_j))(x + r\xi) dS(\xi).
\]
Therefore,
\[
r\partial_r \tilde{p}(x, r) = -\int_{|\xi|=1} \xi_i \partial_\xi_j ((u_i - v_i)(u_j - v_j))(x + r\xi) dS(\xi). \tag{3}
\]

Lemma 1. Let $\Omega$ be an open set in $\mathbb{R}^3$, and let $x \in \Omega$. Let $r < \text{dist}(x, \partial\Omega)$, let $u$ be a divergence-free vector field in $C^2(\Omega)^3$, and let $v \in \mathbb{R}^3$. If $p$ is a solution of (2)
in \( \Omega \), then

\[
\partial_r \left\{ \bar{p}(x, r) + \int_{|\xi|=1} |\xi \cdot (u(x + r\xi) - v)|^2 \, dS(\xi) \right\} 
\]

\[
= -\frac{1}{r} \int_{|\xi|=1} [3|\xi \cdot (u(x + r\xi) - v)|^2 - |u(x + r\xi) - v|^2] \, dS(\xi). \tag{4}
\]

**Proof.** We are going to use the following identities:

\[
\int_{|\xi|=1} \xi_j \partial_{\xi_j} f(x + r\xi) \, dS(\xi) = r \partial_r \left[ \int_{|\xi|=1} \xi_j^2 f(x + r\xi) \, dS(\xi) \right] 
+ \int_{|\xi|=1} (3\xi_j^2 - 1)f(x + r\xi) \, dS(\xi) \tag{5}
\]

for any \( j \) (no summation of repeated indices used in this formula), and

\[
\int_{|\xi|=1} (\xi_i \partial_{\xi_j} + \xi_j \partial_{\xi_i}) f(x + r\xi) \, dS(\xi) = r \partial_r \left[ \int_{|\xi|=1} 2\xi_i \xi_j f(x + r\xi) \, dS(\xi) \right] 
+ \int_{|\xi|=1} 6\xi_i \xi_j f(x + r\xi) \, dS(\xi). \tag{6}
\]

The proofs of these identities are elementary and are given with full detail in §6.

In view of (3), the expression we need to average is (the negative of)

\[
\xi_1 \partial_{\xi_1} w_1 + \xi_2 \partial_{\xi_2} w_2 + \xi_3 \partial_{\xi_3} w_3 + (\xi_1 \partial_{\xi_2} + \xi_2 \partial_{\xi_1})(w_1 w_2) 
+ (\xi_1 \partial_{\xi_3} + \xi_3 \partial_{\xi_1})(w_1 w_3) + (\xi_2 \partial_{\xi_3} + \xi_3 \partial_{\xi_2})(w_2 w_3),
\]

where \( w = u - v \) and the expression is evaluated at \( x + r\xi \). Using (5) and (6), we group together the terms involving \( r \partial_r \), and separately the ones which do not involve \( r \partial_r \), and sum. Then by (3)

\[
r \partial_r \bar{p}(x, r) = -r \partial_r \int_{|\xi|=1} (\xi \cdot w)^2 \, dS(\xi) - \int_{|\xi|=1} [3(\xi \cdot w)^2 - |w|^2] \, dS(\xi), \tag{7}
\]

which is the same as (4). \( \square \)

**Lemma 2.** Let \( x \in \Omega \subset \mathbb{R}^3 \), let \( 0 < r < \text{dist}(x, \partial \Omega) \), and let \( p \) be a solution of (2) with a divergence-free field \( u \in C^2(\Omega)^3 \). If \( v \in \mathbb{R}^3 \), then

\[
p(x) + \frac{1}{3} |u(x) - v|^2 = \bar{p}(x, r) + \int_{|\xi|=1} |\xi \cdot (u(x + r\xi) - v)|^2 \, dS(\xi) 
+ \int_0^r \frac{d\rho}{\rho} \int_{|\xi|=1} [3|\xi \cdot (u(x + \rho\xi) - v)|^2 - |u(x + \rho\xi) - v|^2] \, dS(\xi). \tag{8}
\]

**Proof.** This follows immediately from (4) by the integration \( \int_0^r d\rho \), since

\[
\bar{p}(x, 0) = p(x) \tag{9}
\]
and
\[
\lim_{r \to 0} \int_{|\xi| = 1} |\xi \cdot (u(x + r\xi) - v)|^2 \, dS(\xi) = \frac{1}{3} \lim_{r \to 0} \int_{|\xi| = 1} |u(x + r\xi) - v|^2 \, dS(\xi). \tag{10}
\]
The formula (8) can be specialized by choosing \(v\). But first let us introduce the quantity
\[
\sigma_{ij}(\overrightarrow{y - x}) = 3 \frac{(y_i - x_i)(y_j - x_j)}{|y - x|^2} - \delta_{ij}, \tag{11}
\]
where
\[
\overrightarrow{y - x} = \frac{y - x}{|y - x|}.
\]
Note that
\[
\partial_i \partial_j \left( \frac{1}{|x - y|} \right) = \frac{\sigma_{ij}(\overrightarrow{y - x})}{|y - x|^3}.
\]
By choosing \(v = 0\) in (8) we get that
\[
p(x) + \frac{1}{3} |u(x)|^2 = \overline{p}(x, r) + \int_{|y - x| = r} |\xi \cdot u(y)|^2 \, dS(y)
\]
\[
+ \frac{1}{4\pi} \text{PV} \int_{B(x,r)} \frac{\sigma_{ij}(\overrightarrow{x - y})}{|x - y|^3} (u_i u_j)(y) \, dy. \quad \square \tag{12}
\]
Remark 3. In the case \(\Omega = \mathbb{R}^3\) if we carry out the integration \(R^{-1} \int_R^{2R} dr\) in (12) and let \(R \to \infty\), then we obtain (assuming that \(R^{-1} \int_R^{2R} \overline{p} \, dr\) tends to zero)
\[
p(x) + \frac{1}{3} |u(x)|^2 = \frac{1}{4\pi} \text{PV} \int_{\mathbb{R}^3} \frac{\sigma_{ij}(\overrightarrow{x - y})}{|x - y|^3} (u_i u_j)(y) \, dy, \tag{13}
\]
a fact that follows also from the equality
\[
p(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x - y|} \partial_i \partial_j (u_i u_j)(y) \, dy
\]
by integration by parts. Thus, (12) is a local version of this, valid for any \(r > 0\).

Taking \(v = u(x)\) in (8), we get that
\[
p(x) - \overline{p}(x, r) - \int_{|y - x| = r} |\xi \cdot (u(y) - u(x))|^2 \, dS(y)
\]
\[
= \frac{1}{4\pi} \int_{B(x,r)} \frac{\sigma_{ij}(\overrightarrow{x - y})}{|x - y|^3} (u_i(y) - u_i(x))(u_j(y) - u_j(x)) \, dy. \tag{14}
\]
To clarify the relation between (12) and (14), we observe that
\[
\int_{|y - x| = r} \xi_i (\xi \cdot u(y)) \, dS(y) + \frac{1}{4\pi} \text{PV} \int_{B(x,r)} \frac{\sigma_{ij}(\overrightarrow{x - y})}{|x - y|^3} u_j(y) \, dy = \frac{1}{3} u_i(x). \tag{15}
\]
This follows from the obvious fact that
\[
\frac{1}{4\pi} \int_{B(x,r)} \frac{y_i - x_i}{|y - x|^3} (\nabla \cdot u)(y) \, dy = 0
\]
by integration by parts.
Remark 4. Letting \( r \to \infty \), we deduce from (15) in the whole-space case with \( u \) going to zero that
\[
\frac{1}{4\pi} \text{PV} \int_{\mathbb{R}^3} \frac{\sigma_{ij}(x-y)}{|x-y|^3} u_j(y) \, dy = \frac{1}{3} u_i(x),
\]
a fact that follows also from the equality \( \mathbb{P}u = u \), where \( \mathbb{P} \) is the projection onto the divergence-free functions by the formula
\[
\mathbb{P}v = \frac{2}{3} v + \frac{1}{4\pi} \text{PV} \int_{\mathbb{R}^3} \frac{\sigma_{ij}(x-y)}{|x-y|^3} v_j(y) \, dy.
\]
In the principal value integral in (12) we now write
\[
u_i(y)u_j(y) = (u_i(y) - u_i(x))(u_j(y) - u_j(x)) + u_i(x)u_j(y) + u_j(x)u_i(y) - u_i(x)u_j(x)
\]
and use the fact that the averages of \( \sigma_{ij}(\hat{y} - x)/|y-x|^3 \) on spheres about \( x \) vanish. By (15),
\[
p(x) + \frac{1}{3}|u(x)|^2 = \overline{p}(x, r) + \int_{|y-x|=r} |\xi \cdot u(y)|^2 \, dS(y) + \frac{1}{4\pi} \text{PV} \int_{B(x,r)} \frac{\sigma_{ij}(\hat{x} - y)}{|x-y|^3} (u_i(y) - u_i(x))(u_j(y) - u_j(x)) \, dy
\]
\[- 2 \int_{|y-x|=r} (\xi \cdot u(x))(\xi \cdot u(y)) \, dS(y) + \frac{2}{3}|u(x)|^2.
\]
Rearranging, and noting that
\[
\int_{|y-x|=r} (\xi \cdot u(x))^2 \, dS(y) = \frac{1}{3}|u(x)|^2,
\]
we obtain
\[
p(x) = \overline{p}(x, r) + \int_{|y-x|=r} |\xi \cdot (u(y) - u(x))|^2 \, dS(y)
\]
\[+ \frac{1}{4\pi} \int_{B(x,r)} \frac{\sigma_{ij}(\hat{x} - y)}{|x-y|^3} (u_i(y) - u_i(x))(u_j(y) - u_j(x)) \, dy. \tag{17}
\]
Remark 5. The formula (14) follows directly from (12) by using the formula (15), which is a consequence of the divergence-free condition.

Remark 6. The situation in \( \mathbb{R}^2 \) is completely analogous. Instead of (5) and (6) we have
\[
\int_{\mathbb{S}^1} \xi_j \partial_{\xi_j} f(x + r\xi) \, dS(\xi) = r \partial_r \int_{\mathbb{S}^1} \xi_j^2 f(x + r\xi) \, dS(\xi)
\]
\[+ \int_{\mathbb{S}^1} (2\xi_j^2 - 1)f(x + r\xi) \, dS(\xi) \tag{18}
\]
for fixed \( j = 1, 2 \), and

\[
\int_{S^1} (\xi_1 \partial_{\xi_2} + \xi_2 \partial_{\xi_1}) f(x + r\xi) \, dS(\xi)
= r \partial_r \int_{S^1} 2\xi_1\xi_2 f(x + r\xi) \, dS(\xi) + \int_{S^1} 2\xi_1\xi_2 f(x + r\xi) \, dS(\xi),
\]  

(19)

and hence instead of (7) we have the equality

\[
r \partial_r p(x, r) = -r \partial_r \int_{|\xi|=1} (\xi \cdot w)^2 \, dS(\xi) - \int_{|\xi|=1} [2(\xi \cdot w)^2 - |w|^2] \, dS(\xi),
\]

(20)

where \( w = u(x + r\xi) - v \) and \( v \) is a constant vector. This again leads to a local representation formula

\[
p(x) + \frac{1}{2} |u(x) - v|^2 = \overline{p}(x, r) + \int_{|\xi|=1} |\xi \cdot (u(x + r\xi) - v)|^2 \, dS(\xi)
+ \int_0^r \frac{d\rho}{\rho} \int_{|\xi|=1} [2|\xi \cdot (u(x + \rho\xi) - v)|^2 - |u(x + \rho\xi) - v|^2] \, dS(\xi).
\]

(21)

We conclude this section by mentioning similar formulae for the average of the gradient of the pressure. For instance, starting from the fact that \( \partial_1 p \) is a solution of the equation

\[
- \Delta \partial_1 p = \partial_1 \partial_j (\partial_1 (u_i u_j))
\]

(22)

obtained by differentiating (2), we get that

\[
\partial_r \overline{\partial_1 p} = -\partial_r \int_{|\xi|=1} \xi_i \xi_j (\partial_{x_1} (u_i u_j)(x + r\xi)) \, dS(\xi)
- \frac{1}{r} \int_{|\xi|=1} (3\xi_i \xi_j - \delta_{ij})(\partial_{x_1} (u_i u_j)(x + r\xi)) \, dS(\xi)
= -\partial_r \left( \frac{1}{r} \int_{|\xi|=1} \xi_i \xi_j (\partial_{\xi_1} (u_i u_j)(x + r\xi)) \, dS(\xi) \right)
- \frac{1}{r^2} \int_{|\xi|=1} (3\xi_i \xi_j - \delta_{ij})(\partial_{\xi_1} (u_i u_j)(x + r\xi)) \, dS(\xi).
\]

(23)
We can integrate by parts in (23), using the relations

\[
\int_{|\xi|=1} \xi_1 \xi_2 \partial_{\xi_1} f(x + r \xi) \, dS(\xi) = r \partial_r \int_{|\xi|=1} \xi_1^2 \xi_2 f(x + r \xi) \, dS(\xi) \\
+ \int_{|\xi|=1} (4 \xi_1^2 - 1) \xi_2 f(x + r \xi) \, dS(\xi),
\]

\[
\int_{|\xi|=1} \xi_1 \xi_3 \partial_{\xi_1} f(x + r \xi) \, dS(\xi) = r \partial_r \int_{|\xi|=1} \xi_1^2 \xi_3 f(x + r \xi) \, dS(\xi) \\
+ \int_{|\xi|=1} (4 \xi_1^2 - 1) \xi_3 f(x + r \xi) \, dS(\xi),
\]

\[
\int_{|\xi|=1} \xi_1^2 \partial_{\xi_1} f(x + r \xi) \, dS(\xi) = r \partial_r \int_{|\xi|=1} \xi_1^3 f(x + r \xi) \, dS(\xi) \\
+ \int_{|\xi|=1} (4 \xi_1^2 - 2) \xi_1 f(x + r \xi) \, dS(\xi),
\]

\[
\int_{|\xi|=1} \xi_2^2 \partial_{\xi_1} f(x + r \xi) \, dS(\xi) = r \partial_r \int_{|\xi|=1} \xi_1^2 \xi_2^2 f(x + r \xi) \, dS(\xi) \\
+ \int_{|\xi|=1} 4 \xi_1 \xi_2^2 f(x + r \xi) \, dS(\xi),
\]

\[
\int_{|\xi|=1} \xi_3^2 \partial_{\xi_1} f(x + r \xi) \, dS(\xi) = r \partial_r \int_{|\xi|=1} \xi_1^2 \xi_3^2 f(x + r \xi) \, dS(\xi) \\
+ \int_{|\xi|=1} 4 \xi_1 \xi_3^2 f(x + r \xi) \, dS(\xi),
\]

\[
\int_{|\xi|=1} \xi_2 \xi_3 \partial_{\xi_1} f(x + r \xi) \, dS(\xi) = r \partial_r \int_{|\xi|=1} \xi_1 \xi_2 \xi_3 f(x + r \xi) \, dS(\xi) \\
+ \int_{|\xi|=1} 4 \xi_1 \xi_2 \xi_3 f(x + r \xi) \, dS(\xi),
\]

(24)

which can be proved in a manner similar to the proofs of (5) and (6). After some calculations using the relations above we get that

\[
\partial_r \overline{\partial_1 p} = - \left[ \partial_r^2 + \frac{7}{r} \partial_r + \frac{8}{r^2} \right] \int_{|\xi|=1} \xi_1 (\xi \cdot u(x + r \xi))^2 \, dS(\xi) \\
+ \frac{2}{r} \left[ \partial_r + \frac{2}{r} \right] \int_{|\xi|=1} u_1 (x + r \xi) \left( \xi \cdot u(x + r \xi) \right) \, dS(\xi) \\
+ \frac{1}{r} \left[ \partial_r + \frac{2}{r} \right] \int_{|\xi|=1} \xi_1 |u(x + r \xi)|^2 \, dS(\xi).
\]

(25)

This follows because

\[
\overline{\xi_i \xi_j \partial_{\xi_i} u_i u_j} = [r \partial_r + 4 \xi_1 (\xi \cdot u)]^2 - 2 u_1 (\xi \cdot u)
\]

(26)

and

\[
\overline{\partial_{\xi_1} |u(x + r \xi)|^2} = [r \partial_r + 2 \xi_1 |u|^2].
\]

(27)
3. Representations and bounds

We will take $\Omega = \mathbb{R}^3$ in this section. Let us consider

$$ b(x, r) = \tilde{p}(x, r) + \int_{|\xi|=1} |\xi \cdot u(x + r\xi)|^2 dS(\xi). \quad (28) $$

The equation (4) with $v = 0$ is

$$ \partial_r b(x, r) = r^{-1} [ |u|^2 - 3 |\xi \cdot u(y)|^2 ](x, r), \quad (29) $$

and integration from $r$ to infinity combined with (11) gives us that

$$ b(x, r) = -\frac{1}{4\pi} \int_{|x-y| \geq r} \frac{\sigma_{ij}(x-y)}{|x-y|^3} u_i(y) u_j(y) dy. \quad (30) $$

**Proposition 7.** Let $x \in \mathbb{R}^3$, let $r > 0$, and let $p$ be a solution of (2) in $\Omega = \mathbb{R}^3$ with divergence-free $u \in (C^2(\mathbb{R}^3) \cap L^2(\mathbb{R}^3))^3$. For $b$ defined by (28),

$$ \sup_{x \in \mathbb{R}^3} |b(x, r)| \leq \frac{1}{2\pi r^3} \|u\|_{L^2}^2. \quad (31) $$

If $u \in H^1(\mathbb{R}^3)$, then

$$ \sup_{x \in \mathbb{R}^3} |b(x, r)| \leq \frac{C}{2\pi r} \|
abla u\|_{L^2}^2, \quad (32) $$

where $C$ is the constant of Hardy’s inequality in $\mathbb{R}^3$.

**Remark 8.** Obviously, we do not need $C^2$-smoothness for $u$, but rather enough regularity for $b$ to be defined via (28). Of course, the representation (30) requires only that $u \in L^2$.

**Remark 9.** The corresponding local result in an open set $\Omega$ is a bound of $b(\cdot, r)$ in $L^\infty(dx)$ in terms of local $L^1(dx)$ bounds for $b$ and $L^2$ (or $H^1$) bounds for $u$. This can be derived in a straightforward manner by multiplying (29) by an appropriate compactly supported function of $r$ and integrating with respect to $r$.

**Proof of Proposition 7.** The proof follows directly from the inequality

$$ |\sigma_{ij}(\xi) u_i u_j| \leq 2|u|^2, $$

which is valid for any vectors $u \in \mathbb{R}^3$ and $\xi \in S^2$, and from Hardy’s inequality

$$ \int_{\mathbb{R}^3} \frac{|u(y)|^2}{|x-y|^2} dy \leq C \int_{\mathbb{R}^3} |\nabla u(y)|^2 dy. \quad \square $$

Let us define now

$$ \beta(x, r) = \frac{1}{r} \int_r^{2r} \bar{p}(x, \rho) d\rho. \quad (33) $$
Proposition 10. Let \( x \in \mathbb{R}^3 \) and \( r > 0 \), and assume that \( p \) is a solution of (2) in \( \Omega = \mathbb{R}^3 \) with divergence-free \( u \in (C^2(\mathbb{R}^3) \cap L^2(\mathbb{R}^3))^3 \). For \( \beta \) defined by (33),

\[
\sup_{x \in \mathbb{R}^3} |\beta(x, r)| \leq \frac{3}{4\pi r^3} \|u\|_{L^2}^2. \tag{34}
\]

If \( u \in H^1(\mathbb{R}^3) \), then

\[
\sup_{x \in \mathbb{R}^3} |\beta(x, r)| \leq \frac{3C}{4\pi r} \|\nabla u\|_{L^2}^2, \tag{35}
\]

where \( C \) is the constant in Hardy’s inequality in \( \mathbb{R}^3 \).

Proof. Note that

\[
\beta(x, r) = \frac{1}{r} \int_r^{2r} \left( b(x, \rho) - (\xi \cdot u)^2(x, \rho) \right) \, d\rho.
\]

The inequalities follow directly from the equality

\[
\frac{1}{r} \int_r^{2r} (\xi \cdot u)^2(x, \rho) \, d\rho = \frac{1}{4\pi r} \int_{r \leq |y-x| \leq 2r} \left( \frac{x-y}{|x-y|} \cdot u(y) \right)^2 \frac{dy}{|x-y|^2},
\]

Proposition 7, and Hardy’s inequality. \( \square \)

Remark 11. We introduced the average \( \beta(x, r) \) of \( \overline{p}(x, r) \) in order to pass from the pointwise information (31), (32) about \( b(x, r) \) to the pointwise information (34), (35) about \( \beta(x, r) \) without requiring other conditions besides the \( L^2 \) (or \( H^1 \)) bounds for \( u \).

Let us consider the weight function

\[
w(\lambda) = \begin{cases} 
1, & 0 \leq \lambda \leq 1, \\
2-\lambda, & 1 \leq \lambda \leq 2, \\
0, & \lambda \geq 2.
\end{cases} \tag{36}
\]

We now take the representation formula (14) and take the average with respect to \( r \).

Theorem 12. Let \( x \in \mathbb{R}^3 \) and \( r > 0 \), and assume that \( p \) is a solution of (2) in \( \Omega = \mathbb{R}^3 \) with divergence-free \( u \in (C^2(\mathbb{R}^3) \cap L^2(\mathbb{R}^3))^3 \). Then

\[
p(x) = \beta(x, r) + \pi(x, r), \tag{37}
\]

with \( \beta(x, r) \) given by

\[
\beta(x, r) = \frac{1}{r} \int_r^{2r} \overline{p}(x, \rho) \, d\rho \tag{38}
\]

and \( \pi(x, r) \) given by

\[
\pi(x, r) = \frac{1}{4\pi r} \int_{r \leq |y-x| \leq 2r} \left( \frac{y-x}{|y-x|} \cdot (u(y) - u(x)) \right)^2 \frac{dy}{|y-x|^2} \\
+ \frac{1}{4\pi} \int_{|x-y| \leq 2r} w \left( \frac{|y-x|}{r} \right) \sum_{i,j} \left( u_i(y) - u_i(x) \right) \left( u_j(y) - u_j(x) \right) dy. \tag{39}
\]
Remark 13. Passing to the limit as $r \to \infty$ in (37), we get that
\[
p(x) = \frac{|u(x)|^2}{3} + \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\sigma_{ij}(z)}{|z|^3} (u_i(x + z) - u_i(x))(u_j(x + z) - u_j(x)) \, dz.
\]
(40)

This can be obtained also from (13) using (16).

Proof of Theorem 12. We compute the integral $\frac{1}{r} \int_{r}^{2r} d\rho$ of the representation (14) written as
\[
p(x) = p(x, \rho) + \int_{|y-x|=\rho} |\xi \cdot (u(y) - u(x))|^2 \, dS(y)
\]
\[+ \int_{0}^{\rho} \frac{dl}{l} \int_{|y-x|=l} [3(\xi \cdot (u(y) - u(x)))^2 - |u(y) - u(x)|^2] \, dS(y),
\]
(41)
and we use the fact that
\[
\frac{1}{r} \int_{r}^{2r} \left( \int_{0}^{\rho} f(l) \, dl \right) \, d\rho = \int_{0}^{2r} w \left( \frac{l}{r} \right) f(l) \, dl.
\]

In addition to the bounds (34) and (35) we also have bounds that follow from the Morrey inequality
\[
\int_{\mathbb{R}^3} |u(y)|^6 \, dy \leq C \left[ \int_{\mathbb{R}^3} |\nabla u(y)|^2 \, dy \right]^3,
\]
the representation
\[
p = R_i R_j (u_i u_j)
\]
(42)
of the pressure, where $R_i = \partial_i (-\Delta)^{-1/2}$ are the Riesz transforms, and the boundedness of the Riesz transforms on the $L^p$ spaces. \(\square\)

Proposition 14. Let $p$ be the solution of (2) given by (42). For any $q$ with $1 < q < \infty$ there exist constants $C_q > 0$ independent of $r > 0$ so that for any $r > 0$
\[
\|p(\cdot, \cdot)\|_{L^q(\mathbb{R}^3)} \leq C_q \|u\|_{L^{2q}(\mathbb{R}^3)}^2
\]
(43)
and
\[
\|\beta(\cdot, \cdot)\|_{L^q(\mathbb{R}^3)} \leq C_q \|u\|_{L^{2q}(\mathbb{R}^3)}^2.
\]
(44)

For any $a \in (0, 2)$ there exists a constant $C_a > 0$ such that
\[
\|\beta(\cdot, \cdot)\|_{L^3(\mathbb{R}^3)} \leq C_a r^{-a} \alpha \|u\|_{L^2(\mathbb{R}^3)} \|\nabla u\|_{L^2(\mathbb{R}^3)}^{2-a}.
\]
(45)

There exists a constant $C > 0$ such that
\[
\|\nabla p(\cdot, \cdot)\|_{L^2(\mathbb{R}^3)} \leq C r^{-1} \|u\|_{L^4(\mathbb{R}^3)}^2
\]
(46)
and
\[
\|\nabla \beta(\cdot, \cdot)\|_{L^2(\mathbb{R}^3)} \leq C r^{-1} \|u\|_{L^4(\mathbb{R}^3)}^2.
\]
(47)
Proof. The bounds (44) for $\beta$ follow from the bounds (43) for $\overline{p}$ by averaging with respect to $r$. The bounds (43) follow from (42) and the boundedness of the Riesz transforms on the $L^p$ spaces. The bounds (45) follow from (35), the interpolation inequality
\[
\|\beta\|_{L^3(\mathbb{R}^3)} \lesssim \|\beta\|_{L^\infty(\mathbb{R}^3)}^{a/3} \|\beta\|_{L^3(\mathbb{R}^3)}^{1-a/3},
\]
the bound (44) for $q = 3 - a$,
\[
\|\beta(\cdot, r)\|_{L^{3-a}(\mathbb{R}^3)} \leq C_a \|u\|_{L^{6-2a}(\mathbb{R}^3)}^2,
\]
and from interpolation combined with the Morrey inequality
\[
\|u\|_{L^{6-2a}(\mathbb{R}^3)} \leq C \|u\|_{L^2(\mathbb{R}^3)}^{a/(6-2a)} \|\nabla u\|_{L^2(\mathbb{R}^3)}^{(6-3a)/(6-2a)}.
\]
The bound (47) follows from the bound (46) by averaging with respect to $r$. The bound (46) follows from the inequality
\[
\|\nabla \overline{p}(\cdot, r)\|_{L^2(\mathbb{R}^3)} \leq C r^{-1} \|\overline{p}(\cdot, r)\|_{L^2(\mathbb{R}^3)}
\]
and (43) with $q = 2$. The bound (48) follows from the Plancherel identity and the observation that
\[
\hat{\overline{p}}(\xi, r) = \frac{\sin(r|\xi|)}{r|\xi|} \hat{\overline{p}}(\xi).
\]
Indeed,
\[
\int_{\mathbb{R}^3} e^{-ix\cdot\xi} \overline{p}(x, r) \, dx = \int_{|\omega|=1} dS(\omega) \int_{\mathbb{R}^3} e^{-ix\cdot\xi} p(x + r\omega) \, dx
\]
\[
= \hat{\overline{p}}(\xi) \int_{|\omega|=1} e^{ir\xi\cdot\omega} \, dS(\omega),
\]
and it is convenient to compute the last integral by choosing coordinates so that $\xi$ points to the North pole:
\[
\frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^\pi e^{ir|\xi| \cos \theta} \sin \theta \, d\theta = \frac{\sin(r|\xi|)}{r|\xi|}.
\]

Regarding $\pi$ we have the following result.

Proposition 15. Let $\pi(x, r)$ be defined by (39). Then
\[
|\pi(x, r)| \leq C \int_{|z| \leq 2r} \frac{|u(x + z) - u(x)|^2}{|z|^3} \, dz.
\]
Consequently,
\[
\|\pi(\cdot, r)\|_{L^q(\mathbb{R}^3)} \leq C_q r^2 \|\nabla u\|_{L^{2q}(\mathbb{R}^3)}^2
\]
holds for all $1 < q \leq \infty$. In particular, for $q = 3$ the Morrey inequality implies that
\[
\|\pi(\cdot, r)\|_{L^3(\mathbb{R}^3)} \leq C r^2 \|\Delta u\|_{L^2(\mathbb{R}^3)}^2.
\]
Also,
\[
\|\pi(\cdot, r)\|_{L^q(\mathbb{R}^3)} \leq C_q \|u\|_{L^{2q}(\mathbb{R}^3)}^2.
\]
Proof. The inequality (50) is immediate from the definition. To prove (51) we write

\[ |u(x + z) - u(x)|^2 \leq |z|^2 \int_0^1 |\nabla u(x + \lambda z)|^2 \, d\lambda, \]

and after changing the order of integration we have

\[
\left| \int_{\mathbb{R}^3} \phi(x) \, dx \int_{|z| \leq 2r} \frac{|u(x + z) - u(x)|^2}{|z|^3} \, dz \right| \leq C r^2 \|\phi\|_{L^q} \|\nabla u\|_{L^{2q}}^2,
\]

which proves (51). The bound (53) follows from (37), the corresponding bounds for \( p \), and (44). \( \square \)

4. FGT bounds [4] in the whole space

Let us take the Navier-Stokes equation

\[
\partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla p = 0
\]

with

\[
\nabla \cdot u = 0,
\]

multiply by \( \partial_t u - \nu \Delta u \), and integrate using the incompressibility:

\[
\int_{\mathbb{R}^3} |\partial_t u - \nu \Delta u|^2 \, dx = -\int_{\mathbb{R}^3} (u \cdot \nabla u)(\partial_t u - \nu \Delta u) \, dx.
\]

The Schwartz inequality gives us that

\[
\int_{\mathbb{R}^3} |\partial_t u - \nu \Delta u|^2 \, dx \leq \int_{\mathbb{R}^3} |u \cdot \nabla u|^2 \, dx,
\]

and hence

\[
\int_{\mathbb{R}^3} |\partial_t u - \nu \Delta u|^2 \, dx \leq \|u\|_{L^\infty}^2 \|\nabla u\|_{L^2}^2.
\]

The inequality

\[
\|u\|_{L^\infty}^2 \leq C \|\nabla u\|_{L^2} \|\Delta u\|_{L^2}
\]

(56)

is easy to prove using the Fourier transform. Thus,

\[
\int_{\mathbb{R}^3} |\partial_t u - \nu \Delta u|^2 \, dx \leq C \|\Delta u\|_{L^2} \|\nabla u\|_{L^2}^3.
\]

On the other hand,

\[
\int_{\mathbb{R}^3} |\partial_t u - \nu \Delta u|^2 \, dx = \|\partial_t u\|_{L^2}^2 + \nu^2 \|\Delta u\|_{L^2}^2 + \nu \frac{d}{dt} \|\nabla u\|_{L^2}^2,
\]

and therefore

\[
d \|\nabla u\|_{L^2}^2 + \nu \|\Delta u\|_{L^2}^2 + \frac{1}{\nu} \|\partial_t u\|_{L^2}^2
\]

\[
\leq \frac{C}{\nu} \|\Delta u\|_{L^2} \|\nabla u\|_{L^2}^2 \leq \frac{\nu}{2} \|\Delta u\|_{L^2}^2 + \frac{C}{\nu^3} \|\nabla u\|_{L^2}^6.
\]
We now let \( y(t) = \|\nabla u(\cdot, t)\|_{L^2}^2 \), pick a constant \( A > 0 \), divide by \( (A + y)^2 \), and get that
\[
- \frac{d}{dt} \left( \frac{1}{A + y} \right) + \nu \|\Delta u\|_{L^2}^2 + \frac{\|\partial_t u\|_{L^2}^2}{\nu(A + y)^2} \leq \frac{C}{\nu^3} y.
\]
Integrating with respect to the time, we obtain
\[
\int_0^T \nu \|\Delta u\|_{L^2}^2 \, dt + \int_0^T \frac{\|\partial_t u\|_{L^2}^2}{\nu(A + y)^2} \, dt \leq \frac{C}{\nu^4} \|u_0\|_{L^2}^2 + \frac{1}{A}.
\]
Therefore,
\[
\int_0^T \frac{\|\Delta u\|_{L^2}^2}{(A + y)^2} \, dt \leq \frac{C}{\nu^5} \|u_0\|_{L^2}^2 + \frac{1}{\nu A} = C\nu^{-4}[D + \nu^3 A^{-1}]
\]
and
\[
\int_0^T \frac{\|\partial_t u\|_{L^2}^2}{(A + y)^2} \, dt \leq \frac{C}{\nu^3} \|u_0\|_{L^2}^2 + \frac{\nu}{A} = C\nu^{-2}[D + \nu^{-3} A^{-1}],
\]
where
\[
D = \frac{\|u_0\|_{L^2}^2}{\nu}.
\]
Further,
\[
\int_0^T \|\Delta u\|_{L^2}^{2/3} \, dt \leq \left[ \int_0^T \frac{\|\Delta u\|_{L^2}^2}{(A + y)^2} \, dt \right]^{1/3} \left[ \int_0^T (A + y) \, dt \right]^{2/3}
\]
and
\[
\int_0^T \|\partial_t u\|_{L^2}^{2/3} \, dt \leq \left[ \int_0^T \frac{\|\partial_t u\|_{L^2}^2}{(A + y)^2} \, dt \right]^{1/3} \left[ \int_0^T (A + y) \, dt \right]^{2/3},
\]
so that
\[
\int_0^T \|\Delta u\|_{L^2}^{2/3} \, dt \leq C\nu^{-4/3}[D + \nu^3 A^{-1}]^{1/3}[D + AT]^{2/3}
\]
and
\[
\int_0^T \|\partial_t u\|_{L^2}^{2/3} \, dt \leq C\nu^{-2/3}[D + \nu^3 A^{-1}]^{1/3}[D + AT]^{2/3}.
\]
The constant \( A \) is arbitrary, but it is natural to choose
\[
A^2 = \nu^3 T^{-1},
\]
and then
\[
\int_0^T \|\Delta u\|_{L^2}^{2/3} \, dt \leq C\nu^{-4/3} \left[ \frac{\|u_0\|_{L^2}^2}{\nu} + T^{1/2} \nu^{3/2} \right] \tag{59}
\]
and
\[
\int_0^T \|\partial_t u\|_{L^2}^{2/3} \, dt \leq C\nu^{-2/3} \left[ \frac{\|u_0\|_{L^2}^2}{\nu} + T^{1/2} \nu^{3/2} \right]. \tag{60}
\]
From the inequality (56) it follows immediately that
\[
\int_0^T \|u\|_{L^\infty} \, dt \leq C \nu^{-1} \left[ \frac{\|u_0\|_{L^2}^2}{\nu} + T^{1/2} \nu^{3/2} \right]^{3/4} \left[ \frac{\|u_0\|_{L^2}^2}{\nu} \right]^{1/4}. \tag{61}
\]
Let us now consider the other terms in (54). We start by computing
\[
\int_{\mathbb{R}^3} |u \cdot \nabla u + \nabla p|^2 \, dx = \|u \cdot \nabla u\|_{L^2}^2 + \|\nabla p\|_{L^2}^2 + 2 \int_{\mathbb{R}^3} (u \cdot \nabla u) \cdot (\nabla p) \, dx.
\]
Furthermore,
\[
2 \int_{\mathbb{R}^3} (u \cdot \nabla u) \cdot (\nabla p) \, dx = -2 \int_{\mathbb{R}^3} p \text{Tr}(\nabla u)^2 \, dx = 2 \int_{\mathbb{R}^3} p \Delta p \, dx = -2 \|\nabla p\|_{L^2}^2.
\]
Consequently,
\[
0 \leq \int_{\mathbb{R}^3} |u \cdot \nabla u + \nabla p|^2 \, dx = \|u \cdot \nabla u\|_{L^2}^2 - \|\nabla p\|_{L^2}^2.
\]
On the other hand, it is obvious that
\[
\|u \cdot \nabla u\|_{L^2} \leq \|u\|_{L^\infty} \|\nabla u\|_{L^2}
\]
and in view of the previous result
\[
\int_0^T \|u \cdot \nabla u\|_{L^2}^{2/3} \, dt \leq C \nu^{-2/3} \left[ \frac{\|u_0\|_{L^2}^2}{\nu} + T^{1/2} \nu^{3/2} \right]^{1/2} \left[ \frac{\|u_0\|_{L^2}^2}{\nu} \right]^{1/2}.
\]
Also, the inequality \( \|\nabla p\|_{L^2} \leq \|u \cdot \nabla u\|_{L^2} \) gives us that
\[
\int_0^T \|\nabla p\|_{L^2}^{2/3} \, dt \leq C \nu^{-2/3} \left[ \frac{\|u_0\|_{L^2}^2}{\nu} + T^{1/2} \nu^{3/2} \right]^{1/2} \left[ \frac{\|u_0\|_{L^2}^2}{\nu} \right]^{1/2},
\]
from which the following result.

**Theorem 16.** Let \( u \) be a Leray weak solution of the Navier–Stokes equation on the time interval \([0, T]\). Then the quantities \( \|u\|_{L^\infty(\mathbb{R}^3)}, \|\Delta u\|_{L^2(\mathbb{R}^3)}^{2/3}, \|\nabla u\|_{L^2(\mathbb{R}^3)}^{2/3}, \|u \cdot \nabla u\|_{L^2(\mathbb{R}^3)}^{2/3}, \) and \( \|\nabla p\|_{L^2(\mathbb{R}^3)}^{2/3} \) are almost everywhere finite on \([0, T]\), and their time integrals are uniformly bounded, with the bounds in (59)–(63) depending only on \( T, \|u_0\|_{L^2(\mathbb{R}^3)}, \) and \( \nu \).

The proof for Leray weak solutions follows the same pattern as the proof given above for smooth solutions, except that we mollify the advecting velocity, prove the mollification-uniform bounds, and deduce the results using essentially Fatou’s lemma. For completeness, we mention here some other estimates. Interpolating the inequalities
\[
\int_0^T \|\nabla u\|_{L^2}^2 \, dt < \infty
\]
and
\[
\int_0^T \|\nabla u\|_{L^6}^{2/3} \, dt < \infty,
\]
which follow from the Morrey inequality and (59), we get the inequality
\[
\|\nabla u\|_{L^3} \leq C \|\nabla u\|_{L^6}^{1/2} \|\nabla u\|_{L^2}^{1/2},
\]
as well as the other estimates mentioned in Theorem 16.
which then is integrable by the Hölder inequality:

$$\int_0^T \|\nabla u\|_{L^3} dt < \infty.$$ 

Lastly, we note that by interpolating between $L^\infty(dt;L^2(dx))$ and $L^2(dt;L^6(dx))$ we easily see that $u \in L^p(dt,L^q(dx))$ for $q = 6p/(3p-4)$ if $p \geq 2$. For $p \in [1,2]$ interpolation between $L^2(dt;L^6(dx))$ and $L^1(dt,L^\infty(dx))$ gives us that $q = 3p/(p-1)$.

5. Applications

**Theorem 17.** Let $u$ be a solution of (54) and (55) in $\mathbb{R}^3$ and assume that $u$ belongs to $L^\infty(dt;L^2(\mathbb{R}^3)) \cap L^q(dt;C^\alpha(\mathbb{R}^3))$ for some $q \geq 1$. Then $p \in L^q(dt;C^{2\alpha}(\mathbb{R}^3))$ if $\alpha < 1/2$. If $\alpha = 1/2$ then $p \in L^q(dt;\text{LiplogLip})$, where LiplogLip is the class of functions with modulus of continuity $|x - y| \log(|x - y|^{-1})$. If $\alpha > 1/2$ then $p \in L^q(dt;\text{Lip})$, where Lip is the class of Lipschitz continuous functions.

**Proof.** Let us start with two points $x, y$ at the distance $|x - y|$ and choose $r = 8|x - y|$. The representation (14) implies that

$$\begin{cases} |p(x) - \bar{p}(x, r)| \leq C\|u\|_{C^\alpha}^2 r^{2\alpha}, \\ |p(y) - \bar{p}(y, r)| \leq C\|u\|_{C^\alpha}^2 r^{2\alpha}, \end{cases}$$

and therefore it remains to prove that

$$|\bar{p}(x, r) - \bar{p}(y, r)| \leq Cr^{2\alpha}$$

if $2\alpha < 1$ and $C \sim \|u\|_{C^\alpha}^2$. (For $2\alpha = 1$ we get an estimate with $r \log(r^{-1})$, and an estimate with $r$ for $2\alpha > 1$.) In order to do this, we use (4) with $v = u((x + y)/2)$ and integrate from $r$ to infinity, obtaining

$$\bar{p}(x, r) = -\int_{|\xi| = 1} (\xi \cdot (u(x + r\xi) - v))^2 dS(\xi)$$

$$+ \frac{1}{4\pi} \int_{|x - z| \geq r} \frac{\sigma_{ij}(x - z)}{|x - z|^3} (u_i(z) - v_i)(u_j(z) - v_j) dz$$

(65)

and

$$\bar{p}(y, r) = -\int_{|\xi| = 1} (\xi \cdot (u(y + r\xi) - v))^2 dS(\xi)$$

$$+ \frac{1}{4\pi} \int_{|y - z| \geq r} \frac{\sigma_{ij}(y - z)}{|y - z|^3} (u_i(z) - v_i)(u_j(z) - v_j) dz.$$ 

(66)

It is now clear that

$$\left| \int_{|\xi| = 1} (\xi \cdot (u(x + r\xi) - v))^2 dS(\xi) \right| \leq Cr^{2\alpha}\|u\|_{C^\alpha}^2.$$
and
\[ \left| \int_{|\xi|=1} \left( \xi \cdot \left( u(y + r\xi) - v \right) \right)^2 dS(\xi) \right| \leq Cr^{2\alpha} \| u \|_{C^\alpha}^2, \]
so it remains to estimate
\[ \frac{1}{4\pi} \int_{|x-z| \geq r} \frac{\sigma_{ij}(x-z)}{|x-z|^3} w_i w_j \, dz - \frac{1}{4\pi} \int_{|y-z| \geq r} \frac{\sigma_{ij}(y-z)}{|y-z|^3} w_i w_j \, dz, \]
where \( w = u(y) - v \). Now if \( |x-z| \geq r \) but \( |y-z| \leq r \), then \( |x-z| \leq |y-z| + |x-y| \leq 9r/8 \), and so
\[ \frac{1}{4\pi} \int_{|x-z| \geq r, |y-z| \leq r} \frac{\sigma_{ij}(x-z)}{|x-z|^3} w_i w_j \, dz \leq C \| u \|_{C^\alpha}^2 r^{2\alpha}. \]
Similarly, if \( |y-z| \geq r \) but \( |x-z| \leq r \), then
\[ \frac{1}{4\pi} \int_{|x-z| \geq r, |y-z| \leq r} \frac{\sigma_{ij}(y-z)}{|y-z|^3} w_i w_j \, dz \leq C \| u \|_{C^\alpha}^2 r^{2\alpha}. \]
Finally, there remains the integral
\[ \frac{1}{4\pi} \int_{|x-z| \geq r, |y-z| \geq r} (K_{ij}(x-z) - K_{ij}(y-z)) w_i w_j \, dz, \]
where
\[ K_{ij}(\zeta) = (3\zeta_i\zeta_j|\zeta|^{-2} - \delta_{ij})|\zeta|^{-3}. \]
This is now a classical situation in singular integral theory where the smoothness of the kernel is used. We observe that
\[ |K_{ij}(x-z) - K_{ij}(y-z)| \leq C|x-y| \int_0^1 |z - (y + \lambda(x-y))|^{-4} \, d\lambda \]
and that \( |z - (y + \lambda(x-y))| \geq 7r/8 \). Thus,
\[ \left| \frac{1}{4\pi} \int_{|x-z| \geq r, |y-z| \geq r} (K_{ij}(x-z) - K_{ij}(y-z)) w_i w_j \, dz \right| \leq C|x-y| \int_0^1 \int_{|z-x\lambda| \geq 7r/8} |z - x\lambda|^{-4} \left| u(z) - u \left( \frac{x+y}{2} \right) \right|^2 \, dz \, d\lambda, \]
where \( x\lambda = y + \lambda(x-y) \). Fixing \( R > 0 \) (we could take \( R = 1 \), but we prefer to keep dimensionally correct quantities), we get that
\[ |x-y| \int_0^1 \int_{|z-x\lambda| \geq R} |z - x\lambda|^{-4} \left| u(z) - u \left( \frac{x+y}{2} \right) \right|^2 \, dz \, d\lambda \leq C|x-y| R^{-1} \| u \|_{L^\infty}^2. \]
The integral over \( 7r/8 \leq |z - x\lambda| \leq R \),
\[ |x-y| \int_0^1 \int_{7r/8 \leq |z-x\lambda| \leq R} |z - x\lambda|^{-4} \left| u(z) - u \left( \frac{x+y}{2} \right) \right|^2 \, dz \, d\lambda, \]
can be estimated using the inequality

$$\left| u(z) - u\left(\frac{x + y}{2}\right) \right| \leq C \|u\|_{C^\alpha}^2 (|z - x\lambda|^{2\alpha} + r^{2\alpha}).$$

The resulting bounds obtained by integrating over $7r/8 \leq |z - x\lambda| \leq R$ are

$$C \|u\|_{C^\alpha}^2 |x - y| \left[ \frac{1}{2} r^{2\alpha - 1} + r^{2\alpha - 1} \right]$$

if $2\alpha < 1$,

$$C \|u\|_{C^\alpha}^2 |x - y| \left[ \log \left( \frac{8R}{r} \right) + 1 - \frac{r}{R} \right]$$

if $2\alpha = 1$, and

$$C \|u\|_{C^\alpha}^2 |x - y| \left[ \frac{R^{2\alpha - 1}}{2\alpha - 1} + r^{2\alpha - 1} \right]$$

if $2\alpha > 1$. □

We state now some criteria for regularity. We will write $\pi(x, t, r(t))$ for $\pi$ defined according to the formula (39) for a time dependent $u(x, t)$ and with a time dependent $r = r(t)$. Recall that $\pi$ is small if $u$ is regular and $r$ is small.

**Theorem 18.** Let $u$ be a smooth solution of the Navier–Stokes equation on the interval $[0, T]$.

**First criterion.** Assume that there exist $U > 0$, $R > 0$, and $0 < r(t) \leq R$ such that

$$\int_{\{x \in \mathbb{R}^3 : |u(x, t)| \geq U\}} |u(x, t)| |\pi(x, t, r(t))|^2 \, dx \leq \frac{\nu^2}{4} \int_{\mathbb{R}^3} |u(x, t)| |\nabla u(x, t)|^2 \, dx. \quad (67)$$

Assume that there exists a $\gamma > 4$ such that

$$\int_0^T r(t)^{-\gamma} \, dt < \infty. \quad (68)$$

Then

$$u \in L^\infty([0, T], L^3(\mathbb{R}^3)). \quad (69)$$

**Second criterion.** Assume that there exists an $r(t)$ such that $\pi = \pi(x, r(t))$ satisfies

$$\int_0^T \|\pi\|^2_{L^3(\mathbb{R}^3)} \, dt < \infty \quad (70)$$

and that, as above, there exists a $\gamma > 4$ such that (68) holds. Then again (69) holds.

**Proof.** Let us start with the first criterion. We consider the evolution of the $L^3$ norm of the velocity:

$$\frac{d}{dt} \|u\|^3_{L^3(\mathbb{R}^3)} + \nu \int_{\mathbb{R}^3} |\nabla u|^2 |u| \, dx + \int_{\mathbb{R}^3} |u|(u \cdot \nabla p) \, dx \leq 0.$$
Let $p$ be represented using the formula (37) with $r = r(t)$. We split softly the integral involving $\pi$:

$$
\int_{\mathbb{R}^3} |u|(u \cdot \nabla \pi) \, dx = \int_{\mathbb{R}^3} \phi \left( \frac{|u|}{U} \right) |u|(u \cdot \nabla \pi) \, dx
+ \int_{\mathbb{R}^3} \left( 1 - \phi \left( \frac{|u|}{U} \right) \right) |u|(u \cdot \nabla \pi) \, dx,
$$

where $\phi(q)$ is a smooth scalar function with $0 \leq \phi(q) \leq 1$ that is supported in the interval $0 \leq q \leq 1$. Using the estimate

$$
|\nabla \pi(x)| \leq C \int_0^1 \, d\lambda \int_{|z| \leq 2r} \frac{dz}{|z|^2} \left( |\nabla u(x + z)| + |\nabla u(x)||\nabla u(x + \lambda z)| \right),
$$

which follows from (39) by differentiation, we get that

$$
\left| \int_{\mathbb{R}^3} \phi \left( \frac{|u|}{U} \right) |u|(u \cdot \nabla \pi) \, dx \right| \leq CU^2r|\nabla u|_{L^2(\mathbb{R}^3)}^2.
$$

Integration of the second term by parts gives us that

$$
\int_{\mathbb{R}^3} \left( 1 - \phi \left( \frac{|u|}{U} \right) \right) |u|(u \cdot \nabla \pi) \, dx = -\int_{\mathbb{R}^3} \pi u \cdot \nabla \left[ |u| \left( 1 - \phi \left( \frac{|u|}{U} \right) \right) \right] \, dx.
$$

When the derivative falls on $1 - \phi$, we are in the $|u| \leq U$ regime, and we use (53) and interpolation combined with the Morrey inequality

$$
\|u\|_{L^4(\mathbb{R}^3)}^2 \leq C\|u\|_{L^3(\mathbb{R}^3)} \|\nabla u\|_{L^2(\mathbb{R}^3)}
$$

to deduce that

$$
\left| \int_{\mathbb{R}^3} \pi |u|(u \cdot \nabla |u|)U^{-1} \phi' \left( \frac{|u|}{U} \right) \, dx \right| \leq CU\|\pi\|_{L^2(\mathbb{R}^3)} \|\nabla u\|_{L^2(\mathbb{R}^3)}
\leq CU\|u\|_{L^3(\mathbb{R}^3)} \|\nabla u\|_{L^2(\mathbb{R}^3)}^2.
$$

When the derivative falls on $|u|$, we use the condition (67) and the Schwartz inequality:

$$
\left| \int_{\{|u(x,t)| \geq U\}} \pi (u \cdot \nabla |u|) \left( 1 - \phi \left( \frac{|u|}{U} \right) \right) \, dx \right|
\leq \frac{\nu}{2} \int_{\mathbb{R}^3} |u| |\nabla u|^2 \, dx.
$$

As for the integral involving $\beta$, we integrate by parts and use the Hölder inequality followed by (45):

$$
\left| \int_{\mathbb{R}^3} \beta u \cdot \nabla |u| \, dx \right| \leq \|\beta\|_{L^3(\mathbb{R}^3)} \|u\|_{L^2(\mathbb{R}^3)}^{1/2} \sqrt{\int_{\mathbb{R}^3} |u| |\nabla u|^2 \, dx}
\leq \frac{1}{2\nu} \|\beta\|_{L^3(\mathbb{R}^3)}^2 \|u\|_{L^3(\mathbb{R}^3)} \|u\|_{L^3(\mathbb{R}^3)} + \frac{\nu}{2} \int_{\mathbb{R}^3} |u| |\nabla u|^2 \, dx
\leq C\nu^{-1}r^{-2a} \|u\|_{L^2(\mathbb{R}^3)}^2 \|\nabla u\|_{L^3(\mathbb{R}^3)}^{1-2a} \|u\|_{L^3(\mathbb{R}^3)}^{4-2a} \|u\|_{L^3(\mathbb{R}^3)} + \frac{\nu}{2} \int_{\mathbb{R}^3} |u| |\nabla u|^2 \, dx.
$$
By choosing $a = \gamma/(\gamma - 2)$ we have $a < 2$, and from Young’s inequality we see that
\[
r^{-2a}\|\nabla u\|_{L^2(\mathbb{R}^3)}^{4-2a} \leq C(r^{-\gamma} + \|\nabla u\|_L^2(\mathbb{R}^3))
\]
is time-integrable. The result is that the quantity $y(t) = \|u\|_{L^3(\mathbb{R}^3)}$ satisfies an ordinary differential inequality
\[
y^2 \frac{dy}{dt} \leq C_1(t) + C_2(t)y + C_3(t)y,
\]
with
\[
C_1(t) = CU^2 r\|\nabla u\|_{L^2(\mathbb{R}^3)}^2, \quad C_2(t) = CU\|\nabla u\|_{L^2(\mathbb{R}^3)}^2, \quad \text{and} \quad C_3(t) = C\nu^{-1}r^{-2a}\|\nabla u\|_{L^2(\mathbb{R}^3)}^{4-2a}\|u\|_{L^2(\mathbb{R}^3)}^{2a}.
\]
The positive functions $C_1(t)$, $C_2(t)$, and $C_3(t)$ are known to be time-integrable. The interested reader can check that the inequality above is dimensionally correct: each term has the dimensions $[L]^6[T]^{-4}$. Then it follows that
\[
y^2 \frac{dy}{1 + y} dt \leq C_1(t) + C_2(t) + C_3(t)
\]
(no longer dimensionally correct), and after an easy integration it follows that $y$ is a priori bounded with respect to time. This proves the first criterion.

For the proof of the second criterion we again use the representation $p = \pi(x, r) + \beta(x, r)$ with $r = r(t)$, and we estimate the integral involving $\pi$ using straightforward integration by parts and Hölder inequalities:
\[
\left| \int_{\mathbb{R}^3} (u \cdot \nabla \pi)|u| \, dx \right| = \left| \int_{\mathbb{R}^3} \pi(u \cdot \nabla |u|) \, dx \right| \\
\leq \frac{\nu}{2} \int_{\mathbb{R}^3} |u| |\nabla u|^2 \, dx + \frac{C}{\nu}\|u\|_{L^3(\mathbb{R}^3)}\|\pi\|_{L^3(\mathbb{R}^3)}^2.
\]
We estimate the contribution coming from $\beta$ just as we did for the first criterion. The conclusion is that $y(t) = \|u\|_{L^3(\mathbb{R}^3)}$ satisfies the inequality
\[
y^2 \frac{dy}{dt} \leq C_4(t)y + C_3(t)y,
\]
where $C_4(t) = (C/\nu)\|\pi\|_{L^3(\mathbb{R}^3)}^2$ is time-integrable by assumption. It follows again that $y(t)$ is a priori bounded with respect to time. $\Box$

6. Appendix

Here we prove the identities (5) and (6). Let us introduce the polar coordinates
\[
\xi_1 = \rho \cos \phi \sin \theta = \rho cS, \\
\xi_2 = \rho \sin \phi \sin \theta = \rho sS, \\
\xi_3 = \rho \cos \theta = \rho C,
\]

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where for simplicity of notation we use the abbreviations $s = \sin \phi$, $S = \sin \theta$, $c = \cos \phi$, and $C = \cos \theta$. For a function on the unit sphere $\rho = 1$, but in general $f(\xi) = f(\rho cS, \rho sS, \rho C)$, and we have

$$f_\theta = \partial_\theta f = \rho (cCf_1 + sCf_2 - Sf_3),$$
$$f_\phi = \partial_\phi f = \rho (-sSf_1 + cSf_2),$$
$$\rho f_\rho = \rho \partial_\rho f = \rho (cSf_1 + sSf_2 + Cf_3),$$

where $\rho \partial_\rho f = \xi \cdot \nabla_\xi f$ and $\nabla_\xi f = (f_1, f_2, f_3)$. Note that $\rho \partial_\rho (\xi/|\xi|) = 0$ for $\xi \neq 0$.

We have

$$Cf_\theta + Sf_\rho = \rho (cCf_1 + sf_2),$$
$$C\rho f_\rho - Sf_\theta = \rho f_3,$$

and thus

$$\rho f_1 = c(Cf_\theta + Sf_\rho) - \frac{s}{S}f_\phi,$$
$$\rho f_2 = s(Cf_\theta + Sf_\rho) + \frac{c}{S}f_\phi,$$
$$\rho f_3 = C\rho f_\rho - Sf_\theta.$$

Let us now consider $\rho = 1$ and for simplicity write $D_\rho = \rho \partial_\rho$. We first compute

$$\int_{|\xi|=1} \xi_1 \partial_1 f(x + r\xi) dS(\xi),$$

using of course the fact that

$$dS(\xi) = S d\phi d\theta.$$

We have

$$\xi_1 \partial_1 f = cS \left( c(C\partial_\theta + S\rho) - \frac{s}{S} \partial_\phi \right) f$$
$$= D_\rho (\xi_1^2 f) + c^2 SC\partial_\theta f - sc\partial_\phi f.$$

Here we used the fact that on the unit sphere $\xi = \xi/|\xi|$ and $D_\rho (\xi) = 0$. We multiply by $S$ and integrate, by parts where possible. In view of

$$-c^2 \frac{d}{d\theta} (S^2 C) = c^2 S(S^2 - 2C^2) = c^2 S(3S^2 - 2)$$

and

$$S \frac{d}{d\phi} (sc) = 2c^2 S - S,$$

the coefficient of $f$ is obtained by addition,

$$c^2 S(3S^2 - 2) + 2c^2 S - S = S(3\xi_1^2 - 1),$$

so that

$$\int_{|\xi|=1} \xi_1 \partial_1 f dS(\xi) = \int_{|\xi|=1} [D_\rho (\xi_1^2 f) + 3\xi_1^2 f - f] dS(\xi)$$
$$= D_\rho \left[ \int_{|\xi|=1} \xi_1^2 f dS(\xi) \right] + \int_{|\xi|=1} (3\xi_1^2 - 1) f dS(\xi),$$

\[\text{(71)}\]
which is the first relation in (5). The rest of the formulae in (5) are proved similarly. Indeed,

\[ \xi_2 \partial_{\xi_2} f = sS \left[ s(C \partial_\theta + SD_\rho) + \frac{c}{S} \partial_\theta \right] f = \left[ s^2 S^2 D_\rho + s^2 SC \partial_\theta + sc \partial_\phi \right] f. \]

Multiplying by \( S \) and integrating by parts in the terms containing \( \partial_\theta \) and \( \partial_\phi \), we obtain the coefficient of \( f \):

\[
- s^2 \frac{d}{d\theta} (S^2 C) - S \frac{d}{d\phi} (sc) = s^2 S (3S^2 - 2) + S - 2c^2 S \\
= s^2 S (3S^2 - 2) - S + 2s^2 S = (3\xi_2^2 - 1)S,
\]

and therefore

\[
\int_{|\xi|=1} \xi_2 \partial_{\xi_2} f dS(\xi) = D_\rho \left[ \int_{|\xi|=1} \xi_2^2 f dS(\xi) \right] + \int_{|\xi|=1} (3\xi_2^2 - 1) f dS(\xi)
\]
as above. The third term is

\[ \xi_3 \partial_3 f = C(CD_\rho - S \partial_\theta) f. \]

Multiplying by \( S \) and integrating by parts the term containing \( \partial_\theta \), we compute the coefficient of \( f \):

\[
\frac{d}{d\theta} (CS^2) = (3C^2 - 1)S = 3\xi_3^2 - 1,
\]

and thereby the last relation in (5):

\[
\int_{|\xi|=1} \xi_3 \partial_{\xi_3} f dS(\xi) = D_\rho \left[ \int_{|\xi|=1} \xi_3^2 f dS(\xi) \right] + \int_{|\xi|=1} (3\xi_3^2 - 1) f dS(\xi).
\]

We now prove similarly the relations (6). We start with the equality corresponding to the indices (1, 3):

\[
(\xi_1 \partial_{\xi_3} + \xi_3 \partial_{\xi_1}) f = \left[ cS(CD_\rho - S \partial_\theta) + C(c(C \partial_\theta + SD_\rho) - \frac{s}{S} \partial_\phi) \right] f \\
= \left[ 2cSCD_\rho + (cC^2 - cS^2) \partial_\theta - \frac{Cs}{S} \partial_\phi \right] f.
\]

Multiplying by \( S \) and integrating by parts, we obtain the coefficient of \( f \) from the equality

\[
-c \frac{d}{d\theta} (S(1 - 2S^2)) + C \frac{d}{d\phi} (s) = -c(C - 6S^2 C) + Cc = 6cSCS = 6\xi_1 \xi_3 S,
\]

so that

\[
\int_{|\xi|=1} (\xi_1 \partial_{\xi_3} + \xi_3 \partial_{\xi_1}) f dS(\xi) = \int_{|\xi|=1} \left[ 2\xi_1 \xi_3 D_\rho f + 6\xi_1 \xi_3 f \right] dS(\xi) \\
= D_\rho \left[ \int_{|\xi|=1} 2\xi_1 \xi_3 f dS(\xi) \right] + \int_{|\xi|=1} 6\xi_1 \xi_3 f dS(\xi),
\]
which is the relation in (6) for the indices (1, 3). For the indices (1, 2) we have to compute

\[(\xi_1 \partial_2 + \xi_2 \partial_1) f = \left[ cS \left( sSD_\rho + sC \partial_\theta + \frac{c}{S} \partial_\phi \right) + sS \left( cSD_\rho + cC \partial_\theta - \frac{s}{S} \partial_\phi \right) \right] f \]

\[= 2cSsSD_\rho f + 2csSC \partial_\theta f + (c^2 - s^2) \partial_\phi f.\]

Multiplying by \(S\) and integrating by parts, we obtain the coefficient of \(f\) from the condition

\[-2cs \frac{d}{d\theta} (S^2 C) - S \frac{d}{d\phi} (c^2 - s^2) = 2cs(S^3 - 2SC^2) + 4Scs \]

\[= 2cs(S^3 - 2S + 2S^3) + 4csS = 6csS^3 = 6\xi_1 \xi_2 S.\]

Therefore,

\[\int_{|\xi|=1} (\xi_1 \partial_\xi_2 + \xi_2 \partial_\xi_3) f dS(\xi) = \int_{|\xi|=1} \left[ 2\xi_1 \xi_2 D_\rho f + 6\xi_1 \xi_2 f \right] dS(\xi) \]

\[= D_\rho \left[ \int_{|\xi|=1} 2\xi_1 \xi_2 f dS(\xi) \right] + \int_{|\xi|=1} 6\xi_1 \xi_2 f dS(\xi),\]

which is the relation (6) for the indices (1, 2). Finally, for the indices (2, 3) we have to compute

\[(\xi_2 \partial_3 + \xi_3 \partial_2) f = sS(CD_\rho - S \partial_\theta) f + C \left( sSD_\rho + sC \partial_\theta + \frac{c}{S} \partial_\phi \right) f \]

\[= 2sSCD_\rho f + \left( s(C^2 - S^2) \partial_\theta + C \frac{c}{S} \partial_\phi \right) f.\]

Multiplying by \(S\) and integrating by parts, we get the coefficient of \(f\) from the equality

\[-s \frac{d}{d\theta} (S(C^2 - S^2)) - C \frac{d}{d\phi} c = s(6S^2 C - C) + Cs = 6sSCS = 6\xi_2 \xi_3 S,\]

and thus

\[\int_{|\xi|=1} (\xi_2 \partial_\xi_3 + \xi_3 \partial_\xi_2) f dS(\xi) = \int_{|\xi|=1} \left[ 2\xi_2 \xi_3 D_\rho f + 6\xi_2 \xi_3 f \right] dS(\xi) \]

\[= D_\rho \left[ \int_{|\xi|=1} 2\xi_2 \xi_3 f dS(\xi) \right] + \int_{|\xi|=1} 6\xi_2 \xi_3 f dS(\xi),\]

which is (6) for the indices (2, 3).

Bibliography

[1] L. C. Berselli and G. P. Galdi, “Regularity criteria involving the pressure for the weak solutions to the Navier–Stokes equations”, Proc. Amer. Math. Soc. 130:12 (2002), 3585–3595.
[2] P. Constantin, “An Eulerian–Lagrangian approach to the Navier–Stokes equations”, Comm. Math. Phys. 216:3 (2001), 663–686.

[3] L. Escauriaza, G. Seregin, and V. Šverák, “Backward uniqueness for parabolic equations”, Arch. Ration. Mech. Anal. 169:2 (2003), 147–157.

[4] C. Foiaş, C. Guillopè, and R. Temam, “New a priori estimates for Navier–Stokes equations in dimension 3”, Comm. Partial Differential Equations 6:3 (1981), 329–359.

[5] P. Isett, Regularity in time along the coarse scale flow for the incompressible Euler equations, 2013 (v3 – 2014), 48 pp., arXiv:1307.0565.

[6] G. Seregin and V. Šverák, “The Navier–Stokes equations and backward uniqueness”, Nonlinear problems in mathematical physics and related topics. II, Int. Math. Ser. (N.Y.), vol. 2, Kluwer/Plenum, New York 2002, pp. 353–366.

[7] G. Seregin and V. Šverák, “Navier–Stokes equations with lower bounds on the pressure”, Arch. Ration. Mech. Anal. 163:1 (2002), 65–86.

[8] T. Tao, “Localisation and compactness properties of the Navier–Stokes global regularity problem”, Anal. PDE 6:1 (2013), 25–107; 2012, 95 pp., arXiv: 1108.1165v4.

[9] A. Vasseur, “Higher derivatives estimate for the 3D Navier–Stokes equation”, Ann. Inst. H. Poincaré Anal. Non Linéaire 27:5 (2010), 1189–1204.

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