ANOSOV-KATOK CONSTRUCTIONS FOR QUASI-PERIODIC SL(2, R) COCYCLES

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Abstract. We prove that if the frequency of the quasi-periodic SL(2, R) cocycle is Diophantine, then the following properties are dense in the subcritical regime: for any $\frac{1}{2} < \kappa < 1$, the Lyapunov exponent is exactly $\kappa$-Hölder continuous; the extended eigenstates of the potential have optimal sub-linear growth; and the dual operator associated a subcritical potential has power-law decay eigenfunctions. The proof is based on fibered Anosov-Katok constructions for quasi-periodic SL(2, R) cocycles.

1. Introduction

In this paper, we are concerned with one-dimensional analytic quasi-periodic Schrödinger operators:

$$\mathcal{H}_{V, \alpha, \theta} u_n = u_{n+1} + u_{n-1} + V(\theta + n\alpha) u_n,$$

where $V \in C^\infty(T^d, \mathbb{R})$ is the potential, $\alpha \in T^d$ is the frequency, assumed to be rationally independent, and $\theta \in T^d$ is the phase. Due to the rich implications in quantum physics, quasi-periodic Schrödinger operators have been extensively studied [46]. Starting from the early 1980’s, there was already an almost periodic flu which swept the world, as pointed out by Simon [49]. In 2000’s, adopting a dynamical systems point of view (mainly analytic quasi-periodic cocycles) was found to be useful in the study of such operators (1.1), and much progress has been made since then [2, 3, 6, 12].

We recall that an analytic quasi-periodic cocycle $(\alpha, A) \in T^d \times C^\infty(T^d, SL(2, \mathbb{R}))$ is a linear skew product system:

$$\begin{align*}
(\alpha, A) : & \{ T^d \times \mathbb{R}^2 \to T^d \times \mathbb{R}^2 \\
(\theta, v) \mapsto (x + \alpha, A(\theta) \cdot v)
\end{align*}$$

Among many important advances, we should highlight Avila’s global theory of one-frequency analytic quasi-periodic cocycles [2]: if $d = 1$, any $(\alpha, A)$ which is not uniformly hyperbolic, is either supercritical, subcritical or critical, properties that can be expressed in terms of the growth of the cocycle. More precisely, $(\alpha, A)$ is said to be

1. Supercritical, if $\sup_{z \in T} \| A(z; n) \|$ grows exponentially;
2. Subcritical, if there is a uniform subexponential bound on the growth of $\| A(z; n) \|$ through some band $|\Im z| < \delta$;
3. Critical, otherwise.
Here we define the products of cocycles as \((\alpha, A)^n = (n\alpha, A(\cdot; n))\), where \(A(\cdot, 0) = \text{id}\),

\[ A(\cdot; n) = A(\cdot + (n - 1)\alpha) \cdots A(\cdot), \quad \text{for } n \geq 1, \]

and \(A(\cdot; -n) = A(\cdot - n\alpha; n)^{-1}\) for \(n \geq 1\). We also recall that \((\alpha, A)\) is uniformly hyperbolic if there exists a continuous invariant splitting \(\mathbb{R}^2 = E^s(\theta) \oplus E^u(\theta)\) with the following property: there exist \(C > 0\) and \(\lambda > 0\) such that \(\|A(\theta; n) \cdot v\| \leq Ce^{-\lambda n}\|v\|\) holds for every \(n \geq 0\) and every \(v \in E^s(x)\). Simultaneously \(\|A(\theta; -n) \cdot v\| \leq Ce^{-\lambda n}\|v\|\) holds for every \(v \in E^u(x)\). We will denote the set of such cocycles as \(\mathcal{UH}_\alpha\) for short.

The fundamental observation, allowing for the dynamical systems point of view to be taken, is that (1.1) can be seen as a quasi-periodic \(\text{SL}(2, \mathbb{R})\) cocycle, since a sequence \((u_n)_{n \in \mathbb{Z}}\) is a formal solution of the eigenvalue equation \(H_{V,\alpha,\theta} u = E u\) if and only if it satisfies

\[ (u_{n+1}, u_n) = S^V_E(\theta + n\alpha) \cdot (u_n, u_{n-1}), \]

where

\[ S^V_E(\theta) = \begin{pmatrix} E - V(\theta) & -1 \\ 1 & 0 \end{pmatrix} \in \text{SL}(2, \mathbb{R}). \]

Thus, corresponding to (1.1), there is a family of naturally defined cocycles \((\alpha, S^V_E)\), called quasi-periodic Schrödinger cocycles. It is well-known that \(E \in \Sigma_{V,\alpha}\), the spectrum of (1.1), if and only if \((\alpha, S^V_E) \notin \mathcal{UH}_\alpha\). Therefore energies in the spectrum of \(H_{V,\alpha,\theta}\) can be characterized as supercritical, subcritical or critical in terms of its corresponding cocycle \((\alpha, S^V_E)\).

A cornerstone in Avila’s global theory is the “Almost Reducibility Theorem” (ART) [2, 4, 5], which asserts that \((\alpha, A)\) is almost reducible (denoted by \(\mathcal{AR}_\alpha\) for any fixed \(\alpha\)) if it is subcritical. We recall that a cocycle \((\alpha, A)\) is almost reducible, if the closure of its analytic conjugacy class contains a constant cocycle.

In this paper, we will focus on the dynamical and spectral behavior of \((\alpha, S^V_E)\) in the subcritical regime. We remark that our main results work also in the multifrequency case, \(d \geq 2\), since they are proved in the regime \(\mathcal{AR}_\alpha \setminus \mathcal{UH}_\alpha\). Indeed, by Avila’s global theory and ART [2, 4, 5], if \(\alpha \in \mathbb{R} \setminus \mathbb{Q}\), \(A \in \mathcal{AR}_\alpha \setminus \mathcal{UH}_\alpha\) if and only if \((\alpha, A)\) is subcritical.

1.1. **Regularity of Lyapunov exponent.** Our first result concerns the regularity of Lyapunov exponent (LE):

\[ L(\alpha, A) := \lim_{n \to \infty} \frac{1}{n} \int_{\mathbb{T}^d} \ln \|A(\theta; n)\| d\theta \]

as a function of the mapping \(A(\cdot)\) defining the dynamics in the fibers. The LE is a central topic in the spectral theory of Schrödinger operators, since it relates with integrated density of states through the Thouless formula.
The LE also arise naturally in the study of smooth dynamics, and continuity of LE has been the object of important recent research, see [50] and the references therein.

In the analytic topology, the LE is always continuous with respect to both \( \alpha \) and \( A \) at any given cocycle \((\alpha, A)\), provided that \( \alpha \in \mathbb{T}^d \) is rationally independent, see \([18, 21, 35]\). This holds true even for cocycles in higher dimensional groups \( GL(d, \mathbb{C}) \), with \( \alpha \in \mathbb{R}\setminus\mathbb{Q} \) [9]. If \( \alpha \in DC_d^2 \), and the potential \( V \) is small enough (the smallness of \( V \) depending on \( \alpha \)), then \( L(\alpha, S^V_E) \) is \( \frac{1}{2} \)-Hölder continuous w.r.t \( E \) [15]. Subsequently, Avila-Jitomirskaya in [7] generalized the \( \frac{1}{2} \)-Hölder continuity to the non-perturbative regime, lifting the dependence of the smallness of \( V \) on \( \alpha \). By the ART, see [2, 4], the LE is \( \frac{1}{2} \)-Hölder continuous in the whole subcritical regime [47]. This kind of \( \frac{1}{2} \)-Hölder continuity is sharp, since the LE is exactly \( \frac{1}{2} \)-Hölder continuous at the end of spectral gaps by [48].

On the other hand, if \((\alpha, A)\) is conjugated to constant \( SO(2, \mathbb{R}) \) cocycle or \( A \in UH \), then LE is Lipschitz. Therefore a natural question is whether there exist subcritical cocycles such that the optimal Hölder exponent of \( L(\alpha, A) \) can be any fixed number between \( \frac{1}{2} \) and 1. In fact, we will show that this holds in a dense set in \( AR_\alpha \cap UH_\alpha \) for each admissible Hölder exponent.

**Theorem 1.1.** Let \( \frac{1}{2} < \kappa < 1 \), \( \alpha \in DC_d \). There exists a set \( \mathcal{S} \) which is dense in \( AR_\alpha \cap UH_\alpha \) in the \( C^{\omega}(\mathbb{T}^d, SL(2, \mathbb{R})) \) topology such that Lyapunov exponent is exactly \( \kappa \)-Hölder continuous at each point of \( \mathcal{S} \) in the sense that for any \( A \in \mathcal{S} \) we have for \( B \in C(\mathbb{T}^d, SL(2, \mathbb{R})) \),

\[
\liminf_{\|B-A\|_0 \to 0} \frac{\log |L(\alpha, A) - L(\alpha, B)|}{\log \|B - A\|_0} = \kappa.
\]

Some arithmetic condition on \( \alpha \) is needed for the theorem to be true. As pointed out by Avila-Jitomirskaya [7], the Lyapunov Exponent is discontinuous at rational \( \alpha \), which implies then for generic \( \alpha \in \mathbb{R}\setminus\mathbb{Q} \), LE cannot be Hölder continuous at any order. Recently, Avila-Last-Shamis-Zhou [11] showed that if \( \alpha \) is very Liouvillean, then LE can even be not \( \log \)-Hölder continuous.

In the positive Lyapunov Exponent regime, Goldstein and Schlag proved that the LE is Hölder continuous (for one-frequency cocycles) or weak Hölder continuous (for multi-frequency cocycles) for Schrödinger operators whose frequency satisfies a strong Diophantine condition [30, 31].

Continuity of the LE also depends sensitively on the smoothness of \( A(\cdot) \). In the \( C^0 \) topology, any non uniformly hyperbolic \( SL(2, \mathbb{R}) \) cocycle can be approximated by cocycles with zero LE, see [16, 17], and thus the LE is not

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\[ DC_d^{\kappa'}(\tau) := \left\{ \alpha \in \mathbb{T}^d : \inf_{j \in \mathbb{Z}} |\langle n, \alpha \rangle - j| > \frac{\kappa'}{|n|^\tau}, \quad \forall \ n \in \mathbb{Z}^d \setminus \{0\} \right\}. \]

The set of Diophantine numbers is denoted by \( DC_d := \bigcup_{\kappa' > 0, \tau > d-1} DC_d^{\kappa'}(\tau) \).
In the $C^k$ ($k \in \mathbb{N}$) topology, results are quite different. Wang-You, [51], showed that the LE can be discontinuous. On the other hand, Xu-Ge-Wang, [53], recently constructed a class of $C^{2+\epsilon}$ cos-type potential, and for any $\frac{1}{2} \leq \kappa < 1$ showed the existence of an energy, such that LE is exactly $\kappa$-Hölder continuous. For other results of regularity of LE in the smooth or in the Gervey topology, one can consult [22, 45, 52] and the references therein.

1.2. Optimal sub-linear growth of extended eigenstates. Next, we move to spectral applications. In contrast with 1D random Schrödinger operator, one of the most remarkable phenomena exhibited by quasiperiodic Schrödinger operators is the existence of absolutely continuous (ac) spectrum. If $\alpha \in \text{DC}_d$ and $V$ is small enough, then Dinaburg-Sinai in [23] proved that $H_{V,\alpha,\theta}$ has ac spectrum. Eliasson in [24] proved the stronger result that $H_{V,\alpha,\theta}$ has purely ac spectrum for every $\theta$. Avila’s ART [2, 4, 5] ensures that if the potential is subcritical, then the corresponding operator has purely ac spectrum. From the physicist’s point of view, ac spectrum corresponds to a phase where the material is a conductor, and the corresponding eigenstate is extended. Indeed, one can define the inverse participation ratio (IPR) [26] as

$$\text{IPR}(m) = \frac{\sum_{n=1}^{L} |u_m(n)|^4}{\sum_{n=1}^{L} |u_m(n)|^2},$$

where $u_m$ is the $m$-th eigenstate. If $\Gamma = -\lim_{L \to \infty} \frac{\ln(\text{IPR})}{\ln L} = 1$, then the phase is extended. Thus, conductivity of the material is reflected in the growth of its eigenstate.

For $\alpha \in \text{DC}_d$ and $V$ small enough, Eliasson [24] proved that if the energy lies in the end of spectral gaps, then its extended eigenstates have linear growth. On the other hand, for any $E \in \Sigma_{V,\alpha}$, the spectrum of the operator, the extended eigenstates at most have sub linear growth, i.e. $|u_E(n)| \leq o(n)$. This kind of behavior can be generalized to the whole subcritical regime [2, 4]. Thus it is interesting to ask whether this kind of sub-linear growth was optimal. In this paper, we prove that for a dense subset of subcritical potentials, the extended eigenstates of the associated operator have optimal sub-linear growth.

**Theorem 1.2.** Fix $\alpha \in \text{DC}$. For any non-increasing sequence $\{g(n)\}_{n=1}^{\infty}$ satisfying $0 < g(n) < 1$ and $\lim_{n \to \infty} n^{g(n)} = \infty$, for any $V \in C^\omega_T(\mathbb{T}, \mathbb{R})$, if $E \in \Sigma_{V,\alpha}^{\text{sub}}$, then for any $\varepsilon > 0$, there exist $0 < h' < h$, $V \in C^\omega_T(\mathbb{T}, \mathbb{R})$ with $\|V - V'\|_{h'} \leq \varepsilon$, such that $E \in \Sigma_{V',\alpha}^{\text{sub}}$. Moreover, the eigenstate $u_E(n)$ of

$$(H_{V',\alpha,\theta} u_E)(n) = E u_E(n),$$

has sub linear growth with rate $\{g(n)\}_{n=1}^{\infty}$, i.e. there exist $\{n_j\}_{j=1}^{\infty}$ and $c,C$ not depending on $j$ such that

$$c |n_j|^{1-g(n_j)} < |u_E(n_j)| < C |n_j|^{1-\frac{1}{2}g(n_j)}.$$
Remark 1.1. On the right side of the inequality, one can replace $1 - \frac{1}{2}g(n_j)$ by $1 - (1 - \delta)g(n_j)$ for any $\delta > 0$.

Theorem 1.2 shows that there exists extended eigenfunctions with optimal sub-linear growth. We remark however, that the universal hierarchical structure of any extended quasi-periodic eigenfunctions for almost Mathieu operators was recently obtained in [33].

The study of growth of extended eigenstates is not only interesting from the physicist’s point of view, but also important from the mathematical point of view. By (1.2), the study of the growth of the extended eigenstates is equivalent to study the growth of the Schrödinger cocycle. On one hand, the growth of Schrödinger cocycle is crucial for proving existence of ac spectrum. In the subcritical regime, for almost every $E \in \Sigma_{\mathcal{V},\alpha}$, the cocycles are uniformly bounded, see [24], which is sufficient for establishing the existence of ac spectrum by subordinacy theory [29]. However, whether the spectral measure is purely ac or not really depends on the growth of $\|S^V_\mathcal{E}(\theta; j)\|$ on a Lebesgue zero measure set of energies by [3]. On the other hand, the growth of Schrödinger cocycle is important in proving the regularity of the spectral measure, see [8]. Indeed, using Jitomiskaya-Last’s inequality, see [37], Avila-Jitomirskaya in [8] showed that

$$\mu(E - \epsilon, E + \epsilon) \leq C\epsilon \|P_L\|,$$

where $P_L = \sum_{n=1}^{L} (S^V_\mathcal{E})^r(\theta + \alpha; 2n - 1)S^V_\mathcal{E}(\theta + \alpha; 2n - 1)$ satisfies $\det P_L = 1/4\epsilon^2$. Therefore, by Theorem 1.2, it seems interesting to study high order Hölder continuity of spectral measure for a dense set of subcritical energies.

1.3. Power-law decay eigenfunctions. Our third result concerns the localization property of the quasi-periodic long-range operator:

$$(\mathcal{L}\mathcal{V},\alpha,\phi)u)_n = \sum_{k \in \mathbb{Z}^d} V_k u_{n-k} + 2\cos 2\pi (\phi + \langle n, \alpha \rangle)u_n,$$

where $V_k \in \mathbb{C}$ is the Fourier coefficient of $V \in C^\omega(\mathbb{T}^d, \mathbb{R})$. This operator has received a lot of attention, see [7, 13, 20] since it is the Aubry dual of the quasi-periodic Schrödinger operator defined in eq. (1.1). If $V(\theta) = 2\cos \theta$, then it reduces to the extensively studied almost Mathieu operator:

$$(H_{\lambda,\alpha,\theta}u)_n = u_{n+1} + u_{n-1} + 2\lambda \cos (\theta + n\alpha)u_n.$$

For the almost Mathieu operator, there is a sharp phase transition line $\lambda = e^{\beta(\alpha)}$ from singular continuous spectrum to Anderson localization (pure point spectrum with exponentially decaying eigenfunctions), see [13, 38]. In the transition line $\lambda = e^{\beta(\alpha)}$, for frequencies in a dense set, $H_{\lambda,\alpha,\theta}$ displays pure point spectrum, see [10], but the eigenfunction does not decay exponentially, see [38]. As pointed out in [10], the insights gained from the critical parameters often shed light on the creation, dissipation, and the

$\beta(\alpha) := \lim sup_{n \to \infty} \frac{\ln|q_{n+1}|}{n}$, where $p_n/q_n$ is the continued fraction best rational approximants of $\alpha \in \mathbb{R}\setminus\mathbb{Q}$.
mechanism behind the phases of non critical parameters as well. Thus it is
interesting to ask whether there exists real power law decay eigenfunction,
i.e. if the eigenfunctions can decay polynomially. In this paper, we establish
the following result.

**Theorem 1.3.** Let $\alpha \in DC_d$, $0 < h_s < h$ and $s \in \mathbb{N}^+$. Then, there exist
$\varepsilon_0 = \varepsilon_0(\alpha, h, h_s)$ and $0 < h_s < h$, such that for any $\varepsilon < \varepsilon_0$, any
$V \in B_h(\varepsilon) := \{ V \in C^\omega_h(\mathbb{T}, \mathbb{R}) \mid \|V\|_h < \varepsilon \}$
is accumulated by $V_k \in B_{h_s}(\varepsilon)$ such that $L_{V_k,\alpha,\phi}$ has point spectrum with
eigenfunction $u \in h^s \setminus h^{s+1}$ for some $\phi \in \mathbb{R}$, where we denote $h^s = \{ u \in l^2 : \sum_k |k|^s |u_k| < \infty \}$.

Just as Theorem 1.2, Theorem 1.3 also holds if the dual Schrödinger
operator (1.1) lies in the subcritical regime. However, under the assumption
that $\alpha \in DC_d$, the phenomenon exhibited by Theorem 1.3 was not expected
for Schrödinger operators (1.1). Indeed, if $V(\theta)$ is an even function, then for
a $G_\delta$ dense set of $\theta$, $H_{V,\alpha,\theta}$ has no eigenvalues, see [40]. It is widely believed
for a.e. $\theta$, $H_{V,\alpha,\theta}$ has Anderson localization in the positive LE regime. In
fact, $H_{\lambda,\alpha,\theta}$ even exhibits a sharp transition in the phase $\theta$ between singular
continuous spectrum and Anderson localization [39].

If $d = 1$, Bourgain-Jitomirskaya in [20] proved that for any fixed $\alpha \in DC$, $L_{\lambda V,\alpha,\phi}$ has Anderson localization for sufficiently small $\lambda$ and a.e. $\phi$. If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, not necessary Diophantine, then for dense set of small potentials $V$ and a.e. $\phi$, $L_{V,\alpha,\phi}$ has point spectrum with exponentially decaying
eigenfunctions, see [56], while it is still open whether $L_{V,\alpha,\phi}$ has Anderson localization. If $d \geq 2$, Jitomirskaya-Kachkovskiy in [36] proved that for fixed $\alpha \in DC_d$, $L_{\lambda V,\alpha,\theta}$ has pure point spectrum for sufficiently small $\lambda$ and a.e. $\theta$. Recently, Ge-You-Zhou [32] further proved that under the same assumption, $L_{\lambda V,\alpha,\theta}$ has exponentially dynamical localization.

1.4. **Density cocycles reducible in finite differentiability.** Recall that
a cocycle $(\alpha, A) \in \mathbb{T}^d \times C^\omega(\mathbb{T}^d, SL(2, \mathbb{R}))$ is $C^s$ reducible if there exists
$B \in C^s(2\mathbb{T}^d, SL(2, \mathbb{R}))$ such that
$$B^{-1}(\cdot + \alpha)A(\cdot)B(\cdot) \in SL(2, \mathbb{R}),$$
then by Aubry duality, Theorem 1.3 is an immediately corollary of the following reducibility result.

**Theorem 1.4.** Given $s \in \mathbb{N}^+$, $\alpha \in DC_d$. There exists a set $\mathfrak{F}$ which is
dense in $\mathcal{AR}_\alpha \setminus \mathcal{UH}_\alpha$ in the $C^\omega(\mathbb{T}^d, SL(2, \mathbb{R}))$ topology, such that if $A \in \mathfrak{F}$,
then $(\alpha, A)$ is $C^s$-reducible but not $C^{s+1}$-reducible.

We point out that this result is interesting in itself, from the dynamical
systems point of view. The study of the reducibility of cocycles is related
with the linearization of circle diffeomorphisms. Arnol’d in [1] proved that
if an analytic diffeomorphism $f \in \text{Diff}^{\omega}(\mathbb{T})$ is close to a rotation $T_\rho$, where $\rho$
is the rotation number of $f$, and $\rho \in DC$, then $f$ is analytically linearizable.
One of the great achievements of Herman-Yoccoz, see [34, 54], is the proof of the fact that the sharp arithmetic condition for $C^\infty$ linearizability of $f \in \Diff^\infty(\mathbb{T})$, without any smallness condition imposed a priori on $f$, is that its rotation number be Diophantine. In the Liouvillean regime, Herman [34] (see also [44]) proved that for any $s \in \mathbb{N}^+$ and any Liouvillean number $\rho$, the set of $f \in \Diff^\infty(\mathbb{T})$ with rotation number $\rho$ which is $C^s$ but not $C^{s+1}$ linearizable is locally dense. For a more precise description of the theory, we refer the reader for the recent survey of Eliasson-Fayad-Krikorian [25] on circle diffeomorphisms.

Concerning the reducibility of $\SL(2, \mathbb{R})$ cocycles, if the base frequency $\alpha \in DC_d$, and the cocycle $(\alpha, A)$ is sufficiently close to constants (the closeness depends on $\alpha$), Eliasson [24] proved that if the fibered rotation number is Diophantine w.r.t $\alpha$, then $(\alpha, A)$ is analytically reducible. By global to local reduction, reducibility for a full measure set of frequencies holds in the non-perturbative regime, see [48], and in the subcritical regime [4]. Motivated by the results in circle diffeomorphism, then it is natural to study the density of $C^s$ but not $C^{s+1}$ reducible cocycles. Theorem 1.4 provides a positive answer to this question. The corresponding theorem in the case of $C^\infty$ smooth $SU(2)$ cocycles, which in some sense stands in midway between circle diffeomorphisms and $\SL(2, \mathbb{R})$ cocycles, was proved by the first author in [41].

1.5. Some comments on the proofs. The proof of these results are based on the fast approximation by conjugation method introduced by Anosov and Katok in [14], where they constructed mixing diffeomorphisms of the unit disc arbitrarily close to Liouvillean rotations. We refer the reader to §3 for a more detailed description of the method, but in a nutshell it consists of the following idea.

The transient dynamics of a Liouvillean rotation is, for all practical reasons, periodic: if $\rho$ is Liouville, then there exist sequences $p_n, q_n \in \mathbb{N}^*$ such that

$$\|\rho - \frac{p_n}{q_n}\|_\mathbb{T} < |q_n|^{-n}$$

which means that a diffeomorphism conjugate to the rotation by $\rho$ is practically periodic with period $q_n$, in scales of iteration comparable with arbitrarily big powers of $q_n$. It is tempting, therefore, to try to study diffeomorphisms with rotation number equal to $\rho$ by approximating them from the outside, i.e. with diffeomorphisms that have rational rotation numbers, and to boot, are conjugate to them.

Since the dynamics of a periodic rotation are determined by a finite number of iterations, they are easier to tamper with, and since $|q_{n+1}| \gg |q_n|$, it is also possible to modify the dynamics at the scale $q_{n+1}$ in a way that preserves what was constructed in the previous scale $q_n$. 
The limit object, satisfying the mixing property was thus constructed as follows. A diffeomorphism $f_n$ is constructed such that
\[ f_n = H_n \circ T_{p_n/q_n} \circ H_n^{-1} \]
for some smooth conjugation $H_n$, satisfying some finitary version of mixing at a scale of iteration $q_n$. A $q_n$-periodic conjugation $h_n$ is constructed, such that the diffeomorphism
\[ f_{n+1} = H_n \circ h_n \circ T_{p_{n+1}/q_{n+1}} \circ h_n^{-1} \circ H_n^{-1} \]
satisfying some improved finitary version of mixing at a scale of iteration $q_{n+1} \gg q_n$. This can be achieved by choosing $p_{n+1}/q_{n+1}$ very close to $p_n/q_n$, so that the $C^0$ norm of $h_n$ can be allowed to explode. This will ensure that the limit diffeomorphism $f = \lim f_n$ is smooth, but not measurably conjugate to $T_\rho$. The Liouvillean character of $\rho$ is necessary in order to assure the convergence of $f_n$ despite the divergence of $H_n$.

For more information on the method, results and references we point the reader to [28]. This approximation-by-conjugation method has been useful in producing examples of dynamics incompatible with quasi-periodicity, in the vicinity of quasi-periodic dynamics. It is in some sense the counterpart of the KAM method: KAM tends to prove rigidity in the Diophantine world, while Anosov-Katok is used in order to prove non-rigidity in the Liouvillean world. In the context of cocycles, the concept of reducibility, obtained notably via KAM, allows for studying the rigidity results [12] when the fibered rotation number is Diophantine with respect to the frequency, while Anosov-Katok’s construction will be an efficient method to study wild dynamics when the fibered rotation number is Liouville with respect to the frequency.

The fibered Anosov-Katok construction was introduced by the first author in [41]. In this context, the rotation in the basis is fixed, and the only freedom is in the choice of the mapping in the fibers. The rotation $\alpha$ in the basis can be chosen to be Diophantine, and the role of periodic rotations in the classical constructions is taken by resonant cocycles, i.e. cocycles whose rotation number is $k\alpha$, where $k \in \mathbb{Z}$. The rest of the construction remains the same.

From an almost reducibility point of view, the construction consists in engineering the parameters of the KAM normal form, introduced in [42], and further exploited in [41, 43] in order to study a variety of dynamical phenomena present in the almost reducibility regime for cocycles in $SU(2)$. Using the KAM schemes of [22, 47], we further develop these techniques in order to adapt them to $SL(2, \mathbb{R})$ cocycles and in the context of the analytic category (instead of the smooth one, as in the [42]).

We point out that results obtainable by the fibered Anosov-Katok method can be obtained for cocycles over a fixed Liouvillean rotation, but the case of a Diophantine rotation is more difficult, and therefore more interesting.
2. A lemma from linear algebra

Given \( A \in \text{SL}(2, \mathbb{R}) \) and calling
\[
M := \frac{1}{1 + i} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix},
\]
we have by direct calculation \( MAM^{-1} \in \text{SU}(1, 1) \), where \( \text{SU}(1, 1) \) is the group of special unitary \( 2 \times 2 \) matrices preserving the scalar product of \( \mathbb{C}^2 \) with signature \((1, -1)\) i.e.
\[
B^H \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]
where \( B \in \text{SU}(1, 1) \). Since \( M \) is an isometry between the upper half-plane and the disc models for the hyperbolic plane, we know that
\[
\text{SU}(1, 1) = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \big| ||a||^2 - ||b||^2 = 1 \text{ and } a, b \in \mathbb{C} \right\}.
\]
The Lie algebra of \( \text{SU}(1, 1) \), denoted by \( \text{su}(1, 1) \), is formed by traceless Hermitian \( 2 \times 2 \) matrices,
\[
\text{su}(1, 1) = \left\{ \begin{pmatrix} it & z \\ \bar{z} & -it \end{pmatrix} \big| t \in \mathbb{R}, z \in \mathbb{C} \right\}.
\]
Given two matrices in \( \text{su}(1, 1) \)
\[
A_1 = \begin{pmatrix} it_1 & z_1 \\ \bar{z}_1 & -it_1 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} it_2 & z_2 \\ \bar{z}_2 & -it_2 \end{pmatrix}
\]
their scalar product is defined by
\[
\langle \{A_1\}, \{A_2\} \rangle = t_1t_2 + \mathcal{R}(z_1\bar{z}_2) = t_1t_2 + \mathcal{R}z_1\mathcal{R}z_2 + \mathcal{I}z_1\mathcal{I}z_2
\]
so that the natural semi-Riemannian structure on \( \text{su}(1, 1) \) be defined by
\[
A \rightarrow ||t||^2 - ||z||^2 = \det(A)
\]
In \( \text{su}(1, 1) \) we can therefore distinguish three regimes: the elliptic regime where \( \det(A) > 0 \), the parabolic regime where \( \det(A) = 0 \), and the hyperbolic regime where \( \det(A) < 0 \).

Parabolic matrices are not diagonalizable, and hyperbolic ones are anti-diagonalizable, but since in the present paper we focus on the elliptic regime, we prove the following lemma concerning the diagonalization of elliptic matrices in \( \text{su}(1, 1) \). The diagonalizing conjugation given by the lemma is of optimal norm.

**Lemma 2.1.** Let the matrix
\[
A = \begin{pmatrix} it & z \\ \bar{z} & -it \end{pmatrix} \in \text{su}(1, 1)
\]
satisfy \( \det A > 0 \). Then, calling \( \rho = \sqrt{\det A} \), we have
\[
D^{-1}AD = \begin{pmatrix} i\rho & 0 \\ 0 & -i\rho \end{pmatrix},
\]
where

\[ D = (\cos 2\theta)^{-\frac{1}{2}} \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} e^{-i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix} \]

\[ = (\cos 2\theta)^{-\frac{1}{2}} \begin{pmatrix} \cos \theta & e^{2i\phi} \sin \theta \\ e^{-2i\phi} \sin \theta & \cos \theta \end{pmatrix} . \]

Here \( 2\phi = \arg (z - \frac{\pi}{2}) \) and \( \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \) satisfies

(2.1) \[ 2\theta = -\arctan \frac{|z|}{\sqrt{t^2 - |z|^2}} \]

In addition we have

(2.2) \[ \|D\| = \frac{(1 - \tan \theta)^2}{1 + \tan^2 \theta} = \frac{|t| + |z|}{\rho} \]

Proof. Firstly, the invariance of the determinant forces that

\[ \rho = \sqrt{t^2 - |z|^2} \in (0, \infty) \]

where the choice of the sign of \( \rho \) is of course arbitrary and irrelevant.

Let, now, \( z = a + bi \), so that \( A \) can be seen as an element of \( \mathbb{R}^3 \) parameterized by \( (t, b, a) \). Then \( \det A > 0 \) means \( A \) belongs to the cone \( \{(t, b, a)|t^2 > b^2 + a^2\} \). As was shown in Figure 1, we can rotate \( A \) to \( B = (t, |z|, 0) \) which lies in the \( (t, b) \) plane by the conjugation:

\[ \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} . \]

which is a rotation around the \( t \) axis. We can thus restrict the problem to the diagonalization of matrices of the type

\[ \begin{pmatrix} it & i|z| \\ -i|z| & -it \end{pmatrix} \]

In the \( (t, b) \) plane, conjugacy classes are hyperbola \( t^2 - b^2 = C \) where \( C > 0 \) is the square of the angle of the corresponding rotation. In the full \( \mathbb{R}^3 \) space, the conjugacy classes are the hyperboloids obtained by revolving these hyperbola around the \( t \) axis.

This is achieved by conjugation by

\[ R(\theta) = (\cos 2\theta)^{-\frac{1}{2}} \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \]

which is a hyperbolic rotation in \( (t, b) \) plane. We now calculate the remaining free parameter, the hyperbolic angle \( \theta \).

Direct calculation shows that

\[ R(\theta) \begin{pmatrix} i\rho & 0 \\ 0 & -i\rho \end{pmatrix} R^{-1}(\theta) = \begin{pmatrix} i\rho \cos^{-1} 2\theta & -i\rho \tan 2\theta \\ i\rho \tan 2\theta & -i\rho^{-1} \cos 2\theta \end{pmatrix} \]

and imposing that \( i|z| = -i\rho \tan 2\theta \) proves the lemma. Optimality of the conjugation follows since the path

\[ R(\tau \theta) \begin{pmatrix} i\rho & 0 \\ 0 & -i\rho \end{pmatrix} R^{-1}(\tau \theta), \tau \in [0, 1] \]
is the shortest path in $su(1, 1)$ connecting
\[
\begin{pmatrix} i\rho & 0 \\ 0 & -i\rho \end{pmatrix} \text{ with } \begin{pmatrix} it & i|z| \\ -i|z| & -it \end{pmatrix}
\]

For any $B \in SU(1, 1)$, its operator norm satisfies
\[
\|B\|^2 = \lambda_{\text{max}}(B^T B),
\]
where $\lambda_{\text{max}}(B^T B)$ is the maximum eigenvalue of $B^T B$. Therefore, by the structure of $D$, (2.2) follows by simple calculations. □

The following corollary is immediate.

**Corollary 2.1.** Suppose that
\[
A = i \begin{pmatrix} t & \lambda \\ -\lambda & -t \end{pmatrix} \in su(1, 1)
\]
with $t > \lambda \geq 0$. Then the conjugation $D$ constructed in Lemma 2.2 has the following estimations:

1. $|\cos 2\theta - \frac{1}{2} \cos \theta - 1| < \frac{\left(\frac{\lambda}{t}\right)^2}{\sqrt{1 - (\frac{\lambda}{t})^2}}$,
2. $|\cos 2\theta - \frac{1}{2} \sin \theta| \in \left[\frac{\frac{\lambda}{t}}{2(1 - (\frac{\lambda}{t})^2)} \frac{\lambda}{t}, \frac{\frac{\lambda}{t}}{2(1 - (\frac{\lambda}{t})^2)} \frac{\lambda}{t}\right]$

**Proof.** Direct calculation shows that
\[
\cos 2\theta - \frac{1}{2} \cos \theta - 1 = \frac{1}{1 - \tan^2 \theta - 1} = \frac{1}{\tan^2 \theta} \cdot \frac{1}{1 + \sqrt{1 - \tan^2 \theta}},
\]
then estimate (1) follows from
\[
\frac{\lambda}{2t} < |\tan \theta| < \frac{\lambda}{t} < 1.
\]
Also by the fact that
\[
(\cos 2\theta)^{-\frac{1}{2}} \sin \theta = \sqrt{\frac{\tan \theta}{2 \tan 2\theta}},
\]
then (2) follows from (2.3) and (2.1).

3. The Fibered Anosov-Katok construction

In this section, we give the Anosov-Katok constructions for quasi-periodic $SU(1,1)$ cocycles, the construction is initialized by fixing a minimal rotation $\alpha \in \mathbb{R}^d/\mathbb{Z}^d$ and then inductively constructing the sequences $\{k_n\}_{n=0}^{\infty}$, $\{t_n\}_{n=0}^{\infty}$ and $\{\lambda_n\}_{n=0}^{\infty}$ satisfying $\forall n \in \mathbb{N}$:

1. $k_n \in \mathbb{Z}^d$, $k_0 = 0$, and $\|\langle k_n, \alpha \rangle\|_T \to 0$, which forces $|k_n| \to \infty$,
2. $t_n > \lambda_n \geq 0$,
3. $\sqrt{t_n^2 - \lambda_n^2} = \|\langle k_{n+1}, \alpha \rangle\|_T \to 0$.

Given these parameters, assume that at the $n$-th step of a construction, we have a constant cocycle $(\alpha, \tilde{A}_n)$, where
\[
\tilde{A}_n = \begin{pmatrix} e_{k_n}(\alpha) & 0 \\ 0 & e_{-k_n}(\alpha) \end{pmatrix}
\]
Such a cocycle is said to be $k_n$-resonant with respect to $\alpha$.

In the $(n+1)$-th step of the construction, we perturb this cocycle to $(\alpha, \tilde{A}_n e^{F_n(\cdot)})$, where
\[
F_n(\cdot) = 2\pi i \begin{pmatrix} t_n & \lambda_n e^{2k_n(\cdot)} \\ -\lambda_n e^{-2k_n(\cdot)} & -t_n \end{pmatrix},
\]
and $e_k(\cdot) := e^{2\pi i \langle k, \cdot \rangle}$. This perturbation is spectrally supported in the resonant Fourier mode in the anti-diagonal direction, while the constant $\tilde{A}_n$ is diagonal. The goal of the construction is to exploit the non commutativity arising by this special type of perturbation.

In the strip $|\Re x| < h$, we have the following estimate for the perturbation
\[
\|F_n\|_h \leq 2\pi t_n e^{4\pi |k_n|h}.
\]
Let $H_n(\cdot) = \begin{pmatrix} e_{k_n}(\cdot) & 0 \\ 0 & e_{-k_n}(\cdot) \end{pmatrix}$, then one has
\[
H_n^{-1}(\cdot + \alpha) \tilde{A}_n e^{F_n(\cdot)} H_n(\cdot) = \exp \left( 2\pi i \begin{pmatrix} t_n & \lambda_n \\ -\lambda_n & -t_n \end{pmatrix} \right) = \tilde{A}_n
\]
By assumptions (2) and (3) and direct application of Lemma 2.1, there exists $D_n \in SU(1,1)$ such that
\[
D_n^{-1} \exp \left( 2\pi i \begin{pmatrix} t_n & \lambda_n \\ -\lambda_n & -t_n \end{pmatrix} \right) D_n = \begin{pmatrix} e_{k_{n+1}}(\alpha) & 0 \\ 0 & e_{-k_{n+1}}(\alpha) \end{pmatrix} = \tilde{A}_{n+1}
\]
i.e. $\tilde{A}_{n+1}$ is $k_{n+1}$-resonant. Note that, by Lemma 2.1, $D_n$ can be chosen in the form

$$D_n = (\cos 2\theta_n)^{-\frac{1}{2}} \begin{pmatrix} \cos \theta_n & \sin \theta_n \\ \sin \theta_n & \cos \theta_n \end{pmatrix},$$

with estimate

$$\|D_n\|^2 = \frac{t_n + \lambda_n}{\sqrt{t_n^2 - \lambda_n^2}} \in \left[ \frac{t_n}{\|\langle k_{n+1}, \alpha \rangle \|_T}, \frac{2t_n}{\|\langle k_{n+1}, \alpha \rangle \|_T} \right],$$

the last inequality holds since by our assumption (2) and (3).

Let $G_n(\cdot) = H_n(\cdot)D_n$. Then we have

$$G_n^{-1}(\cdot + \alpha)\tilde{A}_n e^{F_n(\cdot)}G_n(\cdot) = \tilde{A}_{n+1},$$

which means the cocycle $(\alpha, \tilde{A}_n e^{F_n(\cdot)})$ is conjugated to the resonant cocycle $(\alpha, \tilde{A}_{n+1})$, and the construction can be iterated.

Consequently, let $B_0 = \text{Id}$, $B_n(\cdot) = G_0(\cdot) \cdots G_{n-2}(\cdot)G_{n-1}(\cdot)$, then, starting with an arbitrary resonant cocycle, we can construct the desired cocycle sequences:

$$A_n(\cdot) = B_n(\cdot + \alpha)\tilde{A}_n B_n^{-1}(\cdot).$$

Before introducing the application of this kind of Anosov-Katok construction, we first prove that, under some mild conditions on the sequence of parameters, the cocycle $(\alpha, A_n(\cdot))$ converges, and the limit cocycle is almost reducible. The first assumption of the lemma is related to the fact that we work in the real analytic category, and therefore we need to impose an exponentially fast growth condition on the resonances. In the lemma we use the notation established in this paragraph.

**Lemma 3.1.** Suppose that for some $\epsilon > 0$

$$4\pi t_0 + \sum_{n=1}^{\infty} 4\pi t_n e^{4\pi h \Sigma_{j=1}^{n-1} |k_j|} \prod_{j=0}^{n-1} \frac{2t_j}{\|\langle k_{j+1}, \alpha \rangle \|_T} < \epsilon.$$  

Then the cocycle $(\alpha, A_n(\cdot))$ converges to $(\alpha, A_\infty(\cdot))$ with $A_\infty(\cdot) \in C^\omega_h(\mathbb{T}^d, SU(1, 1))$ and

$$\|A_\infty - \text{Id}\|_h < \epsilon.$$  

Moreover, the cocycle $(\alpha, A_\infty(\cdot))$ is almost reducible:

$$B_n^{-1}(\cdot + \alpha)A_\infty(\cdot)B_n(\cdot) = \tilde{A}_n + \tilde{F}_n(\cdot),$$

where

$$\|\tilde{F}_n\|_h \leq 4\pi t_n e^{4\pi |k_n|} h + \sum_{j=n+1}^{\infty} 4\pi t_j e^{4\pi h \Sigma_{i=n}^{j-1} |k_j|} \prod_{i=n}^{j-1} \frac{2t_i}{\|\langle k_{i+1}, \alpha \rangle \|_T},$$

$$\|A_\infty - A_n\|_h \leq \sum_{j=n}^{\infty} 4\pi t_j e^{4\pi h \Sigma_{i=1}^{j-1} |k_i|} \prod_{i=0}^{j-1} \frac{2t_i}{\|\langle k_{i+1}, \alpha \rangle \|_T}.$$
Proof. Notice that by our construction (3.4) and (3.5), we have
\begin{equation}
A_n + 1 - A_n = B_n(\cdot + \alpha)(\tilde{A}_n e^{F_n} - \tilde{A}_n)B_n^{-1}(\cdot),
\end{equation}
then by (3.1) and (3.3), one has
\[
\|A_1 - A_0\|_h \leq 2\|B_0\|^2_h\|F_0\|_h \leq 4\pi t_0
\]
if \(n \geq 1\), one has
\[
\|A_n - A_{n+1}\|_h \leq 2\|B_n\|^2_h\|F_n\|_h \leq 4\pi t_n e^{2\pi h \Sigma_{j=1}^n |k_j|} \prod_{j=0}^{n-1} \frac{2t_j}{\|\langle k_j+1, \alpha \rangle \|_T}.
\]
therefore, by our assumption (3.6), the cocycle \((\alpha, A_n(\cdot))\) converges to some \((\alpha, A_\infty(\cdot))\). Also note our construction (3.5) implies that \(A_0 = \text{Id}\), then
\[
\|A_\infty - \text{Id}\|_h < \epsilon.
\]
Furthermore, by (3.9), we have
\[
\tilde{A}_n + \sum_{j=n}^{\infty} (B_n^{-1}B_j)(\cdot + \alpha)(\tilde{A}_j e^{F_j} - \tilde{A}_j)(\cdot)(B_n^{-1}B_j)^{-1}(\cdot)
\]
then (3.7) follows immediately. By (3.6) we know \(\|\tilde{F}_n\|_h \to 0\), thus the cocycle \((\alpha, A_\infty(\cdot))\) is almost reducible. \[\square\]

Let us now provide some motivation for the precise choice of the structure of the perturbations \(F_n\). Note firstly that \(F_n\) is totally determined by the triple \(\{k_n, t_n, \lambda_n\}\). After conjugation by \(H_n\), the cocycle \((\alpha, A_n(\cdot))\) becomes
\[
\exp 2\pi i \left( \begin{array}{cc} t_n & \lambda_n \\ -\lambda_n & -t_n \end{array} \right) = \tilde{A}_n.
\]
The matrix \(\tilde{A}_n\) is parameterized by \(\{t_n, \lambda_n\}\), which shows that the construction of the matrix \(\tilde{A}_n\) is determined by the choice of the perturbation \(F_n\). This calculation provides the only way to perturb an elliptic constant cocycle so that it becomes conjugate to a parabolic or a hyperbolic one.

If we consider the map from \(\mathbb{R}^2\) to \(G\):
\[
(t, \lambda) \to \exp 2\pi i \left( \begin{array}{cc} t & \lambda \\ -\lambda & -t \end{array} \right),
\]
we see that \(\{(t, \lambda) | \lambda < |t|\}, \{(t, \lambda) | |\lambda| = |t|\}\) and \(\{(t, \lambda) | |\lambda| > |t|\}\) correspond to the elliptic, parabolic and hyperbolic matrices in SU(1, 1), respectively. The phenomena we study can appear only in the elliptic, and not in the parabolic and hyperbolic regimes. This is the reason why the condition \(t_n > \lambda_n \geq 0\) is imposed. A second restriction is related with the convergence
of $A_n$ and almost reducibility of the limit cocycle, which is guaranteed by choosing $t_n$ sufficiently small with respect to $\{k_j\}_{j=0}^n$. In other words, we impose that $(t_n, \lambda_n)$ tend to $O$ in the elliptic region and sufficiently fast. Then, depending on the unexpected behavior that we want to produce, different restrictions concerning the relative size of $\lambda_n$ with respect to $t_n$ are imposed.

In §4, we construct cocycles at which the Lyapunov exponent is exactly $\kappa$-Hölder continuous, with $\frac{1}{2} < \kappa < 1$. The eigenvalues of a constant parabolic matrix are $\frac{1}{2}$-Hölder continuous, while in the hyperbolic regime (regime $I$ of fig. 2) they depend smoothly on the matrix. Therefore, we first chose $t_n \approx \lambda_n$ to be elliptic but close to parabolic, in the regime $II$. We then perturb $A_n$ to $A'_n$ which is now hyperbolic, but close to parabolic. By controlling the distance between $A_n$ and $A'_n$, we are able to obtain $\kappa$-Hölder continuity.

In §5, we construct cocycles with sublinear growth. Since $\|H_n\|_0 = 1$, the $C^0$-norm of the conjugations $B_n$ are determined by $\|D_n\|$. Constant elliptic cocycles do not grow, while constant parabolic ones grow linearly. In order to obtain growth of elliptic cocycles, we construct cocycles that are conjugated arbitrarily close to parabolic ones, in regime $II$ of the figure 2, and in that case the growth of the cocycle is comparable to $\|D_n\| \to \infty$, as $n \to \infty$.

In §6, we construct cocycles that are $C^s$ reducible, but not $C^{s+1}$ reducible. This is obtained by restricting the cocycle to the regime $III$ of the figure, where $\|D_n\| \to 1$, as $n \to \infty$, at a prescribed speed.
4. Optimal Hölder continuity of Lyapunov exponent

4.1. Local density results. We first prove that cocycles whose associated Lyapunov exponents are exactly $\kappa$-Hölder continuous, with $\frac{1}{2} < \kappa < 1$, are locally dense.

**Proposition 4.1.** Fix $h > 0$, $\frac{1}{2} < \kappa < 1$ and $\alpha \in \mathbb{R}^d/\mathbb{Z}^d$ rationally independent. Then, for any $\epsilon > 0$ there exists a cocycle $(\alpha, A(\cdot))$ with $A(\cdot) \in C^\omega(T^d, SL(2, \mathbb{R}))$ and

$$\|A(\cdot) - 1d\|_h < \epsilon,$$

such that Lyapunov exponent is exactly $\kappa$-Hölder continuous at $(\alpha, A(\cdot))$.

**Proof.** Using the isomorphism between $SL(2, \mathbb{R})$ and $SU(1, 1)$, we prove the theorem in the context of $SU(1, 1)$ cocycles. Let us first introduce two auxiliary parameters $q, \tilde{\delta}$. Since $\frac{1}{2} < \kappa < 1$, there exists $q \in \mathbb{N}$ such that

$$q > 10 \frac{1 - \kappa}{2\kappa - 1} + 10.$$  

and $0 < \tilde{\delta} < \frac{1}{q+1}$ satisfying:

$$\frac{(1 - \tilde{\delta})\frac{1 - \kappa}{2\kappa - 1} + 1}{\frac{1 - \kappa}{2\kappa - 1} + 1} > q \frac{q}{q + 1}.$$  

We will also use the auxiliary function

$$f(x) := x^{|\ln x|^{-\frac{1}{2}}}$$

which satisfies $\lim_{x \to 0} f(x) = 0$, and $\lim_{x \to 1} f(x) = 1$ and is monotonic increasing on $(0, 1)$.

We can now construct iteratively the sequence $\{k_n\}_0^\infty$. Let $k_0 = 0$, and choose $k_1$ satisfying:

$$f(\|\langle k_1, \alpha \rangle\|_T)^{\frac{1}{2}} < \frac{\epsilon}{32},$$

and

$$8|\ln \|\langle k_1, \alpha \rangle\|_T|^{-\frac{1}{2}} < \tilde{\delta}.$$  

Assuming we have constructed $k_j, j \leq n$, we choose $k_{n+1} \in \mathbb{Z}^d$ satisfying the following properties:

$$f(\|\langle k_{n+1}, \alpha \rangle\|_T)^{\frac{1}{2}} < \frac{\epsilon}{4n+1} e^{-16\pi|k_n|} \|\langle k_n, \alpha \rangle\|_T^{q+1},$$

$$|k_{n+1}| > e^{|k_n|} + 10.$$  

We now call

$$\delta_n = \frac{1}{8} |\ln \|\langle k_{n+1}, \alpha \rangle\|_T|^{-\frac{1}{2}},$$

$$t_n^{(1 - 4\delta_n)}\frac{1}{2\kappa - 1} + 1 = \|\langle k_{n+1}, \alpha \rangle\|_T,$$

and, finally, let

$$\lambda_n = \sqrt{t_n^2 - \|\langle k_{n+1}, \alpha \rangle\|_T^2}.$$
We also remark that by (4.5)
\[ \| \langle k_{n+1}, \alpha \rangle \|_T^{\delta_n} < \| \langle k_n, \alpha \rangle \|_T^{q+1}, \]
which, combined with (4.7), gives
\[ t_n \delta_n (1 - 4\delta_n) \frac{1 - \kappa}{2\kappa - 1} + 1 < t_n^{-1} (q+1)(1 - 4\delta_n - 1) \frac{1 - \kappa}{2\kappa - 1} + 1. \]
Thus by (4.1) and (4.2), we have
\[ t_n < t_n^{\delta_n} < t_n^{-1} (q+1)(1 - 4\delta_n - 1) \frac{1 - \kappa}{2\kappa - 1} + 1 < t_n^{-1} < t_n^{-10} \frac{1 - \kappa}{2\kappa - 1} + 10. \]

With these parameters, we can construct \((\alpha, \tilde{A}_n e^{F_n(\cdot)})\) and then construct \((\alpha, A_n(\cdot))\) by the Anosov-Katok method of §3. First we check the following equality:
\[ 4\pi t_0 + \sum_{n=1}^{\infty} 4\pi t_n e^{4\pi h \Sigma_{j=1}^{n} |k_j|} \prod_{j=0}^{n-1} \frac{2t_j}{\| \langle k_{j+1}, \alpha \rangle \|_T} < \epsilon. \]
Note that by our selection of parameters and estimate (4.8), we have
\[
\begin{align*}
&\prod_{j=0}^{n} \frac{2t_j}{\| \langle k_{j+1}, \alpha \rangle \|_T} \\
\leq &\left( \prod_{j=0}^{n} \frac{\kappa_j - 1}{2\kappa - 1} t_j^{-1} \right) \frac{1 - \kappa}{2\kappa - 1} \leq (2 \kappa_{n+1} \frac{1 - \kappa}{2\kappa - 1} \Sigma_{j=0}^{n-1} \frac{1}{q})
\end{align*}
\]
(4.10)
\[ \leq 2t_n \frac{\delta_n}{2\kappa - 1} \leq 2t_n \frac{\delta_n}{10}, \]

On the other hand, by (4.7) and (4.1), we have
\[ t_n^{\delta_n} \frac{1 - \kappa}{2\kappa - 1} + 1 \leq \| \langle k_{n+1}, \alpha \rangle \|_T^{\delta_n} < \| \langle k_n, \alpha \rangle \|_T^{q+1}, \]
Moreover, by (4.6), we have \(\Sigma_{j=1}^{n} |k_j| < 2|k_n|\). Thus we have
\[
\begin{align*}
4\pi t_0 + &\sum_{n=1}^{\infty} 4\pi t_n e^{4\pi h \Sigma_{j=1}^{n} |k_j|} \prod_{j=0}^{n-1} \frac{2t_j}{\| \langle k_{j+1}, \alpha \rangle \|_T} \\
\leq &4\pi \| \langle k_1, \alpha \rangle \|_T^{\frac{2}{1 - \kappa}} + \sum_{n=1}^{\infty} 4\pi t_n e^{-8\pi |k_n| h} \frac{\delta_n}{4n+1} t_n \frac{\delta_n}{4} \\
\leq &4\pi \| \langle k_1, \alpha \rangle \|_T^{\frac{2}{1 - \kappa}} + \frac{4\pi t_0^4 \epsilon}{8} \leq \frac{4\pi \epsilon}{32} + \frac{\epsilon}{32} < \epsilon.
\end{align*}
\]
by (4.3), (4.11) and (4.10). Therefore by Lemma 3.1, the limit cocycle \(A(\cdot) \in C^\omega_h (\mathbb{T}^d, SU(1,1))\) exists and satisfies
\[ \| A(\cdot) - \text{Id} \|_h = \| A(\cdot) - A_0 \|_h < \epsilon. \]
Moreover,

\begin{equation}
B_n^{-1} (\cdot + \alpha) A (\cdot) B_n (\cdot) = \tilde{A}_n + \tilde{F}_n (\cdot),
\end{equation}

by (4.10) and (4.11), we have estimate

\begin{equation}
\| \tilde{F}_n \|_h \leq 4 \pi t_n e^{4 \pi |k_n|} + \sum_{j=n+1}^{\infty} 4 \pi t_j e^{4 \pi h \Sigma^j_i = |k_j|} \prod_{i=0}^{j-1} \frac{2t_i}{\| \langle k_{i+1}, \alpha \rangle \|_T} \leq \sum_{j=n}^{\infty} 4 \pi t_j \frac{\epsilon}{4^{n+1}} e^{-16 \pi h |k_j|} \| \langle k_j, \alpha \rangle \|^{2q + 2} \leq \frac{2 \epsilon}{4^{n+1}} e^{-8 \pi |k_n|} \| \langle k_n, \alpha \rangle \|^{2q + 2} t_n^1.
\end{equation}

Similarly, by (3.8) of Lemma 3.1, and the following estimate holds true

\begin{equation}
\| A (\cdot) - A_n (\cdot) \| \leq \sum_{j=n}^{\infty} 4 \pi t_j e^{4 \pi h \Sigma^j_i = |k_i|} \prod_{i=0}^{j-1} \frac{2t_i}{\| \langle k_{i+1}, \alpha \rangle \|_T} < \frac{2 \epsilon}{4^{n+1}} e^{-8 \pi |k_n|} \| \langle k_n, \alpha \rangle \|^{2q + 2} t_n^1.
\end{equation}

Turning to the size of the conjugations, by the construction and equations (3.3) and (4.10), we get the estimate

\begin{equation}
\| B_{n+1} \|^2 \leq \prod_{j=0}^{n} \| D_j \|^2 = \| D_n \|^2 \prod_{j=0}^{n-1} \| D_j \|^2 \leq \frac{2t_{n+1}}{\| \langle k_{n+1}, \alpha \rangle \|_T} \prod_{j=0}^{n-1} \frac{2t_j}{\| \langle k_{j+1}, \alpha \rangle \|_T} \leq 2 n^{-1} (1 - 4 \delta_\alpha) \frac{1 - \kappa}{1 - \kappa} 2 t_n^{-1} \delta_n \frac{1 - \kappa}{1 - \kappa} \leq 8 t_n^{-1} (1 - 2 \delta_\alpha) \frac{1 - \kappa}{1 - \kappa}.
\end{equation}

We now prove that Lyapunov exponent is \( \kappa \)-Hölder continuous at \((\alpha, A (\cdot))\).

**Lemma 4.1.** Let \( \frac{1}{2} < \kappa < 1 \) and \((\alpha, A (\cdot))\) constructed as above. Then, for any \( A' (\cdot) \in C (T^d, SU(1, 1)) \) satisfying \( \| A' - A \|_0 \leq 1 \) there exists \( C \) independent of \( A' \) such that

\[ |L(\alpha, A') - L(\alpha, A)| < C \| A' - A \|_0.\]

**Proof.** For any \( A' (\cdot) \in C (T^d, SU(1, 1)) \), let \( \epsilon := \| A' - A \|_0 \). By (4.8), we have

\[ \lim_{n \to \infty} r_{n+1}^{a_n} = 0, \]

and thus there exists \( N_2 = N_2 (\kappa) \) such that \( t_{N_2}^{a_{N_2}} < \left( \frac{1}{32} \right) \frac{2n - 1}{1 - \kappa} \). Consequently, we have

\[ \left( \frac{1}{32} \right) \frac{1 - \delta_n}{1 - \kappa} \frac{1 - 2 \delta_{n-1}}{1 - \kappa} > t_{N_2}^{1 - \delta_n}, \quad n \geq N_2.\]

This implies that if \( \epsilon \) is small enough, there exists \( n \) such that

\[ \epsilon \in \left( t_{N_2}^{1 - \delta_n}, \left( \frac{1}{32} \right) \frac{1 - \delta_n}{1 - \kappa} \frac{1 - 2 \delta_{n-1}}{1 - \kappa} \right).\]
Recall that by construction, cf. eq. (3.2), we have

\[(4.16)\quad D_n \tilde{A}_{n+1} D_n^{-1} = \exp 2\pi i \begin{pmatrix} t_n & \lambda_n \\ -\lambda_n & -t_n \end{pmatrix} := \tilde{A}_n,\]

which means \(\tilde{A}_n\) can be diagonalized by large conjugacy \(D_n\). However, there always exists \(U_n \in U(2)\) such that \(U_n\) conjugates \(\tilde{A}_n\) into Schur Form, i.e.

\[(4.17)\quad U_n^{-1} \tilde{A}_n U_n = \begin{pmatrix} e_{k_{n+1}}(\alpha) & c_n \\ 0 & e_{-k_{n+1}}(\alpha) \end{pmatrix},\]

where \(|c_n| \leq 4\pi t_n\).

Let now \(\tilde{B}_n(\cdot) := B_{n+1}(\cdot) D_n^{-1} U_n\).

Then by equations (4.12), (4.16) and (4.17), we have

\[\tilde{B}_n^{-1}(\cdot + \alpha) A(\cdot) \tilde{B}_n(\cdot) = U_n^{-1} D_n (\tilde{A}_{n+1} + \tilde{F}_{n+1}(\cdot)) D_n^{-1} U_n \]

\[= \begin{pmatrix} e_{k_{n+1}}(\alpha) & c_n \\ 0 & e_{-k_{n+1}}(\alpha) \end{pmatrix} + U_n^{-1} D_n \tilde{F}_{n+1}(\cdot) D_n^{-1} U_n.\]

Let now

\[B_n'(\cdot) := \tilde{B}_n(\cdot) \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix},\]

where \(s^{-1} = \|\tilde{B}_n\|_0 \varepsilon^{\frac{1-\kappa}{2}}\). By construction, \(\tilde{B}_n\) can be written as

\[\tilde{B}_n(\cdot) = B_n(\cdot) H_n(\cdot) U_n.\]

By (4.15) and the parameter choice \(\varepsilon < \left(\frac{1}{32}\right)^{1-\kappa} \frac{1}{\frac{1}{2} t_n^{1-\kappa}}\), we have

\[\|\tilde{B}_n\|_0 = \|B_n\|_0 \leq 2\sqrt{2} t_n^{-(1-2\delta_n) - \frac{1-\kappa}{2(2\kappa-1)}} \varepsilon^{\frac{1-\kappa}{2}} < \varepsilon^{\frac{1-\kappa}{2}}.\]

which implies that

\[(4.18)\quad \|B_n'(\cdot)\|_0 \leq 2\varepsilon^{\frac{1-\kappa}{2}}.\]

By (4.18), one has

\[(4.19)\quad B_n^{-1}(\cdot + \alpha) A(\cdot) B_n'(\cdot) \]

\[= \begin{pmatrix} e_{k_{n+1}}(\alpha) & c_n s^{-2} \\ 0 & e_{-k_{n+1}}(\alpha) \end{pmatrix} + \begin{pmatrix} s^{-1} & 0 \\ 0 & s \end{pmatrix} U_n^{-1} D_n \tilde{F}_{n+1}(\cdot) D_n^{-1} U_n \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}.\]

Since \(\varepsilon > \frac{t_n^{1-\delta_n}}{t_n^{1-\kappa}}\), then by (4.8), we have

\[s^{-2} c_n \leq \|\tilde{B}_n\|_0^2 \varepsilon^{1-\kappa} 4\pi t_n \]

\[\leq 32\pi \varepsilon^{1-\kappa} t_n^{1-\kappa} \frac{1}{t_n^{1-\kappa}} \]

\[\leq \varepsilon^{\kappa} 32\pi t_n^{1-\kappa} t_n^{\frac{q}{2n-1}} < \varepsilon^{\kappa}\]
Consequently, by (4.13), (4.18), (4.19), we have
\[
\|B_n^{-1}(\cdot + \alpha)A'(\cdot)B_n'(-) - \tilde{A}_{n+1}\|_0 \\
\leq \|B_n^{-1}(\cdot + \alpha)(A'(-) - A(\cdot))B_n'(-)\|_0 + \|B_n^{-1}(\cdot + \alpha)A(\cdot)B_n'(-) - \tilde{A}_{n+1}\|_0 \\
\leq \|B_n'\|_0^2 \varepsilon + s^{-2}(c_n) + 2s^2 \|D_n\|^2 \|\tilde{F}_{n+1}\|_0 \\
\leq 4\varepsilon^\kappa + \varepsilon^\kappa + 2s^2(1-\kappa) \frac{2t_n}{\|\langle k_{n+1},\alpha \rangle \|_T} \|\langle k_{n+1},\alpha \rangle \|_T^{-2} + 2^2 \frac{1}{t_n+1} \\
\leq 5\varepsilon^\kappa + \varepsilon^\kappa \|A_n\| \|A_n\|^{-1} \|A_n\|_0^{-\kappa} \\
\leq 8\varepsilon^\kappa.
\]

Since Lyapunov exponent is invariant under conjugacy, we immediately have
\[
L(\alpha, A') = L(\alpha, B_n^{-1}(\cdot + \alpha)A'(\cdot)B_n'(\cdot)) \leq 16\varepsilon^\kappa.
\]

On the other hand, by (4.12), we have
\[
L(\alpha, A) = L(\alpha, \tilde{A}_n + \tilde{F}_n) \leq 2\|\tilde{F}_n\|
\]
by continuity of Lyapunov exponent [21], we then have \(L(\alpha, A) = 0\), consequently,
\[
|L(\alpha, A') - L(\alpha, A)| = L(\alpha, A') < C\|A - A'\|_0^\kappa.
\]

We now prove that Lyapunov exponent is exactly \(\kappa\)-Hölder continuous at \((\alpha, A(\cdot))\), i.e. that the exponent is not higher than \(\kappa\).

**Lemma 4.2.** There exists a sequence \(\{A_n'\}_{n=1}^\infty\) where \(A_n' \in C^\kappa_H(T^d, SU(1, 1))\) such that
\[
|L(\alpha, A_n') - L(\alpha, A)| > c\|A_n' - A\|_h^\kappa,
\]
where \(c\) independent of \(n\) and \(\lim_{n \to \infty} \kappa_n = \kappa\).

**Proof.** Define for every \(n \in \mathbb{N}\)
\[
\tilde{A}_n' := \exp 2\pi i \left( \begin{array}{cc} \lambda_n & 2t_n - \lambda_n \\ \lambda_n - 2t_n & -\lambda_n \end{array} \right),
\]
and
\[
A_n' + 1(\cdot) := (B_nH_n)(\cdot + \alpha)\tilde{A}_n'(B_nH_n)^{-1}(\cdot).
\]
and notice that
\[
\|\tilde{A}_n' - \tilde{A}_n\| \leq 4\pi(t_n - \lambda_n) \\
= 4\pi t_n^2 - \lambda_n^2 \leq 4\pi \|\langle k_{n+1},\alpha \rangle \|_T^2 \|k_{n+1},\alpha\|_T \leq 4\pi t_n^{2(1-4\delta_n)} \frac{1}{t_n^{\kappa - \kappa}} + 1
\]
\[(4.20)\]
Therefore, we have
\[ \|A'_{n+1} - A_{n+1}\|_h \leq \|B_n\|_0^2 \|\hat{H}_n\|_0^2 \|\hat{A}'_n - \hat{A}_n\|_0 \leq 4\pi t_{n-1}^{-(1-2\delta_n-1)\frac{1-\kappa}{2\kappa-1}} 2(1-4\delta_n)\frac{1-\kappa}{2\kappa-1} + 1 \leq t_n^{(2-(8+\frac{\kappa}{1-\kappa})\delta_n)\frac{1-\kappa}{2\kappa-1} + 1} . \]

By (4.8) and (4.14), we have
\[ \|A'_{n+1} - A\|_h \leq \|A'_{n+1} - A_{n+1}\|_h + \|A_{n+1} - A\|_h \leq t_n^{(2-(8+\frac{\kappa}{1-\kappa})\delta_n)\frac{1-\kappa}{2\kappa-1} + 1} + \|\langle k_{n+1}, \alpha \rangle\|_{T}^{2}\pi^{2\frac{1}{2}} t_{n+1} \leq 2t_n^{(2-(8+\frac{\kappa}{1-\kappa})\delta_n)\frac{1-\kappa}{2\kappa-1} + 1} . \]

On the other hand, by (4.20), we have
\[ L(\alpha, A_n') = \frac{t_n(t_n^2 - \lambda_n^2)}{t_n + \lambda_n} \geq 4\pi \sqrt{t_n^2 - \lambda_n^2} = 2\sqrt{2}\pi t_n^{(1-4\delta_n)\frac{1-\kappa}{2\kappa-1} + 1} \geq \|A'_{n+1} - A\|_h^\kappa_n . \]

where
\[ (4.21) \quad \kappa_n := \frac{(1-4\delta_n)\frac{1-\kappa}{2\kappa-1} + 1}{(2-(8+\frac{\kappa}{1-\kappa})\delta_n)\frac{1-\kappa}{2\kappa-1} + 1} = \frac{\kappa - 4\delta_n(1-\kappa)}{1-(8+\frac{\kappa}{1-\kappa})\delta_n(1-\kappa)} . \]

Direct calculation shows that
\[ \kappa < \kappa_n < \kappa + c\delta_n , \]
where \( c(\kappa) = 8((2\kappa - 1)(1-\kappa) + \kappa^2) \), which ends the proof. \( \square \)

The two lemmas imply the theorem. \( \square \)

4.2. Proof of Theorem 1.1: For any \((\alpha, A) \in \mathcal{AR}_\alpha \setminus \mathcal{U}_\alpha \), we only need to conjugate the cocycle \((\alpha, A)\), such that the conjugated cocycle is close to the identity, then we can apply Proposition 4.1 to finish the proof. However, as we will see, some quantitative estimates are still needed. Therefore, we will first conjugate the global almost reducible cocycle to the local regime (for example, as defined in Proposition 7.1), then apply local KAM to get the desired results.

The following result is standard, but the basis of the proof. We will sketch the proof in the appendix, for the sake of completeness.

**Proposition 4.2.** Let \( \alpha \in DC_d, A \in \text{SL}(2, \mathbb{R}), h > h' > 0 \). There exist \( \epsilon = \epsilon(\|A\|, \alpha, h, h') \) such if \( F \in C^\omega_h(T^d, \text{sl}(2, \mathbb{R})) \) and \( \|F\|_h < \epsilon \) then, for all \( n \geq 1 \), there exist \( F_n \in C^\omega_h(T^d, \text{sl}(2, \mathbb{R})) \), \( B_n \in C^\omega_h(T^d, \text{PSL}(2, \mathbb{R})) \) and \( A_n \in \text{SL}(2, \mathbb{R}) \), satisfying
\[ B_n^{-1}(\cdot + \alpha) Ae^{F_n(\cdot)} B_n(\cdot) = A_n e^{F_n(\cdot)} , \]
for any $\epsilon > 0$,

Let $B \in \mathcal{A}$ with

By Proposition 4.1, there exists $\bar{B}_0 \in C^{\omega}(\mathbb{T}^d, PSL(2, \mathbb{R}))$ such that

$$\bar{B}_0^{-1}(\cdot + \alpha)A(\cdot)\bar{B}_0(\cdot) = A_0e^{F_0(\cdot)}.$$ 

with

$$\|F_0\|_{h_1} < \frac{\epsilon}{4\|A_0\|},$$

where $\epsilon = \epsilon(\|\bar{A}_1\|, \alpha, h, h')$ as in Proposition 4.2. Now by Proposition 4.2, for any $\epsilon > 0$, there exist $B'_N \in C^{\omega}(\mathbb{T}^d, PSL(2, \mathbb{R}))$ such that

$$B'_N^{-1}(\cdot + \alpha)A_0e^{F(\cdot)}B'_N(\cdot) = A_Ne^{F_N(\cdot)},$$

satisfying

$$\|B_N\|_{h'_1} \leq \frac{\epsilon^3}{2\|B_0\|^2_{h}}.$$

Let $B_N := \bar{B}_0B'_N$, then we have

$$\|B_N(\cdot + \alpha)A_NB_N^{-1}(\cdot) - A(\cdot)\|_{h'}$$

$$\leq \|B_N\|_{h'_1}^2\|A_N - B_N^{-1}(\cdot + \alpha)A(\cdot)B_N(\cdot)\|_{h'}$$

$$\leq \|\bar{B}_0\|_{h'}^2\|B'_N\|_{h'_1}^2\|A_N - A_Ne^{F_N}\|_{h'}$$

$$\leq 2(\|A_0\| + 2\epsilon)\|\bar{B}_0\|_{h'}^2\|B'_N\|_{h'_1}^2\|F_N\|_{h'} \leq \frac{\epsilon}{4}.$$ 

We now separate three cases, following the regime to which $A_N$ belongs. 

**Case I**: $A_N$ is elliptic. Then there exists $P \in SL(2, \mathbb{R})$ such that

$$P^{-1}A_NP = R_\theta,$$

where $R_\theta \in SO(2, \mathbb{R})$. Since $\alpha \in \mathbb{R}^d$ is rationally independent, there exists $k \in \mathbb{Z}^d$ such that

$$\|\theta - \langle k, \alpha \rangle\|_\mathbb{T} < \frac{\epsilon}{16\|B_N\|_{h'}^2\|P\|^2}.$$ 

By Proposition 4.1, there exists $\bar{A}(\cdot) \in C^{\omega}(\mathbb{T}^d, SL(2, \mathbb{R}))$ such that Lyapunov exponent is exactly $\kappa$-Hölder continuous at $(\alpha, A(\cdot))$ and

$$\|\bar{A} - \text{Id}\|_{h'} < \frac{\epsilon e^{-16\|k\|_{h}}}{{16\|B_N\|_{h'}^2\|P\|^2}}.$$ 

Let $H := M^{-1} \begin{pmatrix} e_k(\cdot) & 0 \\ 0 & e_{-k}(\cdot) \end{pmatrix} M$. Set

$$A'(\cdot) = B_N(\cdot + \alpha)PH(\cdot + \alpha)\bar{A}(\cdot)H^{-1}(\cdot)P^{-1}B_N^{-1}(\cdot).$$
Then, we have
\[ \| A(\cdot) - A'(\cdot) \|_{h'} \leq \| B_N \|_{h'}^2 \| P \|_{h'}^2 \| H \|_{h'}^2 \| \bar{A} - \text{Id} \|_{h'} \]
\[ + \| B_N \|_{h'}^2 \| P \|_{h'}^2 \| R_{\theta} - R_{k,\alpha} \|_{h'} + \| B_N(\cdot + \alpha) A_N B_N^{-1}(\cdot) - A(\cdot) \|_{h'} \]
\[ \leq \varepsilon. \]

Moreover, since the Lyapunov exponent is invariant under conjugation, one can easily check that
\[ \liminf_{\| B - A' \|_0 \to 0} \frac{\log | L(\alpha, A') - L(\alpha, B) |}{\log \| B - A' \|_0} = \kappa, \]
which means that the Lyapunov exponent is exactly \( \kappa \)-Hölder continuous at \((\alpha, A'(\cdot))\).

**Case II**: \( A_N \) is parabolic. In this case, without loss of generality, we assume the eigenvalues of \( A_N \) are \( \{1, 1\} \). Then there exists \( P \in \text{SL}(2, \mathbb{R}) \) such that
\[ P^{-1} A_N P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \]

Let
\[ A'_N := P \begin{pmatrix} 1 - \delta & 1 \\ -\delta & 1 \end{pmatrix} P^{-1}, \]
where \( \delta := \frac{\varepsilon}{4 \| B_N \|_{h'}^2 \| P \|_{h'}^2} \), then \( A'_N \) is elliptic, and, moreover,
\[ \| A(\cdot) - B_N(\cdot + \alpha) A'_N B_N^{-1}(\cdot) \|_{h'} \leq \| B_N(\cdot + \alpha) A_N B_N^{-1}(\cdot) - A(\cdot) \|_{h'} + \| B_N \|_{h'}^2 \| P \|_{h'}^2 \delta \]
\[ \leq \varepsilon. \]

This situation has been transformed into **Case I**, which ends the proof for this case.

**Case III**: \( A_N \) is hyperbolic. Let the eigenvalues of \( A_N \) be \( \{\lambda, \lambda^{-1}\} \) with \( \lambda > 1 \).

We first consider the case
\[ (4.25) \quad |\lambda - 1| > 2 \| F_N \|_{h'}^\frac{1}{2}. \]

In view of Proposition 18 of [48], there exists \( P \in \text{SL}(2, \mathbb{R}) \), with
\[ \| P \| \leq 2 \left( \| A_N \| \right)^\frac{1}{2 |\lambda - 1|} \]
such that
\[ P^{-1} A_N P = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}. \]

Then
\[ P^{-1} B_N^{-1}(\cdot + \alpha) A(\cdot) B_N(\cdot) P = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} e^{F_N(\cdot)}, \]
with $\| \tilde{F}_N \|_{\mathcal{H}'} \leq \frac{8}{|X-T|} \| F_N \|_{\mathcal{H}'}$, and (4.25) implies that

$$| \lambda - 1 | > \| \tilde{F}_N \|_{\mathcal{H}'}^{1/2}.$$ 

Consequently, $(\alpha, A(\cdot))$ is uniformly hyperbolic by the usual cone criterion [55], which contradicts our assumptions.

Therefore, $| \lambda - 1 | < 2 \| F_N \|_{\mathcal{H}'}^{1/2}$. Consequently, there exists an elliptic matrix $A'_N$, such that

$$\| A_N - A'_N \| \leq 2 | \lambda - 1 | < 4 \| F_N \|_{\mathcal{H}'}^{1/2}.$$ 

Then by (4.24), we have

$$\| A(\cdot) - B_N(\cdot + \alpha)A'_N B^{-1}_N(\cdot) \|_{\mathcal{H}'} \leq \| B_N(\cdot + \alpha)A'_N B^{-1}_N(\cdot) - A(\cdot) \|_{\mathcal{H}'} + 4 \| B_N \|_{\mathcal{H}'} \| F_N \|_{\mathcal{H}'}^{1/2} \leq \varepsilon,$$

which again transforms this case into Case I, which concludes the proof.

5. Sub-linear growth of extended eigenfunction

Proposition 5.1. Let $\alpha \in \mathbb{T}^d$ be rationally independent and fix $h > 0$, $\varepsilon > 0$, and a non-increasing sequence $\{g(n)\}_{n=1}^{\infty}$ satisfying $0 < g(n) < 1$ and $\lim_{n \to \infty} n^{g(n)} = \infty$. Then there exists $(\alpha, A(\cdot)) \in \mathbb{T}^d \times C^\omega_\omega(\mathbb{T}^d, \text{SL}(2, \mathbb{R}))$ which has sub-linear growth with rate $\{g(n)\}_{n=1}^{\infty}$. Moreover, it satisfies

$$\| A(\cdot) - \text{Id} \|_h < \varepsilon.$$

Proof. We construct $A(\cdot)$ iteratively. Firstly we construct the sequence $\{k_n\}_{n=0}^{\infty}$. Let $k_0 = 0$. Assuming we have constructed $k_j, j \leq n$. We choose $k_{n+1} \in \mathbb{Z}^d$ satisfying the following:

$$(5.1) \quad \begin{cases} \langle \| (k_{n+1}, \alpha) \| \rangle_T \frac{2}{3} g(\frac{1}{\| (k_{n+1}, \alpha) \|_T}) < \varepsilon^2 \frac{1}{4^n} \quad n = 0, \\ \langle \| (k_{n+1}, \alpha) \| \rangle_T \frac{2}{3} g(\frac{1}{\| (k_{n+1}, \alpha) \|_T}) < \frac{\varepsilon^2}{4^n} e^{-16\varepsilon |k_n|_h^{1/\| (k_n, \alpha) \|_T^{1/2}}} \quad n \geq 1. 
\end{cases}$$

$$(5.2) \quad |k_{n+1}| > e^{|k_n|} + 10.$$ 

The sequence $k_{n+1}$ always exists since $\alpha \in \mathbb{T}^d$ is rationally independent and $\lim_{n \to \infty} n^{g(n)} = \infty$. We now call

$$(5.3) \quad t_n = \| (k_{n+1}, \alpha) \|_T \frac{2}{3} g(N_n),$$

$$\lambda_n = \sqrt{t_n^2 - \| (k_{n+1}, \alpha) \|_T^2},$$

where

$$(5.4) \quad N_n = \frac{1}{4\| (k_{n+1}, \alpha) \|_T}.$$
With these parameters, one can construct \((\alpha, \tilde{A}_n e^{F_n(\cdot)})\) and then \((\alpha, A_n(\cdot))\) by the Anosov-Katok method as in \(\S 3\). Convergence of \(A_n(\cdot)\) follows from the following inequality

\[
4\pi t_0 + \sum_{n=1}^{\infty} 4\pi t_n e^{4\pi h\sum_{j=1}^{n-1}|k_j|} \prod_{j=0}^{n-1} \frac{2t_j}{\|\langle k_{j+1}, \alpha \rangle \|_T} < \epsilon. \tag{5.5}
\]

To show this, notice that (5.1) directly implies that

\[
\|\langle k_{n+1}, \alpha \rangle \|_T < \|\langle k_{n+1}, \alpha \rangle \|_T \leq \|\langle k_n, \alpha \rangle \|_T^{\frac{1}{2}g(N_n)} < \|\langle k_n, \alpha \rangle \|_T^{\frac{1}{4}},
\]

then by (5.2) and the choice of \(t_n\), we have

\[
\prod_{j=0}^{n-1} \frac{2t_j}{\|\langle k_{j+1}, \alpha \rangle \|_T} \leq \left( \prod_{j=0}^{n-1} \|\langle k_{j+1}, \alpha \rangle \|_T \right)^{-1} \leq \|\langle k_n, \alpha \rangle \|_T^{-(\sum_{j=0}^{n-1} \frac{1}{4})}
\]

\[
\leq \|\langle k_n, \alpha \rangle \|_T^{-2} \leq \|\langle k_{n+1}, \alpha \rangle \|_T^{-\frac{1}{4}g(N_n)}.
\]

Therefore by (5.1), (5.2), (5.3), we have

\[
4\pi t_0 + \sum_{n=1}^{\infty} 4\pi t_n e^{4\pi h\sum_{j=1}^{n-1}|k_j|} \prod_{j=0}^{n-1} \frac{2t_j}{\|\langle k_{j+1}, \alpha \rangle \|_T} \leq 4\pi t_0 + \sum_{n=1}^{\infty} 4\pi t_n e^{8\pi h|k_n|} \|\langle k_n, \alpha \rangle \|^{-2}_T
\]

\[
\leq \frac{\pi \epsilon}{8} + \sum_{n=1}^{\infty} \frac{4\pi \epsilon}{2n+2} \|\langle k_{n+1}, \alpha \rangle \|^{\frac{11}{16}g(N_n)} \leq \epsilon.
\]

which establishes eq. (5.5). We can now apply Lemma 3.1, and thus obtain \(A(\cdot) \in C^\omega_h(T^d, SL(2, \mathbb{R}))\) such that \(M^{-1}A_\infty(\cdot)M = A(\cdot)\) and

\[
\|A(\cdot) - \text{Id}\|_h = \|A(\cdot) - A_0\|_h < \epsilon.
\]

Moreover,

\[
B_n^{-1}(\cdot + \alpha)A_\infty(\cdot)B_n(\cdot) = \tilde{A}_n + \tilde{F}_n(\cdot), \tag{5.7}
\]

satisfying the estimate

\[
\|\tilde{F}_n\|_h \leq 4\pi t_n e^{4\pi|k_n|_h} + \sum_{j=n+1}^{\infty} 4\pi t_j e^{4\pi h\sum_{i=n}^{j-1}|k_i|} \prod_{i=n}^{j-1} \frac{2t_i}{\|\langle k_{i+1}, \alpha \rangle \|_T} < \sum_{j=n}^{\infty} 4\pi t_j e^{8\pi |k_j|_h} \|\langle k_j, \alpha \rangle \|^{-2}_T \leq \sum_{j=n}^{\infty} \frac{4\pi \epsilon}{2n+2} \|\langle k_{j+1}, \alpha \rangle \|^{\frac{11}{16}g(N_j)}
\]

\[
\leq \|\langle k_{n+1}, \alpha \rangle \|^{\frac{11}{16}g(N_n)}_T, \tag{5.8}
\]

The estimates hold because of equations (5.1), (5.2), (5.3) and the estimate (5.6). To estimate the growth of \((\alpha, A(\cdot))\), we first estimate the growth of the approximating cocycle \((\alpha, A_{n+1}(\cdot))\).
Lemma 5.1. Letting

\[ N_n = \left[ \frac{1}{4\|\langle k_{n+1}, \alpha \rangle \|_T} \right], \]

we have

\[ \|M^{-1}A_{n+1}(\cdot; N_n)M\|_0 \in \left[ \frac{\|D_n\|^2}{2\prod_{j=0}^{n-1}\|D_j\|^2}, \prod_{j=0}^n \|D_j\|^2 \right]. \]

Proof. First by our construction (3.5), we have

\[ A_{n+1}(\cdot) = B_{n+1}(\cdot + \alpha)\tilde{A}_{n+1}B_{n+1}^{-1}(\cdot) \]
\[ = B_{n+1}(\cdot + \alpha)D_{n+1}^{-1}(D_n\tilde{A}_{n+1}D_n^{-1})D_nB_{n+1}^{-1}(\cdot) \]
\[ = B_{n+1}(\cdot + \alpha)D_{n+1}^{-1}(D_n\begin{pmatrix} e_{k_{n+1}}(\alpha) & 0 \\ 0 & e_{-k_{n+1}}(\alpha) \end{pmatrix} D_n^{-1})D_nB_{n+1}^{-1}(\cdot). \]

Then for any \( l \in \mathbb{N} \) we have

\[ \|M^{-1}A_{n+1}(\cdot; l)M\|_0 \]
\[ = \|M^{-1}B_{n+1}(\cdot + l\alpha)D_{n+1}^{-1}(D_n\tilde{A}_{n+1}D_n^{-1})D_nB_{n+1}^{-1}(\cdot)M\|_0 \]
\[ \in \left[ \|D_{n+1}^{-1}\|_0^2, \|D_{n+1}^{-1}\|_0^2 \right]. \]

(5.9)

Since \( M^{-1}D_nM \in \text{SL}(2, \mathbb{R}) \), we have the Singular value decomposition of \( M^{-1}D_nM \):

\[ M^{-1}D_nM = R_u \begin{pmatrix} \|D_n\| & 0 \\ 0 & \|D_n\|^{-1} \end{pmatrix} R_s, \]

for some \( u_n, s_n \in [0, 2\pi) \). Thus

\[ M^{-1}(D_n\tilde{A}_{n+1}D_n^{-1})D_n^{(l)} \]
\[ = R_u \begin{pmatrix} \|D_n\| & 0 \\ 0 & \|D_n\|^{-1} \end{pmatrix} R_s \begin{pmatrix} 0 & 0 \\ \|D_n\| & \|D_n\|^{-1} \end{pmatrix} R_{-u} \]
\[ = R_u \begin{pmatrix} \cos \phi & -\sin \phi \|D_n\|^{-2} \\ \sin \phi \|D_n\|^{-2} & \cos \phi \end{pmatrix} R_{-u}. \]

(5.10)

where \( \phi = 2\pi l\|\langle k_{n+1}, \alpha \rangle\|_T \).

The key observation is that if \( l = N_n \), we have

\[ \phi \in \left( \frac{\pi}{4}, \frac{3\pi}{4} \right) \]

and

\[ \|D_{n+1}^{(N_n)}\|_0 \in \left[ \frac{\|D_n\|^2}{2}, \|D_n\|^2 \right]. \]

By construction,

\[ B_{n+1}(\cdot) = B_n(\cdot)G_n(\cdot) = B_n(\cdot)H_n(\cdot)D_n, \]
so that
\[ \|B_{n+1}(\cdot)D_n^{-1}\|_0^2 = \|B_n(\cdot)\|_0^2 \leq \prod_{j=0}^{n-1} \|D_j\|^2. \]

Thus we have
\[ \|M^{-1}A_{n+1}(\cdot;N_n)M\|_0 \in \left[ \frac{1}{2\|\langle k_{n+1}, \alpha \rangle\|^{-1-\frac{4}{7}g(N_n)}} \cdot \frac{4}{\|\langle k_{n+1}, \alpha \rangle\|^{-1-\frac{3}{7}g(N_n)}} \right], \]

which proves the lemma.

Now, by eq. (3.3), we have
\[ \|D_n\|_0^2 \in \left[ \frac{1}{2\|\langle k_{n+1}, \alpha \rangle\|^{-1-\frac{4}{7}g(N_n)}} \cdot \frac{4}{\|\langle k_{n+1}, \alpha \rangle\|^{-1-\frac{3}{7}g(N_n)}} \right], \]

and, by eq. (5.6),
\[ (5.11) \quad \|B_n(\cdot)\|_0^2 \leq \prod_{j=0}^{n-1} \|D_j\|^2 < \prod_{j=0}^{n-1} \|\langle k_{j+1}, \alpha \rangle\|^{-1-\frac{4}{7}g(N_n)} \leq \|\langle k_{n+1}, \alpha \rangle\|^{-1-\frac{3}{7}g(N_n)} \]

Consequently, by Lemma 5.1, we have
\[ (5.12) \quad \|M^{-1}A_{n+1}(\cdot;N_n)M\|_0 \in \left[ \frac{1}{4\|\langle k_{n+1}, \alpha \rangle\|^{-1-\frac{4}{7}g(N_n)}} \cdot \frac{4}{\|\langle k_{n+1}, \alpha \rangle\|^{-1-\frac{3}{7}g(N_n)}} \right]. \]

On the other hand, by (5.7) and (5.8), we have
\[ \|A(\cdot;N_n) - M^{-1}A_{n+1}(\cdot;N_n)M\|_0 \]
\[ = \|B_{n+1}(\cdot + N_n\alpha)((\tilde{A}_{n+1} + \tilde{F}_n)(\cdot;N_n) - \tilde{A}_{n+1}^N)B_{n+1}(\cdot)\|_0 \]
\[ \leq 2N_n\|B_{n+1}\|_0^2 \|\tilde{F}_n\|_h \]
\[ \leq 2N_n\|\langle k_{n+2}, \alpha \rangle\|^{-1-\frac{4}{7}g(N_n+1)} \|\langle k_{n+2}, \alpha \rangle\|^{-1+\frac{3}{7}g(N_n+1)} \]
\[ \leq \|\langle k_{n+1}, \alpha \rangle\|^{4} \|\langle k_{n+1}, \alpha \rangle\|^{-1} \]

Combining (5.12) and (5.13) can finish the proof of the proposition.

**Corollary 5.1.** Given \( \alpha \in DC_d \). For any non-increasing sequence \( \{g(n)\}_{n=1}^\infty \) satisfying \( 0 < g(n) < 1 \) and \( \lim_{n \to \infty} n^{g(n)} = \infty \). There exists a set \( \mathcal{D} \) which is dense in \( \mathcal{AR}_\alpha \setminus \mathcal{UH}_\alpha \) in the \( C^\infty(T^d; \text{SL}(2, \mathbb{R})) \) topology such that \( \|A(\cdot;N)\|_0 \) has sub-linear growth with rate \( \{g(n)\}_{n=1}^\infty \) at each point of \( (\alpha, A) \in \mathcal{D} \).

**Proof.** The proof is same as Theorem 1.1, one only need to replace Proposition 4.1 by Proposition 4.1.

**Proof of Theorem 1.2**

If \( d = 1 \), just note \( (\alpha, A) \in \mathcal{AR}_\alpha \setminus \mathcal{UH}_\alpha \), if and only if \( (\alpha, A) \) is subcritical. In fact, by Avila’s almost reducibility theorem (ART) \([2, 4, 5]\), \( (\alpha, A) \) is subcritical, then it is almost reducible and not uniformly hyperbolic. Conversely, if \( (\alpha, A) \) is almost reducible but not uniformly hyperbolic, then the
Lyapunov exponent vanishes in a band [7], which ensures the cocycle is subcritical.

Therefore, if $E \in \Sigma^{sub}_{V,\alpha}$, then $(\alpha, S^V_E)$ is subcritical, then by Corollary 5.1, one can perturb $(\alpha, S^V_E)$ to $(\alpha, A')$, so that it has sub-linear growth. Then the result follows immediately from the following lemma.

**Lemma 5.2** (Avila-Jitomirskaya [8]). Let $\alpha \in \mathbb{T}^d$ be rationally independent, $A \in C^0_h(\mathbb{T}^d, \text{SL}(2, \mathbb{R}))$ for some $h_* > 0$, such that $(\alpha, A)$ is almost reducible. There exists $h_0 \in (0, h_*)$ such that for any $\eta > 0$, one can find $V \in C^0_h(\mathbb{T}^d, \mathbb{R})$ with $|V|_{h_0} < \eta$, $E \in \mathbb{R}$, and $Z \in C^0_h(\mathbb{T}^d, \text{PSL}(2, \mathbb{R}))$ such that

$$Z(\cdot + \alpha)^{-1} A(\cdot) Z(\cdot) = S^V_E(\cdot).$$

Moreover, for every $0 < h \leq h_0$, there is $\delta > 0$ such that if $A' \in C^0_h(\mathbb{T}^d, \text{SL}(2, \mathbb{R}))$ satisfies $|A - A'|_h < \delta$, then there exist $V' \in C^0_h(\mathbb{T}^d, \mathbb{R})$ with $|V'|_h < \eta$ and $Z' \in C^0_h(\mathbb{T}^d, \text{PSL}(2, \mathbb{R}))$ such that $|Z - Z'|_h < \eta$

$$Z'(\cdot + \alpha)^{-1} A'(\cdot) Z'(\cdot) = S^{V'}_E(\cdot).$$

**Remark 5.1.** Avila-Jitomirskaya [8] state the result for $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. The proof, however, applies equally well to the multifrequency case.

\[\square\]

### 6. Power-law Localized Eigenfunction

**Proposition 6.1.** Given $s \in \mathbb{Z}^+$, $h > 0$ and $\alpha \in \mathbb{R}^d/\mathbb{Z}^d$ which is rationally independent. Then for any $\epsilon > 0$, there exists $(\alpha, A(\cdot)) \in \mathbb{T}^d \times C^0_h(\mathbb{T}^d, \text{SL}(2, \mathbb{R}))$ which is $C^s$-reducible but not $C^{s+1}$-reducible. Moreover, it satisfies

$$\|A(\cdot) - \text{Id}\|_h < \epsilon.$$

We again construct $A(\cdot)$ by the Anosov-Katok method. The construction, however, will be a bit different from Section 3. Since the goal is to construct $(\alpha, A(\cdot))$ which is reducible and not merely almost reducible.

**Proof.** First, we construct a sequence $\{k_n\}_{n=0}^{\infty}$ where $k_n \in \mathbb{Z}^d$. Let $k_0 = 0$, $k'_n = \sum_{j=0}^{n} k_n$. Suppose we have constructed $k_i$, $i \leq n$, then we choose $k_{n+1} \in \mathbb{Z}^d$ satisfying

\begin{align}
(6.1) \quad \|\langle k_{n+1}, \alpha \rangle\|_\mathbb{T} &< \frac{\epsilon}{4^n} e^{-16\pi |k'_n|_h}, \\
(6.2) \quad |k_{n+1}| &> e^{|k'_n|} + 10.
\end{align}

Then we construct

$$t_n = \frac{\|\langle k_{n+1}, \alpha \rangle\|_\mathbb{T} |k'_n|^s (n + 10)^2}{\sqrt{|k'_n|^{2s}(n + 10)^4 - 1}},$$

$$\lambda_n = \frac{\|\langle k_{n+1}, \alpha \rangle\|_\mathbb{T}}{\sqrt{|k'_n|^{2s}(n + 10)^4 - 1}}.$$
By direct calculation we have
\begin{equation}
\sqrt{t_n^2 - \lambda_n^2} = \|\langle k_{n+1}, \alpha \rangle\|_2, \quad \frac{\lambda_n}{t_n} = \frac{1}{|k_n|^s(n+10)^2}.
\end{equation}

Once we have these parameters, we perturb the cocycle \((\alpha, \tilde{A}_n')\), to \((\alpha, \tilde{A}_n'eF_n(\cdot)')\), where
\[
\tilde{A}_n' = \begin{pmatrix} e_{k_n'}(\alpha) & 0 \\ 0 & e_{-k_n'}(\alpha) \end{pmatrix}, \quad F_n'(\cdot) = 2\pi i \begin{pmatrix} t_n & \lambda_n e_{2k_n'}(\cdot) \\ -\lambda_n e_{-2k_n'}(\cdot) & -t_n \end{pmatrix}.
\]

Let
\[
H_n'(\cdot) := \begin{pmatrix} e_{k_n'}(\cdot) & 0 \\ 0 & e_{-k_n'}(\cdot) \end{pmatrix},
\]
then we have
\[
H_n'^{-1}(\cdot + \alpha)\tilde{A}_n'eF_n(\cdot)H_n'(\cdot) = \exp 2\pi i \begin{pmatrix} t_n & \lambda_n \\ -\lambda_n & -t_n \end{pmatrix}.
\]

By (6.3) and Lemma 2.1, there exists \(D_n \in SU(1,1)\) such that
\[
D_n^{-1} \exp 2\pi i \begin{pmatrix} t_n & \lambda_n \\ -\lambda_n & -t_n \end{pmatrix} D_n = \begin{pmatrix} e_{k_{n+1}}(\alpha) & 0 \\ 0 & e_{-k_{n+1}}(\alpha) \end{pmatrix}.
\]

Let \(G_n'(\cdot) := H_n'(\cdot)D_nH_n'^{-1}(\cdot)\), then we have
\[
G_n'^{-1}(\cdot + \alpha)\tilde{A}_n'eF_n(\cdot)G_n'(\cdot) = \tilde{A}_{n+1}',
\]
which means the cocycle \((\alpha, \tilde{A}_n'e^{F_n(\cdot)'}\) is conjugated to \((\alpha, \tilde{A}_{n+1}'\)\), which concludes one step of the iteration.

Finally, let \(B_n' := G_0' \cdots G_{n-2}'G_{n-1}'\) and
\[
A_n'(\cdot) := B_n'(\cdot + \alpha)\tilde{A}_n'B_n'^{-1}(\cdot),
\]
similarly as in Lemma 3.1, one can easily show that there exists \(A(\cdot) \in C^\omega_h(\mathbb{T}^d, SL(2,\mathbb{R}))\) such that \(|A_n' - A|_h \to 0\) and
\[
\|A(\cdot) - \text{Id}\|_h = \|A(\cdot) - A_0\| < \epsilon.
\]

We point out the conjugacy \(G_n'(\cdot)\) used here is the main difference with respect to the construction in Section 3. Indeed, by Lemma 2.1, \(D_n\) can be chosen in the form
\[
D_n = (\cos 2\theta_n)^{-\frac{1}{2}} \begin{pmatrix} \cos \theta_n & \sin \theta_n \\ -\sin \theta_n & \cos \theta_n \end{pmatrix},
\]
therefore \(G_n'\) is the form as
\[
G_n' = (\cos 2\theta_n)^{-\frac{1}{2}} \begin{pmatrix} \cos \theta_n & \sin \theta_n e_{2k_n'} \\ \sin \theta_n e_{-2k_n'} & \cos \theta_n \end{pmatrix}.
\]

By Corollary 2.1 and (6.3) we have
\begin{equation}
|\langle \cos 2\theta_n \rangle^{-\frac{1}{2}} \cos \theta_n - 1| < \frac{1}{|k_n'|^2s(n+10)^4 - 1}.
\end{equation}
\[(6.5) \quad |(\cos 2\theta_n)^{-\frac{1}{2}} \sin \theta_n| \in \left(\frac{1}{|k_n^e|^s(n+1)^2}, \frac{2}{|k_n^e|^s(n+1)^2}\right).\]

Then we can get the following estimation on \(G'_n:\)
\[(6.6) \quad \|G'_n - \text{Id}\|_s < \frac{2}{(n+1)^2},\]

By (6.6) we know that
\[(6.7) \quad \|B'_n - B'_{n+1}\|_s = \|B'_n (G'_n - \text{Id})\|_s \leq \frac{C(s,d)}{(n+1)^2}.\]

Thus, there exists \(B(\cdot) \in C^s(\mathbb{T}^d, \text{SL}(2, \mathbb{R}))\) such that \(\|B'_n - B\|_s \to 0.\) Since \(\|\tilde{A}_n - A'_{n+1}\| < \|(k_{n+1}, \alpha)\|_T,\) one can show that
\[(6.8) \quad B(\cdot + \alpha)^{-1}A(\cdot)B(\cdot) = \tilde{A} = \begin{pmatrix} e^{2\pi i \rho} & 0 \\ 0 & e^{-2\pi i \rho} \end{pmatrix},\]

where \(e^{2\pi i \rho} = \lim_{n \to \infty} e^{k_n^e(\alpha)},\) i.e. \((\alpha, A(\cdot))\) is \(C^s\)-reducible.

Next we will prove \((\alpha, A(\cdot))\) is not \(C^{s+1}\)-reducible. First, we have the following:

**Claim 1.**
\[(6.9) \quad \|B(\cdot)\|_{s+1} = \infty.\]

**Proof.** Suppose that \(\|B(\cdot)\|_{s+1} = C < \infty.\) Then we know \(\|\hat{B}(k)\| \leq C|k|^{-(s+1)}\) for all \(k \in \mathbb{Z}^d.\) By (6.7) we know
\[(6.10) \quad \|\hat{B}(k) - \hat{B}'(k)\| \leq \frac{C(s,d)}{|k|^s(n+1)^2}.\]

We now analyze the structure of \(B_n.\) Let
\[u_n := (\cos 2\theta_n)^{-\frac{1}{2}} \cos \theta_n - 1,\]
\[v_n := (\cos 2\theta_n)^{-\frac{1}{2}} \sin \theta_n,\]

and
\[J_k := \begin{pmatrix} 0 & e_k \\ e_{-k} & 0 \end{pmatrix}, \quad G_k := \begin{pmatrix} e_k & 0 \\ 0 & e_{-k} \end{pmatrix}.\]

Therefore, we have
\[B'_n = \prod_{j=0}^{n-1} ((1 + u_j)I + v_jJ'_k) = \sum_{0 \leq j_1, \ldots, j_{l} \leq n-1, j \neq j_1, \ldots, j_l} (\prod_{m=1}^{l} v_{j_m}J'_{j_m}).\]
By direct calculation we know if $l$ is even
\[ \prod_{m=1}^{l} J'_{jm} = \mathcal{G}_{\sum_{m=1}^{l} (-1)^{m-1} k'_{jm}}, \]
otherwise
\[ \prod_{m=1}^{l} J'_{jm} = J_{\sum_{m=1}^{l} (-1)^{m-1} k'_{jm}}. \]

Now we need the following crucial observation: given two set $Q := \{i_1, \ldots, i_r\}$, $P := \{j_1, \ldots, j_l\}$ where $i_n, j_m \in \mathbb{N}$. Then we have
\[ \sum_{m=1}^{l} (-1)^{m-1} k'_{jm} = \sum_{n=1}^{r} (-1)^{n-1} k'_{in}, \]
if and only if $Q = P$. This holds since by our construction (6.2), we have $|k'_{n+1}| \gg \sum_{j=0}^{n} |k'_j|$, if (6.11) satisfied then $(-1)^{l-1} k'_{j_l} = (-1)^{r-1} k'_{i_r}$ must happen. Iterating this step gives $P = Q$.

This observation implies that for any $0 \leq j_1 < \cdots < j_l \leq n - 1$ we have
\[ \widehat{B}'_n (\sum_{m=1}^{l} (-1)^{m-1} k'_{jm}) = \begin{cases} \prod_{j \neq j_1, \ldots, j_l} (1 + u_j) \prod_{m=1}^{l} v_{jm} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & l \text{ even}, \\ \prod_{j \neq j_1, \ldots, j_l} (1 + u_j) \prod_{m=1}^{l} v_{jm} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & l \text{ odd}. \end{cases} \]

In particular, for $j \leq n - 1$ we have
\[ \widehat{B}'_n (k'_j) = v_j (\prod_{i \neq j, 0 \leq i \leq n-1} (1 + u_i)) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \]

By equations (6.4) and (6.5), for any $n \geq 1$, $j \leq n - 1$ we have
\[ \|\widehat{B}'_n (k'_j)\| > \frac{1}{2 |k'_j|^{s(j+10)^2}}. \]

Because of (6.10) and (6.12), if $n \geq C(s, d)$ which is large enough, we have
\[ \|\widehat{B}(k'_j)\| \geq \|\widehat{B}'_n (k'_j)\| - \|\widehat{B}(k'_j) - \widehat{B}'_n (k'_j)\| > \frac{1}{4 |k'_j|^{s(j+10)^2}}. \]

By the assumption $\|B\|_{s+1} < C$ we have
\[ \frac{1}{4 |k'_j|^{s(j+10)^2}} < \frac{C}{|k'_j|^{s+1}}, \]
which contradicts (6.2). \hfill \Box

Now we finish the proof that the cocycle $(\alpha, A(\cdot))$ is not $C^{s+1}$-reducible, proceeding by contradiction. Suppose that there exists $B_1(\cdot) \in C^{s+1}(2T^d, \text{SL}(2, \mathbb{R}))$ such that
\[ B_1^{-1}(\cdot + \alpha) A(\cdot) B_1(\cdot) = \tilde{B}_1, \]
where $\tilde{B}_1 \in \text{SL}(2, \mathbb{R})$. Then, $\tilde{B}_1$ is diagonalizable. Otherwise $\tilde{B}_1$ can be conjugate to a Jordan block. Then the $A(\cdot; n)$ has linear growth on $n$ which contradicts equation (6.8). Combining equations (6.8) and (6.13) we have
\begin{equation}
B_2(\cdot + \alpha) = \tilde{A}B_2(\cdot)\tilde{B}_1^{-1},
\end{equation}
where $B_2 = B^{-1}B_1$. Define the linear operator on $M(2, \mathbb{C})$:
\[ L(Y) := \tilde{A}Y\tilde{B}_1^{-1}, \]
where $Y \in M(2, \mathbb{C})$. Applying the Fourier transform to eq. (6.14), we have
\[ \epsilon_k(\alpha)\tilde{B}_3(k) = L(\tilde{B}_2(k)). \]
Since $B_2(\cdot) \neq 0$ there exists $2l_0 \in \mathbb{Z}^d$ such that $\tilde{B}_2(l_0) \neq 0$. Thus $\epsilon_{l_0}(\alpha)$ is an eigenvalue of $L$. Therefore, the two eigenvalues of $\tilde{B}_1$ are $\{\exp\{2\pi i(l_0, \alpha) + \rho\}, \exp\{-2\pi i(l_0, \alpha) + \rho\}\}$ or $\{\exp\{2\pi i(\rho - l_0, \alpha)\}, \exp\{2\pi i(l_0, \alpha) - \rho\}\}$. Without loss of generality, we assume the eigenvalues to be the former. Since $\tilde{B}_1$ is diagonalizable there exists $\tilde{D} \in \text{SL}(2, \mathbb{C})$ such that
\[ \tilde{D}^{-1}\tilde{B}_1\tilde{D} = \text{diag}\{\exp\{2\pi i(l_0, \alpha) + \rho\}, \exp\{-2\pi i(l_0, \alpha) + \rho\}\}. \]
Let $B_3 := B_1\tilde{D}\begin{pmatrix} \epsilon_{l_0}(\cdot) & 0 \\ 0 & e^{-\epsilon_{l_0}(\cdot)} \end{pmatrix}$. We have
\begin{equation}
B_3^{-1}(\cdot + \alpha)A(\cdot)B_3(\cdot) = \tilde{A}.
\end{equation}
Then there are two cases:

**Case I:** For all $2k \in \mathbb{Z}^d$ where $k \neq 0$, we have $2\rho \neq \langle k, \alpha \rangle \text{ mod } \mathbb{Z}$.
In this case, combining (6.8) and (6.15) we have
\[ B_4(\cdot + \alpha) = \tilde{A}B_4(\cdot)\tilde{A}^{-1}, \]
where $B_4 = B^{-1}B_3$. In the frequency domain, this implies that for every $2k \in \mathbb{Z}^d$ we have
\[ \epsilon_k(\alpha)\tilde{B}_4(k) = \tilde{A}B_4(k)\tilde{A}^{-1}. \]
However, the eigenvalues of the operator $L'(Y) := \tilde{A}Y\tilde{A}^{-1}$ are $\{1, 1, e^{4\pi i\rho}, e^{-4\pi i\rho}\}$, and since $2\rho \neq \langle k, \alpha \rangle$ this implies that $B_4 \equiv \text{constant}$. Thus we have
\[ B = B_1\tilde{D}\begin{pmatrix} \epsilon_{l_0}(\cdot) & 0 \\ 0 & e^{-\epsilon_{l_0}(\cdot)} \end{pmatrix}B_4^{-1}. \]
Due to $\tilde{D}, B_4 \in \text{SL}(2, \mathbb{C})$ and $\|B_1\|_{s+1} < \infty$, we have $\|B\|_{s+1} < \infty$ which contradicts eq. (6.9).

**Case II:** There exists $2l_1 \in \mathbb{Z}^d, l_1 \neq 0$, such that $2\rho = \langle l_1, \alpha \rangle \text{ mod } \mathbb{Z}$.
In this case, we just need to set $B' = B\begin{pmatrix} \epsilon_{l_1}(\cdot) & 0 \\ 0 & e^{-\epsilon_{l_1}(\cdot)} \end{pmatrix}$. Then we have
\[ B'^{-1}(\cdot + \alpha)A(\cdot)B'(\cdot) = \text{Id}. \]
Moreover, for any $s \in Z^+$, for any $\eta > 0$, there exists $(\alpha, A') \in \mathcal{AR}_{\alpha} \setminus \mathcal{UH}_\alpha$ with $\|A'(-) - S_E^V(-)\|_{h_s} \leq \eta$ such that

$$B(\cdot + \alpha)^{-1}A'(\cdot)B(\cdot) = \begin{pmatrix} e^{2\pi i \rho} & 0 \\ 0 & e^{-2\pi i \rho} \end{pmatrix},$$

Moreover, $\|B(\cdot)\|_s < \infty$ while $\|B(\cdot)\|_{s+1} = \infty$. By Lemma 5.2, there exist $\tilde{V}' \in C^\omega_C(\mathbb{T}^d, \mathbb{R})$ with $|\tilde{V}'|_{h_s} < \eta$ and $Z' \in C^\omega_C(\mathbb{T}^d, \text{PSL}(2, \mathbb{R}))$ such that

$$Z'(\cdot + \alpha)^{-1}A'(\cdot)Z'(\cdot) = S_E^V'(\cdot).$$

Let $B' = Z'B$, then

$$B'(\theta + \alpha)^{-1}S_E^V'(\theta)B'(\theta) = \begin{pmatrix} e^{2\pi i \rho} & 0 \\ 0 & e^{-2\pi i \rho} \end{pmatrix},$$

and write $B'(\theta) = \begin{pmatrix} z_{11}(\theta) & z_{12}(\theta) \\ z_{21}(\theta) & z_{22}(\theta) \end{pmatrix}$, then we have

$$E - \tilde{V}_k(\theta)z_{11}(\theta) = z_{11}(\theta - \alpha)e^{-2\pi i \rho} + z_{11}(\theta + \alpha)e^{2\pi i \rho}.$$  \tag{6.16} \label{eq:6.16}

Applying the Fourier transformation to eq. (6.16), we get

$$\sum_{m \in \mathbb{Z}^d} \tilde{V}'(m)\hat{z}_{11}(n - m) + 2\cos 2\pi(p + n\alpha)\hat{z}_{11}(n) = E\hat{z}_{11}(n),$$

i.e., $L_{\tilde{V}',p,\alpha}\hat{z}_{11} = E\hat{z}_{11}$, moreover, since $\|B'(\cdot)\|_s < \infty$ while $\|B'(\cdot)\|_{s+1} = \infty$, then the eigenfunction $\{\hat{z}_{11}(n)\}_{n \in \mathbb{Z}} \in h^s \setminus h^{s+1}$.

### 7. Appendix: Proof of Proposition 4.2:

We follow the proof of the proposition as given in [22, 47]. An alternative proof can be obtained by use of the KAM normal form, following the method introduced in [41]. We need the following result, proved in [22, 47].

**Proposition 7.1.** Let $\alpha \in DC_\sigma(\gamma, \tau), \sigma > 0$. Suppose that $A \in SL(2, \mathbb{R}), f \in C^\omega_C(\mathbb{T}^d, \text{sl}(2, \mathbb{R}))$. Then for any $h_+ < h$, there exists constants $C_0$ and $D_0 = D_0(\alpha', \tau, d)$ such that if

$$\|f\|_h \leq \epsilon \leq \frac{D_0}{\|A\|_{C_0}}\left(\min\{1, \frac{1}{h}\}(h - h_+)\right)^{C_0},$$

then there exists $B \in C^\omega_C(2\mathbb{T}^d, SL(2, \mathbb{R}))$, $A_+ \in SL(2, \mathbb{R})$ and $f_+ \in C^\omega_C(\mathbb{T}^d, \text{sl}(2, \mathbb{R}))$ such that

$$B^{-1}(\theta + \alpha)Ae^{f(\theta)}B(\theta) = A_+e^{f_+(\theta)}.$$
More precisely, letting \( \text{spec}(A) = \{ e^{2\pi i \xi}, e^{-2\pi i \xi} \} \), \( N = \frac{2}{\eta - \eta_*}|\ln \epsilon| \), we can distinguish two cases:

- **(Non-resonant case)** if for any \( n \in \mathbb{Z}^d \) with \( 0 < |n| \leq N \), we have
  \[
  \|2\xi - < n, \alpha >\|_{R/Z} = e^{\frac{1}{16}}
  \]
  then
  \[
  \|B - id\|_{h+} \leq e^{\frac{1}{16}}, \quad \|f_+\|_{h+} \leq e^{2}.
  \]
  Moreover, \( \|A_+ - A\| < 2\epsilon \).

- **(Resonant case)** if there exists \( n_\ast \) with \( 0 < |n_\ast| \leq N \) such that
  \[
  \|2\xi - < n_\ast, \alpha >\|_{R/Z} < e^{\frac{1}{16}}
  \]
  then
  \[
  \|B\|_{h+} \leq |\ln \epsilon|^\frac{1}{2} e^{-\frac{\eta_+}{\eta - \eta_*}}, \quad \|f_+\|_{h+} < ee^{-\frac{\eta_+}{\eta - \eta_*}}.
  \]
  Moreover, \( A_+ \) can be written as \( A_+ = e^{\tilde{A}_+} \) with \( |\tilde{A}_+| \leq e^{\frac{1}{16}} \).

The proof of proposition 4.2 follows by iteration of the proposition here above. Consider the initial cocycle \( (\alpha, A_0 e^{f_0(\theta)}) \), where \( A_0 \in SL(2, \mathbb{R}) \), \( f_0 \in C^\omega_h(T^d, sl(2, \mathbb{R})) \). Without loss of generality, assume that \( h < 1 \), as well as that

\[
\|f_0\|_h \leq \epsilon \ast \leq D_0 \|A_0\|_{C_0^\omega}(\frac{h - \tilde{h}}{\frac{h}{8}})^{C_0 \tau},
\]

where \( D_0 = D_0(\kappa', \tau, d) \) is the constant defined in Proposition 7.1. Then we can define the sequence inductively. Let \( \epsilon_0 = \epsilon \ast \), \( h_0 = h \), and assume that we are at the \((j + 1)\)th KAM step, i.e. we have already constructed \( B_j \in C^\omega_{h_j}(T^d, PSL(2, \mathbb{R})) \) such that

\[
B_j^{-1}(\theta + \alpha) A_0 e^{f_0(\theta)} B_j(\theta) = A_j e^{f_j(\theta)},
\]

where \( A_j \in SL(2, \mathbb{R}) \) has eigenvalues \( e^{\pm i \xi_j} \) and

\[
\|B_j\|_{h_j} \leq \epsilon_j^{\frac{h - \tilde{h}}{4h}}, \quad \|f_j\|_{h_j} \leq \epsilon_j
\]

for some \( \epsilon_j \leq \epsilon_0^{\frac{2j}{4j + 1}} \), and define

\[
h_j - h_{j+1} = \frac{h - \frac{h + \tilde{h}}{2}}{4j + 1}, \quad N_j = \frac{2|\ln \epsilon_j|}{h_j - h_{j+1}}.
\]

By our choice of \( \epsilon_0 \), one can check that

\[
\epsilon_j \leq \frac{D_0 \|A_j\|_{C_0^\omega}(h_j - h_{j+1})^{C_0 \tau}}{\epsilon \ast}.
\]

Indeed, \( \epsilon_j \) on the left side of the inequality decays at least super-exponentially with \( j \), while \( (h_j - h_{j+1})^{C_0 \tau} \) on the right side decays exponentially with \( j \).
Note that (7.2) implies that Proposition 7.1 can be applied iteratively, consequently one can construct

\[ \bar{B}_j \in C_{\omega_{h_j+1}}(\mathbb{T}^d, PSL(2, \mathbb{R})), \quad A_{j+1} \in SL(2, \mathbb{R}), \quad f_{j+1} \in C_{h_{j+1}}(\mathbb{T}^d, sl(2, \mathbb{R})) \]

such that

\[ \bar{B}_j^{-1} \theta A_j e^{f_j(\theta)} \bar{B}_j(\theta) = A_{j+1} e^{f_{j+1}(\theta)}. \]

More precisely, we can distinguish two cases:

**Non-resonant case:** If for any \( n \in \mathbb{Z}^d \) with \( 0 < |n| \leq N_j \), we have

\[ \|2\xi_j - <n, \alpha>\|_{\mathbb{R}/\mathbb{Z}} \geq \epsilon_j^{\frac{1}{15}}, \]

then

\[ \|\bar{B}_j - id\|_{h_{j+1}} \leq \epsilon_j^{\frac{1}{2}}, \quad \|f_{j+1}\|_{h_{j+1}} \leq \epsilon_j^2 := \epsilon_{j+1}, \quad \|A_{j+1} - A_j\| \leq 2 \epsilon_j. \]

Let \( B_{j+1} = B_j(\theta) \bar{B}_j(\theta) \), we have

\[ B_{j+1}^{-1} \theta A_0 e^{f_0(\theta)} B_{j+1}(\theta) = A_{j+1} e^{f_{j+1}(\theta)}, \]

with estimate

\[ \|B_{j+1}\|_{h_{j+1}} \leq 2 \epsilon_j^{-\frac{h-\tilde{h}}{4k}} \leq \epsilon_{j+1}^{-\frac{h-\tilde{h}}{4k}}. \]

**Resonant case:** If there exists \( n_j \) with \( 0 < |n_j| \leq N_j \) such that

\[ \|2\xi_j - <n_j, \alpha>\|_{\mathbb{R}/\mathbb{Z}} \leq \epsilon_j^{\frac{1}{15}}, \]

with estimate

\[ \|\bar{B}_j\|_{h_{j+1}} \leq |\ln \epsilon_j|^{\frac{h_{j+1}}{h_j - h_{j+1}}}, \quad \|f_{j+1}\|_{h_{j+1}} \leq \epsilon_j e^{-h_j \epsilon_j^{-\frac{1}{15}}} := \epsilon_{j+1}. \]

Moreover, we can write \( A_{j+1} = e^{A_j} \) with estimate

\[ |A_j| < 2 \epsilon_j^{\frac{1}{15}}. \]

Let \( B_{j+1}(\theta) = B_j(\theta) \bar{B}_j(\theta) \), then we have

\[ B_{j+1}^{-1} \theta A_0 e^{f_0(\theta)} B_{j+1}(\theta) = A_{j+1} e^{f_{j+1}(\theta)}, \]

with

\[ \|B_{j+1}\|_{h_{j+1}} \leq \epsilon_j^{-\frac{h-\tilde{h}}{4k}} |\ln \epsilon_j|^{\frac{h_{j+1}}{h_j - h_{j+1}}} \leq \epsilon_{j+1}^{-\frac{h-\tilde{h}}{4k}}. \]

The last inequality is possible since by our choice \( \epsilon_{j+1} = \epsilon_j e^{-h_j \epsilon_j^{-\frac{1}{15}}}. \)
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